HURWITZ STACKS OF GROUPS EXTENSIONS AND IRREDUCIBILITY

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ABSTRACT. We study the irreducible components of special loci of curves whose group of symmetries is given as group extension, and which has rational ramification. Using mixed étale cohomology, we show that the local ramification data joined to the extension class of the group implies a relative irreducible results. We mention some potential arithmetic applications for non-Abelian special loci and Hurwitz spaces.

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1. Introduction

This paper is motivated by the study of the stack arithmetic of the moduli spaces of curves with \( m \)-marked points \( \mathcal{M}_{g, [m]} \), i.e. the study of the Gal(\( \overline{\mathbb{Q}}/\mathbb{Q} \))-action on its stack inertia groups \( I_{\mathcal{M}_g}(\overline{\mathbb{Q}}) \cong G \), and more precisely by its geometric formulation in terms of irreducible components of special loci \( \mathcal{M}_{g, [m]}(G) \) of curves with automorphism group \( G \). As detailed in [CM14] §1.1, this approach relies on determining algebraic invariants in family of the components. Via the identification of the normalisation \( \mathcal{M}_g(G) \cong \mathcal{M}_g[G]/\text{Aut}(G) \), this question can be reformulated at the level of Hurwitz spaces \( \mathcal{M}_g[G] \) of \( G \)-covers, and thus share some similarity, in motivations and techniques, with the role of \( \mathcal{M}_g[G] \) in the Regular Inverse Galois Problem. After a brief reminder on the arithmetic of Hurwitz Spaces and Special Loci, we present our result in relation to the question of determining irreducible components of these space, as well as potential applications in algebraic geometry. This paper is an intermediate step on this on-going project.

1.1. Hurwitz Spaces and Special Loci in Arithmetic Geometry. Let \( G \) be a finite group, and denote by \( \mathcal{M}_g[G] \) and \( \mathcal{M}_g(G) \) respectively the Hurwitz stack of \( G \)-covers and the special loci of \( G \) in \( \mathcal{M}_g \) – i.e. the moduli stack of curves admitting a \( G \)-action. Arithmetic motivations for the study of the irreducible components of those stack comes from two similar questions in the study of the absolute Galois group of rational numbers Gal(\( \overline{\mathbb{Q}}/\mathbb{Q} \)).

The Inverse Galois Problem – that is to realize \( G \) as a quotient of the absolute Galois group of rationals Gal(\( \overline{\mathbb{Q}}/\mathbb{Q} \)) – turned to a more geometric questions of realizing \( G \) as regular Galois cover of \( \mathbb{P}^1 - \{0, \infty \} \) and the existence of a \( K \)-rational point in \( \mathcal{M}_g[G] \) – the Regular Inverse Galois Problem. Weakening the Diophantine condition to the question of the existence of rational irreducible components in \( \mathcal{M}_g[G](K) \) led to the construction of Harbater-Mumford irreducible components that are defined over \( \mathbb{Q} \) – see [Fri95], and also [DE06] for the realization of every center-free group \( G \) over \( \mathbb{F}_q \) – see [Wew98], and also [CM15] for the realization of a projective system of finite groups in terms of Hurwitz towers.

The Geometric Galois Representations study actions of Gal(\( \overline{\mathbb{Q}}/\mathbb{Q} \)) on the étale fundamental group of moduli stacks of curves Gal(\( \overline{\mathbb{Q}}/\mathbb{Q} \)) \( \rightarrow \text{Aut}[\pi_1^\text{et}(\mathcal{M}_{g, [m]} \times \overline{\mathbb{Q}})] \). The question of the \( G \)-arithmetic appears through the stack inertia \( I_{\mathcal{M}_g}(\overline{\mathbb{Q}}) \cong \pi_1^\text{et}(\mathcal{M}_{g, [m]} \times \overline{\mathbb{Q}}) \) or equivalently the irreducible components of special loci \( \mathcal{M}_{g, [m]}(G) \) – see [CM14] §1 for details. This geometric approach leads to the description of the Gal(\( \overline{\mathbb{Q}}/\mathbb{Q} \))-action for respectively \( p \)-groups and cyclic groups [CM15] Theorem 5.9, [CM14] Theorem 4.8, it is given by \( \chi \)-conjugation, i.e for \( G = <\gamma> \):

\[
\sigma.\gamma = h_\sigma.\gamma^{\chi(\sigma)}.h^{-1} \quad \text{for } \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}).
\]

The general context being given by the inertia stratification in local gerbes bounded by the automorphism group of objects, this raises the question of describing this Gal(\( \overline{\mathbb{Q}}/\mathbb{Q} \))-action on higher inertia groups. A first step in this direction is given by the definition of rational invariant of irreducible components of the special loci in family – see [CM15] Theorem 4.3 for the cyclic case and quotient curve of any genus.

Our approach deals exploits the arithmetic of \( G \)-covers through the rationality of the ramification locus. We thus deal with the stack \( \mathcal{M}_g(G)_{\text{rat}} \) of special loci with rational
1.2. Characterizing Irreducible Components of Special Loci. The construction of invariants of irreducible components of $M_g(G)$ relies on a long tradition of geometric invariants of $G$-covers, either by group theoretic methods in terms of equivalence classes of generating vectors – see [Bro90] for $G$-curves of genus 2 and 3, and [BW07] for Abelian groups –, or by geometric methods in terms of equivalence classes of monodromy representations of the quotient curve $D$.

Such numerical invariants include: the genus of the quotient curve $D$, the Nielsen invariants counting the number of local monodromy belonging to a set of conjugacy classes, and more recently a global second homology invariant [CLP15] which refines the previous ones in the case of $G$-action with étale factorization and $G = D_n$ – see ibid.

In the case of $M_g(G)$, i.e. for families of $G$-curves $C/S$, construction of local invariants is achieved in terms of étale cohomology, see [CM15] §3, with similar results for $G$ cyclic (notice that the étale factorization property still appears in the arithmetic study, see Theorem 5.4 ibid.) We here complete this approach by the use of mixed cohomology $H^\bullet(X_{et}, \mathbb{H})$ of [Gro68] Chap. V for $(H; G)$-covers, $G$ being an extension of $P$ by $H$: this determines both the ramification and the global parts of the covers within an algebraic deformation – see §4.2 and §4.3 for the construction of this deformation – and leads to the main result of this paper – see Theorem 4.1.

Theorem (A). Let $G$ be a finite group and $\diamond$ a $G$-Hurwitz data. If $\diamond$ contains $H < G$ as isotropy group, the rational quotient morphism $\Phi_{\diamond}^{rat}: M_g[G]^{rat}/\text{Aut}_{G/H, \diamond}(H; G) \to M_g'[G/H]^{rat}/\text{cores}_{G/H}(\diamond)$ is irreducible.

Our approach relies more precisely on the compactification of $G$-equivariant torsors of [Mau16] to build this deformation, see Proposition 4.2.1 and on the identification of $(H; G)$-covers to a certain stack $[R^1 f_* \mathbb{H}^G/H]$ of $H$-torsors, see §4.2.2. The result follows the irreducibility of a relative quotient morphism, see Theorem 4.1 then a discussion on the existence of a $H$-ramification point on top of a $P$-ramification one. Notice that the deformation of the global part relies on an analytic result that does not seem avoidable since it is already a key ingredient in proving the irreducibility of $M_g$ in [DM69] and of the special loci $M/Z/nZ]_{g,[m],\diamond}$ in [Cor87, Cat12] then [CM15].

This plan of the paper goes as follow. The definition of rational Hurwitz datas and rational Hurwitz loci, as well as moduli properties by (co)restriction and under Aut-quotient are presented in §2; the development of mixed cohomology and their relation with $(H; G)$-covers and torsors occupies §3; the main result is established §4 following the deformation of $(H; G)$-covers with $H$-ramification – with algebraic and analytical results. An Appendix collect the properties of the various groups $\text{Aut}_P(G; H)$.

The question of a similar result for (unrational) special loci $M_g,[m]_n(G)\diamond$ with marked points, as well as arithmetic results for such higher stack inertia stratas, or specific cases such as $G = D_n$, will be the subject of a future study – see [CM14] on the various obstacles that should be overcome. In another direction, an enlightened reader will certainly have noticed the Tate-like appearance of the Galois action of Eq. (1.1.1). This remark gives an
additional motivation to this study for the definition of a “stack weight” on higher stack inertia stratas.

2. Hurwitz Characters and Normality

Let \( G \) be a finite group and \( g \geq 0 \) an integer. We consider the stack \( \mathcal{M}_g[G] \) of smooth proper curves of genus \( g \) with a given faithful \( G \)-action, which is a Deligne-Mumford stack over \( \mathbb{Z}[1/|G|] \). In what follows, the scheme \( S \) belongs to the category of \( \text{Spec} \mathbb{Z}[1/|G|]- \) schemes or \( \text{Spec} \mathbb{Q} \)-schemes to avoid any wild ramification difficulties.

2.1. Rational Hurwitz Invariants. Let \( C/S \) be a smooth proper curve endowed with a faithful \( G \)-action \( \iota : G \hookrightarrow \text{Aut}_S(C) \), i.e. faithful in every fiber. After some reminders on the reduced ramification locus \( \text{Ram}_S(C, \iota) \) of \( (C/S, \iota) \), we define a notion of rational Hurwitz datas over \( S \) which are shown to reflect rational and arithmetic properties of \( (C/S, \iota) \) and is a variant of the geometric ones introduced in [Fri77] and [Ser08].

2.1.1. Let \( \mu_{|G|} \) denote the group scheme of \( |G| \)-roots of unity, and consider a geometric fiber \( C_{\bar{s}} \) of \( C/S \). For every ramified point \( x \in C_{\bar{s}} \), denote by \( G_x \) its isotropy group. This is a cyclic group since the action is tame, and it defines a Hurwitz character via the cotangent representation:

\[
\chi_x : G_x \to \mu_{|G|}(k)
\]

where \( \bar{s} = \text{Spec} (k) \), i.e. locally given by \( \chi_x(h) = h(\varpi)/\varpi \mod \varpi^2 \), where \( \varpi \) is a uniformising parameter of \( C \) at \( \bar{s} \). This representation is primitive, i.e. of order \( |G_x| \).

In the relative case, denoting \( C^H \) the scheme of fixed points, one defines a ramification divisor

\[
\text{Ram}(C, \iota) = \sum_{H \neq 1} \phi(|H|)C^H
\]

which is a relative Cartier divisor over \( S \) – we refer to [BR11] §3 for the original approach and details.

Suppose that the support of the ramification divisor is given by disjoint sections \( (e_i)_{i \in I} \). By the same arguments as above, one attaches to each \( e_i \) a cyclic stabilizer group \( G_i \) and a primitive character \( \chi_i : G_i \to \mu_{|G_i|} \), which are locally constants over \( S \) – see especially Lemma 3.1.3 Ibid. If \( S \) is the spectrum of an algebraically closed field, these characters are induced by the action of \( G_i \) on the tangent space of \( C \) at \( C(e_i) \).

Let us introduce the stack \( \mathcal{M}_{g,I}[G]^{\text{rat}} \) of \( G \)-covers with rational ramification locus.

**Definition 2.1.1.** Let \( G \) be a finite group and \( I \) a finite \( G \)-set of cardinal \( m \). We denote by \( \mathcal{M}_{g,I}[G]^{\text{rat}} \) the stack classifying equivariant \( G \)-curves \( (C/S, \iota) \) together with disjoint sections \( \{e_i : S \to C\}_{i=1,...,m} \).

For such a \( G \)-curve, one has

\[
\text{Ram}(C, \iota) = \bigoplus_{i=1}^{m} \phi(|G_i|)e_i,
\]

where \( G_i \) is the stabiliser of \( e_i \) in \( G \).
Denoting $|I| = m$, the stack $\mathcal{M}_{g,I}[G]^{rat}$ is naturally a closed substack of the algebraic stack of $m$-pointed $G$-curves $\mathcal{M}_{g,m}[G]$. Moreover, as the ramification divisor of every $G$-curve $(C/S, \iota)$ splits up to an étale base change, the morphism given by forgetting the ramification divisor

$$\mathcal{M}_{g,I}[G]^{rat} \to \mathcal{M}_g[G]$$

is étale.

The equation (2.1.1) tells us in particular that the genus $g'$ of the quotient curve $C/G$ is fixed, as it is given by the Hurwitz formula:

$$2g - 2 = (2g' - 2)|G| + \sum_{i \in I} (|G_i| - 1)$$

In a similar way to the moduli stack of curves, one proves that the stack $\mathcal{M}_{g,I}[G]^{rat}$ is a smooth Deligne-Mumford algebraic stack as GIT quotient – see [BR11] §6.3. We recall that $\mathcal{M}_g[G]$ (resp. $\mathcal{M}_{g,I}[G]^{rat}$) is not a substacks of $\mathcal{M}_g$ (resp. of $\mathcal{M}_{g,m}$), and that forgetting the $G$-action corresponds to the normalization morphism $\mathcal{M}_g[G]/\text{Aut}(G) \to \mathcal{M}_g(G)$ (resp. idem for $m$-pointed curves) – see [Rom11] §3.4 and [CM15] §2.1 for details.

The notion of Hurwitz characters encodes some geometric and Galois properties that we formalize in the next section.

2.1.2. Let $(C/S, \iota) \in \mathcal{M}_{g,I}[G]$ be an equivariant curve, with isotropy groups $\{G_i\}_{I}$, to which can be associated some primitive characters $\{\chi_i\}_I$ in a functorial way as above. We recall that for a $\gamma \in G$, the $G$-action on $C$ induces conjugacy morphisms $\psi_{\gamma,i}: G_i \to G_{\gamma,i}$ on the isotropy groups which on the characters translates as follow:

(2.1.2) $$G_{\gamma,i} = \gamma^{-1}G_i\gamma$$ and $\chi_{\gamma,i} = \chi_i \circ \psi_{i,\gamma}$ for $\gamma \in G$.

The following is a variant of [Ser08] [Fri17] which originally defines Hurwitz datas on geometric fiber and as conjugacy classes of Hurwitz datas.

**Definition 2.1.2.** Let $G$ be a finite group and $I$ a finite $G$-set of cardinal $m$. A Hurwitz data $\hat{\phi}$ associated to $G$ and $I$ is a $m$-tuple $((G_1, \chi_1), \ldots, (G_m, \chi_m))$ of cyclic subgroups $G_i < G$ and primitive $G_i$-characters $\chi_i \in \text{Isom}(G_i, \mu_{|G|})$ such that equations (2.1.2) are satisfied.

We denote by $\hat{\phi}_G^I$ the set of $\hat{\phi}_G^I$-Hurwitz data associated to $G$ and $I$. For $\hat{\phi} \in \hat{\phi}_G^I$, we write $\text{stab}(\hat{\phi})$ (resp. $\text{char}(\hat{\phi})$) for the list of $G$-isotropy groups (resp. $G$-primitive character), and we call $I$ the set of indices of $\hat{\phi}$.

The Hurwitz data attached to a $G$-cover of $\mathcal{M}_{g,I}[G]^{rat}$ encodes all its classical elementary invariants (eg. the ramification indices and the genus of the quotient curve $C/G$). For a cyclic group $G$, it also coincides with the $\gamma$-type introduced in [CM14] §2.2 whose relation to the classical branching data $\{k_j\}_I$ of a geometric cover is given by Lemma 2.4 Ibid.

Let $\hat{\phi}$ be a $\hat{\phi}_G^I$-Hurwitz data, we denote by $\mathcal{M}_{g,I}[G]^{rat}_{\hat{\phi}}$ the substack of $\mathcal{M}_{g,I}[G]^{rat}$ whose sections are $G$-curves with Hurwitz datas $\hat{\phi}$.

**Proposition 2.1.3.** The stack $\mathcal{M}_{g,I}[G]^{rat}_{\hat{\phi}}$ is open and union of connected components in $\mathcal{M}_{g,I}[G]^{rat}$. 
This immediately follows first from the local constance of the Hurwitz data by Lemma 3.1.3, then from the stability under specialisation.

Recall that the set $\bigotimes_{G}^{I}$ is naturally endowed with a $G$-action through the action of $G$ on $I$ induced by the conjugacy action of Eq. (2.1.2).

**Definition 2.1.4.** A Hurwitz data $\bigotimes = ((G_{1}, \chi_{1}), \ldots, (G_{m}, \chi_{m}))$ is said to be normal if for all $i \in I$ and $\gamma \in G$ we have $G_{\gamma.i} = G_{i}$ and $\chi_{\gamma.i} = \chi_{i}$.

The choice of a representative in each class in $\bigotimes_{G}^{I}/G$ is then sufficient to recover the whole Hurwitz data up to an order on $I$ only. For a normal Hurwitz data $\bigotimes$ and for $\gamma \in G$, the difference between $\bigotimes$ and $\gamma.\bigotimes$ is only the permutation of $I$ given by $\gamma$.

**Proposition 2.1.5.** Let $\bigotimes = ((G_{1}, \chi_{1}), \ldots, (G_{m}, \chi_{m}))$ be a $\bigotimes_{G}^{I}$-Hurwitz data. If $G_{i} < Z(G)$ for all $i \in I$, then $\bigotimes$ is normal.

In particular, every Hurwitz data of an Abelian group is normal.

**Proof.** It follows from Eq. (2.1.2) that a Hurwitz data $\bigotimes$ is normal if and only if for all $i \in I$ one has:

$$\chi_{i} = \chi_{\gamma.i}, \forall \gamma \in G \text{ i.e. } \begin{cases} G_{i} \leq G \text{ and } \\ \chi_{i}(h) = \chi_{i}(\gamma^{-1}h\gamma), \forall h \in G_{i}, \forall \gamma \in G, \end{cases}$$

which by the injectivity of the characters, is equivalent to $G_{i} < Z(G)$ for all $i \in I$. \hfill $\square$

In the case of curves in $\mathcal{M}_{g}[G]$ with non-rational ramification locus, an unordered Hurwitz data is given as a class $\bar{\bigotimes} \in \bigotimes_{G}^{I}/G$, and we define $\mathcal{M}_{g,[G]}^{\bar{\bigotimes}}$ as the locus of $G$-curves in $\mathcal{M}_{g}[G]$ that are étale locally isomorphic to a curve inside $\mathcal{M}_{g,\{\bigotimes\}}$, for a certain $\bigotimes \in \bar{\bigotimes}$. The stack $\mathcal{M}_{g}[G]^{\bar{\bigotimes}}$ is naturally a Deligne-Mumford algebraic substack of the curves with $m$-marked points $\mathcal{M}_{g,[m]}$. As the ramification divisor splits étale locally in a $G$-equivariant way, we have an étale surjective morphism

$$\prod_{\bigotimes \in \bar{\bigotimes}} \mathcal{M}_{g,\{\bigotimes\}}^{\text{rat}} \to \mathcal{M}_{g}[G]^{\bar{\bigotimes}}.$$
some restricted and quotient morphisms

the normality condition provides a finer arithmetic invariant than the global Gal

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( the previous sections. By controlling the Gal(\bar{k}/k)-action, the ramification divisor \( R_\pi \) of \( \pi \) is globally invariant and \( \phi \) reads as the Gal(\bar{k}/k)-action on the generic unramified fiber over \( x \) of the cover – see Ibid §2.9.

In terms of moduli spaces \( \mathcal{M}_{g,m}[G]^{\text{rat}} \_\chi \) associated to a normal Hurwitz data \( \Diamond \), \( \phi \) defines a Gal(\bar{k}/k)-action on \( \{ \mathcal{M}_{g,I}[G]^{\text{rat}} \_\gamma, \Diamond \}, \gamma \in G \) via the \( G \)-action on \( \Diamond \), and satisfies the following property.

**Proposition 2.1.7.** For \( \Diamond \) a normal \( \Diamond^G \)-Hurwitz data, the moduli stack \( \mathcal{M}_{g,I}[G]^{\text{rat}} \_\Diamond \) is Gal(\( \bar{\mathbb{Q}} / \mathbb{Q} \))-invariant in \( \mathcal{M}_{g,[\ell]} \).

This result is immediate following the properties of normal Hurwitz datas developed in the previous sections. By controlling the Gal(\( \bar{\mathbb{Q}} / \mathbb{Q} \))-action on \( G \)-isotropy groups of \( R_\pi \), the normality condition provides a finer arithmetic invariant than the global Gal(\( \bar{\mathbb{Q}} / \mathbb{Q} \))-invariance of \( R_\pi \).

### 2.2. Functorial Properties and Moduli.

We briefly present some properties of the Hurwitz stack \( \mathcal{M}_{g}[G]^{\text{rat}} \) with rational ramification and of Hurwitz datas \( \Diamond \) under change of group and under quotient by automorphisms. In what follows \( H \triangleleft G \), resp. \( P = G/H \), denotes a normal subgroup of \( G \), resp. the associated quotient.

#### 2.2.1. Let \((E,\iota_G)\) be a curve with \( G \)-action, and let us denote \( C = E/H \) and \( D = E/G \) the various quotients given by the situation above, which induces a restricted \( H \)-action \((E,\iota_H)\) on \( E \) as well as a corestricted \( P \)-action \((C,\iota_P)\).

For a given Hurwitz data \( \Diamond = \{(G_i,\chi_i)\}_i \in I \), one obtains some restricted \( \text{res}_H^G(\Diamond) = \{(H_j,\chi'_j)\}_{j \in J} \) and corestricted \( \text{cores}_H^G(\Diamond) = \{(P_k,\tilde{\chi}_k)\}_{k \in K} \) Hurwitz datas given respectively by

\[
\text{res}_H^G(\Diamond)_i = \begin{cases} G_i \twoheadrightarrow H_i = H \cap G_i & \chi_i \mapsto \chi'_i = \chi_i|_{H \cap G_i} \\
\chi_i \mapsto \chi'_i = \chi_i|_{H \cap G_i}
\end{cases}
\]

whose definition is functorial and is similar to those of [BR11] §2.2.2.

Note that the indices \( J \) defining \( \text{res}_H^G(\Diamond) \) (resp. \( K \) defining \( \text{cores}_H^G(\Diamond) \)) are taken over the indices of \( I \in \Diamond \) with non-trivial induced isotropy groups \( H_i \) (resp. non-trivial \( P_j \)). These Hurwitz datas \( \text{res}_H^G(\Diamond) \) and \( \text{cores}_H^G(\Diamond) \) correspond respectively to those of the restricted and corestricted actions \((C,\iota_H)\) and \((D,\iota_P)\) on the curves.

#### 2.2.2. Let \( \Diamond \) be a \( \Diamond^G \)-Hurwitz data and let us denote by \( g' \) the genus of the quotient as defined by \( \Diamond \). We now write \( \mathcal{M}_{g}[H \triangleleft G] \) instead of \( \mathcal{M}_{g}[G] \) for the Hurwitz stack of \( G \)-curves to remember that a normal subgroup \( H \triangleleft G \) is fixed.

By forgetting the action of \( G \), resp. considering the (tame) quotient by \( H \), one obtains some restricted and quotient morphisms

\[
\mathcal{M}_{g}[H \triangleleft G] \to \mathcal{M}_{g}[H], \text{ resp. } \mathcal{M}_{g}[H \triangleleft G] \to \mathcal{M}_{g'}[P],
\]

\begin{align*}
\psi_\pi : \pi_1^G(Y') \to G & - \text{ see DE06 §2.4.2. Assuming that } Y' \text{ admits a rational point } x \in Y' \text{ gives a splitting of the arithmetic-geometric fundamental exact sequence: } \\
1 \to \pi_1(Y' \times \text{Spec } \bar{k}) \to \pi_1(Y') \to \text{Gal}(\bar{k}/k) \to 1
\end{align*}
as well as their rational variants on $\mathcal{M}_g[G]^{\text{rat}}$. One deduces from the study above their versions with Hurwitz datas, $\mathcal{M}_g[H \triangleleft G]^{\text{rat}}_\circ \to \mathcal{M}_g[H]^{\text{rat}}_{\text{res}G(\circ)}$ and most importantly

\begin{equation}
\Phi : \mathcal{M}_g[H \triangleleft G]^{\text{rat}}_\circ \to \mathcal{M}_g'[P]^{\text{rat}}_{\text{cores}G(\circ)}.
\end{equation}

**Proposition 2.2.1.** The morphism

\[ \tilde{\Phi} : \mathcal{M}_g[H \triangleleft G]^{\text{rat}} \to \mathcal{M}_g'[P]^{\text{rat}} \]

is smooth.

**Proof.** Using Proposition 4.6 of [Mau16], we see that $\Phi$ induces a surjection at the level of tangent spaces because the action of $H$ is tame, so that the group $H$ is reductive. Moreover, this surjectivity is true at any geometric point as we can check it functorially, so that. The result then follows from the smoothness of $\mathcal{M}_g[H \triangleleft G]^{\text{rat}}$ and $\mathcal{M}_g'[P]^{\text{rat}}$, which can be proven using deformation theory using once more the reductiveness of $P$ and $G$, and from the characterisation of smooth morphisms between smooth schemes (cf. [Gro67], Théorème 17.11.1).

2.2.3. Recall that one has a left Aut$(G)$-action on $\mathcal{M}_g[G]$. The effect of an automorphism $\psi \in \text{Aut}(G)$ on a $G$-Hurwitz data $\diamond$ attached to a curve $(C, \iota_G) \in \mathcal{M}_g[G]^{\text{rat}}_\circ$ is read on the $\psi$-twisted $G$-cover $(\tilde{C}, \tilde{\iota}_G) \in \mathcal{M}_g[G]_\psi(\circ)$ as follow:

1. the action of $G$ on $I$ and $C$ is obtained through the action of $G$ twisted by $\psi$,
2. the ramification divisor – with the $G$-action given via the sections through $I$ – is unchanged $\text{Ram}((C, \iota_G)) = \text{Ram}((\tilde{C}, \tilde{\iota}_G))$, although its numbering is, together with the action on $I$,
3. a $G$-isotropy group $G_i \in \diamond$ is sent to $\psi(G_i) \simeq G_i$ of $(\tilde{C}, \tilde{\iota}_G)$,
4. the characters $\chi_i : G_i \to \mu_{|G|}$ are uniformly changed by composition by $\psi$.

For a given $\diamond \in \mathcal{Q}_G$, Hurwitz data, we denote by $\text{Aut}_\diamond(G)$ the subgroups of $\text{Aut}(G)$ that fixes $\diamond$, and one defines the subgroup $\text{Aut}_G/H,\diamond(G; H)$ of $\text{Aut}_G/H(\circ; \diamond)$ fixing $\diamond$ – see Appendix [A]. In particular, we can now consider the $\diamond$-fixing variant $\mathcal{M}_g[H \triangleleft G]^{\text{rat}}_\diamond / \text{Aut}_G/H,\diamond(G; H)$ of $\mathcal{M}_g[H \triangleleft G]^{\text{rat}} / \text{Aut}_G/H(G; H)$ and the relative quotient morphism:

\begin{equation}
\Phi^\diamond : \mathcal{M}_g[H \triangleleft G]^{\text{rat}}_\diamond / \text{Aut}_G/H,\diamond(G; H) \to \mathcal{M}_g'[G/H]^{\text{rat}}_{\text{cores}G/H(\circ)},
\end{equation}

whose irreducibility is established in §4.

3. Mixed Cohomology and Torsors Extensions

Let $X$ be a $S$-scheme endowed with an action of a finite group $P$. We study a certain set of torsors extensions associated to the $P$-torsor $X$ and to some finite Abelian groups $H$ as given in the figure below.

This is achieved in term of mixed étale cohomology following [Gro68 §2.1]. Recall that the mixed étale cohomology groups $H^p_\text{ét}(X, \mathcal{F})$ are defined as the derived functors of $\mathcal{F} \mapsto H^p_\text{ét}(X, \mathcal{F})^P$ for a sheaf of abelian groups with a $P$-action $\mathcal{F} \in \text{ShAb}(X, P)$, and that they are the abutment of two spectral sequences – see Ibid. and [Gro57 Chap. 5].
V] – which relates these extensions to their branch loci and class of group extensions respectively via their étale and group cohomologies.

In what follows X denotes an irreducible S-scheme endowed with a faithful action of a finite abstract group P such that Y = X/P exists as a S-scheme, the group H is an Abelian finite group, and Z → X denotes a torsor extension as above, also designed as an AutY(Z)-torsor when G is not identified.

3.1. From Mixed Cohomology to Group Cohomology. Let Z be a scheme endowed with a G-action denoted iZ. We endow the set of torsors extensions with the equivalence relation given by G-equivariant isomorphisms given by H-conjugacy.

Definition 3.1.1. With the notations and assumptions above, we denote by \( \text{Tors}_{X,P}(H,G) \) the set of classes of \((H;G)\)-torsors extensions of \( P \)-torsors X:

\[
\text{Tors}_{X,P}(H,G) = \left\{ \begin{array}{c}
\text{H-torsor } Z' \rightarrow X | Z' \rightarrow Y \text{ is a } \text{G-torsor and } \\
Z'/H \cong X \text{ as } \text{P-torsor}
\end{array} \right\} / \sim
\]

where two \((H;G)\)-torsors extensions \((Z' \rightarrow X)\) and \((Z'' \rightarrow X)\) are equivalent if and only if there exists a G-equivariant Y-isomorphism \( \phi: Z' \cong Z'' \) and \( h \in H \) such that \( \phi \circ i_{Z''}(\gamma) = i_{Z'}(h^{-1}h) \) for all \( \gamma \in G \).

It follows from [Mau16] Proposition 2.2 that the condition for a scheme \( Z \in H^1_{\text{ét}}(X,H) \) to be an AutY(Z)-torsor over Y is to be a P-equivariant H-torsor over X, i.e. \( Z \in H^1_{\text{ét}}(X,H)^P \); in this case the group AutY(Z) is an extension G of P by the kernel H. As shown below, the mixed cohomology presents a refinement of this situation.

3.1.1. For \( F \in \text{AbSh}(X_{\text{ét}},P) \), let us consider the first spectral sequence of mixed cohomology of \([\text{Gro}68\ ]\ §2.1]

\[
E_2^{p,q} = H^p(P,H^q_{\text{ét}}(X,F)) \Rightarrow H^{p+q}_{\text{ét}}(X,F).
\]

For a finite group \( H \) with a given P-action, we define the constant \( P \)-sheaf \( H_X \) on X, which leads to the long exact sequence

\[
0 \rightarrow H^1(P,H_X) \rightarrow H^1_{\text{ét}}(X,H_X) \rightarrow H^1_{\text{ét}}(X,H_X)^P \rightarrow H^2(P,H_X)
\]

where we identify \( H^1(P,H^0_{\text{ét}}(X,H_X)) = H^1(P,H_X) \) since X is connected.

In particular, one recovers that two mixed \( P \)-covers of X, with difference \( \alpha \in H^1_{\text{ét}}(X,H), \) giving rise to the same element in \( H^1_{\text{ét}}(X,H)^P \) have the same H-conjugacy class of splitting extension of P. Moreover as an AutY(Z)-torsor defines a group extension \( G \cong \text{Aut}_Y(Z) \in H^2(H,P) \) of abelian kernel H, the difference of two \( P \)-invariant \( H \)-torsors of \( H^1_{\text{ét}}(X,H) \) with same image in \( H^2(H,P) \) is actually a \( G \)-torsor over Y for \( G \in H^2(H,P) \).

3.1.2. Following the discussion above, the exact sequence \([2.1.2] \) defines an action of \( H^1_{\text{ét}}(X,H) \) on the set of \((H;G)\)-torsors extensions as defined via the association of a \( G \in H^2(P,H), \) and this action is well-defined on the classes in \( \text{Tors}_{X,P}(H,G) \) via \( H^1(P,H). \)

Theorem 3.1.2. Let P be a finite group, H a finite abelian group, Y a scheme and X/Y a P-torsor. For any extension G of P by Y, the mixed cohomology group \( H^1_{\text{ét}}(X,H) \) acts simply transitively on \( \text{Tors}_{X,P}(H,G) \).

This result is a key ingredient in establishing our main Theorem 4.1 as it allows a fine control of the set of \((H;G)\)-torsor.
3.2. From Mixed Cohomology to Cohomology of the Quotient Space. Let us consider the second spectral sequence converging to the mixed cohomology groups. With the same notations as above, let us denote by $\pi: X \to Y = X/P$ the quotient morphism. For $F \in ShAb(X, P)$, one has as a special case of the Leray spectral sequence:

$$E_2^{p,q} = H^p_{\text{ét}}(Y, R^q\pi_*^P F) \Rightarrow H^{p+q}_{\text{ét}}(X_{\text{ét}}, F).$$

When $\pi$ is étale – i.e. the action of $P$ is free – one has $R^q\pi_*^P F = 0$ for $q > 0$ and we get by degeneracy:

$$H^p_{\text{ét}}(X_{\text{ét}}, F) = H^p_{\text{ét}}(Y, \pi_*^P F).$$

In what follows, we consider a finite abelian group $H$ endowed with a given action of $P$, and we denote by $H_X$ the associated sheaf on $X$. In this case, the sequence (3.2.1) leads to a local-global long exact sequence:

$$0 \to H^1_{\text{ét}}(Y, \pi_*^P H_X) \to H^1_{\text{ét}}(X_{\text{ét}}, H_X) \to H^0_{\text{ét}}(Y, R^1\pi_*^P H_X) \to H^2_{\text{ét}}(Y, \pi_*^P H_X).$$

We establish a lifting property of the global part $H^1_{\text{ét}}(Y, \pi_*^P H_X)$ of the mixed cohomology group which is a key element in controlling the deformation of $(H; G)$-covers in the proof of Theorem [14].

3.2.1. To compute the global part $H^1_{\text{ét}}(Y, \pi_*^P H_X)$ we consider the following norm morphism:

$$(\pi_* H_X)_y = \bigoplus_{x' \in P \cdot x} H_{X,x'} \text{ thus } (\pi_*^P H_X)_y \cong H_{X,x'}^{P_x},$$

so we identify $K_y$ and $C_y$ to the kernel and cokernel of the norm morphism $N_{P}^{\text{loc}}: H_{Y,y} \to H_{X,x'}^{P_x}$. From this we deduce:

**Proposition 3.2.1.** Let $x \in X$ be a ramified point of $\pi$ of stabilizer $P_x$ in $P$ and let $y = \pi(x) \in Y$. Then there exists isomorphisms

$$K_y \cong H^0(P_x, H_{X,x}) \text{ and } C_y \cong H^2(P_x, H_{X,x})$$

where $K_y$ and $C_y$ are the kernel and cokernel of $N_{P}^{\text{loc}}$ above.

3.2.2. Suppose moreover that $X$ is a curve and the action of $P$ is faithfull. Since $K$ and $C$ are supported in the branch locus of $\pi$ which is discrete, their higher cohomology groups vanish and we obtain a long exact sequence:

$$0 \to H^0_{\text{ét}}(Y, K) \to H^0_{\text{ét}}(Y, H_Y) \to H^0_{\text{ét}}(Y, \pi_*^P H_X) \to H^0_{\text{ét}}(Y, C) \to H^1_{\text{ét}}(Y, H_Y) \to H^1_{\text{ét}}(Y, \pi_*^P H_X) \to 0$$

From this we deduce the main property of this subsection.
Proposition 3.2.2. The norm morphism $N_P$ of Eq. (3.2.3) induces a transitive action of $\text{H}^1_{\text{et}}(\mathcal{Y}, H_Y)$ on $\text{H}^1_{\text{et}}(\mathcal{Y}, \pi^P_* H_X)$.

In particular, this reduces the construction of $P$-equivariant $H$-torsors over $\mathcal{Y}$ to the construction of $H \triangleleft G$ torsors over $\mathcal{Y}$ – see Corollary 4.3.1 for the application to deformation of the global part of $(H; G)$-covers.

3.2.3. We now give a group theoretic description of $\text{H}^1_P(X_{\text{et}}, H_X)$.

In terms of group theory, the same approach that in §3.2.1 shows that the fibers at $y \in \mathcal{Y}$ of the sheaf $R^1\pi^P_* H_X$ identifies to

(3.2.4) $$(R^1\pi^P_* H_X)_y \cong \text{H}^1(P, H),$$

where $x \in X$ is a lifting of $y$, via the isomorphism $(R^1\pi^P_* H_X)_y \cong \text{H}^1(P, (\pi_* H_X)_y)$. This implies some refinements on the description of $\text{H}^1_P(X_{\text{et}}, H_X)$ in terms of kernel/cokernel, that depend on the first and second group cohomology class extensions of $P$ by $H$.

For example, for $H$ and $P$ coprime order cyclic groups with trivial class in $H^2(P, H)$, one obtains that $\text{H}^1_P(X_{\text{et}}, H_X) \cong \text{H}^1(Y, H)$. The other cases of class of $G \in H^2(P, H)$ lead to similar identification of $\text{H}^1_P(X_{\text{et}}, H_X)$ that we do not reproduce here, since we do not make use of these computations.

4. Irreducibility of the Relative Quotient Morphism

Let $G$ be a finite group, $H \triangleleft G$ a subgroup and $P = G/H$ its quotient, and let $\diamond \in \Diamond^I_G$ a Hurwitz data. The goal of this section is to prove the irreducibility of the quotient morphism $\Phi^{\text{rat}}_{\diamond}$ of (2.2.2) when $\diamond$ has a $P$-étale point, i.e. when one of the $i \in I$ has an image modulo $H$ with trivial stabilizer.

Theorem 4.1. Let $G$ be a finite group, $I$ a finite $G$-set, $H \triangleleft G$ be a cyclic group of prime order and let $\diamond$ be a $\Diamond^I_G$-Hurwitz character. If $\diamond$ has a $P$-étale point, then the morphism

$$\Phi^{\text{rat}}_{\diamond} : \mathcal{M}_G[G]^{\text{rat}}_{\diamond} / \text{Aut}_{G/H, \diamond}(G; H) \to \mathcal{M}_G[G/H]^{\text{rat}}_{\text{cores}_{G}(\diamond)}$$

is geometrically irreducible.

Recall that $\text{Aut}_{G/H, \diamond}(H; G)$ denotes the subset of $\text{Aut}(G)$ that leaves $G/H$ and $\diamond$ fixed – see A.1.1.

The context, along the line of [Man16], is given by whose of $(H; G)$-torsors, which allows to go back and forth to $G$-curves, as well as to apply the results of mixed cohomology on étale and group cohomology of the previous section §3. The proof follows two steps: first a local deformation of the ramification locus – see Proposition 4.2.1 – up to a global part, then the deformation of the global part in cohomology – see Corollary 4.3.1. The local deformation relies on the arithmetic properties of torsors and their compactification to construct explicit $P$-equivariant deformations of $H$-torsors up to a global torsor. The global deformation then relies on equivariant generalisations of ideas developed in [Cor87].

Before going into the deformation process, we first recall some properties of the stack $[R^1\pi_* H]^{G/H}$ classifying $G/H$-invariant $H$-torsors.

IRREDUCIBILITY OF HURWITZ STACKS
4.1. Stack of Covers and Stack of Torsors. Following [Mau16], a stack of $G$-equivariant curves can be described in terms of a certain stack of torsors $[R^1f_*G]$. Indeed, let $\mathcal{C} \to \mathcal{M}_{g',[m]}[G/H]$ be the universal $G/H$-equivariant curve of genus $g'$ endowed with an equivariant divisor $\mathcal{B}$ of degree $m$. Let us write $f: \mathcal{C} \setminus |\mathcal{B}| \to \mathcal{M}_{g',[m]}[G/H]$. Following ibid. §3.2, one attaches to this situation an algebraic stack $[R^1f_*H]^{G/H}_{g'=g}$ which classifies local $H$-torsors which are $G/H$-invariants. This stack admits a stratification $[R^1f_*H]^{G/H}_{g'=g}$ given by the genus $g$ of the fibers – see ibid. §4.2 – and in our case fits within the diagram – see ibid §4.4:

\[
\begin{align*}
\mathcal{M}[H \triangleleft G] & \longrightarrow [R^1f_*H]^{G/H}_{g=g} \\
\mathcal{M}_g[H \triangleleft G]^{\text{rat}/\text{Aut}_{G/H,\emptyset}(H;G)} & \longrightarrow \Phi \mathcal{M}_{g'}[G/H]^{\text{rat/\core}_{G/H}(\emptyset)}
\end{align*}
\]

where $H \triangleleft G$ is here to recall that we implicitly deal with $(H;G)$-covers. In this diagram, $\text{Aut}_{G/H,\emptyset}(H;G)$ denotes a certain subgroup of the automorphisms of $G$ preserving $H \triangleleft G$ – see Appendix [A] for definition and properties – and $\Psi$ factorizes the upper-left triangle, cf. Eq. (A.2.1).

This implies the identification of the fibers of $\Phi$ to a space of torsors, or more precisely (see §5 of [Mau16]):

**Proposition 4.1.1.** In the situation above, let us consider $\mathbb{Z}/p\mathbb{Z} \simeq H \triangleleft G$. Then $[R^1f_*H]^{G/H}_{g=g}$ is representable by an algebraic stack and $\Psi$ induces an isomorphism onto its image.

This construction allows to identify equivariant space of curves to particular torsors. In terms of the analytification of these spaces, we are able to construct topological paths on $[R^1f_*H]^{G/H}_{g=g}$ that automatically lift to $\mathcal{M}_g[H \triangleleft G]/\text{Aut}_{G/H}(H;G)$ thanks to the isomorphism $\Psi$. We emphasize the importance of the quotient by $\text{Aut}_*(G;H)$ in order for $\Phi$ to be a local immersion – see Appendix [A].

4.2. Deformation of Branch Locus. Let $E$ and $E'$ be two $G$-curves with respective rational ramification $R_E$ and $R_{E'}$, and same $\hat{\text{Hurwitz}}$ data $\hat{\text{Hurwitz}}$. In particular, $E$ and $E'$ have same $G$-quotient $D$. Suppose that $E$ and $E'$ have also isomorphic $H$-quotient $C = \Phi^\text{rat}(E) = \Phi^\text{rat}(E')$ in $\mathcal{M}_{g'}[\text{rat/\core}_{G/H}(\emptyset)]$. Assuming that $\hat{\text{Hurwitz}}$ has a $P$-étale point, our goal is to deform $E$ into an $G$-equivariant curve with quotient $D$, same Hurwitz data $\hat{\text{Hurwitz}}$ and with an $H$-branch locus equal to that of $E'$.

4.2.1. We denote by $J_H$ the indices of $\text{res}^G_H(\hat{\text{Hurwitz}})$. Writing $R_E = (e_i)_{i \in I}$ and $R_{E'} = (e'_i)_{i \in I}$, we consider

\[
Z = E \setminus \bigcup_{i \in J_H} \{e_i\} \quad \text{and} \quad Z' = E' \setminus \bigcup_{i \in J_H} \{e'_i\}
\]

with $X = Z/H$ and $X' = Z'/H$, so that $Z$ (resp. $Z'$) is exactly the étale loci of $\pi_H: E \to C$ (resp. $\pi'_H: E' \to C$). In particular, $Z \to X$ and $Z' \to X'$ are both $H$-torsors.
Let $I^0_H \subset J_H$ be the $P$-étale indices of the restricted $H$-cover, i.e. $I^0_H = \{ i \in J_H, G_i \subset H \}$. By definition $I^0_H$ is exactly the set of indices of $J_H$ for which $e_i \in R_E$ and $e'_i \in R_{E'}$ are sent to étale points in $C \to C/P$. Notice that $I^0_H$ is naturally endowed with a free $P$-action because if $i \in I^0_H$, then $G_i = H$ as $H$ is of prime order.

Let $R^P_E = \{ \pi_H(e_i) \}_{i \in I^0_P}$ and $R^P_{E'} = \{ \pi_H'(e'_i) \}_{i \in I^0_P}$, which are the ramification points of $C \to C/P$, since $\pi_H(e_i) = \pi_H'(e'_i)$ for $i \in I \setminus I^0_H$. In the deformation process, the points of $R^P_E$ and $R^P_{E'}$ are thus the only ones that have to be moved, and we assume $I^0_H \neq \emptyset$ accordingly. This deformation of $R^P_E$ to $R^P_{E'}$ is achieved using torsors and their compactification.

Finally, the schemes $X$ and $X'$ can actually be embedded into $C$ in a $P$-equivariant way since $E/H = E'/H = C$. We then consider $\tilde{X} = X \cap X'$, which inherits a $P$-action, and identifies $Z$ and $Z'$ are elements of the mixed cohomology group $H^1_{et}(\tilde{X}, H)$ by Theorem 3.1.2.

4.2.2. We now prove the existence of a deformation of $E$ whose difference with $E'$ is actually in $H^1(Y, \pi^0_e H)$, i.e. whose image in $H^0(Y, R^1\pi^0_e H)$ is zero according to exact sequence $3.2.2$. This result is obtained by deforming the branch locus in $C$ and relies on the existence of compactification of torsors in family as in [Mau16].

**Proposition 4.2.1.** Let $E$ and $E'$ be two $G$-curves, with $H \triangleleft G$, and under the assumptions above, in particular with a $P$-étale point. Then there exists an algebraic deformation $\hat{E}$ of $E$ such that the difference of the $H$-torsors induced by $E'$ and $\hat{E}$ belongs to in $H^1(\tilde{X}, \pi^0_e H)$.

In other words, the difference of $E'$ and $\hat{E}$ is global. We denote by $E/S$ this algebraic deformation, whose construction occupies the rest of this section and is done in two steps: first with respect to the Hurwitz data, and then to the class of $G$.

We can suppose that there exists a $i \in R^P_E \setminus R^P_{E'}$. Otherwise, $R^P_E = R^P_{E'}$, implies that $X = X'$, and the equality of the Hurwitz data implies that the difference is already in $H^1(\tilde{X}, \pi^0_e H)$ thanks to the exact sequence $3.2.2$. Let us move $t_0 = i \in R^P_E \subset C$ to a point of $R^P_{E'}$, by keeping fixed the points that are already in $R^P_{E'} \cap R^P_{E'}$.

To do so, we first build a family $\hat{Z} \to C \times D^0$ of $H$-torsors that:

1. is $P$-invariant;
2. is branched over each $e_i \times D^0$ with ramification data $(G_i, \chi_i)$, for $i \in I_H \setminus \{ G.i \}$;
3. is branched over $P.\pi_H(e_i)$ over one fiber and over $P.\pi_H'(e'_i)$ over another;
4. is branched over a constant divisor denoted $\infty$.

For this, let us fix a point $\infty \in D$ distinct from all the $\pi_G(e_i)$ and $\pi'_G(e'_i)$, and let us define

\[ \hat{D} = D \times D^0, \text{ where we set } D^0 = D \setminus \left( \{ \pi_G(e_i), i \in I_H \setminus G.i \} \cup \{ \infty \} \right), \]

which is naturally a family of curves over $D^0$. We then consider the relative Cartier divisor $\hat{B}$ defined through the diagonal $D^0 \to D \times D^0$, which by definition does not meet the pullback $\infty$ of $\infty$ along $\hat{D} \to D^0$. 

\[ X \supset \tilde{X} \subset X', \quad C \quad \pi \quad \pi' \]

\[ Y \quad \pi \quad \pi' \quad D \]
It follows from Kummer theory – see [CM15] §4.1 – that, up to an étale base change \( \hat{D}^o \to D^o \), there exists a family of \( H \)-torsors \( \hat{Y}_H \) over \( (\hat{D} \setminus (\hat{B} \cup \infty)) \times_D \hat{D}^o \) that has Hurwitz data equal to \((G_i, \chi_i)\) over \( B \) and equal to \((G_i, \chi_i^{-1})\) over \( \infty \times D^o \). Up to another base change, let us fix some liftings \( f_i \) (resp. \( f'_i \)) of \( \pi_G(e_i) \) (resp. of \( \pi'_G(e'_i) \)) in \( \hat{D}^o \). By pulling back along \( \hat{C} = C \times \hat{D}^o \to D \times D^o \), we get an equivariant family \( \hat{X}_H \) of \( H \)-torsors which are \( P \)-equivariant, and that are ramified along the pullbacks \( \hat{B}_X \) and \( \hat{\infty}_X \) of \( \hat{B} \) and \( \infty \).

Notice that the fiber \((\hat{B}_X)_{f_i}\) is naturally identified to \( \sum_{i \in G,i} e_i \), and that \((\hat{B}_X)_{f'_i}\) is identified to \( \sum_{i \in G,i} e'_i \); the branch points are thus the right ones, and so are the ramification data by construction. However the map \( \hat{X}_H \to \hat{D} \) is actually \( P \times H \)-equivariant, i.e. with respect to the trivial extension of \( P \) by \( H \) only, and one must still recover the proper class of \( G \). We refer to Figure 1 for a summary of the construction at this stage.

The next step is thus to go from \( H \times P \) to \( G \). For this, we define a family \( \hat{Z} \) over \( \hat{X} \) of \( H \)-torsors, where

\[
\hat{Z} = \left( Z - (\hat{X}_H)_{f_i} \right) \times \hat{D}^o + \hat{X}_H \quad \text{and} \quad \hat{X} = \hat{C} \setminus \left( \hat{B}_X \cup \hat{\infty}_X \cup \bigcup_{i \in I_H} e_i \times \hat{D}^o \right).
\]

This family is obtained by base change and addition of \( H \)-torsors over \( \hat{X} \), and it satisfies the previous conditions (1) – (4). By Theorem 3.1.2 this \( P \)-invariant \( H \)-torsor defines an element \( H^o_{f'_i}(\hat{X}_e, H) \), and thus inherits an action of a group \( \hat{G} \) which is an extension of \( P \) by \( H \). Since the restriction of this torsor over \( f_i \) equal to \( Z \), we obtains \( \hat{G} = G \).

The \( S \)-family \( E \) of deformation is finally obtained as a \( G \)-equivariant compactification of \( G \)-torsor. Since the Hurwitz data are constant, so is the genus of the compactification of the fibers of \( \hat{Z} \to \hat{D}^o \). Up to an alteration \( S \to \hat{D}^o \), the existence of such a \( G \)-equivariant compactification \( \hat{E} \to S \) of \( \hat{Z} \times_{\hat{D}^o} Z \) follows from [Mau10] Theorem 4.7.

Regarding the Hurwitz datas, there exists moreover liftings \( f_i \) and \( f_i' \) of \( f_i \), resp. \( f'_i \), in \( S \) such that \( E_{f_i} = E \). The Hurwitz data of \( E_{f_i}' \) over \( \pi_H(e_i) \) and of \( E' \) at \( \pi'_H(e'_i) \) are thus equal, while the Hurwitz datas at the other points of \( R^o_E \) are not changed. Repeating this process over all the points over which the Hurwitz data of \( E \) and \( E' \) differ, one build a similar \( S \)-deformation \( \hat{E} \) of \( E \) into a curve \( \hat{E} \).
The difference between the $H$-torsors induced by $\tilde{E}$ and $E'$ is thus global in $H^1(\tilde{X}, \pi_P^H)$ and this concludes the proof of Proposition 4.2.1.

4.3. Deformation of the Global Part. In the situation of $G$-covers with a $P$-étale point – see §4.2.1 for the context –, we establish a deformation result of the global part of the $(H;G)$-torsors. Joined to the local deformation result of Proposition 4.2.1 this concludes the proof of Theorem 4.1.

The following is a direct application of §4.2.2, and relies on the analytification of the spaces.

**Proposition 4.3.1.** Let $C$ be a complete smooth curve, $X \subset C$ be an open subscheme. Let $Z$ and $Z'$ be two $(H;G)$-torsors over $X$ with same local datas such that $Z/H \simeq Z'/H$ as $P$-torsors. If there exists $x \in C$ with trivial stabiliser in $P$ and over which the completion $\tilde{E}$ of $Z$ and $\tilde{E}'$ of $Z'$ are branched, then $E$ can be deformed continuously and $G$-equivariantly to $E'$ in $\mathcal{M}_{g}[H \triangleleft G]/\text{Aut}_{G/H}(H;G)$.

Under this assumptions, notice that $E$ and $E'$ have in particular a $P$-étale point.

**Proof.** According to exact sequence (3.2.2), the difference $c = Z - Z' \in H^1_\text{ét}(X, H_X)$ being global means that $c \in H^1_\text{ét}(Y, \pi_P^H H_X)$, and it follows from Proposition 3.2.2 that $c$ comes from an element $\bar{c} \in H^1_\text{ét}(Y, H_X)$.

Let us denote by $\pi_P: C \to D$ the $P$-quotient morphism and let us write $y = \pi_P(x)$ that we take as base point of $\pi_1^{\text{top}}(D(C)^\text{an}, y)$. The Betti-étale comparison isomorphism $H^1_\text{ét}(D, H) \simeq H^1(D(C)^\text{an}, H)$ – see [Mil80], Theorem III.3.12 – being induced by the morphisms

$$\pi_1^{\text{top}}(D(C)^\text{an}, y) \to H^1(D(C)^\text{an}, Z) \to H^1_\text{ét}(D, H),$$

the $H$-torsor $\bar{c}$ over $Y$ lifts to a topological loop $\gamma \in \pi_1^{\text{top}}(D(C)^\text{an}, y_i)$.

Using the notations of §4.1 the loop $\gamma$ defines a path of torsors in $[R^1 f_* H]_{G/H}^{G/H}$ by considering a family of $G/H$-equivariants $H$-torsors that is branched over the (moving) path $\gamma \subset C$ and over all the other (fixed) branched points coming from $E \to D$ – they are equal to those of $E' \to D$ as the difference between $Z$ and $Z'$ is global. Finally, the path $\gamma'$ can be lifted into $\mathcal{M}_{g}[H \triangleleft G]/\text{Aut}_{G/H}(H;G)$ by Proposition 4.1.1 and it links $E$ to $E'$.

The proof of Theorem 4.1 is now straightforward under the assumptions of $H$ to be cyclic of prime order and of $\diamond$ to have a $P$-étale point: for a given $E \in \mathcal{M}_{g}[G]_{\diamond}^{\text{rat}}/\text{Aut}_{P,\diamond}(H;G)$, Proposition 4.2.1 gives an algebraic deformation of the ramification locus in $C = E/H$ up to a global torsors, which in turn admits a topological deformation by Corollary 4.3.1 hence $\Phi_\diamond^{\text{rat}}$ is geometrically irreducible.
Appendix A. Erratum to “Quelques calculs d’espaces $R^1 f_*G$ sur des courbes” – by S. Maugéais

Let $f : X \to S$ be a morphism of algebraic stacks and $G$ be a smooth group scheme over $S$ and $H \triangleleft G$. In [Mau16], we define a certain stack $[R^1 f_*G]$ over $S$ which gives a local classification of $G$-torsors – denoted $[R^1 f_*G]$ in the case of the complement of a relative Cartier divisor in a smooth curve $f : U \to S$ – and whose $G/H$-equivariant version $[R^1 f_*H]^{G/H}$ we show is related to a certain moduli stack $M_g[H \triangleleft G]$ – see §4.4 Ibid. It has to be noted that since the action of $G$ is fixed, the group $H$ is not a subgroup of the automorphism groups of elements of $M_g[H \triangleleft G]$, and thus one can not consider the 2-quotient $M_g[H \triangleleft G]/H$ as it is done in loc. cit.

Instead of this 2-quotient by $H$, let us present how the geometric interpretation of $[R^1 f_*H]^{G/H}$ is given by forgetting the action of $H$ while keeping that of $G/H$ on the quotient curves.

A.1. Let $\text{Aut}(G; H)$ denote the subgroup of elements of $\text{Aut}(G)$ sending $H$ onto itself.

**Definition A.1.1.** Let $G$ be a finite group and $H \triangleleft G$. We denote by $\text{Aut}_{G/H}(G; H)$ the kernel of the morphism $\text{Aut}(G; H) \to \text{Aut}(G/H)$.

This group comes with a natural morphism

$$\text{Aut}_{G/H}(G; H) \to \text{Aut}(H)$$

induced by restriction to $H$ (when viewed as elements of $\text{Aut}(G; H)$), which is not injective in general. This groups takes place in an exact sequence

$$0 \to \text{Aut}_{G/H}(G; H) \to \text{Aut}(G; H) \to \text{Aut}(G/H)$$

whose last morphism is not surjective.

Note that in general, the description of the automorphism group $\text{Aut}_{G/H}(G; H)$ is far from being straightforward: see [Wel71].

Considering the Hurwitz stack with rational ramification $M_g[G]^{\text{rat}}$ of §2.1.2 and the quotient morphism of §2.2.2, one establishes in a similar way to Proposition 2.2.1:

**Proposition A.1.2.** The morphism

$$\Phi : M_g[H \triangleleft G]^{\text{rat}}/\text{Aut}_{G/H}(G; H) \to M_g[P]^{\text{rat}}$$

is flat, hence universally open.

A.2. The $M_g[H \triangleleft G]$ stack being endowed with an action of $\text{Aut}(G; H)$, one obtains more precisely a factorization of the natural morphism $\Psi$ as below:

$$M_{g,g'}[H \triangleleft G] \xrightarrow{\Psi} [R^1 f_*H]^{G/H} \xrightarrow{\text{Aut}_{G/H}(G; H)} M_{g,g'}[H \triangleleft G]/\text{Aut}_{G/H}(G; H)$$

which comes with the corresponding corrections p.388 and in Proposition 5.2 Ibid. given by replacing the 2-quotient by $H$ by the quotient by $\text{Aut}_{G/H}(G; H)$. 
The only point that needs verification is actually the point $i$ of Lemma 4.8 Ibid., because all the others are still valid. This can be rephrased as follows: let $i_1$ and $i_2$ be two actions of $G$ on a curve $C/\text{Spec} \ k$ inducing the same $H$-torsor in a $G/H$-equivariant way. The actions $i_1$ and $i_2$ are then equal up to an automorphism of $G$ sending $H$ to $H$ and inducing the identity on $G/H$.

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