Forty years and still counting: A tribute to Tony Gutmann

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Abstract. This short paper is intended to highlight some of the research contributions of Tony Guttmann on the occasion of his sixtieth birthday.

1. Introduction
Tony Guttmann has been counting objects of interest in statistical mechanics and enumerative combinatorics for over forty years. He has worked on many problems in these areas and I shall only touch on a small part of his work. The emphasis will be on areas that I know and, to some extent, on areas where we have worked together.

I first met Tony Guttmann in about 1969 or 1970 when I was visiting Martin Sykes at Kings College, London. Martin introduced me to this young Australian post-doc, we went out for a cup of coffee in a cafe on Kingsway, and that was the beginning of a collaboration and friendship which has lasted now for over thirty five years.

Tony has worked in the area of exact enumeration and series analysis since his PhD work with Colin Thompson and Barry Ninham. He has developed highly efficient methods for enumerating embeddings of graphs in lattices and some of his ideas have changed the way in which these problems are now attacked, especially in two dimensions. Given a reasonably long series (eg a series in a suitable high temperature variable for some property, such as the magnetic susceptibility, for the Ising model) one needs to use this information to estimate properties such as the transition temperature or a critical exponent. This is the field of series analysis and Tony is one of the world experts in this area.

More recently Tony developed an interest in models which can be solved exactly using combinatorial approaches. These models are of interest in their own right but can also give bounds on certain properties of models which have been traditionally studied in statistical mechanics. Tony has been instrumental in forging links between the statistical mechanics and enumerative combinatorics communities with considerable benefits to both.

2. Exact enumeration and series analysis
Phase transitions and critical phenomena have been attracting attention from scientists at least since the work of van der Waals. One wants to know if there is a phase transition, where it occurs, and how thermodynamic properties change as the phase transition is approached. One very powerful and general approach is to derive exactly a number of terms in a high or low temperature expansion of a thermodynamic variable, and then to use series analysis techniques to estimate the location of the singularity and the behaviour of the function close to the singularity. The derivation of the terms in the series often involves counting the embeddings of graphs with given homeomorphism type in a particular lattice.
As an example consider counting self-avoiding walks on the square lattice. Let $c_n$ be the number of $n$-edge self-avoiding walks starting from the origin, say. Then it is easy to see that $c_1 = 4$, $c_2 = 12$, $c_3 = 36$ and $c_4 = 100$. One can continue counting by hand and it is relatively easy to find $c_n$ exactly for $n$ less than about 10. Beyond this computer methods are normally used and a backtracking algorithm will work well up to a value of $n$ of around 30. This involves adding one edge at a time and checking for self-avoidance. A totally different approach has been used by Guttmann and several collaborators (including Enting [1], Jensen [2] and Conway [1]) in which a row of edges is added at each stage. This is called the finite lattice method and has dramatically changed the number of terms in a series expansion which can be obtained exactly in two dimensions. For instance, we know that for the square lattice $c_{71} = 4190893020903935054619120005916$ [2].

In the case of self-avoiding walks one wants to use the exact enumeration data to estimate the rate of growth of the number of walks. It is a classical result due to John Hammersley that the limit

$$
\lim_{n \to \infty} n^{-1} \log c_n = \kappa
$$

exists. $\kappa$ is called the connective constant of the lattice. This tells us that $c_n = e^{\kappa n + o(n)}$ but it doesn’t tell us anything about the $o(n)$ term. It is widely believed (on the basis of convincing physical arguments as well as numerical evidence) that

$$
c_n = An^{\gamma-1} \mu^n (1 + o(1))
$$

where $\mu = e^\kappa$ and $\gamma$ is a critical exponent. Given this assumed functional form, one wants to use the exact values of $c_n$ for $n$ less than some $N$ to estimate the values of $\mu$ and $\gamma$. This area was pioneered by the Kings group who developed several apparently simple methods for carrying out the estimation procedure. The whole process is a mixture of science and art since it takes a good deal of experience to be able to generate reliable estimates. Tony Guttmann is a master of this technique [3, 4]. The review of series analysis techniques that he wrote with David Gaunt [3] in 1974 influenced a generation of people in this field.

The exponent $\gamma$ only depends on the dimension but $\mu$ depends on the particular lattice. In two dimensions we know that $\gamma = 43/32$ [5] though this has not been proved rigorously. (We would know this rigorously if we knew that the continuum limit of self-avoiding walks in two dimensions was $SLE_{8/3}$ [6].) The best estimate of $\mu$ for the square lattice comes from exact enumeration and series analysis work [2]. This gives $\mu = 2.6381\ldots$. In addition the same approach gives an estimate of $\gamma$ which agrees with 43/32 to about one part in 10$^6$. In three dimensions the situation is more difficult because we do not have a precise conjecture for the value of $\gamma$ and because the finite lattice method works much less well, so the available series is much shorter. Nevertheless, exact enumeration and series analysis [7] gives the estimates $\gamma = 1.1585$ and $\mu = 4.68404 \pm 0.00009$ The value of $\gamma$ agrees very well with Monte Carlo estimates [8].

Self-avoiding walks are only one example of a problem where exact enumeration and series analysis can make an important contribution to our understanding. For another recent example see the beautiful paper by Guttmann et al on the susceptibility of the Ising model [9].

These methods can also be used for somewhat more complicated problems. For instance, in two dimensions one can enumerate polygons (say on the square lattice) by both perimeter and area [10]. These can be thought of as a model of vesicles. If we work in an ensemble where the perimeter is fixed and the area can vary this mimics the vesicle situation where the volume of the vesicle is controlled by the relative ionic strengths of the solutions inside and outside the vesicle. For this model we know quite a lot about the qualitative properties of the phase boundary between the inflated and deflated regimes and have reasonable numerical estimates for the location of the phase boundary [11]. Again, exact enumeration and series analysis techniques played an important role in unravelling the complicated behaviour of the model [10, 11]. As computational power increases and algorithms become more efficient we can expect to see these techniques being applied to even more complex systems.
3. Connections to combinatorics

Since problems like counting self-avoiding walks and lattice polygons are currently beyond our analytical approaches it seems natural to address simpler versions of these problems. This is for two reasons. If one counts a subset of self-avoiding walks (or polygons) one can obtain a lower bound on \( \mu \), though this gives no information about \( \gamma \). One might also hope to gain some insight into the behaviour of walks or polygons from studying a subset, even if the subset is exponentially small. For these reasons there has been considerable interest in classes of polygons with a constraint (convex polygons, row-convex polygons, staircase polygons, ...) and classes of directed walks (Dyck paths, Motzkin paths, partially directed walks, ...)

In 1990 Brak, Guttmann and Enting [12] found an explicit expression for the perimeter generating function for row-convex polygons on the square lattice, by using an enumeration idea originally due to Temperley [14]. They showed that the number of row-convex polygons with \( 2n \) edges, \( a_n \), behaves like

\[
a_n = c_0(3 + 2\sqrt{2})^{n-1/2}n^{-3/2}(1 + O(n^{-1})).
\] (3.1)

This has the same general behaviour as that expected for polygons. It also gives a lower bound on \( \mu \) of \( \mu \geq \sqrt{3+2\sqrt{2}} = 2.414 \ldots \) which is not a particularly good bound but indicates that one might find larger classes of polygons which can be counted exactly by these methods, yielding better lower bounds. In the same year Brak and Guttmann [13] counted row-convex and staircase polygons by both perimeter and area. That is, they found an expression for the generating function counting both by perimeter and by area. They used these results to build a simple model of vesicles which can be seen as a precursor to the model studied by Fisher et al [11].

This kind of model can also be studied when interactions are added. For instance, one can study partially directed walks with a vertex-vertex contact interaction as a model of self-interacting polymers [15]. Such models are expected to exhibit a coil-ball transition at a specific temperature, the theta temperature. These walks can be enumerated according to the number of edges and the number of contacts and this is one of the few models in which can prove the existence of a coil-ball phase transition [15].

This connection between combinatorics and statistical mechanics has developed into a flourishing set of collaborations between people working in the two fields, with considerable advantages to both areas.

4. Why are some problems easier than others?

Why are some problems in enumerative combinatorics and statistical mechanics easier than others? Given a specification of a problem, can we tell if it will be easy or hard without actually solving it? In one of my favourites of Tony Guttmann’s contributions, he addresses this issue computationally [16].

For instance, consider polygons on the anisotropic square lattice where one counts polygons by horizontal and vertical edges. If \( p_{mn} \) is the number of polygons with \( m \) horizontal edges and \( n \) vertical edges one can define the generating function

\[
P(x, y) = \sum_{m,n} p_{mn} x^m y^n. \tag{4.1}
\]

If we rewrite this as

\[
P(x, y) = \sum_n R_n(x) y^n \tag{4.2}
\]

where

\[
R_n(x) = \sum_m p_{mn} x^m \tag{4.3}
\]

then we can ask for the zeros of \( R_n(x) \) as a function of \( n \). Regarding \( x \) as a complex variable we find that the zeros lie on the unit circle and appear to become dense (on this circle) as \( n \) increases. This puts
severe restrictions on the generating function $P(x, y)$. In particular it says that $P$ cannot be D-finite. This means that the problem cannot be solved in terms of simple classes of functions and suggests that the solution is likely to be extremely difficult. Several other models, including the susceptibility of the two dimensional anisotropic Ising model, the generating function of hexagonal lattice animals and the generating function of directed animals are also almost certainly not D-finite [16].

Although showing that a problem is hard in this sense doesn’t produce a solution, this is a beautiful example of numerical methods throwing light on a theoretical question which is difficult to attack by other means.

5. Conclusion

Tony Guttmann has made contributions to many areas in statistical mechanics and this short account cannot do justice to his influence and his achievements. I firmly believe that his contributions (and the lengths of his series) will continue to grow over the next forty years.

Acknowledgments

It is a pleasure to write about the research achievements of Tony Guttmann. I would like to thank the organisers of the Dunk Island conference for giving me this opportunity.

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