THE ENDOMORPHISM RING THEOREM FOR GALOIS AND D2 EXTENSIONS

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Abstract. Let $S$ be the left bialgebroid $\text{End}_B A_B$ over the centralizer $R$ of a right D2 algebra extension $A|B$, which is to say that its tensor-square is isomorphic as $A$-$B$-bimodules to a direct summand of a finite direct sum of $A$ with itself. We prove that its left endomorphism algebra is a left $S$-Galois extension of $A^p$. As a corollary, endomorphism ring theorems for D2 and Galois extensions are derived from the D2 characterization of Galois extension. We note the converse that a Frobenius extension satisfying a generator condition is D2 if its endomorphism algebra extension is D2.

1. Introduction

Bialgebroids are generalized weak bialgebras over an arbitrary noncommutative base ring [2, 12, 21]. As in the theory of bialgebras or weak bialgebras, there are associated to a bialgebroid, module and comodule algebras, smash products and Galois extensions [3, 6, 13, 15, 18, 20]. These constructions all occur in the tower over a depth two (D2) extension $A|B$, where the noncommutative base ring is the centralizer $R$ of the ring extension: the extension is D2 if its tensor-square is centrally projective w.r.t. $A$ as a natural $B$-$A$-bimodule (left D2) and $A$-$B$-bimodule (right D2) [8, 12]. An extra condition that the natural module $A_B$ is balanced or faithfully flat ensures that $A|B$ is a right Galois extension w.r.t. the $R$-bialgebroid $T := (A \otimes_B A)^B$ [8]. In section 4 the main theorem of this paper shows that even

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![Figure 1. Galois map $\beta$ factors through various isomorphisms in Theorem 4.4.](image-url)
without this condition the left endomorphism ring $E := \text{End}_B A$ is a left $S$-Galois extension of the subalgebra of right multiplications $\rho(A)$. The proof that the Galois mapping is an isomorphism is summarized in the commutative diagram in Figure 1.

Endomorphism ring theorems date back at least to the 1950’s with Kasch’s theorem for Frobenius extensions, Nakayama and Tzuzuku’s theorem for $\beta$-Frobenius extensions, and similar theorems for QF-extensions by Müller [17]: Morita’s paper [15] was a definitive paper for all such Frobenius extensions. The general idea of these theorems is that important properties such as the Frobenius property of an extension $B \subseteq A$ or a ring homomorphism $B \rightarrow A$ sometimes pass up to the extension $A \hookrightarrow \text{End}_B A$ induced from left multiplication $\lambda(a)(x) := ax$. Also the index of a Frobenius extension, or the Hattori-Stallings rank of the underlying projective module, may be viewed as passing up unchanged to the endomorphism ring extension. Sometimes two properties are dual with respect to this shift of levels such as separability and splitness of an extension, as shown by Sugano, so that a split extension leads to separable endomorphism ring extension, and vice versa under suitable conditions. Finally possessing a certain Frobenius homomorphism compatible, or “Markov,” trace passes up in an endomorphism ring theorem as well. These four curiosities as endomorphism ring theorems had a stunning application to topology in the 1980’s when Jones iterated the endomorphism ring construction of a tower over special split, separable Frobenius algebra extensions (type $II_1$ subfactors), finding a countable set of idempotents that satisfy braidlike relations, which together with a Markov trace lead to the first knot and link polynomials since the classical Alexander polynomial (see [12] for an explanation of this point of view, and [2, 13] for the related coring point of view).

Depth two extensions have their origins in finite depth $II_1$ subfactors. An inclusion of finite-dimensional $C^*$-algebras $B \subseteq A$ can be recorded as a bicolored weighted multigraph called a Bratteli inclusion diagram: the number of edges between a black dot representing an (isomorphism class of a) simple module $V$ of $A$ and a white dot representing a simple module $W$ of $B$ is $\dim \text{Hom}_B(V,W)$. This can be recorded in an inclusion matrix of non-negative integers, which corresponds to an induction-restriction table of irreducible characters of a subgroup pair $H < G$ if $A = C^* G$ and $B = C^* H$ are the group algebras.

If we define the basic construction of a semisimple $C^*$-algebra pair $B \subseteq A$ to be the endomorphism algebra $\text{End} A_B$ of intertwiners, we note that $B$ and $E := \text{End} A_B$ are Morita equivalent via bimodule $\varepsilon A_B$ whence the inclusion diagram of the left multiplication inclusion $A \hookrightarrow \text{End} A_B$ is reflection of the diagram of $B \subseteq A$. Beginning with a subfactor $N \subseteq M$, we build the Jones tower using the basic construction

$$N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

where $M_{i+1} = \text{End} M_{i-1}$, then the derived tower of centralizers or relative commutants are f.d. $C^*$-algebras,

$$C_N(N) \subseteq C_M(N) \subseteq C_{M_1}(N) \subseteq C_{M_2}(N) \subseteq \cdots$$

The subfactor $N \subseteq M$ has finite depth if the inclusion diagrams of the derived tower stop growing and begin reflecting at some point, depth $n$ where counting begins with 0. For example, the subfactor has depth two if $C_{M_2}(N)$ is isomorphic to the basic construction of $C_M(N) \subseteq C_{M_1}(N)$. As shown in [12], depth two is
really a property of ring or algebra extensions if viewed as a property of the tensor-
square. One of the aims in such a generalization is to find an algebraic theorem

corresponding to the Nikshych-Vainerman Galois correspondence (see [18]) between
intermediate subfactors of a depth two finite index subfactor \( M \mid N \) and left coideal
subalgebras of the weak Hopf algebra \( C_M(N) \).

In [12, Theorem 6.1] we noted an endomorphism ring theorem for Frobenius D2
extensions, where the proof shows that a right D2 extension has a left D2 endo-
morphism ring extension (although right and left D2 are equivalent for Frobenius
extensions). In this paper, Corollary 5.2 of the main theorem shows that Frobenius
may be removed from this theorem. We use a characterization of one-sided Galois
extension for bialgebroids in terms of the corresponding one-sided D2 property of
extensions in [8, Theorem 2.1] or Theorem 5.1 in this paper. This characterization
and our main theorem then yields endomorphism ring theorems for D2 extensions
and Galois extensions: a right D2 (or Galois) extension has left D2 (or Galois) left
endomorphism extension \( E \mid A \) (Corollary 5.2). There is a formula for the Galois
inverse mapping in eq. (38) (not depending on an antipode as in e.g. [10, below eq.
(13)]). Keeping track of opposite algebras of D2 extensions and their associated
bialgebroids, we extend this to a right endomorphism theorem in Corollary 5.3.

Asking for converses to the endomorphism ring theorem usually lead to more dif-
ficult questions: we note however a simple proof that a D2 endomorphism algebra
extension implies that a one-sided split or generator Frobenius extension is itself
D2 (Theorem 5.7), which answers [10, Question 1].

Bialgebroids equipped with antipodes are Hopf algebroids, although there is a
scientific discussion about what definition to use [4, 5, 14]. Finding further examples
would tend to throw more weight to one of these non-equivalent definitions. In
Theorem 3.6 we show that the bialgebroid \( T \) of a D2 extension \( A \mid B \) over a Kanzaki
separable algebra \( B \) is a Hopf algebroid of the type in [4, B¨ ohm-Szlach´ anyi]. The
antipode is very naturally given by a twist of \( (A \otimes B)A \) utilizing a symmetric
separability element. These provide further examples of non-dual Hopf algebroids,
in contrast to the dual Hopf algebroids \( S \) and \( T \) of a D2 Frobenius extension \( A \mid B \)
[4]. They are also Hopf algebroids with no obvious counterparts among Lu’s version
of Hopf algebroid [14].

2. Preliminaries on D2 extensions

The basic set-up throughout this paper is the following. We work implicitly with
associative unital algebras over a commutative ground ring \( K \) in the category of
\( K \)-linear maps where bimodules are \( K \)-symmetric. An algebra extension \( A \mid B \) is a
unit-preserving algebra homomorphism \( B \rightarrow A \), proper if this is monomorphic. We
often say that an extension \( A \mid B \) has left or right property \( X \) (such as being finite
projective, i.e. f.g. projective) if the natural module \( _BB \) or \( AB \) has the property \( X \).
Let \( A \mid B \) be an algebra extension with centralizer denoted by \( R := C_A(B) = AB \),
bimodule endomorphism algebra \( S := \text{End}_B(A) \) and \( B \)-central tensor-square \( T :=
(A \otimes B)A \). \( T \) has an algebra structure induced from \( T \cong \text{End}_A(A \otimes B)A \) given by

\[
1_T = 1 \otimes 1,
\]

where \( t = t^1 \otimes t^2 \in T \) uses a Sweedler notation and suppresses a possible summation
over simple tensors. Note the homomorphism \( \sigma : R \rightarrow T \) given by \( \sigma(r) = 1 \otimes r \) and
the anti-homomorphism $\tau: R \to T$ by $\tau(r) = r \otimes 1$: the images of $\tau$ and $\sigma$ in all cases commute in $T$. We will work with the $R$-$R$-bimodule induced by $T_{\sigma, \tau}$, which is given by

$$ rTR: \quad r \cdot t \cdot r' := t\sigma(r')\tau(r) = rt^1 \otimes t^2r' $$

for all $t \in T$, $r, r' \in R$.

There is a category of algebra extensions whose objects are algebra homomorphisms $B \to A$ and whose morphisms from $B \to A$ to $B' \to A'$ are commutative squares with algebra homomorphisms $B \to B'$ and $A \to A'$, an isomorphism if the last two maps are bijective. The notions of left or right D2, balanced, finite projective or Galois extensions introduced below are isomorphism invariants in this category (e.g., left D2 involves applying eq. 4).

Let $\lambda: A \leftrightarrow \text{End}_B A$ denote the left multiplication operator and $\rho: A \leftrightarrow \text{End}_B A$ the right multiplication operator. Let $E$ denote $\text{End}_BA$ and note that $S \subseteq E$, a subalgebra under the usual composition of functions. Note that $\lambda$ restricts to an algebra monomorphism $R \to S$ and $\rho$ restricts to an anti-monomorphism $R \to S$; and the images commute in $S$ since $\lambda(r)\rho(r') = \rho(r')\lambda(r)$ for every $r \in R$. We work with the $R$-$R$-bimodule $\lambda, \rho S$ given by

$$ rSR: \quad r \cdot \alpha \cdot r' := \lambda(r)\rho(r')\alpha = r\alpha(-)r' $$

for all $\alpha \in S$, $r, r' \in R$.

We have the notion of an arbitrary bimodule being centrally projective with respect to a canonical bimodule: we say that a bimodule $A,M_B$, where $A$ and $B$ are two arbitrary algebras, is centrally projective w.r.t. a bimodule $A,N_B$, if $A,M_B$ is isomorphic to a direct summand of a finite direct sum of $N$ with itself; in symbols, if $A,M_B \oplus \star \cong \bigoplus^n A,N_B$. This recovers the usual notion of centrally projective $A$-$A$-bimodule $P$ if $N = A$, the natural bimodule.

We recall the definition of a D2, or depth two, algebra extension $A \mid B$ as simply that its tensor-square $A \otimes_B A$ be centrally projective w.r.t. the natural $B$-$B$-bimodule $A$ (left D2) and the natural $A$-$B$-bimodule $A$ (right D2).

A very useful characterization of D2 extension is that an extension is left D2 if there exist finitely many paired elements (a left D2 quasibase) $\beta_i \in S$, $t_i \in T$ such that

$$ a \otimes_B a' = \sum_i t_i\beta_i(a)a' $$

and right D2 if there are finitely many paired elements (a right D2 quasibase) $\gamma_j \in S$, $u_j \in T$ such that

$$ a \otimes_B a' = \sum_j a\gamma_j(a')u_j $$

for all $a, a' \in A$ [12, 3.7]: we fix this notation, especially for a right D2 extension. We note from these equations and the dual bases characterization of finite projective modules that the modules $rT$ and $S_R$ are finite projective in case $A \mid B$ is right D2, while $T_R$ and $rS$ are finite projective in case $A \mid B$ is left D2. (For in the case of $S_R$, for example, let $a = 1$ in eq. 4 and apply id $\otimes \alpha$ for any $\alpha \in S$. Note that $T \to \text{Hom}(S_R, R_R)$ via $t \mapsto (\alpha \mapsto t^1\alpha(t^2))$, in fact an isomorphism.)

Centrally projective algebra extensions, H-separable extensions and f.g. Hopf-Galois extensions are some of the classes of examples of D2 extension. If $A$ and $B$ are the complex group algebras corresponding to a subgroup $H < G$ of a finite
group, then \( A \mid B \) is D2 iff \( H \) is a normal subgroup in \( G \) \([9]\). In \([8]\) we show that finite weak Hopf-Galois extensions are left and right D2. More generally, left Galois extensions for bialgebroids and their comodule algebras are left D2 \([8]\).

A surprisingly large portion of the Galois theory for bialgebroids accompanying a D2 extension may be built up from just a left or right D2 extension \([8]\). Working with as few hypotheses as necessary, we will therefore work with right D2 extensions in this paper. The proposition below and Props. \([3.1]\) and \([3.5]\) further below indicate how one theory for either left or right D2 extensions may cover both cases via a duality. Let \( A^{\text{op}} \) denote the opposite algebra of an algebra \( A \). We briefly use the notation \( S(A \mid B) \) and \( T(A \mid B) \) to denote the \( S \) and \( T \) constructions above with emphasis on their dependence on the algebra extension \( A \mid B \).

**Proposition 2.1.** The algebra extension \( A \mid B \) is right D2 if and only if its opposite algebra extension \( A^{\text{op}} \mid B^{\text{op}} \) is left D2. Moreover, there are canonical algebra isomorphisms \( S(A \mid B) \cong S(A^{\text{op}} \mid B^{\text{op}}) \) and \( T(A \mid B) \cong T(A^{\text{op}} \mid B^{\text{op}}) \).

**Proof.** Let \( a \mapsto \pi \) be the identity anti-isomorphism \( A \to A^{\text{op}} \). Then \( A AB \cong B^{\text{op}} \cdot A^{\text{op}} \cdot A^{\text{op}} \) are identified via this same mapping, as are \( A \cdot A B \otimes A^{\text{op}} \otimes B^{\text{op}} \) \( A^{\text{op}} A^{\text{op}} \) by \( a \otimes c \mapsto a \otimes c := \pi \otimes \pi \). We also note an isomorphism \( \text{End} \, A_B \cong \text{End} \, A^{\text{op}} \) by \( \pi \mapsto \tilde{\pi} \) where \( \tilde{\pi}(a) := \tilde{f}(a) \); similarly the left endomorphism algebra of \( A \mid B \) is identified with the right endomorphism algebra of \( A^{\text{op}} \mid B^{\text{op}} \). The intersection of left and right endomorphism algebras, \( S((A \mid B) \) and \( S(A^{\text{op}} \mid B^{\text{op}}) \) are therefore isomorphic as algebras. The algebras \( T(A \mid B) \cong T(A^{\text{op}} \mid B^{\text{op}}) \) via \( \pi \mapsto \tilde{\pi} \) is readily checked to be an isomorphism of algebras via eq. \((1)\). It follows that paired elements \( \gamma_j \in S(A \mid B) \), \( u_j \in T(A \mid B) \) form a right D2 quasibase for the \( A \mid B \) iff \( \pi_j \in S(A^{\text{op}} \mid B^{\text{op}}) \), \( \pi_j \in T(A^{\text{op}} \mid B^{\text{op}}) \) form a left D2 quasibase for \( A^{\text{op}} \mid B^{\text{op}} \). \( \square \)

We add two new characterizations of a one-sided D2 extension \( A \mid B \) in terms of restriction, induction and coinduction between \( A \)-modules and \( B \)- or \( R \)-modules.

**Proposition 2.2.** Let \( A \mid B \) be an algebra extension and \( C \) an algebra. \( A \mid B \) is right D2 if and only if either one of the conditions below is satisfied:

1. \( \pi_T \) is finite projective and for every \( C \)-\( A \)-bimodule \( CMA \), there is a \( T \otimes_K C \)-\( B \)-bimodule isomorphism
   \[ M \otimes_B A \cong M \otimes_R T \]

2. \( S_R \) is finite projective and for every \( C \)-\( A \)-bimodule \( CMA \), there is a \( T \otimes_K C \)-\( B \)-bimodule isomorphism
   \[ M \otimes_B A \cong \text{Hom}(S_R, MA) \]

**Proof.** Suppose \( A \mid B \) is right D2 with quasibases \( u_j \in T \) and \( \gamma_j \in S \). To the mapping \( M \otimes_R T \to M \otimes_B A \) defined by

\[ m \otimes_R T \mapsto mt \otimes_B t^2 \]

we define the inverse mapping by

\[ m \otimes_B a \mapsto \sum_j m \gamma_j(a) \otimes_R u_j. \]

The left \( T \)-module structures on \( M \otimes_R T \) and \( M \otimes_B A \) are given by \( t \cdot (m \otimes t') := m \otimes t' \) and \( t \cdot (m \otimes a) := mt \otimes bt^2 a \), both of which commute with the left \( C \)-module structure. It is clear from eq. \((1)\) that the mapping in eq. \((6)\) is left \( T \)-linear. That
the mappings are inverse follows from \( t^1 \gamma_j (t^2) \in R \) for each \( j \) and from applying eq. (5) combined with the well-definedness of some obvious mappings.

Furthermore, to the mapping \( M \otimes_B A \rightarrow \text{Hom} (S_R, M_R) \) defined by

\[
(8) \quad m \otimes_B a \mapsto (\alpha \mapsto ma(a))
\]

we define the inverse mapping by

\[
(9) \quad g \mapsto \sum_j g(\gamma_j) u^1_j \otimes_B u^2_j,
\]

where we note from eq. (6) that \( \sum_j \gamma_j \cdot (u^1_j \alpha (u^2_j)) = \alpha \). The left \( T \)-module structure on \( \text{Hom} (S_R, M_R) \) is given by \( (t \cdot f)(\alpha) := f(t^1 \alpha (t^2 - )) \) for each \( \alpha \in S \), where we make use of a right action of \( T \) on \( \text{End}_B A \) defined below in eq. (17) and restricted to \( S \). It is clear that this module action commutes with the left \( C \)-module structure on \( \text{Hom} (S_R, M_R) \) induced by \( C \cdot M \), and that the mapping in eq. (8) is left \( T \otimes C \)-linear.

Conversely, let \( C = A \) and \( M = A \), the natural \( A \)-\( A \)-bimodule, and \( \Leftarrow \) follows from [9, Theorem 2.1, (3),(4)] applied to the opposite algebra extension. (No use is made of the left \( T \)-module structure in this argument.) This is also easy to see by arriving at the defining condition for right \( D_2 \) \( AA \otimes_B AB \oplus \ast \cong \otimes^n AAB \), using the similar finite projectivity condition for \( _R T \) or \( S_R \), and the functor \( A \otimes_R - \) from left \( R \)-modules to \( A \)-\( B \)-bimodules as well as the functor \( \text{Hom} (-, A_R) \) from right \( R \)-modules to \( A \)-\( B \)-bimodules.

A corresponding characterization of left \( D_2 \) extensions is derivable by the opposite algebra extension technique introduced in Prop. 2.1.

3. Preliminaries on bialgebroids and Galois extensions

In this section we review the theory of bialgebroids, Hopf algebroids, their comodule algebras and Galois extensions. We provide an example of the Hopf algebroid in [1], drawn from separability and finite dimensional algebras.

Recall that a left bialgebroid \( R' \)-bialgebroid \( S' \) is first of all two algebras \( R' \) and \( S' \) with two commuting maps (“source” and “target” maps) \( \tilde{s}, \tilde{t} : R' \rightarrow S' \), an algebra homomorphism and anti-homomorphism resp., commuting in the sense that \( \tilde{s}(r) \tilde{t}(r') = \tilde{t}(r') \tilde{s}(r) \) for all \( r, r' \in R' \). Secondly, it is an \( R' \)-coring \( (S', \Delta : S' \rightarrow S' \otimes_{R'} S', \varepsilon : S' \rightarrow R) \) w.r.t. the \( R' \)-\( R' \)-bimodule \( S' \) given by \( (x \in S; r, r' \in R) \)

\[
(10) \quad r \cdot x \cdot r' = \tilde{t}(r') \tilde{s}(r)x.
\]

Third, it is a generalized bialgebra (and generalized weak bialgebra) in the sense that we have the axioms

\[
(11) \quad \Delta(x)(\tilde{t}(r) \otimes 1_S) = \Delta(x)(1_S \otimes \tilde{s}(r)),
\]

\[
\Delta(xy) = \Delta(x)\Delta(y) \quad \text{(which makes sense thanks to the previous axiom),} \quad \Delta(1) = 1 \otimes 1,
\]

and \( \varepsilon(1_{S'}) = 1_{R'} \) and

\[
(12) \quad \varepsilon(xy) = \varepsilon(x\tilde{s}(\varepsilon(y))) = \varepsilon(x\tilde{t}(\varepsilon(y)))
\]

for all \( x, y \in T', r, r' \in R' \). The \( R' \)-bialgebroid \( S' \) is denoted by \( (S', R', \tilde{s}, \tilde{t}, \Delta, \varepsilon) \).

A right bialgebroid is defined like a left bialgebroid with three of the axioms transposed; viz., eqs. (11), (11), and (12). The left bialgebroid \( (S', R', \tilde{s}, \tilde{t}, \Delta, \varepsilon) \) becomes a right bialgebroid under the opposite multiplication and exchanging source and target maps

\[
S'^{\text{op}} = (S'^{\text{op}}, R', \tilde{s}, \tilde{t}, \Delta, \varepsilon)
\]
and a left bialgebroid under the co-opposite comultiplication, opposite base algebra and exchanging source and target maps

\[ S_{\text{cop}}' = (S', R'^{\text{op}}, \mathfrak{t}, \mathfrak{s}, \Delta^{\text{op}}, \varepsilon), \]

which has the opposite \( R'^{\text{op}} \)-\( R \)-bimodule structure from the \( S' \) one starts with \[12\] section 2. A similar situation holds for opposites and co-opposites of right bialgebroids.

In \[12\] 4.1 we establish that \( S := \text{End}_B A_B \) is a left bialgebroid over the centralizer \( R \) with the \( R-R \)-bimodule structure given by eq. \[3\]. The comultiplication \( \Delta_S : S \rightarrow S \otimes_R S \) is given by either of two formulas, using either a left or right D2 quasibase, equal in case \( A \mid B \) is D2:

\[
\Delta_S(\alpha) := \sum_i \alpha(-t_i^1) t_i^2 \otimes \beta_i
\]

\[
\Delta_S(\alpha) := \sum_j \gamma_j \otimes u_j^1 \alpha(u_j^2 -). \]

Since \( S \otimes_R S \cong \text{Hom}_{B-B}(A \otimes_B A, A) \) via \( \alpha \otimes \beta \mapsto (a \otimes a' \mapsto \alpha(a)\beta(a')) \) \[12\] 3.11, with inverse given by either \( F \mapsto \sum_i F(- \otimes t_i) t_i^2 \otimes_R \beta_i \) or \( \sum_j \gamma_j \otimes_R u_j^1 F(u_j^2 \otimes -) \), our formulas for the coproduct simplify greatly via eq. \[5\] to \( \Delta_S(\alpha)(a \otimes a') = \alpha(aa') \) (thus a generalized Lu bialgebroid). The counit belonging to this coproduct is \( \varepsilon_S : S \rightarrow R \) given by \( \varepsilon_S(\alpha) = \alpha(1_A) \). It is then not hard to see that \( (S, \Delta, \varepsilon) \) is an R-coring, i.e., both maps are \( R \)-bimodule morphisms, that \( \Delta_S \) is coassociative and that \( (\varepsilon_S \otimes \text{id}) \Delta_S = \text{id}_S = (\text{id}_S \otimes \varepsilon_S) \Delta_S \). In \[8\] it is shown that a right D2 quasibase alone leads to a left bialgebroid \( S \) over \( R \) with \( S_R \) finite projective; similarly, a left D2 quasibase on \( A \mid B \) implies \( S \) is a left finite projective left \( R \)-bialgebroid.

The algebra \( T \) defined above for any algebra extension is a right bialgebroid over the centerizer \( R \). The \( R \)-coring structure underlying the right \( R \)-bialgebroid \( T \) is given by an \( R-R \)-bimodule structure \( r \cdot t \cdot r' = rt^1 \otimes t^2 r' \), a comultiplication equivalently definable using either a right D2 quasibase or left D2 quasibase,

\[
\Delta_T(t) := \sum_j (t^1 \otimes_B j)(t^2) \otimes_R u_j
\]

\[
\Delta_T(t) := \sum_i t_i \otimes_R (\beta_i(t^1) \otimes_B t^2)
\]

and counit \( \varepsilon_T(t) = t^1 t^2 \), which is the multiplication mapping \( \mu \) restricted to \( T \). It is also true for \( T \) that it is a right \( R \)-bialgebroid when \( A \mid B \) is in possession of only a left D2 quasibase, or only a right D2 quasibase; the only possible loss being finite \( R \)-projectivity of \( T \) on one side.

We note that \( S \) and \( T \) are \( R \)-dual bialgebroids w.r.t. either of the nondegenerate pairings \( \langle \alpha \mid t \rangle := \alpha(t^1) t^2 \) or \( \langle \alpha \mid t \rangle := t^1 \alpha(t^2) \), both with values in \( R \) \[12\] 5.3.

The opposite multiplication \( \overline{\pi} \) on \( A^{\text{op}} \) is characterized by the equation \( \overline{\pi}(\overline{a} \otimes \overline{b}) = \mu(b \otimes a) \) using the notation in Prop. \[2\] 1. The next proposition shows that passing to opposite algebras does not change the chirality of the bialgebroids \( S \) and \( T \) associated with a D2 extension, but it does change their underlying \( R \)-corings to the co-opposites.

**Proposition 3.1.** Let \( A \mid B \) be an algebra extension. Then \( A \mid B \) is left D2 with \( R \)-bialgebroids \( S \) and \( T \) if and only if \( A^{\text{op}} \mid B^{\text{op}} \) is right D2 with \( R^{\text{op}} \)-bialgebroids \( S_{\text{cop}} \) and \( T_{\text{cop}} \).
Proof. Suppose $A \mid B$ is left D2 with $a \otimes a' = \sum t_i \beta_i(a) a'$. Then we have seen in Prop. 2.1 that $A^\text{op} \mid B^\text{op}$ is right D2; in fact, $a \otimes a' = \sum t_i \beta_i(a) a'$, is the equation of a right D2 quasibase. The centralizer $C_{A^\text{op}}(B^\text{op}) = R^\text{op}$ is the opposite of $R = C_A(B)$. If we denote the comultiplication in $T(A^\text{op} \mid B^\text{op})$ by $\Delta_T$ we have

$$\Delta_T(t) = \sum_i t_i^2 \otimes \delta(t_i) \otimes R \otimes t_i = \Delta_T(t)$$

We conclude that $\Delta_T = \Delta^\text{op}_T$. Similarly $\Delta_S = \Delta^\text{op}_S$. It is also apparent that source and target maps are reversed in passing to the opposite algebras, e.g., $\lambda(a) = \rho(\pi)$ for $a \in A$. Finally we have shown in Prop. 2.1 that $S(A^\text{op} \mid B^\text{op}) \cong S$ and $T(A^\text{op} \mid B^\text{op}) \cong T$ as algebras. The converse is proven similarly.

In [12, 4.1] we observed that $S$ acts on $A$ via evaluation as a left $S$-module algebra (or algebroid); if $A_B$ is a balanced module, then the invariant subalgebra $A^S = B$. In this paper, we will be more concerned with the dual concept, comodule algebra (defined below). As an example of this duality, and a guide to what we are about to do, we dualize, as we would (but more carefully) for Hopf algebra actions, the left action just mentioned $\triangleright : S \otimes_B A \to A$, $\alpha \triangleright a := \alpha(a)$ for $\alpha \in S$, $a \in A$, to a right coaction $\varrho_T : A \to A \otimes_B T$ given by $\rho(a) = a_{(0)} \otimes a_{(1)}$ where $\alpha \triangleright a = a_{(0)}[a \mid a_{(1)}]$. This comes out as $\varrho_T(a) = \sum_j \gamma_j(a) \otimes u_j$, since $\alpha(a) = \sum \gamma_j(a)[a \mid u_j]$ (obtained by applying $\text{id} \otimes \alpha$ to eq. (3)). The resulting right $T$-comodule algebra structure on $A$ is studied in [8], where it is shown that assuming $A_B$ balanced and $A \mid B$ right D2 results in a Galois extension $A \mid B$ in the usual Galois coaction picture.

There is also an action of $T$ on $E$ studied in [12, 5.2]: the $R$-bialgebroid $T$ acts from the right on $E$ by

$$f \triangleleft t := t^1 f(t^2 -)$$

for $f \in E, t \in T$. This action makes $E$ a right $T$-module algebra with invariants $\rho(A)$ (where recall $\rho(a)(x) = xa$ for $x, a \in A$). Thinking in terms of Hopf algebra duality, we then expect to see a left coaction $\varrho : E \to S \otimes_R E$ with Sweedler notation $\varrho(f) = f_{(-1)} \otimes f_{(0)}$ satisfying $f \triangleleft t = [f_{(-1)} \mid t] f_{(0)}$. This comes out in terms of a right D2 quasibase as

$$\varrho(f) = \sum_j \gamma_j \otimes (f \triangleleft u_j)$$

since $f \triangleleft t = \sum_j t^1 \gamma_j(t^2) u_j^1 f(u_j^2 -) = \sum_j [\gamma_j \mid t] (f \triangleleft u_j)$. Since $E$ is a variant of a smash product of $A$ with $S$ (cf. [12, section 3]), we would want to show that $E$ is a Galois extension of a copy of $A^\text{op}$ somewhat in analogy with cleft Hopf algebra coaction theory although there is no antipode in our set-up: see eq. 3. Read for why we may think of the natural inclusion $S \to E$ as a total integral which cleaves the $S$-extension $E \mid \rho(A)$. We next turn to several definitions, lemmas and the main theorem below in which we prove that $\varrho$ is a Galois coaction for the left $S$-extension $E$ over $\rho(A)$.

**Definition 3.2.** Let $S'$ be a left $R'$-bialgebroid $(S', \hat{s}, \hat{t}, \Delta, \varepsilon)$. A left $S'$-comodule algebra $C$ is an algebra $C$ with algebra homomorphism $R' \to C$ together with a coaction $\delta : C \to S' \otimes_R C$, where values $\delta(c)$ are denoted by the Sweedler notation $c_{(-1)} \otimes c_{(0)}$, such that $C$ is a left $S'$-comodule over the $R'$-coring $S'$ [2, 18.1],

$$\delta(1_C) = 1_{S'} \otimes 1_C,$$
Definition 3.4. Let $S'$ be a left $S'$-bialgebroid. An algebra $\bar{\mathcal{A}}$ is said to be a left $S'$-bilegebroid over an algebra $R$. Then $A\mid B$ be an algebra extension and $T$ a right bialgebroid over an algebra $R$. Then $A\mid B$ be a right $T$-extension (or Galois extension) if and only if $A^{op} \mid B^{op}$ is left $T^{op}_{\text{cop}}$-extension (or Galois extension, respectively).

Proof. It is noted in [22], and explored further in the next subsection in this paper, that $T = (T, R, s_R, t_R, \Delta, \varepsilon)$ is a right $R$-bialgebroid if $T^{op}_{\text{cop}} = (T^{op}, R^{op}, s_R, t_R, \Delta^{op}, \varepsilon)$ is a left $R^{op}$-bialgebroid. If $A\mid B$ is a right $T$-extension with coaction $\delta(a) = a(0) \otimes a(1)$, then we may define a $T^{op}$-algebra structure on $A^{op}$ via the induced homomorphism $R^{op} \to A^{op}$,  

$$\overline{\delta(x)} = \overline{a(0)} \otimes a(1)$$

using an obvious extension of the notation in Prop. [24]. All the properties of a left comodule algebra follow routinely, including $A^{\text{co}T} = \text{co}T^{op}_{\text{cop}} A$. The one-to-one correspondence of right and left Galois extensions follows from noting that the Galois mapping $\beta : A^{op} \otimes B^{op} A^{op} \to T^{op}_{\text{cop}} \otimes R^{op} A^{op}$ defined above and the right
Galoiś mapping $\beta_R : A \otimes_B A \to A \otimes_R T$ given by $\beta_R(a \otimes a') := aa'(0) \otimes a'(1)$ are isomorphically related by a commutative square, i.e.,
\[
\beta(a \otimes_B a') = \beta_R(a \otimes_B a')
\]
for $a, a' \in A$.

A new example of Hopf algebroid. One unfortunate side-effect of generalizing Lu’s bialgebroid $A^c$ over an algebra $A$ [14] to the right bialgebroid $T$ of a D2 extension $A \mid B$ (in [12]) is that the antipode is lost, for the flip or twist anti-automorphism on $A \otimes_K A^{op}$ does not extend to a self-mapping of $(A \otimes_B A)^B$. If we require $B$ to be separable with symmetric separability element however, there is a projection of $A \otimes_B A \to T$ which we may apply to define a twist of $T$. However, Lu’s definition of Hopf algebroid [14] makes it necessary to find an appropriate section $T \otimes_B T \to T \otimes_K T$ of the canonical map in the other direction, although the centralizer $R$ is not a priori separable. In this subsection, we carry out this plan using instead the alternative definition of Hopf algebroid in [14, Böhm and Szlachányi]. They define a Hopf algebroid to be a left $R'$-bialgebroid $S'$ with anti-automorphism $\tau : S' \to S'$ (called antipode) satisfying eqs. (28), (29) and (30) below.

Let $K$ be a commutative ring and $B$ a Kanzaki separable $K$-algebra [11] strongly separable algebra. This means that there is a separability element $e = e_1 \otimes e_2 \in B \otimes_K B$ which is symmetric, so that $e_1e_2 = 1 = e_2e_1$ as well as $be = eb$ and $e_1b \otimes e_2 = e_1 \otimes be_2$ for all $b \in B$. (Typically for quantum algebra, we use both these equalities repeatedly below together with $bt = tb$ for $t \in T$ and well-definedness of various mappings on equal commuting elements.) For example, all separable algebras over a field of characteristic zero are Kanzaki separable. Over a field of characteristic $p$ matrix algebras of order divisible by $p$ are separable although not Kanzaki separable. Fix the notation above for the next theorem.

**Theorem 3.6.** Let $A \mid B$ be a D2 extension of $K$-algebras where $B$ is Kanzaki separable. Then the left bialgebroid $T^{}_{\text{cop}}$ is a Hopf algebroid.

**Proof.** The standard right bialgebroid $T = (A \otimes_B A)^B$ with structure $(T, R, s_R, \Delta, \varepsilon)$ becomes a left bialgebroid via the opposite multiplication and co-opposite comultiplication as follows [12, 2.1]:
\[
T_{\text{cop}}^{} := (T^{op}, R^{op}, s_L = s_R, t_L = t_R, \Delta^{op}, \varepsilon).
\]
The multiplication on $T^{op}$ is
\[
tt' = t^1t'^1 \otimes_B t^2t^2
\]
while the target and source maps are $s_L : R^{op} \to T^{op}$, $t_L : R \to T^{op}$ are then $s_L(r) = 1_A \otimes r$, $t_L(r) = r \otimes 1_A$ for $r \in R^{op}$. The $R^{op}$- $R^{op}$-bimodule structure on $T$ is then given by
\[
r \cdot t \cdot r' = s_L(r)t_L(r')t = (r' \otimes r)t = r't^1 \otimes t^2r.
\]
In other words $s_{\text{cop}} T_{\text{cop}}$ is the standard bimodule $R T_R$ with endpoint multiplication after passing to modules over the opposite algebra of $R$. Tensors over $R^{op}$ are the same as tensors over $R$ after a flip; e.g.,
\[
T \otimes_{R^{op}} T \cong T \otimes_R T \cong (A \otimes_B A \otimes_B A)^B.
\]
via the mapping
\[
(25) \quad t \otimes t' \xrightarrow{\sim} t'^1 \otimes t'^2 \otimes t^2, 
\]
which is an $R^{op} \cdot R^{op}$-isomorphism \[5.1\].

The comultiplication $\Delta^{op} : T \rightarrow T \otimes R^{op}$ is given for $t \in T$ by the co-opposite of eq. \[15\],
\[
(26) \quad \Delta^{op}(t) = \sum_j u_j \otimes (t^1 \otimes B \gamma_j(t^2))
\]
which in $(A \otimes_B A \otimes_B A)^B$ is the value $t^1 \otimes 1_A \otimes t^2$ after applying eq. \[26\] and the right $D^2$ quasibases eq. \[11\]. We will denote below $\Delta^{op}(t) = t(1) \otimes t(2)$.

The antipode $\tau : T \rightarrow T$ is a flip composed with a projection from $A \otimes_K A$:
\[
(27) \quad \tau(t) := e^1 t^2 \otimes_B t^1 e^2.
\]
We next note that $\tau$ is an anti-automorphism of order two on $T^{op}$. Let $e = f$ in $B \otimes_K B$ so that
\[
\tau(t')\tau(t) = e^1 t'^2 f^1 t'^1 \otimes_B t^1 f^2 t'^1 e^2 = e^1 t'^2 f^1 t^1 e^2 = \tau(t't') \text{ since } t'^1 e^2 \otimes_K e^1 t^2 \in (A \otimes_K A)^B \text{ and } f^2 f^1 = 1_B. \quad \text{In addition,}
\]
\[
\tau^2(t) = \tau(e^1 t^2 \otimes t^1 e^2) = f^1 t^1 e^2 \otimes_B e^1 t^2 f^2 = t,
\]
since $e^2 e^1 = 1$.

Next, we show that $\tau$ satisfies the three axioms of [11 Def. 4.1] given below in eqs. \[28\]–\[30\]. Note that $\tau^{-1} = \tau$. First,
\[
(28) \quad \tau \circ t_L = s_L
\]
since $\tau(r \otimes 1) = e^1 \otimes_B re^2 = 1 \otimes r$ for $r$ in the centralizer $C_A(B)$.

Second, we have the equality in $T \otimes_R T$,
\[
(29) \quad \tau^{-1}(t(2))(1) \otimes \tau^{-1}(t(2))(2) = \tau^{-1}(t) \otimes 1_T
\]
which follows from eq. \[26\] and (working from left to right):
\[
\sum_{j,k} u_k \otimes (e^1 \gamma_j(t^2) \otimes_B \gamma_k(t^1 e^2))u_j = \sum_{j,k} u_k \otimes (e^1 \gamma_j(t^2)u_j^1 \otimes_B u_j^2 \gamma_k(t^1 e^2))
\]
the last expression mapping via the isomorphism in eq. \[26\] into
\[
e^1 \otimes_B t^2 \gamma_k(t^1)u_k^1 \otimes_B u_k^2 e^2 = 1_A \otimes e^1 t^2 \otimes t^1 e^2
\]
in $(A \otimes_B A \otimes_B A)^B$, which is the image of $\tau^{-1}(t) \otimes 1_T$ under the same isomorphism.

Finally, we have the equality in $T \otimes_R T$,
\[
(30) \quad \tau(t(1))(1)t(2) \otimes \tau(t(1))(2) = 1_T \otimes \tau(t)
\]
for all $t \in T$, which follows similarly from
\[
\sum_{j,k} u_k(t(1) \otimes_B \gamma_j(t^2))u_j^1 \otimes_B \gamma_k(t^1)u_j^2 e^2 = \sum_{j,k} u_k^1 t^1 \otimes_B \gamma_j(t^2)u_k^2 \otimes e^1 u_j^2 \otimes_B \gamma_k(u_j^1) e^2
\]
which maps isomorphically to
\[
\sum_{j,k} e^1 u_j^2 \otimes_B \gamma_k(u_j^1)u_k^1 \otimes_B \gamma_j(t^2)u_k^2 e^2 = \sum_j e^1 u_j^2 \otimes t^1 \otimes \gamma_j(t^2)u_j^1 e^2 =
\]
\[
e^1 t^2 \otimes t^1 e^2 \otimes 1_A \epsilon \xrightarrow{\sim} 1_T \otimes \tau(t)
\]
since $\sum_j \gamma_j(a)u_j^2 e^2 \otimes_K e^1 u_j^2 = e^2 \otimes_K e^1 a$ for $a \in A$ follows from eq. \[35\].
Suppose $K$ is a field, then $B$ is finite dimensional as it is a separable algebra. If moreover $A \mid B$ is a proper algebra extension, then there is bimodule projection $A \to B$ by separability, whence $A_R$ is finitely generated (and projective by the D2 quasibase eq. 5) and so $A$ is finite dimensional as well. The theorem is thus viewed as a natural generalization of Lu’s Hopf algebroid with twist antipode to certain finite-dimensional algebra pairs.

4. Main Theorem: endomorphism algebra extension is Galois

Consider a right D2 extension $A \mid B$ with right D2 quasibases $\gamma_j \in S$, $u_j \in T$, left endomorphism algebra $\mathcal{E} = \text{End}_B A$ and natural algebra embedding $\rho : A \hookrightarrow \mathcal{E}$ by right multiplications. We introduced in eq. 5 a possible coaction $\rho : \mathcal{E} \to S \otimes_R \mathcal{E}$, which in this section we show makes the extension $\mathcal{E} \mid \rho(A)$ into a left Galois extension — a nontrivial proof involving several lemmas at first. The next lemma will be used among other things to show that the coaction $\rho$ is coassociative.

**Lemma 4.1.** Let $A \mid B$ be a right D2 extension. Then we have the isomorphisms

(31) \[ S \otimes_R \mathcal{E} \cong \text{Hom}(B A \otimes_B A, B A) \]

via $\alpha \otimes f \mapsto (a \otimes a' \mapsto \alpha(a)f(a'))$, and

(32) \[ S \otimes_R S \otimes_R \mathcal{E} \cong \text{Hom}(B A \otimes_B A \otimes_B A, B A) \]

via $\alpha \otimes \beta \otimes f \mapsto (a \otimes a' \otimes a'' \mapsto \alpha(a)\beta(a')f(a''))$.

**Proof.** The inverse in (31) is given by $F \mapsto \sum_j \gamma_j \otimes u_j \gamma_i u_j \otimes u_j \otimes -$ by eq. 5.

The inverse in (32) is given by

\[ F \mapsto \sum_{j,k,i} \gamma_j \otimes \gamma_k \otimes u_k \gamma_i u_k \otimes u_k \otimes u_k \otimes - \]

since

\[
\sum_{j,k,i} \gamma_j(a)\gamma_k(a')u_k \gamma_i u_k \otimes u_k \otimes u_k \otimes a'' = \sum_{j,k} \gamma_j(a)u_j \gamma_i (a')u_j \otimes u_j \otimes u_j \otimes a''
\]

\[ = F(a \otimes a' \otimes a'') \quad \text{and} \]

\[
\sum_{j,k,i} \gamma_j \otimes \gamma_k \otimes R u_k \gamma_i u_k \otimes \alpha(u_j \gamma_i u_k) \beta(u_j) \otimes f(-) = \sum_{j,k} \gamma_j \otimes \alpha(u_j \gamma_i (a')u_j) \beta(u_j) \otimes f(-)
\]

\[ = \sum_j \gamma_j(-) u_j \alpha(u_j) \otimes \beta \otimes f = \alpha \otimes \beta \otimes f \quad \text{for} \quad f \in \mathcal{E}, \alpha, \beta \in S. \]

The existence alone of an isomorphism in the next lemma may be seen by letting $M$ be free of rank one.

**Lemma 4.2.** Given any algebras $A$ and $B$, with modules $M_A$, $B U$ and bimodule $B N_A$ with $M_A$ f.g. projective, then

(33) \[ M \otimes_A \text{Hom}(B N, B U) \cong \text{Hom}(B \text{Hom}(M_A, N_A), B U) \]

via the mapping $m \otimes \phi \mapsto (\nu \mapsto \phi(\nu(m)))$.

**Proof.** Let $m_i \in M$, $g_i \in \text{Hom}(M_A, A_A)$ be dual bases for $M_A$. For each $n \in N$ let $ng_i(-)$ denote the obvious mapping in $\text{Hom}(M_A, N_A)$. Then the inverse mapping is given by

(34) \[ F \mapsto \sum_i m_i \otimes (n \mapsto F(ng_i(-))). \]
Note that both maps are well-defined module homomorphisms, and inverse to one another since
\[ \sum_i m_i \otimes \phi(-g_i(m)) = \sum_i m_i \otimes_A (g_i(m) \phi)(-) = \sum_i m_i g_i(m) \otimes = m \otimes \phi \]
and \( \nu \mapsto \sum_i F(\nu(m_i) g_i) = F(\nu) \) for \( \nu \in \text{Hom}(M, N) \).

The lemma above is relevant in our situation since the D2 condition implies that a number of constructions such as the tensor-square and endomorphism algebras are finite projective. For example, \( E_A \) is f.g. projective \([12, 3.13]\), which we may also see directly from eq. (5) by applying \( \text{id}_A \otimes_B f \) for \( f \in E \), viewing \( \gamma_j \in S \subseteq E \) and an obvious mapping of \( A \otimes_B A \) into \( \text{Hom}(E_A, A_A) \) which appears in the next lemma.

**Lemma 4.3.** If \( A | B \) is right D2, then
\[ B_A \otimes_B A \cong \text{BHom}(E_A, A_A) \]
via \( \Psi(a \otimes a')(f) = af(a') \).

**Proof.** Let \( F \in (E_A)^* \) (the right \( A \)-dual of \( E \)). Define an inverse \( \Psi^{-1}(F) = \sum_j F(\gamma_j) u_j \). Then \( \Psi^{-1} \Psi = \text{id}_{A \otimes_B A} \) by eq. (5). Also \( \Psi \Psi^{-1} = \text{id}_{E^*} \) since for \( f \in E \),
\[ \Psi \Psi^{-1}(F)(f) = \sum_j F(\gamma_j) u_j^1 f(u_j^2) = F(\sum_j \gamma_j(-) u_j^1 f(u_j^2)) = F(f). \]

Recall that \( \rho(A) \) denotes the set of all right multiplication operators by elements of \( A \).

**Theorem 4.4.** Let \( A | B \) be a right D2 extension. Then \( E \) is a left \( S \)-comodule algebra with the coaction \([12]\) and a Galois extension of its coinvariants \( \rho(A) \).

**Proof.** The coaction \( \rho \) in eq. \([18]\) has value on \( f \in E = \text{End}_B A \) given by
\[ f_{(-1)} \otimes f_{(0)} = \sum_j \gamma_j \otimes u_j^1 f(u_j^2) \]
where \( \gamma_j \in S, u_j \in T \) is a right D2 quasibase. First, the algebra homomorphism \( R \rightarrow E \) is given by \( \lambda \), so for \( 1_E = \text{id}_A = 1_S \), we have
\[ \rho(\text{id}_A) = \sum_j \gamma_j \otimes_R \lambda(u_j^1 u_j^2) = \sum_j \gamma_j(-) u_j^1 u_j^2 \otimes \text{id}_A = 1_S \otimes 1_E. \]

Secondly, we show that \( E \) forms a left \( S \)-comodule w.r.t. the \( R \)-coring \( S \) and the coaction \( \rho \). The coaction is coassociative, \( (\Delta_S \otimes \text{id}_E) \rho = (\text{id}_S \otimes \rho) \rho \), for we use lemma \([14]\) (as an identification and suppressing the isomorphism) and eqs. \([14]\).
and (11) to check values of each side of this equation, evaluated on $A \otimes_B A \otimes_B A$:

\[
\sum_j \Delta_S(\gamma_j) \otimes (f \triangleleft u_j)(a \otimes a' \otimes a'') = \sum_{k,j} \gamma_k(a)u_j^1 \gamma_j(u_k^2a')u_j^1 f(u_j^2a'')
\]

\[
= \sum_j \gamma_j(aa')u_j^1 f(u_j^2a'') = f(aa'a'')
\]

\[
= \sum_{i,j} \gamma_i(a)\gamma_j(a')u_j^1 u_i^1 f(u_j^2u_i^2a'')
\]

\[
= \left( \sum_j \gamma_j \otimes \varrho(f \triangleleft u_j)(a \otimes a' \otimes a''). \right)
\]

We also show that $\varrho : \mathcal{E} \to S \otimes_R \mathcal{E}$ is a left $R$-module morphism: given $r \in R$, $f \in \mathcal{E}$, we use lemma (11) again to note that for $a, a' \in A$

\[
\varrho(\lambda(r)f)(a \otimes a') = \sum_j \gamma_j(a)u_j^1 rf(u_j^2a') = rf(aa') = (r \cdot f_{(1)} \otimes f_{(0)})(a \otimes a'),
\]

by an application of eq. (15) (inserting an $r \in C_A(B)$). (Note with $r = 1$ that we obtain

\[
(35) \quad \varrho(\alpha) = \Delta_S(\alpha) \quad \forall \alpha \in S,
\]

which should be compared to the total integral and cleft extension approach in [10, 6.1] and [6]. Finally, $\mathcal{E}$ is counital, whence a left $S$-comodule, since for $f \in \mathcal{E}$,

\[
(\varepsilon \otimes \text{id}_S) \varrho(f) = \sum_j \gamma_j(1)u_j^1 f(u_j^2-1) = f.
\]

Next we compute that $\text{Im} \varrho$ lies in a submodule of $S \otimes_R \mathcal{E}$ where tensor product multiplication makes sense: again using lemma (11) and for $a, a' \in A$

\[
(f_{(1)}\tilde{t}(r) \otimes f_{(0)})(a \otimes a') = \sum_j \gamma_j(ar)u_j^1 f(u_j^2a') = f(ar')
\]

\[
= \sum_j \gamma_j(a)u_j^1 f(u_j^2ra') = (f_{(1)} \otimes f_{(0)}\lambda(r))(a \otimes a').
\]

Then multiplicativity of the coaction follows from the measuring axiom satisfied by the right action of $T$ on $\mathcal{E}$ [12, 5.2] and eq. (15) ($f, g \in \mathcal{E}$):

\[
\varrho(fg)(a \otimes a') = \sum_j (\gamma_j \otimes (f \triangleleft u_{j(1)}) \circ (g \triangleleft u_{j(2)}))(a \otimes a') = \sum_j \gamma_j(a)u_j^1 f(g(a'))
\]

\[
= \sum_{j,k} \gamma_j(a)u_j^1 f(\gamma_k(u_j^2)u_k^1 g(u_k^2a')) = f(g(aa')) = \sum_{j,i} \gamma_i(\gamma_j(a))u_i^1 f(u_i^2u_j^1 g(u_j^2a'))
\]

\[
= \sum_{i,j} \gamma_i \circ \gamma_j \otimes (f \triangleleft u_i) \circ (g \triangleleft u_j)(a \otimes a') = \varrho(f) \varrho(g)(a \otimes a').
\]

Next we determine the coinvariants $\text{co}^S \mathcal{E}$. Given $a \in A$, we note that $\rho(a) \in \text{co}^S \mathcal{E}$ since

\[
\varrho(\rho(a)) = \sum_j \gamma_j \otimes u_j^1 u_j^2 (-a) = 1_S \otimes \rho(a).
\]
Conversely, suppose \( \sum_j \gamma_j \otimes (f \circ u_j) = 1_S \otimes f \) in \( S \otimes_R \mathcal{E} \cong \text{Hom} (B_A \otimes_B A, B_A) \), then for \( a, a' \in A \),
\[
\sum_j \gamma_j(a)u_j^1f(u_j^2a') = f(aa') = af(a').
\]
It follows that \( f(a) = af(1_A) \) for all \( a \in A \), so \( f = \rho(f(1)) \in \rho(A) \). Hence, \( \text{co}^S \mathcal{E} = \rho(A) \).

Finally, the Galois mapping
\[
\beta : \mathcal{E} \otimes_{\rho(A)} \mathcal{E} \to S \otimes_R \mathcal{E}, \quad \beta(f \otimes g) = f(-1) \otimes f(0)g
\]
under the identification \( S \otimes_R \mathcal{E} \cong \text{Hom} (B_A \otimes_B A, B_A) \) in lemma \( \Box \) is given by an application of eq. \( 35 \): \( (a, a' \in A, f, g \in \mathcal{E}) \)
\[
(37) \quad \beta(f \otimes g)(a \otimes a') = \sum_j \gamma_j(a)u_j^1f(u_j^2g(a')) = f(ag(a')).
\]

We show \( \beta \) to be a composite of several isomorphisms using the lemmas (and see the commutative diagram in section 1). First note that \( \rho(A) \cong A^{op} \) and \( \rho(A)\mathcal{E}_{\rho(A)} \) given by \( \rho(a') \circ f \circ \rho(a)(a'') = f(a''a)a' \) is equivalent to \( \mathcal{E}_A \) given by \( a \cdot f \cdot d(a'') = f(a''a)a' \). This is the usual \( A \)-bimodule structure on the left endomorphism algebra \( \mathcal{E} \) considered in \( [12, 3.13] \), where \( \mathcal{E}_A \) is shown to be f.g. projective. Consider then the composition of isomorphisms,
\[
\mathcal{E} \otimes_{\rho(A)} \mathcal{E} \cong \mathcal{E} \otimes_A \text{Hom} (B_A, B_A) \cong \text{Hom} (\text{H}om (\mathcal{E}_A, A), B_A) \cong \text{Hom} (B_A \otimes_B A, B_A)
\]
given by
\[
f \otimes g \mapsto g \otimes f \mapsto (\nu \mapsto f(\nu(g))) \mapsto (a \otimes a' \mapsto f(ag(a'))).
\]
This is \( \beta \) as given in eq. \( 37 \), whence \( \beta \) is an isomorphism and the extension \( \mathcal{E} | \rho(A) \) is Galois. \( \square \)

By chasing the diagram in Figure 1 around in the opposite direction, we obtain the inverse Galois mapping, \( \beta^{-1} : S \otimes_R \mathcal{E} \to \mathcal{E} \otimes_{\rho(A)} \mathcal{E} \), given by
\[
(38) \quad \beta^{-1}(\alpha \otimes h) = \sum_j \alpha(-u_j^1)h(u_j^2) \otimes \gamma_j.
\]
We check this directly \( (f, g, h \in \mathcal{E}, \alpha \in S) \):
\[
\beta^{-1}(\beta(f \otimes g)) = \beta^{-1}(\sum_j \gamma_j \otimes u_j^1f(u_j^2g(-)) = \sum_{j,k} \gamma_j(-u_k^1)u_j^1f(u_j^2g(u_k^2)) \otimes \delta_k =
\sum_k f(-u_k^1g(u_k^2)) \otimes_{\rho(A)} \gamma_k = \sum_k f(-) \otimes \rho(u_k^1g(u_k^2)) \circ \gamma_k = f \otimes g,
\]
by eq. \( 14 \) and
\[
\beta(\beta^{-1}(\alpha \otimes h)) = \sum_{j,k} \gamma_j \otimes u_j^1\alpha(u_j^2\gamma_k(-)u_k^1)h(u_k^2) = \sum_j \gamma_j \otimes_R u_j^1\alpha(u_j^2)h(-) = \alpha \otimes h.
\]
5. Corollaries and a converse of the main theorem

In this section we recall the characterization in [8] of Galois extension as one-sided D2 and balanced extensions. Then right D2 extensions are shown to have left endomorphism algebra extensions that are left D2. Right endomorphism algebra extensions are shown to be right D2. A converse endomorphism ring theorem for one-sided split D2 Frobenius extensions is proven in Theorem th-converse-endo.

We next recall the right D2 characterization of a right Galois $T$-extension $A \mid B$ for a right bialgebroid $T$ over an algebra $R$ [8] (with roots in [10, 18]). We show in detail why this is equivalent to a left D2 characterization of a left Galois extension.

**Theorem 5.1 ([8]).** Let $A \mid B$ be a proper algebra extension. Then the following hold and are equivalent:

1. $A \mid B$ is a right $T$-Galois extension for some left finite projective right bialgebroid $T$ over some algebra $R$ if and only if $A \mid B$ is right D2 and right balanced.
2. $A \mid B$ is a left $S$-Galois extension for some right finite projective left bialgebroid $S$ over some algebra $R$ if and only if $A \mid B$ is left D2 and left balanced.

**Proof.** We have established the characterizations in [11] or [2] in [8]. Their equivalence may be seen as follows. Suppose we assume [11] and we are given a left D2 and left balanced extension $A \mid B$. Then by Prop. 2.4, $A^{op} \mid B^{op}$ is right D2, and moreover right balanced since $End A^{op} | B^{op} = End B A$ in which left multiplication $\lambda(\pi)$ is equal to $\rho(a)$, the canonical image of right multiplication. Then $A^{op} \mid B^{op}$ is a right $T_{cop}$-Galois extension where $T_{cop}$ is the associated right $R^{op}$-bialgebroid by Prop. 3.1, whence by Prop. 3.5, $A \mid B$ is a left $T^{op}$-Galois extension for the left $R$-bialgebroid $T^{op}$. Since $T_R$ is f.g. projective by the left D2 hypothesis, it follows that $T^{op}$ is right f.g. $R$-projective as well, since $T$ and $T^{op}$ have the same underlying $R$-$R$-bimodule.

The converse follows by reversing the order of application of Props. 2.4, 3.1 and 3.5 and noting that $S_{cop}^{op}$ is left finite projective over $R^{op}$ iff $S$ is right finite projective over $R$.

The following endomorphism ring theorem is then a corollary of this theorem and the main Theorem 4.4.

**Corollary 5.2 (Endomorphism Ring Theorem for D2 Extensions).** If $A \mid B$ is a right D2 algebra extension, then $E \mid A^{op}$ is a left D2 extension (and left balanced).

This generalizes [12, Theorem 6.7] by leaving out the hypothesis that $A \mid B$ be a Frobenius extension (i.e., $A_B$ be finite projective and $A \cong \text{Hom}(A_B, B_B)$ as natural $B$-$A$-bimodules). Via eq. (38) one arrives at left D2 quasibases for $E \mid A^{op}$ in terms of right D2 quasibases for $A \mid B$:

\begin{equation}
T_j := \sum_k \gamma_j(-u^1_k)u^2_k \otimes \gamma_k, \quad B_j(f) := f \circ u_j = u^1_j f(u^2_j - )
\end{equation}

We note that $T_j = \beta^{-1}(\gamma_j \otimes \text{id}_A) \in (E \otimes_{\rho(A)} E)^{\rho(A)}$ and $B_j \in \text{End}_{\rho(A)}(E_{\rho(A)})$, and verify the essential part of the left D2 quasibase eq. (4) in $E \otimes_{\rho(A)} E$: $(f \in E)$

$$\sum_j T_j B_j(f) = \sum_{j,k} \gamma_j(-u^1_k)u^2_k \otimes_{\rho(A)} \gamma_k(u^1_j f(u^2_j - )) =$$
\[ \sum_{j,k} \gamma_j (-) u_k^j \otimes \gamma_k (u_j^2 f(u_j^2 -) u_k^j) = \sum_j \gamma_j (-) u_j^2 f(u_j^2) \otimes 1_\varepsilon = f \otimes 1_\varepsilon. \]

We also have a corollary for the right endomorphism algebra \( E := \text{End}_A B \) extending \( A \) via \( \lambda : A \mapsto E \).

**Corollary 5.3.** If \( A | B \) is a left \( D_2 \) algebra extension, then \( E | A \) is a left \( D_2 \), left balanced and a left Galois \( S_{\text{cop}} \)-extension.

**Proof.** We pass to the right \( D_2 \) extension \( A^{\text{op}} | B^{\text{op}} \), with left \( D_2 \) extension, left balanced and left Galois \( S_{\text{cop}} \)-extension \( \rho : A^{\text{op}} \mapsto \text{End}_{B^{\text{op}}} A^{\text{op}} \) by the main theorem and Theorem 5.1. Note again that \( \text{End}_B A \simeq E \) via \( f \mapsto \overline{f} \) defined in Prop. 2.4 where \( \lambda(a) f \lambda(a') = \rho(\overline{f}) \rho(a') \) for all \( a, a' \in A \), \( f \in E \), whence the isomorphic extension \( E | A \) is left \( D_2 \), left balanced and left Galois.

The left \( S \)-coaction on \( A \) implied by the corollary is given by \( (g \in E) \)

\[ (40) \quad \delta_L (g) = \sum_i \beta_i \otimes_{B^{\text{op}}} g(-1_i^1)1_i^2 \]

which transferred to the smash product \( A#S \simeq \text{End}_A B \) (via \( a#\alpha \mapsto \lambda(a) \circ \alpha \) [3, 3.8, 4.5]) comes out as

\[ (41) \quad \delta_L (a#\alpha) = \alpha_{(2)} \otimes a#\alpha_{(1)} \]

by comparing with eq. [3]. [3] briefly sketches how a smash product \( A#S \) is a Galois extension over \( A \) if \( S \) (is a left bialgebroid acting on \( A \) and) possesses an antipode (e.g., if \( A | B \) is \( H \)-separable).

Every one-sided \( D_2 \) extension has a canonical \( T \)- and \( S \)-bialgebroid associated to it; the next corollary observes (thanks to Böhm) that the \( S \)-bialgebroid of a right \( D_2 \) extension is isomorphic to the \( T^{\text{op}} \)-bialgebroid of its left \( D_2 \) endomorphism algebra extension.

**Corollary 5.4.** Under the hypotheses in Theorem 5.1 the Galois mapping \( \beta \) in eq. 37 restricts to an isomorphism of left bialgebroids

\[ (42) \quad (E \otimes_{\rho(A)} E)^{\rho(A)} \simeq S \]

where the left-hand structure has the opposite multiplication of that in eq. [1].

**Proof.** The isomorphism of \( K \)-modules follows from noting that \( E^{\rho(A)} = \text{End}_{B^{\text{op}}} A \simeq \text{End}_{A^{\text{op}}} B \) via \( f \mapsto f(1) \), and applying the particular \( \rho(A) \)-\( \rho(A) \)-bimodule linearity of the Galois mapping. If \( T^1 \otimes T^2 \in (E \otimes_{\rho(A)} E)^{\rho(A)} \), then \( \beta(T^1 \otimes T^2) = T^1(-T^2(1)) \), which is in \( \text{End}_{B^{\text{op}}} A_B \) since \( T^1(-T^2(a)) = T^1(-T^2(1))a \) for each \( a \in A \).

The mapping \( \beta \) is an algebra isomorphism since

\[ \beta(U^1 T^1 \otimes T^2 U^2) = U^1 T^1(-T^2(U^2(1))) = U^1(T^1(-T^2(1))U^2(1)) = \beta(U^1 \otimes U^2) \circ \beta(T^1 \otimes T^2), \]

where also \( U^1 \otimes U^2 \in (E \otimes_{\rho(A)} E)^{\rho(A)} \).

The mapping \( \beta \) commutes with the source, target, counit and comultiplication mappings of each bialgebroid; e.g., \( \beta(1_A \otimes \lambda(r)) = \rho(r) \) for each \( r \in R \), and \( \varepsilon_S(\beta(T^1 \otimes T^2) = T^1(T^2(1)) \) to which \( \varepsilon_T(T^1 \otimes T^2) = T^1 T^2 \) maps under the identification \( \text{End}_{B^{\text{op}}} A_B \simeq R \).
There has been a question of whether a right or left progenerator H-separable extension $A \mid B$ is split (i.e., has a $B$-$B$-bimodule projection $A \to B$), whence Frobenius: an affirmative answer implies some generalizations of results of Noether-Brauer-Artin on simple algebras [19]. Unfortunately, the next example of a one-sided free H-separable non-Frobenius extension rules out this possibility.

**Example 5.5.** Let $K$ be a field and $B$ the 3-dimensional algebra of upper triangular $2 \times 2$-matrices, which is not self-injective. Since $B \mid K1$ is trivially D2, the endomorphism algebra $A := \text{End} \, B_K \cong M_3(K)$ is a left D2 extension of $\lambda(B)$, which w.r.t. the ordered basis $\langle e_{11}, e_{12}, e_{22} \rangle \lambda(B)$ is the subalgebra of matrices

$$[x, y, z] := \begin{pmatrix}
x & 0 & 0 \\
0 & x & y \\
0 & 0 & z
\end{pmatrix}$$

The left D2 quasibases for $A \mid B$ obtained from the opposite version of eq. (39) indicate an H-separable extension since the $B_j$ are all of the form $p(r_j)$ for $r_j \in R$. Now the centralizer $R$ is the 3-dimensional algebra spanned by matrix units $e_{11}, e_{21}, e_{22} + e_{33}$. The module $B \Lambda A$ is free with left $B$-module isomorphism $A \to B^3$ given by “separating out the columns”

$$(a_3) \mapsto ([a_{11}, a_{21}, a_{31}], [a_{12}, a_{22}, a_{32}], [a_{13}, a_{23}, a_{33}])$$

whence $A \otimes_B A$ and $\text{Hom} (R_K, A_K)$ are both 27-dimensional. The $A$-$A$-homomorphism $A \otimes_B A \to \text{Hom} (R_K, A_K)$ given by $a \otimes c \mapsto (r \mapsto arc)$ is easily computed to be surjective, therefore an isomorphism, whence $A A \otimes_B A A \cong A A^3$, which shows $A \mid B$ is H-separable (and D2). By [16] 6.1, the extension $A \mid B$ is not Frobenius since $B$ is not a Frobenius algebra; therefore $A \mid B$ is not split. (Alternatively, if there is a $B$-linear projection $E : A \to B$, we note $E(e_{32}) = 0$, so $e_{33} = e_{32} e_{23} \in \ker E$, a contradiction.) By applying the matrix transpose, the results of this example may be transposed to a right-sided version.

The endomorphism ring theorem for Galois extensions below follows directly from the main theorem and Theorem 5.1. Let $T'$ be a right $R'$-bialgebroid with $\nu T'$ finite projective and $S$ is as usual the left bialgebroid $\text{End} \, B \Lambda A_B$ over the centralizer.

**Corollary 5.6 (Endomorphism Ring Theorem for Galois Extensions).** If $A \mid B$ is a right Galois $T'$-extension, then $E \mid A^{op}$ is a left $S$-Galois extension.

Next we note a converse of the endomorphism ring theorem in case of Frobenius D2 extensions. Recall that for a Frobenius extension $A \mid B$, being right D2 is equivalent to being left D2. [22] 9. Also recall our notation $E := \text{End} \, A_B$. We will essentially note that under favorable conditions equal modules are cancellable in reversing the argument in [12] 6.7.

**Theorem 5.7.** Let $A \mid B$ be a right generator Frobenius extension. If $E \mid A$ is D2, then $A \mid B$ is D2.

**Proof.** Since $A \mid B$ is Frobenius with Frobenius coordinate system $\phi \in \text{Hom} (B A_B, B B_B)$, $\sum x_i \otimes y_i \in (A \otimes_B A)^A$, then $A_B$ is finite projective, $E \cong A \otimes_B A$ via $f \mapsto \sum f(x_i) \otimes y_i$, with inverse $a \otimes a' \mapsto \lambda(a) \circ \phi \circ \lambda(a')$, and $\lambda : A \mapsto E$ is itself a Frobenius extension [7]. Then by the hypothesis $A_B$ is a progenerator, so $E$ and $B$ are Morita equivalent with context bimodules $E A_B$ and $B \text{Hom} (A_B, B_B) \mid E$, where we note that $\text{Hom} (A_B, B_B) \otimes_E A \cong B$. 
If $E \mid A$ is right D2, then
\[ EE \otimes_A E_A \oplus * \cong \oplus^n EE_A, \]
where substitution of $EE_A \cong E A \otimes_B A A$ and cancelling $A \otimes_A -$ yields
\[ EA \otimes_B A \otimes_B A A \oplus * \cong \oplus^n EA \otimes_B A A. \]
Tensoring this from the left by $B \text{Hom}(A_B, B_B) \otimes_E -$ and the Morita property yields $B A \otimes_B A A \oplus * \cong \oplus^n B A A$, whence $A \mid B$ is left D2. □

By [10, 6.1], it is enough to assume in the theorem above that $E \mid A$ is D2 Frobenius and $A_B$ a progenerator. The theorem fully answers Question 1 at the end of the paper [10]: the two conditions in the definition of depth two Frobenius extensions are equivalent, not independent, since split extensions automatically satisfy the generator condition (and cf. [9, 2.1(5)]).

Finally, we propose a natural problem from the point of view of this paper and [8]: is there an example of a left D2 extension which is not right D2?

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