Algebraic Bethe ansatz for the $\text{gl}(1|2)$ generalized model and Lieb-Wu equations

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Abstract

We solve the $\text{gl}(1|2)$ generalized model by means of the algebraic Bethe ansatz. The resulting eigenvalue of the transfer matrix and the Bethe ansatz equations depend on three complex functions, called the parameters of the generalized model. Specifying the parameters appropriately, we obtain the Bethe ansatz equations of the supersymmetric $t$-$J$ model, the Hubbard model, or of Yang’s model of electrons with delta interaction. This means that the Bethe ansatz equations of these (and many other) models can be obtained from a common algebraic source, namely from the Yang-Baxter algebra generated by the $\text{gl}(1|2)$ invariant $R$-matrix.

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1 Introduction

An important class of exactly solvable 1+1 dimensional quantum systems is related to the Yang-Baxter algebra. The theory of these systems is called the quantum inverse scattering method [1]. The central object of the quantum inverse scattering method is the monodromy matrix, whose elements generate the Yang-Baxter algebra, and whose trace, the transfer matrix, determines a lattice statistical model, which, in many cases, is of interest in physics. The structure of the Yang-Baxter algebra is fixed by a set of numerical functions, the elements of the R-matrix, which satisfies the famous Yang-Baxter equation [2].

Solving an exactly solvable model means at first instance to calculate the spectrum and the eigenfunctions of the transfer matrix. In many cases of interest this task can be accomplished using the quadratic commutation relations between the elements of the monodromy matrix. The model is then said to be solved by algebraic Bethe ansatz. For the algebraic Bethe ansatz to work it has to be possible to identify the elements of the monodromy matrix as ‘particle’ creation and annihilation operators. In particular, a pseudo vacuum state must exist, which is annihilated by all the annihilation operators. In all known cases, where an algebraic Bethe ansatz was successful so far, the elements of the monodromy matrix could be arranged in such a way, that the monodromy matrix acts as an upper triangular matrix on the pseudo vacuum, and the pseudo vacuum is an eigenstate of its diagonal elements. Still, even if a pseudo vacuum exists, on which the monodromy matrix acts triangularly, no general recipe for an algebraic Bethe ansatz is known, and the remaining calculations may be rather involved (see [3] for a highly non-trivial example).

The algebraic Bethe ansatz is best understood for models with the R-matrix of the spin-\(\frac{1}{2}\) XXX and XXZ chains [1]. In this case there is only one monodromy matrix element below the diagonal, which must annihilate the pseudo vacuum. The particular form of the vacuum eigenvalues of the diagonal elements of the monodromy matrix is not used in the construction of the transfer matrix eigenvectors. The vacuum eigenvalues, usually denoted \(a(\lambda)\) and \(d(\lambda)\), enter the algebraic Bethe ansatz solution as functional parameters. Thus, one may think of a model defined by the functional parameters and the triangular action of the monodromy matrix on the pseudo vacuum. The question whether such kind of ‘generalized model’ can be realized for arbitrary parameters \(a(\lambda)\) and \(d(\lambda)\) was first addressed in [4] and after some refinement was answered affirmatively in [5, 6]. For the models with XXX and XXZ R-matrix the arbitrariness of the functional parameters was the key tool in Korepin’s proof [7] of the norm formula and in Slavnov’s work [8] on form factors.

The simplest models which allow for a nested algebraic Bethe ansatz are the models with \(\text{gl}(n)\) invariant R-matrix [2, 4]. Considering the fundamental rep-
resentations of these models one observes that not only the monodromy matrix elements below the diagonal annihilate the pseudo vacuum, but additional zeros appear above the diagonal \([11]\). This fact simplifies the algebraic Bethe ansatz for the fundamental representation as compared to the more general case, where the action of all the elements of the monodromy matrix above the diagonal is non-trivial. For the solution of this more general case a new concept, the vacuum subspace, was introduced by Kulish and Reshetikhin \([12]\). This new concept enabled to perform the algebraic Bethe ansatz for the models with gl\((n)\) invariant \(R\)-matrix with the same generality as in the gl\((2)\) case. The resulting eigenvalue of the transfer matrix and the Bethe ansatz equations depend on \(n\) functional parameters \(a_1(\lambda), \ldots, a_n(\lambda)\), which, together with the triangular action of the monodromy matrix on the pseudo vacuum, define the so-called gl\((n)\) generalized model \([13]\).

Considering the parameters \(a_1(\lambda), a_2(\lambda), a_3(\lambda)\) as free parameters Reshetikhin derived the norm formula for the gl\((3)\) generalized model \([13]\).

In this work we construct the algebraic Bethe ansatz solution for the gl\((1|2)\) generalized model. This work has three motivations:

(i) To perform the algebraic Bethe ansatz for all models with gl\((1|2)\) invariant graded \(R\)-matrix \([9,14]\).

(ii) To pave the ground for the calculation of the norm for this class of models.

(iii) To point out an interesting relation to the Hubbard model, namely, that the Lieb-Wu equations \([15]\) can be generated by the Yang-Baxter algebra with gl\((1|2)\) invariant \(R\)-matrix.

Let us comment here on point (i) above. Points (ii) and (iii) will be further discussed in section 7.

Numerous articles have appeared about models connected to the Yang-Baxter algebra generated by the gl\((1|2)\) invariant \(R\)-matrix. In first place we have to mention the lattice gas model of Lai and Sutherland \([16,17]\) and the supersymmetric \(t-J\) model \([18]\). Their relation to the graded Yang-Baxter algebra was explored in \([19,20]\). Both models are connected to the fundamental representation of gl\((1|2)\). Their algebraic Bethe ansatz solution was obtained in \([14]\) (see also \([19]\)). More recently there was a wave of interest in models related to higher dimensional representations of gl\((1|2)\) \([21,22]\). A fermionic model related to the four-dimensional typical representation was introduced in \([23–25]\) and was named supersymmetric U model. Algebraic Bethe ansatz solutions of this model were obtained in \([26–28]\) (see also \([29]\)). An impurity \(t-J\) model, where the fundamental representation at one lattice site is replaced by the typical four-dimensional representation, was considered in \([30,31]\). Finally, the latest example in our list is the family of ‘doped Heisenberg chains’ \([32]\) related to the atypical representations of gl\((1|2)\).
The algebraic Bethe ansatz solutions of all the above mentioned models can be obtained as special cases of the solution presented in this work. In this sense, we claim that we performed the algebraic Bethe ansatz for all models with $\mathfrak{gl}(1|2)$ invariant $R$-matrix. We would also like to point out that our result applies to the Bethe ansatz for the quantum transfer matrix of the supersymmetric $t$-$J$ model and can be used to prove the conjectures, presented in [33,34], about the quantum transfer matrix eigenvalue and the corresponding Bethe ansatz equations.

Let us note that one of the Bethe ansatz solutions in our above list is quite different from the others and is not covered by our approach. Ramos and Martins’ solution [28] is based on the Yang-Baxter algebra generated by the intertwiner of two typical four-dimensional representations of $\mathfrak{gl}(1|2)$ and not on the fundamental $R$-matrix. They obtain the same result for the eigenvalue of the transfer matrix of the supersymmetric $U$ model and the same Bethe ansatz equations as in [26], but their construction of the eigenvectors is quite different. It will be interesting to find out the relation between the two seemingly different sets of eigenvectors.

The plan of this work is the following: In section 2 we make more precise what we mean by the $\mathfrak{gl}(1|2)$ generalized model. In preparation of the algebraic Bethe ansatz we work out the block structure of the Yang-Baxter algebra in section 3. Section 4 is the core of this work. It is devoted to the algebraic Bethe ansatz. In section 5 we consider the fundamental graded representation, related to the supersymmetric $t$-$J$ model. This section has been included to give an example of how our slightly abstract formalism relates back to physics. Following the example the reader will be able to work out by himself the cases he is interested in. In section 6 we point out a relation of our general result to the Lieb-Wu equations. Section 7 is devoted to a discussion of possible applications.

In order not to overload this article with formulae we shall restrict ourselves to a fixed choice $(+−−)$ of the grading. The other two possible cases (see e.g. [14,19]) are left to the reader. They present no more difficulty than the case considered in the text.

2 The $\mathfrak{gl}(1|2)$ generalized model

The starting point of our considerations is the $\mathfrak{gl}(1|2)$ invariant rational $R$-matrix [8,14] with matrix elements

$$R^{\alpha\gamma}_{\beta\delta}(\lambda) = a(\lambda)(-1)^{p(\alpha)p(\gamma)}\delta^\alpha_\beta\delta^\gamma_\delta + b(\lambda)\delta^\alpha_\delta\delta^\gamma_\beta,$$  

(1)

$\alpha, \beta, \gamma, \delta = 1, 2, 3$. For simplicity we restrict ourselves to the grading $p(1) = 0, p(2) = p(3) = 1$. The matrix $R(\lambda)$ solves the Yang-Baxter equation and is invariant with respect to the action of the fundamental representation of $\mathfrak{gl}(1|2)$. It
satisfies the compatibility condition [9]

\[ R^{\alpha\gamma}_{\beta\delta}(\lambda) = (-1)^{p(\alpha) + p(\beta) + p(\gamma) + p(\delta)} R^{\alpha\gamma}_{\beta\delta}(\lambda) \]  

(2)

The complex valued functions \( a(\lambda) \) and \( b(\lambda) \) of the spectral parameter \( \lambda \in \mathbb{C} \) are defined as

\[ a(\lambda) = \frac{\lambda}{\lambda + ic}, \quad b(\lambda) = \frac{ic}{\lambda + ic}. \]  

(3)

They satisfy the equation \( a(\lambda) + b(\lambda) = 1 \). The additional complex parameter \( c \) will be called the coupling constant. We shall further need the matrix \( \hat{R}(\lambda) \) defined by

\[ \hat{R}^{\alpha\gamma}_{\beta\delta}(\lambda) = R^{\gamma\alpha}_{\delta\beta}(\lambda). \]  

(4)

The graded Yang-Baxter algebra with \( R \)-matrix \( R(\lambda) \) is the graded, associative algebra (with unit element) generated by the elements \( T^\alpha_\beta(\lambda), \alpha, \beta = 1, 2, 3 \), of the so-called monodromy matrix modulo the relations

\[ \hat{R}(\lambda - \mu) \left( T(\lambda) \otimes_s T(\mu) \right) = \left( T(\mu) \otimes_s T(\lambda) \right) \hat{R}(\lambda - \mu). \]  

(5)

We assume that the elements of the monodromy matrix are of definite parity, \( \pi(T^\alpha_\beta(\lambda)) = p(\alpha) + p(\beta) \). The symbol \( \otimes_s \) denotes the super tensor product associated with the grading \( p(1) = 0, p(2) = p(3) = 1 \). For a definition see appendix A.

The \( \mathfrak{gl}(1|2) \) generalized model is the set of all (linear) representations of the graded Yang-Baxter algebra (5) with highest vector \( \Omega \), defined by the fact that \( T(\lambda) \) acts triangularly on \( \Omega \),

\[ T^1_1(\lambda)\Omega = a_1(\lambda)\Omega, \quad T^2_2(\lambda)\Omega = a_2(\lambda)\Omega, \quad T^3_3(\lambda)\Omega = a_3(\lambda)\Omega, \quad T^\alpha_\beta(\lambda)\Omega = 0, \quad \text{for } \alpha > \beta. \]  

(6)

The complex valued functions \( a_j(\lambda), j = 1, 2, 3 \), are called the parameters of the generalized model. These parameters characterize the representation in a similar manner as the highest weight in a highest weight representation of a Lie algebra.

Let us denote the representation space of a given representation of the generalized model by \( \mathcal{H} \). It is clear from the quadratic commutation relations contained in the graded Yang-Baxter algebra (5) and from (4), (7) that we may assume that \( \mathcal{H} \) is spanned by all vectors of the form

\[ \Phi(\lambda_1, \ldots, \lambda_N) = T^\alpha_\beta(\lambda_1) \cdots T^\alpha_N(\lambda_N)\Omega, \]  

(8)
where \( \alpha_k < \beta_k \), \( k = 1, \ldots, N \). This assumption is at least sensible for a finite dimensional representation space \( \mathcal{H} \).

Let us define the transfer matrix

\[
t(\lambda) = (-1)^{p(\alpha)} T_\alpha^\alpha(\lambda) = \text{str}(T(\lambda)) .
\]

(9)

Since \( \check{R}(\lambda) \) is invertible for generic values of \( \lambda \in \mathbb{C} \), we conclude from (2) and (5) that

\[
[t(\lambda), t(\mu)] = 0 \quad (10)
\]

for all generic \( \lambda, \mu \in \mathbb{C} \). Thus \( t(\lambda) \) and \( t(\mu) \) have a common system of eigenfunctions. In other words, the eigenvectors of \( t(\lambda) \) are independent of the spectral parameter \( \lambda \). The task of the algebraic Bethe ansatz for the generalized model is to diagonalize \( t(\lambda) \), i.e., to solve the eigenvalue problem

\[
t(\lambda) \Phi = \Lambda(\lambda) \Phi .
\]

(11)

It is a remarkable fact that this task can be accomplished by solely using the graded Yang-Baxter algebra (5) and the properties (6) and (7) of the highest vector \( \Omega \).

In particular, it is \emph{not} necessary to require that \( T_2^\lambda(\lambda) \Omega = 0 \) as in case of the fundamental graded representation, which corresponds to the supersymmetric \( t-J \) model.

### 3 The graded Yang-Baxter algebra

The structure of the graded Yang-Baxter algebra (5) becomes much clearer after rewriting it in block form. We introduce the shorthand notations

\[
B(\lambda) = (B_1(\lambda), B_2(\lambda)) \quad , \quad C(\lambda) = \begin{pmatrix} C_1(\lambda) \\ C_2(\lambda) \end{pmatrix} ,
\]

\[
D(\lambda) = \begin{pmatrix} D_1^1(\lambda) & D_1^2(\lambda) \\ D_2^1(\lambda) & D_2^2(\lambda) \end{pmatrix} .
\]

(12)

Then we can write the \( 3 \times 3 \) monodromy matrix \( T(\lambda) \) as

\[
T(\lambda) = \begin{pmatrix} A(\lambda) & B_1(\lambda) & B_2(\lambda) \\ C_1(\lambda) & D_1^1(\lambda) & D_1^2(\lambda) \\ C_2(\lambda) & D_2^1(\lambda) & D_2^2(\lambda) \end{pmatrix} = \begin{pmatrix} A(\lambda) \\ B(\lambda) \\ C(\lambda) \end{pmatrix} .
\]

(13)

The defining relations of the graded Yang-Baxter algebra (5) can be thought of as a \( 9 \times 9 \) matrix equation. Let us denote the \( n \times n \) unit matrix by \( I_n \). A similarity
transformation with the matrix

\[
X = \begin{pmatrix}
I_4 & \\
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix} & I_2
\end{pmatrix},
\]  

(14)

which cyclically permutes the 5th, 6th and 7th row, transforms this \(9 \times 9\) equation into

\[
\begin{pmatrix}
1 & bI_2 & aI_2 \\
bI_2 & aI_2 & bI_2 \\
aI_2 & bI_2 & \tilde{r}
\end{pmatrix}
\begin{pmatrix}
A \otimes \tilde{A} & A \otimes \tilde{B} & B \otimes \tilde{A} & B \otimes \tilde{B} \\
A \otimes \tilde{C} & A \otimes \tilde{D} & -B \otimes \tilde{C} & -B \otimes \tilde{D} \\
C \otimes \tilde{A} & C \otimes \tilde{B} & D \otimes \tilde{A} & D \otimes \tilde{B} \\
-C \otimes \tilde{C} & -C \otimes \tilde{D} & D \otimes \tilde{C} & D \otimes \tilde{D}
\end{pmatrix}
\begin{pmatrix}
1 & bI_2 & aI_2 \\
bI_2 & aI_2 & bI_2 \\
aI_2 & bI_2 & \tilde{r}
\end{pmatrix}.
\]  

(15)

For the formula to fit on the line we suppressed the arguments and adopted the following convention: \(A, \ldots, D\) depend on \(\lambda\), and a bar means here that \(\lambda\) is replaced by \(\mu\). Furthermore, \(a = a(\lambda - \mu)\) and \(b = b(\lambda - \mu)\). The \(4 \times 4\) matrix

\[
\tilde{r} = \begin{pmatrix}
b - a & b & -a & b \\
b & -a & a & b \\
-a & b & b - a
\end{pmatrix}
\]  

(16)

is of ‘six-vertex form’. It satisfies the Yang-Baxter equation and is unitary,

\[
\tilde{r}(\lambda) \tilde{r}(-\lambda) = I_4.
\]  

(17)

We would like to remark that the defining relations of the graded Yang-Baxter algebra of the \(\text{gl}(1|2)\) model, when written in block form (15), resemble the corresponding relations for the \(\text{gl}(1|1)\) model.

Out of the 16 relations contained in (15) we shall need the following 4 for the
algebraic Bethe ansatz,

\[ B(\lambda) \otimes B(\mu) = (B(\mu) \otimes B(\lambda)) \tilde{r}(\lambda - \mu) \quad , \] \hfill (18)

\[ A(\lambda) \otimes B(\mu) = -\frac{b(\mu - \lambda)}{a(\mu - \lambda)} B(\lambda) \otimes A(\mu) + \frac{B(\mu) \otimes A(\lambda)}{a(\mu - \lambda)} \quad , \] \hfill (19)

\[ D(\lambda) \otimes B(\mu) = \frac{b(\lambda - \mu)}{a(\lambda - \mu)} B(\lambda) \otimes D(\mu) - (B(\mu) \otimes D(\lambda)) \frac{\tilde{r}(\lambda - \mu)}{a(\lambda - \mu)} \quad , \] \hfill (20)

\[ \tilde{r}(\lambda - \mu) (D(\lambda) \otimes D(\mu)) = (D(\mu) \otimes D(\lambda)) \tilde{r}(\lambda - \mu) \quad . \] \hfill (21)

Note that, by (18), \( B(\lambda) \) constitutes a representation of the Zamolodchikov algebra, and, by (21), \( D(\lambda) \) is a representation of the Yang-Baxter algebra of the \( gl(2) \) model.

**4 The algebraic Bethe ansatz**

Our goal is to calculate the eigenvectors of the transfer matrix \( t(\lambda) = A(\lambda) - \text{tr}(D(\lambda)) \). In analogy with the \( gl(1|1) \) case we shall first of all consider the commutation relations of a multiple tensor product \( B(\lambda_1) \otimes \cdots \otimes B(\lambda_N) \) with \( A(\lambda) \) and \( \text{tr}(D(\lambda)) \). These commutation relations can be obtained by iterating equations (19) and (20). We first present the result of the iteration and explain our notation below.

\[ A(\lambda) \left[ \bigotimes_{j=1}^{N} B(\lambda_j) \right] = \left[ \bigotimes_{j=1}^{N} B(\lambda_j) \right] A(\lambda) \prod_{j=1}^{N} \frac{1}{a(\lambda_j - \lambda)} \]

\[ - \sum_{j=1}^{N} \left\{ B(\lambda) \otimes \left[ \bigotimes_{k \neq j}^{N} B(\lambda_k) \right] \right\} S_{j-1} A(\lambda_j) \frac{b(\lambda_j - \lambda)}{a(\lambda_j - \lambda)} \prod_{k=1}^{N} \frac{1}{a(\lambda_k - \lambda_j)} \quad , \] \hfill (22)

\[ D(\lambda) \otimes \left[ \bigotimes_{j=1}^{N} B(\lambda_j) \right] = (-1)^N \left\{ I_2 \otimes \left[ \bigotimes_{j=1}^{N} B(\lambda_j) \right] \right\} \tilde{T}(\lambda) \prod_{j=1}^{N} \frac{1}{a(\lambda - \lambda_j)} \]

\[ - (-1)^N \sum_{j=1}^{N} \left\{ I_2 \otimes B(\lambda) \otimes \left[ \bigotimes_{k \neq j}^{N} B(\lambda_k) \right] \right\} P_{01} S_{j-1}^{(0)} : \left( \text{tr}(\tilde{T}(\lambda_j)) \right) b(\lambda - \lambda_j) \prod_{k \neq j}^{N} \frac{1}{a(\lambda_j - \lambda_k)} \quad . \] \hfill (23)

Here the operators \( B(\lambda) \) in the multiple tensor products are multiplied in ascending order. We make use of the usual conventions for the embedding of linear operators.
into tensor product spaces and define

\[ \tilde{r}_{j-1,j}(\lambda) = I_2^{\otimes (j-2)} \otimes \tilde{r}(\lambda) \otimes I_2^{\otimes (N-j)} \]  

(24)

for \( j = 2, \ldots, N \) as an operator in \( (\text{End}(\mathbb{C}^2))^\otimes N \). The operators \( S_{j-1} \) appearing on the right hand side of (22) are then defined as

\[ S_{j-1} = \tilde{r}_{1,2}(\lambda_1 - \lambda_j) \tilde{r}_{2,3}(\lambda_2 - \lambda_j) \ldots \tilde{r}_{j-1,j}(\lambda_{j-1} - \lambda_j) \]  

(25)

for \( j = 2, \ldots, N \). We further define \( S_0 = \text{id} \). Then the notation used in (22) is explained.

It remains to explain the operators \( P_{01}, S_{j-1}^{(0)} \) and \( \tilde{T}(\lambda) \) appearing at the right hand side of equation (23). These operators act in one more auxiliary space ‘zero’.

\[ S_{j-1}^{(0)} = I_2 \otimes S_{j-1}, \quad P_{01} = P \otimes I_2^{\otimes (N-1)} \]  

(26)

where \( P \) is the permutation matrix on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) with matrix elements

\[ P_{\alpha\delta}^{\gamma\beta} = \delta_\alpha^\beta \delta_\gamma^\delta \]

In order to define \( \tilde{T}(\lambda) \) we introduce the \( \text{gl}(2) \) R-matrix

\[ r(\lambda) = P \tilde{r}(\lambda) \]  

(27)

and the monodromy matrix of the corresponding fundamental inhomogeneous model,

\[ T^{(0)}(\lambda) = r_{0,N}^{(0)}(\lambda - \lambda_N) \ldots r_{0,1}^{(0)}(\lambda - \lambda_1) \]  

(28)

We shall interpret this auxiliary monodromy matrix as a \( 2 \times 2 \) matrix in space zero with entries acting on spaces \( 1, \ldots, N \). We further introduce

\[ D_{0}^{(0)}(\lambda) = D(\lambda) \otimes I_2^{\otimes N} \]  

(29)

Eventually, \( \tilde{T}(\lambda) \) is defined as

\[ \tilde{T}(\lambda) = D_{0}^{(0)}(\lambda)T^{(0)}(\lambda) \]  

(30)

This expression, too, can be understood as a \( 2 \times 2 \) matrix, the trace of which appears under the sum on the right hand side of equation (23).

Equations (22) and (23) were obtained by the usual symmetry arguments of the algebraic Bethe ansatz. Their derivation solely relies on (18)-(20). As an immediate consequence of (18) we obtain the equation

\[ \bigotimes_{k=1}^{N} B(\lambda_k) = \left\{ B(\lambda_j) \otimes \left[ \bigotimes_{k=1, \ k \neq j}^{N} B(\lambda_k) \right] \right\} S_{j-1} \]  

(31)
which was repeatedly used in the derivation of (22), (23). A mathematically rigorous proof of (22), (23), which is not too hard to do, may be obtained by induction over $N$. Some useful identities needed in the proof of the less trivial case of (23) are provided in appendix B.

A commutation relation, appropriate for our purposes, of the transfer matrix $t(\lambda)$ with a multiple tensor product $B(\lambda_1) \otimes \cdots \otimes B(\lambda_N)$ now easily follows from (22) and (23). We have to take the trace of equation (23) in space zero and have to subtract the result from equation (22). Using the fact that $b(\lambda)/a(\lambda) = ic/\lambda$ is an odd function of $\lambda$ we obtain

\[
t(\lambda) \left[ \bigotimes_{j=1}^{N} B(\lambda_j) \right] = \left[ \bigotimes_{j=1}^{N} B(\lambda_j) \right] \cdot \left\{ A(\lambda) \prod_{j=1}^{N} \frac{1}{a(\lambda_j - \lambda)} - (-1)^N \text{tr}(\tilde{T}(\lambda)) \prod_{j=1}^{N} \frac{1}{a(\lambda - \lambda_j)} \right\} \
+ \sum_{j=1}^{N} \left\{ B(\lambda) \otimes \bigotimes_{k=1}^{N} B(\lambda_k) \right\} S_{j-1} \frac{b(\lambda - \lambda_j)}{a(\lambda - \lambda_j)} \prod_{k=1, k \neq j}^{N} \frac{1}{a(\lambda - \lambda_k)} \cdot \left\{ A(\lambda_j) \prod_{k=1, k \neq j}^{N} \frac{a(\lambda - \lambda_k)}{a(\lambda - \lambda_j)} + (-1)^N \text{tr}(\tilde{T}(\lambda_j)) \right\}.
\]

(32)

Before proceeding with the Bethe ansatz calculation we have to gain more insight into the mathematical structure of the various expressions in equation (32): The vector $B(\lambda_1) \otimes \cdots \otimes B(\lambda_N)$ may be thought of as a $2^N$ component row vector with entries acting on the representation space $\mathcal{H}$. The components of this vector are

\[
\left[ \bigotimes_{j=1}^{N} B(\lambda_j) \right]_{i_1, \ldots, i_N} = B_{i_1}(\lambda_1) \cdots B_{i_N}(\lambda_N),
\]

(33)

$i_n = 1, 2$. Equivalently, $B(\lambda_1) \otimes \cdots \otimes B(\lambda_N)$ is a vector in $(\mathbb{C}^2)^\otimes N \otimes \text{End}(\mathcal{H})$. Suppose $\hat{F}^{i_1, \ldots, i_N} \in \mathcal{H}$ for all $i_1, \ldots, i_N = 1, 2$. Then $\hat{F}$ can be defined as a column vector with $2^N$ components $\hat{F}^{i_1, \ldots, i_N}$ that are vectors in $\mathcal{H}$, or, equivalently, $\hat{F} \in (\mathbb{C}^2)^\otimes N \otimes \mathcal{H}$. It follows that

\[
\left[ \bigotimes_{j=1}^{N} B(\lambda_j) \right] \hat{F} = B_{i_1}(\lambda_1) \cdots B_{i_N}(\lambda_N) \hat{F}^{i_1, \ldots, i_N} \in \mathcal{H}.
\]

(34)

Following Kulish and Reshetikhin [12] let us define the ‘vacuum subspace’ $\mathcal{H}_0 \subset \mathcal{H}$.
\( \mathcal{H} \) by the conditions

\[
A(\lambda)\Phi = a_1(\lambda)\Phi, \quad \text{for all } \Phi \in \mathcal{H}_0.
\]

Clearly, \( \mathcal{H}_0 \) is a linear subspace of \( \mathcal{H} \). The following lemma holds [13].

**Lemma 1.** \( \mathcal{H}_0 \) is invariant under the action of \( D(\lambda) \).

**Proof.** From the Yang-Baxter algebra written in block form (15) we deduce the equations

\[
-b(\lambda - \mu)B(\lambda) \otimes C(\mu) + a(\lambda - \mu)D(\lambda) \otimes A(\mu) =
\]

\[
- a(\lambda - \mu)A(\mu) \otimes D(\lambda) - b(\lambda - \mu)B(\mu) \otimes C(\lambda),
\]

\[
\tilde{r}(\lambda - \mu)(D(\lambda) \otimes C(\mu)) =
\]

\[
- a(\lambda - \mu)C(\mu) \otimes D(\lambda) + b(\lambda - \mu)D(\mu) \otimes C(\lambda).
\]

Let \( \Phi \in \mathcal{H}_0 \). Acting with (37) and (38) on \( \Phi \) we obtain

\[
(A(\mu) \otimes D(\lambda))\Phi = a_1(\mu)D(\lambda)\Phi,
\]

\[
(C(\mu) \otimes D(\lambda))\Phi = 0,
\]

and the lemma is proven. \( \square \)

**Corollary 1.** The space spanned by all linear combinations of vectors of the form

\[
D_2^1(\mu_1) \ldots D_2^1(\mu_M) \Omega \]

is a linear subspace of \( \mathcal{H}_0 \).

Thanks to our lemma we can now proceed with the so-called second level Bethe ansatz. We introduce the notation

\[
T^{(0)}(\lambda) = \begin{pmatrix} A^{(0)}(\lambda) & B^{(0)}(\lambda) \\ C^{(0)}(\lambda) & D^{(0)}(\lambda) \end{pmatrix}, \quad \tilde{T}(\lambda) = \begin{pmatrix} \tilde{A}(\lambda) & \tilde{B}(\lambda) \\ \tilde{C}(\lambda) & \tilde{D}(\lambda) \end{pmatrix}.
\]

Then, by the definition (30) of \( \tilde{T}(\lambda) \),

\[
\tilde{A}(\lambda) = A^{(0)}(\lambda)D_1^1(\lambda) + C^{(0)}(\lambda)D_2^1(\lambda),
\]

\[
\tilde{B}(\lambda) = B^{(0)}(\lambda)D_1^1(\lambda) + D^{(0)}(\lambda)D_2^1(\lambda),
\]

\[
\tilde{C}(\lambda) = A^{(0)}(\lambda)D_2^2(\lambda) + C^{(0)}(\lambda)D_3^2(\lambda),
\]

\[
\tilde{D}(\lambda) = B^{(0)}(\lambda)D_2^2(\lambda) + D^{(0)}(\lambda)D_3^2(\lambda).
\]
Here we used the fact that
\[
[D^\alpha_{\beta}(\lambda), T^{(0)}_\delta(\lambda)] = 0 , \quad \text{for } \alpha, \beta, \gamma, \delta = 1, 2. \tag{43}
\]
Since both, \(D(\lambda)\) and \(T^{(0)}(\lambda)\), are representations of the Yang-Baxter algebra with \(R\)-matrix \(r(\lambda)\) (see (21)), equation (43) allows us to conclude that
\[
\tilde{r}(\lambda - \mu) (\tilde{T}(\lambda) \otimes \tilde{T}(\mu)) = (\tilde{T}(\mu) \otimes \tilde{T}(\lambda)) \tilde{r}(\lambda - \mu). \tag{44}
\]
It follows that \(\text{tr}(\tilde{T}(\lambda))\) can be diagonalized by the algebraic Bethe ansatz if a pseudo vacuum exists on which \(\tilde{T}(\lambda)\) acts triangularly.

The operators \(\tilde{A}(\lambda), \ldots, \tilde{D}(\lambda)\) act on \((\mathbb{C}^2)^{\otimes N} \otimes \mathcal{H}\). In this space let us define the vector
\[
\hat{\Omega} = (\frac{1}{0})^{\otimes N} \otimes \Omega. \tag{45}
\]
We shall see that \(\hat{\Omega}\) is an appropriate pseudo vacuum for \(\tilde{T}(\lambda)\). Let us denote the elements of the \(\text{gl}(2)\) standard basis by \(e^\beta_\alpha\). Taking into account that
\[
r_{0,0}(\lambda) = \begin{pmatrix} (b-a) e^1_1 - a e^2_2 & b e^1_2 \\ b e^2_1 & -a e^1_1 + (b-a) e^2_2 \end{pmatrix}, \tag{46}
\]
when written as a matrix in auxiliary space, and using the definition (28) of the auxiliary monodromy matrix, we obtain
\[
A^{0}(\lambda) (\frac{1}{0})^{\otimes N} = \prod_{j=1}^{N} (b(\lambda - \lambda_j) - a(\lambda - \lambda_j)) \cdot (\frac{1}{0})^{\otimes N}, \tag{47}
\]
\[
D^{0}(\lambda) (\frac{1}{0})^{\otimes N} = (-1)^N \prod_{j=1}^{N} a(\lambda - \lambda_j) \cdot (\frac{1}{0})^{\otimes N}, \tag{48}
\]
\[
C^{0}(\lambda) (\frac{1}{0})^{\otimes N} = 0. \tag{49}
\]
Hence, using (6), (7) and (42), we conclude that
\[
\tilde{A}(\lambda) \hat{\Omega} = (-1)^N a_2(\lambda) \prod_{j=1}^{N} a(\lambda - \lambda_j) \hat{\Omega}, \tag{50}
\]
\[
\tilde{D}(\lambda) \hat{\Omega} = (-1)^N a_3(\lambda) \prod_{j=1}^{N} a(\lambda - \lambda_j) \hat{\Omega}, \tag{51}
\]
\[
\tilde{C}(\lambda) \hat{\Omega} = 0. \tag{52}
\]
This means that \(\tilde{T}(\lambda)\) acts triangularly on \(\hat{\Omega}\).
Let us briefly recall the algebraic Bethe ansatz solution of the \( \mathfrak{gl}(2) \) generalized model (see e.g. [1]). Out of the 16 relations contained in (44) we pick out the following three,

\[
\begin{align*}
\bar{B}(\lambda)\bar{B}(\mu) &= \bar{B}(\mu)\bar{B}(\lambda), \\
\bar{A}(\lambda)\bar{B}(\mu) &= -\frac{b(\lambda - \mu)}{a(\lambda - \mu)} \bar{B}(\lambda)\bar{A}(\mu) + \frac{\bar{B}(\mu)\bar{A}(\lambda)}{a(\lambda - \mu)}, \\
\bar{D}(\lambda)\bar{B}(\mu) &= -\frac{b(\mu - \lambda)}{a(\mu - \lambda)} \bar{B}(\lambda)\bar{D}(\mu) + \frac{\bar{B}(\mu)\bar{D}(\lambda)}{a(\mu - \lambda)}.
\end{align*}
\]

Moving \( \bar{A}(\lambda) \) and \( \bar{D}(\lambda) \) through a product \( \bar{B}(\mu_1)\ldots\bar{B}(\mu_M) \) by means of (53)-(55) leads to

\[
\bar{A}(\lambda) \left[ \prod_{k=1}^{M} \bar{B}(\mu_k) \right] = \left[ \prod_{k=1}^{M} \bar{B}(\mu_k) \right] \bar{A}(\lambda) \prod_{k=1}^{M} \frac{1}{a(\lambda - \mu_k)} - \sum_{k=1}^{M} \left[ \bar{B}(\lambda) \prod_{l=1, l\neq k}^{M} \bar{B}(\mu_l) \right] \bar{A}(\mu_k) \frac{b(\lambda - \mu_k)}{a(\lambda - \mu_k)} \prod_{l=1, l\neq k}^{M} \frac{1}{a(\mu_l - \mu_k)}.
\]

\[
\bar{D}(\lambda) \left[ \prod_{k=1}^{M} \bar{B}(\mu_k) \right] = \left[ \prod_{k=1}^{M} \bar{B}(\mu_k) \right] \bar{D}(\lambda) \prod_{k=1}^{M} \frac{1}{a(\mu_k - \lambda)} - \sum_{k=1}^{M} \left[ \bar{B}(\lambda) \prod_{l=1, l\neq k}^{M} \bar{B}(\mu_l) \right] \bar{D}(\mu_k) \frac{b(\mu_k - \lambda)}{a(\mu_k - \lambda)} \prod_{l=1, l\neq k}^{M} \frac{1}{a(\mu_l - \mu_k)}.
\]

We add the latter equations and use the fact that \( b(\lambda)/a(\lambda) \) is odd. Then

\[
(\bar{A}(\lambda) + \bar{D}(\lambda)) \left[ \prod_{k=1}^{M} \bar{B}(\mu_k) \right] = \left[ \prod_{k=1}^{M} \bar{B}(\mu_k) \right]
\]

\[
\cdot \left\{ \frac{\bar{A}(\lambda) \prod_{k=1}^{M} \frac{1}{a(\lambda - \mu_k)}}{a(\lambda - \mu_k)} + \frac{\bar{D}(\lambda) \prod_{k=1}^{M} \frac{1}{a(\mu_k - \lambda)}}{a(\mu_k - \lambda)} \right\}
\]

\[
+ \sum_{k=1}^{M} \left[ \bar{B}(\lambda) \prod_{l=1, l\neq k}^{M} \bar{B}(\mu_l) \right] \frac{b(\mu_k - \lambda)}{a(\mu_k - \lambda)} \prod_{l=1, l\neq k}^{M} \frac{1}{a(\mu_l - \mu_k)}
\]

\[
\cdot \left\{ \frac{\bar{A}(\mu_k) \prod_{l=1, l\neq k}^{M} \frac{a(\mu_l - \mu_k)}{a(\mu_k - \mu_l)}}{a(\mu_k - \mu_l)} - \frac{\bar{D}(\mu_k) \prod_{l=1, l\neq k}^{M} \frac{a(\mu_l - \mu_k)}{a(\mu_k - \mu_l)}}{a(\mu_k - \mu_l)} \right\}.
\]

Now \( \hat{\Omega} \) is a joint eigenvector of \( \bar{A}(\lambda) \) and \( \bar{D}(\lambda) \) (see (50), (51)). Thus, acting with both sides of equation (58) on \( \hat{\Omega} \) we see that \( \bar{B}(\mu_1)\ldots\bar{B}(\mu_M) \hat{\Omega} \) is an eigenvector.
of $\text{tr}(\tilde{T}(\lambda)) = \tilde{A}(\lambda) + \tilde{D}(\lambda)$,

$$\text{tr}(\tilde{T}(\lambda)) \tilde{B}(\mu_1) \ldots \tilde{B}(\mu_M) \hat{\Omega} = \tilde{A}(\lambda) \tilde{B}(\mu_1) \ldots \tilde{B}(\mu_M) \hat{\Omega} \quad ,$$ (59)

with eigenvalue

$$\tilde{A}(\lambda) = (-1)^N \prod_{j=1}^{N} a(\lambda - \lambda_j)$$

$$\cdot \left\{ a_2(\lambda) \prod_{j=1}^{N} \frac{1}{a(\lambda_j - \lambda)} \prod_{k=1}^{M} \frac{1}{a(\lambda - \mu_k)} + a_3(\lambda) \prod_{k=1}^{M} \frac{1}{a(\mu_k - \lambda)} \right\} \quad ,$$ (60)

if the Bethe ansatz equations

$$\frac{a_2(\mu_k)}{a_3(\mu_k)} \prod_{j=1}^{N} \frac{1}{a(\lambda_j - \mu_k)} = \prod_{l=1, l \neq k}^{M} \frac{a(\mu_k - \mu_l)}{a(\mu_l - \mu_k)}$$ (61)

are satisfied for $k = 1, \ldots, M$.

Since the vacuum subspace $\mathcal{H}_0 \subset \mathcal{H}$ is invariant under $D(\lambda)$, it follows that $\tilde{B}(\mu_1) \ldots \tilde{B}(\mu_M) \hat{\Omega} \in (\mathbb{C}^2)^{\otimes N} \otimes \mathcal{H}_0$. In other words, $\tilde{B}(\mu_1) \ldots \tilde{B}(\mu_M) \hat{\Omega}$ is a column vector with $2^N$ rows having vectors in $\mathcal{H}_0 \subset \mathcal{H}$ as entries. Hence, by the definition (55), (56) of the vacuum subspace, $\tilde{B}(\mu_1) \ldots \tilde{B}(\mu_M) \hat{\Omega}$ is an eigenvector of $A(\lambda)$, viewed as an operator on $(\mathbb{C}^2)^{\otimes N} \otimes \mathcal{H}$,

$$A(\lambda) \tilde{B}(\mu_1) \ldots \tilde{B}(\mu_M) \hat{\Omega} = a_1(\lambda) \tilde{B}(\mu_1) \ldots \tilde{B}(\mu_M) \hat{\Omega} \quad .$$ (62)

Recall that $B(\lambda_1) \otimes \cdots \otimes B(\lambda_N)$ is a row vector with $2^N$ columns. It follows from (52), (59) and (54) that

$$\left[ \bigotimes_{j=1}^{N} B(\lambda_j) \right] \tilde{B}(\mu_1) \ldots \tilde{B}(\mu_M) \hat{\Omega}$$

$$= B_{i_1}(\lambda_1) \ldots B_{i_N}(\lambda_N) \left[ \tilde{B}(\mu_1) \ldots \tilde{B}(\mu_M) \hat{\Omega} \right]_{i_1 \ldots i_N} \in \mathcal{H}$$ (63)

is an eigenvector of $t(\lambda)$ with eigenvalue

$$\Lambda(\lambda) = a_1(\lambda) \prod_{j=1}^{N} \frac{1}{a(\lambda_j - \lambda)}$$

$$- a_2(\lambda) \prod_{j=1}^{N} \frac{1}{a(\lambda_j - \lambda)} \prod_{k=1}^{M} \frac{1}{a(\lambda - \mu_k)} - a_3(\lambda) \prod_{k=1}^{M} \frac{1}{a(\mu_k - \lambda)} \quad ,$$ (64)
if the Bethe ansatz equations

\[
\frac{a_1(\lambda_j)}{a_2(\lambda_j)} = \prod_{k=1}^{M} \frac{1}{a(\lambda_j - \mu_k)}
\]

are satisfied for \( j = 1, \ldots, N \). Thus we have obtained the eigenvectors (63) and eigenvalues (64) of the transfer matrix \( t(\lambda) \) of the \( \text{gl}(1|2) \) generalized model. The eigenvalues and eigenvectors depend on two sets of Bethe ansatz roots \( \{\lambda_j\}_{j=1}^{N} \) and \( \{\mu_k\}_{k=1}^{M} \) determined by the Bethe ansatz equations (61) and (65). We would like to stress that we did not specify the functions \( a_j(\lambda) \), \( j = 1, 2, 3 \), yet, nor did we require that \( D_1^2(\lambda)\Omega = 0 \). Our calculation solely relied on the graded Yang-Baxter algebra (5) and on equations (6) and (7).

**Remark.** Suppose all \( \lambda_j \) and \( \mu_k \) be mutually distinct, and the parameters \( a_j(\lambda) \) have no singularities at \( \lambda_j, \mu_k \). Then \( \Lambda(\lambda) \), equation (64), has simple poles at \( \lambda = \lambda_j, j = 1, \ldots, N \) and \( \lambda = \mu_k, k = 1, \ldots, M \). The condition for the poles at \( \lambda = \lambda_j \) to vanish is

\[
\text{res}\{\Lambda(\lambda_j)\} = 0 \quad , \quad j = 1, \ldots, N
\]

and is equivalent to (65). Similarly, the equations

\[
\text{res}\{\Lambda(\mu_k)\} = 0 \quad , \quad k = 1, \ldots, M
\]

are equivalent to (61).

## 5 The fundamental representation

In order to illustrate our results with an example let us reconsider the algebraic Bethe ansatz of the supersymmetric \( t-J \) model \([19]\). We have to connect the model to a representation of the graded Yang-Baxter algebra (5) and have to identify the parameters \( a_1(\lambda), a_2(\lambda) \) and \( a_3(\lambda) \) of the representation. The Bethe ansatz equations are then given by (61), (65), the eigenvectors and the eigenvalues of the transfer matrix by (63) and (64). We shall basically follow the account of \([35,36]\).

First of all, we introduce canonically anticommuting creation and annihilation operators \( c_j^{+}, c_k \) of electrons \( (a, b = \uparrow, \downarrow; j, k = 1, \ldots, L) \) and the Fock vacuum \( |0\rangle \), annihilated by all annihilation operators, \( c_{k,b}|0\rangle = 0 \). The elements \( (X_j)_{\alpha\beta} \), \( \alpha, \beta = 1, 2, 3 \), of the matrix

\[
X_j = \begin{pmatrix}
(1-n_{j\uparrow})(1-n_{j\downarrow}) & (1-n_{j\downarrow})c_{j\uparrow} & c_{j\downarrow}(1-n_{j\uparrow}) \\
(1-n_{j\downarrow})c_{j\uparrow}^{+} & (1-n_{j\downarrow})n_{j\uparrow} & -c_{j\downarrow}c_{j\uparrow}^{+} \\
c_{j\uparrow}(1-n_{j\downarrow}) & c_{j\downarrow}c_{j\uparrow} & n_{j\downarrow}(1-n_{j\uparrow})
\end{pmatrix}
\]

are
form a complete set of projection operators on the space of states locally spanned
by the basis vectors $|0\rangle, c_{j_1}^\dagger |0\rangle, c_{j_2}^\dagger |0\rangle$. Double occupancy of lattice sites is forbid-
den on this space. Let $X_j^\alpha_\beta = (X_j^\alpha_\beta)_{\alpha,\beta = 1, 2, 3}$. The operator $X_j^\alpha_\beta = 1 - n_{j\uparrow} n_{j\downarrow}$
projects the local space of lattice electrons onto the space from which double oc-
cupancy is excluded. The corresponding global projection operator is

$$P_0 = \prod_{j=1}^L (1 - n_{j\uparrow} n_{j\downarrow}) \quad (69)$$

Owing to the fact that the operators $X_j^\alpha_\beta$ are local projection operators, it fol-
lows from general considerations [35] that the ‘$L$-matrix’

$$L_j(\lambda) = a(\lambda)I_3 + b(\lambda) \begin{pmatrix}
X_{j_1}^1 & X_{j_2}^1 & X_{j_3}^1 \\
X_{j_1}^2 & -X_{j_2}^2 & -X_{j_3}^2 \\
X_{j_1}^3 & -X_{j_2}^3 & -X_{j_3}^3
\end{pmatrix} \quad (70)$$

is a representation of the graded Yang-Baxter algebra (5). This representation has
been termed fundamental graded representation in [35]. The action of $L_j(\lambda)$ on
the Fock vacuum obviously is

$$L_j(\lambda) |0\rangle = \begin{pmatrix}
1 & b(\lambda) X_{j_2}^1 & b(\lambda) X_{j_3}^1 \\
0 & a(\lambda) & 0 \\
0 & 0 & a(\lambda)
\end{pmatrix} |0\rangle \quad (71)$$

The matrix $L_j(\lambda)$ generates the supersymmetric $t$-$J$ model at a single site. The
corresponding monodromy matrix of the $L$-site model is

$$T(\lambda) = L_L(\lambda) \ldots L_1(\lambda) \quad (72)$$

Its action on the Fock vacuum follows from (71) as

$$T(\lambda) |0\rangle = \begin{pmatrix}
1 & B_1(\lambda) & B_2(\lambda) \\
0 & a^L(\lambda) & 0 \\
0 & 0 & a^L(\lambda)
\end{pmatrix} |0\rangle \quad (73)$$

From the latter equation we can read off the parameters of the representation,
$a_1(\lambda) = 1$, $a_2(\lambda) = a_3(\lambda) = a^L(\lambda)$. And we are done (compare [19], equations
(3.47), (3.48) and (3.50)).

**Remark.** Note that the Hamiltonian of the supersymmetric $t$-$J$ model is

$$H = -ic \partial_\lambda \ln \left\{ \left( \text{str}(T(0)) \right)^{-1} \text{str}(T(\lambda)) \right\} \bigg|_{\lambda=0} \quad (74)$$
Because it acts on the restricted space of electronic states, where no lattice site is doubly occupied, we may replace it with (see [33, 34])

\[ HP_0 = P_0 \left\{ -\sum_{j=1}^{L} (c_{j,a}^* c_{j+1,a} + c_{j+1,a}^* c_{j,a}) + 2 \sum_{j=1}^{L} \left( S_j^\alpha S_{j+1}^\alpha - \frac{n_j n_{j+1}}{4} + n_j \right) \right\} P_0 . \]  

(75)

6 Connection to Lieb-Wu equations

We demonstrate that for an appropriate choice of the parameters of the gl(1|2) generalized model the Bethe ansatz equations (61), (65) turn into the Lieb-Wu equations [15], which are the Bethe ansatz equations of the Hubbard model.

Let us introduce two functions of the spectral parameter, \( k(\lambda) \) and \( v(\lambda) \), defined by

\[ \sin k = \lambda, \quad v = \lambda - ic/2 . \]  

(76)

Setting

\[ k_j = k(\lambda_j), \quad j = 1, \ldots, N, \]  

\[ v_k = v(\mu_k), \quad k = 1, \ldots, M \]  

(77)

(78)

and using the explicit form (3) of the function \( a(\lambda) \), we can rewrite the Bethe ansatz equations (65), (61) as

\[ \frac{a_1(\sin k_j)}{a_2(\sin k_j)} = \prod_{k=1}^{M} \frac{v_k - \sin k_j - ic/2}{v_k - \sin k_j + ic/2} , \]  

(79)

\[ \frac{a_2(v_k + ic/2)}{a_3(v_k + ic/2)} \prod_{j=1}^{N} \frac{v_k - \sin k_j - ic/2}{v_k - \sin k_j + ic/2} = \prod_{l=1}^{M} \frac{v_k - v_l - ic/2}{v_k - v_l + ic/2} , \]  

(80)

for \( j = 1, \ldots, N; k = 1, \ldots, M \). Obviously, the latter two equations turn into the Lieb-Wu equations through the choice

\[ \frac{a_1(\sin k)}{a_2(\sin k)} = e^{ikL} , \quad \frac{a_2(v + ic/2)}{a_3(v + ic/2)} = 1 \]  

(81)

of the parameters of the generalized model.

Similarly, replacing \( \sin k \) with \( k \) in equations (76)-(81) we obtain the Bethe ansatz equations [37] of Yang’s model of electrons interacting via a delta function potential.
7 Conclusions

We have obtained the algebraic Bethe ansatz solution of the gl(1|2) generalized model. The Bethe ansatz equations (61), (65) and the eigenvalue of the transfer matrix (64) depend on three functional parameters, which are determined by the respective representation of the graded Yang-Baxter algebra (5). Choosing the parameters appropriately we obtain the Bethe ansatz equations for the models discussed in the introduction. In order to calculate the transfer matrix eigenvalues for the models based on higher representations of gl(1|2) we have to supply additional arguments based on the ideas of fusion and analyticity of the eigenvalue (see [26]).

What are the further applications of our result? They relate to points (ii) and (iii) of the list in the introduction. We do not have much doubt that Reshetikhin’s work [13] on the norm of the gl(3) generalized model carries over almost literally to the gl(1|2) case (see point (ii) in the introduction). It is, however, point (iii) which enthralls us most. We propose to systematically seek for a representation of the graded Yang-Baxter algebra (5) with parameters $a_1(\lambda)$, $a_2(\lambda)$ and $a_3(\lambda)$ satisfying (81). Such a representation would certainly much improve our understanding of the algebraic structure of the Hubbard model. It would probably enable to prove the norm conjecture proposed in [38] for the Bethe ansatz wave functions of the Hubbard model, and, combined with the results of [36], might eventually lead to form factor formulae for local operators in the finite Hubbard chain.

Unfortunately, for the case of gl(1|2) no theorem is known so far that would guarantee the existence of a representation of the graded Yang-Baxter algebra (5) with arbitrary functional parameters. It would be highly desirable to find out whether or not a generalization of the work of Korepin and Tarasov [4–6] to the gl(1|2) case is possible. As long as this question is not settled it remains unclear whether the appearance of the Lieb-Wu equations in the context of the gl(1|2) generalized model reflects a deep connection with the Hubbard model or is just a coincidence.

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Appendix A: Graded algebras

In this appendix we shall recall the basic concepts of graded vector spaces and graded associative algebras. In the context of the quantum inverse scattering method these concepts were first utilized by Kulish and Sklyanin [9, 14].

Graded vector spaces are vector spaces equipped with a notion of odd and even, that allows us to treat fermions within the formalism of the quantum inverse scattering method (see [35, 36]). Let us consider a finite dimensional vector space $V$, which is the direct sum of two subspaces, $V = V_0 \oplus V_1$, $\dim V_0 = m$, $\dim V_1 = n$. We shall call $v_0 \in V_0$ even and $v_1 \in V_1$ odd. The subspaces $V_0$ and $V_1$ are called the homogeneous components of $V$. The parity $\pi$ is a function $V_i \rightarrow \mathbb{Z}_2$ defined on the homogeneous components of $V$,

$$\pi(v_i) = i \quad i = 0, 1 \quad v_i \in V_i.$$ (A.1)

The vector space $V$ endowed with this structure is called a graded vector space or super space.

Let $A$ be an associative algebra (with unity), which is graded as a vector space. Suppose $X, Y \in A$ are homogeneous. If the product $XY$ is homogeneous with parity

$$\pi(XY) = \pi(X) + \pi(Y),$$ (A.2)

then $A$ is called a graded associative algebra [9].

For any two homogeneous elements $X, Y \in A$ let us define the super-bracket $[X, Y]_\pm = XY - (-1)^{\pi(X)\pi(Y)}YX$, and let us extend this definition linearly in both of its arguments to all elements of $A$.

Let $p : \{1, \ldots, n\} \rightarrow \mathbb{Z}_2$. The set of all $n \times n$ matrices $A, B, C, \ldots$ with entries in $A$, such that $\pi(A^{\alpha}_{\beta}) = \pi(B^{\alpha}_{\beta}) = \pi(C^{\alpha}_{\beta}) = \cdots = p(\alpha) + p(\beta)$ is an associative algebra, say $\text{Mat}(A, n)$, since $\pi(A^{\alpha}_{\beta}B^{\beta}_{\gamma}) = p(\alpha) + p(\gamma)$. For $A, B \in \text{Mat}(A, n)$ we define the super tensor product (or graded tensor product)

$$(A \otimes_s B)^{\alpha\gamma}_{\beta\delta} = (-1)^{p(\alpha)+p(\beta)}A^{\alpha}_{\beta}B^{\gamma}_{\delta}.$$ (A.4)

This definition has an interesting consequence. Let $A, B, C, D \in \text{Mat}(A, n)$, such that $[B^{\alpha}_{\beta}, C^{\gamma}_{\delta}]_\pm = 0$. Then

$$(A \otimes_s B)(C \otimes_s D) = AC \otimes_s BD.$$ (A.5)
Appendix B: Three identities

In this appendix we provide three identities that are useful for the proof of equation (2.3). The first one is

$$\left\{ \bigotimes_{j=1}^{N} B(\lambda_{j}) \right\} \otimes D(\lambda) \right\} \hat{r}_{N-1,N}^{(0)}(\lambda - \lambda_{N}) \hat{r}_{N-2,N-1}^{(0)}(\lambda - \lambda_{N-1}) \ldots \hat{r}_{0,1}^{(0)}(\lambda - \lambda_{1})$$

$$= \left\{ \bigotimes_{j=1}^{N} B(\lambda_{j}) \right\} \otimes I_{2} \right\} \{ I_{2} \otimes D(\lambda) \} \{ P_{N-1,N} \ldots P_{12} P_{01} \} \cdot r_{0,N}^{(0)}(\lambda - \lambda_{N}) \ldots r_{0,1}^{(0)}(\lambda - \lambda_{1})$$

$$= \left\{ I_{2} \otimes \left[ \bigotimes_{j=1}^{N} B(\lambda_{j}) \right] \right\} D_{0}^{(0)}(\lambda) T^{(0)}(\lambda) . \quad \text{(B.1)}$$

The second one is

$$\left\{ B(\lambda) \otimes \left[ \bigotimes_{k=1 \atop k \neq j}^{N} B(\lambda_{k}) \right] \right\} \otimes D(\lambda_{j}) \right\} \hat{r}_{N-1,N}^{(0)}(\lambda - \lambda_{N}) \ldots \hat{r}_{j,j+1}^{(0)}(\lambda_{j} - \lambda_{j+1})$$

$$= \left\{ I_{2} \otimes B(\lambda) \otimes \left[ \bigotimes_{k=1 \atop k \neq j}^{N} B(\lambda_{k}) \right] \right\} \{ P_{01} \ldots P_{N-1,j} \} \{ I_{2} \otimes D(\lambda_{j}) \} \cdot \hat{r}_{N-1,N}^{(0)}(\lambda_{j} - \lambda_{N}) \ldots \hat{r}_{j,j+1}^{(0)}(\lambda_{j} - \lambda_{j+1})$$

$$= \left\{ I_{2} \otimes B(\lambda) \otimes \left[ \bigotimes_{k=1 \atop k \neq j}^{N} B(\lambda_{k}) \right] \right\} \{ P_{01} \ldots P_{j-1,j} \} \cdot D_{j}^{(0)}(\lambda_{j}) \hat{r}_{j,j}^{(0)}(\lambda_{j} - \lambda_{N}) \ldots \hat{r}_{j,j+1}^{(0)}(\lambda_{j} - \lambda_{j+1})$$

$$= \left\{ I_{2} \otimes B(\lambda) \otimes \left[ \bigotimes_{k=1 \atop k \neq j}^{N} B(\lambda_{k}) \right] \right\} \{ P_{01} S_{j-1}^{(0)} r_{j,j-1}^{(0)}(\lambda_{j} - \lambda_{j-1}) \cdot \ldots \hat{r}_{j,j}^{(0)}(\lambda_{j} - \lambda_{N}) D_{j}^{(0)}(\lambda_{j}) \hat{r}_{j,j}^{(0)}(\lambda_{j} - \lambda_{N}) \ldots \hat{r}_{j,j+1}^{(0)}(\lambda_{j} - \lambda_{j+1})$$

$$= \left\{ I_{2} \otimes B(\lambda) \otimes \left[ \bigotimes_{k=1 \atop k \neq j}^{N} B(\lambda_{k}) \right] \right\} \{ P_{01} S_{j-1}^{(0)} \{ I_{2} \otimes \text{tr}_{0} \left( D_{0}^{(0)}(\lambda_{j}) T^{(0)}(\lambda_{j}) \right) \} \} . \quad \text{(B.2)}$$
The third identity is an identity between rational functions that can be easily proven by means of Liouville’s theorem,

$$\frac{b(\lambda - \lambda_{N+1})}{a(\lambda - \lambda_{N+1})} \prod_{k=1}^{N} \frac{1}{a(\lambda_{N+1} - \lambda_k)}$$

$$= \frac{b(\lambda - \lambda_{N+1})}{a(\lambda - \lambda_{N+1})} \prod_{j=1}^{N} \frac{1}{a(\lambda - \lambda_j)} - \sum_{j=1}^{N} \frac{b(\lambda_j - \lambda_{N+1})}{a(\lambda_j - \lambda_{N+1})} \prod_{k=1}^{N} \frac{1}{a(\lambda_j - \lambda_k)}.$$

(B.3)

The induction step in the proof of (2.3) may be done as follows. First, insert (B.1) and (B.2) into (2.3). Second, take the tensor product of the resulting equation with $B(\lambda_{N+1})$. Third, use (18) and (20) to bring the operators into the required order. Finally, use (B.3) to rearrange the resulting terms in the desired way.

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