Essentially finite $G$-torsors

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Abstract

Let $X$ be a smooth projective curve of genus $g$, defined over an algebraically closed field $k$, and let $G$ be a connected reductive group over $k$. We say that a $G$-torsor is essentially finite if it admits a reduction to a finite group, generalising the notion of essentially finite vector bundles to arbitrary groups $G$. We give a Tannakian interpretation of such torsors, and we prove that all essentially finite $G$-torsors have torsion degree, and that the degree is 0 if $X$ is an elliptic curve. We then study the density of the set of $k$-points of essentially finite $G$-torsors of degree 0, denoted $M^\text{ef,0}_G$, inside $M^\text{ss,0}_G$, the $k$-points of all semistable degree 0 $G$-torsors. We show that when $g = 1$, $M^\text{ef}_G \subset M^\text{ss,0}_G$ is dense. When $g > 1$ and when $\text{char}(k) = 0$, we show that for any reductive group of semisimple rank 1, $M^\text{ef,0}_G \subset M^\text{ss,0}_G$ is not dense.

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1 Introduction

Let $X$ be a smooth projective connected curve over an algebraically closed field $k$. Let $g = g(X)$ be the genus of $X$. In 1938 Weil introduced the notion of a finite vector bundle; a vector bundle $E$ is called finite if there are two distinct polynomials, $f, g \in \mathbb{N}[x]$, such that the vector bundle $f(E)$ is isomorphic to $g(E)$ (see [Wei38]). For $k = \mathbb{C}$, he proved that a vector bundle is finite if and only if it arises from a representation of $\pi_1(X)$ which factors through a finite group. Almost 40 years later, in [Nor76], Nori introduced the notion of an essentially finite vector bundle as a subquotient of a finite one. The category of essentially finite vector bundles forms a Tannakian category, and the corresponding group is known as the Nori fundamental group, a pro-group scheme over $k$ whose $k$ points are isomorphic to the étale fundamental group, $\pi_1^{\text{et}}(X)$, when $k$ is of characteristic 0 (see [Sza09, Corollary 6.7.20] and also e.g., [EHS08]).

Viewing a vector bundle as a $\text{GL}_n$-torsor, we are led to the question: can we generalise the notion of an essentially finite vector bundle, to a notion of an essentially finite $G$-torsor, for $G$ an affine algebraic group? Nori proved that a vector bundle $E$ is essentially finite if and only if there exists a finite group scheme $\Gamma$, a $\Gamma$-torsor $F_\Gamma$ and a representation $V$ of $\Gamma$ such that $E \cong F_\Gamma \times^\Gamma V$. Hence, we are led to the following definition

**Definition 1.1.** An essentially finite $G$-torsor is a $G$-torsor over $X$ which admits a reduction to a finite group.

Under the correspondence between vector bundles and $\text{GL}_n$-torsors, this agrees with the known definition of essentially finite vector bundles. We prove the following.

**Theorem 1.2.** Let $G$ be a connected, reductive group. Then for any $G$-bundle $F_G$, the following are equivalent.

1. The $G$-bundle $F_G$ is essentially finite.
2. There exists a faithful representation $\rho: G \to \text{GL}_V$ such that $\rho_* F_G$ is an essentially finite vector bundle.
3. For every representation $\rho: G \to \text{GL}_V$, $\rho_* F_G$ is an essentially finite vector bundle.
4. There exists a proper surjective morphism $f: Y \to X$ such that $f^* F_G$ is trivial.

Note also that since semistability can be checked on the adjoint bundle, every essentially finite $G$-torsor is semistable. We give a self-contained proof of this fact, not using the adjoint representation.

Let now $M^\text{ss}_G$ denote the moduli space of semistable $G$-bundles over $X$, for $G$ a connected reductive group. Recall that the connected components of $M^\text{ss}_G$ are indexed by the algebraic fundamental group of $G$, $\pi_1(G)$. If a $G$-bundle, $F_G$, lies in a component corresponding to
if $\pi_1(G)$, then it is said to have degree $d$. Essentially finite vector bundles always have degree 0. We prove the following.

**Theorem 1.3.** For any connected reductive group $G$, every essentially finite $G$-torsor over $X$ is of torsion degree.

Again this generalises the case for $G = \text{GL}_n$, since in this case $\pi_1(G) = \mathbb{Z}$, which is torsion-free. We also show that if $X$ is an elliptic curve then all essentially finite $G$-bundles have degree 0.

Let now $M_{G}^{\text{ef},0}$ denote the $k$-points of the essentially finite $G$-torsors of degree 0, inside $M_{G}^{\text{ss},0}$, and let $G = \text{GL}_n$. If $n = 1$, then essentially finite $G$-bundles correspond to essentially finite line bundles, which correspond to torsion line bundles (see Lemma 3.1 [Nor76]). Hence, $M_{\text{GL}_1}^{\text{ef},0}$ is dense inside $M_{\text{GL}_1}^{\text{ss},0} = \text{Jac}^0(X)$ since torsion points are dense in any abelian variety.

In positive characteristic Ducrohet and Mehta have shown that $M_{\text{GL}_n}^{\text{ef},0}$ is dense in $M_{\text{GL}_n}^{\text{ss},0}$ for all $n$ when $g \geq 2$, and similarly for vector bundles with trivial determinant (they show in fact that a smaller set of objects, called Frobenius periodic vector bundles, are dense; see [DM10]). However, in characteristic zero much less seems to be known about the density of essentially finite bundles when the rank is greater than 1. Hence, we may ask whether $M_{\text{GL}_n}^{\text{ef},0}$ is dense in $M_{\text{GL}_n}^{\text{ss},0}$ for arbitrary connected reductive groups $G$ over an arbitrary, algebraically closed field $k$.

If $g = 0$, that is if $X \cong \mathbb{P}^1$, then it is well-known that $M_{G}^{\text{ss},0}(k)$ is a singleton. Hence it is clear that every essentially finite $G$-torsor over $\mathbb{P}^1$ is trivial. We give a self-contained proof of this result using a Tannakian interpretation of both the classification of $G$-torsors over $\mathbb{P}^1$ (see [Ans18]) and the definition of essentially finite torsors. If $g = 1$, that is if $X$ is an elliptic curve, then we prove that $M_{G}^{\text{ef},0}$ is dense in $M_{G}^{\text{ss},0}$ for arbitrary connected reductive groups $G$ over an arbitrary, algebraically closed field $k$.

On the contrary, if $g \geq 2$ and $\text{char}(k) = 0$, then we show the following.

**Theorem 1.4.** Let $\text{char}(k) = 0$. For all connected, reductive groups of semisimple rank 1, $M_{G}^{\text{ef},0} \subset M_{G}^{\text{ss},0}$ is not dense.

The main work lies in proving the theorem for $\text{PGL}_2$-torsors. Note also that this shows that $M_{\text{GL}_2}^{\text{ef},0}$ is not dense in $M_{\text{GL}_2}^{\text{ss},0}$. In characteristic 0, Weissman [Wei22] has independently obtained this non-density result for $M_{\text{GL}_n}^{\text{ef}}$ for all $n \geq 1$.

By the theorem of Narasimhan and Seshadri, the points of $M_{\text{GL}_n}^{\text{ss},0}(\mathbb{C})$ are also the isomorphism classes of representations $\pi_1(X) \rightarrow U_n(\mathbb{C})$, i.e., there is an analytic homeomorphism between $M_{\text{GL}_n}^{\text{ss},0}(\mathbb{C})$ and the character variety $\text{Hom}(\pi_1(X), U_n(\mathbb{C}))/\sim$. In particular finite vector bundles correspond to unitary representations of $\pi_1(X)$ which factor through finite groups. As the Zariski topology is coarser than the analytic topology we see as a corollary
to non-density for rank $n$ vector bundles that the set of rank $n$ unitary representations of $\pi_1(X)$ which factor through finite groups is not dense inside $\text{Hom}(\pi_1(X), U_n(\mathbb{C}))/\sim$.

The outline of the text is as follows. In Section 2 we introduce the necessary notations and background. In Section 3 we define essentially finite $G$-torsors, generalising the notion of essentially finite vector bundles. We prove that such torsors are (strongly) semistable of torsion degree. Finally, in Section 4 we prove the above mentioned statements about density of $M_{G}^{\text{ef},0}$ in $M_{G}^{\text{ss},0}$.

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2 Notations, conventions and background

Throughout the text, let $k$ be an algebraically closed field and let $X$ be a smooth, projective, connected curve over $k$. Recall that if $G$ denotes an algebraic group over $k$, then a $G$-torsor over $X$ is a scheme $F_G$ over $X$ with an action of $G$ such that there exists an fppf cover, $(U_i \to X)_{i \in I}$ such that for each $i \in I$ there is a $G|_{U_i}$-equivariant isomorphism $F_G|_{U_i} \cong G|_{U_i}$.

We will also use the term $G$-bundle as synonym for $G$-torsor. If $\varphi : H \to G$ is a group morphism and $F_H$ is an $H$-torsor, then we denote by $\varphi_\ast F_H$ the $G$-torsor $\varphi_\ast F_H := F_H \times^H G$.

In the special case when $\varphi : G \to \text{GL}_V$ is a representation of $G$, we denote $\varphi_\ast F_G$ by $V_{F_G}$ (following [Sch15]). If $F_G$ is a $G$-torsor such that $F_G \cong \varphi_\ast F_H$ for some triple $(H, \varphi, F_H)$ as above, then we say that $F_G$ admits a reduction of structure group to $H$. We denote by $\text{Rep}_k(G)$ the category of finite-dimensional representations of $G$ over $k$. Recall that to give a $G$-torsor over $X$ is equivalent to give an exact, $k$-linear, tensor functor $\text{Rep}_k(G) \to \text{Vec}_X$, where $\text{Vec}_X$ denotes the category of vector bundles over $X$. We will use the same notation for the bundle seen as a functor.

Now suppose that $G$ is a connected, reductive group. Given a maximal torus $T \subset G$ let $X^*(T)$ denote the characters of $T$ and let $X_*(T)$ denote the cocharacters. Let further $\Phi \subset X^*(T)$ denote the corresponding roots and let $\Phi^\vee \subset X_*(T)$ denote the corresponding
coroots. We let $\pi_1(G)$ denote the algebraic fundamental group of $G$, namely,

$$\pi_1(G) = X_*(T)/\text{span}\{\Phi^\vee\}.$$  \hfill (2.1)

Given a parabolic $P \subset G$ with Levi quotient $L$, let $\Phi^\vee_L \subset \Phi^\vee$ denote the coroots of $L$. We write $\pi_1(P) := \pi_1(L)$.

Let $M_G$ denote the stack of $G$-torsors over $X$, let $M_{ss}^G$ denote the substack of semistable $G$-torsors and let $M_{ss}^G$ denote the moduli space of semistable $G$-torsors (see §[Ram96a], [Ram96b] and [GLS+08]). If we consider another curve, $Y$, then for clarity we may also write $M_{G,Y}$ to denote the stack of $G$-torsors over $Y$. We define $M_{ss}^{G,Y}$ and $M_{ss}^{G,Y}$ analogously.

Recall that the connected components of $M_G$ are labeled by $\pi_1(G)$, that is,

$$\pi_0(M_G) = \pi_1(G).$$  \hfill (2.2)

If $\tilde{\lambda} \in \pi_1(G)$, let $M_{G,\tilde{\lambda}} \subset M_G$ denote the corresponding component. Define similarly $M_{ss,\tilde{\lambda}}^G$ and $M_{ss,\tilde{\lambda}}^G$ to be the components in $M_{ss}^G$ respectively $M_{ss}^G$ corresponding to $\tilde{\lambda}$.

**Definition 2.1.** If $F_G$ is an object of $M_{G,\tilde{\lambda}}$, then $F_G$ is said to be of degree $\tilde{\lambda}$.

We also have that $\pi_0(M_P) \cong \pi_0(M_L) = \pi_1(P)$ and we similarly say that a $P$-torsor is of degree $\tilde{\lambda}_P$ if it lies in the component corresponding to $\tilde{\lambda}_P$.

**Lemma 2.2.** Suppose that $\varphi : G \to H$ is a morphism of smooth connected algebraic groups and let $F_G$ be a $G$-torsor of degree 0. Then $\varphi_*F_G$ has degree 0.

**Proof.** By [Hof10] we have a commutative diagram of pointed sets

$$\begin{array}{ccc}
\pi_1(G) & \to & \pi_0(M_G) \\
\downarrow & & \downarrow \\
\pi_1(H) & \to & \pi_0(M_H),
\end{array}$$  \hfill (2.3)

where all morphisms are the natural ones induced by $\varphi$ and where the left vertical map is a group morphism. The statement follows. \hfill $\Box$

**Remark 2.3.** In particular, if $F_G$ is a $G$-bundle of degree 0 then $\text{deg} V_{F_G} = 0$ for all representations $V$ of $G$.

### 2.1 Semistable torsors

Let $T$ be a maximal torus of $G$ and let $B \supset T$ be a Borel containing $T$. Then the center of $G$ can be described as

$$Z(G) = \bigcap_{\alpha \in \Phi} \text{ker}(\alpha) \subset T.$$  \hfill (2.1)
By composition via the inclusion $Z(G) \to T$ we have a natural map

$$X_*(Z(G)) \to X_*(T) \to \pi_1(G). \tag{2.2}$$

Upon tensoring with $\mathbb{Q}$ this induces an isomorphism $X_*(Z(G))_\mathbb{Q} \cong \pi_1(G)_\mathbb{Q}$. Following [Sch15] the definition of the slope map and subsequently the definition of a semistable $G$-torsor is as follows.

**Definition 2.4.** For a parabolic subgroup, $P$, such that $B \subset P \subset G$, with corresponding Levi $L$, the slope map $\phi_P : \pi_1(P) \to X_*(T)_\mathbb{Q}$ is the map given by

$$\phi_P : \pi_1(P) \to \pi_1(P)_\mathbb{Q} \cong X_*(Z(L))_\mathbb{Q} \to X_*(T)_\mathbb{Q}. \tag{2.3}$$

**Example 2.5.** For $G = \text{GL}_n$, we will describe the slope map $\phi_G$. We have that $L = G$ so $Z(L) = \text{scalar}_n$, the scalar matrices of rank $n$. We also have the standard identifications $X_*(\text{scalar}_n) \cong \mathbb{Z}$ and $X_*(T) \cong \mathbb{Z}^n$. Further, we may write $\pi_1(G) = \mathbb{Z} \cdot \mathfrak{t}_1$, where $e_i : t \mapsto \text{diag}(1, \ldots, 1, t, 1, \ldots, 1)$ with $t$ in the $i$th position, and $(\cdot)$ represents the image in $\pi_1(G)$. Then we have that $(a, \ldots, a) = na\mathfrak{t}_1$, hence the morphism $X_*(\text{scalar}_n) \to \pi_1(G)$ is simply

$$X_*(\text{scalar}_n) \to X_*(T) \to \pi_1(G) = \mathbb{Z}\mathfrak{t}_1$$

$$a \mapsto (a, \ldots, a) \mapsto (\frac{a}{n}, \ldots, \frac{a}{n}). \tag{2.4}$$

i.e., multiplication by $n$. Thus, upon tensoring with $\mathbb{Q}$ the morphism $\phi_G$ from (2.3) is given by

$$\pi_1(G) \to \pi_1(G)_\mathbb{Q} \cong X_*(\text{scalar}_n)_\mathbb{Q} \to X_*(T)$$

$$a \mapsto \frac{a}{n} \mapsto (\frac{a}{n}, \ldots, \frac{a}{n}). \tag{2.5}$$

Now let $P$ be an arbitrary parabolic of $G = \text{GL}_n$, with Levi factor $L = \prod_{i=1}^m \text{GL}_{n_i}$. Then $Z(L) = \prod_{i=1}^m \text{scalar}_{n_i} \cong \mathbb{Z}^m$. The isomorphism $\pi_1(P)_\mathbb{Q} \to X_*(Z(L))_\mathbb{Q}$ is the inverse to the morphism

$$X_*(Z(L)) \cong \mathbb{Z}^m \to \mathbb{Z}^n \to \mathbb{Z}^m \cong \pi_1(P)$$

$$(a_1, \ldots, a_m) \mapsto (a_1, a_2, \ldots, a_2, a_m, \ldots, a_m) \mapsto (n_1a_1, n_2a_2, \ldots, n_m a_m), \tag{2.6}$$

where $a_i$ occurs $n_i$ times in the tuple in the middle. Thus, the slope map $\phi_P$ is given by

$$\pi_1(P) \to \pi_1(P)_\mathbb{Q} \cong X_*(Z(L))_\mathbb{Q} \to X_*(T)_\mathbb{Q}$$

$$(a_1, \ldots, a_m) \mapsto \left(\frac{a_1}{1}, \ldots, \frac{a_m}{1}\right) \mapsto \left(\frac{a_1}{n_1}, \ldots, \frac{a_m}{n_m}\right) \mapsto \left(\frac{a_1}{n_1}, \ldots, \frac{a_m}{n_m}, \ldots, \frac{a_m}{n_m}\right). \tag{2.7}$$

**Definition 2.6.** Let $F_G$ be a $G$-torsor of degree $\lambda$. We say that $F_G$ is semi-stable if for each parabolic $P \subset G$ and each reduction $F_P$ of $F_G$ to $P$, of degree $\lambda_P$, we have that

$$\phi_P(\lambda_P) \leq \phi_G(\lambda). \tag{2.8}$$
Remark 2.7. If \( \phi_P(\hat{\lambda}P) < \phi_G(\hat{\lambda}) \) then \( F_G \) is called stable.

Example 2.8. Again let \( G = \text{GL}_n \), we show why this definition gives back the usual slope semi-stability for vector bundles. Recall first that the slope \( \mu(E) \) of a vector bundle \( E \) is defined as \( \mu(E) = \frac{\text{deg}(E)}{\text{rk}(E)} \) and that \( E \) is called slope semi-stable if for any subbundle \( F \) we have that \( \mu(F) \leq \mu(E) \).

Let now \( E \) be a vector bundle, let \( P \subset G \) be a parabolic with Levi factor \( L = \prod_{i=1}^m \text{GL}_{n_i} \) and let \( F_P \) be a reduction of \( E \) to \( P \). This amounts to giving a filtration \( 0 \subset E_1 \subset \ldots \subset E_m = E \), where \( \text{rk}E_i - \text{rk}E_{i-1} = n_i \). Then \( \text{deg}(F_P) = (\text{deg}(\pi_1,F_P), \ldots, \text{deg}(\pi_m,F_P)) \) where \( \pi_i : P \to L \to \text{GL}_{n_i} \) is the composition of the projections \( P \to L \) and \( L \to \text{GL}_{n_i} \). Then we see that

\[
\phi_P(\text{deg}(F_P)) = \left( \frac{\text{deg}(\pi_1,F_P)}{n_1}, \ldots, \frac{\text{deg}(\pi_m,F_P)}{n_m} \right)
= (\mu(E_1), \ldots, \mu(E_1), \ldots, \mu(E_m/E_{m-1}), \ldots, \mu(E_m/E_{m-1})).
\]

Since \( \phi_G(\text{deg}(E)) = (\mu(E), \ldots, \mu(E)) \) we see that Definition 2.6 agrees with the usual slope semi-stability definition.

Now we recall some results of [Sch15] regarding the slope map which we will need to prove that essentially finite torsors are semi-stable. To this end, let \( \lambda \in X^*(T) \) be a dominant character and let \( V \) be a finite-dimensional \( G \)-representation of highest weight \( \lambda \). If \( P \) is a parabolic with Levi factor \( L \), and if \( V = \bigoplus_{\nu \in X^*(T)} V[\nu] \) is the weight space-decomposition of \( V \), then let

\[
V[\lambda + Z\Phi_L] := \bigoplus_{\nu \in \lambda + Z\Phi_L} V[\nu],
\]

where \( \Phi_L \) are the roots of the Levi \( L \). Then we have the following result.

Proposition 2.9 ([Sch15] Proposition 3.2.5(b),(c)). Keep the notation as above. Let \( F_G \) be a \( G \)-torsor of degree \( \lambda_G \). Then the slope of the vector bundle \( V_{F_G} \) is given by

\[
\mu(V_{F_G}) = \langle \phi_G(\hat{\lambda}_G), \lambda \rangle.
\]

Furthermore, if \( F_P \) is a \( P \)-torsor of degree \( \hat{\lambda}_P \) with corresponding Levi bundle \( F_L \), then the vector bundle \( V[\lambda + Z\Phi_L]_{F_L} \) has slope

\[
\mu(V[\lambda + Z\Phi_L]_{F_L}) = \langle \phi_P(\hat{\lambda}_P), \lambda \rangle.
\]

3 Essentially finite torsors

We begin with the main object of study in this article.

Definition 3.1. An essentially finite \( G \)-torsor is a \( G \)-torsor over \( X \) which admits a reduction to a finite group.
Remark 3.2. Although we have fixed a smooth, projective, connected curve $X$ over $k$ for simplicity of the exposition, this definition makes sense over an arbitrary scheme. Similarly, we may use the same definition for arbitrary affine groups, not necessarily connected reductive.

Remark 3.3. Note that if $\varphi : \Gamma \to G$ is a map from a finite group $\Gamma$, then we obtain an injection $\tilde{\varphi} : \Gamma/\ker(\varphi) \hookrightarrow G$. If $F_\Gamma$ is a $\Gamma$-torsor, then $\varphi_* F_\Gamma = \tilde{\varphi}_* (\pi_* F_\Gamma)$ as $G$-torsors, so we can always assume $\Gamma$ to be a subgroup of $G$.

Example 3.4. 1. The trivial $G$-torsor $G \times X$ is essentially finite since it admits a reduction to the trivial group.

2. If $\Gamma$ is finite then every $\Gamma$-torsor $F_\Gamma$ is essentially finite since $F_\Gamma \cong \text{id} \times F_\Gamma$.

3. Note that if $\alpha : G \to G'$ is a morphism of algebraic groups and $F_G$ is an essentially finite $G$-torsor, then $\alpha_* F_G$ is an essentially finite $G'$-torsor.

Let us phrase two equivalent conditions for a $G$-bundle to be essentially finite; one in terms of the Nori fundamental group, and one Tannakian interpretation. Since $k$ is algebraically closed, there is a rational point $x$ of $X$. Let $\pi^N_1(X,x)$ denote the Nori fundamental group of $X$ and let $\tilde{X}$ denote the universal $\pi^N_1(X,x)$-torsor over $X$, introduced in [Nor76].

Proposition 3.5. A $G$-bundle $F_G$ is essentially finite if and only if there exists a morphism $\rho : \pi^N_1(X,x) \to G$ such that $\rho_* \tilde{X} \cong F_G$.

Proof. Let $F_G$ be an essentially finite $G$-torsor, let $\iota : \Gamma \hookrightarrow G$ be a finite subgroup of $G$ and let $j : F_\Gamma \to X$ be a $\Gamma$-torsor such that $\iota_* F_\Gamma \cong F_G$. Let $y$ be a rational point of $F_\Gamma$ such that $j(y) = x$. Then $j$ defines a pointed finite torsor $(F_\Gamma, y) \to (X, x)$. By [Nor76, Proposition 3.11], there is a morphism $\pi^N_1(X,x) \to \Gamma$, which we compose with $\iota$ to get a morphism $\rho : \pi^N_1(X,x) \to G$ such that $F_G \cong \rho_* \tilde{X}$.

Conversely, suppose that we have a morphism $\rho : \pi^N_1(X,x) \to G$ such that $\rho_* \tilde{X} \cong F_G$. Since $\pi^N_1(X,x) = \varprojlim A_i$ is the inverse limit of its finite quotients $A_i$ (see [Nor82]), there is some $i$ and a morphism $\rho_i : A_i \to G$ such that $\rho$ factors

$$
\rho : \pi^N_1(X,x) \xrightarrow{\pi_i} A_i \xrightarrow{\rho_i} G
$$

where $\pi_i$ is the projection. Since $\rho_* \tilde{X} \cong \rho_{i,*}(\pi_{i,*} \tilde{X})$ we see that $F_G$ is essentially finite. \square

Proposition 3.6. A $G$-torsor $F_G$ is essentially finite if and only if there exists a finite group $\Gamma$, a $\Gamma$-torsor $F_\Gamma$, and a tensor functor $\alpha : \text{Rep}_k(G) \to \text{Rep}_k(\Gamma)$ such that:

1. we have that $\omega_\Gamma \circ \alpha = \omega_G$, where $\omega_G : \text{Rep}_k(G) \to \text{Vec}_k$ and $\omega_\Gamma : \text{Rep}_k(\Gamma) \to \text{Vec}_k$ are the forgetful functors; and
2. we have a commutative diagram
\[
\begin{array}{ccc}
\text{Rep}_k(G) & \xrightarrow{F_G} & \text{Vec}_X \\
\alpha \downarrow & & \downarrow F_\Gamma \\
\text{Rep}_k(\Gamma) & & \\
\end{array}
\]
\quad (3.2)

Proof. If $F_G$ is essentially finite, coming from a finite group $\Gamma$, a group morphism $\varphi: \Gamma \to G$ and a $\Gamma$-torsor $F_\Gamma$, then we take $\alpha$ to be the induced functor from $\varphi$. Conversely, every such $\alpha$, by [DM82, Corollary 2.9], comes from a group morphism $\varphi: \Gamma \to G$.

Remark 3.7. If a $G$-torsor $F_G$ is essentially finite then there exists a finite group $\Gamma$ and a $\Gamma$-torsor $F_\Gamma$ such that $j_\ast F_\Gamma$ is trivial.

Proposition 3.8. Under the correspondence between vector bundles of rank $n$ and $GL_n$-torsors, a $GL_n$-torsor is essentially finite if and only if the corresponding vector bundle is essentially finite.

Proof. Let $F_{GL_n}$ be a $GL_n$-torsor, and let $\Gamma$ be a finite subgroup of $GL_n$, $\alpha: \Gamma \to GL_n$ and let $j: F_\Gamma \to X$ be a $\Gamma$-torsor such that $F_{GL_n} = \alpha_\ast F_\Gamma$. Then $F_{GL_n}$ is trivialised by $j: F_\Gamma \to X$ so the corresponding vector bundle $E$ is also trivialised by $j: F_\Gamma \to X$. Thus, $E$ is essentially finite.

Conversely suppose $E$ is an essentially finite vector bundle. Then there is a finite group $\iota: \Gamma \to GL_n$ and a $\Gamma$-torsor $F_\Gamma \to X$ such that $E = F_\Gamma \times^\Gamma \mathbb{A}^n$. Then we have that
\[
E = F_\Gamma \times^\Gamma \mathbb{A}^n \cong F_\Gamma \times^\Gamma GL_n \times GL_n \mathbb{A}^n \cong \iota_\ast F_\Gamma \times GL_n \mathbb{A}^n,
\]
whence the vector bundle associated to $\iota_\ast F_\Gamma$ is $E$. Hence, the bundle corresponding to $E$ is isomorphic to $\iota_\ast F_\Gamma$, hence essentially finite.

Lemma 3.9. Let $Y$ be a proper and connected scheme over $k$. A $G$-bundle $F_G$ over $Y$ is trivial if and only if for any faithful representation $\rho: G \to GL_V$, $\rho_\ast F_G$ is trivial.

Proof. The idea of this can be found in [BD13, Lemma 4.5], but we spell out the details since their assumptions on the base scheme are different from ours. Suppose that $\rho: G \to GL_V$ is any faithful representation. Consider the long exact sequence of pointed sets (see [DG70, III, §4.46])
\[
1 \to G(Y) \xrightarrow{\rho_\ast} GL_V(Y) \xrightarrow{\pi} (GL_V/G)(Y) \xrightarrow{\delta} H^1(Y,G) \xrightarrow{\rho_\ast} H^1(Y,GL_V),
\]
where $\pi: GL_V \to GL_V/G$ is the canonical projection. The morphism $\delta$ takes a $Y$-point $y: Y \to GL_V/G$ to the $G$-bundle $\delta(y) := Y \times_{GL_V/G,y,\pi} GL_V$. Since $G$ is reductive, $GL_V/G$
is affine and hence, using that \( Y \) is proper and connected, \( y \) is constant. That is, we have a factorisation \( y : Y \to \text{Spec} \, k \to GL_V/G \). Since \( k \) is algebraically closed, \((GL_V/G)(k) = GL_V(k)/G(k)\), and hence \( y \) being constant implies that there is a lift \( \tilde{y} : Y \to GL_V \) of \( y \). By the universal propery of fiber products we thus see that \( \delta(y) \) admits a section, whence \( \delta(y) \) is trivial. Hence, by exactness of (3.4) a \( G \)-bundle \( F_G \) is trivial if and only if \( \rho_* F_G \) is trivial.

\[\boxdot\]

**Theorem 3.10.** Let \( G \) be a connected, reductive group and let \( F_G \) be a \( G \)-bundle. The following are equivalent.

1. The \( G \)-bundle \( F_G \) is essentially finite.

2. There exists a faithful representation \( \rho : G \to GL_V \) such that \( \rho_* F_G \) is an essentially finite vector bundle.

3. For every representation \( \rho : G \to GL_V \), \( \rho_* F_G \) is an essentially finite vector bundle.

4. There exists a proper surjective morphism \( f : Y \to X \) such that \( f^* F_G \) is trivial.

**Proof.** By above we see that 1. implies 3., and it is clear that 3. implies 2. By \cite{BdS11} 4. is equivalent to 3. Hence we prove that 2. implies 3. and that 3. implies 1.

First suppose that 2. holds, let \( \varphi : G \to GL_W \) be a faithful representation such that \( \varphi_* F_G \) is essentially finite and let \( \rho : G \to GL_V \) be an arbitrary representation. Since \( \varphi_* F_G \) is essentially finite there is a proper surjective morphism \( f : Y \to X \) such that \( f^* \varphi_* F_G \) is trivial. Since any restriction of \( f^* \varphi_* F_G \) to a connected component of \( Y \) is trivial, we may assume that \( Y \) is connected. Thus, since \( f^* \varphi_* F_G \equiv \varphi_* f^* F_G \), we see from Lemma 3.9 that \( f^* F_G \) is trivial. Hence, \( f^* \rho_* F_G \equiv \rho_* f^* F_G \) is trivial, which implies that \( \rho_* F_G \) is essentially finite (again by \cite{BdS11}). This proves that 2. implies 3.

Now assume that 3. holds. Then the functor \( F_G : \text{Rep}_k(G) \to \text{Vec}_X \) factors through the category of essentially finite vector bundles, hence induces a group morphism \( \rho : \pi^G_1(X,x) \to G \) such that \( \rho_* X \cong F_G \). Thus, by Proposition 3.5 \( F_G \) is essentially finite.

\[\boxdot\]

**Proposition 3.11.** Every essentially finite \( G \)-torsor is semistable.

**Proof.** Let \( F_G \) be such a torsor. Let further \( P \subset G \) be a parabolic of \( G \), let \( \lambda \) be a dominant character and let \( V \) be a representation of highest weight \( \lambda \). Since \( F_G \) is essentially finite, the associated vector bundle \( V_{F_G} \) is essentially finite, hence semistable. Hence, using Proposition 2.9, we have that

\[ \langle \psi_G(\check{\lambda}_G), \lambda \rangle = \mu( V_{F_G} ) \geq \mu( V[\lambda + \mathbb{Z} \Phi_L]_{F_L} ) = \langle \psi_P(\check{\lambda}_P), \lambda \rangle. \]  

(3.5)
That is, for every dominant character \( \lambda \in X^*(T)_\mathbb{Q} \) we have that
\[
\langle \psi_G(\hat{\lambda}_G) - \psi_P(\hat{\lambda}_P), \lambda \rangle \geq 0.
\]
(3.6)

Since the cone of cocharacters with non-negative pairing with all dominant characters is double-dual to the cone of simple coroots, we see that
\[
\psi_G(\hat{\lambda}_G) - \psi_P(\hat{\lambda}_P) \geq 0.
\]
(3.7)

**Theorem 3.12.** Let \( F_G \) be an essentially finite \( G \)-torsor. Then its degree is torsion as an element of \( \pi_1(G) \).

**Proof.** Let \( F_G \) be such a bundle. Let \( j : F_T \to X \) be a finite bundle such that \( F_G \cong F_T \times^T G \). Let \( T \) be a maximal torus and \( B \supset T \) a Borel containing \( T \), and choose a reduction \( F_B \) of \( F_G \) to a Borel. We know that \( j^* F_G \) is trivial. Since
\[
j^* F_B \times^B G = j^* (F_B \times^B G) = j^* F_G,
\]
(3.8)
we see that \( j^* F_B \times^B G \) is trivial. We have that \( \pi_0(M_{B,F_T}) = \pi_0(M_{T,F_T}) = X_*(T) \) and this maps surjectively onto \( \pi_0(M_{G,F_T}) \). The fact that \( j^* F_B \) maps to the trivial torsor means that it corresponds to \( 0 \) in \( \pi_1(G) = X_*(T)/\Phi^\vee = \pi_0(M_{G,F_T}) \). This implies that the degree of \( j^* F_B \), seen as an element in \( X_*(T) \), is a sum of coroots. The equality \( \pi_0(M_B) = \pi_0(M_T) \) is induced by the morphism \( \pi_T : B \to T \), so \( \pi_T_* j^* F_B \) also corresponds to a sum of coroots. Since \( \pi_T_* j^* F_B = j^* \pi_T_* F_B \), the conclusion follows if we can show that the morphism
\[
j^* : M_{T,X} \to M_{T,F_T}
\]
(3.9)
has the property that, if \( j^* F_T \) has degree in \( \Phi^\vee \), then the same holds for a multiple of \( \deg(F_T) \).

If \( F_T \) corresponds to the cocharacter \( \mu_F \), then \( j^* F_T \) corresponds to the cocharacter \( \mu_{j^* F_T} = \deg(j) \mu_F \). Thus if \( \mu_F = \sum_{i=1}^n a_i \alpha_i^\vee + \mu \), where \( \alpha_i \) are the simple roots and \( \mu \in X_* \setminus \Phi^\vee \) then
\[
\mu_{j^* F_T} = \sum_{i=1}^n \deg(j) a_i \alpha_i^\vee + \deg(j) \mu = \sum_{i=1}^n a'_i \alpha_i^\vee
\]
Hence, \( \deg(j) \mu \in \Phi^\vee \).

We now apply this to our situation above, i.e., with \( F_T := \pi_T_* F_B \), and since \( \pi_1(G) = X_*(T)/\Phi^\vee \) we can conclude that \( \deg(F_G) \) is torsion. \( \square \)

**Proposition 3.13.** Let \( G \) be a connected, reductive group. If \( X \) is an elliptic curve, then every essentially finite \( G \)-bundle over \( X \) has degree 0.
Proof. We argue by induction on the dimension of $G$. If $\dim(G) = 1$ then $G \cong \mathbb{G}_m$ and the result follows since it is true for all vector bundles. Suppose now that $\dim(G) = n > 1$. Let $F_G$ be an essentially finite $G$-bundle of degree $d$. By [Fra21] there is a proper Levi $L$ and a degree $d' \in \pi_1(L)$ such that the inclusion $\iota : L \to G$ induces a surjection $\mathcal{M}_{L,X}^d \to \mathcal{M}_{G,X}^d$. Let $F_L$ be a reduction of structure group of $F_G$ to $L$. Since $F_G$ is essentially finite there is a faithful representation $\rho : G \to \text{GL}_V$ such that $\rho_* F_G \cong (\rho \circ \iota)_* F_L$ is essentially finite. By Theorem 3.10 this implies that $F_L$ is essentially finite. Since $L$ is a proper Levi, by induction $d' = 0$, whence $d = 0$. \qed

If the characteristic of $k$ is positive, there is a stronger notion of semistability, defined as follows. Let $\sigma_X : X \to X$ denote the absolute Frobenius of $X$.

**Definition 3.14.** A $G$-torsor $F_G$ is said to be **strongly semistable** if for all $n > 0$, $(\sigma_X^n)^* F_G$ is semistable.

**Proposition 3.15.** Every essentially finite $G$-torsor is strongly semistable.

**Proof.** For any algebraic group $H$, and any $H$-torsor, if $\sigma_H : H \to H$ denotes the absolute Frobenius of $H$, then we have that

$$(\sigma_H)_* F_H \cong \sigma_X^* F_H. \quad (3.10)$$

Let now $F_G$ be an essentially finite $G$-torsor. Let $j : F_\Gamma \to X$ be a finite bundle such that $F_G \cong F_\Gamma \times_\Gamma G$. Then by (3.10) applied to $\Gamma$ and since the push-forward along group morphisms commutes with pullbacks, we have that

$$\iota_* (\sigma_X)^* F_\Gamma \cong \iota_* (\sigma_X)^* F_\Gamma \cong (\sigma_X)^* \iota_* F_\Gamma \cong (\sigma_X)^* F_G. \quad (3.11)$$

Hence $(\sigma_X)^* F_G$ is essentially finite and thus semistable. The statement follows similarly via induction. \qed

### 3.1 The prestack of essentially finite torsors

Let $\mathcal{M}_G^\text{ef}$ denote the functor

$$\mathcal{M}_G^\text{ef} : \text{Aff}_k^{\text{op}} \to \text{Grpd}$$

$$U \mapsto \left\{ \text{essentially finite } G\text{-torsors over } U \times X \right\} + \left\{ \text{isomorphism of } G\text{-torsors} \right\}. \quad (3.1)$$

It is immediate that $\mathcal{M}_G^\text{ef}$ is a subfunctor of $\mathcal{M}_G^\text{ss}$.

**Proposition 3.16.** The functor $\mathcal{M}_G^\text{ef}$ is a $k$-prestack.
Proof. First suppose that $f : U' \to U$ is a morphism in $\text{Aff}_k^{\text{op}}$ and suppose $F_G$ is an essentially finite $G$-torsor over $U \times X$. Let $(U_i \to U)$ be a cover and $(g_{ij} : g_{ij} \in G(U_{ij}))$ a cocycle for $F_G$. Then $(f^*U_i \to U')$ is a cover of $U'$ and $(f^*g_{ij})_{ij}$ is a cocycle for $f^*F_G$. Indeed, since $g_{ij}g_{jk} = g_{ik}$ we see that
\[
 f^*g_{ij}f^*g_{jk}(x) = g_{ij}(f(x))g_{jk}(f(x)) = g_{ik}(f(x)) = f^*g_{ik}(x). \tag{3.2}
\]

The torsor $f^*F_G$ is also essentially finite since if $g_{ij} \in \Gamma(U_{ij}) \subset G(U_{ij})$ for some finite group $\Gamma$, then $f^*g_{ij} = g_{ij} \circ f$ also takes values in $\Gamma$. Since $\mathcal{M}_{G}^{\text{eff}}$ is a lax functor we see that $\mathcal{M}_{G}^{\text{eff}}$ is one as well.

Next it is clear that if $F_G, F'_G \in \mathcal{M}_{G}(U)$, then $\text{Isom}(F_G, F'_G) : \text{Aff}_U \to \text{Set}$ is a sheaf since homomorphisms of finite $G$-torsors are simply homomorphisms of $G$-torsors and $\mathcal{M}_{G}^{\text{ss}}$ is a stack.

Remark 3.17. Note however that $\mathcal{M}_{G}^{\text{eff}}$ is not a stack since the descent data is not necessarily effective. Indeed, let $G = \text{GL}_n$ and let $E$ be a vector bundle which is not essentially finite. Let further $(U_i \to X)$ be a trivialising cover of $E$, with trivialising morphisms $\phi_i : E|_{U_i} \to \mathcal{O}^n_{U_i}$. Then $E|_{X \times U_i}$ with the morphisms $(\text{id} \times \phi_j^{-1}) \circ (\text{id} \times \phi_i)$ form a descent data for $E|_{X \times X} \in \mathcal{M}_{G}(X)$. Now, if $E|_{X \times X}$ is essentially finite, then so is $E$. Indeed, by [BdS11] we have a proper surjective morphism $f : Y \to X \times X$ such that $f^*E|_{X \times X}$ is trivial, and by composing with the projection $X \times X \to X$ we have a proper surjective morphism $g : Y \to X$ such that $g^*E$ is trivial. Since $E$ was assumed not to be essentially finite, we conclude that $E|_{X \times X}$ is not essentially finite and the descent data constructed is not effective.

The following statement is immediate, but will be important for us in the final section.

Proposition 3.18. Let $G$ and $G'$ be reductive groups. The isomorphism $\mathcal{M}_{G \times G'}^{\text{ss}} \xrightarrow{\cong} \mathcal{M}_{G}^{\text{ss}} \times \mathcal{M}_{G'}^{\text{ss}}$ restricts to an isomorphism
\[
\mathcal{M}_{G \times G'}^{\text{eff}} \cong \mathcal{M}_{G}^{\text{eff}} \times \mathcal{M}_{G'}^{\text{eff}}. \tag{3.3}
\]

Proof. The isomorphism on objects is given by
\[
 F_{G \times G'} \mapsto (\pi_*F_{G \times G'}, \pi'_*F_{G \times G'}),
 (F_G, F_{G'}) \mapsto F_G \times F_{G'}, \tag{3.4}
\]
where $\pi : G \times G' \to G$ and $\pi' : G \times G' \to G'$ are the projections. If $\Gamma \subset G \times G'$ is a finite structure group of $F_{G \times G'}$, then $\pi(\Gamma)$ and $\pi'(\Gamma)$ are evidently finite structure groups of $F_G$ and $F_{G'}$, respectively. Similarly, finite structure groups $\Gamma$ and $\Gamma'$ of $F_G$, respectively $F_{G'}$, give a finite structure group, $\Gamma \times \Gamma'$ of $F_G \times F_{G'}$. □
4 Density of essentially finite torsors

In this section we prove the density statements made in the introduction. The section is divided into subsections, depending on the genus of \(X\).

4.1 Preliminaries

Proposition 4.1. Suppose \(\pi : G \to H\) is a morphism of algebraic groups such that \(\pi(Z(G)^0) \subset Z(H)^0\). If \(\pi\) admits a section \(s : H \to G\) such that \(s(Z(H)^0) \subset Z(G)^0\), then density of \(M^\text{ef}_G\) in \(M^\text{ss,0}_G\) implies density of \(M^\text{ef}_H\) in \(M^\text{ss,0}_H\).

Proof. We prove the contrapositive. Thus, suppose that \(M^\text{ef}_H\) is not dense in \(M^\text{ss,0}_H\). Since \(\pi^*\) takes essentially finite \(G\)-torsors to essentially finite \(H\)-torsors, by (2.2) we have a commutative diagram as follows

\[
\begin{array}{ccc}
M^\text{ef}_G & \leftrightarrow & M^\text{ef}_H \\
\downarrow & & \downarrow \\
M^\text{ss,0}_G & \leftrightarrow & M^\text{ss,0}_H \\
\end{array}
\] (4.1)

Since \(\pi_*\) is continuous, \(\pi_*(\overline{M^\text{ef}_G}) \subset \overline{M^\text{ef}_H}\). Suppose now on the contrary that \(M^\text{ef}_G\) is dense in \(M^\text{ss,0}_G\). Pick any \(F \in M^\text{ss,0}_H\). Then \(s_*F \in M^\text{ss,0}_G = \overline{M^\text{ef}_G}\). But since \(\pi_*s_* = \text{id}\) we see that

\[F = \pi_*s_*F \in \pi_*(\overline{M^\text{ef}_G}) \subset \overline{M^\text{ef}_H},\] (4.2)

which implies that \(\overline{M^\text{ef}_G} = M^\text{ss,0}_H\). Contradiction. \(\Box\)

Remark 4.2. The condition on the centers is to make sure that the pushforward of a semistable bundle is semistable.

Corollary 4.3. Let \(G\) be a direct product of reductive groups \(G_1\) and \(G_2\). If \(M^\text{ef}_{G_i}\) is not dense in \(M^\text{ss,0}_{G_i}\) for some \(i = 1, 2\), then \(M^\text{ef}_G\) is not dense in \(M^\text{ss,0}_G\).

Proof. We use the projection \(\pi_i : G \to G_i\) and apply the previous proposition. \(\Box\)

Proposition 4.4. Let \(G = T\) be a torus. Then \(M^\text{ef}_T\) is dense in \(M^\text{ss,0}_T\).

Proof. First suppose \(T = \mathbb{G}_m\). Then \(M^\text{ss,0}_T = \text{Jac}^0(X)\) is the Jacobian of \(X\) and essentially finite \(\mathbb{G}_m\)-torsors corresponds to finite line bundles which corresponds to torsion points on \(\text{Jac}^0(X)\), which are dense. If \(T \cong \mathbb{G}_m^r\) for \(r > 1\), then we apply Proposition 3.18 and the statement follows. \(\Box\)
4.2 Genus 0

Let now $X = \mathbb{P}^1_k$, where $k$ is an arbitrary algebraically closed field. By Proposition 3.5 we immediately have the following statement.

**Proposition 4.5.** Every essentially finite $G$-bundle over $X$ is trivial.

**Proof.** Since $\pi_1^N(X,x)$ is trivial, the statement follows from Proposition 3.5. \qed

It is also well-known that $M_G^{\text{ss},0}(k)$ is a singleton so the density statement is immediate.

For the remainder of this section, we give a different proof of Proposition 4.5, which might be interesting in its own right. We do this by using the Tannakian interpretation of essentially finite $G$-bundles and the classification of $G$-bundles on $X$.

The classification of $G$-bundles on $X$ was initially done by Grothendieck [Gro57] and by Harder [Har68] for characteristic $p$. In [Ans18] Anschütz gives a Tannakian interpretation of this classification. We thus begin by introducing the relevant notions from [Ans18].

Over $X$ there is a canonical $\mathbb{G}_m$-torsor

$$
\eta : \mathbb{A}^2 \setminus \{0\} \to X,
\quad (x_0, x_1) \mapsto [x_0 : x_1],
$$

often called the Hopf bundle. Pushforward along this bundle defines an exact, faithful tensor functor

$$
E : \text{Rep}_k(\mathbb{G}_m) \to \text{Vec}_X
V \mapsto \mathbb{A}^2 \setminus \{0\} \times_{\mathbb{G}_m} V.
$$

Taking the Harder-Narashiman filtration of a vector bundle over $X$ defines a fully faithful tensor functor

$$
HN : \text{Vec}_X \to \text{FilVec}_X
$$
from $\text{Vec}_X$ to the category of filtered vector bundles. Finally we can take the graded pieces of a filtered vector bundle and this defines an exact tensor functor

$$
\text{Gr} : \text{FilVec}_X \to \text{GrVec}_X,
$$
where $\text{GrVec}_X$ is the category of graded vector bundles.

**Proposition 4.6** (Anschütz, [Ans18], Lemma 2.3). The composition

$$
E_{\text{Gr}} : \text{Rep}_k(\mathbb{G}_m) \xrightarrow{E} \text{Vec}_X \xrightarrow{HN} \text{FilVec}_X \xrightarrow{\text{Gr}} \text{GrVec}_X
$$

is an equivalence of tensor categories onto its essential image, which consists of graded bundles $E = \bigoplus_{n \in \mathbb{Z}} E_i$ such that each $E_i$ is semistable of slope $i$. 

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The main Theorem of Grothendieck, restated in the Tannaka language by Anschütz is now given by

**Proposition 4.7** (Anschütz, [An18], Theorem 3.3). Let $G$ be a reductive group over $k$. The composition with $E$ defines a faithful functor

$$
\Phi : \text{Hom}^\otimes(\text{Rep}_k(G), \text{Rep}_k(\mathbb{G}_m)) \to \text{Hom}^\otimes(\text{Rep}_k(G), \text{Vec}_X),
$$

which induces a bijection

$$
\text{Hom}^\otimes(\text{Rep}_k(G), \text{Rep}_k(\mathbb{G}_m)) \cong H_1^{\text{et}}(X, G)
$$
on isomorphism classes.

The inverse of this is given by composition with $\mathcal{E}_{\text{Gr}}^{-1} \circ \text{Gr} \circ HN$. Using this we can now describe all essentially finite $G$-bundles on $X$.

**Proposition 4.8.** Every essentially finite $G$-torsor over $X$ is trivial.

**Proof.** Let $F_G : \text{Rep}_k(G) \to \text{Vec}_X$ be an essentially finite torsor. By Proposition (3.6) there exists a commutative diagram of tensor functors

$$
\begin{array}{ccc}
\text{Rep}_k(G) & \xrightarrow{F_G} & \text{Vec}_X \\
\downarrow{\alpha} & & \nearrow{F_{\Gamma}} \\
\text{Rep}_k(\Gamma)
\end{array}
$$

for some finite group $\Gamma$. By [An18] this sits inside the following larger diagram

$$
\begin{array}{cccccc}
\text{Rep}_k(G) & \xrightarrow{\Phi^{-1}(F_G)} & \text{Rep}_k(\mathbb{G}_m) & \xrightarrow{\mathcal{E}} & \text{Vec}_X & \xrightarrow{HN} & \text{FilVec}_X & \xrightarrow{gr} & \text{GrVec}_X \\
\downarrow{\alpha} & & f & & \nearrow{F_{\Gamma}} & & e & & & & e_{\text{Gr}}^{-1} \\
\text{Rep}_k(\Gamma) & & & & & & & & & &
\end{array}
$$

where $f$ is defined to be the composition

$$f := \mathcal{E}_{\text{Gr}}^{-1} \circ \text{gr} \circ \text{HN} \circ F_{\Gamma}.
$$

Since all functors are tensor functors, so is $f$. By [DM82] $f$ is induced by a morphism

$$\tilde{f} : \mathbb{G}_m \to \Gamma.
$$

Since $\mathbb{G}_m$ is connected and $\Gamma$ is discrete we see that $\tilde{f}$ and thus $f$ is the trivial map. But this implies that

$$F_G \cong \mathcal{E} \circ \Phi^{-1}(F_G) \cong \mathcal{E} \circ \mathcal{E}_{\text{Gr}}^{-1} \circ \text{gr} \circ \text{HN} \circ F_G \cong \mathcal{E} \circ \mathcal{E}_{\text{Gr}}^{-1} \circ \text{gr} \circ \text{HN} \circ F_{\Gamma} \circ \alpha \cong \mathcal{E} \circ f \circ \alpha
$$

is the trivial torsor. \qed
4.3 Genus 1

In the case when $X$ is an elliptic curve, the density result follows almost immediately from known properties of $M_G^{ss}$, studied by Laszlo [Las98] in characteristic 0 and Frăţilă in characteristic $p$ [Fră21].

**Proposition 4.9.** Suppose $X$ is an elliptic curve. Then $M_G^{ef}$ is dense in $M_G^{ss,0}$ for any reductive group $G$.

*Proof.* Let $T$ be a maximal torus of $G$ and let $W$ be the corresponding Weyl group. Then, by [Las98, Theorem 4.16] and [Fră21, Theorem 1.1], we have an isomorphism

$$
\varphi : M_T^{ss,0}/W \to M_G^{ss,0}
$$

(4.1)

induced by the inclusion $\iota : T \hookrightarrow G$. Since $\iota_* (M_T^{ef}) \subset M_G^{ef}$, the result follows from Proposition 4.4. $\square$

4.4 Genus $g \geq 2$

Let now $X$ be of genus $g \geq 2$. Suppose first that $\text{char}(k) = p > 0$ and let $\sigma_X$ denote the absolute Frobenius of $X$. Then a vector bundle $E$ is called periodic under the action of Frobenius if $E \cong (\sigma_X^n)_* E$ for some integer $n \geq 1$. If $E$ is such a vector bundle, then, we know that $E$ is trivialized by an étale cover [BD07, Theorem 1.1]. Hence, $E$ is essentially finite [BdS11, Theorem 1]. In [DM10, Proposition 4.1 and corollary 5.1] the authors proved that, for any $n > 0$, the set of $k$-points in $M_{GL_n}^{ss,0}$ (resp $M_{SL_n}^{ss,0}$) periodic under the action of Frobenius is dense. Hence, the set of $k$-points corresponding essentially finite vector bundles is also dense. Hence, we may state the following.

**Proposition 4.10.** Let $k$ be of characteristic $p > 0$. For any $n > 1$, $M_PGL_n^{ef,0}$ is dense in $M_{PGL_n}^{ss,0}$.

*Proof.* This follows from the previous discussion and the fact that the projection $GL_n \to PGL_n$ induces a surjection $M_{GL_n}^{ss,0} \to M_{PGL_n}^{ss,0}$ (see [Ser58, Proposition 18]) which takes essentially finite GL$_n$-bundles to essentially finite PGL$_n$-bundles. $\square$

Let now $k$ be of characteristic zero. We restrict ourselves to split reductive groups of semisimple rank 1. By classical results (see e.g., [Mil17, Chapter 21]) these are all given by the following list.

**Proposition 4.11.** Let $G$ be a split reductive group of semisimple rank 1. Then, up to isomorphism, $G$ is one of the following groups:

$$
\begin{align*}
\text{GL}_2 \times \mathbb{G}_m^r, & \quad \text{SL}_2 \times \mathbb{G}_m^r, & \quad \text{PGL}_2 \times \mathbb{G}_m^r,
\end{align*}
$$

(4.1)
Hence, by Proposition 4.1 applied to the projection map, if we show non-density for SL$_2$, GL$_2$, and PGL$_2$, we show it for all split reductive groups of semisimple rank 1. Now, by known results ([BLS98, I.3 page 7]), the quotient maps on the respective groups induce dominant morphisms

$$
M^{ss}_{SL_2} \to M^{ss,0}_{PGL_2} \\
M^{ss,0}_{GL_2} \to M^{ss,0}_{PGL_2}.
$$

Thus, to show non-density for split reductive groups of semisimple rank 1 it suffices to show it for PGL$_2$, which we do now.

To do this we need a bound on the dimension of $M^{ss}_{O(2)}$. For a connected reductive group $G$ it is well-known that $\dim \mathcal{M}_G = \dim(G)(g - 1)$ (see e.g. [Sor00]). Since $O(2)$ is not connected, we compute $\dim \mathcal{M}_{O(2)}$ following the approach for connected reductive groups.

**Lemma 4.12.** We have that $\dim \mathcal{M}_{O(2)} = g - 1$.

**Proof.** Let $F_{O(2)}$ be an $O(2)$ bundle and let $\mathfrak{o}_2$ denote the Lie algebra of $O(2)$. Let further $\text{Ad} : O(2) \to \text{GL}(\mathfrak{o}_2)$ denote the adjoint representation and let $E := \text{Ad}_* F_{O(2)}$. By definition we know that the dimension of $\mathcal{M}_{O(2)}$ at the point $F_{O(2)}$ is the rank of the cotangent complex at $F_{O(2)}$, which is equal to $-\chi(X, E)$. By Riemann-Roch we thus have that

$$
\dim \mathcal{M}_{O(2)} = -\deg(E) - \text{rk}(E)\chi(X, \mathcal{O}_X) \\
= -\deg(E) + g - 1.
$$

(4.3)

By identifying $O(2)$ as the matrices

$$
O(2) = T' \coprod T'\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \}, \; T' = \{ \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} : t \in \mathbb{G}_m \},
$$

(4.4)

one sees immediately that the adjoint representation is self dual. Hence, $E \cong E^\vee$ and thus $\deg(E) = -\deg(E)$ whence $\deg(E) = 0$. From equation (4.3) we conclude that $\dim \mathcal{M}_{O(2)} = g - 1$. \qed

**Lemma 4.13.** Let $\iota$ denote an inclusion $\iota : O(2) \hookrightarrow \text{PGL}_2$. If $F_{O(2)}$ is a semistable $O(2)$-bundle then $\iota_* F_{O(2)}$ is a semistable PGL$_2$-bundle.

**Proof.** The proof of [BS02, Proposition 2.6] applies verbatim, since an $O(2)$-bundle $F_{O(2)}$ is semistable if and only if $\iota'_* F_{O(2)}$ is semistable, where $\iota' : O(2) \hookrightarrow \text{GL}_2$ is the standard representation. \qed

**Proposition 4.14.** The subset of essentially finite PGL$_2$-torsors is not dense inside $M^{ss,0}_{PGL_2}$.
Proof. By [NvdPT08] the finite subgroups of PGL₂ are given by S₄, A₅, A₄ and for all 
\( n \in \mathbb{N}, \mu_n \) and Dₙ. Furthermore, for each finite subgroup there is only one conjugacy class 
by Proposition 4.1 in [Bea10]. Hence, for a given finite subgroup \( \Gamma \), we may choose any 
embedding \( \iota : \Gamma \hookrightarrow \text{PGL}_2 \) and unambiguously consider \( \iota_* \mathcal{M}_{\Gamma} \subset \mathcal{M}_{\text{PGL}_2}^{\text{ss},0} \).

Now, for any such group \( \Gamma \), \( \iota_* \mathcal{M}_{\Gamma} \subset \mathcal{M}_{\text{PGL}_2}^{\text{ss},0} \) is a finite number of points. Indeed, we have 
that
\[
H^1_{\text{et}}(X, \Gamma) = \text{Hom}(\pi_1(X), \Gamma)
\]
and since \( \pi_1(X) \) is (pro)finitely generated, we see that \( H^1_{\text{et}}(X, \Gamma) \) is a finite set. Hence, to 
prove the proposition it is enough to show that the essentially finite torsors whose finite 
group is isomorphic to \( D_n \) or \( \mu_n \) for some \( n > 0 \), is not dense. By abuse of notation, we 
still denote this subset by \( \mathcal{M}_{\text{PGL}_2}^{\text{ef},0} \).

Let \( \pi : \text{GL}_2 \to \text{PGL}_2 \) denote the quotient morphism. From [NvdPT08] Section 2 we thus 
see that we may choose the embedding such that for every such \( \Gamma \), we have a commutative 
diagram
\[
\Gamma \xleftarrow{\iota} \pi(O(2)) \xhookrightarrow{\iota'} \text{PGL}_2 ,
\]
where \( O(2) \subset \text{GL}_2 \) is realized as the matrices 
\[
O(2) = T' \prod T' \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad T' = \{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a \in \mathbb{G}_m \}.
\]
Since \( \pi(O(2)) \cong O(2) \), and since \( \iota' : O(2) \cong \pi(O(2)) \hookrightarrow \text{PGL}_2 \) is a closed embedding, the 
induced morphism \( \iota' : \mathcal{M}_{O(2)} \to \mathcal{M}_{\text{PGL}_2} \) is locally of finite type (see e.g., [Hof10, Fact 2.3]).
By Lemma 4.13 this induces a map \( \iota'_* : \mathcal{M}_{O(2)}^{\text{ss}} \to \mathcal{M}_{\text{PGL}_2}^{\text{ss}} \), which induces by the universal 
property of the coarse moduli space a morphism of finite type schemes \( M_{O(2)}^{\text{ss}} \to M_{\text{PGL}_2}^{\text{ss}} \). By 
taking base change along \( M_{\text{PGL}_2}^{\text{ss}} \), we obtain an open subscheme \( U \subset M_{O(2)}^{\text{ss}} \) and a morphism 
of finite type \( f : U \to M_{\text{PGL}_2}^{\text{ss},0} \). We thus obtain a Cartesian diagram
\[
\begin{array}{ccc}
U & \xrightarrow{f} & M_{\text{PGL}_2}^{\text{ss},0} \\
\downarrow & & \downarrow \\
M_{O(2)}^{\text{ss}} & \xrightarrow{\iota'_*} & M_{\text{PGL}_2}^{\text{ss}}
\end{array}
\]
Now, for any essentially finite PGL₂-torsor, \( F_{\text{PGL}_2} \), by (4.6) we may assume that \( F_{\text{PGL}_2} = \iota'_* F_{O(2)} \) where \( F_{O(2)} \) is an essentially finite O(2)-torsor.

Hence, we have a finite type morphism \( f : U \to M_{\text{PGL}_2}^{\text{ss},0} \) of projective varieties such that 
\[
M_{\text{PGL}_2}^{\text{ef},0} \subset f(U).
\]
Thus, it suffices to show that $f$ is not dominant. Suppose it was. Then we obtain an inclusion of functions fields

$$k(M_{\text{PGL}_2}^{\text{ss},0}) \hookrightarrow k(U).$$  \hspace{1cm} (4.10)

This implies that

$$3g - 3 = \dim M_{\text{PGL}_2}^{\text{ss},0} = \text{tr.deg}_k k(M_{\text{PGL}_2}^{\text{ss},0}) \leq \text{tr.deg}_k k(U) = \dim U = \dim M_{\text{O}(2)}^{\text{ss}} \leq g - 1,$$

where the last inequality follows from Lemma 4.12.

From the statement for $\text{PGL}_2$ we obtain the same statement for $\text{SL}_2$.

**Corollary 4.15.** The subset of essentially finite $\text{SL}_2$-torsors is not dense inside $M_{\text{SL}_2}^{\text{ss},0}$.

*Proof.* Since the map $M_{\text{SL}_2}^{\text{ss}} \to M_{\text{PGL}_2}^{\text{ss},0}$ is dominant this follows from Proposition 4.14. \qed

From this we obtain the same statement for $\text{GL}_2$.

**Corollary 4.16.** The subset of essentially finite $\text{GL}_2$-torsors is not dense inside $M_{\text{GL}_2}^{\text{ss},0}$.

*Proof.* The same proof as above applies, or we have the following. Consider the map

$$\det : M_{\text{GL}_2}^{\text{ss},0} \to \text{Jac}^0(X).$$  \hspace{1cm} (4.12)

Since $\det^{-1}(\mathcal{O}_X) = M_{\text{SL}_2}^{\text{ss}}$ by Corollary (4.15) we obtain the desired result. \qed

Finally, the complete statement is the following.

**Corollary 4.17.** For any split reductive group $G$, of semi-simple rank 1, the essentially finite $G$-torsors are not dense in $M_G^{\text{ss},0}$.

*Proof.* This follows from the classification of split reductive groups and Proposition 4.14. \qed
References

[Ans18] Johannes Anschütz. A Tannakian classification of torsors on the projective line. English. *C. R., Math., Acad. Sci. Paris*, 356(11-12):1203–1214, 2018. ISSN: 1631-073X. DOI: 10.1016/j.crma.2018.10.006.

[BD07] Indranil Biswas and Laurent Ducrohet. An analog of a theorem of Lange and Stuhler for principal bundles. English. *C. R., Math., Acad. Sci. Paris*, 345(9):495–497, 2007. ISSN: 1631-073X. DOI: 10.1016/j.crma.2007.10.010.

[BD13] Indranil Biswas and João Pedro Dos Santos. Triviality criteria for bundles over rationally connected varieties. English. *J. Ramanujan Math. Soc.*, 28(4):423–442, 2013. ISSN: 0970-1249.

[BdS11] Indranil Biswas and João Pedro Pinto dos Santos. Vector bundles trivialized by proper morphisms and the fundamental group scheme. English. *J. Inst. Math. Jussieu*, 10(2):225–234, 2011. ISSN: 1474-7480. DOI: 10.1017/S1474748010000071.

[Bea10] Arnaud Beauville. Finite subgroups of $\text{PGL}_2(K)$. English. In *Vector bundles and complex geometry. Conference on vector bundles in honor of S. Ramanan on the occasion of his 70th birthday, Madrid, Spain, June 16–20, 2008.* Pages 23–29. Providence, RI: American Mathematical Society (AMS), 2010. ISBN: 978-0-8218-4750-3.

[BLS98] Arnaud Beauville, Yves Laszlo, and Christoph Sorger. The Picard group of the moduli of $G$-bundles on a curve. English. *Compos. Math.*, 112(2):183–216, 1998. ISSN: 0010-437X. DOI: 10.1023/A:1000477122220.

[BS02] V. Balaji and C. S. Seshadri. Semistable principal bundles. I: Characteristic zero. English. *J. Algebra*, 258(1):321–347, 2002. ISSN: 0021-8693. DOI: 10.1016/S0021-8693(02)00502-1.

[DG70] Michel Demazure and Pierre Gabriel. Groupes algébriques. Tome I: Géométrie algébrique. Généralités. Groupes commutatifs. Avec un appendice ‘Corps de classes local’ par Michiel Hazewinkel. French. Paris: Masson et Cie, Éditeur; Amsterdam: North-Holland Publishing Company. xxvi, 700 p. (1970). 1970.

[DM10] Laurent Ducrohet and Vikram B. Mehta. Density of vector bundles periodic under the action of Frobenius. English. *Bull. Sci. Math.*, 134(5):454–460, 2010. ISSN: 0007-4497. DOI: 10.1016/j.bulsci.2009.11.001.

[DM82] Pierre Deligne and J. S. Milne. Tannakian categories. English. Hodge cycles, motives, and Shimura varieties, Lect. Notes Math. 900, 101-228 (1982). 1982.
[Ram96b] A. Ramanathan. Moduli for principal bundles over algebraic curves. II. English. *Proc. Indian Acad. Sci., Math. Sci.*, 106(4):421–449, 1996. issn: 0253-4142. doi: 10.1007/BF02837697.

[Sch15] Simon Schieder. The Harder-Narasimhan stratification of the moduli stack of \(G\)-bundles via Drinfeld’s compactifications. English. *Sel. Math., New Ser.*, 21(3):763–831, 2015. issn: 1022-1824. doi: 10.1007/s00029-014-0161-y.

[Ser58] Jean-Pierre Serre. Espaces fibrés algébriques. *Séminaire Claude Chevalley*, 3, 1958. url: http://www.numdam.org/item/SCC_1958__3__A1_0/.

[Sor00] Christoph Sorger. Lectures on moduli of principal \(G\)-bundles over algebraic curves. English. In *Moduli spaces in algebraic geometry*. Lecture notes of the school on algebraic geometry, Trieste, Italy, July 26–August 13, 1999, pages 1–57. Trieste: The Abdus Salam International Centre for Theoretical Physics (ICTP), 2000. isbn: 92-95003-00-4. url: www.ictp.trieste.it/~pub_off/lectures/.

[Sza09] Tamás Szamuely. *Galois groups and fundamental groups*. English, volume 117 of *Camb. Stud. Adv. Math*. Cambridge: Cambridge University Press, 2009. isbn: 978-0-521-88850-9.

[Wei22] Dario Weissmann. A functorial approach to the stability of vector bundles, 2022. doi: 10.48550/ARXIV.2211.08260. url: https://arxiv.org/abs/2211.08260.

[Wei38] André Weil. Généralisation des fonctions abéliennes. French. *J. Math. Pures Appl. (9)*, 17:47–87, 1938. issn: 0021-7824.