A REFINEMENT OF THE BURGESS BOUND FOR CHARACTER SUMS

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ABSTRACT. In this paper we give a refinement of the bound of D. A. Burgess for multiplicative character sums modulo a prime number \( q \). This continues a series of previous logarithmic improvements, which are mostly due to H. Iwaniec and E. Kowalski. In particular, for any nontrivial multiplicative character \( \chi \) modulo a prime \( q \) and any integer \( r \geq 2 \), we show that

\[
\sum_{M < n \leq M + N} \chi(n) = O \left( N^{1-1/r} q^{(r+1)/4r} (\log q)^{1/4r} \right),
\]

which sharpens previous results by a factor \((\log q)^{1/4r}\). Our improvement comes from averaging over numbers with no small prime factors rather than over an interval as in previous approaches.

1. Introduction

Given a prime number \( q \) and a multiplicative character \( \chi \) modulo \( q \), we consider bounding the sums

\[
(1.1) \quad \sum_{M < n \leq M + N} \chi(n).
\]

The first nontrivial result in this direction, which is about a century old, is due to Pólya [11] and Vinogradov [13] and takes the form

\[
(1.2) \quad \sum_{M < n \leq M + N} \chi(n) = O \left( q^{1/2} \log q \right)
\]

with an absolute implied constant. Clearly the bound (1.2) is nontrivial provided \( N \geq q^{1/2} (\log q)^{1+\varepsilon} \) for any fixed \( \varepsilon > 0 \).

Several logarithmic improvements of (1.2) have recently been obtained for special characters, see [5,6,8] and references therein.

For large values of \( N \), the Polya-Vinogradov bound (1.2) is still the sharpest result known today although in the special case that \( M = 0 \),
Montgomery and Vaughan [9] have shown that assuming the truth of the Generalized Riemann Hypothesis we have
\[ \sum_{0 < n \leq N} \chi(n) = O \left( q^{1/2} \log \log q \right). \]

The Pólya–Vinogradov bound (1.2) can be thought of as roughly saying that for large \( N \), the sequence \( \{\chi(n)\}_{n=M+1}^{M+N} \) behaves like a typical random sequence chosen uniformly from the image \( \chi(\{1, \ldots, q-1\}) \).

We expect this to be true for smaller values of \( N \) although this problem is much less understood. In the special case \( M = 0 \), the Generalized Riemann Hypothesis implies that
\[ \left| \sum_{0 < n \leq N} \chi(n) \right| \leq N^{1/2} q^{\omega(1)}, \]
which is nontrivial provided \( N \geq q^\varepsilon \) and is essentially optimal. We also note that Tao [12] has shown progress on the generalized Elliott-Halberstam conjecture allows one to bound short character sums in the case \( M = 0 \).

For values of \( N \) below the Pólya–Vinogradov range, the sharpest unconditional bound for the sums (1.1) is due to Burgess [1,2] and may be stated as follows. For any prime number \( q \), nontrivial multiplicative character \( \chi \) modulo \( q \) and integer \( r \geq 1 \) we have
\[ \sum_{M < n \leq M+N} \chi(n) = O \left( N^{1-1/r} q^{(r+1)/4r^2} \log q \right), \]
where the implied constant may depend on \( r \), and is nontrivial provided \( N \geq q^{1/4+\varepsilon} \) for any fixed \( \varepsilon > 0 \). This bound has remained the sharpest for short sums over the past fifty years although slight refinements have been made by improving the factor \( \log q \). For example, by [7, Equation (12.58)] we have
\[ \sum_{M < n \leq M+N} \chi(n) = O \left( N^{1-1/r} q^{(r+1)/4r^2} (\log q)^{1/r} \right), \]
and also announced in [7, Chapter 12, Remark, p. 329], that one can actually obtain
\[ \sum_{M < n \leq M+N} \chi(n) = O \left( N^{1-1/r} q^{(r+1)/4r^2} (\log q)^{1/2r} \right), \]
provided \( r \geq 2 \). In this paper we give a further refinement of the Burgess bound (1.3) and thus contribute to the series of logarithmic improvements (1.4) and (1.5). More specifically, we improve (1.5) by...
a factor \((\log q)^{1/4r}\). Our argument follows previously established techniques which proceed by a certain averaging to reduce the problem to bounding bilinear forms. Our improvement comes from averaging over numbers with no small prime factors rather than averaging over an entire interval which we give in Section 4.

2. Main result

Throughout the paper, the implied constants in the symbols ‘\(O\)’ and ‘\(\ll\)’ may occasionally, where obvious, depend on the integer parameter \(r\) and the real parameter \(A\) and are absolute otherwise (we recall that \(U \ll V\) and \(U = O(V)\) are equivalent to \(|U| \leq cV\) for some constant \(c\)).

Our main result is as follows.

**Theorem 2.1.** Let \(q\) be prime, \(r \geq 2\), \(M\) and \(N\) integers with

\[ N \leq q^{1/2+1/4r}. \]

For any nontrivial multiplicative character \(\chi\) modulo \(q\), we have

\[
\sum_{M<n\leq M+N} \chi(n) \ll N^{1-1/r} q^{(r+1)/4r^2} (\log q)^{1/4r}.
\]

3. Preliminary results

The following is a consequence of the Weil bounds for complete character sums, see for example [7, Lemma 12.8].

**Lemma 3.1.** Let \(q\) be prime and \(\chi\) a nontrivial multiplicative character modulo \(q\). Then we have

\[
\sum_{\lambda=1}^{q} \left| \sum_{1 \leq v \leq V} \chi(\lambda + v) \right|^{2r} \ll V^r q + V^{2r} q^{1/2}.
\]

For any positive real numbers \(w\) and \(z\) we denote

\[
V(w) = \prod_{p<w} \left(1 - \frac{1}{p}\right),
\]

and

\[
(3.1) \quad P(z) = \prod_{p<z} p.
\]

It follows from Mertens formula (see [7, Equation (2.16)]) that

\[
(3.2) \quad \frac{1}{\log w} \ll V(w) \ll \frac{1}{\log w}.
\]
As usual, we use \((u, v)\) to denote the greatest common divisor of two integers \(u\) and \(v\).

For real \(U\) and \(z\), we define the set \(\mathcal{U}_z(U)\) by
\[
\mathcal{U}_z(U) = \{1 \leq u \leq U : (u, P(z)) = 1\}.
\]

The following result follows from combining [3, Theorem 4.1] with arguments from the proof of [3, Lemma 4.3]. We also refer the reader to [4, Equation (6.104)].

**Lemma 3.2.** Let \(C\) be sufficiently large and suppose that
\[
z^C \leq U.
\]
Then for the cardinality of \(\mathcal{U}_z(U)\), we have
\[
\frac{U}{\log z} \ll |\mathcal{U}_z(U)| \ll \frac{U}{\log z}.
\]

**Proof.** Let
\[
\mathcal{A} = \{1, \ldots, U\},
\]
so that with notation as in [3, Theorem 4.1] we have
\[
|\mathcal{U}_z(U)| = S(\mathcal{A}, \mathcal{P}, z),
\]
and hence by [3, Theorem 4.1], for any \(v \geq 1\) we have
\[
|\mathcal{U}_z(U)| = UV(z) \left(1 + O \left(\exp(-v \log v - 3v/2)\right)\right)
+ O \left(\sum_{n < z^{2v}} \frac{3^v(n)|r_{\mathcal{A}}(n)|}{n|P(z)}\right).
\]

Considering the last term on the right
\[
\sum_{n < z^{2v}} \frac{3^v(n)|r_{\mathcal{A}}(n)|}{n|P(z)} \ll \sum_{n < z^{2v}} \frac{3^v(n)}{n} \leq z^{2v} \sum_{n|P(z)} \frac{3^v(n)}{n} \leq z^{2v} \prod_{p < z} \left(1 + \frac{3}{p}\right),
\]
and since
\[
\prod_{p < z} \left(1 + \frac{3}{p}\right) \leq \prod_{p < z} \left(1 - \frac{1}{p}\right)^{-3} = V(z)^{-3},
\]
we obtain
\[
\sum_{n < z^{2v}} \frac{3^v(n)|r_{\mathcal{A}}(n)|}{n|P(z)} \ll z^{2v} V(z)^{-3},
\]
which implies that
\[
|\mathcal{U}_z(U)| = UV(z) \left(1 + ET_1 + ET_2\right).
\]
with the error terms
\[ ET_1 = O\left(\exp\left(-v \log v - 3v/2\right)\right) \quad \text{and} \quad ET_2 = O\left(z^{2v}V(z)^{-3}\right). \]
Let \( \varepsilon \) be sufficiently small and take
\[

v = \frac{(1 - \varepsilon) \log U}{2 \log z}.

\]
Then we have
\[ (3.6) \quad ET_2 \ll U^{1-\varepsilon}(\log z)^3, \]
and by (3.4) we may choose \( C \) such that
\[ (3.7) \quad ET_1 \leq \frac{1}{2}. \]

Combining (3.5) with (3.6) and (3.7), we derive
\[ UV(z) \ll |\mathcal{H}_z(U)| \ll UV(z), \]
and the result follows from the Mertens estimate (3.2).

We recall a simplified form of \[3, \text{Lemma } 4.4\].

**Lemma 3.3.** For any integers \( t, z \) any real \( U \geq 1 \) and for any positive constant \( 0 < A < 1/2 \), we have
\[
\sum_{\substack{u \in \mathcal{H}_z(U) \atop t | u}} 1 \ll Ut^{-1}V(z),
\]
if \( z < (Ut^{-1})^A \) and
\[
\sum_{\substack{u \in \mathcal{H}_z(U) \atop t | u}} 1 \ll Ut^{-1}V(Ut^{-1}),
\]
if \( (Ut^{-1})^A \leq z \).

Note that Lemma 3.3 is nontrivial only if \( (t, P(z)) = 1 \).

4. CONGRUENCES WITH NUMBER WITH SMALL PRIME DIVISORS

The new ingredient underlying our argument is the following:

**Lemma 4.1.** Let \( q \) be prime and \( z, M, N \) and \( U \) integers with
\[ (4.1) \quad U \leq N, \quad UN \leq q. \]
Fix a sufficiently small positive real number \( 0 < A < 1/2 \) and suppose \( z \) satisfies
\[ (4.2) \quad 1 < z \leq U^A. \]
Let $P(z)$ and $\mathcal{U}_z(U)$ be given by \eqref{eq:3.1} and by \eqref{eq:3.3}, respectively, and let $I(z, M, N, U)$ count the number of solutions to the congruence
\begin{equation}
\label{eq:4.3}
n_1 u_1 \equiv n_2 u_2 \pmod{q},
\end{equation}
with integral variables satisfying
\begin{equation}
\label{eq:4.4}
M < n_1, n_2 \leq M + N \quad \text{and} \quad u_1, u_2 \in \mathcal{U}_z(U).
\end{equation}
Then we have
\begin{equation}
\label{eq:4.5}
I(z, M, N, U) \ll N|\mathcal{U}_z(U)| \left(1 + \frac{\log U}{(\log z)^2}\right).
\end{equation}

\textbf{Proof.} For each pair of integers $u_1$ and $u_2$, we let $J(u_1, u_2)$ count the number of solutions to the congruence \eqref{eq:4.3} in variables $n_1, n_2$ satisfying
\begin{equation*}
M < n_1, n_2 \leq M + N,
\end{equation*}
so that
\begin{equation*}
I(z, M, N, U) = \sum_{u_1, u_2 \in \mathcal{U}_z(U)} J(u_1, u_2)
= \sum_{u_1 \in \mathcal{U}_z(U)} J(u_1, u_1) + 2 \sum_{u_1, u_2 \in \mathcal{U}_z(U)} J(u_1, u_2).
\end{equation*}
Since
\begin{equation*}
J(u_1, u_1) = N,
\end{equation*}
we have
\begin{equation*}
I(z, M, N, U) = N \sum_{u_1 \in \mathcal{U}_z(U)} 1 + 2 \sum_{u_1, u_2 \in \mathcal{U}_z(U)} J(u_1, u_2).
\end{equation*}
Using Lemma 3.3 (with $t = 1$), the bound \eqref{eq:3.2} and recalling \eqref{eq:4.2} we see that
\begin{equation*}
\sum_{u_1 \in \mathcal{U}_z(U)} 1 \ll \frac{U}{\log z},
\end{equation*}
and hence
\begin{equation}
\label{eq:4.5}
I(z, M, N, U) \ll \frac{NU}{\log z} + \sum_{u_1, u_2 \in \mathcal{U}_z(U)} J(u_1, u_2).
\end{equation}

Fix some pair $u_1, u_2$ with $u_1 < u_2$ and consider $J(u_1, u_2)$. We first note that $J(u_1, u_2)$ is bounded by the number of solutions to the equation
\begin{equation}
\label{eq:4.6}
u_1(M + n_1) - u_2(M + n_2) = kq,
\end{equation}
with variables $n_1, n_2, k$ satisfying
\begin{equation*}
1 \leq n_1, n_2 \leq N, \quad k \in \mathbb{Z}.
\end{equation*}
Since
\[ |kq - (u_1 - u_2)M| \leq UN < q, \]
there exists at most one value \( k \) satisfying (4.6) and hence \( J(u_1, u_2) \) is bounded by the number of solutions to the equation (4.6) with variables satisfying
\[ 1 \leq n_1, n_2 \leq N. \]
Since we may suppose \( J(u_1, u_2) \geq 1 \), fixing one solution \( n_1^*, n_2^* \) to (4.6), for any other solution \( n_1, n_2 \) we have
\[ u_1(n_1 - n_1^*) = u_2(n_2 - n_2^*). \]
The above equation determines the residue of \( n_1 \) modulo \( u_2/(u_1, u_2) \) and for each value of \( n_1 \) there exists at most one solution \( n_2 \). Since \( U \leq N \) this implies that
\[ J(u_1, u_2) \ll N\left(\frac{u_1}{u_2}\right), \]
and hence by (4.5), we derive
\[ (4.7) \quad I(z, M, N, U) \ll \frac{NU}{\log z} + N \sum_{u_1, u_2 \in \mathcal{U}(U)} \frac{(u_1, u_2)}{u_2}. \]

Considering the last sum on the right hand side and collecting together \( u_1 \) and \( u_2 \) with the same value \( (u_1, u_2) = d \), we have
\[ (4.8) \quad \sum_{u_1, u_2 \in \mathcal{U}(U)} \frac{(u_1, u_2)}{u_2} \leq \sum_{d \in \mathcal{U}(U)} d \sum_{u_2 \in \mathcal{U}(U)} \frac{1}{u_2} \sum_{u_1 \in \mathcal{U}(u_2)} 1 \]
\[ = \Sigma_1 + \Sigma_2, \]
where
\[ \Sigma_1 = \sum_{d \in \mathcal{U}(U)} d \sum_{u_2 \in \mathcal{U}(U)} \frac{1}{u_2} \sum_{u_1 \in \mathcal{U}(u_2)} 1, \]
and
\[ \Sigma_2 = \sum_{d \in \mathcal{U}(U)} d \sum_{u_2 \in \mathcal{U}(U)} \frac{1}{u_2} \sum_{u_1 \in \mathcal{U}(u_2)} 1. \]
Considering $\Sigma_1$, by Lemma 3.3 and the condition $z \leq (u_2/d)^A$ we bound

$$\Sigma_1 \ll \sum_{d \in \mathcal{U}_z(U)} \sum_{u_2 \in \mathcal{W}_z(U)} V(z) \ll V(z) \sum_{d \in \mathcal{U}_z(U)} \sum_{u_2 \in \mathcal{W}_z(U)} 1.$$

The condition $z \leq (u_2/d)^A$ in the innermost summation implies that the outer summation over $d$ is non empty only if $(U/d)^A \geq z$ and hence by Lemma 3.3 we have

$$\sum_{d \in \mathcal{U}_z(U)} \sum_{u_2 \in \mathcal{W}_z(U)} 1 \ll UV(z) \sum_{d \in \mathcal{U}_z(U)} \sum_{d \mid u_2 (u_2/d)^A \geq z} d^{-1}.$$  \hfill (4.9)

Let

$$S(t) = \sum_{d \in \mathcal{W}_z(t)} 1.$$  Hence applying partial summation and Lemma 3.3, we obtain

$$\sum_{d \in \mathcal{U}_z(U)} d^{-1} = \frac{S(U)}{U} + \int_1^U \frac{S(t)}{t^2} dt \ll V(z) + \int_1^{z^{1/A}} \frac{S(t)}{t^2} dt + \int_{z^{1/A}}^U \frac{S(t)}{t^2} dt.$$  \hfill (4.10)

For the first integral, bounding trivially $S(t) \leq t$, we derive

$$\int_1^{z^{1/A}} \frac{S(t)}{t^2} dt \ll \int_1^{z^{1/A}} \frac{1}{t^2} dt \ll \log z.$$  \hfill (4.11)

For the second integral, after applying Lemma 3.3, we have

$$\int_{z^{1/A}}^U \frac{S(t)}{t^2} dt \ll V(z) \int_{z^{1/A}}^U \frac{1}{x} dx \ll V(z) \log U.$$  \hfill (4.12)

Substituting (4.11) and (4.12) in (4.10) we obtain

$$\sum_{d \in \mathcal{W}_z(U)} d^{-1} \ll V(z) + \log z + UV(z).$$

In turn, substituting this inequality in (4.9) and recalling the Mertens estimates (3.2) on $V(z)$, we derive

$$\sum_{d \in \mathcal{W}_z(U)} d^{-1} \ll UV(z) \left( V(z) + \log z + \frac{\log U}{\log z} \right) \ll \frac{U}{\log z} \left( 1 + \frac{\log U}{(\log z)^2} \right).$$  \hfill (4.13)
It remains to bound $\Sigma_2$. Note that

$$\Sigma_2 = \sum_{d \in \mathcal{U}_z(U)} d \sum_{u_2 \in \mathcal{U}_z \left( \min\{U, z^{1/A}d\} \right)} \sum_{u_1 \in \mathcal{U}_z(u_2)} \frac{1}{u_2 u_1} \sum_{d \mid u_1} 1,$$

where

$$\Sigma_{21} = \sum_{d \in \mathcal{U}_z(U) \left( U_z^{-1/A} \right)} d \sum_{u_2 \in \mathcal{U}_z(U) \left( z^{1/A}d \right)} \sum_{u_1 \in \mathcal{U}_z(u_2)} \frac{1}{u_2 u_1} \sum_{d \mid u_1} 1,$$

and

$$\Sigma_{22} = \sum_{d \in \mathcal{U}_z(U) \left( U_z^{-1/A} \right)} d \sum_{u_2 \in \mathcal{U}_z(U) \left( U_z \left( z^{1/A}d \right) \right)} \sum_{u_1 \in \mathcal{U}_z(u_2)} \frac{1}{u_2 u_1} \sum_{d \mid u_1} 1.$$

Bounding the innermost sum of $\Sigma_{21}$ trivially, we have

$$\Sigma_{21} \ll \sum_{d \in \mathcal{U}_z(U) \left( U_z^{-1/A} \right)} 1.$$

Noting that $z \leq (z^{1/A})^A$, an application of Lemma 3.3 gives

$$\Sigma_{21} \ll z^{1/A} V(z) \sum_{d \in \mathcal{U}_z(U) \left( U_z^{-1/A} \right)} 1 \ll V(z) U \ll \frac{U}{\log z}.$$

It remains to bound $\Sigma_{22}$. Recalling that

$$\Sigma_{22} \ll \sum_{d \in \mathcal{U}_z(U) \left( U_z^{-1/A} \right)} \sum_{u_2 \in \mathcal{U}_z(U) \left( U_z \left( z^{1/A}d \right) \right)} \sum_{u_1 \in \mathcal{U}_z(u_2)} \frac{1}{u_2 u_1} \sum_{d \mid u_1} 1,$$

by Lemma 3.3 and noting that $z > (u_2/d)^A$ since $d > U z^{-1/A}$, we obtain

$$\Sigma_{22} \ll \sum_{d \in \mathcal{U}_z(U) \left( U_z^{-1/A} \right)} \sum_{u_2 \in \mathcal{U}_z(U) \left( U_z \left( z^{1/A}d \right) \right)} V \left( \frac{u_2}{d} \right).$$

Let $R_d = \frac{\log(U/d)}{\log z}$ then $R_d \geq 1$ if and only if $d \leq U z^{-1}$ and hence

$$\Sigma_{22} \ll \sum_{d \in \mathcal{U}_z(U) \left( U_z^{-1/A} \right)} \sum_{1 \leq r \leq R_d} \sum_{u_2 \in \mathcal{U}_z(U) \left( U z^{-1}d r \right)} \sum_{u_2 \mid \gcd(d, u_2)} V \left( \frac{u_2}{d} \right).$$
Fixing a value of \( r \) and considering the innermost summation over \( u_2 \), since \( z^{r-1} \leq u_2/d \) we have \( V(u_2/d) \leq V(z^{r-1}) \) and therefore we assert

\[
\Sigma_{22} \ll \sum_{d \in \mathbb{P}(U/z)} \sum_{1 \leq r \leq R_d} V(z^{r-1}) \sum_{u_2 \in \mathbb{P}(dz^r)} 1.
\]

Appealing to Lemma 3.3 and separating the term \( r = 1 \), we have

\[
\Sigma_{22} \ll \sum_{d \in \mathbb{P}(U/z)} \sum_{1 \leq r \leq R_d} V(z^{r-1}) z^r \max\{V(z), V(z^r)\}
\]

\[
\ll V(z) \sum_{d \in \mathbb{P}(U/z)} \sum_{d > U^{z-1/A}} z + V(z)^2 \sum_{d \in \mathbb{P}(U/z)} \sum_{d > U^{z-1/A}} z^r,
\]

so that bounding the first sum trivially gives

\[
\Sigma_{22} \ll UV(z) + V(z)^2 \sum_{d \in \mathbb{P}(U/z)} \sum_{d > U^{z-1/A}} z^r.
\]

Since \( R_d = \frac{\log(U/d)}{\log z} \) we have \( z^{R_d} = Ud^{-1} \) and hence

\[
\sum_{2 \leq r \leq R_d} z^r \ll z^{R_d} = \frac{U}{d},
\]

which implies that

\[
\Sigma_{22} \ll UV(z) + UV(z)^2 \sum_{d \in \mathbb{P}(U/z)} \frac{1}{d},
\]

and hence

\[
\Sigma_{22} \ll UV(z)^2 + UV(z)^2 \sum_{U^{z-1/A} \leq d \leq U} \frac{1}{d}
\]

\[
\ll UV(z)^2 \log z \ll \frac{U}{\log z}.
\]

Combining (4.8), (4.13), (4.14) and (4.15) we get

\[
\sum_{u_1, u_2 \in \mathbb{P}(U)} \frac{(u_1, u_2)}{u_2} \ll \frac{U \log U}{(\log z)^3} + \frac{U}{\log z},
\]

and hence by (4.7)

\[
I(z, M, N, U) \ll \frac{NU}{\log z} \left(1 + \frac{\log U}{(\log z)^2}\right),
\]
which together with Lemma 3.2 completes the proof since $A$ is assumed sufficiently small. □

5. Proof of Theorem 2.1

We fix an integer $r \geq 2$ and proceed by induction on $N$. We formulate our induction hypothesis as follows. There exists some constant $c_1$, to be determined later, such that for any integer $M$ and any integer $K < N$ we have

$$\left| \sum_{M < n \leq M+K} \chi(n) \right| \leq c_1 K^{1-1/r} q^{(r+1)/4r^2} (\log q)^{1/4r},$$

and we aim to show that

$$\left| \sum_{M < n \leq M+N} \chi(n) \right| \leq c_1 N^{1-1/r} q^{(r+1)/4r^2} (\log q)^{1/4r}.$$

Since the result is trivial for $N < q^{1/4}$ this forms the basis of our induction. We define the integers $U$ and $V$ by

$$U = \left\lfloor \frac{N}{16 q^{1/2r}} \right\rfloor \quad \text{and} \quad V = \left\lfloor q^{1/2r} \right\rfloor,$$

and note that

$$UV \leq \frac{N}{16}.$$

For any integers $1 \leq u \leq U$ and $1 \leq v \leq V$ we have

$$\sum_{M < n \leq M+N} \chi(n) = \sum_{M-uv < n \leq M+N-uv} \chi(n+uv) = \sum_{M < n \leq M+N} \chi(n+uv) + \sum_{M-uv < n \leq M} \chi(n+uv) - \sum_{M+N-uv < n \leq M+N} \chi(n+uv).$$

By (5.3) and our induction hypothesis we have

$$\left| \sum_{M-uv < n \leq M} \chi(n+uv) \right| \leq \frac{c_1}{4} N^{1-1/r} q^{(r+1)/4r^2} (\log q)^{1/4r},$$

and

$$\left| \sum_{M+N-uv < n \leq M+N} \chi(n+uv) \right| \leq \frac{c_1}{4} N^{1-1/r} q^{(r+1)/4r^2} (\log q)^{1/4r},$$
which combined with the above implies that
\[
\left| \sum_{M < n \leq M + N} \chi(n) - \sum_{M < n \leq M + N} \chi(n + uv) \right| \leq \frac{c_1}{2} N^{1 - 1/r} q^{(r+1)/4r^2} (\log q)^{1/4r}.
\]
Let
\[
(5.4) \quad z = \exp \left( (\log U)^{1/2} \right),
\]
and let \( P(z) \) and \( \mathcal{U}_z(U) \) be defined by (3.1) and (3.3), respectively. Averaging over \( u \in \mathcal{U}_z(U) \) and \( 1 \leq v \leq V \) we see that
\[
(5.5) \quad \left| \sum_{M < n \leq M + N} \chi(n) \right| \leq \frac{1}{|\mathcal{U}_z(U)|V} W + \frac{c_1}{2} N^{1 - 1/r} q^{(r+1)/4r^2} (\log q)^{1/4r},
\]
where
\[
(5.6) \quad W = \sum_{M < n \leq M + N} \sum_{u \in \mathcal{U}_z(U)} \left| \sum_{1 \leq v \leq V} \chi(n + uv) \right|.
\]
By multiplying the innermost summation in (5.6) by \( \chi(u^{-1}) \) and collecting the values of \( nu^{-1} \pmod{q} \), we arrive to
\[
W = \sum_{\lambda=1}^{q} I(\lambda) \left| \sum_{1 \leq v \leq V} \chi(\lambda + v) \right|,
\]
where \( I(\lambda) \) counts the number of solutions to the congruence
\[
n \equiv \lambda u \pmod{q}, \quad M < n \leq M + N, \ u \in \mathcal{U}_z(U).
\]
Writing
\[
W = \sum_{\lambda=1}^{q} I(\lambda)^{(r-1)/r} (I(\lambda)^2)^{1/2r} \left| \sum_{1 \leq v \leq V} \chi(\lambda + v) \right|,
\]
we see that the Hölder inequality gives
\[
W^{2r} \leq \left( \sum_{\lambda=1}^{q} I(\lambda) \right)^{2r-2} \left( \sum_{\lambda=1}^{q} I(\lambda)^2 \right) \left( \sum_{\lambda=1}^{q} \left| \sum_{1 \leq v \leq V} \chi(\lambda + v) \right|^{2r} \right).
\]
From Lemma 3.2 we have
\[
(5.7) \quad \sum_{\lambda=1}^{q} I(\lambda) = \sum_{M < n \leq M + N} \sum_{u \in \mathcal{U}_z(U)} 1 = N |\mathcal{U}_z(U)| \ll \frac{NU}{\log z}.
\]
We have
\[
\sum_{\lambda=1}^{q} I(\lambda)^2 = I(z, M, N, U),
\]
where $I(z, M, N, U)$ is as in Lemma 4.1.

By Lemma 4.1, the assumption $N < q^{1/2+1/4r}$ and (5.4) we have

\begin{equation}
\sum_{\lambda=1}^{q} I(\lambda)^2 \ll \frac{NU}{\log z} \left(1 + \frac{\log U}{(\log z)^2}\right) \ll \frac{NU}{\log z}. \tag{5.8}
\end{equation}

By Lemma 3.1

\begin{equation}
\sum_{\lambda=1}^{q} \left| \sum_{1 \leq v \leq V} \chi(\lambda + v) \right|^{2r} \ll V^{r} q + V^{2r} q^{1/2}. \tag{5.9}
\end{equation}

Combining (5.7), (5.8) and (5.9) gives

\[ W^{2r} \ll \left( \frac{NU}{\log z} \right)^{2r-1} \left( V^{r} q + V^{2r} q^{1/2} \right), \]

and hence recalling (5.2) and (5.4) we see that

\[ \frac{W}{|\mathcal{U}(U)|V} \ll \frac{(\log z)^{1/2r}}{U^{1/2r}} N^{1-1/2r} q^{1/4r} \leq c_0 N^{1-1/r} q^{(r+1)/4r} (\log q)^{1/4r}, \]

for some constant $c_0$. Substituting the above into (5.5) gives

\[ \left| \sum_{M < n \leq M+N} \chi(n) \right| \leq c_0 N^{1-1/r} q^{(r+1)/4r} (\log q)^{1/4r} + \frac{c_1}{2} N^{1-1/r} q^{(r+1)/4r} (\log q)^{1/4r}, \]

from which (5.1) follows on taking $c_1 = 2c_0$.

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