

VANISHING OF PRINCIPAL VALUE INTEGRALS ON SURFACES

Willem Veys

Abstract. Principal value integrals are associated to multi–valued rational differential forms with normal crossings support on a non–singular algebraic variety. We prove their vanishing on rational surfaces in the context of a conjecture of Denef–Jacobs. As an application we obtain a strong vanishing result for candidate poles of $p$–adic and motivic Igusa zeta functions.

Introduction

0.1. Real and $p$–adic principal value integrals were first introduced by Langlands in the study of orbital integrals [Lan1,Lan2,LS1,LS2]. They are associated to multi–valued differential forms on real and $p$–adic manifolds, respectively.

Let for instance $X$ be a non–singular projective algebraic variety of dimension $n$ over $\mathbb{Q}_p$ (the field of $p$–adic numbers). Denoting by $\Omega^n_X$ the vector space of rational differential $n$–forms on $X$, take $\omega \in (\Omega^n_X)^{\otimes d}$ defined over $\mathbb{Q}_p$; we then write formally $\omega^{1/d}$ and consider it as a multi–valued rational differential form on $X$.

We suppose that the support $|\text{div } \omega|$ of $\text{div } \omega$ has normal crossings (over $\mathbb{Q}_p$) on $X$; say $D_i$, $i \in S$, are its irreducible components. Let $\text{div } \omega^{1/d} := \frac{1}{d} \text{div } \omega = \sum_{i \in S} (\alpha_i - 1) D_i$, where then the $\alpha_i \in \frac{1}{d} \mathbb{Z}$. If $\omega^{1/d}$ has no logarithmic poles, i.e. if all $\alpha_i \neq 0$, the principal value integral $\text{PV } \int_X(\mathbb{Q}_p) |\omega^{1/d}|_p$ of $\omega^{1/d}$ on $X(\mathbb{Q}_p)$ is defined as follows. Cover $X(\mathbb{Q}_p)$ by (disjoint) small enough open balls $B$ on which there exist local coordinates $x_1, \ldots, x_n$ such that all $D_i$ are coordinate hyperplanes. Consider for each $B$ the converging integral $\int_B |x_1 x_2 \cdots x_n|_p^s |\omega^{1/d}|_p$ for $s \in \mathbb{C}$ with $\mathcal{R}(s) \gg 0$, take its meromorphic continuation to $\mathbb{C}$ and evaluate this in $s = 0$; then add all these contributions. One can check that the result is independent of all choices.

In the real setting we proceed similarly but then we also need a partition of unity, and we have to assume that $\omega^{1/d}$ has no integral poles, i.e. the $\alpha_i \notin \mathbb{Z}_{\leq 0}$. Here the independency result is somewhat more complicated; it was verified in detail in [Ja1].

0.2. Denef and Jacobs proved a vanishing result for real principal value integrals, and conjectured a similar statement in the $p$–adic case. In both cases let $\mathcal{L}(\omega^{1/d})$ be the locally constant sheaf of $\mathbb{C}$–vector spaces on $X \setminus |\text{div } \omega|$ associated to $\omega^{1/d}$. It has rank 1, a non–zero section on a connected open being an analytic branch of $\omega^{1/d}$ multiplied with a complex number. (In the $p$–adic case we choose an embedding of $\mathbb{Q}_p$ into $\mathbb{C}$.)

1991 Mathematics Subject Classification. 14J26 (11S80 14E15 28B99 14F17).

Key words and phrases. Principal value integral, multi–valued differential form, surface, motivic zeta function.
Theorem [DJ, 1.1.4],[Ja1]. Let $X$ be a non–singular projective algebraic variety of (complex) dimension $n$, defined over $\mathbb{R}$. If $H^n(X(\mathbb{C}) \setminus \text{div } \omega, \mathcal{L}(\omega^{1/d})) = 0$, then $PV \int_{X(\mathbb{R})} |\omega^{1/d}| = 0$.

Conjecture [DJ, 1.2.2]. Let $X$ be a non-singular projective algebraic variety, defined over $\mathbb{Q}_p$. If $H^i(X(\mathbb{C}) \setminus \text{div } \omega, \mathcal{L}(\omega^{1/d})) = 0$ for all $i \geq 0$, then $PV \int_{X(\mathbb{Q}_p)} |\omega^{1/d}|_p = 0$.

(The authors are cautious and mention that perhaps one has to suppose also some good reduction mod $p$ and that all $\alpha_i \notin \mathbb{Z}$.)

0.3. In [Ve7] we ‘upgraded’ $p$–adic principal value integrals to motivic ones, in the same spirit as how motivic integration and motivic zeta functions were inspired by (usual) $p$–adic integration and $p$–adic Igusa zeta functions. See [DL2,DL3] or the surveys [DL4,Loo,Ve6] for these notions.

More precisely, let $X$ be a non–singular algebraic variety (say over $\mathbb{C}$) of dimension $n$ and $\omega^{1/d}$ a multi–valued differential form on $X$. Let as above $\text{div}(\omega^{1/d}) = \sum_{i \in S} (\alpha_i - 1)D_i$ be a normal crossings divisor. We denote also $D^\circ_I := (\cap_{i \in I} D_i) \setminus (\cup_{i \notin I} D_i)$ for $I \subset S$; so $X = \coprod_{I \subset S} D^\circ_I$. If $\omega^{1/d}$ has no logarithmic poles, then the motivic principal value integral of $\omega^{1/d}$ on $X$ is

$$PV \int_X \omega^{1/d} = L^{-n} \sum_{I \subset S} [D^\circ_I] \prod_{i \in I} \frac{L - 1}{L^{\alpha_i} - 1}.$$ 

Here $[\cdot]$ denotes the class of a variety in the Grothendieck ring of algebraic varieties, and $L := [A^1]$, see (1.1). We refer to [Ve7, §2] for a motivation for this expression. Note for instance that this is just the formula for the converging motivic integral associated to the $\mathbb{Q}$–divisor $\text{div}(\omega^{1/d})$ if all $\alpha_i > 0$.

It is natural to ‘upgrade’ also the conjecture in (0.2) to this setting (maybe also assuming that all $\alpha_i \notin \mathbb{Z}$) : if $H^i(X(\mathbb{C}) \setminus \text{div } \omega, \mathcal{L}(\omega^{1/d})) = 0$ for all $i \geq 0$, then $PV \int_X \omega^{1/d} = 0$.

0.4. In this paper we attack (the motivic version of) the conjecture for surfaces, more precisely rational surfaces.

We note that, in dimension 2, this is the crucial class to study : there is no ‘classification’ for configurations $|\text{div } \omega| \subset X$ with all $H^i(X \setminus |\text{div } \omega|, \mathcal{L}(\omega^{1/d})) = 0$ on a rational surface $X$. For non–rational surfaces it is conceivable that the conjecture can be approached through the classification of such configurations from [GP] and [Ve3]. We plan to report on this later. (Also for the applications that we will prove here, the class of rational surfaces is the crucial one, see (0.7).)

Our main theorem is as follows.

Theorem. Let $X$ be a non–singular projective rational surface and $\omega^{1/d}$ a multi–valued differential form on $X$ without logarithmic poles (in particular $|\text{div } \omega|$ has normal crossings).

1. Suppose that $B := |\text{div } \omega|$ is connected. If $\chi(X \setminus B) \leq 0$, then $PV \int_X \omega^{1/d} = 0$.

2. More generally, let $B$ be any connected normal crossings divisor satisfying $B \supset |\text{div } \omega|$. If $\chi(X \setminus B) \leq 0$, then $PV \int_X \omega^{1/d} = 0$. 
Statement (1) is somewhat weaker than the conjecture because of the connectivity condition. On the other hand it is clearly stronger since we assume only that \( \chi(X \setminus B) \leq 0 \), instead of the vanishing of all \( H^i(X \setminus B, \mathcal{E}(\omega^{1/d})) \). (And we do not need the extra assumption \( \alpha_i \not\in \mathbb{Z} \).) The generalization (2) is important in view of the applications on motivic zeta functions and is natural in that context, see (0.8).

The important ingredients in our proof are the structure theorem of [Ve3] for such configurations \( B \subset X \) with \( \chi(X \setminus B) \leq 0 \), and the notion of a more general principal value integral when \( \omega^{1/d} \) is allowed to have ‘some’ logarithmic poles, see (1.4).

0.5. Real and \( p \)-adic principal value integrals appear as coefficients of asymptotic expansions of oscillating integrals and fibre integrals, and as residues of poles of distributions \( |f|^N \) and \( p \)-adic Igusa zeta functions, respectively. See [Ja2, §1] for more details and [AVG,De2,Ig1,Ig2,Ja3,Lae] for more explanations. In the generic situation that the cancelation of some given candidate pole of a \( p \)-adic Igusa zeta function is essentially equivalent to the vanishing of an associated \( p \)-adic principal value integral. Here we are interested in the analogous phenomenon for poles of the motivic zeta function versus associated motivic principal value integrals.

0.6. Denef and Loeser [DL2] associated to a non–constant regular function \( f \) on a non–singular algebraic variety \( M \) of dimension \( n+1 \) its motivic zeta function \( Z_{\text{mot}}(f; T) \); here \( T \) is a variable. They obtained the following formula for it in terms of an embedded resolution \( h: Y \to M \) of the hypersurface \( \{ f = 0 \} \). Let \( E_j, j \in K \), be the irreducible components of \( h^{-1}\{ f = 0 \} \) and \( \nu_j \) and \( \nu_j - 1 \) the multiplicity of \( E_j \) in the divisors \( \text{div}(f \circ h) \) and \( \text{div}(h^* dx) \), respectively, where \( dx \) is a local generator of the sheaf of \( (n+1) \)-forms on \( M \). Denote \( E_j^\circ := (\cap_{j \in J} E_j) \setminus (\cup_{\ell \notin J} E_{\ell}) \) for \( J \subset K \). Then

\[
Z_{\text{mot}}(f; T) = L^{-(n+1)} \sum_{J \subset K} [E_j^\circ] \prod_{j \in J} \frac{(L-1)T^{N_j}}{L^{\nu_j-1}T^{N_j}}.
\]

Fix a ‘candidate pole’ \( L^{\nu_j/N_j} \) of \( Z_{\text{mot}}(f; T) \); see (3.4) for more explanations. In the generic situation that \( \nu_j/N_j \neq \nu_i/N_i \) for all \( i \neq j \), the cancelation of the candidate pole (of order 1) \( L^{\nu_j/N_j} \) is equivalent to the vanishing of ‘its residue’

\[
L^{-n} \sum_{I \subset S_j} [D_i^\circ] \prod_{i \in I} \frac{L-1}{L^{\alpha_i}-1},
\]

where \( S_j = \{ i \in K \setminus \{ j \} | E_i \text{ intersects } E_j \} \), \( D_i := E_j \cap E_i \) and \( \alpha_i = \nu_i - (\nu_j/N_j)N_i \neq 0 \) for \( i \in S_j \), and \( D_i^\circ \) is as usual. This looks like a motivic principal value integral on \( E_j \), and indeed it is equal to \( PV \int_{E_j} \omega^{1/d}, \) where \( \omega^{1/d} \) is some Poincaré residue, see (3.5).

Suppose now that \( E_j \) is mapped to a point by \( h \). Denote \( B_j := \cup_{i \in S_j} D_i \). The famous Monodromy Conjecture predicts more or less that (generically), if \( (-1)^n \chi(E_j \setminus B_j) \leq 0 \), then \( L^{\nu_j/N_j} \) is no pole of \( Z_{\text{mot}}(f; T) \), see (3.5).

0.7. When \( n = 2 \), we now can prove this expected cancelation of candidate poles as a consequence of our Main Theorem. More precisely, we may suppose that \( h \) is a composition of blowing–ups as in [Hi]. Then the exceptional surface \( E_j \) is created during the
resolution process \( h \) either as a projective plane by blowing up a point, or as a ruled surface by blowing up a non–singular curve. Then, with \( B_j \) as above, we can show as corollary of (0.4):

**Theorem.** Let \( L^{\nu_j/N_j} \) be a candidate pole of order 1 for \( Z_{\text{mot}}(f; T) \), as described above. Suppose that \( \chi(E_j \setminus B_j) \leq 0 \).

1. If \( E_j \) is created by blowing up a point, then we have always that \( L^{\nu_j/N_j} \) is no pole.
2. If \( E_j \) is created by blowing up a rational curve, then \( L^{\nu_j/N_j} \) is no pole whenever \( B_j \) is connected.

In fact we obtain a somewhat stronger statement; see (0.8(i)) below. But first we want to comment on this new result.

Concerning (1), the best general result up to now [Ve2] was the analogous cancelation in the context of \( p \)-adic Igusa zeta functions when the centre of the blowing up is a point of multiplicity at most 4 on the strict transform of \( \{ f = 0 \} \). This was achieved using a lengthy classification of all possible configurations with \( \chi(E_j \setminus B_j) \leq 0 \) under this multiplicity restriction.

Concerning (2), it is important to note that non–connected intersection configurations on \( E_j \) are very rare, see (3.7). And moreover, the case of rational centres is the crucial one. When \( E_j \) is created by blowing up a curve of genus \( g \geq 2 \) we already proved the expected cancelation in [Ve2], and we can now also handle the case \( g = 1 \) completely by combining [Ve2] and [Ve3] with recent work of Rodrigues [Ro2].

**0.8. Remarks.** (i) In fact we assume only that \( \nu_j/N_j \neq \nu_i/N_i \) for the \( i \in S_j \), and we show that ‘the contribution of \( E_j \) to the residue of \( L^{\nu_j/N_j} \)’ vanishes when expected.

(ii) Similar vanishing results hold in the context of much more general motivic zeta functions, for instance those of [Ve4] associated to an effective \( \mathbb{Q} \)-Cartier divisor on any \( \mathbb{Q} \)-Gorenstein threefold (instead of just a hypersurface on \( \mathbb{A}^3 \)). Also, (0.7) specializes to the context of Hodge, topological and \( p \)-adic zeta functions.

(iii) Note that in (0.6) it is possible that some \( \alpha_i = 1 \), and thus that \( B_j := \bigcup_{i \in S_j} D_i \supset \) \( \sum | \text{div} \omega^{1/d} | \). So the more general setting of part (2) of our main theorem pops up naturally in this context of poles of zeta functions.

**0.9.** We will work over the base field \( \mathbb{C} \) of complex numbers. In §1 we introduce the more general principal value integrals on surfaces, allowing ‘some’ logarithmic poles, which we need for the proof of the main theorem in §2. Then in §3 we obtain as a corollary the cancelation of candidate poles for zeta functions.

**Acknowledgement.** We would like to remark that [ACLM1, §2] contributed to our inspiration for the proof of the main theorem. And we thank Bart Rodrigues for his useful remarks and suggestions.

**1. Generalized principal value integrals on surfaces**

**1.1.** We first recall briefly the notion of Grothendieck ring of varieties and related constructions.
(i) The Grothendieck ring $K_0(\text{Var})$ of complex algebraic varieties is the free abelian group generated by the symbols $[V]$, where $V$ is a variety, subject to the relations $[V] = [V']$ if $V$ is isomorphic to $V'$, and $[V] = [V \setminus W] + [W]$ if $W$ is closed in $V$. Its ring structure is given by $[V] \cdot [W] := [V \times W]$. (This ring is quite mysterious; see [Po] for the recent proof that it is not a domain.) Usually, one abbreviates $L := [\mathbb{A}^1]$.

For the sequel we need to extend $K_0(\text{Var})$ with fractional powers of $L$ and to localize. Fix $d \in \mathbb{Z}_{>0}$; we consider

$$K_0(\text{Var})[L^{-1/d}] := \frac{K_0(\text{Var})[T]}{(LT^d - 1)}$$

(where $L^{-1/d} := \bar{T}$). We then localize this ring with respect to the elements $L^{i/d} - 1$, $i \in \mathbb{Z} \setminus \{0\}$. What we really need is the subring of this localization generated by $K_0(\text{Var})$, $L^{-1}$ and the elements $(L - 1)/(L^{i/d} - 1)$, $i \in \mathbb{Z} \setminus \{0\}$; we denote this subring by $\mathcal{R}_d$. (We do not know whether or not $\mathcal{R}_d$ has zero divisors.)

(ii) For a variety $V$, we denote by $h^{p,q}(H^i_c(V, \mathbb{C}))$ the rank of the $(p, q)$–Hodge component in the mixed Hodge structure of the $i$th cohomology group with compact support of $V$. The Hodge polynomial of $V$ is

$$H(V) = H(V; u, v) := \sum_{p,q} (\sum_{i \geq 0} (-1)^i h^{p,q}(H^i_c(V, \mathbb{C}))) u^p v^q \in \mathbb{Z}[u,v].$$

Precisely by the defining relations of $K_0(\text{Var})$, there is a well–defined ring homomorphism $H : K_0(\text{Var}) \to \mathbb{Z}[u,v]$, determined by $[V] \mapsto H(V)$. It induces a ring homomorphism $H$ from $\mathcal{R}_d$ to the ‘rational functions in $u$, $v$ with fractional powers’.

(iii) The topological Euler characteristic $\chi(V)$ of a variety $V$ satisfies $\chi(V) = H(V; 1, 1)$ and we obtain a ring homomorphism $\chi : K_0(\text{Var}) \to \mathbb{Z}$, determined by $[V] \mapsto \chi(V)$. Since $\chi(L) = 1$, it induces a ring homomorphism $\chi : \mathcal{R}_d \to \mathbb{Q}$ by declaring $\chi((L - 1)/(L^{i/d} - 1)) = d/i$ (see e.g. [DL2,Ve4] for similar constructions).

1.2. On surfaces we want to extend the notion of principal value integral of (0.3) in two ways. First we will allow the differential form $\omega^{1/d}$ to have ‘some’ logarithmic poles. Another somewhat technical generalization consists in considering a normal crossings divisor whose support contains $\operatorname{div} \omega$. In view of the application on candidate poles of zeta functions, this is natural; see (0.8(iii)) or (3.5). More precisely we fix the following setting.

1.3. Let $X$ be a non–singular projective surface, $\omega^{1/d}$ a multi–valued differential form on $X$, and $D = \cup_{i \in T} C_i$ a normal crossings divisor on $X$ satisfying $D \supset |\operatorname{div} \omega|$. We write formally $\operatorname{div} \omega^{1/d} = \sum_{i \in T} (\alpha_i - 1) C_i$, where thus $\alpha_i = 1$ if $C_i \not\subset |\operatorname{div} \omega|$. We put the following restriction on the possible logarithmic poles of $\omega^{1/d}$.

If for $i \in T$ we have $\alpha_i = 0$, then

1. $C_i$ is rational,
2. no $C_\ell$ that intersects $C_i$ has $\alpha_\ell = 0$,
3. in all but at most two intersection points of $C_i$ with other $C_\ell$ the intersecting curve $C_\ell$ has $\alpha_\ell = 1$.

We call such a pair $(D, \omega^{1/d})$ allowed.
When $\alpha_i = 0$ the adjunction formula $(K_X + C_i) \cdot C_i = K_{C_i}$ and the restrictions above easily yield that either two curves $C_{i_1}$ and $C_{i_2}$ with $\alpha_{i_1} \neq 1 \neq \alpha_{i_2}$ intersect $C_i$ and then $\alpha_{i_1} + \alpha_{i_2} = 0$, or only one curve $C_{i_1}$ with $\alpha_{i_1} \neq 1$ intersects $C_i$ and then $\alpha_{i_1} = -1$. (Here and further on, when applying the adjunction formula we always consider $\text{div} \omega^{1/d}$ as a representative of $K_X$.)

1.4. Definition. Let $X$ be a non–singular projective surface and $(D, \omega^{1/d})$ an allowed pair on $X$. We write $\text{div} \omega^{1/d} = \sum_{i \in T} (\alpha_i - 1) C_i$ as in (1.3) and $C_i^\circ := (\cap_{i \in I} C_i) \backslash (\cup_{i \notin I} C_i)$ for $I \subset T$ as before. Furthermore for $i \in T$ we denote by $C_i \cdot C_i$ the self–intersection number of $C_i$ and by $C_j, j \in T_i (\subset T)$, the curves that intersect $C_i$. To $(D, \omega^{1/d})$ we associate the invariant

$$E_X (D, \omega^{1/d}) := \sum_{i \in T, \forall i \in I} [C_i^\circ] \prod_{i \in I} \frac{L - 1}{\alpha_i - 1} + \sum_{i \in T} (-C_i \cdot C_i) \prod_{j \in T_i} \frac{L - 1}{\alpha_j - 1},$$

living in $\mathcal{R}_d$. In the last sum the expression $(-C_i \cdot C_i) \prod_{j \in T_i} \frac{L - 1}{\alpha_j - 1}$ is the easiest to write down ‘uniformly’, but, since at most two of the $\alpha_j$ are different from 1, this expression boils down to the following.

(1) If $C_i$ intersects two curves $C_{i_1}$ and $C_{i_2}$ with $\alpha_{i_1} \neq 1 \neq \alpha_{i_2}$, we get

$$(-C_i \cdot C_i) \frac{(L - 1)^2}{(L^{\alpha_{i_1}} - 1)(L^{\alpha_{i_2}} - 1)} = (C_i \cdot C_i) \frac{(L - 1)^2 L^{\alpha_{i_1}}}{(L^{\alpha_{i_1}} - 1)^2} = (C_i \cdot C_i) \frac{(L - 1)^2 L^{\alpha_{i_2}}}{(L^{\alpha_{i_2}} - 1)^2}.$$

(2) If $C_i$ intersects only one curve $C_{i_1}$ with $\alpha_{i_1} \neq 1$, we get

$$(-C_i \cdot C_i) \frac{L - 1}{L^{\alpha_{i_1}} - 1} = (C_i \cdot C_i) L.$$

Note. (i) In fact the $C_i$ with $\alpha_i = 1$, i.e. those $C_i$ not belonging to $|\text{div} \omega|$, play no role in the definition of $E_X (D, \omega^{1/d})$: we could as well consider instead of $T$ only $\{i \in T \mid C_i \subset |\text{div} \omega|\}$. So this invariant is really an invariant of $\omega^{1/d}$ only. However, for the sequel it is useful to introduce it as above.

(ii) As a motivation for the expression for the contribution of $C_i$ with $\alpha_i = 0$: it is a kind of limit of the ‘total contribution of $C_i$’ in the formula of (0.3) for $PV \int_X \omega^{1/d}$ if $\alpha_i \neq 0$, see e.g. [Ve5, 3.3].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{}
\end{figure}
1.5. Lemma. Let $X$ be a non–singular projective surface, $P \in X$ and $h : \tilde{X} \to X$ the blowing–up of $X$ with centre $P$. Let $(D, \omega^{1/d})$ be allowed on $X$, writing as usual $\text{div}(\omega^{1/d}) = \sum_{\ell \in T} (\alpha_{\ell} - 1) C_{\ell}$. Then

1. $(h^{-1}D, h^*\omega^{1/d})$ is allowed on $\tilde{X}$,
2. $\mathcal{E}_X(h^{-1}D, h^*\omega^{1/d}) = \mathcal{E}_X(D, \omega^{1/d})$ except when (on $X$) there exists a curve $C_i$ with $\alpha_i = 0$ and curves $C_{i_1}$ and $C_{i_2}$, intersecting $C_i$, with $\alpha_{i_1} + \alpha_{i_2} = 0$ and $\{\alpha_{i_1}, \alpha_{i_2}\} \neq \{-1, 1\}$, such that $P \in C_i, P \notin C_{i_1}$ and $P \notin C_{i_2}$. (And in this exceptional case we do have inequality.)

Proof. The necessary (easy) computations are essentially in [ACLM1] and [Ve5, 3.5]. We just illustrate the exceptional case, see Figure 1. Note that in this case, since $(D, \omega^{1/d})$ is allowed, if there is another curve $C_j$ passing through $P$, it must satisfy $\alpha_j = 1$. And by Note (1.4(i)) we may as well assume that only $C_{i_1}$ and $C_{i_2}$ intersect $C_i$.

Let $C$ denote the exceptional curve of $h$. Then $h^{-1}D$ consists of the union of $C$ and the strict transforms of the $C_{\ell}$. And since $\alpha_i = 0$ we have that $C$ does not appear in the divisor of $h^*\omega^{1/d}$. So we write formally $\text{div}(h^*\omega^{1/d}) = \sum_{\ell \in T} (\alpha_{\ell} - 1) C_{\ell} + (\alpha - 1) C$ with $\alpha = 1$. We have to compare the contributions of $C_i$ to $\mathcal{E}_X(D, \omega^{1/d})$ and of $C_i \cup C$ to $\mathcal{E}_X(h^{-1}D, h^*\omega^{1/d})$. These are

$$-(C_i \cdot C_i) \frac{(L - 1)^2}{(\ell^{\alpha_{i_1}} - 1)(\ell^{\alpha_{i_2}} - 1)}$$

and

$$-(C_i \cdot C_{i_1} - 1) \frac{(L - 1)^2}{(\ell^{\alpha_{i_1}} - 1)(\ell^{\alpha_{i_2}} - 1)} + L,$$

respectively. Their difference $\frac{(L - 1)^2}{(\ell^{\alpha_{i_1}} - 1)(\ell^{\alpha_{i_2}} - 1)} + L$ is nonzero (in $R_d$); one can ensure the non–nullity for instance using $H$ or $\chi$. (On the other hand, when $\{\alpha_{i_1}, \alpha_{i_2}\} = \{-1, 1\}$, this difference would be $-L + L = 0$.) \(\square\)

2. Rational surfaces

2.1. We first summarize the structure theorem of [Ve3] and some of its refinements, which will be the starting point of the proof of our main theorem. Remember that a non–singular rational curve with self–intersection $-1$ is called a $(-1)$–curve.

Structure Theorem. Let $X$ be a non–singular projective rational surface. Let $B$ be a connected normal crossings curve on $X$ with $\chi(X \setminus B) \leq 0$. Assume that $X$ does not contain any $(-1)$–curve disjoint from $B$.

By [GP, Theorem 3] there is a dominant morphism $\varphi : X \setminus B \to \mathbb{P}^1$; let $h : \tilde{X} \to X$ be the minimal morphism that resolves the indeterminacies of $\varphi$, considered as rational map from $X$ to $\mathbb{P}^1$. 
(1) Then there exists a connected curve $B' \supset B$ with $\chi(X \setminus B') \leq \chi(X \setminus B) \leq 0$, such that the morphism $\varphi \circ h$ decomposes as

$$
\tilde{X} \xrightarrow{g} \Sigma \xrightarrow{\pi} \mathbb{P}^1,
$$

where $g$ is a composition of blowing-downs with exceptional curve in $h^{-1}B'$, and $\pi : \Sigma \to \mathbb{P}^1$ is a ruled surface; see Diagram 1. Moreover, $h^{-1}B'$ has normal crossings in $\tilde{X}$.

(2) We can require the configuration $g(h^{-1}B') \subset \Sigma$ to be one of the configurations in Figure 2.

Here $C_1$ and $C_2$ are sections of $\pi$, $C$ is a non-singular curve for which $\pi|_C : C \to \mathbb{P}^1$ has degree 2 (a ‘bisection’), and the other curves are fibres of $\pi$. The minimal number of fibres in (a) and (b) is 2 and 1, respectively; in (c) there must pass a fibre through each ramification point of $\pi|_C$, and we can have any number of other fibres. Note that in (c) the bisection can be non-rational (and then has more than two ramification points).

(3) Irreducible curves $C \subset h^{-1}B'$, which are not components of $h^{-1}B$, occur only in fibres (= exceptional components) of $g$. Moreover, any fibre of $g$ contains at most one such curve $C$ and

$$
\begin{align*}
\text{card}(C \cap h^{-1}B) &= 1 & \text{if } & \chi(\Sigma \setminus g(h^{-1}B')) < 0 \\
\text{card}(C \cap h^{-1}B) &= 2 & \text{if } & \chi(\Sigma \setminus g(h^{-1}B')) = 0.
\end{align*}
$$

Note that always $\chi(\Sigma \setminus g(h^{-1}B')) = 0$ in cases (b) and (c).

Proof. Combine essentially (3.3), (3.5) and (4.3) in [Ve3]. □
2.2. We will denote by \( p : \tilde{\Sigma} \to \Sigma \) the minimal embedded resolution of the configuration \( D := g(h^{-1}B') \subset \Sigma \) in case (c) of the structure theorem. When \( \omega^{1/d} \) is a multi-valued differential form on \( \Sigma \) with \( D \supset |\text{div} \omega| \), we will slightly abuse the terminology of (1.3) and say that \( (D, \omega^{1/d}) \) is allowed on \( \Sigma \) if \( (p^{-1}D, p^*\omega^{1/d}) \) is allowed on \( \tilde{\Sigma} \). In that case we also put \( \mathcal{E}_\Sigma(D, \omega^{1/d}) := \mathcal{E}_{\tilde{\Sigma}}(p^{-1}D, p^*\omega^{1/d}) \).

**Lemma.** Let \( D := g(h^{-1}B') \subset \Sigma \) in the Structure Theorem and let \( \omega^{1/d} \) be a multi-valued differential form on \( \Sigma \). Assume in case (c) that \( C \) is rational (this is equivalent to \( \pi|_C \) having exactly two ramification points). If \( (D, \omega^{1/d}) \) is allowed on \( \Sigma \), then \( \mathcal{E}_\Sigma(D, \omega^{1/d}) = 0 \).

**Proof.** We write as usual \( D \) as \( \bigcup_{i \in T} C_i \) and \( \text{div} \omega^{1/d} = \sum_{i \in T}(\alpha_i - 1)C_i \). Applying the adjunction formula to a generic fibre of \( \pi \) yields in cases (a), (b) and (c) that \( \alpha_1 = -1, \alpha_1 + \alpha_2 = 0 \) and \( \alpha = 0 \), respectively. We first treat the cases (a) and (b).

If no \( \alpha_i = 0 \) this is well known and easily verified; see e.g. [Ve2] for a similar computation. The point is that the contribution of any fibre of \( \pi \) to \( \mathcal{E}_X(D, \omega^{1/d}) \) is zero. Now if \( \alpha_i = 0 \) for some fibre \( C_i \) of \( \pi \), the contribution of \( C_i \) is still zero, simply because its self-intersection \( C_i \cdot C_i = 0 \). This finishes already case (a).

In case (b) we are left with the following possibility: \( \alpha_1 = \alpha_2 = 0 \) and (omitting the possible fibres \( C_{\ell} \) with \( \alpha_{\ell} = 1 \)) there is either only one fibre \( C_i \subset D \) with then necessarily \( \alpha_i = -1 \), or there are two fibres \( C_i \) and \( C'_i \) in \( D \) with \( \alpha_i + \alpha'_i = 0 \) and \( \alpha_i \neq 0 \neq \alpha'_i \). We compute \( \mathcal{E}_X(D, \omega^{1/d}) \) in this last case:

\[
\mathcal{E}_X(D, \omega^{1/d}) = (L-1)^2 + (L-1)\frac{L-1}{L_{\alpha_1} - 1} + (L-1)\frac{L-1}{L_{\alpha'_1} - 1} \\
+ \frac{(L-1)^2}{(L_{\alpha_1} - 1)(L_{\alpha'_1} - 1)} + \frac{(L-1)^2}{(L_{\alpha'_1} - 1)(L_{\alpha'_1} - 1)}.
\]

Since \( C_1 \cdot C_1 = -C_2 \cdot C_2 \) (see e.g. [Ha, Theorem V.2.17]) the last two terms cancel and, since \( \alpha_i + \alpha'_i = 0 \), we obtain

\[
\mathcal{E}_X(D, \omega^{1/d}) = (L-1)^2 \frac{L_{\alpha_i + \alpha'_i} - 1}{(L_{\alpha_1} - 1)(L_{\alpha'_1} - 1)} = 0.
\]

For case (c) we have to consider \( \mathcal{E}_\Sigma(p^{-1}D, p^*\omega^{1/d}) \). We denote the two curves in \( p^{-1}D \) which intersect \( C(\subset \tilde{\Sigma}) \) in \( P_1 \) and \( P_2 \) by \( C_1 \) and \( C_2 \), respectively, see Figure 3.

![Figure 3](image-url)
Claim. A fibre \( C_j \) in \( \Sigma \) not containing \( P_1 \) or \( P_2 \) must have \( \alpha_j = 1 \). Indeed, by allowedness, if \( \alpha_j \neq 1 \) we must have \( \alpha_1 = \alpha_2 = 1 \) and also \( \alpha_\ell = 1 \) for possible other fibres \( C_\ell \) on \( \Sigma \). But then the adjunction formula for \( C \) on \( \tilde{\Sigma} \) yields \( \alpha_j = 0 \), contradicting the other requirement for allowedness.

Consequently, for the computation of \( \mathcal{E}_\Sigma(p^{-1}D, p^*\omega^{1/d}) \) we can neglect possible fibres not containing \( P_1 \) or \( P_2 \), and we know that \( \alpha_1 + \alpha_2 = 0 \) and \( \alpha_1 \neq 0 \neq \alpha_2 \). The adjunction formula on \( \tilde{\Sigma} \) for \( C_i' \) and \( C_i'' \) yields \( \alpha_i' = \alpha_i'' = \frac{\alpha_i + 1}{2} \) for \( i = 1, 2 \). (Here the notations \( \alpha_i, \alpha_i', \alpha_i'' \) refer to Figure 3.) Using the structure of Pic \( \Sigma \) one computes that the self-intersection \( C \cdot C = 4 \) on \( \Sigma \), see e.g. [Ve1, Remark 6.7]. Then clearly \( C \cdot C = 0 \) on \( \tilde{\Sigma} \).

So in the defining expression for \( \mathcal{E}_\Sigma(p^{-1}D, p^*\omega^{1/d}) \) we only need to sum the terms for \( I \subset T \) with \( \alpha_i \neq 0 \) for all \( i \in I \). A simple computation (or directly the formula [Ve5, Proposition 5.4]) yields

\[
\mathcal{E}_\Sigma(p^{-1}D, p^*\omega^{1/d}) = L(L - 1) \sum_{i=1}^{2} \frac{L - 1}{L_{\alpha_i} - 1}(L - 2 + 2(1 + L_{\frac{\alpha_i + 1}{2}}))
\]

\[
= L(L - 1)(1 + \sum_{i=1}^{2} \frac{1}{L_{\alpha_i} - 1}) + 2(L - 1) \sum_{i=1}^{2} L_{\frac{\alpha_i + 1}{2}} \frac{1}{L_{\alpha_i} - 1}
\]

\[
= 0 + 2(L - 1)L^{1/2} \frac{L_{\alpha_1 + \frac{\alpha_2}{2}} - L_{\frac{\alpha_1}{2}} - L_{\frac{\alpha_2}{2}} + L_{\alpha_2 + \frac{\alpha_1}{2}}}{(L_{\alpha_1} - 1)(L_{\alpha_2} - 1)}
\]

\[
= 2(L - 1)L^{1/2} \frac{(L_{\frac{\alpha_1 + \alpha_2}{2}} - 1)(L_{\frac{\alpha_1}{2}} + L_{\frac{\alpha_2}{2}})}{(L_{\alpha_1} - 1)(L_{\alpha_2} - 1)} = 0,
\]

using twice that \( \alpha_1 + \alpha_2 = 0 \). \( \Box \)

We are now ready to prove our vanishing theorem.

2.3. Theorem. Let \( X \) be a non–singular projective rational surface and \( \omega^{1/d} \) a multi-valued differential form on \( X \) without logarithmic poles. Let \( B = \bigcup_{i \in T} C_i \) be a connected normal crossings divisor on \( X \) satisfying \( B \supset |\text{div } \omega| \). If \( \chi(X \setminus B) \leq 0 \), then \( PV \int_X \omega^{1/d} (= \mathcal{E}_X(B, \omega^{1/d})) = 0 \).

Remark. The generalization with \( B \supset |\text{div } \omega| \) is not only needed for the applications in §3, but is already useful in the proof for \( B = |\text{div } \omega| \).

Proof. We first explain our strategy. We will construct maps \( \Sigma \leftarrow X \stackrel{h}{\rightarrow} X \) as in the Structure Theorem 2.1. If the exceptional situation of Lemma 1.5(2) does not occur in any blowing–up of \( g \) or \( h \), then this lemma implies

\[
\mathcal{E}_X(B, \omega^{1/d}) = \mathcal{E}_\tilde{X}(h^{-1}B, \omega^{1/d}) = \mathcal{E}_\tilde{X}(h^{-1}B', \omega^{1/d}) = \mathcal{E}_\Sigma(g(h^{-1}B'), \omega^{1/d}),
\]

where for simplicity we keep the notation \( \omega^{1/d} \) on each surface (a rational differential form is a birational notion anyway). By Lemma 2.2 this last expression vanishes (the
configuration on Σ turns out to be allowed). The crucial point is that we will show that indeed such an exceptional situation never occurs.

We will denote all curves appearing in the image of $h^{-1}B'$ in any intermediate surface by $C_\ell$.

**Step 1.** The surface $X$ does not contain any $(-1)$–curve disjoint from $B$.

Indeed, suppose that $A$ is such a $(-1)$–curve disjoint from $B$. Applying the adjunction formula for $A$ on $X$ would yield $-2 = \deg K_A = A \cdot A = -1$.

So we can really apply Theorem 2.1 and we obtain the extended configuration $B' \supset B$ on $X$ and the maps $\Sigma \xrightarrow{g} \tilde{X} \xrightarrow{h} X$, using from now on all notations introduced in that theorem.

**Step 2.** The exceptional situation of Lemma 1.5(2) does not occur for any constituting blowing–up of $h : \tilde{X} \to X$.

Suppose that there is such a blowing–up $b : X'' \to X'$ with centre $P_i \in C_i$, $\alpha_i = 0$, and two other curves $C_{i_1}$ and $C_{i_2}$ intersecting $C_i$ outside $P_i$, satisfying $\alpha_{i_1} + \alpha_{i_2} = 0$ and $\{\alpha_{i_1}, \alpha_{i_2}\} \neq \{-1, 1\}$; see Figure 4. Denote by $C$ the exceptional curve of $b$; recall that $\alpha = 1$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Figure 4}
\end{figure}

Since $b$ is really needed to resolve the indeterminacies of $\varphi$ ($h$ is minimal), either $C$ itself or some other curve, created after a chain of blowing–ups starting with centre a point of $C$, will project by $g$ onto a section or bisection on $\Sigma$. Such a curve $C'$ has $\alpha' \in \mathbb{Z}_{>0}$, and hence the configuration on $\Sigma$ must be case (b). This also implies that $C_i$ cannot project onto a section or bisection on $\Sigma$, so it is either blown down during $g$ or becomes a fibre on $\Sigma$. Both possibilities yield that at some stage of $g$ the curve $C_i$ can intersect at most two other components of the total transform of $B'$.

So in order to reach that stage by blowing–downs starting from $\tilde{X}$, it is necessary that at least for one of the points $P_1, P_2$ or $P_i$, considered on $C_i \subset \tilde{X}$, the following holds (recall that we keep the same notations for strict transforms of curves and points): $h^{-1}B' \setminus \{P_\ell\}$ has two connected components, and the one not containing $C_i$ is a tree that gets completely contracted before reaching that stage of $g$. 
This cannot happen for \( P_i \) because, as explained above, either \( C \) itself or another curve in ‘its tree’ must project onto a section of \( \Sigma \).

Consequently there is such a tree contracting to \( P_1 \) or \( P_2 \) during \( g \), say to \( P_1 \). Now, since \( \alpha_i = 0 \), the component \( C_\ell \) of \( h^{-1}B \) through \( P_1 \) (in \( \tilde{X} \)) still has \( \alpha_\ell = \alpha_{i_1} \). Analogously, at the stage of \( g \) just before the last blowing–down needed to contract the tree, the ‘last’ component \( C'_\ell \) through \( P_1 \) still has \( \alpha'_\ell = \alpha_{i_1} (\neq 1) \). But, on the other hand, since \( C'_\ell \) now gets contracted by that blowing–down, we must have \( \alpha'_\ell = 1 \). This contradiction finishes step 2.

**Step 3.** The exceptional situation of Lemma 1.5(2) does not occur for any constituting blowing–up of \( g : \tilde{X} \to \Sigma \).

(To be precise, in case (c) for \( \Sigma \) we consider only constituting blowing–ups of \( \tilde{X} \to \tilde{\Sigma} \), using the notation of (2.2).) Suppose that there is such a blowing–up \( b : X'' \to X' \), where we use again the notations as in Figure 4, but this time ‘during \( g \).’ We consider two cases.

(I) \( P_i \), considered on \( C_i \subset X' \), does not belong to any other component of the total transform of \( B' \).

Here we consider two subcases.

(i) \( C \subset h^{-1}B \) in \( \tilde{X} \).

Since \( C_i (\subset \tilde{X}) \) must be blown down during \( h \) (all \( \alpha_\ell \neq 0 \) on \( X \)) and \( B \) has normal crossings, *at most two* components of the total transform of \( B \) can intersect \( C_i \) just before that blowing–down. As we explained above, since \( \alpha_{i_1} \neq 1 \neq \alpha_{i_2} \), the components intersecting \( C_i \) in \( P_1 \) and \( P_2 \) cannot be blown down before \( C_i \). So the tree that was created during \( g \), starting with the blowing–up with centre \( P_i \), must be contracted again by \( h \). This contradicts the minimality of \( h \).

(ii) \( C (\subset \tilde{X}) \) is a component of (the strict transform of) \( B' \setminus B \).

Because \( h^{-1}B \) is connected, each possible further blowing–up of \( g \) (after \( b \)) with centre in \( C \) must have \( P_i \) as centre. Consequently

\[
\chi(X \setminus B') < \chi(X \setminus B) \leq 0
\]

and \( \Sigma \) must be as in case (a) with at least three fibres. The strict transform in \( \tilde{X} \) of the section \( C_1 \subset \Sigma \) can intersect at most two other components of \( h^{-1}B \). (Indeed, since \( \alpha_i = 0 \), the morphism \( h \) cannot be the identity. Hence \( C_1 \), being the only non–fibre in \( \Sigma \), must be created by the last blowing–up of \( h \).) So \( C_1 \) must intersect in \( \tilde{X} \) at least one component \( C' \) coming from \( B' \setminus B \). But \( C' \) gets contracted during \( g \); so in order to have at least three intersections with \( C_1 \) in \( \Sigma \), \( C' \) must intersect in \( \tilde{X} \) another component of \( h^{-1}B \). This contradicts Theorem 2.1(3).

(II) \( P_i \), considered on \( C_i \subset X' \), also belongs to another component \( C' \) of the total transform of \( B' \). Then this curve \( C' \) should be drawn (in the reader’s mind) through \( P_i \) on the \( X'' \)–side of Figure 4, and its strict transform through another point of \( C \) on the \( X'' \)–side.
Since, by Lemma 1.5, \((h^{-1}B, \omega^{1/d})\) is allowed on \(\tilde{X}\), we then have necessarily \(\alpha' = 1\). Moreover, we claim that \(C_i\) must already live on \(\Sigma\), or on \(\tilde{\Sigma}\) in case (c), where we use the notation of (2.2). Indeed, suppose that \(C_i\) is created by a blowing-up of \(g\), or of \(\tilde{X} \to \tilde{\Sigma}\) in case (c); at that stage \(C_i\) can intersect at most two other components. Thus \(C'\) should be created afterwards during \(g\). But this situation is precisely the previous case (I)(ii), which we already contradicted.

We now investigate the cases (a), (b) and (c) for \(\Sigma\) in our (hypothetical) situation. Case (a) is clearly impossible. In case (b) \(C_i\) must be a section, and in case (c) \(C_i\) must be the bisection in \(\Sigma\), see Figure 5. (In case (c) \(C_i\) could be a priori another curve in \(\tilde{\Sigma}\), but then we would have two intersecting curves with \(\alpha_\ell = 0\) on \(\tilde{\Sigma}\), implying the same phenomenon on \(\tilde{X}\).) Somewhat abusing notation in Figure 5, the curves \(C_1, C_2\) and \(C'\) in \(\Sigma\) could be different from those intersecting \(C_i\) in \(X'\), but their \(\alpha\)–coefficients are also \(\alpha_1, \alpha_2\) and \(\alpha'(= 1)\), respectively.

![Figure 5](image-url)

Case (b). On \(\tilde{X}\) we have that \(C_i\) (being an exceptional curve of \(h\)) can intersect at most two other components of \(h^{-1}B\). So all but at most two intersections of \(C_i\) in \(\Sigma\) with fibres \(C_\ell\) of \(\pi: \Sigma \to \mathbb{P}^1\) must ‘split’ during \(g\), i.e. \(C_i\) must eventually get separated from those \(C_\ell\) by a blowing-up of \(g\) with centre in \(C_i\) and exceptional curve coming from \(B' \setminus B\). (The same is true for \(C'_i\).)

In order to obtain such an exceptional curve \(C_j\) from \(B' \setminus B\), intersecting \(C_i\) in \(P_1\), we should have simultaneously \(\alpha_j = 1\) and \(\alpha_j = \alpha_{i_1} \neq 1\), by the same argument as for the conclusion of step 2. The same contradiction works for \(P_2, P'_1\) and \(P'_2\). Hence we must separate \(C_i\) and \(C'_i\) from \(C'\) with exceptional curves from \(B' \setminus B\), intersecting \(C_i\) and \(C'_i\) in \(P_i\) and \(P'_i\), respectively. But this contradicts the connectivity of \(h^{-1}B\).
Case (c). Analogously we will have to separate $C'$ from $C_i$, contradicting the connectivity of $h^{-1}B$.

**Step 4.** $(g(h^{-1}B'),\omega^{1/d})$ is allowed on $\Sigma$ and $\mathcal{E}_X(B,\omega^{1/d}) = \mathcal{E}_\Sigma(g(h^{-1}B'),\omega^{1/d}) = 0$.

By Lemma 1.5 we have $\mathcal{E}_X(B,\omega^{1/d}) = \mathcal{E}_\tilde{X}(h^{-1}B,\omega^{1/d})$. Since also $h^{-1}B'$ is a normal crossings divisor and the curves $C_i$ in $B' \setminus B$ have $\alpha_i = 1$, we have that also $(h^{-1}B',\omega^{1/d})$ is allowed on $\tilde{X}$ and $\mathcal{E}_\tilde{X}(h^{-1}B,\omega^{1/d}) = \mathcal{E}_\tilde{X}(h^{-1}B',\omega^{1/d})$. It is easy to see that then necessarily $(g(h^{-1}B'),\omega^{1/d})$ is allowed on $\Sigma$, and thus $\mathcal{E}_\Sigma(h^{-1}B',\omega^{1/d}) = \mathcal{E}_\Sigma(g(h^{-1}B'),\omega^{1/d})$, again by Lemma 1.5(2). This last expression equals zero by Lemma 2.2. □

**2.4. Remark.** Theorem 2.3 and its proof are quite subtle. In [Ve7, 3.4] we constructed the following similar example. Let $X = \mathbb{P}^2$ and $\omega^{1/2}$ a multi–valued differential form with $|\text{div}\omega|$ a (non–singular) conic $B$. So $\omega^{1/2}$ has no logarithmic poles on $X$ and one easily computes that $PV \int_X \omega^{1/2} \neq 0$. Taking $B'$ as the union of $B$ and one tangent line to $B$, we can construct $\Sigma \overset{g}{\leftarrow} \tilde{X} \overset{h}{\rightarrow} X$, where $h$ is a composition of three blowing–ups and $g$ a composition of two blowing–downs, all outside of $X \setminus B'$, such that $g(h^{-1}B') \subset \Sigma$ is case (b) in Theorem 2.1 with exactly one fibre. Also on $\Sigma$ we have that $\omega^{1/2}$ has no logarithmic poles, but now $PV \int_\Sigma \omega^{1/2} = 0$. In this example we have that $\chi(X \setminus B') = 0$; the only obstruction with the data of Theorem 2.3 is that here $\chi(X \setminus B) = 1 (> 0)$. And in fact the ‘change’ in principal value integral is caused by $g$ which consists precisely of the exceptional situation of Lemma 1.5(2).

Another example in this context is provided by the minimal embedded resolution of two smooth plane conics with one intersection point, see [ACLM1, Example 2.14].

**2.5.** Our vanishing theorem specializes to the level of Hodge polynomials or Euler characteristics. With the same notations as in Definition 1.4 we can introduce analogously the invariants

$$E_X(D,\omega^{1/d}) := \sum_{I \subseteq T} \sum_{\forall i : \alpha_i \neq 0} H(C_i^o) \prod_{i \in I} \frac{uv - 1}{(uv)^{\alpha_i} - 1} + \sum_{\alpha_i = 0} \sum_{T} (-C_i \cdot C_i) \prod_{j \in T} \frac{uv - 1}{(uv)^{\alpha_j} - 1}$$

and

$$e_X(D,\omega^{1/d}) := \sum_{I \subseteq T} \sum_{\forall i : \alpha_i \neq 0} \chi(C_i^o) \prod_{i \in I} \frac{1}{\alpha_i} + \sum_{\alpha_i = 0} \sum_{T} (-C_i \cdot C_i) \prod_{j \in T} \frac{1}{\alpha_j}.$$ 

Then with the same data as in Theorem 2.3 we obtain that $E_X(B,\omega^{1/d}) = e_X(B,\omega^{1/d}) = 0$.

**2.6.** Let $L$ be a $p$–adic field with valuation ring $O$, maximal ideal $P$ and residue field $\mathcal{O}_p \cong \mathbb{F}_q$. We choose an embedding of $L$ into $\mathbb{C}$. When $X$ and $\omega^{1/d}$ are defined over $L$, we can introduce analogously

$$E_{\mathcal{X}}^L(D,\omega^{1/d}) := \sum_{I \subseteq T} \sum_{\forall i : \alpha_i \neq 0} \text{card}(C_i^o)_{\mathbb{F}_q} \prod_{i \in I} \frac{q - 1}{q^{\alpha_i} - 1} + \sum_{\alpha_i = 0} \sum_{T} (-C_i \cdot C_i) \prod_{j \in T} \frac{q - 1}{q^{\alpha_j} - 1},$$
where \(\text{card}(\cdot)_{\mathbb{F}_q}\) denotes the number of \(\mathbb{F}_q\)-rational points of the reduction mod \(P\) of \(C_i^0\). When \(\omega^{1/d}\) has no logarithmic poles and if suitable conditions about good reduction mod \(P\) are satisfied, a similar proof as for Denef’s formula for the \(p\)-adic Igusa zeta function [De1] yields that \(E_k^X(D, \omega^{1/d})\) is (up to a power of \(q\)) precisely the \(p\)-adic principal value integral \(PV \int_{X(L)} |\omega^{1/d}|\).

When we assume that the field \(L\) is ‘big enough’, the same data as in Theorem 2.3 will again imply that this invariant vanishes. More precisely we want that all constructions in this proof of the theorem can be done ‘over \(L\)’, for instance that all centres of blowing-ups are \(L\)-rational points. This can always be achieved by taking a finite extension of a given \(p\)-adic field.

3. Cancelation of candidate poles for zeta functions

3.1. Fix a polynomial \(f \in \mathbb{C}[X_1, \ldots, X_{n+1}] \setminus \mathbb{C}\), or, more generally, a non–constant regular function \(f : M \to \mathbb{A}^1\) from a non–singular \((n+1)\)-dimensional variety \(M\). Denef and Loeser [DL2] associated to \(f\) its motivic zeta function \(Z_{\text{mot}}(f; T) \in \mathcal{M}[[T]]\), where \(\mathcal{M}\) is the localization of \(K_0(\text{Var})\) with respect to \(L\). It describes the orders of all truncated arcs on \(M\) along the hypersurface \(\{f = 0\}\), and is in fact modeled on the classical \(p\)-adic Igusa zeta function. (See also [DL4] or [Ve6] for an introduction to this topic.) We just mention a formula for \(Z_{\text{mot}}(f; T)\) in terms of an embedded resolution of \(\{f = 0\}\), implying in particular that this invariant belongs to the localization of \(\mathcal{M}[T]\) with respect to \(L^a - T^b\), where \(a, b \in \mathbb{Z}_{>0}\).

3.2. Theorem [DL2, 2.2.1]. Let \(h : Y \to M\) be an embedded resolution of \(\{f = 0\}\). Let \(E_j, j \in K\), be the irreducible components of \(h^{-1}\{f = 0\}\). Denote by \(N_j\) the multiplicity of \(E_j\) in \(\text{div}(f \circ h)\) and by \(\nu_j - 1\) the multiplicity of \(E_j\) in \(\text{div}(h^*dx)\), where \(dx\) is a local generator of the sheaf of differential \((n+1)\)-forms on \(M\). For \(J \subset K\) we put \(E^J := (\bigcap_{j \in J} E_j) \setminus (\bigcup_{\ell \notin J} E_\ell)\); so \(Y = \coprod_{J \subset K} E^J\). Then

\[
Z_{\text{mot}}(f; T) = L^{-(n+1)} \sum_{J \subset K} [E^J] \prod_{j \in J} \frac{(L - 1)T^{N_j}}{L^{\nu_j} - T^{N_j}}.
\]

3.3. This zeta function specializes to the Hodge zeta function \(Z_{\text{Hod}}(f; T)\), replacing in the formula above all classes of varieties in \(\mathcal{M}\) by their Hodge polynomial, and further to the ‘classical’ topological zeta function

\[
z_{\text{top}}(f; s) = \sum_{J \subset K} \chi(E^J) \prod_{j \in J} \frac{1}{\nu_j + sN_j} \in \mathbb{Q}(s)
\]

of [DL1]. We refer to [DL2] or [Ve6] for more details.

3.4. The famous monodromy conjecture, stated originally for the \(p\)-adic Igusa zeta function, can be formulated for \(Z_{\text{mot}}(f; T)\) as follows [DL2, 2.4]:
\(Z_{\text{mot}}(f; T)\) belongs to the localization of \(\mathcal{M}\) with respect to those \(L^a - T^b\), \(a, b \in \mathbb{Z}_{>0}\), such that \(e^{2\pi i a/b}\) is an eigenvalue of the local monodromy on the Milnor fibre of \(f\) at some point of \(\{f = 0\}\).

So if \(L^{\nu/N}\) is a pole of \(Z_{\text{mot}}(f; T)\), then \(e^{2\pi i \nu/N}\) is expected to be an eigenvalue of the local monodromy. Note however that one has to be careful with the notion of pole here, the difficulty being that we do not know whether \(\mathcal{M}\) is a domain. See e.g. [RV2] for a precise definition. For the Hodge and topological zeta function the notion of pole is clear.

The \((p\text{-adic version of})\) conjecture was proved in dimension two \((n = 1)\) by Loeser [Loe1]; a simple proof in the motivic/Hodge/topological setting is in [Ro1]. It is still open in general, with partial results in dimension three [Ve2,RV1,ACLM1] in and other special cases [Loe2,ACLM2].

3.5. We keep using the notation of Theorem 3.2. Fix an exceptional component \(E_j\) of \(h\) which is mapped to a point by \(h\). It induces the candidate pole \(L^{\nu_j/N_j}\) for \(Z_{\text{mot}}(f; T)\), respectively \((uv)^{\nu_j/N_j}\) for \(Z_{\text{Hod}}(f; T)\) and \(-\nu_j/N_j\) for \(z_{\text{top}}(f; s)\).

In order for the monodromy conjecture to hold, looking at A’Campo’s formula for the monodromy zeta function [A’C] one expects the following. Suppose we are in the generic case that \(\nu_j/N_j \neq \nu_i/N_i\) for all \(i \neq j\). If \(\chi(E_j^0) = 0\), maybe even if \((-1)^n \chi(E_j^0) \leq 0\), then ‘in general’ \(L^{\nu_j/N_j}\) should not be a pole of \(Z_{\text{mot}}(f; T)\).

Somewhat more precise, suppose only that \(\nu_j/N_j \neq \nu_i/N_i\) for all \(i \in S_j := \{i \in K \mid E_i \cap E_j \neq \emptyset\}\). Then, if \((-1)^n \chi(E_j^0) \leq 0\), one expects that ‘in general’ \(E_j\) does not contribute to the possible pole \(L^{\nu_j/N_j}\), which means that \(L^{\nu_j/N_j}\) should not be a pole of

\[
\frac{1}{L_{n+1}} \sum_{j \in I \subseteq K} [E_j^0] \prod_{i \in I} \frac{(L - 1)T^{N_i}}{L^{\nu_i} - T^{N_i}}.
\]

We refer to e.g. [Ve2, §1] for a motivation. For \(n = 1\) this expectation is true and is part of the proof of the monodromy conjecture [Loe1,Ro1]. Now, \(L^{\nu_j/N_j}\) not being a pole of

\[
\sum_{j \in I \subseteq K} [E_j^0] \prod_{i \in I \setminus \{j\}} \frac{L - 1}{L^{\alpha_i} - 1}
\]

is zero, where \(\alpha_i := \nu_i - (\nu_j/N_j)N_i\) for \(i \in S_j\). Now (**\) is a motivic principal value integral on \(E_j\). Indeed, let \(dx\) be a local generator of the sheaf of \((n+1)\)-forms on \(M\) around the point \(h(E_j)\). Then the Poincaré residue \(\omega^{1/d}\) of \((f \circ h)^{-\nu_j/N_j} h^*(dx)\) on \(E_j\) is a multi-valued differential form on \(E_j\) with \(\text{div} \omega^{1/d} = \sum_{i \in S_j} (\alpha_i - 1)(E_j \cap E_i)\). This is easily verified with local coordinates, see also [Ja3]. We thus have that (**\) is (up to a non–zero constant) equal to \(PV \int_{E_j} \omega^{1/d}\).

Note however that by assumption all \(\alpha_i \neq 0\), so \(\omega^{1/d}\) indeed has no logarithmic poles, but it is possible that some \(\alpha_i = 1\). So in fact we land in a natural way in the more general framework of (1.3) with \(D := \bigcup_{i \in S_j}(E_j \cap E_i) \supset \text{div} \omega\), where the inclusion may be strict, and (**\) is (essentially) \(\mathcal{E}_{E_j}(D, \omega^{1/d})\).
3.6. Let now $n=2$. We may assume that $h$ is constructed as a composition of blowing-ups with non-singular centre. Fix as above a projective exceptional surface $E_j$ which is mapped to a point by $h$. The surface $E_j$ was created during some blowing-up $\pi$ of the resolution process $h$ as a surface $E_j^{(0)}$, where either $E_j^{(0)} \cong \mathbb{P}^2$ or $E_j^{(0)}$ is a ruled surface, when the centre of $\pi$ is a point or a curve, respectively. And then $E_j$ is obtained from $E_j^{(0)}$ by a composition $\varphi : E_j \rightarrow E_j^{(0)}$ of (point) blowing-ups. Denote again $D := \bigcup_{i \in S_j} (E_j \cap E_i)$. We have that $D$ is the inverse image by $\varphi$ of the intersection of $E_j^{(0)}$ with the other components of the total inverse image of $\{f = 0\}$ at the stage of $h$ when $E_j^{(0)}$ was just created. In particular $D$ is connected if and only if this intersection on $E_j^{(0)}$ is connected. And it is thus always connected if $E_j^{(0)} \cong \mathbb{P}^2$.

3.7. By the considerations above Theorem 2.3 yields the following cancelation result for candidate poles of the motivic zeta function. For ease of reference we recall the notations.

Let $M$ be a three-dimensional non-singular variety and $f : M \rightarrow \mathbb{A}^1$ a non-constant regular function. Let $h : Y \rightarrow M$ be an embedded resolution of $\{f = 0\}$, constructed as a composition of blowing-ups. Denote by $E_j, j \in K$, the irreducible components of $h^{-1}\{f = 0\}$ and let $N_j, \nu_j$ and $E_j^{\circ}$ be as in (3.2). Suppose that $E_j$ is mapped to a point by $h$ and that $\nu_j/N_j \neq \nu_i/N_i$ for all $i \in S_j := \{i \in K \mid E_j \cap E_i \neq \emptyset\}$. Denote

$$R_{E_j} := \sum_{j \in I \subset K} \left[E_j^{\circ}\right] \prod_{i \in I \setminus \{j\}} \frac{L-1}{L^{\alpha_i} - 1},$$

‘the contribution of $E_j$ to the residue of $L^{\nu_j/N_j}$ for $\mathbb{Z}_{mot}(f; T)$’.

**Theorem.** Let $\chi(E_j^{\circ}) \leq 0$.

1. If $E_j$ is created by blowing up a point, then we have always $R_{E_j} = 0$.
2. If $E_j$ is created by blowing up a rational curve, and if $\bigcup_{i \in S_j} (E_j \cap E_i)$ is connected, then $R_{E_j} = 0$.

Recall that in case (2) this connectivity is equivalent to the connectivity of the analogous intersection configuration $D^{(0)}$ on the (rational) ruled surface $E_j^{(0)}$ (3.6). Note then that the exceptions in (2), i.e. those $E_j$ with a non-connected intersection configuration, are very special! Indeed, non-connectivity of $D^{(0)}$ implies for instance that $D^{(0)}$ does not contain any fibre of the ruled surface $E_j^{(0)}$. In the embedded resolution process this is quite rare.

There is a recent vanishing result of Rodrigues [Ro2] in this exceptional case. When $\chi(E_j^{\circ}) = 0$, and assuming a minor extra condition, he classified all possible non-connected $D^{(0)}$ with non-singular irreducible components, and verified that then again $R_{E_j} = 0$.

3.8. The case where $E_j$ is a rational surface is the most difficult one. There is no classification of the possible intersection configurations on $E_j$ with $\chi(E_j^{\circ}) \leq 0$; instead we used our structure theorem 2.1. When $E_j$ is created by blowing up a non-rational curve, we already obtained a classification of the possible configurations with $\chi(E_j^{\circ}) \leq 0$ in [Ve2] and [Ve3].
**Proposition.** We use all notations of (3.7). Let \( \chi(E^o_j) \leq 0 \). If \( E_j \) is created by blowing up a non–rational curve, then we have always \( R_{E_j} = 0 \).

**Proof.** Let as above \( E_j \) be created as the ruled surface \( E_j^{(0)} \) while blowing up a curve of genus \( g \) during \( h \). When \( g \geq 2 \), we classified the possible configurations on \( E_j^{(0)} \) with \( \chi(E^o_j) \leq 0 \) in [Ve2, Proposition 5.13], and verified in [Ve2, Propositions 5.1 and 5.3] that \( R_{E_j} = 0 \) for them. (The calculation there is in the context of \( p \)–adic Igusa zeta functions, but is essentially the same in the motivic setting.)

When \( g = 1 \), the possible configurations \( C \) of curves on the ruled surface \( E_j^{(0)} \) with \( \chi(E_j \setminus C) \leq 0 \) were classified in [Ve3, Theorem 6.5]. Again by [Ve2, Propositions 5.1 and 5.3] we have that \( R_{E_j} = 0 \), except for one annoying case (where \( R_{E_j} \neq 0 \)). More precisely, in this case the curves on the ruled surface consist of a number of disjoint elliptic curves, where either one of the curves is not a section, or there are at least three curves. Now recently Rodrigues showed in [Ro2] that in fact this configuration cannot occur in the context of exceptional surfaces in an embedded resolution. \( \square \)

3.9. The theorem and proposition above provide a strong confirmation for the Monodromy Conjecture for surfaces \( (n = 2) \), and could be a major contribution to a proof. Of course there are still various non–obvious remaining parts within this strategy, for instance extending the results of [Ro2] to all possible non–connected \( D_j^{(0)} \) and handling candidate poles of higher order.

3.10. Theorem 3.7 and Proposition 3.8 specialize to the analogous results in the context of the Hodge and topological zeta functions of (3.3). They are also valid in the context of \( p \)–adic Igusa zeta functions (see e.g. [De1, Ve2]) if the \( p \)–adic field \( L \) is assumed ‘big enough’ as in (2.6), and if suitable conditions concerning good reduction mod \( P \) as in [De1] are satisfied. Alternatively, one can take a big enough number field \( F \). Then our vanishing results will be true for the Igusa zeta functions over all except a finite number of completions \( L \) of \( F \).

**References**

[A'C] N. A’Campo, *La fonction zeta d’une monodromie*, Comment. Math. Helv. **50** (1975), 233–248.

[ACLM1] E. Artal Bartolo, P. Cassou-Noguès, I. Luengo and A. Melle Hernández, *Monodromy conjecture for some surface singularities*, Ann. Scient. Ec. Norm. Sup. **35** (2002), 605–640.

[ACLM2] E. Artal Bartolo, P. Cassou-Noguès, I. Luengo and A. Melle Hernández, *Quasi-ordinary power series and their zeta functions*, [math.NT/0306249](http://arxiv.org/abs/math.NT/0306249).

[AVG] V. Arnold, A. Varchenko and S. Goussein–Zadé, *Singularités des applications différentiables II*, Editions Mir, Moscou, 1986.

[De1] J. Denef, *On the degree of Igusa’s local zeta function*, Amer. J. Math. **109** (1987), 991–1008.

[De2] J. Denef, *Report on Igusa’s local zeta function*, Sém. Bourbaki 741, Astérisque **201/202/203** (1991), 359–386.

[DJ] J. Denef and Ph. Jacobs, *On the vanishing of principal value integrals*, C. R. Acad. Sci. Paris **326** (1998), 1041–1046.

[DL1] J. Denef and F. Loeser, *Caractéristiques d’Euler–Poincaré, fonctions zêta locales, et modifications analytiques*, J. Amer. Math. Soc. **5** (1992), 705–720.

[DL2] J. Denef and F. Loeser, *Motivic Igusa zeta functions*, J. Alg. Geom. **7** (1998), 505–537.
VANISHING OF PRINCIPAL VALUE INTEGRALS ON SURFACES

[DL3] J. Denef and F. Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. 135 (1999), 201–232.

[DL4] J. Denef and F. Loeser, *Geometry on arc spaces of algebraic varieties*, Proceedings of the Third European Congress of Mathematics, Barcelona 2000, Progr. Math. 201 (2001), Birkhäuser, Basel, 327–348.

[GP] R. Gurjar and A. Parameswaran, *Open surfaces with non–positive Euler characteristic*, Compositio Math. 99 (1995), 213–229.

[Ha1] T. Hales, *Can p-adic integrals be computed?*, “Contributions to automorphic forms, geometry and number theory” (2004), Johns Hopkins University Press, Baltimore, 313–329.

[Hi] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. Math. 79 (1964), 109–126.

[Ig1] J. Igusa, *Complex powers and asymptotic expansions I*, J. Reine Angew. Math. 268/269 (1974), 110–130; *II*, ibid. 278/279 (1975), 307–321.

[Ig2] J. Igusa, *Lectures on forms of higher degree*, Tata Inst. Fund. Research, Bombay (1978).

[Ja1] Ph. Jacobs, *Principal value integrals, cohomology and Igusa’s zeta functions*, Ph. D. thesis, Univ. Leuven, 1998.

[Ja2] Ph. Jacobs, *Real principal value integrals*, Monatsch. Math. 130 (2000), 261–280.

[Ja3] Ph. Jacobs, *The distribution |f|^λ, oscillating integrals and principal value integrals*, J. Analyse Math. 81 (2000), 343–372.

[Lae] A. Laeremans, *The distribution |f|^s, topological zeta functions and Newton polyhedra*, Ph. D. thesis, Univ. Leuven, 1997.

[Lan1] R. Langlands, *Orbital integrals on forms of SL(3), I*, Amer. J. Math. 105 (1983), 465–506.

[Lan2] R. Langlands, *Remarks on Igusa theory and real orbital integrals*, The Zeta Functions of Picard Modular Surfaces, Les Publications CRM, Montréal; distributed by AMS, 1992, pp. 335–347.

[Loe1] F. Loeser, *Fonctions d’Igusa p–adiques et polynômes de Bernstein*, Amer. J. Math. 110 (1988), 1–22.

[Loe2] F. Loeser, *Fonctions d’Igusa p–adiques, polynômes de Bernstein, et polyédres de Newton, J. reine angew. Math. 412 (1990), 75–96.*

[Loo] E. Looijenga, *Motivic measures*, Séminaire Bourbaki 874, Astérisque 276 (2002), 267–297.

[LS1] R. Langlands and D. Shelstad, *On principal values on p–adic manifolds*, Lect. Notes Math. 1041 (1984), Springer, Berlin.

[LS2] R. Langlands and D. Shelstad, *Orbital integrals on forms of SL(3), II*, Can. J. Math. 41 (1989), 480–507.

[Po] B. Poonen, *The Grothendieck ring of varieties is not a domain*, Math. Res. Letters 9 (2002), 493–498.

[Ro1] B. Rodrigues, *On the monodromy conjecture for curves on normal surfaces*, Math. Proc. Cambridge Phil. Soc. 136 (2004), 1–18.

[Ro2] B. Rodrigues, *Ruled exceptional surfaces and the poles of motivic zeta functions*, preprint (2004), 31p.

[RV1] B. Rodrigues and W. Veys, *Holomorphy of Igusa’s and topological zeta functions for homogeneous polynomials*, Pacific J. Math. 201 (2001), 429–441.

[RV2] B. Rodrigues and W. Veys, *Poles of zeta functions on normal surfaces*, Proc. London Math. Soc. 87 (2003), 164–196.

[Ve1] W. Veys, *Relations between numerical data of an embedded resolution*, Amer. J. Math. 113 (1991), 573–592.

[Ve2] W. Veys, *Poles of Igusa’s local zeta function and monodromy*, Bull. Soc. Math. France 121 (1993), 545–598.

[Ve3] W. Veys, *Structure of rational open surfaces with non–positive Euler characteristic*, Math. Annalen 312 (1998), 527–548.

[Ve4] W. Veys, *Zeta functions and ‘Kontsevich invariants’ on singular varieties*, Canadian J. Math. 53 (2001), 834–865.

[Ve5] W. Veys, *Stringy invariants of normal surfaces*, J. Alg. Geom. 13 (2004), 115–141.
[Ve6] W. Veys, *Arc spaces, motivic integration and stringy invariants*, Advanced Studies in Pure Mathematics (to appear), 43p., Proceedings of "Singularity theory and its applications, Sapporo (Japan), 16–25 september 2003".

[Ve7] W. Veys, *On motivic principal value integrals*, preprint (2004), 14p.