Module of continuity for the functions
belonging to the Sobolev-Grand Lebesgue Spaces

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Abstract. In this short article we generalize the Sobolev’s inequalities for the module of continuity for the functions belonging to the classical Lebesgue space on the (Bilateral) Grand Lebesgue spaces.
We construct also some examples in order to show the exactness of obtained results.

Key words and phrases: Sobolev’s continuity inequalities, derivative, module of continuity, gradient, norm, natural function, Lebesgue spaces, Sobolev’s and ordinary (Bilateral) Grand Lebesgue spaces, Orlicz spaces, embedding theorems, fundamental and truncated fundamental function, conductivity, slowly varying function, support.

Mathematics Subject Classification (2000): primary 60G17; secondary 60E07; 60G70.

1 Introduction. Notations. Statement of problem.

1. Sobolev’s continuity inequality.
Let $D$ be convex closed bounded non-empty domain with Lipshitz boundary in the Euclidean space $R^d$, $d \geq 2$. For instance, the domain $D$ may be the unit ball $B$

\[ B = \{ x : x \in R^d, \ |x| \leq 1 \}. \]

We will consider in this article only the case when the domain $D$ is bounded.
The classical Sobolev’s inequality (for the domain $D$ or for the whole space $R^d$, ) see, e.g. [15], chapter 11, section 5; [33], [34] etc. asserts that for all weak differentiable functions $f, f : R^d \to R$, $d \geq 3$ from the Sobolev’s space $W^1_p(D)$, $p \in [1, d)$, which may be defined as a closure in the Sobolev’s norm

\[ ||f||W^1_p(R^m) = |f|_p + |\nabla f|_p \]

of the set of all finite continuous differentiable functions $f, f : D \to R$, that
|f|_q \leq K_d(p) \ |\nabla f|_p, \ q = q(p) = dp/(d - p), \ p \in [1, d), \ q \in (d/(d - 1), \infty). \quad (1)

Here the notation |x| denotes ordinary Euclidean norm of the vector x,

\[|f|_p = |f|_{p, D} = \left(\int_D |f(x)|^p \, dx\right)^{1/p},\]

\[\nabla f = \{\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3, \ldots, \partial f/\partial x_d\} = \text{grad } f,\]

\[|\nabla f|_p = \left|\left[\sum_{i=1}^{d} (\partial f/\partial x_i)^2\right]^{1/2}\right|_p,\]

\[|\nabla^2 f|_p = \left|\left[\sum_{i=1}^{d} \sum_{j=1}^{d} (\partial^2 f/\partial x_i \partial x_j)^2\right]^{1/2}\right|_p\]

etc.

The best possible constant in the inequality (1) belongs to G. Talenti [34]:

\[K_d(p) = \pi^{-1/2} d^{-1/p} \left[\frac{p - 1}{d - p}\right]^{1-1/p} \cdot \left[\frac{\Gamma(1 + d/2)}{\Gamma(d/p)} \frac{\Gamma(d)}{\Gamma(1 + d - d/p)}\right]^{1/d}.\]

The case \( p = d \) was considered by Yudovich [37] and after 7 years by Trudinger [36]; see also [32], section 1.2; [35], chapter 13, section 4.

The exact values of constants in the Orlich-Sobolev imbedding theorem in this case \( p = d \) was obtained by Moser [21].

Let us consider hereafter only the case \( p > d, \ d = 1, 2, \ldots; \) and also (for simplicity) only the case when

\[f(x)_{\partial D} \overset{\text{def}}{=} \lim_{x \to \partial D} f(x) = 0;\]

where \( \partial D \) denotes the boundary of the domain \( D \).

In this case, i.e. when \( \nabla f \in L_p(D) \), the function \( f(\cdot) \) up to chaining in the subset of the domain \( D \) of the zero measure is continuous.

More detail, let us define as ordinary the module of continuity \( \omega(f, \delta) \) of arbitrary uniform continuous function \( f : D \to R \)

\[\omega(f, \delta) \overset{\text{def}}{=} \sup_{x, y \in D, |x - y| \leq \delta} |f(x) - f(y)|,\]

\( \delta \in (0, 1/e) \).

It is proved, e.g. in the book [20], chapter 1, p. 60-62 that

\[\omega(f, \delta) \leq K_M(d, D) \ \delta^{1-d/p} \cdot \left[\frac{p - 1}{p - d}\right]^{1-1/p} \cdot |\nabla f|_p, \quad (2a)\]
Analogous result is obtained by Leoni [18], chapter 11, section 11.3 with the exact values of a constant:

$$\omega(f, \delta) \leq \frac{2dp}{p-d} \cdot (2\delta)^{1-d/p} \cdot |\nabla f|_p.$$  \hspace{1cm} (2b)

Some important applications of these inequalities in the theory of Partial Differential Equations are described, e.g., in [9], [35]; in the Calculus of Variations - in [22].

Notice that both the inequalities may be rewritten in the equivalent up to multiplicative constant form:

$$\omega(f, \delta) \leq C(d, D) \frac{p}{p-d} \cdot \delta^{1-d/p} \cdot |\nabla f|_p.$$  \hspace{1cm} (2c)

as long as $p > d$.

In the book [9], chapter 5, section 5.6, p. 280-282 the inequalities (2a), (2b) and (2c) was named as a Morrey inequality; see also [22].

The inequalities (2a), (2b) and (2c) was generalized in the works [1], [6], [7] on the Orlicz-Sobolev spaces, i.e. when $\nabla f$ belongs to some Orlicz space. We intent to improve, in particular, this results.

2. Our aim.

Our aim is generalization of Sobolev’s continuity inequality (2a), (2b) or (2c) on some popular classes of rearrangement invariant (r.i.) spaces, namely, on the so-called Sobolev-Grand Lebesgue Spaces $G(\psi)$. We intend to show also the exactness of offered estimations.

3. Another notations. Hereafter $C, C_j$ will denote any non-essential finite positive constants. We define also for the values $(p_1, p_2)$, where $1 \leq p_1 < p_2 \leq \infty$

$$L(p_1, p_2) = \cap_{p \in (p_1, p_2)} L_p.$$  

We denote also

$$\Omega(d) = \frac{2\pi^{d/2}}{\Gamma(d/2)};$$

and denote as usually an indicator function

$$I(A) = I(A, x) = 1, x \in A, \ I(A) = I(A, x) = 0, \ x \notin A.$$  

4. Content of the paper.

The paper is organized as follows. In the next section we recall the definition and some simple properties of the so-called Grand Lebesgue Spaces (GLS) $G(\psi)$ and introduce its generalizations: the so-called Sobolev’s Grand Lebesgue spaces (SGLS) $W^{1}G(\psi)$.

In the section 3 we formulate and prove the main result: the classical Sobolev’s continuity inequality for $W^{1}G(\psi)$ spaces.
In the fourth section we built some examples in order to show the exactness of obtained inequalities. The section 5 is devoted to the consideration the one-dimensional case, in which we can compute the exact value on embedding constant.

The last section contains some concluding remarks: a weight generalizations of the embedding theorem for the high derivatives etc.

2 Sobolev’s-Grand Lebesgue Spaces.

A. Grand Lebesgue Spaces.

Recently, see [16], [10], [11], [12], [13], [14], [23], [24], [25], [26], [27], [28], [30], [31] etc. appears the so-called Grand Lebesgue Spaces $GLS = G(\psi) = G\psi = G(\psi; A, B)$, $A, B = \text{const}$, $A \geq 1$, $A < B \leq \infty$, spaces consisting on all the measurable functions $f : T \to R$ with finite norms

$$||f||_{G(\psi)} \overset{def}{=} \sup_{p \in (A, B)} ||f||_p / \psi(p). \quad (3)$$

Here $\psi(\cdot)$ is some continuous positive on the open interval $(A, B)$ function such that

$$\inf_{p \in (A, B)} \psi(p) > 0, \quad \psi(p) = \infty, \quad p \notin (A, B).$$

We will denote

$$\text{supp}(\psi) \overset{def}{=} (A, B) = \{ p : \psi(p) < \infty, \} \quad (4)$$

The set of all $\psi$ functions with support $\text{supp}(\psi) = (A, B)$ will be denoted by $\Psi(A, B)$.

This spaces are rearrangement invariant, see [2], and are used, for example, in the theory of probability [17], [16], [23]; theory of Partial Differential Equations [11], [14]; functional analysis [26], [27]; theory of Fourier series [29], theory of martingales [24] etc.

Notice that in the case when $\psi(\cdot) \in \Psi(A, B)$, a function $p \to p \cdot \log \psi(p)$ is convex, and $B = \infty$, then the space $G\psi$ coincides with some exponential Orlicz space.

Conversely, if $B < \infty$, then the space $G\psi(A, B)$ does not coincides with the classical rearrangement invariant spaces: Orlicz, Lorentz, Marzinkievitch etc. [30], [31].

We will use the following two important examples (more exact, the two families of examples) of the $\psi$ functions and correspondingly the GLS spaces.

1. We denote

$$\psi(A, B; \alpha, \beta; p) \overset{def}{=} (p - A)^{-\alpha} (B - p)^{-\beta}, \quad (5)$$

where $\alpha, \beta = \text{const} \geq 0, 1 \leq A < B < \infty; p \in (A, B)$ so that

$$\text{supp} \psi(A, B; \alpha, \beta; \cdot) = (A, B).$$
2. Second example:

\[ \psi(1, \infty; 0, -\beta; p) \overset{\text{def}}{=} p^\beta, \]  

but here \( \beta = \text{const} > 0, \ p \in (1, \infty) \) so that

\[ \text{supp } \psi(1, \infty; 0, -\beta; \cdot) = (1, \infty). \]

The space \( G\psi(1, \infty; 0, -\beta; \cdot) \) coincides up to norm equivalence with the Orlicz space over the set \( D \) with usually Lebesgue measure and with the correspondent \( N(\cdot) \) function

\[ N(u) = \exp \left( \frac{u^{1/\beta}}{\beta} \right), \ u \geq 1. \]

Recall that the domain \( D \) has finite measure; therefore the behavior of the function \( N(\cdot) \) is not essential.

**Remark 1.** If we define the degenerate \( \psi_r(p), r = \text{const} \geq 1 \) function as follows:

\[ \psi_r(p) = \infty, \ p \neq r; \psi_r(r) = 1 \]

and agree \( C/\infty = 0, C = \text{const} > 0 \), then the \( G\psi_r(\cdot) \) space coincides with the classical Lebesgue space \( L_r \).

**Remark 2.** Let \( \xi : D \to R \) be some (measurable) function from the set \( L(p_1, p_2), \ 1 \leq p_1 < p_2 \leq \infty \). We can introduce the so-called natural choice \( \psi_\xi(p) \) as follows:

\[ \psi_\xi(p) \overset{\text{def}}{=} |\xi|_p, \ p \in (p_1, p_2). \]

**B. Sobolev’s-Grand Lebesgue Spaces.**

**Definition.**

Let \( l = 1, 2, \ldots \) be any integer positive number. We introduce the following so-called *Sobolev-Grand Lebesgue Space* \( W^lG(\psi) \), \( \psi \in \Psi(A, B) \) as a space of all weak \( l \)-times differentiable functions (in the Sobolev’s sense) with finite norm

\[ ||f||W^lG(\psi) \overset{\text{def}}{=} ||f||G(\psi) + ||\nabla^l f||G(\psi). \]  

(7a)

Note that in the considered case, i.e. when \( f_{\partial D} = 0 \), the \( W^lG(\psi) \) norm is equivalent the simple norm \( ||\nabla^l f||G(\psi) \):

\[ ||f||W^lG(\psi) \asymp ||\nabla^l f||G(\psi). \]  

(7b)

It is evident that the spaces \( W^lG(\psi) \), \( l \geq 1 \) are not rearrangement invariant.

**C. Fundamental and truncated fundamental functions.**

Recall that if the rearrangement invariant space \( Y \) with the norm \( ||\cdot||_Y \) over the measurable space \( (Z, \Sigma) \) equipped with the (non-trivial) measure \( \mu \), then its fundamental function \( \phi(Y; \delta), \ \delta \in (0, \infty) \) is defined by follows:

\[ \phi(Y; \delta) = \sup_{A, \mu(A) \leq \delta} ||I(A)||_Y. \]
More detail information about the fundamental functions for rearrangement invariant spaces see in the book G.Bennet [2], chapter 3.

The expression for the fundamental function for the Grand Lebesgue spaces \( G(\psi) \) may be written as follows:

\[
\phi(G(\psi), \delta) = \sup_{p \in (A,B)} \frac{\delta^{1/p}}{\psi(p)}.
\] (8)

The fundamental function for the \( G(\psi(a, b; \alpha, \beta); \delta) \) spaces are calculated in the article [30]; see also [31].

We recall that when \( \beta > 0 \) as \( \delta \to 0^+ \)

\[
\phi(G(\psi(a, b; \alpha, \beta); \delta)) \sim \beta^{-\beta} b^{-2\beta} \delta^{1/b} |\log \delta|^{-\beta}
\]

and

\[
\phi(G(\psi(1, \infty; 0, -\beta); \delta)) \sim e^{-\beta} \beta^\beta |\log \delta|^{-\beta}, \quad \delta \to 0^+.
\]

**Definition.**

We define the so-called truncated fundamental function \( \phi_{p_-, p_+}(G(\psi); \delta) \) (only for GLS spaces) as follows. Let \( p_- = \text{const} \geq 1, \ p_+ = \text{const} \in (p_-, \infty) \). We put

\[
\phi_{p_-, p_+}(G(\psi); \delta) \overset{df}{=} \sup_{p \in (p_-, p_+)^{\cap \text{supp}(\psi)}} \frac{\delta^{1/p}}{\psi(p)},
\] (9)

where also \( \delta = \text{const} \in (0, \infty) \).

It is presumed that

\((p_-, p_+) \cap \text{supp}(\psi) \neq \emptyset,\)

as long as in the opposite case

\[
\phi_{p_-, p_+}(G(\psi); \delta) = \infty
\]

and the formulating further main result, theorem 1, is trivial.

It is evident that if \( \text{supp} \psi \subset (p_-, p_+) \), then

\[
\phi_{p_-, p_+}(G(\psi); \delta) = \phi(G(\psi); \delta).
\]

### 3 Main result: Sobolev’s continuity inequality for Grand Lebesgue Spaces.

We suppose in this section that \( d \geq 2 \) (the one-dimensional case \( d = 1 \) will be investigated further) and that the (given) function \( f(\cdot), f_{\delta D} = 0 \), belongs to some \( W^1 G(\psi) \) space

\[
f \in W^1 G(\psi), \ \text{supp}(\psi) = (A, B),
\]
where $B > d$; it may be considered also the case $B = \infty$.

Let us denote $A(1) = \max(A, d)$.

**Theorem 1.** The following Sobolev-type continuity inequality holds:

$$\omega(f, \delta) \leq \frac{C(d, D) \, \delta \, ||\nabla f||G(\psi)}{\phi_{A(1), B}(G(\psi), \delta^d)}, \, \delta \in (0, 1/e).$$

(10)

**Remark 3.** In the when $A \geq d$ the last inequality may be rewritten as follows:

$$\omega(f, \delta) \leq \frac{C(d, D) \, \delta \, ||\nabla f||G(\psi)}{\phi(G(\psi), \delta^d)}, \, \delta \in (0, 1/e).$$

**Proof.**

Let $f \in W^1 G\psi$, $f_{\partial D} = 0$. We can and will assume without loss of generality that

$$||f||W^1 G(\psi) = 1,$$

or equally

$$||f||G(\psi)^p = \psi(p), \, p \in (A, B).$$

We have using the inequality (2c) for the values $\delta \in (0, 1/e)$ and $p \in (A(1), B)$:

$$\frac{\omega(f, \delta)}{\delta} \leq C(d, D) \frac{p - 1}{p - d} \, \delta^{-d/p} \, \psi(p) \leq \frac{C_2(d, D)}{\delta^{d/p}/\{|(p - d)/(p - 1)|\psi(p)|\}} \leq \frac{C_3(d, D)}{\delta^{d/p}/\psi(p)}.$$

Since the last inequality is true for all the values $p$ from the interval $p \in (A(1), B)$, we obtain taking the minimum over $p \in (A(1), B)$:

$$\frac{\omega(f, \delta)}{\delta} \leq \inf_{p \in (A(1), B)} \left[ \frac{C_3(d, D)}{\delta^{d/p}/\psi(p)} \right] = \frac{C_3(d, D)}{\sup_{p \in (A(1), B)} \left[ \delta^{d/p}/\psi(p) \right]} = \frac{C_3(d, D)}{\phi_{A(1), B}(G(\psi), \delta^d)}.$$

This completes the proof of theorem 1.

**4 Examples.**

We consider in this section some examples in order to show the exactness of the assertion of theorem 1.
Theorem 2. For all the values $d = 2, 3, 4, \ldots$ there exists an admissible domain $D$, a function $\psi_0(\cdot) \in \Psi(1, \infty)$ and a non-trivial function $f_0(\cdot) \in G\psi_0$, $f_0 : D \rightarrow R$, for which

$$\lim_{\delta \to 0^+} \left[ \omega(f_0, \delta) : \frac{\delta}{\phi(G\psi_0, \delta)} \right] > 0. \quad (11)$$

Proof. Let us consider the space $\phi(G\psi(1, \infty; 0, -\beta))$ and a function

$$f_0(x) = I(|x| \leq 1) |x| |\log |x||^\beta, \beta = \text{const} > 0.$$  

Note that the function $f_0$ is radial function, i.e. it dependent only on the Euclidean norm of a vector $x$, and that the function $\psi(1, \infty; 0, -\beta; p)$ asymptotically as $p \to \infty$ coincides with the natural function for the function $f_0$.

Here $D = B$. It is evident that as $\delta \to 0^+$

$$\omega(f_0, \delta) \sim \delta |\log \delta|^\beta.$$  

Recall that

$$\phi(G\psi(1, \infty; 0, -\beta); \delta) \sim C_1(d, \beta) |\log \delta|^{-\beta}, \delta \to 0^+.$$  

Further, we find by direct calculation as $p \to \infty$ using the multidimensional polar coordinates:

$$|\nabla f_0|^p_p \sim \Omega(d) \int_0^1 z^{d-1} |\log z|^p_{\beta} \, dz =$$

$$= C_2(d) \int_0^\infty e^{-dy} y^{dp} \, dy = C_2(d) d^{-dp+1} \Gamma(\beta p + 1);$$

we used Stirling’s formula.

We conclude that

$$f_0 \in G\psi_0, \psi_0(p) = \psi(1, \infty; 0, -\beta; p).$$

We obtain substituting into expression for $\delta/\phi(G\psi_0, \delta)$ as $\delta \to 0^+$

$$\frac{\delta}{\phi(G\psi_0, \delta)} \sim C_4(d) \delta |\log \delta|^\beta.$$  

This completes the proof of theorem 1.

Remark 4. Let us consider for comparison the case of the space $G\psi(A, B; \alpha, \beta; \cdot)$. Namely, we consider the following function

$$g(x) = [\alpha/(\alpha - 1)] I(|x| \leq 1) |x|^{-1/\alpha} |\log |x||^\gamma.$$  

Here $D = B \in R^d, d \geq 2, \alpha = \text{const} > 1, \gamma = \text{const} > 0$; and denote
\[ b = \alpha d, \quad \beta = \gamma + 1/b = \gamma + 1/(\alpha d); \]

then \( d < b < \infty \).

We find by direct computation as \( p \to b - 0 : f \in L(1, b); \)

\[ |\nabla g|^p \sim \Omega(d) \int_0^1 z^{d-1-p/\alpha} |\log z|^\gamma p \, dz \sim \]

\[ \Omega(d) \int_0^\infty e^{-y(d-p/\alpha)} y^\gamma p \, dy = \Omega(d) \frac{\Gamma(\gamma p + 1)}{(d-p/\alpha)^{\gamma p+1}}, \quad p \in (1, b); \]

therefore

\[ |\nabla g|^p \sim C_4(d, \alpha, \gamma)(b-p)^{-\gamma-1/b} = C_4(d, \alpha, \gamma)(b-p)^{-\beta}, \quad p \in (1, b). \]

On the other words, the function \( g(\cdot) \) belongs to the space \( G\psi(1, b; 0, \beta) \).

It follows from the theorem 1 that

\[ \omega(g, \delta) \leq C_5(d, \alpha, \gamma) \delta^{1-1/\alpha} |\log \delta|^{\gamma+1/b}, \quad \delta \in (0, 1/e), \]

but really

\[ \omega(g, \delta) \sim C_6(d, \alpha, \gamma) \delta^{1-1/\alpha} |\log \delta|^{\gamma}. \]

Note that the main members in the two last expressions coincides; but the second members coincides only asymptotically, as \( \alpha d \to \infty \).

5 The one-dimensional case.

We consider in this section separately the one-dimensional case \( d = 1 \) and correspondingly the set \( D = [0, 1] \), as long as we can obtain in the considered case the asymptotical exact as \( \delta \to 0+ \) value of an embedding constants.

We suppose as before that \( f(0) = f(1) = 0 \) and that \( |\nabla f| \in G\psi, \psi \in \Psi(1, \infty) \).

Theorem 3.

\[ \omega(f, \delta) \leq 1 \cdot \frac{\delta ||f||_{G\psi}}{\phi(G\psi, \delta)}, \quad (13) \]

when the constant ”1” is the best possible.

1. We obtain first of all the upper bound for Sobolev-Grand Lebesgue continuity inequality in the one-dimensional case. Namely, let \( f(0) = f(1) = 0 \) and \( \nabla f \in G\psi, \psi(\cdot) \in \Psi(A, B) \), i.e.

\[ |f'/p| \leq ||f'||G(\psi) \cdot \psi(p), \quad p \in (A, B), \quad 1 \leq A < B \leq \infty. \]
As long as
\[ f(y) - f(x) = \int_x^y f'(z)dz, \quad 0 \leq x \leq y \leq 1, \]
we have denoting \( \delta = |y - x| \) and using Hölder inequality:
\[ |f(y) - f(x)| \leq |y - x|^{1-1/p} |f'|_p \leq ||f||_p (\psi(p) \cdot \delta^{1-1/p}); \]
\[ \omega(f, \delta) \leq \delta ||f||_p (\delta^{1-1/p} \psi(p)), \]
therefore
\[ \omega(f, \delta) \leq \delta ||f||_p (\inf_{p\in (A,B)} [\delta^{1-1/p} \psi(p)]) = \]
\[ \omega(f, \delta) \leq \delta ||f||_p (\psi(p) \cdot \frac{1}{\sup_{p\in (A,B)} [\delta^{1/p} / \psi(p)]}) = \frac{\delta ||f||_p (\psi(p))}{\phi(G(\psi), \delta)}. \]

2. Let us prove that the last inequality is in general case asymptotically as \( \delta \to 0+ \) exact. Namely, we consider the following example (more exactly, the family of examples) of a view:
\[ f_\Delta(x) = I(x \in [0,1]) \cdot x \cdot |\log x|^\Delta, \quad \Delta = \text{const} > 0. \] (14)
It is evident that as \( \delta \to 0+ \)
\[ \omega(f_\Delta, \delta) \sim \delta |\log \delta|^\Delta, \]
\[ |\nabla f_\Delta|_p \sim \Delta^\Delta e^{-\Delta} p^\Delta, \quad p \to \infty, \]
and we choose as before
\[ \psi_\Delta(p) = |f_\Delta|_p; \]
then
\[ ||f_\Delta||_p G \psi_\Delta = 1. \]
Further,
\[ \phi(G\psi_\Delta, \delta) \sim \sup_{p\in (1,\infty)} \frac{\delta^{1/p}}{p^\Delta \Delta e^{-\Delta}} \sim |\log \delta|^{-\Delta}. \]
Thus,
\[ \lim_{\delta \to 0+} \left[ \omega(f_\Delta, \delta) : \frac{\delta ||f_\Delta||_p G \psi_\Delta}{\phi(G\psi_\Delta, \delta)} \right] = \lim_{\delta \to 0+} \frac{\delta |\log \delta|^\Delta}{\phi(G\psi_\Delta, \delta)} = 1, \]
Q.E.D.
6 Hight derivatives.

Let $k, l$ be any positive integer numbers such that $l - k \geq 1$; and $\psi \in \Psi(A,B)$, where $B > d/(l - k)$. We denote

$$p(1) = d/(l - k), \quad p(2) = d/(l - k - 1); \quad d/0 \overset{def}{=} +\infty;$$

$$(A(3), B(3)) = [(p(1), p(2))] \cap [(A(3), B(3))]$$

and assume that $(A(3), B(3)) \neq \emptyset$.

In this section we suppose for simplicity

$$f_{\partial D} = 0, \nabla f_{\partial D} = 0, \ldots, \nabla^{l-1} f_{\partial D} = 0.$$

**Theorem 4.** The following generalized Sobolev-Grand Lebesgue Space inequality holds:

$$\omega \left( \nabla^k f, \delta \right) \leq C(d; l, k; D) \frac{\delta_{l-k} \| \nabla^l f \| G\psi}{\phi_{A(3),B(3)}(G(\psi), \delta^d)}. \quad (15)$$

**Proof.** Let $\nabla^l f \in G\psi, \psi(\cdot) \in \Psi(A, B)$, or equally

$$|\nabla^l f|_p \leq \| \nabla^l f \| G\psi \cdot \psi(p), \quad p \in (A, B).$$

We will use the following Sobolev’s continuity inequality for the classical $L_p$ spaces, see, e.g., [20], chapter 1, p. 60-64:

$$\frac{|\nabla^k f(x) - \nabla^k f(y)|}{|x - y|^\lambda} \leq C_6^{-1}(k, l; d, D; p) \| \nabla^l f \|_p.$$

Here the constant $C_6(\cdot)$ is bounded in the interval $p \in (p(1), p(2))$,

$$\lambda = l - k - d/p, \quad (l - k - 1)p < d < (l - k)p$$

or equally $p \in (p(1), p(2))$.

The last inequality may be rewritten (under our notations and conditions) as follows:

$$\omega \left( \nabla^k f, \delta \right) \leq C_6^{-1}(\cdot) \delta_{l-k-d/p} \cdot \psi(p) \cdot \| \nabla^l f \| G\psi. \quad (16)$$

The assertion of theorem 4 may be obtained as the proof of theorems 1 and 3 after the dividing over $\delta_{l-k}$ and taking minimum over $p \in (A(3), B(3))$. 

11
7 Concluding remarks. Generalizations.

1. Let us denote
\[ \eta(\delta) = \frac{\delta \| \nabla f \| G\psi}{\phi_{A(1,B)}(G(\psi), \delta^d)}, \quad \delta \in (0, 1/e), \]
and introduce the generalized Hölder space \( H(\eta) \) as a space of continuous a.e. functions with zero boundary values \( f : D \to \mathbb{R} \) with finite norm
\[ \|f\|_{H(\eta)} \overset{\text{def}}{=} \sup_{x \in D} |f(x)| + \sup_{\delta \in (0, 1/e)} \left[ \frac{\omega(f, \delta)}{\eta(\delta)} \right]. \]
Then the assertion of theorem 1 may be reformulated as an continuous embedding theorem \( W^{1,G\psi} \subset H(\eta) \):
\[ \|f\|_{H(\eta)} \leq C \| \nabla f \| G\psi. \] (17)

2. At the same examples as in the section 4 are true in the case when
\[ \psi(p) = \psi_L(p) \overset{\text{def}}{=} p^\beta L(p), \quad p \in (1, \infty), \]
or
\[ \psi^{(L)}(p) \overset{\text{def}}{=} (b - p)^{-\beta} L(1/(b - p)), \quad p \in (1, b), \quad b = \text{const} > d, \]
where \( L = L(u) \) is continuous positive slowly varying as \( u \to \infty \) function.

The corresponding examples of the functions \( \{f = f(x)\} \) for the case when \( D = B \subset \mathbb{R}^d \) are described in [30]; see also [31].

For instance, in the case when \( \psi(p) = \psi^{(L)}(p) \) the example function \( f = f(x) \) has a view
\[ f(x) = |x| |\log |x|| |^\beta L(1 + |\log |x||) I(|x| \leq 1). \]

3. Some slight generalizations.

Let now \( D, D \subset \mathbb{R}^d \) be arbitrary open domain in the space \( \mathbb{R}^d \). We denote for arbitrary subset \( K \) of the region \( D, K \subset D \) by \( c_p(K) \) the \( p-\) conductivity of the set \( K \); see the book of Maz’ja [20], chapter 4, section 4.1, p. 191-194 for the definition and some properties of this notion.

Introduce also as in [20], chapter 5, sections 5.3-5.4 the following functions:
\[ \gamma_p(x, y) = c_p(\{(D \setminus x) \setminus y\})^{-1/p}, \]
\[ \Lambda_p(\delta) = \sup_{x,y \in D, |x-y| \leq \delta} \gamma_p(x, y), \]
\[ \lambda^{(\psi)}(\delta) \overset{\text{def}}{=} \inf_{p \in (A,B]} [\Lambda_p(\delta) \psi(p)]. \]

Theorem 5.
\[
\omega(f, \delta) \leq \lambda^{(\psi)}(\delta) \cdot ||\nabla f||G\psi. \tag{18}
\]

**Proof.** It is proved in [20], chapter 5, sections 5.3-5.4 that
\[
\omega(f, \delta) \leq \Lambda_p(\delta) |\nabla f|_p.
\]

Therefore, if \(|\nabla f| \in G(\psi)\), \(\exists \psi \in \Psi(A, B)\), then
\[
\omega(f, \delta) \leq \inf_{\rho \in \text{supp} \psi} [\Lambda_p(\delta) |\nabla f|_p] \leq \inf_{\rho \in \text{supp} \psi} [\Lambda_p(\delta) \psi(p) ||\nabla f||G(\psi)] = \lambda^{(\psi)}(\delta) \cdot ||\nabla f||G\psi.
\]

**Remark 5.** The last result may be used, e.g., for the domains \(\{D\}\) with complicated boundaries.

4. Non-compactness of an embedding operator.

Let \(b = \text{const} > 1, \beta = \text{const} > 0\),
\[
\psi_{b,\beta}(p) = \psi(1, b; 0, \beta + 1/b; p) = (b - p)^{-\beta - 1/b}, \quad p \in (1, b),
\]
\[
\eta_{b,\beta}(\delta) = I(0 \leq |x| \leq 1) \delta^{1 - 1/b} |\log x|^{\beta}.
\]

Let us denote also by \(E\) the unit embedding operator from the space \(W^1G\psi_{b,\beta}\) into the space \(H(\eta_{b,\beta})\):
\[
Eu = v, \quad u \in W^1G\psi_{b,\beta}, \quad v \in H(\eta_{b,\beta}), \quad u = v.
\]

**Theorem 6.** The operator \(E\) is'nt compact operator.

**Proof.** It is sufficient to consider only the one-dimensional case \(d = 1\), i.e. \(D = [0, 1]\).

Let us consider the function
\[
g(x) = I(0 \leq x \leq 1) x^{-1/b} |\log x|^{\beta},
\]
an introduce the a family of a shift functions
\[
g_h(x) = T_hg(x) = g(x + h), \quad x + h \leq 1; \quad T_hg(x) = g(x + h - 1), \ x + h > 1.
\]

Here \(h \in (0, 1/2)\). It is evident that for both the norms \(W^1G\psi_{b,\beta}\) and \(H(\eta_{b,\beta})\)
\[
||g_h||W^1G\psi_{b,\beta} = ||g||W^1G\psi_{b,\beta},
\]
\[
||g_h||H(\eta_{b,\beta}) = ||g||H(\eta_{b,\beta}),
\]
i.e. both the expressions does not dependent on the variable \(h\).
Let us calculate at first the norm \( \|g\|_{W^1G\psi_{b,\beta}} \). We have as \( p \to b - 0 \):

\[
|g_h|^p_p = |g|^p_p \sim \int_0^1 x^{-p/b} |\log x|^{\beta p} \, dx = b^{\beta p+1} \frac{\Gamma(\beta p + 1)}{(b - p)^{\beta p+1}};
\]

\[
|g_h| \sim b^{\beta+1/b} \frac{\Gamma^{1/b}(\beta b + 1)}{(b - p)^{\beta + 1/b}}.
\]

Therefore, the family of the functions \( \{g_h\} \) belongs to some non-trivial ball in the space \( W^1G\psi_{b,\beta} \). Further,

\[
\omega(g_h, \delta) = \omega(g, \delta) = \eta_{b,\beta}(\delta), \quad \delta \in (0, 1/e).
\]

This means that

\[
\sup_{h \in (0, 1/2)} \|g_h\|_{H(\eta_{b,\beta})} = 1.
\]

It is sufficient to prove that

\[
\lim_{|h(1) - h(2)| \to 0} \|g_h(1) - g_h(2)\|_{H(\eta_{b,\beta})} > 0,
\]

or equally

\[
\lim_{h \to 0^+} \zeta(h) > 0,
\]

\[
\zeta(h) \overset{\text{def}}{=} \|g_h - g\|_{H(\eta_{b,\beta})}.
\]

We get:

\[
\zeta(h) \geq \sup_{\delta \in (0, 1/e)} \omega(g_h - g, \delta) \eta_{b,\beta}(\delta) = \sup_{\delta \in (0, 1/e)} \sup_{|\tau| \leq \delta} \sup_{x \in [0, 1]} \frac{|g(x + \tau + h) - g(x + h) - g(x + \tau) + g(x)|}{\eta_{b,\beta}(\delta)} \geq \sup_{\delta \in (0, 1/e)} \sup_{|\tau| \leq \delta} \frac{|g(\tau + h) - g(h) - g(\tau)|}{\eta_{b,\beta}(\delta)}.
\]

We conclude taking the values \( \tau = \delta = h \) that for all sufficiently small positive values \( h \)

\[
\zeta(h) \geq 2 - 2^{1-1/b} = \text{const} > 0.
\]

This completes the proof of theorem 6.

5. Note that the inequality (10) contains as a particular case the classical result (2c) for ordinary Lebesgue spaces \( L_p \), \( p \geq d \), as long as the fundamental function for these spaces has a view

\[
\phi(L_p, \delta) = \delta^{1/p}.
\]
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