Paths, Virasoro characters and fermionic expressions

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Abstract. Baxter’s corner transfer matrix method enables the one-point functions of certain 2D statistical models to be expressed as generating functions of weighted paths. In the case of the Forrester-Baxter models, the generating functions are Virasoro characters associated with the conformal minimal models $M(p,p')$. By combinatorially manipulating the paths, we obtain a quasiparticle description of the ensemble of paths. This directly leads to fermionic expressions for the Virasoro characters, and thence to interesting $q$-series identities.

1. Background

The models of Andrews, Baxter and Forrester [4], and of Forrester and Baxter [15] are 2-dimensional statistical models in which each grid site can be in one of a finite number of states. As specified in [15, eqn. (1.2.1)], interactions between the sites take place locally, only around each fundamental grid square. The interactions have been chosen so as to satisfy the star-triangle relation (also known as the Yang-Baxter relation). As a consequence, the models can be solved. In particular, the probability that a site takes a specific value may be found at temperatures close to the critical point. This probability is expressed in terms of the so-called one-point function and is most conveniently calculated using Baxter’s corner transfer matrix method [5]. In the limit of an infinite size grid, the corner transfer matrices can be diagonalised, resulting in the one-point function being expressed as the generating function of certain infinite-length weighted paths [15]. In regime III, these one-point functions turn out to be the Virasoro characters $\chi_{r,s}^{p,p'}$ that arise in the minimal models of conformal field theory. In this talk, we combinatorially manipulate the paths to obtain expressions for the characters $\chi_{r,s}^{p,p'}$ that are of fermionic form. Equating these expressions with other well-known expressions for $\chi_{r,s}^{p,p'}$ results in $q$-series identities, the simplest of which are the celebrated Rogers-Ramanujan identities.

2. Paths

A path $h$ of length $L$ is an integer sequence $(h_0, h_1, h_2, \ldots, h_L)$ for which $|h_i - h_{i-1}| = 1$ for $1 \leq i \leq L$. By linking the points $(0, h_0), (1, h_1), \ldots, (N, h_N)$ on the $i$-$j$ plane we obtain a graph that conveniently depicts the path.

The Forrester-Baxter models are parametrised by integers $p, p', a, b, c, L$ for which $1 \leq p < p'$ with $p$ and $p'$ coprime, $1 \leq a, b, c < p'$ with $c = b \pm 1$, $L \geq 0$ and $L \equiv (a - b) \mod 2$. For these, we consider the set $P_{a,b,c}^{p,p'}(L)$ of length $L$ paths $h$ for which $h_0 = a$, $h_L = b$ and $1 \leq h_i < p'$ for $0 \leq i \leq L$.\(^1\)

\(^1\) The values of $p$ and $c$ do not feature here, but they are used later to specify how the paths are weighted.
For $1 \leq k \leq p' - 1$, we refer to the region of the $i$-$j$ plane between $j = k$ and $j = k + 1$ as the $k$th band. Thus, when drawn on the $i$-$j$ plane, the paths from $P_{a,b,c}^{p,p'}(L)$ lie in the $p' - 2$ bands between $j = 1$ and $j = p' - 1$. For $1 \leq r < p$, we shade the $\lfloor rp'/p \rfloor$th band and refer to it as the $r$th odd band. Bands that are not odd bands are referred to as even bands. The band structure for the case $(p,p') = (3,8)$, along with a typical path, is shown in Fig. 1.

![Figure 1. Typical path with shaded bands.](image)

Below in Section 4, we will define a weight function on the set $P_{a,b,c}^{p,p'}(L)$ of paths. However, we first examine one of the simplest cases in detail. This special case leads to one of the most famous results in mathematics.

### 3. A special case

In this section, we will consider the case where $p = 2$, $p' = 5$, $a = b = 2$, $c = 3$ and $L \in 2\mathbb{Z}_{\geq 0}$. Our approach is a recasting of that described in [7]. The band structure appropriate to this case, and a particular path are shown in Fig. 2.

![Figure 2. Typical element of $P_{2,2,3}^{2,5}(14)$.](image)

For this section, we define the weight $wt(h)$ of a path $h \in P_{2,2,3}^{2,5}(L)$ by:

$$wt(h) = \sum_{i=1}^{L} i_{h_i \in \{1,4\}} \tag{1}$$

Then the weight of the path depicted in Fig. 2 is $4 + 6 + 9 + 12 = 31$.

For $h \in P_{2,2,3}^{2,5}(L)$ we say that there is a particle at position $i > 0$ if and only if $h_i \in \{1,4\}$. Thus $wt(h)$ is the sum of the positions of the particles in $h$.

The generating function for $P_{2,2,3}^{2,5}(L)$ is then defined by:

$$\chi_{2,2,3}^{2,5}(L) = \sum_{h \in P_{2,2,3}^{2,5}(L)} q^{wt(h)}. \tag{2}$$
Now, for \( n \geq 0 \), consider the subset \( \mathcal{P}^2_{2,2,3}(L)_n \) of \( \mathcal{P}^2_{2,2,3}(L) \) that comprises those paths having precisely \( n \) particles. It is easy to see that \( \mathcal{P}^2_{2,2,3}(L)_n \) has a unique element of minimal weight. It is the path \( h_{\text{min}} \) having particles at positions 1, 3, 5, \ldots, 2n − 1. Thus \( h_{\text{min}}^i = 2 \) when \( i \) is even, \( h_{\text{min}}^i = 1 \) for \( i \) odd and \( i < 2n \), and \( h_{\text{min}}^i = 3 \) for \( i \) odd and \( i > 2n \). For \( n = 3 \), the path \( h_{\text{min}} \) is shown in Fig. 3. Using (1), we obtain

\[
\text{wt}(h_{\text{min}}) = 1 + 3 + 5 + \cdots + (2n - 1) = n^2. \tag{3}
\]

Other elements of \( \mathcal{P}^2_{2,2,3}(L)_n \) can now be obtained from a sequence of local deformations of the path \( h_{\text{min}} \), with these deformations corresponding to a particle moving one position to its right. For instance, on moving the rightmost particle of the path given in Fig. 3 by a sequence of three steps, we obtain the paths given in Fig. 4. Note that it is possible to move this rightmost particle at most \( L - 2n \) steps and that with each step, the weight of the path increases by exactly one.

Having moved the rightmost particle, the next rightmost can be moved. For example, on moving the second particle in the final path of Fig. 4 by two steps, we obtain the two paths given in Fig. 5. Note that this second particle can move by at most the number of steps that the first particle has moved. This process continues with each of the particles moving at most as far as the previous. This process is akin to the movement of beads on the rungs of an abacus, as depicted in Fig. 6. Here the filled dots represent the particles and the unfilled dots represent
vacancies. We see that there are \( \binom{L-n}{n} \) configurations of the particles and vacancies. Moreover, since each movement of a particle by one step to the right increases the weight by one, we obtain:\(^2\)

\[
\sum_{h \in P_{2,2,3}^5(L)_n} q^{wt(h)} = q^{n^2} \left[ \frac{L-n}{n} \right]_q,
\]

where, as usual, the Gaussian polynomial \( \left[ \frac{A}{B} \right]_q \) is defined to be:

\[
\left[ \frac{A}{B} \right]_q = \begin{cases} 
\prod_{i=1}^{B} \frac{1 - q^{A-B+i}}{1 - q^i} & \text{if } 0 \leq B \leq A; \\
0 & \text{otherwise.}
\end{cases}
\]

By summing (4) over all values of \( n \geq 0 \), (2) yields:

\[
\chi_{2,2,3}^{2.5}(L) = \sum_{n=0}^{\infty} q^{n^2} \left[ \frac{L-n}{n} \right]_q.
\]

The expression for \( \chi_{2,2,3}^{2.5}(L) \) that has just been obtained is known as a fermionic expression. This is due to the interpretation in terms of particles which cannot occupy identical positions.

However, there exist other expressions for \( \chi_{2,2,3}^{2.5}(L) \). For example, it may be shown that:\(^3\)

\[
\chi_{2,2,3}^{2.5}(L) = \sum_{k \in \mathbb{Z}} (-1)^k q^{\frac{1}{2}k(5k+1)} \left[ \frac{L}{\frac{1}{2}(L-5k)} \right]_q.
\]

\(^2\) This relies on the fact that the generating function for weighted partitions in an \( M \times N \) box is given by \( \left[ \frac{M+N}{M} \right]_q \). This fact is proved in [3, Theorem 3.1].

\(^3\) First note that the path picture implies that \( \chi_{2,2,3}^{2.5}(L) = q^{L-1} \chi_{2,2,3}^{2.5}(L-2) + \chi_{2,2,3}^{2.5}(L-1) \) for \( L \in \mathbb{Z}_{>0} \) and \( \chi_{2,2,3}^{2.5}(0) = 1 \); and that \( \chi_{2,3,2}^{2.5}(L) = q^{L-1} \chi_{2,3,2}^{2.5}(L-2) + \chi_{2,2,3}^{2.5}(L-1) \) for \( L \in \mathbb{Z}_{>0} + 1 \) and \( \chi_{2,3,2}^{2.5}(1) = 1 \). These recurrence relations are then uniquely solved with both \( \chi_{2,2,3}^{2.5}(L) \) and \( \chi_{2,3,2}^{2.5}(L) \) given by the right side of (7).
Comparing (6) and (7) yields the identity (see [21] and [3, §9, Ex. 4]):

\[
\sum_{n=0}^{\infty} q^{n^2} \begin{pmatrix} L - n \\ n \end{pmatrix}_q = \sum_{k \in \mathbb{Z}} (-1)^k q^{\frac{1}{2} k(5k+1)} \left[ \frac{L}{\frac{1}{2}(L - 5k)} \right]_q,
\]

(8)

which, although we have only derived it here for even \( L \), holds for all \( L \geq 0 \). Taking the limit \( L \to \infty \) then yields:

\[
\sum_{n=0}^{\infty} q^{n^2} = \frac{1}{(q)_\infty} \sum_{k \in \mathbb{Z}} (-1)^k q^{\frac{1}{2} k(5k+1)},
\]

(9)

where \( (q)_1 = 1, (q)_n = \prod_{i=0}^{n-1} (1 - q^i) \) and \( (q)_\infty = \prod_{i=0}^{\infty} (1 - q^i) \). Note that the right side of this expression is a sum over terms of alternating sign. This is characteristic of a bosonic expression. (9) is then an example of a bosonic-fermionic identity.

By applying Jacobi’s triple product identity [16, eqn. (II.28)] to the right side of (9), we obtain the first of the celebrated Rogers-Ramanujan identities:

\[
\sum_{n=0}^{\infty} q^{n^2} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{5n-4})(1 - q^{5n-1})},
\]

(10)

As we will see below, the \( q \)-series given by either side of (9) or (10) is the Virasoro character \( \chi_{1,2}^{2,5} \). In the next section, we report on efforts to extend the above results to other Virasoro characters \( \chi_{r,s}^{p,p'} \).

4. The general case
For the case of general \( p, p', a, b, c \), the weight function for the paths is best specified in terms of an \((x, y)\)-coordinate system which is inclined at 45° to the original \((i, j)\)-coordinate system and whose origin is at the path’s initial point at \((i = 0, j = a)\). Specifically, \( x = \frac{1}{2}(i - j + a) \) and \( y = \frac{1}{2}(i + j - a) \). Note that at each step in the path, either \( x \) or \( y \) is incremented and the other is constant. In this system, the path depicted in Fig. 1 has its first few vertices at \((0, 1), (0, 2), (1, 2), (1, 3), (1, 4), (2, 4), (2, 5), (2, 6), \ldots \).

Now, for \( 1 \leq i \leq L \), we define the weight \( v_i = v(h_{i-1}, h_i, h_{i+1}) \) of the \( i \)th vertex according to its shape, the shading between \( h_1 \) and \( h_{i+1} \), and its \((x, y)\)-coordinate, as specified in Table 1 (the nature of the vertex at \( i = L \) is determined upon setting \( h_{L+1} = c \)). In this table, the bands highlighted with the \( \sim \) symbol can be either even or odd. We shall refer to those four types of vertices for which, in general, the weight is non-zero, as scoring vertices. The other four types

| Vertex | \( v_i \) |
|--------|-----------|
| \( x \) | 0 |
| \( y \) | 0 |
| 0 | \( v_i \) |
of vertices will be termed non-scoring. In Fig. 7, the path of Fig. 1 has been reproduced with each of its scoring vertices circled.

We now define:

\[ wt(h) = \sum_{i=1}^{L} v_i. \] (11)

To illustrate this procedure, consider again the path \( h \in P_{3,5,6}^{3,8}(26) \) depicted in Figs. 1 and 7. Using Table 1, the scoring vertices at \( i = 2, 3, 4, 8, 9, 12, 14, 15, 16, 17, 18, 19 \) and 26 lead to:

\[ wt(h) = 0 + 2 + 1 + 2 + 6 + 6 + 6 + 8 + 7 + 9 + 8 + 10 + 12 = 77 \]

The generating function \( \chi_{a,b,c}^{p,p'}(L) \) for the set of paths \( P_{a,b,c}^{p,p'}(L) \) is defined to be:

\[ \chi_{a,b,c}^{p,p'}(L; q) = \sum_{h \in P_{a,b,c}^{p,p'}(L)} q^{wt(h)}. \] (12)

Often, we drop the base \( q \) from the notation so that \( \chi_{a,b,c}^{p,p'}(L) = \chi_{a,b,c}^{p,p'}(L; q) \). The same will be done for other functions without comment.

We also define the set \( P_{a,b,c}^{p,p'}(L, m) \) to comprise those elements of \( P_{a,b,c}^{p,p'}(L) \) that have precisely \( m \) non-scoring vertices. The corresponding generating function is \( \chi_{a,b,c}^{p,p'}(L, m; q) = \sum_{h \in P_{a,b,c}^{p,p'}(L, m)} q^{wt(h)} \).

5. Vertex words and path manipulations
The vertex word \( w(h) \) of a path \( h \) is the word in the letters \( S \) and \( N \) corresponding to the sequence of scoring and non-scoring vertices in \( h \). The lengths of \( w(h) \) and \( h \) are thus equal. For example, for the path \( h \in P_{3,5,6}^{3,8}(26) \) depicted in Figs. 1 and 7, we have

\[ w(h) = NSSSNNSNSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSS
change corresponding to the local change \(SSN \rightarrow NSS\) in \(w(h)\), or vice-versa. It is easily checked that the corresponding altered path is also an element of \(P_{a,b,c}^{p,p'}(L)\). In addition, for the change \(SSN \rightarrow NSS\), the weight of the path increases by precisely 1.\(^4\)

As in Section 3, we then again have the analogy with beads moving on an abacus. Among all the paths obtained from \(h\) by consecutive particle movements, there is a unique path \(h_{\text{min}} \in P_{a,b,c}^{p,p'}(L)\) of minimal weight obtained by moving all particles as far to the left as possible. The generating function for this set of paths is then \(q^{wt(h_{\text{min}})} \left[ \frac{n+m}{n} \right]_q\), where \(n\) is the total number of particles, and \(m\) is the number of non-scoring vertices in \(h\) (or \(h_{\text{min}}\)).

In the case of the path \(h \in P_{a,b,c}^{3,8}(26)\) depicted in Fig. 7, \(n = 5, m = 13\) and

\[
w(h_{\text{min}}) = SSSSSSSSSNSNNNNNSNNNNNS.
\] (14)

Correspondingly, \(h_{\text{min}} \in P_{a,b,c}^{3,8}(26)\) here is given in Fig. 8.

\[\text{Figure 8. Typical } h_{\text{min}}.\]

Note that \(h_{\text{min}}\) begins with either \(2n\) (or \(2n+1\)) consecutive scoring vertices. We now define

the path \(h_{\text{cut}} \in P_{a,b,c}^{p,p'}(L - 2n, m)\) by removing the first \(2n\) vertices of \(h_{\text{min}}\). For the case of \(h_{\text{min}}\) depicted in Fig. 8, the resulting \(h_{\text{cut}}\) is shown in Fig. 9. Lemma A.1 shows that:

\[wt(h_{\text{cut}}) = wt(h_{\text{min}}) + n(L - n - m).\] (15)

\[\text{Figure 9. Typical } h_{\text{cut}}.\]

Our construction above has ensured that \(w(h_{\text{cut}})\) contains no consecutive pair \(SS\). Our next aim is to construct, from such a path \(h_{\text{cut}} \in P_{a,b,c}^{p,p'}(L', m)\), a path \(h_{sqz} \in P_{a',b',c'}^{p,p'}(m, 2m - L')\) for

\(^4\) Although not required here, an explicit description of the moves in terms of changes to the path may be found in [11, §2.3] or [23, §3.3].
certain $a', b'$ and $c'$ defined below, by shirking the features of the path. This is done by specifying that $w(h_{sqz})$ is obtained from $w(h_{cut})$ by removing the letter $N$ immediately preceding each letter $S$ (restrictions below will ensure that $w(h_{cut})$ does not begin with $S$). This is possible because, in passing from $h_{cut}$ to $h_{sqz}$, the reduction in the distance between scoring vertices is compatible with the reduced distance between odd bands in the underlying grid: nearest neighbour odd bands become separated by one fewer even bands. The map from the path $h_{cut} \in \mathcal{F}_{3,5,6}(16, 13)$ of Fig. 9 to the path $h_{sqz} \in \mathcal{F}_{2,3,4}(13, 10)$ given in Fig. 10 provides an example of this construction.

![Figure 10. Typical $h_{sqz}$.](image)

However, in some case, complications arise at the leftmost and rightmost extremities of the path. Here, we avoid these by imposing restrictions on the values of $a, b, c$.

To describe the restrictions, and to specify $a', b'$ and $c'$, define:

$$
\begin{align*}
    r_{b,c}^{p,p'} &= \left\lfloor \frac{pc}{p'} \right\rfloor + \frac{b - c + 1}{2}, \\
p_j^{p,p'} &= \left\lfloor \frac{(j + 1)p}{p'} \right\rfloor, \\
\beta_{a,b,c}^{p,p'} &= r_{b,c}^{p,p'} - p_j^{p,p'}.
\end{align*}
$$

First, we set $a' = a - \rho_a^{p,p'}$ and $b' = b - r_{b,c}^{p,p'}$. For $2 \leq j \leq p' - 2$, we designate $j$ as *intereven*, *interfacial* or *interodd* according to whether $[(j + 1)p/p'] - [(j - 1)p/p']$ is 0, 1 or 2 respectively. These cases correspond to $j$ being between two even bands, between an even and an odd band, and between two odd bands respectively. If $j$ is interfacial, then $p_j^{p,p'}$ gives the particular odd band that $j$ borders. The upper and lower bands of the underlying grid are even or odd depending on whether $p' > 2p$ or $p' < 2p$. Correspondingly, in these two cases, we designate $j \in \{1, p' - 1\}$ as *extraeven* or *extraodd*. Note that if $p' > 2p$ and $1 \leq j < p'$ then $j$ is either intereven, interfacial or extraeven.

For $a$, we restrict to the cases where $a$ and $a'$ are both extraeven, where $a$ and $a'$ are both interfacial, or where $a$ is interfacial and $a'$ is extraodd. It may be seen that these restrictions ensure that $w(h_{cut})$ does not start with the letter $S$. For $b$ and $c$, we restrict to the cases where $b$ and $b'$ are of the same nature (intereven, interfacial or extraeven), or where $b$ is interfacial and either $b'$ is interodd or extraodd. In the case where $b$ and $b'$ are of the same nature, we set $c' = c - r_{b,c}^{p,p'}$. In the other cases, we set $c' = b' + 1$ if $pb/p'< r$, and $c' = b - 1$ otherwise (this ensures that if $b$ and $c$ straddle the $r$th odd band, then so do $b'$ and $c'$).

In these restricted cases, it is easily checked that each symbol $S_x$ (resp. $S_y$) in $w^*(h_{cut})$ is immediately preceded by a $N_x$ (resp. $N_y$). Application of Lemma A.2 then shows that:

$$wt(h_{cut}) = wt(h_{sqz}) + \frac{1}{4}((L' - m)^2 - \beta^2),$$

where $\beta = \beta_{a,b,c}^{p,p'}$. With $L' = L - 2n$, Combining (15) and (17) yields $wt(h_{min}) = wt(h_{sqz}) + \frac{1}{4}((L - m)^2 - \beta^2)$.

The bijective nature of the three maps constructed above (taking into account the value of $n$ and the sequence of moves required to obtain $h$ from $h_{min}$) now leads to:

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5 Some of these restrictions can be lifted after extending the constructions described here. These extensions and the ensuing complications are tackled to increasing degrees in [13, 14, 23].
Theorem 5.1 Let \( p' > 2p \). If \( a, b, c, \alpha', \beta' \) and \( c' \) are restricted as above then:

\[
\chi_{a,b,c}^{p',p}(L, m) = \sum_{n=0}^{\infty} q^{\frac{1}{4}((L-m)^2 - \beta^2)} \chi_{a',b',c'}^{p'-p}(m, 2m + 2n - L) \left[ \frac{m+n}{m} \right]_q.
\] (18)

We refer to this transformation between generating functions as a \( B \)-transform.

6. The \( D \)-transform

The \( D \)-transform maps a path \( h \in \mathcal{P}_{a,b,c}^{p,p'}(L, m) \) to a path \( h^{\text{dual}} \in \mathcal{P}_{a,b,c}^{p'-p,p'}(L, L - m) \). This map is defined by specifying that \( w(h^{\text{dual}}) \) is obtained from \( w(h) \) by exchanging each \( S \) for an \( N \) and vice-versa. Note that, since each band has changed parity, the shapes of \( h \) and \( h^{\text{dual}} \) are identical. For example, the \( D \)-transform maps the path \( h_{\text{sqz}} \in \mathcal{P}_{3,5}^{2,3,4}(13) \) given in Fig. 10 to the path \( h^{\text{dual}} \in \mathcal{P}_{2,5}^{1,3,1,2}(13) \) given in Fig. 11.

![Typical \( h^{\text{dual}} \)](image)

Lemma A.3 shows that \( wt(h^{\text{dual}}) = \frac{1}{4}(L^2 - (a - b)^2) - wt(h) \). This leads to:

Theorem 6.1

\[
\chi_{a,b,c}^{p',p}(L, m; q) = q^{\frac{1}{4}(L^2 - (a-b)^2)} \chi_{a,b,c}^{p'-p,p'}(L, L - m; q^{-1}).
\] (19)

Repeated use of the \( B \)- and \( D \)-transforms enables us to write a particular \( \chi_{a,b,c}^{p,p'}(L, m) \) in terms of simpler expressions. In fact, if we define \( s_0 \) and \( s_0' \) to be the smallest non-negative integers such that \( |p'_{s_0} - p_{s_0'}| = 1 \), then we can express \( \chi_{s_0,s_0,s_0+1}^{p,p'}(L, m) \) in terms of the trivial \( \chi_{1,1,2}^{1,3}(L, m) \).\(^6\) This latter generating function is easily seen to be given by:

\[
\chi_{1,1,2}^{1,3}(L, m) = q^{\frac{1}{4}L^2} \delta_{m,0}.
\] (20)

The sequence of \( B \)- and \( D \)-transforms that is used depends on the continued fraction of \( p'/p \).

7. Continued fractions

For \( p \) and \( p' \) coprime integers with \( 1 \leq p < p' \), we say that \([c_0, c_1, c_2, \ldots, c_n]\) is the continued fraction for \( p'/p \) if:

\[
\frac{p'}{p} = c_0 + \frac{1}{c_1 + \frac{1}{c_2 + \cdots + \frac{1}{c_{n-1} + \frac{1}{c_n}}}}.
\]

\(^6\) A minor extension of our definitions enables us to use the even simpler \( \chi_{1,1,2}^{1,3}(L, m) = \delta_{L,0} \delta_{m,0} \).
with $c_i \geq 1$ for $0 \leq i < n$, and $c_n \geq 2$.

We then define $t = c_0 + c_1 + \cdots + c_n - 2$ and also define:

$$t_k = -1 + \sum_{i=0}^{k-1} c_i,$$

for $0 \leq k \leq n + 1$. Then $t_{n+1} = t + 1$ and $t_n \leq t - 1$.

We now use these values to define a $t \times t$ matrix $C$, a $(t - t_1 - 1) \times (t - t_1 - 1)$ matrix $\overline{C}$, and a $t_1 \times t_1$ matrix $B$. First define $C_{ji}$ for $0 \leq i, j \leq t$ by, when the indices are in this range,

$$C_{ji} = \begin{cases} -1, & \text{if } j = k, k = 1, 2, \ldots, n; \\ 0, & \text{if } j < k \text{ otherwise}, \\ 1, & \text{if } j = k, k = 1, 2, \ldots, n. \\ 
\end{cases}$$

and set $C_{ji} = 0$ for $|i - j| > 1$. Also define $B_{ji} = \min\{i, j\}$ for $1 \leq i, j \leq t$. Then define the matrices:

$$C = \{C_{ji}\}_{0 \leq j < t, \ 0 \leq i < t}, \quad \overline{C} = \{C_{ji}\}_{t_1 + 1 \leq j < t, \ t_1 + 1 \leq i < t}, \quad B = \{B_{ji}\}_{1 \leq j \leq t_1}. $$

Note that $C$ and $\overline{C}$ are both tri-diagonal. The matrices $C$, $\overline{C}$ and $B^{-1}$ may be viewed as minor generalisations of Cartan matrices of type $A$.

We now obtain $\chi_{p',p,q}^{s_0,s_0+1}(L, m)$ by applying the following sequence of $B$- and $D$-transforms to the expression for $\chi_{1,1,2}^{1,3}(L, m)$ given in (20):

$$B^{a_0-1}DB_1DB_2 \cdots DB^{s-1}DB^{s-2}. $$

Having thus obtained $\chi_{p',p,q}^{s_0,s_0+1}(L, m)$, a final sum of all even $m$ yields the following result (the proof is detailed in [11]):

**Theorem 7.1** Let $1 \leq p < p'$ with $p$ and $p'$ coprime. If $L$ is even then:

$$\chi_{s_0,s_0,s_0+1}^{p,p'}(L) = \sum_{m \in (\mathbb{Z}_{\geq 0})^{t-1}} q^{\frac{1}{2} m^T C m - \frac{1}{4} L^2 \sum_{j=1}^{t-1} \left( m_j - \frac{1}{2} (C \hat{m})_j \right) \frac{1}{m_j}},$$

where the sum is over all vectors $m = (m_1, m_2, \ldots, m_{t-1})^T$, each of whose integer components is even, and where we set $\hat{m} = (L, m_1, m_2, \ldots, m_{t-1})^T$. Here, the components of $C \hat{m}$ are labelled by $0, \ldots, t - 1$.

8. Minimal model Virasoro characters

For $1 < p < p'$ with $p$ and $p'$ coprime, and $1 \leq r < p$ and $1 \leq s < p'$, the Virasoro character $\chi_{p,s}^{r,p'}$ is given by the following bosonic expression [9, 18, 10]:

$$\chi_{p,s}^{r,p'} = \frac{1}{(q)_{\infty}} \sum_{\lambda = -\infty}^{\infty} q^{\lambda^2 r p' + \lambda (p'r - p s) - q^{(\lambda p + r)(\lambda p' + s)}}. $$

The significance of our generating functions $\chi_{p,s}^{r,p'}(L)$ is that, if we set $r = \chi_{p,s}^{r,p'}$ (using (16)) then for $1 \leq r < p$,

$$\lim_{L \to \infty} \chi_{p,s}^{r,p'}(L) = \chi_{p,s}^{r,p'}(L),$$

assuming that $|q| < 1$.\footnote{As detailed in [11, 23], an explicit (bosonic) generating function for $\chi_{p,s}^{r,p'}(L)$ can be established, from which (26) readily follows.}
In view of (26), we can now derive fermionic expressions for the characters $\chi_{p,s}^{p',s'}_0$ by taking the $L \to \infty$ limit of the expressions for $\chi_{s_0,s_0,s_1+1}(L)$ given in Theorem 7.1. In the two cases $p = 2$ and $p > 2$, this yields:

**Theorem 8.1** If $k \geq 2$ then:

$$\chi_{1,k}^{2,2k+1} = \sum_{n \in \mathbb{Z}_{\geq 0}} q^{nT} Bn \frac{q^{nT} Bn}{(q)_{n_1} (q)_{n_2} \cdots (q)_{n_{k-1}}},$$

(27)

where $n = (n_1, n_2, \ldots, n_{k-1})^T$.

**Theorem 8.2** If $2 < p < p'$ with $p$ and $p'$ coprime then:

$$\chi_{r,s}^{p,p'}_{r,s} = \sum_{n \in \mathbb{Z}_{\geq 0}} q^{\tilde{n}^T \tilde{B}n + \frac{1}{2} \tilde{m}^T \tilde{C} \tilde{m}} \frac{1}{(q)_{m_{t+1}}} \prod_{j=1}^{t_1} \frac{1}{(q)_{m_j}} \prod_{j=t_1+2}^{t-1} \left[ m_j - \frac{1}{2}(\tilde{C} \tilde{m})_j \right]_q,$$

(28)

where $n = (n_1, n_2, \ldots, n_{t_1})^T$ and $\tilde{m} = (m_{t_1+1}, m_{t_1+2}, \ldots, m_{t-1})^T$, using which we define $\tilde{n} = (n_1, n_2, \ldots, n_{t_1} + \frac{1}{2} m_{t+1})^T$. Here, the components of $\tilde{C} \tilde{m}$ are labelled by $t_1 + 1, \ldots, t - 1$.

By equating the expressions given in Theorems 8.1 and 8.2 with the corresponding instances of (25) we obtain $q$-series identities of bosonic-fermionic type.

In addition, for certain cases, a product form is also available for $\chi_{r,s}^{p,p'}$ (see [23, §1.1] for more information).

For example, application of Jacobi’s triple product identity [16, eqn. (II.28)] to the expression for $\chi_{1,k}^{2,2k+1}$ given by (25) leads, via Theorem 8.1, to the following identities of Andrews and Gordon [1, 17]:

$$\sum_{n_1, n_2, \ldots, n_{k-1} \in \mathbb{Z}_{\geq 0}} q^{N_1^2 + N_2^2 + \cdots + N_{k-1}^2} \frac{1}{(q)_{n_1} \cdots (q)_{n_{k-1}}} = \prod_{n=1}^{\infty} \frac{1}{1 - q^n},$$

(29)

where $k \geq 2$ and $N_i = n_i + n_{i+1} + \cdots + n_{k-1}$ for $1 \leq i < k$. The Rogers-Ramanujan identity (10) is the case $k = 2$ here.

In addition, application of Watson’s quintuple product identity [16, ex. 5.6] to the expressions for $\chi_{1,k}^{3,3k+1}$ and $\chi_{1,k+1}^{3,3k+2}$ given by (25) leads, via Theorem 8.2, to the following two sequences of identities:

$$\sum_{n_1, n_2, \ldots, n_k \in \mathbb{Z}_{\geq 0}} q^{N_1^2 + N_2^2 + \cdots + N_{k-1}^2 + 2N_k^2} \frac{1}{(q)_{n_1} (q)_{n_2} \cdots (q)_{n_{k-1}} (q)_{n_k}} = \prod_{n=1}^{\infty} \frac{1}{1 - q^n},$$

(30)

(c.f. [12, eqn. (A.4)] and [8, eqns. (2.1) & (2.5)]), and

$$\sum_{n_1, n_2, \ldots, n_k \in \mathbb{Z}_{\geq 0}} q^{N_1^2 + N_2^2 + \cdots + N_{k-1}^2 + N_k^2} \frac{1}{(q)_{n_1} (q)_{n_2} \cdots (q)_{n_{k-1}} (q)_{n_k}} = \prod_{n=1}^{\infty} \frac{1}{1 - q^n},$$

(31)

(see [2]), where in both cases $k \geq 1$, and $N_i = n_i + n_{i+1} + \cdots + n_k$ for $1 \leq i \leq k$. 
9. Further results
In this talk, we have derived fermionic expressions for the characters $\chi^{p,p'}_{r_0,s_0}$ where $r_0$ and $s_0$ are dependent on $p$ and $p'$. Over the past 15 years, fermionic expressions for a gradually widening range of $\chi^{p,p}_{r,s}$ have been obtained (see [23, §1.1] for references). Recently, in [23], fermionic expressions for all Virasoro characters $\chi^{p,p'}_{r,s}$ have been obtained by extending the techniques presented in this talk. In general, these fermionic expressions are sums over terms similar to those appearing on the right side of (28). Intriguingly, as exhibited in [6, 14], only one such term occurs if and only if $s$ (or $p' - s$) and $r$ are respectively a Takahashi length and a truncated Takahashi length [22] associated with the continued fraction of $p'/p$.

Acknowledgments
I thank Omar Foda, Keith Lee and Yaroslava Pugai for collaboration on [11], upon which part of this talk is loosely based. It is also partly based on [23].

Appendix A
Knowledge of $w(h)$ alone is not sufficient to determine the weight $wt(h)$ of a path $h$. Instead, we can use the augmented vertex word $w^{*}(h)$ obtained from $w(h)$ by appending $x$ or $y$ to each letter in $w(h)$ depending on the direction of the path immediately preceding the corresponding vertex in $h$. For example, for the path $h \in P_{3,5,6}^{3,8}(26)$ depicted in Fig. 7, we have:

$$w^{*}(h) = N_yS_yS_xS_yN_yN_xS_yS_xN_xS_xS_yS_yS_yS_yS_yN_yN_xN_xN_yN_yN_yS_y.$$  (A.1)

The definition of the weight given in (11) then implies that:

$$wt(h) = \#\{N_x \cdots S_y\} + \#\{S_x \cdots S_y\} + \#\{N_y \cdots S_x\} + \#\{S_y \cdots S_x\},$$  (A.2)

where the symbol $\#\{A \cdots B\}$ denotes the number of pairs $A, B$ of subscripted letters in $w^{*}(h)$ with $A$ to the left of $B$.

Lemma A.1 If the leftmost length $2n$ subword of $w^{*}(h^{\text{min}})$ is $S_xS_yS_xS_y \cdots S_xS_y$ or $S_yS_xS_yS_x \cdots S_yS_x$, and $w^{*}(h^{\text{cut}})$ is obtained from $w^{*}(h)$ by removing this subword, then:

$$wt(h^{\text{min}}) = wt(h^{\text{cut}}) + n(L - n - m),$$  (A.3)

where $L$ is the length of $h$ and $m$ the number of its non-scoring vertices.

Proof: Since the two cases are very similar, we only consider the case where $w^{*}(h^{\text{min}})$ begins with $S_x$. Consider the change in the right side of (A.2) in passing from $w^{*}(h^{\text{min}})$ to $w^{*}(h^{\text{cut}})$. From $S_x$ and $S_y$ amongst the first $2n$ subscripted letters, $\#\{S_x \cdots S_y\}$ loses $1 + 2 + \cdots + n = (n+1)/2$ contributions, and $\#\{S_y \cdots S_x\}$ loses $0 + 1 + \cdots + (n - 1) = n(n - 1)/2$ contributions. Apart from the first $2n$ subscripted letters, $w^{*}(h^{\text{min}})$ contains $L - 2n - m$ symbols $S_y$ or $S_x$. The contribution from each of these symbols $S_x$ to $\#\{S_y \cdots S_x\}$ decreases by $n$; and likewise the contribution from each of these symbols $S_y$ to $\#\{S_x \cdots S_y\}$ decreases by $n$. Eqn. (A.3) follows.

In the following two proofs, we let $\#S_x$ be the number of symbols $S_x$ in $w^{*}(h)$ etc.,

Lemma A.2 Let $h \in P_{a,b,c}^{p,p'}(L, m)$ be such that there are no consecutive pairs $SS$ in $w(h)$. If $h^{\text{sqp}} \in P_{a,b,c}^{p,p'}(m, 2m - L)$ is such that $w^{*}(h^{\text{sqp}})$ is formed from $w^{*}(h)$ by removing a symbol $N_x$ immediately preceding each symbol $S_x$, and removing a symbol $N_y$ immediately preceding each symbol $S_y$, then:

$$wt(h^{\text{sqp}}) = wt(h) - \frac{1}{4}((L - m)^2 - \beta^2),$$  (A.4)

where $\beta = \beta_{a,b,c}^{p,p'}$. 
Proof: In passing from $h$ to $h^{\text{dual}}$, (A.2) shows that the weight is reduced by the total number of pairs $S_x$ and $S_y$ in $w^*(h)$. Therefore:

$$\wt(h) - \wt(h^{\text{dual}}) = \#S_x \#S_y = \frac{1}{4}(\#S_x + \#S_y)^2 - (\#S_x - \#S_y)^2).$$

Since $\#S_x + \#S_y = L - m$, the lemma is proved if it can be shown that $\#S_y - \#S_x = \beta$. We do this using induction on path length.

Consider a path $h^\dagger \in \mathcal{P}_{a,b',c'}(L)$ and let $c^\dagger = c^\dagger \pm 1$. Define $h^\dagger_i \in \mathcal{P}_{a,c_i,c_i}(L+1)$ by $h_{L}^\dagger = h_{L}^\dagger$ for $0 \leq i \leq L$ and $h_{L+1}^\dagger = c^\dagger$. Let $\#S^u_i$ and $\#S^v_i$ be the number of symbols $S_x$ and $S_y$ in $w(h^\dagger)$, and define $\#S^u_i$ and $\#S^v_i$ correspondingly. With $\Delta \sigma = (\#S^u_i - \#S^v_i) - (\#S^v_i - \#S^u_i)$ and $\Delta r = r_{b',c}^p - r_{b',c}^{p'} = [c^{u} p/p'] - [c^{v} p/p'] + (2c^1 - b^1 - c^1)/2$, we claim that $\Delta \sigma = \Delta r$. To show this, consider the nature of the $(L + 1)$th vertex of $h^\dagger$. The eight possible cases are those considered in Table 1. In the top left case, we have $\Delta \sigma = 1$. Since $c^1 = b^1 + 1$ and $c^3 = b^3 + 2$, we also have $\Delta r = 1$ establishing the claim in this case. For the bottom left case, we obtain $\Delta \sigma = 0 = \Delta r$ as required. In the third case on the top row, $\Delta \sigma = 0$, and since $c^1 = b^1 + 1$ and $c^3 = b^3 + 2$, also $\Delta r = 0$. For the third case on the bottom row, we obtain $\Delta \sigma = 1 = \Delta r$ as required. The other four cases follow similarly. Since $\beta_{a,b',c^1} - \beta_{a,b',c^3} = \Delta r$, the induction step is complete, and the lemma is proved.

Lemma A.3 For $h \in \mathcal{P}_{a,b,c}^p(L)$, let $h^{\text{dual}}$ be such that $w^*(h^{\text{dual}})$ is obtained from $w^*(h)$ by replacing each $S_x$, $S_y$, $N_x$ and $N_y$ by $N_x$, $N_y$, $S_x$ and $S_y$ respectively. Then:

$$\wt(h^{\text{dual}}) = \frac{1}{4}(L^2 - (a - b)^2) - \wt(h).$$

Proof: With $\wt(h)$ given by (A.2), $\wt(h^{\text{dual}})$ is given by the same expression with the symbols $S$ and $N$ interchanged. Thus each pair of symbols in $w^*(h)$ that have different subscripts contributes to exactly one of $\wt(h)$ or $\wt(h^{\text{dual}})$. Thus $\wt(h) + \wt(h^{\text{dual}})$ is the number of such pairs, which is given by:

$$(\#S_x + \#N_x)(\#S_y + \#N_y) = \frac{1}{4}((\#S_x + \#N_x + \#S_y + \#N_y)^2 - (\#S_x + \#N_x - \#S_y - \#N_y)^2).$$

Since $L = \#S_x + \#N_x + \#S_y + \#N_y$ and $a - b = \#S_x + \#N_x - \#S_y - \#N_y$, this immediately yields (A.5). 

References

[1] Andrews G E 1974 An analytic generalization of the Rogers-Ramanujan identities for odd moduli Proc. Nat. Acad. Sci. USA 71 4082-5
[2] Andrews G E 1984 Multiple series Rogers-Ramanujan type identities Pac. J. Math. 114 267–283
[3] Andrews G E 1976 The Theory of Partitions (Reading MA: Addison-Wesley)
[4] Andrews G E, Baxter R J and Forrester P J 1984 Eight-vertex SOS model and generalized Rogers-Ramanujan-type identities J. Stat. Phys. 35 193–266.
[5] Baxter R J 1982 Exactly Solved Models in Statistical Mechanics (London: Academic Press)
[6] Berkovich A, McCoy B M and Schilling A 1998 Rogers-Schur-Ramanujan type identities for the $M(p,p')$ minimal models of conformal field theory Commun. Math. Phys. 191 325–395
[7] Bressoud D M 1989 Lattice paths and the Rogers-Ramanujan identities Proceedings of the international Ramanujan centenary conference 1987 Madras ed K Alladi Lecture Notes in Mathematics 1395, (New York: Springer)
[8] Christe F 1991 Factorized characters and form factors of descendant operators in perturbed conformal systems Int. J. Mod. Phys. A 6 5271–86
[9] Feigin B L and Fuchs D B 1983 Verma modules over the Virasoro algebra *Funct. Anal. Appl.* **17** 241–2
[10] Felder G 1989 BRST approach to minimal models *Nucl. Phys.* **B317** 215–236
[11] Foda O, Lee K S M, Pugai Y and Welsh T A 2000 Path generating transforms *Contemp. Math.* **254** 157–186
[12] Foda O and Quano Y-H 1997 Virasoro character identities from the Andrews-Bailey construction *Int. J. Mod. Phys. A* **12** 1651–75
[13] Foda O and Welsh T A 1999 Melzer’s identities revisited *Contemp. Math.* **248** 207–234
[14] Foda O and Welsh T A 2000 On the combinatorics of Forrester-Baxter models *Physical Combinatorics* 1999 Kyoto ed M. Kashiwara and T. Miwa *Prog. Math.* **191** (Boston: Birkhäuser) pp 49–103
[15] Forrester P J and Baxter R J 1985 Further exact solutions of the eight-vertex SOS model and generalizations of the Rogers-Ramanujan identities *J. Stat. Phys.* **38** 435–472
[16] Gasper G and Rahman M 1990 *Basic Hypergeometric Series* (Cambridge: Cambridge University Press)
[17] Gordon B 1961 A combinatorial generalization of the Rogers-Ramanujan identities *Amer. J. Math.* **83** 393–9
[18] Rocha-Caridi A 1985 Vacuum vector representations of the Virasoro algebra *Vertex Operators in Mathematics and Physics* ed J Lepowsky *et al* (Berlin: Springer) pp 451–473
[19] Rogers L J 1894 Second memoir on the expansion of certain infinite products *Proc. London Math. Soc.* **25** 318–343
[20] Rogers L J and Ramanujan S 1919 Proof of certain identities in combinatorial analysis *Proc. Cambridge Philos. Soc.* **19** 111–6
[21] Schur I 1917 Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche *S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl.* 302–321 (Reprinted 1973 *Gesammelte Abhandlungen* vol 2 (Springer: Berlin) pp 117–136)
[22] Takahashi M and Suzuki M 1972 One-dimensional anisotropic Heisenberg model at finite temperatures *Prog. Theor. Phys.* **48** 2187–2209
[23] Welsh T A 2005 Fermionic expressions for minimal model Virasoro characters *Mem. Amer. Math. Soc.* **175** (no. 827)