INVARITANTS ASSOCIATED WITH LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We apply a novel method for the equivalence group and its infinitesimal generators to the investigation of invariants of linear ordinary differential equations. First, a comparative study of this method is illustrated by an example. Next, the method is used to obtain the invariants of low order linear ordinary differential equations, and the structure invariance group for an arbitrary order of these equations. Other properties of these equations are also discussed, including the exact number of their invariants.

1. Introduction

Due to promising results obtained on invariants of algebraic functions, and to the similarities in properties between differential equations and algebraic equations, the study of invariants of differential equations began in the middle of the nineteen century. One of the most important studies of these functions was carried out by Forsyth in his very valuable memoir [2], in which he considers the linear ordinary differential equation of general order

\[ y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_0y = 0, \]

(1.1)

where the coefficients \(a_{n-1}, a_{n-2}, \ldots, a_0\) are arbitrary functions of the independent variable \(x\). Earlier writers on the subject, cited here in an almost chronological order, include Laplace [10], Laguerre [9], Brioschi [1], and more importantly Halphen, who made ground-breaking contributions in his celebrated memoir [3].

Methods used up to the middle of the nineteen century for studying invariant functions were very intuitive, and based on a direct analysis, in which most results were obtained by a comparison of coefficients of the equation before and after it was subjected to allowed transformations. However, the application of these techniques has been restricted to linear equations.

Based on ideas outlined by Lie [11], the development of infinitesimal methods for the investigation of invariants of differential equations started probably in [18], and a formal method has been suggested [4], which is being commonly used [15] [5] [6] [8] [7]. However, the latter method requires the knowledge of the equivalence transformations of the equation, and thus a new method that provides these transformations and at the same time the infinitesimal generators for the invariant functions has recently been suggested [14].

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In this paper, we use the method of [14] to derive explicit expressions for the invariants of various canonical forms of the general linear ordinary differential equations of order up to 5. We also use the same method to derive the structure invariance group for a certain canonical form of the equation, and for a general order. Relationships between the invariants found as well as some of their other properties, including their exact number, are also investigated. We start our discussion in Section 2 by an illustrative example comparing the former infinitesimal method of [4, 5] with that of [14].

2. Two methods of determination

We begin this section with some generalities about equivalence transformations. Suppose that $F$ represents a family of differential equations of the form

$$\Delta(x, y^{(n)}; C) = 0,$$

(2.1)

where $x = (x^1, \ldots, x^n)$ is the set of independent variables, $y = (y_1, \ldots, y_q)$ is the set of dependent variables and $y^{(n)}$ denotes the set of all derivatives of $y$ up to the order $n$, and where $C$ denotes collectively the set of all parameters specifying the family element in $F$. These parameters might be either arbitrary functions of $x$, $y$ and the derivatives of $y$ up to the order $n$, or arbitrary constants. Denote by $G$ a connected Lie group of point transformations of the form

$$x = \phi(\bar{x}, \bar{y}; \sigma), \quad y = \psi(\bar{x}, \bar{y}; \sigma),$$

(2.2)

where $\sigma$ denotes collectively the parameters specifying the group element in $G$. We shall say that $G$ is the equivalence group of (2.1) if it is the largest group of transformations that maps every element of $F$ into $F$. In this case (2.2) is called the structure invariance group of (2.1) and the transformed equation takes the same form

$$\Delta(\bar{x}, \bar{y}^{(n)}; \bar{C}) = 0,$$

(2.3a)

where

$$\bar{C}_j = \bar{C}_j(C, C^{(s)}; \sigma),$$

(2.3b)

and where $C^{(s)}$ represents the set of all derivatives of $C$ up to a certain order $s$. In fact, the latter equality (2.3b) defines another group of transformations $G_c$ on the set of all parameters $C$ of the differential equation [12], and we shall be interested in this paper in the invariants of $G_c$. These are functions of the coefficients of the original equation which have exactly the same expression when they are also formed for the transformed equation. We also note that $G$ and $G_c$ represent the same set, and we shall use the notation $G_c$ only when there is need to specify the group action on coefficients of the equation.

The terminology used for invariants of differential equations and their variants in the current literature [15, 6, 8, 7, 13] is slightly different from that of Forsyth [2] and earlier writers on the subject. What is now commonly called semi-invariants are functions of the form $\Phi(C, C^{(r)})$ that satisfies a relation of the form $\Phi(C, C^{(r)}) = w(\sigma) \cdot \Phi(\bar{C}, \bar{C}^{(r)})$. When the weight $w(\sigma)$ is equal to one, $\Phi(C, C^{(r)})$, is called an invariant, or an absolute invariant. Semi-invariants usually correspond to partial structure invariance groups obtained by letting either the depend variable or the independent variables unchanged.

Because the infinitesimal method proposed in [14] is still entirely new and has been, in particular, applied only to a couple of examples, we wish to illustrate
a comparison of this method with the most well-known method of [4]. For this purpose we consider Eq. (1.1) with \( n = 3 \), which is the lowest order for which a linear ODE may have nontrivial invariants. The structure invariance group of (1.1) is given by the change of variables \( x = f(\bar{x}), \ y = T(\bar{x})\bar{y} \), where \( f \) and \( T \) are arbitrary functions, and for \( n = 3 \), this equation takes the form
\[
y^{(3)} + a_2 y'' + a_1 y' + a_0 y = 0,
\]
and it is easy to see that the corresponding infinitesimal transformations associated with the structure invariance group may be written in the form
\[
\bar{x} \approx x + \varepsilon \xi(x), \quad \bar{y} \approx y + \varepsilon (\eta(x)y),
\]
where \( \xi \) and \( \eta \) are some arbitrary functions. If we now let
\[
V^{(3)}_a = \xi \partial_x + y \eta \partial_y + \zeta_1 \partial_{y'} + \zeta_2 \partial_{y''} + \zeta_3 \partial_{y'''}
\]
denote the third prolongation of \( V_a = \xi \partial_x + y \eta \partial_y \), then we may write
\[
y' \approx \bar{y}' - \varepsilon \zeta_1(x, y, \bar{y}'), \quad y'' \approx \bar{y}'' - \varepsilon \zeta_2(x, y, \bar{y}', \bar{y}''), \quad y^{(3)} \approx \bar{y}^{(3)} - \varepsilon \zeta_3(x, y, \bar{y}(3)).
\]

According to the former method [4], in order to find the corresponding infinitesimal transformations of the coefficients \( a_2, a_1, \) and \( a_0 \) of (2.4), and hence to obtain the infinitesimal generator for the associated induced group \( G_c \), the original dependent variable \( y \) and its derivatives are to be expressed in terms of \( \bar{y} \) and its derivatives according to approximations of the form
\[
y' \approx \bar{y}' - \varepsilon \zeta_1(x, y, \bar{y}'), \quad y'' \approx \bar{y}'' - \varepsilon \zeta_2(x, y, \bar{y}', \bar{y}''), \quad y^{(3)} \approx \bar{y}^{(3)} - \varepsilon \zeta_3(x, y, \bar{y}(3)).
\]

These expressions are then used to substitute \( \bar{y} \) and its derivatives for \( y \) and its derivatives in the original differential equation. In certain cases such as that of simple infinitesimal transformations of the form (2.7) can be obtained from (2.6). More precisely, an explicit calculation of \( V^{(3)}_a \) shows that
\[
\zeta_1 = yg' + (g - f')y', \quad \zeta_2 = y'(2g' - f'') + yg'' + (g - 2f')y'', \quad \zeta_3 = 3(g' - f'')yg'' + y'(3g'' - f''') + yg''' + (g - 3f')y'''
\]
The first explicit approximation is readily obtained from the second equation of (2.7) which shows that
\[
y = \bar{y} - \frac{1}{1 + \varepsilon g} = \bar{y}(1 - \varepsilon g),
\]
by neglecting terms of order two or higher in \( \varepsilon \). In a similar way, using the equations (2.7) and (2.8) we obtain after simplification the following approximations
\[
y' = (\bar{y}' - \varepsilon yg')(1 - \varepsilon(g - f')) \quad (2.10a)
y'' = [\bar{y}'' - \varepsilon (y'(2g' - f'') + yg'')(1 - \varepsilon(g - 2f'))] \quad (2.10b)
y''' = \left[ \bar{y}^{(3)} - \varepsilon \left(3(g' - f'')yg'' + y'(3g'' - f''') + yg^{(3)} \right) \right] (1 - \varepsilon(g - 3f')) \quad (2.10c)
Substituting equations (2.9) and (2.10) into (2.4) and rearranging shows that the corresponding infinitesimal transformations for the coefficients of the equation are given by
\[\begin{align*}
\bar{a}_0 &= a_0 + \varepsilon [-3a_0f' - a_1g' - a_2g'' + g^{(3)}] \quad (2.11a) \\
\bar{a}_1 &= a_1 + \varepsilon [-2a_1f' + 2a_2g' + a_2f'' - 3g'' + f^{(3)}] \quad (2.11b) \\
\bar{a}_2 &= a_2 + \varepsilon [-a_2f' - 3g' + 3f'']. \quad (2.11c)
\end{align*}\]
In other words, the infinitesimal generator of the group \(G_c\) corresponding to (2.4) is given by
\[X^0 = f \partial_x + \eta \partial_y + \phi_1 \partial_{a_1} + \phi_2 \partial_{a_2} + \phi_0 \partial_{a_0} + \phi_1 \partial_{\bar{a}_1} + \phi_2 \partial_{\bar{a}_2} \quad (2.12)\]

The next step according to the method is to look at minimum conditions that reduce \(V = \{\xi, \eta, \phi\}\) to a generator \(V^0 = \{\xi^0, \eta^0\}\) of the group \(G\) of equivalence transformations. These conditions are imposed on \(\phi\) to obtain the resulting function \(\phi^0\), and hence the generator \(X^0 = \{\xi^0, \phi^0\}\) of \(G_c\).

In the actual case of equation (2.4) we readily find that the generator
\[X = \xi \partial_x + \eta \partial_y + \phi_1 \partial_{a_1} + \phi_2 \partial_{a_2} + \phi_0 \partial_{a_0} \quad (2.13)\]
is given by
\[\begin{align*}
\xi &= f \\
\eta &= gy + h \\
\phi_2 &= -a_2f' - 3g' + 3f'' \\
\phi_1 &= -2a_1f' - 2a_2g' + a_2f'' - 3g'' + f^{(3)} \\
\phi_0 &= -\frac{1}{y} \left[ a_0h + 3a_0yg' + a_1yg' + a_1h'x + a_2yg'' + a_2h'' + yg^{(3)} + h^{(3)} \right],
\end{align*}\]
where \(f, g\) and \(h\) are arbitrary functions of \(x\). It is clear that, due to the homogeneity of (2.4), we must have \(h = 0\). This simple condition immediately reduces \(V = \{f, gy + h\}\) to the well-known generator \(V^0 = \{f, gy\}\) of \(G\). If we denote by \(\phi^0 = \{\phi^0_2, \phi^0_1, \phi^0_0\}\) the resulting vector when the same condition \(h = 0\) is applied to \(\phi = \{\phi_2, \phi_1, \phi_0\}\), then \(X^0 = \{f, \phi^0\}\), which may be represented as
\[X^0 = f \partial_x + \phi^0_2 \partial_{a_2} + \phi^0_1 \partial_{a_1} + \phi^0_0 \partial_{a_0}, \quad (2.14)\]
is the same as the operator $X^0$ of (2.12).

It should however be noted that the former method involves a great deal of algebraic manipulations, and in the actual case of Eq. (2.4), the method works because of the simplicity of the infinitesimal generator $V_a = \{f, yg\}$. For higher orders of the Eq. (1.1), difficulties with this method become quite serious. In fact, variants of this former method are being increasingly used, which also exploit the symmetry properties of the equation [6], but even these variants still require the knowledge of the equivalence group.

3. INVARIENTS FOR EQUATIONS OF LOWEST ORDERS

We shall use the indicated method of [14] in this section to derive explicit expressions for invariants of linear ODEs of order not higher than the fifth. It should be noted that, not only the infinitesimal method we are using for finding these invariants is new, but also almost all of the invariants found are appearing in explicit form for the first time. The focus of Forsyth and his contemporaries was not only on linear equations of the form (2.1), but also on functions $\Phi(C, C(r))$ satisfying conditions of the form

$$\Phi(C, C(r)) = h(g) \cdot \Phi(\bar{C}, \bar{C}(r)), \quad \text{or} \quad \Phi(C, C(r)) = (d\bar{x}/dx)^\sigma \cdot \Phi(\bar{C}, \bar{C}(r)), \quad (3.1)$$

in the case of linear ODEs, where $h$ is an arbitrary function, $\sigma$ is a scalar and where $g = g(\bar{x})$ and $\bar{x}$ are defined by a change of variables of the form $x = f(\bar{x}), y = g \bar{y}$, where $f$ is arbitrary. Such functions are clearly invariants of the linear equation only if the dependent variable alone, or the independent variable alone, is transformed, but not both as in our analysis. However, in his very talented analysis starting with the investigation of semi-invariants satisfying the second condition of (3.1), Forsyth obtained in his memoir [2] a very implicit expression for true invariants of linear ODEs of a general order, given as an indefinite sequence in terms of semi-invariants. As it is stated in [2], earlier works on the subject had given rise only to semi-invariants of order not higher than the fourth, with the exception of two special functions which satisfy the condition of invariance for equations of all orders.

3.1. Equations in normal reduced form. Invariants of differential equations generally have a prominent size involving several terms and factors, and thus they are usually studied by first putting the equation in a form in which a number of coefficients vanish. By a change of variable of the form

$$x = \bar{x}, \quad y = \exp(- \int a_1 d\bar{x})\bar{y}, \quad (3.2)$$

equation (1.1) can be reduced to the simpler form

$$y^{(n)} + A_{n-2}y^{(n-2)} + A_{n-3}y^{(n-3)} + \cdots + A_0 y = 0, \quad (3.3)$$

after the renaming of variables, and where the $A_j = A_j(x)$ are the new coefficients. Eq. (3.3), which is referred to as the normal form of (1.1), will be as in [2] our starting canonical form for the determination of invariants.
The generic infinitesimal generator $X^0$ of $G_c$ such as the one found in (2.12) linearly depends on free parameters $K_j$, and can be written in terms of these parameters as a linear combination of the form,

$$X^0 = \sum_{j=1}^{\nu} K_j W_j,$$  \hspace{1cm} (3.4)

where the $W_j$ are much simplified vector fields free of arbitrary parameters and with the same number of components as $X^0$, and a function $F = F(C)$ satisfies the condition of invariance $X^0 \cdot F = 0$ if and only if $W_j \cdot F = 0$, for $j = 1, \ldots, \nu$. Consequently, the invariant functions are completely specified by the matrix $M \equiv \{W_1, \ldots, W_\nu\}$ whose $j$th row is represented by the components of $W_j$, plus the coordinate system in which the vector fields $W_j$ are expressed. These considerations also apply to prolongations of $X^0$ and corresponding differential invariants. For a given canonical form of the equation, we will use the symbols $\Psi_{a,b}^{c}$ and $X_{a}^{m}$ to represent explicit expressions of invariants and corresponding infinitesimal generator $X^0$, respectively. In such a representation, the subscript $a$ will denote the order of the equation, while the superscript $b$ will represent the order of prolongation of the original generator $X^0$, and $c$ will represent the actual number of the invariant in some chosen order. The superscript $m$ will represent the canonical form considered, and will be $n$ in the case of the normal form (3.3), s for the standard form (1.1), and w for another canonical form to be introduced below. For consistency, coefficients in all canonical forms considered will be represented as in (1.1) by the symbols $a_j = a_j(x)$.

For $n = 3$, the first nontrivial differential invariants of (3.3) occurs only as from the third prolongation of $X^0$. This third prolongation of the generator $X^0$ has exactly one invariant $\Psi_{3,1}^{3,1}$, and both $X^0 = X^3_3$ and $\Psi_{3}^{3,1}$ have already appeared in the recent literature [14]. However, we recall these results here in a slightly simplified form, together with those we have now obtained for the fourth prolongation of $X^0$, as well as for the case $n = 4$.

For $n = 3$, we have

$$X^0_3 = f \partial_x - 2(a_1 f' + f^{(3)}) \partial_{a_1} + \left(-3a_0 f' - a_1 f'' - f^{(4)}\right) \partial_{a_0},$$  \hspace{1cm} (3.5a)

$$\Psi_{3,1} = \frac{(9a_1 \mu^2 + 7 \mu' \mu - 6 \mu\mu')^3}{1728 \mu^8},$$  \hspace{1cm} (3.5b)

$$\Psi_{3}^{3,1} = \Psi_{3}^{3,1},$$  \hspace{1cm} (3.5c)

$$\Psi_{3}^{4,2} = \frac{-1}{18 \mu^4} \left(216a_0^4 - 324a_0^3 a_1' + 18 a_0^2 (9a_1^2 + 2a_1 \mu') + 9 \mu^2 \mu^{(3)} \right)$$

$$- \frac{1}{18 \mu^4} \left(\mu' \left(28 \mu^2 + 9a_1 (a_1 a_1' - 4 \mu'') \right) - 9a_0 (3a_0^3 + 4a_1 a_1' \mu' - 8 \mu' \mu''), \right)$$  \hspace{1cm} (3.5d)

where $f$ is an arbitrary function and where $\mu = -2a_0 + a_1'$. For $n = 4$, the first nontrivial invariant occurs only as from the second prolongation of $X^0 = X^4_4$, and
we have
\[ X_4 = f \partial_x + (-2a_2 f' - 5f^{(3)}) \partial_{a_2} + (-3a_1 f' - 2a_2 f'' - 5f^{(4)}) \partial_{a_1} + \frac{1}{2} \left[-8a_0 f' - 3(a_1 f'' + 2a_2 f^{(3)})\right] \partial_{a_0} \]  
(3.6a)
\[ \Psi_{4,1}^2 = \frac{1}{\mu}(-100a_0 + 9a_2^2 + 50a_1' - 20a_2') \]  
(3.6b)
\[ \Psi_{4,2}^2 = \mu_1(-1053a_2^2 - 4500a_2(2a_1^2 - a_1a_2^2 - a_2^2) + 180a_2^2(130a_0 - 3(5a_1' + 8a_2')) - 100(1300a_0^2 + 75\mu(-10a_0' + 3a_1') + 90a_1'a_2' + 27a_2'' - 60a_0(5a_1' + 8a_2'))) \]  
(3.6c)
where \( \mu = a_1 - a_2' \), and \( \mu_1 = 1/\left[75000\mu^{(8/3)}\right] \).

### 3.2. Equations with vanishing coefficients \( a_{n-1} \) and \( a_{n-2} \)

A much simpler expression for the invariants is obtained if the two terms involving the coefficients \( a_{n-1} \) and \( a_{n-2} \) in Eq. (1.1) are reduced to zero in the transformed equation. Such a change of variables can be accomplished by a transformation of the form

\[ \{\bar{x}, x\} = \frac{12}{n(n-1)(n+1)}a_{n-2}, \quad y = \exp(-\int a_{n-1} d\bar{x})\bar{y}, \]  
(3.7a)
where

\[ \{\bar{x}, x\} = \left(\bar{x}'\bar{x}^{(3)} - (3/2)\bar{x}''\right)\bar{x}^{-2} \]  
(3.7b)

is the Schwarzian derivative, and where \( \bar{x}' = d\bar{x}/dx \). Thus by an application of (3.7) to (1.1), we obtain after the renaming of variables and coefficients an equation of the form

\[ y^{(n)} + a_{n-3}y^{(n-3)} + a_{n-4}y^{(n-4)} + \cdots + a_0y = 0. \]  
(3.8)

In fact, (3.8) is the canonical form that Forsyth [2], Brioscii [1], and some of their contemporaries adopted for the investigation of invariant functions of linear ODEs. However, Forsyth who studied these equations for a general order did not derive any explicit expressions for the invariants, with the exception of a couple of semi-invariants. There is a number of important facts that occur in the determination of the invariants when the equation is put into the reduced form (3.8), but we postpone the discussion on the properties of invariants to the next Section, where a formal result regarding their exact number is also given.

For \( n = 3 \), nontrivial invariants exist only as from the second prolongation of \( X^0 = X_3^0 \), and we have computed them for the second and the third prolongation. For \( n = 4 \), there are no zeroth order invariants and we have computed these invariants for the first and the second prolongation of \( X_4^2 \). For \( n = 5 \), the invariants are given for all orders of prolongation of \( X_5^4 \), from 0 to 2. Since invariants corresponding to a given generator are also invariants for any prolongation of this generator, its is only necessary to list the invariants for the highest order of prolongation of any given generator. In the canonical form (3.8), and regardless of the order of the equation, the generator \( X^0 \) of \( G_c \) will depend on three arbitrary constants that we shall denote by \( k_1, k_2 \) and \( k_3 \).
For $n = 3$, we have
\[ X_3^w = [k_1 + x(k_2 + k_3 x)] \partial_x - 3a_0(k_2 + 2k_3 x) \partial_{a_0} \]
\[ \Psi_{3,1}^1 = \Psi_{3,1}^2 = \left(-\frac{7a_0'^2}{6a_0} + a_0''/a_0^5 \right)^3 \]
\[ \Psi_{3,2}^2 = \left(\frac{28a_0'^3 - 36a_0 a_0'' + 9a_0^2 a_0'''}{9a_0^4} \right) \]

For $n = 4$, we have
\[ X_4^w = [k_1 + x(k_2 + k_3 x)] \partial_x - 3a_1(k_2 + 2k_3 x) \partial_{a_1} + [-3a_1 k_3 - 4a_0(k_2 + 2k_3 x)] \partial_{a_0} \]
\[ \Psi_{4,1}^1 = \Psi_{4,1}^2 = \frac{(-2a_0 + a_1')^3}{a_1^2} \]
\[ \Psi_{4,2}^2 = \Psi_{4,2}^3 = \frac{a_0' - a_0(a_0 + 3a_1')}{a_1^2} \]
\[ \Psi_{4,3}^3 = \frac{(14a_0^2 - 14a_0 a_1' + 3a_1 a_1'^2)^3}{27a_1^8} \]
\[ \Psi_{4,4}^4 = \frac{16a_0^3 + 48a_1 a_0^2 a_1' + 9a_0^2 a_1'' - 9a_0 a_1(6a_0' + a_1'')}{9a_1} \]

Finally, for $n = 5$ we have
\[ X_5^w = [k_1 + x(k_2 + k_3 x)] \partial_x - 3a_2(k_2 + 2k_3 x) \partial_{a_2} - 2[-3a_2 k_3 + 2a_1(k_2 + 2k_3 x)] \partial_{a_1} + [-4a_1 k_3 - 5a_0(k_2 + 2k_3 x)] \partial_{a_0} \]
\[ \Psi_{5,1}^1 = \Psi_{5,1}^2 = \frac{(3a_0 a_2 - a_1')^3}{27a_2^8} \] (3.9a)
\[ \Psi_{5,2}^2 = \Psi_{5,2}^3 = \frac{(-a_1 + a_2')^3}{a_2^2} \]
\[ \Psi_{5,3}^3 = \Psi_{5,3}^4 = \frac{(6a_2 a_1' - a_1^2 - 6a_1 a_2')^3}{216 a_2^5} \]
\[ \Psi_{5,4}^4 = \Psi_{5,4}^5 = \frac{5a_1^4 + 9a_2^2 a_0' - 3a_1 a_2(5a_0 + 2a_1') + 3a_1^2 a_2'}{9a_2^4} \] (3.9b)

and
\[ \Psi_{5,5}^5 = \frac{(7a_1^2 - 14a_1 a_2' + 6a_2 a_2'')^3}{216 a_2^5} \]
\[ \Psi_{5,6}^6 = \frac{4a_1^3 + 24a_1^2 a_2^2 + 9a_0^2 a_0'' - 9a_1 a_2(3a_1' + a_2''')}{9a_2^4} \]
\[ \Psi_{5,7}^7 = \frac{[18(-a_1^3 - a_1^2 a_2' + a_2^2 a_0'') - 6a_1 a_2(11a_0' + 2a_0'') + a_2^2 a_2(55a_0 + 40a_1' + 6a_2'')]^3}{5832 a_2^{15}} \]

3.3. Equations in the standard form \((1.1)\). No attempt has ever been made to our knowledge, to obtain the invariant functions for equations in the canonical form \((1.1)\), due simply to difficulties associated with such a determination and to the prominence in size that semi-invariants for this canonical form already display. Indeed, semi-invariants for the canonical form \((1.1)\) were calculated by Laguerre \[9\] for equations of the third order. Transformations of the form \((5.2)\) or \((3.7)\) are
usually applied to transform (1.1) to an equation in which one or both of the coefficients \(a_{n-1}\) and \(a_{n-2}\) vanish. Even with the infinitesimal generator \(X^0\) of (2.12) at our disposal, it is still difficult to find directly the corresponding invariants for the third order equation, which is the lowest for which the linear equation may have nontrivial invariants. However we show here by an example how these invariants can be found for the third order from those of equations in a much simpler canonical form.

Under the change of variables (3.2), Eq. (2.4) takes, after elimination of a constant factor, the form

\[
\ddot{y}^{(3)} + B_1 \dot{y}' + B_0 \ddot{y} = 0,
\]

where

\[
B_0 = (27a_0 - 9a_1 a_2 + 2a_3^2 - 9a_4')/27
\]

(3.11)

\[
B_1 = (27a_1 - 9a_3 - 27a_4')/27.
\]

(3.12)

The sole invariant \(\Psi_{3,1}^{4,1}\) for equations of the form (3.10) and corresponding to the third prolongation of the associated generator \(X^0\) is given by (3.5). If in that function we replace \(a_0\) and \(a_1\) by \(B_0\) and \(B_1\) respectively, and then express \(B_0\) and \(B_1\) in terms of the original coefficients \(a_0, a_1\) and \(a_2,\) the resulting function is an invariant of (2.4). However, it does not correspond to the third, but to the fourth prolongation of the corresponding generator \(X^0 = X^4_3\) that was obtained in (2.12).

In other words, in the canonical form (1.1) with \(n = 3,\) we have

\[
X^4_3 = X^0, \text{ as given by (2.12)}
\]

(4.1c)

\[
\Psi_{3,1}^{4,1} = \frac{(9B_1\mu^2 + 7\mu^4 - 6\mu')^3}{1728\mu^8},
\]

(3.13)

where we have \(\mu = -2B_0 + B_1'.\) Using Theorem 2 of [14], it can be shown that there are no invariants corresponding to prolongations of \(X^4_3\) of order lower than the fourth, and that the fourth prolongation has precisely one invariant.

4. A NOTE ABOUT THE INFINITESIMAL GENERATORS AND THE INVARIANTS

The method of [14] that we have used in the previous section provides the infinitesimal generator of the equivalence group \(G\) for a given equation of a specific order. This infinitesimal generator must be integrated in order to obtain the corresponding structure invariance group for the given family of equations of a specific order. However, once the structure invariance group for such a given equation has been found, it is generally not hard to extend the result to a general order. We show how this can be done for equations of the form (3.8).

With the notations already introduced in Section 2 if we denote by \(X\) the full symmetry generator of (3.8) for a specific order and then obtain the infinitesimal generator \(V^0\) of the equivalence group \(G,\) we find that

\[
V^0 = V_{0,3} = [k_2 + x(k_3 + k_4x)] \partial_x + g(k_1 + 2k_4x) \partial_y, \quad \text{for } n = 3,
\]

(4.1a)

\[
V^0 = V_{0,4} = [k_2 + x(k_3 + k_4x)] \partial_x + \frac{y}{3}(2k_1 + 2k_3 + 9k_4x) \partial_y, \quad \text{for } n = 4,
\]

(4.1b)

\[
V^0 = V_{0,5} = [k_2 + x(k_3 + k_4x)] \partial_x + g(k_1 + k_3 + 4k_4x) \partial_y, \quad \text{for } n = 5,
\]

(4.1c)

where the \(k_j\) for \(j = 1, \ldots, 4\) are arbitrary constants. Upon integration of these three vector fields, we find that for \(n = 3, 4, 5,\) the structure invariance group can
be written in the general form
\[ x = \frac{\bar{x} - c(1 + a\bar{x})}{b(1 + a\bar{x})}, \quad y = \frac{\bar{y}}{d(1 + a\bar{x})^{(n-1)}}, \quad (4.2) \]
where \( a, b, c \) and \( d \) are arbitrary constants. To see why \( (4.2) \) holds for any order \( n \) of Eq. \( (3.8) \), we first notice that these changes of variables are of the much condensed form
\[ x = g(\bar{x}), \quad y = T(\bar{x})\bar{y}. \]
It thus follows from the properties of derivations that when \( (4.2) \) is applied to \( (3.8) \), every term of order \( m \) in \( y \) will involve a term of order at most \( m \) in \( \bar{y} \) upon transformation. Since the original equation \( (3.8) \) is deprived of terms of orders \( n-1 \) and \( n-2 \), to ensure that this same property will also holds in the transformed equation, we only need to verify that the transformation of the term of highest order, viz. \( y^{(n)} \), does not involve terms in \( \bar{y}^{(n-1)} \) or \( \bar{y}^{(n-2)} \). However, it is easy to see that under \( (4.2) \), \( y^{(n)} \) is transformed into \( b^n(1 + a\bar{x})^{n+1}\bar{y}^{(n)}/d \). We have thus obtained the following result.

**Theorem 1.** The structure invariance group of linear ODEs in the canonical form \( (3.8) \), that is, in the form
\[ y^{(n)} + a_{n-3}y^{(n-3)} + a_{n-4}y^{(n-4)} + \cdots + a_0y = 0, \]
is given for all values of \( n > 2 \) by \( (4.2) \).

It is easy to prove a similar result using the same method for equations in the other canonical forms \( (1.1) \) and \( (3.3) \), for which the corresponding structure invariance groups are well-known. It is also clear that \( (4.2) \) does not violate in particular the structure invariance group of \( (3.8) \) which is given by \( x = F(\bar{x}) \) and \( y = \alpha F^{(n-1)/2}\bar{y} \), where \( F \) is an arbitrary function and \( \alpha \) an arbitrary coefficient.

All the invariants that we have found for linear ODEs are rational functions of the coefficients of the corresponding equation and their derivatives, and this agrees with earlier results [2]. Another fact about these invariants is that for equations in the form \( (3.8) \), nontrivial invariants that depend exclusively on the coefficients of the equation and not on their derivatives, that is, invariants such as that in \( (3.9a) \) which are zeroth order differential invariants of \( X^0 \) occurs as from the order 5 onwards. Indeed, since \( X^0 \) depends only on three arbitrary constants in the case of the canonical form \( (3.8) \), there will be more invariants, as compared to the other canonical forms in which \( X^0 \) depends on arbitrary functions. More precisely, we have the following result

**Theorem 2.** For equations in the canonical form \( (3.8) \), the exact number \( n \) of fundamental invariants for a given prolongation of order \( p \geq 0 \) of the infinitesimal generator \( X^0 \) is given by
\[ n = \begin{cases} n + p(n-2) - 4, & \text{if } (n, p) \neq (3, 0) \\ 0, & \text{otherwise}. \end{cases} \quad (4.3) \]

**Proof.** We first note that the lowest order for equations of the form \( (3.8) \) is 3, and we have seen that when \( n = 3, 4, 5 \) the generator \( X^0 \) depends on three arbitrary constants alone, that we have denoted by \( k_1, k_2 \) and \( k_3 \), and do not depend on any arbitrary functions. Consequently, this is also true for any prolongation of order \( p \) of \( X^0 \), and it is also not hard to see that this property does not depend on the
order of the equation. It is clear that the corresponding matrix $M$ whose $j$th row is represented by the components of the vector field $W_j$ of (3.4) will always have three rows and maximum rank $r = 3$ when $(n, p) \neq (3, 0)$. On the other hand, $X^0$ is expressed in a coordinate system of the form $\{x, a_{n-3}, a_{n-4}, \ldots, a_0\}$ that has $n - 1$ variables. Its $p$th prolongation is therefore expressed in terms of $M = n - 1 + p(n - 2)$ variables. For $(n, p) \neq (3, 0)$, the exact number of fundamental invariants is $M - r$, which is $n + p(n - 2) - 4$, by Theorem 2 of [14]. For $(n, p) = (3, 0)$ a direct calculation shows that the number of invariants is zero. This completes the proof of the Theorem.

Similar results can be obtained for equations in other canonical forms, but as it is customary to choose the simplest canonical form for the study of invariants, we shall not attempt to prove them here.

It should be noted at this point that invariant functions have always been intimately associated with important properties of differential equations, and in particular with their integrability. Laguerre [9] showed that for the third order equation (2.4), when the semi-invariant $\mu = -2B_0 + B'_1$ given in Eq. (3.13) vanishes, there is a quadratic homogeneous relationship between any three integrals of this equation, and consequently its integrability is reduced to that of a second order equation. Subsequently, Brioschi [1] established a similar result for equations in the canonical form (3.3). More precisely, he showed that for third order equations, when the semi-invariant $\mu = -2a_0 + a'_1$ of Eq. (3.5) vanishes, the equation can be reduced in order by one to the second order equation

$$\dddot{y} + \frac{a_1}{4} \ddot{y} = 0,$$

where $y = \dddot{y}^2$ and he also gave the counterpart of this result for fourth order equations when the corresponding semi-invariant $\mu = a_1 - a'_2$ of Eq. (3.6) vanishes. In fact all reductions of differential equations to integrable forms implemented in [3] are essentially based on invariants of differential equations. More recently, Schwarz [19] attempted to obtain a classification of third order linear ODEs based solely on values of invariants of these equations. He also proposed [20] a solution algorithm for large families of Abel’s equation, based on what is called in that paper a functional decomposition of the absolute invariant of the equation.

5. Concluding Remarks

We have clarified in this paper the algorithm of the novel method that has just been proposed in [14] for the equivalence group of a differential equation and its generators, and we have shown how its application to the determination of invariants differs from the former and well-known method of [4, 5], by treating the case of the third order linear ODEs with the two methods. We have subsequently obtained some explicit expressions of the invariants for linear ODEs of orders three to five, and discussed various properties of these invariants and of the infinitesimal generator $X^0$.

Another fact that has emerged concerning the infinitesimal generator $X^0$ of the group $G_c$ for linear ODEs is that, whenever the equation is transformed into a form in which the number of coefficients in the equation is reduced by one, the number of arbitrary functions in the generator $X^0$ corresponding to the transformed equation is also reduced by one, regardless of the order of the equation. Thus while
 depends on two arbitrary functions, \(X^n_a\) depends only on one such function, and \(X^w_a\) depends on no such function, but on three arbitrary constants. This is clearly in agreement with the expectation that equations with less coefficients are easier to solve, since less arbitrary functions in \(X^0\) means more invariants and hence more possibility of reducing the equation to a simpler one. However, the full meaning of the degeneration of these functions with the reduction of the number of coefficients of the equation is still to be clarified. We also believe that on the base of recent progress on generating systems of invariants and invariant differential operators (see \([16, 17]\) and the references therein), it should be possible to treat the problem of determination on invariants of differential equations in a more unified way, regardless of the order of the equation, or the order of prolongation of the operator \(X^0\).

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