QED calculations of three-photon transition probabilities in H-like ions with arbitrary nuclear charge

T Zalialiutdinov, D Solovyev and L Labzowsky

1 Department of Physics, St.Petersburg State University, Ulianovskaya 1, Petrodvorets, St.Petersburg 198504, Russia
2 Petersburg Nuclear Physics Institute, 188300, Gatchina, St. Petersburg, Russia

Received 5 September 2015, revised 17 October 2015
Accepted for publication 21 October 2015
Published 15 February 2016

Abstract
The quantum electrodynamical theory of three-photon transitions in hydrogen-like ions is presented. Emission probabilities of various three-photon decay channels for $p_{2s}$, $p_{1/2}$ and $s_{1/2}$ states are calculated for $Z$, the nuclear charge value, $1 \leq Z \leq 95$. The results are given in two different gauges. The fully relativistic three-photon decay rates of hydrogen-like ions with half-integer nuclear spin are given for transitions between fine structure components. The results can be applied to the Bose–Einstein statistics tests for multiphoton systems.

Keywords: atomic physics, quantum electrodynamics, multiphoton processes, Landau–Yang, three-photon transitions

In the present work the QED theory is applied to the study of spontaneous three-photon decay processes. Calculations of the various transition probabilities for H-like ions for nuclear charge values $Z$ within the region $1 \leq Z \leq 95$ are performed. Furthermore, fully relativistic calculations for three-photon transitions between fine structure components with account for hyperfine structure of hydrogen-like ions are presented. This means that we have a fixed a certain hyperfine substate of the structure component, not introducing explicitly the hyperfine level splitting. We performed calculations in two different gauges, allowing us to accurately check the gauge invariance of the results. For the summation over the complete Dirac spectrum the B-spline method [11] was employed. Relativistic units are used throughout this paper.

Our paper is organized as follows. In section 2 we present a detailed QED derivation of the general expression for the spontaneous three-photon decay rate in H-like ions for an arbitrary combination of electric and magnetic multipoles and in an arbitrary gauge for the electromagnetic potentials. In section 3 we consider three-photon transitions between fine structure components with account for hyperfine structure (as explained above) of a one-electron ion with arbitrary nuclear spin $I$. Numerical values for the transition probabilities...
Table 1. Transition probabilities for $2p_{1/2} \rightarrow 1s_{1/2} + 3\gamma(E1)$ and $2p_{3/2} \rightarrow 1s_{1/2} + \gamma(E1)$ decay rates in $s^{-1}$ for different Z. The number in parentheses indicates the power of ten. Transition energies in eV are listed in the last column.

| Z  | $W_{el}^{(1)}$ | $W_{en}^{(1)}$ | $W_{ff}^{(1)}$ | $\Delta E$ |
|----|----------------|----------------|----------------|------------|
| 1  | 1.168620(-8)   | 1.168632(-8)   | 6.268(8)      | 10.204393  |
| 5  | 4.561173(-3)   | 4.561219(-3)   | 6.918(11)     | 2.551846(2) |
| 10 | 1.164659       | 1.164670       | 6.272(12)     | 1.021675(3) |
| 20 | 2.950782(2)    | 2.950754(2)    | 1.005(14)     | 4.101827(3) |
| 30 | 7.430658(3)    | 7.430721(3)    | 5.106(14)     | 9.287031(3) |
| 40 | 7.237156(4)    | 7.237208(4)    | 1.621(15)     | 1.665912(4) |
| 50 | 4.170711(5)    | 4.170735(5)    | 3.980(15)     | 2.634226(4) |
| 60 | 1.717285(6)    | 1.717293(6)    | 8.310(15)     | 3.851493(4) |
| 70 | 5.579235(6)    | 5.579255(6)    | 1.552(16)     | 5.342837(4) |
| 80 | 1.514879(7)    | 1.514883(7)    | 2.671(16)     | 7.144008(4) |
| 90 | 3.554091(7)    | 3.554098(7)    | 4.316(16)     | 9.305821(4) |
| 95 | 5.1821200(7)   | 5.1821200(7)   | 5.380(16)     | 1.054361(5) |

considered in sections 2 and 3 are presented in tables 1–3. In section 4 we discuss the application of our results to the study of Bose–Einstein statistics and to the extension of LYT [12–14]. This section also contains the concluding remarks.

2. QED theory of three-photon transitions

We present a computationally convenient fully relativistic form of a general expression for the three-photon decay rate in H-like ions for an arbitrary combination of electric and magnetic multipoles and in an arbitrary gauge for the electromagnetic potentials. Relativistic units $\hbar = c = 1$ are employed. In this section the hyperfine structure of the levels is neglected.

The $S$-matrix element for the emission process $i \rightarrow f + 3\gamma$ ($i$ and $f$ denote the initial and final states, respectively) reads [15–17]

$$S^{(3)} = (-ie)^3 \int d^4x_3d^4x_2d^4x_1 \overline{\psi}_i(x_1) \gamma_{j_1} A_{j_1}^a(\vec{k}_{j_1}) \times \gamma_j \psi_i(x_3) \gamma_{j_2} A_{j_2}^b(\vec{k}_{j_2}) \psi_f(x_2),$$

where

$$\psi_n(x) = \psi_n(\vec{r}) e^{-iE_nt},$$

$\psi_n(\vec{r})$ is the solution of the Dirac equation for the atomic electron, $E_n$ is the Dirac energy, $\overline{\psi}_n = \psi_n^\dagger \gamma_0$ is the Dirac conjugated wave function, $\gamma_\mu = (\gamma_0, \gamma_i)$ are the Dirac matrices and $x \equiv (\vec{r}, t)$ are the space-time coordinates. In this paper the Euclidean metric with an imaginary fourth vector component is adopted. The photon wave function (electromagnetic field potential) is described by

$$A_i^{(\vec{k}, \vec{r})}(x) = \frac{2\pi}{\omega} e^{i\vec{k} \cdot \vec{r} - i\omega t},$$

where $k \equiv (\vec{k}, \omega)$ is the photon momentum four-vector, $\vec{k}$ is the photon wave vector, $\omega = |\vec{k}|$ is the photon frequency, $e_i$ are the components of the photon polarization four-vector, $\vec{r}$ is the three-dimensional polarization vector for real photons, $A_i^{(\vec{k}, \vec{r})}$ corresponds to the emitted photon and $A_i^{(\vec{k}, \vec{r})}$ represents the emitted photon, respectively.

For the real transverse photons

$$\vec{A}(\vec{k}, \vec{r}) = \frac{2\pi}{\omega} \vec{e} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \equiv \frac{2\pi}{\omega} \vec{A}_{\vec{r}, \vec{k}} e^{-i\omega t}.$$

The electron propagator for bound electrons can be presented in the form of the eigenmode decomposition with respect to one-electron eigenstates [15, 16]

$$S(x_i, x_2) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega e^{i(\gamma_1 - \gamma_2)} \sum_n \psi_n(\vec{r}_1) \overline{\psi}_n(\vec{r}_2) E_n (1 - i\theta) + \omega.$$}

Here the summation runs over the entire Dirac spectrum for the atomic electron. Insertion of the expressions (2)–(5) into equation (1) and performing the integrations over time and frequency variables yields

$$S^{(3)} = -2\pi e^3 \delta(E_i - E_f - \omega_3 - \omega_2 - \omega_1) \times \sum_{n_\alpha} \left( \overline{\alpha}_A \bar{\psi}_{i,k_1} \right)_{\vec{r}_1} \left( \alpha A_{\vec{k}_2}^{*} \right)_{\vec{r}_2} \left( \bar{\psi}_{f,k_2} \right)_{\vec{r}_3} \left( \overline{\alpha}_A \right)_{\vec{r}_4},$$

where $\alpha$ are the Dirac matrices, $\ldots_{\vec{r}_n}$ denotes the matrix element with Dirac wave function $\psi_2$, $\psi_3$. The amplitude of the emission process $U_{gf}$ is related to the $S$-matrix element via

$$S_{gf} = -2\pi i \delta(E_i - E_f - \omega_3 - \omega_2 - \omega_1) U_{gf}.$$

The differential probability (transition rate) of the process is defined as

$$\frac{dW_{gf}^{(3)}}{d\omega_3 d\omega_2 d\omega_1} = 2\pi \delta(E_i - E_f - \omega_3 - \omega_2 - \omega_1) \left| U_{gf}^{(3)} \right|^2.$$

Then the differential transition rate in conjunction with the integration over photon directions $\vec{p} = \vec{k}/|\vec{k}|$ and the summation over the photon polarizations $\vec{e}$ of all the emitted
where the frequency $\omega_3$ is defined via the $\delta$ function in equation (8). Therefore the total transition rate is

$$W_{i\rightarrow f} = \frac{1}{3!} \frac{1}{2j_i + 1} \sum_{m_{ei},m_{ef}} \int \frac{dW_{i\rightarrow f}(\omega_1,\omega_2)}{d\omega_1 d\omega_2} d\omega_1 d\omega_2,$$

where $j_{ei}, m_{ei}$, $j_{ef}, m_{ef}$ are the angular momenta and their projections for the initial ($i$) and final ($f$) electron states.

Expanding the plane waves into spherical waves in equation (9), we change to describing photons by the total angular momentum $j$, its projection $m$ and parity (type of the photon). Then we arrive at

$$\frac{dW_{i\rightarrow f}(\omega_1,\omega_2)}{d\omega_1 d\omega_2} = \frac{\omega_3}{(2\pi)^2} \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_{i\rightarrow f}} \sum_{m_{i\rightarrow f}} \left[ \frac{(Q_{j_i,m_{i\rightarrow f}}^{(\lambda_3)})_{j_i,m_{i\rightarrow f}}}{(Q_{j_i,m_{i\rightarrow f}}^{(\lambda_3)})_{j_i,m_{i\rightarrow f}}} \frac{(Q_{j_{i\rightarrow f},m_{i\rightarrow f}}^{(\lambda_1)})_{j_{i\rightarrow f},m_{i\rightarrow f}}}{(Q_{j_{i\rightarrow f},m_{i\rightarrow f}}^{(\lambda_1)})_{j_{i\rightarrow f},m_{i\rightarrow f}}} \right] \left[ (E_{i\rightarrow f} - \omega_1)(E_{i\rightarrow f} - \omega_3 - \omega_2) \right]$$

$$+ \sum_{n,n'} (E_{i\rightarrow f} - \omega_1)(E_{i\rightarrow f} - \omega_3 - \omega_2) \left[ \frac{(Q_{j_{i\rightarrow f},m_{i\rightarrow f}}^{(\lambda_2)})_{j_{i\rightarrow f},m_{i\rightarrow f}}}{(Q_{j_{i\rightarrow f},m_{i\rightarrow f}}^{(\lambda_2)})_{j_{i\rightarrow f},m_{i\rightarrow f}}} \right] \left[ \frac{(Q_{j_{i\rightarrow f},m_{i\rightarrow f}}^{(\lambda_1)})_{j_{i\rightarrow f},m_{i\rightarrow f}}}{(Q_{j_{i\rightarrow f},m_{i\rightarrow f}}^{(\lambda_1)})_{j_{i\rightarrow f},m_{i\rightarrow f}}} \right] \left[ (E_{i\rightarrow f} - \omega_1)(E_{i\rightarrow f} - \omega_3 - \omega_2) \right].$$

In equation (11) we employ the reduction of the matrix elements $(Q)_{ab}$ to the radial integrals developed in [18, 19]

$$\left\{ Q_{j_i,m_{i\rightarrow f}}^{(\lambda_1)} \right\}_{n_{i\rightarrow f},m_{i\rightarrow f}} = (-1)^{J_{i\rightarrow f} - m_{i\rightarrow f}} \left( \begin{array}{c} j_i \gamma_1 j_{i\rightarrow f} \gamma_{i\rightarrow f} \\ j_{i\rightarrow f} \gamma_{i\rightarrow f} j_i \gamma_i \end{array} \right)$$

$$\times \left( \begin{array}{c} j_{i\rightarrow f} \gamma_{i\rightarrow f} - \lambda + 1 \frac{(4\pi)}{2\beta_i + 1} \right)^{1/2}$$

$$\times \left[ (2\beta_{i\rightarrow f} + 1)^{1/2} \right]$$

$$\times \left[ j_i \gamma_{i\rightarrow f} - j_{i\rightarrow f} \gamma_{i\rightarrow f} \right]$$

$$\times \left[ \begin{array}{c} j_{i\rightarrow f} \gamma_{i\rightarrow f} - 1/2 \nu_i \gamma_i - m_{i\rightarrow f} \gamma_i \end{array} \right]$$

$$\times \left[ \begin{array}{c} j_i \gamma_i - 1/2 \nu_i \gamma_i - m_{i\rightarrow f} \gamma_i \end{array} \right].$$

Here $j_{i}, m_{i}$ are the total angular momentum of the photon and its projection, $\lambda$ characterizes the type of photon: $\lambda = 1$ corresponds to electric and $\lambda = 0$ corresponds to magnetic photons. The indices $n_{i\rightarrow f}, l_{i\rightarrow f}, m_{i\rightarrow f}$ present a standard set of one-electron Dirac quantum numbers. The radial matrix elements $M_{n_{i\rightarrow f},l_{i\rightarrow f},m_{i\rightarrow f}}^{(\lambda_1)}$ in equation (12) are equal to

$$M_{n_{i\rightarrow f},l_{i\rightarrow f},m_{i\rightarrow f}}^{(\lambda_1)} = \left[ \begin{array}{c} j_i \gamma_i + 1 \\ j_i \gamma_i - J_{i\rightarrow f} \gamma_{i\rightarrow f} \end{array} \right]$$

$$\times \left[ \begin{array}{c} (\kappa_{i\rightarrow f} - \kappa_{i\rightarrow f})I_{j_i+1}^+ + (j_i + 1)I_{j_i-1}^- \end{array} \right]$$

$$- \left[ \begin{array}{c} j_i + 1 \\ j_i \gamma_i \end{array} \right]$$

$$\times \left[ \begin{array}{c} (\kappa_{i\rightarrow f} - \kappa_{i\rightarrow f})I_{j_i-1}^- - (j_i + 1)I_{j_i+1}^+ \end{array} \right]$$

$$- G \left[ \begin{array}{c} 2j_i + 1 \\ j_i \gamma_i \end{array} \right]$$

$$\times \left[ \begin{array}{c} j_i + 1 \\ j_i \gamma_i \end{array} \right]$$

$$\times \left[ \begin{array}{c} (\kappa_{i\rightarrow f} - \kappa_{i\rightarrow f})I_{j_i+1}^+ - I_{j_i-1}^- \end{array} \right]$$

$$- j_iI_{j_i-1}^- + (j_i + 1)I_{j_i+1}^+.$$
where \( g_\alpha \) and \( f_\alpha \) are the large and small components of the radial Dirac wave function as defined in \([18]\), \( \kappa \) is the Dirac angular number, \( \omega \) is the photon frequency, \( j_1 \) represents the spherical Bessel function, \( G \) is the gauge parameter for the electromagnetic potentials. In our calculations we employ the ‘velocity’ gauge \((G = 0)\) and the ‘length’ \((G = \sqrt{\frac{j+1}{j}})\) gauge for the matrix element equation \((13)\) \([20]\). Note that equation \((4)\) corresponds to \((G = 0)\).

The results can be further simplified by the summations over projections of all the angular momenta. For this purpose we define the radial integral part for a particular combination of multiplopes as

\[
S_{l^*/m^*}(i, j, k) = \sum_{lj_{l^*}m_{l^*}} \sum_{lj_{j^*}m_{j^*}} \sum_{lj_{k^*}m_{k^*}} \times M_{j_{l^*}l_{l^*}}(\lambda_{l^*}) M_{j_{j^*}j_{j^*}}(\lambda_{j^*}) M_{j_{k^*}j_{k^*}}(\lambda_{k^*})/
\]

\[
\times \left( E_{n^*j_{l^*}l_{l^*}} - E_{n^*j_{j^*}j_{j^*}} - \omega_k - \omega_j \right) \times \left( E_{n^*j_{k^*}k_{k^*}} - E_{n^*j_{l^*}l_{l^*}} - \omega_j \right)
\]

\[
\times \Delta^l_{n_{l^*}n_{l^*}} \pi_i^l(i) \pi^j_{j^*}(j) \pi^k_{k^*}(k),
\]

where

\[
\pi^l_i(t) = \begin{cases} \frac{1}{2} \left( n_{j^*} - n_{j^*} \right), & \text{if } n_{j^*} + l + j_1 + \lambda_i \text{ is odd} \\ 0, & \text{if } n_{j^*} + l + j_1 + \lambda_i \text{ is even} \end{cases}
\]

\[
\Delta^l_{n_{l^*}n_{l^*}} = \frac{(4\pi)^{1/2}}{\left| j_{l^*}, j_{j^*}, j_{k^*} \right|^{1/2}} \times \left( \frac{1}{2} \right)_{n_{j^*}j_{j^*}j_{j^*}} \left( -1/2 \right)_{n_{j^*}j_{j^*}j_{j^*}}
\]

\[
\times \left( \frac{1}{2} \right)_{n_{j^*}j_{j^*}j_{j^*}} \Theta(i, j, k)
\]

and

\[
\Theta(i, j, k) = \left| j_{l^*}j_{j^*}j_{k^*} \right|^2 \sum_{m_{l^*}m_{k^*}} \times (-1)^{m_{l^*}+m_{k^*}} \left( \frac{1}{2} \right)_{n_{j^*}j_{j^*}j_{j^*}} \left( -1/2 \right)_{n_{j^*}j_{j^*}j_{j^*}}
\]

\[
\times \left( \frac{1}{2} \right)_{n_{j^*}j_{j^*}j_{j^*}} \left( -m_{l^*} \right)_{n_{j^*}j_{j^*}j_{j^*}} \left( m_{l^*} \right)_{n_{j^*}j_{j^*}j_{j^*}}
\]

The indices \( i, j, k \) denote the serial number of the photon which can take the values 1, 2, 3, the notation \([j, k, \ldots]\) means \((2j + l)(2k + 1)\ldots\).

Finally, the expression for the decay rate can be written in the form

\[
\frac{dW_f}{d\omega_1 d\omega_2} = \frac{\omega_3 \omega_2 \omega_1}{(2\pi)^3} \sum_{l^*_\alpha} \sum_{j_{l^*_\alpha}} \sum_{m_{l^*_\alpha}} \times \sum_{l_{l^*_\alpha}m_{l^*_\alpha}} S_{l^*/m^*}(1, 2, 3) + (5 \text{ permutations}) \times \left( \sum_{l_{l^*_\alpha}m_{l^*_\alpha}} S_{l_{l^*_\alpha}m_{l^*_\alpha}}(1, 2, 3) + (5 \text{ permutations}) \right) \frac{2}{\omega_3 \omega_2 \omega_1}
\]

The permutations in equation \((21)\) are understood as permutations of the indices 1, 2, 3.

The numerical results for the transition rates \(2\Gamma_{f/2} \rightarrow 1\Gamma_{l/2} + 2\gamma_{(E1)} \) and \(2\Gamma_{l/2} \rightarrow 1\Gamma_{l/2} + 2\gamma_{(E2)} \) are presented for the H-like ions with \(1 \leq Z \leq 95\) in tables 1, 2 respectively. Summation over the full set of one-electron states was performed within the B-spline approach \([11]\). The calculations were carried out in two relativistic ‘forms’, corresponding to the nonrelativistic ‘length’ and ‘velocity’ forms \([20]\); the results coincide with three–six digits. All values were checked for stability and convergence for the different length of spline basis set. In order to integrate over photon frequency, the 24 points of the Gauss–Legendre quadrature method was employed. For the summation over the Dirac spectrum set from 40 B-spline basis states of order nine were used. The frequency distributions of some transition probabilities are presented in figures 1–3.

### 3. Three-photon transitions between fine structure components

In this section we derive the expression for the three-photon decay rate in H-like ions for nuclei with nonzero spin \( I \) and consider transitions between different hyperfine sublevels of different fine structure levels. The matrix element between states with total angular momentum \( F_\alpha \) and \( F_\beta \) where \( |F_\alpha - I| \leq F \leq |F_\alpha + I| \) can be reduced to the form \([21]\):

\[
\left\{ n_{\alpha F_\alpha M_\alpha} \left[ Q_{j_{j^*}m_{j^*}} \right] n_{\beta F_\beta M_\beta} \right\} = (-1)^{F_{\alpha} - M_{\alpha} + j_{j^*} + 1} \left( F_{\alpha} \right)_{j_{j^*} M_{j^*} M_{\alpha}} \left( F_{\beta} \right)_{j_{j^*} M_{j^*} M_{\beta}}
\]

\[
\times \left( \frac{F_{\alpha}}{j_{j^*} M_{j^*} M_{\alpha}} \right) \left( \frac{F_{\beta}}{j_{j^*} M_{j^*} M_{\beta}} \right) \left( n_{\alpha j_{j^*} M_{\alpha}} \left| Q_{j_{j^*}m_{j^*}} \right| n_{\beta j_{j^*} M_{\beta}} \right)
\]

\[
\left\{ n_{\alpha j_{j^*} M_{\alpha}} \left| Q_{j_{j^*}m_{j^*}} \right| n_{\beta j_{j^*} M_{\beta}} \right\} = (-1)^{j_{j^*}} \left( F_{\alpha} \right)_{j_{j^*} M_{j^*} M_{\alpha}} \left( F_{\beta} \right)_{j_{j^*} M_{j^*} M_{\beta}}
\]

\[
\times \left( \frac{F_{\alpha}}{j_{j^*} M_{j^*} M_{\alpha}} \right) \left( \frac{F_{\beta}}{j_{j^*} M_{j^*} M_{\beta}} \right) \left( n_{\alpha j_{j^*} M_{\alpha}} \left| Q_{j_{j^*}m_{j^*}} \right| n_{\beta j_{j^*} M_{\beta}} \right)
\]

\[
\left( \frac{2j_{j^*} + 1}{2j_{j^*} + 1} \right) \left( \frac{2j_{j^*} + 1}{2j_{j^*} + 1} \right)
\]

Then the decay rate is

\[
\frac{dW_f}{d\omega_1 d\omega_2} = \frac{\omega_3 \omega_2 \omega_1}{(2\pi)^3} \sum_{l^*_\alpha} \sum_{j_{l^*_\alpha}} \sum_{m_{l^*_\alpha}} \sum_{l_{l^*_\alpha}m_{l^*_\alpha}} S_{l^*/m^*}(1, 2, 3) + (5 \text{ permutations}) \times \left( \sum_{l_{l^*_\alpha}m_{l^*_\alpha}} S_{l_{l^*_\alpha}m_{l^*_\alpha}}(1, 2, 3) + (5 \text{ permutations}) \right) \frac{2}{\omega_3 \omega_2 \omega_1}
\]
Table 2. Transition probabilities for $2s_{1/2} \rightarrow 1s_{1/2} + 3\gamma$ ($E2$) and $2s_{1/2} \rightarrow 1s_{1/2} + 2\gamma$ ($E1$) in s$^{-1}$ for different Z. The number in parentheses indicates the power of ten. Transition energies in eV are listed in the last column.

| Z  | $W_{\text{calc}}^{1s}$ | $W_{\text{calc}}^{2s}$ | $W^{2\gamma}$ | $\Delta E$ |
|----|----------------------|----------------------|--------------|-------------|
| 1  | 1.284270(−27)        | 1.284254(−27)        | 8.229063     | 10.204393   |
| 5  | 7.833773(−18)        | 7.833681(−18)        | 1.284705(5)  | 2.551846(3) |
| 10 | 1.281043(−13)        | 1.284925(−13)        | 8.200646(6)  | 1.021674(3) |
| 20 | 2.082775(−9)         | 2.087035(−9)         | 5.195127(8)  | 4.101829(3) |
| 30 | 6.001071(−7)         | 6.080097(−7)         | 5.821090(9)  | 9.287046(3) |
| 40 | 3.306380(−5)         | 3.307976(−5)         | 3.198616(10) | 1.665919(4) |
| 50 | 7.335633(−4)         | 7.337452(−4)         | 1.186615(11) | 2.634252(4) |
| 60 | 9.122989(−3)         | 9.131310(−3)         | 3.426453(11) | 3.851810(4) |
| 70 | 7.601921(−2)         | 7.602718(−2)         | 8.305995(11) | 5.343122(4) |
| 80 | 0.470047              | 0.470080              | 1.767273(12) | 7.144853(4) |
| 90 | 2.304994              | 2.305110              | 3.393473(12) | 9.308396(4) |
| 95 | 4.744875              | 4.745083              | 4.551134(12) | 1.054819(5) |

Figure 1. Three-dimensional plot for frequency distribution of the transition rate $dW/d\omega_1 = 2p_{1/2} (F = 0) \rightarrow 2s_{1/2} (F = 2) + 3\gamma (E1)$ in H-like ions with Z = 19. Nuclear spin $I = 3/2$. On the vertical axis the transition rate $dW/d\omega_1$ in s$^{-1}$ is plotted; on the horizontal axes the photon frequencies are plotted in units $\omega_1/\Delta$, $\omega_2/\Delta$ where $\Delta$ denotes the energy difference $\Delta = E(2p_{1/2}) - E(2s_{1/2})$. The lowest (zero) point is the point with coordinates $\omega_1/\Delta = 1$ at the bottom of the ‘pit’ in the frequency distribution for the transition rate which arises due to SSSR-2.

where

$$S_{F_{\delta},F_{\delta},\alpha}(i, j, k) = \sum_{l_{\alpha,\delta}} \sum_{n,a}^N$$

$$M_{F_{\delta},\alpha}(\omega_i, l_{\alpha}) M_{\alpha,\delta}(\omega_j, l_{\delta}) M_{\delta,\alpha}(\omega_k, l_{\alpha}) /$$

$$\left( E_{\delta_{\delta,\alpha}} - E_{\delta_{\delta,\alpha}} - \omega_k - \omega_j \right) \left( E_{\delta_{\delta,\alpha}} - E_{\delta_{\delta,\alpha}} - \omega_k - \omega_j \right) \times$$

$$\Delta F_{\alpha,\delta,\alpha,\delta,\alpha}(i, j, k) \pi F_{\delta}(i) \pi F_{\delta}(j) \pi F_{\delta}(k),$$

$$\frac{\Delta F_{\alpha,\delta,\alpha,\delta,\alpha}(i, j, k)}{dW/d\omega_1} =$$

$$\frac{(4\pi)^3/2}{\left[ j_{\gamma}, j_{\gamma}, j_{\gamma} \right]^{1/2}}$$

$$\times \left\{ j_{\delta}, j_{\delta}, j_{\delta} \right\}^{1/2}$$

$$\times \left\{ j_{\delta}, j_{\delta}, j_{\delta} \right\}^{1/2}$$

$$\times \left\{ F_{\delta}, F_{\delta}, F_{\delta} \right\}^{1/2}$$

$$\times \left\{ F_{\delta}, F_{\delta}, F_{\delta} \right\}^{1/2}$$

$$\times \left\{ F_{\delta}, F_{\delta}, F_{\delta} \right\}^{1/2}$$

$$\Theta(i, j, k)$$

(25)

Figure 2. Two-dimensional sectional cut of figure 1 at the frequency point $\omega_2/\Delta = 1/3$. All details are the same as in figure 1. The lowest (zero) point is the point with coordinates $\omega_1/\Delta = 1/3$ at the bottom of the ‘pit’ in the frequency distribution which arises due to SSSR-2.
transition rate which arises due to the nucleus expressed in units of nuclear magneton \( \mu_N = |e| \hbar/(2m_p c) \), \( m_e \) and \( m_p \) are the electron and proton mass, respectively. The values of \( I, \mu \) are also given in table 3.

These transitions could be interesting in view of the possible Bose–Einstein statistics tests for the multiphoton systems [12].

4. Application for the Bose–Einstein statistics tests

In [12] the SSSRs for the multiphoton atomic transitions with equivalent photons which present an extension of the LYT [13, 14] were formulated. These rules consist of the following. 1) SSSR-1: two equivalent photons involved in any atomic transition can have only even values for the total angular momentum \( J \); 2) SSSR-2: three equivalent dipole photons involved in any atomic transition can have only odd values for the total angular momentum \( J = 1, 3 \); 3) SSSR-3: four equivalent dipole photons involved in any atomic transition can have only even values for the total momentum values \( J = 0, 2, 4 \). It was established in [12] that SSSR-2, SSSR-3 do not hold, in general, for the photon multipolarity \( J > 1 \).

In [12], experiments on the testing of SSSRs with absorption processes in He-like ions of uranium were suggested, where the lasers can be used as a source for the equivalent photons. An advantage of the use of the laser source is that all photons will have the same frequency. If we divide this frequency by an integer number \( N \), and adjust the laser frequency \( \omega_l \) to the value of the transitions frequency \( \omega_{\text{trans}} = \omega_l/N \), the number of photons \( N \) in the absorption process will be fixed.

Using transitions with the initial total electron momentum \( J_i = 2 \) and the final momentum \( J_f = 0 \) we will fix the total momentum of the photon system \( J \) equal to \( J_i \). Then, choosing \( N_l = 3 \) and \( J_i = 2 \) we will test SSSR-2. According to SSSR-2 the value \( J = 2 \) for three equal photons is forbidden, so that the absorption of the laser light at the corresponding frequency \( \omega_l = \omega_{\text{trans}}/3 \) should be absent. SSSR-3 can be tested in the same way. Numerical examples with highly charged He-like ions were given in [12]. The photon frequencies in this case are in the x-ray region.

Experiments of this type can be extended to H-like ions with half-integer nuclear spin \( I \). The advantage of such experiments consists of the possibility of using optical range lasers. In the recent experiments with heavy atoms and ions [23] it is possible to measure the frequency distribution for the transition rates. In this case the value of the total angular momentum \( F \) for the \( N_l \)-photon system can be fixed by choosing the appropriate values \( F_f \) and \( F_i \) for the initial (lower) and final (upper) levels in the transition process.

In a laser beam all the possible multipolarities of a photon are presented. Thus it should produce all the transitions with the same total parity: E1E1E1, E1M1E2, E1E1M2 etc. However, the processes with the photons of higher multipolarities are usually strongly suppressed in atoms. Due to this suppression the E1E1E1 transition will be dominant. Measuring the absorption rate at the \( \omega_l = \omega_{\text{trans}}/N \) frequency, one can establish the validity or nonvalidity of the particular SSSRs: the atomic vapour should be transparent for the laser
Table 3. Transition probabilities for $2p_{3/2}(F = 0) \rightarrow 2s_{1/2}(F = 2) + 3\gamma(E1)$ and $2p_{3/2}(F = 0) \rightarrow 2s_{1/2}(F = 2) + \gamma(M2)$ in s$^{-1}$ for different H-like ions. Nuclear spin $I = 3/2$. The number in parentheses indicates the power of ten. Transition energies and hyperfine splitting constants $A$ (equation (29)) in eV are listed in the three last columns.

| Ion   | Abundance % | $\mu$ | $I$    | $W_{\text{el}}^{3/2}$ | $W_{\text{g}}^{3/2}$ | $W^{3/2}$ | $\Delta E$ | $A_{2p_{3/2}}$ | $A_{2s_{1/2}}$ |
|-------|-------------|-------|--------|-----------------------|-----------------------|-----------|------------|--------------|---------------|
| $^{33}\text{S}^{15+}$ | 0.75 | 0.64382 | $3/2$ | $8.944609 \times 10^{-20}$ | $8.944608 \times 10^{-20}$ | $6.406678 \times 10^{-6}$ | $2.992524$ | $2.61179 \times 10^{-4}$ | $2.67850 \times 10^{-4}$ |
| $^{35}\text{C}^{16+}$ | 93.3 | 0.82187 | $3/2$ | $3.404860 \times 10^{-19}$ | $3.404860 \times 10^{-19}$ | $1.917035 \times 10^{-5}$ | $3.817932$ | $4.00118 \times 10^{-4}$ | $4.11687 \times 10^{-4}$ |
| $^{39}\text{K}^{18+}$ | 75.8 | 0.39147 | $3/2$ | $3.963697 \times 10^{-18}$ | $3.963697 \times 10^{-18}$ | $1.434485 \times 10^{-4}$ | $5.971640$ | $5.59230 \times 10^{-4}$ | $5.79554 \times 10^{-4}$ |
light at the frequency $\omega_1 = \omega_0/\sqrt{N_c}$. Note also that, unlike the spontaneous emission which is very weak for multiphoton transitions, multiphoton absorption depends on the laser intensity and can be well observed in the experiments.

In our examples we considered transitions between fine structure components with a fixation of certain fine structure subcomponents. Varying the nuclear charge $Z$ we can find the situation when each photon will be in the optical range ($0.857 \text{ eV} - 3.27 \text{ eV}$). For example, to test the SSSRs, H-like ions with $Z = 16, 17, 19$ can be examined. In this case, transitions between fine structure components $2s_{1/2}(F = 2) \rightarrow 2p_{3/2}(F = 0) + 3\gamma(E1)$ can be chosen. The energy intervals between $2p_{3/2}(F = 0)$ and $2s_{1/2}(F = 2)$ states for $Z = 16, 17, 19$ are listed in the eighth column of table 3 and the corresponding $\omega_1$ for $N_c = 3$ is in the optical region. The nuclei of these ions are stable [24]. It is important that for $Z = 16, 17, 19$ the hyperfine splitting both for $2p_{3/2}$ and $2s_{1/2}$ are resolvable (see table 3) i.e. the transition $2s_{1/2}(F = 2) \rightarrow 2p_{3/2}(F = 0)$ can be well separated out. The $3\gamma$ transition rate value is $10^{13}$ times smaller than the one-photon M2 transition rate. However, tuning the laser frequency to one third of the transition frequency excludes one-photon absorption. Thus the frequency distribution depicted in figure 3 and its two-dimensional sectional cut, figure 2 should be observable in experiments of such type. The same picture arises for $1s_{1/2}(F = 0) \rightarrow 2p_{3/2}(F = 0) + 3\gamma(E1)$ transition in a neutral hydrogen atom ($Z = 1$) with $I = 1/2$ and $\mu = 2.793$ (see figure 3). In this case $\omega_1 = 3.40148 \text{ eV}$. For this transition the virtual states $ns_{1/2}(F = 1)$ and $np_{1/2}(F = 1)$ in the sum in equation (24) for $n = 2$ lie between the initial $2p_{3/2}(F = 2)$ and final $1s_{1/2}(F = 0)$ states, which leads to resonance (the situation where the energy denominator becomes zero). The presence of the cascade-producing states in the sum over the intermediate states in the transition amplitude leads to the arrival of the high, but narrow, 'ridge' in the frequency distribution $\frac{dW(\omega_1, \omega_2)}{d\omega_1 d\omega_2}$ [12]. The 'ridge' does not influence the SSSR-2: it does not correspond to the case of three equivalent photons. This is a general situation for all the possible cascade transitions. While in [6, 7] it was demonstrated that two photons behave like two bosons, the experiments suggested above would demonstrate that three photons also obey the Bose–Einstein statistics.

Acknowledgments

The work was supported by RFBR (grant no. 14-02-00188). TZ, DS and LL acknowledge support from St. Petersburg State University with a research grant 11.38.227.2014. The work of TZ was also supported by the nonprofit foundation ‘Dynasty’ (Moscow).

References

[1] Kramers H H and Heisenberg W 1925 Z. Phys. 31 681
[2] Waller I 1928 Z. Phys. 58 75
[3] Goeppert-Mayer M 1931 Ann. Phys. (Leipzig) 9 273
[4] Seager S, Sasselov D and Scott D 1999 Astrophys. J. Lett. 523 L1
[5] Chubja J and Sunyaev R A 2006 Astron. Astrophys. 446 39
[6] DeMille D, Budker D, Kerr N and Deveney E 1999 Phys. Rev. Lett. 83 3978
[7] English D, Yashchuk V V and Budker D 2010 Phys. Rev. Lett. 104 253604
[8] Dunford R W 2004 Phys. Rev. A 69 062502
[9] Kozlov M G, English D and Budker D 2009 Phys. Rev. A 80 042504
[10] Angom D, Bhattacharya K and Rindani S D 2007 Int. J. Mod. Phys. A 22 707
[11] Johnson W R, Blundell S A and Sapirstein J 1988 Phys. Rev. A 37 307
[12] Zalialiutdinov T, Solovyev D, Labzowsky L and Plunien G 2015 Phys. Rev. A 91 033417
[13] Landau L D 1948 Dokl. Akad. Nauk SSSR 60 207
[14] Yang C N 1950 Phys. Rev. 77 242
[15] Berestetskii V B, Lifshitz E M and Pitaevskii L P 1982 Quantum Electrodynamics (Oxford: Pergamon)
[16] Akhiezer A I and Berestetskii V B 1965 Quantum Electrodynamics (New York: Wiley)
[17] Andreev O Yu, Labzowsky L N, Plunien G and Solovyev D A 2008 Phys. Rep. 455 135
[18] Grant I P 1974 J. Phys. B: Atom. Mol. Phys. 7 1458
[19] Goldman S P and Drake G W F 1981 Phys. Rev. A 24 183
[20] Labzowsky L, Solovyev D, Plunien G and Soff G 2006 Eur. Phys. J. D 37 355
[21] Varshalovich D A, Moskalév A N and Khersonskii V K 1988 Quantum Theory of Angular Momentum (Singapore: World Scientific)
[22] Shabaev V M 1994 J. Phys. B 27 5825
[23] Mokler P H and Dunford R W 2004 Phys. Scr. 69 C1
[24] Stone N J 2005 At. Data Nucl. Data Tables 90 75–176