Natural Response of Non-smooth Oscillators Using Homotopy Analysis Combined with Galerkin Projections

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Abstract

Background Several problems from mechanical engineering, e.g., vibrations of a spring–mass system with unequal restraints, pendulum with impact, a gear-pair with backlash and friction, etc. are modeled using second-order differential equations involving discontinuous mathematical functions such as signum, Heaviside, modulus, etc. Several perturbation-like methods such as parameter expansion, homotopy perturbation, modified Lindstedt–Poincaré, and variational iteration have been applied successfully to get the periodic solution as well as the approximate analytical estimate of the natural frequency. The chief limitation of all the methods mentioned above is the poor approximation with the large value of the perturbation parameter.

Purpose The homotopy analysis method overcomes this limitation to a certain extent. It is usually plagued with a slow convergence issue. The task becomes impossible to handle computationally if one includes too many terms of the approximate series, especially dealing with oscillators involving non-smooth nonlinearities. The key issue now is how to accelerate the convergence of the series solution.

Methods To accelerate the convergence of the series solution, we think of the convergence-control parameter as a function of the embedding parameter and call it a convergence-control function. The usual treatment of the homotopy analysis provides an expression for the natural frequency of the oscillator that also includes free parameters arising due to the convergence-control function. Generating extra equations using Galerkin projections and solving the same numerically gives the approximate natural response of the non-smooth oscillators.

Results The proposed method yields an approximate natural response of non-smooth oscillators involving discontinuities of type Heaviside, signum, modulus, etc. The perturbation parameter range over which the approximate solution differs from the one obtained via numerical integration by less than 2 percent is the largest with our approach compared to other approaches like the method of harmonic balance, the Lindstedt–Poincaré method, non-smooth temporal transform, and the conventional homotopy analysis method. The framework developed has a natural extension to oscillators with no periodic response, e.g., the unilaterally constrained simple pendulum where the solutions are decaying with time but are scalable.

Conclusion The superiority of our approach is well-established over a much larger range of the perturbation parameter compared to the usual homotopy analysis method.

Keywords Non-smooth · Natural response · Homotopy analysis · Galerkin projections

Introduction

Oscillatory motion is typically governed by a second order non-smooth differential equation. Several problems from mechanical engineering, e.g., vibrations of a spring-mass system with unequal restraints, a pendulum with impacts, a gear-pair with backlash and friction etc. are modelled using second order differential equations involving discontinuous mathematical functions such as signum, Heaviside, modulus etc. Natural response of such oscillators may be
obtained with ease using numerical integration. However, some analytical or semi-analytical methods allow us to retain some parameter dependence in our approximations. Several perturbation-like methods such as parameter expansion, homotopy perturbation, modified Lindstedt–Poincaré, variational iteration have been applied successfully to get the periodic solution as well as the approximate analytical estimate of the natural frequency [5, 21]. In the context of non-smooth oscillators, such methods are typically applied to find periodic solutions [12, 20]. For example, the homotopy perturbation method has been successfully applied to obtain the solution of a nonlinear oscillator where the discontinuity is modelled using the modulus function [6]. The variational iteration technique (VIM) is applied to several strongly nonlinear oscillators including ones involving discontinuities [4, 16, 18]. Frequencies of limit cycles of several non-smooth oscillators have been determined using an artificial parameter-based Lindstedt–Poincaré method [17]. Non-periodic responses where amplitude decays with time or a frequency does not remain constant are harder to approximate using such methods. One such example is a unilaterally constrained simple pendulum with rigid collisions modelled using Newton’s impact law. The constraint is usually not accommodated in the governing equation, thereby restricting the applicability of such methods.

Here, we develop a framework that combines the homotopy analysis method (HAM) with Galerkin projections to obtain not only the periodic response of certain non-smooth oscillators, but also that of the unilaterally constrained pendulum where the solutions are not periodic. The deformation is parametrized using an embedding parameter \( p \in [0, 1] \), deforming the solution of a linear operator \( (p = 0) \) to that of the non-smooth oscillator or the desired solution \( (p = 1) \). To approximate the periodic responses, our choice of the linear operator is a simple harmonic oscillator. While obtaining the periodic response via HAM, we construct the solution at every order over only one period and also remove the secular terms over the same. Since the response is periodic, extension to subsequent periods is automatic and requires no further analysis. To improve the solution at \( p = 1 \), we introduce a convergence-control function \( h(p) \), which is usually considered to be a constant. Upon following the homotopy procedure, different derivatives of this function evaluated at \( p = 0 \) appear naturally as unknowns in the expressions for the periodic solution as well as the natural frequency. When applied up to \( n \)-th order deformation equation, our method produces \( n \) unknowns compared to HAM which produces just one \( h \). To determine these unknowns, we apply Galerkin projections with appropriate weighting functions and generate extra equations, usually non-algebraic in the context of non-smooth oscillators. Solving the system of equations numerically yields the desired periodic response and the natural frequency. However, this last step makes the procedure analytical-numerical. Compared against the conventional HAM, the technique developed here accelerates the convergence of the solution. This is important while dealing with problems that cannot be treated beyond second order due to analytical difficulties.

Our work comes closest to the optimal homotopy asymptotic method (OHAM). The method has been applied to many problems including non-smooth oscillators [2, 7–9, 14]. As summarized in [7], OHAM involves an auxiliary function, similar to our convergence-control function \( h(p) \), but more general by assuming it to be the function of both \( p \) and time \( t \). Authors assume it to be a polynomial in \( p \) with unknown coefficients which are to be found using one of the methods of weighted residual. Convergence crucially depends on the assumed form. Also nonlinear operator in the homotopy involves an arbitrary parameter which when determined using the principle of minimal sensitivity gives essentially the natural frequency of the periodic response. Our approach is simpler and less prescriptive. We do not assume any form of the convergence-control function; instead, homotopy analysis naturally provides the unknowns to be found using the chosen method of weighted residual. We introduce a time-stretching function that is assumed to be a function of \( p \), which eliminates the need for any arbitrary parameter in the nonlinear operator. We also differ in the important step of the removal of secular terms from their work, as will be seen below.

We begin by modelling the oscillations of a spring-mass system with asymmetric restraints using the Heaviside function and obtain the approximate periodic solution using Lindstedt–Poincaré perturbation method (“A Spring-Mass System with Asymmetric Elastic Restraints”). We then develop the homotopy analysis and Galerkin projections framework for the same oscillator in “Natural Response of Eq. 1 Using Homotopy Analysis and Galerkin Projections”. We also compare our results with those obtained using several other comparable methods and demonstrate the superiority of the approximations obtained using our approach. In “Natural Response of Oscillators Involving Non-smoothness of Type Signum and Modulus”, we consider periodic solutions of oscillators with non-smoothness of type signum and modulus following the same framework developed in the previous section. Finally in “Natural Response of an Impact Oscillator”, we handle the unilaterally constrained simple pendulum modelling it using a Dirac-delta-like function. The decaying response of this oscillator is captured using a homotopy that involves a carefully chosen damped linear oscillator as the reference problem. The decaying oscillations of the impacting pendulum represent an usual application of our method. We compare the natural frequency obtained for each oscillator with the one obtained via methods such as numerical integration (MATLAB ode45), Lindstedt–Poincaré perturbation, the method of harmonic balance, averaging combined with non-smooth temporal transform and HAM. The comparisons illustrate the usefulness of our framework. Solutions of oscillators discussed in “Natural
We attempt to approximate a periodic solution to
\[ x'' + (k_2 - k_1)H(x) + k_1 x = 0, \]
where \( H \) denotes the Heaviside function. Defining \( \epsilon = \frac{k_2}{k_1} - 1 \) and scaling time by \( \sqrt{\frac{k_2}{m}} \), we simplify the above to
\[ \ddot{x} + (1 + \epsilon H(x))x = 0, \]
where the overdot denotes differentiation w.r.t. to the scaled time. Equation (1) is a non-smooth oscillator: linearisation near \( x = 0 \) is not possible. However, the solutions are scalable. Therefore, unlike a typical nonlinear oscillator, the natural frequency is independent of initial conditions. We attempt to approximate the frequency using Lindstedt–Poincaré perturbation method here, and using the homotopy analysis method combined with Galerkin projections in the next section.

**Fig. 1** A spring-mass system with non-smoothness of type Heaviside

### A Spring-Mass System with Asymmetric Elastic Restraints

#### The System

Consider mass \( m \) kept on a frictionless surface restrained by two springs of stiffness \( k_1 \) and \( k_2 \) \((k_1 \neq k_2)\) as shown in Fig. 1. The mass vibrates about the reference position \( x = 0 \) for any non-zero initial condition. While \( x > 0 \), the mass loses contact with the spring of stiffness \( k_1 \), and likewise loses contact with the spring of stiffness \( k_2 \) while \( x < 0 \). The system stiffness \( k \) is thus given by
\[ k = \begin{cases} \frac{k_1}{k_2} & x \leq 0 \\ \frac{k_2}{k_1} & x > 0. \end{cases} \]

Applying the force balance and modelling the system stiffness discontinuity using the Heaviside function, we arrive at the equation governing the motion,
\[ m\ddot{x} + (k_2 - k_1)H(x) + k_1 x = 0, \]
where \( H \) denotes the Heaviside function. Defining \( \epsilon = \frac{k_2}{k_1} - 1 \) and scaling time by \( \sqrt{\frac{k_2}{m}} \), we simplify the above to
\[ \ddot{x} + (1 + \epsilon H(x))x = 0, \]
where the overdot denotes differentiation w.r.t. to the scaled time. Equation (1) is a non-smooth oscillator: linearisation near \( x = 0 \) is not possible. However, the solutions are scalable. Therefore, unlike a typical nonlinear oscillator, the natural frequency is independent of initial conditions. We attempt to approximate the frequency using Lindstedt–Poincaré perturbation method here, and using the homotopy analysis method combined with Galerkin projections in the next section.

#### Natural Response of Eq. (1) Using the Lindstedt–Poincaré Perturbation Method

Lindstedt–Poincaré (LP) method is a singular perturbation technique, which is employed for obtaining periodic solutions of weakly nonlinear oscillators. It is unusual to apply the LP method to a non-smooth oscillator; however, a few attempts have been made [19]. Here, we assume values of spring stiffness \( k_1 \) and \( k_2 \) such that the perturbation parameter assumes \( 0 < \epsilon \ll 1 \). We attempt to approximate a periodic solution to Eq. (1) with initial condition \((1, 0)\) as the solutions to Eq. (1) are scalable. We begin by scaling time \( t \) to \( \tau = \Omega t \) with
\[ \Omega = 1 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \mathcal{O}(\epsilon^3), \]
where \( \omega_i \)'s are unknowns. Equation (1) in the scaled time is
\[ \Omega^2 \ddot{x}_{\tau\tau} + (\epsilon H(x) + 1)\dot{x}_\tau = 0, \]
where the subscript \( \tau \) denotes differentiation w.r.t. \( \tau \). The solution to Eq. (1) in the scaled time \( \tau \) is assumed to be a series strictly asymptotic in \( \epsilon \) and is expressed as
\[ x(\tau) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + \mathcal{O}(\epsilon^3), \]
where each \( x_i(\tau), i = 0, 1, \ldots \) is periodic and \( \mathcal{O}(1) \) for all \( \tau \). Substituting Eqs. (2) and (4) in Eq. (3), we get
\[ \begin{align*}
&x_{0,\tau\tau} + x_0 + \epsilon \left( x_0 H(x_0) + \epsilon x_1 + \epsilon^2 x_2 + x_1 + 2\omega_1 x_{0,\tau\tau} + x_{1,\tau\tau} \right) + \\
&\epsilon^2 \left( x_1 H(x_0) + \epsilon x_1 + \epsilon^2 x_2 + x_2 + 2\omega_1^2 x_{0,\tau\tau} + 2\omega_2 x_{0,\tau\tau} + \\
&2\omega_1 x_{1,\tau\tau} + x_{2,\tau\tau} \right) + \ldots = 0.
\end{align*} \]

To obtain the coefficients at various orders of \( \epsilon \) explicitly from the above, we use the assumption of \( 0 < \epsilon \ll 1 \), and approximate
\[ H(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) \approx H(x_0) + \epsilon x_1 \delta(x_0) + \epsilon^2 \left( \delta(x_0) x_2 + \frac{1}{2} \delta'(x_0) x_2^2 \right). \]

The above approximation may become exact for certain oscillators, for example, Eq. (28). Simplifying Eq. (5) using Eq. (6) and then collecting the coefficients of different powers of \( \epsilon \), we get at various orders
\[ \begin{align*}
\mathcal{O}(\epsilon^0) & : \quad x_{0,\tau\tau} + x_0 = 0, \\
\mathcal{O}(\epsilon^1) & : \quad x_{1,\tau\tau} + x_1 = -x_0 H(x_0) - 2\omega_1 x_{0,\tau\tau}, \\
\mathcal{O}(\epsilon^2) & : \quad x_{2,\tau\tau} + x_2 = -x_1 H(x_0) - \omega_1^2 x_{0,\tau\tau} - x_0 x_1 \delta(x_0) - 2\omega_2 x_{0,\tau\tau} - 2\omega_1 x_{1,\tau\tau}.
\end{align*} \]
Solving Eq. (7a) for \( x_0(\tau) \) with initial condition \((1, 0)\), we get
\[
x_0(\tau) = \cos \tau.
\]
Substituting the above expression for \( x_0(\tau) \) in Eq. (7b), we obtain
\[
x_{1,\tau} + x_1 = -\cos \tau H(\cos \tau) + 2\omega_1 \cos \tau. \tag{8}
\]
Removal of secular terms, the central aspect of any singular perturbation method is not that straightforward for Eq. (8). Since \( x(\tau) \) and hence \( x_1(\tau) \) are periodic with period \( 2\pi \), it suffices to construct both over one time period, \( \tau \in [0, 2\pi] \). By restricting \( H(\cos \tau) \) over \( \tau \in [0, 2\pi] \), the r.h.s. of Eq. (8) maybe modified as
\[
x_{1,\tau} + x_1 = -\cos \tau \left( 1 - H\left( \tau - \frac{\pi}{2} \right) + H\left( \tau - \frac{3\pi}{2} \right) \right) + 2\omega_1 \cos \tau =: F_1(\tau). \tag{9}
\]
Equation (9) is a restricted version of Eq. (8) over the interval \( \tau \in [0, 2\pi] \). It is forced externally at its natural frequency. We eliminate the secular term from the right hand side of Eq. (9) using
\[
\int_0^{2\pi} F_1(\tau) \cos \tau d\tau = 0. \tag{10}
\]
Solving Eq. (10) for \( \omega_1 \) gives
\[
\omega_1 = \frac{1}{4}.
\]
Solving Eq. (9) for \( x_1(\tau) \) with \( \omega_1 = \frac{1}{4} \) and initial condition \((0, 0)\), we get
\[
x_1(\tau) = \left( \frac{1}{2} \cos \tau + \left( -\frac{\pi}{4} + \frac{\tau}{2} \right) \sin \tau \right) H\left( \tau - \frac{\pi}{2} \right) + \left( -\frac{1}{2} \cos \tau + \left( \frac{3\pi}{4} - \frac{\tau}{2} \right) \sin \tau \right) H\left( \tau - \frac{3\pi}{2} \right) - \frac{\tau}{2} \sin \tau.
\]
Though the expression for \( x_1(\tau) \) is linearly growing w.r.t. \( \tau \), it is valid only over the finite interval \([0, 2\pi]\). Repeating the above steps for Eq. (7c), we get
\[
\omega_2 = -\frac{1}{8}
\] and
\[
x_2(\tau) = \left( \frac{-\pi^2}{32} + \frac{\pi \tau}{16} - \frac{1}{8} \right) \cos \tau + \left( \frac{\pi}{16} - \frac{\tau}{8} \right) \sin \tau \right) H\left( \tau - \frac{\pi}{2} \right) + \left( \frac{3\pi^2}{32} + \frac{\pi \tau}{16} + \frac{1}{8} \right) \cos \tau + \left( -\frac{3\pi}{16} + \frac{\tau}{8} \right) \sin \tau \right) H\left( \tau - \frac{3\pi}{2} \right) - \frac{\tau}{2} \cos \tau + \frac{\tau}{16} \sin \tau.
\]
Substituting for \( \omega_1 \) and \( \omega_2 \) in Eq. (2), we get
\[
\Omega = 1 + \frac{\epsilon}{4} - \frac{\epsilon^2}{8} =: \omega_{LP}. \tag{11}
\]
Finally, substituting the expressions for \( x_0(\tau) \), \( x_1(\tau) \) and \( x_2(\tau) \) in Eq. (4) and re-scaling time back to \( t \), we obtain one period of the periodic solution. Figure (2) shows the solution obtained via LP as well as via MATLAB numerical integrator ode45 for two values of \( \epsilon \). It can be seen clearly that as \( \epsilon \) increases, the match between the two solutions deteriorates. One of the major reasons for this mismatch, apart from the loss of asymptotic nature of the series (Eq. (4)) is the approximation Eq. (6) (see Fig. 2).

**Natural Response of Eq. 1 Using Homotopy Analysis and Galerkin Projections**

Smallness of the perturbation parameter \( \epsilon \) that plays a crucial role in the Lindstedt–Poincaré method is not a restriction as far as the application of homotopy analysis method (HAM) to Eq. (1) is considered. We again attempt to find a periodic solution \( x(t) \) to Eq. (1) with initial condition \((1, 0)\). We begin by introducing an embedding parameter \( p \in [0, 1] \), deformation \( \tilde{x}(t;p) \) and construct homotopy \( \mathcal{H} \) as
\[
\mathcal{H} \equiv (1 - p)\mathcal{L}\left( \tilde{x}(t;p) \right) - h(p)\mathcal{N}\left( \tilde{x}(t;p) \right) = 0. \tag{12}
\]
Here, we choose \( h(0) = 0 \) and the linear operator \( \mathcal{L} \equiv \tilde{x} + x \). Nonlinear operator \( \mathcal{N} \) is set to Eq. (1). As \( p \) varies continuously from 0 to 1, the deformation \( \tilde{x}(t;p) \) varies continuously from the solution of \( \mathcal{L}\left( \tilde{x}(t;0) \right) = 0 \) to the solution of \( \mathcal{N}\left( \tilde{x}(t;1) \right) = 0 \). \( \tilde{x}(t;1) \) is the desired periodic solution \( x(t) \).

Usual application of homotopy analysis considers the convergence-control parameter \( h \) as a constant. We introduce it in Eq. (12) as a function of the embedding parameter \( p \) and hence call it a convergence-control function. Making it a function of \( p \) introduces extra unknowns (its derivatives at \( p = 0 \)) which are to be found using extra equations via
The solution to Eq. (1) in this framework is Galerkin projections. We subject it to conditions

\[ G_0 \text{ and } h(t) \neq 0. \]

We scale time \( t \) to \( \tau = \omega(p)t \) and introduce a time-stretching function \( \lambda(p) = \frac{1}{\omega(p)} \). We also choose \( \lambda(0) = 1 \). The frequency of the solution to Eq. (1) in this framework is \( \omega(1) = \frac{1}{\lambda(1)} \). Homotopy with respect to the scaled time is

\[ \mathcal{H} \equiv (1 - p)\left( \ddot{x} + \lambda(p)^2 \dot{x} \right) - h(p)\left( \ddot{x} + \lambda(p)^2 (\epsilon H(x) + 1) \dot{x} \right) = 0. \]

Taylor-expanding \( \ddot{x}(\tau; p) \) about \( p = 0 \), we have

\[ \ddot{x}(\tau; p) = \ddot{x}^{[0]}(\tau) + \sum_{n=1}^{\infty} \frac{1}{n!} \ddot{x}^{[n]}(\tau) p^n, \]

where \( \ddot{x}^{[0]}(\tau) = \ddot{x}(\tau; 0) \) and \( \ddot{x}^{[n]}(\tau) = \frac{\partial^n \ddot{x}}{\partial p^n} |_{p=0} \).

Here as well, we expect all \( \ddot{x}^{[n]}(\tau) \) to be periodic. Differentiating Eq. (14) w.r.t. \( \tau \), we get

\[ \ddot{x} = \ddot{x}^{[0]}(\tau) + \sum_{n=1}^{\infty} \frac{1}{n!} \ddot{x}^{[n]}(\tau) p^n. \]

We also Taylor-expand the time-stretching and convergence-control functions

\[ \lambda(p) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \lambda^{[n]}(p) p^n \quad \text{where} \quad \lambda^{[n]}(p) = \frac{d^n \lambda(p)}{dp^n} |_{p=0}, \]

\[ h(p) = \sum_{n=1}^{\infty} \frac{1}{n!} h^{[n]}(p) p^n \quad \text{where} \quad h^{[n]}(p) = \frac{d^n h(p)}{dp^n} |_{p=0}. \]

We begin by obtaining \( x^{[0]}(\tau) \). Substituting \( p = 0 \) in Eq. (13), we get the zeroth-order deformation equation

\[ x^{[0]}(\tau) + \dot{x}^{[0]}(\tau) = 0. \]

By choosing the initial conditions for the above oscillator as \((1, 0)\), we get

\[ x^{[0]}(\tau) = \cos \tau. \]

Our choice of the initial condition for zeroth-order deformation equation along with Eqs. (14) and (15) fixes the initial conditions for the higher order deformation equations as

\[ x^{[n]}(0) = 0, \quad x^{[n]}(0) = 0, \quad n \geq 1. \]

To get the first-order deformation equation, we differentiate Eq. (13) w.r.t. \( p \) once. Then substituting \( p = 0 \), and using Eqs. (14), (16), (17) and (19), we get

\[ x^{[1]}(\tau) = \left( H(\cos \tau) e h^{[1]} - 2 \lambda^{[1]} \right) \cos \tau. \]

As we did in the previous section, we simplify the above equation by restricting it to the interval of interest, i.e., \( \tau = [0, 2\pi] \). The first-order deformation equation is then modified to

\[ x^{[1]}(\tau) = \left( 1 - H(\tau - \frac{\pi}{2}) + H(\tau - \frac{3\pi}{2}) \right) e h^{[1]} - 2 \lambda^{[1]} \cos \tau =: F_1(\tau). \]

Removing the secular term using Eq. (10), we fetch

\[ \lambda^{[1]} = \frac{e h^{[1]}}{4}. \]

We solve Eq. (21) using initial condition from Eq. (20) to get
Following the same procedure, we proceed up to second order and eliminate the secular term to obtain

\[ \lambda^{[2]} = \frac{e}{2} \left( h^{[1]} + \frac{h^{[2]} + h^{[1]^2}}{2} \right) + \frac{3e^2 h^{[1]^2}}{8}. \]

Substituting \( p = 1 \) in Eq. (16) and expressions for \( \lambda^{[1]} \), \( \lambda^{[2]} \) from the above, we get

\[ \lambda(1) = 1 + \frac{e}{2} \left( \frac{3h^{[1]} + h^{[2]} + h^{[1]^2}}{2} \right) + \frac{3e^2 h^{[1]^2}}{8} + \ldots. \]

The frequency of the desired solution \( \omega(1) \) is then given by

\[
\omega(1) = \frac{1}{\lambda(1)} \approx 1 - \frac{e}{2} \left( \frac{3h^{[1]} + h^{[2]} + h^{[1]^2}}{2} \right) + e^2 \left( \frac{1}{4} \left( \frac{3h^{[1]} + h^{[2]} + h^{[1]^2}}{2} \right)^2 - \frac{3h^{[1]^2}}{8} \right) =: \omega_{HG}.
\]

Ignoring higher order terms, substituting \( p = 1 \) in Eq. (14), using Eqs. (19) and (22) and scaling time back to \( t \), we get

\[
\ddot{x}(t; 1) \approx x^{[0]}(t) + x^{[1]}(t) =: x_1(t).
\]

Subscript 1 here denotes that we approximate solution by solving up to first-order deformation equation. The solution \( x_1(t) \) as obtained using Eq. (24) is restricted to \( t \in [0, \frac{2\pi}{\omega(1)}] \).

Since we approximate one period of the periodic solution, we have the desired solution for all time \( t \). The resulting \( x_1(t) \) is not only a function of \( \omega(1) \) but also of free parameters \( h^{[1]} \) and \( h^{[2]} \). Therefore, we need more equations in addition to Eq. (23) involving these unknowns. We use the Galerkin method of the weighted residual to obtain these equations. Using the expression for \( x_1(t) \) from Eq. (24), we get the residual

\[ \mathcal{R}_1(t) \equiv \ddot{x}_1(t) + x_1(t)(eH(x_1(t)) + 1). \]

For the Galerkin scheme, we choose two weighting functions

\[ w_1(t) = \cos \left( \frac{\omega(1)t}{2} \right) \quad \text{and} \quad w_2(t) = H\left( \frac{\omega(1)t}{2} - \frac{\pi}{2} \right) \]

and impose

\[ \int_0^{\frac{2\pi}{\omega(1)}} w_i(t) \mathcal{R}_1(t) dt = 0, \quad i = 1, 2. \]

The integrand in the above has a coefficient \( H(x_1(t)) \), whose zeros must be analytically determined for the integration. Since the zeros of \( x_1(t) \) are apriori unknowns, they may be approximated. One straightforward approximation is to replace the argument of Heaviside by the solution to the zeroth-order deformation equation in time \( t \), i.e., \( x^{[0]}(t) \). We propose better approximation of zeros of \( x_1(t) \) over the interval \([0, \frac{2\pi}{\omega(1)}]\) using one iteration of the Newton–Raphson scheme. Denoting the two zeros by \( \ell_1^{[1]} \) and \( \ell_1^{[2]} \) given by

\[
\ell_1^{[1]} = \frac{\pi}{2\omega(1)} - \frac{x_1(t)}{x_1(t)} \bigg|_{x_1(t) = \frac{\pi}{2\omega(1)}} \quad \text{and}
\]

\[
\ell_1^{[2]} = \frac{3\pi}{2\omega(1)} - \frac{x_1(t)}{x_1(t)} \bigg|_{x_1(t) = \frac{3\pi}{2\omega(1)}}
\]

we approximate \( H(x_1(t)) \) by,

\[
H(x_1(t)) \approx 1 - H(t - \ell_1^{[1]}) + H(t - \ell_1^{[2]}).
\]

We assume that the expressions for \( \ell_1^{[1]} \) and \( \ell_1^{[2]} \), which are numerically unknown, lie close to \( \frac{\pi}{2\omega(1)} \) and \( \frac{3\pi}{2\omega(1)} \) respectively. This results in \( H(x_1(t)) = 0 \) between \( \ell_1^{[1]} \) and \( \ell_1^{[2]} \) and \( H(x_1(t)) = 1 \) everywhere else over \([0, \frac{2\pi}{\omega(1)}]\). The weighted residual with the approximate expression for \( H(x_1(t)) \) can now be integrated. Note that all terms in \( \mathcal{R}_1(t) \) are of type \( \sin(\omega(1)t) \) or \( \cos(\omega(1)t) \) or similar. Therefore, after putting the limits of integration, unknown \( \omega(1) \) is eliminated and results in \( -\cos(2\pi) \) or \( \sin(2\pi) \) or similar. The integral so obtained has Heaviside terms with arguments involving \( \omega(1) \), \( h^{[1]} \) and \( h^{[2]} \). Fixing \( e \), we solve the system of above equations and Eq. (23) simultaneously for the unknowns using fsolve command of symbolic algebra package Maple. In order to avoid unwanted roots and reduce the computation burden, we fix the interval for \( \omega(1) \) in the neighbourhood of 1 or –1.

Since the solutions to Eq. (1) are scalable, the frequency of its solution is independent of initial conditions. This may be verified by applying the homotopy analysis and Galerkin projections (HG) framework starting with an arbitrarily chosen initial condition instead of \((1, 0)\).

We find \( x_1(t) \) for two values of \( e \) and compare it with the solution obtained using ode45 (built-in integrator of MATLAB) in Fig. 3. We see that the solution by homotopy analysis and Galerkin projections (HG) mimics the numerical
solution for $\epsilon$ as large as 3.2. As $\epsilon$ increases, we observe the mismatch between the two solutions in terms of both frequency and amplitude, noticeable at $\epsilon = 12.8$. This can be attributed to a number of factors: a significant contribution to the error is due to the approximation of $H(x(t))$ (Eq. 25). This error may be minimized by either finding a better alternative to Eq. (25), for example, by applying the Newton–Raphson iteration more than once. The choice of weighting functions also plays a significant role in minimizing the error. Here, we set the weighting functions as the basis functions of the solution from HG with coefficients that seem to contribute maximum to the solution amplitude. Lastly, the method is carried out only up to the first order; however at higher orders, the computational burden increases significantly. In the LP framework, since the series solution (Eq. 4) is asymptotic in $\epsilon$, the validity of the method is restricted to smaller values of $\epsilon$. For HG method, the Taylor expansion (Eq. 14) involves the free parameters that are determined using a heuristic approach of Galerkin projections. As a result Eq. (14) is not guaranteed to be asymptotic in $\epsilon$.

Although application of Galerkin projections in HG demands the approximation Eq. (25), the contribution to error is not as significant. One of the major differences between LP and HAM/HG is the nature of the expression for frequency. LP yields an expression that is purely analytic. HAM gives an analytical expression involving the free parameters which must be determined numerically using Galerkin projections, thereby making it semi-analytic and computationally more expensive. Nevertheless, there is one possibility to obtain an analytical expression of the frequency $\omega(1)$, solely in terms of $\epsilon$: by applying HG to find $\omega(1)$ numerically for several values of $\epsilon$ (like in Fig. 4) and then interpolating the numerically obtained values using a suitable basis function.

Now we compare our approach (HG) to other well-known approaches like non-smooth temporal transformation (NSTT), harmonic balance and conventional HAM.

We consider the oscillator (1) written in the amplitude-phase co-ordinates $[A(t), \varphi(t)]$ as

$$\dot{A} = \frac{\epsilon A}{2} H(A \cos \varphi) \sin 2\varphi,$$

$$\dot{\varphi} = 1 - H(A \cos \varphi) \cos^2 \varphi. \quad (26)$$

Here NSTT can be applied to $\varphi(t)$ using a triangular sine wave $\epsilon(2\varphi/\pi) = (2/\pi) \arcsin \sin(\varphi)$ and a rectangular cosine wave $\epsilon(2\varphi/\pi) = \text{sgn}(\cos(\varphi))$, transforming Eq. (26) to

$$\dot{A} = \frac{\epsilon A}{4} (1 + \epsilon) \sin(\pi \tau),$$

$$\dot{\varphi} = 1 + \frac{\epsilon(1 + \epsilon)}{2} \cos^2(\pi \tau/2). \quad (27)$$

Proposing the above NSTT transformation, the author in [15] subjects Eq. (27) to Krylov-Bogolyubov averaging, thus leading to a closed-form analytical expression for the phase and amplitude computed up to $O(\epsilon)$. The resultant solution
The method of harmonic balance is a special case of Galerkin projections. We also obtain the natural frequency w.r.t. $\epsilon$ using two-term harmonic balance (2T-HB) to Eq. (3) by taking the ansatz as $A_1 \cos \omega t + A_2 \cos 3\omega t$. To determine the unknowns, we compute the residual, generate equations by fixing the initial condition to $(1, 0)$ and collect the coefficients of $1, \cos 2t, \cdots$ from the residual. The resultant system of algebraic equations has multiple roots. To determine the frequency for different values of $\epsilon \in [0, 5]$, we search for the same in a small neighbourhood of 1 and plot the results in Fig. 4. Alternatively, we can also avoid multiple roots using the method of generalized harmonic balance method [1].

Figure 4 compares the frequencies obtained via ode45, LP, NSTT+KB, HAM and 2T-HB and HG over $\epsilon \in [0, 6]$. Treating the frequency obtained via ode45 as the most exact, we determine the % error in the frequencies given by all the above mentioned techniques. We define $\epsilon$-usability range as the range of parameter $\epsilon$ over which the approximate frequency obtained from a given method is in agreement with the same obtained via ode45 within 2%. This range for LP is $\epsilon \in [0, 0.8]$, for NSTT+KB $\epsilon \in [0, 1]$, for 3-rd order HAM $\epsilon \in [0, 3]$, for 2T-HB $\epsilon \in [0, 2]$ and for HG $\epsilon \in [0, 6]$. This establishes the superiority of the hybrid approach of HG over purely asymptotic as well as purely heuristic approaches, though in the limited context of dealing with non-smooth oscillators. Evidently, we observe that NSTT+KB applied up to $O(\epsilon)$ fares better than LP applied up to $O(\epsilon^2)$. Also, 2-n.d order HG offers $\epsilon$-usability range that is twice that of 3-rd order HAM. Using a suitable method of weighted residual to solve for the higher number of unknowns by assuming convergence-control as a function of the embedding parameter, ensures the tighter control over the error leading to more accurate estimate of the natural frequency compared against conventional HAM, thus establishing the accelerated convergence effectively.

**Natural Response of Oscillators Involving Non-smoothness of Type Signum and Modulus**

**Signum Type Discontinuity**

We consider an oscillator

$$\ddot{x} + \epsilon x^2 \text{sgn}(x) = 0, \quad (28)$$

where $\text{sgn}$ denotes the signum function. Re-writing the above equation using the Heaviside function, we have

$$\ddot{x} + \epsilon x^2 \left(2H(x) - 1\right) = 0. \quad (29)$$
The above is now treated using HG framework developed in “Natural Response of Eq. 1 Using Homotopy Analysis and Galerkin Projections”. Choosing the linear operator as a simple harmonic oscillator and nonlinear operator as Eq. (29), we construct the homotopy $\mathcal{H}$ and scaling time from $t$ to $\tau = \frac{t}{\Delta(p)}$, we get

$$\mathcal{H} \equiv (1 - p)\left(\ddot{x}_{\tau} + \lambda p \dot{x}\right) - h(p)\left(\ddot{x}_{\tau} + \epsilon \lambda p \dot{x}^2 \left(2H(\tilde{x}) - 1\right)\right) = 0.$$  

The solutions to Eq. (28) are not scalable. The oscillator is autonomous though and thereby choosing initial phase to be zero, we attempt a periodic solution with initial condition $(A, 0)$. Computing upto the first order, we get approximate expressions for the natural frequency and the solution as

$$\omega(1) = 1 + \frac{h^{[1]} + \frac{1}{3} h^{[2]} + \frac{1}{9} h^{[1]^2}}{4} - \frac{2\epsilon A}{3\pi} h^{[1]} + \frac{h^{[2]} + 1}{8} - \frac{4\epsilon A^2}{9\pi^2} \left(30 \pi - 32\right) h^{[1]^2} + 4\left(4 h^{[1]} + h^{[2]} + h^{[1]^2}\right)^2,$$

$$x_1(\tau) = A \cos \tau + \epsilon A^2 h^{[1]} \left(\frac{1}{3} \cos 2\tau + \frac{4}{3} \sin \tau - 1\right) H\left(\frac{\tau - \pi}{2}\right) + \left(-\frac{1}{3} \cos 2\tau + \frac{4}{3} \sin \tau + 1\right) H\left(\frac{\tau - 3\pi}{2}\right) - \frac{1}{3} \cos \tau - \frac{1}{6} \cos 2\tau - \frac{4}{3\pi} \sin \tau + \frac{1}{2},$$

where $\tau = \omega(1)t$.

We determine the free parameters $h^{[1]}$ and $h^{[2]}$ using Galerkin projections as before using the weighting functions $\cos(\omega(1)t)$ and $H\left(\omega(1)t - \frac{\pi}{2}\right)$, and then solve a system of equations numerically. These unknowns are computed for two values of $\epsilon$ and are given in Table 2. For the same values of $\epsilon$, Fig. 6 shows the comparison between the solution so obtained and the solution via ode45.

Here we highlight that

The above obtained expression for $x_1(t)$ has zero-crossings at $t = \frac{\pi}{2\omega(1)}$, $\frac{3\pi}{2\omega(1)}$. Hence, $H(x_1(t))$ can be evaluated without any approximation. For $A = 1$, the frequency obtained exhibits low % error for an exceptionally large range of $\epsilon$ (Table 1). Due to finite precision of the numerical solver used, we cannot determine frequencies for higher $\epsilon$ values, and hence we are unable to comment on the upper limit of the $\epsilon$-usability range.

Equation (28) is easily amenable to harmonic balance [13]. Figure 7 shows the natural frequency comparison using ode45, HG and 1T-HB. Note that LP is not applicable to Eq. (28) since the unperturbed version is not an oscillator.

| Table 1 Frequency comparison for Eq. (28) |
| --- |
| $\epsilon$ | $\omega(1)$ | ode45 | HG | % error |
| 3 | 1.58368 | 1.58521 | 0.09661 |
| 12 | 3.16736 | 3.17042 | 0.09661 |
| 50 | 6.46694 | 6.47159 | 0.07190 |
| 200 | 12.92821 | 12.94318 | 0.11579 |
| 800 | 25.87322 | 25.88637 | 0.05082 |
| $8 \times 10^{17}$ | 8.17836 $\times 10^8$ | 8.18598 $\times 10^8$ | 0.09317 |

**Modulus Type Discontinuity**

We now consider an oscillator

$$\ddot{x} + x + \epsilon x|x| = 0.$$  

(30)
The non-smoothness is now of type modulus with an argument of \( \dot{x} \). We re-write the above equation in terms of Heaviside function as

\[
\ddot{x} + A \ddot{x} + \epsilon x \dot{x} (2H(\dot{x}) - 1) = 0. \tag{31}
\]

The change in the argument of Heaviside from \( x \) to \( \dot{x} \) does not alter the framework discussed in “Natural Response of Eq. 1 Using Homotopy Analysis and Galerkin Projections”. Homotopy \( \mathcal{H} \) in the scaled time \( \tau \) is given by

\[
\mathcal{H} \equiv (1-p) \left( \dot{x} - \dot{x}(\dot{x}) + \lambda(p) \ddot{x}(\dot{x}) \right) - h(p) \left( \ddot{x} - \ddot{x}(\dot{x}) + \epsilon \lambda(p) \dot{x}(\dot{x}) \right) (2H(\ddot{x}) - 1) = 0.
\]

The solutions to Eq. (30) are not scalable, we obtain a periodic solution with initial condition \((A, 0)\). We compute up to the first order and obtain approximate expressions for the natural frequency and solution \( x(t) \) as

\[
\omega(1) = 1 - \frac{\epsilon A}{3\pi} \left( 4h^{[1]} + h^{[2]} + 2h^{[1]} \right) - \epsilon^2 A^2 \left( \frac{h^{[1]} \ddot{x}(\dot{x})}{24} - \frac{\left( 4h^{[1]} + h^{[2]} + 2h^{[1]} \right)^2}{9\pi^2} \right),
\]

\[
x(t) = A \cos \tau + \epsilon A^2 \ddot{x}(\dot{x}) \left( \left( \frac{2}{3} \sin \tau + \frac{1}{3} \sin 2\tau \right) H(\tau - \pi) + \left( \frac{2}{3} \sin \tau - \frac{1}{3} \sin 2\tau \right) H(\tau - 2\pi) + \frac{(2\pi - 4\tau)}{6\pi} \sin \frac{\tau}{6} \sin 2\tau \right).
\]

We obtain the values of free parameters \( h^{[1]} \) and \( h^{[2]} \) by applying Galerkin projections with weighting functions \( \cos(\omega(1) \tau) \) and \( \tau \sin(\omega(1) \tau) \). We determine these unknowns for two values of \( \epsilon \) (numerical values in Table 2). We thus obtain \( x(t) \) and compare it with the solution obtained via ode45 in Fig. 8. Here as well, the expression for \( x(t) \) has zero-crossings at \( t = \frac{x}{2\omega(1)} \) making the evaluation of \( H(x(t)) \) easy.

---

**Table 2** Frequency and free parameters

| Equation | \( \epsilon \) | \( \omega(1) \) | \( h^{[1]} \) | \( h^{[2]} \) |
|----------|----------------|----------------|--------------|--------------|
| (28)     | 3.2            | 1.63719⋯      | -0.36174⋯   | -0.07434⋯   |
|          | 32             | 5.17727⋯      | -0.03617⋯   | -0.00365⋯   |
| (30)     | 1.6            | 1.34203⋯      | -0.85251⋯   | -0.00140⋯   |
|          | 3.2            | 1.66725⋯      | -0.76351⋯   | -0.02312⋯   |
| (32)     | 0.8            | 1.28342⋯      | -1.73908⋯   | -0.44176⋯   |
|          | 1.6            | 1.61700⋯      | -1.73908⋯   | -3.36354⋯   |

---
A solution to this oscillator starting with an initial condition, say \((A, 0)\) is periodic. The framework discussed in “Natural Response of Eq. 1 Using Homotopy Analysis and Galerkin Projections” can be applied to Eq. (33) as well. The homotopy \(H\) in the scaled time \(\xi \equiv \frac{\lambda(t)}{L} \frac{\dot{x}}{\lambda(p)}\) is given by

\[
H \equiv \lambda(p)(1 - p) \left( \ddot{x} + \lambda(p)^2 \dot{x} \right) - h(p) \left( \dot{x} \frac{\dot{x}}{\lambda(p)} \left( 2H \left( \frac{\dot{x}}{\lambda(p)} \right) - 1 \right) + e \lambda(p)^3 \dot{x} \right) = 0.
\]

We obtain the expressions for the non-scalable periodic solution of Eq. (32) and corresponding frequency by applying HG upto the first order to get

\[
\omega(1) = 1 + A \frac{4h^{(1)} + h^{(2)}}{3\pi} + A^2 \frac{h^{(1)}^2}{12} - e \left( \frac{h^{(1)} + h^{(2)}}{4} - \frac{3e^2 h^{(1)}^2}{8} \right) \quad \text{and}
\]

\[
x_1(\tau) = A \cos \tau - A^2 \frac{h^{(1)}}{3\pi} \left( \frac{2}{3} \sin \tau + \frac{1}{3} \sin 2\tau \right) H(\tau - \pi) + \left( \frac{2}{3} \sin \tau - \frac{1}{3} \sin 2\tau \right) H(\tau - 2\pi) + \frac{\tau - \pi}{3\pi} \sin \tau - \frac{1}{6} \sin 2\tau,
\]

with \(\tau = \omega(1)t\).

The free parameters \(h^{(1)}\) and \(h^{(2)}\) are determined by applying the Galerkin projections, taking weighting functions as before, i.e., \(\cos(\omega(t))\) and \(t \sin(\omega(t))\). Computing the unknowns for two values of \(e\) (numerical values, Table 2), in Fig. 10 we compare the solution so obtained with the one via MATLAB built-in integrator ode45.

Here again, the evaluation of \(H(x_1(t))\) is easy as \(x_1(t)\) has zero-crossings exactly at \(t = \frac{\pi}{2\omega(1)}, \frac{3\pi}{2\omega(1)}\). Figure 11 compares the frequencies by ode45, HG and 2T-HB for \(e \in [0.1, 5]\). The \(e\)-usability range w.r.t. HG is \(e = [0, 10]\) and the same w.r.t. 2T-HB is \(e = [0, 0.8]\).

With respect to all oscillators, HG method is consistently superior to other methods.

**Natural Response of an Impact Oscillator**

**The System Description**

Consider a pendulum constrained unilaterally and undergoing inelastic impacts as shown in Fig. 12.

Scaling time by \(\sqrt{\frac{2}{ml}}\) and using the Newtonian impact law, i.e.,
where $t^*$ is the instant of impact, $e$ is the coefficient of restitution, $\dot{x}^-(t^*)$ and $\dot{x}^+(t^*)$, denoting the velocity of the pendulum just before and after the collision respectively.

To apply homotopy analysis method, we wish to have an equation governing the motion of the pendulum along with the (inelastic) collision constraint of the form

$$x(t) = 0 : \quad \dot{x}^+(t^*) - \dot{x}^-(t^*) - e\dot{x}^-(t^*) = 0.$$  \hspace{1cm} (35)

The displacement $x(t)$ increasingly away from the constraint is assumed to be positive. Then, velocity just before impact $\dot{x}^-(t^*)$ is negative. We thus modify the above taking $\dot{x}^-(t^*) = -\lvert \dot{x}^-(t^*) \rvert$ as

$$\dot{x}^+(t^*) - \dot{x}^-(t^*) + e\frac{\dot{x}^-(t^*)^2}{\lvert \dot{x}^-(t^*) \rvert} = 0.$$  \hspace{1cm} (36)

Taking an infinitesimal positive number $\mu$, $\dot{x}^-(t^* - \mu)$ and $\dot{x}^+(t^* + \mu)$, in the limit $\mu \to 0$, represents the velocity of the pendulum just before and just after the impact respectively. Assuming $x(t)$ to be $C^1$-continuous at the impact, we assume the left hand side of the above equation as

$$\lim_{\mu \to 0} \left( \dot{x}^+(t^* + \mu) - \dot{x}^-(t^* - \mu) + \int_{t^* - \mu}^{t^* + \mu} \dot{x} \, dt + e\frac{\dot{x}^-(t^*)^2}{\lvert \dot{x}^-(t^*) \rvert} \right) = 0.$$  \hspace{1cm} (36)

To formulate the governing law (34), we find it necessary to introduce a Dirac-delta like distribution, $\delta^+(t)$ defined as

$$\int_{-\infty}^{\infty} x(t)\delta^+(t) \, dt = x^-(0),$$

where $x(t)$ is a non-smooth function and $x^-(0)$ is its left hand limit at $t = 0$. The properties of $\delta^+(t)$ are given in the appendix A with appropriate justification. Taking $y(t) = x(t)$ in the Eq. (A6) with the integral restricted to $[t^* - \mu, t^* + \mu]$ (with the time of impact $t^*$ being the root of $x$), Eq. (36) becomes equivalent to
\[
\lim_{\mu \to 0} \left( \ddot{x}(t^* + \mu) - \dot{x}(t^* - \mu) + \int_{t^* - \mu}^{t^* + \mu} x \, dt \right)
+ \epsilon \int_{t^* - \mu}^{t^* + \mu} \dddot{x}^\delta(x) \, dt = 0.
\]

Re-writing the first two terms as the integral of \(\dot{x}\), we get
\[
\lim_{\mu \to 0} \left( \int_{t^* - \mu}^{t^* + \mu} \dot{x} \, dt + \int_{t^* - \mu}^{t^* + \mu} x \, dt + \epsilon \int_{t^* - \mu}^{t^* + \mu} \ddot{x}^\delta(x) \, dt \right) = 0.
\]
Thus,
\[
\lim_{\mu \to 0} \int_{t^* - \mu}^{t^* + \mu} \left( \dddot{x} + x + c\ddot{x}^\delta(x) \right) \, dt = 0.
\]

The physics of the Newtonian impact suggests to hypothesize that the solution \(x(t)\) is \(C^\infty\)-regular, which enables us to write the above formally as
\[
\dddot{x} + x + c\ddot{x}^\delta(x) = 0, \tag{37}
\]
which when compared to Eq. (34) yields \(F_c(x, \dot{x}) = c\ddot{x}^\delta(x)\).
The above equation is written formally and has to be understood in the sense of distributions, as precised in the Appendix A. Equation (37) holds for any sign convention for \(x(t)\).

Since the solutions to Eq. (37) are scalable, the frequency is independent of initial conditions and is equal to that of the unconstrained version \(c = 0\), i.e., one. This information plays an important role in application of HG as will be shown in the next subsection.

**Homotopy Analysis and Galerkin Projections**

Homotopy analysis method is typically used to find a periodic solution. Here we apply the method to obtain a decaying solution to Eq. (37) for some \(-2 \leq \epsilon < -1\). Unlike an oscillator with viscous damping where the decay rate is exponential, here the same is algebraic. We construct the homotopy with linear operator as the viscously damped linear oscillator and proceed to formulate the homotopy as
\[
\mathcal{H} \equiv (1 - p)\mathcal{L} - h(p)\mathcal{N} = 0,
\]
where \(\mathcal{L} \equiv \dddot{x} + \gamma(p)\dot{x} + \chi(p)x\) and \(\mathcal{N} \equiv \dddot{x} + x + c\ddot{x}^\delta(x)\).

\(\gamma(p)\) and \(\chi(p)\) are damping and stiffness as functions of \(p\), introduced to adjust the frequency and decay rate of the solution to \(\mathcal{H}\). Here, we strongly emphasize that the homotopy is written in the sense of distributions. Taking the assumed regularity of the solution to the homotopy at \(p = 1\) into account, we assume that \(\dddot{x}(t^*)\) is a \(C^0\)-regular function, which results in all terms of the homotopy including \(\dddot{x}(t^*)\ddot{x}^\delta(\dddot{x}(t^*))\) being distributions. Differentiation of each term w.r.t. \(p\) is then justified in a distributional sense. We Taylor-expand the functions \(\gamma(p)\) and \(\chi(p)\) as
\[
\gamma(p) = \gamma(0) + \sum_{n=1}^{\infty} \frac{1}{n!} \gamma[n] p^n \quad \text{and} \quad \chi[p] = \frac{d^n \chi(p)}{dp^n} \bigg|_{p=0}.
\]

Considering the scalability of Eq. (37), we wish to obtain a solution with initial condition \((1, 0, 0)\). We begin by substituting \(p = 0\) in the homotopy and obtain the zeroth order deformation equation
\[
\dddot{x}^0 + \gamma(0)\dddot{x}^0 + \chi(0)x^0 = 0. \tag{40}
\]
As can be seen from the above, \(\gamma(0)\) is the decay rate and the expression \(\omega_d(0) = \sqrt{\gamma(0) - \frac{\omega_0^2}{4}}\) is the damped natural frequency of the oscillator at the zeroth order. However, this is true for all other orders as well. Hence \(\gamma(0)\) is the decay rate and \(\omega_d(0)\) is the frequency of the solution to the homotopy at \(p = 1\), the desired solution. \(\gamma(0)\) and \(\chi(0)\) can be determined by Galerkin projections; however, this not only increases the number of equations to be solved, but also makes the equations non-algebraic in the unknowns \((\gamma(0), \chi(0), h[1], h[2], \ldots)\) thereby complicating the application of Galerkin projections. Hence, we prefer to pre-determine \(\gamma(0)\) and \(\chi(0)\) based on the physics of the impact oscillator as follows. As discussed previously, the frequency of the solution to the unilaterally constrained pendulum is equal to that of the unconstrained one, therefore we set
\[
\omega_d(0) = \sqrt{\chi(0) - \frac{\gamma(0)^2}{4}} = 1.
\]

Equating the decay rate per cycle of the zeroth order deformation equation to that of Eq. (37) and then from the above expression, we obtain
\[
\gamma(0) = \frac{-2 \ln(-\epsilon - 1)}{\sqrt{\ln(-\epsilon - 1)^2 + 4\pi^2}} \quad \text{and} \quad \chi(0) = \sqrt{1 + \frac{\ln(-\epsilon - 1)^2}{\ln(-\epsilon - 1)^2 + 4\pi^2}}. \tag{41}
\]
Choosing the initial condition as \((1, 0, 0)\) for Eq. (40), we solve it and obtain
\[
\dddot{x}^{0}(t) = \exp\left(-\frac{\gamma(0)p}{2}\right) \left( \cos \frac{\gamma(0)}{2} t + \frac{\gamma(0)}{2} \sin \frac{\gamma(0)}{2} t \right), \tag{42}
\]
where \(\gamma(0)\) is given by Eq. (41). Zeroes of \(\dddot{x}^{0}(t)\), important in the subsequent analysis are given by
\[ t_i = i\pi - \tan^{-1}\left(\frac{2}{\gamma(0)}\right), \quad \text{where} \quad i = 1, 2, 3, \ldots. \]  

Differentiating the homotopy once, substituting \( p = 0 \), and using Eqs. (4), (17), (38), (39), we obtain the first order deformation equation,

\[
\dot{x}^{[1]} + \gamma(0)x^{[1]} + \chi(0)x^{[1]} = h^{[1]}(\dot{x}^{[0]} + x^{[0]} + \epsilon x^{[0]} \delta^*(x^{[0]}) - \gamma x^{[0]} - x^{[1]}x^{[0]} = \; F_1(t).
\]

where \( x^{[0]} \) is given by Eq. (42). To obtain a solution to the above with ease, we simplify the term \( \delta^*(x^{[0]})(t) \). For any function \( f(t) \) of class \( C^1 \), the Dirac-delta function satisfies

\[
\delta^*(f(t)) = \sum_{i=-\infty}^{\infty} \frac{\delta(t - t_i)}{|f(t_i)|},
\]

where \( t_i \)'s are given by Eq. (43). Substituting the above expression in the first order deformation equation, we get an oscillator forced at its damped natural frequency, 1. We remove terms from the forcing which have the frequency equal to that of the unforced oscillator, even though amplitudes of such terms are not constant w.r.t. time. Therefore, equating the coefficients of \( \sin t \) and \( \cos t \) in \( F_1(t) \) to zero, we get two equations algebraic in unknowns \( \gamma^{[1]} \) and \( \chi^{[1]} \), which on solving yield

\[
\gamma^{[1]} = -\frac{2}{\gamma(0)^2 - 4} \quad \text{and} \quad \chi^{[1]} = -\gamma(0)h^{[1]}.
\]

These expressions play no role in further analysis. The first order deformation equation thus becomes

\[
\dot{x}^{[1]} + \gamma(0)x^{[1]} + \chi(0)x^{[1]} = h^{[1]}(\dot{x}^{[0]} + x^{[0]} + \epsilon x^{[0]} \delta^*(x^{[0]})) = \; F_1(t),
\]

where \( \delta^*(x^{[0]})(t) \) is as expressed in Eq. (45). The above can be re-written as a system of two first order differential equations such that the homogeneous part satisfies the Carathéodory conditions and the non-homogeneous part is a distribution that is not a Lebesgue integrable function. As per the analysis from Chapter 1 of [3], it follows that the first order equation admits a unique solution of class \( C^0 \). We then solve it with initial condition \((0, 0)\) and obtain

\[
x^{[1]}(t) = e^{\gamma^{[1]}t}x^{[0]}(t) + \sum_{i=1}^{\infty} (-1)^i H(t - t_i),
\]

where \( H \) denotes the Heaviside function and \( t_i \)'s are given by Eq. (43). Adding \( x^{[0]}(t) \) and \( x^{[1]}(t) \), we get the expression for the solution \( x_1(t) \). We now determine the only unknown \( h^{[1]} \) by applying Galerkin projections with weighting function \( w_1(t) = \cos t \sum_{i=1}^{\infty} H(t - t_i) \). Computationally, it suffices to consider the first four zeroes of \( x^{[0]}(t) \) in \( w_1(t) \) to get a good match with the solution obtained via numerical integration. We obtain the residual by substituting \( x_1(t) \) in Eq. (37) as

\[
\mathcal{R}_1(t) = \ddot{x}_1 + x_1 + \epsilon \dot{x}_1^2 \delta^*(x_1).
\]

and minimize the weighted residual for all time \( t \) as

\[
\int_{-\infty}^{\infty} w_1(t)\mathcal{R}_1(t)dt = 0.
\]

To ease the integration of the above, we use the Eq. (A6) with \( y(t) = x_1(t) \) and \( f(y(t), \dot{y}(t)) = 1 \) and obtain

\[
\delta^*(x_1(t)) = \sum_{i=-\infty}^{\infty} \frac{\delta(t - t_i)}{|\dot{x}^1_i(t_i)|}.
\]

Using the above, we evaluate the integral in Eq. (47) and determine \( h^{[1]} \). Simpler choice of \( \gamma(0) \) is nicer to implement in Maple than a simpler choice of \( \epsilon \). For \( \gamma(0) = 0.2 \), inverting Eq. (41), we arrive at \( \epsilon = -1.73 \ldots \). To compute the numerical solution to Eq. (37), we use MATLAB’s ode45 integrator with “event detection” on. While integrating the simple harmonic oscillator, we detect the zero-crossings of \( x(t) \). Every time the event is detected, we change the velocity \( \dot{x}(t) \) according to Newtonian impact law and resume numerical integration of the oscillator with the changed velocity. For \( \epsilon = -1.73 \ldots \), the solution \( x_1(t) \) is compared to the solution obtained using ode45 in Fig. 13. The choice of linear operator ensures that the decay rate as well as the frequency are captured well via HG method. But this also results in the shifting of the zero crossings of the HG solution by some
fixed amount from that of the numerical solution. We believe that this limitation maybe overcome by better choosing the linear operator. Although the global behaviour of the solution to Eq. (37) is like a linearly damped solution, its behaviour between two consecutive impacts is like an undamped solution; which is unlike the solution resulting from our choice of linear operator. An intelligent choice of the linear operator, which is partly though significantly, guided by the physics of the impact oscillator enables to get a good match with first two terms of the series solution. Practical application of HAM and thus HG demand the knowledge of the physical laws governing the problem.

**Conclusion and Further Work**

We have applied homotopy analysis method in combination with the Galerkin projections to several non-smooth oscillators and have obtained periodic responses approximately. The methodology developed for an oscillator with discontinuity of type Heaviside may easily be extended to discontinuities of type signum and modulus, as illustrated. Introduction of the convergence-control function and the time-stretching function make the framework natural and simpler. This renders us with an approximate analytical expression for the natural frequency involving the unknowns, also to be solved simultaneously but numerically. The natural frequency thus obtained matches excellently with the frequency obtained via numerical integration. The above property can be extended to discontinuities of type signum and modulus, as illustrated.

We introduce a Dirac-delta-like function $\delta^*(t)$ and the distribution generated by it $T_{\delta^*}(\cdot)$ as

$$T_{\delta^*}(\varphi) = \int_{\Omega} \varphi(t)\delta^*(t)dt = \varphi^-(0) \quad \forall \varphi \in C^\infty_c(\Omega), \quad (A1)$$

where $\varphi^-(0)$ is the left hand limit of the function $\varphi(t)$ at $t = 0$. To determine if $\delta^*(t)$ is a distribution, we take any convergent sequence $\{\varphi_n\}_{n \geq 1} \subset C^\infty_c(\Omega)$ converging to $\varphi \in C^\infty_c(\Omega)$, and check $|T_{\delta^*}(\varphi_n) - T_{\delta^*}(\varphi)| = |\varphi^*_n(0) - \varphi^*(0)| = |\varphi^*_n(0) - \varphi(0)| \longrightarrow 0$ as $n \longrightarrow \infty$. This implies that $T_{\delta^*}(\varphi_n) \longrightarrow T_{\delta^*}(\varphi)$ as $n \longrightarrow \infty$ and thus $\delta^*(t)$ is a distribution.

Let $\alpha$ be a whole number. The derivative $\partial^\alpha \delta^*(t)$ is given by

$$T_{\partial^\alpha \delta^*}(\varphi) = \int_{\Omega} \varphi(t)(\partial^\alpha \delta^*)(t)dt = (-1)^\alpha \int_{\Omega} \varphi(t)\delta^*(t)dt = (-1)^\alpha \partial^\alpha \varphi^-(0) \quad \forall \varphi \in C^\infty_c(\Omega), \quad (A2)$$

where $\partial^\alpha \varphi^-(0)$ denotes the left hand limit of the function $\partial^\alpha \varphi(t)$ at $t = 0$. The above property can be extended to $\varphi(t) \in C^\infty_c(\Omega)$, (space of $\alpha$-regular functions with compact support in $\Omega$) by a density argument. We remark that this property holds for the Dirac-delta distribution as well.

We observe that like the Dirac-delta distribution, $\delta^*(t)$ satisfies the property

$$\int_{-\infty}^{\infty} \delta^*(at) \, dt = \frac{1}{|a|} \int_{-\infty}^{\infty} \delta^*(t) \, dt \quad \forall a \in \mathbb{R} \setminus \{0\}, \quad (A3)$$

by applying a change of variable $at \mapsto t$.

Let us consider a series $(f_n)_{n \geq 1}$ given by

$$f_n(t) = \begin{cases} \frac{2n}{\sqrt{\pi}} e^{-(nt)^2} & t \leq 0 \\ 0 & 0 < t \end{cases}$$

and check if $f_n(t) \longrightarrow \delta^*(t)$ as $n \longrightarrow \infty$ in the distributional sense. By $T_{f_n}$, we denote the distribution generated by $f_n$ on $C^\infty_c(\Omega)$. For all $\varphi(t) \in C^\infty_c(\Omega)$ extended by 0 outside $\Omega$, we have

$$T_{f_n}(\varphi) = \int_{-\infty}^{\infty} f_n(t)\varphi(t)dt = \int_{-\infty}^{0} \frac{2n}{\sqrt{\pi}} e^{-(nt)^2} \varphi(t)dt = \int_{-\infty}^{0} \frac{2}{\sqrt{\pi}} e^{-x^2} \varphi(x/n)dx, \quad (A4)$$

**Properties of $\delta^*$ Distribution**

Let $\Omega$ be an open set in $\mathbb{R}$ encompassing 0. By $C^\infty_c(\Omega)$, we denote the space of $C^\infty$ regular functions with compact support in $\Omega$. The members of the dual space of $C^\infty_c(\Omega)$ are distributions. We say that an equation $f(x, \dot{x}, \ddot{x}) = 0$ is satisfied in the sense of distribution if

$$\int_{\Omega} f(x, \dot{x}, \ddot{x})\varphi \, dx = 0 \quad \forall \varphi \in C^\infty_c(\Omega).$$
where we have applied a variable transform $t = x/n$ in the second last equality. The function series $\frac{2}{\sqrt{\pi}} e^{-x^2} \phi(x/n)$ is bounded by an integrable function $\frac{2}{\sqrt{\pi}} e^{-x^2} \|\phi\|_{L^1(\Omega)}$. Moreover, the series converges pointwise to $\frac{2}{\sqrt{\pi}} e^{-x^2} \phi(0)$ for $x \in (-\infty, 0]$ as $n \to \infty$. Hence applying dominated convergence theorem to the last integral in Eq. (A4) gives

$$
\lim_{n \to \infty} T_n(\phi) = \lim_{n \to \infty} \int_{-\infty}^{0} \frac{2}{\sqrt{\pi}} e^{-x^2} \phi(x/n) \, dx = \int_{-\infty}^{0} \frac{2}{\sqrt{\pi}} e^{-x^2} \phi(0) \, dx = \phi(0).
$$

Using the approximating function series $\{f_n\}_{n \geq 1}$, we calculate the integral of $\delta^*(t)$ in $t \in [a, b]$, $\alpha < 0 < b$. Or, equivalently,

$$
\int_{-\infty}^{0} \phi(t) \delta^*(t) \, dt = \int_{-\infty}^{0} \phi(t) \delta^*(t) \, dt, \quad \forall \phi \in C^\infty_c(\Omega). \quad (A5)
$$

Now, we consider the distribution $\delta^*(x(t))$, where $x(t)$ is a non-smooth function. Let time instant $t^*$ be such that $x(t^*) = 0$. We take an interval $I \equiv [t^* - e, t^* + e]$ (for some $e > 0$) such that for all $t \in I \setminus \{0\}, x(t) \neq 0$. Assuming $\eta \in [0, e]$, for the left half of the interval $I$, we have

$$
x(t^* - \eta) = x(t^*) - \dot{x}(t^*) \eta + \mathcal{O}(\eta^2) = -\dot{x}(t^*) \eta + \mathcal{O}(\eta^2),
$$

using which, we compute

$$
\int_{t^* - e}^{t^* + e} \delta^*(x(t)) \, dt = \int_{t^* - e}^{t^*} \delta^*(x(t)) \, dt \quad \text{(by Eq. (A5))}
$$

$$
= \int_{0}^{\infty} \delta^*(x(t^* - \eta)) \, d\eta
$$

$$
= \int_{0}^{\infty} \delta^*(-\dot{x}(t^*) \eta + \mathcal{O}(\eta^2)) \, d\eta
$$

$$
= \int_{0}^{\infty} \delta^*(\dot{x}(t^*) \eta + \mathcal{O}(\eta^2)) \, d\eta
$$

$$
= \int_{-\eta/[|\dot{x}(t^*)|]}^{0} \delta^*(\eta) \, d\eta \quad \text{(by Eq. (A3))}
$$

$$
= \frac{1}{|\dot{x}(t^*)|} \int_{t^* - e}^{t^* + e} \delta^*(t - t^*) \, dt.
$$

Generalizing the above property and using the definition of $\delta^*(t)$, we find that for some $f(y(t), \dot{y}(t))$ with non-smooth $y(t)$, we have

$$
\int_{-\infty}^{\infty} f(y(t), \dot{y}(t)) \delta^*(y(t)) \, dt = \sum_{i = -\infty}^{\infty} \frac{\int_{\mathcal{I}} f(y(t_i), \dot{y}(t_i)) \, dt}{|\mathcal{I}|}, \quad (A6)
$$

where $t_i$’s are the roots of $y(t)$. Here as well, we remark that the above property holds in the case of Dirac-delta function when $y(t)$ is $C^1$ regular and $f(\cdot, \cdot)$ is $C^0$ regular.

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Declarations

Conflict of Interest The authors declare that they have no conflict of interest.

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