(δ, χ_{FF})-bounded families of graphs

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Abstract

For any graph $G$, the First-Fit (or Grundy) chromatic number of $G$, denoted by $\chi_{FF}(G)$, is defined as the maximum number of colors used by the First-Fit (greedy) coloring of the vertices of $G$. We call a family $\mathcal{F}$ of graphs $(δ, χ_{FF})$-bounded if there exists a function $f(x)$ with $f(x) → \infty$ as $x → \infty$ such that for any graph $G$ from the family one has $\chi_{FF}(G) ≥ f(δ(G))$, where $δ(G)$ is the minimum degree of $G$. We first give some results concerning $(δ, χ_{FF})$-bounded families and obtain a few such families. Then we prove that for any positive integer $ℓ$, $\text{Forb}(K_{ℓ,ℓ})$ is $(δ, χ_{FF})$-bounded, where $K_{ℓ,ℓ}$ is complete bipartite graph. We conjecture that if $G$ is any $C_4$-free graph then $\chi_{FF}(G) ≥ δ(G) + 1$. We prove the validity of this conjecture for chordal graphs, complement of bipartite graphs and graphs with low minimum degree.

Mathematics Subject Classification (2000): 05C15, 05C07, 05C85, 05C38

Keywords: graph coloring; First-Fit coloring; Grundy number; lower bound; minimum degree

1 Introduction

All graphs in this paper are simple undirected graphs. A family $\mathcal{F}$ of graphs is said to be $(δ, χ)$-bounded if there exists a function $f(x)$ satisfying $f(x) → \infty$ as $x → \infty$, such that for any graph $G$ from the family one has $f(δ(G)) ≤ χ(G)$, where $δ(G)$ and $χ(G)$ denotes the minimum degree and chromatic number of $G$, respectively. Equivalently, the family $\mathcal{F}$ is $(δ, χ)$-bounded if $δ(G_n) → \infty$ implies $χ(G_n) → \infty$ for any sequence $G_1, G_2, \ldots \text{ with } G_n ∈ \mathcal{F}$. Motivated by Problem 4.3 in [4], the author introduced and studied $(δ, χ)$-bounded families of graphs (under the name of $δ$-bounded families) in [9]. The so-called color-bound family of graphs mentioned in the related problem of [4] is a family for which there exists a function $f(x)$ satisfying $f(x) → \infty$ as $x → \infty$, such that for any graph $G$ from the family one has $f(\text{col}(G)) ≤ χ(G)$, where $\text{col}(G)$ is defined as $\text{col}(G) = \max\{δ(H) : H ≤ G\} + 1$. It was shown in [9] that if we restrict ourselves to hereditary (i.e. closed under taking induced subgraph) families then the two concepts

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(δ, χ)-bounded and color-bound are equivalent. The first specific results concerning (δ, χ)-bounded families appeared in [9] where the following theorem was proved (in a somewhat different but equivalent form). In the following theorem for any set C of graphs, Forb(C) denotes the class of graphs that contains no member of C as an induced subgraph.

**Theorem 1.** ([9]) For any set C of graphs, Forb(C) is (δ, χ)-bounded if and only if there exists a constant c = c(C) such that for any bipartite graph H ∈ Forb(C) one has δ(H) ≤ c.

Theorem 1 shows that to decide whether Forb(C) is (δ, χ)-bounded we may restrict ourselves to bipartite graphs. A comprehensive study of (δ, χ)-bounded families was done in [2], where the authors proved the following theorem.

**Theorem 2.** ([2]) Given a finite set of graphs \{H_1, H_2, \ldots, H_k\}. Then Forb(H_1, H_2, \ldots, H_k) is (δ, χ)-bounded if and only if one of the following holds:

(i) For some i, H_i is a star tree.

(ii) For some i, H_i is a forest and for some j \neq i, H_j is complete bipartite graph.

The following result concerns Forb(C), where C contains infinitely many graphs in which one of them is a tree.

**Theorem 3.** ([2]) Let T be any non star tree. Then Forb(T, H_1, \ldots) is (δ, χ)-bounded if and only if at least one of H_i-s is complete bipartite graph.

For other (δ, χ)-bounded families of graphs we refer the reader to [2] and [9]. In this paper we work on the Grundy (or First-Fit) chromatic number of graphs. A Grundy k-coloring of a graph G, is a proper k-coloring of vertices in G using colors \{1, 2, \ldots, k\} such that for any two colors i and j with i < j, any vertex colored j is adjacent to some vertex colored i. The Grundy or First-Fit chromatic number of a graph G, denoted by \chi_{ff}(G) (also denoted by \Gamma(G) in some articles), is the largest integer k, such that there exists a Grundy k-coloring for G. It can be shown that \chi_{ff}(G) is the same as the maximum number of colors used by the First-Fit (greedy) coloring of the vertices of G [7]. To determine \chi_{ff}(G) is NP-complete even for complement of bipartite graphs G [7]. For this reason it is natural to obtain lower and upper bounds for \chi_{ff}(G) in terms of ordinary graph theoretical parameters. In this paper we obtain some lower bounds in terms of the minimum degree of graphs. The Grundy number and First-Fit coloring of graphs were studied widely in the literature, see [7, 8] and its references. Throughout the paper we denote the complete graph on n vertices by K_n and the cycle on n vertices by C_n. For each positive integer ℓ, the complete bipartite graph in which each part has ℓ vertices is denoted by K_{ℓ, ℓ}. 

2
2 Results

Generalizing the concept of $(\delta, \chi)$-bounded graph, we define the following notion. A family $\mathcal{F}$ of graphs is called $(\delta, \chi_{\mathcal{F}})$-bounded if there exists a function $f(x)$ with $f(x) \to \infty$ as $x \to \infty$ such that for any graph $G$ from the family one has $\chi_{\mathcal{F}}(G) \geq f(\delta(G))$. It was shown in [7] that $\chi_{\mathcal{F}}(G) = 2$ if and only if $G$ is a complete bipartite graph. Obviously, complete bipartite graphs may have arbitrary large minimum degree. We conclude that the family of complete bipartite graphs is not $(\delta, \chi_{\mathcal{F}})$-bounded. This example induces that perhaps $C_4$ and other complete bipartite graphs have significant role in $(\delta, \chi_{\mathcal{F}})$-boundedness. Note also that any $(\delta, \chi)$-bounded family is also $(\delta, \chi_{\mathcal{F}})$-bounded.

Another interesting chromatic-related parameter is the so-called coloring number of graphs. The coloring number of a graph $G$ is defined as $\text{col}(G) = \max_{H \subseteq G} \delta(H) + 1$. The coloring number of graphs is a polynomial time parameter. See [4, 5, 9] for more results on the coloring number of graphs. The possible relationships between the coloring number and Grundy number of graphs is an interesting research area. For some graphs $G$ we have $\text{col}(G) < \chi_{\mathcal{F}}(G)$. For example the path on four vertices $P_4$ and infinitely many trees satisfy this inequality. Also, for some graphs $G$ we have $\chi_{\mathcal{F}}(G) < \text{col}(G)$. For example consider complete bipartite graphs $K_{a,b}$, where $a, b \geq 2$. We have the following remark, where by a hereditary family we mean any family $\mathcal{F}$ such that for any graph $G$ from $\mathcal{F}$, if $H$ is an induced subgraph of $G$ then $H \in \mathcal{F}$.

Remark 1. Let $\mathcal{F}$ be any hereditary family of graphs such that $\mathcal{F}$ is $(\delta, \chi_{\mathcal{F}})$-bounded. Then there exists a function $f(x)$ with $f(x) \to \infty$ as $x \to \infty$ such that for any graph $G$ from the family one has $\chi_{\mathcal{F}}(G) \geq f(\text{col}(G))$.

Proof. Let $G \in \mathcal{F}$ and $H$ be any induced subgraph of $G$ with $\text{col}(G) = \delta(H) + 1$. We have $H \in \mathcal{F}$. Let $g(x)$ be such that $\chi_{\mathcal{F}}(H) \geq g(\delta(H))$. The proof completes by taking $f(x) = g(x - 1)$.

As we mentioned before any $(\delta, \chi)$-bounded family is also $(\delta, \chi_{\mathcal{F}})$-bounded. In Theorem 5 we obtain $(\delta, \chi_{\mathcal{F}})$-bounded families which are not $(\delta, \chi)$-bounded. For this purpose we first obtain in Proposition 1 a result concerning $(\delta, \chi)$-boundedness of graphs. In the following, the girth of a graph $G$ is the length of a shortest cycle contained in $G$. When $G$ contains a cycle then we say that $G$ has finite girth. We use also the following two facts. The first fact states that any graph with $m$ edges contains a bipartite subgraph with at least $m/2$ edges. The second one states that any graph with $n$ vertices and $m/2$ edges contains a subgraph with minimum degree at least $m/2n$. For the proof of these facts we refer the reader to standard Graph Theory books such as [1].

Proposition 1. Let $\mathcal{C}$ be any finite collection of graphs such that any member of $\mathcal{C}$ has finite girth. Then $\text{Forb}(\mathcal{C})$ is not $(\delta, \chi)$-bounded. In particular $\text{Forb}(K_3, K_{2,m})$ and $\text{Forb}(K_{\ell, \ell})$ are not $(\delta, \chi)$-bounded.

Proof. Let $g$ be an even integer such that the girth of any graph in $\mathcal{C}$ is at most $g$. For the proof we use the following Turán-type result which is attributed to Erdős in [6]. For
any $k$ and $n$ there exists a graph on $n$ vertices with $\Omega(n^{1+1/2k-1})$ edges that contains no cycle of length at most $2k$. Let $g = 2k$, and recall from the previous paragraph that (1) a graph with $m$ edges contains a bipartite subgraph with at least $m/2$ edges, and (2) a graph with $n$ vertices and $m/2$ edges contains a subgraph with minimum degree at least $m/2n$. We conclude that there exists an infinite sequence $G_1, G_2, \ldots$ of bipartite graphs such that $\delta(G_i) \to \infty$ as $i \to \infty$ and the girth of any $G_i$ is more than $g$. This shows that $G_i$ belongs to $\text{Forb}(\mathcal{C})$. This shows that $\text{Forb}(\mathcal{C})$ is not $(\delta, \chi)$-bounded. □

In opposite direction we show in Theorem 5 that $\text{Forb}(K_3, K_{2,m})$ is $(\delta, \chi_{ff})$-bounded. More generally, Theorem 7 asserts that $\text{Forb}(K_{\ell,\ell})$ is $(\delta, \chi_{ff})$-bounded. Before we proceed, we need to introduce a family of trees $T_k, k = 1, 2, \ldots$. For $k = 1, 2, T_1$ (resp. $T_2$) is isomorphic to $K_1$ (resp. $K_2$). Assume that $T_k$ has been defined. Attach a leaf to any vertex of $T_k$ and denote the resulting tree by $T_{k+1}$. It is easily observed that $\chi_{ff}(T_k) = k$. Note also that $|V(T_k)| = 2k-1$. We need also the following result from [9], Theorem 2, where $\rho(G) = |E(G)|/|V(G)|$.

**Theorem 4.** ([9]) Let $G$ be any triangle-free graph such that $G$ does not contain $K_{2,m}$, where $m > 1$. If $\rho(G) \geq (k - 3)(m - 1) + 1$ then $G$ contains all trees on $k$ vertices as induced subgraphs.

The promised result is as follows.

**Theorem 5.** Let $G \in \text{Forb}(K_3, K_{2,m})$. Then

$$\chi_{ff}(G) \geq \log\left(\frac{\delta(G) + 6m - 8}{2m - 2}\right) + 1.$$ 

**Proof.** First note that $G$ does not contain triangle and $K_{2,m}$. Set $\delta(G) = p$ and $k = (p + 6m - 8)/(2m - 2)$, for simplicity. We have

$$(k - 3)(m - 1) + 1 = \left(\frac{p + 6m - 8}{2m - 2} - 3\right)(m - 1) + 1$$

$$= \left(\frac{p - 2}{2m - 2}\right)(m - 1) + 1$$

$$= \frac{p}{2}.$$ 

Since $\rho(G) \geq (p/2)$ then $G$ satisfies the conditions of Theorem 5 with these $k$ and $m$. Therefore $G$ contains all trees on $k$ vertices as induced subgraph. In particular $G$ contains $T_q$ as induced subgraph, where $q = \log((\delta(G) + 6m - 8)/(2m - 2)) + 1$. We conclude that

$$\chi_{ff}(G) \geq \log\left(\frac{\delta(G) + 6m - 8}{2m - 2}\right) + 1.$$ 

□

By applying Theorem 4 and Theorem 5 more economically when $m = 2$ we obtain the following bound.
Corollary 1. Let \( G \in \text{Forb}(K_3, C_4) \). Then

\[
\chi_{ff}(G) \geq \log(\delta(G) + 1).
\]

We noted before that the family of complete bipartite graphs is not \((\delta, \chi_{ff})\)-bounded. Hence the following proposition is immediate from this fact.

Proposition 2. Let \( C \) be any collection of graphs such that any member of it contains an odd cycle. Then \( \text{Forb}(C) \) is not \((\delta, \chi_{ff})\)-bounded. In particular \( \text{Forb}(K_3) \) is not \((\delta, \chi_{ff})\)-bounded.

In Theorem 7 we prove that \( \text{Forb}(K_{\ell, \ell}) \) is \((\delta, \chi_{ff})\)-bounded. For this purpose we need the following theorem from [2].

Theorem 6. ([2]) For every tree \( T \) and for positive integers \( \ell, k \) there exist a function \( f(T, \ell, k) \) with the following property. If \( G \) is a graph with \( \delta(G) \geq f(T, \ell, k) \) and \( \chi(G) \leq k \) then \( G \) contains either \( T \) or \( K_{\ell, \ell} \) as an induced subgraph.

We shall make use of this theorem in proving the next result.

Theorem 7. For each positive integer \( \ell \), \( \text{Forb}(K_{\ell, \ell}) \) is \((\delta, \chi_{ff})\)-bounded.

Proof. Recall that for each positive integer \( k \), \( T_k \) denotes the only smallest tree of Grundy number \( k \). Let \( \{ G_n \}_{n=1}^{\infty} \) be a sequence of \( K_{\ell, \ell} \)-free graphs such that \( \delta(G_n) \to \infty \) as \( n \to \infty \). Assume on the contrary that for some integer \( N \), \( \chi_{ff}(G_n) \leq N \) holds for all \( n \). It follows that for each \( n \), \( T_{N+1} \) is not an induced subgraph of \( G_n \). Hence \( G_n \) belongs to \( \text{Forb}(T_{N+1}, K_{\ell, \ell}) \). Theorem 6 implies that for each \( n \) either \( \delta(G_n) < f(T_{N+1}, \ell, N) \) or \( \chi(G_n) > N \). But the second case is impossible because \( \chi(G_n) \leq \chi_{ff}(G_n) \leq N \). Therefore for each \( n \), \( \delta(G_n) < f(T_{N+1}, \ell, N) \). This contradicts \( \delta(G_n) \to \infty \). This contradiction completes the proof.

The following result shows that chordal graphs are \((\delta, \chi_{ff})\)-bounded with \( f(x) = x + 1 \). Note that the class of chordal graphs is the same as \( \text{Forb}(C_4, C_5, \ldots) \). In a graph \( G \), by a simplicial vertex we mean any vertex \( v \) such that \( G[N(v)] \) is a clique in \( G \), where \( G[N(v)] \) stands for the subgraph of \( G \) induced by the set \( N(v) \) of the neighbors of \( v \) in \( G \). It is a known fact that any chordal graph \( G \) admits a simplicial elimination ordering (see e.g. [1]). In other words, let \( G \) be a chordal graph. Then there exists a vertex ordering \( v_1, \ldots, v_n \) of \( G \) such that \( v_i \) is simplicial in \( G \setminus \{ v_1, \ldots, v_{i-1} \} \). We shall make use of this fact in the following theorem.

Theorem 8. \( \text{Forb}(C_4, C_5, \ldots) \) is \((\delta, \chi_{ff})\)-bounded with \( f(x) = x + 1 \).
Proof. Let $G$ be any chordal graph $G$ and let $v_1, v_2, \ldots, v_n$ be a simplicial ordering of the vertices of $G$. Since $G[N(v) \cup \{v_1\}]$ is a clique in $G$ with $\text{deg}_G(v_1) + 1$ vertices, then $\omega(G) \geq \text{deg}_G(v) + 1 \geq \delta(G) + 1$. We have also $\chi_{\text{ff}}(G) \geq \chi(G) \geq \omega(G)$. Hence $\chi_{\text{ff}}(G) \geq \delta(G) + 1$.

By strengthening Theorem 8 we propose the following conjecture.

**Conjecture 1.** Let $G$ be a $C_4$-free graph. Then $\chi_{\text{ff}}(G) \geq \delta(G) + 1$.

A natural scenario to prove the above conjecture is as follows. Let $F$ be a hereditary family of graphs satisfying the following property. Any member $G$ from the family contains a maximal independent set (MIS) such as $I$ such that $\delta(G \setminus I) = \delta(G) - 1$. We have the following observation which can be proved by induction.

**Observation 1.** Let $F$ be any hereditary family of graphs such that any graph $G$ from the family contains a MIS, say $I$ such that for any vertex $v$ of $G$ if $\text{deg}_G(v) = \delta(G)$ then $\text{deg}_{G \setminus I}(v) = \text{deg}_G(v) - 1$. Then $\chi_{\text{ff}}(G) \geq \delta(G) + 1$ for any graph $G$ from $F$.

Unfortunately the family of $C_4$-free graphs does not satisfy the above condition. In this regard it is worthy to work on the following problem.

**Problem.** Find families $F$ of graphs satisfying the following property. Any graph $G$ from $F$ contains a MIS, say $I$ such that for any vertex $v$ of $G$ if $\text{deg}_G(v) = \delta(G)$ then $\text{deg}_{G \setminus I}(v) = \text{deg}_G(v) - 1$.

In Theorem 9 we show that Conjecture 1 holds for any graph which is the complement of a bipartite graph. We need some prerequisites. In a graph $H$, a subset $D$ of edges in $H$ is called an edge dominating set if each edge in $E(H) \setminus D$ has a common end point with an edge in $D$. Let $H$ be any bipartite graph. Set $G = \overline{H}$. Let $\gamma'(H)$ be the smallest size of an edge dominating set in $H$. It was proved in [7] that $\chi_{\text{ff}}(G) = |V(G)| - \gamma'(H)$. We have now the following theorem.

**Theorem 9.** Let $H$ be any bipartite graph and $G$ be the complement of $H$ such that $G$ is $C_4$-free. Then $\chi_{\text{ff}}(G) \geq \delta(G) + 1$.

**Proof.** Let $n$ be the order of $G$. Since $\delta(G) = n - \Delta(H) - 1$, then the inequality $\chi_{\text{ff}}(G) \geq \delta(G) + 1$ is equivalent to $\gamma'(H) \leq \Delta(H)$. Now we use the fact that for any edge dominating set $R$ in a bipartite graph, there is a matching $M$ which is also an edge dominating set and $|M| \leq |R|$. This fact can be easily proved and we omit mentioning its proof here, and refer the reader to [3]. Let $R$ be an edge dominating set in $H$ with $|R| = \gamma'(H)$. Using the previous fact we obtain that $R$ is a matching and therefore $\gamma'(H) \leq \alpha'(H)$. Hence to complete the proof we need to show that $\alpha'(H) \leq \Delta(H)$. We prove the latter inequality by induction on the number of edges. Note that since $G$
is $C_4$-free then $H$ is $2K_2$-free, where by $2K_2$ we mean the graph consisting only of two independent edges.

Let $M$ be any matching of maximum size in $H$ and $e = uv$ be any edge of $M$. Define another graph as $H_0 = H \setminus \{u, v\}$. We have $\alpha'(H_0) \leq \Delta(H_0)$. Hence $\alpha'(H) - 1 \leq \Delta(H)$. To finalize the proof we show that $\Delta(H_0) + 1 \leq \Delta(H)$. Otherwise, since $H_0$ is an induced subgraph of $H$, we have $\Delta(H_0) = \Delta(H)$. Let $x$ be any vertex in $H_0$ such that $\deg_{H_0}(x) = \Delta(H)$. Without loss of generality, we may assume that $u$ and $x$ are in the same bipartite part of $H$. We show that $u$ is adjacent to any neighbor of $x$. Let $w$ be any neighbor of $x$. Since $H$ is $2K_2$-free then the subgraph of $H$ consisting of two edges $uw$ and $xw$ can not be induced in $H$. Now, since $x$ has the maximum degree then $x$ can not be adjacent to $v$ in $H$. Hence $u$ should be adjacent to $w$. But $v$ is adjacent to $u$ and not adjacent to $x$. This means that the degree of $u$ is strictly greater that the degree of $x$, a contradiction with our choice of $x$. This completes the proof. \(\square\)

The following theorem shows that Conjecture \([\square]\) holds for all graphs $G$ with $\delta(G) \leq 3$.

**Theorem 10.** Let $G$ be a $C_4$-free graph with $\delta(G) \leq 3$. Then $\chi_{ff}(G) \geq \delta(G) + 1$.

**Proof.** Theorem obviously holds if $\delta(G) = 1$. Assume that $\delta(G) = 2$. Let $v$ be any vertex of $G$ and $a, b$ any two neighbors of $v$. If $a$ and $b$ are adjacent then the resulting triangle shows that $\chi_{ff}(G) \geq 3$. Assume that $a$ and $b$ are not adjacent. Let $c$ be any neighbor of $b$. If $a$ and $c$ are not adjacent then we obtain an induced $P_4$ on the vertex set $\{v, a, b, c\}$. Hence the desired inequality holds in this case. Assume that $a$ and $c$ are adjacent. Then since $G$ is $C_4$-free and $b$ is not adjacent to $a$ then $v$ is adjacent to $c$. This gives rise to a triangle. Hence in this case too $\chi_{ff}(G) \geq 3$.

Assume now that $\delta(G) = 3$. Let $v$ be any vertex of degree 3. Let $a, b, c$ be the neighbors of $v$.

**Case 1.** Three vertices $a, b, c$ are independent.

In this case we first note that no two vertices from $\{a, b, c\}$ have a common neighbor other than the vertex $v$, since otherwise let $u$ be a common neighbor of $a$ and $b$. The two vertices $a$ and $b$ are independent and $G$ is $C_4$-free. Hence $v$ should be adjacent to $u$. This contradicts $\deg_G(v) = 3$. Now, let $x$ and $y$ (resp. $z$ and $t$) be two neighbors of $a$ (resp. $b$). We have $\{x, y\} \cap \{z, t\} = \emptyset$ and $c$ is not adjacent to any vertex in $\{x, y, z, t\}$. Consider a small bipartite graph $H$ consisting of the bipartite sets $\{x, y\}$ and $\{z, t\}$ with all edges from $G$ among these parts. If there are at most two edges in $H$ then we color $v$ by 4, $a$ by 3, $b$ by 2 and $c$ by 1. The vertex $a$ needs two neighbors of colors 1 and 2; and the vertex $b$ needs one neighbor of color 1. We can easily fulfil these conditions by assigning suitable colors 1 and 2 to the vertices of $H$. If there are exactly three edges in $H$ then (assuming that $x$ is adjacent to $z$) we consider the following coloring. We color $x$ by 4, $y$ by 3, $z$ by 3, $y$ and $t$ by 2 and $a$ and $b$ by 1. Finally, we consider the case that $H$ is a complete bipartite graph. In this case we color $x$ by 4, $z$ by 3, $a$ and $b$ by 2; and $t$ and $v$ by 1 (note that $t$ and $v$ are not adjacent). All of these pre-colorings are partial Grundy colorings with 4 colors. This completes the proof in Case 1.

**Case 2.** $a$ and $b$ are adjacent and $c$ is not adjacent to $a$ and also to $b$. 


In this case we color \( v \) by 4, \( a \) by 2, \( b \) by 3 and \( c \) by 1. Let \( d \) be a neighbor of \( b \). We color \( d \) by 1. Note that \( c \) and \( d \) can not be adjacent. If \( a \) is adjacent to \( d \) then we obtain a partial Grundy coloring using four colors. Otherwise, let \( a \) be adjacent to a vertex say \( e \). If \( e \) is adjacent to \( d \) then since \( a \) is not adjacent to \( d \), hence \( b \) should be adjacent to \( e \). In this situation we color \( e \) by 1 and remove the color of \( d \). Now the colors of \( \{ v, a, b, c, e \} \) is a partial Grundy coloring with four colors. But if \( e \) is not adjacent to \( d \), we color both vertices \( e \) and \( d \) by 1. Note that in this case the colors of \( \{ v, a, b, c, d, e \} \) introduce a partial Grundy coloring using four colors.

**Case 3.** \( a \) is adjacent to both \( b \) and \( c \); but \( b \) and \( c \) are not adjacent.

In this case we color \( v \) by 4, \( a \) by 3, \( b \) by 1 and \( c \) by 2. Let \( d \) be a new neighbor of \( c \). If \( b \) and \( d \) are not adjacent then we color both of them by 1. The resulting coloring is a partial Grundy coloring using four colors. But if \( b \) and \( d \) are adjacent, then \( a \) should be adjacent to \( d \). Now consider the 4-cycle on \( \{ v, b, c, d \} \). Since the degree of \( v \) is three then \( v \) can not be adjacent to \( d \). Hence \( b \) and \( c \) should be adjacent. But this is a contradiction.

**Case 4.** The only remaining case is that \( a, b, c \) are all adjacent. But in this case we obtain a clique of size four. It is clear that in this case \( \chi_{FF}(G) \geq 4 \).

We end the paper by mentioning that Conjecture 1 is also valid for graphs with minimum degree four. The proof is by checking too many cases. We omit the details.

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