Abstract. For a strongly continuous operator semigroup on a Banach space, we revisit a quantitative version of Datko’s Theorem and the estimates for the constant $M$ satisfying the inequality $\|T(t)\| \leq Me^{\omega t}$, for all $t \geq 0$, in terms of the norm of the convolution and other operators involved in Datko’s Theorem. We use techniques recently developed by B. Helffer and J. Sjöstrand for the Hilbert space case to estimate $M$ in terms of the norm of the resolvent of the generator of the semigroup in the right half-plane.

1. Introduction

Jerry Goldstein’s book [G] influenced and inspired several generations of analysts working in the area of operator semigroups and evolution equations. In particular, a significant progress has been made in asymptotic theory of strongly continuous semigroups (see [ABHN, Chapter 5], [CL, Chapters 1-4], [EN, Chapters IV, V], [P, Chapter 4], [vN] and the literature cited therein).

One of the major results in this direction is a celebrated theorem saying that a strongly continuous semigroup on a Hilbert space is uniformly exponentially stable if and only if the norm of the resolvent of its infinitesimal generator $A$ is uniformly bounded in the right half-plane $\{z \in \mathbb{C} : \text{Re} z \geq 0\}$, see, e.g., [ABHN, Theorem 5.2.1] or [EN, Theorem V.1.11]. Various versions of this theorem are due to many authors including L. Gearhart, G. Greiner, I. Herbst, J. Howland, F. Huang and J. Prüss, see, e.g., [ABHN, Section 5.7] for a historical account and further references, and it is usually called the Gearhart-Prüss Theorem.

We recall that for any strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ there exist (possibly large) constants $\lambda$ and $L = L(\lambda)$ such that the following inequality holds:

$$\|T(t)\| \leq Le^{\lambda t} \text{ for all } t \geq 0. \quad (1.1)$$

In many problems, it is easy to obtain a rough exponential estimate of this type, but one is interested in decreasing $\lambda$ as much as possible. The infimum of all $\lambda$ for which there exists an $L = L(\lambda)$ such that (1.1) holds is called the semigroup growth bound, and is denoted by $\omega(T)$. The semigroup is called uniformly exponentially stable if $\omega(T) < 0$, that is, if the inequality

$$\|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0 \quad (1.2)$$

holds.
holds for some negative $\omega$ and $M = M(\omega)$. Another useful quantity is the abscissa of uniform boundedness of the resolvent, $s_0(A)$, defined as the infimum of all real $\omega \in \mathbb{R}$ such that $\{z : \text{Re} z > \omega\} \subset \rho(A)$ and $\sup_{\text{Re} z > \omega} ||R(z, A)|| < \infty$. Then, the Gearhart-Prüss Theorem says that if the semigroup acts on a Hilbert space then $\omega(T) = s_0(A)$.

Naturally, one would like to evaluate the constant $M$ in (1.2) via the uniform bound of the norm of the resolvent $R(z, A) = (z - A)^{-1}$ in the half-plane $\{z \in \mathbb{C} : \text{Re} z \geq \omega\}$ provided it is finite, that is, via the quantity

$$N := \sup_{\text{Re} z > \omega} ||R(z, A)|| < \infty,$$  

(1.3)

and via the known constants $\lambda, L$ that enter the general inequality (1.1). In particular, the necessity in this evaluation arises in many applied issues related to stability of traveling waves, cf. [GLS1, GLS2, GLSS]. In fact, the first author of the current note has been asked by several applied mathematicians [GHPS] if it is known how $M$ is related to $N$ and $L$. In a recent beautiful paper, B. Helffer and J. Sjöstrand proved the following result relating these constants (see [HS, Proposition 2.1]).

**Theorem 1.1** (B. Helffer and J. Sjöstrand). Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup on a Hilbert space. If $\omega, \lambda \in \mathbb{R}$ are such that $\omega < \lambda$ and (1.1), (1.3) hold, then (1.2) holds with the constant

$$M = L(1 + 2LN(\lambda - \omega)).$$  

(1.4)

In the current (semi-expository) note we examine the situation when the strongly continuous semigroup acts on a **Banach** space $X$. As it is well known, in the Banach space case the Gearhart-Prüss Theorem does not hold (see [ABHN, Example 5.2.2] for an example of $s_0(A) < \omega(T)$ on $L^p(0, 1)$, $p \neq 2$, and more examples in [vN]). For Banach spaces, there are several known replacements of the Gearhart-Prüss Theorem proved by R. Datko, M. Hieber, Y. Latushkin, S. Montgomery-Smith, A. Pazy, F. Räbiger, R. Shibyndkov, J. van Neerven, L. Weis, and others (see again some historical comments in [ABHN, Section 5.7], [CL, LS], [vN] and the bibliography therein). These results are summarized in Theorem 2.5 below, and sometimes are collectively called the Datko Theorem. It says that a strongly continuous semigroup on a Banach space $X$ is uniformly exponentially stable if and only if a convolution operator, $K^+$, is bounded on $L^p(\mathbb{R}_+; X)$. Throughout, we fixed $p$ such that $1 \leq p < \infty$; here $K^+$ is defined by $(K^+ u)(t) = \int_0^t T(t-s)u(s) \, ds$, $t \geq 0$. A quantitative version of this theorem given below, see Theorem 2.5, relates the norm of the convolution operator and the norms of some other operators whose boundedness is also equivalent to the uniform exponential stability of the semigroup.

One can consider the convolution operator, for an $\omega \in \mathbb{R}$, on the exponentially weighted space $L^p_\omega(\mathbb{R}_+; X)$ of the functions $u : \mathbb{R}_+ \to X$ such that $e^{-\omega(\cdot)}u(\cdot) \in L^p(\mathbb{R}_+; X)$, and ask how to evaluate the constant $M$ in (1.2) via the norm of $K^+$ on $L^p_\omega(\mathbb{R}_+; X)$ provided it is finite, that is, via the quantity

$$K := ||K^+||_{B(L^p_\omega(\mathbb{R}_+; X))} < \infty,$$  

(1.5)

and via the known constants $\lambda, L$ in (1.1) (alternatively, by the Datko Theorem 2.5, the operator $K^+$ in (1.5) can be replaced by any other operator mentioned in this theorem). Using, essentially, the techniques of [HS], we show the following result (see the proof in Section 3).
**Theorem 1.2.** Let \( \{T(t)\}_{t \geq 0} \) be a strongly continuous semigroup on a Banach space, \( p \geq 1 \), \( p^{-1} + q^{-1} = 1 \). If \( \omega, \lambda \in \mathbb{R} \) are such that \( \omega < \lambda \) and (1.1), (1.5) hold, then (1.2) holds with the constant

\[
M = L(1 + 4p^{-1/2}q^{-1/2})K(\lambda - \omega)).
\]

One can derive Theorem 1.1 from Theorem 1.2. For, we note that if \( p = 2 \) then (1.6) becomes \( M = L(1 + 2LK(\lambda - \omega)) \) and if \( X \) is a Hilbert space then \( K \leq N \).

The latter inequality is a consequence of the quantitative Datko Theorem 2.5 and Parseval’s identity. Indeed, let us recall that a bounded operator valued function \( m \in L^\infty(\mathbb{R}; B(X)) \) is called an \( L^p \)-Fourier multiplier if the operator \( \mathcal{M}_m \) defined by \( \mathcal{M}_m = \mathcal{F}^{-1}(m(\cdot)\mathcal{F}u(\cdot)) \) is bounded on \( L^p(\mathbb{R}; X) \); here \( \mathcal{F} \) is the Fourier transform.

By Parseval’s identity, if \( X \) is a Hilbert space then \( \mathcal{F} \) is an \( L^2(\mathbb{R}; X) \)-isomorphism, and thus \( \| \mathcal{M}_m \|_{B(L^2(\mathbb{R}; X))} = \sup_{s \in \mathbb{R}} \| m(s) \|_{B(X)} \). An application of the quantitative Datko Theorem 2.5 shows that \( K \) in (1.5) is equal to the norm of \( \mathcal{M}_m \) on \( L^2(\mathbb{R}; X) \) with \( m(s) = R(is, A - \omega) \), \( s \in \mathbb{R} \), see (2.35). Thus, if \( X \) is a Hilbert space, then

\[
K = \| \mathcal{M}_m \|_{B(L^2(\mathbb{R}; X))} = \sup_{s \in \mathbb{R}} \| R(is, A - \omega) \|_{B(X)} \leq \sup_{\text{Re} \geq 0} \| R(z, A - \omega) \|_{B(X)} = N.
\]

This paper originated as a byproduct of a larger project [GLSY] where we intend to prove a generalization of the spectral mapping theorem from [GLS2] for a strongly continuous (but not analytic) semigroup whose generator is induced by a mixed hyperbolic-parabolic system of partial differential equations arising in stability analysis of travelling fronts. Specifically, we need the following result: Given a family of strongly continuous semigroups on a Hilbert space depending on a complex parameter \( \beta \in \Omega \subset \mathbb{C} \), that is, \( \{T(t, \beta)\}_{t \geq 0} \), \( \beta \in \Omega \), let us assume that \( \beta \)-dependent constants \( L(\beta) \) and \( N(\beta) \) in (1.1), (1.3) are bounded from above uniformly for \( \beta \in \Omega \). We then conclude that the constants \( M(\beta) \) in (1.2) are bounded from above uniformly for \( \beta \in \Omega \). The respective Banach space version of this fact is formulated below in Corollary 2.6.

**Notations.** For an operator \( T \) we denote by \( \text{dom} T, \sigma(T), \rho(T), R(T, z) = (z - T)^{-1}, \) respectively, its domain, spectrum, resolvent set, and the resolvent operator. We denote by \( B(X, Y) \) the set of bounded operators from \( X \) into \( Y \), and abbreviate \( B(X) = B(X, X) \).

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## 2. The Quantitative Datko Theorem

Datko’s Theorem asserts that a strongly continuous semigroup \( \{T(t)\}_{t \geq 0} \) with the infinitesimal generator \( A \) acting on a Banach space \( X \) is uniformly exponentially stable if and only if certain operators are bounded. The quantitative version of this theorem relates the norms of these operators.

### 2.1. Convolutions.** We begin by introducing two convolution operators. The operator \( K^+ \) is defined on the space \( L^p(\mathbb{R}^+; X) \), with some \( p \geq 1 \), as the convolution with the operator valued function \( T(\cdot) \), that is, it is defined by the formula

\[
(K^+ u)(t) = \int_0^t T(t - s)u(s) \, ds, \quad t \geq 0.
\]
For any \( \omega \in \mathbb{R} \) let \( L^p_\omega(\mathbb{R}^+; X) \) denote the space with the exponential weight \( e^{-\omega(\cdot)} \), that is, the space of the functions \( u \) such that \( e^{-\omega(\cdot)}u(\cdot) \in L^p(\mathbb{R}^+; X) \) with the norm
\[
\|u\|_{L^p_\omega} = \left( \int_0^\infty \|u(s)e^{-\omega s}\|^p ds \right)^{1/p}.
\]
Let \( J_\omega \) be the isometry acting from \( L^p(\mathbb{R}^+; X) \) onto \( L^p_\omega(\mathbb{R}^+; X) \) by the rule \((J_\omega u)(s) = e^{\omega s}u(s), s \in \mathbb{R}\). Let \( K^+_\omega \) denote the convolution operator defined as in (2.1) but with the semigroup \( \{T(t)\}_{t \geq 0} \) replaced by the rescaled semigroup \( \{T^*_\omega(t)\}_{t \geq 0} \), where \( T^*_\omega(t) = e^{-\omega t}T(t) \). A trivial calculation shows that the operator \( K^+ \) acting on \( L^p_\omega(\mathbb{R}^+; X) \) is isometrically isomorphic to the operator \( K^+_\omega \) acting on \( L^p(\mathbb{R}^+; X) \).

Remark 2.1. The well-known Datko-van Neerven Theorem says that \( \omega(T) < 0 \) if and only if the operator \( K^+ \) is bounded, that is, \( K^+ \in B(L^p(\mathbb{R}^+; X)) \) (see, e.g., [ABHN, Theorem 5.1.2, (i) ⇔ (iii)] or [vN, Theorem 3.3.1 (i) ⇔ (ii)]). The easy “only if” part of this equivalence follows from the fact that the norm of the convolution operator satisfies the estimate \( \|K^+\|_{B(L^p(\mathbb{R}^+; X))} \leq \|T(\cdot)\|_{L^1(\mathbb{R}^+)} \). The “if” part is usually proved by contradiction. Alternatively, it can be derived using Theorem 1.2. Indeed, as we will show in Remark 2.2 below, if \( K^+ \in B(L^p(\mathbb{R}^+; X)) \) then \( K^+_\omega \in B(L^p(\mathbb{R}^+; X)) \) for negative \( \omega \) with small \( |\omega| \). Theorem 1.2 gives an estimate of \( M \) in (1.2) in terms of \( K \) or \( K^+_\omega \) thus implying that \( \omega(T) < 0 \).

Next, we introduce an operator, \( K \), defined on the space \( L^p(\mathbb{R}; X) \) as convolution with the operator valued function \( T(\cdot) \), that is, by the formula
\[
(Ku)(t) = \int_{-\infty}^t T(t-s)u(s)ds, \quad t \in \mathbb{R}.
\]
Similarly to the semi-line case, for any \( \omega \in \mathbb{R} \) one can consider the operator \( K^\omega \) on the space \( L^p_\omega(\mathbb{R}, X) \) with the norm \( \|u\|_{L^p_\omega} = \left( \int_{-\infty}^\infty \|u(s)e^{-\omega s}\|^p ds \right)^{1/p} \); this operator is again isometrically isomorphic to the operator \( K^\omega \) on \( L^p(\mathbb{R}; X) \), where \( K^\omega \) is defined as in (2.2), but with the semigroup \( \{T(t)\}_{t \geq 0} \) replaced by the rescaled semigroup \( \{T^*_\omega(t)\}_{t \geq 0} \).

It is known that \( K \in B(L^p(\mathbb{R}; X)) \) if and only if \( \omega(T) < 0 \) (see, e.g., [CL, Subsection 4.2.1]). Indeed, we recall that the semigroup \( \{T(t)\}_{t \geq 0} \) is called hyperbolic if \( \sigma(T(1)) \cap \{z \in \mathbb{C} : |z| = 1\} = \varnothing \). Let \( P \) denote the Riesz spectral projection corresponding to the part of the spectrum of \( T(1) \) located inside of the unit disc. In particular, \( P = I \) if and only if \( \omega(T) < 0 \). The main Green’s function \( G \) is defined by the formula \( G(t) = T^tP \) for \( t > 0 \) and \( G(t) = -T^t(I - P) \) for \( t < 0 \), see, e.g., [CL, Section 4.2]. It is known that the semigroup \( \{T(t)\}_{t \geq 0} \) is hyperbolic if and only if the operator of convolution with \( G \) is bounded on \( L^p(\mathbb{R}, X) \), see [CL, Theorem 4.22]. The operator of convolution with \( G \) is equal to \( K \) from (2.2) if and only if \( P = I \). This shows that \( K \in B(L^p(\mathbb{R}; X)) \) if and only if \( \omega(T) < 0 \).

Next, we show that if either one of the operators \( K^+ \) or \( K^\omega \) is bounded (equivalently, the inequality \( \omega(T) < 0 \) holds) then
\[
\|K^+\|_{B(L^p(\mathbb{R}^+; X))} = \|K\|_{B(L^p(\mathbb{R}; X))}.
\]
To establish the inequality “\( \leq \)” in (2.3), let us use the following notation. If \( u \) is a function on \( \mathbb{R} \), then \( u|_{\mathbb{R}^+} \) will denote its restriction on \( \mathbb{R}^+ \); if \( v \) is a function on
Lemma 3.5 shows that 
was used to make sure that Fubini's Theorem applies in the proof of the identity 
where both norms are finite if and only if \( \omega \). 

more information regarding operator valued Fourier multipliers.

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and only if \( \sigma \). 

Fourier multipliers and evolution semigroup generators. Let \( S(\mathbb{R}) \) denote the Schwartz class of rapidly decaying functions. For \( \phi \in S(\mathbb{R}) \) and \( x \in X \) we denote by \( \phi \otimes x \) the \( X \)-valued function from \( S(\mathbb{R}; X) \) defined by \( (\phi \otimes x)(s) = \phi(s) x \). 
The linear span of the functions \( \phi \otimes x \) is dense in \( L^p(\mathbb{R}; X) \). Given an operator-valued function \( m \in L^\infty(\mathbb{R}; B(X)) \), we define \( \mathcal{M}_m \) on \( \phi \otimes x \) by \( \mathcal{M}_m(\phi \otimes x) = F^{-1}(m(\cdot)F(\phi \otimes x)) \), where

\[
(Fu)(t) = \int_{\mathbb{R}} u(s)e^{ist} \, ds, \quad (F^{-1}u)(s) = \frac{1}{2\pi} \int_{\mathbb{R}} u(t)e^{ist} \, dt
\]  

(2.6) are the Fourier transforms. We say that \( m \) is an \( L^p(\mathbb{R}; X) \)-Fourier multiplier, if the operator \( \mathcal{M}_m \) extends to a bounded operator on \( L^p(\mathbb{R}; X) \). We refer to \([A, H1]\) for more information regarding operator valued Fourier multipliers.

A strongly continuous semigroup \( \{T(t)\}_{t \geq 0} \) on a Banach space \( X \) is hyperbolic if and only if \( \sigma(A) \cap i\mathbb{R} = \emptyset \) and \( m(s) = (is-A)^{-1} \) is an \( L^p(\mathbb{R}; X) \)-Fourier multiplier, see \([LS, Theorem 2.7 (1) \Leftrightarrow (2)]\). The assertion \( m \in L^\infty(\mathbb{R}; B(X)) \) here is a part of the definition of \( \mathcal{M}_m \). Moreover, \( \omega(T) < 0 \) if and only if \( \sigma(A) \subseteq \{ z : \text{Re} z < 0 \} \), \( s_0(A) \leq 0 \) and \( m \) is an \( L^p(\mathbb{R}; X) \)-Fourier multiplier, cf. \([LR, Corollary 3.8]\) and \([H2]\).

Assume that \( \omega(T) < 0 \). Then one can apply Fubini's Theorem to check that \( \mathcal{F}K(\phi \otimes x) = m(\cdot)\mathcal{F}(\phi \otimes x) \), for all \( \phi \in S(\mathbb{R}) \) and \( x \in X \). Thus, \( K = \mathcal{M}_m \) via a density argument, and

\[
||K||_{B(L^p(\mathbb{R}; X))} = ||M_m||_{B(L^p(\mathbb{R}; X))}, \tag{2.7}
\]

where both norms are finite if and only if \( \omega(T) < 0 \). Here, the condition \( \omega(T) < 0 \) was made to sure that Fubini's Theorem applies in the proof of the identity \( K(\phi \otimes x) = \mathcal{M}_m(\phi \otimes x) \). However, a simple analytic continuation argument given in \([LR, Lemma 3.5]\) shows that \( K(\phi \otimes x) = \mathcal{M}_m(\phi \otimes x) \) provided \( \sigma(A) \subseteq \{ \text{Re} z < 0 \} \) and \( s_0(A) \leq 0 \). Thus, conditions \( \mathcal{M}_m \in B(L^p(\mathbb{R}; X)) \), \( \sigma(A) \subseteq \{ \text{Re} z < 0 \} \) and \( s_0(A) \leq 0 \) imply \( K \in B(L^p(\mathbb{R}; X)) \) and (2.7). On the other hand, if \( K \in B(L^p(\mathbb{R}; X)) \) then \( K^+ \in B(L^p(\mathbb{R}^+; X)) \) by the proof of the inequality “\( \leq \)” in (2.3). This yields
\(\omega(T) < 0\) which implies \(\sigma(A) \subset \{ \Re z < 0 \}\) and \(s_0(A) \leq 0\) thus proving the equalities \(K = M_m\) and (2.7).

We will now define the evolution semigroups and their generators on the spaces \(L^p(\mathbb{R}_+; X)\) and \(L^p(\mathbb{R}; X)\) and relate the inverses of the generators to the convolution operators introduced in Subsection 2.1.

The evolution semigroup \(\{E^+_A(t)\}_{t \geq 0}\) on \(L^p(\mathbb{R}_+; X)\) is defined as

\[
(E^+_A(t)u)(s) = \begin{cases} e^{tA}u(s - t) & \text{for } s \geq t, \\ 0 & \text{for } 0 \leq s < t. \end{cases}
\]  

(2.8)

Let \(\partial_0\) be the operator of differentiation on \(L^p(\mathbb{R}_+; X)\) with the domain \(\text{dom } \partial_0 = \{ f \in W^1_0(\mathbb{R}_+; X) : f(0) = 0 \}\). We keep notation \(A\) for the operator on \(L^p(\mathbb{R}_+; X)\) defined by \((Au)(s) = Au(s), s \in \mathbb{R}_+,\) for all \(u \in \text{dom } A = \{ u \in L^p(\mathbb{R}_+; X) : u(s) \in \text{dom}_X A\}\). Let \(-\partial_0 + A\) denote the sum of the two operators defined on \(\text{dom}(\partial_0) \cap \text{dom}_L(\mathbb{R}_+; X)\). Let \(G^+_A\) denote the infinitesimal generator of the semigroup \(\{E^+_A(t)\}_{t \geq 0}\). One can view \(E^+_A(t)\) as the product \(T(t)S^+_t\), where \(S^+_t = e^{-t\partial_0}\) is the shift semigroup defined on \(L^p(\mathbb{R}_+; X)\) by \((S^+_tu)(s) = u(s - t)\) for \(s \geq t\) and \((S^+_tu)(s) = 0\) for \(0 \leq s < t\). Since the semigroups \(T(t)\) and \(S^+_t\) commute, \(G^+_A\) is the closure of the operator \(-\partial_0 + A\). [CL, Remark 2.35 and CL, Section 2.2.3].

In particular, if \(\phi = \psi|_{\mathbb{R}_+}\) for some \(\psi \in \mathcal{S}(\mathbb{R})\) with \(\psi(s) = 0\) for \(s \leq 0\), and \(x \in \text{dom } A\) then \(\phi \otimes x \in \text{dom } G^+_A\) and \(G^+_A(\phi \otimes x) = -\phi' \otimes x + \phi \otimes Ax\). Therefore, using integration by parts, for any \(\tau \geq 1\) one has:

\[
(K^+G^+_A(\phi \otimes x))(\tau) = \int_0^\tau (-\phi'(s) T(t-s)x + \phi(s) \otimes AT(t-s)x) \, ds
\]

(2.9)

Also, \(G^+_A(\phi \otimes x) = -\phi \otimes x\) for \(x \in X\) because

\[
(E^+_A(t)K^+(\phi \otimes x))(\tau) = \begin{cases} \int_0^\tau \phi(\tau-s)T(s)x \, ds & \text{for any } \tau \geq t, \\ 0, & \text{for any } 0 \leq \tau < t. \end{cases}
\]

(2.10)

Since the linear span of the functions \(\phi \otimes x\) is dense, it follows that

\(K^+G^+_A = -u\) for all \(u \in \text{dom } G^+_A\) and \(G^+_A K^+ = -u\) for all \(u \in L^p(\mathbb{R}_+; X)\).

Thus, cf. [CL, Section 2.2.3], \(\omega(T) < 0\) if and only if \(K^+ \in \mathcal{B}(L^p(\mathbb{R}_+; X))\) if and only if \(0 \in \rho(G^+_A)\); also,

\[
\|K^+\|_{\mathcal{B}(L^p(\mathbb{R}_+; X))} = \|G^+_A\|_{\mathcal{B}(L^p(\mathbb{R}_+; X))}^{-1}.
\]

(2.11)

Remark 2.2. We conclude this fragment by showing how Theorem 1.2 implies the conclusion \(\omega(T) < 0\) in the Datko-van Neerven Theorem (see [ABHN, Theorem 5.1.2 (iii) \(\Rightarrow \) (i)], [vN, Theorem 3.3.1 (ii) \(\Rightarrow \) (i)] and (i) \(\Rightarrow \) (i)) in Theorem 2.5 below. Indeed, if \(K^+ \in \mathcal{B}(L^p(\mathbb{R}_+; X))\) then \(0 \in \rho(G^+_A)\) because \(K^+ = (G^+_A)^{-1}\). But then, by a standard perturbation argument, \(0 \in \rho(G^+_A - \omega)\) for a sufficiently close to zero negative \(\omega\) because \(G^+_A - \omega = G^+_A - \omega\). Then \(K^+_\omega = (G^+_A - \omega)^{-1} \in \mathcal{B}(L^p(\mathbb{R}_+; X))\). Now (1.2) holds by Theorem 1.2, and thus \(\omega(T) < 0\).

Next, let us introduce the evolution semigroup \(\{E_A(t)\}_{t \geq 0}\), defined on \(L^p(\mathbb{R}, X)\) by \((E_A(t)u)(s) = e^{tA}u(s - t), s \in \mathbb{R}\). The generator of this semigroup will be
denoted by $G_A$. As before, $G_A$ is the closure of the operator $-\partial + A$, where $\partial$ is the operator of differentiation on $L^p(\mathbb{R}, X)$ with the domain $W^1_p(\mathbb{R}, X)$. If $\phi \in S(\mathbb{R})$ and $x \in \text{dom}_X A$ then $G_A(\phi \otimes x) = -\phi' \otimes x + \phi \otimes Ax$. Similarly to the semi-line case, we have

$$\mathcal{K}G_A u = -u$$

for $u \in \text{dom} G_A$ and $G_A \mathcal{K} u = -u$ for $u \in L^p(\mathbb{R}, X)$.

Thus, cf. [CL, Sections 2.2.2, 4.2.1], $\omega(T) < 0$ if and only if $\sigma(G_A) \subset \{\text{Re} z < 0\}$, which turn is true if and only if $\mathcal{K} \in \mathcal{B}(L^p(\mathbb{R}, X))$; also

$$\|\mathcal{K}\|_{\mathcal{B}(L^p(\mathbb{R}, X))} = \|(G_A)^{-1}\|_{\mathcal{B}(L^p(\mathbb{R}, X))}.$$  \hspace{1cm} (2.12)

Indeed, one of the main properties of the evolution semigroup is that the growth bound of the semigroup \{T(t)\}_{t \geq 0} is equal to the spectral bound of the semigroup \{E_A(t)\}_{t \geq 0}, that is, $\omega(T) = \sup\{\text{Re} z : z \in \sigma(G_A)\}$, see [CL, Corollary 2.40]. This yields the first of the two equivalent statements just made. Since $\mathcal{K} \in \mathcal{B}(L^p(\mathbb{R}, X))$ if and only if $\mathcal{K}^+ \in \mathcal{B}(L^p(\mathbb{R}^+_X))$ by (2.3), the second equivalence follows from the Datko-van Neerven Theorem (see [ABHN, Theorem 5.1.2 (ii) $\Rightarrow$ (i)]).

We will now show that condition $\sigma(G_A) \subset \{\text{Re} z < 0\}$ implies

$$\|(G_A)^{-1}\|_{\mathcal{B}(L^p(\mathbb{R}, X))} = \sup_{\text{Re}z \geq 0} \|R(G_A, z)\|.$$ \hspace{1cm} (2.13)

Indeed, another nice property of the evolution semigroup generator is that $\sigma(G_A)$ is invariant with respect to vertical translations and that

$$\|R(G_A, z)\|_{\mathcal{B}(L^p(\mathbb{R}, X))} = \|R(G_A, z + i\zeta)\|_{\mathcal{B}(L^p(\mathbb{R}, X))}$$

for all $z \in \rho(G_A), \zeta \in \mathbb{R}$, \hspace{1cm} (2.14)

see [CL, Proposition 2.36(b)] and its proof. Since $G_A$ is a semigroup generator, $\|R(G_A, s)\|_{\mathcal{B}(L^p(\mathbb{R}, X))} \rightarrow 0$ as $s \rightarrow \infty, s \in \mathbb{R}$. Thus, if $\sigma(G_A) \subset \{\text{Re} z < 0\}$ then

$$\sup_{\text{Re}z \geq 0} \|R(G_A, z)\|_{\mathcal{B}(L^p(\mathbb{R}, X))} < \infty. \hspace{1cm} (2.15)$$

Now (2.13) holds by applying the maximal principle to $R(G_A, \cdot)$ on a long horizontal rectangle with the left side belonging to $i\mathbb{R}$. A similar argument based on [CL, Proposition 3.21] shows that $G_A$ and $\mathbb{R}^+$ in (2.13) can be replaced by $G_A^+$ and $\mathbb{R}^+_X$.

Finally, we prove that the condition $\sigma(G_A) \subset \{\text{Re} z < 0\}$ is equivalent to the assertions $(G_A)^{-1} \in \mathcal{B}(L^p(\mathbb{R}^+_X)), \sigma(A) \subset \{\text{Re} z < 0\}$, and $s_0(A) \leq 0$. Indeed, if $\sigma(G_A) \subset \{\text{Re} z < 0\}$ then $\omega(T) < 0$ as remarked above, which implies the required assertions. On the other hand, if $(G_A)^{-1} \in \mathcal{B}(L^p(\mathbb{R}, X))$ then $\{T(t)\}_{t \geq 0}$ is hyperbolic by [LS, Theorem 2.7 (1) $\Rightarrow$ (2)]. Moreover, cf. [LS, Remark 2.11], using elementary properties of Fourier transform, we have

$$\mathcal{M}_m G_A(\phi \otimes x) = -\phi \otimes x, \text{ for } x \in \text{dom} A, \phi \in S(\mathbb{R}),$$

$$G_A \mathcal{M}_m(\phi \otimes x) = -\phi \otimes x, \text{ for } x \in X, \phi \in S(\mathbb{R}),$$

yielding $\mathcal{M}_m = (G_A)^{-1} \in \mathcal{B}(L^p(\mathbb{R}, X))$. This, combined with the assertions $\sigma(A) \subset \{\text{Re} z < 0\}$ and $s_0(A) \leq 0$, yields $\mathcal{K} \in \mathcal{B}(L^p(\mathbb{R}, X))$, as has been noted before, and therefore $\omega(T) < 0$.

### 2.3. Datko’s constant

Let us consider an operator, $\mathcal{D}$, acting from the Banach space $X$ into $L^p(\mathbb{R}^+_X)$ by the formula $(\mathcal{D}x)(t) = T(t)x$, $x \in X$, $t \geq 0$. This operator is bounded if and only if there is a Datko constant, $\mathcal{D}$, such that the following (Datko-Pazy) inequality is satisfied,

$$\int_0^\infty \|T(t)x\|_X^p \, dt \leq D^p \|x\|_X^p \text{ for all } x \in X,$$ \hspace{1cm} (2.17)
and $\|D\|_{B^{(X;L^p([R_+:X])}}}$ is the infimum of all constants $D$ such that (2.17) holds (cf. [ABHN, eqn. (5.5.4)] or [P, eqn. (4.4.3)]). With a slight abuse of notation we will sometimes denote $\|D\|_{B^{(X;L^p([R_+:X])}}}$ by $D$. By the classical Datko-Pazy Theorem, $D < \infty$ if and only if the semigroup $\{T(t)\}_{t \geq 0}$ is uniformly exponentially stable, see [ABHN, Theorem 5.1.2, (i)$\iff$(ii)] or [P, Theorem 4.1]. We will now obtain an estimate from above for $D$ in terms of $\|K\|_{B^{(L^p([R_+:X])}}}$, and an estimate from above for $\|K\|_{B^{(L^p([R_+:X])}}}$ in terms of $D$.

Given (1.1), fix any positive $w > \lambda$ and denote $g(t) = e^{-wt}T(t)x$ for an $x \in X$ and all $t \geq 0$. Then
\begin{equation}
(1 - e^{-wt})T(t)x = \int_0^t e^{-ws}T(t)x \, ds = \int_0^t (T(t-s)e^{-ws}T(s)x \, ds = (K^+g)(t), \quad t \geq 0,
\end{equation}
and (1.1) imply
\begin{align}
\|T(\cdot)x\|_{L^p([R_+:X])} &= w \left\| \int_0^t \left( T(\cdot)x - e^{-w(\cdot)}T(\cdot)x + e^{-w(\cdot)}T(\cdot)x \right) \right\|_{L^p([R_+:X])} \\
&\leq w \left\| K^+g \right\|_{L^p([R_+:X])} + \|g\|_{L^p([R_+:X])} \\
&\leq \left( w \left\| K^+ \right\|_{B^{(L^p([R_+:X])}}} + 1 \right) \left( \int_0^\infty e^{-wpt}L^p e^{\lambda pt} \, dt \right)^{\frac{1}{p}} \|x\| \\
&= L \left( w \left\| K^+ \right\|_{B^{(L^p([R_+:X])}}} + 1 \right) \(p - \lambda)^{-1/p} \|x\|.
\end{align}
Thus $D = \|D\|_{B^{(X;L^p([R_+:X])}}}$ is majorated by the pre-factor in (2.19).

To obtain an estimate from above for $\|K\|_{L^p([R_+:X])}$ in terms of $D$ satisfying (2.17), we remark that (2.17) implies (1.2) with some $\omega < 0$ by the Datko-Pazy Theorem (see [ABHN, Theorem 5.1.2 (i)$\iff$(ii)]). Since $K^+$ is a convolution operator,
\begin{equation}
\|K^+\|_{B^{(L^p([R_+:X])}}} \leq \|T(\cdot)\|_{L^1([R_+:X])} \leq M \int_0^\infty e^{\omega t} \, dt = -M\omega^{-1}.
\end{equation}
An effective estimate for $M$ in terms of $D$ is given in the next proposition whose proof follows the standard argument in the classical Datko-Pazy Theorem, see [P, Theorem 4.1], [EN, Theorem V.1.8].

**Proposition 2.3.** Assume (1.1) with $\lambda > 0$ and (2.17). Denote, for brevity, $C = L(p\lambda D^p + 1)^{1/p}$. Then (1.2) holds for each $\omega$ satisfying $-C^{-1}(DCE^{1/C})^{-p} < \omega < 0$ and $M = CE^{1/C}$.

**Proof.** *Step 1.* We claim that
\begin{equation}
\|T(t)\|_{B^{(X)}} \leq C := L(p\lambda D^p + 1)^{1/p}, \quad \text{for all } t \geq 0.
\end{equation}
Indeed, for $x \in X$ and all $t \geq 0$ we infer
\begin{align}
\langle p\lambda \rangle^{-1} (1 - e^{-p\lambda t})\|T(t)x\|^p &= \int_0^t e^{-p\lambda s}\|T(s)T(t-s)x\|^p \, ds \\
&\leq L_p \int_0^t \|T(t-s)x\|^p \, ds = L^p \int_0^t \|T(s)x\|^p \, ds \leq L^p D^p \|x\|^p.
\end{align}
This and (1.1) imply
\begin{equation}
\|T(t)\|_{B^{(X)}} \leq \sup_{t \geq 0} \left\{ Le^{\lambda t}, LD \langle (p\lambda)^{-1} (1 - e^{-p\lambda t}) \rangle^{-1/p} \right\} = Le^{\lambda t} = C,
\end{equation}
where \( t = t_0 \) is the unique solution of the equation \( e^{\lambda t} = D((p\lambda)^{-1}(1-e^{-p\lambda}))^{-1/p} \).

Finding this solution proves claim (2.21).

**Step 2.** We claim that
\[
\lim_{t \to \infty} \|T(t)\|_{\mathcal{L}(X)} = 0. \tag{2.24}
\]

Indeed, by (2.17) and (2.21), for any \( x \in X \) and \( t > 0 \),
\[
t \|T(t)x\| = \int_0^t \|T(s)T(t-s)x\|^p ds \leq C^p \int_0^t \|T(t-s)x\|^p ds \leq C^p D^p \|x\|, \tag{2.25}
\]
yielding \( \|T(t)\|_{\mathcal{L}(X)} \leq CDt^{-1} \) and (2.24).

**Step 3.** Denote \( \alpha = C^{-1}e^{-1/C} \). By (2.24), for every \( x \in X \) there exists a finite positive number \( t(x) \) defined as
\[
t(x) = \sup \{ t > 0 : \|T(s)x\| \geq \alpha \|x\|, \text{ for all } 0 \leq s \leq t \}. \tag{2.26}
\]

Using (2.17), we have
\[
t(x)\alpha^p \|x\|^p \leq \int_0^{t(x)} \|T(s)x\|^p ds \leq D^p \|x\|^p, \tag{2.27}
\]
and thus \( t(x) \leq (D/\alpha)^p \). Suppose that \( t_1 > (D/\alpha)^p \). Then
\[
\|T(t_1)x\| \leq \|T(t_1-t(x))\| \|T(t(x))x\| \leq C\alpha \|x\| = e^{-1/C} \|x\| \tag{2.28}
\]
due to (2.21) and (2.26). Fix any \( t_1 > (D/\alpha)^p \) and let \( t = nt_1 + s, \ 0 \leq s < t_1 \), \( n = 1,2,\ldots \). Then
\[
\|T(t)\| \leq \|T(s)\| \|T(t_1)\|^n \leq C(e^{-1/C})^n = C(e^{-1/C})^{-s/t_1} (e^{-1/C})^{t/t_1}
\leq Ce^{1/C} e^{-(t_1/C)^{-1}t}, \text{ for all } t \geq t_1. \tag{2.29}
\]

If \( 0 \leq t < t_1 \) then, by (2.21),
\[
\|T(t)\| \leq C = Ce^{1/C} e^{-(t_1/C)^{-1}t} e^{-\frac{1}{p} + \frac{1}{1-p}} \leq Ce^{1/C} e^{-(t_1/C)^{-1}t}. \tag{2.30}
\]

Thus, we conclude that if \( t_1 > (D/\alpha)^p = (DCe^{1/C})^p \) then
\[
\|T(t)\| \leq Ce^{1/C} e^{-(t_1/C)^{-1}t}, \text{ for all } t \geq 0. \tag{2.31}
\]

**Step 4.** We are ready to finish the proof of the proposition. Assume that
\( 0 > \omega > -C^{-1}(DCe^{1/C})^{-p} \) and choose \( t_1 > 0 \) such that the following inequalities hold:
\[
-C^{-1}(DCe^{1/C})^{-p} < -(t_1C)^{-1} < \omega. \tag{2.32}
\]
Then \( t_1 > (DCe^{1/C})^p \) and (2.31) yields (1.2) with \( M = Ce^{1/C} \), as required.

In particular, letting \( \omega = -\frac{1}{2}C^{-1}(DCe^{1/C})^{-p} \) in (2.20) yields:
\[
\|K^+\|_{\mathcal{B}(L^p(\mathbb{R}^n;X))} \leq 2D^p C^{2+p} e^{(1+p)/C}, \tag{2.33}
\]
where \( C = L(p\lambda D^p + 1)^{1/p} \).

**Remark 2.4.** Of course, the range for \( \omega \) in Proposition 2.3 is not optimal: By the quantitative version of Datko’s Theorem proved by J. Van Neerven, one has \( \omega(T) \leq -1/(pD) \), see. [vN, Theorem 3.18]. It would have been interesting to obtain an optimal estimate in (2.33).

\[\diamond\]
2.4. Summary of the results. We summarize the results obtained earlier in this section as follows. We recall the notation \( m(s) = R(is; A), s \in \mathbb{R} \), and emphasize again that the Fourier multiplier \( M_m = \mathcal{F}^{-1}(m(\cdot)\mathcal{F}) \) is defined provided \( \sup_{s \in \mathbb{R}} \|m(s)\| < \infty \). The convolution operators \( \mathcal{K}^+ \) and \( \mathcal{K} \) are defined in Subsection 2.1. The operators \( G_A = -\partial_0 + A \) and \( G_A^+ = -\partial_0 + A \) are defined in Subsection 2.2. The operator \( (Dx)(t) = T(t)x, t \geq 0, x \in X \), is defined in Subsection 2.3. Also, \( s_0(A) \) denotes the abscissa of uniform boundedness of the resolvent and \( \omega(T) \) denotes the growth bound.

**Theorem 2.5** (Quantitative Datko’s Theorem). For a strongly continuous semigroup \( \{T(t)\}_{t \geq 0} \) on a Banach space \( X \), the following assertions are equivalent:

(i) \( \omega(T) < 0 \);
(ii) \( \mathcal{K}^+ \in \mathcal{B}(L^p(\mathbb{R}_+; X)) \);
(iii) \( \mathcal{K} \in \mathcal{B}(L^p(\mathbb{R}; X)) \);
(iv) \( \mathcal{M}_m \in \mathcal{B}(L^p(\mathbb{R}; X)) \) and \( \sigma(A) \subset \{z : \text{Re}z < 0\} \), \( s_0(A) \leq 0 \);
(v) \( (G_A^+)^{-1} \in \mathcal{B}(L^p(\mathbb{R}_+; X)) \);
(vi) \( \sigma(G_A) \subset \{z : \text{Re}z < 0\} \);
(vii) \( D \in \mathcal{B}(X; L^p(\mathbb{R}_+; X)) \).

Moreover, if one of the equivalent assertions holds then \( \omega(T) \leq -1/(pD) \) and

\[
\|\mathcal{K}^+\|_{\mathcal{B}(L^p(\mathbb{R}_+; X))} = \|\mathcal{K}\|_{\mathcal{B}(L^p(\mathbb{R}; X))} = \|\mathcal{M}_m\|_{\mathcal{B}(L^p(\mathbb{R}; X))} = \|(G_A^+)^{-1}\|_{\mathcal{B}(L^p(\mathbb{R}_+; X))} = \sup_{\text{Re}z \geq 0} \|R(G_A^+; z)\|_{\mathcal{B}(L^p(\mathbb{R}_+; X))} = \|(G_A)^{-1}\|_{\mathcal{B}(L^p(\mathbb{R}; X))} = \sup_{\text{Re}z \geq 0} \|R(G_A; z)\|_{\mathcal{B}(L^p(\mathbb{R}; X))}.
\]

Furthermore, if \( w > \max\{0, \lambda\} \) and (1.1) holds then we have the estimate

\[
\|D\|_{\mathcal{B}(X; L^p(\mathbb{R}_+; X))} \leq L(w\|\mathcal{K}^+\|_{\mathcal{B}(L^p(\mathbb{R}_+; X))} + 1)\left(p(w - \lambda)\right)^{-1/p},
\]

and, abbreviating \( C = L(p\lambda\|D\|_{\mathcal{B}(X; L^p(\mathbb{R}_+; X))}^p + 1)^{1/p} \), we have the estimate

\[
\|\mathcal{K}^+\|_{\mathcal{B}(L^p(\mathbb{R}_+; X))} \leq 2\|D\|_{\mathcal{B}(X; L^p(\mathbb{R}_+; X))}C^{2+p(1+p)/C}.
\]

Finally, if \( p = 2 \) and \( X \) is a Hilbert space then the equal norms of all operators in (2.34) – (2.37) are estimated as follows:

\[
\|\mathcal{K}^+\|_{\mathcal{B}(L^2(\mathbb{R}_+; X))} = \cdots = \|\mathcal{M}_m\|_{\mathcal{B}(L^2(\mathbb{R}_+; X))} = \sup_{s \in \mathbb{R}} \|R(A, is)\|_{\mathcal{B}(X)} \leq \sup_{\text{Re}z \geq 0} \|R(A, z)\|_{\mathcal{B}(X)}.
\]

**Proof.** Clearly, (i) implies (ii), (iii), (vii) because (1.2) holds with some negative \( \omega \). As indicated in the proof of the inequality \( "\leq" \) in (2.3), assertion (iii) implies (ii). That (ii) implies (i) is the Datko-van Neerven characterization of stability via convolutions, see [ABHN, Theorem 5.1.2 (i) ⇔ (iii)] or [vN, Theorem 3.3.1]. However, the fact that (ii) implies (i) also follows from Theorem 1.2 as described in Remark 2.2. Thus, the first three assertions are equivalent.

The equivalence of (i) and (iv) is proven in [LR, Corollary 3.8]. Alternatively, we indicated the proof of (iii) ⇔ (iv) in Subsection 2.2, see (2.7) and subsequent comments. The equivalence of (i) and (v) is proved in [CL, Proposition 2.43]; it
also follows from the fact that $-K^+ = (G_A^+)^{-1}$, see (2.11). The equivalence of (i) and (vi) is contained in [CL, Corollary 2.40]; also, it follows from the fact that $(G_A - \omega)^{-1} \in B(L^p(\mathbb{R}; X))$ if and only if the rescaled semigroup $T_\omega(t)$ is hyperbolic for some $\omega \geq 0$, and, in turn, by (iv), using [LS], if and only if $m_\omega(s) = R(A - \omega, is)$ is an $L^p(\mathbb{R}; X)$-Fourier multiplier. We also recall that $(G_A)^{-1} = -K$. The equivalence of (vi) and (vii) is discussed at the end of Subsection 2.2. The equivalence of (i) and (vii) is the subject of the classical Datko Theorem [ABHN, Theorem 5.1.2 (i) $\Leftrightarrow$ (ii)]; also, inequalities (2.38), (2.39) show that (vii) $\Leftrightarrow$ (ii).

The estimate $\omega(T) \leq -1/(pD)$ is the quantitative version of Datko’s Theorem, see [vN, Theorem 3.1.8]. Equality (2.34) is proved in (2.3); also, see [CL, Theorem 2.4.9 (iii)] for (2.35) is the quantitative version of Datko’s Theorem, see [vN, Theorem 3.1.8], shows that $T(t, \cdot) \in L^\infty(\Omega, \mathcal{B}(X))$ for each $t$ and $\Omega$ is an open domain in $\mathbb{C}$. Assume that there exists uniform in $\beta \in \Omega$ constants $L$ and $\lambda$ such that $\|T(t, \beta)\|_{\mathcal{B}(X)} \leq Le^{\lambda t}$ for all $t \geq 0$ and $\beta \in \Omega$. Furthermore, assume that $\omega < \lambda$ and that

$$\sup_{\beta \in \Omega} \|K^+_{\beta}\|_{\mathcal{B}(L^p(\mathbb{R}; X))} < \infty, \quad (2.41)$$

where $K^+_{\beta}$ is the convolution on $L^p(\mathbb{R}; X)$ with the semigroup $\{T(\cdot, \beta)\}$. Then, with a uniform in $\beta$ constant $M$, one has $\|T(t, \beta)\|_{\mathcal{B}(X)} \leq Me^{\omega t}$ for all $t \geq 0$ and $\beta \in \Omega$. Moreover, there is a uniform in $\beta \in \Omega$ constant $D$ such that $\omega(T(\cdot, \beta)) \leq \omega - 1/(pD)$. Finally, if $X$ is a Hilbert space then assumption (2.41) can be replaced by the assumption

$$\sup_{\beta \in \Omega} \sup_{Rez > \omega} \|R(A(\beta, z))\| < \infty, \quad (2.42)$$

where $A(\beta)$ is the infinitesimal generator of the semigroup $\{T(t, \beta)\}_{t \geq 0}$.

**Proof.** Computing $M(\beta)$ for $T(\cdot, \beta)$ by (1.6) and using the assumptions, we conclude that $M = \sup_{\beta \in \Omega} M(\beta) < \infty$. The quantitative Datko’s Theorem 2.5, see [vN, Theorem 3.1.8], shows that

$$\omega(T_\omega(\cdot, \beta)) = \omega(T(\cdot, \beta)) - \omega \leq -1/(pD(\beta)) \quad (2.43)$$

for the rescaled semigroup $T_\omega(t, \beta) = e^{-\omega t}T(t, \beta)$ and a constant $D(\beta)$. But then assumption (2.41) and inequality (2.38) imply $D(\beta) \leq D$ for $w > \max\{\lambda, 0\}$ and

$$D = \sup_{\beta \in \Omega} L(w\|K^+_{\beta}\|_{\mathcal{B}(L^p(\mathbb{R}; X))} + 1)(w - \lambda)^{-1/p}, \quad (2.44)$$

thus yielding the desired uniform estimate for $\omega(T(\cdot, \beta))$.

Finally, the Hilbert space part of the corollary follows from (2.40) applied to the rescaled semigroup $\{T_\omega(\cdot, \beta)\}$. 

**Corollary 2.6.** Let $\{T(t, \beta)\}_{t \geq 0}$ be a family of strongly continuous semigroups $\{T(\cdot, \beta)\}$ on a Banach space $X$, parameterized by an auxiliary complex parameter $\beta \in \Omega \subseteq \mathbb{C}$ such that $T(t, \cdot) \in L^\infty(\Omega, \mathcal{B}(X))$ for each $t$ and $\Omega$ is an open domain in $\mathbb{C}$.
3. The proof of Theorem 1.2

Proof. Let us fix an $x \in X$, and denote $u(t) = T(t)x$, for $t \geq 0$. If $\phi$ is a differentiable scalar valued function on $\mathbb{R}^+$ with bounded derivative then

$$
\mathcal{K}^+(\phi' u)(t) = \int_0^t T(t-s)\phi'(s)T(s)ds
$$

(3.1)

For each $t > 0$, let us choose a function $\phi$ on $\mathbb{R}^+$ which is monotonically increasing on the interval $(t/2,t)$ and such that $\phi(s) = 0$ for $0 \leq s \leq t/2$ and $\phi(s) = 1$ for $s \geq t$ so that supp$(\phi') \subset (t/2,t)$. Then (3.1) yields

$$
e^{-\omega t\|T(t)x\|} = e^{-\omega t\|u(t)\|} = e^{-\omega t\|\phi(t)u(t)\|} = e^{-\omega t\|\mathcal{K}^+(\phi' u)(t)\|},
$$

(3.2)

and by the definition of the operator $\mathcal{K}^+$,

$$
e^{-\omega t\|T(t)x\|} \leq \int_{\text{supp}(\phi')} e^{-\omega_2\|T(t-s)\phi' (t-s)u(s)\|} ds.
$$

(3.3)

Next, let us introduce another function, $\psi$, as $\psi(s) = \phi(t-s)$ for $0 \leq s \leq t$ and $\psi(s) = 0$ for $s > t$. Of course, $\phi(s) = \psi(t-s)$ for $0 \leq s \leq t$ and $\phi(s) = 1$ for $s \geq t$, so that supp$(\psi') \subset (0,t/2)$. Replacing $\phi$ by $\psi$ in (3.3) results in

$$
\int_{\text{supp}(\phi')} e^{-\omega t\|T(t-s)\psi'(t-s)u(s)\|} ds
$$

\leq \int_{\text{supp}(\psi')} \left( e^{-\omega(t-s)\|T(t-s)\psi'(t-s)\|_{B(X)}} \right) (e^{-\omega_2\|u(s)\|}) ds.
$$

(3.4)

Taking $1 \leq p$, $q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and applying Holder’s inequality together with (1.1), we infer that (3.4) is smaller than or equal to

$$
\left( \int_{\text{supp}(\psi')} e^{-\omega(t-s)\|T(t-s)\psi'(t-s)\|_{B(X)}} ds \right)^{1/q} \left( \int_{\text{supp}(\phi')} e^{-\omega_2\|u(s)\|} ds \right)^{1/p}
$$

\leq \left( \int_{\text{supp}(\psi')} e^{-\omega s\|T(s)\psi'(s)\|_{B(X)}} ds \right)^{1/q} \left( \int_{\text{supp}(\phi')} e^{-\omega\|u(s)\|} ds \right)^{1/p}
$$

\leq \left( \int_0^{t/2} e^{-\omega q s} |Le^{\lambda s}\psi'(s)|^q ds \right)^{1/q} \left( \int_{t/2}^t e^{-\omega q s} \|1 - \psi(s)\| u(s) \|ds \right)^{1/p},
$$

(3.5)

where we used the fact that $\psi(s) = 0$ for all $s \in \text{supp}(\phi')$. Moreover, $\psi(0) = 1$ and (3.1) imply that for any $s > 0$ one has

$$
-\mathcal{K}^+((\psi')u)(s) = (1 - \psi(s)) u(s).
$$

(3.6)

Hence, we can rewrite (3.5) as a product of two norms and, using (1.1) again, estimate the right-hand-side of (3.3) by the expression

$$
L\|e^{(\lambda-\omega)(\cdot)}\psi'(\cdot)\|_{L^p(0,t/2)} \|\mathcal{K}^+((\psi')'u)\|_{L^q((t/2,t);X)}
$$

\leq L \|\mathcal{K}^+\|_{B(L^p(0,t);X)} \|e^{(\lambda-\omega)(\cdot)}\psi'(\cdot)\|_{L^q(0,t)} \|\psi' u\|_{L^q(0,t);X)}
$$

\leq L \|\mathcal{K}^+\|_{B(L^p(0,t);X)} \|e^{(\lambda-\omega)(\cdot)}\psi'(\cdot)\|_{L^q(0,t)} \|e^{(\lambda-\omega)(\cdot)}\psi'(\cdot)\|_{L^q(0,t);X)}
$$

\leq L^2 \|\mathcal{K}^+\|_{B(L^p(0,t);X)} \|e^{(\lambda-\omega)(\cdot)}\psi'(\cdot)\|_{L^q(0,t)} \|e^{(\lambda-\omega)(\cdot)}\psi'(\cdot)\|_{L^q(0,t);X)}
$$

(3.7)
Therefore, by (3.3), (3.7) we end up with the following conclusion:
\[
e^{-\omega t}\|T(t)\|_{\mathcal{B}(X)} \leq L^2 \|K^+\|_{\mathcal{B}(L^p(\mathbb{R}^+;X))} \|e^{(\lambda-\omega)\cdot t}\|_{L^q(0,t)} \|e^{(\lambda-\omega)\cdot t}\|_{L^p(0,t)},
\]
(3.8)

Letting \( g(s) = e^{(\lambda-\omega)s}\psi'(s) \), and combining (3.8) with \( e^{-\omega t}\|T(t)\| \leq Le^{(\lambda-\omega)t} \), \( t \geq 0 \), we obtain the following formula for the constant \( M \) in (1.2):
\[
M = \sup_{t \geq 0} \min_{\tau \leq t} \left\{ Le^{(\lambda-\omega)t}, L^2\|K^+\|_{\mathcal{B}(L^p(\mathbb{R}^+;X))}\|g\|_{L^p(0,t)}\|g\|_{L^q(0,t)} \right\}.
\]
(3.9)

The right-hand side of (3.9) depends on the choice of the function \( \psi \). Let us choose it to allow an easy calculation of the norms in \( L^p(0,t) \) and \( L^q(0,t) \). Suppose we choose \( \psi \) such that
\[
\psi(s) = (e^{2(\omega-\lambda)s} - e^{(\omega-\lambda)t})/2 \quad \text{for} \quad 0 \leq s \leq t/2 \quad \text{and} \quad \psi(s) = 0 \quad \text{for} \quad s \geq t/2.
\]
Then
\[
g = 2(\omega - \lambda)e^{(\omega-\lambda)s}(1 - e^{(\omega-\lambda)t})^{-1} \quad \text{for} \quad 0 \leq s \leq t/2 \quad \text{and} \quad g(s) = 0 \quad \text{for} \quad s \geq t/2.
\]
Computing
\[
\|g(\cdot)\|_{L^p(0,t)} = 2(\lambda - \omega)^{1/p}p^{-1/p}(1 - e^{(\omega-\lambda)t})^{-1/p}(1 - e^{(\omega-\lambda)t})^{-1}
\]
and
\[
\|g\|_{L^q(0,t)} = R(1 - \tau^p)^{1/q}(1 - \tau^q)^{1/q}(1 - \tau^2)^{-2}, \quad \tau \in (0, 1),
\]
we infer that equality (3.9) becomes
\[
M = \sup_{t \geq 0} \min_{\tau \in (0,1)} \left\{ Le^{(\lambda-\omega)t}, R(1 - e^{(\omega-\lambda)\cdot t})^{1/2}(1 - e^{(\omega-\lambda)\cdot t})^{1/2}(1 - e^{(\omega-\lambda)t})^{-2} \right\}
\]
\[= L \sup_{\tau \in (0,1)} \min\{\tau^{-2}, f(\tau)\}.
\]
(3.11)

To finish the proof of Theorem 1.2, we will now show the inequality
\[
\sup_{\tau \in (0,1)} \min\{\tau^{-2}, f(\tau)\} \leq R + 1,
\]
(3.12)
which yields (1.6).

First, we note that
\[
(1 - \tau^p)^{1/p}(1 - \tau^q)^{1/q} \leq 1 - \tau^2, \quad \tau \in (0, 1).
\]
(3.13)
Indeed, letting \( a_1 = (1 - \tau^p)^{1/p}, b_1 = (1 - \tau^q)^{1/q}, a_2 = b_2 = \tau \) and applying Hölder inequality to \( \sum a_i b_i \)
yields (3.13).

Next, we claim that there is a unique \( \tau_0 \in (0,1) \) such that \( \tau_0^{-2} = f(\tau_0) \).

Assuming the claim, we prove (3.12) as follows. If \( \tau_0^{-2} = f(\tau_0) \) then
\[
\tau_0^{-2} - 1 = (1 - \tau_0^p)\tau_0^{-2} = R(1 - \tau_0^p)^{1/p}(1 - \tau_0^q)^{1/q}(1 - \tau_0^2)^{-1} \leq R
\]
(3.14)
by (3.13). Thus, \( \tau_0^{-2} = f(\tau_0) \leq R + 1 \). By inspection, \( f(0) = R \) and \( \lim_{\tau \to 1} = +\infty \).

Using the claim above, it follows that \( \min\{\tau^{-2}, f(\tau)\} \) is equal to \( f(\tau) \) for \( \tau \leq \tau_0 \),
and to \( \tau^{-2} \) for \( \tau \geq \tau_0 \). Thus, to establish (3.12), it remains to show that \( f(\tau) \leq R + 1 \)
provided \( \tau \leq \tau_0 \) that is, provided \( f(\tau) \leq \tau^{-2} \). We will consider two cases: First, if \( \tau^2 \geq (R + 1)^{-1} \) and \( f(\tau) \leq \tau^{-2} \) then \( f(\tau) \leq \tau^{-2} \leq R + 1 \), as required. Second, if
\( \tau^2 < (R+1)^{-1} \) and \( f(\tau) \leq \tau^{-2} \) then \( 1 - \tau^2 > 1 - (R+1)^{-1} = R(R+1)^{-1} \). Using this and (3.13), we infer:

\[
f(\tau) = R(1 - \tau^2)^{-1} \cdot (1 - \tau^p)^{1/p} (1 - \tau^q)^{1/q} (1 - \tau^2)^{-1} \leq (R+1),
\]

which completes the proof of the required inequality \( f(\tau) \leq R + 1 \) for all \( \tau \leq \tau_0 \).

It remains to show the claim. Letting \( h(\tau) = \tau^2 f(\tau) \), we note that \( \tau_0^{-2} = f(\tau_0) \) if and only if \( h(\tau_0) = 1 \). By inspection, \( h(0) = 0 \) and \( \lim_{\tau \to 1^-} h(\tau) = +\infty \), and thus it suffices to show that \( h'(\tau) > 0 \) for \( \tau \in (0,1) \). Logarithmic differentiation yields \( h'(\tau) = h_0(\tau)((1 - 2\tau^p + \tau^q)(1 - \tau^q) + (1 - 2\tau^q + \tau^p)(1 - \tau^p)) \), where \( h_0(\tau) = h(\tau)/(\tau(1 - \tau^2)(1 - \tau^p)(1 - \tau^q)) \). Since \( 1 + \tau^2 > 2\tau > 2\max\{\tau^p, \tau^q\} \) for \( \tau \in (0,1) \) and \( p, q \geq 1 \), one has \( h'(\tau) > 0 \) as needed. \( \square \)

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