A COMPLETELY INTEGRABLE SYSTEM FOR THE DETERMINANT

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Abstract. We investigate topological properties of a completely integrable system on $S^2 \times S^2 \times S^2$ which was recently shown to have a Lagrangian fiber diffeomorphic to $\mathbb{R}P^3$ not displaceable by a Hamiltonian isotopy [15]. This system can be viewed as integrating the determinant, or alternatively, as integrating a classical Heisenberg spin chain. We show that the system has non-trivial topological monodromy and relate this to the geometric interpretation of its integrals.

Contents

1. Introduction 1
2. Notation and Conventions 2
3. An Integrable Heisenberg Spin Chain 4
4. Image of the Moment Map 6
5. Connectedness of Regular Level Sets 8
6. Topological Monodromy Around the Critical Line 9
7. Appendix: Non-Degeneracy Computation 11
References 15

1. Introduction

Given a triple of vectors $(X, Y, Z) \in S^2 \times S^2 \times S^2$, the Hamiltonian

$$H(X, Y, Z) = \langle X, Y \rangle + \langle Y, Z \rangle + \langle Z, X \rangle$$

models the pairwise interaction of three identical spin vectors fixed to the vertices of an equilateral triangle. Systems of this type – called Heisenberg spin chains – are of interest to physicists as they provide a classical model for quantum spin in a fixed lattice.

A second natural Hamiltonian on $S^2 \times S^2 \times S^2$ is the determinant,

$$J(X, Y, Z) = \det(X, Y, Z).$$

Together with a third commuting integral $I$, these Hamiltonians define a completely integrable system $(H, J, I)$ on $S^2 \times S^2 \times S^2$.

Non-displaceability of Lagrangian fibers of the resulting integrable system was recently studied by [15]. In the past few years, similar systems on $S^2 \times S^2$ have provided interesting examples of systems with non-displaceable torus fibers [7, 18].
These systems, which degenerate to toric orbifolds, were shown to contain a continuum of non-displaceable Lagrangian torus fibers which limit to a Lagrangian sphere \([7]\).

In this note we study the topological properties of our integrable system. In particular, we show that there is a line segment of non-degenerate critical values through the interior of the moment map image which we will call the ‘critical line’ (see Figure 1). Using the results of \([19, 10]\), we conclude that the monodromy of this system around the critical line is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix},$$

and thus the system does not admit global action-angle coordinates and cannot arise as a toric degeneration. In some sense, the number 3 in this matrix comes from the system’s natural \(\mathbb{Z}_3\)-symmetry.

In Section 2, we begin by recalling some basic facts and fixing notation. In Section 3, we introduce our integrable system. In Section 4 we describe the moment map image and the set of critical values (or the system’s ‘bifurcation diagram’). Section 5 shows that the regular level sets are connected 3-tori and Section 6 computes the non-degeneracy of the critical line and deduces the system’s topological monodromy.

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2. Notation and Conventions

Let \(G\) be a compact, connected Lie group with semisimple Lie algebra \(\mathfrak{g}\) endowed with an Ad-invariant inner product \(\langle , \rangle\). The Kostant-Kirillov-Souriau symplectic
structure on an adjoint orbit $O_Z$ of $Z \in \mathfrak{g}$ is given by
\[ \omega_Z([X, Z], [Y, Z]) = \langle Z, [X, Y] \rangle, \]
where $X, Y \in \mathfrak{g}$. Hamilton’s equation for a function $H : \mathfrak{g} \to \mathbb{R}$ can be written as
\[ \frac{dZ}{dt} = X_H(Z) = [\nabla H(Z), Z], \]
where $\nabla H$ is the gradient vector field defined by the equation $\langle \nabla H(Z), Y \rangle = dH_Z(Y)$ for all $Y \in T_Z\mathcal{O}$. Hence, the Poisson bracket of two functions $H, F$ can be conveniently written as
\[ \{H, F\}_Z = \omega_Z(X_H, X_F) = \omega_Z([\nabla H, Z], [\nabla F, Z]) = \langle Z, [\nabla H, \nabla F] \rangle. \]

A direct sum of semisimple Lie algebras $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$ is endowed with direct sum Lie brackets and Killing forms. An adjoint orbit in $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$ is a product of adjoint orbits $O_{Z_1} \times \cdots \times O_{Z_N}$ and the symplectic structure coincides with the direct sum of their respective symplectic structures, $\omega = \omega_1 \oplus \cdots \oplus \omega_N$. For example, in $\mathfrak{so}(3) \times \cdots \times \mathfrak{so}(3)$ the orbits are products of spheres.

Under the Ad-invariant identification of $\mathfrak{g}$ with its dual, the moment map for the adjoint action of $G$ on an orbit in $\mathfrak{g}$ is inclusion of the orbit into $\mathfrak{g}$. The moment map for the diagonal adjoint action of $G$ on $O_{Z_1} \times \cdots \times O_{Z_N}$ is hence the map $(X_1, \ldots, X_N) \mapsto \sum X_i$. 

![Figure 2. Projections of the moment map image.](image-url)
3. An Integrable Heisenberg Spin Chain

Consider a product of spheres with radius 1, \( M = S^2 \times S^2 \times S^2 \), whose elements are triples \((X, Y, Z)\) of unit length vectors in \(\mathbb{R}^3\). Define the Hamiltonians

\[
H(X, Y, Z) = |X + Y + Z| = \sqrt{3 - \langle X, Y \rangle - \langle Y, Z \rangle - \langle Z, X \rangle},
\]

\[
J(X, Y, Z) = \langle X, [Y, Z] \rangle = \text{det}(X, Y, Z),\]

and

\[
I(X, Y, Z) = \langle X + Y + Z, e_3 \rangle,
\]

where \([,]\) is the standard cross-product on \(\mathbb{R}^3\), \(\langle,\rangle\) is the standard inner product, and \([e_1, e_2, e_3]\) is the standard oriented orthonormal basis with \([e_i, e_j] = \varepsilon_{ijk}e_k\). By ad-invariance of the inner product, the Hamiltonian vector fields of these functions are:

\[
X_H = [\nabla H, (X, Y, Z)] = \frac{1}{H(X, Y, Z)}([Y + Z, X], [X + Z, Y], [X + Y, Z]),
\]

\[
X_J = [\nabla J, (X, Y, Z)] = ([[Y, Z], X], [[Z, X], Y], [[X, Y], Z]),\]

and

\[
X_I = [\nabla I, (X, Y, Z)] = ([e_3, X], [e_3, Y], [e_3, Z]).
\]

Recall that the flow \(\varphi^t_{[v, X]}\) of a vector field \([v, X]\) acts by rotation of the vector \(X\) around the axis \(v\) with period \(2\pi/|v|\), which we denote as \(R^v_t\),

\[
\varphi^t_{[v, X]}X = R^v_tX.
\]

Thus, the Hamiltonian flow of \(I\) acts by rotating each sphere around the \(e_3\)-axis with period \(2\pi\),

\[
\varphi^t_{X_I}(X, Y, Z) = (R^{e_3}_tX, R^{e_3}_tY, R^{e_3}_tZ).
\]

Where defined, the Hamiltonian flow of \(H\) rotates each sphere around the axis \(X + Y + Z\) with period \(2\pi\),

\[
\varphi^t_{X_H}(X, Y, Z) = (R^v_tX, R^v_tY, R^v_tZ),
\]

where \(v = (X + Y + Z)/|X + Y + Z|\). This is perhaps best visualized as rotating the polygon with edges \(X, Y, Z, -X - Y - Z\) around the edge \(-X - Y - Z\).

3.7. Proposition. \(\{H, J\} = \{H, I\} = \{J, I\} = 0\)

Proof. It is a nice exercise to see that this is true based on the geometric description of the Hamiltonians and their flows given above. More algebraically, one can see this using the Lie algebra structure that is present. For example,

\[
\{J, I\}(X, Y, Z) = \langle (X, Y, Z), [[Y, Z], [Z, X], [X, Y]], (e_3, e_3, e_3) \rangle
\]

\[
= \langle [X, Y, Z] + [Y, Z, X] + [Z, X, Y], e_3 \rangle = 0
\]

by the Jacobi identity and ad-invariance of our inner product. A similar calculation shows that \(\{H, I\} = \{H, J\} = 0\).

\[\square\]

\[\text{Note that this } H \text{ is related to the } H \text{ in the introduction by a square root. We do this so that the flow of the Hamiltonian vector field is a } S^1\text{-action where it is defined.}\]
3.8. Remark. The Hamiltonian flow of $J$ is less straightforward to describe, but we can say something about how it acts on the submanifold

$$L = \{(X, Y, Z) \in M : X + Y + Z = 0 \} = H^{-1}(0).$$

There the vectors $X, Y,$ and $Z$ are coplanar and the vector field

$$X_J = ([\pi, X], [\pi, Y], [\pi, Z])$$

where $\pi = [X, Y] = [Y, Z] = [Z, X]$. The flow of this vector field acts on $L$ by rotation of each vector around the axis $\pi$ with constant period.

In [15] it was shown that the fiber $L$ is a non-displaceable Lagrangian submanifold of $S^2 \times S^2 \times S^2$. To see that it is an embedded Lagrangian $\mathbb{R}P^3$, observe that it is the zero level set for the moment map of the diagonal $SO(3)$-action, $(X, Y, Z) \mapsto X + Y + Z$, and the diagonal action of $SO(3)$ is free and transitive. It’s interesting to observe that the moment map image for our system near this Lagrangian fiber resembles that of the geodesic flow on $\mathbb{R}P^3$.

3.9. Remark. Note that one can think of $H$ as a collective function obtained from a Casimir on $so(3)$ via the moment map for the diagonal $SO(3)$-action,

$$S^2 \times S^2 \times S^2 \xrightarrow{(X,Y,Z)\mapsto X+Y+Z} so(3) \xrightarrow{|\cdot|} \mathbb{R}.$$

If you view this system as integrating the Hamiltonian $H$, then complete integrability is not surprising since $H$ has a $SO(3)$-symmetry, coming from it’s definition as a collective function for the $SO(3)$ moment map. This is not typical of classical Heisenberg spin chains considered in the physics literature, where one often studies a ‘chain with boundary conditions’ of $N$ spin vectors

$$(X_0, \ldots, X_{N-1}) \in S^2 \times \cdots \times S^2,$$

and the Hamiltonian

$$H = \sum_{i=0}^{N-1} \langle X_i, X_{i+1} \rangle,$$

where the indices in the sum are considered modulo $N$. This models $N$ spin vectors placed at the vertices of a regular polygon, if we imagine that each vector $X_i$ only interacts with its nearest neighbours $X_{i-1}$ and $X_{i+1}$. The coincidence for $N = 3$ is that a triangle is also a complete graph, and hence we can re-write $H$ as a Casimir on $so(3)$.

Alternately, one might view this system as integrating $J$, which also has a natural $SO(3)$-symmetry since it is the determinant, and it might be more natural to generalize this system from the perspective of forms on the Lie algebra $so(3) \times \cdots \times so(3)$, using the algebraic identities at hand as we demonstrated in Proposition 3.7.

3.10. Remark. It is important to note that all three Hamiltonians have a $\mathbb{Z}_3$-symmetry coming from cyclic permutations $(X, Y, Z) \mapsto (Z, X, Y)$. As we will see, this symmetry provides an explanation for why the system’s monodromy matrix contains the number 3.
4. Image of the Moment Map

The image of the moment map \( \mathcal{H} = (H, J, I) \) is a solid in \( \mathbb{R}^3 \) that is symmetric about the \((H, I)\)-plane and about the \((H, J)\)-plane (see Figure 1 and 2). It is obvious that \(|I| \leq H\), with equality when \(X + Y + Z \in \text{span}(e_3)\), and that \(H \leq 3\) with equality when \(X = Y = Z\).

Observe that

\[
H = \sqrt{3 + 2(a + b + c)} \quad \text{and} \quad |J| = \sqrt{1 + 2abc - (a^2 + b^2 + c^2)}
\]

where \(a = \langle X, Y \rangle\), \(b = \langle Y, Z \rangle\) and \(c = \langle Z, X \rangle\) (the second formula is the volume of a parallelepiped). If we maximize \(|J|\) with the constraint \(H = \text{const}\), then we must have \(a = b = c\) (the interior angles between the three vectors are the same). Using this we can deduce that the image of the moment map is bounded by the inequalities

\[
|J| \leq \sqrt{1 + 2 \left( \frac{H^2 - 3}{6} \right)^3} - 3 \left( \frac{H^2 - 3}{6} \right)^2, \quad |I| \leq H, \quad \text{and} \quad H \leq 3
\]

and equality is achieved in the first inequality when \(a = b = c\).

To see that the image is the entire region described by these inequalities, we can find a tuple \((X, Y, Z) \in S^2 \times S^2 \times S^2\) for a given value of \(H \) and \(J\), then simultaneously rotate the vectors \(X, Y, Z\), and \(Z\) in the \((e_1, e_2)\)-plane to get the desired value of \(I\).

4.3. Lemma. For any \(0 \leq H \leq 3\), and \(J\) satisfying the equations \(4.2\), there is a point \((a, b, c) \in [-1, 1]^3 \cap \{a + b + c \geq -3/2\}\) that satisfies \(4.1\) above.

Proof. Equivalently, the image of the map \(f : [-1, 1]^3 \cap \{a + b + c = x\} \to \mathbb{R}\), given by \(f(a, b, c) = 2abc - a^2 - b^2 - c^2\) is the interval \([-1, (2x^3 - 9x^2)/27]\), where \(x = \frac{H^2 - 3}{2}\). The maximum is achieved when \(a = b = c = x/3\) and the minimum is achieved when \(a = b\) and \(c = 2a^2 - 1\) and \(2a + 2a^2 - 1 = x\). The result follows by the Intermediate Value Theorem. \(\Box\)

4.4. Proposition. The image of the moment map \(\mathcal{H}\) is the region in \(\mathbb{R}^3\) bounded by the equations \(4.2\).

Next, we turn our attention to the critical set for the system. The critical set consists of several subsets:

(1) The sets where \(H\) is critical:
   (a) the embedded \(SO(3) \cong H^{-1}(0)\), which lies over the ‘orbifold’ vertex,
   (b) the three embedded spheres \(S_1 = \{(-X, X, X)\}\), \(S_2 = \{(X, -X, X)\}\), and \(S_3 = \{(X, X, -X)\}\), which lie over the critical line, and
   (c) the diagonally embedded sphere \(S_4 = \{(X, X, X)\}\), which lies over the edge \(H = 3\).

(2) The set \(C_1 = \{(X, Y, Z) : X + Y + Z \in \text{Span}(e_3)\}\) where \(I\) and \(H\) are dependent. This contains the critical set of \(I\) and maps to two opposite faces of the moment map image.

(3) The set \(C_2 = \{(X, Y, Z) : \langle X, Y \rangle = \langle Y, Z \rangle = \langle Z, X \rangle\}\) where \(H\) and \(J\) are dependent. (they are also dependent on the set \(S_1 \cup S_2 \cup S_3 \cup S_4\), where \(J\) is critical). Note \(S_1 \subset C_2\). This set maps to the other two opposite faces of the moment map image. Points in \(C_1 \cap C_2\) map to edges of the moment map image.
4.5. Proposition. The critical set for $\mathcal{H}$ is

$$C = C_1 \cup C_2 \cup S_1 \cup S_2 \cup S_3.$$ 

The set $\mathcal{H}(C_1 \cup C_2)$ is the boundary of the image $\mathcal{H}(M)$ and the set $\mathcal{H}(S_1) = \mathcal{H}(S_2) = \mathcal{H}(S_3)$ is the line segment $\mathcal{H}(M) \cap \{H = 1, J = 0\}$ (see Figure 1).

In the terminology of integrable systems, the set of critical values is the system’s ‘bifurcation diagram.’ This complete description of the bifurcation diagram will be of use to us in Section 6. Note that the three critical spheres $S_1, S_2, S_3$ are permuted by the system’s $\mathbb{Z}_3$-symmetry.

Proof. Throughout we use the fact that $df = 0$ if and only if $X_f = 0$ (when $f$ is smooth).

(1) Since 0 is the global minimum for $H$, the set $H^{-1}(0)$ is critical. To find the other critical sets of $H$, observe that $X_H = 0$ if and only if $[X + Y + Z, X] = 0$, $[X + Y + Z, Y] = 0$, and $[X + Y + Z, Z] = 0$. If $X + Y + Z \neq 0$, this occurs if and only if $X + Y + Z$ is contained in the lines spanned by $X, Y, Z$, so this happens if and only if $X, Y, Z$ are collinear. This entails cases (1b) and (1c).

(2) Observe that $X_I = \alpha X_H$ for some $\alpha \in \mathbb{R}$ if and only if

$$[e_3, X] = \alpha[Y + Z, X],$$

$$[e_3, Y] = \alpha[X + Z, Y],$$

$$[e_3, Z] = \alpha[X + Y, Z]$$

which is true if and only if $X + Y + Z \in \text{span}(e_3)$.

(3) Observe that $X_J = \alpha X_H$ if and only if

$$[[Y, Z], X] = \alpha[X + Y + Z, X],$$

$$[[Z, X], Y] = \alpha[X + Y + Z, Y],$$

$$[[X, Y], Z] = \alpha[X + Y + Z, Z]$$

for some $\alpha$. If $\alpha = 0$ then the 3-tuple $X, Y, Z$ forms an oriented or anti-oriented orthonormal frame, or are collinear, so $J$ is critical. If $\alpha \neq 0$ then

(a) $[Y, Z], Y + Z,$ and $X$ are coplanar,

(b) $[Z, X], X + Z,$ and $Y$ are coplanar, and

(c) $[X, Y], X + Y,$ and $Z$ are coplanar,

which is true if and only if $\angle XY = \angle YZ = \angle ZX$ since $|X| = |Y| = |Z| = 1$.

Finally, suppose that $\alpha X_I + \beta X_H + \gamma X_J = 0$ for $\alpha, \beta, \gamma \in \mathbb{R}$ not all zero. Then

$$\alpha[e_3, X] + \beta[Y + Z, X] + \gamma[[Y, Z], X] = 0$$

$$\alpha[e_3, Y] + \beta[X + Z, Y] + \gamma[[Z, X], Y] = 0$$

$$\alpha[e_3, Z] + \beta[X + Y, Z] + \gamma[[X, Y], Z] = 0$$

and substituting (4.8a) into (4.8b) into (4.8c) we obtain

$$\alpha[e_3, X + Y + Z] + \beta([X, Z] + [Z, X]) + \gamma([[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]) = 0$$

which by the Jacobi identity reduces to $\alpha[e_3, X + Y + Z] = 0$. If $\alpha = 0$, then we are in case 3. If $[e_3, X + Y + Z] = 0$ then we are in case 2. □
Since we now know that our Hamiltonians are independent on an open dense subset of $M$, we can conclude:

4.10. Corollary. $\mathcal{H}$ is a completely integrable system. In particular, the regular level sets of $\mathcal{H}$ are homeomorphic to a disjoint union of finitely many 3-tori.

4.11. Corollary. The set of regular values is homotopy-equivalent to $S^1$.

In the next section, we will see that the regular level sets are connected and in Section 6 we will describe the structure of the Lagrangian foliation near the critical line.

5. Connectedness of Regular Level Sets

5.1. Proposition. The regular fibers of the system $\mathcal{H}$ are connected.

Proof. The proof will have two parts. For $H \neq 1$ we can make a general argument. For $H = 1$ we will provide a direct proof that the fibers of the original system are connected.

Pick a regular value $(r, s, t)$ of $\mathcal{H} = (H, J, I)$ with $r \neq 1$. Since $H$ generates a Hamiltonian $S^1$-action on $M \setminus H^{-1}(0)$, it is a Morse-Bott function such that all critical sets have even index $[1]$. Hence, the regular level sets of $H$ are connected. Since the level sets of $H$ are compact and connected, the symplectic reductions $M_r \equiv H^{-1}(r)/S^1$ are all compact, connected symplectic manifolds. Since $t$ is a regular value of the reduced Hamiltonian $\tilde{I}$, and $\tilde{I}$ generates a free $S^1$-action, we can reduce once more to obtain the compact and connected manifold $M_{r,t} = \tilde{I}^{-1}(t)/S^1$. The image of the twice-reduced Hamiltonian $\tilde{J}$ on $M_{r,t}$ is a line segment with the only critical values being the maximum and minimum. A quick computation shows that $\tilde{J}$ has exactly two critical points, hence the regular fibers $\tilde{J}^{-1}(s)$ are all connected. This implies that the critical fiber $\mathcal{H}^{-1}(r, s, t) \subset M$ is connected.

Now consider regular values of the form $(1, s, t)$ with $s \neq 0$. Since 1 is a critical value of $H$, the previous argument will not work. Since $J$ does not generate an $S^1$-action on $M$, we cannot argue similarly using reduction by $J$ to avoid the critical values of $H$. Instead, we consider the fibers in the reduced system on $M_t = I^{-1}(t)/S^1$ directly.

First, note that the set

\begin{equation}
S_t = \{(X, Y, Z) \in M : X + Y + Z = ke_1 + te_3 \text{ for some } k \in \mathbb{R}^+ \} \subset I^{-1}(t)
\end{equation}

is a transverse slice for the $S^1$-action of $I$ away from the set $H^{-1}(0)$, so the reduced manifold $M_t \setminus \tilde{H}^{-1}(0)$ can be identified homeomorphically with $S_t$. Therefore the level sets $(\tilde{H}, \tilde{J})^{-1}(1, s)$ of the reduced system are

\begin{equation}
\{(X, Y, Z) \in M : X + Y + Z = \sqrt{1-t^2}e_1 + te_3, \text{ and } \langle X, [Y, Z] \rangle = s \}
\end{equation}

By (3.4), the Hamiltonian flow of the reduced Hamiltonian $\tilde{H}$ acts on these level sets by diagonal rotation around the axis $\sqrt{1-t^2}e_1 + te_3$ and this action is free. To see that the level set $(\tilde{H}, \tilde{J})^{-1}(1, s)$ is connected, we consider the quotient by the $S^1$-action generated by $H$, and show it is homeomorphic to a circle.

For $s \neq 0$, a transverse slice for the action generated by $\tilde{H}$ on $(\tilde{H}, \tilde{J})^{-1}(1, s)$ can be given by fixing $X$ in the upper half of the $(e_1, e_3)$-plane:

\[Y_t = M \cap \{X + Y + Z = \sqrt{1-t^2}e_1 + te_3\}\]
\( \cap \{ \langle X, [Y, Z] \rangle = s \} \cap \{ X = (\cos(\theta), 0, \sin(\theta)), 0 \leq \theta \leq \pi \} \)

We now show that \( Y_t \) is homeomorphic to \( S^1 \) for all \(-1 < t < 1\). Let \( X = (\cos(\theta), 0, \sin(\theta)) \), \( Y = (y_1, y_2, y_3) \) and \( Z = (z_1, z_2, z_3) \). Our set \( Y_t \) consists of all solutions to the system of equations

\[
\begin{align*}
(5.4a) & \quad \cos(\theta) + y_1 + z_1 = \sqrt{1-t^2} \\
(5.4b) & \quad y_2 = -z_2 \\
(5.4c) & \quad \sin(\theta) + y_3 + z_3 = t \\
(5.4d) & \quad \cos(\theta)(y_2z_3 - y_3z_2) + \sin(\theta)(y_1z_2 - y_2z_1) = s \\
(5.4e) & \quad y_1^2 + y_2^2 + y_3^2 = z_1^2 + z_2^2 + z_3^2 = 1 
\end{align*}
\]

For a given \( \theta \), the system is equivalent to the intersection of two circles in the \((y_1, y_3)\)-plane

\[
\begin{align*}
(5.5a) & \quad y_1^2 + y_3^2 = 1 - \frac{s^2}{\sin(\theta)} \\
(5.5b) & \quad (y_1 - \sqrt{1-t^2 + \cos(\theta)})^2 + (y_3 - t + \sin(\theta))^2 = 1 - \frac{s^2}{\sin(\theta)} \\
(5.5c) & \quad \text{for which there are 0, 1, or 2 solutions.} 
\end{align*}
\]

As \( \theta \) varies the circles sweep out two ‘bulging’ cylinders that intersect in a curve diffeomorphic to \( S^1 \). Since this fiber in the twice-reduced system is connected, the original fiber in \( M \) is also connected. \( \square \)

5.6. **Remark.** The image of the invariant Lagrangian \( L = H^{-1}(0) \) in the reduction at 0 by \( I \) is a Lagrangian \( S^2 \).

5.7. **Remark.** In [16] it is shown that if \((M^4, \omega, F = (f_1, f_2))\) is a completely integrable system with two degrees of freedom that has only non-degenerate critical points, and whose bifurcation diagram has no vertical tangent lines, then the system has connected fibers. After rotating the moment map image, (and checking that the boundary of \( \mathcal{H}(M) \) consists of non-degenerate elliptic critical values except \((0,0,0)\)) we can apply this result to deduce connectedness of almost all the fibers of \( \mathcal{H} \). Since the reduced system \((\tilde{H}, \tilde{J})\) has a Lagrangian \( S^2 \) fiber, we cannot use this theorem to deduce connectedness of the regular fibers \( \mathcal{H}^{-1}(1, s, 0), s \neq 0 \).

6. **Topological Monodromy Around the Critical Line**

It was shown in [6] that the singular foliation of an integrable system by Lagrangian submanifolds is determined in a neighbourhood of a nondegenerate critical point by its ‘type’ – the conjugacy class of the corresponding Cartan subalgebra in \( \text{sp}(L^\perp/L) \) (see Appendix). This was later strengthened in [20] to show that, under a stability condition, the topological foliation is almost completely determined in a neighbourhood of a non-degenerate singular leaf \( S \) by its ‘type’ – the type of any critical point in \( S \) of lowest rank. In the case of singularities whose critical type is purely focus-focus, or mixed focus-focus and elliptic, the foliation is completely determined by this data:

\[\text{The set of critical values.}\]
6.1. **Theorem.** [10] Let $S$ be a non-degenerate singular leaf of an integrable system $F$, which is non-splitting. If the type of $S$ does not contain any hyperbolic components, then in a neighbourhood of $S$ the Lagrangian foliation of $F$ is topologically equivalent to a direct product of elliptic singularities, focus-focus singularities, and the trivial foliation $D^* \times T^r$.

As was shown in [10], this result also determines the topological monodromy of a given system in a neighbourhood of a non-degenerate singularity. By the computations in the appendix below, the singular fibers $\mathcal{H}^{-1}(1,0,a)$ over the critical line are rank 1 focus-focus singularities. By Proposition 4.5 each singular fiber contains three critical circles, which are permuted by the system’s $\mathbb{Z}_3$-symmetry. These circles are the intersection of the critical fiber with the three critical spheres $S_1, S_2,$ and $S_3$. The system’s $\mathbb{Z}_3$-symmetry shows that these singularities are non-splitting (see also Proposition 4.5). Hence, in a neighbourhood of a singular fiber over the critical line, the system is topologically equivalent to the product of the trivial foliation $D^1 \times S^1$ and a focus-focus singularity with three critical points. By Theorem 6.1 the topological monodromy of the system is

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 3 & 1
\end{pmatrix}.
$$

6.2. **Remark.** In the fiber $\tilde{\mathcal{H}}^{-1}(1,0)$ of the reduced system at $I = 0$, the three cylinders joining the critical points are the sets

$$
\{(X, e_1, -X) \in S \mid X \in S^2 \setminus \pm e_1\}, \{(X, -X, e_1) \in S \mid X \in S^2 \setminus \pm e_1\}
$$

and

$$
\{(e_1, X, -X) \in S \mid Y \in S^2 \setminus \{\pm e_1\}\}
$$

and the critical points are $(e_1, e_1, -e_1), (e_1, -e_1, e_1), (-e_1, e_1, e_1)$. Hence the critical fiber is explicitly homeomorphic to a torus with three longitudinal pinches. The fiber $\mathcal{H}^{-1}(1,0,0)$ is then explicitly homeomorphic to $\tilde{\mathcal{H}}^{-1}(1,0) \times S^1$.

6.3. **Remark.** The fact that this system has non-trivial monodromy should be unsurprising for the following reason: the topology of the $H$-level sets changes as you pass through the critical value 1. This can be seen directly with Morse theory, but there is also a natural interpretation in terms of the topology of polygon spaces (as introduced by [13]). There is a natural diffeomorphism of the level set $H^{-1}(r)$ with the manifold $M(1,1,1,r)$ of closed 4-gons in $\mathbb{R}^3$ with side lengths 1, 1, 1, and $r$. When $r \neq 1$, it has been observed by Knutson, Hausman [8], and Kapovitch and Millson [13] that the quotient $M(1,1,1,r)/SO(3)$ is homeomorphic to $S^2$, and that the quotient map $\pi : M(1,1,1,r) \to S^2$ is a principal $SO(3)$-bundle. Further, the characteristic classes of this principal $SO(3)$-bundle were described by Knutson and Hausman in their paper [9]. Their result says that for $0 < r < 1$ the bundle is trivial, whereas for $1 < r < 3$ the bundle is non-trivial.

It was observed in [3] that such a change in the topology of the level set $H^{-1}(r)$ as $r$ passes through an interior critical value indicates that there must be non-trivial

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3By non-splitting, we mean that the bifurcation diagram of the system restricted to a neighbour- hood of a critical point of lowest rank in $S$ is the same as the bifurcation diagram of the system restricted to a neighbourhood of $S$. This is equivalent to the topological stability condition of Zung.
monodromy around the associated critical fibres, since this forces the pullback of the torus bundle to any circle around the critical line to be non-trivial.

6.4. Remark. For $-1 < t < 1$, the reduced system on $M_t = I^{-1}(t)/S^1$ has a focus-focus singularity with three critical points, or after a perturbation, three simple focus-focus singularities. The manifold $M_t$ is the blow-up $Bl_3\mathbb{C}P^2$, and the moment map image has 3 vertices (see Figure 2). A quick comparison with the list of almost toric systems in [11] shows that this checks out.

7. Appendix: Non-Degeneracy Computation

In this appendix, we show that the critical line is rank 1 non-degenerate focus-focus. One may refer to [4, 5] for introductory details on the topological monodromy of completely integrable systems, and [20, 2] for more details on non-degenerate singularities.

Let $p$ be a critical point of rank $k$ of an integrable system $(H_1, \ldots, H_n)$. Without loss of generality, assume that $dH_1, \ldots, dH_{n-k} = 0$ and the remaining functions $H_{n-k+1}, \ldots, H_n$ are independent at $p$. The operators $\omega^{-1}d^2H_1, \ldots, \omega^{-1}d^2H_{n-k}$ form a commutative subalgebra of $sp(L^\perp/L)$ where the subspace $L \subset T_pM$ is the span of the vector fields $X_{H_{n-k+1}}, \ldots, X_{H_n}$ and $L^\perp$ is its symplectic orthocomplement. The point $p$ is non-degenerate if this is a Cartan subalgebra. Equivalently, $p$ is non-degenerate if $\omega^{-1}d^2H_1, \ldots, \omega^{-1}d^2H_{n-k}$ are linearly independent and some linear combination of these operators has $2(n-k)$ distinct eigenvalues. It was shown in [17] that in $sp(\mathbb{R}, 4)$ there are four conjugacy classes of Cartan subalgebras corresponding to four possible combinations of eigenvalues for a generic element:

1. elliptic-elliptic: $\pm iA, \pm iB$,
2. elliptic-hyperbolic: $\pm A, \pm iB$,
3. hyperbolic-hyperbolic: $\pm A, \pm B$, and
4. focus-focus: $A \pm iB, -A \pm iB$.

Note that considering the induced operators on the vector space $L^\perp/L$ is equivalent to considering the same question for the reduced system $(\tilde{H}_1, \ldots, \tilde{H}_{n-k})$ on the reduction of $M$ by $H_{n-k+1}, \ldots, H_n$ (provided of course, that one can perform this reduction). In Darboux coordinates the operator $A_H = \omega^{-1}d^2H_p$ is equal to the linearization of
the Hamiltonian vector field $X_H$ at $p$, since

$$\frac{\partial X_H^i}{\partial x^j} = \frac{\partial}{\partial x^j} \left( \omega^{ik} \frac{\partial f}{\partial x^k} \right) = \omega^{ik} \frac{\partial^2 f}{\partial x^j \partial x^k} = (\omega^{-1} d^2 H)_j^i.$$  

We now turn to our system on $S^2 \times S^2 \times S^2$. Consider the cylindrical coordinates $(\theta, z) \in (-\pi/2, 3\pi/2) \times (-1, 1)$ with symplectic form $d\theta_i \wedge dz_i$. The map $\phi: (-\pi/2, 3\pi/2) \times (-1, 1) \to S^2$ given by

$$\phi(\theta, z) = (\sqrt{1 - z^2} \cos(\theta), \sqrt{1 - z^2} \sin(\theta), z)$$

is a symplectomorphism. Put a Darboux chart on $S^2 \times S^2 \times S^2$ with the map $\Phi = \phi \times \phi \times \phi$, coordinates $(\theta_1, z_1, \theta_2, z_2, \theta_3, z_3)$, and symplectic form $d\theta_i \wedge dz_i$. Pulling back our system by $\Phi$ we obtain

$$H = \left( \sum_j \sqrt{1 - z_j^2} \cos(\theta_j) \right)^2 + \left( \sum_j \sqrt{1 - z_j^2} \sin(\theta_j) \right)^2 + \left( \sum_j z_j \right)^2$$

$$J = \sum_{j=1,2,3} z_j(1 - z_{j+1}^2)^{1/2} (1 - z_{j-1}^2)^{1/2} \sin(\theta_{j-1} - \theta_{j+1})$$

$$I = z_1 + z_2 + z_3$$

Differentiating,

$$\frac{\partial I}{\partial z_i} = 1, \quad \frac{\partial I}{\partial \theta_i} = 0,$$

$$\frac{\partial H}{\partial z_i} = \frac{-2z_i \cos(\theta_i)}{(1 - z_i^2)^{1/2}} \left( \sum_j \sqrt{1 - z_j^2} \cos(\theta_j) \right)$$

$$- \frac{2z_i \sin(\theta_i)}{(1 - z_i^2)^{1/2}} \left( \sum_j \sqrt{1 - z_j^2} \sin(\theta_j) \right) + 2,$$

$$\frac{\partial H}{\partial \theta_i} = -2 \sin(\theta_i) (1 - z_i^2)^{1/2} \left( \sum_j \sqrt{1 - z_j^2} \cos(\theta_j) \right)$$

$$+ 2 \cos(\theta_i) (1 - z_i^2)^{1/2} \left( \sum_j \sqrt{1 - z_j^2} \sin(\theta_j) \right),$$

$$\frac{\partial J}{\partial z_i} = -z_i (1 - z_i^2)^{-1/2} \left( z_{i-1}(1 - z_{i+1}^2)^{1/2} \sin(\theta_{i+1} - \theta_i) \right)$$

$$+ z_{i+1}(1 - z_{i-1}^2)^{1/2} \sin(\theta_i - \theta_{i-1}))$$

$$+ (1 - z_{i-1}^2)^{1/2} (1 - z_{i+1}^2)^{1/2} \sin(\theta_{i-1} - \theta_{i+1}),$$

$$\frac{\partial J}{\partial \theta_i} = (1 - z_i)^{1/2} \left( z_{i+1}(1 - z_{i-1}^2)^{1/2} \cos(\theta_i - \theta_{i-1}) - z_{i-1}(1 - z_{i+1}^2)^{1/2} \cos(\theta_{i+1} - \theta_i) \right).$$

Hamilton’s equations tell us that

$$X_f = \frac{\partial f}{\partial z_i} \frac{\partial}{\partial z_i} - \frac{\partial f}{\partial \theta_i} \frac{\partial}{\partial \theta_i}.$$
The linearization of $X_f$ at a fixed point of the flow of $f$ is then

$$A_f = \left( \begin{array}{c} -\frac{\partial^2 f}{\partial z_k \partial \theta_i} \\
\frac{\partial^2 f}{\partial z_k \partial z_i} \\
\frac{\partial^2 f}{\partial \theta_k \partial \theta_i} \\
\frac{\partial^2 f}{\partial \theta_k \partial z_i} \end{array} \right)_{i k}.$$ 

We compute

$$\frac{\partial^2 H}{\partial z_k \partial z_i} = \begin{cases} 
-\frac{2 \cos(\theta_i)}{(1-z_i^2)^{1/2}} \left( \sum_j \sqrt{1-z_j^2} \cos(\theta_j) \right) + \frac{2z_i^2 \cos^2(\theta_i)}{(1-z_i^2)} & k = i \\
-\frac{2 \sin(\theta_i)}{(1-z_i^2)^{1/2}} \left( \sum_j \sqrt{1-z_j^2} \sin(\theta_j) \right) + \frac{2z_i^2 \sin^2(\theta_i)}{(1-z_i^2)} & k \neq i 
\end{cases}$$

$$\frac{\partial^2 H}{\partial \theta_k \partial z_i} = \begin{cases} 
\frac{2z_i \sin(\theta_i)}{(1-z_i^2)^{1/2}} \left( \sum_j \sqrt{1-z_j^2} \cos(\theta_j) \right) - \frac{2z_i \cos(\theta_i)}{(1-z_i^2)^{1/2}} \left( \sum_j \sqrt{1-z_j^2} \sin(\theta_j) \right) & k = i \\
\frac{2z_i \cos(\theta_i)}{(1-z_i^2)^{1/2}} \sqrt{1-z_k^2} \sin(\theta_k) - \frac{2z_i \sin(\theta_i)}{(1-z_i^2)^{1/2}} \sqrt{1-z_k^2} \cos(\theta_k) & k \neq i 
\end{cases}$$

$$\frac{\partial^2 H}{\partial z_k \partial \theta_i} = \begin{cases} 
\frac{2z_i \sin(\theta_i)}{(1-z_i^2)^{1/2}} \left( \sum_j \sqrt{1-z_j^2} \cos(\theta_j) \right) + 2z_i \sin(\theta_i) \cos^2(\theta_i) & k = i \\
-\frac{2z_i \cos(\theta_i)}{(1-z_i^2)^{1/2}} \left( \sum_j \sqrt{1-z_j^2} \sin(\theta_j) \right) - 2z_i \sin(\theta_i) \cos^2(\theta_i) \\
2 \sin(\theta_i)(1-z_i^2)^{1/2} \frac{z_k \cos(\theta_k)}{(1-z_k^2)^{1/2}} - 2 \cos(\theta_i)(1-z_i^2)^{1/2} \frac{z_k \sin(\theta_k)}{(1-z_k^2)^{1/2}} & k \neq i 
\end{cases}$$

$$\frac{\partial^2 H}{\partial \theta_k \partial \theta_i} = \begin{cases} 
-2 \cos(\theta_i)(1-z_i^2)^{1/2} \left( \sum_j \sqrt{1-z_j^2} \cos(\theta_j) \right) + 2 \sin^2(\theta_i)(1-z_i^2) & k = i \\
-2 \sin(\theta_i)(1-z_i^2)^{1/2} \left( \sum_j \sqrt{1-z_j^2} \sin(\theta_j) \right) + 2 \cos^2(\theta_i)(1-z_i^2) \\
2 \sin(\theta_i) \sin(\theta_k)(1-z_i^2)^{1/2}(1-z_k^2)^{1/2} + 2 \cos(\theta_i) \cos(\theta_k)(1-z_i^2)^{1/2}(1-z_k^2)^{1/2} & k \neq i 
\end{cases}$$
The critical line consists of points \( p = (1, 0, a), -1 < a < 1 \) (see Figure 1). As we have seen, the critical points in each fiber \( \mathcal{H}^{-1}(1, 0, a) \) are three embedded circles (these are just the intersections of our fiber with the critical spheres \( S_1, S_2, S_3 \)). To compute degeneracy of these critical points it is sufficient to compute the degeneracy of a single critical point in a single circle (by \( \mathbb{Z}_3 \)-symmetry). Let’s consider the points

\[
\frac{\partial^2 J}{\partial z_k \partial z_i} = \begin{cases} 
(1 - z_i^2)^{-3/2} (z_{i-1}(1 - z_i^2)^{1/2} \sin(\theta_{i+1} - \theta_i) \\
\frac{-z_{i-1}^2}{(1 - z_i^2)^{1/2}} (1 - z_{i+1}^2)^{1/2} \sin(\theta_{i+1} - \theta_i) - \frac{z_{i-1}(z_{i+1}^2 - 1)}{(1 - z_i^2)^{1/2}} \sin(\theta_{i+1} - \theta_{i-1}) \bigg) \quad k = i \\
(1 - z_i^2)^{-3/2} (z_{i+1}(1 - z_i^2)^{1/2} \sin(\theta_{i-1} - \theta_i) \\
\frac{-z_{i+1}^2}{(1 - z_i^2)^{1/2}} (1 - z_{i+1}^2)^{1/2} \sin(\theta_{i-1} - \theta_{i+1}) \bigg) \quad k = i - 1 \end{cases}
\]

\[
\frac{\partial^2 J}{\partial \theta_k \partial z_i} = \begin{cases} 
\frac{-z_{i+1}(1 - z_{i+1}^2)^{1/2} \cos(\theta_{i+1} - \theta_i)}{(1 - z_i^2)^{1/2}} (z_{i+1} - z_{i+1} z_{i+1}^2 - 1) \bigg) \quad k = i \\
\frac{-z_{i-1}(1 - z_{i-1}^2)^{1/2} \cos(\theta_{i-1} - \theta_i)}{(1 - z_i^2)^{1/2}} (z_{i-1} - z_{i-1} z_{i-1}^2 - 1) \bigg) \quad k = i + 1 \end{cases}
\]

\[
\frac{\partial^2 J}{\partial z_k \partial \theta_i} = \begin{cases} 
\frac{-z_{i+1}(1 - z_{i+1}^2)^{1/2} \cos(\theta_{i+1} - \theta_i)}{(1 - z_i^2)^{1/2}} (z_{i+1} - z_{i+1} z_{i+1}^2 - 1) \bigg) \quad k = i \\
(1 - z_i^2)^{1/2} \left( (1 - z_{i+1}^2)^{1/2} \cos(\theta_{i+1} - \theta_i) - \frac{z_{i+1} z_{i+1}^2}{(1 - z_i^2)^{1/2}} \cos(\theta_{i+1} - \theta_{i-1}) \right) \quad k = i - 1 \\
(1 - z_i^2)^{1/2} \left( (1 - z_{i-1}^2)^{1/2} \cos(\theta_{i-1} - \theta_i) + \frac{z_{i-1} z_{i-1}^2}{(1 - z_i^2)^{1/2}} \cos(\theta_{i+1} - \theta_{i-1}) \right) \quad k = i + 1 \end{cases}
\]

\[
\frac{\partial^2 J}{\partial \theta_k \partial \theta_i} = \begin{cases} 
(1 - z_i)^{1/2} (1 - z_i^2)^{1/2} \sin(\theta_i - \theta_{i-1}) \quad k = i \\
\frac{-z_{i-1}(1 - z_{i+1}^2)^{1/2} \sin(\theta_{i+1} - \theta_i)}{(1 - z_i^2)^{1/2}} (z_{i-1} - z_{i-1} z_{i-1}^2 - 1) \bigg) \quad k = i - 1 \\
z_{i+1}(1 - z_{i+1})^{1/2} (1 - z_{i+1}^2)^{1/2} \sin(\theta_{i+1} - \theta_i) \quad k = i + 1 \end{cases}
\]
\[ p = (0, a, 0, a, \pi, -a) \] in cylindrical coordinates. The linearization of \( X_J \) at \( p \) is

\[
A_J(p) = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & -1 & 1 & 0 \\
\end{pmatrix}.
\]

The linearization of \( X_H \) at \( p \) is

\[
A_H(p) = \begin{pmatrix}
0 & 0 & 0 & 0 & -2b^2 & 2b^2 \\
0 & 0 & 0 & -2b^2 & 0 & 2b^2 \\
0 & 0 & 0 & 2b^2 & 2b^2 & -4b^2 \\
-2 & 2a^2 & \frac{2a^2}{b^2} & \frac{2a^2}{b^2} & 0 & 0 \\
\frac{2a^2}{b^2} & -2 & \frac{2a^2}{b^2} & \frac{2a^2}{b^2} & 0 & 0 \\
\frac{2a^2}{b^2} & \frac{2a^2}{b^2} & \frac{2a^2}{b^2} & \frac{2a^2}{b^2} & 0 & 0 \\
\end{pmatrix}.
\]

where \( b^2 = 1 - a^2 \). These operators are independent and a quick computation shows that for any \(-1 < a < 1\) the operator on \( L^1/L \) induced by \( A_J + A_H \) has four distinct complex eigenvalues of the form \( A \pm iB, -A \pm iB \). Hence the critical point is rank 1 non-degenerate focus-focus.

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