Local cosmic strings in Brans-Dicke theory with a cosmological constant

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It is known that Vilenkin’s phenomenological equation of state for static straight cosmic strings is inconsistent with Brans-Dicke theory. We will prove that, in the presence of a cosmological constant, this equation of state is consistent with Brans-Dicke theory. The general solution of the full nonlinear field equations, representing the interior of a cosmic string with a cosmological constant is also presented.

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I. INTRODUCTION

Cosmic strings are linear topological defects which may have been produced in the early universe during phase transitions in general unified theories [1],[2]. They were believed to have important implications on cosmology and astrophysics such as their role in primordial density fluctuations in the early universe where galaxies etc. form. Although accumulation of observational data ruled out this possibility and they cannot contribute more than \( \sim 10\% \) [3, 4] to the density fluctuations, the interest in cosmic strings was renewed [5] due to several reasons, such as their relations with fundamental strings in string theory [6, 7, 8].

An infinitely long, straight, local gauge string is characterized by i) a special form for the energy momentum tensor of the string \( T^0_0 = T^z_z \neq 0 \) and others vanishing; ii) a Lorentz boost invariance along the symmetry (z) axis, due to Vilenkin [9]. The investigation of an Abelian-Higgs type theory for a Nielsen-Olesen type \( U(1) \) vortex [10, 11] strengthens the above prescription for a straight string. Exact interior thick string solutions satisfying the above energy-momentum tensor have been found by Gott [12], Hiscock [13], and Linet [14].

The Brans-Dicke (BD) theory [15] is a natural generalization of Einstein’s general relativity (GR) where the gravitational coupling constant is replaced by a scalar field. This theory has some desirable properties; for instance, it obeys the weak equivalence principle and it is also compatible with the solar system experiments for certain ranges of its parameter [16]. It has been applied to several issues ranging from inflation schemes in cosmology to quantum gravity. The inclusion of scalar fields in GR is also suggested by different theories, for example, as a dilaton field which naturally arises in the low energy limit of string theories or in the Kaluza-Klein theories [17].

In the framework of scalar-tensor theories, cosmic strings are investigated in detail [18]-[27]. Sen et al. [23] and later Arazi et al. [24] demonstrated that the BD field equations are inconsistent with a local string having the above energy-momentum tensor. Moreover, Gregory and Santos [25] studied the cosmic string as an isolated self-gravitating Abelian-Higgs vortex in dilaton gravity, and they showed that the above energy-momentum tensor results in a contradiction in the field equations in general, and one must take into the account the other stresses as nonvanishing. However, their investigation also shows that if there exists a mass term in the theory, than the contradiction is removed and Vilenkin’s prescription might be appropriate. The inconsistency is also removed if we have a more general scalar-tensor theory [26], or the string is nonstatic [27]. In this report, I will show that, in the presence of a cosmological constant, Vilenkin’s prescription is consistent with BD theory. I also present some exact solutions of the BD field equations satisfying the above energy-momentum tensor and representing an infinitely long, static straight, local gauge cosmic string.

II. FIELD EQUATIONS

The Brans-Dicke theory is described, in the presence of a cosmological constant, by the action:

\[
S = \int d^4x \sqrt{-g} \left\{ \phi (-R + 2\Lambda) + \frac{\omega}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\} + S_m[\Psi, g]; \tag{1}
\]

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Here $R$ is the Ricci scalar, $\Lambda$ is cosmological constant, $g$ is the determinant of the spacetime metric $g_{\mu\nu}$, $\phi$ is the Brans-Dicke scalar field, $\omega$ is the Brans-Dicke parameter, and $S_m$ denote the action of matter fields $\Psi$. We use units in which $c = \hbar = 1$ and mostly plus signature. The action (1) yields the field equations

\[ G_{\mu\nu} + A g_{\mu\nu} = \frac{T_{\mu\nu}}{\phi} + \frac{\omega}{\phi^2} \left( \phi,_{\mu} \phi,_{\nu} - \frac{1}{2} g_{\mu\nu} \phi^{\alpha} \phi_{,\alpha} \right) + \frac{1}{\phi} \left( \phi,_{\mu} \phi,_{\nu} - g_{\mu\nu} \Box \phi \right), \quad (2) \]

\[ (2\omega + 3) \Box \phi = -2\Lambda \phi + T^\mu_\mu. \quad (3) \]

Here, $T_{\mu\nu}$ is the energy-momentum tensor of the matter fields. Note that, in this frame (1), the energy conservation equation

\[ \nabla_\mu T^\mu_\nu = 0 \quad (4) \]

holds.

Let us consider a general cylindrically symmetric, static metric

\[ ds^2 = e^{2(K-U)(dr^2 - dt^2) + e^{2U}dz^2 + e^{-2U}W^2d\theta^2}, \quad (5) \]

where $t, r, z, \theta$ denote the time, the radial, the axial and the angular cylindrical coordinates with the ranges $-\infty < t, z < \infty$, $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$ and the metric functions $K, U, W$ and the BD scalar field $\phi$ are the functions of $r$ only. For the string, we consider Vilenkin’s prescription and take the only nonvanishing terms of energy-momentum tensor of the form:

\[ T^\mu_0 = T^\mu_z = -\sigma(r). \quad (6) \]

Then, the field equations are:

\[ -\frac{W''}{W} + K' \frac{W'}{W} - U'^2 = \frac{\omega}{2} \left( \frac{\phi'}{\phi} \right)^2 - (K' - U') \frac{\phi'}{\phi} + \frac{W'}{W} \frac{\phi'}{\phi} + (\Lambda + \frac{\sigma}{\phi}) e^{2(K-U)}, \quad (7) \]

\[ K' \frac{W'}{W} - U'^2 = \frac{\omega}{2} \left( \frac{\phi'}{\phi} \right)^2 - (K' - U') \frac{\phi'}{\phi} - \frac{W'}{W} \frac{\phi'}{\phi} - \Lambda e^{2(K-U)}, \quad (8) \]

\[ \frac{W''}{W} - 2U'' - 2U' \frac{W'}{W} + K'' + U'^2 = -\frac{\omega}{2} \left( \frac{\phi'}{\phi} \right)^2 - \phi'' \frac{\phi'}{\phi} - \frac{W'}{W} \frac{\phi'}{\phi} + (\Lambda + \frac{\sigma}{\phi}) e^{2(K-U)}, \quad (9) \]

\[ K'' + U'^2 = -\frac{\omega}{2} \left( \frac{\phi'}{\phi} \right)^2 - U' \frac{\phi'}{\phi} - \phi'' \frac{\phi'}{\phi} - \Lambda e^{2(K-U)}, \quad (10) \]

\[ \phi'' + \phi' \frac{W'}{W} = -\frac{2}{2\omega + 3} (\Lambda \phi + \sigma) e^{2(K-U)}. \quad (11) \]

The energy conservation equation (4) yields $K'\sigma = 0$. Thus, in order to have a string ($\sigma \neq 0$), we must have

\[ K' = 0, \quad (12) \]

and hereafter we take $e^K = 1$. Using this and Eqs. (8) and (10), we find

\[ \phi'' + \phi' \frac{W'}{W} = -2\Lambda \phi e^{-2U}. \quad (13) \]

Considering this equation with(11) and taking into account (12) we have:

\[ \sigma = 2(\omega + 1)\Lambda. \quad (14) \]

This expression says that, for $\Lambda = 0$, we must have either $\sigma = 0$, i. e. no string at all, or $\omega \rightarrow \infty$, if we have $\sigma \neq 0$. Thus, if we insist on $\sigma \neq 0$, the solution reduces to the corresponding solution in GR. This fact was noticed by [23] and they concluded that BD theory is inconsistent for local strings, since for this case the only solution is the same as GR one. However, for $\Lambda \neq 0, \omega \neq -1$ we have $\sigma \neq 0$, and for BD theory with a cosmological constant, Vilenkin’s prescription for local strings is consistent.
III. SOLUTIONS

Having proved that we can have a nontrivial string for BD theory with a cosmological constant, let me introduce exact solutions representing the interior regions of the string. Note that for local gauge strings we must have boost invariance along $t, z$ directions and this requires $U = 0$. Then, the field equations reduce to the following set:

\[ \phi'' + \frac{\phi'}{W}W' = -2\Lambda \phi, \]
\[ \frac{\omega}{2} \left( \frac{\phi'}{\phi} \right)^2 = W' \frac{\phi'}{\phi} + \Lambda, \]
\[ \frac{(\phi W)''}{\phi W} = -2\Lambda(2 + \omega), \]

(15) (16) (17)


together with (14). Now, we limit our analysis to $\omega > -2$. (For $\omega < -2$, the results are the same as $\Lambda \to -\Lambda$.) With this limitation, we have two different sets of solutions depending on the sign of $\Lambda$.

A. $\Lambda > 0$

For this case the Eq. (17) can be integrated to give

\[ W \phi = A \sin(\alpha r) + B \cos(\alpha r), \quad \Lambda > 0, \]

(18)

where

\[ \alpha = \sqrt{2\Lambda(2 + \omega)} > 0. \]

(19)

We choose $B = 0$ due to cylindrical symmetry. By considering Eqs. (15) and (16) we find

\[ W(r) = A \sin^{\frac{\omega+1}{\omega+2}}(\alpha r) \tan^{\epsilon/(\omega+2)}((\alpha/2)r), \]
\[ \phi(r) = \sin^{1/(\omega+2)}(\alpha r) \tan^{-\epsilon/(\omega+2)}((\alpha/2)r), \]

(20) (21)

where $\epsilon = \pm 1$. The metric has the form

\[ ds^2 = -dt^2 + dr^2 + dz^2 + W^2 d\theta^2. \]

(22)

Regularity conditions on the axis for cylindrical symmetry require $W(0) = 0$ and $W'(0) = 1$. The first one is automatically satisfied and the second one is satisfied if

\[ A = 2^{-\omega/(2\omega+4)} / \sqrt{(\omega + 2)\Lambda}. \]

(23)

Hence the solution is regular and free of a conical singularity on the axis. For $\epsilon = 1$ it is also free of any singularity on the axis since scalars constructed from curvature elements are regular. For example, the Ricci scalar is

\[ R = \frac{2\Lambda [\omega + (\omega + 1)^2(1 + \cos(\alpha r))]}{(\omega + 2) \cos((\alpha/2)r)} \]

(24)

and is regular on the axis. Other scalars have the similar expressions. Thus, we see that the interior string solution is smooth and regular near the axis. However, the solution is singular at $r = r_c = \pi/(2\alpha)$, hence the radius of the string should be smaller than $r_c$.

The mass per unit length of the string having radius $r_0$ can be calculated using

\[ m = \int_0^{2\pi} d\theta \int_0^{r_0} \sigma(r) W(r) dr. \]

(25)

This yields

\[ m = 2\pi \frac{2\omega/(\omega+2)(\omega + 1)}{(\omega + 2)} \sin^2 ((\alpha/2)r_0). \]

(26)

The solution with $\epsilon = -1$ must be excluded since it is singular on the axis.
B. $\Lambda < 0$

For this case, from (17) and cylindrical symmetry we must have

$$W\phi = A \sinh (\beta r)$$

(27)

with

$$\beta = \sqrt{|2\Lambda (\omega + 2)|},$$

(28)

which yields, with the metric of the form (22),

$$W(r) = C \sinh^{(\omega + 1)/2(\omega + 2)}(\beta r) \tanh^{\epsilon/(\omega + 2)}((\beta/2) r),$$

$$\phi(r) = \sinh^{1/(\omega + 2)}(\beta r) \tanh^{-\epsilon/(\omega + 2)}((\beta/2) r),$$

(29)

(30)

where $\epsilon = \pm 1$. The analyses of the regularity conditions and curvature scalars show that the solution with $\epsilon = 1$ is smooth, regular, and free of any singularity. Unlike $\Lambda > 0$, there is no upper limit on the value of the radius of the string. The solution with $\epsilon = -1$ is singular on the axis and must be excluded. The mass per unit length of this solution ($\epsilon = 1$) with radius $r_0$ is

$$m = 2\pi 2^{2/(\omega + 2)}(\omega + 1) \sinh^2 ((\beta/2) r_0).$$

(31)

For all of these solutions $\sigma$ and $m$ are positive.

IV. CONCLUSIONS

In this report we have proved that, unlike the case $\Lambda = 0$ [23, 24], in the presence of a cosmological constant, $\Lambda$, the phenomenological equation of state for local cosmic strings $T_{00} = T_{zz} = -\sigma$ ($\sigma > 0$) is consistent with Brans-Dicke theory (for $\omega \neq -1, -2$). This result is in accordance with the analysis of Gregory and Santos [25]. We have also presented exact solutions for full Brans-Dicke field equations with a $\Lambda$ term for this configuration and discussed some of their properties. Further properties of these solutions with possible exterior solutions, analogues of GR results [28, 29], are under investigation [30] and will be presented in elsewhere.

There is an interesting duality between GR and BD theories for cosmic strings. In GR, Vilenkin’s anzats is a good choice; however, when there is a cosmological constant, one must take into account the other stresses as nonvanishing, otherwise there is no static solution[29]. In BD theory, however, as we have shown, the situation is reversed and if there is no cosmological term, then there is no solution, but if there is a cosmological constant, then we can have strings with above properties in BD theory.

[1] A. Vilenkin and E.P.S. Shellard, Cosmic Strings and other Topological Defects, (Cambridge University Press, Cambridge, 1994).
[2] M.B. Hindmarsh and T.W.B. Kibble, Rep. Prog. Phys. 58, 477 (1995).
[3] L. Pogosian, S.-H.H. Tye, I. Wasserman, and M. Wyman, Phys. Rev. D 68, 023506 (2003); Erratum ibid. 73, 089904 (2006).
[4] N. Bevis, M. Hindmarsh, M. Kunz, and J. Urrestilla, astro-ph/0605018.
[5] T. W. B. Kibble, astro-ph/0410073.
[6] S. Sarangi and S.-H.H. Tye, Phys. Lett. B 536, 185 (2002).
[7] G. Dvali and A. Vilenkin, JCAP 0403, 010 (2004).
[8] E.J. Copeland, R.C. Myers, and J. Polchinski, JHEP 0406, 013 (2004).
[9] A. Vilenkin, Phys. Rev. D 23, 852 (1981).
[10] D. Garfinkle, Phys. Rev. D 32, 1323 (1985).
[11] B. Linet, Phys. Lett. A 124, 240 (1987).
[12] J.R. Gott, III, Astrophys. J. 288, 422 (1985).
[13] W.A. Hiscock, Phys. Rev. D 31, 3288 (1985).
[14] B. Linet, Gen. Relat. Gravit. 17, 1109 (1985).
[15] C.H. Brans and R.H. Dicke, Phys. Rev. 124, 925 (1961).
[16] V. Faraoni, *Cosmology in Scalar Tensor Gravity*, (Kluwer Academic Publishers, Netherlands, 2004).

[17] C.H. Brans, gr-qc/9705069.

[18] C. Gundlach and M.E. Ortiz, Phys. Rev. D 42, 2521 (1990);
    L.O. Pimentel and A.N. Morales, Rev. Mex. Fis. 36, S199 (1990).

[19] A. Barros and C. Romero, J. Math. Phys. 36, 5800 (1995).

[20] M.E.X. Guimaraes, Class. Quantum Grav. 14, 435 (1997).

[21] B. Boisseau and B. Linet, Class. Quantum Grav. 14, 3063 (1997).

[22] F. Dahia and C. Romero, Phys. Rev. D 60, 104019 (1999).

[23] A.A. Sen, N. Banerjee, and A. Banerjee, Phys. Rev. D 56, 3706 (1997).

[24] A. Arazi and C. Simeone, Gen. Relat. Gravit. 32, 2259 (2000).

[25] R. Gregory and C. Santos, Phys. Rev. D 56, 1194 (1997).

[26] A.A. Sen and N. Banerjee, Phys. Rev. D 57, 6558 (1998).

[27] A.A. Sen, Pramana 55, 369 (2000).

[28] B. Linet, J. Math. Phys. 27, 1817 (1986).

[29] Q. Tian, Phys. Rev. D 33, 3549 (1986).

[30] O. Delice, (unpublished).