$SL(4, R)$ Generating Symmetry in Five–Dimensional Gravity Coupled to Dilaton and Three–Form

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We give an explicit formulation of the three–dimensional $SL(4, R)/SO(2, 2)$ σ–model representing the five–dimensional Einstein gravity coupled to the dilaton and the three–form field for spacetimes with two commuting Killing vector fields. New matrix representation is obtained which is similar to one found earlier in the four–dimensional Einstein–Maxwell–Dilaton–Axion (EMDA) theory. The $SL(4, R)$ symmetry joins a variety of 5D solutions of different physical types including strings, 0–branes, KK monopoles etc. interpreting them as duals to the four–dimensional Kerr metric translated along the fifth coordinate. The symmetry transformations are used to construct new rotating strings and composite $(0 - 1)$–branes endowed with a NUT parameter.

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I. INTRODUCTION

Recent interest to soliton solutions in multidimensional supergravities stimulates a search of solution generating techniques relevant to various gravity coupled systems containing antisymmetric forms. Dimensional reduction of these theories reveals global symmetries which can be effectively used for such purposes. But although general structure of global symmetries in dimensionally reduced supergravities is well–known [1,2], their practical application requires additional efforts to be done. One has to construct suitable matrix representations in terms of which these symmetries, acting non–linearly (and perhaps partly non–locally) on the set of initial field variables, could be expressed in a tractable way. In dimension three or in higher dimensions with suitably truncated metric ansatz one is able to express all variables in terms of scalar fields forming some non–linear σ–model, in which case the symmetry transformations are particularly simple. It has been shown recently [3] that all block–diagonal p–brane solutions including black, multiple–center, intersecting ones as well as fluxbranes and their non–linear superpositions with charged branes can be produced within a simple σ–model, the essential part of which has the same structure as the static four–dimensional Einstein–Maxwell (EM) system [4]. This model, however, can not be generalized to include rotating or dyon solutions, contrary to the situation in the stationary four–dimensional EM or Einstein–Maxwell–Dilaton–Axion (EMDA) systems. Certain rotating p–brane solutions were presented by Cvetič and Youm [5] apparently hinted by the Myers and Perry [6] multidimensional Kerr–like metrics, but, to our knowledge, no constructive derivation of such solutions has been given so far.

As a step further in developing generating techniques for larger classes of p–branes, we suggest here a new matrix representation for the Maharana–Schwarz symmetry [7] of the five–dimensional gravity coupled to three–form and dilaton with two commuting Killing vector fields. The novel feature consists in exploiting an exceptional local isomorphism between the Maharana–Schwarz group $SO(3, 3)$ relevant to the case and the group $SL(4, R)$. This opens a way to construct instead of the $6 \times 6$ matrix realization a simpler representation of the symmetry by $4 \times 4$ real matrices, which, in turn, may be regarded as a direct generalization of the $Sp(4, R)$ formulation of the stationary four–dimensional EMDA theory [8]. This framework naturally incorporates rotating strings as well as dyon–type solutions which may be interpreted as composite states of strings and 0–branes.

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Our $\sigma$–model constitutes another example of $4 \times 4$ matrix theories of which it has the maximal dimension of the target space. The three–dimensional reduction of the 4D EMDA theory with one vector field corresponds to the $Sp(4, R)/SO(1, 2)$ coset which is six–dimensional. The same theory with two vector fields may be formulated as $SU(2, 2)/U(1, 1)$ $\sigma$–model with the eight–dimensional target space \cite{14}. It is likely that the present case closes the list of theories whose reduction to three dimensions gives a $4 \times 4$ representation, now for nine–dimensional target space. There is some similarity with the case of EMDA with two vector fields, in fact the present theory in four dimensions also has two vector fields, but it is endowed with an additional scalar modulus. Therefore the target space now is odd–dimensional and is no more Kähler. Still the matrix structure is very similar to that found previously in the $Sp(4, R)$ theory, which can be obtained from the present one by identifying two different $2 \times 2$ blocks. (Although the target space is not Kähler, suitable Ernst–like complex potentials can also be found, in terms of which the target space metrics exhibits a close similarity to that of the EMDA theory.) The standard representation of $SL(4, R)$ in terms of the $1 \times 3$ block decomposition was applied to five–dimensional dilaton–axion gravity in $\cite{15}$. This formulation, however, does not make contact with neither $Sp(4, R)$ nor $SU(2, 2)$ theories and thus is less useful for solution generation.

II. FROM FIVE TO FOUR DIMENSIONS

Our starting point is the five–dimensional action

$$S_5 = \int d^5 x \sqrt{-g_5} \left\{ R_5 - \frac{1}{2}(\partial \phi)^2 - \frac{e^{-\alpha \phi}}{12} \tilde{H}^2 \right\},$$

(1)

where $\phi$ is the dilaton, $\tilde{H} = d\tilde{B}$ is an antisymmetric three–form. We perform a two–step reduction to the three–dimensional theory making first the Kaluza–Klein ansatz for the five–dimensional metric suppressing a spacelike coordinate

$$ds_5^2 = e^{-4\varphi} (dy + A_{\mu} dx^\mu)^2 + e^{2\varphi} g_{\mu\nu} dx^\mu dx^\nu,$$

(2)

with all fields depending only on $x^\mu$. Then

$$\sqrt{-g_5} R_5 = \sqrt{-g_4} \left\{ R_4 - 6 g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{4} e^{-6\varphi} F_{\mu\nu} F^{\mu\nu} - 2 \nabla_\mu (g^{\mu\nu} \partial_\nu \varphi) \right\},$$

(3)

where $A = A_\mu dx^\mu$, $F = dA$. The five–dimensional two–form $\tilde{B}$ can be decomposed into the four–dimensional two–form $B$ and the four–dimensional one–form $A$

$$\tilde{B} = B - dy \wedge A.$$

(4)

Then the four–dimensional $H$ will include the appropriate Chern–Simons terms

$$H = dB - A \wedge F,$$

(5)

where $F = dA$.

Omitting a total divergence one obtains the following four–dimensional action

$$S_4 = \int d^4 x \sqrt{-g_4} \left\{ R_4 - \frac{16 + 3\alpha^2}{18\alpha^2} (\partial \phi)^2 - \frac{8 - 3\alpha^2}{3\alpha^2} (\partial \phi)(\partial \psi) - \frac{4 + 3\alpha^2}{18\alpha^2} (\partial \psi)^2 \right\} - \frac{1}{4} e^{-\psi - \phi} F^2 - \frac{1}{4} e^{-2\psi + \phi} F^2 - \frac{e^{-2\phi}}{12} H^2,$$

(6)

where

$$\phi = \frac{1}{2}(\alpha \tilde{\phi} + 4\varphi), \quad \psi = \frac{1}{2} (\alpha \tilde{\phi} - 8\varphi).$$

(7)

The value of the dilaton coupling constant here is assumed to correspond to dimensional reduction of the eleven–dimensional supergravity: $\alpha^2 = 8/3$. Dualising the three–form

$$e^{-2\phi} H = -^* dk,$$

(8)

one can write the corresponding action as

$$S_4 = \int d^4 x \sqrt{-g} \left\{ R - \frac{1}{2}(\partial \phi)^2 - \frac{1}{4} (\partial \psi)^2 - \frac{1}{2} e^{2\phi} (\partial \kappa)^2 - \frac{1}{4} e^{-\psi - \phi} F^2 - \frac{1}{4} e^{-2\psi + \phi} F^2 - \frac{k}{4} \left( F \tilde{F} + F \tilde{F} \right) \right\},$$

(9)

where

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \sqrt{-g} \epsilon_{\mu\nu\lambda\rho} F^{\lambda\rho}, \quad \tilde{F}_{\mu\nu} = \frac{1}{2} \sqrt{-g} \epsilon_{\mu\nu\lambda\rho} F^{\lambda\rho}.$$ (10)

Thus in four dimensions the theory is equivalent to Einstein gravity coupled to two scalars, one pseudoscalar and two vector fields. The equations of motion corresponding to this action consist of a coupled set of Maxwell equations

$$\nabla_\mu \left( e^{-\psi - \phi} F^{\mu\nu} + \kappa \tilde{F}^{\mu\nu} \right) = 0,$$

(11)

$$\nabla_\mu \left( e^{-\psi - \phi} F^{\mu\nu} + \kappa \tilde{F}^{\mu\nu} \right) = 0,$$

(12)

three equations for scalar fields

$$\nabla^2 \phi - e^{2\phi} (\partial \kappa)^2 + \frac{1}{4} e^{-\phi} \left( e^\psi F^2 + e^{-\psi} F^2 \right) = 0,$$

(13)

$$\nabla^2 \kappa + 2 \partial_\mu \kappa \partial^\mu \phi - \frac{1}{4} e^{-2\phi} \left( F \tilde{F} + F \tilde{F} \right) = 0,$$

(14)

$$\nabla^2 \psi - \frac{1}{2} e^{-\phi} \left( e^\psi F^2 - e^{-\psi} F^2 \right) = 0,$$

(15)

and Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \nabla^2 \phi - \frac{1}{4} \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} e^{2\phi} \partial_\mu \kappa \partial_\nu \kappa + \frac{1}{2} e^{2\phi} \left( F_{\mu\alpha} F^{\alpha\nu} + \frac{1}{4} g_{\mu\nu} F^2 \right) + \frac{1}{2} e^{-\phi} \left( F_{\mu\alpha} F^{\alpha\nu} + \frac{1}{4} g_{\mu\nu} F^2 \right) = 0.$$ (16)
The Bianchi identities for two-forms read
\[ \epsilon^{\mu\nu\lambda\rho} \partial_\mu F_{\nu\lambda} = 0, \quad \epsilon^{\mu\nu\lambda\rho} \partial_\mu F_{\nu\lambda} = 0, \] (17)

It is convenient to introduce the complex “axidilaton”
\[ z = \kappa + ie^{-\phi}, \] (18)

and the following combination of the field forms:
\[ F_+ = e^{\psi/2} F + ie^{-\psi/2} \tilde{F}, \quad F_- = e^{-\psi/2} F + ie^{\psi/2} \tilde{F}, \] (19)
such that \( F_+^2 = F_-^2 \). The action (1) can be rewritten in the form
\[ S_4 = \int d^4 \sqrt{-g} \left\{ R - \frac{1}{24} \frac{\partial z}{(\partial z)^2} - \frac{1}{4} (\partial \psi)^2 \right\} \]
\[ - \frac{1}{8} \left\{ z (F_+^2 + F_-^2) \right\}, \] (20)
exhibiting the \( SO(1,1) \) symmetry (\( T \)-duality)
\[ \psi \rightarrow \psi + \beta, \quad F_\pm \rightarrow F_\pm. \] (21)

The field equations in terms of the complex variables
\[ \nabla_\mu \text{Im} \left( e^{\pm \psi/2} F_{\pm\mu} \right) = 0, \] (22)
\[ \nabla^2 z + i \frac{(\partial z)^2}{\text{Im} z} - i \frac{1}{8} (\text{Im} z)^2 (F_+^2 + F_-^2) = 0, \] (23)
\[ \nabla^2 \psi - 1 \frac{\partial z}{\text{Im} z} (|F_+|^2 - |F_-|^2) = 0, \] (24)
\[ R_{\mu\nu} - \frac{\partial_\mu \partial_\nu z}{2(\text{Im} z)^2} - \frac{1}{4} \partial_\mu \psi \partial_\nu \psi \]
\[ + \frac{1}{4} \text{Im} \text{Re} \left( F_{+\mu\alpha} F_{+\nu}^{\alpha} + F_{-\mu\alpha} \tilde{F}_{+\nu}^{\alpha} \right) = 0, \] (25)

together with Bianchi identities
\[ \nabla_\mu \text{Im} \left( e^{\pm \psi/2} F_{\pm\mu} \right) = 0, \] (26)

exhibit the \( SL(2,R) \) S-duality symmetry
\[ z \rightarrow az + b/cz + d, \quad F_\pm \rightarrow (cz + d) F_\pm, \] (27)

where \( ad - bc = 1 \). Therefore, the duality group in four dimensions is \( SO(1,1) \times SL(2,R) \).

III. REDUCTION TO THREE DIMENSIONS

As a second step we reduce the theory to three dimensions assuming stationarity of the four-metric:
\[ ds^2 = -f(dt - \varpi dx^i)^2 + f^{-1} h_{ij} dx^i dx^j, \] (28)
where \( f, \varpi \) and \( h_{ij} \) depend only on \( x^i \). In three dimensions vector fields can be parameterized by scalar potentials \( v_a, u_a, a = 1,2 \) via
\[ F_{i0} = \partial_i v_1, \quad e^{\psi - \phi} F^{ij} + \kappa \tilde{F}^{ij} = -\frac{f}{\sqrt{h}} e^{ijk} \partial_k u_1, \] (29)
\[ F_{i0} = \partial_i v_2, \quad e^{-\psi - \phi} F^{ij} + \kappa \tilde{F}^{ij} = -\frac{f}{\sqrt{h}} e^{ijk} \partial_k u_2. \] (30)

The three-dimensional KK field \( \varpi_i \) then is dualized as follows
\[ \tau^i = \frac{f^2}{\sqrt{h}} e^{ijk} \partial_j \varpi_k. \] (31)

The action for new variables reads
\[ S_3 = \int d^3 \sqrt{h} \left\{ R_3 - \frac{1}{2f^2} \left( (\partial \varphi)^2 + (\tau^2) + \frac{1}{2} (\partial \phi)^2 \right) \right\} \]
\[ + \frac{1}{4} (\partial \varphi)^2 + \frac{1}{2} e^{2\varphi} (\partial \kappa)^2 - \frac{1}{2f} \left( e^{\psi - \phi} (\partial v_1)^2 + e^{-\psi + \phi} w_1^2 \right) \]
\[ + e^{-\psi - \phi} (\partial v_2)^2 + e^{\psi + \phi} w_2^2 \right\}, \] (32)

where
\[ w_1 = \partial u_1 - \kappa \partial v_2, \quad w_2 = \partial u_2 - \kappa \partial v_1, \] (33)

and all vector contractions correspond to the metric \( h_{ij} \).

One can solve the four-dimensional constraint equations for the Ricci component \( R^i_0 \) by introducing the twist potential \( \chi \)
\[ \tau_i = \partial_i \chi + \frac{1}{2} (v_a \partial_i u_a - u_a \partial_i v_a), \] (34)

so finally we obtain a three-dimensional \( \sigma \)-model with the nine-dimensional target space parameterized by \( \Phi^A = (f, \chi, \phi, \psi, \kappa, v_1, w_1, v_2, w_2), A = 1, \ldots, 9 \) and endowed with the metric
\[ dl^2 = \frac{1}{2f^2} \left\{ df^2 + \left[ d\chi + \frac{1}{2} (v_a du_a - u_a dv_a) \right]^2 \right\} \]
\[ + \frac{1}{2} \dot{\varphi}^2 + \frac{1}{4} \dot{\psi}^2 + \frac{1}{2} e^{2\varphi} \dot{\kappa}^2 \]
\[ - \frac{1}{2f} \left[ e^{\psi - \phi} \dot{v}_1^2 + e^{-\psi + \phi} (du_1 - \kappa dv_2)^2 \right. \]
\[ + e^{-\psi - \phi} (dv_2)^2 + e^{\psi + \phi} (du_2 - \kappa dv_1)^2 \right\}. \] (35)

This is the metric of the symmetric space \( SL(4,R)/SO(2,2) \) on which the \( SL(4,R) \) isometry group acts transitively. As the coset representatives one can choose the symmetric \( SL(4,R) \) matrices and finally the target space metric will read
\[ dl^2 = -\frac{1}{4} \text{Tr} \left( dM dM^{-1} \right). \] (36)

The matrix \( M \) can be chosen in the form
\[ M = \begin{pmatrix} 1 & 0 \\ Q^T & 1 \end{pmatrix} \begin{pmatrix} P^{-1} & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} 1 & Q \\ 0 & 1 \end{pmatrix}, \]
\[ = \begin{pmatrix} P^{-1} & 0 \\ Q^T P_1^{-1} & P_2 + Q^T P_1^{-1} Q \end{pmatrix}, \] (37)
where the $P_1, P_2$ and $Q$ are the real $2 \times 2$ matrices and $P_1, P_2$ are symmetric matrices with the same determinant. This matrix is a direct generalization of the $Sp(4, R)$ matrix found in [8] in the case of the four-dimensional EMDA theory. Its inverse is given by

$$\mathcal{M}^{-1} = \begin{pmatrix} P_1 + Q P_2^{-1} Q^T & -Q P_2^{-1} \\ -P_2^{-1} Q^T & P_2^{-1} \end{pmatrix}. \tag{38}$$

The following identity is useful in establishing the equivalence between (33) and (36):

$$\text{Tr} \left( dM d\mathcal{M}^{-1} \right) = \text{Tr} \left( dP_1 dP_2^{-1} + dP_2 dP_2^{-1} - 2 dQ P_2^{-1} Q^T P_2^{-1} \right). \tag{39}$$

Using it one can find the following parameterization of the desired $2 \times 2$ matrices in terms of the potentials:

$$P_1 = e^{\psi/2} \begin{pmatrix} f e^{-\psi} - (v_1)^2 e^{-\phi} - v_1 e^{-\phi} \\ -v_1 e^{-\phi} - e^{-\phi} \end{pmatrix}, \tag{40}$$

$$P_2 = e^{-\psi/2} \begin{pmatrix} f e^{\psi} - (v_2)^2 e^{-\phi} - v_2 e^{-\phi} \\ -v_2 e^{-\phi} - e^{-\phi} \end{pmatrix}, \tag{41}$$

$$Q = \begin{pmatrix} 1/2 \xi - \chi & u_2 - \kappa v_1 \\ u_2 - \kappa v_1 & -\kappa \end{pmatrix}, \tag{42}$$

where

$$\xi = v_1 (u_1 - \kappa v_2) + v_2 (u_2 - \kappa v_1). \tag{43}$$

Therefore, the three-dimensional action can be rewritten as

$$S_3 = \int d^3 x \sqrt{h} \left\{ R_3 + \frac{1}{4} \text{Tr} \left( \nabla \mathcal{M} \mathcal{N} \mathcal{M}^{-1} \right) \right\}. \tag{44}$$

This action is invariant under the $27$-parametric $SL(4, R)$ transformations $\mathcal{M} \rightarrow g^T \mathcal{M} g$ with constant $g \in SL(4, R)$. For $g$ it is convenient to use the Gauss decomposition into the product of left (triangle), center (block-diagonal) and right (triangle) matrices as follow

$$\mathcal{G} = L S R, \quad \mathcal{L} = \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} S^{-1} & 0 \\ 0 & T \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} 1 & R \\ 0 & 1 \end{pmatrix}, \tag{45}$$

where $L, R$ are arbitrary real $2 \times 2$ matrices, $S, T$ are arbitrary real $2 \times 2$ matrices satisfying $\det S = \det T$.

It is easy to check that $R$-transformations

$$P \rightarrow P, \quad F \rightarrow F, \quad Q \rightarrow Q + R. \tag{46}$$

are pure gauge. $S$-transformations

$$P_1 \rightarrow S P_1 S^T, \quad P_2 \rightarrow T^T P_2 T, \quad Q \rightarrow S Q T. \tag{47}$$

contain two gauge, three different scale and two (electric type) Harrison transformations. This subgroup is seven-parametric and the corresponding parameter matrices read

$$S = \begin{pmatrix} e^{s_1} & s_2 \\ s_3 & e^{s_3} \end{pmatrix}, \quad T = \begin{pmatrix} e^{t_1} & t_2 \\ t_3 & e^{t_3} \end{pmatrix}, \tag{48}$$

where $e^{s_1+s_3} - s_2 s_3 = e^{t_1+t_3} - t_2 t_3$. One can identify the following one-parametric transformations

**Scale transformation** ($t_1 = s_1$)

$$f \rightarrow e^{2s_1} f, \quad \chi \rightarrow e^{2s_1} \chi, \quad v_a \rightarrow e^{s_1} v_a, \quad u_a \rightarrow e^{s_1} u_a, \quad \tag{49}$$

**Dilaton shift** ($t_4 = s_4$)

$$\phi \rightarrow \phi - 2s_4, \quad \kappa \rightarrow e^{2s_4} \kappa, \quad v_a \rightarrow e^{-s_4} v_a, \quad u_a \rightarrow e^{s_4} u_a, \quad \tag{50}$$

**Modulus shift** ($-s_1 = s_4 = t_1 = -t_4 = t/2$)

$$\psi \rightarrow \psi + 2t, \quad (v_1, u_2) \rightarrow e^{-t} (v_1, u_2), \quad (u_1, v_2) \rightarrow e^t (u_1, v_2). \tag{51}$$

**Electric gauge**

$$v_1 \rightarrow v_1 + s_2, \quad \chi \rightarrow \chi - \frac{s_2}{2} u_1, \tag{52}$$

$$v_2 \rightarrow v_2 + t_3, \quad \chi \rightarrow \chi - \frac{t_3}{2} v_2. \tag{53}$$

There are two electric Harrison transformations: one generating the Kaluza–Klein vector

$$\phi \rightarrow \phi - \frac{1}{2} \ln H_1, \quad \psi \rightarrow \psi + \ln H_1, \quad f \rightarrow f H_1^{-1/2}, \tag{54}$$

$$v_1 \rightarrow \frac{H_2}{H_1}, \quad u_1 \rightarrow u_1 - s_3 \chi + \frac{s_3}{2} (v_1 u_1 - v_2 u_2), \tag{55}$$

$$u_2 \rightarrow \frac{1}{H_1} \left[ u_2 (1 + s_3 v_1) - s_3 \kappa e^{-\psi + \phi} \right], \tag{56}$$

$$\kappa \rightarrow \kappa - s_3 (u_2 - \kappa v_1), \tag{57}$$

$$\chi \rightarrow \chi + \frac{s_3 H_2}{4H_1} (v_2 u_2 - v_1 u_1 - 2 \chi) - \frac{s_3 e^{-\psi + \phi}}{2H_1} (u_1 - \kappa v_2), \tag{58}$$

where

$$H_1 = (1 + s_3 v_1)^2 - s_3^2 e^{-\psi + \phi}, \tag{59}$$

$$H_2 = v_1 (1 + s_3 v_1) - s_3 e^{-\psi + \phi}, \tag{60}$$

and another generating the three–form electric component

$$\phi \rightarrow \phi - \frac{1}{2} \ln H_1', \quad \psi \rightarrow \psi - \ln H_1', \quad f \rightarrow f H_1'^{-1/2}, \tag{61}$$

$$v_2 \rightarrow \frac{H_2'}{H_1'}, \quad u_2 \rightarrow u_2 - t_2 \chi + \frac{t_2}{2} (v_2 u_2 - v_1 u_1), \tag{62}$$

$$u_1 \rightarrow \frac{1}{H_1'} \left[ u_1 (1 + t_2 v_2) - t_2 \kappa e^{\psi + \phi} \right], \tag{63}$$

$$\kappa \rightarrow \kappa - t_2 (u_2 - \kappa v_2), \tag{64}$$

$$\chi \rightarrow \chi + \frac{t_2 H_2'}{4H_1'} (v_1 u_1 - v_2 u_2 - 2 \chi) - \frac{t_2 e^{\psi + \phi}}{2H_1'} (u_2 - \kappa v_1), \tag{65}$$

where

$$H_1' = (1 + t_2 v_2)^2 - t_2^2 e^{\psi + \phi}, \tag{66}$$

$$H_2' = v_2 (1 + t_2 v_2) - t_2 e^{\psi + \phi}. \tag{67}$$
The subgroup of the left transformations $L$ is more difficult to write in terms of its action on the potentials. In terms of $2 \times 2$ matrices it reads:

$$
P_1^{-1} \rightarrow (1 + QL)^T P_1^{-1}(1 + QL) + L^T P_2 L,
$$

$$
P_2 + Q^T P_1^{-1} Q \rightarrow P_2 + Q^T P_1^{-1} Q,
$$

$$
P_1^{-1} Q \rightarrow (1 + L^T Q^T) P_1^{-1} Q + L^T P_2.
$$

(57)

Using the parameterization $L = \begin{pmatrix} l_1 & l_2 \\
 l_3 & l_4 \end{pmatrix}$, one can identify $l_1$ as a parameter of the Ehlers transformation, $l_2, l_3$ as parameters of the magnetic Harrison transformations, and $l_4$ as a parameter of the Ehlers–like part of $S$–duality. The first three transformations are rather complicated, so we give here explicitly only the last one:

$$
z^{-1} \rightarrow z^{-1} + l_4
$$

$$
v_1 \rightarrow v_1 - l_4 u_2, \quad v_2 \rightarrow v_2 - l_4 u_1.
$$

(58)

However, if the seed solution satisfies the four-dimensional vacuum Einstein equations (i.e. corresponds to the $SL(2, R)/SO(2)$ subspace of the full target space), the $2 \times 2$ matrices admit a simple form

$$
P_1 = P_2 = \begin{pmatrix} f_0 & 0 \\
 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} -\chi_0 & 0 \\
 0 & 0 \end{pmatrix},
$$

and the left transformations considerably simplify. The Ehlers transformation will take the form

$$
f = \frac{f_0}{(1 - l_1 \chi_0)^2 + l_2^2 f_0^2}, \quad \chi = \frac{\chi_0 (1 - l_1 \chi_0) - l_1 f_0^2}{(1 - l_1 \chi_0)^2 + l_2^2 f_0^2},
$$

(60)

the magnetic Harrison transformation for the KK vector will be

$$
\phi = \frac{1}{2} \ln(1 - l_2^2 f_0), \quad \psi = -\ln(1 - l_2^2 f_0),
$$

$$
f \rightarrow f_0 (1 - l_2^2 f_0)^{-\frac{2}{3}}, \quad \chi = \chi_0 \left[ 1 + \frac{l_2^2 f_0}{2(1 - l_2^2 f_0)} \right],
$$

(61)

$$
u_1 = \frac{-l_2 f_0}{1 - l_2^2 f_0}, \quad v_1 = l_2 \chi_0.$$

while the magnetic Harrison transformation generating the three-form field can be expressed as

$$
\phi = \frac{1}{2} \ln(1 - l_3^2 f_0), \quad \psi = \ln(1 - l_3^2 f_0),
$$

$$
f \rightarrow f_0 (1 - l_3^2 f_0)^{-\frac{2}{3}}, \quad \chi = \chi_0 \left[ 1 + \frac{l_2^2 f_0}{2(1 - l_2^2 f_0)} \right],
$$

(62)

$$
u_2 = \frac{-l_3 f_0}{1 - l_3^2 f_0}, \quad v_2 = l_3 \chi_0.$$

Two other useful subspaces of the full target space correspond to the five-dimensional vacuum ($v_2 = u_2 = \kappa = 0, \psi = -2\phi$, the symmetry group being $SL(3, R)$) and to the four-dimensional EMDA system ($v_2 = v_1, u_2 = u_1, \psi = 0$, with the symmetry $Sp(4, R)$). The corresponding solutions can be used as seeds as well.

**IV. SOLUTION GENERATION**

Our interpretation of the $SL(4, R)$ transformations is based on the four-dimensional picture, therefore two type Harrison transformations correspond to different five-dimensional charges: one pair is related to the non-diagonal components of the five-dimensional metric, while another pair — to the three-form field. Thus, taking as seeds the four-dimensional vacuum solutions, one can generate in the first case the vacuum 5D metrics, and in the second — the three-form supported configurations. Starting with the $y$–translated four-dimensional Schwarzschild solution

$$
ds_5^2 = -(1 - \frac{2m}{r}) dt^2 + dy^2 + (1 - \frac{2m}{r})^{-1} dr^2 + r^2 d\Omega_2,
$$

(63)

and applying the sequences of transformations including the second type electric Harrison transformation one arrives at the electric string solution [11]. More precisely, the three-form electric component is produced by the electric Harrison transformations [65] with the parameter $t_2 = \mu$, then one has to compensate the asymptotic values of variables to make the solution asymptotically flat. One possibility is to use the following chain of transformations: the electric gauge $[64]$, $t_3 = \mu/(1 - \mu^2)$; the scale transformation $[60]$, $t_1 = s_1 = \frac{1}{4} \ln(1 - \mu^2)$; the dilaton shift $[61]$, $t_4 = s_4 = -\frac{1}{4} \ln(1 - \mu^2)$ and the modulus shift $[62]$, $t = \frac{1}{2} \ln(1 - \mu^2)$. The final result is

$$
ds_5^2 = (1 + 2q/r)^{-1/3} \left\{ -(1 - \frac{2m}{r}) dt^2 + dy^2 \right\}
$$

$$
+ (1 + 2q/r)^{2/3} \left\{ (1 - \frac{2m}{r})^{-1} dr^2 + r^2 d\Omega_2 \right\},
$$

$$
\phi = -\sqrt{\frac{2}{3}} \ln(1 + 2q/r),
$$

$$
\hat{B}_{ly} = \sqrt{\frac{q}{m + q}} \left[ 1 - (1 - \frac{2m}{r})(1 + 2q/r)^{-1} \right],
$$

(64)

where

$$
q := \frac{\mu^2 m}{1 - \mu^2}.
$$

(65)

This is the black string, of which the extremal version is achieved in the limit $\mu \rightarrow 1, m \rightarrow 0$, $q$ finite.

One can apply a similar chain of transformations including the second type magnetic Harrison transformation $[63]$ with $t_3 = \mu$. The necessary adjustment includes the magnetic gauge $[67]$, $r_2 = \mu/(1 - \mu^2)$, the scale transformation $[60]$, $t_1 = s_1 = \frac{1}{4} \ln(1 - \mu^2)$, the dilaton shift $[61]$, $t_4 = s_4 = -\frac{1}{4} \ln(1 - \mu^2)$ and the modulus shift $[62]$, $t = \frac{1}{2} \ln(1 - \mu^2)$. The resulting solution has the form
\[ ds_5^2 = -(1 + \frac{2p}{r})^{-2/3}(1 - \frac{2m}{r})dt^2 \\
+ (1 + \frac{2p}{r})^{1/3} \left\{ dy^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2d\Omega_2 \right\}, \]
\[
\phi = \sqrt{\frac{2}{3}} \ln(1 + \frac{2p}{r}), \\
B_{\phi y} = -2\sqrt{p(m + p)} \cos \theta, \quad (66)
\]
where
\[
p = \frac{\mu^2 m}{1 - \mu^2}. \quad (67)
\]
To interpret it physically, one has to note that in five dimensions the three-form is dual to the two-form field \( *H \), which will have non-zero the components \( *H_{\alpha\beta} \) like in the case of the four-dimensional magnetically charged black hole. Thus we deal with the black 5D magnetic \( \theta \)-brane, of which the extremal version corresponds to \( m = 0 \).

Application of the first type Harrison transformations generate vacuum 5D solutions. Note, that a variety of such solutions were found in the past using different techniques \([12, 13]\). All of them, as well as their generalizations including a NUT parameter, can be found via suitable chains of the \( SL(3, R) \) transformations (forming a subgroup of the \( SL(4, R) \) with \( v_2 = u_2 = \kappa = 0, \psi = -2\phi \)) from the four-dimensional Kerr metric. For more detailed analysis of the vacuum 5D metrics with commuting isometries through the the \( SL(3, R) \) symmetry see \([14]\).

One is got by an electric Harrison transformation \([57]\), with a parameter \( s_3 = \mu \) together with an appropriate compensation of asymptotic values:
\[
ds_5^2 = -(1 + \frac{2q}{r})^{-1}(1 - \frac{2m}{r})dt^2 \\
+ (1 + \frac{2q}{r}) \left\{ dy^2 + 2\sqrt{q(m + q)}(r + 2q)^{-1}dt \right\}^2 \\
+ (1 - \frac{2m}{r})^{-1}dr^2 + r^2d\Omega_2. \quad (68)
\]
This is the black version of the Dobiasch–Maison solution \([12]\), the original form of which is recovered when \( m = 0 \).

Another 5D vacuum solution resulting from the first type magnetic Harrison transformation \([57]\) with a parameter \( l_2 = \mu \) is
\[
ds_5^2 = -(1 - \frac{2m}{r})dt^2 \\
+ (1 + \frac{2p}{r})^{-1} \left\{ dy^2 + 2\sqrt{p(m + p)} \cos \theta d\phi \right\}^2 \\
+ (1 + \frac{2p}{r}) \left\{ (1 - \frac{2m}{r})^{-1}dr^2 + r^2d\Omega_2 \right\}. \quad (69)
\]
This is the black version of the Gross–Perry–Sorkin monopole \([13, 14]\), the original form corresponding to \( m = 0 \).

The total number of the independent conserved charges in the present theory is seven: mass, NUT, four charges associated with \( F \) and \( F \), and one rotation parameter. Note that, since we are considering the class of 5D metrics with non-compact Killing orbits, only one rotation parameter is allowed. Therefore, the general string solution should contain seven free parameters. Here we exhibit only the simplest four-parametric subfamilies which can be presented in a concise form.

One can repeat the same calculations as before using as a seed the \( y \)-translated four–dimensional vacuum Kerr–NUT metric
\[
ds_5^2 = -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} (dt - \omega d\phi)^2 \\
+ \Sigma \left( \frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta \sin^2 \theta}{\Delta - a^2 \sin^2 \theta} d\phi^2 \right), \quad (70)
\]
where
\[
\Delta := r^2 - 2mr + a^2 - n^2, \\
\Sigma := r^2 + (a \cos \theta + n)^2, \quad (71)
\]
\[
\omega := \frac{2n \Delta \cos \theta + 2a \sin^2 \theta (mr + n^2)}{a^2 \sin^2 \theta - \Delta}. \quad (72)
\]
This family contains three free parameters \( m, a, n \) (mass, rotation and NUT) and the corresponding non-zero seed potentials read
\[
f_0 = \frac{r^2 - 2mr + a^2 \cos^2 \theta - n^2}{r^2 + (a \cos \theta + n)^2}, \\
\chi_0 = \frac{2m(a \cos \theta + n) - 2nr}{r^2 + (a \cos \theta + n)^2}. \quad (74)
\]

Applying the second type electric Harrison chain one obtains
\[
ds_5^2 = H^{-\frac{1}{2}} \left[ dy^2 - \frac{\Delta - a^2 \sin^2 \theta}{\Sigma} (dt - \cosh \delta \omega d\phi)^2 \right] \\
+ H^{\frac{1}{2}} \Sigma \left( \frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta \sin^2 \theta}{\Delta - a^2 \sin^2 \theta} d\phi^2 \right), \\
\hat{\phi} = -\sqrt{\frac{2}{3}} \ln H, \\
\hat{B}_{ty} = \sinh 2\delta (mr + an \cos \theta + n^2) (\Sigma H)^{-1}, \\
\hat{B}_{\phi y} = -2\sinh \delta [n \Delta \cos \theta + a \sin^2 \theta (mr + n^2)] (\Sigma H)^{-1}, \quad (75)
\]
where \( \delta \) is a three-form charge parameter, and
\[
H := 1 + 2 \sinh^2 \delta (mr + an \cos \theta + n^2) \Sigma^{-1}. \quad (76)
\]
This is a NUT generalization of the black rotating string found previously in \([23]\).

Similarly, applying a magnetic second type Harrison transformation one obtains a four-parametric family of rotating magnetic zero-branes:
\[
ds_5^2 = -H^{-\frac{1}{2}} \frac{\Delta - a^2 \sin^2 \theta}{\Sigma} (dt - \cosh \delta \omega d\phi)^2 \\
+ H^{\frac{1}{2}} \left[ dy^2 + \Sigma \left( \frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta \sin^2 \theta}{\Delta - a^2 \sin^2 \theta} d\phi^2 \right) \right], \\
\hat{\phi} = -\sqrt{\frac{2}{3}} \ln H, \\
\hat{B}_{ty} = \sinh 2\delta (mr + an \cos \theta + n^2) (\Sigma H)^{-1}, \\
\hat{B}_{\phi y} = -2\sinh \delta [n \Delta \cos \theta + a \sin^2 \theta (mr + n^2)] (\Sigma H)^{-1}, \quad (77)
\]
where \( \delta \) is a three-form charge parameter, and
\[
H := 1 + 2 \sinh^2 \delta (mr + an \cos \theta + n^2) \Sigma^{-1}. \quad (78)
\]
\[ \hat{\phi} = \sqrt{\frac{2}{3}} \ln H, \]
\[ \dot{B}_{ty} = 2 \sinh \delta [m(a \cos \theta + n) - n r] \Sigma^{-1}, \]
\[ \dot{B}_{\phi y} = - \sinh 2 \delta \left\{ m \cos \theta + (2 \cos \theta - a \sin^2 \theta) [n r - m (a \cos \theta + n)] \Sigma^{-1} \right\}. \quad (77) \]

Application of the first type Harrison transformations gives the NUT generalization of the rotating solution found in [17] (which is the rotating version of the Doblasch-Maison solution):
\[ ds_5^2 = H [(dy + A_t dt + A_\phi d\phi)^2 - H^{-1} \frac{\Delta - a^2 \sin^2 \theta}{\Sigma} (dt - \cosh \delta \omega d\phi)^2 + \Sigma \left( \frac{d\theta^2}{\Delta} + d\theta^2 + \frac{\Delta \sin^2 \theta}{\Delta - a^2 \sin^2 \theta} d\phi^2 \right), \]
\[ A_t = \sinh 2 \delta (n r + a n \cos \theta + n^2) (\Sigma H)^{-1}, \]
\[ A_\phi = - \sinh 2 \delta n (a \cos \theta + n) (\Sigma H)^{-1}, \]
and the rotating generalization (with NUT) of the Gross-Perry-Sorkin monopole:
\[ ds_5^2 = H^{-1} [(dy + A_t dt + A_\phi d\phi)^2 - \frac{\Delta - a^2 \sin^2 \theta}{\Sigma} (dt - \cosh \delta \omega d\phi)^2 + H \Sigma \left( \frac{d\theta^2}{\Delta} + d\theta^2 + \frac{\Delta \sin^2 \theta}{\Delta - a^2 \sin^2 \theta} d\phi^2 \right), \]
\[ A_t = 2 \sinh \delta [m(a \cos \theta + n) - n r] \Sigma^{-1}, \]
\[ A_\phi = - \sinh 2 \delta \left\{ m \cos \theta + (2 \cos \theta - a \sin^2 \theta) [n r - m (a \cos \theta + n)] \Sigma^{-1} \right\}. \quad (79) \]

V. (0 - 1) BRANES

Combining together electric and magnetic Harrison transformations one can obtain solutions with larger number of free parameters. From these we give explicitly a relatively simple composite string - magnetic zero-brane solution. It can be derived applying a sequence of the second type electric and magnetic Harrison transformations to the seed Schwarzschild metric. The resulting solution is
\[ ds_5^2 = \left( 1 - \frac{2q}{r} \right)^{-\frac{1}{2}} \left( 1 + \frac{2p}{r} \right)^{-\frac{1}{2}} (dt + 2 \sqrt{pq} \cos \theta d\phi)^2 + \left( 1 - \frac{2q}{r} \right)^{-\frac{1}{2}} \left( 1 + \frac{2p}{r} \right)^{-\frac{1}{2}} dy^2 + \left( 1 - \frac{2q}{r} \right)^{\frac{1}{2}} \left( 1 + \frac{2p}{r} \right)^{\frac{1}{2}} (dr^2 + r^2 d\Omega_2), \]
\[ \dot{\phi} = \sqrt{\frac{2}{3}} \left[ \ln \left( 1 + \frac{2p}{r} \right) - \ln \left( 1 - \frac{2q}{r} \right) \right], \]
\[ B_{ty} = \sqrt{\frac{q}{p + q}} \left[ 1 - \left( 1 + \frac{2p}{r} \right) \left( 1 - \frac{2q}{r} \right)^{-1} \right], \]
\[ B_{\phi y} = - 2 \sqrt{p(p + q)} \left( 1 - \frac{2q}{r} \right)^{-1} \cos \theta. \quad (80) \]

When \( p = 0 \) the solution reduces to the electric string [54], while for \( q = 0 \) one recovers the magnetic zero-brane [54]. Note that, when both charges are non-zero, one has also a NUT parameter, so the solution is not strictly asymptotically flat. One can compensate the NUT charge via the Ehlers transformation thus making it an additional free parameter. We intend to discuss this and more general solutions in a separate publication.

VI. CONCLUDING REMARKS

We have shown that large classes of solutions to 5D dilaton-axion gravity admitting two commuting isometries with non-compact orbits can be obtained in a unique way using the \( SL(4, R) \) duality from the vacuum four-dimensional metrics such as Schwarzschild or Kerr-NUT. This class does not include spherically symmetric or rotating five-dimensional black holes which, although possess two commuting Killing symmetries too, have one of them of the rotational type. These solutions are still within the scope of the present \( \sigma \)-model, but the group transformations generically relate them to asymptotically non-flat metrics (e.g. corresponding to black holes in the field of a fluxbrane [3]) and we did not consider them here. The \( SL(4, R) \) duality also connects solutions of the 5D dilaton-axion gravity with solutions of the four-dimensional EMDA system, which constitute a subspace \( (v_1 = v_2, u_1 = u_2, \psi = 0) \) of the full target space and may be used as seeds to generate five-dimensional solutions.

An essentially new solution we presented here is the composite \((0 - 1)\)-brane which is a superposition of an electric string and a 0-brane supported by the magnetic sector of the three-form field. This is an example of a dyonic composite solution in the case when electric and magnetic branes have different world-volume dimensions. The situation is similar to that in the eleven-dimensional supergravity where one has electric two-branes and magnetic five-branes. In the latter case composite electromagnetic solutions are realized either as intersecting two- and five-branes, or as composite \( 2 \subset 5 \) branes [21]. Note that the five-dimensional lagrangian does not contain the Chern-Simons term which is essential for existence of the \( 2 \subset 5 \) brane in the eleven-dimensional supergravity.

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