WKB-expansion of the Harish-Chandra-Itzykson-Zuber integral for arbitrary $\beta$

S. Hikami$^a$ and E. Brézin$^b$

$^a$) Department of Basic Sciences, University of Tokyo, Meguro-ku, Komaba, Tokyo 153, Japan. e-mail:hikami@rishon.c.u-tokyo.ac.jp
$^b$) Laboratoire de Physique Théorique, Ecole Normale Supérieure 24 rue Lhomond 75231, Paris Cedex 05, France. e-mail: brezin@lpt.ens.fr

Abstract

This article is devoted to the asymptotic expansion of the generalized Harish-Chandra-Itzykson-Zuber matrix integral for non-unitary symmetries characterized by a parameter $\beta$ (as usual $\beta = 1, 2$ and $4$ correspond to the orthogonal, unitary and symplectic group integrals).

The results are of the form $\sum_{\text{perm.}} \exp(\sum x_i \lambda_i) f(\tau_{ij})/\prod_{i<j} \tau_{ij}^{\beta-1}$, in which $x_i$ and $\lambda_i$ are the eigenvalues of the two $k \times k$ matrices, and $\tau_{ij} = (x_i - x_j)(\lambda_i - \lambda_j)$. A WKB-expansion for $f$ is derived from the heat kernel differential equation, for general values of $k$ and $\beta$. From an expansion in terms of zonal polynomials, one obtains an expansion in powers of the $\tau$’s for $\beta = 1$, and generalizations are considered for general $\beta$. A duality relation, and a transformation of products of pairs of symmetric functions into $\tau$ polynomials, is used to obtain the expression for $f(\tau_{ij})$ for general $\beta$.

$^1$Unité Mixte de Recherche 8549 du Centre National de la Recherche Scientifique et de l’École Normale Supérieure.
1 Introduction

We study the HIZ-integral $I$

$$I(X, \Lambda) = \int \exp(\text{tr } g X g^{-1} \Lambda) dg$$  \hspace{1cm} (1)

where $g$ is one of the Lie groups $O(k)$, $U(k)$ or $Sp(k)$, and $dg$ is the corresponding Haar measure.

The $k \times k$ matrices $X$ and $\Lambda$ are Hermitian (real, complex and quaternion). This integral appears at several key points in random matrix theory. When $g$ varies over the unitary group, the result is well-known: due to Harish-Chandra [11], it has been rederived in the context of random matrix applications by Itzykson and Zuber [12]. The HIZ result turns out to be "WKB exact" in the unitary case, i.e. to be equal to the sum of Gaussian integrals over the $k!$ saddle-points of the integrand.

For non-unitary cases, this semi-classical property is no longer true. However the full WKB expansion is needed in several problems arising in random matrix theory, for instance in the study of universal short-distance correlations of the eigenvalues. There are several studies of those non-unitary cases. For instance, in the orthogonal case the result is known as an expansion in terms of zonal polynomials $Z_p(x_1, \ldots, x_k)$ [15], which are symmetric polynomials in the $x_i$. However, the expressions in terms of zonal polynomials do not provide the desired WKB expansion around the saddle-points, which involves combinations of the form $\prod_{i<j} (x_i - x_j)$. Furthermore there is no explicit expression for zonal polynomials of arbitrary order.

For general $\beta$, zonal polynomials are generalized, and known under the name of Jack polynomials [13] but the situation is not improved. In this article we extend the HIZ result to non-unitary groups and obtain expressions of the form

$$e^{\sum x_i \lambda_i} \chi(\tau_{ij}) / \prod \tau_{ij}^{\beta/2}$$

or

$$e^{\sum x_i \lambda_i} f(\tau_{ij}) / \prod \tau_{ij}^{\beta-1},$$

where $\tau_{ij} = (x_i - x_j)(\lambda_i - \lambda_j)$, with the parameter $\beta$ corresponding to the $O(k), U(k)$ and $Sp(k)$ group integrals for $\beta = 1, 2$ and 4, respectively. Although the group integrals correspond to those three values, the expressions may be continued to arbitrary $\beta$ through the differential equation for $I(X, \Lambda)$.

The fact that, once the exponential prefactor $e^{\sum x_i \lambda_i}$ is extracted, the result may be expressed in terms of the variables $\tau$’s is far from trivial. Although it follows from this paper, through a first representation in terms of extended zonal polynomials, and further identities on products of symmetric functions in the variables $x_i$ and $\lambda_j$, an a priori proof would be welcome.
Recently, a similar expansion, using the formalism of Baker-Akhiezer functions, has been considered for \( I(X, \Lambda) \) by Berest [2].

However this is still not in terms of the \( \tau_{ij} \), and it involves differential operators which make the calculation of \( I(X, \Lambda) \) difficult, except in simple cases.

In the case of the symplectic group \((\beta = 4)\), we have found earlier that the WKB expansion for \( I(X, \Lambda) \) terminates after a finite number of terms[4, 5]. We have obtained these explicit finite expansions for \( k=2,3 \) and 4. In this article we study the coefficients of this expansion for general \( k \); however the structure for arbitrary \( k \) is still only partially known. Again a proof of the fact that the WKB expansion terminates after a finite number of terms would be welcome. Recently the problem has been considered by Ben Said and Ørsted [1] but they have simply verified like ourselves this amazing property.

For other even integer values of \( \beta \), \((\beta = 6, 8, 10, \ldots)\), similarly the WKB series stops after a finite number of terms. In this article we discuss also the coefficients of these expansions, but again proofs of the finiteness are lacking.

The expressions that we have obtained are of the form

\[
I_\beta(x, \lambda) = \sum_{h} \frac{1}{\prod_{i<j}(x_i - x_j)(\lambda_i - \lambda_j)} e^{\sum_{i=1}^{k} x_i \lambda_i} f(\tau_{ij})
\]

where \( f \) is given as a series in terms of the \( \tau_{ij} = (x_i - x_j)(\lambda_i - \lambda_j) \),

\[
f(\tau_{ij}) = 1 + c_1 \sum_{i<j} \tau_{ij} + O(\tau^2).
\]

and the successive coefficients \( c_1, c_2, \ldots \) are functions of \( \beta \) and \( k \). The "perm." in (2) means a sum over the \( k! \) permutations of the \( \lambda_i \)'s.

In the case \( k = 3, \beta = 4 \), we have

\[
f(\tau_{ij}) = 1 - \frac{1}{3}(\tau_{12} + \tau_{23} + \tau_{13}) + \frac{1}{6}(\tau_{12}\tau_{23} + \tau_{23}\tau_{13} + \tau_{13}\tau_{12})
\]

\[
- \frac{1}{12}\tau_{12}\tau_{23}\tau_{13}.
\]

In the case \( k = 4, \beta = 4 \), the result is again a similar finite polynomial which is given in [4]. Our aim is to find similar compact expressions for general values of \( k \) and \( \beta \).

Let us note that in the cases \( \beta = 2m(m = 2, 3, \ldots) \), since the function \( f(\tau_{ij}) \) is a polynomial, it means that the HIZ integrals may be written as

\[
I_\beta = \sum_{perm.} e^{\sum_{i<j} \lambda_i} \frac{1}{[\Delta(x)\Delta(\Lambda)]^2}[1 + a_1 \sum_{i<j} \frac{1}{\tau_{ij}} + O\left(\frac{1}{\tau^2}\right)]
\]

which is a WKB expansion corrected by a finite number of terms.
The integral $I$ is also known to be expressible as an infinite sum over extended zonal polynomials $Z_p(x)$, with a parameter $\alpha = \frac{2}{\beta}$ [15, 13, 16]

$$I_\beta = \sum_{m=0}^{\infty} \frac{1}{m!} \prod_{q=0}^{m-1} \frac{1}{1 + q\alpha} \sum_p \chi_p(1) \frac{Z_p(X)Z_p(\Lambda)}{Z_p(I)}$$

(6)

where $\chi_p(1)$ is a character, and $Z_p(I)$ is a dimensional constant which depends upon $k$. (Explicit values of $Z_p(x)$, $\chi_p(1)$, $Z_p(I)$ are given in Appendix A for the lower orders).

To extract the WKB-exponential $e^{\sum x_i \lambda_i}$ factor in (2), we shift $X \to \tilde{X} = X - \frac{1}{k} \text{tr}X$. Then, the above expression becomes

$$I_\beta = e^{\sum x_i \lambda_i - \frac{1}{k} \sum_i \tau_{ij}} \sum_{m=0}^{\infty} \frac{1}{m!} \prod_{q=0}^{m-1} \frac{1}{1 + q\alpha} \sum_p \chi_p(1) \frac{Z_p(\tilde{X})Z_p(\tilde{\Lambda})}{Z_p(I)}$$

(7)

In the case $\beta = 1$, the power of the Vandermonde factor in the denominator of (2) vanishes. Therefore one may compare directly the expression of (7) with (2) ; consequently the expression for $f(\tau_{ij})$ is obtained by rewriting the products of symmetric functions in terms of polynomials in the $\tau$'s.

In the other cases, $\beta \neq 1$, one needs to extract the Vandermonde factor. For instance if $\beta$ is an even integer, this factor is antisymmetric under permutations of the $x_i$ and it makes the problem difficult.

In this article a dual representation is derived, which solves the problem of the Vandermonde factor. The duality comes from another expression of the integral $I_\beta$ in terms of extended zonal polynomials. Namely, we can write the series (7) as

$$I_\beta = e^{\sum x_i \lambda_i - \frac{1}{k} \sum_i \tau_{ij}} \sum_{m=0}^{\infty} \frac{1}{m!} \prod_{q=0}^{m-1} \frac{1}{1 + q\alpha} \sum_p \chi_p(1) \frac{Z_p(\tilde{X})Z_p(\tilde{\Lambda})}{Z_p(I)}$$

(8)

where the parameter $\alpha$ is now $\frac{2}{2-\beta}$ instead of $\frac{2}{\beta}$. This is a duality transformation of $\beta \to 2 - \beta$. Note that it is the dual invariant product $\beta(\frac{2}{\beta} - 1)$, which appears as a coefficient of the inverse square potential in the Calogero-Moser model. The expression (8) is close to the desired expression (2). The WKB-exponential term $e^{\sum x_i \lambda_i}$ and the correct power of the Vandermonde term are already present in (8). The two expressions (7) and (8), are dual under the transformation $\beta \to 2 - \beta$. Then if one writes the subsequent series in terms of the variables $\tau$, one obtains the desired WKB expansion.

This article is organized as follows:
In section two, we discuss the differential equation satisfied by the integral $I(X, \Lambda)$.

In section three, the series expansion in terms of $\tau_{ij} = (x_i - x_j)(\lambda_i - \lambda_j)$ is obtained by consideration of residues (residual equations).

In section four, the case $\beta = 4$ is investigated separately; the $\tau$-expansion of $f$ is discussed. (The result up to fourth order is given in Table A).

In section five, we extend the $\tau$ expansion to arbitrary $\beta$. The residual equations are also used at this effect. (The coefficients $C_{[\text{graph}]}$ are given in Table B). We find that the residual equations have no $\beta$ dependence.

In section six, we derive the $\tau$ expansion directly for $\beta = 1$ from the zonal polynomial expansion of the integral $I$, based on the expression (7). (These results are expressed in Table C).

In section seven, we extend the result of the section six to general $\beta$, using the extended zonal polynomials (Jack polynomial). We find a duality relation of the type (8). By the duality trasformation $\alpha = \frac{2}{\beta} \rightarrow \frac{2}{2 - \beta}$, the expressions for $C_{[\text{graph}]}$ given in section five (Table B) are rederived. The residual equation of section five, is shown to be also valid for the coefficients of $C_{[\text{graph}]}$, which is determined from the extended zonal polynomials.

In section eight, we discuss the case $\beta = 4$ by taking $\alpha = -1$ in (8). In the denominator of (8), one has to deal with the divergent factor $\frac{1}{1+\alpha}$. This divergence is cancelled by the sum over the partition of the zonal polynomials. We explicitly evaluate (8) up to order six in the limit $\alpha \rightarrow -1$. We find the large $k$ behavior of this expansion.

In section nine, the case $\beta = 2$, where the duality transformation becomes singular, is discussed.

In section ten, the large $k$ limit and large $\beta$ limits are discussed briefly.

Finally, a summary and discussions are given. The application of this $\tau$ expansion to the random matrix theory is briefly mentioned [10].

In Appendix A, the extended zonal polynomials in terms of the classical symmetric functions $s_n$, a sum of powers, are given up to order six for general $\alpha$. The characters $\chi_p(1)$ and the dimensional constants $Z_p(I)$ are explicitly given.

In Appendix B, the expression of paired products of classical symmetric functions, $s_n(\vec{x})s_n(\vec{\lambda})$, as polynomials in $\tau$, is investigated with the help of differential operators $D_{j_1,j_2;\ldots}$, which act on $\tau$ series. This technique is used to prove various identities which appear in the $\tau$ transformations.

In Appendix C, we discuss some cubic and quartic identities for the variables $\tau_{ij}$. In view of these identities, the $\tau$ expansion has some freedom which is used to fix some of the coefficients in the expansion arbitrarily.

In Appendix D, we characterize each $\tau$ terms by the unique factors $x_{i_1}^{p_1}x_{i_2}^{p_2}\cdots\lambda_{j_1}^{q_1}\lambda_{j_2}^{q_2}\cdots$, which are picked up by the differentials $D_{j_1,j_2;\ldots}$. Using these unique characteristic differentials, we obtain the coefficients of $\tau$ ex-
pansion for $f$.

## 2 Differential equations for the HIZ-integrals

The Laplacian operator with respect to the matrix elements

$$L = \sum_{i,j} \frac{\partial^2}{\partial X_{ij}^2}.$$  \hfill (9)

(a short-hand notation for the appropriate group invariant Laplacian) acts on the integrand of $I(x,\lambda)$ in (1) by

$$L e^{\text{tr}AX} = (\text{tr}A^2) e^{\text{tr}AX}.$$  \hfill (10)

Given that $I(x,\lambda)$ is a function of the eigenvalues $x_i$ of the matrix $X$, the equation (10) reads [3],

$$\sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2} + \beta \sum_{i=1,(i\neq j)}^{k} \frac{1}{x_i - x_j} \frac{\partial}{\partial x_i} I = \epsilon I ,$$  \hfill (11)

with the eigenvalue $\epsilon$

$$\epsilon = \sum_{i=1}^{k} \lambda_i^2.$$  \hfill (12)

Note that the integral $I$ is manifestly symmetric under interchange of the matrices $\Lambda$ and $X$, but the procedure is dissymetric. The solutions will of course restore this property, which is far from obvious if one considers the equation (11) alone. The $x$-dependent eigenfunctions of this Schrödinger-like operator have a scalar product given by the measure

$$\langle \varphi_1 | \varphi_2 \rangle = \int dx_1 \cdots dx_k |\Delta(x_1 \cdots x_k)|^{\beta/2} \varphi_1^*(x_1 \cdots x_k) \varphi_2(x_1 \cdots x_k)$$  \hfill (13)

The measure becomes trivial if one multiplies the wave function by $|\Delta|^{\beta/2}$. Thus if, for some given ordering of the $x_i$'s, one changes $I(x)$ to

$$\psi(x_1 \cdots x_k) = |\Delta(x_1 \cdots x_k)|^{\beta/2} I(x_1 \cdots x_k),$$  \hfill (14)

one obtains the quantum Hamiltonian,

$$\left[\sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2} - \beta(\frac{\beta}{2} - 1) \sum_{i<j} \frac{1}{(x_i - x_j)^2}\right] \psi = \epsilon \psi.$$  \hfill (15)

This Schrödinger equation is a simple Calogero-Moser model.
For a given ordering of the $x_i$'s, pulling out of (15) the exponential factor which is the value of the integrand at its saddle-point, one obtains

$$\psi_0 = e^{\lambda_1 x_1 + \cdots + \lambda_k x_k}$$

(16)

where $\chi$ satisfies

$$\left[ \sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2} + 2 \sum_{i=1}^{k} \lambda_i \frac{\partial}{\partial x_i} - y \sum_{i<j} \frac{1}{(x_i - x_j)^2} \right] \chi = 0$$

(17)

in which $y$ stands for

$$y = \beta \left( \frac{\beta}{2} - 1 \right).$$

(18)

In [5] an expansion of $\chi$ for large $\tau$'s

$$\chi = 1 - \frac{y}{2} \left( \sum_{i<j} \tau_{ij} \right) + \frac{y}{2} \left( -1 + \frac{y}{4} \left( \sum_{i<j} \tau_{ij}^2 \right) \right) + \cdots$$

(19)

has been introduced.

For even integer values of $\beta$, this expansion terminates after a finite number of terms. The term of highest degree is $\prod_{i<j}^{k} \tau_{ij}^{-\beta/2+1}$. Therefore, one may write the whole expression as

$$\chi = \frac{f_\beta}{\prod_{i<j}^{k} \tau_{ij}^{\beta/2-1}}$$

(20)

From the definitions of $\psi$ and $\chi$ in (14,16), the integral $I$ is then expressed as

$$I(\chi, \lambda) = \sum_{\text{perm. of } \lambda_i} \frac{1}{\prod_{i<j}^{k} \tau_{ij}^{\beta-1}} e^{\sum_{i=1}^{k} x_i \lambda_i} f_\beta$$

(21)

in which we have normalized $f_\beta$ such that $f_\beta = 1 + O(\tau_{ij})$.

Continuing from even integer $\beta$'s to arbitrary real numbers, we consider below the expression of $f_\beta$ for arbitrary $\beta$ and arbitrary $k$. 

3 Expansion in powers of $\tau$

We now perform an expansion in powers of the variables $\tau$. Let us begin with the simple case $\beta = 4$, and $k = 3$. The differential equation for $\psi$, defined by (15), is

$$\hat{P} \psi = 0$$

(22)
where the operator \( \hat{P} \) is

\[
\hat{P} = \sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2} - \sum_{i<j} \frac{4}{(x_i - x_j)^2} - \sum_{i=1}^{k} \lambda_i^2. \quad (23)
\]

The expression of \( \psi \) in terms of \( \tau_{ij} \) consists, in this simple \( k=3 \) case, of the sum of four terms of increasing degrees:

\[
\psi = \psi_0 + \psi_1 + \psi_2 + \psi_3 \quad (24)
\]

\[
\psi_0 = \frac{1}{\Delta(x)\Delta(\lambda)} e^{\sum x_i \lambda_i} \quad (25)
\]

\[
\psi_1 = C_1 e^{\sum x_i \lambda_i} \frac{1}{\Delta(x)\Delta(\lambda)} [\tau_{12} + \tau_{23} + \tau_{13}] \quad (26)
\]

\[
\psi_2 = C_2 e^{\sum x_i \lambda_i} \frac{1}{\Delta(x)\Delta(\lambda)} [\tau_{12}\tau_{23} + \tau_{23}\tau_{31} + \tau_{31}\tau_{12}] \quad (27)
\]

\[
\psi_3 = C_3 e^{\sum x_i \lambda_i} \frac{1}{\Delta(x)\Delta(\lambda)} [\tau_{12}\tau_{23}\tau_{13}] \quad (28)
\]

where \( \Delta(x) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \) and \( \tau_{12} = (x_1 - x_2)(\lambda_1 - \lambda_2) \).

The unknown coefficients \( C_n \) may be obtained from the differential equation.

Applying \( \hat{P} \) on \( \psi_0 \) and \( \psi_1 \), we have

\[
\hat{P}\psi_0 = -2 \frac{1}{\Delta(x)\Delta(\lambda)} e^{\sum x_i \lambda_i} \left[ \frac{\lambda_1 - \lambda_2}{x_1 - x_2} + \frac{\lambda_1 - \lambda_3}{x_1 - x_3} + \frac{\lambda_2 - \lambda_3}{x_2 - x_3} \right], \quad (29)
\]

\[
\hat{P}\psi_1 = C_1 e^{\sum x_i \lambda_i} \left[ \frac{2}{(x_1 - x_2)(x_2 - x_3)\tau_{13}\tau_{23}} - \frac{2}{(x_1 - x_2)(x_2 - x_3)\tau_{12}\tau_{23}} - \frac{2}{(x_1 - x_2)^2\tau_{13}\tau_{23}} - \frac{2}{(x_1 - x_2)^2\tau_{12}\tau_{23}} - \frac{2}{(x_2 - x_3)^2\tau_{13}\tau_{12}} - \frac{2}{(x_1 - x_2)^2\tau_{13}\tau_{12}} - \frac{2}{(x_2 - x_3)^2\tau_{13}\tau_{12}} - \frac{2}{(x_1 - x_3)^2\tau_{13}\tau_{12}} \right], \quad (30)
\]

where the first three terms are generated by \( \frac{d^2}{dx_i^2} \), and the next three ones from the factor \(-4 \sum \frac{1}{(x_i - x_j)^2}\) in \( \hat{P} \). The last six terms are obtained by
the differentiations both of \( \exp[\sum \lambda_i x_i] \) and \( \frac{1}{\tau_{ij}} \). Note that \( \hat{P} \psi_1 \) consists of terms of order \( O(1/x^4) \) and \( O(1/x^3) \).

The unknown coefficients \( C_n \) are determined by the condition \( \hat{P} \psi = 0 \). Since \( x_i \) and \( \lambda_i \) are arbitrary variables, we have the freedom of taking any particular limit; for instance \( x_1 \to x_2 \). From the condition \( \hat{P} \psi = 0 \), the pole term of \( 1/(x_1 - x_2) \) present in \( \hat{P} \psi_0 \) should be cancelled by the term coming from \( \hat{P} \psi_1 \). This requirement is sufficient to fix the coefficient \( C_1 \). Indeed in \( \hat{P} \psi_0 \), the residue of this pole is proportional to \( \lambda_1 \), \( \hat{P} \psi_0 \sim -2 \frac{\lambda_1}{(x_1 - x_2)\Delta(x)\Delta(\lambda)} \). From the second and the fourth terms in \( \hat{P} \psi_1 \), we have \(-6C_1 \frac{\lambda_1}{(x_1 - x_2)\Delta(x)\Delta(\lambda)} \) term. Therefore, the required cancellation fixes the coefficient \( C_1 \) to be \(-1/3 \).

The term of \(-2 \frac{C_1}{(x_1 - x_2)^2 \tau_{23}} \) in \( \hat{P} \psi_1 \) is cancelled by the term in \( \hat{P} \psi_2 \). Indeed we have

\[
\hat{P} \psi_2 = C_2 e^{\sum x_i \lambda_i} \left[ -\frac{4}{\tau_{12}} \left( \frac{1}{(x_1 - x_2)^2} + \frac{1}{(x_2 - x_3)^2} \right) \right. \\
- \left. \frac{4}{\tau_{23}} \left( \frac{1}{(x_1 - x_3)^2} + \frac{1}{(x_2 - x_3)^2} \right) - \frac{4}{\tau_{13}} \left( \frac{1}{(x_1 - x_2)^2} + \frac{1}{(x_1 - x_3)^2} \right) \right. \\
- \left. \frac{2}{(x_1 - x_2)^2} - \frac{2}{(x_1 - x_3)^2} - \frac{2}{(x_2 - x_3)^2} \right] \\
\tag{31}
\]

The pole term \(-2 \frac{C_1}{(x_1 - x_2)^2 \tau_{23}} \) should be cancelled, and thus

\[
2C_1 + 4C_2 = 0. \\
\tag{32}
\]

Thus one obtains \( C_2 = 1/6 \). The last term \( \psi_3 \) is

\[
\psi_3 = C_3 e^{\sum x_i \lambda_i} \\
\hat{P} \psi_3 = C_3 e^{\sum x_i \lambda_i} \left( -\frac{4}{(x_1 - x_2)^2} - \frac{4}{(x_2 - x_3)^2} - \frac{4}{(x_1 - x_3)^2} \right) \\
\tag{34}
\]

The cancellation of the double pole \( 1/(x_1 - x_2)^2 \) requires

\[
2C_2 + 4C_3 = 0 \\
\tag{35}
\]

i.e. \( C_3 = -1/12 \).

We have thus determined the coefficients of the various terms in the expansion for \( \beta = 4, k = 3 \). The results obtained in this manner coincide with our earlier results [4]; this justifies the expression (4) given in the introduction.
The remarkable point is that the cancellation conditions connect linearly the term \( C_{n-1} \) and the n-th order term \( C_n \) as
\[
AC_{n-1} + BC_n = 0. \tag{36}
\]
It turns out that this structure holds for arbitrary \( k \) and \( \beta \), and it turns out that the coefficients \( A \) and \( B \) hereabove are functions of \( k \) and not of \( \beta \).

4 The \( \tau \)-expansion for the symplectic case \( \beta = 4 \).

The group parameter \( \beta \) remains equal to 4 in this section. (In the next section, we will consider general \( \beta \)). Generalizing the simple \( k = 3 \) example of the previous section, we examine the cancellation of the pole \( 1/(x_1 - x_2) \) when \( x_1 \to x_2 \) for arbitrary \( k \). Let us focus on a specific pole term in the various pieces of \( \hat{P}\psi \).

(i) In \( \hat{P}\psi_{n-1} \), the action of \( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \) on \( e^{\sum \lambda_i x_i} \) and \( 1/(x_1 - x_2) \) yields
\[
-2\frac{\lambda_1 - \lambda_2}{x_1 - x_2} \left[ e^{\sum \lambda_i x_i} \frac{1}{\Delta(x)\Delta(\lambda)} \prod \tau_{im} \right].
\]

(ii) In \( \hat{P}\psi_n \), the multiplication by \( -4 \frac{1}{(x_1 - x_2)^2} \) gives
\[
-4\frac{\lambda_1 - \lambda_2}{x_1 - x_2} \left[ e^{\sum \lambda_i x_i} \frac{1}{\Delta(x)\Delta(\lambda)} \prod \tau_{im} \right].
\]

(iii) The derivative \( d^2/dx_2^2 \) of \( \psi_n \) yields \( -2 \frac{1}{(x_1 - x_2)(x_2 - x_m)} \) (where \( m \neq 1, 2 \)). This term may be written as
\[
e^{\sum \lambda_i x_i} \frac{1}{\Delta(x)\Delta(\lambda)} \left[ -2 \frac{\tau_{1m}}{(x_1 - x_2)(x_2 - x_m)} \prod \tau_{ij} \right].
\]
Since \( \tau_{1m}/(x_2 - x_m) \) reduces to \( \lambda_1 - \lambda_m \) in the limit \( x_1 \to x_2 \), we are left with the pole term
\[
-2\frac{\lambda_1 - \lambda_m}{x_1 - x_2} e^{\sum \lambda_i x_i} \frac{1}{\Delta(x)\Delta(\lambda)} \prod \tau_{ij} \text{ in this limit.}
\]

(iv) Leaving aside the overall factor \( e^{\sum \lambda_i x_i} \frac{1}{\Delta(x)\Delta(\lambda)} \), the pole terms in (i),(ii) and (iii) must cancel. The residue of \( 1/(x_1 - x_2) \) is the combination of \( \lambda_1 \prod \tau_{ij} \) with unknown coefficients \( C \) for each monomial in \( \psi_n \).

For instance, in the case of \( k=3, n=1 \), from (i),(ii) and (iii), we would find
\[
-2C_0 \frac{\lambda_1}{x_1 - x_2} - 4C_1 \frac{\lambda_1}{x_1 - x_2} - 2C_1 \frac{\lambda_1}{x_1 - x_2} = 0 \tag{37}
\]
and the cancellation imposes \( C_0 + 3C_1 = 0 \) \((C_0 = 1)\). The pole term in \( \hat{P}\psi \) is always a single pole, once we factor out the Vandermonde factor in the denominator.
This method of selecting pole terms is simple but at increasing orders there are many terms and many coefficients. A graphical characterization of those terms is thus necessary, in order to extract the various contributions to the pole terms of \( \hat{P}_\psi n \). For the general problem of order \( k \) one has

\[
\psi = e^{\sum \lambda_i x_i} \frac{1}{\Delta(x) \Delta(\lambda)} \left[ C[0] + C[I] (\tau_{12} + \tau_{13} + \cdots) + C[\Lambda] (\tau_{12} \tau_{13} + \cdots) \right.
\]

\[
+ \left. C[I,I] (\tau_{12} \tau_{34} + \cdots) + O(\tau^3) \right] \tag{38}
\]

We represent the terms which are products of \( \tau_{ij} \) by a graph in which one draws a line between the points (i) and (j) whenever a factor \( \tau_{ij} \) is present. For a given graph, a graph-dependent coefficient characterizes its weight in the function \( f_\beta \). For instance the graph \([\text{graph}] = [I,I]\), has a weight \( C[I,I] \), which is the coefficient of the sum \( \Sigma \tau_{n_1 n_2} \tau_{n_3 n_4} \), where the \( n_i \) are all different.

The graph \([\wedge]\) consists of two lines meeting at one point, and it comes with a coefficient \( C[\wedge] \) in front of the sum \( \Sigma \tau_{n_1 n_2} \tau_{n_1 n_3} \) in which the three \( n_i \) are different.

One may now apply the rules (i)-(iii).

at first order: We consider the pole term of the form \( \lambda_1 x_1 - x_2 \). From \( \psi[0] \) the operation (i) gives \(-2C[0] = -2\). The rule (ii) gives the same pole with the coefficient \(-4C[I] \). The rule (iii), from the terms \(-2(\lambda_1 - \lambda_m) / (x_1 - x_2) \), where \( m \) is not equal to 1 or 2 we obtain a term \(-2(\lambda_1 - \lambda_2) / (x_1 - x_2) \).

Therefore the cancellation yields the relation

\[
-2C[0] - 4C[I] - 2(k-2)C[I] = 0 \tag{39}
\]

which gives \( C[I] = -\frac{1}{k} \).

at order two: There are two coefficients corresponding to the two graphs of that order, namely \( C[\Lambda] \) and \( C[I,I] \).

To determine \( C[\Lambda] \), we compare the pole term \( \lambda_1 \tau_{13} \frac{1}{x_1 - x_2} \) or equivalently

\(-\lambda_1 \lambda_3 x_1 \frac{1}{x_1 - x_2} \) term in \( \psi_1 \) and \( \psi_2 \). Applying (i) for \( \psi_1 \) yields \(-2(\lambda_1 - \lambda_2) \tau_{13} \).

Applying (ii) for \( \psi[\Lambda] \), we have \(-4(\lambda_1 - \lambda_2) \tau_{13} \). Applying (iii) for \( \psi[\Lambda] \) term, we get \(-2(\lambda_1 - \lambda_m) \tau_{13} \), where \( m \) is not equal to 1,2 or 3. Thus we find the relation

\[
-2C[I] - 4C[\Lambda] - 2(k-3)C[\Lambda] = 0 \tag{40}
\]

which leads to \( C[\Lambda] = \frac{1}{k(k-1)} \).

To determine \( C[I,I] \), we collect the pole terms of the form \(-2\lambda_1 - \lambda_2 \frac{1}{x_1 - x_2} \).

In \( \tau_{34} = (\lambda_3 - \lambda_4) (x_3 - x_4) \), we look at the term \( \lambda_3 x_4 \), and thus select the pole term \( \lambda_1 \lambda_3 x_4 / (x_1 - x_2) \). From (i), one obtains a coefficient \( 2C[I] \) for this...
pole (fig.2 (i)). Since we are looking at terms containing $\lambda_1\lambda_3x_4$, they arise from $\psi_{[\Lambda]}$ as $-2(\lambda_1 - \lambda_4)\tau_{34}$ with the coefficient $C_{[\Lambda]}$ (fig.2(iii)). Finally from $\psi_{[I,I]}$, we have three type of contributions; the first one is Fig.2(ii), it gives $-4(\lambda_1 - \lambda_2)\tau_{34}C_{[I,I]}$, the second one is $2(\lambda_2 - \lambda_3)\tau_{14}C_{[I,I]}$ (fig.2 (iv)), and the last one is $-2(\lambda_1 - \lambda_m)\tau_{34}C_{[I,I]}$ (Fig.2 (v)). In this term $m$ should be larger than 4, and a factor $(k - 4)$ arises when one sums over all possible values of $m$.

From these terms (fig.2 (i)-(v)), we obtain the pole terms $\lambda_1\lambda_3x_4/(x_1-x_2)$. Thus adding these terms, one finds the relation

$$2C_{[I]} + (4 + 2 + 2(k - 4))C_{[I,I]} + 2C_{[\Lambda]} = 0$$

From this equation, we find

$$C_{[I,I]} = \frac{k - 2}{k(k - 1)^2}$$

At third order:

At order three, there are five different graphs contributing to $\psi_3$.

$$\psi_3 = \psi_{[Y]} + \psi_{[N]} + \psi_{[\Delta]} + \psi_{[\Lambda,I]} + \psi_{[I,I,I]}$$

Fig.1, three contributions of pole terms from the graph $C_{[\Lambda]}$.
(i) $-2(\lambda_1 - \lambda_2)\tau_{13}$, (ii) $-4(\lambda_1 - \lambda_2)\tau_{13}$, (iii) $-2(\lambda_1 - \lambda_m)\tau_{13}$.

Fig.2, five contributions of pole terms from the graph $C_{[\Lambda]}$ and $C_{[I,I]}$.
(i) $-2(\lambda_1 - \lambda_2)\tau_{34}C_{[I,I]}$, (ii) $-4(\lambda_1 - \lambda_4)\tau_{34}C_{[\Lambda]}$, (iii) $-2(\lambda_1 - \lambda_4)\tau_{34}C_{[\Lambda]}$, (iv) $-2(\lambda_2 - \lambda_3)\tau_{14}C_{[I,I]}$, (v) $-2(\lambda_1 - \lambda_m)\tau_{34}$. 
The method for obtaining the coefficients $C_{[Y]}, C_{[N]}, C_{[\Delta]}, C_{[\Lambda, I]}, C_{[I, I, I]}$ is the same as for the second order calculation. For instance, to obtain $C_{[Y]}$, we look at the pole $\lambda_1 \lambda_3 \lambda_4 x_1^2/(x_1 - x_2)$. This singularity is obtained from the three graphs of fig. 3. Adding these three terms, one finds

$$-2C_{[\Lambda]} - 4C_{[Y]} - 2(k - 4)C_{[Y]} = 0 \quad (44)$$

which leads to

$$C_{[Y]} = -\frac{1}{k - 2} C_{[\Lambda]} = -\frac{1}{k(k - 1)(k - 2)} \quad (45)$$

![Fig. 3](image)

(i) (ii) (iii)

Fig. 3, graphs for the determination of $C_{[Y]}$.

For $C_{[N]}$, we concentrate over the combination $\lambda_1 \lambda_3^2 x_1 x_4$, contained in the term $(\lambda_1 - \lambda_2)\tau_{31}\tau_{34}$ for instance. The relevant graphs are

![Fig. 4](image)

(i) (ii) (iii) (iv)

Fig. 4, graphs for the determination of $C_{[N]}$.

These four graphs give $-2(\lambda_1 - \lambda_2)\tau_{31}\tau_{34}, -4(\lambda_1 - \lambda_2)\tau_{13}\tau_{34}, -2(\lambda_1 - \lambda_m)\tau_{13}\tau_{34}(m > 4)$, and $2(\lambda_2 - \lambda_3)\tau_{31}\tau_{14}$, respectively. We extract the pole term $\lambda_1 \lambda_3^2 x_1 x_4/(x_1 - x_2)$ from these contributions, and the cancellation yields

$$-2C_{[\Lambda]} - 4C_{[N]} - 2(k - 4)C_{[N]} - 2C_{[N]} = 0 \quad (46)$$

or

$$C_{[N]} = -\frac{1}{k - 1} C_{[\Lambda]} = -\frac{1}{k(k - 1)^2} \quad (47)$$

For $C_{[\Delta]}$, we consider the cancellation of the pole terms with the residue $\lambda_1 \lambda_3^2 x_1 x_2$. The cancellation gives

$$2C_{[\Lambda]} + 4C_{[\Delta]} + 2(k - 3)C_{[N]} = 0 \quad (48)$$
which determines $C_{[\Delta]}$

$$C_{[\Delta]} = -\frac{1}{k(k - 1)^2}$$ \hspace{1cm} (49)

For $C_{[\Lambda, I]}$, the second order term gives $-2(\lambda_1 - \lambda_2)\tau_{34}\tau_{35}$ as residue. Therefore, we look at $\lambda_1\lambda_3^2x_4x_5$ for residue. There are six different graphs which contribute to this residue.

Adding these terms, the cancellation gives

$$-2C_{[\Lambda]} - 2C - 2(k - 5)C_{[\Lambda, I]} - 4C_{[\Lambda, I]} + 4C_{[N]} - 4C_{[N]} - 2C_{[Y]} = 0 \hspace{1cm} (50)$$

which determines $C_{[\Lambda, I]}$

$$C_{[\Lambda, I]} = -\frac{k - 3}{k(k - 1)^2(k - 2)}$$ \hspace{1cm} (51)

The last coefficient $C_{[I, I, I]}$ needs 7 graphs when we consider the cancellation of the residue $\lambda_1\lambda_3\lambda_4x_5x_6$. The cancellation reads

$$-4C_{[I]} - 4(k - 2)C_{[I, I, I]} - C_{[\Lambda, I]} = 0 \hspace{1cm} (52)$$

which determines $C_{[I, I, I]}$

$$C_{[I, I, I]} = \frac{1}{k - 2}(-C_{[I, I]} - 2C_{[\Lambda, I]})$$

$$= \frac{k^2 - 6k + 10}{k(k - 1)^2(k - 2)^2}$$ \hspace{1cm} (53)

Note that this coefficient $C_{[I, I, I]}$ has a nontrivial numerator, which is not a product of simple factors. This term is characterized as a non-intersecting, three lines graph. When the graphs are made of non-intersecting lines, the expression for $C$ becomes more complex, and there is no obvious simple structure. In other words, for graphs with non-intersecting lines, the decomposition rule found in ref.

At fourth order:

we have computed the coefficients of every graph up to order four. However, we have not been able to find an a priori rule giving the weight of an arbitrary graph. In Table A, the coefficients $C$ are listed.
Table A: WKB-expansion coefficients for $\beta = 4$

\begin{align*}
1=1 & \quad C[\|] = -\frac{1}{k} \\
1=2 & \quad C[A] = \frac{1}{k(k-1)}, \quad C[I,I] = \frac{k-2}{k(k-1)^2} \\
1=3 & \quad C[\Delta] = C[N] = -\frac{1}{k(k-1)^2}, \quad C[Y] = -\frac{1}{k(k-1)(k-2)} \\
 & \quad C[\Lambda,I] = \frac{k-2}{k(k-1)^2(k-2)}, \quad C[1,1,I] = -\frac{k^2-6k+10}{k(k-1)^2(k-2)^2} \\
1=4 & \quad C[\Xi] = C[\angle] = C_1 = \frac{1}{k(k-1)^2(k-2)} \\
 & \quad C[M] = \frac{k^3-3k^2}{k(k-1)^2(k-2)^2}, \quad C[X] = \frac{1}{k(k-1)(k-2)(k-3)} \\
 & \quad C[\Delta,I] = C[N,I] = \frac{(k-3)^2}{k(k-1)^2(k-2)^4}, \quad C[Y,I] = \frac{k^4-4}{k(k-1)^2(k-2)(k-3)} \\
 & \quad C[\Lambda,\Lambda] = \frac{k-4}{k(k-1)^2(k-2)^2} \\
 & \quad C[\Lambda,1,1] = \frac{k^3-10k^2+34k-38}{k(k-1)^2(k-2)^4(k-3)} \\
 & \quad C[I,I,1,1] = \frac{k^4-14k^3+76k^2-1888+174}{k(k-1)^2(k-2)^4(k-3)^2}
\end{align*}

5 The $\tau$-expansion for $\beta=2m$, ($m=2,3,4,\cdots$)

Hereabove we have dealt with $\beta = 4$. For $\beta = 2m$, $m=2,3,4,\ldots$, again we found that the WKB expansion for $f$ terminates after a finite number of terms. However a given $\tau_{ij}$ may now appear non linearly, up to degree $(m-1)$. Therefore, the graphical representations may have now multiple lines connecting the two points (i) and (j) (with a multiplicity at most equal to $(m-1)$). Therefore there are now new types of graphs which did not occur for $\beta = 4(m = 2)$.

From the equation

$$\hat{P} = \sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2} - \beta \left( \frac{\beta}{2} - 1 \right) \sum_{i<j} \frac{1}{(x_i - x_j)^2} - \sum_{i=1}^{k} \lambda_i^2. \quad (54)$$

$$\hat{P} \psi = 0 \quad (55)$$

We expand $\psi$ as

$$\psi = \frac{e^{\sum \lambda_i x_i}}{[\Delta(x)\Delta(\Lambda)]^{\beta/2-1}} \left[ 1 + C[\|](\tau_{12} + \cdots) + C[I,I](\tau_{12}^2 + \tau_{13}^2 + \cdots) + C[\Lambda](\tau_{12} \tau_{13} + \cdots) + \cdots \right] \quad (56)$$

$\hat{P} \psi = 0$ connects the terms in $\psi$ of degree $(n-1)$ to those of degree $n$. 

14
From $\hat{\psi}_0$, we have a pole term of

$$
\hat{P}\psi_0 = \frac{C_0}{[\Delta(\lambda)\Delta(x)]^{\beta/2-1}}[-2\left(\frac{\beta}{2} - 1\right)\frac{\lambda_1 - \lambda_2}{x_1 - x_2}]
$$

(57)

From $\hat{P}\psi[I]$, we obtain

$$
\hat{P}\psi[I] = \frac{C[I]}{[\Delta(\lambda)\Delta(x)]^{\beta/2-1}}[-4\left(\frac{\beta}{2} - 1\right) - 2(\frac{\beta}{2} - 1)(k-2)]\frac{\lambda_1}{x_1 - x_2}
$$

(58)

Thus we obtain the first order result,

$$
C[I] = -\frac{1}{k}.
$$

(59)

It is remarkable that this result is the same as for $\beta = 4$ but now it holds for arbitrary $\beta$. Every residue had the same factor $\left(\frac{\beta}{2} - 1\right)$ and thus $\beta$ drops from the equation. This property holds to all orders, and this characteristic will become important when we discuss the duality relation for zonal polynomials. Therefore, the cancellation of the pole terms gives the same results as for $\beta = 4$, except that there are now new terms coming with the multiple lines.

We list here the results of the cancellation conditions which determine the coefficients characterizing the $\tau$-expansion. They are valid for arbitrary $\beta$, and in fact they do not depend on $\beta$.

**Second order:**

$$
C[I] + (k - 1)C[I,I] + C[A] = 0
$$

(60)

$$
C[I] + (k - 1)C[A] + 2C[II] = 0
$$

(61)

(60) is same as (41), and (61) is different from (40) by the double line term $C[II]$.

**Third order:**

$$
C[I,I] + (k - 2)C[I,I,I] + 2C[A,I] = 0
$$

(62)

$$
C[II] + 3C[III] + (k - 1)C[\triangle] = 0
$$

(63)

$$
C[A] + (k - 1)C[A,I] + C[Y] = 0
$$

(64)

$$
C[A] + (k - 2)C[Y] + 4C[\triangle] = 0
$$

(65)

$$
C[A] + (k - 2)C[N] + C[\triangle] + C[II,I] + C[\triangle] = 0
$$

(66)

$$
C[A] - C[II] + 2C[\triangle] - C[\triangle] + (k - 3)C[N] - (k - 3)C[II,I] = 0
$$

(67)
Fourth order:

\[ C_{[I,I,I]} + (k-3)C_{[II,I,I]} + 3C_{[A,I,I]} = 0 \quad (68) \]
\[ C_{[A,I]} + (k-2)C_{[A,I,I]} + C_{[Y,I]} + C_{[A,A]} = 0 \quad (69) \]
\[ C_{[A,I]} + (k-3)C_{[A,A]} + 2C_{[M]} + 2C_{[A,II]} = 0 \quad (70) \]
\[ 2C_{[I,I,I]} + 8C_{[N,I]} - C_{[A,I]} + 4C_{[II,I,I]} \]
\[ -C_{[A,A]} - 3C_{[Y,I]} + (k-6)C_{[A,II]} = 0 \quad (71) \]

These equations are obtained from the cancellation conditions of the residues of the pole \( \frac{1}{x_1-x_2} \) proportional to \( \lambda_1 \lambda_3 \lambda_4 \lambda_5 x_6 x_7 x_8 \), \( \lambda_1 \lambda_3 \lambda_4 \lambda_5 x_6^2 x_7 \), \( \lambda_1^2 \lambda_3^2 x_4 x_5 x_6 \), \( \lambda_1^2 \lambda_2 \lambda_3 x_4 x_5 x_6 \), respectively.

Further, from the residue of the form \( \lambda_1^2 \lambda_3^2 x_4 x_5^2 \), one finds

\[ C_{[II,I]} + C_{[N]} + 3C_{[\Xi]} + 2C_{[II,II]} + C_{[\Box]} \]
\[ +(k-2)C_{[A,II]} + (k-2)C_{[M]} = 0 \quad (72) \]

From the vanishing of the residue containing \( \lambda_1^2 \lambda_3 \lambda_4 x_5^2 \), one obtains

\[ C_{[Y]} + 2C_{[\Xi]} - 2C_{[\Xi]} + 2C_{[I,I]} + 2C_{[|=|=]} + (k-3)C_{[\angle \angle]} = 0 \quad (73) \]

and we have

\[ C_{[\angle \angle]} = C_{[II]} \quad (74) \]

However, starting with order three and higher, the coefficients of the \( \tau \) expansion are not determined uniquely, since the \( \tau \) variables are not independent. Starting with degree three there are identities, i.e., polynomials vanishing identically. In the appendix C, we discuss the origin of these identities and derive the cubic identity and the quartic identities.

The cubic one is

\[ I_3 = [II, I] - [N] + (k-3)[\triangle] = 0. \quad (75) \]

and the quartic ones are

\[ 2[II, I, I] - [N, I] + (k-5)[\triangle, I] = 0 \quad (76) \]
\[ [\text{III, I}] + 2[\text{II, II}] - [\triangleright] + (k - 3)[\triangle] - 4[\square] = 0 \quad (77) \]

and

\[ [\Lambda, \Pi] - [M] + 2[\triangle, I] + (k - 4)[\square] - \frac{1}{2}(k - 4)[\Pi, \Pi] = 0 \quad (78) \]

and

\[ [\angle, I] - 2[\angle \angle] - 2[\triangle, I] + (k - 4)[\triangleright] - 2(k - 4)[\square] + (k - 4)[\Pi, \Pi] = 0 \quad (79) \]

Those identities give some freedom in writing the above expansion. Indeed one may add one of those identities multiplied by an arbitrary coefficient to the \( \tau \) expansion of the HIZ-integral. For the third order, the coefficients \( C_{\text{II},I}, C_N \) and \( C_\triangle \) may be shifted as

\[
C_{\text{II},I} \rightarrow C_{\text{II},I} + \alpha \\
C_N \rightarrow C_N - \alpha \\
C_\triangle \rightarrow C_\triangle + (k - 3)\alpha
\]

(80)

where \( \alpha \) is arbitrary.

At order four, from (76), (77),(78) and (79), the following shifts leave the expansion unchanged

\[
\begin{align*}
C_{\text{II},I,I} & \rightarrow C_{\text{II},I,I} + 2\gamma_4 \\
C_{N,I} & \rightarrow C_{N,I} - \gamma_4 \\
C_{\text{III},I} & \rightarrow C_{\text{III},I} + \gamma_1 \\
C_{\text{II},II} & \rightarrow C_{\text{II},II} + 2\gamma_1 - \frac{1}{2}(k - 4)\gamma_2 + (k - 4)\gamma_3 \\
C_\triangleright & \rightarrow C_\triangleright - \gamma_1 + (k - 4)\gamma_3 \\
C_\triangle & \rightarrow C_\triangle + (k - 3)\gamma_1 \\
C_\square & \rightarrow C_\square - 4\gamma_1 + (k - 4)\gamma_2 - 2(k - 4)\gamma_3 \\
C_{\triangle, I} & \rightarrow C_{\triangle, I} + 2(k - 4)\gamma_2 - 2(k - 4)\gamma_3 + (k - 5)\gamma_4 \\
C_{\Pi} & \rightarrow C_{\Pi} - \gamma_1 \\
C_M & \rightarrow C_M - \gamma_2 \\
C_{\angle \angle 1} & \rightarrow C_{\angle \angle 1} + \gamma_3 \\
C_{\angle \angle} & \rightarrow C_{\angle \angle} - \gamma_3
\end{align*}
\]

(81)

where \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) are arbitrary numbers, arising as multiplicative factors of the quartic identities.
In Table B, we give the coefficients \( C_{[\text{graphs}]} \). For the coefficients, which have ambiguities due to the above identities, we use a fixing rule which will be discussed in the next section.

The values of Table B are also derived systematically by the \( \tau \)-transformation of the Jack polynomial series as discussed later in Appendix B and Appendix D. In Table B, we use the parameter \( \alpha \) instead of \( \beta \). The parameter \( \alpha \) is a parameter of Jack polynomial, and in Table B, \( \alpha \) is given by \( \alpha = \frac{2}{2-\beta} \). For \( \beta = 4 \), \( \alpha \) becomes \(-1\), and the values in Table B coincide with the values in Table A.

### Table B: WKB-expansion coefficients for \( \beta = 2m \)

| \( l \) | \( 1 \) | \( 2 \) | \( 3 \) |
|---|---|---|
| \( C_{[I]} \) | \(-\frac{1}{k}\) | \( \frac{1}{k(k+\alpha)} \) | \( \frac{1}{k(k+\alpha)(k+2\alpha)} \) |
| \( C_{[\setminus]} \) | \( \frac{1}{k(k+\alpha)}(1 + \frac{\alpha}{k-1}) \) | \( \frac{1}{k(k+\alpha)(k+2\alpha)}(1 + \frac{2\alpha}{k-1}) \) | \( \frac{1+\alpha}{2k(k+\alpha)} \) |
| \( C_{[\triangledown, I]} \) | \( \frac{1}{k(k+\alpha)(k+2\alpha)}(1 + \frac{2\alpha}{k-1}) \) | \( \frac{1}{k(k+\alpha)(k+2\alpha)}(1 + \frac{2\alpha}{k-1}) \) | \( \frac{1+\alpha}{2k(k+\alpha)(k+2\alpha)} \) |
| \( C_{[\triangledown, I], I} \) | \( \frac{(1+\alpha)(1+2\alpha)}{6k(k+\alpha)(k+2\alpha)} \) | \( \frac{(1+\alpha)(1+2\alpha)}{6k(k+\alpha)(k+2\alpha)} \) | \( \frac{1+\alpha}{2k(k+\alpha)(k+2\alpha)} \) |
| \( C_{[N]} \) | \( \frac{1}{k(k+\alpha)(k+2\alpha)}(1 + \frac{3\alpha}{k-1} + \frac{2\alpha^2}{(k-1)(k-2)}) \) | \( \frac{1+\alpha}{2k(k+\alpha)(k+2\alpha)}(1 + \frac{3\alpha}{k-1} + \frac{2\alpha^2}{(k-1)(k-2)}) \) | \( \frac{1+\alpha}{2k(k+\alpha)(k+2\alpha)}(1 + \frac{3\alpha}{k-1} + \frac{2\alpha^2}{(k-1)(k-2)}) \) |
| \( C_{[\Delta]} \) | \( \frac{1}{k(k+\alpha)(k+2\alpha)}(1 + \frac{\alpha}{k-1}) \) | \( \frac{1}{k(k+\alpha)(k+2\alpha)}(1 + \frac{\alpha}{k-1}) \) | \( \frac{1}{k(k+\alpha)(k+2\alpha)}(1 - \frac{\alpha^2}{k-1}) \) |
1=4\left[C_{[\infty]}\right] = \frac{1}{k(k+\alpha)(k+2\alpha)(k+3\alpha)}

C_{[\infty]} = \frac{1}{k(k+\alpha)(k+2\alpha)(k+3\alpha)} (1 + \frac{3\alpha}{k-1})

C_{[\Lambda,A]} = \frac{1}{k(k+\alpha)(k+2\alpha)(k+3\alpha)} (1 + \frac{\alpha}{k+\alpha-1})(1 + \frac{3\alpha}{k-1})

C_{[\Lambda,A,I]} = \frac{1}{k(k+\alpha)(k+2\alpha)(k+3\alpha)} (1 - \frac{2\alpha}{k+\alpha-1} + \frac{7\alpha}{k-1} + \frac{6\alpha^2}{(k-1)(k-2)} - \frac{2\alpha^2}{(k-2)(k+\alpha-1)})

C_{[\Lambda,\Pi,I]} = \frac{1}{k(k+\alpha)(k+2\alpha)(k+3\alpha)} (1 + \frac{6\alpha}{k+\alpha-1} + \frac{9\alpha^2}{(k-1)(k+\alpha-1)} + \frac{8\alpha^2}{(k-2)(k+\alpha-1)} + \frac{25\alpha^3}{(k+\alpha-1)(k-2)(k-3)} - \frac{8\alpha^3}{(k-1)(k-2)(k-3)(k+\alpha-1)})

C_{[\Xi]} = \frac{1+\alpha}{2k(k+\alpha)(k+2\alpha)(k+3\alpha)} (1 + \frac{2\alpha}{k-1})

C_{[\Xi]} = \frac{(1+\alpha)^2}{4k(k+\alpha)(k+2\alpha)(k+3\alpha)}

C_{[\Xi]} = \frac{1+\alpha}{2k(k+\alpha)(k+2\alpha)(k+3\alpha)}

C_{[\Xi]} = \frac{1+\alpha(1+2\alpha)}{6k(k+\alpha)(k+2\alpha)(k+3\alpha)}

C_{[\Xi]} = \frac{(1+\alpha)(1+3\alpha)}{24k(k+\alpha)(k+2\alpha)(k+3\alpha)}

(These 10 coefficients are determined without ambiguities)
The determinations of these coefficients are given in Appendix D.)

The 13 coefficients involve arbitrariness due to 4 quartic identities.

\[
C_{[\Pi]} = \frac{1+\alpha}{2k(k+\alpha)(k+2\alpha)(k+3\alpha)}(1 + \frac{\alpha}{k-1})
\]

\[
C_{[\Delta]} = \frac{1+\alpha}{2k(k+\alpha)(k+2\alpha)(k+3\alpha)}(1 - \frac{2\alpha^2}{k-1})
\]

\[
C_{[\angle]} = \frac{1}{k(k+\alpha)(k+2\alpha)(k+3\alpha)}(1 + \frac{2\alpha}{k-1})
\]

\[
C_{[\triangleright]} = \frac{1}{k(k+\alpha)(k+2\alpha)(k+3\alpha)}(1 - \frac{\alpha(\alpha-1)}{k-1})
\]

\[
C_{[\rhd]} = \frac{1+\alpha}{2k(k+\alpha)(k+2\alpha)(k+3\alpha)}(1 + \frac{3\alpha}{k-1})
\]

\[
C_{[A,II]} = \frac{1+\alpha}{2k(k+\alpha)(k+2\alpha)(k+3\alpha)}(1 + \frac{\alpha}{k+\alpha-1})(1 + \frac{3\alpha}{k-1})
\]

\[
C_{[I,II,II]} = \frac{1+\alpha}{2k(k+\alpha)(k+2\alpha)(k+3\alpha)}(1 - \frac{2\alpha}{k+\alpha-1} + \frac{7\alpha}{k-1} + \frac{6\alpha^2}{(k-1)(k-2)} - \frac{2\alpha^2}{(k-2)(k+\alpha-1)})
\]

\[
C_{[II,II]} = \frac{(1+\alpha)^2}{4k(k+\alpha)(k+2\alpha)(k+3\alpha)}(1 + \frac{4\alpha}{k-1} + \frac{2\alpha^2(2+\alpha)}{(1+\alpha)(k+\alpha-1)})
\]

\[
C_{[III,II]} = \frac{(1+\alpha)(1+2\alpha)}{6k(k+\alpha)(k+2\alpha)(k+3\alpha)}(1 + \frac{3\alpha}{k-1})
\]

\[
C_{[\odot]} = \frac{1}{k(k+\alpha)(k+2\alpha)(k+3\alpha)}(1 + \frac{2\alpha}{k-1})
\]

\[
C_{[\mathbb{M}]} = \frac{1}{k(k+\alpha)(k+2\alpha)(k+3\alpha)}(1 + \frac{\alpha}{k+\alpha-1})(1 + \frac{2\alpha}{k-1})
\]

\[
C_{[N,II]} = \frac{1}{k(k+\alpha)(k+2\alpha)(k+3\alpha)}(1 + \frac{8\alpha}{k-1} - \frac{4\alpha}{k-2} - \frac{\alpha^2}{(k-1)(k-2)} + \frac{\alpha(1+\alpha)(4k+3\alpha-4)}{(k-1)(k-2)(k+\alpha-1)})
\]

\[
C_{[\Delta,II]} = \frac{1}{k(k+\alpha)(k+2\alpha)(k+3\alpha)}(1 - \frac{\alpha(\alpha-3)}{k+\alpha-1} - \frac{\alpha^2(4\alpha-3)}{(k-2)(k+\alpha-1)} - \frac{\alpha^2(3\alpha^2-2\alpha+3)}{(k-1)(k-2)(k+\alpha-1)})
\]

(These 13 coefficients involve arbitrariness due to 4 quartic identities.

The determinations of these coefficients are given in Appendix D.)

Fig.5. symbols for the several graphs.
6 The $\tau$-expansion from zonal polynomials for $\beta = 1$

The Table B shows the first coefficients of the WKB-expansion for general $\beta$. It was derived pertubatively assuming that $\beta$ was an even integer. However the expression may be analytically continued to all integers. In this section we consider $\beta = 1$, which is of practical importance since it is the case of a measure invariant under the orthogonal group (the matrices $X$ and $\Lambda$ being real and symmetric).

The HIZ integral (1) may be expanded in a series involving products of zonal polynomials $Z_p(X)$ and $Z_p(\Lambda)$, as

$$I_{\beta=1} = \int_{O(k)} \left[ \exp \text{tr}(Xg\Lambda g^{-1}) \right] dg = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{O(k)} [\text{tr}(Xg\Lambda g^{-1})]^m dg$$

$$= \sum_{m=0}^{\infty} \frac{1}{m! (2m)!} \sum_{p(m)} \chi_p(1) \int_{O(k)} Z_p(Xg\Lambda g^{-1}) dg$$

$$= \sum_{m=0}^{\infty} \frac{2^m}{(2m)!} \sum_{p(m)} \chi_p(1) \frac{Z_p(X)Z_p(\Lambda)}{Z_p(I_k)}$$

(82)

where $p(m)$ is the partition of order $m$; $Z_p(X)$, the zonal polynomial, is a symmetric homogeneous polynomial of degree $p$ in the $k$ eigenvalues of $X$ [15]. The third equality in (82) follows from the identity

$$(\text{tr}M)^q = c_q \sum_p \chi_p(1) Z_p(M)$$

(83)

where $c_q = \sum \chi_p(1)$. The sum over $p$ runs over the partitions of the number $q$, i.e. over all Young tableaux with $q$ boxes. For the orthogonal group $c_q = 2^q q!/(2q)!$. The function $Z_p(M)$ is a symmetric function of the eigenvalues of $M$ of degree $q$.

In the unitary case, a similar expression in terms of a character expansion is well known; it is used explicitly in [12]; there $\chi_p(1)$ is the dimension of the representation of the permutation group of $p$ objects corresponding to a given Young tableau. (In an appendix, we give the general expression of this integral for arbitrary $\beta$, including the unitary case.)

The zonal polynomial $Z_p(X)$ being an homogeneous symmetric function of $x_1, ..., x_k$, may be expressed in terms of the sums $s_n$ ($n=1,2,3,...$),

$$s_n = \text{tr}X^n = x_1^n + x_2^n + \cdots + x_k^n.$$}

(84)
The zonal polynomial $Z_p(X)$, the constants $\chi_p(1)$, and $Z_p(I)$, which is obtained by setting all $x_i = 1$, are listed up to order five in the Appendix A, when one takes $\alpha = \frac{2}{3} = 2$.

From (21) and (82), we have the relation,

$$e^{\sum_{i=1}^{k} x_i \lambda_i} f_{\beta=1} = I_{\beta=1}(x_i, \lambda_i)$$ \hspace{1cm} (85)

For $\beta=1$, the Vandermonde factor disappears, since it is raised to the power is $\beta - 1$. For arbitrary $\beta$ there is a sum in (1) over the $k!$ permutations of the $\lambda_i$. However, if one expands in powers both the exponential term, together with $f_{\beta=1}$, it turns out that each term of given order is a symmetric function of the $\lambda_i$ and of the $x_i$. Therefore the sum over permutations in (21) is not necessary for $\beta = 1$.

In the previous section, using the coefficients $C$ of the Table B, setting $\beta = 1$, we have obtained the $\tau$-expansion for $f_{\beta=1}$. The same result may be derived from the zonal polynomial expansion for $\beta = 1$.

$$f_{\beta=1} = 1 - \frac{1}{k} (\tau_{12} + \cdots) + \frac{3}{2k(k+2)} (\tau_{12}^2 + \cdots) + \frac{1}{k(k+2)} (\tau_{12} \tau_{23} + \cdots)$$
$$+ \frac{k + 1}{k(k+2)(k-1)} (\tau_{12} \tau_{34} + \cdots) + O(x^3) \hspace{1cm} (86)$$

Indeed, using the explicit values of the character $\chi_p(1)$, the zonal polynomial expansion (82) reads

$$I_{\beta=1} = 1 + \frac{s_1(x)s_1(\lambda)}{k} + \frac{1}{6} \left[ \frac{(s_1(x))^2 + 2s_2(x))(s_1(\lambda))^2 + 2s_2(\lambda))}{k(k+2)} \right.$$  
$$+ 2 \left[ \frac{(s_1(x))^2 - s_2(x))(s_1(\lambda))^2 - s_2(\lambda))}{k(k-1)} \right]$$
$$+ \frac{1}{90} \left[ \frac{Z_{[3]}(x)Z_{[3]}(\lambda)}{k(k+2)(k+4)} + \frac{Z_{[21]}(x)Z_{[21]}(\lambda)}{k(k+2)(k-1)} + \frac{Z_{[13]}(x)Z_{[13]}(\lambda)}{k(k-1)(k-2)} \right]$$
$$+ \cdots \hspace{1cm} (87)$$

For a comparison of this expansion with our earlier expressions $f_{\beta=1}$, one must still expand the exponential factor $e^{\sum x_i \lambda_i}$. But this exponential factor has no explicit $k$ dependence; thus the $k$-dependent coefficients in $f_{\beta=1}$ will not be modified when one expands the exponential. For instance, at first order, the coefficient $\frac{1}{k}$ is present both in (86) and (87); expanding the exponential $e^{\sum x_i \lambda_i}$, we do find that the two expansions coincide. At order two, one writes the coefficient of $\tau_{12} \tau_{34}$ as

$$\frac{k + 1}{k(k+2)(k-1)} = \frac{1}{3k} \left( \frac{1}{k+2} + \frac{2}{k-1} \right), \hspace{1cm} (88)$$
then the terms of degree two in $e^{\sum_x \lambda_i f_{\beta=1}}$ have coefficients which are either $\frac{1}{k(k+2)}$ or $\frac{1}{k(k-1)}$; they are exactly the inverse of the dimensional constant $Z_{[p]}(I)$ in (87).

At order three, consider for instance $\tau_{1234756}$ (three non-intersecting lines). Taking in Table B the coefficient of this term, one may decompose it as

\[
\frac{k^2 + 3k - 2}{k(k + 2)(k + 4)(k - 1)(k - 2)} = \frac{1}{15k(k + 2)(k + 4)} + \frac{9}{5} + \frac{1}{15k(k - 1)(k - 2)} \tag{89}
\]

which is the sum of

\[
\frac{1}{15Z_{[3]}(I_k)} + \frac{9}{15Z_{[21]}(I_k)} + \frac{5}{15Z_{[13]}(I_k)}.
\]

The numerators of these terms are in the ratios 1:9:5, which are also the ratios of the characters $\chi_{[3]}(1) : \chi_{[21]}(1) : \chi_{[13]}(1)$. (We will soon find the reason for this coincidence.)

Let us return to the derivation of the $\tau$-expansion from zonal polynomials. We first shift the diagonal matrices $X$ and $\Lambda$ to make them traceless,

\[
\Lambda = \frac{1}{k} \text{tr}\Lambda + \tilde{\Lambda} \tag{90}
\]

\[
X = \frac{1}{k} \text{tr}X + \tilde{X} \tag{91}
\]

In terms of eigenvalues, it is

\[
\tilde{\lambda}_a = \frac{1}{k} \sum_b (\lambda_a - \lambda_b) \tag{92}
\]

Noting that $\text{tr} \tilde{\Lambda} = \text{tr} \tilde{X} = 0$, the HIZ integral is

\[
I_{\beta=1} = e^{\frac{1}{k}(\text{tr}\Lambda)(\text{tr}X)} \int dg e^{\text{tr} \tilde{\Lambda} g \tilde{X} g^T} = e^{\frac{1}{k}(\text{tr}\Lambda)(\text{tr}X)} \sum_{m=0}^{\infty} \frac{2m}{(2m)!} \sum \chi_p(1) \frac{Z_p(\tilde{\Lambda})Z_p(\tilde{X})}{Z_p(I)} \tag{93}
\]

From the expressions of zonal polynomials in terms of the symmetric functions $s_n$ given in Appendix A, noting that $s_1 = \sum \tilde{x}_a = 0$, and using simple identities such as $\frac{1}{k}(\sum \lambda_i)(\sum x_i) = \sum \lambda_i x_i - \frac{1}{k} \sum_{i<j} (\lambda_i - \lambda_j)(x_i - x_j),$
the integral becomes

\[ I = e^{\sum \lambda_i x_i - \frac{1}{k} \sum_{i<j} \tau_{ij}} \left[ 1 + \frac{1}{(k + 2)(k - 1)} s_2(\bar{x}) s_2(\bar{\lambda}) \right] + \frac{4k}{3(k + 2)(k + 4)(k - 1)(k - 2)} s_3(\bar{x}) s_3(\bar{\lambda}) \]

\[ + \frac{2k(k^2 + k + 2)}{(k + 1)(k + 2)(k + 4)(k + 6)(k - 1)(k - 2)(k - 3)} s_4(\bar{x}) s_4(\bar{\lambda}) \]

\[ + \frac{2k(k + 1)(k + 2)(k + 4)(k + 6)(k - 1)(k - 2)(k - 3)}{2(2k^2 + 3k - 6)} \]

\[ \times \left( (s_2(\bar{x}))^2 s_4(\bar{\lambda}) + s_4(\bar{x})(s_2(\bar{\lambda}))^2 + (x^5) \right) \] (94)

The paired product \( s_2(\bar{x}) s_2(\bar{\lambda}) \) is expressed in terms of the \( \tau_{ij} \) as

\[ s_2(\bar{x}) s_2(\bar{\lambda}) = \left[ \frac{1}{k} \sum_{i<j} (x_i - x_j)^2 \right] \left[ \frac{1}{k} \sum_{i<j} (\lambda_i - \lambda_j)^2 \right] \]

\[ = \frac{(k - 1)^2}{k^2} \sum_{i<j} \tau_{ij}^2 - \frac{2(k - 1)}{k^2} \sum \tau_{ij} \tau_{ik} + \frac{2}{k^2} \sum \tau_{ij} \tau_{kl} \] (95)

The second sum is restricted to \( j < k \) and \( i, j, k \) all different. The last sum is restricted to \( i < j, k < l \) and \( i, j, k, l \) all different. The identity (95) holds for arbitrary \( x_i \) and \( \lambda_i \). In order to fix the coefficients of this identity, one may choose simple values of \( X \) and \( \Lambda \). Let us, for instance, take \( X = \Lambda = (1, 1, \ldots, 1, 0, \ldots, 0) \), where the eigenvalue one is \( q \)-fold degenerate, and zero \( (k - q) \). Then we find

\[ s_2(\bar{x}) s_2(\bar{\lambda}) = \frac{q^2(k - q)^2}{k^2} \] (96)

Note that \( \frac{q^2(k - q)^2}{k^2} \) is invariant under the replacement \( q \rightarrow k - q \). The three terms of the products of \( \tau \) in the r.h.s. of (95) become \( q(k - q) \), \( q(k - q)(k - 2)/2 \) and \( q(q - 1)(k - q)(k - q - 1)/2 \) respectively. These numbers are also invariant by \( q \rightarrow k - q \). Then with three unknown coefficients, \( a, b \) and \( c \), we write

\[ \frac{q^2(k - q)^2}{k^2} = a[q(k - q)] + b[q(k - q)(k - 2)/2] + c[q(q - 1)(k - q)(k - q - 1)/2], \]

(97)

This relation fixes the three unknown coefficients: \( a = \frac{(k - 1)^2}{k^2}, b = -\frac{2(k - 1)}{k^2} \) and \( c = \frac{2}{k^2} \).
In the Appendix B, we derive this identity with the help of differential operators. In the same appendix, we discuss the general expression of product of symmetric functions \(s_n(\tilde{x})s_n(\tilde{\lambda})\) in terms of \(\tau\).

Remarkably, as may be checked in the explicit expressions hereabove for the second order, although each of the three quadratic expressions in \(\tau\) in (95) is not invariant under the permutations of \(\lambda_i\), the combination of the three types of products of \(\tau_{ij}\) is a totally symmetric function as it should. To study further this invariance under permutation, we consider

\[
X = (1, \ldots, 1, 0, \ldots, 0) \\
\Lambda = (0, \ldots, 0, 1, \ldots, 1) 
\]

where the eigenvalue one is q-fold degenerate, and we choose \(q < k\). This is a particular permutation of the previous choice of \(X = \Lambda = (1, \ldots, 1, 0, \ldots, 0)\). We assume \(q < k/2\). The various \(\tau\) are not the same as in (97); the three quadratic terms are now respectively \([q^2]\), \([q^2(q-1)]\) and \([q^2(q-1)^2]/2\). For this particular permutation of the \(\lambda_i\), the identity

\[
\frac{q^2(k-q)^2}{k^2} = \frac{(k-1)^2}{k^2}[q^2] - \frac{2(k-1)}{k^2}[q^2(q-1)] + \frac{2}{k^2} \left[ \frac{q^2(q-1)^2}{2} \right] 
\]

is indeed consistent with (95).

The equation (99) is interesting since, in this choice of permutation, the value of the \(\tau\)'s are all function of \(q\) and not of \(k\). If we set \(k=q\), this equation shows a non-trivial cancellation for three terms, since the left hand side vanishes; in this case the right hand side of (99) vanishes as

\[
(1 - 2 + 1)q^2(q-1)^2 = 0 
\]

To summarize we have found a method to write \(s_2(\tilde{x})s_2(\tilde{\lambda})\) in terms of \(\tau\)'s. Given that it is a function of the \(\tau\)'s the method consists of (i) evaluation of \(s_2(\tilde{x})s_2(\tilde{\lambda})\) for the choice \(X = (1, \ldots, 1, 0, \ldots, 0)\), \(\Lambda = (0, \ldots, 0, 1, \ldots, 1)\), where 1 is q-fold degenerate. (ii) Decompose this value as \(\frac{q^2}{k^2}[(k-1)^2-2(q-1)(k-1)+(q-1)^2]\). (iii) Evaluate the \(\tau\) terms for this same choice as a function of \(q\). (iv) The comparison with the expansion of (iii) fixes the unknown coefficients.

Similar identities expressing the symmetric functions \(s_n(\tilde{X})s_n(\tilde{\lambda})\) in terms of \(\tau\)'s hold at higher order. At order three since \(s_1(\tilde{X}) = 0\), \(Z_0(\tilde{X})\) is given only by \(s_3(\tilde{X})\) (see Appendix A). In order to express \(s_3(\tilde{x})s_3(\tilde{\lambda})\), into the \(\tau_{ij} = (x_i - x_j)(\lambda_i - \lambda_j)\) polynomials, in the Appendix B, we present a method for deriving these identities based on a differential operator \(D_{l,m,n}^{i,j,k}\). Let us for instance quote the result for the third order

\[
s_3(\tilde{X})s_3(\tilde{\lambda}) = -\frac{1}{k^4}(k-1)^2(k-2)^2(\tau_{12}^3 + \cdots) 
\]
\[
+ \frac{3(k-1)(k-2)^2}{k^4} (\tau_{12}^2 \tau_{23} + \cdots)
+ \frac{3(k-2)^3}{k^4} (\tau_{12} \tau_{23} \tau_{13} + \cdots)
- 12 \frac{(k-1)(k-2)}{k^4} (\tau_{12} \tau_{13} \tau_{14} + \cdots)
- 6 \frac{(k-2)^2}{k^4} (\tau_{12} \tau_{23} \tau_{34} + \cdots)
- 3 \frac{(k-2)^2}{k^4} (\tau_{12} \tau_{34} + \cdots)
+ 12 \frac{(k-2)}{k^4} (\tau_{12} \tau_{23} \tau_{45} + \cdots)
- 24 \frac{2}{k^4} (\tau_{12} \tau_{34} \tau_{56} + \cdots).
\]

(101)

In this expression we have used the freedom given by the cubic identity among the \( \tau \)'s to remove the ambiguities.

However the result may be obtained also by the direct method which was used above for the quadratic case. One first choose \( X = \Lambda = (1, \ldots, 1, 0, \ldots, 0) \) with eignevalue one \( q \)-fold degenerate; then \( s_3(\tilde{X})s_3(\tilde{\Lambda}) = q^2(q-k)^2(2q-k)^2/k^4 \), which is invariant by the substitution of \( q \to k-q \). For instance, the last term of non-intersecting graph of \((\tau_{12} \tau_{34} \tau_{56} + perm.)\) becomes \( q^2(q-1)(q-2)(q-3)(k-q)(k-q-1)(k-q-2)/6 \). Each of the 8 terms is invariant by the substitution of \( q \to k-q \). Therefore the coefficients have to be functions of \( k \) only. Then, we apply again the permutation \( \Lambda = (0, \ldots, 0, 1, \ldots, 1) \). The various \( \tau \) terms are polynomial in \( q \). The remarkable factorization identity,

\[ \frac{q^2(q-k)^2(2q-k)^2}{k^4} = - \frac{24}{k^4} \frac{1}{6} q^2(q-1)^2(q-2)^2 \]
\[ + 12 \frac{k-2}{k^4} [q^2(q-1)^2(q-2)] - 9 \frac{(k-2)^2}{k^4} [q^2(q-1)^2] \]
\[ - 12 \frac{(k-1)(k-2)}{k^4} \frac{1}{3} q^2(q-1)(q-2) + 3 \frac{(k-1)(k-2)^2}{k^4} [2q^2(q-1)] \]
\[ - \frac{(k-1)^2(k-2)^2}{k^4} [q^2] \] (102)

extension of the analogous quartic identity, fixes the unknown coefficients.

As discussed in the previous section, the cubic identity

\[ I_3 = [\tau_{12}^2 \tau_{34}] - [\tau_{12} \tau_{23} \tau_{34}] + (k-3)[\tau_{12} \tau_{13} \tau_{24}] = 0 \] (103)

where \( [\tau_{12}^2 \tau_{34}] = \tau_{12}^2 \tau_{34} + \cdots \) leads to an ambiguity, since one may add \( I_3 \) with
an arbitrary coefficient $\alpha$. At order three, one has from (94),

$$
\bar{I}^{(3)} = \frac{1}{6k(k+2)(k+4)(k-1)(k-2)}
\times [(k^2 + 3k - 2)q^6 + (-12k - 24)q^5 + (6k^2 + 12k + 56)q^4 - 48q^3 + 8k^2q^2]
$$

(104)

By taking the choice $X = (1, ..., 1, 0, ..., 0)$ and $\Lambda = (0, ..., 0, 1, ..., 1)$, we evaluate $\bar{I}_{12}^{(3)} = \frac{1}{6}q^2(q - 1)^2(q - 2)^2$. By comparing the highest order of $q$, $q^6$, we find

$$
C_{[1,1,1]} = -\frac{k^2 + 3k - 2}{k(k+2)(k+4)(k-1)(k-2)}
$$

(105)

Subtracting this $C_{[1,1,1]}[\tau_{12}\tau_{34}\tau_{56}]$ from $\bar{I}^{(3)}$, we have

$$
\tilde{\bar{I}}^{(3)}_{q^5} = \frac{1}{6k(k+2)(k+4)(k-1)(k-2)}
\times [(6k + 18)q^5 + (-7k - 41)q^4 + (12k + 12)q^3 + (4k - 4)q^2]
$$

(106)

The quantity $[\tau_{12}\tau_{13}\tau_{45}]$ becomes $q^2(q - 1)^2(q - 2)$ and it is order $q^5$. Thus we determine its coefficient $C_{[\Lambda,i]}$ from the $q^5$ term in (106),

$$
C_{[\Lambda,i]} = -\frac{(k+3)}{k(k+2)(k+4)(k-1)}
$$

(107)

We subtract $C_{[\Lambda,i]}[\tau_{12}\tau_{13}\tau_{45}]$ from (106), and we obtain

$$
\tilde{\bar{I}}^{(3)}_{q^4} = -\frac{(17k + 31)q^4 + (-18k - 78)q^3 + (16k + 32)q^2}{6k(k+2)(k+4)(k-1)}
$$

(108)

The terms, which give order $q^4$, are $[\tau_{12}\tau_{34}] = q^2(q - 1)^2$, $[\tau_{12}\tau_{34}\tau_{34}] = q^2(q - 1)^2$ and $[\tau_{12}\tau_{13}\tau_{14}] = \frac{1}{3}q^2(q - 1)(q - 2)$. ($[\tau_{12}\tau_{23}\tau_{13}]$ becomes vanishing.) Thus we have

$$
\frac{1}{3}C_{[Y]} + C_{[IL, I]} + C_{[N]} = -\frac{17k + 31}{6k(k+2)(k+4)(k-1)}
$$

(109)

Since $C_{[Y]}$ is

$$
C_{[Y]} = -\frac{1}{k(k+2)(k+4)}
$$

(110)

we have

$$
C_{[IL, I]} + C_{[N]} = -\frac{5k + 11}{2k(k+2)(k+4)(k-1)}
$$

(111)

We write

$$
\frac{5k + 11}{2k(k+2)(k+4)(k-1)} = \frac{9}{10k(k+2)(k+4)} + \frac{8}{5k(k+2)(k-1)}
$$

(112)

27
In principle, the coefficients $C_{[II,I]}$, $C_{[N]}$ and $C_{[\Delta]}$ have ambiguities, since there is an identity of (103). One method of fixing this ambiguity may be large k assumption: We assume

$$C_{[II,I]} = -3 \left[ \frac{a}{2k(k+2)(k+4)} + \frac{1-a}{k(k+2)(k+1)} \right]$$

(113)

and

$$C_{[N]} = -\left[ \frac{c}{k(k+2)(k+4)} + \frac{1-c}{k(k+2)(k-1)} \right]$$

(114)

We assume that $C_{[II,I]} \sim \frac{3}{2} \frac{1}{k^2}$ in the large k limit. The factor $\frac{3}{2}$ is the degeneracy factor $= (\beta/4 - 1)/(\beta/2 - 1)$ for $\beta = 1$ (in the next section, we will discuss this factor by the substitution of $\alpha = 2/(2 - \beta)$). And $C_{[N]} \sim \frac{1}{k^2}$ also. Then, the comparison with (112) gives

$$\frac{3}{2} a + c = \frac{9}{10}$$

(115)

Still one parameter $a$ or $c$ remains undetermined. The coefficient $C_{[\Delta]}$ is determined by the residual equation of (113). However, the ambiguity constant $a$ remains. When we consider the case $\beta = 4$, we have a definite value of $C_{[\Delta]}$, since there is no double line in $\beta = 4$, and no cubic equation exists. For the general value of $C_{[\Delta]}$, extending the case of $\beta = 4$, we write

$$C_{[\Delta]} = -\frac{1}{k(k+\alpha)(k+2\alpha)} \left[ 1 - \frac{\alpha^2}{k-1} \right]$$

(116)

Then, using the residual equation of (66), and (67), we determine the constant $a$ and $c$ in (113) and (114) as

$$a = \frac{1}{5}, \quad c = \frac{3}{5}$$

(117)

which also satisfy the large k limit condition (115). In other words, we have assumed the value of $C_{[\Delta]}$ as (116), since we are free to choose of the coefficients of the cubic identity.

Note that when we subtract $[C_{[II,I]} + C_{[N]}]q^2(q-1)^2$ from (108), we obtain

$$\Delta I^{(3)} = \frac{2q^4 + 12q^3 + q^2}{6k(k+2)(k+4)}$$

(118)

where there is no $\frac{1}{k-1}$ factor. This quantity coincides with the sum of $C_{[III]}[\tau_{12}^3]$, $C_{[\Delta]}[\tau_{12}^2\tau_{13}]$ and $C_{[Y]}[\tau_{12}\tau_{13}\tau_{14}]$, since they are

$$C_{[III]} = -\frac{5}{2} \frac{1}{k(k+2)(k+4)}$$

(119)
\[ C_{[\underline{\underline{\Lambda}}]} = -\frac{3}{2} \frac{1}{k(k+2)(k+4)} \]  
\[ C_{[\Lambda]} = -\frac{1}{k(k+2)(k+4)} \]  

where \([\tau_{12}] = q^2, [\tau_{12} \tau_{34}] = 2q^2(q-1)\) and \([\tau_{12} \tau_{13} \tau_{14}] = \frac{1}{3}q^2(q-1)(q-2)\).

It may be interesting to investigate the fourth order term. In (94), we use the choice that \(X = \Lambda = (1, \ldots, 1, 0, \ldots, 0)\) where 1 entries \(q\)-times, 0 entries \((k-q)\) times. Then, noting that \(s_2(\tilde{x})s_2(\tilde{\lambda}) = \frac{q^2(k-q)^2}{k^2}\), \(s_3(\tilde{x})s_3(\tilde{\lambda}) = \frac{q^2(q-k)^2(2q-k)^2}{k^4}\), \(s_4(\tilde{x})s_4(\tilde{\lambda}) = \frac{q^2(k-q)^2(k^2 - 3kq + 3q^2)^2}{k^6}\), \([s_2(\tilde{x})s_2(\tilde{\lambda})]^2 = \frac{q^4(q-k)^4}{k^4}\), \((s_2(\tilde{x}))^2s_4(\tilde{\lambda}) + s_4(\tilde{x})(s_2(\tilde{\lambda}))^2 = \frac{2q^3(k-q)^3(k^2 - 3kq + 3q^2)}{k^5}\), we have the fourth term together with the expansion of \(e^{-\frac{1}{4} \sum_{i<j} \tau_{ij}} = e^{k^2}\).

\[ \tilde{f}^{(4)} = \frac{1}{24k(k+1)(k+2)(k+4)(k+6)(k-1)(k-2)(k-3)} \times \left[ (k^4 + 7k^3 + k^2 - 35k - 6)q^8 + (-24k^3 - 144k^2 - 72k + 144)q^7 \right. 
\[ + \left. (12k^4 + 72k^3 + 308k^2 + 1064k + 864)q^6 \right. 
\[ + \left. (-240k^3 - 1008k^2 - 2304k - 1728)q^5 \right. 
\[ + \left. (44k^4 + 188k^3 + 1944k^2 + 2064k + 96)q^4 \right. 
\[ + \left. (-576k^3 - 672k^2 - 192k)q^3 + (48k^4 + 48k^3 + 96k^2)q^2 \right] \]  

(122)

There are 23 terms in the fourth order for the \(\tau\) expansion as shown in Table B. For the choice of \(X = (1, \ldots, 1, 0, \ldots, 0), \Lambda = (0, \ldots, 0, 1, \ldots, 1)\), each \(\tau\) terms are evaluated as they are polynomials of \(q\), and each terms have different orders of \(q\). The highest order of \(q\) is obtained from the term \([\tau_{12} \tau_{34} \tau_{56} \tau_{78}]\), (here \([\tau \cdots ]\) means the sum of the permutation of the indecies). It becomes

\[ [\tau_{12} \tau_{34} \tau_{56} \tau_{78}] = \frac{1}{4!}q^2(q-1)^2(q-2)^2(q-3)^2 \]  

(123)

The order \(q^8\) term is only this term. Then, from (122), we obtain the coefficient of this term \([\tau_{12} \tau_{34} \tau_{56} \tau_{78}]\), as

\[ C_{[1,1,1,1]} = \frac{(k+3)(k^2+6k+1)}{k(k+1)(k+2)(k+4)(k+6)(k-1)(k-3)} \]

\[ = \frac{1}{k(k+2)(k+4)(k+6)} \left[ 1 + \frac{12}{k-1} + \frac{40}{(k-3)(k-1)(k-3)} + \frac{4}{(k+1)(k-3)} \right] \]  

(124)
This result is expressed by the use of the dimensional constant $Z_p(1)$ and by the constant $\chi_p(1)$, which appear in the zonal polynomial expansion,

$$C_{[I, I, I]} = \frac{1}{105} \sum_p \frac{\chi_p(1)}{Z_p(1)}$$ (125)

The next highest order term of $q^7$ is $[\tau_{12} \tau_{13} \tau_{45} \tau_{67}]$, which has a factor $C_{[\Lambda, I, I]}$. The value of $[\tau_{12} \tau_{13} \tau_{45} \tau_{67}]$ is $\frac{1}{2} q^2 (q - 1)^2 (q - 2)^2 (q - 3)$. Then from the coefficient of $q^7$ in (122), we obtain the coefficient $C_{[\Lambda, I, I]}$,

$$\frac{1}{2} C_{[\Lambda, I, I]} - \frac{1}{2} C_{[I, I, I]} = \frac{1}{24k(k + 1)(k + 2)(k + 4)(k + 6)(k - 1)(k - 2)(k - 3)} \times (-24k^3 - 144k^2 - 72k + 144)$$ (126)

which reads

$$C_{[\Lambda, I, I]} = \frac{k^3 + 8k^2 + 13k - 2}{k(k + 1)(k + 2)(k + 4)(k + 6)(k - 1)(k - 2)}$$

$$= \frac{1}{k(k + 2)(k + 4)(k + 6)} \left[ 1 + \frac{10}{k - 1} + \frac{20}{(k - 1)(k - 2)} + \frac{4}{(k + 1)(k - 2)} \right]$$ (127)

We note here that the following relation of (68), which was derived for the arbitrary value of $\beta$ in the section 5, is satisfied by above two expressions,

$$C_{[I, I, I]} + 3C_{[\Lambda, I, I]} + (k - 3)C_{[I, I, I]} = 0$$ (128)

since we have from Table B, $(\beta = 1)$, $C_{[I, I, I]} = -\frac{1}{k(k+2)(k+4)(k+6)} \left( 1 + \frac{6}{k - 1} + \frac{8}{(k-1)(k-2)} \right)$. The pole term $\frac{1}{k+1}$ in $C_{[I, I, I]}$ is cancelled by the pole term of $C_{[\Lambda, I, I]}$. This $\frac{1}{k+1}$ factor comes only from the dimension of the zonal polynomial $Z_{[2]}$, and it appears first in the fourth order terms.

For the terms, which give order $q^6$, new 4 terms $[\tau_{12} \tau_{13} \tau_{45} \tau_{56}] = q^2 (q - 1)^2 (q - 2)^2$, $[\tau_{12} \tau_{13} \tau_{45} \tau_{56}] = \frac{1}{2} q^2 (q - 1)^2 (q - 2) (q - 3)$, $[\tau_{12} \tau_{13} \tau_{45} \tau_{46}] = \frac{1}{2} q^2 (q - 1)^2 (q - 2)^2 (2q - 5)$, $[\tau_{12} \tau_{13} \tau_{45} \tau_{56}] = \frac{1}{2} q^2 (q - 1)^2 (q - 2)^2$ appear.

From (122), after the subtraction of the contribution of $[\tau_{12} \tau_{34} \tau_{56} \tau_{78}]$ and $[\tau_{12} \tau_{13} \tau_{45} \tau_{67}]$, the $q^6$ part of (122) becomes

$$C_q^6 = \frac{31k^3 + 196k^2 + 119k - 310}{12k(k + 1)(k + 2)(k + 4)(k + 6)(k - 1)(k - 2)}$$ (129)

which have to be the sum of these four terms. To determine these four coefficients, we first fix two coefficients $C_{[Y, I]}$ and $C_{[\Lambda, A]}$ by the residual equation of (69).
These are satisfied by the values of
\[
C_{[N,I]} = \frac{1}{k(k+2)(k+4)(k+6)} \left(1 + \frac{6}{k-1}\right)
\]  
\[
C_{[A,A]} = \frac{1}{k(k+2)(k+4)(k+6)} \left(1 + \frac{8}{k-1} + \frac{8}{(k-1)(k+1)}\right)
\]  
Then, we subtract these two contributions from (129), and the remaining two coefficients are
\[
C_{[N,I]} + \frac{1}{2} C_{[II,II]} = \frac{7k^3 + 48k^2 + 51k - 30}{4k(k+1)(k+2)(k+4)(k+6)(k-1)(k-2)}
\]  
From the residual equation of (71), we are able to confirm above result. There are ambiguities in the fourth order by the 4 quartic identities as shown in appendix C ((282) ∼ (285)). Due to the quartic identity (282), it is free to choose the value of \(C_{[II,II]}\). Here we assume that \(C_{[II,II]}\) is same as \(C_{[Λ,I,I]}\) except a factor of the multiple line, \(\frac{3}{2}\).

\[
C_{[II,II]} = \left(\frac{3}{2}\right) \frac{k^3 + 8k^2 + 13k - 2}{k(k+1)(k+2)(k+4)(k+6)(k-1)(k-2)}
\]  
Then we have
\[
C_{[N,I]} = \frac{k^3 + 6k^2 + 3k - 6}{k(k+2)(k+4)(k+6)(k+1)(k-1)(k-2)}
\]  
These two values satisfy (132).

The limit of \(q \rightarrow 1\) in (122) determines uniquely the coefficient of the term \((τ_{12})^4\), which is four line degeneracy, and it has a value \(q^2\). Other terms are proportional to \(q - 1\). The coefficient, therefore, becomes
\[
C_{[III]} = \frac{35}{8k(k+2)(k+4)(k+6)}.
\]

The limit \(q \rightarrow 2\), in (122), after the subtractions of \([τ_{12}]^4\), becomes
\[
\tilde{I}^{(4)} = \frac{125k^2 + 320k + 291}{2k(k+1)(k+2)(k+4)(k+6)(k-1)}
\]  
which is the sum of \([τ_{12}τ_{23}τ_{14}τ_{34}], [τ_{12}(τ_{13})^2τ_{34}], [(τ_{12})^2τ_{23}τ_{34}], [[(τ_{12})^2τ_{13}]^2], [[(τ_{12})^2τ_{13}]^2], [(τ_{12})^3τ_{34}], [(τ_{12})^3τ_{34}]^2\), which have values of \(\frac{1}{7}q^2(q-1)^2, q^2(q-1)^2, 2q^2(q-1)^2, q^2(q-1), q^2(q-1)^2, 2q^2(q-1)^2\), respectively. By the summation of these terms with the coefficients we have for \(q=2\),
\[
\tilde{I}^{(4)} = \frac{1}{k(k+2)(k+4)(k+6)} \left[(1 + \frac{4}{k-1}) + 6(1 + \frac{2}{k-1})
\right.
\]  
\[
\left. + 12(1 + \frac{4}{k-1}) + 9 + 20 + 10(1 + \frac{6}{k-1}) + \frac{9}{2}(1 + \frac{8}{k-1} + \frac{32}{3(k-1)(k+1)})\right)
\]  
\[
(137)
\]
where 7 terms are added respectively. We derive the following expression, which involves \( \frac{1}{k+1} \) pole. Other terms are easily derived.

\[
C_{[\text{II,II}]} = \frac{9}{4} \frac{1}{k(k+2)(k+4)(k+6)} (1 + \frac{8}{k-1} + \frac{32}{3(k-1)(k+1)})
\]  
(138)

and

\[
C_{[\text{III,II}]} = \frac{5}{2k(k+2)(k+4)(k+6)} (1 + \frac{6}{k-1})
\]  
(139)

We have following equations,

\[
C_{[\text{III}]} + 2C_{[\text{I, I}]} - C_{[\Delta]} + C_{[\text{II}]} + (k-2)C_{[\text{III,II}]} = 0
\]  
(140)

\[
C_{[\text{I, I}]} - C_{[\text{III}]} + (k-3)C_{[\text{II}]} - (k-3)C_{[\text{III,II}]} + 2C_{[\Delta]} = 0
\]  
(141)

One could eliminate the coefficient of \( C_{[\Delta]} \), the triangle diagram with one double line. Then, we know that \( C_{[\text{III,II}]} \) has no pole of \( \frac{1}{k+1} \). This is consistent with the expression (139).

We have up to now confirmed 14 terms at fourth order. The number of fourth order graphs is 23, and among them, three graphs exist, which contain the triangles. These terms are vanishing for the present choice of \( X \) and \( \Lambda \). Thus, 6 terms still have not yet been determined. These 6 terms are \([\tau_{12}\tau_{13}\tau_{14}\tau_{15}], [\tau_{12}\tau_{13}\tau_{14}\tau_{45}], [\tau_{12}\tau_{23}\tau_{34}\tau_{45}], [(\tau_{12})^2\tau_{13}\tau_{14}], [(\tau_{12})^2\tau_{13}\tau_{45}] \) and \([\tau_{12}\tau_{13}(\tau_{45})^2] \). They have values of \( \frac{1}{12}q^2(q-1)(q-2)(q-3), q^2(q-1)^2(q-2), q^2(q-1)^2(q-2), 2q^2(q-1)(q-2), 2q^2(q-1)^2(q-2), 2q^2(q-1)^2(q-2) \).

The sum of these 6 terms should be the value of (122), once one subtracts the subtractions the previous 14 terms.

From the equations of the residues, we obtain

\[
C_{[\Lambda,\text{II}]} + C_{[\text{M}]} = \frac{(k+3)(5k+21)}{2k(k+1)(k+2)(k+4)(k+6)}
\]  
(142)

which satisfy the residue equation of (70). Note that \( C_{[\text{M}]} \) is not uniquely determined in views of the quartic identity discussed in appendix C.

For fixing this ambiguity, we take the value of \( C_{[\Lambda,\text{II}]} \) to be the same as \( C_{[\Lambda,\Lambda]} \) within a factor \( \frac{3}{2} \),

\[
C_{[\Lambda,\text{II}]} = \frac{3}{2} \frac{k^3 + 6k^2 - k - 38}{2k(k+1)(k+2)(k+4)(k+6)}
\]  
(143)

Note there is no \( \frac{1}{k+1} \) factor in the expression as same as (142). From (142), \( C_{[\text{M}]} \) is determined as

\[
C_{[\text{M}]} = \frac{(k+3)^2}{k(k+2)(k+4)(k+6)(k+1)(k-1)}
\]  
(144)
In the next section and in Appendix D, we discuss other terms for general \( \beta \) (or \( \alpha \)).

As a summary of this section, from the zonal polynomial expansion of the HIZ-integral, we have obtained the series given by the symmetric polynomials of (94). By writing these symmetric polynomials \( s_n(\tilde{x}) \) in terms of \( \tau \), we have obtained the \( \tau \)-expansion of the HIZ-integral in the form \( e^\sum x_i\lambda_i f \), where \( f \) is given as (86). We have found that the coefficients are expressed by sums of the inverse of the dimensional constants. We have shown that \( e^\sum x_i\lambda_i f \) is invariant under the permutations of \( \lambda_i \). Therefore, the summation of the permutation contained in (8) is not necessary for \( \beta = 1 \).

7 Character expansion for general \( \beta \) and a duality

In this section, we generalize the results of the previous section to arbitrary \( \beta \), and we discuss the derivations of the \( \tau \)-expansion using a dual representation.

We write the zonal (Jack) polynomial expansion for the integral \( I_\beta \),

\[
I_\beta = e^{\sum x_i\lambda_i - \frac{1}{k} \sum_{i<j} \tau_{ij}} \left[ \sum_{m=0}^{\infty} \frac{1}{m!} \prod_{q=0}^{m-1} \frac{1}{(1 + q\alpha)} \sum_p \chi_p(1) \frac{Z_p(\tilde{X})Z_p(\tilde{\Lambda})}{Z_p(1)} \right]
\]  

(145)

where we have shifted \( x_i \rightarrow \tilde{x}_i = x_i - \frac{1}{k} \sum_{i=1}^{k} x_i \) (subtraction of the mean).

The generalized zonal polynomial \( Z_p(\tilde{X}) \) has a parameter \( \alpha \), which is equal to the present \( \frac{2}{\beta} \). This polynomial is called as Jack polynomial. As for \( \beta = 1 \), we expand in a power series for small \( X \) and \( \Lambda \). The quantity, which is an generalization of the integral (1) to arbitrary \( \beta \), is a symmetric function by interchange of the \( x_i \) and \( \lambda_i \). We shall write this quantity \( I_\beta \) in terms of \( \tau_{ij} = (x_i - x_j)(\lambda_i - \lambda_j) \).

\[
I_\beta = e^{\sum_{i=1}^{k} x_i\lambda_i} \left[ 1 + D[1][\tau_{12}] + D[1,1][\tau_{12}\tau_{34}] + \cdots \right]
\]  

(146)

This expansion is different from the form discussed in section 2, which was

\[
I_\beta = \sum_{\text{perm.}} \frac{e^{\sum x_i\lambda_i}}{\prod_{i<j}(\tau_{ij})^{\beta-1}} \left[ 1 + C[1][\tau_{12}] + C[1,1][\tau_{12}\tau_{34}] + \cdots \right]
\]  

(147)

The differences are that (147) has a Vandermonde determinant, and also it has a sum about the permutations of the \( \lambda_i \). The expression (146) has no Vandermonde term and no sum over permutations. Therefore, the coefficients \( C[\text{graph}] \) and \( D[\text{graph}] \) are not the same in these two expansion. (The \( \beta = 1 \) case is exceptional since there the two expansions coincide).

To obtain the coefficients \( D \) in the expansion (146), we take \( X \) and \( \Lambda \) as \( X = (1, \ldots, 1, 0, \ldots, 0) \) and \( \Lambda = (0, \ldots, 0, 1, \ldots, 1) \), where 1 is q-fold degenerate,
and $q < k/2$. Since $\sum_{i<j}\tau_{ij}$ becomes $-q^2$, with this choice, we have up to second order,

\[
I = e^{\sum x_i \lambda_i \left[ 1 + \frac{q^2}{k} + \frac{q^4}{2k^2} + \cdots \right]} \\
\times \left[ 1 + \frac{1}{2(1+\alpha)} \left( \chi_{[2]}(1) \frac{Z_{[2]}(\tilde{X}) Z_{[2]}(\tilde{\Lambda})}{Z_{[1]}(1)} + \chi_{[1]}(1) \frac{Z_{[1]}(\tilde{X}) Z_{[1]}(\tilde{\Lambda})}{Z_{[1]}(1)} \right) + \cdots \right] \\
= e^{\sum x_i \lambda_i \left[ 1 + \frac{q^2}{k} + \frac{q^4}{2k^2} + \frac{\alpha}{2(k+\alpha)(k-1)} s_2(\tilde{x}) s_2(\tilde{\lambda}) + O(x^3) \right]} \tag{148}
\]

The symmetric functions $s_n(x) = \sum x^n_i$ are easily evaluated within this choice of $X$, and we obtain $s_2(\tilde{x}) = q(1 - \frac{q}{k})^2 + (k - q)(-\frac{q}{k})^2 = \frac{q(k-q)}{k}$, and thus $s_2(\tilde{x}) s_2(\tilde{\lambda}) = \frac{q^2(k-q)^2}{k^2}$.

Thus we have

\[
I = e^{\sum x_i \lambda_i \left[ 1 + \frac{q^2}{k} + \frac{q^4}{2k^2} + \frac{\alpha q^2(q-k)^2}{2(k+\alpha)(k-1)k^2} + O(x^3) \right]} \tag{149}
\]

We denote $\sum_{\text{perm.}} (\tau_{12} + \cdots)$ by $[\tau_{12}]$. Under this choice of $X$ and $\Lambda$, we find that $[\tau_{ij} \tau_{kl} \cdots]$ are polynomials in $q$.

We evaluate them at order two as $[\tau_{12} \tau_{34}] = \frac{1}{2} q^2(q-1)^2$, $[\tau_{12} \tau_{13}] = q^2(q-1)$, $[(\tau_{12})^2] = q^2$. From (146) and (148), we write

\[
1 + D[I][\tau_{12}] + D[I,I][\tau_{12} \tau_{34}] + D[\Lambda][\tau_{12} \tau_{13}] + D[II][\tau_{12}]^2 + \cdots \\
= 1 + D[I](-q^2) + D[I,I](\frac{1}{2} q^2(q-1)^2) + D[\Lambda]q^2(q-1) + D[II]q^2 + \cdots \tag{150}
\]

By comparing the powers $q^4, q^3$ and $q^2$ in this expression with the expansion of (149), we find,

\[
D[I] = -\frac{1}{k} \tag{151}
\]

\[
D[I,I] = \frac{1}{1+\alpha} \left( \frac{1}{k(k+\alpha)} + \frac{\alpha}{k(k-1)} \right) \tag{152}
\]

\[
D[\Lambda] = \frac{1}{k(k+\alpha)} \tag{153}
\]

\[
D[II] = \frac{1+\alpha}{2k(k+\alpha)} \tag{154}
\]

When $\alpha = 2$, they reduce to the previous $\beta = 1$ result. These coefficients are written in terms of the inverse dimensional constants $\frac{1}{Z_{[p]}(1)}$. The coefficients 1 and $\alpha$ in (152) are the characters $\chi_p(1)$ as shown in the appendix A.
As we remarked, these coefficients $D_{[I]}$, $D_{[\Lambda]}$ and $D_{[II]}$ are the coefficients of (146), and not of (147). For instance, the value of $D_{[\Lambda]}$ at $\beta = 4$ is different from the value of $C_{[\Lambda]}$ of (147) as

$$D_{[\Lambda]} = \frac{1}{k(k + \alpha)}|_{\beta=4} = \frac{1}{k(k + \frac{1}{2})} \neq \frac{1}{k(k - 1)}$$  \hspace{1cm} (155)$$

The residual recursion equations, which gives the recursive relations between the coefficients, are derived for the coefficients of (147). However, we find that the following relations are also valid for the expressions of (151) - (154), which are the coefficients in (146),

$$D_{[I]} + (k - 1)D_{[I,I]} + D_{[\Lambda]} = 0$$  \hspace{1cm} (156)$$

$$D_{[I]} + (k - 1)D_{[\Lambda]} + 2D_{[II]} = 0$$  \hspace{1cm} (157)$$

The reason for the remarkable fact that the residual equations are satisfied, is that the residual equation is independent of the value of $\beta$, and thus of the value of $\alpha$. It means that the residual equation holds independently of the existence of the Vandermonde factor in (147).

Therefore, for an arbitrary parameter $\alpha$, we have found expressions for the coefficients which satisfy the residual equation. The coefficients $C_{[\text{graph}]}$ in (147) satisfies the same recursive equation. We have assumed that $\alpha = \frac{2}{\beta}$. However, if we take this value of $\alpha$ as

$$\alpha = \frac{2}{2 - \beta}$$  \hspace{1cm} (158)$$

the coefficients $D_{[\text{graph}]}$ in (146) for the case of $\beta$ become the coefficient $C_{[\text{graph}]}$ in (147) for the same $\beta$.

For instance, the previous $D_{[\Lambda]}$ becomes, after substitution of $\alpha = \frac{2}{2 - \beta}$,

$$D_{[\Lambda]} = \frac{1}{k(k + \alpha)} = \frac{\frac{\beta}{2} - 1}{k((\frac{\beta}{2} - 1)k - 1)}$$  \hspace{1cm} (159)$$

which is indeed the value of $C_{[\Lambda]}$ in (147) as shown in Table B. Similarly, we find that $D_{[I,I]}$ in (152) and $D_{[II]}$ in (154) are the expressions given in Table B after the substitution of $\alpha = \frac{2}{2 - \beta}$,

$$D_{[I,I]} = \frac{1}{1 + \alpha} \left[ \frac{1}{k(k + \alpha)} + \frac{\alpha}{k(k - 1)} \right] = \frac{\left(\frac{\beta}{2} - 1\right)}{k((\frac{\beta}{2} - 1)k - 1)} \left[ 1 - \frac{1}{(\frac{\beta}{2} - 1)(k - 1)} \right]$$  \hspace{1cm} (160)$$

$$D_{[II]} = \frac{1 + \alpha}{2k(k + \alpha)} = \frac{(\frac{\beta}{2} - 1)}{k((\frac{\beta}{2} - 1)k - 1)}$$  \hspace{1cm} (161)$$

35
The relation (158) is a duality relation, and by this relation, we obtain explicit expressions of the WKB expansion of the integral. This transformation \( \alpha = \frac{2}{\beta} \rightarrow \frac{2}{2-\beta} \) becomes an identity for the case of \( \beta = 1 \). For the value of \( \beta = 2 \), this transformation becomes singular, and it needs special consideration (We will discuss this case in section nine).

At order three, we have to consider the next order in (148). It becomes

\[
\frac{q^6}{6k^3} + \frac{q^2}{k} \frac{\alpha}{2(k + \alpha)(k - 1)} s_2(\tilde{x}) s_2(\tilde{\lambda}) \\
+ \frac{1}{6(1 + \alpha)(1 + 2\alpha)} (\chi_{[\mathcal{I}]}(1) Z_{[\mathcal{I}]}(\tilde{x}) Z_{[\mathcal{I}]}(\tilde{\lambda}) + \chi_{[21]}(1) Z_{[21]}(\tilde{x}) Z_{[21]}(\tilde{\lambda})) \\
+ \chi_{[1]}(1) Z_{[1]}(\tilde{x}) Z_{[1]}(\tilde{\lambda})
\]

\[
= \frac{q^6}{6k^3} + \frac{q^2}{k} \frac{\alpha}{2(k + \alpha)(1 + 2\alpha)} s_2(\tilde{x}) s_2(\tilde{\lambda}) \\
+ \frac{\alpha^2 k}{3(k + \alpha)(k + 2\alpha)(k - 1)(k - 2)} s_3(\tilde{x}) s_3(\tilde{\lambda})
\]

(162)

Using the values of \( s_2(\tilde{x}) = \frac{q(k - q)}{k} \), \( s_3(\tilde{x}) = \frac{q^2(q - k)^2(2q - k)^2}{k^4} \), it becomes

\[
\tilde{J}^{(3)} = \frac{1}{6k(k + \alpha)(k + 2\alpha)(k - 1)(k - 2)} q^2(2\alpha^2 k^2 - 12\alpha^2 kq + 14\alpha^2 q^2 - 6\alpha k q^2 + 6\alpha^2 k q^2 + 3\alpha k^2 q^2 + 12\alpha q^3 - 12\alpha^2 q^3 - 6\alpha k q^3 + 2q^4 - 6\alpha q^4 + 2\alpha^2 q^4 - 3k q^4 + 3\alpha k q^4 + k^2 q^4)
\]

(163)

When we put \( \alpha = 2 \) in the above quantity, it becomes the same as (104). This quantity \( \tilde{J}^{(3)} \) should be equal to the following sum of 8 terms,

\[
\tilde{J}^{(3)} = D_{[1,1,1]} \cdot [\tau_{12} \tau_{34} \tau_{56}] + D_{[\Delta, \Delta]} \cdot [\tau_{12} \tau_{13} \tau_{45}] \\
+ \frac{1}{6} \frac{q^2(q - 1)^2(q - 2)^2}{D_{[1,1,1]} \cdot q^2(q - 1)^2(q - 2)} + D_{[\Delta, \Delta]} \cdot q^2(q - 1)^2(q - 2) \\
+ D_{[\mathcal{I}]} \cdot [\tau_{12} \tau_{13} \tau_{14}] + D_{[\mathcal{H}, \mathcal{I}]} \cdot [\tau_{12} \tau_{34}] + D_{[\mathcal{N}]} \cdot [\tau_{12} \tau_{13} \tau_{34}] \\
+ D_{[\mathcal{I}]} \cdot [\tau_{12} \tau_{13} \tau_{14}] + D_{[\Delta]} \cdot [\tau_{12} \tau_{13} \tau_{14}] + D_{[\Delta]} \cdot [\tau_{12} \tau_{23} \tau_{13}] \\
+ D_{[\mathcal{I}]} \cdot \frac{1}{3} q^2(q - 1)(q - 2) + D_{[\Delta]} \cdot q^2(q - 1)^2 + D_{[\mathcal{N}]} \cdot q^2(q - 1)^2 \\
+ D_{[\mathcal{H}]} \cdot q^2 + D_{[\Delta]} \cdot 2q^2(q - 1) + D_{[\Delta]} \cdot 0
\]

(164)

Comparing \( q^6 \) term in (164) with (163), we have

\[
D_{[1,1,1]} = -\frac{k^2 + 3(\alpha - 1)k + 2(\alpha^2 - 3\alpha + 1)}{k(k + \alpha)(k + 2\alpha)(k - 1)(k - 2)}
\]

(165)
By the dimensional constants \( Z_p(I) \) for general \( \alpha \), it is expressed as

\[
D[I, I, I] = -\frac{1}{(1 + \alpha)(1 + 2\alpha)^2} \left[ \chi_{[3]}(1) Z_{[3]}(I) + \chi_{[21]}(1) Z_{[21]}(I) + \chi_{[13]}(1) Z_{[13]}(I) \right] \tag{166}
\]

where the characters \( \chi_p(I) \) and the dimensional constant \( Z_p(I) \) are given in Table D. \( (\chi_{[3]}(1) = 1, \chi_{[21]}(1) = \frac{6\alpha(1 + \alpha)}{2 + \alpha}, \chi_{[13]}(1) = \frac{\alpha^2(1 + 2\alpha)}{2 + \alpha}) \).

By the dual transformation of (158), this expression becomes

\[
D[I, I, I] = -\frac{\left(\frac{\beta}{2} - 1\right)^2}{k((\frac{\beta}{2} - 1)k - 1)((\frac{\beta}{2} - 1)k - 2)} \left[ 1 - \frac{2}{(\frac{\beta}{2} - 1)(k - 1)} \right] + \frac{2}{(\frac{\beta}{2} - 1)^2(k - 1)(k - 2)} \tag{167}
\]

which coincide, here also with the previous values listed in Table B.

By the comparison of \( q^5 \) terms in (164) and (163), we obtain the expression of \( D_{[\Lambda, I]} \) as

\[
D_{[\Lambda, I]} = -\frac{k + 2\alpha - 1}{k(k + \alpha)(k + 2\alpha)(k - 1)} \tag{168}
\]

By the replacement of \( \alpha = 2/(2 - \beta) \), it becomes

\[
D_{[\Lambda, I]} = -\frac{\left(\frac{\beta}{2} - 1\right)^2}{k((\frac{\beta}{2} - 1)k - 1)((\frac{\beta}{2} - 1)k - 2)} \left[ 1 - \frac{2}{(\frac{\beta}{2} - 1)(k - 1)} \right] \tag{169}
\]

which coincides with the result in Table B.

By putting \( q = 1 \) in (163) and (164), we obtain the expression for \( D_{[\Pi I]} \) as

\[
D_{[\Pi I]} = -\frac{(1 + \alpha)(1 + 2\alpha)}{6k(k + \alpha)(k + 2\alpha)} \tag{170}
\]

and by the dual transformation of (158), it becomes

\[
D_{[\Pi I]} = -\frac{(\frac{\beta}{4} - 1)(\frac{\beta}{6} - 1)}{k((\frac{\beta}{2} - 1)k - 1)((\frac{\beta}{2} - 1)k - 2)} \tag{171}
\]

which agrees with the value in Table B. Note that for \( \beta = 4 \), this quantity is vanishing, and it means that for \( \beta = 4 \), there are no multiple line graphs.

Extracting the contributions of \( D_{[I, I, I]}, D_{[\Lambda, I]} \) and \( D_{[\Pi I]} \) from (163) and (164), we have

\[
D_{[\mathcal{N}]} \cdot \frac{1}{3} q^2(q - 1)(q - 2) + D_{[\Pi I]} \cdot q^2(q - 1)^2 + D_{[\mathcal{N}]} \cdot q^2(q - 1)^2 \\
+ D_{[\mathcal{M}]} \cdot 2q^2(q - 1) \\
= -\frac{q^2(q - 1)}{6k(k + \alpha)(k + 2\alpha)(k - 1)} \left[ 3\alpha kq + 11kq + 6\alpha^2 q \\
+ 9\alpha q - 11q + 3\alpha k - 7k - 6\alpha^2 - 15\alpha + 7 \right] \tag{172}
\]
Dividing both sides by $q^2(q-1)$ factor, we obtain by putting $q = 1$,

$$-\frac{1}{3}D_{[Y]} + 2D_{\triangle} = -\frac{6\alpha + 4}{k(k + \alpha)(k + 2\alpha)} \tag{173}$$

From the simple structure of $[Y]$ (no multiple lines), we assume that

$$D_{[Y]} = -\frac{1}{k(k + \alpha)(k + 2\alpha)} = -\frac{(\frac{\beta}{2} - 1)^2}{k((\frac{\beta}{2}k - 1)((\frac{\beta}{2} - 1)k - 2)} \tag{174}$$

which is also consistent with the known value for $\beta = 4, k = 4$ ($D_{[Y]} = -\frac{1}{24}$ [4]). Then, we get

$$D_{\triangle} = -\frac{1 + \alpha}{2k(k + \alpha)(k + 2\alpha)} \tag{175}$$

The sum of the remaining two terms is

$$D_{[II, I]} + D_{[N]} = -\frac{1}{k(k + \alpha)(k + 2\alpha)(k - 1)} \left[\frac{1}{2}(k - 1)(3 + \alpha) + \alpha(2 + \alpha)\right] \tag{176}$$

Since $D_{[II, I]}$ is a coefficient of the term which has a double line in the graphic representation, it should be proportional to the factor $(1 + \alpha)/2$, which corresponds to the double line multiple factor. Therefore, the sum of (176) is divided into two parts,

$$D_{[II, I]} = -\frac{1 + \alpha}{2k(k + \alpha)(k + 2\alpha)} \left[1 + \frac{2\alpha}{k - 1}\right]$$

$$= -\frac{1}{1 + 2\alpha} \left[\frac{1 + \alpha}{2Z_{[3]}(I)} + \frac{\alpha(1 + \alpha)}{Z_{[21]}(I)}\right] \tag{177}$$

$$D_{[N]} = -\frac{1}{k(k + \alpha)(k + 2\alpha)} \left[1 + \frac{\alpha}{k - 1}\right]$$

$$= -\frac{1}{(1 + 2\alpha)} \left[\frac{1 + \alpha}{Z_{[3]}(I)} + \frac{\alpha}{Z_{[21]}(I)}\right] \tag{178}$$

These two expressions coincide with the values of Table B by the duality transformation (158). Since there is a cubic identity equation for three terms $[\tau_1^2 \tau_{34}], [\tau_1 \tau_{23} \tau_{34}]$ and $[\tau_1 \tau_{23} \tau_{13}]$, the coefficients of these terms $D_{[II, I]}, D_{[N]}$ and $D_{[\triangle]}$ have ambiguities. We have to fix these ambiguities. Here we used a fixing assumption for the form of $D_{[II, I]}$, namely that it has $\frac{1 + \alpha}{2}$ as overall factor, and a pole at $k=1$.

The last coefficient $D_{[\triangle]}$ is not determined by the present choice of $X$ and $\Lambda$. One method to determine $D_{[\triangle]}$ is to use the residual equation,

$$D_{\Lambda} + (k - 2)D_{[N]} + D_{[\triangle]} + D_{[II, I]} + D_{\triangle} = 0 \tag{179}$$
Using the previous expressions, we obtain

\[
D_{[\triangle]} = -\frac{1}{k(k + \alpha)(k + 2\alpha)}[1 - \frac{\alpha^2}{k - 1}]
\]  

(180)

This reads to

\[
D_{[\triangle]} = -\frac{1}{1 + 2\alpha}\left[\frac{(1 + \alpha)^2}{Z_{[3]}(I)} - \frac{\alpha^2}{Z_{[2]}(I)}\right]
\]  

(181)

This expression coincides with the value in Table B after the duality transformation (158).

In Appendix D, we have derived the expressions of the coefficients \(D\) up to order fourth, from the extended zonal polynomial expansions. The results are shown in Table B.

\section{Series expansion for \(\beta = 4\)}

We have found in the previous section that the integral (1) may be expressed in two different dual ways,

\[
I_\beta = e^{\sum x_i \lambda_i - \frac{1}{k} \sum_{i<j} \tau_{ij}} \sum_{m=0}^{\infty} \frac{1}{m!} \prod_{q=0}^{m-1} (1 + q\alpha) \sum_p \chi_p(1) \frac{Z_p(\tilde{X})Z_p(\tilde{\Lambda})}{Z_p(I)}
\]  

(182)

and

\[
I_\beta = \sum_{\text{perm.}} \left[ \frac{e^{\sum x_i \lambda_i}}{\prod_{i<j} (x_i - x_j)(\lambda_i - \lambda_j)^{\beta-1}} \right] \times e^{-\frac{1}{k} \sum_{i<j} \tau_{ij}} \sum_{m=0}^{\infty} \frac{1}{m!} \prod_{q=0}^{m-1} (1 + q\alpha') \sum_p \chi_p(1) \frac{Z_p(\tilde{X})Z_p(\tilde{\Lambda})}{Z_p(I)}
\]  

(183)

where \(\tilde{x}_i = x_i - \frac{1}{k} \sum_{j=1}^{k} x_j\), and \(\alpha = \frac{2}{\beta}\), \(\alpha' = -1/(\frac{\beta}{2} - 1)\). The polynomials \(Z_p(x)\) are the extended zonal polynomials with parameters \(\alpha\) in (182) and \(\alpha'\) in (183), and \(\chi_p(1)\) are the characters.

In the previous section, the zonal polynomials were further expressed as polynomials in \(\tau_{ij}\) as (146) and (147).

The duality means that (182) and (183) give the same expression for the integral \(I_\beta\) under the relation of

\[
\alpha = \frac{2}{\beta}, \quad \alpha' = \frac{2}{2 - \beta}
\]  

(184)
or equivalently,
\[ \beta = \frac{2}{\alpha} = 2 - \frac{2}{\alpha'}. \]  
(185)

In the case \( \beta = 4, 6, ..., 2m \) for the even integers, \( \alpha' \) becomes \(-1, -\frac{1}{2}, ..., \frac{1}{m-1} \). The zonal polynomial for \( \alpha' = -1 \) was discussed before in a different context [16]. In such cases, the value of \( \alpha' \) is negative, and the factor in the denominator \( \prod_{q=0}^{m-1} (1 + q\alpha') \) in (183) vanishes, although the whole expression remains finite. Therefore, we need a special treatment for such \( \beta = 2m \) case, and we briefly discuss the case \( \beta = 4 \) here. The dimensional constants \( Z_p(I) \) become degenerate in such cases.

In the case \( \beta = 4 \), the parameter \( \alpha' \) is \(-1 \). The dimensional constants show the degeneracy; for instance, \( Z_{[3]}(I) = k(k - 1)(k - 2) \), and \( Z_{[13]}(I) = k(k - 1)(k - 2) \). Such degeneracies can be seen in the Table A. In addition to this degeneracy, the multiple line factors are proportional to \( (1 + \alpha) \), and there are no multiple line graphs in \( \beta = 4 \) in the representation of (183), and the expansion becomes rather simple for \( \beta = 4 \) as shown in [4, 5].

Here we study the zonal polynomial expansion in the form (183) for the \( \beta = 4 \) case, without writing it as a function of \( \tau \).

The expression (183) may be converted to more useful series in the case \( \alpha = -1 \). In the previous section, we have written it in terms of the symmetric functions \( s_n \) for general \( \alpha \). Although there is a divergent factor \( \frac{1}{1+\alpha} \) in (183), this divergence is cancelled by the sum of the partitions of \( p \). We investigate here the case \( \alpha = -1 (\beta = 4) \) in higher orders, and obtain a useful expression for the integral \( I \).

We define \( \phi(x, \lambda) \) as

\[
I_\beta = \sum_{\text{perm.}} \frac{e^{\sum x_i \lambda_i}}{\prod_{i<j} [(x_i - x_j)(\lambda_i - \lambda_j)]^{\beta-1}} e^{-\frac{1}{\alpha} \sum_{i<j} \tau_{ij} \phi(x, \lambda)} \]  
(186)

\[
\phi(x, \lambda) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{1}{\prod_{q=0}^{m-1} (1 + q\alpha)} \sum_{p} \chi_p(1) Z_p(\tilde{X}) Z_p(\tilde{\Lambda}) \]  
(187)

Note that we have \( f = e^{\frac{1}{\alpha} \sum \tau_{ij} \phi(x, \lambda)} \). Using the table of extended zonal polynomials and characters given in the appendix, we write the zonal polynomials in terms of the symmetric functions \( s_n \), and find the expression for \( \phi \), by noting that \( s_1(\tilde{x}) = 0 \).

\[
\phi(x, \lambda) = 1 + \frac{\alpha}{2(k + \alpha)(k - 1)} s_2(\tilde{x}) s_2(\tilde{\lambda})
\]
\[\phi = 1 - \frac{1}{2(k-1)^2} s_2(\bar{x}) s_2(\bar{\lambda}) + \frac{k}{3(k-1)^2(k-2)^2} s_3(\bar{x}) s_3(\bar{\lambda}) + \frac{\alpha^2 k}{8k(k-1)(k-2)(k-3)(k+\alpha-1)(k+\alpha)(k+2\alpha)(k+3\alpha)} \]

\times \left[2\alpha k^2 (k^2 + \alpha k - k + \lambda) s_4(\bar{x}) s_4(\bar{y}) + 2\alpha k (2k^2 + 3\alpha k - 3k - 3\alpha)(s_4(\bar{x}) s_2^2(\bar{y}) + s_2^2(\bar{x}) s_4(\bar{y})) + (k^4 + 5\alpha k^3 - 5k^3 + 6\alpha^2 k^2 - 18\alpha k^2 + 6k^2 - 18\alpha^2 k^2 + 18\alpha k + 18\alpha^2) s_2^2(\bar{x}) s_2^2(\bar{y}) \right]

\begin{align*}
&+ O(x^5) \\
&\Rightarrow \text{There is no divergence in the limit } \alpha \to -1 \text{ in this expression. Thus we obtain for } \beta = 4, \text{ from the table in the appendix, up to order six,}
\end{align*}

\[\phi = 1 - \frac{1}{2(k-1)^2} s_2(\bar{x}) s_2(\bar{\lambda}) + \frac{k}{3(k-1)^2(k-2)^2} s_3(\bar{x}) s_3(\bar{\lambda})
\]
\[
+ (s_3(\bar{x})^2 s_3(\bar{\lambda})^2) \frac{1}{18k}[-2400 - 1200 k + 7890 k^2
- 8310 k^3 + 4461 k^4 - 1416 k^5 + 264 k^6 - 26 k^7 + k^8]
+ (s_2(\bar{x})^3 s_6(\bar{\lambda}) + s_2(\bar{\lambda})^3 s_6(\bar{x})) \frac{1}{6}[240 - 769 k + 882 k^2 - 424 k^3 + 90 k^4 - 7 k^5]
+ (s_2(\bar{x})^3 s_3(\bar{\lambda}) + s_2(\bar{\lambda})^3 s_3(\bar{x})) \frac{1}{8k}[4800 - 11180 k + 11480 k^2
- 6775 k^3 + 2384 k^4 - 481 k^5 + 50 k^6 - 2 k^7]
+ s_2(\bar{x})^3 s_3(\bar{\lambda}) \frac{1}{3k}[-1200 + 2270 k
- 1725 k^2 + 640 k^3 - 115 k^4 + 8 k^5]
+ s_2(\bar{x})^3 \frac{1}{48k}[-25200 + 60960 k - 64030 k^2 + 38192 k^3
- 13976 k^4 + 3170 k^5 - 431 k^6 + 32 k^7 - k^8]
+ O(x^7) \tag{189}
\]

We first check the simple \( k = 3 \) case, given in the introduction. Multiplying by \( e^{-\frac{1}{3}(\tau_{12} + \tau_{23} + \tau_{13})} \) to \( \phi \), we recover the known result,

\[
e^{-\frac{1}{3}(\tau_{12} + \tau_{23} + \tau_{13})} \phi = 1 - \frac{1}{3}(\tau_{12} + \tau_{23} + \tau_{13})
+ \frac{1}{6}(\tau_{12}\tau_{23} + \tau_{23}\tau_{13} + \tau_{13}\tau_{12})
- \frac{1}{12}\tau_{12}\tau_{23}\tau_{13} \tag{190}
\]

where the last term is obtained from the identity

\[
-\frac{1}{3!3^3}(\tau_{12} + \tau_{23} + \tau_{13})^3 + \frac{1}{24}(\tau_{12} + \tau_{23} + \tau_{13})s_2(\bar{x})s_2(\bar{\lambda})
+ \frac{1}{4}s_3(\bar{x})s_3(\bar{\lambda}) = -\frac{1}{12}\tau_{12}\tau_{23}\tau_{13}. \tag{191}
\]

The fourth order term in (189) has a divergent coefficient \( \frac{1}{(k-3)^2} \). However the fourth order term is finite in the limit \( k \to 3 \), and with the exponential term, it is vanishing at the end. In the fifth order, the same situation occurs. Therefore, we have recovered exactly (190) to all orders.

The expression (189) is lengthy, however its large \( k \) limit becomes simple. In the large \( k \) limit, for each order \( x^l \), the coefficients of the products of the symmetric functions are of order \( \frac{1}{k^l} \). We take these leading terms,

\[
\phi \sim 1 - \frac{1}{2k^2}s_2(\bar{x})s_2(\bar{\lambda}) + \frac{1}{3k^3}s_3(\bar{x})s_3(\bar{\lambda})
\]

42
\[- \frac{1}{4k^4}s_4(\tilde{x})s_4(\tilde{\lambda}) + \frac{1}{8k^4}s_2(\tilde{x})^2s_2(\tilde{\lambda})^2 \\
+ \frac{1}{5k^5}s_5(\tilde{x})s_5(\tilde{\lambda}) + \frac{1}{6k^5}s_2(\tilde{x})s_3(\tilde{x})s_2(\tilde{\lambda})s_3(\tilde{\lambda}) \\
- \frac{1}{6k^6}s_6(\tilde{x})s_6(\tilde{\lambda}) + \frac{1}{8k^6}s_2(\tilde{x})s_4(\tilde{x})s_2(\tilde{\lambda})s_4(\tilde{\lambda}) + \frac{1}{18k^6}s_3(\tilde{x})^2s_3(\tilde{\lambda})^2 \\
- \frac{1}{48k^6}s_2(\tilde{x})^3s_2(\tilde{\lambda})^3 \\
+ O(x^7) \tag{192}\]

There is a rule for the coefficients in (192). Only the combination of the same symmetric functions for \(\tilde{x}\) and \(\tilde{\lambda}\) give the leading terms in the large \(k\) limit. For instance, \(s_2(\tilde{x})^3s_2(\tilde{\lambda})\) and \(s_2(\tilde{x})s_4(\tilde{x})s_2(\tilde{\lambda})s_4(\tilde{\lambda})\) are of order \(\frac{1}{k^6}\). The coefficient of \(s_2(\tilde{x})^3s_3(\tilde{\lambda})^2\), which has different symmetric functions, is of order \(\frac{1}{k^9}\).

The coefficients of (317) is obtained as follows. For the term of \(s_n(\tilde{x})^ps_m(\tilde{x})^t\), \((n \neq m)\), the coefficient becomes

\[ C = (-1)^{n_p+m_t} \frac{1}{p!n_p!m_t!} \frac{1}{k^{n_p+m_t}} \tag{193} \]

For the general case, \(s_{n_1}^{p_1}s_{n_2}^{p_2} \cdots s_{n_j}^{p_j}\), the coefficient is proportional to \(\frac{1}{p_1!p_2! \cdots p_j!n_1^{p_1}n_2^{p_2} \cdots n_j^{p_j}}\). The order of \(\tilde{x}\) is given by \(n_1^{p_1} + \cdots + n_j^{p_j}\), and if this order is even, the minus sign has to be included. Thus, we have the large \(k\) expression for \(\phi\).

In the large \(k\) limit for fixed \(x_i, \lambda_j\), \(f\) is given by \(e^{-\frac{1}{k} \sum \tau_{ij}}\), and \(\phi(x, \lambda) = 1\), for all values of \(\alpha\), as shown in Appendix D. However, in some problems, in which \(s_n(\tilde{x})\) is not order of one, the above large \(k\) formula might be important.

### 9 The HIZ-integral for \(\beta = 2\) and the character expansion

We use here the same zonal polynomial method as for \(\beta = 1\). We define \(\tilde{x}_a = x_a - \frac{1}{k} \sum_{b=1}^{k} x_b\). The integral becomes (\(\alpha = \frac{2}{\beta} = 1\)) from the expression (146),

\[ I = e^{\sum x_i \lambda_j - \frac{1}{k} \sum \tau_{ij}} \left[ \sum_m \sum_{p(m)} \frac{1}{m! \prod_{q=0}^{m-1} (1 + q)} \frac{Z_p(\tilde{x})Z_p(\tilde{\lambda})}{Z_p(I)} \right] \tag{194} \]
This series expansion has to reduce to the simple closed form

$$I_{\beta=2} = \sum_{\text{perm. of } \lambda_i} \frac{e^{\sum_{i=1}^k x_i \lambda_i}}{\prod_{i<j} \tau_{ij}}$$

(195)

in order to reproduce the original HIZ formula [11, 12]..

In other words one must prove the identity,

$$\sum_{\text{perm.}} \frac{e^{\sum \tau_{ij}}}{\prod \tau_{ij}} = \sum_m \sum_p \frac{1}{m!} \prod_{q=0}^{m-1} \frac{1}{(1+q)} Z_p(\tilde{x}) Z_p(\tilde{\lambda})$$

(196)

The proof of this identity is easily done by writing the zonal polynomials, which are Shur functions, as the ratio of the determinants,

$$Z_p(x) = \frac{\det[x_i^j]}{\det[x_i^{j-1}]}$$

(197)

The product of the determinants is also a determinant, it is simply $\det[e^{x_i \lambda_j}]$, from the Binet-Cauchy theorem [18].

This identity leads to interesting equations. The right hand side of (196) is written by the terms of $\tau$, as shown in the case of $\beta = 1$.

For the case $k=2$, the proposition (196) is easily proved by expanding the exponent. For $k=3$, we have the following identity: for instance,

$$\sum_{\text{perm.}} \frac{(\tau_{12} + \tau_{23} + \tau_{13})^p}{\tau_{12} \tau_{23} \tau_{13}} = 0$$

(198)

for $p=1,2$ and $4$. Let us remind the reader here, that the permutations in this sum are interchanges of the $\lambda_i$’s for fixed $x_j$. For $p=3$, we have

$$\sum_{\text{perm.}} \frac{(\tau_{12} + \tau_{23} + \tau_{13})^3}{\tau_{12} \tau_{23} \tau_{13}} = 81$$

(199)

For $p=5$, it becomes

$$\sum_{\text{perm.}} \frac{(\tau_{12} + \tau_{13} + \tau_{23})^5}{\tau_{12} \tau_{23} \tau_{13}} = \frac{3645}{4} s_2(\tilde{x}) s_2(\tilde{\lambda})$$

$$= 405 [(\tau_{12}^2 + \tau_{23}^2 + \tau_{13}^2) - (\tau_{12} \tau_{23} + \tau_{12} \tau_{13} + \tau_{23} \tau_{13})]$$

(200)

where the second equality comes from the expression of $s_2(\tilde{x})$ of (95). For $p=6$, we have

$$\sum_{\text{perm.}} \frac{(\tau_{12} + \tau_{13} + \tau_{23})^6}{\tau_{12} \tau_{23} \tau_{13}} = (81)^2 s_3(\tilde{x}) s_3(\tilde{\lambda})$$

(201)
For \( p=7 \),
\[
\sum_{\text{perm.}} \frac{(\tau_{12} + \tau_{13} + \tau_{23})^7}{\tau_{12}\tau_{23}\tau_{13}} \tau_{12}\tau_{23}\tau_{13} = 1701\left(\frac{3}{2}\right)^4[s_2(\bar{x})s_2(\bar{\lambda})]^2
\]
where for \( k=3 \), we have \( s_4(\bar{x}) = \frac{1}{2}[s_2(\bar{x})]^2 \). For \( p=8 \),
\[
\sum_{\text{perm.}} \frac{(\tau_{12} + \tau_{13} + \tau_{23})^8}{\tau_{12}\tau_{23}\tau_{13}} = 648 \times \left(\frac{27}{2}\right)^2[s_2(\bar{x})s_3(\bar{x})s_2(\bar{\lambda})s_3(\bar{\lambda})]^2.
\]

From these results, one checks the identity (196).

To understand these identities, we divide the left hand side of (200) into five different types of terms. They are all represented by \( s_2(\bar{x})s_2(\bar{\lambda}) \).

\[
\begin{align*}
\sum_{\text{perm.}} \tau_{12}^5 + \tau_{13}^5 + \tau_{23}^5 & = \frac{450}{4}s_2(\bar{x})s_2(\bar{\lambda}) \\
\sum_{\text{perm.}} \tau_{12}(\tau_{13} + \tau_{23}) + \cdots & = \frac{225}{4}s_2(\bar{x})s_2(\bar{\lambda}) \\
\sum_{\text{perm.}} (\tau_{12}^2 + \tau_{23}^2 + \tau_{13}^2) & = 18s_2(\bar{x})s_2(\bar{\lambda}) \\
\sum_{\text{perm.}} (\tau_{12}^3 + \cdots) & = \frac{9}{4}s_2(\bar{x})s_2(\bar{\lambda}) \\
\sum_{\text{perm.}} (\tau_{12}\tau_{23} + \cdots) & = \frac{9}{2}s_2(\bar{x})s_2(\bar{\lambda})
\end{align*}
\]

After summing over permutations, the \( \tau \)-series are expressible by symmetric functions of \( \bar{x} \) and \( \bar{\lambda} \), and are given by homogeneous polynomials in \( \tau \).

We thus have shown that the zonal (character) polynomial series can give the expression for \( f \), although \( f \) is just one.

The dual representation of (8) is singular for \( \beta = 2 \). The transformation \( \alpha = \frac{2}{2-\beta} \) becomes divergent at \( \beta = 2 \). However, if we write \( f \) as a \( \tau \) expansion for fixed \( k \), and let \( \alpha \to \infty \), we have an infinite series, as can be seen in Table B. There are non-vanishing terms in the limit \( \alpha = \infty \). For instance, at first order, \( C_1 = \frac{-1}{k} \). This looks quite strange, since we know that \( f \) is one. In the following, we consider this puzzling problem.

Collecting the non-vanishing terms in the limit \( \alpha \to \infty \) from Table B, we find the following expression,

\[
I_{\beta=2} = \sum_{\text{perm.of}\lambda} \frac{1}{\prod \tau_{ij}} e^{\sum x_i\lambda_i} \left[ 1 - \frac{1}{k}(\tau_{12} + \cdots) + \frac{1}{k(k-1)}(\tau_{12}\tau_{34} + \cdots) + \frac{1}{2k}(\tau_{12}^2 + \cdots) + O(x^3) \right].
\]
According to this expression, \( f_{\beta=2} \) is not one. This discrepancy may be understood by looking at the simple example \( k=2 \). In this \( k=2 \) case, we have

\[
I_{\beta=2} = \sum_{\text{perm. of } \lambda_i} \frac{e^{x_1 \lambda_1 + x_2 \lambda_2}}{\tau_{12}} (1 - \frac{1}{2} \tau_{12} + \frac{1}{4} \tau_{12}^2 + \cdots)
\]

This expansion is divided into two parts;

\[
I_{\beta=2} = \frac{e^{x_1 \lambda_1 + x_2 \lambda_2} - e^{x_1 \lambda_2 + x_2 \lambda_1}}{\tau_{12}} (1 - \frac{1}{2} \tau_{12} + \frac{1}{4} \tau_{12}^2 + \cdots) - \frac{e^{x_1 \lambda_2 + x_2 \lambda_1}}{\tau_{12}} (1 + \frac{1}{2} \tau_{12} + \frac{1}{4} \tau_{12}^2 + \cdots)
\]

Remarkably, if one expands this expression in powers of the \( x \)'s and the \( \lambda \)'s, one finds cancellations between the first and second term between brackets, and agreement with (195).

Thus we find that the limit \( \beta = 2 \) is somehow singular, and the \( \tau \)-expansion, with the coefficients of Table B for \( \beta = 2 \), gives one for the value of the function \( f_{\beta=2} \) after summing over the permutations.

We have thus recovered the ordinary HIZ formula of (195). The \( \tau \)-expansion for \( \beta = 2 \) of (205) may be useful for a check of the coefficients for general \( \beta \), since it should reduce to one for \( f_{\beta=2} \).

10 Expansion of \( f_\beta \) for large \( \beta \), and for large \( k \) and fixed \( \beta \)

We note that the final result for \( f_\beta \) in the large \( \beta \) limit is simple. The coefficients \( C \) involve the products of the inverse of the multiplicity \( l! \). For instance, the term of \( \tau_{12}^2 \tau_{24} \tau_{13} \tau_{34} \) has a coefficient as

\[
\frac{(-1)^8}{2!3!4!k^8} \tau_{12}^2 \tau_{24} \tau_{13} \tau_{34}
\]

Namely \( f_\beta \) is

\[
f_\beta = \sum \frac{1}{l_1 ! l_2 ! \cdots} \frac{1}{k^n} (-1)^n \prod \tau_{ij}
\]

where \( n \) is the total number of the bonds \( \tau_{ij} \), and \( l_j \) is the multiplicity. The expression (209) is simply equal to

\[
f_\beta = e^{-\frac{1}{k} \sum \tau_{ij}}
\]

46
This formula is valid for large $\beta$ and for arbitrary $k$. It may be interesting to refine above expression. For fixed $\beta$ and in the large $k$ limit, we have an expansion in powers of $\tau$. We may thus expand (8) for large $k$. $f$ is given by

$$f_{\beta} = e^{-\frac{1}{k} \sum_{i<j} \tau_{ij}} \sum_{m=0}^{\infty} \frac{1}{m!} \left( \prod_{q=0}^{m-1} (1 + q\alpha) \right) \sum_p \chi_p(1) \frac{Z_p(\tilde{X}) Z_p(\tilde{\Lambda})}{Z_p(I)}$$  \hspace{1cm} (211)$$

and as shown in Appendix D, or shown in the Table B, it is transformed into a $\tau$-series. We find that $f_{\beta}$ is given by

$$f_{\beta} = \prod_{i<j}^{k} [1 - \frac{\tau_{ij}}{\left(\frac{k}{2} - 1\right)k} \beta^{-1}]$$ \hspace{1cm} (212)$$

Of course it reduces again to (210) in the large $\beta$ limit. The above expression (212) automatically satisfies the condition that the series of $f_{\beta}$ stops at the order $\frac{\beta}{2} - 1$ in $\tau$ when $\beta$ is an even integer. Thus we have obtained the improved expression of (212) for $f_{\beta}$, which is valid in the large $\beta$ limit or in the large $k$ limit. This asymptotic form (212) may be useful for finite fixed $\beta$ (for instance, $\beta = 1$) and large $k$.

In the Appendix D, the large $k$ limit of the coefficients $C$ at $l$-th order is investigated. They are given by

$$C = (-1)^l \frac{g}{\prod_{m=0}^{l-1} (k + m\alpha)} (1 + O(\frac{\alpha}{k}))$$ \hspace{1cm} (213)$$

where $g$ is a degeneracy factor for the multiple lines. It is remarkable that the only dimensional factor which appears in the large $k$ limit, is the first row of the Young tableau, which has the form of $(k + m\alpha)$. For instance $(k - 1),(k + \alpha - 1),...$, which are the second row factors, do not appear in the large $k$ limit.

11 Summary and discussions

In this article, we have given the expression of the integral (1) as series in the variables $\tau_{ij} = (x_i - x_j)(\lambda_i - \lambda_j)$ for the function $f$ defined in (2). As discussed in section seven, from the extended zonal polynomial expansion of the integral $I$, and the use of the dual representation of (158), we have obtained expressions for this same function $f$. The coefficients $D_{[\text{graph}]}$ of this $\tau$ expansion are expressed through the dimensional constant $Z_{[p]}(I)$ of the extended zonal polynomials (Jack polynomials). The results for these $C_{[\text{graph}]}$ coincide with the direct perturbational calculations developed in the section five, where the residual equations among various coefficients $C_{[\text{graph}]}$ are used. It is remarkable that these recursive residual equations are independent of the...
parameter $\beta$. We have found the explicit expression for the WKB expansion by this duality.

In this paper, we have proved that the extended Harish-Chandra-Itzykson-Zuber integral for the general $\beta$ case is expressed by the variables of $\tau_{ij}$. The proof is given in two stages; the first is the expression of the Harish-Chandra-Itzykson-Zuber integral by the zonal polynomial expansion with the parameter $\alpha$, which is $\frac{2}{2-\beta}$ by the duality. The second is the transformations of the products of the symmetric functions $s_n(\tilde{x})$ in the zonal polynomial expansion into the $\tau$ variables. This transformation is discussed in Appendix B, and in Appendix D in detail. We found that there are identities among the $\tau$ terms. For instance, we have found explicitly these cubic and quartic identities in the appendix C. These identities give a sort of gauge freedom to choose the values of the coefficients of the $\tau$ expansion. We have considered a fixing of these ambiguities from the large $k$ behavior by imposing definite asymptotic forms. For the $\beta = 4$ case, only single line graphs appear, and for this reason, there are no ambiguities for the coefficients $C$, which are uniquely determined.

The integral (1) is important for the investigation of the random matrix theory, specially in the presence of an external matrix source, as shown for the $\beta = 2$ in [3, 7, 8, 9]. We will discuss in a separate paper the applications of the present results [10].

Acknowledgements
We thank Dr. A. Okounkov who pointed out us the Baker-Akhiezer formula by Berest. S.H. is supported by a Grant-in-Aid for Scientific Research (B) by JSPS.

Appendix A: Extended zonal polynomials (Jack polynomials)

We have uses the expansions of the extended zonal polynomials (named Jack polynomials) in section seven, and we have derived the expression for $f$ from the extended zonal polynomial expansion of the HCIZ-integral $I_\beta$ by duality. Therefore, in this appendix, we give the needed important quantities, the characters $\chi_p(1)$, and the dimensional constants $Z_p(I)$ [13, 14, 16].

The lower Jack symmetric polynomials $Z_{[p]}(X)$ and their dimensions $Z_{[p]}(I)$ are

\begin{align*}
Z_{[1]}(X) &= s_1, \quad Z_{[1]}(I) = k, \quad \chi_{[1]}(1) = 1 \\
Z_{[2]}(X) &= s_1^2 + \alpha s_2, \quad Z_{[2]}(I) = k(k + \alpha), \quad \chi_{[2]}(1) = 1 \\
Z_{[1^2]}(X) &= s_1^2 - s_2, \quad Z_{[1^2]}(I) = k(k - 1), \quad \chi_{[1^2]}(1) = \chi_{[1^2]}(1) = \alpha \\
Z_{[3]}(X) &= s_1^3 + 3\alpha s_1 s_2 + 2\alpha^2 s_3,
\end{align*}
\[ Z_{[3]}(I) = k(k + \alpha)(k + 2\alpha), \quad \chi_{[3]}(1) = 1 \]

\[ Z_{[21]}(X) = s_1^3 + (\alpha - 1)s_1s_2 - \alpha s_3, \]

\[ Z_{[21]}(I) = k(k + \alpha)(k - 1), \quad \chi_{[21]}(1) = \frac{6\alpha(1 + \alpha)}{2 + \alpha} \]

\[ Z_{[1^3]}(X) = s_1^3 - 3s_1s_2 + 2s_3, \quad Z_{[1^3]}(I) = k(k - 1)(k - 2) \]

\[ \chi_{[1^3]}(1) = \frac{\alpha^2(1 + 2\alpha)}{2 + \alpha} \] (214)

where \( \alpha = \beta / 2 \). The classical symmetric functions are denoted by \( s_n, s_n = \sum x_i^n \).

Then one has the relation,

\[ (\text{tr}X)^q = \frac{1}{\prod_{m=1}^{q-1}(1 + m\alpha)} \left[ \sum_p \chi_p(1)Z_p(x) \right] \] (215)

where \( p \) is a partition of the integer \( q \), when \( \alpha = 2/\beta = 2 \).

The dimensional constants \( Z_p(I) \) are obtained by putting \( X = I \) in the zonal polynomials \( Z_p(X) \). They are factorized as a polynomial in \( k \).

From the constants given in (214), and the sum rule (215), we find the HIZ-integral for general \( \beta \) :

\[ I_{\beta} = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{1}{\prod_{q=0}^{m-1}(1 + q\alpha)} \sum_p \chi_p(1) \frac{Z_p(X)Z_p(\Lambda)}{Z_p(I)} \] (216)

If \( \Lambda = I \), it becomes

\[ I_{\beta} = e^{\text{tr}X} \] (217)

which is the correct expression by definition. The values in the following tables of coefficients of the zonal polynomials come from [14] (with a minor correction). The characters \( \chi_p(1) \), are then evaluated on the basis of these values. It agrees with [15] when \( \alpha = 2 \) up to sixth order.

---

The extended zonal polynomial (Jack polynomial) \( Z_p(x) \) with a parameter \( \alpha \) and its coefficient of the symmetric functions \( \chi_p(1) \) is a character and \( Z_p(I) \) is a dimensional constant.

| \( l \) | \( s_1 \) | \( \chi_p(1) \) | \( Z_p(I) \) |
|-------|--------|----------------|------|
| \( Z_{[1]} \) | 1 | 1 | \( k \) |
\( l = 2 \)

| \( s^2_i \) | \( s_2 \) | \( \chi_p(1) \) | \( Z_p(1) \) |
|--------|--------|----------------|----------------|
| \( Z_{[2]} \) | 1 | \( \alpha \) | 1 | \( k(k + \alpha) \) |
| \( Z_{[12]} \) | 1 | -1 | \( \alpha \) | \( k(k - 1) \) |

\( l = 3 \)

| \( s_1^2 \) | \( s_1 s_2 \) | \( s_3 \) | \( \chi_p(1) \) | \( Z_p(1) \) |
|--------|--------|--------|----------------|----------------|
| \( Z_{[3]} \) | 1 | 3 \( \alpha \) | 2 \( \alpha^2 \) | \( \frac{1}{6 \alpha (1 + \alpha)} \) | \( k(k + \alpha)(k + 2\alpha) \) |
| \( Z_{[2,1]} \) | 1 | \( \alpha - 1 \) | \( -\alpha \) | \( \frac{6 \alpha^2 (1 + 3\alpha)}{(1 + \alpha)(2 + \alpha)} \) | \( k(k + \alpha)(k - 1) \) |
| \( Z_{[13]} \) | 1 | -3 | 2 | \( \frac{6 \alpha^2 (1 + 3\alpha)}{(1 + \alpha)(2 + \alpha)} \) | \( k(k - 1)(k - 2) \) |

\( l = 4 \)

| \( s_1^4 \) | \( s_1^3 s_2 \) | \( s_2^2 \) | \( s_1 s_3 \) | \( s_4 \) | \( \chi_p(1) \) |
|--------|--------|--------|--------|--------|----------------|
| \( Z_{[4]} \) | 1 | 6 \( \alpha \) | 3\( \alpha^2 \) | 8\( \alpha^2 \) | 6\( \alpha^3 \) | \( \frac{1}{6 \alpha (1 + 2\alpha)(1 + 3\alpha)} \) |
| \( Z_{[3,1]} \) | 1 | 3 \( \alpha - 1 \) | \( -\alpha \) | 2\( \alpha^2 - 2\alpha \) | \( -2\alpha^2 \) | \( \frac{6 \alpha^2 (1 + 3\alpha)}{(1 + \alpha)(2 + \alpha)(3 + \alpha)} \) |
| \( Z_{[22]} \) | 1 | 2\( \alpha - 2 \) | \( \alpha^2 + \alpha + 1 \) | \( -4\alpha \) | \( \alpha - \alpha^2 \) | \( \frac{6 \alpha^2 (1 + 2\alpha)(1 + 3\alpha)}{(1 + \alpha)(3 + \alpha)} \) |
| \( Z_{[212]} \) | 1 | \( \alpha - 3 \) | \( -\alpha \) | \( 2 - 2\alpha \) | \( 2\alpha \) | \( \frac{6 \alpha^2 (1 + 3\alpha)}{(1 + \alpha)(3 + \alpha)} \) |
| \( Z_{[14]} \) | 1 | -6 | 3 | 8 | \( -6 \) | \( \frac{6 \alpha^2 (1 + 3\alpha)}{(1 + \alpha)(3 + \alpha)} \) |

\( Z_p(1) \)

| \( Z_{[4]} \) | \( k(k + \alpha)(k + 2\alpha)(k + 3\alpha) \) |
| \( Z_{[3,1]} \) | \( k(k + \alpha)(k + 2\alpha)(k - 1) \) |
| \( Z_{[22]} \) | \( k(k + \alpha)(k + \alpha - 1)(k - 1) \) |
| \( Z_{[212]} \) | \( k(k + \alpha)(k - 1)(k - 2) \) |
| \( Z_{[14]} \) | \( k(k - 1)(k - 2)(k - 3) \) |

\( l = 5 \)

| \( s_1^5 \) | \( s_1^4 s_2 \) | \( s_1^3 s_2^2 \) | \( s_1^2 s_3 \) | \( s_2 s_3 \) | \( s_1 s_4 \) | \( s_5 \) |
|--------|--------|--------|--------|--------|--------|--------|
| \( Z_{[5]} \) | 1 | 10 \( \alpha \) | 15\( \alpha^2 \) | 20\( \alpha^2 \) | 20\( \alpha^3 \) | 30 \( \alpha^3 \) | 24 \( \alpha^4 \) |
| \( Z_{[4,1]} \) | 1 | 6\( \alpha - 1 \) | 3\( \alpha(\alpha - 1) \) | \( \alpha(8\alpha - 3) \) | \( -5\alpha^2 \) | \( 6\alpha^2 (\alpha - 1) \) | \( -6\alpha^3 \) |
| \( Z_{[32]} \) | 1 | 2(\( \alpha - 1 \)) | 3\( \alpha^2 - \alpha + 1 \) | \( 2\alpha(\alpha - 3) \) | \( 2\alpha (\alpha^2 + 1) \) | \( -\alpha(7\alpha - 1) \) | \( -2\alpha^2 (\alpha - 1) \) |
| \( Z_{[31]} \) | 1 | 3(\( \alpha - 1 \)) | -5\( \alpha \) | \( 2(\alpha - 1)^2 \) | \( -2(\alpha - 1)^2 \) | \( -4\alpha (\alpha - 1) \) | \( 4\alpha^2 \) |
| \( Z_{[22,1]} \) | 1 | 2(\( \alpha - 2 \)) | \( \alpha^2 - \alpha + 3 \) | \( -2(3\alpha - 1) \) | \( -2(\alpha^2 + 1) \) | \( -\alpha (\alpha - 7) \) | \( 2\alpha (\alpha - 1) \) |
| \( Z_{[21]} \) | 1 | \( \alpha - 6 \) | -3\( \alpha - 1 \) | \( -3(\alpha - 8) \) | \( 5\alpha \) | \( 6(\alpha - 1) \) | \( -6\alpha \) |
| \( Z_{[15]} \) | 1 | -10 | 15 | 20 | -20 | -30 | 24 |
\[
\begin{array}{|c|c|c|}
\hline
Z_{[5]} & 1 & k(k + \alpha)(k + 2\alpha)(k + 3\alpha)(k + 4\alpha) \\
Z_{[4,1]} & 20\alpha(1+3\alpha) & k(k + \alpha)(k + 2\alpha)(k + 3\alpha)(k - 1) \\
Z_{[32]} & 30\alpha^2(1+4\alpha) & k(k + \alpha)(k + 2\alpha)(k - 1)(k + \alpha - 1) \\
Z_{[31]^2} & 30\alpha^2(1+2\alpha)(1+3\alpha)(1+4\alpha) & k(k + \alpha)(k + 2\alpha)(k - 1)(k - 2) \\
Z_{[22,1]} & 30\alpha(1+3\alpha)(1+3\alpha)(1+4\alpha) & k(k + \alpha)(k - 1)(k + \alpha - 1)(k - 2) \\
Z_{[21]^3} & 20\alpha(1+2\alpha)(1+3\alpha)(1+4\alpha) & k(k + \alpha)(k - 1)(k - 2)(k - 3) \\
Z_{[15]} & \alpha^4(1+2\alpha)(1+3\alpha)(1+4\alpha) & k(k - 1)(k - 2)(k - 3)(k - 4) \\
\hline
\end{array}
\]

\[l=6\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
Z_{[6]} & 1 & 15 & 45\alpha^2 & 15\alpha^3 & 40\alpha^2 \\
Z_{[5,1]} & 1 & 10\alpha - 1 & 3\alpha(5\alpha - 2) & -3\alpha^2 & 4\alpha(5\alpha - 1) \\
Z_{[32]} & 1 & 7\alpha - 2 & 9\alpha^2 - 5\alpha + 1 & \alpha(3\alpha^2 + \alpha + 1) & 8\alpha(\alpha - 1) \\
Z_{[31]} & 1 & 3(2\alpha - 1) & 3(3\alpha^2 - \alpha + 1) & -(5\alpha^2 + 3\alpha + 1) & 4\alpha(\alpha - 3) \\
Z_{[42]} & 1 & 6\alpha - 3 & 3\alpha(\alpha - 4) & -3\alpha^2 & 2(4\alpha^2 - 3\alpha + 1) \\
Z_{[31^2]} & 1 & 4(\alpha - 1) & 3(\alpha - 1)^2 & -\alpha(\alpha - 1) & 2\alpha^2 - 9\alpha + 2 \\
Z_{[31^3]} & 1 & 3\alpha & -3(4\alpha - 1) & 3\alpha & 2(\alpha^2 - 3\alpha + 4) \\
Z_{[23]} & 1 & 3(\alpha - 2) & 3\alpha^2 - \alpha + 3 & \alpha(\alpha^2 + 3\alpha + 5) & -4(3\alpha - 1) \\
Z_{[21^2]} & 1 & 2\alpha - 7 & \alpha^2 - 5\alpha + 9 & -(\alpha^2 + \alpha + 3) & -8(\alpha - 1) \\
Z_{[21]} & 1 & \alpha - 10 & -3(2\alpha - 5) & 3\alpha & -4(\alpha - 5) \\
Z_{[16]} & 1 & -15 & 45 & -15 & 40 \\
\hline
\end{array}
\]

(continued)

\[
\begin{array}{|c|c|c|c|}
\hline
Z_{[6]} & 120\alpha^3 & 40\alpha^4 & 90\alpha^3 & 90\alpha^4 \\
Z_{[5,1]} & 20\alpha^2(\alpha - 1) & -8\alpha^3 & 6\alpha^2(5\alpha - 2) & -18\alpha^3 \\
Z_{[42]} & 4\alpha(2\alpha - 1)(\alpha - 1) & -2\alpha^2(\alpha - 1) & \alpha(6\alpha^2 - 17\alpha + 1) & \alpha^2(6\alpha^2 - \alpha + 5) \\
Z_{[32]} & 12\alpha(\alpha^2 + 1) & 2\alpha^2(2\alpha^2 + 3\alpha + 3) & -3\alpha(7\alpha - 1) & -3\alpha(4\alpha^2 + \alpha + 1) \\
Z_{[41^2]} & -6\alpha(3\alpha - 1) & 4\alpha^2 & 6\alpha(\alpha - 1)^2 & -6\alpha^2(\alpha - 1) \\
Z_{[31^2]} & (\alpha - 1)(\alpha - 2)(2\alpha - 1) & -\alpha(2\alpha^2 + \alpha + 2) & -9\alpha(\alpha - 1) & -2\alpha(\alpha - 1)^2 \\
Z_{[31^3]} & -6\alpha(\alpha - 3) & 4\alpha^2 & -6(\alpha - 1)^2 & 6\alpha(\alpha - 1) \\
Z_{[23]} & -12(\alpha^2 + 1) & 2(3\alpha^2 + 3\alpha + 2) & -3\alpha(\alpha - 7) & -3\alpha(\alpha^2 + \alpha + 4) \\
Z_{[21^2]} & -4(\alpha - 1)(\alpha - 2) & 2\alpha(\alpha - 1) & -(\alpha^2 - 17\alpha + 6) & 5\alpha^2 - \alpha + 6 \\
Z_{[21]} & 20(\alpha - 1) & -8\alpha & 6(2\alpha - 5) & -18\alpha \\
Z_{[16]} & -120 & 40 & -90 & 90 \\
\hline
\end{array}
\]

(continued)
Appendix B: The transformation of the paired products of symmetric functions $s_n(\bar{x})$ (power sum) to $\tau$-polynomials

We consider the transformation of the paired products of the classical symmetric function $s_n(\bar{x})$, and $s_n(\bar{\lambda})$ to the $\tau$-polynomials, where $\bar{x}_i = x_i - \frac{1}{k} \sum x_j$, and $\tau_{ij} = (x_i - x_j) (\lambda_i - \lambda_j)$, and $s_n(\bar{x}) = \sum \bar{x}_i^n$.

At order two, $s_2(\bar{x}) s_2(\bar{\lambda})$ is expressed in terms of $\tau_{ij}$ by

$$s_2(\bar{x}) s_2(\bar{\lambda}) = \sigma_{II} \cdot [\tau_{12}^2] + \sigma_{I1} \cdot [\tau_{12} \tau_{13}] + \sigma_{I1} \cdot [\tau_{12} \tau_{34}] \quad (218)$$

where the coefficients $\sigma$ are functions of $k$. We apply the differential operators $D_{k,\lambda}^{ij} = \frac{\partial^j}{\partial x_i \partial x_j \partial \lambda_i \partial \lambda_j}$ on both sides of the above equation. We obtain

$$D^{11}_{2,2} [\tau_{12}^2] = 4, \quad D^{11}_{2,2} [\tau_{12} \tau_{13}] = D^{11}_{2,2} [\tau_{12} \tau_{34}] = 0 \quad (219)$$

52
We use the notation $[\tau_{12}^2]$ for $\sum_{i<j} \tau_{ij}^2$, $[\tau_{12}\tau_{13}]$ for $\sum_{i<j<k} \tau_{ij}\tau_{ik}$, etc. For the symmetric function $s_2(\bar{x})s_2(\bar{\lambda})$, we have

$$D_{2,2}^{1,1}(s_2(\bar{x})s_2(\bar{\lambda})) = \frac{4(k-1)^2}{k^2}$$ (220)

Therefore, we obtain the coefficient of $[\tau_{12}^2]$ as $\sigma_{II} = \frac{(k-1)^2}{k^2}$. Similarly, we apply $D_{2,3}^{1,1}$ and $D_{2,4}^{1,1}$, and then we find the coefficients of $[\tau_{12}\tau_{13}]$ and $[\tau_{12}\tau_{34}]$,

$$D_{2,3}^{1,1}[\tau_{12}^2] = 0, \quad D_{2,3}^{1,1}[\tau_{12}\tau_{13}] = 2, \quad D_{2,3}^{1,1}[\tau_{12}\tau_{34}] = 0, \quad D_{2,3}^{1,1}(s_2(\bar{x})s_2(\bar{\lambda})) = -\frac{4(k-1)}{k^2}$$ (221)

These equations give the coefficient of $[\tau_{12}\tau_{13}]$ as $\sigma_{II} = -\frac{2(k-1)}{k^2}$.

$$D_{2,4}^{1,1}[\tau_{12}^2] = 0, \quad D_{2,4}^{1,1}[\tau_{12}\tau_{13}] = 0, \quad D_{2,4}^{1,1}[\tau_{12}\tau_{34}] = 2, \quad D_{3,4}^{1,1}(s_2(\bar{x})s_2(\bar{\lambda})) = \frac{4}{k^2}$$ (222)

These equations give the coefficient of $[\tau_{12}\tau_{34}]$ as $\sigma_{1,1} = \frac{2}{k^2}$. These results give (95).

We have examined all possible differential operators of order two for (218); they are $D_{11}^{11}, D_{12}^{11}, D_{12}^{12}, D_{11}^{11}, D_{23}^{11}, D_{13}^{12}, D_{34}^{12}$. These operators confirm the equation (218) with the coefficients determined hereabove. We have thus established the identity (218).

At order three, we transform $s_3(\bar{x})s_3(\bar{\lambda})$ to $\tau$-polynomials.

$$s_3(\bar{X})s_3(\bar{\lambda}) = \sigma_{III} \cdot [\tau_{12}^3] + \sigma_{\bar{\phi}} \cdot [\tau_{12}^2\tau_{13}] + \sigma_{\Delta} \cdot [\tau_{12}\tau_{13}\tau_{23}] + \sigma_{Y} \cdot [\tau_{12}\tau_{13}\tau_{14}] + \sigma_{N} \cdot [\tau_{12}\tau_{13}\tau_{34}] + \sigma_{II} \cdot [\tau_{12}\tau_{34}] + \sigma_{1,1} \cdot [\tau_{12}\tau_{13}\tau_{45}] + \sigma_{1,1,1} \cdot [\tau_{12}\tau_{13}\tau_{456}]$$ (223)

By applying $D_{2,2,2}^{1,1,1}$ on both sides of (223), we find two non-vanishing contributions:

$$D_{2,2,2}^{1,1,1}[\tau_{12}^3] = -36, \quad D_{2,2,2}^{1,1,1}(s_3(\bar{x})s_3(\bar{\lambda})) = 36\frac{(k-1)^2(k-2)^2}{k^4}$$ (224)

From this result, we obtain $\sigma_{III} = -\frac{(k-1)^2(k-2)^2}{k^4}$. From $D_{2,2,3}^{1,1,1}$, we obtain

$$D_{2,2,3}^{1,1,1}[\tau_{12}\tau_{13}] = -12, \quad D_{2,2,3}^{1,1,1}(s_3(\bar{x})s_3(\bar{\lambda})) = -36\frac{(k-1)(k-2)^2}{k^4}$$ (225)

which gives $\sigma_{\bar{\phi}} = \frac{3(k-1)(k-2)^2}{k^4}$. By the differentiation $D_{2,3,4}^{1,1,1}$, we obtain

$$D_{2,3,4}^{1,1,1}[\tau_{12}\tau_{13}\tau_{14}] = -6, \quad D_{2,3,4}^{1,1,1}(s_3(\bar{x})s_3(\bar{\lambda})) = 72\frac{(k-1)(k-2)}{k^4}$$ (226)
which reads $\sigma_Y = -\frac{12}{k^4}(k-1)(k-2)$. For the differentiation $D_{2,3,5}^{1,1,4}$, noting that $[\tau_{12}\tau_{13}\tau_{45}]$ includes a sum of the relevant terms $\tau_{12}\tau_{13}\tau_{45} + \tau_{12}\tau_{15}\tau_{34} + \tau_{13}\tau_{15}\tau_{24}$, we obtain

$$D_{2,3,5}^{1,1,4}[\tau_{12}\tau_{13}\tau_{45}] = -6, \quad D_{2,3,5}^{1,1,4}(s_3(\bar{x})s_3(\bar{\lambda})) = -72\frac{k-2}{k^4}$$

(227)

which reads $\sigma_{\Lambda,1} = 12\frac{k-2}{k^4}$. By the differentiation $D_{2,4,6}^{1,3,5}$, we obtain

$$D_{2,4,6}^{1,3,5}[\tau_{12}\tau_{34}\tau_{56}] = -6, \quad D_{2,4,6}^{1,3,5}(s_3(\bar{x})s_3(\bar{\lambda})) = 144\frac{1}{k^4}$$

(228)

which reads $\sigma_{II,1} = -\frac{24}{k^4}$. Thus we determine 5 coefficients of $\sigma$ as the function of $k$, uniquely. These coefficients are represented in (101). For other 3 coefficients, $\sigma_\triangle, \sigma_N, \sigma_{II,1}$, we need other differential operators, which give coupled equations.

By the differentiation $D_{2,2,4}^{1,1,3}$ of (223), two terms of $\tau$ are non-vanishing.

$$D_{2,2,4}^{1,1,3}\{\sigma_N \cdot [\tau_{12}\tau_{13}\tau_{24}] + \sigma_{II,1} \cdot [\tau_{12}\tau_{34}]\} = -4\sigma_N - 4\sigma_{II,1}$$

$$D_{2,2,4}^{1,1,3}(s_3(\bar{x})s_3(\bar{\lambda})) = 36\frac{(k-2)^2}{k^4}$$

(229)

which reads

$$\sigma_N + \sigma_{II,1} = -9\frac{(k-2)^2}{k^4}.$$  

(230)

By the differentiation $D_{3,3,2}^{1,1,2}$, we obtain from (223)

$$8\sigma_\therefore - 4\sigma_\triangle + 4(k-3)\sigma_{II,1} = 36\frac{(k-2)^2}{k^4}$$

(231)

Using the value of $\sigma_\therefore$, we have from above equation,

$$-\sigma_\triangle + (k-3)\sigma_{II,1} = \frac{(15 - 6k)(k-2)^2}{k^4}.$$  

(232)

By the differentiation $D_{2,2,3}^{1,1,2}$, we obtain

$$\sigma_\therefore + \sigma_\triangle + (k-3)\sigma_N = 9\frac{(k-2)^2}{k^4}$$

(233)

Using the obtained value of $\sigma_\therefore$, we have from above equation,

$$\sigma_\triangle + (k-3)\sigma_N = \frac{3(k-2)^2(4-k)}{k^4}.$$  

(234)
The three relations of (230), (232) and (234) are not linearly independent. Indeed if we shift \( \sigma_{11} \to \sigma_{11} + \alpha \), \( \sigma_N \to \sigma_N - \alpha \) and \( \sigma_\Delta \to \sigma_\Delta + \alpha(k - 3) \), these relations are unchanged. It means that there is a cubic identity,

\[
I_3 = [\Pi, I] - [N] + (k - 3)[\Delta] = 0. \tag{235}
\]

The proof of this identity is discussed in the appendix C. Therefore, the coefficients \( \sigma_{11}, \sigma_N, \sigma_\Delta \) have ambiguities up to a parameter \( \alpha \), which is an arbitrary constant. One can add a term \( \alpha I_3 \), i.e. zero, to the expression of \( s_3(\tilde{x})s_3(\tilde{\lambda}) \).

This may be used to write simple expressions for these coefficients, for instance,

\[
\sigma_{11,1} = -6\frac{(k - 2)^2}{k^4}, \quad \sigma_N = -3\frac{(k - 2)^2}{k^4}, \quad \sigma_\Delta = 3\frac{(k - 2)^2}{k^4} \tag{236}
\]

We have examined the transformation of (223) by all possible differential operators \( D_{i,j,k}^{\beta} \) on (223). These operators are, \( D_{111}^{111}, D_{111}^{112}, D_{112}^{112}, D_{112}^{122}, D_{122}^{122}, D_{122}^{123}, D_{123}^{123}, D_{123}^{134}, D_{124}^{134}, D_{124}^{145}, D_{145}^{145}, D_{145}^{156}, D_{156}^{156}, D_{156}^{167}. \)

This means that we have proved (223) with explicit coefficients.

In fourth order, the products of symmetric functions \( (s_2(\tilde{x})s_2(\tilde{\lambda}))^2 \), \( s_2(\tilde{x})^2s_4(\tilde{\lambda}) + s_4(\tilde{x})s_2(\tilde{\lambda})^2 \), and \( s_4(\tilde{x})s_4(\tilde{\lambda}) \) appear in the zonal polynomial expansions. Since \( s_2(\tilde{x})s_2(\tilde{\lambda}) \) is transformed as (218), \( (s_2(\tilde{x})s_2(\tilde{\lambda}))^2 \) is transformed as the square of (218).

We first consider the transformation of \( s_4(\tilde{x})s_4(\tilde{\lambda}) \). It is written as a sum of 23 possible terms,

\[
s_4(\tilde{x})s_4(\tilde{\lambda}) = \sigma_{[+]}[\tau_{12}\tau_{13}\tau_{14}\tau_{15}] + \sigma_{[\triangle]}[\tau_{12}\tau_{13}\tau_{34}\tau_{35}] + \sigma_{[\triangledown]}[\tau_{12}\tau_{13}\tau_{24}\tau_{34}] + \sigma_{[\square]}[\tau_{12}\tau_{13}\tau_{24}\tau_{34}] + \sigma_{[\nabla]}[\tau_{12}\tau_{13}\tau_{34}\tau_{45}] + \sigma_{[\star]}[\tau_{12}\tau_{13}\tau_{34}\tau_{56}] + \sigma_{[\bigcirc]}[\tau_{12}\tau_{13}\tau_{34}\tau_{56}] + \sigma_{[\bigtriangleup]}[\tau_{12}\tau_{13}\tau_{24}\tau_{35}] + \sigma_{[\bigtriangledown]}[\tau_{12}\tau_{13}\tau_{45}\tau_{56}] + \sigma_{[\bigtriangleleft]}[\tau_{12}\tau_{13}\tau_{45}\tau_{56}] + \sigma_{[\downarrow]}[\tau_{12}\tau_{13}\tau_{34}] + \sigma_{[\biguparrow]}[\tau_{12}\tau_{13}\tau_{24}] + \sigma_{[\bigtriangledown]}[\tau_{12}\tau_{13}\tau_{24}] + \sigma_{[\bigtriangledown]}[\tau_{12}\tau_{13}\tau_{34}] + \sigma_{[\nabla]}[\tau_{12}\tau_{13}\tau_{24}] + \sigma_{[\square]}[\tau_{12}\tau_{13}\tau_{24}] + \sigma_{[\nabla]}[\tau_{12}\tau_{13}\tau_{24}] + \sigma_{[\square]}[\tau_{12}\tau_{13}\tau_{24}] + \sigma_{[\nabla]}[\tau_{12}\tau_{13}\tau_{24}] + \sigma_{[\square]}[\tau_{12}\tau_{13}\tau_{24}] + \sigma_{[\nabla]}[\tau_{12}\tau_{13}\tau_{24}] + \sigma_{[\square]}[\tau_{12}\tau_{13}\tau_{24}] + \sigma_{[\nabla]}[\tau_{12}\tau_{13}\tau_{24}] + \sigma_{[\square]}[\tau_{12}\tau_{13}\tau_{24}]. \tag{237}
\]

A systematic way to determine these coefficients \( \sigma \) consists in classifying the terms by the number \( l \) of the points in the graph.
For \( l = 8 \), we make use of the differential operator \( D_{5678}^{1234} \), and the graph is \([I, I, I, I]\). Then, we find uniquely that
\[
\sigma_{I, I, I, I} = \frac{216}{k^6} \tag{238}
\]

For \( l = 7 \), we have two differential operators, \( D_{4567}^{1234} \) and \( D_{1567}^{1234} \), which act on the possible two graphs \([I, I, I, I]\) and \([\Lambda, I, I]\) (other graphs have less than 7 points and do not contribute). From \( D_{4567}^{1234} \), we obtain
\[
\sigma_{\Lambda, I, I} = -72 \frac{(k - 3)}{k^6} \tag{239}
\]

For \( l = 6 \), in addition to the above two graphs, we have 4 graphs, \([N, I]\), \([Y, I]\), \([\Lambda, \Lambda]\), and \([II, I]\). We have 6 different differential operators, \( D_{3456}^{1112}, D_{2256}^{1122}, D_{1256}^{1123}, D_{2456}^{1123}, D_{3456}^{1123}, \) and \( D_{2456}^{1123} \). From \( D_{3456}^{1122} \), we find uniquely,
\[
\sigma_{[Y, I]} = 72 \left( \frac{k^2 - 3k + 3}{k^6} \right) \tag{240}
\]

From \( D_{3456}^{1122} \), we find
\[
\sigma_{[\Lambda, \Lambda]} = -72 \frac{(2k - 3)}{k^6}. \tag{241}
\]

From \( D_{1456}^{1123} \), we have
\[
-24\sigma_{II,I,I} + 12\sigma_{Y,I} - 48\sigma_{N,I} - 12(k - 6)\sigma_{A,I,1,1} = -24 \cdot 24 \cdot 3 \frac{(k - 3)}{k^6} \tag{242}
\]

which reads to
\[
\sigma_{II,I,I} + 2\sigma_{N,I} = 36 \frac{(2k^2 - 10k + 15)}{k^6} \tag{243}
\]

From \( D_{2256}^{1123} \), we obtain the same relation.

From \( D_{1256}^{1123} \), we obtain
\[
8\sigma_{II,I,I} + 14(k - 6)\sigma_{A,I,1,1} + 2(k - 6)(k - 7)\sigma_{II,I,I} + 4\sigma_{\Lambda, \Lambda} + 16\sigma_{N,I} = 24 \cdot 24 \cdot 9 \frac{1}{k^6} \tag{244}
\]

which gives again,
\[
\sigma_{II,I,I} + 2\sigma_{N,I} = 36 \frac{(2k^2 - 10k + 15)}{k^6} \tag{245}
\]

Thus we are unable to find definite values for \( \sigma_{II,I,I} \) and \( \sigma_{N,I} \). The reason of this indefiniteness is the existence of an identity, which is an extension of the cubic identity. In the appendix C, we have derived the following identity \((282)\) :
\[
[II, I, I] - [N, I] + (k - 5)[\triangle, I] = 0. \tag{246}
\]
This identity gives a freedom in the choice of their respective values.

At next order \( l = 5 \) (five points), in addition to these 6 graphs, we have to consider six new graphs, \([X], [\triangle], [M], [\triangle, I], [\triangle, I], [\Lambda, II]\).

For the differential operators \( D_{ijkl}^{mnst} \), we have the following different kinds for \( l = 5 \),

\[
D_{2345}^{1111}, \quad D_{2345}^{1112}, \quad D_{2345}^{1122}, \quad D_{2345}^{1123}, \quad D_{2345}^{1112}, \quad D_{2345}^{1122},
D_{1235}^{1111}, \quad D_{1235}^{1112}, \quad D_{1235}^{1122}, \quad D_{1235}^{1123}, \quad D_{1235}^{1112}, \quad D_{1235}^{1122}
\]

(247)

where we note that \( D_{1445}^{1123} = D_{1225}^{1134} = D_{2235}^{1123} = D_{1345}^{1123} = D_{1245}^{1123} \). (the equal sign means the equivalence for the operator in this problem).

We have uniquely from \( D_{2345}^{1122} \),

\[
\sigma_X = -72 \frac{(k - 1)(k^2 - 3k + 3)}{k^6}
\]

(248)

We next obtain the coupled equations. From \( D_{2345}^{1122} \), we have

\[
\sigma_M + \sigma_{\Lambda, II} = 36 \frac{(k - 3)(2k - 3)}{k^6}
\]

(249)

From \( D_{2345}^{1112} \), we obtain

\[
\sigma_X + (k - 5)\sigma_{Y,1} = -288 \frac{(k^2 - 3k + 3)}{k^6}
\]

(250)

From \( D_{2345}^{1112} \), we obtain

\[
\sigma_{\triangle \triangle} + 2\sigma_{\triangle, I} = -48 \frac{(k - 3)(k^2 - 3k + 3)}{k^6}
\]

(251)

From \( D_{2345}^{1112} \), we have

\[
\sigma_X - 6\sigma_{\triangle \triangle} - 3(k - 5)\sigma_{Y,1} - 12\sigma_{\triangle, I} = 288 \frac{(k^2 - 3k + 3)}{k^6}
\]

(252)

which reduces to (251).

From \( D_{2345}^{1112} \), we have

\[
2\sigma_{\Lambda, II} + 2\sigma_M + (k - 5)\sigma_{\Lambda, \Lambda} = 144 \frac{(2k - 3)}{k^6}
\]

(253)

This reduces to (249).

From \( D_{2345}^{1112} \), we get

\[
\sigma_{\triangle \triangle} - 6\sigma_M - 2(k - 5)\sigma_{N,1} - 2\sigma_{\triangle, I} - (k - 5)\sigma_{\Lambda, \Lambda} - 2\sigma_{\triangle, I} = 144 \frac{(k - 3)^2}{k^6}
\]

(254)
From $D_{2235}^{1134}$, we have

$$-\sigma_M + 2\sigma_{\perp\perp} + (k - 5)\sigma_{N,1} - \sigma_{\Delta,1} + 2\sigma_{\perp,1} + (k - 5)\sigma_{H,1,1} + \sigma_{A,1,1} = -144\frac{(k - 3)^2}{k^6} \quad (255)$$

From $D_{1135}^{1123}$, we have

$$\sigma_X - 8\sigma_{\perp\perp} + 8\sigma_M + 8(k - 5)\sigma_{N,1} - 4\sigma_{Y,1} + 4\sigma_{\Delta,1}
+ (k - 5)(k - 6)\sigma_{A,1,1} - 8\sigma_{\perp,1} + 2(k - 5)\sigma_{H,1,1} = 144\frac{(k - 3)^2}{k^6} \quad (256)$$

These three equations coincide when we put the known values of the coefficients.

The last differentiation in (247), $D_{1235}^{1234}$, gives

$$\sigma_X - 12\sigma_{\perp\perp} - 12\sigma_M - 24(k - 5)\sigma_{N,1} - 4(k - 5)\sigma_{Y,1}
- 9(k - 5)\sigma_{A,1,1} - 12(k - 5)(k - 6)\sigma_{A,1,1} - (k - 5)(k - 6)(k - 7)\sigma_{I,1,1,1}
- 12\sigma_{A,1,1} - 12(k - 5)\sigma_{H,1,1} - 24\sigma_{\perp,1} = 5184\frac{1}{k^6} \quad (257)$$

However, this leads to known identities rather than to a new relation.

Several coefficients of the terms, which have less than 4 points, are determined uniquely by the differential operators. We list them here,

$$\sigma_{[III]} = \frac{1}{k^6}(k - 1)^2(k^2 - 3k + 3)^2, \quad (D_{222,2}^{111,1,1})$$
$$\sigma_{[\perp]} = \frac{12(k - 1)(k - 3)(k^2 - 3k + 3)}{k^6}, \quad (D_{223,3}^{111,1,1})$$
$$\sigma_{[\perp\perp]} = \frac{6(k - 1)(2k - 3)(k^2 - 3k + 3)}{k^6}, \quad (D_{223,3}^{111,1,1})$$
$$\sigma_{[\perp\perp]} = -\frac{4(k - 1)(k^2 - 3k + 3)^2}{k^6}, \quad (D_{222,3}^{111,1,1})$$
$$\sigma_{[\perp]} = -\frac{12(2k - 3)(k^2 - 3k + 3)}{k^6}, \quad (D_{224,1}^{111,3}) \quad (258)$$

We have coupled equations by other differential operators. We list them

with the differential operator $D_{3}^{\ast}$, which are used,

$$\sigma_{[III]} + \sigma_{[\perp]} = 16\frac{(k^2 - 3k + 3)^2}{k^6}, \quad (D_{222,4}^{111,1,3}) \quad (259)$$

$$-2\sigma_{[\perp]} - (k - 3)\sigma_{[III]} + \sigma_{[\perp\perp]} = 16\frac{(k^2 - 3k + 3)^2}{k^6}, \quad (D_{222,2,3}^{111,1,3}) \quad (260)$$

$$3\sigma_{[\perp]} + 2\sigma_{[II]} + \sigma_{[\perp]} + (k - 4)\sigma_M + (k - 4)\sigma_{A,1,1} = -72\frac{(k - 3)(2k - 3)}{k^6}, \quad (D_{222,3,4}^{111,3,3}) \quad (261)$$
\[2\sigma_{[II,II]} + \sigma_{[\Box]} = 36 \frac{(2k - 3)^2}{k^6}, \quad (D_{2,2,4,3})^{1,1,3,3}. \] (262)

\[-2(k - 3)\sigma_\Box + 6\sigma_\triangleleft - 6\sigma_\triangle + (k - 3)(k - 4)\sigma_{\Lambda, II} + 2(k - 3)\sigma_{II, II} + 4(k - 3)\sigma_\equiv = 72 \frac{(2k - 3)^2}{k^6}, \quad (D_{2,2,3,3})^{1,1,3,3}. \] (263)

\[(k - 4)\sigma_{\Lambda 1} + \sigma_\parallel - \sigma_\geq + \sigma_\equiv = 48 \frac{(k - 3)(k^2 - 3k + 3)}{k^6}, \quad (D_{2,4,3,3})^{1,1,1,2}. \] (264)

\[-2(k - 4)\sigma_{\Lambda \perp} + 2(k - 4)\sigma_\Lambda + (k - 4)(k - 5)\sigma_{\Lambda, \Lambda} + 2(k - 4)\sigma_{\Lambda, II} + 2\sigma_\Box - 4\sigma_\equiv + 8\sigma_\equiv + 4\sigma_\parallel = 144 \frac{(k - 3)(2k - 3)}{k^6}, \quad (D_{2,2,3,4})^{1,1,2,2}. \] (265)

\[2(k - 4)\sigma_{\Lambda \perp} + 2\sigma_\Box + 6(k - 4)\sigma_\Lambda + 8\sigma_{II, II} + 16\sigma_\equiv + 8(k - 4)\sigma_{\Lambda, II} + 2(k - 4)(k - 5)\sigma_{\Lambda, \Lambda} + 8\sigma_\equiv - 4\sigma_\parallel = -432 \frac{(2k - 3)}{k^6}, \quad (D_{1,2,3,4})^{1,1,2,2} \] (266)

\[2\sigma_\equiv + (k - 4)\sigma_{\Lambda \perp} - 2\sigma_\equiv + 2\sigma_\parallel = 48 \frac{(k - 3)(k^2 - 3k + 3)}{k^6}, \quad (D_{2212})^{1112}. \] (267)

\[(k - 3)(k - 4)\sigma_{\Lambda \perp} + 2(k - 3)\sigma_\geq + 4(k - 3)\sigma_\equiv + 6\sigma_\triangle + 24\sigma_\parallel + 4\sigma_\triangle + 2(k - 3)\sigma_\parallel + 2(k - 3)\sigma_\equiv = 48 \frac{(2k - 3)(k^2 - 3k + 3)}{k^6} - 48 \frac{1}{k - 2} \sigma_{III}, \quad (D_{1122})^{1112} \] (268)

These coupled equations are equivalent to the following equations,

\[2\sigma_{II, II} + \sigma_\Box = 36 \frac{(2k - 3)^2}{k^6},\]
\[\sigma_{III, I} + \sigma_\equiv = 16 \frac{(k^2 - 3k + 3)^2}{k^6},\]
\[-(k - 3)\sigma_{III, I} + \sigma_\triangle = -8 \frac{(k - 3)(k^2 - 3k + 3)^2}{k^6},\]
\[\sigma_\triangle + (k - 3)\sigma_\equiv = 8 \frac{1}{k^5}(k - 3)(k^2 - 3k + 3)^2.\] (269)
From (268), we obtain by the use of (267),
\[(k - 3)\sigma_\Pi + \sigma_\Delta = \frac{8(k - 3)(k^2 - 3k + 3)^2}{k^6}\] (270)
which is same as (269).

We have the following equation by considering \(D_{1233}^{1122}\),
\[\frac{6}{k - 3}\sigma_\Delta + \sigma_\sqcup + 2\sigma_\sqcap + (k - 4)\sigma_M + 2\sigma_\sqtriangledown = \frac{72(2k - 3)}{k^6}\] (271)

We find that many coefficients are not determined uniquely. We have 23 coefficients and 10 coefficients are determined uniquely. 13 coefficients are not determined uniquely. There are 9 linear independent relations among these undetermined coefficients. These 9 linear independent relations are (259),(260),(263),(264),(262),(245),(249),(251), (255). We find there are 4 quartic identities in appendix C, (282)∼(285). Therefore, we are able to choose freely 4 coefficients.

For instance, if we choose \(\sigma_M = -36(k - 3)^2\), we have
\[
\sigma_M = -36(k - 3)^2 \\
\sigma_{\Lambda,\Pi} = 108(k - 2)(k - 3) \\
\sigma_{\sqcap,\sqcap} = -24(k - 3)(k^2 - 3k + 3) \\
\sigma_{\sqcup,\sqcup} = -18(k - 1)(2k - 3) \\
\sigma_{\sqtriangledown,\sqtriangledown} = -12(k - 3)(k^2 - 3k + 3) \\
\] (272)

**Appendix C: The cubic and quartic identities**

There is one cubic identity in the case \(k = 4\) for the variables \(\tau_{ij} = (x_i - x_j)(\lambda_i - \lambda_j)\) as shown in [5],
\[I_3 = [\tau_{12}^2 \tau_{34}] - [\tau_{12} \tau_{13} \tau_{24}] + [\tau_{12} \tau_{13} \tau_{23}] = 0.\] (273)
which may be conveniently depicted graphically by
\[I_3 = [\Pi, I] - [N] + [\Delta].\] (274)

For \(k > 4\), this cubic identity is modified by a factor \((k - 3)\) as
\[I_3 = [\tau_{12}^2 \tau_{34}] - [\tau_{12} \tau_{13} \tau_{24}] + (k - 3)[\tau_{12} \tau_{13} \tau_{23}] = 0.\] (275)

We apply the differentiations \(D_{abc}^{klm}\), where the indices take one of values among \((1,2,3,...,k)\). By the symmetry, it is enough to consider the values \((1,2,3,4)\). We find all possible such differentials satisfy this identity,
\[D_{abc}^{klm} I_3 = 0.\] (276)
This is a proof that we have an identity of (275).

The geometric proof is also possible. In the case \( k = 4 \), we consider the coincidence of the variable \( x_4 \to x_1 \) and \( \lambda_4 \to \lambda_1 \). In this limit, we obtain three type 3-point graphs. From \([I, I]\), we obtain \([\bigtriangleup]\), and this \([\bigtriangleup]\) is exactly cancelled by the same graphs generated from \([N]\) in this coincidence limit. The graphs \([N]\) also generate the triangle graphs \([\triangle]\), which are cancelled by the existing triangle graphs of the last term of the identity \( I_3 \). Thus we prove that \( I_3 = 0 \) for \( k = 4 \). For \( k > 4 \), we can prove this identity by the inductive method, namely for \( k = 5 \), when we take the limit \( x_5, \lambda_5 \to x_1, \lambda_1 \), the graphs reduces to the graphs of \( k = 4 \) by the cancellations.

Since we have this identity, the coefficients \( \sigma_{I, I}, \sigma_{N}, \sigma_{\triangle} \) are not uniquely determined as we have seen in the appendix B.

In the fourth order (4 line graphs), the identity \( I_3 = 0 \) at \( k = 4 \) is generalized by the multiplication of \([\tau_{12}] = \tau_{12} + \tau_{13} + \cdots + \tau_{3, 4}\). By writing all types of graphs, we find easily the following identity of \( k = 4 \),

\[
I_4 = I_3 \times [\tau_{12}]
= [III, I] + 2[II, II] + [\bigtriangledown] + [\triangle] - 4[\square] - [I I ] = 0
\] (277)

This identity of \( I_4 = 0 \) at \( k = 4 \) is verified directly by the differentiations \( D_{abcd} \) \((a, b, c, d, k, l, m, n \text{ are } 1, 2, 3 \text{ or } 4)\).

For the general \( k \) \((k > 4)\), we use the cubic identity of (275). Multipling \([\tau_{12}]\) on each terms of the cubic identity, we get (the indices are now from 1 to \( k \)),

\[
[II, I][\tau_{12}] = 2[II, I, I] + [III, I] + 2[II, II] + [\bigtriangleup, I] + [\bigtriangledown] + 2[\Lambda, II]
\] (278)

\[-[N][\tau_{12}] = -2[M] - [N, I] - 2[\bigtriangledown] - [III, I] - [\square] - 4[\square] - 2[\triangle]\]

\[(k - 3)[\triangle][\tau_{12}] = (k - 3)[\triangle, I] + (k - 3)[\triangle] + (k - 3)[\bigtriangledown]
\] (279)

Adding these three equations, we obtain an identity.

\[
2[II, I, I] - [N, I] + (k - 3)[\triangle, I] + [III, I] + 2[II, II] + [\bigtriangleup, I] - 2[M]
+ 2[II, \Lambda] + (k - 5)[\bigtriangledown] - [I I ] - 4[\square] - 2[\triangle]\] + (k - 3)[\triangle] = 0
\] (281)

This long identity is devided into 4 sub-identities.

By the analogy of the cubic identity, we extract from (281) an identity,

\[
2[II, I, I] - [N, I] + (k - 5)[\triangle, I] = 0
\] (282)

This identity is the cubic identity plus a separate line. The factor \( (k - 5) \) means that when \( k = 5 \) all terms are vanishing. \([\triangle, I]\) is nonvanishing,
therefore it should have a factor \((k - 5)\) as a coefficient, since other two terms do not exist for \(k=5\).)

This simple (fundamental) identity at fourth order is proved by all possible differentiations \(D^{x_{1,\ldots,\lambda}}\). For instance, with the differentials of \(D_{1123}^{[I, I, I]}\), we find \(D_{1123}^{[I, I, I]} = 4(k - 3)(k - 4)(k - 4)\), \(D_{1123}^{[N, I]} = 16(k - 3)(k - 4)(k - 5)\) and \(D_{1123}^{[\Delta, I]} = 8(k - 3)(k - 4)\). These values satisfy (282). All differentiations which appeared in appendix B satisfy this identity.

The second identity in (281) is a generalization of (277), obtained by adding a factor \((k - 3)\) for \([\triangle]\),

\[
[\mathrm{III}, I] + 2[\mathrm{II}, II] - [\triangleright] + (k - 3)[\triangle] - 4[\square] - [\mathrm{II}] = 0 \quad (283)
\]

This identity is verified by the differentiations of \(D^{klmn}_{abcd}\), where the indices take values from one to four. Indeed in appendix B we have evaluated various differentiations, and we have found the contributions of every graph which enters in this second identity. We have checked all possible differentiations.

As third identity in (281), we find

\[
2[\Lambda, II] - 2[M] + 4[\Delta, I] + 2(k - 4)[\square] - (k - 4)[II, II] = 0 \quad (284)
\]

This identity is also proved by the consideration of all possible differentiations.

Subtracting these three identities from (281), we obtain the fourth identity,

\[
[\bigtriangleup, I] - 2[\bigtriangleup] - 2[\bigtriangleup, I] + (k - 4)[\triangleright] - 2(k - 4)[\square] + (k - 4)[II, II] = 0 \quad (285)
\]

We have checked this identity by the evaluation of all differentiations \(D^{klmn}_{abcd}\) in (247).

Further, we check them by the differentiations \(D^{klmn}_{abcd}\), \((a,b,c,d,k,l,m,n = 1,2,3,4)\), which appeared in the calculations in appendix B. They are \(D_{2234}^{1112}, D_{1122}^{1112}, D_{1234}^{1112}, D_{2234}^{1112}, D_{2435}^{1112}, D_{2234}^{1113}, D_{2244}^{1113}, D_{2234}^{1113}, D_{2223}^{1113}, D_{2224}^{1113}\).

In the appendix B, we were left with 13 undetermined coefficients \(C_{[\text{graph}]}\). We have 9 linear independent equations for them here. Since \(13 - 9 = 4\), given that one has 4 quartic identities \((282) \sim (285)\) all the coefficients may be determined at the expense of introducing 4 arbitrary parameters.

In the case \(\beta = 4\), there are no double or multiple line graphs. Therefore, we have no cubic or higher order identities, and all coefficients of the \(\tau\) expansion are uniquely determined without ambiguities. In the next appendix D, we discuss the case of \(\beta = 4\) in a more systematic way.

**Appendix D**: Characterization of the \(\tau\) expansion by specific differentiations

62
As discussed in appendix B, the coefficients of the $\tau$ expressions for the products of symmetric functions $s_4(\tilde{x})s_4(\tilde{\lambda})$ are determined by focusing on the monomials $x_{i_1}^{p_1} x_{i_2}^{p_2} \cdots \lambda_{j_1}^{q_1} \lambda_{j_2}^{q_2} \cdots$. Such terms may be selected with the help of the differentiations $D_{ij}$ etc. Many terms are thereby uniquely determined since they are characterized by the existence of particular combinations of $x_i$ and $\lambda_j$. Since each $\tau$ term has a specific topological structure when represented graphically, one may characterize such $\tau$ terms by the differentiations $D_{ij}$. For instance, the term $[\tau_{12}^2] = \sum_{i<j} \tau_{ij}^2$ is uniquely characterized by $x_i^2 \lambda_j^2$, which is equivalent to $D_{22}^{11}$. Other types of $\tau$-terms, such as $[\tau_{12} \tau_{13}]$, $[\tau_{12} \tau_{34}]$, have no such $x_i^2 \lambda_j^2$ factor. Therefore $x_i^2 \lambda_j^2$ is a unique factor which selects $[\tau_{12}]$.

Indeed we have used such properties for the determination of $\sigma$, which is a coefficient of the $\tau$ expression for $s_2(\tilde{x})s_2(\tilde{\lambda})$. We study such correspondences between the unique differentiations and the $\tau$ terms (graphs), because we can apply these correspondences directly to determine the coefficients $C$ for the HIZ integrals.

This one-to-one correspondence between $\tau$ terms and differentiations $D$ may be used as follows. We write the function $f$ in (2) as a sum,

$$f = 1 + f(1) + f(2) + f(3) + \cdots$$  \hspace{1cm} (286)

where $f(l)$ is the $l$-th order term of the $\tau$-expansion. The function $f$ is given by

$$f = e^{-\frac{1}{k} \sum_{i<j} \tau_{ij}} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{1}{m-1} \frac{1}{(1+q\alpha)} \sum_{p} \lambda_p Z_p(\tilde{x}) Z_p(\tilde{\lambda}) Z_0(1)$$  \hspace{1cm} (287)

which is expanded as

$$f(1) = -\frac{1}{k} \sum_{i<j} \tau_{ij}$$

$$f(2) = \frac{1}{2k^2} (\sum_{ij} \tau_{ij})^2 + \frac{\alpha}{2(k+\alpha)(k-1)} s_2(\tilde{x}) s_2(\tilde{\lambda})$$  \hspace{1cm} (288)

Table D-1: Characteristic differentials of $\tau$ terms and $f$

| $l=1$ | $D_{11}^3[I] = -1$, | $D_{1}^3(f(1)) = \frac{1}{k}$ |
|-------|----------------------|-----------------|
| $l=2$ | $D_{23}^1[\Lambda] = 2$, | $D_{23}^1(f(2)) = \frac{2}{k(k+\alpha)}$ |
|       | $D_{12}^{[I]} = 4$, | $D_{12}^{[I]}(f(2)) = \frac{2(1+\alpha)}{k(k+\alpha)}$ |
|       | $D_{34}^{[I]} = 2$, | $D_{34}^{[I]}(f(2)) = \frac{2}{k(k+\alpha)} \left(1 + \frac{\alpha}{k-1}\right)$ |
| $l=3$ | $D_{45}^{[I]} = -6$, | $D_{45}^{[I]}(f(3)) = \frac{6}{k(k+\alpha)(k+2\alpha)} \left[1 + \frac{3\alpha}{k-1} + \frac{2\alpha}{(k-1)(k-2)}\right]$ |
|       | $D_{35}^{[I]} = -6$, | $D_{35}^{[I]}(f(3)) = \frac{6}{k(k+\alpha)(k+2\alpha)} \left(1 + \frac{2\alpha}{k-1}\right)$ |
|       | $D_{24}^{[Y]} = -6$, | $D_{24}^{[Y]}(f(3)) = \frac{6}{k(k+\alpha)(k+2\alpha)}$ |
When the $\tau$-terms (graphs) are selected by a single differentiation $D$, the coefficients of the corresponding term in the $\tau$ expansion are given by

$$C_{[\text{graph}]} = \frac{D(f(\nu))}{D[\text{graph}]}$$

(289)

For instance, from the above table, we obtain

$$C_{\Lambda} = \frac{1}{k(k + \alpha)}$$

(290)

The values of the coefficients $C_{[\text{graph}]}$ coincide with the values in Table B, when we put $\alpha = 2/(2 - \beta)$.

The third order term $f_{(3)}$ in (286) follows from the Jack polynomial expansion,

$$f_{(3)} = \frac{1}{6k^3}(\sum_{i<j} \tau_{ij})^3 - \frac{\alpha}{2k(k + \alpha)(k - 1)}(\sum_{i<j} \tau_{ij}) s_2(\bar{x}) s_2(\bar{\lambda})$$

$$+ \frac{\alpha^2 k}{3(k + \alpha)(k + 2\alpha)(k - 1)(k - 2)} s_3(\bar{x}) s_3(\bar{\lambda})$$

(291)

The coefficients of $[\text{II, I}], [\text{N}]$ have arbitrariness due to the cubic identity, which has been discussed in the appendix C. We take here the reasonable constraint that the double line graphs have $\frac{1 + \alpha}{2}$ as overall factor. This means that $C_{\text{II,I}}$ is proportional to $\frac{1 + \alpha}{2}$, and it is vanishing for $\alpha = -1/4$. For the single line graphs, their coefficients are simply $-\frac{1}{k^3}$ in the large $k$ limit, for the third order terms. With these constraints, we have from the table D-1,

$$C_{\text{II,I}} = -\frac{1 + \alpha}{2k(k + \alpha)(k + 2\alpha)}(1 + \frac{2\alpha}{k - 1})$$

(292)

$$C_{\text{N}} = -\frac{1}{k(k + \alpha)(k + 2\alpha)}(1 + \frac{\alpha}{k - 1})$$

(293)

which coincide with the values in Table B by the substitution of $\alpha = \frac{2}{2-\beta}$.

By the differentiation $D_{332}^{112}$, we have from Table D-1,

$$-4C_\Delta = -4(k - 3)C_{\text{II,I}} - 8C_{\text{II,I}} + D_{332}^{112}(f_{(3)})$$

(294)

Using the value of $C_{\text{II,I}}$ in (292), we obtain

$$C_\Delta = -\frac{1}{k(k + \alpha)(k + 2\alpha)}(1 - \frac{\alpha^2}{k - 1})$$

(295)

which coincides with the value in Table B.
For the fourth order, we have from the expression of Table in the appendix A,

\[ f(4) = \frac{1}{24k^4} \left( \sum_{i<j} \tau_{ij} \right)^4 + \frac{\alpha}{4k^2(k + \alpha)(k - 1)} \left( \sum_{i<j} \tau_{ij} \right)^2 s_2(\tilde{x}) s_2(\tilde{\lambda}) \]

\[ \alpha^2 \frac{(\sum_{i<j} \tau_{ij} s_3(\tilde{x}) s_3(\tilde{\lambda}))}{3(k + \alpha)(k + 2\alpha)(k - 1)(k - 2)} \]

\[ + \frac{\alpha^2(18\alpha^2 + 18\alpha k - 18\alpha^2 k + 6k^2 - 18\alpha k^2 + 6\alpha^2 k^2 - 5k^3 + 5\alpha k^3 + k^4)}{8k(k + \alpha)(k + 2\alpha)(k + 3\alpha)(k + \alpha - 1)(k - 1)(k - 2)(k - 3)} \times (s_2(\tilde{x}) s_2(\tilde{\lambda}))^2 \]

\[ + \frac{\alpha^3 k^2 + \alpha k - k + \alpha}{4(k + \alpha)(k + 2\alpha)(k + 3\alpha)(k + \alpha - 1)(k - 1)(k - 2)(k - 3)} s_4(\tilde{x}) s_4(\tilde{\lambda}) \]

\[- \frac{\alpha^3(2k^2 + 3\alpha k - 3k - 3\alpha)}{4(k + \alpha)(k + 2\alpha)(k + 3\alpha)(k + \alpha - 1)(k - 1)(k - 2)(k - 3)} \times [s_2(\tilde{x})^2 s_4(\tilde{\lambda}) + s_4(\tilde{x})(s_2(\tilde{\lambda}))^2] \]

(296)

If we apply \( D_{2345}^{1111} \) on \( f(4) \), we obtain

\[ D_{2345}^{1111}(f(4)) = \frac{24}{k(k + \alpha)(k + 2\alpha)(k + 3\alpha)} \]

(297)

This \( D_{2345}^{1111} \) is a characteristic differentiation of the \( \tau \) term \([X]\), namely it is the unique differentiation for \([X]\) term. Since we have \( D_{2345}^{1111}[X] = 24 \), one obtains

\[ C_X = \frac{1}{k(k + \alpha)(k + 2\alpha)(k + 3\alpha)} \]

(298)

which coincides with the value given in Table B, if we put \( \alpha = \frac{2}{2 - \beta} \). Other differentiations of \( \tau \) terms and \( f(4) \) are represented in the following table D-2.
Table D-2: Characteristic differentiations of $\tau$ terms and $f(4)$

| Category | Expression |
|----------|------------|
| $l=4[X]$ | $D_{2345}^{1111}[X] = 24$, $D_{2345}^{1111}(f(4)) = \frac{24}{k(k+\alpha)(k+2\alpha)(k+3\alpha)}$ |
| $[Y, I]$ | $D_{3456}^{1112}[Y, I] = 24$, $D_{3456}^{1112}(f(4)) = \frac{24(k+3\alpha-1)}{k(k-1)(k+\alpha)(k+2\alpha)(k+3\alpha)}$ |
| $[\Lambda, \Lambda]$ | $D_{4567}^{1112}[\Lambda, \Lambda] = 24$, $D_{4567}^{1112}(f(4)) = \frac{24(k+2\alpha-1)(k+3\alpha-1)}{k(k+\alpha)(k+2\alpha)(k+3\alpha)(k+\alpha-1)(k-1)}$ |
| $[\Lambda, I, I]$ | $D_{5678}^{1123}[\Lambda, I, I] = 24$, $D_{5678}^{1123}(f(4)) = \frac{24}{k(k+\alpha)(k+2\alpha)(k+3\alpha)} (1 - \frac{2\alpha}{k-1} + \frac{7\alpha}{k(k-2)} + \frac{6\alpha^2}{(k-2)(k-1)(k+\alpha-1)} - \frac{2\alpha^2}{(k-2)(k+\alpha-1)})$ |
| $[I, I, I]$ | $D_{5678}^{1234}[I, I, I] = 24$, $D_{5678}^{1234}(f(4)) = \frac{24}{k(k+\alpha)(k+2\alpha)(k+3\alpha)} (1 + \frac{6\alpha}{k-1} + \frac{9\alpha^2}{(k-1)(k+\alpha-1)} + \frac{8\alpha^2}{(k-2)(k+\alpha-1)} + \frac{8\alpha^2}{(k-2)(k-3)(k+\alpha-1)}$ |
| $[\equiv]$ | $D_{2244}^{2244}[\equiv] = 48$, $D_{2244}^{2244}(f(4)) = \frac{24}{k(k+\alpha)(k+2\alpha)(k+3\alpha)} (1 + \frac{2\alpha}{k-1})$ |
| $[\ll]$ | $D_{2233}^{2233}[\ll] = 96$, $D_{2233}^{2233}(f(4)) = \frac{24(1+\alpha)^2}{k(k+\alpha)(k+2\alpha)(k+3\alpha)}$ |
| $[\equiv]$ | $D_{2233}^{2233}[\equiv] = 48$, $D_{2233}^{2233}(f(4)) = \frac{24}{k(k+\alpha)(k+2\alpha)(k+3\alpha)} (1 + \frac{2\alpha}{k-1})$ |
| $[\equiv]$ | $D_{2233}^{2233}[\equiv] = 144$, $D_{2233}^{2233}(f(4)) = \frac{24(1+\alpha)(1+2\alpha)}{k(k+\alpha)(k+2\alpha)(k+3\alpha)}$ |
| $[\equiv]$ | $D_{2222}^{2222}[\equiv] = 576$, $D_{2222}^{2222}(f(4)) = \frac{24(1+\alpha)(1+2\alpha)(1+3\alpha)}{k(k+\alpha)(k+2\alpha)(k+3\alpha)}$ |

66
Table D-3: Characteristic differentiations of \( \tau \) terms and \( f_{(4)} \)

\[
D_{2345}^{1112}(C_{\Delta\Delta}[\Delta\Delta] + C_{\Delta\Delta}[\Delta\Delta]) = 12C_{\Delta\Delta} + 24C_{\Delta\Delta}
\]

\[
D_{2345}^{1112}(f_{(4)}) = \frac{12}{k(k+\alpha)(k+2\alpha)(k+3\alpha)}[(1+\alpha)(1+\frac{3\alpha}{k-1})] + (1+\frac{2\alpha}{k-1})]
\]

\[
D_{2345}^{1112}(C_{[\Lambda,\Lambda] + [\Lambda,\Lambda]} + C_{[M][M]} + C_{1\Lambda,\Lambda] + [\Lambda,\Lambda]) = -24C_{[\Lambda,\Lambda]} - 24C_{[M]} - 12(k-5)C_{[\Lambda,\Lambda]}
\]

\[
D_{2345}^{1112}(f_{(4)}) = -\frac{12(k+2\alpha-1)(k+4\alpha-3k+3\alpha^2-9\alpha+2)}{k(k-1)(k+\alpha-1)(k+2\alpha)(k+3\alpha)(k+3\alpha)}
\]

\[
D_{2224}^{1111}(C_{III,II} + C_{III,II}] = 36C_{III,II} + 36C_{III,II}
\]

\[
D_{2224}^{1111}(f_{(4)}) = \frac{12(1+\alpha)(k+2\alpha-1)(k+3\alpha)}{k(k+\alpha)(k+2\alpha)(k+3\alpha)(k-1)}
\]

\[
D_{2223}^{1111}(C_{\Delta\Delta}[\Delta\Delta] + C_{\Delta\Delta}[\Delta\Delta]) = -72C_{\Delta\Delta} - 36(k-3)C_{III,II} + 36C_{\Delta\Delta}
\]

\[
D_{2223}^{1111}(f_{(4)}) = -\frac{6(1+\alpha)(k+2\alpha-1)(k+3\alpha)}{k(k+\alpha)(k+2\alpha)(k+3\alpha)(k-1)}
\]

\[
D_{1456}^{1123}(C_{[\Pi,\Pi] + [\Pi,\Pi]} + C_{[\Pi,\Pi] + [\Pi,\Pi]} + C_{[N,\Pi] + [N,\Pi]} + C_{[\Lambda,\Pi] + [\Lambda,\Pi]}) = -24C_{[\Pi,\Pi]} + 12C_{[\Pi,\Pi]} - 48C_{[N,\Pi]} - 12(k-6)C_{[\Lambda,\Pi]}
\]

\[
D_{1456}^{1123}(f_{(4)}) = -\frac{12(1+\alpha)(k+2\alpha)(k+3\alpha)}{k(k+\alpha)(k+2\alpha)(k+3\alpha)(k-1)}
\]

\[
D_{2234}^{1122}(C_{\geq}[\geq] + C_{\leq}[\leq] + C_{[\Pi][\Pi] + C_{[\Pi][\Pi]}) = -24C_{[\Pi][\Pi] - 24(k-4)C_{[\Pi][\Pi] + 24C_{[\Pi][\Pi]}
\]

\[
D_{2234}^{1122}(f_{(4)}) = -\frac{12(1+\alpha)(k+2\alpha)(k+3\alpha)}{k(k+\alpha)(k+2\alpha)(k+3\alpha)(k-1)}
\]

\[
D_{2233}^{1124}(C_{\leq}[\leq] + C_{[M][M]} + C_{[N,\Pi] + [N,\Pi]} + C_{[\Delta,\Pi] + [\Delta,\Pi] + C_{[\Lambda,\Pi] + [\Lambda,\Pi]}
\]

\[
D_{2233}^{1124}(f_{(4)}) = -\frac{4(12-78\alpha+120\alpha^2-36\alpha^3-6\alpha^4-36\alpha+151\alpha-70\alpha^2+70\alpha^3+39\alpha^2-90\alpha k^2+27\alpha^2 k^2-18\alpha^3+17\alpha k^3+3\alpha^4)}{k(k-1)(k-2)(k+\alpha-1)(k+2\alpha)(k+3\alpha)}
\]
The coefficients in Table D-3 have ambiguities due to the quartic identities discussed in Appendix C. For the coefficients \( C_{\mathcal{I}}, C_{\mathcal{II}} \), we have

\[
24C_{\mathcal{I}} + 12C_{\mathcal{II}} = \frac{12}{k(k + \alpha)(k + 2\alpha)(k + 3\alpha)}[(1 + \alpha)(1 + \frac{3\alpha}{k - 1}) + (1 + \frac{2\alpha}{k - 1})]
\]

(299)

If we assume that \( C_{\mathcal{I}} \) has an overall factor \((1 + \alpha)\), which implies that it vanishes for \( \beta = 4 \), we uniquely determine these two coefficients as

\[
C_{\mathcal{I}} = \frac{1 + \alpha}{2k(k + \alpha)(k + 2\alpha)(k + 3\alpha)}(1 + \frac{3\alpha}{k - 1})
\]

(300)

and

\[
C_{\mathcal{II}} = \frac{1}{k(k + \alpha)(k + 2\alpha)(k + 3\alpha)}(1 + \frac{2\alpha}{k - 1}).
\]

(301)

These values coincide with Table B, and for \( \beta = 4 \), they are consistent with Table A.

Subtracting the value of \(-12(k - 5)C_{\Lambda,\Lambda}\), which was obtained in Table D-2, from \( D_{2245}^{12}(f(4)) \), we obtain

\[
C_{\Lambda,\Pi} + C_M = \frac{1 + \alpha}{2k(k + \alpha)(k + 2\alpha)(k + 3\alpha)}(1 + \frac{\alpha}{k + \alpha - 1})(1 + \frac{3\alpha}{k - 1}) + \frac{1}{k(k + \alpha)(k + 2\alpha)(k + 3\alpha)}(1 + \frac{\alpha}{k + \alpha - 1})(1 + \frac{2\alpha}{k - 1})
\]

(302)

Normalizing again the double bond with the factor \((1 + \alpha)/2\) which vanishes for \( \beta = 4 \), we obtain

\[
C_{\Lambda,\Pi} = \frac{1 + \alpha}{2k(k + \alpha)(k + 2\alpha)(k + 3\alpha)}(1 + \frac{\alpha}{k + \alpha - 1})(1 + \frac{3\alpha}{k - 1})
\]

(303)

and

\[
C_M = \frac{1}{k(k + \alpha)(k + 2\alpha)(k + 3\alpha)}(1 + \frac{\alpha}{k + \alpha - 1})(1 + \frac{2\alpha}{k - 1})
\]

(304)

We write \( D_{2224}^{1113}(f(4)) \) as

\[
D_{2224}^{1113}(f(4)) = \frac{6(1 + \alpha)(1 + 2\alpha)}{k(k + \alpha)(k + 2\alpha)(k + 3\alpha)}(1 + \frac{3\alpha}{k - 1}) + \frac{18(1 + \alpha)}{k(k + \alpha)(k + 2\alpha)(k + 3\alpha)}(1 + \frac{\alpha}{k - 1})
\]

(305)

Then we find that

\[
C_{\Pi,\Pi} = \frac{(1 + \alpha)(1 + 2\alpha)}{6k(k + \alpha)(k + 2\alpha)(k + 3\alpha)}(1 + \frac{3\alpha}{k - 1})
\]

(306)
and
\[ C_\Pi = \frac{(1 + \alpha)}{2k(k + \alpha)(k + 2\alpha)(k + 3\alpha)}(1 + \frac{\alpha}{k - 1}) \] (307)

We have from Table D-3,
\[ C_{II,\Pi} = \frac{(1 + \alpha)^2}{4k(k + \alpha)(k + 2\alpha)(k + 3\alpha)}(1 + \frac{4\alpha}{k - 1} + \frac{2\alpha^2(2 + \alpha)}{(1 + \alpha)(k - 1)(k + \alpha - 1)}) \] (308)

and
\[ C_\square = \frac{1}{k(k + \alpha)(k + 2\alpha)(k + 3\alpha)}(1 + \frac{2\alpha}{k - 1}) \] (309)

For the differentiation \( D_{2233}^{1113} \), we use obtained values of \( C_\Pi, C_{III,1} \), then we find uniquely,
\[ C_\Delta = \frac{1 + \alpha}{2k(k + \alpha)(k + 2\alpha)(k + 3\alpha)}(1 - \frac{2\alpha^2}{k - 1}) \] (310)

For \( D_{1465}^{1123} \), we subtract the two known expressions of \( C_{Y,1} \) and \( C_{A,111} \), and obtain the following two coefficients,
\[ C_{II,11} = \frac{1 + \alpha}{(2k + \alpha)(k + 2\alpha)(k + 3\alpha)}\left[1 - \frac{2\alpha}{k + \alpha - 1} + \frac{7\alpha}{k - 1} + \frac{6\alpha^2}{(k - 1)(k - 2)} - \frac{2\alpha^2}{(k - 2)(k + \alpha - 1)}\right] \] (311)
\[ C_{N,1} = \frac{1}{k(k + \alpha)(k + 2\alpha)(k + 3\alpha)}\left[1 + \frac{8\alpha}{k - 1} - \frac{4\alpha}{k - 2} - \frac{\alpha^2}{(k + \alpha - 1)(k - 2)} + \frac{(1 + \alpha)(4\alpha k + 3\alpha^2 - 4\alpha)}{(k - 1)(k - 2)(k + \alpha - 1)}\right] \] (312)

We have chosen the overall factor of \( C_{II,11} \) as \( \frac{1 + \alpha}{2} \). The factor \[ 1 - \frac{2\alpha}{k + \alpha - 1} + \frac{\alpha^2}{k - 1} + \frac{6\alpha^2}{(k - 1)(k - 2)} - \frac{2\alpha^2}{(k - 2)(k + \alpha - 1)} \] in \( C_{II,11} \), is the same as for \( C_{A,111} \).

For \( D_{2234}^{1112} \), we obtain after the subtraction of the known three terms,
\[ C_\triangle = \frac{1}{k(k + \alpha)(k + 2\alpha)(k + 3\alpha)}(1 - \frac{\alpha(\alpha - 1)}{k - 1}) \] (313)

This expression becomes \( \frac{(k - 3)}{(k - 1)^2(k - 2)^2} \) for \( \alpha = -1 \), and it is consistent with the value given in Table A. For \( D_{2235}^{1112} \), using the previously determined 6 coefficients, one obtains \( C_{\Delta,1} \) as
\[ C_{\Delta,1} = \frac{1}{k(k + \alpha)(k + 2\alpha)(k + 3\alpha)}\left[1 - \frac{\alpha(\alpha - 3)}{k + \alpha - 1} - \frac{\alpha^2(4\alpha - 3)}{(k - 2)(k + \alpha - 1)} - \frac{\alpha^2(3\alpha^2 - 2\alpha + 3)}{(k - 1)(k - 2)(k + \alpha - 1)}\right] \] (314)
This expression reduces to \( (k-3)^2 \) for \( \alpha = -1 \), and this agrees with the result given in Table A.

We have thus obtained explicitly all the coefficients up to fourth order. They satisfy a remarkable property. When the parameter \( \alpha \) vanishes

\[
\sum_{m=0}^{\infty} \frac{1}{m! \prod_{q=0}^{m-1} (1 + q\alpha)} \sum_{p} \chi_p Z_p(\alpha) Z_p(\lambda) = \prod_{m=1}^{q-1} (1 + m\alpha) \]

becomes unity, and \( f \) in (287) is simply given by

\[
f = e^{-k/2} \sum_{i<j} \tau_{ij}.
\]

Then the coefficients are given by

\[
C = (-1)^l \frac{g}{k^l}
\]

where \( l \) is the order of the \( \tau \) term, and \( C \) the corresponding coefficient; \( g \) is a degeneracy factor for the multiple lines of the graphs. For the double line, \( g = 1/2! \), and for the triple line, \( g = 1/3! \) etc.

The pole terms with denominators \( (k-1),(k-2),(k+\alpha-1),..., \) which come from \( \frac{1}{Z_p(I)} \) in (287) disappear in the expression of the coefficients in the limit \( \alpha = 0 \). This leads to the fact that the pole terms \( \frac{1}{k-1}, \frac{1}{k-2}, \frac{1}{k+\alpha-1},... \), always appear with a factor proportional to \( \alpha \) or to a power of \( \alpha \). Then, the coefficients \( C \) may be factorized as

\[
C = (-1)^l \frac{g}{k^l} \frac{1}{k(k+\alpha)(k+2\alpha) \cdots (k+(l-1)\alpha)} (1 + O(\alpha))
\]

where \( g \) is a degeneracy factor, and for the single multiple line graph case, it is given by

\[
g = \prod_{m=1}^{q-1} (1 + m\alpha) \frac{q!}{q!}
\]

\( q \) is the number of multiple lines. When there are \( n \) multiple lines, \( (q_1,q_2,...,q_n) \) in a graph, the degeneracy factor \( g \) becomes

\[
g = \prod_{i=1}^{n} \frac{\prod_{m=1}^{q_i-1} (1 + m\alpha)}{q_i !}
\]

For instance, in the case of \( C_{III} \), the degeneracy factor \( g \) is \( (1 + \alpha)^2 \). 2!2!

The correction of order \( \alpha \) in (317) is order of \( \frac{1}{k} \) in the large \( k \) limit. Therefore, in the large \( k \) limit, all the coefficients of the \( \tau \)-expansion are given by (317).
The second remark is about the correction terms to (317). Common corrections appear when graphs have common structures. For instance, star graphs have no corrections. As star graphs, we have up to the fourth order, $C_1, C_A, C_{II}, C_Y, C_{III}, C_{\Lambda}, C_X, C_{\Lambda'}, C_{\Xi}, C_{\Xi'}, C_{\Xi''}, C_{III}$. They are given exactly by the leading term of (317).

For the graphs with one separate line, the correction terms are again all the same if the remaining part is a star graph. For instance, we find a common factor for $C_{\Lambda, I}$ and $C_{II, I}$, which is $1 + \frac{2\alpha}{k - 1}$. For $C_{Y, I}, C_{\Xi', I}, C_{III, I}$, the common is $1 + \frac{3\alpha}{k - 1}$. 
References

[1] S. Ben Said and B.Ørsted, J. Math. Pures et Appli. 84, 1393 (2005).

[2] Yu. Berest, CRM Proc.Lect.Notes. Vol.14, The bispectral problem, 11-30 (1998);
   O.A. Chalykh, M.V. Feigin and A.P.Veselov, Commun. Math. Phys. 206, 533 (1999).

[3] E. Brézin and S. Hikami, Commun. Math. Phys. 214, 111 (2000).

[4] E. Brézin and S. Hikami, Commun. Math. Phys. 223, 363 (2001).

[5] E. Brézin and S. Hikami, Commun. Math. Phys. 235, 125 (2003).

[6] E. Brézin and S. Hikami, J. Phys. A:Math.Gen. 36, 711 (2003).

[7] E. Brézin and S. Hikami, Nucl. Phys. B479, 697 (1996).

[8] E. Brézin and S. Hikami, Phys. Rev. E58, 7176 (1998).

[9] E. Brézin and S. Hikami, Phys. Rev. E56, 264 (1997).

[10] E. Brézin and S. Hikami, preparation.

[11] Harish-Chandra, Proc. Nat. Acad. Sci. 42, 252 (1956).

[12] C. Itzykson and J.-B. Zuber, J. Math. Phys. 21, 411 (1980).

[13] H. Jack, Proc. R. Soc. Edingburgh (A), 69, 1 (1970).

[14] H. Jack, A class of polynomials in search of a definition, or the symmetric group parametrized, manuscript. in ”Jack, Hall-Littlewood and Macdonald polynomials”, Contemporary Mathematics, edited by B. Kuznetsov and S. Sahi, (AMS, to be published in 2006). The table in the Appendix A up to the order six is based on this paper with corrections at  \(Z(2^3)\) for  \(S_3^2S_3\). The value \(-4\alpha(3\alpha - 1)\) is corrected as \(-4(3\alpha - 1)\).

(http://www.amsta.leeds.ac.uk/~vadim/jhlm.htm)

[15] A.T. James, Ann. Math. Stat. 31, 151 (1960); 32, 874 (1961); 35, 475 (1965).

[16] I.G. Macdonald, Symmetric functions and Hall polynomials, (Oxford University Press Inc., New York, 1995).

[17] M.L. Mehta, Random matrices, 2nd ed. (Academic Press, New York, 1991).

[18] P. Zinn-Justin and J.-B. Zuber, J. Phys. A 36, 3173 (2003).