Adaptive sampling for linear state estimation

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Abstract. State estimation under sampling rate constraints is important for Networked control. To obtain the lowest possible estimator distortion under such constraints, the samples must be chosen adaptively based on the trajectory of the signal being sampled, rather than deterministically. We treat the case of perfect observations at the sensor in which it measures a diffusion state process perfectly. The sensor has to choose causally, exactly \( N \) sampling times when it transmits samples to a supervisor which receives the samples without delay or distortion. Based on the causal sequence of samples it receives, the supervisor maintains a continuous MMSE estimate. In this paper we provide the optimal adaptive sampling rules to be used by the sensor that minimize the aggregate, finite-horizon, mean-square error distortion for scalar linear estimation. For the mean squared error distortion criterion, we reduce the problem of joint choice of sampling rules and estimator waveforms to one of choosing the sampling rules alone. We find the optimal sampling times to be first hitting times of time-varying, double-sided and symmetric envelopes by the estimation error signal. When the state is Brownian motion, we characterize the optimal sampling envelopes analytically - they are reverse parabolic. In the case of the Ornstein-Uhlenbeck process, we provide a numerical procedure for computing these envelopes. We also characterize the performance of Delta sampling and furnish rules to recursively choose the successive thresholds optimally. The results of these calculations are surprising. Delta sampling performs worse than even the periodic sampling scheme, except possibly when the sample budget is quite small. Moreover, the estimation distortion due to Delta sampling decreases as the sample budget is increased but is bounded away from zero.

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1. Introduction

Networked control systems involve control loops some of whose constituent links are completed over data networks rather than over dedicated analog wires. In such systems, monitoring and control tasks have to be performed under constraints on the amount of information communicated to the control station. These communication constraints limit the rate of packet transmissions from sensor nodes to the supervisor node(s). Apart from the limits on transmission rates, the packets could also be delivered with delays and sometimes, be lost. All of these communication degradations in the networked problem lower performance and system design must take them into account. In this paper, we study only the effect of constraints on communication rates.

Communication rate constraints can be of the following three types: 1) Average rate limit: This is a ‘soft constraint’ and calls for an upper limit on the average number of transmissions; 2) Minimum waiting time between transmissions: Under this type, there is a mandatory minimum waiting time between two successive transmissions from the same node; 3) Finite transmission budget: This is a ‘hard constraint’ and allows only up to a prescribed number of transmissions from the same node over a given time window. In the simplest application of this type of constraint, we set the constraint’s window to be the problem’s entire time horizon. In a variation, one meant to reduce the jitter in sample times, we partition the problem’s time-horizon into multiple disjoint segments and, for every segment, apply the finite transmission budget constraint.

Notice that these different types of constraints can be mixed in interesting ways. In this work, we will adopt the simple version of the finite transmission budget applied over the entire time horizon of the problem, as the communication constraint, it being natural for finite-horizon problems. We study a problem of state estimation which is an important component of distributed control and monitoring systems. Specifically, a scalar linear signal is continuously and fully observed at a sensor which generates a limited number of samples. A supervisor receives the sequence of samples and, on its basis, maintains a causal estimate. The restriction on the number of samples clearly limits the minimum estimation distortion achievable at the supervisor. The design question is: How should the samples be chosen by the sensor to minimize the estimation distortion ? The answer to this question employs the idea that: Samples should be generated only when they contain ‘sufficiently’ new information. Adaptive sampling schemes, or, event-triggered sampling schemes as they are also called, exploit this idea and generate samples at times determined by the trajectory of the source signal being sampled. In contrast, deterministic sampling chooses sample times according to an extraneous clock. The search for efficient adaptive schemes leads us to a problem of optimal timing, one seeking the minimum distortion of the estimator.

Under adaptive sampling, the sensor chooses sampling times based on information it gathers from the continuous observations; the supervisor maintains its estimate based on information from sample times and samples. This information pattern is non-classical and leads to an estimator whose error statistics are more complicated than the one under deterministically sampled systems. We tackle this complication, simplify optimal timing problem, and, show that adaptive sampling policies that apply (time-varying) thresholds on the estimation error signal achieve the minimum distortion. We also study a suboptimal but well-known and natural adaptive policy: Delta-sampling. We characterize completely the structure of optimal sampling, and, the performances of optimal and Delta-sampling policies for drift-free linear systems (the Brownian motion process). For processes with non-zero drift (the Ornstein-Uhlenbeck process), we characterize the
structure of the optimal policy but have to compute its performance, and, that of Delta-sampling through numerical computations. We indicate the procedures for these computations.

1.1. Packet rate constraints in networked control systems. Sampling rate constraints arise in networked control systems because of constraints on the rate of data packets. We will deduce this by examining the properties of packet-based communication schemes. In most scenarios \[9, 20, 15\], the packets are of uniform size and even when of variable size, have at least a few bytes of header and trailer files. These segments of the packet carry source and destination node addresses, a time stamp at origin, some error control coding, some higher layer (link and transport layers in the terminology of data networks) data blocks and any other bits/bytes that are essential for the functioning of the packet exchange scheme but which nevertheless constitute what is clearly an overhead. The payload or actual measurement information in the packet should then be at least of the same size as these ‘bells and whistles’. It costs only negligibly more in terms of network resources, of time, or of energy to send a payload of five or ten bytes instead of two bits or one byte when the overhead part of the packet is already five bytes. This means that the samples being packetized can be quantized with very fine detail, say with four bytes, a rate at which the quantization noise can be ignored for low dimensional variables. For Markov state processes, this means that all of these bytes of payload can be used to specify the latest value of the state. In other words, in essentially all packet-based communication schemes, the right unit of communication cost is the cost of transmitting a single packet whether or not the payload is longer by a few bytes; the exact number of bits used to quantize the sample is not important. There are of course special situations where the quantization rate as well as the sample generation rate matter. An example occurs in the internet congestion control mechanism called Transmission Control Protocol (TCP) \[14\] where a node estimates the congestion state of a link through congestion bits added to regular data packets it receives. In this case, the real payload of packets is irrelevant to the congestion state, and, the information on the congestion state is derived from the one or two bits riding piggy-back on the data packets. The developments in this paper does not apply to such problems where the effect of quantization is important.

State estimation problems with communication rate constraints arise in a wide variety of networked monitoring and control setups such as sensor networks of various kinds, wireless industrial monitoring and control systems, rapid prototyping using a wireless network, and, multi-agent robotics. A recent overview of research in Networked control systems including a variety of specific applications is available from the special issue \[3\]. Event-triggered sampling may also be used to model the functioning of various neural circuits in the nervous systems of animals. After all, the neuron is a threshold-triggered firing device whose operation is closely related to Delta-sampling. However, it is not presently clear if the communication rate constraint adopted in this paper occurs in Biological Neural networks.

1.2. Relationship to previous works. The problem of choosing the time instants to sample sensor measurements has received early attention in the literature. Kushner \[18\] studied the deterministic choice of measurement times in a discrete-time, finite-horizon LQG optimal control problem. He found that controls can be chosen like in a conventional LQG control problem with fixed deterministic sampling instants. Using this fact, he shows that the best (deterministic) sampling schedule can be found by solving a nonlinear optimization problem. Skafidas & Nerode \[29\] also adopt the same setting
but allow the sensor measurement times to be chosen online by the controller. Their conclusion is that the optimal choice of measurement times can be made offline. Their simplified scheduling problem is the same as Kushner’s deterministic one.

The Sensor scheduling problem is a generalization of the problem of choosing measurement times which has been studied for estimation, detection and control tasks \[22, 21, 5, 30\]. The problem asks for online schedules of when to gather measurements from different sensors available; in some setups, at most one sensor can be scheduled at a time. However, the information pattern for this problem is the same as the the works of Kushner and of Skafidas & Nerode. The notable fact about this information pattern is that the flow of data from sensors to their recipients namely, controllers, estimators etc., is regulated by the recipient. Such sensor sampling is of the “pull” type. An alternative is the “push” type of sampling where sensor itself regulates the flow of its data. When only one sensor is available, it has more information than the controller and hence, its conclusions on when to communicate its measurements can potentially improve performance.

Åström & Bernhardsson \[4\] treat a minimum variance control problem with the push type of sampling. The control consists of impulses which reset the state to the origin, but there is an upper limit on the average rate at which impulses can be applied. Under such a constraint, the design asks for a schedule of the application times for the impulses. For scalar Gaussian diffusions, they perform explicit calculations to show that the application of impulses using Delta sampling, namely triggering by first hitting of fixed thresholds is more efficient than the periodic application of impulses. Their conclusions are relevant for us because, in our estimation problem, the estimation error signal is reset to zero at sampling times, and so, sampling in our problem corresponds in their problem to resetting the state by applying an impulse. With this correspondence, further conclusions can be drawn about the problem of repeated sampling for estimation over an infinite horizon. The thesis \[25\] of the first author of this paper shows that the Delta sampling scheme is indeed optimal for the infinite horizon problem of \[4\]. Recently, Henningsson et.al. \[12\] have extended the results of \[4\] and of \[25\] to include a communication constraint containing both an average rate constraint and a minimum waiting time between sampling instants. We should also mention the interesting fact that Delta sampling of a Wiener process has been found by Berger \[6\] to be the compression scheme achieving the Information theoretic Rate-Distortion limit. The work of Lazar & Tóth \[19\] which attempts to extend the Shannon-Nyquist sampling theorem to the case of Delta sampling of deterministic signals.

The work of Imer & Basar \[13\] considers the finite-horizon adaptive sampling problem but in a discrete time setting. They use the optimal form of the estimator under adaptive sampling with a partial proof (see proposition 1 of \[13\]) and describe dynamic programming equations to be satisfied by the optimal sampling policy. We should point out that their sampling scheme for the quadratic distortion namely the time-varying symmetric threshold for the estimation error, is indeed the optimal form of the sampling policy - this can be proved, along the lines of our work, by converting the optimal stopping problem for minimum estimation distortion into one with just a terminal cost or reward.

The study of optimal adaptive sampling timing leads to Optimal stopping problems of Stochastic control, or equivalently as we have already seen, to impulse control problems. The information pattern of adaptive sampling complicates the picture but methods of solving multiple stopping time problems of standard form which are available in the literature \[8\] are indeed useful.

The work reported in this paper has been announced previously in \[25, 26, 27, 28\].
1.3. Contributions and outline of the paper. For the finite horizon state estimation problem, we cast the search for efficient sampling rules as sequential optimization problem over a fixed number of causal sampling times. This we do in section 2 where, we formulate an optimal multiple stopping problem with the aggregate quadratic distortion over the finite time horizon as its objective function. Although the information pattern for the resulting sequential decision problem is non-classical, we use the facts that the state evolution is linear and the quadratic error is an even function, and show that the optimal sampling policy produces a simple and intuitive estimator. We also suggest some modified design strategies for nonlinear and partially observed cases.

In section 3, we take the simplified optimal multiple stopping problem and solve it explicitly when the state is the (controlled) Brownian motion process. The optimal sampling policies are first hitting times of time-varying envelopes by the estimation error signal. Our analytical solution shows that for each of the sampling times, the triggering envelopes are symmetric around zero and diminish monotonically in a reverse-parabolic fashion as time nears the end of the horizon. We also describe analytically, the performance of the class of modified Delta sampling rules in which, we apply different values of the threshold $\Delta$ for the different sampling times. We point out a simple and recursive procedure for choosing the most efficient of these Delta sampling policies.

For the Ornstein-Uhlenbeck process, in section 4, we derive Dynamic programming equations for optimal sampling policy. We compute the solution to these equations numerically. We are not able to say whether an explicit analytic solution like for the Brownian motion process is possible. The optimal sampling times are first hitting times of time-varying envelopes by the estimation error signal. These envelopes are symmetric around zero and diminish monotonically as time nears the end of the horizon. Also derived are the equations governing the performance of modified Delta sampling rules and the most efficient among them is found through a numerical search. Finally, in section 5, we conclude and speculate on extensions to this work for other estimation, control and detection problems.

2. MMSE estimation and optimal sampling

Under a deterministic time-table for the sampling instants, the Minimum mean square error (MMSE) reconstruction for linear systems is well-known and is straightforward to describe - it is the Kalman filter with intermittent but perfect observations. The error variance of the MMSE estimate obeys the standard Riccati equation. In Delta-sampling $^{23, 10, 11}$, also called Delta-modulation, a new sample is generated when the source signal moves away from the previously generated sample value by a distance $\Delta$. By this rule, between successive sample times, the source signal lies within a ball of radius $\Delta$ which is centered at the sample value at the earlier of the two times. Such information being available about the behaviour of the source signal between sample instants is typical of adaptive sampling schemes but never of deterministic sampling schemes. A reconstruction, causal or not, of a signal based on its samples generated adaptively leads to an estimator structure different from that of deterministic sampling. In this section, we will describe the modified structure for a causal estimator minimizing the MMSE.

We will also set up an optimization problem where we seek an adaptive sampling policy minimizing the distortion of the MMSE estimator subject to a limit on the number of samples. By examining some simple symmetry in the statistics of linear systems perturbed by white noise, we will obtain simplifications for the MMSE estimator under optimal sampling. Consider a state process $x_t$ which is a (possibly controlled) scalar
linear diffusion. It evolves according to the SDE:

\[ \begin{align*}
    dx_t &= ax_t dt + bdB_t + u_t dt, \\
    x_0 &= x,
\end{align*} \]

where, \( B_t \) is a standard Brownian motion process. The control process \( u_t \) is RCLL and of course measurable with respect to the \( x \)-process. In fact, the feedback form of \( u_t \) is restricted to depend on the sampled information only; we will describe this subsequently. We assume that the drift coefficient \( a \), the noise coefficient \( b \neq 0 \), and, the initial value \( x \) are known. Now, we will dwell upon sampling and the estimation process.

The state is sampled at instants \( \{ \tau_i \}_{i \geq 0} \) which are stopping times w.r.t. the \( x \)-process. The samples along with the sampling times are encapsulated into data packets and sent over the communication link. This link, while limited in its average throughput, does not require any minimum interval between packets; It also delivers the packets it handles, reliably and with negligible delay. Essential to this abstraction is an assumption that the transactions on the data network happen on a time scale faster that the dynamics of the plant. Formally, we require that the length of data packets be much smaller than the time-constant of the plant: \( \frac{1}{a} \). The samples are of scalars or low-dimensional vectors and so quantization can be ignored.

Let \( \Sigma_t \) represent the data contained in the packets received at the estimator. It is a growing sequence containing the sequence of samples received and also their timing:

\[ \Sigma_t = \left\{ (\tau_i, x_{\tau_i}) \, | \, \tau_i \leq t \right\}. \]

Notice that the binary-valued process \( 1_{\{\tau_i \leq t\}} \) is measurable w.r.t. \( \mathcal{F}^\Sigma_t \). The MMSE estimate \( \hat{x}_t \) is based on knowledge of the multiple sampling policy and all the information contained in the output of the sampler and so it can be written as:

\[ \hat{x}_t = \mathbb{E}\left[ x_t | \Sigma_t \right]. \]

The control signal \( u_t \) is measurable w.r.t. \( \mathcal{F}^\Sigma_t \). Typically it is restricted to be of the certainty-equivalence type as depicted in figure \( 1(a) \). In that case \( u_t \) is, in addition, measurable w.r.t. \( \mathcal{F}^{\hat{x}}_t \). The exact form of the feedback control is not important for our
work, but the requirement that the estimator knows the feedback control policy (and so has access to the control waveform $u^t$) is essential. With this assumption, at the estimator, the control waveform is a known additive component to the state evolution and can be subtracted out. Hence, there is no loss of generality in considering only the uncontrolled plant.

2.1. MMSE estimation under deterministic sampling. Consider now a deterministic scheme for choosing the sampling times. Let the sequence of non-negative and increasing sampling times be:

$$D = \{d_0 = 0, d_1, \ldots\},$$

where, the times $d_i$ are all statistically independent of all data about the state received after time zero. They could at most be correlated to the initial value of the state $x_0$.

We will now describe the MMSE estimate and its variance. Consider a time $t$ in the semi-open interval $[d_i, d_{i+1})$. We have:

$$\hat{x}_t = \mathbb{E}\left[x_t \mid \Sigma_t\right],$$

where, we have used the Markov property of the state process and the mutual independence, conditional on $x_0$, of the state and the sequence $D$. Furthermore,

$$\hat{x}_t = \mathbb{E}\left[e^{a(t-d_i)} x_{d_i} + \int_{d_i}^t e^{a(t-s)} u_s b dB_s + \int_{d_i}^t e^{a(t-s)} u_s u_s ds \mid d_i, x_{d_i}\right],$$

$$= e^{a(t-d_i)} x_{d_i} + \int_{d_i}^t e^{a(t-s)} u_s u_s ds.$$

Thus, under deterministic sampling, the MMSE estimate obeys a linear ODE with jumps at the sampling times.

$$\frac{d\hat{x}_t}{dt} = a\hat{x}_t + u_t, \text{ for } t \notin D, \text{ and, } \hat{x}_t = x_t, \text{ if } t \in D.$$

The variance $p_t = \mathbb{E}\left[(x_t - \hat{x}_t)^2\right]$ is given by the well-known Riccati equation:

$$\frac{dp_t}{dt} = 2ap_t + b^2, \text{ for } t \notin D, \text{ and, } p_t = 0, \text{ if } t \in D.$$

The above description for the MMSE estimate and its variance is valid even when the sampling times are random provided that they are independent of the state process. Then, the evolution equations for the MMSE estimate and its error statistics remain independent of the policy for choosing the sampling times; the solution to these equations merely get reset with jumps at these times. On the other hand, adaptive sampling modifies the evolution of the the MMSE estimator as we will see next.
2.2. The MMSE estimate under adaptive sampling. Under adaptive sampling policies, the sampling times are stopping times w.r.t. the state process. Between sample times, an estimate of the state using any distortion criterion is an estimate up to a stopping time and this is the crucial difference from deterministic sampling. At time $t$ within the sampling interval $[\tau_i, \tau_{i+1})$, the MMSE estimate is given by:

\[
\hat{x}_t = \mathbb{E}[x_t \mid \Sigma_t],
\]

\[
= \mathbb{E}[x_t \mid \tau_i \leq t < \tau_{i+1}, \{ (\tau_j, x_{\tau_j}) \mid 0 \leq j \leq i \}],
\]

\[
= \mathbb{E}[x_t \mid \tau_i \leq t < \tau_{i+1}, \tau_i, x_{\tau_i}],
\]

\[
= x_{\tau_i} + \mathbb{E}[x_t - x_{\tau_i} \mid t - \tau_i < \tau_{i+1} - \tau_i, \tau_i, x_{\tau_i}].
\]

Similarly, its variance $p_t$ can be written as:

\[
p_t = \mathbb{E}[(x_t - \hat{x}_t)^2 \mid \tau_i \leq t < \tau_{i+1}, \tau_i, x_{\tau_i}].
\]

Between samples, the MMSE estimate is an estimate up to a stopping time because the difference of two stopping times is also a stopping time. In general, it is different from the MMSE estimate under deterministic sampling (see appendix [3]). This simply means that in addition to the information contained in previous samples and sample times, there are extra partial observations about the state. This information is the fact that the next stopping time $\tau_{i+1}$ has not arrived. Thus, in adaptive schemes, the evolution of the MMSE estimator is dependent on the sampling policy. This potentially opens the possibility of a timing channel [2] for the MMSE estimator.

Figure 1(b) describes a particular (sub-optimal) scheme for picking a single sample. There are two time-varying thresholds for the state signal, an upper one and a lower one. The initial state is zero and within the two thresholds. The earliest time within $[0, T]$, when the state exits the zone between the thresholds is the sample time. The evolution of MMSE estimator is dictated by the shape of the thresholds thus utilizing information available via the timing channel.

2.3. An optimal stopping problem. We formalize a problem of sampling for optimal estimation over a finite horizon. On the interval $[0, T]$, we seek for the state process \(x\), with the initial condition $x_0$, an increasing and causal sequence of at most $N$ sampling times $\{\tau_1, \ldots, \tau_N\}$, to minimize the aggregate squared error distortion:

\[
J(T, N) = \mathbb{E}\left[\int_0^T (x_s - \hat{x}_s)^2 ds\right].
\]

Notice that the distortion measure does not depend on the initial value of the state because, it operates on the error signal ($e_t = x_t - \hat{x}_t$) which is zero at time zero. Notice also that the communication constraint is captured by an upper limit on the number of samples. In this formulation, we do not get any reward for using fewer samples than the budgeted limit.

The optimal sampling times can be chosen one at a time using a nested sequence of solutions to optimal single stopping time problems. This is because, for a sampling time $\tau_{i+1}$ which succeeds the time $\tau_i$, using a knowledge of how to choose $\tau_{i+1}$ optimally, we can obtain an optimal choice for $\tau_i$ by solving over $[0, T]$ the optimal single stopping time
problem

$$\text{ess inf}_{\tau \geq 0} \mathbb{E} \left[ \int_0^{\tau_i} (x_s - \hat{x}_s)^2 ds + J^* (T - \tau_i, N - i) \right],$$

where, $J^* (T - \tau_i, N - i)$ is the minimum distortion obtained by choosing $N - i$ sample times $\{\tau_{i+1}, \ldots, \tau_N\}$ over the interval $[\tau_i, T]$. The best choice for the terminal sampling time $\tau_N$ is based on solving a single stopping problem. Hence we can inductively find the best policies for all earlier sampling times. Without loss of generality, we can examine the optimal choice of the first sampling time $\tau_1$ and drop the subscript 1 in the rest of this section.

2.3.1. An optimal stopping problem with non-standard costs. The sampling problem is to choose a single $F_t^\tau$-stopping time $\tau$ on $[0, T]$ to minimize

$$F (T, 1) = \mathbb{E} \left[ \int_0^T (x_s - \hat{x}_s)^2 ds + J^* (T - \tau, N - 1) \right],$$

where,

$$J^* = \text{ess inf}_{\{\tau_2, \ldots, \tau_N\}} \mathbb{E} \left[ J (T - \tau, N - 1) \right].$$

The cost $F$ has a terminal component which is standard. The terminal part is a function only of the number of remaining samples $N - 1$, and, the amount of time left $T - \tau$ because, given $\tau$, the terminal part is independent of $x_\tau$.

But, the cost has a running integral up to the stopping time whose integrand depends on the stopping policy. If, for every stopping policy, we can somehow determine the waveform $\{\hat{x}_t\}_{t=0}^T$, then, we can solve the optimization problem by solving a standard optimal stopping problem. To solve such typical stopping problems, we can use the process of minimum expected distortion, the so-called Snell envelope ($S_t$) (see [17] Appendix D and also [24]):

$$S_t = \text{ess inf}_{\tau \geq t} \mathbb{E} \left[ \int_t^T (x_s - \hat{x}_s)^2 ds + J^* (T - \tau, N - 1) \bigg| F_t^\tau \right],$$

$$= \int_t^T (x_s - \hat{x}_s)^2 ds + \text{ess inf}_{\tau \geq t} \mathbb{E} \left[ \int_t^T (x_s - \hat{x}_s)^2 ds + J^* (T - \tau, N - 1) \bigg| x_t \right].$$

Then, the smallest time when the cost of stopping does not exceed the Snell envelope is an optimal stopping time. Since the Snell envelope depends only on the current value of the state and the current time, we get a simple threshold solution for our problem. Now, we will use some properties of the state process and of the squared error distortion criterion to bring our stopping problem into standard form.

2.4. Linearity simplifies the MMSE estimate under optimal adaptive sampling. The sampling problem for state estimation has been cast so far as a non-standard optimal stopping problem with the filter $\hat{x}_t$ being dependent on the stopping rule being optimized. Now, we will look at this optimization as one over the stopping rule as well as a deterministic waveform $\{\xi_t\}_{t=0}^T$ which the estimator uses to represent $x_t$. For example, the supervisor may want to use a piece-wise linear waveform to keep track of $x_t$ until the sampling time but use the least squares estimate $\hat{x}_t$ after the sample has been received. With the arbitrarily chosen piecewise deterministic estimate waveform, the distortion becomes:

$$\mathcal{H} \left( T, \{\xi_t\}_{t=0}^T \right) = \mathbb{E} \left[ \int_0^T (x_s - \xi_s)^2 ds + J^* (T - \tau, N - 1) \right].$$
All the sensor needs to do now is to tailor its stopping policy $T$ to minimize the aggregate distortion between the state and the supervisor’s estimate process. For a fixed estimator $\xi_t$, let $T^*\left(\{\xi_t\}_{t=0}^T\right)$ be an optimal stopping rule which produces the stopping time $\tau^*$ and minimizes $\mathcal{H}$, i.e.

$$\mathcal{G}\left(\{\xi_t\}_{t=0}^T\right) \overset{a.s.}{=} \mathcal{H}\left(T^*\left(\{\xi_t\}_{t=0}^T\right), \{\xi_t\}_{t=0}^T\right) \leq \mathcal{H}\left(T, \{\xi_t\}_{t=0}^T\right), \forall T.$$

**Lemma 1.** Let $\{\xi_t^*\}_{t=0}^T$ be a waveform that minimizes $\mathcal{G}$. Then, it follows that the pair $(T^*\left(\{\xi_t^*\}_{t=0}^T\right), \{\xi_t^*\}_{t=0}^T)$ minimizes $\mathcal{H}$. The optimal waveform $\{\xi_t^*\}_{t=0}^T$ has the property that it is also the MMSE estimate corresponding to the stopping policy $T^*\left(\{\xi_t^*\}_{t=0}^T\right)$:

$$\xi_t^* \overset{a.s.}{=} \mathbb{E}[x_t | \tau^* > t].$$

**Proof:** If the lemma were not true, we could achieve a lower cost by retaining the sampling policy $T^*\left(\{\xi_t^*\}_{t=0}^T\right)$ and using the conditional mean it generates as the waveform at the estimator. This lemma says that if a pair $(T^*\left(\{\xi_t^*\}_{t=0}^T\right), \{\xi_t^*\}_{t=0}^T)$ minimizes $\mathcal{H}$, then, $\{\xi_t^*\}_{t=0}^T$ is almost surely equal to the MMSE estimate of $x_t$ up to the stopping time generated by $T^*\left(\{\xi_t^*\}_{t=0}^T\right)$. Hence,

$$\mathcal{H}^*\left(T^*\left(\{\xi_t^*\}_{t=0}^T\right), \{\xi_t^*\}_{t=0}^T\right) = J^* (T, N).$$

Now, we will use the linearity of the state evolution to narrow down the kind of pairs of stopping rule and waveform that minimize $\mathcal{H}$. Conditional on the initial condition: $x_0 = x$, the unobserved $x_t$ has at all times, an even density function centered at

$$xe^{at} + \int_0^t e^{a(t-s)}u_s \, ds.$$ 

This follows from the linearity of the Fokker-Planck operator [26]. Now, it turns out that combining the waveform

$$\left\{2xe^{at} + 2 \int_0^t e^{a(t-s)}u_s ds - \xi_t^* \right\}_{t=0}^T$$

with the best stopping rule for the waveform $\{\xi_t^*\}_{t=0}^T$ does not increase the cost! This is because the “innovations” process

$$x_t - \left(xe^{at} + \int_0^t e^{a(t-s)}u_s ds\right)$$

has the same statistics as its negative

$$-x_t + \left(xe^{at} + \int_0^t e^{a(t-s)}u_s ds\right).$$

Since the conditional mean is the minimizing waveform a.s., we have that

$$\xi_t^* \overset{a.s.}{=} 2xe^{at} + 2 \int_0^t e^{a(t-s)}u_s ds - \xi_t^*.$$ 

Because of this, we can be assured of the validity of the following useful Lemma which is the fruit of our labours in this section.
Lemma 2. The optimal MMSE estimate is the mean of the Fokker-Planck equation and is given by:

\[ E[x_t|\tau^* > t] \overset{a.s.}{=} \xi_t^* = x e^{at} + \int_0^t e^{a(t-s)} u_s ds, \quad \forall \ t \in [0, T]. \]

Thus we have shown that under the optimal adaptive sampling strategy, the statistics of the state given the stream of time-stamped samples is such that the conditional mean is the same as that under deterministic sampling. However, because of the threshold nature of the optimal stopping policy, the higher moments of the estimation error are strictly lesser than those under any deterministic sampling scheme. Nevertheless we have succeeded in reducing the problem of optimal timing of samples to an optimal multiple stopping problem with an objective function of standard form.

2.5. Extensions to nonlinear and partially observed systems. For nonlinear plants, it is possible that under optimal sampling, the MMSE estimator is no longer the mean of the Fokker-Planck equation:

\[ \xi_t = E[x_t|\tau_{\text{latest}}, x_{\tau_{\text{latest}}}], \quad \text{where}, \ \tau_{\text{latest}} = \sup \{\tau_i \leq t\}. \]

To obtain a tractable optimization problem we can restrict the kind of estimator waveforms allowed at the control station. Using the Fokker-Planck mean above leads to a tractable stopping problem as does using the zero-order hold waveform:

\[ \xi_t = x_{\tau_{\text{latest}}}. \]

However, even a slightly more general piece-wise constant estimate:

\[ \xi_t = h(x_{\tau_{\text{latest}}}, \tau_{\text{latest}}). \]

leads to a stopping problem of non-standard form.

When the plant sensor has a noisy, and in the vector case case, partial observations, the sampling problem remains unsolved. The important question now is: What signal at the sensor should be sampled? Should the raw sensor measurements be sampled and transmitted, or, is it profitable to process them first? We propose a solution with a separation into local filtering and sampling. Accordingly, the sensor should compute a continuous filter for the state. The sufficient statistics for this filter should take the role of the state variable. This means that the sensor should transmit current samples of the sufficient statistics at, sampling times that are stopping times w.r.t. the sufficient statistics process.

In the case of a scalar linear system with observations corrupted by white noise, the local Kalman filter at the sensor \( \hat{x}_{\text{sensor}}^t \) plays the role of the state signal. The Kalman filter obeys a linear evolution equation and so the optimal sampling policies presented in this paper should be valid. In the rest of the paper, we will investigate and solve the sampling problem first for the Brownian motion process and then for the Ornstein-Uhlenbeck process.

3. Sampling Brownian motion

The sampling problem for Brownian motion with a control term added to the drift is no different from the problem without it. This is because the control process \( \{u_t\}_{t \geq 0} \) is measurable w.r.t. \( \mathcal{F}_t^\Sigma \), whether it is a deterministic feed-forward term or feedback based
on the sampled information. Thus, for the estimation problem, we can safely set the control term to be identically zero:

\[ dx_t = bdB_t, \quad x_0 = x. \]

The diffusion coefficient \( b \) can be assumed to be unity. If it is not, we can simply scale the time variable and in \( \frac{1}{b^2} \)-time, the process obeys a SDE driven by a Brownian motion with a unit diffusion coefficient. We study the sampling problem under the assumption that the initial state is known to the MMSE estimator. Note that as per Lemma 2 the MMSE estimate for this process is a zero order hold type reconstruction from the received sequence of samples.

In this chapter, we study three important classes of sampling. The optimal deterministic one is traditionally used and it provides an upper bound on the minimum distortion possible. The first adaptive scheme we study is Delta sampling which is based on first hitting times of symmetric levels by the error process. Finally, we completely characterize the optimal stopping scheme by recursively solving an optimal multiple stopping problem.

**3.1. Optimal deterministic sampling.** Given that the initial value of the error signal is zero, we will show through induction that uniform sampling on the interval \([0, T]\) is the optimal deterministic choice of \( N \) samples. Set the deterministic set of sample times to be:

\[ D = \{d_1, d_2, \ldots, d_N \mid 0 \leq d_i \leq T, \quad d_i \geq d_{i-1} \text{ for } i = 2, \ldots, N\}. \]

Then, the distortion takes the form:

\[ J_{\text{Deter}}(T, N) = \int_0^{d_1} \mathbb{E}(x_s - \hat{x}_s)^2 ds + \int_{d_1}^{d_2} \mathbb{E}(x_s - \hat{x}_s)^2 ds + \ldots + \int_{d_{N-1}}^{T} \mathbb{E}(x_s - \hat{x}_s)^2 ds. \]

Consider the situation of having to choose exactly one sample over the interval \([T_1, T_2]\) with the supervisor knowing the state at time \( T_1 \). The best choice of the sample time which minimizes the cost \( J_{\text{Deter}}(T_2 - T_1, 1) \) is the midpoint \( \frac{1}{2}(T_2 + T_1) \) of the given interval. We propose that the optimal choice of \( N - 1 \) deterministic samples over \([T_1, T_2]\) is the uniform one:

\[ \{d_1, d_2, \ldots, d_{N-1}\} = \left\{ T_1 + i \frac{T_2 - T_1}{N} \mid i = 1, 2, \ldots, N - 1 \right\}, \]
which leads to a distortion equaling $\frac{1}{2N}(T_2 - T_1)^2$. Now, we carry out the induction step to obtain the minimum distortion over the set of $N$ sampling times over $[T_1, T_2]$ to be:

$$
\min_{\{d_1, d_2, \ldots, d_N\}} J_{\text{Deter}}(T_2 - T_1, N) = \min_{d_1} \left\{ \int_0^{d_1} (x_s - \hat{x}_s)^2 ds + \min_{\{d_2, d_3, \ldots, d_N\}} J_{\text{Deter}}(T_2 - T_1 - d_1, N - 1) \right\},
$$

$$
= \min_{d_1} \left\{ \frac{d_1^2}{2} + \frac{(T_2 - T_1 - d_1)^2}{2N} \right\},
$$

$$
= \min_{d_1} \left\{ \frac{N d_1^2 + d_1^2 - 2d_2 (T_2 - T_1) + (T_2 - T_1)^2}{2N} \right\},
$$

$$
= \min_{d_1} \left\{ \frac{(N + 1)(d_1 - \frac{1}{(N+1)}(T_2 - T_1))^2 + \left(1 - \frac{1}{(N+1)}\right)(T_2 - T_1)^2}{2N} \right\},
$$

$$
= \frac{1}{2(N+1)}(T_2 - T_1)^2,
$$

the minimum being achieved for $d_1 = \frac{1}{N+1}(T_2 - T_1)$. This proves the assertion about the optimality of uniform sampling among all deterministic schemes provided that the supervisor knows the value of the state at the start time.

### 3.2. Optimal Delta sampling

As described before, Delta sampling is a simple event-triggered sampling scheme which generates a new sample whenever the input signal differs from the estimate by a pre-specified threshold. Delta sampling is really meant for infinite horizon problems and it produces inter-sample intervals that are unbounded. Since we have on our hands a finite horizon problem, we will use a time-out at the end time of the problem’s horizon. To make the most of this class of rules, we allow the thresholds to vary with the past history of sample times. This way, the sample times are chosen based on information available only at the sensor while, the sequence of thresholds which characterize sampling times depends on information available to the supervisor as well as the sensor.

More precisely, at any sampling time as well as at the start of the horizon, the threshold for the next sampling time is chosen. This choice is allowed to depend on the number of samples remaining as well as the amount of time left till the end of the horizon. We set $\tau_0 = 0$, and define thresholds and sampling times times recursively. The threshold for the $i^{th}$ sampling time is allowed to depend on the values of the previous sampling times, and so it is measurable w.r.t. $\mathcal{F}^\mathcal{E}_t$. Assume that we are given the policy for choosing causally a sequence of non-negative thresholds $\{\delta_1, \delta_2, \ldots, \delta_N\}$. Then, for $i = 1, 2, \ldots, N$, we can characterize the sampling times $\{\zeta_1, \zeta_2, \ldots, \zeta_N\}$ as follows:

$\mathcal{F}^{\delta_i} \subset \mathcal{F}^{\tau_{t_1}, \ldots, \tau_{t_{i-1}}} \text{ if } i > 1,$

$$
\tau_{t_i, \delta_i} = \inf \left\{ t : t \geq \tau_{t_{i-1}, \delta_{i-1}}, \left| x_t - x_{\tau_{t_{i-1}}} \right| \geq \delta_i \right\},
$$

$$
\zeta_i = \min \left\{ \tau_{t_i, \delta_i}, T \right\} \left( \triangleq \tau_{t_i, \delta_i} \wedge T \right).
$$

The first threshold $\delta_1$ depends only on the length of the time remaining at the start of the horizon namely, $T$.

The optimal thresholds can be chosen one at a time using solutions to a nested sequence of optimization problems each with a single threshold as its decision variable. This is because, for a sampling time $\zeta_{i+1}$ succeeding the time $\zeta_i$, using a knowledge of
how to choose $\zeta_{i+1}$ optimally, we can obtain an optimal choice for $\zeta_i$ by solving over $[0, T]$ the optimization problem:

$$\inf_{\delta_i \geq 0} \mathbb{E} \left[ \int_0^{\zeta_i} (x_s - \hat{x}_s)^2 ds + J^*_\text{Thresh} (T - \zeta_i, N - i) \right],$$

where the cost function $J^*_\text{Thresh} (T - \zeta_i, N - i)$ is the minimum aggregated distortion over $[T - \zeta_i, T]$ achievable using at most $N - i$ samples generated using thresholds for the magnitude of the error signal. Hence, if we know how to generate the last sample efficiently, we can inductively figure out rules governing efficient thresholds for earlier sampling times.

3.2.1. Optimal level for a single sample. We drop the subscript $N$ for the terminal sample time:

$$\tau_\delta = \inf_t \{t : |x_t - \hat{x}_t| = \delta \},$$

and its corresponding threshold $\delta$. Here, $\delta$ is a threshold independent of the data acquired after time 0. Our goal is to compute the performance measure for any non-negative choice of the threshold and then select the one that minimizes the estimation distortion:

$$J_{\text{Thresh}} (T, 1) (\delta) = \mathbb{E} \left[ \int_0^{\tau_\delta \wedge T} x_s^2 ds + \int_{\tau_\delta \wedge T}^T (x_s - x_{\tau_\delta \wedge T})^2 ds \right].$$
By using iterated expectations on the second term, we get:

\[
J_{\text{Threshold}}(T, 1)(\delta) = E \left[ \int_0^{\tau_3 \wedge T} x_3^2 ds + \int_{\tau_3 \wedge T}^T \left( (x_3 - x_{\tau_3 \wedge T})^2 \right) ds \right],
\]

\[
= E \left[ \int_0^{\tau_3 \wedge T} x_3^2 ds + \int_{\tau_3 \wedge T}^T \left( (x_3 - x_{\tau_3 \wedge T})^2 \right) ds \right],
\]

\[
= E \left[ \int_0^{\tau_3 \wedge T} x_3^2 ds + \int_{\tau_3 \wedge T}^T (s - \tau_3 \wedge T) ds \right],
\]

\[
(3)
\]

We have thus reduced the distortion measure to a standard form with a running cost and a terminal cost. We will now take some further steps and reduce it one with a terminal part alone. Notice that:

\[
d \left( (T - t) x_3^2 \right) = -x_3^2 dt + 2 (T - t) x_3 dx_3 + (T - t) dt,
\]

which leads to the following representation for the running cost term:

\[
E \left[ \int_0^{\tau_3 \wedge T} x_3^2 ds \right] = E \left[ (T - \tau_3 \wedge T) x_{\tau_3 \wedge T}^2 + \frac{T^2}{2} - \frac{1}{2} (T - \tau_3 \wedge T)^2 \right],
\]

\[
(4)
\]

Note that equation (4) is valid even if we replace \( \tau_3 \) with a random time that is a stopping time w.r.t. the \( x \)-process. Thus, the cost (3) becomes:

\[
J_{\text{Threshold}}(T, 1)(\delta) = \frac{T^2}{2} - \delta^2 E \left[ (T - \tau_3)^+ \right].
\]

If we can describe the dependence of the expected residual time \( E \left[ (T - \tau_3)^+ \right] \) on the threshold \( \delta \), then, we can parametrize the cost purely in terms of \( \delta \). Had we known the PDF of \( \tau_3 \) the computation of the expectation of the difference \( (T - \tau_3)^+ \) would have been easy. Unfortunately the PDF of the hitting time \( \tau_3 \) does not have a closed form solution. There exists a series representation we can find in page 99 of Karatzas and Shreve [16] which is:

\[
f_{\tau_3}(t) = \delta \sqrt{\frac{2}{\pi t^3}} \sum_{k=-\infty}^{\infty} (4k + 1)e^{-\frac{(4k+1)^2t^2}{2t}}.
\]

This series is not integrable and so it cannot meet our needs. Instead we compute the moment generating function of \( (T - \tau_3)^+ \) and thereby compute the expected distortion. These computations are carried out in appendix B. In particular, equation [17] gives the expression:

\[
J_{\text{Threshold}}(T, 1)(\lambda) = \frac{T^2}{2} \left\{ 1 + \frac{\pi^4}{32\lambda^2} - \frac{\pi^2}{4\lambda} - \sum_{k \geq 0} (-1)^k \frac{e^{-(2k+1)^2\lambda}}{(2k+1)^3} \right\},
\]

where, \( \lambda = \frac{T^2}{8\delta^2} \). Our parametrization in terms of the parameter \( \lambda \) reveals some structural information about the solution. Firstly, note that the length of the time horizon does not really matter. The form of the graph of \( J_{\text{Threshold}}(T, 1) \) as a function of the variable \( \lambda \) is
invariant w.r.t. $T$. It is merely scaled by the factor $\frac{T^2}{2}$. The behaviour of the distortion as $\lambda$ is varied can be seen in figure 3. The minimal distortion incurred turns out to be:

$$c_1 = 0.3952 \frac{T^2}{2},$$

this being achieved by the choice:

$$\delta^* = 0.9391 \sqrt{T}.$$ As compared to deterministic sampling, whose optimum performance is $0.5 \frac{T^2}{2}$, we realize that we have slightly more than 20% improvement by using the optimum thresholding scheme.

**How often does the Delta sampler actually generate a sample?** To determine that, we need to compute the probability that the estimation error signal breaches the threshold before the end time $T$. Equation 19 provides the answer: 98%. Note that this average sampling rate of the optimal Delta sampler is independent of the length of the time horizon.

We have provided a complete characterization of the performance of the Delta sampler with one allowed sample. In what follows, we will characterize the behaviour of multiple Delta sampling. In doing so, we will hit upon the remarkable fact that if the sensor is allowed two or more samples, it is actually more efficient to sample deterministically.

### 3.2.2. Multiple delta sampling.

Like in the single sample case, we will show that the expected distortion over $[0, T]$ given at most $N$ samples is of the form $c_N \frac{T^2}{2}$.

Let $\tau_\delta$ be the level-crossing time as before. Then, given a positive real number $\alpha$, consider the following cost:

$$\Upsilon(T, \alpha, \delta) = \mathbb{E}\left[\int_0^{\tau_\delta \wedge T} x^2 s + \alpha \left[(T - \tau_\delta)^+\right]^2 ds\right].$$

Using the same technique as in the single sample case (precisely, the calculations between and including equations 3, 5), we get:

$$\Upsilon(T, \alpha, \delta) = \frac{T^2}{2} - \delta^2 \mathbb{E}\left[(T - \tau_\delta)^+\right] - \left(\frac{1}{2} - \alpha\right) \mathbb{E}\left[(T - \tau_\delta)^+\right]^2.$$

This demands an evaluation of the second moment: $\mathbb{E}\left[(T - \tau_\delta)^+\right]^2$. The evaluation can be done similarly to the one for the first moment and is also described in appendix B. The results of these calculations let us write (equation 18):

$$\Upsilon(T, \alpha, \delta) = \frac{T^2}{2} \left\{\phi(\lambda) + \left[\frac{1}{2} - \alpha\right] \psi(\lambda)\right\},$$

where, $\lambda = \frac{T\pi}{2\delta^2}$, and we define the functions $\phi, \psi$ as follows:

$$\phi(\lambda) = 1 + \frac{\pi^4}{32\lambda^2} - \frac{\pi^2}{4\lambda} - \frac{\lambda}{\pi^2} \sum_{k \geq 0} \frac{(-1)^k e^{-(2k+1)^2\lambda}}{(2k+1)^3},$$

and,

$$\psi(\lambda) = -\frac{5\pi^4}{96\lambda^2} - \frac{\pi^2}{2\lambda} - 2 + \frac{16}{\pi\lambda^2} \sum_{k \geq 0} \frac{(-1)^k e^{-(2k+1)^2\lambda}}{(2k+1)^5}.$$

The choice of $\lambda$ that minimizes the cost $\Upsilon$ can be determined by performing a grid search for the minimum of the scalar function $\phi(\lambda) + \left[\frac{1}{2} - \alpha\right] \psi(\lambda)$. Since this sum is a fixed
Figure 3. We show the estimation distortion due to Delta sampling as a function of the threshold used. Notice that for a fixed \( \Delta \), the distortion decreases steadily as the maximum number of samples remaining (\( N \)) grows. The distortion however never reaches zero. The minimum distortion reaches its lower limit of \( 0.287 \frac{T^2}{\pi^2} \).

function, we conclude that the minimum cost is a fixed percentage of \( \frac{T^2}{\pi^2} \) exactly as in the case of the single sample. This property of this optimization problem is what enables us to determine optimal multiple Delta samplers by induction.

Let us return to the distortion due to \( N \) samples generated using a Delta sampler, with the budget \( N \) being at least 2. If we have determined the optimal Delta samplers for utilizing a budget of \( N - 1 \) or less, then the optimal distortion with a budget of \( N \) samples takes the form:

\[
J^*_{\text{Thresh}}(T, N) = \inf_{\delta_N \geq 0} \mathbb{E}
\left[
\int_0^\tau_{\delta_N} (x_s - \hat{x}_s)^2 ds + J^*_{\text{Thresh}}(T - \tau_{\delta_N}, N - 1)
\right],
\]

\[
= \inf_{\delta_N \geq 0} \mathbb{E}
\left[
\int_0^{\tau_{\delta_N} \wedge T} (x_s - \hat{x}_s)^2 ds + J^*_{\text{Thresh}}((T - \tau_{\delta_N})^+, N - 1)
\right].
\]

Suppose the sensor is allowed to generate absolutely no samples at all. Then the distortion at the supervisor will be: \( \frac{T^2}{2} \). We know that the minimum distortion due to using a single sample is a fixed fraction of \( \frac{T^2}{2} \) namely: \( c_1 \frac{T^2}{2} \). We will now deduce by mathematical induction that the minimum distortions possible with higher sample budgets are also in the form of fractions of \( \frac{T^2}{2} \). Let the positive coefficient \( c_k \) stand for the the hypothetical fraction whose product with \( \frac{T^2}{2} \) is the minimum distortion \( J^*_{\text{Thresh}}(T, k) \). Continuing the previous set of equations, we get:

\[
J^*_{\text{Thresh}}(T, N) = \inf_{\delta_N \geq 0} \mathbb{E}
\left[
\int_0^{\tau_{\delta_N} \wedge T} (x_s - \hat{x}_s)^2 ds + c_{N-1} \left[(T - \tau_{\delta_N})^+\right]^2
\right],
\]

\[
= \inf_{\delta_N \geq 0} \mathbb{E}
\left[
\int_0^{\tau_{\delta_N} \wedge T} (x_s - \hat{x}_s)^2 ds + \frac{T^2}{2} c_{N-1} \left[\phi(\lambda_N) + \left(\frac{1}{2} - c_{N-1}\right) \psi(\lambda_N)\right]
\right].
\]

Because of the scale-free nature of the functions \( \phi, \psi \), we have proved that the minimum distortion is indeed a fixed fraction of \( \frac{T^2}{2} \). Figure 3 depicts the behaviour of the sum functions to be considered for efficient multiple Delta sampling. The last equation gives
Figure 4. Subfigure (a) shows the probability that a sample is generated (Ξ) as a function of the parameter λ, which is inversely related to the square of the threshold δ. Subfigure (b) demonstrates the fact that delta sampling is meant for repeated sampling over infinite horizons. The utilization of the sample budget by the optimal delta sampling scheme is not monotonic with the size of the budget and is actually somewhat counter-intuitive. Over any fixed finite horizon, it uses fewer than six samples on average.

We have derived the performance characteristics of optimal Delta sampling with a fixed sample budget. The parameters describing the performance are tabulated in Table 1 for small values of the budget. To understand the behaviour of optimal Delta sampling when the sample budget is larger than five, look at subfigure (b) in Figure 6 as well as figure 6.
minimum distortion decreases with increasing sample budgets but it does not approach zero. It stagnates at approximately $0.3 \frac{T^2}{T}$ no matter how large a budget is provided. The expected number of samples is not monotonic in its dependence on the budget. It settles to a value close to $0.3$. Clearly, Delta sampling which is optimal for the infinite horizon version of the estimation problem is far from optimal in the finite horizon version. In fact if the sample budget is at least two, then, even deterministic sampling performs better. There is hence a need for determining the optimal sampling policy.

In optimal Delta sampling, the sensor chooses a sequence of thresholds to be applied on the estimation error signal. The choice of a particular threshold is made at the time of the previous sample and is allowed to depend on the past history of sample times. Suppose now that the sensor is allowed to modify this choice causally and continuously at all time instants. Then we get a more general class of sampling policies with a family of continuously varying envelopes for the estimation error signal. This class of policies contains the optimal sampling policy which achieves the minimum possible distortion. Next, we will obtain the optimal family of envelopes by studying the problem of minimum distortion as an optimal multiple stopping problem.

### 3.3. Optimal sampling.

Now, we go after the optimal multiple sampling policy. Consider the non-decreasing sequence: $\{\tau_1, \tau_2, \ldots, \tau_N\}$ with each element lying within $[0, T]$. For this to be a valid sequence of sampling times, its elements have to be stopping times w.r.t. the $x$-process. We will look for the best choice of these times through the optimization:

$$J^* (T, N) = \text{ess inf}_{\{\tau_1, \tau_2, \ldots, \tau_N\}} \mathbb{E} \left[ \int_0^{\tau_1} x_s^2 \, ds + \int_{\tau_1}^{\tau_2} (x_s - \hat{x}_{\tau_1})^2 \, ds + \cdots + \int_{\tau_{N-1}}^{\tau_N} (x_s - \hat{x}_{\tau_{N-1}})^2 \, ds + \int_{\tau_N}^{T} (x_s - \hat{x}_{\tau_N})^2 \, ds \right].$$

The solution to this optimization parallels the developments for Delta sampling. In particular, the minimum distortion obtained by optimal sampling will turn out to be a fraction of $\frac{T^2}{T}$. We will recursively obtain optimal sampling policies by utilizing the solution to the following optimal (single) stopping problem concerning the objective function $\chi$:

$$\text{ess inf}_{\tau} \chi (T, \beta, \tau) = \text{ess inf}_{\tau} \mathbb{E} \left[ \int_0^{\tau} x_s^2 \, ds + \beta (T - \tau)^2 \right],$$

where, $\tau$ is a stopping time w.r.t. the $x$-process that lies in the interval $[0, T]$, and, $\beta$ is a positive real number. We reduce this stopping problem into one having just a terminal cost using the calculations between and including equations 3, 5:

$$\chi (T, \beta, \tau) = \frac{T^2}{2} - \mathbb{E} \left[ 2x_\tau^2 (T - \tau) + (1 - \beta) [(T - \tau)^2] \right],$$

which can be minimized by solving the following optimal stopping problem:

$$\text{ess sup}_{\tau} \mathbb{E} \left[ 2x_\tau^2 (T - \tau) + (1 - \beta) [(T - \tau)^2] \right]$$

This stopping problem can be solved explicitly by determining it Snell envelope process. We look for a $C^2$ function $g(x, t)$ which satisfies the Free boundary PDE system:

$$\frac{1}{2}g_{xx} + g_t = 0, \quad g(x, t) \geq 2x^2 (T - t) + (1 - \beta) (T - t)^2.$$
Given a solution \( g \), consider the process:

\[
S_t \triangleq g(x_t, t).
\]

This is in fact the Snell envelope. To see that, fix a deterministic time \( t \) within \([0, T]\) and verify using Itô’s formula that:

\[
E[S_\tau(x_\tau)|x_t] - S_t = E\left[\int_t^\tau dS_t|x_t\right] = 0,
\]

for any stopping time \( \tau \in [t, T] \), and hence,

\[
S_t = E[S_\tau|x_t] \geq E\left[x_\tau^2(1 - \tau)|\tau \geq t, x_t\right].
\]

The last equation confirms that \( S_t \) is indeed the Snell envelope. Consider the following solution to the Free-boundary PDE system:

\[
g(x, t) = A \left\{ (T - t)^2 + 2x^2(T - t) + \frac{x^4}{3} \right\}.
\]

where \( A \) is a constant chosen such that \( g(x, t) - 2x^2(T - t) - (1 - \beta)(T - t)^2 \) becomes a perfect square. The only possible value for \( A \) then is:

\[
\frac{(5 + \beta) - \sqrt{(5 + \beta)^2 - 24}}{4}.
\]

Then an optimal stopping time is the earliest time the reward obtained by stopping now equals or exceeds the Snell envelope:

\[
\tau^* = \inf_{t \geq 0} \left\{ t : S_t \leq 2x_t^2(T - t) + (1 - \beta)(T - t)^2 \right\},
\]

\[
= \inf_{t} \left\{ t : x_t^2 \geq \sqrt{\frac{3(A - 1 + \beta)}{A}}(T - t) \right\},
\]

and the corresponding minimum distortion becomes

\[
\chi^* (T, \beta) = (1 - A)\frac{T^2}{2}.
\]

We now examine the problem of choosing optimally a single sample.

3.3.1. Optimal choice of a single sample. The minimum distortion due to using exactly one sample is:

\[
J^* (T, 1) = \essinf_{\tau_1} E\left[\int_0^{\tau_1} x_s^2 ds + \int_{\tau_1}^T (x_s - \tilde{x}_{\tau_1})^2 ds \right],
\]

\[
= \essinf_{\tau_1} E\left[\int_0^{\tau_1} x_s^2 ds + \frac{1}{2}(T - \tau_1)^2 ds \right],
\]

\[
= \essinf_{\tau_1} \chi (T, 1, \tau_1).
\]

We have thus reduced the optimization problem to one whose solution we already know. Hence, we have:

\[
\tau_1^* = \inf_{t \geq 0} \left\{ t : x_t^2 \geq \sqrt{3}(T - t) \right\}, \quad \text{and,} \quad J^* (T, 1) = \left(\sqrt{3} - 1\right)\frac{T^2}{2}.
\]
3.3.2. Optimal multiple sampling. We obtain the family of policies for optimal multiple sampling by mathematical induction. Suppose that the minimum distortions due to using no more than \( k-1 \) samples over \([0,T]\) is given by the sequence of values \( \{\theta_1 \frac{T^2}{2}, \ldots, \theta_{k-1} \frac{T^2}{2}\} \). Then consider the minimal distortion due to using up to \( k \) samples:

\[
J^* (T, k) = \text{ess inf}_{\tau_1} \mathbb{E} \left[ \int_0^{\tau_1} x^2 ds + J^* (T - \tau_1, k - 1) \right],
\]

\[
= \text{ess inf}_{\tau_1} \mathbb{E} \left[ \int_0^{\tau_1} x^2 ds + \frac{1 - \theta_{k-1}}{2} (T - \tau_1)^2 ds \right],
\]

\[
= \text{ess inf}_{\tau_1} \chi (T, \theta_{k-1}, \tau_1).
\]

This proves the hypothesis that the minimum distortions for increasing values of the sample budget form a sequence with the form: \( \{\theta_k \frac{T^2}{2}\}_{k \geq 1} \). The last equation also provides us with the recursion which is started with \( \theta_0 = 1 \):

\[
\begin{align*}
\theta_N &= 1 - \frac{(5 + \theta_{N-1}) - \sqrt{(5 + \theta_{N-1})^2 - 24}}{4}, \\
\gamma_N &= \sqrt{\frac{3(\theta_{N-1} - \theta_N)}{1 - \theta_N}}.
\end{align*}
\]

3.4. Comparisons. In figure 6 we have a comparison of the estimation distortions incurred by the three sampling strategies on the same time interval \([0,T]\). The remarkable news is that Delta sampling which is optimal for the infinite horizon version of the estimation problem is easily beaten by the best deterministic sampling policy. There is something structural about Delta sampling which makes it ill suited for finite horizon problems with hard budget limits. This also means that it is not safe to settle for ‘natural’
Performances of the three sampling methods
Number of samples left: N
(2/T^2) \times \text{Minimum estimation distortion}

Delta sampling
Periodic sampling
Optimal sampling

(a) Comparisons

Percentage gain of Optimal sampling of Periodic sampling
Number of samples left: N
100 \times (J_{\text{periodic}} - J_{\text{Optimal}}) / J_{\text{periodic}}

(b) Optimal vs Periodic

Figure 6. The minimum distortions offered by the three sampling methods. As the number of allowed samples grows, the normalized reduction of the distortion of optimal sampling from periodic sampling keeps growing up to a limit of 67%.

event-triggered sampling policies such as Delta sampling. Also, notice that the relative gain of optimal sampling over periodic sampling consistently grows to about 67%.

4. Sampling the Ornstein-Uhlenbeck process

Now we turn to the case when the signal is an Ornstein-Uhlenbeck process:

\[ dx_t = ax_t dt + dW_t, \quad t \in [0, T], \]

with \( x_0 = 0 \) and \( W_t \) being a standard Brownian motion. Again, the sampling times \( S = \{\tau_1, \ldots, \tau_N\} \) have to be an increasing sequence of stopping times with respect to the \( x \)-process. They also have to lie within the interval \([0, T]\). Based on the samples and the sample times, the supervisor maintains an estimate waveform \( \hat{x}_t \) given by

\[
\hat{x}_t = \begin{cases} 
0 & \text{if } 0 \leq t < \tau_1, \\
\tau_i e^{a(t-\tau_i)} & \text{if } \tau_i \leq t < \tau_{i+1} \leq \tau_N, \\
\tau_N e^{a(t-\tau_N)} & \text{if } \tau_N \leq t \leq T.
\end{cases}
\]

The quality of this estimate is measured by the aggregate squared error distortion:

\[
J^*(T, N) = \mathbb{E} \left[ \int_0^T (x_s - \hat{x}_s)^2 ds \right].
\]

4.1. Optimal deterministic sampling. Just like in the case of Brownian motion, we can show through mathematical induction that uniform sampling on the interval \([0, T]\) is the optimal deterministic choice of \( N \) samples: For the induction step, we assume that the optimal choice of \( N - 1 \) deterministic samples over \([T_1, T_2]\) is the uniform one:

\[
\{d_1, d_2, \ldots, d_N\} = \left\{ T_1 + i \frac{T_2 - T_1}{N+1} \bigg| i = 1, 2, \ldots, N \right\}.
\]

The corresponding minimum distortion becomes:

\[
(N+1) \frac{e^{2a \frac{T_2-T_1}{N+1}} - 1}{4a^2} - \frac{1}{2a} (T_2 - T_1).
\]
4.2. Optimal Delta sampling. We do not have an analytical characterization of the performance of Delta sampling. Let us first address the single sample case. The performance measure then takes the form

\[ J_{\text{Thresh}}(T, 1) = E \left[ \int_0^T x_t^2 + \int_{\zeta_1}^T (x_t - \hat{x}_t)^2 \, dt \right] \]

\[ = E \left[ \int_0^T x_t^2 - 2 \int_{\zeta_1}^T x_t \hat{x}_t \, dt + \int_{\zeta_1}^T (\hat{x}_t)^2 \, dt \right] . \]

Now notice that the second term can be written as follows

\[ \mathbb{E} \left[ \int_{\zeta_1}^T x_t \hat{x}_t \, dt \right] = \mathbb{E} \left[ \int_{\zeta_1}^T \mathbb{E}[x_t | \mathcal{F}_{\zeta_1}] \hat{x}_t \, dt \right] = \mathbb{E} \left[ \int_{\zeta_1}^T (\hat{x}_t)^2 \right] , \]

where we have used the strong Markov property of \( x_t \), and that for \( t > \zeta_1 \) we have \( \mathbb{E}[x_t | \mathcal{F}_{\zeta_1}] = x_\zeta e^{-a(t-\zeta_1)} = \hat{x}_t \). Because of this observation the performance measure takes the form

\[ J_{\text{Thresh}}(T, 1) = \mathbb{E} \left[ \int_0^T x_t^2 \, dt - \int_{\zeta_1}^T (\hat{x}_t)^2 \, dt \right] \]

\[ = e^{2aT} - 1 - 2aT - \mathbb{E} \left[ x_{\zeta_1}^2 \right] + 4a^2 T \left( e^{2a(T-\zeta_1)} - 1 \right) \]

\[ = T^2 \left( e^{2a\bar{T}} - 1 - 2a\bar{T} - \mathbb{E} \left[ x_{\bar{T}}^2 \right] \frac{e^{2a(T-\zeta_1)} - 1}{2a} \right) \]

where,

\[ \bar{t} = \frac{t}{T}; \quad \bar{a} = aT; \quad \bar{x}_t = \frac{x_t}{\sqrt{T}}. \]

We have \( \bar{x} \) satisfying the following SDE:

\[ d\bar{x}_t = -\bar{a} \bar{x}_t dt + dw_t. \]

This suggests that, without loss of generality, we can limit ourselves to the normalized case \( T = 1 \) since the case \( T \neq 1 \) can be reduced to the normalized one by using the transformations in (12). In fact, we can solve the single sampling problem on \([0,1]\) to minimize:

\[ J_{\text{Thresh}}(1, 1) = \left\{ e^{-2a} - 1 + 2a \bar{x}_{\bar{T}}^2 - \mathbb{E} \left[ x_{\bar{T}}^2 \right] \frac{e^{2a(1-\delta_1)} - 1}{2a} \right\} . \]

We carry over the definitions for threshold sampling times from section 3.2. We do not have series expansions like for the case of the Wiener process. Instead we have a computational procedure that involves solving a PDE initial and boundary value problem. We have a nested sequence of optimization problems. The choice at each stage being the non-zero level \( \delta_1 \). For \( N = 1 \), the distortion corresponding to a chosen \( \delta_1 \) is given by:

\[ \frac{e^{2a} - 1}{4a^2} - \frac{1}{2a} - \frac{\delta_1^2}{2a} \mathbb{E} \left[ e^{2a(1-\delta_1)} - 1 \right] = \frac{e^{2a} - 1}{4a^2} - \frac{1}{2a} - \frac{\delta_1^2}{2a} \left\{ e^{2a} \left( 1 + 2aU^1(0,0) \right) - 1 \right\} , \]

where the function \( U^1(x,t) \) defined on \([-\delta_1, \delta_1] \times [0,1] \) satisfies the PDE:

\[ \frac{1}{2} U^1_{xx} + a x U^1_x + U^1_t + e^{-2at} = 0, \]
along with the boundary and initial conditions:

\[
\begin{align*}
U^1(-\delta_1, t) &= U^1(\delta_1, t) = 0 \quad \text{for } t \in [0, 1], \\
U^1(x, 1) &= 0 \quad \text{for } x \in [-\delta_1, \delta_1].
\end{align*}
\]

We choose the optimal \( \delta_1 \) by computing the resultant distortion for increasing values of \( \delta_1 \) and stopping when the cost stops decreasing and starts increasing. Note that the solution \( U(0, t) \) to the PDE furnishes us with the performance of the \( \delta_1 \)-triggered sampling over \([t, 1]\). We will use this to solve the multiple sampling problem.

Let the optimal policy of choosing \( N \) levels for sampling over \([T_1, 1]\) be given where \( 0 \leq T_1 \leq 1 \). Let the resulting distortion be also known as a function of \( T_1 \). Let this known distortion over \([T_1, 1]\) given \( N \) level-triggered samples be denoted \( J_{\text{thresh}}^* (1 - T_1, N) \). Then, the \( N + 1 \) sampling problem can be solved as follows. Let \( U^{N+1}(x, t) \) satisfy the PDE:

\[
\frac{1}{2} \frac{d^2}{dx^2} U_x^{N+1} + ax U_x^{N+1} + U_t^{N+1} = 0,
\]

along with the boundary and initial conditions:

\[
\begin{align*}
U^{N+1}(-\delta_1, t) &= U^{N+1}(\delta_1, t) = J_{\text{thresh}}^* (1 - t, N) \quad \text{for } t \in [0, 1], \\
U^{N+1}(x, 1) &= 0 \quad \text{for } x \in [-\delta_1, \delta_1].
\end{align*}
\]

Then the distortion we are seeking to minimize over \( \delta_1 \) is given by:

\[
\frac{e^{2a} - 1}{4a^2} - \frac{1}{2a} - \frac{\delta_1^2}{2a} \left[ \frac{e^{2a(1-\zeta_1)} - 1}{4a^2} - \frac{1 - \zeta_1}{2a} \right] + \mathbb{E} \left[ J_{\text{thresh}}^* (1 - \zeta_1, N) \right]
\]

\[
= \frac{e^{2a} - 1}{4a^2} - \frac{1}{2a} - \frac{\delta_1^2}{2a} \left\{ e^{2a} \left( 1 + 2a U^1(0, 0) \right) - 1 \right\} + U^{N+1}(0, 0).
\]

We choose the optimal \( \delta_1 \) by computing the resultant distortion for increasing values of \( \delta_1 \) and stopping when the distortion stops decreasing.

**4.3. Optimal Sampling.** We do not have analytic expressions for the minimum distortion like in the Brownian motion case. We have a numerical computation of the minimum distortion by finely discretizing time and solving the discrete-time optimal stopping problems.

By discretizing time, we get random variables \( x_1, \ldots, x_M \), that satisfy the AR(1) model below. For \( 1 \leq n \leq M \),

\[
x_n = e^{a} x_{n-1} + w_n, \quad w_n \sim \mathcal{N} \left( 0, \frac{\epsilon^{2a} - 1}{2a} \right); \quad 1 \leq n \leq M.
\]

The noise sequence \( \{w_n\} \) is i.i.d. and Gaussian.

Sampling exactly once in discrete time means selecting a sample \( x_\nu \) from the set of \( M + 1 \) sequentially available random variables \( x_0, \ldots, x_M \), with the help of a stopping time \( \nu \in \{0, 1, \ldots, M\} \). We can define the optimum cost to go which can be analyzed as follows. For \( n = M, M - 1, \ldots, 0 \), using equation (13),

\[
V_n^1(x) = \sup_{n \leq \nu \leq M} \mathbb{E} \left[ \frac{x^2 e^{2a(M-\nu)} - 1}{2a} | x_n = x \right]
\]

\[
= \max \left\{ \frac{x^2 e^{2a(M-n)} - 1}{2a}, \mathbb{E}[V_{n+1}^1(x_{n+1}) | x_n = x] \right\}.
\]

The above equation provides a (backward) recurrence relation for the computation of the single sampling value function \( V_1^1(x) \). Notice that for values of \( x \) for which the l.h.s.
exceeds the r.h.s. we stop and sample, otherwise we continue to the next time instant. We can prove by induction that the optimum policy is a \textit{time-varying threshold} one. Specifically for every time \(n\) there exists a threshold \(\lambda_n\) such that if \(|x_n| \geq \lambda_n\) we sample, otherwise we go to the next time instant. The numerical solution of the recursion presents no special difficulty if \(a \leq 1\). If \(V_n^1(x)\) is sampled in \(x\) then this function is represented as a vector. In the same way we can see that the conditional expectation is reduced to a simple matrix-vector product. Using this idea we can compute numerically the evolution of the threshold \(\lambda_t\) with time. The minimum expected distortion for this single sampling problem is:

\[
\frac{e^{2aT} - 1 - 2aT}{4a^2} - V_0^1(0).
\]

For obtaining the solution to the \(N+1\)-sampling problem, we use the solution to the \(N\)-sampling problem. For \(n = M, M - 1, \ldots, 0\),

\[
V_{n+1}^{N+1}(x) = \sup_{n \leq \nu \leq M} \mathbb{E} \left[ V_{\nu}^N(0) + x_{\nu}^2 \frac{e^{2a(M-\nu)} - 1}{2a} \bigg| x_{\nu} = x \right]
\]

\[
= \max \left\{ V_{n}^N(0) + x^2 \frac{e^{2a(M-n)} - 1}{2a}, V_{n+1}^{N}(0) + \mathbb{E} \left[ V_{n+1}^1(x_{n+1}) | x_n = x \right] \right\}.
\]

4.4. Comparisons. Figure 7 shows the result of the numerical computations for a few stable plants and a single unstable plant. Again, Delta sampling is not very competitive. In the stable cases, it does provide lower distortion than periodic sampling when the size of the sample budget is small.

5. Summary and extensions

We have set up the problem of efficient sampling as an optimal multiple sampling problem. We characterized the timing channel in adaptive sampling through its modification of the estimator. For MMSE estimation of linear systems, under optimal sampling, we showed that while the timing channel exists, it affects only the evolution of the statistics of the second and higher moments of the estimation error. This reduces the optimization problem into a tractable multiple stopping problem.

We have furnished methods to obtain good sampling policies for the finite horizon state estimation problem. When the signal to be kept track of is a Brownian motion, we have analytic solutions. When the signal is an Ornstein-Uhlenbeck process, we have provided computational recipes to determine the best sampling policies and their performances. In both cases, Delta sampling performs poorly with it distortion staying boundedly away from zero even as the sample budget increases to infinity. This means that the designer cannot just settle for ‘natural’ event-triggered schemes without further investigation. In particular, a scheme optimal in the infinite horizon may be, like Delta sampling, a bad performer for the finite horizon.

The approach adopted in this paper leads us to also consider some sampling and filtering problems with multiple sensors. These can possibly be solved in the same way as the single sensor problem. The case where the samples are not reliably transmitted but can be lost in transmission is computationally more involved. There, the relative performances of the three sampling strategies is unknown. However, in principle, the best policies and their performances can be computed using nested optimization routines like we have used in this paper.

Another set of unanswered questions involve the performance of these sampling policies when the actual objective is not filtering but control or signal detection based on the
samples. It will be very useful to know the extent to which the overall performance is decreased by using sampling designs that achieve merely good filtering performance. The communication constraint we treated in this paper was a hard limit on the number of allowed samples. Instead, we could use a soft constraint: a limit on the expected number of samples. We could also study the effect of mandatory minimum intervals between successive sampling times. Extension to nonlinear systems is needed as are extensions to the case of partial observations at the sensor. One could follow the attack line sketched at the end of section 2.

A. Optimal sampling with non-standard MMSE estimates

The statement of lemma 1 and its consequence viz. equation (2) conform to our intuition. \textit{Was there a real need to investigate the validity of the aforementioned lemma?} Here, we give an example of a well-behaved and widely used stochastic process for which, the optimal sampling policy leads to an MMSE estimate which is different from that under deterministic sampling. For convenience, we consider an infinite horizon repeated

\begin{figure}[h]
\centering
\begin{subfigure}[b]{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{fig1a.png}
\caption{A stable case}
\end{subfigure}\hfill
\begin{subfigure}[b]{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{fig1b.png}
\caption{A stable case}
\end{subfigure}

\begin{subfigure}[b]{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{fig1c.png}
\caption{A stable case}
\end{subfigure}\hfill
\begin{subfigure}[b]{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{fig1d.png}
\caption{An unstable case}
\end{subfigure}
\caption{The minimum distortions offered by the three sampling methods. In the stable regime, Delta sampling is more efficient than deterministic sampling when the number of allowed samples is greater than one but still small. In the unstable regime, deterministic sampling always beats Delta sampling.}
\end{figure}
sampling problem where the communication constraint is a limit on the average sampling rate.

Choose the state process to be the Poisson counter $N_t$, a continuous time Markov chain. This is a non-decreasing process which starts at zero and takes integer values. Its sample paths are piecewise constant and RCLL. The sequence of times between successive jumps are independent and are all exponentially distributed with parameter $\lambda$.

Under any deterministic sampling rule, the MMSE estimate is piecewise linear with slope $\lambda$:

$$\hat{N}_t = N_{d_{latest}} + \lambda (t - d_{latest}),$$

(14)

where, $d_{latest}$ is the latest sampling instant as of time $t$. The optimal sampling policy leads to an MMSE estimate which is different.

Stipulate that the constraint on the average sampling rate is equal to $\lambda$ the parameter of the Poisson process. Consider the following sampling policy whose MMSE estimate is of the zero-order hold type:

$$\hat{N}_t = N_{\tau_{latest}},$$

(15)

$$\begin{cases} 
\tau_0 = 0, \\
\tau_{i+1} = \inf \{t | t > \tau_i, N_t > N_{\tau_i}\}, \forall i \geq 0, \\
\tau_{latest} = \max \{\tau_i | \tau_i \leq t\}
\end{cases}$$

(16)

This sampling rule with its MMSE estimate $\hat{N}_t$ leads to an error signal which is identically zero. We also have that

$$E[\tau_{i+1} - \tau_i] = \frac{1}{\lambda}, \forall i \geq 0,$$

and so, the communication constraint is met. On the other hand, the conventional MMSE estimate (14) would result in a non-zero average squared error distortion.

Suppose now that the distortion criterion is not the average value of the squared error but of the lexicographic distance:

$$D(N_t, \hat{N}_t) = \begin{cases} 
0 & \text{if } N_t = \hat{N}_t, \\
1 & \text{otherwise},
\end{cases}$$

then, under deterministic sampling, the maximum likelihood estimate:

$$\bar{N}_t = N_{d_{latest}} + \lfloor \lambda (t - d_{latest}) \rfloor,$$

minimizes the average lexicographic distortion which will be non-zero. However, the adaptive policy (16) provides zero error reconstruction if the constraint on the average sampling rate is not less than $\lambda$.

**B. Statistics of a threshold Hitting time curtailed by a time-out T**

We start by deriving the Moment generating function of the first hitting time $\tau_\delta$:

$$\tau_\delta = \inf_i \{t : |x_t - \hat{x}_t| = \delta\},$$

with the initial condition $x_0 - \hat{x}_0 = w_o$.

**Lemma 3.** If $\tau_\delta$ is the first hitting time of $|x_t - \hat{x}_t|$ at the threshold $\delta$, then,

$$E[e^{-s\tau_\delta}] = \frac{\cosh(w_0\sqrt{2s})}{\cosh(\delta\sqrt{2s})} = F_\tau(s).$$
Proof: Consider the $C^2$ function $h(w, t) = e^{-st}[1 - \cosh(\sqrt{2}sw)/\cosh(\sqrt{2}s\delta)]$ and apply Itô calculus on $h(w, t)$. We can then conclude that

$$
\mathbb{E}[h(w_{\tau_\delta}, \tau_\delta)] - h(w_0, 0) = \mathbb{E} \left[ \int_0^{\tau_\delta} [h_t(w, t) + 0.5h_{w^2}(w, t)] dt \right] = \mathbb{E}[e^{-s\tau_\delta}] - 1,
$$

from which we immediately obtain the desired relation because of the boundary condition: $h(w_{\tau_\delta}, \tau_\delta) = 0$. $lacksquare$

Lemma 3 suggests that the PDF of the random variable $\tau_\delta$ can be computed as $f_{\tau}(t) = \mathcal{L}^{-1}(F_{\tau}(s))$, that is, the inverse Laplace transform of $F_{\tau}(s)$. Invoking the initial condition $w_0 = 0$, we can then write

$$
\mathbb{E}[(T - \tau_\delta)^+] = \int_0^{T} (T-t) f_{\tau}(t) dt = \frac{1}{2\pi j} \oint F_{\tau}(s) e^{st} ds dt
$$

$$
= \frac{1}{2\pi j} \oint F_{\tau}(s) \left[ \int_0^{T} (T-t) e^{st} dt \right] ds
$$

$$
= \frac{1}{2\pi j} \oint \frac{e^{st} - 1 - sT}{s^2 \cosh(\delta \sqrt{2}s)} ds.
$$

The previous integral is on a path that includes the whole left half of the complex plane.

In order to compute this line integral over the complex plane, we need to find the poles of the integrand and then apply the residue theorem. Notice first that $s = 0$ is not a pole since the numerator has a double zero at zero. The only poles come from the zeros of the function $\cosh(\delta \sqrt{2}s)$. Since $\cosh(x) = \cos(jx)$ we conclude that the zeros of $\cosh(\delta \sqrt{2}s)$ which are also the poles of the integrand are:

$$
s_k = -(2k + 1) \frac{\pi^2}{8\delta^2}, \quad k = 0, 1, 2, \ldots
$$

and they all belong to the negative half plane. This of course implies that they all contribute to the integral. We can now apply the residue theorem to conclude that

$$
\mathbb{E}[(T - \tau_\delta)^+] = \frac{1}{2\pi j} \oint \frac{e^{st} - 1 - sT}{s^2 \cosh(\delta \sqrt{2}s)} ds = \sum_{k \geq 0} \frac{e^{s_k T} - 1 - s_k T}{s_k^2} \lim_{s \rightarrow s_k} \frac{s - s_k}{\cosh(\delta \sqrt{2}s)}.
$$

In order to find the last limit we can assume that $s = s_k(1 + \epsilon)$ and let $\epsilon \rightarrow 0$. Then, we can show that

$$
\lim_{s \rightarrow s_k} \frac{s - s_k}{\cosh(\delta \sqrt{2}s)} = (-1)^{(k+1)} \frac{4s_k}{(2k+1)\pi}.
$$

Using this expression, the performance measure of the stopping time $\tau_\delta$ takes the following form:

$$
J_{\text{Thresh}}(T, 1) = \frac{T^2}{2} - \delta^2 \mathbb{E}[(T - \tau_\delta)^+],
$$

$$
= \frac{T^2}{2} \left\{ 1 - \frac{8\delta^2}{\pi T} \sum_{k \geq 0} (-1)^{(k+1)} \frac{1}{2k+1} \frac{e^{s_k T} - 1 - s_k T}{s_k T} \right\}
$$

$$
= \frac{T^2}{2} \phi(\lambda),
$$
where, with the change of variables: \( \lambda = \frac{T\pi^2}{8a^2} \), we have:

\[
\phi(\lambda) \triangleq 1 - \frac{\pi}{\lambda^2} \sum_{k \geq 0} (-1)^k \frac{e^{-(2k+1)^2\lambda}}{(2k+1)^3} - 1 + (2k+1)^2 \lambda,
\]

\[
= 1 - \frac{\pi}{\lambda^2} \sum_{k \geq 0} (-1)^k \frac{e^{-(2k+1)^2\lambda}}{(2k+1)^3} + \frac{\pi}{\lambda^2} \sum_{k \geq 0} (-1)^k \frac{1}{(2k+1)^3} - \frac{\pi}{\lambda} \sum_{k \geq 0} \frac{(-1)^k}{2k+1}.
\]

The final two series in the last equation can be summed explicitly. To do so, we adopt a summation technique described in the book of Aigner and Ziegler [1]. Consider:

\[
\int_0^1 \frac{dx}{1 + x^2} = \int_0^1 \left( \sum_{k \geq 0} (-1)^k x^{2k} \right) dx = \sum_{k \geq 0} (-1)^k \int_0^1 x^{2k} dx = \sum_{k \geq 0} \frac{(-1)^k}{2k+1}.
\]

By an easy evaluation of the definite integral we started with, we get a sum of \( \frac{\pi}{4} \) for the series \( \sum_{k \geq 0} \frac{(-1)^k}{2k+1} \); this result is useful because the series converges slowly. Proceeding along similar lines [7] and working with the multiple integral:

\[
\int_A \cdots \int \frac{dx_1 \cdots dx_n}{1 + (x_1 x_2 \cdots x_n)^2},
\]

over the unit hypercube \( A = [0,1]^n \) in \( \mathbb{R}^n \), we get an explicit expression for the sum \( \sum_{k \geq 0} \frac{(-1)^k}{(2k+1)^n} \) whenever \( n \) is an odd number. In particular,

\[
\sum_{k \geq 0} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}, \quad \sum_{k \geq 0} \frac{(-1)^k}{(2k+1)^5} = \frac{5\pi^5}{1536}.
\]

This reduces the distortion to:

\[
J_{\text{Thresh}} (T, 1) = \frac{T^2}{2} \phi(\lambda) = \frac{T^2}{2} \left\{ 1 + \frac{\pi^4}{32\lambda^2} - \frac{\pi^2}{4\lambda} \right\}
\]

\[
\sum_{k \geq 0} \frac{(-1)^k e^{-(2k+1)^2\lambda}}{(2k+1)^3} \}
\]

where, \( \lambda = \frac{T\pi^2}{8a^2} \).

The estimation distortion due to using \( N+1 \) samples when \( N \) is non-negative is given through the recursion:

\[
J_{\text{Thresh}} (T, N+1) = \frac{T^2}{2} - \delta^2 \mathbb{E} \left[ (T - \tau_{\delta N+1})^+ \right] - \left( \frac{1}{2} - J_{\text{Thresh}} (T, N) \right) \mathbb{E} \left[ \left( (T - \tau_{\delta N+1})^+ \right)^2 \right].
\]

This recursion decomposes the distortion into an expression involving the performance of the policies for the last \( N \) times of the first of \( N+1 \) sampling times over the horizon \([0,T]\), and, the influence of the first of the sampling times namely, \( \tau_{\delta N+1} \). Hence, for our recursive solution to the multiple stopping problem, we need to minimize costs like:

\[
\Upsilon (T, \alpha, \delta) \triangleq \frac{T^2}{2} - \delta^2 \mathbb{E} \left[ (T - \tau_\delta)^+ \right] - \left( \frac{1}{2} - \alpha \right) \mathbb{E} \left[ \left( (T - \tau_\delta)^+ \right)^2 \right].
\]

where \( \alpha \) is positive but no greater than 0.5. This requires an evaluation of the second moment: \( \mathbb{E} \left[ (T - \tau_\delta)^+ \right]^2 \). The evaluation can be done similarly to the one for the first moment:

\[
\mathbb{E} \left[ (T - \tau)^+ \right]^2 = \frac{1}{\pi^2} \int e^{st} - 1 - sT - \frac{1}{2}s^2T^2 \frac{1}{s^3 \cosh(\delta \sqrt{2s})} \, ds.
\]
This gives the following expression for the cost \( \Upsilon (T, \alpha, \delta) \):

\[
\Upsilon (T, \alpha, \delta) = \frac{T^2}{2} \left\{ \phi(\lambda) + \left[ \frac{1}{2} - \alpha \right] \psi(\lambda) \right\},
\]

where, \( \lambda = \frac{T \pi^2}{8 \delta^2} \), and we define functions \( \phi, \psi \) with \( \phi \) being the same as it was earlier in this part of the appendix:

\[
\phi(\lambda) = 1 - \pi \sum_{k \geq 0} (-1)^k \frac{e^{-(2k+1)^2 \lambda} - 1 + (2k+1)^2 \lambda}{(2k+1)^3},
\]

\[
\psi(\lambda) = \frac{16}{\pi \lambda^2} \sum_{k \geq 0} (-1)^k \frac{e^{-(2k+1)^2 \lambda} - 1 + (2k+1)^2 \lambda - 0.5(2k+1)^4 \lambda^2}{(2k+1)^5},
\]

After replacing the summable series with their sums, the distortion due to multiple samples based on thresholds reduces to the boxed expression below. With \( \lambda = \frac{T \pi^2}{8 \delta^2} \),

(18)

\[
J_{\text{thresh}}(T, N + 1) = \frac{T^2}{2} \left\{ 1 + \frac{\pi^4}{32 \lambda^2} - \frac{\pi^2}{4 \lambda} - \frac{\pi}{\lambda^2} \sum_{k \geq 0} (-1)^k \frac{e^{-(2k+1)^2 \lambda}}{(2k+1)^3} \right\} + (0.5 - J_{\text{thresh}}(T, N)) \left[ \frac{-5 \pi^4}{96 \lambda^2} - \frac{\pi^2}{2 \lambda} - 2 + \frac{16}{\pi \lambda^2} \sum_{k \geq 0} (-1)^k \frac{e^{-(2k+1)^2 \lambda}}{(2k+1)^5} \right].
\]

To characterize the statistics of sample budget utilization by multiple Delta sampling we need to find the probabilities of threshold crossings before the time-out \( T \). Given the limit of the allowed number of samples \( N \), let \( \Xi_N \) be the random number of samples generated under the multiple delta sampling scheme, Then, we have:

\[
\mathbb{E}[\Xi_N] = 0 \cdot \mathbb{P}[\tau_{\delta_N} \geq T] + (1 + \mathbb{E}[\Xi_{N-1}]) \cdot \mathbb{P}[\tau_{\delta_N} < T],
\]

\[
= (1 + \mathbb{E}[\Xi_{N-1}]) \cdot \left( 1 - \mathbb{P}[\tau_{\delta_N} \leq T] \right).
\]

As before, we use the Moment generating function of the hitting time to obtain:

\[
\mathbb{E}[\Xi_1] = \mathbb{E} \left[ 1 \{ \tau_{\delta_1} > T \} \right] = \frac{1}{\pi j} \int_0^{e^{sT} - 1} \frac{ds}{s \cdot \cosh(\delta \sqrt{2s})},
\]

With the notation \( \lambda = \frac{T \pi^2}{8 \delta^2} \), and evaluating this complex line integral as in previous cases, we obtain:

(19)

\[
\mathbb{E}[\Xi_1] = \mathbb{P}[\tau_{\delta_1} \leq T] = 1 - \frac{4}{\pi} \sum_{k \geq 0} (-1)^k \frac{e^{-(2k+1)^2 \lambda}}{2k+1}.
\]
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