On the spatial behaviour in dynamics of porous elastic mixtures *

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Abstract- In this paper we study the spatial and temporal behaviour of the dynamic processes in porous elastic mixtures. For the spatial behaviour we use the time-weighted surface power function method in order to obtain a more precisely determination of the domain of influence and we establish spatial decay estimates of Saint–Venant type with time-independent decay rate for the inside of the domain of influence. For the asymptotic temporal behaviour we use the Cesáro means associated with the kinetic and strain energies and establish the asymptotic equipartition of the total energy. A uniqueness theorem is proved for finite and infinite bodies and we note that it is free of any kind of a priori assumptions of the solutions at infinity.

1 Introduction

Various theories have been proposed in literature for describing the behaviour of the chemically reacting media (see, for example, Truesdell and Toupin [1], Kelly [2], Eringen and Ingram [3, 4], Green and Naghdi [5, 6], Müller [7], Dunwoody and Müller [8], Bedford and Drumheller [9], etc).

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Recently Ieşan [10] has developed a theory for binary mixtures of granular materials in Lagrangian description in which the independent constitutive variables are the displacement gradients, displacement fields, volume fractions and volume fraction gradients. The theory takes into account the results established previously by Nunziato and Cowin [11], Goodman and Cowin [12] and Drumheller [13]. The intended applications for such a theory are to granular composites, solid explosives and geological materials.

In [10] a linear theory is also presented and some uniqueness results for bounded bodies are established for the linear dynamic theory with no definiteness assumptions on the elasticities and without any restriction on the initial stresses.

The present paper studies the spatial and temporal behaviour of the solutions to the boundary–initial value problems in the linear dynamic theory of porous elastic mixtures as developed in [10].

For the spatial behaviour of the dynamic processes in porous elastic mixtures we use the time–weighted surface power method developed in [14]. Thus, we introduce a time–weighted surface measure associated with the dynamic process in question and then we establish a first–order partial differential inequality whose integration gives a good information upon the spatial behaviour. Then we obtain a more precisely version of the domain of influence in the sense that for each fixed $t \in [0, T]$ the whole activity is vanishing at distances to the support of the given data on $[0, T]$ greater than $ct$, where $c$ is a constant characteristic to the elastic mixture. A spatial decay estimate of Saint–Venant’s type is established for describing the spatial behaviour of the dynamic process inside of the domain of influence.

As regards the temporal behaviour of the dynamic processes in porous elastic mixtures, we introduce the Cesáro means of various energies and then establish the relations describing the asymptotic equipartition of energy. In this aim we use some Lagrange identities and the method developed by Day [15] and Levine [16].

The plan of our paper is the following one. In the Section 2 we present the basic equations of the linear dynamic theory of porous mixtures developed in [10]. Some constitutive assumptions and other useful results are also presented. The auxiliary identities are established in the Section 3, while in the Section 4 a time–weighted surface measure is defined and its properties are studied. Moreover, a first–order partial differential inequality is established for this measure. The main result concerning the spatial behaviour is presented in Section 5 and some uniqueness results are obtained as a direct consequence. In the Section 6 we introduce the Cesáro means of various energies and establish the
asymptotic equipartition of the total energy.

2 Basic equations

Throughout this article, the motions of continuum are studied respect to a fixed orthonormal frame in $\mathbb{R}^3$. Then, we deal with functions of position and time. Moreover, it is useful stress that in the following text the tensor components of order $p \geq 1$ will appear with Latin subscripts, ranging over the integers $\{1, 2, 3\}$, and summation over repeated subscripts will be implied. Greek indices are understood to range over $\{1, \ldots, 9\}$ if they are lower case letters, or over $\{1, 2\}$ if they are upper case letters; summation convention is not used for these indices. Occasionally, we shall use bold-face character and typical notations for vectors and operations upon them. Superposed dots or subscripts preceded by a comma will mean partial derivative with respect to the time or the corresponding coordinates.

Let $B$ be a bounded or unbounded regular region in the physical 3–dimensional space, whose boundary $\partial B$ is a piecewise smooth surface. A chemically inert binary mixture of two interacting porous elastic solids, $c_1$ and $c_2$, in a given reference configuration, is into $B$.

The positions of particles of $c_1$ and $c_2$ at time $t$ are $x$ and $y$ respectively, i.e.

$$x = x(X, t), \quad y = y(Y, t) \quad X, Y \in B, \quad t \in I,$$

in which $X$ and $Y$ are reference positions of these particles, $I = [0, \infty)$. By following Bedford and Stern [17], we assume that $X = Y$.

Let the top label $\alpha$ refer the various fields to the constituent $c_\alpha$. Taking into account the linear theory, the behaviour of a binary mixture of elastic solids is governed by the local balance equations (see Iesan [10])

$$
\begin{align*}
S_{ji,j}^{(\alpha)} + (-1)^\alpha p_i + \rho^{(\alpha)} f_i^{(\alpha)} &= \rho^{(\alpha)} \dot{u}_i^{(\alpha)}, \\
\dot{h}_{i,i}^{(\alpha)} + g^{(\alpha)} + \ell^{(\alpha)} &= \rho^{(\alpha)} \chi^{(\alpha)} \dot{\varphi}^{(\alpha)},
\end{align*}
$$

on $B \times (0, +\infty)$ (1)

In these equations, $S^{(\alpha)}$, $f^{(\alpha)}$ are the stress tensor and body force associated to $c_\alpha$; $p$ is the vector field for characterising the mechanical interaction between the constituents $c_1$ and $c_2$; $h^{(\alpha)}$, $g^{(\alpha)}$, $\ell^{(\alpha)}$ are the equilibrated stress vector, intrinsic and extrinsic equilibrated body force associated to $c_\alpha$, respectively.
Moreover, \( \mathbf{u}^{(\alpha)} \) is the displacement vector fields associated to \( c_\alpha \); \( \varphi^{(\alpha)} \) is the changes in volume fraction starting from the reference configuration to \( c_\alpha \).

Finally, \( \rho^{(\alpha)}, \chi^{(\alpha)} \) are the bulk mass density and equilibrated inertia of the material \( c_\alpha \) in the reference state.

According to classical interpretation of system (I), we assume that

i. \( u_i^{(\alpha)}, \varphi^{(\alpha)} \in C^{0,2}(\bar{B} \times I) \);
ii. \( S_{ij}^{(\alpha)}, h_i^{(\alpha)} \in C^{0,1}(\bar{B} \times I) \), \( p_i \in C^{0,0}(\bar{B} \times I) \);
iii. \( f_i^{(\alpha)}, g^{(\alpha)}, \ell^{(\alpha)} \in C^{0,0}(\bar{B} \times I) \), \( \rho^{(\alpha)}, \chi^{(\alpha)} \in C^{0}((\bar{B}), \rho) \),

where, \( \bar{B} \) is the closure of \( B \).

Then, we introduce the 29-dimensional vector field

\[
\mathbf{E}(\mathbf{U}) \equiv \{ e_{ij}(\mathbf{U}), g_{ij}(\mathbf{U}), \varphi^{(1)}, \varphi^{(2)}, d_i(\mathbf{U}), \varphi^{(1)}_i, \varphi^{(2)}_i \},
\]

with

\[
\mathbf{U} \equiv \{ \mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \varphi^{(1)}, \varphi^{(2)} \}
\]

and

\[
e_{ij}(\mathbf{U}) = \frac{1}{2}(u_{ij}^{(1)} + u_{ij}^{(2)}), \quad g_{ij}(\mathbf{U}) = u_{ij}^{(1)} + u_{ij}^{(2)}, \quad d_i(\mathbf{U}) = u_i^{(1)} - u_i^{(2)}, \quad \text{on } \bar{B} \times I.
\]

Now, we define the magnitude of \( \mathbf{E}(\mathbf{U}) \) by

\[
|\mathbf{E}(\mathbf{U})| \equiv \left\{ \sum_{\alpha=1}^{2} \left[ e_{ij}(\mathbf{U})e_{ij}(\mathbf{U}) + g_{ij}(\mathbf{U})g_{ij}(\mathbf{U}) + \varphi^{(\alpha)}(\mathbf{U})\varphi^{(\alpha)}(\mathbf{U}) + \right. \right.
\]
\[
\left. + d_i^2(\mathbf{U}) + \varphi^{(\alpha)}_i(\mathbf{U})\varphi^{(\alpha)}_i(\mathbf{U}) \right] \right\}^{1/2}.
\]

Our attention is focused on homogeneous, centrosymmetric mixture, by supposing the initial continuum is free from stresses.

In the context of our theory, the internal energy density associated to \( \mathbf{U} \) is given by

\[
W(\mathbf{U}) = \frac{1}{2} \left[ A_{ijr}e_{ij}(\mathbf{U})e_{rs}(\mathbf{U}) + C_{ijrs}g_{ij}(\mathbf{U})g_{rs}(\mathbf{U}) + \zeta \varphi^{(1)}\varphi^{(1)} + \right.
\]
\[
+ \mu \varphi^{(2)}\varphi^{(2)} + \alpha_{ij} \varphi^{(1)}_i \varphi^{(1)}_j + \gamma_{ij} \varphi^{(2)}_i \varphi^{(2)}_j + a_{ij}d_i(\mathbf{U})d_j(\mathbf{U}) \right. 
\]
\[
+ B_{ijrs}e_{ij}(\mathbf{U})g_{rs}(\mathbf{U}) + D_{ij}e_{ij}(\mathbf{U})\varphi^{(1)} + E_{ij}e_{ij}(\mathbf{U})\varphi^{(2)} 
\]
\[
+ M_{ij}g_{ij}(\mathbf{U})\varphi^{(1)} + N_{ij}g_{ij}(\mathbf{U})\varphi^{(2)} + \beta_{ij} \varphi^{(1)}_i \varphi^{(1)}_j + 
\]
\[
+ b_{ij}d_i(\mathbf{U})\varphi^{(1)}_i + c_{ij}d_i(\mathbf{U})\varphi^{(2)}_i + \tau \varphi^{(1)}\varphi^{(2)}.
\]
The material coefficients, appearing in previous equations (3), are constants and they obey the following symmetry relations:

\[ A_{ijrs} = A_{jirs} = A_{rsij}, \quad B_{ijrs} = B_{jirs}, \quad C_{ijrs} = C_{rsij}, \quad a_{ij} = a_{ji}, \]

\[ \alpha_{ij} = \alpha_{ji}, \quad \gamma_{ij} = \gamma_{ji}, \quad D_{ij} = D_{ji}, \quad E_{ij} = E_{ji}. \]

The constitutive equations are

\[ S^{(1)}_{ji}(U) = (A_{jirs} + B_{rsji})e_{rs}(U) + (B_{ijrs} + C_{jirs})g_{rs}(U) + (D_{ij} + M_{ij})\varphi^{(1)} + (E_{ij} + N_{ij})\varphi^{(2)}, \]

\[ S^{(2)}_{ji}(U) = B_{rsij}e_{rs}(U) + C_{ijrs}g_{rs}(U) + M_{ij}\varphi^{(1)} + N_{ij}\varphi^{(2)}, \]

\[ g^{(1)}(U) = -D_{rs}e_{rs} - M_{rs}g_{rs} - \zeta\varphi^{(1)} - \tau\varphi^{(2)}, \]

\[ g^{(2)}(U) = -E_{rs}e_{rs} - N_{rs}g_{rs} - \tau\varphi^{(1)} - \mu\varphi^{(2)}, \]

\[ p_i(U) = a_{ij}d_j(U) + b_{ij}\varphi^{(1)} + c_{ij}\varphi^{(2)}, \]

\[ h^{(1)}_i(U) = \alpha_{ij}\varphi^{(1)} + \beta_{ij}\varphi^{(2)} + b_{ji}d_j(U), \]

\[ h^{(2)}_i(U) = \beta_{ji}\varphi^{(1)} + \gamma_{ij}\varphi^{(2)} + c_{ji}d_j(U). \]

Let \( \mathbf{A}_1 = \|\tilde{a}_{KL}\| \) \( (K, L = 1, \ldots, 20) \)

\[ \tilde{a}_{\Gamma\Delta} = A_{\Gamma\Delta}, \quad \tilde{a}_{\Gamma(9+\Delta)} = B_{\Gamma\Delta}, \quad \tilde{a}_{\Gamma 19} = D_{\Gamma}, \quad \tilde{a}_{\Gamma 20} = E_{\Gamma}, \]

\[ \tilde{a}_{(9+\Gamma)\Delta} = B_{\Delta\Gamma}, \quad \tilde{a}_{(9+\Gamma)(9+\Delta)} = C_{\Delta\Gamma}, \quad \tilde{a}_{(9+\Gamma)19} = M_{\Gamma}, \quad \tilde{a}_{(9+\Gamma)20} = N_{\Gamma}, \]

\[ \tilde{a}_{19\Delta} = D_{\Delta}, \quad \tilde{a}_{19(9+\Delta)} = M_{\Delta}, \quad \tilde{a}_{1919} = \zeta, \quad \tilde{a}_{1920} = \tau, \]

\[ \tilde{a}_{20\Delta} = E_{\Delta}, \quad \tilde{a}_{20(9+\Delta)} = N_{\Delta}, \quad \tilde{a}_{2019} = \tau, \quad \tilde{a}_{2020} = \mu, \]

where we have called the nine index combinations \((i, j)\) or \((r, s)\) by capital greek letters \((i.e. \Gamma, \Delta, \text{and so on})\). Now, let \( \mathbf{O} \) be the empty matrix \( 20 \times 9 \) and \( \mathbf{A}_2 = \|\tilde{b}_{KL}\| \) \( (K, L = \ldots, 20) \).
1, \ldots, 9) be
\begin{align*}
\tilde{b}_{ij} &= a_{ij}, & \tilde{b}_{i(3+j)} &= b_{ij}, & \tilde{b}_{i(6+j)} &= c_{ij}, \\
\tilde{b}_{(3+i)j} &= b_{ji}, & \tilde{b}_{(3+i)(3+j)} &= \alpha_{ij}, & \tilde{b}_{(3+i)(6+j)} &= \beta_{ij}, \\
\tilde{b}_{(6+i)j} &= c_{ji}, & \tilde{b}_{(6+i)(3+j)} &= \beta_{ji}, & \tilde{b}_{(6+i)(6+j)} &= \gamma_{ij}.
\end{align*}

Then, the energy density (3) assumes the form
\begin{equation}
W(U) = \frac{1}{2} \sum_{K,L=1}^{29} \tilde{A}_{KL} E_K(U) E_L(U) = \frac{1}{2} \mathbf{E}(U) \cdot \mathbf{AE}(U),
\end{equation}
where the matrix \( \mathbf{A} = \| \tilde{A}_{KL} \| \ (K, L = 1, \ldots, 29) \) is defined by
\begin{equation}
\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathcal{O} \\ \mathcal{O}^T & \mathbf{A}_2 \end{bmatrix},
\end{equation}
and \( \mathcal{O}^T \) is the transposed matrix of \( \mathcal{O} \).

In what follows, we assume that \( \rho^{(\alpha)}, \chi^{(\alpha)} \) are strictly positive and \( W(U) \) is a positive definite quadratic form; thus, there exist the positive constants \( \xi_m \) and \( \xi_M \) so that
\begin{equation}
\xi_m |\mathbf{E}(U)|^2 \geq 2W(U) \geq \xi_M |\mathbf{E}(U)|^2
\end{equation}
where \( \xi_m \) is the minimum elastic modulus and \( \xi_M \) is the maximum elastic moduli.

Let
\begin{equation*}
\mathbf{S}(U) \equiv \{ S_{1j}^{(\alpha)}(U), S_{2j}^{(\alpha)}(U), g^{(1)}(U), g^{(2)}(U), p_i(U), h_i^{(1)}(U), h_i^{(2)}(U) \},
\end{equation*}
then the magnitude of \( \mathbf{S}(U) \) is defined by
\begin{equation*}
|\mathbf{S}(U)| \equiv \left\{ \sum_{\alpha=1}^{2} \left[ S_{1j}^{(\alpha)}(U) S_{2j}^{(\alpha)}(U) + h_i^{(\alpha)}(U) h_i^{(\alpha)}(U) + g^{(\alpha)}(U) g^{(\alpha)}(U) \right] + p_i(U) p_i(U) \right\}^{1/2}.
\end{equation*}

Taking into account the equations (3, 4, 5, 6, 7, 8), we prove that
\begin{equation}
|\mathbf{S}(U)|^2 = \mathbf{AE} \cdot \mathbf{AE} = \mathbf{E} \cdot \mathbf{A}^2 \mathbf{E} \leq \xi_M \mathbf{E} \cdot \mathbf{AE} = 2\xi_M W(U).
\end{equation}

The surface tractions \( \mathbf{s}^{(\alpha)}(\mathbf{U}) \) and \( \mathbf{h}^{(\alpha)}(\mathbf{U}) \) are defined by
\begin{equation}
\begin{aligned}
s_i^{(\alpha)}(\mathbf{U}) &= S_{ji}^{(\alpha)}(\mathbf{U}) n_j, & h_i^{(\alpha)}(\mathbf{U}) &= h_j^{(\alpha)}(\mathbf{U}) n_j,
\end{aligned}
\end{equation}
where \( n_j \) is the unit normal vector.
where n is the outward unit normal vector to boundary surface. The relations (11, 12) imply that
\[ \sum_{\alpha=1}^{2} [s_i^{(\alpha)}(U)s_i^{(\alpha)}(U) + h^{(\alpha)}(U)h^{(\alpha)}(U)] \leq |S(U)|^2 \leq 2\xi MW(U). \] (13)

If we introduce the notations
\[
a_{ijrs} = A_{jirs} + B_{rsji} + B_{jisr} + C_{jisr}, \\
b_{ijrs} = B_{jirs} + C_{jirs}, \\
d_{ijrs} = C_{ijrs}, \\
\tau_{ij} = D_{ij} + M_{ij} \\
\sigma_{ij} = E_{ij} + N_{ij},
\]
the equations (3) become
\[
W(U) = \frac{1}{2} \left[ a_{ijrs}u_{i,j}^{(1)}u_{r,s}^{(1)} + d_{ijrs}u_{i,j}^{(2)}u_{r,s}^{(2)} + \zeta \varphi^{(1)}(1) \varphi^{(1)}(1) + \mu \varphi^{(2)}(2) \varphi^{(2)} + \\
\alpha_{ij} \varphi_{i,j}^{(1)} \varphi_{j,i}^{(1)} + \gamma_{ij} \varphi_{i,j}^{(2)} \varphi_{j,i}^{(2)} + \alpha_{ij} d_{i}(U)d_{j}(U) \right] + b_{ijrs}u_{i,j}^{(1)}u_{r,s}^{(2)} + \\
\tau_{ij}u_{i,j}^{(1)} \varphi_{i,j}^{(1)} + \sigma_{ij}u_{i,j}^{(2)} \varphi_{i,j}^{(2)} + M_{ij}u_{i,j}^{(2)} \varphi_{i,j}^{(1)} + N_{ij}u_{i,j}^{(2)} \varphi_{i,j}^{(2)} + \\
+ \beta_{ij} \varphi_{i,j}^{(1)} \varphi_{j,i}^{(2)} + b_{ij}d_{i}(U)\varphi_{i,j}^{(1)} + c_{ij}d_{i}(U)\varphi_{i,j}^{(2)} + \tau \varphi^{(1)}(1) \varphi^{(2)} .
\]

Using the symmetry relation (4), we get
\[
a_{ijrs} = a_{rsij}, \\
\alpha_{ij} = \alpha_{ji}, \\
\gamma_{ij} = \gamma_{ji}.
\]

(15)
The constitutive equations \((3)\) become

\[
S_{ji}^{(1)}(U) = a_{ijrs}u_{r,s}^{(1)} + b_{ijrs}u_{r,s}^{(2)} + \tau_{ij} \varphi^{(1)} + \sigma_{ij} \varphi^{(2)},
\]

\[
S_{ji}^{(2)}(U) = b_{rsij}u_{r,s}^{(1)} + d_{ijrs}u_{r,s}^{(2)} + M_{ij} \varphi^{(1)} + N_{ij} \varphi^{(2)},
\]

\[
g^{(1)}(U) = -\tau_{rs}u_{r,s}^{(1)} - M_{rs}u_{r,s}^{(2)} - \zeta \varphi^{(1)} - \tau \varphi^{(2)},
\]

\[
g^{(2)}(U) = -\sigma_{rs}u_{r,s}^{(1)} - N_{rs}u_{r,s}^{(2)} - \tau \varphi^{(1)} - \mu \varphi^{(2)},
\]

\[
p_{i}(U) = a_{ij}d_{j}(U) + b_{ij} \varphi^{(1)} + c_{ij} \varphi^{(2)},
\]

\[
h_{i}^{(1)}(U) = \alpha_{ij} \varphi^{(1)} + \beta_{ij} \varphi^{(2)} + b_{ji}d_{j}(U),
\]

\[
h_{i}^{(2)}(U) = \beta_{ji} \varphi^{(1)} + \gamma_{ij} \varphi^{(2)} + c_{ji}d_{j}(U).
\]

It follows from the equations \((15), (16), (17)\) that

\[
2W(U) = \sum_{\alpha=1}^{2} \left[ S_{ji}^{(\alpha)}(U)u_{i,j}^{(\alpha)} + p_{i}(U)d_{i}(U) + h_{i}^{(\alpha)}(U)\varphi_{i}^{(\alpha)} - g^{(\alpha)}(U)\varphi^{(\alpha)} \right]. \tag{18}
\]

and

\[
\dot{W}(U) = \sum_{\alpha=1}^{2} \left[ \dot{S}_{ji}^{(\alpha)}(U)\dot{u}_{i,j}^{(\alpha)} + p_{i}(U)\dot{d}_{i}(U) + \dot{h}_{i}^{(\alpha)}(U)\dot{\varphi}_{i}^{(\alpha)} - \dot{g}^{(\alpha)}(U)\dot{\varphi}^{(\alpha)} \right]. \tag{19}
\]

We consider the initial-boundary value problem \(P\) defined by the equations of motion \((1)\), the geometrical equations \((2)\) and the constitutive equations \((3)\) and the following initial-boundary conditions

\[
u_{i}^{(\alpha)} = a_{i}^{(\alpha)}, \quad \dot{u}_{i}^{(\alpha)} = \dot{a}_{i}^{(\alpha)}, \quad \varphi^{(\alpha)} = \varphi_{0}^{(\alpha)}, \quad \dot{\varphi}^{(\alpha)} = \dot{\varphi}_{0}^{(\alpha)} \quad \text{on } B \times \{0\}. \tag{20}
\]

and

\[
u_{i}^{(\alpha)} = \ddot{u}_{i}^{(\alpha)} \quad \text{on } \Sigma_{1} \times I, \quad s_{i}^{(\alpha)} = \dot{s}_{i}^{(\alpha)} \quad \text{on } \Sigma_{2} \times I, \quad \varphi^{(\alpha)} = \tilde{\varphi}^{(\alpha)} \quad \text{on } \tilde{\Sigma}_{3} \times I, \quad \dot{h}^{(\alpha)} = \tilde{h}^{(\alpha)} \quad \text{on } \Sigma_{4} \times I, \tag{21}
\]

where \(\Sigma_{i} \ (i = 1, \ldots, 4)\) are the subsets of \(\partial B\) such that

\[
\Sigma_{1} \cup \Sigma_{2} = \Sigma_{3} \cup \Sigma_{4} = \partial B, \quad \Sigma_{1} \cap \Sigma_{2} = \Sigma_{3} \cap \Sigma_{4} = \emptyset.
\]
The terms on right-hand in equations (21) and (22) are prescribed continuous functions; along with $f^{(1)}, f^{(2)}, \ell^{(1)}, \ell^{(2)}$ these constitute the external data of the problem $P$.

An array field $U = \{u^{(1)}, u^{(2)}, \varphi^{(1)}, \varphi^{(2)}\}$, meeting all equation (1, 2, 3, 20) and (21), will be referred to as a (regular) solution of the problem $P$.

## 3 Auxiliary identities

In this section we establish some integral identities that we will use in next sections.

**Lemma 3.1** Let $U$ be a solution of initial-boundary-value problem $P$. Then, for every regular region $P \subset B$ with regular boundary $\partial P$, it follows that
\[
\frac{1}{2} \int_P e^{-\lambda t} \sum_{\alpha=1}^{2} \left[ \rho^{(\alpha)} \dot{u}^{(\alpha)}(t) \ddot{u}^{(\alpha)}(t) + \rho^{(\alpha)} \chi^{(\alpha)} \dot{\varphi}^{(\alpha)}(t) \ddot{\varphi}^{(\alpha)}(t) + 2W(U(t)) \right] dv +
\]
\[
+ \frac{1}{2} \lambda \int_{0}^{t} \int_P e^{-\lambda s} \sum_{\alpha=1}^{2} \left[ \rho^{(\alpha)} \dot{u}^{(\alpha)}(s) \ddot{u}^{(\alpha)}(s) + \rho^{(\alpha)} \chi^{(\alpha)} \dot{\varphi}^{(\alpha)}(s) \ddot{\varphi}^{(\alpha)}(s) \right] dv ds +
\]
\[
+ 2W(U(s)) dv ds = \int_{0}^{t} \int_P e^{-\lambda s} \sum_{\alpha=1}^{2} \left[ \rho^{(\alpha)} f^{(\alpha)}(s) \dot{u}^{(\alpha)}(s) + \rho^{(\alpha)} \ell^{(\alpha)}(s) \dot{\varphi}^{(\alpha)}(s) \right] dv ds +
\]
\[
+ \frac{1}{2} \int_P \sum_{\alpha=1}^{2} \left[ \rho^{(\alpha)} \dot{u}^{(\alpha)}(0) \ddot{u}^{(\alpha)}(0) + \rho^{(\alpha)} \chi^{(\alpha)} \dot{\varphi}^{(\alpha)}(0) \ddot{\varphi}^{(\alpha)}(0) + 2W(U(0)) \right] dv +
\]
\[
+ \int_{0}^{t} \int_{\partial P} e^{-\lambda s} \sum_{\alpha=1}^{2} \left[ S^{(\alpha)}(U(s)) \dot{u}^{(\alpha)}(s) + h^{(\alpha)}(U(s)) \dot{\varphi}^{(\alpha)}(s) \right] ds,
\]
where $\lambda$ is a positive parameter and $t \in I$.

**Proof.** The equations (1) and (19) lead to
\[
e^{-\lambda s} \frac{d}{ds} \left[ \frac{1}{2} \sum_{\alpha=1}^{2} \left[ \rho^{(\alpha)} \dot{u}^{(\alpha)}(s) \ddot{u}^{(\alpha)}(s) + \rho^{(\alpha)} \chi^{(\alpha)} \dot{\varphi}^{(\alpha)}(s) \ddot{\varphi}^{(\alpha)}(s) + 2W(U(s)) \right] \right] =
\]
\[
e^{-\lambda s} \sum_{\alpha=1}^{2} \left[ \rho^{(\alpha)} f^{(\alpha)}(s) \dot{u}^{(\alpha)}(s) + \rho^{(\alpha)} \ell^{(\alpha)}(s) \dot{\varphi}^{(\alpha)}(s) +
\]
\[+ \left[ S^{(\alpha)}_{ji}(U(s)) \dot{u}^{(\alpha)}(s) + h^{(\alpha)}_{ji}(U(s)) \dot{\varphi}^{(\alpha)}(s) \right] \right].
\]

By an integration of the equations (23) over $P \times [0, t]$ and by using the divergence theorem, we obtain the equation (22). □

If we introduce
\[
\mathcal{E}(t) = \int_{B} \frac{1}{2} \sum_{\alpha=1}^{2} \left[ \rho^{(\alpha)} \dot{u}^{(\alpha)}(t) \ddot{u}^{(\alpha)}(t) + \rho^{(\alpha)} \chi^{(\alpha)} \dot{\varphi}^{(\alpha)}(t) \ddot{\varphi}^{(\alpha)}(t) + 2W(U(t)) \right] dv,
\]
(24)
then, for $\lambda = 0$ and $P = B$ the equations (22) reduce to

$$
\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_B \sum_{\alpha=1}^2 \left[ \rho^{(\alpha)} f^{(\alpha)}(s) u^{(\alpha)}_i(s) + \rho^{(\alpha)} \ell^{(\alpha)}(s) \dot{\varphi}^{(\alpha)}(s) \right] dv ds +
$$

$$
\int_0^t \int_{\partial B} \sum_{\alpha=1}^2 \left[ s^{(\alpha)}(U(s)) \dot{u}^{(\alpha)}_i(s) + h^{(\alpha)}(U(s)) \dot{\varphi}^{(\alpha)}(s) \right] d a ds +
$$

$$
\int_0^t \int_P \sum_{\alpha=1}^2 \left[ \rho^{(\alpha)} u^{(\alpha)}_i(0) \dot{u}^{(\alpha)}_i(0) + \rho^{(\alpha)} \chi^{(\alpha)} \varphi^{(\alpha)}(0) \dot{\varphi}^{(\alpha)}(0) \right] dv ds +
$$

$$
\int_0^t \int_{\partial P} \sum_{\alpha=1}^2 \left[ s^{(\alpha)}(U(s)) u^{(\alpha)}_i(s) + h^{(\alpha)}(U(s)) \varphi^{(\alpha)}(s) \right] d a ds +
$$

$$
\int_0^t \int_P \sum_{\alpha=1}^2 \left[ \rho^{(\alpha)} f^{(\alpha)}(s) u^{(\alpha)}_i(s) + \rho^{(\alpha)} \ell^{(\alpha)}(s) \varphi^{(\alpha)}(s) \right] dv ds,
$$

$t \in I$.

**Proof.** The relations (11) and (18) imply that

$$
\frac{d}{ds} \left[ \sum_{\alpha=1}^2 \left[ \rho^{(\alpha)} \dot{u}^{(\alpha)}_i(s) \dot{u}^{(\alpha)}_i(s) + \rho^{(\alpha)} \chi^{(\alpha)} \dot{\varphi}^{(\alpha)}(s) \dot{\varphi}^{(\alpha)}(s) \right] \right] =
$$

$$
= \sum_{\alpha=1}^2 \left[ \rho^{(\alpha)} \dot{u}^{(\alpha)}_i(s) \dot{u}^{(\alpha)}_i(s) + \rho^{(\alpha)} \chi^{(\alpha)} \dot{\varphi}^{(\alpha)}(s) \dot{\varphi}^{(\alpha)}(s) \right] +
$$

$$
+ \sum_{\alpha=1}^2 \left[ \rho^{(\alpha)} f^{(\alpha)}(s) u^{(\alpha)}_i(s) + \rho^{(\alpha)} \ell^{(\alpha)}(s) \varphi^{(\alpha)}(s) - 2W(U(s)) +
$$

$$
+ [s^{(\alpha)}(U(s)) u^{(\alpha)}_i(s) + h^{(\alpha)}(U(s)) \varphi^{(\alpha)}(s)] \right].
$$

The relation (23) follows from (27) by an integration of over $P \times [0, t]$ and by using the divergence theorem. ■

**Lemma 3.3** Let $U$ be a solution of initial-boundary-value problem. Then, for every
regular region $P \subset B$ with regular boundary $\partial P$, it follows that

$$2 \int_P \sum_{\alpha=1}^{2} [\rho^{(\alpha)} u^{(\alpha)}_i(t) \dot{u}^{(\alpha)}_i(t) + \rho^{(\alpha)} \chi^{(\alpha)} \varphi^{(\alpha)}(t) \dot{\varphi}^{(\alpha)}(t)] dv =$$

$$= \int_P \sum_{\alpha=1}^{2} [\rho^{(\alpha)} u^{(\alpha)}_i(0) \dot{u}^{(\alpha)}_i(2t) + \rho^{(\alpha)} \dot{u}^{(\alpha)}_i(0) u^{(\alpha)}_i(2t) +$$

$$+ \rho^{(\alpha)} \chi^{(\alpha)} \varphi^{(\alpha)}(0) \dot{\varphi}^{(\alpha)}(2t) + \rho^{(\alpha)} \chi^{(\alpha)} \varphi^{(\alpha)}(2t) \dot{\varphi}^{(\alpha)}(0)] dv +$$

$$+ \int_0^t \int_P \sum_{\alpha=1}^{2} [\rho^{(\alpha)} f^{(\alpha)}_i(t-s) u^{(\alpha)}_i(t+s) - \rho^{(\alpha)} f^{(\alpha)}_i(t+s) u^{(\alpha)}_i(t-s) +$$

$$+ \rho^{(\alpha)} \ell^{(\alpha)}(t-s) \varphi^{(\alpha)}(t+s) - \rho^{(\alpha)} \ell^{(\alpha)}(t+s) \varphi^{(\alpha)}(t-s)] dv ds +$$

$$+ \int_0^t \int_{\partial P} \sum_{\alpha=1}^{2} [s^{(\alpha)}_i(U(t-s)) u^{(\alpha)}_i(t+s) - s^{(\alpha)}_i(U(t+s)) u^{(\alpha)}_i(t-s) +$$

$$+ h^{(\alpha)}(U(t-s)) \varphi^{(\alpha)}(t+s) - h^{(\alpha)}(U(t+s)) \varphi^{(\alpha)}(t-s)] dv ds, \quad t \in I. \quad (28)$$

**Proof.** For every function $\phi \in C^2(I)$, it is holds the following identity

$$\ddot{\phi}(t-s) \phi(t+s) - \dot{\phi}(t+s) \phi(t-s) =$$

$$- \frac{d}{ds} \{ \phi(t-s) \phi(t+s) + \phi(t-s) \dot{\phi}(t+s) \}, \quad s \in [0, t], \quad t \in I, \quad (29)$$

In view of the relation (4), we have

$$\sum_{\alpha=1}^{2} [\rho^{(\alpha)} \ddot{u}^{(\alpha)}_i(t-s) u^{(\alpha)}_i(t+s) - \rho^{(\alpha)} \ddot{u}^{(\alpha)}_i(t+s) u^{(\alpha)}_i(t-s)] =$$

$$\sum_{\alpha=1}^{2} \{ \rho^{(\alpha)} f^{(\alpha)}_i(t-s) u^{(\alpha)}_i(t+s) - \rho^{(\alpha)} f^{(\alpha)}_i(t+s) u^{(\alpha)}_i(t-s) +$$

$$+ [s^{(\alpha)}_{ji}(U(t-s)) u^{(\alpha)}_i(t+s) - s^{(\alpha)}_{ji}(U(t+s)) u^{(\alpha)}_i(t-s)]_{,j} +$$

$$+ [s^{(\alpha)}_{ji}(U(t+s)) u^{(\alpha)}_i(t-s) - s^{(\alpha)}_{ji}(U(t-s)) u^{(\alpha)}_i(t+s)] \} +$$

$$- p_i(U(t-s)) d_i(U(t+s)) + p_i(U(t+s)) d_i(U(t-s)), \quad (30)$$

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and
\[ \sum_{\alpha=1}^{2} \left[ \rho^{(\alpha)} \chi^{(\alpha)} \dot{\varphi}^{(\alpha)}(t-s) \varphi^{(\alpha)}(t+s) - \rho^{(\alpha)} \chi^{(\alpha)} \dot{\varphi}^{(\alpha)}(t+s) \varphi^{(\alpha)}(t-s) \right] = \]
\[ \sum_{\alpha=1}^{2} \left\{ \rho^{(\alpha)} f_i^{(\alpha)}(t-s) u_i^{(\alpha)}(t+s) - \rho^{(\alpha)} f_i^{(\alpha)}(t+s) u_i^{(\alpha)}(t-s) + \right\} \]
\[ + [h_j^{(\alpha)}(U(t-s)) \varphi^{(\alpha)}(t+s) - h_j^{(\alpha)}(U(t+s)) \varphi^{(\alpha)}(t-s)]_j + \]
\[ + [h_j^{(\alpha)}(U(t+s)) \varphi_j^{(\alpha)}(t-s) - h_j^{(\alpha)}(U(t-s)) \varphi_j^{(\alpha)}(t+s)] + \]
\[ + g(U(t-s)) \varphi^{(\alpha)}(t+s) - g(U(t+s)) \varphi^{(\alpha)}(t-s) \right\} = 0. \]

Further, with the help of (13) and (14) we prove that
\[ \sum_{\alpha=1}^{2} \left\{ S_{ji}^{(\alpha)}(U(t+s)) u_i^{(\alpha)}(t-s) - S_{ji}^{(\alpha)}(U(t-s)) u_i^{(\alpha)}(t+s) + \right\} \]
\[ + [h_j^{(\alpha)}(U(t-s)) \varphi^{(\alpha)}(t+s) - h_j^{(\alpha)}(U(t+s)) \varphi^{(\alpha)}(t-s)]_j + \]
\[ + [h_j^{(\alpha)}(U(t+s)) \varphi_j^{(\alpha)}(t-s) - h_j^{(\alpha)}(U(t-s)) \varphi_j^{(\alpha)}(t+s)] + \]
\[ - p_i(U(t-s)) d_i(U(t+s)) + p_i(U(t+s)) d_i(U(t-s)) + \]
\[ + g(U(t-s)) \varphi^{(\alpha)}(t+s) - g(U(t+s)) \varphi^{(\alpha)}(t-s) \right\} = 0. \]

Then, the equations (29)–(32) imply that
\[ \sum_{\alpha=1}^{2} \left\{ \rho^{(\alpha)} f_i^{(\alpha)}(t-s) u_i^{(\alpha)}(t+s) - \rho^{(\alpha)} f_i^{(\alpha)}(t+s) u_i^{(\alpha)}(t-s) + \right\} \]
\[ + [S_{ji}^{(\alpha)}(U(t-s)) u_i^{(\alpha)}(t+s) - S_{ji}^{(\alpha)}(U(t+s)) u_i^{(\alpha)}(t-s)]_j + \]
\[ \rho^{(\alpha)} f_i^{(\alpha)}(t-s) \varphi^{(\alpha)}(t+s) - \rho^{(\alpha)} f_i^{(\alpha)}(t+s) \varphi^{(\alpha)}(t-s) + \]
\[ + [h_j^{(\alpha)}(U(t-s)) \varphi^{(\alpha)}(t+s) - h_j^{(\alpha)}(U(t+s)) \varphi^{(\alpha)}(t-s)]_j \right\} = \]
\[ = - \frac{d}{ds} \sum_{\alpha=1}^{2} \left\{ \rho^{(\alpha)} u_i^{(\alpha)}(t-s) u_i^{(\alpha)}(t+s) + \rho^{(\alpha)} u_i^{(\alpha)}(t-s) \dot{u}_i^{(\alpha)}(t+s) \right\} \]
\[ + \rho^{(\alpha)} \chi^{(\alpha)} \ddot{\varphi}^{(\alpha)}(t-s) \varphi^{(\alpha)}(t+s) + \rho^{(\alpha)} \chi^{(\alpha)} \varphi^{(\alpha)}(t-s) \dot{\varphi}^{(\alpha)}(t+s) \right\} = \]

The equation (28) is reached by performing an integration of the equations (33) over \( P \times [0, t] \) and then by using the divergence theorem. \( \blacksquare \)
4 A time weighted surface measure

Having fixed a time \( T \in I \), for the external given data of the problem \( P \) we define the set \( \hat{D}_T \) by:

i. if \( x \in B \), then

\[
\dot{a}_i^{(1)}(x) \neq 0 \text{ or } \dot{a}_i^{(2)}(x) \neq 0 \text{ or } a_i^{(1)}(x) \neq 0 \text{ or } a_i^{(2)}(x) \neq 0
\]

or

\[
\phi_0^{(1)}(x) \neq 0 \text{ or } \phi_0^{(2)}(x) \neq 0
\]

or there exists \( \tau \in [0, T] \) such that

\[
f_i^{(1)}(x, \tau) \neq 0 \text{ or } f_i^{(2)}(x, \tau) \neq 0 \text{ or } f_i^{(1)}(x, \tau) \neq 0 \text{ or } f_i^{(2)}(x, \tau) \neq 0;
\]

ii. if \( x \in \partial B \), then there exists \( \tau \in [0, T] \) such that

\[
u_i^{(1)}(x, \tau) \neq 0 \text{ or } \phi_i^{(2)}(x, \tau) \neq 0
\]

or

\[
\psi_i^{(1)}(x, \tau) \neq 0 \text{ or } \psi_i^{(2)}(x, \tau) \neq 0
\]

or there exists \( \tau \in [0, T] \) such that

\[
f_i^{(1)}(x, \tau) \neq 0 \text{ or } f_i^{(2)}(x, \tau) \neq 0 \text{ or } f_i^{(1)}(x, \tau) \neq 0 \text{ or } f_i^{(2)}(x, \tau) \neq 0;
\]

The set \( \hat{D}_T \) represents the support of the initial and boundary data and the body force on the time interval \([0, T]\). In what follows, we assume \( \hat{D}_T \) is a bounded set.

We consider a nonempty set \( \hat{D}_T^* \) which is such that \( \hat{D}_T \subset \hat{D}_T^* \subset \bar{B} \) and

i. if \( \hat{D}_T \cap B \neq \emptyset \), then we choose \( \hat{D}_T^* \) to be the smallest bounded regular region in \( \bar{B} \) that includes \( \hat{D}_T \) ; in particular, we set \( \hat{D}_T^* = \hat{D}_T \) if \( \hat{D}_T \) it also happens to be a regular region;

ii. if \( \emptyset \neq \hat{D}_T \subset \partial B \), then we choose \( \hat{D}_T^* \) to be the smallest regular subsurface of \( \partial B \) that includes \( \hat{D}_T \) ; in particular, we set \( \hat{D}_T^* = \hat{D}_T \) if \( \hat{D}_T \) is a regular subsurface of \( \partial B \);

iii. if \( \hat{D}_T = \emptyset \), then we choose \( \hat{D}_T^* \) to be an arbitrary nonempty regular subsurface of \( \partial B \).

Now, we mean the set \( D_r \), by

\[
D_r = \{ x \in \bar{B} : \hat{D}_T^* \cap \Sigma(x, r) \neq \emptyset \}, \quad r \geq 0,
\]

where \( \Sigma(x, r) \) is the closed ball with radius \( r \) and center at \( x \). Further, we use the notation \( B_r \) for the part of \( B \) contained in \( \bar{B} \setminus D_r \) and \( B(r_1, r_2) = B_{r_2} \setminus B_{r_1}, r_1 > r_2; S_r \) denotes
the subsurface of $\partial B_r$ contained into inside of $B$ and whose outward unit normal vector $n$ is forwarded to the exterior of $D_r$. Of course, taking into account that for each $r > 0$, $\hat{D}_T \subset D_r$ and $\hat{D}_T \cap B_r = \emptyset$, we get

$$\tilde{u}^{(\alpha)}_i = 0, \quad \tilde{v}^{(\alpha)}_i = 0, \quad \tilde{\varphi}^{(\alpha)} = 0, \quad \tilde{\zeta}^{(\alpha)} = 0 \quad \text{on } B_r,$$

$$f^{(\alpha)}_i = 0, \quad \ell^{(\alpha)} = 0 \quad \text{on } B_r \times [0, T],$$

$$s_i^{(\alpha)} u^{(\alpha)}_i = 0, \quad h^{(\alpha)} \varphi^{(\alpha)} = 0 \quad \text{on } (B_r \cap \partial B) \times [0, T].$$

(35)

For a fixed positive parameter $\lambda$ and for any $r \geq 0$, $t \in [0, T]$, we define the time–weighted surface power function $P(r, t)$

$$P(r, t) = -\int_0^t \int_{S_r} e^{-\lambda s} \sum_{\alpha=1}^2 |s_i^{(\alpha)}(U(s)) u_i^{(\alpha)}(s) + h^{(\alpha)}(U(s)) \varphi^{(\alpha)}(s)| \, da \, ds.$$

(36)

In the following Lemmas, we show some relevant properties of the function $P(r, t)$.\[4.1\] Let $U$ be a solution of initial-boundary-value problem $\mathcal{P}$ and $\hat{D}_T$ be the bounded support of the external data on the time interval $[0, T]$. Then, the corresponding time–weighted surface power function $P(r, t)$ is a continuous differentiable function on $r \geq 0$, $t \in [0, T]$ and

$$\frac{\partial}{\partial t} P(r, t) = -\int_{S_r} e^{-\lambda s} \sum_{\alpha=1}^2 [s_i^{(\alpha)}(U(t)) u_i^{(\alpha)}(t) + h^{(\alpha)}(U(t)) \varphi^{(\alpha)}(t)] \, da;$$

(37)

$$\frac{\partial}{\partial r} P(r, t) = -\frac{1}{2} \int_{S_r} e^{-\lambda s} \sum_{\alpha=1}^2 \left[ \rho^{(\alpha)} u_i^{(\alpha)}(t) u_i^{(\alpha)}(t) + \rho^{(\alpha)} \varphi^{(\alpha)}(t) \varphi^{(\alpha)}(t) + 2W(U(t)) \right] \, da - \frac{\lambda}{2} \int_0^t \int_{S_r} e^{-\lambda s} \sum_{\alpha=1}^2 \left[ \rho^{(\alpha)} u_i^{(\alpha)}(s) u_i^{(\alpha)}(s) + \rho^{(\alpha)} \varphi^{(\alpha)}(s) \varphi^{(\alpha)}(s) + 2W(U(s)) \right] \, da \, ds.$$\[38\]

Moreover, at a fixed $t \in [0, T]$, $P(r, t)$ is a non–increasing function with respect to $r$.

**Proof.** The equation (37) is an immediate consequence of the definition of $P(r, t)$. The Lemma 3.1 for $B(r_1, r_2)$ and the divergence theorem imply
\[ P(r_1, t) - P(r_2, t) = \]
\[ = \int_0^t \int_{B(r_1, r_2)} e^{-\lambda s} \sum_{\alpha=1}^2 \left[ \rho^{(\alpha)} f_i^{(\alpha)}(s) \dot{u}_i^{(\alpha)}(s) + \rho^{(\alpha)} \chi^{(\alpha)} \ell^{(\alpha)}(s) \dot{\phi}^{(\alpha)}(s) \right] dv ds + \]
\[ -\frac{1}{2} \int_0^t \int_{B(r_1, r_2)} e^{-\lambda s} \frac{\partial}{\partial s} \sum_{\alpha=1}^2 \left[ \rho^{(\alpha)} \dot{u}_i^{(\alpha)}(s) \dot{u}_i^{(\alpha)}(s) + \rho^{(\alpha)} \chi^{(\alpha)} \dot{\phi}^{(\alpha)}(s) \dot{\phi}^{(\alpha)}(s) \right] dv ds + \]
\[ + 2W(U(t)) ds. \] (39)

with \( 0 \leq r_2 \leq r_1 \) and fixed \( t > 0 \). By using equations (35) and performing an integration by parts, the relation (39) becomes

\[ P(r_1, t) - P(r_2, t) = -\frac{1}{2} \int_0^t \int_{B(r_1, r_2)} e^{-\lambda s} \sum_{\alpha=1}^2 \left[ \rho^{(\alpha)} \dot{u}_i^{(\alpha)}(s) \dot{u}_i^{(\alpha)}(s) + \rho^{(\alpha)} \chi^{(\alpha)} \dot{\phi}^{(\alpha)}(s) \dot{\phi}^{(\alpha)}(s) \right] dv ds + \]
\[ + 2W(U(t)) ds. \] (40)

This relation straightly leads to (38).

Since we have assumed that \( \rho^{(\alpha)}, \chi^{(\alpha)} \) are strictly positive, \( \lambda \) is positive and \( W(U) \) is a positive definite quadratic form, the equation (40) gives

\[ P(r_1, t) \leq P(r_2, t) \quad \text{with } r_1 \geq r_2. \] (41)

**Lemma 4.2** Let \( U \) be a solution of initial-boundary-value problem \( P \) and \( \bar{D}_T \) be the bounded support of the external data on the time interval \([0, T]\). Then, the function \( P(r, t) \) satisfies the following first–order differential inequalities, for any \( r \geq 0, \ t \in [0, T] \)

\[ \frac{\lambda}{c} |P(r, t)| + \frac{\partial}{\partial r} P(r, t) \leq 0, \] (42)

\[ \frac{1}{c} \left| \frac{\partial}{\partial t} P(r, t) \right| + \frac{\partial}{\partial r} P(r, t) \leq 0, \] (43)

where

\[ c = \sqrt{\frac{\xi M}{m}} \quad \text{with } m = \min \{ \rho^{(1)}, \rho^{(2)}, \rho^{(1)} \chi^{(1)}, \rho^{(2)} \chi^{(2)} \}. \] (44)

and \( \xi_M \) is the maximum elastic moduli.
Proof. It follows from Schwarz’s inequality and the arithmetic–geometric mean inequality
\[
\left| \sum_{\alpha=1}^{2} \left[ s_{i}^{(\alpha)}(U(t)) u_{i}^{(\alpha)}(t) + h^{(\alpha)}(U(t)) \varphi^{(\alpha)}(t) \right] \right| \leq \frac{1}{2} \sum_{\alpha=1}^{2} \left[ \varepsilon \rho^{(\alpha)}(t) u_{i}^{(\alpha)}(t) + \frac{1}{\varepsilon \rho^{(\alpha)}} s_{i}^{(\alpha)}(U(t)) s_{i}^{(\alpha)}(U(t)) + \varepsilon \rho^{(\alpha)} \chi^{(\alpha)}(t) \varphi^{(\alpha)}(t) + \frac{1}{\varepsilon \rho^{(\alpha)}} h^{(\alpha)}(U(t)) h^{(\alpha)}(U(t)) \right],
\]
where \( \varepsilon \) is an arbitrary positive constant.

Using the relations (36, 45, 44, 13) and taking \( \varepsilon = c \), we deduce
\[
|P(r, t)| \leq \frac{c}{2} \int_{0}^{t} \int_{r} e^{-\lambda t} \sum_{\alpha=1}^{2} \left[ \rho^{(\alpha)}(s) u_{i}^{(\alpha)}(s) + \rho^{(\alpha)} \chi^{(\alpha)}(s) \varphi^{(\alpha)}(s) + 2W(U(s)) \right] ds
\]
for \( r \geq 0 \) and \( 0 \leq t \leq T \).

Similarly, from (37) we obtain
\[
\left| \frac{\partial}{\partial t} P(r, t) \right| \leq \frac{c}{2} \int_{0}^{t} \int_{r} e^{-\lambda t} \sum_{\alpha=1}^{2} \left[ \rho^{(\alpha)}(t) u_{i}^{(\alpha)}(t) + \rho^{(\alpha)} \chi^{(\alpha)}(t) \varphi^{(\alpha)}(t) + 2W(U(t)) \right] ds \leq \frac{c}{2} \int_{0}^{t} \int_{r} e^{-\lambda t} \sum_{\alpha=1}^{2} \left[ \rho^{(\alpha)}(s) u_{i}^{(\alpha)}(s) + \rho^{(\alpha)} \chi^{(\alpha)}(s) \varphi^{(\alpha)}(s) + 2W(U(s)) \right] ds
\]
for \( r \geq 0 \) and \( t \in [0, T] \). By the equation (38) and the relations (46) and (47) we obtain the result. \( \blacksquare \)

Lemma 4.3 Let \( U \) be a solution of initial-boundary-value problem \( P \) and \( D_{T} \) be the bounded support of the external data on the time interval \([0, T]\). Then, it follows that
\[
P(r, t) \geq 0, \quad \text{for } r \geq 0, \quad 0 \leq t \leq T;
\]
moreover
\[
P(r, t) = E(r, t),
\]
where
\[
E(r, t) = \frac{1}{2} \int_{B_{r}} e^{-\lambda t} \sum_{\alpha=1}^{2} \left[ \rho^{(\alpha)}(t) u_{i}^{(\alpha)}(t) + \rho^{(\alpha)} \chi^{(\alpha)}(t) \varphi^{(\alpha)}(t) + 2W(U(t)) \right] dv + \frac{c}{2} \int_{0}^{t} \int_{B_{r}} e^{-\lambda s} \sum_{\alpha=1}^{2} \left[ \rho^{(\alpha)}(s) u_{i}^{(\alpha)}(s) + \rho^{(\alpha)} \chi^{(\alpha)}(s) \varphi^{(\alpha)}(s) + 2W(U(s)) \right] dv ds.
\]
Proof. If B is a bounded body, then the variable \( r \) ranges on \([0, L]\), where
\[
L = \max \{ \min \{ (x_i - y_i)(x_i - y_i) \} : y \in \hat{D}_T^r \} : x \in \bar{B} \} < \infty.
\] (51)

Starting from the definition of \( \hat{D}_T \) and by using the relation (36), we obtain
\[
P(L, t) = 0, \quad 0 \leq t \leq T;
\] (52)
thus, the equation (41) implies the relation (48).

If B is an unbounded body, then the variable \( r \) ranges on \([0, \infty)\). The inequality (43) is equivalent to
\[
\frac{1}{c} \frac{\partial}{\partial t} P(r, t) + \frac{\partial}{\partial r} P(r, t) \leq 0,
\] (53)
and
\[
-\frac{1}{c} \frac{\partial}{\partial t} P(r, t) + \frac{\partial}{\partial r} P(r, t) \leq 0.
\] (54)

If we choose the initial condition \((r_0, t_0)\) such that \( t_0 \in [0, T] \) and \( r_0 \geq ct_0 \) and we put \( t = t_0 + \frac{r - r_0}{c} \) in inequality (53), then
\[
\frac{d}{dr} [P(r, t_0 + \frac{r - r_0}{c})] \leq 0,
\] (55)
thus,
\[
P(r, t_0 + \frac{r - r_0}{c}) \leq P(r_1, t_0 + \frac{r_1 - r_0}{c}) \quad \text{with } r \geq r_1.
\] (56)

For \( r = r_0 \) and \( r_1 = r_0 - ct_0 \), we get
\[
P(r_0, t_0) \leq P(r_0 - ct_0, 0).
\] (57)

Similarly, by setting \( t = t_0 - \frac{r - r_0}{c} \) in (54), it follows
\[
\frac{d}{dr} [P(r, t_0 - \frac{r - r_0}{c})] \leq 0,
\] (58)
so that
\[
P(r_0 + ct_0, 0) \leq P(r_0, t_0).
\] (59)

Taking into account \( P(r_0 - ct_0, 0) = 0 \) and \( P(r_0 + ct_0, 0) = 0 \), the relations (57), (59) imply that
\[
P(r_0, t_0) = 0.
\]
Of course, for \( r_0 \to \infty \) in the above relations, it follows

\[
P(\infty, t_0) = \lim_{r_0 \to \infty} P(r_0, t_0) = 0, \tag{60}\]

and, by (11), we conclude that the relation (18) is true.

The equation (49) follows from the relation (10) by means of the use of the relations (52) and (10).

5 Spatial behaviour

By the properties of the time-weighted surface power function \( P \), we establish the theorem that gives a complete description of the spatial behaviour of the elastic process in question outside of the support of the external data.

**Theorem 5.1** Let \( U \) be a solution of initial-boundary-value problem, \( \mathring{D}_T \) be the bounded support of the external data on the time interval \([0, T]\) and let \( P(r, t) \) be the time-weighted surface power measure associated with \( U \).

i. **Spatial behaviour:** For each fixed \( t \in [0, T] \) and \( 0 \leq r \leq c t \), we have

\[
P(r, t) \leq P(0, t) \exp\left(-\frac{\lambda}{c} r \right); \tag{61}\]

ii. **Domain of influence results:** For each fixed \( t \in [0, T] \) and \( r \geq c t \), we have

\[
u^{(1)}_i = 0, \quad u^{(2)}_i = 0,
\varphi^{(1)} = 0, \quad \varphi^{(2)} = 0 \quad \text{on } B_r \times [0, T]. \tag{62}\]

**Proof.** The equations (12), (48) give

\[
\frac{\partial}{\partial r} \left[ \exp\left(\frac{\lambda}{c} r \right) P(r, t) \right] \leq 0, \quad 0 \leq r \leq c t, \quad 0 \leq t \leq T, \tag{63}\]

so that we obtain the equation (61).

If we choose \( t \in [0, T] \) and we set \( r = c t \) in (38), then

\[
d\frac{d}{dt} [P(ct, t)] \leq 0, \tag{64}\]

and

\[
P(r', t) \leq P(ct, t) \leq P(0, 0) = 0, \quad \forall r' \geq c t. \tag{65}\]
In order to (65), (61), we have for all \( r \geq ct \)

\[ P(r, t) \leq 0, \quad (66) \]

and, taking into account (48), we obtain

\[ P(r, t) = 0. \quad (67) \]

Now, the equations (67), (49), (50) imply

\[ E(r, t) = \frac{1}{2} \int_{B_r} e^{-\lambda t} \sum_{a=1}^{2} \left[ \rho^{(a)} \dot{u}_i^{(a)}(t) \dot{u}_i^{(a)}(t) + \rho^{(a)} \chi^{(a)} \dot{\varphi}^{(a)}(t) \dot{\varphi}^{(a)}(t) \right. \]
\[ + 2W(U(t)) \] 
\[ \left. + \frac{\lambda}{2} \int_0^t \int_{B_r} e^{-\lambda s} \sum_{a=1}^{2} \left[ \rho^{(a)} \dot{u}_i^{(a)}(s) \dot{u}_i^{(a)}(s) + \rho^{(a)} \chi^{(a)} \dot{\varphi}^{(a)}(s) \dot{\varphi}^{(a)}(s) + 2W(U(s)) \right] \, dv \right] \, ds = 0. \quad (68) \]

Since \( \rho^{(a)} \) and \( \chi^{(a)} \) are strictly positive, \( \lambda \) is positive and \( W(U) \) is a positive definite quadratic form, we have

\[ \dot{u}_i^{(1)} = 0, \quad \dot{u}_i^{(2)} = 0, \quad \dot{\varphi}^{(1)} = 0, \quad \dot{\varphi}^{(2)} = 0 \quad \text{on } B_r \times [0, T]; \quad (69) \]

so, by (35), we obtain the equations (62). ■

If we put \( t = T \) and \( r = cT \) in the equations (62) for \( t = T \) and \( r = cT \) they imply that the set \( D_{cT} \) covers a domain of elastic disturbances produced by the the data at time \( T \), i.e.

\[ u_i^{(1)} = 0, \quad u_i^{(2)} = 0, \quad \varphi^{(1)} = 0, \quad \varphi^{(2)} = 0 \quad \text{on } B_{cT} \times [0, T]. \quad (70) \]

This result is known as a so-called domain of influence theorem (see Gurtin [18]).

As an immediate consequence of the equation (62), we establish the following uniqueness result valid for a bounded or unbounded body

**Theorem 5.2 (Uniqueness)** It exists at most one (regular) solution for the boundary-initial-value problem.

**Proof.** Thanks to the linearity of the problem, we have only to show that the null data imply null solution. Let \( \tilde{U} = \{ \tilde{u}_i^{(1)}, \tilde{u}_i^{(2)}, \tilde{\varphi}^{(1)}, \tilde{\varphi}^{(2)} \} \) a solution corresponding to null data. Since the set \( \tilde{D}_T = \emptyset \) for each \( T \in (0, +\infty) \) and the function \( P(r, t) = 0 \), we can conclude that

\[ \tilde{u}_i^{(1)} = 0, \quad \tilde{u}_i^{(2)} = 0, \quad \tilde{\varphi}^{(1)} = 0, \quad \tilde{\varphi}^{(2)} = 0 \quad \text{on } B \times I. \quad ■ \]
We remark that if $B$ is a bounded regular region, for values of $T$ big enough, then it exists a value of $t \in [0, T]$ having the property that $D_{ct} = B$, the relation (62) becomes superfluous and the behaviour of solutions is fully described by the relation (61). On the other hand, for values of $T$ sufficiently small, the behaviour of solutions will be described by the relation (62) almost as in $B$. Similar arguments are valid for an unbounded regular region.

6 Asymptotic equipartition of energy

Throughout this section we study the time asymptotic behaviour of the solutions of the initial-boundary value problem $P_0$ for the bounded regular region $B$ defined by the following equations of motion

$$
\begin{align*}
S_{ji,j}^{(a)} + (-1)^a p_i &= \rho^{(a)} \ddot{u}_i^{(a)}, \\
h_{i,i}^{(a)} + g^{(a)} &= \rho^{(a)} \chi^{(a)} \ddot{\varphi}^{(a)}, \quad \text{on } B \times (0, +\infty), \\
\end{align*}
$$

(71)

the geometrical equations (2) and the constitutive equations (5), the initial conditions (20) and the boundary conditions

$$
\begin{align*}
\dot{u}_i^{(a)} &= 0 \quad \text{on } \Sigma_1 \times I, \quad s_i^{(a)} = 0 \quad \text{on } \Sigma_2 \times I, \\
\varphi^{(a)} &= 0, \quad \text{on } \Sigma_3 \times I, \quad h^{(a)} = 0 \quad \text{on } \Sigma_4 \times I.
\end{align*}
$$

(72)

Now, we introduce the Cesàro means of various energies associated with the solution $U$ of the problem $P_0$:

$$
\begin{align*}
K_C^u(t) &= \sum_{\alpha=1}^{2} \frac{1}{2t} \int_0^t \int_B \rho^{(a)} \dot{u}_i^{(a)}(s) \ddot{u}_i^{(a)}(s) dv ds, \\
K_C^\varphi(t) &= \sum_{\alpha=1}^{2} \frac{1}{2t} \int_0^t \int_B \rho^{(a)} \chi^{(a)} \dot{\varphi}^{(a)}(s) \ddot{\varphi}^{(a)}(s) dv ds, \\
S_C(t) &= \sum_{\alpha=1}^{2} \frac{1}{t} \int_0^t \int_B W(U(s)) dv ds,
\end{align*}
$$

(73)

and

$$
K_C(t) = K_C^u(t) + K_C^\varphi(t),
$$

(74)
If $\text{meas} \Sigma_1 = 0$, then there exists a family of rigid motions and null change in volume fraction which satisfy the equations (71, 2, 5, 20) and (72). We decompose the initial data $a_i^{(a)}$ and $\dot{a}_i^{(a)}$ as

$$a_i^{(a)} = \bar{a}_i^{(a)} + A_i^{(a)}, \quad \dot{a}_i^{(a)} = \dot{\bar{a}}_i^{(a)} + \dot{A}_i^{(a)}, \quad (75)$$

where $\bar{a}_i^{(a)}$ and $\dot{\bar{a}}_i^{(a)}$ are the rigid displacements determined so that $A_i^{(a)}$ and $\dot{A}_i^{(a)}$ satisfy the normalization restrictions

$$\int_B \rho(\alpha) A_i^{(a)} dv = 0, \quad \int_B \rho(\alpha) x_j A_k^{(a)} dv = 0, \quad (76)$$

and $\varepsilon_{ijk}$ is the alternating symbol.

We put

$$\hat{C}^1(B) \equiv \{ \text{whit } v_i \in C^1(\bar{B}) : v_i = 0 \text{ on } \Sigma_1 \text{ if } \text{meas} \Sigma_1 \neq 0, \quad \text{or } \int_B \sum_{\alpha=1}^2 \rho(\alpha) v_i dv = 0 , \quad \int_B \sum_{\alpha=1}^2 \rho(\alpha) x_j v_k dv = 0 \} \text{ if } \text{meas} \Sigma_1 = 0,$$

and

$$\hat{C}^1(B) \equiv \{ \zeta \in C^1(B) : \zeta = 0 \text{ on } \Sigma_3 \},$$

and

$$\hat{W}_1(B) \equiv \text{the completion of } \hat{C}^1(B) \text{ by means of } || \cdot ||_{W_1(B)},$$

and

$$\hat{W}_1(B) \equiv \text{the completion of } \hat{C}^1(B) \text{ by means of } || \cdot ||_{W_1(B)}.$$

The spaces $W_m(B)$ represents the familiar Sobolev space and $W_m(B) \equiv [W_m(B)]^3$.

The equation (10) assures that the following Korn’s inequality (19) holds

$$\int_B \sum_{\alpha=1}^2 2W(\mathbf{v}) dv \geq m_1 \int_B \sum_{\alpha=1}^2 (v_i^{(a)} v_i^{(a)} + x^{(a)} \phi^{(a)} \phi^{(a)}) dv, \quad m_1 = \text{const.} > 0, \quad (77)$$

for every $\mathbf{V} = \{ v^{(1)}, v^{(2)}, \phi^{(1)}, \phi^{(2)} \} : v^{(a)} \in \hat{W}_1(B), \quad \phi^{(a)} \in \hat{W}_1(B)$.

If $\text{meas} \Sigma_1 = 0$, then we shall find it a convenient practice to decompose the solution $U = \{ u^{(1)}, u^{(2)}, \varphi^{(1)}, \varphi^{(2)} \}$ of the problem $\mathcal{P}_0$ in the form

$$u_i^{(a)} = \bar{a}_i^{(a)} + \dot{\bar{a}}_i^{(a)} + v_i^{(a)}, \quad \varphi^{(a)} = \phi^{(a)}, \quad (78)$$
where \( V := \{v^{(1)}, v^{(2)}, \phi^{(1)}, \phi^{(2)}\} \in \tilde{W}_1(B) \times \tilde{W}_1(B) \times \tilde{W}_1(B) \times \tilde{W}_1(B) \) represents the solution of the problem \( P_0 \) with the initial data \( \{\tilde{a}_i^{(a)}, \tilde{\phi}_0^{(a)}\} \) and \( \{\tilde{a}_i^{(a)}, \tilde{\phi}_0^{(a)}\} \).

**Theorem 6.1** Let \( U \) is the solution to the problem \( P_0 \). Then, for all choice of initial data with \( a^{(a)} \in W_1(B) \), \( \dot{a}^{(a)} \in W_0(B) \), \( \phi^{(a)} \in W_1(B) \), \( \dot{\phi}^{(a)} \in W_0(B) \). Then the following asymptotic behaviour of the solution \( U \) holds:

i. if \( \text{meas} \Sigma_1 \neq 0 \), we have

\[
\lim_{t \to \infty} K_C(t) = \lim_{t \to \infty} S_C(t) = \frac{1}{2} \mathcal{E}(0); \tag{79}
\]

ii. if \( \text{meas} \Sigma_1 = 0 \), we have

\[
\lim_{t \to \infty} K_C(t) = \lim_{t \to \infty} S_C(t) + \frac{1}{2} \int_B \sum_{\alpha=1}^{2} \rho^{(a)} \dot{a}_i^{(a)} \dot{\phi}_i^{(a)} dv = \frac{1}{2} \mathcal{E}(0) \tag{80}
\]

where \( \mathcal{E}(t) \) is defined by (74).

**Proof.** Respect to the problem \( P_0 \) it follows that \( f^{(1)} = 0, f^{(2)} = 0, \ell^{(1)} = 0, \ell^{(2)} = 0 \) and the boundary conditions (72) are verified. Thus, the equation (74) becomes

\[
\mathcal{E}(t) = \mathcal{E}(0), \quad t \geq 0. \tag{81}
\]

so that

\[
K_C(t) + S_C(t) = \mathcal{E}(0), \quad \text{for all } \quad t \geq 0. \tag{82}
\]

On the other hand, the relations (26) and (28) imply

\[
K_C(t) - S_C(t) = -\frac{1}{4t} \int_B \sum_{\alpha=1}^{2} \left\{ 2 \rho^{(a)} \dot{a}_i^{(a)}(0) \dot{\phi}_i^{(a)}(0) + 2 \rho^{(a)} \chi^{(a)} \phi^{(a)}(0) \phi^{(a)}(0) + \rho^{(a)} \left\{ \dot{u}_i^{(a)}(0) \dot{u}_i^{(a)}(2t) + \dot{u}_i^{(a)}(0) u_i^{(a)}(2t) + \rho^{(a)} \chi^{(a)} \phi^{(a)}(0) \phi^{(a)}(2t) + \rho^{(a)} \chi^{(a)} \phi^{(a)}(2t) \phi^{(a)}(0) \right\} dv, \quad t > 0. \tag{83}
\]

The relations (10), (24) and (81) imply

\[
\int_B \rho^{(a)} \dot{u}_i^{(a)}(s) \dot{u}_i^{(a)}(s) dv \leq 2 \mathcal{E}(0), \tag{84}
\]

\[
\int_B \rho^{(a)} \chi^{(a)} \phi^{(a)}(s) \phi^{(a)}(s) dv \leq 2 \mathcal{E}(0), \tag{84}
\]

\[
\int_B \phi^{(a)}(s) \phi^{(a)}(s) dv \leq \frac{2}{\xi_m} \int_B \sum_{\alpha=1}^{2} W(U(s)) dv \leq \frac{2}{\xi_m} \mathcal{E}(0), \tag{84}
\]

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thus
\[
\lim_{s \to \infty} \frac{1}{s} \int_B \rho^{(\alpha)} \dot{u}^{(\alpha)}(s) \dot{u}^{(\alpha)}(s) dv = 0,
\]
\[
\lim_{s \to \infty} \frac{1}{s} \int_B \rho^{(\alpha)} \chi^{(\alpha)} \dot{\varphi}^{(\alpha)}(s) \varphi^{(\alpha)}(s) dv = 0,
\]
\[
\lim_{s \to \infty} \frac{1}{s} \int_B \dot{\varphi}^{(\alpha)}(s) \varphi^{(\alpha)}(s) dv = 0.
\]
By using the Schwarz’s inequality and the relation (85) in (83), we obtain
\[
\lim_{t \to \infty} K_C(t) - S_C(t) = \lim_{t \to \infty} \frac{1}{4t} \int_B \sum_{\alpha=1}^{2} \rho^{(\alpha)} \dot{u}^{(\alpha)}(0) u^{(\alpha)}(2t) dv.
\]
When \( \text{meas} \Sigma_1 \neq 0 \), then for \( u \in \hat{W}_1(B) \), \( \varphi^{(\alpha)} \in \hat{W}_1(B) \), the relations (24), (77) and (81) imply
\[
\int_B \sum_{\alpha=1}^{2} u^{(\alpha)}(s) u^{(\alpha)}(s) dv \leq \frac{1}{m_1} \int_B \sum_{\alpha=1}^{2} 2W(U(s)) dv \leq \frac{2}{m_1} \mathcal{E}(0),
\]
and, by means of the Schwarz’s inequality, we obtain
\[
\lim_{t \to \infty} \left\{ \frac{1}{4t} \int_B \sum_{\alpha=1}^{2} \rho^{(\alpha)} \dot{u}^{(\alpha)}(0) u^{(\alpha)}(2t) dv \right\} = 0.
\]
Then, by the equations (88) and (89) we have
\[
\lim_{t \to \infty} K_C(t) - S_C(t) = 0.
\]
The relations (82) and (89) imply (79).

When \( \text{meas} \Sigma_1 = 0 \), then, the equation (73), (74) and (78) lead to
\[
\frac{1}{4t} \int_B \sum_{\alpha=1}^{2} \rho^{(\alpha)} \dot{u}^{(\alpha)}(0) u^{(\alpha)}(2t) dv = \frac{1}{4t} \int_B \sum_{\alpha=1}^{2} \rho^{(\alpha)} \dot{u}_i \dot{a}_i dv +
\]
\[
+ \frac{1}{4t} \int_B \sum_{\alpha=1}^{2} \rho^{(\alpha)} (\dot{\dot{u}}^{(\alpha)}_i + \dot{\varphi}^{(\alpha)}_i) v_i(2t) dv + \frac{1}{2} \int_B \sum_{\alpha=1}^{2} \rho^{(\alpha)} \dot{a}_i \dot{a}_i dv.
\]
Since \( V = \{ v^{(1)}, v^{(2)}, \varphi^{(1)}, \varphi^{(2)} \} \in \hat{W}_1(B) \times \hat{W}_1(B) \times \hat{W}_1(B) \times \hat{W}_1(B) \), from (24), (77) and (81), we deduce that
\[
\int_B \sum_{\alpha=1}^{2} v^{(\alpha)}_i(s) v^{(\alpha)}_i(s) dv \leq \frac{2}{m_1} \int_B \sum_{\alpha=1}^{2} W(V(s)) dv \leq \frac{2}{m_1} \mathcal{E}(0).
\]
Taking into account of the equations (90) and (91), we have

\[
\lim_{t\to\infty} \frac{1}{4t} \int_B \sum_{\alpha=1}^2 \rho^{(\alpha)} \dot{u}_i^{(\alpha)}(0) u_i^{(\alpha)}(2t) dv = \frac{1}{2} \int_B \sum_{\alpha=1}^2 \rho^{(\alpha)} \dot{\bar{a}}_i^{(\alpha)} \dot{\bar{a}}_i^{(\alpha)} dv. \tag{92}
\]

With the aid of the equations (86) and (92), we conclude that

\[
\lim_{t\to\infty} K_C(t) = \lim_{t\to\infty} S_C(t) + \frac{1}{2} \int_B \sum_{\alpha=1}^2 \rho^{(\alpha)} \dot{\bar{a}}_i^{(\alpha)} \dot{\bar{a}}_i^{(\alpha)} dv. \tag{93}
\]

Moreover, the equations (82) and (93) lead to (80).

References

[1] C. Truesdell and R. Toupin The Classical Field Theories in “Handbuch der Physik” (ed. S. Flugge), vol. III /3, Springer–Verlag, Berlin, 1960.

[2] P. Kelly, A reacting continuum. *Int. J. Engng. Sci.* 2, pp. 129–153, 1964.

[3] A. C. Eringen and J. D. Ingram, A continuum theory of chemically reacting media–I. *Int. J. Engng. Sci.* 3, pp. 197–212, 1965.

[4] J. D. Ingram and A. C. Eringen, A continuum theory of chemically reacting media–II. Constitutive equations of reacting fluid mixtures. *Int. J. Engng. Sci.* 4, pp. 289–322, 1967.

[5] A. E. Green and P. M. Naghdi, A dynamical theory of interacting continua. *Int. J. Engng. Sci.* 3, pp. 231–241, 1965.

[6] A. E. Green and P. M. Naghdi, A note on mixtures. *Int. J. Engng. Sci.* 6, pp. 631–635, 1968.

[7] I. Müller, A thermodynamic theory of mixtures of fluids. *Arch. Rat. Mech. Anal.* 28, pp. 1–39, 1968.

[8] N. Dunwoody and I. Müller, A thermodynamic theory of two chemically reacting ideal gases with different temperatures. *Arch. Rat. Mech. Anal.* 29, pp. 344–369, 1968.

[9] A. Bedford and D. S. Drumheller, Theory of immiscible and structured mixtures. *Int. J. Engng. Sci.* 21, pp. 863–960, 1983.
[10] D. Ieșan, On the theory of mixtures of elastic solids. *J. Elasticity* **35**, pp. 251–268, 1994.

[11] J. W. Nunziato and S. C. Cowin, A nonlinear theory of elastic materials with voids. *Arch. Rat. Mech. Anal.* **72**, pp. 175-201, 1979.

[12] M. A. Goodman e S. C. Cowin, A continuum theory for granular materials, *Arch. Rational Mech. Anal.* **44**, pp. 249-266, 1972.

[13] D. S. Drumheller, The theoretical treatment of a porous solids using a mixture theory. *Int. J. Solids Struct.* **14**, pp. 441-456, 1978.

[14] S. Chiriță and M. Ciarletta, Time-weighted surface power function method for the study of spatial behaviour in dynamics of continua. *Eur. J. Mech. A/Solids* **18**, pp. 915–933, 1999.

[15] W. A. Day, Means and autocorrelations in elastodynamics. *Arch. Rational Mech. Anal.* **73**, pp. 243–256, 1980.

[16] H. A. Levine, An equipartition of energy theorem for weak solutions of evolutionary equations in Hilbert space: The Lagrange identity method. *J. Diff. Eqns* **24**, pp. 197–210, 1977.

[17] A. Bedford e M. Stern, A multi-continuum theory for composite elastic materials, *Acta Mechanica*, **14**, pp. 85-102, 1972.

[18] M. A. Gurtin, *Linear theory of elasticity*. In Handbuch der Physik (ed. Flugge S.) Springer–Verlag Berlin vol. VIa/2, pp. 1–295, 1972.

[19] I. Hlaváček and J. Necas, ON inequalities of Korn’s type, *Arch. Rational Mech. Anal.* **36**, pp. 305-334, 1970.