ENDOFUNCTORS AND POINCARÉ–BIRKHOFF–WITT THEOREMS

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ABSTRACT. We determine what appears to be the bare-bones categorical framework for Poincaré–Birkhoff–Witt type theorems about universal enveloping algebras of various algebraic structures. Our language is that of endofunctors; we establish that a natural transformation of monads enjoys a Poincaré–Birkhoff–Witt property only if that transformation makes its codomain a free right module over its domain.

INTRODUCTION

It is well known that the commutator \([a, b] = ab – ba\) in every associative algebra satisfies the Jacobi identity. Thus, every associative algebra may be regarded as a Lie algebra, leading to a functor from the category of associative algebras to the category of Lie algebras assigning to an associative algebra \(A\) a Lie algebra \(A^{(-)}\) with the same underlying vector space and the Lie bracket as above. This functor admits a left adjoint \(U(-)\), the universal enveloping associative algebra of a Lie algebra. The classical Poincaré–Birkhoff–Witt (PBW) theorem identifies the universal enveloping algebra of \(L\), as a vector space, with the symmetric algebra of \(L\); the precise properties of such an identification depend on the chosen strategy of the proof.

More generally, a functor from the category of algebras of type \(S\) to the category of algebras of type \(T\) is called a functor of change of structure if it does not change the underlying object of an algebra, but only changes the structure operations. Slightly informally, one says that a functor of change of structure has the PBW property if, for any \(T\)-algebra \(A\), the underlying vector space of its universal enveloping \(S\)-algebra \(U_S(A)\) admits a description that does not depend on the algebra structure, but only on the underlying vector space of \(A\).

This intuitive view of the PBW property is inspired by the notion of a PBW pair of algebraic structures due to Mikhalev and Shestakov [19]. There, the algebraic setup is that of varieties of algebras. The authors of [19] define, for any \(T\)-algebra \(A\), a canonical filtration on the universal enveloping algebra \(U_S(A)\) which is compatible with the \(S\)-algebra structure, and establish that there is a canonical surjection
\[
\pi: U_S(\text{Ab } A) \rightarrow \text{gr } U_S(A),
\]
where \(\text{Ab } A\) is the Abelian \(S\)-algebra on the underlying vector space of \(A\). They say that the given algebraic structures form a PBW pair if that canonical surjection is an isomorphism. Furthermore, they prove a result stating that this property is equivalent to \(U_S(A)\) having a basis of certain monomials built out of the basis elements of \(A\), where the definition of monomials does not depend on a particular algebra \(A\). This latter definition is slightly more vague than the former one; trying to formalise it, we discovered a pleasant categorical context where PBW theorems belong. The approach we propose is to use the language of endofunctors: so that a fully rigorous way to say “the definition of monomials does not depend on a particular algebra” is to say that the underlying vector space of \(U_S(A)\) is of the form \(\mathcal{X}(A)\), where \(\mathcal{X}\) is an endofunctor on the category of vector spaces, and that identification is natural with respect to algebra maps. Our main result (Theorem 2.1) states that if algebraic structures are encoded by monads, and a functor of change of structure arises from a natural transformation of monads \(\phi: M \rightarrow N\), then the PBW property holds if and only if the right module action of \(M\) on \(N\) via \(\phi\) is free.

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It is perhaps worth noting that our definition of the PBW property exhibits an interesting “before/after” dualism with that of [19]: that definition formalises the intuitive notion that “operations on \(A\) do not matter before computing \(U_\mathcal{S}(A)\)” (so that operations on \(U_\mathcal{S}(A)\) have some canonical “leading terms”, and then corrections that do depend on operations of \(A\), while our approach suggests that “operations on \(A\) do not matter after computing \(U_\mathcal{S}(A)\)” (so that the underlying vector space of \(U_\mathcal{S}(A)\) is described in a canonical way). It turns out that our formalisation is a bit more refined than that of [19], in particular it shows that the extent to which a PBW result may be functorial depends on the characteristic of the ground field (see e.g. Proposition 3.2).

We argue that our result, being a necessary and sufficient statement, should be regarded as the bare-bones framework for studying the PBW property. Most usual approaches to PBW theorems tend to utilise something extrinsic; e.g., in the case of Lie algebras, one may consider only Lie algebras associated to Lie groups and identify the universal enveloping algebra with the algebra of distributions on the group supported at the unit element (see [24], this is probably the closest in spirit to the original proof of Poincaré [20]), or use the additional coalgebra structure on the universal enveloping algebra (like in the proof of Cartier [5], generalised by Loday in [16] who defined a general notion of a “good triple of operads”). Proofs that do not use such \textit{deus ex machina} devices normally rely on an explicit presentation of universal enveloping algebras by generators and relations (the most famous application of Bergman’s Diamond Lemma [3], somewhat in the spirit of the proofs of Birkhoff [4] and Witt [26]); while very efficient, those proofs break functoriality in a rather drastic way, which is highly undesirable for objects defined by a universal property.

1. \textsc{Recollections: Monads, Algebras, Modules}

In this section, we recall some basic definitions and results from category theory used in this paper, referring the reader to [13, 14, 18] for further details.

1.1. \textbf{Monads}. Let \(\mathcal{C}\) be a category. Recall that all endofunctors of \(\mathcal{C}\) form a strict monoidal category \((\text{End}(\mathcal{C}), \circ, 1)\). More precisely, in that category morphisms are natural transformations, the monoidal structure \(\circ\) is the composition of endofunctors, \((\mathcal{F} \circ \mathcal{G})(c) = \mathcal{F}((\mathcal{G}(c)))\), and the unit of the monoidal structure \(1\) is the identity functor, \(1(c) = c\). A \textit{monad} on \(\mathcal{C}\) is a monoid \((\mathcal{M}, \mu_\mathcal{M}, \eta_\mathcal{M})\) in \(\text{End}(\mathcal{C})\); here we denote by \(\mu_\mathcal{M} : \mathcal{M} \circ \mathcal{M} \to \mathcal{M}\) the monoid product, and by \(\eta_\mathcal{M} : 1 \to \mathcal{M}\) the monoid unit.

1.2. \textbf{Algebras}. An \textit{algebra for the monad} \(\mathcal{M}\) is an object \(c\) of \(\mathcal{C}\), and a structure map \(\gamma_c : \mathcal{M}(c) \to c\) for which the two diagrams

\[
\begin{array}{ccc}
\mathcal{M}(\mathcal{M}(c)) & \xrightarrow{\mathcal{M}(\gamma_c)} & \mathcal{M}(c) \\
\downarrow{\mu_\mathcal{M}(c)} & & \downarrow{\gamma_c} \\
\mathcal{M}(c) & \xrightarrow{\gamma_c} & c \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
1(c) & \xrightarrow{\eta_\mathcal{M}(c)} & \mathcal{M}(c) \\
\downarrow{1_c} & & \downarrow{\gamma_c} \\
c & \xrightarrow{\gamma_c} & c \\
\end{array}
\]

commute for all \(c\). The category of algebras over a monad \(\mathcal{M}\) is denoted by \(\mathcal{C}^\mathcal{M}\).

1.3. \textbf{Modules}. The notion of a module over a monad follows the general definition of a module over a monoid in monoidal category. We shall primarily focus on right modules; left modules are defined similarly. A \textit{right module over a monad} \(\mathcal{M}\) is an endofunctor \(\mathcal{R}\) together with a natural transformation \(\rho_\mathcal{R} : \mathcal{R} \circ \mathcal{M} \to \mathcal{R}\).
for which the two diagrams

\[
\begin{array}{ccc}
R(M(M(c))) & \xrightarrow{\mathcal{R}(\mu_M(c))} & R(M(c)) \\
\mathcal{R}(M(c)) \downarrow & & \downarrow \mathcal{R}(\varepsilon(c)) \\
R(M(c)) & \xrightarrow{\mathcal{R}(\varepsilon(c))} & R(c)
\end{array}
\quad
\begin{array}{ccc}
R(\varepsilon(c)) & \xrightarrow{\mathcal{R}(\eta_M(c))} & R(M(c)) \\
\downarrow \mathcal{R}(\varepsilon(c)) & & \downarrow \mathcal{R}(\varepsilon(c)) \\
R(c) & \xrightarrow{\mathcal{R}(\varepsilon(c))} & R(c)
\end{array}
\]

commute for all \( c \). The category of right modules over a monad \( M \) is denoted by \( \text{Mod}_M \). The forgetful functor from the category \( \text{Mod}_M \) to \( \text{End}(C) \) has a left adjoint, called the free right \( M \)-module functor; the free right \( M \)-module generated by an endofunctor \( X \) is \( X \circ M \) with the structure map \( X \circ M \circ M \to X \circ M \) given by \( 1_X \circ \mu_M \).

1.4. Coequalizers in categories of algebras. Recall that a reflexive pair in a category \( C \) is a diagram

\[
\begin{array}{ccc}
c_1 & \xrightarrow{f} & c_2 \\
\downarrow d & & \downarrow g \\
\end{array}
\]

where \( f d = g d = 1_{c_2} \). Throughout this paper, we shall assume the following property of the category \( C \): for every monad \( M \), the category \( C^M \) has coequalizers of all reflexive pairs. There are various criteria for that to happen, see, for instance, [1] and [2, Sec. 9.3] (both relying on the seminal work of Linton on coequalizers in categories of algebras [15]). In particular, this property holds for any complete and cocomplete well-powered regular category where all regular epimorphisms split. This holds, for instance, for the category \( \text{Set} \) and the categories \( \text{Vect}_k \) (the category of vector spaces over \( k \), for any field \( k \)) and \( \text{Vect}_k^\Sigma \) (the category of symmetric sequences over \( k \), for a field \( k \) of zero characteristic), as well as their “super” (\( Z \)- or \( Z/2 \)-graded) versions, which are the main categories where we expect our results to be applied.

2. Categorical PBW theorem

2.1. The adjunction between change of structure and direct image. Suppose that \( M \) and \( N \) are two monads on \( C \), and that \( \phi : M \to N \) is a natural transformation of monads. For such data, one can define the functor of change of algebra structure

\[
\phi^* : C^N \to C^M
\]

for which the algebra map \( \phi(c) : M(c) \to c \) on an \( N \)-algebra \( c \) is computed as the composite

\[
M(c) \xrightarrow{\phi(1_c)} N(c) \xrightarrow{\gamma_c} c.
\]

By [15, Prop. 1], under our assumptions on \( C \) the functor \( \phi^* \) has a left adjoint functor, the direct image functor \( \phi_! \), and for every \( M \)-algebra \( c \), the \( N \)-algebra \( \phi_!(c) \) can be computed as the coequalizer of the reflexive pair of morphisms

\[
\begin{array}{ccc}
N(M(c)) & \xrightarrow{1_N(\phi(1_c))} & N(N(c)) \\
\downarrow 1_N(\gamma_c) & & \downarrow \mu_N(1_c) \\
N(c) & \xrightarrow{1_N(\gamma_c)} & N(c)
\end{array}
\]

which is reflexive with the arrow \( d : N(c) \to N(M(c)) \) given by

\[
N(c) \xrightarrow{\xi} N(\varepsilon(c)) \xrightarrow{1_N(\eta_M(1_c))} N(M(c)).
\]

This construction is well understood and frequently used in the case of analytic functors [12], where the monads in question are actually operads [17]; in that case this formula for the adjoint functor fits into the general framework of relative composite products of operadic bimodules [11, 21]. Relative products of arbitrary endofunctors do not, in general, satisfy all the properties of relative composite products;
however, the proof below shows that in some situations all the necessary coequalizers exist (and are absolute), and, as a consequence, for the PBW purposes there is no need to restrict oneself to analytic endofunctors.

2.2. The main result. As we remarked above, our goal is to give a categorical formalisation of an intuitive view of the PBW property according to which “the underlying object of the universal enveloping algebra of $c$ does not depend on the algebra structure of $c$”. Suppose that $\phi: M \rightarrow N$ is a natural transformation of monads on $C$. We shall say that the datum $(M, N, \phi)$ has the PBW property if there exists an endofunctor $X$ such that the underlying object of the universal enveloping $M$-algebra $\phi(c)$ of any $M$-algebra $c$ is isomorphic to $X(c)$ naturally with respect to morphisms in $C^M$. Using this definition, one arrives at a very simple and elegant formulation of the PBW theorem. Note that using the natural transformation $\phi$, we can regard $N$ as a right $M$-module via the maps $N \circ M \rightarrow N \circ N \rightarrow N$.

**Theorem 2.1.** Let $\phi: M \rightarrow N$ be a natural transformation of monads. The datum $(M, N, \phi)$ has the PBW property if and only if the right $M$-module action on $N$ via $\phi$ is free.

**Proof.** In this proof, we shall utilize the following very well known useful observation: in any split fork diagram

\[
\begin{array}{ccc}
  c_1 & \xrightarrow{f} & c_2 \\
  \downarrow{g} & & \downarrow{e} \\
  d & = & d
\end{array}
\]

where $es = 1_d$, $ft = 1_{c_2}$, and $gt = se$, $d$ is the coequalizer of the pair $f, g$.

Let us first suppose that the datum $(M, N, \phi)$ has the PBW property, and let $X$ be the corresponding endofunctor. Let us take an object $d$ of $C$ and consider the free $M$-algebra $c = M(d)$. Let us first show that the direct image $\phi(c)$ is the free $N$-algebra $N(d)$. For that, we need to prove that $N(d)$ is the coequalizer of the reflexive pair

\[
N(M(M(d))) \xrightarrow{1_N(\phi(1_{M(d)}))} N(N(M(d))) \xrightarrow{\mu_N(1_{M(d)})} N(M(d)).
\]

To establish that, we define the arrow $e: N(M(d)) \rightarrow N(d)$ to be the composite

\[
N(M(d)) \xrightarrow{1_N(\phi(1_d))} N(N(d)) \xrightarrow{\mu_N(1_d)} N(d),
\]

the arrow $s: N(d) \rightarrow N(M(d))$ to be the composite

\[
N(d) \xrightarrow{\cong} N(\mathbb{1}(d)) \xrightarrow{1_N(\eta_{M(\mathbb{1}(d))})} N(M(d)),
\]

and the arrow $t: N(M(d)) \rightarrow N(M(M(d)))$ to be the composite

\[
N(M(d)) \xrightarrow{\cong} N(\mathbb{1}(d)) \xrightarrow{1_N(\eta_{M(\mathbb{1}(d))})} N(M(M(d))),
\]

and note that $es = 1$, $gt = 1$, $se = ft$, which by the above observation indeed implies $\phi(c) \cong N(d)$. Thus, we have a natural isomorphism

\[
N(d) \cong \phi(c) \cong X(c) = X(M(d)) = (X \circ M)(d).
\]

Since the PBW isomorphism is natural with respect to morphisms in $C^M$, this latter isomorphism is natural with respect to morphisms in $C$, and therefore is an isomorphism of endofunctors $N \cong X \circ M$, and hence $N$ is a free right $M$-module.
The other way round, suppose that \( N \) is a free right \( M \)-module, so that \( N \cong X \circ M \) for some endofunctor \( X \). Let us prove that \( \phi_f(c) \equiv X(c) \) for every \( M \)-algebra \( c \). Indeed, the algebra \( \phi_f(c) \) is the coequalizer of the reflexive pair

\[
N(M(c)) \xrightarrow{1_N(\phi_f(1_c))} N(N(c)) \xrightarrow{\mu_N(1_c)} N(c),
\]

Let us define the arrow \( e: N(c) \to X(c) \) to be the composite

\[
N(c) \cong X \circ M(c) \xrightarrow{1_N(1_c)} X(c),
\]

the arrow \( s: X(c) \to N(c) \) to be the composite

\[
X(c) \cong X(\mathbb{1}(c)) \xrightarrow{1_X(\eta_X(1_{M(c)}))} X(M(c)) \cong N(c),
\]

and the arrow \( t: N(c) \to N(M(c)) \) to be the composite

\[
N(c) \cong X(M(c)) \xrightarrow{1_X(\eta_X(1_{M(c)}))} X(M(M(c))) \cong N(M(c)),
\]

so that \( es = 1, ft = 1, se = gt \), and the same observation completes the proof. Thus, we have \( \phi_f(c) \equiv X(c) \) naturally with respect to morphisms in \( \mathcal{C}^M \), so the datum \( (M, N, \phi) \) has the PBW property. \( \square \)

3. Applications

Main applications of our results at the moment deal with the case where the endofunctors \( M \) and \( N \) are analytic [12], so that the monads are in fact operads [17]. In this section, we mainly discuss \( \mathcal{C} = \text{Vect}_k \). In general, for analytic endofunctors to make sense and satisfy various familiar properties, it is enough to require that the category \( \mathcal{C} \) is symmetric monoidal cocomplete (including the hypothesis that the monoidal structure distributes over colimits). In addition, if one requires a homological criterion for freeness like the one we use below, one has to assume that the category of symmetric sequences \( \mathcal{C}^\Sigma \) is a concrete Abelian category where epimorphisms split.

3.1. Classical PBW theorem. As a first simple example, let us outline a proof of the classical Poincaré–Birkhoff–Witt theorem for Lie algebras over a field \( k \) of characteristic zero within our framework. For that, we consider the morphism of operads \( \text{Lie} \to \text{Ass} \) which is defined on generators by the formula

\[
[a_1, a_2] \mapsto a_1 \cdot a_2 - a_2 \cdot a_1.
\]

**Theorem 3.1** (Poincaré [20], Birkhoff [4], Witt [26]). Let \( L \) be a Lie algebra over a field \( k \) of characteristic zero. There is a vector space isomorphism

\[
U(L) \cong S(L)
\]

which is natural with respect to Lie algebra morphisms.

**Proof.** According to Theorem 2.1, it is sufficient to establish freeness of the associative operad as a right Lie-module. For that, one argues as follows. There is a filtration on the operad \( \text{Ass} \) by powers of the two-sided ideal generated by the Lie bracket \( a_1 \cdot a_2 - a_2 \cdot a_1 \). The associated graded operad \( \text{gr Ass} \) is easily seen to be generated by two operations that together satisfy the defining relations of the operad Poisson encoding Poisson algebras and, possibly, some other relations. By a straightforward computation, \( \dim \text{Poisson}(n) = n! = \dim \text{Ass}(n) \), which implies that there can be no other relations. For the operad Poisson, we have \( \text{Poisson} \cong \text{Com} \circ \text{Lie} \) on the level of endofunctors, so it is a free right Lie-module with generators \( \text{Com} \). Now, we note that we are working with connected weight graded operads over a field of characteristic zero, so not only one can do homological algebra in the Abelian category of right modules [10], but one can define the notion of a minimal resolution of a weight graded module and prove its existence and uniqueness up to isomorphism, like it is done for modules over rings in the seminal paper of Eilenberg [9]. This guarantees a homological criterion for freeness of a right \( M \)-module \( R \) via vanishing of higher homology of the corresponding bar construction \( B \cdot (R, M, 1) \). By a spectral sequence
argument, if $\mathcal{M}$ has a filtration compatible with the right action on $\mathcal{R}$, it is enough to prove that vanishing result for the associated graded operad. Thus, the Lie-freeness of Poisson implies the Lie-freeness of $\text{Ass}$, with the same generators $\text{Com}$. Noting that $\text{Com}(L) = S(L)$ completes the proof.

Let us remark that our main result immediately implies that proving a PBW type theorem for a pair of operads acting on $\text{Vect}$, automatically proves the same theorem for these operads acting on a category $\mathcal{C}$ of which the category $\text{Vect}_{\mathcal{K}}$ is a full subcategory. In particular, the result we just established means that the same holds for associative algebras and Lie algebras in various symmetric monoidal categories that extend the category of vector spaces; thus, both the PBW theorem for Lie superalgebras [22] and the PBW theorem for twisted Lie algebras [25] are immediate consequences of Theorem 3.1, and do not need to be proved separately.

Another useful feature of the example of the morphism Lie $\rightarrow \text{Ass}$ is that it highlights a slight difference between our approach and the one of [19]. It turns out that by talking about PBW pairs, one does not detect an important distinction between the case of a field of characteristic zero and a field of positive characteristic; more precisely, the following result holds. (As the proof above shows, in the characteristic zero case, such issues do not arise, and the two approaches are essentially equivalent.)

**Proposition 3.2.** *Let the ground field $\mathcal{K}$ be of characteristic $p > 0$. Then the pair $(\text{Ass}, \text{Lie})$ is a PBW-pair in the sense of [19], so that $S(L) = U(\text{Ab} L) \cong U(L)$ for any Lie algebra $L$, but there is no way to choose vector space isomorphisms $S(L) \cong U(L)$ to be natural in $L$.***

*Proof.* The previous argument shows that $\text{gr Ass} \cong \text{Poisson}$ over any field $\mathcal{K}$. This easily implies that the canonical surjection $\pi$ is an isomorphism, establishing the PBW pair property. However, if we had functorial in $L$ vector space isomorphisms $S(L) \cong U(L)$, then by Theorem 2.1 we would have $\text{Ass} \cong \text{Com} \circ \text{Lie}$ as analytic endofunctors, and as a consequence the trivial submodule of $\text{Ass}(n) \cong \mathcal{K} S_n$ would split as a direct summand, which is false in positive characteristic.

To have a better intuition about the second part of the proof, one may note that the proof of equivalence of two definitions in [19] goes by saying that if we have a PBW pair of algebraic structures, then, first, the universal enveloping algebra of an Abelian algebra has a basis of monomials which does not depend on a particular algebra, and then derive the same for any algebra using the PBW property. The latter step requires making arbitrary choices of liftings that cannot be promoted to an endofunctor.

### 3.2. Further Directions

A more advanced version of the filtration argument used here can be applied in other cases. For example, in [7] the associated graded operad of the operad $\text{PreLie}$ of pre-Lie algebras was studied, also for the filtration by powers of the two-sided ideal generated by the Lie bracket. It follows from the description of the associated graded operad that $\text{PreLie}$ is a right Lie-module, which immediately proves the Poincaré-Birkhoff-Witt theorem for pre-Lie algebras due to Segal [23]. It is interesting that $\text{PreLie}$ is also free as a left Lie-module, which was used by Chapoton [6] to establish that for a free pre-Lie algebra $L$, the result of change of algebra structure $\phi^*(L)$ is free as a Lie algebra.

When the filtration method is not sufficient, one may use shuffle operads to prove freeness of modules, as indicated by the first author in [8]; this also relies on the homological criterion of freeness mentioned above. Blending together the shuffle operad method with the filtration method into a much more technical argument allowed the first author to prove a completely new PBW type result in the case of the well-known morphism of operads from the operad $\text{PreLie}$ to the operad $\text{Dend}$ of dendriform algebras, answering a question he was asked by Jean-Louis Loday around 2009. This result will appear elsewhere; to the best of our knowledge, all attempts to prove it by more "classical" methods have been unsuccessful.

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