An elementary proof of Poincare Duality with local coefficients

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1 Introduction

The statement and proof of the Poincare Duality for (possibly noncompact) orientable manifolds without boundary are abound. The duality has a version for possibly non-orientable manifolds using local coefficients. Several proofs of this result can be found in the literature (e.g. [Sp]). Yet these proof involves either sheaf theory or something equivalent and thus far from elementary. This note is written in the attempt to provide a record of this generalized Poincare Duality available to the general audience.

Readers are assumed to be familiar with basics of the theory of orientation of manifolds (the material of [Ha] Section 3.3 or any equivalent) and the theory of homology with local coefficients (e.g. [Wh] Chapter VI, Section 1,2,3).

Notations of [Ha] and [Wh] will be borrowed and used in this note.

Throughout our discussion, \( M \) will be a manifold of dimension \( n \).

Let \( R \) be a ring with identity such that \( 2 \cdot \text{id} \neq 0 \).

2 Preliminaries

2.1 The Orientation Bundle

Let \( M_R = \bigcup_{x \in M} H_\pi(M|x; R) \), topologized as follows:

For any chart \( \varphi : W \to \mathbb{R}^n \), any \( B \subseteq W \) such that \( \varphi(B) \) is an open ball with finite radius and any element \( \alpha_B \in H_\pi(M|B; R) \), define \( U(\alpha_B) \) as the set of images of \( \alpha_B \) under the canonical \( H_n(M|B; R) \to H_n(M|x; R) \) as \( x \) ranges through \( B \). The collection of \( U(\alpha_B) \) forms a basis for a topology on \( M_R \). The canonical projection \( p : M_R \to M \) is a covering map.

Recall that a bundle of groups/modules/rings over a space \( X \) is defined as a functor from the fundamental groupoid of \( X \) to the category of groups/modules/rings.

There is a bundle of \( R \)-modules on \( M \), denoted also as \( M_R \), such that \( M_R(x) = p^{-1}(x) = H_n(M|x; R) \) (with the obvious module structure) and \( M_R([u]) = L_u \), where \( u \) is a path in \( M \) and \( L_u : p^{-1}(u(1)) \to p^{-1}(u(0)) \) is
the map defined as in [Hatcher p. 69] (this construction can also be found in [Whitehead V.1. Example 5]). $M_R$ is called the $(R)$-orientation bundle of $M$.

2.2 The Canonical Double Cover

Let $\tilde{M} = \{ \pm \mu_x \otimes \text{id} \in H_n(M|x; R) | \mu_x \text{ is a generator of } H_n(M|x; \mathbb{Z}) \}$, here we are using the identification $H_n(M|x; R) \cong H_n(M|x; \mathbb{Z}) \otimes R$. The manifold $\tilde{M}$ is oriented as follows:

For each $\mu_x \otimes \text{id} \in \tilde{M}$ where $\mu_x$ is a generator of $H_n(M|x; \mathbb{Z})$, let $B \subseteq M$ be an open ball with finite radius containing $x$. Let $\mu_B \in H_n(M|B; \mathbb{Z})$ be the element corresponding to $\mu_x$ via the canonical isomorphism $H_n(M|B) \rightarrow H_n(M|x)$. Then $\mu_B \otimes \text{id} \in H_n(M|B; R) \cong H_n(M|B; \mathbb{Z}) \otimes R$.

Let $\tilde{O}_{\mu_x \otimes \text{id}}$ be the element corresponding to $\mu_x \otimes \text{id}$ under the identification (each map below is an isomorphism):

$$H_n(\tilde{M}|\mu_x \otimes \text{id}; R) \leftarrow H_n(U(\mu_B \otimes \text{id})|\mu_x \otimes \text{id}; R) \xrightarrow{p_*} H_n(B|x; R) \rightarrow H_n(M|x; R)$$

Then $\tilde{O}_{\mu_x \otimes \text{id}}$ is independent of $B$ and $\{\tilde{O}_{\mu_x \otimes \text{id}}\}$ is an orientation on $\tilde{M}$. This orientation will be referred to as the chosen orientation in what follows.

It is not hard to see that $\tilde{p} = p|_{\tilde{M}}$ is a covering space of index two.

Denote by $\tau$ the unique Deck transformation of $\tilde{M}$ with no fixed points. Namely, $\tau(y) = -y$ for any $x \in M$ and $y \in H_n(M|x; R)$.

Lemma 1 The involution $\tau$ reverses the chosen orientation on $\tilde{M}$.

Proof This comes from the commutativity of the diagram

$$
\begin{array}{ccc}
H_n(\tilde{M}|\mu_x \otimes \text{id}; R) & \xrightarrow{\tau} & H_n(U(\mu_B \otimes \text{id})|\mu_x \otimes \text{id}; R) \\
\downarrow \tau & & \downarrow \tau \\
H_n(\tilde{M}| -\mu_x \otimes \text{id}; R) & \xleftarrow{\tau} & H_n(U(-\mu_B \otimes \text{id})| -\mu_x \otimes \text{id}; R)
\end{array}
$$

2.3 The Fundamental Class

Next we define a fundamental class for $M$.

Let $C_n(X; R)$ stands for the singular chain of a space $X$ with coefficient in $R$. Define

$$C_n^+(\tilde{M}; R) = \{ \alpha \in C_n(\tilde{M}; R) | \tau(\alpha) = \alpha \}, C_n^-(\tilde{M}; R) = \{ \alpha \in C_n(\tilde{M}; R) | \tau(\alpha) = -\alpha \}$$

Let $K \subseteq M$ be a compact subspace. Define $\tilde{K} = \tilde{p}^{-1}(K)$ and

$$C_n^+(\tilde{M}|K; R) = C_n^+(\tilde{M}; R)/C_n^+(\tilde{M}; R) \cap C_n(\tilde{M} - \tilde{K})$$

$$C_n^-(\tilde{M}|K; R) = C_n^-(\tilde{M}; R)/C_n^-(\tilde{M}; R) \cap C_n(\tilde{M} - \tilde{K})$$


There are exact sequences
\[ 0 \rightarrow C_n^-(\tilde{M}|\tilde{K}; R) \rightarrow C_n(\tilde{M}|\tilde{K}; R) \rightarrow C_n^+(\tilde{M}|\tilde{K}; R) \rightarrow 0 \]
\[ 0 \rightarrow C_n^+(\tilde{M}|\tilde{K}; R) \rightarrow C_n(\tilde{M}|\tilde{K}; R) \rightarrow C_n^-(\tilde{M}|\tilde{K}; R) \rightarrow 0 \]
where \( \Sigma(\alpha) = \alpha + \tau(\alpha) \) and \( \Delta(\alpha) = \alpha - \tau(\alpha) \).

The chain complex \( C_* M[K; R] \) is isomorphic to \( C_*^+ (\tilde{M}|\tilde{K}; R) \) via \( \sigma \otimes r \rightarrow \tilde{\sigma}_1 \otimes r + \tilde{\sigma}_2 \otimes r \) where \( \tilde{\sigma}_1, \tilde{\sigma}_2 \) are the two liftings of \( \sigma : \Delta^n \rightarrow M \).

By Lemma 3.27(b) of [Ha], \( H_k(M[K; R]) = 0, k > n \). Thus (1) produces a long exact sequence
\[ \cdots \rightarrow 0 \rightarrow H_n(C_*^- (\tilde{M}|\tilde{K}; R)) \rightarrow H_n(\tilde{M}|\tilde{K}; R) \rightarrow H_n(M[K; R]) \rightarrow \cdots \] (3)

There is a \( \mathbb{Z}_2 \) group action on \( \tilde{M} \) such that \( \bar{1} \in \mathbb{Z}_2 \) acts by \( \sigma \). \( \mathbb{Z}_2 \) also acts (as a group) on \( R \) with \( \bar{1} \cdot r = -r \). Thus one could define \( C_n(\tilde{M}|\tilde{K}; \mathbb{Z}) \otimes_{\mathbb{Z}_2} R \) using the routine method of treating a left \( \mathbb{Z}[[\mathbb{Z}_2]] \)-module as a right module.

The canonical surjection \( C_n(\tilde{M}|\tilde{K}; R) = C_n(\tilde{M}|\tilde{K}; \mathbb{Z}) \otimes R \rightarrow C_n(\tilde{M}|\tilde{K}; R) \otimes_{\mathbb{Z}_2} R \) has the same kernel as \( \Delta \) in (2): \( C_n^+(\tilde{M}|\tilde{K}; R) \). Thus we obtain an identification
\[ C_n^-(\tilde{M}|\tilde{K}; R) \leftrightarrow C_n(\tilde{M}|\tilde{K}; \mathbb{Z}) \otimes_{\mathbb{Z}_2} R \]

On the other hand, let \( C_* M[K; M_R] \) be the chain complex of \( (M, M - K) \) with coefficient in the bundle \( M_R \). We can define a homomorphism
\[ \phi : C_* (\tilde{M}|\tilde{K}; \mathbb{Z}) \otimes_{\mathbb{Z}_2} R \rightarrow C_* (M[K; M_R], \phi(\bar{\sigma} \otimes_{\mathbb{Z}_2} r) = (r\bar{\sigma}(e_0))\bar{p} \circ \bar{\sigma} \]

where \( \bar{\sigma} : \Delta^n \rightarrow \tilde{M} \) and \( (r\bar{\sigma}(e_0))\bar{p} \circ \bar{\sigma} \) is represented by the element of \( C_* (M; M_R) = \oplus_{\sigma : \Delta^n \rightarrow M} M_R(\sigma(e_0)) \) with \( r\bar{\sigma}(e_0) \) on the \( \bar{p} \circ \bar{\sigma} \) coordinate and 0 otherwise. It is not hard to verify that \( \phi \) is an isomorphism. Hence we obtain an identification \( C_*^- (\tilde{M}|\tilde{K}; R) \leftrightarrow C_* (M[K; M_R]). \) Explicitly this comes from the diagram
\[ \begin{array}{ccc}
C_*^- (\tilde{M}|\tilde{K}; R) & \xrightarrow{\Delta} & C_* (\tilde{M}|\tilde{K}; \mathbb{Z}) \otimes R \\
\downarrow & & \downarrow \\
C_* (\tilde{M}|\tilde{K}; \mathbb{Z}) \otimes_{\mathbb{Z}_2} R & \xrightarrow{\phi} & C_* (M[K; M_R])
\end{array} \] (4)

The above identification is a chain isomorphism, thus we have
\[ H_n(C_*^- (\tilde{M}|\tilde{K}; R)) \leftrightarrow H_n(M[K; M_R]) \]
where \( H_n(M[K; M_R]) \) is the homology group with coefficients in \( M_R \).

Plug this into (3), we get
\[ \cdots \rightarrow 0 \rightarrow H_n(M[K; M_R]) \rightarrow H_n(\tilde{M}|\tilde{K}; R) \xrightarrow{\bar{p}_*} H_n(M[K; R]) \rightarrow \cdots \] (5)
Since (4) is natural with respect to compact subspaces $K_1 \subseteq K_2 \subseteq M$, so is (5).

The chosen orientation on $\tilde{M}$ uniquely determines an element in $\nu_K \in H_n(M|\tilde{K}; R)$ that restrict to the orientation at each $\tilde{x} \in \tilde{K}$ (cf. Lemma 3.27 (a) of [Ha]).

**Lemma 2** $\tilde{p}_*(\nu_K) = 0$ for any compact subspace $K \subseteq M$.

**Proof** By the uniqueness part of Lemma 3.27(a) of [Ha], it suffice to prove $\tilde{p}_*(\nu_K)$ restrict to 0 at each $x \in M$. Let $\tilde{p}^{-1}(x) = \{\tilde{x}, \tilde{x}'\}$. We have a commutative diagram

$$
\begin{array}{c}
H_n(M|\tilde{K}; R) \\ \downarrow \tilde{p}_* \end{array} \xrightarrow{k_*} \begin{array}{c} H_n(\tilde{M}|\{\tilde{x}, \tilde{x}'\}; R) \\ \downarrow \tilde{p}_* \end{array} \xrightarrow{i_*} H_n(M|x; R)
\end{array}
$$

where horizontal maps are induced by inclusions. Hence the goal become showing that $\nu_K$ maps to 0 via $H_n(M|\tilde{K}; R) \rightarrow H_n(\tilde{M}|\{\tilde{x}, \tilde{x}'\}; R) \rightarrow H_n(M|x; R)$.

Take an open neighborhoods $U$ of $x$ such that $\tilde{p}^{-1}(U) = \tilde{U} \sqcup \tilde{U}'$ and $\tilde{x} \in \tilde{U}, \tilde{x}' \in \tilde{U}'$. Consider the following commutative diagrams

$$
\begin{array}{c}
H_n(\tilde{M}|\tilde{x}; R) \oplus H_n(\tilde{M}|\tilde{x}'; R) \\ \downarrow j_* \oplus j'_* \end{array} \xrightarrow{k_*} \begin{array}{c} H_n(\tilde{M}|\{\tilde{x}, \tilde{x}'\}; R) \\ \downarrow \tilde{p}_* \end{array} \xrightarrow{i_*} H_n(M|x; R)
\end{array}
$$

where $j, j', i, k$ are inclusions. By excision and additivity, $\tau_* \oplus \tau'_* \circ j_* \oplus j'_*$ is an isomorphism. On the other hand, it is not hard to observe that $(j_* \oplus j'_*) \circ (\tau_* \oplus \tau'_*) = (j_* \circ \tau_*) \oplus (j'_* \circ \tau'_*)$. Obviously $\tau_*$ is also an isomorphism.

For any $\alpha \in H_n(\tilde{M}|\tilde{K}; R)$, $k_*(\alpha) = \nu_*(\beta) + i'_*(\gamma)$. Hence $(j_* \oplus j'_*) \circ k_*(\alpha) = j_* \circ k_*(\alpha) + i'_* \circ j'_*(\gamma)$. Note that $j_* \circ k_*(\alpha)$ and $j'_* \circ k'_*(\alpha)$ are orientations at $\tilde{x}, \tilde{x}'$ respectively. So $\tau_* \circ j_* \circ k_*(\alpha) = -j'_* \circ k'_*(\alpha)$. This implies $\tau_*(\beta) = -\gamma$. But $\tilde{p}_* \circ \tau = \tilde{p}_*$, whence $\tilde{p}_*(\beta, \gamma) = \tilde{p}_*(\beta) + \tilde{p}_*(\gamma) = \tilde{p}_* \circ \tau_*(\beta) + \tilde{p}_* \circ \tau_*(\gamma) = \tilde{p}_*(-\gamma) + \tilde{p}_*(\gamma) = 0$. Thus $\tilde{p}_* \circ k_*(\alpha) = \tilde{p}_* \circ (i_* \oplus i'_*)(\beta, \gamma) = i_* \circ \tilde{p}_*(\beta, \gamma) = 0$. □
By Lemma 2 and the exactness of (5), the orientation of \( \tilde{M} \) uniquely determines an element of \( H_n(M|K; M_R) \), denoted also as \( \nu_K \). The naturality of (4) show that \( \{ \nu_K | K \subseteq M \text{ compact} \} \) is compatible with respect to inclusion of \( K \)'s and thus define an element \([M]\) of \( \lim_K H_n(M|K; M_R) \) where the inverse limit is taken with respect to all compact \( K \subseteq M \) and inclusions \( K_1 \subseteq K_2 \subseteq M \). \([M]\) is called the fundamental class of \( M \).

2.4 Restricting to open subspaces; Compatibility

Let \( U \subseteq M \) be an open subset. Here is a few elementary facts we shall need:

**Proposition 1**

i) There is a canonical embedding \( \tilde{U} \hookrightarrow \tilde{M} \) induced by the excision \( H_n(U|x; R) \rightarrow H_n(M|x; R), x \in U \).

ii) The bundle \( U_R \) (of \( R \)-modules) is canonically isomorphic to the restriction of the bundle \( M_R \) to \( U \).

iii) The chosen orientation of \( \tilde{M} \) restrict to the chosen orientation on \( \tilde{U} \).

iv) For any \( K \subseteq U \) compact, the excision \( H_n(\tilde{U}|\tilde{K}; R) \rightarrow H_n(\tilde{M}|\tilde{K}; R) \) sends \( \nu_{U|K}^U \) to \( \nu_{M|K}^M \) where \( \nu_{U|K}^U, \nu_{M|K}^M \) are elements determined by the chosen orientation.

v) There diagram

\[
\begin{array}{ccc}
H_n(M|K; M_R) & \rightarrow & H_n(\tilde{M}|\tilde{K}; R) \\
\uparrow & & \uparrow \\
H_n(U|K; M_R) & \rightarrow & H_n(\tilde{U}|\tilde{K}; R)
\end{array}
\]

commutes, where horizontal maps are those in (5) and vertical ones are induced by inclusion.

**Corollary 1** The homomorphism \( H_n(U|K; U_R) \rightarrow H_n(M|K; M_R) \) induced by inclusion sends \( \nu_{U|K}^U \) to \( \nu_{M|K}^M \), where \( \{ \nu_{U|K}^U \}, \{ \nu_{M|K}^M \} \) define \([U], [M]\) respectively.

Thus \( \nu_{U|K}^U \) is compatible with respect to inclusions of open \( U \)'s as well as compact \( K \)'s.

2.5 Cap Products

We start with defining tensor product of bundle of modules. Let \( G \) (resp. \( G' \)) be a bundle of left (resp. right) \( R \)-modules over a space \( X \). The tensor product \( G \otimes_R G' \) is defined as the bundle of abelian groups where \( G \otimes_R G'(x) = G(x) \otimes_R G'(x) \) and \( G \otimes_R G'(\{u\}) = G(\{u\}) \otimes_R G'(\{u\}) \) for any \( x \in X \) and \( u : I \rightarrow X \).

Denote the vertices of \( \Delta^n \) as \( e_0, e_1, \cdots, e_n \). Let \( \sigma : \Delta^n \rightarrow X \) be a continuous map. For \( 0 \leq i_1 < i_2 < \cdots < i_k \leq n \), let \( \sigma_{[i_1, i_2, \cdots, i_k]} \) denote \( \sigma \) restricted to the simplex \( e_{i_1}, e_{i_2}, \cdots, e_{i_k} \).
Now we are able to define the cap product on (absolute) chains. Assume that $G$ (resp. $G'$) is a bundle of left (resp. right) $R$-modules over a space $X$, the cap product is defined as

$$C^k(X; G) \otimes_R C_n(X; G') \longrightarrow C_{n-k}(X; G \otimes_R G')$$

where $c \in C^k(X; G) = \prod_{\rho: \Delta^n \rightarrow X} G(\rho(e_0))$, $g \in G(\sigma(e_0))$, and $g \sigma \in C_n(X; G') = \bigoplus G'(\eta(e_0))$ denotes the element which has value $g$ on the $\sigma$-coordinate and 0 otherwise.

If $A_1, A_2$ are subspaces of $X$, the above absolute cap product induces a relative

$$C^k(X, A_1; G) \otimes_R C_n(X, A_1 + A_2; G') \longrightarrow C_{n-k}(X, A_2; G \otimes_R G')$$

where the relative $C_\ast, C^\ast$ are defined in the obvious way.

The cap product satisfies the identity

$$\partial(c \smile \alpha) = c \smile (\partial\alpha) - (\delta c) \smile \alpha, c \in C^k(X; G), \alpha \in C_n(X; G')$$

Note that the sign appearing in the above equation is a result of our adopting the definition in [Wh].

There is thus an induced cap product on (co)homology

$$H^k(X, A_1; G) \otimes_R H_n(X, A_1 + A_2; G') \longrightarrow H_{n-k}(X, A_2; G \otimes_R G')$$

For the special case where $X = M, A_1 = M - K, A_2 = \emptyset$ and $G' = M_R$, we obtain

$$H^k(M|K; G) \otimes_R H_n(M|K; M_R) \longrightarrow H_{n-k}(M; G \otimes_R M_R) \quad (5)$$

Naturality with respect to inclusion of compact subspaces can be easily verified, thus (5) produces

$$\lim_{K} H^k(M|K; G) \otimes_R \lim_{K} H_n(M|K; M_R) \longrightarrow H_{n-k}(M; G \otimes_R M_R)$$

Note that $\lim_{K} H^k(M|K; G)$ is canonically isomorphic to $H^k(M; G)$, the cohomology of $M$ with compact support and with coefficient in $G$ (the proof of this in the case of ordinary coefficients can be found in [Ha] Section 3.3). So the above becomes

$$H^k_c(M; G) \otimes_R \lim_{K} H_n(M|K; M_R) \longrightarrow H_{n-k}(M; G \otimes_R M_R)$$

If one choose the fundamental class $[M] \in \lim_{K} H_n(M|K; M_R)$, there is a homomorphism

$$H^k_c(M; G) \stackrel{[M]}{\longrightarrow} H_{n-k}(M; G \otimes_R M_R)$$

6
3 Proof of the Duality Theorem

Lemma 3 Let $U, V$ be open subsets of $M$ with $U \cup V = M$. $K \subseteq U, L \subseteq V$ are compact subspaces. Let $G$ be a bundle of right $R$-modules over $M$. Then the following diagram commutes up to sign

\[
\begin{array}{ccc}
\delta & H^k(M|K \cap L; G) & H^k(M|K; G) \oplus H^k(M|L; G) & H^k(M|K \cup L; G) \\
\downarrow & \downarrow & \downarrow & \downarrow \sim \nu^M_{K \cup L} \\
H^k(U \cap V|K \cap L; G) & H^k(U|K; G) \oplus H^k(V|L; G) & \sim \nu^U_{K \cap L} & \sim \nu^V_{K \cap L} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
H^k(U \cap V|K \cap L; G) & H^k(U|K \cap L; G) & H^k(U|K; G) & H^k(U|L; G) \\
\sim \nu^U_{K \cap L} & \sim \nu^U_{K \cap L} & \sim \nu^V_{K \cap L} & \sim \nu^V_{K \cap L} \\
H^k(U \cap V|K \cap L; G) & H_{n-k}(U \cap V; G \otimes_R L) \oplus H_{n-k}(U \cap V; G \otimes_R M_R) & H_{n-k}(U \cap V; G \otimes_R M_R) & H_{n-k}(U \cap V; G \otimes_R M_R) \\
\sim \nu^U_{K \cap L} & \sim \nu^U_{K \cap L} & \sim \nu^V_{K \cap L} & \sim \nu^V_{K \cap L} \\
H^k(U \cap V|K \cap L; G) & H_{n-k}(U \cap V; G \otimes_R (U \cap V) \cap R) & H_{n-k}(U \cap V; G \otimes_R M_R) & H_{n-k}(U \cap V; G \otimes_R M_R) \\
\sim \nu^U_{K \cap L} & \sim \nu^U_{K \cap L} & \sim \nu^V_{K \cap L} & \sim \nu^V_{K \cap L} \\
H^k(U \cap V|K \cap L; G) & H_{n-k}(U \cap V; G \otimes_R U_R) & H_{n-k}(U \cap V; G \otimes_R U_R) & H_{n-k}(U \cap V; G \otimes_R U_R) \\
\end{array}
\]

where the two rows are Mayer-Vietoris sequences (see Appendix), the upper left and upper middle maps are induced by inclusions.

Proof We begin with the two blocks without $\delta$ or $\partial$. Commutativity of the one on the left would follow once we establish the commutativity of the following

\[
\begin{array}{ccc}
H^k(M|K \cap L; G) & H^k(M|K; G) & H^k(U|K; G) \\
\downarrow & \downarrow & \downarrow \sim \nu^U_{K \cap L} \\
H^k(U \cap V|K \cap L; G) & H^k(U|K \cap L; G) & \sim \nu^U_{K \cap L} \\
\downarrow & \downarrow & \downarrow \\
H^k(U \cap V|K \cap L; G) & H_{n-k}(U \cap V; G \otimes_R (U \cap V) \cap R) & H_{n-k}(U \cap V; G \otimes_R U_R) \\
\sim \nu^U_{K \cap L} & \sim \nu^U_{K \cap L} & \sim \nu^V_{K \cap L} \\
H^k(U \cap V|K \cap L; G) & H_{n-k}(U \cap V; G \otimes_R M_R) & H_{n-k}(U \cap V; G \otimes_R M_R) \\
\sim \nu^U_{K \cap L} & \sim \nu^U_{K \cap L} & \sim \nu^V_{K \cap L} \\
H^k(U \cap V|K \cap L; G) & H_{n-k}(U \cap V; G \otimes_R V_R) & H_{n-k}(U \cap V; G \otimes_R V_R) \\
\sim \nu^U_{K \cap L} & \sim \nu^U_{K \cap L} & \sim \nu^V_{K \cap L} \\
\end{array}
\]

in which all arrows that is not a cap product is induced by inclusion.

The above diagram commutes because of the compatibility of $\nu^U_{K \cap L}$'s.

Similarly, commutativity of the block on the right in (6) follows from commutativity of

\[
\begin{array}{ccc}
H^k(M|L; G) & H^k(M|K \cup L; G) \\
\downarrow & \downarrow \sim \nu^M_{K \cup L} \\
H^k(V|L; G) & H_{n-k}(M; G \otimes_R M_R) \\
\sim \nu^V_{L} & \sim \nu^V_{L} \\
H_{n-k}(V; G \otimes_R V_R) & H_{n-k}(V; G \otimes_R V_R) \\
\end{array}
\]

Again such commutativity comes from compatibility of $\nu^V_{L}$'s.

For the block involving $\partial$ and $\delta$, [Ha] presented a detailed proof (p. 246-247) which carries verbatim to the case of twisted coefficients. So I shall not rewrite
it in this note. Due to the difference in the convention of signs defining \( \delta, \partial \), the block commutes up to a factor of \(-1\) instead of \((-1)^{k+1}\).

Corollary 2 Let \( U, V \) be open subsets of \( M \) with \( U \cup V = M \). Let \( G \) be a bundle of right \( R \)-modules over \( M \). Then there is a (up to sign) commutative diagram

\[
\begin{array}{cccc}
H^k_c(U \cap V; G) & \longrightarrow & H^k_c(U; G) \oplus H^k_c(V; G) & \longrightarrow & H^k_c(M; G) \\
\downarrow & & \downarrow & & \downarrow \\
H_{n-k}(U \cap V; G \otimes_R M_R) & \longrightarrow & H_{n-k}(U; G \otimes_R M_R) \oplus H_{n-k}(V; G \otimes_R M_R) & \longrightarrow & \oplus H_{n-k}(V; G \otimes_R M_R)
\end{array}
\]

where vertical maps are cap products with respective fundamental classes and the two rows are Mayer-Vietoris sequences.

Proof This follows from the preceding Lemma by taking the direct limit of (6) with respect to the directed set \( \{(K, L) | K \subseteq U \text{ exact}, L \subseteq V, (K, L) \leq (K', L') \text{ iff } K \subseteq K', L \subseteq L'\} \).

Now we can prove the Poincare Duality.

Theorem 1 For any manifold \( M \) and any bundle of right \( R \)-modules \( G \), the homomorphism

\[
H^k_c(M; G) \xrightarrow{\sim [M]} H_{n-k}(M; G \otimes_R M_R)
\]

is an isomorphism.

Proof The proof of [Ha] Theorem 3.35 applies with one exception. In the case when \( M = \mathbb{R}^n \), one use the fact that \( \mathbb{R}^n \) is contractible to deduce that \( G \) and \((\mathbb{R}^n)_R\) is isomorphic to a constant bundle. Choose and fix a base point of \( \mathbb{R}^n \), say \( 0 \). Define \( G_0 = G(0) \). Identify \((\mathbb{R}^n)_R(0)\) with \( R \). Then \( H^k_c(\mathbb{R}^n; G) \), \( \lim_{K \uparrow} H_\ast(\mathbb{R}^n|K; M_R) \) and \( H_{n-k}(\mathbb{R}^n; G \otimes_R (\mathbb{R}^n)_R) \) can be canonically identified with \( H^k_c(\mathbb{R}^n; G_0) \), \( \lim_{K \uparrow} H_\ast(\mathbb{R}^n|K; R) \) and \( H_{n-k}(\mathbb{R}^n; G_0 \otimes_R R) \) respectively. Such identifications are compatible with the cap product. Under such identification, it is not hard to check from the definition that \([M] = [\mathbb{R}^n] \in \lim_{K \uparrow} H_\ast(\mathbb{R}^n|K; M_R)\) as defined in this note is an fundamental class in \( \lim_{K \uparrow} H_\ast(\mathbb{R}^n|K; R) \), i.e. an element that restrict to the local orientation at each \( x \in \mathbb{R}^n \) for a chosen orientation. Thus the Poincare Duality for the \( R \)-orientable manifold \( \mathbb{R}^n \) (the one we apply here is slightly more general than what appears in [Ha] since the coefficients can be in any \( R \)-module, but the same proof as in the simpler case carries verbatim to prove this generalized case) proves that \( \sim [M] \) is an isomorphism.
It should be mentioned that our result concerns merely cohomology with compact support. There is a version of Poincare duality for ordinary cohomology as recorded in [Sp], which is an isomorphism between Alexander cohomology and locally finite homology. The proof of this result, however, seems to require sheaf theory or some equally sophisticated machinery.

Appendix: The (Relative) Mayer Vietoris sequences with local coefficients

We shall need the following lemma, whose proof is essentially identical to that of corresponding results for (co)homology with constant coefficients:

**Lemma 4** Let \( X = \bigcup \alpha \text{ Int} X_\alpha \) where \( X_\alpha \)'s are subspaces. Let \( G \) be a bundle of groups on \( X \). Define \( C^* \left( \sum \alpha X_\alpha; G \right) = \left\{ \sum_i g_i \sigma_i \in C^* (X; G) \mid \text{each } \sigma_i(\Delta^n) \text{ is contained in some } X_\alpha \right\} \).

Define \( C^* \left( \sum \alpha X_\alpha; G \right) \rightarrow C^* (X; G) \) and \( C^* (X; G) \rightarrow C^* \left( \sum \alpha X_\alpha; G \right) \) induce isomorphism on homology.

Given a pair \((X, Y) = (A \cup B, C \cup D)\) with \( C \subseteq A, D \subseteq B \) and \( X = \text{Int} X A \cup \text{Int} X B, Y = \text{Int} Y C \cup \text{Int} Y D \). For a bundle of groups \( G \) on \( X \), there are Mayer-Vietoris sequences:

\[
\cdots \longrightarrow H_n(A \cap B, C \cap D; G) \longrightarrow H_n(A, C; G) \oplus H_n(B, D; G) \longrightarrow H_n(X, Y; G) \longrightarrow \cdots
\]

and

\[
\cdots \longrightarrow H^n(X, Y; G) \longrightarrow H^n(A, C; G) \oplus H^n(B, D; G) \longrightarrow H^n(A \cap B, C \cap D; G) \longrightarrow \cdots
\]

The sequence for homology is deduced by essentially the same way as ordinary (untwisted) coefficients (cf. Hatcher). On the other hand, the proof for the cohomological Mayer-Vietoris sequence with untwisted coefficients (cf Hathcer pp. 204) almost carries to the twisted case except when proving

\[
0 \longrightarrow C^n (A+B, C+D; G) \xrightarrow{\varphi} C^n (A, C; G) \oplus C^n (B, D; G) \xrightarrow{\varphi} C^n (A \cap B, C \cap D; G) \longrightarrow 0
\]

is exact. This sequence no longer comes from dualizing the corresponding sequence for homology. Thus one has to prove the exactness by hand. The non-trivial part is proving the surjectivity of \( \varphi \). We will show this by constructing for any \( \alpha \in C^n (A \cap B, C \cap D; G) \) a pair \((\beta, \gamma) \in C^n (A, C; G) \oplus C^n (B, D; G)\) as follows:

\[
\beta(\sigma) = \begin{cases} 0 & \sigma(\Delta^n) \not\subseteq A \cap B \\
\sigma(\Delta^n) & \sigma(\Delta^n) \subseteq C \\
\alpha(\sigma) & \text{otherwise}
\end{cases}
\]

\[
\gamma(\sigma) = \begin{cases} -\alpha(\sigma) & \sigma(\Delta^n) \subseteq C \\
0 & \text{otherwise}
\end{cases}
\]

It is not hard to verify that \( \varphi(\beta, \gamma) = \alpha \). Thus \( \varphi \) is surjective.
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5 Bibliography

[Ha] A. Hatcher, Algebraic Topology(2002)
[Sp] E. Spanier, Algebraic Topology(1966), Springer-Verlag
[Wh] G. Whitehead, Elements of Homotopy Theory(1978), Springer-Verlag