ON $THH$ AND ÉTALE BASE-CHANGE

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ABSTRACT. We interpret the base-change theorem for $THH$ of Weibel-Geller and McCarthy-Minasian via the description of $THH$ as tensoring with the circle. We thus obtain a simple categorical proof of étale descent. However, we give a counterexample to show that base-change fails for faithful Galois extensions of ring spectra in the sense of Rognes.

1. Introduction

Let $R$ be an $\mathbb{E}_1$-ring spectrum. The topological Hochschild homology $THH(R)$ of $R$ is a spectrum constructed as the geometric realization of a certain cyclic object built from $R$, a homotopy-theoretic version of the Hochschild complex of an associative ring. Topological Hochschild homology has been studied in particular because of its connections with algebraic $K$-theory via the theory of trace maps. More generally, if $R$ is an $\mathbb{E}_1$-algebra in $A$-modules for an $\mathbb{E}_\infty$-ring $A$, then one can define a relative version $THH^A(R)$.

In [WG91], it is shown that the algebraic version of Hochschild homology for commutative rings satisfies an étale base-change result. Equivalently, if $k$ is a commutative ring and if $A \to B$ is an étale morphism of commutative $k$-algebras with $A$ flat over $k$, then there is a natural equivalence $B \otimes_A THH^k(A) \simeq THH^k(B)$. Weibel-Geller’s result also applies in the non-flat case, although it cannot be stated in this manner. This result was generalized by McCarthy-Minasian in [MM03], who prove the analogous statement for an étale morphism of connective $\mathbb{E}_\infty$-rings [MM03 Lem. 5.7]. In fact, they prove the result more generally for any $THH$-étale morphism of connective $\mathbb{E}_\infty$-rings.

In the setting of structured ring spectra, however, there are additional morphisms of nonconnective ring spectra that behave much like étale morphisms but which are not étale on homotopy groups. The faithful Galois extensions of Rognes [Rog08] are key examples here. It is important that a map $A \to B$ of ring spectra may behave like an étale morphism (e.g., it may be a faithful Galois extension) even though the map on homotopy groups $\pi_*(A) \to \pi_*(B)$ may be far from étale. This in particular includes cases where the homotopy groups $\pi_*(B)$ are much simpler than $\pi_*(A)$, so that invariants of $A$ can often be computed by computing them for $B$ and performing a type of “descent.” We refer to [MS14] for a recent illustration of these techniques.

In this note, we reinterpret the base-change question for $THH$ in terms of the formulation $THH(R) \simeq S^1 \otimes R$ for $\mathbb{E}_\infty$-rings, due to McClure-Schwänzl-Vogt [MSV97].

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1We will use the convention that a morphism $A \to B$ of $\mathbb{E}_\infty$-rings is étale if $\pi_0(A) \to \pi_0(B)$ is étale and the map $\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_*(B)$ is an isomorphism.
As a result, we show:

**Theorem 1.1.** There exists a morphism $A \to B$ of $\mathbb{E}_\infty$-ring spectra which is a faithful $G$-Galois extension, but such that the map $\text{THH}(A) \otimes_A B \to \text{THH}(B)$ is not an equivalence.

We in fact pinpoint exactly what goes wrong from a categorical perspective, and why this phenomenon cannot happen in the étale setting, thus proving a variant of the Weibel-Geller-McCarthy-Minasian theorem:

**Theorem 1.2.** Let $R$ be an $\mathbb{E}_\infty$-ring, and let $A \to B$ be an étale morphism of $\mathbb{E}_\infty$-$R$-algebras (possibly nonconnective). Then the natural map $\text{THH}^R(A) \otimes_A B \to \text{THH}^R(B)$ is an equivalence.

The use of categorical interpretation of $\text{THH}$ in proving such base-change theorems is not new; McCarthy-Minasian use this interpretation in [MM03] in a different manner.

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2. Categorical generalities

Let $\mathcal{C}$ be a cocomplete $\infty$-category, and let $x \in \mathcal{C}$. Given $x \in \mathcal{C}$, we can [Lur09, §4.4.4] construct an object $S^1 \otimes x$.

Choose a basepoint $\ast \in S^1$. Then we have a diagram

\[
\begin{array}{ccc}
S^1 \otimes x & \rightarrow & S^1 \otimes y \\
\downarrow & & \downarrow \\
x & \rightarrow & y
\end{array}
\]

As a result of this diagram, we have a natural map in $\mathcal{C}$,

\[
S^1 \otimes x \sqcup x y \rightarrow S^1 \otimes y.
\]

In order for (2) to be an equivalence, for any object $z \in \mathcal{C}$, the square of spaces

\[
\begin{array}{ccc}
\text{Hom}(S^1, \text{Hom}_\mathcal{C}(y, z)) & \rightarrow & \text{Hom}_\mathcal{C}(y, z) \\
\downarrow & & \downarrow \\
\text{Hom}(S^1, \text{Hom}_\mathcal{C}(x, z)) & \rightarrow & \text{Hom}_\mathcal{C}(x, z)
\end{array}
\]

must be homotopy cartesian. This happens only in very special situations.

**Proposition 2.1.** Let $f : X \to Y$ be a map of spaces. Then the diagram

\[
\begin{array}{ccc}
\text{Hom}(S^1, X) & \rightarrow & X \\
\downarrow & & \downarrow \\
\text{Hom}(S^1, Y) & \rightarrow & Y
\end{array}
\]

is homotopy cartesian if and only if for every point $p \in X$, the map from the connected component of $X$ containing $p$ to that of $Y$ containing $f(p)$ is a homotopy equivalence.
Proof. Without loss of generality, we may assume that $X, Y$ are connected spaces. In this case, choosing compatible basepoints in $X, Y$, we get equivalences
\[ \Omega X \simeq \text{fib} \left( \text{Hom}(S^1, X) \to X \right), \quad \Omega Y \simeq \text{fib} \left( \text{Hom}(S^1, Y) \to Y \right), \]
and the fact that (4) is homotopy cartesian now implies that $\Omega X \to \Omega Y$ is a homotopy equivalence. Since $X$ and $Y$ are connected, this implies that $X \to Y$ is a homotopy equivalence.

\[ \square \]

Definition 2.2. We will say that a map of spaces $X \to Y$ is a \textit{split covering space} if the equivalent conditions of Proposition 2.1 are met. In particular, $X \to Y$ is a covering space, which is trivial on each connected component of $Y$.

Observe that the base-change of a split covering space is still a split covering space.

Corollary 2.3. Suppose $x \to y$ is a morphism in $\mathcal{C}$ as above. Then the natural map $S^1 \otimes x \sqcup_y y \to S^1 \otimes y$ is an equivalence if and only if, for every object $z \in \mathcal{C}$, the map of spaces $\text{Hom}_\mathcal{C}(y, z) \to \text{Hom}_\mathcal{C}(x, z)$ is a split cover.

Proof. Our map is an equivalence if and only if (3) is homotopy cartesian for each $z \in \mathcal{C}$. By Proposition 2.1, we get the desired claim.

\[ \square \]

We now give this class of morphisms a name.

Definition 2.4. A morphism $x \to y$ in an $\infty$-category $\mathcal{C}$ is said to be \textit{strongly $0$-cotruncated} if, for every $z \in \mathcal{C}$, the map $\text{Hom}_\mathcal{C}(y, z) \to \text{Hom}_\mathcal{C}(x, z)$ is a split covering space.

Corollary 2.3 states that $x \to y$ has the property that $S^1 \otimes x \sqcup_y y \to S^1 \otimes y$ is an equivalence if and only if the map is strongly $0$-cotruncated.

For passage to a relative setting, we will find the following useful.

Proposition 2.5. Let $\mathcal{C}$ be a cocomplete $\infty$-category, let $a \in \mathcal{C}$, and let $x \to y$ be a morphism in $\mathcal{C}_a$. If $x \to y$ is strongly $0$-cotruncated when regarded as a morphism in $\mathcal{C}$, then it is strongly $0$-cotruncated when regarded as a morphism in $\mathcal{C}_a$.

Proof. Suppose $a \to z$ is an object of $\mathcal{C}_a$. Then we have
\[
\text{Hom}_{\mathcal{C}_a}(y, z) = \text{fib} \left( \text{Hom}_{\mathcal{C}}(y, z) \to \text{Hom}_{\mathcal{C}}(a, z) \right), \\
\text{Hom}_{\mathcal{C}_a}(x, z) = \text{fib} \left( \text{Hom}_{\mathcal{C}}(x, z) \to \text{Hom}_{\mathcal{C}}(a, z) \right).
\]
Since $\text{Hom}_{\mathcal{C}}(y, z) \to \text{Hom}_{\mathcal{C}}(x, z)$ is a split cover, it follows easily that the same holds after taking homotopy fibers over the basepoint in $\text{Hom}_{\mathcal{C}}(a, z)$.

\[ \square \]

3. $E_\infty$-RING SPECTRA

We let $\text{CAlg}$ denote the $\infty$-category of $E_\infty$-ring spectra. The construction $THH$ in this case can be interpreted (by [MSV97]) as tensoring with $S^1$: that is, we have
\[ THH(A) \simeq S^1 \otimes A, \quad A \in \text{CAlg}. \]
If one works in a relative setting, under an $E_\infty$-ring $R$, then one has $THH^R(A) \simeq S^1 \otimes A$, where the tensor product is computed in $\text{CAlg}^R$. 

Given a morphism in $\text{CAlg}_{R/}$, $A \to B$, we can use the setup of the previous section and obtain a morphism

$$THH^R(A) \otimes_A B \to THH^R(B)$$

which is a special case of (2), since the pushout of $\mathbb{E}_\infty$-rings is the relative tensor product. The base-change problem for $THH$ asks when this is an equivalence.

By Corollary 2.3, this is equivalent to the condition that morphism $A \to B$ in $\text{CAlg}_{R/}$ should be strongly 0-cotruncated.

A morphism $A \to B$ in $\text{CAlg}$ is called étale if $\pi_0(A) \to \pi_0(B)$ is étale and the natural map $\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_*(B)$ is an isomorphism. We can now prove Theorem 1.2 from the introduction, which we restate for convenience.

**Theorem.** Let $R$ be an $\mathbb{E}_\infty$-ring and let $A \to B$ be an étale morphism in $\text{CAlg}_{R/}$. Then the natural morphism $THH^R(A) \otimes_A B \to THH^R(B)$ is an equivalence.

This is closely related to [WG91, Theorem 0.1] and includes it in the case of a flat extension $R \to A$ of discrete $\mathbb{E}_\infty$-rings. For connective $\mathbb{E}_\infty$-rings, this result is [MM03, Lem. 5.7] (who treat more generally the case of a $THH$-étale morphism).

**Proof.** Given an étale morphism $A \to B$ in $\text{CAlg}_{R/}$, we need to argue that it is strongly 0-cotruncated. By Proposition 2.5, we may reduce to the case where $R = S^0$. Given $C \in \text{CAlg}$, we have a homotopy cartesian square

$$\begin{array}{ccc}
\text{Hom}_{\text{CAlg}}(B, C) & \longrightarrow & \text{Hom}_{\text{Ring}}(\pi_0 B, \pi_0 C) \\
\downarrow & & \downarrow \\
\text{Hom}_{\text{CAlg}}(A, C) & \longrightarrow & \text{Hom}_{\text{Ring}}(\pi_0 A, \pi_0 C)
\end{array}$$

by, e.g., [Lur14, §7.5]. Here Ring is the category of rings. Since the right vertical map is a map of discrete spaces and therefore a split covering, it follows that $\text{Hom}_{\text{CAlg}}(B, C) \to \text{Hom}_{\text{CAlg}}(A, C)$ is a split covering as desired.

We also note in passing that the étale descent theorem has a converse.

**Corollary 3.1.** Let $A \to B$ be a morphism of connective $\mathbb{E}_\infty$-rings which is almost of finite presentation [Lur14, §7.2.5]. Suppose the map $THH(A) \otimes_A B \to THH(B)$ is an equivalence. Then $A \to B$ is étale.

**Proof.** Indeed, $B$ defines a 0-cotruncated object (Definition 4.1) of $\text{CAlg}_{A/}$ and it is well-known that this, combined with the fact that $B$ is almost of finite presentation, implies that $B$ is étale. We reproduce the argument for the convenience of the reader.

In fact, since $B$ is 0-cotruncated, one finds that for any $B$-module $M$, the space of maps $\text{Hom}_{\text{CAlg}_{A/}}(B, B \oplus M)$ is homotopy discrete, where the $\mathbb{E}_\infty$-ring $B \oplus M$ is given the square-zero multiplication. Replacing $M$ by $\Sigma M$, it follows that

$$\text{Hom}_{\text{CAlg}_{A/}}(B, B \oplus M) \cong \Omega \text{Hom}_{\text{CAlg}_{A/}}(B, B \oplus \Sigma M)$$

For an $\infty$-category $C$ and a morphism $x \to y$, we let $C_{x/y}$ denote $(C_{x/y})/y$ where $y \in C_{x/y}$ via the given morphism.
is actually contractible. Thus the cotangent complex $L_{B/A}$ vanishes, which implies that $B$ is étale over $A$ by [Lur11a, Lem. 8.9]. □

The above argument also appears in [Rog08, §9.4], where it is shown that a map $A \to B$ which is 0-truncated (which Rognes calls formally symmetrically étale, and which has been called $THH$-étale in [MM03]) has to have vanishing cotangent complex (which is called $TAQ$-étale); see [Rog08, Lem. 9.4.4]. The key point is that in the connective setting, $TAQ$-étaleness plus a weak finiteness condition is enough to imply étaleness. This entirely breaks down when one works with nonconnective $E_\infty$-ring spectra.

4. An example

We begin with a useful weakening of Definition 2.4.

Definition 4.1. A morphism $x \to y$ in an $\infty$-category $C$ is said to be 0-cotruncated if, for every $z \in C$, the map $\text{Hom}_C(y, z) \to \text{Hom}_C(x, z)$ is a covering space (i.e., has discrete homotopy fibers over any basepoint). An object $x \in C$ is said to be 0-cotruncated if $\text{Hom}_C(x, z)$ is discrete for any $z \in C$.

The condition that $x \to y$ should be cotruncated is equivalent to the statement that $y \in C_{x/}$ should define a 0-cotruncated object. Note that an object $x \in C$ is 0-cotruncated if and only if the natural map $x \to \mathbb{S} \otimes x$ is an equivalence.

In the setting of $E_\infty$-ring spectra, étale morphisms are far from the only examples of 0-cotruncated morphisms. For example, any faithful $G$-Galois extension in the sense of Rognes [Rog08] is 0-cotruncated. This is essentially [Rog08, Lemma 9.2.6]. However, we show that faithful Galois extensions need not be strongly 0-cotruncated. Equivalently, base-change for $THH$ can fail for them.

Proof of Theorem 1.1. Consider the degree $p$ map $S^1 \to S^1$, which is a $\mathbb{Z}/p$-torsor. Let $k$ be a separably closed field of characteristic $p$. The induced map of $E_\infty$-rings $\phi: C^*(S^1; k) \to C^*(S^1; k)$ is a faithful $\mathbb{Z}/p$-Galois extension of $E_\infty$-ring spectra. This follows from [Rog08, Prop. 5.6.3(a)] together with the criterion for the faithfulness via vanishing of the Tate construction [Rog08, Prop. 6.3.3]. See also [Mat14, Th. 7.13].

We will show, nonetheless, that $\phi$ does not satisfy base-change for $THH$, or equivalently that it is not strongly 0-cotruncated. It suffices to show this in $\text{CAlg}_{k/}$ in view of Proposition 2.2.

By $p$-adic homotopy theory [Man01] (see also [Lur11b], which does not assume $k = \mathbb{F}_p$), the natural map $S^1 \to \text{Hom}_{\text{CAlg}_{k/}}(C^*(S^1; k), k)$ exhibits $\text{Hom}_{\text{CAlg}_{k/}}(C^*(S^1; k), k)$ as the $p$-adic completion of $S^1$. In particular, $\text{Hom}_{\text{CAlg}_{k/}}(C^*(S^1; k), k) \simeq K(\mathbb{Z}_p, 1)$ and the map given by precomposition with $\phi$

$\text{Hom}_{\text{CAlg}_{k/}}(C^*(S^1; k), k)$ $\xrightarrow{\phi^*}$ $\text{Hom}_{\text{CAlg}_{k/}}(C^*(S^1; k), k)$,

is identified with multiplication by $p$, $K(\mathbb{Z}_p, 1) \to K(\mathbb{Z}_p, 1)$. In particular, while this is a covering map, it is not a split covering map, so that $\phi$ is not strongly 0-cotruncated. □
The use of cochain algebras in providing such counterexamples goes back to an idea of Mandell [MM03 Ex. 3.5], who gives an example of a morphism of $E_\infty$-ring spectra with trivial cotangent complex (i.e., is $TAQ$-étale) which is not $THH$-étale. Namely, Mandell shows that if $n > 1$, then the map $C^*(K(\mathbb{Z}/p, n); \mathbb{F}_p) \rightarrow \mathbb{F}_p$ has trivial cotangent complex.

We close by observing that it is the fundamental group that it is at the root of these problems.

**Proposition 4.2.** Let $X$ be a simply connected, pointed space, and let $A \rightarrow B$ be a faithful $G$-Galois extension of $E_\infty$-rings. In this case, the map of $E_\infty$-rings

$$ (X \otimes A) \otimes_A B \rightarrow X \otimes B, $$

is an equivalence.

In particular, one does have base-change for higher topological Hochschild homology (i.e., where $X = S^n, n > 1$).

**Proof.** Following the earlier reasoning, it suffices to show that whenever $C \in \text{CAlg}$, the square

$$ \begin{CD}
\text{Hom}(X, \text{Hom}_{\text{CAlg}}(B, C)) @>>> \text{Hom}_{\text{CAlg}}(B, C) \\
\downarrow @. \downarrow \\
\text{Hom}(X, \text{Hom}_{\text{CAlg}}(A, C)) @>>> \text{Hom}_{\text{CAlg}}(A, C)
\end{CD} $$

is homotopy cartesian. However, this follows because $\text{Hom}_{\text{CAlg}}(B, C) \rightarrow \text{Hom}_{\text{CAlg}}(A, C)$ is a covering space, and $X$ is simply connected. \qed

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