Analytical and numerical solutions to Stefan problem in model of the glaciation dynamics of the multilayer cylinder in sea water

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Abstract. A mathematical model of the glaciation dynamics of the multilayer cylinder is suggested. The model reflects the specific feature of glaciation in salt water. The choice of effective model parameters and thermal characteristics of the increasing ice is considered. The calculations of problem using front-tracking finite difference method and using analytical solution found in quasi-stationary approximation are presented.

1. Introduction
Many papers have studied the sea-ice formation, for example, [1]–[6]. Extensive studies on the physical and structural properties of sea ice were carried out [7]–[11]. This work continues research of the sea-gas pipelines in the northern seas, which began in "Models of sea gas-pipelines"[12]. The nonstationary problem of the gas-pipeline glaciation, which has multilayer cylindrical wall in sea-water is considered. In contrast to the freshwater, in the salt water the inflow of salt into the water layer adjacent to the front of glaciation occurs. It increases the density of water and causes the additional convective flows. The aggregate effect results in increasing of total heat flux from water to the front of glaciation. This mechanism is taken into account by introduction in the Stefan condition of the additional heat flux \[ \alpha \frac{dy}{dt} \] from the adjacent water layer, that is proportional to the rate of ice growth \[ \frac{dy}{dt} \]. \( \alpha \) is the effective parameter of the model having dimensions of the volumetric heat source, in the fresh water \( \alpha = 0 \). The selection technique of average salinity \( s_i \) of growing ice from which depend all thermal and physical characteristics of sea ice, and the selection of effective parameter \( \alpha \) are explained below.

2. Mathematical model
The model of the cylindrical gas pipeline glaciation with two coating layers in the salt water is written as follows:

\[ \rho_1 c_1 \frac{\partial T_1}{\partial t} = \lambda_1 L(T_1), \quad r \in (R, R_1), \quad t > t_0; \]  
\[ t = t_0, \quad T_1(r) = T_0^0(r); \]  
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where $t > t_0$, $r = R$: $T_1 = T_0(t)$; $r = R_1$: $T_1 = T_2$, $\lambda_1 \frac{\partial T_1}{\partial r} = \lambda_2 \frac{\partial T_2}{\partial r}$; $\rho_2 c_2 \frac{\partial T_2}{\partial t} = \lambda_2 L(T_2)$, $r \in (R_1, R_2)$, $t > t_0$; $t = t_0$, $T_2(r) = T_2^0(r)$; $t > t_0$, $r = R_2$: $T_2 = T_i$, $\lambda_2 \frac{\partial T_2}{\partial r} = \lambda_i \frac{\partial T_i}{\partial r}$; $\rho_1 c_i \frac{\partial T_i}{\partial t} = \lambda_i L(T_i)$, $r \in (R_2, R_2 + y(t))$, $t > t_0$; $t = t_0$, $y = y_0$, $T_i(r) = T_i^0(r)$; $t > t_0$, $r = R_2 + y(t)$: $T_i = T_s$; $\lambda_i \frac{\partial T_i}{\partial r} |_{R_2 + y(t)} - q = \gamma \rho_i \frac{dy}{dt}$, $q = q_w + \alpha \frac{dy}{dt}$, $t > t_0$.

We use the following designations: $L = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right)$ is the Laplace operator in cylindrical coordinate system $(r, z, \varphi)$ under conditions $\frac{\partial}{\partial r} = \frac{\partial}{\partial z} = 0$; $\lambda_k$, $\rho_k$, $c_k$, $T_k = T_k(r, t)$ are the thermal conductivity [W/mK], the density [kg/m$^3$], the specific heat [kJ/kgK] and the temperature distribution [K] in $k$-th layer, indices $k = 1, 2$, $i$ correspond to the following regions: 1 to the first layer of pipeline wall (steel), 2 to the second layer of pipeline wall (concrete), $i$ to the ice layer; $R$ is the inner radius of the gas-pipeline [m], $R_1$, $R_2$ are the outer radii of the pipeline wall layers [m]; $T_0(t)$ is the temperature distribution in the ice layer at the initial moment of time. $\gamma$ is the latent heat (of fusion) for sea ice [kJ/kg]; $q_w$ is the radial component of the heat flux vector from seawater to glaciation front; $r = R_2 + y(t)$ is the coordinate of the glaciation front; $y = y(t)$ is the ice thickness [m] upon the surface of pipeline at the time moment $t$; $y_0$ is the ice thickness [m] at the initial moment of time $t_0$. By $r = R_2 + y_0$ following equalities hold $T_i^0(R_2 + y_0, t_0) = T_s$. Model can be extended to a larger number of coating layers. The equation (11) is the modified Stefan condition, which covers the additional heat flux from the water. In model the parameters $\lambda_i$, $c_i$, $\rho_i$, $T_s$, $\gamma$, $\alpha$, $q_w$ are considered to be constant.

3. Selection of parameters

For selecting parameters of model, which are in agreement with the known experimental data, was used following technique. The values of the effective parameter $\alpha^0$, of the range of the ice temperature changes and of the ice salinity $s_i^0$ were set. The values of $\lambda_i$, $c_i$, $\rho_i$, $\gamma$ were calculated using known experimental data given in view of the empirical formulas and of the tables in [13,14].

The other parameters of model $q_w$, $T_s$, $\rho_1$, $c_1$, $\lambda_1$, $\rho_2$, $c_2$, $\lambda_2$, $R$, $R_1$, $R_2$ were assumed to known and unchanged. Time dependence of the ice growth rate $W = \frac{dy}{dt}$ is calculated using the given below algorithm to numerical solution of system (1)–(11). The average salinity $s_i \left[ ^0_{/oo} \right]$ of growing ice was calculated using semiempirical formula by Tsurikov [13]:

$$s_i = S_w \frac{7 \sqrt{W}}{7 \sqrt{W} + 10.30}. \quad (12)$$

$S_w$ is the salinity of sea water \(^0_{/oo}\), $W = \frac{dy}{dt}$ is the rate of ice growth [mm/h]. The sea water salinity was chosen to be $S_w = 35 \ ^0_{/oo}$. The ice salinity $s_i$ values were found on the Tsurikov formula (12) using dependence $W = \frac{dy}{dt}$ calculated from model. The mean value of salinity
< s_i > is compared to the selected value s^0_i. If the difference between < s_i > and s^0_i was over 2 %, the procedure selecting effective parameters s_i and α was repeated according to the same scheme with new s^0_i and α^0.

The following correlated parameters of model were found according to the technique described above:

\[ q_w = 30.4447 \text{ W/m}^2, \quad T_s = 271.236 \text{ K}, \quad T_0 = (T_s - 6) \text{ K}, \]
\[ R = 0.51 \text{ m}, \quad \rho_1 = 10^4 \text{ kg/m}^3, \quad \rho_2 = 2300 \text{ kg/m}^3, \quad c_1 = 450 \text{ J/(kg} \cdot \text{ K)}, \]
\[ c_2 = 924 \text{ J/(kg} \cdot \text{ K)}, \quad \lambda_1 = 24 \text{ W/(m} \cdot \text{ K)}, \quad \lambda_2 = 1.7 \text{ W/(m} \cdot \text{ K)}, \]
\[ \delta_1 = 0.04 \text{ m}, \quad \delta_2 = 0.12 \text{ m}, \quad \gamma = 303000 \text{ J/kg}, \quad \rho_1 = 931 \text{ kg/m}^3. \]

4. Numerical solution algorithm

If the temperature upon the outer surface of the gas pipeline is lower than the sea water-ice transition temperature, the pipeline glaciation may occur. The thickness of the ice layer increases in time, and the ice growth rate converges to zero. At an equilibrium state the ice thickness y_s reaches the maximum (steady) value and the modified Stefan condition (11) passes to the equality: \[ \lambda_i \frac{\partial T_s}{\partial r} \bigg|_{R_2+y_s} - q_w = 0. \] This equality together with the stationary heat transfer equations in heat-insulation layers and with other conditions of model enables to calculate the maximum thickness of the ice layer y_s from the transcendental equation:

\[ \frac{\lambda_i(T_s - T_0)}{(R_2 + y_s)\left(\frac{\lambda_1}{\lambda_2} \ln \frac{R_1}{R} + \frac{\lambda_1}{\lambda_2} \ln \frac{R_2}{R} + \ln \frac{R_2+y_s}{R_2}\right)} - q_w = 0. \] (14)

For the numerical solution of this Stefan-type problem can be used a method with the explicit tracking of moving surface. We use the method of Douglas and Gallie [15], [16], which is an iterative finite difference method, with variable time steps. In this approach the time step size \( \tau_n+1 \) is variable and at the \( (n+1) \)-th temporal level it is determined so that the ice thickness increased on constant value \( h \) during this time step. The value \( h \) is the space grid step size. The convergence of this numerical method was proved for a Stefan-type problem in [15], [16]. The time step size in the null approximation \( \tau^{(0)} \) at the \( (n+1) \)-th temporal level is set to equal previous step size \( \tau_n \) at the \( n \)-th temporal level: \( \tau^{(0)} = \tau_n \).

The calculation on model begins at the moment of time \( t_0 \), when necessary and sufficient conditions are fulfilled for beginning glaciation of the outer surface of pipeline:

\[ T_2^0(R_2) < T_s, \quad \lambda_2 \frac{\partial T_2}{\partial r} \bigg|_{R_2} > q_w. \]

The first step \( \tau_1 \) was chosen from the difference version of equation (11):

\[ \tau_1 = \frac{(\gamma \rho_i + \alpha)h^2}{(\lambda_i(T_s - T_2^0) - q_w h)}, \]

\( T_2^0[N2] \) is the temperature in second insulation-layer at boundary with ice at initial time moment, \( N2 \) is the array length of temperature of second insulation-layer. Arrays of the temperatures in two insulation-layers and in ice layer at the \( s \)-the iteration is calculated using the implicit finite difference scheme with step size \( \tau^s \).
The step size $\tau_{s+1}$ at the $(s+1)$-th iteration at the $(n+1)$-th temporal level is calculated using obtained array of the temperature $T_s$ in the ice layer according to the formula:

$$
\tau_{s+1} = \tau_s + \left(\gamma \rho_i + \alpha\right)(h - h^s)/(q_{II} - q_w),
$$

$$
h^s = \tau_s (q_I - q_w)/(\gamma \rho_i + \alpha),
$$

$$
q_{II} = \lambda_i(T^{n}[n+1] - T^{s}[n])/h,
$$

where $(n+1)$ is the array length of the temperature in the ice layer at the $(n+1)$-th temporal level. The iterative process is terminated if the inequality $|\tau_{s+1} - \tau_s| \leq \varepsilon$ holds for a specified small quantity $\varepsilon$.

By this algorithm the time step size $\tau$, the overall glaciation time $t$, the ice growth rate $W$ and the temperature distributions are calculated for every thickness of the ice layer $y$.

5. Results

As an example, in Table 1 we present the calculation results of the beginning of the gas pipeline glaciation in seawater with parameter values (13) by the presented algorithm.

| $y$, cm. | 0.3 | 0.5 | 1 | 3 | 5 | 7 | 9 | 11 | 12.5 |
|---------|-----|-----|---|---|---|---|---|----|------|
| $t$, h. | 13.60 | 20.624 | 43.500 | 139.220 | 264.170 | 423.300 | 633.021 | 916.34 | 1219.44 |
| $\tau$, h. | 3.935 | 4.033 | 4.287 | 5.471 | 7.018 | 9.120 | 12.132 | 16.811 | 22.472 |
| $W$, mm/h. | 0.254 | 0.248 | 0.233 | 0.183 | 0.142 | 0.110 | 0.082 | 0.060 | 0.040 |

For parameter values (13) the maximum ice thickness $y_*$ is equal to 18.2378 cm.

The ice salinity calculation $s_i(W)$ on values $W(t)$ from the Table 1 using Tsurikov formula (12) is presented in Table 2.

| $W$, mm/h. | 0.254 | 0.248 | 0.233 | 0.183 | 0.142 | 0.110 | 0.08 | 0.06 |
|-------------|------|------|------|------|------|------|-----|-----|
| $s_4(W)$, % | 8.93 | 8.85 | 8.645 | 7.88 | 7.135 | 6.44 | 5.70 | 5.0 |

As follows in table 2 the mean salinity $< s_i > (W)$ of the growth ice, calculating by the formula (12) is equal to 7.3 %/oo, which confirms that the set parameters (13) is correlated.

It is possible to show that model at quasi-stationary approximation has the analytical solution when the ratio of the ice layer thickness to the outer radius of the gas pipeline is small: $y/R_2 \ll 1$. The analytical solution in this approximation is

$$
t - t_0 = -\frac{(y - y_0)}{a} + b \ln \left(\frac{R_2 - cy_0}{R_2 - cy}\right),
$$

$$
a = \frac{q_w t_x}{r_x (\gamma \rho_i + \alpha)},
$$

$$
b = \frac{\lambda_i (T_* - T_0)}{q_w^2 t_x (1 + d)} (\gamma \rho_i + \alpha) ,
$$

with

$$
T_* = T_{II}[n+1] - T^{s}[n],
$$

$$
T_0 = T_{II}[n] - T^{s}[n].
$$
\[ c = \left( \frac{\lambda_i (T^* - T_0)}{r_x R_2 (1 + d)} q_w \right)^{-1} - \frac{d}{1 + d} \],
\[ d = \frac{\lambda_1}{\lambda_1} \ln \frac{R_1}{R} + \frac{\lambda_2}{\lambda_2} \ln \frac{R_2}{R_1}, \]

the values \( t, t_0, y, y_0, a, b, c, d, R_2 \) in the equation (15) are dimensionless, \( r_x, t_x \) are the characteristic values of variables \( r \) and \( t \) respectively.

In Table 3 we present the comparing the results of the numerical modeling (Table 1) and the calculation using analytical solution (15) with the initial conditions \( t_0 = 20.624 \) h., \( y_0 = 0.5 \) cm.

| \( y \), cm. | 1 | 3 | 5 | 7 | 9 | 11 |
|---|---|---|---|---|---|---|
| \( t \), h. | 43.50 | 139.22 | 264.17 | 423.30 | 633.02 | 916.34 |
| \( \tilde{t} \), h. | 41.49 | 138.67 | 261.92 | 418.73 | 620.28 | 884.51 |

As follows in table 3 the ice layer with the thickness of 1 cm is formed a faster in accordance with the analytical solution, which is based on the quasi-stationary temperature distribution. The calculations of nonstationary problems suggest that at the very beginning of the processes the temperature distributions in the layers are different from the quasi-stationary ones. This is reflected in the fact that the values \( t \) and \( \tilde{t} \) are differ for \( y = 1 \) cm. Further, with a reasonable degree of accuracy the heat processes can be considered the quasi-stationary, which is confirmed by the values \( t, \tilde{t} \) close to each other in domain \( y = 3 \) cm. Distinctions between numerical and analytical solutions increases for \( y \geq 5 \) cm. This is due to the fact that, the ice layer while growing, less and less holds the condition \( y \ll R_2 \). Consequently, as follows in table 3 if that the conditions using to obtain analytical solution are satisfied, then the analytical and numerical solutions to the glaciation dynamic problem are coincident. The calculation by the presented algorithm of the numerical solution of different variants of the gas pipeline glaciation shown that the process of glaciation can estimated as stationary for defined thickness of ice for many sets of parameters and the thickness of ice can calculate using analytical solution obtained (15).

6. Conclusion

The paper offered the modification of Stefan condition, which allows to take into account the feature of glaciation in salt water. The algorithm of the numerical solution of the glaciation model of multilayer cylindrical wall in sea-water and the selection procedure of the average thermal characteristics of the growing ice were suggested. The analytical solution of the problem was given for the approximation of the thin layer. The presented comparing the numerical and analytical solution confirms reliability and acceptable accuracy of the suggested numerical solution to the nonlinear system of equations.

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