Multiple solutions for a generalised Schrödinger problem with “concave–convex” nonlinearities

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Abstract. A class of generalised Schrödinger elliptic problems involving concave–convex and other types of nonlinearities is studied. A reasonable overview about the set of solutions is provided when the parameters involved in the equation assume different real values.

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1. Introduction

We are interested in investigating the following classes of stationary generalised Schrödinger problems

\[
\begin{align*}
- \text{div}(\vartheta(u) \nabla u) + \frac{1}{2} \vartheta'(u)|\nabla u|^2 &= \lambda |u|^{q-2}u + \mu |u|^{p-2}u \quad \text{in } \Omega, \\
\vartheta(u) &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^N \), \( N \geq 3 \), is a bounded smooth domain, \( 1 < q < 4 \), \( \max\{2,q\} < p < 22^* \), \( \lambda \) and \( \mu \) are real parameters and \( \vartheta : \mathbb{R} \to [1, \infty) \) is a general even \( C^1 \)-function whose hypotheses will be posteriorly mentioned.

When \( \Omega = \mathbb{R}^N \), equation \((P_{\lambda,\mu,q,p})\) is related to the existence of solitary wave solutions for the parabolic quasilinear Schrödinger equation

\[
i\partial_t z = -\Delta z + V(x)z - \rho(|z|^2)z - \Delta(l(|z|^2))l'(|z|^2)z, \quad x \in \mathbb{R}^N,
\]

where \( z : \mathbb{R} \times \mathbb{R}^N \to C \), \( V : \mathbb{R}^N \to \mathbb{R} \) is a given potential and \( l, \rho \) are real functions. Equation (1.1) appears naturally as a model for several physical phenomena, depending on the type of function \( l \) considered. In fact, if \( l(s) = s \), (1.1) describes the behaviour of a superfluid film in plasma physics; see [12]. For \( l(s) = (1 + s)^{1/2} \), (1.1) models the self-channelling of a high-power ultrashort laser in matter; see [3–5,13]. Furthermore, (1.1) also appears in plasma physics and fluid mechanics [14], in dissipative quantum mechanics [11], in the theory of Heisenberg ferromagnetism and magnons [18] and in condensed matter theory [16].

If we take \( z(t, x) = e^{-iEt}u(x) \) in (1.1), we get the corresponding steady-state equation

\[
- \Delta u + V(x)u - \Delta(l(u^2))l'(u^2)u = \rho(u) \quad \text{in } \mathbb{R}^N.
\]

In the case that \( \rho(s) = \lambda |s|^{q-2}s + \mu |s|^{p-2}s \) and \( \mathbb{R}^N \) is replaced by \( \Omega \), problem (1.2) can be obtained from \((P_{\lambda,\mu,q,p})\), simply by choosing \( \vartheta(s) = 1 + (l(s^2))^{1/2} \), for some \( C^2 \)-function \( l \).

Many authors have studied stationary Schrödinger problems like \((P_{\lambda,\mu,q,p})\) under different nonlinearities and functions \( \vartheta \), when \( \Omega = \mathbb{R}^N \). Without any intention to provide a complete overview about

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the matter, we just refer the reader to some seminal contributions: In the case $\vartheta(s) = 1 + 2s^2$, see [6,8–10,15,17,22,24]. In the case $\vartheta(s) = 1 + s^2/(1 + s^2)$, see [7,20,21].

The main goal of the present paper is provide a reasonable outline about the existence of multiple solutions for problem $\left( P_{\lambda,\mu,q,p} \right)$, when the parameters involved assume different values and function $\vartheta$ satisfies general conditions which cover some of the cases previously mentioned. More specifically, we assume that

1. $\vartheta(s)$ is decreasing in $(-\infty, 0)$ and increasing in $(0, \infty)$;
2. $\vartheta(s)/s^2$ is nondecreasing in $(-\infty, 0)$ and nonincreasing in $(0, \infty)$;
3. $\lim_{|s| \to \infty} \vartheta(s)/s^2 = \alpha^2/2$, for some $\alpha > 0$.

Some examples of functions satisfying $(\vartheta_1) - (\vartheta_3)$ can be given by:

$$\vartheta_*(s) = 1 + 2s^2, \quad \vartheta_#(s) = 1 + \frac{s^2}{2(1 + s^2)} + s^2$$

and $\vartheta_1(s) = 1 + \ln(1 + e^{s^2})$;

other examples can be found in [19], where the authors consider the problem $\left( P_{\lambda,\mu,q,p} \right)$ with power type nonlinearities.

To the best of our knowledge, this is the first article to treat this type of nonlinearity for such a class of generalised Schrödinger problems in bounded domains. Our approach consists in switching the task to look for solutions of the general semilinear problem $\left( P_{\lambda,\mu,q,p} \right)$, by task to find solutions of

$$\begin{cases} -\Delta v = \lambda f'(v)|f(v)|^{q-2}f(v) + \mu f'(v)|f(v)|^{p-2}f(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in C^2(\mathbb{R})$ is a solution of the ordinary differential equation

$$f'(s) = \frac{1}{\vartheta(f(s))^{1/2}} \quad \text{for } s > 0 \text{ and } f(0) = 0,$$

with $f(s) = -f(-s)$ for $s \in (-\infty, 0)$. Since $f$ is odd and $\vartheta$ is even, equation (ODE) is yet true for negative values. It is well known that $v$ is a weak solution of $\left( P_{\lambda,\mu,q,p}^* \right)$ if and only if $u = f(v)$ is a weak solution of $\left( P_{\lambda,\mu,q,p} \right)$; see [19] or [20]. Naturally, a weak solution of $\left( P_{\lambda,\mu,q,p}^* \right)$ is a function $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \nabla v dx = \lambda \int_\Omega f'(u)|f(u)|^{q-2}f(u)v dx + \mu \int_\Omega f'(u)|f(u)|^{p-2}f(u)v dx, \quad \text{(1.3)}$$

for all $v \in H_0^1(\Omega)$. Moreover, the energy functional $J_{\lambda,\mu} : H_0^1(\Omega) \to \mathbb{R}$ associated with $\left( P_{\lambda,\mu,q,p}^* \right)$ is

$$J_{\lambda,\mu}(u) = \frac{1}{2} \|u\|^2 - \frac{\lambda}{q} \int_\Omega |f(u)|^q dx - \frac{\mu}{p} \int_\Omega |f(u)|^p dx. \quad \text{(1.4)}$$

Lemma 2.1, in the next section, assures that the previous notion of weak solution makes sense, and ensures that functional $J_{\lambda,\mu}$ is well defined and is $C^1$.

Due to the nature of the generalised Schrödinger operator, some interesting phenomena can be observed when one compares $\left( P_{\lambda,\mu,q,p}^* \right)$ to the classical concave–convex problem involving the Laplacian operator. To be more precise, in the case of Laplacian operator, results of the existence of infinitely many high-energy solutions are observed as $1 < q < 2 < p$ and $\mu > 0$; see [2]. On the other hand, the existence of infinitely many low-energy solutions occurs as $1 < q < 2 < p$ and $\lambda > 0$ is small enough; see [1]. In the case of the generalised Schrödinger operator studied in the present paper, an analogous result of infinitely many high-energy solutions for $\mu > 0$ is only occurring for $1 < q < 4 < p$. Moreover, the counterpart of the result of infinitely many low-energy solutions for $\lambda > 0$ requires $1 < q < 2$ and $p \neq 4$. What is noticed in Theorem 1.1 is the existence of a “grey zone”, namely $2 \leq q < p \leq 4$, where the set of solutions has an intermediate behaviour, presenting simultaneously influence of both powers and the length of $\lambda$ and...
μ. In such a zone, one can get a finite number of solutions as large as one want, provided that μ or λ are large enough. More specifically:

**Theorem 1.1.** Suppose that θ satisfies $\theta_1 - \theta_3$. The following claims hold:

(i) let $\lambda \in \mathbb{R}$, $\mu > 0$ and $1 < q < 4$. If $4 < p < 22^*$, then $(P_{\lambda,\mu,p,q})$ has a sequence of solutions $\{u_n\}$ with $J_{\lambda,\mu}(f^{-1}(u_n)) \to \infty$. Furthermore, if $\max\{q,2\} < p < 4$, then for each $k \in \mathbb{N}$ there exists $\mu_k > 0$ such that $(P_{\lambda,\mu,p,q})$ has at least $k$ pairs of nontrivial solutions $u_k$ with $J_{\lambda,\mu}(f^{-1}(u_k)) > 0$, provided that $\mu \in (\mu_k,\infty)$;

(ii) let $\lambda > 0$, $\mu \in \mathbb{R}$ and $p \neq 4$. If $1 < q < 2$, then $(P_{\lambda,\mu,p,q})$ has a sequence of solutions $\{u_n\}$ with $J_{\lambda,\mu}(f^{-1}(u_n)) < 0$ and $J_{\lambda,\mu}(f^{-1}(u_n)) \to 0$. Furthermore, if $2 < q < 4$, then for each $k \in \mathbb{N}$ there exists $\lambda_k > 0$ such that $(P_{\lambda,\mu,p,q})$ has at least $k$ pairs of nontrivial solutions $u_k$ with $J_{\lambda,\mu}(f^{-1}(u_k)) < 0$, provided that $\lambda \in (\lambda_k,\infty)$;

(iii) let $\lambda > 0$, $\mu < \lambda_1\alpha^2/4$ and $p = 4$. Then, for each $k \in \mathbb{N}$ there exists $\lambda_k > 0$ such that $(P_{\lambda,\mu,p,q})$ has at least $k$ pairs of nontrivial solutions $u_k$ with $J_{\lambda,\mu}(f^{-1}(u_k)) < 0$, provided that $\lambda \in (\lambda_k,\infty)$, where $\alpha$ is defined in $(\theta_3)$.

Throughout the paper, $|A|$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}^N$, $[1 < u] := \{x \in \Omega : 1 < u(x)\}$, $\lambda_1$ is the first eigenvalue of Laplacian operator with homogeneous Dirichlet boundary condition and $C, C_0, C_1, C_2$ stand for positive constants whose exact value is not relevant for our purpose.

The paper is organised as follows.

In Sect. 2, we study a suitable change of variable which becomes problem $(P_{\lambda,\mu,p,q})$ in a more manageable one. In Sect. 3, we prove non-existent results. In Sect. 4, we prove existence results.

2. Preliminaries

Although the proof of the next lemma can also be found in [19], for the reader’s convenience and by its relevant role throughout the paper, we provide it here.

**Lemma 2.1.** Let $\theta \in C^1(\mathbb{R})$ and $f$ a solution of (ODE). The following claims hold:

(i) $f$ is uniquely defined and it is an increasing $C^2$-diffeomorphism, with $f''(s) = -\theta'(f(s))/2\theta(f(s))^2$, for all $s > 0$;

(ii) $0 < f'(s) \leq 1$, for all $s \in \mathbb{R}$;

(iii) $\lim_{s \to 0} f(s)/s = 1/\theta(0)^{1/2}$;

(iv) $|f(s)| \leq |s|$, for all $s \in \mathbb{R}$;

(v) suppose $(\theta_1) - (\theta_2)$ hold. Then, $|f(s)|/2 \leq f'(s)|s| < |f(s)|$, for all $s \in \mathbb{R}\{0\}$, and the map $s \mapsto |f(s)|/\sqrt{|s|}$ is nonincreasing in $(-\infty,0)$ and nondecreasing in $(0,\infty)$;

(vi) suppose that $(\theta_1) - (\theta_3)$ hold. Then,

$$\lim_{s \to -\infty} \frac{|f(s)|}{\sqrt{|s|}} = \left(\frac{8}{\alpha^2}\right)^{1/4}$$

and

$$\lim_{s \to -\infty} \frac{f(s)}{s} = 0,$$

where $\alpha$ is given in $(\theta_3)$.

**Proof.**

(i)–(ii) Existence, uniqueness, regularity, monotonicity and (ii) follow directly from (ODE). To see that $f(\mathbb{R}) = \mathbb{R}$, observe that $f(s) = (\Upsilon^{-1})(s)$, where

$$\Upsilon(t) = \int_0^t \theta(r)^{1/2}dr.$$

Since $\theta \geq 1$, $|\Upsilon(t)| \geq |t|$ for all $t \in \mathbb{R}$. Consequently, $\lim_{|t| \to \infty} |\Upsilon(t)| = \infty$. Thence, $\lim_{s \to \infty} |f(s)| = \infty$. 


Notice that, by L’Hôpital rule, we get
\[
\lim_{s \to 0} \frac{f(s)}{s} = \lim_{s \to 0} f'(s) = \frac{1}{\vartheta(0)^{1/2}}.
\]

It follows from (ii). (v) Since \( f \) is odd and \( \vartheta \) is even, it is sufficient to prove the inequalities for \( s > 0 \). For that, let \( r_1 : [0, \infty) \to \mathbb{R} \) defined by
\[
r_1(s) = f(s)\vartheta(f(s))^{1/2} - s.
\]
Notice that \( r_1(0) = 0 \) and by (ODE) and \((\vartheta_1)\), we have
\[
r_1'(s) = \vartheta'(f(s))f(s)/2\vartheta(f(s)) > 0.
\]
Therefore, the second inequality in (v) follows. Now, to prove the first inequality in (v), let \( r_2 : [0, \infty) \to \mathbb{R} \) be defined by
\[
r_2(s) = 2s - f(s)\vartheta(f(s))^{1/2}.
\]
We have that \( r_2(0) = 0 \) and by (ODE) and \((\vartheta_2)\),
\[
r_2'(s) = 1 - \vartheta'(f(s))f(s)/2\vartheta(f(s)) \geq 0,
\]
showing that the inequality in (v) holds. Moreover, since
\[
\left( \frac{f(s)}{\sqrt{s}} \right)' = \frac{2f'(s)s - f(s)}{2s\sqrt{s}} \geq 0, \ \forall \ s > 0,
\]
the second part of (v) follows.

(vi) Observe that from (v), we have
\[
\lim_{|s| \to \infty} \frac{|f(s)|}{\sqrt{|s|}} = l, \ \text{with} \ l \in (0, \infty].
\]
Again, since \( f \) is odd and \( \vartheta \) is even, it is sufficient to consider the case \( s \to \infty \). Suppose that
\[
\lim_{s \to \infty} f(s)/\sqrt{s} = \infty. \hspace{1cm} (2.1)
\]
If this is the case, then, by (i), we get \( f(s) \to \infty \) as \( s \to \infty \). By applying the L’Hôpital rule and using \((\vartheta_3)\), we conclude from (2.1) that
\[
\lim_{s \to \infty} \frac{f(s)}{\sqrt{s}} = \lim_{s \to \infty} 2f'(s)\sqrt{s}
\]
\[
= 2 \lim_{s \to \infty} \sqrt{\frac{s}{\vartheta(f(s))}}
\]
\[
= 2 \sqrt{\lim_{s \to \infty} \left( \frac{\sqrt{s}}{f(s)} \right)^2}
\]
\[
= 2 \sqrt{\lim_{s \to \infty} \vartheta(f(s))/f(s)^2}
\]
\[
= 2 \sqrt{\frac{0}{\alpha^2/2}} = 0,
\]
showing that
\[
\lim_{s \to \infty} f(s)/\sqrt{s} = 0. \hspace{1cm} (2.2)
\]
Since (2.2) contradicts (2.1), it follows that \(0 < \lim_{s \to \infty} \frac{f(s)}{\sqrt{s}} = l < \infty\). Applying one more time the L'Hôpital rule, we have
\[
l = 2 \sqrt{\frac{\lim_{s \to \infty} \left(\frac{\sqrt{s}f(s)}{f(s)}\right)^2}{\lim_{s \to \infty} \frac{d\left(f(s)\right)}{f(s)^2}}} = 2 \sqrt{\frac{1/l^2}{(\alpha^2/2)}}.
\]
Or equivalently,
\[
l = \left(\frac{8}{\alpha^2}\right)^{1/4}.
\] (2.3)

On the other hand, from (2.3),
\[
\lim_{s \to \infty} \frac{f(s)}{s} = \lim_{s \to \infty} \frac{f(s)}{\sqrt{s}} = \left(\frac{8}{\alpha^2}\right)^{1/4} \times 0 = 0.
\]
\(\square\)

Before finishing this section, we are going to introduce two technical lemmas which will be very helpful later on.

**Lemma 2.2.** Let \(\{u_n\}\) be a sequence of measurable functions \(u_n : \Omega \to \mathbb{R}\). Then,
\[
\chi_{\left[1 < \liminf_{n \to \infty} u_n\right]}(x) \leq \liminf_{n \to \infty} \chi_{[1 < u_n]}(x) \text{ in } \Omega,
\]
where, from now on, \(\chi_A\) stands for the characteristic function of a set \(A \subset \Omega\).

**Proof.** Let us define \(u := \liminf_{n \to \infty} u_n\) and \(g : \Omega \to \{0, 1\}\) by
\[
g(x) = \liminf_{n \to \infty} \chi_{[1 < u_n]}(x).
\]
If \(g \equiv 1\), there is nothing to be proved. Otherwise, it is sufficient to prove that if \(g(x) = 0\), then \(\chi_{[1 < u]}(x) = 0\). Indeed, observe that if \(g(x) = 0\) then there exists a subsequence \(u_{n_k}\) where \(\{n_k\} \subset \mathbb{N}\) depends on \(x\), such that
\[
\chi_{[1 < u_{n_k}]}(x) = 0, \forall k \in \mathbb{N}.
\]
Equivalently,
\[
u_{n_k}(x) \leq 1, \forall k \in \mathbb{N}.
\]
Passing to the lower limit as \(k\) goes to infinity, we obtain
\[
u(x) = \liminf_{n \to \infty} u_n(x) \leq \liminf_{k \to \infty} u_{n_k}(x) \leq 1,
\]
or yet
\[
\chi_{[1 < u]}(x) = 0.
\]
\(\square\)

Now on, let us agree that \(\{e_j\}\) stands for a Hilbertian basis of \(H^1_0(\Omega)\) composed by functions in \(L^\infty(\Omega)\) (for example, the basis composed by eigenfunctions of Laplacian operator with Dirichlet boundary condition),
\[
X_j := \text{Span}\{e_j\}, Y_k := \oplus_{j=0}^k X_j \text{ and } Z_k := \oplus_{j=k}^\infty X_j.
\]

Since \(|f(s)|\) behaves like \(|s|\) near the origin and like \(|s|^{1/2}\) at infinity, the next lemma will be very helpful to get some important estimates for the existence results.

**Lemma 2.3.** Let \(S_k\) be the unit sphere of \(Y_k\). There exist positive constants \(\beta_k, \beta_k(r), \alpha_k, \tau_k\) such that
\( (i) \) 
\[ \beta_k \leq ||1 < |su||, \]  
for all \( u \in S_k \) and \( s > \alpha_k \), and 
\[ ||su|| < 1 = \Omega, \]  
for all \( u \in S_k \) and \( 0 < s < \tau_k \).

\( (ii) \) for each \( r \in [1, 2^*] \), 
\[ \beta_k(r) \leq \int_{1 < |su|} |u|^r \, dx, \]  
for all \( u \in S_k \) and \( s > \alpha_k \).

\textbf{Proof.}  \( (i) \) First, we are going to prove that (2.4) holds.

Indeed, suppose that there exist \( \{s_n\} \subset (0, \infty) \) and \( \{u_n\} \subset S_k \) with \( s_n \to \infty \) and 
\[ ||1 < |s_n u_n||| \to 0 \text{ as } n \to \infty. \]  
(2.7)

Since \( Y_k \) has finite dimension, there exists 
\[ u \in S_k \]  
such that, up to a subsequence, \( u_n \to u \) in \( H_0^1(\Omega) \) and 
\[ u_n(x) \to u(x) \text{ a.e. in } \Omega. \]

Therefore,
\[ |s_n u_n| \to \infty \text{ in } [u \neq 0]. \]  
(2.9)

It follows from (2.8), (2.9), Lemma 2.2(1), Fatou lemma and (2.7) that 
\[ 0 < ||u \neq 0|| \leq ||1 < \liminf_{n \to \infty} |s_n u_n||| \]
\[ = \int_{\Omega} \chi_{1 < \liminf_{n \to \infty} |s_n u_n||} \, dx \]
\[ \leq \int_{\Omega} \liminf_{n \to \infty} \chi_{1 < |s_n u_n||} \, dx \]
\[ \leq \liminf_{n \to \infty} \int_{\Omega} \chi_{1 < |s_n u_n||} \, dx \]
\[ = \liminf_{n \to \infty} ||1 < |s_n u_n||| = 0. \]

This is a clear contradiction. Therefore, (2.4) holds. Now, in order to prove (2.5), observe that if \( u \in S_k \) then, by Cauchy–Schwarz inequality 
\[ |u(x)| = \left| \sum_{j=0}^{k} y_j e_j(x) \right| \leq \left( \sum_{j=0}^{k} y_j^2 \right)^{1/2} \left( \sum_{j=0}^{k} e_j(x)^2 \right)^{1/2} \leq (k + 1)M^2, \]  
(2.10)

where \( M := \max_{j=0}^{k} |e_j|_{\infty}. \) Consequently, choosing \( \tau_k := 1/(k + 1)M^2 \) the result follows.
By Fatou lemma, Lemma 2.2 and since $Y_k$ has finite dimension, we have
\[
\liminf_{s \to \infty} \int_{[1<|su|]} |u|^r \, dx = \liminf_{s \to \infty} \int_{\Omega} |u|^r \chi_{[1<|su|]}(x) \, dx \\
\geq \int_{\Omega} |u|^r \liminf_{s \to \infty} \chi_{[1<|su|]}(x) \, dx \\
\geq \int_{\Omega} |u|^r \chi_{[u\neq 0]}(x) \, dx \\
= \int_{\Omega} |u|^r \, dx \geq \zeta_k(r),
\]
for all $u \in S_k$ and some $\zeta_k(r) > 0$. Choosing $0 < \beta_k(r) < \zeta_k(r)$, the result is proved. 

3. Non-existent results

In this section, we are interested in proving some non-existent results which complement Theorem 1.1. More specifically:

**Theorem 3.1.** The following claims hold:

(i) if $\lambda, \mu \leq 0$, then $(P_{\lambda,\mu,q,p})$ does not have any nontrivial solution;
(ii) suppose that $\vartheta$ satisfies $(\vartheta_1) - (\vartheta_2)$, $1 < q \leq 2$ and $p \geq 4$ hold. If $\lambda < 0$, then $(P_{\lambda,\mu,q,p})$ does not have solutions $u$ satisfying $J_{\lambda,\mu}(f^{-1}(u)) \leq 0$. Analogously, if $\mu < 0$, then $(P_{\lambda,\mu,q,p})$ does not have solutions $u$ satisfying $J_{\lambda,\mu}(f^{-1}(u)) \geq 0$;
(iii) suppose that $\vartheta$ satisfies $(\vartheta_1) - (\vartheta_3)$. If $\max\{2, q\} < p \leq 4$ and $\lambda < 0$, then there exists $\mu_* > 0$ such that $(P_{\lambda,\mu,q,p})$ does not have solutions $u$ satisfying $J_{\lambda,\mu}(f^{-1}(u)) \leq 0$, whatever $\mu \in (0, \mu_*);$ Moreover, if $1 < q < 2 < p \leq 4$ and $\mu > 0$, then there exists $s_* > 0$ such that $(P_{\lambda,\mu,q,p})$ does not have solutions $u$ satisfying $J_{\lambda,\mu}(f^{-1}(u)) \geq 0$, whatever $\mu \in (-s_*, s_*);$ if $2 \leq q < p \leq 4$ and $\mu < 0$, then there exists $\nu_* > 0$ such that $(P_{\lambda,\mu,q,p})$ does not have solutions $u$ satisfying $J_{\lambda,\mu}(f^{-1}(u)) \leq 0$, whatever $\mu \in (-\nu_*, \nu_*);$ if $2 \leq q < p \leq 4$, then there exist $r_* > 0$ such that $(P_{\lambda,\mu,q,p})$ does not have any nontrivial solution, whatever $\lambda, \mu \in (-r_*, r_*).$

**Proof.** (i) Indeed, by $f(0) = 0$ and Lemma 2.1(ii) we have $f(s)s \geq 0$ for all $s \in \mathbb{R}$. Thus, if $u$ is a solution, then
\[
\|u\|^2 = \lambda \int_{\Omega} f'(u) \, dx + \mu \int_{\Omega} f(u) \, dx \\
= \lambda \int_{\Omega} f'(u) \, dx + \mu \int_{\Omega} f(u) \, dx \\
\leq 0.
\]
Therefore, $u = 0$.

(ii) Suppose that $\lambda < 0$ and $u$ is a nontrivial weak solution of $(P'_{\lambda,\mu,q,p})$. By previous item, we have $\mu > 0$. By Lemma 2.1(v),
\[
\lambda \int_{\Omega} |f(u)|^q \, dx + \frac{\mu}{2} \int_{\Omega} |f(u)|^p \, dx < \|u\|^2.
\]
If $J_{\lambda,\mu}(u) \leq 0$, then
\[
\frac{1}{2} \|u\|^2 - \frac{\lambda}{q} \int_{\Omega} |f(u)|^q \, dx - \frac{\mu}{p} \int_{\Omega} |f(u)|^p \, dx \leq 0.
\]
Thus,
\[
\|u\|^2 \leq \frac{2\lambda}{q} \int_{\Omega} |f(u)|^q \, dx + \frac{2\mu}{p} \int_{\Omega} |f(u)|^p \, dx.
\] (3.2)
By comparing (3.1) and (3.2), we get
\[
0 \leq \lambda \left(1 - \frac{2}{q}\right) \int_{\Omega} |f(u)|^q \, dx + \mu \left(\frac{1}{2} - \frac{2}{p}\right) \int_{\Omega} |f(u)|^p \, dx < 0,
\]
whenever $1 < q \leq 2$ and $p \geq 4$. This is a clear contradiction.

Now, let $\mu < 0$ and $u$ be a weak solution of $(P'_{\lambda,\mu,q,p})$. In an analogous way to the first part, by using Lemma 2.1(v) and the fact that $J_{\lambda,\mu}(u) \geq 0$, we get
\[
0 \leq \lambda \left(1 - \frac{2}{q}\right) \int_{\Omega} |f(u)|^q \, dx + \mu \left(\frac{1}{2} - \frac{2}{p}\right) \int_{\Omega} |f(u)|^p \, dx \leq 0,
\]
for all $1 < q \leq 2$ and $p \geq 4$. The result follows.

(iii) If $\max\{2, q\} < p \leq 4$, $\lambda < 0$ and $u$ is a nontrivial weak solution of $(P'_{\lambda,\mu,q,p})$, then, by $f(0) = 0$ and Lemma 2.1(ii), $f(s)s \geq 0$ for all $s \in \mathbb{R}$. Moreover, by item (i), we have $\mu > 0$. Thence,
\[
\|u\|^2 \leq \mu \int_{\Omega} f'(u) |f(u)|^{p-1} |u| \, dx.
\]
By Lemma 2.1(v),
\[
\|u\|^2 \leq \mu \int_{\Omega} |f(u)|^p \, dx.
\] (3.3)
It follows from items (v) and (vi) of Lemma 2.1 that
\[
|f(s)| \leq (8/\alpha^2)^{1/4} |s|^{1/2},
\]
for all $|s| > 1$. Thus, by Lemma 2.1(iv) and since $2 \leq p \leq 4$,
\[
\int_{\Omega} |f(u)|^p \, dx \leq \int_{\|u\| \leq 1} |u|^p \, dx + (8/\alpha^2)^{p/4} \int_{\|u\| > 1} |u|^{p/2} \, dx
\leq \int_{\|u\| \leq 1} |u|^2 \, dx + (8/\alpha^2)^{p/4} \int_{\|u\| > 1} |u|^{p/2} \, dx
\leq \int_{\|u\| \leq 1} |u|^2 \, dx + (8/\alpha^2)^{p/4} \int_{\|u\| > 1} |u|^2 \, dx.
\] (3.4)
By (3.3), (3.4) and Sobolev embeddings,
\[
\|u\|^2 \leq \mu [1 + (8/\alpha^2)^{p/4}]\|u\|^2 \leq \mu [1 + (8/\alpha^2)^{p/4}]C_1 \|u\|^2.
\] (3.5)
Since $u$ is a nontrivial solution, we obtain
\[
0 < \frac{1}{[1 + (8/\alpha^2)^{p/4}]C_1} =: \mu_* \leq \mu.
\] (3.6)
To prove the second part, suppose that $\lambda > 0$ and $u$ is a nontrivial solution with $J_{\lambda, \mu}(u) \geq 0$. It follows from Lemma 2.1(v) that
\[
\|u\|^2 \leq \lambda \int_{\Omega} |f(u)|^q \, dx + |\mu| \int_{\Omega} |f(u)|^p \, dx \\
\leq \frac{q}{2} \|u\|^2 + |\mu| \left( 1 + \frac{q}{p} \right) |f(u)|^p \, dx.
\]
Consequently,
\[
\left( 1 - \frac{q}{2} \right) \|u\|^2 \leq |\mu| \left( 1 + \frac{q}{p} \right) |f(u)|^p \, dx.
\]
As $2 \leq p \leq 4$, by (3.4),
\[
\left( 1 - \frac{q}{2} \right) \|u\|^2 \leq |\mu| \left( 1 + \frac{q}{p} \right) [1 + (8/\alpha^2)^{q/4}] C_1 \|u\|^2.
\]
Since $1 < q < 2$, we have
\[
0 < \frac{\left( 1 - \frac{q}{2} \right)}{\left( 1 + \frac{q}{p} \right) [1 + (8/\alpha^2)^{q/4}] C_1} \leq |\mu|.
\]
The result is proved.

(iv) Let $2 \leq q < 4$, $\mu < 0$ and $u$ be a nontrivial weak solution of $(P'_{\lambda, \mu, q, p})$, by Lemma 2.1(v)
\[
\|u\|^2 \leq \lambda \int_{\Omega} |f(u)|^q \, dx.
\]
By item (i), (3.4) and Sobolev embeddings,
\[
\|u\|^2 \leq \lambda [1 + (8/\alpha^2)^{q/4}] C_1 \|u\|^2.
\]
(3.7)
Since $u$ is a nontrivial solution, we obtain
\[
0 < \frac{1}{[1 + (8/\alpha^2)^{q/4}] C_1} =: \lambda_* \leq \lambda.
\]
(3.8)
Finally, suppose that $\mu > 0$ and $u$ is a nontrivial solution with $J_{\lambda, \mu}(u) \leq 0$. It follows from Lemma 2.1(v) that
\[
\|u\|^2 \geq -|\lambda| \int_{\Omega} |f(u)|^q \, dx + \frac{\mu}{2} \int_{\Omega} |f(u)|^p \, dx \\
\geq \frac{p}{4} \|u\|^2 - |\lambda| \left( 1 + \frac{p}{2q} \right) \int_{\Omega} |f(u)|^q \, dx.
\]
Since $p < 4$,
\[
0 < \left( 1 - \frac{p}{4} \right) \|u\|^2 \leq |\lambda| \left( 1 + \frac{p}{2q} \right) \int_{\Omega} |f(u)|^q \, dx.
\]
Since $2 \leq q < 4$, by (3.4)
\[
\left( 1 - \frac{p}{4} \right) \|u\|^2 \leq |\lambda| \left( 1 + \frac{p}{2q} \right) [1 + (8/\alpha^2)^{q/4}] C_1 \|u\|^2.
\]
Therefore,

$$0 < \frac{(1 - \frac{q}{p})}{\left(1 + \frac{q}{2q}\right)[1 + (8/\alpha^2)^{q/4}]C_1} \leq |\lambda|.$$ 

(v) Let $2 \leq q < p \leq 4$ and $u$ be a nontrivial weak solution of $(P_{\lambda,\mu,\rho,\sigma})$. By Lemma 2.1(v) and (3.4),

$$\|u\|^2 \geq |\lambda| \int_{\Omega} |f(u)|^q dx + |\mu| \int_{\Omega} |f(u)|^p dx \geq \left[|\lambda|[1 + (8/\alpha^2)^{q/4}]C_1 + |\mu|[1 + (8/\alpha^2)^{p/4}]C_2\right] \|u\|^2.$$

Since $u$ is nontrivial, the result follows. $\square$

4. Multiplicity of solutions

The proof of the existence results will be divided in several propositions. Before, we need to introduce some definitions. We say that $J_{\lambda,\mu}$ satisfies the $(PS)^*_c$ condition, with respect to $\{Y_n\}$, if any sequence $\{u_n\} \subset H_0^1(\Omega)$, such that

$$u_n \in Y_n, J_{\lambda,\mu}(u_n) \rightarrow c \quad \text{and} \quad (J_{\lambda,\mu}|_{Y_n})'(u_n) \rightarrow 0$$

contains a subsequence converging to a critical point of $J_{\lambda,\mu}$. Any sequence $\{u_n\} \subset H_0^1(\Omega)$ satisfying (4.1) is said to be a $(PS)^*_c$ for $J_{\lambda,\mu}$. It is well known that the $(PS)^*_c$ condition implies the classical $(PS)_c$ condition; see [23].

**Proposition 4.1.** Suppose $(\vartheta_1) - (\vartheta_3)$ hold.

(i) If $p = 4$, then $J_{\lambda,\mu}$ satisfies the $(PS)^*_c$ condition, for all $1 < q < 4$, $\lambda \in \mathbb{R}$ and $\mu < \lambda^2/4$;

(ii) if $p \neq 4$, then $J_{\lambda,\mu}$ satisfies the $(PS)^*_c$ condition, for all $1 < q < \min\{4, p\}$ and $\lambda, \mu \in \mathbb{R}$.

**Proof.** (i) Let $p = 4$ and $\{u_n\}$ be a $(PS)^*_c$ sequence for $J_{\lambda,\mu}$, i.e. (4.1) holds. If $\lambda > 0$ and $\mu \leq 0$, it follows by Lemma 2.1(v) that

$$C + C_0\|u_n\| \geq J_{\lambda,\mu}(u_n) - \frac{1}{p}(J_{\lambda,\mu}|_{Y_n})'(u_n)u_n \geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{2p}\right) \int_{\Omega} |f(u_n)|^q dx.$$

Now, we have to consider two cases: if $1 < q \leq 2$, we conclude from Lemma 2.1(iv) and Sobolev embedding that

$$C + C_0\|u_n\| \geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{2p}\right) C_1\|u_n\|^q. \quad (4.2)$$

Before considering the case $2 < q < 4$, observe that we cannot use Lemma 2.1(iv) in the same way as previously because $|u|^q$ might not be integrable. To overcome this difficulty, we note that, by items (v) and (vi) of Lemma 2.1

$$|f(s)| \leq (8/\alpha^2)^{1/4}|s|^{1/2}, \quad (4.3)$$

for all $s \in \mathbb{R}$. By Lemma 2.1(iv), for each $2 \leq r \leq 22^*$,

$$\int_{\Omega} |f(u)|^r dx \leq (8/\alpha^2)^{r/4} \int_{\Omega} |u|^{r/2} dx. \quad (4.4)$$

Thus, if $2 < q < 4$, it follows from (4.4) and Sobolev embedding that

$$C + C_0\|u_n\| \geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{2p}\right) (8/\alpha^2)^{q/4} C_1\|u_n\|^{q/2}. \quad (4.5)$$
By (4.2) and (4.5), \{u_n\} is bounded in \(H^1_0(\Omega)\). If \(\lambda, \mu > 0\), by Lemma 2.1(v), (4.4) and Sobolev embedding, we have

\[
C + C_0\|u_n\| \geq J_{\lambda, \mu}(u_n) - \frac{1}{4}(J_{\lambda, \mu}|_{Y_n})'(u_n)u_n
\]

\[
\geq \left(\frac{1}{4} - \frac{\mu}{\lambda_1 \alpha^2}\right)\|u\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{8}\right)\int_{\Omega} |f(u_n)|^q \, dx.
\]

Hence \{u_n\} is bounded in \(H^1_0(\Omega)\), if \(\mu < \lambda_1 \alpha^2/4\).

On the other hand, if \(\lambda, \mu \leq 0\) we get

\[
C + C_0\|u_n\| \geq J_{\lambda, \mu}(u_n) - \frac{1}{p}(J_{\lambda, \mu}|_{Y_n})'(u_n)u_n \geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{8}\right)\int_{\Omega} |f(u_n)|^q \, dx,
\]

showing that \{u_n\} is bounded in \(H^1_0(\Omega)\). If \(\lambda \leq 0\) and \(\mu > 0\),

\[
C + C_0\|u_n\| \geq J_{\lambda, \mu}(u_n) - \frac{1}{4}(J_{\lambda, \mu}|_{Y_n})'(u_n)u_n
\]

\[
\geq \left[\frac{1}{4} - \frac{\mu}{\lambda_1 \alpha^2}\right]\|u\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{4}\right)\int_{\Omega} |f(u_n)|^q \, dx.
\]

Therefore, \{u_n\} is again bounded in \(H^1_0(\Omega)\), if \(\mu < \lambda_1 \alpha^2/4\). Hence, up to a subsequence, we have

\[
u_n \rightarrow u \text{ in } H^1_0(\Omega),
\]

\[
\int_{\Omega} f'(u_n)|f(u_n)|^{q-2} f(u_n)(u_n - u) \, dx \rightarrow 0
\]

and

\[
\int_{\Omega} f'(u_n)|f(u_n)|^{p-2} f(u_n)(u_n - u) \, dx \rightarrow 0.
\]

Defining \(v_n := P_{Y_n}u\) as been the orthogonal projection of \(u\) onto \(Y_n\), we have

\[
v_n \rightarrow u \text{ in } H^1_0(\Omega).
\]

Since \(u_n - v_n \in Y_n\) and \{\(u_n - v_n\)\} is bounded in \(H^1_0(\Omega)\), we conclude that

\[
(J_{\lambda, \mu}|_{Y_n})'(u_n)(u_n - v_n) = o_n(1).
\]

Thence,

\[
\int_{\Omega} \nabla u_n \nabla (u_n - v_n) = \lambda \int_{\Omega} f'(u_n)|f(u_n)|^{q-2} f(u_n)(u_n - v_n) \, dx
\]

\[
+ \mu \int_{\Omega} f'(u_n)|f(u_n)|^{p-2} f(u_n)(u_n - v_n) \, dx + o_n(1).
\]

By (4.6), (4.7), (4.8) and (4.9), we conclude that

\[
\|u_n\|^2 = \|v_n\|^2 + o_n(1).
\]

The result follows now from (4.6) and (4.9).

(ii) Let \(p \neq 4\) and \{\(u_n\)\} be a \((PS)^*_c\) sequence for \(J_{\lambda, \mu}\). If \(\lambda > 0\) and \(\mu \leq 0\), we can reason exactly as in the case \(p = 4\). On the other hand, if \(\lambda, \mu > 0\) we have to consider separately two cases: if \(p < 4\), it follows by Lemma 2.1(v), (4.4) and Sobolev embedding that

\[
C + C_0\|u_n\| \geq J_{\lambda, \mu}(u_n) - \frac{1}{p}(J_{\lambda, \mu}|_{Y_n})'(u_n)u_n
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{p} \right) \|u_n\|^2 - \frac{\mu}{2p} (8/\alpha^2)^{p/4} C_1 \|u_n\|^{p/2} - \lambda \left( \frac{1}{q} - \frac{1}{2p} \right) \int_\Omega |f(u_n)|^q \, dx.
\]

By estimating the last part as (4.2) and (4.5), we conclude that \( \{u_n\} \) is bounded in \( H_0^1(\Omega) \). In the case \( p > 4 \), it is sufficient to note that by Lemma 2.1(v)
\[
C + C_0 \|u_n\| \geq J_{\lambda,\mu}(u_n) - \frac{2}{p} (J_{\lambda,\mu}|Y_n)'(u_n) u_n \geq \left( \frac{1}{2} - \frac{2}{p} \right) \|u_n\|^2 - \lambda \left( \frac{1}{q} - \frac{1}{2p} \right) \int_\Omega |f(u_n)|^q \, dx.
\]

Once more time, the boundedness of \( \{u_n\} \) in \( H_0^1(\Omega) \) follows from a reasoning similar to (4.2) and (4.5).

Finally, if \( \lambda, \mu \leq 0 \), we argue exactly as in the case \( p = 4 \), and if \( \lambda \leq 0 \) and \( \mu > 0 \), we have
\[
C + C_0 \|u_n\| = J_{\lambda,\mu}(u_n) - \frac{1}{p} (J_{\lambda,\mu}|Y_n)'(u_n) u_n
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{p} \right) \|u_n\|^2 - \frac{\mu}{2p} (8/\alpha^2)^{p/4} C_1 \|u_n\|^{p/2} - \lambda \left( \frac{1}{q} - \frac{1}{2p} \right) \int_\Omega |f(u_n)|^q \, dx,
\]
when \( p < 4 \), and
\[
C + C_0 \|u_n\| \geq J_{\lambda,\mu}(u_n) - \frac{2}{p} (J_{\lambda,\mu}|Y_n)'(u_n) u_n \geq \left( \frac{1}{2} - \frac{2}{p} \right) \|u_n\|^2 - \lambda \left( \frac{1}{q} - \frac{2}{p} \right) \int_\Omega |f(u_n)|^q \, dx,
\]
when \( p > 4 \). In all cases, we can conclude that \( \{u_n\} \) is bounded in \( H_0^1(\Omega) \). Now the result follows exactly equal to the case \( p = 4 \). \( \square \)

**Proposition 4.2.** Suppose \((\vartheta_1) - (\vartheta_3)\), \( 4 < p < 22^* \) and \( \mu > 0 \). Then there exist \( 0 < r_k < \rho_k \) such that
\[
\max_{u \in Y_k, \|u\| = \rho_k} J_{\lambda,\mu}(u) \leq 0 \quad (4.11)
\]
and
\[
\inf_{u \in Z_k, \|u\| = r_k} J_{\lambda,\mu}(u) \to \infty \text{ as } k \to \infty. \quad (4.12)
\]

**Proof.** To prove (4.11), observe that by Lemma 2.1(v)
\[
|f(s)| \geq f(1)|s|^{1/2}, \text{ if } |s| > 1.
\]
Thus, for each \( u \in S_k \) and \( \rho > 0 \)
\[
J_{\lambda,\mu}(\rho u) \leq \frac{1}{2} \rho^2 + \frac{\lambda}{q} \int_\Omega |f(\rho u)|^q \, dx - \frac{\mu}{p} f(1)^p \rho^{p/2} \int_{|s| < |\rho u|} |s|^{p/2} \, ds.
\]

By Lemma 2.3(ii), there exist positive constants \( \alpha_k, \beta_k(p/2) \) such that, for every \( u \in S_k \) and \( \rho > \alpha_k \), we get
\[
J_{\lambda,\mu}(\rho u) \leq \frac{1}{2} \rho^2 + \frac{\lambda}{q} \int_\Omega |f(\rho u)|^q \, dx - \frac{\mu}{p} f(1)^p \beta_k(p/2) \rho^{p/2}. \quad (4.13)
\]
Now, we are going to consider two cases: if \( 1 < q \leq 2 \), it follows from Lemma 2.1(iv) and Sobolev embedding that
\[
J_{\lambda,\mu}(\rho u) \leq \frac{1}{2} \rho^2 + \frac{\lambda}{q} C_1 \rho^q - \frac{\mu}{p} f(1)^p \beta_k(p/2) \rho^{p/2}.
\]
Since $p > 4$, choosing $\rho_k > \max\{1, \lfloor p/(2 + |\lambda|C_1/q)/\mu f(1)p\beta_k(p/2)\rfloor^{2/(p-4)}\}$, we have

$$J_{\lambda, \mu}(\rho_k u) \leq \left(\frac{1}{2} + \frac{|\lambda|}{q} C_1\right) \rho_k^2 - \frac{\mu}{p} f(1)p\beta_k(p/2)\rho_k^{p/2} < 0,$$

for all $u \in S_k$. On the other hand, if $2 < q < 4$, by (4.13), (4.4) and Sobolev embedding, we have

$$J_{\lambda, \mu}(\rho u) \leq \frac{1}{2} \rho^2 + \frac{|\lambda|}{q} (8/\alpha^2)^{q/4} C_1 \rho^{q/2} - \frac{\mu}{p} f(1)p\beta_k(p/2)\rho^{p/2}.$$

Therefore, choosing $\rho_k > \max\{1, \lfloor p/(2 + |\lambda|C_1/q)/\mu f(1)p\beta_k(p/2)\rfloor^{2/(p-4)}\}$, we have

$$J_{\lambda, \mu}(\rho_k u) \leq \left(\frac{1}{2} + \frac{|\lambda|}{q} (8/\alpha^2)^{q/4} C_1\right) \rho_k^2 - \frac{\mu}{p} f(1)p\beta_k(p/2)\rho_k^{p/2} < 0,$$

for all $u \in S_k$. This proves (4.11).

To prove (4.12), note that for any $1 \leq r < 2^*$, we can define

$$\theta_{r,k} := \sup_{u \in Z_k \setminus \{0\}} \frac{|u|_r}{\|u\|}.$$

It is a straightforward consequence of compact Sobolev embeddings that

$$\theta_{r,k} \to 0 \text{ as } k \to \infty;$$

see Lemma 3.8 in [23].

If $1 < q < 2$, by Lemma 2.1(iv) and (4.4)

$$J_{\lambda, \mu}(u) \geq \frac{1}{2} \|u\|^2 - \frac{|\lambda|}{q} \int_{\Omega} |u|^q \, dx - \frac{\mu}{p} (8/\alpha^2)^{p/4} \int_{\Omega} |u|^{p/2} \, dx.$$

By Sobolev embeddings and (4.14),

$$J_{\lambda, \mu}(u) \geq \frac{1}{2} \|u\|^2 - \frac{|\lambda|}{q} C_1 \|u\|^q - \frac{\mu}{p} (8/\alpha^2)^{p/4} \theta_{p/2,k}^2 \|u\|^{p/2},$$

for all $u \in Z_k$. Since $1 < q < 2$, for $\|u\| \geq R_*$ with $R_* > 0$ large enough,

$$\frac{|\lambda|}{q} C_1 \|u\|^q < \frac{1}{r} \|u\|^2,$$

for some $r > 2p/(p-2)$. Thus, for $\|u\| \geq R_*$, we get

$$J_{\lambda, \mu}(u) \geq \left(\frac{1}{2} - \frac{1}{r}\right) \|u\|^2 - \frac{\mu}{p} (8/\alpha^2)^{p/4} \theta_{p/2,k}^2 \|u\|^{p/2}.\quad(4.16)$$

It follows from (4.15) that, by choosing $r_k = 1/\mu(8/\alpha^2)^{p/4} \theta_{p/2,k}^{2/(p-4)}$, there exists $k_0 \in N$ such that $r_k \geq R_*$ for all $k \geq k_0$. Therefore,

$$J_{\lambda, \mu}(u) \geq \left(\frac{r - 2}{2r} - \frac{1}{p}\right) \rho_k^2,$$

for all $u \in Z_k$ with $\|u\| = r_k$ and $k \geq k_0$. Since $r_k \to \infty$ as $k \to \infty$, the result follows. If $2 \leq q < 4$, it follows from (4.4) that

$$J_{\lambda, \mu}(u) \geq \frac{1}{2} \|u\|^2 - \frac{|\lambda|}{q} (8/\alpha^2)^{q/4} \int_{\Omega} |u|^{q/2} \, dx - \frac{\mu}{p} (8/\alpha^2)^{p/4} \int_{\Omega} |u|^{p/2} \, dx.$$

By Sobolev embeddings and (4.14),

$$J_{\lambda, \mu}(u) \geq \frac{1}{2} \|u\|^2 - \frac{|\lambda|}{q} (8/\alpha^2)^{q/4} C_1 \|u\|^q - \frac{\mu}{p} (8/\alpha^2)^{p/4} \theta_{p/2,k}^2 \|u\|^{p/2}.\quad(4.17)$$
Now, since $1 \leq q/2 < 2$, we can proceed in an analogous way to the case $1 < q < 2$ for the choice of $r_k$. Since we can choose $\rho_k$ even greater, in order to have $\rho_k > r_k$, the result follows. \hfill \Box

**Proposition 4.3.** Suppose that $\vartheta$ satisfies $(\vartheta_1) - (\vartheta_3)$, $1 < q < 2$ and $\lambda > 0$ hold. Then, there exist $0 < r_k < \rho_k$ such

(i) $\inf_{u \in Z_k, \|u\| = \rho_k} J_{\lambda, \mu}(u) \geq 0$;

(ii) $\max_{u \in Y_k, \|u\| = r_k} J_{\lambda, \mu}(u) < 0$;

(iii) $\inf_{u \in Z_k, \|u\| \leq \rho_k} J_{\lambda, \mu}(u) \to 0$ as $k \to \infty$.

**Proof.**

(i) Let us consider $p \geq 4$. Since $1 < q < 2$, by Lemma 2.1(iv), (4.4) and (4.14), we get

$$J_{\lambda, \mu}(u) \geq \frac{1}{2} \|u\|^2 - \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx - \frac{|\mu|}{p} (8/\alpha^2)^{p/4} \int_{\Omega} |u|^{p/2} \, dx$$

$$\geq \frac{1}{2} \|u\|^2 - \frac{\lambda}{q} \theta_{q,k} \|u\|^q - \frac{|\mu|}{p} (8/\alpha^2)^{p/4} \rho_{p/2,k}^{p/2} \|u\|^{p/2},$$

for all $u \in Z_k$. If $p \geq 4$, there exists $\delta > 0$ small enough, such that

$$\frac{|\mu|}{p} (8/\alpha^2)^{p/4} \rho_{p/2,k}^{p/2} \|u\|^{p/2} \leq \frac{1}{4} \|u\|^2,$$

for all $u \in Z_k$ with $\|u\| \leq \delta$ (and $k$ large enough if $p = 4$). Thus, by choosing

$$\rho_k = (4\lambda \theta_{q,k}/q)^{1/(2-q)},$$

we have $(1/4)\rho_k^2 = (\lambda/q)\theta_{q,k}^2 \rho_k^2$. Consequently, $\rho_k \to 0$ as $k \to \infty$, and therefore, there exists $k_0 > 0$ satisfying $\rho_k \leq \delta$ for all $k \geq k_0$. Finally, by (4.19)

$$J_{\lambda, \mu}(u) \geq \frac{1}{4} \|u\|^2 - \frac{\lambda}{q} \theta_{q,k} \|u\|^q = 0,$$

for all $u \in Z_k, k \geq k_0$, with $\|u\| = \rho_k$. On the other hand, if $2 < p < 4$, we conclude from (4.18) that

$$J_{\lambda, \mu}(u) \geq \frac{1}{2} \|u\|^2 - \left[ \frac{\lambda}{q} + \frac{|\mu|}{p} (8/\alpha^2)^{p/4} \right] \eta_k \|u\|^\gamma,$$

for all $u \in Z_k$ with $\|u\| < 1, 1 < \gamma := \min\{q,p/2\} < 2$, $\eta_k := \max\{\theta_{q,k}, \theta_{p/2,k}\}$ and $k \geq k_0$. Thus, by choosing

$$\rho_k = \left( 2(\lambda/q) + |\mu| (8/\alpha^2)^{p/4} / p \eta_k^\gamma \right)^{1/(2-\gamma)},$$

with $k \geq k_0$, the result follows.

(ii) By Lemma 2.1(iii), there exists $\varepsilon > 0$ such that

$$|f(s)| \geq \varepsilon |s|,$$

for all $|s| \leq 1$. Thus,

$$J_{\lambda, \mu}(u) \leq \frac{1}{2} \|u\|^2 - \frac{\lambda}{q} \varepsilon \int_{\|u\| \leq 1} |u|^q \, dx + \frac{|\mu|}{p} \int_{\Omega} |f(u)|^p \, dx.$$

By the second part of Lemma 2.3(i) and Lemma 2.1(iv), we have

$$J_{\lambda, \mu}(ru) \leq \frac{1}{2} r^2 - \frac{\lambda}{q} \varepsilon \int_{\Omega} |ru|^q \, dx + \frac{|\mu|}{p} \int_{\Omega} |ru|^p \, dx,$$
for all $u \in S_k$ and $0 < r < \tau_k$. Since $Y_k$ has finite dimension, there exists $\zeta_k(q) > 0$ such that

$$J_{\lambda,\mu}(ru) \leq \frac{1}{2} r^2 - \frac{\lambda}{q} \varepsilon q \zeta_k(q) r^q + \frac{[\mu]}{p} \int |ru|^2 \, dx,$$

for all $u \in S_k$ and $0 < r < \tau_k$, where in the last part we use the fact that $p > 2$. By Sobolev embeddings,

$$J_{\lambda,\mu}(ru) \leq \frac{1}{2} r^2 - \frac{\lambda}{q} \varepsilon q \zeta_k(q) r^q + \frac{[\mu]}{p} C_1 r^2.$$

Thence,

$$J_{\lambda,\mu}(ru) \leq \left( \frac{1}{2} + \frac{[\mu]}{p} C_1 \right) r^2 - \frac{\lambda}{q} \varepsilon q \zeta_k(q) r^q,$$

for all $0 < r < \min\{1, \rho_k, \tau_k\}$. Since $1 < q < 2$, by choosing

$$0 < r_k < \min\{1, \tau_k, \rho_k, [\lambda \varepsilon q \zeta_k(q)]^{1/(2-q)} \},$$

the item is proved.

(iii) By (4.20) and (4.21), we conclude that

$$o_k(1) \leq b_k := \inf_{u \in Z_k, \|u\| \leq \rho_k} J_{\lambda,\mu}(u) \leq J_{\lambda,\mu}(0) = 0,$$

where $o_k(1) \to 0$ as $k \to \infty$. Consequently, $b_k \to 0$ as $k \to \infty$. \hfill \qedsymbol

Proof of Theorem 1.1(i): Since $J_{\lambda,\mu}$ is an even functional, the first part of Theorem 1.1(i) is a direct consequence of fountain theorem in [23] and Propositions 4.1(ii) and 4.2. To prove the second part, observe that if $1 < q < 2$, it follows from $\mu > 0$, Lemma 2.1(iv), (4.4) and Sobolev embeddings that

$$J_{\lambda,\mu}(u) \geq \frac{1}{2} \|u\|^2 - \frac{[\lambda]}{q} C_1 \|u\|^q - \frac{\mu}{p} (8/\alpha^2)^{p/4} \theta^{p/2}_{p/2,m} \|u\|^{p/2},$$

for all $u \in Z_m$. On the other hand, if $2 \leq q < 4$, it follows from $\mu > 0$, (4.4) and Sobolev embeddings that

$$J_{\lambda,\mu}(u) \geq \frac{1}{2} \|u\|^2 - \frac{[\lambda]}{q} (8/\alpha^2)^{q/4} C_2 \|u\|^q - \frac{\mu}{p} (8/\alpha^2)^{p/4} \theta^{p/2}_{p/2,m} \|u\|^{p/2},$$

for all $u \in Z_m$. Consequently,

$$J_{\lambda,\mu}(u) \geq \frac{1}{2} \|u\|^2 - \frac{[\lambda]}{q} C_3 \|u\|^\alpha(q) - \frac{\mu}{p} (8/\alpha^2)^{p/4} \theta^{p/2}_{p/2,m} \|u\|^{p/2},$$

where $\alpha : (1, 4) \to [1, 2)$ is give by $\alpha(s) = s$ if $1 < s < 2$ and $\alpha(s) = s/2$ if $2 \leq s < 4$. Thence, there exists $R_*$ large enough such that

$$\frac{1}{4} \|u\|^2 \geq \frac{[\lambda]}{q} C_3 \|u\|^\alpha(q),$$

for all $u \in Z_m$ with $\|u\| \geq R_*$. Since $p < 4$,

$$J_{\lambda,\mu}(u) \geq \left[ \frac{1}{4} - \frac{\mu}{p} (8/\alpha^2)^{p/4} \theta^{p/2}_{p/2,m} \right] \|u\|^{p/2},$$

for all $u \in Z_m$ with $\|u\| \geq \max\{R_*, 1\}$. Observe that there exists $m_0 > 0$ such that

$$\frac{1}{4} \geq \frac{\mu}{p} (8/\alpha^2)^{p/4} \theta^{p/2}_{p/2,m},$$

for all $m \geq m_0$. By choosing $r_m = \max\{R_*, m\}$, we have

$$\inf_{u \in Z_m, \|u\| = r_m} J_{\lambda,\mu}(u) \to \infty \text{ as } m \to \infty. \quad (4.22)$$
Finally, by items (iv) and (v) of Lemma 2.1 and (4.4), there exists $C > 0$ such that
\[ J_{\lambda,\mu}(\rho u) \leq \frac{\rho^2}{2} + \frac{|\lambda|}{q} C \rho^\alpha(\|u\|) \int_\omega |u|^\alpha(\|u\|)dx - \frac{\mu}{p} f(1)^p \rho^{p/2} \int_{|\rho u| > 1} |u|^{p/2}dx, \]
for all $u \in S_m$. It follows from Lemma 2.3(ii) and Sobolev embedding that there exists $\alpha_m, \beta_m(p/2) > 0$ such that
\[ J_{\lambda,\mu}(\rho_m u) \leq \frac{\rho_m^2}{2} + \frac{|\lambda|}{q} C \rho_m^\alpha(\|u\|) - \frac{\mu}{p} f(1)^p \beta_m(p/2) \rho_m^{p/2}, \]
for some $\rho_m > \max\{\alpha_m, r_m\}$ and for all $u \in S_m$. Therefore, there exists $\mu_m > 0$ such that
\[ \max_{u \in Y_m, \|u\| = \rho_m} J_{\lambda,\mu}(u) \leq 0, \quad (4.23) \]
for all $\mu > \mu_m$. To finish the proof, let us define
\[ B_m = \{ u \in Y_m : \|u\| \leq \rho_m \}, \]
\[ \Gamma_m = \{ \gamma \in C(B_m, H^1_0(\Omega)) : \gamma \text{ is odd and } \gamma|_{\partial B_m} = id \} \]
and
\[ c_m = \inf_{\gamma \in \Gamma_m} \max_{u \in B_m} J_{\lambda,\mu}(\gamma(u)). \]
By definition of $c_m$ and Lemma 3.4 in [23], we have
\[ \infty > c_m \geq \inf_{u \in Z_m, \|u\| = r_m} J_{\lambda,\mu}(u), \quad (4.24) \]
for all $m$. On the other hand, by (4.22), we conclude that
\[ \inf_{u \in Z_m, \|u\| = r_m} J_{\lambda,\mu}(u) > 0, \]
for all $m \geq m_0$. It is also a consequence of (4.22) and (4.24) that given $k \in \mathbb{N}$, there exists $m(k) > m_0$ with $k \leq m(k) - m_0$, such that we have at least $k$ different numbers $c_j$ when $m_0 \leq j \leq m(k)$. Thus, by (4.23) and Theorem 3.5 in [23], there exist $\mu_k := \mu_{m(k)} > 0$ and a $(PS)_{c_j}$-sequence for $J_{\lambda,\mu}$, for each $m_0 \leq j \leq m(k)$, whenever $\mu > \mu_k$. Finally, by Proposition 4.1(ii), it follows that the numbers $c_j$ are critical points of $J_{\lambda,\mu}$ as $\mu > \mu_k$. 

Proof of Theorem 1.1(ii): Since $J_{\lambda,\mu}$ is an even functional, the proof of first part of Theorem 1.1(ii) follows from dual fountain theorem in [23] and Propositions 4.1(ii) and 4.3. To prove the second part, note that, since $2 \leq q < 4$ and $\lambda > 0$, it follows by (4.4) and Sobolev embeddings that
\[ J_{\lambda,\mu}(u) \geq \frac{1}{2} \|u\|^2 - \frac{\lambda}{q} (8/\alpha^2)^{q/4} \|u\|^{q/2} - \frac{|\lambda|}{p} (8/\alpha^2)^{p/4} \|u\|^{p/2}, \]
for all $u \in Z_m$. Thus, for $m$ large enough, we have $0 < \eta_m := \max\{\theta_{q/2,m}, \theta_{p/2,m}\} < 1$ and
\[ J_{\lambda,\mu}(u) \geq \frac{1}{2} \|u\|^2 - \left( \frac{\lambda}{q} + \frac{|\lambda|}{p} \right) (8/\alpha^2)^{\eta_m^2} \|u\|^{q/2}, \quad (4.25) \]
for all $u \in Z_m$ with $\|u\| < 1$. By choosing $\rho_m = \left[ 2 (\lambda/q + |\lambda|/p) (8/\alpha^2)^{\eta_m^2} \right]^{2/(4-q)}$, it follows that for $m \geq m_*$, with $m_*$ large enough
\[ \inf_{u \in Z_m, \|u\| = \rho_m} J_{\lambda,\mu}(u) \geq 0. \quad (4.26) \]
On the other hand, by Lemma 2.1(iii) and (4.4)
\[ J_{\lambda,\mu}(ru) \leq \frac{r^2}{2} - \frac{\lambda}{q} \varepsilon^q \int_{|ru| \leq 1} |ru|^q dx + \frac{|\lambda|}{p} (8/\alpha^2)^{2\varepsilon^2} \int_{\Omega} |u|^{p/2} dx, \]
for all \( u \in S_m \). It follows from Lemma 2.3(i) that there exists \( \tau_m > 0 \) such that

\[
J_{\lambda,\mu}(r_m u) \leq \frac{r_m^2}{2} - \frac{\lambda}{q} r_m \int_{\Omega} |u|^q dx + \frac{|\mu|}{p} (8/\alpha^2)^p/4 r_m^p/2 \int_{\Omega} |u|^{p/2} dx,
\]

for some \( 0 < r_m < \min\{\tau_m, \rho_m\} \) fixed and for all \( u \in S_m \). Although \( q \) can be greater than \( 2^* \) when the dimension \( N \) is large enough, it is a consequence of definition of \( Y_m \) that \( Y_m \subset L^\infty(\Omega) \), and therefore, \(|.|_q\) defines a norm in \( Y_m \). Since \( Y_m \) has finite dimension,

\[
J_{\lambda,\mu}(r_m u) \leq \frac{r_m^2}{2} - \frac{\lambda}{q} r_m \zeta_m(q) + \frac{|\mu|}{p} (8/\alpha^2)^p/4 C_1 r_m^p/2,
\]

for some \( \zeta_m(q) > 0 \). Therefore, there exists \( \lambda_m > 0 \) such that

\[
b_m := \max_{u \in Y_m, \|u\| = r_m} J_{\lambda,\mu}(u) < 0,
\]

for all \( \lambda > \lambda_m \).

Finally, by (4.25), we conclude that

\[
o_m(1) \leq \inf_{u \in Z_m, \|u\| \leq \rho_m} J_{\lambda,\mu}(u) \leq J_{\lambda,\mu}(0) = 0,
\]

where \( o_m(1) \rightarrow 0 \) as \( m \rightarrow \infty \), showing that

\[
d_m := \inf_{u \in Z_m, \|u\| \leq \rho_m} J_{\lambda,\mu}(u) \rightarrow 0 \text{ as } m \rightarrow \infty.
\]

To finish the proof, for each \( t \geq m \geq m_+ \), we are going to apply Theorem 3.5 in [23] to the functional \(-J_{\lambda,\mu}\) on \( Y_t \); for this, let us define:

\[
Z^t_m = \bigoplus_{j=m}^{t} X_j,
\]

\[
B^t_m = \{ u \in Z^t_m : \|u\| \leq \rho_m \},
\]

\[
\Gamma^t_m = \{ \gamma \in C(B^t_m, Y_m) : \text{\( \gamma \) is odd and } \gamma_{|\partial B^t_m} = id \}
\]

and

\[
c^t_m = \sup_{\gamma \in \Gamma^t_m} \min_{u \in B^t_m} J_{\lambda,\mu}(\gamma(u)).
\]

By definition of \( c^t_m \) and Lemma 3.4 in [23], we have

\[
d_m < c^t_m \leq b_m,
\]

for all \( t \geq m \geq m_+ \). Therefore, up to a subsequence, there exists

\[
c_m \in [d_m, b_m]
\]

such that

\[
c^t_m \rightarrow c_m \text{ as } t \rightarrow \infty.
\]

From (4.27), (4.28) and (4.30), given \( k \in \mathbb{N} \), there exist \( m(k) \) with \( k < m(k) - m_* \) and \( \lambda_k := \lambda_{m(k)} > 0 \) such that we have at least \( k \) different numbers \( c_m \) as \( m_* \leq m \leq m(k) \), whenever \( \lambda > \lambda_k \). Thus, by Theorem 3.5 in [23], for each \( m_* \leq m \leq m(k) \), there exists \( u_t \in Y_t \) such that

\[
c^t_m - 2/t \leq J_{\lambda,\mu}(u_t) \leq c^t_m + 2/t \text{ and } \|(J_{\lambda,\mu}|_{Y_t})(u_t)\| \leq 8/t,
\]

whenever \( \lambda > \lambda_k \). Consequently, by (4.31) and (4.32), up to a subsequence, \( \{u_t\} \) is a \((PS)^*_{c_m}\) sequence. By Proposition 4.1(ii), \( c_m \) is a critical point of \( J_{\lambda,\mu} \) for all \( m_* \leq m \leq m(k) \). The result follows. \( \square \)

**Proof of Theorem 1.1(iii):** It is sufficient to argue exactly as in the proof of the second part of Theorem 1.1(ii) and use Proposition 4.1(i) instead of Proposition 4.1(ii). \( \square \)
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