ON THE CATEGORY OF FINITELY GENERATED FREE GROUPS

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Abstract. It is well known that the opposite $\mathbf{F}^{\text{op}}$ of the category $\mathbf{F}$ of finitely generated free groups is a Lawvere theory for groups, and also that $\mathbf{F}$ is a free symmetric monoidal category on a commutative Hopf monoid, or, in other words, a PROP for commutative Hopf algebras. In this paper, we give a direct, combinatorial proof of the latter fact, without using Lawvere theories.

1. Introduction

A Lawvere theory [9] is a category $\mathcal{T}$ with finite products, equipped with an object $x$ such that every object of $\mathcal{T}$ is isomorphic to the $n$th power $x^n$ of $x$ for some $n \geq 0$. A $\mathcal{T}$-algebra or an algebra over $\mathcal{T}$ is a product preserving functor $A: \mathcal{T} \to \text{Set}$. It is well known that the opposite $\mathbf{F}^{\text{op}}$ of the category $\mathbf{F}$ of finitely generated free groups is a Lawvere theory, and that the $\mathbf{F}^{\text{op}}$-algebras are naturally identified with groups. In other words, $\mathbf{F}^{\text{op}}$ is a Lawvere theory for groups.

A Lawvere theory and its opposite category are (essentially) a special kind of PROPs. A PROP is a symmetric strict monoidal category $\mathcal{P}$ equipped with one object $x$ such that $\text{Ob}(\mathcal{P}) = \{x^\otimes n \mid n \geq 0\}$. For a fixed field $k$, a $\mathcal{P}$-algebra over $k$ is a symmetric monoidal functor $F: \mathcal{P} \to \text{Vect}_k$, where $\text{Vect}_k$ is the category of $k$-vector spaces. There are PROPs corresponding to many notions of “algebras” over $k$ such as (associative, unital) algebras, coalgebras, bialgebras, Hopf algebras, etc., see [10]. In this paper, we consider the PROP for commutative Hopf algebras, which is a PROP $\mathcal{H}$ such that the $\mathcal{H}$-algebras are identified with commutative Hopf algebras. The PROP $\mathcal{H}$ can be defined also as the free symmetric monoidal category generated by a commutative Hopf monoid. See Section 3 for the definition.

Pirashvili [11] proved that $\mathbf{F}^{\text{op}}$ is a PROP for cocommutative Hopf algebras, relying on the fact that $\mathbf{F}^{\text{op}}$ is the Lawvere theory for groups. Equivalently, $\mathbf{F}$ is a PROP for commutative Hopf algebras, i.e., we have an isomorphism of symmetric monoidal categories

$$\mathcal{H} \simeq \mathbf{F}. \quad (1)$$

In this paper, we will give a direct, self-contained proof of the isomorphism (1), without using Lawvere theories. Our proof is combinatorial in the sense that it does not involve the notion of (co)products in categories, which is used in the definition of Lawvere theories.

An advantage of this combinatorial proof is that it admits generalizations to categories, possibly without (co)products. In Section 7 we discuss such generalizations with motivations in topology. Our first motivation of writing this paper is to
provide a prototype for the proof of the results mentioned in Sections 7.2 and 7.3, but we also hope that our proof of (1) is simpler and easier to understand.

1.1. Organization of the paper. The rest of this paper is organized as follows. In Section 2, we recall the notions of Hopf monoids in symmetric monoidal categories and convolutions, and give some basic constructions. In Section 3, we define the category $H$ and a Hopf monoid $H$ in $H$, and prove some necessary results. For $m, n \geq 0$, we define a surjective monoid homomorphism $\alpha_{m,n}: F_{m,n} \rightarrow H(m, n)_{\text{conv}}$, where $F_{m,n}$ is the direct product of $m$ copies of the free group $F_n = \langle x_1, \ldots, x_n \rangle$ of rank $n$, and $H(m, n)_{\text{conv}}$ is the convolution monoid on the Hom set $H(m, n)$. In Section 4, we define the symmetric monoidal category $F$, a Hopf monoid $F$ in $F$, and a symmetric monoidal functor $T: H \rightarrow F$, which maps the Hopf monoid structure of $H$ to that of $F$. We construct a group isomorphism $\tau_{m,n}: F_{m,n} \xrightarrow{\sim} F(m, n)_{\text{conv}}$, where $F(m, n)_{\text{conv}}$ is the convolution monoid on $F(m, n)$. In Section 5, we prove that the functor $T: H \rightarrow F$ is an isomorphism by using the commutative diagram (29). In Section 6, we give a direct proof of the well-known fact that $H$ admits finite coproducts, without relying on the results for $F$. Section 7 is a brief account of categories closely related to $H$, which provides our motivation to study the direct combinatorial proof of $H \simeq F$.

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2. Hopf monoids in symmetric monoidal categories

Let $C = (C, \otimes, I, P)$ be a symmetric strict monoidal category, where $I$ denotes the unit object and

$$P_{X,Y}: X \otimes Y \xrightarrow{\sim} Y \otimes X$$

denotes the symmetry.

2.1. Hopf monoids. A Hopf monoid (also called Hopf algebra) in $C$ is an object $H$ in $C$ equipped with morphisms

$$\mu: H \otimes H \rightarrow H, \quad \eta: I \rightarrow H, \quad \Delta: H \rightarrow H \otimes H, \quad \epsilon: H \rightarrow I, \quad S: H \rightarrow H,$$

called the multiplication, unit, comultiplication, counit and antipode, respectively, satisfying

(1) $\mu(\mu \otimes H) = \mu(H \otimes \mu)$, $\mu(\eta \otimes H) = 1_H = \mu(H \otimes \eta)$,

(2) $(\Delta \otimes H) \Delta = (H \otimes \Delta) \Delta$, $(\epsilon \otimes H) \Delta = 1_H = (H \otimes \epsilon) \Delta$,

(3) $\epsilon \eta = 1_I$, $\epsilon \mu = \epsilon \otimes \epsilon$, $\Delta \eta = \eta \otimes \eta$,

(4) $\Delta \mu = (\mu \otimes \mu)(H \otimes P_{H,H} \otimes H)(\Delta \otimes \Delta)$,

(5) $\mu(H \otimes S) \Delta = \mu(S \otimes H) \Delta = \eta \epsilon$. 

H is said to be commutative if

$$\mu_{P_H,H} = \mu.$$  

As is well known, in a commutative Hopf monoid, the antipode $S$ is involutive: $S^2 = 1_H$.

The iterated multiplications and comultiplications

$$\mu^{[n]} : \mathcal{H}^{\otimes n} \to \mathcal{H}, \quad \Delta^{[n]} : \mathcal{H} \to \mathcal{H}^{\otimes n}$$

for $n \geq 0$ are defined inductively by

$$\mu^{[0]} = \eta, \quad \mu^{[1]} = 1_H, \quad \mu^{[n]} = \mu(\mu^{[n-1]} \otimes H) \quad (n \geq 2),$$

$$\Delta^{[0]} = \epsilon, \quad \Delta^{[1]} = 1_H, \quad \Delta^{[n]} = (\Delta^{[n-1]} \otimes H)\Delta \quad (n \geq 2).$$

They satisfy the generalized (co)associativity relations:

$$\mu^{[m]}(\mu^{[k_1]} \otimes \cdots \otimes \mu^{[k_m]}) = \mu^{[k_1+\cdots+k_m]},$$

$$\Delta^{[k_1]} \otimes \cdots \otimes \Delta^{[k_m]}\Delta^{[n]} = \Delta^{[k_1+\cdots+k_m]}$$

for $m \geq 0, k_1, \ldots, k_m \geq 0$.

For a sequence $n_1, \ldots, n_p \geq 0, p \geq 0$, we set

$$\mu^{[n_1,\ldots,n_p]} = \mu^{[n_1]} \otimes \cdots \otimes \mu^{[n_p]},$$

$$\Delta^{[n_1,\ldots,n_p]} = \Delta^{[n_1]} \otimes \cdots \otimes \Delta^{[n_p]}.$$

2.2. **Convolutions.** A monoid in $\mathcal{C}$ is an object $H$ equipped with morphisms $\mu : H \otimes H \to H$ and $\eta : I \to H$ satisfying (2), and a comonoid in $\mathcal{C}$ is an object $H$ equipped with morphisms $\Delta : H \to H \otimes H$ and $\epsilon : H \to I$ satisfying (3).

Let $A = (A, \mu_A, \eta_A)$ be a monoid in $\mathcal{C}$, and $C = (C, \Delta_C, \epsilon_C)$ a comonoid in $\mathcal{C}$. Then the set $\mathcal{C}(C, A)$ of morphisms from $C$ to $A$ in $\mathcal{C}$ is equipped with a monoid structure, with multiplication given by the convolution

$$f \ast f' := \mu_A(f \otimes f')\Delta_C$$

for $f, f' \in \mathcal{C}(C, A)$, and with unit given by $\eta_A \epsilon_C \in \mathcal{C}(C, A)$. Let $\mathcal{C}(C, A)_{\text{conv}}$ denote this monoid.

In this paper, we use convolutions in the following special situation.

Let $H$ be a Hopf monoid in $\mathcal{C}$. Then, for $m \geq 0$, the tensor power $\mathcal{H}^{\otimes m}$ has a monoid structure

$$\mu^m : \mathcal{H}^{\otimes m} \otimes \mathcal{H}^{\otimes m} \to \mathcal{H}^{\otimes m}, \quad \eta^m := \eta^{\otimes m} : I \to \mathcal{H}^{\otimes m},$$

where $\mu^m$ is defined inductively by $\mu^0 = 1_I$ and

$$\mu^{m+1} = (\mu^m \otimes \mu)(\mathcal{H}^{\otimes m} \otimes \mathcal{P}_{H,H^{\otimes m}} \otimes H) \quad (m \geq 0).$$

Similarly, $\mathcal{H}^{\otimes m}$ has a comonoid structure

$$\Delta^m : \mathcal{H}^{\otimes m} \to \mathcal{H}^{\otimes m} \otimes \mathcal{H}^{\otimes m}, \quad \epsilon^m := \epsilon^{\otimes m} : \mathcal{H}^{\otimes m} \to I,$$

where $\Delta^m$ is defined inductively by $\Delta^0 = 1_I$ and

$$\Delta^{m+1} = (\mathcal{H}^{\otimes m} \otimes \mathcal{P}_{H^{\otimes m},H} \otimes H)(\Delta^m \otimes \Delta) \quad (m \geq 0).$$

(The monoid $\langle \mathcal{H}^{\otimes m}, \mu^m, \eta^m \rangle$ and the comonoid $\langle \mathcal{H}^{\otimes m}, \Delta^m, \epsilon^m \rangle$ are part of the Hopf monoid structure on $\mathcal{H}^{\otimes m}$, with the antipode $S^m := S^{\otimes m} : \mathcal{H}^{\otimes m} \to \mathcal{H}^{\otimes m}$.)
For \( f, g \in H^\otimes m \to H^\otimes n \), \( m, n \geq 0 \), the convolution \( f \ast g \): \( H^\otimes m \to H^\otimes n \) of \( f \) and \( g \) is given by
\[
(10) \quad f \ast g = \mu_n(f \otimes g) \Delta_m.
\]
The operation \( \ast \) is associative and unital with unit \( \eta_n \epsilon_m \). Hence \( \ast \) gives the set \( \mathcal{C}(H^\otimes m, H^\otimes n) \) a monoid structure. Let \( \mathcal{C}(H^\otimes m, H^\otimes n)_{\text{conv}} \) denote this monoid.

2.3. \textbf{The operation } \lor. \text{ We also need the following variant of convolution. For } \( f \): \( H^\otimes m \to H^\otimes n \), \( g \): \( H^\otimes m' \to H^\otimes n \), \( m, m', n \geq 0 \), set
\[
f \lor f' := \mu_n(f \otimes f') : H^\otimes (m + m') \to H^\otimes n
\]
It is easy to see that \( \lor \) is associative and unital with unit \( \eta_n \): \( I \to H^\otimes n \).

\textbf{Lemma 1.} Suppose that \( H \) is commutative. Then we have
\[
(11) \quad (f \lor f') \ast (g \lor g') = (f \ast g) \lor (f' \ast g')
\]
for \( f, g \): \( H^\otimes m \to H^\otimes n \) and \( f', g' \): \( H^\otimes m' \to H^\otimes n \), \( m, m', n \geq 0 \).

\textbf{Proof.} We obtain \( (11) \) as follows.
\[
(f \lor f') \ast (g \lor g') = \mu_n(\mu_n(f \otimes f') \otimes \mu_n(g \otimes g')) \Delta_{m + m'}
= \mu_n(\mu_n \otimes \mu_n)(f \otimes f' \otimes g \otimes g')(H^\otimes m \otimes P_{H^\otimes m, H^\otimes m'} \otimes H^\otimes m') (\Delta_m \otimes \Delta_{m'})
= \mu_n^4(H^\otimes n \otimes P_{H^\otimes n, H^\otimes n} \otimes H^\otimes n)(f \otimes g \otimes f' \otimes g') (\Delta_m \otimes \Delta_{m'}),
\]
where for \( k \geq 0 \), \( \mu_n^k : (H^\otimes n)^k \to H^\otimes n \) is the \( k \)-fold iterated multiplication for the monoid \( H^\otimes n \). Commutativity of \( H \) implies commutativity of \( H^\otimes n \): \( \mu_n P_{H^\otimes n, H^\otimes n} = \mu_n \), and hence \( \mu_n^4(H^\otimes n \otimes P_{H^\otimes n, H^\otimes n} \otimes H^\otimes n) = \mu_n^4 \). Therefore we have
\[
(f \lor f') \ast (g \lor g') = \mu_n^4(f \otimes g \otimes f' \otimes g')(\Delta_m \otimes \Delta_{m'})
= \mu_n(\mu_n \otimes \mu_n)(f \otimes g \otimes f' \otimes g')(\Delta_m \otimes \Delta_{m'})
= \mu_n(\mu_n (f \otimes g) \Delta_m) \otimes (\mu_n (f' \otimes g') \Delta_m)
= (f \ast g) \lor (f' \ast g').
\]
\( \square \)

By Lemma 1 when \( H \) is commutative, \( \lor \) gives rise to a monoid homomorphism
\[
(12) \quad \lor : \mathcal{C}(H^\otimes m, H^\otimes n)_{\text{conv}} \times \mathcal{C}(H^\otimes m', H^\otimes n)_{\text{conv}} \to \mathcal{C}(H^\otimes (m + m'), H^\otimes n)_{\text{conv}}.
\]

2.4. \textbf{Permutation morphisms.} For \( n \geq 0 \), let \( \mathfrak{S}_n \) denote the symmetric group of order \( n \). Define a homomorphism
\[
\mathfrak{S}_n \to \mathfrak{H}(n, n), \quad \sigma \mapsto P_\sigma = P^H_\sigma
\]
by
\[
P_{(i, i + 1)} = H^\otimes i - 1 \otimes P_{H, H} \otimes H^\otimes n - i - 1
\]
for \( i = 1, \ldots, n - 1 \).
2.5. **Generalized $\Delta \mu$-relation.** We have the following generalization of (11), (12):

(13) \[ \Delta^{[n]_H} = (\mu^{[m]}) \otimes P_{t_{m,n}}(\Delta^{[n]}) \otimes \]

for $m, n \geq 0$, where $t_{m,n} \in S_{m,n}$ is defined by

(14) \[ t_{m,n}((l - 1)n + k) = (k - 1)m + l \]

for $1 \leq k \leq n$, $1 \leq l \leq m$.

3. **The category $H$**

3.1. **Definition of $H$.** Let $H$ denote the free symmetric strict monoidal category generated by a commutative Hopf monoid $H$. The objects in $H$ are tensor powers $H^{\otimes n}$, $n \geq 0$, which are usually identified with $n$. The morphisms in $H$ are obtained by taking tensor products and compositions of copies of the morphisms

\[ P_{m,n} = P_{H^{\otimes m}, H^{\otimes n}} : m + n \rightarrow n + m \quad (m, n \geq 0), \]

\[ \mu = \mu_H : 2 \rightarrow 1, \quad \eta = \eta_H : 0 \rightarrow 1, \quad \Delta = \Delta_H : 1 \rightarrow 2, \quad \epsilon = \epsilon_H : 1 \rightarrow 0, \]

\[ S = S_H : 1 \rightarrow 1, \]

satisfying the axioms of symmetric monoidal category and commutative Hopf monoid, and satisfying no relations that are not implied by these axioms.

**Freeness** or **universality** of $H$ is the fact that for any symmetric strict monoidal category $C$ and any commutative Hopf monoid $A = (A, \mu_A, \eta_A, \Delta_A, \epsilon_A, S_A)$ in $C$, there is a unique strict symmetric monoidal functor $F : H \rightarrow C$ mapping the Hopf monoid $H$ to the Hopf monoid $A$, i.e., $F(H) = A$ and $F(\mu_H) = \mu_A, \ldots, F(S_H) = S_A$.

In other words, the category $H$ is a PROP for commutative Hopf algebras.

3.2. **Factorization of $H$.**

**Lemma 2.** Every morphism $f : m \rightarrow n$ in $H$ admits factorization

(15) \[ f = \mu^{[p_1, \ldots, p_m]} P_{e_{1, \ldots, e_s}} \Delta^{[p_1, \ldots, p_m]}, \]

where $s, p_1, \ldots, p_m, q_1, \ldots, q_n \geq 0$ with $s = p_1 + \cdots + p_m = q_1 + \cdots + q_n$, $e_1, \ldots, e_s \in \{0,1\}$ and $\sigma \in S_s$.

**Proof.** Let $H^0$ (resp. $H^+$, $H^-$) denote the monoidal subcategory of $H$ generated by the object $H = 1$ and morphisms $\{P_{1,1}, S\}$ (resp., $\{\mu, \eta\}$, $\{\Delta, \epsilon\}$). We also use the symbols $H^*(*) = 0, +, -$ for categories to denote the set of morphisms, $\prod_{m,n \geq 0} H^*(m,n)$. We can easily verify the following.

(16) \[ H^0 = \{ P_{\sigma}(S_{e_1} \otimes \cdots \otimes S_{e_s}) \mid s \geq 0, \sigma \in S_s, (e_1, \ldots, e_s) \in \{0,1\}^s \}, \]

(17) \[ H^+ = \{ \mu^{[q_1, \ldots, q_n]} \mid n \geq 0, \quad q_1, \ldots, q_n \geq 0 \}, \]

(18) \[ H^- = \{ \Delta^{[p_1, \ldots, p_m]} \mid m \geq 0, \quad p_1, \ldots, p_m \geq 0 \}. \]

We have the following inclusions

(19) \[ H^- H^+ \subset H^+ H^0 H^-, \]

(20) \[ H^0 H^+ \subset H^+ H^0, \quad H^- H^0 \subset H^0 H^- \]

follows from the generalized $\Delta \mu$-relation [13]. To prove the first inclusion in \[(19), it suffices to show that H^0 g \subset H^+ H^0] for any $g = 1_k \otimes f \otimes 1_l$ with $f \in \{\mu, \eta\}$ and $k, l \geq 0$, since the category $H^+$ is generated by such $g$’s. The case $f = \mu$ (resp.
\( f = \eta \) can be checked by using naturality of the symmetry and \( S\mu = \mu P_{1,1}(S \otimes S) \) (resp. \( S\eta = \eta \)). The second inclusion in (20) can be proved similarly.

Using (19) and (20), we see that the set \( H^+ H^0 H^- \) is closed under composition, i.e. \( H^+ H^0 H^- \subset H^+ H^0 H^- \). Since the category \( H \) is generated by \( H^+ H^0 H^- \), we have \( H = H^+ H^0 H^- \). Then the lemma follows from [10]–[13].  

Remark 3. Lemma 2 admits several natural generalizations, which can be proved by similar arguments.

- We do not need the freeness of the category \( H \). Lemma 2 generalizes to any symmetric monoidal category generated by a commutative Hopf monoid possibly with additional relations.
- One can generalize Lemma 2 to a symmetric strict monoidal category generated by a Hopf monoid, which is not assumed to be commutative. In this case, \( e_1, \ldots, e_s \) should be taken as arbitrary nonnegative integers. If we assume that the antipode is invertible, then \( e_1, \ldots, e_s \) should be taken as arbitrary integers.
- One can generalize Lemma 2 to braided monoidal category generated by a Hopf monoid. In this case the permutation morphism \( P_\pi \) should be replaced with a braid.

3.3. The homomorphism \( \alpha_{m,n} : F^m_n \rightarrow H(m,n)_{\text{conv}} \). For \( n \geq 0 \), define a monoid homomorphism

\[
\alpha_{1,n} : F_n \rightarrow H(1,n)_{\text{conv}}
\]

by

\[
\alpha_{1,n} \left( x_j^{(-1)^e} \right) = \eta_{j-1} \otimes S^e \otimes \eta_{n-j}
\]

for \( j = 1, \ldots, n \) and \( e = 0, 1 \). This is well defined since \( \eta_{j-1} \otimes 1 \otimes \eta_{n-j} \) and \( \eta_{j-1} \otimes S \otimes \eta_{n-j} \) are convolution-inverse to each other.

Now we extend the definition of \( \alpha_{1,n} \) as follows. For \( m, n \geq 0 \), define a map

\[
\alpha_{m,n} : F^m_n \rightarrow H(m,n)_{\text{conv}}
\]

by

\[
(21) \quad \alpha_{m,n}(w_1, \ldots, w_m) = \alpha_{1,n}(w_1) \vee \cdots \vee \alpha_{1,n}(w_m)
\]

for \( w_1, \ldots, w_m \in F_n \), where the \( \vee \) are defined in Section 2.3. Clearly, this definition of \( \alpha_{m,n} \) is compatible with that of \( \alpha_{1,n} \) above. If \( m = 0 \), then the right hand side of (21) should be understood as the unit in \( \vee \), which is \( \eta_n \).

Lemma 4. The map \( \alpha_{m,n} \) is a monoid homomorphism.

Proof. We have

\[
\begin{align*}
\alpha_{m,n}(w_1, \ldots, w_m) \ast \alpha_{m,n}(w_1', \ldots, w_m') \\
= (\alpha_{1,n}(w_1) \vee \cdots \vee \alpha_{1,n}(w_m)) \ast (\alpha_{1,n}(w_1') \vee \cdots \vee \alpha_{1,n}(w_m')) \\
= (\alpha_{1,n}(w_1) \ast \alpha_{1,n}(w_1')) \vee \cdots \vee (\alpha_{1,n}(w_m) \ast \alpha_{1,n}(w_m')) \quad \text{(by Lemma 1)} \\
= \alpha_{1,n}(w_1 w_1') \vee \cdots \vee \alpha_{1,n}(w_m w_m') \\
= \alpha_{m,n}(w_1 w_1', \ldots, w_m w_m')
\end{align*}
\]

and

\[
\alpha_{m,n}(1, \ldots, 1) = \alpha_{1,n}(1) \vee \cdots \vee \alpha_{1,n}(1) = \eta_n \epsilon \vee \cdots \vee \eta_n \epsilon = \eta_n \epsilon m.
\]
3.4. Surjectivity of $\alpha_{m,n}$. For $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$ and $e \in \{0,1\}$, set

$$y_{i,j,e} := (1, \ldots, x_{j}^{(-1)^e}, \ldots, 1) \in F_n^m$$

with $x_{j}^{(-1)^e}$ in the $i$th place. As a monoid, $F_n^m$ is generated by the $y_{i,j,e}$. As a group, $F_n^m$ is generated by the $y_{i,j,0}$. Set

$$y_{i,j,e}^H := \alpha_{m,n}(y_{i,j,e}) = (\eta_{j-1} \otimes 1 \otimes \eta_{n-j}) S^e(\epsilon_{i-1} \otimes 1 \otimes \epsilon_{m-i}).$$

Since $y_{i,j,1} = y_{i,j,0}^{-1}$, it follows that $y_{i,j,0}$ and $y_{i,j,1}$ are convolution-inverse to each other.

**Lemma 5.** The homomorphism $\alpha_{m,n}$ is surjective. (Thus, $H(m,n)_{\text{conv}}$ is a group, since it is a homomorphic image of the group $F_n^m$.)

**Proof.** Define the size $s(f)$ of a morphism $f: m \to n$ in $H$ to be the least integer $s \geq 0$ such that $f$ is expressed as in \([15]\). We have $s(f) = 0$ if and only if $f = \eta_{n}e_{m}$, the convolution unit in $H(m,n)$.

To prove $\alpha_{m,n}$ surjective, it suffices to prove the following claim: If $s(f) > 0$, then there is an $f': m \to n$ with $s(f') < s(f)$ such that

$$f = y_{i,j,e}^H * f'$$

for some $i, j, e$.

Let us prove this claim. We assume the situation in Lemma 2. Set

$$i := \min\{i' \in \{1, \ldots, m\} \mid p_{i'} > 0\},$$

which is well-defined since $s = s(f) > 0$. Note that $p_1 = \cdots = p_{i-1} = 0$ and $p_i > 0$.

Set $e := e_1$. Let $j \in \{1, \ldots, n\}$ be the unique integer such that

$$q_1 + \cdots + q_{j-1} + 1 \leq \sigma(1) < q_1 + \cdots + q_j.$$

We have $q_j > 1$. We may assume $\sigma(1) = q_1 + \cdots + q_{j-1} + 1$, since if not we can modify $\sigma$ in \([15]\) using commutativity of $H$ so that we have $\sigma(1) = q_1 + \cdots + q_{j-1} + 1$.

Then one can check \([22]\) with

$$f' = f[p_{1, \ldots, q_{j-1}+1-\sigma(1)}, \ldots, p_{m}] S^{e_2} \otimes \cdots \otimes S^{e_s} \Delta^{[0, \ldots, 0, p_{i-1}, p_{i+1}, \ldots, p_m]},$$

where $\sigma' \in \mathfrak{S}_{s-1}$ is the composition of

$$\{1, \ldots, s-1\} \sim \xrightarrow{\sigma} \{2, \ldots, s\} \sim \xrightarrow{\sim} \{1, \ldots, s\} \setminus \{\sigma(1)\} \sim \{1, \ldots, s-1\},$$

where the unnamed arrows are the unique order-preserving bijections. This completes the proof. \qed

4. The category $F$

4.1. The category $F$ of finitely generated free groups. Let $F$ denote the full subcategory of the category of groups, such that $\text{Ob}(F) = \{F_n \mid n \geq 0\}$. We usually identify the object $F_n$ with the integer $n$.

Define a bijection

$$\tau_{m,n}: F_n^m \xrightarrow{\sim} F(m,n)$$

by

$$\tau_{m,n}(w_1, \ldots, w_m)(x_i) = w_i$$

\(23\)
for \( w_1, \ldots, w_m \in F_n \) and \( i = 1, \ldots, m \). We also use the notation

\[
[w_1, \ldots, w_m]_{m,n} = \tau_{m,n}(w_1, \ldots, w_m).
\]

The composition rule for this bracket notation is as follows.

\[
[w_1, \ldots, w_m]_{m,n} [v_1, \ldots, v_l]_{l,n} = [v_1(w_1, \ldots, w_m), \ldots, v_l(w_1, \ldots, w_m)]_{i,n},
\]

where \( v_i(w_1, \ldots, v_m) \in F_n \) is obtained from \( v_i \) by substituting \( w_j \) for \( x_j \) for \( j = 1, \ldots, m \).

4.2. Symmetric monoidal structure of \( F \). The category \( F \) has a monoidal structure with tensor functor \( \otimes \) given by the free product, and the monoidal unit given by \( 0 = F_0 \). We have \( m \otimes n = m + n \) for all \( m, n \geq 0 \). The tensor product \( f \otimes f' : m + m' \to n + n' \) of \( f : m \to n \) and \( f' : m' \to n' \) is the unique homomorphism \( f \otimes f' : F_{m+m'} \to F_{n+n'} \) such that

\[
(f \otimes f')(x_i) = \begin{cases} f(x_i) & \text{for } i = 1, \ldots, m, \\ s_n(f'(x_{i-m})) & \text{for } i = m + 1, \ldots, m + m'. \end{cases}
\]

Here the homomorphism \( s_n : F_{n'} \to F_{n+n'} \) is defined by \( s_n(x_i) = x_{i+n} \).

In the bracket notation, we have

\[
[w_1, \ldots, w_m]_{m,n} \otimes [w'_1, \ldots, w'_{m'}]_{m',n'} = [w_1, \ldots, w_m, s_n(w'_1), \ldots, s_n(w'_{m'})]_{m+n,m'+n'}.
\]

The symmetry in \( F \) are defined by

\[
P_{m,n} = [x_{m+1}, \ldots, x_{m+n}, x_1, \ldots, x_m]_{m+n,n+m} : m + n \to n + m.
\]

4.3. Commutative Hopf monoid in \( F \). The following is well known.

**Proposition 6.** The object \( 1 \) in \( F \) has a commutative Hopf monoid structure with the multiplication, unit, comultiplication, counit and antipode defined by

\[
\mu_F = [x_1, x_1]_{1,2}, \quad \eta_F = [1]_{0,1}, \quad \Delta_F = [x_1x_2]_{1,2}, \quad \epsilon_F = [1]_{1,0}, \quad S_F = [x_1^{-1}]_{1,1}.
\]

Let \( F = (1, \mu_F, \eta_F, \Delta_F, \epsilon_F, S_F) \) denote this Hopf monoid in \( F \). We often omit the subscript \( F \) from the notation.

**Proof.** The proposition can be checked by easy computations. For example, \( \Delta \mu = (\mu \otimes \mu)(1 \otimes P_{1,1} \otimes 1)(\Delta \otimes \Delta) \) can be checked by

\[
\Delta \mu = [x_1x_2]_{1,2} [x_1, x_1]_{2,1} = [x_1x_2, x_1x_2]_{2,2},
\]

\[
(\mu \otimes \mu)(1 \otimes P_{1,1} \otimes 1)(\Delta \otimes \Delta) = [x_1, x_2, x_2]_{1,2,2} [x_1x_2, x_1x_2, x_1x_2]_{4,2} = [x_1, x_2, x_1x_2]_{4,2} [x_1x_2, x_3x_4]_{2,4} = [x_1x_2, x_1x_2]_{2,2}.
\]

By Proposition 6 and universality of \( H \), there is a unique symmetric monoidal functor

\[
T : H \to F,
\]

which maps the Hopf monoid \( H \) in \( H \) to the Hopf monoid \( F \) in \( F \).
4.4. The group $\mathbf{F}(m,n)_{\text{conv}}$.

**Proposition 7.** For $m, n \geq 0$, the bijection $\tau_{m,n}$ gives rise to a group isomorphism

$$\tau_{m,n}: \mathbb{F}_n^m \xrightarrow{\cong} \mathbf{F}(m,n)_{\text{conv}}.$$  

**Proof.** It suffices to see that $\tau_{m,n}$ is a monoid homomorphism, i.e., we have

$$[w_1, \ldots, w_m]_{m,n} \ast [w'_1, \ldots, w'_m]_{m,n} = [w_1w'_1, \ldots, w_mw'_m]_{m,n},$$

for $w_1, \ldots, w_m, w'_1, \ldots, w'_m \in \mathbb{F}_n$, and

$$\eta_{m,n} = [1, \ldots, 1]_{m,n}.$$  

(27) can be checked as follows:

$$[w_1, \ldots, w_m]_{m,n} \ast [w'_1, \ldots, w'_m]_{m,n} = \mu_n([w_1, \ldots, w_m]_{m,n} \otimes [w'_1, \ldots, w'_m]_{m,n}) \Delta_n$$

$$= [x_1, \ldots, x_n, x_1, \ldots, x_n]_{2m,n} \alpha_n[1, \ldots, w_m, s_n(w'_1), \ldots, s_n(w'_m)]_{2m,2n}$$

$$\times [x_1x_{m+1}, \ldots, x_{m+2m}]_{m,2m}$$

$$= [w_1, \ldots, w_m, w'_1, \ldots, w'_m]_{2m,n} [x_1x_{m+1}, \ldots, x_{m+2m}]_{m,2m}$$

$$= [w_1w'_1, \ldots, w_mw'_m]_{m,n}.$$  

(28) can be checked easily. ∎

5. ISOmorphism between $\mathbf{H}$ and $\mathbf{F}$

**Theorem 8.** The functor $T: \mathbf{H} \rightarrow \mathbf{F}$ is an isomorphism of symmetric strict monoidal categories.

As mentioned in the introduction, the above result is well known [11]. The proof below is a direct, combinatorial proof, not relying on Lawvere theories.

**Proof.** It suffices to prove that $T_{m,n}: \mathbf{H}(m,n) \rightarrow \mathbf{F}(m,n)$ is bijective for $m, n \geq 0$.

Consider the following diagram of group homomorphisms

$$\mathbf{H}(m,n)_{\text{conv}} \xrightarrow{T_{m,n}} \mathbf{F}(m,n)_{\text{conv}}$$

Here, $T_{m,n}$ is a homomorphism since $T$ is a monoidal functor. Since $\tau_{m,n}$ is an isomorphism (Proposition 7) and $\alpha_{m,n}$ is surjective (Lemma 5), it suffices to prove that the diagram commutes.

Since the group $\mathbb{F}_n^m$ is generated by $y_{i,j,0}$ for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, we have only to check

$$T_{m,n}\alpha_{m,n}(y_{i,j,0}) = \tau_{m,n}(y_{i,j,0}).$$  

(30)
Indeed,
\[
T_{m,n} \alpha_{m,n}(y_{i,j,0}) = T_{m,n}(y_{i,j,0}^H) = T_{m,n} \left( (\eta_\mathcal{H}^{\otimes j - 1} \otimes 1_1 \otimes \eta_\mathcal{H}^{\otimes n - j})(\epsilon_\mathcal{H}^{\otimes i - 1} \otimes 1_1 \otimes \epsilon_\mathcal{H}^{\otimes m - i}) \right) = (\eta_\mathcal{F}^{\otimes j - 1} \otimes 1_1 \otimes \eta_\mathcal{F}^{\otimes n - j})(\epsilon_\mathcal{F}^{\otimes i - 1} \otimes 1_1 \otimes \epsilon_\mathcal{F}^{\otimes m - i}) = [x_j]_{1,n}^1, \ldots, x_1, \ldots, 1]_{m,1} \]
\[
= [1, \ldots, x_j, \ldots, 1]_{m,n} = y_{i,j,0}.
\]

\[\square\]

### 6. Coproducts in \( \mathbf{H} \)

As is well known, \( \mathbf{F} \) admits finite coproducts given by free products. Since \( \mathbf{H} \cong \mathbf{F} \), the category \( \mathbf{H} \) admits finite coproducts. Here we provide a direct proof of this fact, without relying on the corresponding result for \( \mathbf{F} \) in Sections 4 and 5.

**Lemma 9.** For \( m, m', n \geq 0 \), there are group homomorphisms
\[
\pi = \pi_{m,m',n} : \mathbf{H}(m + m', n) \to \mathbf{H}(m, n), \quad \pi(f) = f(1_m \otimes \eta_{m'}) ,
\]
\[
\pi' = \pi'_{m,m',n} : \mathbf{H}(m + m', n) \to \mathbf{H}(m', n), \quad \pi(f) = f(\eta_m \otimes 1_{m'}). 
\]

**Proof.** Clearly, we have \( \pi(\eta_m \epsilon_{m + m'}) = \eta_m \epsilon_m \). For \( f, g \in \mathbf{H}(m + m', n) \), we have
\[
\pi(f \ast g) = \mu_n(f \otimes g)(\Delta_{m + m'}(1_m \otimes \eta_{m'})) = \mu_n(f \otimes g)(1_m \otimes P_{m,m'} \otimes 1_{m'})(\Delta_{m} \otimes \Delta_{m'})(1_m \otimes \eta_{m'}) \\
= \mu_n(f \otimes g)(1_m \otimes P_{m,m'} \otimes 1_{m'})(\Delta_{m} \otimes \eta_{m'} \otimes \eta_{m'}) \\
= \mu_n(f \otimes g)(1_m \otimes \eta_{m'} \otimes 1_m \otimes \eta_{m'}) \Delta_m \\
= \mu_n(f(1_m \otimes \eta_{m'}) \otimes g(1_m \otimes \eta_{m'})) \\
= \pi(f) \ast \pi(g).
\]
Hence \( \pi \) is a homomorphism. Similarly, we can prove \( \pi' \) is a homomorphism. \[\square\]

**Proposition 10** (well known). The category \( \mathbf{H} \) admits finite coproducts. More precisely, the object \( 0 \) is the initial object, and for \( m, m' \geq 0, m + m' \) is a coproduct of \( m \) and \( m' \), with a coproduct diagram
\[
(31)\quad m \quad \underbrace{\otimes \eta_{m'}}_{\otimes 1_{m'}} \quad m + m' \quad \eta_m \otimes 1_{m'} \quad m'.
\]

**Proof.** That \( 0 \) is initial in \( \mathbf{H} \) follows easily from Lemma 2. To prove that (31) is a coproduct diagram, it suffices to prove that for every \( n \geq 0 \) the homomorphism
\[
(32)\quad \forall : \mathbf{H}(m, n)_{\text{conv}} \times \mathbf{H}(m', n)_{\text{conv}} \to \mathbf{H}(m + m', n)_{\text{conv}}
\]
from (32) and the homomorphism
\[
(\pi, \pi') : \mathbf{H}(m + m', n)_{\text{conv}} \to \mathbf{H}(m, n)_{\text{conv}} \times \mathbf{H}(m', n)_{\text{conv}}
\]
are inverse to each other. We check this claim on the generators.
First, we check \( \forall (\pi, \pi')(y_{i,j,0}^H) = y_{i,j,0}^H \) for \( i \in \{1, \ldots, m + m'\}, j \in \{1, \ldots, n\} \). If \( 1 \leq i \leq m \), then
\[
\forall (\pi, \pi')(y_{i,j,0}^H) = \forall (y_{i,j,0}^H, \eta_1\epsilon_{m'}) = y_{i,j,0}^H,
\]
and if \( m + 1 \leq i \leq m + n \), then
\[
\forall (\pi, \pi')(y_{i,j,0}^H) = \forall (\eta_n\epsilon_m, y_{i-m,j,0}^H) = y_{i,j,0}^H.
\]
Conversely, let us check \((\pi, \pi') \triangledown (f, f') = (f, f')\) for the generators of \( H(m, n)_{\text{conv}} \times H(m', n)_{\text{conv}} \). \( (f, f') = (y_{i,j,0}^H, \eta_1\epsilon_{m'}) \) with \( i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\} \) and \( (f, f') = (\eta_n\epsilon_m, y_{i,j,0}^H) \) for \( i \in \{1, \ldots, m'\}, j \in \{1, \ldots, n\} \). In the former case,
\[
(\pi, \pi') \triangledown (y_{i,j,0}^H, \eta_1\epsilon_{m'}) = (\pi, \pi')(y_{i,j,0}^H) = (\pi(y_{i,j,0}^H), \pi'(y_{i,j,0}^H)) = (y_{i,j,0}^H, \eta_1\epsilon_{m'}).
\]
The latter case can be checked similarly. \( \Box \)

By the proof of Proposition \([10]\) it follows that we have a group isomorphism
\[
H(m, n)_{\text{conv}} \simeq H(1, n)_{\text{conv}}^m.
\]

7. Motivations from topology

7.1. Category of bouquets of circles. Here we recall well-known facts about the cogroup structure on \( S^1 \) and the Hopf monoid \( F = \mathbb{Z} \) in \( F \).

Let \( \text{Top}_*/h \) denote the category of pointed topological spaces and homotopy classes of pointed continuous maps. Let \( B \) denote the full subcategory of \( \text{Top}_*/h \) with \( \text{Ob}(B) = \{ \forall^n S^1 \mid n \geq 0 \} \). The fundamental group gives an isomorphism of symmetric monoidal categories
\[
\pi_1: B \xrightarrow{\simeq} F
\]
in a natural way.

It is well known that the circle \( S^1 \) has a structure of a cogroup, see \([1]\). This means that there are two maps
\[
\Delta: S^1 \to S^1 \triangledown S^1, \quad \gamma: S^1 \to S^1,
\]
which, together with the natural maps
\[
\mu: S^1 \triangledown S^1 \to S^1, \quad \eta: * \to S^1, \quad \epsilon: S^1 \to *,
\]
form a commutative Hopf monoid structure in \( \text{Top}_*/h \). The maps \( \Delta \) and \( \gamma \) are the familiar ones that appear in the definition of the fundamental group of topological spaces. The fundamental group functor maps this cogroup structure on \( S^1 \) to the commutative Hopf monoid structure on \( \pi_1(S^1) = \mathbb{Z} = \mathbb{F} \) in \( F \).

7.2. Category of handlebody embeddings. Here we discuss the category of handlebody embeddings, denoted by \( \mathcal{H} \), which appeared in \([5]\), and will be studied in detail in \([8]\). The opposite \( \mathcal{H}^{\text{op}} \) is isomorphic to the “category of bottom tangles”, \( B \), in \([4]\), and to the “category of special Lagrangian cobordisms” in \([2]\).

Define \( \mathcal{H} \) as follows. The objects are nonnegative integers. The morphisms from \( m \) to \( n \) are the isotopy classes of embeddings of a genus \( m \) handlebody \( V_m \) into a genus \( n \) handlebody \( V_n \). Here, \( V_m \) is the (3-dimensional) handlebody of genus \( m \) obtained by attaching \( m \) 1-handles on the top of a cube. By an embedding of \( V_m \) into \( V_n \) we mean an embedding which fixes the bottom face of the cube. The isotopy classes are taken through such embeddings.
It is observed in [4] that the category $B$ has a structure of a braided monoidal category, and there is a Hopf monoid in $B$. (In fact, $B$ may be regarded as a subcategory of the category of cobordisms of surfaces with boundaries parameterized by $S^1$, introduced by Crane and Yetter [3] and Kerler [8].) Thus $H \simeq B^{op}$ is a braided monoidal category and admits a Hopf monoid in it.

There is a braided monoidal functor

$$\pi^H_1 : H \longrightarrow F,$$

mapping $n \in \text{Ob}(H)$ to $\pi_1(V_n) \simeq F_n$, such that for $[f] : m \rightarrow n$ with $f : V_m \hookrightarrow V_n$, we have

$$\pi^H_1([f]) = \pi_1(f) : F_m \rightarrow F_n,$$

where we identify $F_m$ with $\pi_1(V_m)$ for $m \geq 0$. For each $m, n \geq 0$, the map

$$\pi^H_1 : \mathcal{H}(m, n) \rightarrow F(m, n)$$

is surjective, and not bijective (if $m \neq 0$). Note that two morphisms $[f], [g] : m \rightarrow n$ in $\mathcal{H}$ satisfy $\pi^H_1([f]) = \pi^H_1([g])$ if and only if the representative embeddings $f, g : V_m \hookrightarrow V_n$ are homotopic fixing the bottom face of the cube. Note also that the quotient category $\mathcal{H}/(\text{homotopy})$ is naturally isomorphic to the category of bouquets of circles, $B$, since $V_m$ is homotopy equivalent to $\bigvee^m S^1$. One may regard $\mathcal{H}$ as a refinement of $\mathcal{H}/(\text{homotopy}) \simeq B \simeq F$, obtained by replacing homotopy with isotopy.

In [6], we will give a presentation of $H \simeq B^{op}$ as a braided monoidal category, which may be regarded as a refinement of the presentation of $F$. In $\mathcal{H}$ there is a Hopf monoid, which is mapped by $\pi^H_1$ into the Hopf monoid $F$ in $F$.

### 7.3. Category of chord diagrams in handlebodies

Here we discuss the category $A$ of chord diagrams in handlebodies, which will appear in a joint work with Massuyeau [7].

As in the previous subsection, let $\mathcal{B} \simeq \mathcal{H}^{op}$ denote the category of bottom tangles in handlebodies [4]. In [7], using the Kontsevich integral, we will construct a functor

$$Z : \mathcal{B} \longrightarrow A,$$

where $A$ is the “category of chord diagrams in handlebodies”, which will be defined in [7]. Here we mention only the following, ignoring some technical details such as completions and non-associative monoidal structures, which usually appears in the category-theoretic study of the Kontsevich integral and its variants. The objects in $A$ are nonnegative integers. The Hom space $A(m, n)$ is the $\mathbb{Q}$-vector space spanned by “chord diagrams” on $n$ strands based at the bottom edge in a 2-dimensional handlebody obtained by attaching $m$ 1-handles on the top of a square. The vector space $A(m, n)$ is graded by the number of chords. Then the degree 0 part $A_0(m, n)$ of $A(m, n)$ is isomorphic to $\mathbb{Q}H^{op}(m, n)$, the $\mathbb{Q}$-vector space with basis $H^{op}(m, n)$. Thus, we may identify the linear category $\mathbb{Q}H^{op}$ as the degree 0 part $A_0$ of the graded linear category $A$. 
The functor $Z$ is related to the functor $\pi^H_1: \mathcal{H} \to \mathbf{F}$ by the following commutative diagram

\[
\begin{array}{cccc}
\mathcal{B} & \cong & \mathcal{H}^{\text{op}} & H^{\text{op}} \\
\mathcal{Z} & \Rightarrow & (\pi^H_1)^{\text{op}} & (T^{\text{op}})^{-1} \\
\mathcal{A} & \Rightarrow & A_0 & \cong \mathbb{Q}H^{\text{op}} \\
\end{array}
\]

(33)

The category $\mathcal{A}$ is a $\mathbb{Q}$-linear symmetric monoidal category, admitting a cocommutative Hopf monoid in the degree 0 part $A_0$. In [7], we will give a presentation of $\mathcal{A}$ as a $\mathbb{Q}$-linear symmetric monoidal category.

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