ON THE $S^1$-FIBRED NIL-BOTT TOWER

MAYUMI NAKAYAMA

Abstract. We shall introduce a notion of $S^1$-fibred nilBott tower. It is an iterated $S^1$-bundles whose top space is called an $S^1$-fibred nilBott manifold and the $S^1$-bundle of each stage realizes a Seifert construction. The nilBott tower is a generalization of real Bott tower from the viewpoint of fibration. In this note we shall prove that any $S^1$-fibred nilBott manifold is diffeomorphic to an infranilmanifold. According to the group extension of each stage, there are two classes of $S^1$-fibred nilBott manifolds which is defined as finite type or infinite type. We discuss their properties.

1. Introduction

Let $M$ be a closed aspherical manifold which is a top space of an iterated $S^1$-bundles over a point:

\begin{equation}
M = M_n \to M_{n-1} \to \ldots \to M_1 \to \{\text{pt}\}.
\end{equation}

Suppose $X$ is the universal covering of $M$ and each $X_i$ is the universal covering of $M_i$ and put $\pi_1(M_i) = \pi_i (i = 1, \ldots, n-1)$ and $\pi_1(M) = \pi$.

Definition 1.1. An $S^1$-fibred nilBott tower is a sequence (1.1) which satisfies I, II and III below ($i = 1, \ldots, n-1$). The top space $M$ is said to be an $S^1$-fibred nilBott manifold (of depth $n$).

I. $M_i$ is a fiber space over $M_{i-1}$ with fiber $S^1$.

II. For the group extension

\begin{equation}
1 \to \mathbb{Z} \to \pi_i \to \pi_{i-1} \to 1
\end{equation}

associated to the fiber space (I), there is an equivariant principal bundle:

\begin{equation}
\mathbb{R} \to X_i \overset{p_i}{\longrightarrow} X_{i-1}.
\end{equation}

III. Each $\pi_i$ normalizes $\mathbb{R}$.

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The purpose of this paper is to prove the following result.

**Theorem 1.2.** Suppose that $M$ is an $S^1$-fibred nilBott manifold.

(I) If every cocycle of $H^2_\phi(\pi_{i-1};\mathbb{Z})$ which represents a group extension (1.2) is of finite order, then $M$ is diffeomorphic to a Riemannian flat manifold.

(II) If there exists a cocycle of $H^2_\phi(\pi_{i-1};\mathbb{Z})$ which represents a group extension (1.2) is of infinite order, then $M$ is diffeomorphic to an infranilmanifold. In addition, $M$ cannot be diffeomorphic to any Riemannian flat manifold.

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**2. Seifert construction**

We shall explain the Seifert construction briefly. It is a tool to construct a closed aspherical manifold for a given extension.

Let

\begin{equation}
(2.1) \quad 1 \longrightarrow \Delta \longrightarrow \pi \overset{\nu}{\longrightarrow} Q \longrightarrow 1
\end{equation}

be a group extension. Then there is a conjugation function $\phi : Q \to \text{Aut}(\Delta)$ defined by a section $s : Q \to \pi$ of $\nu$. The group extension (2.1) is represented by a cocycle $f : Q \times Q \to \Delta$ for which each element $\gamma \in \pi$ is viewed as $(n, \alpha)$ with group law:

$$(n, \alpha)(m, \beta) = (n \cdot \phi(\alpha)(m) \cdot f(\alpha, \beta), \alpha \beta)$$

$(\forall n, m \in \Delta, \forall \alpha, \beta \in Q)$ (cf. [13] for example).

Suppose $\Delta$ is a torsionfree finitely generated nilpotent group. By Mal’cev’s existence theorem, there is a (simply connected) nilpotent Lie group $N$ containing $\Delta$ as a discrete uniform subgroup. Moreover if $Q$ acts properly discontinuously on a contractible smooth manifold $W$
such that the quotient space $W/Q$ is compact, then there is a smooth map $\lambda : Q \to \text{Map}(W,N)$ satisfies $f = \delta^1\lambda$:

$$f(\alpha, \beta) = (\tilde{\phi}(\alpha) \circ \lambda(\beta) \circ \alpha^{-1}) \cdot \lambda(\alpha) \cdot \lambda(\alpha\beta)^{-1} \quad (\alpha, \beta \in Q)$$

Here $\tilde{\phi} : Q \to N$ is the extension of $\phi$. And an action of $\pi$ on $N \times W$ is obtained by

$$\pi \times N \times W \ni (n, \alpha)(x, w) = (n \cdot \tilde{\phi}(\alpha)(x) \cdot \lambda(\alpha)(\alpha w), \alpha w).$$

This action $(\pi, N \times W)$ is said to be a Seifert construction. (See [5] for details.)

Taking a finite group $F$ and $\{pt\}$ as $Q$ and $W$ above;

$$1 \longrightarrow \Delta \longrightarrow \pi \longrightarrow \nu \longrightarrow F \longrightarrow 1,$$

we may put $N$ as $\text{Map}(W,N)$ before respectively. Let $K$ be a maximal compact subgroup of $\text{Aut}(N)$. The group $E(N) = N \rtimes K$ is said to be the euclidean group of $N$. Then there is a discrete faithful representation $\rho : \pi \to E(N)$ which is defined by

$$\rho((n, \alpha)) = (n \cdot \chi(\alpha), \mu(\chi(\alpha)^{-1}) \circ \tilde{\phi}(\alpha)) \quad (n \in \Delta, \alpha \in F),$$

where $\chi : F \to N$ is a map such that $f = \delta^1\chi$:

$$f(\alpha, \beta) = \tilde{\phi}(\alpha)(\chi(\beta)) \cdot \chi(\alpha) \cdot \chi(\alpha\beta)^{-1} \quad (\alpha, \beta \in Q).$$

(See [13].) Note that the action $(\rho(\pi), N)$ is a Seifert construction and $N/\rho(\pi)$ is an infranilmanifold (cf [5] or [13]).

### 3. $S^1$-fibred nilBott tower

This section gives the proof Theorem 1.2.

Suppose that

$$M = M_n \xrightarrow{s^1} M_{n-1} \xrightarrow{s^1} \ldots \xrightarrow{s^1} M_1 \xrightarrow{s^1} \{pt\}$$

is an $S^1$-fibred nilBott tower. By the definition, there is a group extension of the fiber space;

$$1 \to \mathbb{Z} \to \pi_i \to \pi_{i-1} \to 1$$

for any $i$. The conjugate by each element of $\pi_i$ defines a homomorphism $\phi : \pi_{i-1} \to \text{Aut}(\mathbb{Z}) = \{\pm 1\}$. With this action, $\mathbb{Z}$ is a $\pi_{i-1}$-module so that the group cohomology $H^2_\phi(\pi_{i-1}, \mathbb{Z})$ is defined. Then the above group extension (3.2) represents a 2-cocycle in $H^2_\phi(\pi_{i-1}, \mathbb{Z})$, (cf. [13]).
Proof. Given a group extension (3.2), we suppose by induction that there exists a torsionfree finitely generated nilpotent normal subgroup $\Delta_{i-1}$ of finite index in $\pi_{i-1}$ such that the induced extension $\Delta_i$ is a central extension:

$$
1 \longrightarrow Z \longrightarrow \pi_i \longrightarrow \pi_{i-1} \longrightarrow 1
$$

(3.3)

It is easy to see that $\tilde{\Delta}_i$ is a torsionfree finitely generated normal nilpotent subgroup of finite index in $\pi_i$. Then $\pi_i$ is a virtually nilpotent subgroup, i.e. $1 \rightarrow \Delta_i \rightarrow \pi_i \rightarrow F_i \rightarrow 1$ where $F_i = \pi_i/\Delta_i$ is a finite group. Let $\tilde{N}_i$, $N_{i-1}$ be a nilpotent Lie group containing $\tilde{\Delta}_i$, $\Delta_{i-1}$ as a discrete cocompact subgroup respectively. Let $A(\tilde{N}_i) = \tilde{N}_i \rtimes \text{Aut}(\tilde{N}_i)$ be the affine group. If $\tilde{K}_i$ is a maximal compact subgroup of $\text{Aut}(\tilde{N}_i)$, then the subgroup $E(\tilde{N}_i) = \tilde{N}_i \rtimes \tilde{K}_i$ is the euclidean group of $\tilde{N}_i$. Then there exists a faithful homomorphism (see (2.5)):

$$
\rho_i : \pi_i \longrightarrow E(\tilde{N}_i)
$$

(3.4)

for which $\rho_i|_{\tilde{\Delta}_i} = \text{id}$ and the quotient $\tilde{N}_i/\rho_i(\pi_i)$ is an infranilmanifold. The explicit formula is given by the following

$$
\rho_i((n, \alpha)) = (n \cdot \chi(\alpha), \mu(\chi(\alpha)^{-1}) \circ \tilde{\phi}(\alpha))
$$

(3.5)

for $n \in \tilde{\Delta}_i$, $\alpha \in F$ where $\chi : F \rightarrow \tilde{\Delta}_i$, $\tilde{\phi} : F \rightarrow \text{Aut}(\tilde{N}_i)$. As $\tilde{\Delta}_i \leq \tilde{N}_i$, there is a 1-dimensional vector space $R$ containing $Z$ as a discrete uniform subgroup which has a central group extension:

$$
1 \rightarrow R \rightarrow \tilde{N}_i \rightarrow N_{i-1} \rightarrow 1
$$

where $N_{i-1} = \tilde{N}_i/R$ is a simply connected nilpotent Lie group. As $Z \leq R \cap \tilde{\Delta}_i$ is discrete cocompact in $R$ and $R \cap \tilde{\Delta}_i/Z \rightarrow \tilde{\Delta}i/Z \cong \Delta_{i-1}$ is an inclusion, noting that $\Delta_{i-1}$ is torsionfree, it follows that $R \cap \Delta_i = Z$. We obtain the commutative diagram in which the vertical maps are inclusions:

$$
1 \longrightarrow Z \longrightarrow \tilde{\Delta}_i \longrightarrow \Delta_{i-1} \longrightarrow 1
$$

(3.6)

$$
1 \longrightarrow R \longrightarrow \tilde{N}_i \longrightarrow N_{i-1} \longrightarrow 1.
$$
On the other hand, (3.4) induces the following group extension:

\[
\begin{array}{ccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_i & \longrightarrow & \pi_{i-1} & \longrightarrow & 1 \\
\| & & \rho_i & & \phi_i & & \phi_i & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \rho_i(\pi_i) & \longrightarrow & \hat{\rho}_i(\pi_{i-1}) & \longrightarrow & 1.
\end{array}
\]

(3.7)

Since \(\Delta_i\) centralizes \(Z\), \(\tilde{N}_i\) centralizes \(R\). So \(\hat{\rho}_i\) is a homomorphism from \(\pi_{i-1}\) into \(E(N_{i-1})\). The explicit formula is given by the following:

\[
\hat{\rho}_i((\bar{n}, \alpha)) = (\bar{n} \cdot \bar{\chi}(\alpha), \mu(\bar{\chi}(\alpha)^{-1}) \circ \hat{\phi}(\alpha))
\]

for \(\bar{n} \in \Delta_{i-1}, \alpha \in F\) where \(\bar{\chi} = p_i \circ \chi : F \to \Delta_{i-1}, \hat{\phi} : F \to \text{Aut}(N_{i-1}); \hat{\phi}(\alpha)(\bar{x}) = \phi(\alpha)(x)\).

Note that \(\hat{\phi}(\alpha)(R) = R\). As the action \((\pi_{i-1}, N_{i-1})\) is properly discontinuous and \(\pi_{i-1}\) is torsionfree, the representation \(\hat{\rho}_i : \pi_{i-1} \to E(N_{i-1})\) is faithful. (Note that \(\hat{\rho}_i|_{\Delta_{i-1}} = \text{id}\).) Thus we obtain an equivariant fibration:

\[
(Z, R) \longrightarrow (\rho_i(\pi_i), \tilde{N}_i) \longrightarrow (\hat{\rho}_i(\pi_{i-1}), N_{i-1}).
\]

(3.8)

Suppose by induction that \((\pi_{i-1}, X_{i-1})\) is equivariantly diffeomorphic to the infranil-action \((\hat{\rho}_i(\pi_{i-1}), N_{i-1})\) as above. We have two Seifert fibrations from (1.3) where

\[
(Z, R) \to (\pi_i, X_i) \xrightarrow{\rho} (\pi_{i-1}, X_{i-1})
\]

and (3.9) where

\[
(Z, R) \to (\rho_i(\pi_i), \tilde{N}_i) \xrightarrow{\mu_i} (\hat{\rho}_i(\pi_{i-1}), N_{i-1}).
\]

As \(\rho_i : \pi_i \to \rho_i(\pi_i)\) is isomorphic such that \(\rho_i|_Z = \text{id}\), the Seifert rigidity implies that \((\pi_i, X_i)\) is equivariantly diffeomorphic to \((\rho_i(\pi_i), \tilde{N}_i)\). This shows the induction step. Let \(M = X/\pi\). Then \((\pi, X)\) is equivariantly diffeomorphic to an infra-nilaction \((\rho(\pi), \tilde{N})\) for which \(\rho : \pi \to E(\tilde{N})\) is a faithful representation.

We have shown that \(M\) is diffeomorphic to an infranilmanifold \(\tilde{N}/\rho(\pi)\). According to Cases I, II (stated in Theorem 1.2), we prove that \(\tilde{N}\) is isomorphic to a vector space for Case I or \(\tilde{N}\) is a nilpotent Lie group but not a vector space for Case II respectively.

Case I. As every cocycle of \(H_2^\phi(\pi_{i-1}, \mathbb{Z})\) representing a group extension (3.2) is finite, the cocycle in \(H^2(\Delta_{i-1}, \mathbb{Z})\) for the induced extension of (3.3) that \(1 \to \mathbb{Z} \to \Delta_i \to \Delta_{i-1} \to 1\) is also finite. By induction, suppose that \(\Delta_{i-1}\) is isomorphic to a free abelian group \(\mathbb{Z}^{i-1}\). Then the cocycle in \(H^2(\mathbb{Z}^{i-1}, \mathbb{Z})\) is zero, so \(\Delta_i\) is isomorphic to a free abelian
group $Z^i$. Hence the nilpotent Lie group $N_i$ is isomorphic to the vector space $\mathbb{R}^i$. This shows the induction step. In particular, $\pi_i$ is isomorphic to a Bieberbach group $\rho_i(\pi_i) \leq E(\mathbb{R}^i)$. As a consequence $X/\pi$ is diffeomorphic to a Riemannian flat manifold $\mathbb{R}^n/\rho(\pi)$.

Case II. Suppose that $\pi_{i-1}$ is virtually free abelian until $i - 1$ and the cocycle $[f] \in H^2(\pi_{i-1}, \mathbb{Z})$ representing a group extension $1 \to \mathbb{Z} \to \pi_i \to \pi_{i-1} \to 1$ is of infinite order in $H^2(\pi_{i-1}, \mathbb{Z})$. Note that $\pi_{i-1}$ contains a torsionfree normal free abelian subgroup $Z^{i-1}$. As in (3.3), there is a central group extension of $\tilde{\Delta}_i$:

$$
1 \longrightarrow \mathbb{Z} \longrightarrow \pi_i \longrightarrow \pi_{i-1} \longrightarrow 1
$$

(3.10)

$$
1 \longrightarrow \mathbb{Z} \longrightarrow \tilde{\Delta}_i \longrightarrow Z^{i-1} \longrightarrow 1
$$

where $[\pi_{i-1} : Z^{i-1}] < \infty$. Recall that there is a transfer homomorphism $\tau : H^2(Z^{i-1}, \mathbb{Z}) \to H^2(\pi_{i-1}, \mathbb{Z})$ such that $\tau \circ i^* = [\pi_{i-1} : Z^{i-1}] : H^2(\pi_{i-1}, \mathbb{Z}) \to H^2(\pi_{i-1}, \mathbb{Z})$, see [2] (9.5) Proposition p.82 for example. The restriction $i^*[f]$ gives the bottom extension sequence of (3.10).

If $i^*[f] = 0 \in H^2(Z^2, \mathbb{Z})$, then $0 = \tau \circ i^*[f] = [\pi_{i-1} : Z^{i-1}][f] \in H^2(\pi_{i-1}, \mathbb{Z})$. So $i^*[f] \neq 0$. Therefore $\tilde{\Delta}_i$ (respectively $N_i$) is not abelian (respectively not isomorphic to a vector space). As a consequence, $N$ is a simply connected (non-abelian) nilpotent Lie group. □

In order to study $S^1$-fibred nilBott manifolds further, we introduce the following definition:

**Definition 3.1.** If an $S^1$-fibred nilBott manifold $M$ satisfies Case I (respectively Case II) of Theorem 1.2, then $M$ is said to be an $S^1$-fibred nilBott manifold of finite type (respectively of infinite type). Apparently there is no intersection between finite type and infinite type. And $S^1$-fibred nilBott manifolds are of finite type until dimension 2.

**Remark 3.2.** Let $M$ be an $S^1$-fibred nilBott manifold of finite type, then $\rho(\pi)$ is a Bieberbach group (cf. Theorem 1.2). By the Bieberbach Theorem, $\rho(\pi)$ satisfies a group extension

$$
1 \to \mathbb{Z}^n \to \rho(\pi) \longrightarrow H \to 1
$$

(3.11)

where $\mathbb{Z}^n = \rho(\pi) \cap \mathbb{R}^n$, and $H$ is the holonomy group of $\rho(\pi)$. We may identify $\rho(\pi)$ with $\pi$ whenever $\pi$ is torsionfree.

**Proposition 3.3.** Suppose $M$ is an $S^1$-fibred nilBott manifold of finite type. Then the holonomy group of $\pi$ is isomorphic to the power of cyclic group of order two $(\mathbb{Z}_2)^s$ in $0 \leq s \leq n$. 

Proof. Let $M$ be an $S^1$-fibred nilBott manifold of finite type. From (3.2) recall a group extension

$$1 \to \mathbb{Z} \to \pi_i \xrightarrow{p_i} \pi_{i-1} \to 1$$

which associates to the equivariant fibration:

$$(\mathbb{Z}, R) \to (\pi_i, \tilde{N}_i) \xrightarrow{p_i} (\pi_{i-1}, N_{i-1}).$$

If $f$ is a cocycle in $H_2^\phi(\pi_{i-1}, \mathbb{Z})$ for Case I representing (3.12), then there exists a map $\lambda : \pi_{i-1} \to \mathbb{R}$ such that

$$f(\alpha, \beta) = \bar{\phi}(\alpha)(\lambda(\beta)) + \lambda(\alpha) - \lambda(\alpha \beta) \quad (\alpha, \beta \in \pi_{i-1})$$

(see [3]). Moreover let $(n, \alpha) \in \pi_i$ and $(x, w) \in \tilde{N}_i = \mathbb{R} \times N_{i-1}$, then the action of $\pi_i$ is given by

$$(n, \alpha)(x, w) = (n + \bar{\lambda}(\alpha)(x) + \lambda(\alpha), \alpha w)$$

(Remark that $n \in \mathbb{Z}$, $\alpha \in \pi_{i-1}$ and see (2.3).) As we have shown in Case I of Theorem 1.2, $(\pi_i, \tilde{N}_i)$ is a Bieberbach group action. Let $(\pi_{i-1}, N_{i-1}) = (\pi_{i-1}, \mathbb{R}^{i-1})$ where $\pi_{i-1} \leq E(i-1) = \mathbb{R}^{i-1} \rtimes O(i-1)$ is a Bieberbach group such that

$$\alpha w = b_\alpha + A_\alpha w \quad (w \in \mathbb{R}^{m-1})$$

here $b_\alpha \in \mathbb{R}^m$, $A_\alpha \in O(i-1)$ in the above action of (3.14).

Let $L : E(i-1) \to O(i-1)$ be the linear holonomy homomorphism. Suppose inductively that $L(\pi_{i-1}) = \{A_\alpha \mid \alpha \in \pi_{i-1}\} \leq (\mathbb{Z}_2)^{i-1}$. Here

$$(\mathbb{Z}_2)^{i-1} = \left\{ \begin{pmatrix} \pm 1 \\ \vdots \\ \pm 1 \end{pmatrix} \right\} \leq O(i-1).$$

Then the above action (3.14) has the formula:

$$(n, \alpha) \left[ \begin{array}{c} x \\ w \end{array} \right] = \left( \begin{array}{c} n + \lambda(\alpha) \\ b_\alpha \end{array} \right), \left( \begin{array}{cc} \bar{\phi}(\alpha) & 0 \\ 0 & A_\alpha \end{array} \right) \left[ \begin{array}{c} x \\ w \end{array} \right],$$

where $\left[ \begin{array}{c} x \\ w \end{array} \right] \in \tilde{N}_i = \mathbb{R} \times \mathbb{R}^{i-1} = \mathbb{R}^i$. It follows $(n, \alpha) \in E(i)$. Since $\bar{\phi} : \pi_{i-1} \to \{ \pm 1 \} \leq \text{Aut}(\mathbb{R})$ is a unique extension of $\phi : \pi_{i-1} \to \text{Aut}(\mathbb{Z}) = \{ \pm 1 \}$, we see that the $H_i$ isomorphism $(\mathbb{Z}_2)^s, (0 \leq s \leq i)$.

This proves the induction step. \qed

Corollary 3.4. Each $S^1$-fibred nilBott manifold of finite type $M_i$ admits a homologically injective $T^k$-action where $k = \text{Rank } H_1(M_i)$. Moreover, the action is maximal, i.e. $k = \text{Rank } C(\pi_i)$. 
Proof. We suppose by induction that there is a homologically injective maximal \( T^{k-1} \)-action on \( M_{i-1} = T^{i-1}/H \) such that \( k - 1 = \text{Rank } H_1(M_{i-1}) = \text{Rank } C(\pi_{i-1}) \) \((k - 1 > 0)\). Since \( \pi_i, \pi_{i-1} \) are Bieberbach groups, there are two group extensions

\[
1 \to \mathbb{Z}^i \to \pi_i \xrightarrow{h_i} H_i \to 1
\]

\[
1 \to \mathbb{Z}^{i-1} \to \pi_{i-1} \xrightarrow{h_{i-1}} H_{i-1} \to 1
\]

where \( H_i, H_{i-1} \) are holonomy groups of \( \pi_i, \pi_{i-1} \), respectively and \( \mathbb{Z}^i = \pi_i \cap \mathbb{R}^i, \mathbb{Z}^{i-1} = \pi_{i-1} \cap \mathbb{R}^{i-1} \). We have a following diagram

\[
\begin{array}{c}
1 \\
\downarrow \\
1 \\
\downarrow \\
1 \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
1
\end{array}
\begin{array}{c}
\mathbb{Z} \\
\downarrow \\
\mathbb{Z}^i \\
\downarrow \\
\mathbb{Z}^{i-1} \\
\downarrow \\
\mathbb{Z}^i \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
1
\end{array}
\begin{array}{c}
\pi_i \\
\downarrow \\
\pi_{i-1} \\
\downarrow \\
H_i \\
\downarrow \\
H_{i-1} \\
\downarrow \\
1
\end{array}
\]

(3.17)

Let \( p : \mathbb{R}^i = \mathbb{R} \times \mathbb{R}^{i-1} \to T^i = S^1 \times T^{i-1} \) be the canonical projection such that \( \text{Ker } p = \mathbb{Z}^i = \pi_i \cap \mathbb{R}^i \). By Proposition \( \ref{prop:holonomy} \), \( H_i = (\mathbb{Z}_2)^s \) for some \( s \) \((1 \leq s \leq i)\). The action \((\pi_i, \mathbb{R}^i)\) induces an isometric action \((H_i, T^i)\) from \( \ref{prop:action} \). We may represent the action as the following

\[
\hat{\alpha}
\begin{pmatrix}
z_1 \\
z_2 \\
\vdots \\
z_i
\end{pmatrix}
= \begin{pmatrix}
t_{\hat{\alpha}} \cdot \psi(\hat{\alpha})(z_1) \\
z_2' \\
\vdots \\
z_i'
\end{pmatrix}
\]

(3.18)

here \( \hat{\alpha} = h_i((n, \alpha)) \in H_i, t_{\hat{\alpha}} = p(n + \lambda(\alpha)) \in S^1, \) and \( \psi : H_i \to \{\pm 1\}, \)

\[
\psi(\hat{\alpha})(z_1) = \begin{cases} 
z_1 & \text{if } \tilde{\varphi}(\alpha) = 1 \\
z_1^- & \text{if } \tilde{\varphi}(\alpha) = -1. \end{cases}
\]

(3.19)
Note that \((t_\alpha)^2 = p(n + \lambda(\alpha))p(n + \lambda(\alpha)) = p(2n + 2\lambda(\alpha)).\) Suppose \(\hat{\varphi}(\alpha) = 1.\) By (3.16),

\[(3.20) \quad (n, \alpha)^2 \begin{bmatrix} x \\ w \end{bmatrix} = \left( \begin{bmatrix} 2n + 2\lambda(\alpha) \\ b_\alpha + A_\alpha w \end{bmatrix}, \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ w \end{bmatrix}.
\]

Since \(2n + 2\lambda(\alpha) \in \mathbb{Z},\) then \((t_\alpha)^2 = 1\ i.e.\ t_\alpha = \pm 1.\)

If \(H_t = \text{Ker } \psi,\) it follows from (3.21) that the \(S^1\)-action on 
\(T^i = S^1 \times T^{i-1}\) as left translations induces an \(S^1\)-action on \(M_i = T^i/H_i\) so that \(T^k\)-action on \(M_i = T^i/H_i\) follows

\[(3.21) \quad \begin{bmatrix} t \\ t' \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_i \end{bmatrix} = \begin{bmatrix} t \cdot z_1 \\ z_2 \\ \vdots \\ z_i \end{bmatrix} \begin{bmatrix} t' \cdot z_1 \\ z_2 \\ \vdots \\ z_i \end{bmatrix},
\]

where \((t, t') \in S^1 \times T^{k-1}, [z_1, \ldots, z_i] \in M_i = T^i/H_i.\) On the other hand, if there is an element \(\hat{\alpha}\) of \(H_i\) which \(\psi(\hat{\alpha})(z) = \bar{z},\) then \(M_i\) admits a \(T^{k-1}\)-action by the induction hypothesis. The group extension (3.12) gives rise to a group extension:

\[(3.22) \quad 1 \rightarrow \mathbb{Z}/[\pi_i, \pi_i] \cap \mathbb{Z} \rightarrow \pi_i/[\pi_i, \pi_i] \xrightarrow{\nu_i} \pi_{i-1}/[\pi_{i-1}, \pi_{i-1}] \rightarrow 1.
\]

As in the proof of Proposition 3.3 \(\[(0, \alpha), (n, 1) = ((\phi(\alpha) - 1)(n), 1).\) It follows that \([\pi_i, \pi_i] \cap \mathbb{Z} = \{1\}\) or \([\pi_i, \pi_i] \cap \mathbb{Z} = 2\mathbb{Z}\) according to whether \(H_i = \text{Ker } \psi\) or not. So (3.22) becomes

\[(3.23) \quad 1 \longrightarrow \mathbb{Z} \longrightarrow H_1(M_i) \xrightarrow{\nu_i} H_1(M_{i-1}) \longrightarrow 1,
\]

or

\[(3.24) \quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow H_1(M_i) \xrightarrow{\nu_i} H_1(M_{i-1}) \longrightarrow 1.
\]

For (3.23), it follows \(k = \text{Rank } H_1(M_i)\) for which \(M_i\) admits a homologically injective \(T^k\)-action as above. For (3.24), \(k - 1 = \text{Rank } H_1(M_i)\) and \(M_i\) admits a homologically injective \(T^{k-1}\)-action by the induction hypothesis.

Suppose \(H_i = \text{Ker } \psi.\) Noting that the group extension \(1 \rightarrow \mathbb{Z} \rightarrow \pi_i \xrightarrow{p_i} \pi_{i-1} \rightarrow 1\) is a central extension, we obtain a group extension:

\[1 \rightarrow \mathbb{Z} \rightarrow C(\pi_i) \xrightarrow{p_i} p_i(C(\pi_i)) \rightarrow 1.
\]

On the other hand, since \(M_i\) admits the above \(T^k\)-action, \(\mathbb{Z}^k \subset C(\pi_i).\)

Let \(\text{Rank } C(\pi_i) = k + l, (l = 0, 1, 2, \ldots),\) then \(\mathbb{Z}^{k+l-1} \subset p_i(C(\pi_i)).\) By the induction hypothesis, \(k - 1 = \text{Rank } C(\pi_{i-1}) \geq \text{Rank } p_i(C(\pi_i)).\) Therefore as \(l = 0, k = \text{Rank } C(\pi_i).\)

Assume that there exists an element \(\hat{\alpha} \in H_i\) such that \(\psi(\hat{\alpha})(z) = \bar{z}.\)
It is easy to check that \( \mathbb{Z} \cap C(\pi_i) = \phi \), i.e. \( C(\pi_i) \leq C(\pi_{i-1}) \) and since \( M_i \) admits \( T^{k-1} \)-action, \( \mathbb{Z}^{k-1} \leq C(\pi_i) \). By the induction hypothesis, \( k - 1 = \text{Rank} \ C(\pi_i) \).
Therefore in each case the torus action is maximal. \( \square \)

4. 3-DIMENSIONAL \( S^1 \)-FIBRED NILBOTT TOWERS
By the definition of \( S^1 \)-fibre-nilBott manifold \( M_n \), \( M_2 \) is either a torus \( T^2 \) or a Klein bottle \( K \) so that \( M_2 \) is a Riemannian flat manifold.

4.1. 3-DIMENSIONAL \( S^1 \)-FIBRED NILBOTT MANIFOLDS OF FINITE TYPE.
Any 3-dimensional \( S^1 \)-fibre-nilBott manifold \( M_3 \) of finite type is a Riemannian flat manifold. It is known that there are just 10-isomorphism classes \( G_1, \ldots, G_6, B_1, \ldots, B_4 \) of 3-dimensional Riemannian flat manifolds. (Refer to the classification of 3-dimensional Riemannian flat manifolds by Wolf [16].) In particular, for Riemannian flat 3-manifolds corresponding to \( B_2 \) and \( B_3 \), we have shown that there are two \( S^1 \)-fibre-nilBott towers: \( B_2 \to K \to S^1 \to \{ \text{pt} \} \) and \( B_1 \to K \to S^1 \to \{ \text{pt} \} \) in [13]. Remark that every real Bott manifold is an \( S^1 \)-fibre-nilBott manifold of finite type and \( B_2 \) and \( B_4 \) are not real Bott manifolds. And the following Proposition 4.1 have been proved. See [13] for details.

**Proposition 4.1.** The 3-dimensional \( S^1 \)-fibre-nilBott manifold of finite type are those of \( G_1, G_2, B_1, B_2, B_3, B_4 \).

4.2. 3-DIMENSIONAL \( S^1 \)-FIBRED NILBOTT MANIFOLDS OF INFINITE TYPE.
Any 3-dimensional \( S^1 \)-fibre-nilBott manifold \( M_3 \) of infinite type is an infranil-Heisenberg manifold. The 3-dimensional simply connected nilpotent Lie group \( N_3 \) is isomorphic to the Heisenberg Lie group \( N \) which is the product \( \mathbb{R} \times \mathbb{C} \) with group law:

\[
(x, z) \cdot (y, w) = (x + y - \text{Im} \bar{z}w, z + w).
\]

Then the maximal compact Lie subgroup of \( \text{Aut}(N) \) is \( \mathbb{U}(1) \rtimes \langle \tau \rangle \) which acts on \( N \)

\[
e^{i\theta}(x, z) = (x, e^{i\theta}z), \ (e^{i\theta} \in \mathbb{U}(1)).
\]

\[
\tau(x, z) = (-x, \bar{z}).
\]

(4.1)

A 3-dimensional compact infranilmanifold is obtained as a quotient \( N/\Gamma \) where \( \Gamma \) is a torsionfree discrete uniform subgroup of \( \text{E}(N) = N \rtimes (\mathbb{U}(1) \rtimes \langle \tau \rangle) \). (See [4].)
Let

\[
S^1 \to M_3 \to M_2
\]
be an $S^1$-fibred nilBott manifold of infinite type which has a group extension $1 \to \mathbb{Z} \to \pi_3 \to \pi_2 \to 1$. Since $\mathbb{R} \subset \mathbb{N}$ is the center, there is a commutative diagram of central extensions (cf. (3.17)):

$$
\begin{array}{cccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Delta_3 & \longrightarrow & \Delta_2 & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{R} & \longrightarrow & N & \longrightarrow & \mathbb{C} & \longrightarrow & 1.
\end{array}
$$

Using this, we obtain an embedding:

$$
\begin{array}{cccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3 & \longrightarrow & \pi_2 & \longrightarrow & 1 \\
\downarrow & \rho & \downarrow & \hat{\rho} & \downarrow & & & & \\
1 & \longrightarrow & \mathbb{R} & \longrightarrow & E(N) & \longrightarrow & \mathbb{C} \rtimes (U(1) \rtimes \langle \tau \rangle) & \longrightarrow & 1.
\end{array}
$$

Note that $\mathbb{C} \rtimes (U(1) \rtimes \langle \tau \rangle) = \mathbb{R}^2 \rtimes O(2) = E(2)$. Since $\mathbb{R} \cap \pi_3 = \mathbb{Z}$ from (4.3), $\hat{\rho}(\pi_2)$ is a Bieberbach group in $E(2)$ so that $\mathbb{R}^2 / \hat{\rho}(\pi_2)$ is either $T^2$ or $K$.

**Case (i).** Suppose that the holonomy group of $\pi_3$ is trivial. Since $L(\pi_3) = \{1\}$ in $U(1) \rtimes \langle \tau \rangle$, it is noted that $\pi_3 = \hat{\Delta}_3$ from (1.3) and (4.2). As $\Delta_3 \leq N$, $\hat{\Delta}_3$ is isomorphic to $\Delta(k)$ defined below.

Let $k \in \mathbb{Z}$ and define $\Delta(k)$ to be a subgroup of $N$ generated by

$$c = (2k, 0), a = (0, k), b = (0, ki).$$

Put $Z = \langle c \rangle$ which is a central subgroup of $\Delta(k)$. It is easy to see that (4.4)

$$[a, b] = c^{-k}.$$

Since $\mathbb{R}$ is the center of $N$, we have a principal bundle

$$S^1 = \mathbb{R} / \mathbb{Z} \to N / \Delta(k) \to \mathbb{C} / \mathbb{Z}^2.$$

Then the euler number of the fibration is $\pm k$. (See [12] for example.)

**Case (ii).** Suppose that the holonomy group of $\pi_3$ is nontrivial. Then we note that $L(\pi_3) = \mathbb{Z}_2 \leq U(1) \rtimes \langle \tau \rangle$, but not in $U(1)$. By (3.17) $L(\pi_3) = L(\pi_2)$, so first we note that $L(\pi_2)$ is not contained in $U(1)$. Suppose that $(b, A)$ is a element of $\pi_2 \leq \mathbb{R}^2 \rtimes O(2)$. Then for any $x \in \mathbb{R}^2$, $(b, A)x \neq x$, because the action of $\pi_2$ on $\mathbb{R}^2$ is free. Therefore determinant of $(A - E)$ is zero. This implies that if $A \in SO(2)$, then $A = E$. So $L(\pi_2) = L(\pi_3)$ not in $U(1)$. Suppose that there exists an element $g \in \pi_3$ such that $L(g) = (e^{i\theta}, \tau) \in U(1) \rtimes \langle \tau \rangle$. Noting
Note that \( \alpha \) (4.1), it follows \( L(g)^2 = 1 \). Then \( L(\pi_3) = (U(1) \cap L(\pi_3)) \cdot \langle L(g) \rangle \). Let \( \pi'_3 = L^{-1}(U(1) \cap L(\pi_3)) \leq \pi_3 \) which has the commutative diagram:

\[
\begin{array}{cccccc}
1 & \longrightarrow & Z & \longrightarrow & \pi_3 & \longrightarrow & \pi_2 & \longrightarrow & 1 \\
& & \| & & \| & & \| & & \\
1 & \longrightarrow & Z & \longrightarrow & \pi'_3 & \longrightarrow & \pi'_2 & \longrightarrow & 1.
\end{array}
\]

Since \( \pi'_3 \) also acts on \( \mathbb{R}^2 \) free, \( L(\pi'_3) = U(1) \cap L(\pi_3) = \{1\} \). Hence \( L(\pi_3) = \mathbb{Z}_2 = \langle L(g) \rangle \). Note that \( M_2 \) is the Klein bottle \( K \), since \( L(\pi_2) = \mathbb{Z}_2 \). Let \( n = (x, 0) \) be a generator of \( \mathbb{Z} \leq N \). Choose \( h \in \pi_3 \) with \( L(h) = 1 \) such that the subgroup \( \langle p_3(g), p_3(h) \rangle \) is the fundamental group of \( K \). It has a relation \( p_3(g)p_3(h)p_3(g)^{-1} = p_3(h)^{-1} \). Then \( \langle n, g, h \rangle \) is isomorphic to \( \pi_3 \). In particular, those generators satisfy

\[
\begin{align*}
g h g^{-1} &= n^k h^{-1} (\exists \, k \in \mathbb{Z}), \\
g n g^{-1} &= L(g)n = \tau n = n^{-1}, \quad h n h^{-1} = L(h)n = n.
\end{align*}
\]

On the other hand, let \( \Gamma(k) \) be a subgroup of \( E(N) \) generated by

\[
n = ((k, 0), I), \quad \alpha = \left( \begin{pmatrix} 0 \ 1 \\ k \ 1 \end{pmatrix}, \tau \right), \quad \beta = ((0, k), I).
\]

Note that \( \alpha^2 = ((0, k), I) \). Then it is easily checked that

\[
\alpha n \alpha^{-1} = n^{-1}, \quad \alpha \beta \alpha^{-1} = n^k \beta^{-1}, \quad \beta n \beta^{-1} = n.
\]

Then the subgroup generated by \( \hat{\alpha}^2, \hat{\beta} \) is isomorphic to the subgroup of translations of \( \mathbb{R}^2 \); \( t_1 = \left( \begin{pmatrix} k \\ 0 \end{pmatrix} \right), t_2 = \left( \begin{pmatrix} 0 \\ k \end{pmatrix} \right) \). Let \( T^2 = \mathbb{R}^2/\langle t_1, t_2 \rangle \). Then it is easy to see that the element \( \gamma = [\hat{\alpha}] \) of order 2 acts on \( T^2 \) as

\[
\gamma (z_1, z_2) = (-z_1, z_2).
\]

As a consequence, \( \mathbb{R}^2/\langle \hat{\alpha}, \hat{\beta} \rangle = T^2/\langle \gamma \rangle \) turns out to be \( K \). So \( M_3 = N/\Gamma(k) \) is an \( S^1 \)-fibred nilBott manifold:

\[
S^1 \to N/\Gamma(k) \to K
\]

where \( S^1 = \mathbb{R}/\langle n \rangle \) is the fiber (but not an action).
Compared (4.6) with $\Gamma(k)$, $\pi_3$ is isomorphic to $\Gamma(k)$ with the following commutative arrows of isomorphisms:

\[
\begin{array}{cccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3 & \longrightarrow & \pi_2 & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \langle n \rangle & \longrightarrow & \Gamma(k) & \longrightarrow & \langle \hat{\alpha}, \hat{\beta} \rangle & \longrightarrow & 1.
\end{array}
\]

As both $(\pi_3, X_3)$ and $(\Gamma(k), N)$ are Seifert actions, the isomorphism of (4.11) implies that they are equivariantly diffeomorphic, i.e.

\[M_3 = X_3/\pi_3 \cong N/\Gamma(k).\]

This shows the following.

**Proposition 4.2.** A 3-dimensional an $S^1$-fibred nilBott manifold $M_3$ of infinite type is either a Heisenberg nilmanifold $N/\Delta(k)$ or an infranilmanifold $N/\Gamma(k)$.

## 5. Realization

Let $Q = \pi_1(K)$ be the fundamental group of $K$. $Q$ has a presentation:

\[(5.1) \quad \{g, h \mid ghg^{-1} = h^{-1}\}.\]

A group extension $1 \rightarrow \mathbb{Z} \rightarrow \pi \rightarrow Q \rightarrow 1$ for any 3-dimensional $S^1$-fibred nilBott manifold over $K$ represents a 2-cocycle in $H^2(\phi; Q, \mathbb{Z})$ for some representation $\phi$. Conversely, given a representation $\phi$, we may show any element of $H^2(\phi; Q, \mathbb{Z})$ can be realized as an $S^1$-fibred nilBott manifold.

We must consider following cases of a representation $\phi$:

**Case 1.** $\phi(g) = 1$, $\phi(h) = 1$,

**Case 2.** $\phi(g) = 1$, $\phi(h) = -1$,

**Case 3.** $\phi(g) = -1$, $\phi(h) = 1$,

**Case 4.** $\phi(g) = -1$, $\phi(h) = -1$.

Let $\phi_i$ $(i = 1, 2, 3, 4)$ be the representation $\phi$ of the **Casei** above. A 2-cocycle $[f_k] \in H^2(\phi_i; Q, \mathbb{Z})$ gives rise to a group extension

\[1 \rightarrow \mathbb{Z} \rightarrow \pi(k) \xrightarrow{p} G \rightarrow 1,
\]

where $\pi(k)$ is generated by $\tilde{g}, \tilde{h}, n$ such that $\langle n \rangle = \mathbb{Z}$, $p(\tilde{g}) = g$, $p(\tilde{h}) = h$. By (5.1),

\[(5.2) \quad \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1}.
\]
for some \( k \in \mathbb{Z} \). Note that \( [f_0] = 0 \).

**Case 1:** Since \( \phi_1 \) is trivial, \( H^2_\phi(Q, \mathbb{Z}) = H^2(Q, \mathbb{Z}) \approx H^2(K, \mathbb{Z}) \approx \mathbb{Z}_2 \). Moreover \( \pi_1(k) \) satisfies the presentation:

\[
\begin{align*}
\tilde{g} \tilde{n} \tilde{g}^{-1} &= n, \quad \tilde{h} \tilde{n} \tilde{h}^{-1} = n, \quad \tilde{g} \tilde{h} \tilde{g}^{-1} = n^k \tilde{h}^{-1}.
\end{align*}
\]

**Lemma 5.1.** The groups \( \pi_1(0), \pi_1(1) \) are isomorphic to \( \pi_1(B_1), \pi_1(B_2) \) respectively.

**Proof.** First we discuss about \( \pi_1(0) \). Let \( \tilde{g}, \tilde{h}, n \in \pi_1(0) \) be as above. Put \( \varepsilon = \tilde{g}, \quad t_1 = \tilde{g}^2, \quad t_2 = n, \quad \text{and} \quad t_3 = \tilde{h} \). Note that the group which generated by \( \varepsilon, t_1, t_2, t_3 \) coincides with \( \pi_1(0) \). Using the relation (5.3),

\[
\begin{align*}
\varepsilon^2 &= t_1, \\
\varepsilon t_2 \varepsilon^{-1} &= g\tilde{h} \tilde{g}^{-1} = \tilde{h}^{-1} = t_2^{-1}, \\
\varepsilon t_3 \varepsilon^{-1} &= \tilde{g} n \tilde{g}^{-1} = n = t_3.
\end{align*}
\]

Compared this relation with \( \pi_1(B_1), \pi_1(0) \) is isomorphic to \( \pi_1(B_1) \). (in the Wolf’s notation [16])

Second, about \( \pi_1(1) \). Let \( \tilde{g}, \tilde{h}, n \in \pi_1(1) \) be as above. Put \( \varepsilon = \tilde{g}, \quad t_1 = \tilde{g}^2, \quad t_2 = \tilde{g}^{-2} n, \quad \text{and} \quad t_3 = \tilde{h} \). The group which generated by \( \varepsilon, t_1, t_2, t_3 \) coincides with \( \pi_1(1) \). By using the relation (5.3),

\[
\begin{align*}
\varepsilon^2 &= t_1, \\
\varepsilon t_2 \varepsilon^{-1} &= \tilde{g} \tilde{h} \tilde{g}^{-1} = \tilde{g}^{-1} n \tilde{g}^{-1} = \tilde{g}^{-2} n = t_1, \\
\varepsilon t_3 \varepsilon^{-1} &= \tilde{g} \tilde{h} \tilde{g}^{-1} = \tilde{g}^2 \tilde{g}^{-2} n \tilde{h}^{-1} = t_1 t_2 t_3^{-1}.
\end{align*}
\]

This implies that \( \pi_1(1) \) is isomorphic to \( \pi_1(B_2) \). (See [16]) \( \square \)

Remark that the fundamental group \( \pi_1(B_2) \) is isomorphic to \( \pi_1(1) \) so we have a group extension which represents the \( [f_1] \in H^2_\phi(Q, \mathbb{Z}) \) with 2-torsion. Therefore, if \( [f_k] \neq 0 \), then \( k = 1 \).

**Case 2:** Let \( \phi_2(g) = 1, \phi_2(h) = -1 \), then \( \pi_2(k) \) has the following presentation.

\[
\begin{align*}
\tilde{g} n \tilde{g}^{-1} &= n, \quad \tilde{h} n \tilde{h}^{-1} = n^{-1}, \quad \tilde{g} \tilde{h} \tilde{g}^{-1} = n^k \tilde{h}^{-1}.
\end{align*}
\]

for some \( k \in \mathbb{Z} \).

**Proposition 5.2.** The groups \( \pi_1(0), \pi_1(1) \) are isomorphic to \( \pi_1(B_3), \pi_1(B_4) \) respectively.
Proof. Let \( \tilde{g}, \tilde{h}, n \in 2\pi(0) \) be as before. Put \( \alpha = \tilde{h}\tilde{g}, \varepsilon = \tilde{h}^{-1}, t_1 = g^2, t_2 = \tilde{h}^{-2}, \) and \( t_3 = n. \) Note that the group generated by \( \alpha, \varepsilon, t_1, t_2, t_3 \) coincides with \( 2\pi(0). \) Using the relation (5.4),

\[
\alpha^2 = (\tilde{h}\tilde{g})^2 = \tilde{h}\tilde{h}^{-1}\tilde{g}\tilde{g} = \tilde{g}^2 = t_1,
\]

\[
\varepsilon^2 = t_2,
\]

\[
\varepsilon\alpha\varepsilon^{-1} = \tilde{h}^{-1}\tilde{h}\tilde{g}\tilde{h} = \tilde{h}^{-1}\tilde{g} = t_2\alpha.
\]

\[
\alpha t_2\alpha^{-1} = \tilde{h}\tilde{g}h^{-2}\tilde{g}^{-1}\tilde{h}^{-1} = \tilde{h}^{-2} = t_2^{-1},
\]

\[
\alpha t_3\alpha^{-1} = \tilde{h}\tilde{g}\tilde{g}^{-1}\tilde{h}^{-1} = n^{-1} = t_3^{-1},
\]

\[
\varepsilon t_1\varepsilon^{-1} = \tilde{h}^{-1}\tilde{g}^{-2}\tilde{h} = \tilde{h}^{-1}\tilde{g}\tilde{h}^{-1}\tilde{g} = \tilde{h}^{-1}\tilde{h}\tilde{g}\tilde{g} = \tilde{g}^2 = t_1,
\]

\[
\varepsilon t_3\varepsilon^{-1} = \tilde{h}^{-1}n\tilde{h} = n^{-1} = t_3^{-1}.
\]

This relation correspond to of \( \pi_1(B_3). \) (See [16]). So \( 2\pi(0) \) is isomorphic to \( \pi_1(B_3). \)

Let \( \tilde{g}, \tilde{h}, n \in 2\pi(1) \) be as above. Put \( \alpha = \tilde{h}\tilde{g}, \varepsilon = n^{-1}\tilde{h}^{-1}, t_1 = n^{-1}\tilde{g}^2, t_2 = \tilde{h}^{-2}, \) and \( t_3 = n^{-1}. \) Using the relation (5.4), we obtain a presentation:

\[
\alpha^2 = (\tilde{h}\tilde{g})^2 = \tilde{h}n\tilde{h}^{-1}\tilde{g}\tilde{g} = n^{-1}\tilde{g}^2 = t_1,
\]

\[
\varepsilon^2 = t_2,
\]

\[
\varepsilon\alpha\varepsilon^{-1} = n^{-1}\tilde{h}^{-1}\tilde{g}\tilde{h}n = \tilde{h}^{-1}\tilde{g}n = t_3t_3\alpha.
\]

\[
\alpha t_2\alpha^{-1} = \tilde{h}\tilde{g}^{-2}\tilde{g}^{-1}\tilde{h}^{-1} = \tilde{h}^{-2} = t_2^{-1},
\]

\[
\alpha t_3\alpha^{-1} = \tilde{h}\tilde{g}^{-1}\tilde{g}^{-1}\tilde{h}^{-1} = n = t_3^{-1},
\]

\[
\varepsilon t_1\varepsilon^{-1} = n^{-1}\tilde{h}^{-1}n^{-1}\tilde{g}^2\tilde{h}n = n^{-1}t_2^{-1} = t_1,
\]

\[
\varepsilon t_3\varepsilon^{-1} = n^{-1}\tilde{h}^{-1}n^{-1}\tilde{h}n = n = t_3^{-1}.
\]

This implies that \( 2\pi(1) \) is isomorphic to \( B_4. \) (See [16]) \( \Box \)

**Proposition 5.3.** Any element of \( H^2_{\phi_2}(Q, \mathbb{Z}) \) is isomorphic to \( \mathbb{Z}_2 \)

**Proof.** Let \( Q' \) be the subgroup of \( Q \) which is generated by \( g, h^2 \in Q \) with \( gh^2g^{-1} = (ghg^{-1})^2 = h^{-2}. \) We have a commutative diagram:

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & 2\pi(k) & \longrightarrow & Q & \longrightarrow & 1 \\
& & & & & & \downarrow \phi & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi' & \longrightarrow & Q' & \longrightarrow & 1
\end{array}
\]

(5.5)
where $\pi'$ is the subgroup of $2\pi(k)$ which is generated by $n, \tilde{g}, \tilde{h}\cdot$. Note that
\[ \tilde{g}\tilde{h}^{2}\tilde{g}^{-1} = n^{k}\tilde{h}^{-1}n^{k}\tilde{h}^{-1} = \tilde{h}^{-2}. \]

Since the subgroup $\langle \tilde{g}, \tilde{h}\rangle$ of $\pi'$ maps isomorphically onto $Q'$ and a restriction $\phi|Q' = \text{id}$, then $\pi' = \mathbb{Z} \times Q'$. This shows that the restriction homomorphism $\iota' : H^{2}_{\phi}(Q, \mathbb{Z}) \to H^{2}(Q', \mathbb{Z})$ is the zero map, equivalently $\iota'[k] = 0$. Using the transfer homomorphism $\tau : H^{2}(Q', \mathbb{Z}) \to H^{2}_{\phi}(Q, \mathbb{Z})$ and by the property $\tau \circ \iota'(f) = [Q : Q'][f] = 2[f]$ ($\forall [f] \in H^{2}_{\phi}(Q, \mathbb{Z})$), we obtain $2[f] = 0$.

On the other hand, from (5.4)
\[ n^{k} = \tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h} = (0, g)(0, h)(-f_{k}(g^{-1}, g), g^{-1})(0, h) = f_{k}(g, h) + f_{k}(g^{-1}, g) + f_{k}(gh, g) + f_{k}(h^{-1}, h). \]

Since there exists a 2-cocycle $[f_{1}]$ from Proposition 5.2
\[ n = f_{1}(g, h) + f_{1}(g^{-1}, g) + f_{1}(gh, g) + f_{1}(h^{-1}, h), \]
and
\[ n^{k} = kf_{1}(g, h) + kf_{1}(g^{-1}, g) + kf_{1}(gh, g) + kf_{1}(h^{-1}, h). \]

Noting that $[f_{k}]$ represents of $2\pi(k)$ if and only if $f_{k}$ satisfies (5.6), the relation (5.7) shows that
\[ [f_{k}] = k \cdot [f_{1}]. \]

As the consequence, $H^{2}_{\phi}(Q, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$.

This gives the following result:

**Corollary 5.4.** The group extension $2\pi(k)$ is isomorphic to $\pi_{1}(B_{3})$ or $\pi_{1}(B_{4})$ accordance with $k$ is odd or even.

**Case 3:** The group $3\pi(k)$ has the following presentation. For some $k \in \mathbb{Z}$,
\[ \tilde{g}n\tilde{g}^{-1} = n^{-1}, \tilde{h}n\tilde{h}^{-1} = n, \tilde{g}\tilde{h}\tilde{g}^{-1} = n^{k}\tilde{h}^{-1}. \]

**Lemma 5.5.** The groups $3\pi(0)$, $3\pi(k)$ are isomorphic to $\pi_{1}(G_{2})$, $\Gamma(k)$ respectively. (cf. (4.7))

Proof. Let $\tilde{g}, \tilde{h}, n \in 3\pi(0)$ be as before. Put $\alpha = \tilde{g}$, $t_{1} = \tilde{g}^{2}$, $t_{2} = \tilde{h}$, and $t_{3} = n$. Note that the group generated by $\alpha, t_{1}, t_{2}, t_{3}$ coincides with $3\pi(0)$. By using the relation (5.9), it is easy to check that:
\[ \alpha^{2} = t_{1}, \]
\[ \alpha t_{2} \alpha^{-1} = t_{2}^{-1}, \]
\[ \alpha t_{3} \alpha^{-1} = t_{3}^{-1}. \]
Suppose $\tilde{g}, \tilde{h}, n \in \pi(0)$. Put $\alpha = \tilde{g}, \beta = \tilde{h}$. This implies that $\pi(0)$ is isomorphic to $\Gamma(k)$. \hfill \square

**Proposition 5.6.** $H^2_{\phi}(G, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$.

**Proof.** From Theorem 1.2 and Lemma 5.8, $H^2_{\phi}(G, \mathbb{Z})$ is torsionfree. Moreover, it satisfies (5.8). Therefore $H^2_{\phi}(G, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$. \hfill \square

**Case 4.** The group $\pi(k)$ has the following presentation.

(5.10) $\tilde{g}n\tilde{g}^{-1} = n^{-1}, \tilde{h}n\tilde{h}^{-1} = n^{-1}, \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1}$. 

Put $\alpha = gh$. It is easy to check that

(5.11) $\alpha n\alpha^{-1} = n, \tilde{h}n\tilde{h}^{-1} = n^{-1}, \alpha\tilde{h}\alpha = n^k\tilde{h}^{-1}$

Noting that $\pi(k)$ coincide with the group which is generated by $\alpha, \tilde{h}$ and $n$, we can show that $\pi(k)$ is isomorphic to $\pi(0)$.

We have shown that any element of $H^2_{\phi}(Q, \mathbb{Z})$ can be realized an $S^1$-fibred nilBott manifold, and obtain the following table.

| $\pi_1(M_3)$ | $H^2_{\phi}(Q, \mathbb{Z})$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
|--------------|-----------------|---------------|---------------|--------|
| $[f] = 0$    | $\pi_1(B_1)$    | $\pi_1(G_2)$  | $\pi_1(B_3)$  | $\pi_1(B_4)$ |
| $[f] \neq 0$: torsion | $\pi_1(B_2)$    | $\pi_1(B_4)$  | -             | -      |
| $[f]$: torsionfree | -             | -             | $\Gamma(k)$  | -      |

Let $\mathbb{Z}^2 = \pi_1(T^2)$ be the fundamental group of $T^2$ which is generated by $\alpha, \beta$.

Given a representation $\phi$, we may show any element of $H^2_{\phi}(\mathbb{Z}^2, \mathbb{Z})$ can be realized as an $S^1$-fibred nilBott manifold.

We must consider following cases of a representation $\phi$:

**Case 5.** $\phi(\alpha) = 1, \phi(\beta) = 1$,

**Case 6.** $\phi(\alpha) = 1, \phi(\beta) = -1$,

**Case 7.** $\phi(\alpha) = -1, \phi(\beta) = -1$.

Let denote $\phi_i$ as before. In each case, a 2-cocycle $[f_k] \in H^2_{\phi}(\mathbb{Z}^2, \mathbb{Z})$ gives rise to a group extension

$$1 \to \mathbb{Z} \to \pi(k) \to \pi(k) \to 1,$$

where $\pi(k)$ is generated by $\tilde{\alpha}, \tilde{\beta}, m$ such that $\langle m \rangle = \mathbb{Z}, p(\tilde{\alpha}) = \alpha, p(\tilde{\beta}) = \beta$. By (5.1),

(5.12) $\tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1} = m^k\tilde{\beta}$. 

for some \( k \in \mathbb{Z} \).

**Case 5:** The group \( 5\pi(k) \) has the following presentation.

\[
\tilde{\alpha}m\tilde{\alpha}^{-1} = m, \quad \tilde{\beta}m\tilde{\beta}^{-1} = m, \quad \tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1} = m^k\tilde{\beta}.
\]

for some \( k \in \mathbb{Z} \), this relation gives the following proposition. (See (4.4)).

**Proposition 5.7.** The groups \( 5\pi(0), 5\pi(k) \) are isomorphic to \( \pi_1(T^3), \pi_1(\Delta(-k)) \) respectively.

**Case 6:** The group \( 6\pi(k) \) has the following presentation.

\[
\tilde{\alpha}m\tilde{\alpha}^{-1} = m, \quad \tilde{\beta}m\tilde{\beta}^{-1} = m^{-1}, \quad \tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1} = m^k\tilde{\beta}.
\]

for some \( k \in \mathbb{Z} \).

**Proposition 5.8.** The groups \( 6\pi(0), 6\pi(1) \) are isomorphic to \( \pi_1(B_1), \pi_1(B_2) \) respectively.

**Proof.** First let \( k = 0 \). Put \( m = \tilde{h}, \tilde{\alpha} = n, \tilde{\beta} = \tilde{g}, \) then we can check easily that \( 6\pi(0) \) is isomorphic to \( 1\pi(0) \), i.e. \( \tilde{g}n\tilde{g}^{-1} = n, \tilde{h}n\tilde{h}^{-1} = n, \tilde{g}\tilde{h}\tilde{g}^{-1} = \tilde{h}^{-1} \). So \( 6\pi(0) \) is isomorphic to \( \pi_1(B_1) \).

Second suppose \( k = 1 \). Put \( m = n, \tilde{\alpha} = \tilde{g}, m^{-1}\tilde{\beta} = \tilde{h}, \) then we can check easily that \( 6\pi(1) \) is isomorphic to \( \pi_1(B_2) \) in the same way above. \( \square \)

Moreover we can obtain the following after the fashion of proof for the Proposition 5.3.

**Case 7:** The group \( 7\pi(k) \) has the following presentation.

\[
\tilde{\alpha}m\tilde{\alpha}^{-1} = m^{-1}, \quad \tilde{\beta}m\tilde{\beta}^{-1} = n^{-1}, \quad \tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1} = m^k\tilde{\beta}.
\]

for some \( k \in \mathbb{Z} \). Then it easy check that \( 7\pi(k) \) is isomorphic to \( 6\pi(k) \) in the same way for Case 4 above.

As a consequence, we obtain a table:

| \( \pi_1(M_3) \) | Case 1 | Case 2 and 3 |
|------------------|--------|-------------|
| \( H^j_\phi(\mathbb{Z}^2, \mathbb{Z}) \) | \( \mathbb{Z} \) | \( \mathbb{Z}_2 \) |
| \( [f] = 0 \) | \( \mathbb{Z}^2 \) | \( \pi_1(B_1) \) |
| \( [f] \neq 0: \text{torsion} \) | - | \( \pi_1(B_2) \) |
| \( [f]: \text{torsionfree} \) | \( \Delta(k) \) | - |

**Theorem 5.9** (Halperin-Carlsson conjecture [14]). Let \( T^s \) be an arbitrary effective action on an \( m \)-dimensional \( S^1 \)-fibred nilBott manifold \( M \) of finite type. Then

\[
\sum_{j} C_j \leq b_j \text{ (the } j\text{-th Betti number of } M \text{).}
\]
In particular \(2^s \leq \sum_{j=0}^{m} \text{Rank } H_j(M)\).

**Proof.** By Corollary 3.4, \(M\) admits a homologically injective \(T^k\)-action where \(k = \text{Rank } C(\pi)\) where \(\pi = \pi_1(M)\). Then we have shown in [6] that any homologically injective \(T^k\)-actions on any closed aspherical manifold satisfies that \(kC_j \leq b_j\).

It follows from the result of Conner-Raymond[3] that there is an injective homomorphism \(1 \to \mathbb{Z}^s \to C(\pi)\). This shows that \(s \leq k\) so we obtain

\[
(5.17) \quad sC_j \leq b_j (= \text{the j-th Betti number of } M).
\]

\[\square\]

**Remark 5.10.** This result is obtained when \(M_i\) is a real Bott manifold by Masuda, Choi and Oum.

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Department of Mathematics and Information of Sciences, Tokyo Metropolitan University, Minami-Ohsawa 1-1, Hachioji, Tokyo 192-0397, JAPAN

E-mail address: nakayama-mayumi@ed.tmu.ac.jp