Schwarzschild horizon and the gravitational redshift formula

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The gravitational redshift formula is usually derived in the geometric optics approximation. In this note we consider an exact formulation of the problem in the Schwarzschild spacetime, with the intention to clarify under what conditions this redshift law is valid. It is shown that in the case of shocks the radial component of the Poynting vector can scale according to the redshift formula, under a suitable condition. If that condition is not satisfied, then the effect of the backscattering can lead to significant modifications. The obtained results imply that the energy flux of the short wavelength radiation obeys the standard gravitational redshift formula while the energy flux of long waves can scale differently, with redshift being dependent on the frequency.

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I. INTRODUCTION

The gravitational redshift formula in the Schwarzschild spacetime is

\[ \omega' = \sqrt{1 - \frac{2m}{a} \omega} \]  

relates the frequency \( \omega' \) detected by an observer located at \( R \) with the initial frequency \( \omega \) of a photon emitted from an areal distance \( a \) from the gravitational center.

Formula (3) is derived in the approximation of geometric optics (see [3], [4], [5], [6]). The aim of this paper is to present an exact treatment of the problem within the framework of the classical wave theory; the only simplification consists in neglecting the backreaction effect. It is assumed that an isolated pulse of an electromagnetic wave is emitted outward. Its initial support is contained in the annulus \((a, b)\). Initial pulses that are characterized by \((b - a)/a \ll 1\) will be - for the sake of brevity - referred to as shocks. One can define a radial component of the energy flux \( P_R \) by a suitable spherical projection of the Poynting vector. It will be shown that if the relative width \((b - a)/a \) is much smaller than some power of the relative distance \((a - 2m)/a \) from the horizon then the radial energy flux of a shock conforms to a relation like in Eq. (1).

The order of this paper is following. Sec. II brings the main result and outlines its proof. Sec. III is devoted to the estimation of a reduced electromagnetic potential. In Secs IV and V we derive bounds on the radiation amplitudes. Sec. VI proves the desired flux relation. Last section is dedicated to a short summary and discussion on the limitations of the redshift formula.

II. MAIN RESULTS

The space-time geometry is defined by the Schwarzschild line element,

\[ ds^2 = -(1 - \frac{2m}{R})dt^2 + \frac{1}{1 - \frac{2m}{R}} dR^2 + R^2 d\Omega^2, \]  

where \( t \) is a time coordinate, \( R \) is a radial coordinate that coincides with the areal radius and \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) is the line element on the unit sphere, \( 0 \leq \phi < 2\pi \) and \( 0 \leq \theta \leq \pi \). Throughout this paper \( G \), the Newtonian gravitational constant, and \( c \), the velocity of light are put equal to 1. We define the Regge-Wheeler coordinate \( r^* = R + 2m \ln (\frac{R}{2m} - 1) \) and, for the sake of concise notation, \( \eta_R \equiv 1 - \frac{2m}{R} \).

The Maxwell equations read

\[ \nabla_\mu F^\mu = 0, \]  

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( A_\mu \) is the electromagnetic potential. It is convenient to assume \( A_0 = 0 \) and the Coulomb gauge condition \( \nabla_i A^i = 0 \). It is useful to follow Wheeler [7] and employ the multipole expansion. We will consider only the dipole term and, more specifically, choose the potential one-form \( A = \sqrt{3/2} \sin^2 \theta \Psi (r^*, t) d\phi \). A similar analysis with the same conclusions can be done in any multipole order.

We seek, following [8], a reduced dipole potential \( \Psi (r^*, t) \) in the form \( \Psi = \tilde{\Psi} + \delta \), where \( \delta \) satisfies the dipole equation

\[ (-\partial_0^2 + \partial_{r^*}^2) \delta = \eta_R \left[ \frac{2}{R^2} \delta + \frac{6mf}{R^4} \right]. \]  

Here \( \tilde{\Psi} (r^*, t) = \partial_0 f (r^* - t) + \frac{(r^* - t)}{R} \), \( \partial_0 \equiv \partial_t \) and \( f \) is an arbitrary function with support in \((a, b)\). It is well known that \( \tilde{\Psi} \) solves Maxwell equations in Minkowski spacetime [9]. \( f \) can be uniquely determined from initial data corresponding to an initially outgoing radiation. Initially \( \delta = \partial_0 \delta = 0 \).

The stress-energy tensor of the electromagnetic field is

\[ T^{\mu\nu} = (1/2)(F^\mu_\gamma F^{\nu\gamma} - (1/4)g^{\mu\nu} F_{\gamma\delta} F^{\gamma\delta}) \]  

and the time-like translational Killing vector is denoted as \( \zeta \).

We define the projected energy density...
\[ \hat{\rho} = \int_{S(R)} dS(R) T^\alpha_\alpha \sqrt{g_{RR}} = 2\pi \left( \frac{\partial_\alpha \Psi^2}{\eta_r} + \eta_r (\partial_r \Psi)^2 + \frac{2(\Psi)^2}{r^2} \right) \]  

The energy \( E_R(t) \) of the electromagnetic field \( \Psi \) contained in the exterior of a sphere \( S(R) \) of the radius \( R \) reads \( E_R(t) = \int R^\infty ds \hat{\rho} \). Let \( n \) be the unit normal to \( S(R) \). One finds that \( \partial_t E_R(t) = n^R \hat{P}_R \); thus the energy flux \( \frac{dE}{dt}|_{R=\text{const}} \) (here \( ds = \sqrt{g_{RR}} dt \) is the proper time interval) through \( S(R) \) equals to \( \hat{P}_R \). \( \hat{P}_R \) is the surface integral of the normal component of the Poynting vector,

\[ \hat{P}_R(R,t) = \int_{S(R)} dS(R) T^\nu_{\nu} \eta_r = -\frac{4\pi}{\eta_R} \partial_\eta \Psi \partial_{\nu} \Psi. \]

It is convenient to introduce the reduced strength field amplitudes \( h_+ \) and \( h_- \)

\[ h_+ \equiv \frac{1}{\eta_R} \left( -\partial_\eta \Psi + \partial_r \Psi \right), \]
\[ h_- \equiv \frac{1}{\eta_R} \left( \partial_\eta \Psi + \partial_r \Psi \right). \]

\( h_+ \) and \( h_- \) represent the outgoing radiation and ingoing radiation, respectively. Eq. (3) reads, in terms of the amplitudes \( h_+ \) and \( h_- \),

\[ \hat{P}_R(R,t) = \frac{\pi}{\eta_R} \left( (\eta_R h_+)^2 - (\eta_R h_-)^2 \right). \]

We define

\[ \epsilon \equiv (b-a)/(an^2), \]

a quotient of the relative width of the pulse by 5-th power of the relative distance \( \eta \) from the event horizon. This quantity, with this particular power of \( \eta_r \), is needed in order to prove a sufficient condition for the validity of the redshift formula. It is remarkable that \( \epsilon \) depends both on the relative width and on the relative distance from the Schwarzschild horizon. We comment on this point in the last section.

One can prove a number of properties that are valid for shocks, provided that they satisfy the condition \( \epsilon << 1 \). These are

i) the amplitude \( h_+(R) \) is well approximated by \( \tilde{h}_+ \), where

\[ \hat{h}_+(R,t) = \frac{1}{\eta_R} (-\partial_\eta + \partial_r) \left( \delta + \partial_t f \right); \]

ii) the function \( \tilde{h}_+(R(t),t) \) conforms to the scaling law \( \eta_R \tilde{h}_+(R(t),t) = \eta_0 \tilde{h}_+(R(0),t) \);  

iii) \( h_- \) is negligible.

Let us define \( \tilde{\Gamma}_{R_0} \) as a null geodesic directed outward from the point \( R_0 \) of the initial hypersurface and let \( \tilde{\Gamma}_{R_0,(R,t)} \) be a segment of \( \tilde{\Gamma}_{R_0} \) that connects \( R_0 \) and \( (R,t) \).

Comparing the energy fluxes through the spheres \( S(R) \) (where \( R >> 2m \)) and the initial \( S(R_0) \) one obtains, from (5) and properties i) - iii), that

\[ \hat{P}_R(R) \approx \sqrt{\frac{\eta_R}{\eta_0}} \hat{P}_R(R_0), \]

where it is assumed that \( R \) and \( R_0 \) are connected by the null geodesic segment \( \tilde{\Gamma}_{R_0,(R,t)} \).

III. ESTIMATING THE REDUCED ELECTROMAGNETIC POTENTIAL

In this section we sketch briefly derivation of bounds on the electromagnetic potential. There are two bounds. One of them (Eqs. (12 and (13)) applies to initial data (or a fixed Cauchy hypersurface), and it is essentially a Sobolev-type inequality. The second bound (Eq. (21)) is proven by the application of the energy method succeeded by a Sobolev-type argument. A detailed calculation can be found in [1].

One can show, in the case of initial data of compact support, that on the initial hypersurface

\[ \frac{4\pi f^2(R)}{r^2} \leq E_a a_0^2 F(m, y), \]

where \( A \leq R \leq b \) and

\[ F(m, y) \equiv y - 1 - \frac{16\tilde{m}^4}{3(-y + 2m)^3} - \frac{16\tilde{m}^4}{3(-1 + 2m)^3} + \frac{16\tilde{m}^3}{(-1 + 2m)^2} - \frac{16\tilde{m}^3}{(-y + 2m)^2} + \frac{24\tilde{m}^2}{-y + 2m} + \frac{24\tilde{m}^2}{-1 + 2m} + 8\tilde{m} \ln \frac{y - 2\tilde{m}}{1 - 2\tilde{m}}. \]

Here \( \tilde{m} \equiv m/a \) and \( y = b/a \). Eq. (12) is a special case of a result proven in [1].

It is useful to proceed as in [1] and define an energy \( H(R, t) \) of the field \( \delta \),

\[ H(R, t) = \int R^\infty dr \left( \frac{(\partial_t \delta)^2}{\eta_r} + \eta_r (\partial_r \delta)^2 + \frac{2\delta^2}{r^2} \right), \]

The upper integration bound is in fact finite, since the original support of \( f \) was finite and therefore \( \delta \) must also have a finite support if \( t < \infty \). This is, however, irrelevant for us. It can be shown that

\[ \sqrt{H(a_t, t)} \leq 6m \int_0^t dt \left( \int_0^\infty \frac{dr f^2}{r^8} \right)^{1/2}, \]

here the external integration countour (the \( dt \) - integral) coincides with null geodesic \( \tilde{\Gamma}_a \). The internal (\( dr \)-) integration is done on a fixed (\( t = \text{const} \)) Cauchy hypersurface. It is useful to replace the integration parameter \( t \) with \( \delta \).
by the areal radius $a_t$ and to introduce the dimensionless variables $x = a_t/a, y = r/a$. Provided that is done, the insertion of (12) into (15) leads to

$$\sqrt{H(a_t, t)} \leq 6 \frac{m}{a} \sqrt{\frac{E_a F(m, y)}{4\pi}} \int_1^\infty \frac{dx}{1 - \frac{2m}{x}} \left( \int_x^\infty \frac{dy}{y^6} \right)^{1/2}. \quad (16)$$

The internal integration can be done explicitly, so that

$$\sqrt{H(a_t, t)} \leq 6 \frac{m}{a} \sqrt{\frac{E_a F(m, y)}{4\pi}} \times \int_1^\infty \frac{dx}{1 - \frac{2m}{x}} \left( \frac{1}{5x^5} - \frac{\dot{m}}{x^6} + \frac{12\dot{m}^2}{7x^7} - \frac{\dot{m}^3}{x^8} \right)^{1/2}. \quad (17)$$

The right hand side of (17) is bounded from above by

$$\sqrt{H(a_t, t)} \leq 6 \frac{m}{a} \sqrt{\frac{E_a F(m, y)}{4\pi}} \left( \int_1^\infty \frac{dx}{(x - 2\dot{m})^2} \right)^{1/2} \times \left( \int_1^\infty dx \left[ \frac{1}{5x^5} - \frac{\dot{m}}{x^6} + \frac{12\dot{m}^2}{7x^7} - \frac{\dot{m}^3}{x^8} \right] \right)^{1/2}; \quad (18)$$

here the integration has been extended to $\infty$ and the Schwarz inequality has been used. This now can be calculated exactly, but the numerical factors do not really matter - from our point of view the only interesting fact is that

$$\sqrt{H(a_t, t)} \leq C \sqrt{\frac{F(m, y)}{\eta_a}}, \quad (19)$$

where $C$ is some (time-independent) constant. Later on we will always use the same symbol $C$ in order to denote various numerical factors that appear in the course of calculations.

One can derive a bound

$$\frac{[\delta]}{R} \leq \sqrt{\frac{\dot{m}}{\dot{\eta}} \frac{H(a_t, t)}{\eta_R}}; \quad (20)$$

in order to do this, observe that $\frac{[\delta]}{R} = \int_R^\infty \frac{dr}{r} \frac{[\delta]}{r}$. The use of the Schwarz inequality yields now immediately, taking into account (1), $\frac{[\delta]}{R} \leq \sqrt{\int_1^\infty \frac{dr}{(r - 2m)^2}} \sqrt{2H} = \frac{2H(a_t, t)}{\eta_R \eta_a}$.

Formulæ (19) and (20) give an estimate

$$\frac{[\delta]}{R} \leq \frac{C}{\eta_a} \sqrt{\frac{F(m, y)}{\eta}}. \quad (21)$$

### IV. BOUNDS ON THE INGOING RADIATION

The amplitude $h_-$ of the ingoing radiation can be split as follows

$$h_- = \tilde{h}_- + \hat{h}, \quad (22)$$

where

$$\tilde{h}_- = \frac{1}{\eta_R} (\partial_0 + \partial_r) \tilde{\Psi} \quad (23)$$

and

$$\hat{h} = \frac{1}{\eta_R} (\partial_0 + \partial_r) \delta. \quad (24)$$

$\tilde{h}_-$ represents the amplitude related to $\tilde{\Psi}$ while $\hat{h}$ is the term induced by the backscatter. We shall bound $h_-$ within the main flow inside that region of the Schwarzschild spacetime that is bounded by null cones spanned by spherical bundles of geodesics $\Gamma_a$ and $\Gamma_b$, $t > 0$.

A straightforward algebra yields $\tilde{h}_- = \frac{1}{\eta_R} (\partial_0 + \partial_r) \tilde{\Psi} = \frac{h}{R}$. The use of (12) gives

$$|\tilde{h}_-| \leq \frac{\eta_R}{R} \left( \frac{E_a F(m, y)}{4\pi} \right)^{1/2} \leq \frac{C}{R} \sqrt{F(m, y)}. \quad (25)$$

The function $\hat{h}$ vanishes identically at $t = 0$, by the definition of $\tilde{\Psi}$. Below we derive a bound on the function $\hat{h}$.

The evolution equation (4) can be rewritten as

$$(-\partial_0 + \partial_r) \left( \frac{\eta_R R}{R} \hat{h} \right) = \eta_R \left[ \frac{2}{R^2 \delta + 6mf}{\eta} \right]. \quad (26)$$

Define a null geodesic $\Gamma_c$ that is directed inward from a point $c$ of the initial hypersurface. Notice that $\hat{h}$ vanishes identically along the outgoing null geodesic $\Gamma_b$. Taking this into account one obtains, integrating (26) along $\Gamma_c$, $c > b$,

$$\eta_R \hat{h}(R, t) = \int_c^R \frac{dr}{r^2 \delta + 6mf} \left[ \frac{2}{R^2 \delta + 6mf} \right]. \quad (27)$$

The insertion of the estimate (21) on the function $\delta$ as well as of the bound (12) on $\hat{f}$, allows one to deduce from (27)

$$|\hat{h}| \eta_R \leq C \sqrt{\frac{F(m, y)}{\eta_R}}. \quad (28)$$
V. SCALING OF THE AMPLITUDE OF THE OUTGOING RADIATION

First we show that in the case of shocks $h_+$ can be approximated by $\tilde{h}_+$. Indeed, from the definitions of $h_+$ and $\tilde{h}_+$ (\ref{29} and \ref{31}, respectively) follows

$$h_+ = \tilde{h}_+ - \frac{f}{R^2}. \quad \text{(29)}$$

We can conclude, taking into account the bound \ref{12}, that

$$|h_+ - \tilde{h}_+| = \frac{|f|}{R^2} \lesssim \frac{1}{R} \sqrt{F(m, y)}. \quad \text{(30)}$$

One can easily see, by inspection of formulae \ref{29} and \ref{30}, that $\sqrt{F(m, y)} / \eta_a < C \sqrt{\epsilon}$. Therefore the difference in \ref{30} becomes infinitesimally small if $\epsilon << 1$.

The evolution equation \ref{31} can be written in terms of $\tilde{h}_+$ as

$$(\partial_0 + \partial_r)(\eta \tilde{h}_+) = \eta R \left[ \frac{2}{R^2} \delta + 6m f R \right]. \quad \text{(31)}$$

Let us recall, that the right hand side of \ref{31} can be pointwise bounded, using \ref{12} and \ref{21}. The integration of \ref{31} along a null geodesic $\Gamma_{R_0}$, $a \leq R_0 \leq b$ yields now

$$\eta \tilde{h}_+(R, t) - \eta_R \tilde{h}_+(R_0) \leq C \frac{\sqrt{F(m, y)}}{\eta_a}. \quad \text{(32)}$$

Fixing the energy $E_a$, one notices that by choosing $\epsilon << 1$ the right hand side of \ref{32} can be made arbitrarily small (see the remark following Eq. \ref{33}). Thus the product $\eta \tilde{h}_+$ is constant. In this case one clearly sees the manifestation of the redshift - the rescaling of the amplitude $\tilde{h}_+$. If $R >> 2m$ (a distant observer), then $\tilde{h}_+(R) = \eta_R \tilde{h}_+(R, t = 0)$. A similar fact was discussed earlier in the context of a massless scalar field theory \ref{34}.

VI. CALCULATING THE FLUX.

Recall the formula \ref{33}:

$$\hat{P}_R(R, t) = \frac{\pi}{\sqrt{\eta_R}} \left( (\eta R \tilde{h}_+)^2 - (\eta R \tilde{h}_-)^2 \right). \quad \text{(33)}$$

The estimates of the preceding section allow us to write, in the case of shocks, the initial value of the flux

$$\hat{P}_R(R_0, 0) = \frac{\pi}{\sqrt{\eta_R}} \left( (\eta_R \tilde{h}_+)^2 + O(\sqrt{F(m, y)}) \right) \quad \text{(34)}$$

and (if $(r, t) \in \tilde{\Gamma}_{R_0}$)

$$\hat{P}_R(R, t) = \frac{\pi}{\sqrt{\eta_R}} \left( (\eta \tilde{h}_+)^2 + O(\sqrt{F(m, y)}) \right). \quad \text{(35)}$$

One can show, as pointed in Sec. V, that $F(m, y) < C n_\epsilon^2 \epsilon$. The product $\eta_R \tilde{h}_+$ is constant if $\epsilon << 1$, as shown in Sec V. Therefore one obtains, comparing \ref{33} and \ref{35}, the required relation

$$\hat{P}_R(R, t) = \sqrt{\frac{\eta_R}{\eta}} \hat{P}_R(R_0, 0). \quad \text{(36)}$$

VII. CONCLUSIONS

We have shown that if a relative width $(b - a)/a$ of a shock is much smaller than some power of its relative distance $\eta_a$ from the horizon of the Schwarzschild black hole, then the radial energy flux satisfies Eq. \ref{33}. The condition that $\epsilon << 1$ can be interpreted (appealing to the so-called similarity theorem of the Fourier transform theory \ref{11}) as the demand that the radiation is dominated by sufficiently high frequencies. Thus one infers that in the high frequency limit the backscatter is negligible: invoking now to the geometric optics approximation, one can relate the energy fluxes with frequencies \ref{33} and the standard redshift formula \ref{34} follows. It is necessary to point out, that barring backscatter, all of the energy of an outgoing pulse would get to infinity, although its energy flux would have to obey \ref{33}. The only mechanism that can diminish the energy is the backscatter of the radiation off the curvature of the spacetime.

The compactness condition $\epsilon << 1$ is sufficient but not necessary for the validity of the standard redshift formula. The necessary condition is probably weaker - $(b - a)/(an^k) << 1$, where $k$ is some number not bigger than 5 and strictly bigger than 1. In the case of frequencies much lower than a critical value $\eta_a^2 / (b - a)$ the asymptotic energy flux might well disobey the scaling law \ref{33}. That suggests that the effect of the backscattering can be discovered by the observation of discrete spectra. Any attempt to fit observed data to the simple scaling law of \ref{34} would lead to redshifts depending on the frequency, in the case of a significant backscatter.

In the present paper the consideration is focused only on the dipole term, but a similar analysis with the same conclusions can be done in any multipole order. The key part that would require a modification is the Sobolev type estimate of section III; the adaptation of the remaining parts of the procedure is straightforward.

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