An iterative estimation for disturbances of semi-wavefronts to
the delayed Fisher-KPP equation

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Abstract
We give an iterative method to estimate the disturbance of semi-wavefronts of the equa-
tion: \( \dot{u}(t,x) = u''(t,x) + u(t,x)(1 - u(t-h,x)) \), \( x \in \mathbb{R} \), \( t > 0 \); where \( h > 0 \). As a consequence, we show the exponential stability, with an unbounded weight, of semi-
wavefronts with speed \( c > 2\sqrt{2} \) and \( h > 0 \). Under the same restriction of \( c \) and \( h \), the uniqueness of semi-wavefronts is obtained.

Keywords: semi-wavefront, stability, uniqueness, delay, reaction-diffusion equations
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1. Introduction
In this work, we present some results that answer key questions about bounded sol-
lutions of the form \( u(t,x) := \phi_c(x + ct) \) to the diffusive Hutchison equation, also called delayed Fisher-KPP equation, with delay \( h \geq 0 \),

\[
\dot{u}(t,x) = u''(t,x) + u(t,x)(1 - u(t-h,x)), \quad t > 0, \quad x \in \mathbb{R},
\]

(1)

(here \( \dot{\cdot} \) indicates the temporal derive whereas \( \,' \) indicates the spatial derive) satisfying \( \phi_c(-\infty) = 0 \) and \( \lim_{z \to +\infty} \phi_c(z) > 0 \) for some speed \( c > 0 \) and profile \( \phi_c : \mathbb{R} \to \mathbb{R}_+ \). Such solutions are called semi-wavefronts which clearly satisfy the ordinary differential equation with delay

\[
\phi''_c(z) - c\phi'_c(z) + \phi_c(z)(1 - \phi_c(z)(1 - ch)) = 0, \quad z \in \mathbb{R}.
\]

If \( \phi_c(+\infty) = 1 \) semi-wavefronts are called wavefronts.

If \( h = 0 \), the questions on existence, uniqueness, geometry and stability of wavefronts have been satisfactory responded (see, e.g. [11] and [15] and references therein). In this case the general conclusions are: (i) semi-wavefronts are indeed monotone wavefronts existing for all \( c \geq 2 \), (ii) two wavefronts with same speed are unique up to translations and (iii) wavefronts are stable under suitable perturbations.

However, for \( h > 0 \) it has been only recently established the existence of semi-
wavefronts on the domain \( \{(h,c) \in \mathbb{R}^2 : h \geq 0 \text{ and } c \geq 2\} \) (see [7] and [2]). The
study of the existence of wavefronts to (1) was initiated by Wu and Zou [12] who constructed a monotone integral operator for small $h$, depending on $c$, whose iterative application to adequate sub and super-solutions converges to a monotone wavefront. Next, Hasik and Trofimchuk [7] demonstrated that such monotone wavefronts are unique up to translations. Moreover, the existence and uniqueness of monotone wavefronts has been completely established [6, 7]. In [7] and [2] the authors gave significative information about the geometry of semi-wavefronts, they demonstrated the existence of non-monotone wavefronts in the domain $\{(h, c) \in \mathbb{R}^2 : 0 \leq h \leq 1 \text{ and } c \geq 2\}$ and the existence of asymptotically periodic semi-wavefronts whenever $h > \pi/2$, in a neighborhood of $\pi/2$, and $c(h)$ sufficiently large (see [7, Theorem 1.7] and [2, Theorem 2.3]). One of the open problems proposed in [2] is to prove that the semi-wavefronts are unique, up to translation, in $\Omega = \{\phi \in \mathbb{R}^2 : h \geq 0 \text{ and } c \geq 2\}$. In this paper we obtain the uniqueness of semi-wavefronts in a region of parameters $(h, c)$ that includes the region $\{(h, c) \in \mathbb{R}^2 : h \geq 0 \text{ and } c > 2\sqrt{2}\}$, and we provide a result about the stability of these semi-wavefronts: as much as we know, there are no results concerning the stability of semi-wavefronts of (1) when $h > 0$. In our main result, the description of domain the parameters $(h, c)$ to the stability depends on a uniform estimation of semi-wavefronts. Thus, the estimation $\phi_+(z) \leq e^{ch}$, for all $z \in \mathbb{R}$, obtained in [7], turns out very profitable.

The study of the asymptotic behavior of solutions to reaction-diffusion equations with delay normally require some kind of maximum principle. For example, the Mackey-Glass type equations

$$\dot{u}(t, x) = u''(t, x) - u(t, x) + g(u(t - h, x)), \quad t > 0, x \in \mathbb{R},$$

has been widely studied under the KPP condition $g(u) \leq g'(0)u$, $u > 0$, for which, initially, the monotonicity of $g$ was assumed (see, e.g., [8]). Then, it was shown in [12] that the conclusion of stability of semi-wavefronts to (2) can be obtained only assuming the Lipschitz continuity of $g$ (type monostable) by a suitable application of the maximum principle. However, this technique does not work with (1) since this equation is strongly non monotone.

Our technique is closed to the contractive approach used by Travis and Webb [13] which establishes the stability and uniqueness of stationary solutions to nonlinear partial functional differential equation $\dot{u}(t) = Au(t) + F(u(t), t > 0$, where $u(t) := u(t + \theta)$ with $\theta \in [-h, 0]$ and $u : [-h, +\infty) \to X$ for some Banach space $X$, $F : C([-h, 0], X) \to X$ and $A$ is a certain linear operator. The main assumption in [13] is that $A$ generates a strongly continuous semigroup with a growth bound greater than the global Lipschitz constant of $F$. Unfortunately, the Travis-Webb approach does not apply to (1) because the nonlinearity is not Lipschitz in the usual spaces. However, we show that this technique can be adapted using a variation of parameter formula given in terms of a certain fundamental solution to the linearization of (1) around of a semi-wavefront. Here we follow the theory of fundamental solutions constructed by Friedman [4].

So, in order to pose the Cauchy problem we take an initial datum in the space $C := C([-h, 0], C^{1,1}(\mathbb{R}))$ where, as usual,

$$C^{1,1}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \text{ is bounded and Lipschitz continuous}\}$$

with norm

$$|f|_{C^{1,1}} := \sup_{x \in \mathbb{R}} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$
and the norm in $\mathcal{C}$ is the sup-norm, i.e., $|f|_{\mathcal{C}} := \sup_{s \in [-h,0]} |f(s)|_{C^1,1}$. With all this notation, we give the following global existence result to solutions of (1):

**Proposition 1.** If $u_0 \in \mathcal{C}$ then there exists a unique solution $u(t, \cdot)$ to (1) such that $u(t, x) = u_0(t, x)$ for all $(t, x) \in [-h, 0] \times \mathbb{R}$ and $u \in \mathcal{C}$ for all $t \geq 0$. Also, $\lim_{t \to 0} u(t, x) = u_0(0, x)$ for all $x \in \mathbb{R}$ and if $u_0 \geq 0$ then $u(t, x) \geq 0$ for all $(t, x) \in [-h, +\infty) \times \mathbb{R}$.

**Proof.** We denote the fundamental solution to the heat equation by $\Gamma_0(t, \cdot) := e^{-x^2/4t}/2\sqrt{\pi t}$.

Next, from [13, Lemma 4.1] the integral equation

$$u(t, x) = \Gamma_0(t, \cdot) * u_0(0, \cdot)(x) + \int_0^t \Gamma_0(t - s, \cdot) * [1 - u(t - h, \cdot)]u(t, \cdot)(x)ds,$$

has a unique bounded solution on $[0, h] \times \mathbb{R}$ which is a solution to (1) since $u(t - h, \cdot)$ is Lipschitz continuous, uniformly with respect to $t \in [0, h]$. Then, differentiating (3) we conclude that

$$|u_x(t, x)| \leq \text{ess sup}_{y \in \mathbb{R}} |u_x(0, y)| + \frac{\sqrt{t}}{2\sqrt{\pi}} \sup_{s, y \in [0, h] \times \mathbb{R}} |u(s, y)[1 - u(s - h, y)]|$$

on $[0, h] \times \mathbb{R}$ (4)

so $u(t, \cdot)$ is Lipschitz continuous, uniformly with respect to $t \in [0, h]$. Repeating the process to the intervals $[h, 2h], [2h, 3h], \ldots$, the first assertion is true. Next, since the parabolic operator $Lu = u'' + (1 - u_0)u - \dot{u}$ has a fundamental solution on $[0, h] \times \mathbb{R}$ (see, e.g., [4, Chapter 1, Theorem 10]) we have that $u(t, x)$ tends to $u_0(0, x)$ when $t$ tends to 0, for all $x \in \mathbb{R}$. Finally, by the Phragmén-Lindelöf principle [10, Chapter 3, Theorem 10] applied to the operator $L$ we obtain $u(t, x) \geq 0$, for all $(t, x) \in [0, h] \times \mathbb{R}$, whenever $u_0 \geq 0$ and in a similar form we have $u(t, \cdot) \geq 0$ for the intervals $[h, 2h], [2h, 3h], \ldots$.

**Remark 2.** We note that if the initial datum is such that $u_0(s, x) = O(x^2)$ the comparison principle works too, however, the boundedness of derivative in (4) requires the boundedness of $u_0(s, \cdot)$ uniformly in $s \in [0, h]$ and without the uniform boundedness in time the uniqueness of solution fails to the step $[0, h]$ (see, e.g., [12]). Also, we note that the typical results to global existence (see, e.g., [13, Proposition 2.1] and [16, Theorem 2.3]) do not apply to (1).

An interesting open problem is to determine whether the solutions of (1), with initial data in $\mathcal{C}$ for example, are globally bounded in time. As much as we know, this issue has been only dealt to the boundary-initial Cauchy problem on a bounded domain (see, e.g., [3]).

2. Main Result

Now, we denote by $0 < \lambda_1(c) \leq \lambda_2(c)$ the zeros of quadratic function

$$\epsilon_\lambda := \lambda^2 - c\lambda + 1.$$

Next, for $\lambda \in \mathbb{R}$ we define

$$|f|_{\lambda} = \sup_{z \in \mathbb{R}} e^{-\lambda z} |f(z)| = \|e^{-\lambda} f(\cdot)\|_{\infty},$$

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and
\[ L^\infty_\lambda = \{ f : \mathbb{R} \rightarrow \mathbb{R}, |f|_\lambda < \infty \}. \]
Also, denoting by \( z = x + ct \) and \( v(t, z) := u(t, z - ct) \) we have
\[ \dot{v}(t, z) = v''(t, z) - cv'(t, z) + v(t, z)(1 - v(t - h, z - ch)), \quad (t, z) \in \mathbb{R}_+ \times \mathbb{R} \quad (5) \]
so, a semi-wavefront \( \phi_c \) is a stationary solution of (5) and we can establish the following result on exponential stability of semi-wavefronts.

**Theorem 3 (Stability with unbounded weight).** Let \( \lambda_c \in (\lambda_1(c), \lambda_2(c)) \). Suppose that \( \phi_c \) is semi-wavefront to (1) and \( v_0 \in \mathcal{C} \) an initial datum to (5). Then the following assertions are true

(i) If \( v_0 \) is non negative such that \( v_0(0, \cdot) \in L^\infty_\lambda \) then
\[ \lim_{t \rightarrow \infty} v(t, z) = 0 \quad \text{for all} \quad z \in \mathbb{R}. \]
(ii) If
\[ e^{-\lambda_c ch} |\phi_c|_0 \leq \epsilon_{\lambda_c} \]
and \( \delta \leq 0 \) is number satisfying
\[ e^{-\lambda_c ch} |\phi_c|_0 \leq e^{\delta h}(\delta + \epsilon_{\lambda_c}) \quad (6) \]
then
\[ \phi_c(\cdot) - v_0 \in C([-h, 0], L^\infty_\lambda) \quad (7) \]
implies
\[ |v(t, \cdot) - \phi_c|_{\lambda_c} \leq K_{\lambda_c} e^{\delta t} \quad \text{for all} \quad t \geq -h, \quad (8) \]
where \( K_{\lambda_c} = \max_{s \in [-h, 0]} |\phi_c - v_0(s, \cdot)|_{\lambda_c}. \)

Finally, if we take \( \lambda_c = c/2 \) in (7) we obtain (5) on the domain \( \{(h, c) \in \mathbb{R}^2 : h \geq 0 \} \).

Denoting by
\[ \kappa^c_h := \frac{c}{2} - \frac{1}{ch} + \frac{1}{2} \sqrt{c^2 + \frac{4}{c^2 h^2} - 4}, \]
the number \( \kappa^c_h \) is a critical point of the function \( Q : [\lambda_1(c), \lambda_2(c)] \rightarrow \mathbb{R} \) defined by
\[ Q(\lambda) := e^{\lambda ch} (-\lambda^2 + c\lambda - 1), \]
we can establish the following result for the uniqueness of semi-wavefronts to (1) up to translations.
**Corollary 4 (Uniqueness of semi-wavefronts).** Let $c > 2$. If $\phi_c$ and $\psi_c$ are semi-wavefronts such that

$$
\min\{|\phi_c|_{0}, |\phi_c|_{0}\} < Q(\kappa_{\psi})
$$

then $\phi_c(\cdot) = \psi_c(\cdot + z_0)$ for some $z_0 \in \mathbb{R}$. In particular, if $c > 2\sqrt{2}$ the semi-wavefronts to [1] with speed $c$ are unique up to translations for all $h \geq 0$.

Now, for a function $v : [-h, +\infty) \times \mathbb{R} \to \mathbb{R}$ and an initial datum $\eta_0$ we define the following Cauchy problem

$$
\begin{align*}
\dot{\eta}(t, z) &= \eta''(t, z) - c\eta'(t, z) + \eta(t, z - ch) \eta(t, z) \quad (t, z) \in \mathbb{R}_+ \times \mathbb{R}, \\
\eta(0, z) &= \eta_0(z) \quad z \in \mathbb{R}.
\end{align*}
$$

**Lemma 5.** Let $\lambda \in \mathbb{R}$ and $v_t \in C$ for all $t \geq 0$ be. If $\eta_0 \in L^\infty$ then (10)-(11) has a unique solution $\eta(t, \cdot)$ in $L^\infty$ for all $t \geq 0$. Moreover, denoting by $\Gamma_v(t)\eta_0 := \eta(t, \cdot)$ we have the following estimate

$$
|\Gamma_v(t)\eta_0|_{\lambda} \leq e^{-ct}\eta_0|_{\lambda} \quad \text{for all } t \geq 0.
$$

**Proof.** Making the change of variable $p(t, z) := e^{-\lambda z} \eta(t, z)$ we have that $p$ satisfies the Cauchy problem

$$
\begin{align*}
\dot{p}(t, z) &= p''(t, z) + (2\lambda - c)p'(t, z) + (c\lambda - v(t - h, z - ch)p(t, z) \quad (t, z) \in \mathbb{R}_+ \times \mathbb{R} \\
p(0, z) &= e^{\lambda z}\eta_0(z), \quad z \in \mathbb{R}
\end{align*}
$$

If we translate the spatial variable by $(2\lambda - c)t$ then, using the Lipschitz continuity of the function $\tilde{v}(t, \cdot) := v(t, \cdot - (2\lambda - c)t)$ (uniformly in $t \in [0, T]$, for any $T < \infty$), by [12, Lemma 4.1] we conclude that (13) has a unique bounded solution (uniformly in $t \in [0, T]$, for any $T < \infty$), so $\eta(t, \cdot) \in L^\infty$ for all $t \geq 0$. Then, by the Phragmén-Lindelöf principle $\eta(t, \cdot) \geq 0$ for all $t \geq 0$ and the estimation (12) is obtained from [12, Lemma 4.1].

**Proof (Theorem 3).** (i) It follows from Proposition [1] and Lemma [5] with $\eta = v$ and $\lambda = \lambda_c$.

(ii) We denote by $U = [0, h] \times \mathbb{R}$, $R = |\phi_c|_{0}e^{-\lambda \cdot h}$ and $w_k(t, x) := u(t + (k - 1)h, x)$ where $k = 0, 1, 2, \ldots$. If we take $w = v - \phi_c$ then

$$
\dot{w}_k(t, z) = w_k''(t, z) - c\phi_c'(t, z) + [1 - v_k(t, z - ch)]w_k(t, z) + f_{k-1}(t, z), \quad \text{on } U,
$$

where $f_{k-1}(t, \cdot) = -\phi_c(\cdot)w_{k-1}(t, \cdot - ch)$.

Now, we define the sequence $C_k := K_{\lambda_c} e^{(k-1)\delta h}$ for $k = 0, 1, 2, \ldots$. Clearly, $|w_0(t)|_{\lambda_c} \leq C_0 e^{\delta t}$ for $t \in [0, h]$. Next, we assume that

$$
|w_1(t)|_{\lambda_c} \leq C_1 e^{\delta t} \quad \text{for all } t \in [0, h]
$$

and some $l \geq 1$. 

Next, by Lemma 5 there exist $D_1 > 0$ such that

$$|f_i(t, z)| \leq D_1 e^{c_2 z^2/2}, \text{ on } U.$$ (17)

Therefore, using Proposition 1 Chapter 1, Theorem 12 implies that the associated Cauchy problem to (15) with initial datum $u_{l+1}(0, \cdot)$ has a unique bounded solution which is represented by

$$w_{l+1}(t) = \Gamma_{v_l}(t)w_{l+1}(0) + \int_0^t \Gamma_{v_i}(t - \tau)f_i(\tau)d\tau.$$ (16)

Next, using $u_{l+1}(0) = u_l(h)$ and Lemma 5 we have

$$|w_{l+1}(t)|_{\lambda_c} \leq e^{-\epsilon \lambda_c} |w_l(h)|_{\lambda_c} + R \int_0^t e^{-\epsilon \lambda_c (t-\tau)}|w_l(\tau)|_{\lambda_c}d\tau.$$ (18)

Because of (16) we have

$$|w_{l+1}(t)|_{\lambda_c} \leq e^{-\epsilon \lambda_c} \delta C_i + Re^{-\epsilon \lambda_c} C_i [e^{(\epsilon \lambda_c + \delta) t} - 1]/(\epsilon \lambda_c + \delta) = e^{\delta t} C_i [e^{-(\epsilon \lambda_c + \delta) t + \delta h} + R(1 - e^{-(\epsilon \lambda_c + \delta) t})/(\epsilon \lambda_c + \delta)].$$

Now, due to (16), the function $M : [0, h] \rightarrow \mathbb{R}$ defined by

$$M(t) = e^{-(\epsilon \lambda_c + \delta) t + \delta h} + R(1 - e^{-(\epsilon \lambda_c + \delta) t})/(\epsilon \lambda_c + \delta)$$

is non-increasing. Therefore,

$$|w_{l+1}(t)|_{\lambda_c} \leq e^{\delta(t+h)} C_i = e^{\delta t} C_{l+1} \text{ for all } t \in [0, h].$$

So, the general conclusion is

$$|w(t, \cdot)|_{\lambda_c} \leq K_{\lambda_c} e^{\delta t} \text{ for all } t \geq -h.$$ (19)

Finally, due to the estimate $|\phi_c|_0 < e^{c_k}$ [7, Theorem 4.3], taking $\lambda_c = c/2$ we have that (10) is satisfied for all $c > 2\sqrt{2}$.

**Proof (Corollary 4).** By a standard argument of ordinary differential equations the asymptotic behavior of non negative solutions of (2) at $-\infty$ is

$$\phi_\psi(z) = A_{\phi_c} e^{z c} e^{\lambda_1 z} + o(z^c e^{\lambda_1 z})$$

$$\psi_\phi(z) = A_{\psi_c} e^{z c} e^{\lambda_1 z} + o(z^c e^{\lambda_1 z})$$

(18)

where $A_{\phi_c}, A_{\psi_c} > 0, j_c = 0, 1$ and $j_c = 1$ if and only if $c = 2$. So, for some $z_0 \in \mathbb{R}$ we have $A_{\phi_c (+z_0)} = A_{\phi_c (\cdot)}$.

Otherwise, denoting $\Phi_c(z) = \phi_c(z) - \psi_c(z + z_0)$ we have

$$\Phi''_c(z) - c \Phi'_c(z) + (1 - \psi_c(z))\Phi_c(z) - \phi_c(z)\Phi_c(z - ch) = 0 \quad z \in \mathbb{R}.$$ (19)

Next, using (18) and applying [3, Proposition 7.2] we can conclude

$$\Phi_c(z) = O(e^{c_2 z^2}) \quad \text{if } c > 2.$$ (20)

Next, we take, for example, the initial datum $v_0(s, z) = \min\{1, A_{\phi_c} e^{\lambda_1 z}\} \in \mathcal{C}$. Then, applying Theorem 3 with $\lambda_c = \kappa_c^2, w_0^1 = v_0 - \phi_c$ and $w_0^2 = v_0 - \psi_c$ we conclude that

$$\phi_c(\cdot) = \psi_c(\cdot + z_0) \text{ under the condition (19).}$$

Similarly, taking $\lambda_c = c/2$ we conclude that

$$\phi_c(\cdot) = \psi_c(\cdot + z_0) \text{ on the domain } \{(h, c) \in \mathbb{R}^2 : h \geq 0 \text{ and } c > 2\sqrt{2}\}.$$
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