Nonlinear Weakly Curved Rod by $\Gamma$-Convergence

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Abstract We present a nonlinear model of weakly curved rod, namely the type of curved rod where the curvature is of the order of the diameter of the cross-section. We use an approach analogous to the one for rods and curved rods and start from the strain energy functional of three dimensional nonlinear elasticity. We do not impose any constitutional behavior of the material and work in a general framework. To derive the model, by means of $\Gamma$-convergence, we need to set the order of strain energy (i.e., its relation to the thickness of the body $h$). We analyze the situation when the strain energy (divided by the order of volume) is of the order $h^4$. This is the same approach as the one used in Föppl-von Kármán model for plates and the analogous model for rods. The obtained model is analogous to Marguerre-von Kármán for shallow shells and its linearization is the linear shallow arch model which can be found in the literature.

Keywords Weakly curved rod · Gamma convergence · Shallow arch · Asymptotic analysis

Mathematics Subject Classification (2000) 74K20 · 74K25

1 Introduction

The study of thin structures is the subject of many works in the theory of elasticity. There is a vast literature on the subject of rods, plates and shells (see [5, 8, 9]).

The derivation and justification of lower dimensional models, equilibrium and dynamics, of rods, curved rods, weakly curved rods, plates and shells in linearized elasticity, by using formal asymptotic expansion, is well established (see [8, 9] and the references therein). In
all these approaches, one starts from the equations of three-dimensional linearized elasticity and then via formal asymptotic expansion derives the lower dimensional models. One can also obtain convergence results. In [3, 4] the linear model of weakly curved rod (or as it is called shallow arch) is derived and a convergence result is obtained. We call weakly curved rods or shallow arches those characterized by the fact that the curvature of their centerline has the same order of magnitude as the diameter of the cross section, both being much smaller than their length.

A formal asymptotic expansion is also applied to derive non-linear models of rods, plates and shells (see [8, 9, 20] and the references therein), starting from three-dimensional isotropic elasticity (usually Saint-Venant-Kirchoff material). A hierarchy of the models is obtained, depending on the order of the external loads, i.e., their relation to the thickness of the body $h$ (see also [11] for plates).

However, the formal asymptotic expansion does not give a convergence result. The first convergence result, in deriving lower dimensional models from three-dimensional non-linear elasticity, is obtained applying $\Gamma$-convergence, a very powerful tool introduced by De Giorgi (see [6, 10]). Using $\Gamma$-convergence, elastic string models, membrane plate and membrane shell models are obtained (see [1, 16, 17]). It is assumed that the external loads are of order $h^0$. The obtained models are different from those obtained by means of the formal asymptotic expansion in the sense that additional relaxation of the energy functional is made.

Recently, a hierarchy of models of rods, curved rods, plates and shells was obtained via $\Gamma$-convergence (see [12–15, 19, 22, 23, 27, 28]). Influence of the boundary conditions and the order and the type of the external loads is largely discussed for plates (see [13, 18]). Let us mention that $\Gamma$-convergence results provide us the convergence of the global minimizers of the total energy functional. Recently, compensated compactness arguments were used to obtain convergence for the stationary points of the energy functional (see [21, 24, 25]).

Here we apply the tools developed for rods, plates and shells to obtain a weakly curved rod model by $\Gamma$-convergence. It is assumed that we have free boundary conditions and that the strain energy (divided by the order of volume) is of the order $h^4$, where $h$ is the thickness of the rod. This corresponds to the situation when external transversal dead loads are of order $h^3$ (see Remark 9). The order $h^4$ of the strain energy gives Föppl-von Kármán model for plates, Marguerre-von Kármán model for shallow shells and an analogous model for rods (see [14, 23, 29]). The obtained model is non-linear and of the highest order in the hierarchy of models. Its linearization is shallow arch model, obtained in [3, 4] for isotropic, homogeneous case (see for comparison Remark 8 d)). Here we do not assume any constitutive behavior and thus work in a more general framework. The main result is stated in Theorem 5.

Throughout the paper $\overline{A}$ or $\{A\}^-$ denotes the closure of the set. By a domain we call a bounded open set with Lipschitz boundary. $I$ denotes the identity matrix, $\text{SO}(3)$ denotes the manifold of rotations in $\mathbb{R}^3$ and by (3) we denote the set of antisymmetric matrices $3 \times 3$. $\mathbb{R}^{3 \times 3}_{\text{sym}}$ denotes the set of symmetric matrices. By $\text{sym}A$ we denote the symmetric part of the matrix, $\text{sym}A = \frac{1}{2}(A + A^T)$. $e_1, e_2, e_3$ are the vectors of the canonical base in $\mathbb{R}^3$. By $\nabla_h$ we denote $\nabla = \nabla e_1 + \frac{1}{h} \nabla e_2, e_3$. $\|f\|_{C^1(\Omega)}$ stands for $C^1$ norm of the function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, i.e., $\|f\|_{C^1(\Omega)} = \max_{x \in \Omega} |f| + \sum_{i=1}^n \max_{x \in \Omega} |\partial_i f|$. $\to$ denotes the strong convergence and $\rightharpoonup$ the weak convergence.
2 Setting up the Problem

Let \( \omega \subset \mathbb{R}^2 \) be an open set having area equal to \( A \) and Lipschitz boundary. For all \( h \), which satisfy \( 0 < h \leq 1 \), and for given \( L \) we define

\[
\omega^h = h\omega, \quad \Omega^h = (0, L) \times h\omega.
\]  

(2.1)

We shall leave out superscript when \( h = 1 \), i.e., \( \Omega = \Omega^1, \omega = \omega^1 \). Let us by \( \mu(\omega) \) denote

\[
\mu(\omega) = \int_\omega (x_2^2 + x_3^2) \, dx_2 \, dx_3.
\]  

(2.2)

Let us choose coordinate axis such that

\[
\int_\omega x_2 \, dx_2 \, dx_3 = \int_\omega x_3 \, dx_2 \, dx_3 = \int_\omega x_2 x_3 \, dx_2 \, dx_3 = 0.
\]  

(2.3)

For every \( h \) we define the curve \( C^h \) of the form

\[
C^h = \{ \theta^h(x_1) = (x_1, \theta^h_2(x_1), \theta^h_3(x_1)) \in \mathbb{R}^3 : x_1 \in (0, L) \},
\]  

(2.4)

where \( \theta^h_k(x_1) \), for \( k = 2, 3 \), are given functions satisfying \( \theta^h_k \in C^3(0, L) \). Let \( (t^h, n^h, b^h) \) be the Frenet trihedron associated with the curve \( C^h \)

\[
t^h = \frac{1}{\sqrt{1 + ((\theta^h_2)'')^2 + ((\theta^h_3)'')^2}} \begin{pmatrix} 1 & (\theta^h_2)' & (\theta^h_3)' \end{pmatrix},
\]  

(2.5)

\[
n^h = \frac{(t^h)'}{||(t^h)'||},
\]  

(2.6)

\[
b^h = t^h \times n^h.
\]  

(2.7)

We suppose \( n^h \in C^1(0, L) \) which is satisfied if \( (\theta^h_2)'', (\theta^h_3)''' \) do not vanish at the same time (that is equivalent to the fact that the curvature of \( C^h \) is strictly positive for any \( x_1 \in (0, L) \)). The case where \( C^h \) has null curvature points can be treated in the same fashion, provided that we suppose that along these points we have the same degree of smoothness as before with \( t^h, n^h \) and \( b^h \) appropriately chosen (see Remark 4). We define the map \( \Theta^h : \hat{\Omega}^h \to \Theta^h(\hat{\Omega}^h) = \{ \hat{\Omega}^h \}^- \subset \mathbb{R}^3 \), where \( \hat{\Omega}^h := \Theta^h(\hat{\Omega}^h) \), in the following manner:

\[
\Theta^h(x^h) = (x_1, \theta^h_2(x_1), \theta^h_3(x_1)) + x_2^h n^h(x_1) + x_3^h b^h(x_1),
\]  

(2.8)

and we assume that \( \Theta^h \) is a \( C^1 \) diffeomorphism which can be proved if \( h \) is small enough and \( \theta^h_k \), for \( k = 2, 3 \), are of the form considered here. Namely, we take \( \theta^h_2 = h\theta_2, \theta^h_3 = h\theta_3 \) where \( \theta_k \in C^3(0, L) \). Let us assume

\[
((\theta_1)'')^2(x_1) + ((\theta_2)'')^2(x_1) \neq 0,
\]  

(2.9)

for all \( x_1 \in (0, L) \). A generic point in \( \hat{\Omega}^h \) or \( \{ \hat{\Omega}^h \}^- \) will be denoted by \( x^h = (x_1, x_2^h, x_3^h) \).

Like in [12–15, 19, 22, 23] we start from three dimensional non linear elasticity functional of strain energy (see [7] for an introduction to non linear elasticity)

\[
I^h(y) := \frac{1}{h^2} \int_{\hat{\Omega}^h} W^h(x^h, \nabla y) \, dx^h.
\]  

(2.10)
It is natural to divide the strain energy by $h^2$, since the volume is vanishing with the order of $h^2$. We are interested in finding the $\Gamma$-limit (i.e., in characterizing the limits of minimizers) of the functionals $\frac{1}{h^4} I_h$. The reason why we divide by $h^4$ is that we want to obtain the theory analogous to Föppl-von Kármán for plates and rods (see [13, 14, 23]) and Marguerre-von Kármán for shallow shells (see [29]). We do not analyze the part of the energy functional which contains forces because the part with the strain energy contains the derivatives (at least for the external dead loads) and thus makes the most difficult part of the analysis (see Remarks 9 and 11). We assume that the body is free at the boundary. The consideration of the other boundary conditions is also possible. We rewrite the functional $I_h$ on the domain $\Omega$, i.e., we write

$$I_h(y) := \int_\Omega W_h(\Theta^h \circ P^h(x), (\nabla y) \circ \Theta^h \circ P^h) \det((\nabla \Theta^h) \circ P^h(x)) \, dx,$$

where by $P^h : \mathbb{R}^3 \to \mathbb{R}^3$ we denote the mapping $P^h(x_1, x_2, x_3) = (x_1, hx_2, hx_3)$. $(\nabla y) \circ \Theta^h \circ P^h$ denotes $\nabla y$ evaluated at the point $\Theta^h(P^h(x))$. We assume that for the stored energy function $W_h$ we have

$$W_h(\Theta^h \circ P^h(x), F) = W(x, F), \quad \forall x \in \Omega, F \in \mathbb{R}^{3 \times 3}$$

where $W$ satisfies the following assumptions (the same ones as in [23], see Remark 1):

(i) $W : \Omega \times \mathbb{R}^{3 \times 3} \to [0, +\infty]$ is a Carathéodory function; for some $\delta > 0$ the function $F \mapsto W(x, F)$ is of class $C^2$ for $\text{dist}(F, \text{SO}(3)) < \delta$ and for a.e. $x \in \Omega$;

(ii) the second derivative $\frac{\partial^2 W}{\partial F^2}$ is a Carathéodory function on the set $\Omega \times \{F \in \mathbb{R}^{3 \times 3} : \text{dist}(F, \text{SO}(3)) < \delta\}$ and there exists a constant $\gamma > 0$ such that

$$\left| \frac{\partial^2 W}{\partial F^2}(x, F)[G, G] \right| \leq \gamma |G|^2 \quad \text{if} \quad \text{dist}(F, \text{SO}(3)) < \delta \quad \text{and} \quad G \in \mathbb{R}^{3 \times 3}_{\text{sym}},$$

(iii) $W$ is frame-indifferent, i.e., $W(x, F) = W(x, RF)$ for a.e. $x \in \Omega$ and every $F \in \mathbb{R}^{3 \times 3}, R \in \text{SO}(3)$;

(iv) $W(x, F) \equiv 0$ if $F \in \text{SO}(3)$; $W(x, F) \geq C \text{dist}^2(F, \text{SO}(3))$ for every $F \in \mathbb{R}^{3 \times 3}$, where the constant $C > 0$ is independent of $x$.

Under these assumptions we first show the compactness result (Theorem 3), i.e., we take the sequence $y_h^h \in W^{1,2} (\Omega^h; \mathbb{R}^3)$ such that

$$\limsup_{h \to 0} \frac{1}{h^4} I^h < +\infty,$$

and conclude how that fact affects the limit displacement. In Lemma 2 we prove the lower bound, in Theorem 4 we prove the upper bound and that enables us to identify the limit functional (Theorem 5). First we start with some basic properties of the mappings $\Theta^h$ which are necessary for further analysis.

Remark 1 Since the stored energy functions $W^h : \Omega^h \to \mathbb{R}^{3 \times 3} \to [0, \infty]$ change their domains, we must also put the dependence on $h$. In the case of the homogeneous material relation (2.12) is trivially satisfied. Relation (2.12) thus tells us what are the assumptions on the constitutive law in the case of inhomogeneous material (i.e., how the inhomogeneity
of the material changes as $h \to 0$) for the subsequent analysis to be valid. More general assumption would be that $W^h$ satisfies the frame indifference and the following property

$$\limsup_{F \to I, h \to 0} \sup_{x \in \Omega} \frac{|W^h(\Theta^h \circ P^h(x), F) - W(x, F)|}{\|F - I\|^2} = 0,$$

where $W$ satisfies the properties (i)–(iv).

3 Properties of the Mappings $\Theta^h$

We introduce for $k = 2, 3$,

$$p_k(x_1) = \frac{\theta''_k(x_1)}{\sqrt{(\theta''_2(x_1))^2 + (\theta''_3(x_1))^2}}. \quad (3.1)$$

Notice that

$$p_2^2 + p_3^2 = 1, \quad p_2p'_2 + p_3p'_3 = 0. \quad (3.2)$$

Let us denote $p = p_2p'_3 - p'_2p_3$.

**Theorem 1** Let the functions $\theta_k^h$ be such that

$$\theta_k^h(x_1) = h\theta_k(x_1), \quad \text{for all } x_1 \in (0, L), k = 2, 3,$$

where $\theta_k \in C^3(0, L)$ is independent of $h$. Then there exists $h_0 = h_0(\theta) > 0$ such that the Jacobian matrix $\nabla \Theta^h(x^h)$, where the mappings $\Theta^h$ are defined with (2.8), is invertible for all $x^h \in \Omega^h$ and all $h \leq h_0$. Also there exists $C > 0$ such that for $h \leq h_0$ we have

$$\det \nabla \Theta^h = 1 + h^{\delta_h}(x^h), \quad (3.3)$$

and

$$t^h(x_1) = e_1 + h\theta'_2(x_1)e_2 + h\theta'_3(x_1)e_3 + h^2a_1(x_1), \quad (3.4)$$

$$n^h(x_1) = p_2(x_1)e_2 + p_3(x_1)e_3 - h(\theta'_2p_2 + \theta'_3p_3)(x_1)e_1 + h^2a_2(x_1), \quad (3.5)$$

$$b^h(x_1) = -p_3(x_1)e_2 - p_2(x_1)e_3 + h(\theta'_2p_3 - \theta'_3p_2)(x_1)e_1 + h^2a_3(x_1), \quad (3.6)$$

$$\nabla \Theta^h(x^h) = R_e(x_1) + hC(x_1) + x_1^hD(x_1) + x_3^hE(x_1) + h^2O^h(x^h), \quad (3.7)$$

$$(\nabla \Theta^h(x^h))^{-1} = R^T_e(x_1) - hC_1(x_1) - x_1^hD_1(x_1) - x_3^hE_1(x_1) + h^2O^h_2(x^h), \quad (3.8)$$

$$\| (\nabla \Theta^h) - R_e\|_{L^\infty(\Omega^h; R^{3 \times 3})} < Ch, \quad (3.9)$$

$$\| (\nabla \Theta^h)^{-1} - R^T_e\|_{L^\infty(\Omega^h; R^{3 \times 3})} < Ch, \quad (3.10)$$

where

$$R_e = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}, \quad R = \begin{pmatrix} p_2 & -p_3 \\ p_3 & p_2 \end{pmatrix}, \quad (3.11)$$
\[
C = \begin{pmatrix}
0 & -(\theta'_2 p_2 + \theta'_3 p_3) & -(\theta'_3 p_2 - \theta'_2 p_3) \\
\theta'_2 & 0 & 0 \\
\theta'_3 & 0 & 0
\end{pmatrix}, \quad (3.12)
\]
\[
D = \begin{pmatrix}
0 & 0 & 0 \\
p'_2 & 0 & 0 \\
p'_3 & 0 & 0
\end{pmatrix}, \quad E = \begin{pmatrix}
0 & 0 & 0 \\
-p'_3 & 0 & 0
\end{pmatrix}, \quad (3.13)
\]
\[
C_1 = \begin{pmatrix}
0 & -\theta'_2 & -\theta'_3 \\
\theta'_2 p_2 + \theta'_3 p_3 & 0 & 0 \\
\theta'_3 p_2 - \theta'_2 p_3 & 0 & 0
\end{pmatrix}, \quad (3.14)
\]
\[
D_1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
p & 0 & 0
\end{pmatrix}, \quad E_1 = \begin{pmatrix}
0 & 0 & 0 \\
-p & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad (3.15)
\]

and \(\delta^h : \Omega^h \to \mathbb{R}, \ a_i : (0, L) \to \mathbb{R}^3, i = 1, 2, 3, \mathbf{O}_k^h : \Omega^h \to \mathbb{R}^{3 \times 3}, k = 1, 2\) are functions that satisfy

\[
\sup_{0 < h \leq h_0} \sup_{x \in \Omega} |\delta^h(x)| \leq C_0,
\]

\[
\sup_{0 < h \leq h_0} \sup_{x \in \Omega} ||\mathbf{a}_i^h(x) || \leq C_0, \quad \sup_{0 < h \leq h_0} \sup_{x \in \Omega} \|(|\mathbf{a}_i^h)'(x)| \leq C_0
\]

\[
\sup_{0 < h \leq h_0} \sup_{i,j} \sup_{x \in \Omega} \|\mathbf{O}_k^h_{i,j}(x^h) \| \leq C_0, \quad k = 1, 2,
\]

for some constant \(C_0 > 0\).

**Proof** It can be easily seen that

\[
t^h(x_1) = e_1 + h\theta'_2(x_1) e_2 + h\theta'_3(x_1) e_3 - \frac{h^2}{2}((\theta'_2)^2 + (\theta'_3)^2) e_1 + h^3 \mathbf{a}_1^h(x^h), \quad (3.16)
\]

where \(\|\mathbf{a}_i^h\|_{c^2(0, L)} \leq C\). The relations (3.5) and (3.6) are the direct consequences of the relation (3.16). Let us by \(\mathbf{u}^h : (0, L) \to \mathbb{R}^3\) denote the function

\[
\mathbf{u}^h = (1, h\theta'_2, h\theta'_3)^T. \quad (3.17)
\]

It is easy to see that

\[
\nabla \Theta^h(x) = (\mathbf{u}^h(x_1) + x^2_2 (\mathbf{n}^h)'(x_1) + x^2_3 (\mathbf{b}^h)'(x_1) | \mathbf{n}^h(x_1) | \mathbf{b}^h(x_1)). \quad (3.18)
\]

The relations (3.3), (3.7), (3.9), (3.10) are the direct consequences of the relations (3.4)–(3.6) and (3.18). The relation (3.8) is the direct consequence of the fact that, for a regular matrix \(\mathbf{A}\) and arbitrary \(\mathbf{B}\) that satisfies \(\|\mathbf{A}^{-1}\mathbf{B}\| < 1\) (\|·\| is the operational norm), the matrix \(\mathbf{A} + \mathbf{B}\) is invertible and

\[
\| (\mathbf{A} + \mathbf{B})^{-1} - (\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}^{-1}) \| \leq \frac{\|\mathbf{A}^{-1}\mathbf{B}\|^2 \|\mathbf{A}\|^{-1}}{1 - \|\mathbf{A}^{-1}\mathbf{B}\|}. \quad (3.19)
\]

To end the proof observe that

\[
\mathbf{C_1} = \mathbf{R}_e^T \mathbf{C} \mathbf{R}_e^T, \quad \mathbf{D_1} = \mathbf{R}_e^T \mathbf{D} \mathbf{R}_e^T, \quad \mathbf{E_1} = \mathbf{R}_e^T \mathbf{E} \mathbf{R}_e^T.
\]

\(\square\)
Remark 2 By a careful computation it can be seen that $o^i_2$, $o^i_3$ and $o^i_4$ (defined in the relation (3.16)) are dominantly in $e_2$, $e_3$ plane, i.e., that we have for $i = 2, 3, 4$:

$$
o^i_2(x_1) = f^i_2(x_1)e_2 + f^i_3(x_1)e_3 + hr^i_1(x_1),$$  \hspace{1cm} (3.19)

where $f^i_2$, $f^i_3 \in C^1(0, L)$, sup$_{0 < k \leq h_0} ||r^i_1||_{C^1(0, L)} \leq C$, for some $C > 0$.

Remark 3 By a further inspection it can be seen that

$$f^2_4 = -\frac{1}{2} \theta^2_2((\theta^2_2)^2 + (\theta'^2_2)^2), \quad f^3_4 = -\frac{1}{2} \theta^2_3((\theta^2_3)^2 + (\theta'^2_3)^2),$$  \hspace{1cm} (3.20)

$$f^2_5 = p_1\left(f_2(\theta', \theta'') - \frac{1}{2}((\theta'^2_2)^2 + (\theta'^2_3)^2) - \theta'^2_2 p_2 + \theta'^2_3 p_3, \right),$$  \hspace{1cm} (3.21)

$$f^3_5 = p_1\left(f_3(\theta', \theta'') - \frac{1}{2}((\theta'^2_2)^2 + (\theta'^2_3)^2) - \theta'^2_2 p_2 + \theta'^2_3 p_3, \right),$$  \hspace{1cm} (3.22)

$$f^3_5 = p_2((\theta'^2_2)^2 + (\theta'^2_3)^2) - p_2 f_2(\theta', \theta''),$$  \hspace{1cm} (3.23)

$$f^3_5 = -p_2((\theta'^2_2)^2 + (\theta'^2_3)^2) + p_2 f_2(\theta', \theta''),$$  \hspace{1cm} (3.24)

where $f_2(\theta', \theta'') \in C^1(0, L)$ is the expression that includes $\theta'$, $\theta''$:

$$f_2(\theta', \theta'') = \frac{1}{2}\left( (p_2 + p_3)((\theta'^2_2)^2 + (\theta'^2_3)^2) + 2(\theta'^2_2 + \theta'^2_3)(\theta'^2_2 p_2 + \theta'^2_3 p_3) 
- \sqrt{(\theta'^2_2)^2 + (\theta'^2_3)^2}((\theta'^2_2 p_2 + \theta'^2_3 p_3)^2). \right)$$

Remark 4 It is not necessary to impose the condition (2.9). All we need is the existence of the expansions given by (3.4)–(3.6), where $p_2, p_3 \in C^1(0, L)$, including the statement of Remark 2.

Remark 5 Although $\Theta^h$ makes the small perturbation of the central line, $(x_1, 0, 0)$, for $x_1 \in [0, L]$, it is not true that $\nabla \Theta^h$ is close to the identity (like in the shallow shell model, see [29]). In fact, there are torsional effects of order 0 on every cross section. This is the main reason why the change of coordinates introduced in the next chapter is useful.

4 $\Gamma$-Convergence

We shall need the following theorem which can be found in [12].

Theorem 2 (On geometric rigidity) Let $U \subset \mathbb{R}^m$ be a bounded Lipschitz domain, $m \geq 2$. Then there exists a constant $C(U)$ with the following property: for every $v \in W^{1,2}(U; \mathbb{R}^m)$ there is an associated rotation $R \in \text{SO}(m)$ such that

$$||\nabla v - R||_{L^2(U)} \leq C(U)||\text{dist}(\nabla v, \text{SO}(m))||_{L^2(U)}. \hspace{1cm} (4.1)$$

The constant $C(U)$ can be chosen uniformly for a family of domains which are Bilipschitz equivalent with controlled Lipschitz constants. The constant $C(U)$ is invariant under dilations.
The following version of the Korn’s inequality is needed.

**Lemma 1** Let \( \omega \subset \mathbb{R}^2 \) be an open bounded set with Lipschitz boundary and \( u \in L^2(\omega; \mathbb{R}^2) \). Let us by \( e_{ij}(u) \) denote \( e_{ij}(u) = \frac{1}{2}(\partial_i u + \partial_j u) \). Let us suppose that for every \( i, j = 1, 2 \) we have that \( e_{ij}(u) \in L^2(\omega) \). Then we have that \( u \in W^{1,2}(\omega; \mathbb{R}^2) \). Also there exists constant \( C(\omega) \), depending only on the domain \( \omega \), such that we have

\[
\|u\|_{W^{1,2}(\omega; \mathbb{R}^2)} \leq C(\omega) \left( \left\| \int_{\omega} u \, dx_1 \, dx_2 \right\| + \left\| \int_{\omega} (x_1 u_2 - x_2 u_1) \, dx_1 \, dx_2 \right\| \right.
\]

\[
+ \sum_{i,j=1,2} \|e_{ij}(u)\|_{L^2(\omega)} \right) .
\]

(4.2)

Let us suppose that the domains \( \omega_n \) are changing in the sense that they are equal to \( \omega_n = A_n \omega \), where \( A_n \in \mathbb{R}^{2\times2} \), and that there exists a constant \( C \) such that \( \|A_n\|, \|A_n^{-1}\| \leq C \). Then the constant in the inequality (4.2) can be chosen independently of \( n \).

**Proof** The first part of the lemma (the fixed domain) is a version of the Korn’s inequality (see, e.g., [26]). The last part we shall prove by contradiction. Let us suppose the contrary that for each \( n \in \mathbb{N} \) there exists \( s^n \) and \( u^n \in W^{1,2}(\omega_n; \mathbb{R}^2) \) such that we have

\[
\left| \int_{\omega_n} u^n \, dx_1 \, dx_2 \right| + \left| \int_{\omega_n} (x_1 u^n_2 - x_2 u^n_1) \, dx_1 \, dx_2 \right| + \sum_{i,j=1,2} \|e_{ij}^n(u^n)\|_{L^2(\omega_n)}
\]

\[
\leq \frac{1}{n} \|u^n\|_{W^{1,2}(\omega_n; \mathbb{R}^2)},
\]

(4.3)

where we have by \( e_{ij}^n(\cdot) \) denoted the symmetrized gradient on the domain \( \omega_n \). Without any loss of generality we can suppose that \( \|u^n\|_{W^{1,2}(\omega_n; \mathbb{R}^2)} = 1 \). Let us take the subsequence of \( (s^n) \) (still denoted by \( (s^n) \)) such that \( A_n s^n \rightarrow A \) and \( A_n^{-1} s^n \rightarrow A^{-1} \) in \( \mathbb{R}^{2\times2} \).

Let us consider the sequence \( u^n = u^n \circ A_n \circ A^{-1} \). It is clear that there exist \( C_1, C_2 > 0 \) such that

\[
C_1 \leq \|u^n\|_{W^{1,2}(\omega_\infty; \mathbb{R}^2)} \leq C_2,
\]

(4.4)

where we have put \( \omega_\infty := A \omega \). Thus there exists \( u \in W^{1,2}(\omega_\infty; \mathbb{R}^2) \) such that \( u^n \rightharpoonup u \) weakly in \( W^{1,2}(\omega_\infty; \mathbb{R}^2) \). Specially, by the compactness of the embedding \( L^2 \hookrightarrow W^{1,2} \) (see, e.g., [2]), we also conclude the strong convergence \( u^n \rightharpoonup u \) in \( L^2(\omega_\infty; \mathbb{R}^2) \). Since it is valid \( A_n \circ A^{-1} \rightarrow I \), it can be easily seen that, from the weak convergence, it follows \( e_{ij}^n(u^n) \circ (A_n \circ A^{-1}) \rightarrow e_{ij}(u) \), weakly in \( L^2(\omega_\infty; \mathbb{R}^2) \), where we have by \( e_{ij}(\cdot) \) denoted the symmetrized gradient on the domain \( \omega_\infty \). From the weak convergence we can conclude that

\[
\|e_{ij}(u)\|_{L^2(\omega_\infty)} \leq \liminf_{n \to \infty} \|e_{ij}^n(u^n) \circ (A_n \circ A^{-1})\|_{L^2(\omega_\infty)} = 0,
\]

(4.5)

for every \( i, j = 1, 2 \). We can also conclude from (4.3) that

\[
\int_{\omega_\infty} u \, dx_1 \, dx_2 = 0, \quad \int_{\omega_\infty} (x_1 u_2 - x_2 u_1) \, dx_1 \, dx_2 = 0.
\]

(4.6)
Applying the standard Korn’s inequality on the domain $\omega_\infty$, i.e.,

$$\|u - u_e\|_{W^{1,2}(\omega_\infty; \mathbb{R}^2)} \leq C(\omega_\infty) \left( \|u - u_e\|_{L^2(\omega_\infty; \mathbb{R}^2)} + \sum_{i,j=1,2} \|e_{ij}(u) - e_{ij}(u_e)\|_{L^2(\omega_\infty; \mathbb{R}^2)} \right),$$

we conclude that $u_e^n \rightarrow u$ strongly in $W^{1,2}(\omega_\infty; \mathbb{R}^2)$. But then (4.4), (4.5), (4.6) make a contradiction with the version of the Korn’s inequality (4.2) on the domain $\omega_\infty$. □

**Remark 6** The same proof can be done under the assumption that $\omega_s = F_s(\omega)$, where $F_s$ is the family of $\text{Bilipschitz}$ mappings whose $\text{Bilipschitz}$ constants we can control (i.e., the $\text{Lipschitz}$ constants of $FS$ is the family of $\text{Bilipschitz}$ mappings whose $\text{Bilipschitz}$ constants we can control (i.e., the $\text{Lipschitz}$ constants of $FS$ family point in $\omega_s$), and constants $c_n$ denote the change of coordinates $\mu(\omega)$ evaluated at the point $\Theta^h(P^h(x))$.

In the sequel we suppose $h_0 \geq 1$ (see Theorem 1). If this was not the case, what follows could be easily adapted. Using Theorem 2 we can prove the following theorem:

**Theorem 3** Let $y^h \in W^{1,2}(\tilde{\Omega}^h; \mathbb{R}^3)$ and let

$$E^h = \frac{1}{h^2} \int_{\tilde{\Omega}^h} \text{dist}^2(\nabla y^h, \text{SO(3)}) \, dx.$$

Let us suppose that

$$\limsup_{h \to 0} \frac{E^h}{h^4} < +\infty. \quad (4.11)$$

Then there exist maps $R^h : [0, L] \rightarrow \text{SO(3)}$ and $\tilde{R}^h : [0, L] \rightarrow \mathbb{R}^{3 \times 3}$, with $|\tilde{R}^h| \leq C$, $\tilde{R}^h \in W^{1,2}([0, L], \mathbb{R}^{3 \times 3})$ and constants $\tilde{R}^h \in \text{SO(3)}$, $c^h \in \mathbb{R}^3$ such that the functions $\tilde{y}^h := (\tilde{R}^h)^T y^h - c^h$ satisfy

$$\|\nabla \tilde{y}^h \circ \Theta^h \circ P^h - R^h\|_{L^2(\Omega)} \leq Ch^2, \quad (4.12)$$
Moreover if we define

\[
\|R^h - \tilde{R}^h\|_{L^2([0, L])} \leq Ch^2, \quad \|(\tilde{R}^h)'\|_{L^2([0, L])} \leq Ch, \quad (4.13)
\]

\[
\|R^h - I\|_{L^\infty([0, L])} \leq Ch. \quad (4.14)
\]

Moreover if we define

\[
u^h = \frac{1}{A} \int_{\omega} \tilde{y}_1^h \circ \Theta^h \circ P^h - \frac{x_1}{h^2} \, dx_2 \, dx_3, \quad (4.15)
\]

\[
v_k^h = \frac{1}{A} \int_{\omega} \tilde{y}_k^h \circ \Theta^h \circ P^h - h \theta_k \frac{x_1}{h} \, dx_2 \, dx_3, \quad \text{for } k = 2, 3, \quad (4.16)
\]

\[
u^h = \frac{1}{A \mu(\omega)} \int_{\omega} x_2' (\tilde{y}_3 \circ \Theta^h \circ P^h) - x_3' (\tilde{y}_2 \circ \Theta^h \circ P^h) \frac{h^2}{h} - v_3 \frac{h^2}{h} \, dx_2 \, dx_3, \quad (4.17)
\]

then, up to subsequences, the following properties are satisfied:

(a) \(u^h \rightharpoonup u\) in \(W^{1,2}(0, L)\);
(b) \(v_k^h \rightharpoonup v_k\) in \(W^{1,2}(0, L)\), where \(v_k \in W^{2,2}(0, L)\) for \(k = 2, 3\);
(c) \(w^h \rightharpoonup w\) weakly in \(W^{1,2}(0, L)\);
(d) \((\tilde{\nu}^h \circ \Theta^h \circ P^h - I) \rightharpoonup A\), in \(L^2(\Omega)\), where \(A \in W^{1,2}(0, L)\) is given by

\[
A = \begin{pmatrix}
0 & -v_2' & -v_3' \\
v_2' & 0 & -w \\
v_3' & w & 0
\end{pmatrix}; \quad (4.18)
\]

(e) \(\text{sym } R^h - \tilde{R}^h \rightharpoonup \frac{\Lambda^2}{2}\) uniformly on \((0, L)\);
(f) the sequence \(\nu^h\) defined by

\[
\nu_k^h(x) = \frac{1}{h^2} \left( \tilde{y}_k^h \circ \Theta^h \circ P^h(x) - \frac{x_1}{h} u^h(x_1) \right. \\
+ x_2'((v_k^h)' + \theta_k')(x_1) + x_3'((v_3^h)' + \theta_3')(x_1) \left. \right), \quad (4.19)
\]

\[
\nu_k^h(x) = \frac{1}{h^2} \left( \tilde{y}_k^h \circ \Theta^h \circ P^h(x) - h \theta_k - h x_k' \right. \\
- v_k^h(x_1) - h (x_k') \omega^h(x_1) \left. \right), \quad \text{for } k = 2, 3,
\]

where \((x') \perp := (0, -x_3', x_2')\), is weakly convergent in \(L^2(\Omega)\) to a function \(\gamma\) belonging to the space \(C\), where

\[
C = \left\{ \gamma \in L^2(\Omega; \mathbb{R}^3) : \int_{\omega} \gamma = 0, \ \partial_2 \gamma, \ \partial_3 \gamma \in L^2(\Omega; \mathbb{R}^3), \right. \\
\left. \int_{\omega} (x_2' \gamma_2(x_1, \cdot) - x_3' \gamma_3(x_1, \cdot)) \, dx_2 \, dx_3 = 0, \ \text{for a.e. } x_1 \in (0, L) \right\}. \quad (4.19)
\]

Moreover \(\partial_k \gamma^h \rightharpoonup \partial_k \gamma\) in \(L^2(\Omega)\) for \(k = 2, 3\).
Proof. We follow the proof of Theorem 2.2 in [23]. Applying Theorem 2 as in the compactness result of [22] (using the boundedness of $\nabla \Theta^h$ and $(\nabla \Theta^h)^{-1}$ we can find a sequence of piecewise constant maps $R^h : [0, L] \to SO(3)$ such that

$$\int_{\Omega} \| (\nabla y^h) \circ \Theta^h \circ P^h - R^h \|^2 dx \leq C h^4, \quad (4.20)$$

and

$$\int_{I'} \| R^h(x_1 + \xi) - R(x_1) \|^2 dx_1 \leq C h^2 (|\xi| + h)^2, \quad (4.21)$$

where $I'$ is any open interval in $(0, L)$ and $\xi \in \mathbb{R}$ satisfies $|\xi| \leq \text{dist}(I', [0, L])$. Let $\eta \in C^\infty_0(0, 1)$ be such that $\eta \geq 0$ and $\int_{0}^{1} \eta(s)ds = 1$. We set $\eta_h = \frac{1}{h} \eta(\frac{x}{h})$ and we define

$$\tilde{R}^h(x_1) := \int_{-h}^{h} \eta_h(s) R^h(x_1 - s) ds,$$

where we have extended $R^h$ outside $[0, L]$ by taking $R^h(x_1) = R^h(0)$ for every $x_1 < 0$, $R^h(x_1) = R^h(L)$ for every $x_1 > L$. Clearly $\| \tilde{R}^h \| \leq C$ for every $h$ while the properties (4.13) follow from the property (4.21). Moreover since by construction (see [22])

$$\| R^h(x_1 + s) - R^h(x_1) \|^2 \leq C h^3 \int_{-h}^{h} \text{dist}^2(\nabla y^h, SO(3)) \leq C h^3,$$

for every $|s| \leq h$ we have by Jensen’s inequality that

$$\| \tilde{R}^h - R^h \|^2_{L^\infty([0,L];\mathbb{R}^{3x3})} \leq C h^3. \quad (4.22)$$

By the Sobolev-Poincaré inequality and the second inequality in (4.13), there exist constants $Q^h \in \mathbb{R}^{3x3}$ such that $\| \tilde{R}^h - Q^h \|_{L^\infty([0,L];\mathbb{R}^{3x3})} \leq C h$. Combining this inequality with (4.22), we have that $\| R^h - Q^h \|_{L^\infty([0,L];\mathbb{R}^{3x3})} \leq C h$. This implies that $\text{dist}(Q^h, SO(3)) \leq C h$; thus, we may assume that $Q^h$ belongs to $SO(3)$ and by modifying $Q^h$ by order $h$, if needed. Now choosing $\tilde{R}^h = Q^h$ and replacing $R^h$ by $(Q^h)^T R^h$ and $\tilde{R}^h$ by $(Q^h)^T \tilde{R}^h$, we obtain (4.14). By suitable choice of constants $c^h \in \mathbb{R}^3$ we may assume that

$$\int_{\Omega} (\tilde{y}^h \circ \Theta^h \circ P^h - x_1) = 0, \quad \int_{\Omega} (\tilde{y}^h_k \circ \Theta^h \circ P^h - h\theta_k) = 0, \quad \text{for } k = 2, 3. \quad (4.23)$$

Let $A^h = \frac{R^h - I}{h}$. By (4.14) there exists $A \in L^\infty((0, L); \mathbb{R}^{3x3})$ such that, up to subsequences,

$$A^h \to A \text{ weakly * in } L^\infty((0, L); \mathbb{R}^{3x3}). \quad (4.24)$$

On the other hand it follows from (4.13) and (4.14) that

$$\frac{\tilde{R}^h - I}{h} \to A \text{ weakly in } W^{1,2}((0, L); \mathbb{R}^{3x3}). \quad (4.25)$$

In particular, $A \in W^{1,2}((0, L); \mathbb{R}^{3x3})$ and $h^{-1}(\tilde{R}^h - I)$ also converges uniformly. Using (4.22) we deduce that

$$A^h \to A \text{ uniformly.} \quad (4.26)$$
In view of (4.12) this clearly implies the convergence property in (d). Since $R^h \in SO(3)$ we have

$$A^h + (A^h)^T = -hA^h(A^h)^T.$$ 

Hence, $A + A^T = 0$. Moreover, after division by $2h$ we obtain property (e) by (4.26). For adapting the proof to the proof of Theorem 2.2 in [23] it is essential to see that

$$(\nabla \tilde{y}^h) \circ \Theta^h \circ P^h = (\nabla (\tilde{y}^h \circ \Theta^h) \circ P^h)((\nabla \Theta^h)^{-1} \circ P^h)$$

$$= \nabla_h (\tilde{y}^h \circ \Theta^h \circ P^h)((\nabla \Theta^h)^{-1} \circ P^h).$$

(4.27)

From (4.27) it follows that

$$((\nabla \tilde{y}^h) \circ \Theta^h \circ P^h)((\nabla \Theta^h) \circ P^h) = \nabla_h (\tilde{y}^h \circ \Theta^h \circ P^h).$$

(4.28)

and

$$(\nabla \tilde{y}^h) \circ \Theta^h \circ P^h - I = (\nabla_h (\tilde{y}^h \circ \Theta^h \circ P^h - \Theta^h \circ P^h))((\nabla \Theta^h)^{-1} \circ P^h - R_e^T)$$

$$+ (\nabla_h (\tilde{y}^h \circ \Theta^h \circ P^h - \Theta^h \circ P^h))R_e^T.$$ 

(4.29)

Let us notice that from (2.8), (3.5), (3.6) we can conclude

$$\Theta_k = h\theta_k + hx'_k + O_k(h^3) \quad \text{for } k = 2, 3,$$

(4.30)

where $\|O_k(h^3)\|_{C^1(\Omega)} \leq C h^3$.

By multiplying (d) with $(\nabla \Theta^h) \circ P^h = \nabla_h (\Theta^h \circ P^h)$ and using (3.9), (4.28) we obtain

$$\frac{\nabla_h (\tilde{y}^h \circ \Theta^h \circ P^h - \Theta^h \circ P^h)}{h} \rightarrow AR_e \quad \text{in } L^2(\Omega).$$

(4.31)

Property (b) follows immediately from (4.31) by using (3.7), (4.23) and (4.30). Moreover, $v'_k = A_k x'_k$ for $k = 2, 3$ so that $v_k \in W^{2,2}(0, L)$ since $A \in W^{1,2}(0, L)$. By using (e), (3.10), (4.12) and (4.31) from (4.29) we conclude that

$$\left\| \frac{1}{h^2} \text{sym}(\nabla_h (\tilde{y}^h \circ \Theta^h \circ P^h - \Theta^h \circ P^h)R_e^T) \right\|_{L^2((0,L);\mathbb{R}^{3 \times 3})} \leq C.$$ 

(4.32)

The weak convergence of $u^h$ follows from (3.7), (4.32) and the definition of $R_e$. By using the convergence (4.31) and Poincare inequality on each cut $\{x_1\} \times \omega$ we can conclude

$$\frac{\tilde{y}^h \circ \Theta^h \circ P^h - \Theta^h \circ P^h}{h^2} - \frac{1}{h^2 A} \int_{\omega} (\tilde{y}^h \circ \Theta^h \circ P^h - \Theta^h \circ P^h)$$

$$\rightarrow (AR_e)_{22} x_2 + (AR_e)_{23} x_3 \quad \text{in } L^2(\Omega).$$

(4.33)

By using (2.3), (4.7) and (4.30) we conclude from (4.33)

$$w^h_2 := \frac{r \tilde{y}^h \circ \Theta^h \circ P^h - x'_2}{h} - \frac{1}{h^2 A} \int_{\omega} \tilde{y}^h \circ \Theta^h \circ P^h \rightarrow A_{23} x'_3 \quad \text{in } L^2(\Omega).$$

(4.34)
Let us note that since the left hand side of (4.33), i.e., (4.34) is bounded in $W^{1,2}(\Omega)$ the convergence in (4.34) is in fact weak in $W^{1,2}(\Omega)$. The only nontrivial thing to prove is the boundedness of $\partial_1 w_2^h$ in $L^2(\Omega)$. By the chain rule we have for $i = 1, 2, 3$

$$\partial_1(\mathbf{y}_i^h \circ \Theta^h \circ P^h) = ((\partial_1 \mathbf{y}_i^h) \circ \Theta^h \circ P^h)((\partial_1 \Theta_i^h) \circ P^h) + ((\partial_2 \mathbf{y}_i^h) \circ \Theta^h \circ P^h)((\partial_2 \Theta_i^h) \circ P^h) + ((\partial_3 \mathbf{y}_i^h) \circ \Theta^h \circ P^h)((\partial_3 \Theta_i^h) \circ P^h) \quad (4.35)$$

and for $k = 2, 3$

$$\partial_k(\mathbf{y}_i^h \circ \Theta^h \circ P^h) = h\left[((\partial_1 \mathbf{y}_i^h) \circ \Theta^h \circ P^h)((\partial_k \Theta_i^h) \circ P^h) + ((\partial_2 \mathbf{y}_i^h) \circ \Theta^h \circ P^h)((\partial_k \Theta_i^h) \circ P^h) + ((\partial_3 \mathbf{y}_i^h) \circ \Theta^h \circ P^h)((\partial_k \Theta_i^h) \circ P^h)\right]. \quad (4.36)$$

From (4.12), (4.34) and (4.35) we conclude that the boundedness of $\partial_1 w_2^h$ in $L^2(\Omega)$ is equivalent to the boundedness of

$$z_2^h = \frac{R_{21}^h \partial_1 \Theta_1 + R_{22}^h \partial_1 \Theta_2 + R_{23}^h \partial_1 \Theta_3 - h(p_2' x_2 - p_3' x_3)}{h^2} - \frac{1}{h^2 A} \left(R_{21}^h \int_\omega \partial_1 \Theta_1 + R_{22}^h \int_\omega \partial_1 \Theta_2 + R_{23}^h \int_\omega \partial_1 \Theta_3\right), \quad (4.37)$$

in $L^2(\Omega)$. By using (2.3) and (3.18) we conclude

$$z_2^h = \frac{R_{21}^h (x_2 (n_1^h)' + x_3 (b_1^h)') + R_{22}^h (x_2 (n_2^h)' + x_3 (b_2^h)')}{h} + \frac{R_{23}^h (x_2 (n_3^h)' + x_3 (b_3^h)') - (p_2' x_2 - p_3' x_3)}{h}. \quad (4.38)$$

The boundedness of $z_3^h$ in $L^2(\Omega)$ is the consequence of (3.5), (3.6) and (4.14). Now we have proved $w_2^h \rightharpoonup A_{23} x_2'$ weakly in $W^{1,2}(\Omega)$.

Analogously we conclude

$$w_3^h := \frac{1}{h} \mathbf{y}_3^h \circ \Theta^h \circ P^h - x_3' - \frac{1}{h^2 A} \int_\omega \mathbf{y}_3^h \circ \Theta^h \circ P^h \rightarrow -A_{23} x_2', \quad (4.39)$$

weakly in $W^{1,2}(\Omega)$. Now, since $w_3^h$ can be written as

$$w^h(x_1) = \frac{1}{A\mu(\omega)} \int_\omega (x_2' w_3^h - x_3' w_3^h) \, dx_2 \, dx_3, \quad (4.40)$$

it is clear that $w^h$ converges weakly to the function $w = -A_{23} = A_{32}$ in $W^{1,2}(0, L)$. Let us define $\beta^h : \Omega' \rightarrow \mathbb{R}^3$, $\beta^h = \gamma \circ x'^{-1}$. By the chain rule we have

$$\partial_1 \beta_1^h = (\partial_1 \mathbf{y}_1^h) \circ (x')^{-1} + (p_2' x_2' + p_3' x_3')(\partial_2 \mathbf{y}_1^h) \circ (x')^{-1} + (-p_2' x_2' + p_3' x_3')(\partial_3 \mathbf{y}_1^h) \circ (x')^{-1},$$

$$+ (-p_2' x_2' + p_3' x_3')(\partial_3 \mathbf{y}_1^h) \circ (x')^{-1}.$$
\[\partial^2 \beta^h_i = p_2(\partial^2 \gamma_i^h) \circ (x')^{-1} - p_3(\partial \gamma_i^h) \circ (x')^{-1},\]
\[\partial^3 \beta^h_i = p_3(\partial^2 \gamma_i^h) \circ (x')^{-1} + p_2(\partial^3 \gamma_i^h) \circ (x')^{-1}.\]

By differentiating \(\beta_1\) with respect to \(x_k\), with \(k = 2, 3\), we have
\[\partial^2 \beta^h_1 = \frac{1}{h^3} \partial^2 (\tilde{y}_1^h) \circ \Theta^h \circ P^h \circ (x')^{-1} + \frac{1}{h} ((\nu_2^h)' + \theta_2'),\]
\[\partial^3 \beta^h_1 = \frac{1}{h^3} \partial^3 (\tilde{y}_1^h) \circ \Theta^h \circ P^h \circ (x')^{-1} + \frac{1}{h} ((\nu_3^h)' + \theta_3').\]

Let us analyze only \(\partial^2 \beta^h_1\). We have by (3.18), (4.36) and the chain rule
\[\partial^2 \beta^h_1 = \frac{1}{h^2} ((\partial^2 \tilde{y}_1^h) \circ \Theta^h \circ P^h \circ (x')^{-1}) (p_2 n_1^h - p_3 b_1^h) + \frac{1}{h} ((\nu_2^h)' + \theta_2').\]

By using (3.5), (3.6), (3.18), (4.12), (4.14), (4.35) and the definition of \(\nu_k^h\) we can conclude that for proving the boundedness of \(\partial^2 \beta^h_1\) it is enough to prove the boundedness of \(\delta^h_{1,2}\) in \(L^2(\Omega)\) where
\[\delta^h_{1,2} := -\frac{h R_{11}^h \theta_1' + R_{12}^h + R_{12}^h}{h^2} + \frac{h \theta_2'}{h^2} = \frac{1}{h} - \frac{R_{11}^h}{h} \theta_1' + \frac{R_{12}^h + R_{21}^h}{h^2}.\]

The boundedness of \(\delta^h_{1,2}\) in \(L^\infty(\Omega)\) is then the consequence of the property (e). In the same way we can prove the boundedness of \(\partial^3 \beta_1^h\). Using the Poincare inequality and the fact that \(\int_{\omega'/(x_1)} \beta_1^h d x_2 d x_3 = 0\), we deduce that there exists a constant \(C > 0\) such that
\[\int_{\omega'/(x_1)} (\beta_1^h(x))^2 d x_2 d x_3 \leq C \int_{\omega'/(x_1)} [(\partial^2 \beta_1^h(x))^2 + (\partial^3 \beta_1^h(x))^2] d x_2 d x_3\]
for a.e. \(x_1 \in (0, L)\) and for every \(h\). Although the constant \(C\) depends on the domain, since all domains are translations and rotations of the domain \(\omega\), the constant \(C\) can be chosen uniformly. Integrating both sides with respect to \(x_1\), we obtain that the sequence \((\beta_1^h)\) is bounded in \(L^2(\Omega')\) and so, up to subsequences, \(\beta_1^h \rightharpoonup \beta_1\) and \(\partial_k \beta_1^h \rightharpoonup \partial_k \beta_1\) weakly in \(L^2(\Omega')\), for \(k = 2, 3\). From the relations (4.41) it can be concluded that \(\gamma_1^h \rightharpoonup \gamma_1\) and \(\partial_k \gamma_1^h \rightharpoonup \partial_k \gamma_1\) weakly in \(L^2(\Omega')\), for \(k = 2, 3\), where \(\gamma = \beta \circ x'\). For the sequences \((\beta_2^h), (\beta_3^h)\), we have by differentiation that for \(j, k = 2, 3\)
\[\partial^j \beta_k^h = \frac{1}{h^2} \left( \frac{1}{h} \partial^j \tilde{y}_k^h \circ \Theta^h \circ P^h \circ (x')^{-1} - h \delta_{jk} - h w^h (1 - \delta_{jk}) (-1)^k \right).\]

By using the chain rule we see that for \(k = 2, 3\),
\[
\partial_2(\tilde{\mathbf{y}}_k^h \circ \mathbf{\Theta}^h \circ P^h \circ (x')^{-1}) = h \left( (\partial_1 \tilde{\mathbf{y}}_k^h \circ \mathbf{\Theta}^h \circ P^h \circ (x')^{-1})(p_2 n_1^h - p_3 b_1^h) \right.
\]
\[
+ (\partial_2 \tilde{\mathbf{y}}_k^h \circ \mathbf{\Theta}^h \circ P^h \circ (x')^{-1})(p_2 n_2^h - p_3 b_2^h) \left. \right)
\[
+ (\partial_3 \tilde{\mathbf{y}}_k^h \circ \mathbf{\Theta}^h \circ P^h \circ (x')^{-1})(p_2 n_3^h - p_3 b_3^h) \right).
\]
\[
\partial_3(\tilde{\mathbf{y}}_k^h \circ \mathbf{\Theta}^h \circ P^h \circ (x')^{-1}) = h \left( (\partial_1 \tilde{\mathbf{y}}_k^h \circ \mathbf{\Theta}^h \circ P^h \circ (x')^{-1})(p_3 n_1^h + p_2 b_1^h) \right.
\]
\[
+ (\partial_2 \tilde{\mathbf{y}}_k^h \circ \mathbf{\Theta}^h \circ P^h \circ (x')^{-1})(p_3 n_2^h + p_2 b_2^h) \left. \right)
\[
+ (\partial_3 \tilde{\mathbf{y}}_k^h \circ \mathbf{\Theta}^h \circ P^h \circ (x')^{-1})(p_3 n_3^h + p_2 b_3^h) \right).
\]

Now we want to check that for \(j, k = 2, 3\)
\[
e_{jk}(\beta^h) := \frac{1}{2} (\partial_j \beta_k^h + \partial_k \beta_j^h).
\]

is bounded in \(L^2(\Omega')\). In the similar way as for \(\beta_1\) (relations (4.44) and (4.45)) we can conclude using (3.5), (3.6), (4.12), (4.14) and the property (e), that for every \(j, k = 2, 3\), \(e_{jk}(\beta^h) \in L^2(\Omega')\). By using Korn’s inequality (Lemma 1) we have that there exists \(C > 0\) such that
\[
\|\beta_2^h\|^2_{W^{1,2}(\omega'(x_1))} + \|\beta_3^h\|^2_{W^{1,2}(\omega'(x_1))} \leq C \left( \left\| \int_{\omega'(x_1)} \beta_2^h dx' \right\| + \left\| \int_{\omega'(x_1)} \beta_3^h dx' \right\| \right)
\]
\[
+ \left\| \int_{\omega'(x_1)} (x'_2 \beta_2^h - x'_3 \beta_3^h) dx' \right\| + \left\| \sum_{j,k=1,2} e_{jk}(\beta^h) \right\|_{L^2(\omega'(x_1))},
\]

for a.e. \(x_1 \in (0, L)\). From the definition of \(v_k^h\) and \(w^h\) we see that the functions \((\beta_2^h(x_1, \cdot), \beta_3^h(x_1, \cdot))\) belong to the space
\[
\mathcal{B}_{x_1} = \left\{ \beta = (\beta_2, \beta_3) \in W^{1,2}(\omega'(x_1); \mathbb{R}^2) : \int_{\omega'(x_1)} \beta dx_2 dx_3 = 0, \right.
\]
\[
\left. \int_{\omega'(x_1)} (x'_2 \beta_3 - x'_3 \beta_2) dx' = 0 \right\}. \tag{4.49}
\]

for every \(x_1\). By integrating (4.48) with respect to \(x_1\) we conclude that \(\beta_2^h, \beta_3^h\) are bounded in \(L^2(\Omega')\) as well as their derivatives with respect to \(x_2, x_3\). From this we can conclude the same fact about \(\mathbf{\gamma}_2^h, \mathbf{\gamma}_3^h\). The fact that the weak limit belongs to the space \(C\) can be concluded from the fact that for every \(h\) and a.e. \(x_1 (\beta_2^h(x_1, \cdot), \beta_3^h(x_1, \cdot)) \in \mathcal{B}_{x_1}\). This finishes the proof of (f).

4.1 Lower Bound

**Lemma 2** Let \(\mathbf{y}^h, \bar{\mathbf{y}}^h, E^h, R^h, u^h, v^h, w^h, \mathbf{\gamma}^h, \beta^h = \mathbf{y}^h \circ (x')^{-1}, \mathbf{\gamma} = \mathbf{y} \circ (x')^{-1}, A\) be as in Theorem 3 and let us suppose that the condition (4.11) is satisfied and that \(\mathbf{y}^h \rightharpoonup \mathbf{y}, \partial_2 \mathbf{y}^h \rightharpoonup \partial_2 \mathbf{y}, \partial_3 \mathbf{y}^h \rightharpoonup \partial_3 \mathbf{y}\) weakly in \(L^2(\Omega)\), i.e., \(\beta^h \rightharpoonup \beta, \partial_2 \beta^h \rightharpoonup \partial_2 \beta, \partial_3 \beta^h \rightharpoonup \partial_3 \beta\) weakly in \(L^2(\Omega')\). Let us define
\[ \eta^h_1(x) = \frac{1}{h} \left( (\Theta^h \circ P^h)(x) - \Theta^h_1 \circ P^h \right) - u^h(x_1) + x'_2(v^h_2)'(x_1) + x'_3(v^h_3)'(x_1) \right), \] \(4.50\)

\[ \eta^h_k(x) = \frac{1}{h^2} \left( (\tilde{y}^h_k \circ \Theta^h \circ P^h)(x) - \Theta^h_k \circ P^h \right) - v^h_k(x_1) - h(x'_k)^\perp \omega^h(x_1) \right), \text{ for } k = 2, 3, \] \(4.51\)

and \(\kappa^h = \eta^h \circ (x')^{-1}\). Then we have that \(\eta^h \rightharpoonup \eta\) weakly in \(L^2(\Omega)\) and \(\partial_k \eta^h \rightharpoonup \partial_k \eta\) weakly in \(L^2(\Omega)\), i.e., \(\kappa^h \rightharpoonup \kappa, \partial_2 \kappa^h \rightharpoonup \partial_2 \kappa, \partial_3 \kappa^h \rightharpoonup \partial_3 \kappa\) weakly in \(L^2(\Omega')\). Here

\[ \eta_1 = \gamma_1, \] \(4.52\)

\[ \eta_2 = \gamma_2 + f^2_k x_2 + f^3_k x_3 = \gamma_2 + g^2_k x_2 + g^3_k x_3, \] \(4.53\)

\[ \eta_3 = \gamma_3 + f^2_k x_2 + f^3_k x_3 = \gamma_3 + g^3_k x_2 + g^3_k x_3, \] \(4.54\)

\[ \kappa = \eta \circ (x')^{-1}, \] \(4.55\)

\(f^j_k\) are defined in Remark 3 and \(g^j_k\) can be easily defined for the above identities to be valid, i.e., for \(k = 2, 3\), we define

\[ g^2_k = p^2 f^2_k - p^3 f^3_k, \quad g^3_k = p^3 f^2_k + p^2 f^3_k. \] \(4.56\)

The following strain convergence is valid

\[ G^h := \frac{(R^h)^T ((\nabla y^h) \circ \Theta^h \circ P^h) - I}{h^2} \rightharpoonup G \text{ in } L^2(\Omega; \mathbb{R}^{3 \times 3}) \] \(4.57\)

and the symmetric part of \(G\) denoted by \(\tilde{G}\), satisfies

\[ \tilde{G} = \text{sym} \left( J - \frac{1}{2} A^2 + K \right), \] \(4.58\)

where

\[ J = \begin{pmatrix} u' + v'_2 \theta'_2 + v'_3 \theta'_3 & 0 & 0 \\ w \theta'_2 & v'_2 \theta'_2 & v'_2 \theta'_3 \\ -w \theta'_2 & v'_3 \theta'_2 & v'_3 \theta'_3 \end{pmatrix}, \] \(4.59\)

\[ K = \begin{pmatrix} -x'_2 v'_2 - x'_3 v'_3 & \partial_2 \kappa & \partial_3 \kappa \\ -x'_2 w' & \partial_2 \kappa & \partial_3 \kappa \\ x'_2 w' \end{pmatrix}. \] \(4.60\)

Moreover,

\[ \liminf_{h \to 0} \frac{1}{h^6} \int_{\Omega^h} W^h(x^h, \nabla \tilde{y}^h) \, dx \]

\[ = \liminf_{h \to 0} \frac{1}{h^4} \int_{\Omega} W(x, (\nabla \tilde{y}^h) \circ \Theta^h \circ P^h) \det ((\nabla \Theta^h) \circ P^h(x)) \, dx \]
where \( Q_3 \) is twice the quadratic form of linearized elasticity, i.e.,

\[
Q_3(x, F) = \frac{\partial^2 W}{\partial F^2}(I)[F, F].
\]

**Proof** We follow the proof of Lemma 2.3 in [23]. Firstly, using Remark 2, it can be seen that

\[
\eta_1^h = \gamma_1^h + h\alpha_1,
\]

\[
\eta_2^h = \gamma_2^h + f_2^3 x_2 + f_2^3 x_3 + h\alpha_2,
\]

\[
\eta_3^h = \gamma_3^h + f_3^3 x_2 + f_3^3 x_3 + h\alpha_3,
\]

where \( \|\alpha\|_{C^1(\Omega)} \leq C \), for some \( C > 0 \). The convergence of \( \eta^h \) is an easy consequence of the convergence of \( \eta^h \). The estimate (4.12) implies that the \( L^2 \) norm of \( G^h \) is bounded; therefore up to subsequences, there exists \( G \in L^2(\Omega; \mathbb{R}^{3 \times 3}) \) such that (4.57) is satisfied. From the fact that \( R^h \to I \) boundedly in measure we conclude that \( R^h G^h \to G \) in \( L^2(\Omega; \mathbb{R}^{3 \times 3}) \). In order to identify the symmetric part of \( G \) we decompose \( R^h G^h \) as follows:

\[
R^h G^h = \left( \nabla \gamma^h \circ \Theta^h \circ P^h - I \right) \frac{1}{h} + \frac{R^h - I}{h^2},
\]

so that

\[
F^h := \text{sym} \left( \nabla \gamma^h \circ \Theta^h \circ P^h - I \right) \frac{1}{h} = \text{sym}(R^h G^h) + \text{sym} \left( \frac{R^h - I}{h^2} \right).
\]

The right hand side of (4.63) converges weakly to \( \tilde{G} + \frac{A^2}{2} \) by (4.14), (4.57) and property (e) of the Theorem 3. Therefore the sequence \( F^h \) has a weak limit in \( L^2(0, L) \), satisfying \( F = \tilde{G} + \frac{A^2}{2} \). To conclude (4.57) we only need to identify \( F \). Consider the functions

\[
\phi_1^h := \left( \nabla \gamma^h \circ \Theta^h \circ P^h - x_1 \right) \frac{1}{h^2}.
\]

From property (f) of Theorem 3 it follows that the functions \( \phi_1^h - u^h + x_1^3(v_2^3) + x_2^3(v_2^3) + x_3^3(v_3^3) \) converge strongly to \( 0 \) in \( L^2(\Omega) \). Thus by property (a) and (b) of Theorem 3 we conclude that

\[
\phi_1^h \to u - x_1^3(v_2^3) - x_2^3(v_3^3) \quad \text{in } L^2(\Omega).
\]

By using the chain rule, the property (d) of Theorem 3, (3.5), (3.6) and (4.30) we can conclude that

\[
\partial_1 \phi_1^h \to F_{11} - x_1(\theta_2^3 p_2 + \theta_3^3 p_3) + x_3(\theta_2^3 p_3 - \theta_3^3 p_2) - v_2(\partial_1 x_2^3 + \theta_2^3) - v_3(\partial_1 x_3^3 + \theta_3^3), \quad (4.66)
\]

weakly in \( L^2(\Omega) \). From (4.65) and (4.66) we conclude that

\[
\begin{align*}
&u' - \partial_1 x_2^3(v_2^3 + \theta_2^3) - x_2^3(v_2^3 + \theta_2^3) - \partial_1 x_3^3(v_3^3 + \theta_3^3) - x_3^3(v_3^3 + \theta_3^3) \\
&= F_{11} - x_1(\theta_2^3 p_2 + \theta_3^3 p_3) + x_3(\theta_2^3 p_3 - \theta_3^3 p_2).
\end{align*}
\]
\[-v_2'(\partial_1 x_2' + \theta_2') - v_3'(\partial_1 x_3' + \theta_3').\] (4.67)

After some calculation we obtain

\[F_{11} = u' + v_2'\theta_2' + v_3'\theta_3' - x_2''v_2' - x_3''v_3'.\] (4.68)

To identify \(F_{12}\) we have to do some straight forward computations. By using the chain rule, (3.5), (3.6), Remark 2 and property (d) of Theorem (3) we can conclude

\[
\frac{1}{h^2} \partial_1(\tilde{y}_2^h \circ \Theta^h \circ P^h) + \frac{1}{h^3} \left( p_2 \partial_2(\tilde{y}_1^h \circ \Theta^h \circ P^h) - p_3 \partial_3(\tilde{y}_1^h \circ \Theta^h \circ P^h) \right)
= 2F_{12}^h + \frac{\partial_1 x_2'}{h} - w(\theta_3' + \partial_1 x_3') + O_h^h,
\] (4.69)

where \(\lim_{h \to 0} ||O_h^h||_{L^2(\Omega; \mathbb{R}^{3 \times 3})} = 0\). On the other hand it can be easily seen that

\[
\partial_2 \beta_1^h = p_2 \partial_2 \gamma_1^h - p_3 \partial_3 \gamma_1^h = \frac{1}{h^3} \left( p_2 \partial_2(\tilde{y}_1^h \circ \Theta^h \circ P^h) - p_3 \partial_3(\tilde{y}_1^h \circ \Theta^h \circ P^h) \right)
+ \frac{1}{h}((v_2^h)' + \theta_2').
\] (4.70)

From (4.34), (4.69), (4.70) we conclude

\[
2F_{12}^h = \frac{1}{h^2} \partial_1(\tilde{y}_2^h \circ \Theta^h \circ P^h - hx_2') - \frac{1}{h}((v_2^h)' + \theta_2')
+ \partial_2 \beta_1^h + w\theta_3' + w\partial_1 x_3' - O_h^h
= \partial_1 w_2^h + \partial_2 \beta_1^h + w\theta_3' + w\partial_1 x_3' - O_h^h.
\] (4.71)

By using (4.34) we conclude that the right hand side of (4.71) converges in \(W^{1,2}(\Omega)\) to

\[
\partial_1(-wx_3') + \partial_2 \beta_1 + w\theta_3' + w\partial_1 x_3' = -x_3'w' + w\theta_3' + \partial_2 \kappa_1,
\] (4.72)

since \(\beta_1 = \kappa_1\). On the other hand we know that the left hand side of (4.71) converges strongly in \(L^2(\Omega)\) to \(2F_{12}\) and thus we can conclude

\[F_{12} = \frac{1}{2}(-x_3'w' + w\theta_3' + \partial_2 \kappa_1).\] (4.73)

In the same way one can prove

\[F_{13} = \frac{1}{2}(x_2'w' - w\theta_2' + \partial_3 \kappa_1).\] (4.74)

To identify \(F_{22}\) let us observe that by the chain rule, (3.5), (3.6) and the property (d) of Theorem (3) we have

\[
\frac{1}{h^3} \left( p_2 \partial_2(\tilde{y}_2^h \circ \Theta^h \circ P^h - \Theta_2 \circ P^h) - p_3 \partial_3(\tilde{y}_2^h \circ \Theta^h \circ P^h - \Theta_2 \circ P^h) \right)
= F_{22}^h - v_2'\theta_2' + O_h^h,
\] (4.75)
where \( \lim_{h \to 0} \| O^h_2 \|_{L^2(\Omega; \mathbb{R}^{3 \times 3})} = 0 \). On the other hand we can conclude

\[
\partial_2 k^h_2 = p_2 \partial_2 \eta^h_2 - p_3 \partial_3 \eta^h_3
\]

\[
= \frac{1}{h^3} \left( p_2 \partial_2 (\bar{y}^h_2 \circ \Theta^h \circ P^h - \Theta_2 \circ P^h) - p_3 \partial_3 (\bar{y}^h_2 \circ \Theta^h \circ P^h - \Theta_2 \circ P^h) \right).
\]

(4.76)

In the same way as before we conclude that

\[ F_{22} = v_2^t \theta^t_2 + \partial_2 k_2. \tag{4.77} \]

Analogously we can conclude

\[ F_{33} = v_3^t \theta^t_3 + \partial_3 k_3. \tag{4.78} \]

To identify \( F_{23} = F_{32} \) we, by using the chain rule, (3.5), (3.6) and the property (d) of Theorem (3), can conclude:

\[
\frac{1}{h^3} \left( p_3 \partial_2 (\bar{y}^h_2 \circ \Theta^h \circ P^h - \Theta_2 \circ P^h) + p_2 \partial_3 (\bar{y}^h_2 \circ \Theta^h \circ P^h - \Theta_2 \circ P^h) \right)
\]

\[
= \frac{1}{h^2} (\partial_2 \bar{y}^h_2) \circ \Theta^h \circ P^h - v_2^t \theta^t_2 + O^h_4,
\]

(4.79)

where \( \lim_{h \to 0} \| O^h_3 \|_{L^2(\Omega; \mathbb{R}^{3 \times 3})} = 0 \). In the same way we conclude

\[
\frac{1}{h^3} \left( p_2 \partial_3 (\bar{y}^h_2 \circ \Theta^h \circ P^h - \Theta_2 \circ P^h) - p_3 \partial_2 (\bar{y}^h_2 \circ \Theta^h \circ P^h - \Theta_2 \circ P^h) \right)
\]

\[
= \frac{1}{h^2} (\partial_3 \bar{y}^h_3) \circ \Theta^h \circ P^h - v_3^t \theta^t_3 + O^h_4,
\]

(4.80)

where \( \lim_{h \to 0} \| O^h_4 \|_{L^2(\Omega; \mathbb{R}^{3 \times 3})} = 0 \). It can be also concluded that

\[
p_2 \partial_3 \eta^h_2 - p_3 \partial_2 \eta^h_3 = \frac{1}{h^3} \left( p_2 \partial_2 (\bar{y}^h_2 \circ \Theta^h \circ P^h - \Theta_2 \circ P^h) \right.
\]

\[
- p_3 \partial_3 (\bar{y}^h_2 \circ \Theta^h \circ P^h - \Theta_2 \circ P^h)\left) + \frac{1}{h^2} w^h. \tag{4.81} \]

\[
p_3 \partial_2 \eta^h_2 + p_2 \partial_2 \eta^h_3 = \frac{1}{h^3} \left( p_3 \partial_2 (\bar{y}^h_2 \circ \Theta^h \circ P^h - \Theta_2 \circ P^h) \right.
\]

\[
+ p_2 \partial_3 (\bar{y}^h_2 \circ \Theta^h \circ P^h - \Theta_2 \circ P^h)\left) - \frac{1}{h^2} w^h. \tag{4.82} \]

By summing the relations (4.79)-(4.82) and letting \( h \to 0 \) we find that

\[ 2F_{23} = v_2^t \theta^t_2 + v_3^t \theta^t_3 + \partial_2 k_3 + \partial_3 k_2. \tag{4.83} \]

To prove the lower bound we can continue in the same way as in the proof of Lemma 2.3 in [23], by using the Taylor expansion, the cutting and Scorza-Dragoni theorem. \( \square \)
4.2 Upper Bound

**Theorem 4** (Optimality of lower bound) Let \( u, w \in W^{1,2}(0, L) \) and \( v_k \in W^{2,2}(0, L) \) for \( k = 2, 3 \). Let \( \eta \) be a function in \( C \) where

\[
C = \left\{ \eta \in L^2(\Omega; \mathbb{R}^3) : \int_\omega \eta = 0, \ \partial_2 \eta, \partial_3 \eta \in L^2(\Omega; \mathbb{R}^3), \right. \\
\left. \int_\omega (x_3^2 \eta_2(x_1, \cdot) - x_2^2 \eta_3(x_1, \cdot)) \, dx_2 \, dx_3 = 0, \ \forall x_1 \in (0, L) \right\}. \tag{4.84}
\]

Set

\[
\tilde{G} = \text{sym} \left( J - \frac{1}{2} A^2 + K \right). \tag{4.85}
\]

Here \( A, J, K \) are defined by the expressions (4.18), (4.59) and (4.60) and \( \eta, \kappa \) are defined by the expressions (4.52)–(4.55).

Then there exists a sequence \( (\tilde{\eta}^h) \subset W^{1,2}(\Omega^h, \mathbb{R}^3) \) such that for \( u^h, v_k^h, w \) defined by the expressions (4.15)–(4.17) the properties (a)–(d) of Theorem 3 are valid. Also we have that the property (f) of Theorem 3 is valid (which is equivalent that for \( \eta^h \) defined by the expressions (4.50)–(4.51) we have \( \eta^h \rightharpoonup \eta \) weakly in \( L^2(\Omega) \) and \( \partial_k \eta^h \rightharpoonup \partial_k \eta \) weakly in \( L^2(\Omega) \)). Also the following convergence is valid

\[
\lim_{h \to 0} \frac{1}{h^6} \int_{\Omega^h} W^h(x^h, \nabla \tilde{\eta}^h) \, dx^h = \frac{1}{2} \int_{\Omega} Q^3(x, G(x)) \, dx. \tag{4.86}
\]

**Proof** Let us first assume that \( u, w, v_k, \eta \) are smooth. Then we define for \( (x_1, x_2^h, x_3^h) \in \Omega^h \):

\[
\tilde{\eta}^h(\Theta^h(x_1, x_2^h, x_3^h)) = \Theta^h(x_1, x_2^h, x_3^h) + \left( \begin{array}{c} h^2 u(x_1) \\ hv_2(x_1) \\ hv_3(x_1) \end{array} \right) + h^2 \left( \begin{array}{c} -x_2(v_2^h p_2 + v_3^h p_3)(x_1) - x_3(v_2^h p_2 - v_3^h p_3)(x_1) \\ -x_2(p_3 w)(x_1) - x_3(p_2 w)(x_1) \\ x_2(p_2 w)(x_1) - x_3(p_3 w)(x_1) \end{array} \right) + h^3 \eta(x_1, x_2^h/h, x_3^h/h). \tag{4.87}
\]

where \( \eta : \Omega \to \mathbb{R}^3 \) is going to be chosen later. The convergence (a)–(d) and the fact that \( \eta^h \rightharpoonup \eta \) weakly in \( L^2(\Omega) \) and \( \partial_k \eta^h \rightharpoonup \partial_k \eta \) weakly in \( L^2(\Omega) \) can easily be seen to be valid for this sequence. We also have

\[
\nabla \tilde{\eta}^h \nabla \Theta^h = \nabla \Theta^h + \left( \begin{array}{ccc} h^2 u^h & -h(v_2^h p_2 + v_3^h p_3) & -h(v_2^h p_2 - v_3^h p_3) \\ hv_2^h & -h p_2 w & -h p_2 w \\ hv_3^h & h p_3 w & -h p_3 w \end{array} \right) \\
+ h^2 \left( \begin{array}{c} -x_2(v_2^h p_2 + v_3^h p_3)' + x_3(v_2^h p_3 - v_3^h p_2)' \\ -x_2(p_3 w)' - x_3(p_2 w)' \\ x_2(p_2 w)' - x_3(p_3 w)' \end{array} \right) + \left( \begin{array}{c} \partial_2 \eta \\ \partial_3 \eta \end{array} \right). \tag{4.88}
\]
From (4.88), by using (3.8), we conclude

\[
\nabla \hat{y}^h = I + h \begin{pmatrix} 0 & -v'_2 & -v'_3 \\ v'_2 & 0 & -w \\ v'_3 & w & 0 \end{pmatrix} \\
+ h^2 \begin{pmatrix} u' + v'_2 \theta'_2 + v'_3 \theta'_3 & 0 & 0 \\ 0 & v'_2 \theta'_2 & v'_2 \theta'_3 \\ 0 & v'_3 \theta'_2 & v'_3 \theta'_3 \end{pmatrix} \\
+ h^2 \begin{pmatrix} -x_2(v''_2 p_2 + v''_3 p_3) + x_3(v''_2 p_3 - v''_3 p_2) \\ -x_2(p_3 w') - x_3(p_2 w') \\ x_2(p_2 w') - x_3(p_3 w') \end{pmatrix} \begin{vmatrix} \frac{\partial_2 \kappa}{\partial_3 \kappa} \end{vmatrix} \\
+ O(h^3).
\]

(4.89)

Using the identity \((I + M)^T (I + M) = I + 2 \text{ sym } M + M^T M\) we obtain

\[
(\nabla \hat{y}^h)^T (\nabla \hat{y}^h) = I + 2h^2 \text{ sym } J + 2h^2 \text{ sym } K + h^2 A^T A + O(h^3),
\]

where \(\|O(h^3)\|_{L^\infty(\Omega; \mathbb{R}^{3 \times 3})} \leq Ch^3\), for some \(C > 0\).

Taking the square root we obtain

\[
[(\nabla \hat{y}^h)^T (\nabla \hat{y}^h)]^{1/2} = I + h^2 G + O(h^3). \tag{4.90}
\]

We have \(\det(\nabla \hat{y}^h) > 0\) for sufficiently small \(h\). Hence by frame-indifference \(W(x, (\nabla \hat{y}^h) \circ \Theta^h \circ P^h) = W(x, [\nabla \hat{y}^h]^T (\nabla \hat{y}^h))^{1/2} \circ \Theta^h \circ P^h\); thus by (4.90) and Taylor expansion we obtain:

\[
\frac{1}{h^4} W(x, (\nabla \hat{y}^h) \circ \Theta^h \circ P^h) \rightarrow \frac{1}{2} Q_3(x, \tilde{G}(x)) \text{ a.e.}
\]

and by the property (ii) of \(W\) for \(h\) small enough

\[
\frac{1}{h^4} W(x, (\nabla \hat{y}^h) \circ \Theta^h \circ P^h) \leq \frac{1}{2} C(\|J\|^2 + \|K\|^2 + \|A\|^4) + Ch.
\]

The equality (4.86) follows by the dominated convergence theorem. Namely, we have

\[
\frac{1}{h^4} \int_{\Omega^h} W^h(x^h, \nabla \hat{y}^h) \, dx = \frac{1}{h^4} \int_{\Omega} W(x, (\nabla \hat{y}^h) \circ \Theta^h \circ P^h) \det((\nabla \Theta^h) \circ P^h(x)) \, dx \\
\rightarrow \frac{1}{2} \int_{\Omega} Q_3(x, \tilde{G}) \, dx.
\]

In the general case, it is enough to smoothly approximate \(u, w\) in the strong topology of \(W^{1,2}, v_k\) in the strong topology of \(W^{2,2}\), and \(\eta, \partial \eta\) in the strong topology of \(L^2\) and to use the continuity of the right hand side of (4.86) with respect to these convergences.

\[\square\]

**Remark 7** Notice that

\[
K = \left( A' \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} | \partial_2 \kappa | \partial_3 \kappa \right) = L + \left( A' \begin{pmatrix} 0 \\ x'_2 \\ x'_3 \end{pmatrix} | \partial_2 \beta | \partial_3 \beta \right).
\]
Here $\beta = \gamma \circ (x')^{-1}$ and

$$L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & g_2^2 & g_2^3 \\ 0 & g_3^2 & g_3^3 \end{pmatrix}. \quad (4.91)$$

From the fact that $\gamma \in C$ we can conclude $\beta \in B$, where

$$B = \left\{ \beta \in L^2(\Omega'; \mathbb{R}^3) : \int_\omega \beta = 0, \ \partial_2 \beta, \ \partial_3 \beta \in L^2(\Omega'; \mathbb{R}^3), \int_{\omega'(x_1)} (x'_2 \beta_2(x_1, \cdot) - x'_2 \beta_3(x_1, \cdot)) \, dx'_2 \, dx'_3 = 0, \text{ for a.e. } x_1 \in (0, L) \right\}. \quad (4.92)$$

### 4.3 Identification of the $\Gamma$-Limit

Let $Q : (0, L) \times \mathbb{R} \times (3) \to [0, +\infty)$ be defined as

$$Q(x_1, t, F) = \min_{\alpha \in W^{1, 2}(\omega'(x_1); \mathbb{R}^3)} \int_{\omega'(x_1)} Q_3 \left( x, \left( F \begin{pmatrix} 0 \\ x'_2 \alpha_2 \\ x'_3 \alpha_3 \end{pmatrix} + t e_1 \right), \partial_2 \alpha, \partial_3 \alpha \right) \, dx'_2 \, dx'_3, \quad (4.93)$$

where $Q_3$ is the quadratic form defined in (4.61). For $u, w \in W^{1, 2}(0, L)$ and $v_2, v_3 \in W^{2, 2}(0, L)$ we introduce the functional

$$I^0(u, v_2, v_3, w) := \frac{1}{2} \int_0^L Q \left( x_1, u' + v'_2 \theta'_2 + v'_3 \theta'_3 + \frac{1}{2} ((v'_2)^2 + (v'_3)^2), A' \right) \, dx_1, \quad (4.94)$$

where $A \in W^{1, 2}((0, L); (3))$ is defined by (4.18). We shall state the result of $\Gamma$-convergence of the functionals $\frac{1}{h} I^h$ to $I^0$. Before stating the theorem we analyze some properties of the limit density $Q$.

**Remark 8** By using the remarks in the beginning of chapter 4 in [23] the following facts can be concluded:

(a) The functional $Q_3(x, G)$ is coercive on symmetric matrices, i.e., there exists a constant $C > 0$, independent of $x$, such that $Q_3(x, G) \geq C \| \text{sym} \, G \|^2$, for every $G$ (this is the direct consequence of the assumption iv) on $W$). The minimum in (4.93) is attained.

Since the functional $Q_3(x, G)$ depends only on the symmetric part of $G$, it is invariant under transformation $\alpha \mapsto \alpha + c_1 + c_2 (x')^\perp$ and hence the minimum can be computed on the subspace

$$V_{x_1} : = \left\{ \alpha \in W^{1, 2}(\omega'(x_1), \mathbb{R}^3) : \int_{\omega'(x_1)} \alpha = 0, \int_{\omega'(x_1)} (x'_2 \alpha_2 - x'_2 \alpha_3) \, dx'_2 \, dx'_3 = 0 \right\}.$$

Strict convexity of $Q_3(x, \cdot)$ on symmetric matrices ensures that the minimizer is unique in $V_{x_1}$. 

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Fix \( x_1 \in (0, L) \), \( t \in \mathbb{R} \) and \( \mathbf{F} \in (3) \). Let \( \mathbf{a}^{\text{min}} \in V \) be the unique minimizer of the problem (4.93). We set
\[
g(x'_2, x'_3) = \mathbf{F} \begin{pmatrix} 0 \\ x'_2 \\ x'_3 \end{pmatrix} + t \mathbf{e}_1, \quad b_{ij}^{hk} = \frac{\partial^2 W}{\partial F_{ih} \partial F_{jk}}(x, I),
\]
and we call \( B_{hk} \) the matrix in \( \mathbb{R}^{3 \times 3} \) whose elements are given by \( (B_{hk})_{ij} = b_{ij}^{hk} \). Then \( \mathbf{a}^{\text{min}} \) satisfies the following Euler-Lagrange equation:
\[
\int_{\omega'(x_1)} \sum_{h,k=2,3} (B_{hk} \partial_k \mathbf{a}^{\text{min}}, \partial_h \varphi) \, dx_2' \, dx_3' = -\int_{\omega'(x_1)} \sum_{h=2,3} (B_{h1} \mathbf{g}, \partial_h \varphi) \, dx_2' \, dx_3', \tag{4.95}
\]
for every \( \varphi \in W^{1,2}(\omega'(x_1); \mathbb{R}^{3 \times 3}) \). From this equation it is clear that \( \mathbf{a}^{\text{min}} \) depends linearly on \((t, \mathbf{F})\). Moreover \( Q \) is uniformly positive definite, i.e.,
\[
Q(x_1, t, \mathbf{F}) \geq C(t^2 + \| \mathbf{F} \|^2), \quad \forall t \in \mathbb{R}, \forall \mathbf{F} \in (3), \tag{4.96}
\]
and the constant \( C \) does not depend on \( x_1 \).

By mimicking the proof of Remark 4.3 in [23] it can be seen that there exists a constant \( C' \) (independent of \( x_1, t \) and \( \mathbf{F} \)) such that
\[
\| \partial_2 \mathbf{a}^{\text{min}} \|_{L^2(\omega'(x_1); \mathbb{R}^{3 \times 3})} + \| \partial_3 \mathbf{a}^{\text{min}} \|_{L^2(\omega'(x_1); \mathbb{R}^{3 \times 3})} \leq C' \| \mathbf{g} \|^2_{L^2(\omega'(x_1); \mathbb{R}^{3 \times 3})}, \tag{4.97}
\]
for a.e. \( x_1 \in (0, L) \). To adapt the proof we only need to have that the constant in the Korn’s inequality
\[
\int_{\omega'(x_1)} \sum_{j,k=2,3} |\partial_j \mathbf{a}^{\text{min}}|^2 \, dx_2' \, dx_3' \leq C_1 \int_{\omega'(x_1)} \sum_{j,k=2,3} |e_{jk}(\mathbf{a}^{\text{min}})|^2 \, dx_2' \, dx_3' \tag{4.98}
\]
can be chosen independently of \( x_1 \). This is proved in Lemma 1.

When \( Q_3 \) does not depend on \( x_2, x_3 \) we can find a more explicit representation for \( Q \). More precisely \( Q \) can be decomposed into the sum of two quadratic forms
\[
Q(x_1, t, \mathbf{F}) = Q_1(x_1, t) + Q_2(x_1, \mathbf{F}),
\]
where
\[
Q_1(x_1, t) := \min_{a, b \in \mathbb{R}^3} Q_3(x_1, (t \mathbf{e}_1 | a | b)), \tag{4.99}
\]
\[
Q_2(x_1, 0, \mathbf{F}) := Q(x_1, 0, \mathbf{F}). \tag{4.100}
\]
The relations (4.8) are only needed for this. If we assume the isotropic and homogeneous case, i.e.,
\[
Q_3(\mathbf{F}) = 2\mu \left| \frac{\mathbf{F} + \mathbf{F}^T}{2} \right|^2 + \lambda (\text{trace } \mathbf{F})^2,
\]
then after some calculation (see Remark 3.5 in [22]) it can be shown that
\[
Q_1(t) = \frac{\mu (3\lambda + 2\mu)}{\lambda + \mu} t^2,
\]
\[ Q_2(x_1, F) = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \left( F_{12}^2 \int_{\partial \omega(x_1)} (x'_2)^2 \, dx'_2 \, dx'_3 + 2F_{12}F_{13} \int_{\partial \omega(x_1)} x'_2 x'_2 \, dx'_2 \, dx'_3 \right) \]
\[ + F_{13}^2 \int_{\partial \omega(x_1)} (x'_3)^2 \, dx'_2 \, dx'_3 \right) + \mu \tau F_{23}^2, \]

where the constant \( \tau \) is so-called torsional rigidity, defined as
\[ \tau(\omega(x_1)) = \tau(\omega) = \int_{\omega} (x_2^2 + x_3^2 - x_2 \partial_3 \varphi + x_3 \partial_2 \varphi) \, dx_2 \, dx_3, \]

and \( \varphi \) is the torsion function, i.e., the solution of the Neumann problem
\[
\begin{aligned}
\{ & \Delta \varphi = 0 && \text{in } \omega, \\
& \partial_\nu \varphi = -(x_3, -x_2) \cdot \nu && \text{on } \partial \omega.
\end{aligned}
\]

The following theorem can be proved in the same way as Theorem 4.5 in [21] (we need Theorem 3, Lemma 2, Theorem 4, Remark 7 and Remark 8).

**Theorem 5** As \( h \to 0 \), the functionals \( \frac{1}{h^4} I^h \) are \( \Gamma \)-convergent to the functional \( I^0 \) given in (4.94), in the following sense:

(i) (compactness and lim inf inequality) if \( \limsup_{h \to 0} h^{-4} I^h < +\infty \) then there exists constants \( \tilde{R}^h \in \text{SO}(3), \, e^h \in \mathbb{R}^3 \) such that (up to subsequences) \( \tilde{R}^h \to \tilde{R} \) and the functions defined by
\[
\tilde{y}^h := \left( \tilde{R}^h \right)^T y^h - e^h, \quad u^h = \frac{1}{A} \int_{\omega} \tilde{y}^h \circ \Theta^h \circ P^h - x_1 \frac{1}{h^2} \, dx_2 \, dx_3
\]
\[
v^h_k = \frac{1}{A} \int_{\omega} \tilde{y}^h_k \circ \Theta^h \circ P^h - h \theta_k \, dx_2 \, dx_3
\]
\[
w^h = \frac{1}{A \mu(\omega)} \int_{\omega} x'_2 (\tilde{y}^h_2 \circ \Theta^h \circ P^h) - x'_1 (\tilde{y}^h_2 \circ \Theta^h \circ P^h) \, \frac{1}{h^2} \, dx_2 \, dx_3
\]
satisfy

(a) \( (\nabla \tilde{y}^h) \circ \Theta^h \circ P^h \to I \) in \( L^2(\Omega) \).

(b) there exist \( u, \, w \in W^{1,2}(0, L) \) such that \( u^h \rightharpoonup u \) and \( w^h \rightharpoonup w \) weakly in \( W^{1,2}(0, L) \).

(c) there exists \( v_k \in W^{2,2}(0, L) \) such that \( v^h_k \to v_k \) strongly in \( W^{1,2}(0, L) \) for \( k = 2, 3 \). Moreover we have
\[
\liminf_{h \to 0} \frac{1}{h^4} I^h(y^h) \geq I^0(u, v_2, v_3, w). \quad (4.101)
\]

(ii) (lim sup inequality) for every \( u, \, w \in W^{1,2}(0, L), \, v_2, \, v_3 \in W^{2,2}(0, L) \) there exists \( (\tilde{y}^h) \) such that (a)(c) hold (with \( \tilde{y}^h \) replaced by \( y^h \)) and
\[
\lim_{h \to 0} \frac{1}{h^4} I^h(y^h) = I^0(u, v_2, v_3, w). \quad (4.102)
\]

**Remark 9** Let \( f_2, \, f_3 \in L^2(0, L) \). We introduce the functional
\[
I^0 = I^0(u, v_2, v_3, w) - \int_0^L \sum_{k=2,3} f_k v_k. \quad (4.103)
\]
for every $u \in W^{1,2}(0, L)$, $v_2, v_3 \in W^{2,2}(0, L)$, and $w \in W^{1,2}(0, L)$. The functional $J^0$ can be obtained as $\Gamma$-limit of the energies $\frac{1}{h} I^h$ by adding a term describing transversal body forces of order $h^3$ (see [13], see also [29]). For longitudinal body forces see [18]. The problem for longitudinal body forces arises because the longitudinal forces should be of order $h^2$, the same order as for the model in [22]. One needs to impose certain stability condition to see which model describes the behavior of the body for the longitudinal forces of order $h^2$.

**Remark 10** The term $u' + v_2' \theta_2' + v_3' \theta_3' + \frac{1}{2}((v_2')^2 + (v_3')^2)$ in the strain measures the extension of the central line (which is of the second order). Namely, if we approximate the deformation of the weakly curved rod by:

$$
\varphi_1(x_1, x_2, x_3) = x_1 + h^2 u + h^2 x_2'(v_2' + \theta_2') + h^2 x_3'(v_3' + \theta_3')
$$

$$
\varphi_k(x_1, x_2, x_3) = h \theta_k + h x_k' + h v_k + h^2 (x_k')^2 w, \quad \text{for } k = 2, 3,
$$

we see, that the following is valid

$$
\|\partial_1 \varphi b(x_1, 0, 0)\|^2 - \|\partial_1 \varphi^k(x_1, 0, 0)\|^2 = h^2 \left(2u' + 2v_2' \theta_2' + 2v_3' \theta_3' + (v_2')^2 + (v_3')^2\right).
$$

**Remark 11** The existence of the solution for the functional $J^0$ under the Dirichlet boundary condition for $v_k$ at both ends of the rod can be proved directly. It is also enough that we impose $v_2, v_2', v_3, v_3'$ at the one end. The existence can also be proved for the free boundary condition under the hypothesis that $\int_0^L f_k dx_1 = 0$, $\int_0^L x_1 f_k dx_1 = 0$ for $k = 2, 3$. It can be done in the same way as the proof of Lemma 5 in [29].

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