Two-dimensional isotropic harmonic oscillator on a
time-dependent sphere

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Abstract
In this paper, we investigate a two-dimensional isotropic harmonic oscillator on a time-dependent spherical background. The effect of the background can be represented as a minimally coupled field to the oscillator’s Hamiltonian. For a fluctuating background, transition probabilities per unit time are obtained. Transitions are possible if the energy eigenvalues of the oscillator $E_i$ and frequencies of the fluctuating background $\omega_n$ satisfy the following two simple relations: $E_j \simeq E_i - \hbar \omega_n$ (stimulated emission) and $E_j \simeq E_i + \hbar \omega_n$ (absorption). This indicates that a background fluctuating at a frequency of $\omega_n$ interacts with the oscillator as a quantum field of the same frequency. We believe this result is also applicable for an arbitrary quantum system defined on a fluctuating maximally symmetric background.

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1. Introduction

Higgs and Leemon investigated the non-relativistic motion of a particle on a $N$-dimensional fixed sphere (embedded in the Euclidean $(N + 1)$-dimensional space) under conservative central potentials \cite{1, 2}. These central potentials reduce to the familiar Coulomb and isotropic oscillator potentials of a Euclidean geometry when the curvature of the sphere goes to zero. Higgs defined the motion on a sphere of a constant radius by means of a central (or gnomonic) projection from the motion of a $N$-dimensional plane, tangent to the sphere at a given point. The advantage of this projection over all others for the analysis of particle motion on a sphere derives from the fact that free particle motion (uniform motion on a great circle) projects into a rectilinear plane, while its non-uniform motion projects into the tangent plane. In other words, the projected free particle orbits are the same as those in the Euclidean geometry: the curvature affects only the speed of the projected motion. Higgs also showed that this feature persists in
the presence of a central force derived from a potential $V(r)$, i.e. he proved that the dynamical symmetries in a sphere are the same as those in the plane. Using the Hamiltonians as functions of the Casimir operators, the eigenstates and eigenvalues of the two systems were also obtained in [1] and [2], respectively.

Recently, by exploring the Higgs model, we have constructed the generalized (nonlinear) coherent states [3] of a two-dimensional harmonic oscillator constrained to a spherical surface to study some of their quantum optical properties [4]. Accordingly, we have developed a feasible physical model to generate nonlinear coherent states on a sphere in a generalized trapped ion system [5].

The geometry of the sphere coherent states is studied in [6]. It has been shown that the structure and properties of sphere-(nonlinear) coherent states could be explored to studying the curvature effects of physical space on both transition probability and geometric phase. It has been shown that the transition probability decreases with increasing curvature of the physical space which, in turn, affects the geometric phase.

In the present contribution, investigating the behaviour of an isotropic oscillator on a dynamical background is our main purpose. For this purpose, we study a two-dimensional isotropic harmonic oscillator constrained to a spherical surface with a non-constant (time-dependent) radius. Our ambition is to incorporate the variation of the sphere’s radius in the isotropic oscillator’s Hamiltonian as a minimally coupled field. This model could be related to many of the above-mentioned topics and we will study them in the future.

In the next stage, by using the perturbation theory, we will study the transition rate between energy levels to achieve a golden rule. Appreciable probabilities for transition are possible only if the transition and radius fluctuation frequencies are nearly on resonance, e.g., if, $E_f \simeq E_i - \hbar \omega_n$ (stimulated emission) or by $E_f \simeq E_i + \hbar \omega_n$ (absorption). Thus, the isotropic harmonic oscillator on the sphere will be excited via appropriate frequencies of the background fluctuations. The proposed golden rule is expected to be applicable for any quantum system defined in a fluctuating maximally symmetric background. We expect that for a general time-dependent background, the golden rule is still valid where $\omega_n$s are eigenmodes of the oscillation of the time-dependent background. This could also have interesting applications in cosmology and opto-mechanical systems.

2. Isotropic oscillator on a time-dependent background

In this section, we obtain the Hamiltonian of a two-dimensional isotropic harmonic oscillator confined to a spherical surface of non-constant (time-dependent) radius. For this purpose, we use the gnomonic projection, which is the projection onto the tangent plane from the centre of the sphere in the embedding space. The advantage of this projection over all others for the analysis of the motion of a particle on a sphere stems from the fact that the free particle motion (uniform motion on a great circle) projects onto rectilinear, but non-uniform motion on the tangent plane. That is, the projected free particle orbits are the same as in the Euclidean geometry: the curvature affects only the speed of the projected motion. This feature persists in the presence of a central force derived from a potential $V(r)$ [1]. We first obtain the metric of the time-dependent background in the gnomonic coordinates and then calculate the Hamiltonian of the harmonic oscillator.

Let us designate the Cartesian coordinates of the time-dependent spherical background by $(q_0, q_1, q_2)$, and assume that they satisfy the sphere equation all the time:

$$ q_0^2(t) + q_1^2(t) + q_2^2(t) = R^2(t), \quad (1) $$

where $R(t)$ is the radius of the sphere.
If we denote the Cartesian coordinates of the tangent plane to the sphere by \( x \) and \( y \), the relationship between these two coordinates is given by
\[
q_1 = \frac{x}{\sqrt{1 + \lambda(x^2 + y^2)}},
\]
\[
q_2 = \frac{y}{\sqrt{1 + \lambda(x^2 + y^2)}},
\]
\[
q_0 = \pm \frac{1}{\sqrt{\lambda[1 + \lambda(x^2 + y^2)]}},
\]
where \( \lambda = \frac{1}{R(t)} \) is the time-dependent curvature of the sphere and \( + (--) \) in the last line means the upper (lower)-half of the sphere. Accordingly, a point on the sphere can be represented as
\[
\vec{r} = \left( \frac{x}{\Lambda}, \frac{y}{\Lambda}, \frac{1}{\sqrt{\lambda} \Lambda} \right),
\]
where
\[
\Lambda = \sqrt{1 + \lambda(x^2 + y^2)}.
\]
Now the differential of \( \vec{r} \) is given by
\[
d\vec{r} = \vec{r}_x \, dx + \vec{r}_y \, dy + \vec{r}_t \, dt,
\]
where \( \vec{r}_x, \vec{r}_y \) and \( \vec{r}_t \) are partial derivatives of \( \vec{r} \) with respect to \( x, y \) and \( t \), respectively. Thus, after a straightforward calculation, we obtain the metric of the sphere as
\[
d\bar{s}^2 = d\vec{r} \cdot d\vec{r} = \frac{\vec{x} \cdot d\vec{y}}{r^2 \Lambda^2} + \frac{1}{\Lambda} \left[ \vec{x} \cdot d\vec{x} - \frac{1}{r^2}(\vec{x} \cdot d\vec{x}) \right] + 2(\vec{r}_t \cdot \vec{r}_\tau) \, dt \, dx + 2(\vec{r}_t \cdot \vec{r}_\tau) \, dt \, dy + 2(\vec{r}_t \cdot \vec{r}_\tau) \, dt^2,
\]
where
\[
\vec{x} = (x, y), \quad r^2 = x^2 + y^2.
\]
Now, we can obtain \( \dot{s} \) from equation (6) as follows:
\[
\dot{s}^2 = \frac{\vec{x} \cdot \vec{x}}{r^2 \Lambda^2} + \frac{1}{\Lambda} \left[ \vec{x} \cdot \vec{x} - \frac{1}{r^2}(\vec{x} \cdot \vec{x}) \right] + 2(\vec{r}_t \cdot \vec{r}_\tau) \dot{x} + 2(\vec{r}_t \cdot \vec{r}_\tau) \dot{y} + 2(\vec{r}_t \cdot \vec{r}_\tau). \tag{7}
\]
In [1], the authors showed that the Lagrangian of a free particle on a sphere is affected by the curvature of the sphere. By studying the dynamical symmetries of the sphere, they also found that the form of the central potentials (such as isotropic harmonic oscillator and Coulomb potentials) is not affected by the curvature of the sphere if we use gnomonic coordinates. We can, therefore, write the Lagrangian (using its standard definition) of a two-dimensional isotropic oscillator on a time-dependent spherical background by using equation (8) as
\[
\mathcal{L} = \left( \frac{1}{2} \dot{s}^2 - V(x, y) \right) = \frac{1}{2} \left( \frac{\vec{x} \cdot \vec{x}}{r^2 \Lambda^2} + \frac{1}{\Lambda} \left[ \vec{x} \cdot \vec{x} - \frac{1}{r^2}(\vec{x} \cdot \vec{x}) \right] + (\vec{r}_t \cdot \vec{r}_\tau) \dot{x} + (\vec{r}_t \cdot \vec{r}_\tau) \dot{y} + (\vec{r}_t \cdot \vec{r}_\tau) \right) - V(x, y), \tag{9}
\]
where \( V(x, y) \) is the potential of the two-dimensional isotropic oscillator on the tangent plane coordinates (we assume \( m = \omega = 1 \)),
\[
V(x, y) = \frac{1}{2}(x^2 + y^2). \tag{10}
\]
By using the Lagrangian (9), we can calculate the momentum vector as

$$\vec{p} = \frac{\partial L}{\partial \dot{x}} \hat{i} + \frac{\partial L}{\partial \dot{y}} \hat{j}$$

$$= \frac{x(\dot{x} \cdot \ddot{x})}{r^2 \Lambda^2} + \frac{1}{\Lambda} \left[ \ddot{x} - \frac{1}{r^2} (\ddot{x} \cdot \dot{x}) \right] + \vec{A}(t),$$

(11)

where

$$\vec{A}(t) = (\vec{r}_i \cdot \vec{r}_j) \hat{i} + (\vec{r}_i \cdot \vec{r}_j) \hat{j}.$$  

(12)

With proper calculation, we get the Hamiltonian of the two-dimensional isotropic oscillator constrained to a time-dependent spherical background,

$$H = \dot{x} \cdot \vec{p} - L = H_0(x, \vec{p} - \vec{A}(t); \lambda) + \phi(t),$$

(13)

where \(\phi(t) = -(\vec{r}_i \cdot \vec{r}_i)\) and \(H_0(x, \vec{p}; \lambda)\) is the usual Hamiltonian of a two-dimensional isotropic oscillator on a sphere obtained by Higgs [1],

$$H_0(x, \vec{p}; \lambda) = \frac{1}{2} (x^2 + \lambda \vec{L}^2) + \frac{1}{2} (\vec{x}^2 + \vec{y}^2),$$

(14)

where

$$\vec{\pi} = \vec{p} + \lambda \vec{x}(\vec{p} \cdot \vec{x})$$

(15)

and

$$\vec{L} = \vec{x} \times \vec{p}.$$  

(16)

It is obvious from equation (13) that the effect of a time-dependent spherical background appears in the related Hamiltonian as a minimally coupled field. Thus, we may interpret \(\vec{A}(t)\) and \(\phi(t)\) in a similar way as the vector and scalar potentials of an electromagnetic field if we choose \(e = c = 1\).

To consider the quantum harmonic oscillator on a time-dependent background, we quantize the Hamiltonian (13) by replacing classical position and momentum with related operators

$$\hat{H} = \hat{H}_0(\vec{x}, \vec{\pi} - \vec{\pi}(t); \lambda) - (\vec{r}_i \cdot \vec{r}_i),$$

(17)

where

$$\hat{H}_0(\vec{x}, \vec{\pi}; \lambda) = \frac{1}{2} (\hat{\vec{x}}^2 + \lambda \hat{\vec{L}}^2) + \frac{1}{2} (\hat{x}^2 + \hat{y}^2),$$

(18)

\(\hat{\vec{x}}\) and \(\hat{\vec{L}}\) are symmetric forms of (15) and (16) given from the following two equations:

$$\hat{\vec{x}} = \vec{p} + \frac{\lambda}{2} \left[ \vec{x}(\vec{p} \cdot \vec{x}) + (\vec{p} \cdot \vec{x}) \right]$$

(19)

and

$$\hat{\vec{L}}^2 = \frac{1}{2} \hat{L}_{ij} \hat{L}_{ij}, \quad \hat{L}_{ij} = \hat{x}_i \hat{p}_j - \hat{x}_j \hat{p}_i.$$ (20)

3. Harmonic oscillator on a fluctuating background

In this section, we consider a two-dimensional isotropic harmonic oscillator on a fluctuating spherical background. The time-dependent radius of the sphere is defined by

$$R(t) = R_0 + \sum_n \alpha_n \sin(\omega_n t),$$

(21)
where $R_0$ and $\alpha_n$ are real constants and $\omega_n$ are frequencies of different modes of the fluctuations. We also assume that the fluctuation amplitudes $\alpha_n$ are very small compared to $R_0$ such that $\forall n: \frac{\alpha_n}{R_0} \ll 1$. Therefore, we can expand the curvature of the sphere as

$$\lambda \triangleq \frac{1}{R^2} = \lambda_0 \left[1 - 2(\lambda_0)^2 \sum_n \alpha_n \sin(\omega_n t)\right] + O(\lambda_0^3 \alpha_n^2),$$

(22)

where $\lambda_0 = \frac{1}{R_0}$ and $O(\lambda_0^3 \alpha_n^2)$ means the terms of order equal to or greater than $\lambda_0^3 \alpha_n^2$, which are negligible for this system. According to this limiting case, we also obtain

$$\phi(t) \equiv 0,$$

$$\tilde{A}(t, \tilde{\alpha}) \equiv f(t) \tilde{m}(\tilde{\alpha}),$$

(23)

where

$$f(t) = - (\lambda_0)^2 \sum_n \alpha_n \omega_n \cos(\omega_n t)$$

(24)

and

$$\tilde{m}(\tilde{\alpha}) = \frac{\tilde{\alpha}}{1 + \lambda_0 (\tilde{\alpha})^2}.$$

(25)

Thus, from equation (17), the Hamiltonian of the two-dimensional harmonic oscillator on a fluctuating spherical background can be written as

$$\hat{H} \equiv \hat{H}_0(\tilde{x}, \tilde{p}; \lambda_0) + \hat{V}(t, \tilde{x}, \tilde{p}; \alpha; \lambda_0),$$

(26)

where the time-independent Higgs Hamiltonian, $\hat{H}_0(\tilde{x}, \tilde{p}; \lambda_0)$, is given by equation (18) and the time-dependent part $\hat{V}(t, \tilde{x}, \tilde{p}; \alpha; \lambda_0)$ is defined as

$$\hat{V}(t, \tilde{x}, \tilde{p}; \alpha; \lambda_0) = V_0(t; \alpha)\hat{V}_1(\tilde{x}, \tilde{p}; \lambda_0) + \tilde{V}_0(t; \alpha)\tilde{\hat{V}}_1(\tilde{x}, \tilde{p}; \lambda_0).$$

(27)

where

$$V_0(t; \alpha) = (\lambda_0)^2 \sum_n \alpha_n \omega_n \cos(\omega_n t),$$

(28)

$$\tilde{V}_0(t; \alpha) = (\lambda_0)^2 \sum_n \alpha_n \sin(\omega_n t),$$

(29)

$$\hat{V}_1(\tilde{x}, \tilde{p}; \lambda_0) = \tilde{\hat{p}}\tilde{\hat{m}} + \tilde{\hat{m}}\tilde{\hat{p}}$$

$$+ \frac{\lambda_0}{2} \left\{ \tilde{\hat{p}}[\tilde{\hat{m}}(\tilde{x}\tilde{\hat{m}}) + (\tilde{\hat{m}}\tilde{\hat{x}})\tilde{\hat{p}}] + \tilde{\hat{m}}[\tilde{x}(\tilde{\hat{p}}\tilde{\hat{m}}) + (\tilde{\hat{p}}\tilde{\hat{x}})\tilde{\hat{m}}] \right\}$$

$$+ \lambda_0 [\tilde{x}(\tilde{\hat{p}}\tilde{\hat{x}} - \tilde{\hat{m}}\tilde{\hat{x}})\tilde{\hat{p}} - \tilde{\hat{m}}(\tilde{\hat{p}}\tilde{\hat{x}} - \tilde{\hat{m}}\tilde{\hat{x}})]$$

$$+ \lambda_0 \left\{ \tilde{x}[\tilde{\hat{p}}\tilde{\hat{m}} + (\tilde{\hat{m}}\tilde{\hat{x}})\tilde{\hat{p}}] + (\tilde{\hat{m}}\tilde{\hat{x}})\tilde{\hat{p}} \right\}$$

(30)

and

$$\tilde{\hat{V}}_1(\tilde{x}, \tilde{p}; \lambda_0) = - \left\{ L^2 + \frac{1}{2} \tilde{x}^2 + \tilde{x}\tilde{\hat{p}} + \tilde{\hat{m}}\tilde{\hat{x}} + \frac{1}{2} \tilde{\hat{p}}^2 - \frac{\lambda_0}{2} \left( \tilde{x}\tilde{\hat{p}} + \tilde{\hat{m}}\tilde{\hat{x}} \right) \right\}.$$

(31)
4. Transition golden rule

Let us now investigate briefly the question of transition probabilities per unit time between two arbitrary states. Consider the Hamiltonian \( \hat{H} \), which is defined by equation (26) and has two parts \( \hat{H}_0 \) and \( \hat{V} \). The eigenvalues and eigenstates of \( \hat{H}_0 \) are also given in [1]. We calculate the transition probability at time \( t \) from an initial eigenstate \( |i⟩ \) of \( \hat{H}_0 \) to a final state \( |j⟩ \) by treating \( \hat{V} \) as a perturbation,

\[
P_{i \to j} (t) = \left| \frac{-i}{\hbar} \int_0^t e^{i\omega_{ji}t} V_{ji} (t') dt' \right|^2, \tag{32}\]

where \( \omega_{ji} = \frac{E_j - E_i}{\hbar} \) and

\[
V_{ji} (t) = \langle j | \hat{V} | i⟩.
\tag{33}\]

Using (27), the transition probability takes the following form:

\[
P_{i \to j} (t) = \frac{1}{\hbar^2} \left| V_{ji} \int_0^t e^{i\omega_{ji}t} \hat{V}_0 (t'; \alpha) dt' + \tilde{V}_{ji} \int_0^t e^{i\omega_{ji}t} \tilde{V}_0 (t'; \alpha) dt' \right|^2,
\tag{34}\]

where

\[
V_{ji} = \langle j | \hat{V} | \tilde{\chi}, \tilde{p}, \tilde{\lambda}_0⟩ | i⟩,
\tag{35}\]

and

\[
\tilde{V}_{ji} = \langle j | \hat{\tilde{V}} | \tilde{\chi}, \tilde{p}, \tilde{\lambda}_0⟩ | i⟩.
\tag{36}\]

Using equation (28), we obtain

\[
\int_0^t e^{i\omega_{ji}t} \hat{V}_0 (t'; \alpha) dt' = (\lambda_0)^2 \sum_n \alpha_n \alpha_n \int_0^t e^{i\omega_{ji}t} \cos(\omega_{jt}) dt' \\
= -\frac{i}{2} (\lambda_0)^2 \sum_n \alpha_n \alpha_n \left[ e^{i(\omega_{ji} + \omega_n)t} - 1 \right] \left[ e^{i(\omega_{ji} - \omega_n)t} - 1 \right] \\
= \frac{1}{2} (\lambda_0)^2 \sum_n \alpha_n \left[ A_n^+ (t) + A_n^- (t) \right],
\tag{37}\]

where

\[
A_n^\pm (t) = \alpha_n e^{i[(\omega_{ji} \pm \omega_n)t/2]} \sin[(\omega_{ji} \pm \omega_n)t/2] / [(\omega_{ji} \pm \omega_n)/2]. \tag{38}\]

We can also obtain by using equation (29),

\[
\int_0^t e^{i\omega_{ji}t} \tilde{V}_0 (t'; \alpha) dt' = (\lambda_0)^2 \sum_n \alpha_n \int_0^t e^{i\omega_{ji}t} \sin(\omega_{jt}) dt' \\
= \frac{1}{2} (\lambda_0)^2 \sum_n \left[ A_n^+ (t) - A_n^- (t) \right].
\tag{39}\]

Now, to obtain the transition probability, we must calculate the following term:

\[
\left| \sum_n A_n^+ (t) [\alpha_n \tilde{V}_{ji} - i\tilde{\tilde{V}}_{ji}] + \sum_n A_n^- (t) [\alpha_n \tilde{V}_{ji} + i\tilde{\tilde{V}}_{ji}] \right|^2,
\tag{40}\]

which contains the following three terms:

\[
\left| \sum_n A_n^+ (t) [\alpha_n \tilde{V}_{ji} - i\tilde{\tilde{V}}_{ji}] \right|^2,
\tag{41}\]
\[ \sum_n A_n^\pm(t) \left| \omega_n V_{ji} \mp i \tilde{V}_{ji} \right|^2 \]  \hspace{1cm} (42)

and

\[ \sum_{n,m} A_n^\pm(t) A_m^\pm(t)^* \left[ \omega_n V_{ji} - i \tilde{V}_{ji} \right] \left[ \omega_m V_{ji} - i \tilde{V}_{ji} \right] + \text{C.C.} \]  \hspace{1cm} (43)

We can write equations (41) and (42) as the sum of

\[ \sum_n \left| A_n^\pm(t) \right|^2 \left| \omega_n V_{ji} \mp i \tilde{V}_{ji} \right|^2 \]  \hspace{1cm} (44)

and

\[ \sum_{n,m} A_n^\pm(t) A_m^\pm(t)^* \left[ \omega_n V_{ji} \mp i \tilde{V}_{ji} \right] \left[ \omega_m V_{ji} \mp i \tilde{V}_{ji} \right] + \text{C.C.} \]  \hspace{1cm} (45)

It is clear from equation (40) that if \( t \) is large enough, then the probability of finding the system in the state \( |j\rangle \) will only be appreciable if the denominator of one of the terms is close to zero. Moreover, both denominators cannot simultaneously be close to zero. According to similar arguments in the standard quantum mechanics books [7], a good approximation is therefore to neglect the interference terms (43) and (45) in calculating the transition probability. Thus, if \( \omega_{ji} \simeq \pm \omega_n \), only \( n \)th term in (40) will have an appreciable magnitude and we can write (40) as

\[ \left| A_n^\pm(t) \right|^2 \left| \omega_n V_{ji} \mp i \tilde{V}_{ji} \right|^2 \]  \hspace{1cm} (46)

which for large values of \( t \) tends to take the following form:

\[ 2\pi \alpha_n^2 \left| \omega_n V_{ji} \mp i \tilde{V}_{ji} \right|^2 \delta(\omega_{ji} \pm \omega_n) = 2\pi \hbar \alpha_n^2 \left| \omega_n V_{ji} \mp i \tilde{V}_{ji} \right|^2 \delta(E_j - E_i \pm \hbar \omega_n). \]  \hspace{1cm} (47)

The transition probability per unit time will, therefore, be given by

\[ \Gamma_{i \rightarrow j \neq i}(t) = \frac{2\pi}{\hbar} \frac{\lambda_0}{4} \sum_n \alpha_n^2 \left| \omega_n V_{ji} - i \tilde{V}_{ji} \right|^2 \delta(E_j - E_i \pm \hbar \omega_n) + \left| \omega_n V_{ji} + i \tilde{V}_{ji} \right|^2 \delta(E_j - E_i - \hbar \omega_n), \]  \hspace{1cm} (48)

which is the desired golden rule.

To summarize, we obtain the appreciable transition probability only if \( E_j \simeq E_i - \hbar \omega_n \) (stimulated emission) or \( E_j \simeq E_i + \hbar \omega_n \) (absorption). The isotropic oscillator on the sphere may, therefore, be exited from lower states to upper ones by using suitable frequencies of the fluctuating background. If the oscillator’s states are taken to represent a multi-level atomic system, the multi-mode radius fluctuations can be interpreted as a classical multi-mode radiation which interacts with the atomic system [8, 9]. Our results are expected to be equally valid for other quantum systems on more general time-dependent backgrounds.

5. Summary and concluding remarks

In this paper, a two-dimensional isotropic harmonic oscillator was investigated on a sphere with a time-dependent radius. It was shown that variations in the sphere radius could be represented by a minimally coupled Hamiltonian. As a realization of the model, a two-dimensional isotropic harmonic oscillator was considered on a fluctuating background. A simple golden rule was obtained for the transition probabilities per unit time between energy levels. This method can be generalized to other time-dependent maximally symmetric surfaces, such as hyperbolic or...
de-Sitter spaces. Generalization to an arbitrary surface by using the approach of [10] is also interesting. The authors believe that new experimental methods can be used to study quantum systems on time-dependent backgrounds with important results. For example, in [11], the authors experimentally studied the impact of intrinsic and extrinsic curvature of space on the evolution of light. It should be possible to investigate evaluation of light by experimental methods on a time-dependent background in the near future.

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