Nonminimal particle-like solutions in cubic scalar field theory

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Abstract

The cubic scalar field theory admits the bell-shaped solitary wave solutions which can be interpreted as massive Bose particles. We rule out the nonminimal p-brane action for such a solution as the point particle with curvature. When quantizing it as the theory with higher derivatives, it is shown that the corresponding quantum equation has SU(2) dynamical symmetry group realizing the exact spin-coordinate correspondence. Finally, we calculate the quantum corrections to the mass of the bell boson which can not be obtained by means of the perturbation theory starting from the vacuum sector.

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1 Introduction

The kink solutions are known to be the solitary wave solutions appearing in the relativistic $\varphi^4$ or sine-Gordon models and admitting the interpretation in terms of a particle. Their classical and quantum properties are already studied in a lot of works [1, 2, 3]. The aim of this paper is to study another, bell-shaped, solitary wave solution which has to be the only particle-like solution in the $\varphi^3$ theory. The approach developed in [2] consists in the constructing of the p-brane action where the nonminimal terms depending on the world-volume curvature are induced by the field fluctuations in the neighborhood of the static solution. When requiring these fluctuations to be damping at infinity, the effective action evidently arises after nonlinear reparametrization of the initial theory and excluding of zero field oscillations.

The paper is arranged as follows. In Sec. 2 we obtain $\varphi^3$-bell solution and study its properties on the classical level. In Sec. 3 we perform the nonlinear parametrization of the $\varphi^3$ action by means of the Bogolyubov transition to the collective degrees
of freedom. After this, minimizing the action with respect to field fluctuations, we
remove zero modes and obtain the effective action containing the curvature terms.
Sec. 4 is devoted to quantization of this action as the constrained theory with higher
derivatives. In result we obtain the Schrödinger wave equation describing wave
functions and mass spectrum of the quantum bell boson. Then we calculate the zeroth
and first excited levels to rule out the bell particle mass with quantum corrections.
Conclusions are made in Sec. 5.

2 Particle-like solution

Let us consider the (1+1)-dimensional action

\[ S[\phi] = \int L(\varphi) \, d^2 x, \tag{1} \]

\[ L(\varphi) = \frac{1}{2}(\partial^m \varphi)(\partial^m \varphi) - \lambda \varphi \left( \varphi - \frac{m^2}{\lambda} \right)^2 + \frac{2}{\lambda^2} \left( \frac{m^2}{3} \right)^3 \left( 1 - \text{sgn} \left( \frac{m^2}{\lambda} \right) \right), \tag{2} \]

where \( \varphi(x, t) \) is the dimensionless scalar field, \( \lambda \in \mathbb{R} \). The last term in eq. (2) is
introduced in such a way that the potential energy

\[ U(\varphi) = \lambda \varphi \left( \varphi - \frac{m^2}{\lambda} \right)^2 - \frac{2}{\lambda^2} \left( \frac{m^2}{3} \right)^3 \left( 1 - \text{sgn} \left( \frac{m^2}{\lambda} \right) \right) \tag{3} \]

would approach zero in the appropriate local minimum point. For definiteness below
we will assume \( m \in \mathbb{R} \) hence

\[ L(\varphi) = \frac{1}{2}(\partial^m \varphi)(\partial^m \varphi) - \lambda \varphi \left( \varphi - \frac{m^2}{\lambda} \right)^2, \tag{4} \]

and the system has a single local minimum point. Therefore, the state \( \varphi = \frac{m^2}{\lambda} \)
has to be (locally) the most energetically favorable, and stability of it grows as the
barrier’s height,

\[ \Delta U = \frac{1}{3} \left( \frac{2m^3}{\lambda^3} \right)^2, \]

increases. We will find solutions of the corresponding equation of motion,

\[ \partial^m \partial^m \varphi + \lambda (\varphi - \frac{m^2}{\lambda})(3\varphi - \frac{m^2}{\lambda}) = 0, \tag{5} \]

in the class of solitary waves

\[ \varphi(\rho) = \varphi \left( \frac{x - vt}{\sqrt{1 - v^2}} \right), \tag{6} \]

hence

\[ \varphi_{\rho\rho} - \lambda (\varphi - \frac{m^2}{\lambda})(3\varphi - \frac{m^2}{\lambda}) = 0. \tag{7} \]

The general integral of this equation can be expressed in terms of the elliptic functions

\[ \frac{\varphi - \frac{m^2}{\lambda} - \alpha_3}{\alpha_2 - \alpha_3} = \text{sn}^2 \left( \sqrt{-\frac{\lambda}{2k}} \rho, k \right), \tag{8} \]
where $\rho_0$ is supposed zero,

$$k = \frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3},$$

$\alpha_i$ are roots of the cubic equation

$$\alpha^3 + \frac{m^2}{\lambda} \alpha^2 + C = 0,$$  \hspace{1cm} (9)

and $C$ is an integration constant.

Among the solutions (8) we are needed in the regular solitary waves with the localized energy density

$$\varepsilon(x, t) = \frac{\partial L}{\partial (\partial^0 \varphi)} \partial^0 \varphi - L.$$  \hspace{1cm} (10)

It can be checked immediately that the only such a solution is the bell-shaped one:

$$\varphi_b(\rho) = \frac{m^2}{\lambda} \tanh^2 \left( \frac{m\rho}{\sqrt{2}} \right),$$  \hspace{1cm} (11)

having the energy density

$$\varepsilon_b(x, t) = \frac{2(m^3/\lambda)^2}{1 - v^2} \operatorname{sch}^4 \left( \frac{m\rho}{\sqrt{2}} \right) \left[ 1 - \operatorname{sch}^2 \left( \frac{m\rho}{\sqrt{2}} \right) \right].$$  \hspace{1cm} (12)

Therefore, it can be interpreted as the relativistic particle with the energy

$$E_{\text{class}} = \int_{-\infty}^{+\infty} \varepsilon_b(x, t) \, dx = \frac{\mu}{\sqrt{1 - v^2}},$$  \hspace{1cm} (13)

where the classical mass of the bell-particle is

$$\mu = \frac{8\sqrt{2} m^5}{15 \lambda^2}.$$  \hspace{1cm} (14)

One should done the following remarks upon the features of the solution (11). At first, it is topologically trivial unlike, e.g., the kink solutions in the $\varphi^4$ or sin-Gordon theories. It could give rise to certain problems with the stability against small perturbations but following the theorem proven below the corresponding effective p-brane action for the bell boson appears to be stable. Second, unlike the $\varphi^4$ or sin-Gordon cases the function $-\varphi_b$ has not to be the solution of (7). It means that the particle-antiparticle interpretation in the $\varphi^3$ theory differs from that in the $\varphi^4$ one and should be modified. One can show that the model (2) at $m^2 < 0$ also admits the single particle-like solution

$$\bar{\varphi}_b(\rho) = \frac{m^2}{\lambda} \left[ \operatorname{sch}^2 \left( \frac{|m|\rho}{\sqrt{2}} \right) + \frac{1}{3} \right],$$  \hspace{1cm} (15)

having the same energy (12) - (14). From the figure 1 one can see that the bell and anti-bell solitons occupy the different level lines shifted with respect to each other. It evidently has to be the consequence of the fact that the symmetry $\varphi \rightarrow -\varphi$ in the cubic SFT is broken initially.


3 Effective action

In this section we will construct the nonlinear effective action of the $\varphi^3$ theory (4), (4) about the static bell solution (11). Let us introduce the set of the collective coordinates $\{\sigma_0 = s, \sigma_1 = \rho\}$ such that

$$x^m = x^m(s) + e^m_{(1)}(s)\rho, \quad \varphi(x, t) = \tilde{\varphi}(\sigma),$$

(16)

where $x^m(s)$ turn to be the coordinates of a (1+1)-dimensional point particle, $e^m_{(1)}(s)$ is the unit spacelike vector orthogonal to the world line, hence the components of the Frenet basis are

$$e^m_{(0)} = \frac{\dot{x}^m}{\sqrt{\dot{x}^2}}, \quad e^m_{(1)} = -\frac{1}{\sqrt{\dot{x}^2}}\frac{\dot{e}^m_{(0)}}{k},$$

where the dot means the derivative with respect to $s$. Then the action (4) can be rewritten in the new coordinates as

$$S[\tilde{\varphi}] = \int L(\tilde{\varphi}) \Delta \, d^2\sigma,$$

(17)

where

$$\Delta = \det \left| \frac{\partial x^m}{\partial \sigma^k} \right| = \sqrt{\dot{x}^2}(1 - \rho k),$$

$$L(\tilde{\varphi}) = \frac{1}{2} \left[ \frac{(\partial_s \tilde{\varphi})^2}{\Delta^2} - (\partial_\rho \tilde{\varphi})^2 \right] - \lambda \varphi^2 \left( \varphi - \frac{m^2}{\lambda} \right)^2,$$

and $k$ is the curvature of a particle world line

$$k = \frac{\varepsilon_{mn} \dot{x}^m \ddot{x}^n}{(\sqrt{\dot{x}^2})^3},$$

(18)
where \( \varepsilon_{mn} \) is the unit antisymmetric tensor. This new action contains the redundant degree of freedom which eventually leads to appearance of the so-called “zero modes”. To eliminate this nonphysical degree of freedom we must constrain the model by means of the conditions of both the vanishing of the functional derivative with respect to field fluctuations about some chosen static solutions and damping of the fluctuations at infinity, and in result we will obtain the required effective action.

So, the fluctuations of the field \( \tilde{\varphi}(\sigma) \) in the neighborhood of some static solution \( \varphi_{st}(\rho) \) are given by the expression

\[
\tilde{\varphi}(\sigma) = \varphi_{st}(\rho) + \delta \tilde{\varphi}(\sigma).
\]

Substituting it into the action (17) and considering the static equation of motion (4) for \( \varphi_{st} \) we have

\[
S[\delta \tilde{\varphi}] = \int d^2\sigma \left\{ \Delta \left[ L(\varphi_{st}) + \frac{(\partial_s \delta \tilde{\varphi})^2}{2\Delta^2} - \frac{1}{2} (\partial_\rho \delta \tilde{\varphi})^2 - \lambda \left( 3\varphi_{st} - 2\frac{m^2}{\lambda} \right) \delta \tilde{\varphi}^2 \right] + \varphi_{st}' \delta \tilde{\varphi} \partial_\rho \Delta + O(\delta \tilde{\varphi}^3) \right\},
\]

\[
L(\varphi_{st}) = -\frac{1}{2} (\varphi_{st}')^2 - \lambda \varphi_{st} \left( \varphi_{st} - \frac{m^2}{\lambda} \right)^2,
\]

(20)

where the prime means the derivative with respect to \( \rho \). Extremalizing this action with respect to \( \delta \tilde{\varphi} \) one can obtain the equation in partial derivatives for field fluctuations:

\[
\left[ \partial_s \Delta^{-1} \partial_s - \partial_\rho \Delta \partial_\rho + 2\lambda \Delta \left( 3\varphi_{st} - 2\frac{m^2}{\lambda} \right) \right] \delta \tilde{\varphi} + \varphi_{st}' k \sqrt{\dot{x}^2} = O(\delta \tilde{\varphi}^2),
\]

which when substituting

\[
\varphi_{st} = \varphi_b, \rho = \epsilon u, \delta \tilde{\varphi} = \frac{m^2}{\lambda} \delta X,
\]

in the linear approximation has the form:

\[
\left[ \epsilon^2 \partial_s \Delta^{-1} \partial_s - \partial_\rho \Delta \partial_\rho + 4\Delta \left( 3X_0 - 2 \right) \right] \delta X + \epsilon X_0' k \sqrt{\dot{x}^2} = 0,
\]

(21)

where

\[
\epsilon = \frac{\sqrt{2}}{m}, \quad X_0 = \tanh^2 u, \quad \Delta = \sqrt{x^2(1 - \epsilon uk)}.
\]

To resolve this very complicated equation we suppose

\[
\delta X(s, u) = \epsilon k(s) f(u),
\]

(22)

and, expanding eq. 21 in the Taylor series with respect to \( \epsilon \) in such a way that

\[
\lim_{\epsilon \to 0} \epsilon^2 \frac{1}{\sqrt{\dot{x}^2}} \frac{d}{ds} \frac{1}{\sqrt{\dot{x}^2}} \frac{d}{ds} \neq 0,
\]

we have in the linear approximation the partitioned system of the two ordinary differential equations

\[
\epsilon^2 \frac{1}{\sqrt{\dot{x}^2}} \frac{d}{ds} \frac{1}{\sqrt{\dot{x}^2}} \frac{d}{ds} + ck = 0,
\]

(23)
\[-f_{uu} + 4 (3X_0 - 2) f - cf + X'_0 = 0. \]  

(24)

First of all, we are needed to find the solution of the last equation such that field fluctuations vanish at both infinities, i.e., we suppose the next boundary conditions:

\[f(+\infty) = f(-\infty) = 0,\]

(25)

evidently corresponding to the singular Stourm-Liouville problem which can be completely resolved in our case.

**Theorem.** The differential equation (24) has the two eigenfunctions and eigenvalues satisfying with the boundary conditions (25):

\[f_+ = \frac{C_+(\cosh^2 u - 5/4) + 2 \sinh u}{3 \cosh^3 u}, \quad c_+ = 3,\]
\[f_- = \frac{C_- - 2 \sinh u }{5 \cosh^3 u}, \quad c_- = -5,\]

where \(B_\pm\) are arbitrary integration constants.

**Proof.** Firstly we consider the case \(c = 0\). Then the general integral of eq. (24) is

\[f_{c=0} = \frac{\tanh u}{15} \left[ 3 + \cosh^2 u + \frac{C_1}{\cosh^2 u} \right] + C_2 \cosh^4 u \, \text F \left( 1, 3, \frac{9}{2}; \cosh^2 u \right), \]

(26)

where \(\text F (a, b, c; z)\) is the hypergeometric function, \(C_1\) are integration constants. As one can see this solution does not satisfy with (25). Therefore, below we will assume \(c \neq 0\).

Performing the variable change \(z = \cosh^2 u\), we can rewrite (24) in the form

\[2z(z-1)f_{zz} + (2z-1)f_z - 2 \left( 1 - \frac{c}{4} - \frac{3}{z} \right) f = \frac{\sqrt{z-1}}{z^{3/2}}, \]

(27)

hence after the shifting substitution

\[f = \bar{f} + \frac{2 \sqrt{z-1}}{c z^{3/2}} \]

we obtain the homogeneous equation

\[2z(z-1)\bar{f}_{zz} + (2z-1)\bar{f}_z - 2 \left( \bar{c}^2 - \frac{3}{z} \right) \bar{f} = 0, \]

(28)

where

\[c = 4(1 - \bar{c}^2), \quad |\bar{c}| \neq 1. \]

(29)

Then the bound state condition (25) should be rewritten as

\[\bar{f}(1) = 0, \quad \bar{f}(+\infty) = 0. \]

(30)

The general integral of eq. (29) can be expressed in the form:

\[\bar{f} = \frac{C_1}{z^{3/2}} \, \text F \left( -\frac{3}{2} - \bar{c}, -\frac{3}{2} + \bar{c}, -\frac{5}{2}; z \right) + C_2 z^2 \, \text F \left( 2 - \bar{c}, 2 + \bar{c}, \frac{9}{2}; z \right). \]

Expanding the hypergeometric functions in series in the neighborhood of \(z = 1\), it is straightforward to check that the first from the conditions (30) will be satisfied if we suppose

\[C_2 = \frac{8 \bar{c}}{1575} (\bar{c}^2 - 1)(4\bar{c}^2 - 1)(4\bar{c}^2 - 9) \tan(\pi \bar{c}) C_1 \equiv -C^{(\text{reg})} C_1, \]

where \(C^{(\text{reg})}\) is the regularized part of \(C_1\).
hence

\[ |\tilde{c}| \neq n + 1/2 \text{ except } |\tilde{c}| = 1/2, 3/2. \]

Therefore, the solution of eq. (29) vanishing at \( z = 1 \) is the function

\[
\frac{1}{C_1} \tilde{f}^{(\text{reg})} = z^{-3/2} F \left( -\frac{3}{2} - \tilde{c}, -\frac{3}{2} + \tilde{c}, -\frac{5}{2}; z \right) - C^{(\text{reg})} z^2 \Gamma \left( 2 - \tilde{c}, 2 + \tilde{c}, \frac{9}{2}; z \right). \tag{31}
\]

To specify the parameters at which \( f^{(\text{reg})} \) would satisfy with the second condition \( (30) \) we should consider the asymptotical behavior of \( \tilde{f}^{(\text{reg})} \) at large \( z \). We have

\[
\frac{1}{C_1} \tilde{f}^{(\text{reg})}(z \to \infty) = \frac{\sin^2(2\pi \tilde{c})}{4\pi^{7/2}} \left[ A(\tilde{c}, z) + A(-\tilde{c}, z) \right], \tag{32}
\]

where

\[
A(a, x) = \Gamma(2a) \left[ \frac{8i}{15} \Gamma(a + 2) \Gamma(a + 5/2) + \frac{315}{48} C^{(\text{reg})} \Gamma(a - 3/2) \Gamma(a - 1) \right] (-x)^a (1 + O(1/x)).
\]

From this expression and eqs. (29) and (31) it can easily be seen that \( \tilde{f}^{(\text{reg})} \) does not satisfy with the Stourm-Liouville conditions everywhere except the points:

\[ |\tilde{c}| = 1/2, 3/2, \]

or, following (29),

\[ c = 3, -5, \]

respectively. The corresponding eigenfunctions can be obtained directly from (31), Q.E.D.

By virtue of this theorem we have recently obtained all the necessary functions to construct the effective action for the \( \varphi^3 \) theory about the static bell solution. Taking into account eqs. (21) and (22) and the theorem, the action (20) can be rewritten in the explicit p-brane form

\[
S_{\text{eff}} = S_{\text{eff}}^{(\text{class})} + S_{\text{eff}}^{(\text{fluct})} = - \int ds \sqrt{\dot{x}^2} \left( \mu + \alpha_\pm k^2 \right), \tag{33}
\]

where

\[
\mu = - \int_{-\infty}^{+\infty} (1 - \rho k) L(\varphi_b) d\rho \left[ \epsilon_b(\rho) \right]_{\nu=0},
\]

see eq. (14), and

\[
\alpha_\pm = \frac{e}{2} \left( \frac{m^2}{\lambda} \right) \int_{-\infty}^{+\infty} f_\pm X_0' du = \frac{\mu}{c_\pm m^2}. \tag{34}
\]

Thus, one can see that the fluctuational corrections lead to appearance of two different \( \alpha \) (unlike the single \( \alpha \) in [2]), therefore, we have the bifurcation of the bell solution (11) as the particle with curvature. However, in the following section it will be shown that on the quantum level nonminimal term, leading to the quantum mass corrections, is required to be such that \( c > 0 \).
Finally, the action \((33)\) yields the equations of motion for the bell field solution as a p-brane:

\[
e^2 \frac{1}{\sqrt{x^2}} \frac{d}{ds} \frac{1}{\sqrt{x^2}} \frac{dk}{ds} + \left( c_{\pm} - \frac{e^2}{2} k^2 \right) k = 0. \tag{35}
\]

Considering the proven theorem, one can see that eq. \((23)\) was just the linearized version of this expression.

4 Quantization

In the previous section we obtained classical effective actions for the model in question. Thus, to quantize them we must consecutively construct the Hamiltonian structure of the point particle with curvature. From eqs. \((18)\) and \((33)\) one can see that we have the theory with higher derivatives \([5]\). Hence, below we will treat the coordinates and momenta as the canonically independent coordinates of phase space. The phase space consists of the two pairs of canonical variables:

\[
x_m, \quad p_m = \frac{\partial L}{\partial \dot{q}^m} - \dot{\Pi}_m, \tag{36}
\]

\[
q_m = \dot{x}_m, \quad \Pi_m = \frac{\partial L}{\partial \dot{q}^m}, \tag{37}
\]

hence we have

\[
p^n = -e^n(0) \mu \left[ 1 - \frac{1}{c_{\pm} m^2 k^2} \right] + \frac{2\mu}{c_{\pm} m^2} \frac{e^n(1)}{\sqrt{q^2}} \dot{k}, \tag{38}
\]

\[
\Pi^n = -\frac{2\mu}{c_{\pm} m^2} \frac{e^n(1)}{\sqrt{q^2}} k. \tag{39}
\]

Besides, the Hessian matrix constructed from the derivatives with respect to accelerations appears to be singular that points out the presence of the constraints on the phase variables of the theory. There exist the two primary constraints of first kind

\[
\Phi_1 = \Pi^n q_m \approx 0, \tag{40}
\]

\[
\Phi_2 = p^n q_m + \sqrt{q^2} \left[ \mu + \frac{c_{\pm} m^2}{4\mu} q^2 \Pi^2 \right] \approx 0, \tag{41}
\]

besides we should add the proper time gauge condition,

\[
G = \sqrt{q^2} - 1 \approx 0, \tag{42}
\]

to remove the non-physical gauge degree of freedom. Then, when introducing the new variables,

\[
\rho = \sqrt{q^2}, \quad v = \text{arctanh} \left( \frac{p(1)}{p(0)} \right), \tag{43}
\]

the constraints can be rewritten in the form

\[
\Phi_1 = \rho \Pi_\rho, \tag{44}
\]

\[
\Phi_2 = \rho \left[ -\sqrt{p^2} \cosh v + \mu \left( 1 - \frac{4\mu}{c_{\pm} m^2} \right) \right],
\]

\[
G = \rho - 1,
\]
hence finally we obtain the constraint

\[ \Phi_2 = -\sqrt{p^2 \cosh v + \mu - \frac{c_+ m^2}{4\mu}} \Pi_v^2 \approx 0, \]  

(45)

which in the quantum theory \((\Pi_v = -i\partial/\partial v)\) yields

\[ \hat{\Phi}_2|\Psi\rangle = 0. \]

As was shown by Kapustnikov et al, the constraint \(\Phi_2\) on the quantum level admits several coordinate representations that, generally speaking, lead to different nonequivalent theories, therefore, the choice between the different forms of \(\hat{\Phi}_2\) should be based on the physical relevance. Then the physically admissible equation determining quantum dynamics of the quantum bell particle has the form:

\[ [\hat{H} - \varepsilon]|\Psi(\zeta)\rangle = 0, \]

(46)

\[ \hat{H} = -\frac{d^2}{d\zeta^2} + \frac{B^2}{4} \sinh^2 \zeta - B \left( S + \frac{1}{2} \right) \cosh \zeta, \]

(47)

where \(S = 0\) in our case, and

\[ \zeta = v/2, \quad \sqrt{p^2} = M, \]

\[ B = \frac{8\sqrt{2}}{m} \sqrt{\frac{\mu M}{c_\pm}}, \]

(48)

\[ \varepsilon = \frac{16\mu^2}{c_\pm m^2} \left( 1 - \frac{M}{\mu} \right). \]

It can readily be seen that for the case \(c_-\) the ratio \(M/\mu\) should be negative that seems to be unphysical because we expect that in ground state \(M = \mu\). In this connection even the interpretation in terms of antiparticles does not save a situation because, as was mentioned above, the solution which could be interpreted as the anti-bell boson state lies in the rather different parameter space (unlike the \(\varphi^4\) theory where the p-branes with negative \(\alpha\) admit physical interpretation). Therefore, below we will consider the case \(c_+\).

As was established in the works \([6, 7]\), \(SU(2)\) has to be the dynamical symmetry group for this Hamiltonian which can be rewritten in the form of the spin Hamiltonian

\[ \hat{H} = -S_z^2 - BS_x, \]

(49)

where the spin operators,

\[ S_x = S \cosh \zeta - \frac{B}{2} \sinh^2 \zeta - \sinh \zeta \frac{d}{d\zeta}, \]

\[ S_y = i \left\{ -S \sinh \zeta + \frac{B}{2} \sinh \zeta \cosh \zeta + \cosh \zeta \frac{d}{d\zeta} \right\}, \]

(50)

\[ S_z = \frac{B}{2} \sinh \zeta + \frac{d}{d\zeta}, \]

satisfy with the commutation relations

\[ [S_i, S_j] = i\epsilon_{ijk} S_k, \]
besides

\[ S_x^2 + S_y^2 + S_z^2 \equiv S(S + 1). \]

At \( S \geq 0 \) there exists an irreducible \((2S + 1)\)-dimensional subspace of the representation space of the su(2) Lie algebra, which is invariant with respect to these operators. Determining eigenvalues and eigenvectors of the spin Hamiltonian in the matrix representation which is realized in this subspace, one can prove that the solution of (46) is the function

\[ \Psi(\zeta) = \exp \left( -\frac{B}{2} \cosh \zeta \right) \sum_{\sigma=-S}^{S} \frac{c_{\sigma}}{\sqrt{(S-\sigma)! (S+\sigma)!}} \exp(\sigma \zeta), \quad (51) \]

where the coefficients \( c_\sigma \) are the solutions of the system of linear equations

\[
\left( \epsilon + \sigma^2 \right) c_\sigma + \frac{B}{2} \left[ \sqrt{(S-\sigma)(S+\sigma+1)} \ c_{\sigma+1} + \sqrt{(S+\sigma)(S-\sigma+1)} \ c_{\sigma-1} \right] = 0,
\]

\[ c_{S+1} = c_{-S-1} = 0, \quad \sigma = -S, \ -S + 1, \ ..., \ S. \]

Regrettably, these expressions give only the finite number of exact solutions which is equal to the dimensionality of the invariant subspace. Therefore, for the spin \( S = 0 \) we can find only the ground state wave function and eigenvalue:

\[ \Psi_0(\zeta) = C_1 \exp \left( -\frac{B}{2} \cosh \zeta \right), \ \epsilon_0 = 0, \quad (52) \]

i.e., we have obtained the expected result that the mass of the quantum bell \( c_+ \)-boson in the ground state coincide with the classical one,

\[ M_0 = \mu. \quad (53) \]

In absence of exact wave functions for more excited levels let us find the first quantum correction to mass of the bell particle in the approximation of the quantum harmonic oscillator. It is easy to see that at \( B \geq 1 \) the (effective) potential

\[ V(\zeta) = \left( \frac{B}{2} \right)^2 \sinh^2 \zeta - \frac{B}{2} \cosh \zeta \quad (54) \]

has the single minimum

\[ V_{\text{min}} = -B/2 \quad \text{at} \quad \zeta_{\text{min}} = 0. \]

Then following to the \( \hbar \)-expansion technique we shift the origin of coordinates (to satisfy \( \epsilon = \epsilon_0 = 0 \) in absence of quantum oscillations) in the point of minimum, and expand \( V \) in the Taylor series to second order near the origin thus reducing the model to the oscillator of the unit mass, energy \( \epsilon/2 \) and oscillation frequency

\[ \omega = \frac{1}{2} \sqrt{B(B-1)}. \]

Therefore, the quantization rules yield the discrete spectrum

\[ \epsilon = \sqrt{B(B-1)(n+1/2)} + O(\hbar^2), \quad n = 0, \ 1, \ 2, \ ..., \quad (55) \]
and the first quantum correction to particle mass will be determined by the lower
energy of oscillations:
\[ \varepsilon = \frac{1}{2} \sqrt{B(B - 1)} + O(h^2), \]  
(56)
that gives the algebraic equation for \( M \) as a function of \( m \) and \( \lambda \).

We can easily resolve it in the approximation of weak coupling. Assuming \( \lambda \to 0 \)
(or, equivalently, \( B \gg 1 \)) we find eq. (56) in the form
\[ \varepsilon = \frac{B}{2} + O(\lambda h^2), \]  
(57)
which after considering of eqs. (58) and (53) yields
\[ (M - \mu)^2 = \frac{c_+ m^2 M}{8\mu} + O(\lambda h^2), \]

hence we obtain
\[ M = \mu \pm \frac{1}{2} \sqrt{\frac{c_+}{2} m} + O(\lambda h^2). \]  
(58)
Thus, the mass of the bell boson with quantum corrections at first order of \( \hbar \) is
\[ M = \frac{8\sqrt{2} m^5}{15 \lambda^2} \pm \frac{1}{2} \sqrt{\frac{3}{2} m}, \]  
(59)
The quantum term in this expression has to be in good agreement with that obtained
in [1] and numerically justified in [2] for the kink solution. We just point out that the
main (classical) term turns to be singular at \( \lambda \to 0 \), therefore, the obtained results are
nonperturbative and can not be ruled out from the vacuum sector of the \( \phi^3 \) theory
through the series of the perturbation theory.

5 Conclusion

It was shown that the cubic scalar field theory admits the localized bell-shaped soli-
tary wave interpreted as a massive quantum mechanical particle. This solution has
the counterpart of the same mass which can be imagined as the antiparticle. The
differences in the particle-antiparticle interpretation from that for the models con-
serving the parity symmetry were studied in this connection. Further, considering
field fluctuations in the neighborhood of the static solution, we ruled out the p-brane
action for the \( \phi^3 \)-bell solutions as the nonminimal point particle with curvature.

When quantizing this action as the constrained theory with higher derivatives, it
was shown that the resulting Schröedinger equation has the potential from the Razavi
class. This equation has SU(2) dynamical symmetry group in the ground state and
can be written by virtue of spin operators. Note, that though the reformulation
of some interaction concerning the coordinate degrees of freedom in terms of spin
variables is widely used (e.g., in the theories described by the Heisenberg Hamiltonian,
see [8]), it has to be merely the physical approximation as a rule, whereas in our
case the spin-coordinate correspondence is exact. Finally, we found the quantum
corrections to the mass of the bell bosons which could not be obtained by means of
the perturbation theory starting from the vacuum sector.
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