P-V Criticality of the FRW Universe in the Novel 4D Gauss-Bonnet Gravity

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In this paper, we study the thermodynamics especially the P-V criticality of the Friedmann-Robertson-Walker (FRW) universe in the novel 4-dimensional Gauss-Bonnet gravity, where we define the thermodynamic pressure \( P \) from the cosmological constant \( \Lambda \) as \( P = -\frac{\Lambda}{8\pi} \). We obtain the first law of thermodynamics and equation of state of the FRW universe. We find that, if the Gauss-Bonnet coupling constant \( \alpha \) is positive, there is no P-V phase transition. If \( \alpha \) is negative, there are P-V phase transitions and critical behaviors within \(-1/3 \leq \omega \leq 1/3\). Particularly, there are two critical points of the P-V criticality in the case \( \alpha < 0 \), \(-1/3 < \omega < 1/3\). We investigate these P-V criticality around the critical points, and calculate the critical exponents. We find that these critical exponents in the \(-1/3 < \omega \leq 1/3\) case are consistent with those in the mean field theory, and hence satisfy the scaling laws.

I. INTRODUCTION

Black holes are exact solutions of the Einstein field equation in general relativity, and have been confirmed by astronomical observations recently \([1, 2]\). Besides their existence, black holes have also shown some astonishing properties, particularly their thermodynamics. In the 1970s, the four laws of black hole thermodynamics \([3-5]\) were established by Hawking, Bekenstein, et al. In black hole thermodynamics, the Hawking temperature \([6]\) is proportional to the surface gravity near the horizon, and the entropy is proportional to the horizon area instead of the volume, which is one of the inspirations of holography. Black holes also have Hawking radiation or evaporation by considering quantum effects. For the dynamical black hole case, i.e. there is no global time-like killing vector in the spacetime, the event horizons may not exist, but the apparent horizons always exist. Therefore, thermodynamics of dynamical black holes are usually established at the apparent horizons instead of event horizons. Hayward found a unified first law \([7, 8]\) of thermodynamics of dynamical spacetime from the Einstein field equations, where the quasi-local Misner-Sharp energy \([9, 10]\) is used. Projecting the unified first law of thermodynamics on the apparent horizons, one can get the first law of thermodynamics of the dynamical black holes.

On the other hand, black holes are also found to have various kinds of phase transitions. The first black hole phase transition was discovered by Hawking and Page in asymptotic anti-de Sitter (AdS) black holes \([11]\), which corresponds to the quark confinement/deconfinement through AdS/CFT unveiled by Witten \([12]\). Later, many other phase transitions were found, one of them is the P-V phase transition \([13]\) in AdS black holes analogous to the gas/liquid phase transition in the van der Waals system. In asymptotic AdS spacetime, the thermodynamic pressure \( P \) comes from the negative cosmological constant as \( P = -\frac{\Lambda}{8\pi} \) inspired by the treatment in cosmology \([14]\). One can further get the equation of state or express \( P \) as a function of the temperature \( T \) and the thermodynamic volume \( V \): \( P = P(T, V) \), and discuss P-V criticality from it. The P-V phase transitions have been found in many black hole systems in Einstein gravity as well as some modified gravities such as Horndeski gravity \([15]\), and de Rham-Gabadadze-Tolley (dRGT) massive gravity \([16, 17]\), etc. Most of the P-V phase transitions have critical phenomena, while the critical exponents are consistent with those in the mean field theory and hence satisfy the scaling laws. However, in a recent paper, clues have been found that the scaling laws may be violated under certain conditions \([17]\).

The Friedmann-Robertson-Walker (FRW) universe is a dynamical spacetime characterized by a time-dependent scale factor \( a(t) \). Note that, the FRW universe is not asymptotically flat, and may not have an event horizon \([7]\). However, the apparent horizon or inner trapping horizon always exists in the FRW universe, which makes it very similar to a dynamical black hole \([18]\). Hence, many concepts and methods from (dynamical) black holes or trapping surfaces can be transplanted to the research of the FRW universe, such as the Hawking temperature, Bekenstein-Hawking entropy, Misner-Sharp energy, the first law of thermodynamics and the unified first law of thermodynamics, etc. One obtains

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the first law of thermodynamics on the apparent horizon of the FRW universe in the same manner as that for black holes [18]. What’s more, starting from the first law of thermodynamics and the Clausius relations, one can also derive the Friedmann’s equations in Einstein gravity [19] and even in some modified gravities [20], which indicates that there is an essential relationship between thermodynamics and gravity. In our recent paper [21], we further investigated the thermodynamics of the FRW universe in the Einstein gravity, where we got the equation of state $P = P(V, T)$ in the FRW universe. From the equation of state, we found that there is no $P-V$ phase transition in the FRW universe under Einstein gravity. However, as $P-V$ phase transitions of black holes have been found in many modified theories of gravity, we probably also present in the FRW universe under modified gravities.

The Gauss-Bonnet gravity is a famous modified gravity, which is a gravitational theory with high order curvature, and has a very close relationship with quantum gravity or string theory [22, 23]. One of its advantages is that there are no more than second-order derivative terms of metrics in the equations of motion as Einstein gravity. However, the well-known Gauss-Bonnet term in the action is a topological invariant (or Euler characteristic class), and does not contribute to the gravitational field equations in the 4-dimensional spacetime [24], thus not dynamical. To recover its dynamical effect, a new method was proposed by Galvin and Lin [25], which is often called the novel 4-dimensional (4D) Gauss-Bonnet gravity. Recently this novel theory has received some discussions about its regularization method or limiting procedure, and hence the action may be not well-defined for an arbitrary 4D metrics [26–32]. However, it is important to note that the action of 4D Gauss-Bonnet gravity is well-defined and determinant for the static spherical symmetric (SSS) metric fields or the FRW universe [33, 34], because it is always possible to extract the dimensional factor from the equation of state and leave a nonzero Gauss-Bonnet correction to general relativity. Therefore, the novel 4D Gauss-Bonnet gravity has been widely used to the study of black holes, such as the thermodynamics, phase transitions and microstructure of the 4D charged black hole in AdS space [35, 36], the Hawking evaporation of AdS black holes [37], etc. If applied to cosmology, the novel 4D Gauss-Bonnet gravity can help to explain inflation [38] and the current cosmic acceleration even alleviate the Hubble tension [39]. In this paper, we apply the novel 4D Gauss-Bonnet gravity to the FRW universe and study its thermodynamics especially $P-V$ criticality, and find that there are indeed $P-V$ phase transitions and critical phenomena in this framework.

Our paper is organized in the following way. In Sec.II, we study the thermodynamics of the FRW universe in the novel 4D Gauss-Bonnet gravity. In Sec.III, we investigate the $P-V$ criticality of the FRW universe in the novel 4D Gauss-Bonnet gravity. Sec.IV is for conclusion and discussion.

II. THERMODYNAMICS OF THE FRW UNIVERSE IN THE NOVEL 4D GAUSS-BONNET GRAVITY

In this section, we mainly study the thermodynamics of the FRW universe in the novel 4D Gauss-Bonnet gravity. After making a brief warm-up of the Freedman’s equations of the FRW universe obtained in the novel 4D Gauss-Bonnet gravity, we further investigate the first law of thermodynamics on the apparent horizon, and then derive the equation of state of the FRW universe in this case.

A. Warm-up: Freedman’s Equations of the FRW Universe in the Novel 4D Gauss-Bonnet Gravity

The action in the traditional Gauss-Bonnet gravity is written as [19]

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} (R - 2\Lambda + \alpha R_{GB}) + S_m, \quad (2.1)$$

where $R_{GB}$ is the Gauss-Bonnet term with high order of curvature

$$R_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta}, \quad (2.2)$$

which is a topological term, and usually does not has dynamical effect in the 4-dimensional spacetime case.

The action in the novel Gauss-Bonnet gravity is [25]

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} (R - 2\Lambda + \frac{\alpha}{D-4}R_{GB}) + S_m, \quad (2.3)$$

where the coupling constant $\alpha$ is replaced by $\frac{\alpha}{D-4}$, and it is actually the dimensional regularization method especially the so-called minisuperspace regularization [32]. The novel 4D Gauss-Bonnet gravity is defined as the $D \to 4$ limit, and in this way the Gauss-Bonnet term has dynamical effects on the equation of motion of the FRW universe as can be seen in the following Friedmann’s equations.
After variation of the action (2.3), one gets the field equation in the novel 4D Gauss-Bonnet gravity [19]

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} - \frac{\alpha}{D-4}H_{\mu\nu} = 8\pi T_{\mu\nu}, \]  

(2.4)

where

\[ H_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R_{GB} - 2R_{\mu\nu} + 4R_{\gamma\delta}R_{\nu\gamma} + 4R_{\delta\gamma}R_{\mu\gamma\nu} - 2R_{\mu\gamma\delta\lambda}R_{\nu}\gamma\delta\lambda. \]  

(2.5)

For the FRW universe, the line element is

\[ ds^2 = -dt^2 + a^2(t)\left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \]  

(2.6)

where \(\{t, r, \theta, \phi\}\) is the co-moving coordinate system of the universe. After considering the matter filled in the FRW universe as the perfect fluid with stress-tensor

\[ T_{\mu\nu} = (\rho_m + p_m)u_\mu u_\nu + p_m g_{\mu\nu}, \]  

(2.7)

where \(\rho_m, p_m\) are its energy density and pressure, and \(u_\mu\) is the dual 1-form of the 4-velocity \(u^\mu\), one obtains the corresponding Friedmann’s equations in the novel 4D Gauss-Bonnet Gravity [19, 39]

\[ H^2 + \frac{k}{a^2} + \tilde{\alpha}(H^2 + \frac{k}{a^2})^2 = \frac{1}{3}(8\pi \rho_m + \Lambda), \]  

\[ [1 + 2\tilde{\alpha}(H^2 + \frac{k}{a^2})](\dot{H} - \frac{k}{a^2}) = -4\pi(\rho_m + p_m). \]  

(2.8)

(2.9)

Here \(\tilde{\alpha} \equiv \frac{\alpha}{D-7}(D-3)(D-4) = (D-3)\alpha\), so the dimensional factor \(1/(D-4)\) cancels with \(D-4\). In the 4-dimensional FRW universe, \(\tilde{\alpha}\) is exactly the nonzero \(\alpha\), the Gauss-Bonnet corrections are neither zero nor divergent, so its dynamical effects on the evolution of the universe are nontrivial.

**B. The First Law of Thermodynamics of the FRW Universe in the Novel 4D Gauss-Bonnet Gravity**

For convenience to discuss the first law of thermodynamics of the FRW universe in the novel 4D Gauss-Bonnet gravity, we rewrite the FRW metric in (2.6) with another decomposition form

\[ ds^2 = h_{ab}dx^a dx^b + R^2(d\theta^2 + \sin^2\theta d\phi^2), \]  

(2.10)

where \(a, b = 0, 1\) with \(x^0 = t, x^1 = r\) and \(R \equiv a(t)r\) is the physical or areal radius of the universe. In this new form, the apparent horizon of the FRW universe is easily obtained from \(h^{ab}\partial_a R\partial_b R = 0\), while its expression is [19]

\[ R_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}}. \]  

(2.11)

From this expression, one easily derives the time derivative of the apparent horizon as

\[ \dot{R}_A = -HR_A^3(\dot{H} - \frac{k}{a^2}). \]  

(2.12)

In addition, the surface gravity on the apparent horizon is [19]

\[ \kappa = -\frac{1}{R_A}(1 - \frac{\dot{R}_A}{2HR_A}). \]  

(2.13)

We treat the apparent horizon of the universe as an inner trapping horizon like [40], and the surface gravity \(\kappa\) is negative which corresponds to \(\dot{R}_A < 2HR_A\) [8]. Therefore, the Hawking temperature at the apparent horizon of the FRW universe is

\[ T = \frac{|\kappa|}{2\pi} = \frac{1}{2\pi R_A}\left(1 - \frac{\dot{R}_A}{2HR_A}\right). \]  

(2.14)
Note that, the relationship between entropy and area of horizon depends only on gravitational theory, thus the entropy of the FRW universe can be assumed to take the same form as that of a charged black hole in the 4D Gauss-Bonnet gravity \[35, 36\]

$$S = \pi R_A^2 + 4\pi \ln\left( \frac{R_A}{\sqrt{|\alpha|}} \right) = \frac{A}{4} + 2\pi \alpha \ln\left( \frac{A}{A_0} \right), \quad (2.15)$$

where $A_0 \equiv 4\pi|\alpha|$. Note that, the entropy $S$ includes two terms, the usual Bekenstein-Hawking term $A/4$ and the additional logarithmic term $2\pi \alpha \ln(A/A_0)$, which is very similar to the quantum corrected entropy formula \[41\], and this may be a manifestation of the close relationship between Gauss-Bonnet gravity and quantum gravity.

The generalized Misner-Sharp energy inside the apparent horizon of the FRW universe with cosmological constant in the 4D Gauss-Bonnet gravity is \[9, 10\]

$$M \equiv R_A^2 + \frac{\alpha}{2R_A} - \frac{\Lambda}{6} R_A^3, \quad (2.16)$$

where $V = 4\pi R_A^3/3$. The work density of the matter field is defined as \[40\]

$$W \equiv -\frac{1}{2} g^{ab} T_{ab} = \frac{1}{2}(\rho_m - p_m). \quad (2.17)$$

The cosmological constant provides the thermodynamic pressure

$$P = -\frac{\Lambda}{8\pi}. \quad (2.18)$$

Therefore, after using these quantities, one can get the generalized Smarr relation as

$$M = -2TS + 3WV - 2PV + \left[ \frac{1}{R_A} + 8\pi T \ln\left( \frac{R_A}{R_0} \right) - 4\pi T \right] \alpha, \quad (2.19)$$

and the differential form of the Misner-Sharp energy or the Smarr relation is

$$dM = -TdS + WdV + VdP + Ad\alpha, \quad (2.20)$$

where

$$A \equiv 4\pi T \ln\left( \frac{R_A}{R_0} \right) + \frac{1}{2R_A}. \quad (2.21)$$

This differential form is just the first law of the FRW universe in the novel 4D Gauss-Bonnet gravity. From this form, one finds that the Misner-Sharp energy $M$ is better to be interpreted as the enthalpy $H \equiv M$ inside the apparent horizon, and $V$ is the thermodynamic volume.

C. The Equation of State of the FRW Universe in the Novel 4D Gauss-Bonnet Gravity

For a thermodynamics system, besides the laws of thermodynamics, equation of state like the van der Waals system usually also plays a very important role. From the first law of thermodynamics \(2.20\), one can see that $P$ and $V$ are conjugate variables which should satisfy an equation of state together with the temperature $T$. In order to clearly obtain the equation of state of the FRW universe in the novel 4D Gauss-Bonnet gravity, we first rewrite the Friedmann’s equations into two useful equations by using \(2.11\) and \(2.12\)

$$\rho_m + \frac{\Lambda}{8\pi} = \frac{3}{8\pi R_A^2} + \frac{3\alpha}{8\pi R_A^4}, \quad (2.22)$$

$$\rho_m + p_m = \left(1 + \frac{2\alpha}{R_A^2}\right) \frac{R_A}{4\pi H R_A^3}. \quad (2.23)$$

From the Hawking temperature \(2.14\) and one of the above Friedmann’s equation \(2.23\), one can get

$$\rho_m + p_m = \left(1 + \frac{2\alpha}{R_A^2}\right) \left(\frac{1}{2\pi R_A^2} - \frac{T}{R_A}\right). \quad (2.24)$$
We assume that the energy density and the pressure of the perfect fluid satisfy $p_m = \omega \rho_m$, where $\omega$ is a constant.

In our paper, we restrict our discussion in $\omega \in (-1, 1/3]$. Because, if $\omega < -1$, the fluid is called phantom, which does not satisfy the three energy conditions in [42]. In addition, we are also not aware of any kind of fluid that has $\omega > 1/3$. If $\omega = -1$, one can not get an equation of state about $P, T, R_A$, so this case is also not considered. Therefore, for our case with $\omega \in (-1, 1/3]$, from (2.22), (2.24) and (2.18), we finally obtain the equation of state of the FRW universe in the novel 4D Gauss-Bonnet gravity as

$$P = -\frac{T}{(\omega + 1)R_A} - \frac{2\alpha T}{(\omega + 1)R_A^2} + \frac{1 - 3\omega}{8\pi(\omega + 1)R_A^3} + \frac{(5 - 3\omega)\alpha}{8\pi(\omega + 1)R_A^4}. \quad (2.25)$$

One can see that if $R_A$ is very small, the $\alpha$-dependent terms play more important roles than the other two terms, but if $R_A$ is very large, they can be neglected.

### III. THE P-V CRITICALITY OF THE FRW UNIVERSE IN THE NOVEL 4D GAUSS-BONNET GRAVITY

In this section, we investigate the $P-V$ phase transition and the critical behavior of the FRW universe in the novel 4D Gauss-Bonnet gravity. At the critical point, the equation of state should satisfy

$$\frac{\partial P}{\partial V} = (\frac{\partial^2 P}{\partial V^2})_T = 0. \quad (3.1)$$

Since $V = 4\pi R_A^3 / 3$ in the FRW universe in (2.20), the above conditions are equivalent to

$$\frac{\partial P}{\partial R_A} = (\frac{\partial^2 P}{\partial R_A^2})_T = 0. \quad (3.2)$$

Applying the above conditions to the equation of state (2.25), one gets

$$4\pi R_A^3 T + (3\omega - 1)R_A^2 + 24\pi\alpha R_A T + 2(3\omega - 5)\alpha = 0, \quad (3.3)$$

$$8\pi R_A^3 T + 3(3\omega - 1)R_A^2 + 96\pi\alpha R_A T + 10(3\omega - 5)\alpha = 0. \quad (3.4)$$

From the above two equations, one can get the condition of the critical radius of apparent horizon

$$(1 - 3\omega)R_A^4 + 24\alpha R_A^2 + 12(5 - 3\omega)\alpha^2 = 0. \quad (3.5)$$

In our paper, $\omega$ has been restricted within the range $(-1, 1/3]$. Furthermore, if $-1 < \omega < -1/3$, there is no real root of $R_A^2$ since the discriminant $\Delta = -48\alpha^2(3\omega + 1)(3\omega - 7)$ is negative in this case. Hence, for obtaining the real solutions of $R_A^2$, we are only interested in the case with $-1/3 \leq \omega \leq 1/3$ in the following investigations.

Note that, for $-1/3 < \omega < 1/3$, the above equation of $R_A^2$ can have two real roots, and for $\omega = 1/3$ or $\omega = -1/3$, the above equation of $R_A^2$ has only one real root. Therefore, for convenience, we will make our investigations into two parts in the following: two critical points case with $-1/3 < \omega < 1/3$ and one critical point case with $\omega = \pm 1/3$.

#### A. Two Critical Points Case with $-1/3 < \omega < 1/3$

When $-1/3 < \omega < 1/3$, Eq.(3.5) is a cubic equation of $R_A^2$ and has two roots

$$R_A^2 = \frac{12 - 2s}{3\omega - 1} \alpha, \quad (3.6)$$

and

$$R_A^2 = \frac{12 + 2s}{3\omega - 1} \alpha, \quad (3.7)$$

where $s \equiv \sqrt{-3(3\omega - 7)(3\omega + 1)}$. Obviously, if the coupling constant $\alpha$ is positive, the above two roots are negative within $-1/3 < \omega < 1/3$, and there is no real solution of $R_A$. Consequently, there is no critical point. If $\alpha$ is negative,
both of the roots are positive within \(-1/3 < \omega < 1/3\), and the two critical apparent horizons are

\[
R_{c+} = \sqrt{\frac{2(6 - s)}{3\omega - 1}} \alpha,
\]

(3.8)

\[
R_{c-} = \sqrt{\frac{2(6 + s)}{3\omega - 1}} \alpha,
\]

(3.9)

and \(R_{c+} < R_{c-}\). The corresponding critical temperatures are

\[
T_{c+} = \frac{3\omega + 5 + s}{48\pi} \sqrt{\frac{2(6 - s)}{(3\omega - 1)\alpha}} > 0,
\]

(3.10)

\[
T_{c-} = \frac{3\omega + 5 - s}{48\pi} \sqrt{\frac{2(6 + s)}{(3\omega - 1)\alpha}} > 0,
\]

(3.11)

where \(T_{c+} > T_{c-}\), the corresponding critical pressures are

\[
P_{c+} = \frac{3(9\omega^2 - 6\omega - 23) + 2(3\omega - 7)s}{288(\omega + 1)(3\omega - 5)\pi\alpha} < 0,
\]

(3.12)

\[
P_{c-} = \frac{3(9\omega^2 - 6\omega - 23) - 2(3\omega - 7)s}{288(\omega + 1)(3\omega - 5)\pi\alpha} < 0,
\]

(3.13)

and the dimensionless critical coefficients are

\[
\epsilon_{c+} \equiv \frac{P_{c+}R_{c+}}{T_{c+}} = \frac{-3(9\omega - 17) + s}{24(\omega + 1)(3\omega - 5)} < 0,
\]

(3.14)

\[
\epsilon_{c-} \equiv \frac{P_{c-}R_{c-}}{T_{c-}} = \frac{-3(9\omega - 17) - s}{24(\omega + 1)(3\omega - 5)} < 0.
\]

(3.15)

In the following, we will calculate the four critical exponents \((\alpha', \beta, \gamma, \delta)\) defined in the following quantities [36]:

\[
C_{V,\alpha} = T(\frac{\partial S}{\partial T})_V \propto |t|^{-\alpha'},
\]

(3.16)

\[
\eta = \frac{V_l - V_s}{V_c} \propto |t|^\beta,
\]

(3.17)

\[
\kappa_T = -\frac{1}{V} (\frac{\partial V}{\partial T})_P \propto |t|^{-\gamma},
\]

(3.18)

\[
p \propto v^\delta,
\]

(3.19)

where

\[
p = \frac{P - P_c}{P_c}, \quad t = \frac{T - T_c}{T_c}, \quad v = \frac{V - V_c}{V_c},
\]

(3.20)

and check whether they satisfy the following scaling laws

\[
\alpha' + 2\beta + \gamma = 2, \quad \alpha' + \beta(1 + \delta) = 2,
\]

\[
\gamma(1 + \delta) = (2 - \alpha')(\delta - 1), \quad \gamma = \beta(\delta - 1).
\]

(3.21)

It should be noted that there are only two independent relations in the above scaling laws. Obviously, \(C_{V,\alpha}\) is zero, which means the first critical exponent \(\alpha'\) is zero. To calculate another three critical exponents, it is convenient to first expand the equation of state in (2.25) around the critical point. The expansion of the pressure around the first critical point \((R_{c+}, P_{c+})\) is

\[
p_+ = a_{10} t_+ + a_{11} t_+ v_+ + a_{03} v_+^3 + \mathcal{O}(t_+ v_+^2, v_+^4),
\]

(3.22)

where

\[
p_+ = \frac{P - P_c}{P_{c+}}, \quad t_+ = \frac{T - T_c}{T_{c+}}, \quad v_+ = \frac{V - V_c}{V_{c+}},
\]

(3.23)
Therefore, from the above two relations, one can get a nontrivial solution \( \delta \) which gives the last critical exponent is \( s \) where index with

\[
a_{10} \equiv \frac{\partial p}{\partial t} = \frac{T_c}{P_c} \frac{\partial P}{\partial T} = \frac{T_c + (R_{c+}^2 + 2\alpha)}{(1 + \omega)P_c R_{c+}^2} = \frac{2T_c(3\omega + 5 - s)\alpha}{(1 + \omega)(1 - 3\omega)P_c R_{c+}^2} > 0, \tag{3.24}
\]

\[
a_{11} \equiv \frac{\partial^2 p}{\partial t \partial v} = \frac{R_c T_c}{3P_c} \left( \frac{\partial^2 P}{\partial T \partial R_A} \right) = \frac{T_c + (R_{c+}^2 + 6\alpha)}{3(1 + \omega)P_c R_{c+}^2} = \frac{2T_c(9\omega - 3 + s)\alpha}{3(1 + \omega)(1 - 3\omega)P_c R_{c+}^2} > 0, \tag{3.25}
\]

\[
a_{03} \equiv \frac{1}{3!} \left( \frac{\partial^3 p}{\partial v^3} \right) = \frac{R_c^3}{162P_c^2} \left( \frac{\partial^3 P}{\partial P^3 A} \right) = \frac{(1 - 3\omega)R_{c+}^2 + 6(3\omega - 5)\alpha}{648(1 + \omega)\pi P_c R_{c+}^4} = \frac{(9\omega - 21 + s)\alpha}{324(1 + \omega)\pi P_c R_{c+}^4} < 0, \tag{3.26}
\]

all of which are not zero.

According to the Maxwell’s area law, the coexistence phase in the FRW universe with the same pressure \( p^*_c \) satisfies

\[
p^*_c = a_{10} t_+ + a_{11} t_+ v_+ + a_{03} v_+^3 = a_{10} t_+ + a_{11} t_+ v_+ + a_{03} v_+^3,
\]

where index \( s \) and \( l \) represent states with small and large radii \( \rho_A \) in the endpoints of coexistence phase of the FRW universe. Hence, one easily obtains

\[
a_{11}(v_+ - v_+) t_+ + a_{03}(v_+^3 - v_+^3) = 0. \tag{3.28}
\]

On the other hand, the Maxwell’s area law \( \int v dp = \int_v^T v \frac{\partial p}{\partial v} dv = 0 \) still holds, so we have another relation

\[
2a_{11}(v_+^2 - v_-^2) t_+ + 3a_{03}(v_+^4 - v_-^4) = 0. \tag{3.29}
\]

Therefore, from the above two relations, one can get a nontrivial solution

\[
v_+ - v_+ = \sqrt{\frac{a_{11}}{a_{03}}} t_+, \quad v_+ = -\sqrt{\frac{a_{11}}{a_{03}}} t_+,
\]

so

\[
\eta_+ = v_+ - v_+ = 2 \sqrt{\frac{a_{11}}{a_{03}}} t_+ \propto \left| t_+ \right|^{1/2}, \tag{3.31}
\]

which shows that the second critical exponent \( \beta \) is 1/2. Interestingly, \( a_{11} > 0, a_{03} < 0 \), and hence we easily derive \( t_+ > 0 \), which means that the coexistence phase in the FRW universe occurs above the critical temperature with \( T > T_{c+} \). This behavior is significant different from black hole and van der Waals system cases with \( T < T_c \).

The isothermal compressibility at the critical point \( (R_{c+}, P_{c+}) \) is

\[
\kappa_T = -\frac{1}{V_{c+}} \frac{\partial V}{\partial P} \big|_{c+} \propto -\left( \frac{\partial p}{\partial v_+} \right)^{-1} = -\frac{1}{a_{11} t_+} \propto t_+^{-1}, \tag{3.32}
\]

which gives the third critical exponent \( \gamma = 1 \). At \( t_+ = 0 \), the shape of the critical isotherm is

\[
p \propto v^3, \tag{3.33}
\]

which gives the last critical exponent is \( \delta = 3 \). Therefore, we get the four critical exponents around the first critical point \( (R_{c+}, P_{c+}) \)

\[
\alpha' = 0, \quad \beta = \frac{1}{2}, \quad \gamma = 1, \quad \delta = 3, \tag{3.34}
\]

which is the same with those in the mean field theory, and hence satisfies the scaling laws \( (3.21) \).

For the second critical point \( (R_{c-}, P_{c-}) \) case, similarly, the expansion of the equation of state around this critical point is

\[
p_- = b_{10} t_- + b_{11} t_- v_- + b_{03} v_-^3, \tag{3.35}
\]

with

\[
p_- = \frac{P - P_{c-}}{P_{c-}}, \quad v_- = \frac{V - V_{c-}}{V_{c-}}, \quad t_- = \frac{T - T_{c-}}{T_{c-}}, \tag{3.36}
\]
and the coefficients are

\[ b_{10} = -\frac{T_c - (R_c^2 + 2\alpha)}{(1 + \omega)P_c - R_c^3} = \frac{2T_c - (3\omega + 5 + s)\alpha}{(1 + \omega)(1 - 3\omega)P_c - R_c^3} > 0, \]  
\[ b_{11} = \frac{T_c - (R_c^2 + 6\alpha)}{3(1 + \omega)P_c - R_c^3} = \frac{2T_c - (9\omega - 3 - s)\alpha}{3(1 + \omega)(1 - 3\omega)P_c - R_c^3} < 0, \]  
\[ b_{03} = \frac{(1 - 3\omega)R_c^2 + 6(3\omega - 5)\alpha}{648(1 + \omega)\pi P_c - R_c^4} = \frac{(9\omega - 21 - s)\alpha}{324(1 + \omega)\pi P_c - R_c^4} < 0, \]

all of which are not zero. By similar methods, one can find that the critical exponents are the same with the \((R_c+, P_c+\) case, so the scaling laws are still satisfied. Note that, since \(b_{11}, b_{03} < 0\) in this case, the coexistence phase in the FRW universe occurs below the critical temperature with \(T < T_{c-}\). Interestingly, this case is similar with those in the black holes or the van der Waals system, and different from the above case around the first critical point \((R_c+, P_c+\).

In order to more clearly represent our results in the above discussions, we will focus on a special interesting case with \(\omega = 0\) in the following, which is also the matter dominated period of our universe. In this \(\omega = 0\) case, the critical values of the two critical points are

\[ R_{c+} = \sqrt{-(12 - 2\sqrt{21})\alpha}, \quad T_{c+} = \frac{\sqrt{33 + 7\sqrt{21}}}{24\pi\sqrt{\alpha}}, \quad P_{c+} = \frac{69 + 14\sqrt{21}}{1440\pi\alpha}, \]  
\[ R_{c-} = \sqrt{-(12 + 2\sqrt{21})\alpha}, \quad T_{c-} = \frac{\sqrt{33 - 7\sqrt{21}}}{24\pi\sqrt{\alpha}}, \quad P_{c-} = \frac{69 - 14\sqrt{21}}{1440\pi\alpha}. \]

The dimensionless critical coefficients are

\[ \epsilon_{c+} = \frac{P_{c+}R_{c+}}{T_{c+}} = \frac{51 + \sqrt{21}}{120} \approx -0.463, \quad \epsilon_{c-} = \frac{P_{c-}R_{c-}}{T_{c-}} = \frac{51 - \sqrt{21}}{120} \approx -0.387. \]

In this case with \(\omega = 0\), the equation of state is explicitly shown as

\[ P = -\frac{T}{R_A^2} + \frac{1}{8\pi R_A^2} - \frac{2\alpha T}{R_A^3} + \frac{5\alpha}{8\pi R_A^3}. \]

For conveniently and explicitly showing the diagrams, we first define the dimensionless reduced pressure, reduced temperature and reduced apparent horizon as

\[ \hat{P} = \frac{P}{P_{c+}}, \quad \hat{T} = \frac{T}{T_{c+}}, \quad \hat{R} = \frac{R_A}{R_{c+}}, \]

and hence rewrite the equation of state as

\[ \hat{P}_+ = \frac{2(51 - \sqrt{21})\hat{T}_+}{43\hat{R}_+^2} - \frac{6(8 - \sqrt{21})\hat{R}_+^4}{43\hat{R}_+^4} - \frac{2(19 + 3\sqrt{21})\hat{T}_+}{43\hat{R}_+^4} + \frac{27 + 2\sqrt{21}}{43\hat{R}_+^4}, \]  
\[ \hat{P}_- = \frac{2(51 + \sqrt{21})\hat{T}_-}{43\hat{R}_-^2} - \frac{6(8 + \sqrt{21})\hat{R}_-^4}{43\hat{R}_-^4} - \frac{2(19 - 3\sqrt{21})\hat{T}_-}{43\hat{R}_-^4} + \frac{27 - 2\sqrt{21}}{43\hat{R}_-^4}. \]

Therefore, we easily draw the \(\hat{P}-\hat{R}\) diagrams to illustrate the above two equations of state with \(\omega = 0\) in the following Fig.1.
B. One Critical Point Case with $\omega = 1/3$ or $-1/3$

When $\omega$ is equal to 1/3, from (3.5) one obtains

$$R_A^2 = -2\alpha.$$  \hspace{1cm} (3.48)

Obviously, if $\alpha$ is positive, there is no real solution of $R_A$. If $\alpha$ is negative, there is one real solution

$$R_c = \sqrt{-2\alpha},$$  \hspace{1cm} (3.49)

while the corresponding critical temperature and pressure are

$$T_c = \frac{1}{4\pi} \sqrt{\frac{-2}{\alpha}}, \quad P_c = \frac{3}{32\pi\alpha}. \hspace{1cm} (3.50)$$

From the above critical quantities, one can also get a dimensionless critical coefficient

$$\epsilon \equiv \frac{P_c R_c}{T_c} = -\frac{3}{8}, \hspace{1cm} (3.51)$$

which is minus to the value in the van der Waals liquid-gas system. Meanwhile, the equation of state in this case with $\omega = 1/3$ is reduced to

$$P = \frac{-3T}{4R_A} - \frac{3\alpha T}{2R_A^3} + \frac{3\alpha}{8\pi R_A^4}. \hspace{1cm} (3.52)$$

Using the critical values, one can also define the dimensionless reduced pressure, reduced temperature and reduced apparent horizon as

$$\tilde{P} \equiv \frac{P}{P_c}, \quad \tilde{T} \equiv \frac{T}{T_c}, \quad \tilde{R} \equiv \frac{R_A}{R_c}, \hspace{1cm} (3.53)$$

and rewrite the equation of state as

$$\tilde{P} = \frac{2\tilde{T}}{\tilde{R}} - \frac{2\tilde{T}^3}{\tilde{R}^3} + \frac{1}{\tilde{R}^4}. \hspace{1cm} (3.54)$$

One can also see that at the critical radius $\tilde{R} = 1$, the two terms with temperature cancel each other. In addition, the $\tilde{P}-\tilde{R}$ diagram can be drawn from the above equation in Fig. 2.
In the following, we will calculate the four critical exponents in this case with $\omega = 1/3$. Obviously, for fixed $V$ and $\alpha$, the first critical exponent $\alpha'$ is still 0. In order to calculate another three critical exponents, one can also expand the equation of state around the critical point:

$$p = a_{10} t + a_{11} t v + a_{03} v^3 + \mathcal{O}(tv^2, v^4) = \frac{4}{3} tv - \frac{4}{81} v^3 + \mathcal{O}(tv^2, v^4).$$  \hspace{1cm} (3.55)

One can see that in this case, $a_{10} = 0$, $a_{11} > 0$ and $a_{03} < 0$, which means that the temperature term vanishes near the critical point as said before, while calculations of these three critical exponents are the same as those in the above subsection with discussions in $-1/3 < \omega < 1/3$. Therefore, these three critical exponents in this case with $\omega = 1/3$ are still $\beta = \frac{1}{2}$, $\gamma = 1$, $\delta = 3$, and the scaling laws are still satisfied. Note that, in this case, the coexistence phase occurs above the critical temperature.

In the $\omega = -1/3$ case, from (3.5), the critical values are

$$R_c = \sqrt{-6\alpha}, \quad T_c = \frac{1}{2\pi\sqrt{-6\alpha}}, \quad P_c = \frac{5}{96\pi\alpha},$$  \hspace{1cm} (3.56)

and the dimensionless critical coefficient is

$$\epsilon \equiv \frac{P_c R_c}{T_c} = \frac{5}{8}.$$  \hspace{1cm} (3.57)

In this case, the equation of state can be explicitly written as

$$P = -\frac{3T}{2R_A} + \frac{3}{8\pi R_A^2} - \frac{3\alpha T}{R_A^3} + \frac{9\alpha}{8\pi R_A^4},$$  \hspace{1cm} (3.58)

which can also be rewritten by reduced quantities as

$$\tilde{P} = -\frac{12\tilde{T}}{5R} + \frac{6}{5R^2} - \frac{4\tilde{T}}{5R^3} + \frac{3}{5R^4}.$$  \hspace{1cm} (3.59)

The $\tilde{P}$-$\tilde{R}$ diagrams can be drawn from the above equation in this case with $\omega = -1/3$ in Fig. 3.
FIG. 3: Criticality of P-V Phase Transition with \( \omega = -1/3 \). The blue/yellow/green line corresponds to temperature \( 0.8T_c / T_c / 1.2T_c \) respectively.

For the critical exponents in this case with \( \omega = -1/3 \), one can still expand the equation of state as before (3.22), one will find that \( a_{11} \) is zero, i.e. \( tv \) term vanishes. Therefore, the high order term \( tv^2 \) should be considered if its coefficient \( a_{12} \) is not zero, and the expansion in this case is written as

\[
p = a_{10} + a_{03}v^3 + a_{12}tv^2 + O(tv^3, v^4) = \frac{8}{5} t - \frac{8}{135} v^3 - \frac{4}{15} tv^2 + O(tv^3, v^4).
\]

(3.60)

Since this case is not the same as those in the above discussions with nonzero \( a_{11} \) and \( a_{03} \). The critical exponents may be changed, and will be beyond the mean field theory. This result is interesting, however, it is not our main claim in this paper, and we will investigate in details on this issue in our future work.

C. Summary of the P-V Criticality

The numbers of the critical points are listed in the table.

| \( \omega \) | \( -1 < \omega < -1/3 \) | \( -1/3 < \omega < 1/3 \) | \( \omega = -1/3 \) | \( \omega = 1/3 \) |
|---------|-----------------|----------------|-------------|-------------|
| \( \alpha > 0 \) | 0 | 0 | 0 | 0 |
| \( \alpha < 0 \) | 0 | 1 | 2 | 1 |

As can be seen from the table, the P-V phase transitions of the FRW universe in the novel 4D Gauss-Bonnet gravity are very rich. If \( \alpha \) is positive, there is no P-V phase transition within \(-1 < \omega \leq 1/3\). While if \( \alpha \) is negative, there are P-V phase transitions within \(-1/3 \leq \omega \leq 1/3\). Particularly, if \( \alpha < 0, \omega = -1/3 \) or \( 1/3 \), there is one critical point. While if \( \alpha < 0, -1/3 < \omega < 1/3 \), there are two critical points, which implies that there may be three phases in the FRW universe like Helium or black hole system [43–46]. Interestingly, in the \(-1/3 < \omega \leq 1/3\) case, the critical exponents satisfy the scaling laws. However, in the \( \omega = -1/3 \) case, the critical exponents may be different from the values in the mean field theory, which will be investigated in details in our future work.

IV. CONCLUSION AND DISCUSSION

In this work, we have studied the thermodynamics especially the P-V criticality of the FRW universe in the novel 4D Gauss-Bonnet gravity, where we treat the cosmological constant related to the thermodynamic pressure as \( P = -\Lambda / 8\pi \).

From the generalized Misner-Sharp energy and the entropy inside the apparent horizon of the FRW universe in the Gauss-Bonnet gravity, we have got the first law of thermodynamics, where the Misner-Sharp energy is better to be interpreted as the enthalpy of the FRW universe. Particularly, we obtain the thermodynamic equation of state \( P(T, V) \) of the FRW universe in the novel 4D Gauss-Bonnet gravity. From the equation of state, we find that the P-V criticality only occurs in the case with \( \alpha < 0 \) and \(-1/3 \leq \omega \leq 1/3\). Particularly, if \( \omega = 1/3 \) or \(-1/3 \), there is only one critical point. While if \(-1/3 < \omega < 1/3 \), there are two critical points of the P-V phase transitions. We also calculate the four critical exponents \((\alpha', \beta, \gamma, \delta)\) of the critical points. We find that, in the \(-1/3 < \omega \leq 1/3\) case, the four critical
exponents are the same as those in the mean field theory, while the scaling laws are satisfied. However, in the $\omega = -1/3$ case, the critical exponents are quite different.

One can see that in the novel 4D Gauss-Bonnet gravity, the FRW universe has rich phase transitions and critical behaviors compared with Einstein gravity. Particularly, under some conditions, the FRW universe has two critical points, which have been rarely seen in the literature to the best of our knowledge. It means that the FRW universe may have three phases under Gauss-Bonnet gravity like Helium or black hole system. In the $\alpha < 0$ and $\omega = -1/3$ case, the $tv$ term vanishes, so it should have a different expansion. In this case, the critical exponents may be changed, and the scaling laws may be violated, which means the critical behaviors are perhaps beyond the mean field theory. However, its deeper physical meaning is still unknown to us and worth further investigating. In addition, the work density $W$ is also a conjugate variable of the thermodynamic volume $V$, so it should have a different expansion. In this case, the critical exponents may be changed, and the corresponding $p_m$ phase transition or $P_{\text{total}}$ phase transition. One can also discuss the free energy landscape, stability, Joule-Thompson expansion of the FRW universe within the framework of the Gauss-Bonnet gravity. We plan to study these topics in our future work.

V. ACKNOWLEDGMENT

We are grateful for the inspiring discussions with Profs. Hongsheng Zhang, Shao-Wen Wei, Yen Chin Ong, Xiao-Mei Kuang, Dr. Tao-Tao Sui, et al. This work is supported by the National Natural Science Foundation of China (NSFC) under grants Nos. 12175105, 11575083, 11565017, Top-notch Academic Programs Project of Jiangsu Higher Education Institutions (TAPP), and is also supported by “the Fundamental Research Funds for the Central Universities, NO. NS2020054” of China.
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