Poisson Reduction of Controlled Hamiltonian System by Controllability Distribution

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Abstract. In this paper, we first study the Poisson reductions of controlled Hamiltonian (CH) system and symmetric CH system by controllability distributions. These reductions are the extension of Poisson reductions by distribution for Poisson manifolds to that for phase spaces of CH systems with external force and control. We give Poisson reducible conditions of CH system by controllability distribution, and prove that the Poisson reducible property for CH systems leaves invariant under the CH-equivalence. Moreover, we study the Poisson reduction of symmetric CH system by $G$-invariant controllability distribution. Next, we consider the singular Poisson reduction and SPR-CH-equivalence for CH system with symmetry, and prove the singular Poisson reduction theorem of CH system. We also study the relationship between Poisson reduction for singular Poisson reducible CH systems by $G$-invariant controllability distribution and that for associated reduced CH system by reduced controllability distribution. At last, some examples are given to state the theoretical results.

Keywords: controlled Hamiltonian system, Poisson reduction by distribution, CH-equivalence, controllability distribution, singular Poisson reducible CH system.

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1 Introduction

Poisson reduction is an important topic in the study of Poisson geometry for constructing new Poisson manifold and simplifying Hamiltonian systems defined on a Poisson manifold, and it is also a powerful tool in the study of stability and bifurcation theory of mechanical systems. See Abraham et al. [1, 2], Libermann and Marle [10], Marsden et al. [11, 12, 16], Marsden and Ratiu [14], Ortega and Ratiu [18]. Just as we have known that the Poisson reduction is first a generalization of symplectic reduction method to Poisson manifolds and to the singular context. In addition, one can also study the Poisson reduction for Poisson manifolds by pseudo-groups and distributions, since the special Poisson brackets can be induced on the presheaf of Poisson algebras. There have been many results and ways of reduction, such as, optimal point and orbit Poisson reduction, regular Poisson reduction for Hamiltonian systems by using optimal momentum map, are given in Ortega and Ratiu [18], and the singular reductions for Hamiltonian system, Dirac structure and nonlinear control system, are given in Sjamaar and Lerman [20], Jotz et al. [8], Śniatycki [21], as well as the reduction for Poisson manifolds by distributions, are given in Marsden and Ratiu [13], Falceto and Zambon [5], and Jotz and Ratiu [7], and there is still much to be done in this topic.

On the other hand, it is well known that the mechanical control system theory has formed an important subject in recent twenty years. Its research gathers together some separate areas of research such as mechanics, differential geometry and nonlinear control theory, etc., and the emphasis of this research on geometry is motivated by the aim of understanding the structure of equations of motion of the system in a way that helps both analysis and design. So, it is natural to study mechanical control systems by combining with the analysis of dynamic systems and the geometric reduction theory of Hamiltonian and Lagrangian systems. For examples, Birtea et al. in [3] and Sánchez de Alvarez in [22], studied the controllability of Poisson systems; Nijmeijer and Van der Schaft in [17], studied the nonlinear dynamical control systems as well as the use of feedback control to stabilize mechanical systems; Leonard and Marsden in [9] and Bloch and Leonard in [4] gave the underwater vehicle with internal rotors and rigid spacecraft with internal rotors as the practical models of Hamiltonian systems with control. In particular, we note that in Marsden et al. [15], the authors studied regular reduction theory of controlled Hamiltonian systems with symplectic structure and symmetry, as an extension of regular symplectic reduction theory of Hamiltonian systems under regular controlled Hamiltonian equivalence conditions, and Wang in [25] generalized the work in [15] to study the singular reduction theory of regular controlled Hamiltonian systems, and Wang and Zhang in [29] generalized the work in [15] to study
optimal reduction theory of controlled Hamiltonian (CH) systems with Poisson structure and symmetry by using optimal momentum map and reduced Poisson tensor (or reduced symplectic form). In addition, since the Hamilton-Jacobi theory is developed based on the Hamiltonian picture of dynamics, it is natural idea to extend the Hamilton-Jacobi theory to the (regular) controlled Hamiltonian system and its a variety of reduced systems, and it is also possible to describe the relationship between the CH-equivalence for controlled Hamiltonian systems and the solutions of corresponding Hamilton-Jacobi equations. Wang in [26–28] studied this work and applied to give explicitly the motion equations and Hamilton-Jacobi equations of reduced spacecraft-rotor system and reduced underwater vehicle-rotors system on a symplectic leaves by calculation in detail, which show the effect on controls in regular symplectic reduction (by stages) and Hamilton-Jacobi theory.

It is worthy of note that if there is no momentum map for our considered systems, then the reduction procedures given in Marsden et al [15] and Wang and Zhang [29], and hence in Wang [25–28] can not work. One must look for a new way. On the other hand, motivated by the work of Poisson reductions by distribution for Poisson manifolds in Ortega and Ratiu [18], we note that in above research work, the phase space $T^*Q$ of the CH system is a Poisson manifold, and its control subset $W \subset T^*Q$ is a fiber submanifold. If we assume that $D \subset TT^*Q|_W$ is a controllability distribution of the CH system, then it is a natural problem if we could study the Poisson reduction for the CH system by controllability distribution. This is our first goal of research in this paper. Next, Wang and Zhang in [29] give the regular Poisson reduction of CH system, we hope to develop this reduction in the singular context, and to describe the relationship between singular (including regular) Poisson reduction and by controllability distribution Poisson reduction for CH system. This is our second goal of research in this paper. The main contributions in this paper are given as follows. (1) We give Poisson reducible conditions of CH system and symmetric CH system by controllability distributions, and prove the Poisson reducible property for CH systems to keep invariant under the CH-equivalence; (2) We study the singular Poisson reduction and SPR-CH-equivalence for CH system with symmetry, and prove the singular Poisson reduction theorem, which is a generalization of the regular Poisson reduction theorem in [29] to the singular context; (3) We prove a theorem to explain the relationship between Poisson reduction for singular Poisson reducible CH systems by $G$-invariant controllability distribution and that for associated reduced CH system by reduced controllability distribution.

A brief of outline of this paper is as follows. In the second section, we review some relevant definitions and basic facts about Poisson manifolds, generalized distributions, and the Poisson reduction for Poisson manifold by distribution, as well as the CH system defined by using Poisson tensor on the cotangent bundle of a configuration manifold and its CH-equivalence, which will be used in subsequent sections. The Poisson reduction of CH system by controllability distribution and that of symmetric CH system by $G$-invariant controllability distribution are considered respectively, in the third section and the fourth section, and we prove that the property of Poisson reduction for CH system by controllability distribution leaves invariant under CH-equivalence conditions. In the fifth section, the singular Poisson reducible CH system and its SPR-CH-equivalence are considered, and singular Poisson reduction theorem is proved, which shows the relationship between the SPR-CH-equivalence for singular Poisson reducible CH systems with symmetry and the CH-equivalence for associated singular Poisson reduced CH systems. Moreover, in the sixth section, a theorem is given to show the relationship between singular (including regular) Poisson reduction and by controllability distribution Poisson reduction for
CH system. At last, some examples are given to state theoretical results of Poisson reduction for CH systems by controllability distributions. These research work develop the reduction theory of controlled Hamiltonian systems with symmetry and make us have much deeper understanding and recognition for the structure of controlled Hamiltonian systems.

2 Preliminaries

In order to study the Poisson reductions of CH systems by controllability distributions, we first give some relevant definitions and basic facts about Poisson manifolds, generalized distributions, and the Poisson reductions for Poisson manifolds by distributions. We also recall briefly the CH systems defined by using Poisson tensor on a Poisson fiber bundle and on the cotangent bundle of a configuration manifold and their CH-equivalence, which will be used in subsequent sections. we shall follow the notations and conventions introduced in Abraham et al [1,2], Marsden and Ratiu [13,14], Ortega and Ratiu [18], Jotz and Ratiu [7], and Wang and Zhang [29].

2.1 Poisson Manifolds and Generalized Distributions

Let P be a smooth manifold. \( C^\infty(P) \) a set of smooth functions on P, and \( \mathfrak{X}(P) \) a set of smooth vector fields on P. A Poisson bracket (or a Poisson structure) on the manifold P is a bilinear operation \( \{\cdot,\cdot\} \) on \( C^\infty(P) \) such that \( (C^\infty(P),\{\cdot,\cdot\}) \) is a Lie algebra; and \( \{\cdot,\cdot\} \) is a derivation in each factor. A manifold P endowed with a Poisson bracket on \( C^\infty(P) \) is called a Poisson manifold, and denoted by \((P,\{\cdot,\cdot\})\). The pair \( (C^\infty(P),\{\cdot,\cdot\}) \) is also called a Poisson algebra. The derivation property of the Poisson bracket implies that for any two functions \( f, g \in C^\infty(P) \), the value of the bracket \( \{f,g\}(m) \) at an arbitrary point \( m \in P \) depends on \( f \) only through \( df(m) \), which allows us to define a covariant antisymmetric two-tensor \( B \in \Lambda^2(T^*P) \) by \( B(m)(df(m),dg(m)) = \{f,g\}(m), \forall f, g \in C^\infty(P) \). This tensor \( B \) is called the Poisson tensor of \( P \). The vector bundle map \( B^2 : T^*P \to TP \) naturally associated to \( B \) is defined by \( B(\alpha_m,\beta_m) = \langle \alpha_m, B^2(m)\beta_m \rangle, \forall \alpha_m, \beta_m \in T^*_m P \). The range \( D := B^2(T^*P) \subset TP \) is called the characteristic distribution.

A generalized distribution \( D \) on \( P \) is a subset of the tangent bundle \( TP \) such that for any point \( m \in P \), the fiber \( D(m) := D \cap T_m P \) is a vector subspace of \( T_m P \). The dimension of \( D(m) \) is called the rank of \( D \) at \( m \). A point \( m \in P \) is a regular point of the distribution \( D \), if there exists a neighborhood \( U \) of \( m \) such that the rank of \( D \) is constant on \( U \). Otherwise, \( m \) is a singular point of the distribution. A distribution \( D \) is called regular if every point \( m \in P \) is a regular point of the distribution \( D \). A differentiable section of \( D \) is a differentiable vector field \( X \) defined on an open subset \( U \) of \( P \), such that for any point \( z \in U, X(z) \in D(z) \). An immersed connected submanifold \( N \) of \( P \) is said to be an integral manifold of the distribution \( D \) if \( T_z i(T_z N) \subset D(z), \forall z \in N \), where \( i : N \to P \) is the inclusion. \( N \) is said to be of maximal dimension at \( z \in N \) if \( T_z i(T_z N) = D(z) \). The generalized distribution \( D \) is differentiable if for every point \( m \in P \) and for every vector \( v \in D(m) \), there exist a differentiable section \( X \) of \( D \), defined on an open neighborhood \( U \) of \( m \), such that \( X(m) = v \). The generalized distribution \( D \) is completely integrable if for every point \( m \in P \), there exists an integral manifold of \( D \) everywhere of maximal dimension which contains \( m \). The generalized distribution \( D \) is involutive if it is invariant under the (local) flows associated to differentiable sections of \( D \). From Stefan [23] and Sussmann [24] we know that \( D \) is completely integrable if and only if it is involutive.
Let $D$ be an integrable generalized distribution on $P$, then for every point $m \in P$, there exists a unique connected integral manifold $\mathcal{L}_m$ of $D$, which contains $m$ and has maximal dimensions. $\mathcal{L}_m$ is called the maximal integral manifold or the accessible set of $D$ going through $m$, which is a symplectic leaf of $P$. We know that the local structure of Poisson manifolds is more complex than what one obtains in the symplectic case. However, the symplectic stratification theorem shows that if $P$ is a finite dimensional Poisson manifold, then $P$ is the disjoint union of its symplectic leaves. Each symplectic leaf in $P$ is an injectively immersed Poisson submanifold and the induced Poisson structure on the leaf is symplectic.

In the following we will be interested in the specific case in which the generalized distribution is given by an everywhere defined family of vector fields, that is, there is a family of smooth vector fields $\mathcal{F}$ whose elements are vector fields $X$ defined on an open subset $\text{Dom}(X) \subset P$ such that, for any $m \in P$ the generalized distribution $D_{\mathcal{F}}$ is given by

$$D_{\mathcal{F}}(m) = \text{span}\{X(m) \in T_mP \mid X \in \mathcal{F} \text{ and } m \in \text{Dom}(X)\}.$$  

Note that in such a case the distribution $D_{\mathcal{F}}$ is differentiable by construction. We will say that $D_{\mathcal{F}}$ is the generalized distribution spanned by $\mathcal{F}$.

Let $G$ be a Lie group acting properly and canonically on the Poisson manifold $(P, \{\cdot, \cdot\})$ by the map $\Phi : G \times P \to P$, and for any $g \in G$, the map $\Phi_g := \Phi(g, \cdot) : P \to P$ is a diffeomorphism of $P$. A submanifold $S(\subset P)$ is called $G$-invariant, if $\Phi_g(S) = S, \forall g \in G$. A vector field $X \in \mathfrak{X}(P)$ is called $G$-equivariant, if $X \cdot \Phi_g = T\Phi_g \cdot X, \forall g \in G$. We denote the set of $G$-equivariant vector fields on $P$ by $\mathfrak{X}(P)^G$. Let $\mathcal{F}$ be an everywhere defined family of local vector fields on $P$ and $D_{\mathcal{F}}$ the generalized distribution spanned by it. We say that the distribution $D_{\mathcal{F}}$ is $G$-invariant when for any $z \in P$ and any $g \in G$, we have that $T_z\Phi_g D_{\mathcal{F}}(z) = D_{\mathcal{F}}(\Phi_g(z))$. A distribution spanned by a family of $G$-equivariant vector fields is always $G$-invariant, but the reverse implication is not necessarily true, see Ortega and Ratiu [18]. The following propositions are important for the study in this paper.

**Proposition 2.1** Let $(P, \{\cdot, \cdot\})$ be a Poisson manifold with Poisson tensor $B \in \Lambda^2(T^*P)$. Then for any $m \in P$ and any vector subspace $V \subset T_mP$, we have that

(i) $B^2(m)(V^\circ) = (V \cap T_m\mathcal{L})^{\omega_\mathcal{L}(m)}$;

(ii) $B^2(m)(B^2(m)(V^\circ))^\circ = V \cap T_m\mathcal{L},$

where $V^\circ := \{\alpha_m \in T_m^*P \mid \langle \alpha_m, v \rangle = 0, \forall v \in V\} \subset T^*P$ is the annihilator of $V$ in $T_m^*P$, and $\mathcal{L}_m$ is the symplectic leaf of $P$ that contains the point $m$, and $\omega_\mathcal{L}$ is the symplectic form on $\mathcal{L}$.

**Proposition 2.2** Let $(P, \{\cdot, \cdot\})$ be a Poisson manifold with Poisson tensor $B \in \Lambda^2(T^*P)$. Assume that Lie group $G$ acts canonically on $P$, and for any $m \in P$, $G_m$ is the isotropy subgroup of $G$ at point $m$. Then we have that

(i) $B^2(m) : T_m^*P \to T_mP$ is $G_m$-equivariant;

(ii) If the Poisson bracket $\{\cdot, \cdot\}$ on $P$ is induced by a symplectic form $\omega$, then for any vector subspace $V \subset T_mP$, $B^2(m)(V^{G_m}) = (B^2(m)(V))^{G_m}$, where the $G_m$-superscript denotes the set of $G_m$-fixed points in the corresponding space.

In order to give some examples of application for the Poisson reduction of CH systems, we need some specific submanifolds, for examples, coisotropic submanifold, see Libermann and Marle [10]; and cosymplectic submanifold, see Weinstein [30].
Definition 2.3 Let \((P, \{\cdot, \cdot\})\) be a Poisson manifold with Poisson tensor \(B \in \Lambda^2(T^*P)\), and \(S \subseteq P\) an immersed smooth submanifold of \(P\). The conormal bundle of the submanifold \(S\) is
\[
(TS)^\circ := \{\alpha_s \in T^*_s P \mid <\alpha_s, v_s> = 0, \forall s \in S, v_s \in T_s S\},
\]
and it is a vector subbundle of \(T^*P|_S\).

(i) The submanifold \(S\) is called coisotropic, if \(B^2((TS)^\circ) \subseteq TS\);

(ii) The submanifold \(S\) is called cosymplectic, if \(S\) is an embedded submanifold and satisfies that \(B^2((TS)^\circ) \cap TS = \{0\}\), and \(T_s S + T_s \mathcal{L}_s = T_s P\), for any \(s \in S\) and \(\mathcal{L}_s\) the symplectic leaf of \((P, B)\) containing \(s \in S\).

In particular, if \((P, B)\) is a symplectic manifold and its Poisson bracket on \(P\) is induced by a symplectic form \(\omega \in \Omega^2(P)\), then in this case the condition is given by
\[
B^2((TS)^\circ) = \{u \in TS \mid \omega(u, v) = 0, \forall v \in (TS)^\circ\} = (TS)^\omega \subseteq TS,
\]
that is, \(S\) is a usually coisotropic submanifold of the symplectic manifold \((P, \omega)\). Any cosymplectic submanifold of the symplectic manifold \((P, \omega)\) is its symplectic submanifold.

2.2 Reduction of Poisson Manifold by Distribution

Let \((P, \{\cdot, \cdot\})\) be a Poisson manifold, and \(S \subseteq P\) a decomposed subset of \(P\). Let \(\{S_i\}_{i \in I}\) be the pieces of the corresponding decomposition. The topology of \(S\) is not necessarily the relative topology as a subset of \(P\). We say that \(D \subseteq TP|_S\) is a smooth generalized distribution (that is, not necessarily of constant rank) on \(S\) adapted to the decomposition \(\{S_i\}_{i \in I}\), if \(D_{S_i} := D |_{TS_i}\) is a smooth distribution on \(S_i\) for all \(i \in I\). The distribution \(D\) is said to be integrable if \(D_{S_i}\) is integrable for each \(i \in I\). Thus, in this case, we can partition each \(S_i\) into the corresponding maximal integral manifolds, and the resulting equivalence relations on each \(S_i\), whose equivalence classes are precisely these maximal integral manifolds, induce an equivalence relation \(D_S\) on the whole set \(S\) by taking the union of the different equivalence classes corresponding to all the \(D_{S_i}\), and we can define the quotient space \(S/D_S := \bigcup_{i \in I} S_i/D_{S_i}\), and denote by \(\pi_{D_S} : S \rightarrow S/D_S\) the natural projection. We say that a subset \(U \subseteq S\) is \(D_S\)-invariant if it is invariant under the flow of any section of \(D_{S_i}\), for all \(i \in I\).

We define the presheaf of smooth functions \(C^\infty_{S/D_S}\) on \(S/D_S\) as the map that associates to any open subset \(V\) of \(S/D_S\) the set of functions \(C^\infty_{S/D_S}(V)\) characterized by the following property: \(f \in C^\infty_{S/D_S}(V)\) if and only if for any \(z \in V\), there exists \(m \in \pi^{-1}_{D_S}(z)\), \(U_m\) an open neighborhood of \(m \in P\), such that \(U_m \cap S\) is \(D_S\)-invariant, and \(F \in C^\infty_P(U_m)\) satisfying
\[
f \circ \pi_{D_S}|_{\pi^{-1}_{D_S}(V) \cap U_m} = F|_{\pi^{-1}_{D_S}(V) \cap U_m}.
\]
In this case, we say that \(F\) is a local extension of \(f \circ \pi_{D_S}\) at the point \(m\). Moreover, the presheaf \(C^\infty_{S/D_S}\) is said to have the \((D, D_S)\)-local extension property, when the topology of \(S\) is stronger than the relative topology and the local extensions of \(f \circ \pi_{D_S}\) defined in (2.1) can always be chosen to satisfy
\[
dF(u)|_{D(u)} = 0, \text{ for any } u \in \pi^{-1}_{D_S}(V) \cap U_m.
\]
In this case, we say that \(F\) is a local \(D\)-invariant extension of \(f \circ \pi_{D_S}\) at the point \(m\).

At first, we recall the following definition, see Ortega and Ratiu [18].
Definition 2.4 Let \((P, \{\cdot, \cdot\})\) be a Poisson manifold, \(S \subset P\) a decomposed space, and \(D \subset TP|_S\) a smooth distribution adapted to the decomposition \(\{S_i\}_{i \in I}\) of \(S\). The distribution \(D\) is called **Poisson or canonical**, if the condition \(df|_D = dg|_D = 0\), for any \(f, g \in C^\infty_P(U)\) and any open subset \(U \subset P\), implies that \(d\{f, g\}|_D = 0\).

From this definition, it is easy to know immediately the following result. Assume that distribution \(D\) is Poisson and the presheaf \(C^\infty_{S/D_S}\) has the \((D, D_S)\)-local extension property. If \(F\) and \(H \in C^\infty_P(U_m)\) are local \(D\)-invariant extensions of \(f \circ \pi_{D_S}\) and \(h \circ \pi_{D_S}\) at \(m \in \pi^{-1}_D(z)\), then their Poisson bracket \(\{F, H\}\) is also local \(D\)-invariant extension.

Now we can introduce the definition of reduction by distribution for Poisson manifold as follows.

**Definition 2.5** Let \((P, \{\cdot, \cdot\})\) be a Poisson manifold with Poisson tensor \(B \in \Lambda^2(T^*P)\), \(S\) a decomposed subset of \(P\), and \(D \subset TP|_S\) a Poisson integrable distribution that is adapted to the decomposition of \(S\), and \(D_S := D \cap TS\). Assume that the presheaf \(C^\infty_{S/D_S}\) has the \((D, D_S)\)-local extension property. We say that \((P, B, D, S)\) is **Poisson reducible**, when \((S/D_S, C^\infty_{S/D_S}, B^{S/D_S})\) is a well-defined presheaf of Poisson algebras, where for any open set \(V \subset S/D_S\), the Poisson tensor \(B^{S/D_S}\) is defined by the Poisson bracket
\[
\{\cdot, \cdot\}_{V^{S/D_S}} : C^\infty_{S/D_S}(V) \times C^\infty_{S/D_S}(V) \to C^\infty_{S/D_S}(V),
\]
given by
\[
\{f, h\}_{V^{S/D_S}}(\pi_{D_S}(m)) := \{F, H\}_B(m),
\]
for any \(m \in \pi^{-1}_D(z)\), \(z \in V\) and any local \(D\)-invariant extension \(F\) and \(H \in C^\infty_P(U_m)\) of \(f \circ \pi_{D_S}\) and \(h \circ \pi_{D_S}\) at \(m\), respectively, where \(\pi_{D_S} : S \to S/D_S\) is the projection, and \(U_m\) is an open neighborhood of \(m\), such that \(U_m \cap S\) is \(D_S\)-invariant. That is, \(B^{S/D_S}(\pi_{D_S}(m)) := B(m)\).

The following theorem give the Poisson reducible conditions by distribution for Poisson manifold, see Ortega and Ratiu [13] and Jotz and Ratiu [7].

**Theorem 2.6** Assume that \((P, \{\cdot, \cdot\})\) is a Poisson manifold with Poisson tensor \(B \in \Lambda^2(T^*P)\), \(S\) a decomposed set of \(P\), and \(D \subset TP|_S\) a Poisson integrable generalized distribution adapted to the decomposition of \(S\), \(D_S := D \cap TS\), such that the presheaf \(C^\infty_{S/D_S}\) has the \((D, D_S)\)-local extension property. Then \((P, B, D, S)\) is Poisson reducible if and only if for any \(m \in S\)
\[
B^S(\Delta_m) \subset (\Delta_m)^S_{S/D_S}
\]
where
\[
\Delta_m := \left\{ dF(m) \left| F \in C^\infty_P(U_m), dF(s)|_{D(s)} = 0 \text{ for all } s \in U_m \cap S, \text{ and for any open neighborhood } U_m \text{ of } m \text{ in } P \right\},
\]
and
\[
(\Delta_m)^S := \left\{ dF(m) \in \Delta_m \left| F|_{U_m \cap S} \text{ is constant for any open neighborhood } U_m \text{ of } m \text{ in } P \text{ and an open neighborhood } V_m \text{ of } m \text{ in } S \right\}.
\]
In particular, if \(S\) is endowed with the relative topology, then the definition of \((\Delta_m)^S\) simplifies to
\[
(\Delta_m)^S = \{ dF(m) \in \Delta_m \left| F|_{U_m \cap S} \text{ is constant for any open neighborhood } U_m \text{ of } m \text{ in } P \}.\]
When $S$ is an embedded submanifold of $P$ and $D \subset TP|_S$ is a subbundle of the tangent bundle of $P$ restricted to $S$. In this case Ortega and Ratiu in [18] gave the following proposition.

**Proposition 2.7** Assume that $(P, \{\cdot, \cdot\})$ is a Poisson manifold and $S$ is an embedded submanifold of $P$ and $D \subset TP|_S$ is a subbundle of the tangent bundle of $P$ restricted to $S$, such that $D_S = D \cap TS$ is a smooth regular integrable distribution on $S$. Then the presheaf $C^\infty_{S/D_S}$ has the $(D, D_S)$-local extension property.

Moreover, by using Theorem 2.6 and the above proposition, one can also obtain another result on the Poisson reducible conditions by distribution for Poisson manifold, which are given in Falceto and Zambon [5] and Marsden and Ratiu [13]. In particular, for some specific submanifolds, including the coisotropic and cosymplectic submanifolds, by using Proposition 2.1 one can get the following reduction theorem, see Ortega and Ratiu [18].

**Theorem 2.8** Let $(P, \{\cdot, \cdot\})$ be a Poisson manifold with Poisson tensor $B \in \Lambda^2(T^*P)$, and $S \subset P$ an embedded smooth submanifold of $P$ and the distribution $D := B^\sharp((TS)\circ ) \subset TP|_S$. If any one of the following three conditions holds,

(i) $D_S := D \cap TS$ is a smooth and integrable generalized distribution on $S$, such that the presheaf $C^\infty_{S/D_S}$ has the $(D, D_S)$-local extension property;

(ii) $S$ is coisotropic, and the presheaf $C^\infty_{S/D_S}$ has the $(D, D_S)$-local extension property;

(iii) $S$ is cosymplectic;

Then $(P, B, D, S)$ is Poisson reducible.

### 2.3 Controlled Hamiltonian System and CH-equivalence

In this subsection, we shall introduce the CH system defined by using Poisson structure on the cotangent bundle of a configuration manifold, which is a special case of the Definition 3.1 of CH system defined on a Poisson fiber bundle in [29]. We also discuss the controlled Hamiltonian equivalence (CH-equivalence) of such systems. For convenience, we assume that all controls appearing in this paper are the admissible controls.

Let $(E, M, N, \pi, G)$ be a Poisson fiber bundle and $B$ be a Poisson tensor on $E$, then we have a induced bundle map $B^\sharp: T^*E \to TE$ such that for any $\lambda_z, \nu_z \in T^*_zE$, $z \in E$, $B(\lambda_z, \nu_z) = \langle \lambda_z, B^\sharp(\nu_z) \rangle$. If $H: E \to \mathbb{R}$ is a Hamiltonian, then the Hamiltonian vector field $X_H \in TE$ can be expressed by $X_H = B^\sharp dH$, and $(E, B, H)$ is a Hamiltonian system. Moreover, if considering the external force and control, we can define a CH system on the Poisson fiber bundle $E$ as follows.

**Definition 2.9** (CH System) A CH system on $E$ is a 5-tuple $(E, B, H, F, W)$, where $(E, B, H)$ is a Hamiltonian system, and the function $H: E \to \mathbb{R}$ is called the Hamiltonian, a fiber-preserving map $F: E \to E$ is called the (external) force map, and a fiber submanifold $W$ of $E$ is called the control subset.

Sometimes, $W$ also denotes the set of fiber-preserving maps from $E$ to $W$. When a feedback control law $u: E \to W$ is chosen, the 5-tuple $(E, B, H, F, u)$ denotes a closed-loop dynamic system.

Let $Q$ be a smooth manifold, and $T^*Q$ its cotangent bundle. If $T^*Q$ has a Poisson structure $\{\cdot, \cdot\}$, we can define a Poisson tensor $B$ on $T^*Q$ such that $(T^*Q, B)$ is a Poisson vector bundle.
we take that $E = T^*Q$, from above definition we can obtain a CH system on the cotangent bundle $T^*Q$, that is, 5-tuple $(T^*Q, B, H, F, W)$. Where the fiber-preserving map $F : T^*Q \rightarrow T^*Q$ is the (external) force map, that is the reason that the fiber-preserving map $F : E \rightarrow E$ is called an (external) force map in above definition. In particular, the cotangent bundle $T^*Q$ has a canonical symplectic form $\omega$, so $(T^*Q, \omega)$ is a symplectic vector bundle. From above definition we also obtain a RCH system on the cotangent bundle $T^*Q$, that is, 5-tuple $(T^*Q, \omega, H, F, W)$, see Marsden et al. [15].

In order to describe the dynamics of the CH system $(E, B, H, F, W)$ with a control law $u$, we need to give a good expression of the dynamical vector field of CH system. At first, we introduce a notations of vertical lift maps of a vector along a fiber. For a smooth manifold $E$, its tangent bundle $TE$ is a vector bundle, and for the fiber bundle $\pi : E \rightarrow M$, we consider the tangent mapping $T\pi : TE \rightarrow TM$ and its kernel $\ker(T\pi) = \{\rho \in TE | T\pi(\rho) = 0\}$, which is a vector subbundle of $TE$. Denote by $VE := \ker(T\pi)$, which is called a vertical bundle of $E$. Assume that there is a metric on $E$, and we take a Levi-Civita connection $\mathcal{A}$ on $TE$, and denote by $HE := \ker(\mathcal{A})$, which is called a horizontal bundle of $E$, such that $TE = HE \oplus VE$. For any $x \in M$, $a_x, b_x \in E_x$, any tangent vector $\rho(b_x) \in T_{b_x}E$ can be split into horizontal and vertical parts, that is, $\rho(b_x) = \rho^h(b_x) \oplus \rho^v(b_x)$, where $\rho^h(b_x) \in H_{b_x}E$ and $\rho^v(b_x) \in V_{b_x}E$. Let $\gamma$ be a geodesic in $E_x$ connecting $a_x$ and $b_x$, and denotes by $\rho^v_\gamma(a_x)$ a tangent vector at $a_x$, which is a parallel displacement of the vertical vector $\rho^v(b_x)$ along the geodesic $\gamma$ from $b_x$ to $a_x$. Since the angle between two vectors is invariant under a parallel displacement along a geodesic, then $T\pi(\rho^v_\gamma(a_x)) = 0$, and hence $\rho^v_\gamma(a_x) \in V_{a_x}E$. Now, for $a_x, b_x \in E_x$ and tangent vector $\rho(b_x) \in T_{b_x}E$, we can define the vertical lift map of a vector along a fiber given by

$$vlift : TE_x \times E_x \rightarrow TE_x; \quad vlift(\rho(b_x), a_x) = \rho^v_\gamma(a_x).$$

It is easy to check from the basic fact in differential geometry that this map does not depend on the choice of $\gamma$. If $F : E \rightarrow E$ is a fiber-preserving map, for any $x \in M$, we have that $F_x : E_x \rightarrow E_x$ and $T F_x : TE_x \rightarrow TE_x$, then for any $a_x \in E_x$ and $\rho \in TE_x$, the vertical lift of $\rho$ under the action of $F$ along a fiber is defined by

$$(vlift(F_x)\rho)(a_x) = vlift((TF_x\rho)(F_x(a_x)), a_x) = (TF_x\rho)^v_\gamma(a_x),$$

where $\gamma$ is a geodesic in $E_x$ connecting $F_x(a_x)$ and $a_x$.

In particular, when $\pi : E \rightarrow M$ is a vector bundle, for any $x \in M$, the fiber $E_x = \pi^{-1}(x)$ is a vector space. In this case, we can choose the geodesic $\gamma$ to be a straight line, and the vertical vector is invariant under a parallel displacement along a straight line, that is, $\rho^v_\gamma(a_x) = \rho^v(b_x)$. Moreover, when $E = T^*Q$, $M = Q$, by using the local trivialization of $TT^*Q$, we have that $TT^*Q \cong TQ \times T^*Q$. Because of $\pi : T^*Q \rightarrow Q$, and $T\pi : TT^*Q \rightarrow TQ$, then in this case, for any $\alpha_x, \beta_x \in T^*_xQ$, $x \in Q$, we know that $(0, \beta_x) \in V_{\beta_x}T^*_xQ$, and hence we can get that

$$vlift((0, \beta_x)(\alpha_x)) = vlift((0, \beta_x)(\alpha_x)) = \left[\frac{d}{ds}_{|s=0}}(\alpha_x + s\beta_x),$$

which is consistent with the definition of vertical lift map along fiber in Marsden and Ratiu [14].

For a given CH System $(T^*Q, B, H, F, W)$, the dynamical vector field of the associated Hamiltonian system $(T^*Q, B, H)$ is that $X_H = B^2dH$, where $B^2 : T^*T^*Q \rightarrow TT^*Q; dH \rightarrow B^2dH$, such that for any $\lambda \in T^*T^*Q$, $B(\lambda, dH) = <\lambda, B^2dH>$. If considering the external force
$F : T^*Q \to T^*Q$, by using the above notations of vertical lift maps of a vector along a fiber, the change of $X_H$ under the action of $F$ is that

$$\text{vlift}(F)X_H(\alpha_x) = \text{vlift}((TFX_H)(F(\alpha_x)))(\alpha_x) = (TX_H)^{\gamma}(\alpha_x),$$

where $\alpha_x \in T^*_xQ$, $x \in Q$ and $\gamma$ is a straight line in $T^*_xQ$ connecting $F_x(\alpha_x)$ and $\alpha_x$. In the same way, when a feedback control law $u : T^*Q \to W$ is chosen, the change of $X_H$ under the action of $u$ is that

$$\text{vlift}(u)X_H(\alpha_x) = \text{vlift}((TuX_H)(u(\alpha_x)))(\alpha_x) = (TX_H)^{\gamma}(\alpha_x).$$

In consequence, we can give an expression of the dynamical vector field of CH system as follows.

**Proposition 2.10** The dynamical vector field of a CH system $(T^*Q, B, H, F, W)$ with a control law $u$ is the synthetic of Hamiltonian vector field $X_H$ and its changes under the actions of the external force $F$ and control $u$, that is,

$$X_{(T^*Q, B, H, F, u)}(\alpha_x) = X_H(\alpha_x) + \text{vlift}(F)X_H(\alpha_x) + \text{vlift}(u)X_H(\alpha_x),$$

for any $\alpha_x \in T^*_xQ$, $x \in Q$. For convenience, it is simply written as

$$X_{(T^*Q, B, H, F, u)} = B^2dH + \text{vlift}(F) + \text{vlift}(u). \tag{2.3}$$

We also denote that $\text{vlift}(W) = \bigcup\{\text{vlift}(u)X_H | u \in W\}$. For the CH system $(E, B, H, F, W)$ with a control law $u$, we have also a similar expression of its dynamical vector field. It is worthy of note that in order to deduce and calculate easily, we always use the simple expression of dynamical vector field $X_{(T^*Q, B, H, F, u)}$.

Next, we note that when a CH system is given, the force map $F$ is determined, but the feedback control law $u : T^*Q \to W$ could be chosen. In order to describe the feedback control law to modify the structure of CH system, we give the controlled Hamiltonian matching conditions and CH-equivalence as follows.

**Definition 2.11** (CH-equivalence) Suppose that we have two CH systems $(T^*Q_i, B_i, H_i, F_i, W_i)$, $i = 1, 2$, we say them to be CH-equivalent, or simply, $(T^*Q_1, B_1, H_1, F_1, W_1) \underset{\text{CH}}{\sim} (T^*Q_2, B_2, H_2, F_2, W_2)$, if there exists a diffeomorphism $\varphi : Q_1 \to Q_2$, such that the following Hamiltonian matching conditions hold:

**HM-1:** The cotangent lift map of $\varphi$, that is, $\varphi^* : T^*Q_2 \to T^*Q_1$ is a Poisson map, and $W_1 = \varphi^*(W_2)$;

**HM-2:** $\text{Im}[B_i^2dH_1 + \text{vlift}(F_1) - T\varphi^*(B_2^2dH_2) - \text{vlift}((\varphi^*F_2)\varphi_*)] \subset \text{vlift}(W_1)$, where the map $\varphi_* = (\varphi^{-1})^* : T^*Q_1 \to T^*Q_2$, $T\varphi^* : TT^*Q_2 \to TT^*Q_1$ is the tangent map of $\varphi^*$ and $\text{Im}$ means the pointwise image of the map in brackets.

It is worthy of note that our CH-system is defined by using the Poisson tensor on the cotangent bundle of a configuration manifold, we must keep with the Poisson structure when we define the CH-equivalence, that is, the induced equivalent map $\varphi^*$ is Poisson on the cotangent bundle. The following Theorem 2.12 explains the significance of the above CH-equivalence relation, the proof is given in [29].

**Theorem 2.12** Suppose that two CH systems $(T^*Q_i, B_i, H_i, F_i, W_i)$, $i = 1, 2$, are CH-equivalent, then there exist two control laws $u_i : T^*Q_i \to W_i$, $i = 1, 2$, such that the two associated closed-loop systems produce the same equations of motion, that is, $X_{(T^*Q_1, B_1, H_1, F_1, u_1)} \cdot \varphi^* = T\varphi^*X_{(T^*Q_2, B_2, H_2, F_2, u_2)}$. Moreover, the explicit relation between the two control laws $u_i, i = 1, 2$, is given by

$$\text{vlift}(u_1) - \text{vlift}((\varphi^*u_2)\varphi_*) = -B_i^2dH_1 - \text{vlift}(F_1) + T\varphi^*(B_2^2dH_2) + \text{vlift}((\varphi^*F_2)\varphi_*). \tag{2.4}$$
3 Poisson Reduction of CH System by Controllability Distribution

In this section, we first give a definition of controllability distribution of CH system, then give a theorem to show the Poisson reducible conditions of CH system by controllability distribution. Moreover, we prove that the property of Poisson reduction for CH system by controllability distribution leaves invariant under the CH-equivalence.

**Definition 3.1**
For a CH system \((T^*Q, B, H, F, W)\), the fiber submanifold \(W\) of \(T^*Q\) is its control subset. If for each \(u \in W\), the dynamical control system \(\dot{x} = X_{(T^*Q, B, H, F, u)}\) is controllable,\(\) that is, for any two states \(x_0\) and \(x_1\) of this system, there is a finite piecewise smooth integral curve \(x(t)\) of \(\dot{x} = X_{(T^*Q, B, H, F, u)}\), \(t \in [0, 1]\), such that \(x(0) = x_0\) and \(x(1) = x_1\). Then \(W\) is called a controllability submanifold of CH system.

**Definition 3.2**
Assume that \(W\) is a controllability submanifold of CH system \((T^*Q, B, H, F, W)\), and a distribution \(D \subset T(T^*Q)|_W\), and it is a Poisson integrable generalized distribution, then \(D\) is called a controllability distribution of CH system.

For a CH system \((T^*Q, B, H, F, W)\), assume that \(W\) is its controllability submanifold and \(D \subset T(T^*Q)|_W\) is its controllability distribution, such that \(D_W = D \cap TW\) is a smooth regular integrable distribution on \(W\). If the presheaf \(C^\infty_{W/D_W}\) has the \((D, D_W)\)-local extension property, then we may consider that \((T^*Q, B, D, W)\) is Poisson reducible if \((W/D_W, C^\infty_{W/D_W}, B_{W/D_W})\) is a well-defined presheaf of Poisson algebras. Thus, by using the Poisson reduction by controllability distribution \(D\) for the phase space of CH system, we can define the Poisson reducible CH system by controllability distribution as follows.

**Definition 3.3**
A CH system \((T^*Q, B, H, F, W)\) is called to be Poisson reducible by controllability distribution \(D\), if \((T^*Q, B, D, W)\) is Poisson reducible in the sense of Definition 2.5.

From above definition and Theorem 2.6, we can obtain the following theorem.

**Theorem 3.4**
Suppose that \((T^*Q, B, H, F, W)\) is a CH system, and \(W\) is its controllability submanifold and \(D \subset T(T^*Q)|_W\) is its controllability distribution such that \(D_W = D \cap TW\) is a smooth regular integrable distribution on \(W\), and the presheaf \(C^\infty_{W/D_W}\) has the \((D, D_W)\)-local extension property. Then the CH system \((T^*Q, B, H, F, W)\) is Poisson reducible by controllability distribution \(D\), if and only if for any \(z \in W\)

\[
B^z(\Delta_z) \subset (\Delta_z)_W^o
\]

where

\[
\Delta_z := \left\{ df(z) \mid f \in C^\infty_{T^*Q}(U_z), \quad df(y)|_{D(y)} = 0, \quad \text{for all } y \in U_z \cap W, \quad \text{and for any open neighborhood } U_z \text{ of } z \text{ in } T^*Q \right\},
\]

and

\[
(\Delta_z)_W := \left\{ df(z) \in \Delta_z \mid f|_{U_z \cap W} \text{ is constant for any open neighborhood } U_z \text{ of } z \text{ in } T^*Q \right\}.
\]
Proof. In fact, we take that $P = T^*Q$ and $S = W$, the conclusion is a direct consequence of Theorem 2.6. □

It is worthy of note that for convenience, here and in subsequent sections $W$ is endowed with the relative topology. For the general case that the topology of $W$ is stronger than the relative topology, we can also obtain the similar result. Moreover, if considering the CH-equivalence of CH system, we can get the following theorem to state that the property of Poisson reduction for CH system by controllability distribution leaves invariant under CH-equivalence conditions.

**Theorem 3.5** Suppose that two CH systems $(T^*Q_i, B_i, H_i, F_i, W_i)$, $i = 1, 2$, are CH-equivalent with equivalent map $\varphi^*: T^*Q_2 \to T^*Q_1$. Then we have that

(i) $W_1$ is controllability submanifold of CH system $(T^*Q_1, B_1, H_1, F_1, W_1)$ if and only if $W_2$ is controllability submanifold of CH system $(T^*Q_2, B_2, H_2, F_2, W_2)$.

(ii) $D_1 = T\varphi^*(D_2) \subset TT^*Q_1|_{W_1}$ is controllability distribution of CH system $(T^*Q_1, B_1, H_1, F_1, W_1)$, such that $D_{W_1} = D_1 \cap TW_1$ is smooth regular integrable distributions on $W_1$ and the presheaf $C^\infty_{W_1/D_{W_1}}$ has the $(D_1, D_{W_1})$-local extension property if and only if $D_2 \subset TT^*Q_2|_{W_2}$ is controllability distribution of CH system $(T^*Q_2, B_2, H_2, F_2, W_2)$, such that $D_{W_2} = D_2 \cap TW_2$ is smooth regular integrable distributions on $W_2$ and the presheaf $C^\infty_{W_2/D_{W_2}}$ has the $(D_2, D_{W_2})$-local extension property.

(iii) CH system $(T^*Q_1, B_1, H_1, F_1, W_1)$ is Poisson reducible by controllability distribution $D_1$ if and only if CH system $(T^*Q_2, B_2, H_2, F_2, W_2)$ is Poisson reducible by controllability distribution $D_2$.

Proof. (i) Because CH systems $(T^*Q_i, B_i, H_i, F_i, W_i)$, $i = 1, 2$, are CH-equivalent, then there exists a diffeomorphism $\varphi: Q_1 \to Q_2$, such that the cotangent lift map $\varphi^*: T^*Q_2 \to T^*Q_1$ and the tangent lift map $T\varphi^*: TT^*Q_2 \to TT^*Q_1$ are vector bundle isomorphism, and by Theorem 2.12 there exist always a pair of control laws $u_i: T^*Q_i \to W_i$, $i = 1, 2$, such that $X(T^*Q_1, B_1, H_1, F_1, u_1) \cdot \varphi^* = T\varphi^*X(T^*Q_2, B_2, H_2, F_2, u_2)$. Note that $W_1 = \varphi^*(W_2)$, hence from Definition 3.1 we know that (i) holds.

(ii) Notice that $D_1 = T\varphi^*(D_2)$, and $D_2 \subset TT^*Q_2|_{W_2}$. Thus,

$$D_{W_1} = D_1 \cap TW_1 = T\varphi^*(D_2) \cap T\varphi^*(TW_2) = T\varphi^*(D_2 \cap TW_2) = T\varphi^*(D_{W_2}).$$

Since $\varphi^*$ and $T\varphi^*$ are vector bundle isomorphism, and $\varphi^*$ is Poisson, from Definition 2.4, Definition 3.2 and the definition of the presheaf local extension property, we know that (ii) holds.

(iii) If CH system $(T^*Q_1, B_1, H_1, F_1, W_1)$ is Poisson reducible by controllability distribution $D_1$, we shall prove that the CH system $(T^*Q_2, B_2, H_2, F_2, W_2)$ is Poisson reducible by controllability distribution $D_2$. By the above conclusions (i), (ii) and Theorem 3.4, it suffices to show that for any $z_2 \in W_2$, we have that $B_2^1(\Delta^2_{z_2}) \subset (\Delta^2_{z_2})_{W_2}$. In fact, assume that for any $\alpha_2 = df_2(z_2) \in \Delta^2_{z_2}$, that is, $f_2 \in C^\infty_{Q_2}(U_{z_2})$, where $U_{z_2}$ is an open neighborhood of $z_2$ in $T^*Q_2$, such that $U_{z_2} \cap W_2 = D_{W_2}$-invariant and $df_2(s_2)|_{D_2(s_2)} = 0$, for all $s_2 \in U_{z_2} \cap W_2$. Since two CH systems $(T^*Q_i, B_i, H_i, F_i, W_i)$, $i = 1, 2$, are CH-equivalent, and the equivalent map $\varphi^*: T^*Q_2 \to T^*Q_1$ is a Poisson diffeomorphism, then for $z_1 \in T^*Q_1$, such that $z_2 = \varphi^*(z_1)$, there is an open neighborhood $U_{z_1}^1$ of $z_1$ in $T^*Q_1$, such that $U_{z_2}^2 = \varphi^*(U_{z_2}^1)$ and $U_{z_2}^1 \cap W_1 = D_{W_1}$-invariant,
and there exists \( f_1 \in C^\infty T^*Q_1(U^1_{z_1}) \), such that \( f_2 = f_1 \cdot \varphi^* \) and \( df_1(s_1)|_{D_1(s_1)} = 0 \), for all \( s_1 \in U^1_{z_1} \cap W_1 \), and \( \alpha_1 = df_1(z_1) \in \Delta^1_{z_1} \). In the same way, for any \( \beta_2 = dg_2(z_2) \in (\Delta^2_{z_2})_{W_2} \), that is, \( g_2 \in C^\infty T^*Q_2(U^2_{z_2}) \) and \( g_2|_{U^2_{z_2} \cap W_2} \) is constant for any open neighborhood \( U^2_{z_2} \) of \( z_2 \) in \( T^*Q_2 \), then there is a \( g_1 \in C^\infty T^*Q_1(U^1_{z_1}) \), such that \( g_2 = g_1 \cdot \varphi^* \) and \( g_1|_{U^2_{z_2} \cap W_1} \) is constant for the corresponding open neighborhood \( U^1_{z_1} \) of \( z_1 \) in \( T^*Q_1 \), and \( \beta_1 = dg_1(z_1) \in (\Delta^1_{z_1})_{W_1} \). Because CH system \( (T^*Q_1, B_1, H_1, F_1, W_1) \) is Poisson reducible by controllability distribution \( D_1 \), from the above conclusions (i), (ii) and Theorem 3.4 we have that \( B_2^\infty(\Delta^1_{z_1}) \subset (\Delta^1_{z_1})_{W_1} \). It follows that

\[
\{f_1, g_1\}_{B_1}(z_1) = \langle dg_1(z_1), B_2^\infty(df_1(z_1)) \rangle = \langle \beta_1, B_2^\infty(\alpha_1) \rangle = 0.
\]

Notice that the map \( \varphi^*: T^*Q_2 \to T^*Q_1 \) is Poisson, we have that

\[
\langle \beta_2, B_2^\infty(\alpha_2) \rangle = \langle dg_2(z_2), B_2^\infty(df_2(z_2)) \rangle = \{f_2, g_2\}_{B_2}(z_2) = \{f_1 \cdot \varphi^*, g_1 \cdot \varphi^*\}_{B_2}(z_2) = (\varphi^*)^*\{f_1, g_1\}_{B_1}(z_1) = 0.
\]

Thus, we prove that \( B_2^\infty(\alpha_2) \in (\Delta^2_{z_2})_{W_2} \) and \( B_2^\infty(\beta_2) \in (\Delta^2_{z_2})_{W_2} \), and hence CH system \( (T^*Q_2, B_2, H_2, F_2, W_2) \) is Poisson reducible by controllability distribution \( D_2 \).

Conversely, if CH system \( (T^*Q_2, B_2, H_2, F_2, W_2) \) is Poisson reducible by controllability distribution \( D_2 \), by using the same way we can verify that for any \( z_1 \in W_1 \), we have that \( B_1^\infty(\Delta^1_{z_1}) \subset (\Delta^1_{z_1})_{W_1} \). Thus, the CH system \( (T^*Q_1, B_1, H_1, F_1, W_1) \) is Poisson reducible by controllability distribution \( D_1 \) by the above conclusions (i), (ii) and Theorem 3.4. ■

### 4 Poisson Reduction of Symmetric CH System by G-invariant Controllability Distribution

In this section, we shall consider the symmetric CH system and give the Poisson reducible conditions of this system by \( G \)-invariant controllability distribution.

Let \( Q \) be a smooth manifold and \( T^*Q \) its cotangent bundle with a Poisson tensor \( B \). Let \( \Phi : G \times Q \to Q \) be a smooth left action of the Lie group \( G \) on \( Q \), and the cotangent lifted left action \( \Phi^*: G \times T^*Q \to T^*Q \) is Poisson. If Hamiltonian \( H: T^*Q \to \mathbb{R} \) is \( G \)-invariant, then the 4-tuple \( (T^*Q, G, B, H) \) is a symmetric Hamiltonian system. Moreover, if the external force map \( F: T^*Q \to T^*Q \) and control subset \( W \) of \( T^*Q \) are \( G \)-invariant, then we can give the definition of symmetric CH system on \( T^*Q \) as follows.

**Definition 4.1 (Symmetric CH System)** A **symmetric CH system** on \( T^*Q \) is a 6-tuple \( (T^*Q, G, B, H, F, W) \), where the action of \( G \) on \( T^*Q \) is Poisson, and the Hamiltonian \( H: T^*Q \to \mathbb{R} \), the external force map \( F: T^*Q \to T^*Q \), and the control subset \( W \) of \( T^*Q \) are \( G \)-invariant.

For a symmetric CH system \( (T^*Q, G, B, H, F, W) \), if the \( G \)-invariant control subset \( W \) is a controllability submanifold of the symmetric CH system, then \( W \) is called a **G-invariant controllability submanifold** of this system; if a \( G \)-invariant distribution \( D \subset T(T^*Q)|_W \) is a controllability distribution of the symmetric CH system, then \( D \) is called a **G-invariant controllability distribution** of this system, simply written as \( D^G \). In this case \( D_W = D^G \cap TW \) is also \( G \)-invariant distribution on \( W \). For the presheaf \( C^\infty_W(D^G_W) \), the \((D^G, D_W)\)-local extension property shows that \( F \) is a local \( D^G \)-invariant extension of \( f \circ \pi^{-1}_{D_W} \) at the point \( m \in \pi^{-1}_{D_W}(V) \),
for any open $G$-invariant subset of $W/D_W$, that is, the local extensions of $f \circ \pi_{D_W}$ defined in (2.1) can always be chosen to satisfy
\[ dF(n)|_{DG(n)} = 0, \text{ for any } n \in \pi_{D_W}^{-1}(V) \cap U_m. \] (4.1)

From Definition 2.4 it is easy to see that the $G$-invariant distribution $D^G$ is Poisson if the condition $d_f|_{DG} = d|_{DG} = 0$, for any $f, g \in C^{\infty}_T(U)^G$, $U \subset T^*Q$ an open $G$-invariant subset, implies that $d\{f, g\}|_{DG} = 0$. Moreover, assume that the $G$-invariant distribution $D^G$ is Poisson, and the presheaf $C^{\infty}_W/D_W$ has the $(D^G, D_W)$-local extension property. If $F$ and $G \in C^{\infty}_T(U)^G$ are local $D^G$-invariant extensions of $f \circ \pi_{D_W}$ and $g \circ \pi_{D_W}$ at $m \in \pi_{D_W}^{-1}(V)$, then their Poisson bracket $\{F, G\}$ is also local $D^G$-invariant extension.

Now for a symmetric CH system $(T^*Q, G, B, H, F, W)$, assume that $W$ is its $G$-invariant controllability submanifold and $D^G \subset TT^*Q|_W$ is its $G$-invariant controllability distribution, such that $D_W = D^G \cap TW$ is a smooth regular $G$-invariant integrable distribution on $W$, and the presheaf $C^{\infty}_W/D_W$ has the $(D^G, D_W)$-local extension property, then we may consider that $(T^*Q, B, D^G, W)$ is Poisson reducible if $(W/D_W, C^{\infty}_W/D_W, B^{W/D_W})$ is a well-defined presheaf of Poisson algebras. Thus, by using the Poisson reduction by $G$-invariant controllability distribution $D^G$ for the phase space of symmetric CH system, we can define the Poisson reducible symmetric CH system by $G$-invariant controllability distribution as follows.

**Definition 4.2** A symmetric CH system $(T^*Q, G, B, H, F, W)$ is called to be Poisson reducible by $G$-invariant controllability distribution $D^G$, if $(T^*Q, B, D^G, W)$ is Poisson reducible by $G$-invariant distribution $D^G$ in the sense of Definition 2.5, that is, $(W/D_W, C^{\infty}_W/D_W, B^{W/D_W})$ is a well-defined presheaf of Poisson algebras, where the Poisson tensor $B^{W/D_W}$ is defined by, for any open $G$-invariant set $V \subset W/D_W$, the Poisson bracket $\{\cdot, \cdot\}^{W/D_W}_V : C^{\infty}_W(D^G)_V \times C^{\infty}_W(D^G)_V \to C^{\infty}_W(D^G)_V$ is given by
\[ \{k, l\}^{W/D_W}_V(\pi_{D_W}(m)) = \{K, L\}_B(m) \]
for any $m \in \pi_{D_W}^{-1}(V)$ and any local $D^G$-invariant extensions $K, L \in C^{\infty}_T(U)^G$ of $k \circ \pi_{D_W}$ and $l \circ \pi_{D_W}$ at $m$, respectively, where $\pi_{D_W} : W \to W/D_W$ is the projection, and $U_m$ is an open $G$-invariant neighborhood of $m$, such that $U_m \cap W$ is $D_W$-invariant.

By using the above definition and Proposition 2.2, we can obtain the following theorem to give the Poisson reducible conditions of symmetric CH system by $G$-invariant controllability distribution.

**Theorem 4.3** Suppose that $(T^*Q, G, B, H, F, W)$ is a symmetric CH system, $W$ is $G$-invariant controllability submanifold and and $D^G \subset TT^*Q|_W$ is $G$-invariant controllability distribution, such that $D_W = D^G \cap TW$ is a smooth regular $G$-invariant integrable distribution on $W$, and the presheaf $C^{\infty}_W/D_W$ has the $(D^G, D_W)$-local extension property. Then the symmetric CH system $(T^*Q, G, B, H, F, W)$ is Poisson reducible by $G$-invariant controllability distribution $D^G$, if and only if for any $z \in W$
\[ B^G(\Delta^G_z) \subset (\Delta^G_z)_W \] (4.2)
where
\[ \Delta^G_z := \left\{ f \in C^{\infty}_T(U_z)^G, \quad d_f(y)|_{D^G(y)} = 0, \text{ for all } y \in U_z \cap W, \text{ and } \right. \]
for any open $G$-invariant neighborhood $U_z$ of $z$ in $T^*Q$.
and

\[(\Delta_z^G)_W := \left\{ df(z) \in \Delta_z^G \left| f|_{U_z \cap W} \text{ is constant for any open } G\text{-invariant neighborhood } U_z \text{ of } z \text{ in } T^*Q \right. \right\}.\]

**Proof.** If the symmetric CH system \((T^*Q, G, B, H, F, W)\) is Poisson reducible by \(G\)-invariant controllability distribution \(D^G\), then \((W/D_W, C^\infty_{W/D_W}B_{W/D_W})\) is a well-defined presheaf of Poisson algebras and for any open \(G\)-invariant subset \(V \subset W/D_W\), the Poisson tensor \(B^{W/D_W}\) is defined by the following Poisson bracket

\[\{k, l\}^{W/D_W}_{V}(\pi_{D_W}(z)) = \{K, L\}_B(z) \quad (4.3)\]

for any \(z \in \pi^{-1}_{D_W}(V)\) and any local \(D^G\)-invariant extensions \(K, L \in C^\infty_{T^*Q}U_z^G\) of \(k \cdot \pi_{D_W}\) and \(l \cdot \pi_{D_W}\) at \(z\), respectively, where \(\pi_{D_W} : W \to W/D_W\) is the projection, and \(U_z\) is an open \(G\)-invariant neighborhood of \(z\), such that \(U_z \cap W\) is \(D_W\)-invariant. We shall prove that the inclusion \((4.2)\) holds. For any \(\alpha = dK(z) \in \Delta_z^G\), that is, \(K \in C^\infty_{T^*Q}U_z^G\), where \(U_z\) is an open \(G\)-invariant neighborhood of \(z\) in \(T^*Q\), such that \(U_z \cap W\) is \(D_W\)-invariant and \(dK(s)|_{D_W(s)} = 0\), for all \(s \in U_z \cap W\). In the same way, for any \(\beta = dL(z) \in \Delta_z^G\), that is, \(L \in C^\infty_{T^*Q}U_z^G\) and \(L|_{U_z \cap W}\) is constant for the open \(G\)-invariant neighborhood \(U_z\) of \(z\) in \(T^*Q\). Since \(K|_{U_z \cap W}\) and \(L|_{U_z \cap W}\) are \(D_W\)-invariant, they have constant values on the leaves of \(D_W\). Note that \(W\) is endowed with the relative topology, and \(U_z \cap W\) is open in \(W\). Thus, we can take that the open set \(V := \pi_{D_W}(U_z \cap W) \subset W/D_W\), and define functions \(k, l \in C^\infty_{W/D_W}(V)\) by using \(K, L \in C^\infty_{T^*Q}U_z^G\), such that \(k \cdot \pi_{D_W}|_{\pi^{-1}_{D_W}(V)\cap U_z} = K|_{\pi^{-1}_{D_W}(V)\cap U_z}\) and \(l \cdot \pi_{D_W}|_{\pi^{-1}_{D_W}(V)\cap U_z} = L|_{\pi^{-1}_{D_W}(V)\cap U_z}\). Because \(L\) is constant on \(U_z \cap W\), hence the function \(l\) is constant on the neighborhood \(V = \pi_{D_W}(U_z \cap W)\) of \(\pi_{D_W}(z)\) in \(W/D_W\), and we have that \(\{k, l\}^{W/D_W}_{V}(\pi_{D_W}(z)) = 0\). In consequence, from \((4.3)\) we have that

\[< \beta, B^2(\alpha) >= < dL(z), B^2(dK(z)) >= \{K, L\}_B(z) = \{k, l\}^{W/D_W}_{V}(\pi_{D_W}(z)) = 0.\]

Thus, we prove that \(B^2(\alpha) \in (\Delta_z^G)_{W}\) and obtain the desired inclusion \(B^2(\Delta_z^G) \subset (\Delta_z^G)_{W}\).

Conversely, if assume that the inclusion \((4.2)\) holds, we shall prove that the symmetric CH system \((T^*Q, G, B, H, F, W)\) is Poisson reducible by \(G\)-invariant controllability distribution \(D^G\), that is, \((W/D_W, C^\infty_{W/D_W}B_{W/D_W})\) is a well-defined presheaf of Poisson algebras. For any open \(G\)-invariant subset \(V \subset W/D_W\), and functions \(k, l \in C^\infty_{W/D_W}(V)\), as well as \(K, L \in C^\infty_{T^*Q}U_z^G\), which are the local \(D^G\)-invariant extensions of \(k \cdot \pi_{D_W}\) and \(l \cdot \pi_{D_W}\) at \(z \in \pi^{-1}_{D_W}(V)\), respectively. We define the Poisson tensor \(B^{W/D_W}\) by the Poisson bracket

\[\{\cdot, \cdot\}^{W/D_W}_V : C^\infty_{W/D_W}(V) \times C^\infty_{W/D_W}(V) \to C^\infty_{W/D_W}(V),\]

which is given by \((4.3)\). We shall prove that this Poisson bracket has the property of local \(D^G\)-invariant extension, that is,

\[\{k, l\}^{W/D_W}_{V} \cdot \pi_{D_W}|_{\pi^{-1}_{D_W}(V)\cap U_z} = \{K, L\}_B|_{\pi^{-1}_{D_W}(V)\cap U_z}.\]

In order to do this, we have to check that \(\{K, L\}_B(z)\) doesn’t depend on the choice of the point \(z \in \pi^{-1}_{D_W}(V)\) and the local extensions \(K\) and \(L\). Because \(D^G\) is a \(G\)-invariant controllability distribution, it is a \(G\)-invariant Poisson integrable generalized distribution, if the presheaf \(C^\infty_{W/D_W}\) has the \((D^G, D_W)\)-local extension property, and \(K\) and \(L \in C^\infty_{T^*Q}U_z^G\) are local \(D^G\)-invariant extensions of \(k \circ \pi_{D_W}\) and \(l \circ \pi_{D_W}\) at \(z \in \pi^{-1}_{D_W}(V)\), then their Poisson bracket \(\{K, L\}\)
is also local $D^G$-invariant extension. Thus the function $\{K, L\}|_{U_z \cap W}$ is constant along the integral curves of any section of $D_W$. Take $z$, $z' \in T^*Q$, such that $\pi_D W(z) = \pi_D W(z')$. Assume that $K'$, $L' \in C^\infty_T(U_z)^G$, are the local $D^G$-invariant extensions of $k \cdot \pi_D W$ and $l \cdot \pi_D W$ at $z' \in \pi_D^{-1}_W(V)$, respectively, where $U_z'$ is an open $G$-invariant neighborhood of $z'$ in $T^*Q$, such that $U_z' \cap W$ is $D_W$-invariant. Since $z'$ can be connected to $z$ by a finite union of integral curves of sections of $D_W$, and $U_z \cap W$ and $U_z' \cap W$ are both $D_W$-invariant, it follows that $U_z \cap U_z' \cap W$ contains $z$ and $z'$ and it is also $D_W$-invariant. Thus, $\{K', L'\}_B(z) = \{K', L'\}_B(z')$.

Next, we shall check that $\{K', L'\}_B(z) = \{K, L\}_B(z)$, that is, that $\{K, L\}_B(z)$ doesn’t depend on the choice of the extensions $K$ and $L$. Because of the antisymmetry of $\{\cdot, \cdot\}_B$, it suffices to show that it doesn’t depend on the choice of the extension $L$. Let $L' \in C^\infty_T(U_z)^G$, be another local $D^G$-invariant extension of $l \cdot \pi_D W$. Then we have that $(L - L')|_{\pi_D^{-1}_W(V) \cap U_z} = 0$, and $d(L - L')|_{\pi_D(s)} = dL|_{\pi_D(s)} - dL'|_{\pi_D(s)} = 0$, for any $s \in U_z \cap W$. Hence $d(L - L')(z) \in (\Delta^G_W)$. Notice that $dK(z) \in \Delta^G$ by definition, and by using the inclusion (4.2), we have that

$$\{K, L - L'\}_B(z) = d(L - L')(z), B^2(dK(z)) = 0,$$

that is, $\{K, L\}_B(z) = \{K, L'\}_B(z)$. At last, the Leibniz and Jacobi identities for $\{\cdot, \cdot\}_V^{W/DW}$ follow directly from the definition of $\{\cdot, \cdot\}_V^{W/DW}$ and the fact that $\{\cdot, \cdot\}_B$ satisfies these identities.

\section{Singular Poisson Reduction of CH System with Symmetry}

In this section, we shall introduce the singular Poisson reduction of CH system with symmetry $(T^*Q, G, B, H, F, W)$ and its $SP$-reduced system $(M^{(K)}, B^{(K)}, h^{(K)}, J^{(K)}, W^{(K)})$, as well as $SPR$-CH-equivalence by using the controlled Hamiltonian method given in Marsden et al. [15]. These are a generalization of the regular Poisson reduction of CH system with symmetry given in [29] to the singular context. We shall also follow the notations and conventions for singular reduction of a differential space introduced in Jotz, Ratiu and Śniatycki [8], Pflaum [19], Sjamaar and Lerman [20].

Let $Q$ be a smooth manifold and $T^*Q$ its cotangent bundle with a associated Poisson bracket $\{\cdot, \cdot\}$ and Poisson tensor $B$. Let $\Phi : G \times Q \rightarrow Q$ be a smooth left action of the Lie group $G$ on $Q$, and the cotangent lifted left action $\Phi^\pi : \times T^*Q \rightarrow T^*Q$ is canonical, and proper, but may not be free. Thus, the orbit space $T^*Q/G$ is not necessarily smooth manifold, but just a stratified topological space, and the projection $\pi_G : T^*Q \rightarrow T^*Q/G$ is a surjective submersion. In the following we shall describe the structure of the orbit space $T^*Q/G$. For a closed Lie subgroup $K$ of $G$, we define the isotropy type set $(T^*Q)_K = \{m \in T^*Q | G_m = K\}$, where $G_m = \{g \in G | gm = m\}$ is the isotropy subgroup of $m \in T^*Q$. Since the $G$-action on $T^*Q$ is proper, all isotropy groups are compact, and the sets $(T^*Q)_K$, where $K$ ranges over the closed Lie subgroups of $G$ for which $(T^*Q)_K$ is nonempty, form a partition of $T^*Q$. Define the normalizer of $K$ in $G$ by $N(K) := \{g \in G | gKg^{-1} = K\}$, $N(K)$ is a closed Lie subgroup of $G$. Since $K$ is a normal subgroup of $N(K)$, the quotient group $N(K)/K$ is a Lie group. If $m \in (T^*Q)_K$, we have that $G_m = K$, and for all $g \in G$, $G_{gm} = gKg^{-1}$. Thus, $gm$ lies in $(T^*Q)_K$ if and only if $g \in N(K)$, and the action of $G$ on $T^*Q$ restricts to an action of $N(K)$ on $(T^*Q)_K$, which induces a free and proper action of $N(K)/K$ on $(T^*Q)_K$.

Define the orbit type set $(T^*Q)_K = \{m \in T^*Q | G_m \in (K)\}$, where $(K)$ is set of $K$ conjugate classes. Then, $(T^*Q)_K = \{gm | g \in G, m \in (T^*Q)_K\} = \pi_G^{-1}(\pi_G((T^*Q)_K))$. In these cases, the
connected components of \((T^*Q)_K\) and \((T^*Q)_{(K)}\) are embedded submanifolds of \(T^*Q\), therefore \((T^*Q)_K\) is an isotropy type submanifold and \((T^*Q)_{(K)}\) is an orbit type submanifold. Moreover, 

\[ \pi_G((T^*Q)_{(K)}) = \{ gm|m \in (T^*Q)_K \}/G = (T^*Q)_K/N(K) = (T^*Q)_K/(N(K)/K). \]

But the action of \(N(K)/K\) on \((T^*Q)_K\) is free and proper, which implies that \((T^*Q)_K/(N(K)/K)\) is a quotient manifold. Thus, \(\pi_G((T^*Q)_{(K)})\) is a manifold contained in the orbit space \(T^*Q/G\).

From Jotz et al [8] we know that a partition of the orbit space \(T^*M\) and stratification of \(\pi_T\) the free and proper \(G\) action of \(M\), so it induces a flow induced by the bracket of \(G\). If the orbit type stratification of \(T^*Q/G\) is called the orbit type stratification of the orbit space, and the strata, denoted by \(M^{(K)}\), which are the connected components of \(\pi_G((T^*Q)_{(K)})\) of such stratification, are orbits of the family of all vector fields on \(T^*Q/G\). Moreover, the each stratum \(M^{(K)}\) is a Poisson manifold with the Poisson bracket \(\{\cdot,\cdot\}^{(K)}\) uniquely characterized by the relation

\[ \{f_K, g_K\}^{(K)} = \{f_K \cdot \pi^{(K)}(g_K \cdot \pi^{(K)}(\alpha)) = \{f_K \cdot \pi^{(K)}(g_K \cdot \pi^{(K)}(\alpha)) \}B(\alpha), \forall \alpha \in T^*Q, f_K, g_K \in C^\infty(M^{(K)}), \tag{5.1} \]

which is a induced Poisson structure by that of \(T^*Q\), such that the projection \(\pi^{(K)}: T^*Q \to M^{(K)}\) is a Poisson map. On the other hand, from Ortega and Ratiu [15], we know that for the free and proper \(G\)-action on \(T^*Q\), the orbit space \((T^*Q)/G\) is Poisson diffeomorphic to a Poisson fiber bundle with respect to its natural induced Poisson bracket, if a connection \(A\) on the \(G\)-principal bundle \(\pi: Q \to Q/G\) is introduced. We assume that the bundle structures of \(T^*Q\) and \((T^*Q)/G\) is compatible with the stratification of \(T^*Q/G\), such that the each stratum \(M^{(K)}\) is a Poisson fiber bundle with Poisson tensor \(B^{(K)}\), where \(B^{(K)}\) denotes the Poisson tensor induced by the bracket \(\{\cdot,\cdot\}^{(K)}\) on \(M^{(K)}\). From (5.1) we have that \(\pi^{(K)} \cdot B^{(K)} = B\). The pair \((M^{(K)}, B^{(K)})\) is called the singular Poisson reduced stratum of \((T^*Q, B)\).

For the Hamiltonian systems associated to \(G\)-invariant and stratified Hamiltonian functions, we can naturally reduce to Hamiltonian systems on these strata. In fact, if \(H: T^*Q \to \mathbb{R}\) is a \(G\)-invariant and stratified Hamiltonian, the flow \(f_t\) of the Hamiltonian vector field \(X_H\) leaves the orbit type stratification of the orbit space \(T^*Q/G\) invariant and commutes with the \(G\)-action, so it induces a flow \(f_t^{(K)}\) on each stratum \(M^{(K)}\), defined by \(f_t^{(K)} \cdot \pi^{(K)} = \pi^{(K)} \cdot f_t\), and the vector field \(X_{H^{(K)}}\) generated by the flow \(f_t^{(K)}\) on \((M^{(K)}, B^{(K)})\) is Hamiltonian with the associated singular Poisson reduced Hamiltonian function \(h^{(K)}: M^{(K)} \to \mathbb{R}\) defined by \(h^{(K)} \cdot \pi^{(K)} = H\) and the Hamiltonian vector fields \(X_H^{(K)} = \pi^{(K)}\cdot H\) are \(\pi^{(K)}\)-related. Moreover, if consider that the external force map \(F: T^*Q \to T^*Q\) and the controlled subset \(W\) are \(G\)-invariant and stratified, then we can introduce a kind of the singular Poisson reducible CH systems as follows.

**Definition 5.1** (Singular Poisson Reducible CH system) A 6-tuple \((T^*Q, G, B, H, F, W)\), where the Hamiltonian \(H: T^*Q \to \mathbb{R}\), the fiber-preserving map \(F: T^*Q \to T^*Q\) and the fiber submanifold \(W\) of \(T^*Q\) are all \(G\)-invariant and stratified, is called a singular Poisson reducible CH system, if for the isotropy subgroup \(K \subset G\) of the \(G\)-action on \(T^*Q\), the connected components of \(\pi_G((T^*Q)_{(K)})\) form a corresponding orbit type stratification of the orbit space \(T^*Q/G\), and there exists the singular Poisson reduced system, that is, the 5-tuple \((M^{(K)}, B^{(K)}, h^{(K)}, f^{(K)}, W^{(K)})\), where the stratum \(M^{(K)}\) is a connected component of the projection \(\pi_G((T^*Q)_{(K)})\) of the \((K)\)-orbit type submanifold \((T^*Q)_{(K)}\) to the orbit space, \(\pi^{(K)} \cdot B^{(K)} = B\), \(h^{(K)} \cdot \pi^{(K)} = H\), \(f^{(K)} \cdot \pi^{(K)} = \pi^{(K)} \cdot f_t\), \(W^{(K)} = \pi^{(K)}(W)\), is a CH system, which is simply written as SP-reduced CH system. Where \((M^{(K)}, B^{(K)})\) is the singular Poisson reduced stratum, the function \(h^{(K)}: M^{(K)} \to \mathbb{R}\) is called the reduced Hamiltonian, the fiber-preserving map \(f^{(K)}: M^{(K)} \to M^{(K)}\) is called the reduced (external) force map, \(W^{(K)}\) is a fiber submanifold of \(M^{(K)}\) and is called the reduced control subset.
Denote by $X_{(T^*Q,G,B,H,F,u)}$ the vector field of singular Poisson reducible CH system $(T^*Q,G,B,H,F,W)$ with a control law $u$, then

$$X_{(T^*Q,G,B,H,F,u)} = B^i \frac{\partial}{\partial h} + \text{vlift}(F) + \text{vlift}(u)$$

Moreover, for the singular Poisson reducible CH system we can also introduce the singular Poisson reduced controlled Hamiltonian equivalence (SPR-CH-equivalence) as follows.

**Definition 5.2 (SPR-CH-equivalence)** Suppose that we have two singular Poisson reducible CH systems $(T^*Q_i,G_i,B_i,H_i,F_i,W_i)$, $i = 1, 2$, we say them to be SPR-CH-equivalent or simply $(T^*Q_1,G_1,B_1,H_1,F_1,W_1) \overset{\text{SPR-CH}}{\sim} (T^*Q_2,G_2,B_2,H_2,F_2,W_2)$, if there exists a diffeomorphism $\varphi : Q_1 \to Q_2$, such that the following Hamiltonian matching conditions hold:

**SPR-H1:** The cotangent lift map $\varphi^* : T^*Q_2 \to T^*Q_1$ is a Poisson map, and $W_1 = \varphi^*(W_2)$;

**SPR-H2:** For the Lie group actions $\Phi^T_i : G_i \times T^*Q_i \to T^*Q_i$, $i = 1, 2$, the map $\varphi^* : T^*Q_2 \to T^*Q_1$ is $(G_2,G_1)$-equivariant, and for the corresponding orbit type stratification of the orbit spaces $T^*Q_i/G_i$, $i = 1, 2$, the induced orbit space map $\varphi^*_i : T^*Q_2/G_2 \to T^*Q_1/G_1$ is stratified;

**SPR-H3:** $\text{Im} [B^1_i dh_1 + \text{vlift}(F_1) - T \varphi^*(B^2_i dh_2) - \text{vlift}(\varphi^*F_2 \varphi^*)] \subset \text{vlift}(W_1)$.

It is worthy of note that for the singular Poisson reducible CH system, the equivalent map not only keeps the Poisson structure, but also keeps the equivariance of $G$-action on the cotangent bundle and the stratification. If a feedback control law $u^{(K)} : M^{(K)} \to W^{(K)}$ is chosen, the SP-reduced CH system $(M^{(K)},B^{(K)},h^{(K)},f^{(K)},W^{(K)})$ is a closed-loop dynamic system with a control law $u^{(K)}$. Assume that its vector field $X_{(M^{(K)},B^{(K)},h^{(K)},f^{(K)},u^{(K)})}$ can be expressed by

$$X_{(M^{(K)},B^{(K)},h^{(K)},f^{(K)},u^{(K)})} = (B^{(K)})^i \frac{\partial}{\partial h^{(K)}} + \text{vlift}(f^{(K)}) + \text{vlift}(u^{(K)}),$$

where $(B^{(K)})^i \frac{\partial}{\partial h^{(K)}} = X^{h^{(K)}}, \text{vlift}(f^{(K)}) = \text{vlift}(f^{(K)})X^{h^{(K)}}, \text{vlift}(u^{(K)}) = \text{vlift}(u^{(K)})X^{h^{(K)}},$ and satisfies the condition that $X_{(M^{(K)},B^{(K)},h^{(K)},f^{(K)},u^{(K)})}$ and $X_{(T^*Q,G,B,H,F,u)}$ are $\pi^{(K)}$-related, that is,

$$X_{(M^{(K)},B^{(K)},h^{(K)},f^{(K)},u^{(K)})} \cdot \pi^{(K)} = T \pi^{(K)} \cdot X_{(T^*Q,G,B,H,F,u)};$$

where $T \pi^{(K)} : T^*Q \to TM^{(K)}$ is tangent map of the projection $\pi^{(K)} : T^*Q \to M^{(K)}$. Then we can obtain the following singular Poisson reduction theorem for CH system, which explains the relationship between the SPR-CH-equivalence for the singular Poisson reducible CH systems with symmetry and the CH-equivalence for $SP$-reduced CH systems.

**Theorem 5.3** Two singular Poisson reducible CH systems $(T^*Q_i,G_i,B_i,H_i,F_i,W_i)$, $i = 1, 2$, are SPR-CH-equivalent if and only if the associated $SP$-reduced CH systems $(M^{(K)}_i,B^{(K)}_i,h^{(K)}_i,f^{(K)}_i,W^{(K)}_i), i = 1, 2$, are CH-equivalent.

**Proof.** If $(T^*Q_1,G_1,B_1,H_1,F_1,W_1) \overset{\text{SPR-CH}}{\sim} (T^*Q_2,G_2,B_2,H_2,F_2,W_2)$, then there exists a diffeomorphism $\varphi : Q_1 \to Q_2$ such that $\varphi^* : T^*Q_2 \to T^*Q_1$ is a $(G_2,G_1)$-equivariant Poisson map, $W_1 = \varphi^*(W_2)$, and for the corresponding orbit type stratification of the orbit spaces $T^*Q_i/G_i$, $i = 1, 2$, the induced orbit space map $\varphi^*_i : T^*Q_2/G_2 \to T^*Q_1/G_1$ is stratified, and SPR-H3 holds. From the following commutative diagram-11:
We can define a map $\varphi^*_{(K)} : M^2_{(K_2)} \to M^1_{(K_1)}$ such that $\varphi^*_{(K)} \cdot \pi_{(K_2)} = \pi_{(K_1)} \cdot \varphi^*$, and $\varphi^*_G \cdot \pi_{G/G} = \pi_{G \cdot \varphi}$. Because $\varphi^* : T^*Q_2 \to T^*Q_1$ is $(G_2, G_1)$-equivariant and the induced orbit space map $\varphi^*_G : T^*Q_2/G_2 \to T^*Q_1/G_1$ is stratified, the map $\varphi^*_{(K)} = \varphi^*_G|_{M^2_{(K_2)}} : M^2_{(K_2)} \to M^1_{(K_1)}$ is well-defined. We shall show that $\varphi^*_{(K)}$ is a Poisson map and $W^1_{(K_1)} = \varphi^*_{(K)}(W^2_{(K_2)})$. In fact, since $\varphi^* : T^*Q_2 \to T^*Q_1$ is a Poisson map, the map $(\varphi^*)^* : \Lambda^2(T^*Q_1) \to \Lambda^2(T^*Q_2)$ such that $(\varphi^*)^*B_1 = B_2$, and by using (5.1), the map $\pi^*_{(K)} : T^*Q_i \to M^1_{(K_i)}$ is also a Poisson map, $B_i = \varphi^*_{(K_i)}B^1_{(K_i)}$, $i = 1, 2$, from the commutative Diagram-12, we have that

$$\pi^*_{(K_2)} \cdot (\varphi^*_{(K)})^*B^1_{(K_1)} = (\pi^*_{(K_1)} \cdot \varphi^*)^*B^1_{(K_1)} = (\pi^*_{(K_1)} \cdot \varphi^*)^*B^1_{(K_1)} = (\varphi^* \cdot \pi^*_{(K_1)}B^1_{(K_1)}) = (\varphi^*)^*B^1_{(K_1)} = B_2 = \pi^*_{(K_2)} \cdot B^2_{(K_2)}.$$

Notice that $\pi^*_{(K_2)}$ is surjective, thus, $(\varphi^*_{(K)})^*B^1_{(K_1)} = B^2_{(K_2)}$. Because $W^1_{(K_i)} = \pi^*_{(K_i)}(W^2_{(K_i)}), i = 1, 2$ and $W^1 = \varphi^*(W^2)$, we have that

$$W^1_{(K_i)} = \pi^*_{(K_i)}(W^2_{(K_i)}) = \pi^*_{(K_i)}(W^2) = \varphi^*_{(K_i)} \cdot \pi^*_{(K_2)}(W^2) = \varphi^*_{(K_i)}(W^2_{(K_2)}).$$

Next, from (5.2) and (5.3), we know that for $i = 1, 2$,

$$X_{(T^*Q_i, G_i, B_i, H_i, F_i, u_i)} = B^i_{(K_i)} dH_i + \text{vlift}(F_i) + \text{vlift}(u_i),$$

$$X_{(M^1_{(K_i)}, B^1_{(K_i)}, h^1_{(K_i)}, f^1_{(K_i)}, u^1_{(K_i)})} = (B^1_{(K_i)} dH_i + \text{vlift}(f^1_{(K_i)}) + \text{vlift}(u^1_{(K_i)}),$$

and from (5.4), we have that

$$X_{(M^1_{(K_i)}, B^1_{(K_i)}, h^1_{(K_i)}, f^1_{(K_i)}, u^1_{(K_i)})} \cdot \pi^*_{(K_i)} = T\pi^*_{(K_i)} \cdot X_{(T^*Q_i, G_i, B_i, H_i, F_i, u_i)}.$$

Since $H_i, F_i$ and $W_i$ are all $G_i$-invariant and stratified, $i = 1, 2,$ and

$$h^1_{(K_i)} \cdot \pi^*_{(K_i)} = H_i, f^1_{(K_i)} \cdot \pi^*_{(K_i)} = \pi^*_{(K_1)} \cdot F_i, u^1_{(K_i)} \cdot \pi^*_{(K_i)} = \pi^*_{(K_1)} \cdot u_i, i = 1, 2.$$
where \( \varphi(K)_* = (\varphi(K))^{-1} : M_1^{(K_1)} \to M_2^{(K_2)} \). From Hamiltonian matching condition SPR-H3, we have that

\[
\text{Im}[\{(M^{(K_1)}_1, B_1^{(K_1)}, h_1^{(K_1)}, f_1^{(K_1)}, W_1^{(K_1)})\} + \text{vlift}(f_1^{(K_1)}) - T\varphi(K)_*((B_2^{(K_2)})^* d h_2^{(K_2)}) - \text{vlift}(\varphi(K)_* f_2^{(K_2)} \cdot \varphi(K)_*)] \subset \text{vlift}(W_1^{(K_1)}).
\]

(5.5)

So,

\[
(M_1^{(K_1)}, B_1^{(K_1)}, h_1^{(K_1)}, f_1^{(K_1)}, W_1^{(K_1)}) \sim \text{CH} (M_2^{(K_2)}, B_2^{(K_2)}, h_2^{(K_2)}, f_2^{(K_2)}, W_2^{(K_2)}).
\]

Conversely, assume that the SP-reduced CH systems \((M_i^{(K_i)}, B_i^{(K_i)}, h_i^{(K_i)}, f_i^{(K_i)}, W_i^{(K_i)}), i = 1, 2\), are CH-equivalent. Then there exists a diffeomorphism \( \varphi(K)_* : M_2^{(K_2)} \to M_1^{(K_1)} \), which is a Poisson map, \( W_1^{(K_1)} = \varphi(K)_*(W_2^{(K_2)}) \), and (5.5) holds. We can define the maps \( \varphi^* : T^* Q_2 \to T^* Q_1 \), and \( \varphi^*_{/G} : T^* Q_2 / G_2 \to T^* Q_1 / G_1 \), such that \( \pi^{(K_1)} \cdot \varphi^* = \varphi^*(K) \cdot \pi^{(K_2)} \), and \( \varphi^*_{/G} \cdot \pi_{/G_2} = \pi_{/G_1} \cdot \varphi^* \), and \( \varphi^*_{/G|M(K)} = \varphi(K)_* : M_2^{(K_2)} \to M_1^{(K_1)} \), see the commutative Diagram-11, as well as a diffeomorphism \( \varphi^* : Q_1 \to Q_2 \) whose cotangent lift is just \( \varphi^* : T^* Q_2 \to T^* Q_1 \). From definition of \( \varphi^* \), we know that \( \varphi^* \) is \((G_2, G_1)\)-equivariant and the induced orbit space map \( \varphi^*_{/G} : T^* Q_2 / G_2 \to T^* Q_1 / G_1 \) is stratified. In fact, for any \( \alpha_i \in T^* Q_i, g_i \in G_i, i = 1, 2 \), such that \( \alpha_1 = \varphi^*(K)(\alpha_2) \), \( [\alpha_1] = \varphi(K)_*[\alpha_2] \), then we have that

\[
\pi^{(K_1)} \cdot \varphi^* \cdot (\Phi_{2g_2}(\alpha_2)) = \pi^{(K_1)} \cdot \varphi^*(g_2 \alpha_2) = \varphi(K)_* \pi^{(K_2)}(g_2 \alpha_2) = \varphi(K)_*[\alpha_2] = [\alpha_1]
\]

Since \( \pi^{(K_1)} \) is surjective, so \( \varphi^* \Phi_{2g_2} = \Phi_{1g_1} \varphi^* \), and hence \( \varphi^* \) is \((G_2, G_1)\)-equivariant. Notice that \( M_i^{(K_i)} \) form the corresponding orbit type stratification of the orbit space \( T^* Q_i / G_i, i = 1, 2 \), and by construction \( \varphi^*_{/G|M(K)} = \varphi(K)_* : M_2^{(K_2)} \to M_1^{(K_1)} \). Thus, in this case the induced orbit space map \( \varphi^*_{/G} : T^* Q_2 / G_2 \to T^* Q_1 / G_1 \) is stratified. Moreover, \( \pi^{(K_1)}(W_1) = W_1^{(K_1)} = \varphi(K)_*(W_2^{(K_2)}) = \varphi(K)_* \pi^{(K_2)}(W_2) = \pi^{(K_1)} \cdot \varphi^*(W_2) \). Since \( \pi^{(K_1)} \) is surjective, then \( W_1 = \varphi^*(W_2) \). We shall show that \( \varphi^* \) is a Poisson map. Because \( \varphi(K)_* : M_2^{(K_2)} \to M_1^{(K_1)} \) is a Poisson map, the map \( \langle \varphi(K)_* \rangle^* : \Lambda^2(T^* M_1^{(K_1)}) \to \Lambda^2(T^* M_2^{(K_2)}) \) such that \( \langle \varphi(K)_* \rangle^* B_1^{(K_1)} = B_2^{(K_2)} \) and by using (5.1), \( \pi^{(K_1)} : T^* Q_i \to M_1^{(K_i)} \) is also a Poisson map, \( B_i = \pi^{(K_i)} B_i^{(K_i)}, i = 1, 2 \), from the commutative Diagram-12, we have that

\[
B_2 = \pi^{(K_2)} B_2^{(K_2)} = \pi^{(K_2)} \langle \varphi(K)_* \rangle^* B_1^{(K_1)} = (\varphi^*)^* \pi^{(K_1)} B_1^{(K_1)} = (\varphi^*)^* B_1.
\]

Since the vector field \( X_{T^* Q_i, G_i, B_i, H_i, F_i, u_i} \) and \( X_{M_i^{(K_i)}, B_i^{(K_i)}, h_i^{(K_i)}, f_i^{(K_i)}, u_i^{(K_i)}} \) is \( \pi^{(K_i)} \)-related, \( i = 1, 2 \), and \( H_i, F_i \) and \( W_i \) are all \( G_i \)-invariant and stratified, \( i = 1, 2 \), in the same way, from (5.5), we have that

\[
\text{Im}[B_1 dh_1 + \text{vlift}(F_1) - T\varphi^*(B_2 dh_2) - \text{vlift}(\varphi^* F_2 \varphi_*)] \subset \text{vlift}(W_1),
\]

that is, Hamiltonian matching condition SPR-H3 holds. Thus,

\[
(T^* Q_1, G_1, B_1, H_1, F_1, W_1) \overset{SPR-H3}{\sim} CH (T^* Q_2, G_2, B_2, H_2, F_2, W_2).
\]

\[\Box\]

**Remark 5.4** When the \( G \)-action on \( T^* Q \) is free and proper, then \( T^* Q / G \) is a smooth manifold. Itself is only one stratified space. In this case, from Definition 5.1 and Definition 5.2 we can obtain the regular reducible CH system and RPR-CH-equivalence, and from Theorem 5.3 we can get the regular Poisson reduction theorem. Thus, we can recover the regular Poisson reduction results of CH systems given in Wang and Zhang [28].

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6 Relationship Between Singular and by Controllability Distribution Poisson Reductions

For a singular Poisson reducible CH system \((T^*Q,G,B,H,F,W)\), it is a symmetric CH system and \(W\) is a \(G\)-invariant submanifold of \(T^*Q\). Assume that \(W\) is a controllability submanifold and \(D^G \subset TT^*Q|_W\) is a \(G\)-invariant controllability distribution, such that \(D_W = D^G \cap TW\) is a smooth regular \(G\)-invariant integrable distribution on \(W\), and the presheaf \(C^*_W/D_W\) has the \((D^G,D_W)\)-local extension property, then singular Poisson reducible CH system \((T^*Q,G,B,H,F,W)\) is also Poisson reducible by \(G\)-invariant controllability distribution \(D^G\). On the other hand, for the \(SP\)-reduced CH system \((M^{(K)},B^{(K)},h^{(K)},f^{(K)},W^{(K)})\), if \(W^{(K)} = \pi^{(K)}(W)\) is a controllability submanifold and \(D^{(K)} = T\pi^{(K)}(D^G) \subset TM^{(K)}|_{W^{(K)}}\) is controllability distribution, then we may consider that the \(SP\)-reduced CH system \((M^{(K)},B^{(K)},h^{(K)},f^{(K)},W^{(K)})\) is Poisson reducible by the reduced controllability distribution \(D^{(K)}\). Moreover, we can also prove that the property of Poisson reduction for singular Poisson reducible CH systems by \(G\)-invariant controllability distribution leaves invariant under SPR-CH-equivalence conditions.

At first, we give the following theorem to state the relationship between Poisson reduction for singular Poisson reducible CH systems by \(G\)-invariant controllability distribution \(D^G\) and Poisson reduction for associated \(SP\)-reduced CH system by controllability distribution \(D^G/G\).

**Theorem 6.1** Suppose that symmetric CH system \((T^*Q,G,B,H,F,W)\) is a singular Poisson reducible CH system and its associated \(SP\)-reduced CH system is \((M^{(K)},B^{(K)},h^{(K)},f^{(K)},W^{(K)})\). Then we have that

(i) \(W^{(K)} = \pi^{(K)}(W)\) is a controllability submanifold of \(SP\)-reduced CH system \((M^{(K)},B^{(K)},h^{(K)},f^{(K)},W^{(K)})\) if and only if \(W\) is \(G\)-invariant controllability submanifold of symmetric CH system \((T^*Q,G,B,H,F,W)\).

(ii) \(D^{(K)} = T\pi^{(K)}(D^G) \subset TM^{(K)}|_{W^{(K)}}\) is controllability distribution of the \(SP\)-reduced CH system, such that \(D^{(K)}_W = D^{(K)} \cap TW^{(K)}\) is a smooth regular integrable distribution on \(W^{(K)}\), and the presheaf \(C^*_W(D^{(K)}_W/D^{(K)}_W)\) has the \((D^{(K)},D^{(K)}_W)\)-local extension property if and only if \(D^G \subset TT^*Q|_W\) is \(G\)-invariant controllability distribution of the symmetric CH system, such that \(D_W = D^G \cap TW\) is a smooth regular \(G\)-invariant integrable distribution on \(W\), and the presheaf \(C^*_W(D_W)\) has the \((D^G,D_W)\)-local extension property.

(iii) The singular Poisson reducible CH system \((T^*Q,G,B,H,F,W)\) is Poisson reducible by \(G\)-invariant controllability distribution \(D^G\), if and only if the associated \(SP\)-reduced CH system \((M^{(K)},B^{(K)},h^{(K)},f^{(K)},W^{(K)})\) is Poisson reducible by the reduced controllability distribution \(D^{(K)}\).

**Proof.** (i) Because the symmetric CH system \((T^*Q,G,B,H,F,W)\) is a singular Poisson reducible, for \(u : T^*Q \to W\), there exists \(u^{(K)} : M^{(K)} \to W^{(K)}\), such that \(u^{(K)} \cdot \pi^{(K)} = \pi^{(K)} \cdot u\) and

\[
X_{(M^{(K)},B^{(K)},h^{(K)},f^{(K)},u^{(K)})} \cdot \pi^{(K)} = T\pi^{(K)} \cdot X_{(T^*Q,G,B,H,F,u)}
\]

where \(T\pi^{(K)} : TT^*Q \to TM^{(K)}\) is tangent map of the projection \(\pi^{(K)} : T^*Q \to M^{(K)}\). Note that \(W^{(K)} = \pi^{(K)}(W)\), and \(W\) is \(G\)-invariant, hence from Definition 3.1 we know that (i) holds.
Thus, we prove that $D^{(K)} = T\pi^{(K)}(D^G) \subset TM^{(K)}|_{W^{(K)}}$, and $D^G \subset TT^*Q|_W$ is $G$-invariant. Thus,

$$D_{W^{(K)}} = D^{(K)} \cap TW^{(K)} = T\pi^{(K)}(D^G) \cap T\pi^{(K)}(TW) = T\pi^{(K)}(D^G \cap TW) = T\pi^{(K)}(D_W).$$

Since the bundle structures of $T^*Q$ and $(T^*Q)/G$ is compatible with the stratification of $T^*Q/G$, and $\pi^{(K)}$ and $T\pi^{(K)}$ are vector bundle maps, and $\pi^{(K)}$ is Poisson, from Definition 2.4, Definition 3.2 and the definition of the presheaf local extension property, we know that (ii) holds.

(iii) If the singular Poisson reducible CH system $(T^*Q, G, B, H, F, W)$ is Poisson reducible by $G$-invariant controllability distribution $D^G$, we shall prove that the associated $SP$-reduced CH system $(M^{(K)}, B^{(K)}, h^{(K)}, f^{(K)}, W^{(K)})$ is Poisson reducible by the reduced controllability distribution $D^{(K)}$. By the above conclusions (i), (ii) and Theorem 3.4, it suffices to show that for any $[z] = \pi^{(K)}(z) \in W^{(K)}$, $z \in W$ we have that $(B^{(K)})^\sharp(\Delta^{(K)}_{[z]}) \subset (\Delta^{(K)}_{[z]})^{\circ}_{W^{(K)}}$, where

$$\Delta^{(K)}_{[z]} := \left\{ f \in C^{\infty}_{M^{(K)}}(U_{[z]}), \ d\alpha(y)|_{D^{(K)}(y)} = 0, \text{ for all } y \in U_{[z]} \cap W^{(K)} \right\},$$

and $$\left(\Delta^{(K)}_{[z]}\right)^{\circ}_{W^{(K)}} := \left\{ f \in \Delta^{(K)}_{[z]} \left| f|_{U_{[z]} \cap W^{(K)}} \text{ is constant for any open neighborhood } U_{[z]} \text{ of } [z] \in M^{(K)} \right\}.\right.$$ 

In fact, assume that for any $\alpha^{(K)} = d\pi_K([z]) \in \Delta^{(K)}_{[z]}$, that is, $f_K \in C^{\infty}_{M^{(K)}}(U_{[z]})$, where $U_{[z]}$ is an open neighborhood of $[z]$ in $M^{(K)}$, such that $U_{[z]} \cap W^{(K)}$ is $D_{W^{(K)}}$-invariant and $d\pi_K(y)|_{D^G(y)} = 0$, for all $y \in U_{[z]} \cap W^{(K)}$. Since the map $\pi^{(K)} : T^*Q \to M^{(K)}$ is surjective, then for $z \in T^*Q$, such that $[z] = \pi^{(K)}(z)$, there is an open $G$-invariant neighborhood $\tilde{U}_z$ of $z$ in $T^*Q$, such that $U_{[z]} = \pi^{(K)}(\tilde{U}_z)$ and $\tilde{U}_z \cap W$ is $D_W$-invariant, and therefore exists a $G$-invariant function $f \in C^{\infty}_{T^*Q}(\tilde{U}_z)^G$, such that $f = f_K \cdot (\pi^{(K)})$ and $d\pi_K(y)|_{D^G(y)} = 0$, for all $y \in \tilde{U}_z \cap W$, and $\alpha = d\pi_K([z]) \in \Delta^{G}_{\tilde{U}_z}$. In the same way, for any $\beta^{(K)} = dg_K([z]) \in (\Delta^{(K)}_{[z]})^{\circ}_{W^{(K)}}$, that is, $g_K \in C^{\infty}_{M^{(K)}}(U_{[z]})$ and $g_K|_{U_{[z]} \cap W^{(K)}}$ is constant for any open neighborhood $U_{[z]}$ of $[z]$ in $M^{(K)}$, then there is a $G$-invariant function $g \in C^{\infty}_{T^*Q}(\tilde{U}_z)$, such that $g = g_K \cdot (\pi^{(K)})$ and $g|_{\tilde{U}_z \cap W}$ is constant for the corresponding open $G$-invariant neighborhood $\tilde{U}_z$ of $z$ in $T^*Q$, and $\beta = dg_K(z) \in (\Delta^{G}_{\tilde{U}_z})$. Because the singular Poisson reducible CH system $(T^*Q, G, B, H, F, W)$ is Poisson reducible by $G$-invariant controllability distribution $D^G$, from the above conclusions (i), (ii) and Theorem 4.3 we have that $B^\sharp(\Delta^{G}_{\tilde{U}_z}) \subset (\Delta^{G}_{\tilde{U}_z})^{\circ}_W$. It follows that

$$\{ f, g \}_{B(z)} = \langle dg_K([z]), B^\sharp(d\pi_K([z])) \rangle = \langle \beta, B^\sharp(\alpha) \rangle = 0.$$

Notice that the map $\pi^{(K)} : T^*Q \to M^{(K)}$ is Poisson, we have that

$$0 = \{ f, g \}_{B(z)} = \{ f_K \cdot (\pi^{(K)}), g_K \cdot (\pi^{(K)}) \}_{B(z)} = (\pi^{(K)})^\sharp \{ f_K, g_K \}_{B^{(K)}([z])} = 0.$$

Since $(\pi^{(K)})^\sharp$ is injective, then $\{ f_K, g_K \}_{B^{(K)}([z])} = 0$, and

$$\langle \beta^{(K)}, (B^{(K)})^\sharp(\alpha^{(K)}) \rangle = \langle dg_K([z]), (B^{(K)})^\sharp(d\pi_K([z])) \rangle = \{ f_K, g_K \}_{B^{(K)}([z])} = 0.$$

Thus, we prove that $(B^{(K)})^\sharp(\alpha^{(K)}) \in (\Delta^{(K)}_{[z]})^{\circ}_{W^{(K)}}$ and $(B^{(K)})^\sharp(\beta^{(K)}) \subset (\Delta^{(K)}_{[z]})^{\circ}_{W^{(K)}}$, and hence the $SP$-reduced CH system $(M^{(K)}, B^{(K)}, h^{(K)}, f^{(K)}, W^{(K)})$ is Poisson reducible by the reduced
controlability distribution $D^{(K)}$.

Conversely, if the associated $SP$-reduced CH system $(M^{(K)}, B^{(K)}, h^{(K)}, f^{(K)}, W^{(K)})$ is Poisson reducible by the reduced controlability distribution $D^{(K)}$, by using the same way we can verify that for any $z \in W$, we have that $B^2(\Delta^G_1) \subset (\Delta^G_2)_W$. Thus, the singular Poisson reducible CH system $(T^*Q, G, B, H, F, W)$ is Poisson reducible by $G$-invariant controlability distribution $D^G$ by the above conclusions (i), (ii) and Theorem 4.3. ■

Next, if considering the SPR-CH-equivalence of singular Poisson reducible CH systems, we can get the following theorem to state the property of Poisson reduction for singular Poisson reducible CH systems by $G$-invariant controlability distribution leaves invariant under SPR-CH-equivalence conditions.

**Theorem 6.2** Suppose that two singular Poisson reducible CH systems $(T^*Q_1, G_1, B_1, H_1, F_1, W_1)$, $i = 1, 2$, are SPR-CH-equivalent with equivalent map $\varphi^* : T^*Q_2 \to T^*Q_1$. Then we have that

(i) $W_1$ is $G_1$-invariant controlability submanifold of CH system $(T^*Q_1, G_1, B_1, H_1, F_1, W_1)$ if and only if $W_2$ is $G_2$-invariant controlability submanifold of CH system $(T^*Q_2, G_2, B_2, H_2, F_2, W_2)$.

(ii) $D^{G_1}_1 = T\varphi^*(D^{G_2}_2) \subset TT^*Q_1|_{W_1}$ is $G_1$-invariant controlability distribution of CH system $(T^*Q_1, G_1, B_1, H_1, F_1, W_1)$, such that $D_{W_1} = D^{G_1}_1 \cap TW_1$ is smooth regular $G_1$-invariant integrable distributions on $W_1$ and the presheaf $C^\infty_{W_1/D_{W_1}}$ has the $(D^{G_1}_1, D_{W_1})$-local extension property, if and only if $D^{G_2}_2 \subset TT^*Q_2|_{W_2}$ is $G_2$-invariant controlability distribution of CH system $(T^*Q_2, G_2, B_2, H_2, F_2, W_2)$, such that $D_{W_2} = D^{G_2}_2 \cap TW_2$ is smooth regular $G_2$-invariant integrable distributions on $W_2$ and the presheaf $C^\infty_{W_2/D_{W_2}}$ has the $(D^{G_2}_2, D_{W_2})$-local extension property.

(iii) The singular Poisson reducible CH system $(T^*Q_1, G_1, B_1, H_1, F_1, W_1)$ is Poisson reducible by $G_1$-invariant controlability distribution $D^{G_1}_1$ if and only if the singular Poisson reducible CH system $(T^*Q_2, G_2, B_2, H_2, F_2, W_2)$ is Poisson reducible by $G_2$-invariant controlability distribution $D^{G_2}_2$.

**Proof.** (i) Because the singular Poisson reducible CH systems $(T^*Q_i, G_i, B_i, H_i, F_i, W_i)$, $i = 1, 2$, are SPR-CH-equivalent, then there exists a diffeomorphism $\varphi : Q_1 \to Q_2$, such that the cotangent lift map $\varphi^* = T^*\varphi : T^*Q_2 \to T^*Q_1$ and the tangent lift map $T\varphi^* : TT^*Q_2 \to TT^*Q_1$ are vector bundle isomorphism, and $\varphi^*$ is $(G_2, G_1)$-equivariant Poisson map. Note that $W_1 = \varphi^*(W_2)$, and $W_i$ is $G_i$-invariant, $i = 1, 2$, hence from Definition 3.1 and Theorem 3.5 (i) we know that (i) holds.

(ii) Notice that $D^{G_1}_1 = T\varphi^*(D^{G_2}_2) \subset TT^*Q_1|_{W_1}$, and $D^{G_2}_2 \subset TT^*Q_2|_{W_2}$. Thus,

$D_{W_1} = D^{G_1}_1 \cap TW_1 = T\varphi^*(D^{G_2}_2) \cap T\varphi^*(TW_2) = T\varphi^*(D^{G_2}_2 \cap TW_2) = T\varphi^*(D_{W_2})$.

Since $\varphi^*$ and $T\varphi^*$ are vector bundle isomorphism, and $\varphi^*$ is $(G_2, G_1)$-equivariant Poisson map, from Definition 2.4, Definition 3.2, the definition of the presheaf local extension property and Theorem 3.5 (ii), we know that (ii) holds.

(iii) If singular Poisson reducible CH system $(T^*Q_1, G_1, B_1, H_1, F_1, W_1)$ is Poisson reducible by $G_1$-invariant controlability distribution $D^{G_1}_1$, from the above conclusions (i), (ii) and Theorem
we can get the desired conclusion. 

Suppose that symmetric CH system Corollary 6.3 the above theorem and the regular Poisson reduction theorem given in Wang and Zhang [29]. we can consider the regular Poisson reducible CH system and obtain the following corollary from (ii) D

is Poisson reducible by G

system, such that D_1^{(K_i)} = T\pi_1^{(K_1)}(D_1^{G_1}). On the other hand, from Theorem 5.3 we know that two singular Poisson reducible CH systems (T*Q_i, G_i, B_i, H_i, F_i, W_i), i = 1, 2, are SPR-CH-equivalent with equivalent map f_\text{\textsuperscript{*}} : T*Q_2 \rightarrow T*Q_1, if and only if the associated SP-reduced CH systems (M_1^{(K_i)}, B_i^{(K_i)}, h_i^{(K_i)}, f_i^{(K_i)}, W_i^{(K_i)}), i = 1, 2, are CH-equivalent with equivalent map f_\text{\textsuperscript{*}} : M_2^{(K_2)} \rightarrow M_1^{(K_1)}. Moreover, from Theorem 3.5 we know that the property of Poisson reduction for CH systems by controllability distribution leaves invariant under the CH-equivalence. Thus, the SP-reduced CH system (M_2^{(K_2)}, B_2^{(K_2)}, h_2^{(K_2)}, f_2^{(K_2)}, W_2^{(K_2)}) is Poisson reducible by the reduced controllability distribution D_2^{(K_2)}, where D_1^{(K_1)} = Tf_\text{\textsuperscript{*}}(D_2^{(K_2)}), and hence from the above conclusions (i), (ii) and Theorem 6.1 we have that the singular Poisson reducible CH system (T*Q_2, G_2, B_2, H_2, F_2, W_2) is Poisson reducible by G_2-invariant controllability distribution D_2^{G_2}.

Conversely, if the singular Poisson reducible CH system (T*Q_2, G_2, B_2, H_2, F_2, W_2) is Poisson reducible by G_2-invariant controllability distribution D_2^{G_2}, by using the above same argument we can get the desired conclusion. 

In particular, when G-action is free, the orbit space T*Q/G is a smooth manifold, in this case we can consider the regular Poisson reducible CH system and obtain the following corollary from the above theorem and the regular Poisson reduction theorem given in Wang and Zhang [29].

**Corollary 6.3** Suppose that symmetric CH system (T*Q, G, B, H, F, W) is a regular Poisson reducible CH system and its associated RP-reduced CH system is (T*Q/G, B/G, h/G, \text{\textsuperscript{\textbullet}}/G/W). Then we have that

(i) W/G = \pi/G(W) is a controllability submanifold of RP-reduced CH system (T*Q/G, B/G, h/G, f/G, W/G) if and only if W is G-invariant controllability submanifold of symmetric CH system (T*Q, G, B, H, F, W).

(ii) D/G = T\pi/G(D/G) \subset T(T*Q/G)|_{W/G} is controllability distribution of the RP-reduced CH system, such that D/W/G = D/G \cap TW/G is a smooth regular integrable distribution on W/G, and the presheaf C_\\text{\textsuperscript{\infty}}/W/D/W/G has the (D/G, D/W/G)-local extension property, if and only if D/G \subset TT*Q/W is a G-invariant controllability distribution, such that D/W = D/G \cap TW is a smooth regular G-invariant integrable distribution on W, and the presheaf C_\\text{\textsuperscript{\infty}}/W/D/W has the (D/G, D/W)-local extension property.

(iii) The regular Poisson reducible CH system (T*Q, G, B, H, F, W) is Poisson reducible by G-invariant controllability distribution D/G, if and only if the associated RP-reduced CH system (T*Q/G, B/G, h/G, f/G, W/G) is Poisson reducible by the reduced controllability distribution D/G.

(iv) If two regular Poisson reducible CH systems (T*Q_i, G_i, B_i, H_i, F_i, W_i), i = 1, 2, are RPR-CH-equivalent with equivalent map f_\text{\textsuperscript{*}} : T*Q_2 \rightarrow T*Q_1, then CH system (T*Q_1, G_1, B_1, H_1, F_1, W_1) is Poisson reducible by G_1-invariant controllability distribution D_1^{G_1} if and only if CH system (T*Q_2, G_2, B_2, H_2, F_2, W_2) is Poisson reducible by G_2-invariant controllability distribution D_2^{G_2}, where D_1^{G_1} = Tf_\text{\textsuperscript{*}}(D_2^{G_2}).
7 Applications

In order to understand well the abstract theory, in this section, some examples are given to state theoretical results of Poisson reduction for CH systems by controllability distributions.

Example 7.1. (Optimal point reduction of CH system) Let $Q$ be a smooth manifold and $T^*Q$ its cotangent bundle with the Poisson tensor $B$. Let $\Phi : G \times Q \to Q$ be a smooth left action of the Lie group $G$ on $Q$, and the cotangent lifted left action $\Phi^T : G \times T^*Q \to T^*Q$ be canonical, and proper. Let $A'_G$ be the generalized distribution on $T^*Q$ defined by the relation

$$A'_G(\alpha) := \{X_\alpha(\xi) | \xi \in C^\infty(T^*Q)^G\}, \forall \alpha \in T^*Q,$$

and $A'_G$ be the $G$-characteristic distribution and it be smooth and integrable in the sense of Stefan [23] and Sussmann [24]. Moreover, let $J : T^*Q \to T^*Q/A'_G$ be the optimal momentum map associated to the cotangent lifted left action $\Phi^T$, and $\rho \in T^*Q/A'_G$ be a value of $J$. If $m \in T^*Q$ and $J(m) = \rho$, then $J^{-1}(\rho) = A'_G \cdot m$. Denote by $G_\rho$ the isotropy subgroup of $G$ in $\rho$. If $G_\rho$ acts properly on $J^{-1}(\rho)$, then the orbit space $(T^*Q)_\rho = J^{-1}(\rho)/G_\rho$ is a smooth symplectic quotient manifold and the projection $\pi_\rho : J^{-1}(\rho) \to (T^*Q)_\rho$ is a canonical surjective submersion. See Ortega and Ratiu [18]. For the symmetric CH system $(T^*Q,G,B,H,F,W)$, if the control subset $W = J^{-1}(\rho)$ is a $G$-invariant controllability submanifold of $T^*Q$, and the generalized distribution $D$ is given by the tangent spaces to the orbits of the $G$-action. Assume that $D$ is $G$-invariant controllability distribution of symmetric CH system. Notice that $g \cdot J(m) = J(g \cdot m)$, for any $g \in G$, $m \in T^*Q$, this $G$-action is available on the leaf space of any distribution spanned by $G$-equivariant vector fields. For any $\rho \in T^*Q/A'_G$, there is a unique symplectic leaf $L_\rho$ of $(T^*Q,B)$ such that $J^{-1}(\rho) \subset L_\rho$ and $i_{L_\rho} : J^{-1}(\rho) \to L_\rho$ is smooth.

In particular, since $(T^*Q,\omega)$ is a symplectic manifold with a canonical symplectic form $\omega$ and the $G$-action has an associated $Ad^*$-equivariant momentum map $J : T^*Q \to g^*$, then the fiber $W = J^{-1}(\rho)$ of the optimal momentum map is therefore an embedded submanifold of $T^*Q$. In this case, $D_W := D \cap TW$ is a generalized distribution given by the tangent spaces to the orbits of the $G_\rho$-action, and $W/D_W = J^{-1}(\rho)/G_\rho$, and from Ortega and Ratiu [18] we know that the presheaf $C^\infty_{W/D_W}$ has the $(D,D_W)$-local extension property. Thus, the symmetric CH system $(T^*Q,G,B,H,F,J^{-1}(\rho))$ is Poisson reducible by the controllability distribution $D$.

Example 7.2. (Optimal orbit reduction of CH system) Let $\rho \in T^*Q/A'_G$ be a value of the optimal momentum map $J$, and $O_\rho = G \cdot \rho \subset T^*Q/A'_G$ be the $G$-orbit of the point $\rho$. If isotropy subgroup $G_\rho$ acts properly on $J^{-1}(\rho)$, then there is a unique smooth structure on $J^{-1}(O_\rho)$ that makes it into an initial submanifold of $T^*Q$ and the $G$-action on $J^{-1}(O_\rho)$ by restriction of the $G$-action on $T^*Q$ is smooth and proper and all its isotropy subgroups are conjugate to a given compact isotropy subgroup of the $G$-action on $T^*Q$. Notice that $G \cdot J^{-1}(\rho)/G = J^{-1}(O_\rho)/G$, the quotient space $(T^*Q)_{O_\rho} = J^{-1}(O_\rho)/G$ admits a unique smooth structure that makes the projection $\pi_{O_\rho} : J^{-1}(O_\rho) \to (T^*Q)_{O_\rho}$ a surjective submersion. Moreover, note that the map $J^{-1}(\rho)/G_\rho \to J^{-1}(O_\rho)/G$, $[m]_{\rho} \to [m]_{O_\rho}$, is a bijection, so the quotient space $(T^*Q)_{O_\rho} = J^{-1}(O_\rho)/G$ has a smooth symplectic structure $\omega_{O_\rho}$ induced from the optimal point reduced space $(T^*Q)_\rho$. If the $G$-action on $T^*Q$ is free, from the Poisson stratification theorem given in Fernandes et al [6], we know that the symplectic leaves of the $RP$-reduced space $(T^*Q/G,B/G)$ are just given by the $O_\rho$-reduced space $((T^*Q)_{O_\rho},\omega_{O_\rho}), \rho \in T^*Q/A'_G$. For the symmetric CH system $(T^*Q,G,B,H,F,W)$, if the control subset $W = J^{-1}(\rho)$ is a $G$-invariant controllability submanifold of $T^*Q$, and the generalized distribution $D$ is given by the tangent spaces to the orbits of the $G$-action. Assume that $D$ is $G$-invariant controllability distribution.

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of symmetric CH system. For \( \rho \in T^*Q/A'_G \), assume that \( J^{-1}(O_\rho) \) is connected and there is a unique symplectic leaf \( L_{Q_\rho} \) of \((T^*Q, B)\) such that \( J^{-1}(O_\rho) \subset L_{Q_\rho} \) and \( i_{L_{Q_\rho}} : J^{-1}(O_\rho) \to L_{Q_\rho} \) is smooth. In particular, since \((T^*Q, \omega)\) is a symplectic manifold with a canonical symplectic form \( \omega \) and the \( G \)-action has an associated \( \Ad^* \)-equivariant momentum map \( J : T^*Q \to \mathfrak{g}^* \), then the fiber submanifold \( W = J^{-1}(O_\rho) \) of the optimal momentum map is therefore an embedded submanifold of \( T^*Q \). In this case, \( D_W := D \cap TW \) is a generalized distribution given by the tangent spaces to the orbits of the \( G \)-action on \( W \), and \( W/D_W = J^{-1}(O_\rho)/G \), and from Ortega and Ratiu [15] we know that the presheaf \( C^\infty_{W/D_W} \) has the \((D, D_W)\)-local extension property. Thus, the symmetric CH system \((T^*Q, G, B, H, F, J^{-1}(O_\rho))\) is Poisson reducible by the controllability distribution \( D \).

**Example 7.3.** (Poisson reduction of CH system by characteristic distribution) For the CH system \((T^*Q, B, H, F, W)\), assume that the control subset \( W \) is an embedded controllability submanifold of \( T^*Q \), and the distribution \( D := B^\sharp((TW)\circ) \subset TT^*Q|_W \) is controllability distribution, such that \( D_W := D \cap TW \) is a smooth and integrable generalized distribution on \( W \), and the presheaf \( C^\infty_{W/D_W} \) has the \((D, D_W)\)-local extension property. Then from Theorem 2.8, we know that \((T^*Q, B, D, W)\) is Poisson reducible, and hence the CH system \((T^*Q, B, H, F, W)\) is Poisson reducible by the characteristic distribution \( D := B^\sharp((TW)\circ) \).

**Example 7.4.** (Poisson reduction of CH system by controllability coisotropic submanifold) For the CH system \((T^*Q, B, H, F, W)\), assume that the control subset \( W \) is an embedded controllability coisotropic submanifold of \( T^*Q \), and \( D := B^\sharp((TW)\circ) \subset TT^*Q|_W \) is controllability distribution, such that \( D_W := D \cap TW \) and the presheaf \( C^\infty_{W/D_W} \) has the \((D, D_W)\)-local extension property. In this case, by using the coisotropic property of \( W \) we can prove that \( B^\sharp((TW)\circ) \subset TW \), and \( D_W := D \cap TW = D \) is a smooth and integrable generalized distribution on \( W \). Then from Theorem 2.8, we know that \((T^*Q, B, D, W)\) is Poisson reducible, and hence the CH system \((T^*Q, B, H, F, W)\) is Poisson reducible by the controllability coisotropic submanifold.

**Example 7.5.** (Poisson reduction of CH system by the inverse image of a coadjoint orbit) Let \((T^*Q, B)\) be a Poisson fiber bundle with associated Poisson tensor \( B \in \wedge^2(T^*T^*Q) \). Let \( G \) be a Lie group acting freely and canonically on \((T^*Q, B)\) with an associated coadjoint equivariant standard momentum map \( J : T^*Q \to \mathfrak{g}^* \). Let \( \mu \in \mathfrak{g}^* \) be a value of the momentum map \( J \) and \( O_\mu \subset \mathfrak{g}^* \) its coadjoint orbit. We know that \( J^{-1}(O_\mu) \) is an initial submanifold of \( T^*Q \), and for any \( m \in J^{-1}(O_\mu) \),

\[
T_m(J^{-1}(O_\mu)) = \ker T_mJ + \mathfrak{g} \cdot m.
\]

If \( L \) is the symplectic leaf of \((T^*Q, B)\) containing \( m \), from Proposition 2.1 and \( \mathfrak{g} \cdot m \subset T_mL \), we have that

\[
B^\sharp((T_m(J^{-1}(O_\mu)))^\circ) \subset T_m(J^{-1}(O_\mu)).
\]

Thus, \( J^{-1}(O_\mu) \) is a coisotropic submanifold of \((T^*Q, B)\). For the CH system \((T^*Q, B, H, F, W)\), if the control subset \( W = J^{-1}(O_\mu) \) is an embedded controllability coisotropic submanifold of \( T^*Q \), and \( D := B^\sharp((TW)\circ) \subset TT^*Q|_W \) is controllability distribution, then \( D_W := D \cap TW = D \) is a smooth and integrable generalized distribution on \( W \). If the presheaf \( C^\infty_{W/D_W} \) has the \((D, D_W)\)-local extension property, then the CH system \((T^*Q, B, H, F, J^{-1}(O_\mu))\) is Poisson reducible by the inverse image of a coadjoint orbit.
Example 7.6. (Poisson reduction of CH system by controllability cosymplectic submanifold) For the CH system \((T^*Q, B, H, F, W)\), assume that the control subset \(W\) is an embedded controllability cosymplectic submanifold of \(T^*Q\), and \(D := B^\#((TW)^\circ) \subset TT^*Q|_W\) is controllability distribution. In this case, by using the cosymplectic property of \(W\) we have that \(B^\#((TW)^\circ) \cap TW = \{0\}\), and hence \(D_W := D \cap TW = \{0\}\) is a trivially smooth and integrable generalized distribution on \(W\), and the presheaf \(C^\infty_{W/D_W} = C^\infty_W\) has the \((D, D_W)\)-local extension property. Then from Theorem 2.8, we know that \((T^*Q, B, D, W)\) is Poisson reducible, and hence the CH system \((T^*Q, B, H, F, W)\) is Poisson reducible by the controllability cosymplectic submanifold.

The theory of mechanical control system is a very important subject, following the theoretical development of geometric mechanics, a lot of important problems about this subject are being explored and studied. Wang in [26] studies the Hamilton-Jacobi theory of regular controlled Hamiltonian systems with the symplectic structure and symmetry, and Wang in [27, 28] apply the above work to give explicitly the motion equations and Hamilton-Jacobi equations of reduced rigid spacecraft-rotor system and reduced underwater vehicle-rotors system on the symplectic leaves by calculation in detail, which show the effect on controls in regular symplectic reduction (by stages) and Hamilton-Jacobi theory. But if we define a controlled Hamiltonian system on the cotangent bundle \(T^*Q\) by using a Poisson structure, just same as we have done in this paper and in Wang and Zhang [29], then the way given in Wang [26] cannot be used, what and how we could do? This is a problem worthy to be considered in detail. In addition, we also note that there have been a lot of beautiful results of reduction theory of Hamiltonian systems in celestial mechanics, hydrodynamics and plasma physics. Thus, it is an important topic to study the application of reduction theory and Hamilton-Jacobi theory of controlled Hamiltonian systems in celestial mechanics, hydrodynamics and plasma physics. These are our goals in future research.

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Reference

[1] R. Abraham and J. E. Marsden, Foundations of Mechanics, Second edition, Addison-Wesley, Reading, MA, 1978.

[2] R. Abraham, J. E. Marsden and T. S. Ratiu, Manifolds, Tensor Analysis and Applications, Applied Mathematical Science, 75, Springer-Verlag, New York, 1988.

[3] P. Birtea, M. Puta and T. S. Ratiu, Controllability of Poisson systems, SIAM J. Control Optim., 43(3) (2004), 937-954.

[4] A.M. Bloch and N.E. Leonard, Symmetries, conservation laws, and control, In “Geometry, Mechanics and Dynamics, Volume in Honor of the 60th Birthday of J.E. Marsden” (eds. P.Newton, P.Holmes and A. Weinstein), Springer, New York, 2002.

[5] F. Falceto and M. Zambon, An extension of the Marsden-Ratiu reduction for Poisson manifolds, Lett. Math. Phys., 85(3)(2008), 203-219.
[6] R. L. Fernandes, J. P. Ortega and T. S. Ratiu, The momentum map in Poisson geometry, Amer. J. Math., 131(5) (2009), 1261-1310.

[7] M. Jotz and T. S. Ratiu, Poisson reduction by distributions, Lett. Math. Phys., 87(1-2) (2009), 139-147.

[8] M. Jotz, T. S. Ratiu, and J. Śniatycki, Singular reduction of Dirac structures, Trans. Amer. Math. Soc., 363 (2011), 2967-3013.

[9] N.E. Leonard, and J.E. Marsden, Stability and drift of underwater vehicle dynamics: mechanical systems with rigid motion symmetry, Physica D 105(1997), 130-162.

[10] P. Libermann and C. M. Marle, Symplectic Geometry and Analytical Mechanics, Kluwer Academic Publishers, 1987.

[11] J. E. Marsden, Lectures on Mechanics, in: London Mathematical Society Lecture Notes Series, vol. 174, Cambridge University Press, 1992.

[12] J. E. Marsden, G. Misiolek, J. P. Ortega, M. Perlmutter and T. S. Ratiu, Hamiltonian Reduction by Stages, in: Lecture Notes in Mathematics, vol. 1913, Springer, 2007.

[13] J. E. Marsden and T. S. Ratiu, Reduction of Poisson manifolds, Lett. Math. Phys., 11(2) (1986), 161-169.

[14] J. E. Marsden and T. S. Ratiu, Introduction to Mechanics and Symmetry, second edition, Texts in Applied Mathematics, vol. 17, Springer-Verlag, 1999.

[15] J. E. Marsden, H. Wang, and Z. X. Zhang, Regular reduction of controlled Hamiltonian system with symplectic structure and symmetry, (arXiv: 1202.3564, To appear in Diff. Geom. Appl.).

[16] J. E. Marsden and A. Weinstein, Reduction of symplectic manifolds with symmetry, Rep. Math. Phys., 5 (1974), 121-130.

[17] H. Nijmeijer and A. J. Van der Schaft, Nonlinear Dynamical Control Systems, Springer-Verlag, 1990.

[18] J. P. Ortega and T. S. Ratiu, Momentum Maps and Hamiltonian Reduction, Progress in Mathematics, 222, Birkhäuser, 2004.

[19] M. J. Pflaum, Analytic and Geometric Study of Stratified Spaces, Lecture Notes in Mathematics, 1768, Springer-Verlag, 2001.

[20] R. Sjamaar and E. Lerman, Stratified symplectic spaces and reduction, Ann. of Math., 134 (1991), 375-422.

[21] J. Śniatycki, Singular reduction for nonlinear control systems, Reports on Mathemathical Physics, 57(2) (2006), 163-178.

[22] G. Sánchez de Alvarez, Controllability of Poisson control systems with symmetry, Contemp. Math., 97 (1989), 399-412.

[23] P. Stefan, Accessible sets, orbits and foliations with singularities, Proc. Lond. Math. Soc. 29 (1974), 699-713.
[24] H. Sussmann, Orbits of families of vector fields and integrability of distribution, Trans. Amer. Math. Soc. 180 (1973), 171-188.

[25] H. Wang, Singular reduction of regular controlled Hamiltonian system with symmetry, (2012).

[26] H. Wang, Hamilton-Jacobi theorems for regular controlled Hamiltonian system and its reductions, (2013d, arXiv: 1305.3457, To submit to J. Geom. Mech.).

[27] H. Wang, Symmetric reduction and Hamilton-Jacobi equation of rigid spacecraft with a rotor, J. Geom. Symm. Phys., 32 (2013), 87-111, (arXiv: 1307.1606).

[28] H. Wang, Symmetric reduction and Hamilton-Jacobi equation of underwater vehicle with internal rotors, (2013e, arXiv: 1310.3014).

[29] H. Wang and Z. X. Zhang, Optimal reduction of controlled Hamiltonian system with Poisson structure and symmetry, Jour. Geom. Phys., 62 (5) (2012), 953-975.

[30] A. Weinstein, The local structure of Poisson manifolds, Jour. Diff. Geom., 18 (1983), 523-557.