GLOBAL STABILISATION FOR A CLASS OF STOCHASTIC CONTINUOUS NON-LINEAR SYSTEMS WITH TIME-VARYING DELAY

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1 INTRODUCTION

Time-delay phenomenon is widespread in most real plants, such as vehicle suspension systems, information transmission process and manipulator systems. In order to eliminate the negative effects of time delays that may cause system performance to become unstable or even crash, a Lyapunov–Krasovskii (L-K) functional or Lyapunov–Razumikhin stability technique has been adopted [1–3]. Subsequently, a lot of significant achievements have been developed for non-linear systems with constant/time-varying delays [4–9]. As another major obstacle often encountered in non-linear control, stochastic factors exist widely in practical systems in the form of internal errors, external noises, unknown parameter disturbances and so on, which inevitably limit the performance of the system and lead to undesirable inaccuracy or instability [10, 11]. Therefore, numerous efforts have strived for the control design of stochastic time-delay non-linear systems with various structures or growth conditions [12–30]. The problem of state-feedback stabilisation for a class of stochastic time-delay non-linear systems was studied in [12, 13]. Moreover, the output-feedback controller was designed in [14] for a class of stochastic time-delay non-linear systems. Thereupon, [15] promoted the stabilisation result to a more general class of systems. Furthermore, [16, 17] investigated the fuzzy control and neural control of the stochastic time-delay systems. The adaptive control and output-feedback control were discussed in [18] and [19] for a class of stochastic time-delay systems with perturbations or input saturation. Additionally, some excellent results for robust stability or robust $H_{\infty}$ control of stochastic non-linear systems were presented in [20–24].

It is worth emphasising that the systems discussed above are strict-feedback stochastic time-delay systems; that is to say, the powers of systems are equal to one. On the other hand, stochastic high-order non-linear system is also an important class of stochastic systems. Due to the Jacobian linearisation of high-order non-linear systems are neither controllable nor feedback linearisable, with the effect of stochastic noise, the control and analysis is nontrivial. In fact, many excellent works have been done on the stabilisation of stochastic high-order...
Preliminaries

For example, by employing homogeneous domination manner, the global stabilisation of stochastic time-delay non-linear systems with high-order power was studied in [31] in feed-forward form, and then, [32] discussed the finite-time stabilisation of stochastic high-order non-linear systems in strict-feedback form. Moreover, based on the adding a power integrator and the L-K functional, the global tracking control was solved for stochastic high-order non-linear systems with time-varying delay by output-feedback [33] and adaptive design [34], respectively.

However, when a stochastic non-linear system is continuous but not smooth, in which the power of the considered system is less than one, those methods developed in the literature above are inapplicable because they still require some smoothness of the system. Therefore, the stabilisation problem of low-order stochastic time-delay non-linear systems has been rarely studied.

In particular, for the deterministic systems in which the effects of stochastic factors are not considered, the global finite-time stabilisation was settled in [35] for a class of continuous non-linear systems. Moreover, for stochastic delay-free systems, the finite-time control for a class of low-order stochastic non-linear systems was addressed in [38–40] by output feedback or state feedback for a class of low-order stochastic non-linear systems with input delay or state stabilisation was settled in [35] for a class of continuous non-linear systems. If stochastic factors are not considered, the global finite-time stabilisation was settled in [35] for a class of continuous non-linear systems. For example, by employing homogeneous domination manner, the global stabilisation of stochastic non-linear systems was addressed in [38–40] by output feedback or state feedback for a class of low-order stochastic non-linear systems with input delay or state stabilisation was settled in [35] for a class of continuous non-linear systems.

2 PRELIMINARIES AND PROBLEM FORMULATION

2.1 Preliminaries

Consider stochastic time-delay non-linear system

\[ dx = f(t, x(t), x(t - \tau(t)))dt + g(t, x(t), x(t - \tau(t)))d\omega(t), \]

with initial data \( \{x(\cdot) : -\tau \leq \xi \leq 0\} = \xi \in C_{P_0}^{0}([-\tau, 0]; \mathbb{R}^n) \) and \( \forall \tau \geq 0 \), where \( \tau(t) : \mathbb{R}^+ \to [0, \tau] \) is a Borel measurable function; \( \omega(t) \) is an \( r \)-dimensional standard Wiener process defined on a complete probability space \( \{\Omega, \mathcal{F}, P\} \), where \( \Omega \) is a sample space, \( \mathcal{F} \) is a \( \sigma \)-field, \( P \) is the probability measure; \( f(\cdot): \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) and \( g(\cdot): \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{d_\mathbb{R}^n} \) are unknown continuous functions.

Two definitions and four key lemmas are provided in advance.

Definition 1 ([3]). For any given function \( V(x, t) \in C^{2,1} \) of system (1), the differential operator \( \mathcal{L} \) is defined as

\[ \mathcal{L}V = \frac{\partial V}{\partial t} + \frac{1}{2} \text{Tr} \left\{ g^T \frac{\partial^2 V}{\partial x \partial x^T} g \right\}; \]

\[ \frac{1}{2} \text{Tr} g^T \frac{\partial^2 V}{\partial x \partial x^T} g \]

is known as the Hessian term of \( \mathcal{L} \).

Definition 2 ([41]). The equilibrium \( x(\cdot) = 0 \) of system (1) is said to be globally asymptotically stable in probability, if for any \( \varepsilon > 0 \), there exists a function \( \mu(\cdot, \cdot) \in C \mathcal{L} \) such that \( P\{|x(t)| \leq \mu(\|\xi\|, t)\} \geq 1 - \varepsilon \) for any \( t \geq 0 \), \( \xi \in C_{P_0}^{0}([-\tau, 0]; \mathbb{R}^n) \setminus \{0\} \), where \( \|\xi\| = \sup_{\xi \in [-\tau, 0]} |x(\xi)| \).

Lemma 1 ([42]). For \( p, q \in \mathbb{R}, r \geq 1 \), one has

\[ |p - q|^{\frac{1}{r}} \leq 2^{\frac{1}{r-1}} |p^r - q^r|, \quad |\frac{1}{r} - \frac{1}{r-1}| \leq 2^{\frac{1}{r-1}} |p - q|^{\frac{1}{r}} \]

\[ |p|^{\frac{1}{r}} + |q|^{\frac{1}{r}} \leq 2^{\frac{1}{r-1}} |p|^{\frac{1}{r}} + |q|^{\frac{1}{r}} \leq 2^{\frac{1}{r-1}} (|p| + |q|)^{\frac{1}{r}}, \]

where \( r \) is a positive odd integer satisfying \( |p^r - q^r| \geq |p - q|^{\frac{1}{r}} \).
and for any \( r > 0 \) and \( p_1, \ldots, p_n \in \mathbb{R} \), then
\[
(|p_1| + \cdots + |p_n|)^r \leq \max\{n^{r-1}, 1\}(|p_1|^r + \cdots + |p_n|^r).
\]

**Lemma 2** ([42]). For given positive real numbers \( m, n \) and a function \( a(x, y) \), there holds for all \( x, y \in \mathbb{R} \),
\[
|a(x, y)x^{m}y^n|
\leq c(x, y)|x|^{m+n} + \frac{n}{m+n} \left( \frac{m}{m+n+c(x, y)} \right)^{\frac{n}{m}} |a(x, y)|^{\frac{m+n}{m}} |y|^{m+n},
\]
where \( c(x, y) > 0 \).

**Lemma 3** ([42]). If the function \( g : [c, d] \to \mathbb{R} \) is monotone continuous and satisfies \( g(c) = 0 \), then
\[
\int_a^b g(t) dt \leq |g(d)|d - c.
\]

**Lemma 4** ([41]). For system (1), if there exists a \( C^2 \)-function \( V(x(t), t) \), a non-negative function \( W(x) \), and two class \( \mathcal{K}_\infty \) functions \( \mathcal{G}_1(\cdot) \), \( \mathcal{G}_2(\cdot) \) such that
\[
\mathcal{G}_1(\|x(t)\|) \leq V(x(t), t) \leq \mathcal{G}_2(\left( \sup_{-\tau \leq s \leq 0} \|x(t + s)\| \right)),
\]
\[
\mathcal{L}V(x(t), t) \leq -W(x),
\]
then (i) there exists a unique strong solution on \([-\tau, \infty)\);
(ii) the equilibrium \( x(t) = 0 \) is globally asymptotically stable in probability and \( P\{\lim_{t \to \infty} x(t) = 0\} = 1 \) if \( f(t, 0, 0) \equiv 0 \), \( g(t, 0, 0) \equiv 0 \) and \( W(x(t)) = \gamma(\|x(t)\|) \) with a class \( \mathcal{K}_\infty \) function \( \gamma(\|x(t)\|) \).

### 2.2 Problem formulation

Consider the stochastic time-delay non-linear system

\[
\left\{
\begin{array}{l}
dz(t) = \zeta(t) dt + f_1(t, z, z(t - \tau(t)) + u(t - \tau(t))) dt \\
\quad + g_1(t, z, z(t - \tau(t)) + u(t - \tau(t))) d\omega(t), \\
dz(t) = u dt + f_2(t, z, z(t - \tau(t)) + u(t - \tau(t))) dt \\
\quad + g_2(t, z, z(t - \tau(t)) + u(t - \tau(t))) d\omega(t),
\end{array}
\right.
\]
\[
y(t) = \zeta(t), \quad i = 1, \ldots, n - 1,
\]

where \( \zeta = [\zeta_1, \ldots, \zeta_n]^{T} \in \mathbb{R}^n \) is measurable state, \( u \in \mathbb{R} \) and \( y \in \mathbb{R} \) are control input and system output. \( [\zeta(t - \tau(t)) + u(t - \tau(t))]^{T} \in \mathbb{R}^n \). \( r \in \mathbb{R}_{r \leq 1} \). The time-varying delay \( \tau(t) \) is bounded and \( 0 < \tau(t) \leq \phi < 1 \) with a known constant \( \phi \). \( \omega(t) \) and the initial state are defined as that in (1). The drift term \( f_1 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) and the diffusion term \( g_1 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) are the same as (1) and satisfy \( f_1(t, 0, 0, 0) = 0 \) and \( g_1(t, 0, 0, 0) = 0 \), \( i = 1, \ldots, n \).

The objective of this paper is to construct a state-feedback controller such that the solution of the closed-loop system is globally asymptotically stable in probability. To this end, the following assumption is needed:

**Assumption 1.** For each \( i = 1, \ldots, n \), there exist two positive constants \( c_1, c_2 \) such that
\[
|f_i(\cdot)| \leq \rho^{-1} c_1 \sum_{j=1}^{n} (|z_j(\cdot)|^r + |z_j(\cdot - \tau(\cdot))|^r)
\]
\[
+ |u(\cdot)|^r + |u(\cdot - \tau(\cdot))|^r),
\]
and
\[
\|g_i(\cdot)\| \leq \rho^{-1/2} c_2 \sum_{j=1}^{n} (|z_j(\cdot)|^r + |z_j(\cdot - \tau(\cdot))|^r)
\]
\[
+ |u(\cdot)|^r + |u(\cdot - \tau(\cdot))|^r),
\]
where \( \rho > 0 \), \( r \) is the power of system (2), and \( \tau = 1 + r \).

**Remark 1.** Assumption 1 indicates that both state delay and input delay are considered in diffusion and Hessian terms simultaneously, which destroys the original form of lower-triangular where the traditional backstepping method can usually be directly used to control design. Furthermore, the appearance of \( z_j(\cdot - \tau(\cdot)) \) leads to the obstacle of completely non-differentiable, which results in the fact that all the control methods that require smoothness in the existing literature are no longer applicable. Therefore, the results on the global stabilisation for the only continuous but non-smooth stochastic non-linear systems are quite scarce. Hence, it is significant and challenging to study the global stabilisation of only continuous but non-smooth stochastic non-linear systems. On the other hand, “smooth” means “sufficiently often differentiable,” most often \( C^\infty \) (infinitely often differentiable). This indicates that the requirement of “smooth” system is much stronger than “continuous” one. That is to say, a stochastic non-linear system may be continuous but non-smooth, but the smooth stochastic non-linear system must be continuous. Hence, the results proposed in this paper can be applied to a wider range of systems and is more general in theory.

**Remark 2.** Compared with the exiting literature, both the effects of time-varying delays and the problem of completely non-differentiable produced by low-order powers have to be faced in this paper. Actually, the globally adaptive control and output-feedback control were concerned in \([18, 19]\) for a class of stochastic non-linear-time delay systems with the power \( r = 0 \). In \([13, 31-34]\), the global stabilisation and tracking control were investigated for the stochastic high-order non-linear-time delay systems with various structures and conditions. In addition, without involving the effects of time delays, \([38-40]\) designed the finite-time controller for a class of stochastic low-order non-linear systems in a lower-triangular form or upper-triangular form. In view of these facts, it is meaningful and challenging to study the global stabilisation for a more general class of stochastic non-linear systems with both time-varying delays and low-order powers.
3 | MAIN RESULTS

The main results are stated in the following:

**Theorem 1.** If system (2) satisfies Assumption 1, under the state feedback controller of the form

$$ u = - \sum_{i=1}^{n} (\prod_{j=i}^{n} \beta_j) z_i, $$

(5)

with the parameters $\beta_i, i = 1, ..., n$ determined in (9), (11) and (15), then the following hold:

(i) the closed-loop system has a unique strong solution on $[-\tau, \infty)$;
(ii) the closed-loop system is globally asymptotically stable in probability.

**Proof.** The proof is divided into three parts. On the basis of homogeneous domain technique, the controller of the nominal system (6) is constructed in the first part, and the second part designs the controller of the original system (2). Additionally, the third part aims to present the theoretical analysis of the closed-loop system composed by (2) and (5).

Part I: Design the controller $\mathbf{u}$ for nominal system (6).

First of all, the nominal system of (2) can be represented as

$$
\begin{align*}
\frac{d z_i(t)}{dt} &= z_{i+1}(t)dt, & i = 1, ..., n - 1, \\
\frac{d z_n(t)}{dt} &= u'(t)dt.
\end{align*}
$$

(6)

Then, we introduce

$$
\begin{align*}
\xi_1 &= z_1, & \alpha_2 &= -\beta_1 \xi_1, \\
\xi_2 &= z_2 - \alpha_2, & \alpha_3 &= -\beta_2 \xi_2, \\
& \vdots & \\
\xi_n &= z_n - \alpha_n, & \alpha_{n+1} &= -\beta_n \xi_n,
\end{align*}
$$

(7)

where $\alpha_2, ..., \alpha_{n+1}$ are virtual control laws in which $\beta_1, ..., \beta_n$ are positive constants.

Now, we focus on revealing the backstepping recursive procedure.

**Step 1.** Considering the Lyapunov function $V_1 = \frac{\rho}{2} \xi_1^2$, it can be verified from Definition 1 that

$$
\mathcal{L} V_1 \leq -\rho \xi_1^{1+r} + \rho \xi_1 (\xi_2 - \alpha_2),
$$

(8)

where $\rho$ is a positive parameter. Constructing

$$
\alpha_2(z_1) = -\beta_1 \xi_1, \quad \beta_1 \geq \eta^1
$$

(9)

and substituting (9) into (8), it yields that

$$
\mathcal{L} V_1 \leq -\rho \xi_1^{1+r} + \rho \xi_1 (\xi_2 - \alpha_2),
$$

(10)

**Step $k$ ($2 \leq k \leq n - 1$).** Assume at step $k - 1$, there exist a series of virtual control laws

$$
\alpha_l(z_1, z_2, ..., z_{l-1}) = -\beta_{l-1} \xi_{l-1}, \quad l = 2, ..., k
$$

(11)

that can assure the Lyapunov function candidate $V_{k-1} = \sum_{j=1}^{k-1} \frac{\rho}{2} \xi_j^2$ satisfying

$$
\mathcal{L} V_{k-1} \leq -(n - k + 2) \rho \sum_{j=1}^{k-1} \xi_j^{1+r} + \rho \xi_{k-1} (\xi_k - \alpha_k),
$$

(12)

Letting $k = 2$, it is clear to see that (12) becomes (10). Choosing $V_k = V_{k-1} + \frac{\rho}{2} \xi_k^2$, and (12) imply that

$$
\mathcal{L} V_k \leq -(n - k + 2) \rho \sum_{j=1}^{k-1} \xi_j^{1+r} + \rho \xi_{k-1} (\xi_k - \alpha_k) + \rho \xi_k \xi_{k+1}
$$

$$
+ (\beta_{k-1} |z_{k-1}| + ... + \beta_k |z_{k-2}| ... + \beta_1 |z_1|) \rho |z_k|.
$$

(13)

According to Lemmas 1 and 2, one can get

$$
\begin{align*}
\rho \xi_{k-1} (\xi_k - \alpha_k) + (\beta_{k-1} |z_{k-1}| + ... + \beta_k |z_{k-2}| ... + \beta_1 |z_1|) \rho |z_k| \\
& \leq 2^{1-r} \rho |z_{k-1}| \xi_k^{1+r} + \beta_k (|z_{k-1}| \xi_k + \beta_{k-1} |z_{k-1}|) \rho |z_k| \\
& \quad + ... + \beta_k \cdots \beta_1 (|z_{k-1}| \xi_k + \beta_1 |z_1|) \rho |z_k|
\end{align*}
$$

(14)

with the constant

$$
\lambda_k \geq \frac{r}{1+r} \left( \frac{3}{1+r} \right)^{1+\frac{r}{2}} + \beta_{k-1}
$$

$$
+ \frac{1}{1+r} \left( \frac{3r}{1+r} \right)^{\frac{k-1}{k-1}} \prod_{j=1}^{k-1} \beta_j \cdot \beta_{1+r}^{1+r}
$$

$$
+ \frac{1}{1+r} \left( \frac{3r}{1+r} \right)^{\frac{k-2}{k-2}} \prod_{j=1}^{k-2} \beta_j^{1+r}.
$$

(15)

Therefore, the $k$th virtual control law is chosen as

$$
\alpha_{k+1}(z_1, z_2, ..., z_k) = -\beta_k \xi_k, \quad \beta_k \geq (n - k + 1 + \lambda_k)^{1+r}.
$$

(16)

Substituting (14) and (15) into (13), one obtains

$$
\mathcal{L} V_k \leq -(n - k + 1) \rho \sum_{j=1}^{k} \xi_j^{1+r} + \rho \xi_k (\xi_{k+1} - \alpha_{k+1}).
$$

(16)

**Step $n$.** With the recursive manner in mind, positive constants $\beta_1, ..., \beta_n$ can be determined one by one. Select $V_n = \sum_{i=1}^{n} \frac{\rho}{2} \xi_i^2$. 


and

\[ \mathbf{u} = \alpha_{n+1} = -\beta_n \xi_n = -\sum_{j=1}^{n} \left( \prod_{i=j}^{n} \beta_i \right) \zeta_j. \] (17)

It can be deduced that

\[ \mathcal{L} V_u = \sum_{j=1}^{n} \frac{\partial V_u}{\partial \xi_j} \ddot{\xi}_j + \frac{\partial V_u}{\partial \xi_j} \dot{\theta}_j \leq -\rho \sum_{j=1}^{n} \xi_j^{1+r}. \] (18)

**Part II: Construct the actual controller \( u \) for original system (2).**

Take the same Lyapunov function

\[ V_o = V_u = \frac{\rho}{2} \sum_{j=1}^{n} \xi_j^2 \] (19)

and Definition 1 into account, it follows from (2) and (19) that

\[ \mathcal{L} V_o = \sum_{j=1}^{n} \frac{\partial V_o}{\partial \xi_j} \ddot{\xi}_j + \frac{\partial V_o}{\partial \xi_j} \dot{\theta}_j + \frac{1}{2} \text{Tr} \left( g_j^T \frac{\partial^2 V_o}{\partial \xi_j \partial \xi_j} g_j \right) + \frac{\partial V_o}{\partial \xi_j} \dot{u}. \] (20)

Then, analyse each term of the last inequality (20) one by one. Firstly, by using the same controller as in the Part I, that is to say, by choosing

\[ u = \mathbf{u} = -\sum_{j=1}^{n} \left( \prod_{i=j}^{n} \beta_i \right) \zeta_j, \] (21)

one obtains

\[ \sum_{j=1}^{n} \frac{\partial V_o}{\partial \xi_j} \ddot{\xi}_j + \frac{\partial V_o}{\partial \xi_j} \dot{\theta}_j \leq -\rho \sum_{j=1}^{n} \xi_j^{1+r}. \] (22)

To proceed further, there exist positive constants \( \theta_{11}, \theta_{12}, \theta_{21}, \theta_{22} \) such that

\[ \sum_{j=1}^{n} \frac{\partial V_o}{\partial \xi_j} \dot{\theta}_j \leq \theta_{11} \sum_{j=1}^{n} \xi_j^{1+r} + \theta_{12} \sum_{j=1}^{n} \xi_j^{1+r}(t - \tau), \]

\[ \sum_{j,k=1}^{n} \frac{1}{2} \text{Tr} \left( g_j^T \frac{\partial^2 V_o}{\partial \xi_j \partial \xi_k} g_k \right) \leq \theta_{21} \sum_{j=1}^{n} \xi_j^{1+r} + \theta_{22} \sum_{j=1}^{n} \xi_j^{1+r}(t - \tau). \] (23)

Please see Appendix for the specified proof of (23). Thus, by substituting (22) and (23) into (20), one has

\[ \mathcal{L} V_o \leq -\rho \sum_{j=1}^{n} \xi_j^{1+r} + (\theta_{11} + \theta_{21}) \sum_{j=1}^{n} \xi_j^{1+r} \]

\[ + (\theta_{12} + \theta_{22}) \sum_{j=1}^{n} \xi_j^{1+r}(t - \tau) \]

\[ \leq \left( \rho - \theta_{11} - \theta_{21} \right) \sum_{j=1}^{n} \xi_j^{1+r} \]

\[ + (\theta_{12} + \theta_{22}) \sum_{j=1}^{n} \xi_j^{1+r}(t - \tau). \] (24)

Define

\[ V'(\cdot) = V_o(\cdot) + W_o, \quad W_o := \frac{\varphi_1}{1 - \varphi} \int_{-\tau(t)}^{t} \xi_j^{1+r}(s) \, ds, \] (25)

with \( \varphi_1 = \theta_{12} + \theta_{22} \) is a positive parameter. It follows from (24) and (25) that \( V' \in C^2 \) and

\[ \mathcal{L} V' \leq -\left( \rho - \theta_{11} - \theta_{21} - \frac{\varphi_1}{1 - \varphi} \right) \sum_{j=1}^{n} \xi_j^{1+r}(t). \] (26)

Moreover, if one selects appropriate parameter \( \rho \) to satisfy

\[ \rho - \theta_{11} - \theta_{21} - \frac{\varphi_1}{1 - \varphi} \geq 1, \] (27)

there holds

\[ \mathcal{L} V' \leq -\sum_{j=1}^{n} \xi_j^{1+r}. \] (28)

**Remark 3.** The condition \( 0 < \tau(t) \leq \varphi < 1 \) is necessary, which ensures that the positive time-delay terms in (26) can be canceled out by selecting the appropriate L-K functional in (25), so that (28) can be obtained. Furthermore, the condition is general, which also can be found in a lot of literature with stochastic time-delays, such as [16, 19, 33, 34].

**Part III: Theoretical analysis of the closed-loop system composed by (2) and (5).**

On the one hand, it can be inferred from (25) and (28) that \( V' \) is \( C^2 \), non-negative, radially unbounded and \( \mathcal{L} V' \leq 0 \). Thus, for any initial data, Lemma 4 implies that the closed-loop system (2) and (5) has a solution. On the other hand, defining \( \xi = [\xi_1, \ldots, \xi_n]^T \), and \( \pi(\xi) \triangleq \sum_{j=1}^{n} \xi_j^{1+r} \), and considering the fact that \( 1 + r \) is an even num-
ber, it is clear to see that $\pi(\xi)$ is continuous, positive definite, and radially unbounded. With the aid of lemma 4.3 in [43], there exist two class $\mathcal{K}_\infty$ functions $\alpha_1(\cdot), \alpha_2(\cdot)$ that satisfy

$$\alpha_1(\|\xi\|) \leq \pi \leq \alpha_2(\|\xi\|).$$

(29)

It follows from (28) and (29) that

$$\mathcal{L}V \leq -\sum_{j=1}^{n} \xi_1^{1+\varepsilon} = -\pi \leq -\alpha_1(\|\xi\|).$$

(30)

Hence, there exists a function $\varpi_1(\|\xi\|) \in \mathcal{K}_\infty$ such that

$$V(\xi, t) \geq \frac{\rho}{2} \sum_{j=1}^{n} \xi_2^{2} \geq \varpi_1(\|\xi\|).$$

(31)

In addition, letting $\xi(t) = 0, \xi \in [-\tau, 0)$, with the mean value theorem and Lemma 3 in mind, one has

$$V(\xi(t), t)$$

$$= \frac{\rho}{2} \sum_{j=1}^{n} \xi_2^{2} + \frac{\alpha_0}{1-\delta} \sum_{j=1}^{n} \int_{t-\tau(t)}^{t} \xi_1^{1+\varepsilon}(\tilde{\Theta}) d(\tilde{\Theta})$$

$$= \frac{\rho}{2} \sum_{j=1}^{n} \xi_2^{2} + \frac{\alpha_0}{1-\delta} \sum_{j=1}^{n} \tau \cdot \xi_1^{1+\varepsilon}(\tilde{\Theta})$$

$$\leq \frac{\rho}{2} \sum_{j=1}^{n} \sup_{\tau \leq 0} \xi_2^{2} (\tau + \varepsilon) + \frac{\alpha_0}{1-\delta} \sum_{j=1}^{n} \tau \cdot \sup_{\tau \leq 0} \xi_1^{1+\varepsilon}(\tau + \varepsilon)$$

$$\leq \varpi_2(\sup_{\tau \leq 0} \|\xi(\tau + \varepsilon)\|)$$

(32)

with $\tilde{\Theta} \in [t-\tau(t), t]$ and $\varpi_2 \in \mathcal{K}_\infty$.

Consequently, with the aid of Lemma 4, from (28), (31), (32), one can obtain that the closed-loop system (2) and (5) has a unique strong solution on $[-\tau, \infty)$. Moreover, the equilibrium $\xi = 0$ is globally asymptotically stable in probability. From (7) and Definition 2, it can be directly verified that the origin $z = 0$ of system (2) is also globally asymptotically stability in probability. This completes the whole proof.

**Remark 4.** On the basis of the homogenous domination approach, a concise but effective controller is constructed for the nominal system, avoiding the explosion of computation caused by the backstepping recursive control to the diffusion and Hessian terms directly. Then, the control parameter $\rho$ is presented deliberately, to regulate those non-linearities produced in the controller design for the original complete system. Afterward, an appropriate Lyapunov–Krasovskii functional is chosen so that the globally asymptotic stability of the closed-loop system can be assured with less control cost. Furthermore, the saving of control cost is reflected in three aspects: (i) The Equation (5) shows that the controller is designed to be more concise in form; (ii) Comparing with [34, 38], the value of control $u$ is reduced to ensure the stability of the closed-loop system in about the same time; (iii) The Equation (5) shows that the controller is designed to be more concise in form; (ii) Comparing with [34, 38], the value of control $u$ is reduced to ensure the stability of the closed-loop system in about the same time; (iii) Accordingly, the closed-loop system is clarified.

### 4 Simulation Example

Consider the stochastic continuous time-delay non-linear system

$$\begin{cases}
    d\zeta = \frac{5}{7} \zeta + \left( \frac{1}{10} \sin \zeta + \frac{5}{14} (t - \tau(t)) \right) dt + \frac{1}{6} \zeta d\omega(t), \\
    d\eta = \frac{5}{7} \eta + \left( \frac{1}{5} \cos \zeta + \frac{5}{10} \zeta (t - \tau(t)) \right) dt \\
    + \frac{1}{6} \sin \zeta \sin \zeta d\omega(t).
\end{cases}$$

(33)

Apparently, Assumption 1 is satisfied with $c_1 = \frac{1}{5}$ and $c_2 = \frac{1}{6}$. With the aid of the control design in last section, there hold

$$r = \frac{5}{7}, \quad \beta_1 \geq \nu^2 = \frac{27}{5},$$

$$\beta_2 \geq (1 + A_2) \frac{1}{7} = \left( 1 + \frac{5}{12} \left( \frac{6}{7} \right)^2 \right)^{\frac{1}{7}} + \left( \frac{5}{12} \left( \frac{4}{7} \right)^7 \right)^{\frac{1}{7}} \beta_1^{\frac{1}{49}} \frac{7}{5}$$

and the controller can be designed as

$$u = -\beta_1 \beta_2 \zeta_1 - \beta_2 \zeta_2 = -135 \zeta_1 - 45 \zeta_2.$$  

(34)

For demonstration, we choose the initial condition of the system as $[\zeta_1(0), \zeta_2(0)]^T = [-0.5, 0.6]^T$. Without control input, Figure 1 presents the trajectories of states $\zeta_1, \zeta_2$ under the controller $u = 0$. Under the controller in (34), Figures 2 and 3 show that the controller (34) stabilises the closed-loop system and reveal the responses of (33) and (34) where the effectiveness of the provided control strategy is clarified.
By choosing same initial values as $[x_1(0), x_2(0)]^T = [-0.5, 0.6]^T$, Figures 4–6 show that the convergent time of this paper is much less than that of existing result [40], and although the convergent time of this paper is roughly the same as in [38], the value of control required in this paper is much less than [38]. This paper also deals with the undesired negative effects of time-varying delay in addition. Therefore, the controller proposed in this paper is more efficient than those in existing results [38] and [40].

Besides, the relation between the convergent time and control $u$ and the values of $\beta_1, \beta_2$ is verified through Figures 7 and 8. Specifically, Figure 7 shows that the convergence rate of the states becomes faster if $\beta_1, \beta_2$ become larger, but it can be seen from Figure 8 that the cost is the huge increase of con-
control value. Therefore, in order to save control cost, we choose smaller $\beta_1, \beta_2$ in (34) to achieve satisfactory results.

5 | CONCLUSION

This note has studied the global stabilisation problem for a class of stochastic low-order non-linear systems corrupted by both state delay and input delays. Based on the homogeneous domination approach, a delay-independent controller has been constructed for the nominal system, which avoided the disadvantage of the explosion of computation caused by backstepping recursive method to drift and diffusion terms. Then, by introducing the appropriate Lyapunov–Krasoviskii functional and the design parameter, the globally asymptotic stability of the closed-loop system has been guaranteed. A natural problem under investigation is how to achieve the finite-time stabilisation of the stochastic time-delay non-linear systems. In addition, how to design the output-feedback controller for stochastic low-order non-linear systems is also a difficult but valuable topic. Besides, another open problem is whether the scheme could be used to control the physical systems.

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APPENDIX A

Proof of inequality (23). Firstly, the controller (5) indicates that

$$
\begin{align*}
&c_i \sum_{j=1}^{n} |u^j(t) + u(t - \tau(t))| \\
&\leq n\beta_{\infty}(\|\xi_0^j\| + \|\xi(t - \tau(t))\|).
\end{align*}
$$

(A.1)

It follows from (7) that

$$
\begin{align*}
\dot{\xi}_k &= \ddot{\xi}_k + \beta_k \dot{\xi}_{k-1} \\
&= \ddot{\xi}_k + \beta_k \dot{\xi}_{k-1} + \cdots + \beta_{k-1} \beta_k \cdots \beta_1 \zeta_1 \\
&= \ddot{\xi}_k + \sum_{j=1}^{k-1} \prod_{i=j}^{k-1} \beta_i \zeta_i, \quad 1 \leq k \leq n.
\end{align*}
$$

(A.2)

Then, for each $i = 1, \ldots, n$, Assumption 1 implies that there exists a positive constant $\gamma_i$ such that

$$
|f_i(t)| \leq \rho^{-1} i \sum_{j=1}^{i} (|\zeta_j| + |\zeta_j(t - \tau(t))|) + |\zeta_i^j| + |u(t - \tau(t))|.
$$

\begin{align*}
&\leq \rho^{-1} i \sum_{j=1}^{i} (|\zeta_j| + \beta_j \zeta_{j-1} + |\zeta_j(t - \tau(t))|) + |\zeta_i^j| + |u(t - \tau(t))| \\
&\quad + \beta_{j-1} \zeta_{j-1}(t - \tau(t)) + \beta_2 |\zeta_j| + \beta_1 \zeta_j(t - \tau(t)) | \\
&\leq \rho^{-1} i \sum_{j=1}^{n} (|\zeta_j| + |\zeta_j(t - \tau(t))|).
\end{align*}

(A.3)

where $\gamma_i = c_i \cdot \max_{1 \leq k \leq i} \{1 + \beta_k^2, 1 + n\beta_k^2\}$.

In view of the fact that $V_i = \frac{\beta_i}{2} n \sum_{j=1}^{n} \eta_j^2$ and $\beta_1, \ldots, \beta_n$ are positive, for $i = 1, \ldots, n$, there exists a suitable constant $\gamma_1$ such that

$$
\frac{\partial V_i}{\partial t} \leq \rho \left( |\eta_j| + \sum_{j=1}^{n} \left( \prod_{j=1}^{i-1} \beta_j \right) |\xi_k| \right) \leq \rho \gamma_1 \sum_{j=1}^{n} |\eta_j|. \quad (A.4)
$$

With the help of Lemma 2, one can arrive at

$$
\begin{align*}
\gamma_1 \gamma_1 n \sum_{k=1}^{n} |\eta_j| \sum_{k=1}^{n} |\xi_k| &\leq \frac{1}{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \xi_{1+r}^j + \frac{\gamma_2}{n} \sum_{j=1}^{n} \xi_{1+r}^j, \quad (A.5)
\end{align*}
$$

$$
\begin{align*}
\gamma_1 \gamma_1 \gamma_1 \sum_{j=1}^{n} |\eta_j| \sum_{k=1}^{n} |\xi_k(t - \tau(t))| &\leq \frac{1}{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \xi_{1+r}^j + \gamma_2 \sum_{j=1}^{n} \xi_{1+r}^j(t - \tau(t)), \quad (A.6)
\end{align*}
$$

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where $\gamma_2$ is a positive constant. Thus,

$$\sum_{j=1}^{n} \frac{\partial V}{\partial y_j} f_j \leq \theta_1 \sum_{j=1}^{n} \frac{\partial \xi_j}{\partial y_j} \psi_{1} + \theta_2 \sum_{j=1}^{n} \frac{\partial \xi_j}{\partial y_j} (t - \tau(t)), \quad (A.7)$$

where $\theta_1 = 2 + \gamma_2$ and $\theta_2 = \gamma_2$.

Now, we prove the second inequality of (23). With the similar mind, for $l = 1, \ldots, n, \kappa = 1, \ldots, n$, it is obviously to see that

$$|\xi_k|^{\frac{1}{2}} \leq |\xi_k|^{\frac{1}{2}} + \beta_{k-1} |\xi_{k-1}|^{\frac{1}{2}},$$

$$|\xi_k(t - \tau(t))^{\frac{1}{2}} \leq |\xi_k(t - \tau(t))|^{\frac{1}{2}} + \beta_{k-1} |\xi_{k-1}(t - \tau(t))|^{\frac{1}{2}},$$

$$|\nu(t)|^{\frac{1}{2}} = \beta u^{\frac{1}{2}} |\xi_u|^{\frac{1}{2}},$$

$$|\nu(t - \tau(t))|^{\frac{1}{2}} = \beta u^{\frac{1}{2}} |\xi_u(t - \tau(t))|^{\frac{1}{2}}.$$  

Assumption 1 indicates that

$$\|g\| \leq \rho^{-\frac{1}{2}} \sum_{k=1}^{n} \left( |\xi_k|^{\frac{1}{2}} + |\xi_k(t - \tau(t))|^{\frac{1}{2}} + |\nu(t)|^{\frac{1}{2}} + |\nu(t - \tau(t))|^{\frac{1}{2}} \right)$$

$$\leq \rho^{-\frac{1}{2}} \sum_{2 \leq k \leq n} \left( 1 + \beta_{k}^{\frac{1}{2}} \right) \sum_{j=1}^{n} \left( |\xi_j|^{\frac{1}{2}} + |\xi_j(t - \tau(t))|^{\frac{1}{2}} \right).$$

Furthermore, for $1 \leq l \leq \kappa \leq n$, one has

$$\|g\| \|g\| \leq \rho^{-\frac{1}{2}} \sum_{k=1}^{n} \left( |\xi_k^{1+r} + |\xi_k^{1+r}(t - \tau(t))|^{\frac{1}{2}} \right) \quad (A.9)$$

with $\tilde{\gamma}_2 = 2m_{\gamma}^2 \max_{2 \leq k \leq n} \left\{ (1 + \beta_k)^{1+r} \right\}$, and (7) shows that

$$\frac{\partial^2 V_{\gamma}}{\partial y_{\gamma} \partial \gamma_{\gamma}} \leq \rho^{-\frac{1}{2}} \sum_{j=1}^{n} \left( |\xi_j^{1+r} + |\xi_k^{1+r}(t - \tau(t))|^{\frac{1}{2}} \right) \quad (A.10)$$

Therefore,

$$\sum_{k=1}^{n} \frac{1}{2} \text{Tr} \left\{ g^T \frac{\partial^2 V_{\gamma}}{\partial y_{\gamma} \partial \gamma_{\gamma}} g \right\} \leq \sum_{1 \leq l < \kappa \leq n} \left| \frac{\partial^2 V_{\gamma}}{\partial y_{\gamma} \partial \gamma_{\gamma}} \right| \|g\| \|g\| \leq \frac{n(n + 1)}{2} \gamma_2 \left( \sum_{k=1}^{n} \left( |\xi_k^{1+r} + |\xi_k^{1+r}(t - \tau(t))|^{\frac{1}{2}} \right) \right) \leq \theta_2 \sum_{k=1}^{n} \xi_k^{1+r} + \theta_21 \sum_{j=1}^{n} \xi_j^{1+r}(t - \tau), \quad (A.11)$$

where $\theta_21 = \theta_2 = \frac{n(n + 1)\tilde{\gamma}_2}{2}$. \hfill $\blacksquare$