EXISTENCE OF EF-EQUIVALENT NON ISOMORPHIC MODELS

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Abstract. We prove the existence of pairs of models of the same cardinality $\lambda$ which are very equivalent according to EF games, but not isomorphic. We continue the paper [4], but we don’t rely on it.

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There had been much study of equivalence relations between models. When we study such an equivalence relation, one of the basic questions is - is this relation actually trivial - equivalent models are isomorphic? For example, countable models which are elementary equivalent in $L_{\omega_1,\omega}$ are isomorphic. (Scott showed this in [9] for countable vocabulary, and Chang generalized it in [1] for any vocabulary). For $\lambda = \text{cf}(\lambda) > \aleph_0$, Morley gave (without publishing) a counter example - a pair of $L_{\infty,\lambda}$ equivalent models of size $\lambda$ which are not isomorphic. Shelah([8]) gave such an example for almost every singular $\lambda$.

Those questions also relate to classification theory: The existence of ”strongly” equivalent models which are not isomorphic is a non-structure property for a class of models. On the other side, if ”not too strong” equivalence relation is actually the isomorphism relation, this is a structure property. (See [8] and [2]).

One of the equivalence relations studied in this context, is equivalence under $E(\text{hrenfeucht-Fraisse})$ games. A detailed discussion of $E(\text{F})$ games and their history, can be found in [3]. The general structure of an $E(\text{F})$ game on a pair of models is as follows:

There are two players - isomorphism player, who we call ISO and anti-isomorphism player, who we call AIS. During the game, AIS chooses members of the models, and ISO defines ”interactively” a partial isomorphism between the models - in every move he has to extend that partial isomorphism, such that the elements chosen by AIS will be contained in the domain or in the range. The isomorphism player looses the game if at some point, he cannot find a legal move. If he doesn’t lose, he wins. We limit the length of the game and the number of elements that AIS may choose at each move. (Because, if AIS can list all the members of one of the models, then the game is not interesting). In [4], the games were with fixed length. In this paper, we deal with $E(\text{F})$ games approximated by trees - the length of the game is limited by adding the demand that in each move, AIS has to choose a node in some fixed tree $T$ (with certain properties), such that the sequence of nodes formed by his choices, is strictly increasing in the order $<^T$. If AIS cannot choose such node - he looses.

We say that two models are equivalent with respect to some $E(\text{F})$ game $\mathcal{D}$, if ISO has a winning strategy in $\mathcal{D}$ played on those models.

In [4] it was proved that if $\lambda = \text{cf}(\lambda) = \lambda^{\aleph_0}$, then there are non isomorphic models of size $\lambda$ which are $E(\text{F})_{\alpha,\lambda}$ equivalent for every $\alpha < \lambda$. Where $E(\text{F})_{\alpha,\lambda}$ equivalence means that they are equivalent under every $E(\text{F})$ game with $\alpha$ stages, such that AIS has to choose $< \lambda$ members of the models at each stage. There was also a result for $\lambda$ singular, with a necessary change of the
equivalence relation.

Here we generalize the results in 2 ways: first, we move to EF games approximated by trees instead of fixed-length games (See Hyttinen and Tuuri in [2] who investigated such games in the context of classification theory). Second, we give results also for $\lambda > \beth_{\omega}$ without the assumption $\lambda = \lambda^{\aleph_0}$, where we use PCF theory to have some "approximation" instead of $\lambda = \lambda^{\aleph_0}$.

In section 1 we prove that for regular $\lambda = \lambda^{\aleph_0}$ for some class of reasonably large trees (See detailed discussion justifying the choice, in the beginning of section 1) for every tree from that class there are non isomorphic models of size $\lambda$ which are equivalent under EF games approximated by that tree, such that in each move AIS is allowed to choose $< \lambda$ members of the models (See definition 1.1).

In section 2 we do the parallel for singular $\lambda$. But for singular $\lambda$, if we allow AIS to choose $< \lambda$ elements in each move, and the tree has a branch of length $\text{cf}(\lambda)$, then the game is not interesting, because AIS can choose all the members of the models during the game. So we have to be more careful - we allow AIS to choose only one element in each move. This is still a generalization of the result for such $\lambda$ in [3] - see the discussion at the beginning of section 2.

In section 3 we prove that for regular $\lambda > \beth_{\omega}$, for every tree of size $\lambda$ without a branch of length $\lambda$, there are non isomorphic models of size $\lambda$ which are equivalent under the EF game approximated by that tree, such that in each move AIS is allowed to choose $< \lambda$ members of the models.

In section 4 we prove a similar result for $\lambda > \text{cf}(\lambda) > \beth_{\omega}$. As we explained above, because of the singularity of $\lambda$, we have to restrict the number of elements that AIS is allowed to choose at each move - in stage $\alpha$, AIS has to choose $< 1 + \alpha$ members of the models.
1. Games with trees for regular $\lambda = \lambda^{\aleph_0}$

In [2] there is a construction of non-isomorphic models of size $\lambda$ which are equivalent under EF games approximated by trees of size $\lambda$ with no $\lambda$ branch, when $\lambda = \lambda^{<\lambda}$. In [4] there is such a construction under a weaker assumption on $\lambda - \lambda = \text{cf}(\lambda) = \lambda^{\aleph_0}$, but there the result is for games of any fixed length $< \lambda$, not for games which are approximated by trees. We want to generalize this result to games approximated by trees.

Now, which trees should we consider? If we limit ourselves only to trees of size $\lambda$, it seems that the set of trees will be "small". Why? - assume for example that $\lambda = \text{cf}(\lambda) = \lambda^{\aleph_0} < \lambda^{\aleph_1}$. A tree of size $\lambda$ must drop at least one of the following conditions:

1. above every node there is an antichain of size $\lambda$
2. every chain of size $\leq \aleph_1$ has an upper bound

If $\lambda \gg \aleph_1$, this kind of trees seem to be too degenerate. We could have demanded that the size of the tree will be $\leq 2^{\aleph_0}$. But it is possible that $2^{<\lambda} = 2^{\lambda}$ and it is reasonable to assume that the result will not be true in this case.

We take the middle road: we don’t limit explicitly the size of the tree, but we demand that the tree will be "definable" enough - the cause of not having a branch of length $\lambda$, is that the nodes of the tree are actually partial functions from $\lambda$ to $\lambda$ which satisfy a certain local condition. By "local" we mean that a function $f$ satisfies the condition iff any restriction of the $f$ to a countable set satisfies it. The tree order is inclusion, and there is no function from $\lambda$ to $\lambda$ which satisfies the condition. By [3] this result is indeed generalization of "for every tree of size $\lambda$ and no $\lambda$ branch".

**Definition 1.1.** For a tree $T$, a cardinal $\mu$, and models with common vocabulary $M_1, M_2$, the game $\mathcal{G}_{T,\mu}(M_1, M_2)$ between the players ISO and AIS is defined as follows:

After stage $\alpha$ in the game we have the sequence $\langle f_\beta : \beta \leq \alpha \rangle$, which is an increasing continuous sequence of partial isomorphisms from $M_1$ to $M_2$, and the sequence $\langle z_\beta : \beta \leq \alpha \rangle$ which is an increasing continuous sequence in $T$.

Stage $\alpha$ in the game is as follows:

First, AIS chooses $z_\alpha$ of level $\alpha$ of $T$, such that for every $\beta < \alpha$ $\; z_\alpha \succ^T z_\beta$.

Then,

1. if $\alpha = 0$ then $f_\alpha = \emptyset$
2. if $\alpha$ is limit then $f_\alpha = \bigcup_{\beta < \alpha} f_\beta$
3. if $\alpha = \beta + 1$ then AIS chooses $A_1 \subseteq M_1, A_2 \subseteq M_2$ such that $|A_1 \cup A_2| < 1 + \mu$. Then ISO should choose $f_\alpha$ such that:
   - $f_\alpha$ is a partial isomorphism from $M_1$ to $M_2$, $f_\beta \subseteq f_\alpha$
   - $A_1 \subseteq \text{Dom}(f_\alpha), A_2 \subseteq \text{Rang}(f_\alpha)$

The first player who cannot find a legal move loses the game. If ISO has a winning strategy for $\mathcal{G}_{T,\mu}(M_1, M_2)$, we say that $M_1, M_2$ are $EF_{T,\mu}$ equivalent.

**Definition 1.2.** We say that $\exists_{T,\lambda}$ holds, if:
(1) $\mathcal{F}$ is a set of partial functions from $\lambda$ to $\lambda$
(2) if $f$ is a partial function from $\lambda$ to $\lambda$ then $f \in \mathcal{F}$ iff for every countable $u \subseteq \text{Dom}(f)$ $f\mid u \in \mathcal{F}$
(3) there is no $f \in \mathcal{F}$ such that $\text{Dom}(f) = \lambda$

**Definition 1.3.** If $\boxdot F, \lambda$ holds, we define a tree $T_{\mathcal{F}}$ in the following way:

- the nodes are functions $f$ such that $f \in \mathcal{F}$ and $\text{Dom}(f)$ is an ordinal.
- the order is inclusion

Note that this tree does not have a branch of length $\geq \lambda$

**Remark 1.4.** If $T$ is a tree of size $\lambda$ with no $\lambda$-branch, we can assume without loss of generality that $T \subseteq \lambda$. Define $\mathcal{F}$ by $f \in \mathcal{F}$ if $f$ is a partial function from $\lambda$ to $\lambda$ such that $x < y \Rightarrow f(x) < T f(y)$. We get that $\boxdot F, \lambda$ holds, and $T$ can be embedded (as a partial order) in $T_{\mathcal{F}}$

**Theorem 1.5.** Suppose :

$(1)$ $\text{cf}(\lambda) = \lambda = \lambda^{\aleph_0}$

$(2)$ $\boxdot F, \lambda$ holds

$(3)$ $T = T_{\mathcal{F}}$

then :

There are non-isomorphic models $M_1, M_2$ of size $\lambda$ which are $EF_{T, \lambda}$ equivalent.

Proof: First, we shall define a tool for constructing models.

**Definition 1.6.** $\mathfrak{r}$ is a structure parameter if it consists of the following objects:

$(1)$ a set $I$

$(2)$ a set $J_s$ for each $s \in I$, such that if $s_1 \neq s_2$ then $J_{s_1} \cap J_{s_2} = \emptyset$.

- denote $J = \bigcup_{s \in I} J_s$

$(3)$ sets $S, T$ such that $S \subseteq I \times I$, $T \subseteq J \times J$

**Definition 1.7.** For a given structure parameter $\mathfrak{r}$ we define a model $M = M_{\mathfrak{r}}$ in the following way:

First for each $s \in I$ let $G_s$ be an abelian group generated freely by $\{x_t : t \in J_s\}$ except of the relation $\forall x (2x = 0)$. (We could have also used free group or free abelian group, But this choice makes the proof a bit simpler). We demand also that if $s_1 \neq s_2$ then $G_{s_1} \cap G_{s_2} = \emptyset$.

For $(s_1, s_2) \in S$, let $G_{s_1, s_2}$ be the subgroup of $G_{s_1} \times G_{s_2}$ generated by $\{(x_{t_1}, x_{t_2}) : (t_1, t_2) \in T \cap (J_{s_1} \times J_{s_2})\}$.

The universe of $M$ is $\bigcup_{s \in I} G_s$. The vocabulary of $M$ consists of :

$(1)$ For each $a \in M$, a unary function symbol $F_a$

$(2)$ For each $s \in I$, a unary relation symbol $P_s$

$(3)$ For each $(s_1, s_2) \in S$, a binary relation symbol $Q_{s_1, s_2}$

The interpretation of the symbols in $M$ is as follows :

$(1)$ For each $b \in M$, $s \in I$, $a \in G_s$ if $b \in G_s$ then $F_a^M(b) = a + b$; else $F_a^M(b) = b$
(2) For each \( s \in I \), \( P^M_s = G_s \)
(3) For each \((s_1, s_2) \in S\), \( Q^M_{s_1, s_2} = G_{s_1, s_2} \)

**Lemma 1.8.** Suppose \( I' \subseteq I \) and \( f \) is a function, \( f : \bigcup_{s \in I'} G_s \to M \). Then \( f \) is a partial automorphism of \( M \) iff:

1. For each \( s \in I' \), \( f(0_{G_s}) \in G_s = P^M_s \Rightarrow f(0_{G_s}) \in P^M_s = G_s \)
2. For each \( s \in I' \), \( a \in G_s \) we have \( f(a) = f(0_{G_s}) + a \)
3. For each \( s_1, s_2 \in I' \) if \((s_1, s_2) \in S\) then \((f(0_{G_{s_1}}), f(0_{G_{s_2}})) \in G_{s_1, s_2} \)

**Proof:** Suppose \( f \) is a partial automorphism then:

1. For each \( s \in I' \), \( 0_{G_s} \in G_s = P^M_s \Rightarrow f(0_{G_s}) \in P^M_s = G_s \)
2. For each \( s \in I' \), \( a \in G_s \) we have \( f(a) = f(F^M_a(0_{G_s})) = F^M_a(f(0_{G_s})) = f(0_{G_s}) + a \)
3. For each \( s_1, s_2 \in I' \) if \((s_1, s_2) \in S\) then \((0_{G_{s_1}}, 0_{G_{s_2}}) \in G_{s_1, s_2} \) (because it’s a subgroup of \( G_{s_1} \times G_{s_2} \)) but \( G_{s_1, s_2} = Q^M_{s_1, s_2} \), therefore we have \((f(0_{G_{s_1}}), f(0_{G_{s_2}})) \in G_{s_1, s_2} \)

Similar arguments show the other direction. \( \blacksquare \)

Now we shall define a structure parameter \( \mathfrak{r} \). Then define \( M = M_\mathfrak{r} \). Then we will choose elements \( a_\ast, b_\ast \in M \), define \( M_1 = (M, a_\ast), M_2 = (M, b_\ast) \) and show that \( M_1, M_2 \) are as required in theorem 1.5.

Let \( \mathfrak{r} = r_{\lambda, \mathcal{F}} \) be the following structure parameter:

1. \( I = [\lambda]^{\aleph_0} \)
2. For \( u \in I \), \( J_u \) consists of the quadruples \( t = (u, g, h, \zeta) \) where:
    a. \( g, h \) are functions from \( u \) into \( \lambda \)
    b. \( \zeta \) is a function from \( \text{supRang}(g) \cap u \) into \( \lambda \)
    c. \( \zeta \in \mathcal{F} \)
    d. \( g, h \) are weakly increasing
    e. \( g(x) = g(y) \Rightarrow h(x) = h(y) \)
    f. \( h(x) > x \)

For \( t = (u, g, h, \zeta) \) we will denote \( u = u^t \), \( g = g^t \), \( h = h^t \), \( \zeta = \zeta^t \)

3. \( S = \{ (u_1, u_2) : u_1, u_2 \in I \text{ and } u_1 \subseteq u_2 \} \)
4. \( T = \{ (t_1, t_2) : t_1, t_2 \in J, u^{t_1} \subseteq u^{t_2}, g^{t_1} \subseteq g^{t_2}, h^{t_1} \subseteq h^{t_2}, \zeta^{t_1} \subseteq \zeta^{t_2} \} \)

Let \( M = M_{\mathfrak{r}, \mathcal{F}} = M_\mathfrak{r} \) be the corresponding model. Note that \( |I| = \lambda^{\aleph_0} = \lambda \) and for each \( u \in I \), \( |J_u| = \lambda^{\aleph_0} = \lambda \), therefore \( ||M|| = \lambda \). Define:\( a_\ast = \emptyset_{G_0}, b_\ast = x_{(0,0,0,0)} \).

**Claim 1.9.** \( M_1, M_2 \) are \( \text{EF}_{\mathcal{T}, \lambda} \) equivalent

**Proof:**

**Definition 1.10.** We define a set of functions \( \mathcal{G} = \mathcal{G}(\lambda) \) with a partial order \( \leq^{\mathcal{G}} \) in the following way:

1. For an ordinal \( \alpha < \lambda \), \( \mathcal{G}_\alpha \) is the set of functions \( g \) which satisfy:
   a. \( g : \gamma \to \alpha, \gamma < \lambda \)
   b. \( g \) is weakly increasing
(2) \( G = \bigcup_{\beta < \lambda} G_{\alpha} \)

(3) For each \( g \in G \) such that \( \text{Dom}(g) = \gamma \) we define \( h_g : \gamma \rightarrow \gamma + 1 \) by:
\[
h_g(x) = \operatorname{Min}(\{ y : y < \gamma \land g(y) > g(x) \} \cup \{ \gamma \})
\]

(4) \( g_1 \leq G g_2 \) if \( g_1 \subseteq g_2 \) and \( h_{g_1} \subseteq h_{g_2} \)

Claim 1.11. \( g(x) = g(y) \Rightarrow h_g(x) = h_g(y) \)

(2) \( h_g(x) > x \)

(3) \( h_g \) is weakly increasing

(4) For every \( g_1, g_2 \in G \) \( g_1 \leq G g_2 \) iff

(a) \( \text{Dom}(g_1) = \gamma_1 \leq \gamma_2 = \text{Dom}(g_2) \), \( g_1 \subseteq g_2 \)

(b) \( \gamma_1 < \gamma_2 \) then \( g_2(\gamma_1) > g_2(x) \) for every \( x < \gamma_2 \)

(5) If \( g_1 \in G_{\alpha} \) and \( \text{Dom}(g_1) < \gamma < \lambda \) then there is \( g_2 \in G_{\alpha+1} \) such that
\[
g_1 \leq G g_2 \text{ and } \text{Dom}(g_2) = \gamma
\]

(6) If \( \delta < \lambda \) and we have \( g_{\alpha} : \alpha < \delta \) such that \( g_{\alpha} \in G_{\alpha} \) and \( \beta < \alpha \Rightarrow g_\beta \leq G g_{\alpha} \),
then \( g = \bigcup_{\alpha < \delta} g_{\alpha} \) satisfies \( g \in G_{\delta} \) and \( g \leq G g \) for each \( \alpha < \delta \)

Proof:

(1)-(3) Easy

(4) If there is \( x < \gamma_1 \) such that \( g_2(\gamma_1) = g_2(x) \) then \( h_{g_2}(x) = h_{g_2}(\gamma_1) > \gamma_1 \geq h_{g_1}(x) \)
so \( g_1 \not\leq G g_2 \). On the other direction, if \( g_1 \subseteq g_2 \) and \( g_2(\gamma_1) > g_2(x) \)
for every \( x < \gamma_1 \), then for every such \( x \): If there is \( y < \gamma_1 \) such that \( g_1(y) > g_1(x) \), let \( y' \) be the minimal \( y \) which satisfies this. We get \( h_{g_1}(x) = h_{g_2}(x) = y' \). If there is no such \( y \), we get \( h_{g_1}(x) = h_{g_2}(x) = \gamma_1 \).
Therefore we have \( h_{g_1} \subseteq h_{g_2} \).

(5) Define \( g_2 : \gamma \rightarrow \alpha + 1 \) by:

For \( x \in \text{Dom}(g_1) \) \( g_2(x) = g_1(x) \).
For \( x \in \gamma \setminus \text{Dom}(g_1) \) \( g_2(x) = \alpha \).
By (4) we get that \( g_1 \leq G g_2 \)

(6) Remember that \( \lambda \) is regular therefore \( \bigcup_{\alpha < \delta} \text{Dom}(g_{\alpha}) < \lambda \)

Now we will describe a winning strategy for ISO in the game \( \odot_{T,\lambda}(M_1, M_2) \).
In stage \( \alpha \) of the game ISO will choose a function \( g_{\alpha} \) such that:

(1) \( g_{\alpha} \in G_{\alpha} \)

(2) \( \beta < \alpha \Rightarrow g_\beta \leq G g_\alpha \)

(3) If \( \alpha \) is a successor ordinal and in stage \( \alpha \) AIS chose the sets \( A_1, A_2 \)
then for each \( u \in I \) such that \( (A_1 \cup A_2) \cap G_u \neq \emptyset \) we have \( u \subseteq \text{Dom}(g_{\alpha}) \)

The choice of \( g_{\alpha} \) is done in the following way:

(1) \( g_0 = \emptyset \)

(2) If \( \alpha \) is limit, then \( g_{\alpha} = \bigcup_{\beta < \alpha} g_{\beta} \).
By \( g_{\alpha} \in G_{\alpha} \) and \( \beta < \alpha \Rightarrow g_\beta \leq G g_\alpha \)

(3) If \( \alpha = \beta + 1 \) and in stage \( \alpha \) AIS chose the sets \( A_1, A_2 \), ISO will choose \( \gamma < \lambda \) such that \( \text{Dom}(g_\beta) < \gamma \) and \( u \subseteq \gamma \) for every \( u \in I \) such that \( (A_1 \cup A_2) \cap u \neq \emptyset \) (Such \( \gamma \) exists because \( |A_1 \cup A_2| + \aleph_0 < \lambda \)) . By \( g_{\alpha} \) there is \( g \in G_{\alpha} \) such that \( \text{Dom}(g) = \gamma \) and \( g_\beta \leq G g \). ISO will choose such a function as \( g_{\alpha} \).
Now remember that if $\alpha = \beta + 1$, then in stage $\alpha$ AIS has to choose a node on level $\alpha$, which is actually a function $\zeta_\alpha : \alpha \to \lambda$, $\zeta_\alpha \in \mathcal{F}$. Then he chooses $A_1 \subset M_1$, $A_2 \subset M_2$. Then ISO has to choose partial isomorphism $f_\alpha$ from $M_1$ to $M_2$ such that $f_\beta \subseteq f_\alpha$, $A_1 \subseteq \text{Dom}(f_\alpha)$, $A_2 \subseteq \text{Rang}(f_\alpha)$ (See 1.1). So, ISO chooses $g_\alpha$, and then defines $f_\alpha$ according to $f_\beta$, $A_1$, $A_2$, $g_\alpha$, $\zeta_\alpha$ in the following way:

$$\text{Dom}(f_\alpha) = \text{Dom}(f_\beta) \cup \bigcup \{ G_u : u \in I, (A_1 \cup A_2) \cap G_u \neq \emptyset \}.$$  

For each $u \in I$ we have $G_u \subseteq \text{Dom}(f_\alpha)$ or $G_u \cap \text{Dom}(f_\alpha) = \emptyset$. If $G_u \subseteq \text{Dom}(f_\alpha)$ we define $f_u(0) = x_t$, where $t = (u, g_\alpha | u, h_\alpha | u, \zeta_\alpha | (u \cap \text{supRang}(g_\alpha | u)))$ (Note that because $g_\alpha \in G_\alpha$, we have $\text{Rang}(g_\alpha) \subseteq \alpha = \text{Dom}(\zeta_\alpha)$).

For every $a \in G_u$ we define $f_u(a) = f_u(0) + a$. By the construction we get that if $(u_1, u_2) \in S$ then $(f_u(0), f_u(0)) \in G_{u_1} \cap G_{u_2}$ (because the corresponding couple of $t$-ies lays in $T$). Therefore by 1.8 $f_\alpha$ is a partial automorphism of $M$. We also have:

(1) For $\beta < \alpha$ $g_\beta \subseteq g_\alpha$, $h_\beta \subseteq h_\alpha$, $\zeta_\beta \subseteq \zeta_\alpha$. Therefore $f_\beta \subseteq f_\alpha$.

(2) For each $\alpha > 0$ $f_\alpha(u) = f_\alpha(0) = x_{(0,0,0,0)} = b_*$. Therefore $f_\alpha$ is a partial isomorphism from $M_1 = (M, a_*)$ into $M_2 = (M, b_*)$.

Claim 1.12. $M_1, M_2$ are not isomorphic.

Proof: It is enough to show that $M$ is rigid (doesn’t have a non-trivial automorphism).

Assume toward contradiction that $f \neq id$ is an automorphism of $M$. For each $u \in I$ we define $c_u = f(0)$. By 1.8 for each $u \subseteq w \in I$ we have $(c_u, c_w) \in G_{u,w}$.

For each $u \subseteq w \in I$ and $t = (w, g, h, \zeta) \in J_w$ we define $\pi_{w,u}(t) \in J_u$ by $\pi_{w,u}(t) = : (w, g | u, h | u, \zeta | (c_u | u))$. By the definition of $T$ we have that if $t \in J_w$, $r \in J_u$ then $(r, t) \in T$ iff $r = \pi_{w,u}(t)$. We define homomorphism $\bar{\pi}_{w,u} : G_w \to G_u$ by $\bar{\pi}_{w,u}(x_t) = x_r$ where $r = \pi_{w,u}(t)$. We get that $G_{u,w}$ is the subgroup of $G_u \times G_w$ generated by $\{(\bar{\pi}_{w,u}(x_t), x_t) : t \in J_w\}$.

Since $\{x_t : t \in J_w\}$ generate $G_w$, we get that $G_{u,w} = \{(\bar{\pi}_{w,u}(c), c) : c \in G_w\}$. Define $n(w)$ to be the length of the reduced representation of $c_w$ as a sum of the generators $\{x_t : t \in J_u\}$. For $u \subseteq w \in I$ we get $n(w) \leq n(u)$ since $c_u = \bar{\pi}_{w,u}(c_w)$ and $\bar{\pi}_{w,u}$ sends one generator to one generator. If for every $u \in I$ there is $w \in I$ such that $n(w) > n(u)$ we can find a sequence $\langle u_n : n < \omega \rangle$ such that $u_n \in I$ and $n(u_n) < n(u_{n+1})$. Define $w = \cup_{n<\omega} u_n$, we get that $n(w)$ is infinite - contradiction. Therefore, there is $u_* \in I$ such that $n(u_*)$ is maximal. Since we assumed $f \neq id$, $n(u_*) > 0$.

Choose $t_* \in J_{u_*}$ such that $x_{t_*}$ appears in the reduced representation of $c_{u_*}$. For each $u_* \subseteq w \in I$ there is a unique $t(w) \in J_w$ such that $\pi_{w,u_*}(t(w)) = t_*$ and $x_{t(w)}$ appears in the reduced representation of $c_w$. Such $t(w)$ exists because $c_{u_*} = \bar{\pi}_{w,u_*}(c_w)$. It is unique because if there were two such $t$-ies, $t_1, t_2$ then $\pi_{w,u_*}(x_{t_1}) = \pi_{w,u_*}(x_{t_2}) = x_{t_*}$. Since in $G_{u_*}$ $\forall x(2x = 0)$ it implies
$n(w) > n(u_*)$ which contradicts the maximality of $n(u_*)$.

Note that if $u \subseteq w \subseteq z \in I$ then $\pi_{z,u} = \pi_{w,u} \circ \pi_{z,w}$. Therefore, by uniqueness of $t(w)$ if $u_* \subseteq w \subseteq z \in I$ we have $t(w) = \pi_{z,w}(t(z))$. For each $u_* \subseteq w \in I$, define $g^w = g^{t(w)}$, $h^w = h^{t(w)}$, $\zeta^w = \zeta^{t(w)}$. If $u_* \subseteq w_1, w_2 \in I$ then the functions $g^{w_1}, h^{w_1}, \zeta^{w_1}$ and $g^{w_2}, h^{w_2}, \zeta^{w_2}$ are respectively compatible, since $t(w_1) = \pi_{z,w_1}(t(z))$ and $t(w_2) = \pi_{z,w_2}(t(z))$ where $z = w_1 \cup w_2$.

Define $g = \cup\{g^w : u_* \subseteq w \in I\}$
$h = \cup\{h^w : u_* \subseteq w \in I\}$
$\zeta = \cup\{\zeta^w : u_* \subseteq w \in I\}$.

We get:

(1) $\text{Dom}(g) = \text{Dom}(h) = \lambda$
(2) $g, h$ are weakly increasing
(3) $h(x) > x$
(4) $g(x) = g(y) \Rightarrow h(x) = h(y)$
(5) $\zeta \in \mathcal{F}$ (this is by $\text{1.2(2)}$)
(6) $\sup \text{Rang}(g) \subseteq \text{Dom}(\zeta)$

By $\text{1.2(3)}$ $\text{Dom}(\zeta) \neq \lambda$. Therefore by (6) $\sup \text{Rang}(g) < \lambda$. Since $g$ is weakly increasing and $\lambda$ is regular, there is $\alpha_0 < \lambda$ such that for every $\alpha_0 < \alpha < \lambda$ $g(\alpha) = g(\alpha_0)$. By (4) we get that for every $\alpha_0 < \alpha < \lambda$ $h(\alpha) = h(\alpha_0)$.

Choose $\alpha > h(\alpha_0) > \alpha_0$ and get that $h(\alpha) < \alpha$ contradicting (3). □
2. Games with trees for singular $\lambda = \lambda^{\aleph_0}$

It is clear that for $\lambda$ singular we cannot expect the same result as in the previous section, since the AIS player would be able to list all the members of $M_1, M_2$. Thus, we prove a weaker result - we allow AIS to choose only one element in each turn. We also remark in [2.2] that this result generalizes the result in [4] for such $\lambda$.

**Theorem 2.1.** Suppose :

1. $\text{cf}(\lambda) < \lambda = \lambda^{\aleph_0}$
2. $\mathbb{F}_{F, \lambda}$ holds
3. $\mathcal{T} = \mathcal{T}_F$

then :
There are non-isomorphic models $M_1, M_2$ of size $\lambda$ which are $\mathcal{E}_{F, 1}$ equivalent.

**Remark 2.2.** We can show that Theorem 2.1 generalizes the result in [4] by choosing appropriate $\mathcal{F}$. The result there shows the existence of two non-isomorphic models of size $\lambda$ which are equivalent under every $\mathcal{E}$ game of length $< \text{cf}(\lambda)$, which consists of sub-games of length $< \lambda$, such that AIS chooses the length of each sub-game before it starts, and in every sub-game he chooses one element in each move - see the definitions there. Now, an appropriate $\mathcal{F}$ can be chosen by looking at the proof there, but we will take a shortcut - we will use the result instead of the proof. Let us choose a pair of models $M_1, M_2$ as in the result in [4]. Without loss of generality assume that the universe of $M_1$ is $\lambda \times \{1\}$, and the universe of $M_2$ is $\lambda \times \{2\}$. We can take $\mathcal{F}$ to be the set of functions $f$ which satisfy the following conditions:

1. $\text{Dom}(f) \subseteq \lambda$, $\text{Rang}(f) \subseteq \lambda$
2. define a partial function $f'$ from $M_1$ to $M_2$ by:
   (a) $\text{Dom}(f') = \text{Dom}(f) \times \{1\}$
   (b) for every $\alpha \in \text{Dom}(f)$, $f'((\alpha, 1)) = (f(\alpha), 2)$

then, $f'$ is a partial isomorphism.

Now, it is not hard to see that $\mathcal{E}_{F, 1}$ equivalence implies equivalence as in the result of [4].

**Proof of Theorem 2.1.**

Denote $\kappa = \text{cf}(\lambda)$. ($\kappa > \aleph_0$ because $\lambda = \lambda^{\aleph_0}$). Let $\langle \mu_i : i < \kappa \rangle$ be an increasing and continuous sequence such that: $\mu_0 = 0$, $\mu_i^+ < \mu_{i+1} = \text{cf}(\mu_{i+1})$, $i > 0 \Rightarrow \mu_i > \aleph_0$, $\bigcup_{i< \kappa} \mu_i = \lambda$. For every $\alpha < \lambda$ there is a unique $i < \kappa$, such that $\alpha \in [\mu_i, \mu_{i+1})$. We denote $i = 1(\alpha)$.

We define a structure parameter $\mathfrak{r} = \mathfrak{r}_{F, \lambda}$ in the following way:

1. $I = [\lambda]^{\aleph_0}$
2. for $u \in I$ $J_u$ is the collection of quadruples $t = (u, g, h, \zeta)$ such that :
   (a) $g, h$ are functions from $u$ into $\lambda$, $\zeta$ is a function from some subset of $u$ into $\lambda$. 
   (b) $\zeta \in \mathcal{F}$
(c) for every \( x \in u \), \( g(x) \in [\mu_i(x), \mu_i^+] \), \( h(x) \in [\mu_i(x), \mu_i(x+1)] \)
(d) \( g, h \) are weakly increasing
(e) \( g(x) = g(y) \Rightarrow h(x) = h(y) \)
(f) \( h(x) > x \)
(g) \( \text{Dom}(\zeta) = u \cap \bigcup \{ \mu_i(x) : x \in u \text{ and } h(x) = \mu_i(x+1) \} \)
For \( t = (u, g, h, \zeta) \) we denote \( u = u^t, g = g^t, h = h^t, \zeta = \zeta^t \)
(3) \( S = \{(u_1, u_2) : u_1, u_2 \in I, u_1 \subseteq u_2 \} \)
(4) \( T = \{(t_1, t_2) \in J : u_1^t \subseteq u_2^t, \mu_1^t \subseteq \mu_2^t, \zeta_1^t \subseteq \zeta_2^t \} \)

Let \( M = M_{F, \lambda} = M_{\lambda} \) be the corresponding model. Define: \( a_* = 0_{G_0}, b_* = x_{\langle 0, \emptyset, \emptyset, \emptyset \rangle} \).
\( M_1 = (M, a_*), M_2 = (M, b_*) \).

**Claim 2.3.** \( M_1, M_2 \) are \( \text{EF}_{T,1} \) equivalent

**Proof:**

**Definition 2.4.** We define a partially ordered set of functions \((W, \leq W)\), which depends on the sequence \( \langle \mu_i : i < \kappa \rangle \) in the following way:

1. we define a set \( B \) such that \( \bar{\beta} \in B \) iff:
   - \( \bar{\beta} = (\beta_i : i < \kappa), \mu_i \leq \beta_i \leq \mu_{i+1} \)
   - there is \( j = j(\bar{\beta}) \) such that \( i < j(\bar{\beta}) \) if \( \beta_i = \mu_{i+1} \)

2. for \( \bar{\beta} \in B \) we define \( W_{\bar{\beta}} \) to be the set of functions \( g \) which satisfy:
   - \( \text{Dom}(g) = \bigcup_{i < \kappa} \{ \mu_i, \beta_i \} \)
   - \( g \) is weakly increasing
   - for every \( i < \kappa, x \in [\mu_i, \beta_i] \) we have \( g(x) \in [\mu_i, \mu_i^+] \), and if \( g(x) = \mu_i^+ \) then \( i < j(\bar{\beta}) \)

3. for \( j < \kappa \) we define \( W_j = \bigcup \{ W_{\bar{\beta}} : j(\bar{\beta}) \leq j \} \)

4. for \( g \in W_{\bar{\beta}} \) we define a function \( h_g \) in the following way:
   \[ \text{Dom}(h_g) = \text{Dom}(g) \text{ and for } i < \kappa, x \in [\mu_i, \beta_i] \text{ we define } h_g(x) = \min([y : \mu_i \leq y < \beta_i \land g(y) > g(x)] \cup \{ \beta_i \}) \]

**Claim 2.5.**

1. \( g(x) = g(y) \Rightarrow h_g(x) = h_g(y) \)
2. \( h_g(x) > x \)
3. \( h_g \) is weakly increasing
4. \( x \in [\mu_i, \mu_{i+1}] \Rightarrow h_g(x) \in [\mu_i, \mu_{i+1}] \)
5. Suppose that \( g_1 \in W_{\beta_1}, g_2 \in W_{\beta_2} \) then \( g_1 \leq W g_2 \) iff:
   - \( g_1 \subseteq g_2 \) (therefore for every \( i < \kappa, \beta_i^1 \leq \beta_i^2 \))
   - for every \( i < \kappa \) if \( \beta_i^1 < \beta_i^2 \) then for every \( x \in [\mu_i, \beta_i^1] \), \( g_2(x) < g_2(\beta_i^1) \)
6. if \( g_1 \in W_j \) and \( \beta \in B, j(\beta) \leq j \) then there is \( g_2 \in W_j \) such that:
   - \( g_1 \leq W g_2 \) and \( \bigcup_{i < \kappa} [\mu_i, \beta_i] \subseteq \text{Dom}(g_2) \)
7. if \( \delta < \mu_{\gamma}^+ \) and \( \langle \alpha : \alpha < \delta \rangle \) satisfy \( \alpha < \beta \Rightarrow g_\alpha \leq W g_\beta, g_\alpha \in W_j \) then there is \( g \in W_j \) such that \( \alpha < \delta \Rightarrow g_\alpha \leq W g \)

**Proof:**

(1) - (4) easy
In every stage

We may assume that $\text{Dom}(g_1) \subseteq \cup_{i<\kappa} [\mu_i, \beta_i)$. Define for $i < \kappa$

$$\gamma_i = \mu_i + \text{Sup}\{g_1(x) : x \in \text{Dom}(g_1) \cap [\mu_i, \mu_{i+1})\}.$$

Since $g_1 \in W_j$ we have $i \geq j \Rightarrow \gamma_i < \mu_i^+$. Define for $i < \kappa$

$$\gamma_i^* = \begin{cases} 
\mu_i^+ & \text{if } i < j \\
\gamma_i & \text{if } i \geq j 
\end{cases}$$

Now define $g_2$ by: $\text{Dom}(g_2) = \cup_{i<\kappa} [\mu_i, \beta_i)$, and for every $i < \kappa$, $x \in [\mu_i, \beta_i)$ we define:

$$g_2(x) = \begin{cases} 
g_1(x) & x \in \text{Dom}(g_1) \\
\gamma_i^* & x \notin \text{Dom}(g_1) 
\end{cases}$$

Since $j(\bar{\beta}) \leq j$ we have $g_2 \in W_j$. By (5) we have $g_1 \leq^W g_2$.

(7) Define for every $i < \kappa$: 

$$\beta_i = \sup(\cup_{\alpha<\delta} \text{Dom}(g_\alpha) \cap [\mu_i, \mu_{i+1}]) + \mu_i$$

$$\gamma_i = \sup(\cup_{\alpha<\delta} \text{Rang}(g_\alpha | [\mu_i, \mu_{i+1}]) + \mu_i$$

For every $\alpha < \delta$ $g_\alpha \in W_j$. Therefore for every $i \geq j$

- $\sup(\text{Dom}(g_\alpha) \cap [\mu_i, \mu_{i+1})) < \mu_{i+1}$
- $\sup\text{Rang}(g_\alpha | [\mu_i, \mu_{i+1}]) < \mu_i^+.$

Therefore, since $\delta < \mu_i^+ \leq \mu_i^+ < \mu_{i+1} = \text{cf}(\mu_{i+1})$, we get that for $i \geq j$

$$\beta_i < \mu_{i+1} \text{ and } \gamma_i < \mu_i^+.$$

Define for $i < \kappa$: $\beta_i^* = \begin{cases} 
\mu_{i+1} & i < j \\
\beta_i & i \geq j 
\end{cases}$

Denote $g' = \cup_{\alpha<\delta} g_\alpha$.

Define $g \in W_j$ by:

$$\text{Dom}(g) = \cup_{i<\kappa} [\mu_i, \beta_i^*)$$

For $i < \kappa, x \in [\mu_i, \beta_i^*)$ $g(x) = \begin{cases} 
g'(x) & x \in \text{Dom}(g') \\
\gamma_i^* & x \notin \text{Dom}(g') 
\end{cases}$

By (5) we get that $\alpha < \delta \Rightarrow g \geq^W g_\alpha$. \(\square\)

Now we will describe a winning strategy for ISO:

In every stage $\alpha$ in the game ISO will choose a function $g_\alpha$ such that:

1. $g_\alpha \in W_{i(\alpha)+1}$
2. $\varepsilon < \alpha \Rightarrow g_\varepsilon \leq^W g_\alpha$
3. If in stage $\alpha$ AIS chose an element from $G_u$ then $u \subseteq \text{Dom}(g_\alpha)$

ISO can choose such $g_\alpha$ in the following way:

1. for $\alpha = 0$ $g_0 = \emptyset$
2. for $\alpha$ limit, since $\alpha < \mu_{i(\alpha)+1}$ and for every $\varepsilon < \alpha$ $g_\varepsilon \in W_{i(\alpha)+1}$, we can use (7) of 2.5
3. If $\alpha = \varepsilon + 1$ and in stage $\alpha$ AIS chose element from $G_u$, then we choose $\beta = \langle \beta_i : i < \kappa \rangle$ in the following way:

   If $i < i(\alpha) + 1$ then $\beta_i = \mu_{i+1}$. Else, $\mu_{i+1} > \alpha$, we choose $\beta_i < \mu_{i+1}$ such that $u \cap [\mu_i, \mu_{i+1}) \subseteq [\mu_i, \beta_i)$. Now $j(\bar{\beta}) = i(\alpha) + 1,$
so by (6) of [2.5] we can find \( g \in W_{i(\alpha)+1} \) such that \( g_{\varepsilon} \cong \gamma \) and 
\[ \cup_{i<\kappa} [\mu_{i}, \mu_{i}^{+}] \subseteq \text{Dom}(g). \]
Define \( g_{\alpha} = g \).

Now if \( \alpha = \varepsilon + 1 \) and in stage \( \alpha \) AIS chose an element from \( G_{u} \) and the node \( \zeta_{\alpha} \in T \), then ISO will define the automorphism \( f_{\alpha} \) according to \( g_{\alpha}, \zeta_{\alpha} : \text{Dom}(f_{\alpha}) = \text{Dom}(f_{\varepsilon}) \cup G_{u} \). For every \( w \) such that \( G_{w} \subseteq \text{Dom}(f_{\alpha}) \), 
\[ f_{\alpha}(t_{G_{w}}) = x_{t} \text{ where } t = (w, g_{\alpha} | w, h_{\alpha} | w, \zeta_{\alpha} | v) \]
where \( v = w \cap \{ \mu_{i}(x) : x \in w \wedge h_{\alpha}(x) = \mu_{i}(x)+1 \} \)
(Note that \( v \subseteq \alpha = \text{Dom}(\zeta_{\alpha}) \), because \( g_{\alpha} \in W_{i(\alpha)+1} \).

As in section 1, we get that \( f_{\alpha} \) is a partial isomorphism and \( \varepsilon < \alpha = f_{\varepsilon} \subseteq f_{\alpha} \).

\[ \square \]

**Claim 2.6.** \( M_{1}, M_{2} \) are not isomorphic.

Proof: We imitate the proof of [1.12]. It is enough to show that \( M \) is rigid. Assume toward contradiction that \( f \neq \text{id} \) is an automorphism of \( M \). For each \( u \subseteq w \in I \) and \( t = (w, g, h, \zeta) \in J_{w} \) we define \( \pi_{w,u}(t) \in J_{u} \) by \( \pi_{w,u}(t) = (u, g^{t} | u, h^{t} | u, \zeta^{t} | v) \)
where \( v = \cup \{ \mu_{i}(x) : x \in u \wedge h^{t}(x) = \mu_{i}(x)+1 \} \cap u \).

We proceed as in the proof of [1.12] and we get that we can find functions \( g, h, \zeta \) such that:

1. \( \text{Dom}(g) = \text{Dom}(h) = \lambda, \text{Dom}(\zeta) \subseteq \lambda \)
2. if \( i(x) = i \) then \( g(x) \in [\mu_{i}, \mu_{i}^{+}] \), \( h(x) \in [\mu_{i}, \mu_{i+1}] \)
3. \( g, h \) are weakly increasing
4. \( g(x) = g(y) \Rightarrow h(x) = h(y) \)
5. \( h(x) > x \)
6. \( h(x) = \mu_{i}(x)+1 \Rightarrow \mu_{i}(x) \subseteq \text{Dom}(\zeta) \)
7. \( \zeta \in F \)

By (7) we get that \( \text{Dom}(\zeta) \neq \lambda \), therefore by (6) there is \( i < \kappa \) such that 
such that \( i(x) = i \Rightarrow i(h(x)) = i \). By (2) \( i(x) = i \Rightarrow g(x) \leq \mu_{i}^{+} \). By (3) \( g \) is weakly increasing. Since \( \mu_{i+1} = \text{cf}(\mu_{i}+1) > \mu_{i}^{+} \), we can find \( \alpha_{0} \) such that 
\( \alpha_{0} \leq x < \mu_{i}+1 \Rightarrow g(x) = g(\alpha_{0}) \). By (5) \( h(\alpha_{0}) > \alpha_{0} \). By the choice of \( i \) we get that \( h(\alpha_{0}) < \mu_{i}+1 \). Choose \( h(\alpha_{0}) < x < \mu_{i}+1 \). We get \( h(x) > x > h(\alpha_{0}) \)
but \( g(x) = g(\alpha_{0}) \). This contradicts (4). Therefore we proved that \( M \) is rigid.  
\[ \square \]
3. λ regular > ℳω

In this section we show a result which holds for every λ regular > ℳω. In the previous sections we used the assumption λ = λ^ℵ₀. Here we use instead of it the existence of a set \( \mathcal{P} \subset [\lambda]^{ℵ₀} \) of size \( \lambda \) which is ”dense”. By ”dense” we mean that for every \( A \in [\lambda]^ℶ_ω \) there is \( B \subset A, B \in \mathcal{P} \).

Remark 3.1. (1) Looking at the proof, one can see that instead of \( \lambda > ℳ_ω \), it is enough to assume the following:
(a) \( \lambda > 2^{ℵ₀} \)
(b) There is \( \mathcal{P} \subset [\lambda]^{ℵ₀} \) such that:
   (i) |\( \mathcal{P} \)| = \( \lambda \)
   (ii) For every \( A \in [\lambda]^{ℵ₀} \), there is \( B \in \mathcal{P} \), \( B \subset A \).
(2) It is possible that it can be proved in ZFC that every \( \lambda > 2^{ℵ₀} \) satisfies
   (1)(b) (It is a problem in cardinal arithmetic).

Theorem 3.2. Suppose:
(1) \( \lambda = \text{cf}(\lambda) > ℳ_ω \)
(2) \( \mathcal{T} \) is a tree of size \( \lambda \) with no branch of length \( \lambda \)

Then: there are models \( M_1, M_2 \) of size \( \lambda \) which are EF\( _{\mathcal{T},\lambda} \) equivalent but not isomorphic

Proof: Let \( \chi \) be large enough cardinal (for example \( \chi = ℳ_7(\lambda) \)).

Claim 3.3. We can find \( M \) such that :
(1) \( M \) is elementary sub-model of \( H(\chi) \)
(2) \( \lambda + 1 \subseteq M \)
(3) |\( M \)| = \( \lambda \)
(4) for every \( \langle (x_i, z_i) : i < \lambda \rangle \) such that \( x_i \in M, z_i \in \mathcal{T} \) for every \( i < \lambda \) there is an increasing sequence \( \langle i_n : n < ω \rangle \) such that:
   (a) \( \langle (x_{i_n}, z_{i_n}) : n < ω \rangle \in M \)
   (b) if in addition, for \( i < j < \lambda \) the level of \( z_i \) (in \( \mathcal{T} \)) is strictly less then the level of \( z_j \), then \( \langle z_{i_n} : n < ω \rangle \) is an antichain in the order \( \leq \mathcal{T} \)

Proof:
We use part of the RGCH theorem(see Shelah [5])

RGCH Theorem (partial version) 3.4. if \( \lambda \geq ℳ_ω \) then there is regular \( \kappa < ℳ_ω \) and \( \mathcal{P} \subseteq [\lambda]^{<ℳ_ω} \) such that :
(1) |\( \mathcal{P} \)| = \( \lambda \)
(2) for every \( A \in [\lambda]^{ℳ_ω} \), We can find \( \langle A_i : i < \epsilon \rangle \) such that:
   \( \epsilon < \kappa \), \( A_i \in \mathcal{P} \) for every \( i < \epsilon \) and \( A = \bigcup_{i < \epsilon} A_i \)

Corollary 3.5. if \( \lambda \geq ℳ_ω \) then we can find a set \( \mathcal{P}^* \subseteq [\lambda]^{ℵ₀} \) such that
|\( \mathcal{P}^* \)| = \( \lambda \) and for every \( A \in [\lambda]^{ℳ_ω} \) there is \( B \in \mathcal{P}^* \) such that \( B \subseteq A \)

Proof:
Choose \( \kappa \) and \( \mathcal{P} \) as in 3.4 and define \( \mathcal{P}^* = \bigcup \{ [A]^{ℵ₀} : A \in \mathcal{P} \} \)
We construct $\mathcal{M}_n$ for every $n < \omega$ such that:

1. $\mathcal{M}_0$ is an elementary sub-model of $\mathcal{H}(\chi)$ such that $|\mathcal{M}_0| = \lambda$, $\lambda + 1 \subseteq \mathcal{M}_0$ and for every $A \in [\lambda]^\omega$, there is $B \in \mathcal{M}_0 \cap [\lambda]^{\aleph_0}$, $B \subseteq A$ (This is possible by 3.5)
2. $|\mathcal{M}_n| = \lambda$
3. $\mathcal{M}_n$ is an elementary sub-model of $\mathcal{H}(\chi)$
4. if $A \in \mathcal{M}_n$ and $|A| \leq \lambda$, then $A \subseteq \mathcal{M}_{n+1}$
5. $\mathcal{M}_n \in \mathcal{M}_{n+1}$, $\mathcal{M}_n \subset \mathcal{M}_{n+1}$

Now, let $\mathcal{M} = \bigcup_{n<\omega} \mathcal{M}_n$. We will prove that $\mathcal{M}$ satisfies the conclusion of claim 3.3.

Suppose that $\langle (x_i, z_i) : i < \lambda \rangle \subseteq \mathcal{M} \times \mathcal{T}$ satisfies $x_i \in \mathcal{M}$, $z_i \in \mathcal{T}$ for every $i < \lambda$. We may assume without loss of generality, that there is $n_0 < \omega$ such that $\{ (i, x_i, z_i) : i < \lambda \} \subseteq \mathcal{M}_{n_0}$. If the condition in 3.3 4(b) is not satisfied, then we are done, because we can find $A \in [\lambda]^{\aleph_0}$ such that $\{ (i, x_i, z_i) : i \in A \} \in \mathcal{M}_{n_0+1}$. (Because in $\mathcal{M}_{n_0+1}$ there is one to one correspondence between $\lambda \times \mathcal{M}_{n_0} \times \mathcal{T}$ and $\lambda$, and every subset of $\lambda$ of size $\beth_\omega$ has infinite countable subset that is a member of $\mathcal{M}_0$).

If the condition in 3.3 4(b) is satisfied, then we have 2 cases:

1. We can find $A \in [\lambda]^{\omega}$ such that $\langle z_i : i \in A \rangle$ is an antichain in $\leq \mathcal{T}$

2. We cannot find such $A$.

If we are in case 1, then we are done in the same way as before.

Suppose we are in case 2.

**Claim 3.6.** for every $j < \lambda$, we can find $j < i_0 < i_1 < i_2 < \lambda$, such that $z_{i_0} <\mathcal{T} z_{i_1}, z_{i_2}$ and $z_{i_1}, z_{i_2}$ are not comparable in $\leq \mathcal{T}$.

Proof: assume toward contradiction that there is $j^* < \lambda$, such that we can’t find $j^* < i_0 < i_1 < i_2 < \lambda$ which are as in the claim.

Define $C = \{ z_i : j^* < i < \lambda \}$. Then, being comparable in $\leq \mathcal{T}$ is an equivalence relation on $C$. Since $\lambda$ is regular, either there are $\lambda$ equivalence classes or there is an equivalence class of size $\lambda$. In other words, $C$ contains an antichain or a chain of size $\lambda$, both options are not possible, the first since we are in case (2) and the second since $\mathcal{T}$ doesn’t have a $\lambda$ branch -contradiction.

By claim 3.6 we can choose for every $j < \lambda$ a triple $i_0(j), i_1(j), i_2(j)$ such that:

1. $i_0(j) < i_1(j) < i_2(j) < \lambda$
2. $j < j' \Rightarrow i_2(j) < i_0(j')$
3. $z_{i_0(j)} <\mathcal{T} z_{i_1(j)}, z_{i_2(j)}$
4. $z_{i_1(j)}$ and $z_{i_2(j)}$ are not comparable in $\leq \mathcal{T}$
We choose $A \in [\lambda]^{\aleph_0}$, such that $\{\langle j, i_0(j), i_1(j), i_2(j), x_j, z_j \rangle : j \in A \} \in \mathcal{M}_{n_0+1}$

Using Ramsey theorem in $\mathcal{M}_{n_0+1}$, we can find an increasing sequence

$\langle j_n : n < \omega \rangle$ such that:

1. for every $n < \omega$, $j_n \in A$
2. $\langle j_n : n < \omega \rangle \in \mathcal{M}_{n_0+1}$
3. $\{z_{i_1(j_n)} : n < \omega \}$ is a chain or an antichain in $\mathcal{T}$
4. $\{z_{i_2(j_n)} : n < \omega \}$ is a chain or an antichain in $\mathcal{T}$

Now we are done, since either $\{z_{i_1(j_n)} : n < \omega \}$ or $\{z_{i_1(j_n)} : n < \omega \}$ must be an antichain. Because if both are chains, we get that $z_{i_1(j_0)} <^T z_{i_1(j_1)}$, $z_{i_2(j_0)} <^T z_{i_2(j_1)}$. Since $z_{i_0(j_1)}$ is on higher level then $z_{i_1(j_0)}$, $z_{i_2(j_0)}$ and it is $<^T z_{i_1(j_1)}$, $z_{i_2(j_1)}$. We get that $z_{i_1(j_0)}$, $z_{i_2(j_0)} <^T z_{i_0(j_1)}$ - contradiction, since by the construction they are not comparable. $\square$

We choose $\mathfrak{M}$ as in claim 3.3.

We define a structure parameter $\mathfrak{r} = \mathfrak{r}(\mathfrak{M})$ in the following way:

**Definition 3.7.**  
(1) $I$ consists of the objects of the form $(u, \Lambda)$ where:

(a) $u \in \lambda^{<\aleph_0}$

(b) $\Lambda \in \mathfrak{M}$, $|\Lambda| \leq \aleph_0$, $\Lambda$ is a set of partial functions with finite domain, from $\lambda$ to $\lambda$.

for $s = (u, \Lambda)$ we denote $u = u^s$, $\Lambda = \Lambda^s$.

We define $\Gamma(s) = u^s \cup \{\text{Dom}(f) : f \in \Lambda^s\}$. Note that this a countable set.

(2) For $s = (u, \Lambda) \in I$, $J_s$ consists of all the objects of the form $t = (u, \Lambda, g, h, F, z)$ where:

(a) $g, h$ are functions from $u$ to $\lambda$

(b) $F$ is a function from $\Lambda^2$ to $\{0, 1\}$

(c) $z \in \mathcal{T}$

(d) Let $\alpha$ be the level of $z$ in the tree $\mathcal{T}$, then $\alpha$ is minimal under the condition $\alpha > y$ for every $y$ such that: $y \in \text{Rang}(g)$ or there are $f_1, f_2 \in \Lambda$ such that $F(f_1, f_2) = 1$ and $y \in \text{Rang}(f_1)$.

(e) There is a witness $(g, h)$ for $t$, which means that :

(i) $\text{Dom}(g) = \text{Dom}(h) \subseteq \lambda$, $\text{Rang}(g) \cup \text{Rang}(h) \subseteq \lambda$

(ii) $\Gamma(s) \subseteq \text{Dom}(g)$

(iii) $g, h$ are weakly increasing

(iv) $h(x) > x$

(v) $g(x) = g(y) \Rightarrow h(x) = h(y)$

(vi) $g \subseteq g, h \subseteq h$

(vii) for every $(f_1, f_2) \in \Lambda^2$

$F(f_1, f_2) = 1$ iff $f_1 \subseteq g \wedge f_2 \subseteq h$

(3) $S = I^2$

(4) $T$ consists of the pairs $(t_1, t_2) \in J^2$ where :

(a) $t_1, t_2$ have a common witness

(b) $z^{t_1}, z^{t_2}$ are comparable in the order $\leq^T$
Fact 3.8. if:

(1) \( s \in I, z \in T \)
(2) \( g, h \) satisfy conditions (i)-(v) from 3.7(e)
(3) \( \text{Dom}(g) \subseteq \alpha \) where \( \alpha \) is the level of \( z \)

then:

(1) there is unique \( t \in J_s \) such that \((g, h)\) is a witness for \( t \), and \( z^t \leq_T z \). we denote \( t = t(s, g, h, z) \)
(2) if :
   (a) \( g', h', z' \) also satisfy the conditions in (1)
   (b) \( z, z' \) are comparable in \( \leq_T \)
   (c) \( g', h' \) are compatible with \( g, h \) respectively

then :

\( t(s, g, h, z) = t(s, g', h', z') \)

Let \( M = M_\ell \) be the corresponding model. We can check that \( ||M|| = \lambda \).
Let \( a_\ast = 0_{G(\emptyset, \emptyset)} \), \( b_\ast = x_{(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, z_\ast)} \) where \( z_\ast \) is the root of \( T \) (without loss of generality there is a root).
Define \( M_1 = (M, a_\ast), M_2 = (M, b_\ast) \).

Claim 3.9. \( M_1, M_2 \) are EF\( T, \lambda \) equivalent.

We describe a winning strategy for ISO - this is very similar to the proof of 1.9, so we will omit the details. We are using the definitions in 1.10.

In every stage \( \alpha \) of the game ISO will choose a function \( g_\alpha \) such that :

(1) \( g_0 = \emptyset \)
(2) \( g_\beta \in G_\alpha \) (See definition of \( G_\alpha \) and \( \leq^G \) in 1.10)
(3) \( \beta < \alpha \Rightarrow g_\beta \leq^G g_\alpha \)
(4) If in stage \( \alpha \) AIS chose the sets \( A_1, A_2 \) then

   for each \( s \in I \), if \( G_s \cap (A_1 \cup A_2) \neq \emptyset \) then \( \Gamma(s) \subseteq \text{Dom}(g_\alpha) \)

Now if \( \alpha = \beta + 1 \) and in stage \( \alpha \) AIS chose the sets \( A_1, A_2 \) and the node \( z_\alpha \), ISO will define \( h_\alpha = h_{g_\alpha} \) and then define \( f_\alpha \) by :

(1) \( \text{Dom}(f_\alpha) = \bigcup \{G_s : \Gamma(s) \subseteq \text{Dom}(g_\alpha)\} \)
(2) for each \( s \) such that \( G_s \subseteq \text{Dom}(f_\alpha), f_\alpha(0_{G_s}) = x_t \),

   where \( t = t(s, g_\alpha, h_\alpha, z_\alpha) \) \( \square \)

Claim 3.10. \( M_1, M_2 \) are not isomorphic

Proof:
It is enough to show that \( M \) is rigid. Assume toward contradiction that \( f \neq id \) is an automorphism of \( M \). Denote for \( s \in I \), \( c_s = f(0_{G_s}) \). Denote \( W_s = \{t \in J_s : x_t \) is in the reduced representation of \( c_s\} \). Since \( f \neq id \) there is \( s^* = (u^*, \Lambda^*) \) such that \( W_{s^*} \neq \emptyset \). Note also that if \( u^{s^*} \subseteq u^s \) and \( \Lambda^* \subseteq \Lambda^s \), then there is a natural projection \( \pi_{s, s^*} \) from \( J_s \) into \( J_{s^*} \) such that \( W_{s^*} \subseteq \text{Rang}(\pi_{s, s^*} \mid W_s) \) (see the proof of 1.12) therefore \( W_s \neq \emptyset \).

Choose for \( i < \lambda \ s_i, t_i, \alpha_i \) such that :
Proof:

\begin{enumerate}
\item $s_i \in I$, $s_i = (u^* \cup \{\alpha_i\}, \Lambda^*)$
\item $i_t \in W_{s_i}$
\item $\alpha_i < \lambda$
\item $i < j \Rightarrow h^{i_t}(\alpha_i) < \alpha_j$
\end{enumerate}

Case (*1) : $\text{Sup}\{g^{i_t}(\alpha_i) : i < \lambda\} = \lambda$. Then, since the level of $z^{i_t}$ in $\mathcal{T}$ must be greater then $g^{i_t}(\alpha_i)$, we may assume that if $i < j$ then the level of $z^{i_t}$ is strictly less then the level of $z^{j_t}$.

Case (*2) : $\text{Sup}\{g^{i_t}(\alpha_i) : i < \lambda\} < \lambda$. Then by regularity of $\lambda$, we may assume that for every $i, j < \lambda$ $g^{i_t}(\alpha_i) = g^{j_t}(\alpha_j)$

Now, no matter in which case we are, we proceed in the following way:

By the properties of $\mathfrak{M}$ (see claim 3.3) we can find a set $A \subset \lambda$ such that:

\begin{enumerate}
\item $|A| = \aleph_0$
\item $\{W_{s_i} : i \in A\} \in \mathfrak{M}$
\item if we are in case (*1) $\{z^{i_t} : i \in A\}$ is an antichain (We can have that because in case(*2) the level of $z^{i_t}$ is strictly increasing with $i$ - See 3.3)
\end{enumerate}

We define $s^+ = (u^*, \bigcup_{i \in A} W_{s_i} \cup \Lambda^*)$. (Note that $\bigcup_{i \in A} W_{s_i} \in \mathfrak{M}$, therefore $s^+ \in I$)

Claim 3.11. For every $i \in A$, if $r \in J_{s^+}$, $t \in W_{s_i}$, $(r, t) \in T$

then :

\begin{enumerate}
\item if $(g, h)$ is a witness for $r$ then $g^t \subseteq g$, $h^t \subseteq h$
\item if $t \neq t' \in J_{s_i}$ then $(r, t') \notin T$
\end{enumerate}

Proof:

(1) Let $(g_0, h_0)$ be a common witness for $r, t$. Then $g^t \subseteq g_0$, $h^t \subseteq h_0$. Now $g^t, h^t \in \Lambda^+$ therefore $(g^t, h^t) \in \text{Dom}(F^r)$. since $(g_0, h_0)$ is a witness for $r$ and $g^t \subseteq g_0$, $h^t \subseteq h_0$ then $F^r(g^t, h^t) = 1$. Therefore for any witness $(g, h)$ of $r$, we have $g^t \subseteq g$, $h^t \subseteq h$.

(2) There are 3 cases :

\begin{enumerate}
\item $g^t \neq g''$ or $h^t \neq h''$. Then, since all those functions have the same domain, we get that $r, t'$ cannot have a common witness $(g, h)$ because by (1) we must have $g^t \subseteq g$, $h^t \subseteq h$.
\item $F^t \neq F^{t'}$. Then, since $\text{Dom}(F^t) = \Lambda^+ \subseteq \Lambda^{s^+} = \text{Dom}(F^{r'})$ and $(r, t) \in T$ we know that $F^t \subseteq F^{r'}$. Since $F^t \neq F^{t'}$ and $\text{Dom}(F^{t'}) = \text{Dom}(F^{r'})$, we get that $F^r$ and $F^{t'}$ aren’t compatible (and therefore there is no common witness)
\item $z^t \neq z^{t'}$. By the previous cases we may assume that $F^t = F^{t'}$, $g^t = g^{t'}$, $h^t = h^{t'}$ therefore $z^t, z^{t'}$ are on the same level (See 3.7 (d)). We can also see that $z^{t'}$ must be on a greater level (Remember that $F^t \subseteq F^{r'}$ and $F^{r'}(g^t, h^t) = 1$). Since $(r, t) \in T$, $z^t, z^{t'}$ are comparable in $\leq_T \Rightarrow z^{t'}, z^{t'}$ are not $\Rightarrow (r, t') \notin T$
\end{enumerate}
Claim 3.12. For every $i \in A$ there is $r \in W_{s^+}$ such that $(r, t_i) \in T$

Proof:
Since $(c_s, c_{s^+}) \in G_{s,s^+}$ and this group is generated by $\{(x_t, x_{t'}) : (t, t') \in T \cap (J_s \times J_{s^+})\}$, there are representations (not necessarily reduced) $c_{s_i} = x_{w_1} + \cdots + x_{w_n}$ and $c_{s^+} = x_{r_1} + \cdots + x_{r_n}$ such that $(r_n, w_n) \in T$.

We may assume that if $1 \leq \ell_1 < \ell_2 \leq n$, then either $r_{\ell_1} \neq r_{\ell_2}$ or $w_{\ell_1} \neq w_{\ell_2}$.

(Otherwise, we can reduce both representations - remember that in those groups $2x = 0$). Since $x_{t_i}$ appears in the reduced representation of $c_{s_i}$, $t_i$ must appear among the $w$-ies. Let $\ell$ be such that $w_\ell = t_i$. Now we show that if $\ell_1 \neq \ell$, then $r_{\ell_1} \neq r_{\ell}$. Assume toward contradiction that $r_{\ell_1} = r_{\ell}$. By our assumption, $w_{\ell_1} \neq w_{\ell}$. Now, we have:

1. $(r_{\ell_1}, w_{\ell_1}), (r_{\ell}, w_{\ell}) \in T$
2. $w_{\ell} \in W_{s_i}$
3. $w_{\ell} \neq w_{\ell_1}$

This contradicts 3.11.

We got that for every $\ell_1 \neq \ell$, $r_{\ell_1} \neq r_{\ell}$. This implies that $x_{r_{\ell}}$ does not cancel, so $r_{\ell} \in W_{s^+}$ and we are done. □

Now choose for each $i \in A$ $r_i \in W_{s^+}$ such that $(r_i, t_i) \in T$.

Claim 3.13. $i < j \Rightarrow r_i \neq r_j$

Proof:
If we are in case (*1): $\{z^{t_i} : i \in A\}$ is an antichain. So, $z^{t_i}, z^{t_j}$ are not comparable. Since $z^{t_i} \geq_T z^{t_j}$ and $z^{r_j} \geq_T z^{r_i}$ (See the proof of 3.11 - $z^{r_i}, z^{t_i}$ are comparable and $z^{r_i}$ is on greater level), We must have $r_i \neq r_j$.

If we are in case (*2): assume toward contradiction that $r = r_i = r_j$. Let $(g, h)$ be a witness for $r$. By 3.11 $g^{t_i}, g^{t_j} \subseteq g$, $h^{t_i}, h^{t_j} \subseteq h$. Since we are in case (*2) we get that $g(\alpha_i) = g(\alpha_j)$ but by the construction $h(\alpha_i) < \alpha_j < h(\alpha_j)$ which contradicts the definition of a witness (see 3.7 2(e)). □

We got that $W_{s^+}$ is infinite - contradiction. Therefore $M$ must be rigid. □

3.11 3.12 3.13 3.10 3.2
4. \( \lambda > \text{cf}(\lambda) > \aleph_\omega \)

Clearly, for \( \lambda \) singular > \( \aleph_\omega \) we cannot prove the same result as for \( \lambda \) regular > \( \aleph_\omega \) (Since in such game AIS will be able to list all the elements of the two models). Therefore, we define another type of game.

**Definition 4.1.** Let \( M_1, M_2 \) be models with common vocabulary. Let \( \mathcal{T} \) be a tree. We define the game \( \mathcal{D}_\mathcal{T}^*(M_1, M_2) \) in the same way as the definition of \( \mathcal{D}_\mathcal{T}^* \) (See 1.1) except that in stage \( \alpha \) we demand that the sets \( A_1, A_2 \) chosen by AIS will satisfy \( |A_1 \cup A_2| < 1 + \alpha \) instead of \( |A_1 \cup A_2| < 1 + \mu \). We say that \( M_1, M_2 \) are \( EF_{\mathcal{T}}^* \) equivalent if ISO has a winning strategy for \( EF_{\mathcal{T}}^*(M_1, M_2) \).

**Remark 4.2.** Note that in theorem 2.1, if we replace \( EF_{\mathcal{T},1}^* \) with \( EF_{\mathcal{T}}^* \) we don’t get a stronger result, because for every tree \( \mathcal{T} \) which satisfies the conditions there, we can construct another tree \( \mathcal{T}' \) which satisfies the conditions, such that \( EF_{\mathcal{T},1}^* \) equivalence would imply \( EF_{\mathcal{T}}^* \) equivalence.

**Theorem 4.3.** Suppose that:

1. \( \lambda > \text{cf}(\lambda) = \kappa > \aleph_\omega \)
2. \( \mathcal{T} \) is a tree of size \( \lambda \) without a \( \lambda \) branch

then:

There are non-isomorphic models \( M_1, M_2 \) of size \( \lambda \) which are \( EF_{\mathcal{T}}^* \) equivalent.

Proof:
Let \( \chi \) be a large enough cardinal (for example \( \chi = \aleph_7(\lambda) \)).

**Claim 4.4.** We can find \( \mathcal{M} \) such that:

1. \( \mathcal{M} \) is elementary sub-model of \( \mathcal{H}(\chi) \)
2. \( \lambda + 1 \subseteq \mathcal{M} \)
3. for every \( \langle (x_i, z_i) : i < \kappa \rangle \) such that \( x_i \in \mathcal{M}, z_i \in \mathcal{T} \) for every \( i < \lambda \) there exists an increasing sequence \( \langle i_n : n < \omega \rangle \) such that:
   a. \( \langle (x_{i_n}, z_{i_n}) : n < \omega \rangle \in \mathcal{M} \)
   b. if in addition, for every \( \alpha < \lambda \) there is \( i < \kappa \) such that the level of \( z_i \) is greater then \( \alpha \), then we can also have that \( \langle z_{i_n} : n < \omega \rangle \) is an antichain in \( \leq \mathcal{T} \)

Proof:
The same proof as the proof of 3.3 (We are using the fact that \( \kappa \) is regular and \( \kappa > \aleph_\omega \) )

Let \( \mathcal{M} \) be as in claim 4.4. Let \( \langle \mu_i : i < \kappa \rangle \) be an increasing and continuous sequence such that \( \mu_0 = 0, \mu_i^+ + \aleph_0 < \mu_{i+1} = \text{cf}(\mu_{i+1}), \bigcup_{i<\kappa} \mu_i = \lambda \).

For every \( \alpha < \lambda \) there is a unique \( i < \kappa \), such that \( \alpha \in [\mu_i, \mu_{i+1}) \). We denote this \( i \) by \( i(\alpha) \).

We define a structure parameter \( \mathfrak{r} \) in the following way:
Definition 4.5. (1) $I$ consists of the objects of the form $(u, \Lambda)$ where:
(a) $u \in \lambda^\omega$,
(b) $\Lambda \in 2\mathcal{M}, |\Lambda| \leq \aleph_0$, $\Lambda$ is a set of partial functions with finite domain, from $\lambda$ to $\lambda$.
for $s = (u, \Lambda)$ we denote $u = u^s, \Lambda = \Lambda^s$.
We define $\Gamma(s) = u^s \cup \bigcup \{\text{Dom}(f) : f \in \Lambda^s\}$. Note that this a countable set.
(2) For $s = (u, \Lambda) \in I$, $J_s$ consists of the objects of the form $t = (u, \Lambda, g, h, F, z)$ where:
(a) $g, h$ are functions from $u$ to $\lambda$
(b) $F$ is a function from $\Lambda^2$ to $\{0, 1\}$
(c) $z \in T$
(d) Let $\alpha$ be the level of $z$ in the tree $T$, then $\alpha$ is minimal under the condition that $\alpha \geq \mu_i(x)$ for every $x$ such that:
$h(x) = \mu_i(x)+1$ or there are $f_1, f_2 \in \Lambda$ such that $F(f_1, f_2) = 1$ and $f_2(x) = \mu_i(x)+1$
(e) There is a witness $(g, h)$ for $t$, which means that :
(i) $\text{Dom}(g) = \text{Dom}(h) \subseteq \lambda, \text{Rang}(g) \cup \text{Rang}(h) \subseteq \lambda$
(ii) $\Gamma(s) \subseteq \text{Dom}(g)$
(iii) $g \subseteq g, h \subseteq h$
(iv) for every $(f_1, f_2) \in \Lambda^2$
$F(f_1, f_2) = 1$ iff $f_1 \subseteq g \land f_2 \subseteq h$
(v) $g, h$ are weakly increasing
(vi) $h(x) > x$
(vii) $g(x) = g(y) \Rightarrow h(x) = h(y)$
(viii) $g(x) \in [\mu_i(x), \mu_i(x)+]$
(ix) $h(x) \in [\mu_i(x), \mu_i(x)+1]$
(3) $S = I^2$
(4) $T$ consists of the pairs $(t_1, t_2) \in J^2$ where :
(a) $t_1, t_2$ have a common witness
(b) $z^{t_1}, z^{t_2}$ are comparable in the order $\leq T$

Fact 4.6. if:
(1) $s \in I, z \in T$
(2) $g, h$ satisfy (i)-(v)
(3) $\bigcup \{\mu_i(x) : h(x) = \mu_i(x)+1\} \subseteq \alpha$ where $\alpha$ is the level of $z$
then:
(1) there is unique $t \in J_s$ such that $(g, h)$ is a witness for $t$, and $z^t \leq T z$.
we denote $t = t(s, g, h, z)$
(2) if :
(a) $g', h', z'$ satisfy the conditions in (1)
(b) $z, z'$ are comparable in $\leq T$
(c) $g', h'$ are compatible with $g, h$ respectively
Let $M = M_s$ be the corresponding model. We can check that $||M|| = \lambda$.

Let $a_\ast = 0_{G(\emptyset, \emptyset)}$, $b_\ast = x(\emptyset, \emptyset, \emptyset, z_\ast)$ where $z_\ast$ is the root of $T$ (without loss of generality there is a root).

Define $M_1 = (M, a_\ast), M_2 = (M, b_\ast)$.

**Claim 4.7.** $M_1, M_2$ are $EF^*_\pi$ equivalent.

We describe a winning strategy for ISO - this is very similar to the proof of 2.3, so we omit the details. We use the definitions in 2.4. In every stage $\alpha$ of the game, ISO will choose a function $g_\alpha$, such that:

1. $g_0 = \emptyset$
2. $g_\alpha \in W_{i(\alpha) + 1}$
3. $\beta < \alpha \Rightarrow g_\beta \subseteq W g_\alpha$
4. if in stage $\alpha$ AIS chose the sets $A_1, A_2$ then for each $s \in I$, if $G_s \cap (A_1 \cup A_2) \neq \emptyset$ then $\Gamma(s) \subseteq \text{Dom}(g_\alpha)$

Now if $\alpha = \beta + 1$ and in stage $\alpha$ AIS chose the sets $A_1, A_2$ and the node $z_\alpha$, ISO will define $h_\alpha = h_{g_\alpha}$, and then define $f_\alpha$ by:

1. $\text{Dom}(f_\alpha) = \bigcup\{G_s : \Gamma(s) \subseteq \text{Dom}(g_\alpha)\}$
2. for each $s$ such that $G_s \subseteq \text{Dom}(f_\alpha)$,
   
   \[ f_\alpha(0_{G_s}) = x_t \text{ where } t = t(s, g_\alpha, h_\alpha, z_\alpha) \]

**Claim 4.8.** $M_1, M_2$ are not isomorphic.

Proof:

It is enough to show that $M$ is rigid. The proof is very similar to the proof of 3.10. Assume toward contradiction that $f \neq id$ is an automorphism of $M$. Denote $W_s = \{t \in J_s : x_t \text{ is in the reduced representation of } c_s\}$. Since $f \neq id$ there is $s^* = (u^*, \Lambda^*)$ such that $W_{s^*} \neq \emptyset$.

Case (*1):

We can find $(s_\theta, t_\theta, \alpha_\theta : \theta < \kappa)$ such that:

1. $s_\theta \in J, s_\theta = (u^* \cup \{\alpha_\theta\}, \Lambda^*)$
2. $t_\theta \in W_{s_\theta}$
3. $h^i(\alpha_\theta) = \mu_{i(\alpha_\theta) + 1}$
4. $\theta < \varepsilon < \kappa \Rightarrow i(\alpha_\theta) < i(\alpha_\varepsilon)$

In this case, note that the level of $z^{t_\theta}$ must be $\geq \mu_{i(\alpha_\theta)}$.

Case (*2):

We cannot find such a sequence. Therefore, for every large enough $i < \kappa$ for every $\alpha$ such that $i(\alpha) = i$, for $s(\alpha) = (u^* \cup \{\alpha\}, \Lambda^*)$, for every $t \in W_{s(\alpha)}$, $h^i(\alpha) < \mu_{i+1}$.

Choose $i^*$ which satisfies this and $\mu_{i^*} > \mu$.

We can find $(t_{\theta}, s_\theta, \alpha_\theta : \theta < \mu_{i^*} + 1)$ such that:

1. $s_\theta \in I, t_\theta \in W_{s_\theta}$
(2) $1(\alpha_\theta) = i^*$

(3) $\theta < \varepsilon \Rightarrow h^{\theta}(\alpha_\theta) < \alpha_\varepsilon \ (< h^{\varepsilon}(\alpha_\varepsilon))$

Since $\mu_{i+1} = cf(\mu_{i+1}) > \mu_i^+$, and for every $\theta$ we have $g^{\theta}_\theta(x) \leq \mu_i^+$, (This is by 4.5(2)(e)(viii)), we may assume that $g^{\theta}(\alpha_\theta)$ is constant.

Now, in both cases, we proceed in a similar way to the proof of 3.10. Using 4.4, we choose $A \subset \kappa$ such that:

1. $|A| = \aleph_0$
2. $(W_{s_\theta} : \theta \in A) \in \mathfrak{M}$
3. if we are in case (*1) then $(z^{s_\theta} : \theta \in A)$ is an antichain in $\leq^T$ (We can demand this because in case (*1) the levels of the $z^{s_\theta}$-ies aren’t bounded in $\lambda$ - See 4.4)

Define $s^+ \in I$ by $s^+ = (\emptyset, \Lambda^* \cup \{g^t, h^t : t \in W_{s_\theta}, \theta \in A\})$.

**Claim 4.9.** For every $\theta \in A$, if $r \in J_{s^+}, t \in W_{s_\theta}, (r, t) \in T$ then:

1. if $(g, h)$ is a witness for $r$ then $g^t \subseteq g, h^t \subseteq h$
2. if $t \neq t' \in J_{s_\theta}$ then $(r, t') \notin T$

Proof: see the proof of 3.11 \[\square.1.9\]

**Claim 4.10.** For every $\theta \in A$ there is $r \in W_{s^+}$ such that $(r, t_\theta) \in T$

Proof: see the proof of 3.12 \[\square.1.10\]

Now, using 4.10, we choose for each $\theta \in A$, $r_\theta \in W_{s^+}$, such that $(t_\theta, r_\theta) \in T$.

**Claim 4.11.** $\theta < \varepsilon \Rightarrow r_\theta \neq r_\varepsilon$

Proof:

If we are in case (*1):

$z^{s_\theta}, z^{s_\varepsilon}$ are not comparable. But, $z^{r_\theta} \geq^T z^{s_\theta}$ because they are comparable and $z^{r_\theta}$ is on greater level, since that level is determined by 3.5 2(d). By the same argument, $z^{r_\varepsilon} \geq^T z^{s_\varepsilon}$. Therefore, $z^{r_\varepsilon}, z^{r_\theta}$ aren’t comparable, so $r_\theta \neq r_\varepsilon$.

If we are in case (*2):

Assume toward contradiction that $r = r_\theta = r_\varepsilon$.

Let $(g, h)$ be a witness for $r$. By 4.9 $g^{s_\theta}, g^{s_\varepsilon} \subseteq g, h^{s_\theta}, h^{s_\varepsilon} \subseteq h$. Since we are in case (*2) we get that $g(\alpha_\theta) = g(\alpha_\varepsilon)$ and $h(\alpha_\theta) < \alpha_\varepsilon < h(\alpha_\varepsilon)$ which contradicts the definition of a witness (see 4.5 2(e)). \[\square.4.11\]

We got that $W_{s^+}$ is infinite - contradiction. Therefore, $M$ must be rigid. \[\square.4.12\]
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