A Simultaneous Generalization of Independence and Disjointness in Boolean Algebras

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Abstract We give a definition of some classes of boolean algebras generalizing free boolean algebras; they satisfy a universal property that certain functions extend to homomorphisms. We give a combinatorial property of generating sets of these algebras, which we call n-independent. The properties of these classes (n-free and ω-free boolean algebras) are investigated. These include connections to hypergraph theory and cardinal invariants on these algebras. Related cardinal functions, in, the minimum size of a maximal n-independent subset and iω, the minimum size of an ω-independent subset, are introduced and investigated. The values of in and iω on P(ω)/fin are shown to be independent of ZFC.

Keywords Boolean algebra · Independence · Delta system · Forcing

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1 Definitions

A boolean algebra A is free over its subset X if it has the universal property that every function f from X to a boolean algebra B extends to a unique homomorphism. This is equivalent to requiring that X be independent and generate A (uniqueness). A generalization, ⊥-free, is introduced in Heindorf [5], and some of its properties are dealt with. I follow his notation for some of its properties, but that of Koppelberg [6] for the operations +, ·, −, 0, 1 on Boolean Algebras, with the addition that for an element a of a boolean algebra, we let a0 = −a and a1 = a. An elementary product
of $X$ is an element of the form $\prod_{x \in R} x^{e_x}$ where $R$ is a finite subset of $X$ and $e \in R^2$.

We further generalize the notion of freeness to $n$-freeness for $1 \leq n \leq \omega$.

It is nice to have a symbol for disjointness; we define $a \perp b$ if and only if $a \cdot b = 0$.

**Definition 1.1** Let $n$ be a positive integer, $A$ and $B$ be nontrivial boolean algebras, and $U \subseteq A$. A function $f : U \rightarrow B$ is $n$-preserving if and only if for every $a_0, a_1, \ldots, a_{n-1} \in U$, $\prod_{i<n} a_i = 0$ implies that $\prod_{i<n} f(a_i) = 0$.

An infinite version of this is also important.

**Definition 1.2** Let $A$ and $B$ be nontrivial boolean algebras, and $U \subseteq A$. A function $f : U \rightarrow B$ is $\omega$-preserving if and only if for every finite $H \subseteq U$, $\prod H = 0$ implies that $\prod f[H] = 0$.

Then we say that $A$ is $n$-free over $X$ if every $n$-preserving function from $X$ into arbitrary $B$ extends to a unique homomorphism. The uniqueness just requires that $X$ be a generating set for $A$.

The existence of such extensions is equivalent to an algebraic property of $X$, namely that $X^+$ is $n$-independent. This notion is defined below, and the equivalence is proved. (For $n = 1$, this is the usual notion of free and independent; for $n = 2$, the notions are called $\perp$-free and $\perp$-independent by Heindorf [5]; Theorem 1.3 in the same paper shows that a 2-free boolean algebra has a 2-independent generating set. We differ from Heindorf in that he allows 0 to be an element of a $\perp$-independent set.) Since any function that is $n$-preserving is also $m$-preserving for all $m \leq n \leq \omega$, so that an $m$-free boolean algebra is also $n$-free over the same set; in particular, any $n$-free boolean algebra is $\omega$-free. It’s also worth noting that a function is $\omega$-preserving if and only if it’s $n$-preserving for all finite $n$.

Freeness over $X$ implies that no elementary products over $X$ can be 0. $n$-independence weakens this by allowing products of $n$ or fewer elements of $X$ to be 0. This requires some other elementary products to be 0 as well—if $x_1 \cdot x_2 \cdot \ldots \cdot x_m = 0$, then any elementary product that includes $x_1, \ldots, x_m$ each with exponent 1 must also be 0.

**Definition 1.3** Let $A$ be a boolean algebra. For $n$ a positive integer, $X \subseteq A$ is $n$-independent if and only if $0 \notin X$ and for all nonempty finite subsets $F$ and $G$ of $X$, the following three conditions hold:

\begin{align*}
(\perp 1) & \quad \sum F \neq 1. \\
(\perp 2) & \quad \text{If } \prod F = 0, \text{ there is an } F' \subseteq F \text{ with } |F'| \leq n \text{ such that } \prod F' = 0. \\
(\perp 3) & \quad \text{If } 0 \neq \prod F \leq \sum G, \text{ then } F \cap G \neq \emptyset.
\end{align*}

**Definition 1.4** Let $A$ be a boolean algebra. $X \subseteq A$ is $\omega$-independent if and only if $0 \notin X$ and for all nonempty finite subsets $F$ and $G$ of $X$, the following two conditions hold:

\begin{align*}
(\perp 1) & \quad \sum F \neq 1. \\
(\perp 3) & \quad \text{If } 0 \neq \prod F \leq \sum G, \text{ then } F \cap G \neq \emptyset.
\end{align*}
We note that in both the above definitions, if $X$ is infinite, then $(\perp 3) \implies (\perp 1)$; suppose $(\perp 1)$ fails; take a finite $G$ with $\sum G = 1$, then take some $x \not\in G$ and let $F \defeq \{x\}$; then $0 < \prod F \leq \sum G$ and $F \cap G \neq \emptyset$.

$(\perp 3)$ has several equivalent forms which will be useful in the sequel.

**Proposition 1.5** The following are equivalent for a subset $X$ of a boolean algebra $A$:

1. For all nonempty finite $F, G \subseteq X$, $(\perp 3)$.
2. For all nonempty finite $F, G \subseteq X$ such that $F \cap G = \emptyset$ and $\prod F \neq 0$, $\prod F \neq \sum G$.
3. For all nonempty finite $F, G \subseteq X$ such that $F \cap G = \emptyset$ and $\prod F \neq 0$, $\prod F \cdot \prod G \neq 0$, where $-G \defeq \{-g : g \in G\}$.
4. Let $X$ be bijectively enumerated by $I$ such that $X = \{x_i : i \in I\}$. For all nonempty finite $R \subseteq I$ and all $\varepsilon \in R^2$ such that $1 \in \text{rng} \varepsilon$ and $\prod_{i \in R} x_{\varepsilon_i} \neq 0$, $\prod_{i \in R} x_{\varepsilon_i} \neq 0$.

In words, the final equivalent says that no elementary product of elements of $X$ is 0 unless the product of the non-complemented elements is 0. We note that in the presence of $(\perp 2)_n$, the words “of $n$” may be inserted after “product.”

**Proof** We begin by pointing out that $(\perp 3)$ has two hypotheses, $0 \neq \prod F$ and $\prod F \leq \sum G$. Thus the contrapositive of $(\perp 3)$ is “If $F \cap G = \emptyset$, then $0 = \prod F$ or $\prod F \neq \sum G$,” which is equivalent to (2).

(2) and (3) are equivalent by some elementary facts: $a \leq b \iff a \cdot -b = 0$ and de Morgan’s law that $- \sum G = \prod -G$.

$(3) \implies (4)$: Assume (3) and the hypotheses of (4). If $\text{rng} \varepsilon = \{1\}$, the conclusion is clear. Otherwise, let $F \defeq \{x_i : i \in R$ and $\varepsilon_i = 0\}$. Then (3) implies that $\prod_{i \in R} x_{\varepsilon_i} \neq 0$, as we wanted.

$(4) \implies (3)$: Assume (4) and the hypotheses of (3). Let $R \defeq \{i \in I : x_i \in F \cup G\}$ and let $\varepsilon_i = 1$ if $x_i \in F$ and $\varepsilon_i = 0$ otherwise. Then (4) implies that $\prod F \cdot \prod -G \neq 0$, as we wanted. □

**Proposition 1.6** The following are equivalent for a subset $X$ of a boolean algebra $A$:

1. $X$ is $\omega$-independent
2. Let $X$ be bijectively enumerated by $I$ such that $X = \{x_i : i \in I\}$. For all nonempty finite $R \subseteq I$ and all $\varepsilon \in R^2$ such that $\prod_{i \in R} x_{\varepsilon_i} \neq 0$, $\prod_{i \in R} x_{\varepsilon_i} \neq 0$.

**Proof** The proof is similar to that of Proposition 1.5. $(\perp 1)$ is taken care of since products over an empty index set are taken to be 1 by definition. □

In the same spirit, we have an equivalent definition of $n$-independent.

**Proposition 1.7** Let $n$ be a positive integer or $\omega$, $A$ a nontrivial boolean algebra and $X \subseteq A^+$. $X$ is $n$-independent if and only if for every $R \in [X]^{<\omega}$ and every $\varepsilon \in R^2$, 
if $\prod_{x \in R} x^{e_x} = 0$ then there is an $R' \subseteq R$ with $|R'| \leq n$ such that $\varepsilon[R'] = \{1\}$ and $\prod R' = 0$.

Proof If $n = \omega$, this is part of Proposition 1.6.

Let $n$ be a positive integer, $A$ a boolean algebra, and $X \subseteq A^+$. We first show that $n$-independent sets have the indicated property.

Assume that $X$ is $n$-independent; take $R \in [X]^<\omega$ and $\varepsilon \in {}^R 2$ such that $\prod_{x \in R} x^{e_x} = 0$. Let $F = \{x \in R : \varepsilon_x = 1\}$ and $G = \{x \in R : \varepsilon_x = 0\}$. $F \neq \emptyset$; otherwise $\sum R = - \prod R = 1$, contradicting $(\perp 1)$. Since $\prod_{x \in R} x^{e_x} = \prod F \cdot \prod \neg G$, we have that $\prod F \leq \sum G$. If $G = \emptyset$, then $\sum G = 0$ and so $\prod F = 0$ as well. If $G \neq \emptyset$, then $\prod F = 0$ since $F \cap G = \emptyset$, using $(\perp 3)$. Then $R'$ is found by $(\perp 2)_n$. Now we show that sets with the indicated property are $n$-independent.

Assume that $X$ has the indicated condition and $F, G \in [X]^<\omega \setminus \{\emptyset\}$. We have three conditions to check.

$(\perp 1)$ Suppose that $\sum F = 1$. We let $F$ be the set $R$ in the condition, setting $\varepsilon_x = 0$ for all $x \in F$. Then $\prod_{x \in F} x^{e_x} = \prod \neg F = - \sum F = 0$ and $\{x \in F : \varepsilon_x = 1\} = \emptyset$, thus there is no $R'$ as in the condition, since products over an empty index set are equal to 1.

$(\perp 2)_n$ Suppose that $\prod F = 0$. Again we let $F$ be the set $R$ in the condition, this time setting $\varepsilon_x = 1$ for all $x \in F$. Then the condition gives us the necessary $F'$

$(\perp 3)$ Suppose that $0 \neq \prod F \leq \sum G$ and $F \cap G = \emptyset$. Let $R = F \cup G$ and $\varepsilon \in {}^R 2$ be such that $\varepsilon[F] = \{1\}$ and $\varepsilon[G] = \{0\}$. Then $\prod_{x \in R} x^{e_x} = 0$ and the condition gives $\prod F = 0$, which contradicts the original supposition. \hfill \Box

Lemma 1.8 If $H$ is an $\omega$-independent set that has no finite subset $F$ such that $\prod F = 0$, then $H$ is in fact independent. Furthermore, if $H$ is $n$-independent with no subset $F$ of size $n$ or less with $\prod F = 0$, then $H$ is independent.

Proof We only need show that $(\perp 2)_1$ holds, which it does vacuously. \hfill \Box

2-independence, and thus $n$-independence for $2 \leq n \leq \omega$, is also a generalization of pairwise disjointness on infinite sets.

Theorem 1.9 If $X \subseteq B^+$ is an infinite pairwise disjoint set, then $X$ is 2-independent.

Proof This is clear from Proposition 1.7. \hfill \Box

Some non-trivial examples of 2-free boolean algebras are the finite-cofinite algebras. For infinite $\kappa$, let $A = \text{FinCo}(\kappa)$. At $(A)$ is a 2-independent generating set for $A$.

Having an $n$-independent generating set is equivalent to $n$-freeness. This is known in Koppelberg [6] for $n = 1$ and Heindorf [5] for $n = 2$. Our proof is more elementary than that of Heindorf [5] in that it avoids clone theory.

Theorem 1.10 If $A$ is $\omega$-free over $X$, then $X^+$ is $\omega$-independent.
Proof Let $A$ and $X$ be as in the hypothesis; we show that $X^+$ is $\omega$-independent.
Without loss of generality, we may assume that $0 \not\in X$ so that $X^+ = X$.

$(\perp 1)$ Let $f : X \to \{0, 1\}$ be such that $f(X) = \{0\}$. Clearly $f$ is $\omega$-preserving and thus extends to a homomorphism $\overline{f}$. Take $F \in \{X\}^{<\omega}$; then $\overline{f}(\sum F) = \sum f[F] = 0$, so that $\sum F \neq 1$.

$(\perp 3)$ Take $F, G \in \{X\}^{<\omega}$ such that $F \cap G = \emptyset$ and $\prod F \neq 0$. Let $f : X \to \{0, 1\}$ be such that $f[F] = \{1\}$ and $f[X \setminus F] = \{0\}$. We claim that $f$ is $\omega$-preserving.
If $H \subseteq X$ is finite such that $\prod f[H] \neq 0$, then it must be that $H \subseteq F$, and hence $\prod H \neq 0$. Thus $f$ extends to a homomorphism $\overline{f}$. Then

$$\overline{f}(\prod F \cdot \prod \neg G) = \prod f[F] \cdot \prod \overline{f}[\neg G] = 1,$$

and so $\prod F \cdot \prod \neg G \neq 0$.  

Theorem 1.11 Let $n$ be a positive integer and $A$ a boolean algebra. If $A$ is $n$-free over $X$, then $X^+$ is $n$-independent.

Proof Again, without loss of generality $X = X^+$.

From Theorem 1.10, $X$ is $\omega$-independent, so we need only show that $(\perp 2)_{\omega}$ holds for $X$. We do this by contradiction; assume that $F \subseteq X$ is finite, of cardinality greater than $n$, $\prod F = 0$, and every subset $F' \subseteq F$ where $F'$ is of size $n$ is such that $\prod F' \neq 0$.

Define $f : X \to \{0, 1\}$ by letting $f[F] = \{1\}$ and $f[X \setminus F] = \{0\}$.

Then $f$ is $n$-preserving. Let $G \subseteq X$ be of size $n$ and have $\prod G = 0$. Then $G \not\subseteq F$, so some $x \in G$ has $f(x) = 0$, so $\prod f[G] = 0$. Thus $f$ must extend to a homomorphism, but then $f(0) = f(\prod F) = \prod f[F] = \prod \{1\} = 1$, which is a contradiction.  

Theorem 1.12 Let $A$ be generated by its $\omega$-independent subset $X$. Then $A$ is $\omega$-free over $X$.

Proof Let $f$ be an $\omega$-preserving function with domain $X$; we will show that $f$ extends to a unique homomorphism.

Take a finite $H \subseteq X$ and $\varepsilon \in H^2$ such that $\prod_{h \in H} h^{\varepsilon_h} = 0$. Then by $(\perp 3)$ and $(\perp 1)$, $\prod_{i=1}^k h = 0$. Then since $f$ is $\omega$-preserving, $\prod_{i=1}^k f(h) = 0$ and thus $\prod_{h \in H} f(h)^{\varepsilon_h} = 0$.

Thus by Sikorski’s extension criterion, $f$ extends to a homomorphism.

Uniqueness is clear as $X$ is a generating set.  

Theorem 1.13 Let $n$ be a positive integer. If $X$ generates $A$ and $X$ is $n$-independent, then $A$ is $n$-free over $X$.

Proof Let $f$ be an $n$-preserving function with domain $X$.

Take any distinct $x_0, x_1, \ldots, x_{k-1}$ and $\varepsilon \in \mathbb{K}^2$ such that $\prod_{i<k} x_i^{\varepsilon_i} = 0$.

Then by Proposition 1.7, there is an $F' \subseteq \{x_i : \varepsilon_i = 1 \text{ and } i < k\}$ such that $|F'| \leq n$ and $\prod F' = 0$. Since $f$ is $n$-preserving, it must be that $\prod f[F'] = 0$, and thus $\prod_{i<k} f(x_i)^{\varepsilon_i} = 0$. Thus, by Sikorski’s extension criterion, $f$ extends to a homomorphism.

Uniqueness is clear as $X$ is a generating set.  

\[ \square \]
So we have shown that the universal algebraic property defining \( n \)-free boolean algebras is equivalent to having an \( n \)-independent generating set.

**Theorem 1.14** \( \omega \)-free boolean algebras (and thus all \( n \)-free boolean algebras) are semigroup algebras.

A semigroup algebra is a boolean algebra that has a generating set that includes \( \{0, 1\} \), is closed under the product operation, and is disjunctive when 0 is removed.

**Proof** Let \( A \) be \( \omega \)-free over \( G \). Then let \( H' \) be the closure of \( G \cup \{0, 1\} \) under finite products, that is, the set of all finite products of elements of \( G \), along with 0 and 1. Clearly \( H' \) generates \( A \), includes \( \{0, 1\} \), and is closed under products, so all that remains is to show that \( H = H' \setminus \{0\} \) is disjunctive. From Proposition 2.1 of Monk [8], \( H \) is disjunctive if and only if for every \( M \subseteq H \) there is a homomorphism \( f \) from \( \langle H \rangle \) into \( \mathcal{P}(M) \) such that \( f(h) = M \downarrow h \) for all \( h \in H \).

To this end, given \( M \subseteq H \), let \( f : G \to \mathcal{P}(M) \) be defined by \( g \mapsto M \downarrow g \). We claim that \( f \) is \( \omega \)-preserving. Suppose \( G' = [G]^{<\omega} \) is such that \( \prod G' = 0 \). Then \( \prod_{g \in G} f(g) = \prod_{g \in G'} (M \downarrow g) = \{ a \in M : \forall g \in G' \{ a \leq g \} \} = \emptyset \). So \( f \) extends to a unique homomorphism \( \hat{f} \) from \( A \) to \( \mathcal{P}(M) \). If \( h \in H \setminus \{1\} \), then \( h = g_1 \cdot g_2 \cdot \ldots \cdot g_n \) where each \( g_i \in G \). So

\[
\hat{f}(h) = \hat{f}(g_1 \cdot g_2 \cdot \ldots \cdot g_n) = \hat{f}(g_1) \cap \hat{f}(g_2) \cap \ldots \cap \hat{f}(g_n)
\]

\[
= (M \downarrow g_1) \cap (M \downarrow g_2) \cap \ldots \cap (M \downarrow g_n)
\]

\[
= M \downarrow (g_1 \cdot g_2 \cdot \ldots \cdot g_n) = M \downarrow h.
\]

Likewise, \( \hat{f}(1) = M = M \downarrow 1 \). Thus \( H \) is disjunctive and \( A \) is a semigroup algebra over \( H' \).

\( \square \)

### 2 Hypergraphs and their Anticlique Algebras

There is a correspondence with hypergraphs for \( \omega \)-free boolean algebras. We recall that a hypergraph is a pair \( \mathcal{G} = \langle V, E \rangle \) where \( V \) is called the vertex set, and \( E \subseteq \mathcal{P}(V) \setminus \{\emptyset\} \) is called the hyperedge set; an element of \( E \) is called a hyperedge. We will insist on loopless hypergraphs, that is, \( E \subseteq \mathcal{P}(V) \setminus [V]^1 \). A hypergraph is \( n \)-uniform if \( E \subseteq [V]^n \). For a given hypergraph, we call a set \( A \subseteq V \) an anticlique if it includes no hyperedges; that is, for all \( e \in E \), \( e \setminus A \neq \emptyset \), and call the set of anticliques \( A(\mathcal{G}) \). Given a hypergraph \( \mathcal{G} \), we define an \( \omega \)-free boolean algebra as a subalgebra of \( \mathcal{P}(A(\mathcal{G})) \). For \( v \in V \), let \( v_+ \triangleq \{ A \in A(\mathcal{G}) : v \in A \} \), which is an element of \( \mathcal{P}(A(\mathcal{G})) \), and for a set \( H \) of vertices, \( H_+ \triangleq \{ v_+ : v \in H \} \). We then define the anticlique algebra of \( \mathcal{G} \) as \( \mathcal{B}_a(\mathcal{G}) \triangleq (V_+) \).

We do not consider cliques in general hypergraphs; it’s not clear which way to define them. For an \( n \)-uniform hypergraph, a clique may be non-controversially defined as a set \( C \) where \( [C]^n \subseteq E \), but for a hypergraph with hyperedges of different cardinalities, it is not clear how many hyperedges must be included in a clique. This difficulty stems from a lack of a reasonable way to define “complement hypergraph.”
A few possibilities for the hyperedge set of $\overline{G}$ are $\mathcal{P}(G) \setminus E$, $[G]^{<\omega} \setminus E$, and $[G]^{\mathcal{P}(E \setminus \{e\})} \setminus E$. For an $n$-uniform hypergraph $(G, E)$, the complementary hypergraph is $(G, [G]^n \setminus E)$, and then a clique in $\mathcal{G}$ is an anticlique in $\overline{G}$. Each possible definition for complement hypergraph results in a different definition for clique, all of which are more complicated than our definition of anticlique. Since anticliques suffice for our study, we do not choose a side on what a clique ought to be.

**Theorem 2.1** For any hypergraph $\mathcal{G} = (V, E)$, $\mathcal{R}_a(\mathcal{G})$ is $\omega$-free over $V_+$. 

**Proof** We need only show that $V_+$ is $\omega$-independent; we will use Proposition 1.7.

Suppose that $R \subseteq [V]^{<\omega}$, $\varepsilon \in \mathbb{S}_2$, and $\bigcap_{v \in R} v_+^\varepsilon = \emptyset$. Let $S = \{v \in R : \varepsilon_v = 1\}$. If $\bigcap_{v \in S} v_+^\varepsilon \neq \emptyset$, let $T$ be a member of $\bigcap_{v \in S} v_+^\varepsilon$. So then $T$ is an anticlique, and $S \subseteq T$. We note that clearly every subset of an anticlique is again an anticlique, so $S$ is also an anticlique, and $S \subseteq \bigcap_{v \in R} v_+^\varepsilon$. \hfill $\square$

If the hypergraph is somewhat special, we have more:

**Theorem 2.2** For any hypergraph $\mathcal{G} = (V, E)$ where $E \subseteq [V]^{\leq n}$, $\mathcal{R}_a(\mathcal{G})$ is $n$-free.

**Proof** We show that $V_+$ is $n$-independent.

From the previous theorem, we need only show that $(\perp 2)_n$ holds for $V_+$. Let $F$ be a finite subset of $V$ such that $\prod F_+ = 0$. Using the observation that $\prod F_+$ is the set of anticliques that include $F$, $F$ is not an anticlique. Thus some hyperedge $e$ is a subset of $F$. Then $\prod e_+ = 0$ as no anticlique can include that hyperedge. Since all hyperedges have at most $n$ vertices, $|e| \leq n$, which is what we wanted. \hfill $\square$

We also reverse this construction. Given a boolean algebra $A$ with an $\omega$-independent generating set $H$, we construct a hypergraph $\mathcal{G}$ such that $A \cong \mathcal{R}_a(\mathcal{G})$; we call it the $\perp$-hypergraph of $A, H$. The vertex set is $H$, and the hyperedge set is defined as follows; a subset $e$ of $H$ is a hyperedge if and only if the following three conditions are all true:

1. $e$ is finite.
2. $\prod e = 0$.
3. If $f \subseteq e$, then $\prod f \neq 0$.

We have only finite hyperedges in this graph, and no hyperedge is contained in another. Note that if $H$ is $n$-independent, the hyperedge set is included in $[H]^{\leq n}$.

**Theorem 2.3** Let $n$ be a positive integer or $\omega$, $X \subseteq A$ be $n$-independent and generate $A$, and $\mathcal{G} = (X, E)$ be the $\perp$-hypergraph of $A$. Then $A \cong \mathcal{R}_a(\mathcal{G})$.

**Proof** Let $f : X \to X_+$ be defined so that $v \mapsto v_+$ for $v \in X$. We claim that $f$ is an $n$-preserving function. If $G \subseteq X$ is of size $\leq n$ such that $\prod G = 0$, then it has a subset $G'$ minimal for the property of having 0 product; thus $G' \in E$, so that $\prod G'_+ = 0$, and so $\prod f[G] = 0$.

$f$ is bijective, and its inverse is also $n$-preserving; the image of $f$ is a generating set, so that $f$ extends to an isomorphism. \hfill $\square$
Definition 2.4 Let $\mathcal{G}_i (V_i, E_i)$ be hypergraphs for $i \in \{0, 1\}$. A hypergraph homomorphism is a function $f : V_0 \rightarrow V_1$ such that if $e \in E_0$, then $f[e] \in E_1$.

Notice that a graph homomorphism is a hypergraph homomorphism when the graphs are considered as 2-uniform hypergraphs.

In the rest of this section we consider ordinary graphs, that is, hypergraphs for which $E \subseteq V^2$. In this case, “clique” is not ambiguous, so we can define the clique algebra of a graph. We let $C(\mathcal{G})$ be the set of cliques in $\mathcal{G}$, and $v_+$ be the set of cliques including vertex $v$. (This conflicts with an earlier use of $v_+$, but context will make it clear which is meant.) Then $\mathcal{B}_c(\mathcal{G})$ is the subalgebra of $\mathcal{P}(C(\mathcal{G}))$ generated by $\{v_+ : v \in G\}$.

We give some examples of 2-free boolean algebras with unusual properties.

For a 2-free algebra of the form $\mathcal{B}_c(T)$ for a tree (in the graph-theoretical sense—a connected acyclic graph) or a forest $T$ of size $\kappa$, there are further conclusions that can be drawn. As a forest has no triangles, all the cliques in $T$ are of size at most 2.

So any subset of $T_+$ of size 3 or more has a disjoint pair.

If $T$ is a $\kappa$-tree (in the order theoretic sense, that is, of height $\kappa$ and each level of size $< \kappa$), and we take the edge set to consist of pairs $\{u, v\}$ where $v$ is an immediate successor of $u$, then $T_+$ has a pairwise disjoint subset of size $\kappa$—take an element of every other level—so that FinCo($\kappa$) $\leq$ $\mathcal{B}_c(T)$, and Fr($\kappa$) $\leq$ $\mathcal{B}_a(T)$.

It seems difficult to avoid one of FinCo($\kappa$) and Fr($\kappa$) as a subalgebra, as it is necessary to find a graph of size $\kappa$ with no clique or anticlique of size $\kappa$. A witness to $\kappa \not\rightarrow \langle \kappa \rangle^2_2$ is the edge set of such a graph, but we do not know about the variety of such witnesses. If $\kappa$ is weakly compact, then there are no such witnesses and so for any graph of size $\kappa$, FinCo($\kappa$) or Fr($\kappa$) is a subalgebra of $\mathcal{B}_c(\mathcal{G})$.

As a graph can be characterized as a symmetric non-reflexive relation, for any non-reflexive relation $R$, we may form algebras $\mathcal{B}_a(R \cup R^{-1})$ and $\mathcal{B}_c(R \cup R^{-1})$. When $R$ is an ordering of some sort, $R \cup R^{-1}$ is usually called the (edge set of the) comparability graph of $R$. Thus for a (non-reflexive ) ordering $\langle P, \prec \rangle$, it has comparability graph $\mathcal{G}_P = \{P, < \cup <^{-1}\}$ and we define its comparability algebra $\mathcal{B}_{\text{co}}(P) \overset{\text{def}}{=} \mathcal{B}_c(\mathcal{G}_P)$ and its incomparability algebra $\mathcal{B}_{\text{aco}}(P) \overset{\text{def}}{=} \mathcal{B}_a(\mathcal{G}_P)$. Since points in the partial order are vertices of the comparability graph, we may use the $p_+$ notation without fear of confusion. When $P$ is a partial order in the strict sense, $C \subseteq P$ is a clique in $\mathcal{G}_P$ if and only if $C$ is a chain in $\leq$ if and only if $C_+$ is an independent subset of $\mathcal{B}_{\text{co}}(P)$, and $A \subseteq P$ is an anticlique in $\mathcal{G}_P$ if and only if $A$ is an antichain in $\leq$ if and only if $A_+$ is a pairwise disjoint set in $\mathcal{B}_{\text{aco}}(P)$. So if $\langle T, \leq \rangle$ is a $\kappa$-Suslin tree, in both $\mathcal{B}_{\text{co}}(T)$ and $\mathcal{B}_{\text{aco}}(T)$, $T_+$ is a 2-independent set of size $\kappa$, but has no independent subset of size $\kappa$, nor a pairwise disjoint subset of size $\kappa$ since $T$ has neither chains nor antichains of size $\kappa$.

Proposition 2.5 If $f : P \rightarrow Q$ is a strictly order-preserving function, that is, a morphism in the category of partial orders, then there is a homomorphism $f^* : \mathcal{B}_{\text{aco}}(P) \rightarrow \mathcal{B}_{\text{aco}}(Q)$ such that $f^*(p_+) = f(p)_+$.

Proof By the universal property of 2-free boolean algebras, we need only show that $g$ is 2-preserving where $g(p_+) = f(p)_+$. Then $g$ extends to the $f^*$ of the conclusion.

Fix distinct $p, p' \in P$; if $p_+ \perp p'_+$ in $\mathcal{B}_{\text{aco}}(P)$, then $p$ and $p'$ are comparable in $P$, without loss of generality, $p < p'$. Then $f(p) < f(p')$, so that $f(p)_+ \perp f(p')_+$. \(\square\)
Similarly, an incomparability-preserving map from $P$ to $Q$ gives rise to a homomorphism of $\mathcal{B}_{co}(P)$ and $\mathcal{B}_{co}(Q)$.

3 Hypergraph Spaces

The dual spaces to $\omega$-free boolean algebras are also interesting. Like with graphs, a hypergraph space may be defined in terms of a hypergraph—the definition generalizes that of a graph space.

**Definition 3.1** Let $\mathcal{G} = (G, E)$ be a hypergraph and $A(\mathcal{G})$ its set of anticliques. For each $v \in G$, we define two sets:

$$v_+ \overset{\text{def}}{=} \{ A \in A(\mathcal{G}) : v \in A \}$$

$$v_- \overset{\text{def}}{=} \{ A \in A(\mathcal{G}) : v \notin A \} .$$

Then the hypergraph space of $\mathcal{G}$ is the topology on $A(\mathcal{G})$ with $\bigcup_{v \in G} \{ v_+, v_- \}$ as a closed subbase.

Any topological space $\mathcal{T}$ for which there is a hypergraph $\mathcal{G}$ such that $\mathcal{T}$ is homeomorphic to the hypergraph space of $\mathcal{G}$ is called a hypergraph space.

**Theorem 3.2** The Stone dual of an $\omega$-free boolean algebra is a hypergraph space.

**Proof** Let $A$ be an $\omega$-free boolean algebra. Thus by Theorem 2.3, there is a hypergraph $\mathcal{G}$ such that $A \cong \mathcal{B}_a(\mathcal{G})$. Let $\mathcal{T}$ be the hypergraph space of $\mathcal{G}$. We claim that $\text{Clop}(\mathcal{T}) \cong A$.

In fact, $\text{Clop}(\mathcal{T}) = \mathcal{B}_a(\mathcal{G})$. On both sides here, elements are sets of anticliques of $\mathcal{G}$. As $\mathcal{T}$ is defined by a clopen subbase, elements of $\text{Clop}(\mathcal{T})$ are finite unions of finite intersections of elements of that subbase $\bigcup_{v \in G} \{ v_+, v_- \}$. Elements of the right hand side are sums of elementary products of elements of $\bigcup_{v \in G} \{ v_+, v_- \}$, that is, sums of finite products of elements of $\bigcup_{v \in G} \{ v_+, v_- \}$. As the operations are the usual set-theoretic ones on both sides, they are in fact the same algebra.

The topological result follows by duality. \hfill $\square$

We repeat a few definitions from Bell and van Mill [3] needed for some topological applications.

**Definition 3.3** Let $n \in \omega$ for all these definitions.

A set $S$ is $n$-linked if every $X \in [S]^n$ has non-empty intersection.

A set $P$ is $n$-ary if every $n$-linked subset of $P$ has non-empty intersection.

A compact topological space $\mathcal{T}$ has compactness number at most $n$, written $\text{cmpn}(\mathcal{T}) \leq n$, if and only if it has an $n$-ary closed subbase. $\mathcal{T}$ has compactness number $n$, written $\text{cmpn}(\mathcal{T}) = n$, if and only if $n$ is the least integer for which $\text{cmpn}(\mathcal{T}) \leq n$. $\text{cmpn}(\mathcal{T}) = \omega$ if there is no such $n$.

The following generalizes and algebraizes Proposition 3.1 of Bell [1].
Proposition 3.4 If a boolean algebra $A$ is $n$-free for some $2 \leq n \leq \omega$, then $\text{cmpn} (\text{Ult} \ A) \leq n$.

Proof This is vacuous if $n = \omega$. If $n < \omega$, then $\text{Ult} \ (A)$ is a hypergraph space for a hypergraph $\mathcal{G}$ with all hyperedges of size $\leq n$.

We take the clopen subbase $S = \bigcup_{v \in G} \{v_+, v_-\}$ of the hypergraph space of $\mathcal{G}$ and show that it is $n$-ary. Let $\mathcal{F} \subseteq S$ be $n$-linked. We may write

$$\mathcal{F} = \{v_+ : v \in A\} \cup \{v_- : v \in B\}$$

for some $A, B \subseteq G$. Since $v_+ \cap v_- = \emptyset$ and $n \geq 2, A \cap B = \emptyset$. Let $A'$ be a finite subset of $A$. Since any product of $n$ or fewer elements of $\mathcal{F}$ is non-zero, $A'$ must be an anticlique in $\mathcal{G}$; if not, then $\prod A'_+ = 0$, so then $A'_+$ would have a subset of size $n$ with empty intersection, contradicting that $\mathcal{F}$ is $n$-linked. Thus $A' \in \bigcap \mathcal{F}$, that is, $\mathcal{F}$ has non-empty intersection and thus $S$ is $n$-ary. $\square$

Bell’s [2] Corollary 5.2 shows that certain topologies on $[\omega_1]^\leq n$ have compactness number $n$ for certain $n, m \leq \omega$. These topologies are the hypergraph spaces of $[\omega_1, [\omega_1]^{2n-3}]$ and $[\omega_1, [\omega_1]^{3n-2}]$.

Theorem 3.5 For infinitely many $n \in \omega$, there is a boolean algebra which is $n$-free and is not $(n-1)$-free.

Proof Let $k$ be the least integer for which $\mathcal{B}_a \left([\omega_1, [\omega_1]^{2n-3}]\right)$ is $k$-free and $\ell$ be the least integer for which $\mathcal{B}_a \left([\omega_1, [\omega_1]^{2n-2}]\right)$ is $\ell$-free. We have that $n \leq k \leq 2n - 3$ and $n \leq \ell \leq 2n - 2$. The lower bounds are a consequence of the compactness numbers of those spaces (Bell’s [2] result and Proposition 3.4), while the upper bounds are a consequence of Theorem 2.2.

Thus we have, for arbitrary $n \in \omega$, an $\omega$-free boolean algebra of finite freeness at least $n$. $\square$

4 Constructions

In this section, we consider the categories of $n$-independently generated boolean algebras and of hypergraph spaces and their behavior under some constructions.

If a boolean algebra $A$ is $\omega$-free, it is isomorphic to $\mathcal{B}_a (\mathcal{G})$ for some hypergraph $\mathcal{G}$; we’ll call this the $\perp$-hypergraph of $A$. If a boolean algebra is 2-free, this $\perp$-hypergraph is a graph, so we can just call it the $\perp$-graph. Such a boolean algebra is also isomorphic to $\mathcal{B}_c (\mathcal{G})$ for a graph $\mathcal{G}$, which is called the intersection graph of $A$.

We will show in Section 5 that complete boolean algebras are not $\omega$-free. As $\mathcal{P} (\kappa)$ is isomorphic to $^\kappa 2$, the class of $\omega$-free boolean algebras is not closed under infinite products.

Theorem 4.1 Let $2 \leq n \leq \omega$. If $H \subseteq A$ and $K \subseteq B$ are $n$-independent, then $L \overset{\text{def}}{=} (H \times \{0\}) \cup (\{0\} \times K)$ is $n$-independent in $A \times B$.

Proof We will apply Proposition 1.7. Suppose that $F \in [H]^{<\omega}, G \in [K]^{<\omega}, \varepsilon \in ^F 2, \delta \in ^G 2$, and $\prod_{x \in F} (x, 0)^{\varepsilon_x} \cdot \prod_{y \in G} (0, y)^{\delta_y} = 0$. If there are $x \in F$ and $y \in G$
such that $\varepsilon_x = \delta_y = 1$, then $(x, 0) \cdot (0, y) = 0$ as desired. Otherwise, without loss of generality, we may assume that $\varepsilon [F] \subseteq \{0\}$. Then $\prod_{x \in F} (x, 0)^{\varepsilon_x} = (\prod_{x \in F} -x, 1)$, so that $\prod_{y \in Y} y^{\delta_y} = 0$; then the $n$-independence of $K$ gives the result.

It is important to note that $L$ does not generate $\langle H \rangle \times \langle K \rangle$; in fact (Theorem 4.4), the product of $n$-free boolean algebras is not in general $n$-free. However, it is the case that $\langle H \rangle \times \langle K \rangle$ is a simple extension of the subalgebra generated by $L$; $\langle L \rangle \cdot (1, 0) = \langle H \rangle \times \langle K \rangle$.

This result generalizes to infinite products quite easily, though the notation is considerably more cumbersome.

**Theorem 4.2** For $2 \leq n \leq \omega$, if $\langle A_i : i \in I \rangle$ is a system of boolean algebras and for every $i \in I$, $H_i \subseteq A_i$ is $n$-independent in $A_i$, then the set $H^\text{def} = \bigcup_{i \in I} p_i [H_i]$, where

$$p_i (h) (j) = \begin{cases} h \; i = j & 0 \; i \neq j \end{cases}$$

is $n$-independent in $A^\text{def} = \prod_{i \in I} A_i$ and $\prod_{i \in I} A_i$.

**Proof** This is essentially the same as Theorem 4.1 with more cumbersome notation.

$p_i (h)$ is the function in $A$ that is 0 in all but the $i$th coordinate and is $h$ in the $i$th coordinate, so that the projections $\pi_i [p_i [H_i]] = H_i$ and for $i \neq j$, $\pi_j [p_i [H_i]] = \{0\}$.

We apply Proposition 1.7. Suppose that $R \in [H]^{<\omega}, \varepsilon \in R^2$, and $\prod_{x \in R} x^{\varepsilon_x} = 0$. Let $J^\text{def} = \{i \in I : R \cap p_i [H_i] \neq \emptyset\}$. If $J$ is a singleton, say $J = \{i\}$, then the $n$-independence of $H_i$ clearly makes $H$ $n$-independent. So we now concern ourselves with the case that $|J| > 1$, that is, we have distinct $i, j \in J$. If there are $x \in p_i [H_i]$ and $y \in p_j [H_j]$ with $\varepsilon_x = \varepsilon_y = 1$, then $x \cdot y = 0$ and we have our conclusion. So we may assume that there is at most one $i \in J$ for which there is an $x \in p_i [H_i]$ such that $\varepsilon_x = 1$. Then for any particular $i \in I$, $\prod \{x^{\varepsilon_x} : x \in R, x \neq p_i [H_i]\}$ has $i$-th coordinate 1, and so the facts that

$$0 = \prod_{x \in R} x^{\varepsilon_x} = \prod \{x^{\varepsilon_x} : x \in R \cap p_i [H_i]\},$$

and that all the $H_i$ are $n$-independent make $H$ $n$-independent.

When $n = 2$, we can also consider the $\perp$-graph and intersection graph of $H$ in the above theorem. The intersection graph is easily described: two elements of $H$ have non-zero product if and only if they have non-zero product in one of the factors, so that the intersection graph is the disjoint union of the intersection graphs of the $H_i$. The $\perp$-graph is more complex. The $\perp$-graph of each $H_i$ is an induced subgraph, but these subgraphs are connected to each other—each vertex in $H_i$ is connected to every vertex in $H_j$ for $i \neq j$. This construction is the “join”.

In other words, for any collection $\{G_i\}$ of graphs, $Bc (\bigcup_{i \in I} G_i) \leq \prod_{i \in I} Bc (G_i)$ and $Ba (\bigcup_{i \in I} G_i) \leq \prod_{i \in I} Ba (G_i)$.
The use of the word “free” in $n$-free is warranted by the following:

**Theorem 4.3** Suppose that $A \defeq \bigoplus_{i \in I} A_i$ is an amalgamated free product of subalgebras $A_i$ for $i \in I$, where $C \leq A_i$ for each $i \in I$, $A_i \cap A_j = C$ for $i \neq j$, $A_i$ is $n$-free over $H_i$, and $C \leq \{ H_i \cap H_j \}$. Then $A$ is $n$-free over $\bigcup_{i \in I} H_i$.

**Proof** For convenience, assume that each $A_i \leq A$, $C \leq A_i$ and that, for $i \neq j$, $A_i \cap A_j = C$, and that $H_i$ is a set over which $A_i$ is $n$-free. We show that $A$ is $n$-free over $H \defeq \bigcup_{i \in I} H_i$.

Let $B$ be a boolean algebra, and $f : H \to B$ be $n$-preserving. Then for each $i \in I$, $f_i \defeq f \restriction H_i$ is also $n$-preserving. So each $f_i$ extends to a unique homomorphism $\varphi_i : A_i \to B$. That $\varphi_i \restriction C = \varphi_j \restriction C$ is clear as $C \subseteq \{ H_i \cap H_j \}$.

Then the universal property of amalgamated free products gives a unique homomorphism $\varphi : A \to B$ that extends every $\varphi_i$. Note that

$$\varphi \restriction H = \bigcup_{i \in I} (\varphi \restriction H_i) = \bigcup_{i \in I} (\varphi_i \restriction H_i) = \bigcup_{i \in I} f_i = f.$$ 

So we have a unique extension of $f$ to a homomorphism, which is what we wanted. \qed

This course includes free products.

An example where $C \neq \{0, 1\}$ is as follows: Let $\mathcal{G}$ be the complete graph on the ordinal $\omega_1 + \omega$ and $\mathcal{H}$ the complete graph on the ordinal interval $(\omega_1, \omega_1 \cdot 2)$. Then $\mathcal{B}_a(\mathcal{G}) \cong \mathcal{B}_a(\mathcal{H}) \cong \text{Fr}(\omega_1)$. Note that $G \cap H = (\omega_1, \omega_1 + \omega)$ so that $G_+ \cap H_+ = (\omega_1, \omega_1 + \omega)$. We let $C = (\omega_1, \omega_1 + \omega) \subseteq \text{Fr}(\omega_1)$. It is clear that $C$ is as required in Theorem 4.3. Then we have that $\mathcal{B}_a(\mathcal{G}) \otimes \mathcal{B}_a(\mathcal{H})$ is $2$-free over $G_+ \cup H_+$.

If $C$ is $2$-free over $\bigcup_{i \in I} H_i$, the $\perp$-graph of $\bigcup_{i \in I} H_i$ is easily described in terms of those of $H_i$. It is the “amalgamated free product” or “amalgamated disjoint union” in the category of graphs—the same universal property holds. More concretely, given a set of graphs $\mathcal{G}_i = (G_i, E_i)$, each of which has $\mathcal{F} = (F, E)$ as a subgraph, the amalgamated disjoint union of the $\mathcal{G}_i$ over $\mathcal{F}$ is a graph on the union of the vertex sets where two vertices are adjacent if and only if they are adjacent in some $\mathcal{G}_i$. That is, elements of $G_i \setminus F$ and $G_j \setminus F$ are not adjacent for $i \neq j$.

In case $C = 2$ and we have a free product, the $A_i$ form a family of independent subalgebras, so two elements of $H$ (constructed in the proof above) have product zero if and only if they are in the same $H_i$ and have zero product in $A_i$. So the $\perp$-graph of $H$ is the disjoint union of the $\perp$ graphs of the $H_i$. The intersection graph of $H$ is similarly constructed from those of the $H_i$: the independence of the $A_i$ means that the intersection graph of $H$ is the join of the intersection graphs of the $H_i$.

That is, $\bigoplus_{i \in I} \mathcal{B}_a(\mathcal{G}_i) \cong \mathcal{B}_a(\bigcup_{i \in I} \mathcal{G}_i)$ and $\bigoplus_{i \in I} \mathcal{B}_c(\mathcal{G}_i) \cong \mathcal{B}_c(\bigcup_{i \in I} \mathcal{G}_i)$.

Products of $n$-free boolean algebras behave in a somewhat more complicated manner. As discussed previously, infinite products of $\omega$-free boolean algebras are not necessarily $\omega$-free.

**Theorem 4.4** FinCo $(\omega_1) \times \text{Fr}(\omega_1)$ is not $2$-free.
Proof We use subscript function notation for the coordinates of tuples; i.e. \((a, b)_0 = a\) and \((a, b)_1 = b\). We also extend this to sets of tuples; \(((a, b), (c, d))_0 = (a, c)\).

We proceed by contradiction; suppose that \(A \eqdef \text{FinCo}(\omega_1) \times \text{Fr}(\omega_1)\) is 2-free over \(X\), where \(0 \notin X\), that is, \(X\) is 2-independent.

Consider \(a_a \eqdef ((a), 0)\) for \(a < \omega_1\). \(a_a\) is an atom in \(A\), so it must be an elementary product of \(X\), that is, \(a_a = \prod_{x \in H_a} x^{e(xa)}\), with \(H_a \in [X]^{<\omega_0}\). So let \(M \in [\omega_1]^{\omega_1}\) be such that \(\{H_a : a \in M\}\) is a \(\Delta\)-system with root \(F\). Let \(G_a \eqdef H_a \setminus F\). Since

\[
M = \bigcup_{\delta \in \tau^2} \{\alpha \in M : \forall x \in F [\varepsilon(\alpha, x) = \delta]\},
\]

there is an uncountable \(N \subseteq M\) such that \(\varepsilon(\alpha, x) = \varepsilon(\beta, x)\) for all \(\alpha, \beta \in N\) and all \(x \in F\), so that we may write, for \(\alpha \in N\), \(a_a = \prod_{x \in F} x^{\delta_x} \cdot \prod_{x \in G_a} x^{\varepsilon(\alpha, x)}\). For each \(\alpha \in N\), let \(G_a^{\alpha} \eqdef \{x \in G_a : \varepsilon(\alpha, x) = 1\}\). If \(\alpha, \beta \in N\) with \(\alpha \neq \beta\), then there are \(x \in G_a^\alpha\) and \(y \in G_a^\beta\) such that \(x \cdot y = 0\), by the 2-independence of \(X\) and the fact that

\[
0 = a_a \cdot a_\beta = \prod_{x \in F} x^{\delta_x} \cdot \prod_{x \in G_a} x^{\varepsilon(\alpha, x)} \cdot \prod_{x \in G_{a\beta}} x^{\varepsilon(\beta, x)},
\]

thus \(\prod G_a^\alpha \cdot \prod G_{a\beta} = 0\). Since \(\text{Fr}(\omega_1)\) has cellularity \(\omega\), the set \(\{\alpha \in N : (\prod G_a^\alpha) \neq 0\}\) is countable, hence \(P \eqdef N \setminus \{\alpha \in N : (\prod G_a^\alpha) \neq 0\}\) is uncountable and for \(\alpha \in P\), \((\prod G_a^\alpha) = 0\). Since \((\prod G_a^\alpha) \cdot (\prod G_{\beta})_0 = 0\) for distinct \(\alpha, \beta \in P\), each \((\prod G_a^\alpha)\) is finite when \(\alpha \in P\).

\(X\) must generate \((1, 0)\); let \(b_j\) for \(j < n\) be disjoint elementary products of \(X\) such that \(\sum_{j<n} b_j = (1, 0)\). Thus there must be exactly one \(i < n\) such that \(b_{i0}\) is cofinite; without loss of generality, \(i = 0\) so that \(b_{i0}\) is cofinite and \(b_{i0} = 0\). Write \(b_0\) as an elementary product, that is, \(b_0 = \prod_{j<m} c_j^{e_j}\) with each \(c_j \in X\). Then choose an \(\alpha \in P\) such that \(\prod G_a^\alpha \leq b_0\) and \(G_a^\alpha \cap \{c_j : j < m\} = \emptyset\). Then \(\prod G_a^\alpha \cdot \sum_{j<n} c_j^{1-e_j} = 0\), so \(\text{rng} \xi = \{0\}\); that is, \(b_0 = \prod_{j<m} -c_j\).

Note that \(X_1\) generates \(\text{Fr}(\omega_1)\), so it must be uncountable, thus \((X \setminus \{c_j : j < m\} \setminus F)\) is also uncountable; let \(Y \subseteq X\) be such that \(Y_1\) is an uncountable independent subset of \((X \setminus \{c_j : j < m\} \setminus F)\); such a \(Y\) exists by Theorem 9.16 of Koppelberg [6]. Note that no finite product of elements of \(Y\) is 0. Let \(\theta : Y \to (0, 1)\) be such that \(d_y \eqdef (y^{\theta_y})_0\) is finite for each \(y \in Y\).

Consider \(\{d_y : y \in Y\}\); Each \(d_y\) is finite and \(Y\) is an uncountable set, and thus there is an uncountable \(Z \subseteq Y\) where \(\{d_y : y \in Z\}\) is a \(\Delta\)-system with root \(r\). Let \(y, z, t \in Z\) be distinct. Then let \(e_y \eqdef d_y \setminus r, e_z \eqdef d_z \setminus r,\) and \(e_t \eqdef d_t \setminus r\). Then

\[
d_y \cdot d_z \cdot -d_t = (e_y \cup r) \cap (e_z \cup r) \cap (\omega_1 \setminus (e_t \cup r)) = r \cap (\omega_1 \setminus (e_t \cup r)) = \emptyset,
\]

Then \(\prod_{j<n} -c_j \cdot y^{\theta_y} \cdot z^{\theta_z} \cdot t^{1-\theta_t} = 0\) and again, the only elements with exponent 1 are elements of \(Y\) and thus there is no disjoint pair, contradicting Proposition 1.7.

So we have a contradiction and thus there is no 2-independent generating set for \(A\). \(\square\)

This is also an example of a simple extension of a 2-free boolean algebra that is not 2-free; the full product is a simple extension by \((0, 1)\) of the subalgebra generated by the set in Theorem 4.1.
The dual of this theorem is that we have two graph spaces whose disjoint union is not a graph space; in fact we can say a bit more since the disjoint union of two supercompact spaces is supercompact. We show a slightly more general result here:

**Proposition 4.5** If \( X \) and \( Y \) are \( n \)-compact spaces, then \( X \cup Y \) is \( n \)-compact.

**Proof** Suppose that \( S \) and \( T \) are \( n \)-ary subbases for the closed sets of \( X \) and \( Y \) respectively; that is, for any \( S' \subseteq S \) with \( \bigcap S' = \emptyset \), there are \( n \) members \( a_1, a_2, \ldots, a_n \) of \( S' \) such that \( a_1 \cap a_2 \cap \ldots \cap a_n = \emptyset \), and similarly for \( T \). Then \( W \defeq S \cup T \cup \{ X, Y \} \) is an \( n \)-ary subbase for the closed sets of \( X \cup Y \).

\( \Box \)

So, letting \( n = 2 \), the dual space of \( \text{FinCo}(\omega_1) \times \text{Fr}(\omega_1) \) is supercompact, but is not a graph space.

**Theorem 4.6** \( \text{FinCo}(\omega_1) \times \text{Fr}(\omega_1) \) is \( 3 \)-free.

**Proof** Let \( \{ x_\alpha : \alpha < \omega_1 \} \) be an independent generating set for \( \text{Fr}(\omega_1) \). Then the set \( X \defeq \{ (\{ \alpha \}, x_\alpha) : \alpha < \omega_1 \} \cup \{ (1, 0) \} \) is a \( 3 \)-independent generating set for \( \text{FinCo}(\omega_1) \times \text{Fr}(\omega_1) \). That \( X \) generates \( \text{FinCo}(\omega_1) \times \text{Fr}(\omega_1) \) is clear. We use Proposition 1.7 to show that \( X \) is \( 3 \)-independent. Take any \( R \in [X]^{\omega} \) and \( \varepsilon \in \mathbb{R}^2 \) such that \( \prod_{x \in R} x^{\varepsilon_x} = (0, 0) \). Since there is no elementary product of elements of \( \{ x_\alpha : \alpha < \omega_1 \} \) that is \( (0, 1, 0) \in R \) and \( \varepsilon_{(1, 0)} = 1 \). Then there is a pair \( a, b \) of elements in \( R \) such that \( \pi_2(a) \perp \pi_2 b \) and \( \varepsilon_a = \varepsilon_b = 1 \), so that \( \{ (1, 0) \cup a, b \} \subseteq R \) and \( (1, 0) \cdot a \cdot b = 0 \).

\( \Box \)

5 Cardinal Function Results

Cellularity and independence have been considered earlier. Here we give a few results relating other cardinal functions to properties of \( \perp \)-graphs and intersection graphs. We will always assume that the graphs and algebras are infinite in this section.

**Lemma 5.1** Let \( A \) be \( \omega \)-free and \( \omega \leq \kappa = |A| \). Then \( B \defeq \text{FinCo}(\kappa) \) is a homomorphic image of \( A \).

**Proof** Let \( G \) be a set over which \( A \) is \( \omega \)-free. Any bijective function \( f : G \to \text{At}(B) \) is \( \omega \)-preserving as all elements of \( \text{At}(B) \) are disjoint. Since \( A \) is \( \omega \)-free, \( f \) extends to a homomorphism \( \tilde{f} \) from \( A \) to \( B \). Since the image of \( f \) includes a set of generators, \( \tilde{f} \) is surjective as well; that is, \( B \) is a homomorphic image of \( A \).

\( \Box \)

The first use of this is that no infinite \( \omega \)-free boolean algebra has the countable separation property. The countable separation property is inherited by homomorphic images (5.27(c) in Koppelberg [6]), so if any infinite \( \omega \)-free boolean algebra of size \( \kappa \) has the countable separation property, then by Lemma 5.1, \( \text{FinCo}(\kappa) \) has the countable separation property, which is a contradiction. In particular, \( \mathcal{P}(\omega) / \text{fin} \) is not \( \omega \)-free.

We show that the spread of an \( \omega \)-free boolean algebra is equal to its cardinality.

Theorem 13.1 of Monk [8] gives several equivalent definitions of spread, all of
which have the same attainment properties; the relevant one to our purposes is the following.

\[ s(A) = \sup \{ c(B) : B \text{ is a homomorphic image of } A \} . \]

**Theorem 5.2** For \( A \) \( \omega \)-free, \( s(A) = |A| \). Furthermore, it is attained.

**Proof** From Lemma 5.1, \( B = \text{FinCo}(|A|) \) is a homomorphic image of \( A \). Since \( c(B) = |B| = |A| \), an element of the set in the above definition of \( s(A) \) is \( |A| \). Thus \( s(A) = |A| \) is attained. \( \square \)

As they are greater than or equal to \( s \), Inc, Irr, h-cof, hL, and hd are also equal to cardinality for \( \omega \)-free boolean algebras. Incomparability and irredundance are also attained by the \( \omega \)-free generating set. This result also determines that \( |\text{Id}A| = 2^{|A|} \) as \( 2^{|A|} \leq |\text{Id}A| \). Then since \( s \) is attained, \( |\text{Sub}A| = 2^{|A|} \) as well.

**Theorem 5.3** If \( A \) is infinite and \( \omega \)-free, then \( \pi(A) = |A| \).

Here \( \pi \) is the density of \( A \), the minimum of the cardinalities of dense subsets of \( A \).

**Proof** Take \( H \) to be a set over which \( A \) is \( \omega \)-free and let \( D \subseteq A^+ \) be dense.

For each \( d \in D \), we can find a non-zero elementary product of elements of \( H \) below \( d \); write it as \( \prod F_d \cdot \prod -G_d \) for finite disjoint \( F_d, G_d \subseteq H \).

Now we show that \( H = \bigcup_{d \in D} F_d \). Obviously \( \bigcup_{d \in D} F_d \subseteq H \), so we need only show \( H \subseteq \bigcup_{d \in D} F_d \). Choose an \( h \in H \). Since \( D \) is dense, there is a \( d \in D \) with \( d \leq h \). So \( \prod F_d \cdot \prod -G_d \leq d \leq h \). Thus \( \prod F_d \leq h + \sum G_d \), and since \( H \) is \( \omega \)-independent, \( h \in F_d \).

Since all the \( F_d \) are finite, \( |D| = |H| = |A| \). \( \square \)

We claim that the length (and therefore depth) of an \( \omega \)-free boolean algebra is \( \aleph_0 \). This uses several preceding results.

**Theorem 5.4** If \( A \) is \( \omega \)-free, then \( A \) has no uncountable chain.

**Proof** Let \( A \) be \( \omega \)-free over \( G \).

Recall from Theorem 1.14 that \( A \) is a semigroup algebra over the set \( H \) of finite products of elements of \( G \cup \{0, 1\} \). For \( h \in H \setminus \{0, 1\} \), choose \( g_1, \ldots, g_n \in G \) such that \( h = g_1 \cdot \ldots \cdot g_n \) and set \( h_G \overset{\text{def}}{=} \{g_1, \ldots, g_n\} \).

Due to the result of Heindorf [4], if there is an uncountable chain in \( A \), there is an uncountable chain in \( H \). So by way of contradiction, we assume that there is an uncountable chain \( C \subseteq H \). Without loss of generality, we may assume that \( 0, 1 \notin C \) so that every element of \( C \) is a finite product of elements of \( G \).
Let $C_G \overset{\text{def}}{=} \{ h_G : h \in C \}$. We note that

$$\bigcup_{h \in C} C_G = \bigcup_{h \in C} h_G \subseteq G$$

is the set of all elements of $G$ that are needed to generate the elements of $C$, that is, $C \subseteq \bigcup C_G$, so $C$ is a chain in that subalgebra of $A$ as well.

In order to reach a contradiction, we first show that there are no finite subsets of $\bigcup C_G$ with zero product. Take $F \in \left( \bigcup C_G \right)^{<\omega}$. Therefore a $v \in F$, there is a $c_v \in C_G$ such that $v \in c_v$. Note that $\prod c_v \in C$ and $\prod c_v \leq v$. Thus $\left\{ \prod c_v : v \in F \right\} \subseteq C$, so $0 \neq \prod \{ \prod c_v : v \in F \} \leq \prod F$, and hence $\prod F \neq 0$.

Thus $\bigcup C_G$ has no finite subset with zero product. As $\bigcup C_G \subseteq G$ is $\omega$-independent, by Lemma 1.8, it is independent. Thus $\langle \bigcup C_G \rangle$ is free and hence has no uncountable chain, contradicting our original assumption. \qed

**Theorem 5.5** Let $A$ be $\omega$-free over $H$. Then $|\text{End } A| = 2^{|A|}$.

**Proof** For each $x \in H$, choose $y_x \in A$ such that $y_x < x$. For each $J \subseteq H$, define $f_J : H \to A$ as

$$f_J (x) = \begin{cases} y_x & x \in J \\ x & \text{otherwise.} \end{cases}$$

$f_J$ is 2-preserving and extends to an endomorphism. So we have exhibited $2^{|A|}$ endomorphisms. \qed

### 6 Maximal $n$-Independence Number

We can look at $n$-independent sets in boolean algebras that aren’t $n$-free. The natural thing to do is introduce a cardinal function, $n\text{Ind}$, that measures the supremum of the cardinalities of those sets. Since $n\text{Ind}$ is a regular sup-function, we can define a spectrum function and a maximal $n$-independence number of a boolean algebra in the standard way.

**Definition 6.1** Let $1 \leq n \leq \omega$.

$$i_{\text{ns}} (A) \overset{\text{def}}{=} \{|X| : X \text{ is a maximal } n\text{-independent subset of } A\}$$

$$i_n (A) \overset{\text{def}}{=} \min \left( i_{\text{ns}} (A) \right).$$

This could be written as $n\text{Ind}_{mm}$ according to the notation of Monk [8]. Note that $i_1 = i$ where $i$ is the minimal independence number as seen in Monk [9].

This is defined for every boolean algebra; from the definition it is easily seen that the union of a chain of $n$-independent sets is $n$-independent, so Zorn’s Lemma shows that there are maximal $n$-independent sets. $i_n (A)$ is infinite for all $n \leq \omega$ if $A$ is atomless (shown in Lemma 6.3), and has value 1 if $A$ has an atom.

**Proposition 6.2** For all $n$ with $1 \leq n \leq \omega$, if $A$ has an atom, then $i_n (A) = 1$.  

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Proof If \( a \) is an atom of \( A \), then we claim that \( \{a\} \) is a maximal \( n \)-independent subset of \( A^+ \). That \( \{a\} \) is \( n \)-independent is clear as any singleton other than \( \{0\} \) and \( \{1\} \) is independent.

Let \( x \in A^+ \setminus \{a\} \), we show that \( \{a, x\} \) is not \( n \)-independent. There are two cases.

If \( a \leq x \), then \( 1 = a + a \leq x + a \), so that \((\bot 1)\) fails.
If \( a \leq -x \), then \( x \leq -a \), so that \( 0 \neq \prod x \leq \sum \{a\} \), but \( \{x\} \cap \{a\} = \emptyset \), so that \((\bot 3)\) fails.

\[ \qed \]

Lemma 6.3 Let \( B \) be a boolean algebra, \( 2 \leq n \leq \omega \), and \( H \subseteq B^+ \) be \( n \)-independent. If \( H \) is maximal among \( n \)-independent subsets of \( B^+ \), then \( H \) is infinite and \( \sum H = 1 \) or \( H \) is finite and \( -\sum H \) is an atom.

Proof We prove the contrapositive. First, the case that \( H \) is infinite. Let \( H \subseteq B^+ \) be \( n \)-independent and have \( b < 1 \) as an upper bound. We show that \( H \cup \{b\} \) is \( n \)-independent:

Note that \( -b \notin H \), as \( -b \notin b \). Now we will apply Proposition 1.7. So, assume that \( R \in [H \cup \{b\}]^{<\omega} \), \( \varepsilon \in R^2 \), and \( \prod_{x \in R} x^\varepsilon = 0 \). If \( -b \notin R \), the conclusion follows since \( H \) is \( n \)-independent. So suppose that \( -b \in R \). Let \( R' \defeq R \setminus \{b\} \). Then we have two cases:

Case 1. \( \varepsilon_{-b} = 1 \). If there is an \( x \in R' \) such that \( \varepsilon_x = 1 \), then \( x \leq b \) and so \( x \cdot -b = 0 \) as desired. So assume that \( \varepsilon[R'] = \{0\} \). Then \( -b \leq \sum_{x \in R} x \leq b \), which is a contradiction.

Case 2. \( \varepsilon_{-b} = 0 \). If \( \varepsilon_x = 1 \) for some \( x \in R' \), then

\[
0 = \prod_{y \in R} y^\varepsilon_y = \prod_{y \in R'} y^\varepsilon_y \cdot b = \prod_{y \in R'} y^\varepsilon_y
\]

and the \( n \)-independence of \( H \) gives the result. So assume that \( \varepsilon[R'] = \{0\} \). Then \( b \leq \sum R' \leq b \), so \( b = \sum R' \). Then \( b \cdot \prod_{x \in R'} -x = 0 \), contradicting the \( n \)-independence of \( H \).

So we have that if \( H \) is infinite and maximal \( n \)-independent, it has no upper bound other than 1, so \( \sum H = 1 \).

Now we consider the case that \( H \) is finite. If \( -\sum H \) is not an atom, let \( 0 < a < -\sum H \), then we claim that \( H \cup \{a\} \) is \( n \)-independent. Again we use Proposition 1.7. Assume that \( R \in [H \cup \{a\}]^{<\omega} \), \( \varepsilon \in R^2 \), and \( \prod_{x \in R} x^\varepsilon = 0 \). Without loss of generality, \( a \in R \).

Case 1. \( \varepsilon_a = 1 \). If \( \varepsilon_x = 1 \) for some \( x \in R \setminus \{a\} \), then \( a \cdot x \leq a \cdot \sum H = 0 \), as desired. Otherwise

\[ a \leq \sum (R \setminus \{a\}) \leq \sum H \]

and so \( a = 0 \), contradiction.

Case 2. \( \varepsilon_a = 0 \). If \( \varepsilon_x = 1 \) for some \( x \in R \setminus \{a\} \), then \( a \cdot x = 0 \), hence \( x \leq -a \), and then

\[
\prod_{y \in R} y^\varepsilon_y = \prod \{y^\varepsilon_y : y \in R \setminus \{a\}\}
\]
and the conclusion follows. Otherwise
\[-a \leq \sum (R \setminus \{a\}) \leq \sum H,\]
so \(-\sum H \leq a\), contradicting \(a < -\sum H\).

The converse of Lemma 6.3 does not hold. An example due to Monk is in \(\text{Fr}(X \cup Y)\) where \(X \cap Y = \emptyset\) and \(|X| = |Y| = \kappa \geq \omega\). \(X\) is independent, is not maximal for 2-independence, and has sum 1. Here \(\sum X = 1\) is the only non-trivial part–by way of contradiction, let \(b\) be a non-1 upper bound for \(X\). Then \(-b\) has the property that \(x \cdot -b = 0\) for all \(x \in X\), so let \(a\) be a elementary product of elements of \(X \cup Y\) where \(a \leq -b\). Take some \(x \in X\) that does not occur in that elementary product. Then since \(X \cup Y\) is independent, \(a \cdot x \neq 0\), but since \(a \leq -b\), \(a \cdot x = 0\).

Theorem 6.4 For \(B\) atomless, and \(2 \leq n \leq \omega\), \(p(B) \leq i_n(B)\).

Here \(p(B)\) is the pseudo-intersection number, defined in Monk [9] as
\[p(A) \overset{\text{def}}{=} \min \{|Y| : Y \subseteq A \text{ and } \sum Y = 1 \text{ and } \sum Y' \neq 1 \text{ for every finite } Y' \subseteq Y\}.
\]

Proof Since \(B\) is atomless, a maximal \(n\)-independent set \(Y\) has \(\sum Y = 1\), and by (⊥1), if \(Y' \subseteq Y\) is finite, \(\sum Y' \neq 1\). That is, the maximal \(n\)-independent sets are included among the \(Y\) in the definition of \(p\).

We do not know if strict inequality is possible.

Corollary 6.5 For all \(n\) with \(1 \leq n \leq \omega\), \(i_n(P(\omega)/\text{fin}) \geq \aleph_1\).

Proof \(\aleph_1 \leq p(P(\omega)/\text{fin}) \leq i_n(P(\omega)/\text{fin})\).

We also recall that under Martin’s Axiom, \(p(P(\omega)/\text{fin}) = \beth_1\), so the same is true of \(i_n\).

Proposition 6.6 Any \(B\) with the strong countable separation property has, for all \(2 \leq n \leq \omega\), \(i_n(B) \geq \aleph_1\).

Proof Such a \(B\) is atomless, so let \(H \subseteq B^+\) be \(n\)-independent and countably infinite, that is \(H = (h_i : i \in \omega)\). Then let \(c_m \overset{\text{def}}{=} \sum_{i \leq m} h_i\). Each \(c_m\) is a finite sum of elements of \(H\), thus by (⊥1), \(c_m < 1\). Then \(C = \{c_i : i \in \omega\}\) is a countable chain in \(B \setminus \{1\}\), so by the strong countable separation property, there is a \(b \in B\) such that \(c_i \leq b < 1\) for all \(i \in \omega\). Then as \(h_i \leq c_i\), \(h_i \leq b\) for all \(i \in \omega\) as well, that is, \(b\) is an upper bound for \(H\). Thus by Lemma 6.3, \(H\) is not maximal.

In addition, we show that maximal \(n\)-independent sets lead to weakly dense sets.

We use the notation \(-X = \{-x : x \in X\}\) frequently in the sequel.

Theorem 6.7 Let \(1 \leq n \leq \omega\). If \(X \subseteq A\) is maximal \(n\)-independent in \(A\), then the set \(Y\) of nonzero elementary products of elements of \(X\) is weakly dense in \(A\).
Recall that $Y$ is weakly dense in $A$ if and only if $Y \subseteq A^+$ and for every $a \in A^+$, there is a $y \in Y$ such that $y \leq a$ or $y \leq -a$.

**Proof** If $a \in X$, this is trivial, so we may assume that $a \notin X$ and hence $X \cup \{a\}$ is not $n$-independent.

By Proposition 1.7, there exist $R \in [X \cup \{a\}]^{<\omega}$ and $\varepsilon \in R^2$ such that $\prod_{x \in R} x^{\varepsilon_x} = 0$ while for every $R' \in [R \setminus \{a\}]^{<\omega}$, if $\varepsilon[R'] \subseteq \{1\}$ then $\prod R' \neq 0$. This last implication holds for every $R' \in [R \setminus \{a\}]^{<\omega}$, and so $\prod \{x^{\varepsilon_x} : x \in R \setminus \{a\}\} \neq 0$ since $X$ is $n$-independent. But $\prod \{x^{\varepsilon_x} : x \in R \setminus \{a\}\} \leq a$ or $\leq -a$, as desired. $\Box$

**Corollary 6.8** If $A$ is atomless and $1 \leq n \leq \omega$, then $\tau(A) \leq i_n(A)$.

Recall the definition of the reaping number:

$$\tau(A) \overset{\text{def}}{=} \min \{|X| : X \text{ is weakly dense in } A\}$$

**Proof** Since $A$ is atomless, all maximal $n$-independent sets are infinite, and thus there is a set of size $i_n(A)$ weakly dense in $A$. $\Box$

We do not know if strict inequality is possible.

We do not currently have any results for the behavior of $i_n$ on any type of product or its relationship to $u$.

We show the consistency of $i_n(\mathcal{P}(\omega)/\text{fin}) < \aleph_1$ for $1 \leq n \leq \omega$. The argument is similar to exercises (A12) and (A13) in Chapter VIII of Kunen [7]; the main lemma follows.

**Lemma 6.9** Let $M$ be a countable transitive model of ZFC and $1 \leq k \leq \omega$. For a subset $a$ of $\omega$, let $[a]$ denote its equivalence class in $\mathcal{P}(\omega)/\text{fin}$. Suppose that $\kappa$ is an infinite cardinal and $\langle a_i : i < \kappa \rangle$ is a system of infinite subsets of $\omega$ such that $\langle [a_i] : i < \kappa \rangle$ is $k$-independent in $\mathcal{P}(\omega)/\text{fin}$. Then there is a generic extension $M[G]$ of $M$ using a ccc partial order such that in $M[G]$ there is a $d \subseteq \omega$ with the following properties:

1. $\langle [a_i] : i < \kappa \rangle \setminus \langle [\omega \setminus d] \rangle$ is $k$-independent.
2. If $x \in (\mathcal{P}(\omega) \cap M) \setminus (\{a_i : i < \kappa\} \cup \{\omega \setminus d\})$,

   then

   $$\langle [a_i] : i < \kappa \rangle \setminus \langle [\omega \setminus d] , [x] \rangle$$

   is not $k$-independent.

**Proof** We work within $M$ here.

Let $B$ be the $k$-independent subalgebra of $\mathcal{P}(\omega)/\text{fin}$ generated by $\{[a_i] : i < \kappa\}$. By Sikorski’s extension criterion, let $f$ be a homomorphism from $\langle [a_i] : i < \kappa \rangle \cup \{[m] : m \in \omega\}$ to $\overline{B}$ such that $f([a_i]) = [a_i]$ and $f([m]) = 0$. Then let $h : \mathcal{P}(\omega) \rightarrow \overline{B}$ be a homomorphic extension of $f$ as given by Sikorski’s extension theorem.
Claim 1. If $R$ is a finite subset of $\kappa$ and $\varepsilon \in R^2$ is such that $\bigcap_{i \in R} a_i$ is infinite, then
$$
\bigcap_{i \in R} a_i^n \cap d \text{ is infinite.}
$$
Let $R$ and $\varepsilon$ be as given, then for each $n \in \omega$, let
$$
E_n \overset{\text{def}}{=} \{(b, y) \in P : \exists m > n \left[ m \in \bigcap_{i \in R} a_i^n \cap y \right] \}.
$$
First, we show that each $E_n$ is dense. Take $(b, y) \in P$. Then $c \overset{\text{def}}{=} (\bigcap_{i \in R} (a_i^n)) \setminus b$ is infinite; if not, then $c$ is finite (thus in $\ker(h)$, as is $b$) and $\bigcap_{i \in R} a_i^n \subseteq b \cup c$. Applying $h$ to both sides gives $\prod_{i \in R} [a_i]^n = 0$, which is a contradiction of Proposition 1.7. So we choose an $m \in c \setminus y$ such that $m > n$; then $(b, y \cup \{m\}) \leq (b, y)$ and $(b, y \cup \{m\}) \in E_n$, showing that $E_n$ is dense. This shows the claim, as for each $n \in \omega$, $E_n \cap G \neq \emptyset$, so that we have an integer larger than $n$ in $\bigcap_{i \in R} a_i \cap d$.

Claim 2. If $R$ is a finite subset of $\kappa$ and $\varepsilon \in R^2$ such that $\bigcap_{i \in R} a_i$ is infinite, then
$$
\bigcap_{i \in R} a_i^n \setminus b \cap d \text{ is infinite.}
$$
Let $R$ and $\varepsilon$ be as given, then for each $n \in \omega$, let
$$
D_n \overset{\text{def}}{=} \{(b, y) \in P : \exists m > n \left[ m \in \bigcap_{i \in R} a_i^n \setminus b \cap y \right] \}.
$$
To show that $D_n$ is dense, take any $(b, y) \in P$. Since $\bigcap_{i \in R} a_i^n$ is infinite from Proposition 1.7, it follows that we may choose $m > n$ such that $m \in \bigcap_{i \in R} a_i^n \setminus y$. Then $(b \cup \{m\}, y) \leq (b, y)$ and $(b \cup \{m\}, y) \in D_n$, as desired.

Take some $(b, y) \in D_n \cap G$. Then there is an $m > n$ such that $m \notin d$ (thus proving the claim). In fact, choose $m > n$ such that $m \in \bigcap_{i \in R} a_i^n \setminus y$. We claim that $m \notin d$. Suppose that $m \notin d$; then we have a $(c, z) \in G$ with $m \in z$ and $(e, w) \in G$ that is a common extension of $(b, y)$ and $(c, z)$. Then $m \in w \cap b \setminus y$, contradicting that $(e, w) \leq (b, y)$.

Claim 3. $\langle [a_i] : i < \kappa \rangle \cap ([\omega \setminus d])$ is $k$-independent.

Suppose that $R \in [\kappa]^{<\omega}$, $\varepsilon \in R^2$, $\delta \in 2$, and $\prod_{i \in R} [a_i]^{\varepsilon_i} \cdot [\omega \setminus d]^{\delta} = 0$. By claims 1 and 2 (depending on $\delta$), $\prod_{i \in R} [a_i]^{\varepsilon_i} = 0$. Since $\langle [a_i] : i < \kappa \rangle$ is $k$-independent, there is a subset $R' \subseteq \{i \in R : \varepsilon_i = 1\}$ of size at most $k$ such that $\prod_{i \in R'} [a_i] = 0$, as desired.

Claim 4. If $b \in \ker(h)$, then $b \cap d$ is finite.

$\langle (c, y) \in P : b \subseteq c \rangle$ is dense in $P$, so that there is a $(c, y) \in G$ such that $b \subseteq c$. We show $b \cap d \subseteq y$ and thus is finite. Let $m \in b \cap d$ and choose an $(e, z) \in G$ such that $m \in z$. Let $(r, w) \in G$ be a common extension of $(e, z)$ and $(c, y)$; then (recalling the definition of the order) $m \in w \cap c \subseteq y$.

Claim 5. If
$$
x \in \left( \wp(\omega) \cap M \right) \setminus ([a_i] : i < \kappa) \cup ([\omega \setminus d]),
$$

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then

\[ s \overset{\text{def}}{=} \langle [a_i] : i < \kappa \rangle \backslash \big[ [\omega \setminus d], [x] \big] \]

is not \( k \)-independent.

We have two cases here. The slightly easier is if \( x \in \ker(h) \); then by claim 4, \( x \cap d \) is finite, so that \([x] \leq [\omega \setminus d]\), causing \( s \) to fail to even be ideal-independent. If \( x \notin \ker(h) \), then there is a \( b \in B \) with \( 0 < b \leq h(x) \).

Since \( B \) is \( k \)-freely generated by \( \langle [a_i] : i < \kappa \rangle \), we may take \( b \) to be an elementary product of elements of \( \langle [a_i] : i < \kappa \rangle \). Then \( b = [c] \), where \( c = \bigcap_{i \in R} a_i^{R_i} \) is infinite. Then \( c \setminus x \in \ker(h) \). By claim 4, this gives \( \prod_{i \in R} [a_i]^{R_i} \cdot [x] \cdot [d] = 0 \), contradicting Proposition 1.7 for \( s \).

\[ \square \]

**Theorem 6.10** For each \( 1 \leq k \leq \omega \), it is consistent with \( \Box_1 > \aleph_1 \) that \( i_k (\mathcal{P}(\omega)/\text{fin}) = \aleph_1 \).

**Proof** We begin with a countable transitive model \( M \) of \( \text{ZF} + \Box_1 > \aleph_1 \), then iterate the construction of Lemma 6.9 \( \omega_1 \) times as in Lemma 5.14 of chapter VIII of Kunen [7]. This results in a model of \( \text{ZF} + \Box_1 > \aleph_1 + i_k (\mathcal{P}(\omega)/\text{fin}) = \aleph_1 \).

This shows that \( i_k (\mathcal{P}(\omega)/\text{fin}) = \Box_1 \) is independent of \( \text{ZF} \).

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