Heavy quark masses from lattice QCD

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Based on a lot of past, recent and ongoing work of the LPHA Collaboration, in particular

- **Precision computation of the strange quark’s mass in quenched QCD**
  J. Garden, J. H., R. Sommer and H. Wittig, Nucl. Phys. B 571 (2000) 237

- **A precise determination of the charm quark’s mass in quenched QCD**
  J. Rolf and S. Sint, J. High Energy Phys. 0212 (2002) 007

- **The $D_s$-meson decay constant in the continuum limit of quenched QCD**
  J. H. and A. Jüttner, in preparation

- **Computation of the strong coupling in QCD with two dynamical flavours**
  M. Della Morte, R. Frezzotti, J. H., J. Rolf, R. Sommer and U. Wolff, Nucl. Phys. B 713 (2005) 378

- **Non-perturbative quark mass renormalization in two-flavor QCD**
  M. Della Morte, R. Hoffmann, F. Knechtli, J. Rolf, R. Sommer, I. Wetzorke and U. Wolff, Nucl. Phys. B 729 (2005) 117

- **Non-perturbative heavy quark effective theory**
  J. H. and R. Sommer, J. High Energy Phys. 0402 (2004) 022

- **Effective heavy-light meson energies in small-volume quenched QCD**
  J. H. and J. Wennekers, J. High Energy Phys. 0402 (2004) 064

- **Heavy Quark Effective Theory computation of the mass of the bottom quark**
  M. Della Morte, N. Garron, M. Papinutto and R. Sommer, J. High Energy Phys. 0701 (2007) 007

- **Towards a non-perturbative matching of HQET and QCD with dynamical light quarks**
  M. Della Morte, P. Fritzsch, J. H., H. Meyer, H. Simma and R. Sommer, PoS LAT2007 (2007) 246

- LPHA Collaboration, in progress
Outline

1. Computing quark masses with lattice QCD
   - Generic strategy & Illustration
   - The charm quark’s mass

2. The bottom quark’s mass
   - Non-perturbative Heavy Quark Effective Theory
   - Application: Computation of $M_b$
   - Status in two-flavour QCD

3. Challenges & Outlook
Lattice QCD

‘Ab initio’ approach to determine phenomenologically relevant key parameters

\[ \mathcal{L}_{\text{QCD}} [g_0, m_f] = -\frac{1}{2g_0^2} \text{Tr} \{ F_{\mu\nu} F_{\mu\nu} \} + \sum_{f=u,d,s,...} \bar{\psi}_f \left( \gamma_\mu (\partial_\mu + g_0 A_\mu) + m_f \right) \psi_f \]

Experiment

\[ \begin{bmatrix} F_\pi \\ m_\pi \\ m_K \\ m_D \\ m_B \end{bmatrix} \]

QCD parameters (RGI)

\[ \begin{bmatrix} \Lambda_{\text{QCD}} \\ M_u, M_d \\ M_s \\ M_c \\ M_b \end{bmatrix} \]

Predictions

\[ \begin{bmatrix} F_D \\ F_B \\ B_K, B_B \\ \xi \end{bmatrix} \]

\[ \mathcal{L}_{\text{QCD}} [g_0, m_f] \rightarrow \text{means formulation of QCD on a Euclidean lattice with:} \]

- Gauge invariance
- Locality
- Unitarity
- Applicable for all scales
- Technical issues/obstacles
  - Continuum limit & Renormalization
  - Computing resources of > 1 TFlop/s
Lattice QCD
‘Ab initio’ approach to determine phenomenologically relevant key parameters

\[ \mathcal{L}_{\text{QCD}}[g_0, m_f] = \frac{-1}{2g_0^2} \text{Tr}\{F_{\mu\nu}F_{\mu\nu}\} + \sum_{f=u,d,s,\ldots} \bar{\psi}_f \{\gamma_\mu (\partial_\mu + g_0 A_\mu) + m_f\} \psi_f \]

\[
\begin{bmatrix}
F_\pi \\
m_\pi \\
m_K \\
m_D \\
m_B
\end{bmatrix}
\xrightarrow{\mathcal{L}_{\text{QCD}}[g_0, m_f]}
\]

\[
\begin{bmatrix}
\Lambda_{\text{QCD}} \\
M_u, M_d \\
M_s \\
M_c \\
M_b
\end{bmatrix}
\]

\[
\begin{bmatrix}
F_D \\
F_B \\
B_K, B_B \\
\xi \\
\ldots
\end{bmatrix}
\]

\[\mathcal{L}_{\text{QCD}}[g_0, m_f]\]

means formulation of QCD on a Euclidean lattice with:

- Gauge invariance
- Locality
- Unitarity
- Applicable for all scales
- Technical issues/obstacles
  - Continuum limit & Renormalization
  - \(O\left(\frac{1}{\sqrt{t_{\text{CPU}}}}\right)\) statistical errors
\[ U_\mu(x) = e^{i a g_0 A_\mu(x)} \]

\[ \psi(x) \]

- **Lattice cutoff** \( a^{-1} \sim \Lambda_{UV} \)
- **Finite volume** \( L^3 \times T \)
- **Lattice action**

\[ S[U, \bar{\psi}, \psi] = S_G[U] + S_F[U, \bar{\psi}, \psi] \]

\[ S_G = \frac{1}{g_0^2} \sum_p \text{Tr} \{1 - U(p)\} \]

\[ S_F = a^4 \sum_x \bar{\psi}(x) D[U] \psi(x) \]

**EVs:** represented as path integrals

\[ Z = \int D[U] D[\bar{\psi}, \psi] e^{-S[U, \bar{\psi}, \psi]} = \int D[U] \prod_f \det (\mathcal{D} + m_f) e^{-S_G[U]} \]

\[ \langle O \rangle = \frac{1}{Z} \int \prod_{x,\mu} dU_\mu(x) O \prod_f \det (\mathcal{D} + m_f) e^{-S_G[U]} \triangleq \text{thermal average} \]

**Stochastic evaluation with Monte Carlo methods**

\[ \langle O \rangle = \frac{1}{N} \sum_{n=1}^N O_n \pm \Delta_O \] from numerical simulations
\[ U_\mu(x) = e^{i a g_0 A_\mu(x)} \]

\[ \psi(x) \]

- Lattice cutoff \( a^{-1} \sim \Lambda_{\text{UV}} \)
- Finite volume \( L^3 \times T \)
- Lattice action

\[ S[U, \overline{\psi}, \psi] = S_G[U] + S_F[U, \overline{\psi}, \psi] \]

\[ S_G = \frac{1}{g_0^2} \sum_p \text{Tr} \{1 - U(p)\} \]

\[ S_F = a^4 \sum_x \overline{\psi}(x) D[U] \psi(x) \]

**EVs:** represented as path integrals

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**Towards realistic QCD simulations — Systematics**

1. *Dynamical* fermion effects \( \leftrightarrow \) quenched approximation: \( \det(D + m_f) = 1 \)
2. Lattice artefacts cause discretization errors \( \rightarrow \) \( \langle O \rangle_{\text{lat}} = \langle O \rangle_{\text{cont}} + O(a^p) \)
3. Quark mass restrictions \( \rightarrow \) \( a \ll m_q^{-1} \ll L \): extrapolate by ChPT & HQET
4. Energy range restrictions \( \rightarrow \mu < a^{-1} \lesssim 4 \text{ GeV} \) if (typically) \( a \gtrsim 0.05 \text{ fm} \)
Computing quark masses with lattice QCD

- Generic strategy & Illustration
- The charm quark’s mass
Multiple scale problems always difficult for a numerical/lattice treatment

$1/L \ll \mu, E, |p| \ll 1/\alpha$

$\mu_{\text{pert}} \gg \Lambda_{\text{QCD}}$

$100\text{ MeV}$

$10\text{ GeV}$

$1\text{ GeV}$

$100\text{ GeV}$
Scale problem and generic strategy

Multiple scale problems always difficult for a numerical/lattice treatment

\[ \mu \rightarrow \infty \]

intermediate = Schrödinger functional, finite-volume scheme
Scale dependence of QCD parameters

- **Renormalization group (RG) equations**

\[ \mu \frac{\partial \bar{g}}{\partial \mu} = \beta (\bar{g}) \]

\[ \mu \frac{\partial \bar{m}}{\partial \mu} = \tau (\bar{g}) \bar{m} \]

\[ \bar{g} \rightarrow 0 \sim -\bar{g}^3 \left\{ b_0 + b_1 \bar{g}^2 + b_2 \bar{g}^4 + \ldots \right\} \]

\[ \bar{g} \rightarrow 0 \sim -\bar{g}^2 \left\{ d_0 + d_1 \bar{g}^2 + \ldots \right\} \]

- **Solution leads to exact equations in a mass-independent scheme**

\[ \Lambda \equiv \mu (b_0 \bar{g}^2)^{-b_1/(2b_0^2)} e^{-1/(2b_0 \bar{g}^2)} \exp \left\{ -\int_0^{\bar{g}} dg \left[ \frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g} \right] \right\} \]

\[ M \equiv \bar{m} (\mu) (2b_0 \bar{g}^2)^{-d_0/(2b_0)} \exp \left\{ -\int_0^{\bar{g}} dg \left[ \frac{\tau(g)}{\beta(g)} - \frac{d_0}{b_0 g} \right] \right\} \]

*RG invariant quark mass*
Scale dependence of QCD parameters

Renormalization group (RG) equations, $\bar{g} \equiv \bar{g}(\mu)$

$$\mu \frac{\partial \bar{g}}{\partial \mu} = \beta(\bar{g}) \quad \bar{g} \to 0 \quad -\bar{g}^3 \{ b_0 + b_1 \bar{g}^2 + b_2 \bar{g}^4 + \ldots \}$$

$$\mu \frac{\partial \bar{m}}{\partial \mu} = \tau(\bar{g}) \bar{m} \quad \tau(\bar{g}) \to 0 \quad -\bar{g}^2 \{ d_0 + d_1 \bar{g}^2 + \ldots \}$$

Solution leads to exact equations in a mass-independent scheme

$$\Lambda \equiv \mu \left( b_0 \bar{g}^2 \right)^{-b_1/(2b_0^2)} e^{-1/(2b_0 \bar{g}^2)} \exp \left\{ - \int_0^{\bar{g}} dg \left[ \frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g} \right] \right\}$$

$$M \equiv \bar{m}(\mu) \left( 2b_0 \bar{g}^2 \right)^{-d_0/(2b_0)} \exp \left\{ - \int_0^{\bar{g}} dg \left[ \frac{\tau(g)}{\beta(g)} - \frac{d_0}{b_0 g} \right] \right\}$$

Simple relations between different renormalization schemes:

$$S \rightarrow S' \quad \alpha \rightarrow \alpha' = \alpha + c \alpha^2 + O(\alpha^3)$$

$\Rightarrow M$ is scale & scheme independent
Scale dependence of QCD parameters

- Renormalization group (RG) equations, $\bar{g} \equiv \bar{g}(\mu)$
  \[ \mu \frac{\partial \bar{g}}{\partial \mu} = \beta(\bar{g}) \quad \bar{g} \to 0 \sim -\bar{g}^3 \{ b_0 + b_1 \bar{g}^2 + b_2 \bar{g}^4 + \ldots \} \]
  \[ \mu \frac{\partial \bar{m}}{\partial \mu} = \tau(\bar{g}) \bar{m} \quad \tau(\bar{g}) \to 0 \sim -\bar{g}^2 \{ d_0 + d_1 \bar{g}^2 + \ldots \} \]

- Solution leads to exact equations in a mass-independent scheme
  \[ \Lambda \equiv \mu \left( b_0 \bar{g}^2 \right)^{-b_1/(2b_0^2)} e^{-1/(2b_0 \bar{g}^2)} \exp \left\{ - \int_0^{\bar{g}} dg \left[ \frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g} \right] \right\} \]
  \[ M \equiv \bar{m}(\mu) \left( 2b_0 \bar{g}^2 \right)^{-d_0/(2b_0)} \exp \left\{ - \int_0^{\bar{g}} dg \left[ \frac{\tau(g)}{\beta(g)} - \frac{d_0}{b_0 g} \right] \right\} \quad \text{RG invariant quark mass} \]

- Simple relations between different renormalization schemes:
  \[ \frac{\Lambda_{S'}}{\Lambda_S} = e^{\frac{c}{4\pi b_0}} , \quad \frac{\bar{m}_{S'}(\mu)}{\bar{m}_S(\mu)} = 1 + O(\alpha(\mu)) \quad \mu \to \infty \quad 1 \Rightarrow M_{S'} = M_S \equiv M \]

⇒ Choose convenient scheme (i.e. physical coupling) to compute $M$
The basic equation for the RGI quark mass

Definition of the running mass through the PCAC relation:

\[
\partial_\mu A^{su}_\mu = (m_u + m_s) p^{su}
\]

\[
\begin{cases}
\lambda^{su}_\mu = \overline{u} \gamma_\mu \gamma_5 s : \text{ axial vector current} \\
p^{su} = \overline{u} \gamma_5 s : \text{ pseudoscalar density}
\end{cases}
\]

Upon renormalization in the lattice regularized theory \((g_0 \leftrightarrow a)\):

\[
\overline{m_u}(\mu) + \overline{m_s}(\mu) = \frac{Z_A(g_0)}{Z_P(g_0, \mu)} \times \frac{\langle 0 | \partial_\mu A^{su}_\mu | K^-(p = 0) \rangle}{\langle 0 | p^{su} | K^-(p = 0) \rangle} \quad m_u + m_s = \frac{F_K m_K^2}{G_K} : \text{ bare PCAC}
\]
The basic equation for the RGI quark mass

Definition of the running mass through the PCAC relation:

\[ \partial_\mu A^{su}_\mu = (m_u + m_s) p^{su} \]

\[ \begin{cases} A^{su}_\mu = \bar{u} \gamma_\mu \gamma_5 s: & \text{axial vector current} \\ p^{su} = \bar{u} \gamma_5 s: & \text{pseudoscalar density} \end{cases} \]

Upon renormalization in the lattice regularized theory \((g_0 \leftrightarrow \alpha)\):

\[ \bar{m}_u(\mu) + \bar{m}_s(\mu) \quad \text{renormalized & running} \]

\[ m_u + m_s = \frac{F_K m_K^2}{G_K} : \text{bare PCAC} \]

- Scale & scheme dependence via \(Z_P\), which is poorly convergent in PT
  \(\rightarrow\) NP determination needed \([\text{\textsc{Alpha} Collaboration} 1999 (N_f = 0) & 2005 (N_f = 2)]\)
  (\(Z_A\) fixed by requiring a chiral Ward identity from Euclidean current algebra)

- Analogously for the charm sector:

\[ \bar{m}_s(\mu) + \bar{m}_c(\mu) = \frac{Z_A(g_0)}{Z_P(g_0, \mu)} \frac{\langle 0 | \partial_\mu A^{cs}_\mu | D_s^+(p = 0) \rangle}{\langle 0 | P^{cs} | D_s^+(p = 0) \rangle} \]

\[ \begin{cases} A^{cs}_\mu = \bar{s} \gamma_\mu \gamma_5 c \\ P^{cs} = \bar{s} \gamma_5 c \end{cases} \]

\(m_s + m_c: \text{bare PCAC}\)

with same \(Z_P\) in a mass/flavour independent renormalization scheme
Now split up the problem according to the generic strategy:

\[ M_s = \frac{M}{\overline{m}(\mu)} \overline{m}_s(\mu) = \frac{M}{\overline{m}(\mu)} \frac{Z_A(g_0)}{Z_P(g_0, \mu)} \underbrace{m_s(g_0)}_{\text{bare PCAC}} \]

\[ \equiv Z_M(g_0) \times m_s(g_0) \]

\[ Z_M(g_0) = \frac{M}{\overline{m}(\mu)} \frac{Z_A(g_0)}{Z_P(g_0, \mu)} = \frac{M}{\overline{m}(\mu_{\text{pert}})} \overline{m}(\mu_{\text{had}}) \frac{Z_A(g_0)}{Z_P(g_0, \mu_{\text{had}})} \]

\[ \begin{array} \text{PT} \\ \text{NP} \end{array} \]

\[ \begin{array} \text{"easy"} \end{array} \]
Basic equation for the RGI quark mass

\[ Z_M(g_0) = \frac{Z_A(g_0)}{Z_P(g_0, \mu_{\text{had}})} \frac{\overline{m}(\mu_{\text{pert}})}{\overline{m}(\mu_{\text{had}})} \frac{M}{\overline{m}(\mu_{\text{pert}})} \]

\[ M_i = Z_M(g_0) m_i(g_0) \quad \text{(here: } i = s, c) \]
NP calculation of $\bar{m}(\mu_{\text{pert}})/\bar{m}(\mu_{\text{had}})$ in the intermediate SF scheme

*Schrödinger Functional* = QCD partition function in a cylinder (Dirichlet BCs in $x_0$)

$\Rightarrow$ NP running from (finite-volume) correlation functions & a recursive technique
Physics inputs & Lattice QCD computation

- Set bare strange mass in the Lagrangian s.th. $m_K/F_K = \text{experiment}$
  $\Rightarrow$ amounts to slightly extrapolate to physical quark mass: $m_{PS} \rightarrow m_K$

- MeV-scale by fixing the static quark potential to phenomenology: $r_0 = 0.5 \text{ fm}$

- Evaluate ratio $R \equiv F_K/G_K$ of K-meson matrix elements in numerical lattice simulations and combine this with $Z_M$:

$$M_s + M_u = Z_M \frac{F_K}{G_K} m_K^2$$

[NP $Z_M$: \textbf{\textsc{Alpha}} Collaboration 1999 & 2005 ($N_f = 0, 2$)]

(CHPT “cheat”: We actually compute $M_s + M_\ell$, $M_\ell = \frac{1}{2}(M_u + M_d)$, $\frac{M_s}{M_\ell} = 24.4$)

$
\begin{align*}
N_f = 0 \text{ result:} \\
\overline{m}_s^{\overline{MS}}(2 \text{ GeV}) &= 97(4) \text{ MeV (~15% quenching error)}
\end{align*}
$
The charm quark’s mass

Calculation in same spirit as for strange

- **Physics input**: bare charm mass in $\mathcal{L}_{\text{QCD}}$ s.th. $m_{D_s}/F_K = \text{experiment}$
- Additional complication in the charm sector:
  - $O(\alpha m_{q,c})$ cutoff effects become relevant

$$M_c = Z_M \left[ 1 + (b_A - b_P) \alpha m_{q,c} \right] m_c = Z_M \frac{Z_m Z_P}{Z_A} m_{q,c} (1 + b_m \alpha m_{q,c})$$

$\rightarrow$ remove $O(\alpha m_{q,c})$ NP’ly

Rolf & Sint, JHEP0212(2002)007

LPHA Collaboration & INFN/TOV, in progress

[ LPHA Collaboration 2000 ($N_f = 0$) & in progress ($N_f = 2$) ]
The charm quark’s mass

Calculation in same spirit as for strange

- **Physics input:** bare charm mass in $\mathcal{L}_{QCD}$ s.th. $m_{D_s}/F_K = \text{experiment}$
- **Additional complication in the charm sector:**
  - $O(\alpha m_{q,c})$ cutoff effects become relevant
  
  $$M_c = Z_M \left[ 1 + (b_A - b_P) \alpha m_{q,c} \right] m_c = Z_M \frac{Z_m Z_P}{Z_A} m_{q,c} \left( 1 + b_m \alpha m_{q,c} \right)$$

  $\rightarrow$ remove $O(\alpha m_{q,c})$ NP’ly

  [ LPHA Collaboration 2000 ($N_f = 0$) & in progress ($N_f = 2$) ]

- **$N_f = 0, r_0 = 0.5 \text{ fm}$:** $M_c = 1654(45) \text{ MeV}$

  $\Rightarrow \quad m_c^{\overline{MS}}(m_c) = 1301(34) \text{ MeV}$

- **Similar analysis for $N_f > 0$ in progress**
The charm quark’s mass

Calculation in same spirit as for strange

- **Physics input:** bare charm mass in $\mathcal{L}_{\text{QCD}}$ s.th. $m_{D_s}/F_K = \text{experiment}$
- **Additional complication in the charm sector:**
  - $O(a m_{q,c})$ cutoff effects become relevant

\[
M_c = Z_M \left[ 1 + (b_A - b_P) a m_{q,c} \right] m_c = Z_M \frac{Z_m Z_P}{Z_A} m_{q,c} (1 + b_m a m_{q,c})
\]

→ remove $O(a m_{q,c})$ NP’ly

\[
\text{[ \text{ALPHA Collaboration} 2000 (N_f = 0) & in progress (N_f = 2) ]}
\]

- $N_f = 0$, $r_0 = 0.5 \text{ fm}$: $M_c = 1654(45) \text{ MeV}$
  \[
  \Rightarrow \quad \overline{m}_{c}^{\overline{\text{MS}}} (\overline{m}_c) = 1301(34) \text{ MeV}
  \]
- Similar analysis for $N_f > 0$ in progress

- Charm *just* doable, but $\alpha M_b \simeq 4 \alpha M_c$!
  \[
  \Rightarrow \quad \text{for b-quarks, continuum limit } \alpha \to 0 \text{ can’t be controlled in this way}
  \]
  \[
  \Rightarrow \quad \text{need for an effective theory}
  \]
The bottom quark’s mass

- Non-perturbative Heavy Quark Effective Theory
- Application: Computation of $M_b$
- Status in two-flavour QCD
Lattice HQET — Why?

- **Light quarks: too light**
  - Widely spread objects
  - Finite-volume errors due to light pions
- **b-quark: too heavy**
  - Extremely localized object
  - B-mesons with a propagating b-quark on the lattice require very small $a < \frac{1}{m_b}$, beyond today’s computing resources

⇒ Recourse to an *effective* theory for the b-quark:

**Heavy Quark Effective Theory**

[ Eichten, 1988; Eichten & Hill, 1990 ]

\[
\bar{\psi}_b \{ \gamma_\mu D_\mu + m_b \} \psi_b \quad \rightarrow
\]
Lattice HQET — Why?

- Light quarks: too light
  - Widely spread objects
  - Finite-volume errors due to light pions

- b-quark: too heavy
  - Extremely localized object
  - B-mesons with a propagating b-quark on the lattice require very small $a < \frac{1}{m_b}$, beyond today’s computing resources

⇒ Recourse to an effective theory for the b-quark:

**Heavy Quark Effective Theory**

\[
\mathcal{L}_{\text{HQET}}(x) = \overline{\psi}_h(x) \left[ D_0 + m_b \right] \psi_h(x) - \frac{\omega_{\text{kin}}}{2m_b} D^2 - \frac{\omega_{\text{spin}}}{2m_b} \sigma \cdot B \psi_h(x) + \ldots
\]

[ Eichten, 1988; Eichten & Hill, 1990 ]
Non-perturbative formulation of HQET

Beyond the static approximation

\[ \overline{\psi}_h \left[ \nabla_0^* + \delta m \right] \psi_h \rightarrow \text{Eichten-Hill action} \]

\[ \overline{\psi}_h \left( - \frac{1}{2} D^2 \right) \psi_h \equiv \mathcal{O}_{\text{kin}} \rightarrow \left\{ \text{kinetic energy from heavy quark's residual motion} \right\} \]

\[ \overline{\psi}_h \left( - \frac{1}{2} \sigma \cdot B \right) \psi_h \equiv \mathcal{O}_{\text{spin}} \rightarrow \left\{ \text{chromomagnetic interaction with the gluon field} \right\} \]

- \( \delta m, \omega_i (g_0, m) \) must be determined such that HQET matches QCD

- **Analogously:** Composite fields in the effective theory, e.g.

\[
A_{0}^{\text{HQET}}(x) = Z_{A}^{\text{HQET}} \overline{\psi}_l(x) \gamma_0 \gamma_5 \psi_h(x) + c_{A}^{\text{HQET}} \overline{\psi}_l(x) \gamma_j \gamma_5 \overleftarrow{D}_j \psi_h(x) + \ldots
\]

\[ \propto 1/m \]
Beyond the static approximation

\[ \overline{\psi}_h \left( \nabla^*_0 + \delta m \right) \psi_h \quad \rightarrow \quad \text{Eichten-Hill action} \]

\[ \overline{\psi}_h \left( - \frac{1}{2} D^2 \right) \psi_h \equiv \mathcal{O}_{\text{kin}} \quad \rightarrow \quad \{ \text{kinetic energy from heavy quark's residual motion} \} \]

\[ \overline{\psi}_h \left( - \frac{1}{2} \sigma \cdot B \right) \psi_h \equiv \mathcal{O}_{\text{spin}} \quad \rightarrow \quad \{ \text{chromomagnetic interaction with the gluon field} \} \]

**EVs** = Functional integral representation at the quantum level:

\[ \langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\varphi] \, O[\varphi] \, e^{- (S_{\text{rel}} + S_{\text{HQET}})} \]

\[ \mathcal{Z} = \int \mathcal{D}[\varphi] \, e^{- (S_{\text{rel}} + S_{\text{HQET}})} \]

Now the *integrand* is expanded in a *power series* in \(1/m\)

\[ \exp \{- S_{\text{HQET}}\} = \]

\[ \exp \left\{ - a^4 \sum_x \mathcal{L}_{\text{stat}}(x) \right\} \]

\[ \times \left\{ 1 - a^4 \sum_x \mathcal{L}^{(1)}(x) + \frac{1}{2} \left[ a^4 \sum_x \mathcal{L}^{(1)}(x) \right]^2 - a^4 \sum_x \mathcal{L}^{(2)}(x) + \ldots \right\} \]
Non-perturbative formulation of HQET

Beyond the static approximation

\[ \overline{\psi}_h \left[ \nabla_0^* + \delta m \right] \psi_h \rightarrow \text{Eichten-Hill action} \]
\[ \overline{\psi}_h \left( - \frac{1}{2} \mathbf{D}^2 \right) \psi_h \equiv \mathcal{O}_{\text{kin}} \rightarrow \left\{ \begin{array}{l} \text{kinetic energy from heavy quark's residual motion} \\ \text{chromomagnetic interaction with the gluon field} \end{array} \right. \]
\[ \overline{\psi}_h \left( - \frac{1}{2} \mathbf{\sigma} \cdot \mathbf{B} \right) \psi_h \equiv \mathcal{O}_{\text{spin}} \rightarrow \left\{ \begin{array}{l} \text{kinetic energy from heavy quark's residual motion} \\ \text{chromomagnetic interaction with the gluon field} \end{array} \right. \]

\[ \Rightarrow \langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\varphi] e^{-S_{\text{rel}} - a^4 \sum_x \mathcal{L}_{\text{stat}}(x)} \mathcal{O} \left\{ 1 - a^4 \sum_x \mathcal{L}^{(1)}(x) + \ldots \right\} \]

Important (but not automatic) implications of this definition of HQET

- $1/m$ – terms appear only as *insertions* of local operators
  \[ \Rightarrow \text{Power counting: Renormalizability at any given order in } 1/m \]
- Equivalent Existence of the *continuum limit with universality*
- Effective theory = *Continuum asymptotic expansion in* $1/m$
Mass renormalization pattern in HQET

Already at the level of $\mathcal{L}_{\text{stat}}(x) = \bar{\psi}_h(x) \left[ \nabla_0^* + \delta m \right] \psi_h(x)$:

Linear divergence $\delta m \propto a^{-1}$ originates from mixing of $\bar{\psi}_h D_0 \psi_h$ with $\bar{\psi}_h \psi_h$.

$$\begin{align*}
m_b^{\overline{\text{MS}}} &= Z_{\text{pole}}^{\overline{\text{MS}}} \times m_{\text{pole}} \\
m_{\text{pole}} &= m_B - E_{\text{stat}} - \delta m
\end{align*}$$

\[
\begin{bmatrix}
m_B : \text{(exp.) B-meson mass} \\
E_{\text{stat}} : \text{static binding energy}
\end{bmatrix}
\]

$$\delta m = \frac{c(g_0)}{a} \sim e^{1/(2b_0 g_0^2)} \times \left\{ c_1 g_0^2 + c_2 g_0^4 + \ldots \right\}$$
Mass renormalization pattern in HQET

Already at the level of \( \mathcal{L}_{\text{stat}}(x) = \overline{\psi}_h(x) \left[ \nabla^*_0 + \delta m \right] \psi_h(x) \):

Linear divergence \( \delta m \propto a^{-1} \) originates from mixing of \( \overline{\psi}_h D_0 \psi_h \) with \( \overline{\psi}_h \psi_h \)

\[
m_b^{\overline{\text{MS}}} = Z_{\text{pole}}^{\overline{\text{MS}}} \times m_{\text{pole}} \quad m_{\text{pole}} = m_B - E_{\text{stat}} - \delta m
\]

\[
\delta m = \frac{c(g_0)}{a} \sim e^{1/(2b_0 g_0^2)} \times \{ c_1 g_0^2 + c_2 g_0^4 + \ldots \}
\]

- **In PT:**
  
  uncertainty = truncation error \( \sim e^{1/(2b_0 g_0^2)} c_{n+1} g_0^{2n+2} \xrightarrow{g_0 \to 0} \infty \)

  \( \Rightarrow \) Non-perturbative \( c(g_0) \) needed

  \( \Rightarrow \) NP renormalization (resp. matching to QCD) of HQET required for the continuum limit to exist

- **Power-law divergences even worse at the level of** \( 1/m \) – corrections:
  
  \( a^{-1} \to a^{-2} \)
Need to treat the b as particle with finite mass (while $\alpha m_b \ll 1$ to keep $\alpha$-effects small) and $L m_b \gg 1$ to apply HQET

⇒ Trick: start with QCD in a small volume, $V = L^4 \approx (0.4 \text{ fm})^4$

Matching conditions

$\Phi^\text{QCD}_k = \Phi^\text{HQET}_k$

for observables $\Phi_k$

(renormalized quantities, computable for $\alpha \to 0$)

HQET parameters fixed by relating them to QCD observables in small $V$

Rather than simulating a propagating ‘real’ relativistic b-quark, one aims at determining the NP heavy quark mass dependence of $\Phi^\text{QCD}_k$ in finite $V$
Connecting small and large volumes

- Matching volume: $L = L_1 \simeq 0.4$ fm, very small lattice spacings
- Gap to large volumes and practicable lattice spacings, where physical quantities (e.g. $m_B$, $F_{B_s}$) may be extracted, bridged by a . . .

Finite-size scaling step

[Lüscher, Weisz & Wolff, 1991; \textit{ALPHA Collaboration}, 1993-2006]
Finite-size scaling step

[ Lüscher, Weisz & Wolff, 1991; Λlpha Collaboration, 1993-2006 ]

- Fully non-perturbative, continuum limit can be taken everywhere
- Use the QCD Schrödinger Functional, \( L \rightarrow 2L \) via Step Scaling Functions
  \[ \Phi_k^{HQET}(2L) = \sigma_k \left( \{ \Phi_j^{HQET}(L), j = 1, \ldots, N \} \right) \]
  \[ 2L = 2L_1 \approx 0.8 \text{ fm} \]
- Large \( V \) (\( L \approx 2 \text{ fm} \)) at same resolution, where a B-meson fits comfortably
Non-trivial matching problem:

\{m_{\text{bare}} + \delta m, \omega_{\text{kin}}, \omega_{\text{spin}}\} \text{ from QCD s.th. } m_{\text{pole}} + \delta m \leftrightarrow M_b

⇒ N = 3 matching conditions:

\[ \Phi_{k}^{QCD}(L, M) = \Phi_{k}^{HQET}(L, M) \quad k = 1, 2, 3 \]

[M : RGI heavy quark mass]
Non-trivial matching problem:

\[
\{ m_{\text{bare}} + \delta m, \omega_{\text{kin}}, \omega_{\text{spin}} \} \text{ from QCD s.th. } m_{\text{pole}} + \delta m \leftrightarrow M_b
\]

\[\Rightarrow N = 3 \text{ matching conditions:}\]

\[\Phi_{k}^{QCD}(L, M) = \Phi_{k}^{HQET}(L, M) \quad k = 1, 2, 3 \]

[ \( M : \) RGI heavy quark mass ]

Basic equation in leading order of HQET (static approximation)

\[m_B = E_{\text{stat}} - E_{\text{stat}} + E_{\text{stat}}\]

\[= E_{\text{stat}} - E_{\text{stat}}(L_2) + E_{\text{stat}}(L_2) - E_{\text{stat}}(L_1) + E(L_1, M_b) \quad (\ast)\]

\( \alpha \to 0 \text{ in HQET} \quad \alpha \to 0 \text{ in HQET (}\sigma_m) \quad \alpha \to 0 \text{ in QCD} \]

- Divergent static quark’s self-energy \( \delta m \) cancels in differences!
- \( \Phi_2(L_1, M) \) carries entire (relativistic) heavy quark mass dependence
Basic equation in leading order of HQET (static approximation)

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with \( E_{\text{stat}}(L_1, M_b) \) [matching to QCD]

- Divergent static quark’s self-energy \( \delta m \) cancels in *differences*!
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\[ L_1 \simeq 0.4 \text{ fm}, \quad L_2 = 2L_1 \]

\[ \text{Use } r_0 m_B^{(e x p)}, \quad r_0 = 0.5 \text{ fm} \& \text{ solve } (\star) \]

\[ \Rightarrow M_{b}^{\text{stat}} = (6771 \pm 99) \text{ MeV} \]

- NP renormalization & Continuum limit
- Error dominated by that on \( Z_M \) (\( \simeq 1\% \)) in \( L_1 M = Z_M Z (1 + b_m a m_q) \times L_1 m_q \)
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with \( E_{\text{stat}} = E(L_1, M_b) \) (matching to QCD)

\[ \alpha \to 0 \text{ in HQET} \]

Divergent static quark’s self-energy \( \delta m \) cancels in \( \text{differences} \)!

\( \Phi_2(L_1, M) \) carries entire (relativistic) heavy quark mass dependence

Quenched result
Della Morte, Garron, Sommer & Papinutto
JHEP01(2007)007

\[ L_1 \approx 0.4 \text{ fm}, \ L_2 = 2L_1 \]

\[ \text{Use } r_0 m_B^{(\text{exp})}, r_0 = 0.5 \text{ fm } \& \text{solve } (\ast) \]

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Inclusion of $1/m$–terms

$m_B$ at next-to-leading order of HQET

$$m_B = m_{\text{bare}} + E_{\text{stat}} + \omega_{\text{kin}} E_{\text{kin}} + \omega_{\text{spin}} E_{\text{spin}}$$

- $E_{\text{kin}}, E_{\text{spin}}$ associated with $\bar{\psi}_h (-\frac{1}{2} D^2) \psi_h$ and $\bar{\psi}_h (-\frac{1}{2} \sigma \cdot B) \psi_h$ in $\mathcal{L}^{(1)}$

→ Three observables $\Phi_1, \Phi_2, \Phi_3$ required in the matching step

- Considering the spin-averaged $B$-meson instead, $\omega_{\text{spin}}$ cancels:

$$m_B^{(\text{av})} = \frac{1}{4} m_B + \frac{3}{4} m_B^* = m_{\text{bare}} + E_{\text{stat}} + \omega_{\text{kin}} E_{\text{kin}}$$

→ Only two observables $\Phi_1, \Phi_2$ necessary
Inclusion of $1/m$–terms

$m_B$ at next-to-leading order of HQET

\[ m_B = m_{\text{bare}} + E_{\text{stat}} + \omega_{\text{kin}} E_{\text{kin}} + \omega_{\text{spin}} E_{\text{spin}} \]

- $E_{\text{kin}}, E_{\text{spin}}$ associated with $\bar{\psi}_h (-\frac{1}{2} D^2) \psi_h$ and $\bar{\psi}_h (-\frac{1}{2} \sigma \cdot B) \psi_h$ in $\mathcal{L}^{(1)}$
  → Three observables $\Phi_1, \Phi_2, \Phi_3$ required in the matching step
- Considering the spin-averaged B-meson instead, $\omega_{\text{spin}}$ cancels:
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  → Only two observables $\Phi_1, \Phi_2$ necessary

Strategy

| Experiment | Lattice with $am_b \ll 1$ |
|------------|--------------------------|
| $m_B = 5.4$ GeV | $\Phi_1(L_1, M), \Phi_2(L_1, M)$ |
| $\Phi_{HQET}^1(L_2), \Phi_{HQET}^2(L_2)$ | $\Phi_{HQET}^1(L_1), \Phi_{HQET}^2(L_1)$ |
| $L_1 \simeq 0.4$ fm, $L_2 = 2L_1$ | $\sigma_m(u_1)$ |
| $u_1 = \bar{g}^2(L_1)$ | $\sigma_{1}^{\text{kin}}(u_1), \sigma_{2}^{\text{kin}}(u_1)$ |
Matching formula = Static part + \(1/m_b\)–correction:

\[
L_2 m_B^{(av)} = L_2 m_B^{stat} = L_2 [E_{stat} - \Gamma_1^{stat}(L_2)] + \sigma_m(u_1) + 2\Phi_2(L_1, M_b) + L_2 [E_{kin} - \Gamma_1^{kin}(L_2)] \omega_{kin}
\]

Most difficult piece encountered in the calculation

Large-volume HQET matrix element \([E_{kin} - \Gamma_1^{kin}(L_2)]\) in the \(1/m_b\)–contribution

Result in the quenched approximation

\[
\bar{m}_{b}^{MS} = 4.374(64) \text{ GeV} - 0.027(22) \text{ GeV} + \frac{O(\Lambda^3/m_b^2)}{O(\Lambda^2/m_b)} + \text{negligible}
\]

Further ingredients:

- Finite- and large-volume B-meson energies extracted from static-light SF correlators
- HYP-smeared static actions [Hasenfratz & Knechtli, 2001; \text{\textit{ALPHA}} Collaboration 2004/05]
- Consistency with results from different matching observables/conditions checked
Matching to QCD in finite $V$ with a relativistic $b$-quark

- Evaluation of QCD heavy-light valence quark correlation functions
  \( m_{\text{val}} = m_{\text{sea}} = 0 \) in the SF

- NP quark mass renormalization in \( V = L_1^4 \approx (0.5 \text{ fm})^4 \) finished:

  \[
  z \equiv L_1 M = Z_M Z (1 + b_m a m_q) \times L_1 m_q \quad \text{known in terms of bare parameters}
  \]

  \( \rightarrow \) enables to calculate the heavy quark mass dependence of observables
Status in two-flavour QCD

Della Morte, Fritzsch, H., Meyer, Simma & Sommer, PoS LAT2007(2007)246

Matching to QCD in finite $V$ with a relativistic $b$-quark

$N_f = 2$ degenerate massless sea quarks

- Evaluation of QCD heavy-light valence quark correlation functions ($m_{\text{light}}^{\text{val}} = m_{\text{sea}} = 0$ in the SF)
- NP quark mass renormalization in $V = L_1^4 \approx (0.5 \text{ fm})^4$ finished:
  
  $$z \equiv L_1 M = Z_M Z (1 + b_m \alpha m_q) \times L_1 m_q \quad \text{known in terms of bare parameters}$$

  $\rightarrow$ enables to calculate the heavy quark mass dependence of observables

Example for a CL extrapolation: spin-averaged B-energy $\equiv L_1 \times \left[ \frac{1}{4} \Gamma_{PS} + \frac{3}{4} \Gamma_V \right]$

\[\Phi_3^{\text{CL}}(z) = L(\Gamma_{PS} + 3 \Gamma_V) / 4\]
Challenges:

- $\langle M_s \rangle$, $M_c$ for $N_f = 2$:
  - smaller lattice spacings and smaller quark masses are under way by various groups/collaborations
  - appear feasible owing to algorithmic/technical improvements: domain decomposition, low-mode deflation & all-to-all quark propagators
    - [Lüscher, '03–'05; Del Debbio et al., '07; Lüscher, '07; Foley et al., '05]
Challenges & Outlook

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    - with current results (i.e. status reported @ Lattice 2007), it is still quite a bit early for a continuum limit (e.g. B. Blossier for ETMC: \(\overline{m}_c^{\text{MS}}(\overline{m}_c) = 1478(75)\text{ MeV}\))
Challenges:

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- \(M_b\) for \(N_f = 2\) through non-perturbative HQET is in progress:
  - error for \(M_b\) is dominated by \(Z_M(g_0)\) → reduce by a factor 2 ?!
  - alternative/complementary approach: combination of static results and (TOV) step scaling method to turn QCD extrapolations into interpolations
    [de Divitiis et al., ’03; Guazzini, Sommer & Tantalo, ’07]
Challenges & Outlook

**Challenges:**

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- **$M_b$ for $N_f = 2$ through non-perturbative HQET is in progress:**
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    - [de Divitiis et al., '03; Guazzini, Sommer & Tantalo, '07]

- **Light up & down quarks, large volumes and chiral extrapolation**

**Why should we face the challenges?**

- **LQCD:** “No” assumptions to determine fundamental SM parameters
- **Superb experimental input available:** $m_p, m_K, m_D, m_B, \ldots$
Bottomline conclusions:

- Non-perturbative (NP) tools for renormalization are fully developed
- Quenched computations are under good control, particularly thanks to
  - NP $O(\alpha)$ improvement
  - NP renormalization
- LQCD with dynamical quarks, $N_f \geqslant 2$:
  
  Significant progress & even more expected in the next 2–3 years