Subelliptic $\text{Spin}_C$ Dirac Operators, IV
Proof of the Relative Index Conjecture

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Abstract
We prove the relative index conjecture, which in turn implies that the set of embeddable deformations of a strictly pseudoconvex CR-structure on a compact 3-manifold is closed in the $C^\infty$-topology.

1 Proof of the Relative Index Conjecture

In this short paper, which continues the analysis presented in [3], we show how the formula for the relative index between two Szegö projectors $S_0, S_1$, defined by two embeddable CR-structures on a contact 3-manifold $(Y, H)$, gives a proof of the relative index conjecture:

Theorem 1. Let $(Y, H)$ be a compact 3-dimensional co-oriented, contact manifold, and let $S_0$ be the Szegö projector defined by an embeddable CR-structure with underlying plane field $H$. There is an $M$ such that for the Szegö projector $S_1$ defined by any embeddable deformation of the reference structure with the same underlying plane field, we have the upper bound:

$$\text{R-Ind}(S_0, S_1) \leq M.$$  (1)

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Recall that the deformations of a reference CR-structure, $T^0_0 Y$, on $(Y,H)$ are parameterized by

$$\text{Def}(Y,H,S_0) = \{ \Phi \in C^\infty(Y; \text{Hom}(T^0_0 Y, T^1_0 Y)) : \| \Phi \|_{L^\infty} < 1 \},$$

via the prescription:

$$\Phi T^0_0 Y = \{ Z_y + \Phi_y(Z_y) : Z_y \in T^0_0 Y \}.$$  \hfill (3)

Here and in the sequel we often use the Szegő projector to label a CR-structure. Let $E \subset \text{Def}(Y,H,S_0)$ consist of the embeddable deformations, that is, CR-structures arising as pseudoconvex boundaries of complex surfaces. In [2] we showed that if $S_0$ is Szegő projector defined by the reference CR-structure and $S_1$ that defined by an embeddable deformation, then the map

$$S_1 : \text{Im } S_0 \rightarrow \text{Im } S_1$$

is a Fredholm operator. $R\text{-Ind}(S_0, S_1)$ denotes its Fredholm index, which we call the relative index. In the proof of Theorem E in [2] we showed that, for each $m \in \mathbb{N} \cup \{0\}$ and any $\delta > 0$, the subsets of $\text{Def}(Y,H,S_0)$ given by

$$\mathcal{E}^\delta_m = \{ S_1 \in \text{Def}(Y,H,S_0) : -\infty < R\text{-Ind}(S_0, S_1) \leq m \} \text{ and } \| \Phi \|_{L^\infty}^2 \leq \frac{1}{2} - \delta,$$

are closed in the $C^\infty$-topology. In fact, we show that there is an integer $k_0$, so that this conclusion holds for a sequence $< \Phi_n >$ converging to $\Phi$ in the $C^{k_0}$-norm.

Combining (1) with Theorem E of [2] we prove:

**Corollary 1.** Under the hypotheses of Theorem 1, the set of embeddable deformations of the CR-structure on $Y$ is closed in the $C^\infty$-topology.

**Proof of the Corollary.** Suppose that $< \Phi_n >$ is a sequence of embeddable deformations in $\mathcal{E} \subset \text{Def}(Y,H,S_0)$ converging to $\Phi \in \text{Def}(Y,H,S_0)$, in the $C^\infty$-topology. We first observe that $\| \Phi \|_{L^\infty} < 1$.

Let $\Psi_1$ and $\Psi_2$ be deformations of the reference structure, with local representations

$$\Psi_j = \psi_j Z \otimes \bar{\omega}.$$  \hfill (6)

The local representation of $\Psi_2$ as a deformation of $\Psi_1$ is given by

$$\psi_{21} = \frac{\psi_2 - \psi_1}{1 - \psi_1 \psi_2},$$

see equation (5.5) in [2][II]. We can represent $\Phi$ as a deformation of any of the structures in the sequence. From equation (7) it is clear that there an integer $N$ so
that, as deformations of $\Phi_N$, a tail of the sequence and its limit lie in the $L^\infty$-ball in $\text{Def}(Y, H, S_N)$, centered at 0, of radius $\frac{1}{2}$. Theorem 1 shows that there is an $M$ so that

$$\text{R-Ind}(S_N, S_n) \leq M, \text{ for all } n \in \mathbb{N}. \quad (8)$$

Theorem E from [2] then implies that the limiting structure $\Phi$ is also embeddable, completing the proof of the corollary. \square

Before proving Theorem 1 we recall the formula for the relative index proved in [3]:

**Theorem 2.** Let $(Y, H)$ be a compact 3-dimensional co-oriented, contact manifold, and let $S_0, S_1$ be Szegő projectors for embeddable CR-structures with underlying plane field $H$. Suppose that $(X_0, J_0), (X_1, J_1)$ are strictly pseudoconvex complex manifolds with boundaries $(Y, H, S_0), (Y, H, S_1)$, respectively, then

$$\text{R-Ind}(S_0, S_1) = \dim H^{0,1}(X_0, J_0) - \dim H^{0,1}(X_1, J_1) + \frac{\text{sig}[X_0] - \text{sig}[X_1] + \chi[X_0] - \chi[X_1]}{4}. \quad (9)$$

Here $\text{sig}[X]$ is the signature of the non-degenerate quadratic form,

$$([\alpha], [\beta]) \mapsto \int_X \alpha \wedge \beta, \quad (10)$$

defined for $[\alpha], [\beta] \in \tilde{H}^2(X)$, the image of $H^2(X, bX)$ in $H^2(X)$, and $\chi[X]$ is the topological Euler characteristic:

$$\chi[X] = \sum_{j=0}^{4} b_j(X) (-1)^j, \text{ where } b_j(X) = \dim H_j(X; \mathbb{Q}). \quad (11)$$

**Proof of Theorem 1.** Let $X_1$ be a minimal resolution of the normal Stein space with boundary $(Y, H, S_1)$. It follows from a theorem of Bogomolov and De Oliveira that there is a small perturbation of the complex structure on $X_1$ making it into a Stein manifold, see [1]. Hence it follows that $X_1$, with a deformed complex structure, has a strictly plurisubharmonic exhaustion function, and therefore $X_1$ has the homotopy type of a 2-dimensional CW-complex. Thus expanding the formula in (9) gives:

$$\text{R-Ind}(S_0, S_1) = C_0 - \dim H^{0,1}(X_1, J_1) - \frac{\text{sig}[X_1] + 1 - b_1(X_1) + b_2(X_1)}{4}, \quad (12)$$
where $C_0$ denotes the contribution of the terms from the reference structure:

$$C_0 = H^{0,1}(X_0, J_0) + \frac{\text{sig}[X_0] + \chi(X_0)}{4}. \quad (13)$$

The fact that $X_1$ is homotopic to a 2-complex implies that $b_1(X_1) \leq b_1(Y)$, see [5]. As $\text{sig}[X_1]$ is the signature of the cup product pairing on $\check{H}^2(X_1)$, it is evident that

$$|\text{sig}[X_1]| \leq \dim \check{H}^2(X_1) \leq \dim H^2(X_1, bX_1) = b_2(X_1). \quad (14)$$

The last equality is a consequence of the Lefschetz duality theorem. Hence

$$0 \leq b_2(X_1) + \text{sig}[X_1],$$

and therefore

$$\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) \leq C_0 + \frac{b_1(Y) - 1}{4}. \quad (15)$$

This completes the proof of the theorem. \hfill \Box

**Remarks on the Ozbagci-Stipsicz Conjecture:** Note that

$$\text{sig}[X_1] + b_2(X_2) = 2b_2^+(X_1) + b_0^0(X_1),$$

where $b_2^+(X_1)$ is the dimension of the space on which the pairing in (10) is positive and $b_0^0(X_1)$ is the dimension of the kernel of the map $H^2(X_1, bX_1) \to \check{H}^2(X_1)$. A global bound on $|\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1)|$, among all Szegő projectors $\mathcal{S}_1$ defined by elements of $\mathcal{E}$, is therefore equivalent to an upper bound for $b_2^+(X_1) + b_0^0(X_1) + \dim H^{0,1}(X_1)$, among all Stein spaces, $X_1$ filling $(Y, H)$. The existence of an upper bound on $b_2^+(X_1) + b_0^0(X_1)$ was conjectured by Ozbagci and Stipsicz, and proved in some special cases, see [5].

The fact, proved in [2], that $\text{R-Ind}(\mathcal{S}_0, \mathcal{S}_1) \geq 0$, for sufficiently small deformations shows that, for such deformations:

$$\dim H^{0,1}(X_1) + \frac{2b_2^+(X_1) + b_0^0(X_1)}{4} \leq \dim H^{0,1}(X_0) + \frac{2b_2^+(X_0) + b_0^0(X_0) + b_1(Y) - b_1(X_0)}{4}. \quad (16)$$

In [5] Stipsicz shows that for any Stein filling of $(Y, H)$, we have the estimate $b_0^0(X_1) \leq b_1(Y)$, as well as the existence of a constant $K_{(Y,H)}$ so that

$$b_2^-(X_1) \leq 5b_2^+(X_1) + 2 - K_{(Y,H)} + 2b_1(Y). \quad (17)$$

These estimates, along with (16) prove a “germ” form of the Ozbagci–Stipsicz conjecture: among sufficiently small, embeddable deformations of the CR-structure on the boundary of a strictly pseudoconvex surface, the set of numbers

$$\{b_1(X_1), \sigma(X_1), \chi(X_1)\}$$
is finite. The notion of smallness here depends in a complicated way on the reference CR-structure.

Our results suggest a strategy for proving a lower bound on \( R\text{-Ind}(S_0, S_1) \), among deformations \( \Phi \) with \( \|\Phi\|_{L^\infty} < 1 - \epsilon \), for an \( \epsilon > 0 \). Suppose that no such bound exists, one could then choose a sequence \( \langle \Phi_n \rangle \subset \mathcal{E} \) for which \( R\text{-Ind}(S_0, S_n) \) tends to \( -\infty \). A contradiction would follow immediately if we could show that \( \langle \Phi_n \rangle \) is bounded in the \( C^{k_0+1} \)-norm.

While such an \textit{a priori} bound seems unlikely for the original sequence, it would suffice to replace the sequence \( \langle \Phi_n \rangle \) with a “wiggle-equivalent” sequence. Let \( M_n \) denote a projective surface containing \( (Y, \Phi_n T^0_0 Y) \) as a separating hypersurface, see [4]. An equivalent sequence with better regularity might be obtained by wiggling the hypersurfaces defined by \( (Y, \Phi_n T^0_0 Y) \) within \( M_n \), perhaps using some sort of heat-flow. After composing the resultant deformations with contact transformations, we might be able to obtain a sequence \( \langle \Phi_n' \rangle \) with \( R\text{-Ind}(S_0, S_n') = R\text{-Ind}(S_0, S_n) \) that does satisfy an \textit{a priori} \( C^{k_0+1} \)-bound. Such an argument would seem to require an improved understanding of the metric geometry of \( \text{Def}(Y, H, S_0) \), as well as the relationship of an abstract deformation to the local extrinsic geometry of \( Y \) as a hypersurface in \( M_n \).

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