NEW INEQUALITIES FOR WEAVING FRAMES IN HILBERT SPACES

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Abstract. In this paper, we establish Parseval identities and surprising new inequalities for weaving frames in Hilbert space, which involve scalar $\lambda \in \mathbb{R}$. By suitable choices of $\lambda$, one obtains the previous results as special cases. Our results generalize and improve the remarkable results which have been obtained by Balan et al. and Găvruţa.

1. Introduction

Frames in Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [8] to study some deep problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer [6], and today frames play important roles in many applications in several areas of mathematics, physics, and engineering, such as coding theory [14, 17], sampling theory [23, 20], quantum measurements [9], filter bank theory [13] and image processing [7].

Let $\mathcal{H}$ be a separable space and $I$ a countable index set. A sequence $\{\phi_i\}_{i \in I}$ of elements of $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, \phi_i \rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$  

The number $A, B$ are called lower and upper frame bounds, respectively. If $A = B$, then this frame is called an $A$-tight frame, and if $A = B = 1$, then it is called a Parseval frame.

Suppose $\{\phi_i\}_{i \in I}$ is a frame for $\mathcal{H}$, then the frame operator is a self-adjoint positive invertible operators, which is given by

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{i \in I} \langle f, \phi_i \rangle \phi_i.$$  

The following reconstruction formula holds:

$$f = \sum_{i \in I} \langle f, \phi_i \rangle S^{-1} \phi_i = \sum_{i \in I} \langle f, S^{-1} \phi_i \rangle \phi_i,$$

where the family $\{\tilde{\phi}_i\}_{i \in I} = \{S^{-1} \phi_i\}_{i \in I}$ is also a frame for $\mathcal{H}$, which is called the canonical dual frame of $\{\phi_i\}_{i \in I}$. The frame $\{\tilde{\phi}_i\}_{i \in I}$ for $\mathcal{H}$ is called an alternate dual frame of $\{\phi_i\}_{i \in I}$ if the following formula holds:

$$f = \sum_{i \in I} \langle f, \phi_i \rangle \varphi_i = \sum_{i \in I} \langle f, \varphi_i \rangle \phi_i.$$  

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for all $f \in \mathcal{H}$ [11].

Let $\{\phi_i\}_{i \in I}$ be a frame for $\mathcal{H}$, for every $J \subset I$, we define the operator

$$S_J = \sum_{i \in J} \langle f, \phi_i \rangle \phi_i,$$

and denote $J^c = I \setminus J$.

The concept of discrete weaving frames for separable Hilbert spaces was introduced by Bemrose, Casazza et al. [4], which is motivated by distributed signal processing. For example, in wireless sensor networks where frames may be subjected to distributed processing under different frames. Thus, weaving frames have potential applications in wireless sensor networks that require distributed processing under different frames, as well as preprocessing of signals using Gabor frames. Recently, weaving frames in Hilbert spaces have been studied intensively, for more details see [5, 12, 21]

**Definition 1.** Two frames $\{\phi_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ for a Hilbert space $\mathcal{H}$ is said to be woven if there are universal constants $A$ and $B$ so that for every partition $\sigma \subset I$, the family $\{\phi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is a frame for $\mathcal{H}$ with lower and upper frame bounds $A$ and $B$, respectively. The family $\{\phi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is called a weaving.

If $A = B$, we call that $\{\phi_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ are $A$-woven, and if $A = B = 1$, then we call them 1-woven.

Suppose that $\{\phi_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ are woven, the frame operator of $\{\phi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is defined by

$$S_{\mathcal{W}} f = \sum_{i \in \sigma} \langle f, \phi_i \rangle \phi_i + \sum_{i \in \sigma^c} \langle f, \psi_i \rangle \psi_i,$$

then $S_{\mathcal{W}}$ is a bounded, invertible, self-adjoint and positive operator. A frame $\{\varphi_i\}_{i \in I}$ is called an alternate dual frame of $\{\phi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ if for all $f \in \mathcal{H}$ the following identity holds:

$$f = \sum_{i \in \sigma} \langle f, \phi_i \rangle \varphi_i + \sum_{i \in \sigma^c} \langle f, \psi_i \rangle \varphi_i.$$  \hfill (1.1)

For every $\sigma \subset I$, define the bounded linear operators $S_{\mathcal{W}}^\sigma$, $S_{\mathcal{W}}^{\sigma^c} : \mathcal{H} \to \mathcal{H}$ by

$$S_{\mathcal{W}}^{\sigma} f = \sum_{i \in \sigma} \langle f, \phi_i \rangle \phi_i, \quad S_{\mathcal{W}}^{\sigma^c} f = \sum_{i \in \sigma^c} \langle f, \psi_i \rangle \psi_i.$$

It is easy to check that $S_{\mathcal{W}}^\sigma$ and $S_{\mathcal{W}}^{\sigma^c}$ are self-adjoint.

In [1], the authors solved a long-standing conjecture of the signal processing community. They showed that for suitable frames $\{\phi_i\}_{i \in I}$, a signal $f$ can (up to a global phase) be recovered from the phase-less measurements $\{|\langle f, \phi_i \rangle|\}_{i \in I}$. Note, that this only shows that reconstruction of $f$ is in principle possible, but there is not an effective constructive algorithm. While searching for such an algorithm, the authors of [2] discovered a new identity for Parseval frames [3]. The authors in [10, 24] generalized these identities to alternate dual frames and got some general results. The study of inequalities has
interested many mathematicians. Some authors have extended the equalities and inequalities for frames in Hilbert spaces to generalized frames [16, 18, 22, 19]. The following form was given in [3] (See [2] for a discussion of the origins of this fundamental identity).

**Theorem 1.** Let \( \{ \phi_i \}_{i \in I} \) be a Parseval frame for \( \mathcal{H} \). For every \( J \subset I \) and every \( f \in \mathcal{H} \), we have

\[
\sum_{i \in J} |\langle f, \phi_i \rangle|^2 + \left\| \sum_{i \in J^c} \langle f, \phi_i \rangle \phi_i \right\|^2 = \sum_{i \in J} |\langle f, \phi_i \rangle|^2 + \left\| \sum_{i \in J} \langle f, \phi_i \rangle \phi_i \right\|^2 \geq \frac{3}{4} \|f\|^2. \tag{1.2}
\]

Later on, the author in [10] generalized Theorem 1 to general frames.

**Theorem 2.** Let \( \{ \phi_i \}_{i \in I} \) be a frame for \( \mathcal{H} \) with canonical dual frame \( \{ \tilde{\phi}_i \}_{i \in I} \). Then for every \( J \subset I \) and every \( f \in \mathcal{H} \), we have

\[
\sum_{i \in J} |\langle f, \phi_i \rangle|^2 + \sum_{i \in J^c} \left| \left( S_{J^c} f, \tilde{\phi}_i \right) \right|^2 = \sum_{i \in J} |\langle f, \phi_i \rangle|^2 + \sum_{i \in J^c} \left| \left( S_{J^c} f, \tilde{\phi}_i \right) \right|^2 \geq \frac{3}{4} \sum_{i \in I} |\langle f, \phi_i \rangle|^2. \tag{1.3}
\]

**Theorem 3.** Let \( \{ \phi_i \}_{i \in I} \) be a frame for \( \mathcal{H} \) and \( \{ \varphi_i \}_{i \in I} \) be an alternate dual frame of \( \{ \phi_i \}_{i \in I} \). Then for every \( J \subset I \) and every \( f \in \mathcal{H} \), we have

\[
\text{Re} \left( \sum_{i \in J} \langle f, \varphi_i \rangle \overline{\langle f, \phi_i \rangle} \right) + \left\| \sum_{i \in J^c} \langle f, \varphi_i \rangle \phi_i \right\|^2 = \text{Re} \left( \sum_{i \in J^c} \langle f, \varphi_i \rangle \overline{\langle f, \phi_i \rangle} \right) + \left\| \sum_{i \in J} \langle f, \varphi_i \rangle \phi_i \right\|^2 \geq \frac{3}{4} \|f\|^2. \tag{1.4}
\]

Motivated by these interesting results, the authors in [24] generalized the Theorem 3 to a more general form which does not involve the real parts of the complex numbers.

**Theorem 4.** Let \( \{ \phi_i \}_{i \in I} \) be a frame for \( \mathcal{H} \) and \( \{ \varphi_i \}_{i \in I} \) be an alternate dual frame of \( \{ \phi_i \}_{i \in I} \). Then for every \( J \subset I \) and every \( f \in \mathcal{H} \), we have

\[
\left( \sum_{i \in J} \langle f, \varphi_i \rangle \overline{\langle f, \phi_i \rangle} \right) + \left\| \sum_{i \in J^c} \langle f, \varphi_i \rangle \phi_i \right\|^2 = \left( \sum_{i \in J} \langle f, \varphi_i \rangle \overline{\langle f, \phi_i \rangle} \right) + \left\| \sum_{i \in J^c} \langle f, \varphi_i \rangle \phi_i \right\|^2 \geq \frac{2}{4} \|f\|^2. \tag{1.5}
\]

In this paper, we generalize the above mentioned results for weaving frames in Hilbert spaces. We generalize the above inequalities to a more general form which involve a scalar \( \lambda \in \mathbb{R} \) which is different from the scalar \( \lambda \in \{0, 1\} \) in [19]. Since a frame is woven with itself, the previous equality and inequalities in frames can be obtained as a special case of the results we establish on weaving frames.

### 2. Results and their proofs

We first state a simple result on operators, which is a distortion of [24, Lemma 2.1].

**Lemma 1.** If \( P, Q \in \mathcal{L}(\mathcal{H}) \) satisfying \( P + Q = I_{\mathcal{H}} \), then \( P + Q^*Q = Q^* + P^*P \).

**Proof.** A simple computation shows that

\[
P + Q^*Q = I_{\mathcal{H}} - Q + Q^*Q = I_{\mathcal{H}} - (I_{\mathcal{H}} - Q^*)Q = I_{\mathcal{H}} - P^*(I_{\mathcal{H}} - P) = I_{\mathcal{H}} - P^* + P^*P = Q^* + P^*P.
\]

\(\square\)
Now we state and prove a Parseval weaving frame identity.

**Theorem 5.** Suppose $\{\phi_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ for a Hilbert space $\mathcal{H}$ are 1-woven. Then for all $\sigma \subset I$ and all $f \in \mathcal{H}$, we have

$$\sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2 + \left\| \sum_{i \in \sigma} (f, \psi_i) \psi_i \right\|^2 = \sum_{i \in \sigma} |\langle f, \psi_i \rangle|^2 + \left\| \sum_{i \in \sigma} (f, \phi_i) \phi_i \right\|^2 \geq \frac{3}{4} \|f\|^2. \quad (2.1)$$

**Proof.** Since $\{\phi_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ are 1-woven, the weaving frame $\{\phi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is a Parseval frame for $\mathcal{H}$. Then the frame operator of $\{\phi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c}$ is $S_W = I_{\mathcal{H}}$. For every $\sigma \subset I$, we have $S_W^\sigma + S_W^{\sigma^c} = I_{\mathcal{H}}$. Note that $S_W^\sigma$ is a self-adjoint operator, and therefore $(S_W^{\sigma^c})^* = S_W^\sigma$. By applying Lemma 1 to the operator $S_W^\sigma$ and $S_W^{\sigma^c}$, for all $f \in \mathcal{H}$, we obtain

$$\langle S_W^\sigma f, f \rangle + \left( (S_W^{\sigma^c})^* S_W^\sigma f, f \right) = \left( (S_W^\sigma)^* S_W^\sigma f, f \right) + \left( (S_W^{\sigma^c})^* S_W^{\sigma^c} f, f \right).$$

Thus

$$\langle S_W^\sigma f, f \rangle + \left\| S_W^\sigma f \right\|^2 = \left( S_W^\sigma f, f \right) + \left\| S_W^{\sigma^c} f \right\|^2.$$

Hence

$$\sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2 + \left\| \sum_{i \in \sigma} (f, \psi_i) \psi_i \right\|^2 = \sum_{i \in \sigma} |\langle f, \psi_i \rangle|^2 + \left\| \sum_{i \in \sigma} (f, \phi_i) \phi_i \right\|^2.$$

Next, we prove the inequality of (2.1). A simple computation shows that

$$(S_W^\sigma)^2 + (S_W^{\sigma^c})^2 = (S_W^\sigma)^2 + (I_{\mathcal{H}} - (S_W^\sigma))^2 = 2(S_W^\sigma)^2 - 2S_W^\sigma + I_{\mathcal{H}} = 2 \left( S_W^\sigma - \frac{1}{2} I_{\mathcal{H}} \right)^2 + \frac{1}{2} I_{\mathcal{H}},$$

and so

$$(S_W^\sigma)^2 + (S_W^{\sigma^c})^2 \geq \frac{1}{2} I_{\mathcal{H}}.$$

Since $S_W^\sigma + S_W^{\sigma^c} = I_{\mathcal{H}}$, it follows that

$$S_W^\sigma + (S_W^{\sigma^c})^2 + S_W^{\sigma^c} + (S_W^\sigma)^2 \geq \frac{3}{2} I_{\mathcal{H}}. \quad (2.2)$$

Notice that operator $S_W^\sigma$ is also self-adjoint and therefore $(S_W^{\sigma^c})^* = S_W^\sigma$. Applying Lemma 1 to the operators $P = S_W^\sigma$ and $Q = S_W^{\sigma^c}$, we obtain

$$S_W^\sigma + (S_W^{\sigma^c})^2 = S_W^\sigma + (S_W^\sigma)^2.$$

Then equation (2.2) means that

$$(S_W^\sigma + (S_W^{\sigma^c})^2) \geq \frac{3}{4} I_{\mathcal{H}}.$$

Therefore for all $f \in \mathcal{H}$, we have

$$\sum_{i \in \sigma} |\langle f, \psi_i \rangle|^2 + \left\| \sum_{i \in \sigma} (f, \phi_i) \phi_i \right\|^2 = \langle S_W^\sigma f, f \rangle + \langle S_W^{\sigma^c} f, S_W^\sigma f \rangle = \left( (S_W^{\sigma^c} + (S_W^\sigma)^2) f, f \right) \geq \frac{3}{4} \|f\|^2.$$

This completes the proof. □

**Remark 6.** If we take $\phi_i = \psi_i$ for all $i \in I$ in Theorem 5, we can obtain the Theorem 1.
Lemma 2. Let $P, Q \in L(\mathcal{H})$ be two self-adjoint operators such that $P + Q = I_{\mathcal{H}}$. Then for any $\lambda \in \mathbb{R}$, and all $f \in \mathcal{H}$, we have

$$
\|Pf\|^2 + \lambda \langle Qf, f \rangle = \|Qf\|^2 + (2 - \lambda) \langle Pf, f \rangle + (\lambda - 1) \|f\|^2
$$

$$
\geq (\lambda - \frac{\lambda^2}{4}) \|f\|^2.
$$

Proof. For all $f \in \mathcal{H}$, we have

$$
\|Pf\|^2 + \lambda \langle Qf, f \rangle = \langle (P^2 - \lambda P + \lambda I_{\mathcal{H}})f, f \rangle
$$

$$
= \langle (P^2 - \lambda P + \lambda I_{\mathcal{H}})f, f \rangle
$$

$$
= \langle (I_{\mathcal{H}} - P)^2 f, f \rangle + (2 - \lambda) \langle Pf, f \rangle + (\lambda - 1) \langle f, f \rangle
$$

$$
= \|Qf\|^2 + (2 - \lambda) \langle Pf, f \rangle + (\lambda - 1) \|f\|^2.
$$

A simple computation of (2.3), we have

$$
\langle (P^2 - \lambda P + \lambda I_{\mathcal{H}})f, f \rangle = \langle (P - \frac{\lambda}{2} I_{\mathcal{H}})^2 - \frac{\lambda^2}{4} I_{\mathcal{H}} + \lambda I_{\mathcal{H}} \rangle f, f \rangle
$$

$$
= \langle (P - \lambda I_{\mathcal{H}})^2 + (\lambda - \frac{\lambda^2}{4}) I_{\mathcal{H}} \rangle f, f \rangle
$$

$$
\geq (\lambda - \frac{\lambda^2}{4}) \|f\|^2.
$$

This proves the desired result. \qed

Theorem 7. Suppose two frames $\{\phi_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ for a Hilbert space $\mathcal{H}$ are woven. Then for any $\lambda \in \mathbb{R}$, for all $\sigma \subset I$ and all $f \in \mathcal{H}$, we have

$$
\sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2 + \sum_{i \in \sigma} \left| \left\langle S_{W}^{c} f, S_{W}^{-1} \phi_i \right\rangle \right|^2 + \sum_{i \in \sigma} \left| \left\langle S_{W}^{c} f, S_{W}^{-1} \psi_i \right\rangle \right|^2
$$

$$
= \sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2 + \sum_{i \in \sigma} \left| \left\langle S_{W}^{c} f, S_{W}^{-1} \phi_i \right\rangle \right|^2 + \sum_{i \in \sigma} \left| \left\langle S_{W}^{c} f, S_{W}^{-1} \psi_i \right\rangle \right|^2
$$

$$
\geq \left( 1 - \frac{\lambda^2}{4} \right) \sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2 + (1 - \frac{\lambda^2}{4}) \sum_{i \in \sigma} |\langle f, \psi_i \rangle|^2.
$$

Proof. Since $\{\phi_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ are woven, for all $\sigma \subset I$, $\{\phi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma}$ is a frame for $\mathcal{H}$. Let $S_{W}$ be the frame operator for $\{\phi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma}$. Since $S_{W}^{c} + S_{W}^{c} = S_{W}$, it follows that

$$
S_{W}^{-1/2} S_{W}^{-1/2} S_{W}^{-1/2} + S_{W}^{-1/2} S_{W}^{-1/2} S_{W}^{-1/2} = I_{\mathcal{H}}.
$$

Considering $P = S_{W}^{-1/2} S_{W}^{-1/2}$, $Q = S_{W}^{-1/2} S_{W}^{-1/2}$, and $S_{W}^{1/2} f$ instead of $f$ in Lemma 2, we obtain

$$
\|S_{W}^{-1/2} S_{W}^{c} f\|^2 + \lambda \left\langle S_{W}^{-1/2} S_{W}^{c} f, S_{W}^{1/2} f \right\rangle
$$

$$
= \|S_{W}^{-1/2} S_{W}^{c} f\|^2 + (2 - \lambda) \left\langle S_{W}^{-1/2} S_{W}^{c} f, S_{W}^{1/2} f \right\rangle + (\lambda - 1) \|S_{W}^{1/2} f\|^2
$$

$$
\geq (\lambda - \frac{\lambda^2}{4}) \|S_{W}^{1/2} f\|^2.
$$
\[ \geq (\lambda - \frac{\lambda^2}{4}) \|S_W^{-1/2}f\|^2, \]

thus

\[
\langle S_W^{-1}S_W^{-1}f, S_W^{-1}f \rangle + \lambda \langle S_W^{-1}f, f \rangle \\
= \langle S_W^{-1}S_W^{-1}f, S_W^{-1}f \rangle + 2 \langle S_W^{-1}f, f \rangle - \lambda \left( \langle (S_W^{-1} + S_W^{-1})f, f \rangle + (\lambda - 1) \langle S_W f, f \rangle \right) \\
\geq (\lambda - \frac{\lambda^2}{4}) \langle S_W f, f \rangle .
\]

Then

\[
\langle S_W^{-1}S_W^{-1}f, S_W^{-1}f \rangle \\
= \langle S_W^{-1}S_W^{-1}f, S_W^{-1}f \rangle + 2 \langle S_W^{-1}f, f \rangle - \lambda \left( \langle (S_W^{-1} + S_W^{-1})f, f \rangle + (\lambda - 1) \langle S_W f, f \rangle \right) \\
\geq (\lambda - \frac{\lambda^2}{4}) \langle S_W f, f \rangle - \lambda \langle S_W^{-1}f, f \rangle ,
\]

thus

\[
\langle S_W^{-1}S_W^{-1}f, S_W^{-1}f \rangle = \langle S_W^{-1}S_W^{-1}f, S_W^{-1}f \rangle + 2 \langle S_W^{-1}f, f \rangle - \lambda \langle S_W^{-1}f, f \rangle \\
\geq \lambda \langle S_W^{-1}f, f \rangle - \frac{\lambda^2}{4} \langle S_W f, f \rangle .
\] (2.4)

hence

\[
\langle S_W^{-1}S_W^{-1}f, S_W^{-1}f \rangle + \langle S_W^{-1}f, f \rangle = \langle S_W^{-1}S_W^{-1}f, S_W^{-1}f \rangle + \langle S_W^{-1}f, f \rangle \\
\geq (\lambda - \frac{\lambda^2}{4}) \langle S_W^{-1}f, f \rangle + (1 - \frac{\lambda^2}{4}) \langle S_W^{-1}f, f \rangle .
\] (2.5)

We have

\[
\langle S_W^{-1}S_W^{-1}f, S_W^{-1}f \rangle = \langle S_W S_W^{-1}S_W f, S_W^{-1}S_W f \rangle \\
= \left( \sum_{i \in \Omega} \langle S_W^{-1}S_W f, \phi_i \rangle \phi_i + \sum_{i \in \Omega} \langle S_W^{-1}S_W f, \psi_i \rangle \psi_i, S_W^{-1}S_W f \rangle ight) \\
= \left( \sum_{i \in \Omega} \langle S_W^{-1}S_W f, \phi_i \rangle \phi_i, S_W^{-1}S_W f \rangle + \left( \sum_{i \in \Omega} \langle S_W^{-1}S_W f, \psi_i \rangle \psi_i, S_W^{-1}S_W f \rangle ight) \\
= \sum_{i \in \Omega} |\langle S_W^{-1}S_W f, \phi_i \rangle|^2 + \sum_{i \in \Omega'} |\langle S_W^{-1}S_W f, \psi_i \rangle|^2 .
\] (2.6)

Similarly

\[
\langle S_W^{-1}S_W^{-1}f, S_W^{-1}f \rangle = \sum_{i \in \Omega} |\langle S_W^{-1}f, \phi_i \rangle|^2 + \sum_{i \in \Omega'} |\langle S_W^{-1}f, \psi_i \rangle|^2 .
\] (2.7)

\[
\langle S_W^{-1}f, f \rangle = \sum_{i \in \Omega} |\langle f, \phi_i \rangle|^2 .
\] (2.8)
\[ \langle S_W^* f, f \rangle = \sum_{i \in \sigma^c} |\langle f, \phi_i \rangle|^2. \] (2.9)

Using equations (2.5)-(2.9) in the inequality (2.3), we obtain
\[
\sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2 + \sum_{i \in \sigma} |\langle S_W^* f, S_W^{-1} \phi_i \rangle|^2
\geq \sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2 + \sum_{i \in \sigma} |\langle S_W^* f, S_W^{-1} \phi_i \rangle|^2
\geq (\lambda - \frac{\lambda^2}{4}) \sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2 + (1 - \frac{\lambda^2}{4}) \sum_{i \in \sigma} |\langle f, \psi_i \rangle|^2.
\]

Remark 8. If we take \( \phi_i = \psi_i \) for all \( i \in I \) and \( \lambda = 1 \) in Theorem 7, we can obtain Theorem 2 with scalar \( 3/4 \).

Lemma 3. If \( P, Q \in L(\mathcal{H}) \) satisfy \( P + Q = I_H \). Then for any \( \lambda \in \mathcal{R} \), we have
\[ P^* P + \lambda(Q^* + Q) = Q^* Q + (1 - \lambda)(P^* + P) + (2\lambda - 1)I_H \geq (1 - (\lambda - 1)^2)I_H. \]

Proof.
\[ P^* P + \lambda(Q^* + Q) = P^* P + \lambda(I_H - P^* + I_P) = P^* P - \lambda(P^* + P) + 2\lambda I_H. \]
and
\[ Q^* Q + (1 - \lambda)(P^* + P) + (2\lambda - 1)I_H = (I_H - P^*)(I_H - P) + (1 - \lambda)(P^* + P) + (2\lambda - 1)I_H \]
\[ = P^* P - \lambda(P^* + P) + 2\lambda I_H \]
\[ = (P - \lambda I_H)^*(P - \lambda I_H) + (1 - (\lambda - 1)^2)I_H \]
\[ \geq (1 - (\lambda - 1)^2)I_H. \]
Hence the result follows. \( \square \)

Theorem 9. Suppose two frames \( \{\phi_i\}_{i \in I} \) and \( \{\psi_i\}_{i \in I} \) for a Hilbert space \( \mathcal{H} \) are woven and \( \{\phi_i\}_{i \in I} \) is an alternate dual frame of the weaving frame \( \{\phi_i\}_{i \in \sigma} \cup \{\psi_i\}_{i \in \sigma^c} \). Then for any \( \lambda \in \mathcal{R} \), for all \( \sigma \subset I \) and all \( f \in \mathcal{H} \), we have
\[
\text{Re} \left( \sum_{i \in \sigma} \langle f, \phi_i \rangle \overline{\langle f, \phi_i \rangle} \right) + \left\| \sum_{i \in \sigma^c} \langle f, \phi_i \rangle \psi_i \right\|^2
\geq (2\lambda - \lambda^2) \text{Re} \left( \sum_{i \in \sigma} \langle f, \phi_i \rangle \overline{\langle f, \phi_i \rangle} \right) + (1 - \lambda^2) \text{Re} \left( \sum_{i \in \sigma^c} \langle f, \phi_i \rangle \overline{\langle f, \phi_i \rangle} \right). \] (2.10)
Proof. For all \( f \in \mathcal{H} \) and all \( \sigma \subset I \), define the operators
\[
E_\sigma f = \sum_{i \in \sigma} \langle f, \varphi_i \rangle \phi_i, \quad E_{\sigma^c} f = \sum_{i \in \sigma^c} \langle f, \varphi_i \rangle \psi_i.
\]
Then the series converge unconditionally and \( E_\sigma, E_{\sigma^c} \in \mathcal{L}(\mathcal{H}) \). By (1.1), we have \( E_\sigma + E_{\sigma^c} = I_\mathcal{H} \).

Applying Lemma 3 to the operators \( P = E_\sigma \) and \( Q = E_{\sigma^c} \), for all \( f \in \mathcal{H} \), we obtain
\[
\langle E_\sigma^* E_\sigma f, f \rangle + \lambda \langle (E_{\sigma^c}^* + E_{\sigma^c}) f, f \rangle
\]
\[
= \langle E_\sigma^* E_\sigma f, f \rangle + \lambda \langle E_{\sigma^c}^* f, f \rangle + \lambda \langle E_\sigma f, f \rangle \quad (2.11)
\]
\[
= \langle E_\sigma^* E_\sigma f, f \rangle + (1 - \lambda) \langle (E_{\sigma^c}^* + E_{\sigma^c}) f, f \rangle + (2\lambda - 1)\|f\|^2
\]
\[
= \langle E_\sigma^* E_\sigma f, f \rangle + (1 - \lambda) (\langle E_{\sigma^c} f, f \rangle + \langle E_\sigma f, f \rangle) + (2\lambda - 1) \langle I_\mathcal{H} f, f \rangle. \quad (2.12)
\]

A simple computation of (2.11) and (2.12), we have
\[
\|E_\sigma f\|^2 + 2\lambda \text{Re} \langle E_{\sigma^c} f, f \rangle = \|E_{\sigma^c} f\|^2 + 2(1 - \lambda) \text{Re} \langle E_\sigma f, f \rangle + (2\lambda - 1) \text{Re} \langle I_\mathcal{H} f, f \rangle.
\]

Then,
\[
\|E_\sigma f\|^2 = \|E_{\sigma^c} f\|^2 + 2(1 - \lambda) \text{Re} \langle E_\sigma f, f \rangle - 2\lambda \text{Re} \langle E_{\sigma^c} f, f \rangle + (2\lambda - 1) \text{Re} \langle I_\mathcal{H} f, f \rangle
\]
\[
= \|E_{\sigma^c} f\|^2 + 2\text{Re} \langle E_\sigma f, f \rangle - 2\lambda \text{Re} \langle (E_{\sigma^c} + E_{\sigma^c}) f, f \rangle + (2\lambda - 1) \text{Re} \langle I_\mathcal{H} f, f \rangle
\]
\[
= \|E_{\sigma^c} f\|^2 + 2\text{Re} \langle E_\sigma f, f \rangle - \text{Re} \langle (E_{\sigma^c} + E_{\sigma^c}) f, f \rangle
\]
\[
= \|E_{\sigma^c} f\|^2 + \text{Re} \langle E_\sigma f, f \rangle - \text{Re} \langle E_{\sigma^c} f, f \rangle.
\]

Hence,
\[
\|E_\sigma f\|^2 + \text{Re} \langle E_{\sigma^c} f, f \rangle = \|E_{\sigma^c} f\|^2 + \text{Re} \langle E_\sigma f, f \rangle. \quad (2.13)
\]

Since,
\[
\|E_\sigma f\|^2 = \left\| \sum_{i \in \sigma} \langle f, \varphi_i \rangle \phi_i \right\|^2. \quad (2.14)
\]
\[
\text{Re} \langle E_{\sigma^c} f, f \rangle = \text{Re} \left( \sum_{i \in \sigma^c} \langle f, \varphi_i \rangle \overline{\langle f, \psi_i \rangle} \right). \quad (2.15)
\]
\[
\|E_{\sigma^c} f\|^2 = \left\| \sum_{i \in \sigma^c} \langle f, \varphi_i \rangle \psi_i \right\|^2. \quad (2.16)
\]
\[
\text{Re} \langle E_\sigma f, f \rangle = \text{Re} \left( \sum_{i \in \sigma} \langle f, \varphi_i \rangle \overline{\langle f, \phi_i \rangle} \right). \quad (2.17)
\]

Using equations (2.13)-(2.17), we have
\[
\text{Re} \left( \sum_{i \in \sigma} \langle f, \varphi_i \rangle \overline{\langle f, \phi_i \rangle} \right) + \left\| \sum_{i \in \sigma^c} \langle f, \varphi_i \rangle \psi_i \right\|^2 = \text{Re} \left( \sum_{i \in \sigma^c} \langle f, \varphi_i \rangle \overline{\langle f, \psi_i \rangle} \right) + \left\| \sum_{i \in \sigma} \langle f, \varphi_i \rangle \phi_i \right\|^2.
\]
We now prove the inequality of (2.10). From Lemma 3, we have
\[
(E_\sigma^* E_\sigma f, f) + \lambda (E_\sigma^* f, f) + \lambda (E_\sigma f, f) \geq (2\lambda - \lambda^2) \langle I_\mathcal{H}, f, f \rangle.
\]
(2.18)

Then
\[
\|E_\sigma f\|^2 + 2\lambda\text{Re} (E_\sigma^* f, f) \geq (2\lambda - \lambda^2)\text{Re} (I_\mathcal{H}, f, f),
\]
hence
\[
\|E_\sigma f\|^2 \geq (2\lambda - \lambda^2)\text{Re} (I_\mathcal{H}, f, f) - 2\lambda\text{Re} (E_\sigma^* f, f)
\]
\[
= (2\lambda - \lambda^2)\text{Re} ((E_\sigma + E_\sigma^*) f, f) - 2\lambda\text{Re} (E_\sigma f, f)
\]
\[
= (2\lambda - \lambda^2)\text{Re} (E_\sigma f, f) - \lambda^2\text{Re} (E_\sigma f, f)
\]
\[
= (2\lambda - \lambda^2)\text{Re} (E_\sigma f, f) + (1 - \lambda^2)\text{Re} (E_\sigma^* f, f) - \text{Re} (E_\sigma f, f).
\]

Therefore,
\[
\|E_\sigma f\|^2 + \text{Re} (E_\sigma^* f, f) \geq (2\lambda - \lambda^2)\text{Re} (E_\sigma f, f) + (1 - \lambda^2)\text{Re} (E_\sigma f, f).
\]
(2.19)

Using equations (2.14)-(2.17) and (2.19), we have
\[
\text{Re} \left( \sum_{i \in \sigma} \langle f, \varphi_i \rangle \overline{\langle f, \phi_i \rangle} \right) + \left\| \sum_{i \in \sigma^c} \langle f, \varphi_i \rangle \psi_i \right\|^2 \geq (2\lambda - \lambda^2)\text{Re} \left( \sum_{i \in \sigma} \langle f, \varphi_i \rangle \overline{\langle f, \phi_i \rangle} \right) + (1 - \lambda^2)\text{Re} \left( \sum_{i \in \sigma^c} \langle f, \varphi_i \rangle \overline{\langle f, \psi_i \rangle} \right).
\]
The proof is completed. \( \square \)

**Remark 10.** Theorem 3 can be obtained from Theorem 9 by taking \( \phi_i = \psi_i \) for all \( i \in I \) and \( \lambda = \frac{1}{2} \).

**Theorem 11.** Suppose \( \Phi = \{\phi_i\}_{i \in I} \) and \( \Psi = \{\psi_i\}_{i \in I} \) for a Hilbert space \( \mathcal{H} \) are woven and \( \{\varphi_i\}_{i \in I} \) is an alternate dual frame of the weaving frame \( \{\phi_i\}_{i \in \sigma} \cup \{\phi_i\}_{i \in \sigma^c} \). Then for any \( \lambda \in \mathbb{R} \), for all \( \sigma \subset I \) and all \( f \in \mathcal{H} \), we have
\[
\left( \sum_{i \in \sigma} \langle f, \varphi_i \rangle \overline{\langle f, \phi_i \rangle} \right) + \left\| \sum_{i \in \sigma^c} \langle f, \varphi_i \rangle \psi_i \right\|^2 = \left( \sum_{i \in \sigma} \langle f, \varphi_i \rangle \overline{\langle f, \phi_i \rangle} \right) + \left\| \sum_{i \in \sigma} \langle f, \varphi_i \rangle \phi_i \right\|^2.
\]
(2.20)

**Proof.** For \( \sigma \subset I \) and \( f \in \mathcal{H} \), we define the operator \( E_\sigma \) and \( E_\sigma^* \) as in Theorem 9. Therefore, we have
\[
E_\sigma + E_\sigma^* = I_\mathcal{H}.
\]
By Lemma 1, we have
\[
\left( \sum_{i \in \sigma} \langle f, \varphi_i \rangle \overline{\langle f, \phi_i \rangle} \right) + \left\| \sum_{i \in \sigma^c} \langle f, \varphi_i \rangle \psi_i \right\|^2 = \langle E_\sigma f, f \rangle + \langle E_\sigma^* E_\sigma f, f \rangle
\]
\[
= \langle E_\sigma^* f, f \rangle + \langle E_\sigma E_\sigma f, f \rangle
\]
\[
= \langle E_\sigma^* f, f \rangle + \| E_\sigma f \|^2
\]
\[
= \left( \sum_{i \in \sigma^c} \langle f, \varphi_i \rangle \overline{\langle f, \psi_i \rangle} \right) + \left\| \sum_{i \in \sigma} \langle f, \varphi_i \rangle \phi_i \right\|^2.
\]
Hence (2.20) holds. The proof is completed. \( \square \)
Theorem 12. Suppose two frames $\{\phi_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ for a Hilbert space $\mathcal{H}$ are woven and $\{\varphi_i\}_{i \in I}$ is an alternate dual frame of the weaving frame $\{\phi_i\}_{i \in \sigma} \cup \{\phi_i\}_{i \in \sigma^c}$. Then for every bounded sequence $\{a_i\}_{i \in I}$ and every $f \in \mathcal{H}$, we have

$$
\left( \sum_{i \in \sigma} a_i \langle f, \varphi_i \rangle \langle f, \phi_i \rangle \right) + \left( \sum_{i \in \sigma^c} a_i \langle f, \varphi_i \rangle \langle f, \psi_i \rangle \right) + \left( \sum_{i \in \sigma} (1 - a_i) \langle f, \varphi_i \rangle \psi_i + \sum_{i \in \sigma} (1 - a_i) \langle f, \varphi_i \rangle \phi_i \right)^2
$$

$$
= \left( \sum_{i \in \sigma} a_i \langle f, \varphi_i \rangle \phi_i + \sum_{i \in \sigma^c} a_i \langle f, \varphi_i \rangle \psi_i \right)^2 + \left( \sum_{i \in \sigma} (1 - a_i) \langle f, \varphi_i \rangle \langle f, \phi_i \rangle \right) + \left( \sum_{i \in \sigma} (1 - a_i) \langle f, \varphi_i \rangle \langle f, \psi_i \rangle \right)^2.
$$

Proof. For all $\sigma \subset I$ and $f \in \mathcal{H}$, we define the operators

$$
E_\sigma f = \sum_{i \in \sigma} a_i \langle f, \varphi_i \rangle \phi_i, \quad E_{\sigma^c} f = \sum_{i \in \sigma^c} a_i \langle f, \varphi_i \rangle \psi_i,
$$

and

$$
F_\sigma f = \sum_{i \in \sigma} (1 - a_i) \langle f, \varphi_i \rangle \phi_i, \quad F_{\sigma^c} f = \sum_{i \in \sigma} (1 - a_i) \langle f, \varphi_i \rangle \psi_i.
$$

Note that these series converge unconditionally. Also we have $E_\sigma, E_{\sigma^c}, F_\sigma, F_{\sigma^c} \in L(\mathcal{H})$ and $E_\sigma + E_{\sigma^c} + F_\sigma + F_{\sigma^c} = I_\mathcal{H}$. Applying Lemma 1 to the operators $P = E_\sigma + E_{\sigma^c}$ and $Q = F_\sigma + F_{\sigma^c}$ and for every $f \in \mathcal{H}$, we have

$$
\left( \sum_{i \in \sigma} a_i \langle f, \varphi_i \rangle \langle f, \phi_i \rangle \right) + \left( \sum_{i \in \sigma^c} a_i \langle f, \varphi_i \rangle \langle f, \psi_i \rangle \right) + \left( \sum_{i \in \sigma} (1 - a_i) \langle f, \varphi_i \rangle \psi_i + \sum_{i \in \sigma} (1 - a_i) \langle f, \varphi_i \rangle \phi_i \right)^2
$$

$$
= (E_\sigma f, f) + (E_{\sigma^c} f, f) + (F_\sigma + F_{\sigma^c})^*(F_\sigma + F_{\sigma^c}) f, f
$$

$$
= ((E_\sigma + E_{\sigma^c}) f, f) + ((F_\sigma + F_{\sigma^c})^*(F_\sigma + F_{\sigma^c}) f, f)
$$

$$
= ((F_\sigma + F_{\sigma^c})^* f, f) + ((E_\sigma + E_{\sigma^c})^*(E_\sigma + E_{\sigma^c}) f, f)
$$

$$
= ((F_\sigma + F_{\sigma^c}) f, f) + \|(E_\sigma + E_{\sigma^c}) f\|^2
$$

$$
= \|(E_\sigma + E_{\sigma^c}) f\|^2 + (F_\sigma f, f) + (F_{\sigma^c} f, f)
$$

$$
= \left( \sum_{i \in \sigma} a_i \langle f, \varphi_i \rangle \phi_i + \sum_{i \in \sigma^c} a_i \langle f, \varphi_i \rangle \psi_i \right)^2 + \left( \sum_{i \in \sigma} (1 - a_i) \langle f, \varphi_i \rangle \langle f, \phi_i \rangle \right) + \left( \sum_{i \in \sigma} (1 - a_i) \langle f, \varphi_i \rangle \langle f, \psi_i \rangle \right)^2.
$$

Hence the relation holds.

Observe that if we consider $\sigma \subset I$ and

$$
a_i = \begin{cases} 
0, & \text{if } i \in \sigma \\
1, & \text{if } i \in \sigma^c,
\end{cases}
$$

then Theorem 11 follows from Theorem 12.

Remark 13. If we take $\phi_i = \psi_i$ for all $i \in I$ in Theorem 11 and Theorem 12, we can obtain Theorem 4 and Theorem 2.3 of [24].
Theorem 14. Suppose two frames $\{\phi_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ for a Hilbert space $\mathcal{H}$ are woven. Then for any $\lambda \in \mathbb{R}$, $\sigma \subset I$ and $f \in \mathcal{H}$, we have

$$0 \leq \sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2 - \sum_{i \in \sigma} |\langle S_W^{-1}f, S_W^{-1}\phi_i \rangle|^2 - \sum_{i \in \sigma^c} |\langle S_W^{-1}f, S_W^{-1}\psi_i \rangle|^2$$

$$\leq \frac{\lambda^2}{4} \sum_{i \in \sigma^c} |\langle f, \psi_i \rangle|^2 + (1 - \frac{\lambda}{2})^2 \sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2.$$

Proof. Considering positive operators $P = S_W^{-1/2}S_W^{-1/2}$, $Q = S_W^{-1/2}S_W^{-1/2}$, then $P + Q = I_H$, and

$$PQ = P(I_H - P) = P - P^2 = (I_H - P)P = PQ,$$

then

$$0 \leq PQ = P(I_H - P) = P - P^2 = S_W^{-1/2}(S_W - S_W S_W S_W)S_W^{-1/2},$$

which follows $S_W^\sigma - S_W^{-1}S_W^{-1}S_W^\sigma \geq 0$. Then for all $f \in \mathcal{H}$, we have

$$\sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2 - \sum_{i \in \sigma} |\langle S_W^\sigma f, S_W^{-1}\phi_i \rangle|^2 - \sum_{i \in \sigma^c} |\langle S_W^\sigma f, S_W^{-1}\psi_i \rangle|^2$$

$$= \langle S_W^\sigma f, f \rangle - \langle S_W^{-1}S_W^\sigma f, S_W^\sigma f \rangle$$

$$= \langle (S_W^\sigma - S_W^{-1}S_W^\sigma) f, f \rangle \geq 0.$$ 

By (2.4), we have

$$\langle S_W^{-1}S_W^\sigma f, S_W^\sigma f \rangle = \langle S_W^\sigma f, f \rangle \leq 4 \langle S_W f, f \rangle - \frac{\lambda^2}{4} \langle S_W f, f \rangle,$$

and then

$$\sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2 - \sum_{i \in \sigma} |\langle S_W^\sigma f, S_W^{-1}\phi_i \rangle|^2 - \sum_{i \in \sigma^c} |\langle S_W^\sigma f, S_W^{-1}\psi_i \rangle|^2$$

$$= \langle S_W^\sigma f, f \rangle - \langle S_W^{-1}S_W^\sigma f, S_W^\sigma f \rangle$$

$$\leq \langle S_W^\sigma f, f \rangle - \lambda \langle S_W^\sigma f, f \rangle + \frac{\lambda^2}{4} \langle S_W f, f \rangle$$

$$= (1 - \lambda) \langle S_W^\sigma f, f \rangle + \frac{\lambda^2}{4} \langle S_W f, f \rangle$$

$$= (1 - \lambda) \left\langle (S_W - S_W^\sigma) \right\rangle + \frac{\lambda^2}{4} \langle S_W f, f \rangle$$

$$= (\lambda - 1) \langle S_W^\sigma f, f \rangle + (1 - \frac{\lambda}{2})^2 \langle S_W f, f \rangle$$

$$= \frac{\lambda^2}{4} \langle S_W^\sigma f, f \rangle + (1 - \frac{\lambda}{2})^2 \langle S_W f, f \rangle$$

$$= \frac{\lambda^2}{4} \sum_{i \in \sigma^c} |\langle f, \psi_i \rangle|^2 + (1 - \frac{\lambda}{2})^2 \sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2.$$  

□
Theorem 15. Suppose two frames \( \{ \phi_i \}_{i \in I} \) and \( \{ \psi_i \}_{i \in I} \) for a Hilbert space \( \mathcal{H} \) are woven. Then for any \( \lambda \in \mathcal{R} \), \( \sigma \subset I \) and \( f \in \mathcal{H} \), we have

\[
(2\lambda - \frac{\lambda^2}{2} - 1) \sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2 + (1 - \frac{\lambda^2}{2}) \sum_{i \in \sigma^c} |\langle f, \psi_i \rangle|^2 
\leq \sum_{i \in \sigma} |\langle S_{W} f, S_{W}^{-1} \phi_i \rangle|^2 + \sum_{i \in \sigma^c} |\langle S_{W} f, S_{W}^{-1} \psi_i \rangle|^2 + \sum_{i \in \sigma} \left| \langle S_{W}^\sigma f, S_{W}^{-1} \phi_i \rangle \right|^2 + \sum_{i \in \sigma^c} \left| \langle S_{W}^\sigma f, S_{W}^{-1} \psi_i \rangle \right|^2 
\leq \sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2 + \sum_{i \in \sigma^c} |\langle f, \psi_i \rangle|^2.
\]

Proof. By (2.5), we have

\[
\langle S_{W}^{-1} S_{W}^\sigma f, S_{W}^\sigma f \rangle \geq (\lambda - \frac{\lambda^2}{4}) \langle S_{W} f, f \rangle - \frac{\lambda^2}{4} \langle S_{W}^\sigma f, f \rangle. \tag{2.21}
\]

\[
\langle S_{W}^{-1} S_{W}^\sigma f, S_{W}^\sigma f \rangle \geq (\lambda - \frac{\lambda^2}{4} - 1) \langle S_{W} f, f \rangle + (1 - \frac{\lambda^2}{4}) \langle S_{W}^\sigma f, f \rangle. \tag{2.22}
\]

From (2.21) and (2.22), we obtain

\[
\sum_{i \in \sigma} |\langle S_{W} f, S_{W}^{-1} \phi_i \rangle|^2 + \sum_{i \in \sigma^c} |\langle S_{W} f, S_{W}^{-1} \psi_i \rangle|^2 + \sum_{i \in \sigma} \left| \langle S_{W}^\sigma f, S_{W}^{-1} \phi_i \rangle \right|^2 + \sum_{i \in \sigma^c} \left| \langle S_{W}^\sigma f, S_{W}^{-1} \psi_i \rangle \right|^2 
= \langle S_{W}^{-1} S_{W}^\sigma f, S_{W}^\sigma f \rangle + \langle S_{W}^{-1} S_{W}^\sigma f, S_{W}^\sigma f \rangle 
\geq (2\lambda - \frac{\lambda^2}{2} - 1) \langle S_{W} f, f \rangle + (1 - \frac{\lambda^2}{2}) \langle S_{W}^\sigma f, f \rangle 
\geq (2\lambda - \frac{\lambda^2}{2} - 1) \sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2 + (1 - \frac{\lambda^2}{2}) \sum_{i \in \sigma^c} |\langle f, \psi_i \rangle|^2.
\]

Next, we prove the last part. Let \( P = S_{W}^{-1/2} S_{W}^\sigma S_{W}^{-1/2} \), \( Q = S_{W}^{-1/2} S_{W}^\sigma S_{W}^{-1/2} \). Since \( PQ = QP \), we have

\[ P - P^2 = P(I - P) = PQ \geq 0, \]

then for all \( f \in \mathcal{H} \), \( \| P f \|^2 \leq \langle P f, f \rangle \). Similarly, \( \| Q f \|^2 \leq \langle Q f, f \rangle \). Hence,

\[
\sum_{i \in \sigma} |\langle S_{W} f, S_{W}^{-1} \phi_i \rangle|^2 + \sum_{i \in \sigma^c} |\langle S_{W} f, S_{W}^{-1} \psi_i \rangle|^2 + \sum_{i \in \sigma} \left| \langle S_{W}^\sigma f, S_{W}^{-1} \phi_i \rangle \right|^2 + \sum_{i \in \sigma^c} \left| \langle S_{W}^\sigma f, S_{W}^{-1} \psi_i \rangle \right|^2 
= \langle S_{W}^{-1} S_{W}^\sigma f, S_{W}^\sigma f \rangle + \langle S_{W}^{-1} S_{W}^\sigma f, S_{W}^\sigma f \rangle 
= \langle S_{W}^{-1/2} S_{W}^\sigma f, S_{W}^{-1/2} S_{W}^\sigma f \rangle + \langle S_{W}^{-1/2} S_{W}^\sigma f, S_{W}^{-1/2} S_{W}^\sigma f \rangle 
= \langle S_{W}^{-1/2} S_{W}^\sigma S_{W}^{-1/2} S_{W}^\sigma f, S_{W}^{-1/2} S_{W}^\sigma f \rangle + \langle S_{W}^{-1/2} S_{W}^\sigma S_{W}^{-1/2} S_{W}^\sigma f, S_{W}^{-1/2} S_{W}^\sigma S_{W}^{-1/2} S_{W}^\sigma f \rangle 
\leq \langle S_{W}^{-1/2} S_{W}^\sigma S_{W}^{-1/2} S_{W}^\sigma f, S_{W}^{-1/2} S_{W}^\sigma S_{W}^{-1/2} S_{W}^\sigma f \rangle 
= \langle S_{W} f, f \rangle + \langle S_{W}^\sigma f, f \rangle 
= \sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2 + \sum_{i \in \sigma^c} |\langle f, \psi_i \rangle|^2.
\]

\[ \square \]
By Theorem 14 and Theorem 15, we immediately get the following results.

**Corollary 1.** Suppose two frames \( \{ \phi_i \}_{i \in I} \) and \( \{ \psi_i \}_{i \in I} \) for a Hilbert space \( \mathcal{H} \) are \( A \)-woven. Then for any \( \lambda \in \mathbb{R} \), \( \sigma \subseteq I \) and \( f \in \mathcal{H} \), we have

\[
0 \leq A \sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2 - \left\| \sum_{i \in \sigma} \langle f, \phi_i \rangle \phi_i \right\|^2 \leq \frac{A \lambda^2}{4} \sum_{i \in \sigma^c} |\langle f, \psi_i \rangle|^2 + (1 - \frac{\lambda^2}{2}) A \sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2
\]

and

\[
(2\lambda^2 - 1) A \sum_{i \in \sigma} |\langle f, \phi_i \rangle|^2 + (1 - \frac{\lambda^2}{2}) A \sum_{i \in \sigma^c} |\langle f, \psi_i \rangle|^2 \leq \left\| \sum_{i \in \sigma} \langle f, \phi_i \rangle \phi_i \right\|^2 + \left\| \sum_{i \in \sigma^c} \langle f, \psi_i \rangle \psi_i \right\|^2 \leq A \|f\|^2.
\]

**Proof.** Since \( \{ \phi_i \}_{i \in I} \) and \( \{ \psi_i \}_{i \in I} \) are \( A \)-woven, we have \( S_W^{-1} = \frac{1}{A} I_\mathcal{H} \), and then the results hold by Theorem 14 and Theorem 15. \( \square \)

**Remark 16.** If we take \( \lambda = 1 \) and \( \phi_i = \psi_i \) for all \( i \in I \) in Theorem 14 and Theorem 15, we obtain the similar inequalities in Theorem 5 and Theorem 6 of [15].

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