Control aspects of holonomic quantum computation

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Abstract

A unifying framework for the control of quantum systems with non-Abelian holonomy is presented. It is shown that, from a control theoretic point of view, holonomic quantum computation can be treated as a control system evolving on a principal fiber bundle. An extension of methods developed for these classical systems may be applied to quantum holonomic systems to obtain insight into the control properties of such systems and to construct control algorithms for two established examples of the computing paradigm.

1 Introduction

Geometric phases have long been a source of fascination and insight into classical and quantum physical theories [1]. In recent years, they have proven to be useful in describing the dynamics and control of certain nonholonomic mechanical systems with symmetry [2]. Inspired by the appearance of geometric phases in biology, engineers have sought to create motion in machines via cyclic variations in shape space. These endeavors [3]-[6] and the characterization of optimal trajectories [7]-[10] remain active areas of research.

Most recently, the quantum geometric phase has been realized as a way of constructing logic gates in a quantum computer [11]. Holonomic Quantum Computation (HQC) employs non-Abelian geometric phases (holonomies) for the purpose of quantum information processing. Here we present a unified framework for the control of quantum holonomic systems with applications to quantum computing by casting the model as a control system evolving on a principal bundle. The integration of control theoretic ideas into the HQC paradigm sharpens existing results for these systems and reveals computational techniques for solving two separate but related fundamental problems in quantum computing. First, the well known conditions for determining universal quantum computation must be translated to the holonomic framework. The determination of universality, however, is existential in nature and generally not constructive. Quantum logic gate synthesis or constructive controllability is the process of determining from the system dynamics the construction or concatenation from available transformations some desired dynamic transformation of the state. This latter task is required for executing quantum algorithms. Both of these issues have subtleties in the holonomic framework not encountered in the usual dynamical approach to quantum computing.

Since the introduction of this novel approach to quantum computing [11], there has been considerable interest from the research community in proposing physical systems capable of performing HQC [12]-[19] and exploring its mathematical foundations [20]-[30]. This paper draws on much of this work to provide new a characterization of the problem, introduce novel computational techniques and present new results for two classes of control architectures related to HQC. In particular, we state simple conditions for determining the holonomy group of a principal fiber bundle with connection. These conditions, well known and reported...
elsewhere \[8, 5, 27\], circumvent the difficulties in directly applying the Ambrose-Singer theorem which was originally stated as the technique for determining universality of HQC. For quantum holonomy groups of dimension greater than 3, a direct application of Ambrose-Singer essentially neglects the contribution of nested Lie brackets of horizontally lifted vector fields, however these vector fields can provide new transformations available for manipulating quantum information. From the product bundle representing single qubit rotations and two qubit interactions, conditions for universality can then be derived. Having established the decisive condition for determining universality, we explore the difficult inverse problem associated to constructing holonomies. Namely, given a desired holonomy what is the loop in parameter space that generates it? In principal, this information is sufficient for the experimentalist to construct particular holonomies in the laboratory. For the the $\mathbf{CP}^n$ model, holonomic logic gate synthesis has been recently addressed with a numerical optimization scheme in Refs. \[29, 28\]. Moreover, Ref. \[28\] refines the method to minimize the length of the loop in parameter space. Since the parameters must be driven sufficiently slowly for the adiabatic approximation to hold, minimizing length also minimizes the time to construct the logic gate. This criterion is perhaps relevant for combatting decoherence. Logic gate synthesis has also been treated analytically in Ref \[30\], however these loops are characterized in the Grassmann manifold and not in parameter space. Characterizations of length minimizing loops can also be found in Refs. \[8, 9\].

We apply the theory to two well studied models of HQC. We provide a complete analysis of the so-called Optical Holonomic Computer \[12, 21, 22, 23, 24, 25, 26\]. We extend the results of Ref. \[27\] by carrying out the universality analysis and explicitly characterizing a parametric loop that can be employed to construct an arbitrary two-qubit logic gate. These results surmount a negative result reported for this model \[22\]. In a similar manner, we use the Cartan decomposition of the unitary group to solve the constructive controllability problem for holonomic systems involving a conditional Berry phase. To the best of our knowledge, aside from the $\mathbf{CP}^n$ model, the two control models treated here encompass all proposed holonomic computing schemes.

This paper is organized as follows: In Section 2, following \[22\] the geometry of holonomic quantum computation is reviewed. Also in Section 2, we introduce the product bundle describing single qubit holonomies and two qubit interaction holonomies and state conditions for universal holonomic quantum computation. In Section 3, we introduce methods for solving the path ordered integral associated with logic gate synthesis in the holonomic framework. The main contributions of the paper are contained in Section 4, where we apply the theory of the previous sections.

## 2 Holonomic Quantum Computation

If a quantum state undergoes adiabatic evolution subject to a periodic Hamiltonian, it acquires a phase after one complete cycle. Berry’s surprising discovery \[31\] was that, in addition to the well known dynamical phase associated to the evolution, there is a phase of purely geometric origin. Berry’s phase was then understood as the holonomy or geometric phase corresponding to a principal bundle with connection over a parameter space \[32\]. This phenomenon has been generalized in a variety of ways, most notably to non-adiabatic evolutions \[33\] and to degenerate systems possessing a non-Abelian phase factor \[34\].

### 2.1 Preliminaries

We construct a family of degenerate Hamiltonians parameterized by elements of a parameter space $M$ that govern the quantum dynamics. To allow for the possibility of a countably infinite dimensional Hilbert space, we consider universal classifying bundles. For further details see \[22, 24, 35, 36\].
Let $\mathcal{H}$ be a separable (possibly infinite dimensional) Hilbert space, and define the manifolds

$$
St_k(\mathcal{H}) = \{ V = (|v_1\rangle, \ldots, |v_k\rangle) \in \mathcal{H} \times \cdots \times \mathcal{H} \mid V^\dagger V = 1 \} \quad (1)
$$

$$
Gr_k(\mathcal{H}) = \{ X \in B(\mathcal{H}) \mid X^2 = X, X^\dagger = X, \text{tr} X = n \}. \quad (2)
$$

where $B(\mathcal{H})$ denotes the set of bounded linear operators on $\mathcal{H}$. These manifolds are known as the (universal) Stiefel and Grassmann manifolds respectively. The space $Gr_k$ is also known as a classifying space and can be defined as the union of Grassmann manifolds

$$
Gr_k(\mathcal{H}) \equiv \bigcup_{n=k}^{\infty} Gr_{k,n}(\mathcal{H}). \quad (3)
$$

Denote this $U(k)$-bundle by $P_k$. Note that when $\mathcal{H} \cong \mathbb{C}^n$ and the system has a $k$-dimensional degeneracy, the bundle of interest is the more familiar $U(k)$-bundle $St_{k,n}(\mathbb{C}^n) \rightarrow Gr_{k,n}(\mathbb{C}^n)$ which can be written in terms of coset spaces as

$$
\frac{U(n)}{U(n-k)} \rightarrow \frac{U(n)}{U(n-k) \times U(k)}. \quad (4)
$$

We continue with the infinite dimensional case with the understanding that the development specializes to this case when $\mathcal{H}$ is finite dimensional.

Let $M$ be a finite dimensional parameter space and suppose the classifying map $\Pi_k : M \rightarrow Gr_k$ (to be defined below) is given. Then form the pullback bundle $Q_k = \Pi_k^* P_k$.

$$
\begin{array}{ccc}
Q_k & \rightarrow & St_k(\mathcal{H}) \\
\downarrow & & \downarrow \\
M & \xrightarrow{\Pi_k} & Gr_k(\mathcal{H}).
\end{array} \quad (5)
$$

Let $H_0$ be a Hamiltonian with a $k$-dimensional degeneracy spanned by the orthogonal basis $\{|v_j\rangle\}_{j=1}^k$. To simplify notation, let the degenerate eigenvalue be 0. In holonomic quantum computation, the degenerate subspaces of $H_0$ encode the quantum information. Suppose we have at our disposal a set of $U(k)$ unitary transformations

$$\{W_1(x), W_2(x), \ldots, W_m(x)\} \quad (6)$$

parametrized by the base coordinate $x$. These are the (exponentiated) analogues of control Hamiltonians. Setting

$$U_k(x) = \prod_j W_j(x) \quad (7)$$

we obtain the isospectral family of Hamiltonians given by

$$\mathcal{O}(H_0) \equiv U_k(x) H_0 U_k^\dagger(x). \quad (8)$$

In the adiabatic approximation, the adjoint orbit $\mathcal{O}(H_0)$ forms a family of Hamiltonians that govern the system since there are no energy level crossings. The classifying map is then be defined as

$$\Pi_k(x) \equiv U_k(x) \left( \sum_{j=1}^k |v_j\rangle \langle v_j| \right) U_k^\dagger(x). \quad (9)$$

### 2.2 Control systems on principal fiber bundles

In general, let $Q$ be a principal fiber bundle with structure group $G$ over a base manifold $M$. Recall that a connection on $Q$ defines a $G$-invariant distribution $\mathbb{H}$ such that $T_qQ = \mathbb{H}_q \oplus \mathcal{V}_q$, where $\mathcal{V}_q \equiv T_q(\mathcal{O}(q)) \cong g$.
(the Lie algebra of \( G \)). Alternatively, a connection can be characterized by an Ad-equivariant \( g \)-valued one-form \( \mathcal{A} \) on \( Q \) such that \( \mathcal{A} \cdot \xi_q = \xi \), where \( \xi_q \) is the infinitesimal generator of the group action and \( \xi \in g \). The horizontal subspace at a point \( q \in Q \) is then defined as the kernel \( H_q = \{ v_q \mid \mathcal{A} \cdot v_q = 0 \} \). The local connection one-form, \( \mathcal{A} \), is defined with respect to a local section \( \sigma \) by \( \mathcal{A} \equiv \sigma^* \mathcal{A} \). Using Ad-equivariance and the fact that \( \mathcal{A} \) is the identity on vertical vectors, we can obtain the local connection form in terms of the base variables only \[ 10 \]

\[
\mathcal{A} \cdot \dot{q} = \text{Ad}_g(g^{-1} \dot{g} + A(x) \cdot \dot{x}).
\]

We note that the term \( g^{-1} \dot{g} \) is in the Lie algebra \( g \), by interpreting \( g^{-1} \dot{g} \) as the lifted action of \( g^{-1} \) on \( \dot{g} \in T_q G \). Restricting the connection \( \mathcal{A} \) to act on horizontal vectors yields an equation for the evolution of the group elements given by

\[
g^{-1} \dot{g} = -A(x) \cdot \dot{x}.
\]

Returning to the quantum setting, we note that the canonical connection on the bundle \( Q_k \) is given by \( A_wz \equiv U_k^{-1} dU_k \). The matrix elements of the connection form are given by

\[
A_{\bar{v}v}^\mu \equiv \langle \bar{v} | \frac{\partial}{\partial x^\mu} U_k(x) | v \rangle.
\]

This is commonly known as the Wilczek-Zee connection \[ 34 \].

Assuming direct control over the base variables, we may interpret the quantum control system as a control system evolving on a principal bundle and write it locally as

\[
g^{-1} \dot{g} = -A_{wz}(x) \cdot \dot{x}
\]

\[
\dot{x} = u
\]

where \( u \) is a vector of control inputs describing the controlled evolution in parameter space.

A formal solution to this system of equations corresponding to a particular path in parameter space is given by the path ordered integral

\[
P \exp \int_{\gamma} -A_{wz} \, dx.
\]

When \( \gamma \) is a closed curve in \( M \), then \( P \exp \oint_{\gamma} -A_{wz} \, dx \) lies in \( G \) and is known as the holonomy of \( \gamma \). It is well known that the set of all such group elements taken over the set of closed curves in \( M \) is a subgroup of \( G \) and is known as the holonomy group. In holonomic quantum computation, quantum logic gates are implemented by holonomies acting on the degenerate subspaces.

### 2.3 Universality

A control system evolving on a principal fiber bundle is said to be **locally controllable** if any group element can be implemented on the state of the system. In the context of quantum computing, a system with this property is said to be (exactly) **universal**. This property is a fundamental requirement for building a quantum processor. Loosely speaking, in the usual dynamical approach to quantum computing (as opposed to the geometric approach addressed here), the Lie algebra generated by the system Hamiltonian and the control Hamiltonians determines the universality of the system. For an \( N \) qubit system, it is sufficient for the Lie algebra to span \( su(2^N) \). We now show how this condition translates to the holonomic framework.

Let \( X^h \) denote the horizontal lift of a vector field \( X \) on \( M \). This is the unique vector on \( TQ \) such that \( T\pi(X^h) = X \) where \( \pi \) is the projection \( \pi : Q \to M \). Then the **curvature** can be defined as a \( g \)-valued 2-form
on $Q$ given by

$$
A([X^h_i, X^h_k]) = -\mathcal{F}(X^h_i, X^h_k)
$$

(15)

where $[\cdot, \cdot]$ denotes the Lie bracket on $TQ$. Thus evaluating the curvature determines the vertical component of the Lie bracket of horizontally lifted vector fields. Now let $f : Q \to \mathfrak{g}$ be an Ad-equivariant function on $Q$, then

$$
A([X^h, f]) = -dA(X^h, f) + X^h(A(f)) + f(A(X^h))
$$

(16)

$$
= X^h f
$$

(17)

since the function $f$ is $\mathfrak{g}$-valued and $dA(\cdot, \cdot)$ is zero if either argument is vertical [35]. Using the correspondence between covariant derivatives of the associated adjoint bundle and Lie derivatives of Ad-equivariant functions [38], we obtain

$$
A([X^h, f]) = X^h f = D_{X^h} f.
$$

(18)

Now, the curvature itself is an Ad-equivariant function on $Q$ [38], so setting $\mathcal{F} = f$ and using the previous expression (18) to evaluate iterated Lie brackets of horizontally lifted vector fields, we can obtain the corollary to the well known Chow-Rashevski theorem from control theory.

**Theorem 1 (Ambrose-Singer-Chow-Rashevski) The system (13) is locally controllable at $q \in Q$ if the curvature $F(X_{i_1}, X_{i_2})$ and all of its covariant derivatives $D_{X_{i_k}} \cdots D_{X_{i_3}} F(X_{i_1}, X_{i_2})$ evaluated at the point $x = \pi(q)$ span the entire Lie algebra of $G$.

Following [39], we refer to the theorem as Ambrose-Singer-Chow-Rashevski since it can be considered to be a corollary to the Ambrose-Singer theorem from the theory of holonomy [38]. We note also that we have stated the theorem in terms of base vector fields and the local curvature. All the necessary ingredients of the theorem, although not explicitly stated, can be found in [38]. In fact, the *infinitesimal holonomy algebra* is spanned by elements of the form

$$
X^h_{i_k} \cdots X^h_{i_1} \cdot \mathcal{F}(X^h_{i_1}, X^h_{i_2}).
$$

(19)

We can then use the correspondence [1] to relate this to covariant derivatives of the associated adjoint bundle. This statement is used in our applications, since in some holonomic quantum computation problems the relevant holonomy algebra does not span the entire Lie algebra. However, it does contain *non-local* operations which together with holonomies corresponding to *local* operations do indeed span the entire Lie algebra. This is the usual local/non-local analysis often encountered in quantum information science.

For the purposes of building a quantum processor, the quantum information is stored in the $\mathbb{C}^2$ vector bundle associated to $Q^2$ and single qubit rotations are performed by $SU(2)$ holonomies acting on the fiber $\mathbb{C}^2$. Interactions among qubits are modeled as $SU(4)$ holonomies acting on the fibers of the vector bundle associated to $Q^4$.

Thus we may treat the control problems separately and form the product bundle (and its pullbacks)

$$
\cdots \ St_2(\mathcal{H}_i) \times St_4(\cdots \otimes \mathcal{H}_i \otimes \mathcal{H}_j \otimes \cdots) \times St_2(\mathcal{H}_j) \cdots
$$

$$
\cdots \ Gr_2(\mathcal{H}_i) \times (Gr_4)^{int}(\cdots \otimes \mathcal{H}_i \otimes \mathcal{H}_j \otimes \cdots) \times Gr_2(\mathcal{H}_j) \cdots
$$

(20)

To set notation, let

$$
I_x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_y = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad I_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

(21)
We define the local algebra generated by the elements

\[ I_{k1} = I_k \otimes 1 \quad I_{k2} = 1 \otimes I_k \]  

(22)

where \( k \in \{x, y, z\} \). The local algebra is the Lie algebra corresponding to the local group \( SU(2) \otimes SU(2) \).

To conclude exact universality (controllability) of the system, one should compute the control Lie algebra with the constituent holonomy algebras. For example in the two-qubit system, compute the Lie algebra generated by \( su(2) \otimes 1, 1 \otimes su(2) \) and the interaction holonomy algebra \( \text{hol}_{\text{int}} \) associated with the bundle \( St_4(H_i \otimes H_j) \rightarrow (Gr_4)^{\text{int}}(H_i \otimes H_j) \). In the generic case, the control Lie algebra will generate \( su(4) \) provided that \( \text{hol}_{\text{int}} \) is not isomorphic to the local group or is trivial \([39, 40]\).

**Theorem 2** The two-qubit holonomic system is exactly universal if the Lie algebra generated by the local algebra and \( \text{hol}_{\text{int}} \) spans \( su(4) \).

### 3 Constructive Controllability

Having established conditions for determining universality in HQC, we now present various ways of solving or approximating the solution to the path ordered integral arising for the differential equation defining the group displacement

\[ g^{-1}\dot{g} = -A(x) \cdot \dot{x} . \]  

(23)

For control systems on principal bundles, this equation describes the group transformation obtained from a controlled cyclic variation of the parameters in the base manifold. Recall, that we assume direct access and complete controllability over the base variables. We endeavor to ascertain the desired group transformation resulting from a particular choice of loop in the base space. This is the notion of constructive controllability in the context of a control system on a principal bundle. In HQC these procedures provide explicit methods for logic gate synthesis. This requires dealing with the path ordered integral obtained from \([28]\).

#### 3.1 Path Ordered Integral

We define the path ordered integral as a product integral. Let \( \gamma \) be a curve in the base manifold \( M \) and let \( x^\mu \) be local coordinates. We may express the local connection form \( A \) in terms of coordinates as

\[ A(x) = A_\mu(x)dx^\mu . \]  

(24)

The curve \( \gamma \) is parameterized by an intrinsic parameter \( s \), which is naturally considered to be time. In terms of \( s \), \( A \) takes the form

\[ A_\mu(x(s))dx^\mu = A(s)ds \]  

(25)

where

\[ A(s) \equiv A_\mu(x(s)) \frac{dx^\mu(s)}{ds} . \]  

(26)

Let \( [s_0, s_T] \) be a real interval over which the curve \( \gamma \) is defined. Consider a partition of the interval \( P = \{s_0, s_1, \ldots, s_n\} \) such that \( \Delta s_k = s_k - s_{k-1} \) and \( s_n = s_T \). Then path ordering operator may be defined as

\[ \mathbf{P} \exp \int_\gamma -A(s)ds \equiv \lim_{n \to \infty} \prod_{s_0}^{s_n} \exp(-A(s_k)\Delta s_k) . \]  

(27)
This definition clearly shows the dependence on the ordering of the exponentials and the difficulties associated with its solution, given that we are naturally interested in the case where the relevant group is non-Abelian. When $G$ is Abelian, then one can directly integrate the connection coefficients and apply the usual exponential operator.

### 3.2 Abelian Substructures

A common technique in holonomic quantum computation for tackling the integral (27) is to restrict the class of loops and exploit Abelian substructures in the connection components [12, 21]. The strategy is briefly described as follows. Choose a particular 2-manifold of $M$ spanned by the coordinates $(\sigma, \tau)$ such that the associated connection components commute, that is $[A_{\sigma}, A_{\tau}] = 0$ but for which the local curvature form is not identically zero. For these restricted loops the path ordering in (27) can be avoided and the line integral

$$\oint_{\gamma} A(x)dx = \oint_{\gamma} (A_{\sigma}d\sigma + A_{\tau}d\tau)$$

(28)

can be integrated directly and exponentiated.

Alternatively, one can use a non-Abelian Stokes theorem [41] for evaluating holonomies corresponding to curves lying in a 2-submanifold of parameter space.

### 3.3 Averaging

The exact results of the previous section were accompanied by restrictions on the set of loops available to the controller or by exploiting Abelian substructures in the connection components. Here we review local approximations that can be used for any system evolving on a principal bundle.

Approximate control algorithms have been developed for left invariant control systems on Lie groups of the form

$$g^{-1}\dot{g} = \epsilon U(t)$$

(29)

where $\epsilon$ is a (small) parameter and $U(t) = T_\alpha u^\alpha(t)$ for a basis $\{T_\alpha\}$ of $g$ [42].

A Magnus expansion is employed for a representation of the solution

$$g(t) = g(0) \exp(\xi(t))$$

(30)

given by

$$\xi(t) = \epsilon \int_0^t U(\tau)d\tau + \frac{\epsilon^2}{2} \int_0^t [\tilde{U}(\tau), U(\tau)]d\tau$$

$$+ \frac{\epsilon^3}{4} \int_0^t \left[ \int_0^\tau [\tilde{U}(\sigma), U(\sigma)]d\sigma, U(\tau) \right] d\tau + \ldots$$

(31)

where $\tilde{U}(t)$ is the effective input “averaged” over the time period [42].

This expansion has been generalized for systems evolving on principal fiber bundles [5]. Let $\gamma : [0, T] \to M$ be a closed curve in the base space parameterized by $x \in M$, then it is shown in [5] that the holonomy associated to $\gamma$ can be locally approximated by

$$g(T) = g(0) \exp(\xi(\gamma))$$

(32)
where
\[ \xi(\gamma) = -\frac{1}{2} F(X_i, X_j) \int_\gamma dx_i dx_j + \frac{1}{3} D_{X_i}(F(X_j, X_k)) \int_\gamma dx_i dx_j dx_k + \ldots . \] (33)

Here \( F(X_i, X_j) \) is the local curvature form evaluated on the base coordinate vectors \( X_i = \frac{\partial}{\partial x_i} \) evaluated at \( \gamma(0) \), \( D_{\frac{\partial}{\partial x_i}} \) is the covariant derivative of the curvature along the base coordinate vector \( \frac{\partial}{\partial x_i} \) and the area integrals are defined by
\[ \mathcal{I}_{x_i x_j x_k} = \int_\gamma dx_i dx_j dx_k \equiv \int_0^T \int_0^{t_k} \int_0^{t_j} \dot{x}_i(t_j) dt_i \dot{x}_j(t_j) dt_j \dot{x}_k(t_k) dt_k . \] (34)

Higher order terms are given by higher order covariant derivatives of the curvature. This is plausible given expression (33) and the fact that iterated Lie brackets of horizontally lifted vector fields appear as covariant derivatives of the curvature.

### 4 Applications

In this section we apply the results of the preceding sections. We first review the \( \mathbb{C}P^n \) model of quantum holonomy. This was the original system discovered by Wilczek and Zee \[34\] and subsequently proposed as a model for HQC. We then consider two very different models of quantum holonomic systems. Holonomic quantum computation with squeezed coherent states has a rich interaction holonomy group that can be exploited to obtain constructive controllability algorithms. On the other hand, quantum computation based on the conditional phase shift has become the dominant control strategy for a wide range of holonomic quantum computing schemes.

#### 4.1 The \( \mathbb{C}P^n \) Model

The \( \mathbb{C}P^n \) \[\Pi\] model gives a concrete example illustrating how non-Abelian holonomies can occur in highly degenerate systems. In this model, we assume that the Hilbert space \( \mathcal{H} \) is finite dimensional from the outset. That is, we have the isomorphism \( \mathcal{H} \cong \mathbb{C}^{n+1} = \{ |\alpha\rangle \}_{\alpha=1}^{n+1} \). We further assume an \( n \)-dimensional degenerate subspace with eigenvalue 0. We may write the degenerate Hamiltonian \( H_0 \) as
\[ H_0 = \epsilon |n+1\rangle \langle n+1| = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \epsilon \end{pmatrix} . \] (35)

Let \( \{ T_{j,n+1}(x) \}_{j=1}^l \) denote a basis of \( \mathfrak{u}(n) \) parameterized by \( x \in M \) embedded in \( \mathfrak{u}(n+1) \) and let \( \mathcal{U}_n = \prod_{j=1}^l \exp(T_{j,n+1}(x)) \). Given these control operations, it is perhaps not surprising that the holonomy group can be shown to be \( U(n) \) by considering the curvature coefficients only (and not its covariant derivatives) \[33\] \[\Pi\]. We have the isomorphism
\[ \mathcal{O}(H_0) \cong \frac{U(n+1)}{U(n) \times U(1)} \cong \frac{SU(n+1)}{U(n)} \cong \mathbb{C}P^n . \] (36)

This system requires control over \( 2n = \dim \mathbb{C}P^n \) real parameters to control an \( n \)-level system. For high dimensional systems, this may be an unrealistic requirement.
4.2 Squeezed Coherent States

In this section we revisit the mathematical foundations of holonomic quantum computation with squeezed coherent states. There is considerable literature already on this model [12, 21, 22, 23, 25, 26], here we exploit the methods of geometric control. Originally, this model was proposed in the context of quantum optics [12] with displacers and squeezers operating on coherent laser beams in a non-linear Kerr medium and thus known as the Optical Holonomic Computer. However, other physical systems have quantum states that may be displaced and squeezed. As far as the control analysis is concerned, these systems are identical. Pachos has recently adapted the model to perform trapped ion quantum computation [16]. We, therefore, refer to this model generically as holonomic quantum computation with squeezed coherent states.

4.2.1 Harmonic Oscillator

Recall that the commutation relations of the creation, annihilation and number operator $N \equiv a^\dagger a$, are given by

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = a, \quad [a, a^\dagger] = 1.$$  \hspace{1cm} (37)

The underlying Hilbert space $\mathcal{H}$ is a Fock space and takes the form

$$\mathcal{H} = \{ |n \rangle, \alpha \in \mathbb{N} \cup 0 \}. \hspace{1cm} (38)$$

The creation and annihilation operators act on $\mathcal{H}$ according to

$$a |n \rangle = \sqrt{n} |n - 1 \rangle, \quad a^\dagger |n \rangle = \sqrt{n+1} |n+1 \rangle, \quad a |0 \rangle = 0.$$ \hspace{1cm} (39)

Thus $a$ and $a^\dagger$ create and destroy quanta.

Since we are interested in the two-qubit system, we will use the subscript $i$ to distinguish the creation and annihilation corresponding to the $i$-th field of the harmonic oscillator. That is, we set $N_i = a_i^\dagger a_i$ and

$$a_1 = a \otimes 1, \quad a_1^\dagger = a^\dagger \otimes 1,$$

$$a_2 = 1 \otimes a, \quad a_2^\dagger = 1 \otimes a^\dagger.$$ \hspace{1cm} (40) \hspace{1cm} (41)

To provide a concrete example, we will use the degenerate Hamiltonian

$$H^i = N_i(N_i - 1)$$ \hspace{1cm} (42)

to encode the $i$-th qubit in the degenerate subspace $\{ |0_i \rangle, |1_i \rangle \}$ and

$$H^{12} = N_1(N_1 - 1) + N_2(N_2 - 1)$$ \hspace{1cm} (43)

to obtain controlled interactions on the computation basis $\{ |00 \rangle, |01 \rangle, |10 \rangle, |11 \rangle \}$ where $|ij \rangle = |i \rangle \otimes |j \rangle$. In the optics context, this Hamiltonian corresponds (up to a constant) to placing lasers in a non-linear Kerr medium [12, 21]. However, the form of the degenerate Hamiltonian does not affect the control analysis. For our purposes, it is used only to encode the quantum information. With slight modification of some constants, the results in this section apply to the trapped ion model proposed in [16].

4.2.2 Single Qubit

We consider first single qubit rotations. Consider the eigenvalue problem for a single creation operator. The state $|\alpha \rangle$ can be written in terms of the basis $\{ |n \rangle \}$,

$$|\alpha \rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n \rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n a^n}{n!} |0 \rangle.$$ \hspace{1cm} (44)
Which is equal to
\[ |\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle \] (45)
and allows for the definition of the displacement operator
\[ |\alpha\rangle = D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a} |0\rangle . \] (46)
Note that the introduction of \( e^{-\alpha^* a} \) does nothing since \( a |0\rangle = |0\rangle \).
By using the Campbell-Baker-Hausdorff formula and noting that \([a a^\dagger, -\alpha^* a] = |\alpha|^2\), we may write
\[ |\alpha\rangle = D(\alpha)|0\rangle = e^{\alpha a^\dagger - \alpha^* a} |0\rangle . \] (47)
We see that the displacement operator creates a coherent state \( |\alpha\rangle \) from the vacuum state \( |0\rangle \).
In HQC with squeezed coherent states, the displacement operator will be a control operator.
The other transformation we have at our disposal is the squeezing operator \( S(\beta) \), defined by
\[ S(\beta) = e^{\beta \Lambda_+ - \beta^* \Lambda_-} . \] (48)
where
\[ \Lambda_+ \equiv \frac{1}{2} a^{12} \quad \Lambda_- \equiv \frac{1}{2} a^{2} . \] (49)
If we define,
\[ \Lambda_3 = \frac{1}{4}(aa^\dagger + a^\dagger a) \] (50)
then we have the commutation relations,
\[ [\Lambda_3, \Lambda_+] = \Lambda_+ \quad [\Lambda_3, \Lambda_-] = -\Lambda_- \quad [\Lambda_+, \Lambda_-] = 2\Lambda_3 . \] (51)
These are the commutation relations for \( su(1,1) \); thus we see that the squeeze operator is a representation of the non-compact group \( SU(1,1) \). With these two unitary transformations, we form the product
\[ U_2(\alpha, \beta) = D(\alpha)S(\beta) \] (52)
and the isospectral family of Hamiltonians
\[ U_2(\alpha, \beta)H_i U_2^\dagger(\alpha, \beta) . \] (53)
The holonomy group for the single qubit system has been shown to be \( U(2) \) \cite{12 21 22 23}. Thus we have complete control over the single qubit.

### 4.2.3 Two-qubit

To obtain universality over the entire quantum register it suffices to show non-trivial \( U(4) \) transformations on the computational basis and check the control Lie algebra. Analogously to the single qubit case, we employ displacement and squeeze operators as our control operations. Let
\[ J_+ = a^{1\dagger}a_2 , \quad J_- = a^{\dagger 2}a_1 , \quad J_3 = \frac{1}{2}(a^{1\dagger}a_1 - a^{\dagger 2}a_2) . \] (54)
These generate \( SU(2) \) with the commutation relations,
\[ [J_3, J_+] = J_+ , \quad [J_3, J_-] = -J_- , \quad [J_+, J_-] = 2J_3 . \] (55)
The two-mode displacement operator is defined as
\[ N(\xi) = \exp (\xi a^{1\dagger}a_2 - \xi^* a^{\dagger 2}a_1) . \] (56)
Similarly, we may define the two-mode squeeze operator as a representation of $SU(1,1)$. Let

\[ K_+ = a_1^\dagger a_2, \quad K_- = a_1 a_2, \quad K_3 = \frac{1}{2} (a_1^\dagger a_1 + a_1 a_2^\dagger), \]

(57)

and

\[ [K_3, K+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3. \]

(58)

The two-mode squeeze operator is defined as

\[ M(\zeta) = \exp (\zeta a_1^\dagger a_2 - \zeta a_1 a_2) \]

(59)

where $\xi, \zeta \in \mathbb{C}$. Set

\[ U_4 = N(\xi) M(\zeta). \]

(60)

Setting $\zeta = r_2 e^{\imath \theta_2}$ and $\xi = r_3 e^{\imath \theta_3}$ and we obtain the two-qubit connection coefficients \[21\] listed in Appendix A. The interaction holonomy algebra spans $su(2) \times su(2) \times u(1)$ \[27\] (also listed in Appendix B). Higher order covariant derivatives do not yield independent group directions. The matrices in $\mathfrak{hot}_{int}$ sit in $u(4)$ in a manner that allows for non-local $U(4)$ transformations on the computational basis. By the reduction theorem for connections \[58\], the connection is reducible to a $su(2) \times su(2) \times u(1)$-valued connection and we may reduce the total space to $St_{2,4}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. To determine the reduced base manifold, we form the quotient

\[ \frac{U(4)}{SU(2) \times SU(2) \times U(1)} \cong \frac{SU(4)}{SU(2) \times SU(2)} \cong SGr_{2,4} \cong (Gr_{2,4})^{int}. \]

(61)

In a similar manner, we can reduce the bundles $St_{2}(\mathcal{H}_i) \rightarrow Gr_{2}(\mathcal{H}_i)$ corresponding to the single qubit rotations. The $U(2)$ holonomies act in the product space $\mathcal{H}_1 \otimes \mathcal{H}_2$ as $U(2) \otimes 1$ and $1 \otimes U(2)$. The bundles reduce to $St_{2,4}(\mathcal{H}_1 \otimes \mathcal{H}_2) \rightarrow Gr_{2,4}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. For the full two-qubit system, we have the reduced product bundle

\[ \begin{array}{c}
\text{St}_{2,4}(\mathcal{H}_1 \otimes \mathcal{H}_2) \times \text{St}_{2,4}(\mathcal{H}_1 \otimes \mathcal{H}_2) \\
\downarrow \\
\text{Gr}_{2,4}(\mathcal{H}_1 \otimes \mathcal{H}_2) \times (Gr_{2,4})^{int}(\mathcal{H}_1 \otimes \mathcal{H}_2) \\
\downarrow \\
\text{Gr}_{2,4}(\mathcal{H}_1 \otimes \mathcal{H}_2). 
\end{array} \]

(62)

4.2.4 Control Algebra

To be complete, we will now demonstrate that all of $SU(4)$ may be obtained from the single qubit rotations and the two-qubit transformations above. Of course, as we have mentioned earlier, this is generically true provided the two-qubit holonomy group is not isomorphic to the local group. Nonetheless, it is useful to go through the computations.

From the single qubit analysis, we know that we can perform local transformations of the form $SU(2) \otimes SU(2)$. To simplify matters further, we use linear combinations of the the two-qubit curvature forms and covariant derivatives and consider only

\[ \{ F_{r_2 \theta_3} , F_{r_3 \theta_3}, F_{r_3 \theta_2}, D_{\frac{\partial}{\partial r_2}} F_{r_3 \theta_2}, \tilde{D}_{\frac{\partial}{\partial r_2}} F_{r_2 \theta_2} \} \]

(63)

where

\[ \tilde{D}_{\frac{\partial}{\partial r_2}} F_{r_2 \theta_2} = \begin{pmatrix} 0 & 0 & 0 & -e^{-i \theta_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -e^{i \theta_2} & 0 & 0 & 0 \end{pmatrix} 4i \sinh 2r_2. \]

(64)

From this set of matrices and the local algebra \[22\], we may build a set of holonomic transformations spanning $su(4)$. After taking iterated brackets from these sets, we find that one choice of spanning elements is given by

\[ su(4) = \{ C_1 \cup C_2 \cup C_3 \} \]

(65)
where \( C_1 = \{ I_{x1}, I_{y1}, I_{z1}, I_{x2}, I_{y2}, I_{z2} \} \) and
\[
C_2 = \{ F_{r_{2r_3}}, F_{r_{2}r_3}, D_{\frac{\partial}{\partial r_2}}F_{r_2r_3}, \hat{D}_{\frac{\partial}{\partial r_3}}F_{r_2r_3} \}
\]
\[
C_3 = \left\{ [I_{x1}, F_{r_{2r_3}}], [I_{x1}, F_{r_2r_3}], [I_{x2}, F_{r_2r_3}], [I_{x1}, F_{r_2r_3}], [I_{x1}, F_{r_{2r_3}}], I_{x2} \right\}.
\]

Please see Appendix C for the matrix representation of these elements.

### 4.2.5 An approximate holonomy in the Cartan subalgebra of \( su(4) \)

In the preceding section, we showed that it is indeed possible to create holonomic transformations spanning the full unitary group on two qubits. This was not a constructive procedure. In this section, we show that by using a combination of the methods in the previous sections, we can solve the logic gate synthesis problem completely. We use the local expansion of the holonomy procedure to construct an element in the Cartan subalgebra of \( su(4) \) and use the Cartan decomposition of \( SU(4) \) to obtain the result.

The Cartan decomposition of the unitary groups is a useful technique that has been used for constructing quantum control algorithms [43, 44], deriving time optimal control laws for quantum spin systems [45] and understanding the entanglement content of 2-qubit unitaries [46, 47]. Here we review the decomposition for the purposes of constructing control algorithms.

Let \( K \) denote a closed and compact subgroup of a Lie group \( G \). Assume that \( g \) admits a vector space decomposition
\[
g = \mathfrak{k} \oplus \mathfrak{p}
\]
where \( \mathfrak{k} \) is the Lie algebra of \( K \) and \( \mathfrak{p} \) is vector space orthogonal to \( \mathfrak{k} \) with respect to a bi-invariant metric \( \langle \cdot, \cdot \rangle \) on \( g \). Further assume that \( \mathfrak{k} \) and \( \mathfrak{p} \) satisfy the following commutation relations
\[
[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k} \quad [\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}.
\]

We refer to this decomposition as a Cartan decomposition of the Lie algebra \( g \).

Let \( a \) denote a maximal Abelian subalgebra contained in \( \mathfrak{p} \). The algebra \( a \) is often called the Cartan subalgebra of \( g \). Then one can write \( G \) as
\[
G = KAK
\]
where \( A = \exp(\mathfrak{a}) \).

In a two-qubit system, interactions among the qubits are modeled by the products
\[
I_{kl} = 2I_k \otimes I_l
\]
where \( k, l \in \{ x, y, z \} \). For example,
\[
I_{yy} = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.
\]

The Cartan decomposition of \( su(4) \) is given by
\[
\mathfrak{k} = i\{ I_{x1}, I_{y1}, I_{z1}, I_{x2}, I_{y2}, I_{z2} \}
\]
\[
\mathfrak{p} = i\{ I_{xx}, I_{xy}, I_{xz}, I_{yx}, I_{yy}, I_{yz}, I_{zx}, I_{zy}, I_{zz} \}
\]
\[
\mathfrak{a} = i\{ I_{xx}, I_{yy}, I_{zz} \}.
\]
Thus we can write any $g \in SU(4)$ as
\[
g = K_1 \exp(-i \phi_1 I_{xx} - i \phi_2 I_{yy} - i \phi_3 I_{zz}) K_2
\] (74)
where $\phi_j$ is a real parameter and $K_j \in SU(2) \otimes SU(2)$.

By inspection of the two-qubit curvature forms and their covariant derivatives, it seems possible that $I_{yy}$ can be obtained by linear combinations the elements,
\[
\{ F_{r_2 \theta_2}, F_{r_2 \theta_3}, D_{\frac{\partial}{\partial r_2}} F_{r_2 \theta_2}, D_{\frac{\partial}{\partial r_2}} F_{r_2 \theta_3} \}.
\] (75)

Equivalently, $I_{yy}$ is contained in the real span of
\[
\left[ \frac{\partial^h}{\partial x_k}, \left[ \frac{\partial^h}{\partial x_{k-1}}, \ldots \left[ \frac{\partial^h}{\partial x_2}, \frac{\partial^h}{\partial x_1} \right] \ldots \right] \right]
\] (76)
where $x \in \{ r_2, \theta_2, \theta_3 \}$.

We therefore choose a candidate loop, $\gamma^*$, of the form
\[
\begin{align*}
\theta_2(t) &= \theta_2(0) + \Theta_2 \sin(t) \\
r_2(t) &= r_2(0) + R_2 \cos(t) - R_2 \\
\theta_3(t) &= \theta_3(0) + \Theta_3 \sin(nt), \quad n \neq 1 \\
r_3 &= \text{constant}
\end{align*}
\] (77, 78, 79, 80)
with the parameters $\{ n, r_2(0), \theta_2(0), \theta_3(0), R_2, \Theta_2, \Theta_3 \}$ to be determined. We compute the integrals appearing in the expansion (33), with the period $T = 2\pi$ and choose some parameters to yield the expressions,
\[
\begin{align*}
F_{r_2 \theta_2} \cdot \mathcal{I}_{r_2 \theta_2} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \cdot 2i \sin 2r_2(0) \cdot R_2 \Theta_2 \pi \\
F_{r_2 \theta_3} \bigg|_{\theta_3(0) = \pi} \cdot \mathcal{I}_{r_2 \theta_3} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot i \sin 2r_3 \sin 2r_2(0) \cdot \frac{R_2 \Theta_3 \sin 2n\pi}{n^2 - 1} \\
D_{\frac{\partial}{\partial r_2}} F_{\theta_3} \bigg|_{\theta_3(0) = \pi} \cdot \mathcal{I}_{r_2 \theta_3, r_2} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot 2i \sin 2r_3 \cosh 2r_2(0) \cdot \frac{-6R_2^2 \Theta_3 \sin 2n\pi}{n^2 - 5n^2 + 4} \\
D_{\frac{\partial}{\partial r_2}} F_{\theta_2} \bigg|_{\theta_2(0) = 0} \cdot \mathcal{I}_{r_2 \theta_2, r_2} &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \cdot 4i \sin 2r_2(0) \cdot 2 \Theta_2 R_2^2 \pi \\
&+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \cdot 4i \cosh 2r_2(0) \cdot 2 \Theta_2 R_2^2 \pi
\end{align*}
\] (81, 82, 83, 84, 85)
We also have,

\[ I_{\theta_2 \theta_2} = 0 \]
\[ F_{\theta_2 \theta_3} = 0 \]
\[ D_{\frac{\partial}{\partial \theta_3}} F_{\theta_2 \theta_2} = 0 \]

\[ D_{\frac{\partial}{\partial \theta_2}} F_{\theta_2 \theta_2} \bigg|_{r_3(0) = \pi/4} = 0 \]
\[ F_{\theta_2 \theta_2} \cdot I_{r_2 \theta_2} = F_{\theta_2 \theta_2} \cdot I_{\theta_2 \theta_2} \]
\[ F_{\theta_3 \theta_3} \bigg|_{\theta_3(0) = \pi} = F_{\theta_3 \theta_2} \bigg|_{\theta_3(0) = \pi} \cdot I_{\theta_3 \theta_2} \]
\[ -2D_{\frac{\partial}{\partial \theta_2}} F_{\theta_2 \theta_3} \bigg|_{\theta_3(0) = \pi} \cdot I_{r_2 \theta_2 \theta_3} = D_{\frac{\partial}{\partial \theta_2}} F_{\theta_2 \theta_2} \bigg|_{\theta_2(0) = 0} \cdot I_{r_2 \theta_2 \theta_2} \]

The strategy now is to choose parameters so that the \( F_{\theta_2 \theta_2} \) terms kill the terms along the diagonal in the expressions \( D_{\frac{\partial}{\partial \theta_2}} F_{\theta_2 \theta_2} \) and \( D_{\frac{\partial}{\partial \theta_3}} F_{\theta_2 \theta_2} \). Then, with those parameters chosen, we choose the rest of the parameters so that the remaining terms combine to yield \(-i\theta I_{yy}\), where \( \theta \) is a free parameter. Remembering to include the coefficients in the expansion, the first objective leads to the following equation,

\[ r_2(0) = \frac{1}{2} \arctanh(2R_2) \]  

Setting \( \Theta_2 = \Theta_3 = \theta \), and substituting the previous equation defining \( r_2(0) \) and \( R_2 \), the second objective yields

\[ R_2 = -\frac{\sin 2n\pi}{4\pi(n^2 - 4)} \]  

If \( n \) is chosen to as a non-integer so that \( \Theta_2 = \Theta_3 = \theta \), the loop \( \gamma^* \) determines the holonomy (up to third order)

\[ \Gamma(\gamma^*) = \exp(-i\theta I_{yy}) \]  

where \( \theta \) is a free parameter.

Loops generating single qubit \( SU(2) \) holonomies can be characterized by \textit{abelianizing} the dynamics [21]. Thus with this single two-qubit holonomy, we can construct any \( SU(4) \) transformation on the computational basis. To see this, recall that any \( SU(4) \) transformation may be written with the Cartan decomposition

\[ g = K_1 \exp(-i\theta_1 I_{xx} - i\theta_2 I_{yy} - i\theta_3 I_{zz}) K_2 \]

where \( K_j \in SU(2) \otimes SU(2) \).

We may obtain the transformations \( \exp(-i\theta_1 I_{xx}) \) and \( \exp(-i\theta_1 I_{zz}) \) by noting that

\[ K_z(\frac{\pi}{2}) \exp(-i\theta I_{yy}) K_z(\frac{\pi}{2}) = \exp(-i\theta I_{xx}) \]
\[ K_x(\frac{\pi}{2}) \exp(-i\theta I_{yy}) K_x^{-1}(\frac{\pi}{2}) = \exp(-i\theta I_{zz}) \]

where

\[ K_z(\theta) = \exp(-i\theta I_{z1}) \exp(-i\theta I_{z2}) \]
\[ K_x(\theta) = \exp(-i\theta I_{z1}) \exp(-i\theta I_{z1}) \]

Thus any \( g \in SU(4) \) can be approximated up to third order by,

\[ g = K_1 K_z^{-1}(\frac{\pi}{2}) \Gamma(\gamma^*) K_z(\frac{\pi}{2}) \Gamma(\gamma^*) K_x(\frac{\pi}{2}) \Gamma(\gamma^*) K_x^{-1}(\frac{\pi}{2}) K_2 \]
One can also use the *abelianization* procedure of the previous section to construct a holonomy of the form

$$\tilde{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}.$$  \hspace{1cm} (103)

One can then show that the Lie algebra element $\tilde{\xi}$ such that $\exp(\tilde{\xi}) = \tilde{U}$ along with the local algebra generates $su(4)$ under repeated bracketing. Thus $\tilde{U}$ is a so called *universal logic gate*. However, this procedure does not give a prescription for building an arbitrary $SU(4)$ transformation.

### 4.3 Conditional Berry Phases

An interesting hybrid scheme to quantum computing involving *dynamical* $SU(2)$ rotations and conditional Berry phases has been realized as a universal set of gates for several physical systems proposed for quantum computing. To date, there have been HQC implementations using this control paradigm with NMR [13], trapped ions [14], neutral atoms [15], semiconductor nanostructures [17], and Josephson junction networks [18, 19]. We refer the reader to the literature for a description of the physical systems underlying these proposed quantum computing schemes.

Here we are interested in the control strategy of the experimentalist with the gates available in systems of this type. Namely, how does one build an arbitrary unitary transformation on two coupled qubits given only single qubit rotations and the conditional phase shift? It is perhaps surprising that indeed this is possible and one can entangle qubits with only the conditional phase shift as the non-local operation. For this model we do not concentrate on the generation of the fundamental logic gates since the $SU(2)$ transformations are typically not holonomic and the Abelian Berry phase contributing to the conditional phase gate can be computed with Stokes theorem.

Since the conditional phase gate is not an element of $SU(4)$, we employ a Cartan decomposition of $U(4)$. To this end, recall the notation of the previous section and note that the real span of the sets

$$\mathfrak{k} = i\{I_{x1}, I_{y1}, I_{z1}, I_{x2}, I_{y2}, I_{z2}\}$$  \hspace{1cm} (104)

$$\mathfrak{p} = i\{1_4, I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{xz}, I_{yx}, I_{yz}, I_{zx}, I_{zy}\}$$  \hspace{1cm} (105)

form a basis of $u(4)$ in the tensor product representation. Moreover, one can check the commutation relations \cite{67} to confirm that the set forms a Cartan decomposition of $u(4)$ where $u(4) = \mathfrak{k} \oplus \mathfrak{p}$. Since the maximal Abelian subalgebra $\mathfrak{a}$ contained in $\mathfrak{p}$ is just

$$\mathfrak{a} = i\{1_4, I_{xx}, I_{yy}, I_{zz}\},$$  \hspace{1cm} (106)

we obtain the decomposition for any $G \in U(4)$

$$G = K_1 \exp(-i\phi_0 1_4 - i\phi_1 I_{xx} - i\phi_2 I_{yy} - i\phi_3 I_{zz})K_2$$  \hspace{1cm} (107)

where $\phi_i \in \mathbb{R}$ and $K_j \in K = SU(2) \otimes SU(2)$.

### 4.3.1 Control Algorithms

Proceeding along the lines of \cite{43, 45, 44}, we develop control algorithms with single qubit rotations and the conditional phase shift. The action of the conditional phase shift of the computational basis is as follows,

$$U_\phi \ket{00} = \ket{00}, \quad U_\phi \ket{01} = \ket{01}$$  \hspace{1cm} (108)

$$U_\phi \ket{10} = \ket{10}, \quad U_\phi \ket{11} = e^{-i\phi} \ket{11}.$$  \hspace{1cm} (109)
Under the isomorphism \( C^2 \otimes C^2 \cong C^4 \), the conditional phase shift can be written as

\[
U_\phi = \begin{pmatrix}
0 & 0 & 0 & e^{i\phi} \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

In terms of the basis (104) we can write

\[
U_\phi = \exp(-i\frac{\phi}{4}(21_4 + I_{zz} - (I_{z1} + I_{z2})))
\]

Let

\[
\tilde{K}_z = \exp(-i\frac{\phi}{4}(I_{z1} + I_{z2})),
\]

since

\[
[21_4, I_{z1}] = [I_{zz}, I_{z1}] = 0
\]

we have

\[
\tilde{K}_z U_{\phi/2} = \exp(-i\frac{\phi}{4}(I_{z1} + I_{z2})) \cdot \exp(-i\frac{\phi}{4}(21_4 + I_{zz} - (I_{z1} + I_{z2})))
\]

\[
= \exp(-i\frac{\phi}{4}(21_4 + I_{zz})).
\]

Let

\[
K_j(\theta) = \exp(-i\theta I_{j1}) \exp(-i\theta I_{j2}) \in K
\]

for \( j \in \{x, y\} \) and \( \theta \in \mathbb{R} \) is a real parameter. We have the commutation relations,

\[
[ -i(I_{z1} + I_{z2}), -i(21_4 + I_{zz}) ] = [ -i(I_{z1} + I_{z2}), -iI_{zz} ]
\]

and

\[
[ -iI_{zz}, [-i(I_{j1} + I_{j2}), -iI_{zz}] ] = -i(I_{j1} + I_{j2}).
\]

Thus by the Campbell-Baker-Hausdorff formula, we obtain,

\[
K_j(\theta)\tilde{K}_z U_{\phi/2} K_j^{-1}(\theta) = K_j(\theta) \exp(-i\frac{\phi}{4}(21_4 + I_{zz})) K_j^{-1}(\theta)
\]

\[
= \exp(-i\frac{\phi}{4}(21_4 + I_{zz}) \cos(\theta) + \left[ (-iI_{j1} - iI_{j2}), -iI_{zz} \right] \sin(\theta))).
\]

We employ a \( \pi \)-rotation to achieve the necessary decoupling. Using the preceding expression we get,

\[
K_j(\pi)\tilde{K}_z U_{\phi/2} K_j^{-1}(\pi) = K_j(\pi) \exp(-i\frac{\phi}{4}(21_4 + I_{zz})) K_j^{-1}(\pi)
\]

\[
= \exp(-i\frac{\phi}{4}(21_4 - I_{zz})).
\]

So we obtain

\[
\tilde{K}_z U_{\phi/2} : K_j(\pi)\tilde{K}_z U_{\phi/2} K_j^{-1}(\pi) = \exp(-i\phi 1_4).
\]

Similarly,

\[
\tilde{K}_z' U_{\phi} : K_j(\pi)\tilde{K}_z' U_{\phi} K_j^{-1}(\pi) = \exp(-i\phi I_{zz})
\]

where \( \tilde{K}_z' = \exp(-i\frac{\phi}{2}(I_{z1} + I_{z2})). \)

Finally, by noting that for

\[
K_x(\theta) = \exp(-i\theta I_{x1}) \exp(-i\theta I_{x2})
\]

\[
K_y(\theta) = \exp(-i\theta I_{y1}) \exp(-i\theta I_{y2})
\]
we have
\[
K_y(\frac{\pi}{2}) \exp(-i\phi I_{zz}) K_y^{-1}(\frac{\pi}{2}) = \exp(-i\phi I_{xx}) \tag{126}
\]
\[
K_x^{-1}(\frac{\pi}{2}) \exp(-i\phi I_{zz}) K_x(\frac{\pi}{2}) = \exp(-i\phi I_{yy}). \tag{127}
\]

Using (126) and (127), we can construct the desired decomposition for any \( G \in U(4) \),
\[
G = K_1 \cdot \exp(-i\phi_0 I_4 - i\phi_1 I_{xx} - i\phi_2 I_{yy} - i\phi_3 I_{zz}) \cdot K_2 \tag{128}
\]
\[
= K_1 \cdot \exp(-i\phi_0 I_4) \cdot K_y(\frac{\pi}{2}) \exp(-i\phi_1 I_{zz}) K_y^{-1}(\frac{\pi}{2}) \cdot K_x^{-1}(\frac{\pi}{2}) \exp(-i\phi_2 I_{zz}) K_x(\frac{\pi}{2}) \tag{129}
\]
\[
\cdot \exp(-i\phi_3 I_{zz}) \cdot K_2.
\]

This can now be written in terms of just elements of \( SU(2) \otimes SU(2) \) and the conditional phase shift \( U_\phi \),
\[
G = K_1 \cdot \tilde{K}_z U_{\phi_0/2} K_j(\pi) \tilde{K}_z U_{\phi_0/2} K_j^{-1}(\pi) \cdot K_y(\frac{\pi}{2}) \tilde{K}_y U_{\phi_1} K_j(\pi) \tilde{K}_y U_{-\phi_1} K_j^{-1}(\pi) K_y^{-1}(\frac{\pi}{2}) \cdot K_x^{-1}(\frac{\pi}{2})
\]
\[
\tilde{K}_x U_{\phi_2} K_j(\pi) \tilde{K}_x U_{-\phi_2} K_j^{-1}(\pi) K_x(\frac{\pi}{2}) \tilde{K}_x U_{\phi_3} K_j(\pi) \tilde{K}_x U_{-\phi_3} K_j^{-1}(\pi) \cdot K_2. \tag{130}
\]

Given the freedom of choosing \( j \) in \( K_j \) this sequence can be simplified somewhat. For example, choose \( j = y \) in the product \( K_j^{-1}(\pi) \cdot K_y(\frac{\pi}{2}) \) to obtain \( K_j^{-1}(\pi) \cdot K_y(\frac{\pi}{2}) = K_y(-\pi) \cdot K_y(\frac{\pi}{2}) = K_y(-\frac{\pi}{2}) \). Using this substitution and two others, the decomposition simplifies to
\[
G = K_1 \cdot \tilde{K}_z U_{\phi_0/2} K_j(\pi) \tilde{K}_z U_{\phi_0/2} K_y(-\frac{\pi}{2}) \tilde{K}_y U_{\phi_1} K_j(\pi) \tilde{K}_y U_{-\phi_1} K_y(-\frac{\pi}{2}) K_x^{-1}(\frac{\pi}{2}) \tilde{K}_x U_{\phi_2} K_j(\pi) \tilde{K}_x U_{-\phi_2} K_x(\frac{\pi}{2}) \tilde{K}_x U_{\phi_3} K_j(\pi) \tilde{K}_x U_{-\phi_3} K_j^{-1}(\pi) \cdot K_2. \tag{131}
\]

Finally, absorbing \( \tilde{K}_z \) and \( K_j^{-1}(\pi) \) into \( K_1 \) and \( K_2 \) respectively, we get
\[
G = K_1 U_{\phi_0/2} K_j(\pi) \tilde{K}_z U_{\phi_0/2} K_y(\frac{\pi}{2}) \tilde{K}_y U_{\phi_1} K_j(\pi) \tilde{K}_y U_{-\phi_1} K_y(-\frac{\pi}{2}) K_x^{-1}(\frac{\pi}{2}) \tilde{K}_x U_{\phi_2} K_j(\pi) \tilde{K}_x U_{-\phi_2} K_x(\frac{\pi}{2}) \tilde{K}_x U_{\phi_3} K_j(\pi) \tilde{K}_x U_{-\phi_3} K_2. \tag{132}
\]

Some remarks are appropriate. This sequence of unitary transformations is exact and a precise prescription for building any \( U(4) \) logic gate with just local operations and the conditional phase shift. We make no claim that this decomposition is optimal with respect to number of elements nor time. In the holonomic framework, time optimality is constrained by the adiabatic requirement. In this case, one should then focus primarily on minimizing the number of loops necessary to build an arbitrary gate.

## 5 Conclusion

In this paper, we have considered holonomic quantum computation from a control theoretic point of view. A general framework for the control analysis is obtained by casting the relevant problems as control systems evolving on principal fiber bundles. We have applied this framework to two well established models of the computing scheme. To the best of our knowledge, all holonomic computing schemes proposed thus far fall into one of the two models considered here. From a control perspective, an interesting avenue for future work would be extending these ideas to the control of molecular systems in the Born-Oppenheimer approximation (as mentioned in [11]). Holonomies can be realized in this regime [36] and it is reasonable to expect that a similar analysis can be carried out for these systems. However, a direct application of the methods proposed here will not suffice since the control parameters themselves are quantum degrees of freedom and therefore possess a non-trivial uncontrolled evolution of their own. In other words, the state equations analogous to those considered here [13] will be coupled quantum control problems.
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Appendix

A Two-Qubit Connection Components

\[ A_{r_2} = \begin{pmatrix} 0 & 0 & 0 & -e^{-i\theta_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{i\theta_2} & 0 & 0 & 0 \end{pmatrix} \]

\[ A_{r_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & e^{i\theta_3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (2\cosh^2 r_2 - 1) \]

\[ A_{\theta_2} = \begin{pmatrix} 0 & 0 & 0 & e^{-i\theta_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{i\theta_2} & 0 & 0 & 0 \end{pmatrix} \frac{i}{2} \sinh 2r_2 + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \frac{i}{2} (\cosh 2r_2 - 1) \]

\[ A_{\theta_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-i\theta_3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & e^{i\theta_3} & 0 & 0 \end{pmatrix} \frac{i}{2} \cosh 2r_2 \sin 2r_3 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{i}{2} \sin^2 r_3 \]

B Interaction Holonomy Algebra

\[ F_{r_2 r_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{-i\theta_2} \\ 0 & e^{i\theta_3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} 2\sinh 2r_2 \]

\[ F_{r_2 \theta_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} 2i \sinh 2r_2 \]

\[ F_{r_2 \theta_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\theta_3} \\ 0 & e^{i\theta_3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} i \sin 2r_3 \sinh 2r_2 \]

\[ F_{r_3 \theta_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} i \sin 2r_3 \sinh^2 2r_2, \]
\[
D \frac{\partial^2}{\partial \theta^2} F_{r_2 \theta_2} = \begin{pmatrix} 0 & 0 & 0 & -e^{-i \theta_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{i \theta_2} & 0 & 0 & 0 \end{pmatrix} \ 2 \sinh^2 2r_2
\]

\[
D \frac{\partial^2}{\partial \theta^2} F_{r_2 \theta_2} = \begin{pmatrix} 0 & 0 & 0 & -e^{-i \theta_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -e^{i \theta_2} & 0 & 0 & 0 \end{pmatrix} \ 4i \sinh 2r_2 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \ 4i \cosh 2r_2 ,
\]

\[
D \frac{\partial^2}{\partial \theta^2} D \frac{\partial^2}{\partial \theta^2} F_{r_2 \theta_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ 2i \sinh^3 r_2 + \begin{pmatrix} 0 & 0 & 0 & e^{-i \theta_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{i \theta_2} & 0 & 0 & 0 \end{pmatrix} \ 2i \sinh^2 2r_2 \cosh 2r_2
\]

### C Higher Order Brackets

\[
[ I_{x1}, F_{r_2 r_3} ] = \begin{pmatrix} 0 & e^{-i \theta_3} & 0 & 0 \\ e^{i \theta_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{-i \theta_3} \\ 0 & 0 & -e^{i \theta_3} & 0 \end{pmatrix} \ i \sinh 2r_2
\]

\[
[ I_{x1}, F_{r_2 \theta_3} ] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{-i \theta_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{-i \theta_3} \end{pmatrix} \ \frac{1}{2} \sin 2r_2 \sinh 2r_2
\]

\[
[ I_{x2}, F_{r_2 r_3} ] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{-i \theta_3} & 0 & 0 & 0 \\ 0 & e^{i \theta_3} & 0 & 0 \end{pmatrix} \ i \sinh 2r_2
\]

\[
[ I_{x2}, F_{r_2 \theta_3} ] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{i \theta_3} & 0 & 0 & 0 \\ 0 & -e^{i \theta_3} & 0 & 0 \end{pmatrix} \ \frac{1}{2} \sin 2r_2 \sinh 2r_2
\]

\[
[ [ I_{x1}, F_{r_2 r_3} ], I_{x2} ] = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \ i \sin \theta_3 \sinh 2r_2
\]