Dependence of fragmentation in self-gravitating accretion discs on small scale structure

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\textbf{ABSTRACT}

We propose a framework for understanding the fragmentation criterion for self-gravitating discs which, in contrast to studies that emphasise the ‘gravoturbulent’ nature of such discs, instead focuses on the properties of their quasi-regular spiral structures as derived from simulations. This interpretative framework is shown to be consistent with existing 2D and 3D numerical studies as well as with the 2D grid based and SPH simulations conducted here. We propose two evolutionary paths to fragmentation and argue that the correct simulation of each of these involves different numerical requirements: i) collapse on the free-fall time, which requires that the ratio of cooling time to dynamical time ($\beta < 3$) and ii) quasistatic collapse on the cooling time at a rate that is sufficiently fast that fragments are compact enough to withstand disruption when they encounter spiral features in the disc. We argue that the previous finding of Paardekooper et al. (2011) (in which 2D grid based simulations demonstrate a numerically converged fragmentation limit of $\beta < 3$ and which we reproduce here with both 2D grid based and 2D SPH simulations) is a consequence of the fact that such simulations smooth the gravitational force on a scale of $H$, the scale height of the disc. Such simulations thus only allow fragmentation via route i) above since they suppress the quasistatic contraction of fragments on scales $< H$; the inability of fragments to contract to significantly smaller scales then renders them susceptible to disruption at the next spiral arm encounter. On the other hand, 3D simulations (along with 2D simulations that, with questionable realism, smooth gravity on smaller scales) indeed show fragmentation at higher $\beta$ via route ii). We derive an analytic prediction for fragmentation by route ii) based on the requirement that fragments can contract sufficiently to withstand disruption by spiral features, basing this calculation on the properties of spiral structures derived from simulations. We find that this leads to a predicted maximum $\beta$ for fragmentation of $\sim 12$, in good agreement with all previous well resolved 3D simulations. We also discuss the necessary numerical requirements on both grid based and SPH codes if they are to model fragmentation via route ii).

\section{1 INTRODUCTION}

Observations have identified massive planets orbiting at large distances from their host stars (Marois et al. 2008). Such planets cannot have formed at their present location via the core-accretion mechanism and migration after formation at smaller radii is unlikely in some cases (Fabrycky & Murray-Clay 2010). An alternative explanation is that such planets formed near their present locations via direct gravitational collapse (Boss 2000). However, the exact disc properties necessary to form planets via the gravitational instability are still unknown.

The gravitational stability of a disc has typically been understood in terms of the toomre $Q$ parameter (Toomre 1964), given by

$$Q = \frac{c_s \kappa}{\pi G \Sigma}$$

where $c_s$ is the speed of sound, $\Sigma$ is the surface density of the disc and $\kappa$ is the epicyclic frequency ($\Omega$ for a Keplerian disc). Perturbation analysis of an axisymmetric, infinitely thin disc indicates that $Q > 1$ is required for stability.

When $Q \lesssim 1$ the gravitational instability creates spiral waves. The shock heating produced by these spiral waves stabilises the disc by increasing $Q$. This leads to a self-regulated $Q \sim 1$ state in which cooling is balanced by spiral wave heating. However, if the gas can cool efficiently, such a state is not established and over-densities instead collapse into bound objects.

More than a decade of computer simulations have confirmed that discs with $Q \lesssim 1$ are unstable, but can only fragment when cooling is efficient (Gammie 2001; Rice et al. 2003, 2005; Meru & Bate 2011, 2012; Paardekooper et al. 2011).

Because the evolution of $Q$ is driven most rapidly by
changes in temperature (Lodato 2008), much effort has gone into defining the cooling rate necessary for discs to fragment. The simplest approach parametrises the cooling rate via a constant $\beta$ as

$$\frac{du}{dt} = -\frac{u}{\beta M}$$

where $u$ is the internal energy per unit mass.

Early studies found that, for equations of state with $P \sim \rho^\gamma$ and $\gamma = 5/3$, $\beta > \beta_{\text{crit}} \approx 6$ was sufficient to prevent fragmentation (Rice et al. 2005). It was later realised by Meru & Bate (2011) that this result was not numerically converged, implying that fragmentation for slower cooling (higher $\beta$) may be possible. Such non-convergence is unexpected, since even the lowest resolution simulations resolve the disc scale height $H$, which is expected to be the most unstable wavelength.

Several possible explanations of this non-convergence have been proposed. Lodato & Clarke (2011), Meru & Bate (2012) suggested that unwanted artificial viscosity heating (which is resolution dependent) could stabilise discs against fragmentation. Rice et al. (2012) suggested that a modification of the implementation of $\beta$ cooling in SPH could resolve the convergence issue. Paardekooper et al. (2011) found that when starting from commonly used smooth initial conditions, a boundary layer develops between laminar and gravitationally structured flows and can induce fragmentation.

Fundamentally, the non-convergence of the fragmentation boundary can only be explained if the results are corrupted by numerical effects, or if it is physically necessary to resolve length scale small than the Jeans' length. This latter interpretation would be at odds with expectations based on the dispersion relation for gravitationally unstable discs (Section 4.2), which implies that the most unstable wavelength is $\sim H$ and that the required amplitude for triggering collapse increases at smaller and larger spatial scales. Hopkins & Christiansen (2013) suggested that rare high amplitude fluctuations on scale $< H$ can lead to gravitational collapse and used this to argue for a stochastic fragmentation model that could operate even at very large values of $\beta$. Part of the motivation for the present study is thus to examine whether, in high resolution studies (with up to 30 resolution elements across $H$), we see any evidence of this second mode. In Section 4 of this paper we present a range of new simulations examining the fragmentation boundary in 2D using both SPH and FARGO. We find that fragmentation is never initiated on scales $\ll H$ (in contrast to the hypothesis above), but that these scales become important in modelling the contraction of an unstable region to a size where it can survive disruption by spiral shocks. In Section 5 we expand on this argument, providing estimates based on characterisation of the quasi-regular spiral shock structure in the simulations. Finally, in Section 6 we discuss how this relates to the results presented in the literature and its implications for the physical process of fragmentation.

2 NUMERICAL METHODS

2.1 Disc model

We initialise all our simulations using a power law in surface density and temperature and a vertically isothermal Gaussian with scale height $H$. We fix the index of the surface density and choose the temperature scaling so that $Q$ is initially constant. That is,

$$\Sigma = \frac{M_D}{\pi R_i^2} \left( \frac{R}{R_i} \right)^{-p} \frac{\Gamma}{2}$$

$$c_s = \sqrt{\frac{GM}{R_i}} \left( \frac{R}{R_i} \right)^{-p+3/2} \frac{q\Omega_i\Gamma}{2}$$

where $\Sigma$ is the surface density, $M$ is the star’s mass, $M_D$ the disc mass, $R_i$ the inner radius of the disc and $c_s$ is the sound speed of the gas. $q = M_D/M$, $Q_0$ is the initial value of $Q$ in the disc, $p$ is a parameter to be specified and $\Gamma$ is given by,

$$\Gamma^{-1} = \begin{cases} \log(\xi) & \text{if } p = 2 \\ \frac{\log(\xi)}{q(2-p)} & \text{if } p \neq 2 \end{cases}$$

where $\xi = R_o/R_i$ and $R_o$ the is the outer radius of the disc. The disc aspect ratio $H/R$ is then given by

$$H/R = \frac{\Gamma Q_0 \xi^2}{2} \left( \frac{R}{R_i} \right)^{2-p}$$

while the ratio of the resolution scale $h$ to the scale height $H$ is,

$$\frac{h}{H} = \left( \frac{8\pi}{N\gamma q^2 \xi^2 \Gamma^3} \right)^{1/d} \left( \frac{R}{R_i} \right)^{3(p-2)/d}$$

where $d$ is the number of dimensions. Note that the normalisation of $h/H$ will change by a factor of order unity between different types of code.

We have slightly belaboured this point to emphasise the motivation for our parameter choices. The canonical choice in the literature is to choose $p = 1$. Equations 6 and 7 show that this choice results in a radial gradient in both aspect ratio and resolution, which breaks the otherwise scale free nature of the problem. To allow all radii in our simulations to be treated equally, we instead set $p = 2$. We also choose $q = 2$ and $\xi = 5$ to reduce the computational expense of our simulations. Following Paardekooper et al. (2011) we use the equation of state $P = K \Sigma^\gamma$ with the adiabatic exponent $\gamma = 5/3$.

Our SPH simulations were performed using a modified version of the popular code GADGET2 (Springel 2005). The code was modified to include artificial conductivity (Monaghan 1997), $\beta$ cooling, particle accretion and the correct treatment of softening with variable smoothing lengths (Price & Monaghan 2007). We also implemented the improved artificial viscosity method of Cullen & Dehnen (2010), which aims to minimise the amount of artificial viscosity present away from shock fronts, while still resolving shocks correctly. We set the strength of the artificial viscosity (and conductivity) to the values that produce the best results in test problems where the correct result is known (e.g., shock tube test, Sedov blast wave, Kelvin-Helmholtz instability). For artificial conductivity we use $\alpha_{\text{cond}} = 1.0$. For the standard implementation of artificial viscosity, the appropriate values are $\alpha_{\text{SPH}} = 1.0$ and $\beta_{\text{SPH}} = 2.0$ (Monaghan 2005).
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For the artificial viscosity method of Cullen & Dehnen (2010), we use \( \alpha_{\text{max}} = 5.0, \alpha_{\text{min}} = 0.0 \) and \( T = 0.05 \) (Cullen & Dehnen 2010). Note that although \( \alpha_{\text{max}} \) is quite high, in practice the average per-particle value is only \( \alpha_{\text{SPH}} \sim 0.1 \) away from shocks.

Grid based simulations were performed using the FARGO code (Masset 2000). FARGO uses a polar grid, which is logarithmically spaced in the radial direction and linearly spaced in the azimuthal direction. The numbers of radial and azimuthal bins were chosen so that each cell is approximately square (i.e. \( R \Delta \phi \approx \Delta R \)).

All of the simulations in this paper are performed in 2D. This is partially to allow comparison with FARGO, which only operates in 2D with self-gravity, but also to maximise the simulation resolution (\( h/H \)). This is partially to allow comparison with FARGO, which only operates in 2D with self-gravity, but also to maximise the simulation resolution (\( h/H \)), which scales as \( N^{-1/4} \) (\( N \) being the number of dimensions).

After initialising each simulation as described above, we run each simulation for 700\( t_{\text{dyn}} \) at the inner edge (10 outer rotation periods) with \( \beta = 30 \) to allow the disc to settle into the \( Q \sim 1 \) state without fragmenting. We then linearly decrease \( \beta \) from 30 to the desired value over the next 700\( t_{\text{dyn}} \). A simulation is deemed to be “non-fragmenting” if it does not fragment in a further 700\( t_{\text{dyn}} \). This procedure follows Paardekooper et al. (2011) & Clarke et al. (2007) and is designed to prevent fragmentation being induced by a boundary layer between regions of smooth and turbulent flow.

### 2.2 Gravity in 2D

The aim of two dimensional simulations is to capture the physics of the three dimensional system as accurately as possible with a two dimensional representation. The treatment of the gravitational force requires particular care. In 3D, the gravitational potential is given by Poisson’s equation

\[
\nabla^2 \Phi = 4\pi G \rho \nabla^2 \Phi = 4\pi G \Sigma \delta (z - \lambda)
\]

(8)

In 2D, the code does not have access to the volume density and so we must obtain an expression in terms of \( \Sigma \) and other disc quantities. How best to do this depends on the details of the numerical code. FARGO, which calculates the gravitational force by directly solving Poisson’s equation uses the expression

\[
\nabla^2 \Phi = 4\pi G \Sigma \delta (z - \lambda)
\]

(9)

where \( \lambda \) is a softening factor typically set to some multiple of the disc scale height \( H \). If \( \lambda = 0 \) then the code behaves as if \( \lambda = h \) where \( h \) is the grid spacing.

SPH calculates the gravitational force by summing the contribution from each particle (typically with the aid of a tree). To account for the vertical extent of the disc in 2D, we calculate the gravitational force using a gravitational potential given by,

\[
\Phi = -G \sum_{b=1}^{N} m_b \phi (|r - r_b|, h)
\]

(10)

where \( m_b \) is the mass of each particle \( h \) is some softening factor and

\[
\phi = \begin{cases} 1/h_G^2 (\frac{1}{2} q - \frac{1}{2} q^3 + \frac{1}{2} q^4), & 0 \leq q < 1; \\ 1/h_G^2 (\frac{3}{2} q^2 - 3 q^2 + \frac{8}{5} q^3 - \frac{3}{2} q^4 - \frac{1}{15} q^5), & 1 \leq q < 2; \\ \frac{1}{h^2}, & q \geq 2 \end{cases}
\]

(11)

where \( q = r/h_G \). (Price & Monaghan 2007). Adopting this form for \( \phi \), it follows that

\[
\nabla^2 \Phi = 4\pi G \left( \frac{7}{10 h_G^2 \Sigma} \right) \approx 4\pi G \rho \left( \frac{H}{h_G} \right)
\]

(12)

where \( \Sigma \) is estimated using SPH interpolation with kernel length \( h_G \).

If we set \( h_G = h_{\text{SPH}} \) in SPH or \( \lambda = 0 \) in FARGO, the right hand side of Equations (9) and (12) will increase with resolution. That is, unless an explicit gravitational softening is provided in 2D simulations of self-gravitating discs, the gravitational force will increase with resolution, suggesting that the value of the cooling time required to suppress fragmentation should increase with resolution. On the other hand, if the force is softened on the scale \( H \) one would expect simulations to converge because the suppression of gravitational effects on small scales means that these scales play no role in the fragmentation problem. To test this we ran all our simulations twice, once with no softening except on the resolution scale and once with softening on the scale \( H \).

### 3 RESULTS

#### 3.1 SPH simulations

To measure the fragmentation boundary, we performed a series of SPH simulations at different values of \( \beta \) using the gradually settled initial conditions described in Section 2. We repeated this experiment with the gravitational force softened according to \( h_G = H \) and \( h_G = h_{\text{SPH}} \). All simulations were classified as “fragmenting” or “non-fragmenting” as described in Section 2.

Figure 1 shows the results of these simulations with runs softened on \( H \) and \( h \) shown in blue and black respectively. Clearly the amount of gravitational softening used has a dramatic impact on the fragmentation boundary. The
softened simulations are physically incorrect (as they fail to
scales $\sim FARGO$), the fragments form from structures with length
simulations at high resolution.

$\beta$ converge and fragment at similar values of $\beta$, we turn off gravitational softening the simulations do not
and the results of our 2D SPH simulations. Once again, when
fragmentation is likely, we investigate the evolution of grav-
itionally unstable over-densities in a $Q \sim 1$ disc using
simplified models for the self-regulated disc structure.

3.2 FARGO simulations

To better understand the non-convergence of the SPH simu-
ations and to verify that the SPH results are not an artefact
of numerical technique, we ran the same simulations using
the grid code FARGO. Once again, we softened the gravita-
tional force on either the resolution scale of the simulation
or $H$. The simulations were initialised and classified into
"fragmenting" and "non-fragmenting" in the same way as
the SPH simulations.

The results are shown in Figure 2. The runs with gravity
softened on $H$ are shown in blue while those without any
extra softening are shown in black. The simulations which
are softened on the scale of $H$ are numerically converged to
$\beta_{crit} \approx 3$, in good agreement with previous 2D simulations
and the results of our 2D SPH simulations. Once again, when
we turn off gravitational softening the simulations do not
converge and fragment at similar values of $\beta$ to the SPH
simulations at high resolution.

For those simulations that fragment (in both SPH and
FARGO), the fragments form from structures with length
scales $\sim H$. This remains true even for the unsoftened sim-
ulations that do not converge. That is, the non-convergence
of our simulations without gravitational softening is not a
consequence of instabilities triggered on scales significantly
smaller than $H$, but of how resolution affects the evolution
of regions of size $\sim H$ once they become unstable.

4 COLLAPSE CRITERIA

The only difference between the converging simulations softened on $H$ and the non-converging simulations softened on $h$ in Section 3 is on scales smaller than $H$. Although un-
softened simulations are physically incorrect (as they fail to
account for the disc’s 3D structure), it is at first sight sur-
prising that processes on scales much less than the Jeans’
length ($H$ for a $Q = 1$ disc) should affect the fragmentation
process.

For a disc to fragment, gravitationally unstable over-
densities must be able to collapse into gravitationally bound
objects that can resist disruption (Kratter & Murray-Clay
2011) & Paardekooper et al. 2012. To determine the role of
scales smaller than $H$ in fragmentation and estimate when
fragmentation is likely, we investigate the evolution of grav-
ationally unstable over-densities in a $Q \sim 1$ disc using
simplified models for the self-regulated disc structure.

4.1 Free fall collapse

The collapse of an over-density into a bound fragment can
happen in two different ways. If the cooling rate is fast
enough, any extra heat generated by compression of the collaps-
ing over-density is radiated away. In this regime, the over-
density can never become pressure supported and so collapses in free-fall. This will be the case whenever the cool-
ing time is less than the free-fall collapse time.

For a self-gravitating disc, the free-fall collapse time is
given by Kratter & Murray-Clay (2011)
\[
t_{ff} = \frac{1}{\sqrt{G \rho}} = \sqrt{2\pi Q P_{dyn}} \tag{13}
\]

Therefore, an over density will undergo free-fall collapse
whenever $t_{cool} < t_{ff}$, which is the case whenever
$\beta < \sqrt{2\pi Q} \approx 3 \tag{14}$

In this regime, the collapse time $t_{ff} \ll t_{spi}$ (the time
scale on which condensations encounter the next spiral
shock; see below) for all reasonable values of $Q$. Disc frag-
mentation is therefore inevitable and a self-regulated steady-
state cannot form if over-densities collapse in free-fall.

4.2 Pressure supported collapse

If $\beta > 3$ then gravitationally unstable over-densities become
pressure supported before they can completely collapse. Fur-
ther collapse can only occur when the extra heat generated by
collapse is removed, which happens on the cooling time
scale. Such pressure supported clumps will only survive if
they can collapse before being disrupted by their environ-
ment.

Although the $Q \sim 1$ quasi-equilibrium is constantly
changing, it is still possible to define the average geometric
properties of the spiral waves that are created by the
gravitational instability (Cossins et al. 2009). In particular,
we describe the spiral waves using an azimuthal wavenumber
$m$ and a radial wavenumber $k$, which may vary as a function
of $R$.

Let us now consider a patch of disc that has become
gravitationally unstable. On average, the longest it will sur-
vive for before encountering a spiral shock is
\[
t_{spi} = \frac{2\pi}{m |\Omega - \Omega_p|} = \frac{2\pi}{m \xi} t_{dyn} \tag{15}
\]
where $\xi = |\Omega - \Omega_p|/\Omega$ and $\Omega_p$ is the pattern speed of the
spiral arms.
The core of our argument is that the survival of overdensities (and hence the ability of a self-gravitating disc to progress unstable over-densities to gravitationally bound fragments) depends on an over-density’s ability to collapse before being disrupted by encountering a spiral shock. To first order, collapse of an over-density takes place on the cooling time scale $t_{\text{cool}} = \beta_{\text{diss}}$ and over-densities live for at most $t_{\text{spi}}$ before encountering a spiral wave. Therefore, collapse and fragmentation requires that $t_{\text{cool}} < t_{\text{spi}}$, which is true if

$$\beta < \frac{2\pi}{m \xi}$$

Provided that the tight-winding assumption holds ($|kR| \gg m$), it must be the case that

$$M = \frac{m \xi}{kH}$$

where $M$ is the Mach number of the spiral shock.

Substituting Equation 17 into Equation 16 implies that

$$\beta_{\text{crit}} = \frac{2\pi}{m \xi} = \frac{2\pi}{MkH}$$

Given a dispersion relation, it is possible to re-write $kH$ in terms of the Toomre parameter $Q$ and the Mach number $M$. If we assume an infinitely thin, tightly wound disc, the standard dispersion relation can be re-written as (Binney & Tremaine 2008),

$$m^2 \xi^2 = H^2 k^2 - \frac{2H|k|}{Q} + 1$$

substituting in Equation 17 and solving for $kH$ gives

$$kH = -1 + \sqrt{1 + \frac{Q^2 (M^2 - 1)}{Q(M^2 - 1)}} \approx \frac{Q}{2}$$

where the last equality holds when $M - 1 \ll 1$.

Substituting Equation 20 into Equation 18 gives

$$\beta_{\text{crit}} = \frac{4\pi}{MQ}$$

where $Q$ is to be determined numerically.

Although it may seem that $Q$ is unconstrained in this model, it is actually equivalent to setting the most unstable wavelength. For example, if the most unstable wavelength is similar to that of the axisymmetric disc (where $kH = 1/Q$), then it must be the case that $Q = \sqrt{2}$. Numerical simulations typically find that $1 < Q < \sqrt{2}$. Together with the expectation that $M \approx 1$, this leads us to predict that $\beta_{\text{crit}} \approx 9 - 13$ for well resolved simulations.

Equations 18 & 21 are important as they provide a simple analytic estimate of $\beta_{\text{crit}}$ (the first, to our knowledge). That said, the arguments made in reaching this estimate are all “first-order” and so we expect Equation 21 to be accurate only up to a factor of order unity. Furthermore, Equation 19 assumes an infinitely thin disc. If we use a dispersion relation that accounts for disc thickness (see Equation 31 of Cossins et al. 2009), the resulting estimate of $\beta_{\text{crit}}$ is reduced slightly. More explicitly,

$$\beta_{\text{crit}} = \frac{4\pi}{MQ} \left(1 - \frac{Q}{2}\right)$$

and the most unstable wavelength changes so that we expect $Q = 1$.

### 4.3 Resolution requirements

To understand the resolution requirements of resolving the collapse process described in the previous section, we need to know how “collapsed” a clump needs to be in order to survive a collision with a spiral arm. The average amount of energy per unit mass lost to cooling between spiral wave encounters is roughly $ut_{\text{spi}}/t_{\text{cool}}$. If shock heating balances cooling (as it must in the $Q \sim 1$ state), material will gain

$$\Delta u = ut_{\text{spi}}/t_{\text{cool}} = \frac{2\pi}{m \xi}$$

in internal energy per encounter with a spiral.

In order for a clump to survive a collision with a spiral arm, it must still be smaller than its initial size, once this extra $\Delta u$ energy has been deposited. That is, collapse begins with a patch of size $\sim H$, which then contracts until it is hit by a spiral wave. This spiral wave deposits energy in the clump, causing it to expand. If the expanded size of the post-shock clump is greater than the size when collapse began, the clump will no longer be unstable and will not survive.

The binding energy of the clump with size $x$, per unit mass, is given by,

$$e_{\text{bind}} = \frac{2GMu}{3x}$$

and so the post-shock clump size, $x_s$, is given by,

$$\frac{2GMu}{3x_s} = \frac{2GMu}{3x} - \frac{2\pi u}{m \xi \beta}$$

If we require that $x_s < H$, this implies that

$$\frac{x}{H} < \left(1 + \frac{3\pi Q}{m \xi (\gamma - 1) \beta}\right)^{-1}$$

For reasonable values of $\gamma = 5/3$, $Q = 1$ and $\beta = 6$, this yields the requirement that $x/H < .25$. Furthermore, if we derive a simple expression for $x(t)$ (see Kratter & Murray-Clay 2011 for example), substitute $x(t_{\text{spi}})$ for $x$, using Equation 15 and solve for the $\beta$ (for the same values of $\gamma$ and $Q$), we find that fragmentation occurs when

$$\beta < \frac{2\pi}{m \xi}$$

a requirement that is remarkably similar to Equation 18 which we obtained from simple time scale arguments. The fact that Equations 27 and 18 are so similar means that requiring $t_{\text{cool}} < t_{\text{spi}}$ ensures that the clump has contracted sufficiently to satisfy Equation 25 and hence survive.

However, the derivation of Equation 20 has required a number of simplifying assumptions regarding the non-linear evolution of the unstable over-density. The true utility of Equation 20 is not in providing a more precise constraint on $\beta_{\text{crit}}$ than Equation 18 (both are subject to an uncertainty of order a factor of 2), but in understanding what physical scales and processes need to be resolved in order to accurately model the non-linear evolution of the clump and the formation of fragments.

More specifically, we find that to survive a collision with a spiral wave, a fragment must have $x/H < 1$. The exact value of $x/H$ required depends on $\beta$ and like all other arguments in this section is uncertain to within a factor of order unity. Nevertheless, this reasoning shows why resolving scales smaller than $H$ is necessary to accurately model
the physics of disc fragmentation. It is necessary to resolve the physical scale of the initial instability and the scale to which that instability must shrink in order to survive collisions with spiral waves.

5 DISCUSSION

Most simulations that have investigated the fragmentation boundary in 2D have found a numerically converged fragmentation boundary of $\beta_{\text{crit}} \approx 3$ (Gammie 2001, Paardekooper et al. 2011). These 2D simulations softened their gravitational force on $\sim H$ similar to our simulations in Section 3, which also found $\beta_{\text{crit}} \approx 3$. The only major exception to this is Meru & Bate (2012), who used a very weak gravitational softening of $3 \times 10^{-3} H$ and obtained a much higher fragmentation boundary and much weaker numerical convergence. The fragmentation boundary also appears to be independent of $\gamma$ in softened 2D simulations, with simulations using $\gamma = 2$ (Gammie 2001) and $\gamma = 5/3$ (Paardekooper et al. 2011), Section 3 all yielding $\beta_{\text{crit}} \approx 3$.

In contrast to this, all 3D simulations performed to date fragment at well above $\beta = 3$ (Rice et al. 2005, 2012, 2014, Cossins et al. 2009, Meru & Bate 2011, 2012, Rice et al. 2014) and have been convincingly shown to depend on $\gamma$ (Rice et al. 2005).

A consequence of softening gravity on the scale $H$ is that the force between mass elements separated by less than $H$ goes to zero (to first order). This suppression of the gravitational force (which uses a Fourier transform of Poisson’s equation) all yielding $\beta_{\text{crit}} \approx 3$.

3D simulations also permit fragmentation via pressure supported collapse, which does depend on $\gamma$ (see Equation 26). It is possible that a more sophisticated model for approximating the gravity in two dimensions, such as the technique proposed by Müller et al. (2012), would reduce the fragmentation suppressing effects of gravitational softening. However, it is inevitable that some differences with the full 3D solution to Poisson’s equation will remain. In particular, models that assume a vertical structure for the disc are likely to be least accurate at modelling the quasi-spherical over-densities that must be modelled for fragmentation. As some gravitational softening is necessary to approximate the three dimensional force in two dimensions, any two dimensional simulation must either fail to model the gravitational force 100% correctly or suppress fragmentation via the collapse of pressure supported clumps.

Three dimensional simulations do not suffer from this limitation and so are able to model both pressure supported and free-fall collapse. However, there remains some disagreement as to the converged value of the fragmentation boundary in 3D (Meru & Bate 2011, 2012, Paardekooper et al. 2011, Lodato & Clarke 2011, Rice et al. 2012, 2014). The model for pressure supported collapse developed in Section 4 estimates $\beta_{\text{crit}} \sim 12$. Given that we do not expect this estimate to be more accurate than a factor of 2, this does not allow us to discriminate between the $\beta_{\text{crit}} \sim 8$ favoured by Rice et al. (2014) and the $\beta_{\text{crit}} = 20 - 30$ of Meru & Bate (2012). However, we argue in Section 5.2 that the quasi-regular nature of the spiral structure in self-gravitating discs lead us to expect that fragmentation should not be possible at value of $\beta$ significantly higher than the latter range ($\beta_{\text{crit}} = 20 - 30$), even allowing for the possibility of stochastic fragmentation.

5.1 Numerical technique specific issues

A fixed grid, such as the one used by FARGO, is only able to follow a contracting clump down to the resolution scale of the simulation. Given this, care must be taken that the resolution scale is much smaller than the smallest scale of physical importance. We have shown in Section 4 that this length scale can be significantly smaller than the disc scale height $H$. Failure to resolve small enough scales will suppress

\[ \nabla^2 \Phi = 4\pi G(\Sigma/h) \] (28)

$\nabla^2$ does not use an explicit gravitational softening, but is forced to omit small wavelength modes in his calculation of the gravitational force (which uses a Fourier transform of Poisson’s equation). Discarding these modes is equivalent to a gravitational softening on the scale $\sim 0.3H$. Where $h \sim N^{-1/2}$, that is, the gravitational force will continue to increase with resolution. This increased gravitational pull will make fragmentation easier as the resolution increases, as is demonstrated in Figures 1 and 2. Even if a converged value of the fragmentation boundary can be reached with gravity softened on the resolution scale, its physical interpretation is unclear. [Müller et al. 2012]. Rice et al. (2005) showed that the fragmentation boundary varied as $(\gamma(\gamma - 1))^{-1}$ in 3D simulations. However, the softened 2D simulations of Gammie (2001), Paardekooper et al. (2011) and those performed here all agree on the fragmentation boundary, despite using significantly different values of $\gamma$. This finding is also consistent with different types of simulations (softened 2D, realistic 3D), probing different modes of fragmentation (free-fall collapse, pressure supported collapse). Softened 2D simulations are only able to probe fragmentation by free-fall collapse, which does not directly depend on $\gamma$ (since the free-fall time is independent of $\gamma$). 3D simulations also permit fragmentation via pressure supported collapse, which does depend on $\gamma$ (see Equation 26).
fragmentation by preventing collapse from proceeding to the point where it can resist disruption by interaction with spiral arms.

By contrast, the adaptive nature of SPH ensures that so long as the resolution is high enough to resolve the initial instability \((h/H < 1)\), it will continue to resolve the clump as it contracts. However, modelling the collapse of pressure supported clumps is challenging in SPH because of the effect of particle noise. Any random “thermal” motions acquired by SPH particles will supply an additional pressure that will prevent collapse. Unlike pressure derived from internal energy, this pressure due to particle noise can not be removed by cooling.

Failure to provide adequate artificial viscosity at shock fronts is a well known source of particle noise in SPH. Strong shear, as is present in differentially rotating discs, can also lead to particle noise. Consistent with this interpretation, Meru & Bate (2012) found that lowering the strength of the artificial viscosity creates particle noise and inhibits fragmentation. It is also worth noting that there is abundant evidence from tests with analytic solutions that the suppression of particle noise at shocks also requires a rather large value of \(\alpha_{SPH} > 0.7\) (Lattanzio et al. 1986; Lombardi et al. 1999; Thacker et al. 2000). This is significantly larger than the value of \(\alpha_{SPH} = 0.1\) this is commonly employed in disc fragmentation calculations (Rice et al. 2005; Cossins et al. 2009; Meru & Bate 2011; Forgan et al. 2011; Rice et al. 2014). Unfortunately, such a high value of \(\alpha_{SPH}\) produces significant heating away from shock fronts unless the resolution is very high and this can suppress fragmentation, as has been discussed extensively in the literature (Meru & Bate 2011; Lodato & Clarke 2011; Rice et al. 2014).

Therefore, special care must be taken to ensure that shock-fronts are well modelled and particle noise suppressed in order to accurately model fragmentation. Unfortunately, doing so greatly increases the number of particles required to prevent excessive artificial viscosity heating away from shock-fronts (although this computational cost can be reduced by using modern SPH techniques such as improved artificial viscosity triggers (Cullen & Dehnen 2010) and kernels with increased neighbour number (Dehnen & Aly 2012)).

5.2 Stochastic fragmentation

It was shown by Paardekooper (2012) that even when simulations settle into the \(Q \sim 1\) state without fragmenting immediately, they can still fragment stochastically over the lifetime of the disc. Paardekooper (2012) argued that this occurred when a clump “got lucky” and survived for long enough to resist disruption and form a fragment. This process can also be understood within the framework of quasi-regular, but intermittent, spiral structure.

The expression which relates the geometry of the spiral waves to the fragmentation boundary, \(m_\xi\), describes the average geometry of the disc. The spiral arms present at any particular time will fluctuate about this average geometry. When \(\beta > \beta_{crit}\), the average time between spiral wave encounters is shorter than the collapse time of an over-density. As such, the disc is resistant to fragmentation on average. However, if the gravitational instability happens to be triggered in a part of the disc with the right random fluctuations in spiral geometry, it can survive longer than would be expected from the mean geometry and collapse enough to fragment.

Precise numerical studies are required to quantify the exact likelihood that this random process will lead to fragmentation, which is beyond the scope of the present study.

5.3 General implications

This paper has argued that two dimensional simulations cannot be trusted to model the fragmentation process correctly. Nonetheless, we are now in a position to provide some important upper limits on when fragmentation can occur.

We argue that fragmentation is impossible if the initial instability cannot contract significantly before it encounters a spiral arm. Since collapse occurs on the cooling time scale, this occurs whenever

\[
 t_{cool} \geq \frac{2\pi}{m_\xi} t_{dyn} \tag{29}
\]

Furthermore, \(m_\xi\) can be related to \(Q\) and the Mach number via the dispersion relation to give,

\[
 t_{cool} \geq \frac{4\pi}{MQ} t_{dyn} \approx 12 t_{dyn} \tag{30}
\]

Equation (30) is uncertain to within a factor of a few. Nevertheless, we can confidently say that our model predicts that fragmentation will not occur whenever \(\beta\) is greater than a few tens.

There is some possibility that stochastic fragmentation can lead to fragmentation at higher cooling times. Over the self-gravitating lifetime of a disc, this may shift the “effective fragmentation boundary” a factor of two or so higher (although detailed numerical studies are needed to confirm this).

The opacity regime in the outer parts of protoplanetary discs leads to \(\beta \propto R^{-9/2}\) (Clarke 2009; Cossins et al. 2010; Paardekooper 2012). Since \(\beta\) is such a strong function of radius, a factor of a few uncertainty in \(\beta_{crit}\) makes little practical difference to planet formation theory. That is, the upper bounds provided by Equation (30) already place the allowable radii for fragmentation with sufficient precision.

6 CONCLUSIONS

In this paper we have presented a new interpretation of the fragmentation process in discs, which emphasises the role of the quasi-regular spiral structure. In this interpretation, a disc can only fragment when an over-density is able to contract to a size small enough to survive being disrupted when it encounters a spiral arm. We show that this requirement means that simulations of fragmenting discs need to resolve both the initial instability at scale \(H\) and the subsequent collapse of the resulting over-densities to a size \(x \ll H\).

We have further shown that the fragmentation boundary obtained in 2D simulations depends strongly on the type of gravitational softening employed. We have found that although gravitational softening on the scale \(H\) is necessary to account for the vertical structure of the disc and obtain a converged fragmentation boundary, it also suppresses fragmentation by preventing the collapse of pressure supported clumps. As such, it is perhaps unlikely that any 2D model...
of a self-gravitating disc will be able to capture all the processes necessary to model disc fragmentation in its entirety.

The necessity of modelling the contraction of a clump from instability through to disruption resistant fragment also leads to additional numerical requirements for simulations of fragmenting discs. For codes employing a fixed-grid, it imposes the requirement that scales \( x \ll H \) be resolved so that clumps can collapse enough to resist disruption by spiral waves. For SPH calculations, where this resolution requirement is automatically satisfied if \( H \) is well resolved, it leads to the requirement that particle jitter be eliminated. To achieve this, the artificial viscosity must be calibrated to produce the correct results in test problems where the answer is known. Additionally, the widely discussed requirement that artificial viscosity heating be small compared to the imposed cooling must also be satisfied.

Unfortunately, satisfying all these requirements (3D simulations, no particle-jitter, minimal artificial viscosity heating) requires a very high particle number. Even for the parameters used in this paper, which were chosen to maximise computational efficiency, achieving an artificial viscosity heating less than 10% of the cooling rate when \( \beta = 20 \) with the standard fixed artificial viscosity method (\( \alpha_{SPH} = 1.0 \)) requires of order 400 million particles in 3D.

Despite these severe numerical limitations, the time scale arguments presented in this paper allow us to predict that \( \beta_{crit} \) can be at most a few tens. Given the strong scaling of \( \beta \) with radius in realistic protoplanetary discs, a more precise constraint on \( \beta_{crit} \) is unlikely to be of physical importance for disc fragmentation as a mechanism for giant planet formation.

7 MATERIALS & METHODS

In the interests of reproducibility and transparency, all code and data used in performing this work have been made freely available online at https://bitbucket.org/constantAmateur/discfragmentation

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