Damage detection using modal frequency curve and squared residual wavelet coefficients-based damage indicator

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A B S T R A C T

A theoretical and experimental study of the frequency-based damage detection method has been presented in this paper. Based on the eigenvalue problem and perturbation assumption of defect in modal response, the theoretical basis of the modal frequency curve method is established. The extraction of defect characteristics from the modal frequency curve via discrete wavelet transform is illustrated. The above background leads to the development of a new multiple-mode damage indicator for damage localisation and a damage estimator for size prediction. Then, the proposed method has been applied to aluminium samples with pre-defined damage sections. Finite element modelling and experimental testing results are presented to demonstrate the performance of the method. Additionally, detectability with respect to the various mass ratios is investigated to support the ability of the method in real applications. The numerical and experimental results suggest that the use of the damage indicator provides a more robust and unambiguous damage identification than the sole use of the wavelet coefficients of the modes investigated. In addition, the damage estimator predicts the defect size to a satisfactory level.

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1. Introduction

There has been an increasing development and use of light, stiff, multi-layer and multi-functional components and structures in modern engineering designs [1]. In order to ensure the state of structural health of these components and structures, suitable inspection and evaluation approaches are necessary. Several traditional inspection methods (e.g. visual inspection, magnetic particle inspection) for surface defect detection, such as cracks and notches, are not available for internal defect in multi-layer structures. Despite the relative high cost of operation of other non-destructive detection methods, such as ultrasonic inspection and thermography, there are also limitations due to specimen surface types and qualities. As a cost-effective direct structural integrity assessment method, vibration-based structural damage detection methods have received increasing attention for real applications [2].

Several researchers have studied the effect of frequency change of a structure in the damaged state with respect to the intact state. Earlier works on the frequency change approach for beam-like structures have been reported in [3–6]. Gadelrab [7] studied the frequency change due to damage in a two-layer laminate beam using finite element method. He showed that frequency shift depends on the defect locations and boundary conditions. Kessler et al. [8] showed that the reduction of the...
frequency is generally proportional to the reduction of global stiffness due to interface defect in structures at lower vibration modes. Several numerical modelling techniques of damaged structures have been reviewed by Della and Shu [9]. They stated that the effect of natural frequency change is generally found to be a function of the size and locations of the defects. Alnafaie [10] showed that the defect-induced variation of mode shape is more distinct for higher flexural vibration modes. Salawu [11] reviewed the numerous literatures available on damage identification based on the shifting of natural frequencies. The review showed the feasibility of using accurately measured modal frequency data in damage detection and condition monitoring for engineering structures. Comprehensive reviews of the structural health monitoring approaches using vibration-based methods are also reported in [12]. Although the modal frequencies of structures can be precisely measured with cost-effective arrangement of sensors (using few sensors) in real applications, the change of the modal frequency may not provide enough information for defect locations compared with mode shapes, as stated by Yam et al. [13,14].

To improve the localisation of the damage using frequencies, a modified frequency-based damage detection method has been proposed by Zhong and Oyadiji [15,16]. The method uses the variations of the modal frequency data as a roving mass is traversed to various locations on a beam in order to detect and localise cracks in beam-like structures through vibration testing. The concept of using the modal frequency curves and its high-order spatial derivatives for crack detection has been demonstrated using measured frequency data. Recently, the above frequency-based approach has been studied by Zhang et al [17,18] on a damaged plate and cylinder by calculating the residual values from the 2D gapped smoothing method (GSM). The above studies have been focused on the application of the method and revealed the applicability in practice. However, the theoretical basis of the frequency curve (or surface) concept in damage detection is not well established. Especially, the assumption of unchanged mode shape under applied concentrated mass loading in [15,17] is only valid for sufficiently small mass ratio condition. Moreover, the previous works based on the numerical differentiation or gapped smoothing approaches are known to be less sensitive compared to the wavelet approaches. The connection of the frequency curve method with wavelet transform is illustrated in the present paper.

In the past few years, wavelet-based approaches have attracted increasing attention for vibration-based damage detection methods. Hong et al. [19] investigated damage detection based on Mexican hat wavelet coefficients of the fundamental mode shape that were derived using continuous wavelet transform (CWT) and discrete wavelet transform (DWT). They suggested that the number of vanishing moments of wavelets in crack detection should be at least two. However, it is well known that wavelets with lower vanishing moments provide less accurate results in terms of the location of the damage [20]. Chang and Chen [20] used DWT to derive Gabor wavelet coefficients of mode shape data for crack detection and localisation in a beam-like structure. The Gabor wavelet coefficients of the mode shapes are used as the location indicator and the frequency reduction used as the depth estimator of the crack. Nevertheless, the Gabor wavelet coefficients manifested the influences of the boundary which required further smoothing procedure to remove the boundary effects. Doula et al. [21] experimentally studied the symmetrical wavelet sym4 in damage detection of beam-like structures using CWT. They introduced a method to suppress experimental noise in measured mode shapes based on the knowledge of the noise level. Rucka and Wilde [22] proposed a 2D DWT method with biorthogonal filter bior5.5 in damage detection for clamped beam-like and plate-like structures using the fundamental mode shape. The performance of the biorthogonal wavelet in processing the measured mode shape data was validated experimentally. The main limit of the mode shape method is the number of sensors required to capture the data. Qiao et al. [23] reported the solution to this issue using the Laser Doppler Vibrometer in displacement shape measurement for laminated composite plate. Nevertheless, the high expense of the measurement device limits its general engineering application in reality. Zhong and Oyadiji [24] studied the stationary wavelet transform (SWT) method for crack detection in beam-like structures using measured and reconstructed mode shape data. Their approach determines the difference between the SWT of two sets of mode shape data of beam-like structures with simply-supported boundary conditions. They showed that the use of SWT, which is an up-sampling procedure, gives more crack information than the use of DWT, which is a down-sampling procedure. However, the application of their proposed method is restricted to structures with symmetrical boundary conditions.

A comparison study of several signal processing approaches of mode shape-based damage detection method for a plate-like structure has been reported by Fan and Qiao [25]. Based on the numerical and experimental results for damage with different shapes, the authors pointed out the robust performance of the 2D CWT approach with Mexican hat wavelet compared to such methods as 2D GSM and 2D SEM (strain energy method). Recently, Gökdag and Kopmaz [26] reported the use of DWT and CWT approach for crack detection in beam-like structures. The method uses DWT approximation coefficient to reconstruct the intact mode shape, which is regarded as the baseline. It then estimates the damage-induced mode shape difference by CWT coefficients. However, the assumption that the DWT approximation coefficient of the damaged data is equivalent to the baseline data may not be generally valid. Recently, Katunin [27] compared the use of 2D B-spline wavelets with respect to other wavelet in the literature under 2D DWT process. The author showed that the 2D B-spline wavelet gave a less noisy performance compared with other wavelets investigated. However, the 2D DWT method does not produce comprehensive damage information in one coefficient surface. It usually needs additional steps to overlay the directional images (coefficients) to obtain the final results. In general, the DWT method produces satisfactory results for damage location for beam-like structures with less computation time compared to CWT. For plate-like structures, the 2D DWT methods require additional steps in image composition. Therefore, it is not as direct as the 2D CWT methods. It is also worth to note that, several previous methods rely on the wavelet coefficients of fundamental mode shape. However, it is well-established that a damage close to a nodal line has less effect on the particular mode. Therefore, the use of single mode data

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may not be sufficient to detect the damage near a nodal line. To overcome this limitation, in this paper a multiple-mode wavelet-index is developed based on biharmonic wavelet using DWT.

The literature has also reported damage detection methods which do not use the local structural information (e.g. mode shapes or frequency curve). Wei et al. [28] reported an energy spectrum method for estimating the modal property deviation with respect to the size of internal damage in laminated plate structures through wavelet analysis. Instead of working in the frequency domain, their results showed the additional energy dissipation caused by damage that can be used as a damage indicator by applying wavelet transform of the time signals. Hein and Feklistova [29], recently, proposed a computational efficient online damage detection method based on machine learning. They applied artificial neural network (ANN) to establish the mapping relationship between frequencies, Haar wavelet expansion of fundamental mode shapes of vibrating beams and delamination status based on numerical models. Zhang et al. [30] examined three different inverse algorithms for solving the non-linear equations to predict the interface, lengthwise location and size of delamination using a graphical method, ANN and surrogate-based optimization. More detailed references on the applications of wavelet transforms and machine learning methods in structural damage detection can be found in [31]. The methods based on machine learning and inverse algorithms have the advantage of less processing time compared to the methods that require the knowledge of mode shapes or frequency curves. Nevertheless, the methods that employ local structural information do not require baseline information.

In this paper, the theoretical basis of modal frequency curve detection method is firstly established based on the concept of eigenvalue problem. The modification of the structural responses due to mass loading are taken into account. From the literature review, it was found that the fastest wavelet processing algorithm is DWT, which requires relatively few scales with down-sampling. Although SWT has a similar scale to DWT, it is slower than DWT because it involves up-sampling. The slowest wavelet processing algorithm is CWT which involves many continuous scales. A multiple-mode wavelet-index is developed based on biorthogonal wavelet using DWT of the modal frequency curves. DWT was selected for this work for fast multi-mode processing of the modal frequency curves data. The reason for the choice of the biorthogonal wavelet is described in detail in Section 2.2. The feasibility of damage detection using the wavelet coefficients of the frequency curves is illustrated. To enhance the noise-robustness of the approach, a multiple-mode index based on the squared residual of the mean wavelet coefficients along spatial locations is proposed to locate the defect. A corresponding damage estimator is developed based on the damage indicator to evaluate the size of the defect. Three two-layer bonded aluminium beam samples with damaged sections at different longitudinal locations are studied. Finite element models are employed to simulate the modal frequency data of the damaged samples to obtain the frequency curves. The three numerical cases are also validated through experimental testing based on the proposed damage indicator and estimator. The sensitivity of the proposed method with respect to the ratio of the additional mass to the mass of the structure is investigated. The effect of the wavelet filter selection is also discussed. The frequency curve method used in this paper is based on the modal frequencies which can be determined experimentally using few sensors in applications. This is a major advantage over mode shape methods which require an array of sensors to determine the mode shapes accurately.

2. Background

Section 2.1 focuses on the establishment of the theoretical basis of the modal frequency curve based on the concept of eigenvalue problem. The use of wavelet analysis for damage detection based on frequency curve is illustrated in Section 2.2. The proposed damage indicator and estimator are presented in Section 2.3.

2.1. Theoretical basis of the modal frequency curve

In the finite element method, the modal parameters of the structure are obtained by solving the eigenvalue problems in matrix form. The eigenvalue problem of a free vibrating structure without extra point-mass loading can be expressed as,

\[-\omega^2 \mathbf{m} V + \mathbf{k} V = 0\]

where \( \mathbf{m} \) and \( \mathbf{k} \) are the mass and stiffness matrices, \( \mathbf{V} = [V_1, V_2, \ldots, V_n] \), is the eigenvector, \( V_i = V(x_i) \) is the normalised displacement response at location \( x_i \), \( \omega \) is angular frequency and the modal matrix is given by \( \mathbf{V} = [V_1, V_2, \ldots, V_n] \). The stiffness and mass matrices can be transformed into the modal space to give the modal mass and modal stiffness matrices as,

\[
\begin{bmatrix}
\mathbf{M} \\
\mathbf{K}
\end{bmatrix} = \mathbf{V}^T \begin{bmatrix}
\mathbf{m} & \mathbf{k} \\
\mathbf{k}^T & \mathbf{V}
\end{bmatrix} \mathbf{V}
\]

When a point-like mass \( m_0 \) is attached on the structure at location \( x \), the modified mass matrix of the structure becomes \( \mathbf{m}' \), the modal responses of the structure also changes from the original eigenvector matrix \( \mathbf{V} \) to a modified eigenvector matrix \( \mathbf{V}' \), which depends on both the mass magnitude \( m_0 \) and location \( x \). The modified modal stiffness matrix \( \mathbf{K}' \) and the modified modal mass matrix \( \mathbf{M}' \) for the modified structure become,
\[
\hat{M} = [\hat{V}]^T [\hat{m}] [\hat{V}]
\]
\[
\hat{K} = [\hat{V}]^T [\hat{k}] [\hat{V}]
\]

The modified modal mass matrix \([\hat{M}]\) can be represented as the superposition of the unmodified modal mass matrix \([M]\) and the modal point mass matrix \([M_0]\), that is
\[
[\hat{M}] = [M] + [M_0]
\]
\[
[M_0] = [\hat{V}]^T [m_0] [\hat{V}]
\]

For the \(n\)th vibration mode, the uncoupled equation of motion from Eq. (1) can be written as,
\[
-\omega_n^2 m_n + K_n = 0
\]
where
\[
M_n = M_n + M_0 + \left( [V]^T [m_0] [V] \right) = M_n + m_0 V_n^2 (x)
\]

\(\bar{K}_n\) and \(\bar{M}_n\) are the modified modal mass and modal stiffness of the structure for the mode \(n\) and the scalar term \(V_n^2 (x)\) is the square of the modal constant extracted from the modified mode shape vector \(\hat{V}_n (x)\) of mode \(n\) at location \(x\). The variable \(M_0\) shows the contribution of the point mass attached at the location \(x\) to the mode \(n\). The corresponding modal angular frequency \(\omega_n\) is then derived for the uncoupled single degree of freedom system,
\[
\omega_n (V_n (x), m_0) = \sqrt{\frac{\bar{K}_n}{\bar{M}_n + m_0 V_n^2 (x)}}
\]

From Eq. (6), the modal frequency is the function of both mode shape and the extra mass. The variation of the modal frequency with respect to location \(x\) can be determined by differentiation of both sides of Eq. (6) with respect to \(x\) which yields,
\[
\frac{\partial \omega_n}{\partial x} = \frac{-2 \omega_n m_0 V_n (x)}{M_n + m_0 V_n^2 (x)} \frac{\partial V_n (x)}{\partial x}
\]

As can be seen from Eq. (7), the first order partial derivative is a polynomial function which contains the same order of derivatives as the modified mode shape. Further, the minus sign in Eq. (7) shows that the modal frequency decreases as the magnitude of the mode shape increases. It is worth to note that, the concentrated mass loading term in the numerator serves as a magnifier of the change of modal frequency due to the change of the mode shape of the modified structure. For higher-order frequency derivatives, the expression is generally a polynomial function with different orders of mode shape derivatives where the modified mode shape with highest order derivatives equal to the order of frequency derivatives, that is,
\[
\frac{\partial^k \omega_n}{\partial x^k} = P_1 \frac{\partial^k V_n (x)}{\partial x^k} + P_2 \frac{\partial^{k-1} V_n (x)}{\partial x^{k-1}} + \ldots + P_k \frac{\partial V_n (x)}{\partial x} + \ldots + P_m \left( M_n, m_0, \omega_n, V_n (x) \right)
\]

By the general definition of smoothness, the above results indicate that the modal frequency has the same order of smoothness as the modified mode shape. Furthermore, it should be noted that the mode shapes of the original and the modified structures are different orthogonal basis vectors for the same vector space. The modified mode shape vector for the \(n\)th vibration mode \(\hat{V}_n (x)\) can be expressed by the expansion of the original mode shape \(V_n (x)\) as,
\[
\hat{V}_n (x) = \sum_{i=1}^{n} \beta_n V_n (x)
\]
where \(\beta_n\) is a scalar coefficient. Substituting Eq. (9) into Eqs (6) and (8), the modal frequency curve and its spatial derivatives are given by
\[
\omega_n (V_n (x), m_0) = \sqrt{\frac{\bar{K}_n}{\bar{M}_n + m_0 \sum_{i=1}^{n} \beta_n V_n^2 (x)}}
\]
\[
\frac{\partial^k \omega_n}{\partial x^k} = \sum_{j=1}^{k} P_j \frac{\partial^k \left[ \sum_{i=1}^{n} \beta_n V_n (x) \right]}{\partial x^k}, P_j = P_j \left( M_n, m_0, \omega_n, V_n (x) \right)
\]

For a damaged structure, it is known that the local damage (e.g. void or delamination) will result in an increase of the local flexibility of the structure [1]. The damage-induced flexibility can be considered as a local perturbation \(\delta u(x)\) in the structural displacement responses. Using the orthogonal property of the mode shapes, the displacement field which contains the perturbation can be expanded using the linear combination of the modal vectors for a given structure as,
\[ u(x) + \delta u(x) = \sum_{i=1}^{n} a_n V_n(x) = \sum_{i=1}^{n} a_n[V_n(x) + \delta V_n(x)] \]

where \( V_n(x) \) is the mode shape with perturbation term for the modified structure, \( V_n(x) \) and \( \delta V_n(x) \) are the mode shape vector and perturbation term, respectively. \( a_n \) is the coefficient of expansion (modal participation factor). It is well-known that the defect-induced response of each mode will be more significantly manifested at the spatial location of the damage. This behaviour is unique and is distinct from the random noise-induced perturbations. Using the same procedures illustrated above, the modal frequency and its high-order derivative for the damaged structure can be expressed as,

\[ \omega_n(\bar{V}_n(x), m_0) = \frac{K_n}{M_n + m_0\left[\sum_{i=1}^{n} \beta_i(\bar{V}_n(x) + \delta V_n(x))^2\right]} \]

\[ \frac{\partial^2 \omega_n}{\partial k^2} = \sum_{j=1}^{k} P_j \frac{\partial^k}{\partial k^j} \left[ \beta_j (\bar{V}_n(x) + \delta V_n(x)) \right] P_j = \left[ M_n, m_0, \omega_n, V(x) \right] \]

In general, the Eq. (13) can be regarded as a linear map with mass-dependent magnification factor between corresponding unmodified mode shapes and modal frequency curves. It is worth to note that the assumption that the mode shape does not change under the mass loading is only true for very small mass magnitude and may lead to error for a more general condition. Also, the equation shows that the modal frequency curve is dependent on the linear combination of the unmodified mode shapes. The effect of perturbation also contributes to the derivatives of modal frequency, as shown in Eq. (14). In addition, the contributions of the variations of the mode shape and local mode shape perturbation can be presented as separable as shown in Eq. (14). Therefore, the higher order oscillations at the spatial location of the damage can be directly linked to the presence of the damage of the structure. This enables damage detection using modern analysing methods such as wavelet transforms.

The aims of the proposed method are the detection and localisation of subsurface damage, such as bonding failure and delamination, in laminated beam and plate structures. An example of using the local mass loading for structural damage detection is presented in Fig. 1. The global view presents the scenario of the damage detection on a laminated beam with clamped ends. The local view shows the influence of the local mass loading which magnifies the dynamic response of the subsurface damage. By using the local frequency deviation caused by the interaction between the local inertia and damage, the proposed method can be used as an intact-free method in structural damage detection. For example, it can be applied to damage detection in civil structures such as bridges. The added mass can be due to a test vehicle which is driven across the bridge. Similarly, the method can be applied to mechanical structures and components such as laminated composite beams, blades, panels and shells.

2.2. Wavelet analysis of the modal frequency curve

In this section, the one-dimensional wavelet transform of the frequency curves is presented. In the paper, the Wavelet Toolbox [32] is employed to perform the DWT through the software package MATLAB (2014a). According to [33], wavelet transform for a one-dimensional signal can be expressed as,

\[ c_{b,a} = \langle \psi, f \rangle = \int \psi(x) f(x) dx \quad \text{where} \quad \psi(x)_{b,a} = \frac{1}{\sqrt{a}} \psi \left( \frac{x-b}{a} \right) \]

![Fig. 1. Illustration of the proposed method for a beam-like structure with subsurface damage.](image-url)
where the function $\psi$ is the piecewise smooth spline for a given order. The scaling factor $a$ and translation parameter $b$ are used to generate wavelets for multiresolution analysis. Generally, the damage-induced deviation is a local effect and therefore the detail wavelet coefficients (obtained from the high-pass wavelet filter) are more useful for damage detection propose. A general wavelet has $N$ vanishing moment and is defined as,

$$\langle \psi, x^k \rangle = \int x^k \psi(x) dx = 0, k < N-1$$

To see the effect of vanishing moments on the modal frequency curve given in the Eq. (13), applying Eq. (15) and expanding the modal frequency curve in Eq. (14) by Taylor series,

$$c_{n,a} = \langle \psi, \omega_n \rangle = \int \left[ \sum_{i=1}^{k} \frac{\partial^n \omega_n}{\partial x^k} \frac{1}{k!} \Delta x^k \psi(x) \right]$$

$$= \int \left[ \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{1}{k!} \left( p_i \frac{\partial^j \sum_{i=1}^{n} \beta_n \omega_n(x) }{\partial x^k} + p_j \frac{\partial^j \sum_{i=1}^{n} \beta_n \omega_n(x) }{\partial x^k} \right) \Delta x^k \psi(x) \right]$$

From Eq. (17), the first term in the bracket is the smooth term which does not contains high-order oscillations which are associated with damage detection according to Eq. (16). The second term contains the high order oscillations due to the damage that will be extracted and used for damage detection purpose.

For any wavelet-based damage detection method, the selection of wavelet generally has an influence on the performance of the method, as discussed by many authors [19–23]. Previous researches have shown that the minimum number of vanish moments for a wavelet filter should be two [19]. Also, wavelets with longer support showed better performance in terms of the stability for mode shape decompositions [21]. It is known that, the biorthogonal wavelet shows some attractive properties such as high-order smoothness, small approximation error as well as analogies to mode shape profile [34,35]. Researchers have shown that the use of the biorthogonal wavelet with vanish moment of four shows a good performance in damage localisation from mode shapes for beams and plates using CWT algorithm [22]. Instead of using CWT, the DWT algorithm has been shown to satisfactorily detect damage in 1D and 2D structures with less computation time [27]. Based on the literature above, the biorthogonal wavelet with four vanishing moments (bior 4.4) has been employed with DWT to offer a good approximation of the original signal in the present study. A more detailed introduction to biorthogonal wavelet and its properties can be found in Unser [34]. In addition, a comprehensive guide of wavelet selection in terms of classes of function and number of vanishing moments and its effect in signal processing is reported by [35].

In practice, due to the limited number of sampling points in applications, the modal frequency curves need to be interpolated to obtain its general profile. To find the suitable interpolation method, the cubic and biharmonic splines are investigated for use in interpolating the frequency curves before the wavelet transform. The data interpolated using cubic and biharmonic splines interpolated data for a given set of frequency data are compared with reference data (without interpolation) in Fig. 2(a). The absolute error between the two interpolation methods is given in Fig. 2(b). The global error of the two methods is general less than 0.5% from the reference. However, a significant error (about 2%) occurs at the boundaries using the cubic spline method. Compared to the spline method, the biharmonic spline method gives a better approximation of the boundary region.

2.3. Squared residual damage indicator and estimator

Due to the presence of random noise in measurement, the local damage feature of interest obtained from the detailed coefficients may be due to the noise. Usually, threshold values are employed to perform noise suppression before further analysis as reported in [16,17]. However, as recently stated by [36], the noise threshold values are problem-dependent and

![Fig. 2. Comparison of biharmonic and cubic spline interpolated signal with reference data.](image-url)
not easy to estimate in real applications. Instead of using the threshold values in processing, an attempt has been made to use the statistical properties of the multiple mode measured frequency data to indicate the damage location. Fig. 3 shown the basic procedures of the proposed damage index and the size metric function.

Firstly, the measured frequency data points are interpolated with biharmonic spline. First-order smooth signal extension mode is selected to match the data set length of the wavelet transform. The obtained signal is then decomposed using biorhogonal wavelet (bior 4.4) and its absolute normalised detail coefficients are extracted. The absolute detailed coefficients from each mode are superposed with equal weight to enable the accumulation of the responses from multiple modes which can be defined as,

$$c(x)_{sum} = \sum_{i=1}^{n} \left| \psi_{\alpha i} \left( \omega_{a i} \right) \right|$$ \hspace{1cm} (18)

where the function $c_{sum}$ represents superposed absolute detailed wavelet coefficients of the first $n$ modes. Assume that the noise contribution in the detailed coefficients is distributed at different spatial locations for each measured mode. Also note that the response of the defect is more localised and its contribution to the frequency deviation repeatedly occurs at the same location. Therefore, it is natural to use the divergence of the superposed coefficients along spatial location to highlight the sign of the damage. To achieve this, the mean value of the superposed coefficients is estimated and the bias of each data point from its mean value is calculated to yield the squared residual value of the signal as the damage indicator $d$ as follows,

$$d_{indicator} = (c(x)_{sum} - \bar{c}_{sum})^2 = \left( \frac{\sum_{i=1}^{m} c(x)_{sum}}{m} - \bar{c}_{sum} \right)^2$$ \hspace{1cm} (19)

where $\bar{c}_{sum}$ represents the mean value of the superposed wavelet coefficients and $m$ represents the length of the dataset after wavelet transform. The standard deviation is calculated to serve as a guide to obtain the confidence index level towards the background components. Fig. 4 illustrates the squared residual index and the metric function for damage localisation and size estimation purpose from one of the case studies. The maxima value of the damage indicator is employed to define the defect location, as shown in Fig. 4(a). In addition, the size of the damage region can be determined by finding the distance of the intersection points between the damage indicator and damage estimator (red dashed line) and the damage index curve, as shown in the Fig. 4(b).

The defect size is defined as the distance between the interaction points of the damage estimator and the damage indicator, as shown in Fig. 4(b), that is

$$d_{size} = x2 - x1$$

$$d(x1)_{indicator} = d(x2)_{indicator} = \sigma_{estimator}$$ \hspace{1cm} (20)

Fig. 3. Logical diagram of the processing procedures used in the present study.

Fig. 4. An example of the proposed damage indicator and damage estimator for delamination detection.
3. Numerical studies

3.1. Damage at different locations

This section presents the configurations of the finite element models of the damaged beams that were investigated and the obtained numerical results. The damage was induced as debonding at selected sections of two sheets of aluminium that were bonded together. It should be noted that perfect bonding was assumed at the bonded sections. In the present work, cohesive elements were not used to bind the sheets together. Fig. 5 presents the geometrical parameters, boundary conditions and the locations of damage defined in the finite element models. The ABAQUS (v6.13) numerical modelling software package was used to generate the original modal frequency data. The software predicted the natural frequencies of the debonded beams for different axial locations of the debonding as well as different axial locations of the additional point mass. The material property of aluminium was used for all the elements (Young’s module = 70 GPa, Poisson ratio = 0.33 and Density = 2700 kg/m³). In detail, each simulated beam had 50 × 10 × 2 elements along the length, width and thickness directions, which gave a plane aspect ratio equal to one. To reduce the locking effect in linear element for bending modes, the 3D solid quadratic hexahedral elements (C3D20) were employed in the study.

The debonded sections of the damaged beams were simulated using node separation at pre-defined locations of each case which created a zero-volume void. The effect of inter lamina contact at high frequency was not simulated in the present study. However, for high frequency analysis, the contact effect should be considered in the model to give accurate prediction [37]. As discussed earlier in section 2, the magnitude of the mass magnified the change of modal frequency curves. Detailed studies of the effect of mass magnitude on modal frequency curves have been reported in [16,18]. In the present study, the point-mass is modelled as a point element with fix magnitudes of between 5% and 25% of the mass of the structure. The connection between the mass and structure is assumed to be rigid in the simulation. The ABAQUS input file scripting approach was used for model generation. For each debonded beam, a total of 21 input files was generated to simulate mass loading at 21 points with interspacing of 25 mm. The frequencies of the first four bending modes of the beam are extracted from the output results and then processed in the MATLAB programming environment.

Figs 6–8 present the numerical modal frequency data, wavelet coefficients and the damage indicators for the three case studies. In each case, the modal frequencies for mode 1 to mode 4 are presented in subplot (a1) to (a4). The wavelet coefficients for mode 1 to mode 4 are presented in subplot (b1) to (b4). The subplot (c) denotes the damage indicator and a zoomed view of the damage indicator and estimator is presented in subplot (d).

Generally, the results show that the effects of the damage are directly visible in the frequency curves of certain modes. For example, for case 1, as shown in Fig. 6, when the defect is near one of the boundaries, a distinct local frequency reduction (around 7% lower than intact reference) is observed directly in the frequency curves at the delaminated section for the third and fourth modes (Fig. 6(a3) and (a4)). When the damage is located about one-quarter length from one end, (Fig. 7), a certain degree of distortion is observed in the frequency curve as shown by modes 2 to 4 in Fig. 7(a2) to (a4). However, there are no distinct abnormal frequency deviations in mode 1 (Fig. 7(a1)). For the case where the damage is located at the centre, sharp peaks are indicated by modes 2 to 4, as shown in Fig. 8(a2–a4). As stated in earlier the local abnormal distortions observed above in the modal frequency curves are related to the defect-induced perturbation, which can be used to identify the defect. Moreover, the magnitude of these perturbations depend on the location of the defect. This dependence reflects the different sensitivities of the flexural modes.

In order to clearly deduce the effect of the damage from all the modes, it is necessary to apply the wavelet analysis to the frequency curves. For case 1 given in Fig. 6(b1–b4), the peak of the wavelet coefficients along the beam is shown at the debonded section for all the modes. It can be seen that many responses, which are due to higher order terms, also occur at different locations of the beam. This effect will be suppressed if the sampling density is increased but this will increase the processing time in practise. For case 2, the peak coefficients are located at the debonded section in modes 1 to 3, as shown in Fig. 7(b1–b3). The figures show that the largest peaks occur at the location of the debonding. However, Fig. 7(b4) shows that for mode 4, the wavelet peak is not

![Fig. 5. Geometrical parameters, boundary conditions and location of defects for finite element modelling.](image-url)
significantly greater than the other wavelet peaks at other axial locations of the beam. For this mode, the damage-induced response is relatively much weaker than that of modes 1 to 3. For case 3, the peak coefficients for the first four modes provide the exact location information of the defect, as shown in Fig. 8(b1–b4). Moreover, as shown in Fig. 8(d), the location of the defect is at the
However, the damage-induced deviations of the frequency curve are still observable in the detail wavelet coefficients. This feature suggests the existence of additional local perturbation term of the modal frequency curve due to the mode shape perturbation described in Eq. (17).

Fig. 7. Case 2: Numerical modal frequency curves, wavelet coefficients and damage indicator with defect at [125–175 mm].
The derived damage indicator for the three case studies is shown in Figs 6(c), 7(c) and 8(c). According to these figures, the damage indicator is identified by the largest wavelet peaks that occur at the same locations across the four modes of vibration considered. The other peaks are filtered out (as random effects) as they do not repeat across the four modes. Thus, this distinct feature can be used to identify damage.

**Fig. 8.** Case 3: Numerical modal curves, wavelet coefficients and damage indicator with defect at [225–275 mm].

The derived damage indicator for the three case studies is shown in Figs 6(c), 7(c) and 8(c). According to these figures, the damage indicator is identified by the largest wavelet peaks that occur at the same location across the four modes of vibration considered. The other peaks are filtered out (as random effects) as they do not repeat across the four modes. Thus, this distinct feature can be used to identify damage.
identify and locate the damage. The zoomed view shows clearly the intersection between the proposed damage estimator (mean value) which is based on the damage indicator (peak value) as shown in Figs 6(d), 7(d) and 8(d). When the defect-induced responses are relatively strong compared to other noise effects (e.g., numerical noise and boundary effect), the level of damage estimator (red dashed

Fig. 9. Numerical modal frequency curves, wavelet coefficients and damage indicator of mass ratio at 5%.

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line) is much higher than the background wavelet level which occurs at other locations of the beam that are not in the damaged section, as shown in Figs. 6(d) and 8(d). The gap between the damage estimator and the background level reduces if the defect occurs at non-sensitive sections such as a quarter of length from the end (case 2) which coincides with the nodal region of mode 4.

Fig. 10. Numerical modal frequency curves, wavelet coefficients and damage indicator of mass ratio at 10%.
Although the frequency curve of the intact structure is not necessary as a reference or base-line in the above procedures, the intact reference offers the advantage of quickly locating the local deviation of the modal frequency of the damaged structure in practice. As suggested from the above analysis, the difference between the intact and defect responses can

Fig. 11. Numerical modal frequency curves, wavelet coefficients and damage indicator of mass ratio at 20%.

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predict approximately the defect location. Nevertheless, the deduced damage size is generally not as accurate as the index based on the wavelet coefficients.

3.2. Influence of the magnitude of the point mass loading

In this section, the influence of the mass ratio is analysed using FEM. To see how the mass loading magnitude influences the modal frequency curve as well as the wavelet-based damage index, mass ratios of 5%, 10%, 15%, 20% and 25% are employed for a damage at the central location of the beam. The results of the frequency curves and wavelet coefficients for mass ratios 5%, 10% and 20% respectively are shown in Figs. 9–11, respectively.

As can be seen from the modal frequency curves, increasing the mass ratio enhances the structural dynamics at the damaged section. The deviation of the frequency values at the damaged section between the intact and the damaged beams increases as the mass ratio increases. This feature is also reported in the literature [16,18]. This effect agrees with Eq. (13) which shows that the additional mass loading magnifies local perturbation caused by damage (reduction of the stiffness). Furthermore, from the numerical results, for mass ratio equal to or greater than 10%, the damage-induced frequency deviation is directly visible from the frequency curves when the intact data is employed as a reference. It is also worth to note that the visibility of the damage also depends on experimental noise level in real applications.

In terms of the wavelet coefficients, the damage section is manifested in the peak coefficients for various mass ratios from 5% to 20%. As reported in the literature, the wavelet coefficients measure the local deviation from the smoothness of the signal. Thus, in principle, the detected damage location as well as the predicted length of the damage does not vary much for different mass ratios. Fig. 12 presents the peak magnitude of the detail coefficients at the damage section with respect to various mass ratios. A quasi-proportional relationship between mass ratio and magnitude of the wavelet coefficient is revealed by the curves. This shows the dynamic enhancement due to the use of the large mass loading at the damage region. It is worth to note that the above sensitivity analysis does not consider the influences of the experimental errors due to the environment and instruments. The real application of vibration testing may contain sources of errors such as human factors, sensitivity of instruments, and the spectrum analysis parameters (influence of the frequency resolution can be also be important).

4. Experimental validation

The manufacturing processes of the delaminated samples are shown in Fig. 13. Aluminium sheet samples were cut into exact sizes and were cleaned using acetone in order to remove dirt and grease before bonding. To create the damage, the
defined section is coated with mould release agent to prevent adhesive bonding at the defined regions. Then, high strength epoxy adhesive (3M DP-190) (the black medium shown in Fig. 13) is mixed and applied on the prepared bonding faces of both the top and bottom sheets which constitute the samples during the pot-life period. The defined damaged sections can be seen from Fig. 13. The bonded samples were then loaded with weights for 72 hours at room temperature to reach the full bonding strength.

Fig. 14 shows views of the experimental setup. The full length of a sample is 550 mm with 25 mm clamped in the support at each end. The effective length, width and thickness of the samples are the same as given in Fig. 5. In order to minimise external contact effect, the samples are subjected to impact excitation using an impact hammer (PCB 086C03). A calibrated weight of 100 g is used to simulate the point-mass in the experiment. In order to reduce the effect of sensor weight, a low weight (0.5 g) accelerometer (PCB 352C22) is attached to the central line of the structure to measure the out-of-plane frequency response function (FRFs) of the sample. The effect of accelerometer cable in terms of the mass contribution is not considered as the cable is generally light compared to the structure. The mounting of both the mass and the accelerometer are achieved by using normal adhesive petro wax (PCB 080A109). The whole experiment is performed using LMS Test.Lab system. The frequency span used is from 0–600 Hz, with a frequency resolution of around 0.3 Hz throughout the test. For each damage case, the mass loading is imposed at 21 locations along the sample surface. The responses of the samples when the positions of the mass coincide with the fix ends are measured without mass loading. The FRFs of the first four bending modes of the beam are recorded for further analysis.

Figs 15–17 present the experimental results for cases 1 to 3. From Fig. 15, it can be seen that the experimental modal frequency curves of the first three modes do not show any noticeable local deviations at the defect region for the near boundary defect. However, the fourth mode shows significant deviations at the defect region. The lowest point in mode 4 (Fig. 15(a4)) shows a 6% drop or deviation in frequency compared to the corresponding symmetrical location. A similar drop in frequency is also predicted by the finite element analysis as shown in Fig. 6. For case 2 presented in Fig. 16, a slight deviation is indicated in the frequency curve of the fundamental mode (Fig. 16(a1)), and fairly sharp peaks are manifested in modes 2 and 4 (Fig. 16(a3–a4)) at the edges of the damage. Although, these abnormal effects are fairly observable, they are not as strong as in case 1, and this location-dependent behaviour agrees with finite element prediction. When damage occurs at the centre location, Fig. 17 shows that the central section of the frequency curve of mode 2 is distorted. Similar to Fig. 6(a3) and (a4), sharp regions are exhibited at the delaminated section in both frequency curves of modes 3 and 4, as shown in Fig. 17(a3) and (a4), respectively.

These experimental case studies suggest that the damage severity is location-dependent. The highest severity is demonstrated for damage that occurs near the clamped end due to the large stress. It is worth to note that the damage may not always produce a significant local frequency reduction as observed in the first three modes of case 1. In fact, as in cases 2 and 3, the delaminated section produces either a sharp or a distorted segment of the frequency curve without a large frequency reduction. By definition, such local non-smooth features indicate the presence of local high-order terms in the frequency curve when approximated by polynomial functions. Eqs. (13) and (14) have shown the connection between modal frequency curve and mode shapes and its spatial derivatives for the original (unmodified) structure. The order of smoothness change is related to the perturbation of the mode shape of multiple modes and results in a deviation of the frequency curves. This deviation is magnified by the interaction of the additional local inertia with the subsurface damage.

From the experimental frequency curves of the cases presented, it is seen that the frequency curve of some modes generally do not show significant abnormality. In order to enhance the detectability of defects, the frequency curves were subjected to wavelet analyses. From Fig. 15(b1–b4), the peak values of the wavelet coefficients for case 1 occur around the defined delaminated section for all the modes. For case 2, the highest wavelet coefficient for each mode still corresponds to the defect location as shown in Fig. 16(b1–b4). Although, the predominant peak is still defect-dependent, the defect-free region also indicates secondary peaks whose locations seems to be related to the turning point of the frequency curves. This behaviour was observed to be similar in both numerical and experimental results. For example, the magnitude of these secondary peaks can reach up to 0.8 of the magnitude of the predominant peak due to damage as shown in mode 3 (Fig. 16(b3)). However, the oscillations seen in the wavelet coefficients in Fig. 17(b1) for the fundamental mode of case 3 are more random in nature. The results show three sets of peak magnitudes of wavelet coefficients namely: small medium and large coefficients, whose location are not related to the turning point but seem to be
influenced by random measurement noise effects. These results show that the use of only a single mode, especially the fundamental mode, does not provide a robust defect identification. On the other hands, the defect responses obtained from modes 2 to 4 in Fig. 17 (b2) to (b4) show that the use of a multiple mode strategy gives a better identification and localisation of defects in structures.

Fig. 15. Case 1: Experimental modal frequency curves, wavelet coefficients and damage indicator with defect at [25–75 mm].
In practice, there are a number of sources of experimental measurement errors such as environmental noise, electrical noise, errors in modal parameter estimation and, the precision of the measurement instruments. These sources of measurement errors increase the difficulty in practise for an accurate measurement of the corresponding noise threshold under

Fig. 16. Case 2: Experimental modal frequency curves, wavelet coefficients and damage indicator with defect at [100–150 mm].
different environmental conditions. In addition, the detail coefficients only evaluate a relative smoothness change for a given signal along different spatial locations. In the worst condition, the damage-induced frequency deviation may be suppressed artificially if the frequency deviation is relatively small or the threshold values are overestimated. Another risk
may come from the use of one or two modes in measurement. As illustrated in case 3, the detailed coefficient of the fundamental mode contains the superposition of the damage-induced response and the noise-induced response, which may lead to the wrong estimation of the defect location if only one mode has been employed in damage localisation.

According to the orthogonal property of mode shapes, the damage-induced frequency deviation should be projected into each mode at the same spatial location. Hence, the summation of individual mode coefficients can be employed to accumulate the defect-induced response. Figs 15(c), 16(c) and 17(c) present the multiple mode damage indicator based on the experimental data. The maximum values of the damage indicators can successfully predict the location of the defects in the above case studies. Moreover, comparing the damage indicator to the single mode detailed coefficients, the enhancement of the damage detectability by using the squared residual indicator is demonstrated. It can be seen from case 2 that the damage identification is enhanced by using the squared residual damage indicator which provides a much more distinguishable feature than conventional detail coefficients alone.

The detailed view of the size prediction for the three case studies are presented in Figs 15(d), 16(d) and 17(d). The experimentally derived sizes from the damage estimator generally match well with the pre-defined sections. The variation of the damage severity in different cases is also suggested from the experimental results in terms of the variation of the difference between the damage estimator and background level. The experimental results presented indicate that the current measurement configuration used in this work enables the approximate size of the defect to be evaluated satisfactorily.

Generally, the accuracy of size evaluation depends on both the damage severity as well as the sensitivity of the measurement instruments. Due to random effects present in real measurements, the accurate estimation of the size may require the superposition of frequency data from sufficient number of modes to suppress the random effect. Higher modes tend to show more noticeable defect-induced effect for some defect configurations, as discussed in case 1. However the large number of sampling points that are required to obtain accurate profiles of the frequency curves for higher modes will generally increase the time cost in practice.

The present study focuses on the establishment of the theoretical background of the proposed wavelet-index based on local frequency deviation and its numerical and experimental investigation. As a wavelet-based index, the performance of the method may vary slightly for various selected wavelet filters. Several researchers have discussed the influence of the wavelet filter on damage detection for beam-like and plate-like structures and some of the studies can be found in references [22,25,27,36]. Nevertheless, it is expected that the result will generally be consistent in spite of the slightly variations.

In summary, some key features of the present study are as follows:

1. The theoretical basis of the modal frequency curve is established on the basis of the eigenvalue problem and the effect of mode shape modification due to concentrated mass loading is considered.
2. The connection between the wavelet detail coefficients and defect-induced responses has been illustrated using the concept of vanishing moments of the wavelet and the analytical properties of the modal frequency curve.
3. The biharmonic spline interpolation method shows a better performance than normal cubic spline method in the boundary treatment. The better approximation of the frequency curves near a boundary reduces the need for additional smoothing operations.
4. A squared residual damage indicator using multiple vibration modes data has been developed. The damage indicator is threshold independent and provides the enhanced detection and localisation of the defect at any longitudinal section. A damage estimator using the standard deviation of the damage indicator is defined to evaluate the size of the defect. The damage estimator shows the ability to estimate the approximate size of a defect based on the damage indicator.
5. The influence of the mass ratio in the proposed method is analysed. The mass ratio has a direct effect on the frequency deviation caused by damage and which agrees with previous studies [18]. A quasi-linear relationship is observed between the mass ratio and wavelet coefficient magnitude at a damaged section. The wavelet-based index shows stable performance for mass ratios from 5% to 25% of the mass of the structure.

5. Conclusions

In this paper, a novel damage detection method using modal frequency curve has been presented. The theoretical basis of the modal frequency curve and its application for damage detection combined with wavelet analysis has been derived. The biharmonic spline interpolation method studied in the present work has provided a better treatment of boundary effects. A new damage indicator and a damage estimator based on multiple mode biorthogonal wavelet coefficients are designed to locate and evaluate the internal defects. Numerical models have been employed to study the theoretical performance of the proposed method. The simulated cases are validated through experimental vibration testing. The theoretical concept of the defect-induced local frequency distortions are supported by numerical and experimental results. A reliable performance in terms of defect localisation and approximate size estimation has been observed from the proposed damage indicator and estimator. The damage indicator provides greater enhancement of the damage detectability than the use of only the pure wavelet detailed coefficients of individual modes. In addition, the damage indicator does not require any threshold value. The developed frequency method has the potential ability in damage detection under environmental loading using single point sensor measurement for large size structures.
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