ADDENDUM TO “BMO: OSCILLATIONS, SELF IMPROVEMENT, GAGLIARDO COORDINATE SPACES AND REVERSE HARDY INEQUALITIES”

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Abstract. We provide a precise statement and self contained proof of a Sobolev inequality (cf. [A, page 236 and page 237]) stated in the original paper. Higher order and fractional inequalities are treated as well.

1. Introduction

One of the purposes of the original paper (cf. [A]) was to highlight some connections between interpolation theory, and inequalities connected with the theory of BMO and Sobolev spaces. This resulted in a somewhat lengthy paper and as consequence many known results were only stated, and the reader was referred to the relevant literature for proofs. It has become clear, however, that a complete account of some of the results could be useful. In this expository addendum we update and correct one paragraph of the original text by providing a precise statement and proof of a Sobolev inequality which was stated in the original paper (cf. [A, (13) page 236, and line 10, page 237]). Included as well are the corresponding results for higher order and fractional inequalities.

All the results discussed in this note are known\textsuperscript{1}. The only novelty is perhaps in the unified presentation.

We shall follow the notation and the ordering of references of the original paper [A] to which we shall also refer for background, priorities, historical comments, etc. Newly referenced papers will be labeled with letters.

2. The Hardy-Littlewood-Sobolev-O’Neil program

We let

\[
\|f\|_{L(p,q)} = \left\{ \left\{ \int_0^\infty \left( (f^+(t))^{1/p} \right)^{q \frac{dt}{t}} \right\}^{1/q} : 1 \leq p < \infty, 1 \leq q \leq \infty \right\}^{1/q}
\]

where

\[
\|f\|_{L(\infty,q)} := \left\{ \int_0^\infty \left( (f^{**}(t) - f^*(t))^{q \frac{dt}{t}} \right)^{1/q} \right\}^{1/q}.
\]

In particular we note that in this notation

\[
\|f\|_{L(\infty,\infty)} = \{ f : \sup_{t>0} \{ f^{**}(t) - f^*(t) \} < \infty \},
\]

\[
L(1,1) = L^1.
\]

\textsuperscript{1}In presenting the results yet again we have followed in part advise from Rota [52].
Moreover, if \( f \) has bounded support
\[
\|f\|_{L^{(\infty, 1)}} = \|f\|_{L^\infty}.
\]

In [A] (13) page 236 we stated that “it was shown in [7] that (2.3)
\[
\|f\|_{L^{(\bar{p}, q)}} \leq c_n \|\nabla f\|_{L^{(p, q)}}, 1 \leq p \leq n, \quad \frac{1}{p} = \frac{1}{n} - \frac{1}{q}, \quad 1 \leq q \leq \infty, f \in C_0^\infty(R^n).
\]

However, to correctly state what was actually shown in [7], the indices in the displayed formula need to be restricted when \( p = 1 \). The precise statement reads as follows:

**Theorem 1.** Let \( n > 1 \). Let \( 1 \leq p \leq n, 1 \leq q \leq \infty \), and define \( \frac{1}{p} = \frac{1}{n} - \frac{1}{q} \). Then, if \((p, q) \in (1, n] \times [1, \infty) \) or if \( p = q = 1 \), we have
\[
\|f\|_{L^{(\bar{p}, q)}} \leq c_n(p, q) \|\nabla f\|_{L^{(p, q)}}, f \in C_0^\infty(R^n).
\]

**Remark 1.** If \( n = 1 \), then \( p = q = 1 \), and (2.4) is an easy consequence of the fundamental theorem of Calculus.

The corresponding higher order result (cf. [A] line 10, page 237) reads as follows,

**Theorem 2.** Let \( k \in N, k \leq n, 1 \leq p \leq \frac{n}{k}, 1 \leq q \leq \infty \). Define \( \frac{1}{p} = \frac{1}{n} - \frac{k}{n} \). Then, (i) if \( k < n \), and \((p, q) \in (1, \frac{m}{k}) \times [1, \infty), \) or \((p, q) \in \{1\} \times \{1\}, \) or (ii) if \( n = 1 \), and \( p = q = 1 \), we have
\[
\|f\|_{L^{(\bar{p}, q)}} \leq c_{n,k}(p, q) \|\nabla^k f\|_{L^{(p, q)}}, f \in C_0^\infty(R^n),
\]
where \( |D^k f| \) is the length of the vector whose components are all the partial derivatives of order \( k \).

**Remark 2.** Observe that when \( p = \frac{n}{k}, p > 1, \) and \( q = 1 \), we have
\[
\|f\|_{L^\infty} = \|f\|_{L^{(\infty, 1)}} \leq \|D^k f\|_{L^{(\frac{m}{k}, 1)}}, f \in C_0^\infty(R^n).
\]

We also obtain an \( L^\infty \) estimate when \( p = \frac{n}{k} = 1, \) and \( q = 1 \)
\[
(2.5) \quad \|f\|_{L^\infty} = \|f\|_{L^{(\infty, 1)}} \leq \|D^k f\|_{L^1}, f \in C_0^\infty(R^n).
\]

In the particular case when we are working with \( L^p \) spaces, i.e. \( p = q \), there is no need to separate the cases \( p = 1 \) and \( p > 1 \), and Theorems 1 and 2 give us what we could call the “completion” of the Hardy-Littlewood-Sobolev-O’Neil program, namely

**Corollary 1.** Let \( 1 \leq k \leq n, 1 \leq p \leq \frac{n}{k}, \frac{1}{p} = \frac{1}{n} - \frac{k}{n} \). Then
\[
(2.6) \quad \|f\|_{L^{(\bar{p}, p)}} \leq c_n(p) \|D^k f\|_{L^{(p, p)}}, f \in C_0^\infty(R^n).
\]

**Proof. (of Theorem 1).** The case \( 1 < p \leq n \). We start with the inequality (cf. [A] (58) page 263)),
\[
(2.7) \quad f^{**}(t) - f^*(t) \leq c_n t^{1/n}(\nabla f)^{**}(t),
\]
which yields
\[
(2.8) \quad (f^{**}(t) - f^*(t)) t^{1/p} t^{-1/n} \leq c_n t^{1/p}(\nabla f)^{**}(t).
\]
If \( q < \infty \), we integrate (2.8) and find
\[
\left\{ \int_0^\infty [(f^{**}(t) - f^*(t)) t^{1/p}]^{1/q} \frac{dt}{t} \right\}^{1/q} \leq c_n \left\{ \int_0^\infty [t^{1/p}(\nabla f)^{**}(t)]^q \frac{dt}{t} \right\}^{1/q}
\]
\[
\leq C_n(p,q) \|\nabla f\|_{L(p,q)},
\]
where in the last step we used Hardy’s inequality (cf. [St, Appendix 4, page 272]).

To identify the left hand side we consider two cases. If \( p = n \), then \( \bar{p} = \infty \) and the desired result follows directly from the definitions (cf. (2.1)). If \( p < n \), then we can write
\[
f^{**}(t) = \int_t^\infty f^{**}(s) - f^*(s) \frac{ds}{s},
\]
and use Hardy’s inequality (cf. [St, Appendix 4, page 272]) to get
\[
\|f\|_{L(\bar{p},\infty)} \leq \left\{ \int_0^\infty |(f^{**}(t) t^{1/\bar{p}})|^{1/q} \frac{dt}{t} \right\}^{1/q} \leq \left\{ \int_0^\infty [(f^{**}(t) - f^*(t)) t^{1/\bar{p}}]^{1/q} \frac{dt}{t} \right\}^{1/q}.
\]
The case \( q = \infty \) is easier. Indeed, if \( p = n \), the desired result follows taking a sup in (2.8), while if \( p < n \), from (2.9) we find
\[
f^{**}(t) \leq \int_t^\infty (f^{**}(s) - f^*(s)) s^{1/\bar{p}} - 1/\bar{p} \frac{ds}{s}
\]
\[
\leq t^{-1/\bar{p}} \sup_s (f^{**}(s) - f^*(s)) s^{1/\bar{p}}.
\]
Consequently
\[
\|f\|_{L(\bar{p},\infty)} \leq \sup_s (f^{**}(s) - f^*(s)) s^{1/\bar{p}}.
\]

Therefore, combining the estimates we have obtained for the right and left hand sides, we obtain
\[
\|f\|_{L(\bar{p},\infty)} \leq \|\nabla f\|_{L(p,q)}, 1 < p \leq n, 1 \leq \bar{p} \leq \infty.
\]

Finally, we consider the case when \( p = q = 1 \). In this case we have \( \frac{1}{p} = 1 - \frac{1}{n} \).

At this point recall the inequality (cf. [A] page 264)]
\[
\int_0^t (f^{**}(s) - f^*(s)) s^{1/p} \frac{ds}{s} = \int_0^t (f^{**}(s) - f^*(s)) s^{1-1/n} \frac{ds}{s}
\]
\[
\leq \int_0^t (\nabla f)^*(s) ds.
\]

Let \( t \to \infty \), to find
\[
\int_0^\infty (f^{**}(s) - f^*(s)) s^{1/p} \frac{ds}{s} \leq c_n \int_0^\infty (\nabla f)^*(s) ds = c_n \|\nabla f\|_{L^1} = c_n \|
\]

We conclude the proof remarking that, as we have seen before,
\[
\|f\|_{L(\bar{p},1)} \leq \int_0^\infty (f^{**}(s) - f^*(s)) s^{1/\bar{p}} \frac{ds}{s}.
\]

□
3. Higher Order

We will only deal in detail with the case \( k = 2 \) (i.e. the case of second order derivatives) since the general case follows by induction, *mutatis mutandi*.

**Proof.** (i) Suppose first that \( n > 2 \). Let \( \bar{p}_1 \) and \( \bar{p}_2 \) be defined by \( \frac{1}{\bar{p}_1} = \frac{1}{p} - \frac{1}{n} \) and \( \frac{1}{\bar{p}_2} = \frac{1}{p} - \frac{2}{n} = \frac{1}{p} - \frac{2}{2} = \frac{1}{p} \). The first step of the iteration is to observe (cf. [75]) the elementary fact:

\[
|\nabla(\nabla f)| \leq |D^2(f)|.
\]

Therefore, by (2.4) we have

\[
(\nabla f)^{**}(t) - (\nabla f)^*(t) \leq t^{1/n}[|\nabla(\nabla f)|^{**}(t)
\leq t^{1/n} |D^2(f)|^{**}(t).
\]

Consequently, we find

\[
(3.1) \quad ((\nabla f)^{**}(t) - (\nabla f)^*(t)) t^{1/\bar{p}_1} \leq t^{1/\bar{p}_1} |D^2(f)|^{**}(t).
\]

Suppose that \( 1 < p \leq \frac{n}{2} \), and let \( 1 \leq q < \infty \). Then, from (3.1) and a familiar argument, we get

\[
\|\nabla f\|_{L(\bar{p}_1,q)} \leq \left\{ \int_0^\infty t^{1/p} t^{1/\bar{p}_1} |D^2(f)|^{**}(t) dt \right\}^{1/q}.
\]

Thus,

\[
\|\nabla f\|_{L(\bar{p}_1,q)} \leq \|D^2(f)\|_{L(p,q)}.
\]

Now, combining the previous inequality with the already established first order case (cf. Theorem [1]) we find,

\[
\|f\|_{L(\bar{p}_2,q)} \leq \|\nabla f\|_{L(\bar{p}_1,q)} 
\leq \|D^2(f)\|_{L(p,q)}.
\]

Likewise we can treat the case when \( q = \infty \). The analysis also works in the case \( p = 1 = q \). In this case we replace (3.1) with (2.11):

\[
\int_0^t (\nabla f)^{**}(s) - (\nabla f)^*(s) s^{1-1/n} ds \leq \|D^2 f\|_1(s) ds,
\]

which yields

\[
\int_0^\infty (\nabla f)^{**}(s) - (\nabla f)^*(s) s^{1-1/n} ds \leq \|D^2 f\|_1(s) ds.
\]

Therefore

\[
\|\nabla f\|_{L(\bar{p}_1,1)} \leq \|D^2 f\|_1.
\]

At this point recall that the first order case gives us

\[
\|f\|_{L(\bar{p}_2,1)} \leq \|\nabla f\|_{L(\bar{p}_1,1)}.
\]

Thus,

\[
\|f\|_{L(\bar{p}_2,1)} \leq \|D^2 f\|_1.
\]
Finally consider the case when \( n = 2 = k \), this means that \( p = \frac{2}{2} = 1 \), and we let \( q = 1 \). Then, from
\[
\int_0^t \left( (Df)^*(s) - (Df)* (s) \right) s^{1/2} \frac{ds}{s} \lesssim \int_0^t (D^2f)^*(s) ds
\]
we once again derive
\[
\| \nabla f \|_{L^2(1,1)} \lesssim \| D^2f \|_{L^1}.
\]
Moreover, since
\[
(f**(t) - f^*(t)) \lesssim t^{1/2} (\nabla f)**(t)
\]
ingenerating we get
\[
\| f \|_{L^(\infty,1)} \lesssim \| \nabla f \|_{L^2(1,1)},
\]
consequently, we see that,
\[
\| f \|_{L^\infty} = \| f \|_{L^{(\infty,1)}} \lesssim \| D^2f \|_{L^1}.
\]

**Example 1.** In the case \( n > 2, p = \frac{n}{2}, q = 1 \), we have
\[
\| f \|_{L^{(\infty,1)}} \lesssim \| \nabla f \|_{L^{(n,1)}} \lesssim \| D^2f \|_{L^{(\frac{n}{2},1)}},
\]
in other words
\[
(3.2) \quad \| f \|_{L^\infty} \lesssim \| D^2f \|_{L^{(\frac{n}{2},1)}}.
\]

**Remark 3.** Sobolev inequalities involving only the Laplacian are usually referred to as *reduced Sobolev inequalities* and there is a large literature devoted to them. For example, in the context of the previous Example, since \( n/2 > 1 \) it is possible to replace \( D^2 \) by the Laplacian in (3.2) (cf. the discussion in [St, Chapter V]). The correct *reduced* analog of (3.2) when \( n\neq 2 \) involves a stronger condition on the Laplacian, as was recently shown by Steinerberger [Stef], who, in particular, shows that for a domain \( \Omega \subset R^2 \) of finite measure, and \( f \in C^2(\Omega) \cap C(\Omega) \), there exists an absolute constant \( c > 0 \) such that
\[
\max_{x \in \Omega} |f(x)| \leq \max_{x \in \partial \Omega} |f(x)| + c \max_{x \in \Omega} \int_{\Omega} \max \left\{ 1, \log \frac{|\Omega|}{|x-y|^2} \right\} |\Delta f(y)| dy.
\]
In particular, when \( f \) is zero at the boundary, Steinerberger’s result gives
\[
(3.3) \quad \max_{x \in \Omega} |f(x)| \leq c \max_{x \in \Omega} \int_{\Omega} \max \left\{ 1, \log \frac{|\Omega|}{|x-y|^2} \right\} |\Delta f(y)| dy.
\]

By private correspondence Steinerberger showed the author that (3.3) implies an inequality of the form
\[
(3.4) \quad \| f \|_{L^\infty(\Omega)} \leq \| \Delta f \|_{L^1(\Omega)} + \| \Delta f \|_{L^\infty(\log L)^1(\Omega)}.
\]

Let us informally put forward here that one can develop an approach to Steinerberger’s result (3.4) using the symmetrization techniques of this paper, if one uses a variant of symmetrization inequalities for the Laplacian, originally obtained by Maz’ya-Talenti, that was recorded in [Mm, Theorem 13 (ii), page 178]. We hope to give a detailed discussion elsewhere.
4. The Fractional Case

In this section we remark that a good deal of the analysis can be also adapted to the fractional case (cf. [59]). Let us go through the details. Let \( X(R^n) \) be a rearrangement invariant space, and let \( \phi_X(t) = \|X_{(0,t)}\|_X \), be its fundamental function. Let \( w_X \) be the modulus of continuity associated with \( X \):

\[
w_X(t, f) = \sup_{|h| \leq t} \|f(h) - f(0)\|_X.
\]

Our basic inequality will be (cf. [50] and [59])

\[(4.1) \quad f^{**}(t) - f^*(t) \leq c_n \frac{w_X(t^{1/n}, f)}{\phi_X(t)}, f \in C_0^\infty(R^n).
\]

Let \( \alpha \in (0, 1), 1 \leq p \leq \frac{n}{\alpha}, 1 \leq q \leq \infty. \) Let (with the usual modification if \( q = \infty \))

\[
\|f\|_{\check{B}^\alpha_{p,q}} = \left\{ \int_0^\infty [t^{-\alpha}w_{L^p}(t, f)]^q \frac{dt}{t} \right\}^{1/q}.
\]

**Theorem 3.** Suppose that \( \alpha \in (0, 1), 1 \leq p \leq \frac{n}{\alpha}, \frac{1}{p} = \frac{1}{\alpha} - \frac{\alpha}{n} \). Then, we have

\[
\|f\|_{L(\check{p}, q)} \leq \|f\|_{\check{B}^\alpha_{p,q}}, f \in C_0^\infty(R^n).
\]

**Proof.** Consider first the case \( q < \infty. \) Let \( X = L^p \), then \( \phi_X(t) = t^{1/p} \), consequently \( (4.1) \) becomes

\[
f^{**}(t) - f^*(t) \leq c_n \frac{w_{L^p}(t^{1/n}, f)}{t^{1/p}}, f \in C_0^\infty(R^n),
\]

which yields

\[
\left\{ \int_0^\infty [(f^{**}(t) - f^*(t))t^{\frac{\alpha}{n}}]^q \frac{dt}{t} \right\}^{1/q} \leq c_n \left\{ \int_0^\infty [t^{-\alpha/n}w_{L^p}(t^{1/n}, f)]^q \frac{dt}{t} \right\}^{1/q}
\]

\[
\simeq \left\{ \int_0^\infty [t^{-\alpha}w_{L^p}(t, f)]^q \frac{dt}{t} \right\}^{1/q},
\]

\[
\simeq \|f\|_{\check{B}^\alpha_{p,q}}.
\]

It follows readily that

\[
\|f\|_{L(\check{p}, q)} \leq \|f\|_{\check{B}^\alpha_{p,q}}, f \in C_0^\infty(R^n).
\]

For the case \( q = \infty \) we simply go back to

\[(4.2) \quad f^{**}(t) - f^*(t)t^{\frac{\alpha}{n}} \leq c_n t^{-\alpha/n}w_{L^p}(t^{1/n}, f),
\]

and take a sup over all \( t > 0. \)

**Example 2.** Note that when \( p = \frac{n}{\alpha}, \) then \( \frac{1}{p} = 0, \) consequently if \( 1 \leq q \leq \infty, \) we have that for \( f \in C_0^\infty(R^n),
\]

\[(4.3) \quad \|f\|_{L(\infty, q)} = \left\{ \int_0^\infty [(f^{**}(t) - f^*(t))^q \frac{dt}{t} \right\}^{1/q}
\]

\[
\leq c_n \|f\|_{\check{B}^\alpha_{p,q}}.
\]
In particular, if \( q = 1 \),
\[
\|f\|_{L^\infty} = \|f\|_{L(\infty, 1)} \leq c_n \|f\|_{\dot{B}^{n-1}_\infty}, f \in C_0^\infty(R^n).
\]

The corresponding result for Besov spaces anchored on Lorentz spaces follows the same analysis. Let \( 1 \leq p \leq \infty, 1 \leq r \leq \infty, 1 \leq q \leq \infty, 0 < \alpha < 1 \). We let (with the usual modification if \( q = \infty \))
\[
\|f\|_{\dot{B}^{\alpha,q}_{L(p,r)}} = \left\{ \int_0^\infty [t^{1-\alpha} w_{L(p,r)}(t, f)]^{q} \frac{dt}{t} \right\}^{1/q}.
\]

Note that since \( \phi_{L(p,r)}(t) \sim t^{1/p}, 1 \leq p < \infty, 1 \leq r \leq \infty \), our basic inequality now takes the form
\[
(4.4) \quad f^{**}(t) - f^{*}(t) \leq c_n \frac{w_{L(p,r)}(t^{1/n}, f)}{t^{1/p}}, f \in C_0^\infty(R^n), 1 \leq p < \infty, 1 \leq r \leq \infty.
\]

Then, *mutatis mutandi* we have

**Theorem 4.** Suppose that \( \alpha \in (0, 1), 1 \leq p \leq \frac{d}{d-1} \frac{d}{p} = \frac{1}{p} - \frac{d}{n} \). Then, if \( p > 1, 1 \leq r < \infty \), or \( p = r = 1 \), we have
\[
\|f\|_{L(p,q)} \leq \|f\|_{\dot{B}^{\alpha,q}_{L(p,r)}}, f \in C_0^\infty(R^n).
\]

5. More Examples and Remarks

5.1. **On the role of the** \( L(\infty,q) \) **spaces.** In the range \( 1 < p < n \), (2.4) and (2.6) yield the classical Sobolev inequalities. Suppose that \( p = n \). Then \( \frac{1}{p} = 0 \), and (2.4) becomes
\[
(5.1) \quad \|f\|_{L(\infty,q)} \leq \|\nabla f\|_{L(n,q)}, 1 \leq q \leq \infty.
\]

When dealing with domains \( \Omega \) with \( |\Omega| < \infty \), from (2.7) we get, \( 1 \leq q \leq \infty \),
\[
(5.2) \quad \left\{ \int_0^{|\Omega|} (f^{**}(s) - f^{*}(s))^q \frac{ds}{s} \right\}^{1/q} \leq \|\nabla f\|_{L(n,q)}, f \in C_0^\infty(\Omega).
\]

To compare this result with classical results it will be convenient to normalize the *norm* as follows
\[
\|f\|_{L(\infty,q)(\Omega)} = \left\{ \int_0^{|\Omega|} (f^{**}(s) - f^{*}(s))^q \frac{ds}{s} \right\}^{1/q} + \frac{1}{|\Omega|} \int_\Omega |f(x)| \, dx.
\]

**Remark 4.** Note that this does not change the nature of (5.2) since if \( f \) has compact support on \( \Omega \), then if we let \( t \to |\Omega| \) in
\[
f^{**}(t) - f^{*}(t) \leq c_n t^{1/n} \nabla f^{**}(t)
\]
we find that
\[
\frac{1}{|\Omega|} \int_\Omega |f(x)| \, dx = f^{**}(|\Omega|) \leq |\Omega|^{1/n-1} \|\nabla f\|_{L^1(\Omega)} \leq \|\nabla f\|_{L(n,q)}.
\]
Let us consider the case \( q = n \). It was shown in [7, page 1227] (the so called Hansson-Brezis-Wainger-Maz’ya embedding) that

\[
\left\{ \int_0^{|\Omega|} \left( \frac{f^{**}(s)}{1 + \log \frac{|\Omega|}{s}} \right)^n \frac{ds}{s} \right\}^{1/n} \lesssim \|f\|_{L(\infty,n)}(\Omega) \\
\lesssim \|\nabla f\|_{L(q,q)} + \|f\|_{L(\infty)}.
\]

Therefore, (5.1) implies an improvement on the Hansson-Brezis-Wainger-Maz’ya embedding. The connection with BMO appears when \( q = \infty \), for then we have

\[
\|f\|_{L(\infty,\infty)} \lesssim \|\nabla f\|_{L(n,\infty)}, f \in C_0^\infty(\mathbb{R}^n).
\]

In the case \( p = n, q = 1 \). Then, (2.4) gives

\[
\|f\|_{L(\infty,1)} \lesssim \|\nabla f\|_{L(n,1)}, f \in C_0^\infty(\mathbb{R}^n),
\]

which ought to be compared with the following (cf. [St1])

\[
\|f\|_{L\infty} \lesssim \|\nabla f\|_{L(n,1)}, f \in C_0^\infty(\mathbb{R}^n).
\]

Indeed, let us show that (5.3) gives (5.4). From

\[
\frac{d}{dt}(tf^{**}(t)) = \frac{d}{dt}(\int_0^t f^*(s)ds) = f^*(t),
\]

it follows (by the product rule of Calculus) that

\[
\frac{d}{dt}(f^{**}(t)) = -\left(\frac{f^{**}(t) - f^*(t)}{t}\right).
\]

Therefore, if \( f \) has compact support,

\[
\|f\|_{L(\infty,1)} = \lim_{t \to \infty} \int_0^t (f^{**}(s) - f^*(s)) \frac{ds}{s} = \lim_{t \to \infty} (f^{**}(0) - f^{**}(t)) \\
= \|f\|_{L\infty} - \lim_{t \to \infty} \frac{1}{t} \|f\|_{L1} \\
= \|f\|_{L\infty}.
\]

5.2. The Gagliardo-Nirenberg Inequality and Weak type vs Strong Type.
It is well known that the Sobolev inequalities have remarkable self improving properties. In this section we wish to discuss the connections of these self improving effects with symmetrization. The study is important when trying to extend Sobolev inequalities to more general contexts.

We consider three forms of the Gagliardo-Nirenberg inequality. The strong form of the Gagliardo-Nirenberg inequality

\[
\|f\|_{L(n',1)} \lesssim \|\nabla f\|_{L1}, f \in C_0^\infty(\mathbb{R}^n),
\]

which implies the classical version of the Gagliardo-Nirenberg inequality

\[
\|f\|_{L\infty} \lesssim \|\nabla f\|_{L1}, f \in C_0^\infty(\mathbb{R}^n).
\]

which in turn implies the weaker version of the Gagliardo-Nirenberg inequality

\[
\|f\|_{L(n',\infty)} \lesssim \|\nabla f\|_{L1}, f \in C_0^\infty(\mathbb{R}^n).
\]
Let us now show that (5.7) implies (5.5). In [A, (55) page 261] we showed that (5.7) implies the symmetrization inequality

\[ f^{**}(t) - f^{*}(t) \preceq t^{1/n}(\nabla f)^{**}(t). \]

Conversely, (5.8) can be rewritten as

\[ (f^{**}(t) - f^{*}(t))t^{1-1/n} \preceq \int_0^t (\nabla f)^{*}(s)ds. \]

Consequently, taking a sup over all \( t > 0 \) we see that (5.8) in turn implies (5.5).

Moreover, let us show that (5.8) implies the isoperimetric inequality (here we ignore the issue of constants to simplify the considerations). To see this suppose that \( E \) is a bounded set with smooth border and let \( f_n \) be a sequence of smooth functions with compact support such that \( f_n \to \chi_E \) in \( L^1 \), with

\[ \|\nabla f_n\|_{L^1} \to Per(E) \]

where \( Per(E) \) is the perimeter of \( E \). Selecting \( t > |E| \), we see that \( (f^{**}_n(t) - f^{*}_n(t)) \to \frac{1}{t} |E| \), therefore from (5.9) we find

\[ \frac{1}{t} |E| t^{1-1/n} \preceq Per(E) \]

therefore letting \( t \to |E| \), gives

\[ |E|^{1-1/n} \preceq Per(E). \]

This concludes our proof that (5.7) is equivalent to (5.5) since it is a well known consequence of the co-area formula that the isoperimetric inequality is equivalent to (5.5) (cf. [67]). At the level of symmetrization inequalities we have shown in [A, page 263] that (5.5) implies the symmetrization inequality

\[ \int_0^t (f^{**}(s) - f^{*}(s))s^{1-1/n}ds \preceq \int_0^t (\nabla f)^{*}(s)ds. \]

Moreover, conversely, taking a sup over all \( t > 0 \) in (5.10), shows that (5.10) implies (5.5).

A direct proof of the fact that (5.10) implies (5.8) is straightforward. Indeed, starting with

\[ \int_{t/2}^t (f^{**}(s) - f^{*}(s))s^{1-1/n}ds \preceq \int_0^t (\nabla f)^{*}(s)ds, \]

and using the fact that \( (f^{**}(t) - f^{*}(t))t = \int_t^\infty \lambda f(s)ds \) increases, we see that

\[ (f^{**}(t/2) - f^{*}(t/2))t^{1-1/n} \preceq \int_0^t (\nabla f)^{*}(s)ds, \]

and (5.5) follows readily. The proof that we give now, showing that (5.8) implies (5.10) is indirect. First, as we have seen (5.8) is equivalent to the validity of (5.7) which in turn implies the following inequality due to Maz’ya-Talenti (cf. [65]),

\[ t^{1-1/n}[-f^{*}(t)]' \leq \frac{d}{dt} \int_{\{|f(x)|>f^{*}(t)\}} |\nabla f(x)| dx. \]

\(^2\)Note that by Pólya-Szegö, \( f^{*} \) is absolutely continuous
To proceed further we need a new expression for $f^{**}(t) - f^*(t)$, which we derive integrating by parts:

$$
\begin{align*}
&f^{**}(t) - f^*(t) = \frac{1}{t} \int_0^t [f^*(s) - f^*(t)] ds \\
&= \frac{1}{t} \left( sf^*(s) - f^*(t) \right|_{s=0}^{s=t} + \frac{1}{t} \int_0^t [-f^*(s)]' ds \\
&= \frac{1}{t} \int_0^t [-f^*(s)]' ds.
\end{align*}
$$

(5.12)

Therefore,

$$
\begin{align*}
\int_0^t (f^{**}(s) - f^*(s)) s^{-1/n} ds &= \int_0^t \frac{1}{s} \int_0^s u[-f^*(u)]' du s^{-1/n} ds \\
&= -n \int_0^t \left( \int_0^s u[-f^*(u)]' du \right) ds^{-1/n} \\
&= -n \left( \int_0^s u[-f^*(u)]' du \right) s^{-1/n} \bigg|_{s=0}^{s=t} + n \int_0^t [-f^*(s)]' s^{-1/n} ds.
\end{align*}
$$

We claim that we can discard the integrated term since its contribution makes the right hand side smaller. To see this note that, since (5.8) holds, (5.12) implies

$$
\left( \int_0^s u[-f^*(u)]' du \right) s^{-1/n} = (f^{**}(s) - f^*(s)) s^{-1/n} \lesssim \int_0^s (\nabla f)^*(u) du,
$$

which in turn implies that $(\int_0^s u[-f^*(u)]' du) s^{-1/n} \to 0$ when $s \to 0$. Consequently, we can continue our estimates to obtain,

$$
\begin{align*}
\int_0^t (f^{**}(s) - f^*(s)) s^{-1/n} ds &\lesssim n \int_0^t [-f^*(s)]' s^{-1/n} ds \\
&\leq \int_0^t [-f^*(s)]' s^{-1/n} ds \\
&\leq \int_0^t \frac{d}{dt} \left( \int_{\{|f(x)| > f^*(s)\}} |\nabla f(x)| \ dx \right) ds & \text{(by (5.11))} \\
&\leq \int_{\{|f(x)| > f^*(t)\}} |\nabla f(x)| \ dx \\
&\leq \int_0^t (\nabla f)^*(s) ds.
\end{align*}
$$

Underlying these equivalences between weak and strong inequalities is the Maz’ya truncation principle (cf. [34]) which, informally, shows that, contrary to what happens for most other inequalities in analysis, in the case of Sobolev inequalities: weak implies strong!

In [A] we showed the connection of the truncation method to a certain form of extrapolation of inequalities initiated by Burkholder and Gundy. The import of these considerations is that the symmetrization inequalities hold in a very general context and allow for some unification of Sobolev inequalities. For example, the preceding analysis and the corresponding symmetrization inequalities can be extended for gradients defined in metric measure spaces using a variety of methods. One method, often favored by probabilists, goes via defining the gradient by
suitable limits, in this case, under suitable assumptions, we can use isoperimetry to reformulate the symmetrization inequalities and embeddings (cf. [63], [64], and the references therein). In the context of metric probability spaces with concave isoperimetric profile $I$, the basic inequality takes the form
\begin{equation}
  f^{**}(t) - f^{*}(t) \leq \frac{t}{I(t)} |\nabla f|^{**}(t).
\end{equation}

For example, if we consider $\mathbb{R}^n$ with Gaussian measure, the isoperimetric profile satisfies
\begin{equation}
  I(t) \sim t \left(\log \frac{1}{t}\right)^{1/2}, \text{ } t \text{ near zero}.
\end{equation}

Thus in the Gaussian case (5.13) yields logarithmic Sobolev inequalities (cf. [Mm], [63], [64], for more on this story). A somewhat different approach, which yields however similar symmetrization inequalities, obtains if we define the gradient indirectly via Poincaré inequalities and then derive the symmetrization inequalities using maximal inequalities. The analysis here depends on a large body of classical research on maximal functions and Poincaré inequalities (for the symmetrization inequalities that result we refer to [17], and Kalis' 2007 PhD thesis at FAU).

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