Abstract

A copula of continuous random variables $X$ and $Y$ is called an implicit dependence copula if there exist functions $\alpha$ and $\beta$ such that $\alpha(X) = \beta(Y)$ almost surely, which is equivalent to $C$ being factorizable as the $\ast$-product of a left invertible copula and a right invertible copula. Every implicit dependence copula is supported on the graph of $f(x) = g(y)$ for some measure-preserving functions $f$ and $g$ but the converse is not true in general.

We obtain a characterization of copulas with implicit dependence supports in terms of the non-atomicity of two newly defined associated $\sigma$-algebras. As an application, we give a broad sufficient condition under which a self-similar copula has an implicit dependence support. Under certain extra conditions, we explicitly compute the left invertible and right invertible factors of the self-similar copula.

Keywords: Markov operators; non-atomic; bivariate copulas; implicit dependence; self-similar copulas; fractal support

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1 Introduction

It is well-known that the bivariate copula of two random variables completely captures their dependence structure. Notable examples are the independence copula $\Pi(u, v) = uv$, which corresponds to independent random variables, and the copulas of completely dependent random variables, called complete dependence copulas. Since it was discovered [10, 9, 8] that there are complete dependence copulas arbitrarily closed to $\Pi$ in the uniform norm, many norms have been introduced and investigated in the literature [3] giving rise to measures of dependence such as $\omega$ in [17] and $\zeta_1$ in [18]. These dependence measures defined in terms of the copula's first partial derivatives attain the maximum value 1 at least for complete dependence copulas and the minimum value 0 when and only when the copula is $\Pi$. However, with respect to the measure of mutual complete
dependence (MCD) \( \omega \), the independence copula can still be approximated by \textit{implicit dependence copulas} \( [1] \), defined as copulas of random variables \( X \) and \( Y \) which are implicitly dependent in the sense that \( f(X) = g(Y) \) a.s. for some Borel measurable functions \( f \) and \( g \). For Rényi-type measures of dependence \( [13] \) such as \( \omega^* \) in \( [15] \) and \( \nu_\star \) in \( [7] \), with respect to which all complete dependence copulas have measure 1, it can be proved that all implicit dependence copulas also have maximum measure 1. It is then evident that implicit dependence copulas play a crucial role in understanding as well as comparing and contrasting measures of MCD and Rényi-type dependence measures. Closely related and constituting a much larger class than the implicit dependence copulas are the copulas whose supports are that of an implicit dependence copula.

We shall investigate copulas with implicit dependence supports via their corresponding Markov operators and associated pairs of \( \sigma \)-algebras. Our approach is motivated by the characterization of idempotent copulas via their \( \sigma \)-algebras of invariant sets in \( [4] \). In particular, they proved that non-atomic idempotent copulas must be of the form \( L \ast L' \) for some left invertible copula \( L \) (where \( C^T \) denotes the transpose of \( C \), i.e. \( C^T(x,y) = C(y,x) \), and \( \ast \) is the product of copulas first defined and studied in \( [2, 12] \)). We define \( \sigma \)-algebras \( \sigma_C \) and \( \sigma_C^\ast \) associated with a copula \( C \) for which the corresponding Markov operator \( T_C \) maps indicator functions of sets in \( \sigma_C \) to indicator functions of sets in \( \sigma_C^\ast \). We prove that copulas with implicit dependence supports are exactly copulas whose both \( \sigma \)-algebras are non-atomic. Our main result finds an application in copulas with fractal supports introduced by Fredricks, Nelsen and Rodríguez-Lallena \( [6] \).

Given a transformation matrix \( A \), there is a unique \textit{self-similar copula} \( C_A \) such that \( [A](C_A) = C_A \), where \( [A] \) maps the class of bivariate copulas into itself according to the weights given by the entries in \( A \). As a consequence, we obtain a broad sufficient condition on a transformation matrix \( A \) under which the self-similar copula \( C_A \) is non-atomic and hence has an implicit dependence support. Working directly with the transformation matrix \( A \), a sufficient condition under which \( C_A \) is an implicit dependence copula is also given. It would be even more interesting if we had a characterization of implicit dependence copulas via behaviors of their \( \sigma \)-algebras. We are hopeful that our future attempts will not be futile as such a characterization would be beneficial in studies involving products of implicit dependence copulas.

The manuscript is organized as follows. Section \( 2 \) lays the necessary background on copulas and Markov operators for the rest of the paper. We then define the associated \( \sigma \)-algebras of a copula and prove their basic properties in Section \( 3 \). Section \( 4 \) gives a definition of non-atomic copulas and some of their fundamental properties summarizing in a characterization of non-atomic copulas. In the final section, the characterization is used in an investigation of copulas with fractal support. We also give a sufficient condition on a transformation matrix under which the induced invariant copula can be written as the product of a left invertible copula and a right invertible copula.
2 Background on copulas and Markov operators

Let $\lambda$ denote the Lebesgue measure on $\mathbb{R}$, $I \equiv [0, 1]$ and $\mathcal{B} \equiv \mathcal{B}(I)$ the Borel $\sigma$-algebra on $I$. Since we always consider $\lambda$-integrable functions on $I$ that are measurable with respect to various sub-$\sigma$-algebras $\mathcal{M}$ of $\mathcal{B}$, we will denote by $L^1(\mathcal{M})$ the class of $\lambda$-integrable $\mathcal{M}$-measurable functions on $I$. Borel measurable functions are called Borel functions. For $A \in \mathcal{B}$, $1_A$ denotes the indicator function of $A$ and $1 \equiv 1_I$.

A (bivariate) copula $C$ is a function from $I^2$ to $I$ which is the joint distribution function of two random variables uniformly distributed on $[0, 1]$. For random variables $X, Y$ with joint distribution $F_{X,Y}$ and continuous marginal distributions $F_X$ and $F_Y$, there exists, by the Sklar’s theorem, a unique copula $C$, called the copula of $X$ and $Y$, for which $F_{X,Y}(x, y) = C(F_X(x), G_Y(y))$ for all $x, y$. The independence copula is the product copula $\Pi(u, v) = uv$. Complete dependence copulas are either the copulas $C_{\mu_f} \equiv C_{e,f}$ or $C_{\mu_e} \equiv C_{f,e}$ where $e(x) = x$ and $f$ is a measure-preserving function on $I$ in the sense that $\lambda(f^{-1}(B)) = \lambda(B)$ for all $B \in \mathcal{B}$. Here, $C_{\mu_f}(u, v) = \lambda(f^{-1}([0, u]) \cap g^{-1}([0, v]))$ for $u, v \in I$. The comonotonic and countermonotonic copulas are $M = C_{e,e}$ and $W = C_{e,1-e}$, respectively. Each copula $C$ induces a doubly stochastic measure $\mu_C$ by $\mu_C((a, b] \times (c, d]) = C(b, d) - C(b, c) - C(a, d) + C(a, c)$. The support of $C$ is then defined as the support of the induced measure $\mu_C$, i.e. the complement of the union of all open sets having zero $\mu_C$-measure. One can construct a new copula by taking any convex combinations of two copulas. Any two copulas $C, D$ also give rise to a new copula via the *-product: $(C * D)(u, v) = \int_0^1 \partial_x C(u, t) \partial_t D(t, v) dt$. The binary operation $*$ makes the class of copulas a monoid with null element $\Pi$ and identity $M$. If $C * D = M$ then $C$ is a left inverse of $D$ and $D$ is a right inverse of $C$. The left invertible copulas are exactly the complete dependence copulas $C_{e,f}$, while the right invertible copulas are exactly the complete dependence copulas $C_{f,e}$. See [11, 5] for comprehensive introductions to many aspects of copulas.

A linear operator $T$ on $L^1(\mathcal{B})$ is called a Markov operator if

i. $T1 = 1$,

ii. $\int_0^1 T\psi d\lambda = \int_0^1 \psi d\lambda$ for all $\psi \in L^1$, which is equivalent to $T^*1 = 1$, and

iii. $T\psi \geq 0$ for all $\psi \geq 0$, which means $T$ is positive.

So a Markov operator must be a bounded linear operator on $L^1$ (and $L^\infty$). From [12], for each copula $C$, there corresponds a Markov operator $T_C$ defined by

$$(T_C\psi)(x) = \frac{d}{dx} \int_0^1 \partial_x C(x, t) \psi(t) dt.$$  

In fact, the mapping $\Phi: C \mapsto T_C$ is an isomorphism from the set of copulas endowed with the *-product onto the set of Markov operators under the composition. In particular, $T_{C*D} = T_C \circ T_D$. Define also $(T_C^*\varphi)(y) = \frac{d}{dy} \int_0^1 \partial_y C(t, y) \varphi(t) dt$.  

3
The copula $C$ and the Markov operators $T_C$ and $T_C^*$ are also related by the identities

$$C(x, y) = \int_0^x T_C 1_{[0,y]}(t) \, dt \quad \text{and} \quad C(x, y) = \int_0^y T_C^* 1_{[0,x]}(t) \, dt. \quad (1)$$

In fact, it is a good exercise in functional analysis to verify that the Markov operator $T_C^*$ coincides with the extension of the adjoint of $T_C$ to a Markov operator on $L^1$, i.e. $T_C^* = (T_C)^\ast$.

Let us quote a very useful result from \cite{b} Theorem 2.11 where, for brevity, we denote $T_{f,g} = T_{f,g}$.

**Theorem 2.1.** Let $f$ be a measure-preserving Borel function and $\psi \in L^1(\mathcal{B})$. Then $[T_f \psi](x) = \psi \circ f(x)$ and $[T_f(\psi \circ f)](x) = \psi(x)$ for almost all $x \in [0,1]$.

## 3 Associated $\sigma$-algebras

Unless stated otherwise, all equalities of two functions hold $\lambda$-almost everywhere and all equalities of two sets mean that their symmetric difference has Lebesgue measure zero. The integral on the whole unit interval $I$ is denoted simply by $\int$.

Let $C$ be the copula of random variables $X$ and $Y$ which are uniformly distributed on $[0,1]$. Then for any Borel sets $R, S \subseteq [0,1]$, we have

$$T_C 1_S(x) = P(Y \in S | X = x) \quad \text{and} \quad T_C^* 1_R(y) = P(X \in R | Y = y).$$

They are called transition probabilities. See \cite{c} We also have $T_C^* = T_C^\ast$ and $(T_C^\ast)^\ast = T_C$. Roughly speaking, if $T_C 1_S = 1_R$, then it happens with probability one that if $X \in R$ then $Y \in S$. For each copula $C$ or Markov operator $T = T_C$, let us define

$$\sigma_C = \sigma_T \equiv \{ S \in \mathcal{B} : T 1_S = 1_R \text{ for some } R \} \quad \text{and} \quad \sigma_C^\ast = \sigma_T^\ast \equiv \{ R \in \mathcal{B} : T^* 1_R = 1_S \text{ for some } S \}.$$ 

**Example 1.** Let us explicitly compute $T_C$ for $C = \Pi$, $C$ is some complete dependence copulas and $C = \frac{M+W}{2}$.

1. $T_\Pi 1_S = 1_R$ is equivalent to $1_R(x) = \frac{d}{dx} \int_S \partial_2 \Pi(x,t) \, dt = \lambda(S)$ for a.e. $x \in [0,1]$. So $\lambda(S) = 0$ or 1 and hence $\lambda(R) = \lambda(S) = 0$ or 1. Thus, $\sigma_\Pi = \sigma_\Pi^\ast = \{ S \in \mathcal{B} : \lambda(S) = 0 \text{ or } 1 \}$.

2. With essentially the same arguments as that in \cite{d}, $\sigma_M, \sigma_M^\ast, \sigma_W$ and $\sigma_W^\ast$ are the Borel sets.

3. For $C = \frac{M+W}{2}$, $T_C 1_S(x) = \frac{1}{2} 1_S(x) + \frac{1}{2} 1_S(1-x)$ a.e. $x$. If $S$ is symmetric with respect to $\frac{1}{2}$, i.e. $x \in S$ if and only if $1 - x \in S$, then $T_C 1_S = 1_S$. Conversely, $T_C 1_S(x) = 1_R(x)$ implies that $x \in S$ if and only if $1 - x \in S$. Moreover, for such a symmetric set $S$, $(1_R - 1_S)(x) = (T_C 1_S - 1_S)(x) = \frac{1}{2} (1_S(x) - 1_S(1-x)) = 0$ a.e. $x$. That is, $R = S$. Hence, $\sigma_{\frac{M+W}{2}} = \sigma_{\frac{M+W}{2}}^\ast = \{ S \subseteq [0,1] : x \in S \iff 1 - x \in S \}$. 

4
4. For $0 < \alpha < 1$, let $L_\alpha$ denote the complete dependence copula whose support consists of the line segments $y = \frac{x}{\alpha}$, $0 \leq x \leq \alpha$, and $y = \frac{x}{1-\alpha}$, $\alpha < x \leq 1$. Then a direct computation yields $T_{L_\alpha}I_S = I_{(\alpha S) \cup (\alpha+(1-\alpha)S)}$ and hence $\sigma_{L_\alpha} = \mathcal{B}$ and $\sigma_{L_\alpha}^* = \{ \alpha S \cup (\alpha+(1-\alpha)S) : S \in \mathcal{B} \}$.

Listed below are basic properties of sets $S$ and $R$ linked by $T$.

**Proposition 3.1.** Let $T$ be a Markov operator and $S_1, S_2, R_1, R_2 \in \mathcal{B}$. Then

1. if $T I_{S_1} = I_{R_1}$ then $T^* I_{R_1} = I_{S_1}$;
2. if $T I_{S_1} = I_{R_1}$ then $T I_{S_1^c} = I_{R_1^c}$;
3. if $T I_{S_i} = I_{R_i}$ for $i = 1, 2$, then $T (I_{S_1 \cup S_2} = I_{R_1 \cap R_2}$;
4. the classes $\sigma_T$ and $\sigma_T^*$ are $\sigma$-algebras; and
5. $\sigma_T^* = \sigma_{T^*}$, that is $\sigma_T^* = \sigma_{C^*}$.

**Proof.**

By the definition of $T^* I_R$ applied to $I_S$ with the canonical identification between $(L^1)^*$ and $L^\infty$, $\int_S T^* I_{R} d\lambda = \int_S I_{S} T^* I_R d\lambda = \int I_R I_R d\lambda = \lambda(R) = \int_S I_S d\lambda$. Since $0 \leq T^* I_R \leq 1$ (a.e.), $T^* I_R = I_S$ as desired. If $T I_S = I_R$ then it follows from $T (I_S + I_{S^c}) = T (1) = 1$ that $T I_{S^c} = I_{R^c}$. Suppose $T I_{S_i} = I_{R_i}$ for $i = 1, 2$ and let $S = S_1 \cap S_2$, $R = R_1 \cap R_2$. Since $S \subseteq S_1, I_{S_1} - I_S \geq 0$ and so $I_{R_1} - T I_{S} = T (I_{S_1} - I_S) \geq 0$. Therefore $T I_S \leq \min(I_{R_1}, I_{R_2}) = I_{R_1 \cap R_2} = I_R$. But $\int (I_R - T I_S) d\lambda = \int I_R d\lambda - \int I_S d\lambda = \lambda(R) - \lambda(S) = 0$, where the last equality follows from considering $T^* I_{R_1}(I_{S_1}) = I_{R_2}(T I_{S_1})$. Hence, $T I_S = I_R$. Suppose $T I_{S_i} = I_{R_i}$ for all $i \in \mathbb{N}$. If $S_1, S_2, \ldots$ are mutually disjoint then so is the sequence $R_1, R_2, \ldots$ because $\sum_{i=1}^{\infty} I_{R_i} = \sum_{i=1}^{\infty} T I_{S_i} = T (\sum_{i=1}^{\infty} I_{S_i}) = T (I_{\cup_{i=1}^{\infty} S_i}) \leq I$. Thus, $T (I_{\cup_{i=1}^{\infty} S_i}) = I_{\cup_{i=1}^{\infty} R_i}$. Generally, we write $\cup_i S_i$ as the disjoint union $\cup_i S_i$, where $\hat{S}_i = S_i \setminus \cup_{j<i} \hat{S}_j$. Letting $T I_{\hat{S}_i} = I_{R_i}$, it follows from the disjoint union case above that $T I_{\hat{S}_i \cup \hat{S}_j} = I_{R_i \setminus \cup_{j<i} R_j}$. Hence, $R_i = R_i \setminus \cup_{j<i} R_j$ by induction. Consequently, $T (I_{\cup_{i=1}^{\infty} S_i}) = I_{\cup_{i=1}^{\infty} R_i}$. This clearly follows from the fact that $T^{**} = T$. □

**Remark.** By Theorem 2.3, $T_L^* \circ T_L = I$, the identity map, on $L^1(\mathcal{B})$ if the copula $L$ is left invertible. However, it follows from the above Proposition that for all copula $C$, $T_C^* \circ T_C = I$ on $\{ I_S : S \in \sigma(C) \}$, or equivalently on the class of $\sigma(C)$-measurable functions.

**Theorem 3.2.** Let $T$ be a Markov operator with associated $\sigma$-algebras $\sigma_T$ and $\sigma_T^*$. If $\psi$ is $\sigma_T$-measurable then $T \psi$ is $\sigma_T^*$-measurable.

**Proof.** By the linearity of $T$ and the definition of $\sigma_T$ and $\sigma_T^*$, if $\psi$ is a simple $\sigma_T$-measurable function then $T \psi$ is simple and $\sigma_T^*$-measurable. The case when $\psi$ is $\sigma_T$-measurable follows from the continuity of $T$. □
4 Non-atomic copulas

Definition 1. Let $\mathcal{S}$ be a sub-$\sigma$-algebra of $\mathcal{B}$. A set $S$ in $\mathcal{S}$ is called an atom in $\mathcal{S}$ if $\lambda(S) > 0$ and for all $E \in \mathcal{S}$, either $\lambda(S \cap E) = \lambda(S)$ or $\lambda(S \cap E) = 0$. The $\sigma$-algebra $\mathcal{S}$ is said to be non-atomic if there are no atoms in $\mathcal{S}$; otherwise, it is called atomic.

The $\sigma$ is called totally atomic if there is a (countable) collection of essentially disjoint atoms $E_1, E_2, \ldots$ in $\mathcal{S}$ such that $\sum_i \lambda(E_i) = 1$. We say that a bivariate copula $C$ is non-atomic if both $\sigma$-algebras $\sigma_C$ and $\sigma_{C^*}$ are non-atomic. And $C$ is called totally atomic if both $\sigma_C$ and $\sigma_{C^*}$ are totally atomic.

Note that the non-atomicity is a generalization of the notion of the same name in [4], which is defined only for idempotent copulas $C$ via their invariant sets defined as Borel sets $S$ for which $T_C 1_S = 1_S$. However, the two notions agree for idempotent copulas.

Proposition 4.1. If a copula $C$ is non-atomic and idempotent then $\sigma_C = \sigma_{C^*}$ is the $\sigma$-algebra of invariant sets.

Proof. Since idempotent copula is symmetric, $T^*_C = T_{C^*} = T_C$. Consequently, $T^*_C \circ T_C = T_C \circ T_C = T_{C \circ C} = T_C$. Now, if $T_C 1_S = 1_R$ then $T^*_C 1_R = 1_S$ and so $T^*_C \circ T_C 1_S = 1_S$. Hence $S$ is an invariant set of $C$ and so is $R$ as $1_R = T_C 1_S = 1_S$. That is, $\sigma_C$ and $\sigma_{C^*}$ are subsets of the class of invariant sets. The converse inclusions are clear. \hfill \Box

Example 2. It is evident from the computations of $\sigma_C$ and $\sigma_{C^*}$ in Example 1 that $\Pi$ is totally atomic but $M, W, \frac{M \wedge W}{2}$ and $L_\alpha$ are non-atomic. In fact, every complete dependence copula is non-atomic.

Proposition 4.2. Let $C$ be a copula.

1. $\sigma_C = \mathcal{B}$ if and only if $C$ is left invertible.

2. $\sigma_{C^*} = \mathcal{B}$ if and only if $C$ is right invertible.

Proof. We shall prove only 1 as 2 can be proved in a similar manner. If $C = C_{eg}$ then $T_C = T_{eg}$ maps $\psi$ to $\psi \circ g$. So for every $B \in \mathcal{B}$, $T_C 1_B = 1_B \circ g = 1_{g^{-1}(B)}$. Hence, $\sigma_C = \mathcal{B}$. Conversely, Theorem 5 in [10] implies in particular that $T$ is multiplicative, meaning $T(\psi \cdot \phi) = T\psi \cdot T\phi$ for all $\psi, \phi \in \mathcal{L}^\infty$, if and only if $T = T_{eg}$ for some measure-preserving function $g$. By the linearity and continuity of $T = T_{eg}$, it then suffices to show that $T(1_B \cdot 1_E) = 1_B \cdot T1_E$ for all $B, E \in \mathcal{B}$. In fact, if $T1_B = 1_{B'}$ and $T1_E = 1_{E'}$ then $T1_{B \cap E} = 1_{B' \cap E'}$ (by Proposition 3.1[10]) and the desired equality follows. \hfill \Box

Lemma 4.3. Let $C$ be a copula with associated Markov operator $T_C$ and doubly stochastic measure $\mu_C$ and $R, S \subseteq \mathcal{B}$. $T_C 1_S = 1_R$ if and only if $\mu_C(R \times S) = \lambda(R) = \lambda(S)$.

Proof. Note the fact that $\mu_C(R \times S) = \int_R T_C 1_S d\lambda$. If $T_C 1_S = 1_R$ then $\mu_C(R \times S) = \lambda(R)$ and, by the property of $T_C$, $\lambda(R) = \int 1_R d\lambda = \int T_C 1_S d\lambda$. 

6
\[ \int 1_S \, d\lambda = \lambda(S). \] Conversely, if \( \mu_C(R \times S) = \lambda(R) = \lambda(S) \) then \( \int_R T_C 1_S \, d\lambda = \mu_C(R \times S) = \lambda(S) = \int 1_S \, d\lambda = \int T_C 1_S \, d\lambda \) and \( \int_R T_C 1_S \, d\lambda = \mu_C(R \times S) = \lambda(R) = \int 1_R \, d\lambda. \) Since \( T_C 1_S(t) \in [0,1], T_C 1_S(t) = 0 \) for all \( t \notin R \) and \( T_C 1_S(t) = 1 \) for all \( t \in R \), i.e. \( T_C 1_S = 1_R. \) \hfill \Box

**Proposition 4.4.** Let \( T \) be a Markov operator and \( f \) and \( g \) be measure-preserving functions on \([0,1]\). Then the following are equivalent.

1. \( T[1^g_f] = 1^g_{f^{-1}(B)} \) for all \( B \in \mathcal{B}. \)
2. \( T(\theta \circ g) = \theta \circ f \) for all Borel functions \( \theta \in L^1. \)
3. \( T\psi = T_{\text{ef}} \circ T_{\text{ge}} \psi \) for all \( g^{-1}(\mathcal{B}) \)-measurable functions \( \psi. \)

**Proof.** \( \Box \) For every Borel set \( B \subseteq I, T(1_B \circ g) = T[1^g_f] 1_B = 1_B \circ f. \) So, \( T(\psi \circ g) = \psi \circ f \) for all simple Borel functions \( \psi. \) By the standard measure-theoretic argument, \( T(\theta \circ g) = \theta \circ f \) for every Borel function \( \theta. \)

2 \( \Rightarrow \) 3 Using Theorem 2.1 \( 2 \) implies that \( T_{fe} \circ T_{ge} \theta = T_{fe}(T_{ge}(\theta \circ g)) = T_{fe}(\theta \circ f) = \theta \) for every Borel function \( \theta, \) and hence \( T_{fe} \circ T_{ge} = I = T_M. \) This means that \( C_{ef} \ast C = C_{ge}. \) Left multiplying by \( C_{ef} \) yields \( C_{ef} \ast C_{ef} = C_{ef} \ast C_{ge}, \) i.e. \( T_{ef} \circ T_{fe} \circ T = T_{ef} \circ T_{ge}. \) On the other hand, by Theorem 2.1 for any Borel function \( \theta, T_{ef} \circ T_{fe}(\theta \circ f) = (\theta \circ f) \) which gives \( T_{ef} \circ T_{fe} \circ T(\theta \circ g) = T(\theta \circ g). \) Since \( \{ \theta \circ g; \theta \) is a Borel function\} coincides with the class of \( g^{-1}(\mathcal{B}) \)-measurable functions, \( T_{ef} \circ T_{fe} \circ T = T \) on the class of \( L^1 \)-functions which are \( g^{-1}(\mathcal{B}) \)-measurable. Therefore, \( T\psi = (T_{ef} \circ T_{ge}) \psi \) for all \( (g^{-1}(\mathcal{B}) = \sigma_C) \)-measurable \( \psi \in L^1. \)

3 \( \Rightarrow \) 1 This is clear from taking \( \psi = 1^g_{g^{-1}(B)}. \) \hfill \Box

The equivalence relation \( \approx \) on \( \mathcal{B} \) is defined as follows: \( E \approx F \) if and only if the symmetric difference \( E \Delta F \) has Lebesgue measure zero. Of course, \( \approx \) is still an equivalence relation on \( \mathcal{J} \subseteq \mathcal{B}. \) The equivalence class of \( S \) in \( \mathcal{J} \) is denoted by \([S]_\mathcal{J}\) or just \([S]\) if no confusion can arise. The collection of equivalence classes in \( \mathcal{J} \) is denoted by \([\mathcal{J}]\). \([\mathcal{J}]\) is in fact a measure algebra induced by the Lebesgue measure \( \lambda. \) That is, \([\mathcal{J}]\) is a Boolean \( \sigma \)-algebra with respect to the operations \([S] \lor [R] = [S \cup R] \) and \([S] \land [R] = [S \cap R]\) together with \( \lambda: [\mathcal{J}] \to [0,1]\) defined overadditively by \( \lambda([S]) = \lambda(S) \) and satisfying \( \lambda(\bigvee_{i=1}^\infty \{A_i\}) = \sum_{i=1}^\infty \lambda([A_i]) \) if \([A_i] \land [A_j] = [\emptyset]\) for \( i \neq j. \) See [14, p.398].

\( T_C \) induces a well-defined equivalence class function \( \Upsilon_C: [\sigma_C] \to [\sigma_C^R] \) mapping \([S]\) to \([R]\) if and only if \( T_C 1_S = 1_R. \) It follows from the defining property \( \text{ii} \) of \( T \) that \( \lambda(R) = \lambda(S) \) and hence \( \Upsilon_C \) is measure-preserving. The following lemma summarizes results from Theorem 3.2 and the proof of Theorem 3.3 in [4].

**Lemma 4.5** ([4]). Let \( \mathcal{J} \subseteq \mathcal{B} \) be a non-atomic \( \sigma \)-algebra. Then

1. there exists a surjective isomorphism \( \Psi: [\mathcal{J}] \to [\mathcal{B}] \) which means \( \Psi([S]) = \Psi([S]) \) and \( \Psi([S]_\mathcal{J}) = \Psi([S]) \) for all \( [S]_\mathcal{J} \in [\mathcal{J}] \) and \( \Psi \) is onto (it follows that \( \Psi \) is one-to-one and an isometry with respect to the
metric $\rho([S],[R]) = \lambda([S] \triangle [R])$ and preserves countable unions and intersections); and

2. for any surjective isomorphism $\Psi : [\mathcal{S}] \to [\mathcal{B}]$, there exists a unique (a.e.) measure-preserving Borel function $h : [0,1] \to [0,1]$ such that $h^{-1}(\mathcal{B}) \subseteq \mathcal{S}$ (in fact, they are essentially equivalent,) and that $h^{-1}(B) \in \Psi^{-1}([B])$ for all $B \in \mathcal{B}$.

Proof. Enumerate $\mathbb{Q} \cap [0,1] = \{r_n\}_{n \in \mathbb{N}}$ and set $I_n = [0,r_n]$. For each $n \in \mathbb{N}$, choose $S_n$ in the equivalence class $\Psi^{-1}([I_n])$ so that $S_k = [0,1]$ if $r_k = 1$ and $S_k \subseteq S_l$ whenever $r_k < r_l$. Define a Borel measure-preserving function $h(x) = \inf \{r_k : x \in S_k\}$. For $S \in \mathcal{S}$, choose $B_0 \in \Psi([S])$ and set $S_0 = h^{-1}(B_0)$. Since $\Psi$ is 1-1, $[S] = \Psi^{-1}([B_0])$. We prove that $h^{-1}(B_0) \in \Psi^{-1}([B_0])$ by considering $\mathcal{M} = \{B \in \mathcal{B} : h^{-1}(B) \in \Psi^{-1}([B])\}$. Since $\mathcal{M}$ is a monotone class containing all $[0,r_k]$, it contains all Borel sets and we have the claim. To prove uniqueness, if $k : [0,1] \to [0,1]$ is a measure-preserving Borel function such that $k^{-1}(B) \in \Psi^{-1}([B])$ for all $B \in \mathcal{B}$, then $k^{-1}([0,r_n]) \in \Psi^{-1}([0,r_n])$. So $\lambda(\{h^{-1}(0,r_n) \triangle k^{-1}(0,r_n)\}) = 0$ for all $n$. By Proposition 10 in Chapter 11 (p. 261), $h = k$ a.e. \[\square\]

**Theorem 4.6.** Let $C$ be a copula with associated Markov operator $T_C$ and doubly stochastic measure $\mu_C$. For measure-preserving functions $f$ and $g$ on $[0,1]$, the following are equivalent.

1. $\mu_C(\text{graph}\{f = g\}) = 1$, where $\text{graph}\{f = g\} = \{(x,y) : f(x) = g(y)\}$.

2. $T_C = T_{ef} \circ T_{ge}$ on the class of $g^{-1}(\mathcal{B})$-measurable functions.

3. $C$ is non-atomic with $\sigma_C \supseteq g^{-1}(\mathcal{B})$ and $\sigma_C \supseteq f^{-1}(\mathcal{B})$.

Proof. $[1] \Rightarrow [2]$ By Proposition 4.4 Lemma 4.3 and the measure-preserving property of $f$ and $g$, it suffices to show that $\mu_C(f^{-1}(B) \times g^{-1}(B)) = \lambda(f^{-1}(B))$ for all $B \in \mathcal{B}$. This follows from $\mu_C(f^{-1}(B) \times g^{-1}(B)) \leq \mu_C(f^{-1}(B) \times I) = \lambda(f^{-1}(B))$ and

$$\mu_C(f^{-1}(B) \times g^{-1}(B)) \geq \mu_C(\{(x,y) : f(x) = g(y) \in B\})$$

$$= \mu_C(\{(x,y) : f(x) = g(y) \} \cap \{(x,y) : f(x) \in B\})$$

$$= \lambda(f^{-1}(B)).$$

$[2] \Rightarrow [1]$. For each $n \in \mathbb{N}$ and $i = 1, 2, \ldots, 2^n$, put $I_{i,n} = \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]$ and $B_n = \bigcup_{i=1}^{2^n} f^{-1}(I_{i,n}) \times g^{-1}(I_{i,n})$. Then $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \supseteq \cdots$. And

$$\mu_C(B_n) = \sum_{i=1}^{2^n} \mu_C(f^{-1}(I_{i,n}) \times g^{-1}(I_{i,n})) = \sum_{i=1}^{2^n} \lambda(f^{-1}(I_{i,n})) = \sum_{i=1}^{2^n} \lambda(I_{i,n}) = 1,$$

where we have used Lemma 4.3 in the second equality and the measure-preserving property of $f$ in the third. Set $B = \bigcap_{n=1}^{\infty} B_n$. We have $\mu_C(B) = \lim_{n \to \infty} \mu_C(B_n) = \lambda(f^{-1}(B))$. 

8
1. It then suffices to show that \( B = \{ (x,y) : f(x) = g(y) \} \). First, if \( f(x) \neq g(y) \) then there exists \( n \in \mathbb{N} \) such that \( \frac{1}{2^n} < |f(x) - g(y)| \) and hence \((x,y) \notin B_n\). Conversely, it is clear that \( \{ (x,y) : f(x) = g(y) \} = \bigcup_i \{ (x,y) : f(x) = g(y) \in I_i,n \} \subseteq B_n \) for all \( n \).

\( \Rightarrow \) By Proposition 4.4, \( T_C \mathbb{1}_{g^{-1}(B)} = \mathbb{1}_{f^{-1}(B)} \) for all \( B \in \mathcal{B} \), which implies that \( g^{-1}(\mathcal{B}) \subseteq \sigma_C \) and \( f^{-1}(\mathcal{B}) \subseteq \sigma_C^* \). But both \( g^{-1}(\mathcal{B}) \) and \( f^{-1}(\mathcal{B}) \) are non-atomic because \( g \) and \( f \) are measure-preserving. Therefore the finer \( \sigma \)-algebras \( \sigma_C \) and \( \sigma_C^* \) are non-atomic as well.

\( \Rightarrow \) Our proof is in three steps. i) By applying Lemma 4.5(1) to \( \sigma_C \) and \( \sigma_C^* \), it follows that \( \mathcal{Y}_C : [\sigma_C] \to [\sigma_C^*] \) defined earlier is one-to-one, onto, measure-preserving and preserves order, complementation and the lattice operation on equivalence classes corresponding to countable unions of monotonic sequence of sets, i.e. \( \mathcal{Y}_C \) is a surjective isomorphism. ii) By Lemma 4.5(2), there exists \( \mathcal{E}_C : [\sigma_C^*] \to [\mathcal{B}] \) and a unique (a.e.) Borel function \( f : I \to I \) such that \( f^{-1}(B) \in \mathcal{E}_C^{-1}([B]) \) for all \( B \in \mathcal{B} \). So the composition \( \mathcal{E}_C \circ \mathcal{Y}_C : [\sigma_C] \to [\mathcal{B}] \) satisfies the same properties and induces a unique Borel function \( g : I \to I \) such that

\[
g^{-1}(B) \in (\mathcal{E}_C \circ \mathcal{Y}_C)^{-1}([B]) \quad \text{for all } B \in \mathcal{B}. \tag{2}
\]

iii) From (2), \( \mathcal{E}_C \left( \{ g^{-1}(B) \} \right) = \mathcal{E}_C^{-1}([B]) \) and, by the definition of \( \mathcal{Y}_C, T_C \mathbb{1}_{g^{-1}(B)} = \mathbb{1}_{f^{-1}(B)} \) for all \( B \in \mathcal{B} \). And the proof is done by Proposition 4.4. □

In particular, for measure-preserving functions \( f,g \) on \( I \), the support of \( C_{ef} \ast C_{ge} \) "is" the graph of \( f(x) = g(y) \), that is \( \mu(C_{ef} \ast C_{ge} (\text{graph } \{ f = g \})) = 1 \), and its mass is distributed uniformly in the sense that \( T_{C_{ef} \ast C_{ge}} \mathbb{1}_B = T_{ef}(T_{ge} \mathbb{1}_B) \) is \((\sigma_C^* = f^{-1}(\mathcal{B}))-\text{measurable for all } B \in \mathcal{B}.

Example 3. For a fixed \( \alpha \in (0,1) \), consider \( C_\alpha = \alpha M + (1-\alpha)W \) with Markov operator \( T_\alpha \). It is readily verified that \( \sigma_{C_\alpha} = \sigma_{C_\alpha}^* = \{ S \in \mathcal{B} : S = 1-S \} \) and \( T_\alpha \mathbb{1}_S = \mathbb{1}_S \) for all \( S \in \sigma_{C_\alpha} \). But only \( T_\frac{1}{2} \) has the property that \( T_\frac{1}{2} \mathbb{1}_B \) is \( \sigma_{C_\frac{1}{2}} \)-measurable for all \( B \in \mathcal{B} \). Note also that \( C_\frac{1}{2} = L_\frac{1}{2} \ast L_\frac{1}{2}. \)

5 Copulas with fractal support

Let us recall the construction of copulas with self-similar supports in Fredricks et al. [2] put in the context of patched copulas [21, 4].

Definition 2. A transformation matrix is a matrix \( A \) with nonnegative entries, for which the sum of all entries is 1 and every row and column has at least one non-zero entry.

Given a transformation matrix \( A = [a_{ij}]_{k \times \ell} \) where the first index \( i \) is the column number from left \( (i = 1) \) to right \( (i = k) \) and the second index \( j \) is the row number from bottom \( (j = 1) \) to top \( (j = \ell) \), the matrix multiplication of \( A \) and \( B = [b_{jm}]_{\ell \times n} \) is defined by \( [a_{ij}] [b_{jm}] = \sum_{j=1}^{\ell} a_{ij} b_{jm} \). \( A^t = [a_{ji}]_{\ell \times k} \).
This unconventional entry arrangement syncs well with the product of copulas denoted by \( [A] \) as
\[
C \ast D(u,v) = \int_0^1 \partial_2 C(u,t) \partial_1 D(t,v) \, dt.
\]
Let \( p_i \) denote the sum of the entries in the first \( i \) columns of \( A \) and let \( q_j \) denote the sum of the entries in the first \( j \) rows of \( A \) where \( p_0 = 0 \) and \( q_0 = 0 \). The partitions \( \{ p_i \}_{i=0}^k \) and \( \{ q_j \}_{j=0}^\ell \) of \([0,1]\) yield a rectangular partition of \([0,1]^2\) consisting of \( k \cdot \ell \) rectangles with vertices \((p_i,q_j)\). Given a copula \( C \), we construct a new copula denoted by \([A](C)\) by placing a scaled copy of \( C \) in each of the \( k \cdot \ell \) rectangles weighted according to the mass given by the corresponding entry in \( A \). It is defined for \((u,v) \in R_{ij} = [p_{i-1},p_i] \times [q_{j-1},q_j]\) by
\[
[A](C)(u,v) = \sum_{i'<i,j'<j} a_{i'j'} + \frac{u-p_{i-1}}{p_i-p_{i-1}} \sum_{j'<j} a_{ij'} + \frac{v-q_{j-1}}{q_j-q_{j-1}} \sum_{i'<i} a_{i'j} + a_{ij} C \left( \frac{u-p_{i-1}}{p_i-p_{i-1}}, \frac{v-q_{j-1}}{q_j-q_{j-1}} \right).
\]
Here, empty sums are zero by convention. See [4] for more details. It will be more convenient to view \([A](C)\) as the so-called *patched copula* [21][1] defined as
\[
[A](C)(u,v) = \sum_{i=1}^k \sum_{j=1}^\ell a_{ij} C(F_i(u),G_j(v))
\]
where \( F_i(u) = \min \left( 1, \frac{u-p_{i-1}}{p_i-p_{i-1}} \right) \mathbb{1}_{(p_{i-1},\infty)}(u) \) is the uniform distribution on \([p_{i-1},p_i]\) and \( G_j(v) = \min \left( 1, \frac{v-q_{j-1}}{q_j-q_{j-1}} \right) \mathbb{1}_{(q_{j-1},\infty)}(v) \) is the uniform distribution on \([q_{j-1},q_j]\). Denote \( \Delta p_i = p_i - p_{i-1} \) and \( \Delta q_j = q_j - q_{j-1} \).

We then investigate how this rectangular patching of a copula \( C \) according to the transformation matrix \( A \) affects \( C \) in terms of their Markov operators.

\[
T_{[A](C)} f(x) = \sum_{i=1}^k \sum_{j=1}^\ell a_{ij} \frac{d}{\Delta q_j} \int_{q_{j-1}}^{q_j} \partial_2 C(F_i(x),G_j(t)) \, f(t) \, dt \\
= \sum_{i=1}^k \sum_{j=1}^\ell a_{ij} \frac{d}{\Delta p_i} \int_0^1 \partial_2 C(F_i(x),s) \, f \circ G_j^{-1}(s) \, ds \\
= \sum_{i=1}^k \sum_{j=1}^\ell \frac{a_{ij}}{\Delta p_i} T_C(f \circ G_j^{-1})(F_i(x)). \tag{3}
\]

Consequently, \( T_{[A](C)}(f)(F_i^{-1}(x)) = \sum_{j=1}^\ell \frac{a_{ij}}{\Delta p_i} T_C(f \circ G_j^{-1})(x) \), which can be written in matrix form as
\[
\left[ T_{[A](C)}(f)(F_i^{-1}(x)) \right]_{k \times 1} = \left[ \frac{a_{ij}}{\Delta p_i} \right]_{k \times \ell} \left[ T_C(f \circ G_j^{-1})(x) \right]_{\ell \times 1}.
\]
See [19, 20] for essentially the same identity in terms of Markov kernels.

Define inductively \( |A|^n(C) = |A|\left( |A|^{n-1}(C) \right) \) for \( n \geq 2 \). Fredricks et al. [6] showed that for any transformation matrix \( A \neq [1] \) and copula \( C, |A|^n(C) \) converges (pointwise and hence uniformly) to a unique copula \( C_A \), as \( n \to \infty \). Moreover, \( C_A \) is the fixed point of \( |A| \), i.e. \( |A|(C_A) = C_A \). However, since uniform convergence will not suffice for our purposes, we shall investigate the convergence of \( |A|^k(D) \) with respect to a stronger norm with respect to which the \( \ast \)-product is jointly continuous. We choose the modified Sobolev norm defined as \( \|C\|_S^2 = \|C\|_1^2 + \|C\|_2^2 \) where \( \|C\|_1^2 = \int_0^1 \int_0^1 |\partial SC(u, v)|^2 \, du \, dv \). See [3].

**Proposition 5.1.** Let \( A \) be a transformation matrix whose dimension is at least \( 2 \times 2 \) and let \( C \) and \( D \) be copulas. Then

\[
\| [A](C) - [A](D) \|_S \leq r \| C - D \|_S
\]

where \( r^2 = \max_{i,j} \left( \sum_{j=0}^{k} \frac{a_{ij}^2 q_j - q_j - 1}{p_i - p_i - 1} \sum_{j=0}^{k} \frac{a_{ij}^2 p_i - p_i - 1}{q_j - q_j - 1} \right) < 1 \) and \( \| \cdot \|_S \) is the modified Sobolev norm.

**Proof.** Denote \( A = [a_{ij}]_{k \times \ell} \) with \( k, \ell \geq 2 \) and let \( \{p_i\}_{i=0}^{k}, \{q_j\}_{j=0}^{\ell} \) be the induced partitions of \([0, 1]\) on the \( x \)-axis and \( y \)-axis, respectively. For \( u \in (p_i - 1, p_i) \) and \( v \in (q_j - 1, q_j) \), \( \partial_1 [A](C)(u, v) = a_{ij} F_i^j(u) \partial_1 C(F_i(u), G_j(v)) + \sum_{k=1}^{k} a_{ik} F_i^j(u) \) and so

\[
|\partial_1 [A](C)(u, v) - \partial_1 [A](D)(u, v)| = \frac{a_{ij}}{p_i - p_i - 1} |\partial_1 C(F_i(u), G_j(v)) - \partial_1 D(F_i(u), G_j(v))|.
\]

Hence, \( \| [A](C) - [A](D) \|_S^2 \) is equal to

\[
\sum_{i=1}^{k} \sum_{j=1}^{\ell} \frac{a_{ij}^2 q_j - q_j - 1}{p_i - p_i - 1} \int_{q_j - 1}^{q_j} \int_{p_i - 1}^{p_i} |\partial_1 C(F_i(u), G_j(v)) - \partial_1 D(F_i(u), G_j(v))|^2 \, du \, dv
\]

\[
= \left( \sum_{i=1}^{k} \sum_{j=1}^{\ell} \frac{a_{ij}^2 q_j - q_j - 1}{p_i - p_i - 1} \right) \| C - D \|_1^2.
\]

A similar proof yields \( \| [A](C) - [A](D) \|_2^2 = \left( \sum_{i=1}^{k} \sum_{j=1}^{\ell} \frac{a_{ij}^2 p_i - p_i - 1}{q_j - q_j - 1} \right) \| C - D \|_2^2 \).

Now, since \( a_{ij} \leq p_i - p_i - 1 \) and \( \sum_i a_{ij} = q_j - q_j - 1 \),

\[
\sum_{j=1}^{\ell} \sum_{i=1}^{k} a_{ij}^2 \frac{q_j - q_j - 1}{p_i - p_i - 1} \leq \sum_{j=1}^{\ell} \sum_{i=1}^{k} a_{ij} (q_j - q_j - 1) = \sum_{j=1}^{\ell} (q_j - q_j - 1)^2 < \sum_{j=1}^{\ell} (q_j - q_j - 1) = 1,
\]

where we have used the assumption that \( \ell \geq 2 \) in the last inequality. In fact, the same assumption implies that \( a_{ij} < p_i - p_i - 1 \) for some \( j \). The same line of proof using \( k \geq 2 \) shows \( \sum_{j=1}^{\ell} \sum_{i=1}^{k} a_{ij}^2 \frac{p_i - p_i - 1}{q_j - q_j - 1} < 1 \) and the desired result follows.\( \square \)
We now have that the mapping \([A]\) is a contraction on the class of copulas \(C_2\) with respect to the modified Sobolev norm. Using the fact that \(C_2\) is complete with respect to the modified Sobolev norm, it follows from the Contraction-Mapping Theorem that:

**Theorem 5.2.** For each transformation matrix \(A\) of dimension at least \(2 \times 2\), and for any initial copula \(D\), the sequence \(\{[A]^r(D)\}\), converges to the copula \(C_A\) in the modified Sobolev norm.

**Definition 3.** Let \(A = [a_{ij}]_{k \times \ell}\) be a transformation matrix, \(0 \neq I \subset \{1, 2, \ldots, k\}\) and \(\emptyset \neq J \subset \{1, 2, \ldots, \ell\}\). The pair \((I, J)\) is called an invariant pair of \(A\) if \(a_{ij} = 0\) for all \((i,j) \in (I \times J^c) \cup (J^c \times I)\) and \(a_{ij} > 0\) for some \((i,j) \in I \times J\). Two invariant pairs \((I_1, J_1)\) and \((I_2, J_2)\) of \(A\) are called disjoint if \(I_1 \cap J_2 = \emptyset\) and \(I_1 \cap J_2 = \emptyset\). It follows that if they are not disjoint then both \(I_1 \cap I_2\) and \(J_1 \cap J_2\) are not empty.

We say that \(A\) is disjointly decomposable if \(A\) has a finite number of pairwise disjoint invariant pairs \((I_1, J_1), (I_2, J_2), \ldots, (I_N, J_N)\) such that \(\bigcup_{n=1}^N I_n = \{1, 2, \ldots, k\}\) and \(\bigcup_{n=1}^N J_n = \{1, 2, \ldots, \ell\}\). For each \(n = 1, 2, \ldots, N\), let us denote by \(A_n\) the \(k \times \ell\) matrix whose \((i,j)\)th entry is \(a_{ij}\) if \(i \in I_n\) and \(j \in J_n\) and is equal to zero otherwise. Observe that \(A = \sum_{n=1}^N A_n\) and in particular, every non-zero entry in \(A\) appears in exactly one \(A_n\). We also say that \(A\) is disjointly decomposable as the sum \(\sum_{n=1}^N A_n\) or disjointly decomposable by \(N\) invariant pairs.

Such a disjoint decomposition of \(A\) gives rise to two partitions of \([0, 1]\):

\[
\{Q_n\}_{n=1}^N\text{ and } \{P_n\}_{n=1}^N
\]

defined by \(Q_n = \bigcup_{j \in J_n} (q_{j-1}, q_j)\) and \(P_n = \bigcup_{i \in I_n} (p_{i-1}, p_i)\). Then \(\lambda(Q_n) > 0\) and \(\lambda(P_n) > 0\) for all \(n = 1, \ldots, N\). Each pair \((I_n, J_n)\) induces set functions \(\mathscr{G}_n\) and \(\mathscr{F}_n\) mapping \(B \in \mathscr{B}\) to

\[
\mathscr{G}_n(B) = \bigcup_{j \in J_n} G^{-1}_j(B) \subseteq Q_n \text{ and } \mathscr{F}_n(B) = \bigcup_{i \in I_n} F^{-1}_i(B) \subseteq P_n.
\]

Note that \(G^{-1}_j((0,1)) = (q_{j-1}, q_j)\) and \(F^{-1}_i((0,1)) = (p_{i-1}, p_i)\).

**Remark.** If \(A\) has \(N\) invariant pairs \((I_1, J_1), (I_2, J_2), \ldots, (I_N, J_N)\), not necessarily disjoint, such that \(\bigcup_{n=1}^N I_n = \{1, 2, \ldots, k\}\) and \(\bigcup_{n=1}^N J_n = \{1, 2, \ldots, \ell\}\) then \(A\) is disjointly decomposable by \(N' \geq N\) invariant pairs. In fact, without loss of generality, if \((I_1, J_1)\) and \((I_2, J_2)\) are not disjoint then \((I' = I_1 \cap I_2 = \emptyset)\) and \((J' = J_1 \cap J_2 = \emptyset)\) and it can be shown that \((I_1 \setminus I_2, J_1 \setminus J_2, (I_2 \setminus I_1, J_2 \setminus J_1), (I', J'))\) are pairwise disjoint invariant pairs. Replacing \((I_1, J_1), (I_2, J_2)\) with these three pairs gives us a finer list of invariant pairs. Then repeat the process until all invariant pairs are pairwise disjoint.

**Lemma 5.3.** If \((I, J)\) is an invariant pair of a transformation matrix \(A\) and \(S, R \in \mathscr{B}\) are such that \(T_C 1_S = 1_R\), then \(T_{[A]^r} 1_{\bigcup_{j \in J} G^{-1}_j(S)} = 1_{\bigcup_{i \in I} F^{-1}_i(R)}\).
Theorem 5.5. Suppose a transformation matrix \( A \). Let \( \sigma \).

Lemma 5.4. For \( j \in J \), we have \( \mathbb{1}_{\bigcup_{j' \in J} G_{j'}^{-1}(S)} \circ G_j^{-1}(x) = \mathbb{1}_S(x) \); otherwise, \( \mathbb{1}_{\bigcup_{j' \in J} G_{j'}^{-1}(S)} \circ G_j^{-1}(x) = 0 \). So, by (3),

\[
T_{[A][C]} \mathbb{1}_{\bigcup_{j \in J} G_j^{-1}(S)} = \sum_{i=1}^k \sum_{j \in J} \frac{a_{ij}}{\Delta p_i} T_C \left( \mathbb{1}_{\bigcup_{j' \in J} G_{j'}^{-1}(S)} \circ G_j^{-1} \right) \left( F_i(x) \right)
\]

where we have used the assumption that \( a_{ij} = 0 \) if \( i \notin I \) and \( j \in J \). Since for each \( i \in I \), \( a_{ij} = 0 \) for all \( j \notin J \), the sum \( \sum_{j \in J} a_{ij} \) is equal to \( \Delta p_i \) and

\[
T_{[A][C]} \mathbb{1}_{\bigcup_{j \in J} G_j^{-1}(S)} = \sum_{i \in I} T_C \left( \mathbb{1}_S \right) \left( F_i(x) \right) = \sum_{i \in I} 1_R \left( F_i(x) \right) = \mathbb{1}_{\bigcup_{i \in I} F_i^{-1}(R)} (x).
\]

Proof. For \( j \in J \), we have \( \mathbb{1}_{\bigcup_{j' \in J} G_{j'}^{-1}(S)} \circ G_j^{-1}(x) = \mathbb{1}_S(x) \); otherwise, \( \mathbb{1}_{\bigcup_{j' \in J} G_{j'}^{-1}(S)} \circ G_j^{-1}(x) = 0 \). So, by (3),

\[
T_{[A][C]} \mathbb{1}_{\bigcup_{j \in J} G_j^{-1}(S)} = \sum_{i=1}^k \sum_{j \in J} \frac{a_{ij}}{\Delta p_i} T_C \left( \mathbb{1}_{\bigcup_{j' \in J} G_{j'}^{-1}(S)} \circ G_j^{-1} \right) \left( F_i(x) \right)
\]

Proof. For each \( n = 1, \ldots, N \), the positivity of \( T_{[A][C]} \mathbb{1}_S \) implies that \( T_{[A][C]} \mathbb{1}_S \leq T_{[A][C]} \mathbb{1}_R \). If it holds that \( T_{[A][C]} \mathbb{1}_S = 0 \) for a.e. \( x \) not in \( P_n \), then \( T_{[A][C]} \mathbb{1}_S \leq \mathbb{1}_{R \cap P_n} = \mathbb{1}_{R_n} \). Summing over all \( n \) gives

\[
\mathbb{1}_R = T_{[A][C]} \mathbb{1}_S = \sum_n T_{[A][C]} \mathbb{1}_S \leq \sum_n \mathbb{1}_{R_n} = \mathbb{1}_R
\]

and hence \( T_{[A][C]} \mathbb{1}_S = \mathbb{1}_{R_n} \) for all \( n \). We then prove the claim. If \( x \notin P_n \equiv \bigcup_{i \in I_n} (p_i^{-1}, p_i) \), then \( F_i(x) = 0 \) or \( 1 \) for all \( i \in I_n \) and

\[
T_{[A][C]} \mathbb{1}_{S_n} (x) = \sum_{i \in I_n} \sum_{j \in J_n} \frac{a_{ij}}{\Delta p_i} T_C \left( \sum_{j' \in J_n} \mathbb{1}_{S \cap (q_{j',-1}, q_{j',})} \circ G_j^{-1} \right) \left( F_i(x) \right)
\]

We use the convention that 0 and 1 are not in the support of all functions. □

Theorem 5.5. Suppose a transformation matrix \( A \) is disjointly decomposable by \( N \geq 2 \) invariant pairs. Then the invariant copula \( C_A \) is non-atomic.

Proof. Let \( S, R \in \mathcal{B} \) be such that \( T_{C_A} \mathbb{1}_S = \mathbb{1}_R \) and \( \lambda(S) = \lambda(R) > 0 \). Since \( C_A = |A|[C_A] \), by Lemma 5.4, \( T_{C_A} \mathbb{1}_S = \mathbb{1}_{R_n} \) for all \( n = 1, 2, \ldots, N \) where \( S_n \equiv \mathcal{G}_n(S) \subseteq Q_n \) and \( R_n \equiv \mathcal{F}_n(R) \subseteq P_n \) are such that \( S = \bigcup_n S_n \) and
If one of the $\lambda(S_n)$’s is strictly between 0 and $\lambda(S)$ then $S$ and $R$ are not atoms of $C_A$ and $C_A^*$, respectively. Otherwise, there is an $n_1$ such that $\lambda(S_{n_1}) = \lambda(S)$ and we repeat the process by applying Lemma 5.4 to $T_{C_A} 1_{S_{n_1}} = 1_{R_{n_1}}$ and obtain $S_{n_1} = \mathcal{G}_n(S_{n_1}) \subseteq \mathcal{G}_n(Q_{n_1})$ and $R_{n_1} = \mathcal{F}_n(R_{n_1}) \subseteq \mathcal{F}_n(P_{n_1})$ are such that $S_{n_1} = \bigcup_n S_{n_1,n}$, $R_{n_1} = \bigcup_n R_{n_1,n}$ and $T_{C_A} 1_{S_{n_1,n}} = 1_{R_{n_1,n}}$ for all $n = 1, 2, \ldots, N$. If some $\lambda(S_{n_1,n})$ lies between 0 and $\lambda(S)$ then we are done. Otherwise, there must be an $n_2$ such that $\lambda(S_{n_1,n_2}) = \lambda(S)$. This process will certainly stop because $\lambda(\mathcal{G}_n(\cdots(\mathcal{G}_{n_2}(Q_{n_1})))) = \lambda(Q_{n_2}) \cdots \lambda(Q_{n_1}) \lambda(Q_{n_1}) \to 0$ as $k \to \infty$ and $\lambda(S) > 0$. 

**Example 4.** Let $A_1 = K_0 + K_1$ and $A_2 = K_0 + K_2$ where

$$K_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 1/12 & 0 & 1/4 \\ 0 & 0 & 0 \\ 1/4 & 0 & 1/12 \end{bmatrix} \quad \text{and} \quad K_2 = \begin{bmatrix} 1/6 & 0 & 1/6 \\ 0 & 0 & 0 \\ 1/6 & 0 & 1/6 \end{bmatrix}.$$

Then both transformation matrices $A_1$ and $A_2$ are disjointly decomposable by 2 invariant pairs $\{(1, 3), (1, 3)\}$ and $\{(2), (2)\}$. By Theorem 5.5, the invariant copulas $C_{A_1}$ and $C_{A_2}$ are non-atomic and hence, by Theorem 4.6, it is supported in the support of an implicit dependence copula. In fact, they share the same support shown in Figure 1. Note also that both copulas are symmetric as the corresponding matrices are. We will see later that (only) $C_{A_2}$ can be factored as $L_1 * L_1^*$ for some left invertible copulas $L_1, L_2$.

A transformation matrix $L = [\lambda_{in}]_{k \times N}$ is said to be a left complete dependence matrix if there is exactly one nonzero entry in each column, i.e. for each $i$, there exists a positive integer $n_i \leq N$ such that $\lambda_{in} = 0$ for $n \neq n_i$. Right complete dependence matrices are defined similarly.
Lemma 5.6. Let $L$ be a left complete dependence matrix with dimension at least $2 \times 2$ and $C$ be a left invertible copula. Then $[L](C), [L]^2(C), \ldots, [L]^r(C), \ldots$ are left invertible copulas converging to a left invertible copula $C_L$. The same statement also holds if all occurrences of “left” are replaced by “right.”

Proof. Since the general case can be proved by induction on $r$, it suffices to show that $[L](C)$ is left invertible. By definition, the $x$-partition of $[L](C)$ is determined by the only non-zero entry in each column: $p_0 = 0$ and $p_i = \sum_{i' \leq i} \lambda_{i'n}$, $i = 1, \ldots, k$. So $[L](C)(u, v) = \sum_{i=1}^k \lambda_{in} C(F_i(u), G_n(v))$ and

$$[L](C)^t \ast [L](C)(u, v) = \sum_{i'=1}^k \sum_{i=1}^k \lambda_{in} \lambda_{i'n} \int_0^1 \frac{\partial}{\partial t} C(F_i(t), G_n(t))(u) \frac{\partial}{\partial t} C(F_i(t), G_n(t))(v) \, dt$$

$$= \sum_{i=1}^k \lambda_{in}^2 \int_0^1 \frac{\partial}{\partial t} C(F_i(t), G_n(t))(u) \frac{\partial}{\partial t} C(F_i(t), G_n(t))(v) F_i(t)^2 \, dt$$

$$= \sum_{i=1}^k \lambda_{in} C^t \ast C(G_n(u), G_n(v))$$

$$= \sum_{i=1}^k \lambda_{in} M(u, v).$$

Therefore, $[L](C)$ is left invertible. In general, we have $[L]^r(C)^t \ast [L]^r(C) = M$ and so, by taking the limit as $r \to \infty$ with respect to the modified Sobolev norm (see Theorem 5.2), $C_L^t \ast C_L = M$. □

Theorem 5.7. Let $A$ be a $k \times \ell$ transformation matrix, with $k, \ell \geq 2$, which is disjointly decomposable by $N \geq 2$ invariant pairs. If all $A_1, A_2, \ldots, A_N$ have rank one, then $C_A = L \ast R$ for some left invertible copula $L$ and right invertible copula $R$.

Proof. If $A_n$ has rank one, then there exist a row matrix $L_n = [\lambda_{in}]_{k \times 1}$ and a column matrix $R_n = [\rho_{nj}]_{1 \times \ell}$ such that $|L_n| = |A_n| = |R_n|$ and

$$A_n = \frac{1}{|A_n|} L_n R_n,$$

(4)

where $|A|$ denotes the sum of the absolute values of all entries in a matrix $A$. Stacking up $L_n$’s vertically and $R_n$’s horizontally, we obtain transformation matrices

$$L = [\lambda_{in}]_{k \times N} = \begin{bmatrix} L_N \\ \vdots \\ L_1 \end{bmatrix} \quad \text{and} \quad R = [\rho_{nj}]_{N \times \ell} = \begin{bmatrix} R_1 & \cdots & R_N \end{bmatrix}.$$ 

Since $L_n$’s are disjoint, each column of $L$ has exactly one non-zero entry. Similarly, each row of $R$ has exactly one non-zero entry.
We then show that \([L](C_1) \ast [R](C_2) = [A](C_1 \ast C_2)\) for any copulas \(C_1\) and \(C_2\). For \(m = 1, 2, \ldots, N\), denote by \(H_m\) the uniform distribution on \(\sum_{n=1}^{m-1} |A_n|, \sum_{n=1}^{m} |A_n|\). So \(H'_m = \frac{1}{|A_m|}\) on its support. For \(u, v \in [0, 1]\),

\[
[L](C_1) \ast [R](C_2)(u, v) = \int_0^1 \left( \sum_{i=1}^{\ell} \sum_{j=1}^{N} \lambda_{in} \partial_2 C_1(F_i(u), H_n(t)) \right) \left( \sum_{j=1}^{\ell} \sum_{n=1}^{N} \lambda_{nj} \rho_j \partial_1 C_2(H_m(t), G_j(v)) \right) \, dt
\]

\[
= \sum_{i=1}^{\ell} \sum_{j=1}^{N} \left( \sum_{n=1}^{N} \lambda_{in} \lambda_{nj} \frac{\rho_j}{|A_n|} \right) C_1 \ast C_2(F_i(u), G_j(v)) = [A](C_1 \ast C_2)(u, v),
\]

(5)

It is left to verify that the sum over \(n\) in (5) is equal to the \((i, j)\)th-element in \(A\). Since \(\sum_{i=1}^{N} A_n\) is a disjoint decomposition of \(A = [a_{ij}]_{k \times \ell}\), the equation (4) implies that if \((i, j) \in I_m \times J_m\) for some (unique) \(m\) then

\[
a_{ij} = \frac{\text{the (i, j)th element of } L_m R_m}{|A_m|} = \frac{\lambda_{im} \rho_{mj}}{|A_m|} = \sum_{n=1}^{N} \lambda_{in} \rho_{nj},
\]

where the last equality follows from the fact that \(\lambda_{in} \rho_{nj} = 0\) if \((i, j) \notin I_n \times J_n\). It also clearly follows from this fact that if \((i, j) \notin I_m \times J_m\) for all \(m\) then \(\sum_{n=1}^{N} \lambda_{in} \rho_{nj} = 0\) if \((i, j) \notin I_n \times J_n\).

By (6), a proof by induction on \(r = 1, 2, \ldots\) yields \([L]^r(C_1) \ast [R]^r(C_2) = [A]^r((L)^{r-1}(C_1) \ast (R)^{r-1}(C_2)) = \cdots = [A]^r(C_1 \ast C_2)\). So \([L]^r(E) \ast [R]^r(E) = [A]^r(E)\) for all \(r \geq 1\) if \(E\) is an idempotent copula, e.g. \(E = M\) or \(\Pi\). By Lemma 5.6 \([L]^r(M)\) are left invertible copulas and \([R]^r(M)\) are right invertible copulas.

By Theorem 5.2 with respect to the modified Sobolev norm, \([L]^r(M)\), \([R]^r(M)\) and \([A]^r(M)\) converge respectively to a left invertible copula \(C_L\), a right invertible copula \(C_R\) and a copula \(C_A\) satisfying \([L](C_L) = C_L, [R](C_R) = C_R\) and \([A](C_A) = C_A\). As a consequence of the joint continuity of the \(*\)-product with respect to \(\|\|_S\), we obtain \(C_L \ast C_R = C_A\) as desired.

**Example 5.** Fix \(r \in (0, 1/2)\) and consider the transformation matrix

\[
\begin{bmatrix}
r/2 & 0 & r/2 \\
0 & 1 - 2r & 0 \\
r/2 & 0 & r/2
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 - 2r & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
r/2 & 0 & r/2 \\
0 & 0 & 0 \\
r/2 & 0 & r/2
\end{bmatrix}.
\]

Both matrices on the right hand side have rank one which implies by Theorem 5.7 that the invariant copula can be factored as the product of a left invertible copula and a right invertible copula. When \(r = 1/3\), the transformation matrix is \(A_2\) in Example 4. The copula factors, \(C_{L_2}\) and \(C_{R_2}\), of \(C_{A_2}\) are shown (approximately) in Figure 2 where \(L_2 = \begin{bmatrix}
0 & 1/3 & 0 \\
1/3 & 0 & 1/3
\end{bmatrix} = R_2'.\)
Figure 2: Supports of $[A_2]^5(\Pi)$, $[L_2]^5(\Pi)$ and $[R_2]^5(\Pi)$ in Example 6

Figure 3: Supports of $[A_3]^5(\Pi)$, $[L_3]^5(\Pi)$ and $[R_3]^5(\Pi)$ in Example 6

Example 6. Let us also consider the factorization of a non-symmetric invariant copula by first setting

$$K_3 = \begin{bmatrix} 3/28 & 0 & 27/140 \\ 0 & 0 & 0 \\ 1/7 & 0 & 9/35 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0 & 3/10 & 0 \\ 1/4 & 0 & 9/20 \end{bmatrix} \quad \text{and} \quad R_3 = \begin{bmatrix} 3/10 & 0 \\ 0 & 3/10 \\ 4/10 & 0 \end{bmatrix}.$$

Then the non-symmetric transformation matrix $A_3 = K_0 + K_3$ is disjointly decomposable by 2 invariant pairs ($\{1,3\}, \{1,3\}$) and ($\{2\}, \{2\}$). Since $K_0$ and $K_3$ have rank one, it follows from Theorem 5.7 and its proof that $C_{A_3} = C_{L_3} \ast C_{R_3}$. Approximations of their supports are illustrated in Figure 3.

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