GENERALIZATIONS OF THE RIPS FILTRATION FOR QUASI-METRIC SPACES WITH PERSISTENT HOMOLOGY STABILITY RESULTS

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ABSTRACT. Rips filtrations over a finite metric space and their corresponding persistent homology are prominent methods in Topological Data Analysis to summarize the “shape” of data. Crucial to their use is the stability result that says if $X$ and $Y$ are finite metric space then the (bottleneck) distance between persistence diagrams, barcodes or persistence modules constructed by the Rips filtration is bounded by $2d_{GH}(X, Y)$ (where $d_{GH}$ is the Gromov-Hausdorff distance). Using the asymmetry of the distance function we construct four different constructions analogous to the Rips filtration that capture different information about the quasi-metric spaces. The first method is a one-parameter family of objects where, for a quasi-metric space $X$ and $a \in [0, 1]$, we have a filtration of simplicial complexes $\{R^a(X)_t\}_{t \in [0, \infty)}$ where $R^a(X)_t$ is clique complex containing the edge $[x, y]$ whenever $a \min\{d(x, y), d(y, x)\} + (1 - a) \max\{d(x, y), d(y, x)\} \leq t$. The second method is to construct a filtration $\{R^{dir}(X)_t\}$ of ordered tuple complexes where tuple $(x_0, x_2, \ldots x_p) \in R^{dir}(X)_t$ if $d(x_i, x_j) \leq t$ for all $i \leq j$. Both our first two methods agree with the normal Rips filtration when applied to a metric space. The third and fourth methods use the associated filtration of directed graphs $\{D(X)_t\}$ where $x \rightarrow y$ is included in $D(X)_t$ when $d(x, y) \leq t$. Our third method builds persistence modules using the the connected components of the graphs $D(X)_t$. Our fourth method uses the directed graphs $D_t$ to create a filtration of posets (where $x \leq y$ if there is a path from $x$ to $y$) and corresponding persistence modules using poset topology. The Gromov-Hausdorff distance can be naturally extended to quasi-metric spaces and we prove that these new constructions enjoy the same Gromov-Hausdorff stability as the original persistent homology for Rips filtrations.

1. INTRODUCTION

The Rips filtration is an increasing sequence of simplicial clique complexes built on a metric space to add topological structure to an otherwise disconnected set of points. The persistent homology of the Rips filtration is widely used in applied topology because it encodes useful information about the topology of the underlying metric space ([3, 4, 9, 12]). There are many potential applications for studying data whose structure is a quasi-metric space. Examples includes the web hyperlink quasi-metric space and
quasi-metrics induced from weighted directed graphs found throughout science (for example, biological interaction graphs [8] or the connections in neural systems [7, 4]). We wish to extend existing methods in applied topology for use on quasi-metric spaces. In this paper we utilise the asymmetry of the distance function to construct four different analogs of the persistent homology of the Rips filtration for use on quasi-metric spaces.

The Rips filtration over finite metric space $X$ is a filtration of simplicial complexes $\{\mathcal{R}(X)_t\}_{t \in [0, \infty)}$ where $\mathcal{R}(x)_t$ is the clique complex over the graph containing edges $\{[x, y] : d(x, y) \leq t\}$. From the Rips filtration is we produce a persistence module (or a discrete summary of the persistence module in the form of a barcode or persistence diagram) which describes its persistent homology.

A persistence module is a family of vector spaces $\{V_t : t \in \mathbb{R}\}$ equipped with linear maps $\phi^t_s : V_s \to V_t$ for each pair $s \leq t$ with $\phi^t_t = \text{id}$ and $\phi^t_s = \phi^t_r \circ \phi^r_s$ whenever $s \leq r \leq t$. Two persistence modules, $\{(V_t, \{\phi^t_s\})\}$ and $\{(U_t, \{\psi^t_s\})\}$, are $\epsilon$-interleaved when there exist families of linear maps $\{\alpha_t : V_t \to U_{t+\epsilon}\}$ and $\{\beta_t : U_t \to V_{t+\epsilon}\}$ satisfying natural commutation conditions. There is a pseudo-metric on the space of persistence modules called the interleaving distance, $d_{\text{int}}$, which is the infimum of the set of $\epsilon > 0$ such that there exists an $\epsilon$-interleaving.

The persistence module we construct from the persistent homology of a Rips filtration over $X$ has vector spaces $\{H_\ast(\mathcal{R}(X)_t)\}_{t \in [0, \infty)}$ along with maps on homology induced by inclusions; $\phi^t_s = \iota_s : H_\ast(\mathcal{R}(X)_s) \to H_\ast(\mathcal{R}(X)_t)$ when $s \leq t$. Arguable the most important result in Topological Data Analysis is stability theorem which states that if $X, Y$ are finite metric spaces, with corresponding persistence modules $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ computed via the persistent homology of the Rips complexes over $X$ and $Y$, then

$$d_{\text{int}}(\mathcal{P}(X), \mathcal{P}(Y)) \leq 2d_{\text{GH}}(X, Y)$$

where $d_{\text{int}}$ is the interleaving distance and $d_{\text{GH}}$ is the Gromov-Hausdorff distance.

Using the asymmetry of the distance function we construct four different constructions analogous to the Rips filtration that capture different information about the quasi-metric spaces. The Gromov-Hausdorff distance can be naturally extended to quasi-metric spaces and we prove that these new constructions enjoy the same Gromov-Hausdorff stability as the standard persistent homology for Rips filtrations.

For increased flexibility regarding order we will use ordered tuple complexes (shortened to OT-complexes) instead of simplicial complexes. An OT-complex $K$ is a sets of ordered tuples $(v_0, v_1, \ldots v_p)$ such that if $(v_0, v_1, \ldots v_p)$ is in $K$ then $(v_0, v_1, \ldots, \hat{v}_i, \ldots, v_p) \in K$ for all $i$. Note that the repetitions of the $v_j$ are allowed in the same ordered tuple. OT-complexes can be thought of as a generalization of simplicial complexes. Chain
complexes, boundary maps, homology and persistent homology are easily defined for OT-complexes. There are canonical OT-complexes associated to any simplicial complex which have isomorphic homology groups. Furthermore, the persistence modules constructed using simplicial or OT-complexes are isomorphic.

In section 2 we provide the definitions of quasi-metric spaces (and pseudo-quasi-metric spaces, etc). We also define Gromov-Hausdorff distances between metric spaces and show, through the definition using correspondences, how it is naturally extends to a pseudo metric between pseudo-quasi-metric spaces. We also prove a useful proposition about, given two quasi-metric spaces $X$ and $Y$, the existence of expanded pseudo-quasi-metric spaces $\tilde{X}$ and $\tilde{Y}$, with $d_{GH}(X, \tilde{X}) = 0$ and $d_{GH}(Y, \tilde{Y}) = 0$, between which the Gromov-Hausdorff distance achieving correspondence determines a bijection.

In section 3 we provide some background material about persistence modules (and how they relate to persistence diagrams), we define the persistence module constructed from the Rips filtration and state the related stability theorem.

In section 4 we introduce OT-complexes, and filtrations of OT-complexes. We define persistence modules built from filtrations of OT-complexes. We also define when an OT-complex is an expansion of another and prove that expansions of filtrations of OT-complexes produce isomorphic persistence modules. We also describe the filtration of OT-complexes associated to a filtration of simplicial complexes and show they produce isomorphic persistence modules.

In section 5 we define our first generalization of the Rips filtration over a pseudo-quasi-metric space $X$. For each $a \in [0, 1]$ let

$$f_a(x, y) = a \min\{d(x, y), d(y, x)\} + (1 - a) \max\{d(x, y), d(y, x)\}.$$  

This is a symmetric function in $x$ and $y$ and as such we can construct a filtration of graphs $G_t$ by including the edge $[x, y]$ when $f_a(x, y) \leq t$. We can construct a filtration of OT-complexes, $\mathcal{R}^a(X)$, where each of the OT-complexes is that constructed from a clique complex over $G_t$. Notably $\mathcal{R}^a(X) = \mathcal{R}(X)$ when $X$ is a metric space as in that case $f_a(x, y) = d(x, y)$. We show that if $a \in [0, 1]$, and $X, Y$ are pseudo-quasi-metric spaces $X, Y$ with corresponding persistence modules $\mathcal{P}(\mathcal{R}^a(X))$ and $\mathcal{P}(\mathcal{R}^a(Y))$, then

$$d_{int}(\mathcal{P}(\mathcal{R}^a(X)), \mathcal{P}(\mathcal{R}^a(Y))) \leq 2d_{GH}(X, Y).$$

In section 6 we define the directed Rips filtration of OT-complexes over pseudo-quasi-metric space $X$, which is a filtration $\{\mathcal{R}^{dir}(X)_t\}$ of ordered tuple complexes where tuple $(x_0, x_2, \ldots x_p) \in \mathcal{R}^{dir}(X)_t$ if $d(x_i, x_j) \leq t$ for all $i \leq j$. This filtration of OT-complexes agrees with that built over the standard Rips filtration when $X$ is a metric space. We show that if $X, Y$ are pseudo-quasi-metric spaces $X, Y$ with corresponding
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persistence modules $\mathcal{P}(\mathcal{R}^{\text{dir}}(X))$ and $\mathcal{P}(\mathcal{R}^{\text{dir}}(Y))$, then $d_{\text{int}}(\mathcal{P}(\mathcal{R}^{\text{dir}}(X)), \mathcal{P}(\mathcal{R}^{\text{dir}}(Y))) \leq 2d_{\text{GH}}(X,Y)$.

Our third and fourth constructions of persistence modules are via filtrations of directed graphs. For $X$ a pseudo-quasi-metric space, and $t \geq 0$, let $D(X)_t$ be the directed graph whose vertices are the points in $X$ and where the arrow $x \to y$ is included when $d(x,y) \leq t$. We call the family $\{D(X)_t\}_{t \in [0,\infty)}$ of directed graphs the associated filtration of directed graphs to $X$.

Completely analogous to equivalence classes of connected components within a graph defining the standard 0th dimensional homology we can consider the vector spaces whose elements are linear combination of strongly connected components in a directed graph. In section 7 we define the strongly connected persistence module for a filtration. Given pseudo-quasi-metric spaces $X$ and $Y$, we can create strongly connected persistence modules $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ through their associated filtrations of directed graphs. We prove that $d_{\text{int}}(\mathcal{P}(X), \mathcal{P}(Y)) \leq 2d_{\text{GH}}(X,Y)$. Note that when $X$ is a metric space then $\mathcal{P}(X)$ is the 0th dimensional persistence module for the Rips filtration $\mathcal{R}(X)$.

Our fourth method uses the directed graphs to create a filtration of posets. Given a directed graph $D$ over vertices $X$ we say $x \leq y$ if there is a path from $x$ to $y$. From a filtration of directed graphs we obtain a filtration of posets. This provides natural persistence modules using poset topology. When $X$ is a metric space then the poset topology is not very informative. Its 0th dimensional persistent homology is the same as that of the persistent homology of the standard Rips filtration $\mathcal{R}(X)$ and its higher dimensional homology groups are always trivial. Given pseudo-quasi-metric spaces $X$ and $Y$, and a dimension $k$. Let $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ denote the corresponding $k$th dimensional poset persistence modules built through their associated filtration of directed graphs. Then $d_{\text{int}}(\mathcal{P}(X), \mathcal{P}(Y)) \leq 2d_{\text{GH}}(X,Y)$.

2. DIRECTED GRAPHS, QUASI AND PSEUDO METRIC SPACES AND GROMOV-HAUSDORFF DISTANCE

The existing stability results involve the the Gromov-Hausdorff distance which is a metric on the space of compact metric spaces. We will wish to generalize this distance function to the space of quasi-metric spaces in order to use it in stability theorems later. Before introducing quasi-metric spaces we will first define (weighted) directed graphs as these provide a wide array of examples.

Definition 1. A directed graph is a ordered pair $D = (V,A)$ where $V$ is a set whose elements are called vertices, and $A$ is a set of ordered pairs of vertices called arrows or...
directed edges. A **weighted directed graph** is a directed graph where each arrow is given a non-negative weight.

Note that a graph can be thought of as a directed graph such that whenever an arrow \( v \to w \) is in \( A \) then its opposite direction \( w \to v \) must also be in \( A \).

**Definition 2.** Let \( X \) be a non-empty set and \( d : X \times X \to \mathbb{R} \). Consider the following potential properties of \( d \):

1. \( d(x, x') \geq 0 \) for all \( x, x' \in X \)
2. \( d(x, x') = d(x', x) \) for all \( x, x' \in X \)
3. For all \( x, x', x'' \in X \), \( x = x' \) if and only if \( d(x, x') = 0 \) and \( d(x', x) = 0 \)
4. \( d(x, x'') \leq d(x, x') + d(x', x'') \) for all \( x, x', x'' \in X \).

If \( (X, d) \) satisfies (1), (2), (3) and (4) it is called a **metric space**. If \( (X, d) \) satisfies (1), (3) and (4) it is called a **quasi-metric space** and we can call \( d \) a **quasi-metric**. If \( (X, d) \) satisfies (1), (2) and (4) it is called a **pseudo-metric space** and we can call \( d \) a **pseudo-metric**. If \( (X, d) \) satisfies (1) and (4) it is called a **pseudo-quasi-metric space** and we can call \( d \) a **pseudo-quasi-metric**.

We can build examples of these different types of spaces using weighted directed graphs. Given a weighted directed graph \( D = (V, A) \), and two vertices \( x, y \in V \) we call \( x = v_0, v_1, v_2, \ldots, v_m = y \) a path from \( x \) to \( y \) if all of the arrows \( v_i \to v_{i+1} \) are in \( A \). The length of that path \( (x = v_0, v_1, v_2, \ldots, v_m = y) \) is the sum of the weights \( \sum_{i=0}^{m-1} w(v_i \to v_{i+1}) \). Construct \( d : V \times V \to \mathbb{R} \) by setting \( d(x, y) \) to be the length of the shortest path from \( x \) to \( y \) (and \( \infty \) if no path exists). Since each arrow has non-negative in weight, the function \( d \) automatically satisfies (1) in Definition 2. By considering the concatenation of paths, we can easily see that \( d \) also automatically satisfies (4) in Definition 2. Thus \( (V, d) \) must always be a quasi-pseudo metric space.

The **Gromov-Hausdorff distance** between metric spaces \((X, d_X)\) and \((Y, d_Y)\) is defined by

\[
\text{d}_{GH}(X, Y) = \inf_{Z, f: X \to Z, g: Y \to Z} \text{d}_{H,Z}(f(X), g(Y))
\]

where the infimum is taken over all metric spaces \( Z \) and isometric embeddings \( f \) and \( g \) to \( Z \) from \( X \) and \( Y \), respectively. \( \text{d}_{H,Z} \) is the Hausdorff distance between subsets of \( Z \). It is a standard result that the Gromov-Hausdorff distance is a metric on the space of compact metric spaces.

A useful alternate but equivalent formula for the Gromov-Hausdorff distance can be given through correspondences. The set \( \mathcal{M} \subset X \times Y \) is a correspondence between \( X \) and \( Y \) if for all \( x \in X \) there exists some \( y \in Y \) with \( (x, y) \in \mathcal{M} \) and for all \( y \in Y \) there
is some \( x \in X \) with \((x, y) \in M \). Using correspondences we know can write

\[
(2.1) \quad d_{GH}(X, Y) = \frac{1}{2} \inf_{\mathcal{M}} \max_{(x_1, y_1), (x_2, y_2) \in \mathcal{M}} |d_X(x_1, x_2) - d_Y(y_1, y_2)|.
\]

The definition of Gromov-Hausdorff distance now naturally extends to pseudo-quasi-metric spaces.

**Definition 1.** For \( X, Y \) quasi-metric spaces with quasi-metrics \( d_X \) and \( d_Y \) the Gromov-Hausdorff distance between them can be defined as

\[
d_{GH}(X, Y) = \frac{1}{2} \inf_{\mathcal{M}} \sup_{(x_1, y_1), (x_2, y_2) \in \mathcal{M}} |d_X(x_1, x_2) - d_Y(y_1, y_2)|.
\]

This is identical to the definition for metric spaces, as shown in (2.1), we just need to be careful about being consistent with the order of the points.

It is straightforward to verify that \( d_{GH} \) is a pseudo-metric on the space of finite pseudo-quasi-metric spaces, and a metric on the space of finite quasi-metric spaces.

Given a finite pseudo-quasi-metric space \( X \) let \( \sim \) be an equivalence relation in \( X \) where \( x \sim y \) if \( d_X(x, y) = 0 = d_X(y, x) \). Let \( Y = X/\sim \), containing exactly one representative from each equivalence class under \( \sim \). Set \( \pi : X \to Y \) by \( \pi(x) = [x] \). We can see that \( d_{GH}(X, Y) = 0 \) as the correspondence

\[
\mathcal{M} := \{(x, [x]) : x \in X \} \subset X \times Y
\]
satisfies \( d_X(x_1, x_2) = d_Y(y_1, y_2) \) for all \((x_1, x_2), (y_1, y_2) \in \mathcal{M}\). \( Y \) is the unique quasi-metric space such that \( d_{GH}(X, Y) = 0 \)

We will later wish to “expand” our quasi-metric spaces so that the distance achieving correspondence determines a bijection.

**Proposition 2.** Let \( X, Y \) be finite pseudo-quasi-metric spaces. Then there exist finite pseudo-quasi-metric spaces \((\tilde{X}, d_{\tilde{X}})\) and \((\tilde{Y}, d_{\tilde{Y}})\), and a bijection \( \psi : \tilde{X} \to \tilde{Y} \) such that

1. \( X \subset \tilde{X} \) and there exists a projection \( \pi : \tilde{X} \to X \) such that
   \[
   d_{\tilde{X}}(x, \hat{x}) = d_{\tilde{X}}(\pi(x), \pi(\hat{x})) = d_X(\pi(x), \pi(\hat{x}))
   \]
   for all \( x, \hat{x} \in \tilde{X} \).
2. \( Y \subset \tilde{Y} \) and there exists a projection \( \pi : \tilde{Y} \to Y \) such that
   \[
   d_{\tilde{Y}}(y, \hat{y}) = d_{\tilde{Y}}(\pi(y), \pi(\hat{y})) = d_Y(\pi(y), \pi(\hat{y}))
   \]
   for all \( y, \hat{y} \in \tilde{Y} \).
3. \( d_{\tilde{X}}(\psi(x), \psi(\hat{x})) \leq d_{\tilde{X}}(x, \hat{x}) + 2d_{GH}(X, Y) \) for all \( x, \hat{x} \in \tilde{X} \)
4. \( d_{\tilde{X}}(\psi^{-1}(y), \psi^{-1}(\hat{y})) \leq d_{\tilde{Y}}(y, \hat{y}) + 2d_{GH}(X, Y) \) for all \( y, \hat{y} \in \tilde{Y} \)
Proof. Fix an indexing on the points in $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$. Also fix a correspondence $\mathcal{M}$ between $X$ and $Y$ such that

$$|d_X(\hat{x}, x) - d_Y(\hat{y}, y)| \leq 2d_{GH}(X, Y)$$

for all $(x, y), (\hat{x}, \hat{y}) \in \mathcal{M}$. Such a correspondence exists since $X$ and $Y$ are finite, so only finitely many correspondences exist and the infimum in Definition 1 must be achieved.

Let $\tilde{X} = \{x_{(i,j)} : (x_i, y_j) \in \mathcal{M}\}$. For each $i$, let $s_i = \min\{j : (x_i, y_j) \in \mathcal{M}\}$. By relabelling each $x_i \in X$ as $x_{(i,s_i)}$ we can see $X \subset \tilde{X}$ and that

$$\pi_X : \tilde{X} \to X$$

$$x_{(i,j)} \mapsto x_{(i,s_i)}$$

is a projection. Define the function $d_{\tilde{X}}$ on $\tilde{X} \times \tilde{X}$ by

$$d_{\tilde{X}}(x_{(i,j)}, x_{(k,l)}) = d_X(x_{(i,s_i)}, x_{(k,s_k)}).$$

It is easy to check $d_{\tilde{X}}$ is a pseudo-quasi-metric and

$$d_{\tilde{X}}(x, \hat{x}) = d_{\tilde{X}}(\pi(x), \pi(\hat{x})) = d_X(\pi(x), \pi(\hat{x}))$$

for all $x, \hat{x} \in \tilde{X}$. Thus the conditions of (1) are satisfied.

Let $\tilde{Y} = \{y_{(i,j)} : (x_i, y_j) \in \mathcal{M}\}$. For each $j$, let $r_j = \min\{i : (x_i, y_j) \in \mathcal{M}\}$. By relabelling each $y_j \in Y$ by $y_{(r_j,j)}$ we can see $Y \subset \tilde{Y}$ and that

$$\pi_Y : \tilde{Y} \to Y,$$

$$y_{(i,j)} \mapsto y_{(r_j,j)}$$

defines a projection. Symmetrically to above we can define a pseudo-quasi-metric function $d_{\tilde{Y}}$ on $\tilde{Y}$ by $d_{\tilde{Y}}(y_{(i,j)}, y_{(k,l)}) = d_Y(y_{(r_j,j)}, y_{(r_l,l)})$ and that $d_{\tilde{Y}}$ satisfies the conditions of (2).

Define a bijection $\psi : \tilde{X} \to \tilde{Y}$ by $\psi(x_{(i,j)}) = y_{(i,j)}$. For $x_{(i,j)}, x_{(k,l)} \in \tilde{X}$ we have $d_{\tilde{X}}(x_{(i,j)}, x_{(k,l)}) = d_X(x_{(i,s_i)}, x_{(k,s_k)})$ and $d_{\tilde{Y}}(\psi(x_{(i,j)}), \psi(x_{(k,l)})) = d_Y(y_{(r_j,j)}, y_{(r_l,l)})$. Since $(x_i, y_j), (x_k, y_l) \in \mathcal{M}$ (and that $x_i, x_k, y_j,$ and $y_l$ have been relabelled $x_{(i,s_i)}, x_{(k,s_k)}, y_{(r_j,j)},$ and $y_{(r_l,l)}$ respectively) we know that

$$|d_X(x_{(i,s_i)}, x_{(k,s_k)}) - d_Y(y_{(r_j,j)}, y_{(r_l,l)})| \leq 2d_{GH}(X, Y).$$
This implies
\[ d_\tilde{Y} \left( \psi(x_{(i,j)}), \psi(x_{(k,l)}) \right) = d_Y(y_{(r_j,j)}, y_{(r_l,l)}) \]
\[ \leq d_X(x_{(i,s_i)}, x_{(k,s_k)}) + 2d_{GH}(X,Y) \]
\[ = d_\tilde{X}(x_{(i,j)}), x_{(k,l)}) + 2d_{GH}(X,Y) \]

This proves part (3) and the proof to part (4) is symmetric. \qed

It is worth observing that the spaces $\tilde{X}$ and $\tilde{Y}$ in the above Proposition satisfy $d_{GH}(\tilde{X},\tilde{Y}) = d_{GH}(X,Y)$ and that the correspondence $\{(x_{(i,j)}, y_{(i,j)})\}$ achieves the infimum in the definition of the Gromov-Hausdorff distance.

3. Background - persistence modules and existing stability results

In this section we will cover some background theory on persistence modules and the interleaving distance between persistence modules. This is important because the interleaving distance between persistence modules bounds the bottleneck distance between their corresponding persistence diagrams. We also state standard results bounding the interleaving distance of persistence modules of the persistent homology of Rips filtrations of finite metric spaces by the Gromov-Hausdorff distance.

To introduce and motivate the concepts we will provide a brief summary of the theory of persistent homology. We will omit most of the details as we will be phrasing all results in later sections in terms of persistence modules. From more details about the history and applications of persistent homology we refer the reader to [11, 6, 5, 1].

Persistent homology describes how the homology groups evolve over an increasing family of topological spaces. Throughout this section let $K = \{K_t\}$ denote a family of reasonable topological spaces such that $K_s \subset K_t$ whenever $s \leq t$. Given $s \leq t$ the $k$-th dimensional persistent homology group for $K$ from $s$ to $t$ are the $k$-th dimensional homology classes in $K_s$ that “persist” until $K_t$, that is $Z_k(K_s)/(Z_k(K_t) \cup B_k(K_s))$. This is isomorphic to the image of the induced map on homology $\iota_* : H_k(K_s) \to H_k(K_t)$ from the inclusion $K_s \subset K_t$.

Barcodes and persistence diagrams were introduced as discrete summaries of persistent homology information. Each barcode consists of a multiset of real intervals called bars. The barcode corresponding to the $k$-th dimensional persistent homology of $K$ is $\{I_1, I_2, \ldots, I_n\}$ if, for all $s \leq t$, the dimension of $\text{im}(\iota_* : H_k(K_s) \to H_k(K_t))$ equals the number of bars in $\{I_1, I_2, \ldots, I_n\}$ that span $[s,t)$. The corresponding persistence diagram is the multiset of points in $\mathbb{R}^2$, $\{(a_i, b_i)\}$, where $a_i, b_i$ are the endpoints of the bar.
$I_i$, alongside infinite copies of every point along the diagonal (acting the role of empty intervals).

Barcodes and persistence diagrams have played a prominent role in applied topology as topological summaries of data. In particular, they can provide insight into the “shape” of point cloud data through the persistent homology of the Rips filtration over that point cloud. Much of the power behind the use of barcodes and persistence diagrams comes from stability theorems such as the stability theorem for the persistent homology of the Rips filtration over a finite metric space.

Theoretical advances have increased the flexibility of the algebraic structures used. Persistence, such as persistent homology of a filtration of simplicial complexes, can be defined directly at an algebraic level. In [13], Zomorodian and Carlsson introduced the concept of a persistence module and proved that barcodes (and equivalently persistence diagrams) can be defined for persistence modules satisfying reasonable finiteness conditions. It was shown in [2] that we can define a distance between persistence modules (called the interleaving distance) and that the interleaving distance between persistence modules is a bound on the bottleneck distance of their corresponding persistence diagrams. Throughout this paper we will work directly with persistence modules.

**Definition 3.** Let $R$ be a commutative ring with unity. A persistence module over $A \subset \mathbb{R}$ is a family $\{P_t\}_{t \in A}$ of $R$-modules indexed by real numbers, together with a family of homomorphism $\{i_s^t : P_t \to P_s\}$ such that $i_r^t = i_r^s \circ i_s^t$ for all $t \leq s \leq r$, and $i_t^t = \text{id}_{P_t}$.

In general persistence modules can be defined over any subset of $\mathbb{R}$. However, as we will be considering the effect of distance, all our persistence modules will be over the interval $[0, \infty)$. If $R$ is a field then the $P_t$ are all vector fields and the $i_s^t$ are linear maps. As is standard in Topological Data Analysis, we will assume throughout that $R$ is the fixed field $F$ (usually taken to be $\mathbb{F}_2$ for computational reasons). In the theory of persistence modules there are technical requirements about “tameness”. We say $\mathcal{P}$ is tame if rank $i_t^t$ is always finite. Every persistence module in this paper will automatically be tame because we are working with finite pseudo-quasi-metric spaces.

**Definition 4.** Two persistence modules $\mathcal{P}^X$ and $\mathcal{P}^Y$ are $\epsilon$-interleaved if there are families of homomorphisms $\{\alpha_t : P_t^X \to P_t^Y\}_{t \in \mathbb{R}}$ and $\{\beta_t : P_t^Y \to P_t^X\}_{t \in \mathbb{R}}$ such that the diagrams in (3.1) and (3.2) commute.
Definition 5. Two persistence modules $\mathcal{P}^X$ and $\mathcal{P}^Y$ are isomorphic if they are 0-interleaved.

The diagrams in (3.1) and (3.2) are slightly different to those given in [2] but the diagrams here commuting will imply that theirs also commute. We also differ from [2] by only defining an $\epsilon$-interleaving. In [2] they define both strongly and weakly $\epsilon$-interleaved, both of which are weaker than our notion of interleaving.

If $\mathcal{P}^X$ and $\mathcal{P}^Y$ are $\epsilon_1$-interleaved and $\mathcal{P}^Y$ and $\mathcal{P}^Z$ are $\epsilon_2$-interleaved then composing homomorphisms shows that $\mathcal{P}^X$ and $\mathcal{P}^Z$ are $(\epsilon_1 + \epsilon_2)$-interleaved. We can define a pseudo-distance on the space of persistence modules, called the interleaving distance, where the interleaving distance between $\mathcal{P}^X$ and $\mathcal{P}^Y$ is the infimum of the set of $\epsilon > 0$ such that $\mathcal{P}^X$ and $\mathcal{P}^Y$ are $\epsilon$-interleaved. It is worth noting that two persistence modules might have interleaving distance 0 and yet not be 0-interleaved (and thus not isomorphic).

More details about the pseudo-metric space structure of persistence modules and how the interleaving distance between persistence modules relates to the distances between corresponding persistence diagrams can be found in [2, 6, 13].

Definition 6. Given a finite metric space $X$ the Rips filtration of $X$ is a family of finite simplicial complexes $\mathcal{R}(X) = \{\mathcal{R}(X)_t\}_{t \geq 0}$ with $\mathcal{R}(X)_t$ the clique complex on the graph with vertices $\{x \in X\}$ and edges $\{[x_1, x_2] : d_X(x_1, x_2) \leq t\}$.

The standard stability theorem for persistent homology of Rips complexes built from finite metric spaces is the following.

Theorem 7. Let $X$ and $Y$ be finite metric spaces and $\mathcal{R}(X), \mathcal{R}(Y)$ the corresponding Rips filtrations. Then the $k$th homology persistence modules of $\mathcal{R}(X)$ and $\mathcal{R}(Y)$ are $2d_{GH}(X,Y)$ interleaved.
Sometimes the Rips filtration is defined by adding the edge \([x_1, x_2]\) when \(d_X(x_1, x_2) \leq t/2\) instead of \(d_X(x_1, x_2) \leq t\), so some of the results may differ from here by a corresponding factor of 2.

We will be proving generalizations of this theorem for the setting when \(X\) and \(Y\) are quasi-metric spaces and \(\mathcal{R}(X)\) and \(\mathcal{R}(Y)\) are one of the generalizations of Rips filtrations.

4. Persistent homology of OT-complexes

Ordered tuple complexes can be thought of as generalizations of simplicial complexes. We will find them useful in the analysis of quasi-metric spaces as they have more flexibility with regard to order (allowing asymmetric roles within the same tuple).

**Definition 3.** An ordered tuple is a sequence of \((v_0, v_1, v_2, \ldots v_n)\) potentially including repeats. A ordered tuple complex (shortened to OT-complex) is a collection \(K\) of ordered tuples such that if \((v_0, v_1, v_2, \ldots v_n) \in K\) then \((v_0, \ldots, \hat{v}_i, \ldots v_n) \in K\) for all \(i\) (where \((v_0, \ldots, \hat{v}_i, \ldots v_n)\) is the ordered tuple with \(v_i\) removed).

It is worth emphasizing that each ordered tuple is determined by the ordered sequence and not just the underlying vertices; \((v_1, v_2, v_3)\) and \((v_3, v_1, v_2)\) are distinct.

The ideas of homology and persistent homology naturally extend to OT-complexes.

**Definition 8.** Given an OT-complex \(K\) we can build a chain complex \(C_\ast(K)\) where \(C_p(K)\) is the set of all the \(F\) linear combinations of the ordered tuples in \(K\) with length \(p+1\). This is an \(F\) vector space whose basis vectors are the ordered sets in \(K\) of length \(p+1\). We define a boundary map \(\partial_p : C_p(K) \to C_{p+1}(K)\) by

\[
\partial_p((v_0, v_1, v_2, \ldots v_p)) = \sum_{i=0}^{k} (-1)^i (v_0, \ldots, \hat{v}_i, \ldots v_p)
\]

and extending linearly. Then we can define the \(k\)-th homology group of OT-complex \(K\) as \(H_k(K, F) = \ker(\partial_{k-1})/\im(\partial_k)\).

When \(K_1 \subset K_2\) are both OT-complexes then the inclusion of chains induces a map on their homology groups: \(\iota_* : H_\ast(K_1) \to H_\ast(K_2)\).

**Definition 4.** We say \(\mathcal{K} = \{K_t\}\) is filtration of OT-complexes if \(K_t \subset K_r\) whenever \(t \leq r\). We define the \(k\)th dimensional ordered tuple persistence module corresponding to \(\mathcal{K}\) as follows:

- for each \(t\) set the vector space \(V_t = H_k(K, F)\)
for each pair $s \leq t$ we have a linear map induced from inclusion
\[ \iota_{t \to s} : H_s(K_s) \to H_s(K_t). \]

It is easy to check that this does satisfy the requirements of a persistence module.

A common way to build OT-complexes is through a simplicial complex. For a simplicial complex $K$ there is an OT-complex $K^{OT}$ where $(v_0, v_1, \ldots, v_p) \in K^{OT}$ whenever $[v_0, v_1 \ldots v_p]$, after removing any repeats, is a simplex in $K$. In [10], Munkres calls the chain complex $C_*(K^{OT})$ the ordered chain complex of $K$, and shows that the simplicial homology of $K$ and the OT-complex homology of $K^{OT}$ are isomorphic. This isomorphism result holds also for persistence modules of filtrations of simplicial complexes as the isomorphisms on homology groups commute with the induced maps on homology by inclusions.

**Definition 5.** Let $K$ be an OT-complex. We say that $K$ is closed under adjacent repeats if whenever $(v_0, v_1, \ldots, v_p) \in C_p(K)$ then $(v_0, \ldots, v_i, v_i, \ldots, v_p) \in C_{p+1}(K)$ for all $i = 0, 1, \ldots p$.

**Definition 6.** Let $K$ and $\tilde{K}$ be OT-complexes, both closed under adjacent repeats, over vertex sets $V$ and $\tilde{V}$ respectively. We say that $\tilde{K}$ is an expansion of $K$ if there exists a projection $\pi : \tilde{V} \to V$ such that $(v_0, v_1, \ldots, v_p) \in \tilde{K}$ if and only if $(\pi(v_0), \pi(v_1), \ldots, \pi(v_p)) \in K$.

Let $\mathcal{K} = \{K_t\}$ and $\tilde{\mathcal{K}} = \{\tilde{K}_t\}$ be filtrations of OT-complexes over vertex sets $V$ and $\tilde{V}$ respectively. We say that $\tilde{\mathcal{K}}$ is an expansion of $\mathcal{K}$ if there exists a projection $\pi : \tilde{V} \to V$ such that, for all $t$, $(v_0, v_1, \ldots, v_p) \in \tilde{K}_t$ if and only if $(\pi(v_0), \pi(v_1), \ldots, \pi(v_p)) \in K_t$.

**Proposition 9.** If $\mathcal{K} = \{K_t\}$ and $\tilde{\mathcal{K}} = \{\tilde{K}_t\}$ are filtrations of OT-complexes such that $\tilde{\mathcal{K}}$ is an expansion of $\mathcal{K}$ then the persistence modules of $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are isomorphic.

**Proof.** Both $\pi$ and the inclusion map $i : K_t \to \tilde{K}_t$ induce chain maps $\pi_# : C_*(K_t) \to C_*(\tilde{K}_t)$ and $i_# : C_*(K_t) \to C_*(\tilde{K}_t)$. Observe that $\pi_# \circ i_# = \text{id} : C_*(K_t) \to C_*(K_t)$, so $\pi_* \circ i_* = \text{id} : H_*(K_t) \to H_*(K_t)$ for all $t$.

Suppose $(v_0, v_1, \ldots, v_i, \ldots, v_p) \in C_p(\tilde{K}_t)$. We want to show that $(v_0, v_1, \ldots, v_i, \pi(v_i), \ldots, \pi(v_p)) \in C_{p+1}(\tilde{K}_t)$. To do this we use that $\tilde{K}_t$ is closed under adjacent repeats, the definition of expansions (twice), and the property that $\pi$ is a projection (so $\pi(\pi(v_j) = \pi(v_j)$).

\[
(v_0, v_1, \ldots, v_i, \ldots, v_p) \in C_p(\tilde{K}_t) \implies (v_0, v_1, \ldots, v_i, v_i, \ldots, v_p) \in C_{p+1}(\tilde{K})
\]

\[
\implies (\pi(v_0), \pi(v_1), \ldots, \pi(v_i), \pi(v_i), \ldots, \pi(v_p)) \in C_{p+1}(K_t)
\]

\[
\implies (\pi(v_0), \pi(v_1), \ldots, \pi(v_i), \pi(\pi(v_i)), \ldots, \pi(\pi(v_p))) \in C_{p+1}(K_t)
\]

\[
\implies (v_0, v_1, \ldots, v_i, \pi(v_i), \ldots, \pi(v_p)) \in C_{p+1}(\tilde{K}_t)
\]
Consider the prism operator
\[ P((v_0, v_1, \ldots, v_p)) = \sum_{i=0}^{p} (-1)^{-i}((v_0, v_1, \ldots, v_i, \pi(v_i), \pi(v_{i+1}), \ldots, \pi(v_p))). \]

Routine algebra shows that \( \partial P + P\partial = i_{\#} \circ \pi_{\#} - \text{id} \) and thus \( i_{\#} \circ \pi_{\#} \) is chain homotopic to the identity. This implies \( i_{\#} \circ \pi_{\#} : H_*(\tilde{K}_t) \rightarrow H_*(\tilde{K}_t) \) is the identity function.

The chain maps \( \pi_{\#} \) and \( i_{\#} \) commute with the inclusion maps for the filtrations of OT-complexes and hence the following diagrams commute
\[
\begin{array}{ccc}
H_*(\tilde{K}_s) & \xrightarrow{i_*} & H_*(\tilde{K}_t) \\
\downarrow \pi_* & & \downarrow \pi_* \\
H_*(K_s) & \xrightarrow{i_*} & H_*(K_t)
\end{array}
\quad
\begin{array}{ccc}
H_*(\tilde{K}_s) & \xrightarrow{i_*} & H_*(\tilde{K}_t) \\
\downarrow \pi_* & & \downarrow \pi_* \\
H_*(K_s) & \xrightarrow{i_*} & H_*(K_t)
\end{array}
\]

Since \( i_* \circ \pi_* = \text{id} : H_*(\tilde{K}_t) \rightarrow H_*(\tilde{K}_t) \) and \( \pi_* \circ i_* = \text{id} : H_*(K_t) \rightarrow H_*(K_t) \) for all \( t \) we see that \( K \) and \( \tilde{K} \) are isomorphic.

5. Rips Filtrations of OT-complexes built using the distance function

Due to the asymmetry of the distance function there are multiple options for analogous objects. In this section we will consider a family of filtrations of OT-complexes built using symmetric functions related to the distance function.

**Definition 7.** Let \( X \) be a finite pseudo-quasi-metric space \( X = \{x_1, \ldots, x_N\} \). For any \( a \in [0, 1] \) we can define a symmetric function
\[ f_a : X \times X \rightarrow \mathbb{R} \]
\[ (x, y) \mapsto a \min\{d(x, y), d(y, x)\} + (1 - a) \max\{d(x, y), d(y, x)\} \]

Set \( R^a(X)_t \) be the OT-complex containing \((x_0, x_1, \ldots, x_p)\) whenever \( f_a(x_i, x_j) \leq t \) for all \( i, j \).

We call the filtration of OT-complexes \( R^a(X) = \{R^a(X)_t\}_{t \in [0, \infty)} \) the Rips filtration under \( f_a \).

Using the isomorphism between simplicial and ordered tuple homology mentioned in section 4 we can observe that the persistent homology of \( R^a(X) \) is isomorphic to the persistent homology of the filtration of simplicial complexes \( \{\text{Rips}^a(X, r)\} \), where \( \text{Rips}^a(X, r) \) is the clique complex over the graph containing vertices \( \{x_i\} \) and edges \( \{[x_i, x_j] | f_a(x_i, x_j) \leq t\} \).
**Theorem 10.** Fix $a \in [0,1]$ and a homology dimension $k$. Let $X$ and $Y$ be finite pseudo quasi-metric spaces. Let $P^X$ and $P^Y$ be the corresponding $k$th dimension homology persistence modules constructed from the corresponding Rips filtrations under $f_a (\{R^a(X)_t\}$ and $\{R^a(Y)_t\}$ respectively). Then $P^X$ and $P^Y$ are $2d_{GH}(X,Y)$ interleaved.

**Proof.** Let $\tilde{X}, \tilde{Y}, \pi_X : \tilde{X} \to X, \pi_Y : \tilde{Y} \to Y$ and $\psi : \tilde{X} \to \tilde{Y}$ be as constructed in Proposition 2. Let $\{R^a(\tilde{X})_t\}$ and $\{R^a(\tilde{Y})_t\}$ be the corresponding Rips filtrations under $f_a$. Let $P^X$ and $P^Y$ be their corresponding $k$th dimension homology persistence modules.

First observe that $\{R^a(\tilde{X})_t\}$ and $\{R^a(\tilde{X})_t\}$ are both closed under adjacent repeats as $f_a(x,x) = 0$ for all $x \in \tilde{X}$. For all $x, \hat{x} \in \tilde{X}$ we know $f_a(x,\hat{x}) = f_a(\pi(x),\pi(\hat{x}))$ as $f_a$ is a function of distances which are preserved under $\pi$. Since the condition for finite chains is determined by $f_a$, this implies that $\{R^a(\tilde{X})_t\}$ is an expansion of $\{R^a(X)_t\}$. By Proposition 3 we know $P^X$ and $P^Y$ are 0-interleaved.

A symmetric argument shows $P^Y$ and $P^X$ are 0-interleaved.

Finally we will use $\psi$ to build a $2d_{GH}(X,Y)$ interleaving between $P^X$ and $P^Y$. From Proposition 2 we know that for all $x, \hat{x} \in \tilde{X}$

$$\min\{d_X(\psi(x),\psi(\hat{x})), d_X(\psi(x),\psi(x))\} \leq \min\{d_X(x,\hat{x}), d_X(\hat{x},x)\} + 2d_{GH}(X,Y)$$

and

$$\max\{d_X(\psi(x),\psi(\hat{x})), d_X(\psi(x),\psi(x))\} \leq \max\{d_X(x,\hat{x}), d_X(\hat{x},x)\} + 2d_{GH}(X,Y),$$

and thus that $f_a(\psi(x),\psi(\hat{x})) \leq f_a(x,\hat{x}) + 2d_{GH}(X,Y)$. This implies that

$$\psi(R^a(\tilde{X})_t) \subset R^a(\tilde{Y})_{t+2d_{GH}(X,Y)}.$$

Similar we can argue that $\psi^{-1}(R^a(\tilde{Y})_t) \subset R^a(\tilde{X})_{t+2d_{GH}(X,Y)}$.

The induced maps $\psi_* : H_*(R^a(\tilde{X})_t) \to H_*(R^a(\tilde{Y})_{t+2d_{GH}(X,Y)})$ and $\psi_*^{-1} : H_*(R^a(\tilde{Y})_t) \to H_*(R^a(\tilde{X})_{t+2d_{GH}(X,Y)})$ determine a $2d_{GH}(X,Y)$-interleaving between the persistence modules. The diagrams (3.1) and (3.2) commute as they are the induced maps on homology from maps between OT-complexes which commute by construction.

□

An alternative proof that draws for previous stability results for compact metric spaces could works in the case that $a \leq 1/2$. This involves both showing that $f_a$ determines and metric and comparing the Gromov-Hausdorff distances of $d_{GH}((X,d_X), (Y,d_Y))$ and $d_{GH}((X,f^X_a), (Y,f^Y_a))$. This alternative proof is included in the appendix.
6. Directed Rips Filtration

Another method of construction of filtrations of OT-complexes that agree with Rips filtration when restricted to metric spaces is to use the asymmetry of the distance function to restrict the order of the vertices in admissible tuples.

**Definition 8.** Let $X$ be a pseudo-quasi-metric space. Set $\{R_{\text{dir}}(X)_t\}$ to be the filtration of OT-complexes where $(v_0, v_1, \ldots, v_p) \in R_{\text{dir}}(X)_t$ when $d_X(v_i, v_j) \leq t$ for all $i \leq j$. We call $\{R_{\text{dir}}(X)_t\}$ the *directed Rips filtration* of $X$.

We can see immediately from the definition that the directed Rips filtration is the normal Rips filtration when defined for a metric space. We can prove that the persistence modules for the directed Rips filtrations enjoy the Gromov-Hausdorff stability.

**Theorem 11.** Let $X$ and $Y$ be finite pseudo-quasi-metric spaces. Let $P^X$ and $P^Y$ be the corresponding $k$th dimension homology persistence modules constructed from the corresponding directed Rips filtrations $\{R_{\text{dir}}(X)_t\}$ and $\{R_{\text{dir}}(Y)_t\}$. Then $P^X$ and $P^Y$ are $2d_{GH}(X,Y)$ interleaved.

**Proof.** The proof is very similar to that of Theorem 10, so we will omit most of the details.

Just as in the proof of Theorem 10 let $\tilde{X}, \tilde{Y}, \pi_X : \tilde{X} \to X, \pi_Y : \tilde{Y} \to Y$ and $\psi : \tilde{X} \to \tilde{Y}$ be as constructed in Proposition 2. Let $\{R_{\text{dir}}(\tilde{X})_t\}$ and $\{R_{\text{dir}}(\tilde{Y})_t\}$ be the filtrations of OT-complexes as defined in 8 and $\mathcal{P}^{\tilde{X}}$ and $\mathcal{P}^{\tilde{Y}}$ be their corresponding $k$th dimension homology persistence modules.

From the distance preserving property of $\pi$, and since all these OT-complexes are naturally closed under adjacent repetitions, $\{R_{\text{dir}}(\tilde{X})_t\}$ is an expansion $\{R_{\text{dir}}(\tilde{X})_t\}$ and hence $\mathcal{P}^{\tilde{X}}$ and $\mathcal{P}^{\tilde{X}}$ are 0-interleaved.

A symmetric argument shows $\mathcal{P}^{\tilde{Y}}$ and $\mathcal{P}^{\tilde{Y}}$ are 0-interleaved.

Again $\psi$ again induces a $2d_{GH}(X,Y)$ interleaving between $\mathcal{P}^{\tilde{X}}$ and $\mathcal{P}^{\tilde{Y}}$. Part (3) ensures $d_{\tilde{Y}}(\psi(x), \psi(\tilde{x})) \leq d_{\tilde{X}}(x, \tilde{x}) + 2d_{GH}(X,Y)$ for all $x, \tilde{x} \in \tilde{X}$ which implies

$$\psi(\{R_{\text{dir}}(\tilde{X})_t\}) \subset \{R_{\text{dir}}(\tilde{Y})_{t+2d_{GH}(X,Y)}\},$$

and part (4) ensures $d_{\tilde{X}}(\psi^{-1}(y), \psi^{-1}(	ilde{y})) \leq d_{\tilde{Y}}(y, \tilde{y}) + 2d_{GH}(X,Y)$ for all $y, \tilde{y} \in \tilde{Y}$, which implies that

$$\psi(\{R_{\text{dir}}(\tilde{X})_t\}) \subset \{R_{\text{dir}}(\tilde{Y})_{t+2d_{GH}(X,Y)}\}. $$

$\square$
7. Strongly connected component persistence

Zeroth dimensional persistent homology is all about tracking the evolution of connected components. For directed graphs, unlike graphs, there is choice in how to interpret what a connected component is, and each interpretation has their own corresponding persistence module. We consider the persistence of weakly and strongly connected components. Weakly connected components are the components of the graph when the directions are forgotten. Given a filtration of a directed graph by edge weights, the weakly connected persistence would be the same as the zeroth dimensional homology of the Rips filtration under $f_1(x, y) = \min\{d(x, y), d(y, x)\}$ (see section 5) and to the zeroth dimensional directed homology of the directed Rips complex in section 6.

The strongly connected components of a directed graph are the equivalence classes of points where $v \sim w$ when there exists both a path from $v$ to $w$ and a path from $w$ to $v$. Given a filtration of directed graphs we can construct a persistence module based on linear combinations of strongly connected components (analogous to zeroth dimensional homology being based on linear combinations of connected components).

**Definition 12.** Let $\mathcal{D} = \{D_t : t \in \mathbb{R}\}$ be a filtration of directed graphs. Let $[v]_t$ denote the strongly connected component of $D_t$ containing $v$. We define the *strongly connected persistence module* corresponding to $\mathcal{D}$ as follows:

- for each $t$ set the vector space $V_t$ to be the set of finite linear combinations of strongly connected component (elements are of the form $\sum_{i=1}^{k} \lambda_i [v_i]_t$ with $\lambda_i \in \mathbb{F}$)
- for each pair $s \leq t$ we have a linear map induced from inclusion $\iota_{t \rightarrow s}(\sum_{i=1}^{k} \lambda_i [v_i]_t) = \sum_{i=1}^{k} \lambda_i [v_i]_s$

We will now check that the strongly connected persistence module does satisfy the requirements of a persistence module. Whenever we have an inclusion of directed graphs $D_t \subset D_s$, if there is a path from $v$ to $w$ in $D_t$ then there is a path from $v$ to $w$ in $D_s$. This implies that the maps $\iota_{t \rightarrow s}$ are well defined. Furthermore for $u \leq t \leq s$ we have $\iota_{t \rightarrow s}(\iota_{u \rightarrow t}(\sum_{i=1}^{k} \lambda_i [v_i]_u)) = \sum_{i=1}^{k} \lambda_i [v_i]_s$ Whenever the directed graphs $D_t$ are all finite (which is always true in almost any application) we automatically know that the $V_t$ are all finite dimensional and hence the strongly connected persistence module is $q$-tame.

We can create strongly connected persistence modules from pseudo-quasi-metric spaces by constructing a relevant filtration of directed graphs.

**Definition 9.** Given a finite pseudo-quasi-metric space $X$ there is a natural filtration of directed graphs $\{\mathcal{D}(X)_t : t \in [0, \infty)\}$ associated to $X$ by setting $\mathcal{D}(X)_t$ to the
the directed graph with vertices the points in \( X \) and including arrow \( x \to y \) when \( d(x, y) \leq t \). We will call this the associated filtration of directed graphs to \( X \).

**Theorem 13.** Let \( X \) and \( Y \) be finite pseudo-quasi-metric spaces and let \( D(X) = \{ D(X)_t \} \) and \( D(Y) = \{ D(Y)_t \} \) be the associated filtrations of directed graphs. Similarly define a filtration of directed graphs \( \{ Y_i \} \) from \( Y \). Let \( \mathcal{P}^X \) and \( \mathcal{P}^Y \) be the strongly connected component persistence modules for \( D(X) \) and \( D(Y) \). Then \( \mathcal{P}^X \) and \( \mathcal{P}^Y \) are \( 2d_{GH}(X, Y) \) interleaved.

**Proof.** Fix a correspondence \( \mathcal{M} \) between \( X \) and \( Y \) such that

\[
\sup_{(x_1, y_1), (x_2, y_2) \in \mathcal{M}} |d(x_1, x_2) - d(y_1, y_2)| \leq 2d_{GH}(X, Y).
\]

Construct a map \( \alpha : X \to Y \) where for each \( x \) we arbitrarily fix a representative from \( \{ y \in Y : (x, y) \in \mathcal{M} \} \) and construct a map \( \beta : Y \to X \) where for each \( y \) we arbitrarily fix a representative from \( \{ x \in X : (x, y) \in \mathcal{M} \} \).

We will first show, for every \( x \in X \), that \( \beta(\alpha(x)) \) and \( x \) lie in the same strongly connected component in \( D(X)_{2d_{GH}(X,Y)} \). From our construction of \( \alpha \) and \( \beta \) we know that \( (x, \alpha(x)) \) and \( (\beta(\alpha(x)), \alpha(x)) \) are both in \( \mathcal{M} \) and hence

\[
d_X(x, \beta(\alpha(x)) - d_Y(\alpha(x), \alpha(x)) \leq 2d_{GH}(X,Y),
\]

that is \( d_X(x, \beta(\alpha(x)) \leq 2d_{GH}(X,Y) \). Similarly \( d_X(\beta(\alpha(x)), x) \leq 2d_{GH}(X,Y) \) and hence both \( x \to \beta(\alpha(x)) \) and \( \beta(\alpha(x)) \to x \) are in \( D(X)_{2d_{GH}(X,Y)} \).

A symmetric argument shows, for every \( y \in Y \), that \( \alpha(\beta(y)) \) and \( y \) lie in the same strongly connected component in \( D(Y)_{2d_{GH}(X,Y)} \).

Suppose that there is a path from \( x_1 \) to \( x_2 \) in \( D(X)_t \). This means that there exists a sequence of points \((x_1 = a_1, a_2, \ldots, a_k = x_2)\) in \( X \) such that \( d(a_i, a_{i+1}) \leq t \). Since each \( (a_i, \alpha(a_i)) \in \mathcal{M} \) we have

\[
|d_X(a_i, a_{i+1}) - d_Y(\alpha(a_i), \alpha(a_{i+1}))| \leq 2d_{GH}(X,Y)
\]

for each \( i \) and hence \((\alpha(x_1) = \alpha(a_1), \alpha(a_2), \ldots, \alpha(a_k) = \alpha(x_2))\) is a path in \( D(Y)_{t + 2d_{GH}(X,Y)} \).

If \([x_1]_t = [x_2]_t\) then there exist paths in \( D(X)_t \) from \( x_1 \) to \( x_2 \) and from \( x_2 \) to \( x_1 \). Our construction guarantees that there exist paths in \( D(Y)_{t + 2d_{GH}(X,Y)} \) from \( \alpha(x_1) \) to \( \alpha(x_2) \) and from \( \alpha(x_2) \) to \( \alpha(x_1) \). This means that \( \alpha_t \) determines a linear map \( \alpha_* : P_t^X \to P_{t + 2d_{GH}(X,Y)}^Y \) where \([x]_t \mapsto [\alpha(x)]_{t + 2d_{GH}(X,Y)}\).

Symmetric arguments show that \( \beta_t \) determine a linear map \( \beta_* : P_t^Y \to P_{t + 2d_{GH}(X,Y)}^X \) where \([y]_t \mapsto [\beta(y)]_{t + 2d_{GH}(X,Y)}\).
It only remains to show that \( \alpha_* \) and \( \beta_* \) satisfy an \( 2d_{GH}(X,Y) \) interleaving. That \( (3.1) \) commutes follows directly from the construction of \( \alpha \) and \( \beta \). For \( (3.2) \), observe that if \( x \) and \( \beta(\alpha(x)) \) are in the same strongly connected component in \( D(X)_{2d_{GH}(X,Y)} \) then they are still in the same connected component in \( D(X)_{t+2d_{GH}(X,Y)} \) for any \( t \geq 0 \). \( \square \)

8. OT-complexes constructed using the poset structure

In the theory of partially ordered sets ("posets"), the order complex of a poset is the set of all finite chains. Its homology contains important information about the poset. From a directed graph we can build a poset over its vertices by saying \( x \leq y \) whenever there is a path from \( x \) to \( y \). From the associated filtration of directed graphs to a quasi-metric space we can create a filtration of posets which we will call the poset Rips filtration. We will show that the persistent homology of the poset Rips filtration enjoys stability with respect to the Gromov-Hausdorff distance.

Definition 14. A partially ordered space (which from now on we will abbreviate to "poset") is a set \( X \) equipped with a binary relation \( \leq \) which is reflexive and transitive (\( x \leq x \) for all \( x \in X \) and if \( x \leq y \) and \( y \leq z \) then \( x \leq z \)). A filtration of posets is a set \( \mathcal{X} = \{ (X_t, \leq t) : t \in \mathbb{R} \} \) such that whenever \( s \leq t \) implies \( X_s \subset X_t \) and if \( x \leq y \) then \( x \leq_t y \).

Definition 10. Given a poset \( (X, \leq) \) let \( \mathcal{O}(X, \leq) \) be the OT-complex containing \( (x_0, x_1, \ldots, x_p) \) when \( x_0 \leq x_1 \leq \ldots \leq x_p \).

We define the \( k \)th dimensional poset persistence module corresponding to the filtration of posets \( \mathcal{X} = \{ (X_t, \leq_t) \} \) as the \( k \)th dimensional homology persistence module for the filtration \( \mathcal{O}(\mathcal{X}) = \{ \mathcal{O}(X_t, \leq_t) \} \).

The use of the symbol \( \mathcal{O} \) is motivated by the isomorphism between the homology of \( \mathcal{O}(X, \leq) \) and the simplicial homology of the order complex for the the poset \( (X, \leq) \).

The filtration of associated graphs to a quasi-metric space provide a natural filtration of preordered spaces by saying that \( x \leq y \) if there exists a path from \( x \) to \( y \).

Definition 11. Let \( X \) be a quasi-metric space and let \( \{ D(X)_t \} \) be its associated filtration of directed graphs. For each \( t \geq 0 \) construct a preordered space \( (X_t, \leq_t) \) with \( X_t \) the set of points in \( X \) and \( x \leq_t y \) when there exists a path in \( D(X)_t \) from \( x \) to \( y \).

Let \( \mathcal{O}(X) = \{ \mathcal{O}(X)_t \} \) be the filtration of OT-complexes corresponding to the filtration of posets \( \{ (X_t, \leq_t) \} \). We call \( \mathcal{O}(X) \) the poset Rips filtration of \( X \).
Theorem 15. Let $X$ and $Y$ be finite pseudo-quasi-metric spaces with poset Rips filtrations $O(X)$ and $O(Y)$. Let $P^X$ and $P^Y$ be the $k$th dimensional persistence modules for $O(X)$ and $O(Y)$ respectively. Then $P^X$ and $P^Y$ are $2d_{GH}(X,Y)$ interleaved.

Proof. The proof is repeats arguments from Theorem 10 and Theorem 13 so we will omit most of the details.

Just as in the proof of Theorem 10 let $\tilde{X}, \tilde{Y}, \pi_X : \tilde{X} \to X, \pi_Y : \tilde{Y} \to Y$ and $\psi : \tilde{X} \to \tilde{Y}$ be as constructed in Proposition 2. Let $\{O(\tilde{X})_t\}$ and $\{O(\tilde{Y})_t\}$ be their poset Rips filtrations and $P^{\tilde{X}}$ and $P^{\tilde{Y}}$ their corresponding $k$th dimension homology persistence modules.

First observe that all these OT-complexes are naturally closed under adjacent repetitions. From the distance preserving property of $\pi$, there path from $x_i$ to $x_j$ in $D(\tilde{X})_t$ if and only if path from $\pi(x_i)$ to $\pi(x_j)$ in $D(\tilde{X})_t$. Thus we can say they $\pi$ preserves order.

From the order preserving property of $\pi$, and since all these OT-complexes are naturally closed under adjacent repetitions, we conclude $\{O(\tilde{X})_t\}$ is an expansion of $\{O(\tilde{X})_t\}$. By Proposition 9, $P^{\tilde{X}}$ and $P^X$ are $0$-interleaved. A symmetric argument shows $P^{\tilde{Y}}$ and $P^Y$ are $0$-interleaved.

Just as in the proof to Theorem 13 we can show that whenever theres a path from $x$ to $\hat{x}$ in $D(\tilde{X})_t$ then there is a path from $\alpha(x)$ to $\alpha(\hat{x})$ is $D(\tilde{Y})_{t+2d_{GH}(X,Y)}$. This implies that when $(x_0, x_1, \ldots, x_p) \in C_p(O(\tilde{X})_t)$ then $(\alpha(x_0), \alpha(x_1), \ldots, \alpha(x_p))$ is in $C_p(O(\tilde{Y})_{t+2d_{GH}(X,Y)})$. Thus $\psi : \tilde{X} \to \tilde{Y}$ induces a family of homomorphisms $\{\psi_* : H_*(O(\tilde{X})_t) \to H_*(O(\tilde{Y})_{t+2d_{GH}(X,Y)})\}$. Similarly $\psi^{-1}$ induces a family of homomorphism $\{\psi_*^{-1} : H_*(O(\tilde{Y})_t) \to H_*(O(\tilde{X})_{t+2d_{GH}(X,Y)})\}$. Since

$$\psi_*^{-1} \circ \psi_*[(x_0, \ldots, x_p)]_t = [(\psi^{-1}(\psi(x_0)), \ldots, \psi^{-1}(\psi(x_p)))]_{t+2d_{GH}(X,Y)} = [(x_0, \ldots, x_p)]_{t+2d_{GH}(X,Y)}$$

we see that

$$\psi_*^{-1} \circ \psi_* : H_*(O(\tilde{X})_t) \to H_*(O(\tilde{X})_{t+2d_{GH}(X,Y)})$$

is the map on homology induced by the inclusion $O(\tilde{X})_t \subset O(\tilde{X})_{t+2d_{GH}(X,Y)}$. Similarly $\psi_* \circ \psi_*^{-1} : H_*(O(\tilde{Y})_t) \to H_*(O(\tilde{Y})_{t+2d_{GH}(X,Y)})$ is the map on homology induced by inclusion.

We thus conclude that $\psi : \tilde{X} \to \tilde{Y}$ induces a $2d_{GH}(X,Y)$ interleaving between $P^{\tilde{X}}$ and $P^{\tilde{Y}}$. □
9. Future Directions

In this paper we introduce four different ways to construct persistence modules from quasi-metric spaces. A future direction is to apply these methods to quasi-metric spaces to see what they reveal about their quasi-metric structure, or to use as a method of getting a lower bound on the Gromov-Hausdorff distance between different quasi-metric spaces.

Another further direction is to adapt these methods to construct persistence modules for sub-level set filtrations of functions on quasi-metric spaces and proving related stability results. For example, we believe that the all four constructions built from a suitably defined sublevel sets of the extremity function (analogous to constructions in [3]) could have Gromov-Hausdorff stability. This would provide another way of capturing the “shape” of a quasi-metric space.

10. Appendix

Lemma 16. Let $a \in [0, 1/2]$ and $d(\cdot, \cdot)$ be a pseudo-quasi-metric on $X$. Define $f_a : X \times X \to \mathbb{R}$ by

$$f_a(x, y) = a \min\{d(x, y), d(y, x)\} + (1 - a) \max\{d(x, y), d(y, x)\}.$$ 

Then $f_a$ is a pseudo-metric on $X$. If we further assume that $d(\cdot, \cdot)$ is a quasi-metric then $f_a$ defines a metric in $X$.

Proof. First observe that both $d(x, y) \geq 0$ and $d(y, x) \geq 0$ for all $x, y \in X$ which implies that $f_a(x, y) \geq 0$ for all $x, y \in X$ and thus $f_a$ satisfies property (1). By construction $f_a$ is symmetric and hence satisfies property (2). Thus all we need to show the triangle inequality; that is $f_a(x, z) \leq f_a(x, y) + f_a(y, z)$ for all $x, y, z$.

We can rewrite $f_a$ as

$$f_a(x, y) = a(d(x, y) + d(y, x)) + (1 - 2a) \max\{d(x, y), d(y, x)\}$$

since $a, 1 - 2a \geq 0$ it is thus enough to show that $g(x, y) = d(x, y) + d(y, x)$ and $h(x, y) = \max\{d(x, y), d(y, x)\}$ both satisfy the triangle inequality.

$$g(x, z) = d(x, z) + d(z, x)$$

$$\leq (d(x, y) + d(y, z)) + (d(z, y) + d(y, x))$$

$$= (d(x, y) + d(y, x)) + (d(y, z) + d(z, y))$$

$$= g(x, y) + g(y, z)$$
\[ d(x, z) \leq d(x, y) + d(y, z) \leq \max\{d(x, y), d(y, x)\} + \max\{d(y, z), d(z, y)\} = h(x, y) + h(y, z) \]

and

\[ d(z, x) \leq d(z, y) + d(y, x) \leq \max\{d(z, y), d(y, z)\} + \max\{d(y, x), d(x, y)\} = h(y, z) + h(x, y). \]

Together they imply that \( h(x, z) \leq h(x, y) + h(y, z). \)

\[ \square \]

It is important to note that if \( a > 1/2 \) then \( f_a \) is not necessarily define a metric. For example, consider a quasi-metric space of three points \( \{x, y, z\} \) and a quasi-metric distance function \( d \) tabulated in Table 1.

**Table 1.** If \( a > 1/2 \) and \( K > 2a/(2a - 1) \) then \( f_a(x, y) + f_a(y, z) = 2a + 2K(1 - a) < K = f_a(x, z) \) and thus \( f_a \) does not satisfy the triangle-inequality.

| d   | x  | y  | z  |
|-----|----|----|----|
| x   | 0  | 1  | K  |
| y   | K  | 0  | K  |
| z   | K  | 1  | 0  |

| f_a | x    | y   | z    |
|-----|------|-----|------|
| x   | 0    | a+K(1-a) | K   |
| y   | a+K(1-a) | 0    | a+K(1-a) |
| z   | K    | a+K(1-a) | 0   |

**Lemma 17.** Let \( a \leq 1/2 \). \( d_{GH}((X, f^X_a), (Y, f^Y_a)) \) \( \leq 2d_{GH}((X, d_X), (Y, d_Y)) \).

**Proof.** Take some correspondence \( \mathcal{M} \subset X \times Y \) such that

\[
\max_{(x_1, y_1), (x_2, y_2) \in \mathcal{M}} |d_X(x_1, x_2) - d_Y(y_1, y_2)| \leq 2d_{GH}((X, d_X), (Y, d_Y))
\]

Let \((x_1, y_1), (x_2, y_2) \in \mathcal{M},

\[
f_a(x_1, x_2) - f_a(y_1, y_2) = a(d(x_1, x_2) + d(x_2, x_1) - d(y_1, y_2) - d(y_2, y_2)) + (1 - 2a)(\max\{d(x_1, x_2), d(x_2, x_1)\} - \max\{d(y_1, y_2), d(y_2, y_1)\})
\]

If \( d(y_1, y_2) \geq d(y_2, y_1) \) and \( d(x_1, x_2) \leq d(x_2, x_1) \) then

\[
\max\{d(x_1, x_2), d(x_2, x_1)\} - \max\{d(y_1, y_2), d(y_2, y_1)\} = d(x_2, x_1) - d(y_1, y_2) \leq 2d_{GH}((X, d_X), (Y, d_Y)).
\]
If \( d(y_1, y_2) \geq d(y_2, y_1) \) and \( d(x_1, x_2) \geq d(x_2, x_1) \) then

\[
\max \{d(x_1, x_2), d(x_2, x_1)\} - \max \{d(y_1, y_2), d(y_2, y_1)\} = d(x_2, x_1) - d(y_2, y_1) \\
\leq 2d_{GH}((X, d_X), (Y, d_Y)).
\]

The other two cases are treated similarly.

Thus \( f_a(x_1, x_2) - f_a(y_1, y_2) \leq a4d_{GH}((X, d_X), (Y, d_Y)) + (1 - 2a)2d_{GH}((X, d_X), (Y, d_Y)) = 2d_{GH}((X, d_X), (Y, d_Y)) \). We can interchange the roles of \( X \) and \( Y \) to show that \( |f_a(x_1, x_2) - f_a(y_1, y_2)| \leq 2d_{GH}((X, d_X), (Y, d_Y)) \). Since this holds for all \((x_1, y_1), (x_2, y_2) \in M\), we have

\[
d_{GH}((X, f_a^X), (Y, f_a^Y)) = \frac{1}{2} \inf_{M_{correspondence}} \max_{(x_1, y_1), (x_2, y_2) \in \hat{M}} |f_a(x_1, x_2) - f_a(y_1, y_2)| \\
\leq \frac{1}{2} \max_{(x_1, y_1), (x_2, y_2) \in M} |f_a(x_1, x_2) - f_a(y_1, y_2)| \\
\leq d_{GH}((X, d_X), (Y, d_Y))
\]

\[\square\]

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