How much classical information is carried by a quantum state? An approach inspired by Kolmogorrov complexity.

Doriano Brogioli
Universität Bremen, Energiespeicher– und Energiewandlersysteme, Bibliothekstraße 1, 28359 Bremen, Germany.

Abstract

In quantum mechanics, a state is an element of a Hilbert space whose dimension exponentially grows with the increase of the number of particles (or qubits, in quantum computing). The vague question “is this huge Hilbert space really there?” has been rigorously formalized inside the computational complexity theory; the research suggests a positive answer to the question. Along this line, I give a definition of the (classical) information content of a quantum state, taking inspiration from the Kolmogorov complexity. I show that, for some well-known quantum circuits (having a number of gates polynomial in the number of qubits), the information content of the output state, evaluated according to my definition, is polynomial in the number of qubits. On the other hand, applying known results, it is possible to devise quantum circuits that generate much more complex states, having an exponentially-growing information content. A huge amount of classical information can be really present inside a quantum state, however, I show that this property is not necessarily exploited by quantum computers, not even by quantum algorithms showing an exponential speed-up with respect to classical computation.

1 Introduction

One of the surprising aspects of quantum mechanics is the enormous size of the data necessary to represent a quantum state, in comparison with the classical view of the same system. Although it underlies every physical phenomenon, most of this astonishing complexity is lost when we observe the system at macroscopic level. As an example, performing an \textit{ab initio} calculation of a semiconductor electronic structure, by a brute-force numerical solution of the Schrödinger equations, is practically unfeasible; complex approximation methods are needed to simulate the system in a reasonable time. Notwithstanding the complexity of the calculation, the result is just expressed by a few calcu-
lated parameters, such as the band-gap: a few digits, which hardly show the astonishing complexity of the underlying system.

The discussion above hinges around the word “calculation”: actually, a deep insight into the complexity of quantum mechanics is obtained by experiments on quantum computers. They can be seen as laboratory instruments for testing the most unusual quantum phenomena, rather than as machines for solving difficult mathematical problems. In the field of quantum computing, the concepts presented in the previous paragraph are well formalized. The information contained in a quantum state is actually huge: the quantum equivalent of $n$ bits, i.e. $n$ qubits (the bits of quantum computers), is described by $2^n$ complex numbers (actually, rational numbers suffice [1]). This notwithstanding, a measurement of the quantum state just gives us $n$ classical bits. What is surprising is rather that quantum computers are exponentially faster (in $n$) than classical ones. More formally, this is expressed in computational complexity theory [2] by the fact that the class $\text{BQP}$ (roughly, problems that can be solved in polynomial time by a quantum computer with good probability) is believed to strictly include the class $\text{P}$ (roughly, the problems that can be solved in polynomial time by classical computers).

The exploration of an exponentially large number of classical configuration is often described as the reason of the exponential speed-up in the calculation time: generating a large superposition of states, with a large entanglement, and make them interfere are necessary ingredients [3] (although not sufficient [4]) of quantum circuits showing superior performances with respect to classical computers. This can be seen as a quite indirect proof of the existence of an exponentially large amount of information in a quantum state.

The quite vague question: “are all these components of the quantum state really there?” arised in various contexts, in particular connected with the interpretations of quantum mechanics. More practically and rigorously, this question can been formalized in computational complexity theory. S. Aaronson mentioned this question as one of the “ten grand challenges for quantum computing theory” [5] and suggested to address it by studying the substitution of classical certificates and advices instead of their quantum equivalents in two quantum complexity classes, $\text{QMA}$ and $\text{BQP/qpoly}$ [6]. An even more practical reason for asking the question is to devise memory-compressed (zipped) representations of quantum states, aimed at a more efficient simulation of quantum systems [7].

Actually, there are clues suggesting that the information contained in a quantum state is not so huge. One of such clues is that $\text{BQP}$ is a strict subset of $\text{PSPACE}$: roughy, the quantum computers can be simulated using a polynomial amount of memory, although a very long calculation time could be needed. Unfortunately, this fact does not help us to identify a compact structure representing the quantum state. Another clue is that the quantum computers are less powerful than “non-deterministic Turing machines”, i.e. they do not simultaneously explore every possible classical state (as sometimes erroneously said in science popularization): this is formally expressed by the belief that $\text{BQP}$ does not contain $\text{NP-complete}$ problems, or by the fact that the Grover’s search algorithm only gives a polynomial (instead of an exponential) speed-up.
Figure 1: Scheme of the approach to the evaluation of the information content of a quantum state generated by a circuit. In the upper part, the circuit $C$ is implemented in a quantum computer, applied to the initial state $|0\rangle$, and the output state is measured in the computational basis. In the lower part, a compressed (zipped) representation $I$ of $|0\rangle$ is fed as input to an algorithm, which generates samples with the same probability distribution of the quantum computer.
In order to systematically connect the results available in literature to the above-described question, here I propose an approach inspired by the Kolmogorov complexity to evaluate the classical information content of a quantum state generated by a circuit. In short, the idea behind the Kolmogorov complexity is that the amount of information carried by a string equals the length of the shortest algorithm able to generate it. The approach must however be adapted, as schematically shown in Fig. 1. Indeed, in order to extend this idea to quantum states, two problems have to be solved.

The first problem is that the quantum state is not a string but rather gives a probability distribution. There are two possible meanings of “simulating a quantum system”: either generate the numbers representing the probabilities of measuring a given output, or to sample from that distribution, i.e. generate random numbers with the same probability of the quantum state measurement. These two operations are called strong and weak simulation, respectively. The two operations do not have the same complexity, e.g. the weak is faster than the strong simulation [8]. According to the discussions reported in literature [9, 10], I choose the weak simulation: as shown in Fig. 1, the algorithm will be a probabilistic algorithm generating samples; such samples will have the same probability distribution of the measurement of the quantum state. A similar approach has been proposed for circuit complexity classes [11].

The second problem is that, in many cases, the shortest algorithm that samples the required probability distribution is just the algorithm that simulates the quantum circuit $C$ acting on $|0\rangle$, by brute force. Indeed, such an algorithm is quite short, although it could take an extremely long time. However, this solution does not express at all the intuitive concept of “information content of the output quantum state”. We can get closer to the intuitive idea by imposing a limit to the calculation time. As shown in Fig. 1 I consider an algorithm running in polynomial time in the number $n$ of qubits and taking an input $I$ representing the quantum state in a compressed (zipped) form. This input $I$ will then represent the classical information contained in the quantum state.

Summarizing, we aim at classically simulating the measurements of the quantum state $C|0\rangle$, i.e. the application of a circuit $C$ to the initial state $|0\rangle$ (upper part of Fig. 1). We devise a procedure to describe $C|0\rangle$ in a compressed form $I$. This representation $I$ is fed as input to the probabilistic sampling algorithm, which generates samples with the same probability distribution of the quantum computer. The aim is to study (lower bound of) the size of $I$, after imposing that the sampling algorithm works in polynomial time in the number of qubits $n$. The formalization of this procedure is described in Sect. 2.

In general, any state $|\Psi\rangle$ can be obtained by the application of a suitable circuit $C$ to $|0\rangle$; the size of $I$ can be thus very large, exponential in the number of qubits. However, many of the surprising features of quantum computers arise in relatively short circuits: calling $m$ the number of 1- and 2-qubit gates of the circuit, I assume that $m$ is polynomial in the number of qubits $n$. Under this assumption, it makes sense to wonder if $I$ really needs to be exponentially large in $n$, or if a polynomially-sized $I$ suffices. The discussion of this question, for various circuits, is the main topic of this paper.
Literature results on boson sampling and instantaneous quantum polynomial protocol [12, 13, 14] show that it is actually possible to devise specific circuits that generate a quantum state with a complexity that is exponentially-growing in the number of qubits \(n\).

On the other hand, in Sect. 3 I show that some well-known quantum circuits \(C\) lead to quantum states with a polynomial amount of information, i.e. \(I\) has polynomial size. Some of the cases are not trivial and quite surprising: it could be naively expected that an exponentially-sized \(I\) is needed. Such quantum circuits are used by quantum algorithms showing an exponential speed-up with respect to classical computation.

I finally suggest that the final answer to the question “is this huge Hilbert space really there?” is positive, although the possibility of actually using this huge amount of information is not necessarily exploited by quantum computers, not even in algorithms showing superior performances with respect to classical computers. This fact can be compared with the above-mentioned fact that generating a large superposition of states, with a large entanglement, and make them interfere are necessary ingredients [3] for getting superior performances from quantum computers, although they are not sufficient [4] for making a quantum circuit hard to be simulated.

### 2 Definition of classical information contained in the quantum state generated by a quantum circuit

The problems that I will discuss concern the possibility of simulating the result of a measurement of the quantum state generated by a quantum circuit \(C\). The “simulation” is done by a classical probabilistic algorithm (also called “randomized”), i.e. a classical algorithm that has a generator of random uniformly distributed bits. This concept is captured by the following definition.

**Definition 1** (Sampling algorithm). A set of quantum circuits \(\mathcal{C}\) is given, with the property that every quantum circuit \(C \in \mathcal{C}\), operating on \(n\) qubits, has \(m\) 1- and 2-qubit gates, with \(m\) polynomial in \(n\). A sampling algorithm \(A_C\) is a classical probabilistic algorithm with the following property. For there exists an input \(I\) to \(A_C\), such that the output of \(A_C\) is a string \(x\) having the same probability distribution of measuring \(x\) after applying \(C\) to the quantum state |0\rangle.

Since \(A_C\) is probabilistic, its output is different every time it is run, hence we can speak of the probability of the output. It is worth remarking that the term “classical” is omitted from the name “sampling algorithm” defined above and will be implicitly assumed in the following.

There is a trivial sampling algorithm working for every \(\mathcal{C}\): giving a representation of \(C\) as input \(I\), the sampling algorithm is any algorithm that performs a weak simulation of the quantum circuit. In this case, the size of input \(I\)
is polynomial in \( n \), but the calculation time is likely exponential in \( n \). Indeed, it is widely believed that \textbf{BQP} contains problems outside \textbf{P}; this means that the weak simulation of some quantum circuits requires super-polynomial time. Actually, all the known simulation algorithms operate in exponential time, although the base of the exponential can be quite small in specialized algorithms \cite{8}.

On the other hand, it is possible to devise a sampling algorithm that works in polynomial time in \( n \), by providing a suitable representation of \( C|0\rangle \) as input \( I \). The procedure is based on the so-called “inverse transform sampling” and is described, for the present application, in Sect. III of Ref. \cite{7}.

The procedure is now briefly sketched. All the probabilities \( P(x) \) are first calculated based on the strong simulation of \( C \) on the initial state \( |0\rangle \). Then the probabilities are summed to get the cumulative probabilities \( P_C(x) = \sum_i^x P(i) \) and their representation is passed to the sampling algorithm \( A_C \) as input \( I \). Finally, \( A_C \) generates a random number \( p \); applies the bisection algorithm to find the largest \( x \) such that \( P_C(x) \leq p \); and returns this \( x \).

Although this method works for every circuit class \( C \), the size of input \( I \) is exponential in \( n \). It is also worth noting that, in this case, the complexity of the calculation performed by the quantum circuit \( C \) shows up during the calculation of \( I \) from \( C \), which requires exponential time.

These two simple examples of sampling algorithms show that there is a trade-off between space and time resources: if one between input size and calculation time is polynomial in \( n \), then the other is super-polynomial. For this reason, I suggest that it is interesting to define the information content of the quantum state generated by a set of quantum circuits as follows.

**Definition 2.** A set of quantum circuits \( C \) is given, with the properties described in Def. 1. Considering the sampling algorithms \( A_C \) that operates in polynomial time in \( n \) on average, we say that the information content of the output quantum state \( C|0\rangle \) is polynomial in \( n \) if there is a sampling algorithms \( A_C \) for which the input \( I \) has polynomial size in \( n \). Analogously, we define “super-polynomial” and “exponential”.

Literature results on boson sampling and instantaneous quantum polynomial protocol \cite{12, 13, 14} show that it is actually possible to devise specific circuit sets \( C \) that generate a quantum state with a complexity that is exponentially-growing in the number of qubits \( n \). An explicit construction of such quantum circuits can be found in Sect. 4.3 of Ref. \cite{12}; the proof is based on the conjecture that the polynomial hierarchy does not collapse (information based on a personal communication by the Author of the cited paper).

On the other hand, in the next section I will present some cases in which the information content is polynomial. These cases include quantum circuits used in quantum algorithms that show exponential speed-up with respect to classical computation.
3 Examples

3.1 Quantum circuits that can be efficiently simulated by classical algorithms

It is known that some classes of quantum circuits can be efficiently simulated by classical algorithms, although it is believed that this cannot be done in general. In such cases, the input $I$ to the sampling algorithm $A_C$ can be - trivially - a representation of the circuit.

A very simple example is the class of circuits formed by a sequence of Hadamard gates, followed by a sequence of Toffoli gates; these circuits are called HT in Ref. [9]. Each Hadamard gate puts a qubit in an uniform superposition of states $|0\rangle$ and $|1\rangle$: this operation can be simulated by tossing a coin, obtaining a random $x$. The subsequent Toffoli gates are, actually, classical reversible gates: applying them to $x$, the desired sample is obtained. The HT circuits actually correspond to the probabilistic classical computation, in which a sequence of classical operations is applied to an input state which also contains random coins.

Various classes of circuits can be efficiently simulated by classical algorithms [10]; they include circuits composed by Clifford gates [15] and by nearest-neighbor matchgates [16], optical quantum information circuits [17], 2-local commuting circuits [18], circuits with small entanglement [3], tree-like circuits [19]. In several cases, the weak simulation of the circuit class has been explicitly addressed, see the discussion in Refs. [9, 10].

In this context, a particularly important class is represented by the circuits composed by the Clifford gates: the Gottesmann-Knill theorem [15] states that they can be efficiently simulated by classical algorithms. The proof is based on the “normalizer” formalism; it will be discussed in more in detail in Sect. 3.4, together with its extensions. A different proof, given in Ref. [9], shows more explicitly that it is possible to make a weak simulation, i.e. to sample the outcome of the measurements.

In the generic formulation exposed at the beginning of the section, the input $I$ is a representation of the circuit itself. In the case of circuits composed by Clifford gates, there is an alternative possibility. Indeed, following the various proofs of Gottesmann-Knill theorem, it appears that the quantum state (after the application of any number of Clifford gates) can be represented by means of a data structure with polynomial size in $n$: either the so-called “tableau” [20], or a “graph-state” representation [21]. The sampling algorithm can then operate in polynomial time on such structures for generating the random samples. The above-mentioned alternative possibility is thus to use one of such structures as input $I$ for the sampling algorithm.

It must be remarked that the “tableau” and the “graph-state” representation allows the sampling algorithm to generate samples independently of the number of gates $m$ of the circuit, which are thus not limited to be polynomial in $n$: this is a stronger property than required by Def. [2] which only refers to $m$ polynomial in $n$. This property is shared with other circuits that can be classically simulated;
the discussion is reported in Ref. [10], where the quantum states that can be efficiently sampled by classical circuits (independently of the number of gates \( m \)) are called "computationally tractable states".

### 3.2 Period-finding circuit in Shor’s algorithm.

It could be argued that quantum circuits that cannot be simulated classically should exploit quantum states with a super-polynomial information content. The factorization of a large number \( N \) is believed to be impossible for a classical algorithm working in polynomial time in \( \log N \). The famous Shor’s factorization algorithm uses a quantum circuit that operates in polynomial number of gates. We thus expect that the quantum states involved in this quantum circuit cannot be sampled by a classical algorithm. I will show that this is actually not the case, i.e., also in the case of Shor’s algorithm, the information content is polynomial in \( n \).

A fundamental element of Shor’s factorization algorithm is the “modulus exponentiation” function \( f \):

\[
f_{a,N}(x) = a^x \pmod{N}
\]  

defined based on \( N \) and an integer \( a \). This function can be efficiently calculated by a classical reversible circuit (thus, also by a quantum one).

The quantum circuit used in Shor’s algorithm has the role of finding the period \( r \) of \( f_{a,N}(x) \), for given \( a \) and \( N \). The function is indeed periodic: there are two numbers \( r \) and \( x_{\text{min}} \) such that \( f_{a,N}(x) = f_{a,N}(x+r) \) for every \( x \geq x_{\text{min}} \). I define “period” as the smallest \( r \).

In Shor’s algorithm, \( a \) and \( N \) are coprime: this condition ensures that \( f_{a,N}(x) \) is always periodic, i.e. \( x_{\text{min}} = 0 \). In the following, I will also focus on this case. In more general cases, \( f_{a,N}(x) \) is still “ultimately periodic”, i.e., there are values of \( a \) and \( N \) such that \( f_{a,N}(x) \) is periodic for \( x \) larger than a \( x_{\text{min}} > 0 \). There are no reasons why such values cannot be used anyway in the period-finding quantum circuit. This case can be addressed by a trivial extension of the methods discussed in this section and is reported in detail in Appendix A.

I describe now the quantum circuit. It uses two registers, of \( n_x \) and \( n_f \) qubits, respectively, initialized to \( |0\rangle |0\rangle \).

**Quantum circuit 1.**

1. Apply Hadamard gates to the second register to get a uniform superposition of \( x \):

\[
|\Psi_1\rangle = 2^{-n_x} \sum_x |x\rangle |0\rangle
\]  

2. Calculate the modulus exponentiation:

\[
|\Psi_2\rangle = 2^{-n_x} \sum_x |x\rangle |f_{a,N}(x)\rangle
\]
3. Apply the quantum Fourier transform to the first register:

$$|\Psi_3\rangle = 2^{-n_x} \sum_x |\tilde{x}\rangle |f_{a,N}(x)\rangle$$  \hspace{1cm} (4)

The circuit is hard-wired to perform the modulus exponentiation $f$ with given $a$ and $N$. It is worth noticing that the quantum Fourier transform used here is the one on $\mathbb{Z}_{2^{n_x}}$, and that it operates on an interval of numbers whose length is $2^{n_x}$.

When the measurement is finally performed, the second register results in a value of $f_0$, equal to $f_{a,N}(\tilde{x})$ for a random $\tilde{x}$. In $|\Psi_2\rangle$, the first register thus contains a superposition of all the $|x\rangle$ such that $f_{a,N}(x) = f_0$ (all the $x$ in the pre-image of $f_0$, including $\tilde{x}$): they are exactly spaced by $r$. Applying the Fourier transform to the first register then transforms this periodicity into peaks, at multiples of $2^{n_x}/r$. The final measurement of the first register will give an $\tilde{x}$ close to one of such peaks: the periodicity $r$ of $f$ is found.

Before discussing the tricky details, it is worth considering the special case when $r$ is a power of 2 (smaller than $2^{n_x}$). In this case, the peaks in the Fourier transform are exactly at integer numbers $2^{n_x}/r$ and are perfectly sharp: the Fourier transform is 0 everywhere, except for values of $\tilde{x}$ multiples of $2^{n_x}/r$, where it takes complex values with same modulus. Measuring the first register thus always gives exactly a $\tilde{x}$ multiple of $r$.

In this special case, it is thus easy to weakly simulate the output of the measurement of the quantum circuit Circ.\[ by the following procedure:

**First register** output a random multiple of $2^{n_x}$ less than $2^{n_x}$;

**Second register** generate a random $\tilde{x}$ and output $f_{a,N}(\tilde{x})$

Unfortunately, the situation is less simple when $r$ is not a power of 2: the position of the peaks in the Fourier transform, $2^{n_x}/r$, do not exactly fall on an integer number and the peaks become broad. The measurement of the first register will give an integer value close to a multiple of $2^{n_x}/r$.

As discussed above, assuming that the second register will be measured, after the calculation of the modulus exponentiation, the first register contains a superposition of states $|rm + x_0\rangle$, where $0 \leq m \leq M$ and $M$ is the maximum $m$ such that $rm + x_0 < 2^{n_x}$; the number of the elements in the superposition is $M + 1$. The Fourier transform is the superposition of all the $|\Psi(\tilde{x})\rangle |\tilde{x}\rangle$ with:

$$\Psi(\tilde{x}) = \frac{1}{\sqrt{2^{n_x}(M+1)}} \sum_{m=0}^{M} \exp\left(2\pi i \frac{\tilde{x} (rm + x_0)}{2^{n_x}}\right)$$ \hspace{1cm} (5)

I remind that the frequency $\tilde{x}$ is in the same range of $x$, i.e. $0 \ldots 2^{n_x} - 1$. The expression, up to an irrelevant phase term depending on $x_0$, is a geometrical series, which can be explicitly calculated:

$$\Psi(\tilde{x}) = \frac{\exp\left(2\pi i \frac{\tilde{x} x_0}{2^{n_x}}\right) \exp\left[2\pi i \frac{\tilde{x} (M+1)}{2^{n_x}}\right] - 1}{\sqrt{2^{n_x}(M+1)}} \frac{\exp\left[2\pi i \frac{\tilde{x}}{2^{n_x}}\right] - 1}{\exp\left[2\pi i \frac{\tilde{x}}{2^{n_x}}\right] - 1}$$ \hspace{1cm} (6)

9
This expression is not defined for $\tilde{x} = 0$; here and in the following, I will implicitly assume that the undefined values are extrapolated by continuity, considering a real $\tilde{x} \to 0$.

The probability $P(\tilde{x}) = |\Psi(\tilde{x})|^2$ is now calculated; it is useful to write it as:

$$P(\tilde{x}) = \frac{q}{2^n_x} \rho(\tilde{x}p) \tag{7}$$

where:

$$\rho(v) = \frac{1}{q(M+1)} \frac{1 - \cos \left[ \frac{2\pi v(M+1)}{q} \right]}{1 - \cos \left[ \frac{2\pi v}{q} \right]} \tag{8}$$

and $p/q = r/2^n_x$; the two natural numbers $p$ and $q$ are chosen so that they are coprime.

Thanks to the periodicity of the trigonometric functions, $\rho$ is periodic with period $q$. This suggests to “wrap” it as:

$$P(\tilde{x}) = \frac{q}{2^n_x} \rho(\tilde{x}p \mod q) \tag{9}$$

It is possible to express $\tilde{x}$, in the range $0 \ldots 2^n_x - 1$, as

$$\tilde{x} = s(\tilde{x}) + z(\tilde{x}) q \tag{10}$$

where:

$$s(\tilde{x}) = \tilde{x} \mod q \tag{11}$$

is in the range $0 \ldots q - 1$ and

$$z(\tilde{x}) = \left\lfloor \frac{\tilde{x}}{q} \right\rfloor \tag{12}$$

is in the range $0 \ldots 2^n_x/q - 1$. The result is:

$$P(\tilde{x}) = \frac{q}{2^n_x} \rho \left[ s(\tilde{x}) p \mod q \right] \tag{13}$$

This expression can be interpreted by identifying the two terms on the right-hand side as two probabilities: and

$$P(\tilde{x}) = \rho_s \left[ z(\tilde{x}) \right] \rho_s \left[ s(\tilde{x}) \right] \tag{14}$$

where the probability distributions of $s$ and $z$ are:

$$\rho_s(z) = \frac{q}{2^n_x} \tag{15}$$

and

$$\rho_s(s) = \rho \left[ sp \mod q \right] \tag{16}$$

Since $p$ and $q$ are coprime, $v = sp \mod q$ is a bijection between $v$ and $s$ on the range $0 \ldots q - 1$; moreover it is possible to efficiently calculate $s$ given $v$. Sampling $s$ from $\rho_s(s)$ can thus be accomplished by first sampling $v$ from $\rho(v)$ and then calculating $s$ using the above-mentioned bijection.

Summarizing, sampling from $P(\tilde{x})$, i.e. simulating Circ. $\boxed{}$, can be accomplished by the following algorithm:
Algorithm 1.

1. Sample a natural number $v$ from the distribution $\rho(v)$, with $v$ in the range $0 \ldots q - 1$;
2. Calculate $s$ such that $v = sp \mod q$;
3. Uniformly sample a natural number $z$ in the range $0 \ldots 2^n s / q - 1$;
4. Return $\tilde{x} = s + zq$.

It remains to show how to sample from $\rho(v)$. Figure 2 shows an example of the graph of $\rho(v)$: intuitively, it represents the probability between two consecutive peaks. It is possible to efficiently sample from $\rho(v)$ by means of Monte Carlo rejection sampling, using the proposal function $\eta(v)$:

$$
\eta(v) = \frac{1}{q(M + 1)} \min \left[ (M + 1)^2, \frac{2}{1 - \cos \left( \frac{2\pi v}{q} \right)} \right]
\tag{17}
$$

The proposal function $\eta(v)$ is approximately the envelope of $\rho(v)$; both functions are shown in Fig. 2. The application of the Monte Carlo procedure by means of the reported proposal function $\eta(v)$ does not present remarkable features and is sketched in Appendix B for completeness sake, together with the (quite long but trivial) proof of its efficiency.
3.3 Circuit used in Grover’s algorithm

The Grover’s algorithm is presented as a quantum algorithm for finding a database entry $x_0$ with a desired property among the $n$-bit strings $[22, 23]$. The requested property is expressed by a function $f(x) : [0, 1]^n \rightarrow [-1, 1]$; it returns -1 if $x$ has the requested property, else 1. The goal of the algorithm is to find $x_0 \in [0, 1]^n$ such that $f(x_0) = -1$. I assume that there is only a single $x_0$ with this property.

The function $f(x)$ is seen as a black box, i.e. nothing is known about it, except that it is implemented as a subroutine that returns $f(x)$ on input $x$. A classical algorithm should make an exhaustive search on $2^n$ elements to find $x_0$: on average, it will take $2^{n-1}$ attempts. Grover’s algorithm shows a polynomial speed-up, namely the number of steps is of the order of $2^{n/2}$.

In this section, I show that the quantum states arising in the quantum circuit used in Grover’s algorithm carry a polynomial classical information amount: actually, I give two different possible sampling algorithms $A_C$ and input states $I$, exploiting different resources. Rigorously, Grover’s algorithm prescribes the use of a quantum circuit with length exponential in the number of bits $n$, which would not fit into Def. 1. Thus I must considered a shortened version of that circuit. A complete description of Grover’s algorithm is outside the scope of this paper; I describe the shortened quantum circuit only to the extent that is needed to discuss the information content according to Def. 2.

The shortened version of the quantum circuit is the following.

Quantum circuit 2 (Shortened version of the circuit used in Grover’s algorithm). The function $f(x)$ is used to define the operator $U_f$:

$$U_f |x⟩ = f(x) |x⟩$$  \hspace{1cm} (18)

A second operator, called “diffusion operator”, $U_g$, is defined analogously, with a function $g(x)$, which takes the value -1 for $x = 0$, else 1.

The initial quantum state is $|0⟩$. Hadamard gates are applied to every qubit. Then the following sets of gates are repeatedly applied $t$ times:

- $U_f$ is applied
- Hadamard gates are applied to all the qubits
- $U_g$ is applied
- Hadamard gates are applied to all the qubits

The number $t$ is a polynomial in $n$, $t(n)$.

The output of the circuit is finally measured in the computational basis.

Definition 1 requires that the number of 1- and 2-qubit gates of the circuit is polynomial in $n$. The diffusion operator $U_g$ can be built using a polynomial (in $n$) number of 1- and 2-qubit gates [23]. Moreover, I assume that $f(x)$ can be calculated by a classical reversible circuit with a polynomial (in $n$) number.
of 1- and 2-qubit gates; also $U_f$ can be realized with a polynomial number of
gates. In Grover’s algorithm, $t$ is exponential in $n$; here I assume instead that it
is polynomial, so that the total number of 1- and 2-qubit gates $m$ is polynomial
in $n$, as requested in Def. [1].

It can be shown that, at any $m$, the quantum state is a superposition of $|0\rangle$
and $|x_0\rangle$ states [22]. The probability $P(x)$ of getting the required result $x_0$ can
be explicitly calculated:

$$P(t) = \sin^2 \left[ (2t + 1) \arcsin \left( \frac{1}{2^{n/2}} \right) \right]$$  \hspace{1cm} (19)

It is worth remarking that $t$ is considered as a fixed polynomial in $n$.

A possible sampling algorithm $A_C$ can be the following.

**Algorithm 2.**

**Input** $x_0$ such that $f(x_0) = -1$ and the number of steps $t$.

**Output** $x_0$ with probability $P(t)$, else 0.

(The algorithm is trivial)

The described procedure requires to first find $x_0$, thus the algorithm that
generates $I$ must perform an exhaustive search on $[0, 1]^n$. A more lazy alternative consists to pass a representation of $f(x)$ (or $U_f$) as input $I$.

**Algorithm 3.**

**Input** A representation of the circuit that calculates $f(x)$ and the number
of steps $t$.

**Output** $x_0$ with probability $P(t)$, else 0.

1. Randomly generate $N$ numbers $x_i$ in the range $0\ldots2^n$, where:

$$N = \left\lceil \frac{\log (1 - P(t))}{\log(1 - \frac{1}{2^n})} \right\rceil \hspace{1cm} (20)$$

2. Calculate $f(x_i)$;

3. If $f(x_i)$ is 1 for every $x_i$, return 0.

4. Call $x_0$ the $x_i$ such that $f(x_i) = -1$.

5. With probability $P'$ return $x_0$, where

$$P' = \frac{P(t)}{1 - (1 - \frac{1}{2^n})^N} \hspace{1cm} (21)$$

6. Else return 0.
This algorithm is correct and efficient. The proof is sketched in Appendix C. The appendix also shows that \( N \) (proportional to the running time) is polynomial in \( t \), with leading order two.

Both algorithms, Alg. 3 and Alg. 2, can be used as sampling algorithm \( A_C \) in Def. 2. The case of Alg. 3 is similar to the case discussed in Sect. 3.1: it shows that Circ. 2 can be efficiently simulated by classical algorithms, although it takes quadratic time in \( n \) and \( m \). The alternative sampling algorithm, Alg. 2, requires that the solution of the database search, \( x_0 \), is given as input \( I \); it reminds the situation of Sect. 3.2, where the solution of the problem (there, the period \( r \)) is used as input to the sampling algorithm \( A_C \).

The two algorithms also differ in the used resources: there is a trade-off between them. Algorithm 3 requires a possibly longer \( I \), representing the circuit of \( U_f \), runs in quadratic time in \( t \), and the preparation of \( I \) takes polynomial time in the size of the circuit of \( U_f \). By contrast, Alg. 2 requires a shorter \( I \), just containing \( x_0 \) and \( t \), runs in fixed time independently on \( t \), but the preparation of \( I \) requires an exhaustive search to find \( x_0 \).

### 3.4 Generalization of Clifford gates

The Clifford gates, mentioned above in Sect. 3.1, can be generalized by a method based on group theory [24]. A complete and detailed discussion of the generalization of Clifford gates is outside the scope of this paper. Here I discuss the topic only to the extent that it is relevant for the information content of Def. 2. The discussion follows the lines of Ref. 24.

A finite Abelian group \( G \) is given. The quantum states are defined on a computational basis formed by states \( |x⟩ \), with \( x \in G \). The “generalized Clifford gates” \( C_i \) are a set of unitary operators, defined based on \( G \). These gates include the Fourier transforms over \( G \). Roughly, they operate on many bits, so they look more like whole circuits. The traditional Clifford gates are generated by the group \( G = \mathbb{Z}_2^n \). The definition of these gates is not reported here and is not necessary for the discussion.

Based on \( G \), the following definition is given.

**Definition 3** (Coset state of \( G \)). An Abelian finite group \( G \) is given. \( K \) is a subgroup of \( G \) and \( x \) is an element of \( G \). The coset state \( |ψ(K,x)⟩ \) is:

\[
|ψ⟩ = \frac{1}{\sqrt{|K|}} \sum_{k \in K} |k + x⟩
\]

where \(|·|\) is the cardinality of a set.

The interest in the generalized Clifford gates and in the coset states stems from the fact that it is possible to efficiently classically sample from the output of any sequence of generalized Clifford gates \( C_i \) applied to a coset state \( |ψ(K,x)⟩ \). However this operation (the efficient classical sampling) requires to know a suitable representation of \( |ψ(K,x)⟩ \): in particular, a generating set of \( K \) must be known, with polynomial size in \( \log |G| \) (roughly analogous to the
number of qubits, although here we do not have qubits at all). The proof is
given in Ref. [24].

In order to apply this result to the discussion performed in this paper, I
introduce a class of circuits.

**Quantum circuit 3** (Circuits ending with generalized Clifford gates). An
Abelian finite group $G$ is given. The circuit is composed by two stages. The
first, $U$, is applied to $|0\rangle$ and generates a coset state $|\psi(K, x)\rangle$. The second is
composed by a sequence of generalized Clifford gates over $G$.

According to the above-mentioned results of Ref. [24], the following efficient
sampling algorithm can be used.

**Algorithm 4.**

- **Input** The generating set of $K$, with polynomial size in $|G|$. 
- **Output** The requested samples 
  The algorithm is described in Ref. [24].

We see that the quantum circuits of the family Circ. 3 can be efficiently
sampled by Alg. 4, thus also in this case the amount of information is polynomial
in $n$ according to Def. 2.

Quantum circuits of the family Circ. 3 are used in quantum algorithms for
solving the “Abelian hidden subgroup problems” [25]. A problem in this class
(rigorously, with some limitations) can be expressed in terms of a decision prob-
lem in $\mathbf{NP}$ that is in $\mathbf{BQP}$ but is believed to lie outside $\mathbf{P}$. The certificate is the
generating set of the hidden subgroup. Thus we see that the certificate itself,
which is polynomially large in $\log |G|$, is the required input $I$. This situation
reminds what happens with the factorization discussed in Sect. 3.2, where the
period $r$ is used to calculate the factors, constituting the certificate of the $\mathbf{NP}$
problem, and is given as input $I$ to the sampling algorithm $A_C$. Actually, the
factorization can be seen as an Abelian hidden subgroup problems by means of
a quantum circuit of the family Circ. 3. Shor’s algorithm is indeed an approxi-
mated version of such a quantum algorithm.

### 3.5 Period-finding circuit with uniform superposition of $N$

In the previous section, the value of $N$ is hard-wired inside the quantum circuit
and the information needed by the sampling algorithm, in order to describe the
quantum circuit, is the period $r$. It is however possible to make an equivalent
quantum circuits that takes $N$ as an input, in a separate register. This single
change would not significantly modify the situation: depending on the input $N$,
the sampling algorithm is provided by a different $r$. Apparently, the situation
would dramatically change if the input $N$ is constructed as a superposition of all
the possible values of qubits, by means of Hadamard gates: providing the value
of $r$ for each $N$ would require an amount of bits exponentially large in $\log N$.
This would be a case of polynomial size of information according to Def. 2. I will
show that this is not the case and the difficulty can be circumvented, leading to a polynomial amount of information also in this case.

We use three registers, initially containing 0, representing \( N \), \( x \), and the result of the modulus exponentiation \( f_{a,N}(x) \); the number of qubits are \( n_N \), \( n_x \) and \( n_f \), respectively. The stages of the circuit are:

**Quantum circuit 4.**

1. Apply Hadamard gates to the first register to get a uniform superposition of \( N \):

   \[
   |\Psi_1\rangle = 2^{-n_N} \sum_N |N\rangle |0\rangle
   \]

   (23)

2. Apply Hadamard gates to the second register to get a uniform superposition of \( x \):

   \[
   |\Psi_2\rangle = 2^{-n_N n_x} \sum_{N,x} |N\rangle |x\rangle |0\rangle
   \]

   (24)

3. Calculate the modulus exponentiation:

   \[
   |\Psi_3\rangle = 2^{-n_N n_x} \sum_{N,x} |N\rangle |x\rangle |f_{a,N}(x)\rangle
   \]

   (25)

4. Apply the quantum Fourier transform to the second register:

   \[
   |\Psi_4\rangle = 2^{-n_N n_x} \sum_{N,x} |N\rangle |\tilde{x}\rangle |f_{a,N}(x)\rangle
   \]

   (26)

Sampling the measurement of \( |\Psi_4\rangle \) would give a random \( N \) in the first register, a value \( f_{a,N}(x) \) for a random \( x \) in the third, and an \( \tilde{x} \) close to a multiple of \( r \). Sampling first a random \( N \) and then trying to sample \( \tilde{x} \) would require to calculate \( r \) from \( N \), which is likely not feasible in polynomial time, or to provide the sampling algorithm with the value of \( r \) for every \( N \), which is not feasible in exponential space in \( \log N \).

However, it turns out that it is possible to sample the couple \( (N,r) \), with a uniform distribution of \( N \), such that \( r \) is the period of \( f_{a,N}(x) \), in polynomial time in \( \log N \). Although surprising, it is actually easier to sample the couple \( (N,r) \) rather than finding the period \( r \) corresponding a number \( N \). I give a proof of this curious fact in the following of the section. This fact allows the sampling algorithm of Sect. 3.2 to be adapted to this new case, thus the quantum states involved in the above described quantum circuit, Circ. 4 have a polynomial amount of information according to Def. 2.

Curiously, with the described procedure, no information must be passed to the sampling algorithm, except the number of qubits \( n_N \), \( n_x \) and \( n_f \): the outcome of the measurement can be sampled by a fixed algorithm, with a fixed length, whatever is the number of involved qubits. By seeking for a more complex quantum state, we actually devised a much simpler one.

In the rest of this section, I sketch the algorithm to sample the couple \( (N,r) \) in polynomial time. This algorithm profits on a method for uniformly sampling
numbers in factorized form \([26, 27]\), operating in polynomial time: although uniformly sampling a number \(N\) and then factorizing it is believed to be a difficult task, directly generating the factors \(p_i\), such that their product \(N\) is uniformly distributed, can be accomplished in polynomial time in the number of desired bits.

The algorithm given by Kalai \([27]\) is extremely simple.

**Algorithm 5 (Kalai’s algorithm).**

**Input:** maximum number \(N_{\text{max}}\)

**Output:** a random \(N\), uniformly distributed between 1 and \(N_{\text{max}}\), along with its prime factorization.

1. Generate a sequence \(N_{\text{max}} \geq s_1 \geq s_2 \geq \ldots s_l = 1\) by choosing \(1 \leq s_1 \leq N_{\text{max}}\) and \(1 \leq s_{l+1} \leq s_l\), until reaching 1.

2. Select the \(s_i\) that are prime.

3. Let \(N\) be the product of the prime \(s_i\)’s.

4. If \(N \leq N_{\text{max}}\), output \(N\) (along with its factors) with probability \(N/N_{\text{max}}\).

5. Otherwise, repeat from 1.

First, we notice that the Kalai’s algorithm can be adapted to generate an uniformly distributed \(N\) along with the factorization of \(N - 1\), with a slight modification. Moreover, we notice that the prime numbers generated by the algorithm are the results of a uniform sampling of numbers \(s_i\).

The algorithm for sampling the couples \((N, r)\) first makes use of the Kalai’s algorithm (Alg. 5) for uniformly sampling \(N\) (along with its prime factors) in the range \(1 \ldots N_{\text{max}} = 2^n N\). In turn, the uniform sampling of the \(s_i\)’s, required in step 1 of Kalai’s algorithm, is also performed by using the Kalai’s algorithm, so that the obtained prime factors \(p_i\) of \(N\) are supplied with the factorization of \(p_i - 1\). So, summarizing, we uniformly sample \(N\), in the form:

\[
N = \prod_i p_i^{\nu_i} \tag{27}
\]

where the \(p_i\)’s are prime; moreover, we know the prime factorization of \(p_i - 1\):

\[
p_i = \prod_j q_i^{d_{i,j}} + 1 \tag{28}
\]

where the \(q_i\)’s are prime. It is worth highlighting that Kalai’s algorithm (Alg. 5) is used at two levels, not recursively, i.e. we need the prime factorization of \(N\) and of \(p_i - 1\), but we do not need the prime factorization of \(q_i - 1\). The information obtained by the procedure, the \(p_i\)’s and the \(q_i\)’s, is sufficient to calculate \(r\) as follows.
In group theory, the quantity $r$ is known as the *order of $a$* in the multiplicative group $\mathbb{Z}_N^*$. The *order* of the group, $\varphi(N)$, is defined as the number of invertible elements of $\mathbb{Z}_N^*$. It can be computed from the prime factorization of $N$:

$$\varphi(N) = \prod_{i} p_i^{\nu_i-1} (p_i - 1)$$  \hspace{1cm} (29)

By using Eq. 28:

$$\varphi(N) = \prod_{i} p_i^{\nu_i-1} \prod_{j} q_{i,j}^{\mu_{i,j}}$$  \hspace{1cm} (30)

Some of the $p_i$’s and $q_i$’s can be equal. In polynomial time, it is possible to rearrange the two products as:

$$\varphi(N) = \prod_{i} q_i^{\lambda_i}$$  \hspace{1cm} (31)

where the $q_i$’s are distinct primes and $Q'$ is the number of distinct $q_i$’s.

We then proceed by finding the largest $\tau_i \leq \lambda_i$ such that:

$$\frac{\varphi(N)}{a^{\tau_i}} \equiv 1 \pmod{N}$$  \hspace{1cm} (32)

This operation can be also done in polynomial time in $n_N$. The order of $a$ is then:

$$r = \prod_{i} q_i^{\lambda_i - \tau_i}$$  \hspace{1cm} (33)

This works since $\mathbb{Z}_N^*$ is an Abelian group which decomposes as a direct group of Abelian groups of orders $q_1^{\lambda_1}, \ldots, q_{Q'}^{\lambda_{Q'}}$.

Summarizing, the reported procedure first generates uniformly distributed random numbers $N$ along with the prime factors $p_i$ and $q_{i,j}$; then it calculates $r$ from $p_i$ and $q_{i,j}$. We thus conclude that the quantum states involved in the above-described circuit, Circ. 4, also contain a polynomial amount of information according to Def. 2.

## 4 Conclusion

I proposed how to define the information content of a quantum state, inspired to the Kolmogorov complexity. The formal definition is given in Def. 2. It is trivial to see that the information content of any quantum state is at most exponential in the number of qubits $n$. I discuss the information content of quantum states generated by applying $m$ quantum gates to the initial state $|0\rangle$, with $m$ polynomial in $n$, as often done in quantum computation.

Trivially, the information content is polynomial in the case of quantum circuits that can be simulated classically. More interesting is the case of quantum
circuits that are believed to be difficult to classically simulate, in particular the ones used in algorithms that are believed to show superior performances in quantum computers than in classical ones. In particular, I analyzed quantum circuits that are used to solve **NP** problems and thus cannot be simulated classically. They include the algorithms for solving the Abelian hidden subgroup problem and the factorization (Shor’s algorithm). Grover’s algorithm is also discussed. It turns out that the information content roughly corresponds to the certificate of the corresponding **NP** problem, which has polynomial size in \( n \).

Looking for a more complex situation, it can be suggested to try with problems in which such a certificate does not exist. I created such a case by using the quantum circuit used to factorize \( N \) and feeding it with a superposition of every \( N \) with \( n \) bits. In this case, the information content should contain the factorizations of every \( N \), thus an exponential amount of information. Instead, even in this case the amount of information is polynomial in \( n \): actually, it turns out that the sampling algorithm requires even less information than for the case of the factorization with a fixed \( N \).

We know that there are specially designed quantum circuits giving rise to quantum states with a super-polynomial information amount, according to Def. 2. However, from the examples discussed above, we see that the use of quantum states with this huge amount of information is not necessarily needed by quantum algorithms showing an exponential speed-up with respect to classical ones. It seems that the huge Hilbert space is really there, but the speed-up of quantum computation is not necessarily require to store a correspondingly huge amount of information in the quantum state.
A Modulus exponentiation with generic $a$ and $N$

In this section, I discuss the period-finding quantum circuit used in Shor’s algorithm, for the case in which the state is prepared using the modulus exponentiation $f_{a,N}(x)$ with generic $a$ and $N$. In other words, I relax the condition (present in Shor’s algorithm) that $a$ and $N$ are coprime.

I remind that, in general, the modulus exponentiation is “ultimately” periodic: there are two numbers $r$ and $x_{min}$ such that $f_{a,N}(x) = f_{a,N}(x + r)$ for every $x \geq x_{min}$. If $a$ and $N$ are coprime (as in Shor’s algorithm), then $x_{min} = 0$. In this section I consider the more general case, in which $x_{min}$ can be larger than 0, i.e. the function is “ultimately periodic” but not necessarily “periodic”.

In order to sample the result of the measurement, it is first necessary to calculate $x_{min}$; following the discussion in Sect. 3.2, we assume that $r$ is known. It is of course possible to find $x_{min}$ by simply trying all the values starting from 0 and verifying if $f_{a,N}(x_{min}) = f_{a,N}(x_{min} + r)$. This procedure is actually efficient, thanks to the following proposition:

**Proposition 1.** For every $N$ and $a < N$, $x_{min}$ is smaller than the number of binary digits of $N$, $x_{min} \leq \log_2(N)$.

**Proof.** Let us define:

$$d_k = \gcd (a^k, N) \tag{34}$$

With increasing $k$, $d_k$ increases; however, it cannot exceed $N$, thus it will stabilize. This means that there is a $\bar{k}$ such that $d_k = d_{\bar{k}}$ for every $k \geq \bar{k}$. I call $d = d_{\bar{k}}$ this gcd.

In order to proceed with the discussion, it is worth expressing $a$ and $N$ in terms of their prime factorization:

$$a = \prod_i p_i^{\mu_i} q_i^{\nu_i} \tag{35}$$

and

$$N = \prod_i p_i^{\lambda_i} s_i^{\tau_i} \tag{36}$$

where $p_i$, $q_i$, and $s_i$ are distinct primes and $\mu_i$, $\nu_i$, $\lambda_i$, and $\tau_i$ are the multiplicities. The primes $p_i$ are the only common prime factors between $a$ and $N$.

I highlight that the prime factorization of $a$ and $N$ is only used in this proof, in order to prove the thesis, but it is not proposed as a part of an algorithm. Indeed, such an algorithm would not be efficient.

Using the prime factorization, $d_k$ can be expressed as:

$$d_k = \prod_i p_i^{\min(k\mu_i, \lambda_i)} \tag{37}$$

For large enough $k$ we get $d$:

$$d = \prod_i p_i^{\lambda_i} \tag{38}$$
The ratio between $N$ and $d$ is thus:

$$\frac{N}{d} = \prod_i s_i^\tau_i$$

(39)

It can be noticed that $N/d$ does not have any prime factor in common with $a$, thus they are coprime. According to Euler’s theorem, the function $a^\tau \mod N/d$ is periodic (not just ultimately periodic), thus there is an integer $r' > 0$ such that:

$$a^{r'} \mod \frac{N}{d} = a^0 \mod \frac{N}{d} = 1$$

(40)

In other words, there is a natural number $p > 0$ such that:

$$a^{r'} = \frac{Np}{d} + 1$$

(41)

Moreover, according to the definition of $d$, there is a natural number $q > 0$ such that:

$$a^k = dq$$

(42)

Combining the last two equations:

$$a^{k+r'} = Np + a^k$$

(43)

Applying the modulus:

$$a^{k+r'} \mod N = a^k \mod N$$

(44)

By comparing this expression with the definitions of $r$ and $x_{min}$, we conclude that $r'$ is a multiple of $r$ and

$$x_{min} \leq \bar{k}.$$  

(45)

From Eq. (37), if $k$ is equal or larger than all the $\lambda_i$, then $d_k = d$. As a consequence, $\bar{k}$ is at most the multiplicity of any prime in the factorization of $N$. In turn, this multiplicity is at most $\log_2(N)$. This can be easily seen by noticing that the multiplicity of 2 in any number up to $N$ is up to $\lfloor \log_2(N) \rfloor$; it is not possible to have a larger multiplicity with prime factors larger than 2. The conclusion is thus that:

$$\bar{k} \leq \log_2(N)$$

(46)

The combination of this last equation and Eq. (45) leads to the thesis.

With the knowledge of $x_{min}$ and $r$, it is now possible to describe the sampling procedure. Following the procedure described in Sect. 3.2 we can still generate a random $\bar{x}$ and output $f_{a,N}(\bar{x})$ as the measurement of the second register, even if $x_{min}$ does not vanish. However, two different cases must be taken into consideration to properly sample the measurement of the first register.

$\bar{x} < x_{min}$ The pre-image of $f_{a,N}(\bar{x})$ under $f_{a,N}(x)$ is constituted by the single point $\bar{x}$ ($\bar{x}$ belongs to the non-periodic part of $f_{a,N}(x)$). The Fourier transform of the state $|\bar{x}\rangle$ is the uniform superposition of all the $\bar{x}$, thus the requested sample is a uniformly random $\bar{x}$.

Appendix – 21
The pre-image of $f_{a,N}(x)$ under $f_{a,N}(x)$ is composed by a sequence of $M + 1$ points, spaced by $r$, represented as $mr + x_0$ for $m$ in the range $0 \ldots M$. The values of $M$ and $x_0$ can be easily calculated based on $x_{\min}$ and $r$. The Fourier transform corresponds to Eq. 6; the procedure reported in Sect. 3.2 can thus be applied, with the suitable values of $M$.

B Sampling from the probability $\rho(v)$

In this section I show how to sample from $\rho(v)$. This operation can be efficiently performed by means of Monte Carlo rejection sampling. For clarity sake and for introducing the notation, I shortly summarize the Monte Carlo rejection sampling procedure. A “proposal function” $\eta(v)$, such that $\eta(v) \geq \rho(v)$, is suitably chosen; then the following steps are performed:

1. A natural number $v$ (the proposed sample) is sampled from the proposal probability $P_p(v) = \eta(v) / \sum v \eta(v)$; 
2. The number $v$ is accepted (i.e., returned as the valid sample) with the acceptance probability $P_a(v) = \rho(v) \eta(v)$; 
3. In case of rejection, the procedure is repeated from point 1.

The chosen $\eta(v)$ is reported in Eq. 17; it is approximately the envelope of $\rho(v)$. The two functions are shown in Fig. 2. It is easy to prove that $\eta(v) \geq \rho(v)$; this inequality can be also seen in the graph. Under this condition, the Monte Carlo rejection sampling gives the correct results. However, it is still necessary to prove that the procedure is efficient, i.e. if it operates in polynomial time in $\log(q)$ on average. This is ensured if [28]:

- It is possible to efficiently sample from the proposal probability $P_p(v)$.
- The number

$$v = \sum v \eta(v), \quad (47)$$

representing the average number of proposed samples needed to obtain one accepted sample, is at most polynomial in $\log(q)$;

The efficient method for sampling the proposal probability $P_p(v)$ that I propose is based on the algorithm for sorted (i.e. monotonically decreasing) probability distributions [29]. Our proposal probability $P_p(v)$ is not sorted, but it is symmetrical around the central point $v = q/2$, i.e. $P_p(v) = P_p(q - v)$, and it is monotonically decreasing on the left side and increasing on the right side (see Fig. 2). It is thus possible to sort the probabilities in decreasing order, by sequentially taking one point from the left and one point from the right. This is done by defining:

$$v(\bar{v}) = \begin{cases} \left\lfloor \frac{\bar{v}}{2} \right\rfloor & \bar{v} \text{ even} \\ q - 1 - \left\lfloor \frac{\bar{v}}{2} \right\rfloor & \bar{v} \text{ odd} \end{cases} \quad (48)$$

Appendix – 22
The resulting $P_p[v(\bar{v})]$ monotonically decreases with increasing $\bar{v}$; moreover, the function $v(\bar{v})$ is a bijection on the domain $0 \ldots q - 1$. It is thus possible to apply the efficient algorithm of Ref. [29] to the sorted probability distribution $P_p[v(\bar{v})]$; the algorithm returns $\bar{v}$, which is then used to calculate $v(\bar{v})$ and return it as the desired sample.

The proof of the second point is now sketched. We start by identifying the point $\delta$ in which the transition between the two arguments of the $\min(\cdot, \cdot)$ operator in Eq. 17 takes place:

$$\delta = \frac{q}{2\pi} \arccos \left[ 1 - \frac{2}{(M + 1)^2} \right]$$

(49)

This allows us to rewrite Eq. 17 as:

$$\eta(v) = \begin{cases} 
\frac{M+1}{q(M+1)} & v \in R_A \cup R_E \\
\frac{2}{1 - \cos(\frac{2\pi v}{q})} & v \in R_B \cup R_C \cup R_D 
\end{cases}$$

(50)

where the five ranges are $R_A = 0 \ldots \lfloor \delta \rfloor$, $R_B = \lfloor \delta \rfloor + 1 \ldots q/2 - 1$, $R_C = \{q/2\}$ (a single number), $R_D = q/2 + 1 \ldots q - \lfloor \delta \rfloor - 1$, and $R_E = q - \lfloor \delta \rfloor \ldots q - 1$. The reason for splitting the central part of the domain into the three ranges $R_B$, $R_C$, and $R_D$ is related to the symmetry of $\eta(v)$ and will be discussed below.

According to the definition, $M$ is in the range $1 \ldots q - 1$; using Eq. 49 we get $\delta \in (0, q/6]$. These bounds will be used in the following calculations.

The calculation of $\nu$ by Eq. 51 is now split into the ranges:

$$\nu = S_1 + S_2 + S_3$$

(51)

where $S_1$, $S_2$, and $S_3$ are the summations of $\eta(v)$ over $R_A \cup R_E$, $R_B \cup R_D$, and $R_C$, respectively.

The addends $\eta(v)$ are constant over $R_A$ and $R_E$, and equal to $(M + 1)/q$:

$$S_1 = \sum_{v \in R_A \cup R_E} \eta(v) = \frac{M + 1}{q} \left(2 \lfloor \delta \rfloor + 1 \right)$$

(52)

The range $R_C$ is composed by a single number:

$$S_3 = \sum_{v \in R_C} \eta(v) = \frac{1}{q(M + 1)}$$

(53)

Thanks to the symmetry $\eta(v) = \eta(q - v)$, the summations of $\eta(v)$ over $R_B$ and $R_D$ are equal. The result is:

$$S_2 = \sum_{v \in R_B \cup R_D} \eta(v) = \frac{4}{q(M + 1)} \sum_{\nu = \lfloor \delta \rfloor + 1}^{q/2 - 1} \frac{1}{1 - \cos \left(\frac{2\pi \nu}{q} \right)}$$

(54)

The sum is further split into two parts, separating the first term, which gives the largest contribution:

$$S_2 = S_{2a} + S_{2b}$$

(55)
where:

\[ S_{2a} = 4 \frac{q}{q(M+1)} \frac{1}{1 - \cos \left(2\pi \frac{\delta}{q} + 1\right)} \]  

(56)

and

\[ S_{2b} = 4 \frac{q}{q(M+1)} \sum_{v=\lfloor \delta \rfloor + 2}^{q/2-1} \frac{1}{1 - \cos \left(2\pi \frac{v}{q}\right)} \]  

(57)

It is now necessary to find quantities, larger than \( S_1, S_{2a}, S_{2b}, \) and \( S_3, \) that are independent of \( M. \)

Since \( M \) and \( q \) are positive and integer, from Eq. 53

\[ S_3 \leq 1 \]  

(58)

\( S_1 \) is defined in Eq. 52. Since \( M \leq q - 1: \)

\[ S_1 \leq 2 \frac{M+1}{q} \lfloor \delta \rfloor + 1 \]  

(59)

Applying the inequality:

\[ \arccos (1-x) \leq \sqrt{2x + x} \]  

(60)

to Eq. 49 we get:

\[ \delta \leq \frac{q}{\pi} \left( \frac{1}{M+1} + \frac{1}{(M+1)^2} \right) \]  

(61)

Using this inequality to further elaborate Eq. 59 gives:

\[ S_1 \leq 1 + \frac{3}{\pi} \]  

(62)

Now I discuss the term \( S_{2a}, \) defined in Eq. 56. The inequality

\[ 1 - \cos (x) \geq \frac{x^2}{4} \]  

(63)

holds for \( 0 \leq x \leq 3\pi/4. \) From Eq. 61, \( \delta \leq 3q/(4\pi); \) the argument of \( \cos(\cdot) \) in Eq. 56, i.e. \( 2\pi \lfloor \delta \rfloor + 1 \)/q, is within the range for large enough \( q, \) thus the inequality can be used:

\[ S_{2a} \leq \frac{4}{\pi^2} \frac{q}{M+1} \frac{1}{(\lfloor \delta \rfloor + 1)^2} \]  

(64)

Applying the inequality:

\[ \arccos (1-x) \geq \sqrt{x} \]  

(65)

to Eq. 49 we get:

\[ \delta \geq \frac{q}{2\pi} \frac{\sqrt{x}}{M+1} \]  

(66)

Appendix – 24
Moreover, $\lfloor \delta \rfloor + 1 \geq \delta$. These inequalities are used to elaborate Eq. 64:

$$S_{2a} \leq \frac{8M + 1}{q}$$  \hspace{1cm} (67)

Remembering that $M \leq q - 1$:

$$S_{2a} \leq 8.$$  \hspace{1cm} (68)

The last term to be discussed is $S_{2b}$, defined in Eq. 57. The addends of the summation decrease with increasing $v$; this allows us to calculate a quantity greater than $S_{2b}$ in terms of an integral:

$$S_{2b} \leq \frac{4}{q(M + 1)} \int_{\lfloor \delta \rfloor + 1}^{q/2 - 2} \frac{d\xi}{1 - \cos \left(\frac{2\pi \xi}{q}\right)}$$  \hspace{1cm} (69)

The calculation of the integral is straightforward:

$$S_{2b} \leq \frac{2}{\pi(M + 1)} \left[ \cot \left(\frac{\pi \lfloor \delta \rfloor + 1}{q}\right) - \tan \left(\frac{2\pi}{q}\right) \right]$$  \hspace{1cm} (70)

For $q > 2$ the second term in parenthesis is positive:

$$S_{2b} \leq \frac{2}{\pi(M + 1)} \cot \left(\frac{\pi \lfloor \delta \rfloor + 1}{q}\right)$$  \hspace{1cm} (71)

The argument of the cot(\cdot) is between 0 and $\pi$ (actually, less than $\pi/2$); in this range:

$$\cot (x) \leq \frac{1}{x}$$  \hspace{1cm} (72)

Using this inequality:

$$S_{2b} \leq \frac{2}{\pi^2(M + 1) \lfloor \delta \rfloor + 1}$$  \hspace{1cm} (73)

Using Eq. 66:

$$S_{2b} \leq \frac{2\sqrt{2}}{\pi}$$  \hspace{1cm} (74)

Summarizing, the terms in which $\nu$ is decomposed, i.e. $S_1$, $S_{2a}$, $S_{2b}$, and $S_3$ (see Eqs. 51 and 55) are smaller than constant quantities, as shown in Eqs. 58, 62, 68, and 74. The sampling of $\rho(\nu)$ is thus efficient.

C Proof of the validity and efficiency of Algorithm

In this section I prove that Alg. 3 is valid and efficient.

The probability of not finding $x_0$ among the $N$ generated samples $x_i$ is $(1 - 2^n)^N$, corresponding to the probability of returning 0 in step 3. The probability of reaching step 4 is

$$P_4 = 1 - (1 - 2^n)^N$$  \hspace{1cm} (75)

Appendix – 25
It represents the probability of finding \( x_0 \). According to Eq. 20, \( N \) is the smallest integer such that \( P_4 \geq P(t) \): the algorithm finds \( x_0 \) with a slightly larger probability than the quantum circuit (but with a polynomially longer time, see below). This reasoning also ensures that \( P' \) in Eq. 21 is smaller than 1.

The total probability of returning \( x_0 \) in step 5 is \( P_4 P' \), i.e. the composed probability of reaching step 5 and outputting \( x_0 \). Using \( P' \) defined in Eq. 21:

\[
P_4 P' = P(t)
\]

Thus I have shown that the algorithm outputs \( x_0 \) with the desired probability. Necessarily 0 is also output with the correct probability \( 1 - P(t) \).

It remains to show that the sampling is efficient. This depends on how many attempts must be done for finding \( x_0 \), i.e. on \( N \). By using the definition of \( N \), Eq. 20 and the definition of \( P(t) \), Eq. 19:

\[
N \leq \frac{\log \left\{1 - \sin^2 \left[ (2t + 1) \arcsin \left( \frac{1}{2^{n/2}} \right) \right] \right\}}{\log \left(1 - \frac{1}{2^n}\right)} + 1
\]

(77)

For large enough \( n \), \( \arcsin \left(1/2^{n/2}\right) < 1/2^{n/2} - 1 \):

\[
N \leq \frac{\log \left\{1 - \sin^2 \left( \frac{2t+1}{2^{n/2}-1} \right) \right\}}{\log \left(1 - \frac{1}{2^n}\right)} + 1
\]

(78)

For positive \( x \), \( \sin(x) < x \):

\[
N \leq \frac{\log \left[1 - \left( \frac{2t+1}{2^{n/2}-1} \right)^2 \right]}{\log \left(1 - \frac{1}{2^n}\right)} + 1
\]

(79)

Since \( -\log(1 - x) > x \):

\[
N \leq -2^n \log \left[1 - \left( \frac{2t+1}{2^{n/2}-1} \right)^2 \right] + 1
\]

(80)

For any polynomial \( t(n) \) and for large enough \( n \):

\[
N \leq 8 \left(2t + 1\right)^2 + 1
\]

(81)

Since \( t(n) \), the number of iterations, is a polynomial in \( n \), and each iteration is polynomial in \( n \), the running time is polynomial in \( n \), i.e. the algorithm is efficient.

**Contents**

1 Introduction

2 Definition of classical information contained in the quantum state generated by a quantum circuit

Appendix – 26
3 Examples

3.1 Quantum circuits that can be efficiently simulated by classical algorithms 7
3.2 Period-finding circuit in Shor’s algorithm 8
3.3 Circuit used in Grover’s algorithm 12
3.4 Generalization of Clifford gates 14
3.5 Period-finding circuit with uniform superposition of $N$ 15

4 Conclusion 18

A Modulus exponentiation with generic $a$ and $N$ Appendix – 20

B Sampling from the probability $\rho(v)$ Appendix – 22

C Proof of the validity and efficiency of Algorithm 3 Appendix – 25

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Appendix – 28
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