Abstract

Determinant representations of form factors are used to represent the spontaneous magnetization of the Heisenberg XXZ chain (\( \Delta > 1 \)) on the finite lattice as the ratio of two determinants. In the thermodynamic limit (the lattice of infinite length), the Baxter formula is reproduced in the framework of Algebraic Bethe Ansatz. It is shown that the finite size corrections to the Baxter formula are exponentially small.
1 Introduction

The aim of this paper is the computation, within algebraic Bethe ansatz method, of the spontaneous magnetization of the XXZ spin-$\frac{1}{2}$ chain (see [1, 2]), by taking the thermodynamic limit of the form factor formulas of local spin operators for the finite chain obtained in [3].

We consider the XXZ Heisenberg chain of finite length $M$ [4, 5], the Hamiltonian of which is given by

$$H_\Delta = J \sum_{m=1}^{M} \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta (\sigma_m^z \sigma_{m+1}^z - 1) \right\},$$

(1.1)

with periodic boundary conditions

$$\sigma_1^a = \sigma_{M+1}^a, \quad a = x, y, z.$$

This Hamiltonian acts in the quantum space $\mathcal{H}$ of the chain, which is the tensor product of $M$ local quantum spin-$\frac{1}{2}$ spaces $\mathcal{H}_m$ isomorphic to $\mathbb{C}^2$ and $\sigma_m^a$, $a = x, y, z$, are the standard Pauli matrices acting in $\mathcal{H}_m$. In the anisotropic case ($\Delta \neq \pm 1$) this model was solved by means of the Bethe ansatz in [6, 7] (see also [8]).

We use the algebraic Bethe ansatz solution for this model [9], taking the XXZ $R$-matrix in the following normalization (as a matrix acting in $\mathbb{C}^2 \otimes \mathbb{C}^2$):

$$R(\lambda, \mu) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & b(\lambda, \mu) & c(\lambda, \mu) & 0 \\
0 & c(\lambda, \mu) & b(\lambda, \mu) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$ 

(1.2)

The functions $b(\lambda, \mu)$ and $c(\lambda, \mu)$ are defined as

$$b(\lambda, \mu) = \frac{\sin(\lambda - \mu)}{\sin(\lambda - \mu - i\zeta)}, \quad c(\lambda, \mu) = -\frac{i\sinh \zeta}{\sin(\lambda - \mu - i\zeta)},$$

where the parameter $\zeta$ is related to the anisotropy parameter $\Delta$ of the Hamiltonian as

$$\Delta = \frac{1}{2}(q + q^{-1}), \quad \text{with} \quad q = e^{\zeta}.$$

The $R$-matrix is a linear operator in the tensor product of two two-dimensional linear spaces $V_1 \otimes V_2$, where each $V_i$ is isomorphic to $\mathbb{C}^2$, and depends generically on two spectral parameters $\lambda_1$ and $\lambda_2$ associated to these two vector spaces. It is denoted by $R_{12}(\lambda_1, \lambda_2).$
The monodromy matrix is constructed as an ordered product, in an auxiliary space \( V_0 \) isomorphic to \( \mathbb{C}^2 \), of such \( R \)-matrices \( R_{0m} \) acting in \( V_0 \otimes \mathcal{H}_m \):

\[
T(\lambda) = R_{0M}(\lambda + \frac{i\zeta}{2}) \ldots R_{02}(\lambda + \frac{i\zeta}{2}) R_{01}(\lambda + \frac{i\zeta}{2}) = \begin{pmatrix}
A(\lambda) & B(\lambda) \\
C(\lambda) & D(\lambda)
\end{pmatrix}^{[0]}.
\]

In the last formula, the monodromy matrix is represented as a \( 2 \times 2 \) matrix in the auxiliary space \( V_0 \), whose entries \( A(\lambda), B(\lambda), C(\lambda), \) and \( D(\lambda) \) are operators in the quantum space \( \mathcal{H} \) of the chain.

Since we will further use some results of [3], let us note, to compare notations, that the spectral parameters are equal here to the corresponding spectral parameters of [3] multiplied by \( i \), and that \( \zeta \) corresponds to \( -\eta \) of [3]. Moreover, we consider here the homogeneous case where all the inhomogeneity parameters \( \xi_m \) of [3] are equal to \( -\zeta/2 \).

The eigenstates of the Hamiltonian (1.1) can be constructed by the action of the operators \( B(\lambda) \) on the ferromagnetic state \( |0\rangle \) (which is the state with all the spins up).

More precisely, the state

\[
|0\rangle B(\lambda_1) \ldots B(\lambda_N) |0\rangle
\]

is an eigenstate of the Hamiltonian (1.1) if the set of spectral parameters \( \{\lambda_j\}_{1 \leq j \leq N} \) is a solution of the Bethe equations

\[
\frac{\sin(\lambda_j + \frac{i\zeta}{2})}{\sin(\lambda_j - \frac{i\zeta}{2})} \prod_{k=1}^{N} \frac{\sin(\lambda_j - \lambda_k + i\zeta)}{\sin(\lambda_j - \lambda_k - i\zeta)} = -1, \quad 1 \leq j \leq N.
\]

The norm of Bethe eigenstates is given by the Gaudin formula [8, 10] (see also [3]):

\[
\langle 0 | \prod_{j=1}^{N} C(\lambda_j) \prod_{k=1}^{N} B(\lambda_k) | 0 \rangle = (-\sinh \zeta)^N \prod_{\alpha \neq \beta} \frac{\sin(\lambda_\alpha - \lambda_\beta + i\zeta)}{\sin(\lambda_\alpha - \lambda_\beta)} \det \mathcal{N}(\{\lambda_\alpha\}),
\]

where \( \mathcal{N} \) is an \( N \times N \) matrix the elements of which are given by

\[
\mathcal{N}_{ab} = -i \frac{\partial}{\partial \lambda_b} \ln \left\{ \frac{\sin(\lambda_a - \frac{i\zeta}{2})}{\sin(\lambda_a + \frac{i\zeta}{2})} \prod_{k=1}^{N} \frac{\sin(\lambda_a - \lambda_k + i\zeta)}{\sin(\lambda_a - \lambda_k - i\zeta)} \right\}.
\]

The ground state of the XXZ model in the region \( \Delta = \frac{1}{2}(q + q^{-1}) > 1, q > 1, \) is degenerated in the thermodynamic limit \( (M \to \infty) \), namely there are two states with the same energy which we will call the ground state \( |\Psi_1\rangle \) and the quasi-ground state \( |\Psi_2\rangle \) (on the finite lattice, these states possess different energies). These two states can be obtained in the algebraic Bethe ansatz framework (see [3]).

The spontaneous magnetization is defined as the modulus of the normalized matrix element of the local spin operator \( \sigma^z_m \) between these two states:

\[
s_0 = \left| \frac{\langle \Psi_1 | \sigma^z_m | \Psi_2 \rangle}{\langle \Psi_1 | \Psi_2 \rangle} \right|^\frac{1}{2}.
\]
In [2,11], Baxter proved that the spontaneous magnetization of the infinite chain could be expressed as an infinite product:

\[ s_0 = \left( \prod_{n=1}^{\infty} \frac{1 - q^{-2n}}{1 + q^{-2n}} \right)^2, \quad (M = \infty). \] (1.7)

This formula in the thermodynamic limit was also reproduced by means of the q-vertex operator approach (see [12]).

In this paper, we study the spontaneous magnetization \( s_0 \) in the framework of the algebraic Bethe ansatz. From the results of paper [3], the corresponding form factor on the finite chain is represented as a ratio of two determinants of \( M \times M \) matrices. In the thermodynamic limit, it becomes the ratio of two Fredholm determinants of linear integral operators, which can be computed explicitly. This leads directly to the Baxter formula (1.7). Thus, we obtain here an independent proof of the Baxter formula from algebraic Bethe ansatz. We prove also that there are no perturbative (\( \frac{1}{M^k} \)) finite size corrections so that for the finite chain the Baxter formula has only exponentially small corrections.

This paper is organized as follows.

In section 2, the expression for the form factor of operator \( \sigma^z_m \) obtained in [3] is recalled. It is reformulated in a form suitable for the thermodynamic limit and particularized to the ground and quasi-ground states.

In section 3, the procedure of taking the thermodynamic limit is discussed.

In section 4, the spontaneous magnetization is represented in terms of Fredholm determinants, which gives a proof of the Baxter formula in the thermodynamic limit. The finite size corrections are also evaluated.

## 2 The form factor

Let us consider the function

\[ s_z(m) = \frac{\langle 0 \mid \prod_{j=1}^{N} C(\mu_j) \sigma^z_m \prod_{k=1}^{N} B(\lambda_k) \mid 0 \rangle}{\langle 0 \mid \prod_{j=1}^{N} C(\lambda_j) \prod_{k=1}^{N} B(\lambda_k) \mid 0 \rangle}, \] (2.1)

where \( \{\lambda_k\}_{1 \leq k \leq N} \) and \( \{\mu_j\}_{1 \leq j \leq N} \) are solutions of the Bethe equations

\[ \left( \frac{\sin(\lambda_j + \frac{i\zeta}{2})}{\sin(\lambda_j - \frac{i\zeta}{2})} \right)^M \prod_{k=1}^{N} \frac{\sin(\lambda_j - \lambda_k - i\zeta)}{\sin(\lambda_j - \lambda_k + i\zeta)} = -1, \quad 1 \leq j \leq N. \] (2.2)
These equations can also be written in a logarithmic form:

\[ Mp_0(\lambda_j) + \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = 2\pi n_j, \quad 1 \leq j \leq N, \tag{2.3} \]

where \( n_j \) are integers for \( N \) odd and half integers for \( N \) even. The bare momentum \( p_0(\lambda) \) and the scattering phase \( \theta(\lambda) \) are defined as

\[ p_0(\lambda) = i \ln \frac{\sin(\lambda + \frac{i\zeta}{2})}{\sin(\lambda - \frac{i\zeta}{2})}, \]
\[ \theta(\lambda) = i \ln \frac{\sin(i\zeta - \lambda)}{\sin(i\zeta + \lambda)}. \]

In the recent paper [3], an expression for any arbitrary matrix element (form factor) of the \( \sigma^z_m \) operator between two Bethe states has been obtained in the framework of algebraic Bethe ansatz. Using this result (see proposition 5.2 of [3]) as well as the Gaudin-Korepin [8, 10] formula for the norm of Bethe states, one can represent the function \( s_z(m) \) as follows:

\[ s_z(m) = \exp \left\{-i(m-1)\sum_{k=1}^{N} (p_0(\lambda_k) - p_0(\mu_k))\right\} \prod_{j>k} \frac{\sin(\lambda_k - \lambda_j)}{\sin(\mu_k - \mu_j)} \frac{\det(S - 2Q)}{\det N}, \tag{2.4} \]

where the \( N \times N \) matrix \( S \) is given by

\[ S_{ab} = \Phi_{ab}^+ + \Phi_{ab}^- , \]
\[ \Phi_{ab}^\pm = \frac{\sinh \zeta}{\sin(\lambda_b - \mu_a) \sin(\lambda_b - \mu_a \pm i\zeta)} \prod_{k=1}^{N} \frac{\sin(\lambda_b - \mu_k \pm i\zeta)}{\sin(\lambda_b - \lambda_k \pm i\zeta)}, \]

and \( Q \) is the following \( N \times N \) rank one matrix:

\[ Q_{ab} = \frac{\sinh \zeta}{\sin(\mu_a + i\frac{\zeta}{2}) \sin(\mu_a - i\frac{\zeta}{2})} \prod_{k=1}^{N} \frac{\sin(\mu_k - i\frac{\zeta}{2})}{\sin(\lambda_k - i\frac{\zeta}{2})}. \]

\( N \) is the Gaudin matrix:

\[ N_{ab} = \delta_{ab} \left\{ Mp_0'(\lambda_a) - 2\pi \sum_{k=1}^{N} K(\lambda_a - \lambda_k) \right\} + 2\pi K(\lambda_a - \lambda_b), \tag{2.5} \]
where the function $K(\lambda)$ is proportional to the derivative of the scattering phase

$$K(\lambda) = -\frac{1}{2\pi} \theta'(\lambda) = \frac{\sinh 2\zeta}{2\pi \sin(\lambda + i\zeta) \sin(\lambda - i\zeta)}. \tag{2.6}$$

Note that the fact that $Q$ is a rank one matrix together with the orthogonality of two different Bethe states (i.e. $\det S = 0$) allowed us to insert the global factor $\prod_{k=1}^{N} \frac{\sin(\mu_k - i\zeta/2)}{\sin(\lambda_k - i\zeta/2)}$ in the matrix elements of $Q$.

Thus, we will derive the Baxter formula from the expression (2.4). In order to take the thermodynamic limit in (2.4), we first rewrite it in a more convenient form by introducing the following matrix

$$\mathcal{X}_{ab} = \frac{e^{i(\lambda_a - \mu_b)}}{\sin(\lambda_a - \mu_b)} \prod_{k \neq b}^{N} \sin(\mu_k - \lambda_k), \tag{2.7}$$

the determinant of which is

$$\det \mathcal{X} = \exp \left\{ i \sum_{j=1}^{N} (\lambda_j - \mu_j) \right\} \prod_{j>k}^{N} \frac{\sin(\lambda_k - \lambda_j)}{\sin(\mu_k - \mu_j)}. \tag{2.8}$$

Hence (2.4) can be rewritten as

$$s_z(m) = \exp \left\{ -i(m - 1) \sum_{k=1}^{N} (p_0(\lambda_k) - p_0(\mu_k)) \right\} \exp \left\{ i \sum_{j=1}^{N} (\mu_j - \lambda_j) \right\} \frac{\det(\mathcal{X}S - 2\mathcal{X}Q)}{\det N}. \tag{2.9}$$

The products of matrices $\mathcal{M} \equiv \mathcal{X}S$ and $\mathcal{V} \equiv \mathcal{X}Q$ can be calculated using some identities for the rational functions. The procedure is similar to the one used in Appendix A. Here we merely give the results:

$$\mathcal{M}_{ab} = -2\pi K(\lambda_a - \lambda_b) + \frac{i\delta_{ab}}{N} \sum_{k=1}^{N} \frac{\sin(\lambda_a - \lambda_k)}{\sin(\lambda_a - \mu_k)} \left( \prod_{k=1}^{N} \frac{\sin(\lambda_a - \mu_k + i\zeta)}{\sin(\lambda_a - \lambda_k + i\zeta)} - \prod_{k=1}^{N} \frac{\sin(\lambda_a - \mu_k - i\zeta)}{\sin(\lambda_a - \lambda_k - i\zeta)} \right), \tag{2.8}$$

$$\mathcal{V}_{ab} = i \left\{ \frac{\exp\{i(\lambda_a - i\zeta/2)\}}{\sin(\lambda_a - i\zeta/2)} - \frac{\exp\{i(\lambda_a + i\zeta/2)\}}{\sin(\lambda_a + i\zeta/2)} \right\} \exp \left\{ i \sum_{j=1}^{N} (p_0(\mu_j) - p_0(\lambda_j)) \right\}. \tag{2.9}$$
Note that these expressions are valid for any two different solutions \( \{ \lambda_j \}_{1 \leq j \leq N} \) and \( \{ \mu_k \}_{1 \leq k \leq N} \) of Bethe equations. To calculate the spontaneous magnetization (1.6), one has to particularize them to the ground and quasi-ground states \( |\Psi_1\rangle \) and \( |\Psi_2\rangle \), that is to compute the quantity

\[
s_z(m) = \frac{\langle \Psi_1 | \sigma_z^m | \Psi_2 \rangle}{\langle \Psi_2 | \Psi_2 \rangle}.
\] (2.10)

From now on, \( \{ \mu_1, \ldots, \mu_N \} \) and \( \{ \lambda_1, \ldots, \lambda_N \} \) will denote the sets of spectral parameters solutions of the Bethe equations (2.3) corresponding respectively to the ground state \( |\Psi_1\rangle \) and the quasi-ground state \( |\Psi_2\rangle \). For both states, \( N \) is equal to \( M/2 \). The ground state \( |\Psi_1\rangle \) is parametrized by the following set of \( n_j \):

\[
n_j = -\frac{N+1}{2} + j, \quad j = 1, 2, \ldots, N,
\] (2.11)

whereas the quasi-ground state \( |\Psi_2\rangle \) is parametrized by the shifted set:

\[
\tilde{n}_j = -\frac{N+1}{2} + j - 1, \quad j = 1, 2, \ldots, N,
\] (2.12)

(see, e.g., [8]). We will moreover use the following notations:

\[
\delta \lambda_j = \lambda_{j+1} - \lambda_j, \quad \delta \mu_j = \mu_{j+1} - \mu_j, \quad \delta \tilde{\lambda}_j = \mu_{j} - \lambda_j.
\]

The energies of these two states are equal in the thermodynamic limit, the difference of their total momenta being equal to \( \pi \).

Thus, one has expressed the quantity (2.10) in the form:

\[
s_z(m) = (-1)^{m-1} \exp \left\{ i \sum_{j=1}^{N} \delta \tilde{\lambda}_j \right\} \frac{\det(M - 2V)}{\det N},
\] (2.13)

where \( M \) and \( N \) are the \( N \times N \) matrices the elements of which are given by (2.8) and (2.5), whereas the elements of the matrix \( V \) are simply (since the difference of the total momenta of the ground and quasi-ground state is \( \pi \)):

\[
V_{ab} = \frac{e^{2i\lambda_a} - \cosh \zeta}{\sin(\lambda_a + i\frac{\zeta}{2}) \sin(\lambda_a - i\frac{\zeta}{2})}.
\] (2.14)

The expression (2.13) turns out to be very convenient to take the thermodynamic limit since the functional form of the non-diagonal terms of the matrices \( M \) and \( N \), as well as the matrix elements of \( V \), do not depend on the number of sites \( M = 2N \) of the chain. Nevertheless, the diagonal terms of the matrices \( M \) and \( N \) are more complicated and should be treated separately.
3 The thermodynamic limit

3.1 General procedure

In the previous section, we obtained an expression of the spontaneous magnetization as a quotient of two determinants. Most of the matrix elements involved are independent of $N$, which makes the thermodynamic limit ($M = 2N \to \infty$) quite obvious. The only terms we have to take care of (i.e. which depend explicitly of $N$) are on the one hand the global coefficient, and on the other hand the diagonal matrix elements of $M$ and $N$. Both kinds of terms consist of sums (or products) on the different spectral parameters in the ground or in the quasi-ground states. Thus, the first step in taking the thermodynamic limit is to know how to deal with such sums. This leads us to formulate the two following propositions.

As previously, $\{\lambda_j\}_{1 \leq j \leq N}$ and $\{\mu_j\}_{1 \leq j \leq N}$ denote respectively the set of spectral parameters corresponding to the quasi-ground state, solution of (2.3)-(2.12), and the set of spectral parameters corresponding to the ground state, solution of (2.3)-(2.11).

**Proposition 3.1.** Let $f$ be a $C^\infty_\pi$-periodic function. Then the sum of all the values $f(\lambda_j)$ where the set of the spectral parameters $\{\lambda_j\}_{1 \leq j \leq N}$ parametrizes the quasi-ground state can be replaced by an integral in the thermodynamic limit according to the following rule:

$$
\frac{1}{M} \sum_{j=1}^{N} f(\lambda_j) = \int_{-\pi/2}^{\pi/2} d\lambda f(\lambda) \rho(\lambda) + O(M^{-\infty}),
$$

(3.1)

where $\rho(\lambda)$ is defined as the solution of the Lieb equation [13]

$$\rho(\lambda) + \int_{-\pi/2}^{\pi/2} d\nu K(\lambda - \nu) \rho(\nu) = \frac{p'_0(\lambda)}{2\pi}.
$$

(3.2)

The same equations (3.1)-(3.2) stand concerning the sum on the values $f(\mu_j)$ where $\{\mu_j\}_{1 \leq j \leq N}$ parametrizes the ground state.

The proof of the proposition is given in the Appendix B.

There exists a generalization of this result in the case that $f$ is not itself a periodic function but that its derivative is periodic:

**Proposition 3.2.** Let $f$ be a $C^\infty$ function such that $f'$ is $\pi$-periodic. Then the sums of the values of $f$ at the points $\lambda_j$, $1 \leq j \leq N$ parametrizing the quasi-ground state, and respectively at the points $\mu_j$, $1 \leq j \leq N$ parametrizing the ground state, are given in the thermodynamic limit as the corresponding integrals, the only rational finite-size
correction being of order $\frac{1}{M}$:

$$\frac{1}{M} \sum_{j=1}^{N} f(\lambda_j) = \int_{-\pi/2}^{\pi/2} d\lambda f(\lambda) \rho(\lambda) + \frac{c_1(f)}{M} + O(M^{-\infty}),$$  \hspace{1cm} (3.3)$$

$$\frac{1}{M} \sum_{j=1}^{N} f(\mu_j) = \int_{-\pi/2}^{\pi/2} d\mu f(\mu) \rho(\mu) + \frac{c_2(f)}{M} + O(M^{-\infty}),$$  \hspace{1cm} (3.4)$$

where $c_1$ and $c_2$ are constants independent of $M$.

3.2 The norm

Using the proposition 3.1 and the Lieb equation (3.2) one can immediately calculate the diagonal terms of the Gaudin matrix $N$:

$$N_{aa} = 2\pi M \left\{ \rho(\lambda_a) + \frac{1}{M} K(0) \right\} + O(M^{-\infty}).$$  \hspace{1cm} (3.5)$$

Thus the matrix elements of $N$ have the following form in the thermodynamic limit:

$$N_{ab} = 2\pi M \left\{ \delta_{ab} \rho(\lambda_a) + \frac{1}{M} K(\lambda_a - \lambda_b) \right\} + O(M^{-\infty}).$$  \hspace{1cm} (3.6)$$

3.3 Diagonal matrix elements of $M$

In order to calculate the diagonal elements of the matrix $M$, let us consider the following $\pi$-periodic functions:

$$\phi_{\pm}(\lambda_a) = \prod_{k=1}^{N} \frac{\sin(\lambda_a - \mu_k \pm i\zeta)}{\sin(\lambda_a - \lambda_k \pm i\zeta)},$$  \hspace{1cm} (3.7)$$

$$\phi(\lambda_a) = M \prod_{k=1}^{N} \frac{\sin(\lambda_a - \mu_k)}{\sin(\lambda_a - \lambda_k)},$$  \hspace{1cm} (3.8)$$

and those defined similarly for the second set of parameters:

$$\psi_{\pm}(\mu_k) = \prod_{a=1}^{N} \frac{\sin(\mu_k - \lambda_a \pm i\zeta)}{\sin(\mu_k - \mu_a \pm i\zeta)},$$  \hspace{1cm} (3.9)$$

$$\psi(\mu_k) = M \prod_{a=1}^{N} \frac{\sin(\mu_k - \lambda_a)}{\sin(\mu_k - \mu_a)},$$  \hspace{1cm} (3.10)$$
The diagonal elements of the matrix $M$ (2.8) are actually:

$$M_{aa} = -2\pi K(0) + iM\phi^{-1}(\lambda_a) (\phi_+(\lambda_a) - \phi_-(\lambda_a)).$$ (3.11)

These functions are related by remarkable identities:

$$\frac{1}{M} \sum_{a=1}^{N} \frac{\sinh \zeta}{\sin(\mu_k - \lambda_a \pm i\zeta) \sin(\mu_k - \lambda_a)} \phi(\lambda_a) = \pm i\psi_\pm^{-1}(\mu_k),$$ (3.12)

the detailed proof of which, based on identities for rational functions, is given in Appendix A.

It is to mention that the factors $\phi(\lambda)$ and $\psi(\mu)$ were used earlier by N. Slavnov (we thank N. Slavnov for communicating us the results contained in his Ph. D. thesis).

Summing up the identities (3.12) with plus and minus signs one obtains the following equation:

$$\frac{2\pi}{M} \sum_{a=1}^{N} K(\mu_k - \lambda_a) \phi(\lambda_a) = i(\psi_+^{-1}(\mu_k) - \psi_-^{-1}(\mu_k),$$ (3.13)

which can be rewritten in an integral form by means of proposition 3.1:

$$\int_{-\pi}^{\pi} K(\mu - \lambda) \phi(\lambda) \rho(\lambda) d\lambda = \frac{i}{2\pi} (\psi_+^{-1}(\mu) - \psi_-^{-1}(\mu)) + O(M^{-\infty}).$$ (3.14)

An analogous equation can be derived similarly for the functions $\psi(\mu)$ and $\phi_\pm(\lambda)$:

$$\int_{-\pi}^{\pi} K(\lambda - \mu) \psi(\mu) \rho(\mu) d\mu = \frac{i}{2\pi} (\phi_+^{-1}(\lambda) - \phi_-^{-1}(\lambda)) + O(M^{-\infty}).$$ (3.15)

The computation of the main order of the functions $\phi_\pm(\lambda)$, $\psi_\pm(\mu)$ in the thermodynamic limit is rather simple but it is more complicated to prove that there are no rational perturbative finite-size corrections. We present this calculation in the Appendix C. The result is

$$\phi_\pm(\lambda) = \psi_\pm(\lambda) = \pm i + O(M^{-\infty}).$$ (3.16)

Equation (3.14) is used to calculate the function $\phi(\lambda)$. By means of (3.16), it can be rewritten as

$$\int_{-\pi}^{\pi} K(\mu - \lambda) \phi(\lambda) \rho(\lambda) d\lambda = -\frac{1}{\pi} + O(M^{-\infty}),$$
which admits a unique solution

$$\phi(\lambda)\rho(\lambda) = -\frac{1}{\pi} + O(M^{-\infty}).$$

Thus, one obtains the following expression for the diagonal terms (3.11) of the matrix $\mathcal{M}$ in the thermodynamic limit:

$$\mathcal{M}_{aa} = 2\pi \rho(\lambda_a) M - 2\pi K(0) + O(M^{-\infty}).$$

Hence the elements of $\mathcal{M}$ have the following form:

$$\mathcal{M}_{ab} = 2\pi M \left\{ \delta_{ab} \rho(\lambda_a) - \frac{1}{M} K(\lambda_a - \lambda_b) \right\} + O(M^{-\infty}).$$

(3.18)

4 Fredholm determinant representation

In the thermodynamic limit, the determinants of the matrices $\mathcal{N}$ and $\mathcal{M} - 2\mathcal{V}$ are replaced by Fredholm determinants:

$$\det \mathcal{N} = (2\pi M)^N \left( \prod_{j=1}^{N} \rho(\lambda_j) \right) \left\{ \det (\hat{I} + \hat{K}) + O(M^{-\infty}) \right\},$$

(4.1)

$$\det (\mathcal{M} - 2\mathcal{V}) = (2\pi M)^N \left( \prod_{j=1}^{N} \rho(\lambda_j) \right) \left\{ \det (\hat{I} - \hat{K} - 2\hat{V}) + O(M^{-\infty}) \right\},$$

(4.2)

where $\hat{I}$ is the identity operator and $\hat{K}$ is the integral operator acting on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with the kernel $K(\lambda - \mu)$ (2.9). The kernel of the integral operator $\hat{V}$ is

$$V(\lambda, \mu) \equiv V(\lambda) = \frac{e^{2i\lambda} - \cosh \zeta}{2\pi \sin(\lambda + i\frac{\zeta}{2}) \sin(\lambda - i\frac{\zeta}{2})}.$$ 

(4.3)

To calculate the coefficient in the representation (2.13) one has to use the proposition 3.2, and the first order of the Taylor developpement of $\delta \tilde{\lambda}_j$ which has been computed in Appendix C:

$$\exp \left\{ i \sum_{j=1}^{N} \delta \tilde{\lambda}_j \right\} = i + O(M^{-\infty}).$$

This leads to the following Fredholm determinant representation for the function $s_z(m)$ (2.10):

$$s_z(m) = i(-1)^{m-1} \frac{\det(\hat{I} - \hat{K} - 2\hat{V})}{\det(\hat{I} + \hat{K})} + O(M^{-\infty}).$$

(4.4)
The function
\[ \tilde{s}_z(m) = \frac{\langle \Psi_2 | \sigma_z^m | \Psi_1 \rangle}{\langle \Psi_1 | \Psi_1 \rangle} \]
can be computed similarly, which gives
\[ \tilde{s}_z(m) = -s_z(m). \]

Therefore the spontaneous staggered magnetization admits the following representation in terms of Fredholm determinants:
\[ s_0 = \left| \frac{\det(\hat{I} - \hat{K} - 2\hat{V})}{\det(\hat{I} + \hat{K})} \right| + O(M^{-\infty}). \tag{4.5} \]

What remains to do now is to compute explicitly these determinants in order to obtain the Baxter formula. This can be done by means of Fourier transformation.

As the kernel of the integral operator \( \hat{K} \) depends only on the difference of the two variables, and since we are in the regime \( q > 1 \), this operator can be diagonalized by Fourier transformation. Its eigenvalues are thus obtained as the Fourier coefficients \( k_n \) of the function \( K(\lambda) \):
\[ k_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K(\lambda)e^{2i\lambda n} d\lambda = e^{-2\zeta|n|} = q^{-2|n|}. \tag{4.6} \]

Therefore the determinant of the operator \( \hat{I} + \hat{K} \) is only the infinite (convergent) product of its eigenvalues, namely \( (q > 1) \):
\[ \det(\hat{I} + \hat{K}) = \prod_{n=-\infty}^{\infty} (1 + q^{-2|n|}) = 2 \prod_{n=1}^{\infty} (1 + q^{-2n})^2. \tag{4.7} \]

After Fourier transformation, the operator \( \hat{V} \) becomes an infinite matrix which admits only one nonzero column \( v_{n0} \):
\[ v_{n0} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V(\lambda)e^{2i\lambda n} d\lambda. \tag{4.8} \]

Since the operator \( \hat{I} - \hat{K} \) is diagonalized by the Fourier transformation, only the diagonal element \( v_{00} \) contributes to the determinant, namely
\[ \det(\hat{I} - \hat{K} - 2\hat{V}) = -2v_{00} \prod_{n=1}^{\infty} (1 - q^{-2n})^2. \tag{4.9} \]
The computation of the integral \((4.8)\) leads to \(v_{00} = -1\). Finally, from the expression \((4.7)\) for the spontaneous magnetization, one obtains the following result:

\[
s_0 = \left( \prod_{n=1}^{\infty} \frac{1 - q^{-2n}}{1 + q^{-2n}} \right)^2 + O(M^{-\infty}),
\]

which coincides with the Baxter formula.

**Appendix A**

In this appendix we give the proof of identity \((3.12)\), which provides a typical example of a proof for identities involving rational functions. In particular, \((2.8)\) and \((2.14)\) can be proved similarly.

We thus consider the following identity

\[
\sum_{a=1}^{N} \frac{\sinh \zeta}{\sin(\mu_k - \lambda_a + i\zeta) \sin(\mu_k - \lambda_a)} \prod_{m=1}^{N} \frac{\sin(\lambda_a - \mu_m)}{\sin(\lambda_a - \lambda_m)} = i \prod_{m=1}^{N} \frac{\sin(\mu_k - \mu_m + i\zeta)}{\sin(\mu_k - \lambda_m + i\zeta)}. \tag{A.1}
\]

To prove this relation it is convenient to introduce the exponential variables:

\[
v_k = e^{i\mu_k}, \quad u_a = e^{i\lambda_a}, \quad q = e^\zeta.
\]

The left hand side of \((A.1)\) can be represented using these variables as

\[
\text{l.h.s.} = i v_k^2 \prod_{m=1}^{N} u_m \sum_{a=1}^{N} \frac{q^2 - 1}{(v_k^2 - u_a^2)(v_k^2 - u_a^2 q^2)} \prod_{m \neq a}^{N} \frac{u_a^2 - v_m^2}{u_a^2 - u_m^2}. \tag{A.2}
\]

For the right hand side one obtains

\[
\text{r.h.s.} = i \prod_{m=1}^{N} \frac{u_m(v_k^2 - v_m^2q^2)}{v_m(v_k^2 - u_m^2 q^2)}. \tag{A.3}
\]

Consider the sum in \((A.2)\) as a function of \(v_k^2\)

\[
W_1(v_k^2) = \sum_{a=1}^{N} \frac{q^2 - 1}{(v_k^2 - u_a^2)(v_k^2 - u_a^2 q^2)} \prod_{m \neq a}^{N} \frac{u_a^2 - v_m^2}{u_a^2 - u_m^2}.
\]
which is to compare with the following function

\[ W_2(v_k^2) = v_k^{-2} \prod_{m=1}^{N} \frac{v_k^2 - v_m^2 q^2}{v_k^2 - u_m^2 q^2}. \]

Both of them are rational functions. When \( v_k^2 \to \infty \) one can easily see that \( W_J(v_k^2) \to 0 \), for \( J = 1, 2 \). Also both functions have only simple poles in the points \( v_k^2 = v_m^2 q^2 \) and their residues in these points are equal. Hence one can conclude that

\[ W_1(v_k^2) = W_2(v_k^2), \]

which leads immediately to the identity (A.1).

**Appendix B**

We present here the proof of the propositions 3.1 and 3.2.

Let us begin with the proof of proposition 3.1. Note first that, in case of variables \( q_j, 1 \leq j \leq N \) which are homogeneously distributed inside an interval of length \( \pi \) (i.e. \( q_{j+1} - q_j = \frac{\pi}{N} \)), it is easy to prove the analogous proposition (with \( \rho(\lambda) = \frac{1}{2\pi} \)):

\[ \frac{1}{M} \sum_{j=1}^{N} f(q_j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dq \, f(q) + O(M^{-\infty}). \]

We will reduce to this case by means of the Bethe equations in the logarithmic form (2.3):

\[ p_0(\lambda_j) + \frac{1}{M} \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = \frac{2\pi n_j}{M}. \]  

(B.1)

For a set of variables \( \{\lambda_j\}_{1 \leq j \leq N} \) parametrizing the ground or the quasi-ground state, let us introduce the function

\[ q_M(\lambda) = p_0(\lambda) + \frac{1}{M} \sum_{k=1}^{N} \theta(\lambda - \lambda_k), \]

and the set of variables \( q_j = q_M(\lambda_j) = \frac{2\pi n_j}{M}, 1 \leq j \leq N \). These new variables have a homogeneous distribution. Moreover, \( q_M(\lambda) \) is an invertible function for any \( \Delta > 1 \), at least for \( M > M_0(\Delta) \) with \( M_0 \) large enough. For \( \Delta > 2, M_0(\Delta) = 0 \) (and does not depend on \( \Delta \)). It follows from the estimate

\[ q_M'(\lambda) = p_0'(\lambda) - \frac{2\pi}{M} \sum_{k=1}^{N} K(\lambda - \lambda_k) \geq p_0'(\pi/2) - \pi K(0) \]

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(the r.h.s. is positive for $\Delta > 2$). For $1 \leq \Delta \leq 2$, the existence of $M_0(\Delta)$ follows from the fact that the solution for the ground state does exist and is unique for any $M$ \[17\], and the density $\rho(\lambda) = (1/2\pi) \lim_{M \to \infty} q_M(\lambda)$ is positive in the thermodynamic limit. Note at last that, since $q_M(\lambda + \pi) = q_M(\lambda) + \pi$, the function $f \circ q_M^{-1}$ is $\pi$-periodic. Hence we can represent any sum on $j$ as an integral:

$$\frac{1}{M} \sum_{j=1}^{N} f(\lambda_j) = \frac{1}{M} \sum_{j=1}^{N} f(q_M^{-1}(q_j)), \quad (B.2)$$

which we can rewrite as an integral on $\lambda$:

$$\frac{1}{M} \sum_{j=1}^{N} f(\lambda_j) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} d\lambda q'_M(\lambda) f(\lambda) + O(M^{-\infty}). \quad (B.4)$$

One has thus to evaluate the derivative of $q_M$ in the thermodynamic limit. We look for it in the following form:

$$q'_M(\lambda) = p'_0(\lambda) - \frac{2\pi}{M} \sum_{k=1}^{N} K(\lambda - \lambda_k) = 2\pi(\rho(\lambda) + \varphi_M(\lambda)), \quad (B.5)$$

where the correction $\varphi_M$ is to be determined. An integral equation for it can be obtained taking $f(\lambda) = K(\mu - \lambda)$: from (B.4) one obtains

$$\int_{-\pi/2}^{\pi/2} d\lambda K(\mu - \lambda) \varphi_M(\lambda) = -\varphi_M(\mu) + O(M^{-\infty}).$$

As all the eigenvalues \[16\] of the integral operator $\hat{K}$ are positive, the only function satisfying this constraint is 0 up to order $O(M^{-\infty})$. Hence we proved that

$$\frac{1}{M} \sum_{j=1}^{N} f(\lambda_j) = \int_{-\pi/2}^{\pi/2} d\lambda \rho(\lambda) f(\lambda) + O(M^{-\infty}). \quad (B.6)$$

The proposition 3.2 can be derived directly from this proposition and the analogous fact for the homogeneous distribution of momenta. Really for any function $g(q)$ such that $g'(q)$ is a periodic function one can easily prove that

$$\frac{1}{M} \sum_{j=1}^{N} g(q_j) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} dq \ g(q) + \frac{C_1}{M} + O(M^{-\infty}),$$
(as all further corrections are proportional to the integrals of second and higher derivatives of $g(q)$ and these integrals are evidently zero). Now one can continue the proof as for the first proposition. One should just mention that if $g'(\lambda)$ is a periodic function then $\frac{dg(\lambda(q))}{dq}$ is also a periodic function.

Appendix C

In this appendix, we present the calculation of the functions $\phi_{\pm}(\lambda)$ (3.7) and $\psi_{\pm}(\lambda)$ (3.9).

The functions $\phi_{\pm}(\lambda)$ can be expressed in the exponential form,

$$\phi_{\pm}(\lambda_a) = \exp \left\{ \sum_{k=1}^{N} \left( \ln \sin(\lambda_a - \lambda_k \mp i\zeta) - \ln \sin(\lambda_a - \lambda_k \pm i\zeta) \right) \right\}. \quad (C.1)$$

One should note that if all $\tilde{\delta}_{\zeta} \lambda_k$ are replaced by the corresponding $\delta \lambda_k$, this expression is exactly equal to $-1$. We will use this property further to calculate the function $\phi_{\pm}(\lambda)$ in the thermodynamic limit.

Let us at first calculate the “shift functions” $\delta \tilde{\lambda}_k$ and $\delta \lambda_k$. The difference of two successive Bethe equations in the logarithmic form (2.3) leads to:

$$M \left\{ p_0(\lambda_j + \delta \lambda_j) - p_0(\lambda_j) \right\} + \sum_{k=1}^{N} \left\{ \theta(\lambda_j + \delta \lambda_j - \lambda_k) - \theta(\lambda_j - \lambda_k) \right\} = 2\pi. \quad (C.2)$$

Representing the differences in the left hand side as Taylor series and using proposition 3.1 and Lieb equation (3.2), we obtain the following equation for $\delta \lambda$:

$$M \sum_{k=1}^{\infty} \frac{1}{k!} \rho^{(k-1)}(\lambda)(\delta \lambda)^k = 1 + O(M^{-\infty}). \quad (C.3)$$

This equation defines uniquely (at order $O(M^{-\infty})$) the shift function $\delta \lambda$ as a functional of $\rho(\lambda)$. The solution of this equation can be found as a series on $\frac{1}{M}$:

$$\delta \lambda = \frac{1}{\rho(\lambda) M} \left( 1 + \frac{1}{2M} \left( \frac{1}{\rho(\lambda)} \right)' + \frac{1}{12M^2} \left( \frac{1}{\rho^2(\lambda)} \right)'' + O \left( \frac{1}{M^3} \right) \right).$$

We will not use the explicit form of this solution in further calculations.

The calculation of the function $\delta \tilde{\lambda}_k$ is more complicated. Taking the difference between the Bethe equations in the logarithmic form for the ground state and quasi ground state one obtains:

$$M \left\{ p_0(\lambda_j + \delta \tilde{\lambda}_j) - p_0(\lambda_j) \right\} + \sum_{k=1}^{N} \left\{ \theta(\lambda_j + \delta \tilde{\lambda}_j - \lambda_k - \delta \tilde{\lambda}_k) - \theta(\lambda_j - \lambda_k) \right\} = 2\pi. \quad (C.4)$$
Let us introduce the function $z(\lambda_k) = M\rho(\lambda_k)\delta\tilde{\lambda}_k$. Using the proposition \[3.1\] and Lieb equation \[3.2\], one obtains the following equation for this function

$$z(\lambda) + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K(\lambda - \mu)z(\mu)d\mu = 1 + O\left(\frac{1}{M}\right).$$ (C.5)

This equation admits a unique solution $z(\lambda) = \frac{1}{2} + O(M^{-1})$, hence the expression for the first order of $\delta\tilde{\lambda}_k$:

$$\delta\tilde{\lambda}_k = \frac{1}{2\rho(\lambda_k)M} + O\left(\frac{1}{M^2}\right).$$

To obtain the next orders we compare the Bethe equations for the ground and quasi-ground state, namely the sums $\sum_{k=1}^{N} \theta(\lambda_j - \lambda_k)$ and $\sum_{k=1}^{N} \theta(\lambda_j - \mu_k)$. The function $\theta(\lambda)$ is not periodic but its derivative is $\pi$-periodic, and thus proposition 3.2 can be applied:

$$\frac{1}{M} \left( \sum_{k=1}^{N} \theta(\lambda_j - \mu_k) - \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) \right) = \frac{c}{M} + O(M^{-\infty})$$

where $c$ does not depend on $M$. As only the first order correction is non-zero, we can use merely the first order of $\delta\tilde{\lambda}_k$ to calculate $c$. Finally one obtains

$$\sum_{k=1}^{N} \theta(\lambda_j - \mu_k) - \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = \pi + O(M^{-\infty}).$$

The Bethe equations for the ground and quasi-ground states can now be written as

$$Mp_0(\mu_j) + \sum_{k=1}^{N} \theta(\mu_j - \mu_k) = 2\pi n_j,$$ (C.6)

$$Mp_0(\lambda_j) + \sum_{k=1}^{N} \theta(\lambda_j - \mu_k) = 2\pi n_j - \pi + O(M^{-\infty}).$$ (C.7)

Taking the difference of these equations and using once again the Taylor series and the Lieb equation we obtain the equation defining $\delta\tilde{\lambda}$ as a functional of $\rho(\lambda)$:

$$M \sum_{k=1}^{\infty} \frac{1}{k!} \rho^{(k-1)}(\lambda)(\delta\tilde{\lambda})^k = \frac{1}{2} + O(M^{-\infty}).$$ (C.8)

This result is to be compared with (C.3). It follows that

$$\delta\tilde{\lambda}[\rho(\lambda)] = \delta\lambda[2\rho(\lambda)] + O(M^{-\infty}).$$ (C.9)
To calculate the functions $\phi_{\pm}(\lambda)$ we use our first remark: replacing $\delta \tilde{\lambda}_k$ by $\delta \lambda_k$ in the expression (C.1) and developing it as a series on $\frac{1}{M}$ leads to

$$-1 = \exp \left\{ \sum_{k=1}^{N} \left( \ln \sin(\lambda_a - \lambda_k - \delta \lambda_k \pm i \zeta) - \ln \sin(\lambda_a - \lambda_k \pm i \zeta) \right) \right\},$$

$$= \exp \left\{ - \sum_{k=1}^{N} \left( \frac{\cot(\lambda_a - \lambda_k \pm i \zeta)}{\rho(\lambda_k) M} \right) + \sum_{n=1}^{\infty} \frac{A_n[\rho(\lambda)]}{M^n} + O(M^{-\infty}) \right\},$$

where $A_n[\rho(\lambda)]$ are homogeneous functionals of $\rho(\lambda)$ which do not depend on $M$. The first term of the development can be easily computed using proposition 3.1 once again:

$$\sum_{k=1}^{N} \left( \frac{\cot(\lambda_a - \lambda_k \pm i \zeta)}{\rho(\lambda_k) M} \right) = \int_{-\pi/2}^{\pi/2} d\lambda \cot(\lambda_a - \lambda \pm i \zeta) + O(M^{-\infty}) = \mp i \pi + O(M^{-\infty}).$$

Hence it follows that

$$A_n[\rho(\lambda)] = 0.$$

Thus, due to the relation (C.9), the functions $\phi_{\pm}(\lambda)$ can be represented as the following series on $\frac{1}{M}$:

$$\phi_{\pm}(\lambda) = \exp \left\{ - \sum_{k=1}^{N} \left( \frac{\cot(\lambda_a - \lambda_k \pm i \zeta)}{2 \rho(\lambda_k) M} \right) + \sum_{n=1}^{\infty} \frac{A_n[2 \rho(\lambda)]}{M^n} + O(M^{-\infty}) \right\}.$$

Since $A_n[\rho(\lambda)]$ are homogeneous functionals of $\rho(\lambda)$, we obtain the very simple result:

$$\phi_{\pm}(\lambda) = \pm i + O(M^{-\infty}). \quad (C.10)$$

The functions $\psi_{\pm}(\mu)$ can be computed similarly:

$$\psi_{\pm}(\mu) = \mp i + O(M^{-\infty}). \quad (C.11)$$

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