Chemical Potential in the Gravity Dual of a 2+1 Dimensional System

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We study probe D5 branes in D3 brane AdS$_5$ and AdS$_5$-Schwarzschild backgrounds as a prototype dual description of strongly coupled 2+1 dimensional quasi-particles. We introduce a chemical potential through the U(1)$_R$ symmetry group, U(1) baryon number and a U(1) of isospin in the multi-flavour case. We find the appropriate D5 embeddings in each case - the embeddings do not exhibit the spontaneous symmetry breaking that would be needed for a superconductor. The isospin chemical potential does induce the condensation of charged meson states.

INTRODUCTION

Recently there has been interest in whether the AdS/CFT Correspondence [1, 2, 3] can be used to understand 2+1 dimensional condensed matter systems (for example [4, 5, 6, 7, 8, 9, 10]). The typical UV degrees of freedom in these systems are electrons in the presence of a Fermi surface and a gauged U(1), QED. When brought together in certain 2d states they can become relativistic and strongly coupled - possibly such systems might induce superconductivity too by breaking the gauge symmetry. The philosophy, which may be overly naive, is to find relativistic strongly coupled systems that show these behaviours and hope they share some universality with the physical systems. Whether or not that linkage becomes strong, it is interesting to study the AdS duals of 2+1d systems.

In this paper we will study the dynamics of the theory on the world volume of a mixed D3 and D5 brane construction with a 2+1 dimensional intersection, which has previously been studied at zero temperature in the absence of chemical potential in [11, 12, 13]. The gravity dual of the D3s, at zero temperature, is AdS$_5 \times S^5$, which is dual to the 3+1 dimensional $\mathcal{N} = 4$ super Yang Mills theory. Here these interactions will be used to loosely represent strongly coupled “phonons”. We will introduce 2+1d “quasi-particles” via D5 branes (with a 2+1d intersection with the D3s) - states connecting the two set of branes should be expected to carry quantum numbers that interact with the D3 brane dynamics and flavour quantum numbers associated with the number of D5 branes - the full field theory can be found in [13]. We will work in the probe approximation for the D5 branes which corresponds to quenching quasi-particle loops in the phonon background [14]. At zero chemical potential the theory has $\mathcal{N} = 4$ supersymmetry and at zero quasi-particle mass is conformal [11, 12, 13]. The system is related to the higher dimensional D3-D7 intersection where the $\mathcal{N} = 4$ gauge theory on the D3 branes has been used to describe gluon dynamics and the D3-D7 strings quarks - some progress in the study of the properties of mesons in 3+1d strongly coupled gauge theories has been achieved [15]. The D3-D5 defect system seems a natural starting point therefore for 2+1 dimensional systems.

To attempt to mimic a solid state system one must weakly gauge a U(1) symmetry of the system and introduce an associated chemical potential (by setting $A_\mu = \mu$). There are a number of possible U(1)s that can play this role.

Firstly the D3-D5 world volume theory has an unbroken SO(3) global symmetry corresponding to rotations in the 4-plane transverse to the D5 brane. We will introduce a chemical potential for the quasi-particles with respect to a U(1) subgroup of the SO(3) - this can be done by simply spinning the D5 branes in an SO(2) plane [16]. The embedding of the D5 brane is described by a scalar that is charged under this U(1) symmetry so one naively expects to trigger superconductivity in the spirit described in [5] - but here we would have an explicit understanding of the UV degrees of freedom the scalar describes. Naively one expects the scalar describing the D5 embedding to be destabilized by the presence of a chemical potential which gives the scalar a negative mass squared. An equivalent statement is that one expects the centrifugal force associated with the rotational motion of the brane to force it off the spin axis. In fact though we find the minimum area embedding for such spinning probe D5 branes and find this is not the case.

The crucial physics is that the speed of light decreases as one moves into the centre of AdS - eventually it becomes less than the rotation speed of the D5 brane. We show, following the higher dimensional analysis in [16, 17, 19] that there are regular D5 embeddings into the interior which have a more complicated embedding structure. The branes bend in the direction of the rotation so that there are two linked scalar fields describing the embedding - this richer scalar fields turns out to not include superconductivity, a subtlety on top of the arguments in [5].

We can introduce mass terms for the quasi-particles that explicitly break the U(1) symmetry and we discuss the embeddings in these cases. There is a first order phase
transition when the R-charge chemical potential grows above the mass of the quasi-particle bound states - below the transition the quasi-particles exist as deconfined particles whilst above it they are confined into bound states. This transition is analogous to the meson melting transition seen in this system and the D3-D7 system at finite temperature [20, 21, 22, 23]. We also analyze the finite temperature behaviour of these solutions by using the AdS$_{5}$ Schwarzschild geometry as the background.

Next we study a chemical potential for the U(1) associated with baryon number for the quark fields - this seems the most natural candidate for how QED would manifest in the effective relativistic theory of a solid state system. The U(1) appears in the gravity dual as the U(1) gauge symmetry on the surface of the D5 branes - we allow configurations with non-zero profiles for these fields on the D5. Here we are again led by results in the D3-D7 system [24].

At zero temperature the presence of the gauge field on the brane naively adds in an additional constraint on the solutions that that gauge field should be regular as the brane passes from positive to negative values of the effective radial coordinate on the brane. This criterion rules out small perturbations of the standard flat embeddings of the D5 brane - instead the true solutions become ones where the D5 brane kinks through the origin of the space. The kink, which for large quark mass is rather sharp, has been interpreted in [24] as a tube of strings connecting the asymptotic branes. In [20] it has been argued that an external charge could be responsible for the irregularity of the gauge field and that the flat embeddings should be retained. In either interpretation, amongst these configurations is one for zero quark mass which turns out to simply lie straight through the origin of the space - the chemical potential does not induce any R-charged operators to condense. The other fields on the D5 world volume carry no net baryon number and so do not couple to the chemical potential - there is no condensation. The system does not therefore act like a superconductor.

The finite temperature behaviour of these solutions with non-zero baryon number chemical potential is also explored. The D5 brane embeddings kink on to the black hole event horizon in this case. As in the D3-D7 case there is a first order phase transition between two different sorts of in-falling solutions which is analogous to the meson melting transition at finite temperature but zero chemical potential [20, 21, 22, 23].

Finally we turn to isospin chemical potential in the case of multiple (but still probe) D5 branes. This theory seems less relevant to solid state physics because the quasi-particles have a U(2) flavour symmetry - electrons don’t! On the other hand it is natural to discuss in this context and as advocated in [6, 8] may provide some lessons for p-wave superconductors. The embeddings of any individual brane is simply the same as for an equal magnitude baryonic chemical potential and there is no induced R-charge breaking at zero quark mass. Where the theories differ is that there are vector bosons ('$W^{\pm}$') on the branes’ world volume that couple to the chemical potential - we work in a truncated version of the DBI action that is just Yang Mills theory on the D5 world volume. We show, with an analysis very similar to that of [6, 8] and recent work in the D3-D7 system [27, 28], that below some critical value of the temperature the W-bosons condense at a second order transition. This is dual to the formation of a spin one condensate which is charged under U(1) isospin number in the gauge theory. Were one to identify that U(1) with QED we would have a superconductor.

THE D3 THEORY

We will represent the strong interaction dynamics with the large N $N = 4$ super Yang Mills theory on the surface of a stack of D3 branes. It is described at zero temperature by AdS$_{5} \times S^{5}$

$$ds^2 = \frac{(u^2 + r^2)}{L^2} dx^2_{5+1} + \frac{r^2}{(u^2 + r^2)} (dp^2 + \rho^2 d\Omega^2_3 + dr^2 + r^2 d\Omega^2_2)$$

where we have written the geometry to display the directions the D3 lie in ($x_{3+1}$), those we will embed the D5 on ($x_{2+1}$, $\rho$ and $\Omega_2$) and those transverse ($r$ and $\Omega_2$). $L$ is the AdS radius.

At finite temperature the description is given by the AdS-Schwarzschild black hole

$$ds^2 = \frac{u^2}{L^2} (-h(u)dt^2 + dx^2_3) + \frac{L^2}{u^2 h(u)} du^2 + L^2 d\Omega^2_5$$

$$h(u) = 1 - \frac{u^4}{w^4}$$

It is helpful to make the change of variables to isotropic coordinates

$$\frac{u \ du}{\sqrt{u^4 - u_0^4}} = \frac{dw}{w}$$

and choose the integration constant such that if $u_0 = 0$ the zero-temperature geometry is recovered

$$2u^2 = u^2 + \sqrt{u^4 - u_0^4}$$
The metric can now be written as

\[ ds^2 = \frac{1}{R} \left( w^2 + \frac{w^4}{4e^2} \right) \left( -\frac{w^4 - v^4}{w^4 + v^4} dt^2 + dx_3^2 \right) + \frac{L^2}{w} \left( dp^2 + \rho^2 d\Omega_2^2 + dr^2 + r^2 d\Omega_2^2 \right) \]

with \( w^2 = \rho^2 + r^2 \), which shares the coordinate structure of (1).

QUENCHED MATTER FROM A D5 PROBE AT \( T=0 \)

We will introduce quenched matter via a probe D5 brane. The underlying brane configuration is as follows:

\[
\begin{array}{ccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
D3 & - & - & - & - & - & - & - & - & - \\
D5 & - & - & - & - & - & - & - & - & - \\
\end{array}
\]

In polar coordinates the D5 fills the radial direction of AdS5 and is wrapped on a two sphere.

The action for the D5 is just its world volume

\[ S \sim T \int d^8\xi \sqrt{-\det G} \sim \int d\rho \rho^2 \sqrt{1 + \rho^2} \]

where \( T \) is the tension and we have dropped angular factors on the two-sphere.

This is clearly minimized when \( r \) is constant so the D5 lies straight. The value of the constant is the size of the mass gap for the quasi-particles. We will mainly be interested in the conformal case where the constant is zero. Note the general large \( \rho \) solution is of the form

\[ r = m + \frac{c}{\rho} + .. \]

Here \( m \) is an explicit mass term for the quasi-particles in the Lagrangian and \( c \) the expectation value for a bi-quasi-particle operator - note \( m \) has dimension one and \( c \) dimension two adding to three as required for a Lagrangian term in 2+1d. The solution with non-zero \( c \) is not normalizable in pure AdS5. Note that when \( m = c = 0 \) the theory is conformal. Including a non-zero \( m \) or \( c \) breaks the SO(3) symmetry i.e it breaks one transverse SO(2) symmetry. From this it is apparent that \( m \) and \( c \) carry charge under that U(1). Were \( c \) to be non-zero when \( m = 0 \) it would be an order parameter for the spontaneous breaking of the U(1) symmetry.

R-CHARGE CHEMICAL POTENTIAL/SPIN

Our theory as yet lacks the relevant perturbation of the Fermi surface and the U(1) of QED. We will associate the U(1) with a subgroup of the SO(3) of the \( \Omega_2 \) - for concreteness we will use the angle in the \( x_7 - x_8 \) directions.

To include a chemical potential we will spin the D5 brane in the angular direction \( \phi \) of this U(1) with angular speed \( \mu \).

The spinning of the D5 branes implies that the quasi-particles see a chemical potential. This is in fact a little bit of a peculiar limit since the background D3 theory also has fields, including scalars, charged under the U(1). We are not allowing that geometry to backreact to the chemical potential. In fact we had better not - the pure D3 theory has a moduli space for separating the D3s in the transverse 6-plane. Were we to set them spinning they would scatter to infinity since there is no central force to support rotation. In the theory on the D3 surface there is a run away Bose-Einstein condensation. We simply wish to switch off this physics - it is not what we are interested in - so we forbid such backreaction. The D3 theory is in an unstable state but will nevertheless provide some strongly coupled interactions for the quasi-particles that do see the chemical potential.

An Overly Naive Ansatz

We first look for solutions where the D5 embedding has \( \phi = \mu t \) and we will allow the position \( r \) (the radial distance in \( x_7 - x_8 \)) to be a function of \( \rho \). The action is

\[ S \sim \int d\rho \rho^2 \sqrt{(1 + \rho^2)(1 - \frac{L^4}{(\rho^2 + r^2)^2} r^2 \mu^2)} \]

Naively one is expecting the centrifugal force from the spinning to eject the brane from the axis at all but the end points where the boundary conditions hold the brane. This would lead to a spontaneous symmetry breaking or superconducting state. We will see that this is what this naive system tries to achieve.

The equation of motion for \( r \) as a function of \( \rho \) is easily computed but unrevealing. At large \( \rho \) the solutions tend to the no-rotation limit \( r \sim m + \frac{c}{\rho} \).

The (pair of) circle(s) in the \( (\rho, r) \) plane described by \( L^4 \mu^2 r^2 = (\rho^2 + r^2)^2 \) is clearly a zero of the action so branes wrapped there provide a solution to the equation of motion. Anything going within the locus described by the two circles is moving faster than the local speed.
of light and is presumably not physical. This locus is a stationary limit surface - we call it the ergosurface below.

There exist “Karch-Katz” type solutions \[14\] for D5-branes that do not encounter the stationary limit surface - these solutions essentially lie flat above everything plotted in Fig.1. We want to know what happens to those solutions when the curves which minimize the action like those actually exist down to the top of the ergosurface) (top).

The relation between \(m\) and \(c\) for curves impacting on the stationary limit surface in this way is shown in Fig.1. We find it numerically convenient to switch coordinates and write the AdS geometry as

\[
ds^2 = \frac{w^2}{\ell^2} ds_{3+1}^2 + \frac{L^2}{w^2} (dw^2 + w^2 \left( d\theta^2 + \sin^2 \theta d\Omega_2^2 + \cos^2 \theta d\tilde{\Omega}_2^2 \right))
\]

(10)

The D5 will now be embedded in the \(x_{3+1}, w\) and \(\Omega_2\) directions - the naive solutions above are recovered by looking for solutions that have \(\theta(w)\) and \(\phi = \mu t\) where \(\phi\) is the ‘first’ angle of the \(\Omega_2\).

In these coordinates the Lagrangian for our more ambitious ansatz for the rotating D5 embedding is

\[
\mathcal{L} = \frac{w^2 \sin^2 \theta}{\ell^2} \times \\
\sqrt{(1 - \frac{L^4 \mu^2 \cos^2 \theta}{w^2}) (1 + w^2 \theta')^2 + w^2 \cos^2 \theta \phi'^2}
\]

(11)

If \(\phi' \sim \mu\) the two \(\mu^2\) terms compete against each other removing the naive intuition about centrifugal force.

Since the action only depends on \(\phi'\) and not \(\phi\) one can integrate the equation of motion for \(\phi'\). One could then substitute back in for \(\phi'\) in terms of the integration constant - this though gives an action with a “zero over zero” form at the ergosurface that is hard to work with. Instead, following \[17, 18\], we Legendre transform to \(\mathcal{L}' = \mathcal{L} - \phi' \frac{\partial \mathcal{L}}{\partial \phi'}\). This gives (setting \(\frac{\partial \mathcal{L}}{\partial \phi'} = J\)

\[
\mathcal{L}' = \frac{1}{w \cos \theta} \sqrt{(1 - \frac{L^4 \mu^2 \cos^2 \theta}{w^2}) (1 + w^2 \theta')^2}
\times \sqrt{w^6 \sin^4 \theta \cos^2 \theta - J^2}
\]

(12)

This has a “zero times zero” form at the ergosurface which is simpler to work with numerically.

For a solution that crosses the ergosurface we demand that the action be positive everywhere and this fixes \(J\) - the two terms must pass through zero and switch signs together. Having fixed \(J\) in this way one can then look at the \(\theta\) equation of motion near the ergosurface. Expanding near the surface, and after some algebra, one finds the following consistency equation for the \(\theta\) derivative

A More Sophisticated Ansatz

We will now try a more sophisticated ansatz where the brane has in addition some profile \(\phi(w)\) where \(\phi\) is the angle on which they spin (ie \(\phi = \mu t + \phi(w)\)). The ansatz is inspired by the work in \[17\] where similar issues are encountered when a magnetic field is switched on on the brane’s world-volume.

In these coordinates the Lagrangian for our more ambitious ansatz for the rotating D5 embedding is

The problem of course here is that the solutions are singular at the ergosurface where they kink. This is a sign that our ansatz is wrong - none of this is the right physics.
In the three-dimensional \((w, \theta, \phi)\) subspace the ergosurface is the torus given by \(L^2 \mu \cos \theta \pm w\), which in a plane of constant \(\phi\) gives two adjacent circles of radius \(\frac{w}{L^2}\). Fig. 2 shows a sequence of regular solutions in the \((\rho, r(\rho))\) coordinates of the previous section. To obtain regular solutions one should make an odd continuation to the negative quadrant as shown. We show a full D5 embedding in Fig. 3 with both the \(\theta(w)\) and \(\phi(w)\) dependence plotted - note the D5 rotates at speed \(\mu\) in the \(\phi\) direction (around the axis of the torus).

Clearly there is no spontaneous symmetry breaking in these solutions - the solutions smoothly map onto the solution which lies along the axis as the mass parameter \(m\) is taken to zero. In the field theory presumably the conformal symmetry breaking parameter \((\mu)\) which might trigger symmetry breaking is the same parameter as that telling us there’s a plasma density cutting off the theory - there’s no room for dynamics. This model turns out not to be an example of the behaviour studied in [5].

The presence of a non-trivial profile \(\phi(w)\) for the embeddings that penetrate the ergosurface indicates on the field theory side of the duality that there is a vev for the scalar field associated with the phase of the condensate \(c\) - this would be the Goldstone mode if there were spontaneous symmetry breaking. Note that the regular Karch Katz embeddings, away from the ergosurface, have \(\phi(w)\) constant so there is no such vev.

Again we see there is a first order transition between the Karch-Katz type solutions and those that enter the ergosurface region. We plot the values of \(c\) vs \(m\) for these solutions in Fig. 4 - it shows the same spiral structure around the first order transition as we saw with the naive ansatz. We will discuss the meaning of this transition below in the thermal context.

\[
w^2 \theta'^2 + \tan \theta \ w \theta' - 1 = 0 \quad (13)
\]

There are thus two allowed gradients at the ergosurface. In fact numerically we find choosing any gradient focuses on to the same flow both within and outside the ergosurface. We can numerically shoot in and out from a point near the ergosurface in order to generate regular embeddings.
Thermal behaviour

One can perform the same analysis in the thermal background. Writing $b^4 = \frac{w^4}{w^4 + b^4}$, there is again a torus-like ergosurface given by the equation

$$L^2 \mu \cos \theta = \pm \frac{1}{w} \frac{w^4 - b^4}{\sqrt{w^4 + b^4}} \quad (14)$$

and also a spherical horizon at $w = b$. One finds the horizon always lies within the ergosurface because the local speed of light is zero at the horizon. Note, below we find no phase transition when raising the temperature through the scale of the chemical potential. There would be a transition from a runaway Bose-Einstein condensation to a stable theory were we to allow the chemical potential to backreact on the geometry.

One can form the Legendre-transformed Lagrangian (which recovers the $T = 0$ case for $b = 0$)

$$\mathcal{L} = \frac{1}{w w^4 + b^4} \sqrt{\frac{(w^4 - b^4)^2}{w^4 (w^4 + b^4)} \left( 1 - L^4 \mu^2 c_g^2 w^2 \frac{(w^4 + b^4)}{(w^4 - b^4)^2} \right)}$$

$$\sqrt{(1 + w^2 \theta^2)} \left( s^3 c_g - \frac{(w^4 - b^4)^2}{w^2} - J^2 \frac{w^4}{w^4 + b^4} \right) \quad (15)$$

FIG. 5: A selection of solution curves for D5 branes in the thermal geometry (with $\theta_0 = 1$). The grey region is the interior of the ergosurface and the black region is the interior of the event horizon.

The embeddings which extremize the action fall into two types - Karch-Katz type embeddings and those which hit the ergosurface. Fluctuations of the former would reveal a bound state spectrum. The latter embeddings inevitably fall onto the event horizon (a selection of these is plotted in Fig. 5 for $\theta_0 = 1$). In addition for these embeddings that pass through the ergosurface $\theta_{tt}$ switches sign on the world volume - the ergosurface acts like a horizon for the world volume fields. Here fluctuations would have a quasinormal spectrum along the lines of.

There is therefore a first order transition in the behaviour of the theory as the quasi-particle mass goes through the scale of the chemical potential or temperature. This transformation is explored in detail in [19]. Note here it seems the transition is always a meson melting transition at finite temperature. At zero temperature the transition is driven by quantum rather than thermal fluctuations and has been described in terms of a metal-insulator transition in [25].

The D3-D7 System

Much of the above parallels results already found in the D3-D7 system [16, 19]. That system describes an $\mathcal{N} = 2$ 3+1d gauge theory with fundamental matter hypermultiplets in the gauge background of $\mathcal{N} = 4$ super-Yang Mills theory. In [16] an analysis similar to our “naive ansatz” was performed suggesting spontaneous symmetry breaking. Those authors have since refined their analysis in a related system with a background electric field [17] and concluded that if regular embeddings are insisted upon the symmetry breaking is not present (see also [18]). Were they to update [16] they would find embeddings analogous to our D5 embeddings above as they indicate in [19].

BARYON NUMBER CHEMICAL POTENTIAL

Another, and perhaps the most likely, way in which to embed the U(1) symmetry of QED into the brane set up is through the quasi-particle number global symmetry (essentially baryon number). The conserved vector current and its source, which is effectively a background gauge field configuration for this symmetry, manifest holographically as the U(1) gauge symmetry living on the world volume of the D5 brane. We can introduce a chemical potential for baryon number by switching on a constant $A_t$ component for this U(1) gauge field. We will study the embeddings of such a configuration at zero and non-zero temperature. Much of this analysis again mirrors that for the D3/D7 system which can be found in [24].
Zero temperature

The DBI action for the D5 brane including the surface gauge field is

\[ S \sim T_5 \int d^6\xi \sqrt{\det(F^{ab}) + 2\pi\alpha' F_{ab}} \]  

(16)

We consider embeddings of the D5 brane in the \( \rho - r \) plane with in addition \( 2\pi\alpha' A_0(\rho) = A(\rho) \) to represent the chemical potential. The action is then of the form

\[ \mathcal{L} \sim \rho^2 \sqrt{1 + r^2} - A^2 \]  

(17)

Since the action is independent of \( A \) the equation of motion for \( A \) implies \( \frac{\partial\mathcal{L}}{\partial A} \) is a constant, \( Q \). We find

\[ A^2 = Q^2 \frac{1 + r^2}{\rho^4 + Q^2} \]  

(18)

It is useful to perform a Legendre transform again (\( \mathcal{L}' = \mathcal{L} - A' \frac{\partial\mathcal{L}}{\partial A} \)) and work with

\[ \mathcal{L}' = \sqrt{(1 + r^2)(\rho^4 + Q^2)} \]  

(19)

The \( r \) independence again gives a simple equation for the embedding that is

\[ r' = \frac{c_1}{\sqrt{\rho^4 + Q^2 - c_1^2}} \]  

(20)

with \( c_1 \) a constant.

To interpret this equation it is helpful to initially turn off the chemical potential, \( Q = 0 \). It is then clear that the solutions are singular at \( \rho = \sqrt{c_1} \) and the only regular case is \( c_1 = 0 \) so that \( r' = 0 \) - we recover the usual flat embeddings.

When we allow non-zero \( Q \) there become a bigger set of regular solutions - those with \( c_1 \leq Q \). These solutions are plotted in Fig.6 for varying \( c_1 \) and provide an alternative embedding, that crosses through the origin, for each value of \( m \) in the large \( \rho \) asymptotics of the embedding. In [24] it was argued (in the D3-D7 case) that these are the true embeddings when there is a surface gauge field on the brane. We can see that the flat embeddings (\( c_1 = 0 \)) are not regular from (18) - they have a none zero gradient \( A' \) at \( \rho = 0 \) so there will be a kink in the \( A \) field as it crosses over the \( r \) axis. For the solutions that pass through the origin though the \( A \) field is regular. In [26] it has been argued that the irregular solutions should be maintained with the irregularity interpreted as the presence of an external source.

As can be seen from Fig.6, whichever interpretation is taken, the embedding for the case of massless quarks is unchanged from the usual flat embedding. The embedding does not therefore spontaneously break any symmetry with the introduction of a baryon chemical potential. Away from \( m = 0 \) for the case of the regular embeddings the embeddings do change and there is a condensate present. From (20) we can see that up to a sign \( c_1 \) is just the asymptotic parameter \( c \) that determines the condensate. For small \( c_1 \) the quark mass grows linearly but as \( c_1 \) approaches \( Q \) \( m \) rises sharply - the resulting plot of \( c \) vs \( m \) therefore shows that the condensate asymptotes to a constant value for large mass - see Fig.7.

Another possible source of spontaneous breaking would be if the gauge field vev on the D5 led to other fields in the D5 brane world volume condensing. In fact though all the fields on the D5 are in the adjoint representation of, generically, a \( U(N_f) \) flavour symmetry. Adjoint fields of the \( U(1) \) of baryon number are chargeless and hence have no interaction with the gauge field. There is no possibility for such condensation and the system is not superconducting.
Finite temperature

We can also study the theory with baryon number chemical potential at finite temperature by finding D5 embeddings with a non-zero surface $\mathcal{A}_t$ gauge field in the black hole geometry \([5]\). We again set $b^4 \equiv \frac{v_0^4}{4}$. The DBI Lagrangian for such an embedding is

$$\mathcal{L} = \rho^2 w^4 + b^4 \sqrt{\frac{1}{w^4} \left( \frac{(w^4 - b^4)^2}{w^4 + b^4} \right) (1 + r'^2) - A'^2} \quad (21)$$

This time we have the gauge field

$$A'^2 = \frac{Q^2 \, \frac{1}{w^4} \left( \frac{(w^4 - b^4)^2}{w^4 + b^4} \right) (1 + r'^2) - Q^2}{\rho^4 \left( \frac{w^4 + b^4}{w^4} \right)^2} \quad (22)$$

The Legendre-transformed version of the Lagrangian is

$$\mathcal{L}' = \sqrt{\frac{1}{w^4} \left( \frac{(w^4 - b^4)^2}{w^4 + b^4} \right) (1 + r'^2) \left( \rho^4 \left( \frac{w^4 + b^4}{w^4} \right)^2 + Q^2 \right)} \quad (23)$$

One can shoot out from the horizon attempting to fill out the $m$ parameter space asymptotically. The solutions are very similar to those in the D3-D7 case as outlined in [24] and see also [26]. There is no spontaneous symmetry breaking. There is a first order phase transition between the large $m$ embeddings, that are essential flat except for a spike down onto the black hole, and smoother embeddings that fall into the black hole at lower $m$. This transition persists until the chemical potential becomes large, where there is no transition and the two phases coexist. In Fig.9 we plot the quark condensate, $c$ versus quark mass, $m$, for varying values of $Q$ (which determines the chemical potential) around the critical value of $Q$ where the phase transition between “spike” embeddings and smooth horizon entering embeddings ends. The disappearance of the phase transition is evident - the physics closely resembles that in the D3-D7 case discussed in detail in [24].

**ISOSPIN CHEMICAL POTENTIAL**

The final possible source of a chemical potential in the D3-D5 set up is from the isospin symmetry present when there are two or more flavours of quasi-particle (D5) present. In contrast to the discussion of baryon number above, there are clearly operators which carry isospin charge eg. $\langle \bar{\psi} \gamma_0 \gamma_3 \psi \rangle$. These can be expected to condense at zero isospin chemical potential and break the symmetry spontaneously in the spirit of the zero temperature work in [29] and the phenomenological holographic model of p-wave condensation in [24, 30]. Here we will work first at finite temperature and only consider the case of zero quark mass - the embeddings of the branes are identical to those above for baryon number, where one uses the modulus of the isospin as the chemical potential, so at zero quark mass the D5s lie straight along the axis as shown in Figure 6. The mesons of the theory are therefore melted by the thermal bath but the operators can nevertheless condense. One is perhaps making a departure from any obvious connection to a solid state system at this point since one would require a system with a $U(2)$ or greater flavour symmetry on the quasi-particles - presumably there is only one sort of electron in a solid state system!

The full theory of flavours on the D5 brane is expected to be unstable in the presence of a chemical potential at zero temperature. The situation is analogous to the D3-D7 system discussed in [30] - the squarks have a moduli
space in the gauge theory at zero temperature and chemical potential which shows up on the gravity side as a moduli space for the size of instanton configurations on the D7 (here D5s) world-volume [31]. An isospin chemical potential will induce a negative mass squared for the scalars forcing the vev or instanton size to infinity. We will simply neglect this runaway behaviour here, fix the scalar vevs to zero and study the fermionic operators of the theory - hopefully this tells us about the behaviour of a theory with fermions but no scalars.

The full DBI action to all orders in the surface gauge field is not fully known - an attempt to use the full DBI on a similar problem in the D3-D7 setup has recently appeared [27]. We though will follow the path of the D3-D7 analysis in [28] and just use the first order expansion of the action

\[ S \sim T_5 \int d^6\xi \sqrt{-\text{det}G} \left( 1 - \frac{1}{4} \text{Tr} \left( F^2 \right) \right) \] (24)

We expect this action to represent the dynamics well. We will write the ansatz for the gauge fields as

\[ A = \Phi(\rho) \tau^3 dt + w(\rho) \tau^1 dx_1 \] (25)

as in [8]. The coordinate representation of the Yang-Mills equation is

\[ \partial_\rho \left( \sqrt{-g} F^{\rho \alpha i} \right) = -g_{YM} \left( \delta^{im} \delta^{jl} - \delta^{il} \delta^{jm} \right) A^j_\mu A^\mu_l A^{\nu m} \] (26)

For our ansatz there are two equations of motion

\[ (\sqrt{-g} g^{00} g^{rr} \Phi')' = g_{YM} \sqrt{-g} g^{00} g^{11} w^2 \Phi \] (27)

\[ (\sqrt{-g} g^{11} g^{rr} w')' = g_{YM} \sqrt{-g} g^{00} g^{11} \Phi^2 w \] (28)

Restricting ourselves to a massless quark D5 brane so the induced brane metric is given by isotropic AdS-Schwarzschild, the equations are (having absorbed factors of \( g_{YM} \) and \( L \) into the definitions of the fields)

\[ \left( \frac{r^4 + 1}{r^4 - 1} \Phi' \right)' = \frac{\sqrt{r^4 + 1}}{r^4 - 1} w^2 \Phi \] (29)

\[ \left( \frac{r^4 - 1}{\sqrt{r^4 + 1}} w' \right)' = \frac{-\sqrt{r^4 + 1}}{r^4 - 1} \Phi^2 w \] (30)

In the isotropic coordinates one should shoot out from a small displacement \( x \) from the horizon using the initial condition \( w = w_0 \) and \( \Phi = \Phi_2 x^2 \) for constants \( w_0 \) and \( \Phi_2 \). Near the boundary of AdS the solutions behave as

\[ \Phi \sim \mu - \frac{\rho}{r} \] (31)

\[ w \sim \mu' + \frac{c}{r} \] (32)

We search for solutions which have \( \mu' = 0 \) because these are solutions which are normalizable and hence describe a condensate of the charged mesonic operator. Requiring this to be the case one can solve the coupled nonlinear equations to yield multiple branches of solutions. The lowest, monotonic branch is presumably the stable solution and for these solutions one can plot the dependence of the condensate on the chemical potential. Since the quarks we put in were massless, the only two scales are the chemical potential and the temperature and so large chemical potential can be equivalently viewed as low-temperature.

The results are plotted in Fig.10 - for \( \mu \ll T \) there is no condensation. For large \( \mu \) though there is a second order phase transition to a phase with a charged vector condensate. The behaviour is similar to that previously observed in [6, 8, 27, 28]. As \( \mu/T \) goes to infinity the condensate \( c \) tends to a finite constant (\( \approx 0.3 \)) when measured in units of \( \mu^2 \), which is similar to the behaviour found in [28] for a pair of D7 probe branes - in their 3+1 dimensional theory the condensate tends to a constant in units of \( \mu^3 \).

**FIG. 10**: Plot of the charged vector condensate (the parameter \( c \) from [31]) in units of chemical potential squared versus \( 1/\mu \) in the case of isospin chemical potential at zero quark mass.

**SUMMARY**

We have proposed probe D5 branes in D3 brane backgrounds as a plausible dual for a strongly coupled quasiparticle theory in 2+1 dimensions - at zero temperature and chemical potential the theory is supersymmetric and conformal. We introduced a chemical potential with respect to the global U(1) symmetries associated with R-charge and baryon number and found the resulting regular D5 embeddings. These embeddings do not display spontaneous symmetry breaking and, indeed, at
zero temperature and zero intrinsic mass the theory is essentially indifferent to the chemical potential remaining as a state of conformal quasi-particles.

For the R-charge case we show a first order phase transition in the massive theory as the quasi-particle mass crosses the value of the chemical potential - on one side the quasi-particles are confined on the other they are not. At finite temperature the transition is between solutions that fall into the black hole and those that don’t.

At finite temperature in the baryon number case there is also a phase transition between two different black hole embeddings which is the equivalent of the meson melting transition at finite temperature but zero chemical potential. If the chemical potential becomes too large then the transition ceases to occur for any quark mass.

Finally we introduced isospin chemical potential in the case of two probe D5 branes and reproduced the second order transition to a phase with a charged vector condensate previously seen in other systems in [6, 8, 27, 28].

We hope that these explorations will form a useful platform from which to find a holographic model of some real solid state system. We note that many of the transport properties of this system have also been recently explored in [32].

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