Topological Phase Transition Independent of System Non-Hermiticity

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Non-Hermiticity can vary the topology of system, induce topological phase transition, and even invalidate the conventional bulk-boundary correspondence. Here, we show the introducing of non-Hermiticity without affecting the topological properties of the original chiral symmetric Hermitian systems. Conventional bulk-boundary correspondence holds, topological phase transition and the (non)existence of edge states are unchanged even though the energy bands are inseparable due to non-Hermitian phase transition. Chern number for energy bands of the generalized non-Hermitian system in two dimension is proved to be unchanged and favorably coincides with the simulated topological charge pumping. Our findings provide insights into the interplay between non-Hermiticity and topology. Topological phase transition independent of non-Hermitian phase transition is a unique feature that beneficial for future applications of non-Hermitian topological materials.

Introduction.—Parity-time ($\mathcal{PT}$) symmetry stimulates the development of non-Hermitian physics \cite{hermitic}. Non-Hermitian systems \cite{nonhermitic} exhibit many intriguing features and applications that not limited to power oscillation \cite{power}, coherent perfect absorption \cite{coherent}, unidirectionality behaviors \cite{unidirectional}, single-mode laser \cite{single-mode}, robust energy transfer \cite{robust}, and exceptional point (EP) enhanced sensing \cite{EP} due to its nonorthogonal eigenstates and the exotic topology of EPs \cite{EP-topology}. The scope of topological phase of matter has also been extended to non-Hermitian region \cite{nonhermitian} and stimulates several interesting discussions on $\mathcal{PT}$-symmetric topological interface states \cite{interface}, non-Hermitian bands theory \cite{bands}, topological invariants \cite{invariants}, EP lines and surfaces \cite{EP-lines,EP-surfaces}, high-order topological phases \cite{high-order}, and symmetry protected non-Hermitian topological phases \cite{symmetry}. Topological classification are discussed for general non-Hermitian systems \cite{general}, for non-Hermitian systems with reflection symmetry \cite{reflection}, and alteratively classified by the geometric features of singularity ring \cite{singularity}. The non-Hermiticity and non-Abelian gauge potentials can create interesting topological phases \cite{non-Abelian}. The robust and efficient topological edge state lasing is an interesting application of non-Hermitian topological systems \cite{lasing,non-hermitian-lasing}. Furthermore, chiral inversion symmetry is uncovered to protect the CBBC in non-Hermitian topological systems \cite{chiral}, and the CBBC and the skin modes are elucidated in the viewpoint of non-Hermitian Aharonov-Bohm effect; alternatively, they are elucidated from a transfer matrix perspective \cite{transfer} and the Green’s function method \cite{Green}. Besides, the non-Hermiticity can solely induce topological phase, which has been demonstrated in trivial Hermitian systems associated with staggered gain and loss \cite{staggered}, asymmetric coupling amplitude \cite{asymmetric}, and imaginary coupling \cite{imaginary}, respectively. Therefore, the non-Hermiticity can alter the topology of system, induce topological phase transition, and even ruin the CBBC; in contrast to the topology changed by non-Hermiticity, retaining the topology of Hermitian system in the non-Hermitian generalization is a critical and meaningful challenge for non-Hermitian topological phase of matter.

In this work, we systematically elucidate the introducing of non-Hermiticity without altering the topological phase transition in the original chiral symmetric Hermitian system; the proposed non-Hermitian topological system holds the CBBC and shares identical topological properties including the (non)existence of topologically protected edge states with their parent Hermitian system, even though the energy bands are deformed into the complex domain and inseparable. The complete set of eigenstates of the non-Hermitian system is exactly mapped from the eigenstates of the original Hermitian system by a set of local transformations; the mapping allows direct projections of their geometric quantities. In the non-Hermitian generalization, five symmetry classes with chiral symmetry are mapped to the other five symmetry classes without chiral symmetry, respectively; the Chern number in two dimension (2D) is proved to be unchanged, and the numerically simulated topological charge pumping favorably agrees with the Chern number.

Mapping the topology.—Chiral symmetric systems can be written in the block off-diagonal form \cite{block-off-diagonal}:

$$H = \begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix},$$

(1)
where we consider $D$ as an arbitrary $n \times n$ matrix. The basis in $H$ can be any degree of freedom, such as the real space coordinate, spin, or other orthogonal complete set. $H$ is referred to as the original Hermitian Hamiltonian, from which a non-Hermitian Hamiltonian $\mathcal{H}$ is created

$$\mathcal{H} = H + i\gamma \sigma_z \otimes I_n,$$  

where $\sigma_z$ is the Pauli matrix, and $I_n$ denotes the $n \times n$ identity matrix. The non-Hermitian term $i\gamma$ in $\mathcal{H}$ does not only play the role of on-site potential, but also the non-Hermitian hopping with asymmetric amplitudes in the real space. For chiral symmetric systems not in the bipartite lattice form, taken the Creutz ladder as an illustration [105], the gain and loss introduced in the block off-diagonal form of $H$ [Eq. (1)] are equivalent to introducing asymmetric couplings in the ladder legs [114]. In addition, introducing non-Hermiticity breaks the chiral symmetry.

Equation (2) provides a way of non-Hermitian generalization without altering the topological phase transition in original Hermitian systems. To characterize the topological properties, we consider the Hamiltonian $H(k)$ in the momentum space, which is the core matrix of a Bloch or a BdG system [118]. $k$ is the momentum and all the information of the topological system is encoded in $H(k)$. The Schrödinger equation is $H(k)|\phi(k)\rangle = \epsilon(k)|\phi(k)\rangle$ and $|\phi(k)\rangle = e^{i\Delta(k)|\phi(k)\rangle}$, where $\rho = \pm$ represents the upper or lower energy band, and $\lambda \in [1, l]$ denotes the band index, assuming the total number of energy bands $2l$. The eigenenergy of $\mathcal{H}(k)$ is

$$\epsilon_{\lambda}^\rho(k) = \rho[\epsilon_{\lambda}^\rho(k)^2 - \gamma^2]^{1/2}. \quad (3)$$

The energy is either real or imaginary. The eigenstate of $\mathcal{H}(k)$ for eigenenergy $\epsilon_{\lambda}^\rho(k)$ is obtained through a mapping (see Appendix A)

$$|\varphi_{\lambda}^\rho(k)\rangle = M_{\lambda}^\rho(k, \gamma) |\phi_{\lambda}^\rho(k)\rangle. \quad (4)$$

with the mapping matrix

$$M_{\lambda}^\rho(k, \gamma) = \begin{pmatrix} a_{\lambda}^\rho(k, \gamma) I_n & 0 \\ 0 & I_n \end{pmatrix}, \quad (5)$$

where $a_{\lambda}^\rho(k, \gamma) = (\epsilon_{\lambda}^\rho(k) + i\gamma) / \epsilon_{\lambda}^\rho(k)$ is a unit modulus complex number for real $\epsilon_{\lambda}^\rho(k)$, and $|a_{\lambda}^\rho(k, \gamma)| \neq 1$ for imaginary $\epsilon_{\lambda}^\rho(k)$. The mapping $M_{\lambda}^\rho(k, \gamma)$ acts as a local transformation, which is essential for the inheritance of topological features from the original Hermitian system in the non-Hermitian generalization.

**Bulk-boundary correspondence.**—CBBC does not always hold in non-Hermitian topological systems [91] [103] [106] [109] [111], where effective imaginary gauge field induces a non-Hermitian Aharonov-Bohm effect that invalidates the CBBC [81] [105] [109] [114]. Considering a bipartite lattice Hamiltonian [118], the gain and loss are respectively introduced in two sublattices for the proposed manner [Eq. (3)]. $\mathcal{H}$ is a 2-site non-Hermitian lattice constituted by $n$ coupled $PT$-symmetric dimers. Applying a unitary transformation, the intra dimer coupling $J$ associated with the gain and loss $\gamma$ in a $PT$-symmetric dimer changes into the asymmetric intra dimer couplings $J \pm \gamma$ (see Appendix B), which appears as inter sublattice couplings; and the effective imaginary gauge field is absent along the translational invariant direction of the sublattice, the non-Hermitian Aharonov-Bohm effect does not occur, and the CBBC holds.

Alternatively, the validity of CBBC can be straightforwardly understood from the mapping between the original Hermitian topological system and the generalized non-Hermitian topological system. Although the energy bands are tightened [Eq. (3)] after introduced the non-Hermiticity, the band structures and their topologies are unchanged. The mapping matrix [Eq. (5)] retains the profile of the eigenstates; the Dirac probability distribution of the eigenstates inside each sublattice is unchanged after mapping [Eq. (4)]. The CBBC is valid for the non-Hermitian system $\mathcal{H}$ and does not require the symmetry protection, this differs from that in Ref. [114].

**Mapping of symmetry classes.**—In the ten Altland-Zirnbauer classes [120], topological systems with chiral symmetry include five symmetry classes and satisfy $SH(k)S^{-1} = -H(k)$ [118], where the chiral operator is $S = \sigma_z \otimes I_n = S^{-1}$. The symmetry class AII does not have additional discrete symmetries, only a combined time-reversal ($T$) and particle-hole ($C$) symmetry $S = TC$ is present. The symmetry classes BDI, CII, CI, and DIII have additional time-reversal and particle-hole symmetries under $TH(-k)T^{-1} = H(k)$ and $CH(-k)C^{-1} = -H(k)$. After introducing the non-Hermiticity, the chiral symmetry vanishes in non-Hermitian topological systems $\mathcal{H}(k) = H(k) + i\gamma \sigma_z \otimes I_n$.

From $S = TC$, we obtain $T^{-1} = CS^{-1} = CS$. Then we have $T(i\gamma S)T^{-1} = -i\gamma T(TC)(CS) = -i\gamma T^2CS$. From $T = SC^{-1}$, we obtain $C(i\gamma S)C^{-1} = -i\gamma T^2C^{-1} = -i\gamma T^2(CS) = -i\gamma T^2CS$. Thus under the action of time-reversal and particle-hole operators, the non-Hermitian term $i\gamma (\sigma_z \otimes I_n)$ is $i\gamma S$ satisfies

$$T(i\gamma S)T^{-1} = C(i\gamma S)C^{-1} = -i\gamma T^2C^2S. \quad (6)$$

After the non-Hermitian generalization, the symmetry class AII $H(k)$ changes to symmetry class A $\mathcal{H}(k)$. The chiral orthogonal (BDI) class has $T^2C^2S = (+1)(+1)S$. Thus,

$$T\mathcal{H}(k)T^{-1} = H(-k) - i\gamma (\sigma_z \otimes I_n) \neq \mathcal{H}(-k), \quad (7)$$

but

$$\mathcal{C}\mathcal{H}(k)\mathcal{C}^{-1} = -H(-k) - i\gamma (\sigma_z \otimes I_n) = -\mathcal{H}(-k). \quad (8)$$

The time-reversal symmetry breaks, but the particle-hole symmetry holds for the non-Hermitian topological systems; the symmetry class BDI is mapped to the symmetry class D. The chiral symplectic (CII) class has $T^2C^2S = (-1)(-1)S$; similarly, only the particle-hole
symmetry holds \( \mathcal{H}(-k)C^{-1} = -\mathcal{H}(k) \) and the symmetry class CII is mapped to the symmetry class C. For the other two symmetry classes with chiral symmetry, the symmetry class CI has \( \mathcal{T}^2\mathcal{C}^2S = (+1)(-1)S \) and the symmetry class DIII has \( \mathcal{T}^2\mathcal{C}^2S = (-1)(+1)S \); both two classes satisfy

\[
\mathcal{T}\mathcal{H}(k)\mathcal{T}^{-1} = \mathcal{H}(-k) + i\gamma(\sigma_z \otimes I_n) = \mathcal{H}(-k),
\]

but

\[
\mathcal{C}\mathcal{H}(k)\mathcal{C}^{-1} = -\mathcal{H}(-k) + i\gamma(\sigma_z \otimes I_n) \neq -\mathcal{H}(-k).
\]

The time-reversal symmetry holds, but the particle-hole symmetry breaks in the non-Hermitian generalization. The mappings of symmetry classes are CI \( \rightarrow \) AI and DIII \( \rightarrow \) AII. In summary, the introduced non-Hermiticity breaks the chiral symmetry and one of the time-reversal and particle-hole symmetries; the five symmetry classes with chiral symmetry shift to the other five symmetry classes without chiral symmetry

AIII \( \rightarrow \) A, BDI \( \rightarrow \) D, CII \( \rightarrow \) C, CI \( \rightarrow \) AI, DIII \( \rightarrow \) AII.

\[ (11) \]

\textbf{Chern number in 2D systems.} — Considering a 2D topological system, the (first) Chern numbers for each band of the two Hamiltonians \( H(k) \) and \( \mathcal{H}(k) \) are exactly identical. In the absence of EPs, the energy bands are separable, the four types of Chern numbers defined under the right and left eigenstates of \( \mathcal{H}(k) \) are identical \[61, 65\] (see Appendix C). For separable bands of the Hermitian Hamiltonian, the bands of non-Hermitian Hamiltonian are “separable” in practice even if the energy bands merge in the presence of EPs. To see that the Chern number is a topological invariant and does not change in the mapping, we employ the conventional definition: unlike the lack of biorthonormal basis at EPs \[121\], whose biorthonormal probability vanishes at the exceptional point \( k \) for certain bands, the Berry connections \( A_{\lambda}^\kappa \equiv i\langle \phi_\lambda^\kappa(k) | \nabla_k | \phi_\lambda^\kappa(k) \rangle \) for \( H(k) \) and \( \tilde{A}_{\lambda}^\kappa \equiv i\langle \phi_\lambda^\kappa(k) | \nabla_k | \phi_\lambda^\kappa(k) \rangle \) for \( \mathcal{H}(k) \) based on the right eigenstates are always well-defined. The nabla operator is \( \nabla_k = \left( \partial_{k_x}, \partial_{k_y} \right) \).

Direct derivation yields

\[
\tilde{A}_{\lambda}^\kappa = A_{\lambda}^\kappa - (1/2) \nabla_k \vartheta \left[ \tilde{A}_{\lambda}^\kappa = \tilde{A}_{\lambda}^\kappa + \langle \varphi_\lambda^\kappa(k) | \nabla_k | \varphi_\lambda^\kappa(k) \rangle e_\delta^\kappa(k)/i(\gamma/\epsilon) \right]
\]

for real (imaginary) spectrum, in which \( \vartheta = \text{arctan}[\gamma/\epsilon]^\kappa(k) \); the relation is gauge dependent (see Appendix C), however, the Berry curvatures are gauge independent \( B_{\lambda}^\kappa = \nabla_k \times A_{\lambda}^\kappa \), \( \tilde{B}_{\lambda}^\kappa = \nabla_k \times \tilde{A}_{\lambda}^\kappa \), and obey \( B_{\lambda}^\kappa = \tilde{B}_{\lambda}^\kappa \) (\( B_{\lambda}^\kappa \neq \tilde{B}_{\lambda}^\kappa \)) for real (imaginary) spectrum; and the contribution of the later term in \( \tilde{A}_{\lambda}^\kappa \) for the Chern number \( c_\delta^\kappa \) is zero.

\[
c_\delta^\kappa = \frac{1}{2\pi} \oint B_{\lambda}^\kappa d^2k = \frac{1}{2\pi} \oint \tilde{B}_{\lambda}^\kappa d^2k.
\]

This is referred to as the topological invariant mapping. Notably, the mapping Eq. \[1\] is directly applicable to the edge states. These conclusions are not relevant to the reality of energy bands and the presence of EPs. The Chern numbers for each energy band of both systems are identical even if the bands merge in the presence of EPs (see Appendix C).

Ultracold atomic gases \[122, 123\], acoustic lattices \[124, 125\], electrical circuits \[126, 128\], and various microwave, optical, and photonic systems \[129, 131\] have become fertile platforms for studying topological phase of matter. Through introducing additional losses, passive non-Hermitian topological systems are created \[74, 75\]: the properties of \( P \mathcal{T} \)-symmetric systems with balanced gain and loss are exacted from the passive systems by shifting a common loss rate. Nowadays, the non-Hermitian topological systems are experimentally realized via sticking absorbers in the dielectric resonator array \[83\], cutting the waveguides in coupled optical waveguide lattice \[75\], and fabricating the radiative loss in open systems of photonic crystals \[54, 74\]. Active elements are required to realize robust topological edge state lasing \[98, 103\], where external pumping is implemented to acquire the gain. The prototypical non-Hermitian topological system is the 1D complex Su–Schrieffer–Heeger (SSH) model \[52\]; here we consider a simple extension \[98–103\], where external pumping is implemented to create \[74, 75\]; the properties of non-Hermitian topological systems are created \[74, 75\]. The prototypical non-Hermitian topological system is the 1D complex Su–Schrieffer–Heeger (SSH) model \[52\]; here we consider a simple extension \[98–103\], where external pumping is implemented to create \[74, 75\]; the properties of non-Hermitian topological systems are created \[74, 75\].
Fig. 1. (a) Schematic of the comb lattice. Numerical simulations of $Q_N(t)$ for the band $(\rho, \lambda) = (\rho, \lambda)$ in two quasi-adiabatic processes (insets) of (b) $\delta_0 = 0$ and (c) $\delta_0 = 1$ at $\gamma = 0.5$, $\omega = 10^{-3}$, and $T = 2\pi/\omega$. Real (imaginary) part of $Q_N(t)$ is in black (red); the corresponding $j_N(t)$ is shown in Appendix E.

equals to the winding number of loop around the band touching point $(\delta, \kappa) = (0, 0)$ in the parameter space.

The biorthonormal current \[ j_N(t) = -i \sum_{m=1}^{N} \langle \eta_m(t) | [J | 1 - \delta(t) | a_N^\dagger b_{N-1}^\dagger \text{-H.c.}] | \varphi_m(t) \rangle, \] (15)

where $m$ is the number of energy levels in the concerned energy band for the $4N$ size system with periodic boundary condition. The parameters vary as $\theta = \omega t$ in the numerical simulation under a quasi-adiabatic process, where the speed of time evolution $\omega \ll 1$, and $t$ varies from 0 to a period of $T = 2\pi/\omega$. To demonstrate a quasi-adiabatic process, we keep $f(t) = [| \tilde{\eta}_m(t) \rangle | \varphi_m(t) \rangle] \to 1$ during the whole process by taking sufficient small $\omega$, where $| \tilde{\eta}_m(t) \rangle$ is the corresponding instantaneous eigenstate of $\hat{H}_L(t)$. For the given initial eigenstates $| \varphi_m(t) \rangle = | \varphi_{-m} \rangle$ and $| \eta_m(0) \rangle = | \eta_{-m} \rangle$, the time evolved states are $| \varphi_m(t) \rangle = T_t \exp[-i \int_0^t \hat{H}(t') dt'] | \varphi_m(t) \rangle$ and $| \eta_m(t) \rangle = T_t \exp[-i \int_0^t \hat{H}_L(t') dt'] | \eta_m(t) \rangle$, where $T_t$ is the time ordering operator and $\hat{H}(t)$ is the Hamiltonian in the real space. The accumulated charge pumping \[ j_N(t) \] passing the dimer $a_N b_{N-1}$ during the interval $t$ is

\[ Q_N(t) = \int_0^t j_N(t') dt'. \] (16)

The topological charge pumping favorably agrees with the Chern number $c_\omega = 1$ [Fig. 1(b)] in the nontrivial phase or $c_\omega = 0$ [Fig. 1(c)] in the trivial phase for real energy band as that in the Hermitian topological systems 136,138. For imaginary energy band without EPs, the amplitude of evolved states $| \eta_m(t) \rangle$ and $| \varphi_m(t) \rangle$ exponentially increase (or decrease); performing the quantization of transport in a counterpart Hamiltonian $\hat{H}'(t) = i \hat{H}(t)$ with corresponding real energy band is feasible to verify the Chern number and the topological properties of imaginary energy band of $\hat{H}(t)$, since $\hat{H}'(t)$ has identical topology and eigenstate with $\hat{H}(t)$.

Alternatively, the topological charge pumping can be retrieved from the dynamical evolution of edge states in the edge Hamiltonian $\hat{H}_{edge}$ (see Appendix E), which is generated by truncating a coupling $J(1 + \delta)$ at the lattice boundary of the bulk Hamiltonian $\hat{H}$ in the real space [Fig. 1(a)]. $\hat{H}_{edge}$ and $\hat{H}_{edge}$ meet the condition of mapping since Eq. (2) still holds. Two pairs of edge states exist

\[ | \varphi_L^\pm \rangle = \frac{1}{\sqrt{2}} \sum_{j=1}^{N} \zeta^{N-j} (e^{\pm i\theta} a_{2j}^\dagger \pm b_{2j}^\dagger) | \text{vac} \rangle, \] (17)

\[ | \varphi_R^\pm \rangle = \frac{1}{\sqrt{2}} \sum_{j=1}^{N} \zeta^{j-1} (e^{\pm i\theta} a_{2j-1}^\dagger \pm b_{2j-1}^\dagger) | \text{vac} \rangle, \] (18)

associated with the energies $\epsilon_{R}^\pm = (\kappa_{\omega}^2 - \gamma^2)^{1/2}$ and $\epsilon_{L}^\pm = (\kappa_{\omega}^2 - \gamma^2)^{1/2}$, respectively; where $\zeta = (\delta - 1)/(\delta + 1)$, $\Omega = 2(1 - \zeta^2)/(1 - \zeta^2)$, $e^{i\theta} = (\epsilon_{R}^0 \pm i\gamma)/\kappa_+$ and $\epsilon^{\pm i\theta} = (\epsilon_{L}^0 \pm i\gamma)/\kappa_-$. The explicit expressions of edge states reveal a fact that the mapping matrix only changes the local phase or amplitude. For real $\epsilon_{R/L}^\pm$, the edge state profiles are independent of $\gamma$ and $\kappa_\pm$ similar as that in Hermitian \[134\] and non-Hermitian \[69\] Rice-Mele models; for imaginary $\epsilon_{R/L}^\pm$, the probability becomes dense in the sublattice with gain (loss) for $| \varphi_L^+ \rangle$ ($| \varphi_R^- \rangle$). The topological charge pumping of an edge state for a loop $L$ in the $\kappa$-$\delta$ plane equals to the Chern number \[139\] (see Appendix E). The energy bands are gapped and real at $\gamma = 0$; as $\gamma$ increasing, imaginary energy levels appear and non-Hermitian phase transition occurs. In Figs. 2(a) and 2(b), the energy bands are depicted at weak and strong $\gamma$, respectively. The not shown imaginary part for real band is zero and vice versa. The edge states retain although energy bands become imaginary. Recently, we notice an experimental work that reported the existence of topological edge states in both unbroken and broken $\mathcal{PT}$-symmetric phases \[130\].
Discussion and conclusion.—For chiral symmetric systems not in the form of a bipartite lattice \[105\], we can first apply a unitary transformation to get the block off-diagonal form Hamiltonian [Eq. (1)]; then, introduce the non-Hermiticity [Eq. (2)]; after the inversion unitary transformation, a non-Hermitian system possessing identical topology to the chiral symmetric Hermitian system is generated. Notably, the mapping theory is applicable for \( H(q) \) instead of the core matrix \( H(k) \), where \( q \) is a set of periodic parameters instead of the momentum \( k \). In addition to the gapped topological systems, the non-Hermitian generalizations are applicable for gapless topological systems \[41 - 44, 141 - 145\]. After introducing the non-Hermiticity in the proposed manner, the gapless degeneracy points may change into pairs of EPs, EP rings, or EP surfaces \[35, 37, 51, 75\]; although the non-Hermitian phase transition occurs, the topology remains unchanged and can be characterized by winding number as indicated in Refs. \[47, 51, 72\].

Our findings provide insights into the interplay between non-Hermiticity and topology. In contrast to the topological phase transition induced by the non-Hermiticity, we propose the non-Hermitian generalization that completely retains the topological phase transition and the (non)existence of edge states in the original chiral symmetric Hermitian systems; and the non-Hermitian phase transition does not alter or destroy the original topology. This dramatically differs from the non-Hermiticity induced topological phase transition \[92 - 117\], differs from the breakdown of CBBC induced by gain and loss or non-Hermitian asymmetric coupling \[32, 39, 41 - 105, 112\], and differs from the situation that the strong non-Hermiticity destroys the topological edge states due to the non-Hermitian phase transition associated with the appearance of band touching EPs \[114\].

The topological phase transition and the non-Hermitian phase transition are independent and separately controllable. This unique feature is valuable for the explorations of novel non-Hermitian topological phases and topologically protected edge state lasing.

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APPENDIX

A. Mapping matrix

The Schrödinger equation for the original Hermitian Hamiltonian \( H(k) \) is \( H(k) |\varphi^\rho_{\lambda}(k)\rangle = \varepsilon^\rho_{\lambda}(k) |\varphi^\rho_{\lambda}(k)\rangle \). \( H(k) \) is the parent Hamiltonian in the non-Hermitian generalization, and has the chiral symmetry,

\[
H(k) = \begin{pmatrix} 0 & D(k) \\ D^\dagger(k) & 0 \end{pmatrix}, \tag{A1}
\]

where we consider \( D(k) \) as an arbitrary \( n \times n \) matrix.

The eigenstates of \( H(k) = H(k) + i\gamma (\sigma_3 \otimes I_n) \) are

\[
|\varphi^\rho_{\lambda}(k)\rangle = M^\rho_{\lambda}(k, \gamma) |\phi^\rho_{\lambda}(k)\rangle, \tag{A2}
\]

with eigenvalues

\[
\varepsilon^\rho_{\lambda}(k) = \rho \sqrt{|\varepsilon^\rho_{\lambda}(k)|^2 - \gamma^2}, \tag{A3}
\]

where the mapping matrix has the form

\[
M^\rho_{\lambda}(k, \gamma) = \begin{pmatrix} a^\rho_{\lambda}(k, \gamma) I_n & 0 \\ 0 & I_n \end{pmatrix}, \tag{A4}
\]

and \( a^\rho_{\lambda}(k, \gamma) = [\varepsilon^\rho_{\lambda}(k) + i\gamma] / \varepsilon^\rho_{\lambda}(k) \) fulfills \( |a^\rho_{\lambda}(k, \gamma)| = 1 \) for real \( \varepsilon^\rho_{\lambda}(k) \) and is pure imaginary for imaginary \( \varepsilon^\rho_{\lambda}(k) \). For real \( \varepsilon^\rho_{\lambda}(k) \), the factor \( a^\rho_{\lambda}(k, \gamma) \) can be written in the form of \( a^\rho_{\lambda}(k, \gamma) = e^{i\vartheta} \), with \( \vartheta = \arctan[\gamma / \varepsilon^\rho_{\lambda}(k)] \).

Notice that,

\[
H(k) M^\rho_{\lambda}(k, \gamma) = \begin{pmatrix} I_n & 0 \\ 0 & a^\rho_{\lambda}(k, \gamma) I_n \end{pmatrix} H(k), \tag{A5}
\]

we have \( H(k) |\varphi^\rho_{\lambda}(k)\rangle = \left[H(k) M^\rho_{\lambda}(k, \gamma)\right] |\phi^\rho_{\lambda}(k)\rangle \), then,

\[
H(k) |\varphi^\rho_{\lambda}(k)\rangle = \varepsilon^\rho_{\lambda}(k) \begin{pmatrix} I_n & 0 \\ 0 & a^\rho_{\lambda}(k, \gamma) I_n \end{pmatrix} |\phi^\rho_{\lambda}(k)\rangle = \varepsilon^\rho_{\lambda}(k) \begin{pmatrix} 0 & a^\rho_{\lambda}(k, \gamma) \end{pmatrix} |\phi^\rho_{\lambda}(k)\rangle, \tag{A6}
\]

therefore, we obtain

\[
H(k) |\varphi^\rho_{\lambda}(k)\rangle = \left[H(k) + i\gamma \sigma_3 \otimes I_n\right] |\varphi^\rho_{\lambda}(k)\rangle = \left(\begin{pmatrix} \varepsilon^\rho_{\lambda}(k) & a^\rho_{\lambda}(k, \gamma) \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & a^\rho_{\lambda}(k, \gamma) \end{pmatrix} - i\gamma \right) |\varphi^\rho_{\lambda}(k)\rangle. \tag{A7}
\]
From \( a_\gamma^e(k) = [\epsilon_\gamma^e(k) + i\gamma] / \epsilon_\gamma^e(k) \) and \( e_\gamma^e(k) = \rho \sqrt{|\epsilon_\gamma^e(k)|^2 - \gamma^2} \), we have \( \epsilon_\gamma^e(k) [a_\gamma^e(k, \gamma)]^{-1} + i\gamma = e_\gamma^e(k) a_\gamma^e(k, \gamma) - i\gamma. \) Thus,\[ \mathcal{H}(k) |\varphi_\gamma^e(k)\rangle = e_\gamma^e(k) |\varphi_\gamma^e(k)\rangle. \] (A8)

In parallel, the eigenstate of \( \mathcal{H}^\dagger(k) \) is given by \( |\eta_\gamma^e(k)\rangle = [M_\gamma^e(k, \gamma)]^\dagger |\varphi_\gamma^e(k)\rangle \) with eigenvalue \( [\epsilon_\gamma^e(k)]^* \).

**B. Unitary transformation**

Figure [B1](a) depicts a one-dimensional (1D) bipartite lattice. The lines are the couplings between sublattices \( A \) and \( B \). Each pair of upper and lower sites constitute a dimer. For a dimer with coupling \( J \) and balanced gain and loss \( \pm i\gamma \), the dimer is \( \mathcal{PT} \)-symmetric described by \( J\sigma_x + i\gamma\sigma_z \). Applying a unitary transformation

\[ U = (I_2 + i\sigma_x) / \sqrt{2}, \] (B1)

we obtain a non-Hermitian dimer with asymmetric couplings \( J \pm \gamma \) in the form of

\[ U (J\sigma_x + i\gamma\sigma_z) U^{-1} = J\sigma_x + i\gamma\sigma_y. \] (B2)

Similarly, the lattice in Figure [B1](a) changes into the lattice in Figure [B1](b) with asymmetric inter sublattice couplings. The gain and loss change into asymmetric intra dimer couplings (vertical arrows); the inter dimer couplings (slant lines) change to the Hermitian couplings including the inter sublattice reciprocal cross-stitch couplings, and the intra sublattice nonreciprocal couplings with symmetric amplitude (horizontal arrows). The nonreciprocal couplings vanish if inter sublattice couplings in the Hermitian system \( H \) are symmetric (the situation that the dashed and solid slant lines are identical). The imaginary gauge field is created at \( J\gamma \neq 0 \) along the vertical direction, but not along the horizontal direction (translational invariant direction); thus, non-Hermitian AB effect is absent and the bulk-boundary correspondence is valid. The conclusion is applicable in a general situation for systems with nonreciprocal couplings and for higher dimensional bipartite lattices.

In a general case, the topological system may have complex coupling. For a nonreciprocal coupling \( J e^{\pm i\phi} \) with Peierls phase \( e^{\pm i\phi} \) and coupling amplitude \( J \), the \( \mathcal{PT} \)-symmetric dimer changes to

\[ U \begin{pmatrix} i\gamma & J e^{-i\phi} \\ J e^{i\phi} & -i\gamma \end{pmatrix} U^{-1} = \begin{pmatrix} 0 & e^{-i\phi} (J + \gamma) \\ e^{i\phi} (J - \gamma) & 0 \end{pmatrix}, \] (B3)

where the coupling with symmetric amplitude is changed into coupling with asymmetric amplitude \( J + \gamma \) and \( J - \gamma \) associated with nonreciprocal Peierls phase \( e^{-i\phi} \) and \( e^{i\phi} \), respectively. The unitary transformation \( U \) applied is

\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & ie^{-i\phi} \\ ie^{i\phi} & 1 \end{pmatrix}. \] (B4)

In two-dimensional topological systems with chiral symmetry, for instance, a two layer system with inter layer couplings as shown in Figure [B1](c), which is a typical bipartite lattice. The non-Hermitian extension is to introduce gain and loss in the upper and lower layers, respectively. Then, the unitary transformation \( U \) applied to each corresponding upper and lower sites yields a new two layer lattice as shown in Figure [B1](d), the asymmetric couplings only exist between the new two layers (sublattices) after unitary transformation. For higher dimensional systems, the asymmetric couplings still only exist between the two new sublattices after the unitary transformation, which is similar as the one-dimensional and two-dimensional cases. Thus, the nonzero Aharonov-Bohm effect is absent in any translational direction of the topological systems, and the conventional bulk-boundary correspondence holds in the non-Hermitian generalization. The conclusion coincides with that of the mapping theory.

**C. Mapping of geometric phase and Chern number**

In this section, we show that the Berry connection, Berry curvature and Chern number of the non-Hermitian Hamiltonian in the momentum space \( H(k) = H(k) + i\gamma (\sigma_i \otimes I_n) \) can be mapped from the Hermitian Hamiltonian \( H(k) \) with chiral symmetry by using the mapping matrix. We prove that two topological systems \( H(k) \) and \( \mathcal{H}(k) \) share an identical Chern number, although their Berry connection and Berry curvature are different. The conclusion is independent of the presence of exceptional points (EPs) in the energy bands.

For the chiral symmetric system \( H(k) \), we have

\[ S H(k) S^{-1} = -H(k), \] (C1)

with \( S = (\sigma_z \otimes I_n) \). Then, from \( H(k) |\phi_\lambda^e(k)\rangle = \epsilon_\lambda^e(k) |\phi_\lambda^e(k)\rangle \), we have \( S H(k) S^{-1} S |\phi_\lambda^e(k)\rangle = \epsilon_\lambda^e(k) S |\phi_\lambda^e(k)\rangle \), which gives \( H(k) [S |\phi_\lambda^e(k)\rangle] = -\epsilon_\lambda^e(k) [S |\phi_\lambda^e(k)\rangle] \). Thus,

\[ |\phi_\lambda^e(k)\rangle = S |\phi_\lambda^e(k)\rangle, \] (C2)

is the eigenstate for energy \( \epsilon_\lambda^e(k) = -\epsilon_\lambda^e(k) \).

We introduce the conventional Berry connection and Berry curvature, which are called the RR Berry connection and Berry curvature \[ \text{[63].} \] The RR Berry connection for non-Hermitian system \( \mathcal{H}(k) \) is

\[ \left( \mathbf{A}_{\lambda}^\rho \right)_{RR} = i \langle \varphi_\lambda^e(k) | \nabla_k |\phi_\lambda^e(k)\rangle, \] (C3)

in which

\[ |\varphi_\lambda^e(k)\rangle = M_\lambda^e(k, \gamma) |\phi_\lambda^e(k)\rangle, \] (C4)

and the normalization condition is satisfied,

\[ \langle \varphi_\lambda^e(k) | \varphi_\lambda^e(k) \rangle = 1 \] under the mapping matrix

\[ M_\lambda^e(k, \gamma) = \Pi_\lambda^e(k, \gamma) \left( \begin{array}{cc} a_\lambda^e(k, \gamma) I_n & 0 \\ 0 & I_n \end{array} \right), \] (C5)
The definition of RR Berry connection is independent of the biorthonormal basis. Although the eigenstates with $\Pi_k^e(k, \gamma) = 1$ for real (imaginary) $\epsilon^e_k(k)$ to guarantee the normalization condition. Then the RR Berry connection can be written as

$$\left( \mathcal{A}_k^e \right)_{\text{RR}} = i \langle \phi^e_k(k) | [M^e_k(k, \gamma)]^\dagger \nabla_k [M^e_k(k, \gamma) | \phi^e_k(k)] \rangle$$

and

$$[M^e_k(k, \gamma)]^\dagger \nabla_k M^e_k(k, \gamma) = \begin{pmatrix} e^{-i\theta} I_n & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} e^{i\theta} I_n & 0 \\ 0 & I_n \end{pmatrix} = I_{2n}, \quad (C7)$$

Thus, the RR Berry connection is

$$\left( \mathcal{A}_k^e \right)_{\text{RR}} = \mathbf{A}_k^e = \frac{1}{2} \nabla_k \theta, \quad (C9)$$

For imaginary $\epsilon^e_k(k)$, we have $[a^e_k(k, \gamma)]^* = -a^e_k(k, \gamma)$,

$$\left[ M^e_k(k, \gamma) \right]^* M^e_k(k, \gamma) \nabla_k - I_{2n} \nabla_k$$

$$= 1 + [a^e_k(k, \gamma)]^2 \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \nabla_k$$

$$= \frac{\epsilon^e_k(k)}{i\gamma} - i \gamma \nabla_k, \quad (C10)$$

and

$$[M^e_k(k, \gamma)]^\dagger [\nabla_k M^e_k(k, \gamma)] = [\Pi^e_k(k, \gamma)]^* \begin{pmatrix} [a^e_k(k, \gamma)]^* I_n & 0 \\ 0 & I_n \end{pmatrix} \nabla_k \begin{pmatrix} a^e_k(k, \gamma) & 0 \\ 0 & \Pi^e_k(k, \gamma) I_n \end{pmatrix} = -\frac{1}{2} a^e_k(k, \gamma) [\Pi^e_k(k, \gamma)]^4 [\nabla_k a^e_k(k, \gamma)] \mathcal{S}. \quad (C11)$$

Then, the RR Berry connection

$$\left( \mathcal{A}_k^e \right)_{\text{RR}} = i \langle \phi^e_k(k) | \nabla_k | \phi^e_k(k) \rangle + \frac{\epsilon^e_k(k)}{i\gamma} \langle \phi^e_k(k) | \nabla_k | \phi^e_k(k) \rangle - \frac{1}{2} ia^e_k(k, \gamma) [\Pi^e_k(k, \gamma)]^4 [\nabla_k a^e_k(k, \gamma)] \langle \phi^e_k(k) | \phi^e_k(k) \rangle$$

$$= \mathbf{A}_k^e + \frac{\epsilon^e_k(k)}{i\gamma} \langle \phi^e_k(k) | \nabla_k | \phi^e_k(k) \rangle. \quad (C12)$$

The definition of RR Berry connection is independent of the biorthonormal basis. Although the eigenstates...
\[ |\varphi_\lambda^\rho(k)\rangle^\text{EP} \text{ and } |\varphi_\lambda^{-\rho}(k)\rangle^\text{EP} \text{ coalesce at EPs and the biorthonormal basis is absent, the RR Berry connection can still be defined. At EPs, the energy is } \epsilon_\lambda^\rho(k)^\text{EP} = 0, \text{ and the mapping matrix has a simple form} \]
\[
\left[ M_\lambda^\rho(k, \gamma) \right]^\text{EP} = \begin{pmatrix} \tilde{p}_\lambda & 0 \\ 0 & I_n \end{pmatrix}. \tag{C13}
\]

Direct derivation shows that the RR Berry connection at EPs is \( \left( \mathcal{A}_\lambda^\rho \right)^\text{RR} = i \langle \varphi_\lambda^\rho(k) | \nabla_k | \varphi_\lambda^\rho(k) \rangle = A_\lambda^\rho. \)

In conclusion, we have
\[
\left( \mathcal{A}_\lambda^\rho \right)^\text{RR} = \begin{cases} 
A_\lambda^\rho - \frac{i}{2} \nabla_k \vartheta, & \text{real } \epsilon_\lambda^\rho(k), \\
\tilde{A}_\lambda^\rho + \left[ N_\lambda^\rho(k) \right]_{\text{RR}}^{-1}, & \text{imaginary } \epsilon_\lambda^\rho(k), \\
A_\lambda^\rho, & \text{at EPs}
\end{cases}
\]

where \( \left[ N_\lambda^\rho(k) \right]_{\text{RR}} = \epsilon_\lambda^\rho(k) \langle \varphi_\lambda^\rho(k) | \nabla_k | \varphi_\lambda^\rho(k) \rangle / (i\gamma). \)

The RR Berry connection \( \left( \mathcal{A}_\lambda^\rho \right)^\text{RR} \) is gauge dependent. If we take the gauge transformation \( |\varphi_\lambda^\rho(k)\rangle \rightarrow e^{i\chi(k)} |\varphi_\lambda^\rho(k)\rangle \) with real \( \chi(k) \), then we have an additional term \( i \nabla_k \chi(k) \) in \( (\mathcal{A}_\lambda)_{\text{RR}} \). However, \( (\mathcal{B}_\lambda^\rho)_{\text{RR}} = \nabla_k \times (\mathcal{A}_\lambda)_{\text{RR}} \) is gauge independent.

For real \( \epsilon_\lambda^\rho(k) \), we have \( (\mathcal{B}_\lambda^\rho)_{\text{RR}} = \nabla_k \times (\mathcal{A}_\lambda)_{\text{RR}} = \nabla_k 
\times (A_\lambda^\rho - \frac{i}{2} \nabla_k \vartheta) = \nabla_k \times A_\lambda^\rho = B_\lambda^\rho. \)

For imaginary \( \epsilon_\lambda^\rho(k) \), the additional term \( \nabla_k \times [N_\lambda^\rho(k)]_{\text{RR}} \) yields zero integration over the Brillouin zone. We prove this as follows. Notice that the eigenstates \( |\varphi_\lambda^\rho(k)\rangle \) and \( |\varphi_\lambda^{-\rho}(k)\rangle \) can choose an identical gauge in the same region of the Brillouin zone; it is because that the two eigenstates are related through the chiral operator \( \mathcal{S} \), and they are orthogonal \( \langle \varphi_\lambda^{-\rho}(k) | \varphi_\lambda^\rho(k) \rangle = 0 \). If we take a gauge transformation
\[
|\varphi_\lambda^\rho(k)\rangle^1 = e^{i\chi(k)} |\varphi_\lambda^\rho(k)\rangle^\text{II}, \tag{C15}
\]
then, we have
\[
|\varphi_\lambda^{-\rho}(k)\rangle^1 = \mathcal{S} |\varphi_\lambda^\rho(k)\rangle^1 = \mathcal{S} e^{i\chi(k)} |\varphi_\lambda^\rho(k)\rangle^\text{II} \tag{C16}
\]

Unlike the Berry connection, term \( \left[ N_\lambda^\rho(k) \right]_{\text{RR}} \) is gauge independent
\[
\left[ N_\lambda^\rho(k) \right]_{\text{RR}}^\text{II} = \epsilon_\lambda^\rho(k) \langle \varphi_\lambda^{-\rho}(k) | i \nabla_k | \varphi_\lambda^\rho(k) \rangle^1 / (i\gamma) = \epsilon_\lambda^\rho(k) e^{-i\chi(k)} \langle \varphi_\lambda^{-\rho}(k) | i \nabla_k | \varphi_\lambda^\rho(k) \rangle^\text{II} / (i\gamma) \tag{C17}
\]
\[
= \epsilon_\lambda^\rho(k) e^{-i\chi(k)} \left[ i \nabla_k e^{i\chi(k)} \right] \langle \varphi_\lambda^{-\rho}(k) | \varphi_\lambda^\rho(k) \rangle^\text{II} / (i\gamma) \tag{C18}
\]
One can use different gauges to define the eigenstate if the eigenstate under one gauge is not well-defined in certain regions of the Brillouin zone. We consider a case that the eigenstate of the concerned energy band is well-defined under gauge I in the region \( D_1 \) and under gauge II in the rest region \( D_2 \) (the cases with more than two gauges required can be similarly generalized). Applying Stokes theorem, we have
\[
\oint_{\text{BZ}} \nabla_k \times \left[ N_\lambda^\rho(k) \right]_{\text{RR}}^\text{II} \, dk = \oint_{\partial D_1} \left[ N_\lambda^\rho(k) \right]_{\text{RR}}^\text{II} \, dk + \oint_{\partial D_2} \left[ N_\lambda^\rho(k) \right]_{\text{RR}}^\text{II} \, dk = 0. \tag{C19}
\]

The chiral symmetry of the Hermitian system \( H(k) \) plays a crucial role to obtain the above conclusions, the chiral symmetry makes term \( \left[ N_\lambda^\rho(k) \right]_{\text{RR}} \) gauge independent, thus, \( \left[ N_\lambda^\rho(k) \right]_{\text{RR}} \) does not contribute to the Chern number. Based on the above analysis, the RR Chern number of the non-Hermitian system \( H(k) \) is exactly identical to the Chern number of the Hermitian system \( H(k) \), even though there EPs and the energy bands are inseparable (the energy bands of the corresponding Hermitian system are separable, and the Chern number is well defined)
\[
\langle \epsilon_\lambda^\rho \rangle_{\text{RR}} = \frac{1}{2\pi} \oint_{\text{BZ}} \left( \mathcal{B}_\lambda^\rho \right)_{\text{RR}} \, d^2k = \frac{1}{2\pi} \oint_{\text{BZ}} \mathcal{B}_\lambda^\rho \, d^2k. \tag{C21}
\]

Now, we discuss the LR Berry connection and Berry curvature based on the biorthonormal basis [60]. \( |\varphi_\lambda^\rho(k)\rangle \) and \( |\eta_\lambda^\rho(k)\rangle \) are the eigenstates of \( H(k) \) and \( H^\dagger(k) \) with eigenvalues \( \epsilon_\lambda^\rho(k) \) and \( \epsilon_\lambda^{-\rho}(k)^* \), respectively. The Schrödinger equations are
\[
H(k) |\varphi_\lambda^\rho(k)\rangle = \epsilon_\lambda^\rho(k) |\varphi_\lambda^\rho(k)\rangle, \tag{C20}
\]
\[
H^\dagger(k) |\eta_\lambda^\rho(k)\rangle = \epsilon_\lambda^{-\rho}(k)^* |\eta_\lambda^\rho(k)\rangle. \tag{C21}
\]

The eigenstates \( \{ |\varphi_\lambda^\rho(k)\rangle, |\eta_\lambda^\rho(k)\rangle \} \) can be mapped from the eigenstates of the original Hermitian system \( H(k) \),
\[
|\varphi_\lambda^\rho(k)\rangle = M_\lambda^\rho(k, \gamma) |\phi_\lambda^\rho(k)\rangle, \tag{C22}
\]
\[
|\eta_\lambda^\rho(k)\rangle = \left[ M_\lambda^\rho(k, \gamma) \right]^\dagger |\phi_\lambda^\rho(k)\rangle. \tag{C23}
\]

In the absence of EP, the mapping matrix can be written as
\[
M_\lambda^\rho(k, \gamma) = \lambda^\rho_\lambda(k, \gamma) \begin{pmatrix} a_\lambda^\rho(k, \gamma) I_n & 0 \\ 0 & I_n \end{pmatrix}, \tag{C24}
\]
with \( \lambda^\rho_\lambda(k, \gamma) = \sqrt{2/[1 + \langle a_\lambda^\rho(k, \gamma) \rangle^2]} \) to guarantee the biorthonormal normalization.

Based on the orthonormal relation \( \langle \phi_\lambda^\rho(k) | \phi_\lambda^{-\rho}(k') \rangle = \delta_{kk'} \delta_{\lambda\lambda'} \delta_{\rho\rho'} \) for the eigenstates of the original Hermitian system, we have the biorthonormal relation
\[
\langle \eta_\lambda^\rho(k) | \phi_\lambda^{-\rho}(k') \rangle = \delta_{kk'} \delta_{\lambda\lambda'} \delta_{\rho\rho'}, \tag{C25}
\]
for the left and right eigenstates of the non-Hermitian system.

The Berry connection based on the left and right eigenstates is defined as
\[
\left(\vec{A}_\lambda^\rho\right)_{\text{LR}} = i \langle \phi_\lambda^\rho(k) | \nabla_k | \phi_\lambda^\rho(k) \rangle
\]
\[
= i \langle \phi_\lambda^\rho(k) | M_\lambda^\rho(k, \gamma) \nabla_k [M_\lambda^\rho(k, \gamma) | \phi_\lambda^\rho(k) \rangle]
\]
\[
= i \langle \phi_\lambda^\rho(k) | M_\lambda^\rho(k, \gamma) \nabla_k [M_\lambda^\rho(k, \gamma) | \phi_\lambda^\rho(k) \rangle
+i \langle \phi_\lambda^\rho(k) | M_\lambda^\rho(k, \gamma) [\nabla_k M_\lambda^\rho(k, \gamma) | \phi_\lambda^\rho(k) \rangle,
\]
where
\[
\left(\vec{A}_\lambda^\rho\right)_{\text{LR}} = \nabla_k \times \left(\vec{S}_\lambda^\rho\right)_{\text{LR}},
\]
\[
= \nabla_k \times \left[ N_\lambda^\rho(k) \right]_{\text{LR}},
\]
\[
\text{LR Berry curvature has the form}
\]
\[
\left(\vec{B}_\lambda^\rho\right)_{\text{LR}} = \nabla_k \times \left(\vec{A}_\lambda^\rho\right)_{\text{LR}}
\]
\[
= \nabla_k \times \left[ N_\lambda^\rho(k) \right]_{\text{LR}},
\]
\[
\text{Integral in Eq. (C32) is under the assumption that EPs are absent in the Brillouin zone, since the biorthonormal basis does not exist at EPs.}
\]
\[
\text{The RR and LL definitions, one can also define the RL and LL Berry connections and Berry curvatures. For the Hermitian and corresponding non-Hermitian topological systems we concern, the RL and LL Berry connections are}
\]
\[
\left(\vec{A}_\lambda^\rho\right)_{\text{RL}} = i \langle \phi_\lambda^\rho(k) | \nabla_k | \eta_\lambda^\rho(k) \rangle
\]
\[
= \left[ A_\lambda^\rho + \frac{i \gamma}{c_\lambda^\rho(k)} \langle \phi_\lambda^\rho(k) | \nabla_k | \phi_\lambda^\rho(k) \rangle \right],
\]
\[
\left(\vec{A}_\lambda^\rho\right)_{\text{LL}} = i \langle \eta_\lambda^\rho(k) | \nabla_k | \varsigma_\lambda^\rho(k) \rangle
\]
\[
= \left[ A_\lambda^\rho - \frac{i \gamma}{c_\lambda^\rho(k)} \langle \phi_\lambda^\rho(k) | \nabla_k | \phi_\lambda^\rho(k) \rangle / (i \gamma),
\]
\[
\text{real } c_\lambda^\rho(k), \text{ at EPs}
\]
\[
\text{and the four definitions of the Chern number are identical for separated bands (i.e., in the absence of EPs) [65].}
D. Details of the 1D comb lattice model

1. Model and energy bands

The non-Hermitian Hamiltonian of the one-dimensional comb lattice model reads

\[ \mathcal{H} = H + i\gamma \sum_{j=1}^{2N} (a_j^\dagger a_j - b_j^\dagger b_j), \]  

(D1)

which is generated from the Hermitian Hamiltonian

\[
H = \sum_{j=1}^{N} \left[ J (1 - \delta) a_{2j}^\dagger b_{2j-1} + J (1 + \delta) a_{2j}^\dagger b_{2j+1} \right] + \sum_{j=1}^{N} (\kappa_+ a_{2j+1}^\dagger b_{2j-1} + \kappa_- a_{2j}^\dagger b_{2j} + \text{H.c.}) \]  

(D2)

under the periodic boundary condition \( b_{2N+1} = b_1 \), and the system parameters are \( \delta = \delta_0 + R \cos \theta \) and \( k_\pm = (1/2) R \sin \theta \) (set \( \kappa \equiv k_+ - k_- = R \sin \theta \)). We refer to the Hamiltonian with periodic boundary condition as the bulk Hamiltonian, and the edge Hamiltonian is the Hamiltonian under open boundary condition. Taking the Fourier transformation

\[
\begin{pmatrix} a_{2j} \\ a_{2j-1}^\dagger \\ b_{2j-1} \\ b_{2j}^\dagger \end{pmatrix} = \frac{1}{\sqrt{N}} \sum_k e^{ikj} \begin{pmatrix} a_{k,+} \\ a_{k,-}^\dagger \\ b_{k,+}^\dagger \\ b_{k,-} \end{pmatrix}, \]

(D3)

we obtain

\[
H = \sum_k \alpha_k^\dagger \mathcal{H}(k) \alpha_k, \]

(D4)

\[
\mathcal{H} = \sum_k \alpha_k^\dagger \mathcal{H}(k) \alpha_k, \]

(D5)

where \( \alpha_k = (a_{k,+}, a_{k,-}^\dagger, b_{k,+}^\dagger, b_{k,-}) \), and the 4 x 4 matrix \( \mathcal{H}(k) \) and \( H(k) \) has the form

\[
\mathcal{H}(k) = H(k) + i\gamma \sigma_z \otimes I_2 = \begin{pmatrix} i\gamma & 0 & \mu_k & \kappa_+ \\ 0 & i\gamma & -i\gamma & 0 \\ -\mu_k & \kappa_- & 0 & 0 \\ 0 & 0 & -\kappa_- & 0 \end{pmatrix}, \]

(D6)

with \( \mu_k = J (1 - \delta) + J (1 + \delta) e^{ik} \), \( \kappa_0 = 2\pi m/N \), \( m = 1, 2, ..., N \). The eigenstates of \( H(k) \) has the form

\[
|\varphi^0_k(k)\rangle = \frac{1}{\Omega^0_k(k)} \begin{pmatrix} \varepsilon^\rho_+ (k) \kappa_\pm - \mu_k \\ \varepsilon^\rho_- (k) [\varepsilon^\rho_+ (k) - \kappa_\pm^2] \\ -\varepsilon^\rho_- (k) [\varepsilon^\rho_+ (k) - \kappa_\pm^2] \\ \kappa_\pm \varepsilon^\rho_- (k) \kappa_\pm - \mu_k \end{pmatrix}, \]

(D7)

and

\[
|\Phi^0_k(k)\rangle = \frac{1}{\Omega^0_k(k)} \begin{pmatrix} \varepsilon^\rho_+ (k) \kappa_0 \\ \varepsilon^\rho_- (k) [\varepsilon^\rho_+ (k) - \kappa_0^2] \\ -\varepsilon^\rho_- (k) [\varepsilon^\rho_+ (k) - \kappa_0^2] \\ \kappa_0 \varepsilon^\rho_- (k) \kappa_0 - \mu_k \end{pmatrix}, \]

(D8)

where \( \Omega^0_k(k) = \sqrt{2} \kappa_- \varepsilon^\rho_+ (k) - \kappa_+^2 \varepsilon^\rho_- (k)^2 \). The energy eigenvalues is the normalization factor and \( (\rho, \lambda = \pm) \). The corresponding eigenvalue is

\[
\varepsilon^\rho_\lambda (k) = \rho \left( \Upsilon_k + \lambda [\Upsilon_k^2 - (k_+ - k_-)]^{1/2} \right)^{1/2}, \]

(D9)

with \( \Upsilon_k = (|\mu_k|^2 + \kappa^2) / 2 \).

The energy bands are depicted in Fig. [D11] at various \( \gamma \) as the supplementary of Figs. 2(a) and 2(b) in the main text.

2. Zak phase

In the condition of \( \kappa_+ = \kappa_- = \kappa_0 \) (\( \theta = 0 \)), the eigenstates of \( \mathcal{H}(k) \) are

\[
|\varphi_k^0(k)\rangle = M^0_k (k, \gamma) |\varphi^0_k(k)\rangle, \]

(D9)

where

\[
|\varphi_k^0(k)\rangle = \frac{1}{\Omega^0_k(k)} \begin{pmatrix} \varepsilon^\rho_+ (k) \kappa_\pm 0 \\ -\varepsilon^\rho_- (k) [\varepsilon^\rho_+ (k) - \kappa_\pm^2] \\ \kappa_\pm \varepsilon^\rho_- (k) \kappa_\pm 0 \\ \kappa_\pm \varepsilon^\rho_- (k) \kappa_\pm 0 \end{pmatrix}, \]

(D10)

are the eigenstates of \( H(k) \), and

\[
M^0_k (k, \gamma) = \sqrt{\frac{2}{1 + e^{2i\theta}}} \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 \\ 0 & e^{i\theta} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]

(D11)

\[
\theta = \arctan \left[ \gamma / \varepsilon^\rho_+ (k) \right]. \]

(D12)

Similarly, the eigenstates of \( \mathcal{H}_I(k) \) are

\[
|\eta_k^0(k)\rangle = [M^0_k (k, \gamma)]^\dagger |\varphi_k^0(k)\rangle. \]

(D13)

By definition, the Berry connection of the non-Hermitian system reads

\[
A^0_k = i \langle \eta_k^0(k) | \partial_k |\varphi_k^0(k)\rangle, \]

(D14)

in which the right and left eigenstates can be written as

\[
|\varphi_k^+(k)\rangle = S(\delta) M^0_k (k, \gamma) |\Phi_k^0(k)\rangle, \]

(D15)

\[
|\eta_k^+(k)\rangle = S(\delta) [M^0_k (k, \gamma)]^\dagger |\Phi_k^0(k)\rangle, \]

(D16)

with

\[
S(\delta) = \begin{pmatrix} \mu_k & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu_k \end{pmatrix}, \]

(D17)

and

\[
|\Phi_k^+(k)\rangle = \frac{1}{\Omega^0_k(k)} \begin{pmatrix} \varepsilon^\rho_+ (k) \kappa_\pm \kappa_0 \\ -\varepsilon^\rho_- (k) [\varepsilon^\rho_+ (k) - \kappa_\pm^2] \\ \kappa_\pm \varepsilon^\rho_- (k) \kappa_\pm \kappa_0 \\ \kappa_\pm \varepsilon^\rho_- (k) \kappa_\pm \kappa_0 \end{pmatrix}. \]

(D18)

Then the Zak phase \( Z^0_k(\delta) = \int_{\delta}^{\delta + 2\pi} A^0_k dk \).
FIG. D1. Energy bands of the edge Hamiltonian for $\delta_0 = 0$ at (a) $\gamma = 0$, (b) $\gamma = 1.0$, (c) $\gamma = 1.5$, (d) $\gamma = 1.8$, (e) $\gamma = 2.8$, (f) $\gamma = 3.3$. The real (imaginary) part is in black (red), the blue dots are EPs. Other parameters are $J = 1$, $\kappa_0 = 2$, $R = 0.6$, and $N = 10$.

\[ Z_\rho^\lambda (\delta) = i \int_{-\pi}^{\pi} \langle \Phi_\rho^\lambda(k) | M_\rho^\lambda(k, \gamma) S^\dagger(\delta) \partial_k [S(\delta) M_\rho^\lambda(k, \gamma) | \Phi_\rho^\lambda(k)] \rangle dk \]
\[ = i \int_{-\pi}^{\pi} \langle \Phi_\rho^\lambda(k) | M_\rho^\lambda(k, \gamma) S^\dagger(\delta) \partial_k S(\delta) \partial_k S(\delta) M_\rho^\lambda(k, \gamma) | \Phi_\rho^\lambda(k) \rangle dk \]
\[ + i \int_{-\pi}^{\pi} \langle \Phi_\rho^\lambda(k) | M_\rho^\lambda(k, \gamma) S^\dagger(\delta) S(\delta) \partial_k [M_\rho^\lambda(k, \gamma) | \Phi_\rho^\lambda(k)] \rangle dk. \]  \hspace{1cm} (D19)

We note that $|\mu_k(\delta)|^2 = |\mu_k(-\delta)|^2 = 4J^2 \left[ \cos^2(k/2) + \delta^2 \sin^2(k/2) \right]$ and $\varepsilon_\rho^\lambda(k, \delta) = \varepsilon_\rho^\lambda(k, -\delta)$, then

\[ S^\dagger(\delta) S(\delta) = \begin{pmatrix} |\mu_k(\delta)|^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = S^\dagger(|\delta|) S(|\delta|), \]  \hspace{1cm} (D20)

\[ \varepsilon_\rho^\lambda(k, \delta) = \varepsilon_\rho^\lambda(k, |\delta|), \]  \hspace{1cm} (D21)

which means $\langle \Phi_\rho^\lambda(k) | M_\rho^\lambda(k, \gamma) S^\dagger(\delta) \partial_k [M_\rho^\lambda(k, \gamma) | \Phi_\rho^\lambda(k)] \rangle$ is the function of $|\delta|$, so we have

\[ Z_\rho^\lambda(\delta) - Z_\rho^\lambda(-\delta) = i \int_{-\pi}^{\pi} \langle \Phi_\rho^\lambda(k) | M_\rho^\lambda(k, \gamma) [S^\dagger(\delta) \partial_k S(\delta) - S^\dagger(-\delta) \partial_k S(-\delta)] M_\rho^\lambda(k, \gamma) | \Phi_\rho^\lambda(k) \rangle dk. \]  \hspace{1cm} (D22)

Direct derivation shows that

\[ Z_\rho^\lambda(\delta) - Z_\rho^\lambda(-\delta) = -8\delta J^2 \kappa_0^2 \int_{-\pi}^{\pi} \frac{\varepsilon_\rho^\lambda(k)^2 e^{2i\delta} + \kappa_0^2}{\Omega_\rho^\lambda(k)^2 (1 + e^{2i\delta})} dk. \]  \hspace{1cm} (D23)
Furthermore, using $e^{i\theta} = [\epsilon_\lambda^+(k) + i\gamma]/\epsilon_\lambda^+(k)$, $\epsilon_\lambda^+(k) = \rho \sqrt{\epsilon_\lambda^2(k) - \gamma^2}$, $1 + e^{2i\theta} = 2e^{i\theta} \cos \theta$ and $\tan \theta = \gamma/\epsilon_\lambda^+(k)$, $\mathcal{Z}_\lambda^+(\delta) - \mathcal{Z}_\lambda^-(\delta)$ is reduced to

$$
\mathcal{Z}_\lambda^+(\delta) - \mathcal{Z}_\lambda^-(\delta) = -\text{sgn}(\delta) \pi - 2i\gamma \rho \delta J^2 \int_{-\pi}^\pi \frac{[\epsilon_\lambda^+(k)]^2}{\sqrt{\epsilon_\lambda^2(k) - \gamma^2} [\epsilon_\lambda^+(k)^4 - \kappa_0^4]} dk,
$$

where the later term is imaginary and non-vanished for non-Hermitian Hamiltonian; however, for $[\epsilon_\lambda^-(k)]^2 = [\epsilon_\lambda^-(\rho(k))]^2$, the summation $\frac{1}{\pi} \sum_{\rho, \lambda} \mathcal{Z}_\lambda^+(\delta) - \mathcal{Z}_\lambda^-(\delta)$ is an integer [71]. Zak phase is a physical interpretation of the Chern number, since the adiabatic transport of particle is regarded as a manifestation of Zak phase.

3. Edge states

The non-Hermitian Hamiltonian under open boundary condition is the edge Hamiltonian

$$
\mathcal{H}_{\text{edge}} = \mathcal{H} - J(1 + \delta) \left( a_{2N}^\dagger b_1 + b_1^\dagger a_{2N} \right).
$$

The original Hermitian edge Hamiltonian possesses four edge states [133], from which we can obtain the corresponding edge states of the non-Hermitian system by using the mapping method. The four edge states of $\mathcal{H}_{\text{edge}}$ can be expressed as

$$
\begin{align*}
|\varphi_R^\pm\rangle &= \frac{1}{\sqrt{N}} \sum_{j=1}^N \zeta^{N-j}(e^{\pm i\delta \kappa_0^j} a_{2j}^\dagger \pm b_{2j}^\dagger) |\text{vac}\rangle, \\
|\varphi_L^\pm\rangle &= \frac{1}{\sqrt{N}} \sum_{j=1}^N \zeta^{j-1}(e^{\pm i\delta \kappa_0^{N-j}} a_{2j-1}^\dagger \pm b_{2j-1}^\dagger) |\text{vac}\rangle,
\end{align*}
$$

with eigenenergies

$$
\begin{align*}
\epsilon_R^\pm &= \pm \sqrt{(\kappa_+)^2 - \gamma^2}, \\
\epsilon_L^\pm &= \pm \sqrt{(\kappa_-)^2 - \gamma^2}.
\end{align*}
$$

Here $\zeta = (\delta - 1)/(\delta + 1), \Omega = 2(1 - \zeta^2)/(1 - \zeta^2), e^{\pm i\delta \kappa_0} = (\epsilon_R^{\pm \delta \kappa_0} + i\gamma)/\kappa_+ $ and $e^{\pm i\delta \kappa_0} = (\epsilon_L^{\pm \delta \kappa_0} + i\gamma)/\kappa_- $.

E. Topological charge pumping

The non-Hermitian comb lattice Hamiltonian under periodic boundary condition in the real space reads

$$
\mathcal{H}(t) = \sum_{j=1}^N \left\{ J [1 - \delta(t)] a_{2j}^\dagger b_{2j-1} + J [1 + \delta(t)] a_{2j+1}^\dagger b_{2j} + \right\} \\
+ \sum_{j=1}^N \kappa_-(t) a_{2j-1} a_{2j-1} + \sum_{j=1}^N \kappa_+(t) a_{2j}^\dagger b_{2j} + \text{H.c.} \\
+ i\gamma \sum_{j=1}^{2N} (a_j^\dagger a_j - b_j^\dagger b_j).
$$

FIG. E1. Particle current $j_N(t)$ for the numerical simulations in Figs. 1(b) and 1(c) of the main text for $\mathcal{H}(t)$. Numerical simulations are performed for (a) topological nontrivial phase $R > |\delta_0|$ at $\delta_0 = 0$ and (b) topological trivial phase $R < |\delta_0|$ at $\delta_0 = 1$. The speed of time evolution is $\omega = 0.001$ and the period is $T = 2\pi \omega^{-1}$. Other parameters are $\gamma = 0.5, J = 1, R = 0.6$, and $N = 10$. Two quasi-adiabatic processes are illustrated in the insets.

where $\delta(t) = \delta_0 + R \cos(\omega t)$ and $\kappa_{\pm}(t) = \kappa_0 \pm (1/2)R \sin(\omega t)$. To examine how the scheme works in practice, we simulate the quasi-adiabatic process by numerically computing the time evolution for a finite system as discussed in the main text. The computation is performed by using a uniform mesh in the time discretization for the time-dependent Hamiltonian $\mathcal{H}(t)$. In order to demonstrate a quasi-adiabatic process, we keep $f(t) = |\langle \eta_{\text{in}}(t) |\varphi_{\text{in}}(t)\rangle| \rightarrow 1$ during the whole process by taking sufficient small $\omega$, where $|\eta_{\text{in}}(t)\rangle$ is the corresponding instantaneous eigenstate of $\mathcal{H}(t)$. Figures [E1(a) and E1(b)] depict the simulations of particle current for the topological nontrivial and trivial phases, respectively. The corresponding total topological charge pumping can be seen in Figs. 1(b) and 1(c) in the main text. We can see that the imaginary parts of the currents yield zero integration in the interval $T$, and $Q_N(T)$ are 1 or 0. The obtained dynamical quantities are in close agreement with the Chern number.

The topological charge pumping can also be observed from the dynamics of edge states in the quasi-adiabatic process of the lattice under open boundary condition. The non-Hermitian edge Hamiltonian of the comb lattice reads

$$
\mathcal{H}_{\text{edge}}(t) = \mathcal{H}(t) - J[1 + \delta(t)] a_{2N}^\dagger b_1 + b_1^\dagger a_{2N}.
$$

The biorthonormal current pumped by adiabatically varying $\theta(t)$ across sites $a_N$ and $b_{N-1}$ is defined as

$$
j_N(t) = -i(\eta(t)) \langle [J [1 - \delta(t)] a_N^\dagger b_{N-1} - \text{H.c.}] |\varphi(t)\rangle.
$$

[E1]
To describe the process $|\varphi^\pm_L\rangle \rightarrow |\varphi^\pm_R\rangle$, the accumulated Thouless charge pumping passing the dimer $a_N$ and $b_{N-1}$ during the interval $t$ is

$$Q_N(t) = \int_0^t j_N(t')dt'. \quad (E4)$$

Take $\theta(t) = \omega t, R > 0, \omega \ll 1$, and the initial state $|\varphi_{edge}(0)\rangle = |\varphi^+_L\rangle$. If $t$ varies from 0 to $T = 2\pi\omega^{-1}$, $Q_N$ should be $1 \text{ mod } 2\pi$. We simulate the quasi-adiabatic process by numerically computing the time evolution in a finite system. In principle, for a given initial eigenstate $|\varphi_{edge}(0)\rangle$, the evolved state under $\mathcal{H}_{edge}(t)$ and $\mathcal{H}^\dagger_{edge}(t)$ is

$$|\varphi(t)\rangle = \mathcal{T}_i\{\exp(-i\int_0^t \mathcal{H}_{edge}(t')dt')|\varphi_{edge}(0)\rangle\}, \quad (E5)$$

and

$$|\eta(t)\rangle = \mathcal{T}_i\{\exp(-i\int_0^t \mathcal{H}^\dagger_{edge}(t')dt')|\eta_{edge}(0)\rangle\}; \quad (E6)$$

where $\mathcal{T}_i$ is the time ordering operator and $|\eta_{edge}(0)\rangle$ is the edge state for $\mathcal{H}^\dagger_{edge}(0)$ corresponding to $|\varphi_{edge}(0)\rangle$. In low speed limit $\omega \rightarrow 0$, we have $f(t) = (\hat{\eta}(t)|\varphi(t)\rangle|1$, where $\hat{\eta}(t)$ is the corresponding instantaneous eigenstate of $\mathcal{H}^\dagger_{edge}(t)$. The bulk-boundary correspondence is that the topological charge pumping of an edge state for a loop $L$ in the $\kappa-\delta$ plane equals to the Chern number. Figures E2(a) and E2(b) depict the numerical simulations of particle current and topological charge pumping for the edge states under open boundary condition for the topological nontrivial and trivial phases in the interval $T$, and the topological charge pumping is 1 or 0, respectively.

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