Localized Solutions of the Non-Linear Klein-Gordon Equation in Many Dimensions

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Abstract

We present a new complex non-stationary particle-like solution of the non-linear Klein-Gordon equation with several spatial variables. The construction is based on reduction to an ordinary differential equation.

The problem of finding or proving the existence of localized solutions of the non-linear Klein-Gordon equation in many spatial dimensions was discussed in many papers from mathematical, physical and numerical points of view [1]-[5]. The book [3] is devoted to complex asymptotic solutions of non-linear equations. We use the approach to construction of localized solutions of linear equations [6, 7].

Here we give a method of calculating complex localized solutions of the non-linear Klein-Gordon equation. For moderate time this solution has simple explicit exponentially decreasing asymptotic behavior outside some area moving with the group speed. The first term of this asymptotics is the exact solution of the linear Klein-Gordon equation presented earlier in [7] which decrease exponentially away from the point moving along the straight line. Inside the moving area this solution can be found numerically from an ordinary differential equation of some complex variable depending on the time and spatial coordinates.

Particle-like solution on the linear Klein-Gordon equation in two dimensions. We consider the linear Klein-Gordon equation with constant coefficients

\[ c^{-2} v_{tt} - \Delta v + m^2 v = 0, \quad \Delta v = v_{xx} + v_{zz}. \]  

(1)

The equation (1) has the solution depending on a single variable \( s \) (see [7])

\[ v = \frac{\exp(ims)}{s} \]  

(2)

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with $s$ depending on the spatial coordinates and time as follows

$$s = i \sqrt{(z - ikb)^2 + x^2 - (ct - i\frac{\omega}{c})^2} = i \sqrt{m^2b^2 + x^2 + z^2 - c^2t^2 + 2ib(\omega t - kz)}.$$  

(3)

Here $k, b$ are free parameters and $\omega = c\sqrt{k^2 + m^2}$.

It is shown in [7] that the solution (2) has finite energy when $b$ and $k$ are real and $\text{Im} s > 0$. If the time is small enough $|t| \ll bm^2/\omega$ than the solution decreases exponentially for $|x| \to \infty$ and $|z| \to \infty$. If $|x| \ll bm$ and $|z| \ll \min (bm^2/k, bm)$ then the expansion of the form

$$\text{im} s \sim -bm^2 - \frac{(z - v_{gr}t)^2}{\Delta^2_\parallel} - \frac{x^2}{\Delta^2_\perp} - i(\omega t - kz)$$

(4)

holds. We use the following notations $\Delta_\parallel = \sqrt{2bm^2}/\omega$, $\Delta_\perp = \sqrt{2b}$. From (4) and (2) it follows that the solution represents a wave packet with the Gaussian envelope filled with oscillations. It moves with the group speed $v_{gr} = d\omega/dk$ in the positive direction of the $z$ axis. This is demonstrated by the numerical calculations of the solution (2) in successive times, see Fig.1 where the results are presented for the parameters $m = 5$, $c = 1$, $k = 2$, $b = 15$ in the conventional units.

**Non-linear Klein-Gordon equation in two spatial dimensions.** We search now the solution on the non-linear Klein-Gordon equation in two dimensional space

$$c^{-2}u_{tt} - \Delta u + f(u) = 0$$

(5)

depending on the spatial coordinates and time only through the complex variable $s$ defined by (3). Then the partial differential equation (5) reduces to the ordinary differential equation

$$u_{ss} + \frac{2}{s}u_s + f(u) = 0.$$  

(6)

Choosing for the sake of definiteness the function $f(u)$ as follows

$$f(u) = m^2u + \gamma u^3, \quad \gamma = \text{const},$$

(7)

we prove that there exists the exact solution on non-linear equation (5) having an estimate

$$u(s) = C\frac{\exp(\text{im} s)}{s}(1 + O(q \exp(-2a))), \quad C = \text{const},$$

(8)

if $q \exp(-2a)$ is small enough, where

$$\text{Re}(\text{im} s) \leq (-a) < 0, \quad q = \gamma C^2/(m|S|).$$

(9)
Figure 1: Particle-like solution on the Klein-Gordon equation in the successive times in conventional units
We use here the technique of integral equations. In conditions of the validity of (4) the inequality (9) can be written as follows

\[
\frac{(z - v_{gr} t)^2}{\Delta_\parallel^2} + \frac{x^2}{\Delta_\perp^2} \geq a - b m^2.
\] (10)

The asymptotics (8) is valid for the solution of (6) outside the ellipse (10). Inside the ellipse (10) the equation (6) should be solved numerically.

**Non-linear Klein-Gordon equation in many dimensional space.** The Klein-Gordon equation in many dimensional space

\[c^{-2} u_{tt} - \Delta u + f(u) = 0, \quad \Delta u = u_{x_1 x_1} + u_{x_2 x_2} + \ldots + u_{x_n x_n},\] (11)

can be treated analogously to the case of two-dimensional space. Seeking the solution of (11) \(u\) as the function of the single complex variable \(s\)

\[s = i \sqrt{(x_1 - ikb)^2 + x_2^2 + \ldots + x_n^2 - (ct - \frac{i \omega c}{b})^2}.\] (12)

we obtain the ordinary differential equation

\[u_{ss} + \frac{n}{s} u_s + f(u) = 0.\] (13)

We suppose that \(f\) is defined by (7). For moderate values of \(t\) a solution on the equation (13) exists with the asymptotics written in terms of the Hankel function

\[u(s) = s^{-(n-1)/2} H^{(1)}_{(n-1)/2}(ms)(1 + O(\exp(-2a))), \quad a \rightarrow \infty\] (14)

which is valid outside the moving area \((x_1 - v_{gr} t)^2/\Delta_\parallel^2 + (x_2^2 + \ldots + x_n^2)/\Delta_\perp^2 \geq (a - b m^2)\) where \(\Delta_\parallel, \Delta_\perp, a\) are defined above. Localization of the solution for moderate times follows from the asymptotics of the Hankel function.

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