On two Möbius functions for a finite non-solvable group

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ABSTRACT
Let $G$ be a finite group, $\mu$ be the Möbius function on the subgroup lattice of $G$, and $\lambda$ be the Möbius function on the poset of conjugacy classes of subgroups of $G$. It was proved by Pahlings that, whenever $G$ is solvable, the property $\mu(H, G) = |N_G(H) : G' \cap H| \cdot \lambda(H, G)$ holds for any subgroup $H$ of $G$. It is known that this property does not hold in general, the Mathieu group $M_{12}$ being a counterexample. In this paper we investigate the relation between $\mu$ and $\lambda$ for some classes of non-solvable groups, among them, the minimal non-solvable groups. We also provide several examples of groups not satisfying the property.

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1. Introduction
The Möbius function of a locally finite poset $P$ is the function $\mu_P : P \times P \to \mathbb{Z}$ defined by
$$\mu_P(x, y) = 0 \text{ if } x \not\leq y, \quad \mu_P(x, x) = 1, \quad \mu_P(x, y) = -\sum_{x < z \leq y} \mu_P(z, y) \text{ if } x < y.$$ Let $G$ be a finite group and $L$ be the subgroup lattice of $G$. The Möbius function of $G$ is defined as $\mu : L \to \mathbb{Z}$, $H \mapsto \mu_L(H, G)$, and it is simply denoted by $\mu(H)$. The Möbius function $\mu$ of a finite group $G$ was first considered by Hall [11] in order to enumerate generating tuples of elements of $G$. As it turns out, the Möbius function of $G$ has many other applications in different areas of mathematics, in the context of enumerative problems where the Möbius inversion formula ([21, Proposition 3.7.1]) is applicable. These areas include group theory, graph theory, combinatorial designs, algebraic topology, and computer science. For a detailed description, see [2, 5, 9, 13, 20] and the references therein.

Beginning with Hall [11], the Möbius function of a finite group $G$ is involved in problems related to the probability of generating $G$ by a given number of elements. Later, Mann [16] introduced the following complex series for any profinite group $G$:
$$P_G(s) = \sum_{H \leq G} \frac{\mu(H)}{|G : H|^s},$$ where $H$ ranges over all open subgroups of $G$. Mann conjectured that this sum is absolutely convergent in some half complex plane whenever $G$ is a positively finitely generated (PFG) group. The conjecture is nowadays reduced to the following one (see [14, 17]): there exist $c_1, c_2 \in \mathbb{N}$
such that, for any almost simple group $G$, $|\mu(H)| \leq [G : H]^{1/2}$ for every $H < G$, and, for any $n \in \mathbb{N}$, the number of subgroups $H < G$ of index $n$ in $G$ satisfying $G = H \text{ soc}(G)$ and $\mu(H) \neq 0$ is bounded above by $n^{1/2}$.

One could try to attack this and other problems related to $\mu$ by investigating a different Möbius function: instead of the lattice $\mathcal{L}$, consider the poset $C$ of conjugacy classes of subgroups of $G$, where $[H] \leq [K]$ if and only if $H$ is contained in some conjugate $K^g$ of $K$ in $G$. Its Möbius function $\lambda : C \to \mathbb{Z}, [H] \mapsto \mu_C([H],[G])$ is denoted by $\lambda(H)$. Clearly, $C$ is the image of $\mathcal{L}$ under the order-preserving map $H \mapsto [H]$, and $C$ is in general more tractable than $\mathcal{L}$ (see [4, Section 3.1]).

The function $\lambda$ is related to the table of marks of $G$, i.e. the matrix $M$ indexed by the conjugacy classes $[H_1], \ldots, [H_n]$ of subgroups of $G$ whose $(i,j)$-entry is the $[H_j]$-mark of $G/H_i$. In fact, $\lambda([H_i],[H_j])$ is the $(j,i)$-entry of $M^{-1}$ (see [15, Section 2]).

Hawkes, Isaacs, and Özaydin [12] showed that

$$\mu(\{1\}) = |G'| \cdot \lambda(\{1\})$$

holds for any finite solvable group $G$, and later Pahlings [18] proved that

$$\mu(H) = [N_G(H) : G' \cap H] \cdot \lambda(H) \quad (1)$$

holds for every $H \leq G$ whenever $G$ is finite and solvable. We say that $G$ satisfies the $(\mu, \lambda)$-property if Equation (1) holds for every $H \leq G$. It is known that the $(\mu, \lambda)$-property does not hold for every finite group (see [1] and [25]). Nevertheless, one may ask to which extent the result of Pahlings holds true.

In this paper, we extend the work of Pahlings about the $(\mu, \lambda)$-property for finite groups. In Section 3 we show that the $(\mu, \lambda)$-property is valid for some infinite families of non-solvable groups, beginning in Theorem 3.3 with minimal simple groups. This is then generalized in Theorem 3.7 to minimal non-solvable groups (i.e., non-solvable groups whose proper subgroups are all solvable). We also present other infinite families of non-solvable groups satisfying the $(\mu, \lambda)$-property, namely, the simple groups $L_2(q)$, $Sz(q)$, $R(q)$, and the almost simple groups $\text{PGL}_2(q)$. In Section 4 we provide sufficient conditions for a direct product of finite groups to satisfy the $(\mu, \lambda)$-property, and discuss the $(\mu, \lambda)$-property for extensions of finite groups. Finally, Section 5 contains examples and final remarks about groups not satisfying the $(\mu, \lambda)$-property.

### 2. Notations and preliminaries

Throughout the paper, $G$ is a finite group, $\mu(H,G)$ is the Möbius function of $H \leq G$ in the subgroup lattice $\mathcal{L}$ of $G$, and $\lambda(H,G)$ is the Möbius function of the conjugacy class $[H]$ of $H \leq G$ in the poset $C$ of conjugacy classes of subgroups of $G$, ordered as follows: $[H] \leq [K]$ if $H$ is contained in the conjugate $K^g$ of $K$ for some $g \in G$. We denote these functions by $\mu(H)$ and $\lambda(H)$ when the group $G$ is clear. The notation $\mu(n)$ represents the number-theoretic Möbius function on the positive integers, which coincides with $\mu_D(1,n)$ in the divisibility poset $D$ on $\mathbb{N}$; we will use the number-theoretical Möbius function in our computations in Tables 1–6. We will mostly follow the group-theoretical notation of the ATLAS [6].

| $H$ | $S_h$ | $M_n(b)$ | $D_{45}(b)$ | $D_{44}(b)$ | $C_{22}(b)$ | $C_2$ | $(1)$ |
|-----|-------|----------|-------------|-------------|-------------|------|------|
| $h$ | $> 1$ | $> 1$    | $> 1$       | $> 1$       | $> 1$       |      |      |
| $\mu(H)$ | $\mu(\xi)$ | $-\mu(\xi)$ | $-\mu(\xi)$ | $-\mu(\xi)$ | $-\mu(\xi)$ | $-\mu(\xi)$ | $\xi$ |
| $N_6(H)$ | $H$ | $H$ | $H$ | $H$ | $D_{2q-1}$ | $L_q$ | $G$ |
| $\lambda(H)$ | $\mu(\xi)$ | $-\mu(\xi)$ | $-\mu(\xi)$ | $-\mu(\xi)$ | $-2\mu(\xi)$ | $\xi$ | $\mu(\xi)$ |

Table 1. Subgroups $H \leq G = L_2(q), q = 2^r$, with $\mu(H) \neq 0$ or $\lambda(H) \neq 0$. 
Table 2. Subgroups $H \leq G = L_2(q)$ with $\mu(H) \neq 0$ or $\lambda(H) \neq 0$; $q = p^e$; $p$ odd; $e > 2$, or $e = 2$ and $p \equiv \pm 1 \pmod{5}$; $H \not\in \{D_6, C_3, C_2, \{1\}\}$.

| $H$ | $G_h$ | $S_h$ | $M_{h,2\Gamma(h)}$ | $M_{h,\kappa(h)}$ | $D_{\Lambda(h)}$ |
|-----|-------|-------|-----------------|-----------------|-----------------|
| $h \mid e$ with | $\frac{1}{k}$ even | $\frac{1}{k}$ odd | $\frac{1}{k}$ odd | $\frac{1}{k}$ odd | $\frac{1}{k}$ even, $p^h \neq 3$ |
| $\mu(H)$ | $\mu(\frac{1}{k})$ | $\mu(\frac{1}{k})$ | $-\mu(\frac{1}{k})$ | $-\mu(\frac{1}{k})$ | $-2\mu(\frac{1}{k})$ |
| $N_0(H)$ | $H$ | $H$ | $H$ | $H$ | $D_{\gcd(2h, q-1)}$ |
| $\lambda(H)$ | $\mu(\frac{1}{k})$ | $\mu(\frac{1}{k})$ | $-\mu(\frac{1}{k})$ | $-\mu(\frac{1}{k})$ | $-\frac{2\mu(e/h)}{\gcd(2, \frac{h}{e})}$ |
| $h \mid e$ with | $\frac{1}{k}$ odd, $p^h \not\in \{3, 5\}$ | $\frac{1}{k}$ even | $\frac{1}{k}$ odd, $p^h \neq 3$ | $\frac{1}{k}$ even, $p^h \neq 3$ | $\frac{1}{k}$ odd, $p^h \not\in \{3, 5\}$ |
| $\mu(H)$ | $-\mu(\frac{1}{k})$ | $-2\mu(\frac{1}{k})$ | $-\mu(\frac{1}{k})$ | $\frac{2(q-1)}{p^h-1} \mu(\frac{1}{k})$ | $\frac{2(q-1)}{p^h-1} \mu(\frac{1}{k})$ |
| $N_0(H)$ | $H$ | $D_{\gcd(h, q-1)}$ | $H$ | $D_{q-1}$ | $D_{q-1}$ |
| $\lambda(H)$ | $-\mu(\frac{1}{k})$ | $-\frac{2\mu(e/h)}{\gcd(2, \frac{h}{e})}$ | $-\mu(\frac{1}{k})$ | $2\mu(\frac{1}{k})$ | $\mu(\frac{1}{k})$ |

Table 3. Subgroups $H \leq G = L_2(q)$ with $\mu(H) \neq 0$ or $\lambda(H) \neq 0$; $q = p^e$; $p$ odd; $e > 2$, or $e = 2$ and $p \equiv \pm 1 \pmod{5}$; $H \in \{D_6, C_3, C_2, \{1\}\}$.

| $H$ | $D_4$ | $C_3$ |
|-----|-------|-------|
| $\mu(H)$ | $\gamma - \beta$, where $\gamma = \begin{cases} -6, & e = 2^0; \\ 0, & \text{otherwise}; \end{cases}$ | $\begin{cases} \frac{q-1}{3} \mu(e), & p = 7, e \text{ odd}; \\ q \frac{3}{3}, & p = 3; \\ 0, & \text{otherwise}. \end{cases}$ |
| $\beta = \begin{cases} 6\mu(e), & p = 3, e \text{ even}; \\ 3\mu(e), & p \in \{3, 5\}, e \text{ odd}; 0, \text{otherwise}. \end{cases}$ | $E_q, \quad q \equiv 0 \pmod{3}$; |
| $N_0(H)$ | $\begin{cases} S_4, & q \equiv \pm 1 \pmod{8}; \\ A_4, & \text{otherwise}. \end{cases}$ | $D_{q-1}, \quad q \equiv 1 \pmod{3}$; |
| $\lambda(H)$ | $\begin{cases} \frac{1}{6} \mu(D_4), & q \equiv \pm 1 \pmod{8}; \\ \frac{1}{3} \mu(D_4), & \text{otherwise}. \end{cases}$ | $\begin{cases} \mu(e), & p = 7, e \text{ odd}; \\ 1, & p = 3; \\ 0, & \text{otherwise}. \end{cases}$ |
| $H$ | $C_2$ | $\{1\}$ |
| $\mu(H)$ | $\gamma - \delta$, where $\gamma = \begin{cases} \frac{1}{2}(q-1), & e = 2^0; \\ 0, & \text{otherwise}; \end{cases}$ | $\begin{cases} |G|\mu(e), & p = 3, e \text{ odd}; \\ 0, & \text{otherwise}. \end{cases}$ |
| $\delta = \begin{cases} -(q-1)\mu(e), & p = 3, e \text{ even}; \\ \frac{q+1}{2} \mu(e), & p = 3, e \text{ odd}; \\ \frac{q-1}{2} \mu(e), & p = 5, e \text{ odd}; \\ 0, & \text{otherwise}. \end{cases}$ | $D_{q-1}, \quad q \equiv 1 \pmod{4}$; |
| $N_0(H)$ | $\begin{cases} D_{q-1}, & q \equiv 1 \pmod{4}; \\ D_{q+1}, & q \equiv 3 \pmod{4}. \end{cases}$ | $G$ |
| $\lambda(H)$ | $\begin{cases} 2\mu(C_2), & q \equiv 1 \pmod{4}; \\ q-1, & q \equiv 1 \pmod{4}; \\ 2\mu(C_2), & q \equiv 3 \pmod{4}. \end{cases}$ | $\begin{cases} \mu(e), & p = 3, e \text{ odd}; \\ 0, & \text{otherwise}. \end{cases}$ |
Lemma 2.1 follows from the fact that (Hall [11, Theorem 2.3]).

| Table 6. Subgroups $H < G = L_2(q)$ with $\mu(H) \neq 0$ or $\lambda(H) \neq 0$; $q = p^2$; $p \geq 7$ odd; $p \equiv \pm 2 \pmod{5}$; $H$ non-maximal subgroup of $G$. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $H$             | $C_p: C_{p-1}$  | $C_{p+1}$      | $D_{2(p+1)}$   | $D_{2(p-1)}$   | $C_{p-1}$      | $A_4$           |
| $\mu(H)$        | 1               | 2              | 2              | 2              | $-2(p+1)$      | 2              |
| $N_3(H)$        | $H$             | $D_{p^2-1}$    | $D_{(p+1)\gcd(4,p-1)}$ | $D_{(p-1)\gcd(4,p+1)}$ | $D_{p^2-1}$    | $S_4$          |
| $\lambda(H)$    | 1               | 1              | $\frac{1}{\gcd(4,l^2-1)}$ | $\frac{1}{\gcd(4,l^2+1)}$ | $-2$           | 1              |

| Table 5. Subgroups $H \leq G = S_3(q), q = 2$, $e \geq 3$ odd, with $\mu(H) \neq 0$ or $\lambda(H) \neq 0$. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $H$             | $G(h)$          | $F(h)$          | $B_0(h)$        | $A_0(h)$        | $B_1(h)$        | $B_2(h)$        | $C_4$           | $C_2$           |
| $h \mid e$ with | $h > 1$         | $h > 1$         | $h > 1$         | $h > 1$         | $h > 1$         | $-2(\mu(e)$     | $\mu(e)$       | $|G|\mu(e)$     |
| $\mu(H)$        | $\mu(h^2)$     | $-\mu(h^2)$    | $-\mu(h^2)$    | $2\frac{1}{2}h^2$ | $-\mu(h^2)$    | $-\mu(h^2)$    | $-\mu(h^2)$    | $-\mu(h^2)$    |
| $N_3(H)$        | $H$             | $h$             | $H$             | $D_{2(q-1)}$    | $H$             | $h$             | $E_{q-1}$       | $E_{q-1}$       |
| $\lambda(H)$    | $\mu(h^2)$     | $-\mu(h^2)$    | $-\mu(h^2)$    | $\mu(h^2)$     | $-\mu(h^2)$    | $-\mu(h^2)$    | $-\mu(h^2)$    | $-\mu(h^2)$    |

We denote by $\text{MaxInt}(G)$ the set whose elements are $G$ and the subgroups of $G$ which are the intersection of maximal subgroups of $G$. The knowledge of $\text{MaxInt}(G)$ is essential in the study of $\mu$, as Lemma 2.1 shows.

Lemma 2.1 (Hall [11, Theorem 2.3]). Let $H \leq G$ be such that $\mu(H) \neq 0$. Then $H \in \text{MaxInt}(G)$.

Lemma 2.1 follows from the fact that $\mathcal{C}$ is a lattice, and in particular, if $H_1, H_2 \in \mathcal{C}$, then $H_1 \cap H_2 \in \mathcal{C}$. Note that in general the poset $\mathcal{C}$ is not a lattice and the meet of $[H_1]$ and $[H_2]$ does not always exist. Nevertheless, we will prove the analogue of Lemma 2.1 for $\mathcal{C}$.

We say that $G$ satisfies the $(\mu, \lambda)$-property if the following relation holds for every $H \leq G$:

$$\mu(H) = [N_3(H) : G \cap H] \cdot \lambda(H).$$
Recall that a minimal non-solvable group is a non-solvable group whose proper subgroups are all solvable. A minimal non-solvable group which is simple is called minimal simple. We point out that, if $G$ is a minimal non-solvable group and $\Phi(G)$ is its Frattini subgroup, then $G/\Phi(G)$ is minimal simple. Hence, minimal simple groups are exactly the Frattini-free minimal non-solvable groups.

Finite minimal simple groups are classified as follows.

**Lemma 2.2** (Thompson [23]). The finite minimal simple groups are the following ones:

- $L_2(2^r)$, where $r$ is a prime;
- $L_2(3^r)$, where $r$ is an odd prime;
- $L_2(p)$, where $p > 3$ is a prime such that $5 \mid (p^2 + 1)$;
- $Sz(2^r)$, where $r$ is an odd prime;
- $L_3(3)$.

### 3. $(\mu, \lambda)$-property for some families of non-solvable groups

#### 3.1. $(\mu, \lambda)$-property for minimal non-solvable groups

Here we show that the $(\mu, \lambda)$-property holds for many classes of finite groups, including minimal non-solvable groups. A key role in our proofs (as well as in the explicit computations of the $\lambda$ function for any group) is played by Lemma 3.1.

**Lemma 3.1.** Let $H \leq G$ be such that $\lambda(H) \neq 0$. Then $H \in \text{MaxInt}(G)$.

**Proof.** We use induction on $[G : H]$. The result is clearly true for $H = G$. Let $H < G$ and let $K \in \text{MaxInt}(G)$ be the intersection of all maximal subgroups of $G$ which contain $H$. Thus, $K \leq M$ for every $M \in \text{MaxInt}(G)$ containing $H$. Suppose $H \not\leq \text{MaxInt}(G)$, so that $H < K$. Let $N \leq G$ be such that $H < N$ and $\lambda(N) \neq 0$. By induction $N \in \text{MaxInt}(G)$, and hence $K \leq N$. Therefore,

$$\lambda(H) = - \sum_{[H] < [N] \leq [G], \lambda(N) \neq 0} \lambda(N) = - \sum_{[K] \leq [N] \leq [G]} \lambda(N) = 0.$$

It follows from Lemmas 2.1 and 3.1 that only subgroups of $G$ containing $\Phi(G)$ have to be considered:

**Corollary 3.2.** If $H \leq G$ and $\Phi(G) \not\leq H$, then $\mu(H) = \lambda(H) = 0$.

**Theorem 3.3** proves that the $(\mu, \lambda)$-property holds for minimal simple groups. Clearly, if $G$ is minimal simple, then $G = G'$ and the $(\mu, \lambda)$-property for $H \leq G$ reads $\mu(H) = (N_G(H) : H) \cdot \lambda(H)$.

**Theorem 3.3.** Let $G$ be a minimal simple group. Then $G$ satisfies the $(\mu, \lambda)$-property.

**Proof.** By Lemma 2.2, $G = L_2(q)$ for some $q$ and the claim follows from Proposition 3.5 below, $G = Sz(q)$ for some $q$ and the claim follows from Proposition 3.4 below, or $G = L_3(3)$ and the claim follows by direct inspection with Magma [3].

The proofs of the next propositions are quite technical. We provide full details in the proof of Proposition 3.4, while we just sketch the proofs of the remaining Propositions 3.5, 3.8 and 3.9.

Note that Propositions 3.4 and 3.5 refer to larger classes of simple groups than just the minimal ones.
Proposition 3.4. Let $q = 2^e$ for some odd $e \geq 3$, and $G$ be the simple Suzuki group $Sz(q)$. Then $G$ satisfies the $(\mu, \lambda)$-property.

Proof. Downs and Jones computed $\text{MaxInt}(G)$ in [10, Theorem 6], as well as the normalizer $N_G(H)$ ([10, Table 2]) and the value $\mu(H)$ ([10, Table 1]) for every $H \in \text{MaxInt}(G)$. Hence, we compute $\lambda(H)$. By Lemma 3.1, we restrict to the subgroups $H \in \text{MaxInt}(G)$. The results are summarized in Table 5. If a subgroup $H < G$ does not appear in Table 5, then $\mu(H) = \lambda(H) = 0$. We use the results and the techniques performed in [10]. These techniques rely on a detailed analysis of the subgroups of $G$ and the 2-transitive action of $G$ on the Suzuki-Tits ovoid $\Omega$ of $\text{PG}(3, q)$, as described by Suzuki [22].

The same notation as in [10] is used. In particular, for any divisor $h \mid e$, $G(h)$ denotes a subgroup $Sz(2^h)$. The stabilizer in $G(h)$ of one point $\infty \in \Omega$ is denoted by $F(h)$ and acts as a Frobenius group on $\Omega$. Then $F(h) = Q(h)A_0(h) = (E_2^h)^{1+1} : C_{2^h-1}$, where $Q(h) = (E_2^h)^{1+1}$ and $A_0(h) = C_{2^h-1}$ are the Frobenius kernel and a Frobenius complement in $F(h)$, respectively. The center of $Q(h)$ is $Z(h) = E_2^h$ if $h > 1$, while $C_2 \cong Z(1) \leq Q(1) \cong C_4$. $B_0(h)$ is a dihedral group of order $2(2^h - 1)$ normalizing $A_0(h)$, and for $h > 1$, $N_G(A_0(h)) = B_0(e)$. We denote by $A_1(h), A_2(h) \leq G(h)$ two cyclic groups such that $\{[A_1(h)]_i, [A_2(h)]_i\} = \{2^h + 2^{(h+1)/2} + 1, 2^h - 2^{(h+1)/2} + 1\}$ with $5 \mid [A_1(h)], 5 \mid [A_2(h)]$. For $i = 1, 2$, $B_i(h)$ is a subgroup of $G(h)$ with $B_i(h) \cong A_i(h) = C_4$. We have $B_1(h) = N_G(A_1(h))$ and, if $h > 1$, then $B_2(h) = N_G(A_2(h))$. Finally, $F(e) = B_2(1) \cong C_4$, $B_0(1) \cong C_2$, $G(1) = B_1(1) \cong AGL(5)$, and $A_0(1) \cong \{1\}$.

By [10, Theorem 6], every $H \in \text{MaxInt}(G)$ is conjugate to one of the groups

$$G(h), \ F(h), \ Q(h), \ Z(h), \ B_i(h), \ A_i(h)$$

for some $i \in \{0, 1, 2\}$ and $h \mid e$, and each of them yields a single conjugacy class in $G$.

1. First, consider the case $h > 1$.
   - Suppose that $H$ is conjugate to one of the following groups:
     $$G(h), \ B_1(h), \ F(h), \ B_0(h), \ B_2(h).$$
     Then $H = N_G(H)$ by [10, Theorem 9] and hence $[N_G(H) : H] = 1$. By [10, Table 2], if $K \leq \text{MaxInt}(G)$ satisfies $H \leq K$, then $K$ is the unique element of its conjugacy class $[K]$ which contains $H$. This implies that the subgroups $\{K \in \mathcal{C} : K \in \text{MaxInt}(G), \ H \leq K \leq G\}$ and $\{[K] \in \mathcal{C} : K \in \text{MaxInt}(G), \ [H] \leq [K] \leq [G]\}$ are isomorphic. This implies $\mu(H) = \lambda(H)$. Thus, the $(\mu, \lambda)$-property holds for $H$.
   - Suppose that $H$ is conjugate to $Q(h)$ for some $h$. By [10, Tables 2], if $h \mid k$ the overgroups of $H$ in $\text{MaxInt}(G)$ are conjugate to $G(k), F(k)$ or $Q(k)$ (obviously, $Q(k) \neq Q(h)$ implies $k \neq h$). Since $\lambda(G(k)) = \mu(e/k)$ and $\lambda(F(k)) = -\mu(e/k)$, we have $\lambda(H) = 0$ by induction. By [10, Table 1], the $(\mu, \lambda)$-property holds for $H$.
   - Suppose that $H$ is conjugate to $Z(h)$. By [10, Table 2], the overgroups of $H$ in $\text{MaxInt}(G)$ are conjugate to $G(k), F(k), Q(k)$ with $h \mid k$, or to $Z(k)$ with $h \mid k$ and $h \neq k$. By [10, Table 1] and induction, we have that $\lambda(H) = 0$ and the $(\mu, \lambda)$-property holds for $H$.
   - Suppose that $H$ is conjugate to $A_0(h)$. By [10, Theorem 9], we have $[N_G(H) : H] = [B_0(e) : A_0(h)] = \frac{2^{(2^e-1)}}{2^e-1}$. By [10, Table 2], the overgroups of $H$ in $\text{MaxInt}(G)$ are conjugate to $G(k), F(k), B_0(k)$ with $h \mid k$, or to $A_0(k)$ with $h \mid k$ and $h \neq k$. Since $\lambda(G(k)) = \lambda(F(k)) = 0$ and $\lambda(B_0(k)) = -\mu(e/k)$, we have by induction $\lambda(H) = \mu(e/h)$. By [10, Table 1] we have $\mu(A_0(h)) = \frac{2^{(2^e-1)}}{2^e-1} \mu(e/h)$. Thus, the $(\mu, \lambda)$-property holds for $H$.
   - Suppose that $H$ is conjugate to $A_1(h)$ for some $h$. By [10, Tables 1 and 2], the overgroups of $H$ in $\text{MaxInt}(G)$ are conjugate to $G(k)$ with $h \mid k$ satisfying $\lambda(G(k)) = \mu(e/k)$, to $B_1(k)$
with \( h \mid k \) satisfying \( \lambda(B_1(k)) = -\mu(e/k) \), or to \( A_1(k) \) with \( h \mid k \) and \( h \neq k \). By induction, we have \( \lambda(H) = 0 \). As \( \mu(H) = 0 \), the \((\mu, \lambda)\)-property holds for \( H \).

- Suppose that \( H \) is conjugate to \( A_2(h) \). By [10, Table 2], the overgroups of \( H \) in \( \text{MaxInt}(G) \) are conjugate to \( G(k) \) with \( h \mid k \) satisfying \( \lambda(G(k)) = \mu(e/k) \), to \( B_2(k) \) with \( h \mid k \) satisfying \( \lambda(B_2(k)) = -\mu(e/k) \), or to \( A_2(k) \) with \( h \mid k \) and \( h \neq k \). By induction, we have \( \lambda(H) = 0 \). As \( \mu(H) = 0 \) by [10, Table 1], the \((\mu, \lambda)\)-property holds for \( H \).

(2) Now, consider the case \( h = 1 \).

- Suppose that \( H \) is conjugate to \( A_1(1) \) or \( G(1) = B_1(1) \). Then the proof is analogous to the case \( h > 1 \).

- Suppose that \( H \) is conjugate to \( F(1) = Q(1) = B_2(1) \cong C_4 \). By [10, Theorem 9], we have \( [N_C(H) : H] = \frac{q^2}{2} \). By [10, Table 2], the overgroups of \( H \) in \( \text{MaxInt}(G) \) are conjugate to \( G(k) \) with \( k \mid e \) satisfying \( \lambda(G(k)) = \mu(e/k) \), to \( F(k) \) with \( 1 \neq k \mid e \) satisfying \( \lambda(F(k)) = -\mu(e/k) \), to \( Q(k) \) with \( 1 \neq k \mid e \) satisfying \( \lambda(Q(k)) = 0 \), to \( B_1(k) \) with \( k \mid e \) satisfying \( \lambda(B_1(k)) = -\mu(e/k) \), or to \( B_2(k) \) with \( 1 \neq k \mid e \) satisfying \( \lambda(B_2(k)) = -\mu(e/k) \). The subposets \( \{[K] : K \in \text{MaxInt}(G) \mid G(1) \leq K \leq G(e) \} \) and \( \{[K] : K \in \text{MaxInt}(G) \mid G(1) \leq K \leq G(e) \} \) are both isomorphic to the divisibility poset \( D_e = \{n \in \mathbb{N} : 1 \mid n \mid e \} \). Thus, \( \sum_{1 \mid k \mid e} \lambda(G(k)) = \sum_{1 \mid k \mid e} \lambda(B_1(k)) = \sum_{1 \mid k \mid e} \mu(e/k) = 0 \). The subposet \( \{[F(k)] : 1 \neq k \mid e \} \) is isomorphic to the divisibility poset \( D_e \setminus \{1\} \). Therefore, \( \sum_{1 \neq k \mid e} \lambda(F(k)) = \sum_{1 \neq k \mid e} -\mu(e/k) = +\mu(e/1) = \mu(e) \). In the same way, we have \( \sum_{1 \neq k \mid e} \lambda(B_2(k)) = \mu(e) \). Hence, \( \lambda(H) = -\mu(e/1) = -2\mu(e) \). By [10, Table 1], \( \mu(H) = -2\mu(e) \). Therefore, the \((\mu, \lambda)\)-property holds for \( H \).

- Suppose that \( H \) is conjugate to \( Z(1) = B_0(1) \cong C_2 \). By [10, Theorem 9], we have \( [N_C(H) : H] = \frac{q^2}{2} \). When \( K \) runs over the overgroups of \( H \) having a subgroup isomorphic to \( C_4 \), the conjugacy classes \([K]\) form a subposet of \( C \) with minimum element \([B_2(1)]\). Hence, the sum \( \sum \lambda(K) \) over this set is equal to zero. The overgroups of \( H \) in \( \text{MaxInt}(G) \) without subgroups isomorphic to \( C_4 \) are the groups conjugate to \( B_0(k) \) for some \( k \mid e \) with \( k \neq 1 \). Since \( \lambda(B_0(k)) = -\mu(e/k) \), and the subposets \( \{[B_0(k)] : 1 \mid k \mid e \} \) are isomorphic, we have that \( \lambda(H) = -\sum_{1 \neq k \mid e} \lambda(B_0(k)) = -\mu(e/1) = -\mu(e) \). By [10, Table 1], \( \mu(H) = -\frac{q^2}{2} \mu(e) \). Therefore, the \((\mu, \lambda)\)-property holds for \( H \).

- Suppose that \( H = A_2(1) = A_0(1) = \{1\} \). As in the previous point, we may compute

\[
\lambda(H) = -\sum_{[K] \in C, K \in \text{MaxInt}(G), 2 \nmid [K]} \lambda(K),
\]

because the sublattice of \( C \) given by the subgroups of even order of \( G \) has a minimum element \([B_0(1)], B_0(1) \cong C_2 \). Hence, we only consider the subgroups \( A_1(k) \). For \( i \in \{1, 2\} \), we have \( \lambda(A_i(k)) = 0 \) for any overgroup \( A_i(k) \) of \( H \). The subgroups \( A_0(k) \) form a sublattice of \( C \) isomorphic to \( D_e \), and satisfy \( \lambda(A_0(k)) = \mu(e/k) \) for every \( k \neq 1 \). Therefore, \( \lambda(A_0(1)) = -\sum_{1 \neq k \mid e} \mu(e/k) = \mu(e/1) = \mu(e) \). Since \( \mu(H) = |G|\mu(e) \) by [10, Table 1], the \((\mu, \lambda)\)-property holds for \( H \). □

**Proposition 3.5.** Let \( q \geq 4 \) be a prime power and \( G = L_2(q) \). Then \( G \) satisfies the \((\mu, \lambda)\)-property.

**Proof.** For every \( H \subseteq G \), \( \mu(H) \) was computed by Downs in [8]. When \( q \) is prime, \( \lambda(H) \) has been computed in [18], and the claim was already proved\(^1\) in [18, Proposition 3].

---

\(^1\)Pay attention to a misprint in the proof of [18, Proposition 3]: the right value of \( a_n \) is 1 or 2 according respectively to \( p \equiv \pm 1 \) or \( p \not\equiv \pm 1 \) modulo \( 4n \), not modulo \( 2n \) as it is written.
Hence, we restrict to the computation of $\lambda(H)$ with $q$ non-prime. The value $\mu(C_3)$ is not computed correctly in [8] when $q = 3^e$. The correct value is $\mu(C_3) = \frac{q}{2}$ for $e > 2$, and is given in Table 3. Also, the case $q = 25$ is not considered in [8]. However, in the cases $q = 9$ and $q = 25$, the claim follows by direct computation with Magma [3].

The techniques used are similar to the ones used in [7, 8, 18], together with Lemma 3.1. The explicit computations are quite long and follow closely the steps performed in the cited papers. Therefore, we have chosen to omit them and to summarize the results in Tables 1–4. If a subgroup $H < G$ does not appear in the tables and is not a maximal subgroup of $G$, then $\mu(H) = \lambda(H) = 0$.

The same notation as in [8] is used. In particular, $q = p^e$ with $p$ prime and $e > 1$. For any divisor $1 \neq h \mid e$, $r(h) = \frac{p^h - 1}{h}$, $s(h) = \frac{p^h + 1}{2h}$, $G_h = \text{PGL}_2(p^h)$, and $S_h = \text{PSL}_2(p^h)$. $A_n$ and $S_n$ are alternating and symmetric groups of degree $n$. $C_m, D_m, E_m$ are respectively cyclic, dihedral, elementary abelian groups of order $m$. For every $1 \neq d \mid (p^h - 1)$, $M_{h,d} \cong E_{ph} : C_d$ is contained in a subgroup $\text{PGL}_2(p^h)$ or $L_2(p^h)$ of $L_2(q)$ and stabilizes a point in the natural action on the projective line over $\mathbb{F}_q$.

We conclude this section proving the validity of the $(\mu, \lambda)$-property for minimal non-solvable groups. To this aim, we recall the corresponding result proved by Pahlings for solvable groups.

**Theorem 3.6.** ([18, Main Theorem]) Let $G$ be a solvable group. Then $G$ satisfies the $(\mu, \lambda)$-property.

We extend this result as follows.

**Theorem 3.7.** Let $G$ be a minimal non-solvable group. Then $G$ satisfies the $(\mu, \lambda)$-property.

**Proof.** If $\Phi(G) = 1$, then $G$ is minimal simple and Theorem 3.3 proves the claim. Let $\Phi(G) \neq 1$ and $H \leq G$. If $\Phi(G) \leq H$, the claim follows by Corollary 3.2. If $\Phi(G) \leq H$, then let $G = G/\Phi(G)$ and $\tilde{H} = H/\Phi(G)$. As noted by Pahlings in the proof of Theorem 3.6 (see [18, Page 10]), the correspondence between subgroups of $G/\Phi(G)$ and overgroups of $\Phi(G)$ in $G$ implies that $\mu(H, G) = \mu(\tilde{H}, \tilde{G}), \lambda(H, G) = \lambda(\tilde{H}, \tilde{G})$, and $[N_{\tilde{G}}(\tilde{H}) : \tilde{H} \cap \tilde{G}] = [N_G(H) : H \cap G']$. As $\tilde{G}$ is minimal simple, the result follows from Theorem 3.3.

**3.2. $(\mu, \lambda)$-property for other families of groups**

In this section, we collect the results about the validity of $(\mu, \lambda)$-property for other groups, namely the almost simple groups $\text{PGL}_2(q)$ and the simple Ree groups $R(q)$.

**Proposition 3.8.** For any prime power $q$, the group $\text{PGL}_2(q)$ satisfies the $(\mu, \lambda)$-property.

We omit the proof of Proposition 3.8. We also omit the tables, which are very similar to the ones of $L_2(q)$. Note that $\text{PGL}_2(q) = L_2(q)$ when $q$ is even. The arguments are simplified when $q$ is odd by the fact that for every $H \leq \text{PGL}_2(q)$ there is just one conjugacy class in $\text{PGL}_2(q)$ of subgroups isomorphic to $H$, except when $H$ is cyclic of order 2 or a dihedral group $D_{2m}$ with $2m$ dividing either $\frac{q - 1}{2}$ or $\frac{q + 1}{2}$.

**Lemma 3.9.** Let $q = 3^e$ with $e \geq 3$ odd. The simple small Ree group $R(q)$ satisfies the $(\mu, \lambda)$-property.

**Proof.** Pierro exhibited in [19] a set $M$ of subgroups of $G = R(q)$ such that $M \subseteq \text{MaxInt}(G)$ ([19, Table 4]). He also computed the normalizer $N_G(H)$ ([19, Lemmas 2.5 to 2.8]) and the value $\mu(H)$ ([19, Theorem 1.11]) for every $H \in M$. As described in [19, Section 3.1], the subgroups $K \in
MaxInt\((G) \setminus M\) are the intersections of maximal subfield subgroups (i.e., of subgroups isomorphic to \(R(3^l)\) for some odd divisor \(f\) of \(e\)) and satisfy \(\mu(K) = 0\) (see [19, Lemma 3.12]). It is straightforward to see that such subgroups \(K\) also satisfy \(\lambda(K) = 0\), by using arguments similar to those provided in Lemmas 3.6, 3.10 and 3.12 in [19].

Therefore, we only have to compute \(\lambda(H)\) for every \(H \in M\). The results are summarized in Table 6. We omit the explicit computations, which make use of Lemma 3.1 and arguments that are similar to the ones used in [19]. They rely on a detailed analysis of the subgroups of \(G\) and the 2-transitive action of \(G\) on the Ree-Tits ovoid \(\Omega\) of PG\((6, q)\), as described by Tits [24].

The notation is the same as in [19], which is the same as in the ATLAS [6]. Note that, for every \(H\) in Table 6, the isomorphism type of \(H\) identifies a unique conjugacy class in \(G\). □

4. Products and extensions

In this section, we consider the \((\mu, \lambda)\)-property for direct products of a finite number of finite groups and for finite extensions of a finite group.

Proposition 4.1 gives a sufficient condition for the Möbius functions of a direct product to split as the product of the corresponding Möbius functions of the factors, generalizing [11, Result 2.8].

**Proposition 4.1.** Let \(n \geq 2\) and \(G = \prod_{i=1}^{n} G_i\) be a direct product of groups \(\{G_i\}\) such that every maximal subgroup \(M\) of \(G\) splits as a direct product \(M = \prod_{i=1}^{n} M_i\), with \(M_i \leq G_i\) for every \(i\). Let \(H = \prod_{i=1}^{n} H_i \leq G\) with \(H_i \leq G_i\) for every \(i\). Then

\[
\mu_G(H) = \prod_{i=1}^{n} \mu_{G_i}(H_i), \quad \lambda_G(H) = \prod_{i=1}^{n} \lambda_{G_i}(H_i).
\]

**Proof.** From the assumptions it follows immediately that, if \(K \in \text{MaxInt}(G)\), then \(K = \prod_{i=1}^{n} K_i\) with \(K_i \leq G_i\) for every \(i\). Hence, in the computation of \(\mu(H)\) and \(\lambda(H)\) we only consider the groups \(H < \prod_{i=1}^{n} K_i \leq G\) with \(K_i \leq G_i\) for every \(i\). Let \(I \subseteq \{1, \ldots, n\}\) be such that \(H_i \neq G_i\) for \(i \in I\), and \(H_i = G_i\) for \(i \not\in I\). Then the subposet of \(\mathcal{L}\) made by the groups \(K = \prod_{i=1}^{n} K_i\) satisfying \(H \leq K \leq G\) is isomorphic to the subgroup poset of groups \(K = \prod_{i \in I} K_i\) satisfying \(\prod_{i \in I} H_i \leq K \leq \prod_{i \in I} G_i\). An analogous poset isomorphism holds for the posets of conjugacy classes. Hence, \(\mu_G(H) = \mu \prod_{i \in I} \mu_i(\prod_{i \in I} H_i)\) and \(\lambda_G(H) = \lambda \prod_{i \in I} \lambda_i(\prod_{i \in I} H_i)\). Then we can assume that \(H_i \neq G_i\) for all \(i = 1, \ldots, n\). We use induction on \([G:H]\), the claim being true when \(H\) is a maximal subgroup of \(G\). We have

\[
\mu_G\left(\prod_{i=1}^{n} H_i\right) = - \sum_{\prod_{i=1}^{n} K_i \leq G} \mu_G\left(\prod_{i=1}^{n} K_i\right) = - \sum_{\prod_{i=1}^{n} K_i \leq G} \prod_{i=1}^{n} \mu_{G_i}(K_i). \tag{2}
\]

Fix \(i_0 \in \{1, \ldots, n\}\), and for every \(i \neq i_0\) fix \(K_i\) with \(H_i < K_i \leq G_i\). Then

\[
\sum_{H_{i_0} < K_{i_0} \leq G_{i_0}} \prod_{i=1}^{n} \mu_{G_i}(K_i) = \left(\prod_{i \neq i_0} \mu_{G_i}(K_i)\right) \cdot \left(\sum_{H_{i_0} < K_{i_0} \leq G_{i_0}} \mu_{G_{i_0}}(K_{i_0})\right) = -\mu_{G_{i_0}}(H_{i_0}) \cdot \prod_{i \neq i_0} \mu_{G_i}(K_i).
\]

We apply this argument to Equation (2), dividing the summation according to the distinct \(\epsilon\) indexes \(i_\epsilon \in \{1, \ldots, n\}\) such that \(K_{i_\epsilon}\) varies with \(H_{i_\epsilon} < K_{i_\epsilon} \leq G_{i_\epsilon}\), while \(K_{j_\epsilon} = H_{j_\epsilon}\) for the remaining \(n - \epsilon\) indexes \(j_\epsilon\). We obtain
\[
\sum_{H \leq \prod_{i=1}^{n} K_i \leq H, K_i = H_{i_1}, \ldots, K_{i_{m_i}} = H_{i_{m_i}}} \prod_{i=1}^{n} \mu_{G_i}(K_i) = (-1)^s \prod_{i=1}^{n} \mu_{G_i}(H_i).
\]

Therefore,
\[
\mu_{G}\left(\prod_{i=1}^{n} H_i\right) = -\left(\prod_{i=1}^{n} \mu_{G_i}(H_i)\right) \cdot \left(\sum_{k \in \{1, \ldots, n\} \text{ even}} \binom{n}{k} - \sum_{k \in \{1, \ldots, n\} \text{ odd}} \binom{n}{k}\right) = \prod_{i=1}^{n} \mu_{G_i}(H_i),
\]

where the last equality is obtained using \(\binom{n}{k} = \left(\frac{n-1}{k}\right) + \left(\frac{n-1}{k-1}\right)\). The claim on \(\lambda_G(H)\) follows similarly. \(\Box\)

We now apply Proposition 4.1 to the direct product of groups satisfying the \((\mu, \lambda)\)-property.

**Proposition 4.2.** Let \(G = \prod_{i=1}^{n} G_i\) be a direct product of groups \(\{G_i\}\) such that every maximal subgroup \(M\) of \(G\) splits as a direct product \(M = \prod_{i=1}^{n} M_i\), with \(M_i \leq G_i\) for every \(i\). If \(G_1, \ldots, G_n\) satisfy the \((\mu, \lambda)\)-property, then \(G\) satisfies the \((\mu, \lambda)\)-property.

**Proof.** If \(H \in \text{MaxInt}(G_i)\), then \(H = \prod_{i=1}^{n} H_i\) with \(H_i \leq G_i\) for every \(i\). In fact, for every \(j = 1, \ldots, m\), let \(M_j\) be a maximal subgroup of \(G\) such that \(M_j = \prod_{i=1}^{n} M_{j,i}\) with \(M_{j,i} \leq G_i\) for every \(i = 1, \ldots, n\). Then \(\prod_{j=1}^{m} M_j = \prod_{i=1}^{n} \prod_{j=1}^{m} M_{j,i}\). We have \(G' = \prod_{i=1}^{n} G_i'\) and \(G' \cap H = \prod_{i=1}^{n} G_i' \cap H_i\). This implies \([N_{G_i}(H) : G_i' \cap H_i] = [N_{G_i}(H_i) : G_i' \cap H_i]\). The claim follows using Lemma 4.1. \(\Box\)

**Example 4.3.** Let \(G = G_1 \times G_2\) where no non-trivial quotients of \(G_1\) and \(G_2\) are isomorphic. Then it is easily seen from Goursat’s lemma that every maximal subgroup \(M\) of \(G\) splits as \(M = M_1 \times M_2\) with \(M_i \leq G_i\), \(i = 1, 2\).

For instance, Proposition 4.2 applies to \(G = G_1 \times G_2\) where \(G_1\) is minimal non-solvable and \(G_2\) is solvable, or \(G_1\) and \(G_2\) are minimal non-solvable with non-isomorphic Frattini quotients.

We give an elementary remark about extensions of groups each of which satisfies the \((\mu, \lambda)\)-property. The proof of Remark 4.4 follows from the same arguments as in the proof of Theorem 3.7.

**Remark 4.4.** The \((\mu, \lambda)\)-property holds for every Frattini extension of a group satisfying the \((\mu, \lambda)\)-property.

**Remark 4.5** shows how to form groups not satisfying the \((\mu, \lambda)\)-property from a given one.

**Remark 4.5.** If a group \(G\) is a finite extension of a group \(\bar{G}\) which does not satisfy the \((\mu, \lambda)\)-property, then \(G\) does not satisfy the \((\mu, \lambda)\)-property.

In fact, arguing as in the proof of Theorem 3.7, it is clear that a subgroup \(\bar{H}\) for which the \((\mu, \lambda)\)-property fails in \(\bar{G}\) is the homomorphic image of a subgroup \(H\) for which the \((\mu, \lambda)\)-property fails in \(G\).

**5. Final remarks**

We start this section by considering a class of finite groups which contains the minimal nonsolvable ones, namely the finite nonsolvable N-groups. Recall that an N-group is a group whose local subgroups are all solvable. Finite nonsolvable N-groups were classified by Thompson [23]. They are almost simple groups \(G\), where \(S \leq G \leq \text{Aut}(S)\) and \(S\) is one of the following simple groups:
• the linear group $L_2(q)$, for some prime power $q \geq 4$;
• the Suzuki group $Sz(q)$, for some non-square power $q \geq 8$ of $2$;
• the linear group $L_3(3)$;
• the unitary group $U_3(3)$;
• the alternating group $A_7$;
• the Mathieu group $M_{11}$;
• the Tits group $^2F_4(2)'$.

**Proposition 5.1.** Every finite simple $N$-group $G$ other than $U_3(3)$ satisfies the $(\mu, \lambda)$-property.

**Proof.** Unless $G$ is cyclic of prime order, $G$ is contained in the above list. If $G = L_2(q)$ or $G = Sz(q)$, the claim follows by Propositions 3.4 and 3.5. By direct inspection with Magma [3], it follows that the groups $L_3(3)$, $A_7$, $M_{11}$ and $^2F_4(2)'$ satisfy the $(\mu, \lambda)$-property, while $U_3(3)$ does not. □

By performing some computations for groups of small order, we notice the following facts:

• all groups of order at most 2000 satisfy the $(\mu, \lambda)$-property;
• the group $U_3(3)$, of order 6048, is the smallest simple group which does not satisfy the $(\mu, \lambda)$-property;
• the subgroups $H \leq U_3(3)$ at which the $(\mu, \lambda)$-property fails are those isomorphic to $C_2$, $S_3$, $D_8$, or $S_4$.

We do not know whether $U_3(3)$ is the smallest non-solvable group not satisfying the $(\mu, \lambda)$-property.

The inspection of $U_3(3)$ may give hints about the structure of groups for which the $(\mu, \lambda)$-property fails, as explained by the following remark.

**Remark 5.2.** Let $G$ be a finite group, $H$ be a subgroup of $G$, $S = \{K \leq G : H \leq K\}$ be the subposet of the subgroup lattice of $G$ made by the overgroups of $H$, and $\bar{S} = \{[K] \leq [G] : [H] \leq [K]\}$ be the corresponding subposet of the conjugacy classes $[K]$ with $[H] \leq [K]$. Suppose that, for every $K \in S \setminus \{G\}$, we have $N_G(K) = G' \cap K$. This holds, for instance, if $G$ is perfect and every $K \in S$ is self-normalizing in $G$. Then the $(\mu, \lambda)$-property for $H$ holds if and only if $\mu(H) = \lambda(H)$, and hence, if and only if the posets $S$ and $\bar{S}$ are isomorphic.

In the case $S_3 \cong H \leq G = U_3(3)$, we have that $S = \{H, M_1, M_2, M_3, G\} \not\cong \bar{S} = \{[H], [M_1], [M_2] = [M_3], [G]\}$, where $M_i$ is a maximal subgroup of $G$. For every $K \in S$, $K$ is self-normalizing in the simple group $G$. Thus, $S \not\cong \bar{S}$ implies that the $(\mu, \lambda)$-property fails at $H$.

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