Non-standard Null Lagrangians and Forces in Classical Mechanics

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Abstract

Non-standard Lagrangians are a family of Lagrangians with terms that do not have clearly discernable energy-like forms. A special class among these Lagrangians are non-standard null Lagrangians and their gauge functions, which are used to introduce forces to the laws of dynamics. It is shown that the non-standard Lagrangian for the law of inertia is Galilean invariant but the forces resulting from the non-standard null Lagrangian are not Galilean invariant, except special only time-dependent forces.

1. Introduction

The role of standard Lagrangians, whose the kinetic and potential energy-like terms can easily be identified, has been well established in Classical Mechanics (CM) (e.g., [1-5]). On the other hand, the so-called non-standard Lagrangians, in which neither kinetic nor potential energy-like terms exist, have been introduced to CM in recent years (e.g., [6-12]). There are also null Lagrangians, whose two main characteristics are: (i) that they identically satisfy the Euler-Lagrange (E-L) equation, and (ii) that they can be expressed as the total derivative of any scalar function, called here a gauge function. The properties and applications of these null Lagrangians have been extensively explored in different fields of mathematics (e.g., 14-20]), and in physics (e.g., [21,22]).

Specifically, recent physical applications involved restoring Galilean invariance of Lagrangians in Newtonian dynamics [23,24], and introducing forces to CM [25-27], which was done independently from the original Newton approach and others [28,29]. Most of the previous work was done by using standard null Lagrangians [24,25]. However, more recently the existence of non-standard null Lagrangians (NSNLs) has been demonstrated [30], and the main objective of this Letter is to use these newly introduced NSNLs to define forces in CM, and convert the law of inertia into the second law of dynamics. Moreover, Galilean invariance of the presented results is investigated and the physical implications are discussed.

2. Non-standard Lagrangian for the law of inertia

A general non-standard Lagrangian [7-11] can be written as

$$L_{ns}[\dot{x}(t), x(t), t] = \frac{1}{g_1(t)\dot{x}(t) + g_2(t)x(t) + g_3(t)} ,$$

(1)
where \( g_1(t), g_2(t), \) and \( g_3(t) \) are arbitrary, but at least twice differentiable, scalar functions of time \( t \). By evaluating these functions for the law of inertia, the following non-standard Lagrangian is obtained \([30]\)

\[
L_{ns}[\dot{x}(t), x(t), t] = \frac{1}{C_1(a_o t + v_o)^2[(a_o t + v_o)\dot{x}(t) - a_o x(t) + C_2]},
\]

with \( C_1 \) and \( C_2 \) being constants of integration, and \( v_o \) and \( a_o \) being specified by the initial conditions for solving the auxiliary differential equation \([30]\). An interesting result is that this Lagrangian gives the law of inertia, which is conservative, despite being explicitly dependent on time \( t \) \([31]\). This non-standard Lagrangian will be used in the next section to introduce forces to the law of inertia.

### 3. General gauge function and non-standard null Lagrangian

The previous work \([30]\) shows that a general gauge function can be constructed

\[
\Phi_{gn}(t) = \frac{h_1(t)}{h_2(t)} \ln |h_2(t)x(t) + h_4(t)|,
\]

where \( h_1(t), h_2(t) \) and \( h_4(t) \) are twice differentiable but otherwise arbitrary functions of \( t \), and in addition \( h_2(t) \neq 0 \). This gauge function gives the following general NSNL

\[
L_{nsn}[\dot{x}(t), x(t), t] = \frac{h_1(t)[h_2(t)\dot{x}(t) + \dot{h}_2(t)x(t)] + \dot{h}_4(t)}{h_2(t)[h_2(t)x(t) + h_4]}
\]

\[
+ \left[ \frac{\dot{h}_2(t)}{h_2(t)} - \frac{h_1(t)\dot{h}_2(t)}{h_2(t)} \right] \ln |h_2(t)x(t) + h_4(t)|,
\]

which can be added to any standard Lagrangian, including the one given by Eq. (2), and the resulting form of the equation of motion will not be altered.

### 4. Introducing forces to the law of inertia

Having obtained the standard Lagrangian for the law of inertia (see Eq. 2) and the general NSNL, we may add them together and obtain the total Lagrangian given by

\[
L_t[\dot{x}(t), x(t), t] = L_{ns}[\dot{x}(t), x(t), t] + L_{nsn}[\dot{x}(t), x(t), t]
\]

Since \( L_t[\dot{x}(t), x(t), t] \) depends explicitly on time, we calculate the energy function corresponding to this Lagrangian \([4,5]\), and obtain

\[
E_t[\dot{x}(t), x(t), t] = \dot{x} \frac{\partial L_t[\dot{x}(t), x(t), t]}{\partial \dot{x}} - L_t[\dot{x}(t), x(t), t]
\]
The resulting energy function is
\[
E_t[\dot{x}(t), x(t), t] = \frac{-\dot{x}(t)}{C_1(a_o t + v_o)[(a_o t + v_o)\dot{x}(t) - a_o x(t) + C_2]^2} - \frac{1}{C_1(a_o t + v_o)[(a_o t + v_o)\dot{x}(t) - a_o x(t) + C_2]}
\]
\[
+ \frac{h_1(t)}{h_2(t)x(t) + h_4(t)} - \frac{h_1(t)[h_2(t)\dot{x}(t) + \dot{h}_2(t)x(t)] + \dot{h}_4(t)}{h_2(t)[h_2(t)x(t) + h_4(t)]}
\]
\[
- \left[ \frac{\dot{h}_1(t)}{h_2(t)} - \frac{h_1(t)\dot{h}_2(t)}{h_2^2(t)} \right] \ln |h_2(t)x(t) + h_4(t)| . \tag{7}
\]

The derived energy function can now be used to obtain the equation of motion from the following well-known relationship \[4,5\]
\[
\frac{dE_t}{dt} = -\frac{\partial L_t}{\partial t} , \tag{8}
\]
and the result is
\[
\ddot{x}(t) = F(x, t) , \tag{9}
\]
where
\[
F(x, t) = \frac{-h_1(t)\dot{h}_2(t)x(t) - \dot{h}_4(t)}{h_2(t)x(t) + h_4(t)} + h_1(t) , \tag{10}
\]
is the force introduced to the law of inertia by the non-standard null Lagrangian. This shows that the force depends explicitly on \( x(t) \) and on \( t \) through arbitrary functions \( h_1(t) \), \( h_2(t) \) and \( h_4(t) \). Typically in classical dynamics, the considered forces do not depend on the dependent variable but only on time, and they are periodic in time. The dependence on \( x(t) \) is easily eliminated by taking \( h_2(t) = \text{const} \) and \( h_4(t) = \text{const} \). Then, we may consider \( h_1(t) = A \cos(\Omega t) \), where \( A \) is the force amplitude and \( \Omega \) is the driving frequency, and get the force that is commonly used to drive dynamical systems \[4,5\].

The derived equation of motion demonstrates how the NSNL can be used to turn an undriven system to a driven one. The main difference between the previous results \[25-27\] and those presented in this Letter is that we show here how classical forces can be defined using the non-standard Lagrangians given by Eqs. (2) and (4). The obtained results demonstrate that the first Newton law of dynamics can be converted into the second law by the means of non-standard null Lagrangians. This gives an independent way to introduce forces in CM, and supplements Newton’s definition of forces that directly relates them to object’s acceleration and mass \[28,29\]. Our results show a deeper connection between Newton’s first and second laws of dynamics.
5. Galilean invariance

The problem of Galilean invariance of the law of inertia has been addressed in most textbooks on CM (e.g., [4,5]). Some authors also pointed out the lack of Galilean invariance of the standard Lagrangian for the law of inertia [2,3,23]; however, a solution to this problem that involves standard null Lagrangians was recently proposed [24]. An interesting question is whether or not the non-standard Lagrangian and the resulting forces are Galilean invariant. Let us now discuss the Galilean invariance of the presented results, and refer to all observers who agree on the Galilean invariance as the Galilean observers.

The Galilean group of the metric is composed of rotations, translations, and boosts between two inertial frames of reference [32,33]. Let \((x, t)\) be an inertial frame moving at a constant velocity, \(V_0\), with respect to a second inertial frame, \((x', t')\), and let their origins coincide at \(t = t' = t_0\). Thus, there are the following transformations between the systems: \(x' = x - V_0t\) and \(t' = t\). By applying these transformations to the law of inertia \(\ddot{x}(t) = 0\), it is seen that Newton’s first law is Galilean invariant. However, its standard Lagrangian is not [2,4,23] and requires a special procedure that involves standard null Lagrangians to restore its Galilean invariance [24].

Let us now investigate Galilean invariance of the non-standard Lagrangian for the law of inertia (see Eq. [2] and written here as

\[
L_{ns}[\dot{x}(t), x(t), t] = \frac{1}{C_1 A^2(t)\{A(t)\dot{x}(t) - a_o x(t) + C_2\}},
\]

where \(A(t) = a_o t + v_o\). After the Galilean transformation, this Lagrangian becomes

\[
L'_{ns}[\dot{x}'(t'), x'(t'), t'] = \frac{1}{C'_1 A'^2(t')\{A'(t')\dot{x}'(t') - a'_o x'(t') + C'_2 + v'_o V_0\}}.
\]

Galilean invariance of \(L_{ns}[\dot{x}(t), x(t), t]\) requires that its form is the same as \(L'_{ns}[\dot{x}'(t'), x'(t'), t']\). For the original and transformed Lagrangians to be of the same form in the variables \(x(t)\) and \(x'(t')\), the following conditions must be satisfied: (i) \(A'(t') = A(t)\), which requires that \(a'_o = a_o\) and \(v'_o = v_o\), and that \(t' = t\) as guaranteed by the Galilean transformation; (ii) \(C'_1 = C_1\) is satisfied in all inertial frames; and (iii) \(C'_2 + v_o V_0 = C_2\) to be valid for all Galilean observers.

Since \(a_o\) and \(v_o\) are the integration constants for the auxiliary equation [30], and \(C_1\) and \(C_2\) are the constants of integration for the law of inertia, these constants are determined by the initial conditions to be specified for a physical problem to be solved. However, both the auxiliary equation and the law of inertia are Galilean invariant; thus, the solutions to these equations must also be the same (Galilean invariant) for all Galilean observers. The latter is equivalent to the requirement that the specified initial conditions are also the same for all Galilean observers, which validates the above conditions (i) and (ii). The condition (iii) shows that \(C'_2 \neq C_2\) and that the constant \(C'_2\) must be modified by adding another constant \(v_o V_0\) to it as compared to \(C_2\). This addition is known in advance by all Galilean observers, who by their definition already agreed on the Galilean invariance. Therefore, the non-standard Lagrangian for the law of inertia given by Eq. [2] is Galilean invariant.
Having demonstrated the Galilean invariance of the law of inertia and its non-standard Lagrangian, we now investigate the Galilean invariance of the driven equation of motion given by Eq. (9) with the force defined by Eq. (10), which was obtained using the non-standard null Lagrangian, whose explicit form is presented by Eq. (4). Using the Galilean transformations, the driven equation of motion becomes

$$\ddot{x}'(t) = \frac{-h_1'(t)\dot{h}_2'(t)x'(t) - h_1'(t)\dot{h}_2'(t)V_0 t - \dot{h}_4'(t)}{\dot{h}_2'(t)x'(t) + h_2'(t)V_0 t + h_4'(t)} + h_1'(t),$$  \hspace{1cm} (13)$$

with the Galilean transformation of time, $t' = t$, being already implemented in this equation.

Comparison of this transformed equation of motion to the original equations (see Eqs. 9 and 10) leads to the following two conditions that must be satisfied in order for the both equations to be of the same form

$$-h_1'(t)\dot{h}_2'(t)V_0(t) t - \dot{h}_4'(t) = -\dot{h}_4'(t),$$  \hspace{1cm} (14)$$

and

$$h_2'(t)V_0 t + \dot{h}_4'(t) = h_4'(t).$$  \hspace{1cm} (15)$$

Since $V_0 \neq 0$ and $t > 0$ for any time $t > t_0$, the conditions require that either $h_1'(t)$ or $h_2'(t)$ must be zero, which makes either $h_1(t)$ or $h_2(t)$ to be zero and this contradicts the definition of the gauge function and the NSNL given by Eqs. (3) and (4), respectively. Moreover, the condition $\dot{h}_2(t) = 0$ would also make the definition of the energy function given by Eq. (7) invalid. Therefore, the general force derived from the non-standard null Lagrangian is not Galilean invariant, which is consistent with the force previously derived using the standard null Lagrangians [25].

As shown in Section 4, the dependence of the force given by Eq. (10) on $x(t)$ can be eliminated by taking $h_2(t) = \text{const}$ and $h_4(t) = \text{const}$, which reduces the force to be only time-dependent. In this special case the resulting force $F(t) = h_1(t)$ is of the same form as the transformed force $F'(t) = h_1'(t)$, and therefore this force is Galilean invariant. Since most commonly used forces in classical dynamics are only time-dependent, such forces are Galilean invariant. Moreover, the equations of motion of the form $\ddot{x} = F(t)$ are also Galilean invariant.

The results of this Letter demonstrate that forces in CM can also be defined using non-standard null Lagrangians, which is a novel way to view forces and it significantly extends the previous work on standard null Lagrangians [25]. Similarly to the previous work, the resulting general forces are not Galilean invariant. However, in a special case when the forces depend on time only, they are Galilean invariant. The presented results also show that the non-standard Lagrangian for the law of inertia is Galilean invariant, and therefore it is different than the standard Lagrangian for the first Newton law, which is not Galilean invariant [2,3,23] and requires a special procedure to restore its invariance [24].

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