A random-matrix theory is presented for the reflection of light by a disordered medium backed by a phase-conjugating mirror. Two regimes are distinguished, depending on the relative magnitude of the inverse dwell time of a photon in the disordered medium and the frequency shift acquired at the mirror. The qualitatively different dependence of the reflectance on the degree of disorder in the two regimes suggests a distinctive experimental test for cancellation of phase shifts in a random medium.

The phase-conjugating mirror we consider consists of a material with a large non-linear susceptibility, which is pumped by two counter-propagating beams at frequency \( \omega_0 \). In such a medium, the propagation is completely diffusive. Conjugation drastically modifies the reflectance even if the random medium itself.) We distinguish a degenerate regime, while in the non-degenerate regime it first decreases and then increases. The disappearance of the reflectance minimum on reducing \( \Delta \omega \) provides a distinctive experimental test for phase-shift cancellation in a random medium.

Our theoretical approach applies techniques of random-matrix theory which were originally developed for the study of phase conjugation of electrons by a superconductor. To reduce the optical problem to the scattering of a scalar wave, we choose a two-dimensional geometry. The scatterers consist of dielectric rods in the \( x \)-direction, randomly placed in the \( x-y \)-plane. The electric field points in the \( z \)-direction and varies in the \( x-y \)-plane only. Two-dimensional scatterers are somewhat artificial, but can be realized experimentally. We believe that our results apply qualitatively to a three-dimensional geometry as well, because the randomization of the polarization by the disorder renders the vector character of the light insignificant.

The \( z \)-component of the electric field at the frequencies \( \omega_{\pm} \) is given by

\[
E_{\pm}(x, y, t) = \text{Re} \, \mathcal{E}_{\pm}(x, y) \exp(-i \omega_{\pm} t).
\]

The phase-conjugating mirror (at \( x = 0 \)) couples the two frequencies via the wave equation

\[
\begin{pmatrix}
\mathcal{H}_0 & -\gamma^* \\
-\gamma & -\mathcal{H}_0
\end{pmatrix}
\begin{pmatrix}
\mathcal{E}_+ \\
\mathcal{E}_-
\end{pmatrix}
= \frac{2\varepsilon \Delta \omega}{\omega_0}
\begin{pmatrix}
\mathcal{E}_- \\
\mathcal{E}_+
\end{pmatrix}.
\]
The complex dimensionless coupling constant $\gamma$ is zero for $x < 0$ and for $x > L_c$, with $L_c$ the length of the non-linear medium forming the phase-conjugating mirror. For $0 < x < L_c$ we put $\gamma = \gamma_0 e^{i\phi}$. The Helmholtz operator $H_0$ at frequency $\omega_0$ is given by

$$H_0 = -k_0^{-2} \nabla^2 - \varepsilon,$$

where $k_0 = \omega_0/c$ and $\varepsilon(x, y)$ is the relative dielectric constant of the medium. We take $\varepsilon = 1$ everywhere except in the disordered region $-L < x < 0$, where $\varepsilon = 1 + \delta \varepsilon(x, y)$. The spatial fluctuations $\delta \varepsilon$ lead to scattering with mean free path $l \gg 1/k_0$. The validity of Eq. (3) requires $\Delta \omega/\omega_0 \ll 1$ and $\gamma_0 \ll 1$. The ratio of these two small parameters

$$\delta \equiv 2\Delta \omega/\gamma_0 \omega_0$$

is arbitrary.

To define finite-dimensional scattering matrices we embed the disordered medium in a waveguide (width $W$), containing $N_{\pm} = \text{Int}(\omega_0 W/c\pi) \gg 1$ propagating modes at frequency $\omega_{\pm}$. A basis of scattering states consists of the complex fields

$$E_{\pm, n}^> = k_{\pm, n}^{-1/2} \sin \left(\frac{n\pi y}{W}\right) \exp(i k_{\pm, n} x - i \omega_{\pm} t), \quad (5a)$$

$$E_{\pm, n}^< = k_{\pm, n}^{-1/2} \sin \left(\frac{n\pi y}{W}\right) \exp(-i k_{\pm, n} x - i \omega_{\pm} t). \quad (5b)$$

Here $n = 1, 2, \ldots, N_{\pm}$ is the mode index and the superscript $>$ ($<$) indicates a wave moving to the right (left), with frequency $\omega_{\pm}$ and wavenumber $k_{\pm, n}$. The normalization in Eq. (5) is such that each wave carries the same flux.

With respect to this basis, incoming and outgoing waves are decomposed as

$$E_{\text{in}} = \sum_{n=1}^{N_+} u_{+, n} E_{+, n}^> + \sum_{n=1}^{N_-} u_{-, n} E_{-, n}^>,$$  

$$E_{\text{out}} = \sum_{n=1}^{N_+} v_{+, n} E_{+, n}^< + \sum_{n=1}^{N_-} v_{-, n} E_{-, n}^<.$$  

The complex coefficients are combined into two vectors

$$u = (u_{+, 1}, u_{+, 2}, \ldots, u_{+, N_+}, u_{-, 1}, u_{-, 2}, \ldots, u_{-, N_-})^T, \quad (7a)$$

$$v = (v_{+, 1}, v_{+, 2}, \ldots, v_{+, N_+}, v_{-, 1}, v_{-, 2}, \ldots, v_{-, N_-})^T. \quad (7b)$$

The reflection matrix $r$ relates $u$ to $v$,

$$v = ru, \quad r = \begin{pmatrix} r_{++} & r_{+-} \\ r_{-+} & r_{--} \end{pmatrix}. \quad (8)$$

The submatrices $r_{\pm\pm}$ have dimensions $N_{\pm} \times N_{\pm}$. The reflectances $R_{\pm}$ are defined by

$$R_- = N_+^{-1} \text{Tr}r_{--}r_{++}^T, \quad R_+ = N_-^{-1} \text{Tr}r_{++}r_{--}^T. \quad (9)$$

For $\Delta \omega \ll \omega_0$ we may neglect the difference between $N_+$ and $N_-$. and replace both by $N = \text{Int}(k_0 W/c\pi)$.

We construct $r$ from the reflection matrix $r_{\text{PCM}}$ of the phase-conjugating mirror and the scattering matrix $S$ of the disordered medium. In the absence of disorder, an incoming plane wave in the direction $(\cos \phi, \sin \phi)$ is retroreflected at the phase-conjugating mirror in the direction $(- \cos \phi, - \sin \phi)$, with a different frequency and amplitude. The $2N \times 2N$ reflection matrix $r_{\text{PCM}}$ is

$$r_{\text{PCM}} = \begin{pmatrix} 0 & -ia e^{i\psi} \\ i a e^{i\psi} & 0 \end{pmatrix}, \quad (10a)$$

$$a_{nm} = \delta_{nm} a(\phi_n), \quad \phi_n = \arcsin(n\pi y/k_0 W), \quad (10b)$$

$$a(\phi) = \sqrt{1 + \delta^2} \cot \left(\frac{\alpha \sqrt{1 + \delta^2}}{\cos \phi} + i \delta\right)^{-1}, \quad (10c)$$

with $\alpha \equiv \frac{1}{2} \gamma_0 k_0 L_c$. The disordered medium in front of the phase-conjugating mirror does not couple $\omega_+ + \omega_-$. Its scattering properties at frequency $\omega$ are described by two $N \times N$ transmission matrices $t_{21}(\omega)$ and $t_{12}(\omega)$ (transmission from left to right and from right to left), plus two $N \times N$ reflection matrices $r_{11}(\omega)$ and $r_{22}(\omega)$ (reflection from left to left and from right to right). Taken together, these four matrices constitute a $2N \times 2N$ scattering matrix

$$S(\omega) = \begin{pmatrix} r_{11}(\omega) & t_{12}(\omega) \\ t_{21}(\omega) & r_{22}(\omega) \end{pmatrix}. \quad (11)$$

which is unitary (because of flux conservation) and symmetric (because of time-reversal invariance). (In contrast, $r_{\text{PCM}}$ is not flux conserving.) Without loss of generality the reflection and transmission matrices of the disordered region can be decomposed as

$$r_{11}(\omega_\pm) = i U_\pm \sqrt{\rho_\pm} U_\pm^T, \quad t_{21}(\omega_\pm) = V_\pm \sqrt{\tau_\pm} V_\pm^T, \quad (12a)$$

$$r_{22}(\omega_\pm) = i V_\pm \sqrt{\rho_\pm} V_\pm^T, \quad t_{12}(\omega_\pm) = U_\pm \sqrt{\tau_\pm} U_\pm^T. \quad (12b)$$

Here $U_{\pm}$ and $V_{\pm}$ are $N \times N$ unitary matrices, and $\tau_{\pm} \equiv 1 - \rho_{\pm}$ is a diagonal matrix with the transmission eigenvalues $T_{\pm, n} \in [0, 1]$ on the diagonal.

Combining Eqs. (8)–(12) we find expressions for $R_{\pm}$ in terms of $\tau_{\pm}$ and $\Omega = V_{\perp} a V_{\perp}$. The expression for $R_-$ is

$$R_- = N_+^{-1} \text{Tr} \tau_{\perp} \Omega \left(1 - \sqrt{\rho_+ \rho_-} \sqrt{\tau_+} \sqrt{\Omega_{\perp}} \right)^{-1} \tau_{\perp} \cdot \left(1 - \Omega_{\perp} \sqrt{\rho_- \rho_+} \sqrt{\tau_+} \right)^{-1} \Omega_{\perp}. \quad (13)$$

The expression for $R_+$ is similar (but more lengthy). To compute the averages $\langle R_{\pm} \rangle$ we have to average over $\tau_{\pm}$ and $\Omega_{\perp}$. We make the isotropy approximation that the matrices $V_{\pm}$ are uniformly distributed over the unitary group $U(N)$. For $\tau_{\text{dwell}} \Delta \omega \ll 1$ we may identify $V_{\perp} = V_{\perp}$, while for $\tau_{\text{dwell}} \Delta \omega \gg 1$ the matrices $V_{\perp}$ and $V_{\perp}$ are independent. In each case the average over $U(N)$
with $N \gg 1$ can be done using the large $N$-expansion of Ref. [12]. The remaining average over $T_{A,n}$ can be done using the known density $\rho(T)$ of the transmission eigenvalues in a disordered medium [1].

In the non-degenerate regime ($\tau_{\text{dwell}}\Delta \omega \gg 1$) the result is

$$
\langle R_- \rangle = \frac{T_0^2 A}{1 - (1 - T_0)^2 A^2},
$$

and

$$
\langle R_+ \rangle = 1 - T_0 + \frac{T_0^2 (1 - T_0) A^2}{1 - (1 - T_0)^2 A^2},
$$

where $T_0 = (1 + 2L/\pi l)^{-1}$ is the transmittance at frequency $\omega_0$ of the disordered medium in the large-$N$ limit [13]. The quantity $A = N^{-1} \text{Tr} a a^\dagger$ is the modal average of the reflectance of the phase conjugating mirror ($A \to \int_0^{\pi/2} d\phi |a(\phi)|^2 \cos \phi$ for $N \to \infty$). Eq. (14) can also be obtained within the framework of radiative transfer theory, in which interference effects in the disordered medium are disregarded [14].

$$
\langle R_- \rangle = 2T_0 \text{Re} \frac{\alpha_0^* (\alpha_0^2 - 1)}{\alpha_0^2 - \alpha_0^* 2} \text{arctanh} \alpha_0,
$$

(15a)

$$
\langle R_+ \rangle = 1 - 2T_0 \text{Re} \frac{\alpha_0^* (\alpha_0^2 - 1)}{\alpha_0^2 - \alpha_0^* 2} \text{arctanh} \alpha_0^*,
$$

(15b)

where the complex number $\alpha_0$ is determined by

$$
\int_0^{\pi/2} d\phi \frac{\cos \phi \ a(\phi)}{1 - \alpha_0 a(\phi)} = \frac{a_0}{1 - a_0^*}
$$

(15c)

When $\delta \to 0$, $\alpha_0 \to 1.284 - 0.0133 i$ for $\alpha = \pi/4$. Both $\langle R_- \rangle$ and $\langle R_+ \rangle$ have a monotonic $L$-dependence, tending to 0 and 1, respectively, as $1/L$ for $L \to \infty$.

To test the analytical predictions of random-matrix theory we have carried out numerical simulations. The Helmholtz equation

$$
(\nabla^2 + \varepsilon \omega_+^2/c^2) E = 0
$$

is discretized on a square lattice (lattice constant $d$, length $L$, width $W$). The relative dielectric constant $\varepsilon$ fluctuates from site to site between $1 \pm \delta \varepsilon$. Using the method of recursive Green functions [13] we compute the scattering matrix $S(\omega)$ of the disordered medium at frequencies $\omega_-$ and $\omega_+$. The reflection matrix $r_{\text{PCM}}$ of the phase-conjugating mirror is calculated by discretizing Eq. (8). From $S(\omega_{\pm})$ and $r_{\text{PCM}}$ we obtain the reflection matrix $r$ of the entire system, and hence the reflectances $R_{\pm}$ using Eq. (8).

We took $W = 51d$, $\delta \varepsilon = 0.5$, $\alpha = \pi/4$, and varied $\delta$ and $L$. For the degenerate case we took $\omega_+ = \omega_+ = 1.252 c/d$, and for the non-degenerate case $\omega_+ = 1.252 c/d$, $\omega_- = 1.166 c/d$. These parameters correspond to $N_+ = 22$, $l_+ = 15.5 d$ at frequency $\omega_+$. (The mean free path is determined from the transmittance of the disordered region.) In the non-degenerate case we have $N_- = 20$, $l_- = 20.1 d$. For comparison with the analytic theory, where the difference between $N_+$ and $N_-$ and between $l_+$ and $l_-$ is neglected, we use the values $N_+$ and $l_+$. Results for the average reflectances (averaged over 150 realizations of the disorder) are shown in Fig. 2 and are in good agreement with the analytical predictions.

A striking feature of the degenerate regime is the absence of the minimum in $\langle R_- \rangle$ as a function of $L/l$ for $A > 1$. A qualitative explanation for the disappearance of the reflectance minimum goes as follows. To first order in $L/l$, disorder reduces the intensity of light reflected with frequency shift $2\Delta \omega$, because some light is scattered back before it can reach the phase-conjugating mirror and undergo a frequency shift. To second order in

\[ \text{FIG. 2. Average reflectances } \langle R_\pm \rangle \text{ as a function of } L/l \text{ for } \alpha = \pi/4 \text{ and } \delta = 0.6, 0.9. \text{ The dashed curves are the non-degenerate case, given by Eq. (14). The solid curves are the degenerate case, given by Eq. (15) for } L/l \gtrsim 3. \text{ Data points are results from numerical simulations (open symbols for the non-degenerate case, filled symbols for the degenerate case). Error bars are the statistical uncertainty of the average over 150 disorder configurations. (When the error bar is not shown it is smaller than the size of the marker.)} \]
disorder increases the intensity because it traps the light near the mirror, where it is amplified by interaction with the pump beams. This explains the initial decrease of $\langle R \rangle$ followed by an increase in the non-degenerate regime. The decrease persists in the degenerate regime, because resonant transmission through the disordered region makes trapping inefficient. The resonant transmission is the result of constructive interference of multiply scattered light, which is made possible by phase-shift cancellation.

In conclusion, we have studied the interplay of optical phase-conjugation and multiple scattering in a random medium. The theoretical predictions of a reflectance minimum provides a clear signature for experimentalists in search for effects of phase-shift cancellation in strong inhomogeneous media. The random-matrix approach presented here is likely to have a broad range of applicability, as in the analogous electronic problem. One direction for future research is to include a second phase-conjugating mirror opposite the first, with a different phase of the coupling constant. Such a system is the optical analogue of a Josephson junction, and it would be interesting to see how far the analogy goes.

This work was supported by the Dutch Science Foundation NWO/FOM.