Creating and preserving multi-partite entanglement with spin chains

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We show how multi-partite entanglement, such as of W-state form, can be created in branched spin chain systems. We also discuss the preservation of such entanglement, once created. The technique could be applied to actual spin chain systems, or to other physical systems such as strings of coupled quantum dots, molecules or atoms.

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INTRODUCTION

The past few years has seen much interest in the propagation of quantum information through spin chains. From a practical quantum technology perspective, such controlled propagation of quantum information has potential use. For macroscopic distances, quantum states of light form the best vehicle for quantum communication—these can propagate through optical fibres or free space with high fidelity\(^1\). However, for much shorter distances it is quite possible that other media, such as spin chains, could make a very useful contribution. Future solid state quantum information devices could require microscopic quantum communication links, between separate quantum processors or registers, or between processors and memory, analogous to the communication links that exist within the computer chips of today.

We use the terminology “spin chain” for any physical system that can be modelled using the Hamiltonian of a coupled spin chain, introduced in the next section. In reality, this system could be a chain of actual spins (produced chemically, or fabricated) connected through nearest-neighbour interactions, or it could be a string of quantum dots or molecules (like fullerenes), containing exciton or spin qubits. A string of trapped atoms is another possibility.

To quantify how well quantum processors or registers might communicate, there has been emphasis on the quality of quantum state propagation through a spin chain or network. It has been shown that a single spin qubit can transfer with decent fidelity along a constant nearest-neighbour exchange-coupled chain\(^2\). The fidelity can approach unity for a qubit encoded into a packet of spins\(^3\). More complicated systems, with different geometry or chosen unequal couplings\(^4\),\(^5\),\(^6\),\(^7\),\(^8\) can also achieve perfect state transfer, as can parallel spins chains\(^9\),\(^10\), or a chain used as a wire, with controlled couplings at the ends\(^11\). Related studies have been made on chains of coupled quantum dots\(^12\), or oscillators\(^13\),\(^14\) and spin chains connected through longer range magnetic dipole interactions\(^15\). State transfer and operations via spin chains using adiabatic dark passage have also been considered\(^16\),\(^17\).

In this work, rather than investigating state transfer, we consider instead the approach of creating an entangled resource, using the natural dynamics of a branched spin chain system and a very simple initial state. The consequences of branching were first studied in the context of divided bosonic chains, composed of coupled harmonic oscillators\(^13\),\(^19\) and have since been studied in systems of propagating electrons\(^20\). In both of these works, the dynamics of Gaussian wave packet type excitations were investigated. In contrast, our work considers the propagation of a single spin-down excitation localized on a single site, and how this moves through a network of spin-up states, generating entanglement. Clearly such entanglement is a useful and flexible resource\(^16\),\(^18\),\(^19\),\(^20\),\(^21\),\(^22\),\(^23\) for quantum information. For example, it could be used for quantum teleportation\(^24\), which enables the transfer of a quantum state, or to connect separated quantum registers or processors. One advantage of this approach is that decoherence or imperfections, which can lead to imperfect entanglement production, could be countered by purification techniques\(^25\) prior to use of the entangled resource. This clearly requires some sort of preservation of the entanglement, once created. We briefly discuss this issue.

SPIN CHAIN FORMALISM

Consider a one-dimensional chain of \(N\) spins, each coupled to their nearest neighbours. The Hamiltonian for the system is

\[
H = - \sum_{i=1}^{N} \frac{E_i}{2} \sigma_i^z + \sum_{i=1}^{N-1} \frac{J_{i,i+1}}{2} (\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+) \tag{1}
\]

where \(\sigma_i^z\) is the \(z\) Pauli spin matrix for the spin at site \(i\) and similarly \(\sigma_i^\pm = \sigma_i^x \pm i \sigma_i^y\). For actual spins, \(E_i/2\) is the local magnetic field (in the \(z\)-direction) at site \(i\) and \(J_{i,i+1}/2\) is the local XY coupling strength between neighbouring sites \(i\) and \(i+1\). Alternatively, for a chain of quantum dots where each qubit is represented by the presence or absence of a ground state exciton, \(E_i\) is the exciton energy and \(J_{i,i+1}\) is the Förster coupling between dots \(i\) and \(i+1\)\(^12\). The spin chain formalism applies to both these physical systems, and to any others that can be modelled by the Hamiltonian\(^1\). The computational basis notation for the spin states at each site is \([0]_i \equiv |\uparrow_z\rangle_i\) and \([1]_i \equiv |\downarrow_z\rangle_i\).
The total $z$-component of spin (magnetization), or total exciton number, is a constant of motion as it commutes with $H$. It is therefore useful to use a state of the system which is the ground state, except for a single flipped spin. This state is straightforward to prepare, assuming local control somewhere (such as at the end of a chain) over a spin. The flipped spin can be regarded as a (conserved) travelling qubit as it moves around under the action of the chain dynamics. An efficient way of representing such states in an $N$ spin network is the site basis defined as $|\ell\rangle = |0_1,0_2,\ldots,\ell_{k-1},1_k,\ell_{k+1},\ldots,0_N\rangle$. A system prepared in this subspace remains in it, with the detailed dynamics depending on the local magnetic fields or exciton energies. However, if these are independent of location $i$ then the dynamics favour quantum state transfer processes, as already mentioned. We adopt this limit in this work.

**CREATION OF MULTI-PARTITE ENTANGLEMENT**

In other works we have considered the distribution and freezing of bipartite entanglement, using a simple one-to-two branched spin chain system and chosen couplings along the chain. A simple spin flip at the end of the input branch can evolve into a superposition of being at one output or the other, thus generating maximal bipartite entanglement at the outputs. Furthermore, this entanglement can be frozen against further dynamical evolution if the final link of each output branch is an additional one-to-two branch, and rapid phase flips are applied when the state reaches the output. Such branched systems can be analysed simply and elegantly using the results of Christandl et al. in [5]. They show that perfect transfer along a spin chain is possible, using $N-1$ unequal couplings for an $N$ site chain—between site $i$ and site $i+1$ given by

$$J_{i,i+1} = \alpha \sqrt{i(N-i)} \ . \quad (2)$$

The size of the interaction is set by the constant $\alpha$. The authors of [5] also demonstrate the projection of an array of spins with equal couplings $j$ onto a 1D chain of unequal couplings. For an array arranged as a series of columns, with each node of column $i$ connected to $n$ different nodes of adjacent column $i+1$, and each node of column $i+1$ connected to $m$ different nodes of column $i$, each column can be projected onto a single node of a one-dimensional spin chain, with coupling $j\sqrt{mn}$ between nodes $i$ and $i+1$ [5]. The consequence of this is that at a one-to-two branching node, the two output couplings pick up a factor of $1/\sqrt{2}$ compared to the value in the equivalent 1D chain (which should satisfy (2) for perfect transmission). This same $1/\sqrt{2}$ branching factor also prevents hub reflection for propagating wave packets in divided chains of oscillators [19], as well as Gaussian wave packets of propagating electrons or magnons [20].

To give an explicit example of how these ideas extend to multi-partite entanglement generation, we consider the spin chain system of figure . Here, using the results of [5], the couplings are chosen so that the 1D equivalent spin chain follows eq. (2) and thus perfect state transfer occurs after a time $\pi/(2\alpha)$. This corresponds to the creation of a tripartite W-state [26]

$$|W_0\rangle = \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle) \quad (3)$$

for the three output spins of the actual structure of figure.

Clearly the creation of the W-state is transient, as the dynamics continues. The entanglement reappears at integer multiples of $\pi/\alpha$ after its initial creation. However, the entanglement can be preserved if fast (compared to $\pi/\alpha$) phase rotations are applied to two of the output qubits at a time when the W-state exists. From the symmetry of the Hamiltonian for the branched system, it is clear that the two states orthogonal to $|W_0\rangle$ are eigenstates and thus play no role in the state transfer evolution. Therefore, if a fast relative phase rotation of $2\pi/3$ is applied to one output spin, and the reverse rotation applied to another of the state $|W_0\rangle$, the W-entanglement will be frozen. Further non-trivial dynamics will occur. This demonstrates how multi-partite entanglement, once created, could be preserved for future use.

A further example of this is the use of the W-state to prepare a singlet. Suppose that the third output spin is measured in the computational basis. With probability 1/3 the first two output spins are projected to $|00\rangle$ and with probability 2/3 they are projected to $|01\rangle + |10\rangle$, the state being heralded by the measurement result. If a fast phase flip is now applied to one of these qubits, the singlet state results. As this is a linear superposition of $|W_+\rangle$ and $|W_-\rangle$, with no $|W_0\rangle$ component, it is frozen.

![FIG. 1: A five-node, one-to-three branched spin chain. The couplings are chosen to effect perfect state transfer in the 1D equivalent three-node chain, thus creating a tripartite W-state in this system.](image)
and $p$ output branches. Examples of their members are represented in figure 2.

Panel (a) corresponds to a “star-like” family [27], in which there is just one hub and all output branches have the same length. We denote members of this family by $(m, l, l, ..., l)$, with an input branch of $m$ spins and $p$ output branches of $l$ spins. Extending the discussion for one-to-two and one-to-three systems, for a one-to-$p$ system the coupling on the outputs from the hub spin are all pick up a factor of $1/\sqrt{p}$ compared to the effective one-dimensional spin chain value, that would be chosen according to eq. (2) to achieve perfect state transfer. (Again, an analogous factor also exists for the condition of no hub reflection for a wave packet excitation on the bosonic or propagating fermion systems discussed previously [19, 20].) The entangled state which can be created by using this family is a $p$-way W-state, consisting of an excitation shared between $p$ separated sites, with the symmetric form $(|1_{n_2}, 0_{n_3}, ..., 0_{n_p}⟩ + |0_{n_2}, 1_{n_3}, ..., 0_{n_p}⟩ + ... + |0_{n_2}, 0_{n_3}, ..., 1_{n_p}⟩)/\sqrt{p}$. In real systems it is likely to be impractical to prepare $p$-way entangled states using this approach for large values of $p$. However, if three-dimensional physical structures can be built—and for example there is the potential for this with quantum dot systems—modest values of $p > 2$, such as 3, 4 and 5 could be possible. For the case $l = 1$, the entanglement could again be frozen by the application of fast relative phase factors. The phases $\theta_i$ for each output $i$ are chosen such that the resultant states are orthogonal to the dynamically produced W-type state; i.e. $\sum_i \exp(i\theta_i) = 0$. There are always $(p - 1)$ possible methods of achieving this, one of which is to apply phases equal to the $p$-th roots of unity to the output spins. Alternatively, for an even $p$ one can apply a $\pi$ phase to $p/2$ outputs.

For the case $l > 1$ the freezing is not complete, although the wait time for the entanglement to reappear is shortened due to destructive interference preventing the spin flip from propagating back beyond the hub, as discussed in [28] for the bipartite case.

Panel (b) gives an example of what we term a “multiple bifurcation” structure. These structures contain more than one hub—and for perfect transfer the branching rule must be implemented at each. Note that for timing of the dynamical evolution to produce complete and simultaneous excitation transfer to the outlying sites, the number of sites along all output paths from the initial hub must be the same. The entangled states which can be created by using this family share the excitation between the different separated outlying sites with different unequal weights. For example, the structure represented in panel (b) gives the form $|1_{n_2}, 0_{n_3}, 0_{n_4}⟩/\sqrt{2} + |0_{n_2}, 1_{n_3}, 0_{n_4}⟩/2 + |0_{n_2}, 0_{n_3}, 1_{n_4}⟩/2$. With only modest branching at each hub, a bifurcating structure potentially could be fabricated in a planar arrangement. This could in principle give rise to a significant number of output branches, whose outlying spins can be entangled, sharing an excitation in an asymmetric manner, where the type of asymmetry depends on the number of hubs and on their position.

FIG. 2: (a) Example of a star-like family member $(m, l, l, l, l)$. (b) Example of a bifurcation-like family member.

CONCLUSION

We have illustrated how branching spin chain systems can be used to generate and distribute multi-partite entanglement from their natural dynamics. Clearly once it is distributed, this entanglement could be isolated through swapping the states of the spins at the ends of the output branches into other adjacent qubits. An alternative approach considered in this work is the application of simple single-qubit operations to the output branch ends. Given its simplicity, this latter approach may be particularly useful in initial experimental investigations of such branched systems. We illustrated this by showing how a tripartite W-state could be created and completely frozen in a very simple branched system. W-states are robust and immune against global dephasing [29]; research is on-going into specific applications of W-states, including in teleportation [30] secure communication [31] and leadership election protocols [32]. Generally, distributed entanglement provides a useful resource, for example for teleportation [24] or distributed quantum processing. In contrast to the use of spin chains to propagate quantum states from one place to another with as high a fidelity as possible, there could be some advantage in building up a high fidelity entangled resource “off line”. Real systems, with their inevitable imperfections, will almost certainly degrade transmission fidelities, even if in principle these approach unity. Certainly, with the “off line” resource approach, purification [25] could be applied to build...
up a higher fidelity resource that can be achieved by direct transmission, provided that the entanglement can be preserved once it is created. High fidelity purified entanglement could be used to transfer quantum states, or perform some other form of quantum communication. In effect, the concept of a quantum repeater could be employed in a solid state, spin chain scenario.

We comment finally that all these related effects result from the basic dynamics of the branched spin chain systems and simply prepared initial states. Some control is needed over the couplings to achieve entanglement creation, distribution and preservation, but there is certainly significant potential for branched spin chain systems to find application in solid state quantum processing and communication. This potential will continue to grow, as fabrication or creation of solid state systems that can operate as spin chains continues to progress.

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