Abstract

Non-abelian higher gauge theory has recently emerged as a generalization of standard gauge theory to higher dimensional (2-dimensional in the present context) connection forms, and as such, it has been successfully applied to the non-abelian generalizations of the Yang-Mills theory and 2-form electrodynamics. (2+1)-dimensional gravity, on the other hand, has been a fertile testing ground for many concepts related to classical and quantum gravity, and it is therefore only natural to investigate whether we can find an application of higher gauge theory in this latter context. In the present paper we investigate the possibility of applying the formalism of higher gauge theory to gravity in (2+1) dimensions, and we show that a nontrivial model of (2+1)-dimensional gravity coupled to scalar and tensorial matter fields - the $\Sigma\Phi E A$ model - can be formulated both as a standard gauge theory and as a higher gauge theory. Since the model has a very rich structure - it admits as solutions black-hole BTZ-like geometries, particle-like geometries as well as Robertson-Friedman-Walker cosmological-like expanding geometries - this opens a wide perspective for higher gauge theory to be tested and understood in a relevant gravitational context. Additionally, it offers the possibility of studying gravity in (2+1) dimensions coupled to matter in an entirely new framework.

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I. INTRODUCTION

Higher gauge theory\(^1\) (HGT), in its non-abelian version, has emerged in the last few years as the natural generalization - from both the mathematical and the physical perspective - of the more traditional standard gauge theory (SGT) to consistently include non-abelian higher 2-form connections. Alternatively, HGT can be viewed, in the integral formulation, as the non-abelian generalization to "surfaces" of the SGT, which involves only curves.

It is not our intention to develop or present general arguments in support of the above statement. Excellent such arguments are already extant in the literature [1], [2]. Instead, we will focus on illustrating how the concept of a HGT can be applied in the context relevant for this paper - namely in (2+1)-dimensional gravity - and on exploring its implications.

For this purpose, we will begin by considering pure gravity in (2+1)-dimensions (with vanishing cosmological constant) as an \(SO(2,1)\) gauge theory. The action of the theory is given by the expression:

\[
S[E, A] = \int_M Tr(E \wedge R[A])
\]  

where \(M\) is the spacetime manifold, \(E\) and \(A\) are \(so(2,1)\) Lie algebra valued 1-forms associated with the triad\(^2\) fields and the Lorentz connection respectively, \(R[A] = dA + A \wedge A\) is the curvature 2-form of the Lorentzian connection \(A\), and \(Tr(...)\) is the non-degenerate invariant bilinear form that can be defined on the \(so(2,1)\) Lie algebra. It is well known that the theory has only topological degrees of freedom and as such it is exactly solvable [3], [4].

The first order variation of the action (1) yields the equations of motion \(R[A] = DE = 0\), where "D" stands for the covariant derivative of the triad field. These equations of motion state in fact that the spin-connection is locally flat \((R[A] = 0)\), and that the spacetime has vanishing torsion \((DE = 0)\). Assuming now that the spacetime has a topology \(M = R \times S\) with \(S\) a spacelike surface, the equations of motion split canonically into evolution equations for the fields (the components of the equations that involve the time derivatives of the fields) and into constraints (the components of the equations involving only the spatial derivatives.

\(^1\) Although we will be using in all of the following the term "higher gauge theory" to conform to the terminology already in use in the literature, we are in fact referring to "2-gauge" theory, as it will become obvious shortly.

\(^2\) The \(E\) fields can be identified with the spacetime triads only if they are invertible, but in the following, for simplicity reasons, we will assume that the \(E\) fields are in fact invertible and we will refer to them as the triads.
of the fields), the latter becoming the generators of the gauge symmetries of the theory. The constraints resulting from the flatness condition become the generators of the Poincaré translational symmetries, while the constraints resulting from the vanishing of the torsion become the generators of Lorentz symmetry transformations. The Poisson algebra of these constraints is the Poincaré algebra, and hence they generate the Poincaré (ISO(2,1)) group as the gauge group for (2+1)-dimensional gravity.

Under these circumstances, one can immediately construct a very important quantity for the theory, namely the holonomy of the spin-connection \( A \) along curves embedded in the spacetime manifold. The importance of the holonomy resides in the fact that on one hand, for closed curves, it provides - upon tracing - observables for the theory due to its invariance under the action of the Poincaré gauge group, and on the other hand - and this is the aspect of relevance for the present discussion - it provides a natural way for labeling curves and loops in the spacetime manifold. It is quite straightforward to understand this latter issue of labeling. As a path-ordered product of exponentials of \( so(2,1) \) Lie algebra elements, the holonomy of the spin connection is itself an \( SO(2,1) \) group element, and as such it can be naturally associated with its underlying curve. In this way, every curve embedded in the spacetime manifold can be labeled by the \( SO(2,1) \) group element corresponding to the holonomy of the spin connection along the curve, as one would have expected to be the case for a SGT. So the picture of (2+1)-dimensional gravity as an SGT is quite clear.

The next issue that one has to consider is the generalization of (2+1)-dimensional gravity to the framework of HGT. According to the fundamentals of HGT, in the differential picture such a generalization should involve - in addition to the Lorentzian connection \( A \) - a 2-form field that satisfies the requirements of a 2-connection, i.e. a 2-connection whose 3-curvature must satisfy certain mathematical conditions. Since such a 2-connection can naturally be integrated over a 2-dimensional manifold (over a surface), in the integral picture the holonomy of the 2-connection would offer a natural way for the surfaces enclosed by curves to also be labeled in a consistent way by group elements.

Our approach to the generalization of (2+1)-dimensional gravity to the HGT framework is based on the main idea that the HGT formalism should reduce to the classical SGT formalism in the limit of a vanishing connection 2-form. Such a generalization, while not universal - one could devise other ways to generalize gravitation in (2+1) dimensions in the present context - has several practical advantages, as will be discussed in more detail.
below. Also, from the principal viewpoint, such a generalization is justified by the fact that it is the most natural and consistent way of extending a known theory to a more general framework.

With the above considerations, we can immediately make a few very important observations that will help us with the construction of the generalized model.

1. Under the assumption that the HGT generalization of (2+1) dimensional gravity reduces to the classical SGT formalism, the limit of vanishing 2-connection implies that the Lagrangian density of the action of the generalized model should necessarily contain the term $Tr(E \wedge R[A])$. Under these circumstances the 2-connection should enter the generalized Lagrangian in a separate term. It is now rather obvious why we have chosen this approach for the generalization of (2+1)-dimensional gravity to the HGT formalism. The term $Tr(E \wedge R[A])$ present in the in the generalized lagrangian will yield upon first order variation of the fields equations of motion involving the 2-curvature of the lorentzian 1-connection and the torsion of the triad, and as such, it will allow one to determine the geometry of the spacetime - in principle at least - directly from the equations of motion of the generalized model. Of course, these equations of motion will be more complicated than their pure gravity counterparts since they will contain additional terms involving the 2-connection whose explicit form will be dependent on how the 2-connection is actually coupled to pure gravity. Nevertheless, they will offer a clear picture of how the spacetime geometry will affected by the presence of the 2-connection of the HGT formalism.

2. By introducing the 2-form connection in the generalized action, we have in fact introduced a new canonical variable, and if we want the classical formulation to be consistent, it is absolutely necessary to introduce an additional field/variable, which should be canonically conjugate to the 2-connection. Furthermore, since canonical conjugacy implies the presence of an exterior derivative operator acting on one of these variables, a simple counting argument shows that the variable conjugate to the 2-connection can only be a 0-form.

3. We have established that the additional term that has to be added to the Lagrangian density of pure gravity in order to generalize the latter to the framework of HGT has to be a functional of a 2-form and a 0-form, and it must include at the very least
an exterior derivative operator acting on one of these two forms. However, since we want the 2-form to be a 2-connection, and not some arbitrary field, the structure of this additional term must also be such that upon a first order variation of the fields it should yield an equation of motion that involves the 3-curvature of the 2-form. Under these circumstances, an obvious choice for the term to be added to the Lagrangian density of pure gravity - and by analogy with the latter - is a functional linear in the 3-curvature of the 2-connection and also linear in the conjugate 0-form field. With these observations, we now have a quite clear picture of the theory that generalizes pure (2+1)-dimensional gravity to the formalism of HGT. If Φ are 0-form Lie algebra valued fields and Σ is the Lie algebra valued 2-connection with the 3-curvature given by the expression:

\[ G[\Sigma] = d\Sigma + A \triangleright \Sigma \]  

where "\triangleright" is an action of (Lie) algebras that will be discussed in more detail in the next section, then the action for (2+1)-dimensional gravity in the HGT formalism has the general form:

\[ S_{HGT} = \int_M \left\{ \text{Tr}_1 (E \wedge R[A]) + \text{Tr}_2 (\Phi \wedge G[\Sigma]) \right\} \]  

Of course, there are several other issues that must be made explicit, like the underlying 2-algebra structure of the fields and the two traces in (3), but nevertheless, under the assumptions of our approach, (3) represents the most natural generalization of pure gravity in (2+1)-dimension to the formalism of HGT.

At this time, there is one more and extremely important observation that we need to make. The expression (3) of the generalized action is reminiscent of a model that has already been studied in the literature, namely the so-called ΣΦEA model \[5\] whose action is given by the expression³:

\[ S_{\Sigma\Phi EA} = \int_M \text{Tr} \left\{ E \wedge R[A] + \Phi \wedge D\Sigma \right\} \]  

where in (4) Φ, Σ are 0-form fields and 2-form fields respectively, the trace is the non-degenerate invariant bilinear form defined on the Lie algebra of \( SO(2,1) \) and "\( D \)" is the

³ The original action of the ΣΦEA model differs from the action in (4) by a surface term, but this is irrelevant for the content of the present paper.
classical exterior covariant derivative.

The resemblance between the two models is not merely a matter of appearance. As we will show, the $\Sigma \Phi EA$ model can be reformulated as a higher gauge theory over the Poincaré 2-algebra, with an action having exactly the form $\Box$. As such, and since it has non-trivial solutions like the BTZ black-hole geometry and the point-particle geometry, the $\Sigma \Phi EA$ constitutes the sought generalization of pure gravity in (2+1) dimensions to the HGT formalism, and to the best knowledge of the authors, the first non-trivial application of the HGT formalism to gravity.

The remainder of the paper is organized as follows. In Section II and Section III we review the basic notions of the HGT formalism and of the $\Sigma \Phi EA$ model respectively, as they pertain to the purpose of this paper. In Section IV we show explicitly how the $\Sigma \Phi EA$ model can be reformulated to become the generalization of pure gravity in (2+1) dimensions to the HGT formalism, and in Section V we provide a discussion of the results and some concluding remarks regarding the future applications of the HGT formalism in gravity.

II. 2-GROUPS, 2-ALGEBRAS AND HIGHER GAUGE THEORY

In this section, we will present the fundamentals of the HGT formalism, limiting ourselves only to those concepts that are relevant for the contents of this paper for reasons of simplicity. Under these circumstances, the presentation of these concepts is by no means self-contained, and for more details, the interested reader is referred to [1], [2], [6] and the references therein.

A. Lie 2-groups and Lie 2-algebras

The relevant mathematical structure underlying the HGT formalism in the integral formulation is the "strict" (Lie) 2-group, which can be viewed as the generalization of the (Lie) group structure underlying the classical SGT. The beauty of the HGT formalism is that due to its categorical nature, there are several equivalent ways in which a "strict" 2-group can be defined [7], [8], and consequently one can choose the definition that best fits one’s purpose. In our particular case, the definition best fitted for a "strict" 2-group is that of a crossed module, and we will give this definition below without insisting on its explicit relation with the (original) categorical definition beyond the statement that the two definitions
are equivalent.

**Definition 1** A Lie 2-group as a crossed module is a quadruple \((G, H, t, \cdot)\) consisting of:

- **a.** Two Lie groups \(G, H\).

- **b.** A smooth group homomorphism \(t : H \rightarrow G\), i.e. a map \(t\) satisfying the relations:

\[
\begin{align*}
t(h_1 \cdot h_2) &= t(h_1) \cdot t(h_2) \\
t(1_H) &= 1_G
\end{align*}
\]

for any \(h_1, h_2 \in H\).

- **c.** A smooth action \(\cdot\) of \(G\) on \(H\) by automorphisms (a smooth homomorphic map \(G \rightarrow \text{Aut}[H]\)), i.e. a group action satisfying the relations:

\[
\begin{align*}
(g_1 \cdot g_2) \cdot h &= g_1 \cdot (g_2 \cdot h) \\
1_G \cdot h &= h \\
g \cdot (h_1 \cdot h_2) &= (g \cdot h_1) \cdot (g \cdot h_2) \\
g \cdot 1_H &= 1_H
\end{align*}
\]

for any \(g, g_1, g_2 \in G\) and any \(h, h_1, h_2 \in H\).

The map \(t\) and the operation \(\cdot\) are required to satisfy the following two sets of compatibility conditions:

- **i)** For any \(g \in G\) and for any \(h, h' \in H\) we have:

\[
\begin{align*}
t(g \cdot h) &= g \cdot t(h) \cdot g^{-1} \\
t(h) \cdot h' &= h \cdot h' \cdot h^{-1}
\end{align*}
\]

- **ii)** For any \(g \in G\) and for any \(h \in H\) there exists \(g' \in G\) such that:

\[
g' = t(h) \cdot g
\]

The compatibility condition (ii), which is related to the generalized notion of flatness and therefore plays a fundamental role in the HGT formalism, is a direct consequence of the construction of the "strict" Lie 2-group as a crossed module. However, there is a
simpler and more intuitive way to understand it: since according to the definition, \( t(h) \in G \), then for any \( g \in G \), the product \( t(h) \cdot g \) is an element of \( G \), or in other words, there is an element \( g' \in G \) such that \( g' = t(h) \cdot g \), which is exactly what the compatibility condition \((ii)\) above states. The conditions in \((7)\) have the role of ensuring the compatibility\(^4\) between the homomorphism \( t \) and the operation \( "\circ" \), or in other words, of ensuring the consistency of the labeling and composition of surfaces.

Analogous to the case of standard Lie groups, one can associate with any ”strict” Lie 2-group a ”differential” object, called a ”strict” Lie 2-algebra, which can be thought of in the present context as the generalization of the notion of Lie algebra to the formalism of higher gauge theory. Also, similar to the case of the Lie 2-groups, there are several equivalent ways in which a Lie 2-algebra can be defined. For our purposes the most appropriate definition is that involving the concept of differential crossed module (which in turn can be thought of as the ”Lie algebra” of a Lie crossed module). Once again, we will only give the definition of the Lie 2-algebra as a differential crossed module without insisting on the relation between this definition and the original categorical definition beyond the statement that the two definitions are equivalent.

**Definition 2** A Lie 2-algebra as a differential crossed module is a quadruple \((g, h, \tau, \triangleright)\) consisting of:

- **a.** Two Lie algebras \( g \) and \( h \).

- **b.** A homomorphism of Lie algebras \( \tau : h \rightarrow g \), i.e. a map satisfying the relation:

\[
\tau([Y_1, Y_2]) = [\tau(Y_1), \tau(Y_2)]
\]

for any \( Y_1, Y_2 \in h \).

- **c.** An action \( "\triangleright" \) of \( g \) on \( h \) by derivations, i.e. a bilinear operation satisfying the

\(^4\) It should be noted that the two conditions in \((7)\) are not independent. For example, by applying the homomorphism \( t \) to the second equation in \((7)\) and by making use of \((5)\) and \((6)\), it is straightforward to recover the first condition in \((7)\). However, the presentation of the compatibility relation between the map \( t \) and the operation \( "\circ" \) by two separate conditions is clearer and more convenient from the practical viewpoint.
relations:

\[ [X_1, X_2] \triangleright Y = X_1 \triangleright (X_2 \triangleright Y) - X_2 \triangleright (X_1 \triangleright Y) \]
\[ X \triangleright [Y_1, Y_2] = [X \triangleright Y_1, Y_2] + [Y_1, X \triangleright Y_2] \]

(10)

for any \( X, X_1, X_2 \in g \) and for any \( Y, Y_1, Y_2 \in h \).

The map \( \tau \) and the operation \( \triangleright \) must satisfy the following two sets of compatibility conditions:

i) For any \( X \in g \) and for any \( Y, Y' \in h \) we must have:

\[ \tau(X \triangleright Y) = [X, \tau(Y)] \]
\[ \tau(Y) \triangleright Y' = [Y, Y'] \]

(11)

ii) On a hypercubic lattice, for any \( X_1 \in g \) and for any \( Y \in h \) there exists \( X_1 \in g \) such that:

\[ \tau(Y) = X_2 - X_1 \]

(12)

Similar to the case of 2-groups, the compatibility condition (12) arises from the construction of a Lie 2-algebra from a differential crossed module. It is straightforward to see that is the lattice counterpart of (8), and as such, it has a similar intuitive explanation and it plays an equally important role - in the differential picture - in the concept of generalized flatness.

The relation between Lie 2-groups and Lie 2-algebras is similar to the relation between standard Lie groups and Lie algebras. Once again, we will not expound upon the details of this relation beyond stating that every Lie 2-group has a Lie 2-algebra \[1\], and that the maps and operations that enter the above definitions are formally related through the expressions \( \tau = d(t) \) and \( \triangleright = d(\blacklozenge) \) \[1, 2\]. Under these circumstances, the Lie 2-algebras emerge intuitively as the ”differentials” of Lie 2-groups in the neighborhood of the 2-group identity, which is pretty much what one would have expected.

B. Higher gauge theory

Once we have introduced the necessary mathematical structure, we can now proceed with the setting of the HGT formalism. Since in all of the following considerations we will be
more interested in the "differential" picture rather than in the "integral" one, we will mainly limit ourselves to use the latter for pictorial purposes only.

As mentioned in the introduction, the general purpose of higher gauge theory is to allow for the possibility of labeling both curves and surfaces in a (spacetime) manifold by group elements, and hence develop - intuitively speaking - a gauge theory of curves and surfaces. In the integral picture, this is done as follows. Given a 2-group \((G, H, t, \cdot)\), we label the curves in the manifold by elements \(g \in G\) and the surfaces of the manifold that are bound by the curves with elements \(h \in H\). The composition of curve labels is done as in classical SGT, and in addition, one can develop in a consistent manner composition rules for surface labels. Using these composition laws for curve and surface labels, one can then generalize the classical SGT concepts of gauge transformations, gauge invariance, etc. to obtain a consistent gauge theory - a 2-gauge theory - whose description and symmetries are determined by the underlying 2-group structure \([1],[2]\).

Alternatively, in the differential description, the underlying structure of HGT is a Lie 2-algebra \((\mathfrak{g}, \mathfrak{h}, \tau, \triangleright)\), and on this Lie 2-algebra we define a \(\mathfrak{g}\)-Lie algebra valued connection 1-form \(A\) and a \(\mathfrak{h}\)-Lie algebra valued connection 2-form\(^5\) \(\Sigma\) (the 2-connection mentioned in the introduction). Of course, since the connection \(A\) is a 1-form, it can naturally be integrated along curves, and as such it can be used to label the curves in the (spacetime) manifold as in the case of the classical SGT. Similarly, since the connection \(\Sigma\) is a 2-form, it can naturally be integrated on surfaces, providing the labels for the surfaces of the (spacetime) manifold.

Let \(\{\gamma^a\}\) be the generators of the Lie algebra \(\mathfrak{g}\), and in all of the following, lower case latin indices form the beginning of the alphabet will denote \(\mathfrak{g}\)-algebra indices. We define the covariant derivative of a \(\mathfrak{g}\)-valued form \(V = V_a \gamma^a\) as:

\[
D_A V = dV + [A, V] = dV + A_a \wedge [\gamma^a, \gamma^b] 
\]

and with this definition, the SGT curvature of the 1-connection \(A\) is given by the expression:

\[
R[A] = dA + \frac{1}{2} [A, A] = da + \frac{1}{2} A_a \wedge [\gamma^a, \gamma^b] 
\]

\(^5\) The notation that we use for the 2-connection is different from the traditional notation used in \([1],[2]\). Instead of \(B\) for the 2-connection, we denote the 2-connection \(\Sigma\) in order to make clear the relationship between the HGT formalism and the \(\Sigma\Phi\text{EA}\) model.
The generalized HGT curvature of the 1-connection $A$ is given by the expression:

$$F[A] = R[A] + \tau(\Sigma)$$

and in the HGT formalism, the generalized curvature $F[A]$ must always be vanishing, i.e. one must always have $F[A] = 0$. This vanishing of the generalized curvature should not be confused with the concept of flatness of the connection 1-from $A$ in SGT. They are different concepts. As can be seen from (15) the vanishing of the generalized curvature implies in the general case a non-flat 1-connection with the SGT curvature given by the expression:

$$R[A] = -\tau(\Sigma)$$

Similarly, if we let $\chi^m$ be the generators of the Lie algebra $\mathfrak{h}$, and we consider latin lower case letters from the end of the alphabet to be $\mathfrak{h}$-algebra indices, we can define the covariant derivative of an $\mathfrak{h}$-valued form $W = W_m\chi^m$ with respect to the $\mathfrak{g}$-valued 1-connection $A$ through the expression:

$$D_A W = dW + A \triangleright W = dW + A_a \wedge W_m(\gamma^a \triangleright \chi^m)$$

Furthermore, in the HGT formalism we can also define a curvature 3-form (the 3-curvature) of the 2-connection $\Sigma$ through the relation:

$$G[\Sigma] = D_A \Sigma = d\Sigma + A \triangleright \Sigma$$

and using this definition, we can introduce the concept of 2-flatness the 3-connection to vanish.

The infinitesimal gauge transformations of these two connections in the HGT formalism are given by the expressions:

$$\delta A = D_A \alpha - \tau(\lambda)$$

$$\delta \Sigma = D_A \lambda - \alpha \triangleright \Sigma$$

where $\alpha$ is a $\mathfrak{g}$-valued 0-from gauge parameter and $\lambda$ is a $\mathfrak{h}$-valued 1-form gauge parameter.

With these considerations we have finished the review of all the fundamental concepts of the HGT formalism that will the necessary for the generalization of (2+1)-dimensional gravity to the HGT formalism. However, before proceeding with the latter task, it is necessary to present a similar review of the $\Sigma\Phi E A$ model, since, as mentioned in the introduction, this model is the principal candidate for the generalization of (2+1)-dimensional gravity to the HGT framework.
III. THE $\Sigma\Phi EA$ MODEL

In this section, we briefly review the basic aspects of the $\Sigma\Phi EA$ model as they are relevant to the generalization of the model to the HGT formalism. For more details about this model the interested reader is referred to [5]. In all of the following we will use, whenever possible, the same notation that we used in the previous section, in order to emphasize the similarities that exist between this model and the HGT formalism.

As mentioned in the introduction, the action of the $\Sigma\Phi EA$ model is given by the expression:

$$S_{\Sigma\Phi EA} = \int_M Tr\{E \wedge R[A] + \Phi \wedge D_A\Sigma\} = \int_M \{E_i \wedge R^i[A] + \Phi_i \wedge D_A\Sigma^i\}$$

(20)

where $E, A$ are the $so(2,1)$-valued triad and (spin) connection 1-form fields, and $\Sigma, \Phi$ are $so(2,1)$-valued 2-form and respectively 0-form fields. The trace in (20) is the non-degenerate invariant bilinear form defined on the $so(2,1)$ Lie algebra through the relation:

$$Tr(J^i J^j) = \eta^{ij}$$

(21)

where $\{J^i, i = 0, 1, 2\}$ are the generators of the $so(2,1)$ Lie algebra, satisfying the commutation relations:

$$[J^i, J^j] = \epsilon^{ijk} J_k$$

(22)

lower case latin indices from the middle of the alphabet are $so(2,1)$ algebra indices, $\eta = diag(-, +, +)$, and for the structure constants in (22) we use the convention $\epsilon^{012} = 1$. The fields $\Sigma$ and $\Phi$ are coupled to gravity through the connection $A$ in the covariant derivative which is given by the expression:

$$D_A\Sigma^i = d\Sigma^i + \epsilon^{ijk} A_j \wedge \Sigma_k$$

(23)

Up to surface terms, the first order variation of the action (20) yields the equations of motion:

$$R^i[A] = 0$$

$$D_A E^i + \epsilon^{ijk} \Sigma_j \wedge \Phi_k = 0$$

$$D_A\Sigma^i = 0$$

$$D_A\Phi^i = 0$$

(24)
and these equations of motion are invariant under the following infinitesimal gauge transformations:

\[
\begin{align*}
\delta A^i &= D_A \alpha^i \\
\delta \Phi^i &= \epsilon^{ijk} \Phi_j \alpha_k \\
\delta \Sigma^i &= D_A \lambda^i - \epsilon^{ijk} \alpha_j \Sigma_k \\
\delta E^i &= D_A \beta^i + \epsilon^{ijk} (E_j \alpha_k - \Phi_j \lambda_k)
\end{align*}
\]  

(25)

with \( \alpha^i, \beta^i \) 0-form and \( \lambda^i \) 1-form \( so(2,1) \)-valued gauge parameters. Furthermore, the infinitesimal gauge transformations for the 2-form field \( \Sigma \) in (25) are themselves invariant under the infinitesimal "translations":

\[
\delta \lambda^i = D_A \rho^i
\]

(26)

where now \( \rho^i \) are 0-form \( so(2,1) \)-valued parameters, and under these circumstances, the number of independent gauge parameters describing the symmetries of the model reduces from 15 to 12.

We conclude the review of the \( \Sigma \Phi EA \) model with a few brief remarks regarding its solvability and solutions. Coupling the fields \( \Sigma, \Phi \) to pure gravity through the connection - as opposed to coupling them through the triads - has the main advantage that the resulting \( \Sigma \Phi EA \) theory preserves the topological character inherited from pure gravity [3], [4], [9]. Indeed, it can be shown [5] that the \( \Sigma \Phi EA \) is a topological model with no local degrees of freedom, and as such it is solvable both classically and in the quantum framework. Consequently, its solutions are non-trivial for non-trivial topologies of the spacetime manifold \( M \) and in the case of a spacetime manifold having the topology \( M = \mathbb{R} \times S \), with \( S \) a spacelike surface whose topology is that of a punctured plane, the model has as solutions the BTZ black-hole geometry [5] and the point particle geometry.

We now have all the required mathematical and physical background for the generalization of \( (2+1) \)-dimensional pure gravity with vanishing cosmological to the HGT formalism. Furthermore, by now it should be quite clear not only that there are close similarities between the \( \Sigma \Phi EA \) model and the HGT formalism, but also what these similarities are and how they could be exploited to reformulate the model in the HGT framework. In the next section, we will investigate in detail these similarities, and we will determine in detail a set of
exact circumstances under which the $\Sigma\Phi EA$ model becomes the sought HGT generalization of pure gravity in (2+1) dimensions.

IV. (2+1)-DIMENSIONAL GRAVITY AS A HIGHER GAUGE THEORY

As mentioned in Section I, with our choice of approach to this issue, generalizing (2+1)-dimensional gravity to the HGT formalism is equivalent with reformulating the $\Sigma\Phi EA$ model within the framework of HGT. In turn, this means on one hand that we must explicitly determine - in the differential picture - the components of the quadruple $(g, h, \tau, \triangleright)$ involved in the definition of a Lie 2-algebra and on the other hand, that we must also determine a suitable action defined over this Lie 2-algebra which reduces to the action of the $\Sigma\Phi EA$ model.

Determining the Lie algebra $g$ is straightforward. Based on the fact that in pure (2+1)-dimensional gravity the 1-connection is a lorentzian connection, the obvious choice for $g$ is the Lie algebra $so(2, 1)$. With this choice, and using the notation in Section III, the 1-connection of the HGT formalism becomes now an $so(2, 1)$-valued connection 1-form $A = A_i J^i$.

We still have to determine the remaining three components of the Lie 2-algebra quadruple - namely the Lie algebra $h$, the homomorphism $\tau$ and the action” $\triangleright$” - and in order to do so, it is useful to summarize the relevant similarities between the $\Sigma\Phi EA$ model and the HGT formalism. The main reason behind this idea is that, as mentioned earlier, we can use these similarities as a guide to establish the remaining details of the reformulation of the $\Sigma\Phi EA$ model as an HGT.

With the above choice for the Lie algebra $g$, the similarities between the two formulations can be summarized as follows:

S1. Both the $\Sigma\Phi EA$ model and the HGT formalism involve a connection 1-form, defined on the Lie algebra $g \equiv so(2, 1)$. Furthermore, they both impose strict conditions upon its SGT curvature, as given by (16) and (24).

S2. Both the $\Sigma\Phi EA$ model and the HGT formalism involve a 2-form $\Sigma$, which in the HGT formalism is a 2-connection, and as such it is defined on the Lie algebra $h$ yet to be determined. If we ignore for the moment the explicit form of the action ” $\triangleright$”, the third equation of motion for the $\Sigma\Phi EA$ model in (24) resembles very much a 2-flatness
condition on the 3-curvature \[18\] of the 2-connection.

**S3.** Once again, ignoring for the moment the explicit form of action "\(\triangleright\)" and of the homomorphism \(\tau\), as it can be seen from \[19\] and \[25\], the infinitesimal gauge transformations for the 1-connection \(A\) and the 2-form \(\Sigma\) in both formulations are given by very similar expressions.

Consider now the observation (S2). It is obvious that if we want the equation of motion for \(\Sigma\) in \[24\] to become a 2-flatness condition in the HGT formalism, we must define the action "\(\triangleright\)" of the Lie algebra \(g\) on \(h\) at the level of their generators as:

\[
J^i \triangleright \chi^m = \epsilon^{imn} \chi^n
\]  

(27)

where for the generators of the Lie algebra \(h\) we have used the notation in Section II, and \(\epsilon^{imn}\) in the rhs of \[27\] is the totally antisymmetric 3-dimensional Levi-Civita symbol as defined in Section III. Furthermore, since the rhs of \[27\] is reminiscent of an adjoint action, in the following we will actually require that the action "\(\triangleright\)" be the adjoint action of the Lie algebra \(g \equiv so(2, 1)\) on \(h\):

\[
J^i \triangleright \chi^m = ad[J^i](\chi^m) = [J^i, \chi^m] = \epsilon^{imn} \chi^n
\]  

(28)

This definition of the action "\(\triangleright\)" drastically restrict our choices for the Lie algebra \(h\). According to \[28\] the algebra \(h\) must be a 3-dimensional algebra such that the adjoint action of \(so(2, 1)\) on \(h\) is characterized by the Levi-Civita symbol \(\epsilon^{ijk}\). Under these circumstances, the choices are obvious: the Lie algebra \(h\) is either the Lorentz algebra \(so(2, 1)\) or the algebra of Poincaré translations \(t^3\). Furthermore, it should be also noted that with the choice \[28\] for the action "\(\triangleright\)" the infinitesimal gauge transformations for the 2-connection/2-form \(\Sigma\) in both \[19\] and \[25\] become identical.

Consider now the observation (S1). The restrictions imposed by the two formulations on the SGT curvature of the Lorentzian 1-connection \(A\) are given by the relations:

\[
R[A] = 0 \quad (\Sigma \Phi EA) \\
R[A] = -\tau(\Sigma) \quad (HGT)
\]  

(29)

It is clear that in order for these restrictions to match in both formulations, one should either have the 2-connection \(\Sigma\) belonging to the kernel of the homomorphism \(\tau\) or alternatively,
one should have the homomorphism $\tau$ be the trivial homomorphism $\tau(Y) = 0$ for any $Y \in h$.

Similarly, if one considers the observation (S3), then the infinitesimal gauge transformations for the Lorentzian 1-connection $A$ are given by the expressions:

\[
\delta A = D_A \alpha \ (\Sigma \Phi E A) \\
\delta A = D_A \alpha - \tau(\lambda) \ (HGT)
\]

and once again, in order to have the two relations match, one should either have the gauge parameter $\lambda$ belonging to the kernel of the homomorphism $\tau$ or alternatively, one should have the homomorphism $\tau$ be the trivial homomorphism.

With these considerations, we can now return to the issue of determining the Lie algebra $h$ of the Lie 2-algebra quadruple. It is straightforward to see that the Lie algebra $h$ cannot be $so(2,1)$, since if that were the case, the quadruple $\left( so(2,1), so(2,1), \tau, ad[so(2,1)](so(2,1)) \right)$ would not satisfy the requirements of a Lie 2-algebra. Explicitly, it is the second condition in (11), namely:

\[
\tau(Y) \triangleright Y' = [Y, Y']
\]

for any $Y, Y' \in so(2,1)$ that is not satisfied by the 2-connection by the 2-connection $\Sigma$ and its 1-form gauge parameter $\lambda$. The reason why these two forms do not satisfy (31) are obvious: according to this condition, if $\Sigma, \lambda$ belong to the kernel of the homomorphism $\tau$, they must also belong to the center of the Lie algebra $h \equiv so(2,1)$, and $so(2,1)$ has only a trivial center. This would imply in turn that the 2-connection and its gauge parameter can only be null forms, which is not a useful result for our purposes.

On the other hand, the condition (31) is identically satisfied for $\Sigma$ and $\lambda$ in the kernel of $\tau$ if $h \equiv t^3$, since the algebra of Poincaré translations is abelian. Furthermore, in this case (31) together with (28) implies that in fact the homomorphism $\tau$ can only be the trivial homomorphism $\tau = 0$.

Under these circumstances, the appropriate HGT mathematical structure underlying the generalization of (2+1)-dimensional gravity to the HGT formalism is in the differential picture the Lie 2-algebra given by the quadruple $\left( so(2,1), t^3, \tau = 0, ad[so(2,1)](t^3) \right)$, i.e. the adjoint or tangent Poincaré 2-algebra [1], [2].

It is useful at this time to give a brief summary of the above considerations. According to the previous arguments, the $\Sigma \Phi E A$ model could be reformulated as an HGT with the
Poincaré 2-algebra. In this case, the two connections would satisfy the equations of motion:

\[ R[A] = 0 \]
\[ G[\Sigma] = 0 \] (32)
i.e. the reformulation implies an HGT that is simultaneously 1-flat and 2-flat, and they should be invariant under the infinitesimal gauge transformations:

\[ \delta A = D_A \alpha \]
\[ \delta \Sigma = D_A \lambda - \alpha \triangleright \Sigma \] (33)

with \( \alpha \) an \( so(2,1) \)-valued 0-form gauge parameter and \( \lambda \) a \( t^3 \)-valued 1-form parameter. Furthermore, due to the 1-flatness of the model, the gauge transformation for the 2-connection \( \Sigma \) is itself invariant under the infinitesimal "translation":

\[ \delta \lambda = D_A \rho \] (34)

where \( \rho \) is a \( t^3 \)-valued 0-form parameter.

What still remains to be done is to find an HGT action which upon first order variation yields as equations of motion the equations (32) for the 2-connections and the remaining equations in (24), and which is invariant under the infinitesimal transformations (33), (34) and under the remaining gauge transformations in (25). Explicitly, and based on the considerations in Section I regarding the generalized form of the HGT action, this means that we need to assign the remaining fields \( E \) and \( \Phi \) to the appropriate Lie algebras of the 2-algebra quadruple, and that we must also determine an appropriate invariant trace operator for the action integral.

The easiest way to solve these remaining problems is to first consider the issue of the trace operator in the action integral. As mentioned in the introduction, the generalized action should be given by an expression of the form:

\[ S_{HGT} = \int_M \left\{ Tr_1 (E \wedge R[A]) + Tr_2 (\Phi \wedge G[\Sigma]) \right\} \] (35)

and we already know that the HGT theory should be formulated with the Poincaré 2-algebra such that the 1-connection is an \( so(2,1) \)-valued 1-form and the 2-connection is a \( t^3 \)-valued 2-form. Furthermore, it is known [3] that on the Poincaré Lie algebra one can define two
non-degenerate invariant bilinear forms, given by the expressions:

\[ \text{Tr}_g(J^iJ^j) = \text{Tr}_h(P^iP^j) = \eta^{ij} \]

\[ \text{Tr}(J^iP^j) = 0 \quad (36) \]

and

\[ \text{Tr}_P(J^iJ^j) = \text{Tr}_P(P^iP^j) = 0 \]

\[ \text{Tr}_P(J^iP^j) = \eta^{ij} \quad (37) \]

where in (36) and (37) we have used the traditional notation for the generators of the Poincaré algebra, and we have used the formal notation \( g \equiv so(2,1) \) and \( h \equiv t^3 \). Under these circumstances, we can only have two choices for assigning the remaining fields \( E \) and \( \Phi \) the algebras of the 2-algebra quadruple:

A. If we use the bilinear form (36), then \( E \) must be an \( g \)-valued 1-form, \( \Phi \) must be a \( h \)-valued 0-form, and the generalized action will be given by the formal expression:

\[ S_{\text{HGT-A}} = \int_M \left\{ \text{Tr}_g(E \wedge R[A]) + \text{Tr}_h(\Phi \wedge G[\Sigma]) \right\} \quad (38) \]

B. If we use the bilinear form (37), then \( E \) must be an \( h \)-valued 1-form, \( \Phi \) must be a \( g \)-valued 0-form, and the generalized action will be given by the formal expression:

\[ S_{\text{HGT-B}} = \int_M \text{Tr}_P \left\{ E \wedge R[A] + \Phi \wedge G[\Sigma] \right\} \quad (39) \]

Of course, both these actions yield the same equations of motion and have the same gauge symmetries as the \( \Sigma \Phi EA \) model. As such they represent the HGT reformulation(s) of this model, and hence the sought generalization(s) of (2+1)-dimensional pure gravity with vanishing cosmological constant to the HGT formalism. The only difference between the two generalization is that for a vanishing 2-connection, the actions (38) and (39) reduce to pure gravity as a gauge theory formulated over the \( so(2,1) \) Lie algebra and respectively over the Poincaré algebra.

We conclude this section with a few remarks regarding the non-triviality of these generalizations. As mentioned earlier, as a topological model the \( \Sigma \Phi EA \) model has non-trivial solutions for non-trivial topologies, and as such one would expect the above HGT generalization(s) to behave similarly since they are just reformulation(s) of the model in different
framework. In other words, we expect that the HGT generalization(s) of our model - and in general any HGT with no local degrees of freedom - should also have non-trivial solutions for non-trivial topologies of the spacetime manifold \( M \), and while ultimately this is indeed the case, it is useful at this time to analyze in more detail the circumstances under which this statement is valid.

To better illustrate the issue that we intend to address, it is useful to revert back to the integral picture of the HGT formalism and recall the labeling procedure for curves and surfaces in the spacetime manifold. In order to preserve the analogy with the SGT formulation of the \( \Sigma \Phi EA \) model, we will confine our considerations to the spatial leafs \( S \) of the foliation of the spacetime \( M = R \times S \). Under these circumstances, the labeling procedure for curves and surfaces embedded in \( S \) can be summarized as follows: if \( P, Q \in S \) are two distinct points in \( S \) and we connect these points by two distinct curves \( \gamma_1 \) and \( \gamma_2 \) such that the surface bound by these curves has the topology of a disc \([2]\), then we can label the two curves by the holonomies \( g_1, g_2 \in G \) of the 1-connection and the surface bound by these curves by a group element \( h \in H \) which can be interpreted as the analog of the Wilson line for the 2-connection. In the HGT framework, these labels are constrained to obey the condition \([8]\), which can be rewritten in the equivalent form \([2]\):

\[
t(h) = g_2 \cdot g_1^{-1}
\]  

(40)

It is useful to note that in this form, and as far as the labeling process is concerned, \([10]\) can be given the following interpretation: surfaces are labeled by group elements \( h \in H \) associated through the homomorphism \( t \) to the (inverse) holonomy of the 1-connection along the boundary of the surface. Loosely speaking, the homomorphism \( t \) associates to each 1-holonomy of the 1-connection a 2-holonomy of the 2-connection.

As it clear from the above considerations, the labeling procedure relies heavily on the requirement that the surface bound by the source and target curves \( \gamma_1 \) and respectively \( \gamma_2 \) should have the topology of a disk. In a sense, this requirement is not at all surprising, since in fact the group elements \( h \in H \) labeling surfaces are in fact given by (the exponential of) a surface integral of the 2-connection \([2]\). However, while this requirement is definitely compatible with a trivial topology of the spatial surface \( S \), it is not at all clear if and how the requirement - and hence the condition \([10]\) - is also compatible with a non-trivial topology of \( S \). This is the issue that needs to be addressed in order to establish whether or not the HGT
generalizations of the $\Sigma \Phi EA$ model admit non-trivial solutions. In particular, and since our immediate purpose is to answer the question of whether or not these generalizations admit the same solutions as the SGT formulation of the model, we will confine our considerations only to the case where the surface $S$ has the topology of a punctured plane. However, for clarity purposes, it is more convenient to develop our arguments for a surface $S$ having the topology of an open annulus. Since the open annulus and the punctured plane are homeomorphic and hence homotopically equivalent, the conclusions of the analysis developed for the former (which is easier to visualize) apply identically to the latter.

Under these circumstances, if we consider now the annular topology, the fundamental group of this surface contains two principal classes of loops, namely the class of contractible loops (homotopic to the null loop) and the class of non-contractible loops surrounding the inner circumference of the annulus$^6$. For the class of contractible loops, the surfaces bound by these loops have the topology of a disc, and as such the condition (40) is satisfied, albeit in a trivial manner.

For the class of non-contractible loops surrounding the inner circumference of the annulus, the situation is quite different. Such loops do not enclose a surface, since such a surface simply does not exist (it is exactly the surface that has been "removed" from the open disk in order to create the annular surface), and hence there exists no label $h \in H$ that can be associated to the holonomies of the 1-connection along these loops by the homomorphism $t$ in accordance to the prescription described earlier. The obvious conclusion to these arguments is that for such non-contractible loops the condition (40) becomes invalid. Loosely speaking once again, this conclusion states that there exists no 2-holonomy $h \in H$ that can be associated to the 1-holonomies corresponding to this class of loops through the homomorphism $t$. And as mentioned above, this conclusion is also valid for the punctured plane topology.

Far from being an inconvenient result, the above conclusion solves in fact the issue of whether or not the HGT generalizations of the $\Sigma \Phi EA$ model admit non-trivial solutions for a surface $S$ having the topology of a punctured plane or an annulus. We can state at this time that the HGT generalizations of the $\Sigma \Phi EA$ model - and HGT in general - are

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$^6$ For the purpose of the present argument, it suffices to consider only the class of loops surrounding one time the inner circumference of the annulus
compatible with a non-trivial topology of the open surface $S$ (i.e. have non-trivial solutions) provided that (40) is valid for all the contractible curves in $S$ and provided that there exist non-contractible loops in $S$ for which the condition (40) fails to be valid in the sense discussed above\(^7\).

We can conclude based on the above considerations that, as stated earlier, the HGT generalizations of the $\Sigma\Phi EA$ model developed in this section are non-trivial in the sense that they are compatible with the punctured plane topology of the surface $S$. Furthermore, it becomes now clear that for this topology of $S$ they also admit the same solutions (e.g. the point-particle solution and the BTZ black-hole solution) as the SGT formulation of the model \(^5\).

V. CONCLUSIONS

In this paper we have considered the issue of the generalization of $(2+1)$-dimensional gravity to the HGT formalism, and we have shown that one such possible generalization arises from the reformulation of the $\Sigma\Phi EA$ model as a higher gauge theory. The generalized theory is an HGT with the adjoint Poincaré 2-group, and it inherits from the $\Sigma\Phi EA$ model its non-trivial solutions that include the BTZ black-hole geometry and the point-particle geometry.

While it is very interesting that the HGT formalism can also find its application to the study of classical gravity - in $(2+1)$ dimensions at least - this is just a first simple step in exploring its potential and implications in this latter context. Further investigation is required in order to determine if more complex such generalizations of classical gravity can be developed in $(2+1)$ dimensions (e.g. generalizations involving non-abelian 2-connections, non-vanishing cosmological constants) and also if the formalism can be successfully applied to the more realistic theory of gravity in $(3+1)$ dimensions.

Nevertheless, even in the context of $(2+1)$-dimensional gravity, the existence of an HGT generalization of pure gravity with vanishing cosmological constant opens a whole new

\(^7\) It must be emphasized that this statement regarding the existence of non-trivial solutions of HGT for non-trivial topologies of the surface $S$ is only valid if $S$ is an open surface. If $S$ is a closed surface, the statement becomes invalid since for example if $S$ has the topology of a torus, it is straightforward to see that the condition (40) remains valid for both non-contractible curves surrounding one time the large and small circumferences of the torus.
range of issues that are worth exploring. In particular, it would be extremely interesting to explore the differences between the spin-foam quantum theories of the $\Sigma \Phi E A$ model as a traditional SGT and as an HGT, and we hope to be able to further study this issue in a future paper.

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