\textbf{L\textsuperscript{1}}-Convergence of Double Trigonometric Series

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\textbf{Abstract.} In this paper we study the pointwise convergence and convergence in \( L\textsuperscript{1} \)-norm of double trigonometric series whose coefficients form a null sequence of bounded variation of order \((p, 0), (0, p)\) and \((p, p)\) with the weight \((jk)^{p-1}\) for some integer \( p > 1 \). The double trigonometric series in this paper represents double cosine series, double sine series and double cosine sine series. Our results extend the results of Young [9], Kolmogorov [4] in the sense of single trigonometric series to double trigonometric series and of Móricz [6, 7] in the sense of higher values of \( p \).

1. Introduction

Consider the double trigonometric series

\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} \psi_j(x) \psi_k(y)
\]  \hspace{1cm} (1.1)

on positive quadrant \( T = [0, \pi] \times [0, \pi] \) of the two dimensional torus. The double trigonometric series (1.1) represents

(a) double cosine series \( \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \lambda_j \lambda_k a_{jk} \cos jx \cos ky \) where \( \lambda_0 = \frac{1}{2} \) and \( \lambda_j = 1 \) for \( j = 1, 2, 3, \ldots \).

(b) double sine series \( \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} \sin jx \sin ky \)

(c) double cosine-sine series \( \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \lambda_j \lambda_k a_{jk} \cos jx \sin ky \) where \( \lambda_0 = \frac{1}{2} \) and \( \lambda_j = 1 \) for \( j = 1, 2, 3, \ldots \).

The rectangular partial sums \( \psi_{mn}(x, y) \) and the Cesàro means \( \sigma_{mn}(x, y) \) of the series (1.1) are defined as

\[
\psi_{mn}(x, y) = \sum_{j=0}^{m} \sum_{k=0}^{n} a_{jk} \psi_j(x) \psi_k(y),
\]
Many authors like M. M. M. M. \cite{6, 7, 2, 3, 5} studied integrability and concerned where as in \cite{6} he studied complex double trigonometric series under \cofactor. Let the sum of the series (1.1) be denoted by \( \sum \psi_{jk}(x, y) \) and for \( \lambda > 1 \), the truncated Cesàro means are defined by

\[
V_{mn}^\lambda(x, y) = \frac{1}{([\lambda m] - m)((\lambda n) - n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} \psi_{jk}(x, y).
\]

Assuming the coefficients \( |a_{jk} : j, k \geq 0| \) in (1.1) be a double sequence of real numbers which satisfy the following conditions which may be called as conditions of bounded variation for some positive integer \( p \):

\[
|a_{jk}|(jk)^p \to 0 \quad \text{as} \quad \max(j, k) \to \infty,
\]

\[
\lim_{k \to \infty} \lim_{j \to \infty} |\Delta_{pq} a_{jk}|(jk)^{p-1} = 0,
\]

\[
\lim_{j \to \infty} \lim_{k \to \infty} |\Delta_{pq} a_{jk}|(jk)^{p-1} = 0,
\]

\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\Delta_{pq} a_{jk}|(jk)^{p-1} < \infty.
\]

For some integers \( p \) and \( q \), the finite order differences \( \Delta_{pq} a_{jk} \) are defined by

\[
\Delta_{00} a_{jk} = a_{jk};
\]

\[
\Delta_{pq} a_{jk} = \Delta_{p-1, q} a_{jk} - \Delta_{p-1, q} a_{j+1, k} \quad (p \geq 1, q \geq 0);
\]

\[
\Delta_{pq} a_{jk} = \Delta_{p, q-1} a_{jk} - \Delta_{p, q-1} a_{jk+1} \quad (p \geq 0, q \geq 1).
\]

Also a double induction argument gives

\[
\Delta_{pq} a_{jk} = \sum_{s=0}^{p} \sum_{t=0}^{q} (-1)^{p+t} \binom{p}{s} \binom{q}{t} a_{j+s, k+t}.
\]

The above mentioned (1.2)-(1.5) conditions generalise the concept of monotone sequences. Also any sequence satisfying (1.5) with \( p = 2 \) is called a quasi-convex sequence \cite{4, 7}. Clearly the conditions (1.2) and (1.5) implies (1.3) and (1.4) for \( p = 1 \) and moreover for \( p = 1 \), the conditions (1.2) and (1.5) reduce to

\[
|a_{jk}| \to 0 \quad \text{as} \quad \max(j, k) \to \infty \quad \text{and} \quad \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\Delta_{11} a_{jk}| < \infty.
\]

Generally the pointwise convergence of the series (1.1) is defined in Pringsheim’s sense \cite{10, vol. 2, ch. 17}. Let the sum of the series (1.1) be denoted by \( f(x, y) \) (provided it exists).

Also let \( \|f\| \) denotes the \( L^1(\mathbb{T}^2) \)-norm, i.e.,

\[
\|f\| = \pi \int_0^\pi \int_0^\pi |f(x, y)| dx \, dy.
\]

Many authors like Móricz \cite{6, 7}, Chen \cite{2}, K. Kaur et al. \cite{3} and Krasniqi \cite{5} studied integrability and \( L^1 \)-convergence of double trigonometric series under different classes of coefficients. In \cite{7}, Móricz studied both double cosine series and double sine series as far as their integrability and convergence in \( L^1 \)-norm is concerned where as in \cite{6} he studied complex double trigonometric series under coefficients of bounded variation.

These authors mainly discussed the case for \( p = 1 \) or \( p = 2 \) and preferred the condition of bounded variation on coefficients. Our aim in this paper is to extend the above results from \( p = 1 \) or \( p = 2 \) to general cases for double trigonometric series of all types as mentioned above.

For convenience, we write \( \lambda n = [\lambda n] \) where \( n \) is a positive integer, \( \lambda > 1 \) is a real number and \( [\ ] \) means greatest integral part and in the results, \( C_p \) denote constants which may not be the same at each occurrence.

Our first main result is as follows:
Theorem 1.1. Assume that conditions (1.2) – (1.5) are satisfied for some integer \( p \geq 1 \), then
(i) \( \psi_{mn}(x, y) \) converges pointwise to \( f(x, y) \) for every \( (x, y) \in T^2 \setminus \{(0, 0)\}; 
(ii) \( ||\psi_{mn}(x, y) - f(x, y)|| = o(1) \) as \( \min(m, n) \to \infty \).

The results mentioned in above theorem has been proved by Móricz [6, 7] for \( p = 1 \) and \( p = 2 \) using suitable estimates for Dirichlet’s kernel \( D_j(x) \) and Fejér kernel \( K_j(x) \) where as in the case of a single series for \( p = 2 \), the results regarding convergence have been proved by Kolmogorov [4].

Obviously, condition (1.5) implies any of the following conditions:

\[
\lim_{\lambda \to 1} \lim_{n \to \infty} \sum_{j=0}^{\lambda_n} \sum_{k=0}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} |\Delta_p a_j (j, k)|^{p-1} = 0; \tag{1.6}
\]

\[
\lim_{\lambda \to 1} \lim_{m \to \infty} \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{\lambda_m} \frac{\lambda_m - j + 1}{\lambda_m - m} |\Delta_p a_j (j, k)|^{p-1} = 0. \tag{1.7}
\]

We introduce the following three sums for \( m, n \geq 0 \) and \( \lambda > 1 \):

\[
S_{10}^j(m, n, x, y) = \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{\lambda_m} \frac{\lambda_m - j + 1}{\lambda_m - m} a_j \psi_j(x) \psi_2(y);
\]

\[
S_{01}^j(m, n, x, y) = \sum_{j=0}^{\lambda_n} \sum_{k=0}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} a_j \psi_j(x) \psi_2(y);
\]

\[
S_{11}^j(m, n, x, y) = \sum_{j=m+1}^{\lambda_m} \sum_{k=m+1}^{\lambda_m} \frac{\lambda_m - j + 1}{\lambda_m - m} \frac{\lambda_n - k + 1}{\lambda_n - n} a_j \psi_j(x) \psi_2(y);
\]

and we have

\[
S_{11}^j(m, n; x, y) = \frac{1}{(\lambda_m - m)} \sum_{u=m+1}^{\lambda_m} \left( S_{01}^j(u, n; x, y) - S_{10}^j(m, n; x, y) \right);
\]

\[
S_{11}^j(m, n; x, y) = \frac{1}{(\lambda_n - n)} \sum_{v=n+1}^{\lambda_n} \left( S_{10}^j(m, v; x, y) - S_{01}^j(m, n; x, y) \right).
\]

This implies

\[
S_{11}^j(m, n; x, y) \leq \left\{ \begin{array}{ll}
2 \sup_{m \leq u \leq \lambda_m} \left| S_{01}^j(u, n; x, y) \right| & \text{if } m \leq n, \\
2 \sup_{n \leq u \leq \lambda_n} \left| S_{10}^j(m, u; x, y) \right| & \text{if } m > n.
\end{array} \right. \tag{1.8}
\]

The second result of this paper is the following theorem:

Theorem 1.2. Let \( E \subset T^2 \). Assume that the following conditions are satisfied:

\[
\lim_{\lambda \to 1} \lim_{m,n \to \infty} \left( \sup_{(x,y) \in E} \left| S_{10}^j(m, n; x, y) \right| \right) = 0; \tag{1.9}
\]

\[
\lim_{\lambda \to 1} \lim_{m,n \to \infty} \left( \sup_{(x,y) \in E} \left| S_{01}^j(m, n; x, y) \right| \right) = 0. \tag{1.10}
\]

If \( V^j_{mn}(x, y) \) converges uniformly on \( E \) to \( f(x, y) \) as \( \min(m, n) \to \infty \), then so does \( \psi_{mn} \).
We will also prove the following theorem:

**Theorem 1.3.** Assume that the conditions (1.2)-(1.4) and (1.6)-(1.7) are satisfied for some integer \( p \geq 1 \), then (i) if \( V^\lambda_{mn}(x, y) \) converges uniformly to \( f(x, y) \) as \( \min(m, n) \to \infty \), then \( \psi_{mn} \) will also converge uniformly to \( f(x, y) \) as \( \min(m, n) \to \infty \). (ii) If \( \|V^\lambda_{mn} - f\| \to 0 \) then \( \|\psi_{mn} - f\| \to 0 \) as \( \min(m, n) \to \infty \).

2. Notations and formulas

The Cesàro sums of order \( \alpha \) of the sequence \( \{\psi_j(t)\} \) for any real number \( \alpha \) are denoted by \( \psi_j^\alpha(t) \). Thus we have

\[
\psi_j^\alpha(t) = \sum_{s=0}^{j} \psi_s^{\alpha-1}(t) \quad (\alpha \geq 1, j \geq 0) \tag{2.1}
\]

In this paper \( \psi_j^\alpha(t) \) either represents \( D_j(t) \) or \( D^*_j(t) \) where \( D_j(t) \) and \( D^*_j(t) \) represents Dirichlet and conjugate Dirichlet Kernels respectively. Also from [8], we have following estimates

(i) \( |\psi_j^\alpha(x)| = O((j+1)^\alpha) \) for all \( \alpha \geq 1, -\pi \leq x \leq \pi \). \tag{2.2}

(ii) \( \psi_j^\alpha(x) = O\left(\frac{1}{x^\alpha}\right) \) for all \( p \geq 2, \ (0 < x \leq \pi) \) \tag{2.3}

3. Lemmas

We require the following lemmas for the proof of our results:

**Lemma 3.1.** For \( m, n \geq 0 \) and \( p > 1 \), the following representation holds:

\[
\psi_{mn}(x, y) = \sum_{j=0}^{m} \sum_{k=0}^{n} a_{jk} \psi_j(x) \psi_k(y) = \sum_{j=0}^{m} \sum_{k=0}^{n} \Delta_{p} a_{jk} \psi_j^p(x) \psi_k^p(y) + \sum_{j=0}^{m} \sum_{l=0}^{p-1} \Delta_{p} a_{j,l+1} \psi_j^{p-1}(x) \psi_{l+1}^p(y) + \sum_{k=0}^{n} \sum_{s=0}^{p-1} \Delta_{p} a_{m+1,k} \psi_k^{p-1}(x) \psi_{m+1}^s(y) + \sum_{r=0}^{p-1} \sum_{l=1}^{n} \Delta_{p} a_{r+1,n+1} \psi_{r+1}^s(x) \psi_{n+1}^l(y).
\]

**Lemma 3.2.** [2] For \( m, n \geq 0 \) and \( \lambda > 1 \), the following representation holds:

\[
\psi_{mn} - \sigma_{mn} = \lambda_m + 1 \left( \frac{\lambda_n + 1}{\lambda_m - n} \sigma_{m,n} - \frac{1}{\lambda_n} \sigma_{m,n} \right) - \lambda_m + 1 \left( \frac{\lambda_n + 1}{\lambda_m - n} \sigma_{m,n} \right) - S^1_{11}(m, n, x, y) - S^1_{10}(m, n, x, y) - S^1_{01}(m, n, x, y).
\]

**Lemma 3.3.** For \( m, n \geq 0 \) and \( \lambda > 1 \), we have the following representation:

\[
V^\lambda_{mn} - \psi_{mn} = S^1_{11}(m, n, x, y) + S^1_{10}(m, n, x, y) + S^1_{01}(m, n, x, y).
\]
Proof. We have

\[ V_m^\lambda(x, y) = \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \psi_{jk}(x, y) \]

Now we can write

\[ \frac{1}{(\lambda_m - m)} \sum_{j=m+1}^{\lambda_m} \psi_{jk}(x, y) = \frac{1}{(\lambda_m - m)} \left[ \sum_{j=0}^{\lambda_m} \psi_{jk}(x, y) - \sum_{j=0}^{m} \psi_{jk}(x, y) \right] \]

Thus

\[ V_m^\lambda(x, y) = \frac{1}{(\lambda_n - n)} \sum_{k=n+1}^{\lambda_n} \left[ \frac{1}{(\lambda_m - m)} \sum_{j=m+1}^{\lambda_m} \psi_{jk}(x, y) \right] \]

\[ = \frac{1}{(\lambda_n - n)} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_m + 1}{(\lambda_m - m) \lambda_m + 1} \sum_{j=0}^{\lambda_m} \psi_{jk}(x, y) - \frac{m + 1}{m + 1} \sum_{j=0}^{m} \psi_{jk}(x, y) \]

\[ = \frac{1}{(\lambda_n - n)} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_m + 1}{(\lambda_m - m) \lambda_m + 1} \sum_{j=0}^{\lambda_m} \sum_{k=0}^{\lambda_n} \psi_{jk}(x, y) - \frac{1}{(\lambda_n - n)} \sum_{j=0}^{m} \sum_{k=0}^{\lambda_n} \psi_{jk}(x, y) \]

\[ = S11 + S22 \]

Now

\[ S11 = \frac{1}{(\lambda_n - n)} \frac{\lambda_m + 1}{(\lambda_m - m) \lambda_m + 1} \sum_{j=0}^{\lambda_m} \sum_{k=0}^{\lambda_n} \psi_{jk}(x, y) - \sum_{j=0}^{m} \sum_{k=0}^{\lambda_n} \psi_{jk}(x, y) \]

\[ = \frac{\lambda_m + 1}{(\lambda_m - m) \lambda_m + 1} \sum_{j=0}^{\lambda_m} \sum_{k=0}^{\lambda_n} \psi_{jk}(x, y) - \sum_{j=0}^{m} \sum_{k=0}^{\lambda_n} \psi_{jk}(x, y) \]

Similarly we get

\[ S22 = \frac{m + 1}{m + 1} \frac{\lambda_n + 1}{(\lambda_n - n) \lambda_n + 1} \sum_{j=0}^{\lambda_m} \sum_{k=0}^{\lambda_n} \psi_{jk}(x, y) - \sum_{j=0}^{m} \sum_{k=0}^{\lambda_n} \psi_{jk}(x, y) \]

Thus we have

\[ V_m^\lambda(x, y) = \frac{\lambda_m + 1}{(\lambda_m - m) \lambda_m + 1} \sum_{j=0}^{\lambda_m} \sum_{k=0}^{\lambda_n} \psi_{jk}(x, y) - \sum_{j=0}^{m} \sum_{k=0}^{\lambda_n} \psi_{jk}(x, y) \]

\[ = \frac{\lambda_m + 1}{(\lambda_m - m) \lambda_m + 1} \sum_{j=0}^{\lambda_m} \sum_{k=0}^{\lambda_n} \psi_{jk}(x, y) - \sum_{j=0}^{m} \sum_{k=0}^{\lambda_n} \psi_{jk}(x, y) \]

\[ \text{(by rearrangement of terms)} \]

The use of Lemma 3.2 gives

\[ V_m^\lambda(x, y) = \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_m - j + 1}{\lambda_m - m} \frac{\lambda_n - k + 1}{\lambda_n - n} \psi_{jk}(x) \psi_{jk}(y) \]

\[ + \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{\lambda_n} \frac{\lambda_m - j + 1}{\lambda_m - m} \psi_{jk}(x) \psi_{jk}(y) + \sum_{j=0}^{m} \sum_{k=0}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} \psi_{jk}(x) \psi_{jk}(y). \]

\[ \square \]
Lemma 3.4. For $m, n \geq 0$ and $\lambda > 1$, we have the following representation:

$$S_{10}^3(m, n; x, y) = \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \frac{\lambda_m - j + 1}{\lambda_m^j - m} a_{jk} \psi_j(x) \psi_k(y)$$

$$= \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \frac{\lambda_m - j + 1}{\lambda_m^j - m} \Delta_{npa} \lambda_j^{p-1} \psi_j^{(p-1)}(x) \psi_k^{p-1}(y) + \sum_{j=m+1}^{\lambda_m} \sum_{i=0}^{p-1} \frac{\lambda_m - j + 1}{\lambda_m^j - m} \Delta_{p\lambda} \lambda_{j+1,n+1} \psi_j^{p-1}(x) \psi_n^{p-1}(y)$$

$$+ \frac{1}{\lambda_m^j - m} \sum_{j=m+1}^{\lambda_m} \sum_{i=0}^{p-1} \sum_{k=0}^{n} \Delta_{\lambda} \lambda_{j+1,k} \psi_j^{(p-1)}(x) \psi_k^{p-1}(y) + \frac{1}{\lambda_m^j - m} \sum_{j=m+1}^{\lambda_m} \sum_{i=0}^{p-1} \sum_{k=0}^{n} \Delta_{\lambda} \lambda_{j+1,n+1} \psi_j^{p-1}(x) \psi_k^{p-1}(y)$$

$$- \sum_{i=0}^{p-1} \sum_{k=0}^{n} \Delta_{p\lambda} \lambda_{m+1,k} \psi_m^{p-1}(y) \psi_n^{p-1}(y) - \sum_{i=0}^{p-1} \sum_{k=0}^{n} \Delta_{\lambda} \lambda_{m+1,n+1} \psi_m^{p-1}(x) \psi_n^{p-1}(y).$$

Proof. We have by summation by parts,

$$S_{10}^3(m, n; x, y) = \sum_{k=0}^{n} \psi_k(y) \left( \sum_{j=m+1}^{\lambda_m} \frac{\lambda_m - j + 1}{\lambda_m^j - m} a_{jk} \psi_j(x) \right)$$

$$= \sum_{k=0}^{n} \psi_k(y) \left( \sum_{j=m+1}^{\lambda_m} \frac{\lambda_m - j + 1}{\lambda_m^j - m} \Delta_{npa} \lambda_j^{p-1} \psi_j^{(p-1)}(x) + \frac{1}{\lambda_m^j - m} \sum_{j=m+1}^{\lambda_m} \sum_{i=0}^{p-1} \Delta_{\lambda} \lambda_{j+1,k} \psi_j^{(p-1)}(x) \psi_k^{p-1}(y) - \sum_{i=0}^{p-1} \sum_{k=0}^{n} \Delta_{\lambda} \lambda_{m+1,k} \psi_m^{p-1}(x) \psi_k^{p-1}(y) \right)$$

$$= \sum_{j=m+1}^{\lambda_m} \frac{\lambda_m - j + 1}{\lambda_m^j - m} \psi_j^{(p-1)}(x) \left( \sum_{k=0}^{n} \Delta_{npa} \lambda_j^{p-1} \psi_k(y) + \frac{1}{\lambda_m^j - m} \sum_{j=m+1}^{\lambda_m} \sum_{i=0}^{p-1} \sum_{k=0}^{n} \Delta_{\lambda} \lambda_{j+1,k} \psi_j^{p-1}(x) \psi_k^{p-1}(y) \right)$$

$$- \sum_{i=0}^{p-1} \sum_{k=0}^{n} \Delta_{p\lambda} \lambda_{m+1,k} \psi_m^{p-1}(x) \psi_k^{p-1}(y)$$

Similarly we can have representation for $S_{01}^3(m, n; x, y)$. □

4. Proof of Theorems

Proof of Theorem 1.1

For $m, n \geq 0$ and $p > 1$, we have from Lemma 3.1

$$\psi_m(x, y) = \sum_{k=0}^{m} \sum_{l=0}^{n} \Delta_{p\lambda} \lambda_j^{p-1} \psi_j^{p-1}(x) \psi_k^{p-1}(y) + \sum_{k=0}^{m} \sum_{l=0}^{n} \Delta_{\lambda} \lambda_{j+1,n+1} \psi_j^{p-1}(x) \psi_k^{p-1}(y)$$
and similarly

$$
+ \sum_{k=0}^{n} \sum_{s=0}^{p-1} \Delta_{\alpha, \alpha+1,k} \psi_{\alpha+1}^{s}(x) \psi_{\alpha+1}^{p-1}(y) + \sum_{s=0}^{p-1} \sum_{t=0}^{t} \Delta_{\alpha, \alpha+1,n+1, \alpha+1} \psi_{\alpha+1}^{s}(x) \psi_{\alpha+1}^{p-1}(y) = \sum_{1} + \sum_{2} + \sum_{3} + \sum_{4}.
$$

Using (2.3), that is, \( \psi_{j}^{p}(x) = O\left( \frac{1}{x^{p}} \right) \) for all \( p \geq 2, (0 < x \leq \pi) \) etc, we have for \( (0 < x, y \leq \pi) \),

$$
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\Delta_{\alpha, \alpha+1,k} \psi_{\alpha+1}^{p-1}(x) \psi_{\alpha+1}^{p-1}(y)| < \infty \quad \text{(by (1.2))}
$$

and also by (1.3) - (1.5), we have

$$
\sum_{j=0}^{m} \sum_{k=0}^{p-1} \Delta_{\alpha, \alpha+1,k} \psi_{\alpha+1}^{p-1}(x) \psi_{\alpha+1}^{p-1}(y) \rightarrow 0 \quad \text{as} \quad \min(m, n) \rightarrow \infty.
$$

and similarly

$$
\sum_{s=0}^{n} \sum_{t=0}^{t} \Delta_{\alpha, \alpha+1,s} \psi_{\alpha+1}^{s}(x) \psi_{\alpha+1}^{p-1}(y) \rightarrow 0 \quad \text{as} \quad \min(m, n) \rightarrow \infty.
$$

Thus

$$
\sum_{j=0}^{m} \sum_{k=0}^{p-1} \Delta_{\alpha, \alpha+1,k} \psi_{\alpha+1}^{p-1}(x) \psi_{\alpha+1}^{p-1}(y) \rightarrow 0 \quad \text{as} \quad \min(m, n) \rightarrow \infty.
$$

Also

$$
\sum_{s=0}^{n} \sum_{t=0}^{t} \Delta_{\alpha, \alpha+1,n+1} \psi_{\alpha+1}^{s}(x) \psi_{\alpha+1}^{p-1}(y) \rightarrow 0 \quad \text{as} \quad \min(m, n) \rightarrow \infty.
$$

Consequently series (1.1) converges to the function \( f(x, y) \) where

$$
f(x, y) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Delta_{\alpha, \alpha+1,k} \psi_{\alpha+1}^{p-1}(x) \psi_{\alpha+1}^{p-1}(y) \quad \text{and} \quad \lim_{m,n \rightarrow \infty} \psi_{m,n}(x, y) = f(x, y).
$$
Now we will calculate \( \| \sum_{1} \|, \| \sum_{2} \|, \| \sum_{3} \| \) and \( \| \sum_{4} \| \) in the following way:

\[
\| \sum_{1} \| = \| \sum_{j=0}^{m} \sum_{k=0}^{n} \Delta_{pp, j, k} \psi_{j}^{p-1}(x) \psi_{k}^{p-1}(y) \|
\leq \sum_{j=0}^{m} \sum_{k=0}^{n} |\Delta_{pp, j, k}| \int_{0}^{\pi} \int_{0}^{\pi} |\psi_{j}^{p-1}(x) \psi_{k}^{p-1}(y)| \, dx \, dy
\leq C_{p} \sum_{j=0}^{m} \sum_{k=0}^{n} |\Delta_{pp, j, k}| k^{p-1} \int_{0}^{\pi} \int_{0}^{\pi} \, dx \, dy \quad \text{(by (2.2))}
\leq C_{p} \sum_{j=0}^{m} \sum_{k=0}^{n} |\Delta_{pp, j, k}| k^{p-1}.
\]

\[
\| \sum_{2} \| = \| \sum_{j=0}^{m} \sum_{k=0}^{n} \Delta_{pp, j, k+1} \psi_{j}^{p-1}(x) \psi_{k}^{p}(y) \|
\leq \sum_{j=0}^{m} \sum_{k=0}^{n} \left( \frac{1}{2} \right)^{j} \sum_{n' \leq j+n} |\Delta_{pp, j, n'+1}| \int_{0}^{\pi} \int_{0}^{\pi} |\psi_{j}^{p-1}(x) \psi_{n'}^{p}(y)| \, dx \, dy
\leq C_{p} \sup_{n \leq k \leq m+n} \sum_{j=0}^{m} |\Delta_{pp, j, k}| k^{p-1} \left( \sum_{n' \leq j+n} n' \right) \quad \text{(by (2.2))}
\leq C_{p} \sup_{n \leq k \leq m+n} \sum_{j=0}^{m} |\Delta_{pp, j, k}| k^{p-1}.
\]

\[
\| \sum_{3} \| = \| \sum_{j=0}^{m} \sum_{k=0}^{n} \Delta_{pp, m+1, k} \psi_{m}^{p}(x) \psi_{k}^{p-1}(y) \|
\leq \sum_{s=0}^{m-1} \sum_{u=0}^{n} \left( \frac{s}{u} \right) \sum_{k=0}^{n} |\Delta_{pp, m+u+1, k}| u^{s} k^{p-1}
\leq C_{p} \sup_{m \leq j \leq m+n} \sum_{k=0}^{n} |\Delta_{pp, j, k}| k^{p-1} \left( \sum_{s=0}^{m} m^{s} \right)
\leq C_{p} \sup_{m \leq j \leq m+n} \sum_{k=0}^{n} |\Delta_{pp, j, k}| k^{p-1}.
\]

\[
\| \sum_{4} \| = \| \sum_{j=0}^{m} \sum_{k=0}^{n} \Delta_{pp, m+1, n+1} \psi_{m}^{p}(x) \psi_{n+1}^{p}(y) \|
\leq \sum_{s=0}^{m-1} \sum_{u=0}^{n} \sum_{v=0}^{1} \left( \frac{s}{u} \right) \sum_{k=0}^{n} |\Delta_{pp, m+u+1, v+1}| m^{s} n^{v}
\leq C_{p} \sup_{j \geq m, k \geq n} |\Delta_{pp, j, k}| k^{p-1}.
\]
Now let $R_{mn}$ consists of all $(j,k)$ with $j > m$ or $k > n$, that is,
\[
\sum \sum_{(j,k) \in R_{mn}} = \sum_{j=m+1}^{\infty} \sum_{k=0}^{n} + \sum_{j=0}^{m} \sum_{k=n+1}^{\infty} + \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty}.
\]

Then
\[
\|f - \psi_{mn}\| = \left( \int_{0}^{\pi} \int_{0}^{\pi} |f(x,y) - \psi_{mn}(x,y)| \, dx \, dy \right)^{1/2}
\]
\[
\leq \sum_{(j,k) \in R_{mn}} \Delta_{p}^{|a| \lambda} \psi_{j}^{p-1}(x) \psi_{k}^{p-1}(y) + \| \sum_{j=0}^{m} \sum_{t=0}^{p-1} \Delta_{p}^{|a| \lambda} \psi_{j}^{p-1}(x) \psi_{t}^{p-1}(y) \|
\]
\[
+ \sum_{k=0}^{n} \sum_{s=0}^{p-1} \Delta_{p}^{|a| \lambda} \psi_{m}^{p-1}(x) \psi_{s}^{p-1}(y) + \| \sum_{k=0}^{n} \sum_{s=0}^{p-1} \Delta_{p}^{|a| \lambda} \psi_{m}^{p-1}(x) \psi_{s}^{p-1}(y) \|
\]
\[
\leq C_{p} \left( \sum_{(j,k) \in R_{mn}} | \Delta_{p}^{|a| \lambda} | p^{-1} k^{p-1} \right) + \left( \sup_{n < s < n + p} \sum_{j=0}^{m} | \Delta_{p}^{|a| \lambda} | p^{-1} k^{p-1} \right)
\]
\[
+ \left( \sup_{m < s < m + p} \sum_{k=0}^{n} | \Delta_{p}^{|a| \lambda} | p^{-1} k^{p-1} \right) + \left( \sup_{m < s < m + p} \sum_{k=0}^{n} | \Delta_{p}^{|a| \lambda} | p^{-1} k^{p-1} \right)
\]
\[
\rightarrow 0 \text{ as } \min(m,n) \rightarrow \infty \quad \text{(As discussed above)}
\]

which proves (ii) part.

**Proof of Theorem 1.2**

Using the relation (1.8), we find that (1.9) or (1.10) implies
\[
\lim_{\lambda \downarrow 1} \lim_{m,n \to \infty} \left( \sup_{(x,y) \in E} |S_{11}^{\lambda}(m,n;x,y)| \right) = 0. \quad (4.1)
\]

Assume that $V_{mn}(x,y)$ converges uniformly on $E$ to $f(x,y)$. Then by Lemma 3.3, we get
\[
\lim_{m,n \to \infty} \left( \sup_{(x,y) \in E} |\psi_{mn}(x,y) - V_{mn}(x,y)| \right)
\]
\[
\leq \lim_{m,n \to \infty} \left( \sup_{(x,y) \in E} |S_{10}^{\lambda}(m,n;x,y)| \right)
\]
\[
+ \lim_{m,n \to \infty} \left( \sup_{(x,y) \in E} |S_{01}^{\lambda}(m,n;x,y)| \right)
\]
\[
+ \lim_{m,n \to \infty} \left( \sup_{(x,y) \in E} |S_{11}^{\lambda}(m,n;x,y)| \right).
\]

After taking $\lambda \downarrow 1$ the result follows from (1.9), (1.10) and (4.1).

**Proof of Theorem 1.3**
Using the Lemma 3.4, we can write the expression for $S_{01}^1(m,n;x,y)$ as

$$
S_{01}^1(m,n;x,y) = \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} \Delta_{\mu \nu \delta \alpha \beta}^j \psi_{\lambda_k}^j(x) \psi_{\lambda_k}^j(y)
$$

$$
= \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} \Delta_{\mu \nu \delta \alpha \beta}^j \psi_{\lambda_k}^j(x) \psi_{\lambda_k}^j(y) + \sum_{k=n+1}^{\lambda_n} \sum_{s=0}^{\lambda_n - 1} \frac{\lambda_n - k + 1}{\lambda_n - n} \Delta_{\mu \nu \delta \alpha \beta}^j \psi_{\lambda_m}^j(x) \psi_{\lambda_m}^j(y)
$$

$$
+ \frac{1}{\lambda_n - n} \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \sum_{i=0}^{p-1} \Delta_{\mu \nu \delta \alpha \beta}^j \psi_{\lambda_k}^j(x) \psi_{\lambda_k}^j(y) + \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} \sum_{s=0}^{\lambda_n - 1} \Delta_{\mu \nu \delta \alpha \beta}^j \psi_{\lambda_m}^j(x) \psi_{\lambda_m}^j(y)
$$

Now by using (1.2)-(1.4) and (1.6) along with estimates of $\psi_{\lambda_k}^j(x)$ etc., as mentioned in [8], we have the following estimates:

$$
\left| \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} \Delta_{\mu \nu \delta \alpha \beta}^j \psi_{\lambda_k}^j(x) \psi_{\lambda_k}^j(y) \right|
$$

$$\leq \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} |\Delta_{\mu \nu \delta \alpha \beta}^j| \psi_{\lambda_k}^j(x) \psi_{\lambda_k}^j(y) \to 0 \text{ as min}(m,n) \to \infty.
$$

Consequently

$$
\left| \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} \Delta_{\mu \nu \delta \alpha \beta}^j \psi_{\lambda_k}^j(x) \psi_{\lambda_k}^j(y) \right|
$$

$$\leq \sum_{s=0}^{p-1} \sum_{j=0}^{m} \frac{\lambda_n}{\lambda_n - n} |\Delta_{\mu \nu \delta \alpha \beta}^j| \psi_{\lambda_m}^j(x) \psi_{\lambda_m}^j(y) \to 0 \text{ as min}(m,n) \to \infty.
$$

So

$$
\left| \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \Delta_{\mu \nu \delta \alpha \beta}^j \psi_{\lambda_k}^j(x) \psi_{\lambda_k}^j(y) \right|
$$

$$\leq \sup_{m<\lambda_n} \sum_{s=0}^{p-1} \sum_{j=0}^{m} |\Delta_{\mu \nu \delta \alpha \beta}^j| \psi_{\lambda_m}^j(x) \psi_{\lambda_m}^j(y) \to 0 \text{ as min}(m,n) \to \infty.
$$

And

$$
\left| \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \Delta_{\mu \nu \delta \alpha \beta}^j \psi_{\lambda_k}^j(x) \psi_{\lambda_k}^j(y) \right|
$$

$$\leq \sup_{m<\lambda_n} \sum_{s=0}^{p-1} \sum_{j=0}^{m} |\Delta_{\mu \nu \delta \alpha \beta}^j| \psi_{\lambda_m}^j(x) \psi_{\lambda_m}^j(y) \to 0 \text{ as min}(m,n) \to \infty.
$$
\[ \leq \sup_{n < k \leq L + p} \sum_{j=0}^{m} |A_{ij}a_{jk}|^{p-1}k^{p-1} \rightarrow 0 \quad \text{as} \min(m,n) \rightarrow \infty. \]

which implies \( \lim_{\lambda_{11} m,n \rightarrow \infty} \left( \sup_{x,y \in E} |\Sigma_{13}| \right) \rightarrow 0 \quad \text{as} \min(m,n) \rightarrow \infty. \)

Similarly we estimate others in brief

\[ |\Sigma_{14}| \leq \sup_{n < k \leq L + p} \sum_{t=0}^{p-1} \sum_{r=0}^{p-1} \sum_{s=0}^{r-1} \sum_{u=0}^{s-1} \sum_{v=0}^{u-1} |A_{ij}a_{jk}|^{p-1}k^{p-1} \]

\[ \leq \sup_{j > m,k > n} |a_{jk}|^{p-1}k^{p-1} \rightarrow 0 \quad \text{as} \min(m,n) \rightarrow \infty. \]

Thus \( \lim_{\lambda_{11} m,n \rightarrow \infty} \left( \sup_{x,y \in E} |\Sigma_{14}| \right) \rightarrow 0 \quad \text{as} \min(m,n) \rightarrow \infty. \)

which implies \( \lim_{\lambda_{11} m,n \rightarrow \infty} \left( \sup_{x,y \in E} |\Sigma_{15}| \right) \rightarrow 0 \quad \text{as} \min(m,n) \rightarrow \infty. \)

\[ |\Sigma_{16}| \leq \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \sum_{u=0}^{r-1} \sum_{v=0}^{s-1} \sum_{w=0}^{t-1} \sum_{y=0}^{u-1} |A_{ij}a_{jk}|^{p-1}k^{p-1} \]

\[ \leq \sup_{j > m,k > n} |a_{jk}|^{p-1}k^{p-1} \rightarrow 0 \quad \text{as} \min(m,n) \rightarrow \infty. \]

So \( \lim_{\lambda_{11} m,n \rightarrow \infty} \left( \sup_{x,y \in E} |\Sigma_{16}| \right) \rightarrow 0 \quad \text{as} \min(m,n) \rightarrow \infty. \)

Thus combining all these, we have

\[ \lim_{\lambda_{11} m,n \rightarrow \infty} \left( \sup_{x,y \in E} |\Sigma_{17}(m,n;x,y)| \right) = 0. \]

Similarly (1.2)-(1.4) and (1.7) results in

\[ \lim_{\lambda_{11} m,n \rightarrow \infty} \left( \sup_{x,y \in E} |\Sigma_{18}(m,n;x,y)| \right) = 0; \]

Thus first part of theorem follows from Theorem 4.2

**Proof of (ii)** We have

\[ \|\psi_{mn} - f\| \leq \|\psi_{mn} - V_{mn}^{\lambda}\| + \|V_{mn}^{\lambda} - f\|. \]

By assumption \( \|V_{mn}^{\lambda} - f\| \rightarrow 0 \), so it is sufficient to show that

\[ \|\psi_{mn} - V_{mn}^{\lambda}\| \rightarrow 0 \quad \text{as} \min(m,n) \rightarrow \infty. \]
By Lemma 3.3, we have

$$\|\psi_{nn} - V_{nn}^\lambda\| \leq \|S_{01}^\lambda(m, n; x, y)\| + \|S_0^\lambda(m, n; x, y)\|$$

Now in order to estimate $\|S_{01}^\lambda(m, n; x, y)\|$, we first find $\|\Sigma_{11}\|$, $\|\Sigma_{12}\|$

$$\|\Sigma_{13}\|, \|\Sigma_{14}\|, \|\Sigma_{15}\|$$

and $\|\Sigma_{16}\|$, so we have

$$\|\Sigma_{11}\| = \|\sum_{j=0}^m \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} \Delta_{pp}a_{jk} \psi_j^{p-1}(x) \psi_k^{p-1}(y)\|$$

$$\leq \sum_{j=0}^m \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} \Delta_{pp}a_{jk} \left|\psi_j^{p-1}\right| \left|\psi_k^{p-1}\right| \int_0^\pi \int_0^\pi dx dy$$

$$\leq C_p \sum_{j=0}^m \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} |\Delta_{pp}a_{jk}| \left|\psi_j^{p-1}\right| \left|\psi_k^{p-1}\right| \left|\psi_m^{p-1}\right| \left|\psi_n^{p-1}\right|$$

$$\|\Sigma_{12}\| = \|\sum_{j=0}^{p-1} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} \Delta_{pp}a_{m+k, j} \psi_j^{p-1}(x) \psi_k^{p-1}(y)\|$$

$$\leq C_p \sup_{m \leq \lambda} \left( \sum_{k=n+1}^{\lambda_n} |\Delta_{pp}a_{jk}| \right) \left( \sum_{j=0}^{p-1} m^p \right)$$

$$\leq C_p \sup_{m \leq \lambda} \sum_{k=n+1}^{\lambda_n} |\Delta_{pp}a_{jk}| \left|\psi_j^{p-1}\right| \left|\psi_k^{p-1}\right| \left|\psi_m^{p-1}\right| \left|\psi_n^{p-1}\right|$$

$$\|\Sigma_{13}\| \leq C_p \sup_{n \leq \lambda} \sum_{k=n+1}^{\lambda_n} |\Delta_{pp}a_{jk}| \left|\psi_j^{p-1}\right|$$

$$\leq C_p \sup_{n \leq \lambda} \sum_{k=n+1}^{\lambda_n} \left( \sum_{j=0}^{p-1} \sum_{j=0}^{p-1} m \right) |\Delta_{pp}a_{jk}| \left|\psi_j^{p-1}\right| \left|\psi_k^{p-1}\right|$$

$$\leq C_p \sup_{n \leq \lambda} \sum_{k=n+1}^{\lambda_n} |\Delta_{pp}a_{jk}| \left|\psi_j^{p-1}\right| \left|\psi_k^{p-1}\right| \left|\psi_m^{p-1}\right| \left|\psi_n^{p-1}\right|$$

$$\|\Sigma_{14}\| \leq C_p \sup_{n \leq \lambda} \sum_{k=n+1}^{\lambda_n} |\Delta_{pp}a_{m+k,k+1}| \left|\psi_j^{p-1}\right|$$

$$\leq C_p \sup_{n \leq \lambda} \sum_{k=n+1}^{\lambda_n} |\Delta_{pp}a_{jk}| \left|\psi_j^{p-1}\right| \left|\psi_k^{p-1}\right| \left|\psi_m^{p-1}\right| \left|\psi_n^{p-1}\right|$$

$$\|\Sigma_{15}\| \leq C_p \sum_{t=0}^{p-1} \sum_{v=0}^{p-1} \left( \sum_{j=0}^{p-1} \sum_{j=0}^{p-1} m \right) |\Delta_{pp}a_{jk}| \left|\psi_j^{p-1}\right| \left|\psi_k^{p-1}\right|$$

$$\leq C_p \sum_{t=0}^{p-1} \sum_{v=0}^{p-1} |\Delta_{pp}a_{jk}| \left|\psi_j^{p-1}\right| \left|\psi_k^{p-1}\right| \left|\psi_m^{p-1}\right| \left|\psi_n^{p-1}\right|$$
Thus we can estimate

\[ \| S_{\alpha \beta}^1(m,n;x,y) \| \leq C_p \sum_{k=0}^{\lambda} \sum_{j=0}^{m} \frac{\lambda_n - k + 1}{\lambda_n - n} |\Delta_{p\alpha} a_{\beta j}|^{p-1} k^{p-1} + C_p \left( \sup_{m<0} \sum_{k=0}^{\lambda} |\Delta_{\alpha j} a_{\beta k}|^{p-1} k^{p-1} \right) \]

\[ + C_p \left( \sup_{n<k<\lambda+p} \sum_{j=0}^{m} |\Delta_{p\alpha} a_{\beta j}|^{p-1} k^{p-1} \right) \]

\[ + C_p \left( \sup_{n<k<\lambda+p} \sum_{j=0}^{m} |\Delta_{\alpha j} a_{\beta k}|^{p-1} k^{p-1} \right) \]

By (1.2)-(1.4) and (1.6), we conclude that

\[ \lim_{\lambda \uparrow 1} \lim_{m,n \to \infty} \left( \| S_{\alpha \beta}^1(m,n;x,y) \| \right) = 0. \]

Similarly by conditions (1.2)-(1.4) and (1.7), we get

\[ \lim_{\lambda \uparrow 1} \lim_{m,n \to \infty} \left( \| S_{\alpha \beta}^2(m,n;x,y) \| \right) = 0. \]

Also by (1.8), we have

\[ \lim_{\lambda \uparrow 1} \lim_{m,n \to \infty} \left( \| S_{\alpha \beta}^3(m,n;x,y) \| \right) = 0. \]

Thus \[ \| \psi_{mn} \| \infty \to 0 \quad \text{as} \quad \min(m,n) \to \infty. \]

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