A GENERALIZATION OF THE PICARD-BRAUER EXACT SEQUENCE

CRISTIAN D. GONZÁLEZ-AVILÉS

ABSTRACT. We extend an argument of S. Lichtenbaum involving codimension one cycles to higher codimensions and obtain a generalization of the well-known Picard-Brauer exact sequence for a smooth variety $X$. The resulting exact sequence connects the codimension $n$ Chow group of $X$ with a certain “Brauer-like” group.

1. Introduction.

Let $k$ be a field and let $X$ be a geometrically integral algebraic $k$-scheme. We write $\overline{k}$ for a fixed separable algebraic closure of $k$ and set $\Gamma = \text{Gal}(\overline{k}/k)$. The $\overline{k}$-scheme $X \times_{\text{Spec } k} \text{Spec } \overline{k}$ will be denoted by $\overline{X}$. Let $k[X]^* = H_0^{\text{ét}}(\overline{X}, \mathbb{G}_m)$ and $Br'X = H_2^{\text{ét}}(X, \mathbb{G}_m)$ be, respectively, the group of invertible regular functions on $\overline{X}$ and the cohomological Brauer group of $X$. The exact sequence mentioned in the title is the familiar exact sequence

\begin{equation}
0 \to H^1(k, k[X]^*) \to \text{Pic } X \to (\text{Pic } \overline{X})^\Gamma \to H^2(k, k[X]^*) \to Br'_X \to H^1(k, \text{Pic } \overline{X}) \to H^3(k, k[X]^*)
\end{equation}

where $H^i(k, -) = H^i(\Gamma, -)$ and $Br'_X = \text{Ker } (Br'X \to Br' \overline{X})$. This sequence may be obtained from the exact sequence of terms of low degree belonging to the Hochschild-Serre spectral sequence

$$H^r(k, H_s^{\text{ét}}(\overline{X}, \mathbb{G}_m)) \Rightarrow H^{r+s}_{\text{ét}}(X, \mathbb{G}_m).$$

When $X$ is smooth (which we assume from now on), there exists an alternative derivation of (1) which makes use of the following (no less familiar) exact sequence:

\begin{equation}
0 \to k[X]^* \to \overline{k}(X)^* \to \text{Div } \overline{X} \to \text{Pic } \overline{X} \to 0
\end{equation}

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where $k(X)^*$ (resp. $\text{Div}(X)$) is the group of invertible rational functions (resp. Cartier divisors) on $X$. This approach, seemingly first used by S.Lichtenbaum in [4] and then reconsidered by Yu.Manin [5, p.403], consists in splitting (2) into two short exact sequences of $\Gamma$-modules and then taking $\Gamma$-cohomology of these sequences. The resulting long $\Gamma$-cohomology sequences are then appropriately combined to produce (1). This paper is a generalization of this idea. The key observation to make is that (2) may be seen as arising from the Gersten-Quillen complex corresponding to the Zariski sheaf $K_1$, which is the sheaf on $X$ associated to the presheaf $U \mapsto K_1(U) = H^0(U,\mathcal{O}_U)^\ast$. In Section 2 we work with the Gersten-Quillen complex corresponding to the Zariski sheaf $K_n$, associated to the presheaf $U \mapsto K_n(U)$, where $K_n$ is Quillen’s $n$-th $K$-functor ($1 \leq n \leq d = \dim(X)$), and obtain the following result. Let $\partial^{n-1}: \bigoplus_{y \in X^{n-1}} k(y)^* \to Z^n(X)$ be the “sum of divisors” map and let $B_n(X)$ be the kernel of the induced map

$$H^2\left(k, \bigoplus_{y \in X^{n-1}} k(y)^*\right) \to H^2(k, Z^n(X)).$$

**Main Theorem.** Let $X$ be a smooth, geometrically integral, algebraic $k$-scheme. Then there exists a natural exact sequence

$$0 \to H^1(k, \ker \partial^{n-1}) \to CH^n(X) \to CH^n(X)^T \to H^2(k, \ker \partial^{n-1})$$

$$\to B_n(X) \to H^1(k, CH^n(X)) \to H^3(k, \ker \partial^{n-1}).$$

The case $n = 1$ of the theorem is precisely the exact sequence (1).

In Section 4, which concludes the paper, we show that the group $B_n(X)$ in the exact sequence of the theorem is “Brauer-like”, in the sense that it contains a copy of $\text{Br}_1 Y = \ker [\text{Br} Y \to \text{Br} \overline{Y}]$ for every smooth closed integral subscheme $Y \subset X$ of codimension $n - 1$.

2. Preliminaries

We keep the notations of the Introduction. In particular, $X$ is a smooth, geometrically integral algebraic $k$-scheme of dimension $d$ and $n$ denotes a fixed integer such that $1 \leq n \leq d$.

There exists a natural bijection between the set of schematic points of $X$ and the set of closed integral subschemes of $X$. This is defined by associating to a point $x \in X$ the schematic closure $V(x)$ of $x$ in $X$. The codimension (resp. dimension) of $x$ is by definition the codimension (resp. dimension) of $V(x)$. The set of points of $X$ of codimension (resp. dimension) $i$ will be denoted by $X^i$ (resp. $X_i$), and $\eta$ (resp. $\overline{\eta}$) will denote the generic point of $X$ (resp. $\overline{X}$). If $x \neq \eta$, the function field of $V(x)$ will be denoted by $k(x)$. We use the standard notation $k(X)$.
for the function field of $X = V(\eta)$. For each $x \in X$, $i_x$ will denote the canonical map $\text{Spec } k(x) \to X$. The function field of $\overline{X}$ will be denoted $\overline{\text{Spec } k}$. For simplicity, we will write $V(\overline{x})$ for $V(x) \times_{\text{Spec } k} \text{Spec } \overline{k}$.

Since $X$ is regular [3, 6.7.4], the sheaf $K_n, X$ admits the following flasque resolution, known as the Gersten-Quillen resolution (see [7, p.72]):

$$0 \to K_n, X \to (i_\eta)_* K_n, X \to \bigoplus_{y \in \overline{X}^1} (i_y)_* K_{n-1} \overline{k}(y) \to \ldots$$

$$\to \bigoplus_{y \in \overline{X}^{n-1}} (i_y)_* \overline{k}(y)^* \to \bigoplus_{y \in \overline{X}^n} (i_y)_* \mathbb{Z} \to 0$$

where, for $y \in \overline{X}^i$, $K_{n-i} \overline{k}(y)$ is regarded as a constant sheaf on $\overline{k}(y)$. It follows that the groups $H^i(X, K_n, X) = H^i(X, K_n)$ are the cohomology groups of the complex

$$K_n \overline{\mathbb{Z}}(X) \xrightarrow{\partial^0} \bigoplus_{y \in \overline{X}^1} K_{n-1} \overline{\mathbb{Z}}(y) \xrightarrow{\partial^1} \ldots \xrightarrow{\partial^{n-2}} \bigoplus_{y \in \overline{X}^{n-1}} \overline{k}(y)^* \xrightarrow{\partial^{n-1}} \bigoplus_{y \in \overline{X}^n} \mathbb{Z}.$$ 

Now, if $q: \overline{X} \to X$ is the canonical morphism and $x \in X$, we write $\overline{X}^{n-i}$ for the set of points $y \in \overline{X}^{n-i}$ such that $q(y) = x$. For $i = 1, 2, \ldots, n-1$ and $x \in X^{n-i}$, set

$$\overline{K}_i(x) = \bigoplus_{y \in \overline{X}^{n-i}} K_i \overline{k}(y).$$

Further, write $Z^n(\overline{X})$ for the group of codimension $n$ cycles on $\overline{X}$, i.e.,

$$Z^n(\overline{X}) = \bigoplus_{y \in \overline{X}^n} \mathbb{Z}.$$ 

Then (3) may be written as

$$K_n \overline{\mathbb{Z}}(X) \xrightarrow{\partial^0} \bigoplus_{x \in X^1} \overline{K}_{n-1}(x) \xrightarrow{\partial^1} \ldots \xrightarrow{\partial^{n-2}} \bigoplus_{x \in X^{n-1}} \overline{K}_1(x) \xrightarrow{\partial^{n-1}} Z^n(\overline{X}).$$

The differential $\partial^{n-1}$ equals $\sum_{x \in X^{n-1}} \partial_x^{n-1}$, where, for each $x \in X^{n-1}$,

$$\partial_x^{n-1}: \overline{K}_1(x) = \bigoplus_{y \in \overline{X}^{n-1}} \overline{k}(y)^* \to Z^n(\overline{X})$$

is the sum of the divisor maps

$$\text{div}_y: \overline{k}(y)^* \to Z^n(\overline{X}).$$
For definition of the latter, see [7, p.72]. We note that each of the maps \( \text{div}_y \) factors through \( Z^1(V(y)) \), whence each \( \partial^{n-1} \) factors through \( Z^1(V(x)) \).

We will write \( CH^n(X) \) for the Chow group of codimension \( n \) cycles on \( X \) modulo rational equivalence. Then \( H^n(X, K_n) = CH^n(X) \) (“Bloch’s formula”).

3. Proof of the main theorem

The complex (4) induces the following short exact sequences of \( \Gamma \)-modules:

\[
0 \to \text{Im} \partial^{n-1} \to Z^n(X) \to CH^n(X) \to 0
\]

and

\[
0 \to \text{Ker} \partial^{n-1} \to \bigoplus_{x \in X^{n-1}} K_1(x) \to \text{Im} \partial^{n-1} \to 0.
\]

Observe that the natural morphism \( q: X \to X \) induces a homomorphism \( CH^n(X) \to CH^n(\overline{X})^\Gamma \).

**Lemma 3.1.** There exist canonical isomorphisms

\[
\text{Ker} \left[ CH^n(X) \to CH^n(\overline{X})^\Gamma \right] = H^1(k, \text{Ker} \partial^{n-1})
\]

\[
\text{Coker} \left[ CH^n(X) \to CH^n(\overline{X})^\Gamma \right] = H^1(k, \text{Im} \partial^{n-1})
\]

and a canonical exact sequence

\[
0 \to H^1(k, CH^n(X)) \to H^2(k, \text{Im} \partial^{n-1}) \to H^2(k, Z^n(\overline{X})).
\]

**Proof.** This follows by taking \( \Gamma \)-cohomology of (5), using the fact that \( Z^n(\overline{X}) \) is a permutation \( \Gamma \)-module and arguing as in [1, proof of Proposition 3.6] to establish the first isomorphism.

**Lemma 3.2.** The exact sequence (6) induces an exact sequence

\[
0 \to H^1(k, \text{Im} \partial^{n-1}) \to H^2(k, \text{Ker} \partial^{n-1}) \to \bigoplus_{x \in X^{n-1}} H^2(k, K_1(x))
\]

\[
\to H^2(k, \text{Im} \partial^{n-1}) \to H^3(k, \text{Ker} \partial^{n-1}).
\]

**Proof.** By Shapiro’s Lemma, for each \( x \in X^{n-1} \) there exists a (non-canonical) isomorphism

\[
H^*(k, K_1(x)) \simeq H^*(\text{Gal}(\overline{k}(y)/k(x)), \overline{k}(y)^*)
\]

where, on the right, we have chosen a point \( y \in \overline{X}^{n-1} \) such that \( q(y) = x \). The result now follows by taking \( \Gamma \)-cohomology of (6), using Hilbert’s Theorem 90.

Combining Lemmas 3.1 and 3.2, we obtain

**Proposition 3.3.** There exists a canonical exact sequence

\[
0 \rightarrow H^1(k, \text{Ker } \partial^{n-1}) \rightarrow CH^n(X) \rightarrow CH^n(X)^F \\
\rightarrow H^2(k, \text{Ker } \partial^{n-1}) \rightarrow \bigoplus_{x \in X^{n-1}} H^2(k, \overline{K}_1(x)). \quad \square
\]

Now define

\[
B_n(X) = \text{Ker } [H^2(k, \bigoplus_{x \in X^{n-1}} \overline{K}_1(x)) \rightarrow H^2(k, Z^n(X))],
\]

where the map involved is induced by \( \partial^{n-1} \). Since the composite

\[
\text{Ker } \partial^{n-1} \rightarrow \bigoplus_{x \in X^{n-1}} \overline{K}_1(x) \xrightarrow{\partial^{n-1}} Z^n(X)
\]

is zero, the natural map \( H^2(k, \text{Ker } \partial^{n-1}) \rightarrow \bigoplus_{x \in X^{n-1}} H^2(k, \overline{K}_1(x)) \) factors through \( B_n(X) \). Thus Proposition 3.3 yields a natural exact sequence

\[
0 \rightarrow H^1(k, \text{Ker } \partial^{n-1}) \rightarrow CH^n(X) \rightarrow CH^n(X)^F \\
\rightarrow H^2(k, \text{Ker } \partial^{n-1}) \rightarrow B_n(X). \quad (8)
\]

We will now extend the above exact sequence by defining a map \( B_n(X) \rightarrow H^1(k, CH^n(X)) \) whose kernel is exactly the image of the map \( H^2(k, \text{Ker } \partial^{n-1}) \rightarrow B_n(X) \) appearing in (8).

It is not difficult to check that the map

\[
\bigoplus_{x \in X^{n-1}} H^2(k, \overline{K}_1(x)) \rightarrow H^2(k, \text{Im } \partial^{n-1})
\]

intervening in the exact sequence of Lemma 3.2 maps \( B_n(X) \) into the kernel of the map \( H^2(k, \text{Im } \partial^{n-1}) \rightarrow H^2(k, Z^n(X)) \). The latter is naturally isomorphic to \( H^1(k, CH^n(X)) \) (see Lemma 3.1). Thus there exists a canonical map \( B_n(X) \rightarrow H^1(k, CH^n(X)) \). Again, it is not difficult to check that the kernel of the map just defined is exactly the image of the map \( H^2(k, \text{Ker } \partial^{n-1}) \rightarrow B_n(X) \) appearing in (8). Thus we obtain a natural exact sequence

\[
0 \rightarrow H^1(k, \text{Ker } \partial^{n-1}) \rightarrow CH^n(X) \rightarrow CH^n(X)^F \rightarrow H^2(k, \text{Ker } \partial^{n-1}) \\
\rightarrow B_n(X) \rightarrow H^1(k, CH^n(X)).
\]

Finally, the homomorphisms \( H^1(k, CH^n(X)) \rightarrow H^2(k, \text{Im } \partial^{n-1}) \) and \( H^2(k, \text{Im } \partial^{n-1}) \rightarrow H^3(k, \text{Ker } \partial^{n-1}) \) from Lemmas 3.1 and 3.2 induce a map \( H^1(k, CH^n(X)) \rightarrow H^3(k, \text{Ker } \partial^{n-1}) \) whose kernel is exactly the image of the map \( B_n(X) \rightarrow H^1(k, CH^n(X)) \) defined above. Thus the following holds.
Theorem 3.4. Let $X$ be a smooth $k$-variety. Then there exists a natural exact sequence

$$
0 \to H^1(k, \text{Ker } \partial^{n-1}) \to CH^n(X) \to CH^n(\overline{X})^T \to H^2(k, \text{Ker } \partial^{n-1}) \to B_n(X) \to H^1(k, CH^n(\overline{X})) \to H^3(k, \text{Ker } \partial^{n-1}),
$$

where $B_n(X)$ is the group $(7)$.

Remark 3.5. When $n = 1$, there are natural isomorphisms $CH^1(X) = \text{Pic } X$ and $CH^1(\overline{X}) = \text{Pic } \overline{X}$ [3, 21.6.10 and 21.11.1]. Further, $X^{n-1} = \{\eta\}$, $\partial^{n-1} \eta = \partial_x^{n-1} \eta$, $K_1(\eta) = k(X)^*$ $\to \text{Div } \overline{X}$ is the usual divisor map (whose kernel equals $H^0(\overline{X}, G_m) \overset{\text{def.}}{=} k[\overline{X}]^*$) and

$$
B_n(X) = B_1(X) = \text{Ker } [H^2(k, k(X)^*) \to H^2(k, \text{Div } \overline{X})] = \text{Br}_1 X,
$$

where $\text{Br}_1 X = \text{Ker } (\text{Br } X \to \text{Br } \overline{X})$ (see the next section). Thus the exact sequence of the theorem is indeed a generalization of (1).

4. The group $B_n(X)$

In this Section we show that the group $B_n(X)$ appearing in the exact sequence of Theorem 3.4 contains a copy of $\text{Br}_1 Y = \text{Ker } (\text{Br } Y \to \text{Br } \overline{Y})$ for every smooth closed integral subscheme $Y \subset X$ of codimension $n - 1$.

Recall that $\partial^{n-1} = \sum_{x \in X^{n-1}} \partial_x^{n-1}$, where, for each $x \in X^{n-1}$,

$$
\partial_x^{n-1} : K_1(x) = \bigoplus_{y \in X^{n-1}_x} k(y)^* \to Z^1(V(\overline{x}))
$$

is the sum of divisors map. For each $x \in X^{n-1}$, set

$$
B_n(x) = \text{Ker } [H^2(k, K_1(x)) \to H^2(k, Z^1(V(\overline{x})))],
$$

where the map involved is induced by $\partial_x^{n-1}$, and let

$$
\Sigma : \bigoplus_{x \in X^{n-1}} H^2(k, Z^1(V(\overline{x}))) \to H^2(k, Z^n(\overline{X})),
$$

be the natural map $(\xi_x) \mapsto \sum c_x(\xi_x)$, where $c_x : H^2(k, Z^1(V(\overline{x}))) \to H^2(k, Z^n(\overline{X}))$ is induced by the inclusion $Z^1(V(\overline{x})) \subset Z^n(\overline{X})$. Then there exists a canonical exact sequence

$$
0 \to \bigoplus_{x \in X^{n-1}} B_n(x) \to B_n(X) \to \text{Ker } \Sigma.
$$

We will relate the groups $B_n(x)$ to more familiar objects.
Fix $x \in X^{n-1}$ and set $Y = V(x)$. Then $Y$ is a geometrically reduced algebraic $k$-scheme [3, 4.6.4]. Further, the map $\overline{K}_1(x) \to Z^1(Y)$ factors through $\text{Div}Y$, the group of Cartier divisors on $Y$. Consider

$$B_n'(x) = \text{Ker} \left[ H^2(k, \overline{K}_1(x)) \to H^2(k, \text{Div}Y) \right] \subset B_n(x).$$

Let $\mathcal{R}_{\overline{Y}}'$ denote the étale sheaf of invertible rational functions on $\overline{Y}$. Note that $\overline{K}_1(x) = H^0(Y, \mathcal{R}_Y')$. Now, since $\overline{Y}$ is reduced, there exists an exact sequence of étale sheaves

$$0 \to \mathbb{G}_{m,Y} \to \mathcal{R}_{\overline{Y}}' \to \text{Div}_{\overline{Y}} \to 0,$$

where $\text{Div}_{\overline{Y}}$ is the sheaf of Cartier divisors on $\overline{Y}$ [3, 20.1.4 and 20.2.13]. This exact sequence gives rise to an exact sequence of étale cohomology groups

$$0 \to H^1_{\text{ét}}(\overline{Y}, \text{Div}_{\overline{Y}}) \to \text{Br}' \overline{Y} \to H^2_{\text{ét}}(\overline{Y}, \mathcal{R}_{\overline{Y}}') \to H^2_{\text{ét}}(\overline{Y}, \text{Div}_{\overline{Y}})$$

where $\text{Br}' \overline{Y} = H^2_{\text{ét}}(\overline{Y}, \mathbb{G}_m)$ is the cohomological Brauer group of $\overline{Y}$ [2, II, p.73]. Similarly, there exists an exact sequence

$$0 \to H^1_{\text{ét}}(Y, \text{Div}_Y) \to \text{Br}' Y \to H^2_{\text{ét}}(Y, \mathcal{R}_Y') \to H^2_{\text{ét}}(Y, \text{Div}_Y).$$

We will regard $H^1_{\text{ét}}(\overline{Y}, \text{Div}_{\overline{Y}})$ (resp. $H^1_{\text{ét}}(Y, \text{Div}_Y)$) as a subgroup of $\text{Br}' \overline{Y}$ (resp. $\text{Br}' Y$).

Now the exact sequence of terms of low degree

$$0 \to E^{1,0}_2 \to E^1 \to E^{0,1}_2 \to E^{2,0}_2 \to \text{Ker} (E^2 \to E^{0,2}_2) \to E^{1,1}_2 \to E^{3,0}_2$$

belonging to the Hochschild-Serre spectral sequence

$$E^{p,q}_2 = H^p(k, H^q_{\text{ét}}(\overline{Y}, \mathcal{R}_Y')) \Rightarrow H^{p+q}_{\text{ét}}(Y, \mathcal{R}_Y')$$

yields, using [2, II, Lemma 1.6, p.72], an exact sequence

$$0 \to H^2(k, \overline{K}_1(x)) \to H^2_{\text{ét}}(Y, \mathcal{R}_Y') \to H^2_{\text{ét}}(\overline{Y}, \mathcal{R}_Y').$$

Similarly, the spectral sequence

$$H^p(k, H^q_{\text{ét}}(\overline{Y}, \text{Div}_{\overline{Y}})) \Rightarrow H^{p+q}_{\text{ét}}(Y, \text{Div}_Y)$$

yields a complex

$$0 \to H^1(k, \text{Div}Y) \to H^1_{\text{ét}}(Y, \text{Div}_Y) \xrightarrow{\psi} H^1_{\text{ét}}(\overline{Y}, \text{Div}_{\overline{Y}}) \xrightarrow{\varphi} H^2(k, \text{Div}Y) \to H^2_{\text{ét}}(Y, \text{Div}_Y) \to H^2_{\text{ét}}(\overline{Y}, \text{Div}_{\overline{Y}})$$

which is exact except perhaps at $H^2_{\text{ét}}(Y, \text{Div}_Y)$. The map labeled $\psi$ in (13) is induced by the canonical morphism $\overline{Y} \to Y$, while the map $\varphi$ is the differential $d^0_{2,1}$ coming from the spectral sequence (see [6, II.4, pp.39-52]). Now we have a commutative diagram
(14)
\[
\begin{array}{ccccccc}
0 & \rightarrow & H^2(k, \mathcal{R}_1(x)) & \rightarrow & H^2_\text{ét}(Y, \mathcal{R}_Y^*) & \rightarrow & H^2_\text{ét}(\overline{Y}, \mathcal{R}_{\overline{Y}}^*) \\
& & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^2(k, \text{Div} \overline{Y})/\text{Im} \varphi & \rightarrow & H^2_\text{ét}(Y, \text{Div}_Y) & \rightarrow & H^2_\text{ét}(\overline{Y}, \text{Div}_{\overline{Y}}) \\
\end{array}
\]

in which the top row is the exact sequence (12), the bottom row (which is only a complex) is derived from (13), and the middle and right-hand vertical maps are the maps in (11) and (10), respectively. Set
\[\widehat{\text{Br}}^1_Y = \ker \left[ \text{Br}'Y/H^1_\text{ét}(Y, \text{Div}_Y) \rightarrow \text{Br}'\overline{Y}/H^1_\text{ét}(\overline{Y}, \text{Div}_{\overline{Y}}) \right].\]

Then the above diagram yields a natural isomorphism
(15) \[\widehat{\text{Br}}^1_Y = \ker \left[ H^2(k, \mathcal{R}_1(x)) \rightarrow H^2(k, \text{Div} \overline{Y})/\text{Im} \varphi \right].\]
(Note: only the exactness of the top row of (14) is needed to obtain the above isomorphism.) On the other hand, there exists an obvious exact sequence
\[0 \rightarrow B'_n(x) \rightarrow \ker \left[ H^2(k, \mathcal{R}_1(x)) \rightarrow H^2(k, \text{Div} \overline{Y})/\text{Im} \varphi \right] \rightarrow \text{Im} \varphi,\]
where \(B'_n(x)\) is the group (9). Using (15) and the fact that \(\text{Im} \varphi\) is naturally isomorphic to \(\text{Coker} \psi\), where \(\psi\) is the map appearing in (13), we conclude that there exists a natural exact sequence
\[0 \rightarrow B'_n(x) \rightarrow \widehat{\text{Br}}^1_Y \xrightarrow{h} \text{Coker} \psi.\]

The map labeled \(h\) in the above exact sequence can be briefly described as \(\varphi^{-1} \circ h^2(\text{div}) \circ u^{-1}\nu\), where \(u: H^2(k, \mathcal{R}_1(x)) \rightarrow H^2_\text{ét}(Y, \mathcal{R}_Y^*)\) is the map intervening in (14) and \(h^2(\text{div}): H^2(k, \mathcal{R}_1(x)) \rightarrow H^2(k, \text{Div} \overline{Y})\) is induced by \(\text{div}: \mathcal{R}_1(x) \rightarrow \text{Div} \overline{Y}\). Next, set
\[\text{Br}'_1Y = \ker \left[ \text{Br}'Y \rightarrow \text{Br}'\overline{Y} \right].\]

There exists a natural exact commutative diagram
\[
\begin{array}{ccccccc}
0 & \rightarrow & H^1_\text{ét}(Y, \text{Div}_Y) & \rightarrow & \text{Br}'_1Y & \rightarrow & \text{Br}'_1Y/H^1_\text{ét}(Y, \text{Div}_Y) \\
& & \downarrow \psi & & \downarrow & & \\
0 & \rightarrow & H^1_\text{ét}(\overline{Y}, \text{Div}_{\overline{Y}})^\Gamma & \rightarrow & (\text{Br}'Y)^\Gamma & \rightarrow & (\text{Br}'\overline{Y}/H^1_\text{ét}(\overline{Y}, \text{Div}_{\overline{Y}}))^\Gamma \\
\end{array}
\]

An application of the snake lemma to the above diagram yields a natural exact sequence
(17) \[0 \rightarrow H^1(k, \text{Div} \overline{Y}) \rightarrow \text{Br}'_1Y \rightarrow \widehat{\text{Br}}'_1Y \xrightarrow{\delta} \text{Coker} \psi.\]
Now using the explicit description of the map $\delta$ [8, Lemma 1.3.2, p.11] together with the description of the map $\varphi = d_2^{0,1}$ from [6, §II.4], it can be shown (with some work) that the maps $h$ in (16) and $\delta$ in (17) are the same. Thus we obtain

**Proposition 4.1.** There exists a canonical isomorphism

$$B'_n(x) = Br'_1Y / H^1(k, \text{Div}\overline{Y}).$$

**Corollary 4.2.** Let $x \in X^{n-1}$ be such that $\overline{Y} = V(\mathfrak{p})$ is locally factorial (this holds, for example, if $Y = V(x)$ is regular). Then there exists a canonical isomorphism

$$B_n(x) = Br'_1Y.$$

**Proof.** The hypothesis implies that $\text{Div}\overline{Y} = Z^1(\mathfrak{p})$ [3, 21.6.9], so $B_n(x) = B'_n(x)$. On the other hand, since $Z^1(\mathfrak{p})$ is a permutation $I$-module, $H^1(k, \text{Div}\overline{Y}) = H^1(k, Z^1(\mathfrak{p})) = 0$. The result is now immediate from the proposition. $\square$

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**Departamento de Matemáticas, Universidad de La Serena, La Serena, Chile**

E-mail address: cgonzalez@usserena.cl