A GENERALIZED SCHUR COMPLEMENT FOR NON-NEGATIVE OPERATORS ON LINEAR SPACES

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Abstract. Extending the corresponding notion for matrices or bounded linear operators on a Hilbert space we define a generalized Schur complement for a non-negative linear operator mapping a linear space into its dual and derive some of its properties.

1. Introduction

In case of $2 \times 2$ block matrices the notion of Schur complement and its generalizations have a long history. We refer to [24], where it is given a comprehensive exposition of the history, theory and versatility of Schur complements. In the case of non-negative matrices the notion of generalized Schur complement can be extended to matrices whose entries are bounded operators in Hilbert spaces, cf. [17]. Moreover, generalized Schur complements are closely related to shorted operators which were introduced by M.G. Kreǐn [11] and have found interesting applications in electrical network theory, cf. [2]. Since there is a natural way to define non-negativity for a linear operator, which maps a linear space into its dual, one can ask for a generalized Schur complement of such an operator. A first attempt was made in [6], where non-negative bounded linear operators from a Banach space into its topological dual were discussed. The present paper deals with a generalized Schur complement and a shorted operator of a non-negative $2 \times 2$ block matrix, whose entries are linear operators on linear spaces. Thus, topological questions, particularly continuity problems, play a minor role. To define a generalized Schur complement of a non-negative operator on a linear space, one needs the notion of a square root. Section 3 deals with this important and useful concept, which was studied by many authors. Section 4 contains definitions and basic properties of the Schur complement and the shorted operator for slightly more general than non-negative operators. In Section 5 further results on generalized Schur complements are derived. Among other things we extend the Crabtree-Haynsworth quotient formula [4]. One of the most useful results concerning $2 \times 2$ block matrices is Albert’s non-negativity criterion [1]. A generalization to non-negative operators on linear spaces and some of its consequences are given in Section 6. The special class of extremal operators, which was introduced by M.G. Kreǐn [11] is the subject of Section 7.

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For bounded linear operators on Hilbert spaces many results concerning the
generalized Schur complement were obtained by Yu. L. Shmulyan. A large part
of them was proved independently or rediscovered later on by several mathema-
ticians from western countries. The present paper is strongly influenced by
Shmulyan’s work and was written to illustrate his contribution to the theory of
generalized Schur complement. In this way most of our assertions of Sections 4.5
are generalizations of results contained in [17] to non-negative operators on linear
spaces.

2. Basic definitions and notations

Any linear space of the present paper is a space over \( \mathbb{C} \), the field of complex
numbers, and its zero element is denoted by 0. For a linear space \( X \), let \( X' \) denote
its dual space of all antilinear functionals on \( X \) and \( \langle x', x \rangle_X := \langle x', x \rangle \) the value
of \( x' \in X' \) at \( x \in X \). If \( X^\sim \) is a subspace of \( X' \), an arbitrary \( x \in X \) defines an
element \( jx \) of \( (X^\sim)' \) according to

\[ \langle jx, x^\sim \rangle_{X'} := \overline{\langle x^\sim, x \rangle}_X, \quad x^\sim \in X^\sim, \]

where \( \bar{\alpha} \) stands for the complex conjugate of \( \alpha \in \mathbb{C} \).

Convention (CN): If for all \( x \in X \setminus \{0\} \) there exists \( x^\sim \in X^\sim \) such that
\( \langle x^\sim, x \rangle \neq 0 \), we shall identify \( X \) and its isomorphic image under the map \( j \) and write

\[ \langle jx, x^\sim \rangle_{X'} = \langle x, x^\sim \rangle_{X'}, \quad x \in X, x^\sim \in X^\sim. \]

The linear space of all linear operators from \( X \) into a linear space \( Y \) is denoted by
\( \mathcal{L}(X, Y) \), and \( I \) is the identity operator in case \( X = Y \). If \( A \in \mathcal{L}(X, Y) \) and \( X_1 \)
is a subspace of \( X \), the symbols \( \ker A \), \( \text{ran} A \) and \( A|_{X_1} \) stand for the null space,
range, and restriction of \( A \) to \( X_1 \), resp. Set \( AX_1 := \text{ran} A|_{X_1} \). The dual operator
\( A' \in \mathcal{L}(Y', X') \) is defined by the relation \( \langle y', Ax \rangle_Y = \langle A'y', x \rangle_X \), \( x \in X, y' \in Y' \).

Examples. 1. If \( Z \) is a linear space and \( A \in \mathcal{L}(X, Y) \), \( B \in \mathcal{L}(Y, Z) \), then
\( (BA)' = A'B' \).
2. If \( A \in \mathcal{L}(X, Y) \), then \( A'' \in \mathcal{L}(X'', Y'') \) and \( A = A''|_X \) according to (CN).
3. If \( A \in \mathcal{L}(X, X') \), then \( A' \in \mathcal{L}(X'', X') \). Taking into account (CN), we get
\( \langle x_2, Ax_1 \rangle_{X'} = \overline{\langle Ax_1, x_2 \rangle}_X \) and \( \langle x_2, Ax_1 \rangle_{X'} = \langle A'x_2, x_1 \rangle_X \), hence,

\[ \langle Ax_1, x_2 \rangle_X = \overline{\langle A'x_2, x_1 \rangle}_X, \quad x_1, x_2 \in X. \quad (2.1) \]

An operator \( A \in \mathcal{L}(X, X') \) is called Hermitian, if \( \langle Ax_1, x_2 \rangle = \overline{\langle Ax_2, x_1 \rangle} \) and
non-negative if \( \langle Ax_1, x_1 \rangle \geq 0 \), \( x_1, x_2 \in X \). The sets of all Hermitian and all
non-negative operators are denoted by \( \mathcal{L}^h(X, X') \) and \( \mathcal{L}^\geq(X, X') \), resp.. The
polarization identity implies that \( A \) is Hermitian if and only if \( \langle Ax, x \rangle \) is real
for all \( x \in X \). Thus \( \mathcal{L}^\geq(X, X') \subseteq \mathcal{L}^h(X, X') \) and the space \( \mathcal{L}^h(X, X') \) can be
provided with Loewner’s semi-ordering, i.e. for \( A, D \in \mathcal{L}^h(X, X') \) we shall write
\( A \leq D \) if and only if \( \langle Ax, x \rangle \leq \langle Dx, x \rangle \), \( x \in X \). Recall Cauchy’s inequality

\[ |\langle Ax_1, x_2 \rangle|^2 \leq \langle Ax_1, x_1 \rangle \langle Ax_2, x_2 \rangle, \quad x_1, x_2 \in X, \quad (2.2) \]

if \( A \in \mathcal{L}^\geq(X, X') \).
3. Square roots

Let $H$ be a complex Hilbert space with norm $\| \cdot \| := \| \cdot \|_H$ and inner product $(\cdot | \cdot) := (\cdot | \cdot)_H$, which is assumed to be antilinear with respect to the second component. Let $R \in \mathcal{L}(X, H)$. Identifying $H$ and the space of continuous antilinear functionals on $H$ in the common way, one has $H \subseteq H'$ and

$$
(h | Rx) = (R'h, x), \quad x \in X, h \in H.
$$

(3.1)

Set $R^* := R' |_H$. From (3.1) it can be concluded that $\ker R^*$ is equal to the orthogonal complement of $(\ran R)^c$, where $M^c$ denotes the closure of a subset $M$ of a topological space. It follows that $R^*$ is one-to-one if and only if $\ran R$ is dense in $H$ and that

$$
\ran R^* = R^*(\ran R)^c
$$

(3.2)

Therefore, we can define a generalized inverse $R^{[-1]}$ of $R^*$ by

$$
R^{[-1]}x' := (R^* |_{(\ran R)^c})^{-1}x', \quad x' \in \ran R^*.
$$

**Lemma 3.1.** Let $R \in \mathcal{L}(X, H)$. An element $x' \in X'$ belongs to $\ran R^*$ if and only if the following conditions are satisfied:

(i) If $x \in \ker R$, then $(x', x) = 0$.

(ii) $\sup_{x \in X} \frac{|(x', x)|^2}{\|Rx\|_H^2} < \infty$ (with convention $\frac{0}{0} := 0$ at the left-hand side).

**Proof.** If $x' \in \ran R$ for some $h \in H$, then

$$
|(x', x)| = |(R'h, x)| = |(h | Rx)| \leq \|h\| \|Rx\|,
$$

which yields (i) and (ii). Conversely, assume that (i) and (ii) are satisfied for some $x' \in X'$. Set $\varphi(Rx) := (x', x), \ x \in X$. Because of (i) $\varphi$ is correctly defined and (ii) implies that $\varphi$ is continuous, so that $\varphi$ is a continuous antilinear functional on $\ran R$. Thus, there exists $h \in H$ such that $(x', x) = (h | Rx) = (R'h, x)$ for all $x \in X$, which yields $x' = R^*h \in \ran R^*$.

**Definition 3.2.** Let $A \in \mathcal{L}(X, X')$. A pair $(R, H)$ of a Hilbert space $H$ and an operator $R \in \mathcal{L}(X, H)$ is called a square root of $A$ if $A = R^* R$, and a minimal square root if, additionally, $\ran R$ is dense in $H$.

Note that there exists a square root of $A$ if and only if there exists a minimal one. The following result is basic to our considerations and generalizes the fact concerning the existence of a square root of a non-negative selfadjoint operator in a Hilbert space. Its well known short proof is recapitulated for convenience of the reader.

**Theorem 3.3.** An operator $A \in \mathcal{L}(X, X')$ possesses a square root if and only if it is non-negative.

**Proof.** Let $A \in \mathcal{L}^\geq(X, X')$. Cauchy’s inequality (2.2) implies that

$$
N := \{x \in X : \langle Ax, x \rangle = 0\}
$$

is a subspace of $X$. Define an inner product on the quotient space $X/N$ by

$$(x_1 + N \mid x_2 + N) := \langle Ax_1, x_2 \rangle, \quad x_1, x_2 \in X,$$

and let $R \in \mathcal{L}(X/N, H)$ be a continuous antilinear functional on $X/N$. Then $R$ extends to a continuous antilinear functional $\tilde{R}$ on $X$ such that $\tilde{R}(x + N) = R(x)$ for all $x \in X$.
and denote the completion of the corresponding inner product space by \( H \). Set 
\[
R_x := x + N, \ x \in X.
\]
It follows \( R \in \mathcal{L}(X, H) \), \( (\text{ran} \ R)^c = H \), and 
\[
\langle Ax_1, x_2 \rangle = (Rx_1 \mid Rx_2) = \langle R^* Rx_1, x_2 \rangle, \quad x_1, x_2 \in X.
\]
Therefore, \((R, H)\) is a minimal square root of \( A \). The "only if"-part of the assertion is obvious. \( \square \)

The notion of a square root of a non-negative operator acting between spaces more general than Hilbert spaces was discussed and applied by many authors. Most of them deal with a topological space \( X \) and in this case continuity problems arise additionally. Some properties of square roots for operators of special type were obtained by Vainberg and Engel’son, cf. [23]. For a Banach space \( X \), the construction of the proof of the preceding theorem was published as an appendix to [21] and attributed to Chobanyan, see also [15] and [22]. Another but related construction was proposed by Sebestyén [16], cf. [19]. Görniak [7] and Görniak and Weron [8] dealt with the existence of a continuous square root if \( X \) is a topological linear space. Görniak, Makagon and Weron [9] investigated square roots of non-negative operator-valued measures. Pusz and Woronowicz [14] extended the construction of the proof of Theorem 3.3 to pairs of non-negative sequilinear forms, cf. [20] for further generalizations.

**Lemma 3.4.** If \( A \in \mathcal{L}(X, X') \) and \((R, H)\) is a square root of \( A \), then 
\[
\ker R = \ker A = \{ x \in X : \langle Ax, x \rangle = 0 \}.
\]

**Proof.** The result follows from a chain of conclusions:
\[
\langle Ax, x \rangle = 0 \Rightarrow \langle R^* Rx, x \rangle = 0 \Rightarrow (Rx \mid Rx) = 0 \Rightarrow Rx = 0,
\]
and conversely
\[
Rx = 0 \Rightarrow R^* Rx = 0 \Rightarrow Ax = 0 \Rightarrow \langle Ax, x \rangle = 0. \ \ \square
\]

The preceding results can be used to derive a version of a part of Douglas’ theorem [5], cf. [18].

**Proposition 3.5.** Let \( A, D \in \mathcal{L}(X, X') \) and \((R_A, H_A)\) and \((R_D, H_D)\) be square roots of \( A \) and \( D \), resp.. The following assertions are equivalent:

(i) \( A \leq \alpha^2 D \) for some \( \alpha \in [0, \infty) \),

(ii) there exists a bounded operator \( W \in \mathcal{L}(H_A, H_D) \) with operator norm \( \|W\| \leq \alpha \) and such that \( R_A^* = R_D^* W \).

If (i) or (ii) are satisfied, there exists a unique \( W \) so that \( W \subseteq (\text{ran} \ R_D)^c \). Moreover, \( \ker W = \ker R_A^* \) for this operator \( W \).

**Proof.** Since \( R_A^* = R_D^* W \) yields \( R_A = R_A^* \mid_X = W' R_D^* \mid_X = W^* R_D \) by (CN), from (ii) it follows
\[
\langle Ax, x \rangle = \|R_Ax\|^2 = \|W^* R_Dx\|^2 \leq \alpha^2 \|R_Dx\|^2 = \alpha^2 \langle Dx, x \rangle, \quad x \in X,
\]
hence, (i). Let \( W_j \in \mathcal{L}(H_A, H_D) \) be such that \( R_A^* = R_D^* W_j \) and \( \text{ran} \ W_j \subseteq (\text{ran} \ R_D)^c \), \( j = 1, 2 \). Then \( \text{ran}(W_1 - W_2) \subseteq \ker R_D^* \) and \( \text{ran}(W_1 - W_2) \subseteq (\text{ran} \ R_D)^c \), which shows that \( W_1 = W_2 \). Now assume that (i) is true. One has
ker $R_D \subseteq \ker R_A$ by Lemma 3.4 hence, ran $R_A^* \subseteq \text{ran } R_D^*$ by Lemma 3.1. The operator $W := R_D^{[-1]} R_A^* \in \mathcal{L}(H_A, H_D)$ satisfies $R_D^* W = R_A^*$, ker $W = \ker R_A^*$ and ran $W \subseteq (\text{ran } R_D)^c$. The inclusion ran $R_D^{[-1]} \subseteq (\text{ran } R_D)^c$ implies that $W^* h = R_A^* (R_D^{[-1]*})^* h = 0$ if $h$ is orthogonal to ran $R_D$. Therefore, from

$$
\|W^* R_D x\|^2 = \|R_A x\|^2 = \langle Ax, x \rangle \leq \alpha^2 \langle Dx, x \rangle = \alpha^2 \|R_D x\|^2, \quad x \in X,
$$

one can conclude that $\|W\| = \|W^*\| \leq \alpha$. \hfill \(\square\)

As a by-product of Proposition 3.5 we obtain the following corollary.

**Corollary 3.6.** Let $A$, $D$, $(R_A, H_A)$, and $(R_D, H_D)$ be as in Proposition 3.5.

(i) If $A \leq \alpha^2 D$ for some $\alpha \in [0, \infty)$, then ran $R_A^* \subseteq \text{ran } R_D^*$.

(ii) If $\beta^2 D \leq A \leq \alpha^2 D$ for some $\alpha, \beta \in (0, \infty)$, then ran $R_A^* = \text{ran } R_D^*$.

(iii) If $(S_A, G_A)$ is a square root of $A$, then ran $R_A^* = \text{ran } S_A^*$.

**Corollary 3.7.** Let $H_j$ be Hilbert spaces and $R_j \in \mathcal{L}(X, H_j)$, $j = 1, 2$. If $(R, H)$ is a square root of the non-negative operator $A := R_1^* R_1 + R_2^* R_2$, then ran $R^* = \text{ran } R_1^* + \text{ran } R_2^*$.

**Proof.** Let $G$ be the orthogonal sum of $H_1$ and $H_2$ and $S \in \mathcal{L}(X, G)$ be defined by $S = \begin{pmatrix} R_1^* & R_2^* \end{pmatrix}$. Since $S^* = (R_1^*, R_2^*)$ and $S^* S = A$, we get that ran $S^* = \text{ran } R_1^* + \text{ran } R_2^*$ and that $(S, G)$ is a square root of $A$. Now apply Corollary 3.6 (iii). \hfill \(\square\)

**Lemma 3.8.** Let $(R, H)$ be a square root and $(S, G)$ a minimal square root of $A \in \mathcal{L}^\geq(X, X')$. There exists an isometry $U \in \mathcal{L}(G, H)$ such that $US = R$.

**Proof.** By Lemma 3.4 there exists an operator $\bar{U}$ satisfying $\bar{U} S x = Rx, x \in X$. From $\|\bar{U} S x\|^2 = \|Rx\|^2 = \langle Ax, x \rangle = \|S x\|^2$ it follows that $\bar{U}$ is isometric and can be extended to an isometry $U \in \mathcal{L}(G, H)$. \hfill \(\square\)

4. **Generalized Schur complements and shorted operators of operators of positive type**

Let $X$ and $Y$ be linear spaces.

**Definition 4.1.** A pair $(A, B)$ of an operator $A \in \mathcal{L}^\geq(X, X')$ and $B \in \mathcal{L}(Y, X')$ is called a positive pair if ran $B \subseteq \text{ran } R^*$ for some square root (and, hence, for all square roots) $(R, H)$ of $A$.

The following criterion is an immediate consequence of Lemma 3.1.

**Lemma 4.2.** Let $A \in \mathcal{L}^\geq(X, X')$ and $B \in \mathcal{L}(Y, X')$. The pair $(A, B)$ is a positive pair if and only if for all $y \in Y$ the following conditions are satisfied:

(i) If $x \in \ker A$, then $\langle By, x \rangle = 0$.

(ii) $\sup_{x \in X} \frac{\|\langle By, x \rangle\|^2}{\|\langle Ax, x \rangle\|} < \infty$ (with convention $\frac{0}{0} := 0$).

According to (CN), the space $X$ can be considered as a subspace of the domain of $B'$. To abbreviate the notation we set

$$
B^\sim := B'|_X.
$$
Note that \((By, x) = \langle B^*x, y \rangle\), \(x \in X\), \(y \in Y\). Thus condition (i) of Lemma 4.2 is equivalent to the inclusion \(\ker A \subseteq \ker B^*\).

If \((A, B)\) is a positive pair and \((R, H)\) is a square root of \(A\), the operators
\[
T := R^{*-1}B
\]
and
\[
\omega(A, B) := T^*T = (R^{*-1}B)^*R^{*-1}B
\]
can be defined. Note that \(B = R^*T\). The following lemma is obvious.

Lemma 4.3. If \((A, B)\) is a positive pair and \((R, H)\) is a square root of \(A\), then for all \(y \in Y\)
\[
\inf \{\|Ty - Rx\| : x \in X\} = 0.
\]
Equivalently, \(\text{ran } T \subseteq (\text{ran } R)^c\). □

Recall that the dual space of \(X \times Y\) can be written as a Cartesian product \((X \times Y)' = X' \times Y'\), where \(\langle \cdot, \cdot \rangle_{X \times Y} = \langle \cdot, \cdot \rangle_X + \langle \cdot, \cdot \rangle_Y\). Also, it should not cause confusion if we identify the subspace \(X \times \{0\}\) of \(X \times Y\) with \(X\). An operator \(A\) of \(\mathcal{L}(X \times Y, (X \times Y)')\) can be represented as a \(2 \times 2\) matrix \(A = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\), where \(A \in \mathcal{L}(X, X')\), \(B, C \in \mathcal{L}(Y, X')\), \(D \in \mathcal{L}(Y, Y')\). It is not hard to see that \(A\) is Hermitian if and only if \(A = D\) and \(C = B^*\). To abbreviate the notation we set \(\mathcal{L}^h(X \times Y, (X \times Y)') =: \mathcal{L}^h\) and \(\mathcal{L}^z(X \times Y, (X \times Y)') =: \mathcal{L}^z\).

Definition 4.4. An operator \(\begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \in \mathcal{L}^h\) is called an operator of positive type if \((A, B)\) is a positive pair. The set of operators of positive type is denoted by \(\mathcal{L}^+(X \times Y, (X \times Y)') =: \mathcal{L}^+\).

Definition 4.5. Let \(A = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \in \mathcal{L}^+\). The operator \(\sigma(A) := D - \omega(A, B)\) is called a generalized Schur complement of \(A\) and the operator
\[
\mathcal{J}(A) := \begin{pmatrix} 0 & 0 \\ 0 & \sigma(A) \end{pmatrix}
\]
is called a shorted operator.

The following result is a generalization of [2, Corollary 1 to Theorem 3].

Proposition 4.6. If \(A = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \in \mathcal{L}^+,\) then \(\text{ran } A \cap Y' \subseteq \text{ran } \mathcal{J}(A)\).

Proof. Let \(y' \in Y'\) be such that \(A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y' \end{pmatrix}\) for some \(\begin{pmatrix} x \\ y \end{pmatrix} \in X \times Y\). Let \((R, H)\) be a minimal square root of \(A\). Since
\[
A = \begin{pmatrix} R^*R & R^*T \\ T^*R & T^*T \end{pmatrix} + \mathcal{J}(A),
\]
one has \(R^*Rx + R^*Ty = 0\), hence, \(Rx + Ty = 0\) and \(T^*Rx + T^*Ty = 0\), which yields
\[
\begin{pmatrix} 0 \\ y' \end{pmatrix} = \mathcal{J}(A) \begin{pmatrix} x \\ y \end{pmatrix} \in \text{ran } \mathcal{J}(A).\] □
The next result is a simple but useful consequence of Lemma 4.3.

**Proposition 4.7.** Let \( A = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \in \mathcal{L}^+ \). For \((x\ y) \in X \times Y\),

\[
\langle \mathcal{S}(A) \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \rangle = \inf_{z \in X} \left\langle A \begin{pmatrix} x - z \\ y \end{pmatrix}, \begin{pmatrix} x - z \\ y \end{pmatrix} \right\rangle.
\]

(4.3)

Particularly, \( \sigma(A) \) and \( \mathcal{S}(A) \) do not depend on the choice of the square root of \( A \).

**Proof.** Since (4.3) is independent of \( x \in X \), it is enough to prove it for \( x = 0 \). From Lemma 4.3 it follows

\[
\inf_{z \in X} \left\langle A \begin{pmatrix} -z \\ y \end{pmatrix}, \begin{pmatrix} -z \\ y \end{pmatrix} \right\rangle
= \inf_{z \in X} \left\langle \begin{pmatrix} R^* R & R^* T \\ T^* R & T^* T \end{pmatrix} \begin{pmatrix} -z \\ y \end{pmatrix}, \begin{pmatrix} -z \\ y \end{pmatrix} \right\rangle + \left\langle \mathcal{S}(A) \begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle
= \inf_{z \in X} \|Ty - Rz\|^2 + \left\langle \mathcal{S}(A) \begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle
= \left\langle \mathcal{S}(A) \begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle.
\]

□

**Corollary 4.8.** (i) If \( A \in \mathcal{L}^+ \), then \( \mathcal{S}(A) \leq A \) and \( \ker A \subseteq \ker \mathcal{S}(A) \).

(ii) If \( A, A_1 \in \mathcal{L}^+ \) and \( A \leq A_1 \), then \( \mathcal{S}(A) \leq \mathcal{S}(A_1) \).

**Proof.** The first assertion of (i) as well as (ii) are immediately clear from Proposition 4.7. To prove the second assertion of (i), let \((x\ y) \in \ker A\). If \( z \in X \), one has

\[
\left\langle A \begin{pmatrix} x - z \\ y \end{pmatrix}, \begin{pmatrix} x - z \\ y \end{pmatrix} \right\rangle = \langle Az, z \rangle \geq 0,
\]

which implies that the infimum at the right hand side of (4.3) is equal to 0. Since \( \mathcal{S}(A) \leq A \) and

\[
\left\langle \begin{pmatrix} A - \mathcal{S}(A) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = 0,
\]

it follows \((x\ y) \in \ker (A - \mathcal{S}(A))\) by Lemma 3.4, hence, \((x\ y) \in \ker \mathcal{S}(A)\). □

**Corollary 4.9.** If \( A \in \mathcal{L}^\geq \), then \( \mathcal{S}(A) \in \mathcal{L}^\geq \).

□

5. **Further applications of square roots**

First we express the generalized Schur complement of an operator of \( \mathcal{L}^\geq \) with the aid of its square root and derive a range description, cf. [2, Corollary 4 to Theorem 1]. Let \( A \in \mathcal{L}^\geq \) and \((R, H)\) be a square root of \( A \). Let \( L \) be the orthogonal complement of \((RX)^c\) and \( P \) be the orthoprojection onto \( L \). Note that \( L \) can be characterized by \( L = \{h \in H : R^* h \in Y'\} \), which yields \( R^* L = \text{ran } R^* \cap Y' \).

**Proposition 5.1.** If \( A \in \mathcal{L}^\geq \), then \( \mathcal{S}(A) = R^* PR \).
Proof. Let \((x, y) \in X \times Y\). An application of (4.3) gives
\[
\left\langle \mathcal{F}(A) \left( \begin{array}{c} x \\ y \end{array} \right), \left( \begin{array}{c} x \\ y \end{array} \right) \right\rangle = \inf_{z \in X} \left\| R \left( \begin{array}{c} x \\ y \end{array} \right) - R \left( \begin{array}{c} z \\ 0 \end{array} \right) \right\|^2,
\]
which shows that \(\left\langle \mathcal{F}(A) \left( \begin{array}{c} x \\ y \end{array} \right), \left( \begin{array}{c} x \\ y \end{array} \right) \right\rangle\) is the squared distance of \(R \left( \begin{array}{c} x \\ y \end{array} \right)\) to \(RX\). Therefore,
\[
\left\langle \mathcal{F}(A) \left( \begin{array}{c} x \\ y \end{array} \right), \left( \begin{array}{c} x \\ y \end{array} \right) \right\rangle = \left\| PR \left( \begin{array}{c} x \\ y \end{array} \right) \right\|^2 = \left\langle R^* PR \left( \begin{array}{c} x \\ y \end{array} \right), \left( \begin{array}{c} x \\ y \end{array} \right) \right\rangle
\]
and the assertion follows from the polarization identity. \(\square\)

**Proposition 5.2.** If \(A \in \mathcal{L}^\ge\) and \((R, H)\) and \((S, G)\) are square roots of \(A\) and \(\mathcal{F}(A)\), resp., then \(\text{ran} S^* = \text{ran} R^* \cap Y'\).

**Proof.** Setting \(R_X := R|_X\) and \(R_Y := R|_Y\), we get
\[
A = \left( \begin{array}{c} R_X^* \\ R_Y^* \end{array} \right) (R_X R_Y) = \left( \begin{array}{c} R_X^* R_X & R_X^* R_Y \\ R_Y^* R_X & R_Y^* R_Y \end{array} \right),
\]
hence,
\[
\sigma(A) = R_Y^* R_Y - (R_X^* R_X)^* R_Y^* R_Y - (R_X^* R_X) R_Y^* R_Y = R_Y^* PR_Y
\]
since \(R_X^* R_X = I - P\). Thus \(\mathcal{F}(A) = R^* PR\) and \((PR, H)\) is a square root of \(\mathcal{F}(A)\). If \(\left( \begin{array}{c} 0 \\ y' \end{array} \right) \in X' \times Y'\) is such that \(R^* h = \left( \begin{array}{c} 0 \\ y' \end{array} \right)\) for some \(h \in H\), then \(R_X^* h = 0\), hence, \(Ph = h\) and \((PR)^* h = R^* h = \left( \begin{array}{c} 0 \\ y' \end{array} \right)\), which implies that \(\text{ran} R^* \cap Y' \subseteq \text{ran}(PR)^* = \text{ran} S^*\) by Corollary 3.6 (iii). Since, obviously, \(\text{ran} S^* \subseteq Y'\) and \(\text{ran} S^* \subseteq \text{ran} R^*\) by Corollaries 4.8 (i) and 3.6 (i), the assertion is proved. \(\square\)

Our next result is a generalization of the Crabtree-Haynsworth quotient formula \[.\] To give it a nice form let us denote \(\sigma(A) =: A/A\).

**Proposition 5.3.** Let \(X, Y,\) and \(Z\) be linear spaces,
\[
D := \left( \begin{array}{ccc} A & B & B_X \\ B^\sim & D & B_Y \\ B_X^\sim & B_Y^\sim & D_1 \end{array} \right) \in \mathcal{L}^\ge(X \times Y \times Z, X' \times Y' \times Z'),
\]
and \(A := \left( \begin{array}{c} A \\ B \end{array} \right)\). The operator \(A/A\) is the left upper corner of \(D/A\) and \(D/A\).

\[
D/A = A/A = D/A.
\]

**Proof.** Let \((R, H)\) be a minimal square root of \(A\), \(R_X := R|_X\), \(R_Y := R|_Y\),
\[
E := (R^*)^{-1} \left( \begin{array}{c} B_X \\ B_Y \end{array} \right),
\]
hence, $R_X^*E = B_X$, $R_Y^*E = B_Y$. From $R_X^{[*-1]}R_X^* = I - P$ one obtains

$$D/A = \begin{pmatrix} R_Y^*R_Y & R_Y^*E \\ E^*R_Y & D_1 \end{pmatrix} - \left( R_X^{[*-1]}(R_X^*R_Y, R_X^*E) \right)^*R_X^{[*-1]}(R_X^*R_Y, R_X^*E)$$

$$= \begin{pmatrix} R_Y^*PR_Y & R_Y^*PE \\ E^*PR_Y & D_1 - E^*(I - P)E \end{pmatrix}$$

and

$$A/A = R_Y^*R_Y - \left( R_X^{[*-1]}R_X^*R_Y \right)^*R_X^{[*-1]}R_X^*R_Y = R_Y^*PR_Y,$$

which shows that $A/A$ is the left upper corner of $D/A$. Since $(PR_Y, H)$ is a square root of $A/A$, one can compute

$$D/A \bigg/ A/A = D_1 - E^*(I - P)E - ((PR_Y)^{*[-1]}R_Y^*PE)^*(PR_Y)^{*[-1]}R_Y^*PE$$

$$= D_1 - E^*(I - P)E - E^*Q\varepsilon,$$

where $Q$ denotes the orthoprojection onto $(\text{ran } PR_Y)^c$. Comparing this with

$$D/A = D_1 - ((R^*)^{-1}R^*)E(R^*)^{-1}R^*E = D_1 - E^*E,$$

we can conclude that the assertion will be proved if we can show that the restriction of $I - P + Q$ to $\text{ran } R$ is the identity. If $h \in \text{ran } R$, then

$$h = R_Xx + R_Yy = R_Xx + (I - P)R_Yy + PR_Yy$$

for some $\begin{pmatrix} x \\ y \end{pmatrix} \in X \times Y$. Since $R_Xx + (I - P)R_Yy \in (\text{ran } R_X)^c$, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ of elements of $X$ such that $\lim_{n \to \infty} R_Xx_n = R_Xx + (I - P)R_Yy$. For $h_n := R_Xx_n + PR_Yy$, we have

$$(I - P + Q)h_n = (I - P + Q)(R_Xx_n + PR_Yy) = R_Xx_n + PR_Yy = h_n$$

and therefore

$$(I - P + Q)h = \lim_{n \to \infty} (I - P + Q)h_n$$

$$= \lim_{n \to \infty} (R_Xx_n + PR_Yy) = R_Xx + R_Yy = h. \quad \square$$

We conclude this section with a criterion for non-negativity of operators of $\mathcal{L}^h$.

**Proposition 5.4.** Let $A = \begin{pmatrix} A & B^* \\ B & D \end{pmatrix} \in \mathcal{L}^h$. The operator $A$ is non-negative if and only if the following two conditions are satisfied:

(i) The operators $A$ and $D$ are non-negative.

(ii) For any square roots $(R_A, H_A)$ and $(R_D, H_D)$ of $A$ and $D$, resp., there exists a contraction $K \in \mathcal{L}(H_D, H_A)$ such that $B = R_A^*KR$ and $\text{ran } K \subseteq (\text{ran } R_A)^c$.

**Proof.** If $A$ is non-negative, assertion (i) is trivial. To prove (ii) let $(R, H)$ be a square root of $A$ and $R_X := R|_X$, $R_Y = R|_Y$, hence

$$A = \begin{pmatrix} R_X^*R_X & R_X^*R_Y \\ R_Y^*R_X & R_Y^*R_Y \end{pmatrix}.$$

Let $(S_A, G_A)$ and $(S_D, G_D)$ be minimal square roots of $A$ and $D$, resp. According to Lemma 3.8 there exist isometries $U_A \in \mathcal{L}(G_A, H_A)$, $V_A \in \mathcal{L}(G_A, H)$, $U_D \in \mathcal{L}(G_D, H)$.
\( \mathcal{L}(G_D, H_D), V_D \in \mathcal{L}(G_D, H) \) satisfying \( U_A S_A = R_A, V_A S_A = R_X, U_D S_D = R_D, V_D S_D = R_Y \). It follows
\[
B = R_X^* R_Y = R_A^* U_A V_D U_D^* R_D = R_A^* K R_D,
\]
where \( K := U_A V_A^* V_D U_D^* \in \mathcal{L}(H_D, H_A) \) is a contraction with \( \text{ran } K \subseteq (\text{ran } R_A)^c \).

Conversely, if (i) and (ii) are satisfied, then
\[
A = \left( \begin{array}{cc} R_A^* R_A & R_A^* K R_D \\ (R_A^* K R_D)^\sim & R_D^* R_D \end{array} \right).
\]
Since
\[
(R_A^* K R_D)^\sim = (R_A^* K R_D)^\sim|_X = R_D^* K' R_A^0|_X = R_D^* K' R_A,
\]
one obtains
\[
A = \left( \begin{array}{cc} R_A^* & 0 \\ 0 & R_D^* \end{array} \right) \left( \begin{array}{cc} I & K \\ K^* & I \end{array} \right) \left( \begin{array}{cc} R_A & 0 \\ 0 & R_D \end{array} \right),
\]
which implies that \( A \) is non-negative. \( \square \)

6. Albert's theorem

An application of Proposition 5.4 leads to a generalization of an important criterion for non-negativity [1], which is often called Albert's theorem in matrix theory. It should be mentioned that Shmulyan [17, Theorem 1.7] had proved a similar assertion even for bounded operators in Hilbert spaces ten years earlier. We also mention the papers [3] and [10].

**Theorem 6.1.** An operator \( A = \left( \begin{array}{cc} A & B \\ B^\sim & D \end{array} \right) \in \mathcal{L}^h \) is non-negative if and only if it is of positive type and \( \sigma(A) \) is non-negative.

**Proof.** If \( A \in \mathcal{L}^2 \), Proposition 5.4 implies that \( A \) is of positive type and \( R_A^*[−1] B = K R_D \) for some contraction \( K \in \mathcal{L}(H_D, H_A) \). It follows
\[
\sigma(A) = R_D^* R_D - (K R_D)^* K R_D = R_D^* (I - K^* K) R_D \geq 0.
\]
Conversely, let \( A \in \mathcal{L}^p \) and \( \sigma(A) \in \mathcal{L}^2(Y, Y') \). If \( (R_A, H_A) \) and \( (R_D, H_D) \) are square roots of \( A \) and \( D \), resp., one has
\[
\| R_A^*[−1] B y \|^2 = \langle \omega(A, B) y, y \rangle \leq \langle D y, y \rangle = \| D y \|^2, \quad y \in Y,
\]
which yields \( K R_D = R_A^*[−1] B \), hence, \( R_A^* K R_D = B \) for some contraction \( K \in \mathcal{L}(H_D, H_A) \). An application of Proposition 5.4 completes the proof. \( \square \)

The preceding theorem can be used to study the set \( \mathcal{L}^\geq \) as well as the set \( \mathcal{L}^+ \) and to establish interrelations between these two sets. A first result is the inclusion \( \mathcal{L}^\geq \subseteq \mathcal{L}^+ \). For a positive pair \((A, B)\) set
\[
A_{ex} := \left( \begin{array}{cc} A & B \\ B^\sim & \omega(A, B) \end{array} \right) \in \mathcal{L}^+.
\]

**Corollary 6.2.** Two operators \( A \in \mathcal{L}(X, X') \) and \( B \in \mathcal{L}(Y, Y') \) form a positive pair if and only if the set
\[
\mathcal{A} := \left\{ A \in \mathcal{L}^2 : A = \left( \begin{array}{cc} A & B \\ B^\sim & D \end{array} \right) \text{ for some } D \in \mathcal{L}^2(Y, Y') \right\}
\]
is non-empty. If \((A, B)\) is a positive pair, the operator \(A_{ex}\) is the minimal element of \(A\). \(\square\)

**Corollary 6.3.** If \(A \in \mathcal{L}^2\), the set
\[
\mathcal{A} := \{ A_1 \in \mathcal{L}^2 : A_1 \leq A \text{ and } X \subseteq \ker A_1 \}
\]
is non-empty and \(\mathcal{S}(A)\) is its maximal element.

**Proof.** Corollaries 3.8 (i) and 3.9 imply that \(\mathcal{S}(A) \in \mathcal{A}\). If \(A = \begin{pmatrix} A & B \\ B^\sim & D \end{pmatrix}\) and \(A_1 \in \mathcal{A}\), then \(A_1\) has representation \(A_1 = \begin{pmatrix} 0 & 0 \\ 0 & D_1 \end{pmatrix}\) and \(D - \omega(A,B) - D_1 \geq 0\), hence, \(A_1 \leq \mathcal{S}(A)\) by Theorem 6.1. \(\square\)

**Corollary 6.4.** Let \(A = \begin{pmatrix} A & B \\ B^\sim & D \end{pmatrix}\) \(\in \mathcal{L}^h\). The operator \(A\) belongs to \(\mathcal{L}^+\) if and only if there exists an operator \(A_1 \in \mathcal{L}^h\) satisfying \(X \subseteq \ker A_1\) and \(A_1 \leq A\).

**Proof.** If \(A \in \mathcal{L}^+\), the operator \(A_1 := \mathcal{S}(A)\) has all properties claimed. Conversely, if there exists an operator \(A_1\) satisfying all conditions, it has the form \(A_1 = \begin{pmatrix} 0 & 0 \\ 0 & D_1 \end{pmatrix}\), where \(D_1 \in \mathcal{L}(Y, Y')\) and \(A - A_1 = \begin{pmatrix} A & B \\ B^\sim & D - D_1 \end{pmatrix}\) \(\in \mathcal{L}^2\). It follows from Theorem 6.1 that \((A,B)\) is a positive pair, hence, \(A \in \mathcal{L}^+\). \(\square\)

Another application of Theorem 6.1 gives an expression of the supremum occurring in Lemma 4.2.

**Corollary 6.5.** If \((A, B)\) is a positive pair, then
\[
\sup_{x \in X} \frac{|\langle By, x \rangle|^2}{\langle Ax, x \rangle} = \langle \omega(A,B)y, y \rangle, \quad y \in Y. \tag{6.1}
\]

**Proof.** Let \(y \in Y\). Since \(A_{ex} \in \mathcal{L}^2\) by Corollary 6.2, one has
\[
|\langle By, x \rangle|^2 \leq \langle Ax, x \rangle \langle \omega(A,B)y, y \rangle,
\]
which yields
\[
\frac{|\langle By, x \rangle|^2}{\langle Ax, x \rangle} \leq \langle \omega(A,B)y, y \rangle, \quad x \in X,
\]
if one takes into account the convention \(\frac{0}{0} := 0\). Thus, (6.1) has been proved if \(Ty = R^{*-1}By = 0\), where \((R,H)\) is a minimal square root of \(A\). Now assume that \(Ty \neq 0\). There exists a sequence \(\{x_n\}_{n \in \mathbb{N}}\) of elements of \(X\) such that \(Rx_n \neq 0\), \(n \in \mathbb{N}\), and \(\lim_{n \to \infty} Rx_n = Ty\). It follows
\[
\lim_{n \to \infty} \frac{|\langle By, x_n \rangle|^2}{\langle Ax_n, x_n \rangle} = \lim_{n \to \infty} \frac{|\langle R^*Ty, x_n \rangle|^2}{\langle R^*Rx_n, x_n \rangle} = \lim_{n \to \infty} \frac{|(Ty|R_{x_n})|^2}{\|Rx_n\|^2} = \|Ty\|^2 = \langle \omega(A,B)y, y \rangle. \quad \square
\]

**Corollary 6.6.** Let \((A_j, B_j)\) with \(A_j \in (X, X')\), \(B_j \in (Y, X')\), \(j = 1, 2\), be positive pairs. Then \((A_1 + A_2, B_1 + B_2)\) is a positive pair and
\[
\omega(A_1 + A_2, B_1 + B_2) \leq \omega(A_1, B_1) + \omega(A_2, B_2). \tag{6.2}
\]
Proof. Since the operators \((A_j)_{ex}, j = 1, 2\) are non-negative, it follows
\[
\left( A_1 + A_2 \begin{array}{c} \omega(A_1, B_1) + \omega(A_2, B_2) \\
B_1 + B_2 \end{array} \right) \in \mathcal{L}_{\geq},
\]
hence, \((6.2)\) by Corollary 6.2. \(\square\)

Corollary 6.7. If \(A_j \in \mathcal{L}^+, j = 1, 2\), then \(A_1 + A_2 \in \mathcal{L}^+\) and
\[
\mathcal{I}(A_1) + \mathcal{I}(A_2) \leq \mathcal{I}(A_1 + A_2).
\]

A subset \(\mathcal{A}\) of \(\mathcal{L}^h\) is called bounded below if there exists \(A_1 \in \mathcal{L}^h\) such that \(A_1 \leq A\) for all \(A \in \mathcal{A}\). An operator \(A_0 \in \mathcal{L}^h\) is called an infimum of \(\mathcal{A}\) if the following conditions are satisfied:

a) \(A_0 \leq A\) for all \(A \in \mathcal{A}\),

b) \(A_1 \leq A_0\) for all \(A_1 \in \mathcal{L}^h\) such that \(A_1 \leq A, A \in \mathcal{A}\).

If an infimum of \(\mathcal{A}\) exists, it is unique. Recall that any set \(\mathcal{A}\), which is bounded from below and directed downwards (i.e. for all \(A_1, A_2 \in \mathcal{A}\) there exists \(A \in \mathcal{A}\) such that \(A \leq A_1\) and \(A \leq A_2\)), possesses an infimum. Particularly, if \(\{A_n\}_{n \in \mathbb{N}}\) is a decreasing sequence of operators of \(\mathcal{L}^h\), which is bounded from below, then there exists an infimum \(A_0\) and \(\langle A_{n \uparrow 1}, z_2 \rangle = \lim_{n \to \infty} \langle A_n \uparrow 1, z_2 \rangle\) for all \(z_1, z_2 \in X \times Y\).

Corollary 6.8. Let \(\mathcal{A}\) be a subset of \(\mathcal{L}^h\), which has an infimum \(A_0\). The operator \(A_0\) belongs to \(\mathcal{L}^+\) if and only if the set \(S(\mathcal{A}) := \{\mathcal{I}(A) : A \in \mathcal{A}\}\) is bounded from below. In this case \(\mathcal{I}(A_0)\) is the infimum of \(S(\mathcal{A})\).

Proof. If \(A_0 \in \mathcal{L}^+\), the set \(S(\mathcal{A})\) is bounded from below since \(\mathcal{I}(A_0) \leq \mathcal{I}(A), A \in \mathcal{A}\), by Corollary 4.8 (ii). Conversely, assume that there exists \(A_1 \in \mathcal{L}^h\) such that \(A_1 \leq \mathcal{I}(A)\) for all \(A \in \mathcal{A}\). It follows \(-A_1 \geq -\mathcal{I}(A)\), which yields \(-A_1 \in \mathcal{L}^+\) by Corollary 6.4 and \(\mathcal{I}(-A_1) \geq \mathcal{I}(\mathcal{I}(A)) = \mathcal{I}(A)\), hence, \(-\mathcal{I}(-A_1) \leq \mathcal{I}(A) \leq A, A \in \mathcal{A}\), by Corollary 4.8. One obtains \(-\mathcal{I}(-A_1) \leq A_0\) and therefore \(A_0 \in \mathcal{L}^+\) by Corollary 6.4. Moreover, \(\mathcal{I}(A_0) \leq \mathcal{I}(A), A \in \mathcal{A}\), and
\[
A_1 = -(\mathcal{I}(A_1)) = \mathcal{I}(A) = \mathcal{I}(A) \leq \mathcal{I}(A_0)
\]
by Corollary 4.8 which implies that \(\mathcal{I}(A_0)\) is the infimum of \(S(\mathcal{A})\). \(\square\)

7. Extremal Operators

An operator \(A \in \mathcal{L}^+\) was called an extremal operator by M.G. Kreĭn \[\square\] if \(\mathcal{I}(A) = 0\). Since \(A = \mathcal{I}(A) + A_{ex}\), an operator is extremal if and only if it has the form
\[
A = A_{ex} = \begin{pmatrix} A & B \\ B^* & \omega(A, B) \end{pmatrix}
\]
for some positive pair \((A, B)\). Particularly, any extremal operator is non-negative. Applying Proposition 4.7 we can give several criteria for an operator to be extremal.

Lemma 7.1. Let \(A \in \mathcal{L}^\geq\). The following assertions are equivalent:

(i) The operator is extremal.
(ii) For all \(\begin{pmatrix} x \\ y \end{pmatrix} \in X \times Y\) and arbitrary \(\varepsilon > 0\) there exists \(z \in X\) such that

\[
\langle A \begin{pmatrix} x - z \\ y \end{pmatrix}, \begin{pmatrix} x - z \\ y \end{pmatrix} \rangle < \varepsilon.
\]

(iii) For any square root \((R, H)\) of \(A\) the spaces \((RX)^c\) and \((\text{ran } R)^c\) coincide.

(iv) For any square root \((R, H)\) of \(A\) one has \(\text{ran } R^* \cap Y' = \{0\}\).

**Proof.** The equivalence of (i) and (ii) is an immediate consequence of \((4.3)\). To prove (i) \(\Leftrightarrow\) (iii), choose a minimal square root \((R, H)\) of \(A\) and let \(L\) and \(P\) be defined as in Proposition \([5.1]\). Then \(\mathcal{N}(A) = R^*PR = 0\) if and only if \(P = 0\) or, equivalently, \(L = \{0\}\), which in turn is equivalent to \((RX)^c = (\text{ran } R)^c\). The equivalence of (iii) and (iv) follows from the equality \(R^*L = \text{ran } R^* \cap Y'\). \(\square\)

Let \((A, B)\) be a positive pair and \((R, H)\) a square root of \(A\). Recall the notation \((4.1)\) of the operator \(T := R^{*-1}B\). Moreover, let \(P_B\) be the orthoprojection onto \((\text{ran } T)^c\). Since the operator \(A_{ex}\) is non-negative, from Theorem \([6.1]\) one can conclude that \((\omega(A, B), B^-)\) is a positive pair as well changing the roles of \(X\) and \(Y\). Thus, the operators \(T^{*-1}B^-\) and

\[
\omega(\omega(A, B), B^-) = [T^{*-1}B^-]^*T^{*-1}B^-\]

can be defined.

**Lemma 7.2.** The equalities \(T^{*-1}B^- = P_B R\) and \(\omega(\omega(A, B), B^-) = R^* P_B R\) hold true.

**Proof.** The second equality is an immediate consequence of the first one. To prove the first equality we shall show that

\[
(T^{*-1}B^- x | h) = (P_B R x | h) \quad \text{for all } x \in X \text{ and } h \in H.
\]

\((7.1)\)

Since \(\text{ran } T^{*-1}B^- \subseteq (\text{ran } T)^c\) it is enough to prove \((7.1)\) for \(x \in X\) and \(h \in \text{ran } T\). If \(h = Ty\) for some \(y \in Y\), we get

\[
(T^{*-1}B^- x | h) = (T^{*-1}B^- x | Ty) = \langle T^* T^{*-1}B^- x, y \rangle = \langle B^- x, y \rangle
\]

and

\[
(P_B R x | h) = (P_B R x | Ty) = (R x | Ty) = \langle R^* Ty, x \rangle = \langle By, x \rangle = \langle B^- x, y \rangle,
\]

hence, \((7.1)\). \(\square\)

From Corollary \([6.2]\) it follows that \(\omega(\omega(A, B), B^-)\) is a minimal element of the set

\[
\left\{ A_1 \in \mathcal{L}(X, X') : \begin{pmatrix} A_1 \\ B^- \end{pmatrix} \omega(A, B) \right\} \in \mathcal{L}^\geq
\]

Note also that

\[
\omega(\omega(A, B), B^-) = \omega(A, B),
\]

cf. \([12]\) Proposition 1.4 (A)]. We call an extremal operator \(A_{ex} = \begin{pmatrix} A \\ B^- \end{pmatrix} \omega(A, B)\) doubly extremal if \(\omega(\omega(A, B), B^-) = A\).
The functional

\[ \text{there exists a sequence } (\sigma_n) \]

\[ \text{of elements of } \mathcal{H}. \]

Lemma 7.5.

Proof. An element \( x \in \mathcal{H} \) belongs to \( H_1 \) if and only if there exists a sequence \( \{x_n\}_{n \in \mathbb{N}} \) of elements of \( X \) such that \( \lim_{n \to \infty} Rx_n = h \) and for all \( y \in Y \),

\[ \langle T^* h, y \rangle = \lim_{n \to \infty} \langle Rx_n, Ty \rangle = \lim_{n \to \infty} \langle x_n, By \rangle = \lim_{n \to \infty} \langle B^* x_n, y \rangle = 0. \]

Proposition 7.6. An operator \( A_{ex} \) is doubly extremal if and only if \( H_1 = \{0\} \).
Proof. Since $H_1 = \{0\}$ if and only if $\ker R^* = \ker T^*$ by Lemma 7.5, the assertion follows from Proposition 7.3.

Corollary 7.7. If an operator $A_{ex}$ is doubly extremal, then $\ker A = \ker B^\sim$. If $\ker A = \ker B^\sim$ and $\operatorname{ran} R$ is closed, then $A_{ex}$ is doubly extremal.

Proof. If $A_{ex}$ is doubly extremal, then $(T^*[-1]B^\sim, H)$ is a square root of $A$, hence, $\ker B^\sim \subseteq \ker T^*[-1]B^\sim = \ker A$ by Lemma 3.4. The first assertion of the corollary follows since $\ker A \subseteq \ker B^\sim$ by Lemma 4.2. Now assume that $\ker A = \ker B^\sim$ and $\operatorname{ran} R$ is closed. If $h \in H_1$, there exist $x \in X$ and a sequence $\{x_n\}_{n \in \mathbb{N}}$ of elements of $X$ such that $h = Rx = \lim_{n \to \infty} Rx_n$ and $\lim_{n \to \infty} \langle B^\sim x_n, y \rangle = 0$ for all $y \in Y$. It follows

$$\langle B^\sim x, y \rangle = \overline{\langle R^* Ty, x \rangle} = \overline{(Rx | Ty)} = \lim_{n \to \infty} \langle Rx_n | Ty \rangle = \lim_{n \to \infty} \overline{\langle By, x_n \rangle} = \lim_{n \to \infty} \langle B^\sim x_n, y \rangle = 0, \quad y \in Y,$$

which implies that $x \in \ker B^\sim = \ker A = \ker R$ and $h = 0$. An application of Proposition 7.6 completes the proof.

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