Mixed discrete variable Gaussian states

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The quantum systems with finite-dimensional Hilbert space have several applications and are intensively explored theoretically and experimentally. The mathematical description of these systems follows the analogy with the usual infinite-dimensional case. There exist finite versions for most of the elements used in the continuous case, but (to our knowledge) there does not exist a finite version corresponding to the mixed Gaussian states. Our aim is to fill this gap. The definition we propose for the mixed discrete Gaussian states is based on the explicit formulas we have obtained in the case of pure discrete variable Gaussian states.

The starting point in the description of quantum systems with finite-dimensional Hilbert space (also called finite quantum systems, discrete variable quantum systems or qudits) is the book of Weyl and the remark of Schwinger that two observables whose eigenstates are connected through the finite Fourier transform share a maximum degree of incompatibility. A phase-space approach was initiated by Wooters and continued by Cohen, Gallet, Vourdas, Hadzitaskos, Bužek, Opatrný, Leonhardt, Galetti, Tolar, Hakioğlu, Zhang, Ruzzi, Klimov, Marchiolli, DeBrot, Runde, Wong, Vourdas, etc.

A remarkable discrete version of Hermite-Gauss functions was presented by Mehta and investigated in more details by Ruzzi. The same method can be used in order to define Gaussian functions of discrete variable. From a mathematical point of view, they are particular cases of the Jacobi functions. Some additional results have been obtained by Cotfas and Dragoman by using an approach directly based on the computation with series.

The mathematical formalism available for the description of quantum systems with finite-dimensional Hilbert space contains: finite discrete Fourier transforms, finite discrete position and momentum like operators, discrete position space, discrete momentum space, discrete phase-space, discrete displacement operators, discrete displaced parity (phase-point) operators, discrete Wigner function, discrete Weyl (characteristic) function, discrete parameterized quasiprobability distributions, discrete Husimi function, discrete Glauber-Sudarshan function, discrete quantum state tomography, discrete Weyl-Wigner-Moyal formalism, discrete Hermite-Gauss function, Gaussian functions of discrete variable, discrete vacuum, discrete coherent state, discrete squeezed states, discrete creation and annihilation operator, finite harmonic oscillators, discrete fractional Fourier transform, discrete Fock space, harmonic uncertainty relation, Harper functions, Fourier-Kravchuk transform, orthogonal polynomials of discrete variable, etc.

To our knowledge, a finite version of the mixed Gaussian states is missing, and our wish is to fill this gap. In the case of pure discrete variable Gaussian states we have obtained for the discrete Wigner function the formulas (38) and (46) (see Supplemental Material). The definition we propose for the mixed discrete variable Gaussian states is based on the remark (presented here for the first time) that these formulas can be written in the compact form (13) and respectively (16). Then, we investigate some of the properties of the discrete Gaussian states defined in this way. Particularly, we define for the first time (to our knowledge) discrete thermal states and a subspace of the Hilbert space where a discrete version of the position-momentum commutation relation is approximately satisfied. The orthogonal projection of a discrete Gaussian state on this subspace seems to be much more significant then the projection on the orthogonal complement.

Phase-space formulation of finite quantum systems. In this section, we review the elements of the phase-space formulation of finite quantum systems we use in the next sections. For simplicity, we consider only quantum systems with an odd-dimensional (d = 2s + 1) Hilbert space $\mathcal{H}$, which can be regarded as the space of all the functions $\psi: \{-s, -s + 1, \ldots, s - 1, s\} \rightarrow \mathbb{C}$ with the inner product $\langle \bar{\psi}|\psi \rangle = \sum_{n=-s}^{s} \bar{\varphi}(n) \psi(n)$. Each function $\psi \in \mathcal{H}$ can be extended up to a periodic function $\psi: \mathbb{Z} \rightarrow \mathbb{C}$ of period d, and $\mathcal{H}$ can also be regarded as the space of all the functions $\psi: \mathbb{Z}_{d} \rightarrow \mathbb{C}$, that is $\mathcal{H} = \ell^{2}(\mathbb{Z}_{d})$. The canonical basis $\{\delta_{s}, \delta_{s+1}, \ldots, \delta_{1}, \delta_{0}\}$, where

$$\delta_{m}(n) = \delta_{mn} = \begin{cases} 1 & \text{if } n = m \text{ modulo } d \\ 0 & \text{if } n \neq m \text{ modulo } d \end{cases} \quad (1)$$

is an orthonormal basis. By using Dirac’s notation $|m\rangle$ instead of $\delta_{m}$, we have $\langle m|k \rangle = \delta_{mk}$ and $\sum_{m=-s}^{s} \langle m|n \rangle = \mathbb{I}$, where $\mathbb{I} = \mathcal{H} \otimes \mathcal{H}$, $\mathbb{I} \psi = \psi$, is the identity operator. The discrete Fourier transform

$$\hat{s}|\psi\rangle(k) = \frac{1}{\sqrt{d}} \sum_{n=-s}^{s} e^{-2\pi ink} \psi(n) \quad (2)$$

is a unitary one: $\hat{s}^{-1} = \hat{s}^{*}$, $\hat{s}^{*}|\psi\rangle(k) = \frac{1}{\sqrt{d}} \sum_{n=-s}^{s} e^{2\pi ink} \psi(n)$. The self-adjoint operator $(-s \leq \hat{n} \leq s$ is the representative modulo d of $\hat{n}$: $\mathcal{H} \otimes \mathcal{H} : \psi \rightarrow \hat{n}\psi$, $\langle \hat{n}\psi|\psi \rangle = \mathbb{I} \psi(n)$ corresponds to the position operator, $\hat{p} = \hat{s}^{*} \hat{q}\hat{s}$ corresponds to the momentum operator, $\hat{p}$ from...
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the continuous case. The operators (the addition is modulo $d$)

$$\mathcal{H} \rightarrow \mathcal{H} : \psi \mapsto e^{\frac{2\pi i}{d} \hat{q}} \psi, \quad e^{\frac{2\pi i}{d} \hat{q}} \psi(n) = e^{\frac{2\pi i}{d} n} \psi(n),$$

$$\mathcal{H} \rightarrow \mathcal{H} : \psi \mapsto e^{-\frac{2\pi i}{d} \hat{p}} \psi, \quad e^{-\frac{2\pi i}{d} \hat{p}} \psi(m) = \psi(m-n),$$

are the translation operators.\ref{9,12,16} and

$$\mathcal{D}(n,k) = e^{\frac{2\pi i}{d} n k} e^{-\frac{2\pi i}{d} \hat{p}} \mathcal{D}(n,k) e^{\frac{2\pi i}{d} n k},$$

$$\mathcal{D}(n,k) : \mathcal{H} \rightarrow \mathcal{H},$$

$$\mathcal{D}(n,k) \psi(m) = e^{\frac{2\pi i}{d} n k} e^{-\frac{2\pi i}{d} \hat{p}} \psi(m-n)$$

are the displacement operators.\ref{9,12,16} The displaced parity operators\ref{9,12,16}

$$\Pi(n,k) : \mathcal{H} \rightarrow \mathcal{H}, \quad \Pi(n,k) = \mathcal{D}(n,k) \Pi \mathcal{D}^{-1}(n,k)$$

$$\Pi(n,k) \psi(m) = e^{-\frac{2\pi i}{d} 2k(n-m)} \psi(2n-m)$$

where $\Pi \psi(n) = \psi(-n)$, form an orthogonal basis in the real Hilbert space $\mathcal{A}(\mathcal{H}) = \{ A : \mathcal{H} \rightarrow \mathcal{H} \ | \ A^\dagger = A \}$ of all the self-adjoint operators, namely

$$\langle \Pi(n,k), \Pi(m,\ell) \rangle = \text{tr}(\Pi(n,k) \Pi(m,\ell)) = d \delta_{mn} \delta_{k\ell}. \quad (6)$$

Any density operator $\varrho : \mathcal{H} \rightarrow \mathcal{H}$ can be written as\ref{14}

$$\varrho = \sum_{n,k} \mathcal{D}(n,k) \Pi(n,k), \quad (7)$$

where (the addition is modulo $d$)

$$\mathcal{D}_\varrho : \{-s,-s+1,\ldots,-s+1\} \times \{-s,-s+1,\ldots,-s+1\} \rightarrow \mathbb{R},$$

$$\mathcal{M}_\varrho(n,k) = \frac{1}{d} \text{tr}(\rho \Pi(n,k)) = \frac{1}{d} \sum_{m=-s}^{s} e^{\frac{2\pi i}{d} k m} \langle n+m | \varrho | n-m \rangle \quad (8)$$

is the discrete Wigner function$^{9,12,16}$ of $\varrho$. The Wigner function of a pure state $\varrho = |\psi\rangle \langle \psi|$ is

$$\mathcal{M}_\varrho(n,k) = \frac{1}{d} \langle \psi | \Pi(n,k) | \psi \rangle = \frac{1}{d} \sum_{m=-s}^{s} e^{-\frac{2\pi i}{d} k m} \psi(n+m) \overline{\psi(n-m)} \quad (9)$$

and has the marginal properties:

$$\sum_{k=-s}^{s} \mathcal{M}_\varrho(n,k) = |\psi(n)|^2, \quad \sum_{n=-s}^{s} \mathcal{M}_\varrho(n,k) = |\overline{\psi}(k)|^2. \quad (10)$$

Pure single-mode discrete variable Gaussian states.- The function $g_\kappa : \mathbb{R} \rightarrow \mathbb{R}$, $g_\kappa(q) = e^{-\frac{\kappa q^2}{2}} = e^{-\frac{\kappa}{d} q^2}$, where $\kappa \in (0, \infty)$ is a parameter, is a Gaussian function of continuous variable. The Wigner function of $g_\kappa$ is $W_\kappa : \mathbb{R}^2 \rightarrow \mathbb{R},$

$$W_{g_\kappa}(q,p) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-\frac{2\pi i}{d} p x} g_\kappa(q+x) \overline{g_\kappa(q-x)} dx,$$

$$= \frac{1}{\sqrt{\pi} \kappa^2} \exp \left\{ \frac{2\pi i}{d} q p \left( e^{-\frac{1}{\kappa} 0} \right) \left( e^{-\frac{1}{\kappa} 0} \right)^{-1} q \right\}.$$ \(11\)

Pure two-mode discrete variable Gaussian states.- Let $\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a $2 \times 2$ real symmetric positive-definite matrix.

The real function $g_\tau : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R},$

$$g_\tau(q_1, q_2) = \exp \left\{ -\frac{\pi}{d} q_1 q_2 \left( e^{-\frac{1}{\kappa} 0} \right) \left( e^{-\frac{1}{\kappa} 0} \right)^{-1} \right\}.$$ \(12\)
is a Gaussian function of two continuous variables. The Wigner function of $g_t$ is
\[ W_g: \mathbb{R}^4 \rightarrow \mathbb{R}, \quad W_g(q_1, q_2, p_1, p_2) = \frac{2}{\hbar \sqrt{\det \tau}} \exp \left\{ -\frac{2\pi}{\hbar^2} (q_1, q_2, p_1, p_2) \begin{pmatrix} \tau^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \right\}. \]

The function
\[ g_t : \{ -s, -s+1, \ldots, s \} \times \{ -s, -s+1, \ldots, s \} \rightarrow \mathbb{R}, \quad g_t(n_1, n_2) = \sum_{\alpha_1, \alpha_2 = -\infty}^{\infty} \exp \left\{ -\frac{\pi}{\hbar} (n_1, n_2, n_1 + \alpha_1 \alpha_2) \begin{pmatrix} a & b \\ b & c \end{pmatrix} (n_1, n_2, n_1 + \alpha_1 \alpha_2) \right\}, \]
\[ w(q_1, q_2, p_1, p_2) = \exp \left\{ -\frac{\pi}{\hbar^2} (q_1, q_2, p_1, p_2) \begin{pmatrix} \tau^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \right\}, \]
\[ \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

Pure and mixed discrete variable Gaussian states. In the case of a continuous variable quantum system, a Gaussian state is a state with a Wigner function of the form
\[ W_P(q, p) = \frac{2}{\hbar \sqrt{\det \sigma}} \exp \left\{ -\frac{\pi}{\hbar^2} (q - \tilde{q}, p - \tilde{p}) \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}^{-1} \begin{pmatrix} q - \tilde{q} \\ p - \tilde{p} \end{pmatrix} \right\}, \]
where $\tilde{q}, \tilde{p} \in \mathbb{R}$ are two parameters, and the covariance matrix
\[ \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}, \quad \text{satisfying} \quad \sigma + i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \geq 0, \quad (19) \]
is a real $2 \times 2$ symmetric positive-definite matrix. In the case $\tilde{q} = \tilde{p} = 0$, we shall write $\rho_\sigma$ instead of $\rho$. Inspired by the relations (14) and (13), we propose the following definition.

Definition 1: In the case of a discrete variable quantum system, a state $\rho$ is a single-mode discrete variable Gaussian state if there exists a real $2 \times 2$ symmetric positive-definite matrix $\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$ such that the Wigner function of $\rho$ is
\[ W_\rho(n, k) = C \sum_{\alpha, \beta = -\infty}^{\infty} \begin{pmatrix} -1 \end{pmatrix}^{\alpha \beta} \begin{pmatrix} w_\sigma(n, \alpha, k, \beta) \end{pmatrix}, \]
\[ w_\sigma(q, p) = \exp \left\{ -\frac{\pi}{\hbar^2} (q - \tilde{q}, p - \tilde{p}) \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}^{-1} \begin{pmatrix} q - \tilde{q} \\ p - \tilde{p} \end{pmatrix} \right\}. \]

In the case $\tilde{q} = \tilde{p} = 0$, we write $g_\sigma$ instead of $g$. We do not know what conditions $\sigma$ has to satisfy in order to have
\[ \rho_\sigma = \sum_{n, k} \rho_\sigma(n, k) \Pi(n, k) \geq 0. \]

Our numerical simulations suggest that (19) may be a sufficient condition.

Theorem 1. For any $K \in (0, \infty)$, the pure state $g_\sigma = \frac{1}{\sqrt{\det \sigma}} g_\sigma$ is a discrete variable Gaussian state with $\sigma = \sigma_\sigma = \begin{pmatrix} K & 0 \\ 0 & 1 \end{pmatrix}$. Theorem 2. For any $K \in (0, \infty)$ and any $n_0, k_0 \in \{ -s, -s+1, \ldots, s-1 \}$, the pure state $\rho = D(n_0, k_0) \rho_\sigma$ is also a discrete variable Gaussian state, with $\sigma = \sigma_\sigma$. Theorem 3. If $\rho$ is a discrete variable Gaussian state, then $D'(n_0, k_0) \rho D'(n_0, k_0)$ is also a discrete variable Gaussian state. In the case of the discrete variable Gaussian state $\rho = \rho_\sigma$, the corresponding Wigner function is $w_\rho(q, p)$.

The last two theorems show that the wide sense discrete variable Gaussian states are also discrete variable Gaussian states. The discrete Gaussian state $\rho = | \cdot \rangle \langle \cdot |$ corresponds to the vacuum state, and the discrete Gaussian states satisfy the resolution of the identity $\sum_{n, k = -s}^{s} | n, k \rangle \langle n, k | = \mathbb{I}$, correspond to the canonical coherent states from the continuous variable case. Any pure state $\rho \in \mathcal{H}$ admits the representation $| \psi \rangle = \| \rho \rangle = \sum_{n, k = -s}^{s} | n, k \rangle \langle n, k |$ as a linear superposition of discrete variable Gaussian states.

Theorem 5. Any continuous variable Gaussian state $\rho_\sigma$ is the limit of a sequence of discrete Gaussian states $\rho_\sigma$: \[ \rho_\sigma = \lim_{d \rightarrow \infty} \rho_\sigma. \]

By following the suggestion offered by the formulas (16) and (17), we propose the following definition.

Definition 2: In the case of a discrete variable quantum system, a state $\rho$ is a two-mode discrete variable Gaussian state if there exists a real $2 \times 2$ symmetric positive-definite matrix $\sigma$ such that the Wigner function of $\rho$ is
\[ W_\rho(n_1, n_2, k_1, k_2) = C \sum_{\alpha_1, \alpha_2 = -\infty}^{\infty} \sum_{\beta_1, \beta_2 = -\infty}^{\infty} \begin{pmatrix} -1 \end{pmatrix}^{\alpha_1 \alpha_2 + \beta_1 \beta_2} \begin{pmatrix} w_\sigma(n_1 + \alpha_1 \beta_1, n_2 + \alpha_2 \beta_2, k_1 + \beta_1 \beta_2, k_2 + \beta_2 \beta_1) \end{pmatrix}, \]
where $C$ is a normalizing constant and $w_\sigma(q_1, q_2, p_1, p_2) = \exp \left\{ -\frac{\pi}{\hbar^2} (q_1, q_2, p_1, p_2) \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}^{-1} \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \right\}$.

Purity of a single-mode finite Gaussian state. In the continuous case, the purity of $\rho_\sigma$ is
\[ \text{tr} \rho_\sigma^2 = 2\pi \hbar \int_{\mathbb{R}^2} W^2_\rho(q, p) dq dp = \frac{1}{\sqrt{\det \sigma}}. \]
Consequently, in \( \ell^2(\mathbb{Z}_d) \) there exists an orthonormal basis \( \{ |n\rangle \} \) formed by eigenfunctions of \( \varrho_{\hat{d}} \) with real values.

In the case \( d = 3 \), we have\(^{11} \) \( \varrho_{\hat{d}} \varrho_{\hat{d}} = \varrho_{\hat{d}} \varrho_{\hat{d}} \), for any \( v > 1 \) and \( \mu > 1 \). In the case \( d > 3 \), our numerical data show\(^{11} \) that

\[
\varrho_{\hat{d}} \varrho_{\hat{d}} \approx \varrho_{\hat{d}} \varrho_{\hat{d}},
\]

and the eigenfunctions of all the density operators \( \varrho_{\hat{d}} \) are almost the same. Therefore, the real eigenfunctions \( \{ |n\rangle \} \), considered in the increasing order of the number of sign alternations, can be regarded as a discrete version of the Hermite-Gauss functions, and

\[
H = \sum_{n=0}^{d-1} (n + \frac{1}{2}) |n\rangle \langle n|
\]

as the Hamiltonian of a \( d \)-dimensional counterpart of the harmonic quantum oscillator.

**On the position-momentum commutation relation.** In the continuous case, the usual position-momentum commutation relation

\[
[\hat{q}, \hat{p}] = i \frac{\hbar}{2\pi}
\]

is satisfied in a subspace dense in \( L^2(\mathbb{R}) \). The discrete counterpart

\[
[\hat{q}, \hat{p}] = i \frac{\hbar}{2}\pi
\]

is not satisfied, but for \( d \) large enough, most of the eigenvalues of the operator \( [\hat{q}, \hat{p}] - i \frac{\hbar}{2\pi} \) are almost null. For example, in the cases \( d = 11 \) and \( d = 61 \), the eigenvalues of \( [\hat{q}, \hat{p}] - i \frac{\hbar}{2}\pi \) are:

\[
\begin{array}{cccc}
\text{Case } d = 11 & \text{Case } d = 61 \\
-4.0 \times 10^{-15} & -3.4 \times 10^{-14}, 8.7 \times 10^{-14}, 0.000071 \\
9.3 \times 10^{-15} & -3.7 \times 10^{-14}, -8.9 \times 10^{-14}, -0.000301 \\
7.4 \times 10^{-15} & 4.1 \times 10^{-14}, -9.9 \times 10^{-14}, 0.00127 \\
7.9 \times 10^{-15} & -4.4 \times 10^{-14}, 1.1 \times 10^{-13}, 0.000513 \\
1.1 \times 10^{-15} & 4.0 \times 10^{-14}, -5.7 \times 10^{-14}, 0.00981 \\
2.1 \times 10^{-15} & 4.7 \times 10^{-14}, 3.6 \times 10^{-14}, -0.07335 \\
3.3 \times 10^{-15} & -4.9 \times 10^{-14}, -2.3 \times 10^{-13}, 0.26040 \\
3.9 \times 10^{-16} & -3.9 \times 10^{-14}, -2.5 \times 10^{-13}, 1.4 \times 10^{-12}, -0.68457 \\
3.4 \times 10^{-21} & 8.23716 \\
0.241 & 2.7 \times 10^{-16}, -6.8 \times 10^{-14}, 4.7 \times 10^{-14}, -8.86191 \\
1.33i & 2.7 \times 10^{-16}, -6.8 \times 10^{-14}, 4.7 \times 10^{-14}, -8.86191 \\
-5.45i & 2.7 \times 10^{-16}, -6.8 \times 10^{-14}, 4.7 \times 10^{-14}, -8.86191 \\
19.99i & 2.7 \times 10^{-16}, -6.8 \times 10^{-14}, 4.7 \times 10^{-14}, -8.86191 \\
-34.92i & 2.7 \times 10^{-16}, -6.8 \times 10^{-14}, 4.7 \times 10^{-14}, -8.86191 \\
-3.3 \times 10^{-14} & -7.2 \times 10^{-14}, -6.8 \times 10^{-14}, 189.607 \\
3.3 \times 10^{-14} & -7.2 \times 10^{-14}, -6.8 \times 10^{-14}, 189.607 \\
3.1 \times 10^{-14} & 8.0 \times 10^{-14}, 3.3 \times 10^{-14}, -0.00001 \\
-3.6 \times 10^{-14} & 8.1 \times 10^{-14}, -0.00001, 94.4717 \\
-120.53i & 3.6 \times 10^{-14}, -8.1 \times 10^{-14}, -0.00001, 94.4717 \\
\end{array}
\]
Some numerical results concerning a phase shift and a single-transformation $\hat{\lambda}$ suggests that for mixed discrete variable Gaussian states 5

$$\lambda_k = \lambda_k^2, \ldots, \lambda_k = \epsilon,$$ and let $\phi_1, \phi_2, \ldots, \phi_d$ be the corresponding eigenfunctions. If we expand a discrete variable Gaussian state in terms of the eigenbasis $\{\phi_k\}$, the most significant coefficients seem to be those corresponding to the functions $\phi_k$ with small $|\lambda_k|$. For $\epsilon > 0$, let $d_\epsilon$ be such that $\{\lambda_1, \lambda_2, \ldots, \lambda_d\} = \{\lambda_k \mid |\lambda_k| < \epsilon\}$, and let $\mathcal{H}_\epsilon = \text{span}\{\phi_1, \phi_2, \ldots, \phi_{d_\epsilon}\}$. If $\epsilon$ is small enough, then we can consider that

$$[\hat{q}, \hat{p}] \approx i\frac{\hbar}{\epsilon} \quad \text{in} \quad \mathcal{H}_\epsilon. \quad (30)$$

By following the analogy with the continuous case, we define

$$\hat{a} = \sqrt{\frac{\pi}{\epsilon}}(\hat{q} + i\hat{p}), \quad \hat{a}^\dagger = \sqrt{\frac{\pi}{\epsilon}}(\hat{q} - i\hat{p}), \quad \hat{a}^\dagger \hat{a} = A, \quad \text{similar to} \quad \hat{a} \hat{a}^\dagger = A \in \mathbb{R}, \quad \hat{a} \hat{a}^\dagger \in \mathbb{C}, \quad (31)$$

The discrete variable Gaussian states seem to mainly belong to $\mathcal{H}_\epsilon$ and $\mathcal{A}(\mathcal{H}_\epsilon)$. In the case of continuous variable, the transformations $\hat{U} = e^{-\frac{i}{\hbar}\hat{H}}$, where

$$\hat{H} = (\hat{a}^\dagger \hat{a}) \left( \begin{array}{cc} A & B \\ B & A \end{array} \right) \left( \begin{array}{c} \hat{a} \\ \hat{a}^\dagger \end{array} \right) \quad \text{with} \quad A \in \mathbb{R}, \quad B \in \mathbb{C}, \quad (32)$$

are Gaussian transforms, that is they preserve the Gaussian nature of the states. For $\hat{U} = e^{-\frac{i}{\hbar}\hat{H}}$, there exists $S \in \text{Sp}(2, \mathbb{R})$ such that

$$e^{-\frac{i}{\hbar}\hat{H}} \rho \sigma e^{\frac{i}{\hbar}\hat{H}} = \rho_{S\sigma S^T}. \quad (33)$$

In the case of discrete variable, the relation $\lim_{\epsilon \to 0} \rho_{S\sigma} = \rho_{\sigma}$ suggests that

$$\hat{\sigma} = \left( \hat{a}^\dagger \hat{a} \right) \left( \begin{array}{cc} A & B \\ B & A \end{array} \right) \left( \begin{array}{c} \hat{a} \\ \hat{a}^\dagger \end{array} \right) \quad \text{with} \quad A \in \mathbb{R}, \quad B \in \mathbb{C}, \quad (34)$$

we must have

$$e^{-\frac{i}{\hbar}\hat{\sigma}} \rho_{\sigma} e^{\frac{i}{\hbar}\hat{\sigma}} \approx \rho_{S\sigma S^T} \quad \text{for} \quad d \quad \text{large enough}. \quad (35)$$

Some numerical results concerning a phase shift and a single-mode squeezing transformation are presented in Table III.

**TABLE III.** Norm of $e^{-\frac{i}{\hbar}\hat{\sigma}} \rho_{\sigma} e^{\frac{i}{\hbar}\hat{\sigma}} - \rho_{S\sigma S^T}$ in two particular cases.

| $A$ | $B$ | $\sigma$ | $d$ | $\|e^{\frac{i}{\hbar}\hat{\sigma}} \rho_{\sigma} e^{-\frac{i}{\hbar}\hat{\sigma}} - \rho_{S\sigma S^T}\|$ |
|-----|-----|----------|-----|----------------------------------|
| 7   | 1   | $\sigma_1$ | 5   | 0.1391 |
| 7   | 1   | $\sigma_2$ | 5   | 0.0032 |
| 7   | 1   | $\sigma_3$ | 5   | 0.0135 |
| 7   | 1   | $\sigma_4$ | 5   | 0.0017 |
| 7   | 1   | $\sigma_5$ | 5   | 0.0160 |
| 7   | 1   | $\sigma_6$ | 5   | 0.1217 |
| 7   | 1   | $\sigma_7$ | 5   | 0.0105 |
| 7   | 1   | $\sigma_8$ | 5   | 0.0088 |
| 7   | 1   | $\sigma_9$ | 5   | 0.0072 |
| 7   | 1   | $\sigma_{10}$ | 5   | 0.0053 |

A possible explanation can be obtained by following the analogy with the continuous case.\[14\]

**Conclusion and outlook.** The use of the discrete variable Gaussian states extends the class of the quantum states which can be described analytically. These particular states have some significant properties and may be useful in certain applications. They can also be used as finite-dimensional approximates for the continuous variable Gaussian states. Generally, $\rho_{S\sigma} = \lim_{d \to \infty} \rho_{\sigma}$, the convergence is very fast.

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Mixed discrete variable Gaussian states

Supplemental Material – Mixed discrete variable Gaussian states

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I. Pure single-mode discrete variable Gaussian states

The discrete Fourier transform $\tilde{g}_κ(s) \rightarrow \mathbb{R}$ is also a Gaussian function of discrete variable, and $^{29}$

$$\tilde{g}_κ = \frac{1}{\sqrt{κ}} g_κ^{-1}. \quad (37)$$

The discrete Wigner function of $g_κ$ is $2M_{g_κ} : \{-s, -s+1, \ldots, s-1, s\} \rightarrow \mathbb{R}$,

$$2M_{g_κ}(n, k) = \frac{1}{d} \sum_{m=-s}^{s} e^{-\frac{1}{2}k \cdot m} g_κ(n+m) \overline{g_κ(n-m)}$$

$$= \frac{1}{\sqrt{2κd}} g_κ(n) \left( g_κ^+(k) + g_κ^-(k) \right)$$
$$+ \frac{1}{\sqrt{2κd}} g_κ^+(n) \left( g_κ^+(k) - g_κ^-(k) \right), \quad (38)$$

where $g_κ^+ : \{-s, -s+1, \ldots, s-1, s\} \rightarrow \mathbb{R}$,

$$g_κ^+(n) = \sum_{κ=\infty}^{\infty} e^{-\frac{κ}{2} \left(n + (\alpha + \frac{1}{2})d\right)^2}$$
$$= \sum_{κ=\infty}^{\infty} g_κ \left( n + (\alpha + \frac{1}{2})d \right) \sqrt{\frac{κ}{2π}}. \quad (39)$$

This formula obtained by Cotfas and Dragoman$^{29}$ can be written as

$$2M_{g_κ}(n, k) = C \sum_{α, β = -\infty}^{∞} (-1)^{αβ} w(n + α\frac{d}{2}, k + β\frac{d}{2}), \quad (40)$$

where $C$ is a normalizing factor and

$$w(q, p) = e^{-\frac{1}{2d^2} \left( \begin{array}{c} q \\ p \end{array} \right)^T \frac{1}{4} \left( q \right). \quad (41)$$

that is

$$2M_{g_κ}(n, k) = C \sqrt{\frac{κ}{2π}} \sum_{α, β = -\infty}^{∞} (-1)^{αβ} W_{g_κ}(n + α\frac{d}{2}, k + β\frac{d}{2}) \sqrt{\frac{κ}{2π}}. \quad (42)$$
II. Pure two-mode discrete variable Gaussian states

The function $g_\tau : \{ -s, -s + 1, \ldots, s - 1, s \} \times \{ -s, -s + 1, \ldots, s - 1, s \} \rightarrow \mathbb{R}$,

$$g_\tau(n_1, n_2) = \sum_{\alpha_1, \alpha_2 = -\infty}^\infty \exp \left\{ -\frac{4}{d}(n_1 + \alpha d)(n_2 + \alpha d) \right\}$$

$$= \sum_{\alpha_1, \alpha_2 = -\infty}^\infty g_\tau \left( (n_1 + \alpha d) \sqrt{\frac{d}{2}}, (n_2 + \alpha d) \sqrt{\frac{d}{2}} \right)$$

(43)

can be regarded as a Gaussian function of two discrete variables.

Its discrete Fourier transform $\tilde{g}_\tau[k_1, k_2] : \{ -s, -s + 1, \ldots, s - 1, s \} \times \{ -s, -s + 1, \ldots, s - 1, s \} \rightarrow \mathbb{R}$,

$$\tilde{g}_\tau[k_1, k_2] = \frac{1}{\sqrt{\det \tau}} \sum_{n_1 = -s}^{s} \sum_{n_2 = -s}^{s} \exp \left( -\frac{4}{d}(k_1 n_1 + k_2 n_2) \right) g_\tau(n_1, n_2)$$

$$= \frac{1}{\sqrt{\det \tau}} \sum_{\beta_1, \beta_2 = -\infty}^\infty g_{\tau^{-1}} \left( (k_1 + \beta_1 d) \sqrt{\frac{d}{2}}, (k_2 + \beta_2 d) \sqrt{\frac{d}{2}} \right)$$

(44)

is also a Gaussian function of discrete variable, and

$$\tilde{g}_\tau = \frac{1}{\sqrt{\det \tau}} g_{\tau^{-1}}$$

(45)

The discrete Wigner function of $g_\tau$ is

$$\mathcal{M}_g : \{ -s, -s + 1, \ldots, s - 1, s \} \times \{ -s, -s + 1, \ldots, s - 1, s \} \times \{ -s, -s + 1, \ldots, s - 1, s \} \rightarrow \mathbb{R},$$

$$\mathcal{M}_g(n_1, n_2, k_1, k_2) = \frac{1}{\sqrt{\det \tau}} \sum_{m_1 = -s}^{s} \sum_{m_2 = -s}^{s} \exp \left( -\frac{4}{d}(m_1 n_1 + m_2 n_2) \right) g_\tau(n_1 + m_1, n_2 + m_2) g_{\tau^{-1}}(n_1 - m_1, n_2 - m_2)$$

$$= \frac{1}{\sqrt{\det \tau}} \mathcal{M}_g(n_1, n_2) \left[ g_{2\tau^{-1}}(k_1, k_2) + g_{2\tau^{-1}}^{0+}(k_1, k_2) + g_{2\tau^{-1}}^{0-}(k_1, k_2) - g_{2\tau^{-1}}^{\perp}(k_1, k_2) \right]$$

$$+ \frac{1}{\sqrt{\det \tau}} g_{\tau^{-1}}^{0+}(n_1, n_2) \left[ g_{2\tau^{-1}}^{0+}(k_1, k_2) - g_{2\tau^{-1}}^{0-}(k_1, k_2) + g_{2\tau^{-1}}^{\perp}(k_1, k_2) \right]$$

$$+ \frac{1}{\sqrt{\det \tau}} g_{\tau^{-1}}^{+}(n_1, n_2) \left[ g_{2\tau^{-1}}^{0+}(k_1, k_2) - g_{2\tau^{-1}}^{0-}(k_1, k_2) - g_{2\tau^{-1}}^{\perp}(k_1, k_2) \right]$$

(46)

where

$$g_{\tau^{-1}}^{0}(n_1, n_2) = \sum_{\alpha_1, \alpha_2 = -\infty}^\infty \exp \left( -\frac{4}{d}(n_1 + \alpha_1 d)(n_2 + \alpha_2 d) \right) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{c} n_1 + \alpha_1 d \\ n_2 + \alpha_2 d \end{array} \right),$$

$$g_{\tau^{-1}}^{0+}(n_1, n_2) = \sum_{\alpha_1, \alpha_2 = -\infty}^\infty \exp \left( -\frac{4}{d}(n_1 + \alpha_1 d + \alpha_2 d)(n_2 + \alpha_2 d) \right) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{c} n_1 + \alpha_1 d + \alpha_2 d \\ n_2 + \alpha_2 d \end{array} \right),$$

$$g_{\tau^{-1}}^{+}(n_1, n_2) = \sum_{\alpha_1, \alpha_2 = -\infty}^\infty \exp \left( -\frac{4}{d}(n_1 + \alpha_1 d + \alpha_2 d)(n_2 + \alpha_2 d) \right) \left( \begin{array}{cc} a b \\ c d \end{array} \right) \left( \begin{array}{c} n_1 + \alpha_1 d + \alpha_2 d \\ n_2 + \alpha_2 d \end{array} \right).$$

(47)

This formula, obtained by Cotfas, can be written as

$$\mathcal{M}_g(n_1, n_2, k_1, k_2) = C \sum_{\alpha_1, \alpha_2 = -\infty}^\infty \sum_{\beta_1, \beta_2 = -\infty}^\infty (-1)^{\alpha_1 \beta_1 + \alpha_2 \beta_2} \times w(n_1 + \alpha_1 d, n_2 + \alpha_2 d, k_1 + \beta_1 d, k_2 + \beta_2 d),$$

(48)

where $C$ is a normalizing factor, and

$$w(q_1, q_2, p_1, p_2) = \exp \left\{ -\frac{2\pi}{d}(q_1, q_2, p_1, p_2) \left( \begin{array}{cccc} \tau^{-1} & 0 & 0 & 0 \\ 0 & \tau^{-1} & 0 & 0 \\ 0 & 0 & \tau^{-1} & 0 \\ 0 & 0 & 0 & \tau^{-1} \end{array} \right) \left( \begin{array}{c} q_1 \\ p_1 \\ q_2 \\ p_2 \end{array} \right) \right\}.$$
Mixed discrete variable Gaussian states

III. Pure and mixed discrete variable Gaussian states

Theorem 1. For any $κ ∈ (0, ∞)$, the pure state

$$g_κ = \frac{1}{\sqrt{⟨g_κ, g_κ⟩}} g_κ$$

is a discrete variable Gaussian state with $σ = σ_κ$, where

$$σ_κ = \begin{pmatrix} κ+1 \\ 0 \\ κ \end{pmatrix}.$$

Proof. Direct consequence of the relations (40) and (41).

Theorem 2. For any $κ ∈ (0, ∞)$ and any $n_0, k_0 ∈ \{-s, -s+1, ..., s-1, s\}$, the pure state

$$ψ = D(n_0, k_0)g_κ$$

is a discrete variable Gaussian state with $σ = σ_κ$.

Proof. We have

$$ψ(m) = D(n_0, k_0)g_κ(m) = e^{-\frac{πi}{σ}n_0k_0} e^{\frac{πi}{σ}m_0} g_κ(m-n_0)$$

and

$$Mψ(n, k) = \frac{1}{g} \sum_{m=-ι}^{ι} e^{\frac{4πi}{σ}km} e^{\frac{4πi}{σ}k_0(m+m)} g_κ(n+m-n_0) e^{-\frac{4πi}{σ}k_0(m-m)} g_κ(n-m-n_0)$$

$$= \frac{1}{g} \sum_{m=-ι}^{ι} e^{\frac{4πi}{σ}km} e^{\frac{4πi}{σ}k_0(m-n_0)} g_κ(n-m-n_0)$$

$$= \frac{1}{g} \sum_{m=-ι}^{ι} e^{\frac{4πi}{σ}(k-k_0)m} g_κ(n-n_0+m) g_κ(n-n_0-m)$$

$$= Mg_κ(n-n_0, k-k_0).$$

Theorem 3. If $g$ is a discrete variable Gaussian state, then

$$D(n_0, k_0)gD^+(n_0, k_0)$$

is also a discrete variable Gaussian state, for any $n_0, k_0 ∈ \{-s, -s+1, ..., s-1, s\}$.

Proof. Since $D^+(n_0, k_0)Π(n, k)D(n_0, k_0) = Π(n-n_0, k-k_0)$, we get

$$M_D(n_0, k_0)gD^+(n_0, k_0) = \frac{1}{g} tr(D(n_0, k_0)gD^+(n_0, k_0)Π(n, k))$$

$$= \frac{1}{g} tr(gD^+(n_0, k_0)Π(n, k)D(n_0, k_0))$$

$$= \frac{1}{g} tr(gΠ(n-n_0, k-k_0))$$

$$= Mg_κ(n-n_0, k-k_0).$$

Theorem 4. The Fourier transform $\hat{g}^\dagger \hat{D}^\dagger$ of a discrete variable Gaussian state $g$ is also a discrete variable Gaussian state. If the matrix corresponding to $g$ is $σ$, then the matrix corresponding to $\hat{g}^\dagger \hat{D}^\dagger$ is $ΩσΩ^T$, where

$$Ω = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof. Since $\hat{D}(n, k)\hat{g} = D(-k, n)$, we have $\hat{g}^\dagger Π(n, k)\hat{g} = Π(-k, n)$ and consequently

$$M_\hat{g}g^\dagger(n, k) = \frac{1}{g} tr(\hat{g}^\dagger Π(n, k)\hat{g})$$

$$= \frac{1}{g} tr(\hat{g}^\dagger Π(n, k)\hat{g})$$

$$= \frac{1}{g} tr(\hat{g}^\dagger Π(-k, n))$$

$$= Mg_κ(-k, n) = Mg_κ(Ω^{-1}(n, k)).$$

The last two theorems show that Fourier and displacement transforms preserve the nature of discrete variable Gaussian states, that is, they are discrete Gaussian transforms.
Theorem 5. Each continuous variable Gaussian state \( \rho_\sigma \) is the limit of a sequence of discrete variable Gaussian states \( \varrho_\sigma \), namely

\[
\rho_\sigma = \lim_{d \to \infty} \varrho_\sigma.
\]  

(58)

Proof. Since \( \lim_{q^2 + p^2 \to 0} w_\sigma(q, p) = 0 \), for \( d \) large enough, we have

\[
w_\sigma(n + \alpha \frac{\pi}{d}, k + \beta \frac{\pi}{d}) \approx 0 \quad \text{for} \quad (\alpha, \beta) \neq (0, 0),
\]

and consequently, up to a normalizing constant \( C \),

\[
\mathcal{M}_{\varrho_\sigma} (n, k) \approx C \mathcal{W}_{\rho_\sigma} \left( n \sqrt{\frac{\pi}{d}}, k \sqrt{\frac{\pi}{d}} \right).
\]

(60)

The set

\[
\left\{ \left( n \sqrt{\frac{\pi}{d}}, k \sqrt{\frac{\pi}{d}} \right) \mid d = 2s + 1 \in \{3, 5, 7, \ldots \}, n, k \in \{-s, -s+1, \ldots, s-1, s\} \right\}
\]

(61)

being dense in the phase space \( \mathbb{R}^2 \), the function \( \mathcal{W}_{\rho_\sigma} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is determined by the functions \( \mathcal{M}_{\varrho_\sigma} : \{-s, -s+1, \ldots, s\} \times \{-s, -s+1, \ldots, s\} \to \mathbb{R} \). Corresponding to \( d \in \{3, 5, 7, \ldots \} \). ■

Theorem 6. For any covariance matrix \( \sigma \), we have

\[
\lim_{d \to \infty} \text{tr} \varrho_\sigma^2 = \frac{1}{\sqrt{\det \sigma}}.
\]

(62)

Proof. For \( d \) large enough, we have \( \mathcal{M}_{\varrho_\sigma} (n, k) \approx C \mathcal{W}_{\rho_\sigma} \left( n \sqrt{\frac{\pi}{d}}, k \sqrt{\frac{\pi}{d}} \right) \), and consequently

\[
\text{tr} \varrho_\sigma^2 = d \sum_{n, k = -s}^{s} \mathcal{M}_{\varrho_\sigma} (n, k) \approx d \sum_{n, k = -s}^{s} \mathcal{W}_{\rho_\sigma} \left( n \sqrt{\frac{\pi}{d}}, k \sqrt{\frac{\pi}{d}} \right)^2.
\]

(63)

By considering a partition of the rectangle \( [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \) into \( d^2 \) squares of area \( \frac{\pi}{d^2} \) and regarding the integrals as limits of Riemann sums, we get

\[
\text{tr} \varrho_\sigma^2 \approx \frac{h}{\sqrt{\frac{\pi}{d}}} \frac{h}{\sqrt{\frac{\pi}{d}}} \mathcal{W}_{\rho_\sigma} \left( \sqrt{\frac{\pi}{d}} k \sqrt{\frac{\pi}{d}} \right)^2 \to \frac{\pi}{d^2} \mathcal{W}_{\rho_\sigma} (q, p) dq dp = \text{tr} \rho_\sigma^2 = \frac{1}{\sqrt{\det \sigma}}. \quad \blacksquare
\]

(64)

The Table I contains some numerical data obtained in certain particular cases.

Theorem 7. If \( \det \sigma = 1 \), then the discrete Gaussian state \( \varrho_\sigma \) is a pure state:

\[
\det \sigma = 1 \quad \Rightarrow \quad \text{tr} \varrho_\sigma^2 = 1.
\]

(65)

Proof. If \( \det \sigma = 1 \), then there exist \( \kappa \in (0, \infty) \) and a canonical transformation (rotation)

\[
\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}
\]

(66)

of the phase space such that

\[
\sigma = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \kappa^{-1} & 0 \\ 0 & \kappa \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.
\]

(67)

Consequently

\[
\mathcal{W}_{\rho_\sigma} (q, p) = \frac{2}{\sqrt{\pi}} e^{-\frac{d}{2} (\kappa q^2 + \kappa^{-1} p^2)} \mathcal{W}_{\rho_{\kappa\sigma}} (q', p')
\]

(68)

but, we know that the discrete Wigner function \( \mathcal{M}_{\rho_{\kappa\sigma}} \) corresponding to \( \mathcal{W}_{\rho_{\kappa\sigma}} \) represents a pure state. ■

The Table II contains some numerical data obtained in certain particular cases.
**IV. Position-momentum commutation relation and the discrete variable Gaussian states**

In the continuous case, the usual position-momentum commutation relation

$$[\hat{q}, \hat{p}] = i\frac{\hbar}{2\pi}$$  \hspace{1cm} (69)

is satisfied in a subspace dense in $L^2(\mathbb{R})$. The discrete counterpart

$$[\hat{q}, \hat{p}] = i\frac{\hbar}{2\pi}$$  \hspace{1cm} (70)

is not satisfied, but for $d$ large enough, most of the eigenvalues of the operator $[\hat{q}, \hat{p}] - i\frac{\hbar}{2\pi}$ are almost null.

Let $\lambda_1, \lambda_2, ..., \lambda_d$ be the eigenvalues of $[\hat{q}, \hat{p}] - i\frac{\hbar}{2\pi}$, considered in the increasing order of their modulus ($|\lambda_1| \leq |\lambda_2| \leq \ldots \leq |\lambda_d|)$, and let $\varphi_1, \varphi_2, ..., \varphi_d$ be the corresponding eigenfunctions. If we expand a discrete variable Gaussian state in terms of the eigenbasis $\{\varphi_k\}$, the most significant coefficients seem to be those corresponding to the functions $\varphi_k$ with small $|\lambda_k|$. For example, in the case $d = 11$, the matrix $(|\langle \varphi_k | g_i \rangle |)$ of $g_1$ in the eigenbasis $\{\varphi_k\}$ is

$$\begin{pmatrix}
0.9999 \\
1.0 \times 10^{-10} \\
2.0 \times 10^{-11} \\
2.0 \times 10^{-12} \\
2.0 \times 10^{-13} \\
0.0001 \\
0.0004 \\
6.0 \times 10^{-14} \\
2.0 \times 10^{-15} \\
0.0000
\end{pmatrix}$$ \hspace{1cm} (71)

and for $\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, the matrix $(|\langle \varphi_k | \mathcal{Q}_\sigma | \varphi_i \rangle |)$ of $\mathcal{Q}_\sigma$ is

$$\begin{pmatrix}
0.8954 & 6.0 \times 10^{-11} & 0.2837 & 1.0 \times 10^{-12} & 0.1059 & 1.0 \times 10^{-14} & 0.0393 & 3.0 \times 10^{-16} & 0.0164 & 3.0 \times 10^{-17} & 0.0064 \\
1.0 \times 10^{-10} & 5.0 \times 10^{-11} & 1.0 \times 10^{-17} & 1.0 \times 10^{-15} & 8.0 \times 10^{-18} & 7.0 \times 10^{-12} & 1.0 \times 10^{-17} & 3.0 \times 10^{-17} & 1.0 \times 10^{-17} & 3.0 \times 10^{-17} & 1.0 \times 10^{-17} \\
0.2837 & 2.0 \times 10^{-11} & 0.0899 & 3.0 \times 10^{-13} & 0.0335 & 4.0 \times 10^{-15} & 0.0124 & 1.0 \times 10^{-16} & 0.0525 & 1.0 \times 10^{-17} & 0.0026 \\
2.0 \times 10^{-13} & 1.0 \times 10^{-17} & 7.0 \times 10^{-14} & 3.0 \times 10^{-18} & 2.0 \times 10^{-14} & 3.0 \times 10^{-17} & 1.0 \times 10^{-14} & 4.0 \times 10^{-17} & 4.0 \times 10^{-17} & 1.0 \times 10^{-17} & 2.0 \times 10^{-15} \\
0.1059 & 7.0 \times 10^{-12} & 0.0335 & 1.0 \times 10^{-13} & 0.0125 & 1.0 \times 10^{-15} & 0.0046 & 4.0 \times 10^{-17} & 0.0019 & 3.0 \times 10^{-18} & 0.0010 \\
1.0 \times 10^{-15} & 1.0 \times 10^{-17} & 5.0 \times 10^{-16} & 2.0 \times 10^{-16} & 2.0 \times 10^{-16} & 1.0 \times 10^{-17} & 8.0 \times 10^{-17} & 1.0 \times 10^{-17} & 3.0 \times 10^{-17} & 1.0 \times 10^{-17} & 7.0 \times 10^{-18} \\
0.0393 & 2.0 \times 10^{-12} & 0.0124 & 5.0 \times 10^{-14} & 0.0046 & 5.0 \times 10^{-16} & 0.0017 & 2.0 \times 10^{-17} & 0.0007 & 6.0 \times 10^{-18} & 0.0003 \\
2.0 \times 10^{-16} & 1.0 \times 10^{-17} & 8.0 \times 10^{-17} & 4.0 \times 10^{-17} & 3.0 \times 10^{-17} & 7.0 \times 10^{-18} & 2.0 \times 10^{-17} & 4.0 \times 10^{-18} & 8.0 \times 10^{-18} & 3.0 \times 10^{-18} & 5.0 \times 10^{-18} \\
0.0164 & 1.0 \times 10^{-12} & 0.0052 & 2.0 \times 10^{-14} & 0.0019 & 2.0 \times 10^{-16} & 0.0007 & 1.0 \times 10^{-17} & 0.0003 & 8.0 \times 10^{-19} & 0.0001 \\
2.0 \times 10^{-17} & 1.0 \times 10^{-17} & 8.0 \times 10^{-18} & 1.0 \times 10^{-17} & 2.0 \times 10^{-17} & 1.0 \times 10^{-17} & 5.0 \times 10^{-18} & 3.0 \times 10^{-17} & 1.0 \times 10^{-18} & 4.0 \times 10^{-17} & 5.0 \times 10^{-18} \\
0.0084 & 6.0 \times 10^{-13} & 0.0026 & 1.0 \times 10^{-14} & 0.0010 & 1.0 \times 10^{-16} & 0.00037 & 2.0 \times 10^{-18} & 0.0001 & 5.0 \times 10^{-18} & 0.0000
\end{pmatrix}$$ \hspace{1cm} (72)

For $\varepsilon > 0$, let $d_\varepsilon$ be such that

$$\{\lambda_1, \lambda_2, ..., \lambda_{d_\varepsilon}\} = \{\lambda_k | |\lambda_k| < \varepsilon\},$$ \hspace{1cm} (73)

and let

$$\mathcal{H}_\varepsilon = \text{span}\{\varphi_1, \varphi_2, ..., \varphi_{d_\varepsilon}\}.$$ \hspace{1cm} (74)

If $\varepsilon$ is small enough, then we can consider that

$$[\hat{q}, \hat{p}] \approx i\frac{\hbar}{2\pi} \text{ in } \mathcal{H}_\varepsilon.$$ \hspace{1cm} (75)

By following the analogy with the continuous case, we define

$$\hat{a} = \sqrt{\frac{\pi}{\hbar}}(\hat{q} + i\hat{p}), \quad \text{similar to} \quad \hat{a} = \sqrt{\frac{\pi}{\hbar}}(\hat{q} + i\hat{p}),$$

$$\hat{a}^\dagger = \sqrt{\frac{\pi}{\hbar}}(\hat{q} - i\hat{p}), \quad \text{similar to} \quad \hat{a}^\dagger = \sqrt{\frac{\pi}{\hbar}}(\hat{q} - i\hat{p}),$$ \hspace{1cm} (76)

The discrete variable Gaussian states seem to mainly belong to $\mathcal{H}_\varepsilon$ and $\mathcal{A}(\mathcal{H}_\varepsilon)$.

In the case of continuous variable, the transformations $\hat{O} = e^{-\frac{i}{\hbar}\hat{H}}$, where

$$\hat{H} = (\hat{a}^\dagger \hat{a}) \left( \begin{array}{cc} A & B \\ B & A \end{array} \right) \left( \begin{array}{c} \hat{a} \\ \hat{a}^\dagger \end{array} \right) \text{ with } A \in \mathbb{R}, B \in \mathbb{C},$$ \hspace{1cm} (77)
Mixed discrete variable Gaussian states are Gaussian transforms, that is they preserve the Gaussian nature of the states. For \( \hat{U} = e^{-\frac{i}{d} \hat{H}} \), there exists \( S \in \text{Sp}(2, \mathbb{R}) \) such that

\[
e^{-\frac{i}{d} \hat{H}} \rho \sigma e^{\frac{i}{d} \hat{H}} = \rho_S \sigma S^T. \tag{78}
\]

In the case of discrete variable, the relation \( \lim_{d \to \infty} \rho_S = \rho_\sigma \) suggests that for

\[
\hat{S} = (\hat{a}^\dagger \hat{a}) \begin{pmatrix} A & B \\ \bar{B} & A \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \quad \text{with} \quad A \in \mathbb{R}, \quad B \in \mathbb{C}, \tag{79}
\]

we must have

\[
e^{-\frac{i}{d} \hat{S}} \rho_\sigma e^{\frac{i}{d} \hat{S}} \approx \rho_S S^T \quad \text{for} \quad d \text{ large enough}. \tag{80}
\]

Some numerical results concerning a phase shift and a single-mode squeezing transformation are presented in Table III. A possible explanation can be obtained by following the analogy with the continuous case. From the relation

\[
[a, a^\dagger] \approx 1 \tag{81}
\]

satisfied in \( \mathcal{H}_\varepsilon \), one gets the relations

\[
\left[ \frac{i}{d} \hat{S}, \hat{a} \right] \approx i \left( -A \hat{a} - B \hat{a}^\dagger \right) \quad \text{in} \quad \mathcal{H}_\varepsilon \tag{82}
\]

which can be written together as

\[
\left[ \frac{i}{d} \hat{S}, \left( \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \right) \right] \approx i \left( -A \hat{a} - B \hat{a}^\dagger \right) \quad \text{in} \quad \mathcal{H}_\varepsilon. \tag{83}
\]

By iteration, we obtain

\[
\left[ \frac{i}{d} \hat{S}, \left[ \frac{i}{d} \hat{S}, \left( \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \right) \right] \right] \approx i^2 \left( -A \hat{a} - B \hat{a}^\dagger \right)^2 \left( \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \right) \quad \text{in} \quad \mathcal{H}_\varepsilon. \tag{84}
\]

\[
\left[ \frac{i}{d} \hat{S}, \left[ \frac{i}{d} \hat{S}, \left[ \frac{i}{d} \hat{S}, \left( \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \right) \right] \right] \right] \approx i^3 \left( -A \hat{a} - B \hat{a}^\dagger \right)^3 \left( \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \right), \quad \text{etc.} \quad \text{in} \quad \mathcal{H}_\varepsilon. \tag{85}
\]

From the relation

\[
e^{X}y^{-X} = Y + \frac{1}{1!}[X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + \cdots \tag{86}
\]

it follows that \( \hat{U} = e^{-\frac{i}{d} \hat{S}} \) satisfies the relation

\[
\hat{U}^\dagger \left( \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \right) \hat{U} \approx \exp \left\{ i \left( -A \hat{a} - B \hat{a}^\dagger \right) \right\} \left( \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \right) \quad \text{in} \quad \mathcal{H}_\varepsilon. \tag{87}
\]

Since

\[
\left( \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \right) = \sqrt{\frac{2}{d}} \left( \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \right) \left( \begin{pmatrix} \hat{a} \\ \hat{p} \end{pmatrix} \right), \tag{88}
\]

the previous relation can be written as

\[
\hat{U}^\dagger \left( \begin{pmatrix} \hat{a} \\ \hat{p} \end{pmatrix} \right) \hat{U} \approx S \left( \begin{pmatrix} \hat{a} \\ \hat{p} \end{pmatrix} \right) \quad \text{in} \quad \mathcal{H}_\varepsilon, \tag{89}
\]

where

\[
S = \frac{1}{2} \left( \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \right) \exp \left\{ i \left( -A \hat{a} - B \hat{a}^\dagger \right) \right\} \left( \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \right) = \exp \left\{ -\frac{i}{2} \left( \begin{pmatrix} (B - \bar{B})i & -2A + B + \bar{B} \\ 2A + B + \bar{B} & -(B - \bar{B})i \end{pmatrix} \right) \right\}. \tag{90}
\]
Mixed discrete variable Gaussian states

This matrix with real elements and \( \det S = 1 \) belongs to the symplectic group \( \text{Sp}(2, \mathbb{R}) \).

For example, the phase shift \( U = e^{\frac{i}{2}(\varphi \hat{a}^\dagger \hat{a} + \varphi \hat{a} \hat{a}^\dagger)} \) corresponds to

\[
S_{\varphi} = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}
\]

and the single-mode squeezing \( U = e^{\frac{i}{2}((\alpha \hat{a}^\dagger - \alpha^* \hat{a})^2 - e^{-i\theta} \hat{a}^2)} \) to

\[
S_{\alpha, \theta} = \begin{pmatrix} \cosh s + \cos \theta & \sin \theta \sinh s \\ \sin \theta \sinh s & \cosh s - \cos \theta \sinh s \end{pmatrix}
\]

In \( \mathcal{H}_e \), we have \([\hat{q}, [\hat{q}, \hat{p}]] \approx 0 \) and \([\hat{p}, [\hat{q}, \hat{p}]] \approx 0 \). Consequently,

\[
\mathcal{D}(n, k) = e^{-\frac{2\pi}{\hbar}k} e^{\frac{2\pi}{\hbar}n \hat{p}} e^{-\frac{2\pi}{\hbar}n \hat{p}}
\]

and by denoting \( S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \), we get

\[
\hat{U}^\dagger \mathcal{D}(n, k) \hat{U} \approx e^{-\frac{2\pi}{\hbar}n \hat{p} - \frac{2\pi}{\hbar}k \hat{q}}
\]

Since

\[
w_\sigma(S^{-1}(q, p)) = \exp \left\{ -(q, p)(S^{-1})^T \sigma^{-1} S^{-1} \left( \begin{array}{c} q \\ p \end{array} \right) \right\}
\]

\[
= \exp \left\{ -(q, p)(S\sigma S^T)^{-1} \left( \begin{array}{c} q \\ p \end{array} \right) \right\} = w_{S\sigma S^T}(q, p)
\]

and

\[
\Pi^2 = \mathbb{I}, \quad \Pi \hat{q} = \hat{q} \Pi, \quad \Pi \hat{p} = -\hat{p} \Pi, \quad \Pi \hat{a} = -\hat{a} \Pi, \quad \Pi \hat{\Pi} \hat{q} = \hat{q} \Pi \hat{a}, \quad \Pi \hat{\Pi} \hat{p} = -\hat{p} \Pi \hat{a}, \quad \Pi \hat{\Pi} \hat{a} = \hat{a} \Pi \hat{\Pi},
\]

we get \( \hat{U}^\dagger \Pi \hat{U} = \Pi \) and

\[
\mathcal{M}_{\hat{U} \sigma \hat{U}^\dagger}(n, k) = \frac{1}{4} \text{tr}(\hat{U} \sigma \hat{U}^\dagger \Pi(n, k) \hat{U}^\dagger) = \frac{1}{4} \text{tr}(\sigma \Pi(n, k) \hat{U}^\dagger \hat{U})
\]

If \( d \) is large enough, then \( w_\sigma(S^{-1}(n, k) + (\alpha, \beta)^d) \approx 0 \) for \( (\alpha, \beta) \neq (0, 0) \), and we get the relation

\[
\mathcal{M}_{\hat{U} \sigma \hat{U}^\dagger}(n, k) \approx C w_\sigma(S^{-1}(n, k))
\]

equivalent to

\[
\hat{U} \sigma \hat{U}^\dagger \approx \sigma S_{\sigma S^T} \quad \text{in} \quad \mathcal{H}_e.
\]