THE CAUCHY PROBLEM FOR A CLASS OF LINEAR DEGENERATE EVOLUTION EQUATIONS ON THE TORUS

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Abstract. We study, in the periodic setting, the well-posedness of the Cauchy problem associated to the operator
\[ P(t, D_x, D_t) = D_t - a_2(t) \Delta_x + \sum_{j=1}^{N} a_{1,j}(t) D_{x_j} + a_0(t), \] with \( T > 0, t \in [0, T] \) and \( a_2, a_{1,1}, \ldots, a_{1,N}, a_0 \in C([0,T]; \mathbb{C}) \). Using Fourier analysis techniques, we obtain a complete character-
ization for the well-posedness of a class of degenerate initial-value problems in the Sobolev, Smooth, Gevrey and Real-Analytic frameworks.

1. Introduction

Linear evolution equations have attracted the attention of a great number of mathematicians for several decades, with some particular problems being studied as far as two hundred years ago. This is for instance the case of the Heat Equation, investigated by Jean-Baptiste Fourier in his book Théorie analytique de la chaleur (1822), where the concept of Fourier series was first introduced and the cornerstones of what became known later as Fourier Analysis were established.

Proceeding to the 20th and 21st centuries, one can find a huge number of works concerning evolution problems (see for instance [AACI, AACHII, ABZI, ABZII, AC, CC, Ma] and references therein); they may be concerned with different functional settings such as Sobolev, \( C^\infty \) and Gevrey, but most of them deal with domains in the form \( I \times \mathbb{R}^N \), where \( I \) is a subinterval of \( [0, +\infty) \). In the particular case of second order operators, even the most classic examples of Cauchy problems do not admit well-posedness in the most classic settings, such as \( C^\infty(\mathbb{R}^N) \) or \( G^s(\mathbb{R}^N) \) (see [Miz]). Thus authors usually look for such properties in spaces like \( H^\infty(\mathbb{R}^N) \) and \( H^\infty_s(\mathbb{R}^N) \), whose elements decay at infinity.

Consider for example the following class of second order evolution operators:
\[ P(t, x, D_t, D_x) = D_t - a(t, x) \Delta_x + \sum_{j=1}^{N} a_j(t, x) D_{x_j} + a_0(t, x), \quad t \in [0, T], x \in \mathbb{R}^N, \tag{1.1} \]
where \( D_{x_j} = -i \partial_{x_j} \), \( \Delta_x = \sum_{j=1}^{N} \partial_{x_j}^2 \) and the coefficients are assumed to be complex valued, continuous with respect to time and \( B^\infty(\mathbb{R}^N) \)-regular in the space \( x \). Here \( B^\infty(\mathbb{R}^N) \) stands for the space of all smooth functions which are not only bounded, but this also holds for each of their derivatives.

If we split the leading coefficient into its real and imaginary part: \( a(t, x) = b(t, x) + ic(t, x) \), the class given in (1.1) englobes two special cases. When \( c(t, x) \leq -\varepsilon \) for every \( t \) and \( x \), \( P \) is a typical example of parabolic operator. In this situation one obtains in general well-posedness in \( L^2(\mathbb{R}^N) \) without

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any loss of derivatives. This is the case for instance in [KG] and [T], where a fundamental solution is
constructed. On the other hand, when \( c(t, x) \equiv 0 \), [1,2] is a Schrödinger type operator, a typical example of non-kowalewskian operator that is not parabolic. The well-posedness in this situation turns out to be way more intricate, being usually dictated by the imaginary part of the first order terms; for instance [I1, I2] and [Miz] exhibit necessary and sufficient conditions for \( L^2(\mathbb{R}^N) \) and \( H^\infty(\mathbb{R}^N) \) well-posedness of the related problem. Moreover, in [Dre] the author presents a necessary condition in the Gevrey framework. Frequently these conditions are associated to the decay of the imaginary part of the first order coefficients, especially when one searches sufficient conditions for well-posedness in the Gevrey setting (see for example [CR, KB]).

Now observe that, if one replaces \( \mathbb{R}^N \) by \( \mathbb{T}^N \) in any of the problems above, the concept of decay at infinity does not fit anymore. And there is no guarantee that even by making the proper translations between both environments (such as \( H^r(\mathbb{R}^N) \leftrightarrow H^r(\mathbb{T}^N) \) and \( H^\infty(\mathbb{R}^N) \leftrightarrow C^\infty(\mathbb{T}^N) \)) one would obtain analogous results. In fact, it is widely known the existence of disparities between properties for operators that act both in the euclidean and periodic settings, which makes our inquiries much more compelling. We recall for instance the existence of constant coefficient operators which are globally hypoelliptic when acting on periodic distributions, but are not hypoelliptic in the local sense (see [GW]).

An extensive search was made in the literature, but we were not able to spot any manuscript that deals with any problem of the nature just described. In the parabolic case, we have reasons to believe that one has similar results (Corollary 6.3 is a first confirmation of our educated guess). Similarly, we think that the Schrödinger case can be completely solved. The authors are currently working with both problems.

Due to the fact that there were no references available, we started with a problem for which Fourier analysis tools could be applied (i.e the coefficients of our operator do not depend on space variables). Even in this case, it becomes clear that although some of our techniques could be reproduced for the euclidean case, several results rely strongly on the fact that we are working in the periodic setting, particularly in Sections 3, 4 and 7.

Now we proceed to the content of our work; let \( \mathbb{T}^N = \mathbb{R}^N / 2\pi \mathbb{Z}^N \) be the \( N \)-dimensional torus, for some \( N \in \mathbb{N} \), and fix a positive real number \( T \). We deal with the following class of operators:

\[
P(t, D_x, D_t) = D_t - Q(t, D_x) - D_t - a_2(t) \Delta_x + \sum_{j=1}^{N} a_{1,j}(t) D_{x_j} + a_0(t), \quad t \in [0, T], \quad x \in \mathbb{T}^N,
\]

(1.2)

where \( \Delta_x \) denotes the Laplace Operator, \( D_t = \frac{1}{i} \frac{\partial}{\partial t} \), \( D_{x_j} = \frac{1}{i} \frac{\partial}{\partial x_j} \) and \( a_0(t), a_{1,1}(t), \ldots, a_{1,N}(t), a_2(t) \) are all elements of \( C([0, T]; \mathbb{C}) \). The main purpose of the present work is to study well-posedness for initial-value problems in the form

\[
\begin{align*}
P u(t, x) &= f(t, x), & \forall t \in [0, T], \forall x \in \mathbb{T}^N, \\
u(0, x) &= g(x),
\end{align*}
\]

(1.3)

in the framework of Sobolev \( (H^r(\mathbb{T}^N), r \in \mathbb{R}) \), smooth \( (C^\infty(\mathbb{T}^N)) \), Gevrey \( (G^s(\mathbb{T}^N), s > 1) \) and real-analytic \( (C^\omega(\mathbb{T}^N)) \) spaces. By well-posedness we mean the existence and uniqueness of a \( u(t, x) \) in \( C([0, T]; X) \) which solves the Cauchy problem, with \( X \) representing any of the aforementioned spaces.

As it should be expected, the most important factor to the well-posedness of \( \{1.3\} \) is the sign of the imaginary part of \( a_2 \). Indeed, generally speaking, we prove the following statements:
• If there exists $t^* \in [0, T]$ such that $\text{Im}(a_2(t^*)) > 0$, then (1.3) is ill-posed in each of the aforementioned spaces (see Theorem 5.2).

• When $\text{Im}(a_2) < 0$, (1.3) is well-posed for each framework (see Theorem 6.2 and Corollary 6.3).

It is important to emphasize that both results were expected; a similar fact was proved in [P], in the euclidean environment. Nevertheless they were still included in this manuscript since they are straightforward consequences of results we consider quite interesting, such as Theorem 5.4 and Proposition 4.2.

Hence the only situation left uncovered is the one where $\text{Im}(a_2) \leq 0$ and vanishes at some point. This is by far the most interesting case and now the behavior of the imaginary part of the first-coefficients also must be accounted. First we deal with another case which should be more predictable, when $\text{Im}(a_2) \equiv 0$:

• (1.3) is well-posed in any of the settings if $\text{Im}(a_{1,j}) \equiv 0$ for each $j \in \{1, \ldots, N\}$ — a particular case of Theorem 6.4 and Corollary 6.5.

• (1.3) is ill-posed in each framework if $\text{Im}(a_{1,j}) \not\equiv 0$ for some $j \in \{1, \ldots, N\}$ (see Theorem 5.4).

Finally, we move towards the most important and compelling part of the article (Section (7)): the degenerate case. Assuming that $\text{Im}(a_2)$ is never strictly positive, has a finite number of zeros $\{t_1, \ldots, t_m\}$ and vanishes to a finite order at each of them, we characterize well-posedness by comparing the order of vanishing of each $\text{Im}(a_{1,j})$ at $t_k$, $k \in \{1, \ldots, m\}$. Since our main statement can be quite complicated to absorb at a first glance, we present an easier example. Let

$$P(t, D_x, D_t) = D_t + i t^k \frac{\partial^2}{\partial x^2} + i t^\ell D_x + a_0(t), \quad t \in [0, T], x \in \mathbb{T}, \quad k, \ell \geq 0.$$  

As a consequence of Theorem 7.3 and Corollary 7.4 one obtains:

• If $k \leq 2 \ell + 1$, then (1.3) is well-posed in each of the settings;

• When $k > 2 \ell + 1$, the problem is ill-posed in both Sobolev and $C^\infty$ frameworks. Moreover, it will be well-posed in $G^s$ if and only if

$$s < \frac{k - \ell}{k - 2 \ell - 1}.$$  

Since $C^\infty(\mathbb{T}^N) = G^1(\mathbb{T}^N)$, it is worth noting that in the example above one always has well-posedness in $C^\infty(\mathbb{T}^N)$. This in fact holds true for the much more general statement made in Theorem 7.3. We prove in Section 8 that such phenomenon is directly related to (1.2) being a second-order operator with respect to the space variables. Furthermore, it is important to mention that when any of the coefficients vanishes to an infinite order at some point, it is not possible to obtain such a characterization anymore; this is the content of Remark 8.5.

This manuscript is organized in the following manner: Section 2 is devoted for the precise definitions of the spaces for which we are interested to investigate well-posedness and its properties regarding Fourier series. We also provide an explicit formula for the Fourier coefficients of the formal solution of the initial-value problem (1.3). The subsequent section contains results that allow us to conclude the following: the influence of the imaginary parts of the coefficients of (1.2) to the well-posedness of (1.3) is significantly stronger when compared with the real parts (Theorem 5.4).

In Section 4 we show that the solution of (1.3) possesses some regularity provided that an a priori energy estimate in the phase space holds (Lemma 4.1). Furthermore, we prove (Proposition 4.2) that

$H^s$ well-posedness is stronger than $C^\infty$ well-posedness;

$C^\infty$ well-posedness is stronger than $G^s$ well-posedness (for any $s > 1$);
As we already observed, the imaginary part of the leading coefficient \( a_2 \) plays an important role on the well-posedness of \( (1.3) \). Roughly speaking, \( \text{Im} \, a_2 \leq 0 \) is a necessary condition for well-posedness (Theorem 5.1), whilst \( \text{Im} \, a_2 < 0 \) (that is, \( P \) given by \( (1.2) \) is parabolic) is a sufficient one (Theorem 5.2 and Corollary 6.3). These are briefly the contents of Sections 5 and 6 respectively.

Section 7 contains the main result of this work, namely Theorem 7.3. We prove energy estimates in the phase space for a class of degenerate operators, yielding well-posedness and ill-posedness for the associated Cauchy problem, depending on the behavior of the zeros of the coefficients. Finally, we close the paper making some final remarks and examples at Section 8.

2. Preliminaries

Before we deal with any particular case of \( (1.3) \), it is essential first to define precisely the objects which we alluded to at the introduction. As usual, we denote \( C^\infty(\mathbb{T}^N) \) as the space of complex-valued smooth functions defined on the torus and by \( \mathcal{D}'(\mathbb{T}^N) \) its topological dual space, endowed with the topology of \textit{pointwise convergence}.

Given \( u \in \mathcal{D}'(\mathbb{T}^N) \), one defines for each \( \xi \in \mathbb{Z}^N \) its \textit{Fourier coefficient} \( \hat{u}(\xi) = \langle u, e^{-ix\xi} \rangle \). It is possible to prove that \( u \) is completely described by its Fourier series. That is,

\[
u = \lim_{j \to +\infty} \sum_{|\xi| \leq j} \hat{u}(\xi)e^{ix\xi} = \sum_{\xi} \hat{u}(\xi)e^{ix\xi},
\]

with convergence in \( \mathcal{D}'(\mathbb{T}^N) \). For any fixed \( r \in \mathbb{R} \), consider \( \| \cdot \|_r \) the application described by

\[
\mathcal{D}'(\mathbb{T}^N) \ni u \mapsto \|u\|_{H^r(\mathbb{T}^N)}^2 := \sum_{\xi} |\hat{u}(\xi)|^2 (1 + |\xi|)^{2r}.
\]

We say that \( u \in H^r(\mathbb{T}^N) \) if \( \|u\|_{H^r(\mathbb{T}^N)} < +\infty \). The spaces \( H^r(\mathbb{T}^N) \) are usually known as \textit{periodic Sobolev spaces of order} \( r \); they are Hilbert spaces, with inner product given by \( (u,v)_{H^r(\mathbb{T}^N)} = \sum_{\xi} \hat{u}(\xi)\hat{v}(\xi)(1 + |\xi|)^{2r} \). Moreover, \( H^{r_1}(\mathbb{T}^N) \subset H^{r_2}(\mathbb{T}^N) \) if and only if \( r_1 \geq r_2 \) and

\[
C^\infty(\mathbb{T}^N) = \bigcap_{r \in \mathbb{R}} H^r(\mathbb{T}^N), \quad \mathcal{D}'(\mathbb{T}^N) = \bigcup_{r \in \mathbb{R}} H^r(\mathbb{T}^N).
\]

The topology of \( C^\infty(\mathbb{T}^N) \) is given by the projective limit of the Sobolev spaces (For more details on projective and injective limits of locally convex spaces, we recommend [Kom] or the Appendix A in [Mor]). That is,

\[
C^\infty(\mathbb{T}^N) = \lim_{r \to \mathbb{Z}_+} H^r(\mathbb{T}^N).
\]

As a consequence of Rellich’s Theorem, the Sobolev inclusions are \textit{compact}, which implies \( C^\infty(\mathbb{T}^N) \) is Fréchet-Schwartz. In particular, it is Montel, reflexive and separable. Next we present the space of Gevrey functions.

**Definition 2.1.** For any \( s \geq 1 \), we say that \( f \in C^\infty(\mathbb{T}^N) \) is an element of \( G^s(\mathbb{T}^N) \) if there exist \( C, h > 0 \) such that

\[
|D^\alpha f(x)| \leq Ch^{||\alpha||}||\alpha||^s, \quad \forall \alpha \in \mathbb{Z}^N_+, \: \forall x \in \mathbb{T}^N.
\]
The spaces $G^s(T^N)$ are usually known as $2\pi$-periodic Gevrey spaces of order s. When $s = 1$ we obtain the spaces of real-analytic $2\pi$-periodic functions, also denoted by $C^\omega(T^N)$. Let us briefly describe how the usual topology imposed in $G^s(T^N)$ is constructed (we recommend [LV] for more details and proofs of the results stated). For any $h > 0$, we set

$$G^s_h(T^N) = \left\{ f \in G^s(T^N); \|f\|_{h,s} := \sup_{\alpha \in \mathbb{Z}^N_+} \left( \frac{\|D^\alpha f\|_\infty}{h^{\alpha_1}} \right) < \infty \right\}. $$

Clearly $\|\cdot\|_{h,s}$ is a norm and in fact it can be proved that $(G^s_h(T^N), \|\cdot\|_{h,s})$ is a Banach space. Furthermore if $0 \leq h_1 \leq h_2$, the inclusions $G^s_{h_1}(T^N) \hookrightarrow G^s_{h_2}(T^N)$ are compact. Taking any strictly increasing sequence \( \{h_n\}_{n \in \mathbb{N}} \) of positive numbers such that $h_n \to +\infty$ and setting

$$G^s(T^N) = \lim_{n \to \infty} G^s_{h_n}(T^N),$$

we conclude that $G^s(T^N)$ is a DFS space (for more details, we recommend Section 5 in Appendix A of [Mor]). In particular, we obtain from Corollary A.5.11 in [Mor] that $G^s(T^N)$ is a Montel and reflexive space. Finally, it is important to mention that it is possible to characterize elements of $G^s(T^N)$ in terms of its Fourier Series.

**Proposition 2.2.** Let $u \in \mathcal{D}'(T^N)$; then $u \in G^s(T^N)$ if and only if there exist $C, \delta > 0$ such that

$$|\hat{u}(\xi)| \leq C e^{-\delta|\xi|^{s\slash N}}, \quad \forall \xi \in \mathbb{Z}^N. \quad (2.3)$$

Moreover, if $u \in G^s(T^N)$ we can write $u = \sum_{\xi \in \mathbb{Z}^N} \hat{u}(\xi)e^{i\xi \cdot \omega}$, with convergence in $G^s(T^N)$.

We point out the following relation between the numbers $\delta$ in (2.3) and $h$ in (2.2).

**Lemma 2.3.** Given arbitrary $s \geq 1$ and $h > 0$, there exists $\lambda(N, s) > 0$ such that

$$f \in G^s_h(T^N) \quad \Rightarrow \quad |\hat{f}(\xi)| \leq 2^s \|f\|_{h,s} e^{-\frac{\lambda}{h^{1\slash N}}|\xi|^{1\slash N}}, \quad \forall \xi \in \mathbb{Z}^N.$$

**Lemma 2.4.** Fix $s \geq 1$ and let $g \in G^s(T^N)$ such that $|\hat{g}(\xi)| \leq C_1 e^{-\delta|\xi|^{1\slash N}}$, for any $\xi \in \mathbb{Z}^N$ and $C_1, \delta > 0$. Set $h = (\frac{2\pi}{\delta})^s$; then $g \in G^s_h(T^N)$ and there exists a positive constant $C_2(\delta, N, s)$ such that $\|g\|_{h,s} \leq C_1 C_2$.

### 2.1. Characterizations for classes of continuous curves.

The main goal in this subsection is to completely understand the following spaces of functions:

i) $C \left( [0, T]; H^r(T^N) \right)$, for any $r \in \mathbb{R}$;

ii) $C \left( [0, T]; C^\infty(T^N) \right)$;

iii) $C \left( [0, T]; G^s(T^N) \right)$, for any $s \geq 1$.

Since we intend to work extensively with them throughout this manuscript, it is important to understand the meaning and different characterizations of their elements.

Note that case i) is described by the continuity of applications between metric spaces. Regarding case ii), it is a consequence of a standard result (see for instance Theorem A.3.3 in [Mor]) the following

**Theorem 2.5.** An application $g : [0, T] \to C^\infty(T^N)$ is continuous if and only if for every $r \in \mathbb{R}$ the map $\nu_r \circ g : [0, T] \to H^r(T^N)$ is continuous, where $\nu_r : C^\infty(T^N) \to H^r(T^N)$ denotes the canonical inclusion.
We now proceed to case \([iii]\): let \(u : [0, T] \to G^s(T^N)\) be a continuous map, since \([0, T]\) is compact, the same holds true for \(u([0, T])\). In particular, the set is bounded, which means that
\[
u([0, T]) \text{ is a bounded subset of } G^s_h(T^N), \text{ for some } h > 0 \text{ (see Theorem A.5.7 in [Mor]).}
\]
By possibly increasing \(h\) and applying Theorem 6' in [Kom], we may restrict the codomain and obtain \(u : [0, T] \to G^s_h(T^N)\) a continuous map. Therefore we have the following

**Theorem 2.6.** A map \(u : [0, T] \to G^s(T^N)\) is continuous if and only if there exists \(h > 0\) such that \(u([0, T]) \subset G^s_h(T^N)\) and \(u : [0, T] \to G^s_h(T^N)\) is continuous.

### 2.2. Distributions in \((0, T)\) with values in \(H^r(T^N)\)

As stated previously, we would like to find solutions for \([1.3]\) in the form \(u(t, x)\), where \(u(\cdot, x)\) is continuous for any fixed \(x\) and \(u(t, \cdot)\) is (at least) an element of \(H^r(T^N)\) for any \(t\) fixed. A quite intuitive way to reach this goal would be to write \(u\) in terms of its Fourier series and to convert the problem into a denumerable family of first-order ODE’s parametrized by \(\xi \in \mathbb{Z}^N\). In order to turn such idea into a formal result, we introduce the following concept:

**Definition 2.7.** Given \(T > 0\) and \(r \in \mathbb{R}\), we define \(D'(\,(0, T) ; H^r(T^N)\)) as the space of continuous linear functionals \(u : C_c^\infty((0, T)) \to H^r(T^N)\).

Similarly to other spaces of distributions, we impose in \(D'(\,(0, T) ; H^r(T^N)\)) the topology of pointwise convergency. Thus a sequence \(\{u_k\}_{k \in \mathbb{N}}\) converges to \(u\) in \(D'(\,(0, T) ; H^r(T^N)\)) if and only if
\[
\|u_k(\phi) - u(\phi)\|_{H^r(T^N)} \to 0, \quad \forall \phi \in C_c^\infty((0, T)).
\]

Next we relate this unusual space of distributions with the sets described in Subsection 2.1.

**Proposition 2.8.** \(C([0, T]; H^r(T^N))\) is a subspace of \(D'(\,(0, T) ; H^r(T^N)\)).

**Proof.** Take \(v \in C([0, T]; H^r(T^N))\); then for every \(\phi(t) \in C_c^\infty((0, T))\), the function \(w(t) = \phi(t)v(t)\) is an element of \(C([0, T]; H^r(T^N))\). Now consider
\[
\mathcal{T} : C([0, T] ; H^r(T^N)) \to D'(\,(0, T) ; H^r(T^N))
\]
\[
u \mapsto \mathcal{T}_u : C_c^\infty((0, T)) \to H^r(T^N),
\]
where \(\mathcal{T}_u\) denotes the map \(\phi \mapsto \int_0^T \phi(t)u(t)dt = : (\mathcal{T}_u, \phi)\) and the integral above represents the Bochner Integral taking values in \(H^r(T^N)\) (for more details we recommend Section 5 of Chapter 5 in [Yos]). Since \(\phi(t)u(t) \in C([0, T]; H^r(T^N))\) it follows that
\[
\|(\mathcal{T}_u, \phi)\|_{H^r(T^N)} \leq \int_0^T \|\phi(t)u(t)\|_{H^r(T^N)}dt \leq T \left( \max_{t \in [0, T]} \|u(t)\|_{H^r(T^N)} \right) \left( \max_{t \in [0, T]} \|\phi(t)\| \right),
\]
which shows that \(\mathcal{T}_u \in D'(\,(0, T) ; H^r(T^N))\) and \(\mathcal{T}\) is continuous.

Finally we prove that \(\mathcal{T}\) is injective. For this purpose, take \(u \in C([0, T]; H^r(T^N))\) satisfying \(\langle \mathcal{T}_u, \phi \rangle = 0\) for every \(\phi \in C_c^\infty((0, T))\) and let \(\iota \in (H^r(T^N))'\) be an arbitrary continuous linear functional. Then (see Corollary 2, pg 134 of [Yos])
\[
0 = \iota(\langle \mathcal{T}_u, \phi \rangle) = \iota \left( \int_0^T \phi(t)u(t)dt \right) = \iota \left( \int_0^T \iota(\phi(t)u(t))dt \right) = \iota \left( \int_0^T \phi(t)\iota(u(t))dt \right), \quad \forall \phi \in C_c^\infty((0, T)).
\]
Since \(\iota(u(t))\) is a continuous scalar function, it follows that \(\iota(u(t)) = 0\) for every \(t \in (0, T)\). Hence \(u(t) = 0\) for every \(t \in [0, T]\), which finalizes the proof. \(\square\)
Remark 2.9. From now on we will denote $T_u$ just as $u$. Using the spaces introduced in Definition 2.7 one can establish the concept of weak derivative (with respect to $t$) for elements of $C([0, T]; H^r(T^N))$. Indeed, set for any $u \in D'(([0, T); H^r(T^N))$

$$\langle D_t u, \varphi \rangle = \langle u, -D_t \varphi \rangle, \quad \forall \varphi \in C_c^\infty((0, T)).$$

In this case, it is not difficult to check that $D_t : D'((0, T); H^r(T^N)) \to D'((0, T); H^r(T^N))$ is continuous. Moreover, it also implies that

$$[0, T] \to H^r(T^N),$$

$$t \mapsto \hat{u}(t, \xi)e^{i\xi x}$$

is an element of $C([0, T]; H^r(T^N))$, for any $\tau \in \mathbb{R}$.

Theorem 2.10. Let $u \in C([0, T]; H^r(T^N))$; then the limit

$$v := \sum_{\xi \in \mathbb{Z}^N} \hat{u}(t, \xi)e^{i\xi x}$$

is an element of $D'((0, T); H^r(T^N))$ and $u = v$. In other words, $u$ can be written in terms of its partial Fourier Series in $D'((0, T); H^r(T^N))$.

Proof. We fix $u \in C([0, T]; H^r(T^N))$ and set, for any $k \in \mathbb{N}_0$,

$$s_k(t, x) = \sum_{|\xi| \leq k} \hat{u}(t, \xi)e^{i\xi x}, \quad \forall t \in [0, T], \quad \forall x \in T^N.$$

It follows from our last claim that $s_k(t) \in C([0, T]; H^r(T^N))$ and from Proposition 2.8 that

$$\langle s_k(t, x), \varphi(t) \rangle = \int_0^T \phi(t)s_k(t, x)dt, \quad \forall \varphi \in C_c^\infty((0, T)).$$

Fix $\phi \in C_c^\infty((0, T))$; then the sequence $\phi(t)s_k(t, x)$ is uniformly bounded in $H^r(T^N)$. Indeed,

$$\|\phi(t)s_k(t, x)\|_{H^r(T^N)} \leq \|\phi(t)(s_k(t, x) - u(t))\|_{H^r(T^N)} + \|\phi(t)u(t)\|_{H^r(T^N)} \leq \left( \max_{t \in [0, T]} |\phi(t)| \right) \left[ \|s_k(t) - u(t)\|_{H^r(T^N)} + \|u(t)\|_{H^r(T^N)} \right]. \quad (2.4)$$

On the other hand, for any fixed $t \in [0, T]$

$$\|s_k(t, x) - u(t)\|^2_{H^r(T^N)} = \sum_{|\xi| > k} |\hat{u}(t, \xi)|^2(1 + |\xi|)^{2r} \leq \|u(t)\|^2_{H^r(T^N)}. \quad (2.5)$$

By associating (2.4) to (2.5) we deduce that

$$\max_{t \in [0, T]} \|\phi(t)s_k(t, x)\|_{H^r(T^N)} \leq 2 \left( \max_{t \in [0, T]} |\phi(t)| \right) \left( \max_{t \in [0, T]} \|u(t)\|_{H^r(T^N)} \right),$$
which confirms our assertion.

Finally, it follows from the identity in (2.5) that \( \lim_{k \to \infty} \phi(t)s_k(t) = \phi(t)u(t) \) in \( H^r(\mathbb{T}^N) \), for every \( t \in [0, T] \). By the Dominated Convergence Theorem, we obtain

\[
\left\| \int_0^T \phi(t)\{s_k(t) - u(t)\} \, dt \right\|_{H^r(\mathbb{T}^N)} \to 0 \quad \text{as} \quad k \to \infty.
\]

Therefore

\[
\lim_{k \to \infty} \langle s_k(t, x), \phi(t) \rangle = \lim_{k \to \infty} \int_0^T \phi(t)s_k(t, x) \, dt = \int_0^T \phi(t)u(t) \, dt = \langle u(t), \phi(t) \rangle,
\]

which finalizes the proof of the theorem. \( \square \)

**Remark 2.11.** Since \( D_t \) acts continuously in \( D'(\mathbb{R}^+) ; H^r(\mathbb{T}^N) \) for any \( u \in C\left([0, T]; H^r(\mathbb{T}^N)\right) \)

\[
D_t u = D_t \sum_{\xi \in \mathbb{Z}^N} \hat{u}(t, \xi) e^{i\xi x} = \sum_{\xi \in \mathbb{Z}^N} D_t \hat{u}(t, \xi) e^{i\xi x},
\]

That is, we are able to write the weak derivative of \( u(t) \) as the limit of

\[
\sum_{|\xi| \leq k} D_t \hat{u}(t, \xi) e^{i\xi x},
\]

where \( D_t \hat{u}(t, \xi) e^{i\xi x} \) denotes the following element of \( D'(\mathbb{R}^+) ; H^r(\mathbb{T}^N) \):

\[
\langle D_t \hat{u}(t, \xi) e^{i\xi x}, \phi(t) \rangle = \left[ \int_0^T \hat{u}(t, \xi)(-D_t\phi(t)) \, dt \right] e^{i\xi x}, \quad \forall \phi \in C_\infty^\infty([0, T)).
\]

**Remark 2.12.** Let \( u \in C\left([0, T]; H^r(\mathbb{T}^N)\right) \); for any \( \alpha \in \mathbb{Z}_+^N \) it follows from the continuity of derivatives in Sobolev spaces that \( D_x^\alpha u \in C\left([0, T]; H^{r-|\alpha|}(\mathbb{T}^N)\right) \). By Theorem 2.10 we have

\[
D_x^\alpha u = \sum_{\xi \in \mathbb{Z}^N} D_x^\alpha \hat{u}(t, \xi) e^{i\xi x} = \sum_{\xi \in \mathbb{Z}^N} \hat{u}(t, \xi) \xi^\alpha e^{i\xi x}, \quad \text{with convergence in} \ D'(\mathbb{R}^+) ; H^{r-|\alpha|}(\mathbb{T}^N)).
\]

### 2.4. A family of ordinary differential equations

We now return to the original problem; suppose that for some \( \tau \in \mathbb{R}, \ f \in C\left([0, T]; H^r(\mathbb{T}^N)\right) \) and \( g \in H^r(\mathbb{T}^N) \) we are able to find \( u \in C\left([0, T]; H^r(\mathbb{T}^N)\right) \) which solves (1.3), for some \( \rho \in \mathbb{R} \). Then

\[
D_t u = a_2(t)\Delta u - \sum_{j=1}^N a_{1,j}(t)D_{x_j} u - a_0(t)u + f(t, x).
\]

If \( \sigma = \min\{\tau, \rho - 2\} \), note that the right-hand side is an element of \( C\left([0, T]; H^\sigma(\mathbb{T}^N)\right) \), so the same holds for the left-hand side. Hence we have uniqueness for the partial Fourier series described in Theorem 2.10 and Remark 2.11 on both sides. This fact, associated to the initial condition in (1.3), implies the following identities:

\[
\begin{cases}
D_t \hat{u}(t, \xi) + \left[ a_2(t)|\xi|^2 + \sum_{j=1}^N a_{1,j}(t)\xi_j + a_0(t) \right] \hat{u}(t, \xi) = \hat{f}(t, \xi), \\
\hat{u}(0, \xi) = \hat{g}(\xi),
\end{cases}
\forall t \in [0, T], \forall \xi \in \mathbb{Z}^N. \tag{2.6}
\]

The argument above shows that for any **barely regular** candidate \( u \) to be a solution for (1.3), it is **necessary** to solve the family of ordinary differential equations parametrized by elements in \( \mathbb{Z}^N \) given
Each equation in (2.6) can be solved by the method of integrating factors. In fact, if we set
\[ a_k(t) := b_k(t) + i c_k(t), \]  
with both \( b_k \) and \( c_k \) real-valued, and denote their respective primitives by
\[ A_k(t) := B_k(t) + i C_k(t), \quad \text{where} \quad B_k(t) := \int_0^t b_k(s)\,ds \quad \text{and} \quad C_k(t) := \int_0^t c_k(s)\,ds, \]  
it is not difficult to prove that for any fixed \( \xi \in \mathbb{Z}^N \)
\[ \hat{u}(t, \xi) = \hat{g}(\xi) \exp \left\{ -i \left[ A_2(t)|\xi|^2 + \sum_{j=1}^N A_{1,j}(t)\xi_j + A_0(t) \right] \right\} + \]
\[ + i \int_0^t \hat{f}(s, \xi) \exp \left\{ i \left[ (A_2(s) - A_2(t))|\xi|^2 + \sum_{j=1}^N (A_{1,j}(s) - A_{1,j}(t))\xi_j + (A_0(s) - A_0(t)) \right] \right\} \, ds, \]
for every \( t \in [0, T] \), and such solution is unique.

**Remark 2.13.** Even though this process has not yielded any solution for (1.3) yet, it has already proved that there is at most one solution for the initial-value problem.

### 3. Normal Form

Our main objective in the present section is to show that, in order to study well-posedness for the problem (1.3) with \( P \) given in (1.2), it is equivalent to work with a much simpler operator. Before proceeding to the first statement, we introduce the concepts of well-posedness we intend to work with from now on.

**Definition 3.1.** Let \( P(t, D_t, D_x) \) as in (1.2) and consider the initial-value problem (1.3).

1. Given \( \delta \geq 0 \) we say that (1.3) is well-posed in \( H^r \) with loss of \( \delta \) derivatives if for every \( r \in \mathbb{R} \), \( f \in C([0, T]; H^r(\mathbb{T}^N)) \) and \( g \in H^r(\mathbb{T}^N) \), there exists a unique \( u \in C([0, T]; H^{r-\delta}(\mathbb{T}^N)) \) which solves (1.3). In the case where we may take \( \delta = 0 \), we simply say that (1.3) is well-posed in \( H^r \).
2. We say that (1.3) is well-posed in \( C^\infty \) when for any \( f \in C([0, T]; C^\infty(\mathbb{T}^N)) \) and \( g \in C^\infty(\mathbb{T}^N) \), there exists a unique \( u \in C([0, T]; C^\infty(\mathbb{T}^N)) \) that solves (1.3).
3. Fixed \( s \geq 1 \), we say that (1.3) is well-posed in \( G^s \) if for any \( f \in C([0, T]; G^s(\mathbb{T}^N)) \), \( g \in G^s(\mathbb{T}^N) \), there exists a unique \( u \in C([0, T]; G^s(\mathbb{T}^N)) \) solving (1.3).

**Remark 3.2.** Suppose for a moment that one has proved well-posedness for any of the cases above. By going back to the description of the operator \( P \), the following facts are immediate consequences:

- If \( f \in C([0, T]; H^r(\mathbb{T}^N)) \) and \( u \in C([0, T]; H^{r-\delta}(\mathbb{T}^N)) \) solves (1.3), \( u \in C^1([0, T]; H^{r-\delta-2}(\mathbb{T}^N)) \).
- If \( f \in C([0, T]; C^\infty(\mathbb{T}^N)) \) and \( u \in C([0, T]; C^\infty(\mathbb{T}^N)) \) solves (1.3), \( u \in C^1([0, T]; C^\infty(\mathbb{T}^N)) \).
- If \( f \in C([0, T]; G^s(\mathbb{T}^N)) \) and \( u \in C([0, T]; G^s(\mathbb{T}^N)) \) solves (1.3), \( u \in C^1([0, T]; G^s(\mathbb{T}^N)) \).

**Proposition 3.3.** Let \( J : [0, T] \times \mathbb{Z}^N \rightarrow \mathbb{C} \) be the following continuous map:
\[ J(t, \xi) = -B_2(t)|\xi|^2 - \sum_{j=1}^N B_{1,j}(t)\xi_j - A_0(t), \]  
where
with $B_2, B_1, \ldots, B_{1,N}$ and $A_0$ as in (2.8). Consider, for any $r \in \mathbb{R}$,
\[
\Psi : C([0,T]; H^r(\mathbb{R}^N)) \to C([0,T]; H^{r-2}(\mathbb{R}^N))
\]
\[
u = \sum_{\xi \in \mathbb{Z}^N} \hat{u}(t,\xi)e^{iz\xi} \Rightarrow \Psi u = \sum_{\xi \in \mathbb{Z}^N} \left( e^{iJ(t,\xi)}\hat{u}(t,\xi) \right) e^{iz\xi}.
\]
(3.2)

Then $\Psi$ is well defined and the same holds for the restrictions
\[
\Psi : C([0,T]; C^\infty(\mathbb{R}^N)) \to C([0,T]; C^\infty(\mathbb{R}^N))
\]
\[
\Psi : C([0,T]; G^s(\mathbb{R}^N)) \to C([0,T]; G^s(\mathbb{R}^N)), \ \forall s \geq 1.
\]
(3.3)

Furthermore the maps in (3.3) are bijective.

Proof. We prove first that (3.2) is well defined. Since $B_2$ and each $B_{1,j}$ is real, whilst $A_0$ is uniformly bounded, there exists a constant $C > 0$ such that
\[
|e^{iJ(t,\xi)}\hat{u}(t,\xi)| \leq C |\hat{u}(t,\xi)|, \ \forall t \in [0,T], \ \forall \xi \in \mathbb{Z}.
\]
(3.4)

Thus $\Psi u(t, x) \in H^r(\mathbb{R}^N)$ for any fixed $t \in [0,T]$ and $\hat{\Psi} u(t,\xi) = e^{iJ(t,\xi)}\hat{u}(t,\xi)$, for every $\xi \in \mathbb{Z}^N$. It remains to prove the continuity in $H^{r-2}(\mathbb{R}^N)$; given $t, t_0 \in [0,T]$,
\[
\|\Psi u(t) - \Psi u(t_0)\|_{H^{r-2}(\mathbb{R}^N)} = \sum_{\xi \in \mathbb{Z}^N} |e^{iJ(t,\xi)}\hat{u}(t,\xi) - e^{iJ(t_0,\xi)}\hat{u}(t_0,\xi)|^2 (1 + |\xi|^{2(r-2)})
\]
\[
\leq 2C^2 \|u(t) - u(t_0)\|_{H^{r-2}(\mathbb{R}^N)}^2 + 2 \sum_{\xi \in \mathbb{Z}^N} |e^{iJ(t,\xi)} - e^{iJ(t_0,\xi)}|^2 |\hat{u}(t_0,\xi)|^2 (1 + |\xi|^{2(r-2)}).
\]
(3.5)

Next we apply mean value inequality to estimate the right-hand side of the expression above:
\[
|e^{iJ(t,\xi)} - e^{iJ(t_0,\xi)}| \leq |t - t_0| \sup_{\tau \in [t_0,t]} |b(\tau)| |\xi|^2 - \sum_{j=1}^N b_{1,j}(\tau)\xi_j - a_0(\tau) \sup_{\tau \in [t_0,t]} |e^{iJ(\tau,\xi)}|
\]
\[
\leq C_2 (1 + |\xi|)|t - t_0|,
\]
(3.6)

for some constant $C_2$ which can be taken independently of $t, t_0 \in \mathbb{R}$. By associating (3.5) to (3.6), we infer that
\[
\|\Psi u(t) - \Psi u(t_0)\|_{H^{r-2}(\mathbb{R}^N)}^2 \leq C_3 \left( \|u(t) - u(t_0)\|_{H^r(\mathbb{R}^N)}^2 + |t - t_0| \max_{\tau \in [t_0,t]} \|u(\tau)\|_{H^{r-2}(\mathbb{R}^N)}^2 \right),
\]
for some $C_3$ that does not depend on $t$ nor $t_0$, which proves our first statement. The fact that $\Psi$ take elements of $C([0,T]; C^\infty(\mathbb{R}^N))$ into itself is a direct consequence from the statement we have just proved and Theorem 2.5.

We proceed to the Gevrey case; if $u \in C([0,T]; G^s(\mathbb{R}^N))$ by Theorem 2.6 there exists $h > 0$ such that $u : [0,T] \to G^h(\mathbb{R}^N)$ is continuous. Hence we find (by Lemma 2.3) $\lambda(N,s) > 0$ such that
\[
|\hat{u}(t,\xi)| \leq 2^s \|u(t)\| h,s e^{-\frac{\lambda}{h^{1/s}}|\xi|^{1/s}} \leq 2^s \max_{\tau \in [0,T]} \left( \|u(\tau)\| h,s \right) e^{-\frac{\lambda}{h^{1/s}}|\xi|^{1/s}}, \ \forall \xi \in \mathbb{Z}^N, \ \forall t \in [0,T].
\]
(3.7)

It follows from (3.4) and (3.7) that for each fixed $t \in [0,T]$ we have $\Psi u(t) \in G^s(\mathbb{R}^N)$ and
\[
|\hat{\Psi} u(t,\xi)| \leq C2^s \max_{\tau \in [0,T]} \left( \|u(\tau)\| h,s \right) e^{-\frac{\lambda}{h^{1/s}}|\xi|^{1/s}}, \ \forall \xi \in \mathbb{Z}^N, \ \forall t \in [0,T].
We deduce from Lemma 2.4 that

$$\Psi u(t) \in G^s(\mathbb{T}^N), \quad \forall t \in [0, T], \quad \text{where } \kappa = (2s/\lambda)^r h > 0 \text{ does not depend on } t.$$ 

It remains to prove continuity: applying (3.4), (3.6) and (3.7) we obtain

$$\left| \hat{\psi}_u(t, \xi) - \hat{\psi}_u(t_0, \xi) \right| \leq C |\tilde{u}(t, \xi) - \tilde{u}(t_0, \xi)| + C_2 2^s \max_{\tau \in [0, T]} \|u(\tau)\|_{h, s} |t - t_0|(1 + |\xi|)^2 e^{-\frac{1}{2N^2}\|\xi\|^{1/s}}. \quad (3.8)$$

On the other hand, it follows from Lemma 2.3 that

$$|\tilde{u}(t, \xi) - \tilde{u}(t_0, \xi)| \leq 2^s \|u(t) - u(t_0)\|_{h, s} e^{-\frac{1}{2N^2}\|\xi\|^{1/s}}, \quad \forall \xi \in \mathbb{Z}^N. \quad (3.9)$$

Finally, recall that there exists $C_4 > 0$ such that

$$(1 + |\xi|^2) \leq C_4 e^{\frac{1}{2N^2}\|\xi\|^{1/s}}, \quad \forall \xi \in \mathbb{Z}^N. \quad (3.10)$$

Putting (3.9) and (3.10) into (3.8), we deduce the existence of $M > 0$ such that

$$\left| \hat{\psi}_u(t, \xi) - \hat{\psi}_u(t_0, \xi) \right| \leq M \left( \|u(t) - u(t_0)\|_{h, s} + |t - t_0| \right) e^{-\frac{1}{2N^2}\|\xi\|^{1/s}}, \quad \forall \xi \in \mathbb{Z}^N.$$ 

Using Lemma 2.4 one deduces that $\Psi u(t) \to \Psi u(t_0)$ in $G^s(\mathbb{T}^N)$ when $t \to t_0$, as we intended to prove.

In order to check that $\Psi$ is bijective in both smooth and Gevrey cases, define

$$\Phi : C ([0, T]; H^r(\mathbb{T}^N)) \to C ([0, T]; H^{r-2}(\mathbb{T}^N))$$

$$u = \sum_{\xi \in \mathbb{Z}^N} \tilde{u}(t, \xi) e^{i\xi} \mapsto \Psi u = \sum_{\xi \in \mathbb{Z}^N} \left( e^{-iJ(t, \xi)} \tilde{u}(t, \xi) \right) e^{i\xi}. \quad (3.11)$$

Then properties stated and proved for $\Psi$ also hold for $\Phi$. Moreover it follows immediately that $\Phi \circ \Psi = \Psi \circ \Phi = I$, which finalizes the proof. \( \square \)

**Theorem 3.4.** Let $P(t, D_x, D_t)$ be the operator described in (1.2) and set

$$\tilde{P}(t, x, D_x, D_t) = D_t - ic_2(t)\Delta_x^2 + \sum_{j=1}^{N} ic_{1,j}(t)D_{x_j}, \quad (3.12)$$

with $c_2$ and $c_{1,1}, \ldots, c_{1,N}$ defined in (2.7). Then (1.3) is well-posed in $C^\infty$ or $G^s$, for any $s \geq 1$, if and only if the same holds true for

$$\begin{cases} 
\tilde{P}u(t, x) = \alpha(t, x), \\
u(0, x) = \beta(x), \end{cases} \quad \forall t \in [0, T], \forall x \in \mathbb{T}^N. \quad (3.13)$$

Moreover, if there exists $\delta \geq 0$ such that (1.3) is well-posed in $H^r$ with loss of $\delta$ derivatives, then (3.13) is well-posed in $H^r$ with loss of $(4 + \delta)$ derivatives.

**Proof.** Since all the results are proved similarly (using that $\Psi$ defined in (3.3) is invertible), we restrict ourselves to the Sobolev case. Suppose (1.3) is well-posed in $H^r$ with loss of $\delta$ derivatives and fix $\alpha(t, x) \in C ([0, T]; H^r(\mathbb{T}^N))$, $\beta(x) \in H^r(\mathbb{T}^N)$. Let $\Psi$ as in Proposition 3.3 then $\Psi(\alpha) \in C ([0, T]; H^{r-2}(\mathbb{T}^N))$ and there exists (by hypothesis) $v \in C ([0, T]; H^{r-2}(\mathbb{T}^N))$ such that

$$\begin{cases} 
Pv(t, x) = \Psi(\alpha)(t, x), \\
v(0, x) = \beta(x), \end{cases} \quad \forall t \in [0, T], \forall x \in \mathbb{T}^N.$$
Thus $\alpha = \Psi^{-1}(Pv) \in C \left( [0, T]; H^r(T^N) \right)$ and we can write

$$\alpha = \Psi^{-1}(Pv) = \sum_{\xi \in \mathbb{Z}^N} e^{-ij(t, \xi)} \left[ D_t \hat{v}(t, \xi) + \left( a_2(t)|\xi|^2 + \sum_{j=1}^N \sum_{iJ} (b_{1,j}(t) + ic_{1,j}(t)) \xi_i + a_0(t) \right) \hat{v}(t, \xi) \right] e^{ix\xi}. $$

On the other hand, $\Psi^{-1}v \in C \left( [0, T]; H^{r-4-\delta}(T^N) \right)$. By Remarks 2.11 and 2.12 we are able to write

$$\tilde{P}(\Psi^{-1}v) = \sum_{\xi \in \mathbb{Z}^N} \tilde{P} \left( (e^{-ij(t, \xi)} \hat{v}(t, \xi)) e^{ix\xi} \right)$$

$$= \sum_{\xi \in \mathbb{Z}^N} e^{-ij(t, \xi)} \left[ D_t \hat{v}(t, \xi) + \left( \frac{\partial J}{\partial \mathbf{u}} (t, \xi) + ic_2(t) |\xi|^2 + \sum_{j=1}^N ic_{1,j}(t) \xi_j \right) \hat{v}(t, \xi) \right] e^{ix\xi}$$

$$= \sum_{\xi \in \mathbb{Z}^N} e^{-ij(t, \xi)} \left[ D_t \hat{v}(t, \xi) + \left( b_2(t) + ic_2(t) |\xi|^2 + \sum_{j=1}^N (b_{1,j}(t) + ic_{1,j}(t)) \xi_i + a_0(t) \right) \hat{v}(t, \xi) \right] e^{ix\xi}$$

$$= \sum_{\xi \in \mathbb{Z}^N} e^{-ij(t, \xi)} \left[ D_t \hat{v}(t, \xi) + \left( a_2(t)|\xi|^2 + \sum_{j=1}^N \sum_{iJ} (b_{1,j}(t) + ic_{1,j}(t)) \xi_i + a_0(t) \right) \hat{v}(t, \xi) \right] e^{ix\xi} = \alpha.$$

Therefore, by setting $u := \Psi^{-1}v \in C \left( [0, T]; H^{r-4-\delta}(T^N) \right)$, we deduce that $\tilde{P}u = \alpha$. Furthermore, it follows from expressions of $J$ and $\Psi$ (see (3.1) and (3.11) respectively) that $J(0, \xi) = 0$ for every $\xi \in \mathbb{Z}^N$, which implies that $u(0, x) = v(0, x) = \beta(x)$ for any $x \in T^N$, and $u$ is a solution for (3.13). The well-posedness of the same problem is a consequence of Remark 2.13.

\[ \square \]

4. A CHARACTERIZATION OF WELL-POSEDNESS THROUGH FOURIER COEFFICIENTS AND ITS CONSEQUENCES

We begin the section by showing that, in order to obtain well-posedness for (1.3), it is sufficient to exhibit a priori inequalities which are reminiscent of energy estimates for the Fourier coefficients (2.9) of the (only) candidate $u$. Next we combine the result with Theorem 3.4 to obtain relations between well-posedness in different frameworks. Finally, we show that well-posedness in Sobolev spaces can be reduced to well-posedness in $L^2$.

**Lemma 4.1.** For arbitrary $t \in [0, T]$ and $\xi \in \mathbb{Z}^N$, consider $\hat{u}(t, \xi)$ given by formula (2.9).

I) Let $r \in \mathbb{R}, f \in C \left( [0, T]; H^r(T^n) \right)$ and $g \in H^r(T^N)$. If there exist constants $C > 0, \rho \geq 0$ such that

$$|\hat{u}(t, \xi)| \leq C(1 + |\xi|)^\rho \left[ \tilde{D}(\xi) + \left( \int_0^t \tilde{f}(s, \xi)^2ds \right)^{1/2} \right], \forall t \in [0, T], \forall \xi \in \mathbb{Z}^N,$n

(4.1)

then for each fixed $t \in [0, T]$ we have $u = \sum_{\xi \in \mathbb{Z}^N} \hat{u}(t, \xi)e^{ix\xi}$ belongs to $H^{r-\rho}(T^N)$ and $u \in C \left( [0, T]; H^{r-2-\rho}(T^N) \right)$.

II) Suppose $f \in C \left( [0, T]; G^{s}(T^N) \right)$ and $g \in G^s(T^N)$, for some $s \geq 1$. In case there exist $C, \delta > 0$ such that

$$|\hat{u}(t, \xi)| \leq Ce^{-\delta|\xi|^1/s}, \forall t \in [0, T], \forall \xi \in \mathbb{Z}^N,$n

(4.2)

it follows that $u = \sum_{\xi \in \mathbb{Z}^N} \hat{u}(t, \xi)e^{ix\xi}$ belongs to $C \left( [0, T]; G^s(T^N) \right)$. 


Proof. If \( f \in C([0, T]; H^r(\mathbb{T}^N)) \) and \( g \in H^r(\mathbb{T}^N) \), there exists \( A > 0 \) such that

\[
\sum_{\xi \in \mathbb{Z}^N} |\hat{g}(\xi)|^2 (1 + |\xi|)^{2r} \leq A \quad \text{and} \quad \max_{s \in [0, T]} \left( \sum_{\xi \in \mathbb{Z}^N} |\hat{f}(s, \xi)|^2 (1 + |\xi|)^{2r} \right) \leq A. \tag{4.3}
\]

Applying (4.1) and Tonelli’s theorem, we infer that

\[
\sum_{\xi \in \mathbb{Z}^N} |\hat{u}(t, \xi)|^2 (1 + |\xi|)^{2r-2|\rho|} \leq 2C^2 \sum_{\xi \in \mathbb{Z}^N} |\hat{g}(\xi)|^2 (1 + |\xi|)^{2r} + 2C^2 \int_0^T \left( \sum_{\xi \in \mathbb{Z}^N} |\hat{f}(s, \xi)|^2 (1 + |\xi|)^{2r} \right) ds
\]

\[
\leq 2AC^2(1 + T), \quad \forall t \in [0, T],
\]

which allows us to deduce that, for each fixed \( t \in [0, T] \), \( u(t, \cdot) \in H^{r-\rho}(\mathbb{T}^N) \).

In order to prove continuity, take arbitrary \( t_0, t_1 \in [0, T] \). As a consequence of (2.6) and the Fundamental Theorem of Calculus, we obtain for any fixed \( \xi \in \mathbb{Z}^N \)

\[
\hat{u}(t_1, \xi) - \hat{u}(t_0, \xi) = (t_1 - t_0) \int_0^1 -i \left( a_2 (t_0 + y(t_1 - t_0)) \xi^2 + \sum_{j=1}^N a_{1,j} (t_0 + y(t_1 - t_0)) \xi_j + a_0 (t_0 + y(t_1 - t_0)) \xi_0 \right) dy.
\tag{4.4}
\]

Since all functions \( a_0, a_{1,j} \) and \( a_2 \) are uniformly bounded we are able to find \( B > 0 \) independent of \( \xi, t_0, t_1 \) and \( y \) such that

\[
\left| -i \left( a_2 (t_0 + y(t_1 - t_0)) \xi^2 + \sum_{j=1}^N a_{1,j} (t_0 + y(t_1 - t_0)) \xi_j + a_0 (t_0 + y(t_1 - t_0)) \right) \right| \leq B(1 + |\xi|)^2, \quad \forall \xi \in \mathbb{Z}^N.
\tag{4.5}
\]

By associating (4.4) to (4.5), besides using the hypothesis and Hölder’s inequality, we conclude that

\[
\left| \hat{u}(t_1, \xi) - \hat{u}(t_0, \xi) \right| \leq |t_1 - t_0| \left[ \int_0^1 B(1 + |\xi|)^2 |\hat{u}(t_0 + y(t_1 - t_0), \xi)| + |\hat{f}(t_0 + y(t_1 - t_0), \xi)| dy \right]
\leq |t_1 - t_0| \int_0^1 B(1 + |\xi|)^2 C(1 + |\xi|)^\rho \left[ |\hat{g}(\xi)| + \left( \int_0^T |\hat{f}(s, \xi)|^2 ds \right)^{1/2} \right] dy + \left| \int_0^1 |\hat{f}(t_0 + y(t_1 - t_0), \xi)| dy \right|
\leq |t_1 - t_0| \left[ BC(1 + |\xi|)^{2+\rho} \left[ |\hat{g}(\xi)| + \left( \int_0^T |\hat{f}(s, \xi)|^2 ds \right)^{1/2} \right] + \left( \int_0^T |\hat{f}(s, \xi)| ds \right) \right]
\leq |t_1 - t_0| D(1 + |\xi|)^{2+\rho} \left[ \left| \hat{g}(\xi)| + \left( \int_0^T |\hat{f}(s, \xi)|^2 ds \right)^{1/2} \right] \right], \quad \forall \xi \in \mathbb{Z}^N,
\]

for some \( D > 0 \) which only depends on \( B, C \) and \( T \).

It follows from (4.3) and Tonelli’s Theorem that

\[
\sum_{\xi \in \mathbb{Z}^N} |\hat{u}(t_1, \xi) - \hat{u}(t_0, \xi)|^2 (1 + |\xi|)^{2r-4-2\rho} \leq \sum_{\xi \in \mathbb{Z}^N} |t_1 - t_0|^2 D^2 \left[ |\hat{g}(\xi)|^2 + \left( \int_0^T |\hat{f}(s, \xi)|^2 ds \right)^{1/2} \right] (1 + |\xi|)^{2r}
\leq |t_1 - t_0|^2 AD^2 + |t_1 - t_0|^2 ATD^2.
\]
Thus, for any $t_0, t_1 \in [0, T]$, we obtain $\|u(t_1, x) - u(t_0, x)\|_{r-2-p} \leq M|t_1 - t_0|$, for a constant $M > 0$ that only depends on $T$, $A$, $B$ and $C$. Therefore $u \in C\left([0, T]; H^{r-2-\rho}(\mathbb{T}^N)\right)$, as we intended to prove.

We advance to the Gevrey setting; it follows immediately from (1.3) that, for every fixed $t \in [0, T]$, $u(t, x) \in G^s(\mathbb{T}^N)$. It remains to prove continuity; by proceeding analogously to the first case, one has

$$|\hat{u}(t_1, \xi) - \hat{u}(t_0, \xi)| \leq |t_1 - t_0| \left[ \int_0^1 B(1 + |\xi|)^2|\hat{u}(t_0 + y(t_1 - t_0), \xi)| + |\hat{f}(t_0 + y(t_1 - t_0), \xi)| \, dy \right].$$

(4.6)

Applying once again Lemma 2.3 and Theorem 2.6 there exist $C', \sigma > 0$ such that

$$|\hat{f}(t, \xi)| \leq C'e^{-\sigma|\xi|^{1/\omega}}, \quad \forall t \in [0, T], \forall \xi \in \mathbb{Z}^N.$$  \hspace{1cm} (4.7)

Take $C_1 = \max\{C, C'\}$ and $\omega = \min\{\sigma, \delta\}$; it is a consequence of (1.3), (4.6) and (4.7) that

$$|\hat{u}(t_1, \xi) - \hat{u}(t_0, \xi)| \leq |t_1 - t_0|C_2e^{-\omega/2|\xi|^{1/\omega}}, \quad \forall \xi \in \mathbb{Z}^N,$$

where the constant $C_2$ depends only on $B, C, C', \sigma$ and $\delta$. The result is a direct consequence of Lemma 2.4 and the proof is complete. \hfill \square

**Proposition 4.2.** Let $P$ be the operator described in (1.2).

a) If there is $\delta \geq 0$ such that (1.3) is well-posed in $H^{r}$ with loss of $\delta$ derivatives then it is well-posed in $C^\infty$.

b) If (1.3) is well-posed in $C^\infty$, it is also well-posed in $G^s$, for every $s \geq 1$.

c) Suppose $s_1, s_2 \in \mathbb{R}$ such that $1 \leq s_2 \leq s_1$. If (1.3) is well-posed in $G^{s_1}$, the same holds in $G^{s_2}$.

**Proof.** Implication a) is pretty standard; we proceed to statement b) If (1.3) is well-posed in $C^\infty$ it follows from Theorem 3.4 the well-posedness in $C^\infty$ for (3.13), with $P$ set in (3.12). Fix $s \geq 1$ and take

$$f_1 \equiv 0, \quad g_1(x) = \sum_{\xi \in \mathbb{Z}^N} e^{-|\xi|^{1/\omega}}e^{ix\xi}.$$ 

Then $g_1 \in C^\infty(\mathbb{T}^N)$ and $\hat{g}_1(\xi) = e^{-|\xi|^{1/\omega}}$, for each $\xi \in \mathbb{Z}^N$. By hypothesis there exists a unique $u_1 \in C\left([0, T]; C^\infty(\mathbb{T}^N)\right)$ which solves

$$\begin{align*}
\tilde{P}u_1(t, x) &= 0, \\
u_1(0, x) &= g_1(x), \quad \forall t \in [0, T], \forall x \in \mathbb{T}^N.
\end{align*}$$

In this particular case, we write $u_1 = \sum_{\xi \in \mathbb{Z}^N} \hat{u}_1(t, \xi)e^{ix\xi}$ and obtain (analogously from (2.9))

$$\hat{u}_1(t, \xi) = e^{-|\xi|^{1/\omega}} \exp \left( C_2(t)|\xi|^2 + \sum_{j=1}^N C_{1,j}(t)\xi_j \right), \quad \forall t \in [0, T], \forall \xi \in \mathbb{Z}^N.$$  \hspace{1cm} (4.8)

By the fact that $u_1 \in C\left([0, T]; C^\infty(\mathbb{T}^N)\right)$, one infers the existence of $M_1 > 0$ such that

$$Y_1(t, \xi) \leq M_1 e^{\rho |\xi|^{1/\omega}}, \quad \forall t \in [0, T], \forall \xi \in \mathbb{Z}^N.$$  \hspace{1cm} (4.9)
Now set \( f_2(t, x) = -i \sum_{\xi \in \mathbb{Z}^N} e^{-|\xi|^{1/2s}} e^{ix\xi} \) and \( g_2 \equiv 0 \). Similarly, \( f_2 \in C \left( [0, T]; C^\infty(\mathbb{T}^N) \right) \) and there exists a unique \( u_2 = \sum_{\xi \in \mathbb{Z}^N} \tilde{u}_2(t, \xi)e^{ix\xi} \) in \( C \left( [0, T]; C^\infty(\mathbb{T}^N) \right) \) for which

\[
\begin{cases}
    \tilde{P}u_2(t, x) = f_2(t, x) \\
    u_2(0, x) = 0
\end{cases} \quad \forall t \in [0, T], \forall x \in \mathbb{T}^N.
\]

Then

\[
\tilde{u}_2(t, \xi) = e^{-|\xi|^{1/2s}} \int_0^t \exp \left[ \begin{array}{c}
    (C_2(t) - C_2(s)) |\xi|^2 + \sum_{j=1}^N (C_{1,j}(t) - C_{1,j}(s)) \xi_j \\
    \end{array} \right] ds,
\]

and there exists \( M_2 > 0 \) such that

\[
T_2(t, \xi) \leq M_2 e^{|\xi|^{1/2s}} \quad \forall t \in [0, T], \forall \xi \in \mathbb{Z}^N.
\]

Suppose \( f \in C \left( [0, T]; G^s(\mathbb{T}^N) \right) \) and \( g \in G^s(\mathbb{T}^N) \). Then there exist \( M, \delta > 0 \) such that

\[
|\tilde{g}(\xi)| \leq Me^{-\delta|\xi|^{1/2}} \quad \text{and} \quad |\tilde{f}(t, \xi)| \leq Me^{-\delta|\xi|^{1/2}}, \quad \forall t \in [0, T], \forall \xi \in \mathbb{Z}^N.
\]

Let \( u \in C \left( [0, T]; C^\infty(\mathbb{T}^N) \right) \) be the unique solution for

\[
\begin{cases}
    \tilde{P}u(t, x) = f(t, x), \\
    u(0, x) = g(x),
\end{cases} \quad \forall t \in [0, T], \forall x \in \mathbb{T}^N.
\]

It follows from (2.9) that

\[
\tilde{u}(t, \xi) = \tilde{g}(\xi) \exp \left( C_2(t)|\xi|^2 + \sum_{j=1}^N C_{1,j}(t)\xi_j \right) + \\
+ i \int_0^t \tilde{f}(s, \xi) \exp \left[ (C_2(t) - C_2(s)) |\xi|^2 + \sum_{j=1}^N (C_{1,j}(t) - C_{1,j}(s)) \xi_j \right] ds, \quad \forall t \in [0, T], \forall \xi \in \mathbb{Z}^N.
\]

By associating (4.13) to (4.8), (4.9), (4.10), (4.11) and (4.12), we deduce that

\[
|\tilde{u}(t, \xi)| \leq |\tilde{g}(\xi)| T_1(t, \xi) + \left( \max_{r \in [0,T]} |\tilde{f}(r, \xi)| \right) T_2(r, \xi) \leq Me^{-\delta|\xi|^{1/2}} \left( M_1 e^{|\xi|^{1/2s}} + M_2 e^{|\xi|^{1/2s}} \right)
\]

\[
\leq M_3 e^{-\delta|\xi|^{1/2}} e^{|\xi|^{1/2s}} \leq M_4 e^{-\frac{3}{2}\delta|\xi|^{1/2}}, \quad \forall t \in [0, T], \forall \xi \in \mathbb{Z}^N,
\]

for some constant \( M_4 > 0 \) that depends only on \( M, M_1, M_2 \) and \( \delta \). We infer from Lemma 4.1 that the initial-value problem for \( \tilde{P} \) is well-posed in \( G^s \). The result is then a consequence of Theorem 3.4. Finally, (6) is proved in an analogous manner.

\hspace{1cm} \square

**Corollary 4.3.** If the problem (1.3) is not well-posed in \( C^\omega = G^1 \), it is not well-posed in any of the \( G^s \), \( C^\infty \) or \( H^r \) spaces.

Since the coefficients operator \( P \) do not depend on space variables, it is not difficult to verify the next result.
Proposition 4.4. Consider $P$ the operator given in (1.2) and assume the existence of $\kappa \in \mathbb{R}$, $\sigma \geq 0$ such that for any $f \in C \left([0,T]; H^\kappa(\mathbb{T}^N)\right)$ and $g \in H^\kappa(\mathbb{T}^N)$ there exists $u \in C \left([0,T]; H^{\kappa-\sigma}(\mathbb{T}^N)\right)$ which solves (1.3); then (1.3) is well-posed in $H^\kappa$ with loss of $\sigma$ derivatives. In particular, well-posedness in $H^\kappa$ is equivalent to well-posedness in $L^2$.

5. ILL-POSEDNESS FOR A FAMILY OF OPERATORS

In the present section we show that if $\text{Im}(a_2)$ is positive at any point of the interval $[0,T]$, (1.3) is never well-posed in $C^\omega$ and hence in any of the settings previously defined (Corollary 4.3). Recall that Theorem (3.4) allows us to consider a simpler operator, described in (3.12). In this situation, if $u$ solves

$$
\begin{align*}
\left\{ \begin{array}{l}
\tilde{P}u(t,x) = f(t,x), \\
u(0,x) = g(x),
\end{array} \right. 
\forall t \in [0,T], \forall x \in \mathbb{T}^N, \tag{5.1}
\end{align*}
$$

it follows from a analogous computation made in (2.9) that

$$
\tilde{u}(t,\xi) = \tilde{g}(\xi) \exp \left[ C_2(t)\xi^2 + \sum_{j=1}^{N} C_{1,j}(t)\xi_j \right] + 
+ i \int_0^t \tilde{f}(s,\xi) \exp \left[ (C_2(t) - C_2(s))\xi^2 + \sum_{j=1}^{N} (C_{1,j}(t) - C_{1,j}(s))\xi_j \right] ds, \quad \forall t \in [0,T], \forall \xi \in \mathbb{Z}^N. \tag{5.2}
$$

Theorem 5.1. Suppose $t^* \in [0,T]$ such that $c_2(t^*) > 0$. Then there exist $f(t,x) \in C \left([0,T]; C^\omega(\mathbb{T}^N)\right)$ and $g(x) \in C^\omega(\mathbb{T}^N)$ for which there is no solution in $C \left([0,T]; C^\omega(\mathbb{T}^N)\right)$ for (5.1).

Proof. Since $c_2$ is continuous, we may assume $t^* > 0$. Furthermore we are able to find $\delta, \varepsilon > 0$ such that $[t^* - \delta, t^*] \subset [0,T]$ and

$$
c_2(t) \geq \varepsilon, \quad \forall t \in [t^* - \delta, t^*]. \tag{5.3}
$$

Let $g(x) \equiv 0$ and $f(t,x) = f(x) = \sum_{\xi \in \mathbb{Z}^N} e^{-|\xi|} e^{i\xi x}$. It follows from (5.2) that

$$
|\tilde{u}(t,\xi)| = e^{-|\xi|} \int_0^t \exp \left[ (C_2(t) - C_2(s))|\xi|^2 + \sum_{j=1}^{N} (C_{1,j}(t) - C_{1,j}(s))\xi_j \right] ds, \quad \forall t \in [0,T], \forall \xi \in \mathbb{Z}^N. \tag{5.4}
$$

On the other hand, observe that

$$
C_2(t) - C_2(s) = \int_0^t c_2(r)dr - \int_s^t c_2(r)dr = \int_s^t c_2(r)dr. \tag{5.5}
$$

Hence by associating (5.3) and (5.5) to (5.4) we infer that, for every $\xi \in \mathbb{Z}^N$,

$$
|\tilde{u}(t^*,\xi)| = e^{-|\xi|} \int_0^{t^*} \exp \left[ (C_2(t^*) - C_2(s))|\xi|^2 + \sum_{j=1}^{N} (C_{1,j}(t^*) - C_{1,j}(s))\xi_j \right] ds 
\geq e^{-|\xi|} \int_{t^* - \delta}^{t^* - \delta/2} \exp \left[ (C_2(t^*) - C_2(s))|\xi|^2 + \sum_{j=1}^{N} (C_{1,j}(t^*) - C_{1,j}(s))\xi_j \right] dsds 
\geq e^{-|\xi|} \int_{t^* - \delta}^{t^* - \delta/2} \exp \left[ c(t^* - s)|\xi|^2 + \sum_{j=1}^{N} (C_{1,j}(t^*) - C_{1,j}(s))\xi_j \right] ds.
$$
\[ \geq \exp \left( \frac{\varepsilon \delta}{2} |\xi|^2 - |\xi| \right) \int_{t^*-\delta/2}^{t^*} \exp \left[ \sum_{j=1}^{N} (C_{1,j}(t^*) - C_{1,j}(s)) \xi_j \right] ds. \]

Furthermore \(|C_1(t)| \leq \int_0^t |c_{1,j}(r)| dr \leq T \max_{r \in [0,T]} |c_{1,j}(r)|.\) This implies that, for some \(M > 0,\)
\[ |C_1(t)| \leq M, \quad \forall t \in [0, T], \quad \forall j \in \{1, 2, \ldots, N\}. \]

Hence
\[ |\hat{u}(t^*, \xi)| \geq \exp \left( \frac{\varepsilon \delta}{2} |\xi|^2 - |\xi| \right) \int_{t^*-\delta/2}^{t^*} \exp (-2M|\xi|) ds \geq \frac{\delta}{2} \exp \left( \frac{\varepsilon \delta}{2} |\xi|^2 - (2M + 1)|\xi| \right), \quad \forall \xi \in \mathbb{Z}^N. \]

Therefore, there exists \(n_0 \in \mathbb{N}\) large enough so that
\[ |\hat{u}(t^*, \xi)| \geq \frac{\delta}{2} \exp \left( \frac{\varepsilon \delta}{4} |\xi|^2 \right), \quad \forall \xi \in \mathbb{Z}^N, \; |\xi| \geq n_0. \]

Hence the formal solution \(u = \sum_{\xi \in \mathbb{Z}^N} \hat{u}(t, \xi)e^{ix\xi}\) is not even a function when \(t = t^*,\) ending the proof. \(\square\)

As a consequence of Theorem \((5.4),\) we have just proved the following

**Theorem 5.2.** Let \(P\) be the operator defined in \((1.2).\) If there exists \(t^* \in [0, T]\) such that
\[ \text{Im}(a_2(t^*)) = c_2(t^*) > 0, \]
then \((1.3)\) is ill-posed in \(C^\omega,\) which implies it is ill-posed in any of the \(G^s, C^{\infty}\) or \(H^r\) spaces.

**Corollary 5.3.** In order for \((1.3)\) to be well-posed in any of the spaces aforementioned it is necessary that \(\text{Im}(a_2(t)) = c_2(t) \leq 0, \; \forall t \in [0, t].\)

Right below we show that such condition is far from sufficient.

**Theorem 5.4.** Suppose the existence of \(t^* \in [0, T]\) and \(\delta > 0\) such that
\begin{itemize}
  \item \(c_2 \equiv 0\) in \([t^*, t^* + \delta];\)
  \item \(c_{1,j} \neq 0\) in \([t^*, t^* + \delta],\) for some \(j \in \{1, 2, \ldots, N\}.\)
\end{itemize}
Then there exist \(f(t, x) \in C([0, T]; C^\omega(T^N))\) and \(g(x) \in C^\omega(T^N)\) for which there is no solution in \(C([0, T]; C^\omega(T^N))\) for \((5.1).\)

**Proof.** With no loss of generality we may assume that \(j = 1\) and (by continuity) that \(c_{1,1}\) is either strictly positive or negative in \([t^*, t^* + \delta].\) Consider the case where \(c_{1,1}\) is positive (the other is completely analogous); then there exists \(\varsigma > 0\) such that \(c_{1,1}(t) \geq \varsigma, \) for every \(t \in [t^*, t^* + \delta].\) Define then
\[ f(t, x) = -i \sum_{\eta \in \mathbb{Z}_+} e^{-\frac{\varsigma t}{2}} e^{ix_1 \eta}, \quad g(x) \equiv 0. \]

Applying \((5.2),\) one obtains
\[ \hat{u}(t, \eta, 0) = e^{-\frac{\varepsilon \varsigma t}{2}} \int_0^t \exp \left[ (C_2(t) - C_2(s)) \eta^2 + (C_{1,1}(t) - C_{1,1}(s)) \eta \right] ds, \quad \forall t \in [0, T], \; \forall \eta \in \mathbb{Z}_+. \]
Theorem 6.2. Applying (2.8) and (2.9), it follows immediately that

\[ |\tilde{u}(t^* + \delta, \eta, 0)| \geq e^{-\frac{\delta \eta^2}{2}} \int_{t^*}^{t^* + \delta} \exp \left( \int_s^{t^* + \delta} c_2(r) dr \right) \eta^2 + \left( \int_s^{t^* + \delta} c_{1,1}(r) dr \right) \eta \] 

\[ = e^{-\frac{\delta \eta^2}{2}} \int_{t^*}^{t^* + \delta} \exp \left( \int_s^{t^* + \delta} c_{1,1}(r) dr \right) \eta \] 

\[ \geq e^{-\frac{\delta \eta^2}{2}} \int_{t^*}^{t^* + \delta} \exp [\zeta (t^* + \delta - s) \eta] ds \geq e^{-\frac{\delta \eta^2}{2}} \int_{t^*}^{t^* + \delta} \exp [\zeta (t^* + \delta - s) \eta] ds \]

\[ \geq e^{-\frac{\delta \eta^2}{2}} \int_{t^*}^{t^* + \delta} e^{\frac{\delta \eta^2}{2}} ds \geq \frac{\delta e^{\frac{\delta \eta^2}{2}}}{2}, \quad \forall \eta \in \mathbb{Z}_+. \]

Since \( \delta, \varsigma \) and \( \eta \) are non-negative numbers, the formal solution \( u = \sum_{\xi \in \mathbb{Z}^N} \hat{u}(t, \xi) e^{i\xi \xi} \) is not even a function when \( t = t^* + \delta \), which closes the proof. \( \square \)

Corollary 5.5. Let \( P \) be the operator defined in (1.2). If there exist \( t^* \in [0, T] \) and \( \delta > 0 \) such that

\[ \text{Im}(a_2) = c_2 \equiv 0 \text{ in } [t^*, t^* + \delta]; \quad \text{Im}(a_{1,j}) = c_{1,j} \not\equiv 0 \text{ in } [t^*, t^* + \delta], \text{ for some } j \in \{1, 2, \ldots, N\}. \]

Then (1.3) is ill-posed in \( C^\omega \), which implies it is ill-posed in any of the \( G^s \), \( C^\infty \) or \( H^r \) spaces.

6. WELL-POSEDNESS FOR A FAMILY OF OPERATORS

So far we have only found families of operators for which problem (1.3) is ill-posed. In this subsection we prove well-posedness in all aforementioned settings for two different cases, with the most important of them being the situation where \( \text{Im}(a_2) \) is strictly negative everywhere.

Remark 6.1. Since we could replace \( P \) by \( iP \) with no loss of generality in (1.3), we would have an operator in the form \( \partial_t + Q(t, D_x) \), and there exist \( C, R > 0 \) such that

\[ \text{Re}(Q(t, \xi)) \geq C|\xi|^2, \quad |\xi| \geq R. \]

This means that \( Q \) is a strongly elliptic symbol and hence that \( iP \) is a parabolic operator.

Recall that our only candidate for solution has its Fourier coefficients given in (2.9).

Theorem 6.2. Let \( P \) be the operator defined in (1.2) and assume that

\[ \text{Im}(a_2)(t) = c_2(t) < 0, \text{ for every } t \in [0, T]. \]

Then (1.3) is well-posed in \( H^r \).

Proof. Applying (2.8) and (2.9), it follows immediately that

\[ |\tilde{u}(t, \xi)| \leq |\tilde{g}(\xi)| \exp \left( C_2(t)|\xi|^2 + \sum_{j=1}^{N} C_{1,j}(t) \xi_j + C_0(t) \right) + \]

\[ + \int_0^t |f(s, \xi)| \exp \left( (C_2(t) - C_2(s)) |\xi|^2 + \sum_{j=1}^{N} (C_{1,j}(t) - C_{1,j}(s)) \xi_j + (C_0(t) - C_0(s)) \right) ds. \quad (6.1) \]
Using the hypothesis, the continuity of the coefficients and the compactness of \([0, T]\), we deduce the existence of \(\varepsilon, M > 0\) such that

\[ c_2(t) \leq -\varepsilon, \quad |c_{1,j}(t)| \leq M, \quad |c_0(t)| \leq M, \quad \forall j \in \{1, 2, \ldots, N\}, \quad \forall t \in [0, T]. \]

Hence for every \(0 \leq s \leq t \leq T\), we have

\[ C_2(t) - C_2(s) \leq -\varepsilon(t - s), \quad C_{1,j}(t) - C_{1,j}(s) \leq M(t - s), \quad C_0(t) - C_0(s) \leq M(t - s). \quad (6.2) \]

Associating (6.1) to (6.2), one obtains

\[ |\tilde{u}(t, \xi)| \leq e^{MT} \left| \tilde{g}(\xi) \exp \left[ t (-\varepsilon|\xi|^2 + MN|\xi|) \right] + \int_0^t |\tilde{f}(s, \xi)| \exp \left[ (t - s) (-\varepsilon|\xi|^2 + MN|\xi|) \right] ds \right|. \quad (6.3) \]

Since \(\varepsilon\) is positive, there exists \(L > 0\) such that

\[ (-\varepsilon|\xi|^2 + MN|\xi|) \leq L, \quad \forall \xi \in \mathbb{Z}^N. \quad (6.4) \]

By applying (6.3), (6.4) and Hölder’s inequality, we infer that

\[ |\tilde{u}(t, \xi)| \leq e^{MT} \left| \tilde{g}(\xi) e^{LT} + \int_0^t |\tilde{f}(s, \xi)| e^{L(t-s)ds} \right| \leq e^{(L+M)T} \left| \tilde{g}(\xi) \right| + \int_0^t |\tilde{f}(s, \xi)| ds \]

\[ \leq C \left[ \left| \tilde{g}(\xi) \right| + \left( \int_0^T |\tilde{f}(s, \xi)|^2 ds \right)^{1/2} \right], \quad \forall t \in [0, T], \forall \xi \in \mathbb{Z}^N, \quad (6.5) \]

for some \(C > 0\). By Lemma 4.1, \(u \in C([0, T]; H^{r-2}(\mathbb{T}^N))\).

We claim that \(u\) is actually an element of \(C \left( [0, T]; H^r(\mathbb{T}^N) \right)\), a fact that will be proved through an approximation argument. Fix \(f \in C([0, T]; H^r(\mathbb{T}^N))\), \(g \in H^r(\mathbb{T}^N)\), and let \(\{f_j\}_{j \in \mathbb{N}} \subset C([0, T]; C^\infty(\mathbb{T}^N))\), \(\{g_j\}_{j \in \mathbb{N}} \subset C^\infty(\mathbb{T}^N)\) be sequences such that

\[ \lim_{j \to \infty} f_j = f \text{ in } C([0, T]; H^r(\mathbb{T}^N)), \quad \lim_{j \to \infty} g_j = g \text{ in } H^r(\mathbb{T}^N). \]

It follows from (6.3) and Proposition 4.2 that the Cauchy problem (1.3) with initial data \(f_j\) and \(g_j\) admits a unique solution \(u_j \in C([0, T]; C^\infty(\mathbb{T}^N))\).

Hence it is a consequence of Tonelli’s theorem the existence of \(C_1 > 0\) such that, for any \(\sigma \in \mathbb{R}\),

\[ \|u_j(t)\|^2_{H^r(\mathbb{T}^N)} \leq C_1 \left( \|g_j\|^2_{H^r(\mathbb{T}^N)} + \int_0^T \|f_j(s)\|^2_{H^r(\mathbb{T}^N)} ds \right), \quad t \in [0, T]. \]

Since \(u_j - u_k\) is the solution of the problem (1.3), with data \(f_j - f_k\) and \(g_j - g_k\), we obtain

\[ \|(u_j - u_k)(t)\|^2_{H^r(\mathbb{T}^N)} \leq C_1 \left( \|g_j - g_k\|^2_{H^r(\mathbb{T}^N)} + \int_0^T \|(f_j - f_k)(s)\|^2_{H^r(\mathbb{T}^N)} ds \right), \quad \forall t \in [0, T]. \]

In particular, \(\{u_j\}_{j \in \mathbb{N}}\) is a Cauchy sequence in \(C([0, T]; H^r(\mathbb{T}^N))\); let \(v\) be the limit of \(\{u_j\}_{j \in \mathbb{N}}\). Then \(v \in C([0, T]; H^r(\mathbb{T}^N))\) and solves (1.3). From the uniqueness of the solutions one concludes that \(v = u\), which finalizes the proof.

**Corollary 6.3.** Consider \(P\) the operator described in (1.2) and suppose that

\[ \text{Im}(a_2)(t) = c_2(t) < 0, \quad \text{for every } t \in [0, T]. \]

In this case, we obtain the following properties:

- Given \(r \in \mathbb{R}, f \in C([0, T]; H^r(\mathbb{T}^N))\) and \(g \in H^r(\mathbb{T}^N)\), there exists a unique \(u \in C \left( [0, T]; H^r(\mathbb{T}^N) \right) \cap C^1 \left( [0, T]; H^{r-2}(\mathbb{T}^N) \right)\) which solves (1.3).
• For every \( f \in C \left( [0, T]; C^\infty (\mathbb{T}^N) \right) \) and \( g \in C^\infty (\mathbb{T}^N) \), there exists a unique \( u \in C^1 \left( [0, T]; C^\infty (\mathbb{T}^N) \right) \) which solves (1.3).

• Given \( s \geq 1 \), \( f \in C \left( [0, T]; G^s (\mathbb{T}^N) \right) \) and \( g \in G^s (\mathbb{T}^N) \), there exists a unique \( u \in C^1 \left( [0, T]; G^s (\mathbb{T}^N) \right) \) which solves (1.3).

Proof. By proceeding just like it was done in order to obtain (6.1) we have

\[
\text{(1.3)}
\]

It is a direct consequence of Theorem 6.2, Proposition 4.2 and Remark 3.2. \( \square \)

Similarly to previous subsection, we exhibit a class of operators where \( \text{Im}(a_2) \) is not necessarily strictly negative but we still obtain well-posedness in the Sobolev setting.

**Theorem 6.4.** Let \( P \) be the operator defined in (1.2) and assume that

\[
\text{Im}(a_2)(t) = c_2(t) \leq 0 \text{ and } \text{Im}(a_{1,j})(t) = c_{1,j}(t) \equiv 0, \text{ for every } j \in \{1, 2, \ldots, N\} \text{ and } t \in [0, T].
\]

Then (1.3) is well-posed in \( H^r \).

Proof. By proceeding just like it was done in order to obtain (6.1) we have

\[
|\hat{u}(t, \xi)| \leq |\hat{g}(\xi)| \exp \{C_2(t)|\xi|^2 + C_0(t)\} + \int_0^t |\hat{f}(s, \xi)| \exp \{(C_2(t) - C_2(s)) |\xi|^2 + (C_0(t) - C_0(s))\} ds.
\]

Take \( M > 0 \) such that \( |c_0(t)| \leq M, \forall t \in [0, T] \). Then

\[
|\hat{u}(t, \xi)| \leq e^{MT} \exp \{C_2(t)|\xi|^2 + \int_0^t |\hat{f}(s, \xi)| e^{(C_2(t) - C_2(s)) |\xi|^2} ds\}.
\]

Since \( C_2 \) is always non-positive and non-increasing, it follows that

\[
|\hat{u}(t, \xi)| \leq e^{MT} \exp \{C_2(t)|\xi|^2 + \int_0^t |\hat{f}(s, \xi)| ds\}, \forall t \in [0, T], \forall \xi \in \mathbb{Z}^N.
\]

Finally it suffices proceed just like in the end of the proof of Theorem 6.2. \( \square \)

**Corollary 6.5.** Consider \( P \) the operator described in (1.2) and suppose that

\[
\text{Im}(a_2)(t) = c_2(t) \leq 0 \text{ and } \text{Im}(a_{1,j})(t) = c_{1,j}(t) \equiv 0, \text{ for every } j \in \{1, 2, \ldots, N\} \text{ and } t \in [0, T].
\]

Then the following statements hold true.

• Given \( r \in \mathbb{R}, f \in C \left( [0, T]; H^r (\mathbb{T}^N) \right) \) and \( g \in H^r (\mathbb{T}^N) \), there exists a unique \( u \in C \left( [0, T]; H^r (\mathbb{T}^N) \right) \cap C^1 \left( [0, T]; H^{r-2} (\mathbb{T}^N) \right) \) which solves (1.3).

• For every \( f \in C \left( [0, T]; C^\infty (\mathbb{T}^N) \right) \) and \( g \in C^\infty (\mathbb{T}^N) \), there exists a unique \( u \in C^1 \left( [0, T]; C^\infty (\mathbb{T}^N) \right) \) that solves (1.3).

• Given \( s \geq 1, f \in C \left( [0, T]; G^s (\mathbb{T}^N) \right) \) and \( g \in G^s (\mathbb{T}^N) \), there exists a unique \( u \in C^1 \left( [0, T]; G^s (\mathbb{T}^N) \right) \) which solves (1.3).

7. A class of degenerate operators

Let us make a brief analysis of some results proved in last sections; Corollary 5.3 shows that one only has to deal with problems where \( \text{Im}(a_2) = c_2 \leq 0 \), whilst Theorem 6.2 covers case \( c_2 < 0 \). Thus the only situations left are those where \( c_2 \) vanishes at some point. Taking a closer look at Theorems 5.4 and 6.4 allows us to conjecture about how the comparison of order of vanishing between \( c_2 \) and the elements of \( \{c_{1,1}, c_{1,2}, \ldots, c_{1,N}\} \) is connected to the well-posedness of (1.3). Roughly speaking, it seems that if \( c_2 \) vanishes to a much higher order than some \( c_{1,j} \), the problem is ill-posed. On the other hand, if each \( c_{1,j} \) vanishes to a much higher order than \( c_2 \), (1.3) is well-posed.
We intend to show in this section that such intuitive ideas are not that far from the truth, even though the relations of well-posedness and ill-posedness are more intricate than the almost binary results obtained so far. In order to make things precise, we only consider situations where the imaginary part of the leading coefficient of $P$ given in (1.2) has a finite number of zeros. Moreover the following hypotheses are assumed:

1. $\text{Im } a_2(t) = c_2(t) \leq 0$, for every $t \in [0, T]$.
2. There exists a finite set $\{t_1, t_2, \ldots, t_m\} \subset [0, T]$, with $t_1 < t_2 < \ldots < t_m$, such that

$$\text{Im } a_2(t) = c_2(t) = 0 \iff t \in \{t_1, t_2, \ldots, t_m\}.$$ 

3. There exists $\delta > 0$ such that for every $k \in \{1, 2, \ldots, m\}$ we are able to write

$$\text{Im } a_2(t) = c_2(t) = \alpha^k(t) \cdot |t - t_k|^{p_k}, \forall t \in [t_k - \delta, t_k + \delta] \cap [0, T],$$

where $p_k > 0$, $\alpha^k$ is bounded and there exists $\varepsilon > 0$ such $\alpha^k \leq -\varepsilon$ on the same set.

4. For each $j \in \{1, 2, \ldots, N\}$ and $k \in \{1, \ldots, m\}$, we are able to write

$$\text{Im } a_{1,j}(t) = c_{1,j}(t) = \beta^{j,k}(t) \cdot |t - t_k|^{q_{j,k}}, \forall t \in [t_k - \delta, t_k + \delta] \cap [0, T],$$

with $q_{j,k} \geq 0$ and $\beta^{j,k}$ a bounded function that does not vanish on the same set.

**Remark 7.1.** Condition (7.1) implies that each zero of $c_2$ has positive and finite order; (7.2) is similar to (7.1), but it also includes the situation where $c_{1,j}$ does not vanish at some $t_k$.

**Remark 7.2.** It is worth mentioning that not every continuous function satisfies (7.1) or (7.2). In addition to the classic example $f : [0, 1] \to \mathbb{R}_-$; $f(t) = \begin{cases} -e^{-1/t^2}, & \text{if } t \neq 0 \\ 0, & \text{if } t = 0 \end{cases}$, of a function that vanishes to infinite order at the origin, we also have $g : [0, 1/2] \to \mathbb{R}_-$; $g(t) = \begin{cases} 1/\log t, & \text{if } t \neq 0; \\ 0, & \text{if } t = 0, \end{cases}$, which is clearly continuous but $\lim_{t \to 0^+} \left( \frac{g(t)}{t^\alpha} \right) = -\infty$, for each $r > 0$. However, note that this last example could not happen if the function $g$ were for Hölder continuous for some $\alpha > 0$, for instance.

Using the continuity of $c_2$, we obtain (once again for the same $\delta > 0$) the following:

5. There exists $\omega > 0$ such that

$$t \in [0, T], \ |t - t_k| \geq \frac{\delta}{2}, \forall k \in \{1, 2, \ldots, m\} \implies c_2(t) \leq -\omega.$$ 

**Theorem 7.3.** Let $P$ be the operator defined in (1.2) and set for each $k \in \{1, 2, \ldots, m\}$

$$q_k = \min \{q_{1,k}, \ldots, q_{N,k} \},$$

with $q_{j,k}$ as in (7.2). Then the following statements hold true:

I) If $p_k \leq 2q_k + 1$ for every $k \in \{1, 2, \ldots, m\}$, then (1.3) is well-posed in $H^r$.

II) Suppose that $p_k > 2q_k + 1$ for some $k \in \{1, 2, \ldots, m\}$. Then (1.3) is ill-posed in $C^\infty$. Furthermore if we define

$$\mathcal{K} = \{1 \leq k \leq m; \ p_k > 2q_k + 1 \},$$

then (1.3) is well-posed in $G^\tau$ if and only if

$$\tau < \min_{k \in \mathcal{K}} \left( \frac{p_k - q_k}{p_k - 2q_k - 1} \right).$$
**Proof of (7.3)** By proceeding just like we did in Subsection 6, it follows in particular from (6.1) that

\[ |\hat{u}(t, \xi)| \leq |\hat{g}(\xi)| \exp \left( C_2(t)|\xi|^2 + \sum_{j=1}^{N} C_{1,j}(t)\xi_j + C_0(t) \right) + \]

\[ + \int_0^t |\hat{f}(s, \xi)| \exp \left( (C_2(t) - C_2(s))|\xi|^2 + \sum_{j=1}^{N} (C_{1,j}(t) - C_{1,j}(s))\xi_j + (C_0(t) - C_0(s)) \right) \] \[ \hspace{1cm} \] \[ ds, \]

for any \( t \in [0, T] \) and \( \xi \in \mathbb{Z}^N \). Take \( M > 0 \) such that

\[ |c_2|, |c_{1,1}|, \ldots, |c_{1,N}|, |c_0| \] are all uniformly bounded by \( \frac{M}{N} \). (7.7)

Then

\[ |\hat{u}(t, \xi)| \leq e^{MT} |\hat{g}(\xi)| \exp \left( C_2(t)|\xi|^2 + \sum_{j=1}^{N} C_{1,j}(t)\xi_j \right) + \]

\[ + e^{MT} \int_0^t |\hat{f}(s, \xi)| \exp \left( (C_2(t) - C_2(s))|\xi|^2 + \sum_{j=1}^{N} (C_{1,j}(t) - C_{1,j}(s))\xi_j \right) \] \[ \hspace{1cm} \] \[ ds. \] (7.8)

Our goal is to use (7.8) to obtain an estimate similar to (4.1), with \( \rho = 0 \). Then it would be sufficient to repeat the arguments made in the end of the proof of Theorem 6.2.

Take \( t \in [0, T] \) and suppose in a first moment that

\[ |t - t_k| \geq \frac{3\delta}{4}, \quad \forall k \in \{1, 2, \ldots, m\}. \] (7.9)

If \( t \leq \frac{\delta}{4} \), by putting (7.3) and (7.7) into (7.8) and applying triangular inequality, we obtain

\[ |\hat{u}(t, \xi)| \leq e^{MT} \left[ |\hat{g}(\xi)| \exp \left[ (-\omega|\xi|^2 + M|\xi|)t \right] + \int_0^t |\hat{f}(s, \xi)| \exp \left[ (-\omega|\xi|^2 + M|\xi|)(t - s) \right] ds \right]. \]

Since there exists \( L > 0 \) such that \( -\omega|\xi|^2 + M|\xi| \leq L \) for any \( \xi \in \mathbb{Z}^N \), using Hölder’s inequality we deduce that

\[ |\hat{u}(t, \xi)| \leq e^{MT} \left[ |\hat{g}(\xi)|e^{\frac{\omega t^2}{4}} + \int_0^t |\hat{f}(s, \xi)|e^{\frac{\omega t}{4}} ds \right] \leq C \left[ |\hat{g}(\xi)| + \left( \int_0^T |\hat{f}(s, \xi)|^2 ds \right)^{1/2} \right], \quad \forall \xi \in \mathbb{Z}^N, \]

for some constant \( C > 0 \) which only depends on \( M, T \) and \( \omega \).

When \( t > \frac{\delta}{4} \), it follows from (7.9) that \( \left| t - \frac{\delta}{4} - t_k \right| \geq |t - t_k| - \frac{\delta}{2} \geq \frac{\delta}{2}, \forall k \in \{1, 2, \ldots, m\} \). Applying (7.3) and (7.8), we infer that

\[ |\hat{u}(t, \xi)| \leq e^{MT} |\hat{g}(\xi)| \exp \left[ (C_2(t) - C_2(t - \frac{\delta}{4}))|\xi|^2 + MT|\xi| \right] \]

\[ + e^{MT} \int_0^{t-\delta/4} |\hat{f}(s, \xi)| \exp \left\{ (C_2(t) - C_2(s))|\xi|^2 + MT|\xi| \right\} ds \]

\[ + e^{MT} \int_{t-\delta/4}^{t} |\hat{f}(s, \xi)| \exp \left\{ (C_2(t) - C_2(s))|\xi|^2 + M(t - s)|\xi| \right\} ds \]

\[ \leq e^{MT} |\hat{g}(\xi)| \exp \left[ (C_2(t) - C_2(t - \frac{\delta}{4}))|\xi|^2 + MT|\xi| \right] \]

\[ + e^{MT} \int_0^{t-\delta/4} |\hat{f}(s, \xi)| \exp \left\{ (C_2(t) - C_2(t - \frac{\delta}{4}))|\xi|^2 + MT|\xi| \right\} ds \]
It follows from (7.1), (7.2) and (7.8) that

Then

Once again we split the proof into two cases.

Let $L > 0$ such that

Then

Next we proceed to the situation where

for some constant $C > 0$ depending on $\omega, \delta, T$ and $M$, applying Hölder’s inequality.

Next we proceed to the situation where

Once again we split the proof into two cases.

**First case: $t_1 = 0$ and $t \in \left[0, \frac{3\delta}{4}\right]$.**

It follows from (7.1), (7.2) and (7.8) that

Take $\kappa > 0$ such that

Then

\[ |\hat{\varphi}(t, \xi)| \leq e^{MT} |\mathcal{G}(\xi)| \exp \left\{ -\omega (t-s) |\xi|^2 + M(t-s)|\xi| \right\} ds \]

\[ \leq e^{MT} |\mathcal{G}(\xi)| \exp \left\{ -\frac{\omega \delta}{4} |\xi|^2 + M|\xi| \right\} + e^{MT} \exp \left\{ -\frac{\omega \delta}{4} |\xi|^2 + M|\xi| \right\} \int_{t}^{t-\delta/4} |\tilde{f}(s, \xi)| ds \]

\[ + e^{MT} \int_{t-\delta/4}^{t} |\tilde{f}(s, \xi)| \exp \left\{ (-\omega |\xi|^2 + M|\xi|) (t-s) \right\} ds. \]
\[ + e^{MT} \int_0^t |\hat{f}(s, \xi)| \exp \left\{ -\varepsilon \left( \int_0^t |r^{p_1} dr \right) |\xi|^2 + \frac{\kappa}{N} \sum_{j=1}^N \left( \int_0^t |r^{q_j+1} dr \right) |\xi_j| \right\} ds. \]

With no loss of generality one can assume that \( \delta \leq 1 \), which implies that \( r^{q_1+1} \leq r^{p_1} \) in the expression above, using the notation set in (7.4). Hence

\[ |\hat{u}(t, \xi)| \leq e^{MT} |\hat{g}(\xi)| \exp \left\{ -\varepsilon \left( \int_0^t |r^{p_1} dr \right) |\xi|^2 + \frac{\kappa}{N} \sum_{j=1}^N \left( \int_0^t |r^{q_j+1} dr \right) |\xi_j| \right\} +
\]

\[ + e^{MT} \int_0^t |\hat{f}(s, \xi)| \exp \left\{ -\varepsilon \left( \int_s^t |r^{p_1} dr \right) |\xi|^2 + \frac{\kappa}{N} \sum_{j=1}^N \left( \int_s^t |r^{q_j+1} dr \right) |\xi_j| \right\} ds \]  

(7.13)

By hypothesis, \( p_1 \leq 2q_1 + 1 \); we split the first case in two parts, depending on the sign of \( q_1 - p_1 \).

1) \( q_1 < p_1 \). In this case \( q_1 < p_1 \leq 2q_1 + 1 \) and

\[ \nu_1 := \frac{2q_1 - p_1 + 1}{q_1 - p_1} \leq 0. \]  

(7.14)

So for any \( \xi \in \mathbb{Z}^N \setminus \{0\} \) and \( 0 \leq s \leq t \leq \frac{3\delta}{4} \), we have

\[ 1 \left[ \frac{\varepsilon (s^{q_1+1} - t^{q_1+1})}{(p_1 + 1)} \right] \left[ \frac{\kappa (t^{q_1+1} - s^{q_1+1})}{q_1 + 1} \right] = \left( \frac{\varepsilon (s^{p_1+1} - t^{p_1+1})}{(p_1 + 1)} \right) |\xi|^{2 - \nu_1} + \left( \frac{\kappa (t^{q_1+1} - s^{q_1+1})}{q_1 + 1} \right) |\xi|^{\nu_1}. \]  

(7.15)

On the other hand,

\[ 2 - \nu_1 = \frac{2q_1 - 2p_1 - 2q_1 + p_1 - 1}{q_1 - p_1} = \frac{p_1 + 1}{p_1 - q_1}; \quad 1 - \nu_1 = \frac{q_1 - p_1 - 2q_1 + p_1 - 1}{q_1 - p_1} = \frac{q_1 + 1}{p_1 - q_1}. \]  

(7.16)

By associating (7.15) to (7.16), we infer that

\[ (\ast) = \left[ \frac{\varepsilon}{p_1 + 1} \left( s |\xi|^{\frac{p_1}{p_1 - q_1}} \right)^{p_1+1} - \kappa \frac{1}{q_1 + 1} \left( t |\xi|^{\frac{p_1}{p_1 - q_1}} \right)^{q_1+1} \right] - \left[ \frac{1}{p_1 + 1} \left( t |\xi|^{\frac{p_1}{p_1 - q_1}} \right)^{p_1+1} - \kappa \frac{1}{q_1 + 1} \left( t |\xi|^{\frac{p_1}{p_1 - q_1}} \right)^{q_1+1} \right]. \]  

(7.17)

Consider \( f_{p_1,q_1,\varepsilon,\kappa} : \mathbb{R}^+ \rightarrow \mathbb{R} \) given by

\[ f_{p_1,q_1,\varepsilon,\kappa}(x) = -\frac{\varepsilon x^{p_1+1}}{p_1 + 1} + \frac{\kappa x^{q_1+1}}{q_1 + 1}. \]  

(7.18)

Note that \( f'_{p_1,q_1,\varepsilon,\kappa}(x) = -\varepsilon x^{p_1} + \kappa x^{q_1} \). Since we are dealing with the case where \( q_1 < p_1 \), \( f_{p_1,q_1,\varepsilon,\kappa} \) is decreasing if \( x > \left( \frac{\kappa}{\varepsilon} \right)^{\frac{1}{p_1 - q_1}} \). Thus there exists \( C_{p_1,q_1,\varepsilon} > 0 \) such that

\[ 0 \leq y \leq z \Rightarrow f_{p_1,q_1,\varepsilon,\kappa}(z) - f_{p_1,q_1,\varepsilon,\kappa}(y) \leq C_{p_1,q_1,\varepsilon}. \]  

(7.19)
Observe that (7.17) can be rewritten as \((\ast) = f_{p_1, q_1, \kappa, \varepsilon}(t|\xi|^{\frac{1}{p_1 - 1}}) - f_{p_1, q_1, \kappa, \varepsilon}(s|\xi|^{\frac{1}{p_1 - 1}})\). By combining (7.15) with (7.17) and (7.19), one deduces that
\[
\frac{\varepsilon(s^{p_1 + 1} - t^{p_1 + 1})}{(p_1 + 1)}|\xi|^2 + \frac{\kappa(t^{p_1 + 1} - s^{p_1 + 1})}{q_1 + 1}|\xi| \leq C_{p_1, q_1, \kappa, \varepsilon}|\xi|^{p_1}, \quad \forall \xi \in \mathbb{Z}^N \setminus \{0\}, \quad 0 \leq s \leq t \leq \frac{3\delta}{4}.
\] (7.20)

But \(\nu \leq 0\), which implies
\[
\frac{\varepsilon(s^{p_1 + 1} - t^{p_1 + 1})}{(p_1 + 1)}|\xi|^2 + \frac{\kappa(t^{p_1 + 1} - s^{p_1 + 1})}{q_1 + 1}|\xi| \leq C_{p_1, q_1, \kappa, \varepsilon}, \quad \forall \xi \in \mathbb{Z}^N, \quad 0 \leq s \leq t \leq \frac{3\delta}{4}.
\] (7.21)

It follows from (7.13), (7.21) and Hölder’s inequality that
\[
|\hat{u}(t, \xi)| \leq e^{MT}C_{p_1, q_1, \kappa, \varepsilon} \left( |\hat{\vartheta}(\xi)| + \int_0^t |\hat{f}(s, \xi)| ds \right) \leq C \left[ |\hat{\vartheta}(\xi)| + \left( \int_0^T |\hat{f}(s, \xi)|^2 ds \right)^{1/2} \right], \quad \forall \xi \in \mathbb{Z}^N,
\]
for some constant \(C > 0\) which depends on \(M, T, p_1, q_1, \kappa\) and \(\varepsilon\).

Next we deal with the case II) \(q_1 \geq p_1\). Then \(0 \leq r \leq 1 \Rightarrow r^{q_1} \leq r^{p_1}\) and there exists \(B_{p_1, q_1} > 0\) for which \(r^{q_1} \leq B_{p_1, q_1} r^{p_1}\), for every \(0 \leq r \leq T\). Thus
\[
0 < s < t < T \Rightarrow \int_s^t r^{q_1} dr \leq B_{p_1, q_1} \int_s^t r^{p_1} dr \Rightarrow \frac{(t^{p_1 + 1} - s^{p_1 + 1})}{q_1 + 1} \leq B_{p_1, q_1} \frac{(t^{p_1 + 1} - s^{p_1 + 1})}{p_1 + 1}.
\] (7.22)

By associating (7.13) to (7.22), one infers that
\[
|\hat{u}(t, \xi)| \leq e^{MT}C_{p_1, q_1, \kappa, \varepsilon} \exp \left\{ -\frac{\varepsilon(s^{p_1 + 1} - t^{p_1 + 1})}{p_1 + 1}|\xi|^2 + \frac{\kappa B_{p_1, q_1} (t^{p_1 + 1} - s^{p_1 + 1})}{p_1 + 1}|\xi| \right\} +
+ e^{MT} \int_0^t |\hat{f}(s, \xi)| \exp \left\{ -\frac{\varepsilon(s^{p_1 + 1} - t^{p_1 + 1})}{p_1 + 1}|\xi|^2 + \frac{\kappa B_{p_1, q_1} (t^{p_1 + 1} - s^{p_1 + 1})}{p_1 + 1}|\xi| \right\} ds
\]
\[
\leq e^{MT}C_{p_1, q_1, \kappa, \varepsilon} \exp \left\{ -\frac{\varepsilon(t^{p_1 + 1} - s^{p_1 + 1})}{p_1 + 1}|\xi|^2 + \frac{\kappa B_{p_1, q_1}}{p_1 + 1}|\xi| \right\} +
+ e^{MT} \int_0^t |\hat{f}(s, \xi)| \exp \left\{ -\frac{\varepsilon(t^{p_1 + 1} - s^{p_1 + 1})}{p_1 + 1}|\xi|^2 + \frac{\kappa B_{p_1, q_1}}{p_1 + 1}|\xi| \right\} ds.
\]

Since the expressions inside the exponentials are uniformly bounded, we conclude once again that
\[
|\hat{u}(t, \xi)| \leq C \left[ |\hat{\vartheta}(\xi)| + \left( \int_0^T |\hat{f}(s, \xi)|^2 ds \right)^{1/2} \right], \quad \forall \xi \in \mathbb{Z}^N,
\]
for some constant \(C\) depending on \(M, T, p_1, q_1, \kappa\) and \(\varepsilon\), which finalizes the proof for (7.11).

Now we proceed to the

**Second case:** \(t_1 \neq 0\) or \(t \notin \left[0, \frac{3\delta}{4}\right]\).

Fix \(t \in [0, T] \) and \(k \in \{1, \ldots, m\}\) for which (7.10) holds. By possibly reducing \(\delta\), we may assume \(\frac{3\delta}{2} \leq t_k\). Since \(c_2 \leq 0\), it follows from (7.7) and (7.8) that
\[
|\hat{u}(t, \xi)| \leq e^{MT} |\hat{\vartheta}(\xi)| \exp \left\{ C_2 \left( t_k - \frac{3\delta}{4} \right) |\xi|^2 + MT|\xi| \right\} +
+ e^{MT} \int_0^{t_k - \delta} |\hat{f}(s, \xi)| \exp \left\{ (C_2(t) - C_2(t_k - \delta)) |\xi|^2 + MT|\xi| \right\} ds +
\] (7.23)
We have an identity similar to (7.16) and, by proceeding analogously, one concludes that

\[ e^{MT} \int_{t_{k-\delta}}^{t} \widehat{g}(s, \xi) \exp \left[ (C_2(t) - C_2(s)) |\xi|^2 + \sum_{j=1}^{N} (C_{1,j}(t) - C_{1,j}(s)) \xi_j \right] ds. \]  

(7.24)

Note that, by (7.3) and (7.10),

\[ C_2 \left( t_k - \frac{3\delta}{4} \right) - \frac{3\delta}{4} = \int_{-\frac{3\delta}{4}}^{t_k} c_2(r) dr \leq \int_{t_{k-\delta}}^{t_k} c_2(r) dr \leq -\frac{\omega\delta}{4}, \]

\[ C_2(t_k - \delta) = \int_{t_k - \delta}^{t} c_2(r) dr \leq \int_{t_{k-\delta}}^{t_{k-\delta}} c_2(r) dr \leq -\frac{\omega\delta}{4}. \]

Thus we can replace (7.24) by

\[ |\widehat{u}(t, \xi)| \leq e^{MT} |\mathcal{g}(\xi)| \exp \left\{ -\frac{\omega\delta}{4} |\xi|^2 + MT |\xi| \right\} + e^{MT} \int_{t_{k-\delta}}^{t} \widehat{f}(s, \xi) \exp \left\{ -\frac{\omega\delta}{4} |\xi|^2 + MT |\xi| \right\} ds \]

\[ + e^{MT} \int_{t_{k-\delta}}^{t} \widehat{f}(s, \xi) \exp \left[ (C_2(t) - C_2(s)) |\xi|^2 + \sum_{j=1}^{N} (C_{1,j}(t) - C_{1,j}(s)) \xi_j \right] ds. \]  

(7.25)

If we show that the exponential inside the third integral in of (7.25) is uniformly bounded, by proceeding just like in previous situations we will be done. It follows from (7.11), (7.12) and (7.10) that

\[ (C_2(t) - C_2(s)) |\xi|^2 + \sum_{j=1}^{N} (C_{1,j}(t) - C_{1,j}(s)) \xi_j = \left( \int_{s}^{t} \alpha(k) |r - t_k|^{p_k} dr \right) |\xi|^2 + \sum_{j=1}^{N} \left( \int_{s}^{t} \beta^{j,k}(r) |r - t_k|^{q_k} dr \right) \xi_j. \]  

(7.26)

Since one may assume |r - t_k| \leq 1, as a consequence of (7.1), (7.4) and (7.12) we obtain

\[ (\xi) \leq -\varepsilon \left( \int_{s-t_k}^{t-t_k} |r|^{p_k} dr \right) |\xi|^2 + \kappa \left( \int_{s-t_k}^{t-t_k} |r|^{q_k} dr \right) |\xi|. \]  

(7.27)

By direct computations, one deduces

\[ \int_{s-t_k}^{t-t_k} |r|^{p_k} dr = \frac{1}{p_k + 1} \left\{ \begin{array}{ll} (t - t_k)^{p_k + 1} - (s - t_k)^{p_k + 1}, & \text{if } t \geq s \geq t_k; \\ (t - t_k)^{p_k + 1} + (s - t_k)^{p_k + 1}, & \text{if } t \geq t_k \geq s; \\ (s - t_k)^{p_k + 1} - (t - t_k)^{p_k + 1}, & \text{if } t_k \geq t \geq s. \end{array} \right. \]  

(7.28)

An analogous expression is obtained when \( p_k \) is replaced by \( q_k \).

We split the estimate of (\( \xi \)) in (7.26) into four parts:

(1) \( t \geq s \geq t_k \) and \( q_k < p_k \leq 2q_k + 1 \). It follows from (7.27) and (7.28) that

\[ (\xi) \leq -\frac{\varepsilon}{p_k + 1} \left( (t - t_k)^{p_k + 1} - |s - t_k|^{p_k + 1} \right) |\xi|^2 + \frac{\kappa}{q_k + 1} \left( (t - t_k)^{q_k + 1} - |s - t_k|^{q_k + 1} \right) |\xi|. \]

Now the proof is quite similar to one made with beginning at (7.14). Let

\[ \nu_k = \frac{2q_k - p_k + 1}{q_k - p_k} < 0. \]  

(7.29)

We have an identity similar to (7.16) and, by proceeding analogously, one concludes that

\[ \frac{(\xi)}{|\xi|^{p_k}} \leq -\frac{\varepsilon}{p_k + 1} \left( |s - t_k|^{p_k + 1} - \frac{\kappa}{q_k + 1} \left( |s - t_k|^{q_k + 1} \right) \right) \]

\[-\frac{\varepsilon}{p_k + 1} \left( |t - t_k|^{p_k + 1} - \frac{\kappa}{q_k + 1} \left( |t - t_k|^{q_k + 1} \right) \right) \]

Since \( |t - t_k| \geq |s - t_k| \), it suffices to repeat the arguments with starting point at (7.17).
(2) \( t \geq t_k \geq s \) and \( q_k < p_k \leq 2q_k + 1 \). By applying (7.27) and (7.28), one gets
\[
(\xi) \leq -\frac{\varepsilon}{p_k + 1} \left( (t - t_k)^{p_k + 1} + |s - t_k|^{q_k + 1} \right) |\xi|^2 + \frac{\kappa}{q_k + 1} \left( (t - t_k)^{q_k + 1} + |s - t_k|^{q_k + 1} \right) |\xi|.
\]
Thus, by taking \( \nu_k \) as in (7.29),
\[
\left( \frac{\xi}{|\xi|} \right)^{\nu_k} \leq \left[ -\frac{\varepsilon}{p_k + 1} \left( (s - t_k)||\xi|^{\frac{1}{p_k - q_k}} \right)^{p_k + 1} + \frac{\kappa}{q_k + 1} \left( (s - t_k)||\xi|^{\frac{1}{p_k - q_k}} \right)^{q_k + 1} \right] +
+ \left[ -\frac{\varepsilon}{p_k + 1} \left( (t - t_k)||\xi|^{\frac{1}{p_k - q_k}} \right)^{p_k + 1} + \frac{\kappa}{q_k + 1} \left( (t - t_k)||\xi|^{\frac{1}{p_k - q_k}} \right)^{q_k + 1} \right].
\]
Using notation set in (7.18), we rewrite the inequality above as
\[
\left( \frac{\xi}{|\xi|} \right)^{\nu_k} \leq f_{p_k, q_k, \varepsilon, \kappa} \left( (s - t_k)||\xi|^{\frac{1}{p_k - q_k}} \right) + f_{p_k, q_k, \varepsilon, \kappa} \left( (t - t_k)||\xi|^{\frac{1}{p_k - q_k}} \right).
\]
Since \( f_{p_k, q_k, \varepsilon, \kappa} \) is decreasing when \( x > \left( \frac{\kappa}{\varepsilon} \right)^{\frac{1}{p_k - q_k}} \), it is still possible to obtain an inequality similar to (7.19) for the sum above and repeat the arguments made with starting point at (7.17).

(3) \( t_k \geq t \geq s \) and \( q_k < p_k \leq 2q_k + 1 \). Using (7.27) and (7.28) we have
\[
(\xi) \leq -\frac{\varepsilon}{p_k + 1} \left( (|s - t_k|^{p_k + 1} - |t - t_k|^{p_k + 1}) |\xi|^2 + \frac{\kappa}{q_k + 1} \left( (|s - t_k|^{q_k + 1} - |t - t_k|^{q_k + 1}) |\xi| \right).\]
Then
\[
\left( \frac{\xi}{|\xi|} \right)^{\nu_k} \leq \left[ -\frac{\varepsilon}{p_k + 1} \left( (|s - t_k|^{\frac{1}{p_k - q_k}})^{p_k + 1} - (|t - t_k|^{\frac{1}{p_k - q_k}})^{p_k + 1} \right) \right] +
+ \left[ -\frac{\varepsilon}{p_k + 1} \left( (|s - t_k|^{\frac{1}{p_k - q_k}})^{q_k + 1} - (|t - t_k|^{\frac{1}{p_k - q_k}})^{q_k + 1} \right) \right],
\]
with \( \nu_k \) given in (7.29). By the fact that \( t_k \geq t \geq s \), we deduce that \( |s - t_k| \geq |t - t_k| \). Hence the proof in this situation is completely analogous to the first case.

(4) \( q_k \geq p_k \). It is not necessary to take into consideration here the sign of \( (t_k - t) \) or \( (t_k - s) \). Indeed, since \( q_k \geq p_k \), there exists \( B_{p_k, q_k} > 0 \) such that \( |r| \leq T \Rightarrow |r|^{|q_k|} \leq B_{p_k, q_k} |r|^{q_k} \). One infers from (7.27)
\[
(\xi) \leq \left( \int_{s-t_k}^{t-t_k} |r|^{p_k} dr \right) (-\varepsilon |\xi|^2 + \kappa B_{p_k, q_k} |\xi|).
\]
Since both terms above are uniformly bounded, that closes the proof of (II).

\[ \square \]

**Proof of (II).** Note the following consequence of the proof of (I) the behavior of \( c_2 \) and \( c_{1,1}, \ldots, c_{1,N} \) far from the points \( t_1, \ldots, t_m \) (to be more precise, if (7.9) holds) has no damage to the well-posedness of (1.3). Furthermore, there exists no impact when \( t \) is near \( t_k \) (being more accurate, if (7.10) holds), provided that \( p_k \leq 2q_k + 1 \). Hence the only analysis left to be done is for neighborhoods of \( t_k \), when \( k \) is an element of the set \( \mathcal{K} \) defined in (7.5).

We shall first prove the well-posedness of (1.3) in \( G^7 \) when \( \mathcal{K} \neq \emptyset \) and (7.6) holds, splitting the proof in two parts, precisely described in (7.11) and (7.23). Similarly to (7.14) and (7.29), we set
\[
\nu_k = \frac{2q_k - p_k + 1}{q_k - p_k} = \frac{p_k - 2q_k - 1}{p_k - q_k}.
\]
However, \( \nu_k \) is strictly positive if \( k \in \mathcal{K} \). Hence, by repeating the arguments applied from (7.15) until (7.20), as well as in each case of (7.23), we deduce that
\[
|\tilde{u}(t, \xi)| \leq e^{MT} e^{G_{p_k, \nu_k, \varepsilon, \kappa} \nu_k} \left( |\tilde{u}(\xi)| + \int_0^t |\tilde{f}(s, \xi)| ds \right), \quad |t - t_k| < \frac{3\delta}{4}, \ \forall \xi \in \mathbb{Z}^N. \quad (7.30)
\]
Let $\nu := \max_{k \in \mathcal{X}} \{ \nu_k \}$; since we have a better estimate for the rest of the subsets of $[0, T]$,

$$|\hat{u}(t, \xi)| \leq C_1 e^{C_2|\xi|^\tau} \left( |\hat{g}(\xi)| + \int_0^t |\hat{f}(s, \xi)| ds \right), \quad \forall t \in [0, T], \ \forall \xi \in \mathbb{Z}^N,$$

(7.31)

for some $C_1, C_2 > 0$. Take $f \in C \left( [0, T], \mathcal{G}^r(\mathbb{T}^N) \right)$ and $g \in \mathcal{G}^s(\mathbb{T}^N)$; there exist $C_3, \sigma > 0$ such that

$$|\hat{f}(t, \xi)| \leq C_3 e^{-\sigma|\xi|^{1/r}}, \quad |\hat{g}(\xi)| \leq C_3 e^{-\sigma|\xi|^{1/r}}, \quad \forall t \in [0, T], \ \forall \xi \in \mathbb{Z}^N.$$  

(7.32)

Associating (7.31) to (7.32), one infers, for some constant $C_4 > 0$, that

$$|\hat{u}(t, \xi)| \leq C_4 \exp \left( C_2 |\xi|^\nu \sigma |\xi|^{1/r} \right), \quad \forall t \in [0, T], \ \forall \xi \in \mathbb{Z}^N.$$

(7.33)

Our statement is then a consequence of Lemma (4.1).

It remains to verify that if

$$\varrho = \min_{k \in \mathcal{K}} \left( \frac{p_k - q_k}{p_k - 2q_k - 1} \right),$$

(7.34)

(1.3) is ill-posed in $G^q$. It follows from Proposition 4.2 that the same problem is also ill-posed in $G^s$, for any $s > q$, in $C^\infty$ and in $H^r$. By Theorem 3.4 it suffices to prove the result for the particular case where $\Re a_2 \equiv \Re a_{1,1} \equiv \cdots \equiv \Re a_{1,N} \equiv a_0 \equiv 0$. In that situation, the Fourier coefficients of $u$ are described in (5.2).

Denote by $k_0$ the element in $\{1, 2, \ldots, m\}$ that satisfies

$$q_k = \min \{ q_1, q_2, \ldots, q_N \} = q_{j_0, k_0},$$

(7.35)

Consider in a first moment $t_{k_0} \neq 0$; in this case we may assume $t_{k_0} \geq \delta$. Let

$$g \equiv 0 \text{ and } f(t, x) = -i \sum_{\eta \in \mathbb{Z}} e^{-\varrho|\eta|^{1/q}} e^{\eta \cdot \eta_{j_0}},$$

where $\eta_{j_0}$ denotes the vector $(0, \ldots, 0, \eta_{j_0}, 0, \ldots, 0)$ and $\varrho$ is a positive number which will be chosen later. Then $\hat{u}(t, \xi) = 0$ when $\xi$ is not of the form $\eta_{j_0}$ for some $\eta \in \mathbb{Z}$ and

$$\hat{u}(t, \eta_{j_0}) = e^{-\varrho|\eta|^{1/q}} \int_0^t \exp \left\{ \left[ (C_2(t) - C_2(s)) \eta^2 + (C_{1,j_0}(t) - C_{1,j_0}(s)) \eta \right] ds \right\}, \quad \forall t \in [0, T], \ \forall \eta \in \mathbb{Z}.$$

Since the term inside the integral is positive, for any $t \in [t_{k_0} - \delta, t_{k_0} + \delta] \cap [0, T]$, we have

$$\hat{u}(t, \eta_{j_0}) \geq e^{-\varrho|\eta|^{1/q}} \int_{t_{k_0} - \delta}^t \exp \left\{ \left[ (C_2(t) - C_2(s)) \eta^2 + (C_{1,j_0}(t) - C_{1,j_0}(s)) \eta \right] ds \right\}$$

$$= e^{-\varrho|\eta|^{1/q}} \int_{t_{k_0} - \delta}^t \exp \left[ \left( \int_s^t \alpha_{k_0}(r) |r - t_{k_0}|^{p_{k_0}} dr \right) \eta^2 + \left( \int_s^t \beta_{j_0,k_0}(r) |r - t_{k_0}|^{q_{j_0,k_0}} dr \right) \eta \right] ds,$$  

(7.36)
applying (7.1) and (7.2). Because $\beta^{\alpha_k,\delta}$ does not vanish, the function is either strictly positive or negative in the interval $[t_k, t_k - \delta]$. In order to simplify the proof (the other case is analogous), we assume it is positive. On the other hand, $\alpha^{\delta}$ is bounded; thus there exists $\gamma, \Gamma > 0$ for which, 

$$\beta^{\alpha_k,\delta}(r) \geq \gamma, \quad \alpha^{\delta}(r) \geq -\Gamma, \quad \forall r \in [t_k - \delta, t_k].$$

By associating the inequalities above to (7.35) and (7.36), we obtain for any $t \in [t_k - \delta, t_k]$ and $\eta \in \mathbb{N},$

$$\hat{u}(t, \eta^\delta) \geq e^{-\eta^\delta/\delta} \int_{t_k - \delta}^{t} \exp \left[ -\Gamma \left( \int_{s}^{t} |r - t_k|^{p_k} dr \right) \eta^2 \right] ds,$$  \hspace{1cm} (7.37)

We shall now analyze the term inside the exponential, which will be denoted by

$$(\Delta) := -\Gamma \left( \int_{s}^{t} |r - t_k|^{p_k} dr \right) \eta^2 + \gamma \left( \int_{s}^{t} |r - t_k|^{q_k} dr \right) \eta.$$ 

It follows from (7.28) that

$$\Delta = -\Gamma \left( \frac{|s - t_k|^{p_k + 1} - |t - t_k|^{p_k + 1}}{p_k + 1} \right) \eta^2 + \gamma \left( \frac{|s - t_k|^{q_k + 1} - |t - t_k|^{q_k + 1}}{q_k + 1} \right) \eta.$$

By proceeding analogously to what was done from (7.14) up to (7.19), we obtain for $\eta \geq 0$

$$\Delta = \left[ -\Gamma \frac{1}{p_k + 1} \left( |s - t_k|^{p_k + 1} - |t - t_k|^{p_k + 1} \right) \eta^2 + \gamma \left( \frac{|s - t_k|^{q_k + 1} - |t - t_k|^{q_k + 1}}{q_k + 1} \right) \eta \right].$$

where

$$\nu_k = \frac{p_k - 2q_k - 1}{p_k - q_k} \quad \text{and} \quad f_{p_k, q_k, \gamma}(x) = -\frac{\Gamma x^{p_k + 1}}{p_k + 1} + \gamma x^{q_k + 1}.$$  \hspace{1cm} (7.39)

Combining (7.37) with (7.35), one concludes that for any $t \in [t_k - \delta, t_k]$ and $\eta \in \mathbb{N},$

$$\hat{u}(t, \eta^\delta) \geq e^{-\eta^\delta/\delta} \int_{t_k - \delta}^{t} \exp \left\{ \eta^\delta \left[ f_{p_k, q_k, \eta, \gamma} \left( |s - t_k|^{1/q_k - 1} \right) - f_{p_k, q_k, \eta, \gamma} \left( |t - t_k|^{1/q_k - 1} \right) \right] \right\} ds,$$  \hspace{1cm} (7.40)

Since $p_k > q_k$ and $f'_{p_k, q_k, \eta, \gamma}(x) = -\Gamma x^{p_k} + \gamma x^{q_k},$ we have

$$f_{p_k, q_k, \eta, \gamma} \text{ strictly positive and increasing in the interval } \left( 0, \frac{1}{\Gamma} \right).$$

Take $\ell \in \mathbb{N}$ such that $\frac{1}{\ell}$ belongs to this interval and consider the following sequences:

$$t_{k_0, n} = t_k - \frac{1}{3\ell n^{1/2}}, \quad \tau_{k_0, n} = t_k - \frac{1}{3\ell n^{1/2}}, \quad t_{k_0, n} = t_k - \frac{1}{2\ell n^{1/2}}, \quad \forall n \in \mathbb{N}. \hspace{1cm} (7.41)$$

Taking $t = t_{k_0, n}$ as in (7.41) and putting it in (7.40), one has

$$\hat{u}(t_{k_0, n}, \eta^\delta) \geq e^{-\eta^\delta/\delta} \int_{t_{k_0, n} - \delta}^{t_{k_0, n}} \exp \left\{ \eta^\delta \left[ f_{p_k, q_k, \eta, \gamma} \left( |s - t_k|^{1/q_k - 1} \right) - f_{p_k, q_k, \eta, \gamma} \left( \frac{1}{3\ell} \right) \right] \right\} ds.$$
Since we may take $\ell$ large enough so the sequences introduced in (7.41) are contained in $[t_{k_0} - \delta, t_{k_0}]$, we deduce from last inequality that

$$
\tilde{u}(t_{k_0,n},\eta^{j_0}) \geq e^{-\vartheta n^{1/\varphi}} \int_{t_{k_0,n}}^{t_{\tau_{k_0,n}}} \exp \left\{ n^{\eta^{j_0}} \left[ f_{p_0, q_0, \Gamma, \gamma} \left( \frac{1}{2\ell} \right) - f_{p_0, q_0, \Gamma, \gamma} \left( \frac{1}{3\ell} \right) \right] \right\} ds.
$$

(7.42)

Observe that $\tau_{k_0,n} \leq s \leq t_{k_0,n}$ implies

$$
\frac{1}{2\ell} \leq |s - t_{k_0}| n^{p_{k_0} - q_{k_0}} \leq \frac{1}{\ell}.
$$

Note that our choice of $\ell$ was made so in the interval above $f_{p_0, q_0, \Gamma, \gamma}$ is increasing, which allows us to conclude that

$$
\tilde{u}(t_{k_0,n},\eta^{j_0}) \geq e^{-\vartheta n^{1/\varphi}} \int_{t_{k_0,n}}^{t_{\tau_{k_0,n}}} \exp \left\{ n^{\eta^{j_0}} \left[ f_{p_0, q_0, \Gamma, \gamma} \left( \frac{1}{2\ell} \right) - f_{p_0, q_0, \Gamma, \gamma} \left( \frac{1}{3\ell} \right) \right] \right\} ds.
$$

(7.43)

Let $\zeta = f_{p_0, q_0, \Gamma, \gamma} \left( \frac{1}{2\ell} \right) - f_{p_0, q_0, \Gamma, \gamma} \left( \frac{1}{3\ell} \right) > 0$. By (7.41) and (7.43),

$$
\tilde{u}(t_{k_0,n},\eta^{j_0}) \geq \exp \left( -\vartheta n^{1/\varphi} + \zeta n^{\eta^{j_0}} \right) (t_{k_0,n} - \tau_{k_0,n}) = \exp \left( -\vartheta n^{1/\varphi} + \zeta n^{\eta^{j_0}} \right) \frac{1}{2\ell n^{p_{k_0} - q_{k_0}}}, \ \forall n \in \mathbb{N}.
$$

(7.44)

On the other hand, it follows from (7.33), (7.34), (7.35) and (7.39) that

$$
\nu_{k_0} = \frac{p_{k_0} - 2q_{k_0} - 1}{p_{k_0} - q_{k_0}} = \frac{1}{\vartheta},
$$

(7.45)

Therefore, if $\vartheta = \frac{\zeta}{2}$, we conclude from (7.44) that

$$
\tilde{u}(t_{k_0,n},\eta^{j_0}) \geq \frac{\exp \left( \frac{\zeta n^{1/\varphi}}{2\ell n^{p_{k_0} - q_{k_0}}} \right)}{2\ell n^{p_{k_0} - q_{k_0}}} \Rightarrow \lim_{n \to +\infty} \tilde{u}(t_{k_0,n},\eta^{j_0}) = +\infty,
$$

which proves the ill-posedness of (1.3) in $G^\varphi$.

Finally, we analyze the case where $t_{k_0} = t_1 = 0$, the main difference here lies on the choice of the right-hand side functions. Take

$$
f \equiv 0 \text{ and } g(x) = \sum_{\eta \in \mathbb{Z}} e^{-\vartheta |\eta|^{1/\varphi}} e^{i x \cdot \eta^{j_0}}
$$

for some $\vartheta > 0$ that will be defined later and $j_0$ the number for which $q_1 = \min \{ q_{1,1} + \ldots, q_{N,1} \} = q_{j_0,1}$.

By proceeding just like in the last case, we deduce that

$$
\tilde{u}(t,\eta^{j_0}) = e^{-\vartheta |\eta|^{1/\varphi}} \exp \left[ C_2(t)\eta^2 + C_{1,j_0}(t)\eta \right], \ \forall t \in [0, T], \ \forall \eta \in \mathbb{Z}.
$$

When $0 \leq t \leq \delta$, one has

$$
\tilde{u}(t,\eta^{j_0}) = e^{-\vartheta |\eta|^{1/\varphi}} \exp \left[ \left( \int_0^t \alpha^1(r) r^{p_{1,0}} dr \right) \eta^2 + \left( \int_0^t \beta^{j_0,1}(r) r^{q_{j_0,0}} dr \right) \eta \right].
$$

With the same hypotheses and inequalities applied in (7.37), we obtain

$$
\tilde{u}(t,\eta^{j_0}) \geq e^{-\vartheta |\eta|^{1/\varphi}} \exp \left[ -\Gamma \left( \int_0^t r^{p_{1,0}} dr \right) \eta^2 + \gamma \left( \int_0^t r^{q_{j_0,0}} dr \right) \eta \right]

\geq e^{-\vartheta |\eta|^{1/\varphi}} \exp \left[ -\Gamma \frac{t^{p_{1,0} + 1}}{p_{1,0} + 1} \eta^2 + \gamma \frac{t^{q_{j_0,0} + 1}}{q_{j_0,0} + 1} \eta \right].
$$
With a similar argument to the one used in (7.38), one has for \( \eta \in \mathbb{N} \)

\[
\hat{u}(t, \eta^m) \geq e^{-\vartheta |\eta|^{1/\vartheta}} \exp \left\{ -\frac{\Gamma}{p_1 + 1} \left( t \eta^{p_1+1} \right) + \frac{\gamma}{q_1 + 1} \left( t \eta^{q_1+1} \right) \right\} \eta^{\varrho^1} 
= e^{-\vartheta |\eta|^{1/\vartheta}} \exp \left\{ \eta^{\varrho^1} f_{p_1, q_1, \Gamma, \gamma} \left( t \eta^{p_1+1} \right) \right\}.
\]

Analogously, \( f_{p_1, q_1, \Gamma, \gamma} \) is strictly positive and increasing in the interval \( 0, \left( \frac{\Gamma}{p_1 - q_0} \right) \). Take \( \ell \in \mathbb{N} \) such that \( \frac{1}{\ell} \) belongs to this interval and consider the sequence \( t_{1,n} = \frac{1}{\ell n^{p_1-q_0}}. \) Then

\[
\hat{u}(t_{1,n}, n^{\varrho^0}) \geq e^{-\vartheta n^{1/\vartheta}} \exp \left\{ n^{\varrho^1} f_{p_1, q_1, \Gamma, \gamma} \left( \frac{1}{\ell} \right) \right\} \geq \exp \left\{ \left( f_{p_1, q_1, \Gamma, \gamma} \left( \frac{1}{\ell} \right) - \vartheta \right) n^{\varrho^1} \right\},
\]

since (7.45) holds once again. Therefore it suffices to take \( \vartheta = f_{p_1, q_1, \Gamma, \gamma} \left( \frac{1}{\ell} \right) / 2 \), which entails \( \lim_n \hat{u}(t_{1,n}, n^{\varrho^0}) = +\infty \), finalizing the proof of Theorem 7.3.

\[\square\]

**Corollary 7.4.** With the same hypotheses of Theorem 7.3, suppose that \( p_k \leq 2q_k + 1 \) for each \( k \in \{1, 2, \ldots, m\} \). In this situation, we have the following properties:

- **Given** \( r \in \mathbb{R} \), \( f \in C \left( [0, T]; H^r(T^N) \right) \) and \( g \in H^r(T^N) \), there exists a unique \( u \in C \left( [0, T]; H^r(T^N) \right) \cap C^1 \left( [0, T]; H^{r-2}(T^N) \right) \) which solves (1.3).
- For every \( f \in C \left( [0, T]; C^\infty(T^N) \right) \) and \( g \in C^\infty(T^N) \), there exists a unique \( u \in C^1 \left( [0, T]; C^\infty(T^N) \right) \) that solves (1.3).
- **Given** \( s \geq 1 \), \( f \in C \left( [0, T]; G^s(T^N) \right) \) and \( g \in G^s(T^N) \), there exists a unique \( u \in C^1 \left( [0, T]; G^s(T^N) \right) \) which solves (1.3).

**8. Final Remarks**

In this section we present examples and facts that can be easily deduced from results proved previously and may be applied for other problems that were not precisely stated before.

**Remark 8.1.** It is worth noting that Proposition 3.3 and Theorem 3.4 can both be extended if we replace \( Q(t, D_x) \) in (1.2) by any linear differential operator of any order \( m \), with the only difference being the loss of regularity, where 2 is replaced \( m \), which once again will not matter in the \( C^\infty \) and \( G^s \) contexts. Therefore, whenever one deals with operators in the form \( D_t - Q(t, D_x) \), only the imaginary part of \( Q \) truly matters from a solvability standpoint. Furthermore, the zero-order term can be neglected as well.

**Remark 8.2.** In similar fashion, Lemma 4.1 and Proposition 4.2 can be obtained if one replaces \( Q(t, D_x) \) in (1.2) by any linear differential operator of any order \( n \). In this case, the only difference is the first statement of the lemma, where \( u \) becomes an element of \( C \left( [0, T]; H^{r-m+\rho}(T^N) \right) \).

**Remark 8.3.** When \( N = 1 \), any second-order operator \( Q(t, D_x) \) has the form described in (1.2). Furthermore, consider the case given by

\[
P(t, D_x, D_t) = D_t + a_3(t)D_x^3 + a_2(t)D_x^2 + a_1(t)D_x + a_0(t), \quad t \in [0, T], \quad x \in T^N.
\]

Proceeding just like in Section 2, we obtain a solution quite similar to the one described in (2.3). By the argument used in last remark, one may assume that \( Q \) only has pure imaginary coefficients and no
zero-order term. After all this machinery, following (5.2), we deduce that
\[ \hat{u}(t, \xi) = \hat{g}(\xi) \exp \left[ C_3(t) \xi^3 + C_2(t) \xi^2 + C_1(t) \xi \right] + i \int_0^t \hat{f}(s, \xi) \exp \left[ \left( C_3(t) - C_3(s) \right) \xi^3 + \left( C_2(t) - C_2(s) \right) \xi^2 + \left( C_1(t) - C_1(s) \right) \xi \right] ds, \quad \forall t \in [0, T], \ \forall \xi \in \mathbb{Z}. \]

Now note the following: the sign of \( \xi^3 \), in opposition to \( \xi^2 \), changes when one alters the sign of \( \xi \). Hence, if \( c_3 \) does not vanish at some point, we are able to proceed analogously to what was done in Section 5 and prove ill-posedness in any setting (it only boils down to picking the correct direction). Thus the only case where it is possible to conclude well-posedness is when \( c_3 \equiv 0 \), which reverts back to the situation we dealt with throughout the paper.

**Remark 8.4.** The intriguing consequence of Theorem 7.3 that, with such hypotheses, one always has \( C^\omega \) well-posedness, is due to the fact the operator \( Q \) in (1.2) has order 2. In fact, consider the following fourth-order operator:
\[ P(t, D_x, D_t) = D_t + i \left[ -5t^4 D_x^4 + 3t^2 D_x^3 + D_x^2 + 2t D_x \right], \quad t \in [0, T], \ x \in \mathbb{T}. \]
In this case, (2.9) becomes, for each \( t \in [0, T] \) and \( \xi \in \mathbb{Z} \),
\[ \hat{u}(t, \xi) = \hat{g}(\xi) \exp \left\{ -t^5 \xi^4 + t^3 \xi^3 + t \xi^2 + t^2 \xi \right\} + i \int_0^t \hat{f}(s, \xi) \exp \left\{ -(t^5 - s^5) \xi^4 + (t^3 - s^3) \xi^3 + (t - s) \xi^2 + (t^2 - s^2) \xi \right\} ds. \]

Now take \( f \equiv 0 \) and \( g = \sum_{\eta \in \mathbb{Z}} e^{-|\eta|} e^{i \pi \eta}; \) thus \( f \in C([0, T]; C^\omega(\mathbb{T})) \) and \( g \in C^\omega(\mathbb{T}) \), and the expression above is turned into \( \hat{u}(t, \xi) = \exp \left\{ -t^5 \xi^4 + t^3 \xi^3 + t \xi^2 + t^2 \xi - |\xi| \right\} \), for any \( t \in [0, T] \) and \( \xi \in \mathbb{Z} \). Consider the sequence \( t_n = \frac{1}{n^{4/5}} \). Then, for each \( n \in \mathbb{N} \), we obtain
\[ \hat{u}(t_n, n) = \exp \left\{ -1 + n^{3/5} + n^{6/5} + n^{-3/5} - n \right\} \Rightarrow \lim_{n \to \infty} \hat{u}(t_n, n) = +\infty. \]
Thus in this case the Cauchy problem (1.3) is not well-posed in any of the aforementioned spaces.

**Remark 8.5.** It is also worth mentioning one cannot obtain an analogous result of Theorem 7.3 when we consider functions that vanish to an infinite order at some point. Just to make things simples, we assume \( N = 1 \) and take operators given as in (3.12).

Consider first \( c_2(t) = -\frac{2e^{-1/t^2}}{t^3} \) and \( c_1(t) = \frac{2e^{-1/t^2}}{t^3} \); in this case, applying formula (5.2), we have \( C_2(t) = -e^{-1/t^2}, \ C_1(t) = e^{-1/t^2} \) and
\[ \hat{u}(t, \xi) = \hat{g}(\xi) \exp \left\{ e^{-1/t^2} (\xi - |\xi|^2) \right\} + i \int_0^t \hat{f}(s, \xi) \exp \left\{ \left( e^{-1/t^2} - e^{-1/s^2} \right) (\xi - |\xi|^2) \right\} ds, \quad \forall t \in [0, T], \ \forall \xi \in \mathbb{Z}, \]
which implies that
\[ |\hat{u}(t, \xi)| \leq |\hat{g}(\xi)| + \int_0^t \left| \hat{f}(s, \xi) \right| ds, \quad \forall t \in [0, T], \ \forall \xi \in \mathbb{Z}. \]
Hence this is an example of operator that is well-posed in \( H^r \) regarding the Cauchy Problem (1.3).
On the other hand, let \( c_2(t) = -\frac{2e^{-1/t^2}}{t^3} \) and \( c_1(t) = \frac{2e^{-1/5t^2}}{5t^3} \), which means that \( C_2(t) = -e^{-1/t^2} \) and \( C_1(t) = e^{-1/5t^2} \). Consider \( f \equiv 0 \) and \( g(x) = \sum_{\eta \in \mathbb{Z}} e^{-|\eta|^{1/2}} e^{ix\eta} \); similarly to the first case we have

\[
\hat{u}(t, \eta) = e^{-|\eta|^{1/2}} \exp \left[ -e^{-1/t^2} \eta^2 + e^{-1/5t^2} \eta \right], \quad \forall t \in [0, T], \quad \forall \eta \in \mathbb{Z}.
\]

Consider, for \( n \in \mathbb{N} \) sufficiently large, \( t_n = \frac{1}{\sqrt{2} \log n} \). Then \( t_n^2 = \frac{1}{2 \log n} = \frac{1}{\log n^2} \Rightarrow -\frac{1}{t_n^2} = -\log n^2 \), which implies that

\[
\exp \left( -\frac{1}{t_n^2} \right) = \frac{1}{n^2} \quad \text{and} \quad \exp \left( -\frac{1}{5t_n^2} \right) = \exp \left( -\frac{\log n^2}{5} \right) = \frac{1}{n^{2/5}}.
\]

Therefore, for \( n \in \mathbb{N} \) sufficiently large,

\[
\hat{u}(t_n, n) = e^{-n^{1/2}} \exp \left[ -\frac{1}{n^2} n^2 + \frac{1}{5n^{2/5}} \right] = e^{3n/5 - n^{1/2} - 1} \Rightarrow \hat{u}(t_n, n) \to \infty \text{ when } n \to \infty.
\]

Hence the Cauchy Problem \([1,3]\) is not well-posed in \( C^2 \), which shows our assertion.

**REFERENCES**

[AACI] Arias Junior, A. & Ascanelli, A. & Cappiello M., *Gevrey well posedness for 3-evolution equations with variable coefficients*. Preprint (2021), https://arxiv.org/abs/2106.09511

[AACII] Arias Junior, A. & Ascanelli, A. & Cappiello M., *The Cauchy problem for 3-evolution equations with data in Gelfand-Shilov spaces*. Journal of Evolution Equations, Vol. 22 (33) (2022).

[ABZI] Ascanelli, A. & Boiti, C. & Zanghirati, L., *Well-posedness of the Cauchy problem for p−evolution equations*. J. Differential Equations Vol. 230 (10) (2012), 2765-2795.

[ABZII] Ascanelli, A. & Boiti, C. & Zanghirati, L., *A Necessary condition for \( H^\infty \) well-posedness of p-evolution equations*. Advances in Differential Equations, Vol. 21 (2016), 1165-1196.

[AC] Ascanelli, A. & Cappiello M., *Weighted energy estimates for p-evolution equations in SG classes*. Journal of Evolution Equations, Vol. 15 (3) (2015), 583-607.

[Bea] Beals, R., *Advanced mathematical analysis: periodic functions and distributions, complex analysis, Laplace transform and applications*, Vol. 12, Springer Science & Business Media (2013).

[CRI] Cicognani, M. & Reissig, M. *Well-posedness for degenerate Schrödinger equations*. Evolution Equations and Control Theory, 3 (1) (2014), 15-33.

[CC] Cicognani, M. & Colombini F., *The Cauchy problem for p-evolution equations*. Trans. Amer. Math. Soc. Vol. 362 (9) (2010), 4853-4869.

[Dre] Dreher M., *Necessary conditions for the well-posedness of Schrödinger type equations in Gevrey spaces*. Bull. Sci. Math. Vol. 127 (2003), 485-503.

[GW] Greenfield, S. J., & Wallach, N. R., *Global hypoellipticity and Liouville numbers*. Proceedings of the American Mathematical Society, 31(1) (1972), 112-114.

[I1] Ichinose, W., *Sufficient condition on \( H^\infty \) well-posedness for Schrödinger type equations*. Communications in Partial Differential Equations, 9 (1984), 33-48.

[I2] Ichinose, W., *On \( L^2 \) well-posedness of the Cauchy problem for Schrödinger type equations on the Riemannian manifold and the Maslov theory*. Duke Mathematical Journal, 56 (3) (1988), 549-588.

[KB] Kajitani, K. & Baba A., *The Cauchy problem for Schrödinger type equations*. Bulletin des Sciences Mathématiques, 119 (5) (1995), 459-473.

[KG] Kumano-Go, H. *Pseudo-differential operators*. The MIT Press, Cambridge, London, 1982.

[Kom] Komatsu, H., *Projective and injective limits of weakly compact sequences of locally convex spaces*. Journal of the Mathematical Society of Japan, 19(3), 366-383 (1967).

[LV] de Lessa Victor, B., *Fourier analysis for Denjoy–Carleman classes on the torus*. Annales Fennici Mathematici, 46(2), 869-895 (2021).
[Ma] Matsuzawa T., *On some degenerate parabolic equations*. Nagoya Mathematical Journal, Vol. 51 (1973), pp. 57–77.

[Miz] Mizohata, S. *On the Cauchy problem*, Vol. 3, Academic Press.

[Mor] Morimoto, M., *An introduction to Sato’s hyperfunctions*, American Mathematical Society, Vol. 129, 1993.

[P] Petrowsky, I. G., *Über das Cauchysche Problem für ein System linearer partieller Differentialgleichungen im Gebiete der nicht analytischen Funktionen*, Bull. Univ. Moscow, Ser. Int., 1 (7) (1938), 1-74.

[T] Tsutsumi, C. *The fundamental solution for a degenerate parabolic pseudo-differential operator*, Proceedings of the Japan Academy, 50 (1) (1974), 11-15.

[Yos] Yosida, K., *Functional Analysis* (6th ed.), New York: Springer Berlin Heidelberg, 1965.

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