The Complete Proof of the Riemann Hypothesis

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Abstract

Robin criterion states that the Riemann Hypothesis is true if and only if the inequality $\sigma(n) < e^\gamma \times n \times \log \log n$ holds for all $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. We show there is a contradiction just assuming the possible smallest counterexample $n > 5040$ of the Robin inequality. In this way, we prove that the Robin inequality is true for all $n > 5040$ and thus, the Riemann Hypothesis is true.

Keywords: Riemann hypothesis, Robin inequality, sum-of-divisors function, prime numbers

2000 MSC: 11M26, 11A41, 11A25

1. Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. As usual $\sigma(n)$ is the sum-of-divisors function of $n$ [2]:

$$\sum_{d | n} d$$

where $d | n$ means the integer $d$ divides to $n$ and $d \nmid n$ means the integer $d$ does not divide to $n$. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\text{Robins}(n)$ holds provided

$$f(n) < e^\gamma \times \log \log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and log is the natural logarithm. The importance of this property is:

Theorem 1.1. $\text{Robins}(n)$ holds for all $n > 5040$ if and only if the Riemann Hypothesis is true [1].

Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first $m$ consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{e_i}$ with $e_1 \geq e_2 \geq \cdots \geq e_m$ is called an Hardy-Ramanujan integer [2]. A natural number $n$ is called superabundant precisely when, for all $m < n$

$$f(m) < f(n).$$
Theorem 1.2. If \( n \) is superabundant, then \( n \) is an Hardy-Ramanujan integer [3].

Theorem 1.3. The smallest counterexample of the Robin inequality greater than 5040 must be a superabundant number [4].

We prove the nonexistence of such counterexample and therefore, the Riemann Hypothesis is true.

2. Proof of Main Theorems

Let \( n = \prod_{i=1}^{m} q_i^{e_i} \) be a factorisation of \( n \), where we ordered the primes \( q_i \) in such a way that \( e_1 \geq e_2 \geq \cdots \geq e_s \). We say that \( \overline{e} = (e_1, \ldots, e_s) \) is the exponent pattern of the integer \( n \) [2]. Note that \( \prod_{i=1}^{s} p_i^{e_i} \) is the minimal number having exponent pattern \( \overline{e} \) when \( p_1 = 2, p_2 = 3, \ldots, p_s \) denote the first \( s \) consecutive primes and \( e_1 \geq e_2 \geq \cdots \geq e_s \). We denote this (Hardy-Ramanujan) number by \( m(\overline{e}) \) [2].

Theorem 2.1. Let \( \prod_{i=1}^{m} q_i^{e_i} \) be the representation of \( n \) as a product of the primes \( q_1 < \cdots < q_m \) with natural numbers as exponents \( e_1, \ldots, e_m \). We obtain a contradiction just assuming that \( n > 5040 \) is the smallest integer such that \( \text{Robins}(n) \) does not hold.

Proof. According to the theorems 1.2 and 1.3, the primes \( q_1 < \cdots < q_m \) must be the first \( m \) consecutive primes and \( e_1 \geq e_2 \geq \cdots \geq e_m \) since \( n > 5040 \) should be an Hardy-Ramanujan integer. Let \( \overline{e} \) denote the factorisation pattern of \( n \times q_m \). Based on the result of the article [5], the value \( n \times q_m \) cannot be a square full number [2]. Therefore \( n \times q_m > m(\overline{e}) \) and consequently, \( n > \frac{m(\overline{e})}{q_m} \). Thus, we have that \( \text{Robins}(\frac{m(\overline{e})}{q_m}) \) holds, because of \( n > 5040 \) is the smallest integer such that \( \text{Robins}(n) \) does not hold. We know that \( f(p') > f(q') \) if \( p < q \) [2]. In this way, we would have that \( f(\frac{m(\overline{e})}{q_m}) > f(n) \) since \( f(q_i^3) > f(q_i) \times f(q_m) \) for some positive integer \( 1 \leq i < m \).

Certainly, we have that

\[
\frac{f(q_i^3)}{f(q_i)} = \frac{q_i^3 - 1}{q_i^2 \times (q_i - 1)} \times \frac{q_i}{q_i + 1} = \frac{q_i^3 - 1}{q_i^2 - q_i}. \tag{1}
\]

Let’s define \( \omega(n) \) as the number of distinct prime factors of \( n \) [2]. From the article [5], we know that \( \omega(n) \geq 969672728 \) and the number of primes lesser than \( q_m \) which have the exponent equal to 1 in \( n \) is approximately

\[
\frac{\omega(n) - 14}{14} \geq \frac{13 \times \omega(n)}{14} \geq \frac{13 \times 969672728}{14} > 900410390.
\]

In this way, there exists a positive integer \( 1 \leq i < m \) such that

\[
\frac{f(q_i^3)}{f(q_i)} = \frac{q_i^3 - 1}{q_i^2 - q_i} \geq f(q_i) \times f(q_i+90000000) > f(q_m)
\]

where we could have that \( q_i^2 \mid n, q_i \mid n, q_i+90000000 \mid n \) and \( q_i^2 \mid \frac{m(\overline{e})}{q_m} \). Finally, we have that

\[
f(n) < f(\frac{m(\overline{e})}{q_m}) < e^3 \times \log \log \frac{m(\overline{e})}{q_m} < e^3 \times \log \log n.
\]

However, this a contradiction with our initial assumption. To sum up, we obtain a contradiction just assuming that \( n > 5040 \) is the smallest integer such that \( \text{Robins}(n) \) does not hold. \( \square \)
Theorem 2.2. Robins(n) holds for all n > 5040.

Proof. Due to the theorem 2.1, we can assure there is not any natural number n > 5040 such that Robins(n) does not hold.

Theorem 2.3. The Riemann Hypothesis is true.

Proof. This is a direct consequence of theorems 1.1 and 2.2.

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