Edge-Vertex Dominating Set in Unit Disk Graphs

Vishwanath R. Singireddy, Manjanna Basappa*

CSIS Department, BITS Pilani, Hyderabad Campus, Jawahar Nagar, Hyderabad, 500078, Telangana, India

ARTICLE INFO

Keywords:
Unit disk graph
Edge-vertex dominating set
Approximation scheme

ABSTRACT

Given an undirected graph $G = (V, E)$, a vertex $v \in V$ is edge-vertex (ev) dominated by an edge $e \in E$ if $v$ is either incident to $e$ or incident to an adjacent edge of $e$. A set $S^\text{ev} \subseteq E$ is an edge-vertex dominating set (referred to as ev-dominating set) of $G$ if every vertex of $G$ is ev-dominated by at least one edge of $S^\text{ev}$. The minimum cardinality of an ev-dominating set is the ev-domination number. The edge-vertex dominating set problem is to find the minimum ev-domination number. As defined above, an edge-vertex dominating set (EVDS) is a set of edges such that every $e \in V$ is incident to an edge $e$ in EVDS or adjacent to a vertex that is incident to an edge $e$ in EVDS. The EVDS problem is to find the minimum cardinality of such a set.

The minimum cardinality of an ev-dominating set is either in CDS or adjacent to vertices of CDS, and all vertices in CDS are connected. The CDS problem is to find the minimum cardinality of such set (see Figure 1(c)). As defined above, an edge-vertex dominating set (EVDS) is a set of edges such that every $e \in V$ is incident to an edge $e$ in EVDS or adjacent to a vertex that is incident to an edge $e$ in EVDS. The EVDS problem is to find the minimum cardinality of such a set (see Figure 1(b)).

The minimum cardinality of an ev-dominating set is either in CDS or adjacent to vertices of CDS, and all vertices in CDS are connected. The CDS problem is to find the minimum cardinality of such set (see Figure 1(c)). As defined above, an edge-vertex dominating set (EVDS) is a set of edges such that every $e \in V$ is incident to an edge $e$ in EVDS or adjacent to a vertex that is incident to an edge $e$ in EVDS. The EVDS problem is to find the minimum cardinality of such a set (see Figure 1(b)).

The minimum cardinality of an ev-dominating set is either in CDS or adjacent to vertices of CDS, and all vertices in CDS are connected. The CDS problem is to find the minimum cardinality of such set (see Figure 1(c)). As defined above, an edge-vertex dominating set (EVDS) is a set of edges such that every $e \in V$ is incident to an edge $e$ in EVDS or adjacent to a vertex that is incident to an edge $e$ in EVDS. The EVDS problem is to find the minimum cardinality of such a set (see Figure 1(b)).

The minimum cardinality of a connected dominating set (CDS) of a graph $G = (V, E)$ is defined as the set of vertices such that every vertex $v \in V$

1. Introduction

Given an undirected graph $G = (V, E)$, the edge neighborhood of an edge $e' \in E$ is the set of edges in $E$ which share a common vertex $v \in V$ with $e'$, i.e., the set of all edges which are adjacent to $e'$. The set of these neighbors of $e'$ is represented as the set $N_e(e') = \{ f \in E \mid e' \cap f \neq \emptyset \}$ and $f$ share a common vertex $v \in V$. The closed edge neighborhood of $e'$ is defined as $N_e(e') = N_e(e') \cup \{ e' \}$. The edge neighborhood of a set $S \subseteq E$ is $N_e(S) = \bigcup_{e' \in S} N_e(e')$. Similarly, the closed edge neighborhood of a set $S \subseteq E$ is $N_e(S) = \bigcup_{e' \in S} N_e(e') \cup S$. The edge neighborhood of $e'$ is $N_2(e') = N_e(N^{-1}_e(e'))$. Similarly, the r-th edge neighborhood is $N_r(e') = N_e(N^{-1}_e(e'))$ for an integer $r \geq 1$.

Given an undirected graph $G = (V, E)$, a vertex $v \in V$ is ev (edge-vertex)-dominated by an edge $e \in E$ if $v$ is incident to $e$ (i.e., an endpoint of $e$) or if $v$ is incident to an adjacent edge of $e$. A set $S^\text{ev} \subseteq E$ is an edge-vertex dominating set (EVDS) (referred to as ev-dominating set) of $G$ if at least one edge of $S^\text{ev}$ (at least two edges for double edge-vertex dominating set) dominates every vertex of $G$. The minimum cardinality of an ev-dominating set is the ev-domination number, denoted by $\gamma^\text{ev}(G)$. If $S^\text{ev}$ is an edge-vertex dominating set and every edge in $S^\text{ev}$ have an endpoint in $S^\text{ev}$, then $S^\text{ev}$ is called the total edge-vertex dominating set, and the minimum cardinality of this set is called the total edge-vertex dominating number and it is denoted by $\gamma_t^\text{ev}$. In the literature, analogous to the ev-vertex dominating set there is a variant of the dominating set called vertex-edge dominating set. Another similar domination model is connected domination model. However, these models are different from ev-dominating. We distinguish between connected domination and edge-vertex domination models with counter-examples below. A connected dominating set (CDS) of a graph $G = (V, E)$ is defined as the set of vertices such that every vertex $v \in V$
the distance between the base station and mining machines due to the transport limitation. Hence, we need to establish the number of base stations as minimum as possible so that every mining machine will get service from at least one of them within some specified distance.

We can model a solution to this as follows: Consider each mining machine as a graph node, there is an edge between two machines if they are at most some distance \( d \), and the distance between every mining machine to the nearest base station is at most \(1.5d\) (i.e., the maximum distance between a base station and a mining machine). We can consider this graph as a unit disk graph by setting \( d = 1 \). Now, find the minimum cardinality edge-vertex dominating set \( S^{ev} \) of this graph. The base stations need to be established such that they are away from the mining machines to avoid nuisance generated by these machines. Hence, it is feasible to establish these base stations, one at the midpoint of each edge \( e \in E \).

2. Related Work

The *edge-vertex dominating set* and *vertex-edge dominating set* in a graph were introduced by Peters [1]. The edge-vertex dominating set and vertex-edge dominating set problems are \(\text{NP}\)-complete, even when restricted to bipartite graphs [2]. For every nontrivial tree \( T \), an upper bound on \( \gamma_{ev}(T) \) is \( (\gamma_f(T) + s - 1)/2 \) where \( s \) is the number of support vertices (the vertex adjacent to a leaf) [3]. The *total domination number* \( \gamma_t \) of a tree is equal to the *ev-dominating number* \( \gamma_{ev} \) plus one [4]. The vertex-edge dominating set problem in \( UDG \) is \(\text{NP}\)-complete [5]. Also, in [5], a polynomial time approximation scheme (PTAS) is proposed. Finding \( \gamma_{ve} \) even in cubic planar graphs is \(\text{NP}\)-hard [6]. The vertex-edge domination problem can be solved in linear time on block graphs [7]. In the same paper it is also shown that finding \( \gamma_{ve} \) in undirected path graphs is \(\text{NP}\)-complete. Given a connected graph \( G \) with \( n \) vertices where \( n \geq 6 \), then we have \( \gamma_{ev}(G) \leq \left\lceil \frac{n}{2} \right\rceil \) [8]. Boutrig et al. [9] gave an upper bound for the independent ve-dominating number in terms of the ve-dominating number for connected \( K_{1,k} \)-free graph with \( k \geq 3 \) and also gave an upper bound on the ve-dominating number for connected \( C_4 \)-free graph.

The double vertex-edge domination was introduced by Krishnakumari et al. [10]. They showed that finding \( \gamma_{dev} \) in a bipartite graph is \(\text{NP}\)-complete and also proved that for every non-trivial connected graph \( G \), \( \gamma_{dev}(G) \geq \gamma_{ev}(G) + 1 \), and \( \gamma_{dev}(T) = \gamma_{ev}(T) + 1 \) or \( \gamma_{dev}(T) = \gamma_{ev}(T) + 2 \) for any tree \( T \). Finding \( \gamma_{dev} \) in chordal graphs is \(\text{NP}\)-complete [11]. They gave a linear time algorithm to find \( \gamma_{dev} \) in proper interval graphs and also showed that finding \( \gamma_{dev} \) in general graphs with vertices having degree at most 5 is \(\text{APX}\)-complete. The double version of edge-vertex domination was studied by Sahin and Sahin [12]. They also gave the relationship between \( \gamma_{dev} \) and \( \gamma_{dev}, \gamma_t, \gamma_{ev} \) for trees and graphs, and also gave formulas to determine the *double ev-dominating number* of paths and cycles. Sahin and Sahin [13] proved that the total ev-dominating set problem is \(\text{NP}\)-hard for bipartite graphs. They also showed that \( (n - l + 2s - 1)/2 \) is the upper bound for \( \gamma'_{ev} \) for a tree \( T \) with order \( n \), \( l \) leaves and \( s \) supporting vertices. To the best of our knowledge, in the literature, ev-dominating set problem is not yet studied in the context of geometric intersection graphs.

Figure 1: (a) Graph \( G(V, E) \) (b) The subset \( \{ed, gi\} \in E \) is the minimum edge-vertex dominating set of \( G \) (c) The subset \( \{a, d, e, f, j\} \in V \) is the minimum connected dominating set of \( G \).

Figure 2: EVDS of a graph
2.1. Our Contribution

In this article, we study the EVDS problem on unit disk graphs. We show that the decision version of this problem is \textit{NP}-complete in UDGs. We also prove that this problem on UDG admits a polynomial time approximation scheme (PTAS). We finally present a simple 5-factor linear-time approximation algorithm.

3. Hardness Results

In this section, we show that the decision version of the EVDS is \textit{NP}-complete, as stated below. We describe a polynomial time reduction from the vertex cover problem, which is known to be \textit{NP}-complete in planar graphs with maximum degree 3 [14], to this problem on UDG.

The EVDS problem on UDGs (EVDS-UDG)

\textbf{Instance:} A UDG \( G = (V, E) \) and a positive integer \( k \).

\textbf{Question:} Does there exist an edge-vertex dominating set \( S^{ev} \) of \( G \) such that \( |S^{ev}| \leq k \).

\textbf{Lemma 1.} ([15]) An embedding of a planar graph \( G = (V, E) \) with maximum degree 4 in the plane is possible such that this embedding uses only \( O(|V|^2) \) area and its vertices are at integer coordinates and its edges are drawn so that they are along the grid line segments of the form \( x = i \) or \( y = j \), for some \( i \) and \( j \).

Biedl and Kant [16] gave an algorithm that produces this kind of embedding in linear time (see Figure 3).

\textbf{Corollary 2.} ([5]) An embedding of a planar graph \( G = (V, E) \) with \( |V| \geq 3 \) and maximum degree 3 in the plane can be constructed in polynomial time, where the embedding is such that the vertices of \( G \) are at \((4i, 4j)\) and the edges of \( G \) are drawn as a sequence of consecutive line segments along the lines \( x = 4i \) or \( y = 4j \), for some \( i \) and \( j \).

\textbf{Lemma 3.} Let \( G = (V, E) \) be an instance of the vertex cover problem with number of edges at least 2. An instance \( G' = (V', E') \) of EVDS-UDG can be constructed from \( G \) in polynomial time.

\textbf{Proof.} The construction of \( G' = (V', E') \) from \( G = (V, E) \) is as follows: First we embed the graph \( G = (V, E) \) into the grid of size \( 4n \times 4n \) such that each edge of \( E \) is composed of a sequence of horizontal or vertical line segment(s) each of whose length is four units long, using one of the algorithms discussed in [17, 18]. The points \( \{p_1, p_2, \ldots, p_n\} \) are referred as the node points in the embedding with respect to the vertex set of \( G \) (see Figure 3(a) and 3(b)). For each edge of length greater than four units, we add a joint point at the joining of two line segments in the embedding other than the node points. Name these points as the joint points (see empty circles in Figure 4). Then for each line segment in the embedding we add three extra points such that each at a distance of 1 unit from one another, at least 1 unit from node points and joint points. Name these points as the added points (see filled square points in Figure 4).

Let \( A \) be the set of added points and \( J \) be the set of joint points. We construct an UDG \( G' = (V', E') \) where the vertex set \( V' = V \cup A \cup J \), and there is an edge between two vertices of \( V' \) if and only if the distance between them is at most 1 unit (see Figure 4). If \( I \) is the total number of line segments in the embedding, then \(|A| = 3I \) and \(|J| \) is at most \( I - |E| \). It follows from Lemma 1 that \( I \) is at most \( O(n^2) \). Clearly, the graph defined by the intersection of unit disks centered at points in \( V' \) is a unit disk graph. Since both the sets \(|V'| \) and \(|E'| \) are bounded by \( O(n^2) \), we can construct \( G' \) from \( G \) in polynomial time. \hfill \Box

\textbf{Lemma 4.} The EVDS-UDG problem is in \textit{NP}.

\textbf{Proof.} Given a subset \( S^{ev} \subseteq E \) and a positive integer \( k \), we can verify that \( S^{ev} \) is an edge-vertex dominating set of size at most \( k \) in polynomial time, by checking whether each vertex \( v \in V \) is edge-vertex dominated by an edge \( e \in S^{ev} \) in polynomial time. \hfill \Box

We prove the \textit{NP}-hardness of the EVDS-UDG problem by reducing the decision version of the vertex cover on planar graph with maximum degree 3 to it. Let \( G = (V, E) \) be a planar graph with maximum degree 3. Then from Lemma 3 we can construct an instance \( G' = (V', E') \) of EVDS-UDG in polynomial time.

Figure 3: (a) A planar graph \( G \) with max degree 3. (b) The embedding of \( G \) on a grid.
Lemma 5. \( G \) has a vertex cover of size at most \( k \) if and only if \( G' \) has an edge-vertex dominating set of size at most \( 3k + 1 \).

Proof. Let \( D \subseteq V \) be a vertex cover of \( G \) of cardinality at most \( k \). Let \( D' \) be the set of vertices of \( G' \) that correspond to the vertices in \( D \). Now, choose every edge of \( G' \) that is incident to every vertex in \( D' \). Represent these chosen edges as the set \( N' \). Since \( G' \) is constructed from \( G \), every vertex of \( G' \) has degree at most 3. Therefore, the cardinality of \( N' \) is at most \( 3k \). Now, choose an edge of each line segment apart from the edges of \( N' \), and let \( N'' \) be the set of those chosen edges. The cardinality of \( N'' \) is \( l \) since there are \( l \) line segments in the embedding. Therefore, we have \( |N'| + |N''| \leq 3k + l \). Let \( p_i \mapsto p_j \) be a path of \( G' \) corresponding to an edge \((v_i, v_j) \in E \) of \( G \). Since \( D \) is the set of vertices corresponding to the vertex cover of \( G \), either \( p_i \in D' \) or \( p_j \in D' \) (or both vertices are in \( D' \)). An edge for each line segment will be chosen as follows:

1. If only one vertex of \((p_i, p_j)\) is in \( D' \), then choose the third edge of line segment that is incident to that vertex in \( D' \). If both vertices of \((p_i, p_j)\) are in \( D' \) then choose third edge of line segment that is incident to either \( p_i \) or \( p_j \).

2. Next, choose every fourth edge from the chosen edge on each line segment by traversing through the line segments of \((p_i, p_j)\) until reaching \( p_j \) (see Figure 5 for illustration).

3. Apply the same process for each sequence of line segments of \( G' \) corresponding to each edge in \( G \).

Now, observe that \( S^{ev} = N' \cup N'' \) is the set of edges in \( G' \) such that every vertex in \( G' \) is ev-dominated by at least one edge \( e \in S^{ev} \) and \( |S^{ev}| \leq 3k + l \).

To prove the sufficiency, consider any edge-vertex dominating set \( S^{ev} \subseteq E' \) of size at most \( 3k' + l \) in \( G' \), where \( k' \) is a positive integer.

Consider a simple path \( p_i \mapsto p_j \) in \( G' \) between a pair of node points \( p_i \) and \( p_j \) (say, \( p_i \mapsto p_j \) in Fig. 6) such that the other vertices the path \( p_i \mapsto p_j \) passes through are only joint points and added points. Now we want to ensure that for any such path all the incident edges at one of \( p_i \) and \( p_j \) are part of \( S^{ev} \). Observe that if both \( p_i \) and \( p_j \) are ev-dominated by only edges of \( E(p_i \mapsto p_j) \cap S^{ev} \), then \( |E(p_i \mapsto p_j)|/4 \) edges are not sufficient to ev-dominate all the vertices of \( p_i \mapsto p_j \). Hence, the number of edges in \( E(p_i \mapsto p_j) \cap S^{ev} \) is greater than \( |E(p_i \mapsto p_j)|/4 \). Then, we can remove all the edges in \( E(p_i \mapsto p_j) \cap S^{ev} \) and update \( S^{ev} \) such that at least one edge of them is now incident at \( p_i \) or \( p_j \) (consider the simple path \( p_i \mapsto p_j \) in Figure 6, the dark edges are in \( S^{ev} \), then we need to add one more edge to \( S^{ev} \) to ev-dominate \( p_j \), i.e., either first or second edge from \( p_j \) so that \( p_j \) will be ev-dominated).

We can repeat the above process for other such paths in \( G' \). However, if both or at least one of \( p_i \) and \( p_j \) is ev-dominated by edges outside \( p_i \mapsto p_j \), then we can simply select all the incident edges at one of them and replace the edges in \( E(p_i \mapsto p_j) \cap S^{ev} \) with their adjacent ones so that the other endpoint of \( p_i \mapsto p_j \) is ev-dominated. Note that the cardinality of the updated \( S^{ev} \) is still at most \( 3k' + l \), because \( \sum_{i,j}(|E(p_i \mapsto p_j) \cap S^{ev}| - 1) \leq l \), where \( k' \) is a positive integer.

Now, given such a \( S^{ev} \), we can select the vertices in \( G \) corresponding to those node points in \( G' \), whose all the incident edges are in \( S^{ev} \). Clearly, this set \( D \) of the selected vertices form a vertex cover of size at most \( k' \) in \( G \). The construction of \( D \) takes polynomial time. □

Theorem 6. The EVDS-UDG problem is \( \text{NP} \)-complete.

Proof. Follows from Lemma 4 and 5. □
4. Polynomial Time Approximation Scheme

In this section, we propose a PTAS for the EVDS set problem in a UDG. It is based on the concept of m-separated collection of subsets, which was introduced by Nieberg and Hurink [19]. This concept was used by many other authors to develop PTAS (for e.g., the Roman dominating set [20], minimum Liar’s dominating set [21], vertex-edge dominating set [5]). However, the way we use it here is quite different from these as we have to select edges to dominate vertices in the EVDS problem. Though our proof idea resembles in them, this is a completely different domination model. Let $G = (V, E)$ be a UDG. Let $h(e_1, e_2)$ denote the minimum number of edges in a simple path between the endpoints of the edges $e_1$ and $e_2$. Consider any two subsets $E_1 \subseteq E$ and $E_2 \subseteq E$, $h(E_1, E_2)$ is defined as the minimum number of edges between any two edges $e_1 \in E_1$ and $e_2 \in E_2$. We use $EVD(A)$ to denote an ev-dominating set and $EVD_{opt}(A)$ to denote the optimal ev-dominating set of the edge-induced subgraph corresponding to $A(\subseteq E)$ (i.e., the subgraph induced by the set of edges $A(\subseteq E)$ and the endpoints of edges in $A$).

Let $S$ be a set of $k$ pairwise disjoint subsets of $E$, i.e., $S_i \subseteq E$ for $i = 1, 2, \ldots, k$. If $h(S_i, S_j) \geq m$, for $i \geq 1$, $j \leq k$ and $i \neq j$, then $S$ is called as an $m$-separated collection of subsets of $E$ (see Figure 7 for $m = 4$).

**Lemma 7.** In a graph $G = (V, E)$, if $S = \{S_1, S_2, \ldots, S_k\}$ is a 4-separated collection of $k$ subsets of $E$, then

$$\sum_{i=1}^{k} |EVD_{opt}(S_i)| \leq |EVD_{opt}(E)|.$$  

**Proof.** Let $A_i$ be the set of edges which are adjacent to the edges of $S_i$ for each $i = 1, 2, \ldots, k$ and $R_i$ the set of edges such that $R_i = S_i \cup A_i$. The edges in sets $R_1, R_2, \ldots, R_k$ are pairwise disjoint, since the set $S$ is 4-separated collection of edges i.e., $(R_i \cap R_j) = \emptyset$, where $i \neq j$. Hence, the edges of $EVD_{opt}(E) \cap R_i$ will ev-dominate every vertex in $S_i$, since $EVD_{opt}(E)$ will ev-dominate every vertex $v \in V$. On the other hand, also $EVD_{opt}(S_i) \subseteq R_i$ ev-dominates every vertex of $S_i$, with a minimum number of edges of $G$. This implies that $|EVD_{opt}(S_i)| \leq |EVD_{opt}(E) \cap R_i|$. For all $k$ subsets of edges in the 4-separated collection $S$, we get

$$\sum_{i=1}^{k} |EVD_{opt}(S_i)| \leq \sum_{i=1}^{k} |(EVD_{opt}(E) \cap R_i)|$$

$$\leq |EVD_{opt}(E)|.$$

\[\square\]

The above Lemma 7 states that a 4-separated collection of subsets of edges $S$ will give a lower bound on the cardinality of an EVDS. Hence, we can get an approximation for the EVDS in $G$, if we are able to enlarge $S_i$ to subsets $Q_i \subseteq E$, in such way that EVDS of expansions are bounded locally and dominate every $v \in V$ globally.

**Figure 7:** 4-Seperated collection of edge sets $S = \{S_1, S_2, S_3, S_4, S_5\}$

**Lemma 8.** In a graph $G = (V, E)$, let $S = \{S_1, S_2, \ldots, S_k\}$ be a 4-separated collection of subsets of edges and $Q = \{Q_1, Q_2, \ldots, Q_k\}$ be a collection of subsets of $E$ with $S_i \subseteq Q_i$ for every $i = 1, 2, \ldots, k$. If there is a $\rho \geq 1$ such that

$$|EVD_{opt}(Q_i)| \leq \rho|EVD_{opt}(S_i)|$$

holds for every $i = 1, 2, \ldots, k$, and if $\bigcup_{i=1}^{k} EVD_{opt}(Q_i)$ is an edge-vertex dominating set of $G$, then $\sum_{i=1}^{k} |EVD_{opt}(Q_i)|$ is the $\rho$-approximation of minimum EVDS set of $G$.

**Proof.** From Lemma 7 we have,

$$\sum_{i=1}^{k} |EVD_{opt}(S_i)| \leq |EVD_{opt}(E)|.$$

Hence, $\sum_{i=1}^{k} |EVD_{opt}(Q_i)| \leq \rho \sum_{i=1}^{k} |EVD_{opt}(S_i)| \leq \rho|EVD_{opt}(E)|.$$

\[\square\]
In the following section we discuss a procedure to construct the subsets \( Q_i \subset E \), that contains a 4-separated collection of edges \( S_i \subset Q_i \), in such a way that a local \((1 + \epsilon)\)-approximation can be guaranteed. The union of the respective local \( EVDS \) will ev-dominates the entire vertex set of \( G \), which results in a global \((1 + \epsilon)\)-approximation for the \( EVDS \) problem.

4.1. Subset Construction

Here, we discuss the construction of the 4-separated collection of subsets of edges \( S = \{S_1, S_2, \ldots, S_k\} \) and respective subsets \( Q = \{Q_1, Q_2, \ldots, Q_k\} \) of \( E \) such that \( S_i \subset Q_i \) for every \( i = 1, 2, \ldots, k \). The basic idea of the algorithm is as follows. We start with an arbitrary edge \( e \in E \) and consider the \( r \)-th edge neighborhood of \( e \), for \( r = 0, 1, 2, \ldots, \) with \( N^r_e[e] = e \). We compute the \( EVDS \) for these edge neighborhoods until the following condition holds

\[
|EVD(N^r_e[e])| > \rho|EVD(N^r_e[e])|
\]

Let \( r_1 \) be the smallest \( r \) that violates the above inequality (1). Set \( S_1 = N^r_1[e], Q_1 = N^{r_1+4}_e[e] \) and \( E_{i+1} = E_i \setminus (N^{r_1+4}_e[e]) \), where \( E_1 = E \). We follow this procedure iteratively for each graph induced by \( E_{i+1} \) and until \( E_i = \emptyset \), finally returning the sets \( S = \{S_1, S_2, \ldots, S_k\} \) and \( Q = \{Q_1, Q_2, \ldots, Q_k\} \).

We find the edge-vertex dominating set of the \( r \)-edge neighborhood \( EVD(N^r_e[e]) \) of an edge \( e \), with respect to the graph \( G \) as follows. Find a maximal matching \( M \) for the graph induced by the edges of \( N^r_e[e] \). We can observe that the edges in \( M \) forms an edge-vertex dominating set for the graph induced by \( N^r_e[e] \). Hence, \( EVD(N^r_e[e]) = M \) as the following lemma says.

**Lemma 9.** A maximal matching \( M \) of the graph \( G' = (V', E') \) induced by the edges in \( N^r_e[e] \), is an \( EVDS \) of \( N^r_e[e] \).

**Proof.** For the contradiction, assume that \( M \) is not an \( EVDS \) of the graph \( G' = (V', E') \) induced by the edges in \( N^r_e[e] \). It means that there exist a vertex \( v \in V' \) which is incident to an edge \( e' \in E' \) such that \( N^r_e[e'] \cap M = \emptyset \). It contradicts that \( M \) is a maximal matching in \( G' \) as the set \( M \cup \{e'\} \) is a matching in \( G' \). Thus, the lemma follows.

**Lemma 10.** If \( G' = (V', E') \) is a UDG induced by the edges in \( N^r_e[e] \) and \( M \) is the maximal matching of \( G' \) then \( |EVD(N^r_e[e])| \leq O(r^2) \).

**Proof.** First we find a maximal matching \( M \), before finding the \( EVDS \) in \( G' = (V', E') \) which is induced by the edges of \( N^r_e[e] \). The number of edges in \( M \) of \( G' \) is bounded by the number of unit disks that are packed in a disk of radius \( r + 2 \) and centered at middle of the edge \( e \). Hence, \( |M| \leq (r + 2)^2 \) and cardinality of the \( EVD(N^r_e[e]) \) is bounded by \( |M| \) (see Lemma 9). Therefore, we have

\[
|EVD(N^r_e[e])| \leq |M| \leq (r + 2)^2 \leq O(r^2).
\]

**Theorem 11.** There exists an \( r_1 \) which violates the following inequality.

\[
|EVD(N^{r_1+4}_e[e])| > \rho|EVD(N^{r_1}_e[e])|
\]

where \( \rho = 1 + \epsilon \) and it is bounded by \( O(\frac{1}{\epsilon} \log \frac{1}{\epsilon}) \).

**Proof.** For the contradiction, assume that there exists an edge \( e \in E \) such that

\[
|EVD(N^{r+4}_e[e])| > \rho|EVD(N^r_e[e])|
\]

for all \( r \geq r_1 \). Then, from Lemma 10, we have

\[
(r + 6)^2 \geq |EVD(N^{r+4}_e[e])|.
\]

Hence, when \( r \) is even, we have

\[
(r + 6)^2 = |EVD(N^{r+4}_e[e])| > \rho|EVD(N^r_e[e])| > \cdots > \rho^2 |EVD(N^2_e[e])| \geq \rho^2
\]

and when \( r \) is odd, we have

\[
(r + 6)^2 = |EVD(N^{r+4}_e[e])| > \rho|EVD(N^r_e[e])| > \cdots > \rho^{\frac{r+1}{2}} |EVD(N^1_e[e])| \geq \rho^{\frac{r+1}{2}}
\]

Now, we can observe that in both the inequalities (2), (3) on the left hand side we have a polynomial in \( r \) which is at least the right hand side value which is exponential in \( r \), it is a contradiction. Therefore, for all \( r \geq r_1 \) the inequality (2) cannot hold, hence there exist such \( r_1 \). Ultimately, \( r_1 \) depends only on \( \rho \), not on the size of the edge-induced subgraph by \( N^{r+4}_e[e] \). As in [19], we can argue that \( r_1 \) is bounded by \( O(\frac{1}{\epsilon} \log \frac{1}{\epsilon}) \) where \( \rho = 1 + \epsilon \).

**Lemma 12.** For an edge \( e \in E \), \( EVD_{opt}(Q_i) \) can be computed in polynomial time.

**Proof.** From the way of construction of \( Q_i \), we can see that \( Q_i \subset N^{r+4}_e[e] \). The cardinality of \( N^{r+4}_e[e] \) is bounded by \( O(\sqrt{r}) \) (see Lemma 10), where \( r \) is bounded by \( O(\frac{1}{\epsilon} \log \frac{1}{\epsilon}) \) (see Theorem 11). Hence, we need at most \( O(n^2) \) possible combinations of tuples to check whether the selected tuple is an \( EVDS \) of \( Q_i \).

**Lemma 13.** \( \bigcup_{i=1}^{k} EVD(Q_i) \) is an edge-vertex dominating set in \( G = (V, E) \).

**Proof.** It follows from the construction of the collection of subsets of edges \( \{Q_1, Q_2, \ldots, Q_k\} \) that each edge that is incident to a vertex \( v \in V \) belongs to a specific subset \( Q_i \) and \( EVD(Q_i) \) is an \( EVDS \) of the graph induced by the edges of \( Q_i \). Therefore, every vertex \( v \in V \) is incident to at least one edge \( e \) such that at least one edge of \( N_v[e] \) is in \( \bigcup_{i=1}^{k} EVD(Q_i) \).
Corollary 14. \[ \bigcup_{i=1}^{n} \text{EV}_D(Q_i) \] is an edge-vertex dominating set in \( G = (V, E) \), for the collection of subsets of edges \( Q = \{Q_1, Q_2, \ldots, Q_n\} \).

Theorem 15. For a given unit disk graph and an \( \epsilon > 0 \), there exists a PTAS (an \( (1 + \epsilon) \)-approximation) algorithm for the EVDS problem with running time \( n^{\Omega(\epsilon^2)} \), where \( \epsilon = \left( \frac{1}{e} \right) \).

Proof. Follows from Corollary 14 and Lemma 12.

5. 5-Factor Approximation Algorithm

In this section, we present a 5-factor approximation algorithm for the EVDS problem on UDG. Let \( S \) be a set of \( n \) points given in the Euclidean plane. We join two of these points with an edge if the distance between those two points is less than or equal to 1 unit. Let \( E \) be the set of such edges with cardinality \( m \) and \( V \) be the set of vertices corresponding to points in \( S \). The graph induced by \( V \) and \( E \) will form a UDG since the distance between any two end-points of \( e \in E \) is at most 1. Assume that such an UDG has no isolated vertex, otherwise EVDS does not exist. To present an approximation algorithm, we consider a rectangular region that contains UDG, and then we partition the region into a set of hexagons. Let \( R \) be an axis-parallel rectangle that contains these points. Consider an hexagonal partition of \( R \), where each hexagon is of side length \( \frac{1}{2} \) and hence the maximum distance between any two points inside it is at most 1. Assume that no point in \( V \) lies on the boundary of any hexagon in the partition.

Lemma 16. Any edge \( e \) with its two endpoints lying in adjacent hexagons can ev-dominate every point in those two hexagons.

Proof. It follows from the fact that there will be an edge between any two points that lie in a hexagon since the distance between those two points will be at most 1. Therefore, an edge \( e \in EVDS \) whose endpoint lies in that hexagon will ev-dominate every other point in that hexagon.

The outline of the algorithm is as follows. Initialize the set \( S^{ev} \) of edges that are in EVDS as empty. Now, arbitrarily pick an edge \( e \in E \) whose endpoints lie in different hexagons and add this edge to \( S^{ev} \) and set \( E = E \setminus \{e\} \). Mark all points that are ev-dominated by \( e \). If there are any unmarked vertices, now choose an edge \( e \in E \) that is incident to unmarked vertex and having another endpoint in a different hexagon, add \( e \) to \( S^{ev} \) and mark all unmarked points that are ev-dominated by \( e \). Repeat this process until every point in \( V \) is marked (see Algorithm 1).

Theorem 17. Algorithm 1 gives a factor 5-approximation for EVDS problem on a UDG in \( O(m + n) \) time.

Proof. Algorithm 1 picks an edge \( e \) arbitrarily whose endpoints lie in different hexagons and then repeatedly selects an edge between an unmarked vertex and another vertex in the different hexagon until there are no unmarked vertices (see Figure 8). In Figure 8, we can observe that Algorithm 1 selected \( EVDS \) as \( \{e_1, e_2, e_3, e_4, e_5\} \) whose cardinality is five whereas the optimal solution may have a single edge that will ev-dominate every given point (see edge \( e \) in Figure 8).

Next, one can see that the algorithm may select at most five times the optimal value, since an edge between points in two adjacent hexagons may ev-dominate the points in all of its adjacent eight hexagons. As we look at every edge between points to know whether its endpoints are the marked vertices and select an edge at line 3 and 8 of Algorithm 1, the running time is polynomial in \( m \) and \( n \).

The approximation factor five of Algorithm 1 follows due to the following two facts:

1. If both the endpoints of an edge \( e \) selected by Algorithm 1 lie within the same hexagon, then none of the vertices corresponding to these points are adjacent to a vertex of its adjacent hexagons.

2. Otherwise an edge \( e \) selected by Algorithm 1 ev-dominate all the points in both the adjacent hexagons (Lemma 16).

All the cells (hexagons) in \( R \) can be grouped as a collection of mega-cells (as in Figure 9), where each mega-cell consists of ten adjacent hexagonal cells (cells colored with same color in Figure 9). Algorithm 1 picks at most five edges to ev-dominate all the points in each mega-cell, whereas in optimal solution at least one edge is required.

6. Discussion

We can assign weights \( w_e \) to the edges of UDG \( G = (V, E) \) such that \( w_e \) is the Euclidean distance between the endpoints of the edge \( e \in E \). We then have a weighted unit disk graph \( G \) and the goal is to find an ev-dominating set.
$S^{ev}$ in $G$, where $w(S^{ev}) = \sum_{e \in S^{ev}} w(e)$ is minimized. We believe that a motivation for this is that it is quite common for a pair of monitoring devices placed at the endpoints of an edge that is in the ev-dominating set to be such that the maintenance cost is minimized, where the cost is proportional to distance a maintenance staff has to travel back and forth between the sites of these two devices. It is not clear whether we can use the $m$-separated collection concept to extend our PTAS for the weighted variant of the problem. Hence, this could be a future work.

7. Conclusion

In this paper, we have proved that the edge-vertex dominating set problem in unit disk graphs is $\text{NP}$-complete. We also showed that the $\text{EVDS}$ problem admits a polynomial time approximation scheme (PTAS). We also gave a simple 5-factor approximation algorithm in linear time.

References

[1] K. Peters, Theoretical and algorithmic results on domination and connectivity (nordhaus-gaddum, gallai type results, max-min relationships, linear time, series-parallel). (1987).
[2] J. R. Lewis, Vertex-edge and edge-vertex domination in graphs, Ph.D. thesis, Ph. D. Thesis, Clemson University, Clemson, 2007.
[3] Y. B. Venkatakrishnan, B. Krishnakumari, An improved upper bound of edge-vertex dominating number of a tree, Information Processing Letters 134 (2018) 14–17.
[4] B. Krishnakumari, Y. Venkatakrishnan, M. Krzywkowski, On trees with total domination number equal to edge-vertex domination number plus one, Proceedings-Mathematical Sciences 126 (2016) 153–157.
[5] S. K. Jena, G. K. Das, Vertex-edge domination in unit disk graphs, Discrete Applied Mathematics (2021).
[6] R. Ziemann, P. Żyliński, Vertex-edge domination in cubic graphs, Discrete Mathematics 343 (2020) 112075.
[7] S. Paul, K. Ranjan, On vertex-edge and independent vertex-edge domination, in: International Conference on Combinatorial Optimization and Applications, Springer, 2019, pp. 437–448.
[8] P. Żyliński, Vertex-edge domination in graphs, Aequationes mathematicae 93 (2019) 735–742.
[9] R. Boutrig, M. Chellali, T. W. Haynes, S. T. Hedetniemi, Vertex-edge domination in graphs, Aequationes mathematicae 90 (2016) 355–366.
[10] B. Krishnakumari, M. Chellali, Y. B. Venkatakrishnan, Double vertex-edge domination, Discrete Mathematics, Algorithms and Applications 9 (2017) 1750045.
[11] Y. B. Venkatakrishnan, H. N. Kumar, On the algorithmic complexity of double vertex-edge domination in graphs, in: International Workshop on Algorithms and Computation, Springer, 2019, pp. 188–198.
[12] B. Şahin, A. Şahin, Double edge–vertex domination, in: International Conference on Intelligent and Fuzzy Systems, Springer, 2020, pp. 1564–1572.
[13] A. Şahin, B. Şahin, Total edge–vertex domination, RAIRO-Theoretical Informatics and Applications 54 (2020) 1.
[14] M. R. Garey, D. S. Johnson, Computers and intractability, volume 174, freeman San Francisco, 1979.
[15] L. G. Valiant, Universality considerations in vlsi circuits, IEEE Transactions on Computers 100 (1981) 135–140.
[16] T. Biedl, G. Kant, A better heuristic for orthogonal graph drawings, Computational Geometry 9 (1998) 159–180.
[17] J. Hopcroft, R. Tarjan, Efficient planarity testing, Journal of the ACM (JACM) 21 (1974) 549–568.
[18] A. Itai, C. H. Papadimitriou, J. L. Szwarcfiter, Hamilton paths in grid graphs, SIAM Journal on Computing 11 (1982) 676–686.
[19] T. Nieberg, J. Hurink, A ptas for the minimum dominating set problem in unit disk graphs, in: International Workshop on Approximation and Online Algorithms, Springer, 2005, pp. 296–306.
[20] W. Shang, X. Hu, The roman domination problem in unit disk graphs, in: International conference on computational science, Springer, 2007, pp. 305–312.
[21] R. K. Jallu, S. K. Jena, G. K. Das, Liar’s domination in unit disk graphs, Theoretical Computer Science 845 (2020) 38–49.