Abstract

We introduce a new approach to understand magnetization dynamics in ferromagnets based on the holographic realization of ferromagnets. A Landau-Lifshitz equation describing the magnetization dynamics is derived from a Yang-Mills equation in the dual gravitational theory, and temperature dependences of the spin-wave stiffness and spin transfer torque appearing in the holographic Landau-Lifshitz equation are investigated by the holographic approach. The results are consistent with the known properties of magnetization dynamics in ferromagnets with conduction electrons.
1 Introduction

The Landau-Lifshitz equation \[1\] is the fundamental equation for describing the dynamics of magnetization (density of magnetic moments) in various magnetic materials. It has been also playing a fundamental role in the development of modern spintronics \[2\]: For instance, its extension to the coupled systems of localized magnetic moments and conduction electrons has led to the concepts of spin transfer torque \[3, 4\] and spin pumping \[5\]. So far, the symmetries and reciprocity in electronic systems have been the guiding principles to develop such extensions. In this article, we introduce another guiding principle to explore the new extensions and magnetization dynamics on the basis of the holographic duality.

The holographic duality is the duality between the quantum many body system defined in \(d\)-dimensional space-time and the gravitational theory (with some matter fields) which lives in \((d+1)\)-dimensional space-time \[6, 7, 8\]. \footnote{See \[9\] for a recent review on the applications of the holographic duality to condensed matter physics.} We constructed a holographic dual model of three-dimensional ferromagnetic systems, which exhibits the ferromagnetic phase transition with spontaneous magnetization and the consistent magnetic properties at low temperatures \[10\]. \footnote{Other holographic approaches to ferromagnetic systems have been also discussed in \[11, 12, 13\].} In the holographic duality, finite temperature effect in ferromagnetic systems can be incorporated as the geometrical effect of black holes in higher dimensional bulk gravity, and the Wick rotation at finite temperatures is not required for the analysis in the dual gravitational theory. Thus, the novel analysis for real-time dynamics of quantum many body systems in nonequilibrium situations can be performed using the holographic approach (for a review, see \[14\]). In addition, the holographic duality is known to be a strong-weak duality, which relates strongly correlated quantum systems to classical gravitational theories. From these viewpoints, the holographic approach can provide new useful tools to analyze nonequilibrium and nonlinear dynamics of magnetization in ferromagnets.

In ferromagnets, spin currents are generated by magnetization dynamics. From the holographic dictionary between the quantities of ferromagnets and gravitational theory \[10\], the spin currents in ferromagnets correspond to the \(SU(2)\) gauge fields in the dual gravitational theory. This correspondence indicates that the dynamics of spin currents, consequently the dynamics of magnetization, can be described by the Yang-Mills equation for \(SU(2)\) gauge fields \[15\] in the holographic dual theory. In the following, we derive a Landau-Lifshitz equation for magnetization dynamics from the Yang-Mills equation within the holographic realization of ferromagnets. This derivation can provide novel perspectives for magnetization dynamics from the non-abelian gauge theory.
This article is organized as follows. In Section 2, we summarize the results of the magnetic properties obtained from the holographic realization of ferromagnets in thermodynamic equilibrium. An extension to nonequilibrium situation including the fluctuations of magnetization and spin currents is discussed in the dual gravitational theory, and the holographic equation of magnetization dynamics is derived in Section 3. In Section 4, temperature dependences of the parameters in the resulting holographic equation are investigated by numerical calculations. Finally, we summarize the results in Section 5.

2 Holographic Dual Model of Ferromagnets

We begin with a brief summary on the holographic dual model of ferromagnets [10]. The dual model is the five-dimensional gravitational theory with an SU(2) gauge field $A^a_M$ and a U(1) gauge field $B_M$, whose action is given by

$$
S = \int \sqrt{-g} \, d^5x \left[ \frac{1}{2\kappa^2} (R - 2\Lambda) - \frac{1}{4e^2} G_{MN} G^{MN} - \frac{1}{4g^2} F^a_{MN} F^a^{MN} - \frac{1}{2} (D_M \phi^a)^2 - V(|\phi|) \right].
$$

Here, $R$ is the scalar curvature of space-time, and the field strength is defined by $F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + \epsilon^{abc} A^b_M A^c_N$ and $G_{MN} = \partial_M B_N - \partial_N B_M$, respectively. The index $a$ labels spin directions in the SU(2) space ($a = 1 \sim 3$), the index $M$ labels space-time directions in five dimensions ($M, N = 0 \sim 4$), and $\epsilon^{abc}$ is a totally anti-symmetric tensor with $\epsilon^{123} = 1$. The model also includes a triplet scalar field $\phi^a$ with the covariant derivative $D_M \phi^a = \partial_M \phi^a + \epsilon^{abc} A^b_M \phi^c$, and the SU(2)-invariant scalar potential $V(|\phi|)$ with the norm $|\phi|^2 = \sum_{a=1}^{3} (\phi^a)^2$. Note that the scalar field is neutral under the U(1) gauge transformation. In order to guarantee asymptotic Anti-de Sitter (AdS) backgrounds, the negative cosmological constant $\Lambda = -6/\ell^2$ is introduced. The field-operator correspondence in the holographic duality [7 8] leads to the following holographic dictionary between the fields of the dual gravitational theory and the physical quantities of ferromagnets:

| Dual gravity | Ferromagnet |
|--------------|-------------|
| Scalar field $\phi^a$ | Magnetization $M^a$ |
| SU(2) gauge field $A^a_M$ | Spin current $J^a_\mu$ |
| U(1) gauge field $B_M$ | Charge current $J_\mu$ |
| Metric $g_{MN}$ | Stress tensor $T_{\mu\nu}$ |

Table 1: Holographic dictionary between the dual gravitational theory and ferromagnets.
2.1 Black Hole as Heat Bath

In order to establish the holographic dictionary, thermodynamical properties of the physical quantities of ferromagnets should be calculated in the dual gravitational theory. In Ref. [10], the temperature dependences of magnetic quantities and the behavior of ferromagnetic phase transition are thoroughly discussed. In the context of the holographic duality, finite temperature effects in the ferromagnets can be incorporated by introducing the black holes into the dual gravitational theory as the background space-time. Indeed, the dual gravitational theory has the charged black hole solution which is a solution to the Einstein, Yang-Mills, and Maxwell equations derived from the action (1):

\[
R_{MN} + \left( \Lambda - \frac{1}{2} R \right) g_{MN} = \frac{k^2}{2e^2} \left( 2G_{KM}G^K_N - \frac{1}{2} G_{KL}G^{KL}g_{MN} \right) + \frac{k^2}{2g^2} \left( 2F^a_{KM}F^{aK}_N - \frac{1}{2} F_{aKL}F^{aKL}g_{MN} \right),
\]

(2)

\[
\nabla_M F^{aMN} + \epsilon^{abc} A^b_M F^{cMN} = 0, \quad \nabla_M G^{MN} = 0,
\]

(3)

where \( \nabla_M \) is the covariant derivative for the affine connection, and the space-time indices \( M, N \) are raised or lowered by the bulk metric \( g_{MN} \). Here, we neglect the contribution from the scalar field and set \( \phi^a = 0 \) for the background. The metric of the black hole\(^3\) is given by

\[
ds^2 = g_{MN} dx^M dx^N = \frac{r^2}{\ell^2} \left( - f(r) dt^2 + dx^2 + dy^2 + dz^2 \right) + \frac{\ell^2}{f(r)} \frac{dr^2}{r^2},
\]

(4)

with the radial function,

\[
f(r) = 1 - (1 + Q^2) \left( \frac{TH}{r} \right)^4 + Q^2 \left( \frac{TH}{r} \right)^6.
\]

(5)

Here, we define the parameter \( Q \):

\[
Q^2 = \frac{2k^2}{3} \left( \frac{\mu_e^2}{e^2} + \frac{\mu_s^2}{g^2} \right).
\]

(6)

The U(1) charge \( \mu_e \) and SU(2) charge \( \mu_s \) of the black hole are supported by the time components of the gauge fields,

\[
B_0 = \mu_e \left( \frac{TH}{\ell} \right) \left( 1 - \frac{r_H^2}{r^2} \right) \quad \text{and} \quad A^3_0 = \mu_s \left( \frac{TH}{\ell} \right) \left( 1 - \frac{r_H^2}{r^2} \right).
\]

(7)

Note that the black hole solution (4) is asymptotically AdS at \( r \to \infty \), and has the (outer) horizon \( r = r_H \).

\(^3\)This type of non-abelian black holes has been discussed in the context of the holographic duality, in the literature such as [11, 16].
For the following discussion, we make a coordinate change of the radial coordinate $r$ into $u$ by $u = 1/r$, and the black hole metric becomes

$$ds^2 = \frac{1}{u^2} \left( -f(u) \, dt^2 + dx^2 + dy^2 + dz^2 + \frac{du^2}{f(u)} \right),$$

(8)

and the transformed function $f(u)$ is given by

$$f(u) = 1 - \left(1 + Q^2\right) u^4 + Q^2 u^6,$$

(9)

where we have set the coupling parameters $e = g = 1$ and the black hole parameters $r_H = \ell = 1$, for simplicity.

In the holographic dual model, the black hole (8) plays the role of the heat bath; due to the Hawking radiation, the black hole temperature is given by

$$T = \frac{2 - Q^2}{2\pi},$$

(10)

and the calculations on the black hole background lead to the thermodynamical properties of the corresponding ferromagnet. Since we focus only on the dynamics of magnetization and spin current, the background space-time is fixed to be the black hole metric (8) in the following.

2.2 Thermodynamics of Ferromagnets from Scalar Dynamics on Charged Black Hole

In order to investigate the thermodynamics of magnetization, we examine the equation of motion for the scalar field $\phi^a$, which is also derived from the action (1):

$$\frac{1}{\sqrt{-g}} \partial_M \left( \sqrt{-g} D^M \phi^a \right) + \varepsilon^{abc} A^b_M D^M \phi^c = \frac{\partial V}{\partial \phi^a}.$$ 

(11)

Here, we consider a static and homogeneous solution in the boundary coordinates, $x^\mu = (t, x^1, x^2, x^3)$, which corresponds to the homogeneous magnetization in ferromagnets. Without loss of generality, the ansatz for such a scalar field, which is invariant under the translations on the boundary, is given by

$$\phi^1 = \phi^2 = 0, \quad \phi^3 = \Phi(u) \neq 0.$$

(12)

Inserting this ansatz, the metric (8), and the gauge fields (7) into the equation (11), we obtain the following equation for $\Phi(u)$:

$$u^2 f(u) \frac{d^2 \Phi}{du^2} + \left( u^2 \frac{df(u)}{du} - 3 u f(u) \right) \frac{d\Phi}{du} = \frac{\partial V}{\partial \Phi}.$$ 

(13)
This equation governs the thermodynamics of magnetization in the dual gravitational theory. We can analyze the solution to this equation numerically with a simple quartic potential

\[ V(|\phi|) = \lambda \left( |\phi|^2 - m^2 / \lambda \right)^2 / 4, \]

and the asymptotic behavior of the numerical solution near the boundary \( u \sim 0 \) (or \( r \sim \infty \)) is obtained:

\[ \Phi(u) \simeq H_0 u^{\Delta_-} + M(T) u^{\Delta_+} \quad \left( \Delta_\pm = 2 \pm \sqrt{4 - m^2} \right). \quad (14) \]

According to the standard recipe in the holographic duality [17, 18], the coefficients \( H_0 \) and \( M(T) \) in the asymptotic expansion correspond to an external magnetic field and a magnetization at temperature \( T \) (under \( H_0 \)), respectively. In Ref. [10], the resulting temperature dependences of magnetization, magnetic susceptibility, and specific heat have been shown to reproduce the ferromagnetic phase transition in the mean field theory. Furthermore, the temperature dependences at low temperatures are also consistent with the existence of the spin wave excitations (magnons) and conduction electrons in low-temperature ferromagnets.

For later convenience, we also comment on the solutions for the gauge fields. Assuming the translational and rotational invariance on the boundary, equilibrium solutions for the gauge fields are given by the following form:

\[ B_0 = b(u) \quad \text{and} \quad A_3^0 = a^3(u), \quad (15) \]

where all the other components vanish. Inserting this ansatz and (12), the Maxwell and Yang-Mills equations on the black hole are reduced to the following simple forms:

\[ \frac{d}{du} \left( \frac{1}{u} \frac{db}{du} \right) = 0 \quad \text{and} \quad \frac{d}{du} \left( \frac{1}{u} \frac{d a^3}{du} \right) = 0. \quad (16) \]

The general solutions are given by the forms (7) in terms of \( u \),

\[ b(u) = \mu_e \left( 1 - u^2 \right) \quad \text{and} \quad a^3(u) = \mu_s \left( 1 - u^2 \right). \quad (17) \]

Here, we impose the boundary conditions \( B_0 = 0 \) and \( A_3^0 = 0 \) at the horizon (\( u = 1 \)), which guarantee the regularity of the gauge fields on the horizon. The remaining integral constants, \( \mu_e \) and \( \mu_s \), correspond respectively to the electrochemical potential of underlying electrons and the spin chemical potential (or spin voltage), through the holographic dictionary.

To summarize, the solutions (14) and (17) on the charged black hole describe the thermodynamical property of the holographic dual ferromagnets in the equilibrium.

## 3 Magnetization Dynamics in Holographic Ferromagnets

In this section, we extend the holographic analysis in the equilibrium, summarized in the previous section, to more general situations including the dynamics of magnetization and
spin currents. In order to discuss the dynamics of magnetization and spin currents, the static and homogeneous ansatze for the scalar field (12) and the gauge fields (15) need to be generalized. Here, we focus on the dynamics with the long wave length in the ordered phase (symmetry broken phase) below the Curie temperature, where various phenomena in modern spintronics are intensively studied.

3.1 Generalized Ansatz and Effective Equations of Motion

For the scalar field, following the standard derivation of the equation for magnetization dynamics, we consider the generalized ansatz for the scalar field as a factorized form:

\[ \phi^a(u, t, x) = \Phi(u) n^a(t, x) \quad \text{with} \quad \sum_{a=1}^{3} n^a n^a = 1, \]

where \( \Phi(u) \) is a solution of the equation (13) with the asymptotic behavior (14). Note that, since we focus only on the dynamics of spontaneous magnetization, we fix \( H_0 = 0 \) throughout this article. In this ansatz, \( n^a(t, x) \) corresponds to the (local) direction of magnetization in ferromagnets.

In ferromagnetic systems, the magnetization dynamics generates various dynamics of spin currents [2]. In the holographic dual theory, the scalar dynamics is also expected to induce the dynamics of the corresponding \( SU(2) \) gauge field, and thus we generalize the static and homogeneous ansatz for the \( SU(2) \) gauge fields to the following factorized forms:

\[
\begin{align*}
A_\parallel^\mu(u, t, x) &= (1 - u^2) a_\parallel^\mu(t, x), \\
A_\perp^\mu(u, t, x) &= (1 - u^2) a_\perp^\mu(t, x), \\
A_i^\parallel(u, t, x) &= G^\parallel(u) a_i^\parallel(t, x), \\
A_i^\perp(u, t, x) &= G^\perp(u) a_i^\perp(t, x) \quad (i = 1 \sim 3),
\end{align*}
\]

where we set the radial component \( A_\parallel^\mu \equiv 0 \) by using the gauge degrees of freedom. Due to the nontrivial scalar solution \( \Phi(u) \), corresponding to the spontaneous magnetization, the \( SU(2) \) gauge symmetry is broken to \( U(1) \). The gauge fields can be correspondingly decomposed into an unbroken component \( A_\parallel^\mu \) and two broken components \( A_\perp^\mu \), which are defined by \( A_\parallel^\mu \propto n^a \) and \( n \cdot A_\perp^\mu = 0 \), respectively. As in the case of the static solutions, the time components of gauge fields should satisfy the horizon boundary condition, \( A_0^\mu = 0 \) at \( u = 1 \), for the regularity. Although the spatial components \( A_i^a \) are not required to vanish on the horizon, the regularity (or finiteness) at the horizon is required. The asymptotic solutions to the linearized Yang-Mills equation near the boundary \( (u \sim 0) \) give the asymptotic expansions for
the radial functions \( G^\parallel(u) \), and \( G^\perp(u) \),
\[
G^\parallel(u) = 1 - \sigma^\parallel \, u^2 + \mathcal{O}(u^4),
\]
\[
G^\perp(u) = 1 + \sigma^\perp \, u^2 + \mathcal{O}(u^4).
\]
(20)

We discuss the concrete numerical solutions of \( G^a(u) \) and their physical implications in the next section.

Since the scalar field \( \phi^a \) does not have the \( U(1) \) charge, the fluctuation (or dynamics) of \( \phi^a \) does not induce further dynamics for the \( U(1) \) gauge field, which implies the solution for \( B_\mu \) in (7) is unchanged, and we can neglect the dynamics of \( B_\mu \).

At first, we consider the equation of motion for the scalar field \( \phi^a \). Inserting the generalized ansatz (18) into the equation (11), we obtain the following equation for \( n^a \):
\[
\left[ u^5 \partial_u \left( u^{-3} f(u) \partial_u \Phi \right) - \frac{\partial V}{\partial \Phi} \right] n^a = \left[ \frac{u^2}{f(u)} D_D n^a - u^2 D_i D_i n^a \right] \Phi.
\]
(21)

Here, we have used the gauge condition \( A^a_u = 0 \), and the gauge covariant derivative is defined as
\[
D^\mu n^a = \partial^\mu n^a + \varepsilon^{abc} A^b_\mu n^c.
\]
The left-hand side of the equation (21) is proportional to the equation (13), and thus vanishes for the solution \( \Phi(u) \). Since \( \Phi(u) \) is a non-trivial solution, which is not identically zero, we have the effective equation of motion for \( n^a \):
\[
f^{-1} D_D n^a - D_i D_i n^a = 0.
\]
(22)

Next, the equation of motion for the gauge fields is considered. The Yang-Mills equation for the \( SU(2) \) gauge field \( A_M^a \) is derived by the variation of the holographic action (1) and given by
\[
\frac{1}{\sqrt{-g}} \partial_N \left( \sqrt{-g} F^{NM} A^a_N = J^M a, \right.
\]
(23)
where the \( SU(2) \) current is defined as
\[
J^a_M = \varepsilon^{abc} \phi^b D_M \phi^c = \varepsilon^{abc} \phi^b \left( \partial_M \phi^c + \varepsilon^{cde} A^d_M \phi^e \right).
\]
(24)

Unlike the static case, the generalized ansatz (18) and (19) give the non-vanishing currents:
\[
J^a_\mu = \Phi^2 \left( \varepsilon^{abc} n^b \partial_\mu n^c + \varepsilon^{abc} \varepsilon^{cde} n^b a^d_\mu n^e + \mathcal{O} \left( u^2 \right) \right).
\]
(25)

Note that the radial component of the currents still vanishes, \( J^a_u \equiv 0 \), due to the gauge fixing condition \( A^a_u \equiv 0 \). With this current, we can explicitly write down the Yang-Mills equations on the charged black hole (8), in the boundary direction:
\[
J^a_0 = u^3 f \partial_u \left( u^{-1} F^a_{u0} \right) + u^2 \left( D_i F^a_{i0} \right),
\]
\[
J^a_i = u^3 \partial_u \left( u^{-1} f F^a_{ui} \right) - u^2 f^{-1} \left( D_0 F^a_{0i} \right) + u^2 \left( D_j F^a_{ji} \right),
\]
(26)
(27)
where the gauge covariant derivative for the field strength is defined as $D_\mu F_{\nu\rho} = \partial_\mu F_{\nu\rho} + \epsilon^{abc} A_\mu^b F_{\nu\rho}$. Inserting the ansatz (19), the Yang-Mills equations give the equations for $n^a$ and $a_\mu^a$. In summary, using the generalized ansatze, we have obtained the coupled equations of motion for $n^a$ and $a_\mu^a$, (22), (26), and (27).

### 3.2 Landau-Lifshitz Equation from Yang-Mills Equation

Since it is difficult to find the general solutions for the coupled non-linear partial differential equations, we seek simple trial solutions for $n^a$ and $a_\mu^a$ to obtain the effective equations of motion. At first, instead of looking for general solutions to the equation (22), we consider the solutions to the simpler equations:

$$D_t n^a = 0 \quad \text{and} \quad D_i n^a = 0,$$

which are explicitly given by

$$\partial_\mu n^a + \epsilon^{abc} a_\mu^b n^c = 0 + O(u^2).$$

These equations lead to the ground state solutions for the effective Hamiltonian for $n^a$:

$$H_{\text{eff}} = \frac{f}{2} (\pi^a)^2 + \frac{1}{2} (D_i n^a)^2,$$

where the conjugate momentum is defined by $\pi^a = f^{-1} D_t n^a$. In this article, we wish to discuss the dynamics of magnetization and spin currents in the boundary ferromagnetic system, which is given by the leading terms in the asymptotic expansions at $u \sim 0$. Hence, the higher order terms in the expansion with respect to $u$ are irrelevant, and we neglect them in the following. Dropping the $O(u^2)$ term, we can easily obtain the solution to (29) for $a_\mu^a$ in terms of $n^a$,

$$a_\mu^a = C_\mu n^a - \epsilon^{abc} n^b \partial_\mu n^c,$$

where we have introduced a vector field $C_\mu$, which is arbitrary at this stage. This solution demonstrates the clear separation of the gauge fields:

$$a_\parallel = C_\mu n^a \quad \text{and} \quad a_\perp^\mu = - \epsilon^{abc} n^b \partial_\mu n^c.$$

The relation for the broken components, $a_\perp^\mu$, is nothing but a non-abelian analogue of the relation between the gauge field and the quantum phase of Cooper pair, $A_\mu = \partial_\mu \theta$, in superconductivity, and also corresponds to the Maurer-Cartan one-form of $G/H \sim SU(2)/U(1)$ in terms of the Nambu-Goldstone modes $n^a$ [19] [20]. Requiring the matching condition to
the static solution (17), \(a_0^a = \mu_s \delta^{a3}\) and \(a_i^a = 0\) for \(n^a = (0, 0, 1)\), the vector field \(C_\mu\) should satisfy the condition:

\[
C_0 = \mu_s \quad \text{and} \quad C_i = 0,
\]

(33)
in the static and homogeneous limit. Note that the relation (31) and the ansatz (18) do not induce new contributions of the scalar fields to the energy-momentum tensor \(T_{MN}\) in the Einstein equations, and consequently the analysis in the probe approximation remain intact.

Next, we consider the effective Yang-Mills equations, (26) and (27). It is not difficult to show that the relation (31) leads to vanishing currents \(J_\mu^a\) up to \(\mathcal{O}(u^2)\), using the explicit form (25). Furthermore, the ansatz for gauge fields (19) with \(A_u^a = 0\) implies

\[
\partial_u \left( u^{-1} F_{u0}^a \right) = 0, \quad \text{and} \quad \partial_u \left( u^{-1} f F_{ui}^a \right) = 0 + \mathcal{O}(u^4).
\]

(34)

Dropping the higher order terms such as \(\mathcal{O}(u^4)\), the remaining Yang-Mills equations reduce to

\[
D_i F_i^a = 0, \quad \text{and} \quad D_0 F_{0i}^a + D_j F_{ji}^a = 0.
\]

(35)

From the viewpoint of the boundary theory (on the ferromagnet side), the first equation corresponds to a non-abelian version of Gauss’s law, and the second corresponds to a non-abelian version of Ampere’s law without source and currents, for the spin gauge fields [21]. Using the relation (31), we obtain the \(SU(2)\) field strength:

\[
F_{\mu \nu}^a = n^a \left[ (\partial_\mu C_{\nu} - \partial_\nu C_\mu) - \varepsilon^{bcd} n^b \partial_\mu n^c \partial_\nu n^d \right] \equiv n^a f_{\mu \nu}.
\]

(36)

Note that a component of the field strength, \(f_{\mu \nu}\), parallel to the magnetization \(n^a\) only remains. With the field strength (36), the effective Yang-Mills equations (35) and the Bianchi identity for the \(SU(2)\) gauge field are reduced to the following equations:

\[
\partial_\mu f_{\mu \nu} = 0 \quad \text{and} \quad \varepsilon^{\mu \nu \rho \sigma} \partial_\nu f_{\rho \sigma} = 0.
\]

(37)

The above equations are the same form as the Maxwell equations, and the terms depending on \(n^a\) in the gauge field \(f_{\mu \nu}\) actually corresponds to the so-called spin electromagnetic field discussed in the study on ferromagnetic metals [22, 21]. The gauge field (36) also corresponds to the unbroken \(U(1)\) gauge field upon the symmetry breaking from \(SU(2)\) to \(U(1)\), with a space-dependent order parameter, which is frequently discussed in the context of solitonic monopoles in non-abelian gauge theories [23]. Since the unbroken gauge fields in

\footnote{We used the relation \(\varepsilon^{abc} \partial_\mu n^b \partial_\nu n^c = n^a \varepsilon^{bcd} n^b \partial_\mu n^c \partial_\nu n^d\) due to \(\sum_a n^a n^a = 1\).}
the holographic dual theory are identified as the (exactly) conserved currents in the boundary quantum system, the gauge field $C_\mu$ is naturally identified as the spin current with the polarization parallel to the magnetization $n^a$, which originates from conduction electrons.

So far, we have obtained the relation between the gauge field $a_\mu^a$ and the (normalized) magnetization $n^a$, which implies that the gauge field dynamics can be solely reduced to the dynamics of the magnetization and the spin electromagnetic field $C_\mu$. Finally, we consider the remaining Yang-Mills equation in the radial $u$-direction, in the holographic dual theory:

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} F^{\mu u a} \right) + \epsilon^{abc} A_\mu^b F^{\mu u c} = J_u^a. \tag{38}
\]

This equation is derived by the variation of the radial $u$-component of the $SU(2)$ gauge fields and specifies the dynamics of the gauge fields in the five-dimensional bulk; this equation cannot be seen in the ferromagnetic system on the boundary. With the ansatz (18), the radial component of the current also vanishes ($J_u^a \equiv 0$), and the gauge fixing condition $A_u^a \equiv 0$ leads to the simple $SU(2)$ field strength $F_u^a = - \partial_u A_u^a$ such as

\[
\begin{align*}
F_0^\parallel &= 2 u a_0^\parallel (t, x), \\
F_0^\perp &= 2 u a_0^\perp (t, x), \\
F_i^\parallel &= 2 u \sigma_i^\parallel \dot{a}_t^\parallel (t, x), \\
F_i^\perp &= -2 u \sigma_i^\perp \dot{a}_t^\perp (t, x),
\end{align*}
\tag{39}
\]

where we used the ansatz (19) and discarded the irrelevant $O(u^3)$ terms. From these forms, the second term in the left-hand side of (38) automatically vanishes due to $\epsilon^{abc} a_\mu^b a_\mu^c = 0$. Inserting the forms of field strength (39) and the relation (31), the equation can be recast as the following form:

\[
\partial_0 \left( C_0 n^a - \epsilon^{abc} n^b \partial_0 n^c \right) - \partial_i \left( \sigma_s^a C_i n^a + \sigma_s^\perp \epsilon^{abc} n^b \partial_i n^c \right) = 0, \tag{40}
\]

where the subleading terms are neglected. Here, we can write down the effective equation of motion for the magnetization $n^a$ in our holographic dual model:

\[
C_0 \dot{n}^a - \epsilon^{abc} n^b \dot{n}^c - \sigma_s^\perp \epsilon^{abc} n^b \nabla^2 n^c - \sigma_s^a C_i \partial_i n^a = 0, \tag{41}
\]

where the dot denotes the time-derivative and $\nabla^2 = \partial_i \partial_i$. Here, we consider the condition, $\partial_0 C_0 - \sigma_s^\parallel \partial_i C_i = 0$, on the unbroken gauge field due to the constraint \( \sum_a n^a n^a = 1 \). This condition implies the conservation of the spin current of conduction electrons, which corresponds to the unbroken gauge field $C_\mu$, as seen below. Note that, since the Maxwell equations (37) for $C_\mu$ is gauge invariant, this condition can be consistently imposed as a gauge fixing condition.

\footnote{A similar analysis on effective equations at the linearized level in two-dimensional magnetic systems has been also discussed in [11].}
Considering the matching condition (33), we decompose $C_0$ into $C_0 = \mu_s + \tilde{C}_0$. Finally, we obtain the holographic equation for magnetization dynamics:

$$\mu_s \dot{n}^a - \varepsilon^{abc}n^b \nabla^c n^a - \sigma^\perp_s \varepsilon^{abc}n^b \nabla^2 n^c + \tilde{C}_0 \dot{n}^a - \sigma^\parallel_s C_i \partial_i n^a = 0.$$  

(42)

Here, we take the spin chemical potential to be $\mu_s = -M_s/\gamma$ with the magnitude of spontaneous magnetization $M_s$ and the gyromagnetic ratio $\gamma (> 0)$ and also identify the spin current and spin accumulation due to conduction electrons as $J^\parallel_{si} = -\sigma^\parallel_s C_i$ and $\Delta \mu_s = \tilde{C}_0$, using the holographic dictionary. Then, the holographic equation becomes the same form as the Landau-Lifshitz equation (without damping terms),

$$\frac{M_s}{\gamma} \dot{n}^a + \varepsilon^{abc}n^b \nabla^2 n^c = -\sigma^\perp_s \varepsilon^{abc}n^b \nabla^2 n^c + \Delta \mu_s \dot{n}^a + J^\parallel_{si} \partial_i n^a.$$  

(43)

The last two terms in the right-hand side can be interpreted as the well-known terms from spin transfer torque, which describes the transfer of spin angular momentum between localized magnetic moments and conduction electrons [21]. Furthermore, the holographic Landau-Lifshitz equation (42) also naturally incorporate the spin inertia term proportional to the second time-derivative of the magnetization, which is discussed in metallic ferromagnets [24].

It should be noted that the holographic Landau-Lifshitz equation automatically incorporates the spin transfer torque due to conduction electrons without introducing the corresponding fields to electrons in the dual gravitational theory. This is consistent with the thermodynamical results at low temperatures, which was obtained in the previous paper [10].

4 Phenomenology of Holographic Magnetization Dynamics

In the isotropic ferromagnets sufficiently below the Curie temperature ($T < T_c$), the dynamics of magnetization vector (or density of magnetic moments), $M^a$, is described by the Landau-Lifshitz equation [1, 26]:

$$\frac{\partial M^a}{\partial t} = -\alpha \varepsilon^{abc}M^b \nabla^2 M^c \quad \text{with} \quad \sum_{a=1}^{3} M^a M^a = M(T)^2 = \text{const.}$$  

(44)

In the following discussion, the external magnetic field and the damping term (or relaxation term) are ignored for simplicity. From the quadratic constraint, the magnetization vector can be represented as $M^a(x,t) = M(T)n^a(x,t)$ with the unit vector $n^a(x,t)$. In terms of $n^a(x,t)$, the Landau-Lifshitz equation becomes

$$M(T)\frac{\partial n^a}{\partial t} = -\alpha M(T)^2 \varepsilon^{abc}n^b \nabla^2 n^c.$$  

(45)

---

6The negative sign is introduced due to the negative value of the gyromagnetic ratio for electrons.
Note that the equation has two parameters, the magnitude of spontaneous magnetization, $M(T)$, at the temperature $T$, and the spin stiffness constant, $\alpha$.

Comparing the holographic equation (42) with the Landau-Lifshitz equation (45), we find that the spin chemical potential, $\mu_s$, in the gauge field solution (17) should be proportional to the magnitude of magnetization, and the spin stiffness constant is given by the coefficients $\sigma_s^\perp$ in the gauge field solution (20) in the following way:

$$\mu_s \propto -M(T) \quad \text{and} \quad \sigma_s^\perp \propto \alpha M(T)^2.$$  (46)

In our holographic dual model, the magnitude of magnetization, $M(T)$, at the temperature $T$ is given by the static solution of the scalar field $\Phi(u)$ through the formula (14). The first relation between the magnitude of magnetization and the spin chemical potential in ferromagnets is well-known, and frequently used as the starting point to analyze the various spintronic phenomena [2].

Although the spin chemical potential in the equilibrium, $\mu_s$, is an integration constant, the coefficient, $\sigma_s^\perp$, is the derived quantity from the gauge field equation, and thus the second relation in (46) on the spin stiffness constant is a nontrivial consequence in the holographic dual model. In order to obtain the coefficient, $\sigma_s^\perp$, we consider the linearized equation of motion for gauge fields on the background solution, with the static and homogeneous ansatz, $A_i^\perp = k G^\perp(u)$, where $k = \text{const}$.

Inserting this ansatz into the Yang-Mills equation (27), we have the following linearized equation for $G^\perp(u)$:

$$u^3 \frac{d}{du} \left( f(u) \left( \frac{dG^\perp}{du} \right) u \right) + \left( u a^3(u) \right)^2 f(u) G^\perp = 0,$$  (47)

where the metric (8) and the $SU(2)$ gauge field (17) are assumed to be the background. Note that this is a linear equation for $G^\perp$, and the constant $k$ is irrelevant. Here, we impose the first relation in (46), $\mu_s = -M(T)/M(0)$, which is the magnetization normalized by the saturated magnetization, $M(T=0)$.

Using the numerical results of the holographic spontaneous magnetization, $M(T)$ in [10], which is obtained using the scalar potential $V(|\phi|) = \lambda (|\phi|^2 - m^2 / \lambda)^2 / 4$ with $\lambda = 1$ and $m^2 = 35/9$, we can numerically solve the equation (47) and obtain the asymptotic expansion (20) near the boundary ($u \sim 0$). The numerical results of temperature dependences of the spin-wave stiffness, $D(T) \simeq \sigma_s^\perp / M(T)$, which appears in the dispersion relation of spin-waves, $\omega = D(T) k^2$, and the spin stiffness constant, $\alpha(T) \simeq \sigma_s^\perp / M(T)^2$, are shown in Figure 1.

---

7The nontrivial profile $A_i^\perp(u)$ on the background does not contribute to the energy-momentum tensor in the Einstein equation at the linearized level.

8The proportionality constant is chosen for convenience in numerical calculations.
The results on the spin-wave stiffness in Figure 1(a) clearly show that $D(T) \propto M(T)$, which is consistent with the relation (46) based on the Landau-Lifshitz equation (44). Furthermore, the results in Figure 1(b) imply the slight temperature dependence of the spin stiffness constant, $\alpha = \alpha(T)$, which can be attributed to the nonlinear spin-wave effects [25].

A similar argument also holds for the unbroken (or parallel) component of the gauge fields, $A_{\parallel}^i$, and we can obtain the coefficient $\sigma_{\parallel}^s$, which leads to the spin torque term in the holographic Landau-Lifshitz equation (42). The nontrivial profile of gauge field, $A_{\parallel}^i(u)$, which is the parallel component to the spin chemical potential, $A_0^\parallel$, leads to the non-vanishing off-diagonal contribution in the right-hand side of the Einstein equation (2), and thus induces the fluctuation of the metric $g_{tx}(u) = h_{tx}(u)/u^2$, where $h_{tx}(u)$ parameterizes the fluctuation finite on the boundary. At the linearized level, two fluctuations, $A_{\parallel}^x(u)$ and $h_{tx}(u)$, form the closed equations, which come from the Yang-Mills equation and Einstein equation, respectively [27, 28]:

\[
\begin{align*}
  u \frac{d}{du} \left( f(u) \left( \frac{dA_{\parallel}}{du} \right) \right) + \left( \frac{da^3(u)}{du} \right) \frac{d}{du} \left( u^2 h_{tx} \right) &= 0, \\
  u^{-2} \frac{d}{du} \left( u^2 h_{tx} \right) + 2 \left( \frac{da^3(u)}{du} \right) A_{\parallel}^x &= 0.
\end{align*}
\]

Deleting the metric fluctuation, we can obtain the equation for $G^\parallel(u)$:

\[
\begin{align*}
  u \frac{d}{du} \left( f(u) \left( \frac{dG^\parallel}{du} \right) \right) - 2 u^2 \left( \frac{da^3(u)}{du} \right)^2 G^\parallel &= 0.
\end{align*}
\]

We can numerically solve the equation, and obtain the coefficient $\sigma_{\parallel}^s$ from the asymptotic expansion of the solution in (20). The resulting temperature dependence of the spin torque coefficient, $\tau_s(T) = \sigma_{\parallel}^s/M(T)$, is shown in Figure 2.
The results on the magnitude of the spin transfer torque, $\tau_s(T)$, show that the spin torque effect is approximately constant at low temperatures (in comparison with magnetization curve), and is vanishing towards the Curie temperature as $\tau_s(T) \propto \left(1 - T/T_c\right)^{2/5}$. This property at low temperatures is consistent with the phenomenological form of the spin transfer torque, $(J_\parallel \cdot \nabla n^a)/M(T)$, whose magnitude is independent of the norm of magnetization due to $|J_\parallel| \propto M(T)$ at the leading order [21]. In addition, the finite spin torque coefficient is a consequence of the both fluctuations of the gauge field and metric. In accordance with the holographic dictionary [27], the metric fluctuation $h_{tx}$ corresponds to the temperature gradient, $\nabla_x T/T$, in the ferromagnetic system. This calculation implies that the effect of spin transfer torque appears only in the nonequilibrium situations, where spin transfer is accompanied by heat (or entropy) transfer.

5 Summary and Discussion

We have discussed a novel approach to understand magnetization dynamics in ferromagnets using the holographic realization of ferromagnetic systems. The Landau-Lifshitz equation describing magnetization dynamics was derived from the Yang-Mills-Higgs equations in the dual gravitational theory. This holographic Landau-Lifshitz equation automatically incorporates not only the exchange interaction but also the spin transfer torque effect due to conduction electrons. Furthermore, we numerically investigated the temperature dependences of the spin-wave stiffness and the magnitude of spin transfer torque in the holographic dual theory, and the results obtained so far are consistent with the known properties of magnetization dynamics in ferromagnets with conduction electrons.
This holographic approach to magnetization dynamics can be applied to more generic situations. For instance, the holographic Landau-Lifshitz equation can incorporate the damping term by considering more generic metric fluctuations, which correspond to phonon dynamics in the boundary ferromagnets. Moreover, the holographic dual theory may provide geometric approaches to spin caloritronics [29], where magnetization dynamics is considered under temperature gradients, from higher dimensional perspectives. We thus believe that the holographic approach provides useful tools to analyze nonequilibrium and nonlinear dynamics of magnetization in ferromagnets, and also leads to new perspectives in spintronics from gravitational physics.

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