Abstract

The notion of best approximation mapping (BAM) with respect to a closed affine subspace in finite-dimensional space was introduced by Behling, Bello Cruz and Santos to show the linear convergence of the block-wise circumcentered-reflection method. The best approximation mapping possesses two critical properties of the circumcenter mapping for linear convergence.

Because the iteration sequence of BAM linearly converges, the BAM is interesting in its own right. In this paper, we naturally extend the definition of BAM from closed affine subspace to nonempty closed convex set and from $\mathbb{R}^n$ to general Hilbert space. We discover that the convex set associated with the BAM must be the fixed point set of the BAM. Hence, the iteration sequence generated by a BAM linearly converges to the nearest fixed point of the BAM. Connections between BAMs and other mappings generating convergent iteration sequences are considered. Behling et al. proved that the finite composition of BAMs associated with closed affine subspaces is still a BAM in $\mathbb{R}^n$. We generalize their result from $\mathbb{R}^n$ to general Hilbert space and also construct a new constant associated with the composition of BAMs. This provides a new proof of the linear convergence of the method of alternating projections. Moreover, compositions of BAMs associated with general convex sets are investigated. In addition, we show that convex combinations of BAMs associated with affine subspaces are BAMs. Last but not least, we connect BAM with circumcenter mapping in Hilbert spaces.

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1 Introduction

Throughout this paper, we shall assume that

\[ \mathcal{H} \text{ is a real Hilbert space,} \]

with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, $\mathbb{N} = \{0, 1, 2, \ldots \}$ and $m \in \mathbb{N} \setminus \{0\}$.

In [10], Behling, Bello Cruz and Santos introduced the circumcentered Douglas-Rachford method, which is a special instance of the circumcentered-reflection method (C-RM) and the first circumcentered isometry method in the literature. Then the same authors contributed [11], [12] and [13] on C-RMs. In [12], in order to prove the linear convergence of the block-wise C-RM that is the sequence of iterations of finite composition of circumcentered-reflection operators, they introduced the best approximation mapping (BAM) and proved that the finite composition of BAMs is still a BAM. Our paper is inspired by [12], and we provide the following main results:

R1: Proposition 3.10 states that the sequence of iterations of BAM solves the best approximation problem associated with the fixed point set of the BAM.

R2: Theorem 4.4 generalizes [12, Theorem 1] and shows that the finite composition of BAMs associated with closed affine subspaces in Hilbert space is a BAM. It also provides a new constant associated with the composition of BAMs. In fact, we provide examples showing that our new constant is independent with the one constructed in [12, Lemma 1]. In particular, as a corollary of the Theorem 4.4, in Corollary 5.12(i) we show the linear convergence of the method of alternating projections (MAP).
R3: Theorems 5.4 and 5.10 use two different methods to show that the convex combination of finitely many BAMs associated with affine subspaces is a BAM.

R4: Theorems 6.26 to 6.28 show linear convergence of the iteration sequences generated from composition and convex combination of circumcenter mappings in Hilbert spaces.

The paper is organized as follows. In Section 2, we present some auxiliary results to be used in the sequel. Section 3 includes definition and properties of the BAM in Hilbert spaces. In particular, the comparisons: BAM vs convergent mapping, BAM vs Banach contraction, and BAM vs linear regular operator are provided. In Section 4, we generalize results shown in [12, Section 2] from $\mathbb{R}^n$ to the general Hilbert space and show that the finite composition of BAMs with closed and affine fixed point sets in Hilbert space is still a BAM. In addition, compositions of BAMs associated with general convex sets are considered in Section 4 as well. In Section 5, we use two methods to show that the convex combination of finitely many BAMs with closed and affine fixed point sets is a BAM. In Section 6, we review definitions and facts on circumcenter mapping and circumcenter mappings as BAMs in Hilbert spaces. Moreover, we show linear convergence of sequences generated from composition and convex combination of circumcenter mappings as BAMs in Hilbert spaces.

We now turn to the notation used in this paper. Let $C$ be a nonempty subset of $\mathcal{H}$. The orthogonal complement of $C$ is the set $C^\perp := \{x \in \mathcal{H} \mid \langle x, y \rangle = 0 \text{ for all } y \in C\}$. C is an affine subspace of $\mathcal{H}$ if $C \neq \emptyset$ and $(\forall \rho \in \mathbb{R}) \rho C + (1 - \rho)c = C$. The smallest affine subspace of $\mathcal{H}$ containing $C$ is denoted by aff $C$ and called the affine hull of $C$. An affine subspace $C$ is said to be parallel to an affine subspace $M$ if $C = M + a$ for some $a \in \mathcal{H}$. Suppose that $C$ is a nonempty closed convex subset of $\mathcal{H}$. The projector (or projection operator) onto $C$ is the operator, denoted by $P_C$, that maps every point in $\mathcal{H}$ to its unique projection onto $C$. $R_C := 2P_C - \text{Id}$ is the reflector associated with $C$. Moreover, $d_C(x) := \min_{c \in C} \|x - c\| = \|x - P_C x\|$. Let $x \in \mathcal{H}$ and $\rho \in \mathbb{R}_{++}$. Denote the ball centered at $x$ with radius $\rho$ as $B[x; \rho]$.

Let $T : \mathcal{H} \to \mathcal{H}$ be an operator. The fixed point set of the operator $T$ is denoted by Fix$T$, i.e., Fix$T := \{x \in \mathcal{H} \mid Tx = x\}$. Denote by $B(\mathcal{H}) := \{T : \mathcal{H} \to \mathcal{H} : T \text{ is bounded and linear}\}$. For every $T \in B(\mathcal{H})$, the operator norm $\|T\|$ of $T$ is defined by $\|T\| := \sup_{\|x\| \leq 1} \|Tx\|$.

For other notation not explicitly defined here, we refer the reader to [3].

## 2 Preliminaries

### Projections and Friedrichs angle

**Fact 2.1** [3, Proposition 3.19] Let $C$ be a nonempty closed convex subset of the Hilbert space $\mathcal{H}$ and let $x \in \mathcal{H}$. Set $D := z + C$, where $z \in \mathcal{H}$. Then $P_D x = z + P_C (x - z)$.

**Fact 2.2** [15, Theorems 5.8] Let $M$ be a closed linear subspace of $\mathcal{H}$. Then $\text{Id} = P_M + P_{M^\perp}$.

Note that the case in which $M$ and $N$ are linear subspaces in the following result has already been shown in [15, Lemma 9.2].

**Lemma 2.3** Let $M$ and $N$ be closed affine subspaces of $\mathcal{H}$ with $M \cap N \neq \emptyset$. Assume $M \subseteq N$ or $N \subseteq M$. Then $P_MP_N = P_N P_M = P_{M \cap N}$.

**Proof.** Let $z \in M \cap N$. By [18, Theorem 1.2], the parallel linear subspaces of $M$ and $N$ are par $M = M - z$ and par $N = N - z$ respectively. By assumption, $M \subseteq N$ or $N \subseteq M$, we know, par $M \subseteq \text{par} \ N$ or par $N \subseteq \text{par} \ M$.

Then by Fact 2.1 and [15, Lemma 9.2], for every $x \in \mathcal{H}$, $P_MP_N x = z + P_{\text{par} M}(P_N(x) - z) = z + P_{\text{par} M}(z + P_{\text{par} N}(x - z)) = z + P_{\text{par} M}P_{\text{par} N}(x - z) = z + P_{\text{par} M \cap \text{par} N}(x - z) = P_{M \cap N} x$, which implies that $P_MP_N = P_{M \cap N}$. The proof of $P_N P_M = P_{M \cap N}$ is similar.

**Definition 2.4** [15, Definition 9.4] The Friedrichs angle between two linear subspaces $U$ and $V$ is the angle $\alpha(U, V)$ between 0 and $\frac{\pi}{2}$ whose cosine, $c(U, V) := \cos \alpha(U, V)$, is defined by the expression

$$c(U, V) := \sup \{|\langle u, v \rangle| : u \in U \cap (U \cap V)^\perp, v \in V \cap (U \cap V)^\perp, \|u\| \leq 1, \|v\| \leq 1\}.$$ 

**Fact 2.5** [15, Theorem 9.35] Let $U$ and $V$ be closed linear subspaces of $\mathcal{H}$. Then the following statements are equivalent.

(i) $c(U, V) < 1$.

(ii) $U + V$ is closed.

(iii) $U^\perp + V^\perp$ is closed.

2
Nonexpansive operators

Definition 2.6 [3, Definition 4.1] Let $D$ be a nonempty subset of $\mathcal{H}$ and let $T : D \to \mathcal{H}$. Then $T$ is

(i) nonexpansive if it is Lipschitz continuous with constant 1, i.e., $(\forall x \in D) \ (\forall y \in D) \ ||Tx - Ty|| \leq ||x - y||$;

(ii) quasinonexpansive if $(\forall x \in D) \ (\forall y \in \text{Fix } T) \ ||Tx - y|| \leq ||x - y||$;

(iii) and strictly quasinonexpansive if $(\forall x \in D \setminus \text{Fix } T) \ (\forall y \in \text{Fix } T) \ ||Tx - y|| < ||x - y||$.

Definition 2.7 [3, Definition 4.33] Let $D$ be a nonempty subset of $\mathcal{H}$, let $T : D \to \mathcal{H}$ be nonexpansive, and let $\alpha \in ]0, 1[$. Then $T$ is averaged with constant $\alpha$, or $\alpha$-averaged for short, if there exists a nonexpansive operator $R : D \to \mathcal{H}$ such that $T = (1 - \alpha) \text{Id} + \alpha R$.

Lemma 2.8 Let $T : \mathcal{H} \to \mathcal{H}$ be an affine operator with $\text{Fix } T \neq \emptyset$. Then $T$ is quasinonexpansive if and only if $T$ is nonexpansive.

Proof. By Definition 2.6, $T$ is nonexpansive implies that $T$ is quasinonexpansive. Suppose that $T$ is quasinonexpansive. Because $\text{Fix } T \neq \emptyset$, take $z \in \text{Fix } T$. Define

$$(\forall x \in \mathcal{H}) \quad F(x) := T(x + z) - z. \quad (2.1)$$

Then by [9, Lemma 3.8], $F$ is linear. Because $T$ is quasinonexpansive,

$$(\forall x \in \mathcal{H}) \quad ||Fx|| = ||T(x + z) - z|| \leq ||(x + z) - z|| = ||x||,$$

which, by the linearity of $F$, implies that

$$(\forall x \in \mathcal{H}, \forall y \in \mathcal{H}) \quad ||Fx - Fy|| \leq ||x - y||. \quad (2.2)$$

Now, for every $x \in \mathcal{H}$ and for every $y \in \mathcal{H},$

$||Tx - Ty|| \overset{(2.1)}{=} ||z + F(x - z) - (z + F(y - z))|| = ||F(x - z) - F(y - z)|| \overset{(2.2)}{\leq} ||x - y||,$

which means that $T$ is nonexpansive. $\blacksquare$

3 Best approximation mapping

The best approximation mapping with respect to a closed affine subspaces in $\mathbb{R}^n$ was introduced by Behling, Bello-Cruz and Santos in [12]. In this section, we extend the definition of BAM from closed affine subspace to nonempty closed convex set, and from $\mathbb{R}^n$ to general Hilbert space. Moreover, we provide some examples and properties of the generalized version of BAM.

Definition of BAM

Definition 3.1 Let $G : \mathcal{H} \to \mathcal{H}$, and let $\gamma \in ]0, 1[$. Then $G$ is a best approximation mapping with constant $\gamma$ (for short $\gamma$-BAM), if

(i) $\text{Fix } G$ is a nonempty closed convex subset of $\mathcal{H}$,

(ii) $\text{P}_{\text{Fix } G} G = \text{P}_{\text{Fix } G}$, and

(iii) $(\forall x \in \mathcal{H}) \ ||Gx - \text{P}_{\text{Fix } G} x|| \leq \gamma ||x - \text{P}_{\text{Fix } G} x||$.

In particular, if $\gamma$ is unknown or not necessary to point out, we just say that $G$ is a BAM.

The following Lemma 3.2(ii) illustrates that in [12, Definition 2], the set $C$ is uniquely determined by the operator $G$, and that, moreover, $C = \text{Fix } G$. Hence, our Definition 3.1 is indeed a natural generalization of [12, Definition 2].
Lemma 3.2 Let $G : \mathcal{H} \to \mathcal{H}$, let $C$ be a nonempty closed convex subset of $\mathcal{H}$, and let $\gamma \in [0, 1]$. Suppose that $P_CG = P_C$ and that
\[ (\forall x \in \mathcal{H}) \quad \|Gx - P_Cx\| \leq \gamma \|x - P_Cx\|. \tag{3.1} \]
Then the following hold:

(i) $GP_C = P_C$.

(ii) Fix $G = C$.

(iii) $G$ is a $\gamma$-BAM.

Proof. (i): For every $y \in \mathcal{H}$, use the idempotent property of $P_C$ and apply (3.1) with $x = P_Cy$ to obtain that
\[ \|GP_Cy - P_Cy\| = \|GP_Cy - P_CP_Cy\| \leq \gamma\|P_Cy - P_CP_Cy\| = 0, \]
which implies that $(\forall y \in \mathcal{H})$ $GP_Cy = P_Cy$, that is, $GP_C = P_C$.

(ii): Let $x \in \mathcal{H}$. On the one hand, by (i), $x \in C \Rightarrow x = P_Cx = GP_Cx = Gx \Rightarrow x \in FixG$. On the other hand, $x \in FixG \Rightarrow x = Gx \Rightarrow \|x - P_Cx\| = \|Gx - P_Cx\| \leq \gamma\|x - P_Cx\| \Rightarrow x - P_Cx = 0 \Rightarrow x \in C$, where the second and third implications are from (3.1), and $\gamma < 1$ respectively. Altogether, (ii) is true.

(iii): This is directly from Definition 3.1.

Proposition 3.3 Let $\gamma \in [0, 1]$. Suppose that $G$ is a $\gamma$-BAM. Then $d_{FixG} \circ G \leq \gamma d_{FixG}$.

Proof. Let $x \in \mathcal{H}$. By Definition 3.1(i), Fix $G$ is a nonempty closed convex set, so $d_{FixG}$ is well defined. Moreover, by Definition 3.1(ii)&(iii),
\[ d_{FixG}(Gx) = \|Gx - P_{FixG}Gx\| = \|Gx - P_{FixG}x\| \leq \gamma\|x - P_{FixG}x\| = \gamma d_{FixG}x. \]

Example 3.4 Let $C$ be a nonempty closed convex subset of $\mathcal{H}$. Then for every $\gamma \in [0, 1]$, $(1 - \gamma)P_C + \gamma Id$ is a $\gamma$-BAM with Fix $G = C$. Moreover, Id is a 0-BAM with Fix Id $= \mathcal{H}$.

Proof. Let $\gamma \in [0, 1]$. Then by [3, Proposition 3.21], $P_C((1 - \gamma)P_C + \gamma Id) = P_C$. In addition, $\|(1 - \gamma)P_Cx + \gamma(x - P_Cx)\| = \gamma\|x - P_Cx\|$. The last assertion is clear from definitions.

Remark 3.5 Let $C$ be a nonempty closed convex subset of $\mathcal{H}$ and let $\gamma \in \mathbb{R}$.

(i) Because $(\forall x \in \mathcal{H}) \|(1 - \gamma)P_Cx + (1 - \gamma)x - P_Cx\| = |\gamma|\|x - P_Cx\|$, and $|\gamma| < 1 \iff \gamma \in ]-1, 1[$, by Definition 3.1(iii), we know that $(1 - \gamma)P_C + \gamma Id$ is a BAM implies that $\gamma \in ]-1, 1[$.

(ii) Let $\varepsilon \in \mathbb{R}_{++}$. Suppose that $\mathcal{H} = \mathbb{R}^2$, $C := B[0; 1]$ and $\gamma := -\varepsilon$. Let $x := (1 + \varepsilon, 0)$. Then
\[ P_C((1 - \gamma)P_C + \gamma Id)x = \begin{cases} (1 - \varepsilon^2, 0) & \text{if } \varepsilon \leq \sqrt{2}, \\ (-1, 0) & \text{if } \varepsilon > \sqrt{2}, \end{cases} \]
which implies that $P_C((1 - \gamma)P_C + \gamma Id)x \neq (1, 0) = P_Cx$, which yields that $(1 - \gamma)P_C + \gamma Id$ is not a BAM.

Hence, using the two items above, we conclude that if $(1 - \gamma)P_C + \gamma Id$ is a BAM, then $\gamma \in ]-1, 1[$ and that generally if $\gamma \in ]-1, 0]$, then $(1 - \gamma)P_C + \gamma Id$ is not a BAM. Therefore, the assumption in Example 3.4 is tight.

Example 3.6 Suppose that $\mathcal{H} = \mathbb{R}^n$. Let $T : \mathcal{H} \to \mathcal{H}$ be $\alpha$-averaged with $\alpha \in ]0, 1[$ and let $T$ be linear. Then $\|TP_{(FixT)^+}\| \in ]0, 1[$ and $T$ is a $\|TP_{(FixT)^+}\|$-BAM.

Proof. The items (i), (ii) and (iii) in Definition 3.1 follow from [4, Lemmas 3.12 and 3.14] and [9, Proposition 2.22] respectively.

It is easy to see that $-Id$ is linear and nonexpansive but not a BAM. Hence, the condition “$T$ is $\alpha$-averaged” in Example 3.6 can not be replaced by “$T$ is nonexpansive”.

4
**Proposition 3.7** Let \( T : \mathcal{H} \to \mathcal{H} \) be a Banach contraction on \( \mathcal{H} \), say, there exists \( \gamma \in [0,1] \) such that

\[
(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \|Tx - Ty\| \leq \gamma \|x - y\|.
\]  
(3.2)

Then \( T \) is a \( \gamma \)-BAM.

**Proof.** By \([3, \text{Theorem 1.50(i)})\], Fix \( T \) is a singleton, say Fix \( T = \{z\} \) for some \( z \in \mathcal{H} \). Let \( x \in \mathcal{H} \). Then \( P_{\text{Fix} T} Tx = z = P_{\text{Fix} T} x \), which implies that \( P_{\text{Fix} T} T = P_{\text{Fix} T} \). Moreover, \( \|Tx - P_{\text{Fix} T} x\| = \|Tx - z\| \leq \gamma \|x - P_{\text{Fix} T} x\| \). Altogether, \( T \) is a \( \gamma \)-BAM.

**Remark 3.8**

(i) Proposition 3.7 illustrates that every Banach contraction is a BAM.

(ii) Note that a contraction must be continuous. By Example 6.17 below, a BAM (even with fixed point set being singleton) is generally not continuous. Hence, we know that a BAM is generally not a contraction and that the converse of Proposition 3.7 fails.

**Proposition 3.9** Let \( A \in \mathbb{R}^{n \times n} \) be a normal matrix. Denote by \( \rho(A) \) the spectral radius of \( A \), i.e.,

\[
\rho(A) := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.
\]

(i) Suppose one of the following holds:
   (a) \( \rho(A) < 1 \).
   (b) \( \rho(A) = 1 \), where \( \lambda = 1 \) is the only eigenvalue on the unit circle and semisimple.

Then \( A \) is a BAM.

(ii) The following are equivalent:
   (a) \( \lim_{k \to \infty} A^k \) exists.
   (b) \( \lim_{k \to \infty} A^k = P_{\text{Fix} A} \).
   (c) \( A \) is a BAM.

**Proof.** (i): If \( \rho(A) < 1 \), then by \([3, \text{Example 2.19})\], \( A \) is a Banach contraction. Hence, by Proposition 3.7, \( A \) is a BAM.

Suppose that \( \rho(A) = 1 \) and \( \lambda = 1 \) is the only eigenvalue of \( A \) on the unit circle and semisimple. Then by the Spectral Theorem for Diagonalizable Matrices \([17, \text{page 517})\] and Properties of Normal Matrices \([17, \text{page 548})\],

\[
A = P_{U_1} + \lambda_2 P_{U_2} + \cdots + \lambda_k P_{U_k},
\]

where \( \sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \) with \( \lambda_1 = 1 \) is the spectrum of \( A \) and \( (\forall i \in \{1, \ldots, k\}) \ U_i := \ker(A - \lambda_i \text{Id}) \). Then clearly \( \text{Fix} A = \ker(A - \text{Id}) = U_1 \). Moreover, by the Spectral Theorem for Diagonalizable Matrices \([17, \text{page 517})\] again, it is easy to see that

\[
P_{\text{Fix} A} A = P_{\text{Fix} A} \quad \forall x \in \mathbb{R}^n \quad A x - P_{\text{Fix} A} x \leq |\lambda_2| \|x - P_{\text{Fix} A} x\|,
\]

where \( |\lambda_2| < 1 \). Therefore, \( A \) is a BAM.

(ii): By the Theorem of Limits of Powers \([17, \text{Page 630})\], \( \lim_{k \to \infty} A^k \) exists if and only if \( \rho(A) < 1 \) or \( \rho(A) = 1 \) with \( \lambda = 1 \) being the only eigenvalue of \( A \) on the unit circle and semisimple, which implies that \( \lim_{k \to \infty} A^k = P_{\text{Fix} A} \). Moreover, by Definition 3.1, \( A \) being a BAM implies that \( \lim_{k \to \infty} A^k = P_{\text{Fix} A} \). Combine these results with (i) to obtain (ii).

**Properties of BAM**

The following Proposition 3.10(ii) states that any sequence of iterates of a BAM must linearly converge to the best approximation onto the fixed point set of the BAM. Therefore, we see the importance of the study of BAMs. The following Proposition 3.10 reduces to \([12, \text{Proposition 1})\] when \( \mathcal{H} = \mathbb{R}^n \) and \( \text{Fix} G \) is an affine subspace of \( \mathbb{R}^n \). In fact, there is little difficulty to extend the space from \( \mathbb{R}^n \) to \( \mathcal{H} \) and the related set from closed affine subspace to nonempty closed convex set.
Proposition 3.10 Let $\gamma \in [0, 1]$ and let $G : H \to H$. Suppose that $G$ is a $\gamma$-BAM. Then for every $k \in \mathbb{N},$

(i) $P_{\text{Fix } G} G^k = P_{\text{Fix } G}$, and

(ii) $(\forall x \in H) \|G^k x - P_{\text{Fix } G} x\| \leq \gamma^k \|x - P_{\text{Fix } G} x\|.$

Consequently, for every $x \in H$, $(G^k x)_{k \in \mathbb{N}}$ converges to $P_{\text{Fix } G} x$ with a linear rate $\gamma$.

Proof. Because $G$ is a $\gamma$-BAM, by Definition 3.1, we have that $\text{Fix } G$ is a nonempty closed and convex subset of $H$, and that

\[
P_{\text{Fix } G} G = P_{\text{Fix } G}, \tag{3.3a}
\]

\[
(\forall y \in H) \|G y - P_{\text{Fix } G} y\| \leq \gamma \|y - P_{\text{Fix } G} y\|. \tag{3.3b}
\]

We argue by induction on $k$. It is trivial that (i) and (ii) hold for $k = 0$. Assume (i) and (ii) are true for some $k \in \mathbb{N}$, that is,

\[
P_{\text{Fix } G} G^k = P_{\text{Fix } G}, \tag{3.4a}
\]

\[
(\forall y \in H) \|G^k y - P_{\text{Fix } G} y\| \leq \gamma^k \|y - P_{\text{Fix } G} y\|. \tag{3.4b}
\]

Let $x \in H$. Now

\[
P_{\text{Fix } G} G^{k+1} x = P_{\text{Fix } G} G(G^k x) = P_{\text{Fix } G} G^k x = P_{\text{Fix } G} x.
\]

Moreover, $\|G^{k+1} x - P_{\text{Fix } G} x\| = \|G(G^k x) - P_{\text{Fix } G}(G^k x)\| \leq \gamma \|G^k x - P_{\text{Fix } G}(G^k x)\| \leq \gamma^k \|x - P_{\text{Fix } G} x\|.$

Hence, the proof is complete by the principle of mathematical induction. $\blacksquare$

Proposition 3.11 Let $T : H \to H$ be quasinonexpansive with $\text{Fix } T$ being a closed affine subspace of $H$. Let $\gamma \in [0, 1]$. Suppose that $(\forall x \in H) \|T x - P_{\text{Fix } T} x\| \leq \gamma \|x - P_{\text{Fix } T} x\|$. Then $T$ is a $\gamma$-BAM.

Proof. By assumptions and Definition 3.1, it remains to prove $P_{\text{Fix } T} T = P_{\text{Fix } T}$.

Let $x \in H$. By [3, Example 5.3], $T$ is quasinonexpansive and $\text{Fix } T \neq \emptyset$ imply that $(T^k x)_{k \in \mathbb{N}}$ is Fejér monotone with respect to $\text{Fix } T$. This, the assumption that $\text{Fix } T$ is a closed affine subspace, and [3, Proposition 5.9(i)] imply that

\[
(\forall k \in \mathbb{N}) \quad P_{\text{Fix } T} T^k x = P_{\text{Fix } T} x,
\]

which yields $P_{\text{Fix } T} T = P_{\text{Fix } T}$ when $k = 1$. $\blacksquare$

The following result shows further connection between BAMs and linear convergent mappings.

Corollary 3.12 Let $T : H \to H$ be quasinonexpansive with $\text{Fix } T$ being a closed affine subspace of $H$. Let $\gamma \in [0, 1]$. Then $T$ is a $\gamma$-BAM if and only if $(\forall k \in \mathbb{N}) (\forall x \in H) \|T^k x - P_{\text{Fix } T} x\| \leq \gamma^k \|x - P_{\text{Fix } T} x\|.$

Proof. “$\Rightarrow$”: This is clearly from Proposition 3.10.

“$\Leftarrow$”: This comes from the assumptions and Proposition 3.11. $\blacksquare$

The following result states that BAM with closed affine fixed point set is strictly quasinonexpansive. In particular, the inequality shown in Proposition 3.13(i) is interesting on its own.

Proposition 3.13 Let $G : H \to H$ with $\text{Fix } G$ being a closed affine subspace of $H$. Let $\gamma \in [0, 1]$. Suppose that $G$ is a $\gamma$-BAM. The following hold:

(i) $(\forall x \in H) (\forall y \in \text{Fix } G) \|G x - y\|^2 + (1 - \gamma^2) \|x - P_{\text{Fix } G}(x)\|^2 \leq \|x - y\|^2.$

(ii) $G$ is strictly quasinonexpansive.
Proof. (i): Because $G$ is a $\gamma$-BAM, by Definition 3.1,

\[ P_{\text{Fix} G} G = P_{\text{Fix} G}, \quad (\forall x \in \mathcal{H}) \quad \|Gx - P_{\text{Fix} G} x\| \leq \gamma \|x - P_{\text{Fix} G} x\|. \] (3.5a)

Because $G$ is a closed affine subspace of $\mathcal{H}$, by [7, Proposition 2.10], for every $x \in \mathcal{H}$ and $y \in \text{Fix} G$,

\[ \|Gx - y\|^2 = \|Gx - P_{\text{Fix} G}(Gx)\|^2 + \|P_{\text{Fix} G}(Gx) - y\|^2 \] (3.6a)

\[ \overset{(3.5a)}{=} \|Gx - P_{\text{Fix} G}(x)\|^2 + \|P_{\text{Fix} G}(x) - y\|^2 \] (3.6b)

\[ \overset{(3.5b)}{\leq} \gamma^2 \|x - P_{\text{Fix} G}(x)\|^2 + \|P_{\text{Fix} G}(x) - y\|^2 \] (3.6c)

and, by [7, Proposition 2.10] again,

\[ \|x - y\|^2 = \|x - P_{\text{Fix} G}(x)\|^2 + \|P_{\text{Fix} G}(x) - y\|^2. \] (3.7)

Combine (3.6) with (3.7) to see that

\[ \|Gx - y\|^2 - \|x - y\|^2 \leq (\gamma^2 - 1) \|x - P_{\text{Fix} G}(x)\|^2, \] (3.8)

which yields (i).

(ii): Because $(\forall x \in \mathcal{H} \setminus \text{Fix} G), \|x - P_{\text{Fix} G} x\| > 0$ and $\gamma \in [0, 1]$, by (3.8),

\[ (\forall x \in \mathcal{H} \setminus \text{Fix} G)(\forall y \in \text{Fix} G) \quad \|Gx - y\| < \|x - y\|. \]

Hence, by Definition 2.6(iii), we obtain that $G$ is strictly quasinonexpansive. ■

Corollary 3.14 Let $G : \mathcal{H} \to \mathcal{H}$ be an affine BAM. Then $G$ is nonexpansive.

Proof. By Definition 3.1(i), $G$ is a BAM yields that Fix $G$ is a nonempty closed and convex subset of $\mathcal{H}$. Moreover, because $G$ is affine,

\[ (\forall x \in \text{Fix} G)(\forall y \in \text{Fix} G)(\forall \alpha \in \mathbb{R}) \quad G(\alpha x + (1 - \alpha)y) = \alpha G(x) + (1 - \alpha)G(y) = \alpha x + (1 - \alpha)y, \]

which implies that Fix $G$ is an affine subspace. Hence, by Proposition 3.13(ii), $G$ is strictly quasinonexpansive. Therefore, by Lemma 2.8, $G$ is nonexpansive. ■

Let $T : \mathcal{H} \to \mathcal{H}$ with Fix $T \neq \emptyset$ and let $\kappa \in \mathbb{R}_+$. We say $T$ is linear regular with constant $\kappa$ if

\[ (\forall x \in \mathcal{H}) \quad d_{\text{Fix} T}(x) \leq \kappa \|x - Tx\|. \]

By the following two results, we know that every BAM is linearly regular, but generally linearly regular operator is not a BAM.

Proposition 3.15 Let $G : \mathcal{H} \to \mathcal{H}$ and let $\gamma \in [0, 1[$. Suppose that $G$ is a $\gamma$-BAM. Then $G$ is linearly regular with constant $\frac{1}{1 - \gamma}$.

Proof. Because $G$ is a $\gamma$-BAM, by Definition 3.1, Fix $G$ is a nonempty closed convex subset of $\mathcal{H}$ and

\[ (\forall x \in \mathcal{H}) \quad \|Gx - P_{\text{Fix} G} x\| \leq \gamma \|x - P_{\text{Fix} G} x\|. \] (3.9)

Let $x \in \mathcal{H}$. By the triangle inequality and (3.9),

\[ \|x - P_{\text{Fix} G} x\| \leq \|x - Gx\| + \|Gx - P_{\text{Fix} G} x\| \leq \|x - Gx\| + \gamma \|x - P_{\text{Fix} G} x\|, \]

\[ \Rightarrow (1 - \gamma)\|x - P_{\text{Fix} G} x\| \leq \|x - Gx\| \]

\[ \Leftrightarrow \|x - P_{\text{Fix} G} x\| \leq \frac{1}{1 - \gamma}\|x - Gx\|. \]

Hence, $(\forall x \in \mathcal{H})\ d_{\text{Fix} T}(x) \leq \frac{1}{1 - \gamma}\|x - Gx\|$, that is, $G$ is linearly regular with constant $\frac{1}{1 - \gamma}$. ■
Example 3.16 Suppose that $\mathcal{H} = \mathbb{R}^2$. Let $C = \mathbb{B}[0; 1]$ and $G = R_C$. Let $x = (2, 0)$. $\text{P}_C R_C x = (0, 0) \neq (1, 0) = \text{P}_C x$, which, by Definition 3.1, yields that $R_C$ is not a BAM. On the other hand, apply [3, Example 2.2] with $\lambda = 2$ to obtain that $R_C = (1 - 2) \text{Id} + 2 \text{P}_C$ is linearly regular with constant $\frac{1}{2}$.

Proposition 3.17 Let $I := \{1, \ldots, m\}$. Let $(\forall i \in I) G_i : \mathcal{H} \to \mathcal{H}$ be operators with $\text{Fix} G_i$ being a closed affine subspace of $\mathcal{H}$ and $(\forall i \in I) \gamma_i \in [0, 1]$. Suppose that $(\forall i \in I) G_i$ is a $\gamma_i$-BAM and that $\cap_{i \in I} \text{Fix} G_j \neq \emptyset$. The following hold:

(i) $G_{m} \cdots G_1$ is strictly quasinonexpansive.

(ii) Fix $G_m \cdots G_1 = \text{Fix} \cap_{i \in I} \text{Fix} G_i$.

(iii) Let $(\omega_i)_{i \in I}$ be real numbers in $[0, 1]$ such that $\sum_{i \in I} \omega_i = 1$. Then $\text{Fix} \sum_{i \in I} \omega_i G_i = \text{Fix} \cap_{i \in I} \text{Fix} G_i$.

Proof. Because $(\forall i \in I) G_i$ is a $\gamma_i$-BAM with $\text{Fix} G_i$ being a closed affine subspace of $\mathcal{H}$, by Proposition 3.13, $(\forall i \in I) G_i$ is strictly quasinonexpansive. Moreover, by assumption, $\cap_{i \in I} \text{Fix} G_i \neq \emptyset$.

(i)&(ii): These are from [3, Corollary 4.50].

(iii): This comes from [3, Proposition 4.47].

Proposition 3.18 Let $G : \mathcal{H} \to \mathcal{H}$ with $\text{Fix} G$ being a nonempty closed convex subset of $\mathcal{H}$. Then $G$ is a 0-BAM if and only if $G = \text{P}_{\text{Fix} G}$.

Proof. “⇒”: Assume that $G$ is a 0-BAM. By Definition 3.1, $(\forall x \in \mathcal{H}) \|G x - \text{P}_{\text{Fix} G} x\| \leq 0\|x - \text{P}_{\text{Fix} G} x\| = 0$. Hence, $G = \text{P}_{\text{Fix} G}$.

“⇐”: Assume that $G = \text{P}_{\text{Fix} G}$. Then by Example 3.4, $G$ is a BAM with constant 0.

Corollary 3.19 Let $(\forall i \in \{1, 2\}) G_i : \mathcal{H} \to \mathcal{H}$ be such that $\text{Fix} G_i$ is a closed affine subspace of $\mathcal{H}$. Suppose that $(\forall i \in \{1, 2\}) G_i$ is a BAM and that $\text{Fix} G_1 \cap \text{Fix} G_2 \neq \emptyset$. Then $G_2 G_1$ is a 0-BAM if and only if $G_2 G_1 = \text{P}_{\text{Fix} G_1 \cap \text{Fix} G_2}$.

Proof. Because Fix $G_1$ and Fix $G_2$ are closed affine subspaces and Fix $G_1 \cap$ Fix $G_2 \neq \emptyset$, Fix $G_1 \cap$ Fix $G_2$ is a closed affine subspace.

“⇒”: By Proposition 3.17(ii), Fix $G_2 G_1 = \text{Fix} G_1 \cap \text{Fix} G_2$ is a closed affine subspace. Hence, by Proposition 3.18, $G_2 G_1 = \text{P}_{\text{Fix} G_1 \cap \text{Fix} G_2} = \text{P}_{\text{Fix} G_1} \cap \text{Fix} G_2$.

“⇐”: By Example 3.4, $G_2 G_1 = \text{P}_{\text{Fix} G_1 \cap \text{Fix} G_2}$ is a 0-BAM.

According to the following Example 3.20 and Example 6.18 below, we know that the composition of BAMs is not sufficient to deduce that the individual BAMs are projectors. Hence, the condition “$G_i$ is a BAM” in the Corollary 3.19 above is more general than “$G_i$ is a projector”.

Example 3.20 Let $U_1 := \mathbb{R}(1, 0)$ and $U_2 := \mathbb{R}(0, 1)$. Set $T_1 := \frac{1}{2} P_{U_1}$ and $T_2 := \frac{1}{2} P_{U_2}$. Then neither $T_1$ nor $T_2$ is a projection. Moreover, $T_2 T_1 = P_{\{0,0\}}$.

Corollary 3.21 Let $C_1$ and $C_2$ be closed convex subsets of $\mathcal{H}$ with $C_1 \cap C_2 \neq \emptyset$. Then $P_{C_2} P_{C_1}$ is a 0-BAM if and only if $P_{C_2} P_{C_1} = P_{C_1 \cap C_2}$.

Proof. Because $C_1 \cap C_2 \neq \emptyset$, by [14, Corollary 4.5.2], Fix $P_{C_2} P_{C_1} = C_1 \cap C_2$ is nonempty, closed, and convex. Therefore, the desired result follows from Proposition 3.18.

Proposition 3.22 Let $z \in \mathcal{H}$. Let $(\forall i \in \{1, 2\}) G_i : \mathcal{H} \to \mathcal{H}$ satisfy

$$(\forall x \in \mathcal{H}) \quad G_1 x = z + G_2 (x - z).$$

Then the following assertions hold:

(i) Fix $G_2 = \text{Fix} G_1 - z$.

(ii) Suppose that Fix $G_1$ or Fix $G_2$ is a nonempty closed convex subset of $\mathcal{H}$. Let $\gamma \in [0, 1]$. Then $G_1$ is a $\gamma$-BAM if and only if $G_2$ is a $\gamma$-BAM.

Proof. (i): Let $x \in \mathcal{H}$. Then,

$x \in \text{Fix} G_2 \iff x = G_2 x = G_1(x + z) - z \iff x + z = G_1(x + z) \iff x + z \in \text{Fix} G_1 \iff x \in \text{Fix} G_1 - z$.

(ii): Clearly, by (i), Fix $G_1$ is a nonempty closed convex subset of $\mathcal{H}$ if and only if Fix $G_2$ is a nonempty closed convex subset of $\mathcal{H}$. 

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Note that
\[ P_{\text{Fix} G_1} G_1 = P_{\text{Fix} G_1} \iff (\forall x \in \mathcal{H}) \ P_{\text{Fix} G_1} G_1 x = P_{\text{Fix} G_1} x \]
\[ \iff (\forall x \in \mathcal{H}) \ P_{z + \text{Fix} G_2} G_1 x = P_{z + \text{Fix} G_2} x \quad \text{(by (i))} \]
\[ \iff (\forall x \in \mathcal{H}) \ z + P_{\text{Fix} G_2} (G_1 x - z) = z + P_{\text{Fix} G_2} (x - z) \quad \text{(by Fact 2.1)} \]

and that
\[ (\forall x \in \mathcal{H}) \ |G_1 x - P_{\text{Fix} G_1} x| \leq |x - P_{\text{Fix} G_1} x| \]
\[ \iff (\forall x \in \mathcal{H}) \ |G_1 x - P_{z + \text{Fix} G_2} x| \leq |x - P_{z + \text{Fix} G_2} x| \quad \text{(by (ii))} \]
\[ \iff (\forall x \in \mathcal{H}) \ |G_1 x - (z + P_{\text{Fix} G_2} (x - z))| \leq |x - (z + P_{\text{Fix} G_2} (x - z))| \quad \text{(by Fact 2.1)} \]
\[ \iff (\forall x \in \mathcal{H}) \ |G_2 (x - z) - P_{\text{Fix} G_2} (x - z)| \leq |(x - z) - P_{\text{Fix} G_2} (x - z)| \]
\[ \iff (\forall x \in \mathcal{H}) \ |G_2 x - P_{\text{Fix} G_2} x| \leq |x - P_{\text{Fix} G_2} x|. \]

Altogether, by Definition 3.1, (ii) above is true. ■

Lemma 3.23 Set \( I := \{1, \ldots, m\} \). Let \( (\forall i \in I) F_i : \mathcal{H} \rightarrow \mathcal{H} \). Define
\[ (\forall i \in I) (\forall x \in \mathcal{H}) \quad T_i x := z + F_i (x - z). \tag{3.11} \]

Let \( \gamma \in [0, 1] \). Then the following hold:

(i) Suppose that Fix \( F_m \cdots F_1 \) or Fix \( T_m \cdots T_1 \) is a nonempty closed and convex subset of \( \mathcal{H} \). Then \( F_m \cdots F_1 \) is a \( \gamma \)-BAM if and only if \( T_m \cdots T_1 \) is a \( \gamma \)-BAM.

(ii) Let \( (\omega_i)_{i \in I} \) be in \( \mathbb{R} \) such that \( \sum_{i \in I} \omega_i = 1 \). Suppose that Fix \( \sum_{i \in I} \omega_i F_i \) or Fix \( \sum_{i \in I} \omega_i T_i \) is a nonempty closed and convex subset of \( \mathcal{H} \). Then \( \sum_{i \in I} \omega_i F_i \) is a \( \gamma \)-BAM if and only if \( \sum_{i \in I} \omega_i T_i \) is a \( \gamma \)-BAM.

Proof. Let \( x \in \mathcal{H} \). By (3.11), it is easy to see that
\[ T_m \cdots T_2 T_1 x = T_m \cdots T_2 (z + F_1 (x - z)) = \cdots = z + F_m \cdots F_2 F_1 (x - z) \]
\[ \sum_{i \in I} \omega_i T_i x = \sum_{i \in I} \omega_i (z + F_i (x - z)) = z + \sum_{i \in I} \omega_i F_i (x - z). \]

Therefore, both (i) and (ii) follow from Proposition 3.22(ii). ■

4 Compositions of BAMs

In this section, we study compositions of BAMs and determine whether the composition of BAMs is still a BAM or not.

Compositions of BAMs with closed and affine fixed point sets

In this subsection, we consider compositions of BAMs with with closed and affine fixed point sets.

The following result is essential to the proof of Theorem 4.2 below.

Lemma 4.1 Set \( I := \{1, 2\} \). Let \( (\forall i \in I) G_i : \mathcal{H} \rightarrow \mathcal{H} \), and let \( \gamma_i \in [0, 1] \). Set \( (\forall i \in I) U_i := \text{Fix} G_i \). Suppose that \( (\forall i \in I) G_i \) is a \( \gamma_i \)-BAM and that \( U_i \) is a closed linear subspace of \( \mathcal{H} \). Denote the cosine \( c(U_1, U_2) \) of the Friedrichs angle between \( U_1 \) and \( U_2 \) by \( c_F \). Let \( x \in \mathcal{H} \), and let \( x = P_{U_1 \cap U_2} x \neq 0 \) and \( G_1 x - P_{U_1 \cap U_2} x \neq 0 \). Set
\[ \beta_1 := \frac{\|P_{U_2} G_1 x - P_{U_1 \cap U_2} x\|}{\|G_1 x - P_{U_1 \cap U_2} x\|} \quad \text{and} \quad \beta_2 := \frac{\|P_{U_1} x - P_{U_1 \cap U_2} x\|}{\|x - P_{U_1 \cap U_2} x\|}. \tag{4.1} \]

Then the following statements hold:
(i) \( \|G_2G_1x - P_{U_1\cap U_2}x\|^2 \leq (\gamma_2^2 + (1 - \gamma_2^2)\beta_1^2) (\gamma_1^2 + (1 - \gamma_1^2)\beta_2^2) \|x - P_{U_1\cap U_2}x\|^2 \).

(ii) \( \beta_1 \in [0, 1] \text{ and } \beta_2 \in [0, 1] \).

(iii) Suppose that \( P_{U_1}x - P_{U_1\cap U_2}x \neq 0 \) and \( P_{U_2}G_1x - P_{U_1\cap U_2}x \neq 0 \). Set
\[
 u := \frac{G_1x - P_{U_1\cap U_2}x}{\|G_1x - P_{U_1\cap U_2}x\|}, \quad v := \frac{P_{U_1}x - P_{U_1\cap U_2}x}{\|P_{U_1}x - P_{U_1\cap U_2}x\|}, \quad \text{and} \quad w := \frac{P_{U_2}G_1x - P_{U_1\cap U_2}x}{\|P_{U_2}G_1x - P_{U_1\cap U_2}x\|}.
\]

Then
\[
 \langle v, w \rangle \leq c_F, \quad (4.2a)
\]
\[
 \beta_1 = \langle u, w \rangle \quad \text{and} \quad \beta_2 \leq \langle u, v \rangle, \quad (4.2b)
\]
\[
 \beta_1\beta_2 \leq \frac{1+c_F}{2}, \quad (4.2c)
\]
\[
 \min\{\beta_1, \beta_2\} \leq \sqrt{\frac{1+c_F}{2}}. \quad (4.2d)
\]

Proof. Because \( G_1 \) is a \( \gamma_1 \)-BAM and \( G_2 \) is a \( \gamma_2 \)-BAM, by Definition 3.1 and Lemma 3.2(i), we have that
\[
 P_{U_1}G_1 = P_{U_1} = G_1P_{U_1} \quad \text{and} \quad P_{U_2}G_2 = P_{U_2} = G_2P_{U_2}, \quad (4.3)
\]
and that
\[
 (\forall y \in \mathcal{H}) \quad \|G_1y - P_{U_1}y\| \leq \gamma_1\|y - P_{U_1}y\| \quad \text{and} \quad \|G_2y - P_{U_2}y\| \leq \gamma_2\|y - P_{U_2}y\|. \quad (4.4)
\]

Note that by (4.3) and Fact 2.2, we have that
\[
 G_2G_1x - P_{U_2}G_1x = (\text{Id} - P_{U_2})G_2G_1x = P_{U_2}^\perp G_2G_1x \in U_2^\perp, \quad (4.5a)
\]
\[
 G_1x - P_{U_2} G_1x = (\text{Id} - P_{U_2})G_1x = P_{U_2}^\perp G_1x \in U_2^\perp, \quad (4.5b)
\]
\[
 G_1x - P_{U_1} G_1x = (\text{Id} - P_{U_1})G_1x = P_{U_1}^\perp G_1x \in U_1^\perp, \quad (4.5c)
\]
\[
 x - P_{U_1} x = (\text{Id} - P_{U_1})x = P_{U_1}^\perp x \in U_1^\perp. \quad (4.5d)
\]

Hence, by the Pythagorean theorem, we obtain
\[
 \frac{\|G_2G_1x - P_{U_1\cap U_2}x\|^2}{\|G_1x - P_{U_2} G_1x\|^2} + \frac{\|P_{U_2} G_1x - P_{U_1\cap U_2}x\|^2}{\|G_1x - P_{U_1\cap U_2}x\|^2} = \|G_2G_1x - P_{U_1\cap U_2}x\|^2, \quad (4.6)
\]
\[
 \frac{\|G_1x - P_{U_2} G_1x\|^2}{\|G_1x - P_{U_2} G_1x\|^2} + \frac{\|P_{U_2} G_1x - P_{U_1\cap U_2}x\|^2}{\|G_1x - P_{U_1\cap U_2}x\|^2} = \|G_1x - P_{U_1\cap U_2}x\|^2, \quad (4.7)
\]
\[
 \frac{\|G_1x - P_{U_1} G_1x\|^2}{\|G_1x - P_{U_1} G_1x\|^2} + \frac{\|P_{U_1} G_1x - P_{U_1\cap U_2}x\|^2}{\|G_1x - P_{U_1\cap U_2}x\|^2} = \|G_1x - P_{U_1\cap U_2}x\|^2, \quad (4.8)
\]
\[
 \frac{\|x - P_{U_1} G_1x\|^2}{\|x - P_{U_1} G_1x\|^2} + \frac{\|P_{U_1} G_1x - P_{U_1\cap U_2}x\|^2}{\|x - P_{U_1\cap U_2}x\|^2} = \|x - P_{U_1\cap U_2}x\|^2. \quad (4.9)
\]

(i): Note that
\[
 \|G_2G_1x - P_{U_1\cap U_2}x\|^2 \equiv \|G_2G_1x - P_{U_2} G_1x\|^2 + \|P_{U_2} G_1x - P_{U_1\cap U_2}x\|^2 \]
Hence, using Definition 2.4, we obtain that

\[ \gamma_2^2 \|G_1 x - P_{U_2} G_1 x\|^2 + \|P_{U_2} G_1 x - P_{U_l \cap U_2} x\|^2 \]

\[ = \gamma_2^2 \left( \|G_1 x - P_{U_2} G_1 x\|^2 + \|P_{U_2} G_1 x - P_{U_l \cap U_2} x\|^2 \right) + (1 - \gamma_2^2) \|P_{U_2} G_1 x - P_{U_l \cap U_2} x\|^2 \]

\[ = \gamma_2^2 \|G_1 x - P_{U_l \cap U_2} x\|^2 + (1 - \gamma_2^2) \|P_{U_2} G_1 x - P_{U_l \cap U_2} x\|^2 \]

\[ = \left( \gamma_2^2 + (1 - \gamma_2^2) \beta_1^2 \right) \|G_1 x - P_{U_l \cap U_2} x\|^2 \]

\[ + (1 - \gamma_2^2) \|P_{U_2} G_1 x - P_{U_l \cap U_2} x\|^2 \]

\[ = \left( \gamma_2^2 + (1 - \gamma_2^2) \beta_1^2 \right) \left( \|G_1 x - P_{U_l \cap U_2} x\|^2 + \|P_{U_2} G_1 x - P_{U_l \cap U_2} x\|^2 \right) \]

\[ \leq \left( \gamma_2^2 + (1 - \gamma_2^2) \beta_1^2 \right) \left( \|G_1 x - P_{U_l \cap U_2} x\|^2 + (1 - \gamma_2^2) \|P_{U_2} G_1 x - P_{U_l \cap U_2} x\|^2 \right) \]

\[ \leq \left( \gamma_2^2 + (1 - \gamma_2^2) \beta_1^2 \right) \left( \gamma_1^2 \|x - P_{U_l \cap U_2} x\|^2 \right) \]

\[ + (1 - \gamma_2^2) \|P_{U_2} G_1 x - P_{U_l \cap U_2} x\|^2 \]

\[ \leq \left( \gamma_2^2 + (1 - \gamma_2^2) \beta_1^2 \right) \left( \gamma_1^2 \|x - P_{U_l \cap U_2} x\|^2 + (1 - \gamma_2^2) \|P_{U_2} G_1 x - P_{U_l \cap U_2} x\|^2 \right) \]

\[ = \left( \gamma_2^2 + (1 - \gamma_2^2) \beta_1^2 \right) \left( \gamma_1^2 + (1 - \gamma_1^2) \beta_2^2 \right) \|x - P_{U_l \cap U_2} x\|^2 . \]

(ii): This comes from (4.1), (4.7) and (4.9).

(iii): By Lemma 2.3 and Fact 2.2, we know that

\[ P_{U_l} x - P_{U_l \cap U_2} x = P_{U_l} x - P_{U_l \cap U_2} P_{U_l} x = (\text{Id} - P_{U_l \cap U_2}) P_{U_l} x = P_{(U_l \cap U_2)^\perp} P_{U_l} (x) \in U_l \cap (U_l \cap U_2)^\perp . \]

By Lemma 2.3, (4.3) and Fact 2.2, \( P_{U_l \cap U_2} x = P_{U_l \cap U_2} P_{U_l} x = P_{U_l \cap U_2} P_{U_l} G_1 x = P_{U_l \cap U_2} P_{U_2} G_1 x \), so by Fact 2.2,

\[ P_{U_2} G_1 x - P_{U_l \cap U_2} x = P_{U_2} G_1 x - P_{U_l \cap U_2} P_{U_2} G_1 x = P_{(U_l \cap U_2)^\perp} P_{U_2} G_1 x \in U_2 \cap (U_l \cap U_2)^\perp . \]

Hence, using Definition 2.4, we obtain that

\[ \langle v, w \rangle = \left\langle \frac{P_{U_l} x - P_{U_l \cap U_2} x}{\|P_{U_l} x - P_{U_l \cap U_2} x\|}, \frac{P_{U_2} G_1 x - P_{U_l \cap U_2} x}{\|P_{U_2} G_1 x - P_{U_l \cap U_2} x\|} \right\rangle \leq c_F , \]

which yields (4.2a).

It is easy to see that

\[ \langle G_1 x - P_{U_l \cap U_2} x, P_{U_2} G_1 x - P_{U_l \cap U_2} x \rangle = \langle G_1 x - P_{U_2} G_1 x, P_{U_2} G_1 x - P_{U_l \cap U_2} x \rangle + \langle P_{U_2} G_1 x - P_{U_l \cap U_2} x, G_1 x - P_{U_l \cap U_2} x \rangle . \]

Hence,

\[ \beta_1 = \frac{\|P_{U_2} G_1 x - P_{U_l \cap U_2} x\|}{\|G_1 x - P_{U_l \cap U_2} x\|} = \left\langle \frac{G_1 x - P_{U_l \cap U_2} x}{\|G_1 x - P_{U_l \cap U_2} x\|}, \frac{P_{U_2} G_1 x - P_{U_l \cap U_2} x}{\|P_{U_2} G_1 x - P_{U_l \cap U_2} x\|} \right\rangle = \langle u, w \rangle , \]

Moreover, by (4.4), \( \|G_1 x - P_{U_l \cap U_2} x\| \leq \gamma_1 \|x - P_{U_l \cap U_2} x\| \leq \|x - P_{U_l} x\| \), then using (4.8) and (4.9), we know that

\[ \|G_1 x - P_{U_l \cap U_2} x\| \leq \|x - P_{U_l} x\|. \]

Hence, (4.11) and (4.12) yield (4.2b).

Note that by (4.2b) and \( \|u\| = \|v\| = \|w\| = 1 \),

\[ \beta_1 + \beta_2 \leq \langle u, v + w \rangle \leq \|u\| \|v + w\| = \sqrt{\|v\|^2 + 2\langle v, w \rangle + \|w\|^2} = \sqrt{2(1 + \langle v, w \rangle)} \leq \sqrt{2(1 + c_F)} , \]

so

\[ \beta_1 \beta_2 \leq \frac{(\beta_1 + \beta_2)^2}{4} \leq \frac{1 + c_F}{2} . \]
which shows (4.2c).

We now turn to (4.2d). Suppose to the contrary that \( \min\{\beta_1, \beta_2\} > \sqrt{\frac{1+c_F}{x}} \). Then \( \beta_1\beta_2 > \frac{1+c_F}{2} \), which contradicts (4.2c).

Altogether, the proof is complete.

In the following result, we extend \([12, \text{Lemma 1}]\) from \(\mathbb{R}^n\) to \(\mathcal{H}\) and also provide a new constant associated with the composition of BAMs. Although the following proof is shorter than the proof of \([12, \text{Lemma 1}]\), the main idea of the following proof is from the proof of \([12, \text{Lemma 1}]\).

**Theorem 4.2** Set \(I := \{1, 2\}\). Let \(\forall i \in I\) \(G_i : \mathcal{H} \to \mathcal{H}\), and let \(\gamma_i \in [0, 1]\). Set \(\forall i \in I\) \(U_i := \text{Fix} G_i\). Suppose that \(\forall i \in I\) \(G_i\) is a \(\gamma_i\)-BAM and that \(U_i\) is a closed linear subspace of \(\mathcal{H}\) such that \(U_1 + U_2\) is closed. Denote the cosine \(c(U_1, U_2)\) of the Friedrichs angle between \(U_1\) and \(U_2\) by \(c_F\). Then the following hold:

(i) \(\text{Fix}(G_2 \circ G_1) = \text{Fix} G_1 \cap \text{Fix} G_2\) is a closed linear subspace of \(\mathcal{H}\) and \(P_{U_1 \cap U_2} G_2 G_1 = P_{U_1 \cap U_2}\).

(ii) Let \(x \in \mathcal{H}\). If \(x - P_{U_1 \cap U_2} x = 0\) or \(G_1 x - P_{U_1 \cap U_2} x = 0\), then \(\|G_2 G_1 x - P_{U_1 \cap U_2} x\| = 0\).

(iii) Set

\[
r := \max \left\{ \sqrt{\gamma_1^2 + (1 - \gamma_2^2) \frac{1+c_F}{2}}, \sqrt{\gamma_2^2 + (1 - \gamma_1^2) \frac{1+c_F}{2}} \right\}.
\]

Then \(r \in \left[ \max\{\gamma_1, \gamma_2\}, 1 \right]\). Moreover,

\[
(\forall x \in \mathcal{H}) \quad \|G_2 G_1 x - P_{U_1 \cap U_2} x\| \leq r\|x - P_{U_1 \cap U_2} x\|. \tag{4.14}
\]

(iv) Set

\[
s := \sqrt{\gamma_1^2 + \gamma_2^2 - 2\gamma_1 \gamma_2 + (1 - \gamma_1^2)(1 - \gamma_2^2) \frac{1+c_F}{4}}.
\]

Then \(s \in \left[ \max\{\gamma_1, \gamma_2, \frac{1}{2}\}, 1 \right]\). Moreover,

\[
(\forall x \in \mathcal{H}) \quad \|G_2 G_1 x - P_{U_1 \cap U_2} x\| \leq s\|x - P_{U_1 \cap U_2} x\|. \tag{4.16}
\]

(v) \(G_2 \circ G_1\) is a min \(\{r, s\}\)-BAM.

**Proof.** Because \(U_1 + U_2\) is closed, by \(\text{Fact 2.5}\), we know that

\[
c_F := c(U_1, U_2) \in [0, 1]\).
\]

Because \(G_1\) is a \(\gamma_1\)-BAM and \(G_2\) is a \(\gamma_2\)-BAM, by \(\text{Definition 3.1}\) and \(\text{Lemma 3.2(i)}\), we have that

\[
P_{U_1} G_1 = P_{U_1} G_1 P_{U_1} \quad \text{and} \quad P_{U_2} G_2 = P_{U_2} G_2 P_{U_2}, \tag{4.18}
\]

and that

\[
(\forall x \in \mathcal{H}) \quad \|G_2 x - P_{U_2} x\| \leq \gamma_2\|x - P_{U_2} x\|. \tag{4.19}
\]

(i): By assumptions and by \(\text{Proposition 3.17(ii)}\), \(\text{Fix}(G_2 \circ G_1) = \text{Fix} G_1 \cap \text{Fix} G_2 = U_1 \cap U_2\) is a closed linear subspace of \(\mathcal{H}\). Because \(U_1 \cap U_2 \subseteq U_1\) and \(U_1 \cap U_2 \subseteq U_2\), by \(\text{Lemma 2.3}\), we know that

\[
P_{U_1 \cap U_2} P_{U_2} = P_{U_1 \cap U_2} P_{U_2} \quad \text{and} \quad P_{U_1 \cap U_2} P_{U_1} = P_{U_1 \cap U_2} P_{U_1}. \tag{4.20}
\]

Moreover,

\[
P_{U_1 \cap U_2} G_2 G_1 = P_{U_1 \cap U_2} G_2 G_1 \tag{4.20}
\]

(i): If \(x - P_{U_1 \cap U_2} x = 0\), then by (4.20) and (4.18), then \(G_2 G_1 x = G_2 G_1 P_{U_1 \cap U_2} x = G_2 G_1 P_{U_1} P_{U_1 \cap U_2} x = G_2 P_{U_1} P_{U_1 \cap U_2} x = P_{U_2} P_{U_1 \cap U_2} x = P_{U_1 \cap U_2} x\). Hence, \(\|G_2 G_1 x - P_{U_1 \cap U_2} x\| = 0\).
If \( G_1x - P_{U_1 \cap U_2}x = 0 \), then by (4.20) and (4.19), \( \|G_2 G_1x - P_{U_1 \cap U_2}x\| = \|G_2 P_{U_1 \cap U_2}x - P_{U_2} P_{U_1 \cap U_2}x\| \leq \gamma_2 \|P_{U_1 \cap U_2}x - P_{U_2} P_{U_1 \cap U_2}x\| = 0 \), that is, \( \|G_2 G_1x - P_{U_1 \cap U_2}x\| = 0 \).

(iii): Because \( \gamma_1 \in [0,1] \) and \( \gamma_2 \in [0,1] \), by (4.17), \( r \in \max \{ \gamma_1, \gamma_2 \}, 1 \).

We shall prove (4.14) next. Let \( x \in \mathcal{H} \). By (ii), we are able to assume \( x - P_{U_1 \cap U_2}x \neq 0 \) and \( G_1x - P_{U_1 \cap U_2}x \neq 0 \). We define \( \beta_1 \) and \( \beta_2 \) as in Lemma 4.1.

Note that if \( P_{U_2}x - P_{U_1 \cap U_2}x = 0 \), then \( \beta_2 = 0 \). Moreover, by Lemma 4.1(i)&(ii), \( \|G_2 G_1x - P_{U_1 \cap U_2}x\|^2 \leq (\gamma_2^2 + (1 - \gamma_2^2) \beta_1^2) \gamma_1^2 \|x - P_{U_1 \cap U_2}x\|^2 \leq (\gamma_1^2 + (1 - \gamma_1^2) \beta_2^2) \|x - P_{U_1 \cap U_2}x\|^2 \). If \( P_{U_2} G_1x - P_{U_1 \cap U_2}x = 0 \), then \( \beta_1 = 0 \) and, by Lemma 4.1(i)&(ii), \( \|G_2 G_1x - P_{U_1 \cap U_2}x\|^2 \leq (\gamma_2^2 + (1 - \gamma_2^2) \beta_2^2) \|x - P_{U_1 \cap U_2}x\|^2 \leq \gamma_2^2 \|x - P_{U_1 \cap U_2}x\|^2 \). Because \( \max \{ \gamma_1, \gamma_2 \} \) \( \leq r \), we know that in these two cases, (4.14) is true. So in the rest of the proof, we assume that \( \text{P}_{U_2}x - \text{P}_{U_1 \cap U_2}x \neq 0 \) and \( \text{P}_{U_1} G_1x - \text{P}_{U_1 \cap U_2}x \neq 0 \).

Using (4.2d) in Lemma 4.1(iii), we obtain that

\[
\beta_1 \leq \sqrt{\frac{1 + c_F}{2}} \quad \text{or} \quad \beta_2 \leq \sqrt{\frac{1 + c_F}{2}}.
\]

Therefore, combine Lemma 4.1(i)&(ii) with (4.13) and (4.21) to obtain (4.14). Altogether, (iii) holds.

(iv): \( s = \sqrt{\gamma_1^2 + \gamma_2^2 - \gamma_1^2 \gamma_2^2 + (1 - \gamma_1^2)(1 - \gamma_2^2)(1 + c_F^2)} = \sqrt{\gamma_1^2 + (1 - \gamma_1^2)(1 - \gamma_2^2)(1 + c_F^2)} \) and \( \gamma_1 \in [0,1] \) and \( \gamma_2 \in [0,1] \) are symmetric in the expression of \( s \). So, by (4.17), \( s \in \left[ \max \{ \gamma_1, \gamma_2 \}, 1 \right] \). In addition, some elementary algebraic manipulations yield \( \gamma_1^2 + \gamma_2^2 - \gamma_1^2 \gamma_2^2 + (1 - \gamma_1^2)(1 - \gamma_2^2)(1 + c_F^2) \geq \gamma_1^2 + \gamma_2^2 - \gamma_1^2 \gamma_2^2 + (1 - \gamma_1^2)(1 - \gamma_2^2)(1 + c_F^2) \) \( \frac{1}{4} \geq \frac{1}{4} \). Hence, \( s \in \left[ \max \{ \gamma_1, \gamma_2 \}, 1 \right] \).

We prove (4.16) next. Let \( x \in \mathcal{H} \). Because \( s \geq \max \{ \gamma_1, \gamma_2 \} \), similarly to the proof of (iii), to show (4.16), we are able to assume \( x - P_{U_1 \cap U_2}x \neq 0 \) and \( G_1x - P_{U_1 \cap U_2}x \neq 0 \) and \( P_{U_1}x - P_{U_1 \cap U_2}x \neq 0 \) and \( P_{U_2}x - P_{U_1 \cap U_2}x \neq 0 \).

Define \( \beta_1 \) and \( \beta_2 \) as in Lemma 4.1.

Use Lemma 4.1(ii) and (4.2c) in Lemma 4.1(iii) respectively in the following two inequalities to obtain that

\[
\left( \gamma_2^2 + (1 - \gamma_2^2) \beta_1^2 \right) \left( \gamma_1^2 + (1 - \gamma_1^2) \beta_2^2 \right) = \gamma_2^2 \gamma_1^2 + \gamma_2^2(1 - \gamma_1^2) \beta_2^2 + \gamma_1^2(1 - \gamma_2^2) \beta_1^2 + (1 - \gamma_2^2)(1 - \gamma_1^2) \beta_1^2 \beta_2^2 \\
\leq \gamma_2^2 \gamma_1^2 + \gamma_2^2(1 - \gamma_1^2) \gamma_1^2(1 - \gamma_2^2) + (1 - \gamma_2^2)(1 - \gamma_1^2) \beta_1^2 \beta_2^2 \\
\leq \gamma_1^2 + \gamma_2^2 - \gamma_1^2 \gamma_2^2 + (1 - \gamma_1^2)(1 - \gamma_2^2) \left( \frac{1 + c_F^2}{4} \right) = s^2.
\]

This and Lemma 4.1(i) yield that

\[
\|G_2 G_1x - P_{U_1 \cap U_2}x\|^2 \leq \left( \gamma_2^2 + (1 - \gamma_2^2) \beta_1^2 \right) \left( \gamma_1^2 + (1 - \gamma_1^2) \beta_2^2 \right) \|x - P_{U_1 \cap U_2}x\|^2 \\
\leq s^2 \|x - P_{U_1 \cap U_2}x\|^2.
\]

Hence, (iv) holds.

(v): Combine Definition 3.1 with (i), (iii) and (iv) to obtain that \( G_2 \circ G_1 \) is a min \( \{ r, s \} \)-BAM.

Lemma 4.3 Set \( I := \{ 1, \ldots, m \} \). Let \( U_1, \ldots, U_m \) be closed linear subspaces of \( \mathcal{H} \). The following hold:

(i) Let \( i \in I \setminus \{ m \} \). Then

\[
U_{i+1} + \cap_{j=1}^i U_j \text{ is closed } \iff \ U_{i+1}^\perp + (\cap_{j=1}^i U_j)^\perp \text{ is closed } \iff \ U_{i+1}^\perp + \sum_{j=1}^i U_j^\perp \text{ is closed }.
\]

(ii) \( \forall i \in I \setminus \{ m \} \): \( U_{i+1} + \cap_{j=1}^i U_j \text{ is closed if and only if } (\forall i \in I) \sum_{j=1}^i U_j^\perp \text{ is closed }\)

Proof. (i): The two equivalences follow by Fact 2.5 and [15, Theorem 4.6.5(5)] respectively.

(ii): Note that by [15, Theorem 4.5(1)], \( U_i^\perp \) is a closed linear subspace of \( \mathcal{H} \), that is, \( U_i^\perp = \overline{U_i}^\perp \). Then the asserted result follows from (i) by the principle of strong mathematical induction on \( m \).

Theorem 4.4 Set \( I := \{ 1, \ldots, m \} \). Let \( (\forall i \in I) \gamma_i \in [0,1] \) and let \( G_i : \mathcal{H} \to \mathcal{H} \) be a \( \gamma_i \)-BAM such that \( U_i := \text{Fix} G_i \) is a closed affine subspaces of \( \mathcal{H} \) with \( \cap_{i \in I} U_i \neq \emptyset \). Assume that \( \forall i \in I \sum_{j=1}^i (\text{par } U_j)^\perp \text{ is closed }\). Then the following statements hold:
(i) \(\forall k \in \{1, \ldots, m\}\) Fix\(G_k \circ \cdots \circ G_1 = \cap_{i=1}^k\) Fix\(G_i\) is a closed affine subspaces of \(H\).

(ii) \(G_m \circ \cdots \circ G_2 \circ G_1\) is a BAM.

(iii) Suppose that \(m = 2\). Denote the cosine \(c(\text{par} U_1, \text{par} U_2)\) of the Friedrichs angle between \(\text{par} U_1\) and \(\text{par} U_2\) by \(c_F\). Set
\[
r := \max_{L \subset U} \sqrt{\gamma_2^2 + (1 - \gamma_2^2) \frac{1 + c_F}{2}}, \quad \text{and} \quad s := \sqrt{\gamma_1^2 + \gamma_2^2 - \gamma_1^2 \gamma_2^2 + (1 - \gamma_1^2)(1 - \gamma_2^2) \frac{(1 + c_F)^2}{4}}.
\]
Then \(\min\{r, s\} \in [0, 1]\) and \(G_2 \circ G_1\) is a \(\min\{r, s\}\)-BAM.

(iv) There exists \(\gamma \in [0, 1]\) such that
\[
(\forall x \in H) \quad \|(G_m \circ \cdots \circ G_2 \circ G_1)^k x - P_{U_1} x\| \leq \gamma^k \|x - P_{U_1} x\|.
\]

Proof. (i): This is from Proposition 3.17(ii).

(ii)&(iii): Let \(z \in \cap_{i \in I} U_i\). Define \((\forall i \in I) F_i : H \to H\) by
\[
(\forall x \in H) \quad F_i(x) := G_i(x + z) - z \quad \text{(4.23)}
\]
By the assumptions, (4.23) and Proposition 3.22, \(F_i\) is a \(\gamma_i\)-BAM with Fix\(F_i = \text{par} U_i\) being a closed linear subspace of \(H\). Hence, by (i), (4.23) and Lemma 3.23(i), we are able to assume that \(U_1, \ldots, U_m\) are closed linear subspaces of \(H\). Then (iii) reduces to Theorem 4.2(v).

We prove (ii) next. If \(m = 1\), then there is nothing to prove. Suppose that \(m \geq 2\). We prove it by induction on \(k \in \{1, \ldots, m - 1\}\). By assumption, \(G_1\) is a BAM, so the base case is true. Assume \(G_k \circ \cdots \circ G_1\) is a BAM for some \(k \in \{1, 2, \ldots, m - 1\}\). By the assumption, \((\forall i \in I) \sum_{j=1}^k U_i^j\) is closed, and by (i) and Lemma 4.3(ii), we know that Fix\((G_k \circ \cdots \circ G_1) + \text{Fix} G_{k+1} = (\bigcap_{j=1}^k U_i^j + U_{k+1}\) is closed. Hence, apply Theorem 4.2(v) with \(G_1 = G_k \circ \cdots \circ G_1 \text{ and } G_2 = G_{k+1}\) to obtain that \(G_{k+1} \circ G_k \circ \cdots \circ G_1\) is a BAM. Therefore, (ii) holds as well.

(iv): This comes from (ii) and Proposition 3.10.

The following Remark 4.5(i) and Remark 4.6(i) exhibit a case where the new constant \(s\) associated with the composition of BAMs presented in Theorem 4.2(v) is better than the constant \(r\) from [12]. Moreover, Remark 4.5 illustrates that generally \(\min\{r, s\}\) in Theorem 4.2 is not a sharp constant for the composition of BAMs.

**Remark 4.5** Let \(L_1\) and \(L_2\) be closed linear subspaces of \(H\). Assume that \(L_1 + L_2\) is closed. Denote by \(c_F := c(L_1, L_2)\) the Friedrichs angle between \(L_1\) and \(L_2\). By [14, Corollary 4.5.2], Fix\(P_{L_2} P_{L_1} = L_1 \cap L_2\) is a closed linear subspace of \(H\). By Example 3.4, both \(P_{L_1} \text{ and } P_{L_2}\) are 0-BAM. Moreover, the following hold:

(i) Apply Theorem 4.2(v) with \(G_1 = P_{L_1}, G_2 = P_{L_2}, \gamma_1 = 0, \gamma_2 = 0\) to obtain that \(\min\{\sqrt{\frac{1 + c_F}{2}}, \frac{1 + c_F}{2}\}\) and \(P_{L_2} P_{L_1}\) is a \(\frac{1 + c_F}{2}\)-BAM.

(ii) By [15, Lemma 9.5(7) and Theorem 9.8],
\[
(\forall x \in H) \quad \|P_{L_2} P_{L_1} x - P_{L_1 \cap L_2} x\| \leq c_F \|x - P_{L_1 \cap L_2} x\|,
\]
and \(c_F\) is the smallest constant satisfying the inequality above. Hence, \(P_{L_2} P_{L_1}\) is a BAM with sharp constant \(c_F\).

Recall that \(c_F := c(U_1, U_2) \in [0, 1]\), so \(c_F < \frac{1 + c_F}{2}\). Hence, we know that generally the constant associated with the composition of BAMs provided by Theorem 4.2(v) is not sharp.

The following Remark 4.6(ii) presents examples showing that the constants \(s\) and \(r\) in Theorem 4.2 are independent.

**Remark 4.6** Consider the constants \(r, s\) in Theorem 4.2(v).

(i) Suppose that \(\gamma_1 = 0\) or \(\gamma_2 = 0\), that is, \(G_1 = P_{U_1}\) or \(G_2 = P_{U_2}\). Without loss of generality, let \(\gamma_2 = 0\). Then
\[
r = \sqrt{\gamma_1^2 + (1 - \gamma_1^2) \frac{1 + c_F}{2}}, \quad \text{and} \quad s = \sqrt{\gamma_1^2 + (1 - \gamma_1^2) \frac{(1 + c_F)^2}{4}}.
\]
Therefore, \(s \leq r\).
(ii) Suppose that \( \gamma := \gamma_1 = \gamma_2 \in [0, 1] \) and that \( c_F = 0 \). Then
\[
\begin{align*}
    r &= \sqrt{\gamma^2 + (1 - \gamma^2) \frac{1}{2}} \\
    s &= \sqrt{2\gamma^2 - \gamma^4 + (1 - \gamma^2)^2 \frac{1}{4}}.
\end{align*}
\]
Hence
\[s^2 - r^2 = (1 - \gamma^2) \frac{3}{4} (3\gamma^2 - 1),\]
which implies that
\[s \geq r \iff \gamma \in \left[ \frac{\sqrt{3}}{3}, 1 \right] \quad \text{and} \quad s < r \iff \gamma \in \left[ 0, \frac{\sqrt{3}}{3} \right].\]

**Compositions of BAMs with general convex fixed point sets**

In this subsection, we investigate compositions of BAMs with general closed and convex fixed point sets.

By Example 3.4, the projection onto a nonempty closed convex subset of \( \mathcal{H} \) is the most common BAM. The following results show that the order of the projections does matter to determine whether the composition of projections is a BAM or not. The next result considers the composition of projections onto a cone and a ball.

**Proposition 4.7** Let \( K \) be a nonempty closed convex cone in \( \mathcal{H} \), and let \( \rho \in \mathbb{R}_{++} \). Denote by \( B := B[0; \rho] \).

(i) \( P_B P_K = P_{K \cap B} \) is a 0-BAM.

(ii) Suppose that \( \mathcal{H} = \mathbb{R}^2, K = \mathbb{R}^2_+ \) and \( \rho = 1 \). Then \( P_K P_B \) is not a BAM.

**Proof.**

(i): By [2, Corollary 7.3], \( P_B P_K = P_{K \cap B} \), which, by Corollary 3.21, yields that \( P_B P_K \) is a 0-BAM.

(ii): By [14, Corollary 4.5.2], \( \text{Fix } P_K P_B = K \cap B \). By [2, Example 7.5], we know that
\[P_{K \cap B} P_K P_B(1, -1) = \left( \frac{1}{\sqrt{2}}, 0 \right) \neq (1, 0) = P_{K \cap B}(1, -1),\]
which implies that \( P_{K \cap B} P_K P_B \neq P_{K \cap B} \). So, by Definition 3.1, \( P_K P_B \) is not a BAM.

The following example considers projections onto an affine subspace and a cone.

**Example 4.8** Suppose \( \mathcal{H} = \mathbb{R}^2 \). Let \( U := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = -x_1 + 1\} \) and \( K := \mathbb{R}^2_+ \). Then the following hold (see also Figure 1):

(i) \( P_U P_K \) is not a BAM.

(ii) \( P_K P_U \) is a \( \frac{\sqrt{2}}{2} \)-BAM.

**Proof.** Define the lines \( L_1 := \mathbb{R} \cdot (1, 0), L_2 := \mathbb{R} \cdot (0, 1), l_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = x_1 - 1\} \) and \( l_2 := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = x_1 + 1\} \). It is easy to see that for every \( (x_1, x_2) \in \mathbb{R}^2 \),
\[
P_U(x_1, x_2) = \left( \frac{x_1 - x_2 + 1}{2}, \frac{-x_1 + x_2 + 1}{2} \right), \tag{4.24a}
\]
\[
P_K(x_1, x_2) = \begin{cases} 
    (x_1, x_2), & \text{if } x_1 \geq 0 \text{ and } x_2 \geq 0; \\
    (0, 0), & \text{if } x_1 < 0 \text{ and } x_2 < 0; \\
    (x_1, 0) = P_{L_1}(x_1, x_2), & \text{if } x_1 \geq 0 \text{ and } x_2 < 0; \\
    (0, x_2) = P_{L_2}(x_1, x_2), & \text{if } x_1 < 0 \text{ and } x_2 \geq 0. 
\end{cases} \tag{4.24b}
\]

By [14, Corollary 4.5.2],
\[
\text{Fix } P_U P_K = U \cap K = \text{Fix } P_K P_U. \tag{4.25}
\]
(i): Let \((x_1, x_2) \in \mathbb{R}^2 \setminus (K \cup \mathbb{R}^2_{<})\) such that \(x_1 - 1 < x_2 < x_1 + 1\), that is, \((x_1, x_2)\) is above \(l_1\) and below \(l_2\) but neither in \(K\) nor in the strictly negative orthant. Then by (4.24),

\[ P_{U \cap K}(x_1, x_2) = P_U(x_1, x_2) \neq P_{U \cap K} P_U(x_1, x_2), \]

which, by Definition 3.1 and (4.25), implies that \(P_U P_K\) is not a BAM.

(ii): By Definition 2.4, the cosine of Friedrichs angles between par \(U\) and \(L_1\) and between par \(U\) and \(L_2\) is

\[ c(\text{par } U, L_1) = \left\langle \frac{(1,-1)}{\sqrt{2}}, (1, 0) \right\rangle = \frac{\sqrt{2}}{2} = \left\langle \frac{(1,-1)}{\sqrt{2}}, (0, 1) \right\rangle = c(\text{par } U, L_2) \]  

(4.26)

Let \((x_1, x_2) \in \mathbb{R}^2\). If \((x_1, x_2) \in \{(y_1, y_2) \in \mathbb{R}^2 : y_1 - 1 \leq y_2 \leq y_1 + 1\}\), then \(P_{U \cap K}(x_1, x_2) = P_U(x_1, x_2)\), which yields that

\[ P_K P_U(x_1, x_2) = P_{U \cap K}(x_1, x_2) \quad \text{and} \quad P_{U \cap K} P_K P_U(x_1, x_2) = P_{U \cap K}(x_1, x_2). \]

Assume that \(x_2 < x_1 - 1\). Then

\[ P_K P_U(x_1, x_2) = P_{L_1} P_U(x_1, x_2), \]

\[ P_{U \cap K} P_K P_U(x_1, x_2) = (1, 0) = P_{U \cap K}(x_1, x_2) = P_{U \cap L_1}(x_1, x_2). \]

Moreover, because \(U\) and \(L_1\) are closed affine subspaces with \(U \cap L_1 \neq \emptyset\),

\[ \|P_{U \cap K} P_K P_U(x_1, x_2) - P_{U \cap K}(x_1, x_2)\| = \|P_{L_1} P_U(x_1, x_2) - P_{U \cap L_1}(x_1, x_2)\| \quad \text{(by (4.27a))} \]

\[ \leq \frac{\sqrt{2}}{2} \|(x_1, x_2) - P_{U \cap L_1}(x_1, x_2)\| \quad \text{(by Remark 4.5(ii) and (4.26))} \]

\[ = \frac{\sqrt{2}}{2} \|(x_1, x_2) - P_{U \cap K}(x_1, x_2)\|. \quad \text{(by (4.27a))} \]

Assume that \(x_2 > x_1 + 1\). Then similarly to the case that \(x_2 < x_1 - 1\), we also have that \(\|P_K P_U(x_1, x_2) - P_{U \cap K}(x_1, x_2)\| \leq \frac{\sqrt{2}}{2} \|(x_1, x_2) - P_{U \cap K}(x_1, x_2)\|\).

Altogether, for every \((x_1, x_2) \in \mathbb{R}^2\), we have that

\[ P_{U \cap K} P_K P_U(x_1, x_2) = P_{U \cap K}(x_1, x_2), \]

\[ \|P_K P_U(x_1, x_2) - P_{U \cap K}(x_1, x_2)\| \leq \frac{\sqrt{2}}{2} \|(x_1, x_2) - P_{U \cap K}(x_1, x_2)\|, \]

which combining with (4.25) yield that \(P_K P_U\) is a \(\frac{\sqrt{2}}{2}\)-BAM.

\[\text{Figure 1: Composition of projections onto line and cone}\]
Remark 4.9  By Proposition 4.7 and Example 4.8, we know that in Theorem 4.4, the assumption \( (\forall i \in I) \text{Fix} G_i \) is closed affine subspaces is not tight, and that the order of the operators matters.

The following example examines the composition of projections onto balls and states that generally the composition of BAMs is not a BAM again.

Example 4.10  Suppose that \( H = \mathbb{R}^2 \). Consider the two closed balls \( K_1 = \{(x_1, x_2) : (x_1 + 1)^2 + x_2^2 \leq 4\} \) and let \( K_2 = \{(x_1, x_2) : (x_1 - 1)^2 + x_2^2 \leq 4\} \). Then the following statements hold (see also Figure 2):

(i) For every \( x \in \{(x_1, x_2) \in \mathbb{R}^2 \setminus (K_1 \cup K_2) : x_1 < 0 \text{ and } x_2 \neq 0\} \), \( P_{K_1 \cap K_2} P_{K_2} P_{K_1} x \neq P_{K_2} P_{K_1} x \).

(ii) \( P_{K_2} P_{K_1} \) is not a BAM.

Proof.  By Example 3.4 and Proposition 3.17(ii), \( \text{Fix} P_{K_2} P_{K_1} = K_1 \cap K_2 \). The proof follows by Definition 3.1, the formula shown in [3, Example 3.18] and some elementary algebraic manipulations.

![Figure 2: Composition of BAMs may not be a BAM](image)

5 Combinations of BAMs

In this section, we consider combinations of finitely many BAMs. In the following results, by reviewing Remark 3.5, we obtain constraints for the coefficients constructing the combinations.

Remark 5.1  Let \( C \) be a nonempty closed convex subset of \( H \) and let \( \gamma \in \mathbb{R} \). Note that by Example 3.4, \( P_C \) and \( \text{Id} = P_H \) are BAMs.

(i) Let \( \gamma \in \mathbb{R} \). By Remark 3.5(i) and Definition 3.1, if \( \gamma \text{Id} + (1 - \gamma) P_C \) is a BAM, then \( \gamma \in [-1, 1] \).

(ii) In addition, suppose that \( H = \mathbb{R}^2 \) and \( C := B[0; 1] \). Then by Remark 3.5(ii), \( \gamma \text{Id} + (1 - \gamma) P_C \) is a BAM implies that \( \gamma \in [0, 1] \).

The following results are similar to Example 4.10.

Example 5.2  Suppose that \( H = \mathbb{R}^2 \). Consider the two closed balls \( K_1 := \{(x_1, x_2) : (x_1 + 1)^2 + x_2^2 \leq 4\} \) and let \( K_2 := \{(x_1, x_2) : (x_1 - 1)^2 + x_2^2 \leq 4\} \). Let \( \alpha \in [0, 1] \). Then the following hold:

(i) For every \( x \in \{(x_1, x_2) \in \mathbb{R}^2 \setminus (K_1 \cup K_2) : x_1 < 0 \text{ and } x_2 \neq 0\} \), \( P_{K_1 \cap K_2}(\alpha P_{K_1} + (1 - \alpha) P_{K_2}) x \neq P_{K_1 \cap K_2} x \).

(ii) \( P_{K_2} P_{K_1} \) is not a BAM.

Proof.  Note that by Example 3.4 and Proposition 3.17(iii), \( \text{Fix}(\alpha P_{K_2} + (1 - \alpha) P_{K_2}) = K_1 \cap K_2 \). The remaining part of the proof is similar to the proof of Example 4.10.

In the remaining part of this section, we consider convex combinations of BAMs with closed and affine fixed point sets.
Lemma 5.3  Set $I := \{1, \ldots, m\}$. Let $(\forall i \in I)$ $G_i$ be a BAM such that $\text{Fix} G_i$ is a closed affine subspace of $\mathcal{H}$, and $\cap_{i \in I} \text{Fix} G_i \neq \emptyset$. Let $(\forall i \in I) \omega_i \in [0,1[$ such that $\sum_{i \in I} \omega_i = 1$. Set $G := \sum_{i \in I} \omega_i G_i$. Then

(i) $\text{Fix} G = \cap_{i \in I} \text{Fix} G_i$ is a closed affine subspace of $\mathcal{H}$.

(ii) $P_{\text{Fix} G} = P_{\text{Fix} G_i}$.

Proof. (i): By Proposition 3.17(iii) and the assumptions, $\text{Fix} G = \text{Fix}(\sum_{i \in I} \omega_i G_i) = \cap_{i \in I} \text{Fix} G_i$ is a closed affine subspace of $\mathcal{H}$.

(ii): By (i) and [3, Proposition 29.14(i)], we know that $P_{\text{Fix} G}$ is affine. Note that $(\forall j \in I) \text{Fix} G = \cap_{i \in I} \text{Fix} G_i \subseteq \text{Fix} G_j$ and that both $\cap_{i \in I} \text{Fix} G_i$ and $\text{Fix} G_j$ are closed affine subspaces. Moreover, by Lemma 2.3 and Definition 3.1(i), we have that

$$\tag{5.1} (\forall j \in I) \quad P_{\text{Fix} G} G_j = P_{\cap_{i \in I} \text{Fix} G_i} G_j = P_{\cap_{i \in I} \text{Fix} G_i} P_{\text{Fix} G} G_j = P_{\cap_{i \in I} \text{Fix} G_i} P_{\text{Fix} G} = P_{\cap_{i \in I} \text{Fix} G_i} = P_{\text{Fix} G}.$$ 

Therefore,

$$P_{\text{Fix} G} G = P_{\text{Fix} G} \left( \sum_{i \in I} \omega_i G_i \right) = \sum_{i \in I} \omega_i P_{\text{Fix} G} G_i \overset{(5.1)}{=} \sum_{i \in I} \omega_i P_{\text{Fix} G} = P_{\text{Fix} G}.$$ 

Convex combination of BAMS with closed and affine fixed point sets

Theorem 5.4  Set $I := \{1,2\}$. Let $(\forall i \in I) \gamma_i \in [0,1[$ and let $G_i : \mathcal{H} \to \mathcal{H}$ be a $\gamma_i$-BAM. Suppose that $(\forall i \in I) \text{Fix} G_i$ is a closed linear subspace of $\mathcal{H}$. Suppose that $\text{Fix} G_1 + \text{Fix} G_2$ is closed. Let $\alpha \in [0,1[$. Set $c_F := c(\text{Fix} G_1, \text{Fix} G_2)$ and $\gamma := \max \left\{ \alpha \sqrt{\gamma_1^2 + (1 - \gamma_1^2) \frac{1 + c_F}{2} + (1 - \alpha), \alpha + (1 - \alpha) \sqrt{\gamma_2^2 + (1 - \gamma_2^2) \frac{1 + c_F}{2}} \right\}$. (5.2)

Then $\max \left\{ \alpha \sqrt{\frac{1}{2}(1 + \gamma_1^2) + (1 - \alpha), \alpha + (1 - \alpha) \sqrt{\frac{1}{2}(1 + \gamma_2^2)}} \right\} \leq \gamma < 1$ and $\alpha G_1 + (1 - \alpha)G_2$ is a $\gamma$-BAM.

Proof. Set $(\forall i \in I) U_i := \text{Fix} G_i$. Because $U_1 + U_2$ is closed, by Fact 2.5, we know that $c_F := c(U_1, U_2) \in [0,1[$, which yields that $\gamma < 1$ and that $(\forall i \in I) \gamma_i^2 + (1 - \gamma_i^2) \frac{1 + c_F}{2} \geq \frac{1}{2}(1 + \gamma_i^2)$. Hence, $\gamma \geq \left\{ \alpha \sqrt{\frac{1}{2}(1 + \gamma_1^2) + (1 - \alpha), \alpha + (1 - \alpha) \sqrt{\frac{1}{2}(1 + \gamma_2^2)}} \right\}.$

Let $x \in \mathcal{H}$. By Lemma 5.3 and Definition 3.1, it suffices to show that

$$\| \alpha G_1 x + (1 - \alpha)G_2 x - P_{U_1 \cap U_2} x \| \leq \gamma \| x - P_{U_1 \cap U_2} x \|. \quad (5.3)$$

Because $(\forall i \in I) G_i$ is a $\gamma_i$-BAM, by Definition 3.1 and Lemma 3.2(i), we have that

$$\tag{5.4} (\forall i \in I) \quad P_{U_i} G_i = P_{U_i} = G_i P_{U_i},$$

and that

$$\tag{5.5} (\forall y \in \mathcal{H}) \quad \| G_i y - P_{U_i} y \| \leq \gamma_i \| y - P_{U_i} y \|.$$ 

If $x = P_{U_1 \cap U_2} x$, then $x \in U_1 \cap U_2$ and $\alpha G_1 x + (1 - \alpha)G_2 x - P_{U_1 \cap U_2} x = \alpha G_1 P_{U_1} x + (1 - \alpha)G_2 P_{U_1} x - x = (\alpha P_{U_1} x + (1 - \alpha) P_{U_2} x) - x = x - x = 0$, from which we deduce that (5.3) holds. Therefore, in the rest of the proof, we assume that $x \neq P_{U_1 \cap U_2} x$. Set

$$\beta_1 := \frac{\| P_{U_1} x - P_{U_1 \cap U_2} x \|}{\| x - P_{U_1 \cap U_2} x \|} \quad \text{and} \quad \beta_2 := \frac{\| P_{U_2} x - P_{U_1 \cap U_2} x \|}{\| x - P_{U_1 \cap U_2} x \|}.$$ 

By the triangle inequality,

$$\| \alpha G_1 x + (1 - \alpha)G_2 x - P_{U_1 \cap U_2} x \|^2 \leq \alpha^2 \| G_1 x - P_{U_1 \cap U_2} x \|^2 + (1 - \alpha)^2 \| G_2 x - P_{U_1 \cap U_2} x \|^2 + 2\alpha(1 - \alpha) \| G_1 x - P_{U_1 \cap U_2} x \\| \| G_2 x - P_{U_1 \cap U_2} x \|.$$ 

(5.7)
Note that \((\forall i \in I)\),

\[
G_i x - P_{U_i} x \overset{(5.4)}{=} G_i x - P_{U_i} G_i x = P_{U_i^\perp} G_i x \in U_i^\perp, \quad (5.8a)
\]

\[
P_{U_i} x - P_{U_i \cap U_2} x = P_{U_i} x - P_{U_i \cap U_2} P_{U_i} x = P_{(U_i \cap U_2)^\perp} P_{U_i} x \in U_i \cap (U_i \cap U_2)^\perp. \quad (5.8b)
\]

Now, using (5.8a) and (5.8b) in the following (5.9a) and (5.9d), we know that \((\forall i \in I)\),

\[
\|G_i x - P_{U_i \cap U_2} x\|^2 = \|G_i x - P_{U_i} x\|^2 + \|P_{U_i} x - P_{U_i \cap U_2} x\|^2 \leq \gamma_i^2 \|x - P_{U_i} x\|^2 + \|P_{U_i} x - P_{U_i \cap U_2} x\|^2 \leq (1 - \gamma_i^2) \|P_{U_i} x - P_{U_i \cap U_2} x\|^2. \quad (5.9a)
\]

\[
= \gamma_i^2 \|x - P_{U_i} x\|^2 + (1 - \gamma_i^2) \|P_{U_i} x - P_{U_i \cap U_2} x\|^2 \leq (1 - \gamma_i^2) \|P_{U_i} x - P_{U_i \cap U_2} x\|^2. \quad (5.9b)
\]

\[
= \gamma_i^2 \|x - P_{U_i} x\|^2 + (1 - \gamma_i^2) \|P_{U_i} x - P_{U_i \cap U_2} x\|^2 + (1 - \gamma_i^2) \|P_{U_i} x - P_{U_i \cap U_2} x\|^2 \leq \gamma_i^2 \|x - P_{U_i} x\|^2 + (1 - \gamma_i^2) \|P_{U_i} x - P_{U_i \cap U_2} x\|^2. \quad (5.9c)
\]

\[
= \gamma_i^2 \|x - P_{U_i} x\|^2 + (1 - \gamma_i^2) \|P_{U_i} x - P_{U_i \cap U_2} x\|^2 + (1 - \gamma_i^2) \|P_{U_i} x - P_{U_i \cap U_2} x\|^2 \leq \gamma_i^2 \|x - P_{U_i} x\|^2. \quad (5.9d)
\]

\[
= \gamma_i^2 \|x - P_{U_i} x\|^2 + (1 - \gamma_i^2) \beta_i^2 \|x - P_{U_i \cap U_2} x\|^2. \quad (5.9e)
\]

\[
= \gamma_i^2 \|x - P_{U_i} x\|^2 + (1 - \gamma_i^2) \beta_i^2 \|x - P_{U_i \cap U_2} x\|^2. \quad (5.9f)
\]

Set

\[
(\forall i \in I) \quad \eta_i := \sqrt{\gamma_i^2 + (1 - \gamma_i^2) \beta_i^2}. \quad (5.10)
\]

Combine (5.7) with (5.9) to obtain that

\[
\|aG_1 x + (1 - a)G_2 x - P_{U_i \cap U_2} x\|^2 \leq \left( a^2 \gamma_1^2 + (1 - a)^2 \gamma_2^2 + 2a(1 - a)\eta_1 \eta_2 \right) \|x - P_{U_i \cap U_2} x\|^2 \leq \left( a^2 \gamma_1^2 + (1 - a)^2 \gamma_2^2 + 2a(1 - a)\gamma_1 \gamma_2 \right) \|x - P_{U_i \cap U_2} x\|^2 \leq \left( a^2 \gamma_1^2 + (1 - a)^2 \gamma_2^2 + 2a(1 - a)\gamma_1 \gamma_2 \right) \|x - P_{U_i \cap U_2} x\|^2. \quad (5.11a)
\]

\[
= (a \gamma_1 + (1 - a) \gamma_2) \|x - P_{U_i \cap U_2} x\|^2. \quad (5.11b)
\]

Combining (5.3), (5.2), (5.10) and (5.11), we know that it remains to show that

\[
\min\{\beta_1, \beta_2\} \leq \sqrt{1 + c_f \over 2}. \quad (5.12)
\]

Note that by (5.6), if there exists \(i \in I\) such that \(P_{U_i} x - P_{U_i \cap U_2} x = 0\), then \(\beta_i = 0\) and (5.12) is true. Hence, we assume \((\forall i \in I)\) \(P_{U_i} x - P_{U_i \cap U_2} x \neq 0\) from now on.

Let \(i \in I\). Because \(P_{U_i} x - P_{U_i \cap U_2} x \in U_i^\perp\) and \(x - P_{U_i} x = P_{U_i^\perp} x \in U_i^\perp\), we have \(P_{U_i} x - P_{U_i \cap U_2} x, x - P_{U_i} x = 0\). Hence

\[
\langle P_{U_i} x - P_{U_i \cap U_2} x, x - P_{U_i \cap U_2} x \rangle = \langle P_{U_i} x - P_{U_i \cap U_2} x, x - P_{U_i} x \rangle + \langle P_{U_i} x - P_{U_i \cap U_2} x, P_{U_i} x - P_{U_i \cap U_2} x \rangle = \|P_{U_i} x - P_{U_i \cap U_2} x\|^2
\]

and thus

\[
\beta_i = \|P_{U_i} x - P_{U_i \cap U_2} x\| = \left\langle P_{U_i} x - P_{U_i \cap U_2} x, x - P_{U_i \cap U_2} x \right\rangle \over \|x - P_{U_i \cap U_2} x\|. \quad (5.13)
\]

Set \(u := \frac{P_{U_i} x - P_{U_i \cap U_2} x}{\|P_{U_i} x - P_{U_i \cap U_2} x\|}, v := \frac{P_{U_2} x - P_{U_1 \cap U_2} x}{\|P_{U_2} x - P_{U_1 \cap U_2} x\|}\) and \(w := \frac{x - P_{U_1 \cap U_2} x}{\|x - P_{U_1 \cap U_2} x\|}\). By (5.8b), \(P_{U_i} x - P_{U_i \cap U_2} x \in U_i \cap (U_i \cap U_2)^\perp\) and \(P_{U_2} x - P_{U_1 \cap U_2} x \in U_2 \cap (U_1 \cap U_2)^\perp\). Hence, by Definition 2.4,

\[
\langle u, v \rangle \leq c_f. \quad (5.14)
\]

Using (5.13), the Cauchy-Schwarz inequality, and \(\|u\| = \|v\| = \|w\| = 1\), we deduce that

\[
\beta_1 + \beta_2 = \langle u + v, w \rangle \leq \|u + v\| = \sqrt{\|u\|^2 + 2\langle u, v \rangle + \|v\|^2} = \sqrt{2(1 + \langle u, v \rangle)} \leq \sqrt{2(1 + c_f)}. \quad (5.15)
\]

Suppose to the contrary that (5.12) is not true, that is, \(\beta_1 > \sqrt{1 + c_f} \over 2\) and \(\beta_2 > \sqrt{1 + c_f} \over 2\). Then

\[
\beta_1 + \beta_2 > 2\sqrt{1 + c_f \over 2} = \sqrt{2(1 + c_f)},
\]

which contradicts with (5.15). Altogether, the proof is complete.
The following example illustrates that the constant associated with the convex combination of BAMs provided in Theorem 5.4 is not sharp.

**Example 5.5** Let $U$ be a closed linear subspace of $\mathcal{H}$. Let $a \in [0,1[$. Then the following hold:

(i) $a P_U + (1-a) P_{U^\perp}$ is a BAM with constant $\max\{a \frac{\sqrt 2}{2} + (1-a), (1-a) \frac{\sqrt 2}{2} + a\}$, by Theorem 5.4.

(ii) $\alpha < a P_U + (1-a) P_{U^\perp}$ is a BAM with sharp constant $\max\{a,1-a\}$.

(iii) $\max\{a \frac{\sqrt 2}{2} + (1-a), (1-a) \frac{\sqrt 2}{2} + a\} > \max\{a,1-a\}$.

**Proof.** (i): By Example 3.4, both $P_U$ and $P_{U^\perp}$ are 0-BAMs. Moreover, by Definition 2.4, $c_F = c(U,U^\perp) = 0$. Hence, using Theorem 5.4 directly, we obtain that $\alpha P_U + (1-a) P_{U^\perp}$ is a BAM with constant $\max\{a \frac{\sqrt 2}{2} + (1-a), (1-a) \frac{\sqrt 2}{2} + a\}$.

(ii): Denote by $\alpha = a P_U + (1-a) P_{U^\perp}$. Let $x \in \mathcal{H}$ and $\gamma \in [0,1[$. Using $\langle P_U x , P_{U^\perp} x \rangle = 0$, Fact 2.2, and $P_{\text{Fix } G} x = P(0) x = 0$, we obtain that

\[
\|G x - P_{\text{Fix } G} x\| \leq \gamma \|x - P_{\text{Fix } G} x\|
\]

which implies that $\gamma \geq \max\{a,1-a\}$, since $x \in \mathcal{H}$ is arbitrary. Therefore, the required result follows from Lemma 5.3, (5.16) and Definition 3.1.

(iii): This is trivial from $a \in [0,1[$ and $\frac{\sqrt 2}{2} \in [0,1[$.

**Theorem 5.6** Set $I := \{1,\ldots,m\}$. Let $(\forall i \in I) \omega_i \in [0,1[$. Suppose that $m \geq 2$ and that $(\forall i \in I) G_i$ is a BAM with $U_i := \text{Fix } G_i$ being a closed affine subspace of $\mathcal{H}$ such that $\cap_{i \in I} \text{Fix } G_i \neq \emptyset$. Suppose that $(\forall i \in I) \sum_{j=1}^i (\text{par } U_j)$ is closed. Then $\sum_{i=1}^m \omega_i G_i$ is a BAM.

**Proof.** Let $z \in \cap_{i \in I} U_i$. Define $(\forall i \in I) F_i : \mathcal{H} \to \mathcal{H}$ by

\[
(\forall x \in \mathcal{H}) \quad F_i(x) := G_i(x+z) - z
\]

By the assumptions, (5.17) and Proposition 3.22, $F_i$ is a BAM with $F_i = \text{par } U_i$ being a closed linear subspace of $\mathcal{H}$. By Proposition 3.17(iii) and by assumptions, $\text{Fix } (\sum_{i \in I} \omega_i G_i) = \cap_{i=1}^m U_i$ is a closed affine subspace. Hence, by (5.17) and Lemma 3.23(ii), to show $\sum_{i=1}^m \omega_i G_i$ is a BAM, we are able to assume that $U_1,\ldots,U_m$ are closed linear subspaces of $\mathcal{H}$.

We prove it by induction on $m$. By Lemma 4.3(ii) and Theorem 5.4, we know that the base case in which $m = 2$ holds. Suppose that $m \geq 3$ and that the required result holds for $m - 1$, that is, for any $\{\alpha_1,\ldots,\alpha_{m-1}\} \subseteq [0,1[$ we have that if $(\forall i \in \{1,\ldots,m-1\}) \sum_{j=1}^i U_j$ is closed, then $\sum_{i=1}^{m-1} \alpha_i G_i$ is a BAM. Note that

\[
\sum_{i=1}^m \omega_i G_i = \left( \sum_{j=1}^{m-1} \omega_j \right) \left( \sum_{i=1}^{m-1} \frac{\omega_i}{\sum_{j=1}^{m-1} \omega_j} G_i \right) + \omega_m G_{m+1}.
\]

Because we have the assumption, $(\forall i \in \{1,\ldots,m\}) \sum_{j=1}^i U_j$ is closed, by the inductive hypothesis, $\sum_{i=1}^{m-1} \frac{\omega_i}{\sum_{j=1}^{m-1} \omega_j} G_i$ is a BAM. By the assumption, $(\forall i \in I) \sum_{j=1}^i U_j$ is closed, by Proposition 3.17(iii) and Lemma 4.3(ii), we know that $\text{Fix } \left( \sum_{i=1}^{m-1} \frac{\omega_i}{\sum_{j=1}^{m-1} \omega_j} G_i \right) + \text{Fix } G_m = \left( \cap_{i=1}^{m-1} U_j \right) + U_m$ is closed. Hence, apply Theorem 5.4 with $G_1 = \sum_{i=1}^{m-1} \frac{\omega_i}{\sum_{j=1}^{m-1} \omega_j} G_i$, $G_2 = G_m$, $\alpha = \sum_{i=1}^{m-1} \omega_i$ to obtain that $\sum_{i=1}^m \omega_i G_i$ is a BAM.
New method using the Cartesian product space reformulation

The main result Theorem 5.10 in this subsection is almost the same with the Theorem 5.6 proved in the previous subsection, however, in this subsection, we use a Cartesian product space reformulation.

In the whole subsection, set $I := \{1, \ldots, m\}$. Let $(\omega_i)_{i \in I}$ be real numbers in $[0,1]$ such that $\sum_{i \in I} \omega_i = 1$. Let $\mathcal{H}^m$ be the real Hilbert space obtained by endowing the Cartesian product $\times_{i \in I} \mathcal{H}$ with the usual vector space structure and with the weighted inner product

$$\langle x, y \rangle = \sum_{i \in I} \omega_i \langle x_i, y_i \rangle. \quad (5.18)$$

Denote by

$$D := \{ (x)_{i \in I} \in \mathcal{H}^m : x \in \mathcal{H} \}.$$

The following well-known fact is critical in proofs in this subsection.

**Fact 5.7** Let $x = (x_i)_{i \in I} \in \mathcal{H}^m$. The following hold:

(i) $P_D x = (\sum_{i \in I} \omega_i x_i)_{i \in I}$.

(ii) Let $(\forall i \in I) \mathcal{C}_i$ be nonempty closed and convex subset of $\mathcal{H}$. Then $P_{\times_{i \in I} \mathcal{C}_i} x = (P_{\mathcal{C}_i} x_i)_{i \in I}$.

*Proof.* (i): This is from [3, Proposition 29.16].

(ii): This is similar to [3, Proposition 29.3]. Because the definition of inner product is different, we show the proof next. Clearly, $(P_{\mathcal{C}_i} x_i)_{i \in I} \in \times_{i \in I} \mathcal{C}_i$. Moreover, by [3, Theorem 3.16],

$$\left( \forall (\varepsilon_i)_{i \in I} \in \times_{i \in I} \mathcal{C}_i \right) \left( \langle x_i \rangle_{i \in I} - (P_{\mathcal{C}_i} x_i)_{i \in I}, (\varepsilon_i)_{i \in I} - (P_{\mathcal{C}_i} x_i)_{i \in I} \right) = \sum_{i \in I} \omega_i \langle x_i - P_{\mathcal{C}_i} x_i, \varepsilon_i - P_{\mathcal{C}_i} x_i \rangle \leq 0,$$

which by [3, Theorem 3.16] again, implies that $P_{\times_{i \in I} \mathcal{C}_i} x = (P_{\mathcal{C}_i} x_i)_{i \in I}$. \hfill ■

In the remaining part of this subsection, let $(\forall i \in I) \mathcal{G}_i : \mathcal{H} \to \mathcal{H}$. Define $F : \mathcal{H}^m \to \mathcal{H}^m$, and $G : \mathcal{H}^m \to \mathcal{H}^m$ respectively by

$$\langle x, y \rangle = \langle x_i, y_i \rangle_{i \in I}, \quad \mathcal{F}(x) = (\mathcal{G}_i x_i)_{i \in I}, \quad \mathcal{G}(x) = (\sum_{j \in I} \omega_j \mathcal{G}_j x_j)_{i \in I}. \quad (5.19)$$

**Proposition 5.8**

(i) Fix $F = \times_{i \in I} \mathcal{G}_i$.

(ii) Let $(\forall i \in I) \gamma_i \in [0,1]$. Suppose that $(\forall i \in I) \mathcal{G}_i$ is a $\gamma_i$-BAM. Then $F$ is a $(\max_{i \in I} \{ \gamma_i \})$-BAM.

*Proof.* Let $x = (x_i)_{i \in I} \in \mathcal{H}^m$.

(i): Now

$$x \in \operatorname{Fix} F \iff (x_i)_{i \in I} = (\mathcal{G}_i x_i)_{i \in I} \iff (\forall i \in I) x_i = \mathcal{G}_i x_i \iff (\forall i \in I) x_i \in \operatorname{Fix} \mathcal{G}_i \iff x \in \times_{i \in I} \operatorname{Fix} \mathcal{G}_i.$$

(ii): Because $(\forall i \in I) \mathcal{G}_i$ is a BAM, we know that $(\forall i \in I) \operatorname{Fix} \mathcal{G}_i$ is a nonempty closed and convex subsets of $\mathcal{H}$ and, by Definition 3.1, that for every $i \in I$,

$$P_{\operatorname{Fix} \mathcal{G}_i} \mathcal{G}_i = P_{\operatorname{Fix} \mathcal{G}_i} \quad (5.21)$$

$$\langle x \in \mathcal{H}, \mathcal{G}_i x - P_{\operatorname{Fix} \mathcal{G}_i \mathcal{G}_i} x \rangle \leq \gamma_i \| x - P_{\operatorname{Fix} \mathcal{G}_i \mathcal{G}_i} x \|. \quad (5.22)$$

By (i) and Fact 5.7(ii), Fix $F = \times_{i \in I} \mathcal{G}_i$ is a nonempty closed convex subset of $\mathcal{H}^m$ and

$$P_{\operatorname{Fix} F} (x) = P_{\times_{i \in I} \operatorname{Fix} \mathcal{G}_i} (x) = (P_{\operatorname{Fix} \mathcal{G}_i} x_i)_{i \in I}. \quad (5.23)$$
Now
\[
P_{\text{Fix}\ F}(x) = P_{\text{Fix}\ F}(\{(G_i x_i)_{i \in I}\}) = (P_{\text{Fix}\ G_i}G_i x_i)_{i \in I} = (P_{\text{Fix}\ G_i}x_i)_{i \in I} = P_{\text{Fix}\ F}(x).
\] (5.24)

Note that by (5.19) and (5.23),
\[
\|F(x) - P_{\text{Fix}\ F}x\|^2 = \|\{(G_i x_i)_{i \in I}\} - (P_{\text{Fix}\ G_i}x_i)_{i \in I}\|^2 = \sum_{i \in I}\omega_i \|G_ix_i - P_{\text{Fix}\ G_i}x_i\|^2 \leq \sum_{i \in I}\omega_i \|x_i - P_{\text{Fix}\ G_i}x_i\|^2 \leq \max_{j \in I}\{\gamma_j^2\} \sum_{i \in I}\omega_i \|x_i - P_{\text{Fix}\ G_i}x_i\|^2 \leq \max_{j \in I}\{\gamma_j^2\}\|x - P_{\text{Fix}\ F}x\|^2.
\] (5.25a)

Therefore, combine (5.24) and (5.25) with Definition 3.1 to obtain the asserted result.

Proposition 5.9 Let \(\gamma \in [0, 1]\). Then the following hold:

(i) If \(F\) is a \(\gamma\)-BAM, then \((\forall i \in I) G_i\) is a \(\gamma\)-BAM.

(ii) \(F\) is a BAM if and only if \((\forall i \in I) G_i\) is a BAM.

Proof. (i): Because \(F\) is a \(\gamma\)-BAM, using Definition 3.1 and Proposition 5.8(i), we know that \(\text{Fix } F = \times_{i \in I} \text{Fix } G_i\) is a nonempty closed and convex subset of \(\mathcal{H}^m\), and there exists \(\gamma \in [0, 1]\) such that
\[
P_{\times_{i \in I} \text{Fix } G_i} F = P_{\times_{i \in I} \text{Fix } G_i}
\] (5.26)

\((\forall x \in \mathcal{H}^m) \quad \|Fx - P_{\times_{i \in I} \text{Fix } G_i}x\| \leq \gamma \|x - P_{\times_{i \in I} \text{Fix } G_i}x\|.\) (5.27)

Hence, \((\forall i \in I) \text{Fix } G_i\) is a nonempty closed and convex subset of \(\mathcal{H}\). Let \(x \in \mathcal{H}\).

Set \(y := (x)_{i \in I} \in \mathcal{H}^m\). By Fact 5.7(ii), (5.19), and (5.26),
\[
(P_{\times_{i \in I} \text{Fix } G_i}G_i x)_{i \in I} = P_{\times_{i \in I} \text{Fix } G_i}((G_i x)_{i \in I}) = P_{\times_{i \in I} \text{Fix } G_i} F y = P_{\times_{i \in I} \text{Fix } G_i} y = (P_{\text{Fix } G_i}(x))_{i \in I'}
\]

which yields \((\forall i \in I) P_{\text{Fix } G_i}G_i = P_{\text{Fix } G_i}\).

Let \(j \in I\). Set \(x := (x_i)_{i \in I} \in \mathcal{H}^m\) such that \(x_j = x\) and \((\forall i \in I \{j\}) x_i \in \text{Fix } G_i\). Then \((\forall i \in I \{j\}) x_i = G_i x_i = P_{\text{Fix } G_i} x_i\). Hence, by (5.27), we have that

\[
\|Fx - P_{\times_{i \in I} \text{Fix } G_i} G_j x\|^2 \leq \gamma^2 \|x - P_{\times_{i \in I} \text{Fix } G_i} x\|^2 \quad \Leftrightarrow \quad \|\{(G_i x_i)_{i \in I}\} - (P_{\text{Fix}\ G_i}x_i)_{i \in I}\|^2 \leq \gamma^2 \|\{(x_i)_{i \in I}\} - (P_{\text{Fix}\ G_i}x_i)_{i \in I}\|^2 \\
\overset{(5.18)}{\Leftrightarrow} \sum_{i \in I}\omega_i \|G_ix_i - P_{\text{Fix}\ G_i}x_i\|^2 \leq \gamma^2 \sum_{i \in I}\omega_i \|x_i - P_{\text{Fix}\ G_i}x_i\|^2 \quad \Leftrightarrow \quad \omega_j \|G_j x - P_{\text{Fix}\ G_j} x\|^2 \leq \gamma^2 \omega_j \|x - P_{\text{Fix}\ G_j} x\|^2 \quad \Leftrightarrow \quad \|G_j x - P_{\text{Fix}\ G_j} x\|^2 \leq \gamma^2 \|x - P_{\text{Fix}\ G_j} x\|^2.
\]

Hence, by Definition 3.1, we know that \((\forall i \in I) G_i\) is a \(\gamma\)-BAM.

(ii): The equivalence comes from (i) above and Proposition 5.8(ii).

The following result is inspired by [8, Proposition 5.25]. With consideration of Proposition 3.10 and Example 3.4, we note that Theorem 5.10 is a refinement of [8, Proposition 5.25].

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Theorem 5.10 Let \( (\forall i \in I) \gamma_i \in [0, 1] \). Suppose that \( (\forall i \in I) G_i \) is a \( \gamma_i \)-BAM and that \( \text{Fix} \, G_i \) is a closed affine subspace of \( \mathcal{H} \) with \( \bigcap_{i \in I} \text{Fix} \, G_i \neq \varnothing \). Set \( c_F := c \left( D, (\bigcap_{i \in I} \text{Fix} \, G_i) \right) \). Suppose that \( \sum_{i \in I} (\text{par Fix} \, G_i) \) is closed. Denote by \( \mu := \max_{i \in I} \{ \gamma_i \} \) and \( \gamma := \sqrt{(1 - \mu^2) \left( \frac{11 + \mu^2}{4} \right)} \). Then \( \text{Fix} \, \sum_{i \in I} \omega_i G_i = \bigcap_{i \in I} \text{Fix} \, G_i \) is a closed affine subspace of \( \mathcal{H} \) and \( \sum_{i \in I} \omega_i G_i \) is a \( \gamma \)-BAM. Moreover,

\[
(\forall x \in \mathcal{H}) \quad \| (\sum_{i \in I} \omega_i G_i) x - P_{\bigcap_{i \in I} \text{Fix} \, G_i} x \| \leq \gamma \| x - P_{\bigcap_{i \in I} \text{Fix} \, G_i} x \|.
\]

Proof. By the assumptions and Lemma 5.3, \( \text{Fix} \, \sum_{i \in I} \omega_i G_i = \bigcap_{i \in I} \text{Fix} \, G_i \) is a closed affine subspace of \( \mathcal{H} \) and \( \text{Fix} \, \sum_{i \in I} \omega_i G_i = \bigcap_{i \in I} \text{Fix} \, G_i \). To show \( \sum_{i \in I} \omega_i G_i \) is a \( \gamma \)-BAM, by Definition 3.1, it suffices to show that

\[
(\forall x \in \mathcal{H}) \quad \| \sum_{i \in I} \omega_i G_i x - P_{\bigcap_{i \in I} \text{Fix} \, G_i} x \| \leq \gamma \| x - P_{\bigcap_{i \in I} \text{Fix} \, G_i} x \|. \tag{5.28}
\]

By Proposition 5.8(i)&(ii), \( F = \times_{i \in I} \text{Fix} \, G_i \) is a closed affine subspace of \( \mathcal{H}^m \) and \( F \) is a \( \mu \)-BAM. By Example 3.4, \( P_D \) is a 0-BAM. By Fact 2.5, \( D^\perp + (\text{par} \, (\times_{i \in I} \text{Fix} \, G_i))^\perp \) is closed if and only if \( D + (\text{par} \, (\times_{i \in I} \text{Fix} \, G_i)) \) is closed. Moreover, by [1, Lemma 5.18], \( D + (\text{par} \, (\times_{i \in I} \text{Fix} \, G_i)) \) is closed if and only if \( \sum_{i \in I} (\text{par Fix} \, G_i) \) is closed, which is our assumption. Hence, we obtain that \( D^\perp + (\text{par} \, (\times_{i \in I} \text{Fix} \, G_i))^\perp \) is closed. Then apply Theorem 4.4(iii) with \( \mathcal{H} = \mathcal{H}^m \), \( G_1 = F \) and \( G_2 = P_D \) to obtain that \( P_D \) is a \( \gamma \)-BAM. Note that, by (5.19) and Fact 5.7(i),

\[
(\forall y \in \mathcal{(y_i)_{i \in I} \in \mathcal{H}^m}) \quad P_D F(y) = (\sum_{j \in I} \omega_i G_j y_i)_{i \in I} \in G(y),
\]

that is, \( P_D F = G \). By [14, Corollary 4.5.2],

\[
\text{Fix} \, G = \text{Fix} (P_D F) = D \cap (\times_{i \in I} (\bigcap_{i \in I} \text{Fix} \, G_i)).
\]

Let \( x \in \mathcal{H} \) and set \( x = (x_i)_{i \in I} \in \mathcal{H}^m \). Similarly with the proof of Fact 5.7(ii), by [3, Theorem 3.16], we have that

\[
P_{\text{Fix} \, G} x = (P_{\bigcap_{i \in I} \text{Fix} \, G_i} x)_{i \in I}. \tag{5.29}
\]

Because \( G = P_D F \) is a \( \gamma \)-BAM, by Definition 3.1(iii),

\[
\begin{align*}
\| Gx - P_{\text{Fix} \, G} x \| & \leq \gamma \| x - P_{\text{Fix} \, G} x \| \tag{5.29} \\
\overset{(5.18) \quad \sum_{i \in I} \omega_i \sum_{j \in I} \omega_j G_j x - P_{\bigcap_{i \in I} \text{Fix} \, G_i} x} & \leq 2 \sum_{i \in I} \omega_i \| x - P_{\bigcap_{i \in I} \text{Fix} \, G_i} x \| \\
\overset{(5.18) \quad \sum_{i \in I} \omega_i P_{\bigcap_{i \in I} \text{Fix} \, G_i} x} & \leq \gamma \| x - P_{\bigcap_{i \in I} \text{Fix} \, G_i} x \|,
\end{align*}
\]

which yields (5.28). Hence, the proof is complete. \[\blacksquare\]

Remark 5.11 Consider Theorems 5.6 and 5.10. Although the results from these two theorems are the same, but there are different assumptions: “\((\forall i \in I) \sum_{i=1}^m \text{(par} \, U_i)\) is closed” and “\(\sum_{i=1}^m \text{(par} \, U_i)\) is closed” respectively.

Suppose that \( m = 3 \), that \( \text{(par} \, U_2)(\text{par} \, U_1)\) is closed, and that \( \text{(par} \, U_3) = \mathcal{H} \), say, \( G_3 = P_{\{0\}} \). Then clearly, \( \sum_{i=1}^3 \text{(par} \, U_i) = \mathcal{H} \) is closed. Hence, “\(\sum_{i=1}^m \text{(par} \, U_i)\) is closed” \(\Rightarrow\) “\((\forall i \in I \setminus \{m\}) \text{(par} \, U_i) + \sum_{i=1}^m \text{(par} \, U_i)\) is closed”.

Therefore, we know that the assumptions in Theorem 5.6 are more restrictive than the assumptions in Theorem 5.10. However, comparing the constant \( \gamma \) in Theorem 5.4 and in Theorem 5.10 for \( m = 2 \), we know that the constants associated with the convex combination of two BAMs are independent in these two theorems. Hence, we keep Theorems 5.6 and 5.10 together.

The following Corollary 5.12(i) is a weak version of [15, Theorem 9.33] which shows clearly the convergence rate of the method of alternating projections.
Corollary 5.12 Let $U_1, \ldots, U_m$ be closed affine subspaces of $H$ with $\cap_{i=1}^m U_i \neq \emptyset$. Then the following statements hold:

(i) Assume that $(\forall i \in \{1, \ldots, m\}) \sum_{j=1}^i (\text{par } U_j)^\perp$ is closed. Then $P_{U_m} \cdots P_{U_2} P_{U_1}$ is a BAM; moreover, there exists $\gamma \in [0, 1]$ such that

$$(\forall x \in H) \quad \| (P_{U_m} \cdots P_{U_2} P_{U_1})^k x - P_{r_{r_{m-1}} U_i} x \| \leq \gamma^k \| x - P_{r_{r_{m-1}} U_i} x \|.$$ 

(ii) Suppose that $\sum_{i \in \{\text{par } U_i\}}^{\perp}$ is closed. Let $(\omega_i)_{1 \leq i \leq m}$ be real numbers in $[0, 1]$ such that $\sum_{i=1}^m \omega_i = 1$. Then $\sum_{i=1}^m \omega_i P_{U_i}$ is a BAM. Moreover, there exists $\gamma \in [0, 1]$ such that

$$(\forall x \in H) \quad \| (\sum_{i=1}^m \omega_i P_{U_i})^k x - P_{r_{r_{m-1}} U_i} x \| \leq \gamma^k \| x - P_{r_{r_{m-1}} U_i} x \|.$$ 

Proof. By Example 3.4, we know that $(\forall i \in \{1, \ldots, m\}) P_{U_i}$ is a 0-BAM and $\text{Fix } P_{U_i} = U_i$ is a closed affine subspace.

(i): This comes from Theorem 4.4(ii) & (iv) with $G_1 = P_{U_1}, \ldots, G_m = P_{U_m}$.

(ii): This follows by Theorem 5.10.

6 Connections between BAMs and circumcenter mappings

In this section, we present BAMs which are not projections in Hilbert spaces. In particular, we connect the circumcenter mapping with BAM.

Definitions and facts on circumcentered isometry methods

Before we turn to the relationship between best approximation mapping and circumcenter mapping, we need the background and facts on the circumcenter mapping and the circumcentered method in this section.

By [6, Proposition 3.3], we know that the following definition is well defined.

Definition 6.1 (circumcenter operator) [6, Definition 3.4] Let $\mathcal{P}(H)$ be the set of all nonempty subsets of $H$ containing finitely many elements. The circumcenter operator is

$$CC: \mathcal{P}(H) \to H \cup \{\emptyset\}: K \mapsto \begin{cases} p, & \text{if } p \in \text{aff}(K) \text{ and } \{\|p - y\| : y \in K\} \text{ is a singleton;} \\
\emptyset, & \text{otherwise.} \end{cases}$$

In particular, when $CC(K) \in H$, that is, $CC(K) \neq \emptyset$, we say that the circumcenter of $K$ exists and we call $CC(K)$ the circumcenter of $K$.

Definition 6.2 (circumcenter mapping) [7, Definition 3.1] Let $F_1, \ldots, F_m$ be operators from $H$ to $H$ such that $\cap_{j=1}^m \text{Fix } F_j \neq \emptyset$. Set $S := \{F_1, \ldots, F_m\}$ and $(\forall x \in H) S(x) := \{F_1 x, \ldots, F_m x\}$. The circumcenter mapping induced by $S$ is

$$CC_S: H \to H \cup \{\emptyset\}: x \mapsto CC(S(x)),$$

that is, for every $x \in H$, if the circumcenter of the set $S(x)$ defined in Definition 6.1 does not exist, then $CC_S x = \emptyset$. Otherwise, $CC_S x$ is the unique point satisfying the two conditions below:

(i) $CC_S x \in \text{aff } (S(x)) = \text{aff } \{F_1(x), \ldots, F_m(x)\}$, and

(ii) $\|CC_S x - F_1(x)\| = \cdots = \|CC_S x - F_m(x)\|$.

In particular, if for every $x \in H$, $CC_S x \in H$, then we say the circumcenter mapping $CC_S$ induced by $S$ is proper. Otherwise, we call $CC_S$ improper.

Fact 6.3 [7, Proposition 3.7(ii)] Let $F_1, \ldots, F_m$ be operators from $H$ to $H$ with $\cap_{j=1}^m \text{Fix } F_j \neq \emptyset$. Set $S := \{F_1, \ldots, F_m\}$. Assume that $CC_S$ is proper and that $\text{Id} \in S$. Then $\text{Fix } CC_S = \cap_{j=1}^m \text{Fix } F_j$. 

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Fact 6.4 [7, Proposition 3.3] Let \( F_1, F_2 \) be operators from \( \mathcal{H} \) to \( \mathcal{H} \) and set \( S := \{F_1, F_2\} \). Then
\[
(\forall x \in \mathcal{H}) \quad CC_S x = \frac{F_1 x + F_2 x}{2}.
\]

Let \( x \in \mathcal{H} \) and assume that \( CC_S \) is proper. The circumcenter method induced by \( S \) is
\[
x_0 := x, \text{ and } x_k := CC_S(x_{k-1}) = CC^k_S x, \text{ where } k = 1, 2, \ldots.
\]

Fact 6.5 [16, Definition 1.6-1] A mapping \( T : \mathcal{H} \to \mathcal{H} \) is said to be isometric or an isometry if
\[
(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \|Tx - Ty\| = \|x - y\|.
\]

Fact 6.6 [9, Proposition 3.3 and Corollary 3.4] Let \( T : \mathcal{H} \to \mathcal{H} \) be isometric. Then \( T \) is affine. Moreover, if \( \text{Fix} T \) is nonempty, then \( \text{Fix} T \) is a closed affine subspace.

Note that by Fact 6.6, every isometry must be affine. In the rest of this section, without otherwise statement,
\[
(\forall i \in \{1, \ldots, m\}) \quad T_i : \mathcal{H} \to \mathcal{H} \text{ is affine isometry with } \bigcap_{j=1}^m \text{Fix} T_j \neq \emptyset.
\]

Denote by
\[
S := \{T_1, \ldots, T_{m-1}, T_m\}.
\]

The associated set-valued operator \( S : \mathcal{H} \to \mathcal{P}(\mathcal{H}) \) is defined by
\[
(\forall x \in \mathcal{H}) \quad S(x) := \{T_1 x, \ldots, T_{m-1} x, T_m x\}.
\]

The following Fact 6.7(i) makes the circumcentered method induced by \( S \) defined in (6.1) well-defined. Since every element of \( S \) is isometry, we call the circumcentered method induced by the \( S \) circumcentered isometry method (CIM).

Fact 6.7 [8, Theorem 3.3 and Proposition 4.2] Let \( x \in \mathcal{H} \). Then the following statements hold:

(i) The circumcenter mapping \( CC_S : \mathcal{H} \to \mathcal{H} \) induced by \( S \) is proper; moreover, \( CC_S x \) is the unique point satisfying the two conditions below:
   
   (a) \( CC_S x \in \text{aff}(S(x)) \), and
   
   (b) \( \{\|CC_S x - Tx\| : T \in S\} \) is a singleton.

(ii) Let \( W \) be nonempty closed affine subspace of \( \cap_{i=1}^m \text{Fix} T_i \). Then \( (\forall k \in \mathbb{N}) \ P_W CC^k_S = P_W = CC^k_S P_W \).

Fact 6.8 [8, Theorem 4.15(ii)] Let \( W \) be a nonempty closed affine subspace of \( \cap_{i=1}^m \text{Fix} T_i \). Assume that there exist \( F : \mathcal{H} \to \mathcal{H} \) and \( \gamma \in [0, 1] \) such that \( (\forall x \in \mathcal{H}) F(x) \in \text{aff}(S(x)) \) and \( (\forall x \in \mathcal{H}) \|Fx - P_W x\| \leq \gamma \|x - P_W x\| \). Then
\[
(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \|CC^k_S x - P_W x\| \leq \gamma^k \|x - P_W x\|.
\]

In fact, it is easy to show that \( W = \text{Fix} CC_S \) from the last inequality with \( k = 1 \) in Fact 6.8.

Fact 6.9 [8, Theorem 4.16(ii)] Suppose that \( \mathcal{H} = \mathbb{R}^n \). Let \( T_S \in \text{aff} S \) satisfy \( \text{Fix} T_S \subseteq \cap_{T \in S} \text{Fix} T \). Assume that \( T_S \) is linear and \( \alpha \)-averaged with \( \alpha \in [0, 1] \). Then \( \|T_S P_{(\cap_{T \in S} \text{Fix} T)^{\perp}}\| \in [0, 1] \). Moreover,
\[
(\forall x \in \mathcal{H})(\forall k \in \mathbb{N}) \quad \|CC^k_S x - P_{\cap_{T \in S} \text{Fix} T} x\| \leq \|T_S P_{(\cap_{T \in S} \text{Fix} T)^{\perp}}\|^k \|x - P_{\cap_{T \in S} \text{Fix} T} x\|.
\]
Circumcenter mappings that are BAMs

**Theorem 6.10** Let W be a nonempty closed affine subspace of $\cap_{i=1}^m \text{Fix} T_i$. Assume that $\text{Id} \in S$ and that there exists $F : \mathcal{H} \to \mathcal{H}$ and $\gamma \in [0,1]$ such that $(\forall x \in \mathcal{H}) Fx \in \text{aff}(S(x))$ and $(\forall x \in \mathcal{H}) \|Fx - PW x\| \leq \gamma\|x - PW x\|$. Then $CC_S$ is a $\gamma$-BAM and $\text{Fix} CC_S = \cap_{i=1}^m \text{Fix} T_i$.

**Proof.** By Fact 6.7(i), $CC_S$ is proper. Then by Facts 6.3 and 6.6, Fix $CC_S = \cap_{i=1}^m \text{Fix} T_i$ is a nonempty closed affine subspace of $\mathcal{H}$. Apply Fact 6.7(ii) with $W = \cap_{i=1}^m \text{Fix} T_i$ to obtain that $P_{\text{Fix} CC_S} CC_S = P_{\text{Fix} CC_S}$. Moreover, by the assumptions, Fact 6.8 and Definition 3.1, we know that $CC_S$ is a $\gamma$-BAM.

The following result states that in order to study whether the circumcenter mapping $CC_S$ is a BAM or not, we are free to assume the related isometries are linear.

**Proposition 6.11** Let $z \in \cap_{i=1}^m \text{Fix} T_i$. Define $(\forall i \in \{1, \ldots, m\}) (\forall x \in \mathcal{H}) F_i x := T_i(x + z) - z$. Set $S_F := \{F_1, \ldots, F_m\}$. Then the following statements hold:

(i) $S_F$ is a set of linear isometries.

(ii) Let $\gamma \in [0,1]$. Assume that $\text{Id} \in S$. Then $CC_S$ is a $\gamma$-BAM if and only if $CC_{S_F}$ is a $\gamma$-BAM.

**Proof.** (i): Because $z \in \cap_{i=1}^m \text{Fix} T_i$, by [9, Lemma 3.8], $F_1, \ldots, F_m$ are linear isometries.

(ii): Because both $S$ and $S_F$ are sets of isometries, by Fact 6.7(ii), both $CC_S$ and $CC_{S_F}$ are proper. Clearly, $\text{Id} \in S$ implies that $\text{Id} \in S_F$ as well. So, by Fact 6.3, $\text{Fix} CC_S = \cap_{i=1}^m \text{Fix} T_i$ and $\text{Fix} CC_{S_F} = \cap_{i=1}^m \text{Fix} F_i$. In addition, by [9, Lemma 4.8], $(\forall x \in \mathcal{H}) CC_S x = z + CC_{S_F} (x - z)$. Hence, the desired result comes from Proposition 3.22 and Definition 3.1.

**Theorem 6.12** Suppose that $\mathcal{H} = \mathbb{R}^n$. Let $T_S \in \text{aff} S$ satisfy that $\text{Fix} T_S \subseteq \cap_{T \in S} \text{Fix} T$. Assume that $T_S$ is linear and $\alpha$-averaged with $\alpha \in [0,1]$. Then $\gamma := \|T_S P_{(\cap_{i=1}^m \text{Fix} T_i)}^{-1}\| \in [0,1]$ and $CC_S$ is a $\gamma$-BAM.

**Proof.** By Facts 6.3 and 6.6, and assumptions, $\text{Fix} CC_S = \cap_{i=1}^m \text{Fix} T_i$ is a closed affine subspace. Apply Fact 6.7(ii) with $W = \cap_{i=1}^m \text{Fix} T_i$ to obtain that $P_{\text{Fix} CC_S} CC_S = P_{\text{Fix} CC_S}$. Moreover, by Fact 6.9, $\gamma \in [0,1]$ and $(\forall x \in \mathcal{H}) \|CC_S x - P_{\cap_{i=1}^m \text{Fix} T_i} x\| \leq \gamma \|x - P_{\cap_{i=1}^m \text{Fix} T_i} x\|$. 

Let $t \in \mathbb{N} \setminus \{0\}$ and let $(\forall i \in \{1, \ldots, t\}) F_i : \mathcal{H} \to \mathcal{H}$. From now on, to facilitate the statements later, we denote

$$\Omega(F_1, \ldots, F_t) := \left\{ F_{i_0} \cdots F_{i_t} F_{i_1} \right\} r \in \mathbb{N}, i_1, \ldots, i_r \in \{1, \ldots, t\}$$

which is the set consisting of all finite composition of operators from $\{F_1, \ldots, F_t\}$. We use the empty product convention: $F_{i_0} \cdots F_{i_t} = \text{Id}$.

**Fact 6.13** [9, Theorem 5.4] Suppose that $\mathcal{H} = \mathbb{R}^n$. Let $F_1, F_2, \ldots, F_t$ be linear isometries on $\mathcal{H}$. Assume that $\widetilde{S}$ is a finite subset of $\Omega(F_1, \ldots, F_t)$, where $\Omega(F_1, \ldots, F_t)$ is defined in (6.3). Assume that $(\forall i \in \{1, \ldots, t\}) A_i := (1 - \alpha_i) \text{Id} + \alpha_i F_i$. Then the following statements hold:

(i) $\text{Fix} CC_S = \cap_{T \in \tilde{S}} \text{Fix} T = \cap_{i=1}^t \text{Fix} F_i = \text{Fix} A$.

(ii) $\|A P_{(\cap_{i=1}^t \text{Fix} F_i)}^{-1}\| < 1$. Moreover, 

$$(\forall x \in \mathcal{H}) \|CC_S x - P_{\cap_{i=1}^t \text{Fix} F_i} x\| \leq \|A P_{(\cap_{i=1}^t \text{Fix} F_i)}^{-1}\| \|x - P_{\cap_{i=1}^t \text{Fix} F_i} x\|.$$

**Fact 6.14** [9, Theorem 5.6] Suppose that $\mathcal{H} = \mathbb{R}^n$. Let $F_1, F_2, \ldots, F_t$ be linear isometries. Assume that $\widetilde{S}$ is a finite subset of $\Omega(F_1, \ldots, F_t)$, where $\Omega(F_1, \ldots, F_t)$ is defined in (6.3). Assume that $(\forall i \in \{1, \ldots, t\}) A_i := (1 - \alpha_i) \text{Id} + \alpha_i F_i$ and 

$$(\forall i \in I \setminus \{1\}) A_i := (1 - \alpha_i) \text{Id} + \alpha_i ((1 - \lambda_i) \text{Id} + \lambda_i F_i) F_{i-1} \cdots F_1.$$

Then the following assertions hold:
(i) Fix $\CC S = \cap_{T \in \SS} \Fix T = \cap_{i=1}^l \Fix F_i = \Fix A$.

(ii) $\| AP_{(\cap_{i=1}^l \Fix F_i)^\perp} \| \in [0,1]$. Moreover,

$$(\forall x \in \HH)(\forall k \in \NN) \quad \| \CC S^k x - \PP_{\cap_{i=1}^l \Fix F_i} x \| \leq \| AP_{(\cap_{i=1}^l \Fix F_i)^\perp} \|^k \| x - \PP_{\cap_{i=1}^l \Fix F_i} x \|.$$ 

**Proposition 6.15** Suppose that $\HH = \RR^n$. Let $F_1, \ldots, F_l$ be linear isometries from $\HH$ to $\HH$. Let $\SS$ be a finite subset of $\Omega(F_1, \ldots, F_l)$.

(i) If $\{\Id, F_1, F_2, \ldots, F_l\} \subseteq \SS$, then $\Fix CC S = \cap_{i=1}^l \Fix F_i$ and $\CC S$ is a BAM.

(ii) If $\{\Id, F_1, F_2 F_1, \ldots, F_1 \cdots F_2 F_1\} \subseteq \SS$, then $\Fix CC S = \cap_{i=1}^l \Fix F_i$ and $\CC S$ is a BAM.

**Proof.** (i): By Fact 6.13(i), $\Fix CC S = \cap_{T \in \SS} \Fix T = \cap_{i=1}^l \Fix F_i$ is a nonempty closed linear subspace of $\HH$. Apply Fact 6.7(ii) with $W = \Fix CC S = \cap_{T \in \SS} \Fix T$ yields $\PP_{\Fix CC S} CC S = \PP_{\Fix CC S}$. Apply Fact 6.13(i) to obtain that there exists $\gamma \in [0,1]$ such that

$$(\forall x \in \HH) \quad \| CC S x - \PP_{\Fix CC S} x \| \leq \gamma \| x - \PP_{\Fix CC S} \Fix Tx \|.$$ 

Hence, by Definition 3.1, $\CC S$ is a BAM.

(ii): The proof is similar to the proof of (i), however, this time we use Fact 6.14 instead of Fact 6.13.

**Theorem 6.16** Assume that $\HH = \RR^n$ and that $F_1, \ldots, F_l$ are affine isometries from $\HH$ to $\HH$ with $\cap_{i=1}^l \Fix F_i \neq \emptyset$. Assume that $\SS$ is a finite subset of $\Omega(F_1, \ldots, F_l)$ defined in (6.3) such that $\{\Id, F_1, F_2, \ldots, F_l\} \subseteq \SS$ or $\{\Id, F_1, F_2 F_1, \ldots, F_1 \cdots F_2 F_1\} \subseteq \SS$. Then $\Fix CC S = \cap_{i=1}^l \Fix F_i$ and $\CC S$ is a BAM.

**Proof.** This is from Proposition 6.15(ii) and Proposition 6.11(ii).

The following example shows that BAM is generally neither continuous nor linear.

**Example 6.17** Suppose that $\HH = \RR^2$, set $U_1 := \RR \cdot (1,0)$, and $U_2 := \RR \cdot (1,1)$. Suppose that $\SS = \{\Id, R_{U_1}, R_{U_2}\}$ or that $\SS = \{\Id, R_{U_1}, R_{U_2}, R_{U_1} U_2\}$. Then the following statements hold.

(i) $\CC S$ is a BAM and $\Fix CC S = \{(0,0)\}$.

(ii) $\CC S$ is neither continuous nor linear.

**Proof.** (i): Because $R_{U_1}$ and $R_{U_2}$ are linear isometries and $\Fix R_{U_1} \cap \Fix R_{U_2} = U_1 \cap U_2 = \{(0,0)\}$, by Theorem 6.16, $\CC S$ is a BAM.

(ii): This is from [7, Examples 4.19 and 4.20].

The following example illustrates that the composition of three BAMs is a projector does not imply that the individual BAMs are projectors.

**Example 6.18** Suppose that $\HH = \RR^2$, set $U_1 := \RR \cdot (1,0)$, $U_2 := \RR \cdot (1,1)$ and $U_3 := \RR \cdot (0,1)$. Denote by $\SS_1 := \{\Id, R_{U_1}, R_{U_2}\}$ and $\SS_2 := \{\Id, R_{U_2}, R_{U_1}\}$. Then the following statements hold:

(i) All of $\CC S_1, \CC S_2$ and $\CC S_3 \CC S_1$ are BAMs. Moreover, $\Fix CC S_1 = \{(0,0)\}$, $\Fix CC S_2 = \{(0,0)\}$, and $\Fix (CC S_3 CC S_1) = \{(0,0)\}$.

(ii) None of the $\CC S_1$, $\CC S_2$ or $\CC S_3 \CC S_1$ is a projector.

(iii) $\CC S_1 CC S_2 CC S_1 = \PP_{\{(0,0)\}}$.

**Proof.** By Fact 6.4, it is easy to see that

$$(\forall x \in \HH) \quad \CC S_1 x = \begin{cases} P_{U_2} x, & \text{if } x \in U_1; \\ P_{U_1} x, & \text{if } x \in U_2; \\ 0, & \text{otherwise}. \end{cases} \quad \text{ and } \quad \CC S_2 x = \begin{cases} P_{U_2} x, & \text{if } x \in U_2; \\ P_{U_2} x, & \text{if } x \in U_3; \\ 0, & \text{otherwise}. \end{cases} \quad (6.4)$$
Hence,

\[
CC_S CC_S x = \begin{cases} 
P_{U_{1_j}} P_{U_{2_j}} x, & \text{if } x \in U_{1}; \\
0, & \text{if } x \in U_{2}; \\
0, & \text{otherwise.}
\end{cases}
\]  

(6.5)

(i): Because \(R_{U_{1_j}}, R_{U_{2_j}}, \text{ and } R_{U_{1_k}}\) are linear isometries, \(\text{Fix} R_{U_{1_j}} \cap \text{Fix} R_{U_{1_k}} = \{(0,0)\}, \) and \(\text{Fix} R_{U_{2_j}} \cap \text{Fix} R_{U_{1_k}} = U_2 \cap U_3 = \{(0,0)\}\). By Theorem 6.16, \(CC_S \) and \(CC_S \) are BAMs and \(\text{Fix} CC_S = \text{Fix} CC_S = \{(0,0)\}. \) Hence, by Theorem 4.4(ii), \(CC_S CC_S \) is BAM and \(\text{Fix}(CC_S CC_S) = \{(0,0)\}\).

(ii): Because \(U_{1_j} \) is not orthogonal with \(U_2\), and by (6.4), the range of \(CC_S \) equals \(U_1 \cup U_2\), but \(CC_S \neq P_{U_{1_j} U_{2_j}}\), we know that \(CC_S \) is not a projector. Similarly, neither \(CC_S \) nor \(CC_S CC_S \) is a projector.

(iii): This is clear from the definitions of \(CC_S \) and \(CC_S CC_S \) presented in (6.4) and (6.5) respectively.

\[\blacksquare\]

**Circumcenter and best approximation mappings in Hilbert space**

Because reflectors associated with closed affine subspaces are isometries, we call the circumcenter method induced by a set of reflectors the **circumcentered reflection method** (CRM). Clearly, all facts on CIM are applicable to CRM.

In this subsection, we assume that

\[U_1, \ldots, U_m \text{ are closed linear subspaces in the real Hilbert space } \mathcal{H},\]  

(6.6a)

\[\Omega := \Omega(R_{U_{1}}, \ldots, R_{U_m}) := \left\{ R_{U_{1_r}} \cdots R_{U_{1_2}} R_{U_{i_1}} \middle| r \in \mathbb{N}, \text{ and } i_1, \ldots, i_r \in \{1, \ldots, m\}\right\},\]  

(6.6b)

\[\Psi := \left\{ R_{U_{i_r}} \cdots R_{U_{i_2}} R_{U_{i_1}} \middle| r, i_1, i_2, \ldots, i_r \in \{0, 1, \ldots, m\} \text{ and } 0 < i_1 < \cdots < i_r \right\}.\]  

(6.6c)

We also assume that

\[\Psi \subseteq S \subseteq \Omega \text{ and } S \text{ consists of finitely many elements.}\]  

(6.7)

For every nonempty closed affine subset \(C \) of \(\mathcal{H}, \mathcal{R}_C CC_S = (2 P_C - \text{Id})(2 P_C - \text{Id}) = 4 P_C - 2 P_C + 2 P_C = \text{Id} = \text{Id}\). So, if \(m = 1\), then \(\Omega = \Psi = \{\text{Id}, R_{U_{1}}\}\). Hence, by the assumption, \(S = \{\text{Id}, R_{U_{1}}\}, \) and, by Fact 6.4, \(CC_S = \frac{1}{2}(\text{Id} + R_{U_{1}}) = P_{U_{1}}. \) By Example 3.4, \(CC_S = P_{U_{1}}\) is a 0-BAM. Therefore, \(m = 1\) is a trivial case and we consider only \(m \geq 2\) below.

**Lemma 6.19** Fix \(CC_S = \cap \mathcal{T} \in S \text{ Fix } T = \cap_{i=1}^m U_i.\)

**Proof.** By construction of \(\Psi\) with \(r = 0\) and \(r = 1\), we know that \(\{\text{Id}, R_{U_{1}}, \ldots, R_{U_m}\} \subseteq \Psi \subseteq \Omega, \) so by Fact 6.3, \(\text{Fix} CC_S = \cap \mathcal{T} \in S \text{ Fix } T \subseteq \cap_{i=1}^m \text{ Fix } R_{U_{i}} = \cap_{i=1}^m U_i. \) On the other hand, because \(S \subseteq \Omega\) and \((\forall T \in \Omega) \cap_{i=1}^m U_i \subseteq \text{Fix } T, \) we know that \(\cap_{i=1}^m U_i \subseteq \cap \mathcal{T} \in S \text{ Fix } T = \text{Fix } CC_S. \) Altogether, \(\text{Fix } CC_S = \cap \mathcal{T} \in S \text{ Fix } T = \cap_{i=1}^m U_i. \) \[\blacksquare\]

**Fact 6.20** [9, Theorem 6.6] Set \(\gamma := \|P_{U_{m}} P_{U_{m-1}} \cdots P_{U_{1}} P_{(\cap_{i=1}^m U_i)}\|\). Assume that \(m \geq 2\) and that \(U_{1}^+ + \cdots + U_{m}^+\) is closed. Then \(\gamma \in [0, 1]\) and

\[
(\forall x \in \mathcal{H}) (\forall k \in \mathbb{N}) \quad \|CC_S x - P_{(\cap_{i=1}^m U_i)} x\| \leq \gamma^k \|x - P_{(\cap_{i=1}^m U_i)} x\|.
\]

**Theorem 6.21** Set \(\gamma := \|P_{U_{m}} P_{U_{m-1}} \cdots P_{U_{1}} P_{(\cap_{i=1}^m U_i)}\|\). Assume that \(m \geq 2\) and that \(U_{1}^+ + \cdots + U_{m}^+\) is closed. Then \(\gamma \in [0, 1]\), \(\text{Fix } CC_S = \cap_{i=1}^m U_i, \) and \(CC_S\) is a \(\gamma\)-BAM.

**Proof.** Because \(\Psi \subseteq S \subseteq \Omega, \) by Lemma 6.19, \(\text{Fix } CC_S = \cap \mathcal{T} \in S \text{ Fix } T = \cap_{i=1}^m U_i\) is a closed linear subspace. Apply Fact 6.7(ii) with \(W = \cap \mathcal{T} \in S \text{ Fix } T\) to obtain that \(P_{\text{Fix } CC_S} CC_S = P_{\text{Fix } CC_S}. \) In addition, by Fact 6.20, \(\gamma \in [0, 1]\) and \((\forall x \in \mathcal{H}) \|CC_S x - P_{\text{Fix } CC_S} x\| \leq \gamma \|x - P_{\text{Fix } CC_S} x\|. \) Hence, by Definition 3.1, \(CC_S\) is a \(\gamma\)-BAM. \[\blacksquare\]

**Corollary 6.22** Assume that \(m = 2\) in (6.6), that \(S = \{\text{Id}, R_{U_{1}}, R_{U_{2}}, R_{U_{2}} R_{U_{1}}\}, \) and that \(U_{1} + U_{2}\) is closed. Set \(\gamma := \|P_{U_{2}} P_{U_{1}} P_{(U_{1} \cap U_{2})}\|. \) Then \(\gamma \in [0, 1]\), \(\text{Fix } CC_S = U_{1} \cap U_{2}, \) and \(CC_S\) is a \(\gamma\)-BAM.
Proof. By Fact 2.5, $U_1 + U_2$ is closed if and only if $U_1^+ + U_2^+$ is closed. Note that $m = 2$ in (6.6) implies $\Psi = \{\text{Id}, R_{U_1}, R_{U_2}, R_{U_1} R_{U_2} R_{U_1}\} = S$. Hence, the desired result is from Theorem 6.21 with $m = 2$.

Theorem 6.23 Let $n \in \mathbb{N} \setminus \{0\}$. Assume that $m = 2n - 1$ and that $U_1, \ldots, U_n$ are closed linear subspaces of $\mathcal{H}$ with $U_1^+ + \cdots + U_n^+$ being closed. Set $(\forall i \in \{1, \ldots, n - 1\}) U_{n+i} := U_{n-i}$. Denote $\gamma := \left\| P_{U_1} P_{U_{n-1}} \cdots P_{U_1} P_{(\cap_{i=1}^n U_i)^{\perp}} \right\|$. Then $\gamma \in [0, 1]$, Fix $CC_S = \cap_{i=1}^m U_i$, and $CC_S$ is a $\gamma^2$-BAM.

Proof. Because $\Psi \subseteq S \subseteq \Omega$, by Lemma 6.19, Fix $CC_S = \cap_{T \in S} \text{Fix } T = \cap_{i=1}^m U_i$ is a closed linear subspace. Apply Fact 6.7(ii) with $W = \cap_{T \in S} \text{Fix } T$ to obtain that $P_{\text{Fix } CC_S} CC_S = P_{\text{Fix } CC_S}$. In addition, by [9, Theorem 6.7], $\gamma \in [0, 1]$ and $(\forall x \in \mathcal{H}) \| CC_S x - P_{\text{Fix } CC_S} x \| \leq \gamma^2 \| x - P_{\text{Fix } CC_S} x \|$. Hence, by Definition 3.1, $CC_S$ is a $\gamma^2$-BAM.

Corollary 6.24 Assume that $m = 3$ in (6.6), that $S := \{\text{Id}, R_{U_1}, R_{U_2}, R_{U_1} R_{U_2}, R_{U_1} R_{U_2} R_{U_1}, R_{U_1} R_{U_2} R_{U_1} \}$, and that $U_1 + U_2$ is closed. Set $\gamma := \left\| P_{U_2} P_{U_1} P_{(U_1 U_2)^{\perp}} \right\|$. Then $\gamma \in [0, 1]$, Fix $CC_S = U_1 \cap U_2$, and $CC_S$ is a $\gamma^2$-BAM.

Proof. Let $U_3 = U_1$ in (6.6) with $m = 3$ to obtain that $\Psi = \{\text{Id}, R_{U_1}, R_{U_2}, R_{U_1} R_{U_2}, R_{U_1} R_{U_2}, R_{U_1} R_{U_2} R_{U_1} \} = S$. Hence, the required result comes from Theorem 6.23 with $n = 2$ and $U_3 = U_1$.

Remark 6.25 [9, Theorem 6.8] shows that the sequence of iterations of the $CC_S$ in Theorem 6.23 attains the convergence rate of the accelerated method of alternating projections which is no larger than the $\gamma^2$ presented in Theorem 6.23. Hence, by [9, Theorem 6.8], using the similar proof of Theorem 6.23, one can show that the constant associated with the BAM, the $CC_S$ in Theorem 6.23, is no larger than the convergence rate of the accelerated method of alternating projections.

Compositions and convex combinations of circumcenter mapping

The following Theorems 6.26 and 6.27 with condition (i) are generalizations of [12, Theorem 2] from one class of circumcenter mapping induced by finite set of reflections to two classes of more general circumcenter mappings induced by finite set of isometries. Recall that $T_1, \ldots, T_m$ are affine isometries from $\mathcal{H}$ to $\mathcal{H}$ with $\cap_{i=1}^m \text{Fix } T_i \neq \emptyset$.

Theorem 6.26 Suppose that $\mathcal{H} = \mathbb{R}^n$. Set $S_1 := \{\text{Id}, T_{q_0+1}, T_{q_0+2}, \ldots, T_1\}$, $S_2 := \{\text{Id}, T_{q_1+1}, T_{q_1+2}, \ldots, T_2\}$, $S_3 := \{\text{Id}, T_{q_l+1}, T_{q_l+2}, \ldots, T_3\}$, with $q_0 = 0, q_1 = m$ and $(\forall i \in \{1, \ldots, t\}) q_i - q_{i-1} \geq 1$. Suppose that one of the following holds:

(i) $CC_S = CC_{S_1} \circ CC_{S_{i-1}} \circ \cdots \circ CC_{S_t}$.

(ii) $CC_S = \sum_{i=1}^m \omega_i CC_{S_i}$, where $\{\omega_i\}_{1 \leq i \leq l} \subseteq [0, 1]$ such that $\Sigma_{i=1}^l \omega_i = 1$.

Then Fix $CC_S = \cap_{i=1}^m \text{Fix } T_i$ and $CC_S$ is a BAM. Moreover, there exists $\gamma \in [0, 1]$ such that

$$(\forall x \in \mathcal{H}) (\forall k \in \mathbb{N}) \| CC_{S_i} x - P_{\cap_{i=1}^m \text{Fix } T_i} x \| \leq \gamma^k \| x - P_{\cap_{i=1}^m \text{Fix } T_i} x \|.$$ 

Proof. By Theorem 6.16, $(\forall i \in \{1, \ldots, t\}) CC_{S_i}$ is a BAM with Fix $CC_{S_i} = \cap_{j=q_i-1}^{q_i-1} \text{Fix } T_j$. Using Facts 6.3 and 6.7 and Proposition 3.17(ii)&(iii), we know that Fix $CC_S = \cap_{j=1}^{m-1} \text{Fix } T_j$. Note that every finite-dimensional linear subspace must be closed. Hence, by Theorem 4.4(ii) and Theorem 5.10, we obtain that $CC_S$ is a BAM. The last inequality comes from Proposition 3.10.

Theorem 6.27 Suppose that $\mathcal{H} = \mathbb{R}^n$. Set $I := \{1, \ldots, t\}$ and

$$(\forall i \in I) \quad S_i := \{\text{Id}, T_{q_{i-1}+1}, T_{q_{i-1}+2} T_{q_{i-1}+1}, \ldots, T_{q_i-2} T_{q_{i-1}+1}\},$$ 

with $q_0 = 0, q_t = m$ and $(\forall i \in \{1, \ldots, t\}) q_i - q_{i-1} \geq 1$. Suppose that one of the following holds:

(i) $CC_S = CC_{S_1} \circ CC_{S_{i-1}} \circ \cdots \circ CC_{S_t}$.

(ii) $CC_S = \sum_{i=1}^l \omega_i CC_{S_i},$ where $\{\omega_i\}_{1 \leq i \leq l} \subseteq [0, 1]$ such that $\Sigma_{i=1}^l \omega_i = 1$. 



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Then Fix \( CC_S = \cap_{i=1}^m \text{Fix} T_i \) and \( CC_S \) is a BAM. Moreover, there exists \( \gamma \in [0, 1] \) such that
\[
(\forall x \in \mathcal{H}) (\forall k \in \mathbb{N}) \quad \| CC_S^k x - P_{\cap_{i=1}^m \text{Fix} T_i} x \| \leq \gamma^k \| x - P_{\cap_{i=1}^m \text{Fix} T_i} x \|.
\]

Proof. The proof is similar to that of Theorem 6.26. \( \blacksquare \)

We conclude this section by presenting BAMs from finite composition or convex combination of circumcenter mappings, which is not projections, in Hilbert spaces. In fact, using Theorems 6.21 and 6.23, one may construct more similar BAMs in Hilbert space.

**Theorem 6.28** Let \( U_1, \ldots, U_{2m} \) be closed affine subspaces of \( \mathcal{H} \) with \( \cap_{i=1}^{2m} U_i \neq \emptyset \). Set \( I := \{1, \ldots, m\} \). Assume that \((\forall i \in I) \) \( \text{par} U_{2i-1} + \text{par} U_{2i} \) is closed. Set
\[
S_1 := \{1d, R_{U_{l_{0}+1}}, R_{U_{l_{0}+2}}, R_{U_{l_{0}+1}}\}, \ldots, S_m := \{1d, R_{S_{l_{m-1}}}, R_{S_{l_{m}}}, R_{U_{l_{m-1}}}\},
\]
with \((\forall i \in \{0\} \cup I) q_i = 2i\). Suppose that one of the following holds:

1. \((\forall i \in I) \sum_{j=1}^{2i} (\text{par} U_j) \perp \) is closed, and \( CC_S = CC_{S_1} \circ CC_{S_{l-1}} \circ \cdots \circ CC_{S_1} \).
2. \( \sum_{j=1}^{2m} (\text{par} U_j) \perp \) is closed, and \( CC_S = \sum_{j=1}^{m} \omega_j CC_{S_j} \), where \( \{\omega_j\}_{1 \leq j \leq m} \subseteq [0, 1] \) such that \( \sum_{j=1}^{m} \omega_j = 1 \).

Then Fix \( CC_S = \cap_{i=1}^{2m} \text{Fix} T_i \) and \( CC_S \) is a BAM. Moreover, there exists \( \gamma \in [0, 1] \) such that
\[
(\forall x \in \mathcal{H}) (\forall k \in \mathbb{N}) \quad \| CC_S^k x - P_{\cap_{i=1}^{2m} U_i} x \| \leq \gamma^k \| x - P_{\cap_{i=1}^{2m} U_i} x \|.
\]

Proof. By Proposition 6.11, we are able to assume that \((\forall i \in \{1, \ldots, 2m\}) U_i \) is closed linear subspace of \( \mathcal{H} \). For every \((i \in I)\), because \( U_{2i-1} + U_{2i} \) is closed, by Corollary 6.22, \( CC_{S_i} \) is a BAM with Fix \( CC_{S_i} = U_{2i-1} \cap U_{2i} \) and by Fact 2.5, \( U_{2i-1} + U_{2i} = \overline{U_{2i-1} + U_{2i}} \). Hence, for every \( i \in I \),
\[
\sum_{j=1}^{i} (\text{par} \text{Fix} CC_{S_j}) \perp = \sum_{j=1}^{i} (U_{2j-1} \cap U_{2j}) \perp = \sum_{j=1}^{i} U_{2j-1} \perp + U_{2j} \perp = \sum_{j=1}^{2j} (\text{par} U_j) \perp.
\]

Therefore, the asserted results follow by Theorem 4.4(ii) and Theorem 5.10. \( \blacksquare \)

## 7 Conclusion and future work

We discovered that the iteration sequence of BAM linearly converges to the best approximation onto the fixed point set of the BAM. We compared BAMs with linear convergent mappings, Banach contractions, and linear regular operators. We also generalized the result proved by Behling, Bello-Cruz and Santos that the finite composition of BAMs with closed affine fixed point sets in \( \mathbb{R}^d \) is still a BAM from \( \mathbb{R}^d \) to the general Hilbert space. We constructed new constant associated with the composition of BAMs. Moreover, we proved that convex combinations of BAMs with closed affine fixed point sets is still a BAM. In addition, we connected BAMs with circumcenter mappings.

Although Theorem 4.4 states that the finite composition of BAMs with closed affine fixed point sets is still a BAM, Example 4.10 shows that the composition of BAMs associated with closed Euclidean balls is generally not a BAM. Moreover, Proposition 4.7 and Examples 4.8 and 4.10 illustrate that to determine whether the composition of BAMs is a BAM or not, the order of the BAMs does matter. In addition, although Theorems 5.6 and 5.10 state that the convex combination of BAMs with closed affine fixed point sets is a BAM, we have a little knowledge for affine combinations of BAMs with general convex fixed point sets. It would be interesting to characterize the sufficient conditions for the finite composition of or affine combination of BAMs with general convex fixed point sets. By Remark 4.5, the constant associated with the composition of BAMs in Theorem 4.2(v) is not sharp. Using Example 5.5, we know that the constant associated with the convex combination of BAMs presented Theorem 5.4 is not sharp as well. Hence, we will also try to find better upper bound for the constant associated with the composition of or the convex combination of BAMs. As we mentioned in Remark 5.11, although the assumption of Theorem 5.6 is more restrictive than that of Theorem 5.10, the constants in these results are independent. We will investigate the relation between the constants associated with the convex combination of BAMs in Theorems 5.6 and 5.10. Last but not least, we will try to find more BAMs with general convex fixed point sets and more applications of those BAMs.
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