DEFORMATION QUANTIZATIONS WITH SEPARATION OF VARIABLES ON A KÄHLER MANIFOLD

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Introduction

In [Ka] a simple geometric construction of some formal deformation quantization (see [BFFLS]) on a Kähler manifold was introduced. This construction provides the deformation quantization obtained from Berezin’s $\ast$-product (see [Be]) on the orbits of a compact semisimple Lie group in [Mo2] and [CGR1] and on bounded symmetric domains in [Mo1] and [CGR2].

The formal $\ast$-product on a Kähler manifold $M$ corresponding to the quantization from [Ka] is connected with the separation of variables into holomorphic and antiholomorphic ones in the following sense. For each open subset $U \subset M$, $\ast$-multiplication from the left by a holomorphic function and from the right by an antiholomorphic function on $U$ coincides with the pointwise multiplication by these functions.

It turns out that all such quantizations with separation of variables can be obtained by a slightly generalized construction from [Ka] and are completely parametrized by geometric objects, the formal deformations of the original Kähler metrics.

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1. Definition of deformation quantization with separation of variables

Define a formal deformation quantization on a symplectic manifold $M$ (see [BFFLS]).

Let \{\(C_r(\cdot, \cdot)\)\}, \(r = 0, 1, 2, \ldots\) be a family of bidifferential operators on \(M\), i.e., of differential operators which act from \(C^\infty(M) \otimes C^\infty(M)\) to \(C^\infty(M)\). Define a binary operation \(\star\) in the space of formal power series \(\mathcal{F} = C^\infty(M)[[\nu]]\), posing for \(f = \sum_{r=0}^{\infty} \nu^r f_r\) and \(g = \sum_{r=0}^{\infty} \nu^r g_r\)

\[
f \star g = \sum_{r=0}^{\infty} \nu^r \sum_{i+j+k=r} C_i(f_j, g_k). \tag{1}\]

The operation \(\star\) defines a formal deformation quantization on the symplectic manifold \(M\), if it is associative and for \(f, g \in C^\infty(M)\) holds

\[
C_0(f, g) = fg, \ C_1(f, g) - C_1(g, f) = i\{f, g\}, \tag{2}\]

where \(\{\cdot, \cdot\}\) is a Poisson bracket on \(M\), corresponding to the symplectic structure.

In such a case the operation \(\star\) is called a \(\star\)-product.

All the deformation quantizations considered in this paper are formal, so in the sequel we will not mention it explicitly.

Since a \(\star\)-product is given by differential operators, it is local, that is, it can be restricted to any open subset \(U \subset M\). The restriction of \(\star\) defines a \(\star\)-product in the space \(\mathcal{F}(U) = C^\infty(U)[[\nu]]\).

If there is given a deformation quantization on \(M\) then for each open subset \(U \subset M\) in the space \(\mathcal{F}(U)\) act the algebras \(\mathcal{L}(U)\) and \(\mathcal{R}(U)\) of the left and right \(\star\)-multiplication operators, respectively. For \(f, g \in \mathcal{F}(U)\) define the operators \(L_f \in \mathcal{L}(U)\) and \(R_g \in \mathcal{R}(U)\) by the relations \(L_f g = R_g f = f \star g\).

The operators from \(\mathcal{L}(U)\) commute with the operators from \(\mathcal{R}(U)\), \([L_f, R_g] = 0\).

For \(U = M\) denote \(\mathcal{L} = \mathcal{L}(M), \mathcal{R} = \mathcal{R}(M)\).
Let $D(U)$ be the algebra of the formal series of differential operators of the form $\tilde{A} = \sum_{r=0}^{\infty} \nu^r A_r$, where $A_r$ are differential operators on $U$ with smooth coefficients. These series act as linear operators on the space $\mathcal{F}(U)$, for $\tilde{A} = \sum_{r=0}^{\infty} \nu^r A_r$ and $f = \sum_{r=0}^{\infty} \nu^r f_r$

$$\tilde{A} f = \sum_{r=0}^{\infty} \nu^r \sum_{s=0}^{r} A_{r-s} f_s.$$ Since one can take a pointwise product of the elements of $\mathcal{F}(U)$, $\mathcal{F}(U)$ is included in $D(U)$ as the algebra of pointwise multiplication operators. It follows from the definition of $\star$-product that $L(U)$ and $R(U)$ are subalgebras of $D(U)$.

Further, we will refer sometimes to formal series of functions, operators etc., as to formal functions, operators, or even omit the word formal, which must not lead to a misunderstanding.

Let $M$ be a Kähler manifold of complex dimension $m$ with a Kähler form $\omega_0$ of the type $(1, 1)$.

Definition. A deformation quantization on the Kähler manifold $M$ is called a deformation quantization with separation of variables if, for any open subset $U \subset M$ and functions $a, b, f \in C^\infty(U)$, such that $a$ is holomorphic and $b$ antiholomorphic, holds $a \star f = a \cdot f, \ f \star b = f \cdot b$.

If on $M$ there is defined a deformation quantization with separation of variables, then for a holomorphic function $a$ and antiholomorphic function $b$ on an arbitrary open subset $U \subset M$, the operators $L_a$ and $R_b$ are the operators of pointwise multiplication by the functions $a$ and $b$ respectively, $L_a = a$ and $R_b = b$. If, moreover, $U$ is a coordinate chart with holomorphic coordinates $z^1, \ldots, z^m$, then, since for $f \in \mathcal{F}(U)$ the operator $L_f$ commutes with $R_{\bar{z}^j} = \bar{z}^j$, it contains only partial derivatives by $z^k$. Similarly, the operator $R_f$ contains only partial derivatives by $\bar{z}^j$.

2. Deformation of Kähler metrics corresponding to quantization with separation of variables

With each deformation quantization with separation of variables on a
Kähler manifold $M$ with a Kähler form $\omega_0$, we canonically associate a formal deformation of the Kähler metrics $\omega_0$, i.e., a formal series $\omega = \omega_0 + \nu \omega_1 + \nu^2 \omega_2 + \ldots$ such that $\omega_1, \omega_2, \ldots$ are closed but not necessarily nondegenerate forms of the type $(1,1)$ on $M$.

On a contractible coordinate chart $U$, there exists a Kähler potential $\Phi_0 \in C^\infty(U)$ such that $\omega_0 = i\partial\bar{\partial}\Phi_0 = \sum_{kl} g_{kl} dz_k \wedge d\bar{z}_l$, where $g_{kl} = \partial^2 \Phi_0 / \partial z_k \partial \bar{z}_l$. Here as well as below we use the tensor rule of summation over repeated indices. The Kähler potential $\Phi_0$ is defined up to a summand of the form $a + b$, where $a$ is a holomorphic and $b$ an antiholomorphic function on $U$.

Denote by $(g^{lk})$ the inverse matrix to $(g_{kl})$. The Poisson bracket of the functions $f, g \in C^\infty(U)$ can be expressed as follows:

$$\{f, g\} = ig^{lk}(\frac{\partial f}{\partial z_k} \frac{\partial g}{\partial \bar{z}_l} - \frac{\partial g}{\partial z_k} \frac{\partial f}{\partial \bar{z}_l}).$$

Let on $M$ be defined a deformation quantization. Introduce bidifferential operators $D_r(\cdot, \cdot)$, such that for $u, v \in C^\infty(M)$ holds $D_r(u, v) = C_r(u, v) - C_r(v, u)$. From (2) it follows that $D_0 = 0$, and $D_1 = i\{\cdot, \cdot\}$. Thus for $f = \sum_{r=0}^{\infty} \nu^r f_r$ and $g = \sum_{r=0}^{\infty} \nu^r g_r$

$$f \star g - g \star f = \sum_{r=1}^{\infty} \nu^r \sum_{i+j+k=r} D_i(f_j, g_k).$$

(3)

**Lemma 1.** Let $U$ be a contractible coordinate chart on $M$. The system of equations for an unknown function $u$ on $U$, $D_1(u, z^k) = f^k$, $k = 1, \ldots, m$, where $f^k \in C^\infty(U)$, has a solution if and only if for all $k, k'$ holds $D_1(f^k, z^{k'}) = D_1(f^{k'}, z^k)$. Then the solution $u$ is determined up to a holomorphic summand.

**Proof.** By using the fact that $D_1 = i\{\cdot, \cdot\}$, the lemma can easily be reduced to the assertion that the solvability condition of the equation $\bar{\partial}u = g_{kl} f^k dz^l$ is a $\bar{\partial}$-closedness of the form $g_{kl} f^k dz^l$.

**Proposition 1.** Let on a Kähler manifold $M$ with a Kähler form $\omega_0$ be defined a formal deformation quantization with separation of variables. Then on each contractible coordinate chart $U \subset M$ there exist formal functions
We will construct, say, the function $u = u^1$. Let $u = u_0 + \nu u_1 + \nu^2 u_2 + \ldots$. The coefficients $u_r$ have to satisfy the following system of equations,

$$\sum_{r=0}^{\infty} \nu^r \sum_{s=0}^{r-1} D_{r-s}(u_s, z^k) = \nu \delta^{1k}, \quad k = 1, \ldots, m.$$ (4)

Equating the coefficients at the same powers of $\nu$ on the left-hand and right-hand sides of (4) we get, at $r = 1$, the equations $D_1(u_0, z^k) = \delta^{1k}, \quad k = 1, \ldots, m$. Taking into account that $D_1 = i\{\cdot, \cdot\}$, it is easy to check that the function $u_0 = \partial \Phi_0 / \partial z^1$ satisfies these equations. For $r > 1$, the obtained equations are as follows,

$$\sum_{s=0}^{r-1} D_{r-s}(u_s, z^k) = 0, \quad k = 1, \ldots, m.$$ (5)

We construct the functions $u_s$ step by step using equations (5) and lemma 1. Assume that for $s < n$ the functions $u_s$ are constructed and satisfy equations (5) for $r \leq n$. We are going to show that the function $u_n$ can be found from equations (5) for $r = n + 1$, which we rewrite in the following form,

$$D_1(u_n, z^k) = -\sum_{s=0}^{n-1} D_{n-s+1}(u_s, z^k) = 0, \quad k = 1, \ldots, m.$$ (6)

It follows from lemma 1 that equations (6) can be solved for unknown $u_n$ if the sum

$$\sum_{s=0}^{n-1} D_1(D_{n-s+1}(u_s, z^k), z^{k'})$$

is symmetric with respect to the permutation of the indices $k$ and $k'$. The Jacoby identity for the $\star$-commutator (3) is reduced to the identities

$$\sum_{i=1}^{r-1} D_i(D_{r-i}(f, g), h) + \text{cyclic permutation of } f, g, h = 0$$ (7)

for any smooth functions $f, g, h$. Setting in (7) $f = u_s, \quad g = z^k, \quad h = z^{k'}, \quad r = n-s+1$, and taking into account that since $z^k$ pair-wise $\star$-commute,
\(D_r(z^k, z^{k'}) = 0\) holds, we get

\[
\sum_{i=1}^{n+1-s} (D_i(D_{n-i-s+2}(u_s, z^k), z^{k'}) - D_i(D_{n-i-s+2}(u_s, z^{k'}), z^k)) = 0. \quad (8)
\]

Summing up equations (8) for \(s = 0, 1, \ldots, n - 1\) and changing the order of summation, we get

\[
\sum_{s=0}^{n-1} (D_1(D_{n-s+1}(u_s, z^k), z^{k'}) - D_1(D_{n-s+1}(u_s, z^{k'}), z^k)) =
\]

\[-\sum_{i=2}^{n+1} \sum_{s=0}^{n-i+1} (D_i(D_{n-i-s+2}(u_s, z^k), z^{k'}) - D_i(D_{n-i-s+2}(u_s, z^{k'}), z^k)). \quad (9)
\]

It follows from the fact that \(D_1(u_0, z^k) = \delta^{1k}\) \(\ast\)-commutes with \(z^{k'}\) that \(D_i(D_1(u_0, z^k), z^{k'}) = 0\), therefore the inner sum on the right-hand side of (9) at \(i = n + 1\) is equal to zero. It follows from (5) that the inner sum on the right-hand side of (9) is equal to zero also for \(1 < i < n + 1\), thus the right-hand side of (9) equals zero which proves the solvability of the system (6) for unknown \(s_n\). The proposition is proved.

In a completely analogous way, one can find the formal functions \(v^1, \ldots, v^m \in \mathcal{F}(U)\) such that \(v^l \ast z^{l'} - z^{l'} \ast v^l = \nu \delta^{ll'}\).

Since \(L_{z^k} = z^k\), it follows from proposition 1 that \([L_{u_k}, z^{k'}] = \delta^{kk'}\). Using the fact that the operators from \(\mathcal{L}(U)\) contain only partial derivatives by \(z^k\), the operators \(L_{u_k}\) and similarly, operators \(R_{v^l}\), can be calculated explicitly.

**Lemma 2.** \(L_{u_k} = u^k + \nu \partial / \partial z^k\), \(R_{v^l} = v^l + \nu \partial / \partial z^l\).

Introduce the formal differential forms \(\alpha = -\sum_k u^k dz^k\) and \(\beta = \sum_l v^l d\bar{z}^l\). Since the operators \(L_{u_k}\) and \(R_{v^l}\) commute, one gets \(\partial u^k / \partial z^l = \partial v^l / \partial z^k\), therefore \(\bar{\partial} \alpha = \partial \beta\). Define the closed formal differential form \(\omega = i \bar{\partial} \alpha = i \partial \beta\) of the type \((1, 1)\). As follows from the proof of proposition 1, the first term of the formal series \(\omega\) coincides with \(\omega_0\), therefore \(\omega\) is a deformation of the Kähler form \(\omega_0\).

Assume \(\tilde{u}^1, \ldots, \tilde{u}^m\) is another set of solutions of (4), and set \(\tilde{\alpha} = -\sum_k \tilde{u}^k dz^k\). It follows from lemma 2 and from the fact that the operators \(L_{u_k}\) and \(R_{v^l}\) commute, one gets \(\partial \tilde{u}^k / \partial z^l = \partial v^l / \partial z^k\), therefore \(\bar{\partial} \tilde{\alpha} = \partial \beta\). Define the closed formal differential form \(\omega = i \bar{\partial} \tilde{\alpha} = i \partial \beta\) of the type \((1, 1)\). As follows from the proof of proposition 1, the first term of the formal series \(\omega\) coincides with \(\omega_0\), therefore \(\omega\) is a deformation of the Kähler form \(\omega_0\).
commute, that the form $i\bar{\partial}\alpha$ coincides with $\omega$, that is, $\omega$ does not depend on the concrete choice of the solution of system (4). It is easy to show also that $\omega$ does not depend on the choice of coordinates on $U$.

It follows from the Poincare $\bar{\partial}$-lemma that on a contractible coordinate chart $U \subset M$ there exists a formal series $\Phi = \Phi_0 + \nu \Phi_1 + \ldots \in \mathcal{F}$ which is a potential of the formal Kähler metrics $\omega = \omega_0 + \nu \omega_1 + \nu^2 \omega_2 + \ldots$. That means that for all $r \geq 0$, $\omega_r = i\partial\bar{\partial}\Phi_r = i(\partial^2 \Phi_r / \partial z^k \partial \bar{z}^l) dz^k \wedge d\bar{z}^l$.

Since $\omega = i\bar{\partial}\alpha = i\bar{\partial}(-\partial\Phi)$, then $\alpha + \partial\Phi$ is a $\bar{\partial}$-closed form of the type $(1,0)$. Therefore the coefficients of $\alpha + \partial\Phi$, which are equal to $\partial\Phi / \partial z^k - u^k$, are holomorphic. Now it is straightforward that $L_{\partial\Phi / \partial z^k} = \partial\Phi / \partial z^k + \nu \partial / \partial z^k$ and, similarly, $R_{\partial\Phi / \partial \bar{z}^l} = \partial\Phi / \partial \bar{z}^l + \nu \partial / \partial \bar{z}^l$.

Thus, starting from a given deformation quantization with separation of variables, we construct on each contractible chart $U \subset M$ a formal deformation $\omega$ of the Kähler form $\omega_0$. It follows from the construction of the form $\omega$ that on the intersections of charts the local forms agree with each other and define a global form $\omega$ on $M$.

**Theorem 1.** Each deformation quantization with separation of variables on a Kähler manifold $M$ canonically corresponds to a formal Kähler metrics $\omega$, which is a deformation of the Kähler metrics $\omega_0$ on $M$. If $\Phi$ is a potential of the formal metrics $\omega$ on a coordinate chart $U \subset M$, then $L_{\partial\Phi / \partial z^k} = \partial\Phi / \partial z^k + \nu \partial / \partial z^k$ and $R_{\partial\Phi / \partial \bar{z}^l} = \partial\Phi / \partial \bar{z}^l + \nu \partial / \partial \bar{z}^l$.

**3. A construction of the quantization with separation of variables from deformation of Kähler metrics**

Our goal is to generalize the construction of the deformation quantization announced in [Ka].

Assume that there is given a formal deformation $\omega$ of the Kähler metrics $\omega_0$ on $M$.

**Lemma 3.** Assume that on a contractible coordinate chart $U \subset M$, there is chosen a potential $\Phi = \Phi_0 + \nu \Phi_1 + \ldots \in \mathcal{F}$ of the formal metrics $\omega$. Then the set of formal series of differential operators from $\mathcal{D}(U)$, which commute
with the operators $\bar{z}^l$ and $\partial \Phi / \partial \bar{z}^l + \nu \partial / \partial \bar{z}^l$, depends only on the metrics $\omega$, rather than on the concrete choice of the potential.

**Proof.** If $\Phi' \in \mathcal{F}$ is another potential of the metrics $\omega$, then $\Phi' = \Phi + a + b$, where $a$ and $b$ are formal series of holomorphic and antiholomorphic functions, respectively. An operator which commutes with $\bar{z}^l$ and $\partial \Phi / \partial \bar{z}^l + \nu \partial / \partial \bar{z}^l$, commutes also with multiplication by antiholomorphic functions. Therefore, it commutes with $\partial \Phi'/\partial \bar{z}^l + \nu \partial / \partial \bar{z}^l = (\partial \Phi / \partial \bar{z}^l + \nu \partial / \partial \bar{z}^l) + \partial b / \partial \bar{z}^l$, which implies the assertion of the lemma.

Denote the set of formal operators mentioned in lemma 3 by $\mathcal{L}_\omega(U)$. Notice, that $\mathcal{L}_\omega(U)$ is an operator algebra.

Let $U$ be a coordinate chart on $M$ with a potential $\Phi_0$ of the Kähler metrics $\omega_0$ defined on it. Denote by $S(U)$ the set of differential operators with smooth coefficients on $U$, which commute with multiplication by the antiholomorphic coordinates $\bar{z}^l$, i.e. which contain only partial derivatives by $z^k$.

Define the differential operators $D^l$ on $U$, $D^l = g^{lk} \partial / \partial z^k = i \{ \bar{z}^l, \cdot \}$.

**Lemma 4.** For all $k, l, l' = 1, \ldots, m$ the following relations hold

(i) $[D^l, D^{l'}] = 0$;
(ii) $[D^l, \partial \Phi_0 / \partial \bar{z}^{l'}] = \delta^l_{l'}$;
(iii)$\partial / \partial z^k = g_{kl} D^l$.

The assertion of the lemma can be checked by direct calculations.

It follows from lemma 4 that any operator from $S(U)$ can be canonically represented as a sum of monomials of the form $a_{l_1 \ldots l_s} D^{l_1} \ldots D^{l_s}$, where $a_{l_1 \ldots l_s} \in C^\infty(U)$ is symmetric with respect to $l_i$.

**Definition.** The twisted symbol of an operator $A \in S(U)$, which is represented in the canonical form $A = \sum a_{l_1 \ldots l_s} D^{l_1} \ldots D^{l_s}$, is a polynomial in $\xi_1, \ldots, \xi^m$, $a(\xi) = \sum a_{l_1 \ldots l_s} \xi^{l_1} \ldots \xi^{l_s}$ with coefficients in $C^\infty(U)$.

From lemma 4 easily follows

**Lemma 5.** Let $a(\xi)$ be the twisted symbol of an operator $A \in S(U)$. Then the twisted symbol of the operator $[A, \partial \Phi_0 / \partial \bar{z}^l]$ is equal to $\partial a / \partial \xi^l$. 

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Consider a system of equations for an unknown operator \( A \in S(U) \),
\[
[A, \partial \Phi_0 / \partial \bar{z}^l] = B_l, \quad l = 1, \ldots, m,
\] (10)
where \( B_l \in S(U) \).

**Lemma 6.** System (10) has solutions if and only if for all \( l, l' \)
\[
[B_l, \partial \Phi_0 / \partial \bar{z}^l] = [B_{l'}, \partial \Phi_0 / \partial \bar{z}^{l'}].
\] If \( A_0 \) is a partial solution of the system, then the general solution is of the form \( A_0 + A_1 \), where \( A_1 \) is an arbitrary multiplication operator.

**Proof.** Pass to the twisted symbols \( a, b_l \) of the operators \( A, B_l \), respectively. System (10) transforms to the equation \( da = \sum b_l d\xi^l \), where \( da = \sum (\partial a / \partial \xi^l) d\xi^l \). The assertion of the lemma is now reduced to a standard fact concerning differential forms with polynomial coefficients, which follows from Euler’s identity.

**Proposition 2.** Let \( \omega_0 \) be a Kähler metrics on \( M \) and \( U \subset M \) be a contractible coordinate chart. For each formal function \( f = \sum \nu^r f_r \in \mathcal{F}(U) \) there exists a unique formal series of differential operators \( \tilde{A}_f = \sum \nu^r A_r \) from \( \mathcal{L}_{\omega_0}(U) \), such that \( \tilde{A}_f 1 = f \). In particular, \( A_0 \) is a multiplication operator by the function \( f_0 \).

**Proof.** Since \( \tilde{A}_f \) commutes with antiholomorphic functions, all the operators \( A_r \) are in \( S(U) \). Let \( \Phi_0 \) be a potential of the metrics \( \omega_0 \). The commutation condition of \( \tilde{A}_f \) with \( \partial \Phi_0 / \partial \bar{z}^l + \nu \partial / \partial \bar{z}^l \) is equivalent to the system of equations \( [A_0, \partial \Phi_0 / \partial \bar{z}^l] = 0 \) and
\[
[A_r, \partial \Phi_0 / \partial \bar{z}^l] = [\partial / \partial \bar{z}^l, A_{r-1}].
\] (11)
Find all the terms of the series \( \tilde{A}_f \) step by step. It follows from lemma 5 that \( A_0 \) is a multiplication operator, so \( A_0 1 = f_0 \) implies that \( A_0 = f_0 \). Assume that we have found all the operators \( A_r \) for \( r < s \) which satisfy (11) and such that \( A_r 1 = f_r \). Let us now show that \( A_s \) can be found from (11) for \( r = s \), i.e., that the conditions of lemma 6 on the right-hand side of (11) are satisfied,
\[
[[\partial / \partial \bar{z}^l, A_{s-1}], \partial \Phi_0 / \partial \bar{z}^l] = [[\partial / \partial \bar{z}^l, A_{s-1}], \partial \Phi_0 / \partial \bar{z}^{l'}].
\]
It follows from the Jacoby identity for commutators that

\[
\left[ \frac{\partial}{\partial z^l}, A_{s-1} \right] \Phi_0 = \left[ \frac{\partial}{\partial z^l}, A_{s-1} \right] \Phi_0 = \left[ \frac{\partial^2}{\partial z^l \partial z^{l'}}, A_{s-1} \right] + \left[ \frac{\partial}{\partial z^l}, \left[ A_{s-1}, \frac{\partial}{\partial z^{l'}} \right] \right].
\]

It is easy to check that the last expression is symmetric with respect to the permutation of \(l\) and \(l'\). Thus system (11) is solvable for \(r = s\). Among its solutions there is the only one solution \(A_s\) such that \(A_s 1 = f_s\). The assertion is proved.

**Lemma 7.** For a given formal function \(f = f_0 + \nu f_1 + \ldots \in \mathcal{F}(U)\) there exists a function \(g = g_0 + \nu g_1 + \ldots \in \mathcal{F}(U)\) such that \(\tilde{A}_f g = 1\) if and only if \(f_0\) does not vanish on \(U\). Then \(g\) is defined uniquely and \(g_0 = 1/f_0\).

**Proof.** Let \(\tilde{A}_f = \sum \nu^r A_r\). The condition \(\tilde{A}_f g = 1\) is equivalent to the system of equations \(A_0 g_0 = 1\) and \(A_0 g_r = -\sum_{s=0}^{r-1} A_s g_{r-s}\). According to proposition 2, \(A_0 = f_0\), therefore if \(f_0\) does not vanish, all the functions \(g_r\) can be calculated step by step. That completes the proof.

**Lemma 8.** Let the formal functions \(f, g \in \mathcal{F}(U)\) be such that \(\tilde{A}_f g = 1\). Then the operator \(\tilde{A}_g\) is inverse to \(\tilde{A}_f\) and, in particular, \(\tilde{A}_g f = 1\).

**Proof.** The operator \(\tilde{A}_f \tilde{A}_g\) belongs to \(\mathcal{L}_{\omega_0}(U)\). Since \(\tilde{A}_f \tilde{A}_g 1 = \tilde{A}_f g = 1\), then \(\tilde{A}_f \tilde{A}_g = \tilde{A}_1 = 1\). It follows from lemma 7 that the coefficient at the zero power of \(\nu\) of the formal series \(g\) does not vanish. Therefore, there exists a function \(h \in \mathcal{F}(U)\) such that \(\tilde{A}_g h = 1\), so \(\tilde{A}_g \tilde{A}_h = 1\). Thus the operator \(\tilde{A}_g\) has both left and right inverse operators which immediately implies the assertion of the lemma.

We will use some elementary facts about formal series. Let \(R\) be a vector space and \(\tilde{R} = R[[\nu]]\) be the space of formal series with coefficients in \(R\). There is a decreasing filtration in \(\tilde{R}\), \(\tilde{R} = \tilde{R}_0 \supset \tilde{R}_1 \supset \tilde{R}_2 \ldots\), where \(\tilde{R}_n\) consists of the series of the form \(\tilde{A} = \sum_{r=n}^{\infty} \nu^r A_r\), \(A_r \in R\). An element \(\tilde{A} \in \tilde{R}\) is of the order \(n\), \(\text{ord}({\tilde{A}}) = n\), if \(\tilde{A} \in \tilde{R}_n \setminus \tilde{R}_{n+1}\). A series \(\sum \tilde{A}_n\) with the elements \(\tilde{A}_n \in \tilde{R}\), such that the order \(\text{ord}({\tilde{A}_n}) \to \infty\) as \(n \to \infty\), converges
to an element of $\tilde{R}$ with respect to the topology defined by the filtration. If $\tilde{A} - \tilde{B}$ is of the order $n$, we write $\tilde{A} \equiv \tilde{B} \pmod{\nu^n}$.

For an arbitrary formal function $S = S_0 + \nu S_1 + \ldots \in \mathcal{F}(U)$ define its exponent, $e^S = e^{S_0} \sum_{n=0}^{\infty} (1/n!)(S - S_0)^n \in \mathcal{F}(U)$. The series in the definition of the exponent converges since $\text{ord}((S - S_0)^n) \geq n$.

Lemma 9. For $S \in \mathcal{F}(U)$, $\partial e^S / \partial z^k = (\partial S / \partial z^k)e^S$ and $\partial e^S / \partial \bar{z}^l = (\partial S / \partial \bar{z}^l)e^S$. Moreover, for $S, T \in \mathcal{F}(U)$ holds the equality $e^S \cdot e^T = e^{S+T}$.

The proof is standard.

Proposition 3. Let $\omega$ be a formal deformation of the Kähler metrics $\omega_0$ on $M$ and $U \subset M$ be a contractible coordinate chart. For each formal function $g = \sum \nu^r g_r \in \mathcal{F}(U)$ there exists a unique formal series of differential operators $\tilde{B}_g = \sum \nu^r B_r$ from $\mathcal{L}_\omega(U)$ such that $\tilde{B}_g 1 = g$.

Proof. Let $\Phi = \Phi_0 + \nu \Phi_1 + \Phi_2 + \ldots$ be a potential of $\omega$. Set $S = \Phi_1 + \nu \Phi_2 + \ldots$. It follows from lemma 9 that $e^{-S}(\partial \Phi_0 / \partial \bar{z}^l + \nu \partial / \partial \bar{z}^l)e^S = \partial \Phi / \partial \bar{z}^l + \nu \partial / \partial \bar{z}^l$. Since, moreover, $e^{-S} \bar{z}^l e^S = \bar{z}^l$ we get that $e^{-S} \mathcal{L}_\omega(U)e^S = \mathcal{L}_\omega(U)$. The operator $\tilde{B}_g$ exists if and only if there is a function $f \in \mathcal{F}(U)$ such that $e^{-S}(\tilde{A}_f)e^S = \tilde{B}_g$. It is enough for $f$ to satisfy the relation $e^{-S} \tilde{A}_f(e^S) = g$ or $\tilde{A}_f(e^S) = e^S g$, which is equivalent to the equality $\tilde{A}_f \tilde{A}_e^S = \tilde{A}_e^S g$. From lemmas 7 and 8 it follows that there is a function $h \in \mathcal{F}(U)$ such that the operator $\tilde{A}_h$ is inverse to $\tilde{A}_e^S$. Therefore $\tilde{A}_f = \tilde{A}_e^S_b \tilde{A}_h$, and so $f = \tilde{A}_e^S g h$, which completes the proof.

According to proposition 3, the mapping $f \mapsto \tilde{B}_f$ is a bijection of $\mathcal{F}(U)$ onto $\mathcal{L}_\omega(U)$. Since $\mathcal{L}_\omega(U)$ is an operator algebra, one can define in $\mathcal{F}(U)$ an associative product $\ast$, carrying over to $\mathcal{F}(U)$ the operator product from $\mathcal{L}_\omega(U)$. For $f, g \in \mathcal{F}(U)$ by definition $\tilde{B}_{f \ast g} = \tilde{B}_f \tilde{B}_g$. Applying both sides of the obtained equality to the constant 1, one gets $f \ast g = \tilde{B}_f g$. That means that $\tilde{B}_f$ is a left multiplication operator in the algebra $\mathcal{F}(U)$ with the operation $\ast$. Denote $L_f = \tilde{B}_f$.

Calculate the first two terms of the formal series of operators $L_{z^l}$.

Lemma 10. $L_{z^l} \equiv \bar{z}^l + \nu D^l \pmod{\nu^2}$. 

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Proof. Let \( L_{\bar{z}^l} \equiv A + \nu B \pmod{\nu^2} \), then

\[
[L_{\bar{z}^l}, \frac{\partial \Phi}{\partial \bar{z}^{l'}}] \equiv [A + \nu B, \frac{\partial \Phi_0}{\partial \bar{z}^{l'}} + \nu \left( \frac{\partial \Phi_1}{\partial \bar{z}^{l'}} + \frac{\partial}{\partial \bar{z}^{l'}} \right)] \pmod{\nu^2}. \tag{12}
\]

The operators \( L_{\bar{z}^l} \) and \( \partial \Phi/\partial \bar{z}^{l'} \) commute, therefore the coefficients at the zero and first powers of \( \nu \) on the right-hand side of (12) are equal to zero. First, \( [A, \partial \Phi_0/\partial \bar{z}^{l'}] = 0 \), therefore, according to lemma 5, \( A \) is a multiplication operator. Since \( L_{\bar{z}^l}1 = A1 = \bar{z}^l \), then \( A = \bar{z}^l \). Taking into account that \( A = \bar{z}^l \), we get the equation \( [B, \partial \Phi_0/\partial \bar{z}^{l'}] = \delta_{l'}^l \). Since \( B1 = 0 \), from lemma 5 follows that \( B = D^l \). The lemma is proved.

Now we obtain the formula expressing the operator \( L_f, f \in \mathcal{F}(U) \), via \( L_{\bar{z}^l} \).

Proposition 4.

\[
L_f = \sum_\alpha \frac{1}{\alpha!} \left( \frac{\partial}{\partial \bar{z}} \right)^\alpha f (L_{\bar{z}} - \bar{z})^\alpha, \tag{13}
\]

where \( \alpha \) is a multi-index.

Proof. It follows from lemma 10 that \( \text{ord}(L_{\bar{z}^l} - \bar{z}^l) = 1 \), therefore \( \text{ord}((L_{\bar{z}} - \bar{z})^\alpha) = |\alpha| \) so the series in (13) converges. Denote temporarily the right-hand side of (13) by \( \tilde{A} \). Since \( (L_{\bar{z}^l} - \bar{z}^l)1 = 0 \), then \( \tilde{A}1 = f \), so to prove the proposition it is enough to show that \( \tilde{A} \in \mathcal{L}_\omega(U) \). Let \( \alpha = (i_1, \ldots, i_m) \) be a multi-index. Introduce the following notation, \( \alpha \pm l = (i_1, \ldots, i_l \pm 1, \ldots, i_m) \).

Taking into account that \( L_{\bar{z}^l} \in \mathcal{L}_\omega(U) \), one gets

\[
\left[ \left( \frac{\partial}{\partial \bar{z}} \right)^\alpha f, \frac{\partial \Phi}{\partial \bar{z}^{l'}} + \nu \frac{\partial}{\partial \bar{z}^{l'}} \right] = \nu \left( \frac{\partial}{\partial \bar{z}} \right)^{\alpha+l} f \quad \text{and}
\]

\[
\left[ \frac{1}{\alpha!} (L_{\bar{z}} - \bar{z})^\alpha, \frac{\partial \Phi}{\partial \bar{z}^{l'}} + \nu \frac{\partial}{\partial \bar{z}^{l'}} \right] = -\nu \frac{1}{(\alpha - l)!} (L_{\bar{z}} - \bar{z})^{\alpha-l},
\]

which implies that

\[
[\tilde{A}, \frac{\partial \Phi}{\partial \bar{z}^l} + \nu \frac{\partial}{\partial \bar{z}^l}] = \nu \sum_\alpha \frac{1}{\alpha!} \left( \frac{\partial}{\partial \bar{z}} \right)^{\alpha+l} f (L_{\bar{z}} - \bar{z})^\alpha -
\]

\[
\sum_\alpha \frac{1}{(\alpha - l)!} \left( \frac{\partial}{\partial \bar{z}} \right)^\alpha f (L_{\bar{z}} - \bar{z})^{\alpha-l} = 0.
\]

The proposition is proved.
It immediately follows from proposition 4 and bilinearity of the product $\star$ that the product $\star$ is given by formula (1) for some bidifferential operators $C_r$. Let $u, v \in C^\infty(U)$. Calculate the operators $C_0$ and $C_1$ considering the first two terms of the series $u \star v$ and taking into account lemma 10,

$$u \star v = L_u v \equiv uv + \nu \sum_i \frac{\partial u}{\partial \bar{z}^i} D^i v \, (\text{mod } \nu^2).$$

It follows that $C_0(u, v) = uv$ and $C_1(u, v) = \sum_i \partial u / \partial z^i D^i v$, therefore

$$C_1(u, v) - C_1(v, u) = \sum_i \left( \frac{\partial u}{\partial \bar{z}^i} D^i v - \frac{\partial v}{\partial \bar{z}^i} D^i u \right) =$$

$$g^{lk} \left( \frac{\partial v}{\partial z^k} \frac{\partial u}{\partial \bar{z}^l} - \frac{\partial u}{\partial z^k} \frac{\partial v}{\partial \bar{z}^l} \right) = i \{u, v\}.$$ 

That means that the product $\star$ is a $\star$-product on the chart $U$ with the Kähler metrics $\omega_0$. It is clear from the construction of the product $\star$ from the deformation of Kähler metrics $\omega$ that on the intersections of charts the products $\star$ agree with each other and define a global deformation quantization with separation of variables on the Kähler manifold $M$. From theorem 1 it follows that the deformation of Kähler metrics corresponding to the $\star$-product $\star$, coincides with $\omega$. Thus we have stated the following

**Theorem 2.** Deformation quantizations with separation of variables on a Kähler manifold $M$ are in 1—1 correspondence with formal deformations of the Kähler metrics $\omega_0$ on $M$. If on $M$ there is given a quantization with separation of variables corresponding to a formal deformation $\omega$ of the metrics $\omega_0$, $U$ is a contractible coordinate chart on $M$, and $\Phi$ is a potential of $\omega$ on $U$, then the operators of left $\star$-multiplication $L(U)$ are characterized by the property that they commute with multiplication by antiholomorphic functions and with the operators

$$R_{\partial \Phi/\partial \bar{z}^l} = \partial \Phi / \partial \bar{z}^l + \nu \partial / \partial \bar{z}^l.$$

Similarly, the operators of right $\star$-multiplication $R(U)$ are characterized by the property that they commute with multiplication by holomorphic functions and with the operators

$$L_{\partial \Phi/\partial z^k} = \partial \Phi / \partial z^k + \nu \partial / \partial z^k.$$

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