AN ASYMPTOTICALLY SHARP FORM OF BALL’S INTEGRAL INEQUALITY

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Abstract. We solve the open problem of determining the second order term in the asymptotic expansion of the integral in Ball’s integral inequality. In fact, we provide a method by which one can compute any term in the expansion. We also indicate how to derive an asymptotically sharp form of a generalized Ball’s integral inequality.

1. Introduction
To prove that every \((n-1)\)-dimensional section of the unit cube in \(\mathbb{R}^n\) has volume at most \(\sqrt{2}\), K. Ball [1] made essential use of the inequality

\[
\sqrt{n} \int_{-\infty}^{\infty} \left| \frac{\sin t}{t} \right|^n \, dt \leq \sqrt{2} \pi, \quad n \geq 2,
\]

in which equality holds if and only if \(n = 2\).

Later, Ball’s integral inequality (1) was proved using different methods; see [2][6] (also see [4] for an analogue of Ball’s inequality). Independently of Ball, D. Borwein, J. M. Borwein and I. E. Leonard investigated, in [2], the asymptotic expansion of the left side of (1). They established the existence of real constants, \(c_j\), such that

\[
\sqrt{n} \int_0^{\infty} \left| \frac{\sin t}{t} \right|^n \, dt \sim \sqrt{\frac{3\pi}{2}} - \frac{3}{20} \sqrt{\frac{3\pi}{2}} \frac{1}{n} + \sum_{j=2}^{\infty} \frac{c_j}{n^j}, \quad \text{as} \quad n \to \infty,
\]

and posed the problem of determining the value of \(c_2\).

K. Oleszkiewicz and A. Pelczyński, in [7], proved the following variant of Ball’s inequality, namely,

\[
n \int_0^{\infty} \left( \frac{2 |J_1(t)|}{t} \right)^n \, dt \leq 4, \quad n \geq 2,
\]

involving a special case of

\[
J_\nu(t) := \sum_{j=0}^{\infty} (-1)^j \left( \frac{t}{2} \right)^{2j+\nu} \frac{1}{j! \Gamma(j+\nu+1)}, \quad t \geq 0, \nu \geq 1/2,
\]
the Bessel function of order $\nu$. They showed that with the method used to establish their inequality (3), one can prove (1). Also, they discussed the more general inequality

$$n^\nu \int_0^\infty \left( 2^\nu \Gamma(\nu+1) \frac{|J_\nu(t)|}{t^\nu} \right)^n t^{2\nu-1} dt < 2^\nu \left( \int_0^\infty \left( 2^\nu \Gamma(\nu+1) \frac{|J_\nu(t)|}{t^\nu} \right)^2 t^{2\nu-1} dt \right)^n,$$  

$n > 2$.

They conjectured it holds if and only if $\frac{1}{2} \leq \nu \leq 1$. In this connection, they pointed out that H. König has noticed the inequality is false when $\nu = \frac{k}{2}, k = 3, 4, \ldots$.

Our paper is divided into three sections and an appendix. The first section is an introduction. The second section is devoted to calculating the $c_2$ in (2), thereby solving the problem posed by D. Borwein, J. M. Borwein and I. E. Leonard. The method that gives $c_2$ can be used to derive any term in the asymptotic expansion in (2). In the third section, we indicate how the method of Section 2 enables one to determine the asymptotic expansion of

$$n^\nu \int_0^\infty \left( 2^\nu \Gamma(\nu+1) \frac{|J_\nu(t)|}{t^\nu} \right)^n t^{2\nu-1} dt, \quad n \geq 2,$$

for all $\nu \geq 1/2$.

2. An asymptotically sharp form of Ball’s integral inequality

In this section we answer the open question of D. Borwein, J. M. Borwein and I. E. Leonard in the following theorem.

**Theorem 1.** Let

$$I(n) := \sqrt{n} \int_0^\infty \left| \frac{\sin t}{t} \right|^n dt,$$

$n \geq 2$, and fix $m \in \mathbb{Z}_+, m \geq 3$. Then, there exist constants $c_3, c_4, \ldots, c_m$ such that

$$I(n) = \sqrt{\frac{3\pi}{2}} \left[ 1 - \frac{3}{20} \frac{1}{n} - \frac{13}{1120} \frac{1}{n^2} + \sum_{j=3}^m c_j \frac{1}{n^j} \right] + O \left( \frac{1}{n^{m+1}} \right).$$

**Proof.** We first observe that $I(n)$ can be replaced by

$$J(n) := \sqrt{n} \int_0^\alpha \left| \frac{\sin t}{t} \right|^n dt,$$

for $1 \leq \alpha < \pi$ fixed.

Indeed,

$$\sqrt{n} \int_0^\infty \left| \frac{\sin t}{t} \right|^n dt \leq \sqrt{n} \int_0^\infty t^{-n} dt = \frac{\sqrt{n}}{n-1} \alpha^{1-n}.$$

Therefore, it suffices to show there exist constants $c_3, c_4, \ldots, c_m$ such that

$$J(n) = \sqrt{n} \int_0^\alpha \left( \frac{\sin t}{t} \right)^n dt$$

$$= \sqrt{\frac{3\pi}{2}} \left[ 1 - \frac{3}{20} \frac{1}{n} - \frac{13}{1120} \frac{1}{n^2} + \sum_{j=3}^m c_j \frac{1}{n^j} \right] + O \left( \frac{1}{n^{m+1}} \right).$$
Making the change of variable \( s = \frac{t}{\sqrt{n}} \) in \( \int_0^\alpha \left( \frac{\sin s}{s} \right)^n ds \), we obtain

\[
J(n) = \int_0^{\alpha \sqrt{n}} \left( \frac{\sin(t/\sqrt{n})}{t/\sqrt{n}} \right)^n dt = \int_0^{\alpha \sqrt{n}} e^{-t^2/6n} \left( \frac{e^{t^2/6n} \sin(t/\sqrt{n})}{t/\sqrt{n}} \right)^n dt.
\]

Now, we use

\[
e^{t^2/6n} = \sum_{j=0}^{\infty} \left( \frac{t^2}{6n} \right)^j j!
\]

and

\[
\frac{\sin(t/\sqrt{n})}{t/\sqrt{n}} = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} \left( \frac{t^2}{n} \right)^j,
\]

whence

\[
e^{t^2/6n} \sin(t/\sqrt{n}) = 1 + \sum_{j=2}^{\infty} \frac{a_j}{n^j} t^{2j},
\]

in which

\[
a_j = \sum_{i=0}^{j-1} \frac{1}{i!6^i (2(j-i)+1)!}.
\]

Using Newton’s Binomial Formula we obtain

\[
\left[ e^{t^2/6n} \sin(t/\sqrt{n}) \right]^n = 1 + n \left[ \sum_{j=2}^{\infty} \frac{a_j}{n^j} t^{2j} \right] + \frac{n(n-1)}{2} \left[ \sum_{j=2}^{\infty} \frac{a_j}{n^j} t^{2j} \right]^2 + \ldots + \frac{n(n-1) \ldots (n-m+1)}{m!} \left[ \sum_{j=2}^{\infty} \frac{a_j}{n^j} t^{2j} \right]^m + \ldots.
\]

(4)

We observe that, for \( t \in [0, \alpha \sqrt{n}] \),

\[
\left| \sum_{j=2}^{\infty} \frac{a_j}{n^j} t^{2j} \right| = \left| e^{t^2/6n} \sin(t/\sqrt{n}) - 1 \right| < 1.
\]

Only the first \( m + 1 \) terms on the right-hand side of (4) yield the powers \( \frac{1}{n^0}, \frac{1}{n}, \frac{1}{n^2}, \ldots, \frac{1}{n^m} \). We get

\[
\frac{1}{n^0}, \frac{a_2 t^4}{n} = -\frac{t^4}{180n}, \quad \left( a_3 t^6 + \frac{1}{2} a_2 t^8 \right) \frac{1}{n^2} = \left( -\frac{t^6}{2835} + \frac{t^8}{64800} \right) \frac{1}{n^2}
\]

and so on.

The highest power of \( t \) yielding \( \frac{1}{n^m} \) is \( t^{4m} \). Accordingly, we write

\[
\left[ e^{t^2/6n} \sin(t/\sqrt{n}) \right]^n = \sum_{j=0}^{2m} b_j \left( \frac{t}{\sqrt{n}} \right)^{2j} + R_{2m} \left( \frac{t}{\sqrt{n}} \right), \quad b_j = b_j(n),
\]

(5)
in which
\[
\left| R_{2m} \left( \frac{t}{\sqrt{n}} \right) \right| \leq C \frac{t^{4m+2}}{n^{2m+1}},
\]
the constant \( C > 1 \) being independent of \( t \in [0, \alpha \sqrt{n}] \).

For concreteness, we now work with the polynomial of degree 28 in (5) corresponding to \( m = 7 \). It is given in the Appendix. Formula (5) becomes
\[
\left[ e^{\frac{t^2}{6n}} \frac{\sin(t/\sqrt{n})}{t/\sqrt{n}} \right]^n = \sum_{j=0}^{14} b_j \left( \frac{t}{\sqrt{n}} \right)^{2j} + O \left( \frac{t^{30}}{n^{15}} \right), \quad b_j = b_j(n),
\]
and gives all the correct terms in the asymptotic expansion up to \( \frac{1}{n^7} \). Multiplying the polynomial in (6) by \( e^{-t^2/6} \), integrating the product from 0 to \( \alpha \sqrt{n} \) and using the fact that
\[
\int_0^{\alpha \sqrt{n}} e^{-t^2/6} t^{2j} dt = 6^{j+\frac{1}{2}} \int_0^{\infty} e^{-t^2} t^{2j} dt + O \left( \frac{1}{n^8} \right)
\]
\[
= 3^j (2j - 1)(2j - 3) \cdots 1 \sqrt{\frac{3\pi}{2}} + O \left( \frac{1}{n^8} \right),
\]
j = 1, 2, \ldots, 2m, we obtain, with an error of \( O \left( \frac{1}{n^8} \right) \),
\[
J(n) = \sqrt{\frac{3\pi}{2}} \left[ 1 - \frac{3}{20} \frac{1}{n} - \frac{13}{1120} \frac{1}{n^2} + \sum_{j=3}^{7} \frac{c_j}{n^j} \right].
\]

\( \square \)

**Remark 2.** Working with the polynomial of degree 28 in the Appendix one can show that
\[
c_3 = \frac{27}{3200}, \quad c_4 = \frac{52791}{3942400}, \quad c_5 = \frac{482427}{66560000},
\]
\[
c_6 = -\frac{124996631}{10035200000}, \quad c_7 = -\frac{5270328789}{136478720000}.
\]

**Remark 3.** A proof using splines that \( I(n) \sim \sqrt{\frac{3\pi}{2}} \) is given in [3].

### 3. A Generalized Ball’s Integral Inequality

We indicate how to determine constants \( c_0, c_1, c_2, c_3, \ldots, c_m \) so that, with \( n \geq 2 \),
\[
I_{\nu}(n) := n^{\nu} \int_0^\infty \left( \frac{2^\nu \Gamma(\nu + 1)|J_{\nu}(t)|}{t^\nu} \right)^n t^{2\nu - 1} dt
\]
\[
= c_0 + \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3} + \cdots + \frac{c_m}{n^m} + O \left( \frac{1}{n^{m+1}} \right).
\]
For definiteness, we do this when \( m = 3 \).

Our first observation is that \( I_{\nu}(n) \) may be replaced by
\[
K_{\nu}(n) := n^{\nu} \int_0^{2^\nu \Gamma(\nu + 1)} \left( \frac{2^\nu \Gamma(\nu + 1)|J_{\nu}(t)|}{t^\nu} \right)^n t^{2\nu - 1} dt.
\]
Indeed, using the estimate
\[ |J_\nu(t)| \leq ct^{-\frac{1}{3}}, \quad t \in \mathbb{R}_+, \nu \geq 1, c = 0.7857468704 \ldots, \]
given in [5], we get, for \( n \) sufficiently large,
\[
\begin{align*}
  n^\nu \int_{x}^{\infty} \left( \frac{2^\nu \Gamma(\nu + 1)|J_\nu(t)|}{t^\nu} \right)^n t^{2\nu - 1} dt & \leq n^\nu \int_{x}^{\infty} \left( \frac{2^\nu \Gamma(\nu + 1)ct^{-\frac{1}{3}}}{t^\nu} \right)^n t^{2\nu - 1} dt \\
  & = n^\nu (2^\nu \Gamma(\nu + 1)c)^n \int_{x}^{\infty} t^{-(\nu + \frac{1}{3})n + 2\nu - 1} dt \\
  & = \frac{n^\nu (2^\nu \Gamma(\nu + 1)c)^n x^{-(\nu + \frac{1}{3})n + 2\nu}}{n (\nu + \frac{1}{3}) - 2\nu} \\
  & \leq n^{-1}c^\nu,
\end{align*}
\]
with \( x = 2^\nu \Gamma(\nu + 1) \).

As we did in Section 2 for \( \sin t \), we make the change of variable \( t \to \frac{t}{\sqrt{n}} \) in \( K_\nu(n) \) and get
\[
K_\nu(n) = \frac{2^\nu \Gamma(n + 1)\sqrt{n}}{\Gamma(n)\Gamma(n + 1)\sqrt{n}} \left( \frac{2^\nu \Gamma(\nu + 1)|J_\nu(t/\sqrt{n})|}{(t/\sqrt{n})^\nu} \right)^n t^{2\nu - 1} dt.
\]

Using the Maclaurin expansion of \( \exp \left( \frac{t^2}{4n(\nu + 1)} \right) \), together with the approximation
\[
(9) \quad 2^\nu t^{-\nu} \Gamma(\nu + 1)J_\nu(t) = \sum_{j=0}^{\infty} \left( \frac{-t^2}{4} \right)^j \Gamma(\nu + j + 1),
\]
we obtain
\[
\begin{align*}
  K_\nu(n) & = \int_{0}^{2^\nu \Gamma(n + 1)\sqrt{n}} \exp \left( \frac{-t^2}{4(\nu + 1)} \right) \\
  & \times \left[ \exp \left( \frac{t^2}{4n(\nu + 1)} \right) \frac{2^\nu \Gamma(\nu + 1)|J_\nu(t/\sqrt{n})|}{(t/\sqrt{n})^\nu} \right]^n t^{2\nu - 1} dt \\
  & = \int_{0}^{2^\nu \Gamma(n + 1)\sqrt{n}} \exp \left( \frac{-t^2}{4(\nu + 1)} \right) \left[ 1 + \sum_{j=2}^{\infty} a_j \left( \frac{t^2}{4n} \right)^j \right]^n t^{2\nu - 1} dt \\
  & = \int_{0}^{2^\nu \Gamma(n + 1)\sqrt{n}} \exp \left( \frac{-t^2}{4(\nu + 1)} \right) \left( 1 + n \sum_{j=2}^{\infty} a_j \left( \frac{t^2}{4n} \right)^j \right) \\
  & \quad + \ldots + \frac{n(n-1)\ldots(n-m+1)}{m!} \sum_{j=2}^{\infty} a_j \left( \frac{t^2}{4n} \right)^j \right]^m t^{2\nu - 1} dt,
\end{align*}
\]
in which
\[
a_j = \sum_{i=0}^{j} (-1)^i \frac{\Gamma(\nu + 1)}{(\nu + 1)^{j-i} (j-i)! \Gamma(\nu + i + 1)}.
\]
One finds that
\[
a_2 = \frac{-1}{2(\nu + 1)^2(\nu + 2)}
\]
\[
a_3 = \frac{-2}{3(\nu + 1)^3(\nu + 2)(\nu + 3)}
\]
\[
a_4 = \frac{\nu - 5}{8(\nu + 1)^4(\nu + 2)(\nu + 3)(\nu + 4)}
\]
and that, moreover,
\[
c_0 = \int_0^\infty \exp\left(\frac{-t^2}{4(\nu + 1)}\right) t^{2\nu - 1} dt = \frac{4^\nu}{2} (\nu + 1)^\nu \Gamma(\nu)
\]
\[
c_1 = \frac{a_2}{16} \int_0^\infty \exp\left(\frac{-t^2}{4(\nu + 1)}\right) t^{4\nu - 1} dt = \frac{-4^\nu(\nu + 1)^\nu}{\nu + 2} \Gamma(\nu + 2)
\]
\[
c_2 = \frac{a_3}{64} \int_0^\infty \exp\left(\frac{-t^2}{4(\nu + 1)}\right) t^{6\nu - 1} dt + \frac{a_2^2}{512} \int_0^\infty \exp\left(\frac{-t^2}{4(\nu + 1)}\right) t^{8\nu - 1} dt = 4^\nu(\nu + 1)^\nu \Gamma(\nu + 2) + \frac{3\nu^2 + 2\nu - 5}{3(\nu + 2)(\nu + 3)}
\]
\[
c_3 = \left(\frac{a_4}{256} - \frac{a_2^2}{512}\right) \int_0^\infty \exp\left(\frac{-t^2}{4(\nu + 1)}\right) t^{8\nu - 1} dt + \frac{a_2a_3}{1024} \int_0^\infty \exp\left(\frac{-t^2}{4(\nu + 1)}\right) t^{10\nu - 1} dt + \frac{a_3^2}{24576} \int_0^\infty \exp\left(\frac{-t^2}{4(\nu + 1)}\right) t^{12\nu - 1} dt
\]
\[
= -4^\nu(\nu + 1)^\nu \Gamma(\nu + 2) + \frac{\nu^3 - \nu^2 - 4\nu - 8}{6(\nu + 2)^2(\nu + 4)}
\]
We observe that when \(\nu = 1\), \(c_0 = 4\), that
\[
\lim_{n \to \infty} n \int_0^\infty \left(\frac{2|J_1(t)|}{t}\right)^n t dt = 4,
\]
which means the maximum value of \(I_1(n)\) occurs at \(n = 2\) and in the limit as \(n\) approaches infinity.
However, when \(\nu > 1\) and \(n \geq 2\), the \(c_0\) in (7) is greater than the \(I_\nu(n)\). In particular,
\[
I_\nu(2) = 2^{3\nu} \Gamma(\nu + 1)^2 \int_0^\infty \frac{J_\nu(t)^2}{t} dt = 2^{3\nu - 1} \nu!(\nu - 1)! < 2^{2\nu - 1}(\nu + 1)^\nu(\nu - 1)! = c_0.
\]
APPENDIX A

The polynomial in \([5]\) corresponding to \(m = 7\) is

\[
\begin{align*}
1 & - \frac{1}{180} n^{-4} - \frac{1}{2835} n^{-6} + \left( \frac{1}{64800 n^2} - \frac{1}{37800 n^3} \right) t^8 + \left( \frac{1}{510300 n^3} - \frac{1}{467775 n^4} \right) t^{10} \\
& + \left( - \frac{1}{34992000 n^5} + \frac{128595600 n^6}{3831077250 n^6} \right) t^{12} + \left( - \frac{1}{18370800 n^7} + \frac{1}{47151720 n^8} \right) t^{14} \\
& - \frac{2}{127702575 n^9} t^{16} + \left( \frac{25194420000 n^9}{462944160000 n^9} + \frac{1}{11035304280000 n^9} \right) t^{18} \\
& - \frac{3617}{265013253000 n^{11}} t^{20} + \left( \frac{1}{80922320000 n^{11}} - \frac{5543}{60153806790000 n^{11}} + \frac{43444416015000 n^{11}}{9001} \right) t^{22} \\
& - \frac{43867}{350813659321125 n^{13}} t^{24} + \left( - \frac{2267481600000 n^{13}}{83329948800000 n^{13}} - \frac{143}{13902213124800000 n^{13}} \right) t^{26} \\
& + \frac{1}{62809} t^{28} + \left( \frac{1}{174611} - \frac{509}{1621577} + \frac{90749797}{28377197430341760000 n^9} \right) t^{30} \\
& - \frac{370206979}{29480751081883840000 n^{11}} t^{32} + \left( \frac{27554578491329089000000 n^{11}}{242303486356578262500 n^{11}} - \frac{236364991}{441301082837} \right) t^{34} \\
& + \left( \frac{1}{465813841} - \frac{7241}{1559186619968000000 n^{13}} + \frac{118238322626424000000 n^{13}}{463523000000} \right) t^{36} \\
& - \frac{3500839669973706000000 n^{15}}{15280310366000000 n^{15}} t^{38} + \left( \frac{218073137720857043625000000 n^{15}}{11358620000000000 n^{15}} - \frac{144228265683897515625 n^{15}}{13158620000000000 n^{15}} \right) t^{40} \\
& + \left( \frac{1}{30855886128000000 n^{17}} + 161993420467200000000 n^{17} \right) t^{42} + \left[ \frac{1}{3567393561} - \frac{589}{57078747291} \right] t^{44} \\
& - \frac{1}{14295083712800000000 n^{19}} t^{46} + \left( \frac{27997256387}{46959821284392360020000000 n^{19}} + \frac{3392780147}{395257562119053917503125 n^{19}} \right) t^{48} \\
& + \frac{1509737107292842107125000000 n^{21}}{395257562119053917503125 n^{21}} t^{50}
\end{align*}
\]

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