Scaling Analysis and Renormalisation
Group for General
(Quantum) Many Body Systems in the
Critical Regime

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Abstract

With the help of a smooth scaling and coarse-graining approach of ob-
servables, developed recently by us in the context of so-called fluctuation
operators (inspired by prior work of Verbeure et al) we perform a rigor-
ous renormalisation group analysis of the critical regime. The approach
is quite general, encompassing classical, quantum, discrete and continuous
systems, the main thrust going to quantum many body systems. Our cen-
tral topic is the analysis of the emergent properties of critical systems on
the intermediate scales and in the scaling limit. To mention some particu-
larly interesting points, we show that systems typically lose part of their
quantum character in the scaling limit (vanishing of commutators) and
we rigorously prove, with the help of the KMS-condition, the emergence
of the phenomenon of critical slowing down together with the necessity
of renormalising the time variable. These general features are then il-
lustrated with the help of an instructive class of models and are related
to the singular structure of quasi particle excitation modes for vanishing
energy-momentum.
1 Introduction

One of the central ideas of the renormalization group analysis of, for example, the critical regime, is scale invariance of the system in the scaling limit. This is the famous scaling hypothesis (as to the underlying working philosophy compare any good textbook of the subject matter like e.g. [1] and references therein). Central in this approach is the so-called blockspin transformation, [2]. That is, observables are averaged and appropriately renormalized over blocks of increasing size. At each intermediate scale a new effective theory is constructed and the art consists of choosing (or rather: calculating) the critical scaling exponents, so that the sequence of effective theories converge to a (scale invariant) limit theory, provided that the start theory lay on the critical submanifold in the (in general infinite dimensional) parameter space of theories or Hamiltonians.

We want to mention a slightly different approach to renormalisation (see for example [5], [6] or [7]), which is more in the spirit of renormalisation in quantum field theory. There exist a lot of cross relations but in the following we do not discuss these more technical aspects.

Usually the calculations can only be performed in an approximative way, the main tools being of a perturbative character and being typically model dependent. Frequently, the discussion relies on spin systems to motivate and explain the calculational steps. While the general working philosophy, based on the concepts of asymptotic scale invariance, correlation length and the like, is the result of a deep physical analysis of the phenomena, there is, on the other hand, no abundance of both rigorous and model independent results.

This applies in particular to the control of the convergence of the scaled $l$-point correlation functions to their respective limits if we start from a microscopic theory, lying on the critical submanifold. In this case, correlations are typically long-ranged and the usual heuristic arguments concerning the manipulations of expressions containing block variables of increasing size in the face of long range correlations among the blocks becomes rather obscure as one is usually cavalier as to the interchange of various limit procedures. One knows from examples, that this may be dangerous in such a context.

Furthermore, the clustering of the higher correlation functions in the various channels of phase space may be quite complex and non-uniform. A concise and self-contained discussion of the more general aspects and problems, lurking in the background together with a useful series of notes and references, can be found in [3], section 7.

Usually, the crucial scaling relation (the scaling hypothesis)

$$W^T_l(Lx_1, \ldots, Lx_l; \mu^*) = L^{-l\cdot n} \cdot L^{l \cdot \gamma} \cdot W^T_l(x_1, \ldots, x_l; \mu^*)$$  (1)

which is conjectured to hold at the fixed point (denoted by $\mu^*$ in the parameter space), is the starting point (or physical input) of the analysis. Here, $W^T_l$ denote the truncated $l$-point functions (see below), $L$ is the diameter of the blocks,
\(L x_i\) denote the centers of the blocks, \(n\) is the space dimension, \(\gamma\) the statistical renormalisation exponent. If it is different from \(n/2\) or, rather, the expected naive value, we have an ‘anomalous’ scale dimension (for convenience we have assumed all observables to scale with the same scale dimension).

In the following analysis, one of our aims is a rigorous investigation of such (and similar) scaling relations for the \(l\)-point functions, starting from the underlying microscopic characteristics of the theory. We will do this in a quite general manner, that is, the underlying model theory can be classical or quantum, discrete or continuous. We try to make only very few and transparent assumptions as it is our strategy, to deal only with the really characteristic (almost model independent) aspects of the subject matter. A central goal of our analysis is a rigorous discussion of a number of characteristic properties of both the intermediate and limit states, the observables and dynamics occurring on these levels of renormalisation etc., with special emphasis on the quantum aspects.

In section 2 we develop the conceptual framework and a variety of technical tools. As a technical side aspect we discuss the differences between our smooth scaling approach and the perhaps more common averaging over sharp blocks. In section 3 we show that classical continuum systems can be easily incorporated into our general scheme. In section 4 we rigorously study a large class of models which can be treated from a unified point of view. We exhibit the close connection between the critical exponents and the spectral properties of the correlation functions for vanishing energy-momentum. In section 5 we analyse characteristic properties of the system on the intermediate scales and in the scaling limit. Among other things we show that the system may lose some of its quantum character in the scaling limit (vanishing of commutators). As a particularly interesting result we provide a rigorous proof of the phenomenon called critical slowing down (based on the KMS-condition) together with a renormalisation of the characteristic time scale of the dynamical evolution (see section 5.4). We show, using the class of models studied in section 4, that the dynamic scaling exponent, occurring in the renormalisation of the time variable, is closely related to the energy-dispersion law of some quasi-particle excitation branch for vanishing momentum.

What regards the general working philosophy, one should perhaps mention the framework, expounded in e.g. [4] in the context of the analysis of the ultraviolet behavior in algebraic quantum field theory, or, in the classical regime, the approach of e.g. Sinai ([19]). While our framework also comprises the classical regime (cf. the discussion concerning classical continuous systems in section 3), it is mainly designed to deal with the more complicated quantum case. In so far, it is an extension of the methods, developed by us in [8], which, on their side, were inspired by prior work of Verbeure et al; see the corresponding references in [8]. Recently we became aware of a nice treatment of the block spin approach in the quantum regime in the book of Sewell ([13]), who employs methods which are different from ours, but are complementing them (quantum (non-) central
As there exist presumably several thousand papers in this field, we feel unable to relate our own approach to all the other approaches or to make a detailed analysis of what is entirely new. Our main thrust goes in the direction of a conceptual analysis and the development of a coherent and general point of view. In this respect we think, our presentation is reasonably self-contained and contains a number of original results. We briefly note that, in order to keep the paper within reasonable length, we chose to perform most of the long and intricate technical analysis of the scaling behavior of $l$-point correlation functions on the critical submanifold elsewhere. A preliminary treatment of this particular problem can be found in the second part of [30].

We end this introduction with mentioning a perhaps subtle point. In the following we concentrate most of our analysis on the hierarchy of correlation functions which can be used to define the theory. We generate renormalized limit correlation functions from them which happen to be scale invariant (in a sense clarified below), thus defining a new limit theory via a reconstruction process. On the other hand, we do not openly discuss the flow of, say, the renormalized Hamiltonians through parameter space as a sequence of more and more coarse-grained effective Hamiltonians. The characteristics of these renormalised intermediate theories are however implicitly given by their hierarchy of correlation functions as was already explained in e.g. [8] or [4].

One should therefore emphasize, that this well-known integrating out or decimation of degrees of freedom, which characterizes the ordinary approaches is automatically contained in our approach! The effective time evolution is carried over from the microscopic theory as described in [8] or (in a slightly other context) in [4], see also [9] and is redefined on each intermediate scale, thus implying automatically a rescaling or renormalisation of both the time evolution and the corresponding Hamiltonian; see section 5. In case we work in an scenario, defined by ordinary Gibbs states, our framework would exactly yield these effective Hamiltonians. Nevertheless, it is an interesting task, to apply our method directly to the microscopic Hamiltonian.

## 2 The Conceptual Framework

### 2.1 Concepts and Tools

As to the general framework we refer the reader to [8]. One of our technical tools is a modified (smoothed) version of averaging (modifications of the ordinary averaging procedure are also briefly mentioned in the notes in [3]). Instead of averaging over blocks with a sharp cut off, we employ a smoothed averaging with
smooth, positive functions of the type
\[ f_R(x) := f(|x|/R) \quad \text{with} \quad f(s) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| \geq 2 \end{cases} \]  \hfill (2)

Remark: We will see in the following, that the final result is more or less independent of the particular class of averaging functions!

We note that this class of scaled functions has a much nicer behavior under Fourier transformation, as, for example, functions with a sharp cut off, the main reason being that the tails are now also scaled. We have
\[ \hat{f}_R(k) = \text{const} \cdot R^n \cdot \hat{f}(R \cdot k) \]  \hfill (3)

One might perhaps think that this choice of averaging will lead (as a consequence of the scaled tails) to a limit theory, being different from one, constructed by employing a sharp cut-off. This is however not the case. As the mathematical differences between the two approaches, that is, using either sharp or smooth cut off functions, are technically a little bit subtle and perhaps not so apparent, we discuss some of the technical aspects below.

We briefly describe the implications coming of translation invariance. We have for the correlation functions
\[ W(x_1, \ldots, x_l) = W(x_1 - x_2, \ldots, x_{l-1} - x_l) \]  \hfill (4)

The truncated correlation functions are defined inductively as follows (see [8])
\[ W(x_1, \ldots, x_l) = \sum_{\text{part}} \prod_{P_i} W^T(x_{i_1}, \ldots, x_{i_k}) \]  \hfill (5)

the sum running over all partitions of the set \{x_1, \ldots, x_l\}. The (distributional) Fourier transform reads
\[ \hat{W}^T(p_1, \ldots, p_l) = \hat{W}^T(p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_{l-1}) \cdot \delta(p_1 + \cdots + p_l) \]  \hfill (6)

The dual sets of variables are
\[ y_i := x_i - x_{i+1} , \quad q_i = \sum_{j=1}^i p_j \quad i \leq (l - 1) \]  \hfill (7)

In contrast to the averaging procedure, introduced above, the usual block-variable-averaging is a sharp cutoff averaging, performed for example over balls, \( B_R \), of radius \( \hat{R} \). That is, observables are integrated over balls, \( B_R \), with the help of the incidence functions
\[ \chi_R(x) := \chi_1(|x|/R) \quad \text{with} \quad \chi_1(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases} \]  \hfill (8)
The averaging over operators, leading to the so-called block or fluctuation operators, is performed in the following way:

\[ A_R := (\text{const}) \cdot R^{-\gamma} \cdot \int A(x) \cdot f_R(x) d^n x \quad (9) \]

with the exponent \( \gamma \) suitably chosen and \( \text{const} \) being a possible numerical and unimportant multiplicative constant (related e.g. to the volume of the unit ball or something like that). A corresponding expression holds for \( \chi_R \) replacing \( f_R \).

Remark: Here and in the following, the \( A \)'s are always normalized to \( \langle A \rangle = 0 \), in order to really get the pure fluctuation effects.

At this point we want to state a general principle which allows to choose an appropriate scaling exponent, \( \gamma \). As in most of the discussions in the literature only a particular fixed field, \( \phi(x) \), or spin, \( S(x_i) \), is employed, it is frequently not clear that something has actually to be said in this context. This holds the more so as quite a few different renormalisation schemes (or philosophies) are used in practice, with the tacit understanding that the critical exponents are physically apriori given and insensitive to the concrete decimation procedure. This problem becomes, in our view, more virulent in the quantum regime with, usually, a lot of different non-commuting observables.

We think that, if we adopt a more general viewpoint, the necessary general principles become more apparent. This holds also for what we call a possible problem of consistency. This problem consists of the following points.

Observation 2.1

1. It is reasonable to choose the scaling exponents so that certain two-point auto-correlation functions of block observables survive in the scaling limit. Note that an inappropriate choice drives the auto-correlation functions either to zero or infinity!

2. The Cauchy-Schwarz inequality then guarantees that at least the corresponding mixed two-point functions remain finite in the limit.

3. On the other hand, this shows that one may have a certain freedom in selecting the correlation functions and, by the same token, the observables one wants to survive in the limit. This will of course affect the structure of the possible limit theories.

4. If, on the other hand, we have a lot of different (non-commuting) quantum observables together with their composites, it is presumably not an easy task to make all these (possibly independent) choices in a consistent way so that a coherent Hilbert space structure results in the limit. This
problem becomes virulent if we end up with a theory having non-vanishing higher truncated correlation functions. The reason is that possible obstructions may result from the decay behavior of higher n-point functions in the difference variables which has to be in complete balance with the chosen scaling exponents. These, on their side, are already fixed by the 2-point functions! We briefly discussed this issue in the last section of [8] and we make a more detailed analysis in the second part of our paper [30].

**Conclusion 2.2** We fix the renormalisation exponents, $\gamma_i$, of the respective observables via the non-vanishing and finiteness of (a class of) 2-point auto-correlation functions. This will yield constraints on the scaling behavior of higher correlation functions, the consistency of which we then can check.

In the following we will mainly employ the smooth cut-off procedure which leads to a more transparent behavior of various expressions in Fourier-space. It is satisfying that in the cases, we can actually control, it leads to results being identical to the version with sharp volume cut-offs. In order to compare these two cut-off conventions we study in a first step various peculiar properties of the averaging functions, $\chi_R(x)$. The Fourier transform of the smooth functions, $f_R(x)$, are again smooth, living in the Schwartz-space, $S$, i.e., decrease fast together with all their derivatives. In [8] we crucially employed $L^1$ or $L^2$ properties of various expressions. In contrast to $f_R(k)$, the $\hat{\chi}_R(k)$’s are no longer in $L^1$ as $\chi_R(x)$ has a jump discontinuity. On the other hand, it is in $L^2$ as

$$\int |\chi(x)_R|^2 d^n x = \int |\hat{\chi}_R(k)|^2 d^n k$$

We have the little lemma

**Lemma 2.3** The Fourier transform, $\hat{\chi}_R(k)$, is in $C^\infty \cap C_0$ but not in $L^1$. It is however in $L^2$. We have the same scaling behavior for $\hat{\chi}_R(k)$ as for $f_R(k)$, that is

$$\hat{\chi}_R(k) = \text{const} \cdot R^n \cdot \hat{\chi}_1(R \cdot k)$$

Proof: The first statement follows from the compact support of $\chi_R(x)$ and the Riemann-Lebesgue lemma. The second statement follows as in the smooth case. □

An explicit calculation for $n = 3$ yields:

$$\hat{\chi}_1(k) = \text{const} \cdot |k|^{-3} \cdot \int_0^{|k|} r \cdot \sin(r)dr$$

For $|k| \to 0$ the integral is proportional to $|k|^3$. Furthermore we can show that the expression is in fact infinitely differentiable in $|k| = 0$. For $|k| \to \infty$ a partial
integration yields an expression proportional to \(-|k| \cdot \cos(|k|) + \sin(|k|)\). That is, we have in leading order for \(|k| \to \infty\):

\[
\hat{\chi}_1(k) \sim |k|^{-2} \text{ for } n = 3 \tag{13}
\]

We mention some further peculiar properties of the indicator function, \(\chi_B(x)\), not shown by other functions. From \(\chi_B(x) = \chi_B(x) \cdot \chi_B(x)\) we infer for the Fourier transform

\[
\hat{\chi}_B(k) = \hat{\chi}_B \ast \hat{\chi}_B(k) \tag{14}
\]

and correspondingly for higher powers.

**Corollary 2.4** By Young’s inequality (see e.g. [29]) we know, that in general the convolution of \(L^2\)-functions is only in \(L^\infty\). The preceding formula shows that the convolution of \(\hat{\chi}_B(k)\) with itself is again in \(L^2\).

Note that such a result is not immediately evident from the concrete form of the respective Fourier transforms. In the case \(n = 1\) say, the Fourier transform is essentially of the form \(\sin(k)/k\). The result for the convolution comes about due to the peculiar oscillatory character of the expression and would not hold for e.g. \(|\hat{\chi}_1(k)|\). We will briefly analyse in the following subsection to what extent the renormalisation process is influenced by these slightly nasty features of sharp cut-off functions.

### 2.2 The case of Normal Fluctuations

As in [8], we assume that away from the critical point the truncated \(l\)-point functions are integrable, i.e. \(\in L^1(R^{n(l-1)})\), in the difference variables, \(y_i := x_i - x_{i+1}\). As observables we choose the translates

\[
A_R(a_1), \ldots, A_R(a_l), A_R(a) := R^{-n/2} \cdot \int A(x + a) f(x/R) d^n x \tag{15}
\]

(where, for convenience, the labels 1...\(l\) denote also possibly different observables). We then get (for the calculational details see [8], the hat denotes Fourier transform, translation invariance is assumed throughout, the \(\text{const} \) may change during the calculation but contains only uninteresting numerical factors):

\[
\langle A_R(a_1) \cdots A_R(a_l) \rangle^T = \text{const} \cdot R^{n/2}.
\]

\[
\int \hat{f}(Rp_1) \cdots \hat{f}(-R[p_1 + \cdots + p_{l-1}]) \cdot \hat{W}^T(p_1, \ldots, p_{l-1}) \cdot e^{-i \sum_{i=1}^{l-1} p_i a_i} \cdot e^{ia_i \sum_{i=1}^{l-1} p_i} \prod dp_i = \text{const} \cdot R^{(l-1)n}.
\]

\[
\int \hat{f}(p_1) \cdots \hat{f}(-[p_1' + \cdots + p_{l-1}']) \cdot \hat{W}^T(p_1'/R, \ldots, p_{l-1}'/R) \cdot e^{-i \sum_{i=1}^{l-1} (p_i'/R) a_i} \cdot e^{ia_i \sum_{i=1}^{l-1} p_i'/R} \prod dp_i'
\]

\(\tag{16}\)
We now scale the $a_i$’s like

$$a_i := R \cdot X_i, \ X_i \text{ fixed} \quad (17)$$

This yields

$$\langle A_R(R \cdot X_1) \cdots A_R(R \cdot X_l) \rangle^T = \text{const.} \cdot R^{(2-l)n/2} \cdot \int e^{-i \Sigma_1^{l-1} p'_i X_i} \cdot e^{i X_i \Sigma_1^{l-1} p_i} \cdot \hat{f}(p'_1) \cdots \hat{f}(-[p'_1 + \cdots + p'_{l-1}]) \cdot \hat{W}^T(p'_1/R, \ldots, p'_{l-1}/R) \prod dp'_i \quad (18)$$

As the $\hat{f}$ are of strong decrease and $\hat{W}^T$ continuous and bounded by assumption ($\hat{W}^T \in L^1(\mathbb{R}^{n(l-1)})$!), we can perform the limit $R \to \infty$ under the integral (Lebesgue’s theorem of dominated convergence) and get:

Case 1 ($l \geq 3$):

$$\lim_{R \to \infty} \langle A_R(R \cdot X_1) \cdots A_R(R \cdot X_l) \rangle^T = 0 \quad (19)$$

Case 2 ($l = 2$):

$$\lim_{R \to \infty} \langle A_R(R \cdot X_1)A_R(R \cdot X_2) \rangle^T = \text{const.} \cdot \int \hat{W}^T(0) \cdot e^{-ip'_1(X_1 - X_2)} \cdot \hat{f}(p'_1) \cdot \hat{f}(-p'_1) dp'_1 \quad (20)$$

We arrive at the conclusion

**Conclusion 2.5** Assuming $L^1$-clustering in the normal regime away from the critical point and employing a smooth cut-off, all the truncated correlation functions vanish in the limit $R \to \infty$ apart from the 2-point function. We hence have a quasi free theory in the limit as described in [8] or in the work of Verbeure et al (cf. the references in [9]).

In the case of smooth averaging we employ the transparent behavior of the Fourier transformed expressions. On the other hand, the Fourier transform is inherently non-local, which sometimes makes the analysis more complicated. When using instead the sharp cut-off convention, we described above, the behavior of the respective Fourier transforms becomes opaque in the general case. On the other hand, we can try to stay in coordinate space and perform the analysis there. Proceeding as in the smooth case but avoiding Fourier transformation we get after some straightforward manipulations

$$\langle A_R(RX_1) \cdots A_R(RX_l) \rangle^T = R^{-ln/2} \cdot R^n \cdot \int W_i^T(y'_1, \ldots, y'_{l-1}) \cdot \chi_1 \left( \sum_{j=1}^{l-1} R^{-1} y'_j + x_i'' - \sum_{j=1}^{l-1} Y_j - X_l \right) \cdots \chi_1(x_i'' - X_l) dy'_1 \cdots dx''_l \quad (21)$$

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with $x_i - x_l = \sum_{i}^{l-1} y_j$ by the definition in the preceding subsection and

$$x_i' = x_i + RX_i, \quad x_i' - x_l' = \sum_{i}^{l-1} y_j', \quad x_l'' := x_l'/R$$

(22)

Again, the limit can be performed under the integral and is zero for $l > 2$. For the two-point function we get

$$\lim_{R \to \infty} \langle A_R(RX_1) \cdot A_R(RX_2) \rangle = \int W_T^{(y)} dy \cdot \int \chi_1(x - Y)\chi_1(x) dx$$

(23)

with $Y := X_1 - X_2$. We hence conclude:

**Conclusion 2.6** In the normal situation of $L^1$-clustering of correlation functions and sharp cut-off functions, $\chi_R(x) = \chi_1(x/R)$, we get the same results as in the case of smooth cut-off functions. However, to prove this, we have to perform the analysis in real space and avoid Fourier transformation.

**Corollary 2.7** It is obvious from the preceding discussion that the particular form of the averaging functions, $f_R(x) = f(x/R)$, need not even simulate a volume averaging. For the argument to hold, it is e.g. sufficient that $f(x)$ is bounded with $\hat{f}(0) \neq 0$ and has compact support. What only changes is an unimportant multiplicative factor, depending on the type of function, being chosen.

### 2.3 The Relation to the Heuristic Scaling Hypothesis

In the following sections we develop a rigorous approach to block-spin renormalisation in the realm of quantum statistical mechanics, which tries to implement the physically well-motivated but, nevertheless, to some extent heuristic scaling hypothesis. The analysis will be performed both in coordinate space and Fourier space. In this subsection we restrict our discussion to the two-point correlation function, for which the asymptotic behavior is simpler and more transparent.

**Remark:** In the rest of the paper we replace the exponent $n/2$ in the definition of $A_R(a)$ by a scaling exponent $\gamma'$, which will usually be fixed during or at the end of a calculation. It plays the role of a critical scaling exponent and is model dependent.

Let us hence study the behavior of

$$\langle A_R(R \cdot X_1)A_R(R \cdot X_2) \rangle = R^{-2\gamma'} \cdot \int W_T((x_1 - x_2) + R(X_1 - X_2))$$

$$\cdot f(x_1/R)f(x_2/R)dx_1dx_2$$

$$= R^{-2\gamma'+2n} \int W_T(R[(x_1 - x_2) + (X_1 - X_2)]) \cdot f(x_1)f(x_2)dx_1dx_2$$

(24)
We make the physically well motivated assumption that, in the critical regime, $W_T$ decays asymptotically like some inverse power, i.e.

$$W_T(x_1-x_2) \sim (\text{const} + F(x_1-x_2)) \cdot |x_1-x_2|^{-(n-\alpha)} \quad 0 < \alpha < n \quad F(x) \in L^1$$

for $|x_1-x_2| \to \infty$, $F$ bounded and well-behaved.

From the last line of (24) we see that, as $f$ has compact support, we can replace $W_T$, for $(X_1-X_2) \neq 0$ and $R \to \infty$ by its asymptotic expression and get for $R$ large:

$$\langle A_R(R \cdot X_1)A_R(R \cdot X_2) \rangle_T \approx \text{const} \cdot R^{-2\gamma' + 2n} \cdot R^{-(n-\alpha)} \cdot \int |y+Y|^{-(n-\alpha)} \cdot f \ast f(y)dy$$

(26)

We choose now

$$\gamma' = (n+\alpha)/2$$

(27)

and get a limiting behavior (for $R \to \infty$) as

$$\text{const} \cdot \int |y+Y|^{-(n-\alpha)} \cdot f \ast f(y)dy$$

(28)

with $y = x_1-x_2, Y = Y_1-Y_2$ and

$$f \ast f(y) := \int f(y+x_2) \cdot f(x_2)dx_2$$

(29)

We see that in contrast to some of the general folklore, the limit correlation functions are not automatically strictly scale invariant but still depend in the above integrated manner on the chosen smearing functions, $f$. Full scale invariance is recovered in the regime $Y \to \infty$. Central in the renormalisation group idea is that systems on the critical surface (i.e., critical systems) are driven towards a fixed point, representing a scale invariant theory. This idea is usually formulated in an abstract parameter space of, for instance, Hamiltonians. In our correlation function approach scale invariance at the presumed fixed point would prove its existence via the scaling properties of the correlation functions, that is

$$W_2^T(L \cdot (X-Y); \mu^*) = L^{-2(n-\gamma')}W_2^T(X-Y; \mu^*)$$

(30)

with $\mu^*$ describing the fixed point in the (usually) infinite dimensional parameter space. We see from the above that this picture is asymptotically implemented by our above limiting correlation functions, as we have (with the choice $\gamma = (n+\alpha)/2$):

$$W_2^T(X-Y; \mu^*) \sim |X-Y|^{-(n-\alpha)}$$

(31)

in the asymptotic regime. That is, the above scaling limit leads to a limit (i.e. fixed point) theory, reproducing the asymptotic behavior of the original (microscopic) theory.

One should however note that in the more general situation of $l$-point correlation functions we have to expect a more complex decay behavior and the existence of various channels as varying clusters of observables move to infinity.
2.4 Strategies for a Renormalisation Analysis on the Critical Surface

Typically, the numerical scaling analysis is developed for the system being away from the critical surface. The reason is that away from criticality, under the heuristic assumption of e.g. exponential clustering, the analysis is not beset with technical difficulties as, for example, the interchange of limits and dealing correctly with long range tails in correlation functions. It is then frequently argued that, in case the system is sufficiently near to the critical surface, the orbits of renormalized model systems nevertheless will approach the vicinity of the fixed point, so that one can make a linear stability analysis of eigenvalues of the renormalisation group around the fixed point. The philosophy is that these systems will ultimately leave the vicinity of the fixed point.

In the second part of [30] (see also the last section of [8]) we undertook to sketch a rigorous renormalisation framework for systems, lying on the critical surface. Due to the inherent long-range correlations, one must be extremely careful in performing such an analysis. As such a rigorous analysis is both technically demanding and a little bit tedious and incorporates a variety of interesting mathematical side aspects like e.g. a singularity analysis of distributions and pseudo differential operators, we decided to separate this rather technical investigation off and give only a brief discussion of one of the methods in this subsection, which we exemplify with the help of the pair correlation function.

The general idea is it, to extract and isolate the characteristic singular behavior of the correlation functions which is responsible for the weak decay of correlation. With $W_T(x)$ the truncated two-point function, we, making the preceding analysis more rigorous, assume the existence of a certain exponent, $\alpha$, so that ($x^2$ denoting the vector-norm squared) we can make the following decomposition.

$$G(x) := W_T(x) \cdot (1 + x^2)^{(n-\alpha)/2} = \text{const} + F(x)$$

with a decaying (non-singular) $F$ which is assumed to be in $L^1$. Fourier transformation then yields:

$$R^{-2\gamma} \cdot \int W_T^2((x_1 - x_2) + R(X_1 - X_2)) f(x_1/R) f(x_2/R) dx_1 dx_2$$

$$= R^{-2\gamma} \cdot \int G((x_1 - x_2) + R(X_1 - X_2)) \cdot [1 + ((x_1 - x_2) + R(X_1 - X_2))^2]^{-(n-\alpha)/2} \cdot f(x_1/R) f(x_2/R) dx_1 dx_2$$

$$= R^{-2\gamma} \cdot R^{2n-(n-\alpha)} \cdot \int dp \hat{G}(p) \cdot e^{-iRp(x_1-x_2)} \cdot \left[ \int e^{-iRp(x_1-x_2)} (R^{-2} + ((x_1 - x_2) + (X_1 - X_2))^2)^{-(n-\alpha)/2} f(x_1) f(x_2) dx_1 dx_2 \right]$$

(33)
where we made the substitution \( x \to R \cdot x \).

As the support of \( f \) is in principle arbitrary, we now assume it to be contained in a sufficiently small ball around zero (or, alternatively, \((X_1 - X_2)\) sufficiently large so that \((x_1 - x_2) + (X_1 - X_2) \neq 0\) for \( x_i \) in the support of \( f \)). With

\[ \hat{G}(p) = \text{const} \cdot \delta(p) + \hat{F}(p) \quad (34) \]

the leading part in the scaling limit \( R \to \infty \) is the \( \delta \)-term. Asymptotically we hence get for \( R \to \infty \) (setting \( y := x_1 - x_2 \) \( Y := X_1 - X_2 \)) the result, already conjectured in the preceding subsection:

\[ R^{n+\alpha-2\gamma} \cdot \text{const} \cdot \int |y + Y|^{-(n-\alpha)} \cdot f \ast f(y) dy \quad (35) \]

with

\[ f \ast f(y) := \int f(y + x_2) \cdot f(x_2) dx_2 \quad (36) \]

and \( y + Y \neq 0 \) on \( \text{supp}(f) \).

The reason why the contribution, coming from \( \hat{F}(p) \), can be neglected for \( R \to \infty \) is the following: \( f \) is assumed to be in \( \mathcal{D} \); by assumption the prefactor never vanishes on the support of \( f(x_i) \). Hence the whole integrand in the expression in square brackets is again in \( \mathcal{D} \) and therefore its Fourier transform, \( \hat{g}(p') \), is in \( \mathcal{S} \) (with \( p' := Rp \)), that is, of rapid decrease. We can therefore perform the \( R \)-limit under the integral and get a rapid vanishing of the corresponding contribution in \( R \) for \( R \to \infty \).

\[ \lim_{R \to \infty} R^{-n} \cdot \int \hat{F}(p'/R) \cdot e^{-iy'Y} \cdot \hat{g}(p') d^n p' = 0 \quad (37) \]

As \( f \ast f \) has again a compact support, we have that, choosing

\[ \gamma = (n + \alpha)/2 \quad (38) \]

the limit correlation function behaves as \( \sim |X_1 - X_2|^{-(n-\alpha)} \) as in the more heuristic analysis heuristic analysis of the preceding subsection.

Along these lines, or choosing a slightly different method (see [30]), one can proceed in the more difficult case of \( l \)-point functions. In the course of this analysis an interesting phenomenon does pop up which leads to some remarkable constraints as to the consistency of the whole renormalisation picture. The critical exponents are typically fixed by the assumed non-vanishing of the scaled auto-correlation functions. On the other hand, the truncated \( l \)-point functions may have a much more intricate cluster behavior (having, in particular, a variety of decay channels). If one wants to go beyond quasi-free limit theories, some higher truncated correlation functions have to be non-vanishing in the scaling limit. For this to be the case, there has to be some fine-tuning between their decay behavior and the values of the critical exponents, which one got from the two-point functions.
3 Some Remarks on Classical Statistical Systems

In this section we want to briefly indicate how our framework can be implemented in the regime of classical statistical mechanics. The situation is more or less obvious in the class of spin- or lattice-systems. The translation group is replaced by some discrete lattice group. The Fourier vectors run through some Brillouin zone instead of $\mathbb{R}^n$, while in coordinate space we employ the same kind of smearing and averaging functions as in the continuous case, the only difference being the replacement of integrals by sums. For continuous classical KMS-systems, some more words are perhaps in order (cf. e.g. [20], [21], [22], [23], [24]).

As infinitely extended phase space, $X$, we take the set of sequences, $x$, of points

$$x = (r_i, p_i)_{i=1}^{\infty} = (x_i)_{i=1}^{\infty}$$

having the local finiteness property, i.e., the number of points, $r_i$, occurring in $x$, is finite in each bounded set, $V \subset \mathbb{R}^n$.

As local $m$-particle observables, $A^{(m)}$, we take

$$A^{(m)}(x) := \sum_{i_1 < \cdots < i_m} f^{(m)}(x_{i_1}, \ldots, x_{i_m})$$

$f$ from the class of smooth function with compact support in coordinate space (the details are of course a matter of convenience). Poisson brackets can then be defined as usual and the local finiteness property guarantees that the expression

$$\{A, B\}(x) := \sum_{j=1}^{\infty} (\partial A/\partial r_j \cdot \partial B/\partial p_j - \partial A/\partial p_j \cdot \partial B/\partial r_j)$$

is well-defined.

The thermodynamic equilibrium states are now probability measures on the Borel-$\sigma$-field, defined on the phase space equipped with the topology canonically induced by the class of observables. The classical KMS-condition we usually employ in the form ($A, B$ real):

$$\langle \{A, B\} \rangle = \beta \langle B \{A, H\} \rangle$$

$H$ being the Hamiltonian.

As in quantum statistical mechanics, we can define certain distributional point fields or densities at, say, coordinate $r$ over the phase space, like e.g. particle density:

$$n_r(x) := \sum_i \delta(r - r_i)$$
momentum density:

\[ p_r(x) := \sum_i p_i \cdot \delta(r - r_i) \] (45)

energy density, stress tensor density etc. (see in particular [23] and [24] where these notions have been systematically employed).

The \textit{l-point distribution functions} can hence be expressed as follows:

\[ \rho^{(l)}(r_1, \ldots, r_l) := \sum_{i_1 < \cdots < i_l} \langle \delta(r_1 - r_{i_1}) \cdots \delta(r_l - r_{i_l}) \rangle \] (46)

\(i_\nu\) running through the indices occurring in \(x\). Ordinary observables can be reconstructed by integrating these densities over local test functions. For a one-particle observable we have for example:

\[ A_f := \int a(r) \cdot f(r) d^m r \] (47)

with \(a(r)\) a one-particle density and correspondingly for more complex densities.

From these remarks one sees immediately, that the whole procedure, we develop in the following, can be immediately transferred to the regime of classical statistical mechanics without significant changes.

4 A Class of Examples

We argued that in the case of poor, that is, non-integrable clustering, it appears to be mathematically more reasonable to perform most of the necessary analysis in coordinate space, as the behavior in Fourier space may be quite involved in the vicinity of \((\omega, k) = (0, 0)\).

The situation improves however if one has a more precise knowledge of the form of correlation functions in Fourier space near \((0, 0)\). We note in passing that our approach is by no means restricted to the case of critical systems. It does also apply to systems at zero temperature or systems above or below a phase transition line. One may have more precise information in Fourier space in various situations like e.g. spontaneous symmetry breaking (see [25] and [8] and further literature given there) or for particular correlation functions and/or commutators (so-called \textit{sum-rules}). The relevant contribution in e.g. the 2-point function can stem from sharp excitation branches or excitations, having a finite lifetime, which is the typical situation in interacting many-body systems.

Remark: The assumptions in the following discussion can be considerably weakened and are only made to cover a sufficiently general and coherent class of models.

In order to better understand the effects of our general scaling approach, we deal in this section with a fairly large class of relatively manageable and simple
models at non-zero temperature which belong to the group of *quasifree systems*. Note however that in contrast to, say, relativistic quantum field theory, we have in general no strong covariance properties. That is, even quasifree systems are not completely uninteresting and supply us with a whole bunch of useful model systems approximating important non-trivially interacting systems. As this notion slightly varies from author to author, we make the following assumptions.

**Assumption 4.1** Our class of models is assumed to have the following properties (in addition to the usual standing assumptions, we do not repeat here; see e.g. [16])

The KMS-representations, $\pi_\beta$, of the quasi-local algebra $(\mathcal{A}, \alpha_{t,x})$ are assumed to be quasifree and faithful, that is

1. All n-point functions are products of 2-point functions.
2. $\pi_\beta(A) \neq 0$ if $A \neq 0$ in $\mathcal{A}$.

The second assumption seems to be physically reasonable (and can in principle be weakened) as it avoids redundancies but need not be fulfilled in general. This situation occurs of course when the original algebra has a non-trivial center and one studies representations which are factors, in which central elements are mapped onto c-numbers. Note that $\omega_\beta$, the KMS-state, is always faithful (that is, separating) in $\pi_\beta(\mathcal{A})''$ (the GNS-representation), however this need not be the case with respect to $\mathcal{A}$ itself.

This point is relatively subtle from a more physical point of view and not much seems to be known. There is a discussion in [16], p. 85ff. which is based on the weak closure, $\mathcal{A}''$ of the original algebra. But, typically, an equilibrium state is given via its local restrictions in form of Gibbs-states, that is, it is naturally only defined on quasi-local elements of the algebra and not on the weak closure. If $\mathcal{A}$ is *simple*, the representation is *faithful*. (For an example of a non-faithful representation see [20]). Note that the so-called *order parameters*, the non-vanishing of which usually signal the occurrence of new phases, are typically global “observables” (for example, meanvalues, not belonging to the algebra of quasi-local observables) and are c-numbers in pure phases, i.e. factor states.

The above assumptions have both a simple technical consequence and a consequence which is perhaps remarkable from a more physical point of view.

**Lemma 4.2** Under the assumptions being made the commutators in each representation, $\pi_\beta$, are c-numbers which do not depend on the KMS-state, that is, in contrast to the 2-point functions, they are state-independent.

Proof: i) The c-number property follows immediately from the vanishing of all higher truncated correlation functions and is in fact independent of the other
assumptions. With \(\text{span}(\pi_\beta(A)\Omega_\beta)\) being dense in the GNS-Hilbert space and
\[
(\Omega_\beta, \pi_\beta(A) \cdot [\pi_\beta(B(x,t)), \pi_\beta(C)] \cdot \pi_\beta(A')\Omega_\beta)
\]
being a sum of 2-point functions, this expression can be shown to be equal to
\[
(\Omega_\beta, [\pi_\beta(B(x,t)), \pi_\beta(C)]\Omega_\beta) \cdot (\Omega_\beta, \pi_\beta(A) \cdot \pi_\beta(A')\Omega_\beta)
\]
\[(48)\]
i) The faithfulness of \(\pi_\beta\) implies (with \(C_{\beta BC}(x,t)\) a function, which follows from i))
\[
C_{\beta BC}(x,t) = [\pi_\beta(B(x,t)), \pi_\beta(C)] = \pi_\beta([B(x,t), C]) = c_{BC}(x,t) = [B(x,t), C]
\]
\[(50)\]
with \(c_{BC}(x,t) = C_{\beta BC}(x,t)\) being independent of the concrete KMS-representation as
\[
\pi_\beta([B(x,t), C] - C_{\beta BC}(x,t) \cdot \mathbf{1}) = 0
\]
\[(51)\]
The physical relevance of the above observation is the following. With
\[
\hat{F}_{\beta AB}(\omega, k) := (\Omega_\beta, \pi_\beta(A)(x,t) \cdot \pi_\beta(B)\Omega_\beta)^T
\]
\[(52)\]
and
\[
C_{\beta AB}(x,t) := (\Omega_\beta, [\pi_\beta(A)(x,t), \pi_\beta(B)]\Omega_\beta)
\]
\[(53)\]
we have the general expression for the respective Fourier transforms
\[
\hat{F}_{\beta AB}(\omega, k) = (1 - \exp(-\beta\omega))^{-1} \cdot \hat{C}_{\beta AB}(\omega, k)
\]
\[(54)\]
Usually, both \(\hat{F}\) and \(\hat{C}\) depend on the parameters, fixing the KMS-state. Our assumptions guarantee that for our model class the temperature dependence on the rhs is entirely concentrated in the prefactor, \((1 - \exp(-\beta\omega))^{-1}\), that is, we have

**Corollary 4.3** For our model class it holds
\[
\hat{F}_{\beta AB}(\omega, k) = (1 - \exp(-\beta\omega))^{-1} \cdot \hat{c}_{AB}(\omega, k)
\]
\[(55)\]
with \(\hat{c}_{AB}(\omega, k)\) temperature independent.

As \(\hat{c}_{AB}(\omega, k)\) is universal, it is typically simple to calculate; use e.g. some ground state representation.

For the further analysis we choose \(A, B\) selfadjoint and get for \(\hat{C}(\omega, k)\) (we supress the labels \(A, B\)):
\[
\hat{C}(\omega, k) = \hat{F}(\omega, k) - \hat{F}(-\omega, -k) = (1 - \exp(-\beta\omega)) \cdot F(\omega, k)
\]
\[(56)\]
and hence
\[ Re \hat{F}(-\omega, -k) = \exp(-\beta \omega) \cdot Re \hat{F}(\omega, k) \] (57)
\[ Im \hat{F}(-\omega, -k) = -\exp(-\beta \omega) \cdot Im \hat{F}(\omega, k) \] (58)
thus clearly exhibiting the two-sidedness of the ($\omega, k$)-spectrum in temperature states.

As, in contrast to the relativistic context (cf. e.g. [27] or [26]), we have in general no strong covariance and/or spectrum conditions for the 2-point functions, we have to make some reasonable assumptions which are fulfilled in typical many-body systems (for more details see [25] and [28]).

**Assumption 4.4** We assume that the excitation spectrum of $\hat{F}(\omega, k)$ fulfills
\[ \hat{F}(\omega, k) = \hat{F}(\omega, -k) \] and contains a sharp excitation branch ($e(k) = e(|k|)$), describing stable quasi particles or collective excitations, with the remaining part being integrable and absolutely continuous around ($\omega, k$) = (0, 0). We denote the singular contribution by
\[ \hat{F}_{\text{sing}}(\omega, k) := J_+^\beta(k) \cdot \delta(\omega - (e(k) - \mu)) + J_-^\beta(k) \cdot \delta(\omega + (e(k) - \mu)) \] (59)
Remark: Note that in the translation invaraint case the above Fourier transforms are measures!

From the above relations we conclude that
\[ Re J_-^\beta(k) = \exp(-\beta(e(k) - \mu)) \cdot Re J_+^\beta(k) \] (60)
\[ Im J_-^\beta(k) = -\exp(-\beta(e(k) - \mu)) \cdot Im J_+^\beta(k) \] (61)
with $\mu$ the (temperature dependent; in case temperature and density are chosen as independent parameters) chemical potential. We arrive at the following result:

**Lemma 4.5** We have
\[ J_-^\beta(k) = \exp(-\beta(e(k) - \mu)) \cdot J_+^\beta(k) \] (62)
and
\[ \hat{C}_{\text{sing}}(\omega, k) = (1 - \exp(-\beta(e(k) - \mu))) \cdot J_+^\beta(k) \delta(\omega - (e(k) - \mu)) \]
\[ - (1 - \exp(-\beta(e(k) - \mu))) \cdot J_-^\beta(k) \delta(\omega + (e(k) - \mu)) \] (63)
As $\hat{C}_{\text{sing}}(\omega, k)$ has to be independent of $\beta$ for our class of models, we have furthermore
\[ J_+^\beta(k) = (1 - \exp(-\beta(e(k) - \mu)))^{-1} \cdot j(k) \] (64)
Proof: This follows directly from the preceding formulas.

As our commutator function is universal, it should not contain the typical singularities which show up in connection with phase transitions and critical phenomena. As to this point we refer to the discussion in e.g. [25] and [28]. These phenomena are typically representation dependent. Therefore, on physical grounds, the function \( j(k) \) should be bounded near \( k = 0 \) and \( e(k) \) can be identified with the dispersion law of an elementary excitation which, in the non-relativistic context, for short-range interactions, passes through zero for \( k \to 0 \).

Remark: In [25] we discussed various dispersion laws. Frequently a simple power law behavior prevails.

We now apply our scaling procedure to the class of model systems described above. In a first step we want to choose the scaling exponent, \( \gamma = \gamma_A \), in the expression

\[
A_R = R^{-\gamma} \cdot \int A(x + RX) \cdot f_R(x) \, d^n x
\]

so that the corresponding autocorrelation function (remember the standing assumption \( \langle A \rangle = 0 \))

\[
\langle A_R(RX_1) \cdot A_R(RX_2) \rangle
\]

is both finite and non-vanishing in the limit \( R \to \infty \).

Our above made observations or assumptions about the spectrum of the 2-point functions show that, provided we have a more detailed knowledge of the system under discussion, we can, even in the case of long-range correlations, perform the analysis in Fourier space getting

\[
\langle A_R(RX_1) \cdot A_R(RX_2) \rangle = \text{const} \cdot R^{-2\gamma + n} \cdot \int \exp(-ik'(X_1 - X_2)) \cdot \hat{f}(k') \hat{f}(-k') \cdot \hat{F}_{AA}(\omega, k'/R)d\omega dk'
\]

with \( k' := Rk_1 \) and \( \hat{f}(k') \hat{f}(-k') = |\hat{f}(k')|^2 \) for \( f(x) \) symmetric and real.

In the following we are concerned with the renormalisation of the singular part of the spectral contribution as the absolutely continuous part is (by assumption) harmless. For \( \omega \geq 0 \) we have to consider the term

\[
\lim_{R \to \infty} I_R := \lim_{R \to \infty} R^{-2(\gamma - n/2)} \cdot \int \exp(ik(X_1 - X_2)) \cdot (1 - \exp(-\beta(e(k/R) - \mu)))^{-1} \cdot j(k/R) \cdot |\hat{f}(k)|^2 d^n k
\]

We do not intend to discuss the mathematically most general case but rather concentrate on situations which are reasonable from a physical point of view.
**Assumption 4.6** We assume that in leading order $e(k)$ behaves like

$$e(k) \sim |k|^\alpha \quad \text{for} \quad |k| \to 0$$

(69)

with $\alpha > 0$.

There is the possibility that $j(0)$ is finite but non-vanishing or that $j(k)$ vanishes for $k \to 0$. We begin with the discussion of the case of non-vanishing $j(0)$.

I) $e(k) \sim |k|^\alpha$ near $k = 0$, $j(k)$ continuous and $\neq 0$ in $k = 0$

**Observation 4.7** i) For $\mu \neq 0$ nothing peculiar happens and we are in the normal situation with $\gamma = n/2$.

ii) For $\mu = 0$, the typical situation at or below the critical point, we have the following behavior

$$I_R \sim R^{-2(\gamma-(n+\alpha)/2)} \quad \text{for} \quad R \to \infty$$

(70)

hence, the anomalous scaling dimension is

$$\gamma = (n + \alpha)/2$$

(71)

with $\alpha$ the exponent in the dispersion law of the sharp elementary excitation mode.

II) $j(k)$ vanishing in $k = 0$

This situation is by no means entirely exceptional. Take for example the time derivative at $t = 0$ of the observable $A$. In the spectrum of the autocorrelation function this leads to an additional prefactor, $\omega^2$, in front of $\hat{F}_{AA}(\omega, k)$. In the singular contribution, $J_+(k)$, this results in an additional factor, $e(k)^2$, and hence in an additional contribution in the scaling exponent

$$\gamma_{\partial A} = n/2 + \alpha/2 - \alpha = \gamma_A - \alpha$$

(72)

**Observation 4.8** If one wants $\partial_t A$ to be a non-vanishing observable in the scaling limit, its scale dimension has to be chosen as

$$\gamma_{\partial_t A} = \gamma_A - \alpha$$

(73)

Similar considerations have to be made for other functions of elementary observables. If the spectrum is known qualitatively as in our case, this can in fact be done in every concrete case. Note furthermore that the temperature independence of the commutator is technically convenient but not absolutely necessary. The same conclusions do hold if the spectral weight along the sharp excitation
branch is temperature dependent. However, in that case we do not have an apri-
or knowledge as to its precise form which may vary with $\beta$. One can also treat
the case of excitations having a finite lifetime (cf. [25]). The excitation branch
now has a finite width and the calculations become even more model dependent.
On the other side we proved in [28] that for $\beta \neq 0$ sharp excitation branches
typically belong to elementary excitations having no interaction with the rest of
the system.

**Remark 4.9** As to interesting consequences concerning the fate of commutators
(i.e. the quantum nature) in the scaling limit see the discussion in subsection 5.3.

5 Rigorous Results on the (Quantum) System
in the Intermediate Regime and in the Scaling
Limit

In this section we assume that the theory exists in the scaling limit provided
that the scaling exponents have been appropriately chosen. Under this proviso
we investigate its algebraic and dynamical limit structure.

5.1 The Description of the System at Varying Scales

In algebraic statistical mechanics we describe a system with the help of an observ-
able algebra, $\mathcal{A}$, a state, $\omega$, or expectation functional, $\langle \circ \rangle$, a time evolution,$\alpha_t$. Frequently one also employs the GNS-Hilbert space representation of the
theory, introduced by Gelfand, Naimark, Segal (see e.g. [12]). We already gave
a brief discussion of these points in [8]. But as the approach of the scaling limit
is quite subtle both physically and mathematically, we would like to give a more
complete discussion of some of the topics in the following.

We begin with fixing the notation and introducing some technical and concep-
tual tools. Expectation values of elements of the underlying observable algebra,
$\mathcal{A}$, at scale “0”, are given by

$$\omega(A(1) \cdots A(l)) =: \langle A(1) \cdots A(l) \rangle$$  \hspace{1cm} (74)

where different indices may denote different elements of the algebra, different
times etc. The dynamics is denoted by

$$\alpha_t(A) = A(t) \text{ or } A_t, \ t \in \mathbb{R}$$  \hspace{1cm} (75)

space translations by

$$\alpha_x(A) = A(x) \text{ or } A_x, \ x \in \mathbb{R}^n$$  \hspace{1cm} (76)
\[ \alpha_{t,x}(A) = A(t,x) \]  

(77)

Given such a structure, we can construct a corresponding Hilbert space representation (for convenience, we use the same symbols for the elements of the original algebra and their representations in the GNS-representation).

\[ \omega \rightarrow \Omega, \quad \omega(A(1) \cdots A(l)) = (\Omega|A(1) \cdots A(l)\Omega)_{GNS} \]  

(78)

\[ \alpha_t \rightarrow U_t, \quad \text{with} \quad \alpha_t(A) \rightarrow U_t \cdot A \cdot U_{-t} \]  

(79)

The averaged or renormalized observables, \( A \rightarrow A_R \), at scale \( R \) are a subset of elements contained in the original algebra, \( \mathcal{A} \). We denote the subalgebra, generated by these elements, by \( \mathcal{A}_R \) with \( \mathcal{A}_R \subset \mathcal{A} \). We can decide to forget the finer algebra, \( \mathcal{A} \), and define the algebra on scale \( R \):

**Definition 5.1** We define the system on scale \( R \) by

\[ \omega^{(R)}(A^{(R)}) := \omega(A_R) \]  

(80)

\[ \alpha^{(R)}_t(A^{(R)}) := (\alpha_t(A))^{(R)} \]  

(81)

\[ \alpha^{(R)}_X(A^{(R)}) := (A(RX))^{(R)} \]  

(82)

more specifically, we define the objects on the lhs implicitly (via the GNS-reconstruction) by the following correspondence

\[ \langle A^{(R)}(t_1, X_1) \cdots A^{(R)}(t_l, X_l) \rangle^{(R)} := \langle A_R(t_1, RX_1) \cdots A_R(t_l, RX_l) \rangle \]  

(83)

Remark: Note the different treatment of time and space-translations. We will come back to this point (which has remarkable physical consequences) below in connection with critical slowing down.

**Theorem 5.2** From the above we see that on each scale we have a new theory, \( \mathcal{S}^{(R)} \) (\( \mathcal{S} \) standing for “system”), which we get by reconstruction from the above hierarchy of correlation functions, in particular, a new, non-isomorphic algebra, \( \mathcal{A}^{(R)} \), and a corresponding GNS-Hilbert space representation. We emphasize that the coarse-grained dynamics is also physically different (despite the seeming similarity of the expressions on both sides of the above definitions).

If the scaling limit does exist, we have, by the same token, a scaling limit system denoted by

\[ \mathcal{S}^\infty = (\omega^\infty, \mathcal{A}^\infty, \alpha_t^\infty, \alpha_X^\infty) \]  

(84)

with

\[ \langle A^\infty(t_1, X_1) \cdots A^\infty(t_l, X_l) \rangle = \lim_{R \to \infty} \langle A_R(t_1, RX_1) \cdots A_R(t_l, RX_l) \rangle \]  

(85)
The proof is more or less obvious from what we have said above.

\[ \square \]

**Corollary 5.3** We generally assume that \( \alpha_t \) is strongly continuous on \( \mathcal{A} \). By the above identification process we can immediately infer that both \( \alpha_t^{(R)} \) and \( \alpha_t^{\infty} \) are also strongly continuous on the corresponding algebras, \( \mathcal{A}^{(R)}, \mathcal{A}^{\infty} \). By the same token, we can infer that \( \omega^{(R)} \) and \( \omega^{\infty} \) are KMS-states at the same inverse temperature \( \beta \).

**Remark 5.4** Strong continuity can be generally achieved by going over to smoothed observables, i.e., by averaging the observables with smooth functions of, say, compact support in the time variable.

Proof of Corollary: Note that the original time evolution “commutes” with the scale transformation in the sense described above. This yields the mentioned result for all finite \( R \). We have in particular that for suitable elements (for the technical details see [16])

\[
\langle B^{(R)}(t) \cdot A^{(R)} \rangle_{(R)} = \langle A^{(R)} \cdot B^{(R)}(t + i\beta) \rangle_{(R)} \tag{86}
\]

and there exists an analytic function, \( F_{AB}^{(R)}(z) \), in the strip \( \{ z = t + i\tau, \ 0 < \tau < \beta \} \) with continuous boundary values at \( \tau = 0, \beta \):

\[
F_{AB}^{(R)}(t) = \langle A^{(R)} \cdot B^{(R)}(t) \rangle_{(R)}, \quad F_{AB}^{(R)}(t + i\beta) = \langle A^{(R)} \cdot B^{(R)}(t + i\beta) \rangle_{(R)} \tag{87}
\]

This is equivalent to the following equation (cf. [16]):

\[
\int \omega^{(R)}(A^{(R)} \cdot B^{(R)}(t)) \cdot f(t)dt = \int \omega^{(R)}(B^{(R)}(t) \cdot A^{(R)}) \cdot f(t + i\beta)dt \tag{88}
\]

for \( \hat{f} \in \mathcal{D} \). As \( f(t + i\beta) \) is of strong decrease in \( t \) the limit \( R \to \infty \) can be performed under the integral and we get the same relation in the scaling limit. The above mentioned equivalence of this property with the KMS-condition shows that the limit state is again KMS. This proves the statement.

**Remarks:**

i) Note what we have already said in [8]. One reason for the non-equivalence of the algebras on different scales stems from the observation that, in general,

\[
A_R \cdot B_R \neq (A \cdot B)_R \tag{89}
\]

Furthermore, in the scaling limit, many different observables of \( \mathcal{A} \) converge to the same limit point, for example, all finite translates of a fixed observable.

ii) A corresponding result in a slightly different context was also proved in [4].
5.2 The Scaling Limit Theory as a Quantum Field Theory

We have seen in sect. 2.3 that the scaling limit of the correlation functions for the block spin observables is not fully scale invariant but only asymptotically so (while the short range details of the original microscopic correlations, encoded in the function $F(x_1 - x_2)$, have been integrated out, there remains an integrated effect of the initial block-function, $f(x)$).

This observation runs a little bit contrary to the general folklore, in which the various limit procedures are frequently interchanged and identified without full justification. We will exhibit the true connections between the various expressions in the following.

With $f(x)$ now being a general test function of e.g. compact support, we have from sect. 2.3, making now the dependence on $f$ explicit

$$\lim_{R \to \infty} \langle A_{R,f}(RX_1) \cdot A_{R,f}(RX_2) \rangle = \text{const} \int |y + Y|^{-(n-\alpha)} \cdot f \ast f(y) dy$$ (90)

with

$$A_{R,f}(RX) = R^{-(n+\alpha)/2} \cdot \int A(RX + x) \cdot f(x/R) dx$$ (91)

We now rewrite the limit correlation function as

$$\langle A_f^\infty(X_1) \cdot A_f^\infty(X_2) \rangle = \int \langle \hat{A}^\infty(x_1 + X_1) \cdot \hat{A}^\infty(x_2 + X_2) \rangle \cdot f(x_1)f(x_2)dx_1dx_2$$ (92)

that is, we identify

$$A_f^\infty(X) = \int \hat{A}^\infty(x + X) \cdot f(x)dx$$ (93)

with $\hat{A}^\infty(x)$ now having rather the character of a field or operator valued distribution.

We have that

$$\langle \hat{A}^\infty(X_1) \cdot \hat{A}^\infty(X_2) \rangle =: W^\infty(X_1 - X_2) = \text{const} \cdot |X_1 - X_2|^{-(n-\alpha)}$$ (94)

Corresponding results would hold for the higher correlation functions, that is, we arrive at

**Conclusion 5.5** In contrast to the block observables, $A_f^\infty$, the field, $\hat{A}^\infty(x)$, displays the full scale invariance.

The field, $\hat{A}^\infty(x)$, can, on the other hand, be directly constructed by means of a related limit procedure, which is however not of block variable type. We start instead with the unsmeared observables and take the scaling limit, $R \to \infty$

$$\lim_{R \to \infty} \langle \hat{A}_R(RX_1) \cdot \hat{A}_R(RX_2) \rangle \text{ with } \hat{A}_R(RX) := R^{(n-\gamma)} \cdot A(RX)$$ (95)
and $n - \gamma = (n - \alpha)/2$.

Remark: The extra scaling factor, $R^n$, replaces the missing integration over the test function $f_R$, the support of which increases like $R^n$.

Performing the same calculations, we see that the above limit is equal to $\langle A^\infty(X_1) \cdot A^\infty(X_2) \rangle$. We hence have

**Conclusion 5.6** The fully scale invariant limit theory is achieved by taking the limits

$$\lim_R \langle \hat{A}_R(RX_1) \cdots \hat{A}_R(RX_l) \rangle =: W^\infty(X_1, \ldots, X_l)$$

The same construction holds of course for the intermediate scales; we define $\hat{A}^{(R)}(X)$ by the following identification

$$\langle \hat{A}^{(R)}(X_1) \cdots \hat{A}^{(R)}(X_l) \rangle_{(R)} := R^{l(n-\gamma)} \cdot \langle A(RX_1) \cdots A(RX_l) \rangle$$

and have for the observables, $A^{(R)}_f$, defined above

$$A^{(R)}_f(X) = \int \hat{A}^{(R)}(X + x) f(x) dx$$

(which can e.g. be checked by direct calculation).

### 5.3 The (Non)-Quantum Character in the Scaling Limit

In subsection B of section 3 of [8], we already discussed the limiting behavior of commutators of scaled observables. In the regime of normal scaling, that is, scale dimension $\gamma = n/2$, we found that commutators are non-vanishing in the generic case in the limit. This means that in general the resulting limit theory is non-abelian (but quasi-free!)! Perhaps a little bit surprisingly, the situation changes at the critical point, where the scale-dimensions are, typically, greater than $n/2$ for at least some observables!

We make the same observation as Sewell in [13], namely, commutators of certain “critical” observables vanish in the scaling limit, i.e., the corresponding limit observables are losing (at least) part of their quantum character.

Remark: We think that the observation that fluctuations and critical behavior at the critical point are typically of a thermal and not of a quantum nature, does somehow belong to the general folklore in the field of critical phenomena, but we are not aware at the moment that this fact has been widely discussed in the literature in greater rigor. Some remarks can e.g. be found in connection with so-called (temperature-zero) quantum phase transitions in [17] or [14] and further references given there.

On the other hand, related phenomena were observed in the context of spontaneous symmetry breaking in sect. 6 of [8] and for certain models by Verbeure et
al in [9]. A careful analysis of the behavior of commutators in a slightly different context can also be found in [15].

We have the following result.

**Theorem 5.7** Let $A, B$ be strictly localized observables with $\gamma_A + \gamma_B > n$. We then have

$$\lim_{R} \| [A_R, B_R] \| = 0$$  \hspace{0.5cm} (99)

**Proof:** With $\gamma_A + \gamma_B > n$ we have

$$\| [A_R, B_R] \| \leq R^{-(\gamma_A+\gamma_B)} \cdot \int \| [A(x_1), B(x_2)] \| \cdot f(x_1/R) f(x_2/R) dx_1 dx_2$$
$$= R^{-(\gamma_A+\gamma_B)} \cdot \int \| [A, B(y)] \| \cdot f(x_1/R) f((x_1 + y)/R) dx_1 dy$$ \hspace{0.5cm} (100)

By assumption $A, B$ have bounded supports, $V_A, V_B \subset \mathbb{R}^n$ so that

$$[A, B(x)] = 0 \text{ for } V_B + x \cap V_A = \emptyset$$ \hspace{0.5cm} (101)

From the support assumption we immediately infer that the above double integral is actually a single integral as the commutator on the rhs vanishes outside a set, $S$, of finite diameter. We get

$$\lim_{R} \| [A_R, B_R] \| \leq \text{const}' \cdot R^{-(\gamma_A+\gamma_B)} \int \chi_S(y) \cdot f(x_1/R) f((x_1 + y/R)) dx_1 dy$$
$$= \text{const}' R^{-(\gamma_A+\gamma_B)} \cdot R^n \int \chi_S(y) \cdot f(x_1') f(x_1' + y/R) dx_1' dy \leq \text{const} \cdot \lim_{R} R^{n-(\gamma_A+\gamma_B)} = 0$$ \hspace{0.5cm} (102)

as $\gamma_A + \gamma_B > n$ by assumption. $\square$

**Corollary 5.8** We arrive at the same result if $A, B$ are not strictly localized but fulfill a norm estimate of the form

$$\| [A, B(y)] \| =: F(y) \in L^1(\mathbb{R}^n)$$ \hspace{0.5cm} (103)

**Proof:** We have

$$R^{-(\gamma_A+\gamma_B)} \cdot \int F(y) \cdot f(x_1/R) f((x_1 + y)/R) dx_1 dy =$$
$$R^{-(\gamma_A+\gamma_B)} \cdot R^{2n} \cdot \int \hat{F}(p) \hat{f}(Rp) \cdot \hat{f}(-Rp) dp$$
$$= R^{-(\gamma_A+\gamma_B)} \cdot R^n \cdot \int \hat{F}(p/R) \hat{f}(p) \cdot \hat{f}(-p) dp$$ \hspace{0.5cm} (104)
We can again perform the $R$-limit under the integral and get the limit expression
\begin{equation}
R^{-(\gamma_A+\gamma_B)} \cdot \hat{F}(0) \cdot \int \hat{f}(p) \cdot \hat{f}(-p) dp \to 0 \quad (105)
\end{equation}
for $R \to \infty$.

A simple example where different renormalisation exponents naturally arise is the following. Take a limit observable, $A^\infty(X)$, and consider its spatial derivative, $\partial_X A^\infty(X)$. Then we have in a slightly sloppy notation (the limit being taken in the sense, described above):
\begin{align*}
\partial_X A^\infty(X) &= \lim_R \partial_X (R^{-\gamma_A} \cdot \int A(x + RX) \cdot f_R(x) d^n x) \\
&= \lim_R (R^{(-\gamma_A+1)} \cdot \int (\partial_x A)(x + RX) \cdot f_R(x) d^n x) \quad (106)
\end{align*}
That is, $\partial_x A = i[P,A]$ has to be scaled with a different scale exponent. Physically, this can be understood as follows. With $f_R(x) = f(|x|/R)$ simulating the integration over a ball with radius $R$, a partial integration in the above formula shifts the $\partial_x$ to the test function, $f_R(x)$. As $\partial_x f_R(x) = 0$ in the interior of the ball, the averaging goes roughly only over the sphere of radius $R$ instead of the full ball. This has to be compensated by a weaker renormalisation.

Another result in this direction can be found in [8] sect.6, in connection with the canonical Goldstone pair in the context of spontaneous symmetry breaking.

Further possible candidates are the time derivatives of observables as, for example, in $\langle \dot{A} \dot{A} \rangle$. Fourier transformation yields an additional prefactor, $\omega^2$ in the spectral weight, $\hat{F}_{AA}(\omega, k)$. The $KMS$-condition leads to another constraint:
\begin{equation}
\hat{F}_{AB}(\omega, k) = (1 - e^{-\beta \omega})^{-1} \cdot \hat{C}_{[A,B]}(\omega, k) \quad (107)
\end{equation}
A combination of such properties shows, that in the scaling limit, the vicinity of $(\omega, k) = (0, 0)$ is important (see the discussion in section [4]).

From covariance properties (as e.g. in models of relativistic quantum field theory) one can infer additional information about certain characteristics of the energy-momentum spectrum. For arbitrary models of non-relativistic many-body theory, however, the situation is less generic and typically model dependent.

Remark: We had several discussions with D.Buchholz about this point, which are gratefully acknowledged. This applies also to the following subsection.

\section*{5.4 The Nature of the Limit Time Evolution and the Phenomenon of Critical Slowing Down}

We argued above that the appropriate choice of the respective scaling dimensions of the observables under discussion is a subtle point and perhaps, to some extent,
even a matter of convenience. After all, one may have some freedom in the choice of the subset of observables which is to survive the renormalisation process.

We will not give a complete analysis of all possibilities in the following but rather emphasize one, as we think, particularly remarkable phenomenon, namely, the phenomenon of critical slowing down. As in the preceding discussion, we choose two observables, \( A, B \), with \( \gamma_A + \gamma_B > n \), implying that the limit commutator vanishes. We assume this also to hold for non-equal times (which follows from a \( L_1 \)-cluster condition as in the preceding subsection), at least on the level of two-point functions, i.e.

\[
\langle [A^\infty, B^\infty(t)] \rangle_\infty = 0
\]

and get the following theorem:

**Theorem 5.9** Under the assumptions being made, we have

\[
\langle A^\infty \cdot B^\infty(t) \rangle_\infty = \text{const for all } t \in \mathbb{R}
\]

Proof: As the limit state is again a \( KMS \)-state, the vanishing of the above commutator implies that the analytic function, \( F_{AB}^\infty(z) \), fulfills

\[
F_{AB}^\infty(t) = F_{AB}^\infty(t + i\beta)
\]

for all \( t \). \( F_{AB}^\infty(z) \) can hence be analytically continued to the whole \( \mathbb{C} \)-plane and is, furthermore, a globally bounded analytic function, hence a constant by standard reasoning. \( \square \)

We see that the subclass of limit observables, which has vanishing limit commutators (see the preceding subsection), has, by the same token, time independent limit correlation functions. As these pair-correlation functions are usually connected with characteristic observable properties of the system (generalized susceptibilities, transport coefficients etc.), this has remarkable physical consequences. The corresponding phenomenon is called critical slowing down. For a review of the physical phenomena see e.g. [18]. In physical terms, the phenomenon can be understood as follows.

In the critical regime, the patches of strongly correlated degrees of freedom become very large and extend practically over all scales. That is, a reorientation of such clusters or a response to external perturbations takes, if viewed on the microscopic time scale, a very long time. In the scaling limit this time scale goes to infinity. If one wants to see observable dynamical effects on the macroscopic level one must hence scale the time variable also. For the unscaled time we have in the limit \( R \to \infty \):

\[
d/dt \langle A^\infty \cdot B^\infty(t) \rangle_\infty = 0
\]

This is the same as

\[
\langle A^\infty \cdot [H^\infty, B^\infty(t)] \rangle_\infty = \lim_{R \to \infty} \langle A_R \cdot [H, B_R(t)] \rangle_R = \lim_{R \to \infty} d/dt \langle A_R \cdot B_R(t) \rangle_R
\]
(cf. subsection 5.1, $H$ is the microscopic Hamiltonian).

What one now has to do is obvious. We have to compensate the vanishing of the above expression in the limit by inserting an appropriate scale factor in the time coordinate. Instead of $B(t)$ we take $B(R^δ \cdot t)$ with $δ$ so chosen that the limit expression is non-vanishing. Note that differentiation with respect to $t$ now yields an explicit prefactor $R δ$. This fixes the macroscopic time scale, $t_m$, for these processes. We define

$$\langle A^∞ \cdot B^∞(t_m) \rangle_∞ := \lim_{R} \langle A_R \cdot B_R(R^δ \cdot t_m) \rangle$$

Physically the effect can be understood by inspecting the middle part of equation 112. The support of $B_R(t)$ spreads with time. This spread is more pronounced if we take $R^δ \cdot t$ instead of $t$. By the same token the overlap with the Hamiltonian (which is basically translation invariant) increases with $R \to \infty$ while $t$ is kept fixed, thus yielding the non-vanishing limit.

It may happen that other observables may evolve on different macroscopic time scales so that the construction of a coherent common macroscopic limit time evolution may not be straightforward. Such more detailed questions have to be separately studied for the various model classes. As we have studied a concrete model class in section 4, we can make much more precise statements if we have some information about the energy-momentum spectrum in the vicinity of $(ω, k) = (0, 0)$. In that section we arrived at the following results:

$$\hat{F}_ρ^{AB}(ω, k) = (1 - \exp(-βω))^{-1} \cdot \hat{c}_{AB}(ω, k)$$

and

$$\hat{F}_{\text{sing}}(ω, k) := J_+^β(k) \cdot δ(ω - (e(k) - μ)) + J_-^β(k) \cdot δ(ω + (e(k) - μ))$$

with

$$J_+^β(k) = (1 - \exp(-β(e(k) - μ)))^{-1} \cdot j(k)$$

and

$$J_-^β(k) = \exp(-β(e(k) - μ)) \cdot J_+^β(k)$$

For e.g.

$$\langle A_R(RX_1, t) \cdot A_R(RX_2, 0) \rangle$$

we have to study expressions like

$$R^{-2(γ_A-n/2)} \int e^{i(e(k/R)-μ)t} \cdot (1 - \exp(-β(e(k/R) - μ)))^{-1} \cdot j(k/R) \cdot |\hat{f}|^2 d^n k$$

Again, the situation is normal for $μ \neq 0$ but becomes singular for $μ = 0$ (cf. section 4).
We concentrate on the case \( e(k) \sim |k|^\alpha \) for \( |k| \to 0 \) and \( j(k) \) continuous and \( \neq 0 \) in \( k = 0 \) (cf. section 4 assumption 4.6). In order to have a non-vanishing limit correlation function we have to choose

\[
\gamma_A = (n + \alpha)/2
\]

In the case where \( j(k) \) vanishes polynomially in \( k = 0 \) we have to make a corresponding choice, as has been described in the mentioned section.

If, furthermore, we want to have a non-trivial time evolution in the limit \( R \to \infty \), we have to scale the microscopic time like

\[
t = R^\alpha \cdot \tau \quad \text{so that} \quad |k/R|^\alpha \cdot t = |k|^\alpha \cdot \tau
\]

**Conclusion 5.10** If we are in the situation, described in section 4, having a singular spectral contribution with quasi-particle-like dispersion law \( e(k) \sim |k|^\alpha \) near \( k = 0 \), we have to scale the microscopic time, \( t \), like \( t = R^\alpha \cdot \tau \), in order to arrive at a non-trivial limit time evolution in the variable \( \tau \).

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**References**

[1] S.K.Ma: “Modern Theory of Critical Phenomena”, Benjamin Inc., Reading 1976

[2] L.Kadanoff et al: “Static Phenomena near Critical Points”, Rev.Mod.Phys. 39(1967)395

[3] G.Parisi: “Statistical Field Theory”, Perseus Books, Reading 1998

[4] D.Buchholz,R.Verch: “Scaling Algebra and Renormalisation Group in Algebraic Quantum Field Theory”, Rev.Math.Phys. 7(1995)1195

[5] D.Amit: “Field Theory, the Renormalisation Group, and Critical Phenomena”, Mcgraw-Hill, N.Y. 1978

[6] M.Le Bellac: “Quantum and Statistical Field Theory”, Oxford Science Publ., Oxford 1991

[7] J.Zinn-Justin: “Quantum Field Theory and Critical Phenomena”, Oxford Science Publ., Oxford 1990

[8] M.Requardt: “Fluctuation Operators and Spontaneous Symmetry Breaking”, J.Math.Phys. 43(2002)351, math-ph/0003012
[9] T.Michoel,A.Verbeure: “Goldstone Boson Normal Coordinates”, Comm.Math.Phys. 216(2001)461, math-ph/0001033

[10] I.M.Gelfand,G.E.Schilow: “Verallgemeinerte Funktionen” vol.I, Deutscher Verlag Wissensch., Berlin 1960

[11] D.Ruelle: “Statistical Mechanics”, Benjamin Inc., N.Y.1969

[12] O.Bratteli,D.W.Robinson: “Operator Algebras and Quantum Statistical Mechanics I”,2nd ed., Springer, N.Y. 1987

[13] G.L.Sewell: “Quantum Theory of Collective Phenomena”, Clarendon Pr., Oxford 1986, chapt. 5.4.2

[14] Vojta: Physik Journal March 2002, p.55ff

[15] B.Momont,A.Verbeure,V.A.Zagrebnov: “Algebraic Structure of Quantum Fluctuations”, J.Stat.Phys. 89(1997)633

[16] O.Bratteli,D.W.Robinson: “Operator Algebras and Quantum Statistical Mechanics II”, Springer N.Y. 1981

[17] S.Sachev: “Quantum Phase transitions”, Cambridge Univ.Pr., Cambridge 1999

[18] B.L.Halperin,P.C.Hohenberg: “Theory of Dynamical Critical Phenomena”, Rev.Mod.Phys. 49(1977)435

[19] Y.G.Sinai: “Theory of Phase Transitions: Rigorous Results”, Pergamon Pr., N.Y. 1982

[20] N.D.Mermin: “Absence of Ordering in certain Classical Systems”, J.Math.Phys. 8(1967)1061

[21] G.Gallavotti,E.Verboven: “On the Classical KMS Boundary Condition”, Nuov.Cim. B28(1975)274

[22] M.Aizenmann,G.Gallavotti,S.Goldstein,J.L.Lebowitz: “Stability and Equilibrium States of Infinite Classical Systems”, Comm.Math.Phys. 45(1976)1

[23] M.Requardt: “A Microscopic Proof of a Goldstone Theorem in Classical Statistical Mechanics”, Zeitschr.Phys.B 36(1979)187

[24] M.Requardt,H.J.Wagner: “Poor Decay of Correlations in Inhomogeneous Fluids and Solids”, J.Stat.Phys. 45(1986)815

[25] M.Requardt: “Dynamical Cluster Properties in the Quantum Statistical Mechanics of Phase Transitions”, J.Phys.Math.Gen. 13(1980)1769

[26] J.Bros,D.Buchholz: “Asymptotic Dynamics of Thermal Quantum Fields”, Nucl.Phys. B627(2002)289, hep-ph/0109136
[27] J.Bros, D.Buchholz: “Axiomatic Analyticity Properties in Thermal Quantum Field Theory”, Ann.H.Poinc. 64(1996)495, hep-th/9606046

[28] H.Narnhofer, M.Requardt, W.Thirring: “Quasi Particles at Finite Temperatures”, Comm.Math.Phys. 92(1983)247

[29] M.Reed, B.Simon: “Methods in Mathematical Physics” vol.II, Acad.Pr., N.Y. 1975

[30] M.Requardt: “Scaling Limit and Renormalisation Group in the Critical Point Analysis of General (Quantum) Many Body Systems, e-print, math-ph/0205011