Decentralized Non-Convex Learning with Linearly Coupled Constraints

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Abstract

Motivated by the need for decentralized learning, this paper aims at designing a distributed algorithm for solving nonconvex problems with general linear constraints over a multi-agent network. In the considered problem, each agent owns some local information and a local variable for jointly minimizing a cost function, but local variables are coupled by linear constraints. Most of the existing methods for such problems are only applicable for convex problems or problems with specific linear constraints. There still lacks a distributed algorithm for solving such problems with general linear constraints under the nonconvex setting. To tackle this problem, we propose a new algorithm, called proximal dual consensus (PDC) algorithm, which combines a proximal technique and a dual consensus method. We show that the proposed PDC algorithm can generate an \( \epsilon \)-Karush-Kuhn-Tucker solution in \( O(1/\epsilon) \) iterations, achieving the lower bound for non-convex problems. Numerical results are presented to demonstrate the good performance of the proposed algorithms for solving a regression problem and a neural network based classification problem over a multi-agent learning network.

1 Introduction

Recently, distributed optimization has attracted significant attention in signal processing, control and machine learning societies [2–4] due to the need of achieving decentralized control and decision making in sensor networks, and large-scale data processing and learning. Since the distributed agents/nodes process only local data and messages exchanged from direct neighbors, distributed optimization is well suited for applications where the data are acquired distributively [5] or where data privacy is of primary concern [6].
In this paper we are interested in distributed optimization methods for solving the following problem

\[
(P) \quad \min_{x_i \in \mathbb{R}^n} \sum_{i=1}^{N} f_i(x_i) \quad \text{s.t.} \quad \sum_{i=1}^{N} B_i x_i = q,
\]

where \(N\) denotes the number of agents, \(f_i : \mathbb{R}^n \to \mathbb{R}\) are smooth and possibly non-convex functions, and \(B_i \in \mathbb{R}^{M \times n}\) and \(q \in \mathbb{R}^M\) are constants. The target is to solve \((P)\) over a multi-agent network with \(N\) agents. It is assumed that each agent \(i\) knows only \(f_i, B_i\) and \(q\), and can only communicate with its neighboring agents. The agent variables are coupled due to the linear constraint \(\sum_{i=1}^{N} B_i x_i = q\). Such problem arises, for example, in machine learning with distributed features [7,8], power control in electrical power networks [9,10], interference management in wireless networks [11], and the network utility maximization problem [12].

### 1.1 Related Works

As a linearly constrained problem, problem \((P)\) can be easily solved under the centralized setting, where all information are gathered in a central node. Centralized methods for solving linearly constrained problems are well studied in the literature. For example, the penalty method [13] penalizes the linear constraints in the objective function and minimizes the penalized objective function instead. The augmented Lagrangian (AL) method [14] uses a min-max approach to solve the resulting AL function. Problem \((P)\) can also be efficiently handled by the alternating direction method of multipliers (ADMM) [15–17]. However, solving \((P)\) in a distributed fashion over a multi-agent network is challenging, due to the nonconvexity and coupled variables. In the existing literature, most of the distributed methods for solving \((P)\) assumed convexity of \(f_i\)'s. For example, if \((P)\) is convex, it is well known that the dual decomposition method [18] is an efficient method to solve such problem over a star network where all agents can communicate with a central node. For a multi-agent network with general graph structures, convexity and strong duality are the keys to most of the existing distributed methods for solving \((P)\); see [7,19–25] and references therein.

Interestingly, when \(B = [B_1, \cdots, B_N]\) is the incidence matrix of the network graph and \(q\) is the zero vector, \((P)\) corresponds to a consensus formulation of the finite sum minimization problem \(\min_x \sum_{i=1}^{N} f_i(x)\), and there exist rich results in the literature for both convex and nonconvex cases, see, for example, the recent surveys [4,26,27]. However, since these decentralized methods are designed for consensus problems, they cannot be used to solve \((P)\) which has a general linear equality constraint.

### 1.2 Contributions

In this paper, we propose a new distributed algorithm for handling \((P)\) under the general linearly coupled constraint and non-convex setting. The proposed method, which we call the \textit{proximal dual consensus} (PDC) method [1],\(^1\) is motivated by the proximal technique recently proposed in [17] and the dual consensus method in [7,19]. Specifically, a strongly convex counterpart of \((P)\) is obtained by introducing a quadratic proximal term centered at an auxiliary primal variable. Then, the dual

\(^1\) Compared with [1], this manuscript presents all the proof details and stronger theoretical results. Furthermore, the current manuscript presents an inexact update version of the proposed PDC algorithm as well as new experiment results.
A consensus approach is applied to the proximal problem, resulting in a distributed algorithm for (P). In the proposed algorithm, the agents can choose to solve their local subproblem exactly or inexactly by simply performing a gradient descent.

While the proposed PDC algorithms are seemingly a simple combination of [17] and [7], proving their convergence and convergence rate are far from easy as several new error bounds are needed. Theoretically, we show that the proposed PDC algorithms can converge to a Karush-Kuhn-Tucker (KKT) solution of (P), and generate an $\epsilon$-optimal solution in $O(1/\epsilon)$ iterations. This meets the iteration complexity lower bound of the first-order algorithms for solving a nonconvex problem [28]. Numerical results are presented to show that the proposed algorithms perform well in a non-convex regularized logistic regression problem and a neural network based distributed classification problem. To the best of our knowledge, the proposed distributed algorithms are the first to solve (P) under the non-convex setting over the multi-agent network.

**Synopsis:** Section 2 presents two application examples of (P) about machine learning from distributed features. The proposed PDC algorithms are developed and analyzed in Section 3. The proof details of main theoretical results are shown in Section 4. Numerical results are given in Section 5 and conclusions are drawn in Section 6.

**Notation:** $I_n$ is denoted as the $n$ by $n$ identity matrix. $\langle a, b \rangle$ represents the inner product of vectors $a$ and $b$, $\|a\|$ is the Euclidean norm, and $\|a\|_A^2$ represents $a^\top A a$; $\otimes$ denotes the Kronecker product; $i \in [N]$ denotes $i \in \{1, \ldots, N\}$.

## 2 Application Examples

Problem (P) appears in many engineering applications. In this section, we discuss an important but less discussed learning problem: learning from distributed features (LDF) [29,30]. As shown in Fig. 1, unlike the conventional learning from distributed samples where the data samples are partitioned and distributively owned by the agents [4], the agents for LDF own all data samples but part of their features only. Such problems appear, for example, in collaborative learning between multiple parties who hold different aspects of information for the same set of samples. Using two examples below, we show that the LDF problem can be expressed as the form of (P).
Example 1 Consider a standard empirical risk minimization problem as follows

$$\min_w \sum_{k=1}^{M} \psi(b_k^T w, v_k) + R(w),$$  \hspace{1cm} (2)$$

where $M$ is the number of data samples, $b_k$ and $v_k$ are respectively the feature vector and (one-hot vector) label for the $k$-th sample, $w \in \mathbb{R}^{nN}$ is the parameter to be optimized, $\psi$ is the loss function, and $R(\cdot)$ is a regularization function.

Consider to solve problem (2) over a multi-agent network with $N$ agents. Assume that each sample $b_k$ is partitioned as $b_k = [b_{1,k}^T, \cdots, b_{N,k}^T]^T$, where $b_{i,k} \in \mathbb{R}^n$, and each agent $i$ owns samples with partial features $\{b_{i,k}\}_{k=1}^{M}$, for all $i \in [N]$. Let the parameter vector $w$ be partitioned in the same fashion, i.e., $w = [w_1^T, \cdots, w_N^T]^T$, and assume the regularization term $R(w) = \sum_{i=1}^{N} R_i(w_i)$.

Then, problem (2) can be reformulated as

$$\min_{w_0, \{w_i\}} \sum_{k=1}^{M} \psi(w_{0,k}, v_k) + \sum_{i=1}^{N} R_i(w_i)$$  \hspace{1cm} (3a)$$

s.t. \sum_{i=1}^{N} b_{i,k}^T w_i - w_{0,k} = 0, \hspace{1cm} k \in [M],$$  \hspace{1cm} (3b)$$

where $w_0 = [w_{0,1}, \ldots, w_{0,M}]^T$ is an introduced variable. Further define $B_i = [b_{i,1}, \cdots, b_{i,M}]^T \in \mathbb{R}^{M \times n}$, $i \in [N]$, and let $\Psi(w_0) = \sum_{k=1}^{M} \psi(w_{0,k}, v_k)$. Thus (3) can be written as

$$\min_{w_0, \{w_i\}} \Psi(w_0) + \sum_{i=1}^{N} R_i(w_i)$$  \hspace{1cm} (4a)$$

s.t. \sum_{i=1}^{N} B_i w_i - w_0 = 0,$$

which is an instance of $\mathcal{P}$.

Example 2 Consider to train a neural network (NN) for classification as shown in Fig. 2. The mapping of the NN is expressed by the function $F_\theta(b_k^T W)$, where $W \in \mathbb{R}^{nN \times K}$ denotes the coefficients of the 1st linear layer of the NN, $K$ is the dimension of the 1st layer output, and $\theta$ incorporates the parameters from the 1st layer output to the output layer of the NN. The learning problem is given by

$$\min_{W, \theta} \sum_{k=1}^{M} L(F_\theta(b_k^T W), v_k),$$  \hspace{1cm} (5)$$

where $L$ is a loss function.

Given that $b_k = [b_{1,k}^T, \cdots, b_{N,k}^T]^T$ for all $k \in [M]$ under the LDF setting, we partition $W = [W_1^T, \ldots, W_N^T]^T$ accordingly, and introduce

$$w_{0,k} = \sum_{i=1}^{N} W_i^T b_{i,k} = \sum_{i=1}^{N} (b_{i,k}^T \otimes I_K) w_i,$$  \hspace{1cm} (6)$$

where $\otimes$ denotes the Kronecker product.
Figure 2: A neural network structure used in problem (5).  

where \( w_i = \text{vec}(W_i) \) is obtained by stacking the columns of \( W_i \). Thus, problem (5) can be written as

\[
\begin{align*}
\min_{w_1, \ldots, w_N, (w_0, \theta)} & \sum_{k=1}^{M} L\left(F_{\theta}(w_{0,k}), v_k\right) \\
\text{s.t.} & \sum_{i=1}^{N} B_i w_i - w_0 = 0, 
\end{align*}
\tag{7a}
\]

\[
\begin{align*}
\text{s.t.} & \sum_{i=1}^{N} B_i w_i - w_0 = 0, 
\end{align*}
\tag{7b}
\]

where \( B_i = [ (b_{i,1}^T \otimes I_K)^T, \ldots, (b_{i,M}^T \otimes I_K)^T ]^T \) and \( w_0 = [ w_{0,1}^T, \ldots, w_{0,M}^T ]^T \).

As a result, the distributed optimization problem (P) includes both problems (4) and (7) on special cases.

3 PDC Algorithms

The network models and assumptions are given in Section 3.1. To solve (P), we develop the PDC algorithm in Section 3.2 and further give its convergence analysis in Section 3.3. In Section 3.4, we further propose an inexact version of PDC, referred as the IPDC.

3.1 Network Models and Assumptions

The multi-agent network is modeled as an undirected graph \( G = (\mathcal{E}, \mathcal{N}) \), where \( \mathcal{E} \) is the set of edges and \( \mathcal{N} \) is the set of agents. Each agent can only communicate and exchange information with its neighbors, i.e., agents \( i, j \) can communicate with each other if and only if \( (i, j) \in \mathcal{E} \).

Let us define some useful notations regarding the network graph. The neighboring set of agent \( i \) is defined as \( \mathcal{N}_i = \{ j \in \mathcal{N} | (i, j) \in \mathcal{E} \} \). Let \( \bar{D} = \text{diag}\{ |\mathcal{N}_1|, \ldots, |\mathcal{N}_N| \} \in \mathbb{R}^{N \times N} \) be the degree matrix of \( G \). The agent-edge incidence matrix is defined as \( \bar{A} \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{N}|} \) which has \( \bar{A}_{e,i} = 1 \) and \( \bar{A}_{e,j} = -1 \) if \( j > i \) and \( e = (i, j) \in \mathcal{E} \), and zeros otherwise. \( \bar{L}^- = \bar{A}^T \bar{A} \) is the (signed) Laplacian matrix of \( G \). Then \( \bar{L}^+ = 2\bar{D} - \bar{L}^- \) is the signless Laplacian matrix of \( G \). Both \( \bar{L}^- \) and \( \bar{L}^+ \) are
positive semi-definite and describe the connectivity of the network. For ease of later use, we also define

\[ A = \tilde{A} \otimes I_M = [A_1, \ldots, A_N], \]
\[ L^+ = \tilde{L}^+ \otimes I_M = [(L^+_1)^\top, \ldots, (L^+_N)^\top]^\top, \]
\[ L^- = \tilde{L}^- \otimes I_M = [(L^-_1)^\top, \ldots, (L^-_N)^\top]^\top, \]

and let \( \lambda_{\text{max}} \) be the maximum eigenvalue of \( L^+ \).

We make the following assumption for the network.

**Assumption 1** The graph \( G \) is connected.

It means that for any \( i, j \in \mathcal{N} \), there exists at least one path connecting \( i \) and \( j \). Assumption 1 is commonly made in the literature of distributed optimization.

For problem (P), we make the following assumption.

**Assumption 2** Each \( f_i \) is continuously differentiable and \( \nabla f_i \) is Lipschitz continuous with constant \( \gamma^+ > 0 \).

Under Assumption 2, there exists a (possibly negative) constant \( \gamma^- \) such that

\[ \langle \nabla f_i(x_i) - \nabla f_i(x'_i), x_i - x'_i \rangle \geq \gamma^- \| x_i - x'_i \|^2, \quad (8) \]

for all \( x_i, x'_i \in \mathbb{R}^n \). Note that if \( \gamma^- \) is negative, then \( f_i \) is not a convex function.

### 3.2 Algorithm Development

The proposed PDC method is motivated by the proximal approach recently presented in [17] for non-convex constrained optimization. In particular, like [17], we consider a “proximal” counterpart of (P)

\[
\min_{x_i, i \in \mathcal{N}} \sum_{i=1}^N f_i(x_i) + \frac{p}{2} \sum_{i=1}^N \| x_i - z_i \|^2 \quad (9a)
\]
\[
\text{s.t. } \sum_{i=1}^N B_i x_i = q, \quad (9b)
\]

where \( p > -\gamma^- \) is a penalty parameter, \( x = (x_1^\top, \ldots, x_N^\top)^\top \) and \( z = (z_1^\top, \ldots, z_N^\top)^\top \) is an introduced proximal variable. Then (9) is a strongly convex problem of \( x \) when \( z \) is fixed. Denote \((x(z), y_0(z))\) as a primal-dual optimal solution of (9). One can verify that \((x(z), y_0(z))\) is a KKT solution of (P) if and only if

\[ x(z) = z. \quad (10) \]

In [17], the authors proposed to handle (9) by using one step of primal-dual gradient update in each iteration \( r \), followed by updating \( \{z^r\} \) by

\[ z^{r+1} = z^r + \beta(x^{r+1} - z^r), \quad (11) \]

6
where \(0 < \beta < 1\) is a positive stepsize. Notice that for \(\beta = 1\), the algorithm may oscillate. It is shown that (11) with sufficiently small \(\beta\) can stabilize the primal-dual sequence and the algorithm can achieve a KKT solution of (P). However, the algorithm in [17] is a centralized algorithm and cannot solve (P) over the multi-agent network.

To obtain a distributed algorithm, we consider employing the dual consensus method in [7] to deal with the (strongly convex) proximal problem (9). First, by the Lagrange duality theory [31], (9) is equivalent to its dual problem

\[
\max_{y_0 \in \mathbb{R}^M} \sum_{i=1}^{N} \left( \phi_i(y_0, z_i) - y_0^T \frac{q}{N} \right),
\]

where \(y_0\) is the dual variable associated with the constraint (9b), and

\[
\phi_i(y_0, z_i) = \min_{x_i \in \mathbb{R}^n} f_i(x_i) + \frac{p}{2} \|x_i - z_i\|^2 + y_0^T B_i x_i,
\]

for \(i \in [N]\), which are concave functions of \(y_0\). In order to solve this dual problem in a decentralized manner, we let \(y_i \in \mathbb{R}^M\) be the local copy of \(y_0\) at agent \(i\), then \(\phi_i\) can be written as

\[
\phi_i(y_i, z_i) = \min_{x_i \in \mathbb{R}^n} f_i(x_i) + \frac{p}{2} \|x_i - z_i\|^2 + y_i^T B_i x_i,
\]

for \(i \in [N]\). As (14) is strongly convex in \(x_i\), \(\phi_i(y_i, z_i)\) is smooth and concave in \(y_i\), and \(\nabla \phi_i(y_i, z_i)\) is Lipschitz continuous with constant \(\frac{1}{p+\gamma}\) [32]. Next, we consider the following consensus constrained counterpart

\[
\max_{y=(y_1, \ldots, y_N)\top} \sum_{i=1}^{N} \left( \phi_i(y_i, z_i) - y_i^T \frac{q}{N} \right)
\]

s.t. \(A y = 0\). (15a)

(15b)

Note that under Assumption 1, \(A y = 0\) is equivalent to \(y_1 = y_2 = \ldots = y_N\).

We further employ the inexact augmented Lagrangian (AL) method in [33] to problem (15). In particular, let

\[
\mathcal{L}_\rho(y, \mu; z) = \sum_{i=1}^{N} \left( \phi_i(y_i, z_i) - y_i^T \frac{q}{N} \right) - \mu^T A y - \frac{\rho}{2} \|A y\|^2,
\]

be the augmented Lagrangian function of (15), in which \(\mu \in \mathbb{R}^{M|\mathcal{E}|}\) is the dual variable associated with (15b), and \(\rho > 0\) is a penalty parameter. Then, according to [33], at each iteration \(r\), we perform

\[
\mu^{r+1} = \mu^r + \alpha A y^r,
\]

\[
y^{r+1} = \arg \max_{y} \mathcal{L}_\rho(y, \mu^{r+1}; z^r) - \frac{\rho}{2} \|y - y^r\|_{L^+}^2,
\]

where \(\alpha > 0\) is a step size parameter. The proximal term \(\frac{\rho}{2} \|y - y^r\|_{L^+}^2\) is used to make the objective function of (17b) decomposable across \(y_1, \ldots, y_N\) so as to obtain a fully distributed algorithm, as we delineate below.
Updates of $x^r$ and $y^r$: According to (14) and (16), we define
\[ G(x, y, \mu; z) \triangleq \sum_{i=1}^{N} \left( f_i(x_i) + \frac{p}{2} \|x_i - z_i\|^2 + y_i^T B_i x_i - y_i^T q_N \right) - \mu^T Ay - \frac{\rho}{2} \|Ay\|^2. \] (18)

Then, by (17b), $(y^{r+1}, x^{r+1})$ is the solution of the following max-min problem:
\[ \max_y \min_{x \in \mathbb{R}^n} \left\{ G(x, y, \mu^{r+1}; z^r) - \frac{\rho}{2} \|y - y^r\|^2_{L^+} \right\}. \] (19)

Similar to [7], based on the saddle point theory, the above max-min problem is equivalent to the following min-max problem
\[ \min_x \max_y \left\{ G(x, y, \mu^{r+1}; z^r) - \frac{\rho}{2} \|y - y^r\|^2_{L^+} \right\}. \] (20)

When fixing $x$, the inner maximization problem for $y$ is a strongly concave quadratic problem. Then, one can show that the inner solution $y^{r+1}$ are decomposable and has a closed-form expression
\[ y^{r+1}_i = \frac{1}{2\rho |N_i|} (B_i x_i - q_N - A_i^T \mu^{r+1} + \rho L^+_i y^r), \] (21)
for all $i \in [N]$. By substituting (21) into (20), the outer solution $x^{r+1}$ of (20) can be determined by
\[ x^{r+1}_i = \arg \min_{x_i \in \mathbb{R}^n} \left\{ f_i(x_i) + \frac{p}{2} \|x_i - z_i\|^2 + \frac{1}{4\rho |N_i|} \|B_i x_i - q/N - A_i^T \mu^{r+1} + \rho L^+_i y^r\|^2 \right\}, \] (22)
for all $i \in [N]$, which are convex problems given $p > -\gamma^-$. Notice that
\[ L^+_i y^r = \sum_{j \in N_i} (y^r_j + y^r_j), \] (23)
and therefore both (21) and (22) can be implemented fully distributively using only local data and messages from the neighbors.

Update of $\mu^r$: Since in (21) and (22), one needs $A_i^T \mu$ instead of $\mu$. We define $p^{r}_i = A_i^T \mu$, and according to (17a), obtain
\[ p^{r+1}_i = p^{r}_i + \alpha A_i^T Ay^r \]
\[ = p^{r}_i + \alpha L^-_i y^r \]
\[ = p^{r}_i + \alpha \sum_{j \in N_i} (y^r_j - y^r_j), \] (24)
for $i \in [N]$.

By combining (21), (22), (24), and (11), we obtain the PDC algorithm (see Algorithm 1). One can see that each agent $i$ uses only local information and variables $\{y_j\}_{j \in N_i}$ from its neighbors.
Algorithm 1 Proposed PDC method for solving (P)

1: **Given** parameters $p, \beta, \alpha, \rho$, and initial variables $x_i^0 = z_i^0 \in \mathbb{R}^n$, $y_i^0 \in \mathbb{R}^M$ and $p_i^0 = 0$ for $i \in [N]$. Set $r = 1$.

2: **repeat**

3: For all $i \in [N]$ (in parallel), send $y_i^r$ to neighbour $j \in \mathcal{N}_i$ and receive $y_j^r$ from neighbour $j \in \mathcal{N}_i$. Perform

\[
\begin{align*}
p_i^{r+1} &= p_i^r + \alpha \sum_{j \in \mathcal{N}_i} (y_i^r - y_j^r), \\
x_i^{r+1} &= \arg \min_{x_i} \left\{ f_i(x_i) + \frac{\beta}{2} \|x_i - z_i^r\|^2 + \frac{1}{4\rho|\mathcal{N}_i|} \|B_i x_i - \frac{q}{N} - p_i^{r+1} + \rho L^+_i y^r\|^2 \right\}, \\
y_i^{r+1} &= \frac{1}{2\rho|\mathcal{N}_i|} \left( B_i x_i^{r+1} - \frac{q}{N} - p_i^{r+1} + \rho L^+_i y^r \right), \\
z_i^{r+1} &= z_i^r + \beta (x_i^{r+1} - z_i^r),
\end{align*}
\]

4: **Set** $r \leftarrow r + 1$.

5: **until** a predefined stopping criterion is satisfied.

3.3 Convergence Analysis

We first define the approximate solutions. As mentioned in Section 3.2 when $z = x(z)$, $z$ is a KKT solution of (P). Therefore, $\|z^t - x(z^t)\|$ can measure how close $z^t$ is to the solution set.

**Definition 1** $z$ is said to be an $\epsilon$-solution of (P) if $\|z - x(z)\|^2 < \epsilon$.

The main theoretical results for the proposed PDC method in Algorithm 1 are given in the following theorem.

**Theorem 1** Let $p > -\gamma^-$, $\rho > 0$. Then for sufficiently small $\alpha$ and $\beta$ (see (57) and (58)), the following results hold for Algorithm 1:

(a) Every limit point of the iteration sequence $\{(z^r, y^r)\}$ is a KKT solution of problem (P);

(b) There exists a $t < r$ such that $z^t$ is an $1/r$ solution, i.e,

\[
\|z^t - x(z^t)\|^2 \leq O\left( \frac{1}{r} \right).
\]

**Proof:** See Section 4. \[\blacksquare\]

Theorem 1 implies that, given constants $\rho$ and $p > -\gamma^-$, if step size parameters $\alpha$ and $\beta$ are sufficiently small, $\{(z^r, y^r)\}$ can asymptotically converge to the set of KKT solutions of problem (P). Moreover, (29) shows that an $\epsilon$-KKT solution can be obtained in $O(1/\epsilon)$ iterations. This means our algorithm can achieve the lower bound of iteration complexity of first-order algorithms for solving the nonconvex problem (P) [28]. Note from (28) that $x^{r+1} - z^r \to 0$ when $z^r$ converges, the above results also hold for the sequence $\{(x^{r+1}, y^r)\}$.
Algorithm 2 Proposed IPDC method for solving (P)

1: Given parameters $p$, $\beta$, $\alpha$, $\rho$ and stepsize $\zeta$, and initial variables $x_i^0 = z_i^0 \in \mathbb{R}^n$, $y_i^0 \in \mathbb{R}^M$ and $p_i^0 = 0$ for $i \in [N]$. Set $r = 1$.

2: repeat
3: For all $i \in [N]$ (in parallel), send $y_i^r$ to neighbour $j \in N_i$ and receive $y_j^r$ from neighbour $j$.
   Perform
   \begin{align*}
   p_i^{r+1} &= p_i^r + \alpha \sum_{j \in N_i} (y_i^r - y_j^r), \\
   x_i^{r+1} &= x_i^r - \zeta \left[ \nabla f_i(x_i^r) + p(x_i^r - z_i^r) + \frac{B_i^T}{2\rho|N_i|} \left( B_i x_i^r - \frac{q}{N} - p_i^{r+1} + \rho L_i^+ y_i^r \right) \right], \\
   y_i^{r+1} &= \frac{1}{2\rho|N_i|} \left( B_i x_i^{r+1} - \frac{q}{N} - p_i^{r+1} + \rho L_i^+ y_i^r \right), \\
   z_i^{r+1} &= z_i^r + \beta (x_i^{r+1} - z_i^r),
   \end{align*}
4: Set $r \leftarrow r + 1$.
5: until a predefined stopping criterion is satisfied.

3.4 Inexact PDC Algorithm

Instead of globally solving the subproblem (26), one may choose to solve it inexactly by using one-step gradient descent only. Specifically, we replace (26) by
\begin{align*}
   x_i^{r+1} = x_i^r - \zeta \left[ \nabla f_i(x_i^r) + p(x_i^r - z_i^r) + \frac{B_i^T}{2\rho|N_i|} \left( B_i x_i^r - \frac{q}{N} - p_i^{r+1} + \rho L_i^+ y_i^r \right) \right],
\end{align*}
where $\zeta > 0$ is a stepsize. Then we obtain the inexact PDC (IPDC) as summarized in Algorithm 2.

Theorem 2 Let $\rho > 1/2$ and $p$ is sufficiently large (see (J.96)). Then for sufficiently small $\beta$, $\zeta$ and $\alpha$ (see (58), (J.97), (J.98), respectively), Algorithm 2 has the same convergence properties as Algorithm 1.

Due to limited space, the proof of Theorem 2 is presented in Appendix J. Theorem 2 shows that the IPDC algorithm enjoys the same iteration complexity as the PDC algorithm. In practice, the inexact algorithm may exhibit slower convergence speed than its exact counterpart in terms of the iteration number, but can have a substantially smaller computation time [7, 19].

4 Proof of Theorem 1

This section presents the proof of Theorem 1. The potential function used in the proof is firstly developed, followed by showing some key error bounds and perturbation bounds. Then based on these results, we prove the claims in the theorem.
4.1 The Potential Function

To prove the convergence, we need to build a proper potential function that can monotonically descend with the iteration number. Since our proposed algorithms are primal-dual proximal algorithms, the potential function should involve both primal and dual information of (15). For ease of later use, we define

$$
Y(z) = \arg \max_{y=[y_1^\top, \ldots, y_N^\top]^\top} \sum_{i=1}^{N} \left( \phi_i(y_i, z_i) - y_i^\top \frac{q}{N} \right)
$$

s.t. $Ay = 0$ (35)

as the set of optimal solutions of (15), which is non-empty since (15) is a convex problem satisfying the Slater's condition [34]. By the definition of $\phi_i$ in (14), we define for $i \in [N]

$$
x_i(y, z) = \arg \min_{x_i} f_i(x_i) + \frac{p}{2} \|x_i - z_i\|^2 + y_i^\top B_i x_i,
$$

and let $x(y, z) = (x_1^\top(y_1, z_1), \ldots, x_N^\top(y_N, z_N))^\top$. Besides, recalling $L_\rho$ in (16), the set of optimal primal variables of (15) is given by

$$
\mathcal{Y}(\mu, z) = \arg \max_y L_\rho(y, \mu; z),
$$

which can be shown to be nonempty given (24) (see Appendix A), and the associated objective value is denoted by $d(\mu; z)$.

Next, we show some descent lemmas for $-L_\rho$ and $d$.

**Lemma 1** For the AL function in (16), we have

$$
-L_\rho(y^{r+1}, \mu^{r+1}; z^{r+1}) + L_\rho(y^r, \mu^r; z^r)
\leq \frac{1}{\alpha} \|\mu^{r+1} - \mu^r\|^2 - \frac{1}{2} \|y^{r+1} - y^r\|^2_{pL^*} - \frac{p}{2} \|y^{r+1} - y^r\|^2
\leq \frac{p}{2} \sum_{i=1}^{N} \langle z_i^{r+1} - z_i^r, z_i^{r+1} + z_i^r - 2x_i(y_i^{r+1}, z_i^{r+1}) \rangle.
$$

($\triangleq$ A1)

$$
-\mu^{r+1} - \mu^r - \alpha \langle Ay^r, A\bar{y}^r \rangle.
$$

($\triangleq$ A2)

**Proof:** See Appendix B. ■

**Lemma 2** For the dual function $d(\mu; z)$, we have

$$
d(\mu^{r+1}; z^{r+1}) - d(\mu^r; z^r)
\leq \frac{p}{2} \sum_{i=1}^{N} \langle z_i^{r+1} - z_i^r, -2x_i(y_i^{r+1}, z_i^r) + z_i^{r+1} + z_i^r \rangle - \alpha \langle Ay^r, A\bar{y}^r \rangle.
$$

($\triangleq$ B1)

where $\bar{y}^{r+1} = [(y_1^{r+1})^\top, \ldots, (y_N^{r+1})^\top]^\top \in \mathcal{Y}(\mu^{r+1}, z^{r+1})$ and $\bar{y}^r \in \mathcal{Y}(\mu^{r+1}, z^r)$.
Proof: See Appendix C.

We also consider the descent property of $G$ in (18).

Lemma 3 Define function

\[ \tilde{G}_r(x^r, y^r, \mu^r; z^r) \triangleq G(x^r, y^r, \mu^r; z^r) + \| y^r \|_L^2 + \frac{1}{2} \| y^r - y^{r-1} \|_{\rho L^+}^2. \] (40)

Then, we have

\[ \tilde{G}_{r+1}(x^{r+1}, y^{r+1}, \mu^{r+1}; z^{r+1}) - \tilde{G}_r(x^r, y^r, \mu^r; z^r) \]
\[ \leq -\frac{1}{\alpha} \| \mu^{r+1} - \mu^r \|^2 - \sum_{i=1}^{N} p + \frac{\gamma}{2} \| x_i^{r+1} - x_i^r \|^2 \]
\[ \triangleq (C1) \]
\[ -\frac{1}{2} \| (y^r - y^{r-1}) - (y^{r+1} - y^r) \|_{\rho L^+}^2 \]
\[ - \left( \frac{\rho}{2} - \frac{\alpha}{2} \right) \| y^{r+1} - y^r \|_{L^-}^2 \]
\[ + \| y^{r+1} - y^r \|_{\rho L^+}^2 \]
\[ - \left( \frac{\rho}{2} - \frac{\alpha}{2} \right) \| y^{r+1} - y^r \|_{L^-}^2 \]
\[ - \rho \| y^{r+1} - y^r \|_{L^-}^2. \] (41)

Proof: See Appendix D.

The above three lemmas can be proved by similar techniques in [17, 33]. Based on the above three lemmas, let us define the potential function as

\[ \Phi^r = \tilde{G}_r(x^r, y^r, \mu^r; z^r) - 2L\rho(y^r, \mu^r; z^r) + 2d(\mu^r; z^r). \] (42)

By combining Lemmas 1, 2 and 3, we have

\[ \Phi^{r+1} - \Phi^r \]
\[ \leq 2(A1) + 2(A2) + 2(B1) + 2(B2) + (C1) - \sum_{i=1}^{N} p + \frac{\gamma}{2} \| x_i^{r+1} - x_i^r \|^2 \]
\[ - \frac{1}{2} \| (y^r - y^{r-1}) - (y^{r+1} - y^r) \|_{\rho L^+}^2 \]
\[ - \left( \frac{\rho}{2} - \frac{\alpha}{2} \right) \| y^{r+1} - y^r \|_{L^-}^2 + \frac{\rho}{2} \left( 1 - \frac{2}{\beta} \right) \| z^{r+1} - z^r \|^2 \]
\[ - \rho \| y^{r+1} - y^r \|_{L^-}^2. \] (43)

4.2 Perturbation Bounds and Error Bounds

To show that $\Phi^r$ is a descent function, we will need some perturbation bound and error bounds. The following lemma gives the upper bound on $x(y, z)$ when $y$ and $z$ are perturbed.

Lemma 4 Consider (36) for $i \in [N]$. With $p > -\gamma^-$, we have

\[ \| x_i(y_i; z_i) - x_i(y_i'; z_i') \| \leq \sigma_1 \| z_i - z_i' \| + \sigma_2 \| y_i - y_i' \|, \] (44)
\[ \| x(y^r, z^r) - x(y^{r+1}, z^{r+1}) \| \leq \sigma_3 \| z^r - z^{r+1} \|, \] (45)

where $\bar{y}^r \in \mathcal{Y}(\bar{u}^{r+1}, z^r)$, $\bar{y}^{r+1} \in \mathcal{Y}(\bar{u}^{r+1}, z^{r+1})$, and $\sigma_1 \triangleq \frac{p(p+\gamma^+)}{p+\gamma^+}$, $\sigma_2 \triangleq \frac{B_{\max}(p+\gamma^+)}{p+\gamma^+}$ with $B_{\max} \triangleq \max_{i \in [N]} \| B_i \|$, and $\sigma_3 \triangleq \frac{p}{p+\gamma^-}$.
Proof: See Appendix E. ■

Next, we determine the error bound that characterizes the distance between \( y_{r+1} \) and the solution set \( \mathcal{Y}(\mu, z) \) of (37)

**Lemma 5** With \( p > -\gamma^- \), there exists a \( \bar{y}^r \in \mathcal{Y}(\mu^{r+1}, z^r) \) of (37) such that

\[
\begin{align*}
    ||y^{r+1} - \bar{y}^r||_L^2 &\leq a_1 ||L^+(y^{r+1} - y^r)||^2, \\
    ||y^{r+1} - \bar{y}^r||^2 &\leq a_2 ||L^+(y^{r+1} - y^r)||^2,
\end{align*}
\]

where

\[
\begin{align*}
a_1 &\triangleq \frac{\theta_1^2}{\rho^2} \left( \frac{1}{p \gamma^+} + \rho^2(p + \gamma^+) \right)^2 + \rho, \\
a_2 &\triangleq \left[ \frac{\theta_1^2}{\rho^2} \left( \frac{1}{p \gamma^-} + \rho^2(p + \gamma^+) \right)^2 + 2\theta_1^2 \right],
\end{align*}
\]

and \( \theta_1 > 0 \) is a constant only depending on the problem instance.

*Proof:* See Appendix F. ■

The following lemma describes the change of \( \bar{y} \in \mathcal{Y}(\mu, z) \) when \( z \) is perturbed.

**Lemma 6** Consider (37) with \( p > -\gamma^- \). For any \( \bar{y}^r \in \mathcal{Y}(\mu^{r+1}, z^r) \), there exists a \( \bar{y}^{r+1} \in \mathcal{Y}(\mu^{r+1}, z^{r+1}) \) such that

\[
||y^r - \bar{y}^{r+1}||^2 \leq a_3 ||z^r - z^{r+1}||^2.
\]

where \( a_3 \triangleq \theta_2^2 \left( (p + \gamma^+)^2 \sigma_3^2 + \frac{B_{\max}^2 \sigma^2}{\rho^2} \right) \) and \( \theta_2 > 0 \) only depends on the problem instance.

*Proof:* See Appendix G. ■

From above (47) and (50), we can obtain that for \( y^{r+1} \), there exists a \( \bar{y}^{r+1} \in \mathcal{Y}(\mu^{r+1}, z^{r+1}) \) so that

\[
||y^{r+1} - \bar{y}^{r+1}||^2 \leq 2a_2 ||L^+(y^{r+1} - y^r)||^2 + 2a_3 ||z^{r+1} - z^r||^2.
\]

Next, we consider the error bound between (35) and (37) due to the violation of the consensus constraint.

**Lemma 7** For any \( \bar{y}^r \in \mathcal{Y}(\mu^{r+1}, z^r) \), there exists a \( y(z^r) \in \mathcal{Y}(z^r) \) such that

\[
||y(z^r) - \bar{y}^r|| \leq \sqrt{a_4} ||A\bar{y}^r||,
\]

where \( a_4 \triangleq \frac{1}{2} \left( \frac{\theta_3^2}{p \gamma^-} + \rho^2 \theta_3^2(p + \gamma^+) \right)^2 + \frac{\theta_3^2}{p^2} \) for some \( \theta_3 > 0 \) only depending on the problem instance.
Proof: See Appendix H.

According to Lemma 4, Lemma 5, and Lemma 7, we can bound the difference between \((x^{r+1}, y^{r+1})\) generated by Algorithm 1 and the primal-dual solution \((x(z^r), y(z^r))\) of problem \((9)\) as follows.

**Corollary 1** For \(y^{r+1}\), there exists a \(\bar{y}^r \in Y(\mu^{r+1}, z^r)\) and \(y(z^r) \in Y(z^r)\) such that
\[
\|y^{r+1} - y(z^r)\| \leq \sqrt{a_2} \lambda_{\text{max}} \|y^r - y^{r+1}\| + \sqrt{a_4} \|Ay^r\|, \tag{53}
\]
Moreover,
\[
\|x^{r+1} - x(z^r)\| \leq \sigma_2 \|y^{r+1} - y(z^r)\|. \tag{54}
\]

**Proof:** To show (53), we have
\[
\|y^{r+1} - y(z^r)\| = \|y^{r+1} - \bar{y}^r + \bar{y}^r - y(z^r)\|
\leq \|y^{r+1} - \bar{y}^r\| + \|\bar{y}^r - y(z^r)\|. \tag{55}
\]
By substituting (47) in Lemma 5 and (52) in Lemma 7 into (55), one obtains (53). To prove (54), we first note from (19), (22) and (36) that \(x^{r+1} = x(y^{r+1}, z^r)\). Thus, by applying (44), we obtain
\[
\|x^{r+1} - x(z^r)\| = \|x(y^{r+1}, z^r) - x(z^r)\|
\leq \sigma_2 \|y^{r+1} - y(z^r)\|, \tag{56}
\]
which is (54).

With the above results, we are ready to prove Theorem 1.

### 4.3 Proof of Theorem 1(a)

We first show that the potential function \(\Phi^r\) in (42) is non-increasing and is lower bounded.

**Lemma 8** Let \(p > -\gamma^-\), \(\rho > 0\) and
\[
\alpha < \min \left\{ \frac{\rho}{5}, \frac{\rho}{8a_1 \lambda_{\text{max}}^2} \right\}, \tag{57}
\]
\[
\beta < \left( \frac{1}{2} + \frac{20p\sigma_2^2 a_2 \lambda_{\text{max}}^2}{\rho} + \frac{\rho(\sigma_2^2 + 2\sigma_3^2 a_3)}{10p\sigma_2^2 a_2 \lambda_{\text{max}}^2} \right)^{-1}. \tag{58}
\]

(a) Then, we have
\[
\Phi^{r+1} - \Phi^r \leq -\sum_{i=1}^{N} \frac{p + \gamma^-}{2} \|x_i^{r+1} - x_i^r\|^2 - C_1 \|y^{r+1} - y^r\|_{L_2}^2
\]
\[
-\frac{1}{2} \|(y^r - y^{r-1}) - (y^{r+1} - y^r)\|_{L_2}^2 - \alpha \|Ay^r\|^2
\]
\[
- C_2 \|y^{r+1} - y^r\|^2 - C_3 \|z^{r+1} - z^r\|^2. \tag{59}
\]

where
\[ C_1 \triangleq \frac{\rho - 5\alpha}{2} > 0, \quad C_2 \triangleq \rho - \left(2\alpha\lambda_{\text{max}} + \frac{\rho}{5\lambda_{\text{max}}^{2}}\right)^{2} > 0, \quad C_3 \triangleq p\left(-\frac{1}{2} + \frac{1}{\beta} - \frac{20p\sigma_{2}\lambda_{\text{max}}^{2}}{\rho} - \frac{\rho(\sigma_{1}^{2} + 2\sigma_{2}^{2}\alpha_{1})}{10p\sigma_{2}\alpha_{2}\lambda_{\text{max}}^{2}}\right) > 0. \]

(b) Moreover, \( \Phi^{r+1} \) is lower bounded, i.e., \( \sum_{r=0}^{T} \Phi^{r+1} > \Phi \) for some constant \( \Phi > -\infty \).

**Proof:** Firstly, consider (A1) in (38), (B2) in (39), (C1) in (41), and their combination:

\[
2(A1) + 2(B2) + (C1) = \frac{1}{\alpha} \| \mu^{r+1} - \mu^{r} \|^{2} - 2\alpha \langle Ay^{r}, A\hat{y}^{r} \rangle \leq \alpha \| Ay^{r} \|^{2} - 2\alpha \langle Ay^{r}, A\hat{y}^{r} \rangle = \alpha \| A(y^{r} - \hat{y}^{r}) \|^{2} - H \| A\hat{y}^{r} \|^{2} \leq 2\alpha \| y^{r} - y^{r+1} \|_{L_{-}}^{2} + 2\alpha \| y^{r+1} - \hat{y}^{r+1} \|_{L_{-}}^{2} - \alpha \| A\hat{y}^{r} \|^{2} \leq 2\alpha \| y^{r} - y^{r+1} \|_{L_{-}}^{2} + 2\alpha a_{1} \| L^{+}(y^{r+1} - y^{r}) \|^{2} - \alpha \| A\hat{y}^{r} \|^{2},
\]

where (i) is due to (17a), (ii) is obtained by adding and subtracting a term \( y^{r+1} \) then using inequality \( \| a + b \|^{2} \leq 2\| a \|^{2} + 2\| b \|^{2} \), and lastly (iii) is owing to (46) of Lemma 5.

Secondly, consider

\[
2(A2) + 2(B1) = 2p \sum_{i=1}^{N} \langle z^{r+1}_{i} - z^{r}_{i}, x_{i}(y^{r+1}_{i}, z^{r+1}_{i}) - x_{i}(y^{r+1}_{i}, z^{r}_{i}) \rangle \leq p\delta \| z^{r+1} - z^{r} \|^{2} + \frac{p}{\delta} \sum_{i=1}^{N} \| x_{i}(y^{r+1}_{i}, z^{r+1}_{i}) - x_{i}(y^{r+1}_{i}, z^{r}_{i}) \|^{2},
\]

where the last inequality is obtained by Young’s inequality and \( \delta > 0 \). Further applying (44) of Lemma 4 to (61), we obtain

\[
2(A2) + 2(B1) \leq p\delta \| z^{r+1} - z^{r} \|^{2} + \sum_{i=1}^{N} \frac{2p\sigma_{2}^{2}}{\delta} \| z^{r+1}_{i} - z^{r}_{i} \|^{2} + \sum_{i=1}^{N} \frac{2p\sigma_{2}^{2}}{\delta} \| y^{r+1}_{i} - \hat{y}^{r+1}_{i} \|^{2} \leq \left( p\delta + \frac{2p\sigma_{2}^{2}}{\delta} \right) \| z^{r+1} - z^{r} \|^{2} + \frac{2p\sigma_{2}^{2}}{\delta} \left[ 2a_{2} \| L^{+}(y^{r+1} - y^{r}) \|^{2} + 2a_{3} \| z^{r+1} - z^{r} \|^{2} \right] = \left( p\delta + \frac{2p\sigma_{2}^{2}}{\delta} + \frac{4p\sigma_{2}^{2}a_{3}}{\delta} \right) \| z^{r+1} - z^{r} \|^{2} + \frac{4p\sigma_{2}^{2}a_{2}}{\delta} \| L^{+}(y^{r+1} - y^{r}) \|^{2},
\]

where (i) is obtained by (51).

By substituting (60) and (62) into (43), we thereby obtain

\[
\Phi^{r+1} - \Phi^{r} = -\sum_{i=1}^{N} \frac{p + \gamma_{-}}{2} \| x^{r+1}_{i} - x^{r}_{i} \|^{2} - \frac{\rho - 5\alpha}{2} \| y^{r} - y^{r+1} \|_{L_{-}}^{2},
\]

15
\[-\frac{1}{2} \|(y^r - y^{r-1}) - (y^{r+1} - y^r)\|^2_{\rho L} - \alpha \|A\hat{y}^r\|^2 - [\rho - (2\alpha a_1 + \frac{4 \rho a_2}{\delta})\lambda_{max}^2] \|y^{r+1} - y^r\|^2_{\rho L} - \|\frac{1}{2} + 1 - \frac{2\sigma_1^2}{\delta} - \frac{4 \sigma_2 a_3}{\delta} \|z^{r+1} - z^r\|^2_{\rho L} + \|\frac{1}{2} + 1 - \frac{2\sigma_1^2}{\delta} - \frac{4 \sigma_2 a_3}{\delta} \|^2_{\rho L} \]

\[\triangleq C_2 \]

We see that if \( \alpha < \rho / 5 \), then \( C_1 \) is positive. In addition, choose \( \delta = \frac{20 \rho a_2 \lambda_{max}^2}{\rho} > \frac{16 \rho a_2 \lambda_{max}^2}{\rho} \), i.e., \( \frac{4 \rho a_2 \lambda_{max}^2}{\delta} < \frac{\rho}{\delta} \), then as long as \( \alpha < \frac{\rho}{8 a_1 \lambda_{max}^2} \), we can have \( C_2 > \rho / 2 > 0 \). Lastly, if \( \beta \) satisfies (58), then \( C_3 \) is positive. Thus, Lemma 8(a) is proved.

The proof for showing \( \Phi^r \) is lower bounded (Lemma 8(b)) is referred to Appendix I.

According to Lemma 8, we have

\[ \|x^r - x^{r+1}\| \to 0, \quad \|y^r - y^{r+1}\| \to 0, \quad \|A\hat{y}^r\| \to 0, \quad \|z^r - z^{r+1}\| \to 0. \]

(64)

(65)

By (11) and \( \|z^r - z^{r+1}\| \to 0 \), one can have \( \|x^r - z^{r+1}\| \to 0 \). Besides, combing (47) and \( \|A\hat{y}^r\| \to 0 \), one then obtains \( \|A\hat{y}^{r+1}\| \to 0 \). Therefore, we obtain

\[ \|x^{r+1} - z^{r+1}\| \to 0, \quad \|A\hat{y}^{r+1}\| \to 0. \]

(66)

By applying (64) and (66) to the KKT conditions of (17b), one then concludes that every limit point of \( \{(z^r, y^r)\} \) is a KKT solution of \( (P) \).

**4.4 Proof of Theorem 1(b)**

Since \( \Phi \) is non-increasing and lower bounded by Lemma 8, one can show that for \( r > 0 \), there exists a \( t \leq r \) such that

\[ \Phi^t - \Phi^{t+1} \leq \frac{\Phi^0 - \Phi}{r}. \]

(67)

Then, by (11), (53) and (54), we can bound

\[ \|z^t - x(z^t)\| \leq \|z^t - x^{t+1}\| + \|x^{t+1} - x(z^t)\| \]

\[ \leq \frac{1}{\beta} \|z^t - z^{t+1}\| + \sigma_2 (\sqrt{a_2 \lambda_{max}^2} \|y^t - y^{t+1}\| + a_3 \|A\hat{y}^t\|) \]

\[ \leq \sqrt{\frac{\Phi^0 - \Phi}{r}} \left( \frac{1}{\beta \sqrt{C_3}} + \frac{\sigma_2 \sqrt{a_2 \lambda_{max}^2}}{\sqrt{C_2}} + \frac{\sigma_2 a_3}{\sqrt{a_1}} \right), \]

(68)

where the last inequality is obtained by applying (59).

**5 Numerical Results**

In this section, we present two numerical examples to illustrate the empirical performance of the proposed algorithms.
5.1 Distributed Logistic Regression

Following (2), we formulate a non-convex regularized logistic regression problem [35] with

\[
\psi(b_k^Tw; v_k) = \log \left( 1 + e^{-v_k(b_k^Tw)} \right), \quad \forall k \in [M],
\]

\[
R_i(w_i) = \lambda \sum_{s=1}^{n} \frac{\xi w_{i,s}^2}{1 + \xi w_{i,s}^2}, \quad \forall i \in [N],
\]

where \( v_k \in \{ \pm 1 \} \) are binary labels, and \( \lambda, \xi > 0 \) are the parameters.

We consider the two images D24 and D68 of Brodatz data set\(^2\), and extract \( M/2 \) overlapping patches with dimension \( \sqrt{N} \times \sqrt{N} \) from these two images, respectively. Each patch is vectorized into \( 1 \times Nn \) -vector, and they are randomly shuffled and stacked into the data matrix \( B \). Here we set \( M = 100 \) and \( n = 100 \), and assumed a network with \( N = 25 \) nodes. The generation of the network graph follows the same method as in [36]. For (69), we set \( \lambda = 0.01 \) and \( \xi = 0.5 \).

We employ the proposed PDC algorithm (Algorithm 1) to handle the non-convex LR problem in the form of (4). Our aim is to use this example to examine how the algorithm parameters \( \alpha, \rho, \beta, \) and \( \rho \) can affect the convergence of proposed PDC algorithms. The fast iterative shrinkage thresholding algorithm (FISTA) [37] is used to solve subproblem (26), and the stopping condition of FISTA is that the normalized proximal gradient [37] is smaller than \( 10^{-5} \). The entries of the initial variables \( x^0 \) and \( y^0 \) are randomly generated following a uniform distribution over the interval

\(^2\)http://www.ux.uis.no/tranden/broadatz.html
Figure 4: Convergence performance of the PDC algorithm with various values of $p$ and $\beta$. 

The experiments are performed 10 times, each with a different initial $(x^0, y^0)$, and the averaged results are presented.

In particular, we assess the algorithm performance by calculating the following two terms

\begin{align}
\text{Gradient residue} &= \frac{1}{N_n} \sum_{i=1}^{N} \|\nabla f_i(x^r_i) + B_i^T y^r_i\|^2, \\
\text{Infeasibility} &= \frac{1}{M} \|\sum_{i=1}^{N} B_i x^r_i - q\|^2,
\end{align}

with respect to the iteration number (communication round) $r$. The simulation results are displayed in Fig. 3 and Fig. 4, where we respectively examine the impacts of the four parameters $\alpha$, $\rho$, $p$ and $\beta$ on the algorithm convergence.

In Fig. 3(a), we set $\beta = 0.1$, $\rho = 0.01$, $p = 0.01$, and $\alpha \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\}$. One can see that a smaller $\alpha$ may slow down the proposed PDC algorithm in terms of the gradient residue (left figure), whereas slightly reducing the feasibility level (right figure). It is observed that when $\alpha = 10^{-1} > \rho$, the algorithm cannot satisfy the linear constraint even though it achieves the smallest gradient residue. Such observation is consistent with Theorem 1.

In Fig. 3(b), we set $\beta = 0.1$, $p = 0.01$, $\alpha = 0.01$, and $\rho \in \{10^{-2}, 10^{-1}, 1, 10, 10^2\}$. One can see from the two figures that parameter $\rho$ does not significantly impact the convergence of gradient residue, but can greatly affect the constraint feasibility. In particular, a larger $\rho$ can cause larger infeasibility values.

In Fig. 4(a), we set $\beta = 0.1$, $\rho = 0.01$, $\alpha = 0.01$, and $p \in \{10^{-2}, 10^{-1}, 1, 10, 10^2\}$. One can see from the two figures that parameter $p$ does not significantly impact the constraint feasibility,
whereas a larger $p$ can slow down the convergence of gradient residue. This is expected since with a larger $p$, problem (9) is a more conservative convex approximation, and thus it slows down the algorithm convergence.

Lastly, in Fig. 4(b), we set $p = 0.01, \rho = 0.01, \alpha = 0.01$, and various values of $\rho$ and $\zeta$. Left column: $\zeta = 0.1$; Right column: $\rho = 1$.

In Appendix K, we further present the convergence results for the IPDC algorithm with respect to the algorithm parameters $\alpha$, $\rho$, $p$ and $\beta$. One can observe that they have similar trends as in Fig. 3 and Fig. 4. The IPDC has an additional step size parameter $\zeta$. Fig. 6 shows that larger $\zeta$ will speed up the convergence in general.

### 5.2 Distributed Neural Network

In this simulation, following (5), we consider the use of the NN to classify handwritten digits in the MNIST dataset. The 60000 training images are divided into 12 batches. The dimension of each image is $28 \times 28$ which is vectorized into a $1 \times 784$ vector ($nN = 784$). These vectors of the 5000 samples are stacked as the data matrix $B$. Then this data matrix is column partitioned into 8 equal parts to be distributed to 8 agents ($N = 8$). The network topology is generated in the same way as in [36].
This NN has 2 layers, with $K = 30$ neurons in the hidden layer. The activation function of the hidden layer and output layer are Rectified Linear Unit (ReLU) and softmax function, respectively. The cross-entropy loss function [38] is applied in the last layer. Then, the proposed IPDC algorithm are applied to train the classification NN by solving problem (7). The experiment is performed on the PyTorch platform.

For the IPDC algorithms, it is set that $p = 10$, $\alpha = 0.01$, $\beta = 0.01$, $\rho \in \{10^{-1}, 1, 10, 100\}$ and $\zeta \in \{10^{-3}, 10^{-2}, 10^{-1}\}$. Moreover, the simulations are preformed with 3 randomly generated initial points, and their average results are presented in Fig. 5.

Fig. 5 displays the training loss and testing accuracy of the NN versus the iteration number. From the left column of Fig. 5, one can observe that the IPDC algorithm with a larger $\rho$ has a sharper decrease in the training loss and increase in the testing accuracy in the first few iterations. But the algorithm can eventually achieve similar training loss and testing accuracy for the tested values of $\rho$. In the right column of Fig. 5, one can observe that the step size $\zeta$ can influence the performance of the proposed IPDC algorithm significantly. Though it is shown in Theorem 2 that a sufficiently small $\zeta$ is required for the algorithm convergence of the IPDC algorithm, it is found from the figure that with a larger $\zeta$ the IPDC algorithm converges faster in both training loss and testing accuracy.

6 Conclusions

In this paper, two new distributed algorithms, i.e., the PDC and the IPDC algorithms, for solving the non-convex linearly constrained problem (P) over the multi-agent network have been proposed. These algorithms can be regarded as extensions of the dual consensus ADMM method in [7] to the challenging non-convex settings, by leveraging the proximal technique in [17]. While the algorithm development is rather straightforward, it turns out that proving the convergence and convergence rate of the proposed algorithms are not. By developing some key error bounds and perturbation bounds, we have shown that, under mild conditions on the algorithm parameters, the proposed algorithms both can converge to a KKT solution of problem (P) with a complexity order of $\mathcal{O}(1/\epsilon)$. Through experiments on a non-convex logistic regression problem and an NN-based MNIST classification task, we have demonstrated the good convergence behaviors of the proposed algorithms with respect to various parameter settings.

To the best of our knowledge, the presented distributed algorithms and convergence analyses are the first for the linearly constrained non-convex problem (P). In the future, it is worthwhile to extend the current PDC framework to mini-batch stochastic gradient descent (SGD) [39] and local SGD [40], and to handle more complex learning problems with hybrid partitioned data samples and features [41]. It is also meaningful to consider distributed algorithms for problem (P) with separable constraints or structured non-smooth regularizers.

Appendices

A Proof of nonempty property of $\mathcal{Y}(\mu, z)$

To prove that the set $\mathcal{Y}(\mu, z)$ is not empty, we first give a general lemma and its proof.
Lemma 9 Let $F(x,y) = f(x) + y^T Ax + v^T y - \frac{1}{2}y^T Q y$, where $f(x)$ is a strongly convex, smooth function, $Q$ is a positive semi-definite matrix and $v \in \text{Image}(Q)$. Then the saddle point of $F(x,y)$ exists, i.e., the solution of the problem $P1$ exists

$$P1 : \min_x \max_y F(x,y).$$

Proof of Lemma 9: First, by the basic property of quadratic optimization, $\arg\max_y F(x,y)$ exists if and only if $Ax \in \text{Image}(Q)$.

Let $\text{rank}(Q) = r$. Without loss of generality, we assume $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix}$ and $Q_{11} \in \mathbb{R}^{r \times r}$ is of full rank. Let $P = \begin{bmatrix} Q_{11} \\ Q_{12} \end{bmatrix}$ and $Y = \{ y \in \mathbb{R}^m \mid y_{r+1} = \cdots = y_m = 0 \}$.

Also let $\text{Image}(Q) = \{ w \mid B w = 0 \}$, where the rows of $B \in \mathbb{R}^{(m-r) \times m}$ are the basis for $\text{Null}(Q)$. Define $X = \{ x \in \mathbb{R}^n \mid B A x = 0 \}$. Then, $x \in X$ implies $x \in \text{Image}(Q)$. For any $\tilde{x} \in X$, there exists some $\tilde{y} \in Y$ such that

$$\tilde{y} = \arg\max_{y \in Y} F(\tilde{x}, y).$$

In fact, the matrix $P$ is of rank $r$. Therefore, $\text{Image}(Q) = \text{Image}(P)$. Notice that $-A\tilde{x} - v \in \text{Image}(Q) = \text{Image}(P)$. It implies that there exists some $u \in \mathbb{R}^r$ such that

$$-A\tilde{x} - v = Pu.$$

Hence, if $\tilde{y} = (u^T, 0, \cdots, 0)^T \in Y \subseteq \mathbb{R}^m$, we have

$$\nabla_y F(\tilde{x}, \tilde{y}) = A\tilde{x} + v + Q\tilde{y} = 0.$$

Consequently, For any $\tilde{x} \in X$, there exists some $\tilde{y} \in Y$ such that

$$\tilde{y} = \arg\max_{y \in Y} F(\tilde{x}, y).$$

Then consider the following constrained strongly convex-strongly concave min-max problem:

$$P2 : \min_{x \in X} \max_{y \in Y} F(x, y).$$

Note that $F(x,y)$ is strongly concave for $y$ in $Y$, thus $P2$ is a strongly convex-strongly concave min-max problem. Since a strongly convex (concave) function has a compact sublevel set, by [42, Prop. 3.6.9], there exists a saddle point for problem $P2$. Suppose that $(\tilde{x}, \tilde{y})$ is a saddle point of $P2$. Since $\tilde{x} \in X$, there exists $y \in Y$ such that

$$\tilde{y} = \arg\max_{y \in Y} F(\tilde{x}, y) = \arg\max_{y \in \mathbb{R}^m} F(\tilde{x}, y).$$

Hence, $\nabla_y F(\tilde{x}, \tilde{y}) = 0$. We construct a saddle point of $P1$. Since $\tilde{x} = \arg\min_{x \in X} F(x, \tilde{y})$, by the KKT conditions, there exists some multiplier $\lambda \in \mathbb{R}^{m-r}$ such that

$$\nabla f(\tilde{x}) + A^T \tilde{y} + A^T B^T \lambda = 0.$$
BA\bar{x} = 0.

Let \( x^* = \bar{x} \) and \( y^* = \bar{y} + B^T\lambda \), we have
\[
\nabla_x F(x^*, y^*) = 0
\]
and
\[
\nabla_y F(x^*, y^*) = -Qy^* + v + Ax^* \\
= (-Q\bar{y} + v + A\bar{x}) + Q(\bar{y} - y^*) \\
= \nabla_y F(\bar{x}, \bar{y}) + Q(-B^T\lambda) \\
= 0,
\]
where the last equality is because the rows of \( B \) spans \( \text{Null}(Q) \). We finish the proof. \( \blacksquare \)

Note that this lemma implies the nonempty property of the \( \mathcal{Y}(\mu, z) \), where \( f(x), y^TAx, Q, v \) correspond \( \sum_{i=1}^N f_i(x_i) + \frac{\rho}{2}\|x_i - z_i\|^2 \), \( \sum_{i=1}^N y_i^T B_i x_i - y^T q/N \), \( \rho L^- \), \( A^T \mu \), respectively.

## B  Proof of Lemma 1

Notice that the change of the \( \mathcal{L}_\rho \) function after one iteration can be bounded as
\[
- \mathcal{L}_\rho(y^{r+1}, \mu^{r+1}; z^r) + \mathcal{L}_\rho(y^r, \mu^r; z^r) \\
\leq \left(- \mathcal{L}_\rho(y^r, \mu^{r+1}; z^r) + \mathcal{L}_\rho(y^r, \mu^r; z^r)\right) + \left(- \mathcal{L}_\rho(y^{r+1}, \mu^{r+1}; z^r) + \mathcal{L}_\rho(y^r, \mu^r; z^r)\right) \\
+ \left(- \mathcal{L}_\rho(y^{r+1}, \mu^{r+1}; z^r) + \mathcal{L}_\rho(y^{r+1}, \mu^{r+1}; z^r)\right).
\]
(B.6)

Then, we will analyze this desired bound by following four steps below.

(a) Bound of \(- \mathcal{L}_\rho(y^r, \mu^{r+1}; z^r) + \mathcal{L}_\rho(y^r, \mu^r; z^r)\):
\[
- \mathcal{L}_\rho(y^r, \mu^{r+1}; z^r) + \mathcal{L}_\rho(y^r, \mu^r; z^r) = \langle \mu^{r+1} - \mu^r, Ay^r \rangle = \frac{1}{\alpha} \| \mu^{r+1} - \mu^r \|^2, \quad \text{(B.7)}
\]
where the last equality is obtained from (17a).

(b) Bound of \(- \mathcal{L}_\rho(y^{r+1}, \mu^{r+1}; z^r) + \mathcal{L}_\rho(y^r, \mu^{r+1}; z^r)\): Since \( \mathcal{L}_\rho(y, \mu^{r+1}; z^r) - \frac{1}{2} \|y - y^r\|^2_{\rho L^+} \) is strongly concave with modulus \( \rho \), and
\[
\nabla_y \mathcal{L}_\rho(y^{r+1}, \mu^{r+1}; z^r) - \rho L^+ (y^{r+1} - y^r) = 0,
\]
(B.8)
by the optimality condition of (17b), we have
\[
\mathcal{L}_\rho(y^r, \mu^{r+1}; z^r) - \frac{1}{2} \|y^r - y^r\|^2_{\rho L^+} \leq \mathcal{L}_\rho(y^{r+1}, \mu^{r+1}; z^r) - \frac{1}{2} \|y^{r+1} - y^r\|^2_{\rho L^+} - \frac{\rho}{2} \|y^{r+1} - y^r\|^2.
\]
(B.9)
(c) Bound of \(-L_\rho(y^{r+1}, \mu^{r+1}; z^{r+1}) + L_\rho(y^{r+1}, \mu^{r+1}; z^r)\): Note that

\[
-L_\rho(y^{r+1}, \mu^{r+1}; z^{r+1}) + L_\rho(y^{r+1}, \mu^{r+1}; z^r) \\
= \sum_{i=1}^N \left( \phi_i(y_i^{r+1}, z_i^r) - \phi_i(y_i^{r+1}, z_i^{r+1}) \right) \\
= \sum_{i=1}^N \left( f_i(x_i(y_i^{r+1}, z_i^r)) + \frac{p}{2} \|x_i(y_i^{r+1}, z_i^r) - z_i^r\|^2 + \langle y_i^{r+1}, B_i x_i(y_i^{r+1}, z_i^r) \rangle \right) \\
- \sum_{i=1}^N \left( f_i(x_i(y_i^{r+1}, z_i^{r+1})) + \frac{p}{2} \|x_i(y_i^{r+1}, z_i^{r+1}) - z_i^{r+1}\|^2 + \langle y_i^{r+1}, B_i x_i(y_i^{r+1}, z_i^{r+1}) \rangle \right).
\]

(B.10)

Since \(x_i(y_i^{r+1}, z_i^r)\) is a minimizer of (26), we have the upper bound

\[
-L_\rho(y^{r+1}, \mu^{r+1}; z^{r+1}) + L_\rho(y^{r+1}, \mu^{r+1}; z^r) \\
\leq \sum_{i=1}^N \left( f_i(x_i(y_i^{r+1}, z_i^{r+1})) + \frac{p}{2} \|x_i(y_i^{r+1}, z_i^{r+1}) - z_i^{r+1}\|^2 + \langle y_i^{r+1}, B_i x_i(y_i^{r+1}, z_i^{r+1}) \rangle \right) \\
- \sum_{i=1}^N \left( f_i(x_i(y_i^{r+1}, z_i^r)) + \frac{p}{2} \|x_i(y_i^{r+1}, z_i^r) - z_i^r\|^2 + \langle y_i^{r+1}, B_i x_i(y_i^{r+1}, z_i^r) \rangle \right) \\
= \frac{p}{2} \sum_{i=1}^N \|x_i(y_i^{r+1}, z_i^r) - z_i^r\|^2 - \frac{p}{2} \sum_{i=1}^N \|x_i(y_i^{r+1}, z_i^{r+1}) - z_i^{r+1}\|^2 \\
= -\frac{p}{2} \sum_{i=1}^N (z_i^{r+1} - z_i^r, z_i^{r+1} - z_i^r).
\]

(B.11)

Summing (B.7), (B.9) and (B.11) leads to the desired result.

\[
C \quad \text{Proof of Lemma 2}
\]

For any \(\bar{y}^r \in \mathcal{Y}(\mu^{r+1}, z^r)\) and \(\bar{y}^r \in \mathcal{Y}(\mu^r, z^r),\)

\[
d(\mu^{r+1}; z^r) - d(\mu^r; z^r) = \left[ \sum_{i=1}^N \left( \phi_i(y_i^r, z_i^r) - \langle y_i^r, b/N \rangle \right) - \langle \mu^{r+1}, A y^r \rangle - \frac{p}{2} \|A y^r\|^2 \right] \\
- \left[ \sum_{i=1}^N \left( \phi_i(y_i^r, z_i^r) - \langle y_i^r, b/N \rangle \right) - \langle \mu^r, A y^r \rangle - \frac{p}{2} \|A y^r\|^2 \right] \\
\overset{(i)}{=} \left[ \sum_{i=1}^N \left( \phi_i(y_i^r, z_i^r) - \langle y_i^r, b/N \rangle \right) - \langle \mu^{r+1}, A y^r \rangle - \frac{p}{2} \|A y^r\|^2 \right] \\
- \left[ \sum_{i=1}^N \left( \phi_i(y_i^r, z_i^r) - \langle y_i^r, b/N \rangle \right) - \langle \mu^r, A y^r \rangle - \frac{p}{2} \|A y^r\|^2 \right]
\]

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where (i) is obtained from the fact that $\bar{y}^r$ is a maximizer of $L_\rho(y, \mu^r; z^r)$, and the last equality is deduced by (17a).

On the other hand, we have

$$
d(\mu^{r+1}; z^{r+1}) - d(\mu^{r+1}; z^r)
= \sum_{i=1}^{N} \left( f_i(x_i(\bar{y}_i^{r+1}, z_i^{r+1})) + \frac{p}{2} \|x_i(\bar{y}_i^{r+1}, z_i^{r+1}) - z_i^{r+1}\|^2 + \langle \bar{y}_i^{r+1}, B_i x_i(\bar{y}_i^{r+1}, z_i^{r+1}) \rangle \right)
- \sum_{i=1}^{N} \left( f_i(x_i(y_i^{r+1}, z_i^r)) + \frac{p}{2} \|x_i(y_i^{r+1}, z_i^r) - z_i^r\|^2 + \langle y_i^{r+1}, B_i x_i(y_i^{r+1}, z_i^r) \rangle \right)
\leq \sum_{i=1}^{N} \left( f_i(x_i(\bar{y}_i^{r+1}, z_i^{r+1})) + \frac{p}{2} \|x_i(\bar{y}_i^{r+1}, z_i^{r+1}) - z_i^{r+1}\|^2 + \langle \bar{y}_i^{r+1}, B_i x_i(\bar{y}_i^{r+1}, z_i^{r+1}) \rangle \right)
- \sum_{i=1}^{N} \left( f_i(x_i(y_i^{r+1}, z_i^r)) + \frac{p}{2} \|x_i(y_i^{r+1}, z_i^r) - z_i^r\|^2 + \langle y_i^{r+1}, B_i x_i(y_i^{r+1}, z_i^r) \rangle \right)
= \frac{p}{2} \sum_{i=1}^{N} \left( \|x_i(\bar{y}_i^{r+1}, z_i^r) - z_i^r\|^2 - \|x_i(\bar{y}_i^{r+1}, z_i^{r+1}) - z_i^{r+1}\|^2 \right)
= \frac{p}{2} \sum_{i=1}^{N} (z_i^{r+1} - z_i^r - 2x_i(\bar{y}_i^{r+1}, z_i^r) + z_i^{r+1} + z_i^r),
$$

where (i) is due to the fact that $x_i(\bar{y}_i^{r+1}, z_i^{r+1})$ is a minimizer of $\phi_i(\bar{y}_i^{r+1}, z_i^{r+1})$. 

\[\blacksquare\]
D  Proof of Lemma 3

The proof of Lemma 3 is a direct corollary of the following two lemmas.

Lemma 10  We have

\[ G(x^{r+1}, y^{r+1}, \mu^{r+1}, z^{r+1}) - G(x^r, y^r, \mu^r, z^r) \]
\[ \leq -\frac{1}{\alpha} \| \mu^{r+1} - \mu^r \|^2 - \sum_{i=1}^{N} \frac{p + \gamma}{2} \| x_i^{r+1} - x_i^r \|^2 - \frac{p}{2} \| y^{r+1} - y^r \|_L^2 + \sum_{i=1}^{N} \langle y_i^{r+1} - y_i^r, B_i(x_i^r - x_i^{r+1}) \rangle \]
\[ + \sum_{i=1}^{N} (2\rho |N_i|) \| y_i^{r+1} - y_i^r \|^2 + \frac{p}{2} \left( 1 - \frac{2}{\beta} \right) \| z^{r+1} - z^r \|^2. \]  (D.15)

Lemma 11

\[ \left( \frac{\alpha}{2} \| y^{r+1} \|^2_{L^-} + \frac{1}{2} \| y^{r+1} - y^r \|_{\rho L^+}^2 \right) - \left( \frac{\alpha}{2} \| y^r \|^2_{L^-} + \frac{1}{2} \| y^r - y^{r-1} \|_{\rho L^+}^2 \right) \]
\[ \leq -\sum_{i=1}^{N} \langle y_i^{r+1} - y_i^r, B_i(x_i^r - x_i^{r+1}) \rangle - \frac{1}{2} \| (y^r - y^{r-1}) - (y^{r+1} - y^r) \|_{\rho L^+}^2 - \left( \rho - \frac{\alpha}{2} \right) \| y^{r+1} - y^r \|_{L^-}^2. \]  (D.16)

We then give the proofs of the above two lemmas.

Proof of Lemma 10:  Notice that the change of the function \( G \) after one iteration can be bounded as

\[ G(x^{r+1}, y^{r+1}, \mu^{r+1}, z^{r+1}) - G(x^r, y^r, \mu^r, z^r) \]
\[ \leq \left( G(x^r, y^r, \mu^{r+1}, z^r) - G(x^r, y^r, \mu^r, z^r) \right) + \left( G(x^{r+1}, y^r, \mu^{r+1}, z^r) - G(x^r, y^r, \mu^{r+1}, z^r) \right) \]
\[ + \left( G(x^{r+1}, y^{r+1}, \mu^r, z^{r+1}) - G(x^r, y^{r+1}, \mu^r, z^r) \right) \]
\[ + \left( G(x^{r+1}, y^{r+1}, \mu^{r+1}, z^{r+1}) - G(x^{r+1}, y^{r+1}, \mu^{r+1}, z^r) \right). \]  (D.17)

Next, we will bound four terms in the right hand side (RHS) of the above inequality respectively.

(a)  Bound of \( G(x^r, y^r, \mu^{r+1}, z^r) - G(x^r, y^r, \mu^r, z^r) \):

\[ G(x^r, y^r, \mu^{r+1}, z^r) - G(x^r, y^r, \mu^r, z^r) = -\langle \mu^{r+1} - \mu^r, Ay^r \rangle = -\frac{1}{\alpha} \| \mu^{r+1} - \mu^r \|^2, \]  (D.18)

where the last equality is due to (17a).

(b)  Bound of \( G(x^{r+1}, y^r, \mu^{r+1}, z^r) - G(x^r, y^r, \mu^{r+1}, z^r) \):

Firstly, by the optimality condition of (14), we have

\[ \langle \nabla f_i(x_i^{r+1}) + p(x_i^{r+1} - z_i^r) + B_i^T y_i^{r+1}, x_i - x_i^{r+1} \rangle \geq 0, \forall x_i. \]  (D.19)
Secondly, since \( f_i(x_i) + \frac{p}{2} \|x_i - z_i^r\|^2 \) is strongly convex with modulus \( p + \gamma^- \), we obtain
\[
f_i(x_i^r) + \frac{p}{2} \|x_i^r - z_i^r\|^2 \geq f_i(x_i^{r+1}) + \frac{p}{2} \|x_i^{r+1} - z_i^r\|^2 + \langle \nabla f_i(x_i^{r+1}) + p(x_i^{r+1} - z_i^r), x_i^r - x_i^{r+1} \rangle + \frac{p + \gamma^-}{2} \|x_i^r - x_i^{r+1}\|^2
\]
\[
\geq f_i(x_i^{r+1}) + \frac{p}{2} \|x_i^{r+1} - z_i^r\|^2 + \langle -B_i^\top y_i^{r+1}, x_i^r - x_i^{r+1} \rangle + \frac{p + \gamma^-}{2} \|x_i^r - x_i^{r+1}\|^2,
\]
where the last inequality is due to (D.19).
Thus, we have the upper bound for the difference \( G(x^{r+1}, y^r, \mu^{r+1}, z^r) - G(x^r, y^r, \mu^{r+1}, z^r) \) as
\[
G(x^{r+1}, y^r, \mu^{r+1}, z^r) - G(x^r, y^r, \mu^{r+1}, z^r) = \sum_{i=1}^N \left( f_i(x_i^{r+1}) + \frac{p}{2} \|x_i^{r+1} - z_i^r\|^2 - f_i(x_i^r) - \frac{p}{2} \|x_i^r - z_i^r\|^2 \right) + \sum_{i=1}^N \langle y_i^r, B_i(x_i^{r+1} - x_i^r) \rangle
\]
\[
\leq \sum_{i=1}^N \langle y_i^{r+1} - y_i^r, B_i(x_i^r - x_i^{r+1}) \rangle - \frac{p + \gamma^-}{2} \|x_i^r - x_i^{r+1}\|^2,
\]
where the last equality is obtained by (D.20).

(c) Bound of \( G(x^{r+1}, y^{r+1}, \mu^{r+1}, z^r) - G(x^{r+1}, y^r, \mu^{r+1}, z^r) \):

Since
\[
\frac{p}{2} \|Ay^{r+1}\|^2 \geq \frac{p}{2} \|Ay^r\|^2 + \rho\langle L^-y^r, y^{r+1} - y^r \rangle + \frac{p}{2} \|y^{r+1} - y^r\|^2_{L^-},
\]
we can have
\[
G(x^{r+1}, y^{r+1}, \mu^{r+1}, z^r) - G(x^{r+1}, y^r, \mu^{r+1}, z^r) = \sum_{i=1}^N \langle y_i^{r+1} - y_i^r, B_i x_i - \frac{q}{N} \rangle - \langle \mu^{r+1}, A(y^{r+1} - y^r) \rangle - \frac{p}{2} \|Ay^{r+1}\|^2 + \frac{p}{2} \|Ay^r\|^2
\]
\[
= \sum_{i=1}^N \langle y_i^{r+1} - y_i^r, B_i x_i - \frac{q}{N} - A_i^\top \mu^{r+1} \rangle - \frac{p}{2} \|Ay^{r+1}\|^2 + \frac{p}{2} \|Ay^r\|^2
\]
\[
\leq \sum_{i=1}^N \langle y_i^{r+1} - y_i^r, B_i x_i - \frac{q}{N} - A_i^\top \mu^{r+1} \rangle - \rho\langle L^-y^r, y^{r+1} - y^r \rangle - \frac{p}{2} \|y^{r+1} - y^r\|^2_{L^-}.
\]
Using the fact that \( L^+ = 2D - L^- \), we further have
\[
G(x^{r+1}, y^{r+1}, \mu^{r+1}, z^r) - G(x^{r+1}, y^r, \mu^{r+1}, z^r)
\]
\[
\leq \sum_{i=1}^N \langle y_i^{r+1} - y_i^r, B_i x_i - \frac{q}{N} - A_i^\top \mu^{r+1} + \rho L_i^+y^r \rangle - 2\rho\langle D_y^r, y^{r+1} - y^r \rangle - \frac{p}{2} \|y^{r+1} - y^r\|^2_{L^-}
\]
\[
\leq \sum_{i=1}^N \langle y_i^{r+1} - y_i^r, 2\rho|N_i|^y_i^{r+1} \rangle - 2\rho\langle D_y^r, y^{r+1} - y^r \rangle - \frac{p}{2} \|y^{r+1} - y^r\|^2_{L^-}
\]
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where (i) is obtained by (21).

(d) Bound of $G(x^{r+1}, y^{r+1}, \mu^{r+1}, z^{r+1}) - G(x^{r+1}, y^{r+1}, \mu^{r+1}, z^r)$: It can be obtained as

$$G(x^{r+1}, y^{r+1}, \mu^{r+1}, z^{r+1}) - G(x^{r+1}, y^{r+1}, \mu^{r+1}, z^r) = \frac{p}{2} \|x^{r+1} - z^r\|^2 - \frac{p}{2} \|x^{r+1} - z^r\|^2$$

$$= \frac{p}{2}(-z^{r+1} + z^r, 2(x^{r+1} - z^r) - z^{r+1} + z^r)$$

$$= \frac{p}{2}(-z^{r+1} + z^r, \frac{2}{\beta}(z^{r+1} - z^r) - z^{r+1} + z^r)$$

$$= \frac{p}{2}\left(1 - \frac{2}{\beta}\right)\|z^{r+1} - z^r\|^2,$$

where (i) is due to the updating step (28) for $z$ of algorithm 1.

Summing (D.18), (D.21), (D.23), and (D.24) gives the desired bound in (D.15).

**Proof of Lemma 11:** Firstly, the optimality condition of (17b) for iteration $r$ and $r-1$ are given as

$$\left\langle \nabla_y \left( L_p(y^{r+1}, \mu^{r+1}, z^r) - \frac{p}{2} \|y^{r+1} - y^r\|^2 \right), y - y^{r+1} \right\rangle \leq 0, \forall y,$$

(D.25)

$$\left\langle \nabla_y \left( L_p(y^r, \mu^{r+1}, z^r) - \frac{p}{2} \|y^r - y^{r-1}\|^2 \right), y - y^{r-1} \right\rangle \leq 0, \forall y.$$  

(D.26)

According to Danskin’s theorem [14, Proposition B.22], we have

$$\nabla_{y_i} \phi_i(y_i^{r+1}, z^r) = B_i x_i^{r+1}.$$  

(D.27)

Hence, the gradient of $L_p(y, \mu; z)$ is given by

$$\nabla_{y_i} L_p(y^{r+1}, \mu^{r+1}, z^r) = B_i x_i^{r+1} - \frac{q}{N} - A_i^\top \mu^{r+1} - \rho L_i^- y_i^{r+1}.$$  

Substituting the above equality into (D.25), we have

$$(B_i x_i^{r+1} - b/N - A_i^\top \mu^{r+1} - \rho L_i^- y_i^{r+1} - \rho L_i^+(y_i^{r+1} - y_i^r), y_i - y_i^{r+1}) \leq 0, \forall y_i.$$  

(D.28)

$$(B_i x_i^r - b/N - A_i^\top \mu^r - \rho L_i^- y_i^r - \rho L_i^+(y_i^r - y_i^{r-1}), y_i - y_i^r) \leq 0, \forall y_i.$$  

(D.29)

Letting $y_i = y_i^r$ and $y_i = y_i^{r+1}$ in the first and second equations above respectively, and summing up these two inequalities, we obtain

$$\langle B_i(x_i^r - x_i^{r+1}), y_i^{r+1} - y_i^r \rangle - \langle A_i^\top (\mu^r - \mu^{r+1}), y_i^{r+1} - y_i^r \rangle - \langle \rho L_i^-(y_i^r - y_i^{r+1}), y_i^{r+1} - y_i^r \rangle$$

$$- \langle (\rho L_i^+)((y_i^r - y_i^{r-1}) - (y_i^{r+1} - y_i^r)), y_i^{r+1} - y_i^r \rangle \leq 0,$$

(D.30)
for $\forall i \in [N]$. By summing the $N$ equations, one obtains

$$
\sum_{i=1}^{N} \langle B_i(x_i^r - x_i^{r+1}), y_i^{r+1} - y_i^r \rangle \leq \langle \mu^r - \mu^{r+1}, A(y_i^{r+1} - y_i^r) \rangle - \rho \|y_i^{r+1} - y_i^r\|^2_{L^-} + \langle (y_i^r - y_i^{r-1}) - (y_i^{r+1} - y_i^r), \rho L^+(y_i^{r+1} - y_i^r) \rangle.
$$

(D.31)

Notice that

$$
\langle \mu^r - \mu^{r+1}, A(y_i^{r+1} - y_i^r) \rangle = \langle \alpha A y^r, A(y^r - y^{r+1}) \rangle = \frac{\alpha}{2} \|y_i^{r+1} - y_i^r\|^2_{L^-} + \frac{\alpha}{2} \|y_i^r - y_i^{r-1}\|^2_{\rho L^+} - \frac{\alpha}{2} \|y_i^{r+1} - y_i^r\|^2_{\rho L^-},
$$

(D.32)

in which (i) is obtained from (17a). Moreover, we have

$$
\langle (y_i^r - y_i^{r-1}) - (y_i^{r+1} - y_i^r), \rho L^+(y_i^{r+1} - y_i^r) \rangle = \frac{1}{2} \|y_i^r - y_i^{r-1}\|^2_{\rho L^+} - \frac{1}{2} \|y_i^{r+1} - y_i^r\|^2_{\rho L^+} - \frac{1}{2} \|y_i^{r+1} - y_i^r\|^2_{\rho L^-}.
$$

(D.33)

By substituting (D.32) and (D.33) into (D.31), we obtain

$$
\sum_{i=1}^{N} \langle y_i^{r+1} - y_i^r, B_i(x_i^r - x_i^{r+1}) \rangle
\leq \frac{\alpha}{2} \|y_i^r\|^2_{L^-} - \frac{\alpha}{2} \|y_i^{r+1}\|^2_{L^-} + \frac{1}{2} \|y_i^r - y_i^{r-1}\|^2_{\rho L^+} - \frac{1}{2} \|y_i^{r+1} - y_i^r\|^2_{\rho L^+} - \|y_i^{r+1} - y_i^r\|^2_{\rho L^-}
- \left( \rho - \frac{\alpha}{2} \right) \|y_i^{r+1} - y_i^r\|^2_{L^-}.
$$

(D.34)

After rearranging the inequality above, we have

$$
\left( \frac{\alpha}{2} \|y_i^{r+1}\|^2_{L^-} + \frac{1}{2} \|y_i^{r+1} - y_i^r\|^2_{\rho L^+} \right) - \left( \frac{\alpha}{2} \|y_i^r\|^2_{L^-} + \frac{1}{2} \|y_i^r - y_i^{r-1}\|^2_{\rho L^+} \right)
\leq - \sum_{i=1}^{N} \langle y_i^{r+1} - y_i^r, B_i(x_i^r - x_i^{r+1}) \rangle - \frac{1}{2} \|y_i^r - y_i^{r-1}\|^2_{\rho L^+} - \left( \rho - \frac{\alpha}{2} \right) \|y_i^{r+1} - y_i^r\|^2_{L^-}.
$$

(D.35)

E Proof of Lemma 4

E.1 Proof of (44) in Lemma 4

The proof of (44) is similar to [17, Lemma 3.6]. Recall from (36) that

$$
x_i(y_i, z_i) = \arg \min_{x_i} \left\{ f_i(x_i) + \frac{p}{2} \|x_i - z_i\|^2 + y_i^\top B_i x_i \right\},
$$

(E.36)
which is a strongly convex problem with modulus $p + \gamma^-$. Thus, according to the error bound for strongly convex problems in [43, Theorem 3.1], we have

$$\|x_i - x_i(y_i, z_i)\| \leq \frac{(p + \gamma^+ + 1)}{p + \gamma^-} \|\nabla f_i(x_i) + p(x_i - z_i) + B_i^\top y_i\|, \quad (E.37)$$

for all $x_i \in \mathbb{R}^n$. Since

$$\nabla f_i(x_i(y'_i, z'_i)) + p(x_i(y'_i, z'_i) - z'_i) + B_i^\top y'_i = 0, \quad (E.38)$$

by letting $x_i = x_i(y'_i, z'_i)$ in (E.37), we obtain

$$\|x_i(y'_i, z'_i) - x_i(y_i, z_i)\| \leq \frac{(p + \gamma^+ + 1)}{p + \gamma^-} \|p(z_i - z'_i) + B_i^\top (y_i - y'_i)\| \quad (E.39)$$

By substituting (E.38) into (E.39), we have

$$\|x_i(y'_i, z'_i) - x_i(y_i, z_i)\| \leq \frac{(p(p + \gamma^+ + 1)}{p + \gamma^-} \|p(z_i - z'_i) + \|B_i\|(p + \gamma^+ + 1)}{p + \gamma^-} \|y_i - y'_i\| \quad (E.40)$$

where (i) is obtained by the triangle inequality and Cauchy-Schwarz inequality.

**E.2 Proof of (45) in Lemma 4**

To show (45), let

$$\bar{h}(x; \mu, z) = \max_y \left\{ G(x, y, \mu, z) - \frac{p}{2} \|z\|^2 \right\}. \quad (\text{E.41})$$

By (36), $x(\bar{y}, z)$ is the minimizer of the function $\bar{h}(x; \mu, z)$, where $\bar{y} \in \mathcal{Y}(\mu, z)$. Then for any $\bar{y}^r \in \mathcal{Y}(\mu^{r+1}, z^r)$ and $\bar{y}^{r+1} \in \mathcal{Y}(\mu^{r+1}, z^{r+1})$, we can have $\nabla \bar{h}(x(\bar{y}^r, z^r); \mu^{r+1}, z^r) = 0$ and $\nabla \bar{h}(x(\bar{y}^{r+1}, z^{r+1}); \mu^{r+1}, z^{r+1}) = 0$. Due to the fact that $\bar{h}$ is strongly convex with modulus $(p + \gamma^-)$, we have

$$\bar{h}(x(\bar{y}^{r+1}, z^{r+1}); \mu^{r+1}, z^r) - \bar{h}(x(\bar{y}^r, z^r); \mu^{r+1}, z^r) \geq \frac{p + \gamma^-}{2} \|x(\bar{y}^r, z^r) - x(\bar{y}^{r+1}, z^{r+1})\|^2, \quad (E.41)$$

and

$$\bar{h}(x(\bar{y}^r, z^r); \mu^{r+1}, z^{r+1}) - \bar{h}(x(\bar{y}^{r+1}, z^{r+1}); \mu^{r+1}, z^{r+1}) \geq \frac{p + \gamma^-}{2} \|x(\bar{y}^r, z^r) - x(\bar{y}^{r+1}, z^{r+1})\|^2. \quad (E.42)$$

Then, by summing (E.41) and (E.42), we obtain

$$\bar{h}(x(\bar{y}^{r+1}, z^{r+1}); \mu^{r+1}, z^r) - \bar{h}(x(\bar{y}^r, z^r); \mu^{r+1}, z^r) + \bar{h}(x(\bar{y}^r, z^r); \mu^{r+1}, z^{r+1}) - \bar{h}(x(\bar{y}^{r+1}, z^{r+1}); \mu^{r+1}, z^{r+1})$$

29
\[ (p + \gamma^{-})\|x(\bar{y}^r, z^r) - x(\bar{y}^{r+1}, z^{r+1})\|^2. \quad \text{(E.43)} \]

On the other hand, according to the definition of \( \bar{h} \), we have
\[
\begin{align*}
\bar{h}(x(\bar{y}^r, z^r); \mu^{r+1}, z^r) - \bar{h}(x(\bar{y}^r, z^r); \mu^{r+1}, z^{r+1}) \\
= \frac{p}{2} \|x(\bar{y}^r, z^r) - z^r\|^2 - \frac{p}{2} \|z^r\|^2 - \left( \frac{p}{2} \|x(\bar{y}^r, z^r) - z^{r+1}\|^2 - \frac{p}{2} \|z^{r+1}\|^2 \right) \\
= px(\bar{y}^r, z^r)^\top (z^{r+1} - z^r),
\end{align*}
\]
and
\[
\begin{align*}
\bar{h}(x(\bar{y}^{r+1}, z^{r+1}); \mu^{r+1}, z^r) - \bar{h}(x(\bar{y}^{r+1}, z^{r+1}); \mu^{r+1}, z^{r+1}) = px(\bar{y}^{r+1}, z^{r+1})^\top (z^{r+1} - z^r). \quad \text{(E.45)}
\end{align*}
\]

By taking the difference between (E.44) and (E.45), and substituting it into (E.43), we obtain
\[
\begin{align*}
(p + \gamma^{-})\|x(\bar{y}^r, z^r) - x(\bar{y}^{r+1}, z^{r+1})\|^2 \\
\leq p\|x(\bar{y}^r, z^r) - x(\bar{y}^{r+1}, z^{r+1})\|^\top (z^r - z^{r+1}) \\
\leq p\|x(\bar{y}^r, z^r) - x(\bar{y}^{r+1}, z^{r+1})\|\|z^r - z^{r+1}\|. \quad \text{(E.46)}
\end{align*}
\]
Dividing \((p + \gamma^{-})\|x(\bar{y}^r, z^r) - x(\bar{y}^{r+1}, z^{r+1})\|\) from both sides of the above inequality, we have
\[
\|x(\bar{y}^r, z^r) - x(\bar{y}^{r+1}, z^{r+1})\| \leq \frac{p}{p + \gamma^{-}}\|z^r - z^{r+1}\| \quad \text{(E.47)}
\]
which is (45).

\[ \blacksquare \]

### F Proof of Lemma 5

To prove the dual error bounds, we recap the Hoffman error bound [44, Lemma 3.2.3] given below.

**Lemma 12** Let \( A \in \mathbb{R}^{m \times n}, C \in \mathbb{R}^{k \times n}, b \in \mathbb{R}^m, d \in \mathbb{R}^k \), and the linear set
\[
S = \{x | Ax \leq b, Cx = d\}.
\]
The distance from a point \( \bar{x} \in \mathbb{R}^n \) to \( S \) is bounded by:
\[
\text{dist}^2(\bar{x}, S) \leq \beta^2(\|A\bar{x} - b\|_+ + \|C\bar{x} - d\|),
\]
where \((\cdot)_+ = \max\{\cdot, 0\}\) denotes the projection to the nonnegative orthant and \( \beta \) is a positive constant depending on \( A \) and \( C \) only.

To prove Lemma 5, let us define \( F_i(x_i) = f_i(x_i) + \frac{p}{2}\|x_i - z_i^r\|^2 \) for all \( i \in [N] \), and thus \( \nabla F_i(x_i) = \nabla f_i(x_i) + p(x_i - z_i^r) \). Note that function \( F_i \) is strongly convex with modulus \((p + \gamma^-)\) and its gradient \( \nabla F_i \) is \((p + \gamma^-)\)-Lipschitz continuous. Then by [45, Eqn. (2.1.8)]), we have
\[
\|\nabla F_i(x_i) - \nabla F_i(x_i')\|^2 \leq (p + \gamma^-)(\nabla F_i(x_i) - \nabla F_i(x_i'), x_i - x_i'). \quad \text{(F.48)}
\]
Besides, since function $\phi_i$ in (14) is concave over $y$ and smooth with Lipschitz constant $\frac{1}{p+\gamma}$, we have

$$
\|\nabla \phi_i(x_i) - \nabla \phi_i(x'_i)\|^2 \leq -\frac{1}{p+\gamma} \langle \nabla \phi_i(x_i) - \nabla \phi_i(x'_i), x_i - x'_i \rangle.
$$

(F.49)

Recall (37) and let

$$
Y(\mu^{r+1}, z^r) = \arg \max_y \sum_{i=1}^N \left( \phi_i(y_i, z_i^r) - \frac{1}{N} y_i^T q \right) - \langle \mu^{r+1}, A y \rangle - \frac{\rho}{2} \|Ay\|^2.
$$

(F.50)

For $\bar{y}^r \in Y(\mu^{r+1}, z^r)$, we have the following KKT conditions

$$
\begin{bmatrix}
\nabla \phi_1(y_1^{r+1}, z_1^r) - \frac{q}{N} \\
\vdots \\
\nabla \phi_N(y_N^{r+1}, z_N^r) - \frac{q}{N}
\end{bmatrix} - A^T \mu^{r+1} - \rho L^- \bar{y}^r = 0,
$$

(F.51)

$$
\nabla F_i(x_i(y_i^{r+1}, z_i^r)) + B_i^T y_i^{r+1} = 0, \forall i \in [N],
$$

(F.52)

where $\nabla \phi_i(y_i^{r+1}, z_i^r) = B_i x_i(y_i^{r+1}, z_i^r)$ for all $i$ obtained by the Danskin’s Theorem [14, Proposition B.22]. Analogously, from (17b) the KKT conditions for $y^{r+1}$ are

$$
\begin{bmatrix}
\nabla \phi_1(y_1^{r+1}, z_1^r) - \frac{q}{N} \\
\vdots \\
\nabla \phi_N(y_N^{r+1}, z_N^r) - \frac{q}{N}
\end{bmatrix} - A^T \mu^{r+1} - \rho L^- (y^{r+1} - y^r) = 0,
$$

(F.53)

$$
\nabla F_i(x_i(y_i^{r+1}, z_i^r)) + B_i^T y_i^{r+1} = 0, \forall i \in [N],
$$

(F.54)

where $\nabla \phi_i(y_i^{r+1}, z_i^r) = B_i x_i(y_i^{r+1}, z_i^r)$ for all $i$.

Let $\bar{y}^r$ be the projection point of $y^{r+1}$ onto $Y(\mu^{r+1}, z^r)$, i.e., $\text{dist}(y^{r+1}, Y(\mu^{r+1}, z^r)) = \|y^{r+1} - \bar{y}^r\|$. With fixed $\{x_i(y_i^{r+1}, z_i^r)\}_{i=1}^N$, (F.51)-(F.52) forms a linear system of variable $\rho \bar{y}^r$, and any $\rho \bar{y}^r$ satisfying (F.51)-(F.52) is a point in $Y(\mu^{r+1}, z^r)$. By (F.53)-(F.54) and the Hoffman bound in Lemma 12, we can bound $\text{dist}(y^{r+1}, Y(\mu^{r+1}, z^r))$ for some constant $\theta_1 > 0$ as follows

$$
\rho^2 \text{dist}^2(y^{r+1}, Y(\mu^{r+1}, z^r))
\leq \theta_1^2 N \sum_{i=1}^N \|B_i(x_i(y_i^{r+1}, z_i^r) - x_i(y_i^r, z_i^r))\|^2 + \theta_1^2 \rho^2 \sum_{i=1}^N \|\nabla F_i(x_i(y_i^{r+1}, z_i^r)) - \nabla F_i(x_i(y_i^r, z_i^r))\|^2
\leq -\frac{\theta_1^2}{p+\gamma} \sum_{i=1}^N \langle B_i x_i(y_i^{r+1}, z_i^r) - B_i x_i(y_i^r, z_i^r), y_i^{r+1} - \bar{y}_i \rangle
\leq -\frac{\theta_1^2}{p+\gamma} \sum_{i=1}^N \langle B_i x_i(y_i^{r+1}, z_i^r) - B_i x_i(y_i^r, z_i^r), y_i^{r+1} - \bar{y}_i \rangle
$$

(F.50)

$$
+ \theta_1^2 \rho^2 (p+\gamma) \sum_{i=1}^N \langle \nabla F_i(x_i(y_i^{r+1}, z_i^r)) - \nabla F_i(x_i(y_i^r, z_i^r)), x_i(y_i^{r+1}, z_i^r) - x_i(y_i^r, z_i^r) \rangle
+ \theta_1^2 \rho^2 \|L^+(y^{r+1} - y^r)\|^2
$$

(F.51)

$$
= -\frac{\theta_1^2}{p+\gamma} \sum_{i=1}^N \langle B_i x_i(y_i^{r+1}, z_i^r) - B_i x_i(y_i^r, z_i^r), y_i^{r+1} - \bar{y}_i \rangle
$$

(F.52)
Thus, (16). \n
\[ \nabla \theta_i \rho^2(p + \gamma^+) \sum_{i=1}^{N} (B_i^T y_{i}^{r+1} - B_i^T \bar{y}_i^r, x_i(y_{i}^{r+1}, z_i^r) - x_i(\bar{y}_i^r, z_i^r)) \]
\[ + \theta_i^2 \rho^2 \|L^+(y^{r+1} - y^r)\|^2, \]

where the first term of (i) is due to (F.49), the third term of (i) is directly obtained by (F.48), and (ii) is owing to (F.52) and (F.54).

Rearranging the right hand side (RHS) of (F.55), we further have the upper bound as

\[ \rho^2 \text{dist}^2(y^{r+1}, \theta(y(\mu^{r+1}, z^r))) \]
\[ \leq - \left( \frac{\theta_i^2}{p + \gamma^-} + \theta_i^2 \rho^2(p + \gamma^+) \right) \sum_{i=1}^{N} \langle \nabla \phi_i(y^{r+1}, z^r) - \nabla \phi_i(\bar{y}_i^r, z_i^r), y_i^{r+1} - \bar{y}_i^r \rangle \]
\[ + \theta_i^2 \rho^2 \|L^+(y^{r+1} - y^r)\|^2 \]
\[ \equiv \left( \frac{\theta_i^2}{p + \gamma^-} + \theta_i^2 \rho^2(p + \gamma^+) \right) \sum_{i=1}^{N} \langle \nabla \phi_i(y^{r+1}, z^r) - \nabla \phi_i(\bar{y}_i^r, z_i^r), y_i^{r+1} - \bar{y}_i^r \rangle \]
\[ + \theta_i^2 \rho^2 \|L^+(y^{r+1} - y^r)\|^2 \]
\[ \equiv \left( \frac{\theta_i^2}{p + \gamma^-} + \theta_i^2 \rho^2(p + \gamma^+) \right) \langle \nabla \phi_i(y^{r+1}, z^r) - \nabla \phi_i(\bar{y}_i^r, z_i^r), y_i^{r+1} - \bar{y}_i^r \rangle \]
\[ - \left( \frac{\theta_i^2}{p + \gamma^-} + \theta_i^2 \rho^2(p + \gamma^+) \right) \rho \|y^{r+1} - \bar{y}_i^r\|_L^2 + \theta_i^2 \rho^2 \|L^+(y^{r+1} - y^r)\|^2, \]

where (i) is due to the definition of \( \nabla \phi_i \), and (ii) is obtained by substituting the gradient of \( L_\rho \) in (16).

On the other hand, by the optimality conditions of (37) for \( \bar{y}^r \) and \( \bar{y}^{r+1} \) respectively, we have

\[ \nabla \rho \bar{y}^{r+1} = 0, \]
\[ \nabla \rho \bar{y}^r = -\rho L^+(y^{r+1} - y^r). \]

Thus,

\[ \langle y^{r+1} - \bar{y}^r, \nabla \rho \bar{y}^{r+1}, z^r \rangle - \nabla \rho \bar{y}^r, z^r \rangle = \langle y^{r+1} - \bar{y}^r, -\rho L^+(y^{r+1} - y^r) \rangle. \]

Substituting (F.59) into (F.56), we have

\[ \rho^2 \|y^{r+1} - \bar{y}^r\|^2 \]
\[ \leq - \left( \frac{\theta_i^2}{p + \gamma^-} + \theta_i^2 \rho^2(p + \gamma^+) \right) \langle y^{r+1} - \bar{y}^r, -\rho L^+(y^{r+1} - y^r) \rangle \]
\[ - \left( \frac{\theta_i^2}{p + \gamma^-} + \theta_i^2 \rho^2(p + \gamma^+) \right) \rho \|y^{r+1} - \bar{y}_i^r\|_L^2 + \theta_i^2 \rho^2 \|L^+(y^{r+1} - y^r)\|^2 \]
\[ \leq \theta_i^2 \left( \frac{1}{p + \gamma^-} + \rho^2(p + \gamma^+) \right) \|y^{r+1} - \bar{y}^r\|^2 \]
Then, by applying the Hoffman bound in Lemma 12, the distance from $\bar{y}^r(G.63)$ to the set
where $\tilde{\eta}$

Proof of lemma 7

In addition, by substituting the chosen $\eta_1$ into (F.60), we further obtain the desired result in

G  Proof of Lemma 6

For any $y^r \in \mathcal{Y}(\mu^{r+1}, z^r)$, we have the KKT condition in (F.51) and (F.52).

Analogously, recall from (37) that

In addition, by substituting the chosen $\eta_1$ into (F.60), we further obtain the desired result in

H  Proof of lemma 7

For $\bar{y}^r \in \mathcal{Y}(\mu^{r+1}, z^r)$, recall the KKT conditions in (F.51) and (F.52).
For \( y(z^r) \in \mathcal{Y}(z^r) \), recall from (35) and we have its KKT conditions as
\[
\begin{bmatrix}
B_1x_1(z^r) - q/N \\
\vdots \\
B_Nx_N(z^r) - q/N
\end{bmatrix} - A^\top \mu(z^r) - \rho L^- y(z^r) = 0,
\]
\[
\nabla f_i(x_i(z^r)) + p(x_i(z^r) - z_i^r) + B_i^\top y_i(z^r) = 0, \forall i \in [N],
\]
\[
A y(z^r) = 0.
\]
(H.65) \hspace{1cm} (H.66) \hspace{1cm} (H.67)

With fixed \( \{x_i(\bar{y}_i^r, z_i^r)\}_{i=1}^N \), (F.51)-(F.52) forms a linear system of variable \( \rho \bar{y}^r \) and \( \mu(z^r) \). Then, by using the Hoffman bound in Lemma 12, the distance from \( y(z^r) \) and \( \mu(z^r) \) satisfying (H.65)-(H.67) to the set \( \mathcal{Y}(\mu^{r+1}, z^r) \) can be bounded as
\[
\rho^2 \|\bar{y}^r - y(z^r)\|^2 + \|\mu^{r+1} - \mu(z^r)\|^2 \\
\leq \theta_3^2 \sum_{i=1}^N \|B_i x_i(\bar{y}_i^r, z_i^r) - B_i x_i(z^r)\|^2 \\
+ \rho^2 \theta_3^2 \sum_{i=1}^N \|\nabla f_i(x_i(\bar{y}_i^r, z_i^r)) - \nabla f_i(x_i(z^r)) + p(x_i(\bar{y}_i^r, z_i^r) - x_i(z^r))\|^2 + \theta_3^2 \|A \bar{y}^r\|^2
\]
\[
\leq \left(-\frac{\theta_3^2}{p + \gamma} \right) \sum_{i=1}^N \langle B_i x_i(\bar{y}_i^r, z_i^r) - B_i x_i(z^r), \bar{y}_i^r - y_i(z^r)\rangle \\
+ \rho^2 \theta_3^2 (p + \gamma^+) \sum_{i=1}^N \langle \nabla f_i(x_i(\bar{y}_i^r, z_i^r)) - \nabla f_i(x_i(z^r)) + p[x_i(\bar{y}_i^r, z_i^r) - x_i(z^r)], x_i(\bar{y}_i^r, z_i^r) - x_i(z^r)\rangle \\
+ \theta_3^2 \|A \bar{y}^r\|^2
\]

(i) \hspace{1cm} (ii)

\[
= \left(-\frac{\theta_3^2}{p + \gamma} + \rho^2 \theta_3^2 (p + \gamma^+) \right) \sum_{i=1}^N \langle B_i x_i(\bar{y}_i^r, z_i^r) - B_i x_i(z^r), \bar{y}_i^r - y_i(z^r)\rangle + \theta_3^2 \|A \bar{y}^r\|^2
\]

(H.68)

where (i) is obtained by (F.49), (ii) is due to (F.52) and (H.66). Further, note that we have
\[
- \sum_{i=1}^N \langle B_i x_i(\bar{y}_i^r, z_i^r) - B_i x_i(z^r), \bar{y}_i^r - y_i(z^r)\rangle
\]
\[
\leq \left(-\rho \mu^{r+1} - \mu(z^r) + \rho L^- (\bar{y}^r - y(z^r)), \bar{y}^r - y(z^r)\right)
\]
\[
\langle \mu^{r+1} - \mu(z^r), A \bar{y}^r\rangle - \|\bar{y}^r - y(z^r)\|^2_{\rho L^-}
\]
\[
\leq \left(-\rho \mu^{r+1} - \mu(z^r), A \bar{y}^r\right),
\]
\[
\rho^2 \|\bar{y}^r - y(z^r)\|^2 + \|\mu^{r+1} - \mu(z^r)\|^2
\]

(H.69)
Note that, rearranging the above inequality, we have

\[ -\left( \frac{\theta_3^2}{p+\gamma} + \rho^2 \theta_3^2(p + \gamma^+) \right) \langle A\bar{y}^r, \mu^{r+1} - \mu(z^r) \rangle + \theta_3^2 ||A\bar{y}^r||^2. \]

\[ \text{(H.70)} \]

Rearranging the above inequality, we have

\[ \|\bar{y}^r - y(z^r)\|^2 \]
\[ \leq \frac{\theta_3^2}{\rho^2} \|A\bar{y}^r\|^2 - \frac{1}{\rho^2} \|\mu^{r+1} - \mu(z^r)\|^2 - \left( \frac{\theta_3^2}{\rho^2(p + \gamma^-)} + \theta_3^2(p + \gamma^+) \right) \langle A\bar{y}^r, \mu^{r+1} - \mu(z^r) \rangle, \]
\[ \text{(i)} \]
\[ \leq \frac{\theta_3^2}{\rho^2} \|A\bar{y}^r\|^2 - \frac{1}{\rho^2} \|\mu^{r+1} - \mu(z^r)\|^2 + \left( \frac{\theta_3^2}{\rho^2(p + \gamma^-)} + \theta_3^2(p + \gamma^+) \right) \|A\bar{y}^r\| \|\mu^{r+1} - \mu(z^r)\|, \]
\[ \text{(ii)} \]
\[ \leq \frac{\theta_3^2}{\rho^2} \|A\bar{y}^r\|^2 + \frac{\tau}{2} \left( \frac{\theta_3^2}{\rho^2(p + \gamma^-)} + \theta_3^2(p + \gamma^+) \right) \|A\bar{y}^r\|^2 \]
\[ + \left( \frac{\theta_3^2}{\rho^2(p + \gamma^-)} + \theta_3^2(p + \gamma^+) \right) \leq \frac{1}{\rho^2} \|\mu^{r+1} - \mu(z^r)\|^2, \]
\[ \text{(H.71)} \]

where (i) is obtained by Cauchy-Schwartz inequality, and (ii) is due to Young’s inequality with \( \tau > 0. \)

If we choose \( \tau = \frac{\theta_3^2}{\rho^2} + \rho^2 \theta_3^2(p + \gamma^+) \), then \( \frac{1}{2\tau} \left( \frac{\theta_3^2}{\rho^2(p + \gamma^-)} + \theta_3^2(p + \gamma^+) \right) - \frac{1}{\rho^2} < 0. \) By substituting this choice of \( \tau \) into (H.71), we obtain the desired (52).

\[ \blacksquare \]

I  Proof of Lemma 8(b)

According to Lemma 8(a), the upper bound for \( \Phi^{r+1} - \Phi^r \) is non-positive. Therefore we can have the conclusion that the potential function \( \Phi \) is non-increasing with \( r \).

Recall from (18) that:

\[ G(x^{r+1}, y^{r+1}, \mu^{r+1}, z^{r+1}) \]
\[ = \sum_{i=1}^{N} \left( f_i(x_i^{r+1}) + \frac{p_i}{2} ||x_i^{r+1} - z_i^{r+1}||^2 + (y_i^{r+1})^\top B_i x_i^{r+1} - \frac{1}{N} (y_i^{r+1})^\top b_i \right) - (\mu^{r+1})^\top A_i y_i^{r+1} - \frac{\rho_i}{2} ||A_i y_i^{r+1}||^2. \]
\[ \text{(I.72)} \]

Note that,

\[ \sum_{i=1}^{N} \left( (y_i^{r+1})^\top B_i x_i^{r+1} - \frac{1}{N} (y_i^{r+1})^\top b_i \right) - (\mu^{r+1})^\top A_i y_i^{r+1} \]
\[ = \sum_{i=1}^{N} (y_i^{r+1})^\top (B_i x_i^{r+1} - b_i/N - A_i^\top \mu^{r+1}) \]
\[ = (i) \sum_{i=1}^{N} (y_i^{r+1})^\top (2\rho_i N_i y_i^{r+1} - \rho_i L_i^+ y_i^{r+1}) \]
\[ = (i) \sum_{i=1}^{N} (y_i^{r+1})^\top (2D y_i^{r+1} - L^+ y_i^{r+1}) \]

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\begin{align}
&\text{(ii)} \quad \rho(y^{r+1})^\top (L^+(y^{r+1} - y^r) + L^-y^{r+1}), \\
&= \rho\|y^{r+1}\|^2_{L^-} + \frac{\rho}{2}\|y^{r+1} - y^r\|^2_{L^+} + \frac{\rho}{2}\|y^{r+1}\|^2_{L^+} - \frac{\rho}{2}\|y^r\|^2_{L^+},
\end{align}

where (i) is obtained by (21) and (ii) is due to $L^+ = 2D - L^-$. Thus, recall $G$, then combine the results in (18) and (I.73). We have

\begin{align}
\tilde{G}^{r+1} &\triangleq G(x^{r+1}, y^{r+1}, \mu^{r+1}, z^{r+1}) + \frac{\alpha}{2}\|y^{r+1}\|^2_{L^-} + \frac{\rho}{2}\|y^{r+1} - y^r\|^2_{L^+} \\
&= \sum_{i=1}^{N} f_i(x_i^{r+1}) + \frac{\rho}{2}\|x^{r+1} - z^{r+1}\|^2 + \left(\frac{\alpha}{2} + \frac{\rho}{2}\right)\|y^{r+1}\|^2_{L^-} \\
&\quad + \rho\|y^{r+1} - y^r\|^2_{L^+} + \frac{\rho}{2}\|y^{r+1}\|^2_{L^+} - \frac{\rho}{2}\|y^r\|^2_{L^+},
\end{align}

which implies that

\begin{align}
\sum_{r=0}^{T} (\tilde{G}^{r+1} - f) > -\frac{\rho}{2}\|y^0\|^2_{L^+} > -\infty,
\end{align}

for any $T$. Due to the weak duality, $d(\mu^r; z^r) \geq L_{\rho}(y^r, \mu^r; z^r)$. We further obtain $\sum_{r=0}^{T} (\Phi^{r+1} - f) > -\frac{\rho}{2}\|y^0\|^2_{L^+} > -\infty$. By this fact and the conclusion of (a) that $\Phi^{r+1}$ is non-increasing, $\Phi^{r+1}$ must be lower bounded, which is denoted as $\bar{\Phi}$.

\section{Proof of Theorem 2}

For the inexact case, since for agent $i$ the inexact solution $x_i^{r+1}$ is not equal to $x_i(y_i^{r+1}, z_i^r)$ anymore, then we have different upper bounds for $L_{\rho}$, $G$, and perturbation of $y$ from those in the exact PDC case.

\subsection*{J.1 Upper bound for $L_{\rho}$}

\textbf{Lemma 13} \textit{For the AL function in (16), we have}

\begin{align}
- L_{\rho}(y^{r+1}, \mu^{r+1}; z^{r+1}) + L_{\rho}(y^r, \mu^r; z^r) &
\leq (A1) - \frac{1}{2}\|y^{r+1} - y^r\|^2_{L^-} - \frac{\rho}{2}\|y^{r+1} - y^r\|^2 \\
&\quad + (A2) + \frac{B_{\text{max}}^2}{2} \sum_{i=1}^{N} \|x_i^{r+1} - x_i(y_i^{r+1}, z_i^r)\|^2 + \frac{1}{2}\|y^{r+1} - y^r\|^2.
\end{align}

\textbf{Proof of Lemma 13:} For proving the descent of $-L_{\rho}$, the IPDC algorithm differs from its exact counterpart (Lemma 1) only in the update of $y$, as shown below.

Bound of $-L_{\rho}(y^{r+1}, \mu^{r+1}; z^r) + L_{\rho}(y^r, \mu^{r+1}; z^r)$: Rewrite (B.8) explicitly, we have

\begin{align}
\begin{bmatrix}
B_1x_1(y_1^{r+1}, z_i^r) \\
\vdots \\
B_Nx_N(y_N^{r+1}, z_i^r)
\end{bmatrix} - q/N - A^\top \mu - \rho A^\top Ay - \rho L^+(y^{r+1} - y^r) = 0.
\end{align}
Then, by the optimality condition of (17b) and applying (J.77), we have

\[
\mathcal{L}_\rho(y^r, \mu^{r+1}; z^r) - \frac{1}{2} \|y^r - y^r\|_{\rho L^+}^2 \\
\leq \sum_{i=1}^N (B_i(x_i^{r+1} - x_i(y_i^{r+1}, z_i^r)), y_i^{r+1} - y_i^r) + \mathcal{L}_\rho(y^{r+1}, \mu^{r+1}; z^r) - \frac{1}{2} \|y^{r+1} - y^r\|_{\rho L^+}^2 \\
- \frac{p}{2} \|y^{r+1} - y^r\|_2^2 \\
\leq \frac{B^2_{\max}}{2} \sum_{i=1}^N \|x_i^{r+1} - x_i(y_i^{r+1}, z_i^r)\|^2 + \frac{1}{2} \|y^{r+1} - y^r\|_2^2 + \mathcal{L}_\rho(y^{r+1}, \mu^{r+1}; z^r) - \frac{1}{2} \|y^{r+1} - y^r\|_{\rho L^+}^2 \\
- \frac{p}{2} \|y^{r+1} - y^r\|_2^2 \tag{J.78}
\]

Summing (B.7), (J.78) and (B.11) leads to the desired result. \(\blacksquare\)

### J.2 Upper bound for \(G\)

**Lemma 14** We have

\[
\tilde{G}_{r+1}(x^{r+1}, y^{r+1}, \mu^{r+1}, z^{r+1}) - G_r(x^r, y^r, \mu^r; z^r) \\
\leq \frac{1}{\alpha} \|\mu^{r+1} - \mu^r\|_2^2 - \sum_{i=1}^N \frac{p + \gamma^-}{2} \|x_i^{r+1} - x_i^r\|^2 - \frac{1}{2} \|y^r - y^{r-1}\|_2^2 - \frac{1}{2} \|y^{r+1} - y^r\|_2^2 \\
- \left(\frac{\rho}{2} - \frac{\alpha}{2}\right) \|y^{r+1} - y^r\|_{L^2}^2 + \|y^{r+1} - y^r\|_{\rho L^+}^2 + \frac{p}{2} (1 - \frac{2}{\beta}) \|z^{r+1} - z^r\|_2^2 \\
+ \sum_{i=1}^N (-\frac{1}{\zeta} + \frac{p + \gamma^+}{2}) \|x_i^{r+1} - x_i^r\|^2. \tag{J.79}
\]

**Proof of Lemma 14:** The only difference for proving the descent of \(\tilde{G}\) between the PDC algorithm (Lemma 10) and the IPDC algorithm is the bound of \(G(x^{r+1}, y^r, \mu^{r+1}, z^r) - G(x^r, y^r, \mu^r, z^r)\).

Define

\[
H(x, y^r, \mu; z) = \max_y \left( G(x, y, \mu; z) - \frac{\rho}{2} \|y - y^r\|_{L^+}^2 \right).
\]

Thus, we have \(G(x^{r+1}, y^r, \mu^{r+1}, z^r) - G(x^r, y^r, \mu^{r+1}, z^r)\) to be bounded as

\[
G(x^{r+1}, y^r, \mu^{r+1}, z^r) - G(x^r, y^r, \mu^{r+1}, z^r) \\
= \sum_{i=1}^N \left( f_i(x_i^{r+1}) + \frac{p}{2} \|x_i^{r+1} - z_i^r\|^2 - f_i(x_i^r) - \frac{p}{2} \|x_i^r - z_i^r\|^2 \right) + \sum_{i=1}^N \langle y_i^r, B_i(x_i^{r+1} - x_i^r) \rangle \\
\leq \sum_{i=1}^N \left( (\nabla f_i(x_i^r) + p(x_i^r - z_i^r))^\top (x_i^{r+1} - x_i^r) + \frac{p + \gamma^+}{2} \|x_i^{r+1} - x_i^r\|^2 \right) + \sum_{i=1}^N \langle y_i^r, B_i(x_i^{r+1} - x_i^r) \rangle \\
= \sum_{i=1}^N \left( \nabla H(x_i^r)^\top (x_i^{r+1} - x_i^r) - \langle B_i^\top y_i^{r+1}, x_i^{r+1} - x_i^r \rangle + \frac{p + \gamma^+}{2} \|x_i^{r+1} - x_i^r\|^2 \right)
\]

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Lemma 15 Given \( \zeta < \frac{2}{p+\gamma^+} \), we consider the upper bound of the solution difference between (26) and (32).

\[
\|x_i^{r+1} - x_i(y_i^{r+1}, z_i^r)\| \leq \sigma_4 \|x_i^{r+1} - x_i^r\|
\]

where \( \sigma_4 = \left(1 + \frac{3}{\zeta(p+\gamma^-)}\right) \).

Proof of Lemma 15: The proof is similar to inequality (3.4) of Lemma 3.6 in [17] hence is omitted.

Lemma 16 Consider (37) with \( p > -\gamma^- \). Then for any \( \bar{y} \in \mathcal{Y}(\mu^{r+1}, \mathbf{z}^r) \), we have

(a) \[
\|y^{r+1} - \bar{y}\|^2 \leq a_1 \|L^+(y^{r+1} - y^r)\|^2 + a_5 \|x^{r+1} - x^r\|^2,
\]

(b) \[
\|y^{r+1} - \bar{y}\|^2 \leq a_2 \|L^+(y^{r+1} - y^r)\|^2 + a_6 \|x^{r+1} - x^r\|^2.
\]

Proof of Lemma 16(a): For any \( \bar{y} \in \mathcal{Y}(\mu^{r+1}, \mathbf{z}^r) \), recall its KKT conditions in (F.51) and (F.52).

Analogously, from (17b), we have KKT conditions for \( y^{r+1} \) as

\[
\begin{bmatrix}
B_1 x_1(y_1^{r+1}, z_1^r) - \frac{q}{N} + B_i(x_i^{r+1} - x_1(y_1^{r+1}, z_1^r)) \\
B_N x_N(y_N^{r+1}, z_N^r) - \frac{q}{N} + B_N(x_N^{r+1} - x_N(y_N^{r+1}, z_N^r)) \\
& \vdots \\
& \vdots \\
& \vdots
\end{bmatrix} - A^\top \mu^{r+1} - \rho L^{-} y^{r+1} - \rho L^{+}(y^{r+1} - y^r) = 0,
\]

\[
\nabla F_i(x_i(y_i^{r+1}, z_i^r)) + B_i^\top y_i^{r+1} = 0, \forall i \in [N].
\]

With fixed \( \{x_i(y_i^r, z_i^r)\}_{i=1}^N \), (F.51) and (F.52) forms a linear system of variable \( \rho \bar{y}^r \). Then, by using the Hoffman bound in Lemma 12, the distance from \( y^{r+1} \) satisfying (J.84) and (J.85) to the set \( \mathcal{Y}(\mu^{r+1}, \mathbf{z}^r) \) can be bounded as

\[
\rho^2 \text{dist}^2(y^{r+1}, \mathcal{Y}(\mu^{r+1}, \mathbf{z}^r))
\]
\[ \leq \theta_1^2 \sum_{i=1}^{N} \| B_i(x_i(y_i^{r+1}, z_i^r)) - x_i(y_i^r, z_i^r) \|^2 + \theta_1^2 \rho^2 \| L^+(y^{r+1} - y^r) \|^2 \\
+ \theta_1^2 \rho^2 \sum_{i=1}^{N} \| \nabla F_i(x_i(y_i^{r+1}, z_i^r)) - \nabla F_i(x_i(y_i^r, z_i^r)) \|^2 + \theta_1^2 \sum_{i=1}^{N} \| B_i(x_i^{r+1} - x_i(y_i^{r+1}, z_i^r)) \|^2 \]
\[ \leq \theta_1^2 \frac{1}{2\eta_1} \left( \frac{1}{p + \gamma^{-}} + \rho^2 (p + \gamma^{+}) \right) \| y^{r+1} - y^r \|^2 + \left[ \frac{\eta_1}{2} \left( \frac{1}{p + \gamma^{-}} + \rho^2 (p + \gamma^{+}) \right) + 1 \right] \theta_1^2 \rho^2 \| L^+(y^{r+1} - y^r) \|^2 \\
- \left( \frac{\theta_1^2}{p + \gamma^{-}} + \theta_1^2 \rho^2 (p + \gamma^{+}) \right) \rho \| y^{r+1} - y^r \|_{L^2} + \theta_1^2 D_{\max}^2 \sigma_4^2 \| x^{r+1} - x^r \|^2, \quad (J.86) \]

where the last inequality is due to (F.60) and (J.81).

Analogous to Lemma 5, choose \( \eta_1 = \frac{\theta_1^2}{\rho^2} \left( \frac{1}{p + \gamma^{-}} + \rho^2 (p + \gamma^{+}) \right) \), we can obtain

\[ \| y^{r+1} - y^r \|^2_{L^2} \leq a_1 \| L^+(y^{r+1} - y^r) \|^2 + \frac{\theta_1^2 D_{\max}^2 \sigma_4^2 \| x^{r+1} - x^r \|^2}{\frac{\eta_1}{2} \left( \frac{1}{p + \gamma^{-}} + \rho^2 (p + \gamma^{+}) \right) \rho} \quad \text{(J.87)} \]

**Proof of lemma 16(b):** By lemma 5 and (J.86), we have

\[ \| y^{r+1} - y^r \|^2 \leq a_2 \| L^+(y^{r+1} - y^r) \|^2 + \frac{2\theta_1^2 D_{\max}^2 \sigma_4^2 \| x^{r+1} - x^r \|^2}{\frac{\eta_1}{2} \left( \frac{1}{p + \gamma^{-}} + \rho^2 (p + \gamma^{+}) \right) \rho} \quad \text{(J.88)} \]

By taking the square root on both sides, we have

\[ \| y^{r+1} - y^r \| \leq \sqrt{a_2 \| L^+(y^{r+1} - y^r) \|^2 + \theta_1^2 D_{\max}^2 \sigma_4^2 \| x^{r+1} - x^r \|^2} \leq \sqrt{a_2 \| L^+(y^{r+1} - y^r) \| + \theta_a \| x^{r+1} - x^r \|.} \quad \text{(J.89)} \]

**Lemma 17** Consider \( p > \gamma^{-} \), based on Lemma 6 and Lemma 16, we have

\[ \| y^{r+1} - y^r \|^2 \leq 2a_2 \| L^+(y^{r+1} - y^r) \|^2 + \theta_1^2 D_{\max}^2 \sigma_4^2 \| x^{r+1} - x^r \|^2 + \theta_a \| x^{r+1} - x^r \| + \theta_a \| x^{r+1} - x^r \|^2. \quad \text{(J.90)} \]

**Proof of Lemma 17:** We have

\[ \| y^{r+1} - y^r \|^2 = \| y^{r+1} - y^{\mu^{r+1}, z^r} \|^2 + \| y^r - y^{r+1} \|^2 \leq 2\| y^{r+1} - y^r \|^2 + 2\| y^r - y^{r+1} \|^2 \leq 2a_2 \| L^+(y^{r+1} - y^r) \|^2 + 2a_6 \| x^{r+1} - x^r \|^2 + \theta_a \| x^{r+1} - x^r \|^2 + \theta_a \| x^{r+1} - z^r \|^2, \quad \text{(J.91)} \]

where the last inequality is due to (J.83) and Lemma 6.

Then, we can bound the following differences.
Lemma 18  For any $\bar{y}' \in \mathcal{Y}(\mu^{r+1}, z')$, there exist $y(z') \in \mathcal{Y}(z')$ and $x(z')$ such that
\[
\|y^{r+1} - y(z')\| \leq \sqrt{a_2\lambda_{\max}}\|y^{r+1} - y'\| + \sqrt{a_6}\|x^{r+1} - x'\| + \sqrt{a_4}\|A\bar{y}'\|, \tag{J.92}
\]
\[
\|x^{r+1} - x(z')\| \leq \sigma_2\|y^{r+1} - y(z')\| + \sigma_4\|x^{r+1} - x'\|. \tag{J.93}
\]

Proof of Lemma 18: Firstly,
\[
\|y^{r+1} - y(z')\| = \|y^{r+1} - \bar{y}' + \bar{y}' - y(z')\|
\leq \|y^{r+1} - \bar{y}'\| + \|\bar{y}' - y(z')\|
\leq \sqrt{a_2\lambda_{\max}}\|y^{r+1} - y'\| + \sqrt{a_6}\|x^{r+1} - x'\| + \sqrt{a_4}\|A\bar{y}'\|, \tag{J.94}
\]
where the last inequality is due to (J.88) and (52).

Secondly, we can have
\[
\|x^{r+1} - x(z')\| = \|x^{r+1} - x(y^{r+1}, z') + x(y^{r+1}, z') - x(z')\|
\leq \|x^{r+1} - x(y^{r+1}, z')\| + \|x(y^{r+1}, z') - x(y(z'), z')\|
\leq \sigma_4\|x^{r+1} - x'\| + \sigma_2\|y^{r+1} - y(z')\|, \tag{J.95}
\]
where the last inequality is due to (44).

Similar to the PDC algorithm (Lemma 8), we also have the non-increasing property of the potential function for the IPDC algorithm, which is given in the following lemma.

Lemma 19  Let $\beta$ satisfies (58) in Lemma 8 and $\rho > 1/2$. Then given $p$ meeting
\[
p > \max \left\{ 0, \gamma^+ - 2\gamma^-, \frac{32(2\rho - 1)B_{\max}^2}{5\lambda_{\max}^2\rho^2} - \gamma^-, 32B_{\max}^2 - \gamma^-, \frac{2\gamma^+ - 7\gamma^-}{5} \right\}, \tag{J.96}
\]
for sufficiently small $\zeta$ and $\alpha$ satisfying
\[
\frac{1}{p + \gamma^-} < \zeta < \min \left\{ \frac{5}{2(\gamma^+ - \gamma^-)}, \frac{2}{p + \gamma^+}, \frac{5a_2\lambda_{\max}^2\rho^2}{64(2\rho - 1)\theta_{\max}^2B_{\max}^2}, \frac{1}{32B_{\max}^2} \right\}, \tag{J.97}
\]
\[
\alpha < \min \left\{ \frac{\rho}{5}, \frac{2\rho - 1}{16a_1\lambda_{\max}^2}, \frac{1}{8a_5\zeta} \right\}, \tag{J.98}
\]
and for any $r$,

(a) we have
\[
\Phi^{r+1} - \Phi^r \leq -C_1\|y^{r+1} - y'\|^2_{L^2} - \frac{1}{2}\|\xi^{r+1} - x^{r+1} - y^{r+1} - y'\|^2_{L^2} - \alpha\|Ay(\mu^{r+1}, z')\|^2
- \left( C_2 - \frac{1}{2} \right)\|y^{r+1} - y'\|^2 - C_3\|z^{r+1} - z'\|^2 - C_4\|x^{r+1} - x'\|^2, \tag{J.99}
\]
where $C_1, C_2, C_3$ is defined in Lemma 8, $C_4$ is
\[
C_4 \triangleq \frac{1}{\zeta} - \frac{\gamma^+ - \gamma^-}{2} - 2\alpha a_5 - \frac{4p\sigma_2^2 a_6}{\delta} - \frac{B_{\max}^2\sigma_4^2}{2}, \tag{J.100}
\]
and $C_1, (C_2 - 1/2), C_3, C_4 > 0$. 

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(b) The function \( \Phi^r \) in (42) is lower bounded, i.e., \( \sum_{r=0}^{T} \Phi^{r+1} > \Phi > -\infty \) for some constant \( \Phi \).

It implies that \( \Phi^r \) is non-increasing with \( r \).

**Proof of Lemma 19(a):** This proof is divided into the following three steps:

- **Step 1:** Obtain the bound in (J.99);
- **Step 2:** Solve sufficient conditions on \( p, \rho, \beta, \zeta, \alpha \) for proving positiveness of \( C_1 - C_4 \);
- **Step 3:** Summarize all the conditions of the convergence for the IPDC algorithm.

**Step 1:** Obtain (J.99): We can follow the same proof procedure of Lemma 8(a). Compared with (59), there are two additional terms in (J.99) shown as follows.

(1) The first one is \(- \frac{1}{2} \| y^{r+1} - y^r \|^2 \), which is obtained by (J.76) in Lemma 13.
(2) The second part \(- C_4 \| x^{r+1} - x^r \|^2 \) consists of five terms. Note that the first two term is obtained by the bound in Lemma 14, the third term is by applying lemma 16(a) into (60)(iii), the fourth term is due to the use of Lemma 17 in (62)(i), and the last term is resulted from the bound in Lemma 13.

**Step 2:** Prove the positiveness of \( C_1 - C_4 \) in (J.99):

(1) Prove \( C_1, C_3 > 0 \): According to Lemma 8(a), we obtain that if \( \alpha < \frac{\rho}{2} \) and \( \beta \) satisfies (58), then \( C_1 \) and \( C_3 \) in (63) are both positive.
(2) Prove \( C_2 > \frac{1}{2} \): As for \( C_2 \) in (63), we firstly assume \( \rho > \frac{1}{2} \). Different from the choice of \( \delta \) in the proof of Lemma 8(a), we choose \( \delta = \frac{20\rho p^2 a^2 \lambda_{\max}^2}{\rho - \frac{1}{2}} \), i.e., \( \frac{4p^2 a^2 \lambda_{\max}^2}{\delta} < \frac{\rho - \frac{1}{2}}{4} \). Then, as long as \( \alpha < \frac{\rho - \frac{1}{2}}{8a_1 \lambda_{\max}^2} \) and \( \rho > \frac{1}{2} \), we can have \( C_2 - \frac{1}{2} > \rho - \frac{1}{4} > 0 \).
(3) Prove \( C_4 > 0 \): By inserting \( \delta = \frac{20p^2 a^2 \lambda_{\max}^2}{\rho - \frac{1}{2}} \) into (J.100), we have

\[
C_4 = \frac{1}{\zeta} - \frac{\gamma^+ - \gamma^-}{2} - 2\alpha a_5 - \frac{(\rho - 1/2)a_6}{5a_2 \lambda_{\max}^2} - \frac{B_{\max}^2 \sigma_4^2}{2}. \tag{J.101}
\]

To guarantee \( C_4 > 0 \), we have

\[
\alpha < \frac{1}{2a_5} \left[ \frac{10a_2 \lambda_{\max}^2 - (\gamma^+ - \gamma^-)5\zeta a_2 \lambda_{\max}^2 - 2(\rho - 1/2)a_6 \zeta - 5B_{\max}^2 \sigma_4^2 \zeta a_2 \lambda_{\max}^2}{10\zeta a_2 \lambda_{\max}^2} \right] = \frac{10a_2 \lambda_{\max}^2 - (\gamma^+ - \gamma^-)5\zeta a_2 \lambda_{\max}^2 - 2(\rho - 1/2)a_6 \zeta - 5B_{\max}^2 \sigma_4^2 \zeta a_2 \lambda_{\max}^2}{20\zeta a_2 \lambda_{\max}^2} \triangleq \frac{D_1}{D_2}. \tag{J.102}
\]

Note that if \( D_1 > 0 \) and \( \alpha < \frac{D_1}{D_2} \), then we can complete the proof. To guarantee \( D_1 > 0 \), it suffices to show the following three conditions

\[
(\gamma^+ - \gamma^-)\zeta a_2 \lambda_{\max}^2 < \frac{5a_2 \lambda_{\max}^2}{2}, \tag{J.103a}
\]

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Next, we prove these conditions one by one.

1) To guarantee (J.103a) holds, i.e., \((\gamma^+ - \gamma^-)\zeta a_2\lambda_{\text{max}}^2 < \frac{5a_2\lambda_{\text{max}}^2}{2}\) we require an upper bound on \(\zeta\), i.e.,

\[
\zeta < \frac{5}{2(\gamma^+ - \gamma^-)}.
\]  
(J.104)

2) To guarantee (J.103b) holds, i.e., \(2(\rho - \frac{1}{2})a_6\zeta < \frac{5a_2\lambda_{\text{max}}^2}{2}\), we obtain another bound on \(\zeta\), i.e.,

\[
\zeta < \frac{5a_2\lambda_{\text{max}}^2\rho^2}{8(\rho - \frac{1}{2})\theta_1^2 B_{\text{max}}^2 \sigma_4^2}.
\]  
(J.105)

Note that this bound is implicit, since \(\sigma_4\) defined in Lemma 15 is a function of \(\zeta\). To overcome this, we assume \(\frac{1}{p+\gamma^-} < \zeta < \frac{2}{p+\gamma^-}\) and \(p > \gamma^+ - 2\gamma^-\). Then, we have \(\sigma_4 = (1 + \frac{3}{\zeta(p+\gamma^-)}) < 4\). Therefore, we obtain an explicit sufficient condition of (J.105), i.e.,

\[
\frac{1}{p+\gamma^-} < \zeta < \frac{5a_2\lambda_{\text{max}}^2\rho^2}{128(\rho - \frac{1}{2})\theta_1^2 B_{\text{max}}^2}.
\]  
(J.106)

Since \(\zeta\) has a lower bound \(\frac{1}{p+\gamma^-}\), we need to guarantee that the solution set of \(\zeta\) in (J.106) is non-empty. That is, the lower bound is required to be smaller than the upper bound. Inserting \(a_2\) of (49) into (J.106), we have

\[
\frac{1}{p+\gamma^-} < \frac{5\lambda_{\text{max}}^2\rho^2}{128(\rho - \frac{1}{2})\theta_1^2 B_{\text{max}}^2 \sigma_4^2} \left[ \frac{\theta_1^4}{\rho^2} \left( \frac{1}{p+\gamma^-} + \rho^2(p + \gamma^+)^2 + 2\theta_1^2 \right) \right] = \frac{5\lambda_{\text{max}}^2\rho^2}{128(\rho - \frac{1}{2})\theta_1^2 B_{\text{max}}^2 \sigma_4^2} \left[ \frac{\theta_1^4}{\rho^2} \left( \frac{1}{p+\gamma^-} + \rho^2(p + \gamma^+)^2 + 2\theta_1^2 \right) \right] < \frac{5\lambda_{\text{max}}^2\rho^2}{128(\rho - \frac{1}{2})\theta_1^2 B_{\text{max}}^2 \sigma_4^2} \left[ \frac{\theta_1^4}{\rho^2} \left( \frac{1}{p+\gamma^-} + \rho^2(p + \gamma^+)^2 + 2\theta_1^2 \right) \right] < \frac{32(2\rho - 1)B_{\text{max}}^2}{5\lambda_{\text{max}}^2\rho^2} - \gamma^-.
\]  
(J.107)

which gives a sufficient condition for achieving (J.106).

3) To guarantee (J.103c) holds, i.e., \(B_{\text{max}}^2 \sigma_4^2 \zeta a_2\lambda_{\text{max}}^2 < \frac{a_2\lambda_{\text{max}}^2}{2}\), we use the same operation on \(\sigma_4\) in (ii), we get

\[
\frac{1}{p+\gamma^-} < \zeta < \frac{1}{32 B_{\text{max}}^2}.
\]  
(J.108)

It is obvious that when \(p > 32 B_{\text{max}}^2 - \gamma^-\), (J.108) is feasible.
Note that for \( \zeta < \frac{5}{2(\gamma^+ - \gamma^-)} \) in (J.104) to hold, we also require \( \frac{1}{p + \gamma^+} < \frac{5}{2(\gamma^+ - \gamma^-)} \), which gives a condition of \( p > \frac{2\gamma^+ - 7\gamma^-}{5} \).

Step 3: Thus, combining all the results above for proving positiveness of \( C_1 - C_4 \), we have the following conditions on \( \beta, \zeta \) and \( \alpha \)

\[
\beta < \left( 1 + \frac{20p\sigma_2^2a_2\lambda_{\text{max}}^2}{\rho} + \frac{\rho(\sigma_1^2 + 2\sigma_2^2a_3)}{10p\sigma_2^2a_2\lambda_{\text{max}}^2} \right)^{-1} \quad \text{(Recall (58))}, \tag{J.109}
\]

\[
\frac{1}{p + \gamma^-} < \zeta < \min \left\{ \frac{5}{2(\gamma^+ - \gamma^-)}, \frac{2}{p + \gamma^+}, \frac{5a_2\lambda_{\text{max}}^2\rho^2}{64(2\rho - 1)\sigma_1^2 B_{\text{max}}^2}, \frac{1}{32B_{\text{max}}^2} \right\} \quad \text{(J.110)}
\]

\[
\alpha < \min \left\{ \frac{\rho}{5}, \frac{2p - 1}{16a_1\lambda_{\text{max}}^2}, \frac{1}{8a_3\zeta} \right\} \quad \text{(J.111)}
\]

provided that \( \rho > \frac{1}{2} \) and \( p \) satisfies

\[
p > \max \left\{ 0, \gamma^+ - 2\gamma^-, \frac{32(2\rho - 1)B_{\text{max}}^2}{5\lambda_{\text{max}}^2\rho^2} - \gamma^- , 32B_{\text{max}}^2 - \gamma^-, \frac{2\gamma^+ - 7\gamma^-}{5} \right\}. \tag{J.112}
\]

Therefore, given sufficiently large \( p \) and \( \rho > \frac{1}{2} \), for sufficiently small \( \beta, \zeta, \) and \( \alpha \), the IPDC algorithm can have the same convergence properties as the PDC algorithm.

**Proof of Lemma 19(b):** The proof is similar to the proof of Lemma 8(b) (see Appendix I) hence is omitted.

Similar to the proof of Theorem 1, we can also obtain that every limit point of the iteration sequence \( \{(z^r, y^r)\} \) is a KKT solution of problem (P). There exists a constant \( \hat{C} > 0 \) and such that

\[
\|z^r - x(z^r)|^2 \leq \frac{\hat{C}}{t}, \tag{J.113}
\]

where \( \hat{C} \triangleq (\Phi^0 - \Phi) \left( \frac{1}{\beta \sqrt{C_3}} + \frac{\sigma_2 \sqrt{\sigma_2 \lambda_{\text{max}}}}{\sqrt{C_2}} + \frac{\sigma_4 \sqrt{\sigma_4 + \sigma_2 \lambda_{\text{max}}}}{\sqrt{C_4}} \right)^2. \)

**K Additional Simulation Results**

In this section, besides of the results displayed in the main text, we show the convergence results for the proposed IPDC algorithm with respect to the parameter \( \alpha, \beta, \zeta, p \) and \( \rho \).

In Fig. 6(a), we set \( \beta = 0.1, p = 10, \rho = 1, \zeta = 0.1, \) and \( \alpha \in \{10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}\} \). One can see that a larger \( \alpha \) may speed up the convergence of gradient residue (left figure), while with distinct \( \alpha \) the IPDC algorithm performs similar with respect to infeasibility (right figure). In Fig. 6(b), we set \( \alpha = 0.01, \rho = 1, \zeta = 0.1, p = 10, \) and \( \beta \in \{10^{-3}, 10^{-2}, 10^{-1}, 1\} \). We can see that \( \beta \) influences the performance of the IPDC algorithm significantly. Interestingly, though Theorem 1 indicates that \( \beta \) should be less than one and a small number, it is shown that the algorithm behaves well with \( \beta = 1 \). This phenomenon is similar to that shown in Fig. 4. In Fig. 6(c), we set \( \alpha = 0.01, \rho = 1, \beta = 0.1, \) and \( \zeta \in \{10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}\} \). One can observe that a larger step size \( \zeta \) may speed up the convergence of gradient residue and infeasibility.

In Fig. 7(a), we set \( \alpha = 0.01, \rho = 1, \zeta = 0.1, \beta = 0.1, \) and \( p \in \{1, 10, 100\} \). One can observe that with a too small \( p = 1 \) or a too large \( p = 100 \) the IPDC algorithm may not converge to a
Figure 6: Convergence curves of the IPDC algorithm different step-sizes.

small gradient residue. Moreover, it is expected that a larger $p$ may slow down the convergence with respect to infeasibility, due to a more conservative convex approximation. In Fig. 7(b), we set $\alpha = 0.01, \rho = 1, \beta = 0.1, \zeta = 0.1, p = 10$, and various values of $\beta$. One can see from these two figures that a larger $\rho$ can fasten the convergence of gradient residue, whereas a larger $\rho$ may slow down the speed satisfying the linear constraints.
Figure 7: Convergence curves of the IPDC algorithm with respect to objective value, gradient residue, and infeasibility.

References

[1] J. Zhang, S. Ge, T.-H. Chang, and Z.-Q. Luo, “A proximal dual consensus method for linearly coupled multi-agent non-convex optimization,” in Proc. ICASSP, pp. 5740–5744, 2020.

[2] G. B. Giannakis, Q. Ling, G. Mateos, I. D. Schizas, and H. Zhu, “Decentralized learning for wireless communications and networking,” in Splitting Methods in Communication, Imaging, Science, and Engineering, pp. 461–497, Springer, 2016.

[3] R. Bekkerman, M. Bilenko, and J. Langford, Scaling up Machine Learning- Parallel and Distributed Approaches. Cambridge University Press, 2012.

[4] T.-H. Chang, M. Hong, H.-T. Wai, X. Zhang, and S. Lu, “Distributed learning in the nonconvex world: From batch data to streaming and beyond,” IEEE Signal Processing Magazine, vol. 37, no. 3, pp. 26–38, 2020.

[5] G. Scutari and Y. Sun, “Parallel and distributed successive convex approximation methods for big-data optimization,” in Multi-agent Optimization, pp. 141–308, Springer, 2018.

[6] J. Konečnỳ, H. B. McMahan, F. X. Yu, P. Richtárik, A. T. Suresh, and D. Bacon, “Federated learning: Strategies for improving communication efficiency,” arXiv preprint arXiv:1610.05492, 2016.

[7] T.-H. Chang, M. Hong, and X. Wang, “Multi-agent distributed optimization via inexact consensus ADMM,” IEEE Trans. Signal Process., vol. 63, no. 2, pp. 482–497, 2014.

[8] Y. Hu, P. Liu, L. Kong, and D. Niu, “Learning privately over distributed features: An ADMM sharing approach,” arXiv preprint arXiv:1907.07575, 2019.

[9] D. K. Molzahn, F. Dörrfler, H. Sandberg, S. H. Low, S. Chakrabarti, R. Baldick, and J. Lavaei, “A survey of distributed optimization and control algorithms for electric power systems,” IEEE Trans. Smart Grid, vol. 8, no. 6, pp. 2941–2962, 2017.
[10] S.-C. Tsai, Y.-H. Tseng, and T.-H. Chang, “Communication-efficient distributed demand response: A randomized admm approach,” IEEE Transactions on Smart Grid, vol. 8, no. 3, pp. 1085–1095, 2017.

[11] C. Shen, T.-H. Chang, K.-Y. Wang, Z. Qiu, and C.-Y. Chi, “Distributed robust multicell coordinated beamforming with imperfect CSI: An ADMM approach,” IEEE Trans. Signal Process., vol. 60, no. 6, pp. 2988–3003, 2012.

[12] D. P. Palomar and M. Chiang, “Alternative distributed algorithms for network utility maximization: Framework and applications,” IEEE Transactions on Automatic Control, vol. 52, no. 12, pp. 2254–2269, 2007.

[13] W. Kong, J. G. Melo, and R. D. Monteiro, “Complexity of a quadratic penalty accelerated inexact proximal point method for solving linearly constrained nonconvex composite programs,” arXiv preprint arXiv:1802.03504, 2018.

[14] D. P. Bertsekas, Nonlinear Programming: 2nd Ed. Cambridge, Massachusetts: Athena Scientific, 2003.

[15] M. Hong, T.-H. Chang, X. Wang, M. Razaviyayn, S. Ma, and Z.-Q. Luo, “A block successive upper-bound minimization method of multipliers for linearly constrained convex optimization,” Mathematics of Operations Research, 2020.

[16] Y. Wang, W. Yin, and J. Zeng, “Global convergence of admm in nonconvex nonsmooth optimization,” Journal of Scientific Computing, vol. 78, no. 1, pp. 29–63, 2019.

[17] J. Zhang and Z.-Q. Luo, “A proximal alternating direction method of multiplier for linearly constrained nonconvex minimization,” SIAM Journal on Optimization, vol. 30, no. 3, pp. 2272–2302, 2020.

[18] D. P. Palomar and M. Chiang, “A tutorial on decomposition methods for network utility maximization,” IEEE Journal on Selected Areas in Communications, vol. 24, no. 8, pp. 1439–1451, 2006.

[19] T.-H. Chang, “A proximal dual consensus ADMM method for multi-agent constrained optimization,” IEEE Trans. Signal Process., vol. 64, no. 14, pp. 3719–3734, 2016.

[20] I. Notarnicola, M. Franceschelli, and G. Notarstefano, “A duality-based approach for distributed min-max optimization,” IEEE Transactions on Automatic Control, vol. 64, no. 6, pp. 2559–2566, 2019.

[21] S. Liang, L. Y. Wang, and G. Yin, “Distributed smooth convex optimization with coupled constraints,” IEEE Transactions on Automatic Control, vol. 65, no. 1, pp. 347–353, 2020.

[22] S. A. Alghunaim, K. Yuan, and A. H. Sayed, “Dual coupled diffusion for distributed optimization with affine constraints,” in Proc. IEEE CDC, pp. 829–834, 2018.

[23] X. Li, L. Xie, and Y. Hong, “Distributed continuous-time nonsmooth convex optimization with coupled inequality constraints,” IEEE Transactions on Control of Network Systems, vol. 7, no. 1, pp. 74–84, 2020.

[24] M. Zhu and S. Martínez, “On distributed convex optimization under inequality and equality constraints,” IEEE Trans. Automatic Control, vol. 57, no. 1, pp. 151–164, 2011.

[25] T.-H. Chang, A. Nedić, and A. Scaglione, “Distributed constrained optimization by consensus-based primal-dual perturbation method,” IEEE Trans. Automatic Control, vol. 59, no. 6, pp. 1524–1538, 2014.

[26] A. Nedic, “Distributed gradient methods for convex machine learning problems in networks: Distributed optimization,” IEEE Signal Processing Magazine, vol. 37, no. 3, pp. 92–101, 2020.

[27] G. B. Giannakis, Q. Ling, G. Mateos, I. D. Schizas, and H. Zhu, Decentralized learning for wireless communications and networking. Springer, 2016.

[28] Y. Carmon, J. C. Duchi, O. Hinder, and A. Sidford, “Lower bounds for finding stationary points i,” Mathematical Programming, pp. 1–50, 2019.

[29] B. Ying, K. Yuan, and A. H. Sayed, “Supervised learning under distributed features,” IEEE Transactions on Signal Processing, vol. 67, no. 4, pp. 977–992, 2019.

[30] Y. Hu, D. Niu, J. Yang, and S. Zhou, “Fdmil: A collaborative machine learning framework for distributed features,” in Proc. SIGKDD, pp. 2232–2240, 2019.

[31] S. Boyd and L. Vandenberghe, Convex Optimization. Cambridge, UK: Cambridge University Press, 2004.

[32] Y. Nesterov, “Smooth minimization of non-smooth functions,” Mathematical Programming, vol. 103, no. 1, 2005.

[33] M. Hong, D. Hajinezhad, and M.-M. Zhao, “Prox-PDA: The proximal primal-dual algorithm for fast distributed nonconvex optimization and learning over networks,” in Proc. 34th Inter. Conf. Mach. Learn. (ICML), pp. 1529–1538, JMLR. org, 2017.

46
[34] S. Boyd, S. P. Boyd, and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.

[35] A. Antoniadis, I. Gijbels, and M. Nikolova, “Penalized likelihood regression for generalized linear models with non-quadratic penalties,” *Annals of the Institute of Statistical Mathematics*, vol. 63, no. 3, pp. 585–615, 2011.

[36] M. E. Yildiz and A. Scaglione, “Coding with side information for rate-constrained consensus,” *IEEE Trans. Signal Process.*, vol. 56, no. 8, pp. 3753–3764, 2008.

[37] A. Beck and M. Teboulle, “A fast iterative shrinkage-thresholding algorithm for linear inverse problems,” *SIAM J. Imaging Sci.*, vol. 2, no. 1, pp. 183–202, 2009.

[38] I. Goodfellow, Y. Bengio, and A. Courville, *Deep Learning*. MIT Press, 2016. http://www.deeplearningbook.org.

[39] X. Lian, C. Zhang, H. Zhang, C.-J. Hsieh, W. Zhang, and J. Liu, “Can decentralized algorithms outperform centralized algorithms? A case study for decentralized parallel stochastic gradient descent,” in *Proc. NeuIPS*, (California, USA), p. 5336?5346, Dec. 4-9, 2017.

[40] H. B. McMahan, E. Moore, D. Ramage, S. Hampson, and B. A. Areas, “Communication-efficient learning of deep networks from decentralized data,” in *Proc. ICML*, (Sydney, Australia), pp. 1–10, Aug. 6-11, 2017.

[41] Q. Yang, Y. Liu, T. Chen, and Y. Tong, “Federated machine learning: Concept and applications,” *ACM Transactions on Intelligent Systems and Technology*, vol. 10, pp. 1–19, Jan. 2019.

[42] D. Bertsekas, A. Nedic, and A. Ozdaglar, *Convex analysis and optimization*, vol. 1. Athena Scientific, 2003.

[43] J.-S. Pang, “A posteriori error bounds for the linearly-constrained variational inequality problem,” *Mathematics of Operations Research*, vol. 12, no. 3, pp. 474–484, 1987.

[44] F. Facchinei and J.-S. Pang, *Finite-dimensional variational inequalities and complementarity problems*. Springer Science & Business Media, 2007.

[45] Y. Nesterov, *Introductory lectures on convex optimization: A basic course*, vol. 87. Springer Science & Business Media, 2013.