HAUSDORFF DIMENSIONS OF SETS RELATED TO ERDÖS-RÉNYI AVERAGES IN BETA EXPANSIONS

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Abstract. Let \( \beta > 1 \), \( I \) be the unite interval \([0, 1)\) and \( \phi \) be an integer function defined on \( \mathbb{N} \setminus \{0\} \) satisfying \( 1 \leq \phi(n) \leq n \). Denote by \( A_\phi(x, \beta) \) the Erdös-Rényi average of \( x \in I \) associated with the function \( \phi \) in \( \beta \)-expansion and \( I_\beta \) the range of \( A_\phi(x, \beta) \) for \( x \in I \). For the level set

\[
ER_{\beta}^\phi(\alpha) = \{ x \in I : A_\phi(x, \beta) = \alpha \}, \quad \alpha \in I_\beta,
\]

in this paper we will determine its Hausdorff dimension under the assumption \( \phi(n) \to \infty \) as \( n \to \infty \) and \( \phi \) is the integer part of some slowly varying sequence. Besides, a generalization to the classic work [2] of Besicovitch is also given in \( \beta \)-expansion.

1. Introduction

Let \( \beta > 1 \) be a real number and \( I \) be the unit interval \([0, 1)\). Define the \( \beta \)-transformation \( T_\beta : I \to I \) as

\[
T_\beta(x) = \beta x - \lfloor \beta x \rfloor, \quad x \in I.
\]

Here, \( \lfloor \cdot \rfloor \) is the floor function. It is well-known (see [20]) that each \( x \in I \) can be uniquely expanded into a finite or an infinite series as

\[
x = \frac{\varepsilon_1(x, \beta)}{\beta} + \frac{\varepsilon_2(x, \beta)}{\beta^2} + \cdots + \frac{\varepsilon_n(x, \beta)}{\beta^n} + \cdots,
\]

where \( \varepsilon_n(x, \beta) = \lfloor \beta T_\beta^{n-1}(x) \rfloor \), \( n \geq 1 \), is called the \( n \)-th digit of \( x \) with respect to base \( \beta \). For simplicity, we can also identify \( x \) with the digit sequence

\[
\varepsilon(x, \beta) := (\varepsilon_1(x, \beta), \varepsilon_2(x, \beta), \ldots).
\]

That is, we can rewrite (1.1) as

\[
x = (\varepsilon_1(x, \beta), \varepsilon_2(x, \beta), \ldots).
\]

The formulas (1.1) or (1.2) is called the \( \beta \)-expansion of \( x \) and the system \((I, T_\beta)\) is called the \( \beta \)-dynamical system.

It is clear that each \( n \)-th digit \( \varepsilon_n(x, \beta) \), \( n \geq 1 \), of \( x \) belongs to the alphabet \( \Sigma = \{0, 1, \ldots, \lceil \beta - 1 \rceil \} \), where \( \lceil \cdot \rceil \) is the ceiling function. Denote by \( \Sigma^\infty \) the set of infinite sequences with all digits from \( \Sigma \). It is worth noting that not all sequences belong to \( \Sigma^\infty \) would be the \( \beta \)-expansion of some \( x \in I \). Moreover, endowed with the metric

\[
d(\varepsilon, \eta) = \beta^{-\min(i \geq 1 : \varepsilon_i \neq \eta_i)},
\]

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where \( \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots) \in \Sigma^\infty \) and \( \eta = (\eta_1, \eta_2, \ldots) \in \Sigma^\infty \), the space \( \Sigma^\infty \) is compact.

In the present paper, we will say the terminology word to mean a finite sequence, \( n \)-word to mean a finite sequence of length \( n \) and meanwhile sequence to mean an infinite sequence for the sake of distinction.

Let \( n \geq 1 \). We call an \( n \)-word \( (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \) or a sequence \( (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \ldots) \) \( \beta \)-admissible if there exists an \( x \in I \) such that the \( \beta \)-expansion of \( x \) begins with this word or is just this sequence. Denote by \( \Sigma^n_\beta \), \( n \geq 1 \), the set of all \( \beta \)-admissible words with length \( n \) and \( \Sigma_\beta \) the set of all admissible sequences. That is,

\[
\Sigma_\beta = \{ \varepsilon \in \Sigma^\infty : \varepsilon \text{ is the } \beta \text{-expansion of some } x \in I \}.
\]

After the introduction of the concept of \( \beta \)-expansion, we would like to introduce the concept of Erdős-Rényi average in \( \beta \)-expansion. Let \( x \in I \) and be of infinite \( \beta \)-expansion \( (\varepsilon_1(x, \beta), \varepsilon_2(x, \beta), \ldots) \). Note that in the sequel we only need to deal with those numbers in \( I \) with infinite \( \beta \)-expansions while the set of numbers with finite \( \beta \)-expansions is countable. Denoted by

\[
S_n(x, \beta) = \sum_{i=1}^{n} \varepsilon_i(x, \beta), \quad n \geq 1,
\]

the sum of the first \( n \) digits of \( x \) or the \( n \)-th partial sum of \( x \). Let \( \phi \) be an integer function defined on \( \mathbb{N} \setminus \{0\} \) satisfying \( 1 \leq \phi(n) \leq n \). Put

\[
I_{n, \phi(n)}(x, \beta) = \max_{0 \leq i \leq n-\phi(n)} \{ S_{i+\phi(n)}(x, \beta) - S_i(x, \beta) \}
\]

and call it the \((n, \phi(n))\)-Erdős-Rényi maximum partial sum of \( x \). Here, \( S_0(x, \beta) = 0 \) is set by convention. Accordingly, call

\[
A_{n, \phi(n)}(x, \beta) = \frac{I_{n, \phi(n)}(x, \beta)}{\phi(n)}
\]

the \((n, \phi(n))\)-Erdős-Rényi average of \( x \) and

\[
A_{\phi}(x, \beta) = \lim_{n \to \infty} A_{n, \phi(n)}(x, \beta)
\]

the Erdős-Rényi average of \( x \) associated with \( \phi \) if the limit exists. In particular, take \( \phi(n) = 1 \), then

\[
I_{n, 1}(x, \beta) = \max\{ \varepsilon_i(x, \beta), 1 \leq i \leq n \};
\]

take \( \phi \) to be the identity function \( \phi_I \) on \( \mathbb{N} \setminus \{0\} \), i.e., \( \phi_I(n) = n \), \( n \geq 1 \), then \( I_{n, \phi_I}(x, \beta) = S_n(x, \beta) \). So, we have

\[
A_{n, \phi_I}(x, \beta) = \frac{S_n(x, \beta)}{n} \quad \text{and} \quad A_{\phi_I}(x, \beta) = \lim_{n \to \infty} \frac{S_n(x, \beta)}{n}.
\]

Thus, the Erdős-Rényi average is a more general concept than the usual algebraic average. In addition, in what follows we will write, respectively, \( A_n(x, \beta) \) instead of \( A_{n, \phi_I}(x, \beta) \) and \( A(x, \beta) \) instead of \( A_{\phi_I}(x, \beta) \) for brevity.

Erdős-Rényi average was first introduced by P. Erdős and A. Rényi [11] in 1970, which gave a pioneering work, by establishing a kind of new strong law of large numbers, on the limit behaviors of the length of the longest run of heads in \( n \) independent Bernoulli trials. After that, many work emerged during the last several decades by considering, for examples, the asymptotic distribution of the length of the longest head run [8, 12, 20], the appearances of long repetitive sequences in random sequences [18], the rate of convergence for a stationary sequence [24] and the case of renewal counting process [29], etc. In this paper, we would like
to study the level sets described by the Erdős-Rényi average and determine their
Hausdorff dimensions, which generalizes the work in [5] form binary expansion to
β-expansion. In addition, some generalizations to the classic work [2] of Besicovitch
in β-expansion are included as well.

Let β > 1. It is well-known (see [26]) that there exists a unique invariant
measure µβ, which is equivalent to the Lebesgue measure L, when β is not an
integer. Moreover, Tβ is ergodic with respect to µβ (see [9]). So, by Birkhoff’s
ergodic theorem, we have

\begin{equation}
A(x, \beta) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varepsilon_1(T^i_\beta x, \beta) = \int_I \varepsilon_1(x, \beta) d\mu_\beta =: \alpha^*(\beta), \quad \mu_\beta \text{-a.e. } x \in I.
\end{equation}

Note that if we replace µβ by the Lebesgue measure, then (1.8) is also valid when
β is an integer. In this case, we have \( \alpha^*(\beta) = (\beta - 1)/2 \).

Denote

\[ \Lambda(\beta) = \sup_{x \in I} \tilde{A}(x, \beta) \quad \text{and} \quad I_\beta = [0, \Lambda(\beta)], \]

where \( \tilde{A}(x, \beta) = \limsup_{n \to \infty} A_n(x, \beta) \), see also the definition (3.1) in Section 3. It is
clear that \( I_\beta = [0, \beta - 1] \) when \( \beta \) is an integer and \( I_\beta \subset [0, \lceil \beta - 1 \rceil] \) when \( \beta \) is not
an integer. For a given \( \alpha \in I_\beta \), define the level set

\begin{equation}
ER_\alpha^\beta(x) = \{ x \in I : A_\phi(x, \beta) = \alpha \}.
\end{equation}

In the following, we will show the Hausdorff dimension of \( ER_\alpha^\beta(x) \) after introducing
some notations.

Let \( \beta > 1, \alpha \in I_\beta, n \geq 1 \) and \( \delta > 0 \). Denote

\[ H^\beta(\alpha, n, \delta) = \left\{ (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \Sigma^\beta_n : n(\alpha - \delta) < \sum_{i=1}^{n} \varepsilon_i < n(\alpha + \delta) \right\} \]

and \( h^\beta(\alpha, n, \delta) = \text{Card } H^\beta(\alpha, n, \delta) \), where the symbol Card denotes the cardinality
of a set.

Let \( (\varepsilon_1^+(1, \beta), \varepsilon_2^+(1, \beta), \ldots) \) be the infinite \( \beta \)-expansion of 1 introduced in the
beginning of Section 2. For each \( m \) with \( \varepsilon_m^+(1, \beta) \geq 1 \), define \( \beta_m = \beta_m(\beta) \) to be the
unique positive root of the equation

\begin{equation}
1 = \frac{\varepsilon_1^+(1, \beta)}{\beta_1^m} + \frac{\varepsilon_2^+(1, \beta)}{\beta_2^m} + \cdots + \frac{\varepsilon_m^+(1, \beta)}{\beta_m^m}.
\end{equation}

Then we have \( \beta_m < \beta \) and \( \beta_m \) increases to \( \beta \) as \( m \to \infty \). Here and in the sequel,
we always assume that \( m \) takes value in \( \{ m : \varepsilon_m^+(1, \beta) \geq 1 \} \) unless otherwise noted.
Moreover, if \( m_1 < m_2 \) and both of \( \beta_{m_1} \) and \( \beta_{m_2} \) are roots of the equation (1.10),
then the increasing property

\[ \Sigma_{\beta_{m_1}} \subset \Sigma_{\beta_{m_2}} \subset \Sigma_\beta \]

holds by (2) in Theorem 2.1.

Then, define the \( \beta \)-adic entropy function as

\begin{equation}
h^\beta(\alpha) = \lim_{\delta \to 0} \lim_{m \to \infty} \liminf_{n \to \infty} \frac{\log h^\beta_m(\alpha, n, \delta)}{(\log \beta)n} = \lim_{\delta \to 0} \lim_{m \to \infty} \limsup_{n \to \infty} \frac{\log h^\beta_m(\alpha, n, \delta)}{(\log \beta)n}.
\end{equation}
Note that in (1.11) the limits for $\delta$ and $m$ both exist since $h_{\beta m}(\alpha, n, \delta)$ is increasing for these two variables and the definition is valid since the second equality holds according to Proposition 4.2 in [31].

The function $\phi$, we focus on in this paper, is limited to a kind of special sequence, called slowly varying sequence. For the definition and corresponding properties, one can see Definition 5.2 and Lemma 5.3 in Section 6. Now, we would like to state the following main result in the present paper where $\dim_H$ denotes the Hausdorff dimension of a set.

**Theorem 1.1.** Let $\beta > 1$ and $\alpha \in I_\beta$. Assume the sequence $\{\theta(n)\}_{n \geq 1}$ is slowly varying and $\theta(n) \to \infty$ as $n \to \infty$. If $\phi(n) = \lfloor \theta(n) \rfloor$, $n \geq 1$, then we have

$$
\dim_H ER_{\phi}^\beta(\alpha) = \begin{cases} h_\beta(\alpha), & 0 \leq \alpha \leq \alpha^*(\beta); \\ 1, & \alpha^*(\beta) < \alpha \leq \Lambda(\beta). \end{cases}
$$

(1.12)

In particular, for the case that $\beta$ is an integer we can easily obtain

**Corollary 1.2.** Let $\beta \geq 2$ be an integer and $\alpha \in [0, \beta - 1]$. If $\phi$ satisfies the conditions in Theorem 1.1 then we have

$$
\dim_H ER_{\phi}^\beta(\alpha) = \begin{cases} h_\beta(\alpha), & 0 \leq \alpha \leq (\beta - 1)/2; \\ 1, & (\beta - 1)/2 < \alpha \leq \beta - 1. \end{cases}
$$

Here, $h_\beta(\alpha)$ can be reduced to the definition (2.3).

This paper is organized as follows. The next section is devoted to some notations and basic properties of $\beta$-expansion. In Section 3, we will introduce the lower and upper Besicovitch sets and determine their Hausdorff dimensions, which is prepared for the calculation of the upper bound of Hausdorff dimension of $ER_{\phi}^\beta(\alpha)$. In Section 4, the full Moran sets and the $\alpha$-Moran sets are introduced, which will be used to obtain the lower bound of Hausdorff dimension of $ER_{\phi}^\beta(\alpha)$. In Section 5, two essential lemmas are presented for ready use. The last section is devoted to the proof of Theorem 1.1 and we will give at first the definition and some basic properties of slowly varying sequence.

The readers are assumed to be familiar with the definition and basic properties of Hausdorff dimension. The book [13] of Falconer is highly recommended. For the related topic about the dimensional theory associated with digit average of numbers, one can trace back the history and see the classic work in [2, 3, 14, 15] and the references therein.

2. $\beta$-expansion

In this section, we introduce some notations, definitions and basic properties about $\beta$-expansion, together with some properties about $\beta$-adic entropy function.

First, we would like to give a result of Parry on charactering whether a digit sequence is admissible. For this purpose, we need to introduce the infinite $\beta$-expansion of 1. Let $\beta > 1$ be given. If the $\beta$-expansion of 1, according to (1.1), terminates, in other words there exists an $n \geq 1$ such that $\varepsilon_n(1, \beta) \neq 0$ but $\varepsilon_m(1, \beta) = 0$ for all $m \geq n + 1$, then we call $\beta$ a simple Parry number and put

$$
(\varepsilon_1^*(1, \beta), \varepsilon_2^*(1, \beta), \ldots) = (\varepsilon_1(1, \beta), \varepsilon_2(1, \beta), \ldots, \varepsilon_n(1, \beta) - 1)^\infty.
$$
Here, \((w)\infty\) denotes the periodic sequence \((w, w, w, \ldots)\) when \(w\) is a word. Otherwise, we write \(\varepsilon_i^\ast(1, \beta) = \varepsilon_i(1, \beta), i \geq 1\), and use \((\varepsilon_1^\ast(1, \beta), \varepsilon_2^\ast(1, \beta), \ldots)\) to denote the \(\beta\)-expansion of 1. In both cases, the infinite \(\beta\)-expansion of 1 is set as
\[
\varepsilon^\ast(1, \beta) = (\varepsilon_1^\ast(1, \beta), \varepsilon_2^\ast(1, \beta), \ldots).
\]

Let \(\prec\) be the lexicographical order on \(\Sigma^\infty\) which is defined as:
\[
(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots) \prec (\eta_1, \eta_2, \eta_3, \ldots)
\]
if and only if \(\varepsilon_1 < \eta_1\) or there exists \(n \geq 1\) such that \(\varepsilon_i = \eta_i\) for \(1 \leq i < n\) but \(\varepsilon_n < \eta_n\).

The admissible sequences is characterized in by the following theorem.

**Theorem 2.1** (Parry \[25\]).

1. A non-negative integer sequence \((\varepsilon_1, \varepsilon_2, \ldots)\) is \(\beta\)-admissible if and only if
   \[
   (\varepsilon_i, \varepsilon_{i+1}, \ldots) \prec (\varepsilon_1^\ast(1, \beta), \varepsilon_2^\ast(1, \beta), \ldots), \quad \forall i \geq 1.
   \]
2. If \(1 < \beta_1 < \beta_2\), then \(\Sigma_{\beta_1} \subset \Sigma_{\beta_2}\).
3. A non-negative integer sequence \((\varepsilon_1, \varepsilon_2, \ldots)\) is the expansion of 1 for some \(\beta > 1\) if and only if
   \[
   (\varepsilon_i, \varepsilon_{i+1}, \ldots) \prec (\varepsilon_1, \varepsilon_2, \ldots) \text{ or } (\varepsilon_i, \varepsilon_{i+1}, \ldots) = (\varepsilon_1, \varepsilon_2, \ldots), \quad \forall i \geq 1.
   \]

Moreover, the cardinality of \(\Sigma^n_{\beta}\) and the topological entropy of \(\beta\)-expansion can be characterized by the following theorem.

**Theorem 2.2** (Rényi \[26\]). For any \(\beta > 1\) and \(n \geq 1\), we have
\[
\beta^n \leq \text{Card} \Sigma^n_{\beta} \leq \frac{\beta^{n+1}}{\beta - 1}.
\]

In particular, the topological entropy of the dynamical system \((I, T_\beta)\) is equal to
\[
\lim_{n \to \infty} (\log \text{Card} \Sigma^n_{\beta})/n = \log \beta.
\]

In what follows, we will introduce a crucial subset \(B_0\) of \((1, \infty)\) (see \[24\]). In fact, many problems about \(\beta\)-expansion can be more easily dealt with when \(\beta\) takes value in this set and by using the technique of approximation we can then solve the problems for the general \(\beta > 1\).

Let \(l_n(\beta), n \geq 1\), be the length of the longest consecutive zeros following the digit \(\varepsilon_n^\ast(1, \beta)\). That is,
\[
l_n(\beta) = \max\{k \geq 0 : \varepsilon_{n+j}^\ast(1, \beta) = 0, \text{ for all } 1 \leq j \leq k\}, \quad n \geq 1.
\]

Write
\[
l_n(\beta) = \{\beta > 1 : \{l_n(\beta)\}_{n \geq 1} \text{ is bounded}\}.
\]

Then the set \(B_0\) is just the collection \(C_3\) in \[27\] such that \(S_\beta\) satisfies the specification property. Moreover, we have

**Lemma 2.3** (See \[22\] \[27\]). The set \(B_0\) is uncountable and dense in \((1, \infty)\). In addition, we have that \(\mathcal{L}(B_0) = 0\) and \(\dim_H B_0 = 1\).

Let \((\varepsilon_1, \ldots, \varepsilon_n), n \geq 1\), be an admissible word. We call
\[
I_n(\varepsilon_1, \ldots, \varepsilon_n) = \{x \in I : \varepsilon_i(x, \beta) = \varepsilon_i, 1 \leq i \leq n\}
\]
an \(n\)-th cylinder. Since each cylinder is an interval, we can also call it an \(n\)-th order basic interval. Denote by \(I_n(x) = I_n(\varepsilon_1(x, \beta), \ldots, \varepsilon_n(x, \beta))\) the \(n\)-th cylinder.
containing $x$ and $|I_n(x)|$ the length of $I_n(x)$. It is easy to see from the expansion (1.1) that

$$|I_n(x)| \leq \beta^{-n}, \quad n \geq 1.$$ 

Obviously, we have $I_n(x) = \beta^{-n}$ when $\beta$ is an integer. In addition to this, the full cylinder of rank $n$ in base $\beta$, defined in the following, also meets this situation.

**Definition 2.4.** Let $(\varepsilon_1, \ldots, \varepsilon_n) \in \Sigma_\beta^n$, $n \geq 1$. An $n$-th cylinder $I_n(\varepsilon_1, \ldots, \varepsilon_n)$ is said to be full if its length verifies

$$|I_n(\varepsilon_1, \ldots, \varepsilon_n)| = \beta^{-n}.$$ 

Accordingly, $(\varepsilon_1, \ldots, \varepsilon_n)$ is called a full word.

Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \Sigma_\beta^n$, $n \geq 1$; $\eta = (\eta_1, \ldots, \eta_m) \in \Sigma_\beta^m$, $m \geq 1$. We defined the concatenation of $\varepsilon$ and $\eta$ as

$$\varepsilon \ast \eta = (\varepsilon_1, \ldots, \varepsilon_n, \eta_1, \ldots, \eta_m)$$

if the concatenated word is admissible. The following lemmas characterize the full cylinders.

**Lemma 2.5** (See Lemma 3.1 in [16]). Let $\beta > 1$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \Sigma_\beta^n$, $n \geq 1$. The following are equivalent:

1. $I_n(\varepsilon_1, \ldots, \varepsilon_n)$ is a full cylinder;
2. $T_\beta^l I_n(\varepsilon_1, \ldots, \varepsilon_n) = I$;
3. For any $\eta \in \Sigma_\beta^m$ with $m \geq 1$, the concatenated word $\varepsilon \ast \eta$ is admissible.

**Lemma 2.6** (See Lemma 3.2 and Corollary 3.3 in [16]). Let $\beta > 1$ and $m, n \geq 1$.

1. If $I_n(\varepsilon_1, \ldots, \varepsilon_n)$ is full, then for any word $(\eta_1, \ldots, \eta_m) \in \Sigma_\beta^m$, we have
   $$|I_{n+m}(\varepsilon_1, \ldots, \varepsilon_n, \eta_1, \ldots, \eta_m)| = |I_n(\varepsilon_1, \ldots, \varepsilon_n)| \cdot |I_m(\eta_1, \ldots, \eta_m)|;$$
2. Let $p \in \mathbb{N}$. Then
   $$I_{n+p}(\varepsilon_1, \ldots, \varepsilon_n, 0^p) \text{ is full } \iff \beta^{-(n+p)} \geq \beta^{-(n+\beta)}.$$ 

Here, $0^p$ with $p \geq 1$ is a word of length $p$ composed by $0$’s and $0^0$ is the empty word.

By Lemma 2.6(1) and the definition of full word, we have

**Corollary 2.7.** Let $\beta > 1$ and $m, n \geq 1$. If the two cylinders $I_n(\varepsilon_1, \ldots, \varepsilon_n)$ and $I_m(\eta_1, \ldots, \eta_m)$ are full, then the cylinder $I_{n+m}(\varepsilon_1, \ldots, \varepsilon_n, \eta_1, \ldots, \eta_m)$ is full; equivalently, if the two words $(\varepsilon_1, \ldots, \varepsilon_n)$ and $(\eta_1, \ldots, \eta_m)$ are full, then the concatenation $(\varepsilon_1, \ldots, \varepsilon_n, \eta_1, \ldots, \eta_m)$ is full.

**Lemma 2.8** (See [23]). Let $\beta > 1$ and $n \geq 1$. Write $M_n(\beta) = \max_{1 \leq i \leq n} \{|s_i(\beta)|\}$. Then, for any admissible word $(\varepsilon_1, \ldots, \varepsilon_n)$, the cylinder $I_{n+m}(\varepsilon_1, \ldots, \varepsilon_n, 0^{m+1})$ is full if $m \geq M_n(\beta)$.

Since the sequence $\{s_i(\beta)\}$ is bounded for each parameter $\beta$ in $B_0$, by Lemma 2.5, Lemma 2.6 and Lemma 2.8 we can easily obtain that

**Lemma 2.9.** Let $\beta \in B_0$ and $(\varepsilon_1, \ldots, \varepsilon_n)$, $n \geq 1$, be any admissible word. There exists an integer $M > 0$ such that $I_{n+M}(\varepsilon_1, \ldots, \varepsilon_n, 0^M)$ is full, which leads to that

1. any admissible word $(\eta_1, \ldots, \eta_m)$ can be concatenated behind $(\varepsilon_1, \ldots, \varepsilon_n, 0^M)$;
The length of cylinder $I_n(\varepsilon_1, \ldots, \varepsilon_n)$ satisfies
\[ \beta^{-(n+M)} \leq |I_n(\varepsilon_1, \ldots, \varepsilon_n)| \leq \beta^{-n}. \]

Recall the root $\beta_m = \beta_m(\beta)$ defined in the equation (1.10). Define
\[ B_1(\beta) = \{ \beta_m(\beta) : \text{the root of (1.10)} \text{ where } \varepsilon^*_m(1, \beta) \geq 1\} \]
and
\[ B_1 = \bigcup_{\beta > 1} (B_1(\beta) \setminus \{1\}). \]

Then we have

**Lemma 2.10.** $B_1 \subset B_0$ and $B_1$ is dense in $(1, \infty)$.

**Proof.** It is followed by the fact that the digit sequence of expansion of 1 under base $\beta_m$ is the $m$-periodic sequence $(\varepsilon^*_1(1, \beta), \ldots, \varepsilon^*_m(1, \beta), \varepsilon^*_m(1, \beta) - 1)^\infty$, where the integer $m$ satisfies $\varepsilon^*_m(1, \beta) \geq 1$. The second conclusion is obvious since $\beta_m \to \beta$ as $m \to \infty$. \qed

We point out that the formula (1.11) may turn into a simpler form when $\beta \in B_1$ as below:

\[ h^\beta(\alpha) = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{\log h^\beta(\alpha, n, \delta)}{(\log \beta)n} = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{\log h^\beta(\alpha, n, \delta)}{(\log \beta)n}. \]

Moreover, we would like to collect more in the following lemma.

**Lemma 2.11** (See Proposition 4.2 and Proposition 4.4 in [31]). Let $\beta > 1$.

1. If $\beta_m = \beta_m(\beta) \in B_1$, $m \geq 1$, then
\[
\begin{align*}
\log h^\beta(\alpha) &= \lim_{\delta \to 0} \lim_{m \to \infty} \frac{\log h^\beta_m(\alpha, n, \delta)}{(\log \beta)n} = \lim_{\delta \to 0} \lim_{m \to \infty} \frac{\log h^\beta_m(\alpha, n, \delta)}{(\log \beta)n} \\
&= \lim_{\delta \to 0} \lim_{n \to \infty} \frac{\log h^\beta(\alpha, n, \delta)}{(\log \beta)n} = \lim_{\delta \to 0} \lim_{n \to \infty} \frac{\log h^\beta(\alpha, n, \delta)}{(\log \beta)n}.
\end{align*}
\]

2. $h^\beta(\alpha)$ is a concave function and continuous on $I_\beta$.

The second conclusion in the above lemma also indicates that the function $h^\beta(\alpha)$ is increasing on $[0, \alpha^*(\beta)]$ and decreasing on $[\alpha^*(\beta), \Lambda(\beta)]$. These properties will be used in the following sections when the classified discussions for the corresponding proofs are needed.

3. **Besicovitch sets**

In this section, we will introduce the notions of the lower and upper algebraic average of $\beta$-expansion of numbers. Based on them, we will give the definitions of lower and upper Besicovitch sets and then determine their Hausdorff dimensions.

Let $\beta > 1$ and $x \in I$. Recall the notation $A_n(x, \beta) = S_n(x, \beta)/n$, $n \geq 1$, in the first section. Then denote respectively by

\[ (3.1) \quad \Delta(x, \beta) = \liminf_{n \to \infty} A_n(x, \beta) \quad \text{and} \quad \bar{A}(x, \beta) = \limsup_{n \to \infty} A_n(x, \beta), \]

the lower and upper algebraic averages of $\beta$-expansion of $x$. If $\Delta(x, \beta) = \bar{A}(x, \beta)$, then the common value is just the algebraic average $A(x, \beta) = \lim_{n \to \infty} S_n(x, \beta)/n$. 

being stated in \( \text{(1.7)} \). Let \( \alpha \in I_\beta \). Define the following Besicovitch set in \( \beta \)-expansion

\[
E_\beta^\alpha = \{ x \in I : A(x, \beta) = \alpha \}.
\]

Then, for the size of \( E_\beta^\alpha \) we have

**Lemma 3.1** (See Corollary 1.3 in [21].) Let \( \beta > 1 \) and \( \alpha \in I_\beta \). Then

\[
\dim_H E_\beta^\alpha = h_\beta^\alpha(\alpha).
\]

Recall the definition of \( \alpha^*(\beta) \) in \( \text{(1.8)} \). Since \( A(x, \beta) = \alpha^*(\beta) \) for almost all \( x \in I \) by Birkhoff’s ergodic theory, we have

\[
\dim_H E_\beta^{\alpha^*(\beta)} = \dim_H \{ x \in I : A(x, \beta) = \alpha^*(\beta) \} = 1.
\]

Thus, Lemma 3.1 gives that

**Corollary 3.2.** Let \( \beta > 1 \). Then we have \( h_\beta^\alpha(\alpha^*(\beta)) = 1 \).

Furthermore, define respectively the lower and upper Besicovitch sets in \( \beta \)-expansion as follows:

\[
E_\beta^\alpha = \{ x \in I : A(x, \beta) \geq \alpha \} \quad \text{and} \quad \bar{E}_\beta^\alpha = \{ x \in I : \bar{A}(x, \beta) \leq \alpha \},
\]

where \( \alpha \in I_\beta \). Then, for the sizes of \( E_\beta^\alpha \) and \( \bar{E}_\beta^\alpha \), we have

**Proposition 3.3.** Let \( \beta > 1 \) and \( \alpha \in I_\beta \). Then

\[
\dim_H E_\beta^\alpha = \begin{cases} 1, & 0 \leq \alpha \leq \alpha^*(\beta); \\ h_\beta^\alpha(\alpha), & \alpha^*(\beta) < \alpha \leq \Lambda(\beta), \end{cases}
\]

and

\[
\dim_H \bar{E}_\beta^\alpha = \begin{cases} h_\beta^\alpha(\alpha), & 0 \leq \alpha \leq \alpha^*(\beta); \\ 1, & \alpha^*(\beta) < \alpha \leq \Lambda(\beta). \end{cases}
\]

Note that Proposition 3.3 generalizes the classic work in [2] of Besicovitch on binary expansion. In fact, if \( \beta = 2 \), then we can easily obtain that \( \alpha^*(2) = 1/2 \), \( \Lambda(2) = 1 \), \( I_2 = [0, 1] \) and by the Stirling’s approximation we also have

\[
h^2(\alpha) = \frac{H(\alpha)}{\log 2}, \quad \text{where} \quad H(\alpha) = -\alpha \log \alpha - (1 - \alpha) \log(1 - \alpha), \quad \alpha \in I_2.
\]

To prove Proposition 3.3, we would like to present firstly a lemma about the relation among the \( \beta \)-adic entropy function, the \( \beta \)-adic lower entropy function and the \( \beta \)-adic upper entropy function. The corresponding notations of them are given in the following.

Let \( \beta > 1, \alpha \in I_\beta, n \geq 1 \) and \( \delta > 0 \). Denote

\[
\mathcal{H}_\beta^\alpha(n, \delta) = \left\{ (\varepsilon_1, \ldots, \varepsilon_n) \in \Sigma_\beta^n : \sum_{i=1}^n \varepsilon_i > n(\alpha - \delta) \right\}
\]

and \( h^\beta(n, \delta) = \text{Card} \mathcal{H}_\beta(\alpha, n, \delta) \). Then define the \( \beta \)-adic lower entropy function

\[
h^\beta(\alpha) = \lim_{\delta \to 0} \lim_{m \to \infty} \liminf_{n \to \infty} \frac{\log h^\beta(\alpha, n, \delta)}{(\log \beta)n},
\]

(3.4)
Here and in the sequel, the number $\beta_m$ with $m \geq 1$ is given in (1.10). Symmetrically, denote

$$
\bar{H}^\beta(\alpha, n, \delta) = \left\{ (\varepsilon_1, \ldots, \varepsilon_n) \in \Sigma_\beta^n : \sum_{i=1}^{n} \varepsilon_i < n(\alpha + \delta) \right\}
$$

and $\bar{h}^\beta(\alpha, n, \delta) = \text{Card} \ \bar{H}^\beta(\alpha, n, \delta)$. Then define the $\beta$-adic upper entropy function

$$
(3.5) \quad \bar{h}^\beta(\alpha) = \lim_{\delta \to 0} \lim_{m \to \infty} \limsup_{n \to \infty} \frac{\log \bar{h}^\beta_m(\alpha, n, \delta)}{(\log \beta)n}.
$$

**Lemma 3.4.** Let $\beta > 1$.

(1) For any $\alpha \in I_\beta$, we have

$$
\bar{h}^\beta(\alpha) = \lim_{\delta \to 0} \liminf_{n \to \infty} \frac{\log \bar{h}^\beta(\alpha, n, \delta)}{(\log \beta)n}
$$

and

$$
\bar{h}^\beta(\alpha) = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{\log \bar{h}^\beta(\alpha, n, \delta)}{(\log \beta)n};
$$

(2) If $0 \leq \alpha \leq \alpha^*(\beta)$, then $\bar{h}^\beta(\alpha) = h^\beta(\alpha)$; if $\alpha^*(\beta) < \alpha \leq \Lambda(\beta)$, then $\bar{h}^\beta(\alpha) = 1$;

(3) If $0 \leq \alpha < \alpha^*(\beta)$, then $\bar{h}^\beta(\alpha) = 1$; if $\alpha^*(\beta) \leq \alpha \leq \Lambda(\beta)$, then $\bar{h}^\beta(\alpha) = h^\beta(\alpha)$.

**Proof.** (1) It can be derived similar to the proof of Proposition 4.2 in [31].

(2) For the first part, it is followed by (2) in Lemma 2.11 and the following inequality

$$
\bar{h}^\beta(\alpha, n, \delta) \leq \bar{h}^\beta(\alpha, n, \delta) \leq 2 \left[ \frac{\alpha + \delta}{2\delta} \right] h^\beta(\alpha, n, \delta)
$$

for sufficiently large $n$ and small $\delta$. The second part is followed by the relation $h^\beta(\alpha^*(\beta)) \leq \bar{h}^\beta(\alpha)$ and Corollary 3.2.

(3) It can be dealt with in a similar way as that of (2).

Now, we are ready to give the proof of Proposition 3.3.

**Proof of Proposition 3.3.** It suffices to prove the conclusion (3.3). Since it is clear that $\dim_H E^\beta(\alpha) = 1$ when $\alpha^*(\beta) \leq \alpha \leq \Lambda(\beta)$ according to the ergodic theory, we only need to prove that $\dim_H E^\beta(\alpha) = h^\beta(\alpha)$ when $0 \leq \alpha \leq \alpha^*(\beta)$ in the following.

On the one hand, we have $E^\beta(\alpha) \subset E^\beta(\alpha)$ for any $\alpha \in I_\beta$. It yields that

$$
(3.6) \quad \dim_H E^\beta(\alpha) \geq \dim_H E^\beta(\alpha) = h^\beta(\alpha)
$$

by Lemma 3.4.

On the other hand, for any $\delta > 0$, we have

$$
E^\beta(\alpha) \subset \bigcap_{l=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{(\varepsilon_1, \ldots, \varepsilon_n) \in \bar{H}^\beta(\alpha, n, \delta)} I_n(\varepsilon_1, \ldots, \varepsilon_n),
$$

By (1) in Lemma 3.4 for any $\eta > 0$, there exists an integer $N$ such that

$$
\bar{h}^\beta(\alpha, n, \delta) < \beta^n(\bar{h}^\beta(\alpha) + \eta), \quad \forall n > N.
$$

Then, for any $l > N$, the $(\bar{h}^\beta(\alpha) + \eta)$-Hausdorff measure of $E^\beta(\alpha)$ satisfies

$$
\mu_{\bar{h}^\beta(\alpha) + \eta}(E^\beta(\alpha)) \leq \sum_{n=1}^{\infty} \bar{h}^\beta(\alpha, n, \delta)(\beta^{-n})^s < \sum_{n=1}^{\infty} (\beta^{\frac{s}{\beta}})^n < \infty,
$$

where $s$ is the Hausdorff dimension of $E^\beta(\alpha)$. Therefore, we have

$$
\dim_H E^\beta(\alpha) = h^\beta(\alpha).
$$

This completes the proof of Proposition 3.3.
which implies that $\dim_H \bar{E}^\beta(\alpha) \leq \bar{h}^\beta(\alpha) + \eta$. Thus,
\begin{equation}
\dim_H \bar{E}(\alpha) \leq \bar{h}^\beta(\alpha) = h^\beta(\alpha)
\end{equation}
by the arbitrariness of $\eta$ and (2) in Lemma 3.3.
On combining (3.6) and (3.7), it finishes the proof. \qed

4. Moran sets

In this section, we first recall the structure and a dimensional result about homogeneous Moran sets, then introduce the definition and some properties about full Moran sets and $\alpha$-Moran sets.

4.1. Homogeneous Moran sets. Let $\delta > 0$ and $\{N_k\}_{k \geq 1}$ be a sequence of integers and $\{c_k\}_{k \geq 1}$ be a sequence of positive numbers satisfying $N_k \geq 2, 0 < c_k < 1, k \geq 1$; and $N_k c_1 \leq \delta, N_k c_k \leq 1, k \geq 2$. Put
\begin{align*}
D_0 &= \{\emptyset\}, \quad D_k = \{(i_1, \ldots, i_k) : 1 \leq i_j \leq N_j, 1 \leq j \leq k\} \quad \text{for} \quad k \geq 1, \\
D &= \bigcup_{k \geq 0} D_k.
\end{align*}
Suppose that $J$ is an interval of length $\delta$. A collection $F = \{J_{\sigma} : \sigma \in D\}$ of subintervals of $J$ is said to have homogeneous structure if it satisfies
\begin{enumerate}
\item $J_\emptyset = J$;
\item For any $\sigma \in D_{k-1}$ with $k \geq 1$, $J_{\sigma^*j}, 1 \leq j \leq N_k$, are subintervals of $J_{\sigma}$ and $\text{int}(J_{\sigma^*i}) \cap \text{int}(J_{\sigma^*j}) = \emptyset$ if $i \neq j$, where int denotes the interior of some set;
\item For any $\sigma \in D_{k-1}$ with $k \geq 1$, we have
\begin{equation}
\frac{|J_{\sigma^*j}|}{|J_{\sigma}|} = c_j, \quad 1 \leq j \leq N_k.
\end{equation}
\end{enumerate}

If the collection $F$ is of homogeneous structure, then the set
\begin{equation}
\mathcal{M} = \mathcal{M}(J, \{N_k\}_{k \geq 1}, \{c_k\}_{k \geq 1}) = \bigcap_{k \geq 1} \bigcup_{\sigma \in D_k} J_{\sigma}
\end{equation}
is called a homogeneous Moran set determined by $F$.

Moreover, write
\begin{equation}
s = \liminf_{k \to \infty} \frac{\log(N_1 N_2 \cdots N_k)}{-\log(c_1 c_2 \cdots c_{k+1} N_{k+1})}.
\end{equation}
Then we have

Lemma 4.1 (See Theorem 2.1 and Corollary 2.1 in [17]). Let $\mathcal{M}$ be the homogeneous Moran set defined in (4.1). Then we have $\dim_H \mathcal{M} \geq s$. In addition, if $\inf_{k \geq 1} c_k > 0$, then $\dim_H \mathcal{M} = s$.

4.2. Full Moran sets. Let $\beta \in B_0, N \geq 1$ be sufficiently large and $P$ be an integer satisfying $0 \leq P \leq \Lambda(\beta)(N + M)$. Recall the integer $M$ being given in Lemma 2.9.

Write
\begin{equation}
W^\beta(P, N + M) := \left\{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N, 0^M) \in \Sigma_{\beta}^N + M : \sum_{i=1}^N \varepsilon_i = P\right\}.
\end{equation}
Based on the set $W^\beta(P, N + M)$, define
\[
W^\beta(P, N + M) = \{ x \in I : (\varepsilon_{i(N+M)+1}(x, \beta), \ldots, \varepsilon_{(i+1)(N+M)}(x, \beta)) \in W^\beta(P, N + M), i \geq 0 \}
\]
=: $W^\beta(P, N + M)^\infty$.

That is, $W^\beta(P, N + M)$ is the set of numbers of which the digit sequences consist of the words in $W^\beta(P, N + M)$. Moreover, this definition is valid according to (3) in Lemma 2.5, Corollary 2.7 and Lemma 2.9. Also, in this paper we call all of the words in $W$ to $N$ to $\beta$ and $\Lambda$. Based on the set $W^\beta(P, N + M)$, define
\[
\dim_H W^\beta(P, N + M) = \frac{\log \text{Card} W^\beta(P, N + M)}{(\log \beta)(N + M)}.
\]

**Proof.** Since $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N, 0^M)$ is full, we have $|(|\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N, 0^M)| = \beta^{-(N+M)}$.
Moreover, for any $k \geq 1$, by Corollary 2.7 we have
\[
|(|\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N, 0^M)|^k = \beta^{-k(N+M)}.
\]

This, together with Lemma 4.1 leads to the conclusion
\[
\dim_H W^\beta(P, N + M) = \liminf_{k \to \infty} \frac{k \log \text{Card} W^\beta(P, N + M)}{(\log \beta)(k + 1)(N + M) - \log \text{Card} W^\beta(P, N + M)}
\]
\[
= \frac{\log \text{Card} W^\beta(P, N + M)}{(\log \beta)(N + M)},
\]

which ends the proof. \hfill \Box

**Corollary 4.3.** Let $\beta \in B_0$, $\alpha \in I_\beta$ and $N \geq 1$. Then we have
\[
\dim_H W^\beta([\alpha(N + M)], N + M) = \frac{\log \text{Card} W^\beta([\alpha(N + M)], N + M)}{(\log \beta)(N + M)}.
\]

**Proof.** Take $P = [\alpha(N + M)]$ in Lemma 4.2 \hfill \Box

Moreover, we can even obtain

**Lemma 4.4.** Let $\beta \in B_0$ and $\alpha \in I_\beta$. We have
\[
\lim_{N \to \infty} \dim_H W^\beta([\alpha(N + M)], N + M) = h^\beta(\alpha).
\]

**Proof.** According to Corollary 4.3 we first show that
\[
\liminf_{N \to \infty} \frac{\log \text{Card} W^\beta([\alpha(N + M)], N + M)}{(\log \beta)(N + M)} = h^\beta(\alpha).
\]

The proof is divided into three cases: $0 \leq \alpha < \alpha^*(\beta)$, $\alpha = \alpha^*(\beta)$ and $\alpha^*(\beta) < \alpha \leq \Lambda(\beta)$. Here, we give only the proof of the first case. The other two cases can be dealt with in a similar way.

Define a function with two variables $\alpha$ and $N$:
\[
w^\beta_M(\alpha, N) = \text{Card} W^\beta([\alpha(N + M)], N + M).
\]

Then, by Lemma 2.11 we know that for each $\alpha$, $w^\beta_M(\alpha, N)$ is increasing with respect to $N$; for each $N$, $w^\beta_M(\alpha, N)$ is constant on $[(k - 1)/(N + M), k/(N + M)]$ where
Corollary 4.5. Let \( \beta \in B_0 \) and \( \alpha \in I_\beta \). Then
\[
\lim_{N \to \infty} \dim_H \mathcal{W}^\beta([\alpha(N + M)] + 1, N + M) = h^\beta(\alpha).
\]
Proof. By lemma 4.4 we have
\[
\lim_{N \to \infty} \dim_H W^\beta([\alpha(N + M)] + 1, N + M) = \lim_{N \to \infty} \dim_H W^\beta \left( \left[ \alpha + \frac{1}{N + M} \right](N + M), N + M \right) = \lim_{N \to \infty} h^\beta \left( \alpha + \frac{1}{N + M} \right) = h^\beta(\alpha)
\]
The last equality is followed by the continuity of \( h^\beta(\alpha) \) in (2) of Lemma 2.11. □

More generally, let \( \beta \in B_0, N \geq 1 \) be large enough and take two integers \( P \) and \( Q \) satisfying \( 0 \leq P \leq Q \leq \Lambda(\beta)(N + M) \). Write
\[
W^\beta([P, Q], N + M) := \left\{ (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N, 0^M) \in \Sigma_{\beta}^N : P \leq \sum_{i=1}^{N} \varepsilon_i \leq Q \right\}.
\]
Based on the set \( W^\beta([P, Q], N + M) \), define the full Moran set
\[
W^\beta([P, Q], N + M) := \left\{ x \in I : (\varepsilon_{i(N+M)+1}(x, \beta), \ldots, \varepsilon_{i(N+M)}(x, \beta)) \in W^\beta([P, Q], N + M), i \geq 0 \right\}.
\]

Then, by an analogue discussion to that of Lemma 4.2 we may obtain

Lemma 4.6. Let \( \beta \in B_0 \) and the two integers \( P \) and \( Q \) satisfy \( 0 \leq P \leq Q \leq \Lambda(\beta)(N + M), N \geq 1 \). Then
\[
\dim_H W^\beta([P, Q], N + M) = \frac{\log \text{Card} W^\beta([P, Q], N + M)}{(\log \beta)(N + M)}.
\]

As a matter of fact, for the above formula 4.6 \( P \) and \( Q \) are not necessarily integers. In a more detail, define similarly
\[
W^\beta((P, Q), N + M) := W^\beta((P, Q), N + M)^\infty
\]
where
\[
W^\beta((P, Q), N + M) := \left\{ (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N, 0^M) \in \Sigma_{\beta}^{N+M} : P < \sum_{i=1}^{N} \varepsilon_i < Q \right\},
\]
then we also have

Lemma 4.7. Let \( \beta \in B_0 \) and two numbers \( P \) and \( Q \) satisfy \( 0 \leq P < Q \leq \Lambda(\beta)(N + M), N \geq 1 \). Then
\[
\dim_H W^\beta((P, Q), N + M) = \frac{\log \text{Card} W^\beta((P, Q), N + M)}{(\log \beta)(N + M)}.
\]

4.3. \( \alpha \)-Moran sets. In this subsection, we will construct a set \( W^\beta_\alpha(\alpha, N + M) \), called \( \alpha \)-Moran set, where \( 0 \leq \alpha < \Lambda(\beta), \beta \in B_0 \) and \( N > 1 \). It can be used to construct a suitable Moran subset to obtain the lower bound of Hausdorff dimension of \( ER_\beta^\alpha(\alpha) \). To this end, we first construct recursively two sequences of sets of digit words \( \{W_n^\beta(\alpha, N + M)\}_{n=1}^\infty \) and \( \{V_n^\beta(\alpha, N + M)\}_{n=1}^\infty \). Take \( N \) to be sufficiently large such that
\[
[\alpha(N + M)] + 1 < \Lambda(\beta)(N + M).
\]
Put
\[ W_1^\beta(\alpha, N + M) = \left\{ (\varepsilon_1, \ldots, \varepsilon_N, 0^M) \in \Sigma_\beta^{N+M}: \sum_{i=1}^{N} \varepsilon_i = [\alpha(N + M)] \right\} \]
and
\[ V_1^\beta(\alpha, N + M) = \left\{ (\varepsilon_1, \ldots, \varepsilon_N, 0^M) \in \Sigma_\beta^{N+M}: \sum_{i=1}^{N} \varepsilon_i = [\alpha(N + M)] + 1 \right\}. \]

Suppose that the sets \( W_i^\beta(\alpha, N + M) \) and \( V_i^\beta(\alpha, N + M) \) are well defined for all \( 1 \leq i \leq n \), then define
\[ W_{n+1}^\beta(\alpha, N + M) = \{ (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2^{n-1}(N+1)}(N+M)) \in W([\alpha 2^n(N + M)], 2^n(N + M)) : \]
\[ (\varepsilon_{2^{n-1}(N+1)(i+1)}, \ldots, \varepsilon_{2^{n-1}(N+1)(i+1)}) \in W_n^\beta(\alpha, N + M) \cup V_n^\beta(\alpha, N + M), i = 0, 1 \}
and
\[ V_{n+1}^\beta(\alpha, N + M) = \{ (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2^{n-1}(N+1)}(N+M)) \in W([\alpha 2^n(N + M)] + 1, 2^n(N + M)) : \]
\[ (\varepsilon_{2^{n-1}(N+1)(i+1)}, \ldots, \varepsilon_{2^{n-1}(N+1)(i+1)}) \in W_n^\beta(\alpha, N + M) \cup V_n^\beta(\alpha, N + M), i = 0, 1 \}.

The above definitions are valid since the estimation
\[ (4.8) \quad 2[\alpha 2^n(N + M)] < [\alpha 2^{n+1}(N + M)] + 1 \leq 2([\alpha 2^n(N + M)] + 1) \]
holds for all \( n \geq 0 \).

With this construction, we know that for each \( n \geq 1 \), every word in \( W_n^\beta(\alpha, N + M) \) is of length \( 2^{n-1}(N + M) \) and the sum of digits is \( [\alpha 2^{n-1}(N + M)] \). Likewise, every word in \( V_n^\beta(\alpha, N + M) \) is of length \( 2^{n-1}(N + M) \) and the sum of digits is \( [\alpha 2^{n-1}(N + M)] + 1 \). Moreover, we may even obtain

**Lemma 4.8.** Let \( \beta \in B_0 \) and \( 0 \leq \alpha < \Lambda(\beta) \). For any \( 0 \leq i \leq n - 1 \), we can decompose uniquely each word in \( W_n^\beta(\alpha, N + M) \) or \( V_n^\beta(\alpha, N + M) \) into successive concatenations of digit words of length \( 2^{n}(N + M) \), the sum of digits in each one is either \( [\alpha 2^{n}(N + M)] \) or \( [\alpha 2^{n}(N + M)] + 1 \).

Based on the sequence of \( \{W_n^\beta(\alpha, N + M)\}_{n=1}^{\infty} \), define the so-called \( \alpha \)-Moran set in \( \beta \)-expansion
\[ \mathcal{W}_n^\beta(\alpha, N + M) = \{ x \in I: (\varepsilon_{2^{n-1}(N+1)(x, \beta)}), \ldots, \varepsilon_{2^{n-1}(N+1)(x, \beta)}(x, \beta)) \in W_n^\beta(\alpha, N + M), n \geq 1 \} \]
\[ =: \prod_{n=1}^{\infty} W_n^\beta(\alpha, N + M). \]

That is, \( \mathcal{W}_n^\beta(\alpha, N + M) \) is the set of numbers of which the digit sequences concatenate the words in \( W_n^\beta(\alpha, N + M), n \geq 1 \), one by one in the order of natural numbers.

**Lemma 4.9.** Let \( \beta \in B_0 \) and \( 0 \leq \alpha < \Lambda(\beta) \). Then
\[ (4.9) \quad \lim_{N \to \infty} \dim_H \mathcal{W}_n^\beta(\alpha, N + M) = h^\beta(\alpha). \]

**Proof.** For the case \( 0 \leq \alpha < \alpha^*(\beta) \), take \( N \) to be large enough such that
\[ [\alpha(N + M)] + 1 < [\alpha^*(\beta)(N + M)]. \]
Then by the structures of sequences in \( \mathcal{W}^\beta_\infty(\alpha, N + M) \), described in Lemma 4.8 we have

\[
\dim_H \mathcal{W}^\beta ([\alpha(N + M)], N + M) \leq \dim_H \mathcal{W}^\beta_\infty(\alpha, N + M) \leq \dim_H \mathcal{W}^\beta \left( [[\alpha(N + M)], [\alpha(N + M)] + 1], N + M \right).
\]

(4.10) Since

\[
\dim_H \mathcal{W}^\beta (\left([[\alpha(N + M)], [\alpha(N + M)] + 1], N + M \right)
\]

\[
= \log \left( \frac{2 \max \{ w_M^\beta (\alpha, N), w_M^\beta (\alpha + \frac{1}{N+M}, N) \}}{(\log \beta)(N + M)} \right)
\]

by letting \( N \to \infty \) we have

\[
\lim_{N \to \infty} \dim_H \mathcal{W}^\beta (\left([[\alpha(N + M)], [\alpha(N + M)] + 1], N + M \right) \leq h^\beta(\alpha)
\]

(4.11) according to Lemma 4.3 and Corollary 4.5. Moreover, we have

\[
\lim_{N \to \infty} \dim_H \mathcal{W}^\beta ([\alpha(N + M)], N + M) = h^\beta(\alpha)
\]

by Lemma 4.3. This, together with (4.10) and (4.11), leads to the conclusion (4.9).

On the other hand, for the remainder case \( \alpha^*(\beta) \leq \alpha < \Lambda(\beta) \), we have similarly that

\[
\min \{ \dim_H \mathcal{W}^\beta ([\alpha(N + M)], N + M), \dim_H \mathcal{W}^\beta ([\alpha(N + M)] + 1, N + M) \}
\]

\[
\leq \dim_H \mathcal{W}^\beta_\infty(\alpha, N + M) \leq \dim_H \mathcal{W}^\beta \left( [[\alpha(N + M)], [\alpha(N + M)] + 1], N + M \right).
\]

Then we can prove this case by Corollary 4.3 and the same discussion as that for the foregoing case.

The proof is completed now. \( \square \)

5. Two Lemmas

In this section, we will present two lemmas for the proof of Theorem 1.1 i.e., the following Lemma 5.1 and Lemma 5.4 The first one is related to subsets of \( \mathbb{N} \) with zero density and the second one describes the lower bound of Hausdorff dimension of \( ER^\beta_\delta(\alpha) \).

Let \( \mathcal{M} \) be a subset of \( \mathbb{N} \) and write its complementary set of \( \mathbb{N} \) as \( \mathbb{N} \setminus \mathcal{M} = \{ n_1 < n_2 < \ldots \} \). Suppose there are a set \( D \subset I \) and a mapping \( \phi:\mathcal{M}: D \to I \) such that the corresponding digit sequences satisfy

\[
(\varepsilon_1(x, \beta), \varepsilon_2(x, \beta), \ldots) \in \Sigma_D \mapsto (\varepsilon_{n_1}(x, \beta), \varepsilon_{n_2}(x, \beta), \ldots) \in \Sigma_\beta,
\]

where \( \Sigma_D \) denotes the set of dig sequences of numbers in \( D \), i.e.,

\[
\Sigma_D = \{ (\varepsilon_1(x, \beta), \varepsilon_2(x, \beta), \ldots) \in \Sigma_\beta : x \in D \}.
\]

If there exist such pair of set \( D \) and mapping \( \phi:\mathcal{M} \), then we call the mapping \( \phi:\mathcal{M} \) maps well on the set \( D \). Given such set \( D \subset I \) and mapping \( \phi:\mathcal{M} \), we may obtain another set

\[
\phi:\mathcal{M}(D) = \{ \phi:\mathcal{M}(x) : x \in D \}.
\]
In addition, we call the set $\mathbb{M}$ is of density zero in $\mathbb{N}$ if 

$$\lim_{n \to \infty} \frac{\text{Card}\{i \in \mathbb{M}: i \leq n\}}{n} = 0.$$ 

Then the relation between the sizes of $D$ and $\phi_{\mathbb{M}}(D)$ can be described as follows.

**Lemma 5.1.** Let $\beta \in B_0$, $D \subset I$ and $\mathbb{M}$ be of density zero in $\mathbb{N}$. If the mapping $\phi_{\mathbb{M}}$ maps well on $D$, then we have 

$$\dim_H \phi_{\mathbb{M}}(D) = \dim_H D.$$ 

**Proof.** (1) To show $\dim_H \phi_{\mathbb{M}}(D) \geq \dim_H D$. If $\dim_H \phi_{\mathbb{M}}(D) = r$, then for any $s > t > r$, the $t$-Hausdorff measure of $\phi_{\mathbb{M}}(D)$ is zero, i.e., $\mathbb{H}^t(\phi_{\mathbb{M}}(D)) = 0$. Therefore, there exists a $\delta$-cover $\{I_j(\overline{x}_j)\}_{j \geq 1}$ of $\phi_{\mathbb{M}}(D)$ with $0 < \delta < 1$ such that 

$$\sum_{j \geq 1} |I_j(\overline{x}_j)|^t < \infty,$$

where $\overline{x}_j \in \phi_{\mathbb{M}}(D)$ and $I_j(\overline{x}_j)$ denotes the cylinder $I_j(\overline{\varepsilon}_1, \ldots, \overline{\varepsilon}_1)$, $j \geq 1$. It yields that 

$$\sum_{j \geq 1} \beta^{-(l_j + \mathbb{M})t} < \infty$$

by (2) of Lemma 2.9. For any $x \in D$, assume that $\phi_{\mathbb{M}}(x) = \bar{x} \in \phi_{\mathbb{M}}(D)$ with digit sequence $(\varepsilon_1, \varepsilon_2, \ldots)$. Write $\phi_{\mathbb{M}}(I_N(\varepsilon_1, \ldots, \varepsilon_N)) = I_n(\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_N)$. Since 

$$\frac{N - n}{N} = \frac{\text{Card}\{i \in \mathbb{M}: i \leq N\}}{N} \to 0$$

as $N \to \infty$, for the above $\delta$ there exists an integer $N_1$ such that 

$$\frac{N - n}{N} < \delta, \quad \text{i.e.,} \quad n \leq N < \frac{n}{1 - \delta} \quad \text{for} \quad N > N_1.$$ 

Moreover, we may even require that $\delta$ is small enough such that for all $j \geq 1$, 

$$l_j > N_1 \quad \text{and} \quad \frac{l_j}{l_j + \mathbb{M}(s - \frac{\delta}{1 - \delta})} > t.$$ 

Since $\phi_{\mathbb{M}}(D) \subset \bigcup_{j \geq 1} I_j(\overline{x}_j)$, we have that $D \subset \bigcup_{j \geq 1} \phi_{\mathbb{M}}^{-1}(I_j(\overline{x}_j))$. Note that, by 

$$\frac{N - n}{N} < \delta, \quad \text{i.e.,} \quad n \leq N < \frac{n}{1 - \delta} \quad \text{for} \quad N > N_1.$$ 

Moreover, we may even require that $\delta$ is small enough such that for all $j \geq 1$, 

$$l_j > N_1 \quad \text{and} \quad \frac{l_j}{l_j + \mathbb{M}(s - \frac{\delta}{1 - \delta})} > t.$$ 

Since $\phi_{\mathbb{M}}(D) \subset \bigcup_{j \geq 1} I_j(\overline{x}_j)$, we have that $D \subset \bigcup_{j \geq 1} \phi_{\mathbb{M}}^{-1}(I_j(\overline{x}_j))$. Note that, by 

$$\frac{N - n}{N} < \delta, \quad \text{i.e.,} \quad n \leq N < \frac{n}{1 - \delta} \quad \text{for} \quad N > N_1.$$ 

Moreover, we may even require that $\delta$ is small enough such that for all $j \geq 1$, 

$$l_j > N_1 \quad \text{and} \quad \frac{l_j}{l_j + \mathbb{M}(s - \frac{\delta}{1 - \delta})} > t.$$ 

Thus, by (5.2) and (5.4), we have 

$$\mathbb{H}^s_\beta(D) \leq \sum_{j \geq 1} \frac{\beta^{\frac{4}{\beta - 1} - j}}{\beta - l_j \mathbb{M}} \geq \frac{\beta}{l_j \mathbb{M}} \sum_{j \geq 1} \frac{\beta^{\frac{4}{\beta - 1} - j}}{\beta - l_j \mathbb{M}} < \infty.$$ 

It follows that $\mathbb{H}^s(D) < \infty$ and then $\dim_H D < s$. Since $s > r$ is arbitrary, we obtain that $\dim_H D \leq r = \dim_H \phi_{\mathbb{M}}(D)$. 


(2) To show $\dim_H \phi_M(D) \leq \dim_H D$. For any arbitrary $0 < \epsilon < 1$, since the set $M \subset \mathbb{N}$ is of density zero, we can choose an integer $N_0 > M$ such that

$$\frac{\text{Card}\{i \in M : i \leq n\}}{n} < \epsilon \quad \text{for all } n \geq N_0.$$ 

Take two numbers $x$ and $y$ with $d(x, y) = \beta^{-t}$, where $N \leq t < N + 1$ for some $N \geq N_0$. Then, by (2) of Lemma 2.9, we have

$$d(\phi_M(x), \phi_M(y)) \leq \beta^{-N + \text{Card}\{i \in M : i \leq N\} \frac{\text{Card}\{i \in M : i \leq N\}}{N}} < \left(\beta^{-(N+1)}\frac{N}{N_0^{1-\epsilon}}\right)^{1/N_0^{1-\epsilon}} < d(\varepsilon, \eta) \frac{N}{N^{1-\epsilon}}.$$ 

It means that the mapping $\phi_M$ is $N(1-\epsilon)/(N+1)$-Hölder on $D$. So, by Proposition 2.3 in [13], we have

$$\dim_H \phi_M(D) < \frac{N + 1}{N(1-\epsilon)} \dim_H D.$$ 

This leads to the conclusion $\dim_H \phi_M(D) \leq \dim_H D$ by the arbitrariness of $\epsilon$ and $N$.

Finally, by combining the above two assertions, we finish the proof.

Lemma 5.1 tells us that the Hausdorff dimension of a set will be invariant if the set of deleted positions of digit sequences of numbers is of density zero in $\mathbb{N}$. Certainly, we need to ensure that each new digit sequence, by deleting the set of positions with density zero from the original sequence, is admissible.

Before the presentation of the second lemma, we need to introduce the definition and corresponding properties of slowly varying sequences.

**Definition 5.2** (See [19, 28, 4]). Let $\theta$ be a function satisfying $\theta(n) > 0$ for all $n \geq 1$. We call the sequence $\{\theta(n)\}_{n \geq 1}$ slowly varying if there is a sequence of positive numbers $\{f(n)\}_{n \geq 1}$ satisfying

$$\lim_{n \to \infty} \frac{\theta(n)}{f(n)} = K > 0$$ 

and

$$\lim_{n \to \infty} n \left(1 - \frac{f(n-1)}{f(n)}\right) = 0.$$ 

There are some typical slowly varying sequences such as

$$\{C > 0\}_{n \geq 1}, \{\log n\}_{n \geq 1}, \{\log \log n\}_{n \geq 1}, \{\text{arctan } n\}_{n \geq 1}, \{\exp(\ln^{\nu} n)\}_{n \geq 1},$$

where $0 < \nu < 1$, etc. The following is a list of some basic properties of the slowly varying sequences.

**Lemma 5.3** (See Lemma 2.2 in [5]). Let the sequence $\{\theta(n)\}_{n \geq 1}$ be slowly varying. Then

1. the sequence $\{C\theta(n)\}_{n \geq 1}$, where $C > 0$, is also slowly varying;
2. $\lim_{n \to \infty} \log \theta(n) / \log n = 0$;
3. $\lim_{n \to \infty} \theta(n) / n = 0$.

Now, we are ready to present the second lemma which just explores the lower bound of Hausdorff dimension of $ER_\beta^\psi(\alpha)$ for the special case $\beta \in B_0$. Based on it, in the last section we will prove Theorem 1.1 for general $\beta > 1$ using the method of approximation.
Lemma 5.4. Let $\alpha \in I_\beta$, where $\beta \in B_0$. Let the sequence $\{\theta(n)\}_{n \geq 1}$ be slowly varying and $\theta(n) \to \infty$ as $n \to \infty$. If $\phi(n) = \lfloor \theta(n) \rfloor$, $n \geq 1$, then we have
\[
\dim_H ER_\phi^\beta(\alpha) \geq h^\beta(\alpha) \quad \text{as } 0 \leq \alpha \leq \alpha^*(\beta)
\]
and
\[
\dim_H ER_\phi^\beta(\alpha) = 1 \quad \text{as } \alpha^*(\beta) < \alpha < \Lambda(\beta).
\]

Proof. We will first prove that $\dim_H ER_\phi^\beta(\alpha) \geq h^\beta(\alpha)$ when $0 \leq \alpha \leq \alpha^*(\beta)$ by constructing a suitable subset of $ER_\phi^\beta(\alpha)$ which is related to $\alpha$-Moran sets. And then prove respectively that $\dim_H ER_\phi^\beta(\alpha) = 1$ when $\alpha^*(\beta) < \alpha < \Lambda(\beta)$ and when $\alpha = \Lambda(\beta)$ by constructing Moran subsets with sufficiently large Hausdorff dimensions.

Case I: $0 \leq \alpha \leq \alpha^*(\beta)$. Similar to the definition of $W_\infty^\beta(\alpha, N + M)$, define the set of numbers
\[
W^\beta_+(\alpha, N + M) := W^\beta_1(\alpha, N + M) \times \prod_{n=1}^\infty W^\beta_n(\alpha, N + M).
\]
Then we may obtain
\[
W^\beta_+(\alpha, N + M) \subset ER_\phi^\beta(\alpha).
\]

In fact, for any number $x \in W^\beta_+(\alpha, N + M)$ and integer $r \geq 1$, we can decompose its digit sequence $(\varepsilon_1(x, \beta), \varepsilon_2(x, \beta), \ldots)$ into successive concatenations of digit words of length $2^r(N + M)$. Moreover, the sum of digits in each word is $[\alpha 2^r(N + M)]$ or $[\alpha 2^r(N + M)] + 1$ but the initial one. Fix $r \geq 1$, assume that
\[
K 2^r(N + M) \leq \phi(n) < (K + 1)2^r(N + M)
\]
for some integer $K$, then
\[
\frac{(K - 2)[\alpha 2^r(N + M)]}{(K + 1)2^r(N + M)} \leq A_{\alpha, \phi}(n, x, \beta) \leq \frac{(K + 1)([\alpha 2^r(N + M)] + 1)}{K 2^r(N + M)}.
\]
Let $n \to \infty$, then $\phi(n) \to \infty$ and $K \to \infty$. Thus,
\[
\frac{[\alpha 2^r(N + M)]}{2^r(N + M)} \leq A_\phi(x, \beta) \leq \bar{A}_\phi(x, \beta) \leq \frac{[\alpha 2^r(N + M)] + 1}{2^r(N + M)},
\]
where
\[
A_\phi(x, \beta) = \liminf_{n \to \infty} A_{\alpha, \phi}(n, x, \beta) \quad \text{and} \quad \bar{A}_\phi(x, \beta) = \limsup_{n \to \infty} A_{\alpha, \phi}(n, x, \beta).
\]
Let $r \to \infty$, then we have $A_\phi(x, \beta) = \alpha$, which proves (16). Consequently, we have
\[
\dim_H W^\beta_+(\alpha, N + M) \leq \dim_H ER_\phi^\beta(\alpha).
\]
Moreover, it is clear that
\[
\dim_H W^\beta_+(\alpha, N + M) = \dim_H W_\infty^\beta(\alpha, N + M)
\]
according to the countable stationarity of Hausdorff dimension. Thus,
\[
\dim_H ER_\phi^\beta(\alpha) \geq \dim_H W_\infty^\beta(\alpha, N + M).
\]
By letting $N \to \infty$ and Lemma 11.8, it yields that
\[
\dim_H ER_\phi^\beta(\alpha) \geq h^\beta(\alpha).
\]
there exists a number $\delta_0$ satisfying with $0 < \delta_0 < \alpha^*(\beta) - \alpha$ and a positive integer $N_0$ such that

$$\log \text{Card} \left\{ (\varepsilon_1, \ldots, \varepsilon_n) \in \Sigma_0^n : \left(\alpha^*(\beta) - \frac{\delta_0}{2}\right)n < \sum_{i=1}^n \varepsilon_i < \left(\alpha^*(\beta) + \frac{\delta_0}{2}\right)n \right\} > 1 - \frac{\epsilon}{2}$$

for all $n \geq N_0$. Take an integer $n_0 > N_0$. Denote

$$U^\beta(\alpha^*(\beta), n_0 + M, \delta_0) = \left\{ (\varepsilon_1, \ldots, \varepsilon_n, 0^M) \in \Sigma_0^{n_0+M} : (\alpha^*(\beta) - \delta_0)(n_0 + M) < \sum_{i=1}^{n_0} \varepsilon_i < (\alpha^*(\beta) + \delta_0)(n_0 + M) \right\}.$$

Based on $U^\beta(\alpha^*(\beta), n_0 + M, \delta_0)$, construct the set of numbers

$$U^\beta(\alpha^*(\beta), n_0 + M, \delta_0) := U^\beta(\alpha^*(\beta), n_0 + M, \delta_0)^\infty.$$

Then, by the formula (4.7) and the above estimation, we have

$$\dim_H U^\beta(\alpha^*(\beta), n_0 + M, \delta_0) = \frac{\log \text{Card} U^\beta(\alpha^*(\beta), n_0 + M, \delta_0)}{(\log \beta)(n_0 + M)} > 1 - \epsilon.$$  

for sufficiently large $n_0$.

Next, based on $U^\beta(\alpha^*(\beta), n_0 + M, \delta_0)$, we will construct a set $U^\beta(\alpha^*(\beta), n_0 + M, \delta_0; \alpha, N + M)$, which is also denoted by $U^\beta(\alpha, N + M)$ for simplicity, in the following.

For each $x(n_0, \delta_0) \in U^\beta(\alpha^*(\beta), n_0 + M, \delta_0)$, we first construct, by induction, a number $x^*(\alpha, N + M)$. Write $x^{(0)} = x(n_0, \delta_0) = (\varepsilon_1(x^0, \beta), \varepsilon_2(x^0, \beta), \ldots)$. Suppose we have defined

$$x^{(j)} = (\varepsilon_1(x^{(j)}, \beta), \varepsilon_2(x^{(j)}, \beta), \ldots), \quad \text{for } 0 \leq j \leq k;$$

then set

$$x^{(k+1)} = (\varepsilon_1(x^{(k)}, \beta), \ldots, \varepsilon_{l_k}(x^{(k)}, \beta), 0^M w_{k+1}^\beta(\alpha, N + M), \varepsilon_{l_k+1}(x^{(k)}, \beta), \ldots).$$

Here, $w_{k+1}^\beta(\alpha, N + M) \in W_{k+1}^\beta(\alpha, N + M)$ and

$$l_1 = N, \quad l_k = \left(4^{k-1}N + (2^{k-1} - 2)(N + M) + (k - 1)\right)M, \quad k \geq 2.$$

That is, $x^{(k+1)}$ is obtained by inserting a word $0^M w_{k+1}^\beta(\alpha, N + M)$ of length $M + 2^k(N + M)$ at the position $l_k + 1$ of the digit sequence of $x^{(k)}$. It is easy to see that $\{x^{(j)}\}_{j \geq 0}$ is a Cauchy sequence. Denote by

$$x^*(\alpha, N + M) = (\varepsilon_1(x^*, \beta), \varepsilon_2(x^*, \beta), \ldots)$$

the corresponding limit point of $\{x^{(j)}\}_{j \geq 0}$. In other words, by inserting the sequence of digit words $\{0^M w_{k+1}^\beta(\alpha, N + M)\}_{k \geq 0}$ into the original positions $4^kN, k \geq 0$, of the digit sequence of number $x(n_0, \delta_0)$, we obtain $x^*(\alpha, N + M)$.

Then define the set

$$U^\beta(\alpha, N + M) = \{ x^*(\alpha, N + M) \in I : x(n_0, \delta_0) \in U^\beta(\alpha^*(\beta), n_0 + M, \delta_0) \}.$$  

It has the following properties:
or $\alpha$ value of $\phi$ can reach its maximal value. In fact, this is guaranteed by an estimation of the concatenated words of length $N$ word $w$ by Lemma 5.1. If $N$ is sufficiently large such that 
\[ l > \limsup_{t \to \infty} \frac{(N + M) + \cdots + 2l(N + M) + tM}{l_t} = 0. \]
Now, define the mapping 
\[ \phi(\alpha, n_0, \delta_0): U^3(\alpha, N + M) \to U^3(\alpha^*(\beta), n_0 + M, \delta_0) \]
then the above property (1) implies that 
\[ \dim_H U^3(\alpha, N + M) = \dim_H U^3(\alpha^*(\beta), n_0 + M, \delta_0) \]
by Lemma 5.1.

(2) $U^3(\alpha, N + M) \subset ER_{\phi}^3(\alpha)$. Take $x^* \in U^3(\alpha, N + M)$. Note that every word $w_{k+1}^\beta(\alpha, N + M)$ in $W_{k+1}^\beta(\alpha, N + M)$ can be decomposed into successively concatenated words of length $N + M$, the sum of digits in each word is $[\alpha(N + M)]$ or $[\alpha(N + M)] + 1$. Thus, when the word $(\varepsilon_{i+1}, \varepsilon_{i+2}, \ldots, \varepsilon_{i+\phi(n)})$ appears in some word $w_{k+1}^\beta(\alpha, N + M)$ in the digit sequence of $x^*(\alpha, N + M)$,
\[ I_{n, \phi(n)}(x, \beta) = \max_{0 \leq i \leq n - \phi(n)} \{ S_{i+\phi(n)}(x, \beta) - S_i(x, \beta) \} \]
can reach its maximal value. In fact, this is guaranteed by an estimation of the value of $\phi(n)$ given in the following.
Since the sequence $\{\theta(n)\}_{n \geq 1}$ is slowly varying and \( \lim_{n \to \infty} \theta(n) = \infty \), we have
\[ \lim_{n \to \infty} \frac{\log \phi(n)}{\log n} = \lim_{n \to \infty} \frac{\log \theta(n)}{\log n} = \lim_{n \to \infty} \frac{\log \theta(n)}{\log n} = 0 \]
by the property (2) in Lemma 5.3. Thus, there exists a number $L > 0$ such that
\[ \frac{\log \phi(n)}{\log n} \ll \frac{1}{2}, \quad \text{i.e.,} \quad \phi(n) \ll n^{\frac{1}{2}}, \quad \forall n > L. \]
Take $N$ to be sufficiently large such that
\[ (N + M)^{\frac{1}{2}} > \max\{2, \sqrt{L} \}. \]
If $l_k < n \leq l_{k+1}$ for some integer $k$, then
\[ \phi(n) \leq \phi(4^k(N + M) + (2^k - 2)(N + M) + kM) \]
\[ < \phi(4^{k+1}(N + M)) \ll (4^{k+1}(N + M))^{\frac{1}{2}} \]
\[ = 2^{k+1}(N + M)^{\frac{1}{2}} \]
\[ < 2^k(N + M). \]
It means that the length of digit word $(\varepsilon_{i+1}, \ldots, \varepsilon_{i+\phi(n)})$ is far less than that of the word $w_{k+1}^\beta(\alpha, N + M)$ for sufficiently large $N$. 


For any $r \geq 1$, assume that $K2^r(N + M) \leq \phi(n) < (K + 1)2^r(N + M)$ for some integer $K$. Then we have
\[
\frac{(K - 1)[\alpha 2^r(N + M)]}{(K + 1)2^r(N + M)} \leq A_{n, \phi(n)}(x, \beta) \leq \frac{(K + 1)([\alpha 2^r(N + M)] + 1)}{K2^r(N + M)}.
\]
Let $n \to \infty$, then $\phi(n) \to \infty$ and $K \to \infty$. And then let $r \to \infty$, similar to the discussion in the inequalities (5.9) and (5.10), we may obtain $A(x, \beta) = \alpha$. It leads to the conclusion $UR_\phi(\alpha, N + M) \subset ER_\phi(\alpha)$.

Property (2) implies that
\[
\dim_H ER_\phi(\alpha) \geq \dim_H UR_\phi(\alpha, N + M).
\]
This, together with (5.13) and (5.15), yields that $\dim_H ER_\phi(\alpha) > 1 - \epsilon$. It proves this case since $\epsilon$ is arbitrary.

Case III: $\alpha = \Lambda(\beta)$. The technique for the proof of this case is similar to that of Case I and we would like to only give the outline here. First, by the definition of $\Lambda(\beta)$, for any $j \geq 1$, there exists digit sequence $\eta^{(j)} = (\eta_0^{(j)}, \eta_1^{(j)}, \ldots) \in \Sigma_\beta$ and an integer $n_j$ such that
\[
\Lambda(\beta) - \frac{1}{j} < \frac{\sum_{i=1}^{n_j+1} \eta_i^{(j)}}{n_j} < \Lambda(\beta) + \frac{1}{j}.
\]
Moreover, the sequence $\{n_j\}_{j \geq 1}$ can be chosen to be strictly increasing. Then, according to the sequences $\{\eta^{(j)}\}_{j \geq 1}$ and $\{n_j\}_{j \geq 1}$, construct the following sequence of admissible words:
\[
(0^M, \eta_0^{(1)}, 0^M), (0^M, \eta_1^{(1)}, \eta_0^{(1)}, 0^M), \ldots, (0^M, \eta_1^{(1)}, \eta_2^{(1)}, \ldots, \eta_{n_1}^{(1)}, 0^M)
\]
\[
(0^M, \eta_0^{(2)}, \eta_1^{(2)}, \ldots, \eta_{n_1+1}^{(2)}, 0^M), \ldots, (0^M, \eta_1^{(2)}, \eta_2^{(2)}, \ldots, \eta_{n_2}^{(2)}, 0^M),
\]
\[
\ldots
\]
\[
(0^M, \eta_0^{(j+1)}, \eta_1^{(j+1)}, \ldots, \eta_{n_{j+1}}^{(j+1)}, 0^M), \ldots, (0^M, \eta_1^{(j+1)}, \eta_2^{(j+1)}, \ldots, \eta_{n_{j+1}}^{(j+1)}, 0^M),
\]
Next, for each $x \in UR_\phi(\alpha^*(\beta), n_0 + M, \delta_0)$ which is defined in the previous Case II, construct the sequence $x(\{\eta^{(j)}\}, \{n_j\})$ by inserting the above sequence of words into the positions $2^kN$, $k \geq 1$, of the digit sequence $x$. In other words, in this case we have
\[
l_k = 2^kN + 2(k - 1)M + \frac{(k - 1)k}{2}, \quad k \geq 1.
\]
Next, define the Moran set $UR_\phi(\alpha^*(\beta), n_0 + M, \delta_0; \{\eta^{(j)}\}, \{n_j\}, N + M)$, being denoted by $UR_\phi(\{\eta^{(j)}\}, \{n_j\}, N + M)$ for short, as
\[
UR_\phi(\{\eta^{(j)}\}, \{n_j\}, N + M) = \left\{ x(\{\eta^{(j)}\}, \{n_j\}) \in \Sigma_\beta : x \in UR_\phi(\alpha^*(\beta), n_0 + M, \delta_0) \right\}.
\]
Then we can deduce that
\[
\dim_H UR_\phi(\{\eta^{(j)}\}, \{n_j\}, N + M) = \dim_H UR_\phi(\alpha^*(\beta), n_0 + M, \delta_0)
\]
by Lemma 5.1 and
\[
UR_\phi(\{\eta^{(j)}\}, \{n_j\}, N + M) \subset ER_\phi(\Lambda(\beta)).
\]
It follows that
\[
\dim_H ER_\phi(\Lambda(\beta)) \geq \dim_H UR_\phi(\alpha^*(\beta), n_0 + M, \delta_0) > 1 - \epsilon.
\]
Thus, we have $\dim_H ER_\phi(\Lambda(\beta)) = 1$ since $\epsilon$ is arbitrary.
The proof is finished now. □

6. PROOF OF THEOREM 1.1

This section is devoted to the proof of Theorem 1.1. In what follows, we would like to introduce a mapping \( \pi_\beta \) at first.

Let \( S_\beta \) be the closure of \( \Sigma_\beta \) under the product topology on \( \Sigma^\infty \) and \( \sigma \) be the shift operator on it which is defined as

\[
\sigma(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots) = (\varepsilon_2, \varepsilon_3, \varepsilon_4, \ldots)
\]

for any \((\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots) \in \Sigma^\infty\). Then \((S_\beta, \sigma|_{S_\beta})\) is a subshift of the symbolic space \((\Sigma^\infty, \sigma)\) and the two systems \((S_\beta, \sigma|_{S_\beta})\) and \((I, T_\beta)\) are metrically isomorphism.

Then define the mapping \( \pi_\beta: S_\beta \to I \) as

\[
\pi_\beta(\varepsilon) = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{\beta^i}, \text{ where } \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots) \in S_\beta.
\]

(6.1)

It is easy to see that \( \pi_\beta \) is a one-to-one mapping for all but countable many digit sequences. In fact, the two digit sequences \((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, 0, 0, \ldots)\) and \((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n-1, \varepsilon_n^*(x, \beta), \varepsilon_2^*(x, \beta), \ldots)\) share the same image, where the word \((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)\), with \( \varepsilon_n \geq 1 \) and \( n \geq 1 \), is admissible and \((\varepsilon_1^*(x, \beta), \varepsilon_2^*(x, \beta), \ldots)\) is the \( \beta \)-expansion of 1. Moreover, the mapping \( \pi_\beta \) is continuous on \( S_\beta \) and satisfies \( \pi_\beta \circ \sigma = T_\beta \circ \pi_\beta \).

Now, we are ready to give the proof of Theorem 1.1 in which the approximation technique is applied to obtain the lower bound of Hausdorff dimension of \( ER_\beta^{\phi}(\alpha) \). One can see the applications of this technique, for instance, in [30] and in the proof of Theorem 1.2 in [21].

Proof of Theorem 1.1. Let \( \beta > 1 \). For the case \( 0 \leq \alpha \leq \alpha^*(\beta) \), we will show, respectively, that the upper bound and lower bound of Hausdorff dimension of \( ER_\beta^{\phi}(\alpha) \) are of common value \( h_\beta^{\phi}(\alpha) \). In addition, for the case \( \alpha^*(\beta) < \alpha \leq \Lambda(\beta) \), we will show that \( \dim_H ER_\beta^{\phi}(\alpha) \) is bigger than \( \log \beta_m/\log \beta \) for all \( m \geq 1 \).

Upper bound. Let \( x \in ER_\beta^{\phi}(\alpha) \). Since \( A_\phi(x, \beta) = \alpha \), for any \( \epsilon > 0 \) there exists an integer \( N_0 > 0 \) such that

\[
A_{\phi(n)}(x, \beta) < \alpha + \epsilon, \quad \forall n > N_0.
\]

Fix \( n_0 > N_0 \). Since \( \phi(n) \to \infty \) as \( n \to \infty \), we can take \( m \) to be sufficiently large such that

\[
m > N_0 \quad \text{and} \quad \frac{\phi(n_0)}{\phi(m)} < \epsilon.
\]

Then, by dividing the beginning digit word \((\varepsilon_1(x, \beta), \varepsilon_2(x, \beta), \ldots, \varepsilon_m(x, \beta))\) of \( x \) into \( \lfloor m/\phi(m) \rfloor + 1 \) successive digit words and all, except the last one, are of equal
lengths $\phi(m)$, we have

$$S_m(x, \beta) = \left[ \begin{array}{c} \left( m - \frac{\phi(n_0)}{\phi(m)} \right) \frac{\phi(n_0)}{m} \right] (\alpha + \epsilon) \phi(m) + \left( m - \frac{\phi(n_0)}{\phi(m)} \right) + 1 \end{array} \right) (\alpha + \epsilon) \phi(n_0)$$

$$= (\alpha + \epsilon) + (\alpha + \epsilon) \frac{\phi(n_0)}{m} < (\alpha + \epsilon) + (\alpha + \epsilon) \frac{\phi(n_0)}{\phi(m)}$$

$$< (\alpha + \epsilon) (1 + \epsilon)$$

for any $\alpha \in I_\beta$. It yields that $\bar{A}(x, \beta) \leq \alpha$ by the arbitrariness of $\epsilon$. So, we have $ER_\phi^\beta(\alpha) \subset \bar{E}^\beta(\alpha)$. Then Proposition 3.3 gives that

$$\dim_H ER_\phi^\beta(\alpha) \leq \dim_H \bar{E}^\beta(\alpha) = h^\beta(\alpha)$$

when $0 \leq \alpha \leq \alpha^*(\beta)$.

**Lower bound.** Recall the definition of root $\beta_m$ given in (1.10), which satisfies $\beta_m \in B_1 \subset B_0$ for $m$ large enough and

$$\beta_m \leq \beta, \quad \Sigma_{m_1} \beta_m \subseteq \Sigma_{m_2} \beta_m \subseteq \Sigma_{\beta} \text{ if } m_1 < m_2 \quad \text{and} \quad \lim_{m \to \infty} \beta_m = \beta.$$ 

Put $D_{\beta, \beta_m} = \pi_{\beta}(\Sigma_{\beta_m})$ and define a mapping $g: D_{\beta, \beta_m} \to I$ satisfying

$$g(x) = \pi_{\beta_m}(\epsilon(x, \beta)), \quad x \in D_{\beta, \beta_m}.$$ 

Then we have the following three conclusions:

1. $\epsilon(g(x), \beta_m) = \epsilon(x, \beta)$;
2. $g(E R_\phi^\beta(\alpha) \cap D_{\beta, \beta_m}) = E R_\phi^\beta(\alpha)$;
3. the function $g$ is $(\log \beta_m / \log \beta)$-Lipschitz on $D_{\beta, \beta_m}$.

The first two conclusions are obvious and the last conclusion is followed by Theorem 3.1 in [1] and Lemma 2.10. Thus, we have

$$\dim_H E R_\phi^\beta(\alpha) \geq \dim_H (\{ E R_\phi^\beta(\alpha) \cap D_{\beta, \beta_m} \}) \geq \frac{\log \beta_m}{\log \beta} \dim_H E R_\phi^\beta(\alpha).$$

By Lemma 5.4 we obtain that

$$\dim_H E R_\phi^\beta(\alpha) \geq \frac{\log \beta_m}{\log \beta} \quad \text{as } \alpha^*(\beta) < \alpha \leq \Lambda(\beta)$$

and

$$\dim_H E R_\phi^\beta(\alpha) \geq \frac{\log \beta_m}{\log \beta} h^m(\alpha) \quad \text{as } 0 \leq \alpha \leq \alpha^*(\beta).$$

Let $m \to \infty$, then we have that $\dim_H E R_\phi^\beta(\alpha) \geq h^\beta(\alpha)$ as $0 \leq \alpha \leq \alpha^*(\beta)$ and $\dim_H E R_\phi^\beta(\alpha) = 1$ as $\alpha^*(\beta) < \alpha \leq \Lambda(\beta)$.

The proof is completed now. $\square$

Take two integer functions $\phi(n) = [c \log n]$ and $\phi(n) = [c \arctan n]$, where $n \geq 1$ and $c > 0$, for examples. Let $\beta > 1$. Define

(6.2) $ER_{\log \phi}^\beta(\alpha) = \left\{ x \in I : \lim_{n \to \infty} A_{n, [c \log n]}(x, \beta) = \alpha \right\}, \quad \alpha \in I_\beta$,

and

(6.3) $ER_{\arctan \phi}^\beta(\alpha) = \left\{ x \in I : \lim_{n \to \infty} A_{n, [c \arctan n]}(x, \beta) = \alpha \right\}, \quad \alpha \in I_\beta$.
It is evident that both of $\{c \log n\}_{n \geq 1}$ and $\{c \arctan n\}_{n \geq 1}$ are slowly varying sequences by (1) in Lemma 5.3. Moreover, since $c \log n \to \infty$ and $c \arctan n \to \infty$ as $n \to \infty$, by Theorem 1.1 we have

**Corollary 6.1.** Let $\beta > 1$ and $\alpha \in I_\beta$. Then

$$\dim_H \mathcal{E} R_{\log}^\beta (\alpha) = \dim_H \mathcal{E} R_{\arctan}^\beta (\alpha) = \begin{cases} h^\beta (\alpha), & 0 \leq \alpha \leq \alpha^{*} (\beta); \\ 1, & \alpha^{*} (\beta) < \alpha \leq \Lambda (\beta). \end{cases}$$

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