Abstract—In a current density versus temperature ($J$–$T$) (Miram) curve in thermionic electron emission, experimental measurements often yield a smooth transition between the exponential region and the saturated emission regions, which is sometimes referred to as the “roll-off” or “Miram curve knee.” The shape of the Miram curve knee is an important figure of merit for thermionic vacuum cathodes. Specifically, cathodes with a sharp Miram curve knee at low temperature with a flat saturated emission current are typically preferred. Our previous work on modeling nonuniform thermionic emission revealed that the space charge effect and the patch field effect are key pieces of physics, which set the Miram curve shape. This work provides a more complete understanding of the physical factors connecting these physical effects and their relative impact on the shape of the knee. For our analyses, we use a periodic, equal-width striped (“zebra crossing”) work function distribution as a model system and illustrate how the space charge and patch field effects restrict the emission current density near the Miram curve knee. Our results indicate that there are three main physical parameters that significantly impact the shape of the Miram curve: 1) the normalized work function difference; 2) the patch-diode size ratio; and 3) the patch–diode field ratio. These parameters relate the patch size, work function values, anode–cathode voltage, and anode–cathode gap distance to the shape of the Miram curve, providing new understanding and a guide to the design of thermionic cathodes used as electron sources in vacuum electronic devices.

Index Terms—Electron beam, Miram curve, thermionic emission.

I. INTRODUCTION

Thermionic cathodes provide the electron source in numerous vacuum electronic devices (VEDs) used in civilian, industrial, and scientific applications, such as communication devices, ion thrusters, thermionic energy converters, and free electron lasers [1], [2], [3], [4]. The total emission current density divided by the cathode area, referred to as the cathode-averaged emission current density, $J$, is a key property in thermionic emission. $J$ is affected by many parameters including temperature $T$, anode–cathode voltage $V_{AK}$, diode geometry, and so on. $J$–$T$ curves normalized by the extrapolated full-space-charge-limited (FSCL) emission current density $J_{FSCL}$ are referred to as Miram curves, named after George Miram [5]. Miram (or $J$–$T$) curves are widely used as a figure of merit to evaluate the cathode performance. Experimental observations show that there is a smooth transition, called the “roll off” or Miram curve knee, between the exponentially growing (temperature-limited or (TL)) region and the saturated emission current (FSCL) region. The shape characterizing the Miram curve knee is important as thermionic cathodes are almost always operated at a temperature somewhat above the Miram curve knee [6], [7]. Thus, knowledge of the shape of the Miram curve knee region, including the smoothness, the temperature, and the flatness of the saturated emission current density, can inform the desired operational parameters of VEDs using thermionic cathodes.

Our recent work [8] resulted in the development of a physics-based model of nonuniform thermionic emission, which predicts a smooth Miram curve knee for a model cathode surface consisting of a 2-D checkerboard work function distribution. The results of this model reveal that the smoothness of the knee arises as a natural consequence of the physics of nonuniform thermionic emission and that the space charge and patch field effects have a more significant impact on smoothing the Miram curve knee than the Schottky barrier lowering. Further study [9] showed that when a 2-D work function distribution obtained from a commercial tungsten dispenser cathode is used with this nonuniform emission model, the model is able to predict $J$–$T$ and $J$–$V$ emission curves in near quantitative agreement with experimental data and accurately predicts the shape of the Miram curve knee.

Our previously published work [8] on the nonuniform emission model successfully included the predictions of key characteristics of the Miram curve, including the smooth knee, but it did not include a detailed examination of the different physical factors connecting the effects of space charge and patch fields and their relative impact on the shape of the Miram curve knee. In this work, we use approximations to
our nonuniform emission model to further study the effects of space charge and patch fields on the shape of the Miram curve knee. We find that there are three main physical parameters that significantly affect the shape of the knee: 1) the normalized work function difference; 2) the patch-diode size ratio; and 3) the patch-diode field ratio. Lower values of these three parameters imply a sharper knee at a lower temperature with a flatter saturated emission current.

Knowledge of how the surface nonuniformity affects the shape of the Miram curve knee would provide an understanding on the relation between the knee shape and the values and sizes of each work function patch, therefore providing guidelines for the design of the cathode materials and electron gun fixtures used in VEDs. This work also provides new insights in the estimation of the cathode microstructure at working temperatures by analyzing experimental Miram curves. Such insights are useful for developing and understanding the performance of thermionic cathodes using new materials or for tailoring cathode processing and manufacturing of existing material systems, for example, through refining microstructure to produce larger emitting patches.

II. METHODS AND THEORY

A. Zebra-Crossing Work Function Distribution

Checkerboard work function distributions [8], [10], [11], [12], [13], [14], [15] and equal-width periodic stripes (or “zebra crossing”) work function distributions [6], [14], [16], [17] are the two model work function distributions used in previous studies of nonuniform thermionic emission. As illustrated in [14], many of the fundamental properties of electron emission from the two-dimensional checkerboard distribution are observable with the simpler, 1-D, zebra-crossing distribution. Therefore, in this work, we use the zebra-crossing surface model to analyze the effects of space charge and patch fields on the shape of Miram curve.

Fig. 1 shows the zebra-crossing model of nonuniform work function distribution on a surface. In this work, we let the $x$-axis run along the surface of the cathode, perpendicular to the patch edges, and the $y$-axis run along the cathode–anode direction, i.e., the direction of electron emission away from the surface. The cathode is set to be at $y = 0$, while the anode is set to be at $y = d$. Mathematically, the work function as a function of surface position is

$$
\phi(x) = \begin{cases} 
\phi_1, & 2(N - 1)a < x < 2Na \\
\phi_2, & 2Na < x < (2N + 1)a 
\end{cases}
$$
(1)

where $\phi_1 > \phi_2 > 0$, $N = 0, \pm 1, \pm 2, \ldots$ is any integer.

![Fig. 1. Model heterogeneous emission surface characterized as a “zebra-crossing” spatial distribution of work function in a laterally infinite parallel diode.](image)

![Fig. 2. Schematic plot illustrating the definition of the TL-FSCL intersection temperature $T_i$ (blue dashed line) and the normalized intersection emission parameter $\alpha$. In this plot, $\alpha = J(A)/J(B)$. The black solid curve is the Miram curve $J(T)$. The red dotted curve is the FSCL extrapolation $J_{FSCL}$. The green dotted curve is the TL extrapolation $J_{TL}$.](image)

B. Quantifying the Shape of Miram Curve Knee

We define the TL-FSCL intersection temperature $T_i$ as the intersection temperature of the separate extrapolations of TL $J_{TL}$ and FSCL $J_{FSCL}$ curves (Fig. 2)

$$
J_{TL}(T_i) = J_{FSCL}(T_i).
$$
(2)

As shown in Fig. 2, much of a Miram curve knee may be observed at temperatures greater than the TL-FSCL intersection temperature $T_i$ (more discussions in Section III).

In this work, we use an area-averaged application of the Richardson–Laue–Dushman (RLD) equation to estimate the TL extrapolation. For the zebra-crossing work function distribution with equal widths for the two work function stripes (Fig. 1), the TL extrapolation equation is

$$
J_{TL} = \frac{1}{2} A T^2 \exp\left(-\frac{\phi_1}{kT}\right) + A T^2 \exp\left(-\frac{\phi_2}{kT}\right)
$$
(3)

where $T$ is the temperature, $k$ is Boltzmann’s constant, and the Richardson constant $A = 4\pi \eta m e k^2 / h^3$, where $m$ is the electron mass, $e$ is the elementary charge, and $h$ is Planck’s constant.

The FSCL extrapolation can be estimated using the Child–Langmuir (CL) law with finite temperature correction (CLT). The equation at the TL-FSCL intersection temperature $T_i$ for a uniform cathode with a single cathode work function value $\phi_K$ is [9], [18]

$$
J_{CLT} = \frac{4\epsilon_0}{9} \sqrt{\frac{2e}{m}} V_{AK}^2 \frac{9}{8\sqrt{\pi}} \eta^{-\frac{3}{2}}
$$

$$
\times \left( \int_0^\eta \frac{dn}{\sqrt{\text{erfcx}(\eta - 1 + 2\sqrt{\pi})}} \right)^2
$$
(4)

where $\epsilon_0$ is the vacuum permittivity, $V_{AK} = V_{\text{applied}} - \phi_A/e + \phi_K/e$ is the difference of the electric potential of the vacuum level between the anode surface and cathode surface, $V_{\text{applied}}$ is the applied voltage between the anode and the cathode as measured in experiments, $\phi_A$ is the anode work function, $\phi_K$ is the cathode work function, $\eta = eV_{AK}/(kT)$, and erfcx is the scaled complementary error function.

For a zebra-crossing cathode, we average $J_{CLT}$ over the cathode surface to estimate the FSCL extrapolation $J_{FSCL}$

$$
J_{FSCL} = \frac{1}{2} \left[ J_{CLT}\big|_{\phi_K=\phi_1} + J_{CLT}\big|_{\phi_K=\phi_2} \right].
$$
(5)
To quantify the impact of the space charge and patch field effects on the emission current density at the TL-FSCL intersection temperature \( T_i \), we define and use the intersection-temperature emission normalized by the FSCL current parameter (Fig. 2)

\[
\alpha = \frac{J(T_i)}{J_{TL}(T_i)} = \frac{J(T_i)}{J_{FSCL}(T_i)} = \frac{J(A)}{J(B)} .
\]

A high \( \alpha \) value is indicative of a Miram curve with a sharp knee at a low temperature and a flat saturated emission current and is thus an indicative of good cathode performance (more discussions in Section III).

Our previous study on nonuniform emission \([8]\) indicated that space charge and patch fields have a significant impact on the shape of the Miram curve knee. Considering the definition of the normalized emission parameter \([6]\), we here develop complementary, limiting-case (LC) analytic models to separately study the following effects: a study of the space charge effect to analyze \( J(T)/J_{FSCL}(T) \) and a study of the patch field effect to analyze \( J(T)/J_{TL}(T) \).

### C. Space Charge Effect

Previous studies on 2-D and 3-D space charge \([6], [14]\) illustrate its effects on the emission current. Seminal work from Lau developed a simple theory for the 2-D CL law \([19]\).

We generalize this 2-D CL law to a cathode with nonuniform emission. In this generalized theory, the \( y \)-axis component of the electric field at \((x_0, 0)\) is

\[
E_y(x, 0) = -\frac{\vec{V}_{AK}}{d} + G \int_0^d dy' \int_{-\infty}^{+\infty} dx' \frac{\rho(x', y')y'}{2\pi e_0 [(x'-x)^2 + y'^2]}
\]

where \( G \) is a constant multiplication factor to account for the contributions due to the image charge. For a zebra-crossing model (Fig. 1), the averaged electric potential difference between the anode and cathode surfaces is \( \vec{V}_{AK} = V_{applied} - \phi_1/e + (\phi_1 + \phi_2)/2(e) \).

To solve for the value of \( G \), we use the 1-D CL theory at zero temperature for a uniform cathode. In the space-charge-limited (SCL) region, \( E_y(x, 0) = 0 \) and

\[
\rho(x, y) = J_{CL} \left( \frac{d}{y} \right)^{2/3} \sqrt{\frac{m}{2e\vec{V}_{AK}}}.
\]

where the CL current density

\[
J_{CL} = \sqrt{\frac{2e}{m}} \frac{4e_0 \vec{V}_{AK}^{3/2}}{9d^2}.
\]

Solving (7)–(9), we get \( G = 3/2 \).

We make some additional approximations as detailed in the following to simplify the space charge effect, which makes it possible to find out the key parameters impacting the space charge effect.

**Approximation 1:** The charge density has the same functional form as (8) after replacing \( J_{CL} \) with \( J(x) \)

\[
\rho(x, y) = J(x) \left( \frac{d}{y} \right)^{2/3} \sqrt{\frac{m}{2e\vec{V}_{AK}}}.
\]

This approximation is expected to be valid when \( J(x) \) is close to \( J_{FSCL} \), corresponding to TL-FSCL transition and FSCL regions.

**Approximation 2:** For the zebra-crossing case, we assume the following.

1) The local emission current density over the \( \phi_1 \) patch is spatially uniform: \( J(x) = J_1 \), for \( 2Na - a < x < 2Na \), where \( N \) is any integer.
2) The \( \phi_1 \) patch emits in the TL region when the electric field at the center of the \( \phi_1 \) patch surface is negative, \( E_{1y} = E_y(2Na - a/2, 0) < 0 \).
3) The SCL emission current density is also spatially uniform over the \( \phi_1 \) patch, denoted by \( J_{SCL} \), which is determined by the equation \( E_{1y} = 0 \).
4) The emission current density of the \( \phi_1 \) patch is \( J_1 = \min(J_{TL}, J_{SCL}) \).
5) Analogous assumptions (1)–(4) also apply for the \( \phi_2 \) patch.

This approximation is expected to be valid when \( a \gg d \). In this limit, each patch becomes uniform except for areas near the patch boundaries.

**Approximation 3:** As \( \phi_1 > \phi_2 > 0 \), the TL current densities have a relation \( J_{2TL} > J_{1TL} > 0 \), so the high work function patch (\( \phi_1 \)) will reach SCL at a higher temperature than the low work function patch (\( \phi_2 \)). It is assumed that the SCL current density for the high work function patch is equal to the 1-D CL theory \( J_{SCL} = J_{CL} \), while \( J_{2SCL} \) may be larger than \( J_{CL} \) at certain temperatures. This approximation is valid at \( T \to 0 \) limit when temperature correction is negligible, but, similar to the findings in [5], [13], and elsewhere, this approximation should still provide accurate results at typical thermionic cathode operating temperatures [6], [14].

Under these approximations, the electric field at the center of the low work function patch (\( \phi_2 \)) is

\[
E_{2y} = \frac{V_{AK}}{d} \left( \frac{J_1}{J_{CL}} + \frac{J_2 - J_1}{J_{CL}} f - 1 \right)
\]

where \( f \) is a function of only \( a/d \)

\[
f \left( \frac{a}{d} \right) = \frac{1}{3\pi} \int_0^1 \sum_{N=-\infty}^{\infty} \int_{(2N-1/2)a/d}^{(2N+1/2)a/d} \frac{\tilde{y}^{1/3}}{\tilde{x}^2 + \tilde{y}^2} d\tilde{x} d\tilde{y}
\]

where the symbols with bars are normalized coordinates \( \tilde{x} = x/d \) and \( \tilde{y} = y/d \).

Letting \( E_{2y} = 0 \), we can solve for the SCL current density of the \( \phi_2 \) patch: \( J_{2SCL} = J_1 + (J_{CL} - J_1)/f \). Therefore, we can predict the average emission current density for the zebra-crossing cathode.

1) TL Region \( J_{1TL} < J_{2TL} < J_{2SCL} \) : Neither the \( \phi_1 \) patch nor the \( \phi_2 \) patch reaches SCL. The average current density of the cathode is \( J = (J_{1TL} + J_{2TL})/2 \).
2) FSCL Region \( J_{2TL} > J_{1TL} > J_{1SCL} = J_{CL} \) : Both the \( \phi_1 \) and \( \phi_2 \) patches are SCL, so \( J = (J_{1SCL} + J_{2SCL})/2 = J_{CL} \).
3) TL-FSCL Transition Region (i.e., the Location of the Miram Curve Knee) \( J_{2TL} > J_{2SCL} \) but \( J_{1TL} < J_{1SCL} \) : The low work function patch (\( \phi_2 \)) is SCL, while the
high work function patch \((\phi_1)\) is still not SCL, so \( J = (J_{\text{TL}} + J_{\text{SCCL}})/2 = J_{\text{CL}} - (J_{\text{CL}} - J_{\text{TL}})(2f - 1)/(2f) \).

In particular, at the TL-FSCL intersection temperature \(T_i\), we assume that the temperature satisfies the condition
\[
J_{\text{CL}} = J_{\text{TL}} = \frac{1}{2} \left[ AT^2 \exp \left(-\phi_1/kT_i\right) + AT^2 \exp \left(-\phi_2/kT_i\right) \right].
\]

(13)

The \(\alpha\) value restricted by the space charge effect is
\[
\alpha_{\text{SC}} = \frac{J(T_i)}{J_{\text{CL}}} = 1 - \frac{2f - 1}{2f} \frac{\phi_1 - \phi_2}{2kT_i}
\]

which only depends on two parameters: 1) the normalized work function difference \((\phi_1 - \phi_2)/(kT_i)\) and 2) the patch diode size ratio \(a/d\) [see (12); \(f\) only depends on \(a/d\)].

D. Patch Field Effect

We assume that the additional electric potential due to the patch field effect satisfies \(V^2 V_{\text{PP}}(x, y) = 0\), with the boundary condition for the cathode surface \(V_{\text{PP}}(x, 0) = -\phi(x)/e\). In the condition of \(d \gg a\), we use \(E_x(\infty, 0) = 0\) to approximate the boundary condition for the anode side and use a uniform electric field \(E_0 = -\partial(V_{\text{applied}} + V_{\text{SC}})/\partial y \leq 0\) to estimate the effects of the applied voltage and the space charge on the electric potential. Under these approximations, the total electric potential is \(V = V_{\text{PP}} + V_{\text{applied}} + V_{\text{SC}} = V_{\text{PP}} - E_0 y\).

Considering the periodicity and the symmetry of this problem, here, we only show the results in \(-a/2 < x < a/2\)
\[
V(x, y) = \frac{-\phi_1 + \phi_2}{2e} + \frac{\phi_1 - \phi_2}{e} p(x, y, \beta_E)
\]

where the field ratio
\[
\beta_E = \frac{E_{\text{patch}}}{-E_0} = \frac{\phi_1 - \phi_2}{-ea E_0}
\]

and the parameter
\[
p(x, y, \beta_E) = \frac{\tilde{y}}{\pi \beta_E} + \frac{i}{\pi} \left( \text{arctanh} e^{-i\tilde{x} - \tilde{y}} - \text{arctanh} e^{i\tilde{x} - \tilde{y}} \right)
\]

(17)

where the symbols with breves are normalized coordinates \(\tilde{x} = \pi x/a\) and \(\tilde{y} = \pi y/a\), and \(i\) is the imaginary unit.

The voltage minimum is \(V_m(x) = \min V(x, y) = V(x, y_m(x))\), where the location of the voltage minimum \(y_m(x)\) satisfies
\[
cosh \tilde{y}_m(x) = (\beta_E \sin \tilde{x})/2 + \sqrt{1 + \left[ \frac{(\beta_E)^2}{2} - 1 \right] \sin^2 \tilde{x}}
\]

(18)

for \(0 < \tilde{x} < \min \{\pi/2, \arcsin \beta_E\}\) and \(y_m(x) = 0\) elsewhere.

For a given cathode surface location \(x\), the location of the voltage minimum \(y_m(x)\) only depends on the field ratio \(\beta_E\). Therefore, for a given \(x\), the parameter \(p(x, y_m(x), \beta_E)\) in (17) also only depends on \(\beta_E\).

The local emission current density follows the RLD equation
\[
J(x) = A T^2 \exp \left[ e V_m(x)/(kT) \right].
\]
The cathode-averaged emission current density \(J\) can be obtained by averaging \(J(x)\) over the cathode surface. The ratio of the average emission current density \(J\) to the TL extrapolation \(J_{\text{TL}}\) [(3)] is
\[
\frac{J}{J_{\text{TL}}} = \frac{1}{2} \left[ AT^2 \exp \left(-\frac{\phi_1}{kT} \right) + AT^2 \exp \left(-\frac{\phi_2}{kT} \right) \right]
\]

(19)

which only depends on the normalized work function difference \((\phi_1 - \phi_2)/(kT)\) and the field ratio \(\beta_E\).

In the low-temperature limit \(T \to 0\) where the space charge effect is negligible, \(-E_0 = E_{\text{AK}} = V_{\text{AK}}/d\). To estimate the effect of the space charge on the \(E_0\) value in the condition of \(J(x) \approx J_{\text{FSCL}}\) and \(d \gg a\), we made the similar approximation as (10) but replacing \(J(x)\) with its average value \(J\)
\[
\rho(x, y) = J \left(\frac{d}{y}\right)^{2/3} \frac{m}{2e V_{\text{AK}}}
\]

(20)

Substituting (20) into (7), we get that the electric field \(E_0\) [or \(E_{\text{y}}(x, 0)\) in (7)] due to the applied voltage and the space charge satisfies
\[
1 = -\frac{E_0}{E_{\text{AK}}} + \frac{J}{J_{\text{FSCL}}} = \frac{J_{\text{patch}}}{E_{\text{AK}}} + \frac{J}{J_{\text{TL}}} J_{\text{FSCL}}
\]

(21)

where \(\beta_E = E_{\text{patch}}/(-E_0)\).

When the value of \(J_{\text{TL}}/J_{\text{FSCL}}\) is given, we eliminate the variable \(\beta_E\) by solving the system of (19) and (21), so \(J/J_{\text{TL}}\) only depends on \((\phi_1 - \phi_2)/(kT)\) and \(E_{\text{patch}}/E_{\text{AK}}\).

In particular, at the TL-FSCL intersection temperature \(T_i\), \(J_{\text{TL}}/J_{\text{FSCL}} = 1\) and the normalized intersection emission restricted by the patch field effect \(\alpha_{\text{PF}} = J(T_i)/J_{\text{TL}}(T_i)\) also only depends on two parameters: 1) the normalized work function difference \((\phi_1 - \phi_2)/(kT)\) and 2) the patch diode field ratio \(E_{\text{patch}}/E_{\text{AK}} = [(\phi_1 - \phi_2)/(ea)]/(V_{\text{AK}}/d)\).

III. RESULTS AND DISCUSSION

Previously, we developed complementary, LC theoretical analytic models to separately estimate the effects of space charge and patch fields on the shape of the Miram curve. The approximations used in the LC models are most rigorously valid in the limiting conditions described above. However, we find that these approximations work quite well even outside these limiting extremes. To demonstrate the broad range of the approximations used, here we validate select results against a numerical simulation using the full nonuniform emission model [8] without any of the approximations introduced above. We refer to these numerical simulation results simply as “simulation” results. Our results in the following demonstrate that the LC models agree with simulation results for a wide range of conditions, including typical operating conditions.

A. Shape of Miram Curve Knee

Fig. 3 shows the Miram curves predicted by our analytic models in different cases, compared with the simulation results using the nonuniform emission model [8] and their TL and FSCL extrapolations. The range of \(\phi_1, \phi_2, a, d,\) and \(T\) values
in Fig. 3 covers the typical values for dispenser cathodes. The values of the three main physical parameters, the TL-FSCL intersection temperature \( T_i \) calculated using (2), and the normalized intersection emission \( \alpha \) values defined in (6) for each case in Fig. 3 are listed in Table I.

In Fig. 3(a) and (b), the patch fields have a more significant impact than the space charge in restricting the emission current. The Miram curves predicted by our patch field limiting case theoretical model (dashed magenta curves) are consistent with simulation results (black solid curve) \[8\].

When \( \frac{E_{\text{patch}}}{E_{AK}} \gg 1 \) [Fig. 3(a)], there is a significant discrepancy between the simulated Miram curve (black solid curve) and its TL extrapolation (green dotted curve, which overlaps with the sky-blue dashed space charge limiting case model curve at low temperatures). In this case, the patch field effect significantly lowers the emission current in the TL region, resulting in a knee temperature much higher than the intersection temperature \( T_i \) of the TL and FSCL extrapolations (Points A and B). The normalized intersection emission \( \alpha \) value is very low.

When \( E_{\text{patch}}/E_{AK} \approx 1 \) [Fig. 3(b)], the patch field effect moderately lowers the emission current in the TL region and leads to a very rounded knee. In this case, the knee temperature is slightly higher than \( T_i \), and \( \alpha \) is moderately low.

Fig. 3(c) shows an example where the impacts of both effects are comparable and not negligible. The \( \alpha \) value is high (close to 1).

In Fig. 3(d), the space charge effect restricts the emission current, causing the saturated emission current density to be not flat. In this case, the simulated Miram curve is step-shaped (black solid curve), consistent with the results predicted by our limit case theoretical model of the space charge effect (sky-blue dashed curve), and \( \alpha \) is again moderately low.

The normalized work function difference \( (\phi_1 - \phi_2)/(kT_i) \) governs the impact of both the patch field and space charge effects. Decreasing this parameter is one of the methods to lessen the restriction on the emission current due to both patch fields and space charge effects, resulting in a sharp knee at low temperature and a flat saturated emission current density [Fig. 3(e)]. In this case, the \( \alpha \) value is very high, close to 1.

### B. Physical Factors Impacting Normalized Emission at TL-FSCL Intersection Temperature

From the LC models, we reveal that there are three main physical parameters significantly affecting the shape of the Miram curve knee: 1) the normalized work function difference \( (\phi_1 - \phi_2)/(kT_i) \); 2) the patch-diode size ratio \( a/d \); and 3) the patch-diode field ratio \( E_{\text{patch}}/E_{AK} = [(\phi_1 - \phi_2)/(ea)]/(V_{AK}/d) \).

As shown in Fig. 3, a large value of the normalized emission value \( \alpha \) at TL-FSCL intersection temperature represents good cathode performance, indicating that neither the space charge nor the patch field effects significantly restrict the knee-temperature emission current and that the Miram curve has a sharp knee at a temperature close to \( T_i \) with a flat, saturated emission current. A small \( \alpha \) value represents that the space charge (local to the cathode) and/or patch field effects significantly restrict the knee-temperature emission current, resulting in a rounded knee, an increased knee temperature, and/or an inclined saturated emission current.

To further illustrate the quantitative effects of the physical factors to the Miram curve knee, in Fig. 4, we plot the predicted \( \alpha \) value as a function of the three main physical parameters, using our theoretical studies summarized in Section II. The lower these three parameters are, the higher the normalized intersection emission parameter \( \alpha \) is, implying a sharper knee at a lower temperature with a flatter saturated emission current and a better cathode performance.
Fig. 4. Normalized intersection emission parameter $\alpha$ as a function of the three main physical parameters. (a) Space-charge-restricted $\alpha_{SC}$ only depends on the normalized work function difference ($\phi_2 - \phi_1$)/($kT$) and the patch-diode size ratio $a/d$. (b) Patch-field-restricted $\alpha_{PF}$ only depends on the normalized work function difference ($\phi_2 - \phi_1$)/($kT$) and the patch-diode field ratio $E_{patch}/E_{AK}$ = $([\phi_2 - \phi_1]/[\ell d])/(V_{AK}/d)$. The right vertical axis is the value of the corresponding Longo parameter $n$, obtained from the relationship $\alpha = 2^{-1/n}$.

Fig. 3(a)–(d) shows the results where the values of work function, the anode–cathode distance, and the anode–cathode voltage are fixed, but the patch size $a$ is variable. Fig. 5 shows the relationship between the normalized intersection emission parameter $\alpha$ and patch size $a$. In the cases of a small patch size, $a/d$ is small, but $E_{patch}/E_{AK}$ is large, so the emission tends to be patch-field-restricted [Fig. 3(a) and (b)]. In the cases of a large patch size, $a/d$ is large, but $E_{patch}/E_{AK}$ is small, so the emission tends to be space-charge-restricted [Fig. 3(d)]. For the illustrative parameters of Fig. 3, the boundary between the patch-field-restricted and the space-charge-restricted regimes approximately locates at $a \approx 20 \mu m$ [Fig. 3(c)]. This result indicates that for a zebra-crossing model, when $\phi_1$, $\phi_2$, $d$, and $V$ are fixed, there exists an optimal patch size $a$, which leads to a highest normalized intersection emission $\alpha$. This indicates that for a given cathode material (work function values fixed) and a given diode structure (diode $d$ and $V$ fixed), there may exist an optimal effective emission patch size so that the cathode can have the best emission performance as it relates to the shape of the Miram curve knee. This fact suggests that there may be an opportunity for tuning cathode microstructure or processing to facilitate the formation of optimal size emitting patches.

C. Relationship to Longo–Vaughan Equation

The Longo–Vaughan equation [20] $J_{TL}^{-n} + J_{FSCL}^{-n} = J^{-n}$ is a commonly used empirical equation to describe the smooth rounded knee in Miram curves. Solving the system of the Longo–Vaughan equation and (2), we get a relationship between the Longo–Vaughan parameter $n$ and the normalized knee emission parameter $\alpha$, i.e., $\alpha = 2^{-1/n}$. Such a relation is plotted in the vertical axis on the right of Figs. 4 and 5.

Vaughan [20] pointed out that well-designed and well-made electron guns have $n$ values in the range of 6–10, which corresponds to $\alpha$ values of 0.89–0.93, while diodes with so-called “patchy emission,” ascribed to uneven heating or other defects, have $n$ in the 2–5 range, corresponding to $\alpha$ values of 0.71–0.87.

IV. Conclusion

In this work, we used an equal-width periodic striped (zebra crossing) work function spatial distribution to study nonuniform thermionic emission. We used a normalized emission parameter $\alpha$ at the TL-FSCL intersection temperature to quantify the shape of the Miram curve knee. We developed complementary, LC analytic models to separately estimate the effects of space charge and patch fields and found that there are three main physical parameters, which significantly affect the shape of the knee: 1) the normalized work function difference ($\phi_2 - \phi_1$)/($kT$); 2) the patch-diode size ratio $a/d$; and 3) the patch-diode field ratio $E_{patch}/E_{AK}$. The lower these three parameters are, the higher the $\alpha$ value is, implying a sharper knee at a lower temperature with a flatter saturated emission current, and a better cathode performance.

The physical knowledge revealed in this work directly and quantitatively connects the patch size, work function values, anode–cathode voltage, and anode–cathode gap distance to the shape of the Miram curve knee, providing new understanding and a guide to the design of thermionic cathodes used as electron sources in VEDs.

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