The sTB-B Hierarchy

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Abstract

We construct a new supersymmetric two boson (sTB-B) hierarchy and study its properties. We derive the conserved quantities and the Hamiltonian structures (proving the Jacobi identity) for the system. We show how this system gives the sKdV-B equation and its Hamiltonian structures upon appropriate reduction. We also describe the zero curvature formulation of this hierarchy both in the superspace as well as in components.

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1. Introduction

In the last few years integrable models [1-3] have become increasingly relevant in the study of strings through the matrix models. It was shown that the KdV hierarchy appears in the so called double scaling limit of the one-matrix model. This fact allowed the exact computation of correlation functions on arbitrary topology [4]. However, since supersymmetric string theories are believed to be more fundamental, a generalization of these techniques to the supersymmetric case is in order. A first step in this direction was pursued in [5] and an arbitrary genus solution was found in [6]. In fact, a new supersymmetric extension of the KdV hierarchy was obtained in the double scaling limit. The authors of Ref. [7,8] showed that this new supersymmetric hierarchy, the sKdV-B hierarchy, was a sort of supersymmetric covariantization of the bosonic KdV hierarchy. In fact, their observation provides us with a general scheme of supersymmetrization for any bosonic hierarchy.

In a parallel study, much attention was devoted to the supersymmetric extension of the Two-Boson system (TB) [9-11]. The supersymmetric Two-Boson (sTB) hierarchy [12,13], has a rich structure. One of its more important properties is that it gives rise to many other supersymmetric integrable systems under appropriate reduction. Our aim in this letter is to supersymmetrize the TB system following the procedure in Refs. [7] and [8]. In this way we will obtain a new supersymmetric TB hierarchy, the sTB-B hierarchy and study its properties. Under appropriate reduction we will show that the sKdV-B equation results from this system.

The paper is organized as follows. In Sec. 2 we review briefly the Two-Boson system. In Sec. 3 we construct the new supersymmetric Two-Boson system showing its bi-Hamiltonian structure. We also explain how to prove the Jacobi identity for odd Poisson structures using super prolongation techniques. In Sec. 4 we show how the sKdV-B equation is embedded in this system. We also show that a supersymmetric generalization of the nonlinear Schrödinger equation (NLS) using this scheme fails to give a local equation. In Sec. 5 we construct the zero curvature formulation for the sTB-B hierarchy in the superspace as well as in components. Our conclusions are presented in Sec. 6. We also refer the readers to [13] where details about supersymmetric calculations can be found.
2. The TB Hierarchy

The two boson system is given by the equations \[9-11\] (prime denotes derivative with respect to \(x\))

\[
\frac{\partial J_0}{\partial t} = (2J_1 + J_0^2 - J_0')'
\]
\[
\frac{\partial J_1}{\partial t} = (2J_0J_1 + J_1')'
\]\n
This system has a nonstandard Lax representation \[9,14\]

\[
\frac{\partial L}{\partial t} = [L, (L^2)_{\geq 1}]
\]

with a Lax operator given by

\[
L = \partial - J_0 + \partial^{-1}J_1
\]

In fact, equations (1) are part of an hierarchy of equations which can be expressed in bi-Hamiltonian form

\[
\left(\frac{\partial J_0}{\partial t_k}\right)\left(\frac{\partial J_1}{\partial t_k}\right) = D_1 \left(\frac{\delta H_{k+1}}{\delta J_0}\right) = D_2 \left(\frac{\delta H_{k+1}}{\delta J_1}\right)
\]\n
The conserved charges (Hamiltonians) are given by

\[
H_n = \text{Tr } L^n = \int dx \text{ Res } L^n = \int dx h_n(J_0, J_1)
\]

\(n = 1, 2, 3, \ldots\)

where “Res” stands for the coefficient of the \(\partial^{-1}\) term in the pseudo-differential operator. The first few conserved charges are

\[
H_1 = \int dx \ J_1
\]
\[
H_2 = \int dx \ J_0J_1
\]
\[
H_3 = \int dx \ (J_1^2 - J_0'J_1 + J_1J_0^2)
\]

and the two Hamiltonian structures for the system in eq. (4) are given by

\[
D_1 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}
\]
\[
D_2 = \begin{pmatrix} 2\partial & \partial J_0 - \partial^2 \\ J_0\partial + \partial^2 & J_1\partial + \partial J_1 \end{pmatrix}
\]
For $k = 2$ we obtain the equations (1) from (4).

### 3. The sTB-B Hierarchy and Its Integrability

The standard procedure to obtain the new supersymmetric TB hierarchy (hereafter referred to as the sTB-B hierarchy) is to first replace the variables $J_0$, $J_1$ in the TB hierarchy (4) (which we will denote by TB($J_0$, $J_1$) following Ref. [7]) by the superfields

$$
(D\Phi_0) = J_0 + \theta \psi_0'
$$

$$
(D\Phi_1) = J_1 + \theta \psi_1'
$$

In this way we get a new hierarchy which we will denote by TB($D\Phi_0$, $D\Phi_1$) and the new sTB-B($\Phi_0$, $\Phi_1$) hierarchy is obtained by taking off one derivative from the left and the right sides of the equation. So, from (4) the TB($D\Phi_0$, $D\Phi_1$)) hierarchy is

$$
\left( \frac{\partial (D\Phi_0)}{\partial t_k} \right) = D_1 |_{J_0=(D\Phi_0)}^{J_1=(D\Phi_1)} \left( \frac{\delta H_{k+1}}{\delta J_0} |_{J_0=(D\Phi_0)}^{J_1=(D\Phi_1)} \right) = D_2 |_{J_0=(D\Phi_0)}^{J_1=(D\Phi_1)} \left( \frac{\delta H_{k+1}}{\delta J_1} |_{J_0=(D\Phi_0)}^{J_1=(D\Phi_1)} \right)
$$

(9)

Using, for instance, the result from Lemma 10 in Ref. [8] we have

$$
\frac{\delta}{\delta \Phi_0} \int dx d\theta h_n((D\Phi_0), (D\Phi_1)) = D \frac{\delta}{\delta J_0} \int dx h_n(J_0, J_1) |_{J_0=(D\Phi_0)}^{J_1=(D\Phi_1)}
$$

$$
\frac{\delta}{\delta \Phi_1} \int dx d\theta h_n((D\Phi_0), (D\Phi_1)) = D \frac{\delta}{\delta J_1} \int dx h_n(J_0, J_1) |_{J_0=(D\Phi_0)}^{J_1=(D\Phi_1)}
$$

(10)

and (9) yields

$$
\left( \frac{\partial \Phi_0}{\partial t_k} \right) = J_1 \left( \frac{\delta K_{k+1}}{\delta \Phi_0} \right) = J_2 \left( \frac{\delta K_{k+1}}{\delta \Phi_1} \right)
$$

(11)

where

$$
J_1 = D^{-1} D_1 |_{J_0=(D\Phi_0)}^{J_1=(D\Phi_1)} D^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
$$

$$
J_2 = D^{-1} D_2 |_{J_0=(D\Phi_0)}^{J_1=(D\Phi_1)} D^{-1} = \begin{pmatrix} 2 & D(D\Phi_0)D^{-1} - D^2 \\ D^{-1}(D\Phi_0)D + D^2 & D^{-1}(D\Phi_1)D + D(D\Phi_1)D^{-1} \end{pmatrix}
$$

(12)
and
\[ K_n = \int dx d\theta h_n((D\Phi_0), (D\Phi_1)) \] (13)

Now, \( J_1 \) and \( J_2 \) are odd Poisson structures and this is a characteristic feature of this supersymmetrization scheme. \( J_1 \) and \( J_2 \) are symmetric and \( J_1 \) satisfies the Jacobi identity trivially. To show that \( J_2 \) satisfies the Jacobi identity is slightly involved and we use the super-prolongation techniques [15,16,13] adapted for odd Poisson structures. The modification needed to treat odd Poisson structures is to use a two-component column matrix of fermionic superfields
\[ \vec{\Omega} = \begin{pmatrix} \Omega_0 \\ \Omega_1 \end{pmatrix} \] (14)

and construct the bivector associated with the Hamiltonian structure \( J \)
\[ \Theta_J = \frac{1}{2} \sum_{\alpha, \beta} \int dz \ ((J)_{\alpha\beta} \Omega_{\beta}) \wedge \Omega_\alpha \quad \alpha, \beta = 0, 1 \] (15)

The necessary and sufficient condition for \( J \) to define a Hamiltonian structure is that the prolongation of this bivector should vanish
\[ \text{pr} \vec{\nu}_{J_2\vec{\Omega}}(\Theta_J) = 0 \] (16)

For \( J_2 \) given by (12) we get
\[ \text{pr} \vec{\nu}_{J_2\vec{\Omega}}(\Phi_0) = 2\Omega_0 + (D^2\Phi_0)(D^{-1}\Omega_1) + (D\Phi_0)\Omega_1 - (D^2\Omega_1) \]
\[ \text{pr} \vec{\nu}_{J_2\vec{\Omega}}(\Phi_1) = -\frac{1}{2} (D^{-1}(D\Phi_0)(D\Omega_0)) + (D^2\Omega_0) + (D^{-1}(D\Phi_1)(D\Omega_1)) \]
\[ + (D^2\Phi_1)(D^{-1}\Omega_1) + (D\Phi_1)\Omega_1 \] (17)

and using this it is easy to show that the prolongation of the bivector (15) vanishes
\[ \text{pr} \vec{\nu}_{J_2\vec{\Omega}}(\Theta_J) = 0 \] (18)

leading to the fact that \( J_2 \) satisfies the Jacobi identity. Also, it can be shown that \( J_1 \) and \( J_2 \) are compatible since
\[ \text{pr} \vec{\nu}_{(J_2 + \alpha J_1)\vec{\Omega}}(\Theta_J) = 0 \] (19)
In this way the sTB-B hierarchy (11) is a bi-Hamiltonian system, which implies its integrability according to Magri’s theorem [15].

From the first Hamiltonian structure in (11) we can see that all flows are local. For \( k = 2 (t_2 = t) \) we get the equations

\[
\begin{align*}
\frac{\partial \Phi_0}{\partial t} &= -(D^4 \Phi_0) + (D(D \Phi_0)^2) + 2(D^2 \Phi_1) \\
\frac{\partial \Phi_1}{\partial t} &= (D^4 \Phi_1) + 2(D((D \Phi_0)(D \Phi_1)))
\end{align*}
\]

which we call the sTB-B equation. Note that these equations are different from the sTB [12,13] equation only in the last term in the equation for \( \Phi_1 \). In components (20) yields

\[
\begin{align*}
\frac{\partial J_0}{\partial t} &= (2J_1 + J_0^2 - J_0')' \\
\frac{\partial \psi_0}{\partial t} &= 2\psi_1' + 2\psi_0'J_0 - \psi_0'' \\
\frac{\partial J_1}{\partial t} &= (2J_0J_1 + J_1')' \\
\frac{\partial \psi_1}{\partial t} &= \psi_1'' + 2\psi_1'J_0 + 2J_1\psi_0'
\end{align*}
\]

which, of course, is invariant under the supersymmetry transformations

\[
\begin{align*}
\delta J_0 &= \epsilon \psi_0' \\
\delta J_1 &= \epsilon \psi_1' \\
\delta \psi_0 &= \epsilon J_0 \\
\delta \psi_1 &= \epsilon J_1
\end{align*}
\]

Equation (20) can also be obtained from the Lax operator

\[
L = D^2 - (D \Phi_0) + D^{-2}(D \Phi_1)
\]

with the nonstandard Lax representation (2). The conserved charges (13) can be written as

\[
K_n = \int dz \text{Res} L^n \quad n = 1, 2, 3, \ldots
\]
where “Res” stands for the residue which is defined to be the coefficient of the $D^{-2} = \partial^{-1}$ term in the pseudo super-differential operator. These charges are purely fermionic. The first ones are

\[
K_1 = \int dz (D\Phi_1) \\
K_2 = \int dz (D\Phi_0)(D\Phi_1) \\
K_3 = \int dz \left[ (D\Phi_1)^2 - (D^3\Phi_0)(D\Phi_1) + (D\Phi_1)(D\Phi_0)^2 \right]
\]

and when written in components they assume the form

\[
K_1 = \int dx \psi'_1 = 0 \\
K_2 = \int dx \left( \psi'_0 J_1 + J_0 \psi'_1 \right) \\
K_3 = \int dx \left( 2J_1 \psi'_1 - \psi''_0 J_1 - J'_0 \psi'_1 + \psi'_1 J^2_0 + 2J_1 J_0 \psi'_0 \right)
\]

which can be easily checked to be invariant under supersymmetry transformations (22). Note that, in the bosonic limit these charges vanish.

Let us also note that the charges $K_n$ can be simply written as the supersymmetric variation of $H_n$ of (6) (without $\epsilon$), namely,

\[
K_n = \delta H_n
\]

In fact, this allows us to identify the bosonic TB conserved charges as the $\theta$ independent part of Tr $L^n$. We could also have constructed bosonic conserved charges for the sTB-B system using odd powers of the square root of (23) [17,18]

\[
Q_{\frac{n+1}{2}} = \int dz \text{Res} L^{\frac{2n+1}{2}} \quad n = 1, 2, 3, \ldots
\]

However, as pointed out by Manin and Radul [19], $L^{1/2}$, in such a case, is nonunique in general. On the other hand, if we require the coefficient functions to vanish at spatial infinity, these bosonic charges are unique and we have checked that they are conserved as well. These are, in fact, a new set of charges distinct from $H_n$ [20].
4. Reductions of the sTB-B Hierarchy

The TB as well the sTB systems reduce to various known integrable systems. Let us show that we can embed the sKdV-B equation into the sTB-B system. Following Refs. [13,21] and using (23), we obtain

\[ (L^3)_{\geq 1} = D^6 + 3(D\Phi_1)D^2 - 3(D^3\Phi_0)D^2 - 3(D\Phi_0)D^4 + 3(D\Phi_0)^2D^2 \]  

(29)

The nonstandard Lax equation

\[ \frac{\partial L}{\partial t} = [L, (L^3)_{\geq 1}] \]  

(30)

leads to the equations

\[ \frac{\partial \Phi_1}{\partial t} = -(D^6\Phi_1) - 3D \left( (D\Phi_1)^2 + (D\Phi_1)(D\Phi_0)^2 + (D^3\Phi_1)(D\Phi_0) - 2(D\Phi_1)(D^3\Phi_0) \right) \]

\[ \frac{\partial \Phi_0}{\partial t} = -(D^6\Phi_0) - D \left( 6(D\Phi_1)(D\Phi_0) - 3(D\Phi_0)(D^3\Phi_0) + (D\Phi_0)^3 \right) \]  

(31)

which, for

\[ \Phi_0 = 0 \]

\[ \Phi_1 = \Phi \]  

(32)

gives

\[ \frac{\partial \Phi}{\partial t} = -(D^6\Phi) - 3D \left( (D\Phi)^2 \right) \]  

(33)

This is, indeed, the sKdV-B equation considered in [7]. The Hamiltonian structures of the sKdV-B can also be obtained from the sTB-B ones. Following the Dirac reduction [21], let us indicate by \( \mathcal{O} \) quantities \( \mathcal{O} \) constrained to satisfy (32). Then,

\[ \mathcal{J}_{sKdV-B} = \mathcal{J}_{22} - \mathcal{J}_{21} \mathcal{J}_{11}^{-1} \mathcal{J}_{12} \]  

(34)

From (12), it follows that

\[ (\mathcal{J}_2)_{11}^{-1} = \frac{1}{2} \]  

(35)

and, consequently, (34) yields

\[ \mathcal{J}_2^{sKdV-B} = \frac{1}{2}(D^4 + 2D^{-1}(D\Phi)D + 2D(D\Phi)D^{-1}) \]  

(36)
As was noted in [21], the first Hamiltonian for the sKdV-B needs to be obtained from the sTB-B Hamiltonian structure $J_0$ and not from $J_1$. Recursively we have

$$J_0 = R^{-1} J_1$$

(37)

where $R = J_2 J_1^{-1}$ is the recursion operator. It is straightforward to show that

$$R^{-1} = \left(\begin{array}{cc}
-\frac{1}{2} (J_2^{sKdV-B})^{-1} D^2 & (J_2^{sKdV-B})^{-1} \\
\frac{1}{2} - \frac{1}{4} D^2 (J_2^{sKdV-B})^{-1} D^2 & \frac{1}{2} D^2 (J_2^{sKdV-B})^{-1}
\end{array}\right)
$$

(38)

Therefore, it follows from (37) and (34) that

$$J_1^{sKdV-B} = (J_0)_{22} - (J_0)_{21} (J_0)^{-1}_{11} (J_0)_{12} = \frac{1}{2}
$$

(39)

The Hamiltonian structures (36) and (39) are, indeed, the ones derived in [7].

It is well known [10,11,13] that the field redefinitions

$$J_0 = -\frac{q'}{q} = -(\ln q)'$$

$$J_1 = \bar{q}q
$$

(40)

take the TB equation (1) to the nonlinear Schrödinger equation (NLS)

$$\frac{\partial q}{\partial t} = -(q'' + 2(\bar{q}q)q)$$

$$\frac{\partial \bar{q}}{\partial t} = \bar{q}'' + 2(\bar{q}q)\bar{q}
$$

(41)

So, it will be expected that some field redefinition will take the sTB-B equation to a new system, the sNLS-B equation. Let us use the field redefinition

$$(D\Phi_0) = -\frac{(D^3Q)}{(DQ)}$$

$$(D\Phi_1) = (DQ)(D\bar{Q})$$

(42)

where

$$(DQ) = q + \theta \phi'$$

$$(D\bar{Q}) = \bar{q} + \theta \bar{\phi}'
$$

(43)

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Using this in the Lax operator (23), we obtain

\[ L = D^2 + \frac{(D^3Q)}{(DQ)} + D^{-2}(DQ)(D\overline{Q}) = G\tilde{L}G^{-1} \]  

(44)

where

\[ G = (DQ)^{-1} \]

\[ \tilde{L} = D^2 + (DQ)D^{-2}(D\overline{Q}) \]

(45)

So, we have the Lax operators \( L \) and \( \tilde{L} \) related by a gauge transformation. Taking the formal adjoint of \( \tilde{L} \)

\[ \mathcal{L} = \tilde{L}^* = -D^2 + (D\overline{Q})D^{-2}(DQ) \]

(46)

we can obtain consistent equations using the standard Lax representation

\[ \frac{\partial \mathcal{L}}{\partial t} = \left[ \left( \mathcal{L}^2 \right)_+, \mathcal{L} \right] \]  

(47)

and they are

\[ \frac{\partial Q}{\partial t} = -(D^4Q) + 2D^{-1}((D\overline{Q})(DQ)^2) \]  

(48)

\[ \frac{\partial \overline{Q}}{\partial t} = (D^4\overline{Q}) - 2D^{-1}((D\overline{Q})(DQ)) \]

which, unfortunately is nonlocal. The reader can verify that the same result (48) can be obtained directly from the supersymmetrization of the bosonic NLS hierarchy following the same steps from (8) to (11) used for the TB hierarchy.

5. Zero Curvature

As is clear from the discussions of the earlier sections, the Hamiltonian structures of the sTB-B are fermionic. Normally, the Hamiltonian structures of the integrable models correspond to interesting symmetry algebras and dictate the choice of the gauge group in the zero curvature formulation of the problem [22]. It is, therefore, interesting to ask whether a zero curvature formulation of this model can be achieved. To this end, we first discuss the problem in superspace before returning to the component formulation.

From the structure of the Lax operator in (23), we note that the linear problem associated with the sTB-B system is given by (λ is the constant spectral parameter.)

\[ L\chi_1 = \lambda\chi_1 \]

(49)
where $\chi_1$ is assumed to be a bosonic superfield eigenfunction. We note that we can write the equation in a completely local form by introducing a second superfield as

$$\chi_2 = [D^{-2}(D\Phi_1)\chi_1]$$  \hspace{1cm} (50)

so that the linear equation can be written as

$$\partial_x \chi = A_1 \chi$$  \hspace{1cm} (51)

where

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} \lambda + (D\Phi_0) & -1 \\ (D\Phi_1) & 0 \end{pmatrix}$$  \hspace{1cm} (52)

It is interesting to note here that both the superfields $\chi_1$ and $\chi_2$ are bosonic. This is different from the discussion in [23] and it is not clear how a graded symmetry group would arise in such a case.

Writing the time evolution equation as

$$\partial_t \chi = A_0 \chi$$  \hspace{1cm} (53)

where we assume that, in terms of general superfields, $A_0$ has the form

$$A_0 = \begin{pmatrix} A & B \\ C & E \end{pmatrix}$$  \hspace{1cm} (54)

the zero curvature equation

$$\partial_t A_1 - \partial_x A_0 - [A_0, A_1] = 0$$  \hspace{1cm} (55)

is obtained as the compatibility condition between the two. This leads to the constraint conditions

$$A = E + B_x - (\lambda + (D\Phi_0))B$$

$$C = E_x - B(D\Phi_1)$$  \hspace{1cm} (56)

as well as the two dynamical equations

$$\frac{\partial}{\partial t}(D\Phi_0) = B_{xx} + 2E_x - [B(D\Phi_0)]_x - \lambda B_x$$

$$\frac{\partial}{\partial t}(D\Phi_1) = E_{xx} + E_x(D\Phi_0) - 2B_x(D\Phi_1) - B(D^3\Phi_1) + \lambda E_x$$  \hspace{1cm} (57)
It is clear now that if we write a Taylor series expansion of the forms

\[ B = \sum_{n=0}^{N} \lambda^{N-n} B_n, \quad E = \sum_{n=0}^{N} \lambda^{N-n} E_n \]  

(58)

the recursion relations take the form

\[ \begin{pmatrix} (DB_{n+1}) \\ (DE_{n+1}) \end{pmatrix} = \begin{pmatrix} D^2 - D(D\Phi_0)D^{-1} & 2 \\ D(D\Phi_1)D^{-1} + D^{-1}(D\Phi_1)D & -D^2 - D^{-1}(D\Phi_0)D \end{pmatrix} \begin{pmatrix} (DB_n) \\ (DE_n) \end{pmatrix} \]  

(59)

which is the same as the recursion operator following from Eqs. (12) with appropriate identifications. If we now choose

\[ B_0 = 1, \quad E_0 = 0 \]  

(60)

it is straightforward to check that the sTB-B hierarchy follows. In particular, the third order equation yields the sKdV-B equation when \( \Phi_0 = 0 \) as it should.

From the form of \( A_1 \), the symmetry group appears more like \( SL(2) \times U(1) \). A graded symmetry algebra is not at all clear and the reduction of the zero curvature condition to components is even more obscure. We have, however, succeed in obtaining the zero curvature condition for the hierarchy in terms of \( 4 \times 4 \) matrices. Without going into details, we simply note that the matrices

\[ A_1 = \begin{pmatrix} (\lambda + J_0) & -1 & 0 & 0 \\ J_1 & 0 & 0 & 0 \\ \psi'_0 & 0 & (\lambda + J_0) & -1 \\ \psi'_1 & 0 & J_1 & 0 \end{pmatrix} \]  

(61)

\[ A_0 = \begin{pmatrix} F & G & 0 & 0 \\ H & K & 0 & 0 \\ P & Q & F & G \\ R & S & H & K \end{pmatrix} \]

would yield, from the zero curvature condition, the constraints (both \( ' \) and the subscript \( x \) denote derivative with respect to \( x \))

\[ F = K + G_x - (\lambda + J_0)G \]

\[ H = K_x - GJ_1 \]

\[ P = S + Q_x - (\lambda + J_0)Q - \psi'_0G \]

\[ R = S_x - J_1Q - \psi'_1G \]

(62)
as well as the dynamical equations

\begin{align}
J_{0t} &= G_{xx} + 2K_x - J_{0x}G - (\lambda + J_0)G_x \\
J_{1t} &= K_{xx} - J_{1x}G - 2J_1G_x + (\lambda + J_0)K_x \\
\psi_{0t} &= Q_{xx} + 2S_x - J_{0x}Q - (\psi_0'G)' - (\lambda + J_0)Q_x \\
\psi_{1t} &= S_{xx} - J_{1x}Q - 2J_1Q_x - 2\psi_1'G_x - \psi_1''G + \psi_0'K_x + (\lambda + J_0)S_x
\end{align}

Once again, writing a Taylor series as in Eq. (58) and choosing

\begin{align}
G_0 &= 1, \\
K_0 &= 0 = Q_0 = S_0
\end{align}

leads to the sTB-B hierarchy in components. It is worth noting here that the matrix structure, in components, suggests an underlying symmetry algebra of \( OSP(2|2) \) which is also isomorphic to \( SL(2|1) \).

6. Conclusions

We have constructed a new supersymmetric two boson (sTB-B) hierarchy and studied its properties. We derived the conserved charges and have shown that it is a bi-Hamiltonian system. The Jacobi identity for the Hamiltonian structures are verified using the super-prolongation technique. We have shown how this system gives the sKdV-B system as well as its Hamiltonian structures upon appropriate reduction. We have also described the zero curvature formulation of this new hierarchy both in the super space as well as in components and have brought out its distinguishing features.

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References

1. L.D. Faddeev and L.A. Takhtajan, “Hamiltonian Methods in the Theory of Solitons” (Springer, Berlin, 1987).
2. A. Das, “Integrable Models” (World Scientific, Singapore, 1989).
3. L. A. Dickey, “Soliton Equations and Hamiltonian Systems” (World Scientific, Singapore, 1991).
4. D. J. Gross and A. A. Midgal, Phys. Rev. Lett. 64, 127 (1990); D. J. Gross and A. A. Midgal, Nucl. Phys. B340, 333 (1990); E. Brézin and V. A. Kazakov, Phys. Lett. 236B, 144 (1990); M. Douglas and S. H. Shenker, Nucl. Phys. B335, 635 (1990); A. M. Polyakov in “Fields, Strings and Critical Phenomena”, Les Houches 1988, ed. E. Brézin and J. Zinn-Justin (North-Holland, Amsterdam, 1989); L. Alvarez-Gaumé, Helv. Phys. Acta 64, 361 (1991); P. Ginsparg and G. Moore, “Lectures on 2D String Theory and 2D Gravity” (Cambridge, New York, 1993).
5. L. Alvarez-Gaumé and J. L. Manés, Mod. Phys. Lett. A6, 2039 (1991); L. Alvarez-Gaumé, H. Itoyama, J. Manès and A. Zadra, Int. J. Mod. Phys. A7, 5337 (1992).
6. K. Becker and M. Becker, Mod. Phys. Lett. A8, 1205, (1993).
7. J. M. Figueroa-O’Farrill and S. Stanciu, Phys. Lett. B316, 282 (1993).
8. J. M. Figueroa-O’Farrill and S. Stanciu, Mod. Phys. Lett. A8, 2125 (1993).
9. B.A. Kupershmidt, Commun. Math. Phys. 99, 51 (1985).
10. H. Aratyn, L.A. Ferreira, J.F. Gomes and A.H. Zimerman, Nucl. Phys. B402, 85 (1993); H. Aratyn, L.A. Ferreira, J.F. Gomes and A.H. Zimerman, “On $W_{\infty}$ Algebras, Gauge Equivalence of KP Hierarchies, Two-Boson Realizations and their KdV Reductions”, in Lectures at the VII J. A. Swieca Summer School, São Paulo, Brazil, January 1993, eds. O. J. P. Éboli and V. O. Rivelles (World Scientific, Singapore, 1994); H. Aratyn, E. Nissimov and S. Pacheva, Phys. Lett. B314, 41 (1993).
11. L. Bonora and C.S. Xiong, Phys. Lett. B285, 191 (1992); L. Bonora and C.S. Xiong, Int. J. Mod. Phys. A8, 2973 (1993).
12. J. C. Brunelli and A. Das, Phys. Lett. B337, 303 (1994).
13. J. C. Brunelli and A. Das, Int. J. Mod. Phys. A10, 4563 (1995).
14. J. C. Brunelli, A. Das and W.-J. Huang, Mod. Phys. Lett. **9A**, 2147 (1994).
15. P. J. Olver, “Applications of Lie Groups to Differential Equations”, Graduate Texts in Mathematics, Vol. 107 (Springer, New York, 1986).
16. P. Mathieu, Lett. Math. Phys. **16**, 199 (1988).
17. P. Dargis and P. Mathieu, Phys. Lett. **A176**, 67 (1993).
18. J. C. Brunelli and A. Das, Phys. Lett. **B354**, 307 (1995).
19. Y. I. Manin and A. O. Radul, Commun. Math. Phys. **98**, 65 (1985).
20. J. C. Brunelli and A. Das, work in progress.
21. J. C. Brunelli and A. Das, Mod. Phys. Lett. **A10**, 2019 (1995).
22. A. Das and S. Roy, J. Math. Phys. **31**, 2145 (1990); A. Das, W.-J. Huang and S. Roy, Int. J. Mod. Phys. **A7**, 3447 (1992); A. Das and S. Roy, Mod. Phys. Lett. **A11**, 1317 (1996).
23. H. Aratyn, A. Das and C. Rasinariu, “Zero Curvature Formalism for Supersymmetric Integrable Hierarchies in Superspace”, UICHEP-TH-97-5.