Abstract. We construct a configuration space model for spaces of maps into certain subcomplexes \( W \) of product spaces (including polyhedral products). As an application we obtain a single suspension splitting for the loop space \( \Omega W \). In case when \( W \) is a polyhedral product, we show that the summands in these splittings have a very direct bearing on the topology of polyhedral products, in particular of moment-angle complexes. This gives a new way of tackling a conjecture relating Golod complexes and co-\( H \)-polyhedral products, as well as a systematic approach for studying the homotopy theory of more general polyhedral products, such as those over minimally non-Golod complexes.

1. Introduction

Polyhedral products have been the subject of recent interest, starting with their appearance as homotopy theoretical generalisations of various objects studied in toric topology. Of particular importance are the polyhedral products \( (D^2, S^1)^K \), known as moment-angle complexes, and the Davis-Januszkiewicz spaces \( (\mathbb{C}P^\infty, *)^K \), which exist as homotopy theoretical models for topological generalisations of universal toric projective varieties and their homotopy orbit spaces (c.f. [27]). The homotopy theory of these spaces has far reaching applications to complex and symplectic geometry (c.f. [7, 23, 22]) and combinatorial and homological algebra (c.f. [27, 1]). In particular, moment-angle manifolds are diffeomorphic to intersection of quadrics, while Stanley-Reisner rings of simplicial complexes appear as the equivariant cohomology ring of moment-angle complexes (c.f. [27, 8]). Calculating the cohomology ring of moment-angle complexes is closely related to the study of the cohomology of a local ring. In general, one hopes that the combinatorics of the underlying simplicial complex \( K \) encodes a great deal of information about the geometrical and topological properties of polyhedral products. From algebraic and topological point of view the case of Golod complexes is especially relevant. A simplicial complex \( K \) on vertex set \([n] = \{1, \ldots, n\}\) is said to be Golod over a field \( k \) if all cup products and Massey products are trivial in the ring \( H^*((D^2, S^1)^K; k) \), and if this is true for all fields \( k \), we say simply that \( K \) is Golod. It was shown [3] that if all cup products are trivial in the ring \( H^*((D^2, S^1)^K; k) \) over a field \( k \), then all higher Massey products are trivial as well. Fixing \( k \) to be a field or \( \mathbb{Z} \), according to [8, 13, 3] there are isomorphisms of graded commutative algebras

\[
H^*((D^2, S^1)^K; k) \cong \text{Tor}_{k[v_1, \ldots, v_n]}(k[K], k) \cong \bigoplus_{I \subseteq [n]} \tilde{H}^*([K_I]; k)
\]

where \( k[K] \) is the Stanley-Reisner ring of \( K \), \( K_I \) is the restriction of \( K \) to vertex set \( I \subseteq [n] \), and the (cup product) multiplication is induced by maps

\[
\iota_{I,J} : [K_{I \cup J}] \longrightarrow [K_I \ast K_J] \cong [K_I] \ast [K_J] \cong \Sigma[K_I] \wedge [K_J],
\]

that realize the canonical inclusions \( K_{I \cup J} \longrightarrow K_I \ast K_J \) whenever \( I \) and \( J \) are non-empty and disjoint, and are zero otherwise. Thus, the Golod condition can be reinterpreted as \( \iota_{I,J} \) inducing trivial maps.

2010 Mathematics Subject Classification. Primary 55P15, 55P35, 55U10, 13F55.

Key words and phrases. polyhedral product, moment-angle complex, toric topology, Stanley-Reisner ring, Golod ring, configuration space.
on $k$-cohomology for disjoint non-empty $I$ and $J$. Further topological and algebraic interpretations of Golodness exist (see [15] [5]), for example, $H_*(\Omega(D^2, S^1)^K; k)$ being a free graded associative algebra, or the multiplication and all higher Massey products being trivial in $\text{Tor}_*[n, \ldots, n](k[K], k)$ (this being the classical algebraic definition of a Golod complex). From this point of view, the homotopy theory of polyhedral products therefore lends itself into the problem of determining when a simplicial complex is Golod. In the opposite direction, a polyhedral product takes on its simplest topological form when we restrict to Golod complexes $K$. This is therefore a natural starting point for studying their homotopy types.

Considerable work has been done in this direction for various classes of Golod complexes [16] [17] [18] [20] [19] [15]. This has culminated in the following conjectured topological characterisation of the Golod complexes.

**Conjecture 1.1.** A moment-angle complex $(D^2, S^1)^K$ is a co-$H$-space if and only if $K$ is Golod.

The right-hand implication here is clear, while confirmation of the left-hand implication immediately gives an explicit form of the homotopy type of $(D^2, S^1)^K$ in terms of $K$. Namely, $(D^2, S^1)^K$ is a co-$H$-space if and only if

$$(D^2, S^1)^K \simeq \bigvee_{I \subseteq [n]} \Sigma|I|^{|I|+1}|K_I|.$$  

This was shown by Iriye and Kishimoto [19], and follows from a general suspension splitting due to Bahri, Bendersky, Cohen, and Gitler (the **BCG splitting**) [2]

$$\Sigma(D^2, S^1)^K \simeq \bigvee_{I \subseteq [n]} \Sigma|I|^{|I|+2}|K_I|.$$  

(1)

There are several simplifications that make this conjecture more approachable. The first of these is localization. We confirm a large primes version of Conjecture 1.1. As a consequence, we reproduce the rational result of Berglund [4] without making any significant use of rational homotopy theory.

**Theorem 1.2.** Localized at any sufficiently large prime $p$, $(D^2, S^1)^K$ is a co-$H$-space if and only if $K$ is Golod over $\mathbb{Z}_p$.

The second simplification is to strengthen the hypothesis on a simplicial complex being Golod. One way of doing this is to require some suspension of the inclusion $\iota_{I,J}$ to be null-homotopic (instead of only inducing trivial maps on cohomology) for every disjoint and non-empty $I, J \subseteq [n]$. We will see that (after appropriate suspension) this is a necessary condition for $(D^2, S^1)^K$ to be a co-$H$-space. Moreover, we will require transitions between these null-homotopies. For example, taking the inclusions

$$\iota_{I,J} \star \iota_{J_2} : |K_{I,J} \star K_{J_2}| \longrightarrow |K_I \star K_{J_1} \star K_{J_2}|,$$

$$\iota_I \star \iota_{J_1,J_2} : |K_I \star K_{J_1, J_2}| \longrightarrow |K_I \star K_{J_1} \star K_{J_2}|,$$

for disjoint non-empty $I$, $J_1$, and $J_2$, and the map

$$\iota_{I,J} : \text{Cone}(\Sigma|K_{I,J}|) \longrightarrow \Sigma|K_I \star K_J|$$

given by a null-homotopy of $\Sigma \iota_{I,J}$, we require the composite $\Sigma(\iota_I \star \iota_{J_1,J_2}) \circ \iota_{I,J_1,J_2}$ to be homotopic to $\Sigma(\iota_I \star \iota_{J_1,J_2}) \circ \iota_{I,J_1,J_2}$ via a homotopy that is fixed on the base $|K_{I,J_1,J_2}|$ of the cone $\text{Cone}(\Sigma|K_{I,J_1,J_2}|)$, and so on for longer joins. These transitions between null-homotopies bear some resemblance to the coherence of homotopy in Stasheff’s higher homotopy associativity of $A_n$-spaces [28]. In our case, the Stasheff associahedron is replaced with a simplicial complex associated to ordered partitions of the vertex set $[n]$, while $H$-space multiplication maps are replaced with null-homotopies of certain reduced diagonals, which up to homeomorphism are the above inclusions of
cones into joins of full subcomplexes. We make this precise in Section II which we call the homotopy Golod condition. We show that the left-hand implication of the conjecture holds for the subclass of Golod complexes that are homotopy Golod.

**Theorem 1.3.** If $K$ is homotopy Golod, then $(D^2, S^1)^K$ is a co-$H$-space.

The extractible complexes defined in [19] comprise the most general subclass of Golod complexes $K$ for which $(D^2, S^1)^K$ is known to be a co-$H$-space. We will show that extractible complexes are homotopy Golod, and give an example of a class of Golod complexes that are homotopy Golod, but which are not extractible.

The homotopy Golod condition will follow from a weaker condition, which we call the weak homotopy Golod condition, and which happens to be a necessary and sufficient combinatorial condition.

**Theorem 1.4.** A moment-angle complex $(D^2, S^1)^K$ is a co-$H$-space if and only if $K$ is weakly homotopy Golod.

The main idea that drives this paper is the following well-known fact due to Ganea: a space $Y$ is a co-$H$-space if and only if the evaluation map $\Sigma \Omega Y \xrightarrow{ev} Y$ has a right homotopy inverse. Therefore, in order to figure out if $Y$ is a co-$H$-space, one may start by trying to find the finest possible splitting of $\Sigma \Omega Y$. To do this we construct configuration spaces that model spaces of maps into very general subcomplexes $W^\ell$ of a coordinate-wise suspensions of product spaces.

**Theorem 1.5.** Let each $X_i$ be a connected basepointed CW-complex and $W$ a subcomplex of their product $X$. Suppose $M$ is a smooth compact parallelizable $\ell$-manifold, $\ell > 0$, $N$ a submanifold, $M/N$ is connected or $W$ is basepoint connected. Then there exists a homotopy equivalence

$$\gamma: C(M, N; W) \to map(Z - N, Z - M; W^\ell),$$

where $Z = M \cup (\partial M \times [0, 1])$.

These configuration spaces generalize classical labelled configuration spaces by allowing particles to collide under certain rules, and provide models for mapping spaces to a rather wide family of spaces including polyhedral products, and therefore moment-angle manifolds, intersections of quadrics, complements of certain arrangements of coordinate subspaces. We use them to construct a single suspension splitting for the loop space of these subcomplexes $W^\ell$ in terms of quotient labelled configuration spaces. In the case when $W^1 = (D^2, S^1)^K$, the summands $D_i$ in the splitting of $\Sigma \Omega(D^2, S^1)^K$ behave as much more combinatorial objects compared to polyhedral products, allowing us to obtain combinatorial statements about their homotopy type. For example, fixing the particles in a configuration in a summand $D_i$, and looking at the subspace of all possible labels, one obtains a subspace homeomorphic to a suspension of a join of simplicial complexes $K_i$. This is analogous to the summands in the BBCG splitting ([11] being combinatorial objects.

These decomposition techniques can be used to study polyhedral products beyond the $K$ Golod case, in particular, for $K$ minimally non-Golod complexes. We construct a filtration $F$ of $(D^2, S^1)^K$ by taking intersections with fat wedge subspaces of $(D^2)^{\times n}$, and then applying our configuration space models to each space $F_i$ in this filtration, the summands in the resulting splitting of $\Sigma \Omega F_i$ give a filtration for the summands $D_i$ in the splitting of $\Sigma \Omega(D^2, S^1)^K$. We show that the attaching maps $f_i$ that build $F_i$ up from $F_{i-1}$ can be studied by looking at the attaching maps $g_{i,n}$ for the spaces $G_{i,n}$ in the filtration of a particular summand $D_n$. This is due to the existence of a factorisation of $f_i$ as $g_{i,n}$ and a map $G_{i,n} \xrightarrow{h_i} F_i$. In the minimally non-Golod case this is often a particularly nice factorisation - one into something combinatorial (the maps $g_{i,n}$) and something more homotopy
theoretic (the maps $h_i$ as wedge sums involving Whitehead products). A closer analysis of these maps will appear in a sequel.

Throughout this paper we will take $-1$ to be the basepoint of $D^1 = [-1, 1]$. Points in $S^1$ are thought of as real numbers in $D^1$ with $-1$ and 1 identified as the basepoint. The suspension $\Sigma X$ of a space $X$ is taken to be the reduced suspension $D^1 \times X/((-1, 1) \times X \cup D^1 \times \{\ast\})$ whenever $X$ is basepointed with basepoint $\ast$. Otherwise it is the unreduced suspension, in other words, the quotient space of $D^1 \times X$ under identifications $(-1, x) \sim \ast_{-1}$ and $(1, x) \sim \ast_1$. In any case, $\Sigma X$ is always basepointed, in the unreduced case the basepoint taken to be $\ast_{-1}$.

2. Coordinate Suspensions

Fix a product of connected basepointed spaces $X = X_1 \times \cdots \times X_n$ and a subspace $W \subseteq X$. For any product $Y = Y_1 \times \cdots \times Y_n$ of basepointed spaces, define the coordinate smash $Y \wedge_X W$ of $Y$ and $W$ in $X$ to be the subspace of $Y \wedge_X X = (Y_1 \wedge X_1) \times \cdots \times (Y_n \wedge X_n)$ given by

$$Y \wedge_X W = \{(y_1, x_1), \ldots, (y_n, x_n) \in Y \wedge_X X \mid (x_1, \ldots, x_n) \in W\}.$$  

In particular, when each $Y_i$ is an $\ell$-sphere $S^\ell$, we call $Y \wedge_X W$ the $\ell$-fold coordinate suspension of $W$ in $X$ and denote it by $W^\ell$. This is then the subspace of $X^\ell = (\Sigma^\ell X_1) \times \cdots \times (\Sigma^\ell X_n)$ given by

$$W^\ell = \{(t_1, x_1), \ldots, (t_n, x_n) \in X^\ell \mid (x_1, \ldots, x_n) \in W\}$$

for pairs $(t_i, x_i)$ in the reduced suspension $\Sigma^\ell X_i = D^\ell \times X_i / (\partial D^\ell \times X_i \cup D^\ell \times \{\ast\})$, $D^\ell$ being the unit $\ell$-disk. Notice $W^0 = W$ and $(W^{\ell_1})^{\ell_2} = W^{\ell_1 + \ell_2}$. We will say $W$ is basepointed connected if it is connected and contains the basepoint $(\ast, \ldots, \ast) \in X$.

**Example 2.1.** Let $W$ be some union of product subspaces $A_1 \times \cdots \times A_n$ and basepoint connected. Then $W^\ell$ is a union of the subspaces $\Sigma^\ell A_1 \times \cdots \times \Sigma^\ell A_n$. In particular, $W^\ell = X^\ell$ when $W = X$ and $W^\ell \subseteq \Sigma^\ell W$ when $W$ is any basepoint connected subspace of $X_1 \vee \cdots \vee X_n \subseteq X$.

We will construct configuration space models for certain spaces of maps to coordinate suspensions in the coming sections. With this in mind, here are a few more motivating examples.

**Example 2.2.** If $(X, A) = ((X_1, A_1), \ldots, (X_n, A_n))$ is a sequence of pairs of spaces, $K$ is a simplicial complex on $n$ vertices, and $W$ is the polyhedral product

$$(X, A)^K = \bigcup_{\sigma \in K} Y^\sigma_1 \times \cdots \times Y^\sigma_n \subseteq X_1 \times \cdots \times X_n,$$

where

$$Y^\sigma_i = \begin{cases} X & \text{if } i \in \sigma \\ A & \text{if } i \notin \sigma \end{cases},$$

then $W^\ell$ is the polyhedral product $(\Sigma^\ell X, \Sigma^\ell A)^K \subseteq X^\ell$ of the $n$-tuple $(\Sigma^\ell X, \Sigma^\ell A)$ of pairs of spaces $((\Sigma^\ell X_1, \Sigma^\ell A_1), \ldots, (\Sigma^\ell X_n, \Sigma^\ell A_n))$.

**Example 2.3.** Taking each $(X_i, A_i)$ to be the pair $(D^1, \partial D^0) = (D^1, S^0)$ and $W = (D^1, S^0)^K$, the polyhedral product $W^1 = (D^2, S^1)^K$ is homotopy equivalent to the moment-angle complex $Z_K$, which in turn appears as the homotopy fibre in a homotopy fibration sequence

$$Z_K \to DJ(K) \to (CP^\infty)^{\times n},$$

where $DJ(K) = (CP^\infty, \ast)^K$ is the Davis-Januszkiewicz spaces. The connecting map $(S^1)^{\times n} \to Z_K$ is given by the action of the torus $\Omega(CP^\infty)^{\times n} = (S^1)^{\times n}$ on $Z_K$ is null-homotopic. For $K$ a dual of a simple polytope, a quasitoric manifold $Q$ is an orbit space $Z_K / (S^1)^{\times m}$ for a choice of subtorus.
$(S^1)^n \subseteq (S^1)^m$ on which this action is free. Since the inclusion of the fibre in the bundle $(S^1)^m \to Z_K \to Q$ factors through $\partial$, it is null-homotopic, meaning there is a homotopy equivalence
\[ \Omega Q \simeq (S^1)^m \times \Omega Z_K \simeq (S^1)^m \times \Omega(D^2, S^1)^K. \]

Thus, configuration space models for loops spaces of polyhedral products yield models for loop spaces of quasi-toric manifolds.

Many interesting spaces are captured as polyhedral products. $(D^2, S^1)^K$ is a manifold when $K$ is a triangulation of a sphere, and in particular, a connected sum of sphere products when $K$ is, for example, an $n$-gon. More general connected sums of sphere products are coordinate suspensions, as suggested by the following example.

**Example 2.4.** Take two copies $D_1^m$ and $D_2^m$ of the $m$-disk, and two copies of the boundary $(m-1)$-sphere $S_1^{m-1} = \partial D_1^m$ and $S_2^{m-1} = \partial D_2^m$. Let $B_1^m$ and $B_2^m$ be closed $(m-1)$-disks in $S_1^{m-1}$ and $S_2^{m-1}$. Form the $m$-disk $D^m$ by gluing $D_1^m$ and $D_2^m$ along some homeomorphism of $B_1^m$ and $B_2^m$. Identically, for two copies $D_1^n$ and $D_2^n$ of the $n$-disk, form the $n$-disk $D^n$ by gluing $D_1^n$ and $D_2^n$ along some homeomorphism of $B_1^n$ and $B_2^n$.

Form the connected sum $C = (S_1^{m-1} \times S_2^{m-1}) \# (S_1^{n-1} \times S_2^{n-1})$ by deleting the interiors of $B_1^m \times B_1^n$ and $B_2^m \times B_2^n$ and gluing along the boundary. The products $P_1 = S_1^{m-1} \times S_1^{n-1}$ and $P_2 = S_2^{m-1} \times S_2^{n-1}$ are then the subspaces of $D^m \times D^n$ sitting on the boundaries of the subspaces $D_1^m \times D_1^n$ and $D_2^m \times D_2^n$, and intersecting on the $(m+n-2)$-disk $B_1^m \times B_1^n \sim B_2^m \times B_2^n$. Thus $C$ is homeomorphic to $P_1 \cup P_2 - \text{int}(B_1^m \times B_1^n)$ as a subspace of $D^m \times D^n$.

Taking the basepoint of $D^m \times D^n$ to be on the boundary of $B_1^m \times B_1^n$, the coordinate suspensions of $P_1$ and $P_2$ in $D^m \times D^n$ are both homeomorphic to $S^m \times S^n$, and the coordinate suspension of $B_1^m \times B_1^n$ is a $(m+n)$-disk. Thus, we see that the coordinate suspension of $C$ in $D^m \times D^n$ is homeomorphic to the connected sum $(S^m \times S^n) \# (S^m \times S^n)$.

Since our aim is to construct configuration space models for mapping spaces of manifolds to coordinate suspensions, an answer to the following question might be of interest.

**Question 2.5.** When is a manifold a coordinate suspension? Given an embedding of a manifold $W$ into an $m$-disk $D^m \cong (D^1)^m$, when is its coordinate suspension $W^1$ a submanifold of $D^{2m} \cong (\Sigma D^1)^{2m}$?

### 2.1. Cofibrations and Splittings

Given any subset $I = \{i_1, \ldots, i_k\} \subseteq [n] = \{1, \ldots, n\}$ such that $i_1 < \cdots < i_k$, and non-negative integer $l$, let
\[
X^l_I = \Sigma^l X_{i_1} \times \cdots \times \Sigma^l X_{i_k},
\]
\[
\hat{X}^l_I = \Lambda^l X_{i_1} \wedge \cdots \wedge \Lambda^l X_{i_k}.
\]

Let $W^l_I$ be the image of $W^l$ under the projection $X^l \to X^l_I$, and $\hat{W}^l_I$ the image of $W^l_I$ under the quotient map $X^l_I \to \hat{X}^l_I$. For any $0 \leq k \leq n$, take the union of subspaces of $X$:
\[
W^l_k = \bigcup_{l \leq n, \ l \leq k} W^l_I,
\]
In other words, $W^l_k = \{(x_1, \ldots, x_n) \in W^l \mid$ at least $n-k$ coordinates $x_i$ are the basepoint $\ast \in X_i\}$. The equality $W^l_k = (W^l_k)^\ell$ is clear. Take the quotient space
\[
\hat{W}^l_k \cong \frac{W^l_k}{W^l_{k-1}}.
\]
Applying these constructions to $X$ in place of $W$, there is an obvious homeomorphism

$$
\hat{X}_k^\ell = \frac{X_k^\ell}{X_{k-1}^\ell} \cong \bigvee_{I \in [n], |I| = k} \hat{X}_I^\ell.
$$

One can think of $\hat{W}_k^\ell$ as the image of $X_k^\ell$ under the quotient map $X_k^\ell \to \hat{X}_k^\ell$, and this restricts to $W_I^\ell \to \hat{W}_I^\ell \subseteq \hat{X}_I^\ell$ for any $|I| = k$. Thus

$$
\hat{W}_k^\ell \cong \bigvee_{I \in [n], |I| = k} \hat{W}_I^\ell.
$$

Mapping $\hat{W}_I^\ell$ via the homeomorphism $\hat{X}_I^\ell \cong \Sigma^{\ell|I|} \hat{X}_I$ that sends $((t_1, x_1), \ldots, (t_{|I|}, x_{|I|}))$ to the point $((t_1, \ldots, t_{|I|}), (x_1, \ldots, x_{|I|}))$, we see that

$$
\hat{W}_I^\ell \cong \Sigma^{\ell|I|} \hat{W}_I,
$$

$$
\hat{W}_k^\ell \cong \Sigma^{\ell k} \hat{W}_k.
$$

Recall that a basepointed CW-complex is a CW-complex with a fixed basepoint, built up from this basepoint by attaching cells via basepoint preserving attaching maps. Consider the following condition:

(A) each $X_i$ is a connected basepointed CW-complex and $W$ is a subcomplex of their product $X$.

Under these assumptions $W^\ell$ is then a subcomplex of the product $X^\ell$, and in turn, $W_{k-1}^\ell$ is a subcomplex of $W_k^\ell$. This implies that the sequence

$$
W_{k-1}^\ell \to W_k^\ell \to \hat{W}_k^\ell
$$

is a cofibration sequence.

The next two propositions generalize the polyhedral product splittings in [2] and [19]. For example, we may recover the BBCG suspension splitting of $W^1 = (D^2, S^1)^K$ by applying the homeomorphism $\hat{W}_I \cong \Sigma |K_I|$ given in [2].

**Proposition 2.6.** Assuming condition (A), for each $\ell \geq 0$ and $1 \leq k \leq n$ there exists a suspension splitting

$$
\Sigma W_k^\ell \cong \bigvee_{I \subseteq [n], |I| \leq k} \Sigma \hat{W}_I^\ell \cong \bigvee_{I \in [n], |I| \leq k} \Sigma \hat{W}_I^\ell \cong \bigvee_{I \in [n], |I| \leq k} \Sigma^{\ell |I| + 1} \hat{W}_I.
$$

*Proof.* Given a subset $I = \{i_1, \ldots, i_s\} \subseteq [n]$, let

$$
q_I : W_k^\ell \to \hat{W}_I^\ell \to \Sigma^{\ell |I|} \hat{W}_I
$$

be the restriction of the composite $X^\ell \to X_I^\ell \to \hat{X}_I^\ell$ of the projection and quotient map to $W_k^\ell$. Notice that this map is the composite $W_k^\ell \to \hat{W}_k^\ell \to \hat{W}_I^\ell$ when $|I| = k$, where the last map is the quotient onto the summand $\hat{W}_I^\ell$ with respect to homeomorphism (4).

We proceed by induction on $k$. Assume that for some $m < k$ the co-$H$-space sum

$$
h_m : \Sigma W_m^\ell \overset{\text{pinch}}{\longrightarrow} \bigvee_{I \subseteq [n], |I| \leq m} \Sigma W_m^\ell \overset{\nu q_I}{\longrightarrow} \bigvee_{I \subseteq [n], |I| \leq m} \Sigma \hat{W}_I^\ell
$$

is a homeomorphism.
is a homotopy equivalence. This is trivial when $m = 1$. There is a commutative diagram of cofibration sequences

$$
\begin{array}{c}
\Sigma W_f^\ell \\
\downarrow h_m \quad \downarrow h_{m+1} \\
\Sigma \tilde{W}_f^\ell \quad \Sigma \tilde{W}_I^\ell \\
\downarrow \quad \downarrow \\
\Sigma W_{m+1}^\ell \\
\end{array}
$$

where the bottom horizontal maps are respectively the summand-wise inclusion and the quotient map onto the appropriate summands. The top maps are the suspended inclusion and quotient. The composite of the first two maps in the sequence $X_m^\ell \to X_{m+1}^\ell \to \tilde{X}_{m+1}^\ell \to \tilde{X}_I^\ell$ is the constant map, and the last map is the quotient onto the summand in splitting (6) for $|I| = m+1$. Then the restriction $W_m^\ell \to W_{m+1}^\ell \to \tilde{W}_I^\ell$ of this composite is the constant map. Since the suspended inclusion $\Sigma W_m^\ell \to \Sigma W_{m+1}^\ell$ is a co-$H$-map and the composites $h_m$ and $h_{m+1}$ are co-$H$-space sums of the maps $\Sigma q_I$, (with $h_{m+1}$ differing from $h_m$ by including the maps $\Sigma q_I$ where $|I| = m+1$), the left-hand square commutes. The homeomorphism $\hat{h}_{m+1}$ is taken to be the suspension of homeomorphism $[4]$. All maps being suspensions, the composite of $\Sigma W_m^\ell \to \Sigma W_{m+1}^\ell$ and $\hat{h}_{m+1}$ is homotopic to the co-$H$-space sum of composites $\Sigma W_m^\ell \to \Sigma W_{m+1}^\ell \to \Sigma W_I^\ell$ over all $|I| = m+1$, the last map being the suspended quotient onto the summand. By our above remarks, this suspended quotient map is just the map $\Sigma q_I$. Therefore the right-hand square commutes up to homotopy.

The left-hand and right-hand vertical maps being homotopy equivalences, $h_{m+1}$ induces an isomorphism of homology groups. All spaces being simply connected CW-complexes, $h_{m+1}$ is a homotopy equivalence. This completes the induction. □

The following well-known fact will be useful.

**Lemma 2.7.** If $Y$ is a co-$H$-space, and $\psi: Y \to Y \vee Y$ its comultiplication, then $\Sigma \psi$ is homotopic to the pinch map $\Sigma Y \xrightarrow{\text{pinch}} \Sigma Y \vee \Sigma Y$.

**Proof.** Let $H$ and $H'$ be the homotopies to the identity of the composites $Y \xrightarrow{\psi} Y \vee Y \xrightarrow{1 \vee \ast} Y$ and $Y \xrightarrow{\psi} Y \vee Y \xrightarrow{\ast \vee \ast} Y$. The composite $Y \xrightarrow{\Delta} Y \vee Y \xrightarrow{\iota} Y \times Y$, where $\iota$ is the canonical inclusion, is homotopic to the diagonal map $Y \xrightarrow{\Delta} Y \times Y$ via the homotopy $G_I(y) = (H_I(y), H_I'(y))$. Take the composite

$$f: \Sigma(Y \times Y) \xrightarrow{\text{pinch}} \Sigma(Y \times Y) \vee \Sigma(Y \times Y) \to \Sigma Y \vee \Sigma Y$$

where the last map is the wedge sum of the suspended projection maps $\Sigma(Y \times Y) \to \Sigma Y$ onto the first and second factor. The composite $f$ is a left homotopy inverse of $\Sigma \iota$. Therefore $\Sigma Y \xrightarrow{\Sigma \psi} \Sigma(Y \vee Y) \xrightarrow{\Sigma \Delta} \Sigma(Y \times Y) \xrightarrow{\iota} \Sigma Y \vee \Sigma Y$ is homotopic to $\Sigma \psi$, and $\Sigma Y \xrightarrow{\Sigma \Delta} \Sigma(Y \times Y) \xrightarrow{\iota} \Sigma Y \vee \Sigma Y$ is the pinch map $\Sigma Y \xrightarrow{\text{pinch}} \Sigma Y \vee \Sigma Y$. These two composites are homotopic since $\Sigma \iota \circ \Sigma \psi = \Sigma(\iota \circ \psi)$ is homotopic to $\Sigma \Delta$. □

**Lemma 2.8.** Assuming condition (A), $W_k^\ell$ is simply connected for $\ell \geq 2$ and each $k \leq n$. Moreover, if $W$ is connected, then $W_k^1$ is also simply connected.

**Proof.** $X^\ell = \Sigma X_1 \times \cdots \times \Sigma X_n$ inherits a basepointed CW-structure from $X$ simply by shifting the dimension of each cell in $X$ up by $\ell$. Then the same thing happens when we coordinate suspend the subcomplex $W_k$ of $X$ to obtain the subcomplex $(W_k)^\ell = W_k^\ell$ of $X^\ell$. Thus $W_k^\ell$ is simply-connected when $\ell \geq 2$ since it has no 1-cells. Given $W$ is connected, since $W$ is a pointed CW-complex, its
1-skeleton has a single point with 1 cells attached to it (i.e. a bouquet of circles at the basepoint), so \(W_k^\ell\) has no 1-cells here as well.

**Proposition 2.9.** Assuming condition (A), let \(\ell \geq 0\) and \(1 \leq k \leq n\), and consider the following:

(a) \(W_k^\ell\) is a co-H-space;
(b) there is a splitting

\[
W_k^\ell \cong \bigvee_{0 \leq j \leq k} \hat{W}_j^\ell \cong \bigvee_{I \subseteq [n], |I| \leq k} \hat{W}_j^\ell \cong \bigvee_{I \subseteq [n], |I| \leq k} \Sigma I \hat{W}_I;
\]

(c) the quotient map \(W_m^\ell \to \hat{W}_m^\ell\) has a right homotopy inverse for each \(m \leq k\);
(d) \(W_I^\ell\) is a simply connected co-H-space for each \(I \subseteq [n]\) with \(|I| \leq k\);
(e) the quotient map \(W_I^\ell \to \hat{W}_I^\ell\) has a right homotopy inverse for each \(I \subseteq [n]\) with \(|I| \leq k\).

Then (a) \(\iff\) (b) \(\iff\) (c) \(\iff\) (d) when \(\ell \geq 2\), or when \(\ell = 1\) and \(W\) is connected. Generally, (a) \(\implies\) (b) provided \(W_j^\ell\) is simply connected for \(j \leq k\).

**Proof.** For \(\ell \geq 2\) or \(\ell = 1\) and \(W\) connected, our hypotheses ensure that all spaces are simply connected via Lemma 2.8. The implication (b) \(\implies\) (a) is clear when \(\ell \geq 1\). To see that (a) \(\implies\) (b) holds, for \(m \leq k\), consider the composite

\[
h_m^\ell : W_m^\ell \to W_k^\ell \to \bigvee_{I \subseteq [n], |I| \leq m} W_k^\ell \xrightarrow{q_I} \bigvee_{I \subseteq [n], |I| \leq m} \hat{W}_I^\ell,
\]

where the first map is the inclusion and \(\psi^\ell\) is given by iterating the comultiplication \(W_k^\ell \to W_k^\ell \vee W_k^\ell\) in some order. By Lemma 2.7, \(\Sigma \psi^\ell\) is homotopic to the iterated pinch map, so by naturality the composite of \(\Sigma \psi^\ell\) with the suspended inclusion \(\Sigma W_m^\ell \to \Sigma W_k^\ell\) is an iterated pinch map. \(\Sigma h_m^\ell\) is then homotopic to the homotopy equivalence \(h_m^\ell\) in the proof of Proposition 2.6, meaning \(h_m^\ell\) induces isomorphisms of homology groups, and since this is a map between simply-connected CW-complexes, \(h_m^\ell\) is a homotopy equivalence.

Suppose (b) holds. From the commutativity of the right square in diagram (6), we may take as a right homotopy inverse of the suspended quotient map \(\Sigma W_m^\ell \to \Sigma \hat{W}_m^\ell\) the composite

\[
f : \Sigma \hat{W}_m^\ell \xrightarrow{\Sigma h_m^\ell} \bigvee_{I \subseteq [n], |I| \leq m} \Sigma W_I^\ell \to \bigvee_{I \subseteq [n], |I| \leq m} \Sigma \hat{W}_I^\ell
\]

where the second map is the inclusion. Since \(h_m^\ell \cong \Sigma h_m^\ell\) and \(h_m^\ell\) is a homotopy equivalence, we may take \(h_m^{-1} = (h_m^\ell)^{-1}\). All the maps in this composite being suspensions, \(f\) desuspends to a map \(f^\ell\) such that the composite \(\hat{W}_m^\ell \xrightarrow{f^\ell} W_m^\ell \to \hat{W}_m^\ell\) induces isomorphisms on homology groups. Since each of the summands in the splitting of \(\hat{W}_m^\ell\) are summands in the splitting of \(\Sigma W_m^\ell\), this last composite must be homotopic to the identity. Thus (b) \(\implies\) (c) holds.

Since each of the cofibrations \(W_m^\ell \to W_m^{\ell-1}\) for \(m \leq k\) trivializes when there is right homotopy inverse of \(W_m^\ell \to W_m^{\ell-1}\), (c) \(\implies\) (b) holds. Thus, we have shown (a) \(\iff\) (b) \(\iff\) (c).

The restriction \(W_k^\ell \to W_I^\ell\) of the projection map \(X^\ell \to X_I^\ell\) to \(W_k^\ell\) has a right inverse \(W_I^\ell \to W_k^\ell\) whenever \(|I| \leq k\), given by restricting the inclusion \(X_I^\ell \to X^\ell\) to \(W_I^\ell\). Thus (a) \(\implies\) (d).

Suppose (d) holds. By (a) \(\implies\) (c) with \(W_I^\ell\) in place of \(W_k^\ell\), the quotient map \(W_I^\ell \to \hat{W}_I^\ell\) has a right homotopy inverse. Thus (d) \(\implies\) (e).
Suppose (e) holds. For any \(|I| = m\) we have a commutative square

\[
\begin{array}{ccc}
W_I^\ell & \longrightarrow & \hat{W}_I^\ell \\
\downarrow & & \downarrow \\
W_m^\ell & \longrightarrow & \hat{W}_m^\ell \\
& & \bigvee_{J \in [n], |J| = m} \hat{W}_j^\ell 
\end{array}
\]

with the vertical maps the inclusions, the right-hand one being the inclusion into the summand \(\hat{W}_I^\ell\). Therefore the wedge sum of maps \(\hat{W}_I^\ell \to W_I^\ell \to W_m^\ell\) over all \(|I| = m\), where \(s_I\) is our given right homotopy inverse, is a right homotopy inverse of \(W_m^\ell \to \hat{W}_m^\ell\). Thus (e) ⇒ (e).

\[\square\]

3. Configuration Spaces

Let \(\text{map}(A,B;W^\ell)\) be the space of maps \(A \to W^\ell\) that map \(B \subseteq A\) to the basepoint. Equivalently, this is the space of basepointed maps \(A/B \to W^\ell\), which is the space of unbased (or free) maps \(A \to W^\ell\) when \(B\) is empty.

Fix \(W\) to be a subspace of a product \(X = X_1 \times \cdots \times X_n\) as in the previous section. Our goal is to prove the following theorem for \(C(M,N;W)\) a certain subspace of the infinite symmetric product \(SP((M/N) \wedge W^\ell)\) defined in the next section.

**Theorem 3.1.** Let each \(X_i\) be a connected basepointed CW-complex and \(W\) a subcomplex of their product \(X\). Suppose \(M\) is a smooth compact parallelizable \(\ell\)-manifold, \(\ell > 0\), \(N\) a submanifold, \(M/N\) is connected or \(W\) is basepoint connected. Then there exists a homotopy equivalence

\[
\gamma: C(M,N;W) \to \text{map}(Z - N, Z - M; W^\ell),
\]

where \(Z = M \cup (\partial M \times [0,1])\).

This generalizes McDuff’s and Bödghemer’s \[21, 6\] configuration space models of mapping spaces to suspensions, originally studied by Boardman, Vogt, May, Giffen, and Segal in the case of spheres. Several other generalisations have appeared since then. Some of these bear certain similarities to ours in the sense that particles are allowed to intersect under various rules, or exist in much greater generality (see \[21, 25, 10, 11\] for example).

3.1. Configuration Spaces with Collisions. Let \(M\) be any path connected space, \(N \subseteq M\) a subspace, and \(Y\) a basepointed space with basepoint \(*\). Let \(D_0(M,N;Y) = *\) and take the quotient space

\[
D_k(M,N;Y) = \coprod_{i=0}^k M^{\times i} \times Y^{\times i} / \sim
\]

where the equivalence relation \(\sim\) is given by

\[
(z_1, \ldots, z_i; x_1, \ldots, x_i) \sim (z_{\sigma(1)}, \ldots, z_{\sigma(i)}; x_{\sigma(1)}, \ldots, x_{\sigma(i)})
\]

for any permutation \(\sigma\) in the symmetric group \(\Sigma_i\), and

\[
(z_1, \ldots, z_i; x_1, \ldots, x_i) \sim (z_1, \ldots, z_{i-1}; x_1, \ldots, x_{i-1})
\]

whenever \(x_i = *\) or \(z_i \in N\). Then

\[
D(M,N;Y) = \bigcup_{k=0}^\infty D_k(M,N;Y).
\]
Notice $D(M, N; Y)$ is just the infinite symmetric product $SP((M/N) \wedge Y)$. We may think of $D(M, N; Y)$ as the space of (possibly colliding) particles in $M$ with labels in $Y$ that are annihilated in $N$. We thus refer to points in $D(M, N; Y)$ as configurations of particles in $M$ with labels in $Y$, and by this we usually mean, without loss of generality, that the particles in such a configuration are non-degenerate, or in other words, that they are outside $N$ and their labels are not the basepoint.

The empty configuration is vacuously non-degenerate. Since order does not matter, it will sometimes be convenient to think of a configuration $(z_1, \ldots, z_i; x_1, \ldots, x_i)$ as a set $\{(z_1, x_1), \ldots, (z_i, x_i)\}$ where each of the pairs $(z_i, x_i)$ are considered distinct regardless of the values of $z_i$ or $x_i$.

**Definition 3.2.** A multiset of points $S = \{(x_1, \ldots, x_s)\} \subset \prod_i X_i$ is represented by a point in $W$ if and only if there exists a $(\bar{x}_1, \ldots, \bar{x}_n) \in W$ and some injective function $f_S : \{1, \ldots, s\} \rightarrow \{1, \ldots, n\}$ such that $x_i = \bar{x}_{f_S(i)}$.

Notice that the multiset $S$ is not represented by a point if it contains two or more elements in the same $X_i$ for some $i$.

Let $X^\nu = X_1 \lor \cdots \lor X_n$ be the wedge at basepoints $* \in X_i$. Define $C(M, N; W)$ to be the subspace of $D(M, N; X^\nu)$ given by the following rule: a configuration of non-degenerate particles $(z_1, \ldots, z_{s_k}; x_1, \ldots, x_{s_k}) \in D(M, N; X^\nu)$ is in $C(M, N; W)$ if and only if

- for each increasing integer sequence $1 \leq i_1 < \cdots < i_k \leq k$, if $z_{i_1} = \cdots = z_{i_k}$, then the multiset of labels $\{(x_{i_1}, \ldots, x_{i_k})\}$ is represented by some point in $W$.

Note the condition $z_{i_1} = \cdots = z_{i_k}$ is vacuously true when $s = 1$, thus the label of each particle is an element of $W_{(i)} \subseteq X_i$ for some $i$. We filter $C(M, N; W)$ by the subspaces

$$C_k(M, N; W) = C(M, N; W) \cap D_k(M, N; X^\nu)$$

having no more than $k$ non-degenerate particles.

**Remark 3.3.** Let $W$ be the space of sub-multisets of $\prod_i X_i$ that are represented by a point in $W$. This is a partial monoid with partial multiplication given by taking a union of two multisets, and defined whenever a union is also represented. One can then take the configuration space of particles in $M$ vanishing in $N$ with summable labels in $W$, where colliding particles are identified and their labels summed using the partial multiplication (c.f. [25]). This configuration space is clearly homeomorphic to $C(M, N; W)$, but has a canonical filtration different from the one given above, which is not the one we are looking for.

**Example 3.4.** If $W = X_1 \times \cdots \times X_n$, then the space $C(M, N; W)$ is homeomorphic to the product of classical (without collisions) labelled configuration spaces $C(M, N; X_1) \times \cdots \times C(M, N; X_n)$. If $W = X_1 \lor \cdots \lor X_n$, then $C(M, N; W) = C(M, N; X_1 \lor \cdots \lor X_n)$, that is, a configuration space with labels without collisions.

When referring to collided particles in a labelled configuration of particles in $M$, we mean a subset of all particles in that configuration sharing the same point in $M$. We can think of these configuration spaces as being like classical configuration spaces with labels in $X^\nu$, but with certain collisions allowed governed by the way $W$ sits inside its ambient space $X$. Here a single particle is considered to collide with itself, and at most $n$ particles can collide. These configuration spaces satisfy most of the familiar properties of classical labelled configuration spaces.

**Proposition 3.5.** The following hold:

(i) **Excision:** Given an open subspace $U \subseteq N$ of $M$ such that the closure of $U$ is also contained in $N$, the inclusion $C(M - U, N - U; W) \rightarrow C(M, N; W)$ is a homeomorphism.
(ii) **Quasifibrations:** If $N$ is submanifold of $M$, $L$ is a codimension 0 submanifold of $M$, and $\partial L/(\partial L \cap N)$ is connected or $W$ is basepoint connected, then the maps of pairs $(L, L \cap N) \to (M, N)$ induce a quasifibration sequence

$$\mathcal{C}(L, L \cap N; W) \to \mathcal{C}(M, N; W) \xrightarrow{q} \mathcal{C}(M, L \cup N; W).$$

**Proof.** Property (i) is clear. We prove property (ii) by following the proof for classical labelled configuration spaces in [23][4]. Let $B_k = C_k(M, L \cup N; W)$, $F = \mathcal{C}(L, L \cap N; W)$, and filter $\mathcal{C}(M, N; W)$ by $E_k = q^{-1}(B_k)$. The induction step is to assume the restriction $q|_{E_{i-1}}: E_{i-1} \to B_{i-1}$ of $q$ is a quasifibration. There is a map

$$g_i: (E_i - E_{i-1}) \to (B_i - B_{i-1}) \times F$$

defined by sending a configuration in $E_i$ with $i$ non-degenerate particles outside $L \cup N$ and some $i'$ particles inside $L - N$ to the corresponding pair of $i$ and $i'$ particle configurations in $B_i - B_{i-1}$ and $F$, respectively. This is a homeomorphism, with inverse defined as expected. Thus the restriction

$$(E_i - E_{i-1}) \xrightarrow{g_i} (B_i - B_{i-1}) \times F \xrightarrow{1 \times F} (B_i - B_{i-1})$$

of $q$ to $(E_i - E_{i-1})$ is a (trivial) quasifibration with fibre $F$. Using an open tubular neighbourhood $(-1,1) \times \partial L$ of $\partial L$ in $M$, take the open neighbourhood $U = (-1,1) \times \partial L \cup L$ of $L$, with the collars $(-1,0) \times \partial L$ and $[0,1) \times \partial L$ outside and inside $L$. Use this to define the neighbourhood $V_i$ of $B_{i-1}$ in $B_i$ as the subspace of configurations in $B_i$ where at least one particle in a configuration of $i$ non-degenerate particles is inside $U$. Consider the deformation $R$ of $U$ onto $L$ defined by compressing $(-1,1) \times \partial L$ onto the inner collar $[0,1) \times \partial L$ injectively, with the inner collar deformed onto $[\frac{1}{2},1) \times \partial L$, and the outer collar homotoped onto $C = (0, \frac{1}{2}) \times \partial L$. This deformation extends to an isotopy of $M$ (assuming $U$ has been taken small enough), thus induces a deformation retraction $\mathcal{R}$ of $V_i$ onto $B_{i-1}$, and in turn a deformation $\mathcal{R}$ of $q^{-1}(V_i)$ into $q^{-1}(B_{i-1}) = E_{i-1}$ such that $\mathcal{R}$ covers $\mathcal{R}$. For any non-degenerate configuration $\omega \in V_i$, there are homeomorphisms $q^{-1}(\omega) \cong F$ and $q^{-1}(\mathcal{R}_1(\omega)) \cong F$ defined by forgetting those particles in $\omega$ and $\mathcal{R}_1(\omega)$ respectively (which must lie outside $L$), and the self-map

$$f: F \xrightarrow{\cong} q^{-1}(\omega) \xrightarrow{f'} q^{-1}(\mathcal{R}_1(\omega)) \xrightarrow{\cong} F$$

with $f'$ defined by mapping a configuration $\alpha \mapsto \mathcal{R}_1(\alpha)$, is a homotopy equivalence when $\partial L/(\partial L \cap N)$ is connected or $W$ is basepoint connected. To see this last point, notice that after performing the deformation $\mathcal{R}$ on any configuration $\alpha \in q^{-1}(\omega)$, all particles from $\mathcal{R}_1(\alpha) \in q^{-1}(\mathcal{R}_1(\omega))$ that lie in $C$ are those particles $(R_1(z), x) \in \mathcal{R}_1(\omega)$ such that $z \in U - (L \cup N)$. Thus, since these particles only depend on $\omega$, when $\partial L/(\partial L \cap N)$ is connected we can pick some fixed paths lying in $C$ from each these particles to $\partial L \cap N$, and homotope them along these paths to annihilate them without ever colliding with the other particles in $\mathcal{R}_1(\omega)$ (those not lying in $C$). On the other hand, when $W$ is basepoint connected, we can simply homotope their labels $x$ to the basepoint without moving the particles. This is done for each subset of collided particles in $C$ by picking a point $y \in W$ that represents them, and then homotoping the labels of each collided particle by restricting to the appropriate coordinate a homotopy of $y$ to the basepoint. In either situation, once we have annihilated these particles lying in $C$, we apply the reverse of the deformation $R$ to the remaining particles. This defines a homotopy of the self-map $f$ to the identity.

The restriction of $q^{-1}(V_i) \to V_i$ to $E_{i-1}$ is a quasifibration by our induction hypothesis, and so by the above, $q$ satisfies the one of the Dold-Thom criteria [12] for being a quasifibration. Also, the restriction of the trivial quasifibration $(E_i - E_{i-1}) \to (B_i - B_{i-1})$ to $q^{-1}(V_i) \cap (E_i - E_{i-1}) \to$
$U_i \cap (B_i - B_{i-1})$ is also a quasifibration since it also trivialises. It follows that $E_i \xrightarrow{q} B_i$ is a quasifibration for any $i$ using the Dold-Thom criteria, and therefore so is $E \xrightarrow{q} B$. \hfill \square

3.2. Section Spaces. Fix $Z$ to be a smooth $\ell$-manifold without boundary, and $M \subseteq Z$ a smooth compact codimension 0 submanifold. Let $T(Z)$ be the spherical bundle obtained from the unit $\ell$-disk tangent bundle of $Z$ by fiberwise collapsing the boundaries of the disks to a basepoint at infinity. This is an $S^\ell$-bundle over $Z$. Consider the pullback

$$
\begin{array}{ccc}
T_\Delta(Z) & \longrightarrow & \prod^n T(Z) \\
\downarrow & & \downarrow \\
Z & \overset{\Delta}{\longrightarrow} & \prod^n Z
\end{array}
$$

of the product bundle projection and diagonal map $\Delta$. Thus $T_\Delta(Z) \longrightarrow Z$ is the pullback bundle of the product bundle by $\Delta$, with fibre $\prod^n S^\ell$. Take our subspace $W$ of the product $X = X_1 \times \cdots \times X_n$, and construct the $W^\ell$-bundle $\mathcal{T}(Z;W) \xrightarrow{\pi} Z$ by applying the coordinate smash $(\prod^n S^\ell) \wedge_X W$ to the fibre $\prod^n S^\ell$ of $T_\Delta(Z)$. Let $\kappa_\infty; Z \longrightarrow \mathcal{T}(Z;W)$ be the section of $\pi$ that sends a point in $z \in Z$ to the basepoint $*_z$ at infinity on the fibre at $z$, and for any subspaces $B \subseteq A \subseteq Z$, let $\mathcal{Y}(Z;A,B;W)$ denote the space of sections of $\pi$ that are defined on $A$, and agree with $\kappa_\infty$ on $B$, which we denote by $\mathcal{Y}(A,B;W)$ when $Z$ is fixed. They satisfy familiar properties:

(i) **Excision:** If $U \subseteq B$ is open as a subspace of $A$ such that the closure of $U$ in $A$ is contained in $B$, then the inclusion $\mathcal{Y}(Z;A,B;W) \longrightarrow \mathcal{Y}(Z;A-U,B-U;W) \cong \mathcal{Y}(Z-U;A-U,B-U;W)$ is a homeomorphism.

(ii) **Fibrations:** Given $B' \subseteq A$, with $B'/((B' \cap B) \longrightarrow A/B$ satisfying the homotopy extension property, the maps of pairs $(B',B' \cap B) \longrightarrow (A,B) \longrightarrow (A,B' \cup B)$ induce a fibration sequence

$$
\mathcal{Y}(A,B' \cup B;W) \longrightarrow \mathcal{Y}(A,B;W) \longrightarrow \mathcal{Y}(B',B' \cap B;W).
$$

One can identify these spaces with mapping spaces whenever $Z$ is parallelizable. Here the bundle $T(Z)$ is trivial, thus $T_\Delta(Z)$ and $\mathcal{T}(Z;W)$ are trivial, and $\mathcal{Y}(A,B;W)$ becomes the space of sections of the projection map $W^\ell \times Z \longrightarrow Z$ defined on $A$ and mapping on $B$ so that the first coordinate is the basepoint in $W^\ell$. This is precisely the mapping space map($A,B;W^\ell$), and the above fibration sequence becomes the mapping space fibration sequence

$$
\text{map}(A,B' \cup B;W^\ell) \longrightarrow \text{map}(A,B;W^\ell) \longrightarrow \text{map}(B',B' \cap B;W^\ell)
$$

induced by the cofibration sequence $B'/((B' \cap B) \longrightarrow A/B \longrightarrow A/(B' \cup B)$.

3.3. The Scanning Map. Recall that $M$ is a smooth compact $\ell$-manifold, $\ell > 0$. Fix $N$ to be a submanifold of $M$ that is closed as a subspace of $M$, and from now assume that each $X_i$ is a basepointed $CW$-complex and $W$ is a subcomplex of their product $X$. This implies that $W^\ell$ is a connected subcomplex of $X^\ell$. Our goal is to construct the scanning map $\mathcal{C}(M,N;W) \xrightarrow{\gamma} \mathcal{Y}(Z-N,Z-M;W)$, where $Z = M \cup (\partial M \times [0,1])$, and show that it is a weak homotopy equivalence whenever $M/N$ is connected or $W$ is basepoint connected. When $M$ is parallelizable, this is a weak equivalence $\mathcal{C}(M,N;W) \xrightarrow{\gamma} \text{map}(Z-N,Z-M;W^\ell)$ between spaces having the homotopy type of connected $CW$-complexes. Theorem 3.1 follows.

Given a length function $\delta: M \times M \longrightarrow \mathbb{R}$ induced by a Riemannian metric, define the measure of variance of a collection of points $z_1,\ldots,z_s \in M$ by $\text{var}(z_1,\ldots,z_s) = 0$ if $s = 1$, and

$$
\text{var}(z_1,\ldots,z_s) = \frac{1}{s(s-1)} \sum_{i \neq j} \delta(z_i,z_j)
$$
if \( s > 1 \). For any \( \varepsilon > 0 \), let \( C^\varepsilon(M, N; W) \) be the subspace of \( C(M, N; W) \) given by the following rule: a configuration \((z_1, \ldots, z_k; x_1, \ldots, x_k) \in C(M, N; W)\) of non-degenerate particles is in \( C^\varepsilon(M, N; W) \) if and only if

- for each increasing integer sequence \( 1 \leq i_1 < \cdots < i_s \leq k \), if \( \{x_{i_1}, \ldots, x_{i_s}\} \) is not represented by a point in \( W \) then \( \text{var}(z_{i_1}, \ldots, z_{i_s}) \geq \varepsilon \).

This holds vacuously for the empty configuration. Let \( C^\mathbb{R}(M, N; W) \) be the following subspace of \( C(M, N; W) \times \mathbb{R} \):

\[
C^\mathbb{R}(M, N; W) = \bigcup_{0 < \varepsilon < \infty} C^\varepsilon(M, N; W) \times \{\varepsilon\}.
\]

**Lemma 3.6.** The composite

\[
C^\mathbb{R}(M, N; W) \xrightarrow{\iota} C(M, N; W) \times \mathbb{R} \xrightarrow{\text{proj}} C(M, N; W)
\]

of the inclusion and the projection is a weak homotopy equivalence.

**Proof.** This follows from the property that any compact subset of \( C(M, N; W) \) is contained in a subspace \( C^\varepsilon(M, N; W) \) for \( \varepsilon \) sufficiently small. \( \square \)

Consider the inclusion \( \iota: W^\varepsilon \rightarrow C(D^\varepsilon, \partial D^\varepsilon; W) \) sending \((z_1, x_1), \ldots, (z_n, x_n)\) to the configuration \((z_1, \ldots, z_n; x_1, \ldots, x_n)\). Notice that a configuration of non-degenerate particles is in the image of \( \iota \) if and only if the multiset of labels of the particles is represented by a point in \( W \).

**Lemma 3.7.** The inclusion \( W^\varepsilon \rightarrow C(D^\varepsilon, \partial D^\varepsilon; W) \) is a homotopy equivalence.

**Proof.** Let us denote \( C = C(D^\varepsilon, \partial D^\varepsilon; W) \). Think of the \( \varepsilon \)-disk \( D^\varepsilon \) as the closed unit-neighbourhood of the origin in \( \mathbb{R}^\varepsilon \), and consider the radial expansion or contraction \( R_t: D^\varepsilon \rightarrow D^\varepsilon \) given by

\[
R_t(x) = \begin{cases} tx & \text{if } |tx| < 1; \\ \frac{tx}{|tx|} & \text{if } |tx| \geq 1 \end{cases}
\]

for any \( t > 0 \). This defines a map \( \bar{R}_t: C \rightarrow C \) that radially expands or contracts a configuration. Let \( C^\varepsilon = C^\varepsilon(D^\varepsilon, \partial D^\varepsilon; W) \) and \( C^\mathbb{R} = C^\mathbb{R}(D^\varepsilon, \partial D^\varepsilon; W) \) using the length function \( \delta(x, y) = |x-y| \). The map \( C^\mathbb{R} \rightarrow C \) is a weak homotopy equivalence by Lemma 3.6. Each inclusion \( C^\varepsilon \times \{\varepsilon\} \xrightarrow{\iota^\varepsilon} C^\mathbb{R} \) is a homotopy equivalence, with homotopy inverse the retraction map \( r_\varepsilon \) that restricts to \( \bar{R}_\varepsilon \) on \( C^\mathbb{R} \times \{\delta\} \), and a homotopy to the identity of \( \iota^\varepsilon \circ r_\varepsilon \) given by mapping \( C^\delta \times \{\delta\} \) to \( C^{\delta + (1-t)\varepsilon} \) via \( \bar{R}_{\varepsilon} \) at each time \( t \), where \( \delta_t = t\delta + (1-t)\varepsilon \). Therefore the inclusion \( \iota^\varepsilon: C^\varepsilon \rightarrow C \) is also weak homotopy equivalence.

Clearly \( W^\varepsilon \subseteq C^\varepsilon \) for any \( \varepsilon > 0 \). Moreover, \( C^\varepsilon \subseteq W^\varepsilon \) for \( \varepsilon \geq 2 \). To see this, notice the variance of any collection of particles in the interior of \( D^\varepsilon \) is less than 2, so given \( \varepsilon \geq 2 \), any configuration of non-degenerate particles in \( C^\varepsilon \) has its multiset of labels represented by a point in \( W \), thus is under the image of \( \iota \). Therefore \( C^\varepsilon = W^\varepsilon \), \( \iota^\varepsilon = \iota \) for \( \varepsilon \geq 2 \), and \( \iota \) is a weak homotopy equivalence. Since both spaces have the homotopy type of CW-complexes, the map \( \iota \) is a homotopy equivalence. \( \square \)

Let \( \delta: Z \times Z \rightarrow \mathbb{R} \) be the length function induced by a Riemannian metric on \( Z \). Given any \( z \in Z \) and subset \( S \subseteq Z \), let \( \delta(z, S) = \inf \{\delta(z, y) \mid y \in S\} \in \mathbb{R} \cup \{\infty\} \) be the smallest distance between \( z \) and the subset \( S \). Define \( \delta(z, S) = \infty \) when \( S = \emptyset \), and let \( S_\varepsilon = \{y \in Z \mid \delta(y, S) \leq \varepsilon\} \) and \( D_{\varepsilon, z} = \{y \in Z \mid \delta(y, z) \leq \varepsilon\} \) be the closed \( \varepsilon \)-neighbourhood of \( S \) and \( z \) in \( Z \), respectively. We take \( \varepsilon' > 0 \) small enough so that the following hold for all \( 0 < \varepsilon < \varepsilon' \):

(i) \((\partial M)_\varepsilon \) is the tubular neighbourhood homeomorphic to \( \partial M \times [-1,1] \), so \( M_\varepsilon \) is homeomorphic to \( M \cup (\partial M \times [0,1]) \);
(ii) for each $z \in M$, the exponential map induces a homeomorphism between the unit $\ell$-disk in the tangent space at $z$ and $D_{\varepsilon,z}$;
(iii) $N_{\varepsilon}$ is a union of closed tubular neighbourhoods of $\text{int}(N)$ and $\partial N$. Each point in $N_{\varepsilon} - N$ lies on a unique geodesic from a unique nearest point in $N$. Thus $N_{\varepsilon}$ deformation retracts onto $N$.

The homeomorphism $M_{\varepsilon} - N_{\varepsilon} \longrightarrow M - N$ given induces a homeomorphism $\Upsilon(\varepsilon) = \Upsilon(Z - N_{\varepsilon}, Z - M_{\varepsilon}; W) \longrightarrow \Upsilon(Z - N, Z - M; W)$. The scanning map

$$\gamma: C(M,N;W) \longrightarrow \Upsilon(Z - N_{\varepsilon}, Z - M_{\varepsilon}; W) \cong \Upsilon(Z - N, Z - M; W)$$

is defined as follows. For any configuration $\omega \in C(M,N;W)$, $\gamma(\omega)$ is the section $Z - N_{\varepsilon} \longrightarrow T(Z;W)$ defined on any $z \in Z - N_{\varepsilon}$ as the image of $\omega$ under the composite

$$C(M,N;W) \longrightarrow C(D_{\varepsilon,z}, \partial D_{\varepsilon,z}; W) \longrightarrow C(D^\ell, \partial D^\ell; W) \longrightarrow W^\ell \longrightarrow T(Z;W)$$

where the first map forgets those particles from $\omega$ that are in $M - D_{\varepsilon,z}$, the second homeomorphism is induced by the homeomorphism $D_{\varepsilon,z} \cong D^\ell$ using the exponential map, $r$ is a left homotopy inverse of $\ell$ from Lemma 3.7, and the last map is the inclusion of the fibre at $z$. The same naturality properties hold for this map as do for the scanning map for classical labelled configuration spaces. For example, if $L \subseteq M$ is a codimension 0 submanifold, $L \cap N$ and $L \cup N$ are submanifolds of $L$ and $M$, respectively, and $\partial L((\partial L \cap N)$ is connected or $W$ basepoint connected, then the sequence of inclusions $(L,L \cap N) \longrightarrow (M,N) \longrightarrow (M,L \cup N)$ induce a morphism of quasifibrations

$$C(L,L \cap N;W) \longrightarrow C(M,N;W) \longrightarrow C(M,L \cup N;W) \gamma$$

$\Upsilon(Z - (L \cap N), Z - L; W) \longrightarrow \Upsilon(Z - N, Z - M; W) \longrightarrow \Upsilon(Z - (L \cap N), Z - M; W)$$

where $\varepsilon$ is taken small enough so that each of the three vertical maps $\gamma$ exist.

With regards to labels, $C(M,N;\cdot)$ and $\Upsilon(Z - N, Z - M; \cdot)$ are functors from the category of subspaces of length $n$ products of basepointed CW-complexes, and coordinate-wise maps between them. More precisely, given $Y = Y_1 \times \cdots \times Y_n$ a product of basepointed CW-complexes, basepoint preserving maps $f_i: X_i \longrightarrow Y_i$, and a subspace $V \subseteq Y$, suppose the coordinate-wise map $\tilde{f} = (f_1 \times \cdots \times f_n): X \longrightarrow Y$ restricts to a map $f: W \longrightarrow V$. This induces a map $f^\ell: W^\ell \longrightarrow V^\ell$ that sends $((z_1,x_1),\ldots,(z_n, x_n)) \mapsto ((z_1,f(x_1)), \ldots, (z_n,f(x_n)))$, and maps

$$C(f): C(M,N;W) \longrightarrow C(M,V) \gamma$$

$\Upsilon(f): \Upsilon(Z - N, Z - M; W) \longrightarrow \Upsilon(Z - N, Z - M; V)$

$C(f)$ mapping $(z_1,\ldots,z_n; x_1,\ldots, x_n) \mapsto (z_1,\ldots,z_n; f(x_1),\ldots, f(x_n))$, and $\Upsilon(f)$ induced by the map $T(f): T(Z;W) \longrightarrow T(Z;V)$ that restricts to $f^\ell$ on each fibre. The scanning map $\gamma$ is then a natural transformation in the homotopy category.

**Lemma 3.8.** Using the notation above, there is a homotopy commutative square

$$\begin{array}{ccc}
C(M,N;W) & \xrightarrow{c(f)} & C(M,N;V) \\
\downarrow{\gamma} & & \downarrow{\gamma} \\
\Upsilon(Z - N, Z - M; W) & \xrightarrow{\Upsilon(f)} & \Upsilon(Z - N, Z - M; V)
\end{array}$$
Proof. The first two maps in composite (4) are natural for any \( \omega \in \mathcal{C}(M, N; W) \) and \( z \in Z - N_z \), as is the last map (i.e. \( T(f) \circ \tau_z = \tau_z \circ f^\ell \)), while the map \( r \) is only up to homotopy: since the following square commutes
\[
\begin{array}{ccc}
W^\ell & \xrightarrow{f^\ell} & V^\ell \\
\downarrow & & \downarrow \\
\mathcal{C}(D^\ell, \partial D^\ell; W) & \xrightarrow{C(f)} & \mathcal{C}(D^\ell, \partial D^\ell; V),
\end{array}
\]
we may replace the homotopy equivalences \( \iota \) with their homotopy inverse \( r \) to obtain a homotopy commutative square. Then since the homotopy \( r \circ C(f) \simeq f^\ell \circ r \) is independent of \( z \) and \( \omega \), it defines a homotopy of sections \( \gamma(C(f)(\omega)) \simeq T(f)(\gamma(\omega)) \), and in turn a homotopy \( \gamma \circ C(f) \simeq \Upsilon(f) \circ \gamma \). □

3.4. Proof of Theorem 3.1. We start by proving the base case: that
\[
\gamma : \mathcal{C}(D^\ell, \partial D^\ell; W) \to \Upsilon(Z - \partial D^\ell, Z - D^\ell; W)
\]
is a weak homotopy equivalence for any smooth \( \ell \)-manifold \( Z \) without boundary that contains \( D^\ell \) as a submanifold. Using excision and the fact that \( D^\ell - \partial D^\ell \) is parallelizable,
\[
\Upsilon(Z - \partial D^\ell, Z - D^\ell; W) \cong \Upsilon(D^\ell - \partial D^\ell, \varnothing; W)
\cong \text{map}(D^\ell - \partial D^\ell, \varnothing; W^\ell)
\cong W^\ell.
\]
One can check that the composite
\[
W^\ell \xrightarrow{\gamma} \mathcal{C}(D^\ell, \partial D^\ell; W) \xrightarrow{\gamma} \Upsilon(Z - \partial D^\ell, Z - D^\ell; W) \xrightarrow{\gamma} \text{map}(D^\ell - \partial D^\ell, \varnothing; W^\ell)
\]
is a homotopy equivalence, and the inclusion \( \iota \) is a homotopy equivalence by Lemma 3.7. Therefore \( \gamma \) is a homotopy equivalence.

The induction for the general case is identical to the classical case in [24], and is taken mostly verbatim from [6]. At each step \( \mathcal{C}(M, N; W) \to \Upsilon(Z - N, Z - M; W) \) is shown to be a weak homotopy equivalence for progressively more general \((M, N)\).

(1) Let \( (H_k, \bar{H}_k) = (D^k \times D^{\ell-k}, D^k \times S^{\ell-k-1}) \) denote a handle of index \( k \) and dimension \( \ell \). We first prove the case where \((M, N) = (H_k, \bar{H}_k)\) is a handle of index \( k \leq \ell \) by inducting on the index. We have already dealt with the base case \( k = 0 \). Assume the assertion holds for all handles of index less than \( k \). Take the following subspaces of \( D^\ell = [0, 1]^{\ell} \):
\[
H'_k = [0, 1]^{\ell-k} \times [0, 1]^k \quad I_k = \left\{ (t_1, \ldots, t_\ell) \in D^\ell \mid t_k \in \left[ \frac{3}{4}, 1 \right] \text{ or } t_i \in [0, \frac{1}{4}] \cup \left[ \frac{3}{4}, 1 \right] \text{ for some } i \in \{k+1, \ldots, \ell\} \right\}.
\]
Take the sequence of maps of pairs \( (H'_k, H'_k \cap I_k) \to (D^\ell, I_k) \to (D^\ell, H'_k \cup I_k) \), and consider the morphism of quasifibrations in diagram (8) applied to this sequence of pairs. The left-hand pair is obtained from handle of index \( k \) and the right-hand pair from a handle of index \( k - 1 \) after thickening the boundary components \( \bar{H}_k \) and \( \bar{H}_{k-1} \), which has no effect on the homeomorphism type of the corresponding configuration and section spaces. Notice \( \mathcal{C}(D^\ell, I_k; W) \) is contractible by pushing particles in the direction towards the subspace \( \{(t_1, \ldots, t_\ell) \in I_k \mid t_k = 1\} \) to be annihilated. Similarly \( \Upsilon(Z - I_k, Z - D^\ell; W) \) is contractible by contracting a map away from this subspace towards the opposite end (analogously to contracting a path in a path space away from the endpoint towards the basepoint). Thus the middle vertical \( \gamma \) is a homotopy equivalence, and so is the right-hand \( \gamma \) by induction.
Therefore, if \( k < \ell \), the assertion holds for the left-hand \( \gamma \). Since \( I_k \) is empty when \( k = \ell \), \( \partial H'_k/(\partial H'_k \cap I_k) = \partial \Omega D'/\emptyset \) is not connected, meaning we need to assume \( W \) is basepoint connected. Under this additional assumption the induction extends to the handle \((H'_k, H'_k \cap I_k) = (D', \emptyset)\) of index \( \ell \).

(2) For the case where \((M, N) = (M, \partial M)\), we pick a handle decomposition of \( M \), and if \( M/\partial M \) is connected but \( W \) is not basepoint connected, we ensure that this decomposition is in terms of handles of index less than \( \ell \). This gives a sequence

\[
(H_k, H_k \cap \partial M) \to (M, \partial M) \to (M, H_k \cup \partial M)
\]

where \((H_k, H_k \cap \partial M) = (H_k, \bar{H}_k)\) is the last handle attached to the boundary of a submanifold \( M' \) to form \( M \), \((M, H_k \cup \partial M)\) becomes the pair \((M', M')\) after excising \( H_k - \bar{H}_k \), and we consider the morphism of quasifibrations induced by this sequence. Inducting on the number of handles in a decomposition and applying excision and the previous case completes this case.

(3) Now consider the pairs \((M, N)\) where \( N \subseteq \partial M \). Let \( L = \partial M - int(N) \), so \( L \cap N = \partial N = \partial L \), and let \( \bar{M} = M \cup (\partial M \times [0,1]) \). Take the sequence

\[
(\bar{L}, \bar{L} \cap \bar{N}) \to (\bar{M}, \bar{N}) \to (\bar{M}, \bar{L} \cup \bar{N})
\]

where \( \bar{N} = N \times [0,1] \) and \( \bar{L} = L \times [0,1] \). Notice that if \( M/N \) is connected, then \( L/\partial L \) is connected, and therefore \( \partial \bar{L}/(\partial \bar{L} \cap \bar{N}) = \partial \bar{L}/(\partial L \times [0,1]) \) is connected. The assertion holds for the pair \((\bar{M}, \bar{L} \cup \bar{N}) = (\bar{M}, \partial \bar{M})\) by step (2). Then just as before, the assertion will follow for \((\bar{M}, \bar{N})\), and therefore for \((M, N)\) if we can show it for the left-hand pair \((\bar{L}, \bar{L} \cap \bar{N}) = (L \times [0,1], \partial L \times [0,1])\), or equivalently for the pair \((L \times [0,1], T \times [0,1])\) where \( T \) is a collar neighbourhood of \( \partial L \) in \( L \). To do this, let \( \bar{L} = (T \times [0,2]) \cup (L \times [\frac{3}{2}, 2]) \subset L \times [0,2] \), and take the sequence

\[
(L \times [0,1], T \times [0,1]) \to (L \times [0,2], \bar{L}) \to (L \times [0,2], (L \times [0,1]) \cup \bar{L})
\]

Since \( \partial \bar{L}/(\partial L \times [0,1]) \) is connected when \( M/N \) is, then

\[
\partial(L \times [0,1])/(\partial(L \times [0,1]) \cap \bar{L}) = \partial \bar{L}/(\partial L \times [0,1] \cup T \times \{0,1\})
\]

is connected as well. The configuration space and section space for the middle pair is easily seen to be contractible, and the assertion holds for the right-hand pair by step (2). Thus it holds for the left-hand pair.

(4) For the most general case where \( N \) is an arbitrary submanifold of \( M \), pick a small enough \( \varepsilon \)-neighbourhood \( N_\varepsilon \) of \( N \) as in the previous subsection. The assertion holds for the pair \((M - int(N_\varepsilon), N_\varepsilon - int(N_\varepsilon))\) by step (3), and since the respective configuration and section spaces for this pair are homeomorphic to those for \((M, N)\), it also holds for \((M, N)\).

4. Suspension Splittings

We assume from now on that each \( X_i \) is a connected basepointed CW-complex, and \( W \) is a basepoint connected subcomplex of their product \( X \). We denote \( \mathcal{C}(M, \emptyset; W) \) by \( \mathcal{C}(M; W) \), in particular, \( \mathcal{C}(W) \) denotes \( \mathcal{C}(\mathbb{R}; W) \), and \( \mathcal{C}_i(W) \) the subspace \( \mathcal{C}_i(\mathbb{R}; W) \). The scanning map gives a homotopy equivalence \( \gamma: \mathcal{C}(D^1; W) \to map(D^1, \partial D^1; W^1) = \Omega W^1 \) when \( M = D^1 = [-1, 1] \), \( N = \emptyset \), and \( Z = (-2, 2) \). Our goal is to obtain a (single) suspension splitting of \( \Omega W^\ell \) for \( \ell \geq 1 \) by giving a suspension splitting of \( \mathcal{C}(D^1; W) \approx \mathcal{C}(W) \). Take

\[
V_k(W) = \bigvee_{i=1}^k \frac{C_i(W)}{C_{i-1}(W)}
\]
and consider that there exists a map
\[ \zeta : \mathcal{C}(W) \to C(\mathbb{R}; V_\infty(W)), \]
where the right-hand space is the classical configuration space of non-overlapping particles in \( \mathbb{R} \) with labels in \( V_\infty(W) \). We define this map as follows. Since \( X_1 \vee \cdots \vee X_n \) wedged at the basepoint \( * \) is a \( CW \)-complex, we may pick a map
\[ f : X_1 \vee \cdots \vee X_n \to [0, 1] \]
such that \( f(*) = 0 \) and \( f(x) > 0 \) when \( x \neq * \). Let \( \zeta_0 : \mathcal{C}_0(W) \to C(\mathbb{R}; V_\infty(W)) \) be given by mapping the empty configuration to the empty configuration, and define
\[ \zeta_k : \mathcal{C}_k(W) - \mathcal{C}_{k-1}(W) \to C(\mathbb{R}; V_\infty(W)) \]
for \( k > 0 \) as follows. Notice that any configuration \( \omega = \{(t_1, x_1), \ldots, (t_k, x_k)\} \in \mathcal{C}_k(W) - \mathcal{C}_{k-1}(W) \) has only non-degenerate particles, that is, the label \( x_i \) of each particle in \( \omega \) is not the basepoint \( * \). Since any two non-degenerate particles in a configuration in \( \mathcal{C}(W) \) cannot collide when their labels are in the same summand \( X_i \) (so \( t_i \neq t_j \) when \( x_i, x_j \in X_i - \{*\} \)), there is a strict total order on \( \omega \) given for any \( y = (t_i, x_i), y' = (t_j, x_j) \in \omega \) by the relation
- \( y < y' \) if and only if either \( x_i \in X_{i'} - \{*\} \) and \( x_j \in X_{j'} - \{*\} \), or else both \( x_i, x_j \in X_l - \{*\} \) for some \( l \) and \( t_i < t_j \).

This then defines a strict total order on the set \( 2^\omega \) of subsets of \( \omega \), given for any \( S, S' \in \omega \) by the relation
- \( S < S' \) if and only if there exists a \( y' \in S' \) such that for every \( y \in S \), either \( y < y' \), or else \( y' < y \) and \( y \in S' \).

This total order does not change as we move these particles and vary their labels in such a way that \( \omega \) moves along any path in \( \mathcal{C}_k(W) - \mathcal{C}_{k-1}(W) \), and if we take a subconfiguration \( \omega' \subset \omega \) in \( \mathcal{C}_{k'}(W) - \mathcal{C}_{k'-1}(W) \), \( k' < k \), then the ordering on \( 2^{\omega'} \subset 2^\omega \) is the one inherited from \( 2^\omega \). Write \( S \preceq S' \) if either \( S < S' \) or \( S = S' \), and let
\[ \eta_{\omega, S'} = \sum_{S \preceq S'} \left( \prod_{(t_i, x_i) \in S} f(x_i) \right). \]

Since each \( x_i \neq * \), we have \( f(x_i) > 0 \), so \( \eta_{\omega, S} < \eta_{\omega, S'} \) whenever \( S \preceq S' \). Then we may define
\[ \zeta_k(\omega) = \{ (\eta_{\omega, S}, \chi_S) \mid S \preceq \omega \} \in C(\mathbb{R}; V_\infty(W)) \]
where the label \( \chi_S \in V_\infty(W) \) is the image of the configuration \( S \in \mathcal{C}_{|S|}(W) \) under the composite of the quotient map and inclusion
\[ \mathcal{C}_{|S|}(W) \to \mathcal{C}_{|S|}(W)/\mathcal{C}_{|S|-1}(W) \xrightarrow{\text{inclusion}} V_\infty(W) \]
(\( S \preceq \omega \) is indeed a configuration in \( \mathcal{C}_{|S|}(W) \), since every subset of a represented multiset of labels is itself represented by a point in \( W \)). Notice that when \( \omega \) approaches some configuration \( \omega' \in \mathcal{C}_{k'}(W) - \mathcal{C}_{k'-1}(W) \) for some \( k' < k \), that is, when some subset \( L \) of at least \( k - k' \) labels \( x_i \) from \( \omega \) approaches the basepoint, then \( \prod_{y \in (z, t_j) \in S} f(x_i) \) approaches 0 and \( \chi_S \) approaches \( * \in V_\infty(W) \) for all \( S \preceq \omega \) that contain a particle with a label in \( L \). Therefore, we see that \( \zeta_k(\omega) \) approaches \( \zeta_k'(\omega') \). Thus, we can define \( \zeta \) by restricting to \( \zeta_k \) on \( \mathcal{C}_k(W) - \mathcal{C}_{k-1}(W) \), for any \( k > 0 \), and \( \zeta_0 \) on the empty configuration.
Notice $\zeta_k$ maps to the subspace $C(\mathbb{R}; V_k(W)) \subseteq C(\mathbb{R}; V_\infty(W))$. Restricting $\zeta$ to the subspaces $C_i(W)$, there is a commutative diagram

$$
\begin{array}{ccc}
C_{i-1}(W) & \longrightarrow & C_i(W) \\
\zeta & & \zeta \\
C(\mathbb{R}; V_{i-1}(W)) & \longrightarrow & C(\mathbb{R}; V_i(W)) \\
& & \longrightarrow C(\mathbb{R}; (C_i(W)/C_{i-1}(W)))
\end{array}
$$

with the left-hand and right-hand horizontal maps the inclusions and quotient maps respectively, and the bottom maps induced by the inclusion $V_{i-1}(W) \rightarrow V_i(W)$ and the quotient map $V_i(W) \rightarrow C_i(W)/C_{i-1}(W)$ onto the last summand. Since the composite of these two maps is the constant map, the composite of the maps in the bottom sequence is the constant map, which gives the right-hand vertical extension. Notice this extension sends a (quotiented) configuration $\omega$ to a configuration with a single particle, whose label is $\omega$.

Recall from the theory of classical labelled configuration spaces \cite{26} that there exists a homotopy equivalence

$$\lambda: C(\mathbb{R}; V_\infty(W)) \rightarrow \Omega \Sigma V_\infty(W).$$

This is also just a special case of Theorem \ref{thm:main} one takes $\Sigma V_\infty(W)$ as the coordinate suspension of $V_\infty(W)$ on a single coordinate, meaning $C(\mathbb{R}; V_\infty(W)) = C(\mathbb{R}; V_\infty(W))$, and $\lambda$ is the composite $C(\mathbb{R}; V_\infty(W)) \xrightarrow{\gamma} C(D^1; V_\infty(W)) \xrightarrow{\gamma} \Omega \Sigma V_\infty(W)$ of our scanning map $\gamma$ and the homotopy equivalence induced by the map $\mathbb{R} \rightarrow D^1$ sending $x \mapsto \frac{x}{\pi}$. Now let

$$\zeta': \Sigma C(W) \rightarrow \Sigma V_\infty(W)$$

be the adjoint of the composite $\lambda \circ \zeta$. Composing the vertical maps in Diagram \eqref{eq:commutative_diagram} with the corresponding scanning maps, and taking the adjoints of these, we obtain a commutative diagram

$$
\begin{array}{ccc}
\Sigma C_{i-1}(W) & \longrightarrow & \Sigma C_i(W) \\
\zeta' & & \zeta' \\
\Sigma V_{i-1}(W) & \longrightarrow & \Sigma V_i(W) \\
& & \longrightarrow \Sigma (C_i(W)/C_{i-1}(W))
\end{array}
$$

the left and middle vertical maps being the restrictions of \eqref{eq:commutative_diagram}. The right-hand vertical extension can be homotoped to the identity using a homotopy to the identity of the composite $\Sigma V_\infty(W) \rightarrow C(D^1, \partial D^1; V_\infty(W)) \rightarrow \Sigma V_\infty(W)$ of the inclusion from Lemma \ref{lem:suspension_inclusion} and its left homotopy inverse used to define the scanning map. The bottom sequence is a cofibration sequence, and since $W$ is a CW-complex, the top sequence is a cofibration since it satisfies the homotopy extension property. The middle vertical $\zeta'$ is then a homotopy equivalence whenever the left-hand $\zeta'$ is. Thus using an induction argument we see that $\Sigma C(W) \xrightarrow{\zeta'} \Sigma V_\infty(W)$ is a homotopy equivalence. $W^1$ is also a CW-complex, implying that $C(W) \rightarrow \Omega W^1$ is a homotopy equivalence, and

$$\Sigma W^1 \simeq \Sigma C(W) \simeq \bigvee_{i \geq 1} \frac{C_i(W)}{C_{i-1}(W)}.$$ 

We can take $W^\ell$ in place of $W = W^0$ and $W^{\ell+1} = (W^\ell)^1$ in place of $W^1$, and obtain the splitting for $\Sigma W^{\ell+1}$ in terms of summands $\Sigma (C_i(W^\ell)/C_{i-1}(W^\ell))$. These summands take a more familiar form.

**Lemma 4.1.** There is a homeomorphism

$$\frac{C_i(W^\ell)}{C_{i-1}(W^\ell)} \cong \Sigma^l \left( \frac{C_i(W)}{C_{i-1}(W)} \right).$$
Proof. Given \( \omega = (t_1, \ldots, t_i; (s_1, x_1), \ldots, (s_i, x_i)) \in \mathcal{C}_i(W^\ell) / \mathcal{C}_{i-1}(W^\ell) \), \( s_j \in D^\ell \), the homeomorphism is given by mapping \( \omega \) to the point \( ((s_1, \ldots, s_i), (t_1, \ldots, t_i; x_1, \ldots, x_i)) \), where \( (s_1, \ldots, s_i) \in D^\ell \). \( \square \)

In summary, we have the following splitting.

**Proposition 4.2.** If each \( X_i \) is a connected basepointed CW-complex, and \( W \) is a basepoint connected subcomplex of their product \( X \), then for \( \ell \geq 0 \),

\[
\Sigma \Omega W^{\ell+1} = \Sigma \mathcal{C}(W^\ell) = \bigvee_{i \geq 1} \frac{\mathcal{C}_i(W)}{\mathcal{C}_{i-1}(W)}.
\]

\( \square \)

4.1. **The Summands in the Splitting.** We continue to assume that each \( X_i \) is a connected basepointed CW-complex, and \( W \) is a basepoint connected subcomplex of their product \( X \). Then recalling the filtration \( * = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n = W \), where

\[
W_k = \bigcup_{I \subseteq [n], |I| = k} W_I,
\]

we see that each \( W_k \) is subcomplex of \( W \), \( W_1 \) is simply connected by Lemma 2.8 and that they satisfy the following property:

- \( B \) If \( x_1, \ldots, x_n \in W \), then \( (x_1, \ldots, x_{i-1}, *, x_{i+1}, \ldots, x_k) \in W \) for each \( x_i \). Equivalent, for each \( m < k \) a set \( \{x_1, \ldots, x_m\} \) is represented by a point in \( W_k \) if and only it is represented by a point in \( W_m \). This property holds with \( W^\ell \) in place of \( W \) when \( \ell \geq 1 \), without any extra assumptions on \( W \).

Again \( \mathcal{C}_i(W_k) \) denotes \( \mathcal{C}_i(\mathbb{R}; W_k) \), where these configuration spaces are constructed by thinking of \( W_k \) as a subspace of \( X \). Take the quotient space

\[
\mathcal{D}_i(W_k) = \frac{\mathcal{C}_i(W_k)}{\mathcal{C}_{i-1}(W_k)}.
\]

We still refer to elements in \( \mathcal{D}_i(W_k) \) as configurations of labelled particles in the sense that they are under the image of the quotient map \( \mathcal{C}_i(W_k) \rightarrow \mathcal{D}_i(W_k) \). They must have exactly \( i \) particles, and are identified with the empty configuration when one of them is a degenerate particle, that is, when its label is a basepoint. The basepoint is taken to be the empty configuration.

We call a sequence \( S = (s_1, \ldots, s_n) \) an \((m, n)\)-partition of a non-negative integer \( m \) if \( s_1 + \cdots + s_n = m \) and each \( s_i \geq 0 \). Given such a sequence, let \( \mathcal{D}_S(W_k) \) be the subspace of \( \mathcal{D}_m(W_k) \) of configurations \( \omega \) where exactly \( s_i \) non-degenerate particles in \( \omega \) have labels in \( X_i \) for each \( 1 \leq i \leq n \), or else \( \omega \) is the empty configuration. To homotope the label of a particle in a configuration in \( \mathcal{D}_m(W_k) \) from one summand \( X_j \) to another summand \( X_i \), one must go through the basepoint, and therefore the empty configuration in \( \mathcal{D}_m(W_k) \). Thus

\[
\mathcal{D}_m(W_k) \cong \bigvee_{(m, n)\text{-partitions } S} \mathcal{D}_S(W_k)
\]

wedged at the empty configuration, and the inclusion

\[
\iota_{m,k}: \mathcal{D}_m(W_{k-1}) \rightarrow \mathcal{D}_m(W_k)
\]

that extends \( \iota_{m,k}: \mathcal{C}_m(W_{k-1}) \rightarrow \mathcal{C}_m(W_k) \) (in turn, induced by the inclusion \( W_{k-1} \rightarrow W_k \)) restricts to an inclusion

\[
\iota_{S,k}: \mathcal{D}_S(W_{k-1}) \rightarrow \mathcal{D}_S(W_k)
\]

for each \((m, n)\)-partition \( S \).
Lemma 4.3. The inclusion \( \iota_{S,k} \) is a homeomorphism when at most \( k-1 \) elements in \( S \) are nonzero. In particular, \( \iota_{m,k} \) is a homeomorphism when \( m \leq k-1 \).

Proof. Any non-degenerate configuration \( \omega \in \mathcal{D}_S(W_k) \) has \( m \) particles when \( S \) is an \((m,n)\)-partition, but since collisions are not allowed between particles with labels in the same \( X_i \), no more than \( k-1 \) can collide at a point when at most \( k-1 \) elements in \( S \) are nonzero. Since the labels of any \( j \leq k-1 \) particles in \( \omega \) colliding at a point are represented by a point in \( W_k \), by property (B) they are also represented by a point in \( W_j \subseteq W_{k-1} \). Therefore \( \omega \in \mathcal{D}_S(W_{k-1}) \subseteq \mathcal{D}_S(W_k) \). \( \square \)

Given a basepoint preserving map \( f: X \to Y \) of non-degenerate basepointed spaces, let \( M(f) = ([0,1] \times X)/([0,1] \times \{\ast\}) \cup_f Y \) be the reduced mapping cylinder of \( f \), and \( C(f) = M(f)/\{1\} \times X \) the reduced homotopy cofiber of \( f \). We write points in \( C(f) \) as pairs \((t, y)\), where \( t \in [0,1] \), \( y \in Y \) if \( t = 0 \), \( y \in X \) if \( t > 0 \), and \((t, y)\) is the basepoint \(* \) when \( t = 1 \) or \( y \) is the basepoint.

Take the \((n,n)\)-partition \( \mathcal{A} = \{1, \ldots, 1\} \). Then \( \mathcal{D}_\mathcal{A}(W_k) \) is the subspace of \( \mathcal{D}_n(W_k) \) containing those configurations of \( n \) particles whose labels are each in a distinct \( X_i \). Our first goal will be to prove the following.

Proposition 4.4. There exists a homotopy commutative diagram

\[
\begin{array}{cccccc}
\mathbb{C}(\iota_{\mathcal{A},n}) & \xrightarrow{\partial} & \mathbb{C}(\iota_{\mathcal{A},n-1}) & \xrightarrow{\Sigma \iota_{\mathcal{A},n}} & \mathbb{C}(\iota_{\mathcal{A},n}) \\
\Sigma \mathbb{C}(\iota_{\mathcal{A},n}) & \xrightarrow{\Sigma \iota_{\mathcal{A},n}} & \Sigma \mathbb{C}(\iota_{\mathcal{A},n-1}) & \xrightarrow{\Sigma \iota_{\mathcal{A},n}} & \Sigma \mathbb{C}(\iota_{\mathcal{A},n}) \\
\Sigma \mathbb{C}(\iota_{\mathcal{A},n-1}) & \xrightarrow{\Sigma \iota_{\mathcal{A},n}} & \Sigma \mathbb{C}(\iota_{\mathcal{A},n-1}) & \xrightarrow{\Sigma \iota_{\mathcal{A},n}} & \Sigma \mathbb{C}(\iota_{\mathcal{A},n}) \\
\end{array}
\]

where consecutive maps in each row form homotopy cofiber sequences, the right-hand vertical homotopy equivalences are extensions induced by the homotopy commutativity of the middle squares. The top second and third vertical maps are homotopy inverses of the quotient maps coming from the splitting in Proposition 4.2, and the bottom ones are given as the composites \( \Sigma \mathbb{C}(\iota_{\mathcal{A},n}) \xrightarrow{\Sigma \gamma} \Sigma \mathcal{W}_1^{1, \text{ev}} \xrightarrow{\text{ev}} W_1^{1, \text{ev}} \) of the suspended scanning map restricted to \( \Sigma \mathbb{C}(\iota_{\mathcal{A},n}) \) and the evaluation map \( \text{ev} \). The map \( \partial \) is the connecting map, given by collapsing the subspace \( \mathcal{D}_\mathcal{A}(W_n) \) of \( \mathbb{C}(\iota_{\mathcal{A},n}) \).

This is meant to reduce the study of the filtration of \( W_1 = W_1 \) by the spaces \( W_k \) to the corresponding filtration for \( \mathcal{D}_\mathcal{A}(W_n) \), as indicated in the following corollaries.

Corollary 4.5. Considering the diagram in Proposition 4.4, the following are equivalent:

(a) \( W_1 \xrightarrow{f} \hat{W}_1 \) has a right homotopy inverse;
(b) \( \Sigma \mathcal{D}_\mathcal{A}(W_n) \xrightarrow{\Sigma \iota_{\mathcal{A},n}} \mathbb{C}(\iota_{\mathcal{A},n}) \xrightarrow{\Sigma \iota_{\mathcal{A},n}} \Sigma \mathbb{C}(\iota_{\mathcal{A},n}) \) has a right homotopy inverse;
(c) \( \mathbb{C}(\iota_{\mathcal{A},n}) \xrightarrow{\partial} \Sigma \mathcal{D}_\mathcal{A}(W_{n-1}) \) is null-homotopic.

Proof. The implication (b) \( \Rightarrow \) (a) holds by Proposition 4.3 while (c) \( \Rightarrow \) (b) is a standard property of homotopy cofibration sequences.

Suppose (b) holds. Since \( \Sigma \mathcal{D}_\mathcal{A}(W_n) \) is homotopy equivalent to a \( CW \)-complex, the homotopy cofibration sequence

\[
\Sigma \mathcal{D}_\mathcal{A}(W_{n-1}) \xrightarrow{\Sigma \iota_{\mathcal{A},n}} \Sigma \mathcal{D}_\mathcal{A}(W_n) \xrightarrow{\Sigma \iota_{\mathcal{A},n-1}} \Sigma \mathbb{C}(\iota_{\mathcal{A},n})
\]
trivializes. In particular \( \Sigma_{\ell, A, n} \) has a left homotopy inverse \( i \). Since \( \mathbb{C}(\ell, A, n) \xrightarrow{\delta} \Sigma D_A(W_{n-1}) \xrightarrow{\partial} \Sigma A, n \)
\( \Sigma D_A(W_n) \) is a homotopy cofibration sequence, \( \Sigma_{\ell, A, n} \circ \partial \) is null-homotopic, so \( (i \circ \Sigma_{\ell, A, n}) \circ \partial \) is null-homotopic. Thus \( (b) \Rightarrow (c) \).

Suppose \( (a) \) holds. Then there is a left homotopy inverse \( \varrho \) of the inclusion \( W_{n-1}^1 \rightarrow W_n^1 \), since \( \hat{W}_{n}^1 \) is its cofibre. Looping and suspending these maps, and applying Lemma 3.8 and Proposition 4.2, we have a homotopy commutative diagram

\[
\begin{array}{ccc}
\Sigma V_{\infty}(W_{n-1}) & \xrightarrow{\zeta} & \Sigma V_{\infty}(W_n) \\
\Sigma \mathcal{C}(W_{n-1}) & \xrightarrow{\Sigma \gamma} & \Sigma \mathcal{C}(W_n) \\
\Sigma \Omega W_{n-1}^1 & \xrightarrow{\Sigma \gamma} & \Sigma \Omega W_n^1,
\end{array}
\]

with \( \Sigma \Omega \varrho \) a left homotopy inverse of the bottom horizontal inclusion. Since the vertical maps are homotopy equivalences, the top horizontal map has a left homotopy inverse. This map breaks up as a wedge sum of the maps \( \Sigma_{\ell, m, n} \), which in turn break up as a wedge sum of maps \( \Sigma_{\ell, S, n} \) for \((m, n)-\)partitions \( S \). Then \( \Sigma_{\ell, A, n} \) has a left homotopy inverse, and since \( \Sigma D_A(W_n) \) is co-\( H \)-space, the homotopy cofibration \( (14) \) trivializes, so \( (b) \) holds. Thus \( (a) \Rightarrow (b) \).

\[\Box\]

Corollary 4.6. \( W_n^1 \) is a co-\( H \)-space if and only if any one of the equivalent statements in Corollary 3.9 hold for \( W_I \) in place of \( W \) for each \( I \subseteq [n] \).

\[\Box\]

Proof. Combine Corollary 4.5 with Proposition 2.9.

4.2. Proof of Proposition 4.4. The homotopy inverse of \( i \) in Lemma 3.7 that we used to define the scanning \( \gamma \) was not given explicitly. The presence of \( \gamma \) in Proposition 4.1 will therefore present some difficulties when trying to prove it. A more direct construction is possible when we fix \( \varepsilon > 0 \) and restrict to the subspace

\[ C_m^\varepsilon(W_k) = C_m^\varepsilon(\mathbb{R}; W) \cap C_m(W_k), \]

where the length function \( \delta \) is given by \( \delta(x, y) = |x - y| \). Thus, a configuration of non-degenerate particles \( (t_1, \ldots, t_m; x_1, \ldots, x_m) \in C_m(W_k) \) is in \( C_m^\varepsilon(W_k) \) if and only if

- for each sequence \( 1 \leq i_1 < \cdots < i_s \leq m \), if \( \{\{x_{i_1}, \ldots, x_{i_s}\}\} \) is not represented by a point in \( W = W_n \), then \( \text{var}(t_{i_1}, \ldots, t_{i_s}) \geq \varepsilon \).

Any compact subset of \( C_m(W_k) \) is in \( C_m^\varepsilon(W_k) \) for \( \varepsilon \) small enough. Then using an argument similar to the one in the proof of Lemmas 3.6 and 3.7, we see that the inclusion \( C_m^\varepsilon(W_k) \rightarrow C_m(W_k) \) is a homotopy equivalence (here in place of \( R_t \) using the map \( R_t: \mathbb{R} \rightarrow \mathbb{R} \) given by \( R_t(x) = tx \)).

Let

\[ D_m^\varepsilon(W_k) = \frac{C_m^\varepsilon(W_k)}{C_{m-1}^\varepsilon(W_k)}, \]

and consider the commutative diagrams of (homotopy) cofibration sequences

\[
\begin{array}{ccc}
C_m(W_{k-1}) & \xrightarrow{i_m^\varepsilon} & C_m(W_k) & \xrightarrow{}\xrightarrow{D_m^\varepsilon(W_k)} & D_m(W_k) \\
\xrightarrow{C(W_{k-1})} & \xrightarrow{C(W_k)} & \xrightarrow{D_m(W_k)},
\end{array}
\]

\[ (14) \]
\[
\begin{align*}
\mathcal{D}^\varepsilon_m(W_{k-1}) & \xrightarrow{\varepsilon_m,k} \mathcal{D}^\varepsilon_m(W_k) \xrightarrow{} \mathcal{C}(\iota_{m,k}^\varepsilon) \\
\mathcal{D}^\varepsilon_m(W_{k-1}) & \xrightarrow{\varepsilon_m,k} \mathcal{D}^\varepsilon_m(W_k) \xrightarrow{} \mathcal{C}(\iota_{m,k})
\end{align*}
\]

After suspending \([12]\), the right vertical inclusion is a homotopy equivalence since the left and middle inclusions are, then so is the right vertical inclusion in \([13]\). The homotopy equivalence \(\mathcal{D}_m(W_k) \rightarrow \mathcal{D}_m(W_k)\) breaks up as a wedge sum of inclusions \(\mathcal{D}_m^\varepsilon(W_k) \rightarrow \mathcal{D}_m(W_k)\) for \((m,n)\)-partitions \(S\), where \(\mathcal{D}_m^\varepsilon(W_k) = \mathcal{D}_m^\varepsilon(W_k) \cap \mathcal{D}_m(W_k)\). So each of these is a homotopy equivalence. Since \(\iota_{m,k}\) is as a wedge sum of \(\iota_{S,k}\) over \((m,n)\)-partitions \(S\),

\[
\mathcal{C}(\iota_{m,k}) \cong \bigvee_{(m,n)\text{-partitions} \, S} \mathcal{C}(\iota_{S,k}),
\]

and the restriction \(i_{m,k}^\varepsilon\) of \(\iota_{m,k}\) is a wedge sum of maps

\[
i_{S,k}^\varepsilon: \mathcal{D}_m^\varepsilon(W_{k-1}) \rightarrow \mathcal{D}_m^\varepsilon(W_k)
\]

that restrict \(\iota_{S,k}\). Thus

\[
\mathcal{C}(i_{m,k}^\varepsilon) \cong \bigvee_{(m,n)\text{-partitions} \, S} \mathcal{C}(i_{S,k}^\varepsilon),
\]

and the homotopy equivalence \(\mathcal{C}(i_{m,k}^\varepsilon) \rightarrow \mathcal{C}(i_{m,k})\) is as a wedge sum of homotopy equivalences \(\mathcal{C}(i_{S,k}^\varepsilon) \rightarrow \mathcal{C}(i_{S,k})\).

Take a configuration \(\omega = (t_1,\ldots,t_n;x_1,\ldots,x_n) \in \mathcal{D}^\varepsilon(A(W_n))\). Configurations in \(\mathcal{D}^\varepsilon\) have particles whose labels are each in a distinct \(X_i\), so we may assume that \(x_i \in X_i\). Observe that configurations in \(\mathcal{D}^\varepsilon\) have the property where any subset of particles contained in an open interval of length \(\varepsilon\) in \(\mathbb{R}\) has its multiset of labels represented by a point in \(W_n\). Then for each \(0 \leq s \leq \frac{\varepsilon}{2}\) define

\[
q_s: \mathcal{D}^\varepsilon(A(W_n)) \rightarrow \Sigma^{n-1}W_n
\]

by

\[
q_s(\omega) = \begin{cases} 
((\tilde{t}_1,\ldots,\tilde{t}_{n-1}),(x_1,\ldots,x_n)) & \text{if } s > 0; \\
* & \text{if } s = 0,
\end{cases}
\]

where

\[
\tilde{t}_i = \begin{cases} 
\frac{t_i-t_n}{s} & \text{if } t_i \in (t_n-s,t_n+s) \\
-1 & \text{if } t_i \leq t_n-s; \\
1 & \text{if } t_n+s \leq t_i.
\end{cases}
\]

The maps \(q_s\) do not vary continuously with respect to \(s\) near \(s = 0\). But since no more than \(n-1\) non-degenerate particles can collide in any configuration \(\omega \in \mathcal{D}^\varepsilon(A(W_n))\), \(q_s\) maps \(\omega\) to the basepoint for all \(s\) sufficiently small (depending on \(\omega\)), so \(q_s\) does vary continuously when restricted to the subspace \(\mathcal{D}^\varepsilon(A(W_{n-1}))\). We then define

\[
\varphi: \mathcal{C}(i_{A,n}^\varepsilon) \rightarrow \Sigma^{n-1}W_n
\]

to be the extension of \(q_s\) defined by \(\varphi(s',\omega) = q_{(1-s')/s}(\omega)\).

There is an inclusion \(\mathcal{W}_n \rightarrow \mathcal{D}(A(W_n))\) given by mapping \((x_1,\ldots,x_n) \rightarrow (0,\ldots,0;x_1,\ldots,x_n)\). We can extend this to an inclusion

\[
\phi: \Sigma^{n-1}W_n \rightarrow \mathcal{C}(i_{A,n}^\varepsilon)
\]
by mapping
\[(t_1,\ldots,t_{n-1},(x_1,\ldots,x_n)) \mapsto (\max\{|t_1|,\ldots,|t_{n-1}|,0\},(t_1,\ldots,t_{n-1},0;x_1,\ldots,x_n)),\]
where each parameter \(t_i\) is in the unit 1-disk \(D^1 = [-1,1]\) by property (B), \((t_1,\ldots,t_{n-1},0;x_1,\ldots,x_n)\) is in \(D^1(W_{n-1})\) when \(\max\{|t_1|,\ldots,|t_{n-1}|,0\} > 0\).

**Lemma 4.7.** The inclusion \(\phi\) is a homotopy equivalence, and \(\varphi\) a choice of homotopy inverse. Thus, the composite
\[\phi^0: \Sigma^{n-1}W_n \xrightarrow{\phi} C(t_{A,n}) \xrightarrow{\cong} C(t_{A,n})\]
is a homotopy equivalence.

**Proof.** When \(n = 1\), \(\Sigma^{n-1}W_n = W_n = W_n, C(t_{A,n}) = D_{A}(W_n) \cong \mathbb{R} \times W_n\), \(\phi\) is the inclusion \(W_n \rightarrow \mathbb{R} \times W_n \rightarrow W_n\).

Assume \(n \geq 2\). Take \(\sigma = ((t_1,\ldots,t_{n-1},(x_1,\ldots,x_n)) \in \Sigma^{n-1}W_n\), let \(\beta = \max\{|t_1|,\ldots,|t_{n-1}|,0\}\) and \(s = (1-\beta)(\bar{\xi})\). Then \(\varphi \circ \phi\) maps \(\sigma \mapsto *\) when \(\beta = 1\), and maps \((\bar{t}_1,\ldots,\bar{t}_{n-1},(x_1,\ldots,x_n))\) when \(\beta < 1\), where \(\bar{t}_i\) is defined in (13) with respect to \(s\) and \(t_n = 0\). Define a homotopy \(H\) of \(\varphi \circ \phi\) to the identity by
\[H_\delta(\sigma) = \begin{cases} ((t_\delta,1,\ldots,t_{\delta,n-1},(x_1,\ldots,x_n)) & \text{if } \beta < 1; \\ * & \text{if } \beta = 1, \end{cases}\]
where \(t_{\delta,i} = \delta t_i + (1-\delta)\bar{t}_i\) for \(0 \leq \delta \leq 1\). We must show that \(H\) is continuous. Any possible discontinuity happens when \(\beta\) approaches 1 (\(s\) approaches 0), in other words, when \((t_1,\ldots,t_{n-1})\) approaches the boundary of \(D^{n-1} = [-1,1]^{n-1}\). To see that there is no issue here, notice as \(t_i\) approaches 1, \(t_i\) is eventually bounded below by \(t_n + s = s\) (which approaches 0), so \(\bar{t}_i\) also approaches 1, and then so does \(t_{\delta,i}\) for each \(\delta\). Similarly, if \(t_i\) approaches \(-1\), then \(t_{\delta,i}\) approaches \(-1\). In either case \(H_\delta(\sigma)\) approaches *.

Take a point \((s',\omega) \in C(t_{A,n})\) for some configuration \(\omega = (t_1,\ldots,t_n;x_1,\ldots,x_n) \in D_{A}(W_n)\), and any \(s' \in [0,1]\). Let \(s = (1-s')(\bar{\xi})\). Then \(\phi \circ \phi\) maps \((s',\omega)\) to * when \(s' = 1\), and maps \((s',\omega)\) to \((\bar{\beta},\bar{\omega})\) when \(s' < 1\), where \(\bar{\omega} = (\bar{t}_1,\ldots,\bar{t}_{n-1},0;x_1,\ldots,x_n) \in D_{A}(W_n), \bar{\beta} = \max\{\bar{t}_1,\ldots,\bar{t}_{n-1},0\}\), and each \(\bar{t}_i\) is as defined in (13) with respect to \(s\). Notice that the following holds:

- if \(\{x_1,\ldots,x_n\}\) is not represented by a point in \(W_n\), then \(\{t_1,\ldots,t_n\}\) is not a subset of the interval \((t_n-s,t_n+s) \subseteq (t_n-\bar{\xi},t_n+\bar{\xi})\), therefore \(\bar{\beta} = 1\) and \((\bar{\beta},\bar{\omega})\) is the basepoint in \(C(t_{A,n})\).

Since collisions are allowed between particles with represented labels, property (b) in effect means that we do not have to worry about collisions when homotoping \((\bar{\beta},\bar{\omega})\) by moving particles in \(\bar{\omega}\) along the real line. Thus, we define a homotopy \(G\) of \(\phi \circ \phi\) to the identity by
\[G_\delta(s',\omega) = \begin{cases} (s_\delta, (t_\delta,1,\ldots,t_{\delta,n};x_1,\ldots,x_n)) & \text{if } s' < 1; \\ * & \text{if } s' = 1, \end{cases}\]
where \(s_\delta = \delta s' + (1-\delta)\bar{\beta}\) and \(t_{\delta,i} = \delta t_i + (1-\delta)\bar{t}_i\) (let \(\bar{t}_n = 0\)). \(G\) is continuous since \(\bar{\beta}\) approaches 1 as \(s'\) approaches 1 (so \(s_\delta\) approaches 1 as \(s'\) approaches 1).

It will be useful to construct a map \(\Sigma D_{n}(W_n) \xrightarrow{ev_{n}} W_n \cong \Sigma^{n}W_n\) that extends a map \(\Sigma C_{n}^{*}(W_n) \xrightarrow{ev_{n}} W_n\) which resembles the adjoint of the scanning map, and whose restriction to the subspace \(D_{A}(W_n)\) is homotopic to \(\Sigma q_{s}\).

Take a configuration \(\omega = (t_1,\ldots,t_n;x_1,\ldots,x_n) \in C_{n}^{*}(W_n)\). Configurations in \(C_{n}^{*}(W_{n})\) have the property where any subset of particles contained in an open interval of length \(\varepsilon\) in \(\mathbb{R}\) has its multiset
of labels represented by a point in \( W_n \). Then for each \( 0 \leq s \leq \frac{\varepsilon}{2} \) we define \( ev_s \) and \( \tilde{ev}_s \) commuting with quotient maps in a square

\[
\begin{array}{ccc}
\Sigma C^s_n(W_n) & \longrightarrow & \Sigma D^s_n(W_n) \\
\downarrow{ev_s} & & \downarrow{ev_s} \\
W^1_n & \longrightarrow & \hat{W}^1_n
\end{array}
\]

by

\[
ev_s(t, \omega) = \begin{cases} 
((\hat{t}_1, x_1), \ldots, (\hat{t}_n, x_n)) & \text{if } s > 0; \\
+ & \text{if } s = 0,
\end{cases}
\]

where

\[
\hat{t}_i = \begin{cases} 
\frac{t_i - \bar{t}}{\varepsilon} & \text{if } t_i \in (\tilde{t} - s, \tilde{t} + s); \\
-1 & \text{if } t_i \leq \tilde{t} - s; \\
1 & \text{if } \tilde{t} + s \leq t_i,
\end{cases}
\]

and

\[
\tilde{t} = \frac{1 + t}{2} (\min\{t_1, \ldots, t_n\} - s) + \frac{1 - t}{2} (\max\{t_1, \ldots, t_n\} + s).
\]

Similarly, given \( \omega = (t_1, \ldots, t_n; x_1, \ldots, x_n) \in D^s_n(W_n) \),

\[
ev_s(t, \omega) = \begin{cases} 
((\hat{t}_1, x_1), \ldots, (\hat{t}_n, x_n)) & \text{if } s > 0; \\
+ & \text{if } s = 0,
\end{cases}
\]

where each \( \hat{t}_i \) is as defined above. Similarly to \( q_s \), \( ev_s \) maps \( (t, \omega) \) to the basepoint for configurations \( \omega \) in the subspace \( D^s_n(W_n-1) \) and all \( s \) sufficiently small depending on \( \omega \). Then we may define

\[
\rho: \Sigma C(\iota^s_{n,n}) \longrightarrow \hat{W}_n^1
\]

to be the extension of \( ev_s \) given by \( \rho(t, (s', \omega)) = ev_{(1-s')(s)}(t, \omega) \), and take the composites

\[
\rho_A : \Sigma C(\iota^s_{n,n}) \longrightarrow \Sigma C(\iota^s_{n,n}) \rightarrow \rho \longrightarrow \hat{W}_n^1
\]

\[
ev_{A,s} : \Sigma D^s_n(W_n) \longrightarrow \Sigma D^s_n(W_n) \longrightarrow \hat{W}_n^1
\]

where the first map in each composite is the inclusion.

**Corollary 4.8.** \( ev_{A,s} \) is homotopic to \( \Sigma q_s \) and \( \rho_A \) is homotopic to \( \Sigma \varphi \). Thus, \( \rho_A \) is a homotopy equivalence, and \( \varphi \) is a choice of homotopy inverse.

**Proof.** Take a point \((t, \omega) \in D^s_n(W_n)\) for some configuration \(\omega = (t_1, \ldots, t_n; x_1, \ldots, x_n) \in D^s_n(W_n)\) and \(t \in [-1, 1]\). For \( s > 0 \), define a homotopy \( H_s \) of \( \Sigma q_s \) to \( ev_{A,s} \) by

\[
(H_s)_\delta(t, \omega) = \begin{cases} 
((t_{\delta,1}, x_1), \ldots, (t_{\delta,n-1}, x_{n-1}), (t_\delta, x_n)) & \text{if } |t| < 1; \\
+ & \text{if } |t| = 1,
\end{cases}
\]

where \( t_{\delta,i} = \delta \hat{t}_i + (1 - \delta) \bar{t}_i \) and \( t_\delta = \delta t_\delta + (1 - \delta) t \). We must show \( H_s \) is continuous. Possible discontinuities happen as \( |t| \) approaches 1. If \( t \) approaches 1, then \( \hat{t} + s \) approaches \( \min\{t_1, \ldots, t_n\} \). Since this is a lower bound for \( t_n \), \( \tilde{t}_n \) approaches 1, and so \( t_\delta \) also approaches 1. Similarly, \( t_\delta \) approaches \(-1 \) as \( t \) approaches \(-1 \). In either case \((H_s)_\delta(t, \omega)\) approaches \(*\). Therefore \( H_s \) is continuous.
When $n = 1$, $\mathbb{C}(\varepsilon_{A,n}^*) = \mathcal{D}_A(W_n) \cong \mathbb{R} \times W_n$, $e^{v_A} \cong \varphi$, so $\rho_A$ is homotopic to $\Sigma \varphi$. Assume $n \geq 2$. Define a homotopy $G$ of $\Sigma \varphi$ to $\rho_A$ by

$$G_\delta(t, (s', \omega)) = \begin{cases} \left(H_{1-s'}(s')\right)_\delta(t, \omega) & \text{if } s' < 1; \\ \ast & \text{if } s' = 1. \end{cases}$$

We must show that there is no discontinuity when $s'$ approaches 1, that is, when $s = (1 - s') \frac{1}{2}$ approaches 0 (we may as well assume $s' > 0$, so $\omega \in \mathcal{D}_A(W_{n-1})$). This is true when $|t| = 1$, so assume $-1 < t < 1$. Let $\alpha = \min\{t_1, \ldots, t_n\}$ and $\beta = \max\{t_1, \ldots, t_n\}$. Since $\omega \in \mathcal{D}_A(W_{n-1})$, there exists an $i \leq n - 1$ such that $t_i \in \{\alpha, \beta\}$ and $t_i \neq t_n$. If $t_i = \alpha$, then $t_i < t_n$, so as $s$ approaches 0, $t_i$ is eventually bounded above by $t_n - s$, and since $-1 < t < 1$, $t_i$ is eventually bounded above by $\tilde{t} - s$. Therefore both $\tilde{t}_i$ and $\ell_i$ approach 1, implying $t_{\delta,i}$ does as well. Similarly, if $t_i = \beta$, then $t_{\delta,i}$ approaches 1. In either case $G_\delta(t, (s', \omega))$ approaches $\ast$ as $s'$ approaches 1. 

**Lemma 4.9.** Suppose we are given a homotopy commutative diagram of simply connected base-pointed spaces homotopy equivalent to CW-complexes

$$
\begin{array}{ccc}
\mathbb{C}(f) & \xrightarrow{\partial} & \Sigma X \\
\downarrow{g} & & \downarrow{g} \\
A & \xrightarrow{h} & B
\end{array}
\begin{array}{ccc}
\Sigma Y & \xrightarrow{\Sigma f} & \Sigma C(f) \\
\downarrow{\tilde{g}} & & \downarrow{\tilde{g}} \\
B & \xrightarrow{q} & B/A,
\end{array}
$$

where the bottom sequence is a cofibration sequence, the top one is the homotopy cofibration sequence induced by the map $f$ (with being the connecting map that collapses $Y$), and where the homotopy equivalence $\tilde{g}$ is an extension of $g$ induced by the homotopy commutativity of the left square. Then the bottom cofibration sequence extends to the left to a homotopy cofibration sequence

$$
\begin{array}{ccc}
\mathbb{C}(f) & \xrightarrow{\bar{g} \circ \partial} & A \\
\downarrow{\epsilon} & & \downarrow{\epsilon} \\
A & \xrightarrow{h} & B.
\end{array}
= \begin{array}{ccc}
\Sigma X & \xrightarrow{\Sigma' f} & \Sigma C(f) \\
\downarrow{\bar{g}} & & \downarrow{\bar{g}} \\
\mathbb{C}(\partial) & \xrightarrow{\partial'} & \mathbb{C}(\varepsilon')
\end{array}
$$

where the top sequence is given by iterating the mapping cone construction. The subspace of points of the form $(t, (0, x)) \in \mathbb{C}(\partial)$ is homeomorphic to $\Sigma Y \cong ([0, 1] \times Y) / \sim$, which $\mathbb{C}(\partial)$ deformation retracts onto, with the retraction $r$ mapping $(0, y) \mapsto \Sigma f(y)$, $(t, (0, x)) \mapsto (t, x)$, and $(t, (s, x)) \mapsto (t + s(1 - t), f(x))$ for $0 < s \leq 1$, where $0 < t \leq 1$ and $(s, x) \in \mathbb{C}(f)$. The map $\bar{r}$ is just the extension of $r$ to the mapping cone.

The extension $\tilde{g}$ is defined for some choice of homotopy $H$ of $g \circ \Sigma f$ to $h \circ \bar{g}$, and is given by mapping $(0, y) \mapsto q \circ g(y)$ and $(t, y) \mapsto q \circ H_t(y)$ for $0 < t \leq 1$. A choice of null-homotopy of $h \circ \bar{g} \circ \partial$ induces a commutative diagram of homotopy cofibration sequences

$$
\begin{array}{ccc}
\mathbb{C}(f) & \xrightarrow{\tilde{g} \circ \partial} & A \\
\downarrow{\epsilon} & & \downarrow{\epsilon} \\
\bar{g} \circ \partial & \xrightarrow{\partial'} & \mathbb{C}(\varepsilon')
\end{array}
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
\downarrow{\kappa} & & \downarrow{\kappa} \\
B & \xrightarrow{q} & B/A.
\end{array}
$$

(20)
We choose the extensions $\kappa$ and $\hat{\kappa}$ explicitly as follows. Take the commutative diagram of homotopy cofibration sequences

$$
\begin{align*}
\mathbb{C}(f) & \xrightarrow{\partial} \Sigma X \xrightarrow{i} \mathbb{C}(\partial) \\
\mathbb{C}(f) & \xrightarrow{\bar{\gamma} \circ \partial} A \xrightarrow{i} \mathbb{C}(\bar{\gamma} \circ \partial),
\end{align*}
$$

induced by the left square. The right-hand square is then a homotopy pushout square, that is, $\mathbb{C}(\bar{\gamma} \circ \partial)$ is the (reduced) double mapping cylinder of $i'$ and $\bar{\gamma}$. Define $\kappa$ by mapping the mapping cylinder $M_{\bar{\gamma}}$ via $(0, y) \mapsto h(y)$ and $(t, y) \mapsto h \circ \bar{\gamma}(y)$, and the mapping cylinder $M_{i'}$ via $(0, y) \mapsto g \circ \bar{\gamma}(y)$ and $(t, y) \mapsto H_t(y)$, for $0 < t \leq 1$. Since $\kappa$ restricts to $h$ on $A$, define the extension $\hat{\kappa}$ by collapsing the subspace $\text{Cone}(A) \subseteq \mathbb{C}(i)$. Define $\mathbb{C}(i') \xrightarrow{\bar{\rho}} \mathbb{C}(i)$ by restricting to the inclusion $p$ mapping $\mathbb{C}(\partial)$ into $M_\nu$, and extending to $\mathbb{C}(i')$ by mapping $\text{Cone}(\Sigma X)$ via $(t, x) \mapsto (4t, x) \in M_\nu$, $(\frac{1}{4} + t, x) \mapsto (1 - 4t, x) \in M_{\bar{\gamma}}$, and $(\frac{1}{4} + 2t, x) \mapsto (4t, \bar{\gamma}(x)) \in \text{Cone}(A)$, for $0 \leq t \leq \frac{1}{4}$. Notice that $\hat{\kappa} \circ \bar{\rho}$ is homotopic to the composite of homotopy equivalences $\bar{\gamma} \circ \bar{\tau}$. Also, $\bar{\rho}$ is a homotopy equivalence since it extends to a homeomorphism after collapsing the contractible subspaces $\text{Cone}(\Sigma X)$ and $\text{Cone}(A)$ of $\mathbb{C}(i')$ and $\mathbb{C}(i)$ respectively. Therefore $\hat{\kappa}$ is a homotopy equivalence, and in turn, $\kappa$ induces isomorphisms of homology groups, so is a homotopy equivalence. The lemma follows by diagram (20). □

**Proof of Proposition 4.4.** Take the homotopy equivalences $\zeta'$ from (11). It is clear that they make left square commute in the diagram of homotopy cofibration sequences

$$
\begin{align*}
\Sigma \mathbb{C}_n(W_{n-1}) & \xrightarrow{\Sigma \iota_{i,n}} \Sigma \mathbb{C}_n(W_n) \xrightarrow{\Sigma \zeta'} \Sigma \mathbb{C}(i_{i,n}) \\
\Sigma V_n(W_{n-1}) & \xrightarrow{\Sigma V_n(W_n)} \bigvee_{1 \leq i \leq n} \Sigma \mathbb{C}(i_{i,n}),
\end{align*}
$$

where the bottom sequence is the wedge sum of homotopy cofibration sequences $\Sigma D_i(W_{n-1}) \xrightarrow{\Sigma \iota_{i,n}} \Sigma D_i(W_n)$ for $1 \leq i \leq n$, thus giving the right vertical extension $\hat{\zeta}'$. Since the left and middle vertical maps are homotopy equivalences, so is $\hat{\zeta}'$. But since $\iota_{m,k}$ is a homeomorphism when $m < k$, the only non-contractible homotopy cofibre $\Sigma \mathbb{C}(i_{i,n})$ between $1 \leq i \leq n$ is $\Sigma \mathbb{C}(i_{n,n})$, implying that the composite

$$
\hat{\nu}: \Sigma \mathbb{C}(i_{n,n}) \xrightarrow{\text{include}} \bigvee_{1 \leq i \leq n} \Sigma \mathbb{C}(i_{i,n}) \xrightarrow{(\hat{\zeta}')^{-1}} \Sigma \mathbb{C}(i_{n,n})
$$

is a homotopy equivalence, where $(\hat{\zeta}')^{-1}$ is a homotopy inverse of $\hat{\zeta}'$. Replacing the vertical homotopy equivalences $\zeta'$, $\hat{\zeta}'$ with their inverses, and restricting to the homotopy cofibration sequence $\Sigma D_n(W_{n-1}) \xrightarrow{\Sigma \iota_{i,n}} \Sigma D_n(W_n) \rightarrow \Sigma \mathbb{C}(i_{n,n})$, gives a homotopy commutative diagram of homotopy cofibrations

$$
\begin{align*}
\Sigma D_n(W_{n-1}) & \xrightarrow{\Sigma \iota_{n,n}} \Sigma D_n(W_n) \xrightarrow{\Sigma \iota_{n,n}} \Sigma \mathbb{C}(i_{n,n}) \\
\Sigma \mathbb{C}_n(W_{n-1}) & \xrightarrow{\Sigma \iota_{n,n}} \Sigma \mathbb{C}_n(W_n) \xrightarrow{\Sigma \iota_{n,n}} \Sigma \mathbb{C}(i_{n,n}).
\end{align*}
$$

(21)
In turn, the top homotopy cofibration sequence splits as the wedge sum of homotopy cofibrations sequences
\[
\Sigma D_S(W_{n-1}) \xrightarrow{\Sigma \iota_{S,n}} \Sigma D_S(W_n) \rightarrow \Sigma C(\iota_{S,n})
\]
over all \((n,n)\)-partitions \(S\), and since \(\iota_{S,k}\) is a homeomorphism when at most \(k - 1\) elements in an \((m,n)\)-partition \(S\) are non-zero, the only non-contractible homotopy cofibre in this splitting is \(\Sigma C(\iota_{A,n})\), and the inclusion \(\Sigma C(\iota_{A,n}) \hookrightarrow \Sigma C(\iota_{n,n})\) is a homotopy equivalence. Restricting to the sequence where \(S = A\), we obtain the top homotopy commutative diagram of homotopy cofibrations in the statement of the proposition.

As for the bottom diagram of homotopy cofibrations, note that the evaluation maps \(\bar{e}v_{\bar{\bar{\bar{\alpha}}}}\), \(ev_{\bar{\bar{\bar{\alpha}}}}\), and the extension \(\rho\), fit into a commutative diagram
\[
\begin{array}{ccc}
\Sigma C_n^e(W_n) & \rightarrow & \Sigma D_n^e(W_n) \\
\downarrow{\bar{e}v_{\bar{\bar{\bar{\alpha}}}}} & & \downarrow{ev_{\bar{\bar{\bar{\alpha}}}}} \\
W_n^1 & \rightarrow & \bar{W}_n^1
\end{array}
\]
The top left horizontal quotient map fits into a commutative diagram of homotopy cofibrations
\[
\begin{array}{ccc}
\Sigma C_n^e(W_{n-1}) & \xrightarrow{\Sigma \iota_{n,n}} & \Sigma C_n^e(W_n) \\
\downarrow{\bar{e}v_{\bar{\bar{\bar{\alpha}}}}} & & \downarrow{ev_{\bar{\bar{\bar{\alpha}}}}} \\
\Sigma D_n^e(W_{n-1}) & \xrightarrow{\Sigma \iota_{n,n}} & \Sigma D_n^e(W_n) \\
\downarrow{q^e} & & \downarrow{\bar{\bar{\bar{\rho}}}} \\
\Sigma C_n^e(W_{n+1}) & \rightarrow & \Sigma C_n^e(W_n) \\
\end{array}
\]
induced by the left square, where the vertical maps are the quotient maps. Combining the right square with the first diagram, we obtain a commutative diagram of homotopy cofibration sequences
\[
\begin{array}{ccc}
\Sigma C_n^e(W_{n-1}) & \xrightarrow{\Sigma \iota_{n,n}} & \Sigma C_n^e(W_n) \\
\downarrow{\bar{e}v_{\bar{\bar{\bar{\alpha}}}}} & & \downarrow{ev_{\bar{\bar{\bar{\alpha}}}}} \\
W_n^1 & \rightarrow & \bar{W}_n^1
\end{array}
\]
where \(\bar{\bar{\bar{\rho}}}\) is the composite
\[
\bar{\bar{\bar{\rho}}}: \Sigma C(\iota_{n,n}) \xrightarrow{q^e} \Sigma C(\iota_{n,n}) \xrightarrow{\rho} \bar{W}_n^1.
\]
By Lemma [44], the composite \(\rho_{\bar{\bar{\bar{A}}}'} \Sigma C(\iota_{A,n}') \rightarrow \Sigma C(\iota_{n,n}) \xrightarrow{\rho} \bar{W}_n^1\) is a homotopy equivalence, and since the first inclusion is a homotopy equivalence, then so is \(\rho\). Notice \(q^e\) fits into a commutative square
\[
\begin{array}{ccc}
\Sigma C(\iota_{n,n}) & \xrightarrow{\iota_{n,n}} & \Sigma C(\iota_{n,n}) \\
\downarrow{q^e} & & \downarrow{\bar{\bar{\bar{\rho}}}'} \\
\Sigma C(\iota_{n,n}) & \xrightarrow{\iota_{n,n}} & \Sigma C(\iota_{n,n})
\end{array}
\]
where \(\bar{\bar{\bar{\rho}}}\) extends the quotient map \(\Sigma C_n(W_n) \xrightarrow{q} \Sigma D_n(W_n)\) and the horizontal inclusions (as we saw in [13] and [14]) are homotopy equivalences. The composite of \(\Sigma C_n(W_n) \xrightarrow{\zeta} \Sigma V_n(W_n)\) with the quotient map onto the summand \(\Sigma D_n(W_n)\) is the left homotopy inverse of the map \(\nu\) in diagram (21), and by the right-hand commutative square in diagram (11), this composite is equal to the composite
\[
\begin{array}{ccc}
\Sigma C_n^e(W_n) & \xrightarrow{q} & \Sigma D_n^e(W_n) \\
\downarrow{\iota_{n,n}} & & \downarrow{\iota_{n,n}} \\
\Sigma C(\iota_{n,n}) & \rightarrow & \Sigma C(\iota_{n,n})
\end{array}
\]
of $q$ and some self homotopy equivalence. Thus, $q$ is also a left homotopy inverse of $\nu$, and similarly for $W_{n-1}$ in place of $W_n$. Then composing (21) with the commutative diagram

$$
\begin{array}{c}
\Sigma C_n(W_{n-1}) \xrightarrow{\Sigma \tau_{n,n}} \Sigma C_n(W_n) \xrightarrow{\Sigma \epsilon_{n,n}} \Sigma C(\iota_{n,n}) \\
| \quad q \quad | \quad \tilde{q} \\
\Sigma D_n(W_{n-1}) \xrightarrow{\Sigma \tau_{n,n}} \Sigma D_n(W_n) \xrightarrow{\Sigma \epsilon_{n,n}} \Sigma C(\iota_{n,n})
\end{array}
$$

gives a commutative diagram of homotopy cofibrations

$$
\begin{array}{c}
\Sigma D_n(W_{n-1}) \xrightarrow{\Sigma \tau_{n,n}} \Sigma D_n(W_n) \xrightarrow{\Sigma \epsilon_{n,n}} \Sigma C(\iota_{n,n}) \\
| \quad q_{\circ \nu} \quad | \quad \tilde{q}_{\circ \nu} \\
\Sigma D_n(W_{n-1}) \xrightarrow{\Sigma \tau_{n,n}} \Sigma D_n(W_n) \xrightarrow{\Sigma \epsilon_{n,n}} \Sigma C(\iota_{n,n}).
\end{array}
$$

Since both composites $q \circ \nu$ are homotopic to the identity, $\tilde{q} \circ \tilde{\nu}$ is a homotopy equivalence. But $\tilde{\nu}$ is a homotopy equivalence, so in turn $\tilde{q}$, $\tilde{\nu}$, and $\tilde{\rho}$ are homotopy equivalences. Notice that $\Sigma C^\epsilon_n(W_n) \xrightarrow{\Sigma^\epsilon\nu} W_n^{1}$ is homotopic to $\Sigma C_n^\epsilon(W_n) \xrightarrow{\nu} \Sigma C_n(W_n) \xrightarrow{\Sigma \epsilon_n} \Sigma \Omega W_n^{1} \xrightarrow{\epsilon_n} W_n^{1}$. This can be checked directly since the right homotopy inverse of $\iota$ in Lemma (3.7) when restricted to $C^\epsilon(D^1, \partial D^1; W_n)$ is given explicitly by applying the radial expansion $R_2$. Now taking the commutative diagram of homotopy cofibrations

$$
\begin{array}{c}
\Sigma C_n^\epsilon(W_{n-1}) \xrightarrow{\Sigma \tau_{n,n}^\epsilon} \Sigma C_n^\epsilon(W_n) \xrightarrow{\Sigma \epsilon_{n,n}^\epsilon} \Sigma C(\iota_{n,n}) \\
| \quad s \quad | \quad \gamma \\
\Sigma C_n(W_{n-1}) \xrightarrow{\Sigma \tau_{n,n}} \Sigma C_n(W_n) \xrightarrow{\Sigma \epsilon_{n,n}} \Sigma C(\iota_{n,n}),
\end{array}
$$

replacing the vertical inclusions with their inverse homotopy equivalences (we can take the right-hand inverse homotopy equivalence to be an extension induced by the resulting homotopy commutative left square after replacing with inverses), and composing the resulting homotopy commutative diagram with (22), we obtain the right-side of the bottom homotopy commutative diagram of homotopy cofibrations in the statement of the proposition.

It remains to show that the bottom cofibration sequence in the proposition extends to the left, and to obtain the left-hand square involving $\phi^0$. This is clear when $n = 1$, since $W_n^{1} = *$ in this case. Since $\Sigma^{n-1} \tilde{W}_n$ is simply connected when $n \geq 2$, and $\phi^0$ is a homotopy equivalence, the $n \geq 2$ case follows by Lemma (119).

\section{5. The Golod Property}

From now on we fix $W = (D^1, S^0)^K$ as a subspace of $X = (D^1)^{\infty}$. Thus $W^1 = (D^2, S^1)^K$. The splitting of $\Sigma \Omega W^1$ in Section 2 leads us to the large primes confirmation of Conjecture 1.1. This generalizes Berglund’s result [4] concerning the rational homotopy type of $(D^2, S^1)^K$ for Golod $K$.

\begin{theorem}
Localized at any sufficiently large prime $p$, $(D^2, S^1)^K$ is homotopy equivalent to a wedge of spheres if and only if $H^*(D^2, S^1)^K; \mathbb{Z}_p)$ has trivial cup products.
\end{theorem}

\begin{proof}
The right-hand implication is clear. For the left-hand, recall that a space $Y$ is a co-$H$-space if and only if the evaluation map $\Sigma \Omega Y \xrightarrow{\epsilon_n} Y$ has a right homotopy inverse [14], and if in addition $Y$ is homotopy equivalent to a finite CW-complex, then $\Sigma Y$ is homotopy equivalent to a wedge of spheres when localized at sufficiently large primes (depending on $Y$).
\end{proof}
Since $\Sigma \Omega W^1 \cong V_1 \Sigma D_t(W)$ and $D_t(W_k)$ has the homotopy type of a finite $CW$-complex, $\Sigma \Omega W^1$ is homotopy equivalent to a wedge of spheres when localized at sufficiently large primes. Since $H^*(W^1; \mathbb{Z}_p)$ has trivial cup products, the evaluation map $\Sigma \Omega W^1 \xrightarrow{ev} W^1$ induces a surjection on $\mathbb{Z}_p$-homology (see Theorem 3.2 and the proof of Theorem 3.1 in [15]). Thus, we can construct a right homotopy inverse for $ev$ by selecting the appropriate spheres from this splitting of $\Sigma \Omega W^1$. In particular, we see that $W^1$ is homotopy equivalent to a wedge of spheres.

We now focus on the integral case. An ordered partition of a finite set $I$ is a sequence of disjoint subsets $I = (I_1, \ldots, I_m)$ of $I$ such that $I = I_1 \cup \cdots \cup I_m$. If $I$ is a subset of the integers $[n]$ with $k = |I|$ elements, and $S = (s_1, \ldots, s_k)$ is any sequence of real numbers, let $i_\ell$ denote the $\ell$th smallest element in $I$, $s'_j$ denote the $j$th smallest element in $S = \{s_1, \ldots, s_k\}$, and $m = |S|$ be the number of distinct elements in $S$. Then assign to $S$ the partition $I_S = (I_1, \ldots, I_m)$ of $I$ where $I_j = \{i_\ell \in I \mid s_\ell = s'_j\}$. When $I = [n]$ we have $k = n$ and $I_j = \{\ell \mid s_\ell = s'_j\}$, so for example, if $I = [4]$ and $S = (-1, \pi, -1, 0)$, then $[4]_S = \{(1, 3), (4), (2)\}$.

We think of the topological $(n-1)$-simplex $|\Delta^{n-1}|$ and its subcomplexes $|K|$ as being without basepoint. The first suspensions $\Sigma|\Delta^{n-1}|$ and $\Sigma|K|$ are therefore unreduced. Again, the suspensions $\Sigma|\Delta^{n-1}|$ and $\Sigma|K|$ are themselves basepointed, with the basepoint $*\cdot 1$ the tip of the double cone corresponding to the basepoint $-1 \in D^1 = [-1, 1]$. Higher suspensions are therefore reduced. This means that any point $(t_1, \ldots, t_n, z) \in \Sigma^n|K|$ is identified with the basepoint $*$ if and only if $t_n = -1$ or $t_i = \pm 1$ for some $i < n$.

Take the diagonal 

$$\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = \cdots = x_n\}$$

and consider the following subspace of the smash product $P_n = (\mathbb{R}^n - \Delta_n) \wedge \Sigma|\Delta^{n-1}|$:

$$Q_K = \bigcup_{y \in (\mathbb{R}^n - \Delta_n)} \{y\} \wedge \Sigma|K_{I_1} \ast \cdots \ast K_{I_m}|,$$

where $[n]_y$ is a partition $[n]_{y_1, \ldots, y_n}$ of $[n]$ as defined above. Here we took $\mathbb{R}^n - \Delta_n$ to be without a basepoint, so $P_n$ is the half-smash product $((\mathbb{R}^n - \Delta_n) \times \Sigma|\Delta^{n-1}|)/(\mathbb{R}^n - \Delta_n) \times \{\ast\}$, and is basepointed.

**Definition 5.2.** A simplicial complex $K$ on vertex set $[n]$ is weakly homotopy Golod if $K$ is a single vertex, or (recursively) $K \setminus \{i\}$ is weakly homotopy Golod for each $i \in [n]$, and the map

$$\Phi_K : \Sigma^n|K| \longrightarrow \Sigma Q_K$$

given for any $z \in |K|$, $t_1, \ldots, t_{n-1}, t \in [-1, 1]$, and $\beta = \max \{t_1, \ldots, t_{n-1}, 0\}$ by

$$\Phi_K(t_1, \ldots, t_{n-1}, t, z) = (2\beta - 1, (t_1, \ldots, t_{n-1}, 0), (t, z))$$

is null-homotopic.

**Theorem 5.3.** $(D^2, S^1)^K$ is a co-$H$-space if and only if $K$ is weakly homotopy Golod.

Recall that $\tilde{W}$ is the image of $W$ under the quotient map $(D^1)^{\times n} \longrightarrow (D^1)^{\times n}$. To prove this theorem we will need to use a simplicial description of $\tilde{W}$. In [2] the theory of diagrams of spaces was used to show the existence of a homeomorphism $\Sigma|K| \cong \tilde{W}$, but for our purposes, we give an explicit description of this homeomorphism as follows. Take the simplex $\Delta^{n-1}$ to be on vertex set $[n]$, and think of its geometric realization as the subspace

$$|\Delta^{n-1}| = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid 0 \leq t_i \leq 1 \text{ and } t_1 + \cdots + t_n = 1\}$$
with each face \( \sigma \in \Delta^{n-1} \) corresponding to the subspace \( |\sigma| = \{(t_1, \ldots, t_n) \in |\Delta^{n-1}| \mid t_i = 0 \text{ if } i \notin \sigma \}. \)

Take the subspaces of \(|\Delta^{n-1}|\) and \((D^1)^\wedge n; \)

\[
U_i = \{(t_1, \ldots, t_n) \in |\Delta^{n-1}| \mid t_i = \max\{t_1, \ldots, t_n\}\},
\]

\[
V_{i,t} = \{(t'_1, \ldots, t'_n) \in (D^1)^\wedge n \mid t'_t = \min\{t'_1, \ldots, t'_n\}\}.
\]

Consider a homeomorphism

\[
(23) \quad h: \Sigma|\Delta^{n-1}| \xrightarrow{\sim} (D^1)^\wedge n
\]

given as follows. For \(-1 < t < 1\), take the homeomorphism \(h_{i,t}: U_i \to V_{i,t}\) given by mapping

\[
(t_1, \ldots, t_n) \mapsto (t + (1 - t)\frac{t_1 - t_1}{t_1}, \ldots, t + (1 - t)\frac{t_n - t_n}{t_n}),
\]

and its inverse homeomorphism \(h_{i,t}^{-1}: V_{i,t} \to U_i\) is given by mapping

\[
(t'_1, \ldots, t'_n) \mapsto \left(\frac{1 - t'_1}{n - \sum_{k=1}^n t'_k}, \ldots, \frac{1 - t'_n}{n - \sum_{k=1}^n t'_k}\right)
\]

(to see that \(h_{i,t} \circ h_{i,t}^{-1}\) and \(h_{i,t}^{-1} \circ h_{i,t}\) are the identity, use the fact that here \(t'_t = t\) and \(1 = t_1 + \cdots + t_n\)).

Notice that \(h_{i,t}\) agrees with \(h_{i,t}\) on \(U_i \cap U_j\), and \(h_{i,t}^{-1}\) agrees with \(h_{j,t}^{-1}\) on \(V_{i,t} \cap V_{j,t}\). Thus, for each \(t \in (-1,1)\), we can piece together the homeomorphisms \(h_{i,t}\) over all \(i \in [n]\) to give a homeomorphism

\[
h_t: |\Delta^{n-1}| = \bigcup_{i=1}^n U_i \xrightarrow{\sim} \bigcup_{i=1}^n V_{i,t} = V_t
\]

defined unambiguously by mapping \(z \mapsto h_{i,t}(z)\) if \(z \in U_i\), with inverse \(h_t^{-1}\) defined by mapping \(z' \mapsto h_{i,t}^{-1}(z')\) if \(z' \in V_{i,t}\). Notice that \((D^1)^\wedge n\) is the disjoint union of \(V_t\) over all \(t \in [-1,1]\), with \(V_1\) being a single point \((1, \ldots, 1)\), and \(V_{-1}\) being the basepoint, and the maps \(h_t\) vary continuously with respect to \(t\). Thus, we define the homeomorphism \(h\) for any \((t, z) \in \Sigma|\Delta^{n-1}|\) by

\[
h(t, z) = \begin{cases} 
  h_t(z) & \text{if } -1 < t < 1; \\
  (1, \ldots, 1) & \text{if } t = 1; \\
  * & \text{if } t = -1.
\end{cases}
\]

The crucial property of \(h\) is that it restricts on an \(m\)-dimensional face \(\sigma\) of \(\Delta^{n-1}\) to a homeomorphism

\[
h_\sigma: \Sigma|\Delta^m| \xrightarrow{\sim} \bigcup_{\sigma \in L} (Y_1^\sigma \wedge \cdots \wedge Y_n^\sigma) \xrightarrow{\sim} (D^1)^\wedge (m+1),
\]

where

\[
Y_i^\sigma = \begin{cases} 
  D^1 & \text{if } i \in \sigma \\
  \{1\} & \text{if } i \notin \sigma.
\end{cases}
\]

Then piecing these together, for any subcomplex \(L\) of \(\Delta^{n-1}\), \(h\) restricts to a homeomorphism

\[
\Sigma|L| = \bigcup_{\sigma \in L} \Sigma|\sigma| \xrightarrow{\sim} \bigcup_{\sigma \in L} Y_1^\sigma \wedge \cdots \wedge Y_n^\sigma.
\]

In particular, when \(W = (D^1, S^0)^K\),

\[
\tilde{W} = \bigcup_{\sigma \in K} Y_1^\sigma \wedge \cdots \wedge Y_n^\sigma,
\]

so \(h\) restricts to a homeomorphism \(\Sigma|\tilde{K}| \xrightarrow{\sim} \tilde{W}\). More generally, for any partition \((I_1, \ldots, I_m)\) of \([n]\), we have an inclusion \(\tilde{W}_{I_1} \wedge \cdots \wedge \tilde{W}_{I_m} \to (D^1)^\wedge n\) given by mapping elements by permuting
coordinates, so that if a \( j \)th coordinate is the coordinate in a factor \( \tilde{W}_{I_k} \) that corresponds to \( i \in I_k \), upon mapping it becomes the \( i \)th coordinate. This then gives a homeomorphism onto its image
\[
\tilde{W}_{I_1} \wedge \cdots \wedge \tilde{W}_{I_m} \cong \bigcup_{\sigma_1 \in K_{I_1}, \cdots, \sigma_m \in K_{I_m}} Y_{I_1}^{(\sigma_1 \cup \cdots \cup \sigma_m)} \wedge \cdots \wedge Y_{I_m}^{(\sigma_1 \cup \cdots \cup \sigma_m)},
\]
so more generally \( h \) restricts to a homeomorphism
\[
\Sigma|K_{I_1} * \cdots * K_{I_m}| = \bigcup_{\sigma_1 \in K_{I_1}, \cdots, \sigma_m \in K_{I_m}} \Sigma|\sigma_1 \cup \cdots \cup \sigma_m| \to \tilde{W}_{I_1} \wedge \cdots \wedge \tilde{W}_{I_m}.
\]

In this manner, we will think of \( \tilde{W}_{I_1} \wedge \cdots \wedge \tilde{W}_{I_m} \) as a subspace of \((D^1)^n\).

**Proof of Theorem 5.3.** Let \( W = (D^1, S^0)^K \). At most \( n-1 \) particles collide in a configuration in \( \mathcal{D}_A(W_{n-1}) \), and if we look at the subspace of all configurations of \( n \) particles fixed at some \( t_1, \ldots, t_n \in \mathbb{R} \) (that is, all possible labels for particles fixed in this way), then this subspace is homeomorphic to \( \{ y \} \wedge \tilde{W}_{I_1} \wedge \cdots \wedge \tilde{W}_{I_m} \), where \( y = (t_1, \ldots, t_n) \), and \( (I_1, \ldots, I_m) = [n]_y \). This in turn is homeomorphic to \( \{ y \} \wedge \Sigma|K_{I_1} * \cdots * K_{I_m}| \) via the restriction of the homeomorphism \( \Sigma|\Delta^{n-1}| \to (D^1)^{\wedge n} \) in (23). Thus, we see that the map
\[
\tilde{h}: Q_K \to \mathcal{D}_A(W_{n-1})
\]
sending \( ((t_1, \ldots, t_n), x) \mapsto (t_1, \ldots, t_n, x_1, \ldots, x_n) \), where \( (x_1, \ldots, x_n) = h(x) \), is a homeomorphism. Notice this map fits into a commutative square
\[
\begin{array}{ccc}
\Sigma^n|K| & \xrightarrow{\Phi_K} & \Sigma Q_K \\
\downarrow{\tilde{z}} & & \downarrow{\tilde{z}} \\
\Sigma^{n-1}W & \xrightarrow{\phi^0} & \mathcal{C}(\mathcal{D}_{A,n}) \\
\downarrow{\tilde{z}} & & \downarrow{\tilde{z}} \\
\Sigma^{n-1}\tilde{W} & \xrightarrow{\phi^0} & \mathcal{D}_A(W_{n-1}) \\
\end{array}
\]
where \( \tilde{\partial} \) is the connecting map given by collapsing the subspace \( \mathcal{D}_A(W_n) \), and \( \phi^0 \) the homotopy equivalence from Lemma 4.7.

Induct on number of vertices. Assume the statement holds for simplicial complexes on less than \( n \) vertices. For the right-hand implication, suppose \( W^1 \) is a co-\( H \)-space. Then by Proposition 2.9 \( W^1_I \) is a co-\( H \)-space for every \( I \subseteq [n] \), and \( W^1 \to \hat{W}^1 \) has a right homotopy inverse. Then by our inductive assumption \( K \setminus \{i\} \) is weakly homotopy Golod for each \( i \in [n] \), and by Corollary 4.5 \( \tilde{\partial} \) is null-homotopic. Since the vertical maps in the above commutative square are homeomorphisms, \( \Phi_K \) is also null-homotopic. Therefore \( K \) is weakly homotopy Golod.

For the left-hand implication, suppose \( K \) is weakly homotopy Golod. Since by definition, each \( K_I \) is weakly homotopy Golod when \( K \) is, by our inductive assumption \( W^1_I \) is a co-\( H \)-space for \( |I| < n \). Then using Proposition 2.9, the quotient map \( W^1_I \to \hat{W}^1_I \) has a right homotopy inverse for \( |I| < n \), and to show \( W^1 \) is a co-\( H \)-space, it remains to show that \( W^1 \to \hat{W}^1 \) has a right homotopy inverse, or equivalently (by Corollary 4.5), that \( \tilde{\partial} \) is null-homotopic. Since \( \Phi_K \) is null-homotopic here, and \( \phi^0 \) is a homotopy equivalence, this follows from the above commutative square.

\[\square\]

Is there a Golod complex that is not weakly homotopy Golod? In light of Theorem 5.3, the truth of Conjecture 4.1 depends on an answer to this question. A counter-example to this conjecture might be easier to come by using the following necessary condition however.

**Proposition 5.4.** If \((D^2, S^1)^K \) is a co-\( H \)-space, then each one of the suspended inclusions
\[
\Sigma^{n+1}t_{I,J}: \Sigma^{n+1}|K_{I,J}| \to \Sigma^{n+1}|K_I * K_J|
\]
is null-homotopic for every disjoint non-empty \( I, J \subseteq [n] \).
Proof. Recall that if \( Y \) is a co-H-space with comultiplication \( Y \xrightarrow{\psi} Y \vee Y \) if and only if the diagonal map \( Y \xrightarrow{\Delta} Y \times Y \) is homotopic to the composite \( Y \xrightarrow{\psi} Y \vee Y \xrightarrow{\text{inclusion}} Y \times Y \). In particular, this implies the reduced diagonal map
\[
\overline{\Delta} : Y \xrightarrow{\Delta} Y \times Y \longrightarrow (Y \times Y)/(Y \vee Y) \xrightarrow{\cong} Y \wedge Y
\]
is null-homotopic whenever \( Y \) is a co-H-space.

Let \( W^1 = W^1_n = (D^2, S^1)^K \), and suppose \( W^1 \) is a co-H-space. By Proposition 2.9, \( W^1_{I \cup J} \) is a co-H-space for each disjoint non-empty \( I, J \subset [n] \), and so \( W^1_{I \cup J} \xrightarrow{\overline{\Delta}} W^1_{I \cup J} \wedge W^1_{I \cup J} \) is null-homotopic. Then the composite
\[
W^1_{I \cup J} \xrightarrow{\overline{\Delta}} W^1_{I \cup J} \wedge W^1_{I \cup J} \longrightarrow W^1_I \wedge W^1_J \longrightarrow \hat{W}^1_I \wedge \hat{W}^1_J
\]
is null-homotopic (where the second last map is the smash of the coordinate-wise projection maps onto \( W^1_I \) and \( W^1_J \), and the last map is the smash of quotient maps) and moreover, is equal to the composite
\[
W^1_{I \cup J} \xrightarrow{\text{quotient}} W^1_{I \cup J} \xrightarrow{\iota^1} \hat{W}^1_I \wedge \hat{W}^1_J
\]
where \( \iota^1 \) is the coordinate-wise inclusion, so this composite is null-homotopic as well. But since \( W^1_{I \cup J} \) is a co-H-space, by Proposition 2.9 the quotient map \( W^1_{I \cup J} \longrightarrow \hat{W}^1_{I \cup J} \) has a right homotopy inverse, so \( \iota^1 \) must be nullhomotopic. Notice that with respect to the homeomorphisms \( \hat{W}^1_I \xleftarrow{\cong} \Sigma^n\iota^1 \hat{W}^1_I \) given in Section 4, \( \iota^1 \) is the \( n = (|I| + |J|) \)-fold suspension of the coordinate-wise inclusion \( \iota^0 : \hat{W}^1_{I \cup J} \longrightarrow \hat{W}^1_I \wedge \hat{W}^1_J \), and with respect to the homeomorphisms (24), \( \iota^0 \) is just the suspended inclusion \( \Sigma |K_{I \cup J}| \longrightarrow \Sigma |K_I \ast K_J| \). Thus, we see that \( \Sigma^{n+1}I_{I,J} \) is null-homotopic.

\[\square\]

This condition is just a homotopy version of the Golod property after appropriate suspension. We will strengthen a desuspension of it so that it becomes sufficient, though possibly no longer necessary. This is the homotopy Golod condition. Recall that we want the inclusions \( \Sigma I_{I,J} \) to be null-homotopic for disjoint non-empty \( I, J \subset [n] \), and at the same time we want to be able to transition between these null-homotopies as outlined in the introduction. We encode these properties as follows.

Consider the delta set \( \mathcal{K}_n = \{ \mathcal{F}_0, \ldots, \mathcal{F}_{n-1} \} \), whose set of \((m-2)\)-faces \( \mathcal{F}_{m-2} \) is the set of all length \( m \) ordered partitions \((I_1, \ldots, I_m)\) of \([n]\), with each \( I_j \neq \emptyset \), and whose face maps \( d_i : \mathcal{F}_{m-2} \longrightarrow \mathcal{F}_{m-3} \) are given by \( d_i((I_1, \ldots, I_m)) = (I_1, \ldots, I_{i-1}, I_i \cup I_{i+1}, I_{i+2}, \ldots, I_m) \) for \( 1 \leq i \leq m - 1 < n \). These satisfy the required identity \( d_j \circ d_i = d_{j-1} \circ d_i \) when \( i < j \). Since each ordered partition \( \mathcal{S} \in \mathcal{F}_{m-2} \) is uniquely determined by its \((m-3)\)-faces \( d_1(\mathcal{S}), \ldots, d_{m-1}(\mathcal{S}) \), then \( \mathcal{K}_n \) is a simplicial complex of dimension \( n-2 \).

We will see that it is a triangulation of \( S^{n-2}\).

Let \( \text{Cone}(\Sigma |K|) = [0, 1] \times \Sigma |K|/\sim \) under the identifications \((1, x) \sim (t, \ast_{-1}) \sim \ast\).

**Definition 5.5.** A simplicial complex \( K \) on vertex set \([n]\) is **homotopy Golod** if \( K \) is a single vertex, or (recursively) \( K\setminus \{i\} \) is homotopy Golod for each \( i \in [n] \) and there is a map
\[
\overline{\Psi}_K : |\mathcal{K}_n| \times \text{Cone}(\Sigma |K|) \longrightarrow \Sigma |\Delta^{n-1}|
\]
such that for any \( \gamma \in |\mathcal{K}_n| \), \( \overline{\Psi}_K(\gamma, \ast) \) is the basepoint \( \ast_{-1} \in \Sigma |\Delta^{n-1}| \), and:

1. the restriction of \( \overline{\Psi}_K \) to \( \{\gamma\} \times \{(0) \times \Sigma |K|\} \) is the suspended inclusion \( \Sigma |K| \longrightarrow \Sigma |\Delta^{n-1}| \);
2. if \( \gamma \in |\mathcal{S}| \) for some \( \mathcal{S} = (I_1, \ldots, I_m) \in \mathcal{F}_{m-2} \), then \( \overline{\Psi}_K \) maps \( \{\gamma\} \times \text{Cone}(\Sigma |K|) \) to a subspace of \( \Sigma |K_{I_1} \ast \cdots \ast K_{I_m}| \subseteq \Sigma |\Delta^{n-1}| \).

We will give two more equivalent definition of homotopy Golodness, which are a bit less directly related to the Golod condition, but which from our standpoint will be more workable.
Definition 5.6 (Second definition of homotopy Golodness). A simplicial complex $K$ on vertex set $[n]$ is homotopy Golod if $K$ is a single vertex, or (recursively) $K\backslash\{i\}$ is homotopy Golod for each $i \in [n]$ and there is a basepoint preserving map

$$
\Psi_K: \Sigma^n|K| \to \Sigma|\Delta^{n-1}|
$$

such that for any $y = (t_1, \ldots, t_n) \in [-1,1]^{\binom{n}{n-1}}/\partial([-1,1]^{\binom{n}{n-1}}) \simeq S^{n-1}$ that is not the basepoint, $\Psi_K$ maps the subspace $\{y\} \wedge \Sigma|K|$ of $\Sigma^n|K|$ as follows:

1. when $t_1 = \cdots = t_{n-1} = 0$, the restriction of $\Psi_K$ to $\{y\} \wedge \Sigma|K|$ is the suspended inclusion $\Sigma|K| \to \Sigma|\Delta^{n-1}|$;

2. letting $S_y = (t_1, \ldots, t_{n-1}, 0)$ and $(I_1, \ldots, I_m) = [n]_{S_y}$, $\Psi_K$ maps $\{y\} \wedge \Sigma|K|$ to a subspace of $\Sigma|K_{I_1} \ast \cdots \ast K_{I_m}| \subseteq \Sigma|\Delta^{n-1}|$.

Definition 5.7 (Third definition of homotopy Golodness). Same as the second definition, but with $\Sigma|\Delta^{n-1}|$ replaced with $\Sigma(D^1)^n$, the subspaces $\Sigma|K_{I_1} \ast \cdots \ast K_{I_m}|$ and $\Sigma|K|$ of $\Sigma|\Delta^{n-1}|$ replaced with the subspaces $\hat{W}_{I_1} \wedge \cdots \wedge \hat{W}_{I_m}$ of $(D^1)^n$, and $\Psi_K: \Sigma^n|K| \to \Sigma|\Delta^{n-1}|$ becoming a map

$$
\Psi_K: \Sigma^n|\hat{W} \to (D^1)^n.
$$

Lemma 5.8. All three definitions of homotopy Golodness are equivalent.

Proof. Since the homeomorphisms $\Sigma|K| \simeq \hat{W}$ and $\Sigma|K_{I_1} \ast \cdots \ast K_{I_m}| \simeq \hat{W}_{I_1} \wedge \cdots \wedge \hat{W}_{I_m}$ are restrictions of the homeomorphism $\Sigma|\Delta^{n-1}| \simeq (D^1)^n$ defined before, the equivalence of the second and third definitions follows.

Think of $S^{n-1}$ as the unit $(n-1)$-cube quotient boundary $[-1,1]^{\binom{n}{n-1}}/\partial([-1,1]^{\binom{n}{n-1}})$. We can write $S^{n-1} = \bigcup_{0 \leq s \leq 1} U_t$, where $U_t = \{(t_1, \ldots, t_n) \in S^{n-1} | \max|t_1|, \ldots, |t_{n-1}| = t\}$. When $0 < t < 1$, $U_t$ is homeomorphic to $S^{n-2}$, while $U_0 = \emptyset$ and $U_1 = \{0, \ldots, 0\}$, where $* \in S^{n-1}$ is the basepoint. We construct homeomorphisms

$$
\tau_t: [K_n] \to U_t \simeq S^{n-2}
$$

for $0 < t < 1$ as follows. Given any face $S = (I_1, \ldots, I_m) \in F_{m-2}$ of $K_n$, its vertices are the length 2 partitions $S_i = (I_1 \cup \cdots \cup I_i, I_{i+1} \cup \cdots \cup I_m)$ of $[n]$ for $i = 1, \ldots, m - 1$, so we can think of any point $\gamma \in |S|$ as a formal sum

$$
\gamma = S_{1} \gamma_{1} + \cdots + S_{m-1} \gamma_{m-1}
$$

such that each $s_i \geq 0$ and $s_1 + \cdots + s_{m-1} = 1$, and $\gamma$ is on a boundary face $|d_i(S)| \subseteq |S|$ if and only if $s_i = 0$, in which case the summand $s_i \gamma_i$ vanishes. Given $j \in [n]$, let $j_{S}$ be the integer such that $j \in I_{j_{S}}$, and let $t_{j, \gamma} = s_0 + s_1 + \cdots + s_{j_{S}-1}$ where $s_0 = 0$. Notice at least two of these must be distinct, so $\beta_{\gamma} = \max|t_{1, \gamma} - t_{n, \gamma}|, \ldots, |t_{n-1, \gamma} - t_{n, \gamma}| > 0$. Then define $\tau_t$ by

$$
\tau_t(\gamma) = \left(t_{\beta_{\gamma}}(t_{1, \gamma} - t_{n, \gamma}), \ldots, t_{\beta_{\gamma}}(t_{n-1, \gamma} - t_{n, \gamma})\right).
$$

One can see that $\tau_t$ is well-defined by checking this definition agrees on each boundary face $d_i(S)$. If $\gamma$ is in the interior of some $k$-face $|S'| \subseteq |S|$ for $k \leq m$, then $[n]_{\tau_t(\gamma) \times 0} = S'$. Then one can check that $\tau_t$ is a homeomorphism with inverse $\tau_t^{-1}: U_t \to [K_n]$ given by

$$
\tau_t^{-1}((t_1, \ldots, t_{n-1})) = \left(\frac{t_1 - t_1'}{t_1' - t_1}, \ldots, \frac{t_{k-1} - t_{k-1}'}{t_{k-1}' - t_{k-1}}\right) S_{k-1}.
$$

where $t_1'$ is the $\ell$th smallest integer in the set $S = \{t_1, \ldots, t_{n-1}, 0\}$, $k = |S|$, and $S_{k-1}$ is the length 2 partition $(I_1' \cup \cdots \cup I_{k-1}' \cup \cdots \cup I_k')$, given that $S' = (I_1', \ldots, I_k')$ denotes the partition $[n]_{(t_1, \ldots, t_{n-1}, 0)}$.

For $t = 0$ and $t = 1$, we define $\tau_0: [K_n] \to U_0$ and $\tau_1: [K_n] \to U_1$ to be the constant maps sending all points to $(0, \ldots, 0)$ and $*$ respectively.
Take the quotient space 

$$Q = |K_n| \times \text{Cone}(\Sigma|K|) / \sim$$

under the identifications $((\gamma, 0, x)) \sim ((\gamma', 0, x)) \sim (0, x)$ and $(\gamma, *) \sim (*)$ for any $\gamma, \gamma' \in |K_n|$ and $x \in \Sigma|K|$. Let

$$\tau_K: Q \to S^{n-1} \wedge \Sigma|K| \cong \Sigma^n|K|$$

be given by mapping $((\gamma, (t, x))) \mapsto (\tau_t(\gamma), x)$. This is a homeomorphism, with inverse

$$\tau_K^{-1}: S^{n-1} \wedge \Sigma|K| \to Q$$

given by mapping $(0, \ldots, 0, x) \mapsto (0, x) \sim (*, x) \sim *$, and $(t_1, \ldots, t_{n-1}, x) \mapsto (\tau_t^{-1}(t_1, \ldots, t_{n-1}), (t, x))$ when $(t_1, \ldots, t_{n-1}) \in U_t$ for some $0 < t < 1$. From the first definition of homotopy Golodness, we see $\Psi_K$ factors as

$$\hat{\Psi}_K: |K_n| \times \text{Cone}(\Sigma|K|) \xrightarrow{\text{quotient}} Q \xrightarrow{\psi_{K}} \Sigma|\Delta^{n-1}|.$$ 

Thus, given $\hat{\Psi}_K$ exists as in the first definition, we can construct $\Psi_K$ as in the second definition by taking it as the composite $\hat{\Psi}_K \circ \tau_K^{-1}$. Conversely, given $\Psi_K$ exists, we can define $\hat{\Psi}_K = \Psi_K \circ \tau_K$, and then take $\hat{\Psi}_K$ to be the above composite of the quotient map and $\hat{\Psi}_K$. Therefore, the first and second definitions are equivalent.

We will use the second and third definitions of homotopy Golodness from now on.

**Theorem 5.9.** If $K$ is homotopy Golod, then $(D^2, S^1)^K$ is a co-$H$-space.

**Proof.** Induct on number of vertices. Assume the statement holds for simplicial complexes on less than $n$ vertices. Since by definition, each $K_I$ is homotopy Golod when $K$ is, by our inductive assumption $W_I^1 = |I| < n$. Then using Proposition 2.3, the quotient map $W_I^1 \to \hat{W}_I^1$ has a right homotopy inverse for $|I| < n$, and to show $W_I^1$ is a co-$H$-space, it remains to show that $W_I^1 \to \hat{W}_I^1$ has a right homotopy inverse.

Given our map $\Psi_K: \Sigma^{n-1}\hat{W} \to (D^1)^\wedge n$, define a map $g: \Sigma^{n-1}\hat{W} \to \mathcal{D}_A(W)$ by mapping a point $\sigma = (t_1, \ldots, t_{n-1}, x)$ via

$$g(\sigma) = (t_1, \ldots, t_{n-1}, 0; \Psi_K(\sigma)).$$

The composite of $g$ and the inclusion $\mathcal{D}_A(W) \to \mathcal{C}(t_{A,n})$ is homotopic to the homotopy equivalence $\Sigma^{n-1}\hat{W} \xrightarrow{\phi^0} \mathcal{C}(t_{A,n})$ from Lemma 4.7 by homotoping $(0, g(\sigma))$ to $(\max(|t_1|, \ldots, |t_{n-1}|, 0), g(\sigma))$ as one might expect, and then in turn homotoping this to $(\max(|t_1|, \ldots, |t_{n-1}|, 0), (t_1, \ldots, t_{n-1}, 0; x))$ by homotoping $\Psi_K(\sigma)$ to $x$. This last homotopy is given by sending $\Psi_K(\sigma)$ to $\Psi_K(\sigma_t)$ at each time $t$, where

$$\sigma_t = ((1 - t) t_1, \ldots, (1 - t) t_{n-1}, x).$$

Thus, since $\phi^0$ is a homotopy equivalence, $\mathcal{D}_A(W) \to \mathcal{C}(t_{A,n})$ has a right homotopy inverse, and then by Corollary 4.5, so does $W_I^1 \to \hat{W}_I^1$. 

**Example 5.10.** Recall from [19] that a complex $K$ on vertex set $[n]$ is *extractible* if either $K \setminus \{i\}$ is a simplex for some $i \in [n]$, or else (recursively) $K \setminus \{i\}$ is extractible for each $i \in [n]$ and the wedge sum of inclusions

$$\Sigma|K\setminus\{1\}| \vee \cdots \vee \Sigma|K\setminus\{n\}| \to \Sigma|K|$$

has a right homotopy inverse. We show that $K$ is homotopy Golod by inducting on number of vertices.

Suppose any extractible complex on less than $n$ vertices is homotopy Golod. Therefore, since the extractible property is closed under vertex deletion, each $K \setminus \{i\}$ is homotopy Golod.
To show that $K$ is homotopy Golod, first consider the case where $K \setminus \{i\}$ is a simplex for some $i \in [n]$. Without loss of generality, assume $i = n$. Let $W = (D^1, S^0)^K$ and define $\Psi_K : \Sigma^{n-1} W \to (D^1)^{\wedge n}$ by mapping $\sigma = (t_1, \ldots, t_{n-1}, (x_1, \ldots, x_n))$ to

$$\Psi_K(\sigma) = ((1 - |t_1|)x_1 - |t_1|, \ldots, (1 - |t_{n-1}|)x_{n-1} - |t_{n-1}|, x_n).$$

The last coordinate $x_n$ that is unchanged corresponds to the vertex $\{n\}$. This clearly satisfies the first condition of homotopy Golodness. One can see that it satisfies the second condition precisely because $K_I$ is a simplex for any $I$ not containing $n$, that is, $W_I = (D^1)^{\wedge |I|}$. So for the second condition to hold, each of the first $n - 1$ coordinates of $\Psi_K(\sigma)$ has the freedom to take on any value in $D^1$ as long as the corresponding $t_i$ is non-zero.

For the general case, let $I_i = [n] - \{i\}$, and consider the map

$$\Psi'_K : \Sigma^n |K| \to \bigvee_{i \in [n]} \Sigma^n |K\setminus\{i\}| \to \bigvee_{i \in [n]} \Sigma |\Delta^I| \to \Sigma |\Delta^{n-1}|$$

where $\Delta^I$ denotes the $(|I| - 1)$-simplex on vertex set $I$, the last map is the wedge sum of inclusions, $\Psi_{K\setminus\{i\}}$ is the map coming from homotopy Golodness of $K\setminus\{i\}$, and $s$ is the right homotopy inverse from the definition of extractible. Notice that $\Psi'_K$ satisfies the second condition of homotopy Golodness since each $\Psi_{K\setminus\{i\}}$ does and the first map $\Sigma^{n-1}s$ is an $(n - 1)$-fold suspension (thus, does not change the first $n - 1$ coordinates). However, $\Psi'_K$ does not necessarily satisfy the first condition. Instead it maps the subspace $\{(0, \ldots, 0)\} \wedge \Sigma |K|$ to $\Sigma |K| \subseteq \Sigma |\Delta^{n-1}|$, such that this map is homotopic to the identity in $\Sigma |K|$, via the homotopy to the identity of the composite of $s$ and the wedge sum of inclusions $\bigvee_i \Sigma |K\setminus\{i\}| \to \Sigma |K|$. Denote this homotopy by $H$. Pick $0 < \varepsilon < 1$, and for any $t \in [-1, 1]$ let

$$\chi(t) = \begin{cases} 0 & \text{if } |t| \leq \varepsilon; \\ \frac{t - \varepsilon}{1 - \varepsilon} & \text{if } t > \varepsilon; \\ \frac{t + \varepsilon}{1 + \varepsilon} & \text{if } t < -\varepsilon. \end{cases}$$

Define $\Psi_K$ for any $\sigma = (t_1, \ldots, t_{n-1}, x)$ by

$$\Psi_K(\sigma) = \Psi'_K(\chi(t_1), \ldots, \chi(t_{n-1}), H_t(x))$$

where

$$t_\sigma = \begin{cases} \frac{\varepsilon - \beta}{\varepsilon} & \text{if } \beta \leq \varepsilon; \\ 0 & \text{if } \beta > \varepsilon, \end{cases}$$

and $\beta = \max\{|t_1|, \ldots, |t_{n-1}|, 0\}$. Now $\Psi_K$ satisfies both of the conditions of homotopy Golodness.

**Example 5.11.** Define a simplicial complex $K$ on vertex set $[n]$ to be $m$-complete if every subset of at most $m$ vertices in $[n]$ is a face in $K$, and we will say $K$ is highly complete if it is $\left\lfloor \frac{n}{2} \right\rfloor$-complete, where $n$ is the number of vertices of $K$.

Suppose $K$ is highly complete. Since $|I| \leq \left\lfloor \frac{n}{2} \right\rfloor$ or $|J| \leq \left\lfloor \frac{n}{2} \right\rfloor$ for any disjoint non-empty $I, J \subseteq [n]$, then at least one of $|K_I|$ or $|K_J|$ is a simplex, implying $|K_I \ast K_J| \simeq \Sigma |K_I| \wedge |K_J|$ is contractible. Therefore each inclusion $\Sigma |K| \to \Sigma |K_I \ast K_J|$ is null-homotopic, implying $K$ is Golod. We show $K$ is homotopy Golod by inducting on number of vertices as follows.

Suppose any highly complete complex on less than $n$ vertices is homotopy Golod. Since each $K \setminus \{i\}$ is highly complete when $K$ is, $K \setminus \{i\}$ is homotopy Golod by induction. Now define

$$\Psi_K : \Sigma^{n-1} W \to (D^1)^{\wedge n}$$
by mapping $\sigma = (t_1, \ldots, t_{n-1}, (x_1, \ldots, x_n))$ to

$$
\Psi_K(\sigma) = \begin{cases} (f_1(x_1), \ldots, f_n(x_n)) & \text{if } \beta < 1; \\
* & \text{if } \beta = 1,
\end{cases}
$$

where $\beta = \max\{|t_1|, \ldots, |t_{n-1}|, 0\}$,

$$
f_i(x_i) = \begin{cases} (1 - 2\alpha_i\beta)(x_i + 1) - 1 & \text{if } 0 \leq \alpha_i < \frac{1}{2}; \\
(1 - \beta)(x_i + 1) - 1 & \text{if } \alpha_i \geq \frac{1}{2},
\end{cases}
$$

and (letting $t_n = 0$)

$$
\alpha_i = \min \left\{ \sum_{i \in S} |t_i - t_j| \bigg| S \subseteq [n] - \{i\} \text{ and } |S| = \left\lfloor \frac{n}{2} \right\rfloor \right\}.
$$

We must show $\Psi_K$ is continuous. Any possible discontinuity happens when $\beta$ approaches 1, in other words, when $(t_1, \ldots, t_{n-1})$ approaches the boundary of $D^{n-1} = [-1,1]^{(n-1)}$. To see that there is no issue here, notice $S \cap S'$ is non-empty for any $S \subseteq [n] - \{i\}$ and $S' \subseteq [n] - \{n\}$ such that $i \neq n$ and $|S| = |S'| = \left\lfloor \frac{n}{2} \right\rfloor$, implying $\max\{\alpha_n, \alpha_i\} \geq \frac{|t_i - t_n|}{2}$. So if $|t_i|$ approaches 1 for some $i \neq n$, since $t_n$ is always 0, at least one of $\alpha_n$ or $\alpha_i$ approaches some value $L \geq \frac{1}{2}$, so one of $f_i(x_i)$ or $f_n(x_n)$ approaches $-1$. In either case $\Psi_K(\sigma)$ approaches $*$. Clearly $\Psi_K$ satisfies the first condition of homotopy Golodness. Since $\hat{\Psi}_I = (D^1)^{|I|}$ whenever $|I| \leq \left\lfloor \frac{n}{2} \right\rfloor$, in order for $\Psi_K$ satisfy the second homotopy Golod condition, each $f_i(x_i)$ has freedom to taken on any value in $[-1,1]$ whenever there are no more than $\left\lfloor \frac{n}{2} \right\rfloor$ values of $j$ for which $t_j = t_i$, so defining $f_i$ as we did presents no difficulties here. On the other hand, when there are strictly more than $\left\lfloor \frac{n}{2} \right\rfloor$ values of $j$ for which $t_j = t_i$, we have $f_i(x_i) = x_i$, so there is no issue here either. Thus $\Psi_K$ satisfies the second condition, and $K$ is homotopy Golod.

**Example 5.12.** Let $K = (\partial \Delta^2 \ast \partial \Delta^1) \cup_{\partial \Delta^1} \Delta^1$ on vertex set $[5]$. Since every pair of distinct vertices in $K$ is connected by an edge, $K$ is highly complete, and therefore homotopy Golod. But $H_2(|K|) \cong \mathbb{Z}$, while $H_2(|K\setminus\{i\}|) = 0$ for each $i \in [5]$, so $K$ cannot be extractible.
11. ______, Configuration spaces with labels and loop spaces on $K$-products, Uspekhi Mat. Nauk 63 (2008), no. 6(384), 161–162. MR 2492776 (2011a:55016)

12. Albrecht Dold and René Thom, Quasifaserungen und unendliche symmetrische Produkte, Ann. of Math. (2) 67 (1958), 230–281. MR 0097062 (20 #3542)

13. M. Franz, The integral cohomology of toric manifolds, Tr. Mat. Inst. Steklova 252 (2006), no. Geom. Topol., Diskret. Geom. i Teor. Mnozh., 61–70. MR 2255969 (2007f:14050)

14. Tudor Ganea, Cogroups and suspensions, Invent. Math. 9 (1969/1970), 185–197. MR 0267582 (42 #3542)

15. Jelena Grbić, Taras Panov, Stephen Theriault, and Jie Wu, Homotopy types of moment-angle complexes for flag complexes, preprint, arXiv:1211.0873, to appear in Trans. Amer. Math. Soc.

16. Jelena Grbić and Stephen Theriault, Homotopy type of the complement of a coordinate subspace of codimension two, Uspekhi Mat. Nauk 59 (2004), no. 6(360), 203–204. MR 2138475 (2007f:55023)

17. ______, The homotopy type of the complement of a coordinate subspace arrangement, Topology 46 (2007), no. 4, 357–396. MR 2321037 (2008j:55023)

18. ______, The homotopy type of the polyhedral product for shifted complexes, Adv. Math. 245 (2013), 690–715. MR 3084441

19. Kouyemon Iriye and Daisuke Kishimoto, Topology of polyhedral products and the golod property of stanley-reisner rings.

20. ______, Decompositions of polyhedral products for shifted complexes, Adv. Math. 245 (2013), 716–736. MR 3084442

21. Sadok Kallel, Spaces of particles on manifolds and generalized Poincaré dualities, Q. J. Math. 52 (2001), no. 1, 45–70. MR 1820902 (2002g:55026)

22. Yael Karshon and Susan Tolman, Classification of Hamiltonian torus actions with two-dimensional quotients, Geom. Topol. 18 (2014), no. 2, 669–716. MR 3180483

23. Mikiya Masuda, Equivariant cohomology distinguishes toric manifolds, Adv. Math. 218 (2008), no. 6, 2005–2012. MR 2431667 (2009j:14067)

24. Dusa McDuff, Configuration spaces of positive and negative particles, Topology 14 (1975), 91–107. MR 0358766 (50 #11225)

25. Paolo Salvatore, Configuration spaces with summable labels, Cohomological methods in homotopy theory (Belaterra, 1998), Progr. Math., vol. 196, Birkhäuser, Basel, 2001, pp. 375–395. MR 1851264 (2002f:55039)

26. Graeme Segal, Configuration-spaces and iterated loop-spaces, Invent. Math. 21 (1973), 213–221. MR 0331377 (48 #9710)

27. Richard P. Stanley, Combinatorics and commutative algebra, second ed., Progress in Mathematics, vol. 41, Birkhäuser Boston, Inc., Boston, MA, 1996. MR 1453579 (98h:05001)

28. James Dillon Stasheff, Homotopy associativity of $H$-spaces. I, II, Trans. Amer. Math. Soc. 108 (1963), 275–292; ibid. 108 (1963), 293–312. MR 0158400 (28 #1623)

School of Mathematics, University of Southampton, Southampton SO17 1BJ, United Kingdom

E-mail address: P.D.Beben@soton.ac.uk

School of Mathematics, University of Southampton, Southampton SO17 1BJ, United Kingdom

E-mail address: J.Grbic@soton.ac.uk