Reduced qKZ equation: general case

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Abstract
We use the quantum group approach for the investigation of correlation functions of integrable vertex models and spin chains. For the inhomogeneous reduced density matrix in case of an arbitrary simple Lie algebra we find functional equations of the form of the reduced quantum Knizhnik–Zamolodchikov equation. This equation is the starting point for the investigation of correlation functions at arbitrary temperature and notably for the ground state.

Keywords: density operator, quantum Knizhnik–Zamolodchikov equation, quantum loop algebras

(Some figures may appear in colour only in the online journal)

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1. Introduction

In this paper we derive a difference-type functional equation, called the discrete reduced quantum Knizhnik–Zamolodchikov equation, for the density operator of a quantum integrable vertex model related to an arbitrary complex simple Lie algebra. Our setting allows for the study of correlation functions at finite and zero temperature in the thermodynamic limit, or alternatively of ground-state correlators on finite ring shaped and infinite chains. Throughout this paper we use methods based on the notion of a quantum group introduced by Drinfeld [1] and Jimbo [2]. To be precise, we consider quantum integrable systems related to a special class of quantum groups, namely the quantum loop algebras, see section 2.2 for the definition.

Our work aims at extending the previous work, see [3, 4] and later developments, related to systems based on the quantum loop algebra $U_q(\mathcal{L}(\mathfrak{sl}_2))$ which enjoys a simple crossing symmetry due to the equivalence of any representation with its dual. Some explorative investigations for a system based on the first fundamental representation of $U_q(\mathcal{L}(\mathfrak{sl}_3))$ allowed for the computation of nearest and next-nearest neighbour correlators for the associated quantum spin chain of XXX-type in the ground-state [5, 6]. In these works the necessity of dealing simultaneously with at least two different representations of the same quantum group became obvious. Furthermore, unitarity conditions involving different representations, and crossing relations for representations dual to each other appeared. Here we put such constructions on solid systematic grounds valid for arbitrary representations of any quantum group. Our constructions will allow for a uniformized investigation of correlation functions making ad-hoc constructions obsolete.

The central object of the quantum group approach is the universal $R$-matrix being an element of the tensor product of two copies of the quantum loop algebra. The integrability objects are constructed by choosing representations for the factors of that tensor product. The consistent application of the method for constructing integrability objects and proving their properties was initiated by Bazhanov, Lukyanov and Zamolodchikov [8–10]. They studied the quantum version of KdV theory. Later on the method proved to be efficient for studying other quantum systems.
integrable models. Accordingly, within the framework of this approach, $R$-operators \cite{11–17}, monodromy operators and $L$-operators were constructed \cite{7, 16–20}. The corresponding sets of functional relations were found and proved \cite{7, 18, 21–24}.

To derive the reduced qKZ equation one needs some special properties of the integrability object related to the quantum loop algebra under consideration. Namely, one uses the unitarity relations, crossing relations and the so-called initial condition. It appears that these relations, apart from the initial condition, follow from the properties of the universal $R$-matrix. The detailed discussion can be found in paper \cite{25}, see also paper \cite{26}.

The plan of the paper is as follows. In section 2 we introduce a quantum loop algebra, its universal $R$-matrix, and define the basic integrability objects called $R$-operators. Then we describe the properties of $R$-operators, such as the unitarity and crossing relations, necessary for the subsequent derivation of the reduced qKZ equation. This section is concluded by the definition of monodromy and transfer operators.

In section 3 we discuss the construction of the Hamiltonian of the system as a member of the system of commuting quantities. The aforementioned initial condition is also given here. We introduce a convenient normalization of the $R$-operators which leads to a simple form of the crossing and unitarity relations. Also the initial condition becomes simple. Then we remind of the definition of the density operator and represent it as the Trotter limit of some sequence of operators. Such a representation allows us to relate the density operator to the partition sum of some square lattice vertex model with the free horizontal boundaries.

A graphical derivation of the reduced qKZ equation is described in section 4, and the corresponding pictures with appropriate comments are placed in the appendix.

2. Quantum loop algebras and integrability objects

2.1. Preliminaries on Lie algebras

Let $\mathfrak{g}$ be a complex finite dimensional simple Lie algebra of rank $l$ \cite{27, 28}, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and $\Delta$ the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. Fix a system of simple roots $\alpha_i, i \in [1..l]$. It is known that the corresponding coroots $h_i$ form a basis of $\mathfrak{h}$, so that

$$\mathfrak{h} = \bigoplus_{i=1}^l \mathbb{C} h_i.$$  

The Cartan matrix $A = (a_{ij})_{i,j \in [1..l]}$ of $\mathfrak{g}$ is defined by the equation

$$a_{ij} = \langle \alpha_j, h_i \rangle.$$  

Denote by $\theta$ the highest root of $\mathfrak{g}$ \cite{27, 28}. We have

$$\theta = \sum_{i=1}^l a_i \alpha_i, \quad \theta^* = \sum_{i=1}^l a_i^* h_i$$  

for some positive integers $a_i$ and $a_i^*$ with $i \in [1..l]$. These integers, together with

$$a_0 = 1, \quad a_0^* = 1,$$

are the Kac labels and the dual Kac labels of the Dynkin diagram associated with the extended Cartan matrix $A^{(1)}$. Recall that the sums
\[ h = \sum_{i=0}^{l} a_i, \quad h^* = \sum_{i=0}^{l} a_i^* \]

are called the Coxeter number and the dual Coxeter number of \( g \).

Denote by \( \mathfrak{h} \) the Cartan subalgebra of \( g \) extended by a one dimensional center \( \mathbb{C} K \). We consider the simple roots \( \alpha_i, i \in [1..l] \), as elements of \( \mathfrak{h}^* \) assuming that

\[ \langle \alpha_i, K \rangle = 0. \]

Introduce an additional ‘root’

\[ \alpha_0 = -\theta \]

and an additional ‘coroot’

\[ h_0 = K - \theta^*. \]

After that for the entries of the extended Cartan matrix \( A^{(1)} = (a_{ij})_{i,j \in [0..l]} \) of \( g \) we have the expression

\[ a_{ij} = \langle \alpha_j, h_i \rangle. \]

### 2.2. Quantum loop algebras

Let \( \hbar \) be a nonzero complex number such that \( q = \exp \hbar \) is not a root of unity. We assume that

\[ q^\nu = \exp(\hbar \nu) \]

for any \( \nu \in \mathbb{C} \). As usually, we define the \( q \)-deformation of a number \( \nu \in \mathbb{C} \) as

\[ [\nu)_q = \frac{q^\nu - q^{-\nu}}{q - q^{-1}}. \]

Note that the extended Cartan matrix \( A^{(1)} \) is symmetrizable. It means that there exists a diagonal matrix \( D = \text{diag}(d_0, d_1, \ldots, d_l) \), where \( d_i, i \in [0..l] \), are positive integers, such that the matrix \( DA^{(1)} \) is symmetric. Such a matrix \( D \) is defined up to a nonzero scalar factor. We fix the integers \( d_i \) assuming that they are relatively prime and denote

\[ q_i = q^{d_i}. \]

The quantum loop algebra \( U_q(\mathcal{L}(g)) \) is a unital associative \( \mathbb{C} \)-algebra generated by the elements

\[ e_i, \quad f_i, \quad i = 0, 1, \ldots, l, \quad q^x, \quad x \in \mathfrak{h}, \]

satisfying the relations

\[ q^{\nu K} = 1, \quad \nu \in \mathbb{C}, \quad q^i q^j = q^{i+j}, \quad (2.1) \]

\[ q^e_i q^{-x} = q^{(\alpha_i,x)} e_i, \quad q^f_i q^{-x} = q^{-(\alpha_i,x)} f_i, \quad (2.2) \]

\[ [e_i, f_j] = \delta_{ij} q_i^h - q_i^{-h} q_i - q_i^{-1} h, \quad (2.3) \]
\[ \sum_{n=0}^{1-a_{ij}} (-1)^n \frac{c_{ij}^{1-a_{ij}+n}}{[1-a_{ij}+n]_q} f_{ij}^{n} = 0, \quad \sum_{n=0}^{a_{ij}} (-1)^n \frac{f_{ij}^{1-a_{ij}+n}}{[1-a_{ij}+n]_q} c_{ij}^{n} = 0. \]  

(2.4)

Here, relations (2.2) and (2.3) are valid for all \( i, j \in [0..l] \). The last line of the relations is valid for all distinct \( i, j \in [0..l] \).

The quantum loop algebra \( U_q(\mathcal{L}(\mathfrak{g})) \) is a Hopf algebra. Here the multiplication mapping \( \mu: U_q(\mathcal{L}(\mathfrak{g})) \otimes U_q(\mathcal{L}(\mathfrak{g})) \to U_q(\mathcal{L}(\mathfrak{g})) \) is defined as

\[ \mu(a \otimes b) = ab, \]

and for the unit mapping \( \iota: \mathbb{C} \to U_q(\mathcal{L}(\mathfrak{g})) \) we have

\[ \iota(\nu) = \nu \cdot 1. \]

The comultiplication \( \Delta \), the antipode \( S \), and the counit \( \varepsilon \) are given by the relations

\[ \Delta(q^i) = q^i \otimes q^i, \quad \Delta(e_i) = e_i \otimes 1 + q_i^h \otimes e_i, \quad \Delta(f_i) = f_i \otimes q_i^{-h} + 1 \otimes f_i, \]  

(2.5)

\[ S(q^i) = q^{-i}, \quad S(e_i) = -q_i^{-h} e_i, \quad S(f_i) = -f_i q_i^h. \]  

(2.6)

\[ \varepsilon(q^i) = 1, \quad \varepsilon(e_i) = 0, \quad \varepsilon(f_i) = 0. \]  

(2.7)

For the inverse of the antipode one has

\[ S^{-1}(q^i) = q^{-i}, \quad S^{-1}(e_i) = -e_i q_i^{-h}, \quad S^{-1}(f_i) = -q_i^h f_i. \]  

(2.8)

### 2.3. Universal R-matrix

Let \( \Pi \) be the automorphism of the tensor square of the algebra \( U_q(\mathcal{L}(\mathfrak{g})) \) defined by the equation

\[ \Pi(a \otimes b) = b \otimes a. \]

It is known that the mapping

\[ \Delta' = \Pi \circ \Delta \]

is a comultiplication in \( U_q(\mathcal{L}(\mathfrak{g})) \) called the opposite comultiplication.

Let \( U_q(\mathcal{L}(\mathfrak{g})) \) be a quantum loop algebra. There exists a unique element \( \mathcal{R} \) of the tensor product of \( U_q(\mathcal{L}(\mathfrak{g})) \otimes U_q(\mathcal{L}(\mathfrak{g})) \) connecting the two comultiplications as

\[ \Delta'(a) = \mathcal{R} \Delta(a) \mathcal{R}^{-1} \]

for any \( a \in U_q(\mathcal{L}(\mathfrak{g})) \), and satisfying in \( U_q(\mathcal{L}(\mathfrak{g})) \otimes U_q(\mathcal{L}(\mathfrak{g})) \otimes U_q(\mathcal{L}(\mathfrak{g})) \) the equations

\[ (\Delta \otimes \text{id})(\mathcal{R}) = \mathcal{R}^{(13)} \mathcal{R}^{(23)}, \quad (\text{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}^{(12)} \mathcal{R}^{(13)}. \]

The meaning of the superscripts in the above relations is explained in any textbook on quantum groups, see also the appendix of paper [25]. The element \( \mathcal{R} \) is called the universal \( R \)-matrix.

One can show that it satisfies the universal Yang–Baxter equation

\[ \mathcal{R}^{(12)} \mathcal{R}^{(13)} \mathcal{R}^{(23)} = \mathcal{R}^{(23)} \mathcal{R}^{(13)} \mathcal{R}^{(12)}. \]

in \( U_q(\mathcal{L}(\mathfrak{g})) \otimes U_q(\mathcal{L}(\mathfrak{g})) \otimes U_q(\mathcal{L}(\mathfrak{g})). \)
There are two main approaches to the construction of the universal $R$-matrix for a quantum loop algebra. One of them was proposed by Khoroshkin and Tolstoy [11, 29–31], and another one is related to the names of Beck and Damiani [32, 33]. It should be noted that we define the quantum loop algebra as a $C$-algebra. It can be also defined as a $C[[\hbar]]$-algebra, where $\hbar$ is considered as an indeterminate. In this case one really has the universal $R$-matrix. In our case, the universal $R$-matrix exists only in some restricted sense, see, for example, paper [34], and the corresponding discussion in paper [25] for the case of $U_q(L(sl_{l+1}))$.

As for any Hopf algebra, starting from two representations of $U_q(L(g))$, say $\varphi_1$ and $\varphi_2$, we construct a new representation $\varphi_1 \otimes_\Delta \varphi_2$ of $U_q(L(g))$ by the relation

$$\varphi_1 \otimes_\Delta \varphi_2 = (\varphi_1 \otimes \varphi_2) \circ \Delta.$$ 

The corresponding $U_q(L(g))$-module is denoted by $V_1 \otimes_\Delta V_2$, where $V_1$ and $V_2$ are the modules corresponding to the representations $\varphi_1$ and $\varphi_2$.

### 2.4. Spectral parameter

In applications to the theory of quantum integrable systems, one usually considers families of representations of a quantum loop algebra parametrized by a complex parameter called a spectral parameter. We introduce a spectral parameter in the following way. Assume that the quantum loop algebra $U_q(L(g))$ is $\mathbb{Z}$-graded,

$$U_q(L(g)) = \bigoplus_{m \in \mathbb{Z}} U_q(L(g))_m,$$

so that any element $a \in U_q(L(g))$ can be uniquely represented as

$$a = \sum_{m \in \mathbb{Z}} a_m, \quad a_m \in U_q(L(g))_m.$$

Given $\zeta \in \mathbb{C}^\times$, we define the grading automorphism $\Gamma_\zeta$ by the equation

$$\Gamma_\zeta(a) = \sum_{m \in \mathbb{Z}} \zeta^m a_m.$$

It is worth noting that

$$\Gamma_{\zeta_1 \zeta_2} = \Gamma_{\zeta_1} \circ \Gamma_{\zeta_2} \quad (2.9)$$

for any $\zeta_1, \zeta_2 \in \mathbb{C}^\times$. Now, for any representation $\varphi$ of $U_q(L(g))$ we define the corresponding family $\varphi_\zeta$ of representations as

$$\varphi_\zeta = \varphi \circ \Gamma_\zeta.$$

If $V$ is the $U_q(L(g))$-module corresponding to the representation $\varphi$, we denote by $V_\zeta$ the $U_q(L(g))$-module corresponding to the representation $\varphi_\zeta$.

The common way to endow $U_q(L(g))$ by a $\mathbb{Z}$-gradation is to assume that

$$q^i \in U_q(L(g))_0, \quad e_i \in U_q(L(g))_s, \quad f_i \in U_q(L(g))_{-s},$$

where $s$ are arbitrary integers. It is clear that for such a $\mathbb{Z}$-gradation one has

$$\Gamma_\zeta(q^i) = q^i, \quad \Gamma_\zeta(e_i) = \zeta^s e_i, \quad \Gamma_\zeta(f_i) = \zeta^{-s} f_i.$$

We denote

$$s = \sum_{i=0}^{l} a_i s_i,$$
where, as above, \( a_i \) are the Kac labels of the Dynkin diagram associated with the extended Cartan matrix of \( g \).

It follows from the explicit expression for the universal \( R \)-matrix [11, 29–33] that

\[
(\Gamma_\zeta \otimes \Gamma_\zeta)(R) = R
\]

for any \( \zeta \in \mathbb{C} \). Besides, equations (2.6) and (2.8) give

\[
S \circ \Gamma_\zeta = \Gamma_\zeta \circ S, \quad S^{-1} \circ \Gamma_\zeta = \Gamma_\zeta \circ S^{-1}.
\]

2.5. \( R \)-operators

Now recall the definition of an \( R \)-operator. Let \( V \) and \( W \) be \( U_q(\mathcal{L}(g)) \)-modules and \( \varphi \) and \( \psi \) the corresponding representations of \( U_q(\mathcal{L}(g)) \). The \( R \)-operator \( R_{V|W}(\zeta|\eta) \) is defined as

\[
\rho_{V|W}(\zeta|\eta)R_{V|W}(\zeta|\eta) = (\varphi \otimes \psi)(R).
\]

Here \( \zeta \) and \( \eta \) are spectral parameters, and \( \rho_{V|W}(\zeta|\eta) \) the normalization factor. We choose the normalization factor so that

\[
\rho_{V|W}(\zeta\nu|\eta\nu) = \rho_{V|W}(\zeta|\eta)
\]

for any \( \nu \in \mathbb{C}^\times \). In this case

\[
R_{V|W}(\zeta|\eta\nu) = R_{V|W}(\zeta|\eta),
\]

see paper [25].

Using (2.9) and (2.10), one can demonstrate that

\[
(\varphi_{\zeta\nu} \otimes \psi_{\eta\nu})(R) = (\varphi_\zeta \otimes \psi_\eta)(R)
\]

for any \( \nu \in \mathbb{C}^\times \). Therefore, under an appropriate choice of the normalization factor, \( R_{V|W}(\zeta|\eta) \) depends only on the combination \( \zeta\eta^{-1} \) and one can use \( R \)-operators depending on only one spectral parameter. Below we always use this choice of the normalization, however, for our purposes it is more convenient to consider \( R \)-operators as depending on two spectral parameters.

We use for the matrix elements of \( R_{V|W}(\zeta_1|\zeta_2) \) the depiction which can be seen in figure 1. Here we associate with \( V \) and \( W \) a single and a double lines respectively. It is worth to note that the indices in the graphical image go clockwise.

For the matrix elements of the inverse \( R_{V|W}(\zeta|\eta)^{-1} \) of the \( R \)-operator \( R_{V|W}(\zeta|\eta) \) we use the depiction given in figure 2. Here we use a grayed circle for the operator and the counter-clockwise order for the indices. This allows one to have a natural graphical form of the equation

\[
R_{V|W}(\zeta|\eta)^{-1}R_{V|W}(\zeta|\eta) = 1_{V\otimes W},
\]

see figure 3.

2.6. Unitarity relations

Let the \( U_q(\mathcal{L}(g)) \)-modules \( V \) and \( W \) be such that the module \( V_\zeta \otimes_\Delta W_\eta \) is simple for general values of the spectral parameters \( \zeta \) and \( \eta \). In this case the following unitarity relation holds:

\[
\tilde{R}_{V|W}(\zeta|\eta)\tilde{R}_{W|V}(\eta|\zeta) = C_{V|W}(\zeta|\eta)1_{W\otimes V}
\]

where \( C_{V|W}(\zeta|\eta) \) are such that all \( U_q(\mathcal{L}(g)) \)-modules under consideration are finite dimensional.
is valid. Here and in similar cases below we use the notations

$$R_{V|W}(\zeta|\eta)^{ia}j\beta$$

**Figure 1.** The matrix elements of an $R$-operator.

$$(R_{V|W}(\zeta|\eta)^{-1})^{ia}j\beta$$

**Figure 2.** The matrix elements of the inverse of an $R$-operator.

is valid. Here and in similar cases below we use the notations

$$\hat{R}_{V|W}(\zeta|\eta) = P_{V|W}R_{V|W}(\zeta|\eta), \quad \hat{R}_{W|V}(\eta|\zeta) = P_{W|V}R_{W|V}(\eta|\zeta)$$

with $P_{V|W}$ and $P_{W|V}$ being the permutation operators on the corresponding tensor products.

### 2.7. Crossing relations

For any finite dimensional $U_q({\mathcal{L}}(g))$-module $V$ one has two dual modules. One dual module is denoted by $V^*$ and is defined with the help of the antipode $S$, another one is denoted by $^*V$ and is defined with the help of the inverse of the antipode $S^{-1}$.

By a crossing relation we mean any relation connecting an $R$-operator $R_{V|W}(\zeta|\eta)$ with an $R$-operator for which one of the modules $V$ and $W$ (or both) is (are) replaced by a dual module. In this paper we will use the following three crossing relations. The first one is

$$R_{V^*|W}(\zeta|\eta) = \rho_{V^*|W}(\zeta|\eta)^{-1} \rho_{V|W}(\zeta|\eta)^{-1}(R_{V|W}(\zeta|\eta)^{-1})^\beta, \quad (2.11)$$

and the second one is

$$R_{V^*|W^*}(\zeta|\eta) = \rho_{V^*|W^*}(\zeta|\eta)^{-1} \rho_{V|W}(\zeta|\eta) R_{V|W}(\zeta|\eta)^\beta. \quad (2.12)$$
The double dual representation \( \varphi^{{**}} \) is isomorphic to \( \varphi_\zeta \) up to a redefinition of the spectral parameter. This leads to the third crossing relation. To describe it, we introduce the following element

\[
x = - \sum_{j=1}^{l} (2d_i - (\theta|\theta)h^i s_i/s) b_j h_j
\]

of \( \tilde{h} \), see [25]. Here \( b_j \) are the matrix elements of the matrix \( B \) inverse to the Cartan matrix \( A \) of the Lie algebra \( \mathfrak{g} \), and \((\cdot|\cdot)\) denotes invariant nondegenerate symmetric bilinear form on \( \mathfrak{g} \) normalized by the equation

\[
(\alpha|\alpha) = 2d_i.
\]

Now one can demonstrate that

\[
(X_V \otimes 1_W) R_{V|W} (q^{-\varepsilon} \zeta|\eta) (X_V^{-1} \otimes 1_W)
\]

\[
= \rho_{V|W} (q^{-\varepsilon} \zeta|\eta)^{-1} \rho_{V|W} (\zeta|\eta)^{-1} (R_{V|W} (\zeta|\eta)^{-1})^\ast.
\]

(2.13)

Here and below

\[
\varepsilon = (\theta|\theta)h^i/s
\]

and

\[
X_V = \varphi(q^\ast).
\]

This is the third crossing relation we need. More crossing relations and the corresponding proofs can be found in paper [25].

2.8. Monodromy and transfer operators

In the theory of quantum integrable statistical systems the matrix elements of an \( R \)-operator are treated as weights of the vertices of a square lattice. To find the corresponding partition function one introduces monodromy operators and the corresponding transfer operators. To define a monodromy operator we use instead of the \( U_q(\mathcal{L}(\mathfrak{g})) \otimes U_q(\mathcal{L}(\mathfrak{g})) \)-module \( V_\zeta \otimes W_\eta \), used in the definition of the \( R \)-operators, the \( U_q(\mathcal{L}(\mathfrak{g})) \otimes U_q(\mathcal{L}(\mathfrak{g})) \)-module

\[
V_\zeta \otimes (W_{1|\eta_1} \otimes \Delta W_{2|\eta_2} \otimes \Delta \cdots \otimes \Delta W_{l|\eta_l}),
\]

(2.14)

and define the monodromy operator \( M_{V|W_1,\ldots,W_l}(\zeta|\eta_1,\eta_2,\ldots,\eta_l) \) as

\[
\rho_{V|W_1}(\zeta|\eta_1) \rho_{V|W_2}(\zeta|\eta_2) \cdots \rho_{V|W_l}(\zeta|\eta_l) M_{V|W_1,\ldots,W_l}(\zeta|\eta_1,\eta_2,\ldots,\eta_l)
\]

\[
= (\varphi_\zeta \otimes (\psi_{1|\eta_1} \otimes \Delta \psi_{2|\eta_2} \otimes \Delta \cdots \otimes \Delta \psi_{l|\eta_l})) (\mathcal{R}).
\]

Using properties of the universal \( R \)-matrix one can see that

\[
M_{V|W_1,\ldots,W_l}(\zeta|\eta_1,\eta_2,\ldots,\eta_l) = R_{V|W_2}^{(1|L+1)} (\zeta|\eta_l) \cdots R_{V|W_2}^{(13)} (\zeta|\eta_2) R_{V|W_2}^{(12)} (\zeta|\eta_1).
\]

(2.15)

Here the meaning of the superscripts can be found again in any textbook on quantum groups. The factors of the tensor product (2.14) are numbered from right to left. The graphical representation of the matrix elements of the monodromy operator for the case \( W_1 = W_2 = \cdots = W_L = W \) can be found in figure 4. The modification needed for the general case is evident.

The transfer operator corresponding to the monodromy operator (2.15) is defined by the equation
$M_{V|W}^L(\zeta|\eta_1, \eta_2, \ldots, \eta_L) = \operatorname{tr}_V(M_{V|W_1, W_2, \ldots, W_L}(\zeta|\eta_1, \eta_2, \ldots, \eta_L))$

**Figure 4.** The matrix elements of a monodromy operator.

$T_{V|W_1, W_2, \ldots, W_L}(\zeta|\eta_1, \eta_2, \ldots, \eta_L) = \operatorname{tr}_V(M_{V|W_1, W_2, \ldots, W_L}(\zeta|\eta_1, \eta_2, \ldots, \eta_L))$

with the depiction for the case $W_1 = W_2 = \cdots = W_L = W$ given in figure 5. Here $\operatorname{tr}_V$ means the partial trace with respect to the space $V$, see, for example, the appendix of paper [25], and hooks at the ends of the line mean that it is closed in an evident way. The most important property of transfer operators is their commutativity

$[T_{V|W_1, W_2, \ldots, W_L}(\zeta|\eta_1, \eta_2, \ldots, \eta_L), T_{V_2|W_1, W_2, \ldots, W_L}(\zeta_2|\eta_1, \eta_2, \ldots, \eta_L)] = 0$. (2.17)

It is the source of commuting quantities of quantum integrable systems.

### 3. Density operator

#### 3.1. Commuting quantities and Hamiltonian

The transfer operator (2.16) acts on the $U_q(\mathfrak{g})$-module $W_{\eta_1} \otimes_\Delta \cdots \otimes_\Delta W_{\eta_L}$. As a vector space it is just $W^{\otimes L}$. We assume that $W = V$ and construct commuting quantities on $V^{\otimes L}$ as follows. First of all we denote

$T_L(\zeta) = T_{V|V}(\zeta|1, 1, \ldots, 1)$. 

It follows from (2.17) that the quantities

$I_m = \frac{1}{m!} \left( \zeta \frac{d}{d\zeta} \right)^m \log T_L(\zeta) \bigg|_{\zeta = 1}$

commute,

$[I_m, I_n] = 0, \quad m, n \in \mathbb{Z}_{>0}$. (3.1)

In fact we have one more operator $T_L(1)$ which commutes with all $I_m$. 

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The usual choice for the Hamiltonian is
\[ H_L = I_1. \]
Assume that the initial condition
\[ \mathcal{R}_{V|V}(\zeta|\xi) = c_V P_{V|V} \]
(3.2)
is valid for some nonzero constant \( c_V \). Here, as above, \( P_{V|V} \) is the permutation operator on \( V \otimes V \). One can demonstrate that in this case
\[ H_L = \sum_{i \in [1..L]} \left. \frac{d\mathcal{R}_{V|V}(\zeta|1)^{i+1}}{d\zeta} \right|_{\zeta=1}, \]
(3.3)
where we assume that
\[ \mathcal{R}_{V|V}(\zeta|1)^{L+1} = \mathcal{R}_{V|V}(\zeta|1)^{L+1}. \]
Thus, we have a local Hamiltonian. For the well known simple graphical derivation of relation (3.3) we refer to paper [25].

### 3.2. Normalization

In this paper we work with a fixed \( U_q(\mathfrak{g}) \)-module \( V \) and its dual \( V^* \). We choose the normalization of \( R_{V^*|V}(\zeta|\eta), R_{V|V^*}(\zeta|\eta) \) and \( R_{V^*|V^*}(\zeta|\eta) \) assuming that
\[ \rho_{V^*|V}(\zeta|\eta) = \rho_{V|V}(\zeta|\eta)^{-1}, \quad \rho_{V|V^*}(\zeta|\eta) = \rho_{V|V}(\zeta|\eta)^{-1}, \]
\[ \rho_{V^*|V^*}(\zeta|\eta) = \rho_{V|V}(\zeta|\eta). \]
In this case the crossing relation (2.11) implies
\[ R_{V^*|V}(\zeta|\eta) = (R_{V|V}(\zeta|\eta)^{-1})^h, \quad R_{V^*|V^*}(\zeta|\eta) = (R_{V|V^*}(\zeta|\eta)^{-1})^h. \]
The graphical image of these relations can be found in figures 6 and 7. Here and below, for the representation \( \varphi^* \) we use the dotted variant of the line used for the representation \( \varphi \).

It is clear that the crossing relation (2.12) takes the form
\[ R_{V^*|V^*}(\zeta|\eta) = R_{V|V}(\zeta|\eta)^f \]
(3.4)
and has the graphical image given in figure 8.

Starting from the crossing relation (2.13), we obtain in the case under consideration two equations

\[(X_V \otimes 1_V) R_{V|V'}(q^{-\epsilon} \zeta|\eta) (X_V^{-1} \otimes 1_V) = D(\zeta|\eta) (R_{V'|V}(\zeta|\eta)^{-1})^n, \tag{3.5}\]
\[(X_V \otimes 1_V) R_{V|V'} (q^{-\epsilon} \zeta|\eta) (X_V^{-1} \otimes 1_V) = D(\zeta|\eta)^{-1} (R_{V'|V'}(\zeta|\eta)^{-1})^n, \tag{3.6}\]

where

\[D(\zeta|\eta) = \rho_V (q^{-\epsilon} \zeta|\eta)^{-1} \rho_V (\zeta|\eta). \tag{3.7}\]

To give a graphical interpretation of these equations, we use for the matrix elements of the operator $X_V$ and its inverse the depiction given in figures 9 and 10. It can be demonstrated now that figures 11 and 12 represent the crossing relations (3.5) and (3.6).

We choose the normalization factor so that the matrix elements of $R_{V|V'}(\zeta|\eta)$ are rational functions of the spectral parameters, and $R_{V|V'}(\zeta|\eta)$ satisfies the unitarity relation

\[\tilde{R}_{V|V'}(\zeta|\eta) \tilde{R}_{V|V'}(\eta|\zeta) = 1_V \otimes 1_V, \tag{3.8}\]

see [35, propositions 9.5.3 and 9.5.5]. We give the graphical form of this relation and the equivalent one in figures 13 and 14.

The crossing relation (3.4) and the unitarity relation (3.8) lead to the unitarity relations in figures 15 and 16.

Using the equations depicted in figures 6, 14, 8 and 7 we come to the chain of equalities given in figure 17. We see that the $R$-operators $R_{V'|V}(\zeta|\eta)$ and $R_{V|V'}(\zeta|\eta)$ satisfy the unitarity relation given in figure 18, or the equivalent relation in figure 19.

Finally we assume that the initial condition (3.2) is satisfied. It follows from the unitarity relation (3.8) that in our case

\[c_V^2 = 1.\]

Possibly changing the sign of $R_{V|V'}(\zeta|\eta)$, without destroying the form of the unitarity and crossing relations, we make $c_V$ equal to 1. The resulting initial condition is depicted in figure 20. The crossing relation (2.12) has now the simple form

\[R_{V'|V'}(\zeta|\eta) = R_{V|V'}(\zeta|\eta)^2, \tag{3.9}\]

and it leads to another initial condition represented by figure 21.

3.3. Density operator

The density operator of a quantum statistical system with the Hamiltonian $H_L$ is given by the equation

\[\rho = \sum \{ ... \}.\]
Figure 9. The matrix elements of the operator $X_v$.

Figure 10. The matrix elements of the inverse of the operator $X_v$.

Figure 11. The crossing relation (3.5).

Figure 12. The crossing relation (3.6).

Figure 13. The unitarity relation (3.8).

Figure 14. The unitarity relation equivalent to (3.8).

Figure 15. A combination of (3.4) and (3.8).
Figure 16. Another combination of (3.4) and (3.8).

Figure 17. The proof of the equation given in figure 18.

Figure 18. The equation which follows from the chain of equalities in figure 17.

Figure 19. Another version of the equation given in figure 18.

Figure 20. The first version of the initial condition.

Figure 21. The second version of the initial condition.
\[ D_L = \frac{1}{Z_L} e^{-\beta H_L}, \quad \beta = \frac{1}{kT}, \]

where \( Z_L \) is the partition function of the system defined as
\[ Z_L = \text{tr} e^{-\beta H_L}. \]

The expectation value of an arbitrary observable \( F \) is
\[ \langle F \rangle = \frac{1}{Z_L} \text{tr} (F e^{-\beta H_L}). \]

Let us exploit the relation of the Hamiltonian \( H_L \) with the transfer operator \( T_L(\zeta) \). To this end we introduce the ‘additive’ spectral parameter \( u \) related to the ‘multiplicative’ spectral parameter \( \zeta \) by the relation
\[ q^u = e^{hu} = \zeta. \]

Slightly abusing notation, we denote by \( T_L(u) \) the transfer operator \( T_L(\zeta) \) expressed as a function of \( u \). Now we have
\[ I_m = \frac{1}{m! h^m} \frac{d^m}{d u^m} \log T_L(u) \bigg|_{u=0}. \]

and it is not difficult to see that
\[ T_L(u) = T_L(0) \exp \left( \sum_{m=0}^{\infty} (h u)^m I_m \right). \]

We consider one more transfer operator related to the module \( V^* \) and defined as
\[ T_L^*(\zeta) = T_{V^*} \mid_{\zeta \mid_{1,1,...,1}}. \]

It generates one more set of commuting quantities
\[ I_m^* = \frac{1}{m!} \left( \zeta \frac{d}{d \zeta} \right)^m \log T_L^*(\zeta) \bigg|_{\zeta=1}. \]

In fact, in addition to (3.1), we have
\[ [I_m^*, I_n^*] = 0, \quad [I_m^*, I_n^*] = 0, \quad m, n \in \mathbb{Z}_{>0}. \]

The operators \( T_L(1) \) and \( T_L^*(1) \) commute with all \( I_m \) and all \( I_m^* \). In fact, \( T_L(1) \) is the left shift and \( T_L^*(1) \) is the right shift, and we have
\[ T_L(1) T_L^*(1) = 1_{V^* L}. \]

In terms of the additive spectral parameter \( u \) we have
Using the crossing relation given in figure 6, we obtain the equation represented by figure 22. Starting from this equation, we determine that

\[ I_1^* = -I_1 = -H_L. \]

For any positive integer \( K \) we can write

\[
T_L(0)^{-K} T_L(u/2\hbar K)^K = \exp \left( \frac{\mu}{2} I_1 + \sum_{m=1}^{\infty} K^{-m} \left( \frac{\mu}{2} \right)^{m+1} I_{m+1} \right),
\]

\[
T_L^*(0)^{-K} T_L^*(-u/2\hbar K)^K = \exp \left( -\frac{\mu}{2} I_1^* - \sum_{m=1}^{\infty} K^{-m} \left( \frac{\mu}{2} \right)^{m+1} (-1)^m I_{m+1}^* \right).
\]

These equations give

\[
T_L^*(-u/2\hbar K)^K T_L(u/2\hbar K)^K
= \exp \left( \frac{1}{2} \mu (I_1 - I_1^*) + \sum_{m=1}^{\infty} K^{-m} \left( \frac{\mu}{2} \right)^{m+1} (I_{m+1} - (-1)^m I_{m+1}^*) \right),
\]

and we see that

\[
\lim_{K \to \infty} (T_L^*(-u/2\hbar K)T_L(u/2\hbar K))^K = \exp(uH_L).
\]

Denote

\[
D_{L,K} = \frac{(T_L^*(\beta/2\hbar K)T_L(-\beta/2\hbar K))^K}{Z_{L,K}},
\]

where

\[
Z_{L,K} = \text{tr}(T_L^*(\beta/2\hbar K)T_L(-\beta/2\hbar K))^K.
\]

Finally, using the multiplicative spectral parameter, we obtain

\[
Z_{L,K} D_{L,K} = (T_L^*(q^{\beta/2K})T_L(q^{-\beta/2K}))^K. \tag{3.10}
\]

It is clear that

\[
D_L = \lim_{K \to \infty} D_{L,K}.
\]

3.4. Density operator as the partition function of a vertex model

It follows from (3.10) that the matrix elements of the operator \( Z_{L,K} D_{L,K} \) can be represented as the partition function of a vertex model on a square lattice, see figure 23. Here we have the periodic boundary conditions in the horizontal direction and open top and bottom boundaries. The thermodynamic limit would be obtained when \( L, K \to \infty \). However, the existence of the limit over \( L \) is quite problematic. Therefore, we proceed to the density operator which allows one to find expectation values only for local observables. To this end we assume that \( L = 2m + n \), where \( m \) and \( n \) are positive integers. We consider \( n \) as fixed and take the trace.
of $Z_{L,K} D_{L,K}$ over the first and the last $m$ spaces associated with vertical directions of the lattice. We denote the corresponding ‘density operator’ as $D_{n,K,m}$ and the corresponding ‘partition function’ as $Z_{n,K,m}$. The density operator of interest is certainly the limit at $m \to \infty$ and $K \to \infty$. We assume that these limits commute, see a discussion in paper [36], so that

$$D_n = \lim_{m \to \infty} \lim_{K \to \infty} D_{n,K,m} = \lim_{K \to \infty} \lim_{m \to \infty} D_{n,K,m}.$$ 

The operator $D_n$ is called the reduced density operator. It can be used to find the average value of any observable localized on a subchain of length $n$. To go further we generalize the objects under consideration in the following way.

We have horizontal transfer operators and vertical monodromy and transfer operators defined in an evident way. We supply a horizontal transfer operator with the spectral parameters $\zeta_1, \ldots, \zeta_K$ or $\xi_1, \ldots, \xi_K$ in dependence on whether it is the operator $T$ or the operator $T^\ast$, see figure 24, the meaning of the boxes is explained below. The vertical monodromy operators are endowed with the spectral parameters $\eta_1, \ldots, \eta_n$. Thus, we consider the generalized density operator

$$D_{n,K,m}(\zeta_1, \ldots, \zeta_K, \xi_1, \ldots, \xi_K | \eta_1, \ldots, \eta_n).$$

Below, if it does not lead to misunderstanding, we omit the explicit designation of dependence on $\zeta_1, \ldots, \zeta_K$ and $\xi_1, \ldots, \xi_K$.

After all we introduce some twisting for the vertical transfer and monodromy operators. In the framework of the quantum group approach a twisting is defined by a choice of a group-like element [7, 25]. Remember that an element $a$ of a Hopf algebra is called group-like if

$$\Delta(a) = a \otimes a.$$ 

In our case the element

$$a = q^{\sum_{i=1}^n \nu_i h_i}$$

is group-like for any complex number $\nu_i$. We denote
and use for the matrix elements of the operator \( A_V(\nu) \) and its inverse the depiction given in figures 25 and 26. One can demonstrate the validity of the graphical equations represented by figures 27 and 28.

It follows from the definition of a group-like element that the operator \( A_V(\nu) \) satisfies a useful equation whose graphical image is represented by figure 29. It is also clear that

\[
A_V(\nu_1 + \nu_2) = A_V(\nu_1)A_V(\nu_2) = A_V(\nu_2)A_V(\nu_1).
\]

This relation leads to a modified version of the graphical equation 29 which can be seen in figure 30. We also need the commutativity equation given in figure 31.

We introduce disorder parameters \( \alpha_1, \ldots, \alpha_l \) and twist the first \( m \) vertical transfer operators. The introduction of disorder parameters regularizes the problem in the case of \( U_q(\mathfrak{s}\mathfrak{l}_2) \) [37, 38]. Further, we introduce parameters \( \kappa_1, \ldots, \kappa_l \) and twist all vertical transfer and monodromy operators, see figure 24. This can be interpreted as turning on a ‘magnetic field’. It should be noted that all equally twisted transfer operators commute.

Denote by \( \mathcal{V} \) the vertical space,

\[
\mathcal{V} = V \otimes V^* \otimes \cdots \otimes V \otimes V^*.
\]
\[ A_{V}(\nu)_{ij} \]

**Figure 25.** The matrix elements of the operator \( A_V(\nu) \).

\[ (A_{V}(\nu)^{-1})_{ij} \]

**Figure 26.** The matrix elements of the inverse of the operator \( A_V(\nu) \).

\[ \begin{array}{c}
\vspace{10pt}
\end{array} \]

**Figure 27.** The first version of the ‘crossing relation’ for the operator \( A_V(\nu) \).

\[ \begin{array}{c}
\vspace{10pt}
\end{array} \]

**Figure 28.** The second version of the ‘crossing relation’ for the operator \( A_V(\nu) \).

\[ \begin{array}{c}
\vspace{10pt}
\end{array} \]

**Figure 29.** The commutativity of the twisting and an \( R \)-operator.

\[ \begin{array}{c}
\vspace{10pt}
\end{array} \]

**Figure 30.** Modified form of the commutativity of the twisting and an \( R \)-operator.

\[ \begin{array}{c}
\vspace{10pt}
\end{array} \]

**Figure 31.** The commutativity of the twisting and a transfer operator.
We use for a vertical monodromy operator, twisted with the parameters \(\nu_1, \ldots, \nu_l\) the notation \(\mathcal{M}^\nu(\zeta_1, \xi_1, \ldots, \zeta_K, \xi_K|\eta)\). It acts on the space \(\mathcal{V} \otimes \mathcal{V}\). In fact, we have
\[
\mathcal{M}^\nu(\zeta_1, \xi_1, \ldots, \zeta_K, \xi_K|\eta) = \left(\\left( (\mathcal{V} \otimes \Delta' \cdot \mathcal{V}^{\eta} \otimes \Delta' \cdot \mathcal{V} \mathcal{V}^{\eta} \otimes \mathcal{V}^{\eta} \otimes \mathcal{V}^{\eta}) \otimes \varphi_\eta \right)(1_\mathcal{V} \otimes \Delta_\nu(\nu)) \right).
\]

It is useful to represent a vertical monodromy operator as
\[
\mathcal{M}(\zeta_1, \xi_1, \ldots, \zeta_K, \xi_K|\eta) = \sum_{i,j} \mathcal{M}(\zeta_1, \xi_1, \ldots, \zeta_K, \xi_K|\eta)_{ij} E_i^j,
\]
where \(E_i^j\) are unit operators on \(\mathcal{V}\) associated with the used basis, and the operators \(\mathcal{M}(\eta)_{ij}\) act on \(\mathcal{V}\). The vertical transfer operator \(\mathcal{T}(\eta)\) is defined as
\[
\mathcal{T}(\zeta_1, \xi_1, \ldots, \zeta_K, \xi_K|\eta) = \text{tr}_\mathcal{V} \mathcal{M}(\zeta_1, \xi_1, \ldots, \zeta_K, \xi_K|\eta) = \sum_i \mathcal{M}(\zeta_1, \xi_1, \ldots, \zeta_K, \xi_K|\eta)_{ii}.
\]

It acts on the vertical space \(\mathcal{V}\). We extend to the vertical monodromy and transfer operators the convention to omit the explicit designation of dependence on \(\zeta_1, \ldots, \zeta_K\) and \(\xi_1, \ldots, \xi_K\).

Looking at figure 24, it is easy to see that
\[
D_{\alpha, K, m}(\eta_1, \ldots, \eta_l)^{1 \cdots 1}_{j_1 \cdots j_l} = \frac{\text{tr}\left((T^\infty)^m \mathcal{M}^{\infty}(\eta_1)^{1}_{j_1} \cdots \mathcal{M}^{\infty}(\eta_l)^{1}_{j_l} (T^{\infty+\alpha})^m\right)}{\text{tr}\left((T^\infty)^m \mathcal{T}^{\infty}(\eta_1) \cdots \mathcal{T}^{\infty}(\eta_l) (T^{\infty+\alpha})^m\right)}.
\]

Here and below we write instead of \(\mathcal{T}^\nu(1)\) just \(\mathcal{T}\).

Generalizing the conjecture made in [39], we assume that the transfer operators \(\mathcal{T}^\nu(\eta)\) and \(\mathcal{T}^{\infty+\alpha}(\eta)\) are diagonalizable. Due to the commutativity of the vertical transfer operators \(\mathcal{T}^\nu(\eta)\) with different spectral parameters \(\eta\), their eigenvectors can be chosen independently of \(\eta\). Let the eigenvectors \(v^\nu_a\) of \(\mathcal{T}^\nu(\eta)\) form a basis of the vertical space. We have
\[
\mathcal{T}^\nu(\eta)v^\nu_a = \lambda^\nu_a(\eta) v^\nu_a,
\]
where \(\lambda^\nu_a(\eta)\) are the corresponding eigenvalues. Denote by \(v_a^\nu\) the vectors forming the dual basis, so that
\[
\langle v_a^\nu, v_b^\nu \rangle = \delta_{ab}.
\]

Now we have
\[
\text{tr}\left((T^\infty)^m \mathcal{M}^{\infty}(\eta_1)^{1}_{j_1} \cdots \mathcal{M}^{\infty}(\eta_l)^{1}_{j_l} (T^{\infty+\alpha})^m\right) = \sum_a \lambda^\nu_a(\eta) \langle v_a^\nu, v_b^\nu \rangle = \sum_a \lambda^\nu_a(\eta)
\]

where instead of \(\lambda^\nu_a(1)\) we write just \(\lambda^\nu_a\). In a similar way we obtain
\[
\text{tr}\left((T^\infty)^m \mathcal{T}^{\infty}(\eta_1) \cdots \mathcal{T}^{\infty}(\eta_l) (T^{\infty+\alpha})^m\right) = \sum_{a,b} \lambda^\nu_a(\eta) \lambda^\nu_b(\eta) \langle v_a^\nu, v_b^\nu \rangle = \sum_{a,b} \lambda^\nu_a(\eta) \lambda^\nu_b(\eta) \langle v_a^\nu, v_b^\nu \rangle.
\]

Following again paper [39], we assume that the eigenvalues \(\lambda^\nu_a\) of \(\mathcal{T}\) and \(\mathcal{T}^{\infty+\alpha}\) with the maximal absolute value are non-degenerate. In this case in the limit \(m \to \infty\) we get
Here we assume also that
\[\langle \psi_0^{\kappa, \alpha}, v_0^{\alpha} \rangle \neq 0, \quad \langle \psi_0^{\kappa+\alpha}, v_0^\alpha \rangle \neq 0.\]

4. Reduced qKZ equation

In this section we describe a graphical derivation of the discrete reduced qKZ equation for an arbitrary quantum loop algebra and consider the zero temperature limit. It appears that in the general case it is convenient to split the equation into two equations and consider them separately.

4.1. First equation

The graphical derivation of the first equation is given in figure A1–A8 with appropriate comments. The sought equation arises from comparison of figures A1 and A8. Looking at figure A8, we see that it is constructive to generalize the concept of density operator. Namely, new operators are also described by the picture similar to figure 24. However, some vertical cut lines can be associated with the dual representation \(\varphi^*\), which is reflected by using a dotted line. We denote the corresponding monodromy and transfer operators as \(\tilde{\mathcal{M}}^{*\nu}(\eta)\) and \(\tilde{T}^{*\nu}(\eta)\).

To be more precise, we illustrate our definition by the following analytical expression

\[D_{n,k}(\eta_1, \ldots, \eta_n)^{\gamma_1 \ldots \gamma_k}_{j_1 \ldots j_n} = \frac{\langle \psi_0^{\kappa}, \mathcal{M}^\kappa(\eta_1)^{\gamma_1}_{j_1} \cdots \mathcal{M}^\kappa(\eta_k)^{\gamma_k}_{j_k} v_0^{\kappa+\alpha} \rangle}{\lambda_0^\kappa(\eta_1) \cdots \lambda_0^\kappa(\eta_k) \langle \psi_0^{\kappa}, v_0^{\kappa+\alpha} \rangle}.\]

Here we use for the corresponding spectral parameter \(\eta_k\) the notation \(\eta_k^*\), having in mind that it is actually \(\eta_k\) but associated with the dual representation. Using the commutativity of the vertical transfer matrices \(T^{\nu}(\eta)\) and \(\tilde{T}^{*\nu}(\eta)\), we assume that \(v_0^\nu\) are also eigenvectors of \(\tilde{T}^{*\nu}(\eta)\) and mark the corresponding eigenvalues by an asterisk, so that

\[\tilde{T}^{*\nu}(\eta)v_0^\nu = \lambda_{n}^{*\nu}(\eta)v_0^\nu.\]

If we take the operator graphically described by figure A1, divide it by the scalar \(Z_{n,k,m}(\eta_1, \ldots, \eta_n)\), put \(\eta_l = \zeta_l\) and take the limit \(m \to \infty\), we obtain the action of some linear operator \(A_0(\eta_1, \ldots, \eta_{n-1}, \zeta_1)\) on the operator \(D_{n,k}(\eta_1, \ldots, \eta_{n-1}, \zeta_1)\). Applying this procedure to the operator given in figure A8, we come to the expression

\[\lambda_0^\kappa(\zeta_1)\lambda_0^{*\kappa + \alpha}(\zeta_1)\frac{\langle \psi_0^{\kappa}, \mathcal{M}^\kappa(\eta_1)^{\gamma_1}_{j_1} \cdots \mathcal{M}^\kappa(\eta_{n-1})^{\gamma_{n-1}}_{j_{n-1}} \cdots \mathcal{M}^\kappa(\eta_k)^{\gamma_k}_{j_k} v_0^{\kappa+\alpha} \rangle}{\lambda_0^\kappa(\zeta_1)\lambda_0^\kappa(\eta_{n-1}) \cdots \lambda_0^\kappa(\eta_1) \langle \psi_0^{\kappa}, v_0^{\kappa+\alpha} \rangle} = \lambda_0^{*\kappa + \alpha}(\zeta_1)\lambda_0^{*\kappa}(\eta_1)D_{n,k}(\eta_1, \ldots, \eta_{n-1}, (q\zeta_1)^{\gamma_1 \ldots \gamma_k}_{j_1 \ldots j_k}).\]

It is worth to remind here that \(\varepsilon = (\theta/\theta)h^\nu/s\).

Consider now the product \(\tilde{T}^{*\nu}(q\zeta_1)\tilde{T}^{\nu}(\zeta_1)\). It is represented by the left picture in figure 32. Successively applying the crossing relations represented by figures 11, 12 and 27, the unitarity relations represented by figures 14 and 18, the commutativity of the operators \(X_\nu\) and \(A_\nu(\nu)\), and the initial condition represented by figure 20, we come to the middle picture. Here we acquire the scalar factor \(\prod_{l=1}^K D^{-1}(q\zeta_1|\zeta_1)D(q\zeta_1|\zeta_1)\). Finally, using the unitarity relations represented by figures 13 and 19, we get the right picture. Thus, we have the equation
\( T^{*\nu}(q^\nu \zeta_1) T^{\nu}(\zeta_1) = \left( \prod_{i=1}^{K} D_{i}^{-1}(q^\nu \zeta_1|\zeta_i) D(q^\nu \zeta_1|\zeta_i) \right) 1_{V}. \) \hspace{1cm} (4.1)

or in terms of the eigenvalues

\( \lambda^{*\nu}_n(q^\nu \zeta_1) \lambda^{\nu}_n(\zeta_1) = \prod_{i=1}^{K} D_{i}^{-1}(q^\nu \zeta_1|\zeta_i) D(q^\nu \zeta_1|\zeta_i). \)

Remembering now about the factor we acquired in transition from figure A1 to figure A8 described in the appendix, we conclude that the comparison of these figures gives the equation

\[ \frac{\lambda_0^\nu(\zeta_1)}{\lambda_0^{n+\alpha}(\zeta_1)} A_n(\eta_1, \ldots, \eta_{n-1}, \zeta_1)(D_{n,K}(\eta_1, \ldots, \eta_{n-1}, \zeta_1)) \]

\[ = D_{n,K}(\eta_1, \ldots, \eta_{n-1}, (q^\nu \zeta_1)^*). \] \hspace{1cm} (4.2)

This equation, together with (4.5), can be used, in particular, for the investigation of the correlation functions at finite non-zero temperature. However, the necessity to fix some spectral parameters leads to some problems and the additional work is required. Here we consider the zero temperature limit which is obtained as follows. We put \( \zeta_i = q^{\beta/2K} \) and \( \xi_i = q^{\beta/2K} \) and take the limit \( \beta \to \infty, K \to \infty \), keeping the ratio \( \beta/K \) fixed and equal to \(-2\log \eta_n/\hbar\). The resulting equation is

\[ \phi(\eta_n) A_n(\eta_1, \ldots, \eta_{n-1}, \eta_n)(D_n(\eta_1, \ldots, \eta_{n-1}, \eta_n)) = D_n(\eta_1, \ldots, \eta_{n-1}, (q^\nu \zeta_n)^*). \] \hspace{1cm} (4.3)

where

\[ \phi(\eta) = \lim_{K \to \infty} \lambda_0^\nu(\eta|\eta, \eta^{-1}, \ldots, \eta, \eta^{-1})/\lambda_0^{n+\alpha}(\eta|\eta, \eta^{-1}, \ldots, \eta, \eta^{-1}). \]

Figure 32. The proof of equation (4.1).
Certainly, in the case $\alpha_i = 0$, $i \in [1..l]$, we have $\phi(\eta) = 1$.

It is constructive to give the graphical image of equation (4.3). Below, using pictures, we assume that $n = 3$. It is enough for understanding the general situation. It is clear that figure 33 depicts equation (4.3). Fat dots in the picture mean changing the interpretation of the type of line. Namely, an input line corresponding to a representation is treated as the output line corresponding to the dual representation and so on. Note that the order of the vector spaces is from the right to the left. Cut the red line below the box with the label $\alpha$ in the left hand side of this equation and slightly deform the picture to obtain figure 34.

Remember that if $V$ is finite dimensional, the space $\text{End}(V)$ of linear operators on $V$ can be identified with the space $V \otimes V^*$. To this end one defines the mapping $\iota_V: V \otimes V^* \to \text{End}(V)$ by the equation

$$\iota_V(v \otimes \psi)u = v \langle \psi, u \rangle.$$

One can show that it is a bijective mapping. Using it, one defines the mapping from $\text{End}(V)$ to $\text{End}(V \otimes V^*)$ which sends an operator $F$ to the operator

$$F^\ast = \iota_V^{-1}(F^{-1}) \otimes \iota_V^{-1}(F^\ast).$$

We numerate the vector spaces as $0, 1, \ldots, n$. Now we can write the analytical expression for the figure 34. It is not difficult to generalize it to the case of an arbitrary $n$. Taking the trace over the additional space, we come to the following analytical expression for the first equation

$$\phi(\eta) tr_0 (A^{(0)}_V(\alpha) \tilde{R}^{(0)}_V(\eta_1 | \eta_n) \ldots \tilde{R}^{(n-2, n-1)}_V(\eta_{n-1} | \eta_n) D^{(0, 1, \ldots, n-1)}(\eta_1, \ldots, \eta_{n-1}, \eta_n)) X^{n-1, n-1}_V \tilde{R}^{(n-2, n-1)}_V(\eta_n | \eta_{n-1}) \ldots \tilde{R}^{(0)}_V(\eta_n | \eta_1) = D_n(\eta_1, \ldots, \eta_{n-1}, (q^\ast \eta_n)^\ast).$$

(4.4)

4.2. Second equation

The graphical proof of the second equation is very similar to the proof of the first one. The initial and the final points can be found in figures A9 and A10. If we take the operator...
depicted in figure A9, divide it by $Z_{η,K,m}(η_1, \ldots, η_{n-1}, η^*_1)$, put $η_m = ξ_1$ and take the limit $m \to ∞$, we obtain the action of some linear operator $B_n(η_1, \ldots, η_{n-1}, ξ^*_1)$ on the operator $D_{n,K}(η_1, \ldots, η_{n-1}, ξ^*_1)$. Applying this procedure to the operator of figure A10, we come to the expression

$$\lambda^∗(ξ_1)B_n(η_1, \ldots, η_{n-1}, ξ^*_1)(D_{n,K}(η_1, \ldots, η_{n-1}, ξ^*_1) = D_{n,K}(η_1, \ldots, η_{n-1}, ξ_1).$$

(4.5)

The zero temperature limit is obtained as follows. We put $ξ_i = q^{-β/2K}$ and $ξ_1 = q^{β/2K}$ and take the limit $β \to ∞, K \to ∞$, keeping the ratio $β/K$ fixed and equal to $2 \log η_m/h$. The resulting equation is

In a similar way as for equation (4.1) we obtain

$$\mathcal{T}^ν(ξ_1)\mathcal{T}^{ν^∗}(ξ_1) = 1, \nu,$$

or in terms of the eigenvalues

$$\lambda^ν(ξ_1)\lambda^{ν^∗}(ξ_1) = 1.$$

Using this relation, we see that the comparison of figures A9 and A10 leads to the equation

$$\frac{\lambda^∗(ξ_1)}{\lambda^{∗∗}(ξ_1)}B_n(η_1, \ldots, η_{n-1}, ξ^*_1)(D_{n,K}(η_1, \ldots, η_{n-1}, ξ^*_1)) = D_{n,K}(η_1, \ldots, η_{n-1}, ξ_1).$$

Figure 34. A preparation for proving equation (4.4).
\[ \phi^*(\eta_1) = \lim_{K \to \infty} \lambda_{0}^{\kappa}(\eta_{0}^{-}\eta_{1}, \eta_{0}, \ldots, \eta_{0}^{-}, \eta_{0}^{-})/\lambda_{0}^{\kappa_{\alpha}}(\eta_{0}^{-}, \eta_{1}, \eta_{0}^{-}, \ldots, \eta_{0}^{-}, \eta_{0}^{-}). \]
In the case $\alpha_i = 0$, $i \in [1 \ldots l]$, we have $\phi^*(\eta) = 1$.

It is clear that figure 35 depicts equation (4.6). Cut the dotted red line in the left hand side of this equation below the box with the label $\alpha$ and slightly deform the picture to obtain figure 36. Writing the analytical expression for the figure 36, generalizing to the case of an arbitrary $n$, and taking the trace over the additional space, we come to the following analytical expression for the second equation

$$
\phi^*(\eta_n) \prod_0 (A^{(0)}_V (\alpha) R^{(01)}_{V|V^*} (\eta_l | \eta_l) \ldots R^{(n-2,n-1)}_{V|V^*} (\eta_{n-1} | \eta_{n-1}))
\prod_0 (D_n (\eta_l, \ldots \eta_{n-1} | \eta_l)) = D_n (\eta_l, \ldots \eta_{n-1} | \eta_l).
$$

### 4.3. Full rqKZ equation

Combining equations (4.3) and (4.6), we come to the final reduced qKZ equation

$$
\phi^*(q^* \eta_n) \phi(\eta_n) R_n (\eta_1, \ldots \eta_{n-1}, q^* \eta_n) (A_n (\eta_1, \ldots \eta_{n-1}, \eta_n) (D_n (\eta_1, \ldots \eta_{n-1}, \eta_n)))
= D_n (\eta_1, \ldots \eta_{n-1}, q^* \eta_n).
$$

The graphical image of this equation can be obtained by combining the graphical equations given in figures 33 and 35, see figure 37. Now we have two additional spaces, $V$ and $V^*$, and numerate the spaces as $0', 0, 1, \ldots, n$. Combining equations (4.4) and (4.7), we obtain the following full reduced qKZ equation

$$
\phi(\eta_n) \phi^*(q^* \eta_n) \prod_0 (A^{(0)}_V (\alpha) \prod_{0'} (R^{(01)}_{V|V^*} (\eta_l | q^* \eta_l) \ldots R^{(n-2,n-1)}_{V|V^*} (\eta_{n-1} | q^* \eta_{n-1})))
\prod_0 (D_n (\eta_l, \ldots \eta_{n-1} | \eta_l)) = D_n (\eta_1, \ldots \eta_{n-1}, q^* \eta_n).
$$
This is the main result of the present paper. In fact, we have the equation satisfied by the zero temperature correlation functions of the chain associated with the loop Lie algebra $U_q(L(g))$. To investigate correlation functions at arbitrary temperature one should return to equations (4.2) and (4.5).

5. Conclusions

We have derived the reduced qKZ equation for the quantum integrable system related to an arbitrary quantum loop algebra. The main feature of the general case compared to the simplest $\mathfrak{sl}_2$-case is that the first fundamental representation does not coincide with its dual, and so, to obtain a closed form of the reduced qKZ equation, two successive steps are needed. We have demonstrated that all necessary unitarity and crossing relations follow from the properties of the algebra. The status of the initial condition is not completely clear. From one side, we do not see how it can be obtained from the properties of the algebra. From the other side, as we know, all $R$-operators found in the framework of the quantum group approach satisfy this condition.

For the case of $U_q(L(\mathfrak{sl}_2))$ the reduced qKZ equation was derived in the thesis [40], see also [41]. The case of $U_q(L(\mathfrak{sl}_3))$ for $q = 1$ was treated in [5] and, using an alternative approach, in [6]. For comparison with the results of the present paper, the following points should be kept in mind. In the case of $U_q(L(\mathfrak{sl}_2))$ one meets the situation when the initial representation $\varphi_\zeta$ is isomorphic to the dual representation $\varphi_\zeta^*$ up to a rescaling of the spectral parameter. It is clear that in this case after the corresponding redefinitions one can use only the first equation. As the result one comes to the equation obtained in the thesis [40]. Note that in the general case, instead of the dual representation, one can use any representation isomorphic to it. In paper [5] as the initial representation the first fundamental representation was used, and the second fundamental representation instead of its dual. Taking this in mind, one can demonstrate that the equation obtained in [5] can be reduced to our equation.

Our result refers to the zero temperature case. In fact, intermediate equations (4.2) and (4.5) can be used as a starting point to investigate the nonzero temperature case. The corresponding consideration for the quantum loop algebra $U_q(L(\mathfrak{sl}_2))$ can be found in paper [41]. It should be noted that in the general case some additional problems arise. We hope to return to this later.

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Figure A1. The initial point of the graphical derivation of the first part of the reduced qKZ equation. We put $\eta_0 = \zeta_1$, use the initial condition (3.15) and proceed to figure A2.

Figure A2. We pull off the emerging loop and raise the arising corner of the red line to the free corner above. This leads us to figure A3.
Figure A3. We move the ‘swing seat’ down, then back up and front down again.

Figure A4. Now we restore all split vertices and reverse the direction of the red vertical line which goes from top to bottom.
Figure A5. We move the leftmost red line behind the scene to the rightmost position and use the initial condition (3.15).

Figure A6. We use iteratively the graphical equation given in figure 30 and proceed to the next figure.
Figure A7. The use of the commutativity equation (3.26) allows us to order all twists. We move the horizontal line with the spectral parameter $\zeta_1$ to the initial position.

Figure A8. This is the last point of the graphical derivation of the first equation.
Figure A9. This is the initial point of the graphical derivation of the second equation.

Figure A10. This is the last point of the graphical derivation of the second equation.
Appendix. Graphical derivation of rqKZ equation

A.1. First equation

The initial configuration for the graphical derivation of the first part of the reduced qKZ equation is given in figure A1. The figure represents the action of an operator, which we denote by \( A_n(\eta_1, \ldots, \eta_n) \), on \( Z_n, K, m(\eta_1, \ldots, \eta_n) \). We mark by red the lines with the spectral parameter \( \zeta_1 \). The triangle and the filled triangle corresponding to the operator \( X_V \) and its inverse are introduced to use further the crossing relations depicted in figures 11 and 12. We put \( \eta_n = \zeta_1 \), use the initial condition figure 20 and proceed to figure A2.

We pull off the emerging loop and raise the arising corner of the red line to the free corner above. This leads us to figure A3.

We move the ‘swing seat’ down, then back up and front down again. To pass through horizontal lines we use the unitarity relations figures 13 and 18. After that we insert the product \( A_V^{-1}(\kappa)A_V(\kappa) \) into the ‘swing seat’ and go to figure A4.

Now we restore all split vertices and, using the equations represented by figures 11, 12 and 27, and the unitarity relations given in figures 18 and 16, reverse the direction of the red vertical line which goes from top to bottom. The commutativity of the operators \( X_V \) and \( A_V(\nu) \) is also used. We acquire the overall factor \( \prod_{i=1}^{K} D(q^\epsilon \zeta_1|\zeta_i) D(q^\epsilon \zeta_1|\xi_i)^{-1} \), where \( D(\zeta, \eta) \) is defined by equation (3.7). We keep this factor in mind. It is clear that the arising red dotted line is associated with the spectral parameter \( q^\epsilon \zeta_1 \). After all that we come to figure A5.

We move the leftmost red line behind the scene to the rightmost position, use the initial condition figure 20 and obtain the configuration given in figure A6.

The next task is to find the right place for the red box with the label \( \alpha \). We use iteratively the graphical equation given in figure 30 and proceed to the next figure.

The use of the commutativity equation figure 31 allows us to order all twists. The last step is pretty cosmetic. We move the horizontal line with the spectral parameter \( \zeta_1 \) to the position where it was at the very beginning and stop at figure A8.

A.2. Second equation

The initial point of the graphical derivation of the second part of the reduced qKZ equation is given in figure A9. Note that to apply the corresponding initial condition we make some rearrangement of the horizontal transfer operators using their commutativity. The figure represents the action of an operator, which we denote by \( B_n(\eta_1, \ldots, \eta_h) \), on the operator \( Z_n, K, m(\eta_1, \ldots, \eta_{n-1}, \eta_h') D_n, K, m(\eta_1, \ldots, \eta_{n-1}, \eta_h') \). Now we mark in red the lines with the spectral parameter \( \zeta_1 \). We put \( \eta_h = \zeta_1 \) and perform transformations similar to those which we made in the derivation of the first part.

The final point of the graphical derivation of the second part of the reduced qKZ equation can be seen in figure A10.

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