ON 2-STAR-PERMUTABILITY IN REGULAR MULTI-POINTED CATEGORIES

MARINO GRAN AND DIANA RODELO

Abstract. 2-star-permutable categories were introduced in a joint work with Z. Janelidze and A. Ursini as a common generalisation of regular Mal’tsev categories and of normal subtractive categories. In the present article we first characterise these categories in terms of what we call star-regular pushouts. We then show that the $3 \times 3$ Lemma characterising normal subtractive categories and the Cuboid Lemma characterising regular Mal’tsev categories are special instances of a more general homological lemma for star-exact sequences. We prove that 2-star-permutability is equivalent to the validity of this lemma for a star-regular category.

Mathematics Subject Classification 2010: 18C05, 08C05, 18B10, 18E10

Introduction

The theory of Mal’tsev categories in the sense of A. Carboni, J. Lambek and M.C. Pedicchio [6] provides a beautiful example of the way how categorical algebra leads to a structural understanding of algebraic varieties (in the sense of universal algebra). Among regular categories, Mal’tsev categories are characterised by the property of 2-permutability of equivalence relations: given two equivalence relations $R$ and $S$ on the same object $A$, the two relational composites $RS$ and $SR$ are equal:

$$RS = SR.$$ 

In the case of a variety of universal algebras this property is actually equivalent to the existence of a ternary term $p(x, y, z)$ satisfying the identities $p(x, y, y) = x$ and $p(x, x, y) = y$ [20]. In the pointed context, that is when the category has a zero object, there is also a suitable notion of 2-permutability, called “2-permutability at 0” [21]. In a variety this property can be expressed by requiring that, whenever for a given element $x$ in an algebra $A$ there is an element $y$ with $xRyS0$ (here 0 is the unique constant in $A$), then there is also an element $z$ in $A$ with $xSzR0$. The validity of this property is equivalent to the existence of a binary term $s(x, y)$ such that the identities $s(x, 0) = x$ and $s(x, x) = 0$ hold true [21]. Among regular categories, the ones where the property of 2-permutability at 0 holds true are precisely the subtractive categories introduced in [14].

The aim of this paper is to look at regular Mal’tsev and at subtractive categories as special instances of the general notion of 2-star-permutable categories introduced in collaboration with Z. Janelidze and A. Ursini in [9]. This generalisation is achieved by working in the context of a regular multi-pointed category, i.e. a regular category equipped with an ideal $N$ of distinguished morphisms [7]. When

---

Key words and phrases. Regular multi-pointed category; star relation; Mal’tsev category; subtractive category; varieties of algebras; homological diagram lemma.
\(\mathcal{N}\) is the class of all morphisms, a situation which we refer to as the total context, regular multi-pointed categories are just regular categories, and 2-star-permutable categories are precisely the regular Mal’tsev categories. When \(\mathcal{N}\) is the class of all zero morphisms in a pointed category, we call this the pointed context, regular multi-pointed categories are regular pointed categories, and 2-star-permutable categories are the regular subtractive categories.

This paper follows the same line of research as in [9] which was mainly focused on the property of 3-star-permutability, a generalised notion which captures Goursat categories in the total context and, again, subtractive categories in the pointed context.

In this work we study two remarkable aspects of the property of 2-star-permutability. First we provide a characterisation of 2-star-permutable categories in terms of a special kind of pushouts (Proposition 2.4), that we call star-regular pushouts (Definition 2.2). Then we examine a homological diagram lemma of star-exact sequences, which can be seen as a generalisation of the 3 \(\times\) 3 Lemma, whose validity is equivalent to 2-star-permutability. We call this lemma the Star-Upper Cuboid Lemma (Theorem 3.3). The validity of this lemma turns out to give at once a characterisation of regular Mal’tsev categories (extending a result in [11]) and, in the pointed context, a characterisation of those normal categories which are subtractive (this was first discovered in [17]).

Acknowledgement. The authors are grateful to Zurab Janelidze for some useful conversations on the subject of the paper.

1. Star-regular categories

1.1. Regular categories and relations. A finitely complete category \(\mathbb{C}\) is said to be a regular category [1] when any kernel pair has a coequaliser and, moreover, regular epimorphisms are stable under pullbacks. In a regular category any morphism \(f: X \rightarrow Y\) has a factorisation \(f = m \cdot p\), where \(p\) is a regular epimorphism and \(m\) is a monomorphism. The corresponding (regular epimorphism, monomorphism) factorisation system is then stable under pullbacks.

A relation \(\varrho\) from \(X\) to \(Y\) is a subobject \(\langle \varrho_1, \varrho_2 \rangle: R \rightarrow X \times Y\). The opposite relation, denoted \(\varrho^o\), is given by the subobject \(\langle \varrho_2, \varrho_1 \rangle: R \rightarrow Y \times X\). We identify a morphism \(f: X \rightarrow Y\) with the relation \(\langle 1_X, f \rangle: X \rightarrow X \times Y\) and write \(f^o\) for the opposite relation. Given another relation \(\sigma\) from \(Y\) to \(Z\), the composite relation of \(\varrho\) and \(\sigma\) is a relation \(\varrho \circ \sigma\) from \(X\) to \(Z\). With this notation, we can write the above relation as \(\varrho = \varrho_2 \varrho_1\). The following properties are well known (see [5], for instance); we collect them in a lemma for future references.

**Lemma 1.1.** Let \(f: X \rightarrow Y\) be any morphism in a regular category \(\mathbb{C}\). Then:

(a) \(f f^o f = f\) and \(f^o f f^o = f^o\);

(b) \(f f^o = 1_Y\) if and only if \(f\) is a regular epimorphism.

A kernel pair of a morphism \(f: X \rightarrow Y\), denoted by

\[(\pi_1, \pi_2): \text{Eq}(f) \rightarrow X,\]

is called an effective equivalence relation; we write it either as \(\text{Eq}(f) = f^o f\), or as \(\text{Eq}(f) = \pi_2 \pi_1^o\), as mentioned above. When \(f\) is a regular epimorphism, then \(f\) is
the coequaliser of $\pi_1$ and $\pi_2$ and the diagram

$$\xymatrix{ \text{Eq}(f) \ar[r]^-{\pi_1} \ar@{<->}[r]_-{\pi_2} & X \ar[r]^f & Y}$$

is called an exact fork. In a regular category any effective equivalence relation is the kernel pair of a regular epimorphism.

1.2. Star relations. We now recall some notions introduced in [10], which are useful to develop a unified treatment of pointed and non-pointed categorical algebra.

Let $\mathcal{C}$ denote a category with finite limits, and $\mathcal{N}$ a distinguished class of morphisms that forms an ideal, i.e. for any composable pair of morphisms $g, f$, if either $g$ or $f$ belongs to $\mathcal{N}$, then the composite $g \cdot f$ belongs to $\mathcal{N}$. An $\mathcal{N}$-kernel of a morphism $f : X \to Y$ is defined as a morphism $n_f : \mathcal{N}_f \to X$ such that $f \cdot n_f \in \mathcal{N}$ and $n_f$ is universal with this property (note that such $n_f$ is automatically a monomorphism).

A pair of parallel morphisms, denoted by $\sigma = (\sigma_1, \sigma_2) : S \rightrightarrows X$ with $\sigma_1 \in \mathcal{N}$, is called a star; it is called a monic star, or a star relation, when the pair $(\sigma_1, \sigma_2)$ is jointly monomorphic.

Given a relation $\rho = (\rho_1, \rho_2) : R \rightrightarrows X$ on an object $X$, we denote by $\rho^* : R^* \rightrightarrows X$ the biggest subrelation of $\rho$ which is a (monic) star. When $\mathcal{C}$ has $\mathcal{N}$-kernels, it can be constructed by setting $\rho^* = (\rho_1 \cdot n_{\rho_1}, \rho_2 \cdot n_{\rho_2})$, where $n_{\rho_1}$ is the $\mathcal{N}$-kernel of $\rho_1$. In particular, if we denote the discrete equivalence relation on an object $X$ by $\Delta_X = (1_X, 1_X) : X \rightrightarrows X$, then $\Delta_X^* = (n_{1_X}, n_{1_X})$, where $n_{1_X}$ is the $\mathcal{N}$-kernel of $1_X$.

The star-kernel of a morphism $f : X \to Y$ is a universal star $\sigma = (\sigma_1, \sigma_2) : S \rightrightarrows X$ with the property $f \cdot \sigma_1 = f \cdot \sigma_2$; it is easy to see that the star-kernel of $f$ coincides with $\text{Eq}(f)^* \rightrightarrows X$ whenever $\mathcal{N}$-kernels exist.

A category $\mathcal{C}$ equipped with an ideal $\mathcal{N}$ of morphisms is called a multi-pointed category [10]. If, moreover, every morphism admits an $\mathcal{N}$-kernel, then $\mathcal{C}$ will be called a multi-pointed category with kernels.

Definition 1.2. [10] A regular multi-pointed category $\mathcal{C}$ with kernels is called a star-regular category when every regular epimorphism in $\mathcal{C}$ is a coequaliser of a star.

In the total context stars are pairs of parallel morphisms, $\mathcal{N}$-kernels are isomorphisms, star-kernels are kernel pairs and a star-regular category is precisely a regular category. In the pointed context, the first morphism $\sigma_1$ in a star $\sigma = (\sigma_1, \sigma_2) : S \rightrightarrows X$ is the unique null morphism $S \to X$ and hence a star $\sigma$ can be identified with a morphism (its second component $\sigma_2$). Then, $\mathcal{N}$-kernels and star-kernels become the usual kernels, and a star-regular category is the same as a normal category [18], i.e. a pointed regular category in which any regular epimorphism is a normal epimorphism.

1.3. Calculus of star relations. The calculus of star relations [9] can be seen as an extension of the usual calculus of relations (in a regular category) to the regular multi-pointed context. First of all note that for any relation $\rho : R \rightrightarrows X$ we have

$$\rho^* = \rho \Delta_X^*.$$  

Inspired by this formula, for any relation $\rho$ from $X$ to an object $Y$, we define

$$\rho^* = \rho \Delta_X^* \quad \text{and} \quad ^* \rho = \Delta_Y^*/\rho.$$
Note that associativity of composition yields
\[ *(g^*) = (*g)^* \]
and so we can write \( *g \) for the above.

For any relation \( \sigma \) (from some object \( Y \) to \( Z \)), the associativity of composition also gives
\[ (\sigma^*)_g = \sigma(*g), \]
and
\[ (\sigma g)^* = \sigma g^*. \]

It is easy to verify that for any morphism \( f : X \to Y \) we have
\[ f^* = *f^* \quad \text{and} \quad f^o = *f^o. \]

2. 2-STAR-PERMUTABILITY AND STAR-REGULAR PUSHOUTS

Recall that a finitely complete category \( C \) is called a Mal’tsev category when any reflexive relation in \( C \) is an equivalence relation [6, 5]. We recall the following well-known characterisation of the regular categories which are Mal’tsev categories:

**Proposition 2.1.** A regular category \( C \) is a Mal’tsev category if and only if the composition of effective equivalence relations in \( C \) is commutative:
\[ \text{Eq}(f)\text{Eq}(g) = \text{Eq}(g)\text{Eq}(f) \]
for any pair of regular epimorphisms \( f \) and \( g \) in \( C \) with the same domain.

There are many known characterisations of regular Mal’tsev categories (see Section 2.5 in [2], for instance, and references therein). The one that will play a central role in the present work is expressed in terms of commutative diagrams of the form
\[
\begin{array}{ccc}
C & \xrightarrow{c} & A \\
g \downarrow & & \downarrow f \\
D & \xrightarrow{d} & B,
\end{array}
\]
where \( f \) and \( g \) are split epimorphisms \( f \cdot s = 1_B, g \cdot t = 1_D, f \cdot c = d \cdot g, s \cdot d = c \cdot t \), and \( c \) and \( d \) are regular epimorphisms. A diagram of type (1) is always a pushout; it is called a regular pushout [4] (alternatively, a double extension [15, 13]) when, moreover, the canonical morphism \( \langle g, c \rangle : C \to D \times_B A \) to the pullback \( D \times_B A \) of \( d \) and \( f \) is a regular epimorphism. Among regular categories, Mal’tsev categories can be characterized as those ones where any square (1) is a regular pushout: this easily follows from the results in [4], and a simple proof of this fact is given in [12].

Observe that a commutative diagram of type (1) is a regular pushout if and only if \( cg^o = f^o d \) or, equivalently, \( gc^o = d^o f \). This suggests to introduce the following notion:

**Definition 2.2.** A commutative diagram (1) is a star-regular pushout if it satisfies the identity \( cg^o = f^o d^* \) (or, equivalently, \( gc^o = d^o f^* \)).
Diagrammatically, the property of being a star-regular pushout can be expressed as follows. Consider the commutative diagram

\[
\begin{array}{ccccccc}
N_g & \downarrow & N_a & \rightarrow & N_x & \downarrow & A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C & \downarrow & M & \rightarrow & D \times_B A & \downarrow & B \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
D & \downarrow & \leftarrow & & \leftarrow & & \leftarrow \\
\end{array}
\]

where \((D \times_B A, x, y)\) is the pullback of \((f, d)\), \(m \cdot p\) is the (regular epimorphism, monomorphism) factorisation of the induced morphism \((g, c) : C \rightarrow D \times_B A\). Then the identity \(cg^o = ba^o\) allows one to identify \(cg^o\) with the relation \((a \cdot n_a, b \cdot n_a)\), while \(f^o d = yx^o\) says that \(f^o d^\ast\) can be identified with the relation \((x \cdot n_x, y \cdot n_x)\).

Accordingly, diagram (1) is a star-regular pushout precisely when the dotted arrow from \(N_a\) to \(N_x\) is an isomorphism. Notice that in the total context the \(N\)-kernels are isomorphisms, so that \(m\) is an isomorphism if and only if (1) is a regular pushout, as expected.

The “star-version” of the notion of Mal’tsev category can be defined as follows:

**Definition 2.3.** A regular multi-pointed category with kernels \(\mathbb{C}\) is said to be a 2-star-permutable category if

\[Eq(f)Eq(g)^* = Eq(g)Eq(f)^*\]

for any pair of regular epimorphisms \(f\) and \(g\) in \(\mathbb{C}\) with the same domain.

One can check that the equality \(Eq(f)Eq(g)^* = Eq(g)Eq(f)^*\) in the definition above can be actually replaced by \(Eq(f)Eq(g)^* \leq Eq(g)Eq(f)^*\).

In the total context the property of 2-star-permutability characterises the regular categories which are Mal’tsev. In the pointed context this same property characterises the regular categories which are subtractive [16] (this follows from the characterisation of subactivity given in Theorem 6.9 in [17]).

The next result gives a useful characterisation of 2-star-permutable categories. Given a commutative diagram of type (1), we write \(gEq(c)\) and \(gEq(c)^*\) for the direct images of the relations \(Eq(c)\) and \(Eq(c)^*\) along the split epimorphism \(g\). The vertical split epimorphisms are such that both the equalities \(gEq(c) = Eq(d)\) and \(gEq(c)^* = Eq(d)^*\) hold true in \(\mathbb{C}\).

**Proposition 2.4.** For a regular multi-pointed category with kernels \(\mathbb{C}\) the following statements are equivalent:

(a) \(\mathbb{C}\) is a 2-star-permutable category;

(b) any commutative diagram of the form (1) is a star-regular pushout.
Proof. (a) ⇒ (b) Given a pushout (1) we have

\[ f \circ d^* = cc_0 f \circ d^* \quad (\text{Lemma } 1.1(2)) \]
\[ = cg_0 d^* \quad (f \cdot c = d \cdot g) \]
\[ = cg_0 gc_0 c^* g^* \quad (\text{Eq}(d)^* = g(\text{Eq}(c)^*)) \]
\[ = cc_0 cg_0 g^* g^* \quad (\text{Eq}(g)\text{Eq}(c)^* = \text{Eq}(c)\text{Eq}(g)^* \text{ by Definition } 2.3) \]
\[ \leq cc_0 cg_0 g g^* \quad (g^* \leq g) \]
\[ = cg_0. \quad (\text{Lemma } 1.1(1)) \]

Since \( cg_0^* \) is the largest star contained in \( cg_0^* \), it follows that \( f \circ d^* \leq cg_0^* \). The inclusion \( cg_0^* \leq f \circ d^* \) always holds, so that \( cg_0^* = f \circ d^* \).

(b) ⇒ (a) Let us consider regular epimorphisms \( f : X \to Y \) and \( g : X \to Z \). We want to prove that \( \text{Eq}(f)\text{Eq}(g)^* = \text{Eq}(g)\text{Eq}(f)^* \). For this we build the following diagram

\[
\begin{array}{c}
\text{Eq}(f) \\
\downarrow \pi_1 \\
X \\
\downarrow f \\
Y \\
\end{array}
\begin{array}{c}
\downarrow \pi_2 \\
\downarrow \rho_1 \\
\downarrow \rho_2 \\
\downarrow g \\
Z \\
\end{array}
\]

that represents the regular image of \( \text{Eq}(f) \) along \( g \). The relation \( g(\text{Eq}(f)) = (\rho_1, \rho_2) \) is reflexive and, consequently, \( \rho_1 \) is a split epimorphism. By assumption, we then know that the equality

\[(A) \quad \rho_1^* g^* = c\pi_1^* \]

holds true. This implies that

\[
\begin{align*}
\text{Eq}(f)\text{Eq}(g)^* &= \pi_2 \pi_1^* g^* \\
&= \pi_2 c^2 \rho_1^* g^* \quad (g \cdot \pi_1 = \rho_1 \cdot c) \\
&= \pi_2 c^2 c\pi_1^* \quad (A) \\
& \leq \pi_2 c^2 c\pi_2^* \pi_2^* \quad (\Delta_{\text{Eq}(f)} \leq \pi_2^* \pi_2) \\
&= \text{Eq}(g)\pi_2^* \pi_2^* \quad (\pi_2(\text{Eq}(c)) = \text{Eq}(g)) \\
&= \text{Eq}(g)\text{Eq}(f)^*,
\end{align*}
\]

where the equality \( \pi_2(\text{Eq}(c)) = \text{Eq}(g) \) follows from the fact that the split epimorphisms \( \pi_2 \) and \( \rho_2 \) induce a split epimorphism from \( \text{Eq}(c) \) to \( \text{Eq}(g) \).

In the total context, Proposition 2.4 gives the characterisation of regular Mal’tsev categories through regular pushouts (see [4] and Proposition 3.4 of [12]), as expected. In the pointed context, condition (b) of Proposition 2.4 translates into the pointed version of the right saturation property [9] for any commutative diagram of type (1): the induced morphism \( \bar{c} : \text{Ker}(g) \to \text{Ker}(f) \), from the kernel of \( g \) to the kernel of \( f \) is also a regular epimorphism. This can be seen by looking at diagram (2), where the \( N \)-kernels now represent actual kernels, so that \( \text{Ker}(a) = \text{Ker}(x) = \text{Ker}(f) \). \( \square \)
2.1. **The star of a pullback relation.** Consider the pullback relation \( \pi = (\pi_1, \pi_2) \) of a pair \((g, \delta)\) of morphisms as in the diagram

\[
\begin{array}{ccc}
W \times_D C & \xrightarrow{\pi_2} & C \\
\downarrow \pi_1 & & \downarrow g \\
W & \xrightarrow{\delta} & D.
\end{array}
\]

The **star of the pullback relation** \( \pi \) is defined as \( \pi^* = \pi \Delta_W^* \). It can be described as the universal relation \( \nu = (\nu_1, \nu_2) \) from \( W \) to \( C \) such that \( \nu_1 \in N \) and \( \delta \cdot \nu_1 = g \cdot \nu_2 \) as in the diagram

\[
\begin{array}{ccc}
(W \times_D C)^* & \xrightarrow{\nu_2} & C \\
\downarrow n_{\pi_1} & & \downarrow g \\
W \times_D C & \xrightarrow{\pi_2} & C \\
\downarrow \pi_1 & & \downarrow g \\
W & \xrightarrow{\delta} & D,
\end{array}
\]

where \( n_{\pi_1} \) is the \( N \)-kernel of \( \pi_1 \), \( \nu_1 = \pi_1 \cdot n_{\pi_1} \) and \( \nu_2 = \pi_2 \cdot n_{\pi_1} \).

By using the composition of relations one has the equalities \( \pi = \pi_2 \pi_1^* = g^* \delta \), so that

\( \pi^* = \pi_2 \pi_1^{\circ^*} = g^* \delta^* \).

In the total context, the star of a pullback relation is precisely that pullback relation. In the pointed context, the star of the pullback (relation) of \((g, \delta)\) is given by \( \pi^* = (0, \ker(g)) \).

A morphism \( f : X \to Y \) in a multi-pointed category with kernels is said to be **saturating** \([9] \) when the induced dotted morphism from the \( N \)-kernel of \( 1_X \) to the \( N \)-kernel of \( 1_Y \) making the diagram

\[
\begin{array}{ccc}
N_{1_X} & \xrightarrow{n_{1_Y}} & N_{1_Y} \\
\downarrow n_{1_X} & & \downarrow n_{1_Y} \\
X & \xrightarrow{f} & Y
\end{array}
\]

commute is a regular epimorphism. All morphisms are saturating in the pointed context. This is also the case for any **quasi-pointed category** \([3] \), namely a finitely complete category with an initial object \( 0 \) and a terminal object \( 1 \) such that the arrow \( 0 \to 1 \) is a monomorphism. As in the pointed case, it suffices to choose for \( N \) the class of morphisms which factor through the initial object \( 0 \). In this case we shall speak of the **quasi-pointed context**. In the total context, any regular epimorphism is saturating. The proof of the following result is straightforward:

**Lemma 2.5.** \([9] \) Let \( C \) be a regular multi-pointed category with kernels. For a morphism \( f : X \to Y \) the following conditions are equivalent:

(a) \( f \) is saturating;
(b) \( \Delta_Y^* = f^* f^\circ \).

The next result gives a characterisation of 2-star-permutable categories which will be useful in the following section.

**Proposition 2.6.** For a regular multi-pointed category \( C \) with kernels and saturating regular epimorphisms the following statements are equivalent:
(a) $C$ is a 2-star-permutable category;
(b) for any commutative diagram

\[
\begin{array}{cccc}
W & \times & D & C \\
\downarrow{\nu_1} & & \downarrow{c} & \downarrow{\chi_1} \\
Y & \times & B & A \\
\downarrow{\beta} & & \downarrow{\gamma} & \downarrow{\delta} \\
D & \times & W & \downarrow{d} \\
\end{array}
\]

where the front square is of the form \((11)\), $\beta \cdot w = d \cdot \delta$, $w$ is a regular epimorphism, \(((W \times_D C)^*, \nu_1, \nu_2)\) and \(((Y \times_B A)^*, \chi_1, \chi_2)\) are stars of the corresponding pullback relations, then the comparison morphism $\lambda: (W \times_D C)^* \to (Y \times_B A)^*$ is also a regular epimorphism.

**Proof.** (a) $\Rightarrow$ (b) To prove that the arrow $\lambda$ in the cube above is a regular epimorphism, we must show that

\[
\begin{array}{cccc}
(W \times_D C)^* & \overset{\lambda}{\longrightarrow} & (Y \times_B A)^* \\
\downarrow{(\nu_1, \nu_2)} & & \downarrow{(\chi_1, \chi_2)} \\
W \times C & \overset{w \times c}{\longrightarrow} & Y \times A \\
\end{array}
\]

is the (regular epimorphism, monomorphism) factorisation of the morphism \((w \cdot \nu_1, c \cdot \nu_2): (W \times_D C)^* \to Y \times A\). That is, we must have $cg \circ \delta^* \circ w^* = f^* \beta^*$, since $\nu_2 \nu_1^* = \nu^* = g^* \delta^*$ and $\chi_2 \chi_1^* = \chi^* = f^* \beta^*$ (see Section 2.1).

The front square of diagram (3) is a star-regular pushout by Proposition 2.4, which means that the equality

\[
(B) \quad cg^* = f^* d^*
\]

holds true. Now, we always have

\[
\begin{align*}
cg^* \delta^* w^* & \leq f^* \delta^* w^* \quad \text{(commutativity of the front face of (3))} \\
& = d^* \beta^* w^* \quad \text{(d \cdot \delta = \beta \cdot w)} \\
& = f^* \beta^* \Delta^*_Y \quad \text{(Lemma 2.5)} \\
& = f^* \beta^*. 
\end{align*}
\]

The other inequality follows from

\[
\begin{align*}
\delta^* w^* & \geq cg^* \delta^* w^* \quad \text{(g \geq g^*)} \\
& = d^* \delta^* w^* \quad \text{(B)} \\
& = f^* \delta^* w^* \quad \text{(* \delta^* = \delta^*; Section 1.3)} \\
& = f^* \beta^*. \quad \text{(as in the inequality above)}
\end{align*}
\]
(b) ⇒ (a) A commutative diagram of type (1) induces a commutative cube

where \( \nu = (g \cdot n_g, n_g) \) is the star of the pullback (relation) of \((g, 1_D)\). By assumption, \( \lambda \) is a regular epimorphism which translates into the equality \( cg^o1_D = f^o d^* \), as observed in the first part of the proof. We get the equality \( cg_1^* = f_1^o d^* \), and this proves that diagram (1) is a star-regular pushout and, consequently, that \( C \) is a 2-star-permutable category by Proposition 2.4.

In the total context, Proposition 2.6 is the "star version" of Proposition 3.6 in [12] (see also Proposition 4.1 in [4]). In the pointed context condition (b) of Proposition 2.6 also reduces to the pointed version of the right saturation property (in the sense of [9]). Indeed, in this context that condition says that, in the following commutative diagram

the induced arrow \( \overline{c} : \text{Ker}(g) \to \text{Ker}(f) \) is a regular epimorphism.

We conclude this section with the pointed version of Propositions 2.4 and 2.6:

**Corollary 2.7.** (see Theorem 2.12 in [9]) For a pointed regular category \( C \) the following statements are equivalent:

(a) \( C \) is a subtractive category;

(b) any commutative diagram of the form (1) is right saturated, i.e. the comparison morphism \( \overline{c} : \text{Ker}(g) \to \text{Ker}(f) \) is a regular epimorphism.

3. The Star-Cuboid Lemma

In [12] it was shown that regular Mal’tsev categories can be characterised through the validity of a homological lemma called the Upper Cuboid Lemma, a strong form of the denormalised \( 3 \times 3 \) Lemma [4, 19, 11]. We are now going to extend this result to the star-regular context. We shall then observe that, in the pointed
context, it gives back the classical Upper $3 \times 3$ Lemma characterising subtractive normal categories.

3.1. $\mathcal{N}$-trivial objects. An object $X$ in a multi-pointed category is said to be $\mathcal{N}$-trivial when $1_X \in \mathcal{N}$. If a composite $f \cdot g$ belongs to $\mathcal{N}$ and $g$ is a strong epimorphism, then also $f$ belongs to $\mathcal{N}$. This implies that $\mathcal{N}$-trivial objects are closed under strong quotients. One says that a multi-pointed category $\mathcal{C}$ has enough trivial objects [8] when $\mathcal{N}$ is a closed ideal [14], i.e. any morphism in $\mathcal{N}$ factors through an $\mathcal{N}$-trivial object and, moreover, the class of $\mathcal{N}$-trivial objects is closed under subobjects and squares, where the latter property means that, for any $\mathcal{N}$-trivial object $X$, the object $X^2 = X \times X$ is $\mathcal{N}$-trivial. An equivalent way of expressing the existence of enough trivial objects is recalled in the following:

**Proposition 3.1.** [8] Let $\mathcal{C}$ be a regular multi-pointed category with kernels. The following conditions are equivalent:

(a) if $(\sigma_1, \sigma_2) : S \rightarrowtail X$ is a relation on $X$ such that $\sigma_1 \cdot n \in \mathcal{N}$ and $\sigma_2 \cdot n \in \mathcal{N}$, then $n \in \mathcal{N}$;

(b) $\mathcal{C}$ has enough trivial objects.

In the following we shall also assume that $\mathcal{N}$-trivial objects are closed under binary products. Remark that in the total and in the (quasi-)pointed contexts there are enough trivial objects, and $\mathcal{N}$-trivial objects are closed under binary products.

Under the presence of enough trivial objects the assumption that $\mathcal{N}$-trivial objects are closed under binary products is equivalent to the following condition:

(a’) if $(\sigma_1, \sigma_2) : S \rightarrowtail X \times Y$ is a relation from $X$ to $Y$ such that $\sigma_1 \cdot n \in \mathcal{N}$ and $\sigma_2 \cdot n \in \mathcal{N}$, then $n \in \mathcal{N}$.

Whenever the category has enough trivial objects, condition (a’) implies that star-kernels “commute” with stars of pullback relations:

**Lemma 3.2.** Let $\mathcal{C}$ be a multi-pointed category with kernels, enough trivial objects, and assume that $\mathcal{N}$-trivial objects are closed under binary products. Given a commutative cube

in $\mathcal{C}$, consider the star-kernels of $c$, $d$ and $w$, and the induced morphisms $\mathcal{F} : \text{Eq}(w)^* \rightarrow \text{Eq}(d)^*$ and $\mathcal{G} : \text{Eq}(c)^* \rightarrow \text{Eq}(d)^*$. Then the following constructions are equivalent (up to isomorphism):

- taking the horizontal star-kernel of $\lambda$ and then the induced morphisms $\text{Eq}(\lambda)^* \rightarrow \text{Eq}(w)^*$ and $\text{Eq}(\lambda)^* \rightarrow \text{Eq}(c)^*$;
• taking the star of the pullback (relation) of \( \overline{\delta} \) and then the induced morphisms \((\Eq(w)^* \times_{\Eq(d)} \Eq(c)^*)^* \Rightarrow (W \times_D C)^*\).

Proof. This follows easily by the usual commutation of kernel pairs with pullbacks and condition (a'). □

In a star-regular category, a (short) star-exact sequence is a diagram

\[
\begin{array}{ccc}
Eq(f)^* & \xrightarrow{f_1} & X \\
\xrightarrow{f_2} & Y
\end{array}
\]

where \(\Eq(f)^*\) is a star-kernel of \(f\) and \(f\) is a coequaliser of \(f_1\) and \(f_2\) (which, by star-regularity, is the same as to say that \(f\) is a regular epimorphism). In the total context, a star-exact sequence is just an exact fork, while in the (quasi-)pointed context it is a short exact sequence in the usual sense.

The Star-Upper Cuboid Lemma

Let \(\mathbb{C}\) be a star-regular category. Consider a commutative diagram of morphisms and stars in \(\mathbb{C}\)

\[
\begin{array}{c}
P \\
\xrightarrow{\pi} (W \times_D C)^* \\
\xrightarrow{\lambda} (Y \times_B A)^*
\end{array}
\]

\[
\begin{array}{c}
\Eq(w)^* \\
\xrightarrow{\nu_1} W \\
\xrightarrow{\nu_2} C \\
\xrightarrow{\chi_1} Y \\
\xrightarrow{\chi_2} A
\end{array}
\]

\[
\begin{array}{c}
\Eq(c)^* \\
\xrightarrow{\delta} C \\
\xrightarrow{\overline{\delta}} \overline{D} \\
\xrightarrow{\overline{\delta}} \overline{A}
\end{array}
\]

\[
\begin{array}{c}
S \\
\xrightarrow{\sigma} D \\
\xrightarrow{\delta} \overline{B}
\end{array}
\]

where the three diamonds are stars of pullback (relations) of regular epimorphisms along arbitrary morphisms (so that \(P = (\Eq(w)^* \times_S \Eq(c)^*)^*\)) and the two middle rows are star-exact sequences. Then the upper row is a star-exact sequence whenever the lower row is.

Note that, in the diagram (5) above, \(d\) is necessarily a regular epimorphism, \(d \cdot \sigma_1 = d \cdot \sigma_2\) since \(\overline{\delta}\) is an epimorphism, and \(\lambda \cdot \pi_1 = \lambda \cdot \pi_2\), because the pair of morphisms \((\chi_1, \chi_2)\) is jointly monomorphic.

Theorem 3.3. Let \(\mathbb{C}\) be a star-regular category with saturating regular epimorphisms, enough trivial objects, and assume that \(\mathcal{N}\)-trivial objects are closed under binary products. The following conditions are equivalent:

(a) \(\mathbb{C}\) is a 2-star-permutable category;
(b) the Star-Upper Cuboid Lemma holds true in \(\mathbb{C}\).

Proof. (a) \(\Rightarrow\) (b) Suppose that the lower row is a star-exact sequence. The fact that \(\pi = \Eq(\lambda)^*\) follows from Lemma 3.2. As explained in Proposition 2.6 \(\lambda\) is a
regular epimorphism if and only if \( cg_0 \delta^* w^o \geq f_0 \beta^* \). In fact we have

\[
\begin{align*}
    cg_0 \delta^* w^o &= cc_0 \alpha_0 \delta^* w^o \quad \text{(Lemma 1.1(1))} \\
    &\geq cc_0 \alpha_0 \delta^* w^o \quad \text{(Eq}(g) \geq \text{Eq}(g)^*) \\
    &= cg_0 \delta^* w^o \quad \text{(Eq}(g) \geq \text{Eq}(g)^*) \\
    &= f^0 \beta^* w^o \quad \text{(Lemma 1.1(2))} \\
    &= f^0 \beta^*.
\end{align*}
\]

(b) \( \Rightarrow \) (a) Consider a commutative cube of the form (5). We construct a commutative diagram of type (5) by taking the star-kernels of \( c, w, d \), and \( \lambda \), so that \( \gamma, \delta, \tau_1 \), and \( \tau_2 \) are the induced arrows between the star-kernels. By Lemma 3.2 we know that \( (\tau_1, \tau_2) \) is the star above the pullback (relation) of \( (\bar{g}, \delta) \). By applying the Star-Upper Cuboid Lemma to this diagram we conclude that the upper row is a star-exact sequence and, consequently, \( \lambda \) is a regular epimorphism. By Proposition 2.6, \( C \) is a 2-star-permutable category.

In the total context, Theorem 5.3 is precisely Theorem 4.3 in [12], which gives a characterisation of regular Mal’tsev categories through the Upper Cuboid Lemma, as expected. In the pointed context, the Star-Upper Cuboid Lemma gives the classical Upper 3 \( \times 3 \) Lemma: in the pointed version of diagram (5), the back part is irrelevant (like in diagram (4)). Then the front part is a 3 \( \times 3 \) diagram where all columns and the middle row are short exact sequences. The Star-Upper Cuboid Lemma claims that the upper row is a short exact sequence whenever the lower row is, i.e. the same as the Upper 3 \( \times 3 \) Lemma. The pointed version of Theorem 5.3 is Theorem 5.4 of [18] which characterises normal subtractive categories. Note that in the pointed context, the Upper 3 \( \times 3 \) Lemma is also equivalent to the Lower 3 \( \times 3 \) Lemma as shown in [18].

References

[1] M. Barr, Exact Categories, in: Lecture Notes in Math. 236 Springer (1971) 1-120.
[2] F. Borceux and D. Bourn, Mal’cev, protomodular, homological and semi-abelian categories, Math. and its Appl. 566, Kluwer, (2004).
[3] D. Bourn, 3 \( \times 3 \) lemma and protomodularity. J. Algebra 236 (2001) 778-795.
[4] D. Bourn, The denormalized 3 \( \times 3 \) lemma. J. Pure Appl. Algebra 177 (2003) 113-129.
[5] A. Carboni, G.M. Kelly, and M.C. Pedicchio, Some remarks on Mal’tsev and Goursat categories, Appl. Cat. Structures 1 (1993) 385-421.
[6] A. Carboni, J. Lambek, and M.C. Pedicchio, Diagram chasing in Mal’cev categories. J. Pure Appl. Alg. 69 (1991) 271-284.
[7] C. Ehresmann, Sur une notion générale de cohomologie, C. R. Acad. Sci. Paris 259 (1964) 2050-2053.
[8] M. Gran, Z. Janelidze and D. Rodelo, 3 \( \times 3 \)-Lemma for star-exact sequences, Homology, Homotopy Appl. 14 (2) (2012) 1-22.
[9] M. Gran, Z. Janelidze, D. Rodelo, and A. Ursini, Symmetry of regular diamonds, the Goursat property, and subtractivity, Theory Appl. Categ. 27 (2012) 80-96.
[10] M. Gran, Z. Janelidze and A. Ursini, A good theory of ideal in regular multi-pointed categories. J. Pure Appl. Algebra 216 (2012) 1905-1919.
[11] M. Gran and D. Rodelo, A new characterisation of Goursat categories, Appl. Categ. Structures 20 (2012) 229-238.
[12] M. Gran and D. Rodelo, The Cuboid Lemma and Mal’tsev categories, published online in Appl. Categ. Structures, DOI: 10.1007/s10485-013-9352-5.
[13] M. Gran and V. Rossi, Galois Theory and Double Central Extensions, Homology, Homotopy Appl. 6 (1) (2004) 283-298.
[14] M. Grandis, On the categorical foundations of homological and homotopical algebra, Cah. Top. Géom. Diff. Catég. 33 (1992) 135-175.
[15] G. Janelidze, What is a double central extension? Cah. Top. Géom. Diff. Catég. 32 (3) (1991) 191-201.
[16] Z. Janelidze, Subtractive categories, Appl. Categ. Struct. 13 (2005) 343-350.
[17] Z. Janelidze, Closedness properties of internal relations I: A unified approach to Mal’tsev, unital and subtractive categories, Theory Appl. Categ. 16 (2006) 236-261.
[18] Z. Janelidze, The pointed subobject functor, 3 × 3 lemmas and subtractivity of spans, Theory Appl. Categ. 23 (2010) 221-242.
[19] S. Lack, The 3-by-3 lemma for regular Goursat categories, Homology, Homotopy Appl., 6 (1) (2004) 1-3.
[20] J.D.H. Smith, Mal’cev Varieties, Lecture Notes in Math. 554, Springer (1976).
[21] A. Ursini, On subtractive varieties, I, Algebra Univ. 31 (1994) 204-222.