SECOND HANKEL DETERMINANT FOR A CLASS OF ANALYTIC FUNCTIONS DEFINED BY A LINEAR OPERATOR

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Abstract. By making use of the linear operator $\Theta^{\lambda, n}_m$, $m \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $\lambda, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ given by the authors, a class of analytic functions $S^{\lambda, n}_m(\alpha, \sigma) (|\alpha| < \pi/2, 0 \leq \sigma < 1)$ is introduced. The object of the present paper is to obtain sharp upper bound for functional $|a_2a_4 - a_3^2|$.

1. Introduction

Let $\mathcal{A}$ denote the class of normalised analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

where $z \in U := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $S$ denote the class of all functions in $\mathcal{A}$ which are univalent.

Robertson [14] introduced the class of starlike functions of order $\sigma$ as follows:

**Definition 1.1** ([14]). Let $\sigma \in [0, 1]$, $f \in S$ and

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \sigma, \; z \in U.$$

Then, we say that $f$ is a starlike function of order $\sigma$ on $U$ and we denoted this class by $S^*(\sigma)$.

Spacek [15] introduced the class of spirallike functions of type $\alpha$ as follows:

**Theorem 1.1** ([15]). Let $f \in S$ and $-\pi/2 < \alpha < \pi/2$. Then $f(z)$ is a spirallike function of type $\alpha$ on $U$ if

$$\Re \left\{ e^{i\alpha} \frac{zf'(z)}{f(z)} \right\} > 0, \; z \in U.$$

We denoted this class by $S_\alpha$. 

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From Definition 1.1 and Theorem 1.1, it is easy to see ([17]) that starlike functions of order \( \sigma \) and spirallike functions of type \( \alpha \) have some relationships on geometry. Starlike functions of order \( \sigma \) map \( U \) into the right half complex plane whose real part is greater than \( \sigma \) by the mapping \( z f'(z)/f(z) \), while spirallike functions of type \( \alpha \) map \( U \) into the right half complex plane by the mapping \( e^{i\alpha} z f'(z)/f(z) \). Since \( \lim_{z \to 0} e^{i\alpha} z f'(z)/f(z) = e^{i\alpha} \), we can deduce that if we restrict the image of the mapping \( e^{i\alpha} z f'(z)/f(z) \) in the right complex plane whose real part is greater than a certain constant, then the constant must be smaller than \( \cos \alpha \).

Libera [16] introduced and studied the class \( S^\alpha_\sigma \) given as follows:

**Definition 1.2** ([16]). Let \( \sigma \in [0, 1], -\pi/2 < \alpha < \pi/2 \) and \( f \in S \). Then \( f \in S^\alpha_\sigma \) if and only if
\[
\Re \left\{ e^{i\alpha} z f'(z)/f(z) \right\} > \sigma \cos \alpha, \quad z \in U.
\]

Obviously,
\[
S^0_\sigma = S^\ast(\sigma) \quad \text{and} \quad S^\sigma_0 = S_\sigma.
\]

For \( f_j \in \mathcal{A} \) given by
\[
f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \quad (j = 1, 2),
\]
the Hadamard product (or convolution) \( f_1 \ast f_2 \) of \( f_1 \) and \( f_2 \) is defined by
\[
(f_1 \ast f_2)(z) = z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k \quad (z \in U).
\]

We recall that a family of the Hurwitz-Lerch Zeta functions \( \Phi^{(\rho, \sigma)}_{\mu, \nu}(z, s, a) \) ([12]) is defined by
\[
\Phi^{(\rho, \sigma)}_{\mu, \nu}(z, s, a) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{(\nu)_\sigma (n + a)^s},
\]
where
\[
(\mu \in \mathbb{C}; \ a, \nu \in \mathbb{C}\backslash \mathbb{Z}^-; \ \rho, \sigma \in \mathbb{R}^+; \ \rho < \sigma \ \text{when} \ s, z \in \mathbb{C}; \ \rho = \sigma \ \text{and} \ s \in \mathbb{C} \ \text{when} \ |z| < 1; \ \rho = s \ \text{and} \ \Re(s - \mu + \nu) > 1 \ \text{when} \ |z| = 1),
\]
contains as its special cases, not only the Hurwitz-Lerch Zeta function
\[
\Phi^{(\rho, \sigma)}_{\mu, \nu}(z, s, a) = \Phi^{(0, 0)}_{\mu, \nu}(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n + a)^s},
\]
but also the following generalized Hurwitz-Zeta function introduced and studied earlier by Goyal and Laddha ([13]),
\[
\Phi^{(1, 1)}_{\mu, 1}(z, s, a) = \Phi_{\mu}(z, s, a) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n + a)^s}, \quad \text{(1.2)}
\]
which, for convenience, are called the Goyal-Laddha-Hurwitz-Lerch Zeta function. Here \((x)_k\) is Pochhammer symbol (or the shifted factorial, since \((1)_k = k!\)) and \((x)_k\) given in terms of the Gamma functions can be written as

\[
(x)_k = \Gamma(x + k) / \Gamma(x) \begin{cases} 
1, & \text{if } k = 0 \text{ and } x \in \mathbb{C}\setminus\{0\}; \\
x(x+1)...(x+k-1), & \text{if } k \in \mathbb{N} \text{ and } x \in \mathbb{C}.
\end{cases}
\]

It follows that the authors [1] introduced the linear operator \(\Theta^{\lambda,n}_m f(z)\) as the following.

For \(a = 1\), in (1.2), we consider the function

\[
G(z) = z\Phi\mu(z, s, 1) = z + \sum_{k=2}^{\infty} \frac{(\mu)_{k-1}}{(k-1)!} \frac{z^k}{k^s}.
\]

Thus

\[
G(z) * G(z)^{(-1)} = z + \sum_{k=2}^{\infty} \frac{(\lambda+1)_{k-1}}{(k-1)!} \frac{z^k}{(m)_k}.
\]

Now for \(s = n, \lambda \in \mathbb{N}_0\) and \(\mu = m \in \mathbb{N}\), we define the linear operator

\[
\Theta^{\lambda,n}_m f(z) = G(z)^{(-1)} * f(z). \quad \left( f \in \mathcal{A} \right)
\]

\[
= z + \sum_{k=2}^{\infty} \frac{(\lambda+1)_{k-1}}{(m)_{k-1}} a_k z^k.
\]  

(1.3)

In [10], Noonan and Thomas stated that the \(q\)th Hankel determinant of the function \(f\) of the form (1.1) is defined for \(q \in \mathbb{N}\) by

\[
H_q(k) = \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+q+1} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+q+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+q-1} & a_{k+q} & \cdots & a_{k+2q-2} \end{vmatrix}.
\]

We now introduce the following class of functions.

**Definition 1.3.** The function \(f \in \mathcal{A}\) is said to be in the class \(S^{\lambda,n}_m(\alpha, \sigma), \quad (|\alpha| < \pi/2, 0 \leq \sigma < 1)\) if it satisfies the inequality

\[
\Re \left\{ e^{i\alpha} \frac{\Theta^{\lambda,n}_m f(z)}{z} \right\} > \sigma \cos \alpha \quad (z \in \mathcal{U}).
\]  

(1.4)

As is usually the case, we let \(P\) be the family of all functions \(p\) analytic in \(U\) for which \(\Re\{p(z)\} > 0\) and

\[
p(z) = 1 + c_1 z + c_2 z^2 + ..., \quad z \in \mathcal{U}.
\]  

(1.5)
It follows from (1.4) that
\[ f \in S_{m}^{\lambda,n}(\alpha,\sigma) \Leftrightarrow e^{\mu_{\lambda,n}} \frac{f(z)}{z} = [(1 - \sigma)p(z) + \sigma] \cos \alpha + i \sin \alpha, \tag{1.6} \]
where \( \alpha \) is real, \( |\alpha| < \pi/2 \) and \( p(z) \in P \).

We note that
\[
S_{1}^{0,0}(\alpha,\sigma) = \left\{ f : f \in A \text{ and } \Re \left\{ e^{i\mu} \frac{f(z)}{z} \right\} > \sigma \cos \alpha \right\},
\]
\[
S_{1}^{0,1}(\alpha,\sigma) = \left\{ f : f \in A \text{ and } \Re \left\{ e^{i\mu} f'(z) \right\} > \sigma \cos \alpha \right\},
\]
\[
S_{1}^{1,0}(0,0) = S_{1}^{1,1}(0,0) = S_{2}^{1,1}(0,0) = \mathcal{R} := \left\{ f : f \in A \text{ and } \Re \{ f'(z) \} > 0 \right\}.
\]

**Remark 1.1 ([6]).** The subclass \( \mathcal{R} \) was studied systematically by MacGregor ([11]) who indeed referred to numerous earlier investigations involving functions whose derivative has a positive real part.

It is well known ([9]) that for \( f \in S \) and given by (1.1) the sharp inequality \( |a_{3} - a_{2}^{2}| \leq 1 \) holds. This corresponds to the Hankel determinant with \( q = 2 \) and \( k = 1 \). For a given family \( \mathcal{F} \) of functions in \( A \), the sharp bound for the nonlinear functional \( |a_{2}a_{4} - a_{3}^{2}| \) is popularly known as the second Hankel determinant. This corresponds to the Hankel determinant with \( q = 2 \) and \( k = 2 \). The second Hankel determinant for some subclasses of analytic and univalent functions has been studied by many authors (see [2]-[6], [18], [19]).

In the present paper, we seek upper bound for the functional \( |a_{2}a_{4} - a_{3}^{2}| \left( f \in S_{m}^{\lambda,n}(\alpha,\sigma) \right) \).

Our investigation includes a recent result of Janteng et al. [2].

To prove our main result, we need the following lemmas.

**Lemma 1.2 ([9]).** Let the function \( p \in P \) and be given by the series (1.5). Then, the sharp estimate
\[ |c_{k}| \leq 2 \quad (k \in \mathbb{N}) \]
holds.

**Lemma 1.3 ([7] and [8]).** Let the function \( p \in P \) be given by the series (1.5). Then
\[ 2c_{2} = c_{1}^{2} + x(4 - c_{1}^{2}) \tag{1.7} \]
for some \( x, |x| \leq 1 \) and
\[ 4c_{3} = c_{1}^{3} + 2(4 - c_{1}^{2})c_{1}x - c_{1}(4 - c_{1}^{2})x^{2} + 2(4 - c_{1}^{2})(1 - |x|^{2})z \tag{1.8} \]
for some \( z, |z| \leq 1 \).
2. Main results

We prove the following.

**Theorem 2.1.** Let the function \( f \) given by (1.1) be in the class \( S_m^{\lambda,n}(\alpha, \sigma) \). Then

\[
|a_2a_4 - a_3^2| \leq \frac{4m^2(1-\sigma)^2(1+m)^2 \cos^2 \alpha}{3^{2n}(\lambda + 1)^2(\lambda + 2)^2}.
\]

(2.1)

The estimate (2.1) is sharp.

**Proof.** Let \( f \in S_m^{\lambda,n}(\alpha, \sigma) \). Then from (1.6) we have

\[
e^{ia} \Theta_m^{\lambda,n} f(z) = [(1-\sigma)p(z) + \sigma] \cos \alpha + i \sin \alpha,
\]

where \( p \in P \) and is given by (1.5). Then

\[
e^{ia} \left\{ 1 + \sum_{k=2}^{\infty} \frac{(k+\lambda-1)!(m-1)!}{\lambda!(k+m-2)!} k^n a_k z^{k-1} \right\} = [(1-\sigma)(1 + \sum_{k=1}^{\infty} c_k z^k) + \sigma] \cos \alpha + i \sin \alpha.
\]

Comparing the coefficients, we get

\[
\begin{align*}
\frac{\lambda+1}{m} 2^n e^{ia} a_2 &= (1-\sigma)c_1 \cos \alpha, \\
\frac{(\lambda+2)(\lambda+1)}{m(m+1)} 3^n e^{ia} a_3 &= (1-\sigma)c_2 \cos \alpha, \\
\frac{(\lambda+3)(\lambda+2)(\lambda+1)}{m(m+1)(m+2)} 4^n e^{ia} a_4 &= (1-\sigma)c_3 \cos \alpha.
\end{align*}
\]

(2.2)

Therefore, (2.2) yields

\[
|a_2a_4 - a_3^2| = \frac{m^2(1-\sigma)^2(1+m) \cos^2 \alpha}{(\lambda + 1)^2(\lambda + 2)} \left| \frac{(m+2)c_1 c_3}{2^{3n}(\lambda + 3)} - \frac{c_2^2(m+1)}{3^{2n}(\lambda + 2)} \right|.
\]

Since the functions \( p(z) \) and \( p(e^{i\theta} z) \), \((\theta \in \mathbb{R})\) are members of the class \( P \) simultaneously, we assume without loss of generality that \( c_1 > 0 \). For convenience of notation, we take \( c_1 = c \), \( c \in [0,2] \). Using (1.7) along with (1.8), we get

\[
|a_2a_4 - a_3^2| = \frac{m^2(1-\sigma)^2(1+m) \cos^2 \alpha}{4(\lambda + 1)^2(\lambda + 2)} \left\{ \frac{(m+2)}{2^{3n}(\lambda + 3)} [c^4 + 2c^2(4-c^2)x - c^2(4-c^2)x^2] + 2c(4-c^2)(1-|x|^2)z - \frac{(m+1)}{3^{2n}(\lambda + 2)} [c^4 + 2c^2(4-c^2)x + x^2(4-c^2)^2] \right\}
\]

\[
= \frac{m^2(1-\sigma)^2(1+m) \cos^2 \alpha}{4(\lambda + 1)^2(\lambda + 2)} \left\{ \frac{(m+2)}{2^{3n}(\lambda + 3)} - \frac{(m+1)}{3^{2n}(\lambda + 2)} \right\} c^4 + \frac{(m+2)}{2^{3n}(\lambda + 3)} - \frac{(m+1)}{3^{2n}(\lambda + 2)} c^4 x
\]

\[
+ \left\{ \frac{(m+2)}{2^{3n}(\lambda + 3)} - \frac{(m+1)}{3^{2n}(\lambda + 2)} \right\} 2c^2(4-c^2)x
\]
\[-\left\{ \frac{c^2(m + 2)}{2^{3n}c(\lambda + 3)} + \frac{(m + 1)(4 - c^2)}{3^{2n}\lambda(\lambda + 2)} \right\} x^2(4 - c^2) + \frac{2(m + 2)}{2^{3n}(\lambda + 3)}c(4 - c^2)(1 - |x|^2)z \right].

An application of triangle inequality and replacement of $|x|$ by $y$ give

$$|a_2a_4 - a_5^2| \leq \frac{m^2(1 - \sigma)^2(1 + m)c^2}{3\alpha} \left\{ \frac{(m + 2)}{2^{3n}(\lambda + 3)} - \frac{(m + 1)}{3^{2n}(\lambda + 2)} \right\} c^4 + \frac{2c^2y(4 - c^2)}{2^{3n}(\lambda + 3)} + \frac{2(m + 2)}{2^{3n}(\lambda + 3)}c(4 - c^2)(1 - y^2) \right\}\frac{y^2(4 - c^2)}{2^{3n}(\lambda + 3)} + \frac{2(m + 2)}{2^{3n}(\lambda + 3)}c(4 - c^2)$$

$$= G(c, y), \quad 0 \leq c \leq 2 \text{ and } 0 \leq y \leq 1. \quad (2.3)$$

We next maximize the function $G(c, y)$ on the closed square $[0, 2] \times [0, 1]$. Since

$$\frac{\partial G}{\partial y} = \frac{m^2(1 - \sigma)^2(1 + m)c^2}{3\alpha} \left\{ \frac{(m + 2)}{2^{3n}(\lambda + 3)} - \frac{(m + 1)}{3^{2n}(\lambda + 2)} \right\} c^4 + \frac{2c^2y(4 - c^2)}{2^{3n}(\lambda + 3)} + \frac{2(m + 2)}{2^{3n}(\lambda + 3)}c(4 - c^2)(1 - y^2) \right\}\frac{y^2(4 - c^2)}{2^{3n}(\lambda + 3)} + \frac{2(m + 2)}{2^{3n}(\lambda + 3)}c(4 - c^2)$$

$c - 2 < 0, 3^{2n}(m + 2)(\lambda + 2) > 2^{3n}(m + 1)(\lambda + 3)$, we have $\partial G/\partial y > 0$ for $0 < c < 2, 0 < y < 1$. Thus $G(c, y)$ cannot have a maximum in the interior of the closed square $[0, 2] \times [0, 1]$. Moreover, for fixed $c \in [0, 2], \max_{0 \leq y \leq 1} G(c, y) = G(c, 1) = F(c)$. Since

$$F(c) = \frac{m^2(1 - \sigma)^2(1 + m)c^2}{3\alpha} \left\{ \frac{(m + 2)}{2^{3n}(\lambda + 3)} - \frac{(m + 1)}{3^{2n}(\lambda + 2)} \right\} c^4 + \frac{2c^2y(4 - c^2)}{2^{3n}(\lambda + 3)} + \frac{2(m + 2)}{2^{3n}(\lambda + 3)}c(4 - c^2)(1 - y^2) \right\}\frac{y^2(4 - c^2)}{2^{3n}(\lambda + 3)} + \frac{2(m + 2)}{2^{3n}(\lambda + 3)}c(4 - c^2)$$

Then $F'(c) = \frac{2m^2(1 - \sigma)^2(1 + m)c^2}{(\lambda + 1)^2(\lambda + 2)^2} \left\{ \frac{(m + 2)}{2^{3n}(\lambda + 3)} - \frac{(m + 1)}{3^{2n}(\lambda + 2)} \right\} c^4 + \frac{2c^2(y(4 - c^2))}{2^{3n}(\lambda + 3)} + \frac{2(m + 2)}{2^{3n}(\lambda + 3)}c(4 - c^2)(1 - y^2) \right\}\frac{y^2(4 - c^2)}{2^{3n}(\lambda + 3)} + \frac{2(m + 2)}{2^{3n}(\lambda + 3)}c(4 - c^2)$, so that $F'(c) < 0$ for $0 < c < 2$ and has real critical point at $c = 0$. Also $F(c) > F(2)$. Therefore, $\max_{0 \leq c \leq 2} F(c)$ occurs at $c = 0$. Therefore, the upper bound of (2.3) corresponds to $y = 1, c = 0$. Hence

$$|a_2a_4 - a_5^2| \leq \frac{4m^2(1 - \sigma)^2(1 + m)^2c^2}{2^{3n}(\lambda + 1)^2(\lambda + 2)^2}. \quad (2.3)$$
which is the assertion (2.1). Equality holds for the function
\[ f(z) = \left( \sum_{k=1}^{\infty} \frac{(m)_{k-1}}{(\lambda + 1)_{k-1}} k^n z^k \right) \ast e^{-i\alpha} \left[ z \left( \frac{1 + (1 - 2\sigma)z^2}{1 - z^2} \cos \alpha + i \sin \alpha \right) \right]. \]

This completes the proof of the Theorem 2.1.

**Remark 2.1.** For \( \alpha = 0, \sigma = 0, \lambda = m = 1, n = 0 \) and for \( \alpha = 0, \sigma = 0, \lambda = 1, m = 2, n = 1 \) we get a resent result due to Janteng et al. [2] as in the following corollary.

**Corollary 2.1.** If \( f \in \mathcal{R} \) then
\[ |a_2 a_4 - a_3^2| \leq \frac{4}{9}. \]
The result is sharp.

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