Instability of scalar charges in 1+1D and 2+1D

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(Dated: November 2, 2018)

PACS numbers: 04.25.-g

I. INTRODUCTION AND SUMMARY

The absence of observations of scalar charges — point-like particles which act as sources of non-interacting scalar fields — is an interesting fact, in particular because of the simplicity of the scalar field theory. This raises the question of whether the model of scalar charges is self consistent. Specifically, one can ask whether scalar charges are stable against self interaction. As we shall show in this paper, scalar charges in (1+1)-dimensions and in (2+1)-dimensions cannot have a constant rest mass already in flat spacetime. Instead, they lose their mass through the emission of monopole waves.

The variable rest mass is an intersting phenomenon, specific to scalar charges, which surprisingly has attracted only little attention, despite its being noted long ago [1]. Recently, the mass loss by a scalar charge has been studied in the spacetime of an expanding universe [2]. It is shown in Ref. [2] that scalar charges must have variable mass in a class of cosmological spacetimes, which includes the de Sitter spacetime and a spatially-flat matter dominated cosmology. It is also shown that the properties of such particles are invariably coupled to the cosmological parameters. In this paper we shall study a similar effect in flat spacetime in (1+1)-dimensions and in (2+1)-dimensions.

Interestingly, similar phenomena never happen with electric charges. Specifically, the force on an electric charge $e$ is given by the Lorentz force law, $f_a = eF_{a\beta}u^\beta$, where $F_{a\beta}$ is the Maxwell field-strength tensor, and $u^\beta$ is the particle’s four velocity. In the rest frame of the particle this reduces to $f_a = eF_{at}$, such that the $t$ component has to vanish because of the skew symmetry of Maxwell’s tensor.

Because of the lack of observations, the scalar field theory is not unique. In particular, there is a number of possible ways to couple sources to scalar fields. Some of the action principles — notably action principles which lead to nonlinear couplings — do not share the phenomenon of variable rest mass [3, 4]. We use here the simplest action principle, which is also the most popular one. This action principle leads to the simplest, linear wave equation. It also yields a force law in which the force is proportional to the field’s gradient. This implies that the force is not necessarily orthogonal to the particle’s world line. Our results are restricted to this action principle. Although other action principles may be interesting to consider, it is interesting to study the implications of a given theory, especially because it is an unusual one. Further, similar phenomena in realistic spacetimes may provide an explanation to the lack of observations of such particles [5].

It is interesting to note that in (3+1)-dimensions this phenomenon occurs only in certain classes of cosmological spacetimes (including de Sitter spacetime and spatially-flat matter dominated cosmology [2]), but it does not occur in wide classes of other spacetimes, including spacetimes of stationary black holes [6]. It is easy and instructive to see why nothing interesting happens in (3+1)-dimensions Minkowski spacetime. The retarded Green’s function in (3+1)-dimensions for a source at the origin of the coordinates is given by $G(t,r;t’,0) = \delta(t-t’)-r/\sqrt{r}$. Because the Green’s function has support only on the light cone, there is no “tail”, or wake, which has support inside the light cone. Consequently, the field $\Phi$ of a static source is strictly static, such that the temporal component of its derivative vanishes. The situation in (2+1)-dimensions and (1+1)-dimensions is markedly different already in flat spacetime. The Green’s function has support inside the light cone, and this leads to non-trivial self interaction.

This paper is organized as follows. In Section II we write the action principle and the equations of motion. In Section III we study the self interaction in (1+1)-dimensions, and in Section IV we study it in (2+1)-dimensions. We find that scalar particles in either (1+1)-dimensions or (2+1)-dimensions must lose their rest mass. The lost energy is radiated to infinity, in addition to some of the energy which is stored in the field at finite distances, such that globally energy is conserved.

II. ACTION PRINCIPLE AND EQUATIONS OF MOTION

We take the action principle in $D$ dimensions to be [3]

$$S = \int \left\{-\frac{1}{8\pi}g^{\alpha\beta}\Phi_{,\alpha}\Phi_{,\beta} - \int (E_0 - q\Phi)\right\}$$
where $E_0$ is the particle’s bare mass, $q$ is the scalar charge which is the source for the scalar field $\Phi$, $g_{\alpha\beta}$ is the fixed metric of the background whose determinant is $g$, and $\lambda$ is an arbitrary parametrization of the particle’s world line $z^\alpha(\lambda)$. Variation of the action with respect to $\Phi$ yields the wave equation

$$\nabla_\mu \nabla^\mu \Phi(x^\alpha) = -4\pi \rho(x^\alpha), \quad (2)$$

where the scalar charge density is given by

$$\rho(x^\alpha) = \int q(\tau) \frac{\delta^D [x, z(\tau)]}{\sqrt{-g}} \, d\tau, \quad \text{(3)}$$

and $\tau$ is the particle’s proper time. Variation of the action with respect to the world line yields the force law $f_\alpha = q\Phi_{,\alpha}$. Notice that the force $f_\alpha$ is not necessarily orthogonal to the world line. This implies that the mass of the particle does not have to be conserved.

Identical force law and wave equation can be obtained also from alternative action principles. For example, some authors use the action

$$S' = \int \left\{ \frac{1}{8\pi} g^{\alpha\beta} \Phi_{,\alpha} \Phi_{,\beta} + \int \left[ \frac{m(\tau)}{2} g_{\alpha\beta} u^\alpha u^\beta + q\Phi \right] \frac{\delta^D [x, z(\tau)]}{\sqrt{-g}} \, d\tau \right\} \sqrt{-g} \, d^D x. \quad \text{(4)}$$

Note, however, that this action is not invariant under reparametrization of the world line.

We note that it is easy to construct action principles which lead to equations of motion where the force is always orthogonal to the world line. For example, consider the action principle

$$S'' = \int \left\{ \frac{1}{8\pi} g^{\alpha\beta} \Phi_{,\alpha} \Phi_{,\beta} - \int E_0 \exp(-q\Phi/E_0) \right\} \sqrt{-g} \, d^D x. \quad \text{(5)}$$

Variation of $S''$ with respect to the world line yields

$$E_0 \frac{Du^\mu}{d\tau} = q(g^{\mu\nu} + u^\mu u^\nu)\Phi_{,\nu}, \quad \text{(6)}$$

which is explicitly orthogonal to the world line. Variation of $S''$ with respect to the field $\Phi$ yields the wave equation

$$\nabla_\mu \nabla^\mu \Phi = -4\pi \rho \exp(-q\Phi/E_0). \quad \text{(7)}$$

The wave equation (7) includes a source term which is coupled nonlinearly to the field $\Phi$. We note that several authors have used the projected force law (4) in tandem with the wave equation (6) [instead of (5)]. This choice is obviously inconsistent.

III. \((1+1)\)-D FLAT SPACETIME

The metric is given simply by

$$ds^2 = -dt^2 + dx^2. \quad \text{(8)}$$

The scalar field equation with this metric is given by Eq. (\ref{eq:scalar_field_equation}), where $\nabla_\mu \nabla^\mu \Phi = -\partial_\mu^2 \Phi + \partial_\mu \Phi$, and where the charge density $\rho(t, x) = q(t)\delta(x)\Theta(t - t_0)$.

Normally, one may expect the charge to be a conserved quantity. For scalar charges, however, no such conservation law exists, and thus we generally allow $q$ to vary with the time $t$. Also, we assume here that the charge density vanished for $t < t_0$. (Later, we can take $t_0 \to -\infty$.) For simplicity we assume that the charge is static, because the phenomenon of interest occurs already there, and we place the charge at the center of the coordinates without loss of generality.

One could argue the following: Spacetime is static, the source for the field is static, and therefore we could expect the field itself to satisfy the same symmetry, i.e., be static. This argument fails for two reasons. First, the charge was born at $t = t_0$, hence it is not strictly static. However, even if $t_0 \to -\infty$, the static solution is untenable. To illustrate this point, let us assume for now that $q(t) = q_0 = \text{const}$. It is simple to find the static solution, which is given by $\Phi_{\text{static}}(x) = -2\pi q_0 |x|$. The problem with this solution is that, in a certain sense, it requires infinite energy. To see that let us consider scalars built from the stress-energy tensor. Two possible scalars are $T = (T_{\alpha\beta}T^{\alpha\beta})^{1/2}$ and $T = T_{\alpha\beta}T^{\alpha\beta}$. In any dimensions $D$, $T = \frac{1}{8\pi} (2 - D)\Phi_{,\alpha} \Phi^{,\alpha}$, and $T = \frac{1}{8\pi} D^{1/2} \Phi_{,\alpha} \Phi^{,\alpha}$. As in $\text{(1+1)-dimensions}$ $T$ is always zero, we shall consider here $T$. With the simple static solution, $\Phi_{,\alpha} \Phi^{,\alpha} = 4\pi q_0^2$, which never decays. Namely, even if we go to infinity, the stress-energy never drops off, such that its integral over the entire space diverges. Note, that unlike the infinite potential energy of a classical electron, this divergence comes from the behavior of the field at infinity, and not from the extrapolation of the field to the coincidence limit with its source. The reason for this ill behavior is that with the static solution we did not require appropriate boundary conditions at infinity.

In order to find the correct physical solution, let us require that the boundary conditions are that there is no incoming radiation at infinity. Then the retarded Green’s function is given by

$$G(t, x; t', x') = 2\pi \Theta(t - t' - |x - x'|). \quad \text{(9)}$$

An interesting property of this Green’s function is that it has support inside the light cone. (Note, that one can always add a constant to the Green’s function. In particular, one can make $G$ vanish inside the light cone. Then, however, $G$ becomes non-zero outside the light cone, in such a way that the solution for the field is unchanged.) The field is obtained by convolving the source with the Green’s function. Specifically,

$$\Phi(t, x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt' dx' \rho(t, x')G(t, x; t', x'). \quad \text{(10)}$$
\[ Q(z) = \int z q(x) \, dx \]  
Note, that this solution is time dependent. Recall, that the assumption of no ingoing radiation at infinity is implicit in the Green’s function. This assumption breaks the time invariance of the problem, even though we do not stipulate outgoing radiation at infinity. We remark that this solution has no tail: the field propagates strictly along the characteristics. It is the time dependence which leads to the effect in question.

We next study the self interaction of the source for the field, the charge \( q \). The self force acting on this charge is given simply by \( f_\alpha(t) = q(t) \partial_\alpha \Phi(t, x) \big|_{x=0} \). Substituting the solution for the field we find that

\[ f_\alpha = 2\pi q^2(t) \partial_\alpha \delta(x) \]  
(11)

This self force has only a temporal component. From symmetry, it is clear why the spatial component has to be zero. However, the temporal component is non-zero in the rest frame of the charge. This implies that the mass of the particle cannot remain constant: it must vary with time. As \( f_t = dq \big| / dt = -dE \big/ dt \), we find that \( dE \big/ dt = -2\pi q^2(t) \), which is a negative-definite quantity.

Here, \( p_\mu \) is the particle’s four-momentum, and \( E \) is its energy. Consequently, the mass of the particle necessarily decreases with time.

In the simplest case, where the energy of the particle has no scalar-charge origin, the charge \( q = q_0 \) does not vary with time. The particle’s rest energy is then given by \( E(t) = E_0 - 2\pi q_0^2 t \), which decreases linearly with time. At the time \( \tau = E_0/2\pi q_0^2 \) the energy of the particle vanishes, but \( E \) is not zero then. Strictly speaking, the particle could continue to lose mass, and at times \( t > \tau \) the mass of the particle would become negative, and continue to grow ever more negative. In order to avoid such runaway problems, we recall that our purely classical treatment cannot predict what happens when a particle loses all its mass. A way to resolve this difficulty is the following. Recall that there is no charge conservation for scalar charges. That is, the charge does not have to be constant. One could consider models in which \( q \) varies with time, such that \( E, \dot{E} \) and \( \dot{q} \) vanish simultaneously. In such models the particle just ceases to exist when both its charge and rest mass vanish. Alternative possibilities are that the charge vanishes before the mass does. This is the case when \( E_0 \) is very large. In that case we would be left with an uncharged remnant.

However, also the original possibility could have an interesting implication. Because of the non-conservation of scalar charges, the charge could abruptly disappear at the time that \( E \) vanishes, such that no negative-energy particles would be produced \[ \text{ }. \]

Two simple models for a scalar-field origin of the particle’s mass are the following. In both models we assume to the particle at time \( t_0 \) a charge \( q_0 \). First, in analogy with the classical electron models, we can introduce a new length scale, the classical radius of a scalar particle, and postulate that the particle’s mass \( E = \alpha q^2 \). This implies that \( q(t) = q_0 \exp[-(\pi/\alpha)(t-t_0)] \). Second, a simpler model, which does not require the introduction of a new length scale, is the one where the particle’s mass is proportional to its charge, i.e., \( E = \beta |q| \). (As motivation for this model consider the case of a charged black hole, for which the maximal mass is equal to its charge.) This implies that \( q(t) = q_0/|1 + \beta^2 t_0(t-t_0)| \). In both models the particle’s charge and mass vanish only at infinitely late times.

Although the local energy of the particle decreases with time, global energy is still conserved. Naturally, we expect energy flux to infinity. Indeed, we find that that flux is given by

\[ \mathcal{F}^\pm = -T^t_t = \pi q^2(t - |x|)(x/|x|), \]  
(12)

where \( \mathcal{F}^\pm \) is the flux to the positive or negative directions, respectively. The total radiated flux is given then by \( \mathcal{F} = \mathcal{F}^+ - \mathcal{F}^- = 2\pi q^2(t - |x|) \). It is interesting to note that \( \dot{E} + \mathcal{F} \neq 0 \), except for the special case where the charge \( q \) is constant. This is the case because of a retardation effect: The lost mass depends on the charge evaluated at a time \( t \), whereas the flux depends on the charge evaluated at the time \( t - |x| \). Unless the charge does not change with time, these two quantities will not balance each other to guarantee global energy conservation. In fact, we find that the flux to infinity is greater than the rate at which the particle loses mass. The extra flux comes from the energy stored in the field at finite distances, which decreases with time. To find the rate of change of the energy stored in the field, we compute

\[ \frac{d}{dt} \int_{-x}^x T^{tt} \, dx = -2\pi |q^2(t - x) - q^2(t)| \]  
(13)

where \( x > 0 \). Now, we do indeed find that

\[ \frac{d}{dt} E + \mathcal{F} + \frac{d}{dt} \int_{-x}^x T^{tt} \, dx = 0 \]  
(14)

such that global energy is explicitly conserved. Note that this is the case for any value of \( x \), and in particular for \( x \rightarrow \infty \), i.e., the entire space.

**IV. (2 + 1)-D FLAT SPACETIME**

The metric is now

\[ ds^2 = -dt^2 + dx^2 + r^2 \, d\theta^2 \]  
(15)

such that the field equation is given by Eq. (3), where the wave operator is \( \nabla_\mu \nabla^\mu = -\partial_t^2 + \partial_r^2 + (1/r) \partial_r \), and the charge density is given by \( \rho = q(t)e(r)\delta(t-t_0) \). For brevity we shall consider here only the case of constant charge, i.e., \( q = q_0 = \text{const.} \). [Generalization to variable
charge is analogous to the case of (1 + 1)-dimensions.]
In analogy with (1 + 1)-dimensions, we seek boundary conditions with no incoming radiation from infinity. The Green’s function then is given by

\[
G(t, r'; t', r) = \frac{2\Theta(t - t') - (r - r')}{(t - t')^2 - (r - r')^2}.
\]

(16)

Convolving the charge density with this Green’s function we find that

\[
\Phi(t, r) = 2q_0 \ln \left[ \frac{r}{(t - t_0) - \sqrt{(t - t_0)^2 - r^2}} \right].
\]

(17)

In (2 + 1)-dimensions the solution has support inside the light cone, and it is time dependent. It is indeed a well known phenomenon, that in odd spacetime dimensions the Huygens principle is violated. Notice that \( \Phi \) vanishes on the light cone, and that it is undefined outside the light cone. This solution satisfies \( \Phi_{,\alpha} \Phi^{,\alpha} = 4q_0 / r^2 \). The self force acting on the particle is \( f = 2q_0^2 / (t - t_0) \), such that the rate at which mass is dissipated is given by \( E = -2q_0^2 / (t - t_0) \). Notice that any finite mass particle must had infinite mass at the time \( t_0 \). Assuming that at the time \( t > t_0 \) the particle had mass \( E_1 \), we find that \( E(t) = E_1 - 2q_0^2 \ln[(t - t_0)/(t_1 - t_0)] \), such that at the finite time \( \tau = t_1 \exp[E_1/(2q_0)] \) the particle loses all its mass.

Again, global energy is conserved: The flux through a circle of radius \( r \) is given by

\[
\mathcal{F} = -\int_0^{2\pi} T_{\tau} r d\theta = 2q_0^2 \frac{t - t_0}{(t - t_0)^2 - r^2}.
\]

(18)

As in the case of (1 + 1)-dimensions, the flux to infinity is greater than the rate at which the particle loses mass. The rest of the flux comes from a decrease in the energy stored by the field, at the rate of

\[
\frac{d}{dt} \int_0^{2\pi} \int_0^r dr T^{\tau t} r = -2q_0^2 \frac{r^2}{(t - t_0)^2 - r^2}[(t - t_0)].
\]

(19)

such that

\[
\dot{E} + \mathcal{F} + \frac{d}{dt} \int_0^{2\pi} \int_0^r dr T^{\tau t} r = 0.
\]

(20)

This energy conservation is valid for any radius \( r \), and in particular for \( r \to \infty \).

V. CONCLUSIONS

We have shown that scalar charges cannot maintain constant mass in (1 + 1)-dimensions and in (2 + 1)-dimensions, already in flat spacetime. In fact, the mass of the particle must always decrease with time. The particle loses its mass because of its self interaction, through the mechanism of emission of monopole waves. In this sense scalar charges are unstable against self interaction. In (3 + 1)-dimensions, however, scalar charges are stable in flat spacetime, and also in certain non-cosmological spacetimes, e.g., in black hole spacetimes. The situation is different when cosmological spacetimes (e.g., de Sitter spacetime, or a matter dominated cosmology) are considered [2], where a similar mass loss is found.

We remark that we have only studied this effect classically. It is interesting to see how scalar particles behave quantum mechanically or in dimensions greater than four.

Acknowledgments

I thank Abraham Harte, Poghos Kazarian, Karel Kuchař, Amos Ori, and Richard Price for discussions. Initial work on this project was done at the California Institute of Technology, where it was supported by NSF grants AST-9731698 and PHY-9900776, and by NASA grant NAG5-6840. This research was supported at the University of Utah by NSF grant PHY-9734871.

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