WEIGHTED ENERGY PROBLEM ON THE UNIT CIRCLE

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Abstract. We solve the weighted energy problem on the unit circle, by finding the extremal measure and describing its support. Applications to polynomial and exponential weights are also included.

1. Introduction and main results

Let $w \not\equiv 0$ be a continuous nonnegative function on the unit circle $T := \{ z : |z| = 1 \}$, and set

$$Q(z) := - \log w(z).$$

Let $\mathcal{M}(T)$ be the space of positive unit Borel measures supported on $T$. For any measure $\mu \in \mathcal{M}(T)$, we define the energy functional

$$I_w(\mu) := \iint \log \frac{1}{|z - t|w(z)w(t)} \, d\mu(z)d\mu(t)$$

$$= \iint \log \frac{1}{|z - t|} \, d\mu(z)d\mu(t) + 2 \int Q(t) \, d\mu(t),$$

and consider the minimum energy problem

$$V_w := \inf_{\mu \in \mathcal{M}(T)} I_w(\mu).$$

For a general reference on potential theory with external fields, or weighted potential theory, one should consult the book of Saff and Totik \[14\]. It follows from Theorem I.1.3 of \[14\] that $V_w$ is finite, and there exists a unique equilibrium measure $\mu_w \in \mathcal{M}(T)$ such that $I_w(\mu_w) = V_w$. Thus $\mu_w$ minimizes the energy functional \[122\] in presence of the external field $Q$ generated by the weight $w$. Furthermore, we have for the potential of $\mu_w$ that

$$U_{\mu_w}(z) + Q(z) \geq F_w, \quad z \in T,$$

and

$$U_{\mu_w}(z) + Q(z) = F_w, \quad z \in S_w,$$

where $U_{\mu_w}(z) := - \int \log |z - t| \, d\mu_w(t)$, $F_w := V_w - \int Q(t) \, d\mu_w(t)$ and $S_w := \text{supp} \mu_w$ (see Theorems I.1.3 and I.5.1 in \[14\]). The weighted capacity of $T$ is defined by

$$\text{cap}(T, w) := e^{-V_w}.$$

If $w \equiv 1$ on $T$, then we obtain the classical logarithmic capacity $\text{cap}(T) = 1$, and the equilibrium measure $dt/(2\pi)$, $e^t \in T$.

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The energy problems with external fields on subsets of the real line were treated in many papers, see [14] for a survey and references. The purpose of this paper is twofold: we provide a general solution to the weighted energy problem on the unit circle, and we also simplify the previously known arguments used in the real line case. Our method applies on the real line too, which leads to shorter proofs and generalizations of the results in [3] and [14].

We give below an explicit form of the equilibrium measure and describe its support for the weighted energy problem on $\mathbb{T}$. Throughout the paper, we use the notation $Q(t) := Q(e^{it})$.

**Theorem 1.1.** Suppose that $Q \in C^{1+\varepsilon}(U)$, where $U$ is an open neighborhood of $S_w$ in $\mathbb{T}$. Then $d\mu_w(e^{i\theta}) = f(\theta) \, d\theta$, where $f \in L_\infty([0, 2\pi))$. Furthermore, the density $f(\theta)$ satisfies the equation

$$f^2(\theta) = \left(\frac{Q'(\theta)}{\pi}\right)^2 - \frac{1}{\pi^2} \int_0^{2\pi} Q'(t)f(t) \cot \frac{\theta - t}{2} \, dt + \frac{1}{4\pi^2}$$

for a.e. $e^{i\theta} \in S_w$, where the integral in (1.7) is understood in the principal value sense.

**Corollary 1.2.** Theorem 1.1 also holds with (1.7) replaced by

$$f^2(\theta) = \frac{1}{\pi^2} \int_0^{2\pi} (Q'(\theta) - Q'(t))f(t) \cot \frac{\theta - t}{2} \, dt - \frac{1}{4\pi}$$

for a.e. $e^{i\theta} \in S_w$.

Let $p(\theta)$ be the right hand side of (1.8). If $Q \in C^2(U)$ then $p \in C(U)$ and $f := \sqrt{p} \in C(S_w)$. Furthermore, $S_w$ is the closure in $\mathbb{T}$ of the open set \{e^{i\theta} \in \mathbb{T} : p(\theta) > 0\}. Hence $f(\theta)$ vanishes at the endpoints of $S_w$.

Moreover, if $Q$ is real analytic on $U$, then $S_w$ is a finite union of closed arcs of $\mathbb{T}$.

**Corollary 1.3.** If $S_w = \mathbb{T}$ under the assumptions of Theorem 1.1, then

$$f(\theta) = \frac{1}{2\pi} - \frac{1}{2\pi^2} \int_0^{2\pi} Q'(t) \cot \frac{\theta - t}{2} \, dt, \quad e^{i\theta} \in \mathbb{T}.$$  

In particular, $f$ is Hölder continuous on $\mathbb{T}$.

We mention another instance when the structure of $S_w$ is clear, which parallels the real line case (see Theorem IV.1.10(b) of [14]).

**Proposition 1.4.** If $Q(t)$ is convex on $(\alpha, \beta) \subset \mathbb{R}$, $\beta - \alpha \leq 2\pi$, then the intersection of $S_w$ with the arc $(e^{i\alpha}, e^{i\beta}) \subset \mathbb{T}$ is either an arc or empty set.

The support $S_w$ plays crucial role in determining the equilibrium measure $\mu_w$ itself, as well as other components of this weighted energy problem. Indeed, if $S_w$ is known then $\mu_w$ can be found as a solution of the singular integral equation

$$\int \log \frac{1}{|z - \zeta|} d\mu(\zeta) - \log w(z) = F, \quad z \in S_w,$$

where $F$ is a constant (cf. [13] and [14] Ch. IV]). Applying the results of [8] (see also [13]), we solve this integral equation and obtain the following theorem.
Theorem 1.5. Let $Q \in C^2(U)$, where $U$ is an open neighborhood of $S_w$ in $T$. Assume that $S_w$ consists of $K$ arcs $\Gamma_k \subset T$, $K \geq 1$, with the endpoints $a_k = e^{i\alpha_k}$ and $b_k = e^{i\beta_k}$ such that $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_K < \beta_K$, $\beta_K - \alpha_1 < 2\pi$. Set $R(z) := \prod_{k=1}^{K} (z - a_k)(z - b_k)$, and consider the branch of $\sqrt{R(z)}$ defined in the domain $\mathbb{C} \setminus S_w$ by $\lim_{z \to \infty} \frac{\sqrt{R(z)}}{z^K} = 1$. By the values of $\sqrt{R(z)}$ on $S_w$, we understand the limiting values from inside the unit disk. Let

\begin{equation}
F(z) := \frac{\sqrt{R(z)}}{\pi i} \int_{S_w} \frac{g(\zeta) d\zeta}{\sqrt{R(\zeta)(\zeta - z)}},
\end{equation}

where
\begin{equation}
g(e^{it}) := iQ'(t) + \frac{1}{2\pi}, \quad e^{it} \in S_w,
\end{equation}

and the integral in (1.9) is the Cauchy principal value. Then the density $f$ of $\mu_w$ is given by
\begin{equation}
f(t) = F(e^{it}), \quad e^{it} \in S_w.
\end{equation}

Furthermore, the following equations are satisfied
\begin{equation}
\int_{S_w} z^k g(z) \frac{dz}{\sqrt{R(z)}} = 0, \quad k = 0, \ldots, K - 1,
\end{equation}

(1.12)
\begin{equation}
\int_{S_w} f(t) dt = 1,
\end{equation}

and
\begin{equation}
\int_{\beta_k}^{\alpha_{k+1}} F(e^{it}) dt = \frac{\alpha_{k+1} - \beta_k}{2\pi} + \frac{Q(\alpha_{k+1}) - Q(\beta_k)}{\pi} i, \quad k = 1, \ldots, K,
\end{equation}

where we assume that $\alpha_{K+1} = \alpha_1 + 2\pi$.

Note that equations (1.11)-(1.13) may be used to find the endpoints of $S_w$.

2. Applications

Consider the weight function
\begin{equation}
w(z) := \prod_{j=1}^{J} |z - z_j|^{\lambda_j},
\end{equation}

where $\lambda_j > 0$, $z_j \in \mathbb{C}$ and $z_j \neq 0$, $j = 1, \ldots, J$. In some special cases, such weights on disks were previously treated in [13], with applications to the weighted polynomial approximation. In general, one can express the equilibrium measure $\mu_w$ for $w$ of (2.1) as a linear combination of harmonic measures, which was done in [9] and [10]. The complete solution of the weighted energy problem on the unit circle given below was first found in [11], in connection with a number theoretic problem on heights of some subspaces of polynomials (see [2] for background).

Theorem 2.1. For the weight $w$ of (2.1), the support $S_w$ consists of $K \leq J$ arcs. If $S_w = T$ then we have that
\begin{equation}
d\mu_w(e^{it}) = \frac{1}{2\pi} \left( 1 + \sum_{j=1}^{J} \lambda_j - \sum_{j=1}^{J} \lambda_j \frac{|z_j|^2 - 1}{|e^{it} - z_j|^2} \right) dt.
\end{equation}
If $S_w \neq \mathbb{T}$ then we set

$$F_1(z) := \frac{\sqrt{R(z)}}{2\pi} \sum_{j=1}^{J} \lambda_j \left( \frac{z_j}{(z_j - z)\sqrt{R(z_j)}} + \frac{\bar{z}_{j}^{-1}}{(\bar{z}_{j}^{-1} - z)\sqrt{R(\bar{z}_{j}^{-1})}} \right),$$

where we use the notation of Theorem 1.5. The equilibrium measure $\mu_w$ is given in this case by

$$d\mu_w(e^{it}) = F_1(e^{it}) \, dt, \quad e^{it} \in S_w,$$

where the values of $\sqrt{R(z)}$ on $S_w$ are the limiting values from inside the unit disk.

Furthermore, the endpoints of $S_w$ satisfy the equations

$$\sum_{j=1}^{J} \lambda_j \left( \frac{z_{j}^{k}}{\sqrt{R(z_{j})}} + \frac{\bar{z}_{j}^{-k}}{\sqrt{R(\bar{z}_{j}^{-1})}} \right) = \frac{1}{1 + \sum_{j=1}^{J} \lambda_j},$$

and the equations

$$\int_{\beta_k}^{\alpha_{k+1}} F_1(e^{it}) \, dt = 0, \quad k = 1, \ldots, K.$$

Our second application is related to the exponential weights of the form

$$w(e^{i\theta}) = e^{-t_M(\theta)}, \quad t_M(\theta) := \sum_{m=-M}^{M} c_m e^{im\theta}$$

is a real valued trigonometric polynomial of degree $M$. A typical example of such weight is given by $w(z) = |e^{-cz}|$, $c \in \mathbb{R}$, which was studied in [12] and [13] on disks and on the Szegő domain. The same weight appeared in a problem on the longest increasing subsequence of random permutations, see [1]. The general solution of the weighted energy problem on the unit circle is given below.

**Theorem 2.2.** For the weight $w$ of (2.6), the support $S_w$ consists of $K \leq M$ arcs. If $S_w = \mathbb{T}$ then we have that

$$d\mu_w(e^{it}) = \left( \frac{1}{2\pi} - \frac{1}{\pi} \sum_{m=-M}^{M} \text{sgn}(m) c_m e^{im\theta} \right) \, dt.$$ If $S_w \neq \mathbb{T}$ then we set

$$F_2(z) := \frac{\sqrt{R(z)}}{\pi} \left( \sum_{m=K}^{M} mc_ms_{m+1}(z) - \sum_{m=-M}^{-1} mc_mr_{m-1}(z) \right),$$

in the notation of Theorem 1.5, where

$$\frac{1}{\sqrt{R(\zeta)}(\zeta - z)} = \sum_{k=K+1}^{\infty} \frac{s_k(z)}{\zeta^k}\quad \text{and} \quad \frac{1}{\sqrt{R(\zeta)}(\zeta - z)} = \sum_{k=0}^{\infty} r_k(z)\zeta^k,$$
respectively near $\infty$ and near 0. The measure $\mu_w$ is given by
\begin{equation}
\frac{d\mu_w(e^{it})}{dt} = F_2(e^{it})
\end{equation}
where the values of $\sqrt{R(z)}$ on $S_w$ are the limiting values from inside the unit disk.

Furthermore, the endpoints of $S_w$ satisfy (1.11) with
\begin{equation}
g(z) = \frac{1}{2\pi} - \frac{1}{\pi} \sum_{m=-M}^{M} mc_m z^m,
\end{equation}
and (1.13) with $F = F_2 + g$.

Note that
\begin{equation}
r_k(z) = \frac{1}{k!} \left| \frac{d^k}{d\zeta^k} \left( \frac{1}{\sqrt{R(\zeta)}(\zeta - z)} \right) \right|_{\zeta=0}, \quad k \geq 0,
\end{equation}
and
\begin{equation}
s_k(z) = \frac{1}{k!} \left| \frac{d^k}{d\zeta^k} \left( \frac{\zeta^{K+1}}{\sqrt{R(1/\zeta)}(1 - z\zeta)} \right) \right|_{\zeta=0}, \quad k \geq 0.
\end{equation}
Hence $s_k(z)$ is a polynomial in $z$ of degree at most $k$, and $r_k(z)$ is a polynomial in $1/z$ of degree at most $k + 1$. Also, it is clearly possible to evaluate the integrals in (1.11) (for this function $g$) by using the residues at 0 and $\infty$.

3. Proofs

The proof of Theorem 1.1 requires the following lemma.

**Lemma 3.1.** The weighted equilibrium measure $\mu_w$ is absolutely continuous with respect to the arclength on $\mathbb{T}$, and
\begin{equation}
\frac{d\mu_w(e^{it})}{dt} = f(t)
\end{equation}
where $f \in L_\infty([0, 2\pi])$.

**Proof.** We shall show that $U^{\mu_w}$ is Lipschitz continuous in $\mathbb{C}$, which implies that the directional derivatives of $U^{\mu_w}$ exist a.e. on $\mathbb{T}$, and
\begin{equation}
\frac{d\mu_w(e^{it})}{dt} = -\frac{1}{2\pi} \left( \frac{\partial U^{\mu_w}}{\partial \mathbf{n}_-}(e^{it}) + \frac{\partial U^{\mu_w}}{\partial \mathbf{n}_+}(e^{it}) \right)
\end{equation}
for a.e. $e^{it} \in \mathbb{T}$, by Theorem II.1.5 of [14], where $\mathbf{n}_-$ and $\mathbf{n}_+$ are the outer and the inner normals to $\mathbb{T}$. Clearly, the normal derivatives of $U^{\mu_w}$ are bounded by the Lipschitz constant, so that we obtain $f \in L_\infty([0, 2\pi])$.

Recall that $Q(t) = -\log w(e^{it})$ is a $C^{1+\epsilon}$ function in an open neighborhood $U$ of $S_w$ in $\mathbb{T}$. We can modify $w(e^{it})$ so that for the resulting function $v(e^{it})$ we still have
\begin{equation}
U^{\mu_w}(z) - \log v(z) = F_w,
\end{equation}
and
\begin{equation}
U^{\mu_w}(z) - \log v(z) \geq F_w,
\end{equation}
and we also have $\log v(e^{it}) \in C^{1+\epsilon}(\mathbb{T})$. Theorem I.3.3 of [14] then implies that $\mu_v = \mu_w$ and $F_v = F_w$. Thus we can work with $v$ instead. Indeed, such modification is possible by (1.4) and (1.5), because $S_w$ is a compact set contained in $U$. Hence we can find an open cover $O \subset U$ for $S_w$, consisting of finitely many open arcs.
Then we set \( v|_O = w|_O \), and modify \( w \) to \( v \) on \( \mathbb{T} \setminus O \) (consisting of finitely many closed arcs) in such a way that (cf. (3.1))

\[
U^\mu w(z) - \log v(z) \geq F_w, \quad z \in \mathbb{T} \setminus O,
\]

and \( \log v(e^\mu) \in C^{1+\varepsilon}(\mathbb{T}) \).

Let \( u \) be a solution of the Dirichlet problem in the unit disk \( D \) for the boundary data \( \log v(e^\mu) + F_w \). Then \( u \in C^{1+\varepsilon}(\overline{D}) \) by Privalov’s theorem (see §5 of Chap. IX in [5]). Since \( u|_\mathbb{T} \leq U^\mu w|_\mathbb{T} \) by our construction, we obtain that

\[
u(z) \leq U^\mu w(z), \quad z \in \overline{D},
\]

as \( U^\mu w \) is superharmonic. In the proof of Lipschitz continuity of \( U^\mu w \), we first consider \( z \in S_w \) and \( \zeta \in \mathbb{C} \), and follow an idea of Götz [6]. Since \( u|_{S_w} = U^\mu w|_{S_w} \), it is immediate that

\[
U^\mu w(z) - U^\mu w(\zeta) \leq u(z) - u(\zeta) \leq C|z - \zeta|, \quad z \in S_w, \ \zeta \in \overline{D},
\]

where \( C \) is the Lipschitz constant for \( u \) on \( \overline{D} \). Note that \( U^\mu w(1/\zeta) = U^\mu w(\zeta) + \log |\zeta|, \ \zeta \neq 0 \), because \( S_w \subset \mathbb{T} \). Hence we have from the above estimate that

\[
U^\mu w(z) - U^\mu w(\zeta) \leq (C + 1)|z - \zeta|, \quad z \in S_w, \ \zeta \in \mathbb{C}.
\]

In order to prove a matching estimate from below, we consider a nearest point \( \zeta^* \in S_w \) for \( \zeta \), i.e., \( \operatorname{dist}(\zeta, S_w) = |\zeta - \zeta^*| =: r \). Then

\[
U^\mu w(z) - U^\mu w(\zeta) = u(z) - u(\zeta^*) + M^\mu w(\zeta^*) - U^\mu w(\zeta)
\]

\[
\geq -C|z - \zeta^*| + U^\mu w(\zeta^*) - U^\mu w(\zeta), \quad z \in S_w, \ \zeta \in \mathbb{C}.
\]

Using the area mean-value inequality, we obtain that

\[
U^\mu w(\zeta^*) \geq \frac{1}{\pi(2r)^2} \int_{D_r(\zeta^*)} U^\mu w(x + iy) \, dx \, dy
\]

\[
= \frac{1}{4\pi r^2} \int_{D_r(\zeta^*)} U^\mu w(x + iy) \, dx \, dy + \frac{1}{4} U^\mu w(\zeta),
\]

where the second term comes from the mean-value property for the harmonic function \( U^\mu w \) in \( D_r(\zeta) \). Note that \( U^\mu w(\xi) \geq \frac{1}{2} U^\mu w(\zeta^*) - (C + 1)|\zeta^* - \xi|, \ \xi \in \mathbb{C} \), by (3.1). Hence (3.3) implies that

\[
U^\mu w(\zeta^*) \geq \frac{4\pi r^2 - \pi r^2}{4\pi r^2} \left( U^\mu w(\zeta^*) - (C + 1)2r \right) + \frac{1}{4} U^\mu w(\zeta),
\]

and that

\[
U^\mu w(\zeta^*) - U^\mu w(\zeta) \geq -6(C + 1)r.
\]

Applying this in (3.2), we have

\[
U^\mu w(z) - U^\mu w(\zeta) \geq -C|z - \zeta^*| - 6(C + 1)|\zeta - \zeta^*|
\]

\[
\geq -C|z - \zeta| - 7(C + 1)|\zeta - \zeta^*| \geq -8(C + 1)|z - \zeta|.
\]

Consequently,

\[
|U^\mu w(z) - U^\mu w(\zeta)| \leq 8(C + 1)|z - \zeta|, \quad z \in S_w, \ \zeta \in \mathbb{C}.
\]

We now show that (3.4) is true for any \( z, \zeta \in \mathbb{C} \). Observe that

\[
\sup\{|U^\mu w(z) - U^\mu w(\zeta)| : |z - \zeta| \leq \delta, \ z, \zeta \in \mathbb{C} \} = |U^\mu w(z_0) - U^\mu w(\zeta_0)|
\]
for some $z_0, \zeta_0 \in \mathbb{C}$, $|z_0 - \zeta_0| \leq \delta$, because
\[
\lim_{z, \zeta \to \infty, \zeta \neq 0} \frac{(U^{\mu_w}(z) - U^{\mu_w}(\zeta))}{|z - \zeta|} = 0.
\]
Consider $h(\xi) := U^{\mu_w}(\xi) - U^{\mu_w}(\xi - z_0 + \zeta_0)$, which is continuous on $\mathbb{C}$ (cf. Theorem I.5.1 of [14]) and harmonic in $\mathbb{C} \setminus S_w$. By the maximum-minimum principle, we have
\[
|U^{\mu_w}(z_0) - U^{\mu_w}(\zeta_0)| = |h(\zeta_0)| \leq \max_{\xi \in S_w} |h(\xi)| = |h(\xi_0)| \leq 8(C+1)|z_0 - \zeta_0|,
\]
where $\xi_0 \in S_w$, and where the last inequality follows from (3.4). Thus $\sup\{|U^{\mu_w}(z) - U^{\mu_w}(\zeta)| : |z - \zeta| \leq \delta, \; z, \zeta \in \mathbb{C}\} \leq 8(C+1)\delta$, i.e., $U^{\mu_w}$ is Lipschitz continuous in $\mathbb{C}$.

Proof of Theorem 1.1. The equilibrium equation (1.5) and Lemma 3.1 give that
\[f \left( \frac{\theta - t}{2} \right) \left| 2\sin \frac{\theta - t}{2} \right| dt = Q(\theta) - F_w, \quad e^{i\theta} \in S_w.
\]
Differentiating this equation with respect to $\theta$, we obtain that
\[
\frac{1}{2} \int_0^{2\pi} f(t) \cot \frac{\theta - t}{2} dt = Q'(\theta)
\]
for almost every $\theta$, $e^{i\theta} \in S_w$. The justification of differentiation under the integral is done as in Lemma 2.45 of [3]. Indeed, consider the harmonic conjugate of $f$ (cf. [4, Chap. 3])
\[
\tilde{f}(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \cot \frac{\theta - t}{2} dt,
\]
where the integral is understood in the principal value sense. Since $f \in L_\infty([0, 2\pi])$, we have that $\tilde{f} \in L_p([0, 2\pi])$ for any $p < \infty$, by M. Riesz’s theorem. From the Fundamental Theorem of Calculus and Fubini’s theorem, we obtain that
\[
\int_0^{2\pi} f(t) \log \left| 2\sin \frac{\theta - t}{2} \right| dt = \pi \int_0^\theta \tilde{f}(s) ds + c, \quad e^{i\theta} \in \mathbb{T},
\]
where $c$ is a constant. It follows that
\[
\pi \int_0^\theta \tilde{f}(s) ds = Q(\theta) - F_w - c, \quad e^{i\theta} \in S_w.
\]
Using the Fundamental Theorem of Calculus again, we can differentiate the above equation, so that
\[
(3.5) \quad \tilde{f}(\theta) = \frac{Q'(\theta)}{\pi} \quad \text{a.e. on } S_w.
\]
Consider the analytic in $D$ function
\[H(z) := \frac{1}{2\pi} \int_0^{2\pi} f(t) \frac{e^{it} + z}{e^{it} - z} dt, \quad z \in D,
\]
and recall that $H(z) = u(z) + i\bar{u}(z)$, $\bar{u}(0) = 0$, where $u$ and $\bar{u}$ have the boundary values (cf. [4])
\[u|_{\mathbb{T}} = f \quad \text{and} \quad \bar{u}|_{\mathbb{T}} = \tilde{f}.
\]
Clearly, the function
\[-iH^2(z) = 2u(z)\bar{u}(z) + i(u^2(z) - u^2(z)), \quad z \in D,
\]
is analytic in $D$. Hence the harmonic conjugate of $2u\tilde{u}$ is

$$(2u\tilde{u})^* = \tilde{u}^2 - u^2 + c,$$

where $c$ is selected by the standard convention $(2u\tilde{u})'(0) = 0 = \tilde{u}^2(0) - u^2(0) + c$. Since $\tilde{u}(0) = 0$, we have

$$c = u^2(0) = \left(\frac{1}{2\pi} \int_0^{2\pi} f(t) \, dt\right)^2 = \left(\frac{\mu_w(T)}{2\pi}\right)^2 = \frac{1}{4\pi^2}.$$ 

Passing to the boundary values, we obtain that

$$2(f\tilde{f})^* = \tilde{f}^2 - f^2 + \frac{1}{4\pi^2}, \quad \text{a.e. on } T,$$

and that

$$(3.6) \quad \frac{2}{\pi} Q'\tilde{f} - \left(\frac{Q'}{\pi}\right)^2 - 2(f\tilde{f})^* + \frac{1}{4\pi^2} = f^2 - \left(\tilde{f} - \frac{Q'}{\pi}\right)^2, \quad \text{a.e. on } U.$$

Observe that the right hand side of (3.6) gives a decomposition for the left hand side on $U$ into the positive part $f^2$ and the negative part $-\left(\tilde{f} - \frac{Q'}{\pi}\right)^2$, because of (3.5) and $f(t) = 0$, $e^{it} \notin S_w$. Hence

$$(3.7) \quad f^2(\theta) = \left(\frac{Q'\theta}{\pi}\right)^2 - \frac{2}{\pi} (Q'\tilde{f}(\theta)) + \frac{1}{4\pi^2}$$

$$= \left(\frac{Q'\theta}{\pi}\right)^2 - \frac{1}{\pi^2} \int_0^{2\pi} Q'(t)f(t) \cot \frac{\theta - t}{2} \, dt + \frac{1}{4\pi^2}$$

for a.e. $\theta$ such that $e^{i\theta} \in S_w$. 

\[ \square \]

**Proof of Corollary 1.2** Using (3.5), we obtain that

$$\frac{1}{\pi^2} \int_0^{2\pi} \left(\frac{Q'}{\pi}\right)^2 - f(t) \cot \frac{\theta - t}{2} \, dt - \left(\frac{Q'\theta}{\pi}\right)^2 + \frac{1}{4\pi^2}$$

$$= \left(\frac{Q'\theta}{\pi}\right)^2 - \frac{1}{\pi^2} \int_0^{2\pi} Q'(t)f(t) \cot \frac{\theta - t}{2} \, dt + \frac{1}{4\pi^2}$$

$$= \left(\frac{Q'\theta}{\pi}\right)^2 - \frac{1}{\pi^2} \int_0^{2\pi} (Q'(t) - Q'(t)) \, dt + \frac{1}{4\pi^2}$$

which is the right hand side of (3.6). If $Q \in C^2(U)$ then $p \in C(U)$, because the function

$$\Phi(\theta, t) := (Q'(\theta) - Q'(t)) \cot \frac{\theta - t}{2}$$

can be extended continuously to $U \times U$ by setting $\Phi(\theta, \theta) := 2Q''(\theta)$. Hence $f$ has a continuous extension on $S_w$ by (1.3), which satisfies $f(\theta) = \sqrt{\rho(\theta)}$, $e^{i\theta} \in S_w$. Using this extension in (3.6), we obtain that

$$p = \frac{2}{\pi} Q'\tilde{f} - \left(\frac{Q'}{\pi}\right)^2 - 2(f\tilde{f})^* + \frac{1}{4\pi^2} = f^2 - \left(\tilde{f} - \frac{Q'}{\pi}\right)^2$$

everywhere on $U$. Therefore, $S_w = \{e^{i\theta} : p(\theta) > 0\}$. 


Hence we obtain from (3.5) for $S$ many zeros on $\mathcal{S}$ see [4, Chap. III] and [8, u value) that

Proof of Theorem 1.5. We start with the observation that $\theta$ are two points $\theta \in e$ for a.e. is a strictly convex function of $\theta$ and $\theta$ so that the claim follows. Suppose that $u(\theta) := U^{\mu_w}(e^{i\theta}) + \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{it} dt \quad \text{a.e. on } \mathbb{T},$

see [4] Chap. III] and [8] §28]. But the latter integral is equal to the mass $\mu_w(\mathbb{T}) = 1.$ Hence we obtain from (3.5) for $S_w = \mathbb{T}$ that

$$f(\theta) = \frac{1}{2\pi} - \frac{1}{\pi} (Q')'(\theta) = \frac{1}{2\pi} - \frac{1}{2\pi^2} \int_0^{2\pi} Q'(t) \cot \frac{\theta - t}{2} dt,$$

for a.e. $e^{it} \in \mathbb{T}.$ Since $Q'$ is Hölder continuous on $\mathbb{T},$ we conclude that the same is true for its conjugate $(Q')'$ by [4] Chap. III]. Therefore, $f$ has a Hölder continuous extension to $\mathbb{T},$ and the above equation holds for all $\theta.$

Proof of Corollary 1.5. We first note that

$$u(\theta) := U^{\mu_w}(e^{i\theta}) = - \int \log|e^{i\theta} - e^{it}| d\mu_w(t) = - \int \log \left| \frac{\theta - t}{2} \right| d\mu_w(t)$$

is a strictly convex function of $\theta$ on each arc of $\mathbb{T} \setminus S_w.$ Indeed, we have

$$u''(\theta) = \frac{1}{4} \int \csc^2 \frac{\theta - t}{2} d\mu_w(t) > 0, \quad e^{i\theta} \in \mathbb{T} \setminus S_w,$$

so that the claim follows. Suppose that $S_w \cap (e^{ia}, e^{ib})$ is not an arc. Then there are two points $\theta_1, \theta_2 \in (\alpha, \beta)$ such that $e^{i\theta_1}, e^{i\theta_2} \in S_w,$ and $e^{i\theta} \not\in S_w$ for $\theta_1 < \theta < \theta_2.$ Since $u(\theta) + Q(\theta)$ is a strictly convex function on $(\theta_1, \theta_2),$ which takes values $u(\theta_1) + Q(\theta_1) = u(\theta_2) + Q(\theta_2) = F_w,$ by (1.3), we have that $u(\theta) + Q(\theta) < F_w$ for $\theta \in (\theta_1, \theta_2).$ But this contradicts (1.2), which is true for any $e^{i\theta} \in \mathbb{T}.$

Proof of Theorem 1.6. We start with the observation that

$$\frac{e^{it} + e^{i\theta}}{e^{it} - e^{i\theta}} = i \cot \frac{\theta - t}{2}.$$

Hence we obtain for the singular Schwarz integral (understood as the principal value) that

$$\frac{1}{2\pi} \int_0^{2\pi} f(t) \frac{e^{it} + e^{i\theta}}{e^{it} - e^{i\theta}} dt = i \tilde{f}(\theta), \quad \text{for a.e. } \theta \in [0, 2\pi).$$

It follows from (3.5) that

$$\frac{1}{2\pi} \int_0^{2\pi} f(t) \frac{e^{it} + e^{i\theta}}{e^{it} - e^{i\theta}} dt = \frac{i}{\pi} Q'(\theta), \quad \text{a.e. on } S_w,$$

and

$$\frac{1}{\pi i} \int_0^{2\pi} f(t) \frac{d(e^{it})}{e^{it} - e^{i\theta}} - \frac{1}{2\pi} = \frac{i}{\pi} Q'(\theta), \quad \text{a.e. on } S_w.$$
Consequently, the density function $f$ satisfies the following singular integral equation with Cauchy kernel:

\[
\frac{1}{\pi i} \int_{S_w} \frac{f(z) \, dz}{z - \zeta} = \frac{i}{\pi} Q'(\theta) + \frac{1}{2\pi}, \quad \zeta = e^{i\theta} \in S_w,
\]

where we set $f(z) = f(e^{it}) := f(t)$. Since $S_w$ consists of finitely many arcs $\Gamma_k$, $k = 1, \ldots, K$, and $f$ is a continuous function vanishing at the endpoints of $S_w$ by Corollary 1.2, we obtain from the results of [8, Chap. 11, §88] (see also [7]) that $f$ must be the unique solution of (3.8) given by the singular integral

\[
f(z) = \frac{\sqrt{R(z)}}{\pi i} \int_{S_w} \frac{g(\zeta) \, d\zeta}{\sqrt{R(\zeta)(\zeta - z)}}, \quad z \in S_w,
\]

where $g(\zeta)$ denotes the right hand side of (3.8), and $\sqrt{R(z)}$ is defined in the statement of Theorem 1.5. Furthermore, vanishing of $f$ at the endpoints of $S_w$ implies that $g$ must satisfy the following moment conditions

\[
\int_{S_w} z^k g(z) \, dz = 0, \quad k = 0, \ldots, K - 1,
\]

see [8] pp. 251, 256] and [7, p. 14]. Hence (1.9)-(1.11) are proved. Equation (1.12) simply expresses the fact that $d\mu_w(e^{it}) = f(t) \, dt$ is a probability measure.

Observe that the equilibrium equation (1.5) gives

\[
U^\mu_w(a_{k+1}) - U^\mu_w(b_k) = Q(\beta_k) - Q(\alpha_{k+1}), \quad k = 1, \ldots, K.
\]

On the other hand, we have that

\[
U^\mu_w(a_{k+1}) - U^\mu_w(b_k) = \int_{\beta_k}^{\alpha_{k+1}} \frac{d}{dt} U^\mu_w(e^{it}) \, dt
\]

\[
= -\frac{1}{2} \int_{\beta_k}^{\alpha_{k+1}} \int_0^{2\pi} f(u) \cot \frac{t - u}{2} \, du \, dt
\]

\[
= -\frac{1}{2i} \int_{\beta_k}^{\alpha_{k+1}} \int_0^{2\pi} f(u) \frac{e^{iu} + e^{it}}{e^{iu} - e^{it}} \, du \, dt
\]

\[
= -\frac{1}{2i} \int_{\beta_k}^{\alpha_{k+1}} \frac{1}{\pi i} \int_{S_w} \frac{F(e^{iu}) \, d(e^{iu})}{e^{iu} - e^{it}} - \frac{1}{2\pi} \, dt.
\]

Note that $\lim_{r\to1-} \sqrt{R(re^{iu})} = -\lim_{r\to1+} \sqrt{R(re^{iu})}$ for $e^{iu} \in S_w$. Thus the limiting boundary values of $\sqrt{R(z)}$ on the arcs of $S_w$, from inside and outside the unit circle, are opposite in sign. Hence the same is true for the function $F(z)$, which is analytic in $\mathbb{C} \setminus S_w$. Passing to the contour integral over both sides of the cut $S_w$, we obtain by the Cauchy integral theorem that

\[
\frac{1}{\pi i} \int_{S_w} \frac{F(e^{iu}) \, d(e^{iu})}{e^{iu} - e^{it}} = \frac{1}{2\pi i} \oint_{S_w} \frac{F(e^{iu}) \, d(e^{iu})}{e^{iu} - e^{it}} = F(e^{it}) - \lim_{z\to\infty} F(z).
\]
The latter limit is equal to 0, in fact, which follows from the moment conditions (1.11). Indeed, we have in a neighborhood of $\infty$ that

$$F(z) = \frac{\sqrt{R(z)}}{\pi i} \int_{S_w} \frac{g(\zeta) \, d\zeta}{\sqrt{R(\zeta)}(\zeta - z)} = -\frac{\sqrt{R(z)}}{\pi i} \sum_{k=0}^{\infty} z^{-k-1} \int_{S_w} \frac{\zeta^k g(\zeta) \, d\zeta}{\sqrt{R(\zeta)}},$$

which gives $F(\infty) = 0$. Consequently,

$$U^w(a_{k+1}) - U^w(b_k) = -\frac{\pi}{i} \int_{\beta_k}^{\alpha_k} \left( F(e^{it}) - \frac{1}{2\pi} \right) \, dt = \frac{\pi}{i} \int_{\beta_k}^{\alpha_k} F(e^{it}) \, dt + \frac{\alpha_{k+1} - \beta_k}{2i}.$$

Combining this equation with (3.10), we prove (1.13).

\[\square\]

**Proof of Theorem 2.1.** Note that $S_w$ cannot contain the zeros of $w$ by (1.3), i.e., we have that

$$Q(\theta) = -\sum_{j=1}^{J} \lambda_j \log |e^{i\theta} - z_j|$$

is infinitely differentiable in a neighborhood of $S_w$ in $\mathbb{T}$. Thus the results of Section 1 apply here. It is clear that

$$Q'(\theta) = -\sum_{j=1}^{J} \frac{\lambda_j r_j \sin(\theta - \phi_j)}{1 + r_j^2 - 2r_j \cos(\theta - \phi_j)} = -\sum_{j=1}^{J} \frac{\lambda_j r_j \sin(\theta - \phi_j)}{|e^{i\theta} - z_j|^2},$$

where $z_j = r_j e^{i\phi_j}, \ j = 1, \ldots, J$. Applying elementary trigonometric identities, we obtain that

$$Q'(\theta) - Q'(t) = \sum_{j=1}^{J} \frac{2\lambda_j r_j^2 \sin(\theta - t) - 2\lambda_j r_j (1 + r_j^2) \cos(\frac{\theta + t}{2} - \phi_j) \sin \frac{\theta - t}{2}}{|e^{i\theta} - z_j|^2 |e^{it} - z_j|^2},$$

and that

$$(Q'(\theta) - Q'(t)) \cot \frac{\theta - t}{2} = \sum_{j=1}^{J} \frac{2\lambda_j r_j^2 (\cos(\theta - t) + 1) - \lambda_j r_j (1 + r_j^2) (\cos(\theta - \phi_j) + \cos(t - \phi_j))}{|e^{i\theta} - z_j|^2 |e^{it} - z_j|^2}.$$

If we insert the above representations in (1.8), it becomes clear that

$$f^2(\theta) = \frac{L_{2J}(e^{i\theta})}{\pi^2 \prod_{j=1}^{J} |e^{i\theta} - z_j|^4}, \quad e^{i\theta} \in S_w,$$

where $L_{2J}(e^{i\theta})$ is a trigonometric polynomial of degree at most $2J$. Since $L_{2J}(e^{i\theta})$ has at most $4J$ zeros on $[0, 2\pi)$, and $S_w = \{e^{i\theta} : L_{2J}(e^{i\theta}) > 0\}$, we conclude that $S_w$ consists of at most $2J$ arcs of $\mathbb{T}$. Naturally, $L_{2J}$ and $f$ vanish at the endpoints of those arcs.
Alternatively, we can write
\[
Q'(\theta) = -\sum_{j=1}^{J} \frac{\lambda_j r_j \sin(\theta - \phi_j)}{1 + r_j^2 - 2r_j \cos(\theta - \phi_j)} = \frac{1}{2i} \sum_{j=1}^{J} \lambda_j (z_j e^{-i\theta} - \bar{z}_j e^{i\theta}) |e^{i\theta} - z_j|^2
\]
\[
= \frac{1}{2i} \sum_{j=1}^{J} \lambda_j \frac{z_j - \bar{z}_j \zeta^2}{(\zeta - z_j)(1 - \bar{z}_j \zeta)}. \quad \zeta = e^{i\theta}.
\]
Thus we obtain from Theorem 1.3 that
\[
g(\zeta) = \frac{1}{2\pi} \sum_{j=1}^{J} \lambda_j \frac{z_j - \bar{z}_j \zeta^2}{(\zeta - z_j)(1 - \bar{z}_j \zeta)} + \frac{1}{2\pi} \sum_{j=1}^{J} \frac{\lambda_j z_j}{\zeta - z_j} + \frac{1}{2\pi} \sum_{j=1}^{J} \frac{\lambda_j \bar{z}_j^{-1}}{\zeta - \bar{z}_j^{-1}}.
\]
This form of \( g \) is convenient for evaluation of the integrals as in (1.9).

We first consider the case \( S_w = T \). Then we have from (3.8) that
\[
\frac{1}{\pi i} \int_{T} \frac{f(z) \, dz}{z - \zeta} = g(\zeta), \quad \zeta \in T.
\]
Applying the inversion formula to this Cauchy singular integral, we obtain
\[
f(z) = \frac{1}{\pi i} \int_{T} \frac{g(\zeta) \, d\zeta}{\zeta - z}, \quad z \in T,
\]
by §27 of [8]. It follows from Plemelj’s formulas (cf. [8], §17) that
\[
f(z) = \lim_{\xi \to z} \frac{1}{\pi i} \int_{|z|<1} \frac{g(\zeta) \, d\zeta}{\zeta - \xi} + \lim_{\xi \to z} \frac{1}{\pi i} \int_{|z|>1} \frac{g(\zeta) \, d\zeta}{\zeta - \xi},
\]
where both integrals are taken in the counterclockwise direction. Evaluating the above integrals via residues and passing to the limits, we immediately have that
\[
f(z) = \frac{1}{2\pi} \sum_{j=1}^{J} \frac{\lambda_j z_j}{z - z_j} + \frac{1}{2\pi} \sum_{|z_j|>1} \frac{\lambda_j z_j}{z - z_j} + \frac{1}{2\pi} \sum_{|z_j|<1} \frac{\lambda_j \bar{z}_j^{-1}}{z - \bar{z}_j^{-1}}
\]
\[
- \frac{1}{2\pi} \sum_{|z_j|>1} \frac{\lambda_j z_j}{z - z_j} - \frac{1}{2\pi} \sum_{|z_j|<1} \frac{\lambda_j \bar{z}_j^{-1}}{z - \bar{z}_j^{-1}}
\]
\[
= \frac{1}{2\pi} \sum_{j=1}^{J} \lambda_j \left( \frac{z_j}{z - z_j} - \frac{\bar{z}_j^{-1}}{z - \bar{z}_j^{-1}} \right)
\]
\[
+ \frac{1}{2\pi} \sum_{|z_j|>1} \lambda_j \left( \frac{z_j^{-1}}{z - z_j} - \frac{\bar{z}_j}{z - \bar{z}_j} \right)
\]
Thus (2.2) follows by a simple algebraic manipulation. Another proof of (2.2) can be produced by using Corollary 1.3 and finding the harmonic conjugate of \( Q' \) from its trigonometric form.

We now assume that \( S_w \) consists of \( K \geq 1 \) proper arcs of \( T \), and apply (1.9) of Theorem 1.5. As in the proof of Theorem 1.5 we observe that the limiting boundary values of \( \sqrt{R(\zeta)} \) on those arcs, from inside and outside the unit circle, are opposite in sign. Hence the same is true for the function \( g(\zeta)/(\sqrt{R(\zeta)}(\zeta - z)) \),
Hence we have that the values of $S$ of (3.12) is analytic in $\mathbb{C} \setminus S_w$, except for the simple poles at $z_j$ and $\bar{z}_j^{-1}$. Here, we used the natural extension of $g(\zeta)$ from $S_w$ to $\mathbb{C}$ as a rational function. Passing to the contour integral over both sides of the cut $S_w$, and computing the residues at $z_j$ and $\bar{z}_j^{-1}$, we obtain that

$$f(z) = \frac{\sqrt{R(z)}}{2\pi i} \int_{S_w} \frac{g(\zeta) d\zeta}{\sqrt{R(\zeta)(\zeta - z)}} = \frac{\sqrt{R(z)}}{2\pi} \sum_{j=1}^{J} \frac{\lambda_j z_j}{\sqrt{R(z_j)(z_j - z)}} + \frac{\sqrt{R(z)}}{2\pi} \sum_{j=1}^{J} \frac{\lambda_j \bar{z}_j^{-1}}{\sqrt{R(\bar{z}_j^{-1})(\bar{z}_j^{-1} - z)}}, \quad z \in S_w.$$

Thus (2.3) is proved. Our next goal is to show that the number of intervals for $S_w$ is at most $J$. Observe that the previous equation gives

$$f(t) = \frac{\sqrt{R(e^{it})}}{\pi} \sum_{j=1}^{J} \frac{-z_j}{\sqrt{R(z_j)(e^{it} - z_j)}} + \frac{1}{\sqrt{R(\bar{z}_j^{-1})(1 - \bar{z}_je^{it})}} = \frac{\sqrt{R(e^{it})}P(e^{it})}{\pi \prod_{j=1}^{J} (e^{it} - z_j)(1 - \bar{z}_je^{it})} = \frac{\sqrt{R(e^{it})}P(e^{it})}{\pi e^{itj} \prod_{j=1}^{J} |e^{it} - z_j|^2}, \quad e^{it} \in S_w,$$

where $P$ is an algebraic polynomial. Comparing this with the previously obtained form for $f^2(t)$, we conclude that

$$R(e^{it})P^2(e^{it}) = e^{i2jt}L_{2J}(e^{it}), \quad (3.11)$$

for $e^{it} \in S_w$, so that these polynomials in $e^{it}$ coincide. It follows that $\deg(RP^2) \leq 4J$, i.e., $2K + 2 \deg P \leq 4J$ and

$$K + \deg P \leq 2J. \quad (3.12)$$

Since $S_w = \{e^{it} : L_{2J}(e^{it}) > 0\}$, we have that $L_{2J}(e^{it})$ takes real values for all $t$, and that $L_{2J}(e^{it}) \leq 0$ for $e^{it} \in \mathbb{T} \setminus S_w$. Equation (3.11) and the one before it suggest that

$$\sqrt{R(e^{it})}P(e^{it})e^{-ijt} \geq 0, \quad e^{it} \in S_w, \quad (3.13)$$

and that $\sqrt{R(e^{it})}P(e^{it})e^{-ijt}$ is pure imaginary on $\mathbb{T} \setminus S_w$. Assume that $P(e^{it}) \neq 0$ for $t \in (\beta_k, \alpha_{k+1})$, that is $P$ does not vanish on $\mathbb{T}$ between a pair of neighboring arcs of $S_w$. Note that the argument of $\sqrt{R(e^{it})}$ decreases by $\pi/2$ when we pass over an endpoint of $S_w$ (when moving in the positive direction on $\mathbb{T}$), while the argument of $P(e^{it})e^{-ijt}$ remains continuous everywhere on $\mathbb{T}$, except for the zeros of $P$. Hence we have that the values of $\sqrt{R(e^{it})}P(e^{it})e^{-ijt}$ should have opposite signs on these neighboring arcs of $S_w$, which immediately contradicts (3.13). It follows that $P(e^{it})$ has a zero on $\mathbb{T}$ between each pair of arcs of $S_w$, so that $\deg P \geq K$. Finally, $2K \leq K + \deg P \leq 2J$, by (3.12), and $K \leq J$.

In the remaining part, we prove (2.4) and (2.5). Applying Theorem 1.5 we see that $g$ must satisfy the moment conditions of (1.11). Again, these integrals are found by passing to the contour integrals over both sides of the cut $S_w$, and
computing the residues at $z_j$, $\bar{z}_j^{-1}$ and $\infty$:

$$\frac{1}{\pi i} \int_{S_w} \frac{z^k g(z) \, dz}{\sqrt{R(z)}} = \frac{1}{2\pi i} \oint_{S_w} \frac{z^k g(z) \, dz}{\sqrt{R(z)}}$$

$$= \frac{1}{2\pi} \sum_{j=1}^{J} \lambda_j \left( \frac{z_j^{k+1}}{\sqrt{R(z_j)}} + \frac{\bar{z}_j^{-k-1}}{\sqrt{R(\bar{z}_j^{-1})}} \right)$$

$$- \frac{1 + \sum_{j=1}^{J} \lambda_j}{2\pi} \lim_{z \to \infty} \frac{z^{k+1}}{\sqrt{R(z)}}.$$

Note that

$$\lim_{z \to \infty} \frac{z^{k+1}}{\sqrt{R(z)}} = \begin{cases} 0, & k = 0, \ldots, K - 2, \\ 1, & k = K - 1, \end{cases}$$

so that (2.4) follows from (1.11).

We deduce (2.5) from (1.13). For this purpose, we evaluate $F(z)$ for $z \in \mathbb{C} \setminus S_w$, by using the residues at $z_j$, $\bar{z}_j^{-1}$ and $z$:

$$F(z) = \frac{\sqrt{R(z)}}{2\pi i} \oint_{S_w} \frac{g(\zeta) \, d\zeta}{\sqrt{R(\zeta)(\zeta - z)}}$$

$$= \frac{\sqrt{R(z)}}{2\pi} \sum_{j=1}^{J} \frac{\lambda_j z_j}{\sqrt{R(z_j)(z_j - z)}} + \frac{\sqrt{R(z)}}{2\pi} \sum_{j=1}^{J} \frac{\lambda_j \bar{z}_j^{-1}}{\sqrt{R(\bar{z}_j^{-1})(\bar{z}_j^{-1} - z)}} + g(z)$$

$$= F_1(z) + g(z).$$

Hence

$$\int_{\beta_k}^{\alpha_{k+1}} F_1(e^{it}) \, dt + \int_{\beta_k}^{\alpha_{k+1}} g(e^{it}) \, dt = \frac{\alpha_{k+1} - \beta_k}{2\pi} + \frac{Q(\alpha_{k+1}) - Q(\beta_k)}{\pi} i,$$

where $k = 1, \ldots, K$, by (1.13). We next compute that

$$\int_{\beta_k}^{\alpha_{k+1}} g(e^{it}) \, dt = \int_{\beta_k}^{\alpha_{k+1}} \left( \frac{1}{2\pi} \sum_{j=1}^{J} \lambda_j \left( \frac{z_j}{e^{it} - z_j} + \frac{\bar{z}_j^{-1}}{e^{it} - \bar{z}_j^{-1}} \right) + \frac{1 + \sum_{j=1}^{J} \lambda_j}{2\pi} \right) \, dt$$

$$= \frac{1}{2\pi i} \sum_{j=1}^{J} \lambda_j \left( z_j \int_{b_k}^{a_k} \frac{dz}{z(z - z_j)} + \bar{z}_j^{-1} \int_{b_k}^{a_k} \frac{dz}{z(z - \bar{z}_j^{-1})} \right)$$

$$+ \frac{\alpha_{k+1} - \beta_k}{2\pi} \left( 1 + \sum_{j=1}^{J} \lambda_j \right).$$
It is immediate to see that
\[
\frac{1}{2\pi i} \sum_{j=1}^{J} \lambda_j \left( z_j \int_{b_k}^{a_{k+1}} \frac{dz}{z(z - z_j)} + z_j^{-1} \int_{b_k}^{a_{k+1}} \frac{dz}{z(z - z_j)} \right)
\]
\[
= \sum_{j=1}^{J} \frac{\lambda_j}{2\pi i} \left( \log \frac{a_{k+1} - z_j}{b_k - z_j} + \log \frac{a_{k+1} - z_j^{-1}}{b_k - z_j^{-1}} - 2 \log \frac{a_{k+1}}{b_k} \right)
\]
\[
= \sum_{j=1}^{J} \frac{\lambda_j}{2\pi i} \left( \log \frac{a_{k+1} |a_{k+1} - z_j|^2}{b_k |b_k - z_j|^2} - 2 \log \frac{a_{k+1}}{b_k} \right)
\]
\[
= \frac{Q(\alpha_{k+1}) - Q(\beta_k)}{\pi} - \frac{\alpha_{k+1} - \beta_k}{2\pi} \sum_{j=1}^{J} \lambda_j.
\]
Substituting the results of the above computations in (3.14), we immediately obtain (2.5).

Proof of Theorem 2.2. We follow essentially the same scheme as in the previous proof. Note that $Q(\theta) = t_M(\theta)$ is infinitely differentiable on $T$. It is clear that
\[
Q'(\theta) = \sum_{m=-M}^{M} imc_m e^{im\theta}
\]
and
\[
g(e^{i\theta}) = \frac{1}{2\pi} - \frac{1}{\pi} \sum_{m=-M}^{M} mc_m e^{im\theta},
\]
where $g$ is defined in Theorem 1.3. Inserting $Q'(\theta)$ into (1.8), we observe that the right hand side $p(\theta)$ is a trigonometric polynomial of degree at most $2M$. Thus we have from Theorem 1.1 and Corollary 1.2 that $d\mu_w(e^{i\theta}) = f(\theta) d\theta$ with $f^2(\theta) = L_{2M}(e^{i\theta})$, where $L_{2M}(z)$ is a Laurent polynomial of degree at most $2M$. It follows that $S_w$ consists of at most $2M$ arcs, because $L_{2M}$ has at most $4M$ zeros on $T$.

When $S_w = T$, we obtain (2.7) from Corollary 1.3 after substituting the conjugate
\[
(Q')^{\bar{\theta}} = \frac{1}{2\pi} \int_{0}^{2\pi} Q'(t) \cot \frac{\theta - t}{2} dt = \sum_{m=-M}^{M} m \text{sgn}(m) c_m e^{im\theta}.
\]
Suppose that $S_w$ consists of $K \geq 1$ proper arcs of $T$, and apply (1.9) of Theorem 1.5. Using that the limiting boundary values of $\sqrt{R(\zeta)}$ on $S_w$ are opposite in sign, we again pass to the contour integral over both sides of the cut $S_w$:
\[
(3.15) \quad F_2(z) = F(z) = \frac{\sqrt{R(z)}}{2\pi i} \int_{S_w} \frac{g(\zeta) d\zeta}{\sqrt{R(\zeta)}(\zeta - z)}
\]
\[
= \frac{\sqrt{R(z)}}{2\pi i} \int_{S_w} \frac{1}{\sqrt{R(\zeta)}(\zeta - z)} \left( \frac{1}{2\pi i} - \frac{1}{\pi} \sum_{m(-M}^{M} mc_m \zeta^m \right) d\zeta
\]
\[
= -\frac{\sqrt{R(z)}}{\pi} \sum_{m=-M}^{M} mc_m \frac{1}{2\pi i} \int_{S_w} \frac{\zeta^m d\zeta}{\sqrt{R(\zeta)}(\zeta - z)}, \quad z \in S_w.
\]
The latter contour integrals are found by the Cauchy integral theorem and evaluation of residues at \( \infty \) and at 0, with the help of the series expansions for \( 1/(\sqrt{R(\zeta)(\zeta - z)}) \). This immediately gives the stated form of \( F_2(z) = F(z), \) \( z \in S_w, \) so that (3.16) follows from (3.10). The expansion coefficients \( r_k(z) \) and \( s_k(z) \) can be expressed in the standard way

\[
r_k(z) = \frac{1}{k!} \frac{d^k}{dz^k} \left( \frac{1}{\sqrt{R(\zeta)(\zeta - z)}} \right) \bigg|_{\zeta = 0}, \quad k \geq 0,
\]
and

\[
s_k(z) = \frac{1}{k!} \frac{d^k}{dz^k} \left( \frac{\zeta^{K+1}}{\sqrt{2^K R(1/\zeta)(1 - z\zeta)}} \right) \bigg|_{\zeta = 0}, \quad k \geq 0.
\]

It transpires now that \( s_k(z) \) is a polynomial in \( z \) of degree at most \( k \), and that \( r_k(z) \) is a polynomial in \( 1/z \) of degree at most \( k + 1 \). Hence we have

\[
F_2(z) = \frac{P(z)\sqrt{R(z)}}{z^M}, \quad z \in S_w,
\]
where \( P(z) \) is a polynomial in \( z \). Comparing this with the previously obtained form for \( f^2(t) \), we conclude that

\[
R(e^{it})P^2(e^{it}) = e^{i2Mt}L_{2M}(e^{it}), \quad e^{it} \in S_w,
\]
so that these polynomials in \( e^{it} \) coincide. It follows that \( \text{deg}(RP^2) \leq 4M \), i.e., \( 2K + 2 \text{deg} P \leq 4M \) and

\[
K + \text{deg} P \leq 2M.
\]

Since \( S_w = \{e^{it} : L_{2M}(e^{it}) > 0\} \), we have that \( L_{2M}(e^{it}) \) takes real values for all \( t \), and that \( L_{2M}(e^{it}) \leq 0 \) for \( e^{it} \in \mathbb{T} \setminus S_w \). Therefore,

\[
\sqrt{R(e^{it})}P(e^{it})e^{-iMt} \geq 0, \quad e^{it} \in S_w,
\]
being the density of \( \mu_w \), and \( \sqrt{R(e^{it})}P(e^{it})e^{-iMt} \) is pure imaginary on \( \mathbb{T} \setminus S_w \).

Assume that \( P \) does not vanish on \( \mathbb{T} \) between a pair of neighboring arcs of \( S_w \). Note again that the argument of \( \sqrt{R(e^{it})} \) decreases by \( \pi/2 \) when we pass over an endpoint of \( S_w \), while the argument of \( P(e^{it})e^{-iMt} \) is continuous on \( \mathbb{T} \), except for possible zeros of \( P \). Hence the values of \( \sqrt{R(e^{it})}P(e^{it})e^{-iMt} \) should have opposite signs on these neighboring arcs of \( S_w \), which immediately contradicts (3.17). It follows that \( P(e^{it}) \) has a zero on \( \mathbb{T} \) between each pair of arcs of \( S_w \), so that \( \text{deg} P \geq K \). Finally, \( 2K \leq K + \text{deg} P \leq 2M \), by (3.16), and \( K \leq M \).

Repeating the same evaluation as in (3.15), but for \( z \notin S_w \), we obtain that

\[
F(z) = \frac{\sqrt{R(z)}}{2\pi i} \oint_{S_w} \frac{g(\zeta) d\zeta}{\sqrt{R(\zeta)(\zeta - z)}} = F_2(z) + g(z).
\]

\[\square\]

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