Ninebrane Structures

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Abstract

String structures in degree four are associated with cancellation of anomalies of string theory in ten dimensions. Fivebrane structures in degree eight have recently been shown to be associated with cancellation of anomalies associated to fivebranes in string theory and M-theory. We introduce and describe Ninebrane structures in degree twelve and demonstrate how they capture some anomaly cancellation phenomena in M-theory. Along the way we also define certain variants, considered as intermediate cases in degree nine and ten, which we call 2-Orientation and 2-Spin structures, respectively. As in the lower degree cases, we also discuss the natural twists of these structures and characterize the corresponding topological groups associated to each of the structures, which likewise admit refinements to differential cohomology.

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1 Introduction

The study of higher connected covers of Lie groups in the context of string theory and M-theory, as advocated in [31, 32, 33], leads to interesting mathematical structures as well as means for canceling anomalies in string theory and M-theory. Beyond String structures in degree four, obtained by killing the third homotopy group of the orthogonal group, we have Fivebrane structures in degree eight obtained by killing the next homotopy group which is in degree seven.

We will consider killing – more precisely, co-killing – further homotopy groups. From the homotopy theoretic point of view one can continue the process of killing indefinitely in a systematic way. However, no systematic understanding of the relevance of all cases exists. What we do is advocate is a natural setting, a description of the higher geometry, as well as provide several examples from M-theory and string theory for which performing such killings in the next few degrees is natural. We highlight the structures we consider here in the following table.

| $k$ | 7 | 8 | 9 | 10 | 11 | 12 |
|-----|---|---|---|----|----|----|
| Homotopy groups $\pi_k(O(n))$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | 0 |
| Connected covers $O(n)(k)$ | String($n$) | Fivebrane($n$) | $O(9)(n)$ | $O(10)(n)$ | Ninebrane($n$) |

The point of view we take here is that the group $O(9)(n)$ is a ‘shift by 8’ analog of the special orthogonal group $SO(n)$. The group $O(10)(n)$ is a ‘shift by 8’ analog of the Spin group $Spin(n)$. The mod 8 periodicity of the homotopy groups of the orthogonal group motivates the following for the corresponding $G$-structures: The classifying spaces $BO(10) = B(O(9))$ and $BO(11) = B(O(10))$ correspond to a ‘shift by 8’ analog of orientation and of Spin structure, respectively. Thus to identify these structures in the second period in the mod 8 periodicity we indicate these as 2-Orientations and 2-Spin structures. We encapsulate the theme in the following diagram of lifts, extending the ones in [32] [33], i.e. the higher part of the Whitehead tower of the orthogonal group:
We identify the obstructions $x_i$, $i = 9, 10$ in Sec. 2 and $i = 12$ in Sec. 8 and characterize the set of lifts in Sec. 4. The first two will be, in a sense, exotic classes. Then in Sec. 5 we twist these structures, and in Sec. 6 we consider variant structures in which the higher obstructions vanish without lower classes having to be zero. We characterize the corresponding groups in Sec. 7, and finally in Sec. 8 we construct differential refinements, and provide a natural M-theoretic setting in way of motivation and examples throughout.

2 \textit{BO}(10) and \textit{BO}(11) structures

The topology and geometry of a manifold can be studied via the structures related to its tangent bundle. Starting with a Riemannian manifold $X^n$, its tangent bundle with structure group $O(n)$, can be lifted to further structures which in turn imposes topological conditions on $X^n$. The first step is to lift the structure group to $SO(n)$ by equipping $X^n$ with an orientation, which is allowed provided the first Stiefel-Whitney class $w_1$ of $TX^n$ is zero. Further structures can be conditionally given. The structure group can be further lifted from $SO(n)$ to the double cover Spin$(n)$ which allows the existence of spinors provided that the second Stiefel-Whitney class $w_2$ is zero.

Note that the process does not stop here, and we can further continue equipping the tangent bundle with higher structures. Due to the homotopy type of the orthogonal group, the next step in the process is consider the lifting to the seventh connected cover denoted $O(7)$, which occurs when the cohomology class $\frac{1}{2}p_1$ is zero, where $p_1$ is the first Pontrjagin class of the tangent bundle. The notation $G(n)$ means that all the homotopy groups of order $0, \cdots, n$ of the original group $G$ are killed.

In fact, the way to obtain the above structures is by pulling back from the universal classifying space to our spacetime $X^n$. Since $SO(n)$ is obtained from $O(n)$ by killing the first homotopy group, then $BSO \simeq BO(2)$, which is also denoted $BO(w_1)$ to highlight the condition imposed by such a structure. Note that in going from $G$ to $BG$ there is a shift in homotopy, i.e. $\pi_i(G) \cong \pi_{i+1}(BG)$. Similarly, $BSpin \cong BO(4) \simeq BO(w_1, w_2)$, this time emphasizing the Spin condition $w_1 = 0 = w_2$. Finally, $BO(8)$, sometimes also denoted $BString$, can be written in the same notation as $BO(w_1, w_2, \frac{1}{2}p_1)$. Notice that in this last case, the additional condition is no longer a mod 2 condition but rather is one on integral cohomology. The requirement $\frac{1}{2}p_1 = 0$ is not quite the same as setting $p_1 = 0$, because the latter is a rational condition which misses the torsion classes in the former integral condition.

Note that $\frac{1}{2}p_1$ is related to the Stiefel-Whitney classes, namely its mod 2 reduction is given by $w_4$. We will see that, in a sense, not all Stiefel-Whitney classes are relevant, but a special role is played by the ones of the form $w_{2i}$. For instance, starting from $w_1 = w_2 = w_4 = 0$ leads to $w_i = 0$ for $i = 1, \cdots, 7$. Thus the only new condition after $w_4 = 0$ is $w_8 = 0$. In terms of classifying spaces, what this implies is that there is a lift from $BO(2^n)$ to $BO(w_{2j})$ for $j < r$.

We now consider the Stiefel-Whitney classes in relatively higher degrees, $w_i$ for $2^3 \leq i < 2^4$. For applications in even higher degrees, see [21]. We start with the following observation for the higher Stiefel-Whitney classes as they arise in the context of M-theory.

**Lemma 2.1** Let $Y^{11}$ be an orientable eleven-manifold. Then we have $w_{11}(Y^{11}) = w_{10}(Y^{11}) = w_9(Y^{11}) = 0$. 

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Proof. Let \( \nu_i \in H^i(Y^{11};\mathbb{Z}_2) \) be the Wu class, i.e. the unique cohomology class such that \( Sq^i(x) = x \cdot \nu_i \) for any \( x \in H^{11-i}(Y^{11};\mathbb{Z}_2) \). From the properties of the Wu classes, we have \( \nu_0 = 1, \nu_i = 0 \) for \( i > 5 \). Wu’s formula relates the Stiefel-Whitney classes to the Wu classes via

\[
w_k = \sum_i Sq^{k-i} \nu_i .
\]  

Using the Adem relation \( Sq^i = Sq^1 Sq^{i-1} \) for \( i \) odd, and that \( Sq^1 : H^{10}(Y^{11};\mathbb{Z}_2) \to H^{11}(Y^{11};\mathbb{Z}_2) \) is trivial for orientable \( Y^{11} \), gives that \( Sq^i : H^{11-i}(Y^{11};\mathbb{Z}_2) \to H^{11}(Y^{11};\mathbb{Z}_2) \) is zero for \( i \) odd. This implies, from the definition of the Wu classes, that \( \nu_i = 0 \) for \( i \) odd. Hence, for orientable \( Y^{11} \), \( \nu_i \) is zero unless \( i \) is even and \( 0 \leq i \leq 4 \). From this and expression (2.1) it follows that \( \nu_i = 0 \) for \( i > 8 \). Therefore, \( w_9(Y^{11}) = w_{10}(Y^{11}) = w_{11}(Y^{11}) = 0 \).

\[ \square \]

**Remarks. 1.** Note that the M-theory fivebrane anomaly cancelation requires an \( MO(9) \) orientation, i.e. a Fivebrane structure \([33]\).

**2.** Note that in general for orientable \( Y^{11} \) with no extra structure, the class \( w_8 \) will not be zero.

**Example.** Consider \( Y^{11} = \mathbb{C}P^2 \times \mathbb{C}P^2 \times (S^1)^3 \) or \( Y^{11} = P(1,4) \times (S^1)^2 \), where \( P(1,4) \) is the Dold manifold defined as follows. \( P(r,s) \) is the quotient \( (S^r \times \mathbb{C}P^s) / \sim \), where \( (x,y) \sim (x',y') \) if and only if \( x' = -x \) and \( y' = -y \). The Dold manifold is the total space of \( \mathbb{C}P^s \) bundle of complex projective spaces over real projective space \( \mathbb{R}P^r \) whose total Stiefel-Whitney class is given by

\[
w(P(r,s)) = (1+e_1)^r (1+e_1+e_2)^{s+1},
\]  

where \( e_1 \) and \( e_2 \) are the generators in the cohomology groups of the corresponding projective spaces \( H^i(\mathbb{R}P^r;\mathbb{Z}_2) \) and \( H^2(\mathbb{C}P^s;\mathbb{Z}_2) \), respectively. In particular, \( P(1,4) \) is orientable and has non-vanishing \( w_8 \).

We have seen above that for orientable \( Y^{11} \), the class \( w_8 \) is not necessarily zero. However, integrality of the one-loop polynomial \( I_8 = \frac{1}{24} [p_2 - (\frac{1}{2} p_1)^2] \) appearing in anomaly cancellation in M-theory requires that \( w_8 \) be in fact zero. This is because it is the mod 2 reduction of the second Spin characteristic class. Recall that the integral cohomology of the classifying space of the Spin group is \([38]\)

\[
H^*(BSpin;\mathbb{Z}) = \mathbb{Z}[Q_1, Q_2, \ldots] \oplus \gamma ,
\]  

with \( \gamma \) a 2-torsion factor, i.e. \( 2\gamma = 0 \). The two relevant degrees are

\[
H^4(BSpin;\mathbb{Z}) \cong \mathbb{Z} \quad \text{with generator} \quad Q_1 \\
H^8(BSpin;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \quad \text{with generators} \quad Q_1^2, Q_2 ,
\]

where the Spin classes \( Q_1 \) and \( Q_2 \) are determined by their relation to the Pontrjagin classes

\[
p_1 = 2Q_1 , \quad p_2 = Q_1^2 + 2Q_2 .
\]  

Obviously, when inverting is possible, the Spin generators are given by \( Q_1 = p_1/2 \) and \( Q_2 = \frac{1}{2} p_2 - \frac{1}{4}(p_1/2)^2 \). The mod 2 reductions of \( Q_1 \) and \( Q_2 \) are \( w_4 \) and \( w_8 \), respectively. It was explained in \([19]\) that it is useful to write the one-loop term in terms of the Spin characteristic classes

\[
I_8 = \frac{Q_2}{24} .
\]  

4
Now $I_8$ is integral when $w_4 = 0$. The latter condition allows to define a “Membrane structure” \[^1\] i.e. a structure defined by the condition $w_4 = 0$. Then $Q_2$ is certainly divisible by 2, and hence we have

**Lemma 2.2** For a manifold $Y^{11}$ with a Membrane structure, we have $w_8(Y^{11}) = 0$.

Next, putting together the above discussion, we have the following observation, still motivated within the context of M-theory.

**Proposition 2.3** Let $Y^{11}$ be a manifold which admits a Fivebrane structure. Then all classes in $Y^{11}$ pulled back from universal classes in $H^n(BO; R)$, for $n \leq 11$ and $R = \mathbb{Z}$ or $\mathbb{Z}_2$, are trivial.

Proof. This follows from statements that can be directly verified. Orientation requires that $w_1(Y^{11}) = 0$. A Fivebrane structure amounts to $\frac{1}{2}p_2(Y^{11}) = 0$ and requires a String structure, i.e. $\frac{1}{2}p_1(Y^{11}) = 0$, which in turn requires a Spin condition, i.e. $w_2(Y^{11}) = 0$, as well as a Membrane condition $w_4(Y^{11}) = 0$. The Fivebrane condition further implies $w_8(Y^{11}) = 0$ via mod 2 reduction from the Fivebrane condition. Finally, all odd Stiefel-Whitney classes up to that degree are zero. This follows from the Wu formula for the action of the Steenrod algebra which takes the general form $\text{Sq}^i w_j = \sum_{t=0}^i \binom{i}{t-1} (-1)^{i-t} w_{i-t} w_{j+t}$ for $i < j$ (see [36]). First, the Wu formula $\text{Sq}^1 w_2 = w_1 w_2 + w_3$ gives that if $w_2 = 0$ then $w_3 = 0$. Second, the formulae $\text{Sq}^1 w_4 = w_1 w_4 + w_5$, $\text{Sq}^2 w_4 = w_2 w_4 + w_6$ and $\text{Sq}^3 w_4 = w_3 w_4 + w_2 w_5 + w_1 w_6 + w_7$ imply that if $w_4 = 0$ then the three classes $w_5$, $w_6$ and $w_7$ are zero. In the next degree, starting with $w_8$, the formulae $\text{Sq}^1 w_8 = w_1 w_8 + w_9$, $\text{Sq}^2 w_8 = w_2 w_8 + w_{10}$ and $\text{Sq}^3 w_8 = w_3 w_8 + w_2 w_9 + w_1 w_{10} + w_{11}$ imply that if $w_8 = 0$ then the classes $w_9$, $w_{10}$ and $w_{11}$ are zero. These last three degrees can also be deduced by appealing to the dimension of the manifold $Y^{11}$, i.e. by using Lemma \[2.4\] \[\square\]

As we saw above, these obstructions mostly vanish for dimension reasons in our range of dimensions. However, we will consider bundles other than the tangent bundle; for example bundles with structure group $\text{SO}(32)$ rather than $\text{SO}(10)$ or $\text{SO}(11)$.

**Example: Orthogonal gauge structure groups.** Consider the orthogonal group $G = \text{SO}(32)$ as a structure group of a gauge bundle over our manifold. This is relevant in type I and heterotic string theory in ten dimensions. \[^2\] The topological role of this group in relation to global anomalies is highlighted in [11], which we recast in our language (cf. [31] [32]). The degree seven generator $\pi_7(\text{SO}(32)) = \mathbb{Z}$ is used to show invariance of theory on $S^3 \times S^7$ and to derive a quantization condition on the H-field, which can be thought of as a curvature of a gerbe in degree three. The next homotopy group $\pi_9(\text{SO}(32)) = \mathbb{Z}_2$ correspond to a Yang-Mills instanton on $S^{10}$ via the embedding $\text{SO}(10) \hookrightarrow \text{SO}(32)$ by viewing the Spin connection, arising from a spinor representation of the natural structure group $\text{SO}(10)$ of the tangent bundle, as a gauge field, by viewing this in the vector representation of $\text{SO}(32)$.

What about $\pi_{10}(\text{SO}(32)) = \mathbb{Z}_2$? We interpret this in essentially the same way as for the case of $\pi_9(\text{SO}(32))$. However, our setting will be M-theory in eleven dimensions rather than string theory in

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\[^1\]This name is introduced in Ref. [25] and justified there by the fact that it arises in connection with anomalies associated with the membrane in M-theory.

\[^2\]Note that the group is more precisely $\text{Spin}(32)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the complement of the component of the center which leads to $\text{SO}(32)$. However, since we are considering connected covers, such differences at the level of the fundamental group (which is killed) will not matter for us.
ten dimensions. We start with the eleven-dimensional sphere $S^{11}$ with structure group $SO(11)$ and embed this in the group $SO(32)$ and, as above, view the Spin connection of the former as a gauge field of the latter. One justification for enlarging of the structure group is to form a generalized connection, taking into account the C-field terms. For instance, in [9] $SO(32)$ is described as a generalized holonomy group, while in [15] the group $SL(32,\mathbb{R})$ played that role; homotopically, this is simply the same as $SO(32)$.

We now define the first two of the new structures and afterwards we will explain the connection to the above classes.

**Definition 2.4** *(2-Orientation structure.)* A 2-Orientation structure is defined by the lift from $BO(9) = BFivebrane$ to $BO(10)$ in the following diagram

$$
\begin{array}{ccc}
X & \overset{f}{\rightarrow} & BFivebrane \\
\downarrow & & \downarrow \\
BO(10) & \rightarrow & K(\pi_9(BO),9)
\end{array}
$$

(2.6)

**Remarks** *(i)* The existence of the above fibration, as well as all the fibrations that we introduce below, follows from the work of Stong [36] [37].

*(ii)* Corresponding to this diagram is a class $x_9 \in H^9(BFivebrane, \pi_9(BO)) = H^9(BFivebrane, \mathbb{Z}_2)$. The map $f : X \rightarrow BFivebrane$ lifts to $\hat{f} : X \rightarrow BO(10)$ if and only if we have the vanishing of the obstruction class

$$
f^*x_9 = 0 \in H^9(X; \pi_9(BO)) = H^9(X; \mathbb{Z}_2).
$$

(2.7)

*(iii)* One might think that we can identify $x_9 = w_9$ as the cohomology ring $H^*(BO; \mathbb{Z}_2)$ is generated by the Stiefel-Whitney classes. However, as we will see shortly, this is not the case.

*(iv)* $BO(10)$ can also be described as a bundle pulled back from the path fibration in the following diagram (as in [33], which builds on [30])

$$
\begin{array}{ccc}
K(\pi_9(B),8) & \overset{f}{\rightarrow} & BO(10) \\
PK(\pi_9(BO),9) & \rightarrow & \Omega K(\pi_9(BO),9) \\
X & \overset{f}{\rightarrow} & BFivebrane \\
\rightarrow & & \rightarrow \\
& & \\
& & K(\pi_9(BO),9)
\end{array}
$$

(2.8)

In the next degree we have:

**Definition 2.5** *(2-Spin structure.)* A 2-Spin structure is defined by the lift from $BO(10)$ to $BO(11)$ in the following diagram

$$
\begin{array}{ccc}
X & \overset{f}{\rightarrow} & BO(10) \\
\downarrow & & \downarrow \\
BO(11) & \rightarrow & K(\pi_{10}(BO),10)
\end{array}
$$

(2.9)
Remarks  
(i) Corresponding to this diagram is a class \( x_{10} \in H^{10}(BO(10), \pi_{10}(BO)) = H^{10}(BO(10), \mathbb{Z}_2) \). The map \( f : X \to BO(10) \) lifts to \( \tilde{f} : X \to BO(11) \) if and only if we have the vanishing of the obstruction class

\[
f^* x_{10} = 0 \in H^{10}(X; \pi_{10}(BO)) = H^{10}(X; \mathbb{Z}_2).
\]  

(ii) One might think that we can identify \( x_{10} = w_{10} \) as the cohomology ring \( H^*(BO; \mathbb{Z}_2) \) is generated by the Stiefel-Whitney classes. However, again, as we will see this is not the case.

(iii) \( BO(11) \) can also be described as a bundle pulled back from the path fibration in the following diagram

\[
\begin{array}{ccc}
K(\pi_{10}(B)), 9 & \longrightarrow & BO(11) \\
\downarrow f & & \downarrow \hspace{1cm} \\
X & \longrightarrow & BO(10)
\end{array}
\hspace{1cm}
\begin{array}{ccc}
PK(\pi_{10}(BO), 10) & \longrightarrow & \Omega K(\pi_{10}(BO), 10) \\
\downarrow & & \downarrow \\
K(\pi_{10}(BO), 10) & & \\
\end{array}
\]  

Identifying the generators \( x_9 \) and \( x_{10} \). As mentioned above, one might be very tempted to identify the obstruction classes \( x_9 \) and \( x_{10} \) with the generators of \( H^i(BO; \mathbb{Z}_2) \) for \( i = 9, 10 \), i.e. with the Stiefel-Whitney classes \( w_9 \) and \( w_{10} \), respectively. However, this would be too simplistic and, as we will see, is not true. The main subtlety here is that beyond Fivebrane in the Whitehead tower of \( BO \), maps from \( BO(9) \) to \( BO \) are no longer surjective (see \[36\] \[37\]). Thus upon close inspection one realizes that these will be exotic characteristic classes not arising from the cohomology of \( BO \) but rather of \( BO(9) \) and \( BO(10) \). For instance, \( x_9 \) arises from pulling back along the map \( K(\mathbb{Z}, 7) \to BFivebrane \). This is analogous to the map \( K(\mathbb{Z}, 3) \to BO(7) \) relating gerbes to String structures and provides interesting geometry. It would be very interesting to identify these obstructions via examples, which we expect to be related to the ones on orthogonal structure groups above.

One can, in fact, study the generators \( x_9 \) and \( x_{10} \) a little more precisely, by specializing the general discussion in \[36\] to our context to relate to more standard classes, namely the fundamental classes of Eilenberg-MacLane spaces. We start with 2-Orientations. The spectral sequence of the fibration \( BO(9) \to BO(8) \to K(\mathbb{Z}, 8) \) gives the long exact sequence of cohomology groups

\[
\cdots \longrightarrow H^0(BO(9); \mathbb{Z}_2) \longrightarrow H^1(BO(8); \mathbb{Z}_2) \longrightarrow H^{10}(BO(8); \mathbb{Z}_2) \longrightarrow \cdots.
\]  

Here the transgression \( \tau \) is given by \( x_9 \mapsto Sq^2 \iota_8 \), where \( \iota_8 \) is the fundamental cohomology class of the Eilenberg-MacLane space \( K(\mathbb{Z}, 8) \). Next, for 2-Spin structures, the spectral sequence of the fibration \( BO(10) \to BO(9) \to K(\mathbb{Z}, 9) \) gives the long exact sequence of cohomology groups

\[
\cdots \longrightarrow H^{10}(BO(9); \mathbb{Z}_2) \longrightarrow H^1(K(\mathbb{Z}, 9); \mathbb{Z}_2) \longrightarrow H^{11}(BO(9); \mathbb{Z}_2) \longrightarrow \cdots.
\]  

The transgression \( \tau \) is given by \( x_{10} \mapsto Sq^2 \iota_9 \), with \( \iota_9 \) the fundamental cohomology class of \( K(\mathbb{Z}, 9) \).

Next we consider the cohomology rings the classifying spaces of 2-Orientations and 2-Spin structures. Let \( A_2 \) be the mod 2 Steenrod algebra and \( f_9 : BO(9) = BFivebrane \to K(\pi_9(BO), 9) = K(\mathbb{Z}, 9) \), \( f_{10} : BO(10) = B2\text{-Orient} \to K(\pi_{10}(BO), 10) = K(\mathbb{Z}, 10) \), and \( f_{12} : BO(12) = BO(11) = B2\text{-Spin} \to K(\pi_{12}(BO), 12) = K(\mathbb{Z}, 12) \) be the maps realizing the lowest nontrivial homotopy groups. The induced maps on cohomology are then given by \( f_9^* : H^i(K(\mathbb{Z}, 9); \mathbb{Z}_2) \to H^i(BFivebrane; \mathbb{Z}_2) \), \( f_{10}^* : H^i(K(\mathbb{Z}, 10); \mathbb{Z}_2) \to H^i(BFivebrane; \mathbb{Z}_2) \), \( f_{12}^* : H^i(K(\mathbb{Z}, 12); \mathbb{Z}_2) \to H^i(BFivebrane; \mathbb{Z}_2) \).
$H^i(\text{B2-Orientation}; \mathbb{Z}_2)$, and $f^*_2 : H^i(K(\mathbb{Z}, 12); \mathbb{Z}_2) \cong H^i(\text{B2-Spin}; \mathbb{Z}_2)$, respectively. Conditions on the subjectivity of these maps can be deduced from [37], and which we will record momentarily. We can also deduce from the general results of Stong [37] that the cohomology rings with $\mathbb{Z}_2$ coefficients for our spaces are

$$H^i(BO(9); \mathbb{Z}_2) \cong (\mathcal{A}_2/\mathcal{A}_2Sq^2)f^*_9(\iota_9),$$
$$H^i(BO(10); \mathbb{Z}_2) \cong (\mathcal{A}_2/\mathcal{A}_2Sq^3)f^*_9(\iota_{10}),$$
$$H^i(BO(12); \mathbb{Z}_2) \cong (\mathcal{A}_2/\mathcal{A}_2Sq^3 + \mathcal{A}_2Sq^2)f^*_9(\iota_{12}),$$

where $\iota_j$ is the fundamental class of the appropriate Eilenberg-MacLane space in degree $j$.

We summarize the above discussion with the following

**Proposition 2.6** (i) The generators $x_9$ and $x_{10}$ are related to the fundamental classes $\iota_8$ and $\iota_9$ of $K(\mathbb{Z}, 8)$ and $K(\mathbb{Z}_2, 9)$ via $\tau(x_9) = Sq^2\iota_8$ and $\tau(x_{10}) = Sq^2\iota_9$, with $\tau$ the transgression in $\tau$ and $\langle 2.12 \rangle$, respectively.

(ii) The maps $f^*_9$, $f^*_{10}$ and $f^*_{12}$ are surjective for $i < 18$, $i < 20$ and $i < 24$, respectively.

Note that the above inequalities are certainly within the range of dimensions of interest in string theory and M-theory.

We now go back to the original question on whether $x_9$ and $x_{10}$ have to do with Stiefel-Whitney classes. We will deduce from the results of Bahri-Mahowald [3] and Stong [36] that there is no simple direct relation. We consider the covering map $p : BO(\phi(r)) \to BO$, where $\phi(r)$ is some specific function of $r$, the three relevant values of which are given as $\phi(3) = 8$, $\phi(4) = 9$ and $\phi(5) = 10$. A result of [36] asserts that the covering map $p$ maps $w_i \in H^*(BO; \mathbb{Z}_2)$ to the generators in $H^*(BO(\phi(r)); \mathbb{Z}_2)$ if $i - 1$ has at least $r$ ones in its dyadic (binary) expansion, and the remaining classes are mapped to decomposables. For our three relevant cases, $i = 8, 9$ and 10, we see that the numbers 7, 8, and 9 have numbers of ones in their binary expansions which are certainly smaller than $r = 4, 5$, and 6, respectively.

Furthermore, a result of [3] states that the class $p^*w_n$ is nonzero in $H^*(BO(\phi(r)); \mathbb{Z}_2)$ if and only if a certain Poincaré series has a nonzero entry in dimension $n$. For $r = 3$ this series is given by $1 + t^8 + t^{12} + \cdots$, which explicitly contains a $t^8$-term. This is the Fivebrane case. Next, for the 2-Orientation case, we have for $r = 4$ the series $\frac{1}{125}(1 + t^{16})(1 + t^{24})(1 + t^{28})(1 + t^{30})(1 + t^{31})$, which does not have a $t^9$ term. Similarily for the case of 2-spun structures, the Poincaré series for $r = 5$ is likewise sparse and is explicitly seen to no have a $t^{10}$ term. Therefore, from both results of [3] and [36] we have

**Proposition 2.7** (i) The covering maps $p : BO(9) \to BO$ and $p : BO(10) \to BO$ send the Steifel-Whitney classes $w_9 \in H^9(BO; \mathbb{Z}_2)$ and $w_{10} \in H^{10}(BO; \mathbb{Z}_2)$ to decomposables.

(ii) The classes $p^*w_9$ and $p^*w_{10}$ are zero in $H^9(BO(9); \mathbb{Z}_2)$ and $H^{10}(BO(10); \mathbb{Z}_2)$, respectively.

### 3 Ninebrane structures

In this section we shift from Stiefel-Whitney classes to Pontrjagin classes to define our third main structure. It is easy to see that $p_1$ is divisible by 2 when we have a String structure. In this case, since $w_4$ is the mod 2 reduction of the first Spin characteristic class $Q_1 = \frac{1}{2}p_1$, we have that
$w_4 = 0$ in the presence of a String structure. From the congruence $p_3 = w_3^2 \mod 2$ and the fact that $w_8 = Sq^2 w_4$ we get that $p_3$ is even under these conditions. In the four-dimensional case, this was enough to determine the obstruction. However, in this twelve-dimensional case, we will see an extra division by $2^2$. Also, as in the case of the Fivebrane structure, there is an extra division by $3$ and, additionally, by the next prime $5$ for a Ninebrane structure. Another distinction to make is that, while $w_4 = 0$ does not imply $w_8 = 0$, having $w_8 = 0$ does imply $w_{12} = 0$. The follows from the Wu formula $Sq^4 w_8 = w_4 w_8 + w_{12}$, and what distinguishes it from the former is the relatively low power of the Steenrod square.

**Definition 3.1** A Ninebrane structure on a 2-Spin manifold $M$ is a lift $\hat{f}$ of the classifying map $f$ in the following diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\hat{f}} & BO\langle 13 \rangle = B\text{Ninebrane} \\
\downarrow{f} & & \downarrow{\pi} \\
BO\langle 11 \rangle & \xrightarrow{x_{12}} & K(Z,12).
\end{array}
$$

The obstruction class for lifting the classifying map $f : X \to BO\langle 12 \rangle$ to $\hat{f} : X \to BO\langle 13 \rangle = B\text{Ninebrane}$ is obtained by pulling back the universal class $x_{12} \in H^{12}(BO\langle 12 \rangle; Z) \cong H^{12}(BO\langle 11 \rangle; Z)$. Thus a manifold $X$ admits a Ninebrane structure if and only if $f^* x_{12} \in H^{12}(X; Z)$ vanishes. This is a fraction of the third Pontrjagin class $p_3$ and can be characterized as follows. For vector bundles over the sphere $S^{12}$ the best possible result on the divisibility of the Pontrjagin class $p_3(\xi)$, for $\xi$ a vector bundle of rank 12 over $S^{12}$, is that $p_3(\xi)$ can be any multiple of (see [18] p. 244, [5] [16]) $(2 \cdot 3 - 1)! \gcd(3 + 1, 2)u$, that is $240u$, where $u$ is the standard generator of $H^{12}(S^{12})$. It follows that the relevant fraction is given by $1/240$ so that, therefore, we straightforwardly have

**Proposition 3.2** The obstruction to a Ninebrane structure is given by $\frac{1}{240} p_3$.

**Remarks.** (i) Note that having simultaneously a String structure, a Fivebrane structure, and positive scalar curvature leads to a Ninebrane structure. This follows from the Lichnerowicz theorem and the index theorem: the obstruction to positive scalar curvature is given by the $\hat{A}$-genus, which in dimension 12 is a combination of the String obstruction $\frac{1}{2} p_1$, the Fivebrane obstruction $\frac{1}{2} p_2$ and the Ninebrane obstruction $\frac{1}{240} p_3$. See [25] for extensive discussions in the Spin case.
(ii) Note that there is a path fibration $K(\pi_{12}(BO), 11) \to PK(\pi_{12}(BO), 12) \to K(\pi_{12}(BO), 12)$, i.e. $\Omega K(Z,12) \to PK(Z,12) \to K(Z,12)$, which induces the fibration

$$K(Z,11) \to B\text{Ninebrane} \to B2\text{-Spin}. \quad (3.1)$$

**Example.** Global considerations in M-theory require extension to a 12-dimensional bounding Spin manifold $Z^{12}$. Supersymmetry implies the existence of a Rarita-Schwinger field, i.e. a spinor-valued one-form, which can be viewed as a section of the Spin bundle tensored with the virtual bundle $TZ^{12} - 4O$, where the subtraction of $4O$ accounts for two Faddeev-Popov ghosts as well as the two extra directions in relating to the Spin bundle in ten dimensions [8]. The action can be written in terms of indices of twisted Dirac operators, one of which being the Rarita-Schwinger operator [42]. The Chern character $ch(TZ^{12} - 4O)$ is given by

$$8 + p_1 + \frac{1}{12}(p_1^2 - 2p_2) + \frac{1}{360}(p_1^3 - 3p_1p_2 + 3p_3). \quad (3.2)$$
Equipping our manifold with a String and a Fivebrane structure, i.e. requiring the vanishing of $\frac{1}{2}p_1$ and $\frac{1}{6}p_2$, we get in dimension twelve the term $\frac{1}{120}p_3$. This is twice the Ninebrane obstruction $\frac{1}{240}p_3$. The situation here is, in some sense, analogous to the case of the first Pontrjagin class $p_1$ vs. the first Spin characteristic class $Q_1 = \frac{1}{2}p_1$. Note that, concentrating on the prime 2, the Ninebrane obstruction is $\frac{1}{2}p_3$, whose mod 2 reduction is the Stiefel-Whitney class $w_{12} = 0$. The similar statement in degree eight is that the 2-adic part of the Fivebrane obstruction which is $\frac{1}{2}p_2$ admits $w_8$ as its mod 2 reduction.

Note, however, that the above example does not amount to a full anomaly cancellation requirement, but merely that the Ninebrane obstruction appears in the expressions of part of the anomaly or effective action. This is then slightly weaker that the statements in the String and Fivebrane cases, which amounted e.g. to the Green-Schwarz anomaly cancellation condition and its dual [31]-[32].

**Invariances of the structures under homotopy equivalences.** We consider whether having one of the three structures defined above is a property that is invariant under automorphisms. The topological invariance of String structures is considered in [19]. It is interesting to note that the obstructions $\frac{1}{2}p_1$ and $\frac{1}{6}p_2$ for String and Fivebrane structures, and the class $p_3$ mod 120 are homotopy invariant. This follows from the results in [35]. Furthermore, we note that the intersection form on a closed Spin 12-dimensional manifold is always even, so we have a further division by 2 for $p_3$. Therefore, we have

**Proposition 3.3** Having a Fivebrane, Ninebrane, 2-Orient, or 2-Spin structure is a homotopy invariant property. So if $f : X \to Y$ is a homotopy equivalence then there is such a structure on $X$ if and only if there is the same one on $Y$.

**Remark.** In [32]-[33], the identification of anomalies in M-theory and string theory with Fivebrane structures required some modification to account for further congruences. For example, the class $\frac{1}{4}p_2$ appeared instead of $\frac{1}{6}p_2$. That further division by 8 was accounted for by defining a variant structure, denoted $F(8)$. Here in the case of Ninebrane structures, the same kind of argument applies and we can similarly account for further divisions of $\frac{1}{240}p_3$ as needed for applications.

**4 The set of lifts**

The set of lifts of familiar structures, such as orientations, Spin structures and String structures is given generally by a torsor over a cohomology group of one dimension less than the dimension of the obstruction. This was also shown for Fivebrane structures in [32]. Such a characterization continues to hold in our case of 2-Orientation = BO(10), 2-Spin = BO(11), and Ninebrane = BO(13) structures. We will describe these lifts in a uniform fashion. Let $A$ denote $\mathbb{Z}_2$ for the first and second structures and $\mathbb{Z}$ for the third structure and let $n = 9, 10, \text{ and } 12$, respectively, and $m = n + 1$. In the case of the 2-Spin structure we have an automatic further lift one more level to $BO(12)$. We
encode all this succinctly in the diagram

\[
\begin{array}{c}
K(A, n - 1) \\
\downarrow \\
BO(m) \\
\downarrow \\
X \\
\downarrow \\
BO(n),
\end{array}
\]  

(4.1)

in which the fibrations are induced from the path fibrations, including the ones in (2.8) and (2.11). The set of structure is given in the three cases by a torsor for the cohomology group \(H^{n-1}(X; A)\).

Therefore, with an equivalence relation on each of the sets given by homotopy of sections, we have

**Proposition 4.1**

(i) The set of 2-Orientation structures on a given Fivebrane structure is given by a torsor for the group \(H^8(X; \mathbb{Z}_2)\).

(ii) The set of 2-Spin structures on a given 2-Orientation structure is given by a torsor for the group \(H^9(X; \mathbb{Z}_2)\).

(iii) The set of Ninebrane structures on a given 2-Spin structure is given by a torsor for the group \(H^{11}(X; \mathbb{Z})\).

**Remarks.** We note the following:

1. On manifolds \(Y^{11}\) of dimension eleven, i.e. as in M-theory, the Ninebrane obstruction vanishes identically by dimension reasons. However, it is still interesting to consider Ninebrane structures on \(Y^{11}\) as those are parametrized by the group \(H^{11}(Y^{11}; \mathbb{Z})\). This is analogous to the case of String structures on 3-dimensional manifolds \(M^3\), where these structures are parametrized by \(H^3(M^3; \mathbb{Z})\), corresponding to a gerbe on the worldvolume and is captured by the volume; see [23] for a characterization and application to the M2-brane. The second case is having a Fivebrane structure on the worldvolume \(M^6\) of the M5-brane. Again this is automatic, but the structures are interestingly enumerated by the 5-gerbe on the worldvolume. See [33][10][30] for detailed accounts.

2. In lower degrees, the lift to a \(BO(n + 1)\)-structure does not depend on the choice of a \(BO(n)\)-structure. This is the case for \(n = 2\), where the existence of a Spin structure does not depend on choice of orientation from which to lift, because of homotopy invariance of the second Steifel-Whitney class \(w_2\). The case \(n = 4\) is similar, where a lift to a String structure does not depend on a choice of underlying Spin structure from which to lift, which can be shown via obstruction theory (see [6]). However, in higher degrees this changes – see The Manifold Atlas Project [17], which we follow in the ensuing discussion. Starting with Fivebrane structures and going up, one has dependence of the higher structure on the choice of lower structures. That is, among the set of String structures there might exist one which does not lift to a Fivebrane structure. Let us illustrate the statement in this degree, and the next degrees which we consider in this article will follow analogously. Consider two String structures given by two classifying maps \(f, g : X \to B\text{String}\) for which the composition \(\tilde{f}, \tilde{g} : X \to B\text{Fivebrane} \pi \to B\text{String}\) are homotopic. Then the two maps
In this section we define twisted versions of the structures defined above, using the approach in [40].

5 Twisted structures

In this section we define twisted versions of the structures defined above, using the approach in [40] [33] [25] [28].

We do not claim that we know the structure of the cohomology ring $H^*(BO(9);\mathbb{Z})$, but it is enough for us to know the first generator and that there is an H-space structure. For rational coefficients, this is studied in [32].
Definition 5.1  (Twisted 2-orientation) A twisted 2-orientation on a submanifold (a brane) $M$ embedded in spacetime $X$ is a homotopy in the following diagram, where $f$ is the classifying map for Fivebrane bundles and $\alpha_9$ is a cocycle of degree 9

$$
\begin{array}{ccc}
M & \xrightarrow{f} & BO(9) \\
\downarrow i & & \downarrow x_9 \\
X & \xrightarrow{\eta} & K(\mathbb{Z}_2, 9) \\
\end{array}
$$

Remark. The obstruction to having a twisted 2-orientation on $M$ is given by

$$f^*x_9 + i^*\alpha_9 = 0. \quad (5.1)$$

Definition 5.2  (Twisted 2-Spin structure) A twisted 2-Spin structure on a submanifold (a brane) $M$ embedded in spacetime $X$ is a homotopy in the following diagram, where $f$ is the classifying map of 2-oriented bundles and $\alpha_{10}$ is a cocycle of degree 10

$$
\begin{array}{ccc}
M & \xrightarrow{f} & BO(10) \\
\downarrow i & & \downarrow x_{10} \\
X & \xrightarrow{\eta} & K(\mathbb{Z}_2, 10) \\
\end{array}
$$

Remark. The obstruction to having a twisted 2-Spin structure on $M$ is given by

$$f^*x_{10} + i^*\alpha_{10} = 0. \quad (5.2)$$

It would be interesting to provide examples of twisted 2-Spin structures and twisted 2-Orientation structures, along the lines of [25] [28].

Definition 5.3  (Twisted Ninebrane structure) A twisted Ninebrane structure is defined by the following diagram

$$
\begin{array}{ccc}
M & \xrightarrow{f} & BO(12) \\
\downarrow i & & \downarrow \frac{1}{240}p_3 \\
X & \xrightarrow{\eta} & K(\mathbb{Z}, 12) \\
\end{array}
$$

Remark. The obstruction to having a twisted Ninebrane structure is given by

$$f^*(\frac{1}{240}p_3) + i^*\alpha_{12} = 0. \quad (5.3)$$

Example: Twisted Ninebrane structures via embeddings. We consider a brane embedded in spacetime via $i : M \hookrightarrow Z$. Then the $\hat{A}$-genera of $M$ and $Z$ can be related via a Riemann-Roch formula. In the simplest case where this embedding is a homotopy equivalence, one has that $\hat{A}(Z)/\hat{A}(M)$ is an element in the real Chern character $\text{cho}(Z)$, that is a Pontrjagin class of some orthogonal bundle [2]. Considering degree four components gives $\frac{1}{2}p_1(M) = \frac{1}{2}p_1(Z) + 12p_1(E)$,
where $E$ is an orthogonal bundle on $Z$. Then a String structure on $M$ leads to a twisted String structure on $Z$ as $12p_1(E) \in Z$. This can also be reversed; by writing $\frac{1}{6}p_1(M) - 12p_1(E) = \frac{1}{2}p_1(Z)$, a String structure on $Z$ amounts to a twisted String structure on $M$. In the presence of a Spin structure, the statement can be improved to a further divisibility by two due to the congruence $p_2^2 = u_2^2 \mod 2$. By the above general formula of Atiyah-Hirzebruch [2] we can deduce similar statements in higher degree cases (and following the approach of [24] with sufficiently high dimensions):

1. **Degree eight:** In the presence of a String structure on both $M$ and $Z$, the degree eight components give

$$\frac{1}{2}p_2(M) = \frac{1}{2}p_2(Z) + 240p_2(E).$$  

(5.4)

We view this as an example of a twisted Fivebrane structure on $Z$ determined by a Fivebrane structure on $M$ and vice versa.

2. **Degree twelve:** Assuming String structures $\frac{1}{6}p_1(M) = 0 = \frac{1}{2}p_1(Z)$ and Fivebrane structures $\frac{1}{6}p_2(M) = 0 = \frac{1}{2}p_2(Z)$ on both the brane and spacetime, we have in degree twelve

$$\frac{1}{240}p_3(M) = \frac{1}{240}p_3(Z) + 252p_3(E).$$  

(5.5)

Therefore, upon setting each side to zero, this gives an equivalence between a Ninebrane structure on $M$ and a twisted Ninebrane structure on $Z$, and vice versa.

**Example: The $E_8$ index in M-theory.** The index of the Dirac operator coupled to an $E_8$ bundle in M-theory on a Spin manifold $Y^{11}$ lifted to twelve dimensions is given by

$$\frac{1}{2}I_{E_8} = \frac{1}{6}G_4 \cup G_4 \cup G_4 - \frac{1}{38}(p_2 - (\frac{1}{2}p_1)^2) \cup G_4 - \frac{31}{15120}p_4 + \frac{13}{30240}p_1p_2 - \frac{1}{15120}p_1^3.$$  

(5.6)

Assuming a String structure, the C-field quantization condition $[12] G_4 + \frac{1}{4}p_1 = a \in H^4(Y^{11}; Z)$ reduces to $G_4 = a$, the characteristic class of the $E_8$ bundle. If we further assume a Fivebrane structure, i.e. $\frac{1}{6}p_2 = 0$, then expression (5.6) reduces to

$$\frac{1}{2}I_{E_8} = \alpha \cdot \frac{1}{240}p_3 - \frac{1}{6}a \cup a \cup a,$$  

(5.7)

where $\alpha = \frac{31}{15120}$. For any value of $\alpha$, the second term can viewed as a rational twist for a rational Ninebrane structure. However, when $1/\alpha$ times the last term is integral, this latter term serves as an integral twist for the first term, which is an obstruction to the would-be Ninebrane structure. Therefore, the triviality of the Dirac index for $E_8$ bundles with classes $a = 186n$ for $n \in Z$, in M-theory on a twelve-dimensional Fivebrane manifold $Z^{12}$ is equivalent to a twisted Ninebrane structure on $Z^{12}$, with the twist given by the cubic term in the $E_8$ characteristic class. Note that this kind of twist is composite and is a cubic analog in degree twelve of the composite quadratic twists giving rise to a String$^*$ structure in degree four [24] and a Fivebrane$^K(Z,4)$ structure in degree eight [25].

Again this example highlights the fact that due to the relative high dimension of the Ninebrane obstruction relative to the dimensions of the applications considered, the situation is not as optimal as one had in the cases of lower obstructions, namely of the String and the Fivebrane, in [32] [33]. Furthermore, the anomalies encountered should necessarily not be of the usual Green-Schwarz type, since these are always of a factorized form: a product of a degree four piece and a degree eight piece. However, we will see in Sec. 8 that there is a natural explanation of a new phenomenon, namely the existence of a Chern-Simons term and a top form in M-theory that lends itself to a natural description in terms of Ninebrane structures.
6 Structures not directly defined via the Whitehead tower

We know that one can define structures arising from vanishing of (multiples of) higher obstructions, without the lower obstructions necessarily vanishing. Examples of such are abundant: A Pin structure requires the vanishing of the would-be Spin obstruction $w_2$ without necessarily having the orientation obstruction $w_1$ vanish. Also, we can have the first Pontrjagin class $p_1$ vanishing without the lower obstruction, the Spin obstruction $w_2$, being zero. Such a structure is called a $p_1$-structure and is important in Chern-Simons theory and low-dimensional topology. See [25] [28] [30] for many examples of structures defined via this general phenomenon.

Let $X = BO(p_i)$ be the homotopy fiber of the map $p_i : BO \to K(\mathbb{Z}, 4i)$ corresponding to the first Pontrjagin class of the universal stable bundle $\gamma$ over the classifying space $BO$. Let $\gamma_X$ be the pullback of $\gamma$ over $X$.

**Definition 6.1** A $p_i$-structure on a submanifold (a brane) $M$ is a fiber map from the stable tangent bundle $TM$ of $M$ to $\gamma_X$. That is, there is the following lifting diagram

$$
\begin{array}{c}
X = BO(p_i) \\
\downarrow \\
M \rightarrow BO \rightarrow K(\mathbb{Z}, 4i).
\end{array}
$$

The Spin/String version of this construction is explained in our context in [23]. So a $p_2$-structure is defined when $p_2 = 0$ but $p_1 \neq 0$, and a $p_3$-structure is defined when $p_3 = 0$ while $p_i \neq 0$, $i = 1, 2$.

**Remark.** We can similarly consider structures defined by Stiefel-Whitney classes and Wu classes, as in [28]. For instance, we define a 2-Pin structure via $x_{10} = 0$ but $x_9 \neq 0$.

We now consider twists of the above structures, generalizing the definition in [30] from degree four to other (higher) degrees.

**Definition 6.2** An $\alpha$-twisted $p_i$-structure on a submanifold (a brane) $\iota : M \to Y$ with a Riemannian structure classifying map $f : M \to BO$, is a 4$\alpha$-cocycle $\alpha : Y \rightarrow K(\mathbb{Z}, 4i)$ and a homotopy $\eta$ in the diagram

$$
\begin{array}{c}
M \xrightarrow{f} BO(n) \\
\downarrow \iota \\
Y \xrightarrow{\alpha_{4i}} K(\mathbb{Z}, 4i).
\end{array}
$$

**Remarks.** (i) The obstruction is then $p_i(M) + [\alpha_{4i}] = 0 \in H^{4i}(M; \mathbb{Z})$. As in the case of twisted String, Fivebrane, or Ninebrane structures, the set of such structures will be a torsor for $H^{4i-1}(M; \mathbb{Z})$.

(ii) Similarly we can define a twisted 2-Pin structure and other variants as the case of those given by Stiefel-Whitney classes and Wu classes.
7 The (twisted) groups

We have defined the structures directly via classifying spaces in previous sections. A natural question then is whether and how to describe the corresponding groups (in the homotopy sense). Here we build the groups as the deloopings of the classifying spaces as in previous cases \[^{33}\]. The general machinery there allows similarly that we define new groups here as follows.

**Definition 7.1** The homotopy fibers of the structures \( B_{2\text{-Orient}} \), \( B_{2\text{-Spin}} \), and \( BN_{\text{Ninebrane}} \) define the groups \( 2\text{-Orient} \), \( 2\text{-Spin} \), and \( \text{Ninebrane} \), respectively.

**Remarks.**
1. Working not necessarily in the stable range, we have the groups \( 2\text{-Orient}(n) \), \( 2\text{-Spin}(n) \), and \( \text{Ninebrane}(n) \). In the notation for connected covers with conventions as in \[^{33}\], these groups are \( O(8)(n) \), \( O(9)(n) \) and \( O(11)(n) \).
2. The group \( 2\text{-Orient}(n) \) is the \( \mathbb{Z}_2 \) double cover (in the homotopy sense, and with a mod 8 shift from the classical notion) of the group \( \text{Fivebrane}(n) \). We have \( \pi_8(\text{Fivebrane}(n)) \cong \mathbb{Z}_2 \) while \( \pi_8(2\text{-Orient}(n)) = 0 \).
3. The group \( 2\text{-Spin}(n) \) is the \( \mathbb{Z}_2 \) double cover (also in the above sense) of the group \( 2\text{-Orient}(n) \). We have \( \pi_9(2\text{-Orient}(n)) \cong \mathbb{Z}_2 \) while \( \pi_9(2\text{-Spin}(n)) = 0 \).
4. For any of the above groups \( G \) we have \( \pi_{10}(G) = 0 \). This is a mod 8 shift of the classical fact that \( \pi_2(G) = 0 \) for any Lie group (and hence also for its connected covers).

Similarly, we can define groups (again in the homotopy sense) as the homotopy fibers of the corresponding twisted structures, again as in \[^{33}\] (see also \[^{24}\]).

**Definition 7.2** The twisted groups \( G^c = O(8)^c(n) \), \( O(9)^c(n) \) and \( \text{Ninebrane}^c(n) \) are the homotopy fibers of the corresponding twisted structures \( \text{BG}^c \).

**Remark.** As in the cases of \( \text{String}(n) \) and \( \text{Fivebrane}(n) \), the Whitehead tower construction allows us to describe the group \( \text{Ninebrane}(n) \) via a fibration with an Eilenberg-MacLane space as a fiber

\[
K(\mathbb{Z}, 10) \longrightarrow \text{Ninebrane}(n) \longrightarrow 2\text{-Spin}(n),
\]

obtained by looping the fibration \(^{(3.1)}\).

It would be interesting to find explicit geometric/categorical models for the above groups.

8 Differential refinement

It is desirable for physics to have differential versions of the topological structures that arise. As in the case of \( \text{String}(n) \) and \( \text{Fivebrane}(n) \), one can consider differential refinements of the higher \( \text{BO}(m) \)-structures to higher stacks. Via the formulation in \[^{14}\] \[^{34}\] we refine the classifying spaces \( \text{BG} \) as topological spaces to \( \text{BG} \) as stacks. This also requires refining Eilenberg-MacLane spaces \( K(\mathbb{Z}, n) = B^{n-1}U(1) \) to stacks \( B^{n-1}U(1) \). Consequently, we have
Proposition 8.1 The above structures refine to moduli stacks described in the diagram

\[
\begin{array}{ccc}
\text{BNinebrane} & \xrightarrow{1/240p_3} & \text{B}^{11}U(1) \\
\text{NinebraneStruc} & \downarrow & \\
\text{BO}(11) & \xrightarrow{x_{10}} & \text{B}^{10}Z_2 \\
\text{2-SpinStr} & \downarrow & \\
\text{BO}(10) & \xrightarrow{x_{9}} & \text{B}^9Z_2 \\
\text{2-OrientStruc} & \downarrow & \\
\text{BFivebrane} & \xrightarrow{1/6p_2} & \text{B}^7U(1) \\
\text{FivebraneStruc} & \downarrow & \\
\text{BString} & \xrightarrow{1/240p_3} & \text{B}^{11}U(1) \\
\end{array}
\]

(8.1)

Remarks. 1. Strictly speaking, the construction used in [14] to Lie integrate the first two invariant polynomials of \(\mathfrak{so}(n)\) to the smooth \(1/2p_1\) and the smooth \(1/6p_2\) would yield for the third invariant polynomial some multiple of \(p_3\) whose homotopy fiber is the result of killing \(\pi_{11} = \mathbb{Z}\) in Fivebrane, but leaving the \(\pi_8 = \pi_9 = \mathbb{Z}_2\) alone. The smooth \(1/240p_3\) as displayed above exists, but this does not follow from the construction in [14]. That construction only kills cocycles at the level of \(L_\infty\) algebras and then integrates that up to smooth higher stacks, but so it cannot kill torsion groups. We thank Urs Schreiber for illuminating discussions on these matters (see also [34]).

2. We can also provide further refinement to \(\text{B}^nU(1)_{\text{conn}}\) by including connections, giving a diagram as above but with the moduli stacks of \(n\)-bundles with connections, using the machinery developed in [14] [34]. The corresponding diagram will be one replacing the above, with \(\text{B}^nA_{\text{conn}}\) replacing \(\text{B}^nA\) and the refined classes \(\hat{c}\) replacing the classes \(c\) via [14] [33] [34].

Trivialization of the Ninebrane obstruction class. Recall that in the case of String and Fivebrane structures we had trivializations of the corresponding forms given by a degree three class \(H_3\) and a degree seven class \(H_7\), respectively, in essentially the following form

\[
dH_3 = \frac{1}{2}p_1(A) , \quad dH_7 = \frac{1}{6}p_2(A) .
\]

(8.2)

Furthermore, such expressions arise in physical settings, e.g. essentially in the Green-Schwarz anomaly cancellation and its dual, as explained in [32] [33]. It makes sense to consider for the cohomology class obstructing the Ninebrane structure a trivialization at the level of differential form representatives given by

\[
dH_{11} = \frac{1}{240}p_3(A) ,
\]

(8.3)

for some 11-form \(H_{11}\). We investigate whether the form \(H_{11}\) has some physical interpretation. Note that because of the relatively low dimensions in M-theory and string theory in comparison to our increasing levels in the Whitehead tower, such an interpretation becomes harder to get. However, we propose a conjectural relation. Recently, existence of top forms was discovered in string theory (see [4] and references therein); these are “potentials” rather than field strengths, i.e. are higher
connections rather than higher curvatures. So such a top form $H_{11}$ in M-theory can be taken to satisfy

$$dH_{11} = -\frac{1}{2} G_4 \wedge G_8 + \cdots.$$  \hspace{1cm} (8.4)

Not much is known about the dynamics or the geometry associated with this form. We propose that $H_{11}$ in (8.4) is the trivialization of the Ninebrane form (8.3), i.e. the two expressions are compatible in the sense that the second admits a correction term by the first. This is analogous in degree twelve to the correction of the equations of motion of the C-field by the one-loop polynomial $I_8$ in degree eight. We hope that more investigations are made on such forms so that the above proposal can be verified. We do, however, provide another possible interpretation. Chern-Simons terms $CS_{11}$ of degree 11 appear in the M-theory action when formulated via the signature (which is equivalent to formulation via Dirac operators) in [27]. We have the relation to the Ninebrane structures and to $p_3$-structures as $p_3(A) \sim dCS_{11}(A)$.

**Secondary 9-brane structures.** Note that while the 12-class $\sim p_3$ necessarily vanishes in 11 dimensions, it is noteworthy that the differential refinement to $\hat{p}_3$ is a differential cohomology class in degree 12, it is a secondary class/invariant which need not vanish even if its underlying (topological) 12-class vanishes. This is of course just the statement that there may be a non-trivial connection 11-form, even if its curvature vanishes. So while in 11 dimensions any bare Fivebrane structure always has a lift to a Ninebrane structure, if one considers differential Fivebrane structures (i.e. maps to $B\text{Fivebrane}_{\text{conn}}$) then there is a actual condition to lift to $B\text{Ninebrane}_{\text{conn}}$, namely that not just the curvature 12-class but also the connection 11-form itself vanishes. As indicated above, that 11-form is just the Lagrangian for the 11-dimensional Chern-Simons term.

**Relation to the M-algebra and the M9-brane.** In the discussion of the algebra corresponding to eleven-dimensional supergravity, and its associated cohomology, it was found in [13] in the super geometric setting that there exists a spacetime-filling brane in M-theory. In the case of the M9-brane the relation to generalized Wess-Zumino-Witten (WZW) models and generalized Chern-Simons (CS) theories is the last column in the following table, which completes the first two cases studied extensively in [7] [1] [39] [23] and [23] [11] [12] [13] [34], respectively.

| Structure | String | Fivebrane | Ninebrane |
|-----------|--------|-----------|-----------|
| Worldvolume | $\Sigma_2$ | $\Sigma_6$ | $\Sigma_{10}$ |
| WZW | WZW$_2$ | WZW$_6$ | WZW$_{10}$ |
| Handlebody | $M^3$ | $M^7$ | $M^{11}$ |
| Chern-Simons | $CS_3$ | $CS_7$ | $CS_{11}$ |
| String or $p_1$ | Fivebrane or $p_2$ | Ninebrane or $p_3$ |

The WZW$_{10}$ theory was studied in [22] [20] [29] [13]. Both of the theories (WZW$_{10}$ and $CS_{11}$) associated to the ninebrane can be refined to the level of moduli stacks of higher bundles with higher connections, as lower degrees [14] [10] [11] [12] [13] [34]. Analogously to the degree four and degree eight cases, i.e. for String and Fivebrane structures, respectively, from the above works, the geometric and topological ingredients associated to the ninebrane are described by the following
Note that this diagram is such that all squares (and hence all composite rectangles) are homotopy cartesian (i.e. are homotopy pullback squares) in the \(\infty\)-topos of smooth \(\infty\)-groupoids (i.e. \(\infty\)-stacks over smooth manifolds). This is a much stronger statement than just that the diagram exists, as each item in the top-left of a square is in fact uniquely characterized (up to equivalence) as completing that square to a homotopy pullback square. Similarly, for the case of \(p_3\)-structures with \(\text{Ninebrane}'\) referring to a \(p_3\)-structure. The main difference between the two diagrams above is that the second does not involve killing the two \(\mathbb{Z}_2\)'s in degrees 9 and 10. Note that the latter
diagram, unlike the former, is a corollary of the theorem in [14] – see the Remarks at the beginning of this section.

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