Minimization of a Ginzburg-Landau type energy with weight and with potential having a zero of infinite order

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Abstract

In this paper, we study the asymptotic behaviour of minimizing solutions of a Ginzburg-Landau type functional with potential having a zero at 1 of infinite order and we estimate the energy. We generalize in this case a lower bound for the energy of unit vector field given by Brezis-Merle-Rivi`ere.

Keywords: Ginzburg-Landau functional, lower bound, variational problem.

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Introduction

Let $G$ be a bounded, simply connected and smooth domain of $\mathbb{R}^2$, $g : \partial G \to S^1$ a smooth boundary data of degree $d$ and $p$ a smooth positive function on $\overline{G}$. We set $p_0 = \min \{ p(x) : x \in \overline{G} \}$ and $\Lambda = p^{-1}(p_0)$. Let us consider a $C^2$ functional $J : \mathbb{R} \to [0, \infty)$ satisfying the following conditions :

(H1) $J(0) = 0$ and $J(t) > 0$ on $(0, \infty)$,

(H2) $J'(t) > 0$ on $(0, 1]$, 

(H3) there exists $\rho_0 > 0$ such that $J''(t) > 0$ on $(0, \rho_0)$.

For each $\varepsilon > 0$ let $u_\varepsilon$ be a minimizer for the following Ginzburg-Landau type functional

$$E_\varepsilon(u) = \int_G p|\nabla u|^2 + \frac{1}{\varepsilon^2} \int_G J\left(1 - |u|^2\right)$$

(0.1)

defined on the set

$$H^1_g(G, \mathbb{C}) = \{ u \in H^1(G, \mathbb{C}) : u = g \text{ on } \partial G \}. \quad (0.2)$$

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It is easy to prove that \( \min_{u \in H^1_0(G, \mathbb{C})} E_\varepsilon(u) \) is achieved by some smooth \( u_\varepsilon \) which satisfies

\[
-\text{div}(p \nabla u_\varepsilon) = \frac{1}{\varepsilon^2} J'(1 - |u_\varepsilon|^2)u_\varepsilon \quad \text{in } G \\
u_\varepsilon = g \quad \text{on } \partial G
\]

where \( j(t) = J'(t) \). In this paper, we are interested in studying the asymptotic behaviour of \( u_\varepsilon \) and estimate the energy \( E_\varepsilon(u_\varepsilon) \) as \( \varepsilon \to 0 \) under the assumptions that \( p \) has a finite number of minima all lying in \( G \) and that it behaves in a "good" way in a neighborhood of each of its minima. More precisely, throughout this paper we shall assume

\[
\Lambda = \{b_1, \ldots, b_N\} \subset G
\]

and there exist real numbers \( \alpha_k, \beta_k, s_k \) satisfying \( 0 < \alpha_k \leq \beta_k \) and \( s_k > 1 \) such that

\[
\alpha_k |x - b_k|^{s_k} \leq p(x) - p_0 \leq \beta_k |x - b_k|^{s_k}
\]

in a neighborhood of \( b_k \) for every \( 1 \leq k \leq N \).

The presence of a nonconstant weight function is motivated by the problem of pinning the vortices of \( u_\varepsilon \) to some restricted sites, see [10] and [16] for more detailed physical motivations. Without loss of generality we assume \( d \geq 0 \). By the way we treat only the case \( d > 0 \), being the case \( d = 0 \) trivial.

The case when \( J(|u|) = \frac{(1 - |u|^2)^2}{4} \) and \( p = \frac{1}{2} \) corresponding to the Ginzburg-Landau energy, was studied by several authors since the groundbreaking works of Béthuel-Brezis and Hélein. More precisely they dealt with the case with boundary data satisfying \( d = 0 \) and \( d \neq 0 \) respectively in [2] and [3]. In this latter work only the case of \( G \) starshaped was treated. Eventually in [18], Struwe gave an argument which works for an arbitrary domain and later del Pino and Felmer in [9] gave a very simple argument for reducing the general case to the starshaped one. More in particular the method of Struwe is found to be very useful for the case of nonconstant \( p \).

The case \( J(|u|) = \frac{(1 - |u|^2)^2}{4} \) and \( p \) not a constant function was studied in [1] [4] [5] [6]. More precisely in [4] [5] the authors considered the cases \( \text{card} \Lambda = 1 \) and \( d \geq 1 \), \( \text{card} \Lambda \geq d \) and the case where \( p \) has minima on the boundary of the domain. In the second case, they showed that actually \( N = d \), the degree around each \( b_k \) is equal to 1 and for a subsequence \( \varepsilon_n \to 0 \)

\[
u_{\varepsilon_n} \to u_* = e^{i\phi} \prod_{j=1}^{d} \frac{z - b_j}{|z - b_j|} \quad \text{in } C^{1,\alpha}_\text{loc}(\overline{G} \setminus \{b_1, \ldots, b_d\})
\]
the configuration \( \{b_1, ..., b_d\} \) being minimizing for a certain renormalized energy defined in \( \Lambda^d \). Moreover they proved the asymptotics \( E_\varepsilon(u_\varepsilon) = \pi p_0 d |\log \varepsilon| + O(1) \). In the first case, if \( \Lambda = \{b\} \subset G \), they proved

\[
u_{\varepsilon_n} \to u_* = e^{i\phi} \left( \frac{z - b}{|z - b|} \right)^{2d} \quad \text{in} \quad C^{1,\alpha}_{\text{loc}}(G \setminus \{b\}),\]

where \( \phi \) is determined by the boundary data \( g \). In [1], the authors studied the case \( \text{card}\Lambda < d \) and established the convergence of a subsequence \( u_{\varepsilon_n} \to u_* \) in \( C^{1,\alpha}_{\text{loc}}(G \setminus \{b_1, ..., b_N\}) \) for every \( \alpha < 1 \), where the \( N \) distinct points \( \{b_1, ..., b_N\} \) lie in \( \Lambda \) and \( u_* \in C^\infty(G \setminus \{b_1, ..., b_N\}, S^1) \) is a solution of

\[-\text{div}(p|\nabla u_*|^2 u_*) = g \quad \text{in} \quad G \setminus \{b_1, ..., b_N\}, \quad u_* = g \quad \text{on} \quad \partial G.\]

Moreover, the degree \( d_k \) of \( u_* \) around each \( b_k \) satisfies \( d_k \geq 1 \) and \( \sum_{k=1}^N d_k = d \).

In the current paper we will suppose that \( \text{card}\Lambda = N < d \) as this is the more interesting case. Indeed, as already observed in [1], singularities of degree \( \geq 1 \) must occur and in some cases they could be on the boundary. Following the same argument as in [2] or in [1], we prove that \( u_{\varepsilon_n} \) has its zeroes located in \( d \) discs of radius \( \sim \varepsilon_n \) called ”bad discs”. Outside this discs \( |u_{\varepsilon_n}| \) is close to 1. For \( n \) large each bad discs contains exactly one zero. Thus there are exactly \( d_k \) zeroes approaching each \( b_k \) (as \( n \to \infty \)). In the case \( d_k > 1 \) (this must be the case of at least one \( k \) if \( N < d \)), one expects to observe an ”interaction energy” between zeroes approaching the same limit \( b_k \). A complete understanding of this process requires a study of the mutual distances between zeroes of \( u_{\varepsilon_n} \) which approach the same \( b_k \). It turns out that these distances depend in a crucial way on the behaviour of the weight function \( p \) around its minima points. In [1] if \( s_k = 2 \), it is showed that each \( b_k \) with \( d_k > 1 \) contributes an additional term to the energy, namely \( \pi p_0 \left( d_k^2 - d_k \right) \log \left( |\log \varepsilon|^{\frac{1}{2}} \right) \) which is precisely the mentioned interaction energy. In our paper the energy cost of each vortex of degree \( \geq 1 \) is much less than the previous one.

In [11] we study the effect of the presence of \( |u| \) in the weight \( p(x, u) = p_0 + s|x|^k|u|^l \) where \( s \) is a small, \( k \geq 0 \) and \( l \geq 0 \). The method of [2, 3, 18] can be adapted without any difficulties to the case of \( J \) satisfying \( H1 - H3 \) with a zero of finite order at \( t = 0 \). This applies for example to \( J(t) = |t|^k, \forall k \geq 2 \). In this article, we are interested in different types of generalization, starting from the case where the potential \( J \) satisfies \( H1 - H2 - H3 \), and \( p \) is non constant. Significative examples are

\[
J(t) = J_h(t) = \begin{cases} 
\exp(-1/t^h) & \text{for } t > 0, \\
0 & \text{for } t \leq 0,
\end{cases}
\]

for \( h > 0 \). In the present paper, a main new feature is that certain potentials with
sufficiently slow growth allow for a vortex energy that is not $\pi |\log\varepsilon| + O(1)$ but instead

$$2\pi p_0 d_k|\log\varepsilon| + 2\pi p_0\frac{d_k^2 - d_k}{s_k}\log|\log\varepsilon| - 2\pi p_0 d_k I\left(\frac{1}{\varepsilon}(|\log\varepsilon|)^{\frac{1}{s_k}}\right) + o\left(I\left(|\log\varepsilon|\right)^{\frac{1}{s_k}}\right).$$

(see also [12, 13]). More precisely we want to prove the following result

**Theorem 1.** For each $\varepsilon > 0$, let $u_\varepsilon$ be a minimizer for the energy (0.1) over $H^1_0(G, C)$, with $G, g$ as above, $d > 0$ and $J$ satisfying $H1$-$H2$-$H3$. Then

i) for a subsequence $\varepsilon_n \to 0$ we have

$$u_{\varepsilon_n} \to u_* = e^{i\phi} \prod_{j=1}^N \left(\frac{z - b_j}{|z - b_j|}\right)^{d_j} \text{ in } C^{1,\alpha}_{loc}(\overline{G} \setminus \{b_1, ..., b_N\})$$

for every $\alpha < 1$, where the $N$ distinct points $\{b_1, ..., b_N\}$ lie in $\Lambda$, $\sum_{j=1}^N d_j = d$ and $\phi$ is a smooth harmonic function determined by the requirement $u_* = g$ on $\partial G$.

ii) Setting

$$I(R) = \frac{1}{2} \int_0^1 \frac{j(t)}{t} \frac{J^{-1}(t)}{t} dt$$

we have

$$E_{\varepsilon_n}(u_{\varepsilon_n}) = 2\pi p_0 d \log \frac{1}{\varepsilon_n} + 2\pi p_0 \left(\sum_{k=1}^N \frac{d_k^2 - d_k}{s_k}\right) \log \log \frac{1}{\varepsilon_n}

- 2\pi p_0 d I\left(\frac{1}{\varepsilon_n} \left(\log \frac{1}{\varepsilon_n}\right)^{\frac{1}{s_k}}\right) + o\left(I\left(|\log\varepsilon|\right)^{\frac{1}{s_k}}\right).$$

(0.8)

As it is showed in [12], $\lim_{R \to \infty} \frac{I(R)}{\log R} = 0$ hence the leading term in the energy is always of order $o(|\log\varepsilon|)$. Moreover it is easy to see that $I(R)$ is a positive, monotone increasing, concave function of $\log R$ for $R$ large (see [12]). The proof of Theorem 1 consists of two main ingredients: the method of Struwe [18] as used also in [9] in order to locate the "bad discs", (i. e. a finite collection of discs of radius $O(\varepsilon)$ which cover the set $\{x : |u_\varepsilon(x) < \frac{1}{2}\}$) and the generalization of a result of Brezis, Merle and Rivi`ere [7] which will play an important role in finding the lower bound of the energy. More precisely in Theorem 2 we will bound from below the energy of a regular map defined away from some points $a_1, a_2, ..., a_m$ in $B_R(0)$ such that $0 < a \leq |u| \leq 1$ in $\Omega$, $\deg(u, \partial B_R(a_j)) = d_j$ and with a bound potential by using the reference map $u_0(z) = \left(\frac{z - a_1}{|z - a_1|}\right)^{d_1} \left(\frac{z - a_2}{|z - a_2|}\right)^{d_2} \cdots \left(\frac{z - a_m}{|z - a_m|}\right)^{d_m}$. After the results of [7], Han and Shafrir, Jerrard, Sandier, Struwe obtained the essential lower bounds for the Dirichlet energy of a unit vector field, see [14], [15], [17] and [18]. The paper is organized as follows. In Section 1, we recall some definitions and results contained in [12]. Section 2 is devoted to prove the generalization of Theorem 4 of [7]
which will be useful for obtaining a precise lower bound of the energy for our case. In Section 3 we prove Theorem \ref{thm:main} by stating an upper and a lower bound for the energy \eqref{eq:energy bound}. In particular we estimate the distance between each singularity and centers of bad discs. Finally, as a corollary of upper and lower bounds of the energy, we find an estimate of the mutual distances between bad discs approaching the same singularity $b_k$.

1 Recalls

In this section we recall some results proved in \cite{ref12} useful in the sequel. Let us consider the following quantity, introduced in \cite{ref12} which will play an important role in our study

$$I(R, c) = \sup \left\{ \int_1^R \frac{1 - f^2}{r} \, dr : \int_1^R J (1 - f^2) r \, dr \leq c \right\} \quad (1.1)$$

for any $R > 1$ and $c > 0$.

**Lemma 1.1.** For every $R > 0$ and $c > 0$, there exists a maximizer $f_0 = f_0^{(R)}$ in \eqref{eq:energy bound} satisfying $0 \leq f_0(r) \leq 1$ for every $r$ such that $f_0(r)$ is nondecreasing. Moreover, if $r_0 = r_0(c)$ is defined by the equation

$$c = J(1) \left( \frac{r_0^2 - 1}{2} \right),$$

then there exists $\tilde{r}_0 = \tilde{r}_0(c, R) \in [1, r_0]$ such that

$$f_0(r) \begin{cases} 
= 0 & \text{if } r \in [1, R] \text{ and } r < \tilde{r}_0, \\
> 0 & \text{if } r > \tilde{r}_0.
\end{cases}$$

Furthermore

$$\int_1^R J (1 - f_0^2) r \, dr = c, \ \forall R > r_0$$

and

$$j (1 - f_0^2) = \frac{1}{\lambda r^2}, \ r > \tilde{r}_0$$

for some $\lambda = \lambda(R, c) > 0$.

Moreover it holds

**Lemma 1.2.** There exist two constants $\kappa_1 > 0, \kappa_2 > 0$ such that

$$\kappa_1 \min (1, \frac{1}{c}) \leq \lambda \leq \kappa_2 (1 + \frac{1}{c}), \ R \geq r_0 + 1.$$ 

Actually, the proof of the previous lemma shows that the estimate of $\lambda$ is uniform for $c$ lying in a bounded interval.
Lemma 1.3. There exists $\kappa > 0$ such that for every $c_1, c_2 \neq 0$ we have

$$|I(R, c_1) - I(R, c_2)| \leq \kappa \max \left( |\ln\left(\frac{1 + \frac{1}{c_1}}{\min(1, \frac{1}{c_2})}\right)|, |\ln\left(\frac{\min(1, \frac{1}{c_1})}{1 + \frac{1}{c_2}}\right)| \right) \quad \forall R \geq 1.$$  

In view of Lemma 1.3 it is natural to set

$$I(R) = I(R, 1)$$

and we have

$$|I(R, c) - I(R)| \leq \kappa \ln(1 + \frac{1}{c}), \quad \forall R \geq 1.$$ 

We recall some properties of $I(R)$.

Lemma 1.4. We have

$$I(R) = \frac{1}{2} \int_{\frac{1}{R^2}}^{\frac{1}{j(\eta_0)}} \frac{j^{-1}(t)}{t} dt \quad \forall R \geq 1. \quad \text{(1.2)}$$

In particular,

$$\lim_{R \to \infty} \frac{I(R)}{\log R} = 0. \quad \text{(1.3)}$$

Moreover for every $\alpha > 0$ there exists a constant $C_1(\alpha)$ such that

$$|I(\alpha R) - I(R)| \leq C_1(\alpha) \quad \text{(1.4)}$$

for $R > \max \left(1, \frac{1}{\alpha}\right)$ and $c \in (0, c_0]$. The next lemma provides an estimate we shall use in the proof of the upper bound in subsection 3.1.

Lemma 1.5. We have

$$\int_{\mu_0}^{R} \left(f_0’\right)^2 \leq C, \quad \forall R > \mu_0$$

where $\mu_0 = \max \left(r_0(1), \frac{1}{\sqrt{aj(\rho_0)}}\right)$ being $r_0(1)$ and $\rho_0$ defined respectively as in Lemma 1.1 and Lemma 1.2.

In Theorem 1 we will need a similar functional to that of (1.1). Hence for $R > 1$ and $c > 0$ we set

$$\tilde{I}(R, c) = \sup \left\{ \int_{1}^{R} \frac{1 - f^2}{r} dr + 4 \int_{1}^{R} \frac{(1 - f^2)^2}{r} dr : \int_{1}^{R} J(1 - f^2) r dr \leq c \right\}. \quad \text{(1.5)}$$

Now, let us recall an important relation between the two functionals (1.1) and (1.5).
Lemma 1.6. There exists a constant $C = C(c)$ such that
\[ |\tilde{I}(R,c) - I(R,c)| \leq C \] (1.6)
for $R > 1$.

The next two propositions will play an important role in the proof of our lower bound in subsection 3.2.

Proposition 1.1. Let $A_{R_1,R_2}$ denote the annulus $\{ R_1 < |x| < R_2 \}$ and let $u \in C^1(A_{R_1,R_2}, \mathbb{C}) \cap C(A_{R_1,R_2}, \mathbb{C})$ satisfy
\[ \deg (u, \partial B_R(0)) = d, \quad j = 1, 2, \]
\[ \frac{1}{2} \leq |u| \leq 1 \text{ on } A_{R_1,R_2} \]
and
\[ \frac{1}{R_2^2} \int_{A_{R_1,R_2}} J (1 - |u|^2) \leq c_0, \]
for some constant $c_0$. Then there exists a constant $c_1$ depending only on $c_0$ such that
\[ \int_{A_{R_1,R_2}} |\nabla u|^2 \geq 2\pi d^2 \left( \log \frac{R_2}{R_1} - I \left( \frac{R_2}{R_1} \right) \right) - d^2 c_1. \]
Moreover, by Proposition 11 in [12] is established a lower bound in a more general perforated domain.

Proposition 1.2. Let $x_1, x_2, ..., x_m$ be $m$ points in $B_\sigma(0)$ satisfying
\[ |x_i - x_j| \geq 4\delta, \forall i \neq j \text{ and } |x_i| < \frac{\sigma}{4}, \forall i, \]
with $\delta \leq \frac{\sigma}{32}$. Set $\Omega = B_\sigma(0) \setminus \bigcup_{j=1}^{m} B_\delta(x_j)$ and let $u$ be a $C^1$-map from $\Omega$ into $\mathbb{C}$, which is continuous on $\partial \Omega$ satisfying
\[ \deg (u, \partial B_\sigma(x_j)) = d_j, \quad \forall j \]
\[ \frac{1}{2} \leq |u| \leq 1 \text{ in } \Omega \]
and
\[ \frac{1}{\delta^2} \int_{\Omega} J (1 - |u|^2) \leq K. \]
Then denoting $d = \sum_{j=1}^{m} d_j$ we have
\[ \int_{\Omega} |\nabla u|^2 \geq 2\pi |d| \left( \log \frac{\sigma}{\delta} - I \left( \frac{\sigma}{\delta} \right) \right) - C \]
with $C = C \left( K, m, \sum_{j=1}^{m} |d_j| \right)$.
2 Lower bound for the energy of unit vector fields

In this section we will generalize Theorem 4 of [7].

Theorem 2. Let \( a_1, a_2, \ldots, a_m \) be \( m \) points in \( B_R(0) \) such that

\[
|a_i - a_j| \geq 4R_0, \quad \forall i \neq j, \tag{2.1}
\]
\[
|a_i| \leq \frac{R}{2}, \quad \forall i, \tag{2.2}
\]
\[
R_0 \leq \frac{R}{4}. \tag{2.3}
\]

Set \( \Omega = B_R(0) \setminus \bigcup_{j=1}^m B_{R_0}(a_j) \) and let \( u \) be a \( C^1 \) map from \( \Omega \) into \( \mathbb{C} \) which is continuous on \( \partial \Omega \) satisfying

\[
0 < \alpha \leq |u| \leq 1 \quad \text{in} \quad \Omega \quad \text{and} \quad \deg(u, \partial B_R(a_j)) = d_j \quad \forall j \tag{2.4}
\]

and

\[
\frac{1}{R_0^2} \int_\Omega J \left( 1 - |u|^2 \right) \leq K. \tag{2.5}
\]

Moreover let \( d = \sum_{j=1}^m |d_j| \).

We consider the map

\[
\begin{align*}
\quad u_0(z) = & \left( \frac{z - a_1}{|z - a_1|} \right)^{d_1} \left( \frac{z - a_2}{|z - a_2|} \right)^{d_2} \cdots \left( \frac{z - a_m}{|z - a_m|} \right)^{d_m} \quad \tag{2.6}
\end{align*}
\]

Then we have

\[
\int_\Omega |\nabla u|^2 \geq p_0 \int_\Omega |\nabla u_0|^2 - 2\pi p_0 \sum_{i=1}^m d_i^2 \left( \frac{R}{R_0} - 2\pi (1 - \alpha^2) p_0 \sum_{i \neq j} |d_i| |d_j| \log \frac{R}{|a_i - a_j|} \right) \tag{2.7}
\]

where \( C \) is a constant depending only on \( p_0, \alpha, d, m \) and \( K \).

Proof Let us set \( \rho = |u| \) so that \( u = \rho e^{i\varphi} \) locally in \( \Omega \). Of course we have \( |\nabla u|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \varphi|^2 \). Now let us set \( u_0 = e^{i\varphi_0} \) locally in \( \Omega \) so that \( |\nabla u_0| = |\nabla \varphi_0| \) where

\[
\nabla \varphi_0(z) = \sum_{i=1}^m d_i \frac{V_i(z)}{|z - a_i|}
\]

\( V_i(z) \) being the unit tangent to the circle of radius \( |z - a_i| \) centered in \( a_i \), namely

\[
V_i(z) = \left( \frac{-y - a_i}{|z - a_i|}, \frac{x - a_i}{|z - a_i|}, \frac{z - a_i}{|z - a_i|} \right).
\]
Finally we can write \( u = \rho u_0 e^{i\psi} \) where \( \psi = \varphi - \varphi_0 \). Then

\[
|\nabla u|^2 = |\nabla \rho|^2 + \rho^2 |\nabla \varphi_0 + \nabla \psi|^2
\]  

(2.8)

By (2.8) we have

\[
\int_{\Omega} p |\nabla u|^2 \geq p_0 \int_{\Omega} |\nabla u|^2 = p_0 \int_{\Omega} |\nabla \rho|^2 + p_0 \int_{\Omega} \rho^2 |\nabla \varphi_0|^2 + p_0 \int_{\Omega} \rho^2 |\nabla \psi|^2 + 2p_0 \int_{\Omega} \rho^2 \nabla \varphi_0 \nabla \psi.
\]

By adding and subtracting one in the second and fourth integral we get

\[
\int_{\Omega} p |\nabla u|^2 \geq p_0 \int_{\Omega} (\rho^2 - 1) |\nabla \varphi_0|^2 + p_0 \int_{\Omega} |\nabla \varphi_0|^2 + p_0 \int_{\Omega} \rho^2 |\nabla \psi|^2 + 2p_0 \int_{\Omega} (\rho^2 - 1) \nabla \varphi_0 \nabla \psi + 2p_0 \int_{\Omega} \nabla \varphi_0 \nabla \psi.
\]

By (2.4) we can write

\[
\int_{\Omega} p |\nabla u|^2 \geq -p_0 \int_{\Omega} (1 - \rho^2) |\nabla \varphi_0|^2 + p_0 \int_{\Omega} |\nabla \varphi_0|^2 + p_0 a^2 \int_{\Omega} |\nabla \psi|^2
\]

\[+ 2p_0 \int_{\Omega} (\rho^2 - 1) \nabla \varphi_0 \nabla \psi + 2p_0 \int_{\Omega} \nabla \varphi_0 \nabla \psi.
\]

(2.9)

Using \( 2AB \geq -|A|^2 - |B|^2 \), for \( A = 2(\rho^2 - 1) \nabla \varphi_0 \) and \( B = \frac{\nabla \psi}{2} \), (2.9) becomes

\[
\int_{\Omega} p |\nabla u|^2 \geq -p_0 \int_{\Omega} (1 - \rho^2) |\nabla \varphi_0|^2 + p_0 \int_{\Omega} |\nabla \varphi_0|^2 + p_0 a^2 \|\nabla \psi\|^2_2
\]

\[- 4p_0 \int_{\Omega} (\rho^2 - 1)^2 |\nabla \varphi_0|^2 - \frac{p_0}{4} \|\nabla \psi\|^2_2 + 2p_0 \int_{\Omega} \nabla \varphi_0 \nabla \psi.
\]

(2.10)

As in Theorem 4 of [8]

\[
\int_{\Omega} \nabla \varphi_0 \nabla \psi \leq C m |d| \|\nabla \psi\|^2_2,
\]

(2.11)

for some universal constant \( C \), (2.10) becomes

\[
\int_{\Omega} p |\nabla u|^2 \geq p_0 \int |\nabla u_0|^2 - \left[ p_0 \int (1 - \rho^2) |\nabla \varphi_0|^2 + 4p_0 \int (\rho^2 - 1)^2 |\nabla \varphi_0|^2 \right]
\]

\[+ p_0 \left( a^2 - \frac{1}{4} \right) \|\nabla \psi\|^2_2 - 2p_0 C m |d| \|\nabla \psi\|^2_2.
\]

(2.12)

Now let us consider the following function

\[
y = \left( a^2 - \frac{1}{4} \right) X^2 - 2C m |d| X.
\]

If \( a > \frac{1}{2} \), it reaches its minimum value \( y_{\text{min}} = -\frac{C^2 m^2 |d|^2}{a^2 - \frac{1}{4}} \) at \( X = \frac{C m |d|}{a^2 - \frac{1}{4}} \). Then we get

\[
\int_{\Omega} p |\nabla u|^2 \geq p_0 \int |\nabla u_0|^2 - p_0 \left[ \int (1 - \rho^2) |\nabla \varphi_0|^2 + 4 \int (\rho^2 - 1)^2 |\nabla \varphi_0|^2 \right] - C
\]

(2.13)
where \( C \) is a constant depending only on \( p_0, a, d \) and \( m \).

Taking into account definitions of functionals (1.1) and (1.3) and relation (1.6) it is enough to estimate

\[
\int_{\Omega} (1 - \rho^2) |\nabla \varphi_0|^2. \tag{2.14}
\]

To this aim let us observe that

\[
\nabla \varphi_0(z) = \frac{d_1}{|z - a_1|} V_1(z) \Rightarrow |\nabla \varphi_0(z)|^2 = \sum_{i=1}^{m} \frac{d_i^2}{|z - a_i|^2} + \sum_{i \neq j} d_id_j \frac{d_i^2}{|z - a_i||z - a_j|}.
\]

Then (2.14) can be written as

\[
\int_{\Omega} (1 - \rho^2) |\nabla \varphi_0|^2 = \int_{\Omega} (1 - \rho^2) \left[ \sum_{i=1}^{m} \frac{d_i^2}{|z - a_i|^2} + \sum_{i \neq j} \frac{d_id_j}{|z - a_i||z - a_j|} \right] dz
\]

\[
\leq \sum_{i=1}^{m} d_i^2 \int_{\Omega} \frac{1 - \rho^2}{|z - a_i|^2} dz + \sum_{i \neq j} d_id_j \int_{\Omega} \frac{1 - \rho^2}{|z - a_i||z - a_j|} dz = \sum_{i=1}^{m} d_i^2 A_i + B. \tag{2.15}
\]

Now we want to analyze each term separately. Then for every \( i = 1, \ldots, m \) let us introduce

\( \delta_i = dist (a_i, \partial B_R(0)) \) and observe that by (2.3) we get \( \frac{R}{2} \leq \delta_i \leq R \).

Therefore let us fix \( i \) and by definition (1.5) get

\[
A_i = \int_{\Omega} \frac{1 - \rho^2}{|z - a_i|^2} dz \leq \int_{B_R(0) \setminus B_{R_0}(a_i)} \frac{1 - \rho^2}{|z - a_i|^2} dz \leq 2\pi I \left( \frac{\delta_i}{R_0} \right) \leq 2\pi I \left( \frac{R}{R_0} \right) + C \tag{2.16}
\]

where \( C \) is independent of \( R, R_0 \) and \( a_i \). For the second term, acting as in Theorem 3 of [7] and using (2.4) we obtain

\[
|B| \leq \sum_{i \neq j} |d_i||d_j| \int_{\Omega} \frac{1 - \rho^2}{|z - a_i||z - a_j|} dz \leq 2\pi (1 - a^2) \sum_{i \neq j} |d_i||d_j| \log \frac{R}{|a_i - a_j|} + C. \tag{2.17}
\]

where \( C \) depends only on \( m \) and \( d \). Then by putting together (2.16) and (2.17) into (2.15) we get

\[
\int_{\Omega} (1 - \rho^2) |\nabla \varphi_0|^2 \leq \frac{2}{2\pi} \left( \sum_{i=1}^{m} d_i^2 \right) I \left( \frac{R}{R_0} \right) + 2\pi (1 - a^2) \sum_{i \neq j} |d_i||d_j| \log \frac{R}{|a_i - a_j|} + C. \tag{2.18}
\]

where \( C \) doesn’t depend on \( R, R_0 \) and \( a_i \) for every \( i = 1, \ldots, m \).

Finally, by (2.13) and (2.18) we get (2.7)
Corollary 2.1. With the same hypotheses of Theorem 2 we get
\[
\int_{\Omega} p|\nabla u|^2 \geq 2\pi p_0 \left( \sum_{i=1}^{m} d_i^2 \right) \left( \log \frac{R}{R_0} - I \left( \frac{R}{R_0} \right) \right) \\
+ 2\pi p_0 \sum_{i \neq j} \left( - (1 - a^2) |d_i| |d_j| + d_id_j \right) \log \frac{R}{|a_i - a_j|} - C,
\]
(2.19)
in particular, if \( d_i \geq 0 \) for \( i = 1, ..., m \) then we obtain
\[
\int_{\Omega} p|\nabla u|^2 \geq 2\pi p_0 \left( \sum_{i=1}^{m} d_i^2 \right) \left( \log \frac{R}{R_0} - I \left( \frac{R}{R_0} \right) \right) \\
+ 2\pi p_0 a^2 \sum_{i \neq j} d_id_j \log \frac{R}{|a_i - a_j|} - C
\]
where \( C \) is a constant depending only on \( p_0, a, d, m \) and \( K \).

Proof It is an immediate consequence of (2.7) and Theorem 5 of [7].

3 Proof of Theorem 1

Throughout this section, for any subdomain \( D \) of \( G \) we shall use the notation
\[
E_{\varepsilon} (u, D) = \int_D p|\nabla u|^2 + \frac{1}{\varepsilon^2} \int_D J (1 - |u|^2)
\]
(3.1)
and if \( D = G \) we simply write \( E_{\varepsilon}(u) \).

Our main result of this section is the asymptotic behavior of the energy for minimizers

Proposition 3.1. Assume \[.4] and \[.5] hold true. Then for a subsequence \( \varepsilon_n \to 0 \) we have
\[
E_{\varepsilon_n} (u_{\varepsilon_n}) = 2\pi p_0 d \log \frac{1}{\varepsilon_n} + 2\pi p_0 \left( \sum_{k=1}^{N} \frac{d_k^2}{s_k} - \frac{d_k}{s_k} \right) \log \log \frac{1}{\varepsilon_n} \\
- 2\pi p_0 a \left( \frac{1}{\varepsilon_n} \left( \log \frac{1}{\varepsilon_n} \right) - \frac{1}{s_k} \right) + o \left( I \left( \log \log \frac{1}{\varepsilon_n} \right) \right).
\]
(3.2)
This gives \[.8\] of Theorem 1

3.1 An upper bound for the energy

Throughout this subsection we shall assume that \( p(x) \) satisfies the two conditions \[.4\] and \[.5\] (see the introduction). If \( G \) is starshaped
\[
\frac{1}{\varepsilon^2} \int_{G} J (1 - |u_{\varepsilon}|^2) \leq C_0, \ \forall \varepsilon > 0.
\]
(3.3)
In the following we shall show that the assumption of starshapeness of the domain can be dropped, by applying an argument of del Pino and Felmer [9].

Let us prove an upper bound for the functional \(0.1\). To this aim, we fix a positive \(\eta_0\) satisfying

\[
0 < \eta_0 < \frac{1}{4} \min \left( \min_{i \neq j} |b_i - b_j|, \min_{i=1,\ldots,N} \text{dist} \left( b_i, \partial G \right) \right).
\]

**Proposition 3.2.** Let us suppose that \(0.4\) and \(0.5\) hold true. Then for a subsequence \(\varepsilon_n \to 0\) we have

\[
E_{\varepsilon_n}(u_{\varepsilon_n}) \leq 2\pi p_0 d \log \frac{1}{\varepsilon_n} + 2\pi p_0 \left( \sum_{i=1}^{N} \frac{d_i^2 - d_k}{s_k} \right) \log \log \frac{1}{\varepsilon_n} - 2\pi p_0 d I \left( \frac{1}{\varepsilon_n}, \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \right) + O(1). \tag{3.4}
\]

**Proof.** Let us fix \(k = 1,\ldots,N\). From this point onwards the proof will develop into three steps.

**Step 1.** Let \(\vartheta_k\) denote a polar coordinate around \(b_k\). Set \(T_{k,\varepsilon_n} = \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}}\). We define

\[
U^k_{\varepsilon_n}(x) = \left( \frac{x - b_k}{|x - b_k|} \right)^{d_k} \quad \text{on } B_{\varepsilon_0}(b_k) \setminus \overline{B_{T_{k,\varepsilon_n}}(b_k)}.
\]

It is very easy to show that

\[
E_{\varepsilon_n} \left( U^k_{\varepsilon_n}, B_{\varepsilon_0}(b_k) \setminus \overline{B_{T_{k,\varepsilon_n}}(b_k)} \right) \leq 2\pi p_0 \frac{d_k^2}{s_k} \log \log \frac{1}{\varepsilon_n} + O(1). \tag{3.6}
\]

**Step 2.** Let us fix \(d_k\) equidistant points \(x_{n_1}^k, x_{n_2}^k, \ldots, x_{n_{d_k}}^k\) on the circle \(\partial B_{T_{k,\varepsilon_n}}(b_k)\). On \(A_{\varepsilon_n} = B_{T_{k,\varepsilon_n}}(b_k) \setminus \bigcup_{j=1}^{d_k} B_{\varepsilon_n / 10d_k} \left( x_{n_j}^k \right)\) we define \(U^k_{\varepsilon_n}\) as an \(S^1\)-valued map which minimizes the energy

\[
\int_{A_{\varepsilon_n}} p |\nabla u|^2 \quad \text{among } S^1\text{-valued maps for the boundary data } \left( \frac{x - b_j}{|x - b_j|} \right)^{d_k} \quad \text{on } \partial B_{T_{k,\varepsilon_n}}(b_k), \quad \frac{x - x_j}{|x - x_j|} \quad \text{on } \partial B_{\varepsilon_n / 10d_k} \left( x_{n_j}^k \right), \quad j = 1,\ldots,d_k.
\]

Clearly we have

\[
E_{\varepsilon_n} \left( U_{\varepsilon_n}(x), A_{\varepsilon_n} \right) \leq C. \tag{3.7}
\]

Now, let us fix \(j \in \{1,\ldots,d_k\}\), let \(\vartheta_j\) denote a polar coordinate around \(x_j\) and let \(f_0(r)\) be a maximizer for \(I \left( \frac{1}{\varepsilon_n}, \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \right)\) as given by Lemma [11]. On each \(B_{\varepsilon_n / 20d_k} \left( x_j \right)\) we define the following function

\[
U^{j,k}_{\varepsilon_n}(x) = \begin{cases} \left( \frac{|x - x_j|}{\lambda_x} \right) f_0(\lambda_x) e^{i\vartheta_j} & \text{on } B_{\lambda_x,\varepsilon_n}(x_j) \\ f_0 \left( \frac{|x - x_j|}{\varepsilon_n} \right) e^{i\vartheta_j} & \text{on } B_{\varepsilon_n / 20d_k} \left( x_j \right) \setminus B_{\lambda_x,\varepsilon_n}(x_j) \\ \left( f_0 \left( \frac{T_{n} \varepsilon_n}{20d_k \varepsilon_n} \right) + \left( 1 - f_0 \left( \frac{T_{n} \varepsilon_n}{20d_k \varepsilon_n} \right) \right) \right)^{e^{i\vartheta_j}} & \text{on } B_{\varepsilon_n / 10d_k} \left( b_k \right) \setminus B_{\varepsilon_n / 20d_k} \left( b_k \right). \end{cases} \tag{3.8}
\]
In this step we prove that

$$E_{\varepsilon_n} \left( U_{\varepsilon_n}^{j,k}, B_{\lambda \varepsilon_n} (x_j) \right) \leq -2 \pi p_0 \frac{1}{s_k} \log \log \frac{1}{\varepsilon_n} + 2 \pi p_0 \log \frac{1}{\varepsilon_n} - 2 \pi p_0 I \left( \frac{1}{\varepsilon_n} \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \right) + O(1).$$

(3.9)

To this aim let us observe that of course we have

$$E_{\varepsilon_n} \left( U_{\varepsilon_n}^{j,k}, B_{\lambda \varepsilon_n} (x_j) \right) = O(1).$$

(3.10)

By putting $U_{\varepsilon_n}^{j,k}(x)$ in the energy we obtain

$$E_{\varepsilon_n} \left( U_{\varepsilon_n}^{j,k}, B_{\lambda \varepsilon_n} (x_j) \right) = 2 \pi \int_{\lambda \varepsilon_n}^{\lambda_{10}} p f_0^2 r dr + 2 \pi \int_{\lambda \varepsilon_n}^{\lambda_{10}} \frac{f_0^2}{r} dr + \frac{2 \pi p_0}{\varepsilon_n} \int_{\lambda \varepsilon_n}^{\lambda_{10}} J \left( 1 - f_0^2 \right) r dr.$$ 

(3.13)

Using Lemma 1.3 and (3.3) we have

$$\int_{\lambda \varepsilon_n}^{\lambda_{10}} p f_0^2 r dr \leq C$$

(3.11)

and

$$\frac{1}{\varepsilon_n} \int_{\lambda \varepsilon_n}^{\lambda_{10}} J \left( 1 - f_0^2 \right) r dr \leq C.$$ 

(3.12)

So let us consider only the term (a)

$$(a) = 2 \pi \int_{\lambda \varepsilon_n}^{\lambda_{10}} p f_0^2 r dr + 2 \pi \int_{\lambda \varepsilon_n}^{\lambda_{10}} \frac{f_0^2}{r} dr + \frac{2 \pi p_0}{\varepsilon_n} \int_{\lambda \varepsilon_n}^{\lambda_{10}} J \left( 1 - f_0^2 \right) r dr.$$ 

(3.13)

By (3.5), Lemma ?? and as $|x - b_k|^s_k \leq 2^s_k (|x - x_j|^s_k + |x_j - b_k|^s_k)$ we have

$$(1) \leq 2^{s_k+1} \pi \beta_k \int_{\lambda \varepsilon_n}^{\lambda_{10}} \left( \log \frac{1}{\varepsilon_n} \right)^{-1} \frac{f_0^2}{r} dr$$

$$\leq \frac{2^{s_k+1}}{(20d_k)^{s_k}} \pi \beta_k \left( \log \frac{1}{\varepsilon_n} \right)^{-1} \int_{\lambda \varepsilon_n}^{\lambda_{10}} \frac{f_0^2}{r} dr$$

$$= 2 \pi \beta_k \left( \frac{1}{10^s k d_k^{s_k} + 1} \right) \left( \log \frac{1}{\varepsilon_n} \right)^{-1} \int_{\lambda \varepsilon_n}^{\lambda_{10}} \frac{f_0^2}{r} dr$$

$$= - 2 \pi \beta_k \left( \frac{1}{10^s k d_k^{s_k} + 1} \right) \left( \log \frac{1}{\varepsilon_n} \right)^{-1} \int_{\lambda \varepsilon_n}^{\lambda_{10}} \frac{1 - f_0^2}{r} dr.$$
Let us observe that

\[
\lim_{n \to +\infty} \left( \log \frac{1}{\varepsilon_n} \right)^{-1} \log \log \frac{1}{\varepsilon_n} = 0
\]

and again by (1.3) that

\[
\lim_{n \to +\infty} \left( \log \frac{1}{\varepsilon_n} \right)^{-1} I \left( \frac{1}{\varepsilon_n} \left( \log \frac{1}{\varepsilon_n} \right)^{-1} \right) = 0.
\]

Then we can conclude

\[ (1) \leq O(1). \]

Now let us consider the other term in (3.13)

\[
(2) = 2\pi p_0 \int_{\lambda \varepsilon_n}^{\tau_{\varepsilon_n}} f_0^2 \frac{dr}{r} = -2\pi p_0 \int_{\lambda \varepsilon_n}^{\tau_{\varepsilon_n}} \frac{1 - f_0^2}{r} \, dr + 2\pi p_0 \int_{\lambda \varepsilon_n}^{\tau_{\varepsilon_n}} \frac{dr}{r}
\]

\[
= -2\pi p_0 I \left( \frac{1}{\varepsilon_n} \left( \log \frac{1}{\varepsilon_n} \right)^{-1} \right) + 2\pi p_0 \left( -\frac{1}{s_k} \log \log \frac{1}{\varepsilon_n} + \log \frac{1}{\lambda \varepsilon_n} \right) + O(1)
\]

\[
= -2\pi p_0 I \left( \frac{1}{\varepsilon_n} \left( \log \frac{1}{\varepsilon_n} \right)^{-1} \right) - 2\pi p_0 \frac{1}{s_k} \log \log \frac{1}{\varepsilon_n} + 2\pi p_0 \log \frac{1}{\varepsilon_n} + O(1).
\]

By collecting together, we have

\[ (a) = (1) + (2) \leq -2\pi p_0 I \left( \frac{1}{\varepsilon_n} \left( \log \frac{1}{\varepsilon_n} \right)^{-1} \right) - 2\pi p_0 \frac{1}{s_k} \log \log \frac{1}{\varepsilon_n} + 2\pi p_0 \log \frac{1}{\varepsilon_n} + O(1). \]

(3.14)

Let us observe that (3.9) will follow from (3.10), (3.11), (3.12) and (3.14) once we prove that

\[ E_{\varepsilon_n} \left( U_{\varepsilon_n}^{j,k} B_{\varepsilon_n} \left( x_j \right) \right) \leq C. \]

(3.15)

In order to verify (3.15) we write,

\[ U_{\varepsilon_n}^{j,k} \left( x_j + re^{i\theta} \right) = z(r)e^{i\theta} \text{ on } B_{\varepsilon_n} \left( x_j \right) \]

(3.16)
where
\[ z(r) = f_0 \left( \frac{T_{\varepsilon_n}}{20d_k\varepsilon_n} \right) + \left( \frac{r - \frac{T_{\varepsilon_n}}{20d_k}}{\frac{T_{\varepsilon_n}}{20d_k}} \right) \left( 1 - f_0 \left( \frac{T_{\varepsilon_n}}{20d_k\varepsilon_n} \right) \right) \]

Acting as in Proposition 3.1 in [12], by the properties of \( f_0 \) of Lemma 1.1 and as \( T_{\varepsilon_n} \) go to zero when \( \varepsilon_n \) tends to zero, we compute

\[
\int_{B_{\frac{T_{\varepsilon_n}}{10d_k}}(x_j) \setminus B_{\frac{T_{\varepsilon_n}}{20d_k}}(x_j)} |\nabla U_{\varepsilon_n}^{j,k}|^2 = \int_{B_{\frac{T_{\varepsilon_n}}{10d_k}}(x_j) \setminus B_{\frac{T_{\varepsilon_n}}{20d_k}}(x_j)} z^2|\nabla \vartheta_k|^2 + 2\pi \int_{\frac{T_{\varepsilon_n}}{20d_k}} (z')^2 \, rdr
\]

\[
= O(1) + 2\pi \left( 1 - f_0 \left( \frac{T_{\varepsilon_n}}{20d_k\varepsilon_n} \right) \right)^2 \int_{\frac{T_{\varepsilon_n}}{20d_k}} rdr \leq C.
\]

About the second term of the energy, using the inequality \( J(t) \leq tj(t) \), Lemma 1.1 and Lemma 1.2, we obtain

\[
\frac{1}{\varepsilon_n^2} \int_{B_{\frac{T_{\varepsilon_n}}{10d_k}}(x_j) \setminus B_{\frac{T_{\varepsilon_n}}{20d_k}}(x_j)} J \left( 1 - |U_{\varepsilon_n}^{j,k}|^2 \right) \leq C \int_{B_{\frac{T_{\varepsilon_n}}{10d_k}}(x_j) \setminus B_{\frac{T_{\varepsilon_n}}{20d_k}}(x_j)} j \left( 1 - |U_{\varepsilon_n}^{j,k}|^2 \right)
\]

\[
\leq C \frac{j \left( 1 - f_0 \left( \frac{T_{\varepsilon_n}}{20d_k\varepsilon_n} \right) \right) \left( \frac{T_{\varepsilon_n}^2}{100d_k^2} - \frac{T_{\varepsilon_n}^2}{400d_k^2} \right)}{\varepsilon_n^2 \lambda}\n
\]

\[
= C \frac{3}{\varepsilon_n^2 \lambda} \left( \frac{T_{\varepsilon_n}}{20d_k\varepsilon_n} \right)^2 = O(1).
\]

Hence by (3.17) and (3.18) we get (3.15).

Finally, by (3.10), (3.11), (3.12), (3.13) and (3.15) we can write

\[
E_{\varepsilon_n} \left( U_{\varepsilon_n}^{j,k}, B_{\frac{T_{\varepsilon_n}}{10d_k}}(x_j) \right) \leq -2\pi p_0 \frac{1}{s_k} \log \log \frac{1}{\varepsilon_n} + 2\pi p_0 \log \frac{1}{\varepsilon_n} - 2\pi p_0 I \left( \frac{1}{\varepsilon_n} \left( \log \frac{1}{\varepsilon_n} \right) \right) + O(1).
\]

**Step 3.** We construct a function \( U_{\varepsilon_n}^k(x) \) defined in \( \bigcup_{j=1}^{d_k} B_{T_{\varepsilon_n}}(x_j) \) such that

\[
U_{\varepsilon_n}^k(x) = U_{\varepsilon_n}^{j,k}(x) \quad \text{if} \quad x \in B_{T_{\varepsilon_n}}(x_j).
\]

As the discs centered in \( x_j \) are disjoint and as they are exactly \( d_k \) discs we get

\[
E \left( U_{\varepsilon_n}^k(x), \bigcup_{j=1}^{d_k} B_{T_{\varepsilon_n}}(x_j) \right) \leq -2\pi p_0 d_k I \left( \frac{1}{\varepsilon_n} \left( \log \frac{1}{\varepsilon_n} \right) \right) - 2\pi p_0 \frac{d_k}{s_k} \log \log \frac{1}{\varepsilon_n} + 2\pi p_0 d_k \log \frac{1}{\varepsilon_n} + O(1).
\]
By (3.6), (3.7) and (3.20) we have

\[
E_{\varepsilon_n}(U_{\varepsilon_n}^k, B_{\eta_0}(b_k)) \leq 2\pi p_0 \frac{d_k^2}{s_k} \log \log \frac{1}{\varepsilon_n} - 2\pi p_0 d_k I \left( \frac{1}{\varepsilon_n} \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \right) - 2\pi p_0 \frac{d_k}{s_k} \log \log \frac{1}{\varepsilon_n} + 2\pi p_0 d_k \log \frac{1}{\varepsilon_n} + O(1). \tag{3.21}
\]

We construct a function $U_{\varepsilon_n}(x)$ defined in $\bigcup_{k=1}^N B_{\eta_0}(b_k)$ such that

\[
U_{\varepsilon_n}(x) = U_{\varepsilon_n}^k(x) \quad \text{if} \quad x \in B_{\eta_0}(b_k)
\]

and we pose $U_{\varepsilon_n}(x) = w$ on $G \setminus \bigcup_{k=1}^N B_{\eta_0}(b_k)$ where $w$ is any $S^1$-valued map of class $C^1$ on this domain which equals $g$ on $\partial G$ and $(\frac{x-b_k}{|x-b_k|})^d_k$ on $\partial B_{\eta_0}(b_k)$ for $k = 1, \ldots, N$. Then $U_{\varepsilon_n} \in H^1_g(G, C)$ and we get

\[
E_{\varepsilon_n}(u_{\varepsilon_n}) \leq E_{\varepsilon_n}(U_{\varepsilon_n}) \leq 2\pi p_0 d \log \frac{1}{\varepsilon_n} + 2\pi p_0 \sum_{k=1}^N \frac{d_k^2 - d_k}{s_k} \log \log \frac{1}{\varepsilon_n} - 2\pi p_0 d I \left( \frac{1}{\varepsilon_n} \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \right) + O(1) \tag{3.22}
\]

which is (3.4).

3.2 A lower bound for the energy

Let us note that by following the same arguments of Lemma 3.1 and Lemma 3.2 in [12] we get

\[
\|u_{\varepsilon}\|_{L^\infty(G)} \leq 1 \quad \text{and} \quad \|\nabla u_{\varepsilon}\|_{L^\infty(G)} \leq \frac{C}{\varepsilon}. \tag{3.23}
\]

Using the construction in [3] we know that there exists $\lambda > 0$ and a collection of balls $\left\{ B_{\lambda\varepsilon}(y_j^\varepsilon) \right\}_{j \in J}$ such that

\[
\left\{ x \in G : |u_{\varepsilon}(x)| \leq \frac{3}{4} \right\} \subset \bigcup_{j \in J} B_{\lambda\varepsilon}(y_j^\varepsilon), \tag{3.24}
\]

\[
|y_i - y_j| \geq 8\lambda \varepsilon \quad \forall i, j \in J, i \neq j
\]

and

\[
\text{card} J \leq N_b.
\]

By our construction all the degrees

\[
\nu_j = \text{deg} \left( u_{\varepsilon}, \partial B_{\lambda\varepsilon}(y_j) \right), j \in J
\]
are well defined. Given any subsequence $\varepsilon_n \to 0$ we may extract a subsequence (still denoted by $\varepsilon_n$) such that

$$\text{card}J_{\varepsilon_n} = \text{cost} = N_1$$

and

$$y_j = y_{\varepsilon_n}^j \to l_j \in \overline{G}, \ j = 1, \ldots, N_1. \quad (3.25)$$

Let $\overline{b}_1, \overline{b}_2, \ldots, \overline{b}_{N_2}$ be the distinct points among the $\{l_j\}_{j=1}^{N_1}$ and set

$$I_k = \{ j \in \{1, \ldots, N_1 \}; y_{\varepsilon_n}^j \to \overline{b}_k \}, \ k = 1, \ldots, N_2.$$ 

Denoting by $d_k = \sum_{j \in I_k} \nu_j$ for every $k = 1, \ldots, N_2$, we clearly have and $\sum_{k=1}^{N_2} d_k = d$. By following the same arguments as in [1], thanks to the previous upper bound and Proposition 1.2, we get

$$d_k > 0 \text{ for every } k = 1, \ldots, N_2 \quad (3.26)$$

and

$$\overline{b}_k \in p^{-1}(p_0) \text{ for every } k = 1, \ldots, N_2. \quad (3.27)$$

Hence $N_2 = N$. Moreover acting as in [1], Lemma 2.1 by Proposition 1.1 and Proposition 1.2, we get $\nu_j = +1$ for every $j \in I_k$.

As in the previous subsection, we fix $\eta$ satisfying

$$0 < \eta < \frac{1}{2} \min \left( \min_{i \neq j} \mid \overline{b}_i - \overline{b}_j \mid, \ \min_{i=1,\ldots,N_2} \text{dist} (\overline{b}_i, \partial G) \right).$$

Let us recall $T_{\varepsilon_n} = \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}}$. We now are able to prove the following lower bound

**Proposition 3.3.** We have, for a subsequence $\varepsilon_n \to 0$

$$E_{\varepsilon_n} (u_{\varepsilon_n}) \geq 2\pi p_0 d \log \frac{1}{\varepsilon_n} + 2\pi p_0 \sum_{k=1}^{N} \frac{d_k^2 - d_k}{s_k} \log \log \frac{1}{\varepsilon_n} - 2\pi p_0 d I \left( \frac{1}{\varepsilon_n} \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \right)$$

$$- 2\pi p_0 \sum_{k=1}^{N} d_k^2 I \left( \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \right) + \frac{9\pi p_0}{8} \sum_{k=1}^{N} \sum_{i \neq j} \log \frac{T_{\varepsilon_n}}{y_i - y_j} + O(1).$$

**Proof.** The proof developes into two steps.

**Step 1.** At first we prove

$$\max_{i \in I_k} |\overline{b}_k - y_i| \sim |\log \varepsilon_n|^{-\frac{1}{s_k}} \quad (3.29)$$
for every \( i \in I_k \ (1 \leq k \leq N_2) \) with \( |I_k| = d_k > 1 \).

We know that \( B_{\eta} (\overline{b}_k) \) contains exactly \( d_k \) bad discs \( B_{\lambda \varepsilon_n} (y_i), \ i \in I_k \) with \( |y_i - y_j| > \varepsilon_{n}^\alpha \ \forall i \neq j, \forall \alpha \in (0, 1) \). Let us denote by \( R_n = \max_{i \in I_k} |y_i - \overline{b}_k| \). Fixing \( \alpha \in (0, 1) \) we have

\[
E \left( u_{\varepsilon_n}, B_{\eta} (\overline{b}_k) \right) \geq E \left( u_{\varepsilon_n}, B_{\eta} (\overline{b}_k) \setminus B_{2R_n} (\overline{b}_k) \right) + E \left( u_{\varepsilon}, B_{2R_n} (\overline{b}_k) \setminus \bigcup_{i \in I_k} B_{\varepsilon_n} (y_i) \right) + E \left( u_{\varepsilon_n}, \bigcup_{i \in I_k} B_{\varepsilon_n} (y_i) \right) = (a) + (b) + (c).
\]

(3.30)

By Proposition 1.1 there exist two constants \( C_1 \) and \( C_3 \) depending only on \( C_0 \) and a constant \( C_2 \) depending on \( C_0 \) and \( d_k \), such that

\[
(a) \geq 2\pi d_k^2 p_0 \left[ \log \frac{\eta}{2R_n} - I \left( \frac{\eta}{2R_n} \right) \right] - d_k^2 C_1
\]

(3.31)

\[
(b) \geq 2\pi d_k p_0 \left[ \log \frac{2R_n}{\varepsilon_{\alpha}} - I \left( \frac{2R_n}{\varepsilon_{\alpha}} \right) \right] - C_2
\]

(3.32)

\[
(c) \geq 2\pi (d_k - 1) p_0 \left[ \log \frac{\varepsilon_{\alpha}}{\lambda \varepsilon_n} - I \left( \frac{\varepsilon_{\alpha}}{\lambda \varepsilon_n} \right) \right] + 2\pi \left( p_0 + \alpha_k \frac{R_{n}^{s_k}}{4} \right) \left[ \log \frac{\varepsilon_{\alpha}}{\lambda \varepsilon_n} - I \left( \frac{\varepsilon_{\alpha}}{\lambda \varepsilon_n} \right) \right] - C_3,
\]

(3.33)

being \( C_0 \) as in (3.3);

Let us denote

\[
f (R_n) = 2\pi p_0 d_k \log \frac{1}{\varepsilon_{n}} + 2\pi p_0 (d_k^2 - d_k) \log \frac{1}{R_n} + \pi \frac{1}{2} \alpha_k (1 - \alpha) R_{n}^{s_k} \log \frac{1}{\varepsilon_{n}}
\]

and

\[
g (R_n) = 2\pi d_k^2 p_0 I \left( \frac{1}{R_n} \right) + 2\pi d_k p_0 I \left( \frac{R_n}{\varepsilon_{n}} \right) + 2\pi \left( p_0 + \alpha_k \frac{R_{n}^{s_k}}{4} \right) I \left( \frac{1}{\varepsilon_{n} - \alpha} \right) + C_4.
\]

where \( C_4 \) is a constant depending only on \( C_0 \) and \( d_k \). Then

\[
E \left( u_{\varepsilon_n}, B_{\eta} (\overline{b}_k) \right) \geq f (R_n) - g (R_n) - C_4
\]

(3.34)

Note that, for \( n \) large enough, \( R_n \) satisfies the following

\[
\frac{\eta}{2R_n} \geq 1 \quad \text{and} \quad \frac{2R_n}{\varepsilon_{n}^\alpha} \geq 1
\]

which is

\[
\frac{\varepsilon_{n}^\alpha}{2} \leq R_n \leq C_5.
\]

(3.35)
Indeed, the first inequality follows since $R_n$ tends to 0. The second inequality comes from the fact $\varepsilon_n < |y_i - y_j| \leq |y_i - \overline{b}_k| + |y_j - \overline{b}_k| \leq 2R_n$ for $i \neq j$.

Now, let us pose $R_n = c_n \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}}$. Then we have

$$
[f (R_n) - g (R_n)] - \left[ f \left( \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \right) - g \left( \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \right) \right] =
$$

$$
\underbrace{\left[ f (R_n) - f \left( \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \right) \right]}_{(1)} + \underbrace{\left[ g \left( \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \right) - g (R_n) \right]}_{(2)}.
$$

By previous calculation we get

$$
(1) = 2\pi p_0 \left( d_k^2 - d_k \right) \log \frac{1}{c_n} + \pi \alpha_k \left( 1 - \alpha \right) (c_n^{s_k} - 1) \quad (3.36)
$$

and

$$
(2) = \left( I \left( \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \right) - I \left( \frac{1}{R_n} \right) \right) 2\pi p_0 d_k^2 + \left( I \left( \frac{1}{c_n} \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \right) - I \left( \frac{R_n}{c_n} \right) \right) 2\pi p_0 d_k
$$

$$
+ \frac{\pi \alpha_k}{2} \left( \left( \log \frac{1}{\varepsilon_n} \right)^{-1} - R_n^{s_k} \right) I \left( \frac{1}{c_n^{1-\alpha}} \right) = \left( I \left( \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \right) - I \left( \frac{1}{c_n} \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \right) \right) 2\pi d_k^2 p_0
$$

$$
+ \left( I \left( \frac{1}{c_n} \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \right) - I \left( \frac{c_n}{c_n} \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \right) \right) 2\pi p_0 d_k + \frac{\pi \alpha_k}{2} \left( 1 - c_n^{s_k} \right) \left( \log \frac{1}{\varepsilon_n} \right)^{-1} I \left( \frac{1}{c_n^{1-\alpha}} \right) \quad (3.37)
$$

Let us consider the case $c_n > 1$. Therefore we have

$$
R_n > \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}}. \quad (3.38)
$$
By (1.2), (3.3) and as the functions \( j^{-1} \) and \( I \) are increasing, we get

\[
(2) \geq 2 \pi p_0 d_k \left( \int_{\log \frac{1}{\varepsilon_n}}^{\log \frac{1}{\varepsilon_n} \frac{2}{\varepsilon_n} \alpha n} j^{-1}(t) \frac{t}{t} \, dt - \int_{\log \frac{2}{\varepsilon_n} d_k}^{\log \frac{2}{\varepsilon_n} d_k} \frac{t}{t} \, dt \right) + \frac{\pi}{2} \alpha_k (1 - c_n^{s_k}) \left( \log \frac{1}{\varepsilon_n} \right)^{-1} I \left( \frac{1}{\varepsilon_n^{1-\alpha}} \right)
\]

\[
= -2 \pi p_0 d_k \int_{\log \frac{1}{\varepsilon_n}}^{\log \frac{1}{\varepsilon_n} \frac{2}{\varepsilon_n} \alpha n} j^{-1}(t) \frac{t}{t} \, dt + \frac{\pi}{2} \alpha_k (1 - c_n^{s_k}) \left( \log \frac{1}{\varepsilon_n} \right)^{-1} I \left( \frac{1}{\varepsilon_n^{1-\alpha}} \right)
\]

\[
\geq -2 \pi p_0 d_k j^{-1} \left( \log \frac{1}{\varepsilon_n} \frac{2}{\varepsilon_n} \alpha n \right) \log \left( \frac{R_n^2}{\log \frac{1}{\varepsilon_n}} \frac{2}{\varepsilon_n} \alpha n \right) + \frac{\pi}{2} \alpha_k (1 - c_n^{s_k}) \left( \log \frac{1}{\varepsilon_n} \right)^{-1} I \left( \frac{1}{\varepsilon_n^{1-\alpha}} \right)
\]

Since

\[
\lim_{n} \left( \log \frac{1}{\varepsilon_n} \right)^{-2} \frac{2}{\varepsilon_n} \alpha n = 0
\]

and by (1.3)

\[
\lim_{n} \left( \log \frac{1}{\varepsilon_n} \right)^{-1} I \left( \frac{1}{\varepsilon_n^{1-\alpha}} \right) = 0,
\]

by regularity of function \( j^{-1} \) and as \( j^{-1}(0) = 0 \), there exists \( n_0 \) such that for \( n \geq n_0 \) we have

\[
(2) \geq 2 \delta \pi p_0 d_k \log \frac{1}{c_n} + \frac{\pi}{2} \alpha_k (1 - c_n^{s_k}) \gamma.
\]

(3.39)

Then if we pose

\[
h \left( R_n \right) = f \left( R_n \right) - g \left( R_n \right),
\]

by (3.36) and (3.39) by choosing \( \delta = \frac{1}{2} \) and \( \gamma = \frac{1}{2} - \alpha \) we get

\[
h \left( R_n \right) - h \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{\varepsilon_n}} \geq 2 \pi p_0 \left( \frac{d_k^2 - d_k}{2} \right) \log \frac{1}{c_n} + \frac{\pi}{2} \alpha_k (1 - \alpha) (c_n^{s_k} - 1) - \frac{\pi}{2} \alpha_k (c_n^{s_k} - 1) \gamma
\]

\[
\geq 2 \pi p_0 \left( \frac{d_k^2 - d_k}{2} \right) \log \frac{1}{c_n} + \frac{\pi}{8} \alpha_k (1 - \alpha) (c_n^{s_k} - 1).
\]

Hence we can conclude as in (1)

\[
h \left( R_n \right) - h \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{\varepsilon_n}} \rightarrow +\infty \text{ as } c_n \rightarrow +\infty.
\]

(3.40)

Now let us consider the case where there exists a subsequence \( (c_{n_k})_k \), which we still denote by \( (c_n) \), such that \( c_n < 1 \). Therefore we have up to a subsequence

\[
R_n < \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}}.
\]

(3.41)
By (1.2), (3.3) and as the functions \( j^{-1} \) and \( I \) are increasing, we get

\[
\begin{align*}
(2) \geq & 2 \pi p_0 d_k^2 \left( \int_{(\log \frac{1}{\varepsilon_n})}^{\eta_n} \frac{j^{-1}(t)}{t} dt - \int_{R^2_n}^{\eta_n} \frac{j^{-1}(t)}{t} dt \right) + \frac{\pi}{2} \alpha_k (1 - c_n^{\delta_k}) \left( \log \frac{1}{\varepsilon_n} \right)^{-1} I \left( \frac{1}{\varepsilon_n^{1-\alpha}} \right) \\
= & -2 \pi p_0 d_k^2 \int_{R^2_n} \left( \frac{1}{\varepsilon_n} \right)^{\frac{1}{\alpha_k}} \frac{j^{-1}(t)}{t} dt + \frac{\pi}{2} \alpha_k (1 - c_n^{\delta_k}) \left( \log \frac{1}{\varepsilon_n} \right)^{-1} I \left( \frac{1}{\varepsilon_n^{1-\alpha}} \right) \\
\geq & -2 \pi p_0 d_k^2 j^{-1} \left( \left( \log \frac{1}{\varepsilon_n} \right)^{\frac{1}{\alpha_k}} \right) \log \left( \frac{1}{\varepsilon_n} \right) + \frac{\pi}{2} \alpha_k (1 - c_n^{\delta_k}) \left( \log \frac{1}{\varepsilon_n} \right)^{-1} I \left( \frac{1}{\varepsilon_n^{1-\alpha}} \right) \\
\geq & -2 \pi p_0 d_k^2 j^{-1} \left( \left( \log \frac{1}{\varepsilon_n} \right)^{\frac{1}{\alpha_k}} \right) \log \frac{1}{c_n} + \frac{\pi}{2} \alpha_k (1 - c_n^{\delta_k}) \left( \log \frac{1}{\varepsilon_n} \right)^{-1} I \left( \frac{1}{\varepsilon_n^{1-\alpha}} \right).
\end{align*}
\]

Since

\[
\lim_n \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{\alpha_k}} = 0
\]

and by (1.3)

\[
\lim_n \left( \log \frac{1}{\varepsilon_n} \right)^{-1} I \left( \frac{1}{\varepsilon_n^{1-\alpha}} \right) = 0,
\]

similarly to the previous case, by regularity of function \( j^{-1} \) and as \( j^{-1}(0) = 0 \) there exists \( n_0 \) such that for \( n \geq n_0 \) we have

\[
(2) \geq -2 \delta \pi p_0 d_k^2 \log \frac{1}{c_n} + \frac{\pi}{2} \alpha_k (1 - c_n^{\delta_k}) \gamma \geq -2 \delta \pi p_0 d_k^2 \log \frac{1}{c_n}. \tag{3.42}
\]

Then if we pose

\[
h(R_n) = f(R_n) - g(R_n),
\]

by (3.36) and (3.42) we get

\[
h(R_n) - h \left( \left( \log \frac{1}{\varepsilon_n} \right)^{\frac{1}{\alpha_k}} \right) \geq 2 \pi p_0 \left( d_k^2 - d_k - \delta d_k^2 \right) \log \frac{1}{c_n} + \frac{\pi}{2} \alpha_k (1 - \alpha) (c_n^{\delta_k} - 1).
\]

Let us choose \( \delta > 0 \) such that \( d_k^2 - d_k - \delta d_k^2 > 1 \) hence \( \delta < 1 - \frac{1 + d_k}{d_k^2} \). This is possible as \( d_k > 1 \) and then \( 1 - \frac{1 + d_k}{d_k^2} > 0 \)

\[
h(R_n) - h \left( \left( \log \frac{1}{\varepsilon_n} \right)^{\frac{1}{\alpha_k}} \right) \rightarrow +\infty \text{ as } \frac{1}{c_n} \rightarrow +\infty. \tag{3.43}
\]

In both cases we can conclude as in [AS]

\[
h(R_n) - h \left( \left( \log \frac{1}{\varepsilon_n} \right)^{\frac{1}{\alpha_k}} \right) \rightarrow +\infty \text{ as } \max \left( c_n, \frac{1}{c_n} \right) \rightarrow +\infty. \tag{3.44}
\]
By (3.34) we get

\[ h(R_n) - h\left(\left(\log \frac{1}{\varepsilon_n}\right)^{-\frac{1}{s_k}}\right) \leq E(u_{\varepsilon_n}, B_{\eta}(b_k)) + C(\alpha) - h\left(\left(\log \frac{1}{\varepsilon_n}\right)^{-\frac{1}{s_k}}\right). \]

We know that \( b_k = b_j \) for some \( j \in \{1, \ldots, N\} \). Hence by using the upper bound found in the previous section, we obtain

\[
\begin{align*}
    h(R_n) - h\left(\left(\log \frac{1}{\varepsilon_n}\right)^{-\frac{1}{s_k}}\right) & \leq 2\pi p_0 d_k \log \frac{1}{\varepsilon_n} + 2\pi p_0 \frac{d_k^2 - d_k}{s_k} \log \log \frac{1}{\varepsilon_n} - 2\pi p_0 d_k I\left(\frac{1}{\varepsilon_n} \left(\log \frac{1}{\varepsilon_n}\right)^{-\frac{1}{s_k}}\right) + O(1) \\
    & \quad + C_4 - 2\pi p_0 d_k \log \frac{1}{\varepsilon_n} - 2\pi p_0 \frac{d_k^2 - d_k}{s_k} \log \log \frac{1}{\varepsilon_n} - \frac{\pi}{2} \alpha_k (1 - \alpha) \\
    & \quad + 2\pi p_0 d_k I\left(\left(\log \frac{1}{\varepsilon_n}\right)^{-\frac{1}{s_k}}\right) + 2\pi p_0 d_k I\left(\frac{1}{\varepsilon_n} \left(\log \frac{1}{\varepsilon_n}\right)^{-\frac{1}{s_k}}\right) \\
    & \quad + 2\pi \left(p_0 d_k + \frac{\alpha_k}{4} \left(\log \frac{1}{\varepsilon_n}\right)^{-1}\right) I\left(\frac{1}{\varepsilon_n^{-\alpha}}\right) + C_4.
\end{align*}
\]

(3.45)

As \( \alpha < 1 \) we get

\[
\begin{align*}
    h(R_n) - h\left(\left(\log \frac{1}{\varepsilon_n}\right)^{-\frac{1}{s_k}}\right) & \leq -2\pi p_0 d_k I\left(\frac{1}{\varepsilon_n} \left(\log \frac{1}{\varepsilon_n}\right)^{-\frac{1}{s_k}}\right) + 2\pi p_0 d_k^2 I\left(\left(\log \frac{1}{\varepsilon_n}\right)^{-\frac{1}{s_k}}\right) + \\
    & \quad + 2\pi p_0 d_k I\left(\frac{1}{\varepsilon_n^\alpha} \left(\log \frac{1}{\varepsilon_n}\right)^{-\frac{1}{s_k}}\right) + 2\pi \left(p_0 d_k + \frac{\alpha_k}{4} \left(\log \frac{1}{\varepsilon_n}\right)^{-1}\right) I\left(\frac{1}{\varepsilon_n^{-\alpha}}\right) + O(1).
\end{align*}
\]

(3.46)

By monotonicity of the functional \( I \) we get for \( n \) large enough

\[
I\left(\frac{1}{\varepsilon_n} \left(\log \frac{1}{\varepsilon_n}\right)^{-\frac{1}{s_k}}\right) \geq I\left(\frac{1}{\varepsilon_n^\alpha} \left(\log \frac{1}{\varepsilon_n}\right)^{-\frac{1}{s_k}}\right),
\]

\[
I\left(\frac{1}{\varepsilon_n^\alpha} \left(\log \frac{1}{\varepsilon_n}\right)^{-\frac{1}{s_k}}\right) \geq I\left(\left(\log \frac{1}{\varepsilon_n}\right)^{-\frac{1}{s_k}}\right)
\]

and

\[
I\left(\frac{1}{\varepsilon_n} \left(\log \frac{1}{\varepsilon_n}\right)^{-\frac{1}{s_k}}\right) \geq I\left(\frac{1}{\varepsilon_n^{-\alpha}}\right).
\]

Moreover as previously, by (1.3) we have

\[
\lim_{n} \left(\log \frac{1}{\varepsilon_n}\right)^{-1} I\left(\frac{1}{\varepsilon_n^{-\alpha}}\right) = 0.
\]

(3.47)
Hence the leading term of the second member in (3.46) is the negative one and we can conclude that

\[ h(R_n) - h \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \rightarrow -\infty \text{ as } n \rightarrow +\infty. \] (3.48)

This is a contradiction and arguing as in [1], (3.44) directly implies (3.29).

**Step 2.** Let \( \eta \) and \( T_{\varepsilon_n} \) as in Proposition (3.2). We know that \( B_{\eta} (\overline{b}_k) \) contains exactly \( d_k \) bad discs \( B_{\lambda \varepsilon} (y_j), j \in I_k \) satisfying (3.29).

We have

\[
E_{\varepsilon_n} (u_{\varepsilon_n}, B_{\eta} (\overline{b}_k)) \geq E_{\varepsilon_n} (u_{\varepsilon_n}, B_{\eta} (\overline{b}_k) \setminus B_{T_{\varepsilon_n}} (\overline{b}_k)) + \sum_{j \in I_k} E_{\varepsilon_n} (u_{\varepsilon_n}, B_{T_{\varepsilon_n}} (\overline{b}_k) \setminus B_{\lambda \varepsilon} (y_j))
\]

\[= E_1 + E_2. \] (3.49)

By Proposition [1.1] we have

\[ E_1 \geq 2\pi p_0 d_k^2 \log \frac{\eta}{T_{\varepsilon_n}} - 2\pi p_0 d_k I \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} - d_k C_6. \]

where \( C_6 \) is a constant depending only on \( C_0 \).

Then

\[ E_1 \geq 2\pi p_0 d_k^2 \log \log \frac{1}{\varepsilon_n} - 2\pi p_0 d_k I \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} + O(1). \] (3.50)

By Corollary [2.1] applied to \( y_1, ..., y_{d_k} \), as \( \nu_j = \deg (u_{\varepsilon}, \partial B (y_j, \lambda \varepsilon)) = +1 \) for every \( j = 1, ..., d_k \) and \( a = \frac{3}{4} \), we have

\[ E_2 \geq 2\pi p_0 d_k \left( \log \frac{T_{\varepsilon_n}}{\lambda \varepsilon} - I \left( \frac{T_{\varepsilon_n}}{\lambda \varepsilon} \right) \right) + \frac{9\pi p_0}{8} \sum_{i \neq j} \log \frac{T_{\varepsilon_n}}{|y_i - y_j|} - C_7 \]

where \( C_7 \) is a constant depending only on \( d_k, C_0 \), and \( p_0 \) where \( C_0 \) is introduce in (3.3).

Then

\[ E_2 \geq -2\pi p_0 d_k \log \log \frac{1}{\varepsilon_n} + 2\pi p_0 d_k \log \frac{1}{\varepsilon_n} - 2\pi p_0 d_k I \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} \]

\[ + \frac{9\pi p_0}{8} \sum_{i \neq j} \log \frac{T_{\varepsilon_n}}{|y_i - y_j|} + O(1). \] (3.51)

By collecting together (3.50) and (3.51) we obtain

\[
E_{\varepsilon_n} (u_{\varepsilon_n}, B_{\eta} (b_k)) \geq 2\pi p_0 \frac{d_k^2 - d_k}{s_k} \log \log \frac{1}{\varepsilon_n} - 2\pi p_0 d_k I \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} + 2\pi p_0 d_k \log \frac{1}{\varepsilon_n}
\]

\[ - 2\pi p_0 d_k I \left( \log \frac{1}{\varepsilon_n} \right)^{-\frac{1}{s_k}} + \frac{9\pi p_0}{8} \sum_{i \neq j} \log \frac{T_{\varepsilon_n}}{|y_i - y_j|} + O(1). \] (3.52)
Summing over $k$ we have

$$
E_{\varepsilon_n}(u_{\varepsilon_n}) \geq E_{\varepsilon_n}\left(u_{\varepsilon_n}, \bigcup_{k=1}^{N} B_\eta(b_k)\right) \geq 2\pi p_0 d \log \frac{1}{\varepsilon_n} + 2\pi p_0 \sum_{k=1}^{N} \frac{d_k^2 - d_k}{s_k} \log \log \frac{1}{\varepsilon_n}
$$

$$
- 2\pi p_0 \sum_{k=1}^{N} d_k^2 \left(\left(\frac{\log \frac{1}{\varepsilon_n}}{s_k}\right)^{\frac{1}{s_k}}\right) - 2\pi p_0 d I \left(\frac{1}{\varepsilon_n} \left(\log \frac{1}{\varepsilon_n}\right)^{\frac{1}{s_k}}\right)
$$

$$
+ \frac{9\pi p_0}{8} \sum_{k=1}^{N} \sum_{i \neq j} \log \frac{T_{\varepsilon_n}}{|y_i - y_j|} + O(1)
$$

which is (3.28).

By collecting together Proposition 3.2 and Proposition 3.3 we obtain Proposition 3.1.

An argument of del Pino and Felmer in [9] can now be used to show that (3.3) holds without the assumption on the starshapeness of $G$. It is enough to use (3.4) and (3.28) and act as in [12]. Thanks to estimate (3.3), we can now follow the construction of bad discs as in [2] and complete the convergence assertion of Theorem 1. Since the arguments are identical to those of [2] we omit the details. Now Theorem 1 is completely proved.

Finally as a consequence of both the upper and the lower bound respectively written as in (3.21) and (3.52), we get the following estimate of the distance between the centers of bad discs.

**Corollary 3.1.** For every $i \neq j$ in $I_k$ ($1 \leq k \leq N_2$) with $|I_k| = d_k > 1$, we have

$$
\exp\left(-C_8 I\left(\left(\frac{\log \frac{1}{\varepsilon_n}}{s_k}\right)^{\frac{1}{s_k}}\right)\right) \log \varepsilon_n \left|^{\frac{1}{s_k}}\right. |y_i - y_j| \leq C_9 \log \varepsilon_n \left|^{\frac{1}{s_k}}\right.
$$

where $C_8$ and $C_9$ are two constants independent of $\varepsilon$.

**Proof** By lower bound (3.52) we have

$$
\int_{\Omega} p |\nabla u|^2 \geq 2\pi p_0 d_k \log \frac{1}{\varepsilon_n} + 2\pi p_0 \frac{d_k^2 - d_k}{s_k} \log \log \frac{1}{\varepsilon_n} - 2\pi p_0 \Sigma_1^{N} d_k^2 I \left(\left(\frac{\log \frac{1}{\varepsilon_n}}{s_k}\right)^{\frac{1}{s_k}}\right)
$$

$$
- 2\pi p_0 d I \left(\frac{1}{\varepsilon_n} \left(\log \frac{1}{\varepsilon_n}\right)^{\frac{1}{s_k}}\right) + \frac{9\pi p_0}{8} \sum_{i \neq j} \log \frac{T_{\varepsilon_n}}{|y_i - y_j|} + O(1).
$$

By the upper bound (3.21) we get

$$
\sum_{i \neq j} \log \left(\frac{|\log \varepsilon_n|^{\frac{1}{s_k}}}{|y_i - y_j|}\right) \leq C_8 I\left(\left(\frac{\log \frac{1}{\varepsilon_n}}{s_k}\right)^{\frac{1}{s_k}}\right)
$$

now, using (2.27) we obtain the claimed result.
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