On the integrability aspects of nonparaxial nonlinear Schrödinger equation and the dynamics of solitary waves

K. Tamilselvan\textsuperscript{a}, T. Kanna\textsuperscript{a,*}, A. Govindarajan\textsuperscript{b}

\textsuperscript{a}Nonlinear waves Research Lab, PG & Research Department of Physics, Bishop Heber college (Affiliated to Bharathidasan University), Tiruchirappalli-620017, Tamil Nadu, India
\textsuperscript{b}Centre for Nonlinear Dynamics, School of Physics, Bharathidasan University, Tiruchirappalli - 620 024, Tamil Nadu, India

Abstract

The integrability nature of a nonparaxial nonlinear Schrödinger (NNLS) equation, describing the propagation of ultra-broad nonparaxial beams in a planar optical waveguide, is studied by employing the Painlevé singularity structure analysis. Our study shows that the NNLS equation fails to satisfy the Painlevé test. Nevertheless, we construct one bright solitary wave solution for the NNLS equation by using the Hirota’s direct method. Also, we numerically demonstrate the stable propagation of the obtained bright solitary waves even in the presence of an external perturbation in a form of white noise. We then numerically investigate the coherent interaction dynamics of two and three bright solitary waves. Our study reveals interesting energy switching among the colliding solitary waves due to the nonparaxiality.

Keywords:
Bright solitary waves, Integrability, Painlevé analysis, Hirota’s bilinearization method, Nonparaxial NLS, Solitary wave interaction

1. Introduction

The advent of the universal nonlinear Schrödinger (NLS) equation in nonlinear optics has opened an avenue to explore various nonlinear waves like solitons, breathers, rogue waves, shock waves, vortices and so on [1, 2, 3]. This ubiquitous model can be derived from the famous Maxwell’s equations by employing the so-called slowly varying envelope approximation (SVEA) alias paraxial approximation (PA), which is justified only if the scale of spatial and temporal variations is larger than the optical wavelength and optical cycle, respectively. Under this approximation, the second-order derivative of the normalized envelope field, with respect to its longitudinal (propagation) co-ordinate, can be ignored. From a practical point of view, the SVEA holds good when the optical modes are propagating along (or at near-negligible angles with) the reference axis.
with its pulse/beam width being greater than the carrier wavelength. Though the NLS equation naturally describes the pulse propagations in optical fibers [4] with such limitations, the pulses encounter a catastrophic collapse when higher order traverse dimensions are included [5, 6]. It should be noted that the inclusion of nonparaxiality (or spatial group velocity dispersion (S-GVD)) has led to the stable propagation of localized pulses even in higher dimensional NLS equations [5].

In addition to nonlinear optics, this S-GVD or nonparaxial effect appears naturally in the dynamics of exciton-polaritons in a waveguide of semiconductor material such as ZnCdSe/ZnSe superlattice [7]. The underlying governing equation is the NLS equation with spatial dispersion term. This system is formally equivalent to the nonparaxial NLS equation (also referred as nonlinear Helmholtz (NLH) equation) which is routinely used to study nonparaxial localized modes in optical wave guides [8]. In the earlier work of Lax et al., [9], it was attempted to investigate the nonparaxial effect by means of expanding field components as a power series in terms of a ratio of the beam diameter to the diffraction length. Following this work, many studies have been carried out to investigate the dynamics of nonparaxiality in various optical settings like nonparaxial accelerating beams [10], optical and plasmonic sub-wavelength nanostructures devices [11, 12, 13], and in the design of Fresnel type diffractive optical elements [14].

Furthermore, the propagation of nonparaxial solitons has stimulated extensive studies in different nonlinear optical settings such as Kerr media [8], cubic-quintic media [15], power-law media [16], and saturable nonlinear media [17]. The soliton theory has also been formulated in the NLH equation with distinct nonlinearities based on relativistic and pseudo-relativistic framework [18, 19, 20]. The coupled version of the NLH equation has been studied to explore various kinds of nonlinear waves, including elliptic waves and solitary waves [21, 22]. Recently, the study of nonparaxiality has been extended to the intriguing area of PT-symmetric optics [23]. Also, quite recently, the present authors have done a systematic analysis of the modulational instability for the cubic-quintic NLH equation and reported various interesting chirped elliptical and solitary waves with nontrivial chirping behavior [24].

In nonlinear dynamical systems, the challenging problem is to identify new nonlinear integrable/nearly integrable models. This has an important consequence for exploring nontrivial localized nonlinear waves with intriguing dynamical features in different physical media [2, 25, 26]. Moreover, investigations of the integrability nature of dynamical systems have been extended to multiple areas of physics, including fluid dynamics, nonlinear optics, Bose-Einstein condensates, bio-physics and so on. Specifically, one can verify the integrability nature of a nonlinear dynamical equation by using a powerful mathematical tool, namely, Painlevé analysis [27, 28]. The Painlevé analysis is a potential tool among many integrability indicators such as the linear eigenvalue problem, bilinear transformation, Bäcklund transformation, Lax-pair method, and inverse scattering method [2, 29]. Through the Painlevé analysis, the integrability nature has been tested for various nonlinear models [30, 31, 32]. As mentioned earlier, the NNLS equation can serve as a fertile platform for studying dynamics of a wide range of physical systems. In a recent work, the symmetry reductions of the NNLS equation have been obtained by the Lie symmetry analysis [33]. However, the integrability nature of this NNLS equation is yet to be investigated. One of the objectives of this paper is to inspect the integrability nature of the following dimensionless NNLS
\[
\dot{\Psi} + k \frac{\partial^2 \Psi}{\partial z^2} + \frac{1}{2} \frac{\partial^2 \Psi}{\partial t^2} + \gamma |\Psi|^2 \Psi = 0, \tag{1}
\]

where \( \Psi \) is the normalized complex envelope field and normalized space \( z \) and retarded time \( t \) are expressed as \( \frac{Z}{L_D} \) and \( \frac{T}{T_0} \), in which the dispersion length \( L_D \) is determined by \( \frac{T_0^2}{|\beta^2|} \). The parameters \( \beta^2 \) and \( T_o \) account for group velocity dispersion (GVD) and input pulse width, respectively. The term \( \kappa \) refers to nonparaxial parameter which ranges from \( 10^{-2} \) to \( 10^{-4} \) with \( \kappa = \frac{1}{2} k_0 L_D \) (where \( k_0 = \frac{2\pi n_0}{\lambda} \) stands for wavenumber, in which \( n_0 \) is refractive index) \[13\]. Also, the term \( \gamma \) indicates the Kerr nonlinearity co-efficient. In the limit, \( \kappa \to 0 \) the equation (1) reduces to the standard NLS equation.

The paper is organized as follows. In Sec. 2, we carry out the investigation of the integrability nature of the NNLS equation with the aid of Painlevé analysis which consists of three steps, namely calculating the leading-order equation, finding the resonance values and verifying the existence of sufficient number of arbitrary functions without the movable critical singularity manifolds. In Sec. 3, we obtain the bright solitary wave solution by employing the Hirota’s bilinearization method. Following that, the stability of the bright solitary wave solution in the presence of external perturbation is examined by numerical simulation in Sec. 4. In addition, the interaction of nonparaxial solitary waves is analyzed by executing split step Fourier (SSF) method. Finally, we conclude our findings in Sec. 5.

2. Painlevé Singularity Structure Analysis

In order to apply the Painlevé singularity structure analysis to equation (1), we consider the dependent variable and its complex conjugate as \( \Psi = r, \Psi^* = s \). Then the equation (1) and its complex conjugate equation can be rewritten as,

\[
ir_z + \kappa r_{zz} + \frac{1}{2} r_{tt} + \gamma (r^2 s) = 0, \tag{2a}
\]
\[
-is_z + \kappa s_{zz} + \frac{1}{2} s_{tt} + \gamma (s^2 r) = 0. \tag{2b}
\]

The singularity structure analysis of the above equations (2a)–(2b) is carried out by seeking the following generalized Laurent series expansion for the dependent variables in the neighbourhood of the non-characteristic singular manifold \( \phi(z, t) = 0 \) with non-vanishing derivatives i.e., \( \phi(z, t) \neq 0 \) and \( \phi_j(z, t) \neq 0 \):

\[
r = \phi^\alpha \sum_{j=0}^{\infty} r_j(z, t) \phi^j, \quad r_0 \neq 0, \tag{3a}
\]
\[
s = \phi^\beta \sum_{j=0}^{\infty} s_j(z, t) \phi^j, \quad s_0 \neq 0, \tag{3b}
\]

where \( \alpha \) and \( \beta \) are integers yet to be determined. Next, in order to analyze the leading order solution, we restrict the above series as, \( r = r_0 \phi^\alpha \) and \( s = s_0 \phi^\beta \). By using these relations in
equation (2) and balancing the most dominant terms, the unknown values $\alpha$ and $\beta$ are determined as, $\alpha = -1$, and $\beta = -1$, accompanied by the following condition

$$2\kappa \phi_z^2 + \phi_t^2 = -\gamma(r_0s_0).$$  \hspace{1cm} (4)

In equation (4), out of two functions $r_0$ and $s_0$, one is arbitrary.

Next, the resonances (powers at which arbitrary functions can enter into the Laurent series (3)) are obtained by determining the values of $j$ upon substitution of the following equations

$$r = r_0 \phi^{-1} + \cdots + r_j \phi^{j-1},$$  \hspace{1cm} (5a)

$$s = s_0 \phi^{-1} + \cdots + s_j \phi^{j-1}$$  \hspace{1cm} (5b)

into equations (2). By collecting the coefficients of $\phi^{j-3}$, we get

$$\begin{pmatrix} \gamma r_0^2 \\ \gamma s_0^2 \end{pmatrix} - \begin{pmatrix} \gamma r_0^2 & \gamma \gamma r_0^2 \\ \gamma s_0^2 & \gamma s_0^2 \end{pmatrix} = 0,$$

where $\delta = \kappa \phi_z^2 + \frac{1}{2} \phi_t^2$. By setting the above matrix determinant to be zero, we obtain a quartic equation for $j$ as follows

$$j^4 - 6j^3 + 5j^2 + 12j = 0.$$  \hspace{1cm} (7)

The roots of equation (7) are the resonance values and are found to be $j = -1, 0, 3, 4$, where the resonance value $j = -1$ denotes the arbitrariness of the singular manifold $\phi(z, t)$. Except this, all other resonance values are positive as required by the Painlevé test.

2.1. Arbitrary Analysis

The third step is to examine the existence of sufficient number of arbitrary functions at these resonance values without introducing movable critical singular manifolds of the singularity structure analysis. To this end, we expand the dependent variables as follows:

$$r = \frac{r_0}{\phi} + r_1 + r_2 \phi + r_3 \phi^2 + r_4 \phi^3,$$  \hspace{1cm} (8a)

$$s = \frac{s_0}{\phi} + s_1 + s_2 \phi + s_3 \phi^2 + s_4 \phi^3.$$  \hspace{1cm} (8b)

Then, substituting the above equations (8) in equations (2) and collecting the co-efficients at various orders of $\phi$, one can study the arbitrariness of the singularity.

First, collecting the terms at the order $\phi^{-3}$ which corresponds to the resonance value $j = 0$, we obtain

$$2\kappa \phi_z^2 + \phi_t^2 = -\gamma(r_0s_0).$$  \hspace{1cm} (9)

This equation is exactly the same as the leading order equation (4).
Second, collecting the coefficients at the order $\phi^{-2}$, we obtain the following equations which are expressed in matrix form as

$$
\begin{pmatrix}
-2(1 + 2\kappa^2) + \gamma r_0^2 \\
\gamma s_0^2
\end{pmatrix}
\begin{pmatrix} r_1 \\ s_1 \end{pmatrix} = -\begin{pmatrix}
ir_0 \rho_z + \kappa r_0z + 2\kappa r_0z \\
-ir_0 \rho_z + \kappa s_0z + 2\kappa s_0z
\end{pmatrix}.
$$

(10)

Here, we have used the Kruskal ansatz of the form $\phi(z, t) = t + \rho(z)$, with $\rho(z)$ being an arbitrary analytic function to simplify the calculations and the $r_j$ and $s_j$ are functions of $z$ only. We obtain the following expressions for $r_1$ and $s_1$ from the above equation (10).

$$r_1 = \frac{1}{3(1 + 2\kappa^2)^2}[(ir_0 \rho_z + \kappa r_0z + 2\kappa r_0z)(2(1 + 2\kappa^2)) + \gamma r_0^2(-is_0 \rho_z + \kappa s_0z + 2\kappa s_0z)],$$

(11a)

$$s_1 = \frac{1}{3(1 + 2\kappa^2)^2}[(2(1 + 2\kappa^2))(-is_0 \rho_z + \kappa s_0z + 2\kappa s_0z) + \gamma s_0^2(ir_0 \rho_z + \kappa r_0z + 2\kappa r_0z)].$$

(11b)

Thus, there is no arbitrary function at this order.

Similarly, collecting the coefficients at the order $\phi^{-1}$, we obtain

$$
\begin{pmatrix}
-2(1 + 2\kappa^2) + \gamma r_0^2 \\
\gamma s_0^2
\end{pmatrix}
\begin{pmatrix} r_2 \\ s_2 \end{pmatrix} = -\begin{pmatrix}
ir_0 \rho_z + \kappa r_0z + \gamma(r_1^2 s_0 + 2r_0 r_1 s_1) \\
-ir_0 \rho_z + \kappa s_0z + \gamma(s_1^2 r_0 + 2s_0 r_1 s_1)
\end{pmatrix}.
$$

(12)

By solving the above set of algebraic equations, we find that $r_2$ and $s_2$ can be expressed in terms of $r_0$ and $s_0$ as below

$$r_2 = \frac{1}{3(1 + 2\kappa^2)^2}[2(ir_0z + \kappa r_0z + \gamma(r_1^2 s_0 + 2r_0 r_1 s_1))(1 + 2\kappa^2) + \gamma r_0^2(-is_0z + \kappa s_0z + \gamma(s_1^2 r_0 + 2s_0 r_1 s_1))],$$

(13a)

$$s_2 = \frac{1}{3(1 + 2\kappa^2)^2}[2(-is_0z + \kappa s_0z + \gamma(s_1^2 r_0 + 2s_0 r_1 s_1))(1 + 2\kappa^2) + \gamma s_0^2(ir_0z + \kappa r_0z + \gamma(r_1^2 s_0 + 2r_0 r_1 s_1))],$$

(13b)

where the expressions for $r_1$ and $s_1$ are as given in equations (11). The above equations (13) indicate that $r_2$ and $s_2$ are not arbitrary functions. Further, collecting the coefficients at the order $\phi^0$ corresponding to the resonance value $j = 3$, we obtain,

$$s_0 r_3 + r_0 s_3 = -\frac{1}{\gamma r_0} [ir_{1z} + \kappa r_{1zz} + ir_{2z} + \kappa r_{2zz} + 2\kappa r_{2zz} + \gamma r_1(r_1 s_1 + 2r_0 s_2 + 2r_2 s_0) + 2\gamma r_0 r_2 s_1],$$

(14a)

$$s_0 r_3 + r_0 s_3 = -\frac{1}{\gamma s_0} [-is_{1z} + \kappa s_{1zz} - is_{2z} + \kappa s_{2zz} + 2\kappa s_{2zz} + \gamma s_1(r_1 s_1 + 2r_0 s_2 + 2r_2 s_0) + 2\gamma s_0 s_2 r_1].$$

(14b)
By carefully analyzing the right hand sides of expressions (14) by symbolic computation, we note that they become non-identical except for the choice $\kappa = 0$, which corresponds to the result of the standard integrable NLS equation. This clearly indicates the violation of arbitrariness for the resonance $j = 3$, as there is no any arbitrary function. Hence the NNLS equation (1) fails to satisfy the Painlevé property at this stage.

Finally, we move on to collect the coefficients at the order $\phi^1$ and one obtains

$$\begin{pmatrix} 1 + 2\kappa r_2^2 \\ 1 + 2\kappa r_2^2 \end{pmatrix} \begin{pmatrix} r_4 \\ r_0 \end{pmatrix} = A,$$

where the column matrix $A$ is given by

$$A = -\begin{pmatrix} i(r_{2,z} + 2r_3 \rho_z) + \kappa(r_{2,zz} + 2r_3 \rho_{zz} + 4r_{3,\rho z}) + \gamma(r_2 s_2 + 2r_0 r_2 s_2 + 2r_1 s_1 r_2 \\
-\frac{r_0^2}{(1 + 2\kappa r_2^2)}(i(s_{2,z} + 2s_3 \rho_z) + \kappa(s_{2,zz} + 2s_3 \rho_{zz} + 4s_{3,\rho z}) + \gamma(s_2 r_0 + 2s_0 s_2 r_2 + 2s_1 s_1 r_2) \\
+2s_1 s_3 r_0 + 2s_0 s_3 r_1 + 2s_0 s_1 r_3 + s_2^2 r_0) \end{pmatrix}.$$ 

As before, here also a rigorous analytical calculation involving symbolic computation shows that the above two equations remain distinct as long as $\kappa$ is non-zero. However, they become identical for $\kappa = 0$, as expected. Thus, due to the failure of existence of sufficient number of arbitrary functions (see equations (14) to (16)), we conclude that the NNLS equation (1) is not free from movable critical singular manifolds. The above singularity structure analysis clearly indicates that the NNLS equation (1) is not integrable in the Painlevé sense.

3. Solitary wave solutions for the NNLS equation

3.1. Hirota’s Bilinearization method

As established in the previous section, the NNLS equation (1) fails to satisfy the Painlevé test for integrability. Hence, one has to consider quasi-analytical methods or numerical analysis to find special solutions in the equation (1) [34, 35, 36]. However, we here attempt to find special solutions in equation (1) by using the well-known Hirota’s bilinearization method, in spite of the equation (1) being non-integrable. The NNLS equation (1) is expressed in a bilinear form by employing the following transformation

$$\Psi = \frac{g(z, t)}{f(z, t)},$$

where $g$ and $f$ are complex and real functions, respectively, and * indicates complex conjugation. The resulting bilinear equations are

$$(iD_z + \kappa D_z^2 + \frac{1}{2} D_z^2)(g \cdot f) = 0,$$

$$(\kappa D_z^2 + \frac{1}{2} D_z^2)(f \cdot f) = \gamma(g g^*).$$
One can obtain the single solitary wave solution by expression $g = \chi g_1$, and $f = 1 + \chi^2 f_2$ in equation (17), where $\chi$ is a formal expression parameter. Solving the resulting linear partial differential equations (17) at various orders of $\chi$ recursively, we obtain

$$
\Psi = \Delta e^{i\eta_1} \text{sech}\left(\eta_{1r} + \frac{R}{2}\right),
$$

(18)

Here

$$
\eta_{1r} = a_r t + b_r z, \quad \eta_{1i} = a_i t + b_i z + \theta, \quad a_i = \sqrt{-b_i + \kappa(b_i^2 - b_r^2) \pm \sqrt{(b_i^2 + b_r^2)(1 + 2kb_i + \kappa(b_r^2 + b_i^2))}},
$$

$$
a_r = -\frac{b_r(2kb_i + 1)}{a_i}, \quad \theta = \tan^{-1}\left(\frac{a_i}{a_r}\right), \quad R = 2 \log \left(\frac{\gamma(\alpha a^*)}{(8kb_r^2 + 4a_i^2)}\right),
$$

(19a)

where $a_r, a_i, b_r, b_i, \alpha_r$, and $\alpha_i$ are real parameters. By direct substitution, we have also verified that the solution (18) indeed satisfies the NNLS equation (1). This one bright solitary wave (18) is characterized by four arbitrary real parameters $b_r, b_i, \alpha_r$ and $\alpha_i$. The amplitude and velocity of one bright solitary wave (18) can be expressed as

$$
\Delta = \frac{a}{2 e^\frac{a}{2}} = \frac{4b_r \sqrt{2 \kappa a_i^2 + (1 + 2kb_i^2)}}{\sqrt{\gamma a_i}}, \quad \text{and} \quad v = \frac{a_i}{(-2kb_i - 1)},
$$

(19b)

respectively. Also, the phase part of the solitary wave is given by $a_i(t + \frac{b_i}{a_i} z)$. Here, the amplitude, velocity and phase of the bright solitary wave are affected significantly by the nonparaxial effect due to the explicit appearance of the nonparaxial parameter $\kappa$ in their corresponding expressions. Note that, the solution (18) reduces to the standard NLS soliton in the paraxial limit (i.e., when $\kappa \to 0$). So, one can conclude that the nonparaxial parameter influences all the physical parameters of bright solitary wave of equation (1). This is one of the distinct features of the obtained solitary wave solution (18). tried to extend the above bilinearization procedure to obtain general two-soliton solution, but unsuccessful. This suggests that the three soliton solution of NNLS system (1) with a sufficient number of parameters does not exist. This conclusion is in support of the Painlevé analysis carried out in the previous section (2.1) showing the NLS system to be non-integrable.

First, we show the propagation of the one bright solitary wave as in Fig. 1 which mimics the typical soliton propagation in integrable systems. Then, in order to reveal the impact of nonparaxiality on the bright solitary wave, we display the intensity plots of the bright solitary wave for different values of the nonparaxial parameter $\kappa$ in Fig. 2. In the absence of the nonparaxial parameter (i.e. $\kappa = 0$), it retraces the standard intensity profile as that of the NLS equation (solid black curve). Upon the onset of the nonparaxial parameter $\kappa$, the bright solitary wave undergoes significant changes, not only in its amplitude and width but also in its central position. These are signatures of the nonparaxiality [19,20]. The influence of the nonparaxial parameter on physical quantities such as amplitude and speed of the bright solitary wave is presented in Fig. 3. It clearly shows that the increase in the nonparaxial parameter enhances the speed of the bright solitary wave.
wave. For the $\kappa$ values lying in the window [-1,1], the amplitude decreases until $\kappa$ becomes zero and then it starts to increase.

We have also investigated the stable dynamics of obtained bright solitary wave of the NNLS equation by employing the split-step Fourier method based on Feit-Flock algorithm [5]. To do so, we add a random uniform white noise as a perturbation at a rate of 10% in the initial solution of bright solitary wave solution [18] [57]. Figure 4 demonstrates that the solitary pulse remains stable for a long propagation distance which is quite larger for optical waveguides, without (see Fig. 4(a)) and with noise (see Fig. 4(b)). Hence the numerical evolution clearly demonstrates that the pulse is robust against small perturbations in the form of uniform white noise for the given system parameters.

4. Scattering dynamics of bright solitary waves in the NNLS system

Interaction of solitary waves is a key feature that determines their potential applications in nonlinear optical systems. It is interesting to study the interaction between two solitons by launch-
Figure 3: Plot depicts the speed and amplitude of the bright solitary wave as a function of the nonparaxial parameter $\kappa$. The parameters are same as given in Fig. 2.

Figure 4: Numerical evolution of stable bright solitary waves, in the absence of perturbation (a) and in the presence of white noise 10% (b). The parameters are the same as in Fig. 1.

ing the soliton pulses far enough from each other, at least with a separation distance around ten times of their pulse-width [38, 39]. The implication of such criteria has really helped to overcome multiple issues like pulse distortion, deteriorations of the data transmission and synchronization in the high-bit-rate systems. In general, interaction of solitons can be classified into two main categories as coherent and incoherent based on their relative phases [40, 41]. In practice, the coherent type of interactions takes place when the interference effects between the overlapping beams are taken into account. It requires the medium to respond instantaneously. On the contrary, incoherent interactions exist when the time response varies slower than the relative phase between solitons. Ultimately, solitons experience periodic collapse with neighboring solitons. It must be hence noted that the incoherent interactions are undesirable in the practical viewpoint [42].

Interaction of various types of solitons has been intensively discussed both from experimental and theoretical aspects [1]. In particular, these studies considered interaction between solitons/solitary waves in the NLS and NLS-like equations [43, 44, 45, 46]. The multicomponent versions of these scalar NLS type equations support bright solitons with interesting shape changing (energy sharing/switching) collisions [47, 48, 49, 50, 51]. These interesting energy sharing
collisions find applications in the context of realizing gates based on soliton collisions \[52, 53, 54, 55, 56\]. However, to date, the intriguing process of soliton interactions remains unexplored in the context of nonparaxial regime except a work that showed a glimpse of the former \[57\]. We are hence interested to study the interactions of bright solitons numerically. The split-step Fourier method based on Feit-Flock algorithm is adopted here to investigate the interaction between two temporally separated bright solitons in the NNLS equation. To study the scattering dynamics of bright solitons in the NNLS equation, we assume the following two temporal bright solitary pulses with equal amplitudes \( \Delta \approx 1 \), in the normalized sense:

\[
\Psi(0, t) = \Psi(0, t + \Delta t_0) \exp(i\phi) + \Psi(0, t - \Delta t_0),
\]

where \( \Psi(0, t + \Delta t_0) \) denotes the bright solitary wave solution given by equation (18) with amplitude \( \Delta \approx 1 \), and \( \phi \) indicates an initial phase difference between the two temporally solitary pulses initially separated by a distance \( \Delta t_0 \). For the simulations performed here, we choose the boundary conditions to minimize the undesired effects such as reflection of radiation at the boundaries of the computational window. In what follows, we present the coherent interactions of nonparaxial bright solitons with different parametric choices of obtained solutions and qualitatively discuss the physics behind the interaction dynamics in detail.

To start with, we consider the collision for the parametric choice \( \kappa = 0.01, \Delta t_0 = 2 \), and vary phase from \( \phi = 0 \) to \( \pi \) as presented in Fig. 5. For \( \phi = 0 \), it exhibits an in-phase interaction dynamics and forms oscillating bound solitary waves as shown in Fig. 5(a). Note that, these localized structures maintain their velocity throughout the propagation and retain their shape throughout the medium. The scenario is changed for the choice of phase \( \phi = \pi/2 \) as observed in Fig. 5(b), where the bound solitary waves execute oscillations and deviate away from the central position. Also, the intensities of the interacting pulses are decreased considerably compared to Fig. 5(a), while their widths are extended. The interacting solitary waves experience a significant drift in their path after collision which indicates a strong repulsion between them. This leads to an increase in their separation distance after collision. For the case, \( \phi = \pi \), the interacting pulses become unstable and dispersion radiations are created [see Fig. 5(c)]. Thus, when the pulses are separated by short distance, stable solitary waves are formed when their phases are correlated \[57\].
Next, we increase the separation distance between the solitary waves and investigate the collision dynamics for in-phase, out-of-phase, and $\phi = \pi/2$ choices for comparative purpose, as shown in Figs. 6(a)-(c). Surprisingly, in this non-integrable system, we obtain standard soliton like shape-preserving collision for the case $\phi = 0$. For $\phi = \pi/2$, there is a non-trivial energy switching among the interacting solitary waves during the collision accompanied by bending (drifting) of the interacting solitary waves resulting an increase in their relative separation distance. For the choice, $\phi = \pi$, also similar behavior takes place [see Figs. 6(a)-(c)]. For larger separation distance there is no passing through collision and we observe only parallel propagation of bound solitons as noticed in Fig. 7(a). It is to be noted that in contrast to the standard NLS and Manakov systems where the solitons exhibit conservation of energy during their collisions, the solitary waves of the present system (1) do not preserve the total energy, rather it conserves the quantity $c_1 = \int_{-\infty}^{\infty} \left( |\Psi|^2 - i\kappa (\Psi^* \frac{\partial \Psi}{\partial z} - \Psi \frac{\partial \Psi^*}{\partial z}) \right) dt$. This constant of motion $c_1$ can be easily obtained by considering the NNLS equation and its complex conjugate equation and by using the asymptotic behavior of bright solitary waves $\left( \Psi \xrightarrow{t \to \pm \infty} 0 \right)$, with a simple algebra. Here the nonparaxial parameter ($\kappa$) is responsible for the violation of the conservation of energy, which is also corroborated through numerical simulations. In particular, as shown in Fig. 5(c), some amount of energy of the solitary wave gets radiated which further proves that the norm of the solitary wave denoting the total intensity is non-conserved.

Finally, it is interesting to reveal the scattering nature of three bright solitary waves by altering the phase of solitary waves from in-phase (zero) to out-of-phase ($\pi$) as shown in Fig. 8. In the case of in-phase, three solitary waves are attracted to each other at $z \sim 7$ ($z$ denotes the propagation distance of the medium), afterwards they get separated symmetrically to each other and also retain their shape after collision (see Fig. 8(a)). We also observe a non-trivial energy switching in the second (middle) solitary wave. By tuning the phase to $\pi/2$, we identify an interesting interaction dynamics, where one solitary pulse (left side) is completely separated from other solitary waves and is deviated away from the remaining solitary waves. The rest of the two solitary pulses initially propagate within a very short separation distance and after the collision due to repulsion between the solitary pulses the separation distance between them is increased. In particular, two solitary pulses have distinct intensity profiles, featuring an energy transfer from one solitary pulse.
Figure 7: The evolution of interaction of two bright solitary waves. The parameters are the same as in Fig. 5 except $\Delta t_0 = 5$.

(a) $\phi = 0$

Figure 8: The density plots of interaction of three bright solitary pulses. The parameters are the same as in Fig. 5 except $\Delta t_0 = 4$.

(a) $\phi = 0$
(b) $\phi = \pi/2$
(c) $\phi = \pi$

to another one as presented in Fig. 8(b). A similar collision behavior with a significant energy switching in the right most solitary wave (before collision) can be observed for the case $\phi = \pi$ with a slight modification as given in Fig. 8(c). Note that after collision, there is a corresponding decrease in the intensities of other two solitary pulses.

5. Conclusion

To conclude, we have investigated the integrability aspects of the NNLS equation by employing the Painlevé singularity structure analysis. Based on this analysis, we have proven that the NNLS equation fails to satisfy the Painlevé test as it is not free from the movable singularity at the resonance $j = 3$. Nevertheless, we have then constructed bright solitary wave for the NNLS equation by using the Hirota’s bilinearization method. We have demonstrated stable propagation over long distance even in the presence of external perturbation, which is seeded in the form of a white noise, by employing the SSFM. The scattering dynamics of bright solitary waves has also been investigated by numerical simulation for different values of separation distance and relative phase. This numerical study reveals that there is an energy/intensity switching among the colliding solitons after collision, due to the nonparaxiality/spatial dispersion. Also, the collision leads to a stronger repulsion between the solitons which results in an increase in the separation distance between the solitons after interaction. Ultimately, there is a significant deviation in the trajectory
of solitary wave. We anticipate that the results will shed light in the formation, propagation and collision of solitary waves in nonparaxial nonlinear media. The energy switching phenomenon during collision in the NNLS system can find applications in optical switching devices, beam steering and in soliton collision based optical computing.

Acknowledgement

The work of K T is supported by a Senior Research Fellowship from Rajiv Gandhi National Fellowship (Grant No. F1-17.1/2016-17/R GNF-2015-17-SC -TAM-8989), University Grants Commission (UGC), Government of India. The work of T K was supported by the Science and Engineering Research Board, Department of Science and Technology (DST-SERB), Government of India, in the form of a major research project (File No. EMR/2015/001408). A G acknowledges the support of DST-SERB for providing a Distinguished Fellowship (Grant No. SB/DF/04/2017) to M L in which A G was a Visiting Scientist. A G is now supported by University Grants Commission (UGC), Government of India, through a Dr. D. S. Kothari Postdoctoral Fellowship (Grant No. F.4-2/2006 (BSR)/PH/19-20/0025).

References

[1] Y. Kivshar, G. Agrawal, Optical Solitons: From Fibers to Photonic Crystals, Academic Press, USA, 2003.
[2] M. Ablowitz, H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981.
[3] C. Kharif, E. Pelinovsky, A. Slunyaev, Rogue Waves in the Ocean, Springer-Verlag, Berlin, Heidelberg, 2009.
[4] G. Fibich, The Nonlinear Schrödinger Equation: Singular Solutions and Optical Collapse, Springer, Switzerland, 2015.
[5] M. D. Feit, J. A. Fleck, Beam nonparaxiality, filament formation, and beam breakup in the self-focusing of optical beams, J. Opt. Soc. Am. B 5 (1988) 633–640.
[6] P. L. Kelley, Self-Focusing of Optical Beams, Phys. Rev. Lett. 15 (1965) 1005–1008.
[7] F. Biaclalana, C. Creatore, Instabilities and solitons in systems with spatiotemporal dispersion, Opt. Express 16 (2008) 14882–14893.
[8] J. M. Christian, G. S. McDonald, T. F. Hodgkinson, P. Chamorro-Posada, Wave envelopes with second-order spatiotemporal dispersion. I. Bright Kerr solitons and cnoidal waves, Phys. Rev. A 86 (2012) 023838.
[9] M. Lax, W. H. Louisell, W. B. McKnight, From Maxwell to paraxial wave optics, Phys. Rev. A 11 (1975) 1365–1370.
[10] Y. Zhang, H. Zhong, M. R. Belić, C. Li, Z. Zhang, F. Wen, Y. Zhang, M. Xiao, Fractional nonparaxial accelerating talbot effect, Opt. Lett. 41 (14) (2016) 3273–3276.
[11] D. K. Gramotnev, S. I. Bozhevolnyi, Plasmonics beyond the diffraction limit, Nat. Photonics 4 (2010) 83–91.
[12] Y. Liu, G. Bartal, D. A. Genov, X. Zhang, Subwavelength Discrete Solitons in Nonlinear Metamaterials, Phys. Rev. Lett. 99 (2007) 153901.
[13] A. V. Gorbachev, D. V. Skryabin, Spatial solitons in periodic nanostructures, Phys. Rev. A 79 (2009) 053812.
[14] G.-N. Nguyen, K. Heggarty, A. Bacher, P.-J. Jakobs, D. Häringer, P. Gérard, P. Pfeiffer, P. Meyrueis, Iterative scalar nonparaxial algorithm for the design of fourier phase elements, Opt. Lett. 39 (2014) 5551–5554.
[15] J. M. Christian, G. S. McDonald, P. Chamorro-Posada, Bistable dark solitons of a cubic-quintic Helmholtz equation, Phys. Rev. A 81 (2010) 053831.
[16] J. M. Christian, G. S. McDonald, R. J. Potton, P. Chamorro-Posada, Helmholtz solitons in power-law optical materials, Phys. Rev. A 76 (2007) 033834.
[17] J. M. Christian, G. S. McDonald, P. Chamorro-Posada, Bistable Helmholtz bright solitons in saturable materials, J. Opt. Soc. Am. B 26 (2009) 2323–2330.
[18] J. M. Christian, G. S. McDonald, T. F. Hodgkinson, P. Chamorro-Posada, Spatiotemporal dispersion and wave envelopes with relativistic and pseudorelativistic characteristics, Phys. Rev. Lett. 108 (2012) 034101.

[19] J. M. Christian, G. S. McDonald, A. Kotsampaseris, Relativistic and pseudorelativistic formulation of nonlinear envelope equations with spatiotemporal dispersion. I. Cubic-quintic systems, Phys. Rev. A 98 (2018) 053842.

[20] J. M. Christian, G. S. McDonald, M. J. Lundie, A. Kotsampaseris, Relativistic and pseudorelativistic formulation of nonlinear envelope equations with spatiotemporal dispersion. II. Saturable systems, Phys. Rev. A 98 (2018) 053843.

[21] S. Blair, Nonparaxial one-dimensional spatial solitons, Chaos 10 (2000) 570–583.

[22] K. Tamilselvan, T. Kanna, A. Khare, Nonparaxial elliptic waves and solitary waves in coupled nonlinear Helmholtz equations, Commun. Nonlinear Sci. Numer. Simulat. 39 (2016) 134–148.

[23] C. Huang, F. Ye, Y. V. Kartashov, B. A. Malomed, X. Chen, PT symmetry in optics beyond the paraxial approximation, Opt. Lett. 39 (2014) 5443–5446.

[24] K. Tamilselvan, T. Kanna, A. Govindarajan, Cubic-quintic nonlinear Helmholtz equation: Modulational instability, chirped elliptic and solitary waves, Chaos 29 (2019) 063121.

[25] Y. V. Kartashov, B. A. Malomed, L. Torner, Metastability of quantum droplet clusters, Phys. Rev. Lett. 122 (2019) 193902.

[26] C. Huang, F. Ye, B. A. Malomed, Y. V. Kartashov, X. Chen, Solitary vortices supported by localized parametric gain, Opt. Lett. 38 (2013) 2177–2180.

[27] J. Weiss, M. Tabor, G. Carnevale, The Painlevé property for partial differential equations, J. Math. Phys. 24 (1983) 522–526.

[28] M. Tabor, J. Weiss, Analytic structure of the lorenz system, Phys. Rev. A 24 (1981) 2157–2167.

[29] M. Lakshmanan, S. Rajasekar, Nonlinear Dynamics: Integrability, Chaos and Patterns, Springer-Verlag Berlin Heidelberg, 2003.

[30] M. J. Ablowitz, A. Ramani, H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of P-type, I., J. Math. Phys. 21 (1980) 715–721.

[31] M. Lakshmanan, R. Sahadevan, Painlevé analysis, Lie symmetries, and integrability of coupled nonlinear oscillators of polynomial type, Phys. Rep. 224 (1993) 1–93.

[32] A. Ramani, B. Grammaticos, T. Bountis, The Painlevé property and singularity analysis of integrable and non-integrable systems, Phys. Rep. 180 (1989) 159–245.

[33] K. Sakkaravarthi, A. Johnpillai, A. D. Devi, T. Kanna, M. Lakshmanan, Lie symmetry analysis and group invariant solutions of the nonlinear Helmholtz equation, Appl. Math. Comput. 331 (2018) 457–472.

[34] E. Tzirtzilakis, M. Xenos, V. Marinakis, T. Bountis, Interactions and stability of solitary waves in shallow water, Chaos, Solitons and Fractals 14 (1) (2002) 87–95.

[35] E. Tzirtzilakis, V. Marinakis, C. Apokis, T. Bountis, Soliton-like solutions of higher order wave equations of the Korteweg–de Vries type, J. Math. Phys. 43 (12) (2002) 6151–6165.

[36] K. Andriopoulos, T. Bountis, K. V. der Weele, L. Tsigaridi, The shape of soliton-like solutions of a higher-order kdv equation describing water waves, J. Nonlinear Math. Phys. 16 (sup1) (2009) 1–12.

[37] A. Govindaraji, A. Mahalingam, A. Uthayakumar, Interaction dynamics of bright solitons in linearly coupled asymmetric systems, Opt. Quant. Electron. 48 (2016) 563.

[38] K. J. Blow, N. J. Doran, Bandwidth limits of nonlinear (soliton) optical communication systems, Electron. Lett. 19 (1983) 429–430.

[39] C. Desem, P. L. Chu, Soliton propagation in the presence of source chirping and mutual interaction in single-mode optical fibres, Electron. Lett. 23 (1987) 260–262.

[40] S. Tsang, K. S. Chiang, K. W. Chow, Soliton interaction in a two-core optical fiber, Opt. Commun. 229 (2004) 431 – 439.

[41] D. W. C. Lai, K. W. Chow, K. Nakkeeran, Multiple-Pole soliton interactions in optical fibres with higher-order effects, J. Mod. Opt. 51 (2004) 455–460.

[42] G. I. Stegeman, M. Segev, Optical Spatial Solitons and Their Interactions: Universality and Diversity, Science 286 (1999) 1518–1523.

[43] X. Ma, R. Driben, B. A. Malomed, T. Meier, S. Schumacher, Two-dimensional symbiotic solitons and vortices in binary condensates with attractive cross-species interaction, Sci. Rep. 6 (2016) 34847.
[44] M. Shalaby, F. Reynaud, A. Barthelemy, Experimental observation of spatial soliton interactions with a $\pi/2$ relative phase difference, Opt. Lett. 17 (1992) 778–780.
[45] H. Triki, T. R. Taha, Solitary wave solutions for a higher order nonlinear Schrödinger equation, Math. Comput. Simulat. 82 (2012) 1333–1340.
[46] H. Triki, K. Porsezian, A. Choudhuri, P. T. Dinda, Chirped solitary pulses for a nonic nonlinear Schrödinger equation on a continuous-wave background, Phys. Rev. A 93 (2016) 063810.
[47] R. Radhakrishnan, M. Lakshmanan, J. Hietarinta, Inelastic collision and switching of coupled bright solitons in optical fibers, Phys. Rev. E 56 (1997) 2213–2216.
[48] T. Kanna, M. Lakshmanan, Exact Soliton Solutions, Shape Changing Collisions, and Partially Coherent Solitons in Coupled Nonlinear Schrödinger Equations, Phys. Rev. Lett. 86 (2001) 5043.
[49] T. Kanna, M. Vijayajayanthi, M. Lakshmanan, Coherently coupled bright optical solitons and their collisions, J. Phys. A: Math. Theor. 43 (2010) 434018.
[50] M. Vijayajayanthi, T. Kanna, M. Lakshmanan, Bright-dark solitons and their collisions in mixed N-coupled nonlinear Schrödinger equations, Phys. Rev. A 77 (2008) 013820.
[51] T. Kanna, K. Sakkaravarthi, M. Vijayajayanthi, Novel energy sharing collisions of multicomponent solitons, Pramana 85 (2015) 881–897.
[52] K. Jakubowski, M. H. Steiglitz, R. Squier, State transformations of colliding optical solitons and possible application to computation in bulkmedia, Phys. Rev. E 58 (1998) 6752.
[53] K. Steiglitz, Time-gated Manakov spatial solitons are computationally universal, Phys. Rev. E 63 (2000) 016608.
[54] T. Kanna, M. Lakshmanan, Exact soliton solutions of coupled nonlinear Schrödinger equations: Shape-changing collisions, logic gates, and partially coherent solitons, Phys. Rev. E 67 (2003) 046617.
[55] M. Vijayajayanthi, T. Kanna, K. Murali, M. Lakshmanan, Explicit construction of single input–single output logic gates from three soliton solution of Manakov system, Commun. Nonlinear Sci. Numer. Simulat. 36 (2016) 391–401.
[56] M. Vijayajayanthi, T. Kanna, K. Murali, M. Lakshmanan, Harnessing energy-sharing collisions of Manakov solitons to implement universal NOR and OR logic gates, Phys. Rev. E 97 (2018) 060201(R).
[57] H. Wang, W. She, Modulation instability and interaction of non-paraxial beams in self-focusing Kerr media, Opt. Commun. 254 (2005) 145 – 151.