A Bayesian approach to source separation

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Abstract. Source separation is one of the signal processing’s main emerging domain. Many techniques such as maximum likelihood (ML), Infomax, cumulant matching, estimating function, etc. have been used to address this difficult problem. Unfortunately, up to now, many of these methods could not account completely for noise on the data, for different number of sources and sensors, for lack of spatial independence and for time correlation of the sources. Recently, the Bayesian approach has been used to push farther these limitations of the conventional methods. This paper proposes a unifying approach to source separation based on the Bayesian estimation. We first show that this approach gives the possibility to explain easily the major known techniques in sources separation as special cases. Then we propose new methods based on maximum a posteriori (MAP) estimation, either to estimate directly the sources, or the mixing matrices or even both.

Key words: Sources separation, Bayesian estimation

1. Introduction

The simplest model for a source separation is

\[ \mathbf{x}(t) = \mathbf{A} \mathbf{s}(t), \]

where \( \mathbf{A} \) is a mixing matrix, \( \mathbf{s}(t) \) is a vector of sources and \( \mathbf{x}(t) \) a vector of independent measurements. The main task is then to recover \( \mathbf{s}(t) \), but one may instead be interested in recovering a separating matrix \( \mathbf{B} \) such that \( \mathbf{s}(t) = \mathbf{B} \mathbf{x}(t) \). When \( \mathbf{A} \) is invertible, it is natural to assume that \( \mathbf{B} = \mathbf{A}^{-1} \) or \( \mathbf{B} = \Sigma \Lambda \mathbf{A}^{-1} \) where \( \Sigma \) is a permutation matrix and \( \Lambda \) a diagonal scaling matrix.

Many source separation algorithms have been recently proposed based on likelihood [1, 2, 3, 4, 5, 6, 7, 8], contrast function [9, 10, 11, 12], estimating function [13, 14, 15, 16], information theory [17, 18, 19], and more.

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generally on principle component analysis (PCA) [20]. Independent factor analysis (IFA) [21, 22, 23] and independent component analysis (ICA) [16, 24, 25]. All these methods assume that the mixing matrix \( A \) is invertible and mainly search for a separating matrix \( B \) such that the components of \( y(t) = Bx(t) \) be independent. This means that all these methods implicitly assume that the sources \( s(t) \) are independent. This may not be the case in some applications. However, the main differences between these methods are in the way they try to insure this independence.

- Maximum likelihood (ML) techniques use directly the independence property by assuming

\[
p(s) = \prod_i p_i(s_i)
\]

and as a result

\[
p(x|A) = \frac{1}{\det(A)} \prod_i p_i \left( [A^{-1}x_i] \right),
\]

or equivalently

\[
p(x|B) = |\det(B)| \prod_i p_i ([Bx]_i) = |\det(B)| \prod_i p_i(y_i),
\]

where \( y_i = [A^{-1}x_i] = [Bx]_i \) and where \( p(s) \) is the probability density function of the source vector \( s \). The ML estimate of the separating matrix is defined as

\[
\hat{B} = \arg \max_B \{ \log p(x|B) \}
\]

\[
= \arg \max_B \left\{ \sum_i r_i(y_i) + \log |\det(B)| \right\} \quad \text{with} \quad r_i(y_i) = \log p_i(y_i).
\]

A great number of algorithms have been proposed to perform this optimization [17, 4].

- Infomax techniques use the entropy of \( y = Bx \) as a measure of independence [20, 24, 28, 29, 30]:

\[
S = -\sum_i p_i(y_i) \log p_i(y_i).
\]

Thus \( S \) is a function of the separating matrix \( B \) and one tries to optimize \( S \) with respect to \( B \).
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– M-estimation techniques define an estimate for the separating matrix $B$ such that

$$\frac{1}{T} \sum_t H(y(t)) = [0] \quad \text{with} \quad y(t) = Bx(t), \quad (7)$$

where it is assumed to have $T$ independent observations $\{x(1), \ldots, x(T)\}$ and where $H$ is an appropriately defined matrix valued function. $[0]$ represents a matrix whose elements are all equal to zero. We can note that M-estimate methods generalize the ML estimation method since the latter can be obtained by taking

$$H(y) = \frac{\partial \log p(y)}{\partial B}. \quad (8)$$

– Contrast function minimization techniques are based on the optimization of a contrast function $c(y) = c(Bx)$ which takes its extremal value when $B$ is a separating matrix $[9, 10]$. Typical examples are the contrast functions measuring, in some way, the independence of the components of $y$, sometimes subject to the constraint that $y(t)$ be spatially white

$$\frac{1}{T} \sum_t y(t) y^\dagger(t) = I. \quad (9)$$

– Higher order statistics (HOS) techniques try to insure the independence of the components of $y$ by minimizing, under the whiteness constraint, a contrast function related to the statistics of the order greater that two such as the cumulants $[31, 32, 33]$. The main limitations of these techniques are the following:

– None of these techniques consider the possible errors on the model or the measurement (sensor) noises;

– All these methods assume that the mixing matrix $A$ is invertible and cannot account for the cases in which $A$ is rectangular (number of sensors different from the number of sources);

– All these methods assume that the sources are independent. Some assume the sources to be also temporally white.

Recently, a few works using the Bayesian approach have been presented to push farther the limits of these methods $[34, 35, 36, 37, 38, 39]$. In the following, we first present the basics of the Bayesian approach, then we show how some of the preceeding techniques can be obtained as special cases, and finally, we propose new ideas to account for spatial correlation between neighbor sources or time correlation of the sources.
2. Bayesian approach
The main idea in the Bayesian approach is to use not only the likelihood 
\( p(x(1), \ldots, x(T)|A) \) but also some prior knowledge about the sources \( s \) and the mixing matrix \( A \) through the assignment of prior probabilities \( p(s) \) and \( p(A) \). Then, noting \( x_{1:T} = \{x(1), \ldots, x(T)\} \) and \( s_{1:T} = \{s(1), \ldots, s(T)\} \) and using these direct probability laws we determine the posterior law

\[
\log p(A, s_{1:T}|x_{1:T}) = \log p(x_{1:T}|A, s_{1:T}) + \log p(A) + T \log p(s) + cte, 
\]

(10)

where we assumed the independence of the sources \( s \) and the mixing matrix \( A \).

From this posterior probability law we can deduce any inference about \( A \) and \( s \). For example, we can estimate both \( A \) and \( s \) by a joint maximum \( a \) posteriori (JMAP) criterion using an alternate maximization algorithm.

We can also focus on the estimation of the mixing matrix \( A \) by marginalizing this posterior law with respect to \( s \) to obtain \( p(A|x_{1:T}) \) and use the resulting MAP criterion to estimate \( A \). Finally, we can integrate \( A \) from the joint law to obtain \( p(s_{1:T}|x_{1:T}) \) and estimate the sources from this marginal posterior law.

Now, before going further in details of these three methods, we are going to illustrate some special cases which result in some classical techniques.

2.1. Exact invertible model and independent sources
If we assume that the model \( x = As \) is exact and that there is not any measurement noise and that the mixing matrix \( A \) is invertible and well conditioned, then we can only look for a separating matrix \( B = A^{-1} \). Indeed, as in conventional methods, if we assume that the sources \( s \) are independent, we have the following relations:

\[
p(s) = \prod_i p_i(s_i),
\]

(11)

and so

\[
p(x|B) = |\text{det}(B)| \prod_i p_i(\{Bx\}_i),
\]

(12)

where \( p_i(s_i) \) is the probability density function of the source component \( i \).

Using these relations, and noting by \( y(t) = Bx(t) \) we have

\[
\log p(B|x_{1:T}) = \log p(x_{1:T}|B) + \log p(B) + cte,
\]

(13)
where
\[ \log p(x_{1:T}|B) = T \log |\det(B)| + \sum_t \sum_i \log p_i(y_i(t)) \]
and \( p(B) \) is a probability distribution on the separating matrix \( B \). Here we assume that we can assign a probability law \( p(A) \) to the mixing matrix \( A \) or equivalently \( p(B) \) to the separating matrix \( B \) to translate any prior knowledge we have about (or we wish to impose to) them. For example, we may know (or assume) that the mixing matrix is such that
\[ \|A\|^2 \leq \epsilon, \]
for some \( \epsilon \); or we may wish that the separating matrix \( B \) be such that its determinant \( |\det(B)| \neq 0 \) and not very far from one. In the first case we can choose
\[ p(A) \propto \exp \left[ -\frac{1}{2\sigma_A^2} \|A\|^2 \right] = \exp \left[ -\frac{1}{2\sigma_A^2} \sum_k \sum_l a_{k,l}^2 \right] \]
and in the second case
\[ p(B) \propto |\det(B)|. \]
Some other possibilities are:
\[ p(A) \propto \exp \left[ -\frac{1}{2\sigma_A^2} \|I - A\|^2 \right] = \exp \left[ -\frac{1}{2\sigma_A^2} \sum_k \left( \sum_l a_{k,l}^2 \right) \right] \]
which tries to impose \( |a_{k,l}| \approx 1 \), \( k = l \) and \( |a_{k,l}| \approx 0 \), \( k \neq l \);
\[ p(A) \propto \exp \left[ -\frac{1}{2\sigma_A^2} \|I - AA^t\|^2 \right] = \exp \left[ -\frac{1}{2\sigma_A^2} \sum_l \left( \sum_{k \neq l} a_{k,l}^2 \right) \right] \]
when the number of sources is less than the number of the sensors; and
\[ p(A) \propto \exp \left[ -\frac{1}{2\sigma_A^2} \|I - A' A\|^2 \right] = \exp \left[ -\frac{1}{2\sigma_A^2} \sum_l \left( \sum_{k \neq l} a_{k,l}^2 \right) \right] \]
when the number of sources is greater than the number of the sensors.
These two last expressions have been proposed and used by Knuth [35].
Other choices based on prior knowledge of the geometrical positions of the sources and receivers and knowledge of the signal propagation law for an acoustical application have been used by [35, 38].

Now, if we consider the MAP estimation, the MAP criterion to optimize becomes

$$J(B) = \log p(B|x_{1..T}) = T \log |\det(B)| + \sum_t \sum_i \log p_i(y_i(t)) + \log p(B) + \text{cte}. \tag{20}$$

Searching now for the MAP solution, the necessary condition is

$$\frac{\partial J(B)}{\partial B} = [0] \rightarrow - \sum_t H(y(t)) = [0], \tag{21}$$

where $H$ is a matrix valued function given by

$$H(y) = \frac{\partial}{\partial B} \left[ \sum_i \log p_i(y_i) + \log |\det(B)| + \frac{1}{T} \log p(B) \right]. \tag{22}$$

As an example, consider a uniform $a$ priori law for $B$. Then we obtain the classical ML estimate which satisfies

$$\sum_t H(y(t)) = [0] \quad \text{with} \quad H(y) = \phi(y) y^t - I, \tag{23}$$

where $\phi(y) = [\phi_1(y_1), \ldots, \phi_n(y_n)]^t$ with

$$\phi_i(z) = -\frac{p_i'(z)}{p_i(z)}. \tag{24}$$

One can add some extra constraints to this optimization. For example, we can optimize the MAP criterion subject to the constraint $\frac{1}{T} \sum_t y(t) y^t(t) = I$ which leads again to

$$\sum_t H(y(t)) = [0] \quad \text{with} \quad H(y) = \alpha(y y^t - I) + \beta \left( \phi(y) y^t + y \phi'(y) \right). \tag{25}$$

Note that in all these relations, $\phi_i(z)$ is related to the probability distribution of the source number $i$. The following table gives the expression of this function for a few known cases.
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| Distribution      | $p(z) \propto$ | $\phi(z) =$ |
|-------------------|----------------|------------|
| Gauss             | $\exp[-\alpha z^2]$ | $2\alpha z$ |
| Laplace           | $\exp[-\alpha |z|]$ | $\alpha \text{sign}(z)$ |
| Cauchy            | $\frac{1}{1 + (z/\alpha)^2}$ | $\frac{2z}{\alpha} - \frac{2z}{\alpha} \tanh(\frac{\alpha z}{2})$ |
| Gamma             | $z^\beta \exp[-\beta z]$ | $-\frac{\alpha}{z} + \beta$ |
| sub-Gaussian law  | $\exp[-\frac{1}{2} z^2 \text{sech}^2(z)]$ | $z + \tanh(z)$ |
| Mixture of Gaussians | $\exp\left[-\frac{1}{2}(z - \alpha)^2\right] + \exp\left[-\frac{1}{2}(z + \alpha)^2\right]$ | $\alpha z - \alpha \tanh(\alpha z)$ |

**Remark:**

$H(y)$ in equations (21) and (23) corresponds to the gradient of MAP and ML criteria. A common technique to obtain the MAP or the ML solutions is then to use a gradient based algorithm such as

$$B^{(k+1)} = B^{(k)} - \gamma H(y)$$

(26)

where $(k)$ and $(k + 1)$ stand for two successive iterations in static case or two successive time instants for dynamic case. This equation forms the main body of a great number of neural network (NN) based algorithms for source separation.

### 2.2. Accounting for Errors

Here we relax the inversibility of the matrix $A$ and take also account of the errors on the data. As an example, we consider the case where the errors can be modelled by an additive term $\epsilon(t)$:

$$x(t) = A s(t) + \epsilon(t), \quad t = 1, \ldots, T.$$  

(27)

We assume also that we can assign a probability law $p(\epsilon)$ to $\epsilon$. In general, it is natural to assume that $\epsilon(t)$ has independent components and is centered and white, i.e.

$$\log p(\epsilon(1), \ldots, \epsilon(T)) = \sum_i \sum_t \log p_i(\epsilon_i(t)).$$  

(28)

From this assumption, we obtain

$$\log p(x_{1:T}|A, s_{1:T}) = \sum_t \sum_i q_i(x_i(t) - [As]_i(t))$$  

(29)
with $q_i(.) = \log p_i(.)$.

Now, we can give the expression of the posterior law which is

$$\log p(A, s_{1..T}|x_{1..T}) = \log p(x_{1..T}|A, s_{1..T}) + \log p(s_{1..T}) + \log p(A) + \text{cte}$$

$$= \sum_t \sum_i q_i(x_i(t) - [As]_i(t)) + \log p(s_{1..T}) + \log p(A) + \text{cte}.$$  \hspace{1cm} (30)

As mentioned before, from here, we can go in at least three directions:

- First integrate $p(A, s_{1..T}|x_{1..T})$ with respect to $A$ to obtain $p(s_{1..T}|x_{1..T})$ and estimate $s_{1..T}$ by

$$\hat{s}_{1..T} = \arg \max_{s_{1..T}} \{ p(s_{1..T}|x_{1..T}) \}.$$  \hspace{1cm} (31)

- Second integrate $p(A, s_{1..T}|x_{1..T})$ with respect to $s_{1..T}$ to obtain $p(A|x_{1..T})$ and estimate $A$ by

$$\hat{A} = \arg \max_A \{ p(A|x_{1..T}) \}.$$  \hspace{1cm} (32)

But, here, when $\hat{A}$ is obtained, we still have to obtain $\hat{B} = \hat{A}^{-1}$ and $\hat{A}$ may not be invertible.

- Third, optimize $p(A, s_{1..T}|x_{1..T})$ simultaneously with respect to both $s_{1..T}$ and $A$ by using an alternating optimization procedure such as

$$\begin{cases} 
\hat{s}^{(k)}_{1..T} = \arg \max_{s_{1..T}} \{ p(A^{(k-1)}, s_{1..T}|x_{1..T}) \} \\
\hat{A}^{(k)} = \arg \max_A \{ p(A, \hat{s}^{(k-1)}_{1..T}|x_{1..T}) \} 
\end{cases}$$  \hspace{1cm} (33)

In the two first cases, the integrations can be done analytically only in the Gaussian case. We then obtain closed form expressions for the solutions.

In any case, before applying any optimization, we have to ensure that the criterion to be optimized has at least an optimum and that this optimum is unique.

2.3. Spatially independent and white sources

The case where we can assume that the sources are independent and white is the simplest one. We have:

$$\log p(s_{1..T}) = \sum_t \sum_j r_j(s_j(t))$$
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and

\[
\log p(A, s_{1..T} | x_{1..T}) = \sum_t \sum_i q_i (x_i(t) - z_i(t)) + \sum_t \sum_j r_j(s_j(t)) + \ln p(A) + \text{cte.}
\]

(34)

with \( z_i = [As]_i \). Then, we can omit the time summation. To simplify the details of the derivations, let first assume

\[
p(A) \propto \exp \left[ -\frac{1}{2\sigma_a^2} \|A\|^2 \right] = \exp \left[ -\frac{1}{2\sigma_a^2} \sum_k \sum_l a_{k,l}^2 \right].
\]

(35)

Later, we will also consider other possibilities such as (17), (18) or (19).

Joint MAP estimation First we consider the joint estimation of \( A \) and \( s \) where the alternating optimization algorithm becomes

\[
\begin{align*}
\hat{s}^{(k)} &= \arg \max_s \left\{ \sum_i q_i (x_i - z_i) + \sum_j r_j(s_j) \right\} \\
\hat{A}^{(k)} &= \arg \max_A \left\{ \sum_i q_i (x_i - z_i) - \frac{1}{2\sigma_a^2} \sum_k \sum_l a_{k,l}^2 \right\}
\end{align*}
\]

(36)

The solution at each iteration has to satisfy

\[
\begin{align*}
\frac{\partial}{\partial s_j} &= -\sum_i a_{ij} q_i'(x_i - z_i) + r_j'(s_j) = 0 \\
\frac{\partial}{\partial a_{ij}} &= -s_j q_i'(x_i - z_i) - \frac{1}{\sigma_a^2} a_{ij} = 0
\end{align*}
\]

(37)

These equations are in general nonlinear and depend on the expressions of \( q \) and \( r \). One exception is the particular case of Gaussian laws

\[
p_i(n) \sim \mathcal{N}(0, \sigma_e^2) \rightarrow q_i(n) = -\frac{1}{2\sigma_e^2} n^2 \rightarrow q_i'(n) = -\frac{1}{\sigma_e^2} n
\]

and

\[
p_j(s) \sim \mathcal{N}(0, \sigma_s^2) \rightarrow r_j(s) = -\frac{1}{2\sigma_s^2} s^2 \rightarrow r_j'(s) = -\frac{1}{\sigma_s^2} s
\]

where we obtain two sets of linear equations to solve for \( s_j \) and \( a_{ij} \) :
\[
\left\{ \begin{array}{l}
\frac{1}{\sigma^2} \sum_i a_{ij} (x_i - [As]_i) - \frac{1}{\sigma^2} s_j = 0 \\
\frac{1}{\sigma^2} s_j (x_i - [As]_i) - \frac{1}{\sigma^2} a_{ij} = 0
\end{array} \right. \rightarrow \left\{ \begin{array}{l}
\sum_i a_{ij} (x_i - [As]_i) - \lambda s_j = 0 \\
\frac{1}{\sigma^2} s_j (x_i - [As]_i) - \mu a_{ij} = 0
\end{array} \right.
\] (38)

with \( \lambda = \frac{\sigma^2}{\sigma^2_s} \) and \( \mu = \frac{\sigma^2}{\sigma^2_a} \).

These two equations have to be solved in each iteration of alternating optimization procedure. Two strategies can be used:

- Solve these equations for each \( s_j \) and then for each \( a_{ij} \) at each iteration:

\[
\left\{ \begin{array}{l}
s_j = \frac{\sum_i a_{ij} (x_i - \hat{x}_i)}{\lambda + \|a_{j*}\|^2} \\
a_{ij} = \frac{s_j (x_i - \hat{x}_i)}{s_j^2 + \mu}
\end{array} \right.
\] (39)

with \( \hat{x}_i = \sum_{k \neq j} a_{ik} s_k \), and \( \|a_{i*}\|^2 = \sum_j a_{i,j}^2 \). This is a single coordinate-wise gradient descent based algorithm.

- Solve these equations for all \( s_j \) and then for all \( a_{ij} \) at each iteration:

\[
\left\{ \begin{array}{l}
A^t(x - As) - \lambda s = 0 \\
(x - As)s^t - \mu A = 0
\end{array} \right. \rightarrow \left\{ \begin{array}{l}
s = (A^t A + \lambda I)^{-1} A^t x \\
A = x s^t (ss^t + \mu I)^{-1} \\
\quad = \frac{x s^t}{\mu} \left[ I - \frac{s s^t}{\mu} \right]
\end{array} \right.
\] (40)

This is a bloc coordinate-wise gradient descent based algorithm.

**Remark 1:**

These two last closed form expressions give us the possibility to discuss the convergency of the joint MAP algorithm in the considered Gaussian case. We may immediately note that \( A \) obtained by this algorithm is not invertible. This means that, in the Gaussian hypothesis, this algorithm does not really separate the signals. Actually, we could remark this from the expression of the joint criterion in this case which is

\[
J(A, s) = -\log p(A, s|x) = \|x - As\|^2 + \lambda \|s\|^2 + \mu \|A\|^2 + cte.
\] (41)

As we can see, in this case, \( J(A, s) \) is a quadratic function of \( s \) for given \( A \) and a quadratic function of \( A \) for given \( s \), but it is a biquadratic function of both \( A \) and \( s \). This symmetry property means that the joint MAP solution in this case is not unique. This criterion may even have an infinite equivalent.
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The proposed iterative algorithm may then converge to any of these solutions depending on the initialization.

Unfortunately, with any non Gaussian hypothesis, we can not obtain any closed form solution and the existence and the uniqueness of a global optimum is very hard to study. However, we can always propose either a fixed point or a gradient descent based algorithm to compute them numerically. For example, if we assume a Gaussian law for the noise, but non Gaussian prior laws for $s$ and for $A$, we have

$$J(A, s) = -\log p(A, s | x) = \| x - As \|^2 + \lambda \phi(s) + \mu \psi(\| A \|) + cte. \quad (42)$$

where $\phi(s) \propto -\log p(s)$ and $\psi(s) \propto -\log p(A)$. Note that the choice of these prior laws is then important if we want to eliminate the above mentioned symmetry property and to be able to find a unique solution to the problem. Then, a gradient based algorithm writes:

$$\begin{align*}
\hat{s}^{(k+1)} &= \hat{s}^{(k)} - \alpha^{(k)} \frac{\partial J}{\partial \hat{s}}(\hat{A}^{(k)}, \hat{s}^{(k)}) \\
\hat{A}^{(k+1)} &= \hat{A}^{(k)} - \beta^{(k)} \frac{\partial J}{\partial \hat{A}}(\hat{A}^{(k)}, \hat{s}^{(k)})
\end{align*} \quad (43)$$

where $\alpha^{(k)}$ and $\beta^{(k)}$ are two step parameters which can be either constant (fixed step gradient) or adaptive during the iterations $(k)$. Replacing for the gradient expressions we obtain:

$$\begin{align*}
\hat{s}^{(k+1)} &= \hat{s}^{(k)} + \alpha^{(k)} \left[ 2\hat{A}^{(k)}(x - \hat{x}^{(k)}) + \lambda \frac{\partial \phi}{\partial \hat{s}}(\hat{s}^{(k)}) \right] \\
\hat{A}^{(k+1)} &= \hat{A}^{(k)} + \beta^{(k)} \left[ 2\hat{s}^{(k)}(x - \hat{x}^{(k)})^t + \mu \frac{\partial \psi}{\partial \hat{A}}(\hat{A}^{(k)}) \right]
\end{align*} \quad (44)$$

with $\hat{x}^{(k)} = \hat{A}^{(k)} \hat{s}^{(k)}$.

A fixed point based algorithm writes:

$$\begin{align*}
\frac{\partial \phi}{\partial \hat{s}}(\hat{s}^{(k)}) &= \frac{-1}{\lambda} \left[ \hat{A}^{(k)}(x - \hat{x}^{(k)}) \right] \\
\frac{\partial \psi}{\partial \hat{A}}(\hat{A}^{(k)}) &= \frac{-1}{\mu} \left[ \hat{s}^{(k)}(x - \hat{x}^{(k)})^t \right]
\end{align*} \quad (45)$$

One can make the comparison with different neural network based algorithm.

**Remark 2:**
In the Gaussian hypothesis case, if we use the prior law (17), we obtain
similar closed form expressions equivalent to (39)

\[
\begin{align*}
  s_j &= \frac{\sum_i a_{ij} (x_i - \hat{x}_i)}{\lambda + \|a_j\|_2^2} \\
  a_{ij} &= \frac{s_j (x_i - \hat{x}_i)}{s_j^2 + \mu}, \text{ for } i = j \quad \text{and} \quad a_{ij} = \frac{s_j (x_i - \hat{x}_i)}{s_j^2 - \mu}, \text{ for } i \neq j.
\end{align*}
\]

(46)

**Remark 3:**

The main interest of this approach is that we can, at least in theory, account for the existence of any correlation between \(s_j\) and \(s_k\) or to model the temporal behavior of any source \(s_j(t)\), for example, via a markov model. We can also account for any prior information we may have about the mixing matrix \(A\) or impose any desired structure for the separating matrix \(B\). For example, if we know that the sources are labelled in such a way that the sensor \(x_i\) is closer to the sources \(s_i, s_{i-1}\) and \(s_{i+1}\) than to any others, we can use it by choosing a prior probability law

\[
p(A) \propto \exp \left[ -\frac{1}{\sigma^2} \sum_i \sum_j (w_{ij} a_{ij})^2 \right]
\]

(47)

with

\[
w_{ij} = \begin{cases} 
  1 & \text{if } i = j \\
  1/(2|i - j + 1|) & \text{if } i \neq j
\end{cases}
\]

(48)

or \(w_{ii} = 1, w_{i,i-1} = w_{i-1,i} = \alpha\) and \((1 - \alpha)\) for all the other coefficients \(w_{ij}\) for some \(0.5 < \alpha < 1\). Then the equations (39) and (40) become

\[
\begin{align*}
  s_j &= \frac{\sum_i a_{ij} (x_i - \hat{x}_i)}{\lambda + \|a_j\|_2^2} \\
  a_{ij} &= \frac{s_j (x_i - \hat{x}_i)}{w_{ij}^2 (s_j^2 + \mu)}
\end{align*}
\]

(49)

and

\[
\begin{align*}
  \mathbf{s} &= (A^t A + \lambda I)^{-1} A^t \mathbf{x} \\
  \mathbf{A} &= \mathbf{x} s^t (s s^t + \mu \mathbf{W} \mathbf{W}^t)^{-1}
\end{align*}
\]

(50)
Marginal MAP estimations

Now, we consider the two other approaches of marginal MAP estimations. First we note that we can rewrite \( x = As \) with \( A \) is a \((m \times n)\) matrix as

\[
x = As = Sa
\]  

(51)

where \( S \) a \((m \times mn)\) bloc Toeplitz matrix and \( a \) a vector of dimension \( mn \) obtained by piling up all the rows of the matrix \( A \):

\[
S = \begin{pmatrix}
s^t & 0 & \ldots & 0 \\
0 & s^t & \ldots & 0 \\
\vdots \\
0 & \ldots & s^t & 0 \\
0 & 0 & \ldots & s^t
\end{pmatrix}
\]  

and \( a = (a_1^* \ a_2^* \ \cdots \ a_m^*)^t \)  

(52)

To be able to obtain closed form expression, in the following we consider only the Gaussian case:

\[
p(x|A, s) \propto \exp\left[-\frac{1}{2\sigma^2_e} \|x - As\|^2\right] \tag{53}
\]

\[
p(A) \propto \exp\left[-\frac{1}{2\sigma^2_a} \psi(A)\right] \quad \text{with} \quad \psi(A) = \|A\|^2 \tag{54}
\]

\[
p(s) \propto \exp\left[-\frac{1}{2\sigma^2_s} \phi(s)\right] \quad \text{with} \quad \phi(s) = \|s\|^2 \tag{55}
\]

which gives

\[
p(A, s|x) \propto \exp\left[-\frac{1}{2\sigma^2_e} J(A, s)\right] \tag{56}
\]

with

\[
J(A, s) = \|x - As\|^2 + \lambda \phi(s) + \mu \psi(A) = \|x - As\|^2 + \lambda \|s\|^2 + \mu \|A\|^2 \tag{57}
\]

\[
= \|x - Sa\|^2 + \lambda \phi(s) + \mu \psi(a) = \|x - Sa\|^2 + \frac{\lambda}{m} \|S\|^2 + \mu \|a\|^2 \tag{58}
\]

Note that, in this case, \( J(A, s) \) is a quadratic function of \( s \) for fixed \( A \) and a quadratic function of \( A \) for fixed \( s \), but it is a bilinear function of both \( A \) and \( s \). This remark means that the joint MAP solution in this case is not unique.

Note also that \( J(A, s) \) can be rewritten as

\[
J(A, s) = (s - \hat{s})^t \hat{P}_s^{-1} (s - \hat{s}) - \hat{s}^t \hat{P}_s^{-1} \hat{s} + x^t x + \mu \|A\|^2 \tag{59}
\]
with
\[
\hat{s} = (A^t A + \lambda I)^{-1} A^t x \quad \text{and} \quad \hat{P}_s = (A^t A + \lambda I)^{-1};
\]
or as
\[
J(a, S) = (a - \hat{a})^t \hat{P}_a^{-1} (a - \hat{a}) - \hat{a}^t \hat{P}_a^{-1} \hat{a} + x^t x + \lambda \|S\|^2 \tag{60}
\]
with
\[
\hat{a} = (S^t S + \mu I)^{-1} S^t x \quad \text{and} \quad \hat{P}_a = (S^t S + \mu I)^{-1}.
\]

With these notations, it is then easy to obtain the expression of the marginals laws \(p(s|x)\) and \(p(A|x)\):
\[
- \ln p(A|x) \propto - \ln \left| \det(\hat{P}_s^{-1}) \right| - J(A, \hat{s})
\]
\[
= - \ln \left| \det(A^t A + \lambda I) \right| - x^t (x - A\hat{s}) + \mu \|A\|^2 \tag{61}
\]
\[
- \ln p(s|x) \propto - \ln \left| \det(\hat{P}_a^{-1}) \right| - J(\hat{a}, s)
\]
\[
= - \ln \left| \det(S^t S + \mu I) \right| - x^t (x - S\hat{a}) + \lambda s^t s \tag{62}
\]

In non Gaussian hypothesis for \(s\) and \(a\) where \(- \ln p(s) = \phi(s) + cte\) and \(- \ln p(a) = \psi(a) + cte\), we can always use the Laplace approximation of the posterior laws. Then, we can use the first lines of these two last equations by replacing \(\det(\hat{P}_s^{-1})\) and \(\det(\hat{P}_a^{-1})\) by, the Jacobians of their respective log probability densities: \(\nabla_2^2 J(A, s)\) and \(\nabla_2^2 J(a, S)\) computed for the MAP estimates \(\hat{s} = \arg \min_s \{J(A, s)\}\) and \(\hat{a} = \arg \min_a \{J(a, S)\}\).

The marginal MAP solutions of \(A\) and \(s\) respectively satisfy \(\frac{\partial \ln p(A|x)}{\partial A} = 0\) and \(\frac{\partial \ln p(s|x)}{\partial s} = 0\). Note that, excepted the Gaussian case where these equations have analytical solutions, we need a numerical optimisation algorithm to compute the solutions. For example, the marginal MAP estimate of \(A\) can be computed by the following iterative algorithm:
\[
\hat{A}^{(k)} = \arg \max_A \left\{ \ln \left| \det(\hat{P}_s^{(k-1)}) \right| + J(A, s^{(k-1)}) \right\}
\]
\[
= \arg \max_A \left\{ \ln \left| \det(A^t A + \lambda I) \right| + x^t A\hat{s}^{(k-1)} + \mu \psi(A) \right\} \tag{63}
\]

where
\[
\hat{s}^{(k)} = \arg \min_s \left\{ J(\hat{A}^{(k-1)}, s) \right\} = \arg \min_s \left\{ \|x - \hat{A}^{(k-1)} s\|^2 + \lambda \phi(s) \right\}.
\tag{64}
\]
In general, neither of these equations have explicit solutions and we have to do the optimization numerically. For example, a gradient based algorithm to compute the marginal MAP estimate of $A$ writes:

$$
\begin{align*}
\Delta \hat{s}^{(k)} & \propto \hat{A}^{(k-1)} (x - \hat{A}^{(k-1)} s) + \lambda \phi'(s) \\
\Delta \hat{A}^{(k)} & \propto A^t (A^t A + \lambda I)^{-1} + \mu \psi'(A)
\end{align*}
$$

(65)

2.4. Spatially correlated sources

As a first extension we still assume that the sources are white, but they are spatially correlated, i.e.

$$
\log p(s_1, \ldots, s_T) = \sum_t r(s_1(t), \ldots, s_N(t))
$$

where $r(s_1, \ldots, s_N)$ represents the joint probability law of the sources. Then we obtain

$$
\log p(A, s_1, \ldots, s_T | x_1, \ldots, x_T) = \sum_t \sum_i [q_i (x_i(t) - y_i(t)) + r(s_1(t), \ldots, s_N(t))] + \log p(A) + \text{cte.}
$$

(66)

But the main difficulty here will be the modeling of these dependencies and simplification of the expression of the joint probability law $r(s_1, \ldots, s_N)$. For example, if the sources are labelled in such a way that only the immediate neighbor sources are correlated, then we can use a first order Markov model and write

$$
r(s_1, \ldots, s_N) = \sum_j r(s_j | s_{j-1}, s_{j+1})
$$

(67)

However, the complexity of the optimization algorithms depend on the expression of the potential function $r(s_j | s_{j-1}, s_{j+1})$. A simple case is a Gaussian model where

$$
\sum_j r(s_j | s_{j-1}, s_{j+1}) = \frac{1}{2\sigma_s^2} |2s_j - (s_{j-1} + s_{j+1})|^2 = \frac{1}{2\sigma_s^2} \|Ds\|^2
$$

with $D$ a tri-diagonal Toeplitz matrix with diagonal elements equal to 2 and off-diagonal elements equal to -1. In this case, the equations (67) become

$$
\begin{align*}
A^t (x - As) - \lambda D^t D s &= 0 \\
(x - As)s^t - \mu A &= 0
\end{align*}
\rightarrow
\begin{align*}
\begin{cases}
\begin{align*}
s &= (A^t A + \lambda D^t D)^{-1} A^t x \\
A &= x s^t (ss^t + \mu I)^{-1}
\end{align*}
\end{cases}
\end{align*}
$$

(68)
2.5. Spatially independent but colored sources

Here we assume that the sources are mutually independent but that they are temporally colored, i.e.

$$\log p(s_{1..T}) = \sum_j r_j(s_j(1), \ldots, s_j(T))$$

where $r_j(s_j(1), \ldots, s_j(T))$ represents the joint probability law of the different samples of source number $j$. Then we obtain

$$\log p(A, s_{1..T}|x_{1..T}) = \sum_t \sum_i q_i (x_i(t) - y_i(t)) + \sum_j r_j(s_j(1), \ldots, s_j(T))$$

$$+ \sum_k \sum_l a_{kl}^2 + \text{cte.} \quad (69)$$

Here again, the main difficulty is the modelization and simplification of the expression of the joint probability laws $r_j(s_j(1), \ldots, s_j(T))$. For example, we can use a first order markov chain model and write

$$r_j(s_j(1), \ldots, s_j(T)) = \sum_t r_j(s_j(t)|s_j(t-1)) \quad (70)$$

As an example here we consider the Gaussian case (or equivalently first order AR models):

$$r_j(s_j(1), \ldots, s_j(T)) = -\sum_t \alpha_j(s_j(t) - s_j(t-1))^2 \quad (71)$$

With this assumption we have

$$\hat{s}_{1..T}^{(k)} = \arg\max_s \left\{ \sum_t \sum_i q_i (x_i(t) - y_i(t)) - \sum_t \sum_j \alpha_j(s_j(t) - s_j(t-1))^2 \right\}$$

$$\hat{A}^{(k)} = \arg\max_A \left\{ \sum_t \sum_i q_i (x_i(t) - y_i(t)) + \frac{1}{\sigma^2_a} \sum_k \sum_l a_{kl}^2 \right\} \quad (72)$$

The solution at each iteration has to satisfy

$$\sum_t \sum_i a_{ij} q_i' (x_i(t) - y_i(t)) - \sum_t \sum_j 2\alpha_j (s_j(t) - s_j(t-1)) = 0$$

$$\sum_t s_j q_i' (x_i(t) - y_i(t)) + \frac{2}{\sigma_a^2} a_{ij} = 0 \quad (73)$$

For the special case of Gaussian noise we obtain
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\[
\begin{align*}
\sum_t \sum_i a_{ij} (x_i(t) - [As]_i(t)) - \sum_t \sum_j 2\lambda_j (s_j(t) - s_j(t-1)) &= 0 \\
\sum_t s_j(t)(x_i(t) - [As]_i(t)) + \mu a_{ij} &= 0
\end{align*}
\]

(74)

with \( \lambda_j = \alpha_j \sigma^2 \) and \( \mu = \frac{2\sigma^2}{\alpha} \).

The two algorithms of (39) and (40) in this case become:

\[
\begin{align*}
s_j(t) &= \frac{\lambda_j s_j(t-1) + \sum_i a_{ij}(x_i - \hat{x}_i)}{\lambda + \|a_i\|^2} \\
a_{ij} &= \frac{s_j(x_i - \hat{x}_i)}{s_j^2 + \mu}
\end{align*}
\]

(75)

and

\[
\begin{align*}
s(t) &= (A^t A + \lambda I)^{-1} [\text{diag} \{\lambda_1, \ldots, \lambda_n\} s(t-1) + A^t x(t)] \\
A &= x s^t (ss^t + \mu I)^{-1}
\end{align*}
\]

(76)

Here, we conclude the presentation of the Bayesian approach to source separation. Beside some of the details of implementation, we showed that the Bayesian approach gives us the possibility to push further some of the limitations of the classical techniques in source separation. In the next section we give a few preliminary numerical results to show the performances of the proposed algorithms.

3. Simulation results

In the following we give a few preliminary examples of simple source separation problem to show some performances of the proposed methods. In all these examples, we used the following algorithm:

\[
\begin{align*}
y &= (A^t A + \lambda I)^{-1} A^t x \\
s &= g(y) \\
\Delta A \propto A^t (A^t A + \lambda I)^{-1} + x s + \mu \psi'(A)
\end{align*}
\]

(77)

with \( \lambda = \mu = .1, N = 100 \) and appropriate \( g \).

3.1. Example 1

Here, we considered two sources

\[
\begin{align*}
s_1(t) &= \sin(500t + 10 \cos(50t)) \\
s_2(t) &= \sin(300t)
\end{align*}
\]

(78)
and used the mixing matrix $A = \begin{pmatrix} 1 & 0.4 \\ -0.6 & 1 \end{pmatrix}$ to obtain the two set of data $x_1(t)$ and $x_2(t)$. Then we applied the algorithm given in (77). The following figures show the obtained results $\hat{s}_1(t)$ and $\hat{s}_2(t)$.

![Figure 1: Results of the source separation in Example 1.](image1)

![Figure 2: Results of the source separation in Example 1: phase space distribution of sources, mixed signals and separated sources.](image2)
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The two sources are well separated.

3.2. Example 2

Hier, we considered three sources

\[
\begin{align*}
  s_1(t) &= \sin(500t + 10\cos(50t)) \\
  s_2(t) &= \sin(300t) \\
  s_3(t) &= \text{sign}(\cos(120t - 5\cos(50t)))
\end{align*}
\]

and used the mixing matrix

\[
  A = 0.3 \times \begin{pmatrix}
    1 & -0.5 & 0.2 \\
    -0.5 & 1 & -0.5 \\
    0.5 & -0.5 & 1
  \end{pmatrix}
\]

to obtain the three set of data \(x_1(t), x_2(t)\) and \(x_3(t)\). The following figures show the obtained results.
Fig. 4: Results of the source separation in Example 2.

Fig. 5: Results of the source separation in Example 2: phase space distribution of sources, mixed signals and separated sources
Here also the three sources are well separated.

3.3. Example 3

Hier, we considered the two sources of the first example, but we simulated the case where there are three receivers using the mixing matrix

$$A = \begin{pmatrix} 1 & -0.5 \\ 0.5 & 1. \\ -0.2 & 0.5 \end{pmatrix}$$

to obtain the three set of data $x_1(t)$, $x_2(t)$ and $x_3(t)$. Then we applied again the algorithm given in (77). The following figures show the obtained results.
Fig. 7: Results of the source separation in Example 3.

Fig. 8: Results of the source separation in Example 3: phase space distribution of sources, mixed signals and separated sources.
Fig. 9: Results of the source separation in Example 3: histograms of sources, mixed signals and separated sources

3.4. Example 4

Hier, we considered the three sources of the Example 2 and simulated the case where there are only two receivers using the mixing matrix

$$A = \begin{pmatrix} 1 & .2 & 1 \\ -.5 & 1 & .2 \end{pmatrix}$$

to obtain the two set of data \(x_1(t)\) and \(x_2(t)\). Then we applied again the algorithm given in (77). The following figures show the obtained results.
Fig. 7: Results of the source separation in Example 4.

Fig. 8: Results of the source separation in Example 4: phase space distribution of sources, mixed signals and separated sources
4. Conclusions

We investigated the use of the Bayesian estimation theory to source separation and showed that this approach has the potential to push farther the limits of the classical methods. This work is not really yet finished. We are going now to compare the performances of the proposed methods to other conventional ones on simulated and real data.

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