A SHARP SYSTOLIC INEQUALITY FOR 3-MANIFOLDS WITH BOUNDARY

EDUARDO LONGA

Abstract. We prove a sharp systolic inequality relating the homological 2-systole of a compact 3-manifold with boundary to its scalar curvature. If equality is attained, then the universal cover of the manifold is isometric to a cylinder over a round hemisphere.

1. Introduction

Let \((M^3, g)\) be a closed and oriented riemannian 3-manifold. The homological 2-systole of \((M, g)\) is defined by

\[
sys_2(M) = \inf \{ \text{Area}(\Sigma) : \Sigma \subset M \text{ embedded}, [\Sigma] \neq 0 \in H_2(M; \mathbb{Z}) \}.
\]

In a recent paper, D. Stern \cite{3} gave a proof of the following systolic inequality, originally proved by Bray-Brendle-Neves \cite{1} in a stronger version:

**Theorem 1.1.** On a closed, connected and oriented riemannian 3-manifold \((M^3, g)\) with positive scalar curvature \(R_M > 0\) and weakly mean-convex boundary \(H_2^\partial(M; \mathbb{Z}) \neq 0\), we have

\[
sys_2(M) \inf_M R_M \leq 8\pi.
\]

Moreover, if equality holds, then the universal cover of \(M\) is isometric to the standard cylinder \(\mathbb{S}^2 \times \mathbb{R}\) up to scaling.

In this paper, we consider 3-manifolds with nonempty boundary. Let \((M, g)\) be such a manifold, and assume that it is compact and oriented. Define its homological 2-systole by

\[
sys_2(M, \partial M) = \inf \{ \text{Area}(\Sigma, \partial \Sigma) : (\Sigma, \partial \Sigma) \subset (M, \partial M) \text{ embedded}, [\Sigma] \neq 0 \in H_2(M, \partial M; \mathbb{Z}) \}.
\]

Inspired by Stern’s ideas, we prove:

**Theorem 1.2.** Let \((M, g)\) a compact, connected and oriented 3-manifold with nonempty boundary. If \(M\) has positive scalar curvature \(R_M > 0\) and weakly mean-convex boundary \(H_2^\partial(M; \mathbb{Z}) \geq 0\), and \(H_2(M, \partial M; \mathbb{Z}) \neq 0\), then

\[
sys_2(M, \partial M) \inf_M R_M \leq 4\pi.
\]

Moreover, if equality holds, then the universal cover of \(M\) is isometric to the cylinder \(\mathbb{S}^2_+ \times \mathbb{R}\) up to scaling, where \(\mathbb{S}^2_+\) is a closed hemisphere of the unit round sphere.

2010 Mathematics Subject Classification. 53C20, 53C24.

Key words and phrases. Systole, scalar curvature, rigidity.

The author was partially supported by grant 2017/22704-0, São Paulo Research Foundation (FAPESP).
2. Proof of Theorem 1.2

We shall make use of the following result:

**Theorem 2.1** ([2, Theorem 1.1]). Let \((M^3, g)\) be a compact, connected and oriented riemannian 3-manifold with boundary \(\partial M \neq \emptyset\). For a harmonic map \(u : M \to S^1 = \mathbb{R}/\mathbb{Z}\) satisfying homogeneous Neumann condition, we have the identity

\[
2\pi \int_{S^1} \chi(\Sigma_\theta) \geq \int_{S^1} \left( \int_{\Sigma_\theta} \frac{1}{2} |(du)^{-2} |Hess(u)|^2 + R_M \right) + \int_{\partial \Sigma_\theta} H^{\partial M},
\]

where \(R_M\) is the scalar curvature of \(M\), \(H^{\partial M}\) is the mean curvature of \(\partial M\), \(\Sigma_\theta = u^{-1}(\theta)\) is a regular level set of \(u\) and \(\chi(\cdot)\) denotes the Euler characteristic.

Recall that Poincaré-Lefschetz duality gives an isomorphism

\([M : S^1] \cong H^1(M; \mathbb{Z}) \cong H_2(M, \partial M; \mathbb{Z}).\)

Since we are assuming that \(H_2(M, \partial M; \mathbb{Z}) \neq 0\), there is a non-trivial homotopy class \([v] \in [M : S^1]\). Applying standard Hodge theory to the cohomology class \(v^* (d\theta) \in H^1_{dR}(M)\) provides an energy-minimising representative \(u : M \to S^1\). It can be shown that this function is harmonic and satisfies homogeneous Neumann condition along \(\partial M\).

To prove Theorem 1.2, fix such a map \(u : M \to S^1\). By Theorem 2.1 and by the fact that \(H^{\partial M} \geq 0\), we have the following inequalities:

\[
2\pi \int_{S^1} \chi(\Sigma_\theta) \geq \int_{S^1} \left( \int_{\Sigma_\theta} \frac{1}{2} |(du)^{-2} |Hess(u)|^2 + R_M \right) + \int_{\partial \Sigma_\theta} H^{\partial M} \geq \frac{1}{2} \inf_M R_M \int_{S^1} \text{Area}(\Sigma_\theta) \geq \frac{1}{2} \text{sys}_2(M, \partial M) \inf_M R_M \int_{S^1} N(\theta).
\]

(1)

Now notice that whenever \(\Sigma_\theta\) is a regular level set of \(u\), then \(|\Sigma| \neq 0 \in H_2(M, \partial M; \mathbb{Z})\). Indeed, if \(S\) is a connected component of \(\Sigma_\theta\) and \(h = u^* (d\theta)\) is the gradient 1-form induced by \(u\), then

\[
\int_S *h = \int_S |h| > 0,
\]

where \(*h\) is the Hodge dual of \(h\).

Also observe that, if \(N(\theta)\) denotes the number of connected components of \(\Sigma_\theta\), then \(\chi(\Sigma_\theta) \leq N(\theta)\). This is simply because \(\chi(S) \leq 1\) for any compact and connected surface with boundary.

Combining these facts with inequality (1), we obtain

\[
2\pi \int_{S^1} N(\theta) \geq 2\pi \int_{S^1} \chi(\Sigma_\theta) \geq \frac{1}{2} \inf_M R_M \int_{S^1} \text{Area}(\Sigma_\theta) \geq \frac{1}{2} \text{sys}_2(M, \partial M) \inf_M R_M \int_{S^1} N(\theta).
\]

Cancelling factors, we get

\[
\text{sys}_2(M, \partial M) \inf_M R_M \leq 4\pi,
\]

as we wanted.

Suppose now that equality holds. Then, analysing all the steps, we have

(i) \(Hess(u) \equiv 0\) on \(M\);
(ii) \(R_M \equiv \inf_M R_M > 0\) is constant along \(M\);
(iii) \(H^{\partial M} \equiv 0\) along \(\partial M\);
(iv) \(\chi(\Sigma_\theta) = N(\theta)\) for every \(\theta \in S^1\).
Firstly, notice that condition (i) implies that $|du|$ has constant norm (different from 0). So, every level set $\Sigma_\theta$ is regular and totally geodesic. Indeed, let $A$ denote the second fundamental form of a level set of $u$, and let $X, Y$ be tangent vectors of that level set. Then

$$A(X, Y) = \left\langle \nabla_X Y, \frac{\nabla u}{|\nabla u|} \right\rangle = \frac{1}{|\nabla u|} \left\langle \nabla_X Y, \nabla u \right\rangle = -\frac{1}{|\nabla u|} \left\langle \nabla_X \nabla u, Y \right\rangle = -1 \frac{1}{|\nabla u|} \text{Hess}(u)(X, Y) = 0.$$ 

Secondly, the Bochner formula for the (harmonic) gradient 1-form $h = u^*(d\theta)$ reads

$$\Delta \frac{1}{2} |h| = |Dh|^2 + \text{Ric}(h, h)$$

Since $|h| = |du|$ is constant and $Dh = \text{Hess}(u) \equiv 0$, we get $\text{Ric}(h, h) = \text{Ric}(\nabla u, \nabla u) = 0$. Now, the Gauss' formula for a level set $\Sigma_\theta$ of $u$,

$$\text{Ric}(N, N) = \frac{1}{2} \left( R_M - R_\theta + H_\theta^2 - \|A_\theta\|_2^2 \right),$$

gives that the sectional curvature of $\Sigma_\theta$ is constant and equal to $\frac{1}{2} R_M$ (which is itself constant by (ii)). Here, $N = \frac{\nabla u}{|\nabla u|}$ denotes the unit normal, $R_\theta$ the scalar curvature, $H_\theta$ the mean curvature and $A_\theta$ the second fundamental form of $\Sigma_\theta$. This way, each component of a level set of $u$ is isometric to a geodesic ball of a round sphere.

Finally, fixing a connected component $S$ of a level set of $u$, the gradient flow of $u$, $\Phi : S \times \mathbb{R} \to M$,

$$\frac{\partial \Phi}{\partial t} = \frac{\nabla u}{|\nabla u|} \circ \Phi,$$

defines a local isometry. It is easy to see that it is also a covering map. Since the boundary of $M$ is minimal, we conclude that $S$ must be a hemisphere of a round sphere. This completes the proof of the theorem.

References

[1] H. Bray, S. Brendle and A. Neves, Rigidity of area-minimizing two-spheres in three-manifolds, Commun. Anal. Geom. 18(4) (2010), 821–830.

[2] H. Bray and D. Stern, Scalar curvature and harmonic one-forms on three-manifolds with boundary, available at arXiv:1911.06803v1 [math.DG]

[3] D. Stern, Scalar curvature and harmonic maps to $S^1$, available at arXiv:1908.09754v2 [math.DG]

Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, R. do Matão 1010, São Paulo, SP 05508-900, Brazil.

E-mail address: eduardo.longa@ufrgs.br