Lagrangian construction of the (\(gl_n, gl_m\))-duality

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Abstract

We give a geometric realization of the symmetric algebra of the tensor space \(C^n \otimes C^m\) together with the action of the dual pair \((gl_n, gl_m)\) in terms of lagrangian cycles in the cotangent bundles of certain varieties. We establish geometrically the equivalence between the \((gl_n, gl_m)\)-duality and Schur duality. We establish the connection between Springer’s construction of (representations of) Weyl groups and Ginzburg’s construction of (representations of) Lie algebras of type \(A\).

0 Introduction

The left action of \(gl_n\) (resp. \(gl_m\)) on the first (resp. second) factor of the tensor product \(C^n \otimes C^m\) induces natural left actions on the \(d\)-th symmetric tensor space \(S^d(C^n \otimes C^m)\). These two actions clearly commute with each other and they form a dual pair in the sense of Howe (cf. [H]). Following Howe, we have the isotypic decomposition

\[
S^d(C^n \otimes C^m) = \bigoplus_{\lambda \in P_d^{\min(n,m)}} nV_\lambda \otimes mV_\lambda
\]

(1)

where \(P_d^k\) is the set of partitions of \(d\) into at most \(k\) parts and \(nV_\lambda\) (resp. \(mV_\lambda\)) is the irreducible module of \(gl_n\) (resp. \(gl_m\)) with highest weight \(\lambda\), and \(\min(n,m)\) denotes the minimum of \(n\) and \(m\).

On the other hand, we consider the set \(n\mathcal{F}\) of \(n\)-step flags \(\mathfrak{F} = (0 = F_0 \subset \ldots \subset F_n = C^d)\) of the vector space \(C^d\) of complex dimension \(d\), with the induced action of the general linear group \(GL_d\). In such a setup, Beilinson, Lusztig and MacPherson [BLM] constructed the quantum group for \(gl_n\). Inspired by their construction, Ginzburg [G] obtained a micro-local version of it for the enveloping algebra \(U(gl_n)\) in terms of lagrangian cycles in the cotangent bundles of \(n\mathcal{F} \times n\mathcal{F}\). This is an analog of the Springer theory for Weyl groups (cf. e.g. [CG, Hu]). Note that statements are made in [G, CG] in terms of \(sl_n\) although the construction indeed yields \(gl_n\). For our purpose, it is important to stick to \(gl_n\).

Let \(\mathcal{N}\) be the nilpotent cone in the general linear Lie algebra \(gl_d\) and let \(\mathcal{N}_n\) be the subset of \(n\)-step nilpotents in \(\mathcal{N}\). Let \(n\mathcal{M}\) be the set

\[
\mathcal{M} := \{(x, \mathfrak{F}) \in \mathcal{N} \times n\mathcal{F} \mid x(F_i) \subset F_{i-1}, i = 1, \ldots, n\}.
\]
One of the key varieties which we introduce in this paper is the fibred product

\[ n^m := n^M \times_{N_{\min(n,m)}} m^M \subset n^M \times m^M. \]

This generalizes the early considerations (cf. [BLM, CG]) of the variety \( n^Z \) in our notation. As is shown in [G, CG], \( H(n^Z) \) admits a canonical associative algebra structure and there exists a surjective algebra homomorphism \( \rho_n : U(gl_n) \rightarrow H(n^Z) \), where \( H(Z) \) denotes the (component-wise) top dimensional Borel-Moore homology of the variety \( Z \).

Our first main result is to give a geometric realization of the \((gl_n, gl_m)\)-duality (1) in terms of the variety \( n^Z_m \), which can be summarized by the following commuting diagram (Theorem 2.4):

\[
\begin{array}{ccc}
U(gl_n) & \leftrightarrow & S^d(C^n \otimes C^m) \\
\downarrow & & \downarrow \\
H(n^Z) & \leftrightarrow & H(n^Z_m) \\
\oplus_{\lambda \in P^d_n} \operatorname{End}(nV_\lambda) & \leftrightarrow & \oplus_{\lambda \in P^d_{\min(n,m)}} \operatorname{Hom}(nV_\lambda, nV_\lambda) \\
& \leftrightarrow & \oplus_{\lambda \in P^d_m} \operatorname{End}(mV_\lambda)
\end{array}
\]

Here \( \leftrightarrow \) and \( \leftrightarrow \) denote left and right algebra actions.

One advantage of our new approach to the \((gl_n, gl_m)\)-duality is that a natural basis analogous to the canonical basis in quantum groups is given by the fundamental classes of the irreducible components of \( n^Z_m \). We can easily identify each isotypic component inside \( H(n^Z_m) \). It will be of great interest to construct other Howe’s duality (cf. [H]) in the spirit of this paper. Such a construction will shed some light on the geometric construction of \((q\text{-deformed})\) enveloping algebras of other classical Lie algebras.

The Springer theory for the symmetric group \( S_d \), regarded as the Weyl group of \( gl_d \), is built on the setup of the \( GL_d \)-diagonal action on \( B \times B \), where \( B \) denotes the variety of all complete flags in \( \mathbb{C}^d \). Grojnowski and Lusztig [GL] considered the fibred product \( n^W := n^F \times_{N_n} B \) and provided a way to obtain the \( q \)-deformed Schur duality on the space of \( d \)-th tensors of \( C^n \) by studying \( n^W \) (also see [GRV] for further applications).

We formulate the classical Schur duality in terms of Borel-Moore homology. Our second main result is to show how the interplay among cycles in \( n^W, Z \) and \( n^Z_m \) gives rise to the equivalence between the \((gl_n, gl_m)\)-duality and Schur duality. We shall also derive Springer’s theorem on the Weyl groups of type \( A \) from the \((gl_n, gl_m)\)-duality. Along the way, we clarify the precise relations between Springer’s original construction of representations of Weyl groups and Ginzburg’s construction of representations of Lie algebras of type \( A \).

The results of this paper can be generalized to the quantum group setting based on constructions over finite fields, or in terms of equivariant K-theory (cf. [BLM, GL, GRV, CG]).
The plan of this paper is as follows. In Sect. 1 we review the convolution in Borel-Moore homology and introduce some new varieties we need later. In Sect. 2 we present the lagrangian construction of the \((gl_n, gl_m)\)-duality. In Sect. 3 we establish the equivalence between the \((gl_n, gl_m)\)-duality and Schur duality.

1 The variety \(nZ^m_k\)

Given a locally compact space \(Z\) the Borel-Moore homology \(H_* (Z)\) with complex coefficient is defined to be the ordinary (relative) homology of the pair \((\hat{Z}, \infty)\) where \(\hat{Z}\) is the one-point compactification of \(Z\). See [CG] for more on the Borel-Moore homology. We denote by \(H(Z)\) the subspace of \(H_* (Z)\) consisting of (component-wise) top homology classes. In all of our applications, each connected component of \(Z\) is of pure dimension, and so \(H(Z)\) is spanned by the fundamental classes of all irreducible components of \(Z\).

Given two closed subsets \(Z\) and \(Z'\) of a smooth oriented manifold \(M\) of real dimension \(s\), we can define an intersection pairing

\[ \cap : H_i (Z) \times H_j (Z') \rightarrow H_{i+j-s} (Z \cap Z'). \]

Now let \(M_1, M_2\) and \(M_3\) be smooth oriented manifolds of real dimension \(m_1, m_2\) and \(m_3\) respectively. Let \(Z_{12} \subset M_1 \times M_2, Z_{23} \subset M_2 \times M_3\) be closed subsets. Assume that the map

\[ p_{13} : p_{12}^{-1} (Z_{12}) \cap p_{23}^{-1} (Z_{23}) \rightarrow M_1 \times M_3 \]

to be proper. Define the set-theoretic composition

\[ Z_{12} \circ Z_{23} = \{ (x_1, x_3) \in M_1 \times M_3 \mid \text{there exists } x_2 \text{ such that } (x_1, x_2) \in Z_{12} \text{ and } (x_2, x_3) \in Z_{23} \}. \]

We observe that \(Z_{12} \circ Z_{23}\) is the image of \(p_{12}^{-1} (Z_{12}) \cap p_{23}^{-1} (Z_{23})\) under the projection \(p_{13}\). We define a convolution

\[ \ast : H_i (Z_{12}) \times H_j (Z_{23}) \rightarrow H_{i+j-m_2} (Z_{12} \circ Z_{23}) \]

by letting

\[ (c_{12}, c_{23}) \mapsto c_{12} \ast c_{23} = (p_{13})_* (c_{12} \otimes [M_3] \cap [M_1] \otimes c_{23}). \]

\(M_1, M_3\) are allowed to be disconnected. Here \([M_1]\) and \([M_3]\) are understood as the sum of the fundamental classes of connected components of \(M_1\) and \(M_3\).

It is easy to check that the convolution in the Borel-Moore homology satisfies the associativity. Furthermore it has an important dimension property: let

\[ p = \frac{m_1 + m_2}{2}, q = \frac{m_2 + m_3}{2}, r = \frac{m_1 + m_3}{2}. \]

Then the convolution induces a map \(H_p (Z_{12}) \times H_q (Z_{23}) \rightarrow H_r (Z_{12} \circ Z_{23})\).
Now let us introduce various geometric objects we will need in the subsequent sections. We fix a positive integer $d$ throughout this paper. We denote by $^n\mathcal{F}$ the set of all $n$-step partial flags

$$\mathfrak{s} = (0 = F_0 \subset F_1 \subset \ldots \subset F_n = \mathbb{C}^d).$$

$n\mathcal{F}$ is a smooth compact manifold with connected components $^n\mathcal{F}_d$ parameterized by the set of all maps $d : [1, n] \rightarrow \mathbb{Z}_+$ such that $\sum_{i=1}^{n} d_i = d$, where $d_i$ denotes the value of $d$ at $i$. Here

$$^n\mathcal{F}_d = \{ \mathfrak{s} = (0 = F_0 \subset F_1 \subset \ldots \subset F_n = \mathbb{C}^d) \mid \dim F_i / F_{i-1} = d_i \}.$$ 

Denote by $\mathcal{N}$ the nilpotent cone in $\mathfrak{gl}_d$ and by $\mathcal{N}_n$ the subset of $n$-step nilpotents in $\mathcal{N}$, namely

$$\mathcal{N}_n = \{ x \in \text{End}(\mathbb{C}^d) \mid x^n = 0 \}.$$ 

Clearly $\mathcal{N}_n$ is a closed subvariety of $\mathcal{N}$. We define

$$^n\mathcal{M} := \{ (x, \mathfrak{s}) \in \mathcal{N}_n \times ^n\mathcal{F} \mid x(F_i) \subset F_{i-1}, i = 1, \ldots, n \}$$

and denote its connected component by $^n\mathcal{M}_d := ^n\mathcal{M} \cap (\mathcal{N}_n \times ^n\mathcal{F}_d)$. $GL_d$ acts on $^n\mathcal{M}, \mathcal{N}_n$ and $^n\mathcal{F}$. This makes the following natural maps induced from the projections $GL_d$-equivariant:

$$\begin{array}{ccc}
^n\mathcal{M} & \xrightarrow{\mu} & \mathcal{N}_n \\
\downarrow & & \downarrow \pi \\
^n\mathcal{F} & & \\
\end{array}$$

One has a natural isomorphism of vector bundles between $^n\mathcal{M}$ and the cotangent bundle $T^*(^n\mathcal{F})$ which makes the following diagram commute:

$$\begin{array}{ccc}
^n\mathcal{M} & \xrightarrow{\pi} & T^*(^n\mathcal{F}) \\
\downarrow & & \downarrow \tilde{\pi} \\
^n\mathcal{F} & & \\
\end{array}$$

Here $\tilde{\pi}$ is the natural projection from $T^*(^n\mathcal{F})$ to the base manifold $^n\mathcal{F}$. We shall denote $^n\mathcal{F}_x := \mu^{-1}(x)$ where $\mu : ^n\mathcal{M} \rightarrow \mathcal{N}_n$ and $x \in \mathcal{N}_n$.

For integers $n, m > 0$ and $k \geq 0$, we introduce the following variety which will play a fundamental role in our geometric construction of $(\mathfrak{gl}_n, \mathfrak{gl}_m)$-duality:

$$^nZ_k^m := ^n\mathcal{M} \times_{\mathcal{N}_k} ^m\mathcal{M}$$

$$= \{ (x, \mathfrak{s}, \mathfrak{s}'), (x, \mathfrak{s}) \in \mathcal{N}_k \times ^n\mathcal{F} \times ^m\mathcal{F} \mid x(F_i) \subset F_{i-1}, x(F'_j) \subset F'_{j-1}, 1 \leq i \leq n, 1 \leq j \leq m \}.$$
Remark 1.1 Clearly \(^nZ_k^m\) is a closed subvariety of \(^{n\!}Z_{k+1}^m\), and \(^nZ_k^m\) stabilizes and is equal to the variety \(\mathbb{n}\! \mathcal{M} \times \mathbb{N} ^m\mathcal{M}\) when \(k \geq \min(n, m)\). We will often implicitly use the identities \(\mathbb{n}\! \mathcal{M}^m_{\min(n, k, m)} = \mathbb{n}\! \mathcal{M}^m_{\min(n, k)} = \mathbb{n}\! \mathcal{M}^m_{\min(k, m)} = \mathbb{n}\! \mathcal{M}^m_k\).

We note that \(\mathbb{n}\! \mathcal{M} \times m\! \mathcal{M} \approx T^*(\mathbb{n}\! \mathcal{F}) \times T^*(m\! \mathcal{F}) \approx T^*(\mathbb{n}\! \mathcal{F} \times m\! \mathcal{F})\). (2)

The second isomorphism above is given by changing the sign of the symplectic structure in the second factor \(T^*(m\! \mathcal{F})\) as usual.

\(\mathbb{n}\! \mathcal{Z}^m_k\) is not connected in general. However we have the following crucial dimension property component-wise. Let \(\mathbb{n}\! \mathcal{Z}^m_k(\mathcal{O})\) be the preimage of a nilpotent \(GL_d\)-orbit \(\mathcal{O}\) under the projection from \(\mathbb{n}\! \mathcal{Z}^m_k\) to \(\mathcal{N}_k\).

**Proposition 1.1** Let \(d_1\) (resp. \(d_2\)) be a partition of \(d\) in \(\mathbb{P}_n^d\) (resp. \(\mathbb{P}_m^d\)). Let \(Z^{\alpha}\) be an irreducible component of \(\mathbb{n}\! \mathcal{Z}^m_k\) contained in \(\mathbb{n}\! \mathcal{M}_{d_1} \times m\! \mathcal{M}_{d_2}\). Then we have

\[
\dim Z^{\alpha} = \frac{1}{2} \dim (\mathbb{n}\! \mathcal{M}_{d_1} \times m\! \mathcal{M}_{d_2}).
\]

**Proof.** Clearly \(\mathbb{n}\! \mathcal{Z}^m_k\) is the union of \(\mathbb{n}\! \mathcal{Z}^m_k(\mathcal{O})\) over all nilpotent conjugacy classes in \(\mathcal{N}_k\). We can write

\[
\mathbb{n}\! \mathcal{Z}^m_k(\mathcal{O}) = GL_d \times G(x) (\mathbb{n}\! \mathcal{F}_x \times m\! \mathcal{F}_x),
\]

where \(x \in \mathcal{O}\) and \(G(x)\) is the stabilizer of \(x\) in \(GL_d\). Thus an irreducible component \(Z^{\alpha}\) of \(\mathbb{n}\! \mathcal{Z}^m_k\) is the closure of a unique component of \(\mathbb{n}\! \mathcal{Z}^m_k(\mathcal{O})\) (say in the connected component \(\mathbb{n}\! \mathcal{F}_{d_1} \times m\! \mathcal{F}_{d_2}\)) of the form

\[
GL_d \times G(x) (\mathbb{n}\! \mathcal{F}^\alpha_x \times m\! \mathcal{F}^\beta_x),
\]

where \(\mathbb{n}\! \mathcal{F}^\alpha_x\) (resp. \(m\! \mathcal{F}^\beta_x\)) is an irreducible component in \(\mathbb{n}\! \mathcal{F}_x \cap \mathbb{n}\! \mathcal{F}_{d_1}\) (resp. \(m\! \mathcal{F}_x \cap m\! \mathcal{F}_{d_2}\)). According to a theorem of Spaltenstein (cf. [S]), the variety \(\mathbb{n}\! \mathcal{F}_x \cap \mathbb{n}\! \mathcal{F}_{d_1}\) is connected and of pure dimension so that

\[
\dim \mathcal{O}_x + 2 \dim (\mathbb{n}\! \mathcal{F}_x \cap \mathbb{n}\! \mathcal{F}_{d_1}) = 2 \dim \mathbb{n}\! \mathcal{F}_{d_1}.
\]

Similarly we have

\[
\dim \mathcal{O}_x + 2 \dim (m\! \mathcal{F}_x \cap m\! \mathcal{F}_{d_2}) = 2 \dim m\! \mathcal{F}_{d_2}.
\]

It follows by comparing with Eq. (2) that

\[
\dim Z^{\alpha} = \dim \mathbb{n}\! \mathcal{Z}^m_k(\mathcal{O}) = \dim \mathcal{O}_x + \dim (\mathbb{n}\! \mathcal{F}_x \cap \mathbb{n}\! \mathcal{F}^\alpha_{d_1}) + \dim (m\! \mathcal{F}_x \cap m\! \mathcal{F}^\beta_{d_2}) = \dim \mathbb{n}\! \mathcal{F}_{d_1} + \dim m\! \mathcal{F}_{d_2} = \frac{1}{2} \dim (\mathbb{n}\! \mathcal{M}_{d_1} \times m\! \mathcal{M}_{d_2}).
\]
This completes the proof.

In the case when \( k \geq \min(n, m) \) and so \( ^nZ^m_k = ^nZ^m \), we have the following description of \( ^nZ^m \) which strengthens Proposition 1.1. The proof is the same as for a similar statement in Springer theory (cf. Proposition 3.3.4, [CG]).

**Proposition 1.2** The variety \( ^nZ^m \) is a union of the conormal bundles to all the \( GL_d \)-orbits in \( ^nF \times ^mF \). Each irreducible component is the closure of the conormal bundle to a unique \( GL_d \)-orbit.

**Remark 1.2** It follows that \( ^nZ^m \) is a lagrangian subvariety of \( ^nM \times ^mM \). So is \( ^nZ^m_k \) since \( ^nZ^m_k \) is a subvariety of \( ^nZ^m \) and \( ^nZ^m_k \) has half the dimension of \( ^nM \times ^mM \) component-wise by Proposition 1.1. Note that in general \( ^nZ^m_k \) is not a union of some conormal bundles to all the \( GL_d \)-orbits in \( ^nF \times ^mF \). \( ^nZ_0^m = ^nF \times ^mF \) for \( k = 0 \) is such an example.

## 2 Lagrangian construction of \((gl_n, gl_m)\)-duality

We make the following observation. The case when \( n = m \) is well known (cf. e.g. [BLM, CG]).

**Lemma 2.1** The orbits of the diagonal action of \( GL_d \) on \( ^nF \times ^mF \) are parameterized by the \( n \times m \) matrices with non-negative integral entries and with the sum of all entries equal to \( d \).

The correspondence is defined as follows: given a pair of flags \((\mathfrak{F}, \mathfrak{F}') \in ^nF \times ^mF\), we let the \((i, j)\)-th entry of the \( n \times m \) matrices to be the dimension of the quotient \( F_i \cap F'_j / (F_{i-1} \cap F'_j + F_i \cap F'_{j-1}) \), \( 1 \leq i \leq n, 1 \leq j \leq m \). In particular, it follows from the lemma that the number of \( GL_d \)-diagonal orbits in \( ^nF \times ^mF \) is equal to \( \dim S^d(C^n \otimes C^m) \).

One easily verifies the set-theoretic composition

\[
^nZ^m_a \circ ^mZ^k_b = ^nZ^k_{\min(a,m,b)}, \quad a, b \geq 0, \ n, m, k > 0.
\]

By applying the convolution in Borel-Moore homology introduced in Sect. 1 to our situation, we obtain a map

\[
H_* (^nZ^m_a) \times H_* (^mZ^k_b) \to H_* (^nZ^k_{\min(a,m,b)}).
\]

Thanks to Proposition 1.1 and the dimension property of the Borel-Moore homology, we obtain the following proposition.

**Proposition 2.1** The convolution induces a map

\[
\star : H (^nZ^k_a) \times H (^kZ^m_b) \to H (^nZ^m_{\min(a,k,b)}).
\]
We list below some important consequences of Proposition 2.1. We recall that 
\( n^Z_m = n^m \) if \( r \geq \min(n,m) \).

**Proposition 2.2**

1) \((H(n^Z_r), \ast)\) is an associative algebra with unit. In particular, \((H(n^Z_r), \ast)\) is an associative algebra.

2) The algebra \(H(n^Z_r)\) acts on \(H(n^Z_k)\) from the left while \(H(m^Z_m)\) acts on \(H(n^Z_k)\) from the right by convolution. These two actions commute with each other.

**Proof.** Putting \( H \) structure on \( n \) Ginzburg [G].

The statement that (Remark 2.1 into at most \( n \) of nilpotent conjugacy classes in the \( \lambda \) we associate to \( \lambda \) an algebra \( A \) H.

Following Ginzburg [G, CG], the algebra \( A \) is an irreducible module over \( H(n^Z_r) \). Given an algebra \( A \) and a left \( A \)-module \( V \), we endow \( V^\vee := \text{Hom}(V, \mathbb{C}) \) a right \( A \)-module structure by letting \((\hat{\cdot}, a)(v) = \hat{a}(a \cdot v)\), where \( \hat{a} \in V^\vee, a \in A, v \in V \). It can be shown (cf. [CG]) that \( H_m^Z F_x \) is isomorphic to \( H^Z F_x \) as a right \( H(n^Z_r) \)-module, where \( R \) denotes right module.

**Remark 2.1** The statement that \((H(n^Z_r), \ast)\) is an associative algebra is due to Ginzburg [G].

We denote by \( P_d \) the set of partitions of \( d \) and by \( P_d \) the set of partitions of \( d \) into at most \( n \) parts. Note that \( P_n \) is in one-to-one correspondence with the set of nilpotent conjugacy classes in the \( n \)-step nilpotent cone \( \mathcal{N}_n \): given a partition \( \lambda \in P_n \), let \( \lambda^t = (a_1, a_2, \ldots) \) be the transpose of \( \lambda \) (\( n \geq a_1 \geq a_2 \geq \ldots \geq 0 \)); we associate to \( \lambda \) the Jordan form \( x_\lambda \) in \( \mathcal{N}_n \) consisting of Jordan blocks of size \( a_1, a_2, \ldots \). We denote by \( c_\lambda \) the nilpotent conjugacy classes of \( x_\lambda \).

One easily shows that 
\[ n^Z_r \circ n^Z_x = n^Z_x, \quad m^Z_r \circ m^Z_m = m^Z_x. \]

Following Ginzburg [CG, CG], \( H(n^Z_x) \) is an irreducible module over \( H(n^Z_r) \). Given an algebra \( A \) and a left \( A \)-module \( V \), we endow \( V^\vee := \text{Hom}(V, \mathbb{C}) \) a right \( A \)-module structure by letting \((\hat{\cdot}, a)(v) = \hat{a}(a \cdot v)\), where \( \hat{a} \in V^\vee, a \in A, v \in V \). It can be shown (cf. [CG]) that \( H_m^Z F_x \) is isomorphic to \( H^Z F_x \) as a right \( H(n^Z_r) \)-module, here \( R \) denotes right module.

**Theorem 2.1** Under the commuting left action of \( H(n^Z_r) \) and the right action of \( H(m^Z_m) \), the space \( H(n^Z_k) \) decomposes as follows:

\[ H(n^Z_k) = \bigoplus_{\lambda \in P_n^{\text{min}(n,k,m)}} \text{Hom}(H(m^Z_x), H(n^Z_x)). \]

Here \( \text{Hom}(H(m^Z_x), H(n^Z_x)) \) is understood as \( H(n^Z_x) \otimes H(m^Z_x)^\vee. \)
Note that in the case when \( k = m = n \), we recover a theorem of Ginzburg (see Theorem 4.1.23 in [CG]). Our theorem in the general case can be proved by following the same line as in [CG] which we sketch below.

**Proof.** We introduce a partial order on the set of nilponent orbits \( O \subset N' \):

\[
O' \leq O \iff O' \subset \overline{O}, \quad O' < O \iff O' \subset \overline{O}, \quad O' \neq O,
\]

where \( \overline{O} \) is the closure of \( O \). Put

\[
A_{\leq O} = \bigcup_{O' \leq O} nZ_k^m(O'), \quad A_{< O} = \bigcup_{O' < O} nZ_k^m(O').
\]

These are closed subvarieties of \( nZ_k^m \). We observe that \( H(A_{< O}) \subset H(A_{\leq O}) \) as a module of \( H(nZ^n) \) and \( H(mZ^m) \). Denote \( H_O = H(A_{\leq O})/H(A_{< O}) \). Then \( H_O \) is spanned by the fundamental classes of irreducible components of \( nZ_k^m(O) \). One can show that as a \((H(nZ^n), H(mZ^m))-\)module there is an isomorphism between \( H_O \) and \( H(nF_{x\lambda}) \otimes H(mF_{x\lambda})_{R} \).

The partial order \( \leq \) gives a filtration of \( H(nZ_k^n) \) by the submodule \( H(A_{< O}) \). The graded space associated to this filtration as a \((H(nZ^n), H(mZ^m))-\)module is isomorphic to \( H(nZ_k^n) \) since \( H(nZ^n) \) and \( H(mZ^m) \) are semisimple algebras [CG]. Therefore

\[
H(nZ_k^n) \approx \bigoplus_{O \subset N'} H_O \\
\approx \bigoplus_{\lambda \in P_{\min(n,k,m)}} H(nF_{x\lambda}) \otimes H(mF_{x\lambda})_{R} \\
\approx \bigoplus_{\lambda \in P_{\min(n,k,m)}} H(nF_{x\lambda}) \otimes H(mF_{x\lambda})^\vee \\
\approx \bigoplus_{\lambda \in P_{\min(n,k,m)}} H_{\text{Hom}}(H(mF_{x\lambda}), H(nF_{x\lambda})).
\]

\( \square \)

The following theorem relates \( H(nZ^n) \) to the enveloping algebra \( U(gl_n) \).

**Theorem 2.2** (Ginzburg) There exists a canonical surjective algebra homomorphism \( \rho_n : U(gl_n) \rightarrow H(nZ^n) \). The \( H(nZ^n) \)-module \( H(F_{x\lambda}) \), regarded as a left \( gl_n \)-module, has highest weight \( \lambda \).

It is known that there exists an anti-involution \( \omega \) on \( H(mZ^m) \) induced from sending \((x, \mathfrak{g}, \mathfrak{g}') \) in \( mZ^m \) to \((x, \mathfrak{g}', \mathfrak{g}) \) which is compatible via \( \rho_m \) with the Cartan anti-involution on \( gl_m \) by taking the transpose (cf. [CG]). By combining the right action of \( H(mZ^m) \) and the anti-involution \( \omega \), we obtain a left \( H(mZ^m) \)-action on \( H(nZ_k^n) \).
Given a left (resp. right) \( gl_m \)-module \( V \), we define a right (resp. left) \( gl_m \) action on \( V \) by letting an element act by its transpose. Denote by \( V^t \) the right (resp. left) \( gl_m \)-module thus obtained. As a consequence of the existence of a non-degenerate invariant form on \( mV_\lambda \), the right \( gl_m \)-module \( mV_\lambda^t \) is isomorphic to \( mV_\lambda' \). Thus we can reformulate Theorem 2.1 as follows.

**Theorem 2.3** Under the left action of \( gl_n \) and the right action of \( gl_m \), the space \( H^{(nZ^m)} \) decomposes as follows:

\[
H^{(nZ^m)} = \bigoplus_{\lambda \in P_{\text{min}}(n,k)} nV_\lambda \otimes mV_\lambda'.
\]

**Remark 2.2** Denote by \( \mathfrak{h}_n \) the Cartan subalgebra of diagonal matrices in \( gl_n \) and \( ^n e_i \) \((i = 1, \ldots, l)\) the standard basis of \( \mathfrak{h}_n \) with 1 in the \((i, i)\)-th entry and 0 elsewhere. By using the explicit formula for the homomorphism \( \rho_n \) (resp. \( \rho_m \)) (cf. \([\mathbb{C} \mathfrak{g}] \)), we can easily show that any fundamental class of an irreducible component of \( ^n Z^m \) is a weight vector with respect to the action of \( \mathfrak{h}_n \) and \( \mathfrak{h}_m \). More precisely, given an irreducible component \( Z_A \) of \( ^n Z^m \) corresponding to an \( n \times m \) matrix \( A = (a_{ij}) \) in Lemma 2.1. Define \( \lambda_i = \sum_{j=1}^m a_{ij}, \mu_j = \sum_{i=1}^n a_{ij} \). Then \( (\lambda_1, \ldots, \lambda_n) \) is the weight for \( \mathfrak{h}_n \subset gl_n \) (with respect to the standard basis \( (^n e_i)_{i=1}^n \)) and \( (\mu_1, \ldots, \mu_m) \) is the weight for \( \mathfrak{h}_m \subset gl_m \).

**Example 2.1** When \( m = 1 \), the variety \( ^1 \mathcal{F} \) is a single point and so \( ^n Z_1^1 = ^n \mathcal{F} \). Note that \( ^n \mathcal{F} \) is also the fiber \( \mu^{-1}(0) \). Theorem 2.3 shows that as a \( gl_n \)-module \( H^{(n\mathcal{F})} \) is irreducible and isomorphic to \( S^d(\mathbb{C}^n) \). On the other hand, if \( d = 1 \) then \( ^n \mathcal{F} \) is a discrete set of \( n \) points and so \( ^n Z^m \) is a set of \( nm \) points. Therefore \( H^{(nZ^m)} \approx \mathbb{C}^n \otimes \mathbb{C}^m \).

The natural left actions of \( gl_n \) and \( gl_m \) on \( \mathbb{C}^n \otimes \mathbb{C}^m \) induces left actions on the \( d \)-th symmetric tensor \( S^d(\mathbb{C}^n \otimes \mathbb{C}^m) \). The decomposition \( (1) \) of the space \( S^d(\mathbb{C}^n \otimes \mathbb{C}^m) \) is exactly the same as \( H^{(nZ^m)} \) in Theorem 2.3 under the action of the dual pair \((gl_n, gl_m)\). Thus we have obtained a left \((gl_n, gl_m)\)-module isomorphism

\[
H^{(nZ^m)} \approx S^d(\mathbb{C}^n \otimes \mathbb{C}^m). \tag{4}
\]

The natural isomorphism above seems to be new even in the case when \( n = m \).

It follows from Theorem 2.3 that we can identify the algebra \( H^{(nZ^n)} \) with a direct sum of the endomorphism algebra \( \sum_{\lambda \in P_n} \text{End}(nV_\lambda) \). The left (resp. right) action of \( H^{(nZ^n)} \) (resp. \( H^{(nZ^m)} \)) on \( H^{(nZ^m)} \) corresponds to the natural left (resp. right) action of \( \text{End}(nV_\lambda) \) (resp. \( \text{End}(mV_\lambda) \)) on \( nV_\lambda \otimes mV_\lambda' \). For the convenience of the reader we summarize the main results of this section as follows. The right action of \( gl_m \) on \( S^d(\mathbb{C}^n \otimes \mathbb{C}^m) \) in the diagram is related to the left action of \( gl_m \) in Eq. (1) by the Cartan anti-involution in \( gl_m \).
Theorem 2.4 \(((gl_n, gl_m)\)-duality) We have the following commutative diagram:

\[
\begin{array}{cccc}
U(gl_n) & \xrightarrow{\cdot} & S^d(C^n \otimes C^m) & \xleftarrow{\cdot} & U(gl_m) \\
\downarrow & & \downarrow & & \downarrow \\
H(nZ^n) & \xrightarrow{\cdot} & H(nZ^m) & \xleftarrow{\cdot} & H(mZ^m) \\
\bigoplus_{\lambda \in P^d_n} \text{End}(nV_\lambda) & \xrightarrow{\cdot} & \bigoplus_{\lambda \in P^d_{\min(n,m)}} \text{Hom}(mV_\lambda, nV_\lambda) & \xleftarrow{\cdot} & \bigoplus_{\lambda \in P^d_m} \text{End}(mV_\lambda)
\end{array}
\]

Here \(\cdot\) and \(\cdot\) denote left and right algebra actions.

Following the same line of the proof of Theorem 2.3, we can have the following identification (which is \((gl_n, gl_m)\)-equivariant):

\[
\begin{align*}
H(nZ^k_a) &= \bigoplus_{\lambda \in P^d_{\min(n,a,k)}} \text{Hom}(kV_\lambda, nV_\lambda) \\
H(kZ^m_b) &= \bigoplus_{\lambda \in P^d_{\min(k,b,m)}} \text{Hom}(mV_\lambda, kV_\lambda) \\
H(nZ^{m}_{\min(a,k,b)}) &= \bigoplus_{\lambda \in P^d_{\min(n,a,k,b,m)}} \text{Hom}(mV_\lambda, nV_\lambda)
\end{align*}
\]

Furthermore, the map in the left column induced by the convolution is identified with the map in the right column given by the obvious composition.

3 The \((gl_n, gl_m)\)-duality and Schur duality

Denote by \(B\) the flag manifold of \(GL_d\), namely

\[
B := \{ \mathfrak{g} = (0 = F_0 \subset F_1 \subset \ldots \subset F_d = \mathbb{C}^d) | \dim F_i/F_{i-1} = 1, i = 1, \ldots, d \}.
\]

Recall that \(N\) is the nilpotent cone of \(gl_d\). We define

\[
\tilde{N} := \{ (x, \mathfrak{g}) \in N \times B | x(F_i) \subset F_{i-1}, i = 1, \ldots, d \}.
\]

The following diagram is commutative and \(GL_d\)-equivariant:

\[
\begin{array}{c}
\tilde{N} \\
\mu \downarrow \\
N \\
\nearrow B
\end{array}
\]

We also have a natural \(GL_d\)-equivariant vector bundle isomorphism between \(\tilde{N}\) and the cotangent bundle \(T^*B\). The projection to \(N\) is called the Springer resolution. \(B_x := \mu^{-1}(x), x \in N\) is the Springer fiber (cf. e.g. [CG]).

We define the fibred products

\[
Z := \tilde{N} \times_N \tilde{N}, \quad ^nW := ^nM \times_{N^n} \tilde{N},
\]
and denote
\[ n W \hat{\circ} m W := (n \mathcal{M} \times_{N_n} \tilde{N}) \circ (\tilde{N} \times_{N_m} m \mathcal{M}). \]

One easily shows that
\[ n W \circ Z = n W, \quad n Z^n \circ n W = n W, \quad n W \hat{\circ} m W = n Z^m. \]

The algebra structure on \( H(Z) \) is given as a special case of Proposition 3.2 below.

**Theorem 3.1** (Springer) The algebra \( H(Z) \) is isomorphic to the group algebra \( \mathbb{C}[S_d] \) of the symmetric group \( S_d \).

**Theorem 3.2** (Schur duality) \( H(n W) \) is isomorphic to the \( d \)-th tensor space of \( \mathbb{C}^n \), such that the following diagram commutes:

\[
\begin{align*}
U(gl_n) & \twoheadrightarrow (\mathbb{C}^n)^{\bigotimes d} & \leftrightarrow & \mathbb{C}[S_d] \\
\downarrow & & & \\
H(n Z^n) & \twoheadrightarrow H(n W) & \leftrightarrow & H(Z) \\
\bigoplus_{\lambda \in P_n} \text{End}(n V_\lambda) & \leftrightarrow \bigoplus_{\lambda \in P_n} n V_\lambda \otimes S_\lambda & \leftrightarrow & \bigoplus_{\lambda \in P_n} \text{End}(S_\lambda)
\end{align*}
\]

where \( S_\lambda \) is the irreducible representation of \( S_d \) parameterized by \( \lambda \).

The variety \( n W \) was earlier introduced by Lusztig and Grojnowski [GL]. Although we cannot find the Schur duality in terms of Borel-Moore homology written down explicitly anywhere, it is certainly well known to experts (see [GL, GRV]). Next, we shall derive Theorem 3.2 and also Springer's theorem from the \((gl_n, gl_m)\)-duality. Given a \( gl_d \)-module \( U \), we shall use \( U^{h_d, \text{det}} \) to denote the zero-weight space in \( U \) of weight \((1, 1, \ldots, 1)\) with respect to the Cartan subalgebra \( h_d \).

Set \( m = d \). Note that \( B \) is a connected component of \( d \mathcal{F} \). By Remark 2.2, we see that the irreducible components in \( n \mathcal{F} \times d \mathcal{F} \) whose fundamental classes are of zero weight \((1, \ldots, 1)\) with respect to \( h_d \) are those coming from \( n \mathcal{F} \times_{N} B \). In this way we see that

\[ H(n W) = H(n Z^d)^{h_d, \text{det}}. \quad (5) \]

In view of the identification (4) and Remark 2.2, this is a geometric interpretation of the following isomorphism of \((gl_n, S_d)\)-modules (cf. [H]):

\[ (\mathbb{C}^n)^{\bigotimes d} = S^d(\mathbb{C}^n \bigotimes \mathbb{C}^d)^{h_d, \text{det}}. \quad (6) \]

The \((gl_n, gl_m)\)-duality identifies the r.h.s. of Eqs. (5) and (6), and so it follows that \( H(n W) = (\mathbb{C}^n)^{\bigotimes d} \).
Note that $B_{x\lambda} \subset dF_{x\lambda}$. An argument similar to what leads to Eq. (5) shows that

$$H(B_{x\lambda}) = H(dF_{x\lambda})^{h_d,\det} = dV_{h_d,\det}^\lambda. \quad (7)$$

We note that the Weyl group $S_d$ acts on the zero-weight vector space of a $gl_d$ module. The following is a well-known fact, cf. [H] and references therein.

**Proposition 3.1** For $\lambda \in \mathcal{P}^d_n$, as an $S_d$-module $\mathcal{V}_\lambda^{h_d,\det}$ is irreducible and isomorphic to $S_\lambda$.

By combining Theorem 2.3, the isomorphism (5) and Proposition 3.1, we obtain

$$H(nW) = H(nZ^d)^{h_d,\det}$$

$$= \bigoplus_{\lambda \in \mathcal{P}^d_n} H(dF_{x\lambda}) \otimes H(B_{x\lambda})$$

$$= \bigoplus_{\lambda \in \mathcal{P}^d_n} \mathcal{V}_\lambda \otimes mV_{h_d,\det}^\lambda$$

$$= \bigoplus_{\lambda \in \mathcal{P}^d_n} n\mathcal{V}_\lambda \otimes S_\lambda. \quad (7)$$

To complete the diagram of Schur duality in Theorem 3.2 it only remains to prove Theorem 3.1.

**Proof of Theorem 3.1.** Let us consider the action of $h_n$ on $H(dZ^d)^{h_d,\det}$ in the case $n = d$. Noting that $Z \subset dZ^d$, we obtain an analog of (7):

$$H(dZ^d)^{h_d,\det} \bigoplus n\mathcal{V}_\lambda \otimes m\mathcal{V}_{h_d,\det}^\lambda = H(Z). \quad (8)$$

By combining the ($gl_d, gl_d$)-duality (Theorem 2.3), the isomorphism (8) and Proposition 3.1 we have

$$H(Z) \approx \bigoplus_{\lambda \in \mathcal{P}^d} S_\lambda \otimes S_\lambda \approx \mathbb{C}[S_d]. \quad \square$$

Indeed the Schur duality also implies the ($gl_n, gl_m$)-duality. The equivalence between ($gl_n, gl_m$)-duality and Schur duality was established in a completely different setting [H]. However we hope that the reader may find it illuminating to see how the interplay between the ($gl_n, gl_m$)-duality and the Schur duality is reflected in our geometric setup.

Let us assume the Schur duality. $nW$ is a union of

$$nW(\mathcal{O}) := GL_d \times_{G(x)} (nF_x \times B_x) \quad (x \in \mathcal{O}) \quad (9)$$

over all nilpotent orbits $\mathcal{O}$ in $N_n$. The composition of sets between $nW(\mathcal{O})$ and $mW(\mathcal{O}')$ (in the sense of $nW \circ mW$) is non-zero if and only if $\mathcal{O}' = \mathcal{O}$, and

$$nW(\mathcal{O}) \circ mW(\mathcal{O}) = GL_d \times_{G(x)} (nF_x \times mF_x) = nZ^m(\mathcal{O}). \quad (10)$$
The Schur duality states that
\[ H(n W) = \bigoplus_{\lambda \in P_n} H(n F_{x_\lambda}) \otimes H(B_{x_\lambda}) = \bigoplus_{\lambda \in P_n} \, {}^n V_\lambda \otimes S_\lambda. \]

It follows from (9) that the isotropic space \( H(n F_{x_\lambda}) \otimes H(B_{x_\lambda}) \) is spanned by the fundamental classes of irreducible components of \( {}^n W(O_\lambda) \), where \( O_\lambda \) denotes the orbit of \( x_\lambda \). In particular \( H(n F_{x_\lambda}) \) is isomorphic to \( {}^n V_\lambda \). Then Eq. (10) and the fact \( {}^n Z^m = \bigsqcup_O {}^n Z^m(O) \) implies the \((gl_n, gl_m)\)-duality
\[ H(n Z_m) = \bigoplus_{\lambda \in P_{\max}(n,m)} H(n F_{x_\lambda}) \otimes H(m F_{x_\lambda}) = \bigoplus_{\lambda \in P_{\max}(n,m)} \, {}^n V_\lambda \otimes {}^m V_\lambda. \]

The rest of this section is the counterpart of results in Section 2 by substituting \( n F, N_n \) etc with \( B, N \) etc. The proofs are totally analogous which we will omit.

We introduce a new variety
\[ Z_k := \Nbar \times_{N} \Nbar \quad (k \geq 0). \]
We have the composition of sets
\[ Z_k \circ Z_l = Z_{\min(k,l)}. \]

We can prove as in the preceding section that \( Z_k \) is a lagrangian subvariety of \( \Nbar \times_{N} \Nbar \). It follows from the dimension property of the convolution in Borel-Moore homology that the convolution induces a map
\[ \star : H(Z_k) \otimes H(Z_l) \longrightarrow H(Z_{\min(k,l)}). \]

In particular we obtain an analog of Proposition 2.2. Note that \( Z_k \subset Z \), and \( Z_k = Z \) iff \( k \geq d \).

**Proposition 3.2** \((H(Z_k), \star)\) is an associative algebra with unit. \( H(Z_k) \) carries a \( H(Z) \) bi-module structure.

One easily shows that
\[ Z \circ B_x = B_x, \quad B_x \circ Z = B_x. \]

The dimension property of \( Z \) ensures that the convolution induces a \( H(Z) \) bi-module structure on \( H(B_x) \). The following theorem is an analog of Theorem 2.1. In the case when \( k \geq d \) so that \( Z_k = Z \) it reduces to Springer’s theorem.

**Theorem 3.3** The algebra \( H(Z_k) \) is isomorphic to \( \bigoplus_{\lambda \in P_k^d} \text{End}(H(B_{x_\lambda})) \). This is also an \( S_d \)-bi-module isomorphism.

In view of Eq. (9) and Proposition 3.1, we reformulate Theorem 3.3 as follows.

**Theorem 3.4** The algebra \( H(Z_k) \) is isomorphic to \( \bigoplus_{\lambda \in P_k^d} \text{End}(S_\lambda) \). This is also an \( S_d \)-bi-module isomorphism.

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