KdV conservation laws for some supersymmetric potentials

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Abstract. It is shown that a limited number of supersymmetric potentials obtained from factorization methods, which are reflectionless, lead to the description of KdV Hamiltonians and their related KdV conservation laws are derived. Also, single-soliton solutions of the KdV equations corresponding to these potentials are presented.

In recent years, supersymmetry has been a rich facet of quantum mechanical solvable problems. This property has appeared in a wide range of solvable problems in quantum mechanics such as in one-, two- and three-dimensional Schrödinger equations [1, 2], the KdV equation [3, 4] and the inverse scattering problem [5, 6]. There are useful and interesting discussions on the different aspects of ‘supersymmetry in quantum mechanics’ in the books introduced in [7]. In [6] the connection between the KdV equation and supersymmetric quantum mechanics of the one-dimensional Schrödinger equation has been obtained by means of Bäcklund transformations. It is shown there that by using $\tau$-function formalism and a vertex operator, one can add a soliton to the multi-soliton solution of the KdV equation or a bound state to a one-dimensional potential. Recently, the higher-order supersymmetric partner potentials have been constructed by iterating a simple finite-difference equation corresponding to the Bäcklund transformations [8]. In this method based on the Bäcklund transformations which operates simply at the level of the $\beta$-functions, wavefunctions, and differential equations etc are not used. In [9], using the supersymmetric quantum mechanics, another procedure is introduced for obtaining the
(n + 1)-soliton solution from the n-soliton solution of the KdV equation and this has been shown to be equivalent to the method known as the ‘standpoint of soliton theory’ introduced in [3]. Miura transformations between the KdV and modified KdV equations, which lead to solution of the eigenvalue problem of the inverse scattering transform for the KdV equation, show that there are infinite conserved quantities [10]. Nonlocal conservation laws for the KdV equation have been obtained in [11] by using the standard method, and by using the supersymmetry method in [12]. The square of the eigenfunction and the related linear operator in the KdV equation also play an important role in obtaining symmetries corresponding to conserved quantities.

In [6], solvable reflectionless one-dimensional potentials, with arbitrary bound-state spectra have been constructed using supersymmetric quantum mechanics. It has been shown in [9] that this kind of model leads to the description of KdV Hamiltonians and their related conservation laws, though this has been considered for only one of the superpotentials. In this paper, all reflectionless solvable one-dimensional quantum mechanical models whose solutions are obtained from good shape invariance symmetry (with supersymmetric structure) are studied. It is shown that only a limited number of superpotentials obtained from the shape invariance with respect to the main quantum number n and the secondary quantum number m lead to description of the KdV Hamiltonian and related conservation laws.

There is a full discussion of the factorization method and algebraic solution of bound-state problems in the review article of Infeld and Hull [13]. They have shown that a wide range of shape-invariant potentials fall into six factorization types. But in our design of the factorization method, shape-invariant potentials fall into two classes. In fact, using the master function theory, it is shown that most of the shape-invariant potentials can be classified into two classes, whether they are shape invariant with respect to the main quantum number n or to the secondary quantum number m. In the first class, the superpotentials are expressed in terms of the master function A(x), the corresponding weight function W(x), and also the main quantum number n. In the second class, the superpotentials are explained in terms of the master function, its weight function, and also the secondary quantum number m. It has been shown that the shape invariance theory represents not only the supersymmetry algebra but also the parasupersymmetry algebra [14, 15].

In previous papers [14, 15], the master function A(x) was introduced as a polynomial of at most degree two. The corresponding non-negative weight function W(x) in the interval (a, b), is chosen such that for a given master function A(x), the expression (A(x)W(x))'/W(x) (or A(x)W'(x)/W(x)) is a polynomial of degree at most one. Also, the interval (a, b) is chosen such that the expression A(x)W(x) and all its derivatives are zero at these points. It has then been proven that the polynomials

\[ \Phi_n(x) = \frac{a_n}{W(x)} \left( \frac{d}{dx} \right)^n (A^n(x)W(x)) \]

are orthogonal with respect to the scalar product defined by the weight function W(x) in the interval (a, b). Here, n is a non-negative integer, and \( a_n \) is the constant of normalization.

The first class of superpotentials, which is obtained from shape invariance with respect to the parameter n, includes the following two factorized Schrödinger equations:

\[ B(n)A(n)\psi_n(\theta) = E(n)\psi_n(\theta) \]
\[ A(n)B(n)\psi_{n-1}(\theta) = E(n)\psi_{n-1}(\theta), \]

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Supersymmetric potentials described by equations (2) are respectively polynomial. The wavefunctions where the explicit form of the change of variable is obtained by solving the following first-order differential equation:

\[ \frac{d\theta}{dx} = \frac{1}{A(x)}. \]  

Equations (2) are the factorized Schrödinger equations \((\hbar = 2M = 1)\) with the raising operators \(B(n)\) and the lowering operators \(A(n)\) as

\[
\begin{align*}
B(n) &= \frac{d}{d\theta} + W_n(\theta) \\
A(n) &= -\frac{d}{d\theta} + W_n(\theta).
\end{align*}
\]

The superpotentials \(W_n(\theta)\) are given in terms of master function \(A(x)\) and weight function \(W(x)\) as

\[
W_n(\theta) = \frac{1}{2} \left[ nA'(x) + \frac{A(x)W''(x)}{W(x)} + n \frac{A'(0)(\frac{A(x)W'(x)}{W(x)})' - A''(0)(\frac{A(x)W'(x)}{W(x)})}{\left(\frac{A(x)W'(x)}{W(x)}\right)'} + nA''(0) \right]_{x=x(\theta)},
\]

where the explicit form of the change of variable \(x = x(\theta)\) is obtained from solving the first-order differential equation (4). The wavefunctions \(\psi_n(\theta)\) are (in terms of \(x\) as a multiplier of the polynomial \(\Phi_n(x)\))

\[
\psi_n(\theta) = [W^{1/2}(x)\Phi_n(x)]_{x=x(\theta)} = \left[ \frac{a_n}{W^{1/2}(x)} \left( \frac{d}{dx} \right)^n (A^n(x)W(x)) \right]_{x=x(\theta)}. \]

Supersymmetric potentials described by equations (2) are respectively

\[
V_{n,\pm}(\theta) = W_n^2(\theta) \pm \frac{d}{d\theta} W_n(\theta).
\]

The second class of superpotentials, obtained by factorization of the Schrödinger equation with respect to the secondary quantum number \(m\), is described as the two following shape-invariant Schrödinger equations \((\hbar = 2M = 1)\) [15]:

\[
\begin{align*}
B(m)A(m)\psi_{n,m}(\theta) &= E(n, m)\psi_{n,m}(\theta) \\
A(m)B(m)\psi_{n,m-1}(\theta) &= E(n, m)\psi_{n,m-1}(\theta),
\end{align*}
\]
with the spectrum $E(n, m)$ as

$$E(n, m) = -(n - m + 1) \left[ \left( \frac{A(x)W'(x)}{W(x)} \right)' + \frac{1}{2}(n + m)A''(x) \right]. \tag{10}$$

Again, note that the secondary quantum number $m$ is a non-negative integer with the maximum value equal to $n$ and that the rhs of equation (10) is independent of $x$. In equations (9) the variable $\theta$ for the given master function $A(x)$ is obtained by solving the following first-order differential equation:

$$\frac{d\theta}{dx} = \frac{1}{\sqrt{A(x)}}. \tag{11}$$

The raising operators $B(m)$ and the lowering operators $A(m)$ are respectively

$$B(m) = \frac{d}{d\theta} + W_m(\theta),$$

$$A(m) = -\frac{d}{d\theta} + W_m(\theta). \tag{12}$$

The superpotentials $W_m(\theta)$ and wavefunctions $\psi_{n,m}(\theta)$, in terms of master function $A(x)$ and its the corresponding weight function $W(x)$ are given as

$$W_m(\theta) = -\left[ \frac{1}{2} \frac{A(x)W'(x)}{W(x)} + \frac{2m-1}{4} A'(x) \right]_{x=x(\theta)} \tag{13}$$

and

$$\psi_{n,m}(\theta) = \left[ \frac{a_n}{A^{(2m-1)/4}(x)W^{1/2}(x)} \left( \frac{d}{dx} \right)^{n-m} \left( A^n(x)W(x) \right) \right]_{x=x(\theta)} \tag{14}$$

where the change of variable $x = x(\theta)$ is substituted by using the solution of the first-order differential equation (11). Equations (8) are Schrödinger equations with supersymmetric potentials, respectively,

$$V_{m,\pm}(\theta) = W_m^2(\theta) \pm \frac{d}{d\theta} W_m(\theta). \tag{15}$$

It should be pointed out that the changes of variables introduced in [2] are special cases of equations (4) and (11).

As mentioned earlier, the end points $a$ and $b$ are selected in such a way that $A(x)W(x)$ and its derivatives vanish at these points and, also, these selections lead to the orthogonality of polynomials $\Phi_n(x)$. Of course, this fact limits the selection of the $\alpha$ and $\beta$ parameters related to weight function $W(x)$. In the first class we can consider the following two superpotentials without breaking the limitation:

1. Eckart superpotential (related to master function $A(x) = \omega^2 x^2 - 1$), and
2. Rosen–Morse II superpotential (related to master function $A(x) = 1 - \omega^2 x^2$).

These superpotentials have the following three properties: (a) they have nonzero value on both $\theta$ semi-axes, (b) their corresponding potentials take a constant value at $\theta \to \infty$ and $-\infty$, and (c) by omitting the independence of only one of the parameters in the weight function, we can obtain the Hamiltonians of the KdV and related conservation laws by considering the transmission coefficient. We recall in table 1 the necessary information for Eckart and Rosen–Morse II superpotentials as spectrum, wavefunction and explicit form of the superpotentials and partner potentials.
Table 1. Information concerning Eckart and Rosen–Morse II potentials with constant values at infinity, obtained from shape invariance with respect to the main quantum number \( n \).

| Name         | \( W(x), x = x(\theta) \) | \( W_n(\theta), V_{n,\pm}(\theta) \) | \( \psi_n^{(\alpha,\beta)}(\theta), E(n) \) |
|--------------|-----------------------------|-----------------------------------|---------------------------------|
| **Eckart**   | \( W(x) = (1 - \omega x)^\alpha (1 + \omega x)^\beta \) | \( W_n(\theta) = -A \coth \omega \theta + \frac{B}{A} \) | \( \psi_n^{(\alpha,\beta)}(\theta) = \left[ a_n (-1)^n (1 - \omega x)^{-\alpha/2} \right] \) |
|              |                             | \( V_{n,\pm}(\theta) = A(\pm \omega) \coth^2 \omega \theta \) |                                  |
| \( A(x) = \omega^2 x^2 - 1 \) | \( \alpha, \beta > -1 \) | \( -2B \coth \omega \theta + \frac{B^2}{A^2} \mp A\omega \) | \( (1 + \omega x)^{-\beta/2} \left( \frac{d}{dx} \right)^n ((1 - \omega x)^{n\pm\alpha} \) |
| \( -\frac{1}{\omega} < x < -1 \) | \( x = -\frac{1}{\omega} \coth \omega \theta \) | |                                  |
| \( \omega > 0 \) | \( -\infty < \theta < +\infty \) | |                                  |
|              | \( B = \frac{1}{4} (\alpha^2 - \beta^2) \omega^2 \) | | \( E(n) = \frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta)^2} \omega^2 \) |
| **Rosen–Morse II** | \( W(x) = (1 - \omega x)^\alpha (1 + \omega x)^\beta \) | \( W_n(\theta) = A \tanh \omega \theta + \frac{B}{A^2} \) | \( \psi_n^{(\alpha,\beta)}(\theta) = \left[ a_n (1 - \omega x)^{-\alpha/2} \right] \) |
|              |                             | \( V_{n,\pm}(\theta) = A(\pm \omega) \tanh^2 \omega \theta \) |                                  |
| \( A(x) = 1 - \omega^2 x^2 \) | \( \alpha, \beta > -1 \) | \( +2B \tanh \omega \theta + \frac{B^2}{A^2} \pm A\omega \) | \( (1 + \omega x)^{-\beta/2} \left( \frac{d}{dx} \right)^n ((1 - \omega x)^{n\pm\alpha} \) |
| \( -\frac{1}{\omega} < x < \frac{1}{\omega} \) | \( x = \frac{1}{\omega} \tanh \omega \theta \) | |                                  |
| \( -\infty < \theta < +\infty \) | \( B = \frac{1}{4} (\alpha^2 - \beta^2) \omega^2 \) | | \( E(n) = \frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta)^2} \omega^2 \) |
As can be seen, in both models in table 1, the shape-invariant potentials have nonzero values in all points on the \( \theta \) axis from \(-\infty \) to \(+\infty \) and tend to a constant value for \( \theta \rightarrow \pm \infty \) limits. By taking \( \alpha = -\beta \), the weight function \( W(x) \) still has a free parameter (as \( \alpha \)) in both cases. If we take \( n = 1 \), we get the following results:

**Eckart:**

\[
\begin{align*}
W_1(\theta) &= -\omega \coth \omega \theta \\
E(1) &= (1 - \alpha^2)\omega^2 \\
V_{1+}(\theta) &= 2\omega^2 \coth^2 \omega \theta - \omega^2 \\
V_{1-}(\theta) &= \omega^2 \\
\psi_{1}^{(\alpha,-\alpha)}(\theta) &= -2a_1\omega(\coth \omega \theta - \alpha)\left(1 + \frac{1}{1 - \coth \omega \theta}\right)^{\alpha/2},
\end{align*}
\]

**Rosen–Morse II:**

\[
\begin{align*}
W_1(\theta) &= -\omega \tanh \omega \theta \\
E(1) &= (1 - \alpha^2)\omega^2 \\
V_{1+}(\theta) &= 2\omega^2 \tanh^2 \omega \theta - \omega^2 \\
V_{1-}(\theta) &= \omega^2 \\
\psi_{1}^{(\alpha,-\alpha)}(\theta) &= -2a_1\omega(\tanh \omega \theta + \alpha)\left(1 - \frac{1}{1 + \tanh \omega \theta}\right)^{\alpha/2}.
\end{align*}
\]

Notice that in both of these examples the spectrum is still a function of \( \alpha \) (i.e. one of the weight function parameters). If \( |\alpha| < 1 \) (which is consistent with the conditions of table 1 together with \( \alpha = -\beta \) limitation), the spectrum \( E(1) \) is positive. If \( |\alpha| > 1 \) (which is not consistent with the conditions of table 1 and \( \alpha = -\beta \) limitation) the polynomials \( \Phi_n^{(\alpha,-\alpha)}(x) \) are not necessarily orthogonal but \( \psi_{1}^{(\alpha,-\alpha)}(\theta) \) will be the solution of the wave equation, and in this case, the energy spectrum \( E(1) \) is negative. This is comparable with the discussion about the bound states of the special example given in [9]. In both the above-mentioned examples following the supersymmetry approach for \( n = 1 \), we take the reflectionless potential \( u(\theta) \):

\[
u(\theta) = V_{1+}(\theta) - V_{1-}(\theta) = \begin{cases} 
\frac{-2\omega^2}{\sinh^2 \omega \theta} & \text{Eckart} \\
\frac{-2\omega^2}{\cosh^2 \omega \theta} & \text{Rosen–Morse II.}
\end{cases}
\]

Clearly, in these two examples, \( V_{1-}(\theta) = \omega^2 \) is a constant potential. Thus, considering a plane wave solution as \( \psi_{1-}(\theta) = e^{ik\theta} \) and taking into account the shape-invariant equations (2), we can obtain a solution for \( V_{1+}(\theta) \). By applying the raising operator \( B(n = 1) = \frac{d}{d \theta} + W_1(\theta) \) on \( \psi_{1-}(\theta) \) in the supersymmetric method, the solution for the partner potential \( V_{1+}(\theta) \) is obtained as

\[
\psi_{+}(\theta) = B(1)\psi_{-}(\theta) = \begin{cases} 
(ik - \omega \coth \omega \theta)e^{ik\theta} & \text{Eckart} \\
(ik - \omega \tanh \omega \theta)e^{ik\theta} & \text{Rosen–Morse II.}
\end{cases}
\]

Moreover, for both the Eckart and Rosen–Morse II models one can obtain in the \( \theta \rightarrow \pm \infty \) limits [9]

\[
\lim_{\theta \rightarrow \pm \infty} \psi_{+}(\theta) = \begin{cases} 
(ik - \omega)e^{ik\theta} & \theta \rightarrow +\infty \\
(ik + \omega)e^{ik\theta} & \theta \rightarrow -\infty.
\end{cases}
\]
Equations (20) explain the fact that the potentials $V_{1,+}(\theta)$ are reflectionless, and since we have $V_{1,-}(\theta) = \omega^2$, so $u(\theta)$ is also a reflectionless potential. Therefore, one can define the transmission coefficient for $u(\theta)$ as

$$T(k) = \frac{\psi_+(\theta \to +\infty)}{\psi_+(\theta \to -\infty)} = \frac{ik - \omega}{ik + \omega}. \quad (21a)$$

Since the difference of $u(\theta)$ and $V_{1,+}(\theta)$ is a constant number, the Schr"odinger equation for reflectionless one-dimensional potential $u(\theta)$ is written as

$$\left(-\frac{d^2}{d\theta^2} + u(\theta)\right)\psi_+(\theta) = k^2 \psi_+(\theta). \quad (22)$$

Now we explain how to derive the KdV Hamiltonians from these discussions. As in [9, 16], we propose the following asymptotic expansion in $\theta \to +\infty$ for the wavefunction in the presence of reflectionless potential $u(\theta)$:

$$\psi_+(\theta) = \exp\left(ik\theta + \int_{-\infty}^{\theta} \Phi(\theta') \, d\theta'\right). \quad (23)$$

It is obvious that transmission coefficient $T(k)$ is

$$T(k) = \lim_{\theta \to +\infty} e^{-ik\theta} \psi_+(\theta) = \exp\left(\int_{-\infty}^{+\infty} \Phi(\theta') \, d\theta'\right). \quad (24)$$

By considering the following expansion for $\Phi(\theta)$:

$$\Phi(\theta) = \sum_{l=1}^{\infty} \frac{f_l(\theta)}{(2ik)^l}, \quad (25)$$

one can conclude from equations (21), (24) that

$$\ln T(k) = \sum_{l=1}^{\infty} \frac{1}{(2ik)^l} \int_{-\infty}^{+\infty} f_l(\theta) \, d\theta = \ln \frac{ik - \omega}{ik + \omega}. \quad (26)$$

By using the Taylor expansion of the right-hand side of equation (26) and comparing the powers $1/k^l$ in both sides, we obtain

$$\int_{-\infty}^{+\infty} f_l(\theta) \, d\theta = \begin{cases} 0 & \text{even } l \\ -\frac{2^{l+1}}{l} \omega^l & \text{odd } l. \end{cases} \quad (27)$$

The integrals in equation (27) regardless of the constant coefficient are related to the Hamiltonians of the KdV. Equation (22), in both the Eckart and Rosen–Morse II superpotential examples, has a similar spectrum for different values of $\omega$. Therefore, the values in the right-hand side of equation (27) are the motion constants of one of the isospectrum deformation potential $u(\theta)$, which construct the KdV Hamiltonians.

There are two approaches for calculating the explicit form of functions $f_l(\theta)$. The first approach is based on direct use of the Schr"odinger equation (22) by substituting expression (23) into it. The result is

$$f_1(\theta) = u(\theta)$$

$$f_{l+1}(\theta) = -\frac{df_l(\theta)}{d\theta} - \sum_{k=1}^{l-1} f_k(\theta) f_{l-k}(\theta) \quad l = 1, 2, 3, \ldots. \quad (28)$$
We can therefore express a few of the $f_l(\theta)$ terms from equations (28) as follows:

\[
\begin{align*}
f_1(\theta) &= u(\theta) & f_2(\theta) &= -u,\theta(\theta) & f_3(\theta) &= u,\theta(\theta) - u(\theta)^2 \\
f_4(\theta) &= (-u,\theta(\theta) + 2u(\theta)^2),\theta & f_5(\theta) &= (u,\theta(\theta) - 3u(\theta)^2),\theta + u,\theta^2(\theta) + 2u^3(\theta).
\end{align*}
\]  

(29)

Since, in equations (29), the functions $f_l(\theta)$ with even $l$ are complete derivatives and all powers and derivatives of $u(\theta)$ in the Eckart and Rosen–Morse II examples have equal values for $\theta \to \pm \infty$, it is evident that the first part of equation (27) can be derived from equations (29). One can, alternatively, obtain the explicit form of the functions $f_l(\theta)$ with odd $l$, and consequently the second part of equation (27) is directly concluded.

The second approach for calculating the $f_l(\theta)$ functions is the supersymmetry procedure, which has results consistent with the above approach. In this case, it is sufficient to equate the wavefunction $\psi_+(\theta)$ given in equation (19) with the proposed asymptotic behaviour given in equation (23):

\[
\exp\left(ik\theta + \sum_{l=1}^{\infty} \frac{1}{2(ik)^l} \int_{-\infty}^{\theta} f_l(\theta') \, d\theta'\right) = N \left\{ \begin{array}{ll}
(ik - \omega \coth \omega \theta)e^{ik\theta} & \text{Eckart} \\
(ik - \omega \tanh \omega \theta)e^{ik\theta} & \text{Rosen–Morse II},
\end{array} \right.
\]  

(30)

where $N$ is the proportion coefficient. By taking the logarithm of both sides and expanding in terms of $k$ powers, we can get by differentiation of both sides with respect to $\theta$ the following equations:

\[
f_l(\theta) = \left\{ \begin{array}{ll}
-\frac{(2\omega)^l}{l} \frac{d}{d\theta} \coth^l \omega \theta & \text{Eckart} \\
-\frac{(2\omega)^l}{l} \frac{d}{d\theta} \tanh^l \omega \theta & \text{Rosen–Morse II}.
\end{array} \right.
\]  

(31)

These results for the functions $f_l(\theta)$, which describe the wavefunction asymptotic behaviour of the corresponding to reflectionless potential $u(\theta)$, coincide with the results of equation (29) for both Eckart and Rosen–Morse II superpotentials. One can also obtain the integrals of the equations (27) (which are the constants of motion) directly from equations (31).

Integrals of functions $f_{2l}(\theta)$ are zero and describe the conservation laws for zero quantities. Considering the functional derivative as

\[
\frac{\delta H_{2l+1}[u]}{\delta u} = \frac{1}{2^{2l+1}} \sum_{k=0}^{\infty} (-d)^k \frac{\partial f_{2l+3}(\theta)}{\partial u^{(k)}}
\]  

(32)

where $u^{(k)}$ is the $k$th derivative of $u(\theta)$. The KdV evolution equation for which the Hamiltonian

\[
H_{2l+1}[u] = \frac{1}{2^{2l+1}} \int_{-\infty}^{+\infty} f_{2l+3}(\theta) \, d\theta,
\]  

(33)

is a conserved quantity reads [3, 16] as

\[
u_l = \frac{\partial}{\partial \theta} \frac{\delta H_{2l+1}[u]}{\delta u}.
\]  

(34)

Using the supersymmetry method, one can conclude from the evolution equation (34) that the transmission coefficient $T(k)$ is a constant of motion, as in [9]. So, considering the expansion equation (26), all of the evolution equations conserve all KdV Hamiltonians, $H_{2l+1}[u]$. In principle, the existence of a conserved quantity in an evolution equation is equivalent on the one hand to the existence of a symmetry and on the other hand to the integrability of the equation of motion. Here, the commutation of various evolution equations is dynamical symmetry. Thus,
the recursion relation between parameters of the time parameters equation (34), one can suppose recursion relation between KdV evolution Hamiltonians [3, 16]: Considering equations (27), the constants where the corresponding conserved functionals H in (31). To obtain single-soliton solutions, by assigning time variable t_{2l+1} in the evolution equation (34), one can suppose u(\theta; t_1, t_3, \ldots) as a function of infinite variables t_1, t_3, \ldots. Each of the time parameters t_1, t_3, \ldots describes one of evolution equations (34). Thus a function like u(\theta; t_1, t_3, \ldots) is obtained by which all KdV evolution equations are described simultaneously:

\[ u_{2l+1} = \frac{\partial}{\partial \theta} \frac{\delta H_{2l+1}[u]}{\delta u}, \quad l = 0, 1, 2, \ldots \]  

(35)

where the corresponding conserved functionals H_{2l+1}[u] are given by (33). Relations (35) describe the family of KdV equations in the Hamiltonian representation. We propose the following solutions, as single-soliton solutions to the KdV hierarchy [17]:

\[ u(\theta; t_1, t_3, \ldots) = \begin{cases} 
-2\omega^2 \\
\sinh^2(\omega \theta + \alpha_1 t_1 + \alpha_3 t_3 + \cdots) \\
\cosh^2(\omega \theta + \alpha_1 t_1 + \alpha_3 t_3 + \cdots) 
\end{cases} \]

Eckart

Rosen–Morse II.

By substituting the proposed solutions (36) in the evolution equation (35) for l = 0, \alpha_1 is calculated, in both Eckart and Rosen–Morse II models, in terms of the second-derivative parameter of the master function, i.e. \omega, as \alpha_1 = \omega. If we substitute these solutions in the recursion relation between KdV evolution Hamiltonians [3, 16]:

\[ \frac{\partial}{\partial \theta} \frac{\delta H_{2l+1}[u]}{\delta u} = \frac{1}{4} \left( \frac{\partial^3}{\partial \theta^3} + 2u, _p(\theta) + 4u(\theta) \frac{\partial}{\partial \theta} \right) \frac{\delta H_{2l-1}[u]}{\delta u}, \]  

(37)

the recursion relation between parameters \alpha_{2l+1} is obtained for both models as

\[ \alpha_{2l+1} = \omega^{2l} \alpha_1 \quad l = 0, 1, 2, \ldots \]  

(38)

Then, the parameters \alpha_{2l+1} are calculated in terms of \omega as

\[ \alpha_{2l+1} = \omega^{2l+1} \quad l = 0, 1, 2, \ldots \]  

(39)

Considering equations (27), the constants \alpha_{2l+1} obtained in (39) play the role of infinite constants of motion which are very important in finding solitary solutions (36).

Now we discuss the second class of superpotentials with constant values at infinite, obtained from the factorization with respect to the secondary quantum number m. We have introduced in table 2 the spectrum, wavefunction and partner potentials of three superpotentials: Scarf II, generalized Pöschl–Teller, and Natanzon. In these three solvable models, the wavefunction is expressed in terms of special functions, not in terms of orthogonal special polynomials. Considering the change of variables of these three models, it is known that partner potentials corresponding to Scarf II have nonzero values for both sides of the \theta axis towards infinity, and tend to constant values for |\theta| \to +\infty. The other two examples are definite at \theta \to +\infty (i.e. they are semi-infinite) and in this limit, partner potentials have a constant value. But, we did not limit the variable \theta, because the derivation method of shape invariance obtained in [15] still holds and equations (9) are still treated as Schrödinger equations. For shape-invariant parameter m = 1, by imposing limitation on both parameters of the weight function, one can obtain the following results to obtain conservation laws for the KdV Hamiltonians:

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Table 2. Information concerning Scarf II, generalized Pöschl–Teller and Natanzon potentials with constant values at infinite, obtained from shape invariance with respect to the secondary quantum number m.

| Name, $A(x)$ | $W(x) = (\omega^2 x^2 + 1)^{\alpha} e^{\beta \tanh^{-1} \omega x}$ | $W_m(\theta), V_{m,\pm}(\theta)$ | $\psi_n^{(\alpha,\beta)}(\theta), E(n, m)$ |
|--------------|-------------------------------------------------|-------------------------------|----------------------------------|
| Scarf II     | $W_m(\theta) = A \tanh \omega \theta + \frac{B}{\cosh \omega \theta}$ | $V_{m,\pm}(\theta) = (B^2 - A^2 \pm \omega^2 A\omega) \frac{1}{\cosh^2 \omega \theta}$ | $\psi_n^{(\alpha,\beta)}(\theta) = a_n \times (\omega^2 x^2 + 1)^{-\frac{2m+2\alpha-1}{4}} \times e^{-\beta \tanh^{-1} \omega x}$ |
| $A(x) = \omega^2 x^2 + 1$ | $\alpha < -1, \beta \in R$ | $B = (2A \pm \omega) \frac{\cosh \omega \theta}{\sinh^2 \omega \theta} + A^2$ | $E(n, m) = -(n - m + 1)(n + m + 2\alpha)\omega^2$ |
| Generalized Pöschl–Teller | $W(x) = (\omega x - 1)^{\alpha}(\omega x + 1)^{\beta}$ | $W_m(\theta) = A \coth \omega \theta - \frac{B}{\sinh \omega \theta}$ | $\psi_n^{(\alpha,\beta)}(\theta) = a_n \times (\omega x - 1)^{-\frac{2m+2\alpha-1}{4}} \times e^{-\beta \tanh^{-1} \omega x}$ |
| $A(x) = \omega^2 x^2 - 1$ | $\alpha > -1, \alpha + \beta + 2 < 0$ | $V_{m,\pm}(\theta) = (A^2 + B^2 \pm \omega^2 A\omega) \frac{1}{\cosh^2 \omega \theta}$ | $E(n, m) = -(n - m + 1)(n + m + \alpha + \beta)\omega^2$ |
| Natanzon     | $W(x) = (2\omega x - 1)^{\alpha}(2\omega x + 1)^{\beta}$ | $W_m(\theta) = A \tanh \omega \theta - B \coth \omega \theta$ | $\psi_n^{(\alpha,\beta)}(\theta) = a_n \times (2\omega x - 1)^{-\frac{2m+2\alpha-1}{4}} \times e^{-\beta \tanh^{-1} \omega x}$ |
| $A(x) = 4\omega^2 x^2 - 1$ | $\alpha > -1, \alpha + \beta + 2 < 0$ | $V_{m,\pm}(\theta) = \frac{B(\pm \omega)}{\sinh^2 \omega \theta} - \frac{A(\pm \omega)}{\cosh^2 \omega \theta}$ | $E(n, m) = -4(n - m + 1)(n + m + \alpha + \beta)\omega^2$ |
| $\frac{1}{2\omega} < x < +\infty$ | $x = \frac{1}{2\omega} \cosh 2\omega \theta$ | $A = (\beta + m - \frac{1}{2})\omega$ | |
Scarf II: \( \alpha = 1/2, \beta = 0 \)

\[
W_1(\theta) = -\omega \tanh \omega \theta \\
E(n, m = 1) = -n(n + 2)\omega^2 \\
\psi_n^{(1/2, 0)}(\theta) = \left[ \frac{a_n}{(\omega^2 \theta^2 + 1)^{n+1/2}} \left( \frac{d}{dx} \right)^n (\omega^2 \theta^2 + 1)^{n+1/2} \right]_{x = \frac{1}{\omega} \sinh \omega \theta}
\]

Generalized Pöschl–Teller: \( \alpha = \beta = 1/2 \)

\[
W_1(\theta) = -\omega \coth \omega \theta \\
E(n, m = 1) = -n(n + 2)\omega^2 \\
\psi_n^{(1/2, 1/2)}(\theta) = \left[ \frac{a_n}{(\omega^2 \theta^2 - 1)^{n+1/2}} \left( \frac{d}{dx} \right)^n (\omega^2 \theta^2 - 1)^{n+1/2} \right]_{x = \frac{1}{\omega} \cosh \omega \theta}
\]

Natanzon: \( \alpha = \beta = 1/2 \)

\[
W_1(\theta) = -2\omega \coth 2\omega \theta \\
E(n, m = 1) = -4n(n + 2)\omega^2 \\
\psi_n^{(1/2, 1/2)}(\theta) = \left[ \frac{a_n}{(4\omega^2 \theta^2 - 1)^{n+1/2}} \left( \frac{d}{dx} \right)^n (4\omega^2 \theta^2 - 1)^{n+1/2} \right]_{x = \frac{1}{2\omega} \cosh 2\omega \theta}.
\]

It is seen that in all of these three examples, the spectra are necessarily negative and quantized in terms of \( n (n \geq m = 1) \).

Using these three models, one can define from supersymmetry approach for \( m = 1 \), the reflectionless potentials \( u(\theta) \) as

\[
u(\theta) = V_{1, +}(\theta) - V_{1, -}(\theta) = \begin{cases} 
\frac{-2\omega^2}{\cosh^2 \omega \theta} & \text{Scarf II} \\
\frac{2\omega^2}{\sinh^2 \omega \theta} & \text{generalized Pöschl–Teller} \\
\frac{2\omega^2}{\sinh^2 \omega \theta} - \frac{2\omega^2}{\cosh^2 \omega \theta} & \text{Natanzon.}
\end{cases}
\]

We extend the functions \( u(\theta) \) given in equations (43) to the region \(-\infty < \theta < 0 \) for the last two models. Thus, the transmission coefficient will be as in equation (21a). But the operator \( B(m = 1) \) which is used to calculate the functions \( f_1(\theta) \) in the supersymmetry method is still used in the \( \theta \to +\infty \) region. The potential \( V_{1, -}(\theta) \) again has a constant value in all the three models, so, considering a plane wave solution as \( \psi_-(\theta) = e^{ik\theta} \) (in all three examples), according to the supersymmetry method induced by shape invariance equations (9), by applying the raising operator \( B(m = 1) = \frac{d}{d\theta} + W_1(\theta) \) on \( \psi_- (\theta) \), we obtain a solution for \( \psi_{1, +}(\theta) \) as

\[
\psi_(\theta) = B(1)\psi_-(\theta) = \begin{cases} 
(ik - \omega \tanh \omega \theta)e^{ik\theta} & \text{Scarf II} \\
(ik - \omega \coth(\omega \theta))e^{ik\theta} & \text{generalized Pöschl–Teller} \\
(ik - 2\omega \coth 2\omega \theta)e^{ik\theta} & \text{Natanzon.}
\end{cases}
\]
The transmission coefficients for the first and second model are calculated as in (21b), but for
the Natanzon model we get the following result:

\[ T(k) = \frac{ik - 2\omega}{ik + 2\omega}. \]  

(45)

By solving the Schrödinger equation (22) for \( u(\theta) \) given in equations (43) for \( \theta \to +\infty \), and
choosing the proposed solution as equation (23), we can again get the conservation laws related
to the KdV equation as (27) for the Scarf II and generalized Pöschl–Teller models. But for
Natanzon model we have

\[ \int_{-\infty}^{+\infty} f_l(\theta) \, d\theta = \begin{cases} 
0 & \text{even } l \\
-\frac{2^{2l+1}}{l} \omega^l & \text{odd } l
\end{cases} \]  

(46)

Direct use of the Schrödinger equation leads again to the same results (29). Application of the
supersymmetry method for calculating functions \( f_l(\theta) \) also gives the following results:

\[ f_l(\theta) = \begin{cases} 
\frac{-(2\omega)^l}{l} \frac{d}{d\theta} (\tanh \omega \theta)^l & \text{Scarf II} \\
\frac{-(2\omega)^l}{l} \frac{d}{d\theta} (\coth \omega \theta)^l & \text{generalized Pöschl–Teller} \\
\frac{-2^{2l}\omega^l}{l} \frac{d}{d\theta} (\coth 2\omega \theta)^l & \text{Natanzon}.
\end{cases} \]  

(47a) \( 47b \) \( 47c \)

Note that the result (47c) is consistent with the result obtained from (29) with regard to
equation (43c). Now let us define the KdV Hamiltonians for the nonzero conserved quantities
such as (33) and consider its motion equation in the form of (34). A single-soliton solution
corresponding to the evolution equation of these three models is proposed similarly to the previous
method as follows :

\[
\begin{align*}
\text{Scarf II:} \\
&= \frac{2\omega^2}{\cosh^2(\omega \theta + \alpha_1 t_1 + \alpha_3 t_3 + \cdots)} \\
&= \frac{-2\omega^2}{\sinh^2(\omega \theta + \alpha_1 t_1 + \alpha_3 t_3 + \cdots)} \\
&= \frac{2\omega^2}{\cosh^2(\omega \theta + \frac{1}{2} \alpha_1 t_1 + \frac{1}{2} \alpha_3 t_3 + \cdots)} \\
&= \frac{-2\omega^2}{\sinh^2(\omega \theta + \frac{1}{2} \alpha_1 t_1 + \frac{1}{2} \alpha_3 t_3 + \cdots)}
\end{align*}
\]  

(48)

For the first and the second proposed solutions in (48), from the evolution equation (35) with
\( l = 0 \), we again get \( \alpha_1 = \omega \), but for the Natanzon model \( \alpha_1 = 2\omega \). By substituting the proposed
solutions (48) in the recursion relations (37) one can obtain the recursion relations (38) for
parameters \( \alpha_{2l+1} \) and then values given in (39) for the two models Scarf II and generalized
Pöschl–Teller. But the recursion relations between parameters \( \alpha_{2l+1} \) for the Natanzon model are
calculated as

\[ \alpha_{2l+1} = (2\omega)^{2l} \alpha_1, \quad l = 0, 1, 2, \ldots \]  

(49)
and therefore the parameters $\alpha_{2l+1}$ are expressed as

$$\alpha_{2l+1} = (2\omega)^{2l+1} \quad l = 0, 1, 2, \ldots$$  \hspace{1cm} (50)

Although soliton solutions of the KdV hierarchy corresponding to Scarf II and generalized Pöschl–Teller models are similar to the Rosen–Morse II and Eckart models, respectively, the result for the Natanzon model is a little different.

To summarize, in this paper we have analysed five quantum mechanical solvable models connected with the KdV equation which have supersymmetric structures and their corresponding asymptotic potentials have constant values and are all reflectionless. Using these facts, one can obtain single-soliton solutions of KdV Hamiltonians for which the conservation laws hold.

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References

[1] Witten E 1981 Nucl. Phys. B 185 513
  Cooper F and Freedman B 1983 Ann. Phys., NY 146 262
  Gendenshtein Zh L 1983 Eksp. Teor. Fiz. Pis. Red. 38 299
  Balantekin A B 1988 Phys. Rev. A 57 4188
[2] Cooper F, Ginocchio J N and Wipf A 1989 J. Phys. A: Math. Gen. 22 3707
[3] Newell A C 1985 Solitons in Mathematics and Physics (Philadelphia, PA: SIAM)
[4] Yamanaka I and Sasaki R 1988 Prog. Theor. Phys. 79 1167
  Mathieu P 1988 Phys. Lett. B 203 287
  Inami T and Kanno H 1991 Commun. Math. Phys. 136 519
  Oevel W and Popowicz Z 1991 Commun. Math. Phys. 139 441
  McArthur I N 1992 Commun. Math. Phys. 148 177
  Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 Phys. Rev. Lett. 19 1095
[5] Eckhaus W and van Harter A 1981 The Inverse Scattering Transformation and the Theory of Solitons (Amsterdam: North-Holland)
  Sukumar C V 1985 J. Phys. A: Math. Gen. 18 L57
  Sukumar C V 1985 J. Phys. A: Math. Gen. 18 2917
  Waikwok Kwong, Riggs H, Rosner J L and Thacker H B 1989 Phys. Rev. D 39 1242
  Qin-mou Wang, Sukhatme U P, Wai-Yee Keung and Imbo T D 1990 Mod. Phys. Lett. A 5 525
  Ablowitz M J and Clarkson P A 1991 Solitons, Nonlinear Evolution Equations and Inverse Scattering (New York: Cambridge University Press)
[6] Waikwok Kwong and Rosner J L 1986 Prog. Theor. Phys. Suppl. 86 366
[7] Kumar B 2001 Supersymmetry in Quantum and Classical Mechanics (New York: Chapman and Hall)
  Cooper F, Khare A and Sukhatme U 2001 Supersymmetry in Quantum Mechanics (Singapore: World Scientific)
[8] Mielnik B, Nieto L M and Rosas-Ortiz O 2000 Phys. Lett. A 269 70
[9] Grant A K and Rosner J L 1994 J. Math. Phys. 35 2142
[10] Miura R M 1968 J. Math. Phys. 9 1202
  Drazin P G and Johnson R S 1989 Solitons (New York: Cambridge University Press)
[11] Manin Yu I and Radul A O 1985 Commun. Math. Phys. 98 65
  Mathieu P 1988 J. Math. Phys. 29 2499
[12] Dargis P and Mathieu P 1993 Phys. Lett. A 176 67

New Journal of Physics 4 (2002) 55.1–55.14 (http://www.njp.org/)
[13] Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21
[14] Jafarizadeh M A and Fakhri H 1997 *Phys. Lett. A* **230** 164
[15] Jafarizadeh M A and Fakhri H 1998 *Ann. Phys., NY* **262** 260
[16] van Groesen E and de Jager E M 1994 *Mathematical Structures in Continuous Dynamical Systems* (Amsterdam: North-Holland)
[17] Fokas A S 1987 *Stud. Appl. Math.* **77** 253