SSP- Structure of Closed Helm and Flower Snark Graph Families

R Mary Jeya Jothi, R Revathi and D Angel
Department of Mathematics, Sathyabama Institute of Science and Technology, Chennai.

jeyajothi31@gmail.com, revathirangan75@gmail.com, angel.zara1001@gmail.com

Abstract. A graph is SSP (Super Strongly Perfect) if all of its (induced) subgraph H in G obsesses a (minimal dominating set) MDS that link up all of its cliques (maximal) in H. In this paper, SSP Structure along with its parameters (counting of maximal cliques, cardinality of dominating set (Minimal) and colourability of closed Helm graph is analysed and also Flower snark graph families are investigated.

1. Introduction
Here, Graphs are observed as connected, undirected, simple and finite. If every vertex in a subset of vertex set is mutually adjacent then it is called a clique. A dominating set M of V is a set in which all vertices in V - M is linked to some vertex of M. A MDS (minimal dominating set) is a set S in V, if for any u ∈ S, S - {u} should not be a dominating set. A path is a chain of group of vertices v_0, e_1, v_1, e_2, v_2, ... v_{l-1}, e_l, v_l, v_0 such that all the vertices are distinct. A cycle is a path which closed and it is denoted by C_n. The cycle is odd or even according to its number of vertices. Colouring of G means allotting different colours to all the vertex v of G in which no two connected vertices receive the identical colour. A Wheel graph is a graph with n vertices constructed by joining a vertex to all remaining vertices of C_{n-1}, an (n-1) cycle and is designated by W_n. A Helm graph H_n is constructed from W_n, an n-wheel graph by joining an edge (pendant) at every vertex of C_{n-1} an (n-1) cycle. That is, it is created by connecting an edge to every vertex of the cycle (outer) of W_n.

2. Overview of the Paper
Many classes of SSP graphs have been characterized already [1]. Along this line of thought, this paper discusses the SSP structure of closed Helm and Flower snark graph families together with the SSP parameters of closed Helm graphs.

2.1. Super Strongly Perfect Graph
A SSP (Super Strongly Perfect) graph is a graph in which all subgraph (induced) H of G obsesses a MDS (minimal dominating set) that link up all cliques (maximal) of H. SSP and non-SSP graphs are demonstrated in figures 1 and 2.
2.1.1. Illustration

![SSP Graph](image1)

Figure 1. SSP Graph

2.1.2. Illustration

![Non-SSP Graph](image2)

Figure 2. Non-SSP Graph

2.1.3. Theorem [1]
G is non-SSP, if G induces a cycle odd length atleast five as a subgraph (induced).

2.1.4. Theorem [3]
A graph G has atleast one maximal clique $K_n$, $n = 2,3,...$, if G is n-colourable $\iff$ if it is SSP.

3. Closed Helm Graph
A closed Helm graph $CH_n$ is constructed from helm $H_n$ and connecting the edges between all the vertices (pendant). The closed helm graph $CH_5$ is demonstrated below in figure 3.
3.1. Illustration

![Diagram of a closed Helm graph CH₅]

3.1.1. Theorem
Every closed Helm graph CHₙ, n is odd, n ≥ 5, is SSP.

Proof:
Let G be a closed Helm graph CHₙ, n is odd, n ≥ 5.
⇒ G is constructed from helm Hₙ and connecting the edges between all the vertices (pendant).
Also, helm graph Hₙ (n is odd) is obtained from Wₙ through adjoining an edge (pendant) at all vertex in the cycle.
It suffices to prove every odd wheel graph is SSP.
Let Wₙ be a wheel graph with odd number of vertices, n ≥ 5. It has 2(n-1) edges and it has maximal cliques K₂ and K₃.
Hence every wheel graph contains two types of induced sub graphs.

i) An induced sub graph with maximal cliques K₃.
ii) An induced sub graph with maximal cliques K₂.

Case: 1
Let G₁ be an induced sub graph with maximal cliques K₃.
⇒ G₁ has a vertex u, which is connected to all of the remaining vertices.
In this case, clearly {u} is the MDS of G₁ and it intersects all the cliques (maximal) K₃ of G₁.
Hence G₁ is SSP.

Case: 2
Let G₂ be a sub graph (induced) with cliques(maximal) K₂.
In this case, G₂ should be either an even cycle or a sub graph of an even cycle.
⇒ The alternate vertices of even cycle will intersect all the cliques (maximal) K₂ of G₂.
⇒ G₂ is SSP.
⇒ In both cases every induced sub graphs H of G induces a MDS that intersects all the cliques (maximal) of H.
Hence Wₙ, n is odd, n ≥ 5 is SSP.
Also, Hₙ (n is odd) is constructed from an n-wheel graph (Wₙ) by adjoining an edge (pendant) at every vertex of the cycle and Flₙ is got from Hₙ by adjoining every vertex (pendant) to the vertex (central) of Hₙ.
⇒ Flₙ is Hₙ along with the adjacency of vertex (pendant) to the vertex (central) of Hₙ will provide another n-set of cliques (maximal) K₃ (already in Wₙ, there exists n-set of cliques (maximal) K₃).
⇒ G has a central vertex say {u}, which is a MDS that covers all the cliques (maximal) K₃ of G.
⇒ G is SSP.
3.1.2. Theorem
Every closed Helm graph \( CH_n \), \( n \) is even, \( n \geq 6 \), is non-SSP.

Proof:
Let \( G \) be a closed Helm graph \( CH_n \), \( n \) is even, \( n \geq 6 \).
\( \Rightarrow \) \( G \) is constructed from helm \( H_n \) and connecting the edges between all the vertices (pendant).
Also, helm graph \( H_n \) (\( n \) is even) is obtained from \( W_n \) through adjoining an edge (pendant) at all vertex in the cycle.
As \( n \) is even, \( W_n \) induces an odd length cycle as a subgraph (induced).
\( \Rightarrow \) \( G \) has a cycle of odd length five (atleast) as a subgraph (induced).
\( \Rightarrow \) \( G \) is non-SSP, by theorem 2.1.3.

3.1.3. Proposition
Let \( G \) be a closed Helm graph \( CH_n \), \( n \) is odd, \( n \geq 5 \), then
1) \( G \) induces \( n-1 \) cliques (maximal) \( K_3 \).
2) \( G \) is 3-colourable.
3) \( G \) has a MDS with cardinality \( 1 + \frac{n-1}{2} \).

Proof:
Let \( G \) be a closed Helm graph \( CH_n \), \( n \) is odd, \( n \geq 5 \).
\( \Rightarrow \) \( G \) is got from \( H_n \) by adding edges between the vertices (pendant).
\( \Rightarrow \) \( H_n \) has \( n-1 \) cliques (maximal) \( K_3 \).
\( \Rightarrow \) \( G \) has \( n-1 \) cliques (maximal) \( K_3 \).
\( \Rightarrow \) \( G \) is 3-colourable (using theorem 2.1.4).
Also, \( G \) has a central vertex say \( \{v\} \), and the vertices (alternate) from the induced cycle \( C_n \) will provide a MDS.
\( \Rightarrow \) \( G \) has a MDS with cardinality \( 1 + \frac{n-1}{2} \).

Proposition 3.1.3 is illustrated below in figure 4.

3.2. Illustration

In figure 4,
1) \( CH_7 \) has 6 cliques (maximal), each of which is a \( K_3 \).
2) \( CH_7 \) has a MDS of 4 vertices.
3) \( CH_7 \) is 3-colourable.
3.1.4. Remark
Let $G$ be a closed Helm graph $\text{CH}_4$. $G$ is SSP and
1) $G$ has 3 cliques (maximal) $K_3$.
2) $G$ is 3 - colourable.
3) $G$ has MDS of 2 vertices.

3.3. Illustration

![Figure 5. CH4](image)

4. Flower Snark
The construction of $J_n$ (flower snark) graph is given below:
1) Form $n$ duplicates of $K_{1,3}$. Mark the middle vertex in each $K_{1,3}$ by $a_i$, vertices (outer) by $b_i$, $c_i$ and $d_i$.
2) Build $(b_1... b_n)$ an $n$-cycle.
3) Now, construct $(c_1... c_n d_1... d_n)$, a $2n$-cycle. One important property in the construction of $J_n$ is, $n$ should be odd [5]. $J_5$ is demonstrated below in figure 6.

4.1. Illustration

![Figure 6. J5](image)
4.1.1. Theorem

Every flower snark $J_n$, $n \geq 5$, is non-SSP.

Proof:

Let $G$ be a flower snark $J_n$, $n \geq 5$.

$\Rightarrow$ $J_n$ is a three regular graph which is cubic hence it always has an odd cycle $C_n$ ($n \geq 5$) at the centre.

$\Rightarrow$ $G$ has an odd length cycle ($C_n$, $n \geq 5$) as a subgraph (induced).

$\Rightarrow$ $G$ is non-SSP (Theorem 2.1.3).

5. Conclusion

In this paper, the structure (cyclic) of closed helm and flower snark graphs are analysed. Also, the SSP parameters of the graphs are given.

6. References

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