Abstract In this paper, Broué’s conjecture is reduced to simple groups, with an additional stability condition.

1. Introduction.

Let \( p \) be a prime and \( k \) an algebraically closed field of characteristic \( p \). Let \( H \) be a finite group with abelian Sylow \( p \)-group \( P \). Let \( B_0(H) \) be the principal block of \( H \) over \( k \). The principal block \( B_0(N_H(P)) \) of \( N_H(P) \) is the Brauer correspondent of \( B_0(H) \) in \( N_H(P) \). Broué conjectures that the blocks \( B_0(H) \) and \( B_0(N_H(P)) \) are derived equivalent. More strongly, Rickard predicts that there is a splendid Rickard equivalence between the two blocks.

Broué’s conjecture has two extended versions. Let \( G \) be a finite group with a normal subgroup \( H \). Set \( G' = \text{N}_G(P) \), \( H' = \text{N}_H(P) \) and \( K = (H \times H')\Delta(G') \), where \( \Delta(G') \) is the diagonal subgroup of \( G' \times G' \). One extended version stated in [9] is

Conjecture A. Assume that the index of \( H \) in \( G \) is coprime to \( p \). Then there is a complex \( C \) of \( kK \)-modules, whose restriction to \( H \times H' \) is a splendid Rickard complex and induces a Rickard equivalence between the blocks \( B_0(H) \) and \( B_0(H') \).

This is equivalent to say that there is a suitable splendid Rickard complex \( C \), which can be extended to \( K \) and induces a splendid Rickard equivalence between the blocks \( B_0(H) \) and \( B_0(H') \). A general reduction of Broué’s conjecture to simple groups was formulated in [4] with the extendibility of complexes and a structure theorem of finite groups with abelian Sylow \( p \)-subgroups. But usually, it is difficult to check whether a complex is extendible or not.

Another extended version stated in [5] is

Conjecture B. There is a \( K \)-stable splendid Rickard complex \( C \) of \( k(H \times H') \)-modules, which induces a splendid Rickard equivalence between the blocks \( B_0(H) \) and \( B_0(H') \).

Obviously, the stability is weaker than the extendibility. So it could be interesting to formulate a reduction of Broué’s conjecture by using the \( K \)-stability condition. For any finite nonabelian simple group \( L \), there is a canonical injective group homomorphism \( L \to \text{Aut}(L) \) induced by the \( L \)-conjugation, where \( \text{Aut}(L) \) denotes the full automorphism group of \( L \). We identify \( L \) with the image of such an injective group homomorphism. Our main result is the following

**Theorem 1.1.** Let \( G \) be a finite group and \( H \) a normal subgroup of \( G \) with \( p' \)-index. Assume that \( P \) is an abelian Sylow \( p \)-subgroup of \( H \). Assume that for any nonabelian composition factor \( L \) of \( H \), Conjecture B holds for \( \text{Aut}(L) \) and its normal subgroup \( L \). Then there is a \( \Delta(N_G(P)) \)-stable splendid Rickard complex inducing a splendid Rickard equivalence between \( B_0(H) \) and \( B_0(N_H(P)) \).

We close this section by pointing out that there is a special reduction for special cases of Broué’s conjecture (see [4], [1] and [3]).
2. Basic properties on Rickard complexes.

In this section, we collect three lemmas, which are more or less known.

2.1. In this paper, all rings are unitary, all modules are left modules and all the modules over \( k \) are finite dimensional, except all the group algebras over the following \( k \)-algebra \( \mathcal{D} \). Let \( A \) be a ring. We denote by \( \mathcal{G} = \mathcal{A}^* \) and \( J(A) \) the multiplicative group of \( A \) and the Jacobson radical of \( A \) respectively. Denote by \( \mathfrak{g} \) the commutative \( k \)-algebra of all the \( k \)-valued functions on the set \( \mathbb{Z} \) of all rational integers. Then \( \mathcal{D} \) is the \( k \)-algebra containing \( \mathfrak{g} \) as a unitary \( k \)-subalgebra and an element \( d \) such that \( \mathcal{D} = \mathfrak{g} \oplus \mathfrak{g}d, \ d^2 = 0 \) and \( df = \text{sh}(f)d \neq 0 \) for any \( f \in \mathfrak{g} \setminus \{0\} \), where \( \text{sh} \) denotes the automorphism on the \( k \)-algebra \( \mathfrak{g} \) mapping \( f \in \mathfrak{g} \) onto the \( k \)-valued function sending \( z \in \mathbb{Z} \) to \( f(z + 1) \). There is a \( k \)-algebra homomorphism \( \mathcal{D} \to k \) mapping \( f + f'd \) on \( f(0) \) for any \( f, f' \in \mathfrak{g} \). Denote by \( i_z \) the \( k \)-valued function mapping \( z' \in \mathbb{Z} \) onto \( \delta_{zz'} \).

2.2. Let \( C \) be a bounded complex of \( k \)-modules with the \( i \)-th differential map \( d_i : C_i \to C_{i-1} \). From the point of view in \([6]\), one can construct a \( \mathcal{D} \)-module \( \mathcal{C} = \oplus_i C_i \) defined by the equalities \( f \cdot (c_i)_i = (f(i)c_i)_i \) and \( d \cdot (c_i)_i = (d_i(c_i))_{i-1} \), where \( f \in \mathfrak{g}, c_i \in C_i \) for any \( i \in \mathbb{Z} \), and \( (c_i)_i \) is an element of \( \mathcal{C} \). Conversely, given a \( \mathcal{D} \)-module \( \mathcal{C} \), one can construct a bounded complex \( C \) of \( k \)-modules, whose \( i \)-th term \( C_i \) is \( i_z(\mathcal{C}) \) and whose \( i \)-th differential map \( d_i : C_i \to C_{i-1} \) is the restriction to \( C_i \) of the linear map \( d_{\mathcal{C}} \), which denotes the image of \( d \) in \( \text{End}_k(\mathcal{C}) \) and maps \( C_i \) into \( C_{i-1} \). The constructions above give a bijective correspondence between bounded complexes of \( k \)-modules and \( \mathcal{D} \)-modules. A bijective correspondence between bounded complexes of \( kG \)-modules and \( \mathcal{D}G \)-modules can be shown in a similar way. Correspondingly, some concepts on complexes of \( kG \)-modules, such as the contractility, the 0-splitness, the tensor product, the \( k \)-dual and so on, are translated into the corresponding ones on \( \mathcal{D}G \)-modules (see \([6, \S 10]\)). These translations are done in an invertible way, and without loss of generality, we can discuss Rickard equivalences in terms of \( \mathcal{D}G \)-modules.

2.3. A \( k \)-algebra \( A \) is a \( \mathcal{D} \)-interior algebra if there is a \( k \)-algebra homomorphism \( \rho : \mathcal{D} \to A \). For any \( x, y \in \mathcal{D} \) and \( a \in A \), we write \( \rho(x)ap(y) \) as \( x \cdot a \cdot y \) for convenience. By \([6, \text{Paragraph 11.2}]\), the \( \mathcal{D} \)-interior algebra structure on \( A \) induces a \( k \)-algebra homomorphism \( \mathcal{D} \to \text{End}_k(A) \), such that for any \( a \in A \) and any \( f \in \mathfrak{g} \), \( f(a) = \sum_{z, z' \in \mathbb{Z}} f(z)iz \cdot a \cdot iz'_{-z} \) and \( d(a) = (d \cdot a - a \cdot d) \cdot s \), where \( s \) is the sign function mapping \( z \in \mathbb{Z} \) onto \( (-1)^z \). Moreover, it is easily checked that for any \( a, a' \in A \) and \( f \in \mathfrak{g} \),

\[
f(aa') = \sum_{z, z' \in \mathbb{Z}} f(z)iz \cdot a \cdot iz'_{-z} (a') \quad \text{and} \quad d(aa') = d(a)s(a') + ad(a').
\]

Therefore \( A \) with the homomorphism \( \mathcal{D} \to \text{End}_k(A) \) is a \( \mathcal{D} \)-algebra (see \([6, \text{11.1}]\)). In a word, a \( \mathcal{D} \)-interior algebra structure on \( A \) induces a \( \mathcal{D} \)-algebra structure on \( A \).

2.4. Let \( H \) be a finite group. A \( k \)-algebra \( B \) is a \( \mathcal{D}H \)-interior algebra if there is a \( k \)-algebra homomorphism \( \mathcal{D}H \to B \). A \( \mathcal{D}H \)-interior algebra \( B \) is obviously a \( \mathcal{D} \)-interior algebra, which induces a \( \mathcal{D} \)-algebra structure on \( B \). Since the images of \( \mathcal{D} \) and \( H \) in \( B \) commute, the \( \mathcal{D} \)-algebra structure of \( B \) and the left and right multiplications of \( H \) on \( B \) determine a \( \mathcal{D}(H \times H) \)-module structure on \( B \). The group algebra \( \mathcal{D}H \) is an obvious \( \mathcal{D}H \)-interior algebra and thus has a \( \mathcal{D}(H \times H) \)-module structure. The subalgebra \( kH \) is a \( \mathcal{D}(H \times H) \)-submodule of \( \mathcal{D}H \) since \( f(a) = f(0)a \) and \( d(a) = 0 \) for any \( a \in kH \) and any \( f \in \mathfrak{g} \). The \( \mathcal{D}(H \times H) \)-module \( kH \) is the same as the \( \mathcal{D}(H \times H) \)-module \( kH \) defined by the homomorphism \( \mathcal{D} \to k \) (see 2.1) and by the left and right multiplications of \( H \). Given a nonzero central idempotent \( e \) of \( kH \), \( kHe \) is a submodule of the \( \mathcal{D}(H \times H) \)-module of \( kH \).

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2.5. Let $H'$ be another finite group and $e'$ a nonzero central idempotent in $kH'$. Let $\mathcal{C}$ be a $\mathcal{D}(H \times H')$-module, whose respective restrictions to $H \times 1$ and $1 \times H'$ are projective. Denote by $\mathcal{C}^*$ the $k$-dual of the $\mathcal{D}(H \times H')$-module $\mathcal{C}$, which is a $\mathcal{D}(H' \times H)$-module. The $\mathcal{D}(H \times H')$-module $\mathcal{C}$ induces a Rickard equivalence between $kHe$ and $kH'e'$ if there are respective isomorphisms of $\mathcal{D}(H \times H)$- and $\mathcal{D}(H' \times H')$-modules

$$\mathcal{C} \otimes_{kH'} \mathcal{C}^* \cong kHe \oplus X$$

and $\mathcal{C} \otimes_{kH} \mathcal{C} \cong kH'e' \oplus \mathcal{Y}$, where the $\mathcal{D}(H \times H)$- and $\mathcal{D}(H' \times H')$-modules $X$ and $\mathcal{Y}$ are contractile. In this case, the $\mathcal{D}(H \times H')$-module $\mathcal{C}$ is a Rickard complex. The endomorphism algebra $\text{End}_{k(1 \times H')}(\mathcal{C})$ is a $\mathcal{D}H$-interior algebra with the obvious algebra homomorphism $s\mathcal{t}_k: \mathcal{D}H \cong \mathcal{D}(H \times 1) \to \text{End}_{k(1 \times H')}(\mathcal{C})$. We denote by $s\mathcal{t}_k^e$ the restriction of $s\mathcal{t}_k$ to $kHe$. Considering the $\mathcal{D}(H \times H)$-module $\text{End}_{k(1 \times H')}(\mathcal{C})$ as in 2.4. Then the homomorphism $s\mathcal{t}_k^e$ is a $\mathcal{D}(H \times H)$-module homomorphism.

Lemma 2.6. Keep the notation in the paragraph 2.5. Then the homomorphism $s\mathcal{t}_k^e$ is a split $\mathcal{D}(H \times H')$-module injection.

Proof. There is a $\mathcal{D}(H \times H)$-module isomorphism $\mathcal{C} \otimes_{kH'} \mathcal{C}^* \cong \text{End}_{k(1 \times H')}(\mathcal{C})$. Since $\mathcal{C}$ induces a Rickard equivalence between $kHe$ and $kH'e'$, we have a $\mathcal{D}(H \times H)$-module isomorphism $\mathcal{h} : \text{End}_{k(1 \times H')}(\mathcal{C}) \cong kHe \oplus X$, where $X$ is a contractile $\mathcal{D}(H \times H)$-module. Denote by $\pi$ and $\tau$ the maps $kHe \oplus X \to kHe$, $(a, x) \mapsto a$ and $kHe \to kHe \oplus X$, $a \mapsto (a, x)$ respectively. Set $\mathcal{g} = \pi \circ \mathcal{h}$ and $\mathcal{f} = \mathcal{h}^{-1} \circ \tau$. For any $x \in H$, any $a \in \text{End}_{k(1 \times H')}(\mathcal{C})$ and any $\mathcal{f} \in \mathcal{g}$, we have $x \cdot \mathcal{f}(e) = \mathcal{f}(xe) = \mathcal{f}(e) \cdot x$.

$$f(\mathcal{f}(e)) = \sum_{x, x' \in \mathcal{Z}} f(x)i_{x,z}(a)i_{z,x'}(\mathcal{f}(e)) = \sum_{x, x' \in \mathcal{Z}} f(z)i_{x',z}(a)i_{z,x}(\mathcal{f}(e)) = \sum_{z \in \mathcal{Z}} f(z)i_{z}(a)\mathcal{f}(e) = f(a)\mathcal{f}(e)$$

and $d(\mathcal{f}(e)) = d(a)s(\mathcal{f}(e)) + ad(\mathcal{f}(e)) = d(a)\mathcal{f}(s(e)) + ad(\mathcal{f}(e)) = d(a)\mathcal{f}(e)$. Therefore the right multiplication by $\mathcal{f}(e)$ determines a $\mathcal{D}(H \times H)$-module endomorphism $r_{\mathcal{f}(e)}$ of $\text{End}_{k(1 \times H')}(\mathcal{C})$. Since $\mathcal{f}$ is the composition of $s\mathcal{t}_k^e$ and $r_{\mathcal{f}(e)}$ and $id_{kHe} = \mathcal{g} \circ \mathcal{f} = (\mathcal{g} \circ r_{\mathcal{f}(e)}) \circ s\mathcal{t}_k^e$, the homomorphism $s\mathcal{t}_k^e$ is a split injection of $\mathcal{D}(H \times H')$-modules.

Lemma 2.7. Keep the notation in the paragraph 2.5. Assume that $e$ and $e'$ are nonzero block idempotents. Then, the $\mathcal{D}(H \times H')$-module $\mathcal{C}$ has up to isomorphism a unique indecomposable noncontractile direct summand.

Proof. By Lemma 2.6, the image $\text{Im}(s\mathcal{t}_k^e)$ of the homomorphism $s\mathcal{t}_k^e$ is a direct summand of the $\mathcal{D}(H \times H)$-module $\text{End}_{k(1 \times H')}(\mathcal{C})$. Since $\mathcal{C}$ induces a Rickard equivalence between $kHe$ and $kH'e'$, any complement of $\text{Im}(s\mathcal{t}_k^e)$ in $\text{End}_{k(1 \times H')}(\mathcal{C})$ is isomorphic to a contractile $\mathcal{D}(H \times H')$-module $X$. Consequently, the center $Z(\text{Im}(s\mathcal{t}_k^e))$ of the algebra $\text{Im}(s\mathcal{t}_k^e)$ is a direct summand of the $\mathcal{D}$-module $\text{End}_{k(1 \times H')}(\mathcal{C})$ and any complement of it is contractile. On the other hand, since $d(kHe) = 0$ (see the paragraph 2.4), $Z(\text{Im}(s\mathcal{t}_k^e))$ is contained in the 0-cycle $C_0(\text{End}_{k(H \times H')}(\mathcal{C}))$ of the $\mathcal{D}$-module $\text{End}_{k(1 \times H')}(\mathcal{C})$. Therefore there is a surjective $k$-algebra homomorphism $C_0(\text{End}_{k(1 \times H')}(\mathcal{C})) \to Z(kHe)$, whose kernel is exactly the 0-boundary $B_0(\text{End}_{k(H \times H')}(\mathcal{C}))$ of the $\mathcal{D}$-module $\text{End}_{k(1 \times H')}(\mathcal{C})$. Since $Z(kHe)$ is local, there is a unique conjugacy class of primitive idempotents in $C_0(\text{End}_{k(1 \times H')}(\mathcal{C}))$, which is not contained in $B_0(\text{End}_{k(H \times H')}(\mathcal{C}))$. By [6, Proposition 10.8], this conjugacy class corresponds to a unique indecomposable noncontractile direct summand of $\mathcal{C}$, up to isomorphism.

The $k$-linear map $kH \to kH$ mapping $x \in H$ onto $x^{-1}$ is an opposite $k$-algebra isomorphism. For any $a \in kH$, we denote by $a^\circ$ the image of $a$ through the opposite isomorphism. The $k$-linear
map \( k(H \times N_H(P)) \to kH \otimes_k kN_H(P) \) mapping \((x, y) \in H \times N_H(P)\) onto \( x \otimes y \) is a \( k \)-algebra isomorphism. We identify \( k(H \times N_H(P)) \) and \( kH \otimes_k kN_H(P) \).

**Lemma 2.8.** Let \( G \) be a finite group, \( H \) a normal subgroup of \( G \) and \( P \) a \( p \)-subgroup of \( H \). Let \( e \) and \( f \) be nonzero idempotents in \( Z(kH) \) and \( Z(kN_H(P)) \) respectively. Assume that the \( N_G(P) \)-conjugation fixes \( e \) and \( f \) and that there is a \( \Delta(N_G(P)) \)-stable Rickard complex \( \mathcal{E} \) inducing a Rickard equivalence between \( kH \) and \( kN_H(P) f \). Then there is an isomorphism \( Z(kH e) \cong Z(kN_H(P) f) \), which is compatible with the \( N_G(P) \)-conjugations on \( Z(kH e) \) and \( Z(kN_H(P) f) \). Moreover, if \( e' \) and \( f' \) are nonzero idempotents in \( Z(kH e) \) and \( Z(kN_H(P) f) \) respectively and they correspond to each other through the isomorphism, then \((e' \otimes f')(\mathcal{E})\) induces a Rickard equivalence between \( kH e' \) and \( kN_H(P) f' \).

**Proof.** By the proof of Lemma 2.7, it is easily seen that the multiplication of \( kH e \) on \( \mathcal{E} \) induces a \( k \)-algebra isomorphism

\[
2.8.1 \quad Z(kH e) \cong H_0(\text{End}_k(H \times N_H(P))(\mathcal{E})) = C_0(\text{End}_k(H \times N_H(P))(\mathcal{E}))/B_0(\text{End}_k(H \times N_H(P))(\mathcal{E})).
\]

Clearly the center \( Z(kH e) \) with the \( N_G(P) \)-conjugation is a \( N_G(P) \)-algebra. Next we endow \( H_0(\text{End}_k(H \times N_H(P))(\mathcal{E})) \) with a \( N_G(P) \)-algebra structure so that the isomorphism 2.8.1 is a \( N_G(P) \)-algebra isomorphism.

Set \( \mathcal{S} = \text{End}_G(\mathcal{E}), \mathcal{T} = \text{End}_D(H \times N_H(P))(\mathcal{E}) \) and \( K = \Delta(N_G(P)) \). Denote by \( \mathcal{S}^* \) and \( \mathcal{T}^* \) the respective multiplication groups of \( \mathcal{S} \) and \( \mathcal{T} \). For any \( x \in K \), since \( \mathcal{E} \) is \( K \)-stable, there is \( f \in \mathcal{S}^* \) such that

\[
2.8.2 \quad f(ya) = yf^{-1}f(a)
\]

for any \( a \in \mathcal{E} \) and any \( y \in H \times N_H(P) \); moreover, if \( f' \) is another such a choice, then there is \( g \in \mathcal{T}^* \) such that \( f' = gf \). We denote by \( \hat{K} \) the set of all such pairs \((x, f)\). Clearly \( \hat{K} \) is a subgroup with respect to the multiplication of the direct product of \( K \times \mathcal{T}^* \), the map \( \mathcal{T}^* \to \hat{K} \) mapping \( f \) onto \((1, f)\) is an injective group homomorphism, and the map \( \hat{K} \to \mathcal{S}^* \) mapping \((x, f)\) onto \( f \) is a group homomorphism. We identify \( \mathcal{T}^* \) with the image of the homomorphism \( \mathcal{T}^* \to \hat{K} \). Then it is easily seen that \( \mathcal{T}^* \) is normal in \( \hat{K} \) and the quotient of \( \hat{K} \) by \( \mathcal{T}^* \) is isomorphic to \( K \).

By the equality 2.8.2, the image of \( \hat{K} \) in \( \mathcal{S}^* \) normalizes \( st(H \times N_H(P)) \). So the \( \hat{K} \)-conjugation induces a \( \hat{K} \)-algebra structure on \( \mathcal{E} = C_0(\text{End}_k(H \times N_H(P)))(\mathcal{E}) \), which induces a \( \hat{K} \)-algebra structure on \( H_0(\text{End}_k(H \times N_H(P)))(\mathcal{E}) \). By the isomorphism 2.8.1, the \( \mathcal{T}^* \)-conjugation on \( H_0(\text{End}_k(H \times N_H(P)))(\mathcal{E}) \) is trivial. Therefore the \( \hat{K} \)-conjugation on \( H_0(\text{End}_k(H \times N_H(P)))(\mathcal{E}) \) induces a \( K \)-algebra structure on \( H_0(\text{End}_k(H \times N_H(P)))(\mathcal{E}) \). By inflating the \( K \)-algebra \( H_0(\text{End}_k(H \times N_H(P)))(\mathcal{E}) \) through the obvious group isomorphism \( N_G(P) \cong K \), we get a \( N_G(P) \)-algebra \( H_0(\text{End}_k(H \times N_H(P)))(\mathcal{E}) \). Then by the equality 2.8.2, it is easily checked that the isomorphism 2.8.1 is a \( N_G(P) \)-algebra isomorphism.

By symmetry, we prove that the homomorphism \( st_{kN_H(P)f} : kN_H(P)f \to \text{End}_k([1 \times H])(\mathcal{E}^*) \) induces a \( k \)-algebra isomorphism \( Z(kN_H(P)f) \cong H_0(\text{End}_{kN_H(P)}(\mathcal{E}^*)) \). By duality, the multiplication of \( Z(kN_H(P)f) \) on \( \mathcal{E} \) induces a \( k \)-algebra isomorphism

\[
2.8.3 \quad Z(kN_H(P)f) \cong H_0(\text{End}_{kN_H(P)}(\mathcal{E}^*)).
\]

Since \( \mathcal{E} \) is \( K \)-stable, the homomorphism 2.8.3 is a \( N_G(P) \)-algebra isomorphism. By composing the isomorphism 2.8.2 and the inverse of the isomorphism 2.8.3, we get the desired \( N_G(P) \)-algebra isomorphism \( Z(kH e) \) and \( Z(kN_H(P)f) \).
3. Proof of Theorem 1.1.

As the title shows, in this section we prove Theorem 1.1. Let $H$ be a finite group and $P$ a Sylow $p$-subgroup of $H$. Assume that $\mathfrak{C}$ is a Rickard complex inducing a Rickard equivalence between $B_0(H)$ and $B_0(N_H(P))$. The complex $\mathfrak{C}$ is splendid if $\mathfrak{C}$ as $k(H \times N_H(P))$-module is projective relative to $\Delta(P)$; in this case, the Rickard equivalence between $B_0(H)$ and $B_0(N_H(P))$ is splendid.

Theorem 3.1. Let $G$ be a finite group, $H$ a normal subgroup of $G$ with $p'$-index, and $P$ a Sylow $p$-subgroup of $H$. Let $L$ be a normal subgroup of $G$ containing $H$. Assume that there is a $\Delta(N_G(P))$-stable splendid Rickard complex $\mathfrak{C}$ inducing a splendid Rickard equivalence between $B_0(H)$ and $B_0(N_H(P))$. Then there is a $\Delta(N_G(P))$-stable splendid Rickard complex inducing a splendid Rickard equivalence between $B_0(L)$ and $B_0(N_L(P))$.

3.2. We begin to prove this theorem. Set $\mathfrak{E} = \text{End}_G(\mathfrak{C})$, $\mathfrak{X} = \text{End}_G(H \times H')(\mathfrak{C})$ and $K = (H \times N_H(P))\Delta(N_G(P))$. As in the proof of Lemma 2.8, we construct the subgroup $\hat{K}$ of $K \times \mathfrak{X}^*$, which consists of all pairs $(x, f)$ in $K \times \mathfrak{X}^*$ satisfying $f(ya) = y^{x^{-1}}f(a)$ for any $a \in \mathfrak{X}$ and any $y \in H \times H'$. Clearly there are two injective group homomorphisms $\mathfrak{X}^* \to \hat{K}$, $a \mapsto (1, a)$ and $H \times H' \to \hat{K}$, $x \mapsto (x, x \cdot \text{id}_G)$. We identify $\mathfrak{X}^*$ and $H \times H'$ with their respective images in $\hat{K}$. It is easy to check that $\mathfrak{X}^*$ and $H \times H'$ are normal in $\hat{K}$, that $\mathfrak{X}^*$ and $H \times H'$ intersect trivially, and that the quotient of $\hat{K}$ by $\mathfrak{X}^*$ is isomorphic to $K$.

3.3. Set $\hat{K} = K/(H \times H')$ and $\hat{K} = \hat{K}/(H \times H')$. The inclusion $\mathfrak{X}^* \subset \hat{K}$ induces an injective group homomorphism $\mathfrak{X}^* \to \hat{K}$. We identify $\mathfrak{X}^*$ and its image in $\hat{K}$. Clearly $\mathfrak{X}^*$ is normal in $\hat{K}$ and the quotient group $\hat{K}$ by $\mathfrak{X}^*$ is isomorphic to $K$. Since $\hat{K}$ acts trivially on the subgroup $k^*$ of $\mathfrak{X}^*$ by conjugation, we can consider the obvious short exact sequence

$$1 \to \mathfrak{X}^*/k^* \to \hat{K}/k^* \to \hat{K} \to 1.$$ 

By Lemma 2.7, we assume without loss of generality that the $\mathfrak{D}(H \times H')$-module $\mathfrak{C}$ is indecomposable. Then $\mathfrak{X}$ is a local algebra and $\mathfrak{X}^* = k^* \times (\text{id}_\mathfrak{X} + J(\mathfrak{X}))$. On the other hand, we have group isomorphisms $G/H \cong N_G(P)/N_H(P)$ and $K/(H \times N_H(P)) \cong \Delta(N_G(P))/\Delta(N_H(P))$. Since the index of $H$ in $G$ is coprime to $p$, the sequence 3.3.1 splits. In particular, $\hat{K}$ has a subgroup $\tilde{K}$ containing $k^*$ such that the quotient of $\tilde{K}$ by $k^*$ is isomorphic to $K$.

3.4. Denote by $\hat{K}$ the inverse image of $\hat{K}$ through the canonical homomorphism $\hat{K} \to \hat{K}$. Then $\hat{K}$ contains the normal subgroup $H \times N_H(P)$ and $\hat{K}$ is a central extension of $K$ by $k^*$. Set $I = \Delta(N_G(P))$ and denote by $\tilde{I}$ the inverse image of $I$ through the canonical homomorphism $\hat{K} \to K$. Then $\tilde{K} = (H \times N_H(P))\tilde{I}$. We choose a subgroup $N'$ of $\tilde{I}$ such that $\tilde{K} = k^*N'$ and that $N'$ contains the subgroup $\Delta(N_H(P))$. Set $\Lambda = k^* \cap N'$. Clearly the quotient $N'/\Lambda$ is isomorphic to $I$. Composing the canonical homomorphism $N' \to N'/\Lambda$, the isomorphism $N'/\Lambda \cong I$ and the isomorphism $I \cong N_G(P)$, $(x, x) \mapsto x$, we get a group homomorphism $N' \to N_G(P)$. Then we lift the conjugation action of $N_G(P)$ on $H$ to an action of $N'$ on $H$ through the homomorphism $N' \to N_G(P)$. The action of $N'$ on $H$ induces an action of $N'$ on $kH$.

3.5. Set $J = \Delta(N_H(P))$. The $k$-linear map $kJ \to kH$ sending any $(x, x) \in J$ onto $x$ is an injective $k$-algebra homomorphism and it obviously preserves the conjugation action of $N'$ on $kJ$ and the action of $N'$ on $kH$. Therefore $kH$ is a $kJ$-interior $N'$-algebra with the action of $N'$ on $kH$ and with the algebra homomorphism $kJ \to kH$ (see [11] §2). Then we construct a crossed product $kH \otimes_{kJ} kN'$ and denote by $N$ the subgroup $H \otimes N'$ in the crossed product. There are injective
group homomorphisms $H \to N$, $x \mapsto x \otimes 1$ and $N' \to N$, $y \mapsto 1 \otimes y$. We identify $H$ and $N'$ and their respective images in $N$. Clearly $H$ is normal in $N$, $N' \cap H = N_H(P)$, $N = HN'$, $N_N(P) = N'$ and the quotient group $N/\Lambda$ is isomorphic to $G$. Let $F$ be the inverse image of $L$ through the canonical homomorphism $N \to G$, and set $F' = N_F(P)$.

3.6. Consider the respective subgroups $(H \times N_H(P))\Delta(F')$ and $(H \times N_H(P))F'$ of $F \times F'$ and $\tilde{K}$. There is an obvious group isomorphism $(H \times N_H(P))\Delta(F') \cong (H \times N_H(P))F'$ which is identical on $H \times N_H(P)$ and maps $(y, y) \in \Delta(F')$ onto $y$. Therefore there is an algebra homomorphism
\[
\mathcal{D}((H \times N_H(P))\Delta(F')) \to \text{End}_k(\mathcal{E}),
\]
extending the structural homomorphism $\mathcal{D}(H \times N_H(P)) \to \text{End}_\mathcal{D}(\mathcal{E})$ of the $\mathcal{D}(H \times N_H(P))$-module $\mathcal{E}$. By [2] Corollary 3.9], the induced module $\text{Ind}_{\mathcal{D}(H \times N_H(P))}^{\mathcal{D}(F \times F')}\mathcal{E}$ induces a splendid Rickard equivalence between $kFb_0(H)$ and $kFb_0(N_H(P))$, where $b_0(H)$ and $b_0(N_H(P))$ are the respective identity elements in $B_0(H)$ and $B_0(N_H(P))$. Since the $\mathcal{D}(H \times N_H(P))\Delta(F')$-module $\mathcal{E}$ is obviously $\Delta(N')$-stable, the module $\text{Ind}_{\mathcal{D}(H \times N_H(P))}^{\mathcal{D}(F \times F')}\mathcal{E}$ is $\Delta(N')$-stable. By Lemma 2.8, there is an $N'$-stable block idempotent $e$ in $Z(kF'\tilde{B})$ such that the $\mathcal{D}(F \times F')$-module $\mathcal{W} = (b_0(F) \otimes e)(\text{Ind}_{\mathcal{D}(F \times F')}^{\mathcal{D}(H \times N_H(P))}\mathcal{E})$ induces a splendid Rickard equivalence between $kB_0(F)$ and $kF'e$.

3.7. Let $M$ be a $kG$-module. Let $Q$ and $R$ be subgroups of $G$ such that $R \leq Q$. We denote by $M^Q$ the submodule of all $Q$-fixed elements of $M$ and by $\text{Tr}_R^Q : M^R \to M^Q$ the usual trace map. Set $M(Q) = M^Q / (\sum_T M^T_Q)$, where $T$ run over all proper subgroups of $Q$. There is an obvious $kN_G(Q)$-module structure on $M(Q)$ and it is known that $M(Q)$ is a $p$-permutation module if the $kG$-module $M$ is so. Furthermore, if $M$ is a $\mathcal{D}$-module, then $M(Q)$ is a $\mathcal{D}N_G(Q)$-module.

3.8. Clearly $b_0(C_F(P)) = b_0(N_F(P))$ and $(P, b_0(C_F(P)))$ is a maximal Brauer pair associated to the block $B_0(F)$. Let $(P, e')$ be a maximal Brauer pair associated to the block $kF'e$. Since $\mathcal{W}$ induces a splendid Rickard equivalence between $B_0(F)$ and $kF'e$, the blocks $B_0(F)$ and $kF'e$ have equivalent Brauer categories; in particular, there is a group isomorphism
\[
N_F(P)/C_F(P) = N_F(P, b_0(C_F(P)))/C_F(P) \cong N_F(P, e')/C_{F'}(P).
\]
Therefore $N_F(P, e')$ is equal to $N_F(P)$ and $e'$ is a block idempotent of $kN_F(P)$.

3.9. Since the $\mathcal{D}(F \times F')$-module $\mathcal{W}$ induces a splendid Rickard equivalence between $B_0(F)$ and $kF'e$, by [2] Proposition 1.5 the $\mathcal{D}(C_F(P) \times C_{F'}(P))$-module $\mathcal{W}' = (b_0(C_F(P)) \otimes e'')(\mathcal{W}(\Delta(P)))$ induces a splendid Rickard equivalence between $B_0(C_F(P))$ and $kC_{F'}(P)e'$. Obviously the module $\mathcal{W}'$ is also a $\Delta(N')$-stable $\mathcal{D}((C_F(P) \times C_{F'}(P))(\Delta(F')))\text{-module and so the induced module}$
\[
\mathcal{W}'' = \text{Ind}_{\mathcal{D}(F' \times F')}^{\mathcal{D}(F \times F')} \mathcal{W}'
\]
is $\Delta(N')$-stable. Since the index of $H$ in $G$ is coprime to $p$, by [2] Corollary 3.9], the module $\mathcal{W}''$ induces a splendid Rickard equivalence between $B_0(F')$ and $kF'e'$.

3.10. Clearly the $\mathcal{D}(F \times F')$-module $\mathcal{W}''' = \mathcal{W} \otimes_{kF'} \mathcal{W}'''$ is $\Delta(N')$-stable, and by the composition of splendid Rickard equivalences, it induces a splendid Rickard equivalence between $B_0(F)$ and $B_0(F')$. On the other hand, since $F$ is a central extension of $L$ by $\Lambda$ and since $\Lambda$ is a $p'$-group, the canonical homomorphism $N \to G$ induces a $k$-algebra isomorphism $B_0(F) \cong B_0(L)$; similarly, the canonical homomorphism $N' \to N_G(P)$ also induces a $k$-algebra isomorphism $B_0(F') \cong B_0(N_L(P))$. 

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Therefore the $\mathcal{O}(L \times N_L(P))$-module $B_0(L) \otimes_{kF} \mathfrak{M}^m \otimes_{kF} B_0(N_L(P))$ is $\Delta(N_G(P))$-stable and induces a splendid Rickard equivalence between $B_0(G)$ and $B_0(N_G(P))$. Up to now, the proof of Theorem 3.1 is finished.

**Remark.** Alternatively, we apply [7 Theorem 1.12] to the $\mathcal{O}(N \times N')$-module $\mathfrak{M}$ and then get a splendid Rickard equivalence between $kN'e$ and $B_0(N')$.

### 3.11. Proof of Theorem 1.1

Denote by $O_{p'}(H)$ the maximal normal $p'$-subgroup of $H$ and by $O_{p'}(H)$ the minimal normal subgroup of $H$ with $p'$-index. Since $H$ has abelian Sylow $p$-subgroups, there are nonabelian simple groups $H_1$, $H_2$, ⋯, $H_n$ and a $p$-subgroup $Q$ such that

$$O_{p'}(H/O_{p'}(H)) \cong H_1 \times H_2 \times \cdots \times H_n \times Q.$$  

Since the canonical homomorphism $H \to H/O_{p'}(H)$ induces a $k$-algebra isomorphism $B_0(H) \cong B_0(H/O_{p'}(H))$, we assume without loss of generality that $O_{p'}(H)$ is trivial. We identify both sides of the isomorphism 3.11.1 and then all $H_i$ and $Q$ are normal subgroups of $O_{p'}(H)$. For each $i$, set $P_i = P \cap H_i$. Clearly $P_i$ is a Sylow $p$-subgroup of $H_i$ and

$$P = P_1 \times P_2 \times \cdots \times P_n \times Q.$$  

The $k$-linear map $kO_{p'}(H) \to kH_1 \otimes_k h_2 \otimes_k \cdots \otimes_k H_n \otimes_k kQ$ sending $(x_1, x_2, \cdots, x_n, u)$ onto $x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes u$ is a $k$-algebra isomorphism, where $x_i \in H_i$ for $1 \leq i \leq n$ and $u \in Q$. We identify both sides of the $k$-algebra isomorphism. Then we have

$$B_0(O_{p'}(H)) = B_0(H_1) \otimes_k B_0(H_2) \otimes_k \cdots \otimes_k B_0(H_n) \otimes_k kQ.$$  

Recall that in the setting of Theorem 1.1, we are identifying $H_i$ and the image of the canonical homomorphism $H_i \to \text{Aut}(H_i)$. For any $x \in H_i$ and any $\varphi \in \text{Aut}(H_i)$, we have $\varphi x \varphi^{-1} = \varphi(x)$. For each $i$, set $X_i = \text{Aut}(H_i)$ and $J_i = (H_i \times N_{H_i}(P_i))\Delta(N_X(P_i))$. According to the hypothesis, there is a $J_i$-stable splendid Rickard complex $\xi_i$ inducing a splendid Rickard equivalence between $B_0(H_i)$ and $B_0(N_{H_i}(P_i))$. It is trivial to see that the complex $\xi_i$ is $J_i$-stable if and only if for any $\varphi \in N_{X_i}(P)$, there is a $\mathcal{O}$-module isomorphism $\mathfrak{f}_{\varphi}^i : \xi_i \cong \xi_i$ such that $\mathfrak{f}_i((x, x')a) = (\varphi(x), \varphi(x'))\mathfrak{f}_i(a)$ for any $x \in H_i$, any $x' \in N_{H_i}(P)$ and any $a \in \mathcal{C}$. Clearly $G$ permutes $H_1$, $H_2$, ⋯, $H_n$ by conjugation. For each $i$, we denote by $G_i$ the stabilizer of $H_i$ in $G$. Set $K_i = (H_i \times N_{H_i}(P_i))\Delta(N_{G_i}(P_i))$. The complex $\xi_i$ is $K_i$-stable, since $\mathfrak{f}_i((x, x')a) = (\varphi_y(x), \varphi_y(x'))\mathfrak{f}_i(a)$, where $x \in H_i$, $x' \in N_{H_i}(P)$, $a \in \mathcal{C}$, $y \in G_i$ and $\varphi_y$ denotes the group isomorphism $H_i \cong H_i$ induced by the $y$-conjugation.

Let $\{H_{i_1}, H_{i_2}, \cdots, H_{i_k}\}$ be a complete representative set of orbits of the action of $G$ on the set $\{H_1, H_2, \cdots, H_n\}$. For any $i$, denote by $\{g_{i_{1,1}}, g_{i_{1,2}}, \cdots, g_{i_{1,\ell}}\}$ a complete representative set of right cosets of $G_{i_{1}}$ in $G$. For any $b$ such that $1 \leq h \leq j_{i_{1}}$, set $H_{i_{1},h} = g_{i_{1,1}}H_{i_{1}}g_{i_{1,1}}^{-1}$ and $P_{i_{1},h} = g_{i_{1,1}}P_{i_{1}}g_{i_{1,1}}^{-1}$. We rearrange the factors in the direct product decompositions 3.11.1 and 3.11.2, so that

$$O_{p'}(H) = H_{i_{1,1}} \times H_{i_{1,2}} \times \cdots \times H_{i_{1,j_{1}}} \times \cdots \times H_{i_{k,1}} \times H_{i_{k,2}} \times \cdots \times H_{i_{k,j_k}} \times Q.$$  

$$P = P_{i_{1,1}} \times P_{i_{1,2}} \times \cdots \times P_{i_{1,j_{1}}} \times \cdots \times P_{i_{k,1}} \times P_{i_{k,2}} \times \cdots \times P_{i_{k,j_k}} \times Q.$$  

Denote by $\xi_{i_{1},h}$ the inflation of $\xi_{i_{1}}$ through the group isomorphism

$$H_{i_{1},h} \times N_{H_{i_{1},h}}(P_{i_{1},h}) \cong H_{i_{1}} \times N_{H_{i_{1}}}(P_{i_{1}}), (x, y) \mapsto (g_{i_{1,1}}^{-1}xg_{i_{1,1}}, g_{i_{1,1}}^{-1}yg_{i_{1,1}}).$$
Then $\mathfrak{c}_{i,\ell,h}$ is a splendid Rickard complex inducing a splendid Rickard equivalence between $B_0(H_{i,\ell,h})$ and $B_0(N_{H_{i,\ell,h}}(P_{i,\ell,h}))$. We define a $\mathfrak{D}(Q \times Q)$-module structure on $kQ$ by the homomorphism $\mathfrak{D} \to k$ (see 2.1) and by the left and right multiplication of $Q$. Set

$$
\mathfrak{c} = \mathfrak{c}_{i,1} \otimes_k \mathfrak{c}_{i,2} \otimes_k \cdots \otimes_k \mathfrak{c}_{i,j_1} \otimes_k \cdots \otimes_k \mathfrak{c}_{i,k,1} \otimes_k \cdots \otimes_k \mathfrak{c}_{i,k,j_k} \otimes_k kQ.
$$

Then it is easy to see that $\mathfrak{c}$ is a splendid Rickard complex inducing a splendid Rickard equivalence between $B_0(O^P(H))$ and $B_0(N_{O^P(H)}(P))$. Set $M = (O^P(H) \times N_{O^P(H)}(P))\Delta(N_G(P))$. Since $N_G(P) \cap G_i = N_{G_i}(P) \subset N_{G_i}(P)$ for each $i$ and since each $\mathfrak{c}_i$ is $K_i$-stable, it is easy to check that the splendid Rickard complex $\mathfrak{c}$ is $M$-stable. Then Theorem 1.1 follows from Theorem 3.1.

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