Effective Field Theory for Finite Systems with Spontaneously Broken Symmetry

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We extend effective field theory to the case of spontaneous symmetry breaking in genuinely finite quantum systems such as small superfluid systems, molecules or atomic nuclei, and focus on deformed nuclei. In finite superfluids, symmetry arguments alone relate the spectra of systems with different particle numbers. For systems with non-spherical intrinsic ground states such as atomic nuclei or molecules, symmetry arguments alone yield the universal features of the low-lying excitations as vibrations that are the heads of rotational bands. The low-lying excitations in deformed nuclei differ from those in molecules because of symmetry properties caused by pairing.

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I. INTRODUCTION

This paper addresses spontaneous symmetry breaking in non-relativistic quantum systems of finite size. Strictly speaking, spontaneous symmetry breaking can only occur in infinite systems. Then the ground state exhibits a lesser degree of symmetry than the Hamiltonian itself, i.e., the ground state is invariant under a symmetry group $\mathcal{H}$ that is a proper subgroup of the Hamiltonian’s symmetry group $\mathcal{G}$. The low-energy excitations are strongly constrained by symmetry and given in terms of (weakly interacting) Nambu-Goldstone modes. These can be calculated within an effective field theory (EFT) that is solely based on the pattern of symmetry breaking. From a technical point of view, the EFT is a non-linear $\sigma$ model with fields that parameterize the coset space $\mathcal{G}/\mathcal{H}$\textsuperscript{1,2}. Examples for spontaneous symmetry breaking are the breaking of spin-rotational symmetry in a ferromagnet, the breaking of translational symmetry in a crystal lattice, and the breaking of chiral symmetry in quantum chromodynamics (QCD). In these examples, the Nambu-Goldstone modes are magnons, phonons, and pions, respectively, and EFTs have been developed for all these cases\textsuperscript{4,10}, see Refs. 11,12 for reviews.

In finite systems, the ground state exhibits the full symmetry of the Hamiltonian, and spontaneous symmetry breaking becomes evident in symmetry-unrestricted mean-field calculations\textsuperscript{13,15}. It is then a major effort (and complication!) to restore the symmetry with the help of projection techniques. The expressions “obscured symmetry breaking”\textsuperscript{10} or “emergent symmetry breaking”\textsuperscript{15} (which we adopt here) have been proposed for such systems. There are two distinct cases for which emergent symmetry breaking plays a role. First, numerical simulations of infinite physical systems are usually limited to a finite volume, and it is then important to understand the finite-size corrections. Some rigorous results are known in this case\textsuperscript{16,17}, and finite-size corrections to partition functions and thermodynamical observables have been worked out within EFTs for simulations of QCD on finite lattices\textsuperscript{18,19}, and for spin systems\textsuperscript{20}. Genuinely finite systems constitute the second and probably physically most interesting case. Prominent examples are the emergence of superfluidity in trapped ultracold Bose gases\textsuperscript{21,22}, pairing in atomic nuclei (both breaking a $U(1)$ phase symmetry), and non-spherical shapes of molecules and atomic nuclei (both breaking $O(3)$ rotational symmetry in the limit of infinite system size). Here, the techniques for constructing EFTs for spontaneous symmetry breaking need to be modified, and the interest is in spectra and transitions rather than in thermodynamical observables.

The EFT for a finite system with emergent symmetry breaking is, of course, related to the EFT for the corresponding infinite system with spontaneous symmetry breaking. The symmetry must be realized nonlinearly, and the Nambu-Goldstone fields parameterize the coset space $\mathcal{G}/\mathcal{H}$\textsuperscript{1,2}. In the infinite system, the proper Nambu-Goldstone fields depend on space and time and exhibit fluctuations of small amplitudes and long wavelengths. A purely time-dependent (and spatially constant) mode is forbidden because it would relate states of inequivalent Hilbert spaces. In the finite system, however, this zero mode, i.e., the spatially constant mode of the Nambu-Goldstone field, must be singled out and treated separately. That mode undergoes large-amplitude fluctuations and upon quantization restores the symmetry. In the finite system, the small-amplitude fluctuations, i.e., the proper Nambu-Goldstone modes with nontrivial temporal and spatial dependence, must likewise be quantized. The theoretical implementation of this program is not trivial and is demonstrated for two interesting and important cases. We first consider as an example the emergent breaking of a $U(1)$ phase symmetry in finite superfluids such as ultracold bosonic atom gases or atomic nuclei. In this case, the proper Nambu-Goldstone modes and the global phase rotations do not couple in leading order, and both modes have the same energy scale. This facilitates the description. Second, we consider the emergent breaking of $SO(3)$ symmetry to its $SO(2)$ subgroup. This case describes the low-energy physics of nonspherical objects with axial symmetry such as linear molecules and deformed atomic nuclei.
The case is more complicated and interesting due to the interactions between global rotations and proper Nambu-Goldstone modes, and the energy scale of the rotational mode differs from the energy scale of the proper Nambu-Goldstone modes. Our detailed presentation of these two problems makes clear how to develop EFTs for systems with emergent symmetry breaking in general.

In this paper we construct EFTs for finite systems with emergent symmetry breaking, and we focus particularly on deformed atomic nuclei. Such nuclei are traditionally described by generalized collective models \[24, 27\] or the interacting boson model \[28, 29\]. For more microscopic approaches to collective motion, we refer the reader to Refs. \[14, 31\]. For deformed nuclei, the presented EFT generalizes the simpler construction of an effective theory proposed recently \[32\]. Based on symmetry principles alone, our model-independent approach re-derives some of the well-known results for collective nuclear models. We expect that extensions of the EFT approach could be useful in addressing well-known and long-standing limitations of the collective models such as, e.g., the significant overprediction of electromagnetic interband transitions, see Refs. \[27, 33\] for recent reviews of this problem.

Our procedure is patterned after the case of the infinite ferromagnet \[3, 7, 10\]. We generalize the expression for unitary transformations in the coset space by including purely time-dependent variables. These account for the dynamics of the finite system. The resulting generators define the Nambu-Goldstone modes as well as the zero modes as classical fields. From these we construct the building blocks of the effective Lagrangian \(L\) using arguments of invariance and energy scaling. Quantization of the Hamiltonian obtained by a Legendre transformation of \(L\) then determines the spectrum.

II. EMERGENT BREAKING OF U(1) PHASE SYMMETRY

Superfluids can be viewed as breaking \(U(1)\) phase symmetry. Examples are infinite Bose-Einstein condensates (BEC) or the paired states of a BCS superconductor. In their mean-field description, these systems exhibit a coherent phase at the expense of a well-defined particle number. In finite systems the particle number is, of course, a good quantum number. While the results we derive in this Section are well know (see, e.g. Ref. \[34\]), their derivation exhibits novel aspects and paves the way for the description of deformed nuclei within an EFT.

In finite superfluids the low-lying excitations are governed by two energy scales. These are the chemical potential \(\mu\) (the energy needed to add a single boson to the system), and the energy \(\Omega\) of long wave-length excitations. These scales are different for noninteracting and for interacting systems. We consider harmonically trapped bosons as an example. For noninteracting bosons, the proper thermodynamic limit is defined \[35\] by keeping the product of particle number \(N\) and the third power of the trap frequency constant. That frequency, in turn, defines \(\mu\) for the condensate. Hence \(\mu \sim \Omega \sim N^{-1/3}\) scale similarly. As \(N \to \infty\), the ground states of systems with different particle numbers \(N\) become quasi degenerate, superpositions of such states (i.e., states with a constant phase and undefined \(N\)) describe the superfluid, and the \(U(1)\) phase symmetry is broken in the thermodynamic limit. For finite uniform systems, both energies scale as \(\mu \sim \Omega \sim N^{-2/3}\) \[36\], and the arguments apply likewise.

For interacting Bose gases of volume \(V\), the chemical potential \(\mu\) typically approaches in the thermodynamic limit the nonzero value \(\mu \sim gN/V\) where \(g\) measures the strength of the interaction \[35\]. Similarly, for BCS superconductors, adding a pair roughly costs an energy of the order of the Fermi energy. The latter becomes constant in the thermodynamic limit. In both cases the breaking of \(U(1)\) symmetry can be understood in the framework of the grand canonical ensemble. The system is coupled to a particle reservoir with external chemical potential \(\mu_{\text{ext}}\), and the Hamiltonian \(H - \mu_{\text{ext}} N\) is minimized. Adjusting \(\mu_{\text{ext}}\) such that \(\mu_{\text{ext}} \approx \mu\) for \(N \gg 1\) introduces a quasi degeneracy between states of different particle numbers, and a superposition of these states then breaks \(U(1)\) symmetry. However, the canonical and the grand canonical ensemble yield different results. For nonextensive quantities the differences decrease as \(N^{-1/2}\) for \(N \to \infty\). It is only within this uncertainty that an isolated finite system can be viewed as equivalent to a finite system coupled to a particle reservoir. Technically, the introduction of a finite external chemical potential \(\mu_{\text{ext}}\) breaks time-reversal invariance, and the resulting effective theory differs from the case \(\mu_{\text{ext}} = 0\). As we will see, the latter can be recovered from the former by simply setting \(\mu_{\text{ext}} = 0\) in the leading-order equations we derive below. For nonzero \(\mu_{\text{ext}}\), the low-energy scales of interest are \(\mu - \mu_{\text{ext}}\) and \(\Omega\), and we assume that both are similar in size.

In the case of a broken \(U(1)\) symmetry we have \(\mathcal{G} = U(1)\) and \(\mathcal{H} = 1\). The Nambu-Goldstone fields parameterize the coset \(\mathcal{G}/\mathcal{H} \sim \mathcal{G}\), which is the group itself. The fields induce local phase transformations, and in a finite system the relevant operator is

\[
U(\alpha, \beta) = e^{i\alpha(t)} e^{V^{1/2} \beta(t, \mathbf{x})} .
\]

Here, \(\beta\) is the Nambu-Goldstone field, the angle \(\alpha\) is the zero mode that needs to be singled out, and \(V\) is the volume. For a proper Nambu-Goldstone field we have \(\int_V d^3 x \beta = 0\). Following Refs. \[11, 12\], we build the invariants of our theory from the derivatives \((\nu = x, y, z)\)

\[
- iU^{-1} \partial_x U = \dot{\alpha} + V^{1/2} \dot{\beta} ,
\]

\[
- iU^{-1} \partial_y U = V^{1/2} \partial_y \beta
\]

which are in the Lie algebra of \(\mathcal{G}\). Here the dot denotes the time derivative. Under a global phase transformation with angle \(\gamma\), the operator \(U\) becomes \(e^{i\gamma} U(\alpha, \beta) = U(\alpha + \gamma, \beta)\). Thus, \(\dot{\alpha}, \dot{\beta}\), and \(\partial_y \beta\) are invariant under
global phase transformations. Note also that $\beta$ is a truly "intrinsic" degree of freedom because it is unaffected by a global phase transformation.

We use these invariants to construct the leading-order terms in the effective Lagrangian, taking account of energy scales. The scale associated with the $\alpha$ degree of freedom is $\mu - \mu_{\text{ext}}$, that associated with $\beta$ is $\Omega$. Assuming also invariance under rotations, we have the invariants

$$L_0 \equiv C_1 \mu_{\text{ext}} \int d^3x \dot{\alpha} = C_1 V \mu_{\text{ext}} \dot{\alpha},$$

$$L_1 \equiv \frac{C_1}{2} \int d^3x \dot{\alpha}^2 = \frac{C_1 V}{2} \alpha^2,$$

$$L_2 \equiv \frac{C_2}{2} \int d^3x \dot{\alpha}^2, \quad L_3 \equiv \frac{D}{2} \int d^3x (\nabla \beta)^2. \quad (4)$$

Here, $C_1$, $C_2$, and $D$ are constants that have to be determined from low-energy data. The Lagrangian is

$$L = \frac{C_1 V}{2} \alpha^2 + C_1 V \mu_{\text{ext}} \dot{\alpha} + \int d^3x \left( \frac{C_2}{2} \dot{\beta}^2 - \frac{D}{2} \left( \nabla \beta \right)^2 \right). \quad (5)$$

The conserved quantity corresponding to invariance under global phase transformations is the particle number

$$N \equiv p_\alpha \equiv \frac{\partial L}{\partial \dot{\alpha}}. \quad (6)$$

This conserved quantity can be derived via the Noether theorem, and $p_\alpha$ is the canonical momentum of $\alpha$.

We expand $\beta(t, \vec{x}) = \sum_j \beta_j \phi_j(\vec{x})$ in a set of orthonormal complex functions $\phi_j(\vec{x}), j = 1, 2, \ldots$ with $\int_V d^3x \phi_j(\vec{x}) = 0$ (absence of zero modes for the Nambu-Goldstone field). As an example, we choose the eigenfunctions of a free particle in a spherical cavity of volume $V$ with von Neumann boundary conditions. The Lagrangian becomes

$$L = \frac{C_1 V}{2} \dot{\alpha}^2 + C_1 V \mu_{\text{ext}} \dot{\alpha} + \sum_{j>0} \left( \frac{C_2}{2} \beta_j^2 - \frac{Dk_j^2}{2} \beta_j^2 \right). \quad (7)$$

Here, $k_j^2$ denotes the squared momentum of the spherical wave $\phi_j(\vec{x})$. A Legendre transformation with $p_{\beta_j} \equiv \partial L / \partial \dot{\beta}_j$ and Eq. (6) yield the Hamiltonian

$$H = \frac{(p_\alpha - C_1 V \mu_{\text{ext}})^2}{2C_1 V} + \sum_{j>0} \left( \frac{p_{\beta_j}}{2C_2} + \frac{Dk_j^2}{2} \beta_j^2 \right). \quad (8)$$

We quantize $H$ by putting $p_\alpha = -i \partial_\alpha$, and $p_{\beta_j} = -i \partial_{\beta_j}$. Then $p_{\beta_j} e^{iN_\alpha} = N e^{iN_\alpha}$. The intrinsic degrees of freedom $\beta_j$ yield harmonic-oscillator spectra with energies

$$\omega_j \equiv k_j \left( \frac{D}{C_2} \right)^{1/2}. \quad (9)$$

In the long-wave-length limit we have $k_j \sim V^{-1/3}$, and the measurement of the low-energy collective excitations of the superfluid determines the ratio $D/C_2$. The amplitude of the Nambu-Goldstone modes (i.e., the oscillator length) is

$$l_j \equiv (C_2 k_j^2)^{-1/4}, \quad (10)$$

and this dimensionless quantity is assumed to be small, $l_j \ll 1$.

For fixed particle number $N$, the $\alpha$-dependent part of the Hamiltonian yields the energy $E_N \equiv (N - C_1 V \mu_{\text{ext}})^2 / (2C_1 V)$. For $N \ll 1$ the energy difference of a system with $N + 1$ and one with $N$ particles is

$$E_{N+1} - E_N \approx \frac{N}{C_1 V} - \mu_{\text{ext}}. \quad (11)$$

We note that $E_{N+1} - E_N = \mu - \mu_{\text{ext}}$ on physical grounds. The emergent breaking of $U(1)$ requires $E_{N+1} - E_N \approx 0$, and Eq. (11) relates the constant $C_1$ to the chemical potential and to the density of the system. We see that the chemical potential and the frequencies and amplitudes of the quantized collective vibrations determine the low-energy constants of the EFT. In addition to the collective vibrations with frequencies $\Omega$ one finds approximately equidistant levels with spacing $\mu - \mu_{\text{ext}}$ belonging to superfluids with different particle numbers. In superfluid atomic nuclei, these harmonic excitations belonging to different numbers of paired nucleons are known as pairing vibrations. In summary we have shown that in finite superfluids, the spectra of systems with different particle numbers are related to each other, and this is a model-independent prediction of the EFT.

We turn to higher-order corrections and establish our power counting. The energy scales used in the construction of $L$ in Eq. (5) are (i) the scale $\mu - \mu_{\text{ext}}$ associated with a change in particle number (we have assumed $\dot{\alpha} \sim \mu - \mu_{\text{ext}}$) and (ii) the scale $\Omega$ of the collective vibrations. We have assumed that both scales are of similar size, $\mu - \mu_{\text{ext}} \sim \Omega$. Moreover, in the low-energy domain the amplitudes of the Nambu-Goldstone modes $\beta_j$ given by Eq. (10) obey $l_j \ll 1$. Thus, we have

$$\beta_j \sim l_j, \quad \dot{\beta}_j \sim \Omega l_j, \quad C_2 \sim \Omega^{-1} l_j^{-2}, \quad p_{\beta_j} \sim l_j^{-1}. \quad (12)$$

In the construction of terms of higher order, we consider only powers of single derivatives because higher-order derivatives of the Nambu-Goldstone field $\beta$ and of the zero mode $\alpha$ can be eliminated via perturbative field redefinitions [38, 39]. The EFT has a breakdown scale $\Lambda$ with $\Lambda \gg \Omega, |\mu_{\text{ext}} - \mu|$. At this scale, $\dot{\alpha}$ (and $\beta, \partial_\mu \beta, \beta$) are a factor $\sqrt{\Lambda / |\mu_{\text{ext}} - \mu|}$ (and $\Lambda / \Omega$, respectively) larger than in the low-energy domain. At the breakdown scale, a Lagrangian term of the form $C_{pqr} \dot{\alpha} \dot{\beta}_p \beta_r \beta_q$ must scale as $\Lambda$ when written in terms of such velocities and fields. This determines the scaling of $C_{pqr}$. When evaluated for velocities and fields in the low-energy domain, that term yields a contribution to $L$ of order $\Lambda (|\mu_{\text{ext}} - \mu| / \Lambda)^{p/2} (\Omega / \Lambda)^{q/2 + r/2}$. Similar considerations
apply for terms containing powers of the spatial derivatives $\partial_{\alpha} \beta$. For energies below the breakdown scale, all these higher-order corrections are perturbatively small. Thus, our procedure yields a perturbative expansion in powers of $|\mu_{\text{ext}} - \mu|/\Lambda$ and $\Omega/\Lambda$.

III. EMERGENT BREAKING OF ROTATIONAL SYMMETRY

For finite objects with an axially symmetric ground-state deformation, the emergent symmetry breaking is from the rotation group to the subgroup of axial symmetry. It is, therefore, useful to recall the breaking of spin-rotational symmetry $O(3)$ to $O(2)$ in infinite ferromagnets \cite{5,7,10}. In the ground state, all spins point in the same direction, violating the spin-rotational symmetry of the Hamiltonian. Ground states with macroscopically different spin directions have zero overlap and define inequivalent Hilbert spaces. As a consequence, the low-lying spectrum of the ferromagnet is dominated by Nambu-Goldstone modes, i.e., spin waves of long wave length that locally induce small rotations of the aligned spins. For a ferromagnet of finite size \cite{20}, the former inequivalent Hilbert spaces are connected by non-vanishing tunneling matrix elements, the ground states belonging to different spin directions have nonzero overlap, and there exist states that are superpositions of these ground states. Such states are, for instance, the Wigner $D$-functions for rotational motion. Physically that implies that the ferromagnet may rotate about an axis perpendicular to the direction of the aligned spins. An analogous situation occurs in linear molecules and in axially symmetric deformed even-even nuclei. In the limit of infinitely large size, the low-lying parts of the spectra of these systems would be determined by Nambu-Goldstone modes. But finite linear molecules or finite axially symmetric deformed even-even nuclei may rotate about an axis perpendicular to the symmetry axis. In all three cases, rotational motion occurs as a consequence of emergent symmetry breaking due to the finite size of the system. Rotational motion is, therefore, distinctly different from the Nambu-Goldstone modes. It plays the same role as the zero mode $\alpha$ in Eq. (11) for the emergent breaking of $U(1)$ phase symmetry.

For a deformed nucleus with $A \gg 1$ nucleons, we can quantify these statements. The linear extension of the system is $\propto A^{1/3}$. The moment of inertia is proportional to mass $\times$ (length)$^2 \propto A^{5/3}$. With increasing $A$, the frequency of rotational motion tends to zero like $A^{-5/3}$, faster than the wave number $\propto A^{-1/3}$ of the massless modes (analogues of the Nambu-Goldstone modes in the infinite system). Hence, there exists a regime of $A$ values where the scale $\xi$ for rotational motion is small compared to the scale $\omega$ for vibrational motion and where $\omega$ in turn is small compared to the breakdown scale $\Lambda$ of the EFT defined by pair-breaking excitations \cite{10}. In the regime $\xi \ll \omega \ll \Lambda$, it is meaningful to consider rotational motion as a small (in energy) correction to the Goldstone theorem. That is the regime we study here. For instance, in rare-earth nuclei we have $\xi \approx 80$ keV from the lowest energy spacing in a rotational band, and $\omega \approx 1$ MeV from the lowest “vibrational” band head, while $\Lambda \approx 2 - 3$ MeV is the cost of pair breaking. The condition $\omega \ll \Lambda$ is met only marginally. Corresponding considerations apply to linear molecules such as CO$_2$. Here the scale of rotational energies is $\xi \approx 1$ cm$^{-1}$, that of vibrational energies is $\omega \approx 500$ cm$^{-1}$, while the breakdown scale $\Lambda \approx 10,000$ cm$^{-1}$ is defined by electronic excitations. The conditions $\xi \ll \omega \ll \Lambda$ are very well fulfilled.

Axially symmetric deformed even-even nuclei consist of nucleons, and linear molecules consist of nuclei. The locations of these constituents have body-fixed spherical coordinates $r, \theta, \phi$. Vibrations of the nucleus/molecule about an axis perpendicular to the symmetry axis act locally on the constituents. The resulting dislocations are assumed to have small amplitude. We expect that Nambu-Goldstone modes related to the coordinate $r$ have higher frequencies than those due to $\theta$ and $\phi$. We, therefore, confine attention to the latter variables although our approach can be straightforwardly generalized. In addition to these small-amplitude vibrations we also consider global rotations of the entire nucleus/molecule.

Our effective theory is universal and applies both to linear molecules and to deformed nuclei. Apart from the magnitude of the low-energy constants, the key difference between both systems is in the symmetry properties of the deformed ground–state wave functions as these define the symmetry properties of the admissible low-lying Nambu-Goldstone modes \cite{11}. We assume that molecular and nuclear ground-state wave functions are axially symmetric about the body-fixed $z$-axis, invariant under time-reversal, and have positive parity. As a consequence of pairing (superfluidity), even-even nuclei differ from linear molecules in that their intrinsic ground states are also invariant under rotations by $\pi$ about any axis perpendicular to the symmetry axis, i.e., possess positive $\mathcal{R}$ parity \cite{20,41}. Hence, low-energetic intrinsic excitations in nuclei must also have positive $\mathcal{R}$ parity.

A. Dynamical Variables and Power Counting

As done for the case of the ferromagnet \cite{5,7,10} and in Sect. III we consider the Nambu-Goldstone modes as classical fields that are later quantized. We prefer to write these fields in the space-fixed (rather than the body-fixed) coordinate system because here the commutation relations of the three generators $J_x, J_y, J_z$ of infinitesimal rotations about the space-fixed $x, y, z$ axes, respectively, are of standard form. The molecular/nuclear ground state is invariant under $SO(2)$ rotations about the body-fixed $z$-axis while $SO(3)$ symmetry is broken by the deformation. Therefore, the Nambu-Goldstone modes lie in the coset space $SO(3)/SO(2)$ \cite{2,3,12}. The modes depend on the angles $\theta, \phi$ defined above and on time $t$. 
and are generated by a unitary transformation $U$. As in Eq. (1), we parameterize the matrix $U$ in product form,

$$U = g(\alpha, \beta)u(x, y),$$

$$g(\alpha, \beta) = \exp\{-i\alpha(t)J_x\} \exp\{-i\beta(t)J_y\},$$

$$u(x, y) = \exp\{-ix(\theta, \phi, t)J_x - iy(\theta, \phi, t)J_y\}.$$  (13)

The purely time-dependent variables $\alpha(t)$ and $\beta(t)$ are the zero modes. They describe global rotations of the finite system and are factored out [18]. They are not Nambu-Goldstone modes but upon quantization generate rotational bands [32, 42]. The fields $x(\theta, \phi, t)$ and $y(\theta, \phi, t)$ with $|x|, |y| \ll 1$ generate small-amplitude vibrations of the constituents. These depend non-trivially on $\theta$ and $\phi$ so that

$$\int d\Omega \ x(\theta, \phi, t) = 0 = \int d\Omega \ y(\theta, \phi, t).$$  (14)

Here $d\Omega$ is the surface element of the three-dimensional unit sphere. In an infinite system $x(\theta, \phi, t)$ and $y(\theta, \phi, t)$ would be genuine Nambu-Goldstone modes. Eqs. (13) and (14) define the dynamical variables of the system. Eq. (13) may look like a rather special ansatz but actually follows from the most general form of $U$.

Further progress hinges on the identification of the energy scales $\xi, \omega$, and $\Lambda$ defined above. The ranges of the variables $\alpha$ and $\beta$ being of order unity, the ratios $\dot{\alpha}/\alpha$ and $\dot{\beta}/\beta$ are governed by the energy scale $\xi$ of rotational motion. We have $|x|, |y| \ll 1$, indicating that the amplitudes of the Nambu-Goldstone fields are small. Then $|x| \sim \omega|x|$ and $|y| \sim \omega|y|$. We are going to show that power counting based upon the inequalities $\xi \ll \omega \ll \Lambda$ together with the symmetry requirements formulated above uniquely determine the leading-order part of the Hamiltonian, except for a small number of constants that have to be determined by fits to data.

Our EFT is characterized by two breakdown scales. The first scale is set by $\Lambda \gg \omega$ and marks the appearance of neglected degrees of freedom. In deformed nuclei these are single-particle degrees of freedom or pair-breaking effects and in linear molecules, electronic excitations. The second scale is related to large-amplitude excitations of the Nambu-Goldstone fields. That scale is reached when these excitations are so large that they practically restore the spherical symmetry of the intrinsically deformed object, or when the energy due to the zero-mode velocities $\dot{\alpha}$ and $\dot{\beta}$ reaches the vibrational scale $\omega$. This second scale is set by $\omega^2/\xi$. For well-deformed nuclei, that scale considerably exceeds $\Lambda$, while both scales are similar in size for linear molecules. We now discuss both breakdown scales separately.

At energies below $\Lambda$ the neglected degrees of freedom cause the appearance of higher-order terms in the effective Lagrangian of the EFT. Such terms involve powers of the leading-order fields and velocities and, possibly, also higher derivatives. The latter can be eliminated via perturbative field redefinitions [35, 39] and are not considered here. At the breakdown scale (where the amplitudes and velocities of the Nambu-Goldstone fields are a factor $(\Lambda/\omega)^{1/2}$ larger than in the low-energy domain), a term in the effective Lagrangian with $\tau$ and $n$ powers of $x$ or $y$ is of order $\Lambda$. At the low-energy scale $\omega$ that term yields a contribution of order $\omega(\omega/\Lambda)^{(n+\tau)/2-1}$. Terms like that give rise to small corrections and, at each order, are finite in number. Similar considerations apply to the spatial derivatives.

The scale for the breaking of emergent symmetry is reached when the amplitudes, velocities and higher derivatives are a factor $(\omega/\xi)^{1/2}$ larger than in the low-energy domain. The arguments of the previous paragraph can essentially be repeated by replacing the scale $\Lambda$ by $\omega^2/\xi$, and the ratio $\Lambda/\omega$ by $\omega/\xi$. At the breakdown scale the contribution of a term in the effective Lagrangian which contains $\sigma$ velocities is of order $\omega$. At the low-energy scale $\xi$ that term scales as $\xi(\xi/\omega)^{\sigma/2-1}$.

These arguments establish our power counting. As a result, our EFT provides an expansion in the two small parameters $\omega/\Lambda$ and $\xi/\omega$, and there is a finite number of terms for each power of these parameters.

B. Effective Lagrangian

The effective Lagrangian is built from invariants. These are constructed from elements $a_{\mu}^x, a_{\mu}^y, a_{\mu}^z$, defined by

$$U^{-1}i\partial_{\mu}U = a_{\mu}^x J_x + a_{\mu}^y J_y + a_{\mu}^z J_z. $$  (15)

The symbol $\partial_{\mu}$ with $\mu = 1, 2, 3$ stands for the partial derivatives with respect to the angles $\theta, \phi$ and time $t$. Explicit expressions for these elements are obtained from Eqs. (13) and (14) in terms of a series expansion in powers of $x, y$, and their partial derivatives where only leading-order terms are kept. We use the Baker–Campbell–Hausdorff expansion and obtain

$$a_{\tau}^x = \dot{x} + \frac{y}{6}(xy - y\dot{x}) - \dot{\alpha}\sin \beta - y\dot{\alpha}\cos \beta + \ldots, $$

$$a_{\tau}^y = \dot{y} - \frac{x}{6}(xy - y\dot{x}) + \dot{\beta} + x\dot{\alpha}\cos \beta + \ldots, $$

$$a_{\tau}^z = -\frac{1}{2}(xy - y\dot{x}) + \dot{\alpha}\cos \beta - y\dot{\alpha}\sin \beta - x\dot{\beta} + \ldots, $$

and

$$a_{\nu}^x = \partial_{\nu} x + \frac{y}{6}(x\partial_{\nu} y - y\partial_{\nu} x) + \ldots, $$

$$a_{\nu}^y = \partial_{\nu} y - \frac{x}{6}(x\partial_{\nu} y - y\partial_{\nu} x) + \ldots, $$

$$a_{\nu}^z = -\frac{1}{2}(x\partial_{\nu} y - y\partial_{\nu} x) + \ldots, $$

To define the invariants we calculate the changes induced on the variables $\alpha, \beta, x, y$ by infinitesimal rotations $r$ of $U$ about angles $\delta \chi$ around the space-fixed $k = x, y, z$ axes. With $h(\gamma) = \exp\{-i\gamma J_z\}$ we have from Eq. (16) that

$$rg(\alpha, \beta) = g(\alpha', \beta')h(\gamma').$$  (18)
Here, the primed angles depend on the angles of the rotation \( r \) and the angles \( \alpha, \beta \). The right-hand side of Eq. \( 15 \) has the form of a general rotation with Euler angles \( (\alpha', \beta', \gamma') \), with explicit expressions given in Ref. \( 32 \). Thus,

\[
r U = g(\alpha', \beta') h(\gamma') u = g(\alpha', \beta') [h(\gamma') uh^{\dagger}(\gamma')] h(\gamma') .
\] (19)

As a result we find that the angles \( \alpha \) and \( \beta \) transform nonlinearly as the azimuthal and polar angle of the two-sphere, respectively, while \( x \) and \( y \) transform linearly as the \( x \) and \( y \) components of a vector under rotations around the \( z \) axis. In other words, under a rotation, the nucleus as a whole changes its orientation and undergoes a rotation around its symmetry axis. This transformation behavior under rotations confirms that \( \alpha \) and \( \beta \) describe the global orientation of the axially symmetric nucleus, while \( x \) and \( y \) are “intrinsic” degrees of freedom. Thus any combination of \( x \) and \( y \) that formally exhibits axial symmetry is indeed fully invariant under rotations. For example, \( x^2 + y^2 \) is invariant under rotations, and the four quantities \( x, y, x', y' \) are transformed into linear combinations of \( x', y', x'', y'' \). These transformation properties are characteristic different from the ones for an infinite system where in Eq. \( 15 \) we would have \( g(\alpha, \beta) = 1 \).

Time-reversal invariance requires that invariants involving time derivatives must contain even powers of \( a_x^\alpha, a_y^\beta, a_z^\gamma \). The lowest-order invariants obtained from Eqs. \( 17 \) are

\[
\begin{align*}
\mathcal{L}_{1a} &= \beta^2 + \dot{\alpha}^2 \sin^2 \beta, \\
\mathcal{L}_{1b} &= \dot{x}^2 + \dot{y}^2 + 2(\dot{x}\dot{y} - \dot{y}\dot{x}) \alpha \cos \beta, \\
\mathcal{L}_{1c} &= (\dot{x} - \dot{y})^2, \\
\mathcal{L}_{1d} &= (x^2 + y^2)[\dot{x}^2 + \dot{y}^2 + 2(x\dot{y} - y\dot{x}) \alpha \cos \beta].
\end{align*}
\] (20)

We note that the invariant \( \mathcal{L}_{1a} \) is essentially the Lagrangian of a rotor, and that the Lagrangian density \( \mathcal{L}_{1b} \) couples global rotations to the Nambu-Goldstone modes. The invariant \( \mathcal{L}_{1c} \) is related to the angular momentum of the Nambu-Goldstone modes, see Eq. \( 29 \). The invariant \( \mathcal{L}_{1d} \) is obtained by multiplying \( \mathcal{L}_{1b} \) with the invariant \( (x^2 + y^2) \) and is of the same order as \( \mathcal{L}_{1c} \). As for the invariants constructed from \( a_x^\nu, a_y^\nu, a_z^\nu \) with \( \nu = \theta \) or \( \nu = \phi \), we use that for fixed \( \nu \) and \( \nu' \), the forms \( (a_x^\nu)^2 + (a_y^\nu)^2 \) of \( a_x^\nu, a_y^\nu \) and \( a_x^\nu, a_y^\nu \) are invariant. Admissible linear combinations of these expressions are defined by the requirement of axial symmetry. Suppressing terms of higher order than \( x^4 \) and multiplying with the additional invariant \( (x^2 + y^2) \), we find the invariants

\[
\begin{align*}
\mathcal{L}_{2a} &= (\dot{L}_x^2 + \dot{L}_y^2)^2, \\
\mathcal{L}_{2b} &= (\dot{L}_x)^2 + (\dot{L}_y)^2, \\
\mathcal{L}_{2c} &= (x\dot{L}_y - y\dot{L}_x)^2, \\
\mathcal{L}_{2d} &= (x^2 + y^2)(\dot{L}_x)^2 + (\dot{L}_y)^2).
\] (21)

Here \( \dot{L}_x \) (\( \dot{L}_z \)) is the vector (the \( z \)-component) operator of orbital angular momentum, respectively, written in terms of \( \theta \) and \( \phi \). The occurrence of the term \( \mathcal{L}_{2b} \) reflects the fact that we impose only axial rather than rotational symmetry on \( \mathcal{L} \). The Lagrangian \( \mathcal{L} \) is given by

\[
L = L_1 + L_2 = \sum_{i=a,b,c,d} \int d\Omega \left( \frac{C_i}{2} \mathcal{L}_{1i} - \frac{D_i}{2} \mathcal{L}_{2i} \right).
\] (22)

Here \( C_i \) and \( D_i \) with \( i = a, b, c, d \) are constants that are determined by low-energy data.

We expand the real variable \( x \) as

\[
x = \sum_{\lambda = 2} \sum_{\nu = -\lambda} x_{\lambda \mu} Z_{\lambda \mu}
\] (23)

and correspondingly for \( y, \dot{x}, \dot{y} \). Aside from normalization constants, the real orthonormal functions \( Z_{\lambda \mu} \) are equal to the real part (for \( \mu \geq 0 \)) and imaginary part (for \( \mu < 0 \)) of the spherical harmonics \( Y_{\lambda \mu} \). The coefficients \( x_{\lambda \mu} \) are real. Terms with \( \lambda = 0 \) and \( \lambda = 1 \) are excluded since \( \lambda = 0 \) violates Eq. \( 14 \) and describes global rotations while \( \lambda = 1 \) describes translations in space \( 26 \). We insert the expansions \( 25 \) into Eq. \( 22 \) and use the resulting expression for \( L \) to define the real canonical momenta

\[
\begin{align*}
p_{\beta} &= \frac{\partial L}{\partial \dot{\beta}}, \\
p_{\alpha} &= \frac{\partial L}{\partial \dot{\alpha}}, \\
p_{\lambda \mu} &= \frac{\partial L}{\partial \dot{x}_{\lambda \mu}}, \\
p_{\nu \lambda \mu} &= \frac{\partial L}{\partial \dot{y}_{\lambda \mu}}.
\end{align*}
\] (24)

From Noether’s theorem we obtain explicit expressions for the three components \( I_x, I_y, I_z \) of angular momentum. These are

\[
\begin{align*}
I_x &= -p_\beta \sin \alpha - p_\alpha \cot \beta \cos \alpha + \frac{\cos \alpha}{\sin \beta} K, \\
I_y &= p_\beta \cos \alpha - p_\alpha \cot \beta \sin \alpha + \frac{\sin \alpha}{\sin \beta} K, \\
I_z &= p_\alpha.
\end{align*}
\] (25) (26) (27)

Here

\[
K = \int d\Omega \ (xp_y - yp_x)
\] (28)

is the angular momentum of the two-dimensional oscillators that describe the intrinsic vibrations. The square of the total angular momentum is

\[
I^2 = p_\beta^2 + \frac{1}{\sin^2 \beta} \left( p_\alpha^2 - 2p_\alpha \cos \beta + K^2 \right).
\] (29)

The terms in Eqs. \( 29 \) obtained by putting \( K = 0 \) can be shown to be equal to the square of the total angular momentum of the rigid rotor.
C. Effective Hamiltonian and Quantization

We use Eqs. (21) and the standard Legendre transformation to transform the effective Lagrangian \( L \) into the effective Hamiltonian \( H \). The scales of the coefficients \( C_{\lambda} \) and \( D_{\lambda} \) (and the terms that are kept in \( H \)) are determined by assuming \( C_\alpha L_{\alpha} \sim \xi \), \( C_\beta L_{\beta} \sim \omega \), \( C_\omega L_{\omega} \sim \sqrt{\omega} \), \( p_\alpha \sim 1 \), and \( p_\gamma \sim |x|^{-1} \). Relevant parts of \( C \) are given below.

For the rigid-rotor part of \( H \), quantization is achieved by symmetrization with respect to \( \alpha, \beta \) and by putting \( p_\beta = -i (\sin \beta)^{-1/2} \partial_\beta (\sin \beta)^{1/2} \), \( p_\alpha = -i \partial_\alpha \). This is the usual quantization for a particle on the sphere. For the remaining canonical momenta we have

\[
p_{\lambda \mu}^x = -i \frac{\partial}{\partial x_{\lambda \mu}}, \quad p_{\lambda \mu}^y = -i \frac{\partial}{\partial y_{\lambda \mu}}.
\]

(30)

Substitution of these expressions into Eq. (29) yields the quantized form of the square \( \hat{I}^2 \) of the operator of total angular momentum. The operator \( \hat{K} \) is given by \( \hat{K} = \sum_{\lambda \mu} (x_{\lambda \mu} p_{\lambda \mu}^y - y_{\lambda \mu} p_{\lambda \mu}^x) \). The three components \( \hat{I}_x, \hat{I}_y, \hat{I}_z \) of the quantized angular momentum can be shown to obey the standard commutation relations. Moreover, every component commutes with \( K \) and with the quantized Hamiltonian \( \hat{H} \). A complete set of commuting operators is, therefore, \( \hat{I}^2, \hat{L}_x, \hat{L}_y, \hat{L}_z \).

D. Spectra

The leading-order \( (O(\omega)) \) contribution to \( \hat{H} \) is given by

\[
\hat{H}_\omega = \sum_{\lambda \mu} \left( \frac{(p_{\lambda \mu}^x)^2 + (p_{\lambda \mu}^y)^2}{2C_\beta} + \frac{C_\omega}{2} \omega_{\lambda \mu} (x_{\lambda \mu}^2 + y_{\lambda \mu}^2) \right)
\]

and describes a set of uncoupled harmonic oscillators with frequencies \( \omega_{\lambda \mu} = [(\lambda+1)D_\alpha + \mu^2 D_\beta]/C_\beta^{1/2} \). We combine the degrees of freedom \( x_{\lambda \mu} \) and \( y_{\lambda \mu} \) into a two-dimensional \( SO(2) \) symmetric harmonic oscillator with quantum numbers \( n_{\lambda \mu} = 0, 1, 2, \ldots \) and \( k_{\lambda \mu} = 0, \pm 1, \pm 2, \ldots \). In units of \( \omega_{\lambda \mu} \) the energies are \( 2n_{\lambda \mu} + |k_{\lambda \mu}| + 1 \). The operator \( \hat{K} \) has eigenvalues \( K = \sum_{\lambda \geq 2} \sum_{\mu = -\lambda}^\lambda k_{\lambda \mu} \).

Next-order corrections to the Hamiltonian \( \hat{H}_\omega \) are either of order \( O(\xi) \) or of order \( O(x^2 \omega) \). The former couple rotations to vibrations. The latter add anharmonicities to the harmonic vibrations and thereby lift the degeneracies. We confine ourselves here to the former. The Hamiltonian is

\[
\hat{H}_{\omega, \xi} = \hat{H}_\omega + \frac{I^2 - \hat{K}^2}{2C_a},
\]

(31)

with \( \hat{I}^2 \) given by the quantized form of Eq. (29). The eigenfunctions of \( (I^2 - \hat{K}^2) \) are Wigner \( D \)-functions \( D_{M,K}^{(\alpha,\beta,0)} \) that depend on total integer spin \( I \) and its projections \( -I \leq M, K \leq I \). The eigenvalues of \( \hat{I}^2 \) are \( I(I+1) \) with \( I \geq |K| \). We see that each vibrational state of the leading-order Hamiltonian \( \hat{H}_\omega \) becomes a band head with spin \( |K| \), and the spectrum exhibits a rotational band on top of each band head. At this order in the EFT, all rotational bands have the same moment of inertia \( C_a \). Differences in the moments of inertia are higher-order effects.

The ground state has quantum numbers \( n_{\lambda \mu} = 0, k_{\lambda \mu} = 0 \) (this implies \( K = 0 \)) and spin \( I = 0 \). It has positive parity and, in the case of nuclei, positive \( R \) parity. For nuclei, this limits the rotational states in the ground-state band to even values of \( I \).

We turn to excited states. Here nuclei and linear molecules differ. For \( D_\beta > 0 \) the lowest single vibrational excitation corresponds to the mode \( (x_{20}, y_{20}) \). The fields \( x \) and \( y \) have positive parity, in keeping with the parity of the axial vectors \( J_x \) and \( J_y \) in Eqs. (13). The lowest excitation has \( |K| = 1 \), negative intrinsic \( R \) parity and, thus, values of \( I = 1, 2, 3, \ldots \). For linear molecules, states with \( |K| = 1 \) are indeed the lowest-lying vibrations. In contradistinction, in nuclei such states are excluded because paired states have positive \( R \) parity. Pair breaking (i.e., generation of states with odd \( R \) parity) happens only at the breakdown scale \( \Lambda \) of our EFT. Thus, pairing excludes low-lying magnetic dipole excitations and more generally any vibrational band head with odd spin and positive parity in the low-energy regime. This essential element provides the only difference in the low-energy spectra of molecules and of deformed nuclei. Indeed, there are no low-lying \( I^\pi = 1^+ \) states in deformed even-even nuclei.

As is well known from data, the ground states of deformed even-even nuclei consistently have quantum numbers \( I = 0 = K \) and positive parity. Low-lying vibrational states have \( K = 0 \) and \( |K| = 2 \) and positive parity. In rare-earth nuclei, both band heads have an excitation energy of about 1 MeV. In the present approach, these states are generated by local rotations around an axis that is perpendicular to the symmetry axis of the ground state. However, the wavefunction corresponding to \( K = 0 \) is symmetric under any \( SO(2) \) rotation of \( x \) and \( y \) and must thus be viewed as an axially symmetric excitation that corresponds to the \( \beta \) excitation of the geometrical model. The \( |K| = 2 \) wave functions exhibit no symmetry under exchange of \( x \) and \( y \) and thus break the axial symmetry. Thus, they correspond to the \( \gamma \) mode of the geometrical model.

Finally, we note that the present approach can be extended to EFTs for other cases such as general molecules. In this case, the ground state fully breaks \( SO(3) \) to the point group \( \mathcal{P} \) of the molecule, and the coset space is thus \( SO(3)/\mathcal{P} \). This case is technically simpler than the breaking from \( SO(3) \) to \( SO(2) \) because the coset essentially has a group structure. The coset can be parameterized by three Euler angles, and the zero modes indeed describe the orientation of the molecule in space while the
Nambu-Goldstone fields describe the intrinsic vibrations.

IV. SUMMARY

In summary, we have shown how to develop effective field theories for finite systems with emergent symmetry breaking. We have applied this approach to two types of systems. (i) Superfluids like infinite Bose-Einstein condensates (BEC) or the paired states of a BCS superconductor that break $U(1)$ phase symmetry. (ii) Systems with non-spherical ground states such as molecules and atomic nuclei that break rotational symmetry. In both cases, symmetry arguments alone yield the universal features of the low-lying excitations. In case (i) these are vibrations. We also relate the spectra of systems with different particle numbers. In case (ii) these are vibrations that are the heads of rotational bands. The moment of inertia is a fit parameter and will, in general, be different for different particle numbers. For instance, superfluid and normal systems. Nuclei and molecules differ in that the ground states of even-even nuclei are paired. This accounts for the absence in nuclei of low-lying band heads with odd spin and positive parity.

In contrast to phenomenological approaches, and except for a small number of constants, our approach yields an explicit expression for the Hamiltonian in leading order, and a systematic procedure to generate terms of higher order. It may, thus, be a useful starting point for the analysis of spectra and electromagnetic transitions. It is textbook knowledge that the traditional collective models of deformed nuclei [25, 26, 29] overpredict transitions between the $\beta$-band ($\gamma$ band) and the ground-state band by a factor of about ten (four) [27]. Furthermore, the interpretation of low-lying collective 0+ states as $\beta$ band heads has been put into question by the available data on $E2$ transitions [33]. This makes it very interesting to study electromagnetic couplings within the EFT approach.

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