Noncommuting coordinates in the Hall effect and in vortex dynamics

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Abstract
Laughlin’s Ansatz to explain the fractional Quantum Hall effect is derived by coupling a particle associated with “exotic” the two-fold central extension of the planar Galilei group. The reduced system is identical to the one used to describe the dynamics of vortices in an incompressible planar fluid.

1 Introduction
One of the most hotly debated issues of present-day theoretical high-energy physics is noncommutative (quantum) mechanics [1], where the coordinates satisfy the nontrivial commutation relation

\[ [\hat{x}, \hat{y}] = i\hbar\theta. \] (1.1)

The real number \( \theta \) here is referred to as the noncommutative parameter. Such a relation may appear rather puzzling at the first sight, and one can wonder about the physical motivations. High-energy physicist usually refer to higher dimensional branes and strings; this can however leave ordinary physicists somewhat sceptical. Below we present some arguments in favor of (1.1) which are, hopefully, more convincing for down-to-earth physicists.

2 The Fractional Quantum Hall Effect
The main experimental result about the Fractional Quantum Hall Effect (FQHE) is that the Hall and diagonal resistivity of some heterostructures is

\[ R_{xy} = \frac{1}{\nu} \frac{\hbar}{e^2}, \quad R_{xx} = 0 \] (2.1)

where the filling factor \( \nu \) is an odd integer, \( \nu = 2n - 1 \) [2, 3]. In his seminal paper Laughlin [4] argues that the FQHE can entirely be explained within the lowest Landau level: the system condensates into a collective ground state, representing an incompressible quantum fluid, based on the “Laughlin” wave functions

\[ \psi(z) = f(z)e^{-B|z|^2/4} \] (2.2)
where $f(z)$ is analytic. The fractional quantization conditions (2.1) is recovered when $f(z) = z^{2n+1}$.

This Note, based on joint work with Christian Duval (and Zalán Horváth) [5, 6], aims to justify the starting point of Laughlin’s description from first principles. Our work ends where that of Laughlin begins.

Before presenting our theory, let us recall the usual treatment of the Landau problem [2]. A charge confined to the plane and moving under the influence of a perpendicular magnetic and a planar electric field is described by the Hamiltonian $H = \vec{p}^2/2m + eV(\vec{x})$ where $[p_1, p_2] = i\hbar eB$. When $V = 0$, the spectrum is $E_n = \hbar(eB/m)(\frac{1}{2} + n)$.

Classically, the particle performs helical motion, as seen from the decomposition

$$\vec{Q} = \vec{x} - \vec{R}, \quad R_{ij} = \frac{1}{eB} \varepsilon_{ij} p_j. \quad (2.3)$$

In fact, $\vec{Q}$ follows the Hall law, and $\vec{R}$ performs a uniform rotation. The remarkable fact [2] is that the guiding center coordinates do not commute but satisfy rather

$$[\hat{Q}_1, \hat{Q}_2] = -i \frac{\hbar}{eB} \quad (2.4)$$

that realize the commutation relation (1.1) with $\theta = -(eB)^{-1}$.

The Landau spectrum is explained by the decomposition (2.3): the guiding center contributes the ground state energy, and the higher Landau levels come from the oscillations of the internal coordinate $\vec{R}$. Semiclassically, $<\vec{R}^2> = (1 + 2n)\frac{\hbar^2}{m^2 e^2 B}$ [2].

It is worth noting that, for very special initial conditions, the guiding center motion can be materialized by actual motions. If the initial postion and velocity are such that the electric field is compensated by the Lorentz force, $eE_i + eB\varepsilon_{ij}v_j = 0$, then the motion is in fact at right angle to $\vec{E}$ i.e., along an equipotential. The generic motion is, of course, the helical one; the initial conditions which satisfy the force-free conditions form indeed a two-dimensional surface in 4D phase space.

Intuitively, our theory presented below has the peculiarity to eliminate these generic, helical motions, leaving us only with those of the guiding center.

### 3 Exotic particles

Let us now present our model. Following Wigner [7], elementary particles correspond to irreducible representations of their fundamental symmetry groups. In the nonrelativistic case, however, the Galilei group is only represented projectively, i.e., only up-to-phase: in spatial dimensions at least 3, it is only a one-parameter central extension of the Galilei group that is unitarily represented. It has been shown furthermore by Bargmann [8] that this phase can not be eliminated by any redefinition, as it corresponds to a nontrivial cohomology class of the group, labeled by the real parameter $m$, interpreted as the mass. Let us record for further reference that a Galilean boost with parameter $\vec{b}$ is implemented, in the momentum representation, as

$$U_{\vec{b}}\phi(\vec{p}) = \phi(\vec{p} - m\vec{b}). \quad (3.1)$$

It follows that the components of the boost generator, $\vec{g}_i = m\vec{p}_i$, commute, $[\vec{g}_1, \vec{g}_2] = 0$. 
The planar case is instead rather peculiar in that the cohomology is \textit{two dimensional} with generators $m$ and $\kappa$, respectively [9]. This has been noticed a long time ago but has not been sufficiently appreciated until recently.

Now the geometric quantization of Kirillov-Kostant-Souriau [10, 11] associates the representations of a group with the coadjoint orbits, endowed with their canonical symplectic structure. The idea of Souriau has been furthermore to consider these orbits as underlying classical models. Explicit calculation [12, 5] yields that the orbit is 4 dimensional, parametrized with the position and momentum, $\vec{x}$ and $\vec{p}$, and carries the “exotic” symplectic structure

$$\omega = d\vec{p} \wedge d\vec{x} + \frac{\theta}{2} \varepsilon_{ij} dp_i \wedge dp_j,$$

where we wrote $\theta = \kappa/m^2$. Using again the momentum representation, a Galilean boost is now represented by

$$U_{\vec{b}}(\phi) = e^{im\theta \vec{b} \times \vec{p}} \phi(\vec{p} - m\vec{b})$$

cf. ordboost. The inclusion of the phase factor implies that the components of new boost generator,

$$\hat{g}_j = m \left[ i \frac{\partial}{\partial p_j} + \frac{1}{2} \theta \varepsilon_{jk} p_k \right],$$

satisfy rather the “exotic” commutation relation

$$[\hat{g}_1, \hat{g}_2] = -i\hbar m^2 \theta.$$  

Having constructed our free model, let us couple it minimally to an electromagnetic field by considering the action

$$\int \left( (\vec{p} - e\vec{A}) \cdot d\vec{x} - \frac{\vec{p}^2}{2m} + eV dt + \frac{\theta}{2} \vec{p} \times d\vec{p} \right),$$

where $(V, \vec{A})$ is an electro-magnetic potential. The associated Euler-Lagrange equations read

$$m^* \dot{x}_i = p_i - m\theta e \varepsilon_{ij} E_j,$$

$$\dot{p}_i = eE_i + eB \varepsilon_{ij} \dot{x}_j,$$

where we have introduced the \textit{effective mass}

$$m^* = m(1 - e\theta B).$$

Let us observe that the velocity and momentum are not proportional if $\theta \neq 0$.

The equations of motions (3.7) can also be written as

$$\omega_{\alpha\beta} \dot{\xi}_\beta = \frac{\partial h}{\partial \xi_\alpha} \quad \text{where} \quad (\omega_{\alpha\beta}) = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & B \\ 0 & -1 & -B & 0 \end{pmatrix}.$$  

Note that the electric and magnetic fields are otherwise arbitrary solutions of the homogeneous Maxwell equation $\partial_i B + \varepsilon_{ij} \partial_j E_j = 0$, which guarantees that the two-form $\omega = i\omega_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta$ is closed, $d\omega = 0$. It follows that the associated Poisson bracket satisfies the Jacobi identity.
When \( m^* \neq 0 \), the determinant
\[
\det (\omega_{\alpha\beta}) = (1 - \theta B)^2 = \left( \frac{m^*}{m} \right)^2
\]
is nonzero; the matrix \((\omega_{\alpha\beta})\) in (3.9) is indeed symplectic and can therefore be inverted. Then the equations of motion (3.9) (or (3.7)) take the Hamiltonian form
\[
\dot{\xi}_\alpha = \{\xi_\alpha, h\},
\]
with the standard Hamiltonian, but with the “exotic” Poisson bracket
\[
\{f, g\} = (\omega^{-1})_{\alpha\beta} \partial_\alpha f \partial_\beta g.
\]
The fundamental commutation relations are in particular
\[
\{x^1, x^2\} = m^* \theta,
\]
\[
\{x^i, p^j\} = m^* \delta_{ij},
\]
\[
\{p_1, p_2\} = m^* B.
\]

Further insight can be gained when the magnetic field \( B \) is a (positive) nonzero constant. The vector potential can then be chosen as \( A_i = \frac{1}{2} B \varepsilon_{ij} x^j \), the electric field \( E_i = -\partial_i V \) being still arbitrary. Let us introduce the new coordinates
\[
Q_i = x_i + \frac{1}{eB} \left( 1 - \sqrt{\frac{m^*}{m}} \right) \varepsilon_{ij} p_j.
\]

Then the equations of motion (3.7) are conveniently presented in terms of the new variables \( \vec{Q} \) and the old momenta \( \vec{p} \), as
\[
\begin{align*}
\dot{Q}_i &= \varepsilon_{ij} \frac{E_j}{B} + \sqrt{\frac{m^*}{m}} \left( \frac{p_i}{m} - \varepsilon_{ij} \frac{E_j}{B} \right), \\
\dot{p}_i &= \varepsilon_{ij} \frac{B m^*}{m} \left( \frac{p_j}{m} - \varepsilon_{jk} \frac{E_k}{B} \right).
\end{align*}
\]

When the magnetic field takes the particular value
\[
B = B_c = \frac{1}{e\theta},
\]
the effective mass (3.8) vanishes, \( m^* = 0 \), so that the system becomes singular. Then the time derivatives \( \dot{\xi}_\alpha \) can no longer be expressed from the variational equations (3.9), and we have resort to “Faddeev-Jackiw” reduction [13]. The result is [5] that the momentum stops to be a dynamical variable,
\[
\frac{p_i}{m} - \varepsilon_{ij} \frac{E_j}{B_c} = 0,
\]
and we end up with the reduced Lagrangian
\[
L_{\text{red}} = \frac{1}{2\theta} \vec{Q} \times \dot{\vec{Q}} - eV(\vec{Q}),
\]
supplemented with the Hall constraint (3.15). Thus, the 4-dimensional phase space is reduced to 2 dimensions, with \( Q_1 \) and \( Q_2 \) as canonical coordinates, and reduced symplectic two-form
\[
\omega_{\text{red}} = \frac{eB_c}{\theta} \varepsilon_{ij} dQ_i \wedge dQ_j.
\]
The new coordinates are therefore again non-commuting,
\[
\{Q_1, Q_2\}_{\text{red}} = -\theta = -\frac{1}{eB_c}.
\]

Remarkably, our new coordinates become, for \( m^* = 0 \), precisely the guiding center coordinates (2.3), as anticipated by the notation.
The equations of motion associated with \((3.16)\), and also consistent with the Hamilton equations \(\dot{Q}_i = \{Q_i, H\}_{\text{red}}\), are given by
\[
\dot{Q}_i = \varepsilon_{ij} \frac{E_j}{B_c},
\]
consistently with the Hall law. Putting \(B_c = 1/e\theta\), the Lagrangian \((3.16)\) becomes formally identical to the one derived by Dunne et al. \([13]\) letting the real mass go to zero.

Quantization of the reduced system is conveniently carried out in the Bargmann-Fock representation \([5]\). Setting \(z = Q_1 + iQ_2\), the (reduced) wave functions are precisely those of Laughlin \((2.2)\). The reduced position operators are
\[
\hat{z} = zf, \quad \hat{\bar{z}} f = 2 \partial_z f,
\]
whose commutator is \([\hat{z}, \hat{\bar{z}}] = 2/eB\). Finally, the reduced Hamiltonian is just the potential \(eV(z, \bar{z})\). In conclusion, we recover the “Laughlin” description \([3]\) of the ground states of the FQHE.

Interestingly, similar ideas to ours have been expressed, independently, by Fosco and Lopez \([15]\).

Let us mention that a fluid model can be built on our “exotic mechanics” following the general principles of plasma physics; in the critical case, it yields an incompressible quantum fluid that moves collectively according to the Hall law \([6]\).

## 4 Dynamics of planar vortices

It has been known for over hundred years that fluid vortices in the plane follow a simple, first-order non-newtonian dynamics \([16, 17, 18]\). For the sake of simplicity, we restrict ourselves to two vortices of identical vorticity. The center-of-vorticity coordinates are constants of the motion. For the relative coordinates \(x = x_1 - x_2\) and \(y = y_1 - y_2\), respectively, the equations of motion become
\[
\gamma \dot{x} = \partial_y H, \quad \gamma \dot{y} = -\partial_x H,
\]
where \(\gamma\) is the vorticity. Let us stress that, in the present purely hydrodynamic context, \(\gamma\) can be any real number. The Hamiltonian, representing the interaction of the vortices, reads
\[
H = -\frac{\gamma^2}{4\pi} \ln r.
\]

These equations can be derived from the hydrodynamics of an incompressible planar fluid \([19, 20]\). The important fact for our purposes is that Eq. \((4.1)\) is a Hamiltonian system,
\[
\dot{\xi} = \{\xi, H\}, \quad \xi = (x, y),
\]
where the Poisson bracket associated with the symplectic structure \(\Omega = \gamma \, dx \wedge dy\). Thus, planar vortex dynamics is exactly of the form of our “reduced dynamics” presented in Section 3. This “coincidence” underlines the fundamental role of vortices in explaining the FQHE.

It is worth noting that the hydrodynamic formulae above apply to the effective dynamics of point-like vortices in a thin film of superfluid \(^4\text{He}\) \([21]\); the only difference being that \(\gamma\), the strength of the vortex, is quantized in multiples of \(h/m\). A more general, 3-dimensional model that takes into account the deformation of the vortex lines has been elaborated by Fetter \([22]\).
An extension of his theory describes vortex dynamics in extreme type II superconductors [23]. The Hall Effect observed in type II superconductors is yet another indication on the rôle of vortices in the Hall context.

Another derivation of the vortex Lagrangian (3.16) is due to Manton, who deduced it, for large separations, from the Landau – Ginzburg theory [24].

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