OPTIMAL DECAY TO THE NON-ISENTROPIC COMPRESSIBLE MICROPOLAR FLUIDS

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ABSTRACT. In this paper, we are concerned with the large-time behavior of solutions to the Cauchy problem on the non-isentropic compressible micropolar fluid. For the initial data near the given equilibrium we prove the global well-posedness of classical solutions and obtain the optimal algebraic rate of convergence in the three-dimensional whole space. Moreover, it turns out that the density, the velocity and the temperature tend to the corresponding equilibrium state with rate \((1 + t)^{-3/4}\) in \(L^2\) norm and the micro-rotational velocity tends to the equilibrium state with the faster rate \((1 + t)^{-5/4}\) in \(L^2\) norm. The proof is based on the detailed analysis of the Green function and time-weighted energy estimates.

1. Introduction and main result. The theory of compressible micropolar fluids was first introduced by Eringen [6]. Physically, micropolar fluids may represent fluids containing of rigid, randomly oriented (or spherical) particle suspended in a viscous medium, where the deformation of fluid particles is ignored. Then the micropolar fluids model which describes the micro-rotational and spin inertia get much more attention. This model is a significant generalization of the Navier-Stokes equations and has been extensively studied and applied for modeling rheologically complex liquids such as blood and suspensions by many engineers and physicists (see, e.g., [6]). In this paper, we consider the viscous non-isentropic compressible micropolar fluids satisfies the following system:

\[
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
\rho(u_t + u \cdot \nabla u) &= (\lambda + \mu - \mu_r)\nabla \text{div} u + (\mu + \mu_r)\Delta u - \nabla P + 2\mu_r \text{rot} w, \\
\rho(w_t + u \cdot \nabla w) &= 2\mu_r (\text{rot} u - 2w) + (c_0 + c_d - c_a)\nabla \text{div} w + (c_a + c_d)\Delta w, \\
\rho(E_t + u \cdot \nabla E) &= -P \text{div} u + \lambda(\text{div} u)^2 + 2\mu \mathcal{D} : \mathcal{D} + 4\mu_r \left(\frac{1}{2} \text{rot} u - w\right)^2 \\
&\quad + c_0 (\text{div} w)^2 + (c_a + c_d)\nabla w : \nabla w^T + (c_d - c_a)\nabla w : \nabla w + \kappa \Delta \theta,
\end{aligned}
\]  

(1.1)
where \( \rho = \rho(t, x) > 0, u = u(t, x) \in \mathbb{R}^3, w = w(t, x) \in \mathbb{R}^3, E = E(t, x) > 0 \) and \( P(\rho, \theta) \) over \( \{ t > 0, x \in \mathbb{R}^3 \} \) are density, velocity, micro-rotational velocity, internal energy and pressure, respectively. The constants \( \mu, \lambda \) are the usual viscosity coefficients (dynamic Newtonian viscosity, second viscosity coefficient), satisfying \( \mu \geq 0 \) and \( 3\lambda + 2\mu \geq 0 \). The positive constant \( \mu_r \) represents the dynamic microrotation viscosity. \( c_0, c_a, c_d \) are constants called coefficients of angular viscosities.

Here the pressure \( P(\rho, \theta) \) satisfies \( P = \kappa \rho \theta, \kappa = \text{const} > 0 \). The internal energy \( E \) is proportional to its temperature, \( E = \frac{\kappa \rho \theta}{\gamma - 1}, \gamma = \text{const} > 1, \) and \( D = \frac{1}{2}(\nabla u + \nabla u^T) \).

\( j_I \) is a positive constant which represents micro-inertia density. The initial data is given by

\[
(\rho, u, w, \theta)(0, x) = (\rho_0, u_0, w_0, \theta_0)(x) \quad \text{for } x \in \mathbb{R}^3, \tag{1.2}
\]

with the far field behavior:

\[
(\rho, u, w, \theta)(t, x) \rightarrow (\bar{\rho}, 0, 0, \bar{\theta}) \quad \text{as } |x| \rightarrow \infty, t \geq 0. \tag{1.3}
\]

Without loss of generality, we set \( \bar{\theta} = \bar{\rho} = j_I = 1 \) throughout the paper.

It is well-known that there is a vast number of articles studying the global existence, uniqueness, regularity, and large time behavior of solutions to the problems connected with the classical compressible fluids model (see [1, 2, 8, 9, 11, 17, 21] and the references therein). In this article, we focus on the compressible micropolar fluids which is still in its beginning stage.

Due to its importance in mathematics and physics, there is a lot of literature devoted to the mathematical theory of the micropolar fluid system. For the compressible equations of the micropolar fluids, Mujaković has made a series of efforts for this model in one dimensional space or with spherical symmetry in three dimensional space. The authors considered the local-in-time existence and uniqueness in [15], the global existence in [14] and regularity in [16] of the solutions to an initial-boundary value problem with homogenous boundary conditions of the compressible one-dimensional micropolar fluid system respectively. Recently, Liu-Zhang [13] derived the long-time behavior of the solutions to the compressible micropolar fluids with external force and [12] proved the optimal decay to the isentropic compressible micropolar fluids. Then, Wang-Wu [23] obtained the pointwise estimates of solutions to the isentropic compressible micropolar fluids through the analysis of the Green function. However, for the non-isentropic micropolar system, the related results are few. Very recently, Huang-Liu-Zhang [7] generalized the isentropic case to the non-isentropic case with additional efforts by taking care of the temperature equation and obtained the global solutions with general initial data and vacuum. Based on these works, in this article, we will focus on the optimal long-time behavior to the non-isentropic micropolar system.

Now, the main results concerning the global existence and large time behavior of solutions to the Cauchy problem (1.1)-(1.3) are stated as follows:

**Theorem 1.1.** For any real number \( s \geq 4 \). There exists small enough \( \delta_1 > 0 \) and \( C > 0 \), such that

\[
\|[(\rho_0 - 1, u_0, w_0, \theta_0 - 1)]_{H^s} \| \leq \delta_1,
\]

then the Cauchy problem (1.1)-(1.3) admits a unique global solution \( U = [n, u, w, m] \) satisfying

\[
[\rho - 1, u, w, \theta - 1] \in C \left( \mathbb{R}^3 \right),
\]

\[
\nabla \rho \in L^2 \left( \mathbb{R}^3 \right), \quad \nabla u \in L^2 \left( \mathbb{R}^3 \right),
\]

\[
[\rho - 1, u, w, \theta - 1] \in C \left( [0, \infty); H^s \left( \mathbb{R}^3 \right) \right),
\]

\[
\nabla \rho \in L^2 \left( [0, \infty); H^{s-1} \left( \mathbb{R}^3 \right) \right), \quad \nabla u \in L^2 \left( [0, \infty); H^s \left( \mathbb{R}^3 \right) \right),
\]

\[
[\rho - 1, u, w, \theta - 1] \in C \left( [0, \infty); H^{s-1} \left( \mathbb{R}^3 \right) \right),
\]

\[
\nabla \rho \in L^2 \left( [0, \infty); H^{s-2} \left( \mathbb{R}^3 \right) \right), \quad \nabla u \in L^2 \left( [0, \infty); H^{s-1} \left( \mathbb{R}^3 \right) \right),
\]

\[
[\rho - 1, u, w, \theta - 1] \in C \left( [0, \infty); H^{s-2} \left( \mathbb{R}^3 \right) \right),
\]

\[
\nabla \rho \in L^2 \left( [0, \infty); H^{s-3} \left( \mathbb{R}^3 \right) \right), \quad \nabla u \in L^2 \left( [0, \infty); H^{s-2} \left( \mathbb{R}^3 \right) \right),
\]
\[ \nabla w \in L^2 \left( [0, \infty); H^s (\mathbb{R}^3) \right), \quad \nabla \theta \in L^2 \left( [0, \infty); H^s (\mathbb{R}^3) \right), \]

and
\[ \| \rho - 1, u, w, \theta - 1 \|_{H^s}^2 + \int_0^t \left( \| \nabla n(\tau) \|_{H^{s-1}}^2 + \| \nabla [u, w, \theta] \|_{H^s}^2 \right) d\tau \]
\[ \leq C \| \rho_0 - 1, u_0, w_0, \theta_0 - 1 \|_{H^s}^2 \| \rho_0 - 1, u_0, w_0, \theta_0 - 1 \|_{H^s}. \]

Moreover, if we further assume that for small enough \( \delta_2 > 0 \),
\[ \| \rho_0 - 1, u_0, w_0, \theta_0 - 1 \|_{H^s \cap L^1} \leq \delta_2, \]
then, there exists a \( C_1 > 0 \) such that the solutions \( \rho, u, w, \theta \) satisfy
\[ \| \rho - 1, u, \theta - 1 \| \leq C_1 (1 + t)^{-\frac{3}{4}}, \quad \| w \| \leq C_1 (1 + t)^{-\frac{5}{4}}. \quad (1.4) \]

Remark 1.1. Here, the algebraic decay rate (1.4) is optimal in sense that this rate coincides with that of the corresponding linear system.

The main strategy to prove the time decay rates stated in Theorem 1.1 relies on the Green function \( G \) of the linearized viscous non-isentropic compressible micropolar fluids. In fact, the solution to the linearized homogenous system can be written as the sum of the fluid part and the electromagnetic part in the form of
\[
\begin{bmatrix}
n(t, x) \\
u(t, x) \\
w(t, x) \\
m(t, x)
\end{bmatrix} =
\begin{bmatrix}
n(t, x) \\
u_\parallel(t, x) \\
w_\parallel(t, x) \\
m_\parallel(t, x)
\end{bmatrix} +
\begin{bmatrix}
0 \\
u_\perp(t, x) \\
w_\perp(t, x) \\
0
\end{bmatrix}.
\]

The decomposition is quite useful in dealing with complex linearized system containing curl. We can refer [1, 3, 4, 5, 23] for the detailed spectrum analysis by using a similar decomposition. Notice that the two terminologies, fluid part and electromagnetic part have been used in [1, 3, 5]. With the help of the above decomposition, we give explicit representations of solutions to the two eigenvalue problems by learning their Green function \( G_1, G_2 \) respectively.

The Fourier transform of \( G_1 \) and \( G_2 \) have different behaviors over different frequency domains. For the fluid part
\[ |\hat{G}_1(t, \xi)| \lesssim \begin{cases} e^{-c|\xi|^2 t} & |\xi| \leq \varepsilon, \\ e^{-c t} & |\xi| \geq K. \end{cases} \]
For the electromagnetic part
\[ |\hat{G}_2(t, \xi)| \lesssim \begin{cases} e^{-c|\xi|^2 t} & |\xi| \leq \varepsilon, \\ e^{-c t} & |\xi| \geq K, \end{cases} \]
with two properly chosen constants \( 0 < \varepsilon \ll 1 \ll K < \infty \). Thus, from the above estimates, it is well known the solution over the high frequency domain decays exponentially while over the low frequency domain it decays polynomially with the rate of the heat kernel. It should be emphasized that those time decay rates are based on the elementary Lyapunov property of the linearized system, which implies
\[ |\hat{U}(t, \xi)| \leq C e^{\frac{c|\xi|^2 t}{1+|\xi|^2}} |\hat{U}_0(\xi)|, \quad \xi \in \mathbb{R}^3. \quad (1.5) \]
However, it will fail to yield the ones for the nonlinear solution given in Theorem 1.1.
Next, we point out some relationships of time rates between the linearized system and nonlinear ones. From Corollary 2.3 the solution \([\rho, u, w, \theta]\) to the linearized homogeneous system corresponding to system (2.3) decays as
\[
\|([\rho-1,u,\theta-1])\| \leq C(1+t)^{-\frac{2}{3}} \|[u_0,\theta_0]\|_{L^1},
\]
\[
\|w\| \leq C(1+t)^{-\frac{2}{3}} \|[w_0,\theta_0]\|_{L^1},
\]
for any \(t \geq 0\), provided that \([\rho_0-1,u_0,\theta_0-1]\) belongs to \(L^1 \cap H^s\) for properly large \(N\).

Moreover, a general approach for obtaining the optimal time decay of solutions in \(L^p\) space with \(p \geq 2\) in any space dimension was developed by Kawashima [10] and Shizuta-Kawashima [18]. Precisely, the main tool used by Kawashima [10] is the Fourier analysis applied to the linearized homogeneous system, and the key part in the proof is to construct some compensation function to capture the dissipation of the hyperbolic component in the solution and then obtain an estimate on the Fourier transform \(\hat{U}\) of the solution \(U\) as (1.5).

The rest of the paper is organized as follows. In Section 2, we reformulate the Cauchy problem on the non-isentropic micropolar fluid system around the constant steady state \([1,0,0,1]\), and study the decay structure of the linearized homogeneous system by the Fourier energy method. In Section 3, we focus on the spectral analysis of the linearized system by two parts. The first part is for the fluid part, the second one for the electromagnetic part. In Section 4, we first prove the global existence of solutions by the energy method, and then show the time asymptotic rate of solutions around the constant states.

**Notation.** Throughout this paper, the norms in the Sobolev Spaces \(H^s(\mathbb{R}^3)\) and \(L^2(\mathbb{R}^3)\) are denoted respectively by \(\|\cdot\|_{H^s}\) and \(\|\cdot\|\) for integer \(N \geq 1\). \(\langle \cdot, \cdot \rangle\) denotes the inner product in \(L^2(\mathbb{R}^3)\); and we set \(\|[A,B]\|_X = \|A\|_X + \|B\|_X\). To simplify the notation, by \(A \sim B\) we mean there exists a positive constant \(C\), such that \(\frac{1}{C}B \leq A \leq CB\). Moreover, \(C\) denotes a general constant which may vary in different estimates.

2. **Decay property of linearized system.** In this section, we shall establish the time-decay estimates of solutions to the linearized system by using the Fourier energy method. The main motivation to present this part is to understand the linear dissipative structure of such complex system based on the direct energy method.

2.1. **Reformulation of the problem.** We assume that the steady state of the micropolar fluids system (1.1) is trivial, taking the form of
\[
\bar{\rho} = 1, \quad \bar{u} = 0, \quad \bar{w} = 0, \quad \bar{\theta} = 1.
\]
Let \(n = \rho - 1, u = u - 0, w = w - 0\) and \(m = \theta - 1\). Then \(\bar{U} := [n,u,w,m]\) satisfies
\[
\begin{aligned}
&n_t + \text{div } u = S_1, \\
&u_t + \nabla u + \nabla m - (\mu + \mu_r)\Delta u - (\mu + \lambda - \mu_r)\nabla \text{div } u - 2\mu_r \nabla \times w = S_2, \\
&w_t + 4\mu_r w - (c_0 + c_d)\Delta w - (c_0 + c_d - c_0)\nabla \text{div } w - 2\mu_r \nabla \times u = S_3, \\
&m_t - (\gamma - 1)\Delta m + (\gamma - 1)\text{div } u = S_4,
\end{aligned}
\]
(2.1)
here the nonhomogeneous source terms \(S_i, (i = 1, 2, 3, 4)\) are defined as
\[
\begin{aligned}
S_1 &= -n \text{div } u - u \cdot \nabla n, \\
S_2 &= -u \cdot \nabla u - f(n)[(\mu + \mu_r)\Delta u + (\mu + \lambda - \mu_r)\nabla \text{div } u + 2\mu_r \nabla \times w] + h(n)\nabla n, \\
S_3 &= -w \cdot \nabla w - f(n)[(\mu + \mu_r)\Delta w + (\mu + \lambda - \mu_r)\nabla \text{div } w + 2\mu_r \nabla \times u] + h(n)\nabla w, \\
S_4 &= -m \cdot \nabla m - f(n)[(\mu + \mu_r)\Delta m + (\mu + \lambda - \mu_r)\nabla \text{div } m + 2\mu_r \nabla \times m] + h(n)\nabla m.
\end{aligned}
\]
its Fourier transform is defined by
\[ S_3 = -u \cdot \nabla w - f(n) [(c_0 + c_d) \Delta w + (c_0 + c_d - c_a) \nabla \text{div} w - 4\mu_\tau w + 2\mu_\tau \nabla \times u], \]
\[ S_4 = -u \cdot \nabla m - \kappa \text{div} u - f(n)(\gamma - 1) \Delta m + \frac{\gamma - 1}{\kappa(n + 1)} \left[ \lambda(\text{div} u)^2 + 2\mu D : D \right. \]
\[ + 4\mu_\tau \left( \frac{1}{2} \text{rot} u - w \right)^2 + c_0(\text{div} w)^2 + (c_0 + c_d) \nabla w : \nabla w + T + (c_0 + c_d - c_a) \nabla w : \nabla w \big], \]
where
\[ f(n) = \frac{n}{n + 1}, \quad h(n) = \frac{m}{n + 1}. \]
The associated initial data is given by
\[ (n, u, w, m)(x, 0) = (n_0, u_0, w_0, m_0)(x). \] (2.2)

2.2. Lyapunov property. In this section, we still use \( U = [n, u, w, m] \) to denote the solutions to the following linearized homogeneous system
\[
\begin{cases}
  n_t + \text{div} u = 0, \\
  u_t + \nabla n + \nabla m - (\mu + \mu_\tau) \Delta u - (\mu + \lambda - \mu_\tau) \nabla \text{div} u - 2\mu_\tau \nabla \times w = 0, \\
  w_t + 4\mu_\tau w - (c_0 + c_d) \Delta w - (c_0 + c_d - c_a) \nabla \text{div} w - 2\mu_\tau \nabla \times u = 0, \\
  m_t - (\gamma - 1) \Delta m + (\gamma - 1) \text{div} u = 0,
\end{cases}
\] (2.3)

with the initial data
\[ (n, u, w, m)(x, 0) = (n_0, u_0, w_0, m_0)(x). \] (2.4)

By applying the Fourier energy method to the Cauchy problem (2.3) and (2.4), we show that there exists a time-frequency Lyapunov functional which is equivalent with \(|\hat{U}(t, \xi)|^2\) and moreover its dissipation rate can also be characterized by the functional itself. The main result will be given as follows.

**Theorem 2.1.** Let \( U(t, x), t > 0, x \in \mathbb{R}^3 \) be a well-defined solution to the system (2.3) and (2.4), then there exists a time-frequency Lyapunov functional \( \mathcal{E}(\hat{U}(t, \xi)) \) with
\[ \mathcal{E}(\hat{U}(t, \xi)) \sim |\hat{U}(t, \xi)|^2 := ||\hat{n}(t, \xi), \hat{u}(t, \xi), \hat{w}(t, \xi), \hat{m}(t, \xi)||^2, \] (2.5)
which satisfies
\[ \frac{d}{dt} \mathcal{E}(\hat{U}(t, k)) + c|\xi|^2||\hat{n}, \hat{w}, \hat{m}||^2 + c \left( \frac{|\xi|^2}{1 + |\xi|^2} \right) |\hat{n}|^2 \leq 0, \] (2.6)
for some suitable positive constant \( c > 0 \). In particular,
\[ \frac{d}{dt} \mathcal{E}(\hat{U}(t, \xi)) + \frac{c|\xi|^2}{1 + |\xi|^2} \mathcal{E}(\hat{U}(t, \xi)) \leq 0 \]
holds for any \( t > 0 \) and \( \xi \in \mathbb{R}^3 \).

**Proof.** We will use the following notations. For an integrable function \( f : \mathbb{R}^3 \to \mathbb{R} \), its Fourier transform is defined by
\[ \hat{f}(\xi) = \int_{\mathbb{R}^3} \exp(-ix \cdot \xi)f(x)dx, \quad x \cdot \xi := \sum_{j=1}^{3} x_j \xi_j, \quad \xi \in \mathbb{R}^3, \]
where \(i = \sqrt{-1} \in \mathbb{C}\). For two complex numbers or vectors \(a\) and \(b\), \((a|b)\) denotes the dot product of \(a\) with the complex conjugate of \(b\). The Fourier transform representation system (2.3) can be written as,

\[
\begin{align*}
\dot{n}_t + i\xi \cdot \dot{u} &= 0, \\
\dot{u}_t + i\xi \dot{n} + i\xi \dot{m} + (\mu + \mu_r)\xi^2 \dot{u} + (\mu + \lambda - \mu_r)\xi \cdot \dot{u} - 2\mu_r i\xi \times \dot{w} &= 0, \\
\dot{\bar{w}}_t + 4\mu_r \dot{\bar{w}} + (c_0 + c_d)\xi^2 \dot{\bar{w}} + (c_0 + c_d - c_a)\xi \cdot \dot{\bar{w}} - 2\mu_r i\xi \times \dot{\bar{w}} &= 0, \\
\dot{\bar{m}}_t + (\gamma - 1)\xi^2 \dot{\bar{m}} + (\gamma - 1)i\xi \cdot \dot{\bar{u}} &= 0.
\end{align*}
\]

Multiplying \((2.7)_1, (2.7)_2, (2.7)_3\) and \((2.7)_4\) by \(\tilde{n}, \tilde{u}, \tilde{\bar{w}}\) and \(\tilde{\bar{m}}\) respectively, taking real part and taking summation of four resultant equations, we finally get

\[
\frac{1}{2} \frac{d}{dt} \left( |\sqrt{\gamma - 1} \tilde{n}, \sqrt{\gamma - 1} \tilde{u}, \tilde{\bar{w}}, \tilde{\bar{m}}|^2 \right) + (\mu + \mu_r)|\xi|^2 |\tilde{u}|^2 + (\mu + \lambda - \mu_r)|\xi \cdot \tilde{u}|^2 + 4\mu_{\bar{m}} \text{Re}(\xi \times \tilde{\bar{w}})|\tilde{\bar{m}}|^2 \leq 0.
\]

By taking the complex dot product of the second equation of system (2.7) with \(i\xi \tilde{n}\), one has

\[
\frac{d}{dt} (\tilde{u} | i\xi \tilde{n}) + |\xi|^2 |\tilde{u}|^2 + (((\mu + \mu_r) |\xi|^2 \tilde{u} | i\xi \tilde{n}) + ((\mu + \lambda - \mu_r) \xi \cdot \tilde{u} | i\xi \tilde{n})
\]

\[
= (\gamma - 1) |\xi \cdot \tilde{u}|^2 + (2\mu_r i\xi \times \tilde{\bar{w}} | i\xi \tilde{n}) - (i\xi \tilde{m} | i\xi \tilde{n})
\]

where we have used the fact

\[
(\tilde{u}_t | i\xi \tilde{n}) = \frac{d}{dt} (\tilde{u} | i\xi \tilde{n}) - (\tilde{u} | i\xi \tilde{n}_t) = \frac{d}{dt} (\tilde{u} | i\xi \tilde{n}) + (i\xi \cdot \tilde{u} | \tilde{n}_t) = \frac{d}{dt} (\tilde{u} | i\xi \tilde{n}) - |\xi \cdot \tilde{u}|^2.
\]

Taking the real part of (2.9) and then using Cauchy-Schwarz inequality, we derive

\[
\frac{d}{dt} \text{Re}(\tilde{u} | i\xi \tilde{n}) + |\xi|^2 |\tilde{u}|^2 \leq \frac{1}{2} |\xi|^2 |\tilde{u}|^2 + C \left( |\xi|^2 |\tilde{u}|^2 + |\xi|^4 |\tilde{u}|^2 + |\xi|^2 |\tilde{w}|^2 + |\xi|^2 |\tilde{m}|^2 \right).
\]

It follows

\[
\frac{d}{dt} \text{Re}(\tilde{u} | i\xi \tilde{n}) + \frac{1}{2} |\xi|^2 |\tilde{u}|^2 \leq C |\xi|^2 |\tilde{u}|^2 + \frac{|\xi|^2}{1 + |\xi|^2} (|\tilde{w}|^2 + |\tilde{m}|^2).
\]

Define

\[
E(\tilde{U}(t, \xi)) = |(\sqrt{\gamma - 1} \tilde{n}, \sqrt{\gamma - 1} \tilde{u}, \tilde{\bar{w}}, \tilde{\bar{m}})|^2 + \frac{\text{Re}(\tilde{u} | i\xi \tilde{n})}{1 + |\xi|^2}
\]

with \(0 < \eta \ll 1\) to be determined so as to guarantee that \(E(\cdot)\) is the desired time-frequency functional satisfying (2.5) and (2.6). Notice that as long as \(0 < \eta \ll 1\) is small enough then

\[
E(\tilde{U}(t, \xi)) \sim |\tilde{U}(t, \xi)|^2
\]

holds true. The sum of (2.8), (2.10) \(\times \eta\) gives

\[
\partial_t E(\tilde{U}(t, k)) + c |\xi|^2 |\tilde{u}, \tilde{\bar{w}}, \tilde{\bar{m}}|^2 + c |\xi|^2 |\tilde{\bar{m}}|^2 \leq 0.
\]
Note that
\[ c|\xi|^2[\hat{u}, \hat{w}, \hat{m}]^2 + c|\xi|^2 = \frac{c|\xi|^2}{1 + |\xi|^2}[\hat{U}]^2. \]
This completes the proof of Theorem 2.1.

Theorem 2.1 exactly shows the dissipative structure of the linearized Micropolar fluids system (1.1). It plays a key role in the study of the nonlinear asymptotic stability of the constant steady state under small perturbations.

2.3. Time-decay property. As in [3, 4] it is now a standard procedure to derive from Theorem 1.1 the \( L^p - L^q \) time decay property of solutions to the linearized system (2.3). Here, we consider it by obtaining the following lemma.

**Lemma 2.2.** Let \( U(t, x), t \geq 0, x \in \mathbb{R}^3 \) be a well-defined solution to the system (2.3) - (2.4). Then, there exists two constants \( c > 0, C > 0 \) such that
\[ |\hat{U}(t, \xi)| \leq C \exp \left( -\frac{c|\xi|^2 t}{1 + |\xi|^2} \right) |\hat{U}_0(\xi)| \]
holds for any \( t \geq 0 \) and \( \xi \in \mathbb{R}^3 \).

**Definition 2.3.** (See [3] for instance.) Let \( 1 \leq p, r \leq 2 \leq q \leq \infty, |\alpha| \geq 0 \) and let \( m \geq 0 \) be an integer. Define
\[ \left[ |\alpha| + 3 \left( \frac{1}{r} - \frac{1}{q} \right) \right]_{+} = \begin{cases} |\alpha|, & \text{if } |\alpha| \text{ is integer and } r = q = 2, \\ \left[ |\alpha| + 3 \left( \frac{1}{r} - \frac{1}{q} \right) \right]_{+} + 1, & \text{otherwise}, \end{cases} \]
where \([\cdot]_{+}\) denotes the integer part of the argument.

Based on the pointwise estimate of Lemma 2.2, it is also straightforward to obtain the \( L^p - L^q \) time-decay property to the Cauchy problem (2.3) - (2.4). By applying Theorem 2.1 together with Lemma 2.2 to the linearized system (2.3), we have

**Proposition 2.1.** Let \( 1 \leq p, s \leq 2 \leq q \leq \infty, |\alpha| \geq 0 \), and let \( j \geq 0 \) be an integer. Let \( U = [n, u, w, m] \) satisfy the system (2.3) for all \( t > 0, x \in \mathbb{R}^3 \). Then, the solution of the linearized homogeneous system satisfies
\[ \left\| \nabla U(t) \right\|_{L^s} \leq C(1 + t)^{-\frac{j}{2} \left( \frac{1}{r} - \frac{1}{q} \right)} \left\| U_0 \right\|_{L^p} + C e^{-ct} \left\| \nabla^{[m + 3 \left( \frac{1}{r} - \frac{1}{q} \right)]} U_0 \right\|_{L^r}, \]
for any \( t \geq 0 \).

For the linearized system (1.1), Proposition 2.1 describes the unified time decay property of the full solution. But, it can not be directly applied to the nonlinear system (4.1)-(4.2) to obtain the time decay rates of the solution. Therefore, we turn to the study of the Green function of the linearized system.

3. Green function. In fact, as in [12, 23] the linearized micropolar fluid system (2.6) can be written as two decoupled subsystems which govern the time evolution of \( n, \nabla \cdot u, \nabla \cdot w \) and \( \nabla \times u, \nabla \times w \) respectively. We decompose the solution to (2.3) - (2.4) into two parts in the form of
\[
\begin{bmatrix}
n(t, x) \\
u(t, x) \\
w(t, x) \\
m(t, x)
\end{bmatrix}
= \begin{bmatrix}
n(t, x) \\
u(t, x) \\
w(t, x) \\
m(t, x)
\end{bmatrix}
+ \begin{bmatrix}
0 \\
u_{\perp}(t, x) \\
w_{\perp}(t, x) \\
0
\end{bmatrix},
\]
(3.1)
where \( u_\parallel, u_\perp \) are defined by
\[
\begin{align*}
  u_\parallel &= \Delta^{-1} \nabla \nabla \cdot u, \\
  u_\perp &= -\Delta^{-1} \nabla \times (\nabla \times u),
\end{align*}
\]
and likewise for \( w_\parallel, w_\perp \). For brevity, the first part on the right of (3.1) is called the fluid part and the second part is called the electromagnetic part, and we also write

\[
U_\parallel = [n, u_\parallel, w_\parallel, m], \quad U_\perp = [u_\perp, w_\perp].
\]

We now derive the equations of system (3.4) is

\[
\text{Green function of the fluid part.}
\]

3.1. **Applying \(-\Delta^{-1}\nabla\) to (3.3) \(U_\parallel\) and noticing \(\nabla \cdot u = \nabla \cdot u_\parallel\), we see that the fluid part \(U_\parallel\) satisfies

\[
\begin{align*}
  \left\{ \begin{array}{l}
  \partial_t n + \text{div} u_\parallel = 0, \\
  \partial_t u_\parallel + \nabla n + \nabla m - (2\mu + \lambda)\Delta u_\parallel = 0, \\
  \partial_t w_\parallel + 4\mu_r w_\parallel - (2c_0 + c_d)\Delta \nabla \cdot w_\parallel = 0, \\
  \partial_t m - (\gamma - 1)\Delta m + (\gamma - 1)\text{div} u_\parallel = 0.
  \end{array} \right.
\end{align*}
\]

Taking the curl of (2.3)\(2, (2.3)_3\) and noticing \(\nabla \times u = \nabla \times u_\parallel\), we see that the fluid part \(U_\parallel\) satisfies

\[
\begin{align*}
  \left\{ \begin{array}{l}
  \partial_t n + \text{div} u_\parallel = 0, \\
  \partial_t u_\parallel + \nabla n + \nabla m - (2\mu + \lambda)\Delta u_\parallel = 0, \\
  \partial_t w_\parallel + 4\mu_r w_\parallel - (2c_0 + c_d)\Delta \nabla \cdot w_\parallel = 0, \\
  \partial_t m - (\gamma - 1)\Delta m + (\gamma - 1)\text{div} u_\parallel = 0.
  \end{array} \right.
\end{align*}
\]

The initial data is given by

\[
[n, u_\parallel, w_\parallel, m]_{t=0} = [n_0, u_\parallel, w_\parallel, m_0].
\]

Applying \(-\Delta^{-1}\nabla\times\) to the above two equations and noticing \(\nabla \times u = \nabla \times u_\perp\), we see that electromagnetic part \(U_\perp\) satisfies

\[
\begin{align*}
  \left\{ \begin{array}{l}
  \partial_t u_\perp - (\mu + \mu_r)\Delta u_\perp - 2\mu_r \nabla \times w_\perp = 0, \\
  \partial_t w_\perp + 4\mu_r w_\perp - (c_0 + c_d)\Delta \nabla \times w_\perp = 0, \\
  \end{array} \right.
\end{align*}
\]

with initial data

\[
[u_\perp, w_\perp]_{t=0} = [u_\perp, w_\perp].
\]

3.1. **Green function of the fluid part.** The Fourier transform for the linear part of system in (3.4) is

\[
\begin{pmatrix}
  \hat{n}_\parallel \\
  \hat{u}_\parallel \\
  \hat{m}_\parallel \\
  \hat{w}_\parallel
\end{pmatrix}_t =
\begin{pmatrix}
  0 & -i|\xi|^2 & 0 & 0 \\
  -i\xi & -(2\mu + \lambda)|\xi|^2 I & -i|\xi|^2 & 0 \\
  0 & -(\gamma - 1)i|\xi|^2 & -(\gamma - 1)|\xi|^2 & 0 \\
  0 & 0 & 0 & (-\sigma|\xi|^2 - 4\mu_r)I_{3 \times 3}
\end{pmatrix}\begin{pmatrix}
  \hat{n}_\parallel \\
  \hat{u}_\parallel \\
  \hat{m}_\parallel \\
  \hat{w}_\parallel
\end{pmatrix},
\]

where \(\sigma = (2(c_0 + c_d) - c_a)\) and \(I\) is identity matrix.
Lemma 3.1. The eigenvalues of the above matrix are \( \lambda_1, \lambda_2, \lambda_3, -(2\mu + \lambda)\|\xi\|^2 \) (with multiplicity 2), \(-\sigma\|\xi\|^2 - 4\mu r\). The solution of the linearized equation (3.4) can be expressed as

\[
\begin{pmatrix}
\hat{n} \\
\hat{u}_\parallel \\
\hat{m}
\end{pmatrix} = \hat{G}^1(\xi, t) \begin{pmatrix}
\hat{n}_0 \\
\hat{u}_\parallel^0 \\
\hat{m}_0
\end{pmatrix},
\]

where

\[
P_1 = \begin{pmatrix}
\lambda_1(\xi^i)^t (\xi^i)^T \\
\xi^i\xi^* \\
0
\end{pmatrix}
\begin{pmatrix}
(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_1) \\
(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_2) \\
0
\end{pmatrix}
\begin{pmatrix}
(\xi^2 + \lambda_1)\xi^T \\
(\xi^2 + \lambda_2)\xi^T \\
0
\end{pmatrix}
\begin{pmatrix}
\lambda_2(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_1) \\
\lambda_3(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3) \\
0
\end{pmatrix}
\begin{pmatrix}
(\xi^2 + \lambda_1)\xi^T \\
(\xi^2 + \lambda_2)\xi^T \\
0
\end{pmatrix}
\begin{pmatrix}
\lambda_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \\
\lambda_2(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1) \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

Here the transformed Green matrix can be written as

\[
\hat{G}^1(\xi, t) = e^{\lambda_1 t} P_1 + e^{\lambda_2 t} P_2 + e^{\lambda_3 t} P_3 + e^{-\sigma\|\xi\|^2 t} P_4 + e^{-(\sigma\|\xi\|^2 - 4\mu r) t} P_5,
\]

where

\[
P_2 = \begin{pmatrix}
\lambda_2(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3) \\
\xi^i\xi^* \\
0
\end{pmatrix}
\begin{pmatrix}
(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_2) \\
(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_1) \\
0
\end{pmatrix}
\begin{pmatrix}
(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_2) \\
(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_1) \\
0
\end{pmatrix}
\begin{pmatrix}
(\xi^2 + \lambda_2)\xi^T \\
(\xi^2 + \lambda_3)\xi^T \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

\[
P_3 = \begin{pmatrix}
\lambda_3(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_1) \\
\xi^i\xi^* \\
0
\end{pmatrix}
\begin{pmatrix}
(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_1) \\
(\lambda_3 - \lambda_1)(\lambda_1 - \lambda_3) \\
0
\end{pmatrix}
\begin{pmatrix}
(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_3) \\
(\lambda_3 - \lambda_1)(\lambda_1 - \lambda_1) \\
0
\end{pmatrix}
\begin{pmatrix}
(\xi^2 + \lambda_3)\xi^T \\
(\xi^2 + \lambda_1)\xi^T \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

In order to get the time-frequency pointwise estimates on \( \hat{n}, \hat{u}_\parallel, \hat{m} \), we now deal with the Green matrix \( \hat{G}^1(\xi, t) \).

From the expression of \( P_i \), as in [22], we can be easy to get the following Lemmas:

**Lemma 3.1.** For each element of \( P_i \) \((i = 1, 2, 3, 4, 5)\), there is a positive constant \( C \) such that

\[
|P_i(j, k)| \leq C, \quad (i = 1, 2, 3, 4, 5, \quad j, k = 1, 2, \cdots, 8).
\]

**Lemma 3.2.** For a sufficiently small \( \varepsilon > 0 \) and a sufficiently large \( K > 0 \), we have

1. when \( |\xi| < \varepsilon \), \( \lambda_1 \) has the following expansion:

\[
\lambda_1 = -\frac{1}{3}\|\xi\|^2 + \sum_{j=2}^{+\infty} a_j|\xi|^{2j},
\]

2. \( \lambda_{2,3} = -\left(\nu + \frac{1}{2} - \frac{1}{6}\right)|\xi|^2 + \sum_{j=2}^{+\infty} b_{2j}|\xi|^{2j} \pm i\left(\sqrt{3}|\xi| + \sum_{j=2}^{+\infty} b_{2j-1}|\xi|^{2j-1}\right),
\]
2. when \( \varepsilon \leq |\xi| \leq K \), \( \lambda_i \) has the following spectrum gap property:
\[
\text{Re} (\lambda_i) \leq -C, \quad \text{for some constant } C > 0,
\]

3. when \( |\xi| > K \), \( \lambda_i \) has the following expansion:
\[
\lambda_1 = -\frac{1}{\nu} + \sum_{j=1}^{\infty} c_j |\xi|^{-2j}, \quad \lambda_2 = -|\xi|^2 + \sum_{j=0}^{\infty} d_j |\xi|^{-2j}, \quad \lambda_3 = -\nu |\xi|^2 + \sum_{j=0}^{\infty} e_j |\xi|^{-2j}.
\]
Here \( \nu = (2\mu + \lambda) \) and all \( a_j, b_j, c_j, d_j, e_j \) are real constants.

Then, we have the following estimates

\textbf{Lemma 3.3.} For a sufficiently small \( \varepsilon > 0 \) and a sufficiently large number \( K > 0 \), we have the following:

1. when \( |\xi| < \varepsilon \), \( \lambda_i \) has the following expansion estimate:
\[
\lambda_1 = -\frac{1}{3} \mathcal{O}(|\xi|^2), \quad \text{Re}\{\lambda_{2,3}\} = - \left( \frac{\nu + 1}{2} - \frac{1}{6} \right) \mathcal{O}(|\xi|^2),
\]

2. when \( \varepsilon \leq |\xi| \leq K \), \( \lambda_i \) has the following spectrum gap property:
\[
\text{Re} (\lambda_i) \leq -C, \quad \text{for some constant } C > 0,
\]

3. when \( |\xi| > K \), \( \lambda_i \) has the following expansion:
\[
\lambda_1 = -\frac{1}{\nu} + \mathcal{O}(|\xi|^{-4}), \quad \lambda_2 = -|\xi|^2 + \mathcal{O}(1), \quad \lambda_3 = -\nu |\xi|^2 + \mathcal{O}(1).
\]

Now, we give some refined \( L^p - L^q \) time-decay properties for \( U_\| = [n, u_\|, w_\|, m] \). For that, we first make the time-frequency pointwise estimates on \( \hat{n}, \hat{u}_\|, \hat{w}_\|, m \) as follows:

\textbf{Lemma 3.4.} Let \( U_\| = [n, u_\|, w_\|, m] \) be the solution to the linearized homogeneous system (2.3) with initial data \( U_{\|0} = [n_0, v_\|0, w_\|0, m_0] \). Then, there exists constants \( \varepsilon > 0, c > 0, C > 0 \) such that for all \( t > 0, |\xi| \leq \varepsilon \)
\[
|\hat{n}(t, \xi), \hat{u}_\|(t, \xi), \hat{w}_\|(t, \xi)| \leq C \exp (-c|\xi|^2 t) \left| [\hat{n}_0(\xi), \hat{u}_\|0(\xi), \hat{w}_\|0(\xi)] \right|,
\]
\[
|\hat{w}_\|(t, \xi)| \leq C \exp (-c t) \left| \hat{w}_\|0(\xi) \right|,
\]
and for all \( t > 0, |\xi| \geq \varepsilon \)
\[
|[\hat{n}(t, \xi), \hat{u}_\|(t, \xi), \hat{w}_\|(t, \xi)]| \leq C \exp(-c t) \left| [\hat{n}_0(\xi), \hat{u}_\|0(\xi), \hat{w}_\|0(\xi)] \right|,
\]
\[
|\hat{w}_\|(t, \xi)| \leq C \exp(-c t) \left| \hat{w}_\|0(\xi) \right|.
\]

\textbf{Proof.} It is suffice to estimate Green function \( \hat{G}(\xi, t) \). Recall that
\[
\hat{G}_1(\xi, t) = e^{\lambda_1 t} P_1 + e^{\lambda_2 t} P_2 + e^{\lambda_3 t} P_3 + e^{-\mu_1 |\xi|^2 t} P_4 + e^{-\sigma_1 |\xi|^2 - 4\mu t} P_5.
\]

For each element of \( \hat{G}(\xi, t) \), combining Lemma 2.2 with Lemma 3.3 gives,
\[
|\hat{G}_1| \leq C e^{-c|\xi|^2 t} \quad (i, j = 1, 2, \cdots, 5), \quad \text{as } |\xi| \leq \varepsilon, \quad (3.13)
\]
\[
|\hat{G}_1| \leq C e^{-c t} \quad (i, j = 1, 2, \cdots, 5), \quad \text{as } \varepsilon \leq |\xi| \leq K, \quad (3.14)
\]
\[
|\hat{G}_1| \leq C e^{-c t} \quad (i, j = 1, 2, \cdots, 5), \quad \text{as } |\xi| \geq K. \quad (3.15)
\]

Then, from (3.14), one has for \( |\xi| \leq \varepsilon \),
\[
|\hat{n}(t, \xi)| = |\hat{G}_{11}\hat{n}_0(\xi) + \hat{G}_{12}\hat{u}_\|0(\xi) + \hat{G}_{13}\hat{w}_\|0(\xi)| \leq C \exp (-c|\xi|^2 t) \left| [\hat{n}_0(\xi), \hat{u}_\|0(\xi), \hat{w}_\|0(\xi)] \right|,
\]
\[ |\hat{u}_\parallel(t, \xi)| = |\hat{G}_{21}\hat{n}_0(\xi) + \hat{G}_{22}\hat{u}_\parallel(\xi) + \hat{G}_{23}\hat{\nu}_0(\xi)| \leq C \exp \left( -c|\xi|^2 t \right) \|\hat{n}_0(\xi), \hat{u}_\parallel(\xi), \hat{\nu}_0(\xi)\|, \]

\[ |\hat{m}(t, \xi)| = |\hat{G}_{31}\hat{n}_0(\xi) + \hat{G}_{32}\hat{u}_\parallel(\xi) + \hat{G}_{33}\hat{\nu}_0(\xi)| \leq C \exp \left( -c|\xi|^2 t \right) \|\hat{n}_0(\xi), \hat{u}_\parallel(\xi), \hat{\nu}_0(\xi)\|, \]

which gives (3.9). And (3.10), (3.11), (3.12) is similar to prove. Thus, the proof is completed. \( \square \)

As [3], it is now a standard procedure to derive from Lemma 3.4 the \( L^p - L^q \) time decay property of the fluid part.

**Proposition 3.2.** Let \( 1 \leq p, r \leq 2 \leq q \leq \infty \) and let \( m \geq 0 \) be an integer. Suppose that \([n, u_\parallel, w_\parallel, m]\) is the solution to the Cauchy problem (2.3)–(2.4). Then \([n, u_\parallel, w_\parallel, m]\) satisfies the following time-decay property:

\[
\| \nabla^k [n, u_\parallel, w_\parallel, m] \|_{L^q} \leq C (1 + t)^{-\frac{3}{2} \left( \frac{1}{r} - \frac{1}{q} \right)} \| [n_0, u_{0\parallel}, m_0] \|_{L^p} + C e^{-ct} \| \nabla^k [n_0, u_{0\parallel}, m_0] \|_{L^r},
\]

\[
\| \nabla^k w_{\parallel} \|_{L^r} \leq C e^{-ct} \| m_0 \|_{L^p} + C e^{-ct} \| \nabla^k [n_0, u_{0\parallel}, m_0] \|_{L^r},
\]

for any \( t \geq 0 \), where \( C = C(m, p, r, q) \) and \( \left[ 3 \left( \frac{1}{r} - \frac{1}{q} \right) \right]_+ \) is defined in Definition 2.3.

### 3.2. Green function of the electromagnetic part.

Recall that the electromagnetic part \( U_\perp = [u_\perp, w_\perp] \) satisfies the following equation

\[
\begin{align*}
\partial_t u_\perp - (\mu + \mu_r) \Delta u_\perp - 2\mu_r \nabla \times w_\perp &= 0, \\
\partial_t w_\perp + 4\mu_r w_\perp - (c_0 + c_d) \Delta w_\perp - 2\mu_r \nabla \times u_\perp &= 0,
\end{align*}
\]

(3.16)

with initial data

\[ [u_\perp, w_\perp]_{t=0} = [u_{0\perp}, w_{0\perp}] . \]

We deduce from (3.32) and (3.35) that

\[
\begin{pmatrix}
\hat{u}_\perp \\
\hat{w}_\perp
\end{pmatrix} = \hat{G}^2(t) \begin{pmatrix}
\hat{u}_{0\perp} \\
\hat{w}_{0\perp}
\end{pmatrix} .
\]

Here \( \hat{G}^2(t) \) is the solution operator of linear equation (3.16),

\[
\hat{G}^2(\xi, t) = \begin{pmatrix}
\hat{G}_{11}^2 & 2\mu_r e^{\kappa_+ t} - e^{\kappa_- t} i\xi \\
2\mu_r e^{\kappa_+ t} - e^{\kappa_- t} i\xi & \hat{G}_{22}^2
\end{pmatrix},
\]

where

\[
\hat{G}_{11}^2 = \frac{\kappa_+ e^{\kappa_+ t} - \kappa_- e^{\kappa_- t}}{\kappa_+ - \kappa_-} - (\mu + \mu_r) |\xi|^2 \frac{e^{\kappa_+ t} - e^{\kappa_- t}}{\kappa_+ - \kappa_-} I_{3 \times 3},
\]

\[
\hat{G}_{22}^2 = \frac{\kappa_+ e^{\kappa_+ t} - \kappa_- e^{\kappa_- t}}{\kappa_+ - \kappa_-} - ((c_0 + c_d) |\xi|^2 + 4\mu_r) \frac{e^{\kappa_+ t} - e^{\kappa_- t}}{\kappa_+ - \kappa_-} I_{3 \times 3},
\]

\[
\kappa_\pm = -\left[ |\mu_1|^2 + 4\mu_r + \sqrt{\mu_1^2 |\xi|^4 + 16\mu_r^2 + 8\mu_r \mu_2 |\xi|^2} \right]_+ + 2 \mu_2 (\mu + \mu_r + (c_0 + c_d) - \mu),
\]

\[
\mu_1 = (\mu + \mu_r + (c_0 + c_d)), \quad \mu_2 = (\mu_r + (c_0 + c_d) - \mu).
As a similar result in [12], we have

**Lemma 3.5** ([12]). Let \( U_\perp = [u_\perp, w_\perp] \) be the solution to the linearized homogeneous system (3.17) with initial data \( U_{\perp,0} = [u_{0\perp}, w_{0\perp}] \). Then, there exist constants \( \varepsilon > 0, c > 0, C > 0 \) such that for all \( t > 0, |\xi| \leq \varepsilon \)

\[
|\hat{u}_\perp(t, \xi)| \leq C \exp(-c|\xi|^2 t) \| [\hat{u}_{0\perp} (\xi), \hat{w}_{0\perp}] \|
\]

\[
|\hat{w}_\perp(t, \xi)| \leq C|\xi| \exp(-c|\xi|^2 t) \| [\hat{u}_{0\perp} (\xi), \hat{w}_{0\perp}] | + C \exp(-ct) \| [\hat{u}_{0\perp} (\xi), \hat{w}_{0\perp}] |
\]

and for all \( t > 0, |\xi| \geq \varepsilon \)

\[
|\hat{u}_\perp(t, \xi)| \leq C \exp(-c t) \| [\hat{u}_{0\perp} (\xi), \hat{w}_{0\perp}] |, \quad |\hat{w}_\perp(t, \xi)| \leq C \exp(-c t) \| [\hat{u}_{0\perp} (\xi), \hat{w}_{0\perp}] |.
\]

We can deduce the \( L^p - L^q \) time-decay property for the electromagnetic part \( U_\perp \) directly from Lemma 3.5.

**Proposition 3.3.** Let \( 1 \leq p, r \leq 2 \leq q \leq \infty \) and let \( m \geq 0 \) be an integer. Suppose that \( [u_\perp, w_\perp] \) is the solution to the Cauchy problem (3.7)-(3.8). Then \( U_\perp = [u_\perp, w_\perp] \) satisfies the following time-decay property:

\[
\| \nabla^k u_\perp \|_{L^q} \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{m}{r}} \| [u_{0\perp}, w_{0\perp}] \|_{L^p} + Ce^{-ct} \left\| \nabla^{k+3(\frac{1}{p} - \frac{1}{q})} [u_{0\perp}, w_{0\perp}] \right\|_{L^r},
\]

\[
\| \nabla^k w_\perp \|_{L^q} \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{m}{r}} \| [u_{0\perp}, w_{0\perp}] \|_{L^p} + Ce^{-ct} \left\| \nabla^{k+3(\frac{1}{p} - \frac{1}{q})} [u_{0\perp}, w_{0\perp}] \right\|_{L^r},
\]

for any \( t \geq 0 \), where \( C = C(m, p, r, q) \) and \( [3\left(\frac{1}{p} - \frac{1}{q}\right)] \) is defined in Definition 2.3.

Now, by combining two representations of the Green functions to the fluid part and the electromagnetic part obtained in Proposition 3.2 and Proposition 3.3 respectively, we obtain

**Theorem 3.6.** Let \( 1 \leq p, r \leq 2 \leq q \leq \infty \) and let \( m \geq 0 \) be an integer. Suppose that \( [n, u, w, m] \) is the solution to the Cauchy problem (2.1)-(2.4) Then \( U = [n, u, w, m] \) satisfies the following time-decay property:

\[
\| \nabla^k n \|_{L^q} \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{m}{r}} \| [u_{0}, w_{0}] \|_{L^p} + Ce^{-ct} \left\| \nabla^{k+3(\frac{1}{p} - \frac{1}{q})} [u_{0}, w_{0}] \right\|_{L^r},
\]

\[
\| \nabla^k u \|_{L^q} \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{m}{r}} \| [n_{0}, u_{0}, w_{0}, m_{0}] \|_{L^p} + Ce^{-ct} \left\| \nabla^{k+3(\frac{1}{p} - \frac{1}{q})} [n_{0}, u_{0}, w_{0}, m_{0}] \right\|_{L^r},
\]

\[
\| \nabla^k m \|_{L^q} \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{m}{r}} \| [u_{0}, m_{0}] \|_{L^p} + Ce^{-ct} \left\| \nabla^{k+3(\frac{1}{p} - \frac{1}{q})} [u_{0}, m_{0}] \right\|_{L^r},
\]

\[
\| \nabla^k w \|_{L^q} \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{m}{r}} \| [u_{0}, w_{0}] \|_{L^p} + Ce^{-ct} \left\| \nabla^{k+3(\frac{1}{p} - \frac{1}{q})} [u_{0}, w_{0}] \right\|_{L^r},
\]

for any \( t \geq 0 \), where \( C = C(k, p, r, q) \) and \( [3\left(\frac{1}{p} - \frac{1}{q}\right)] \) is defined in Definition 2.3.

For later use, we need the following result which is an immediate corollary from Theorem 3.6.
Corollary 1. Under the assumption of Theorem 3.6, there exists a constant $C > 0$, such that
\[
\frac{1}{2} \left\| u_n \right\| L^1 \cap L^2, \\
\frac{1}{2} \left\| u_0 \right\| L^1 \cap L^2, \\
\frac{1}{2} \left\| u_m \right\| L^1 \cap L^2, \\
\frac{1}{2} \left\| w_0 \right\| L^1 \cap L^2,
\]
for any $t \geq 0$.

4. Nonlinear asymptotic stability. 6.1. Global existence. First of all, still using the notation $[n, u, w, m]$ for simplicity, let us write the reformulated nonlinear system (1.1) and (1.2) as
\[
\begin{aligned}
n_t + (n + 1) \, \text{div} \, u &= -u \cdot \nabla n, \\
w_t + m + \mathbf{u} \cdot \nabla m - \frac{(\mu + \mu_r)}{n + 1} \Delta u - \frac{(\mu + \lambda - \mu_r)}{n + 1} \nabla \text{div} \, u - \frac{2 \mu_r}{n + 1} \nabla \times w \\
&= -w \cdot \nabla u, \\
\end{aligned}
\]

where $h = \gamma \frac{1}{\kappa (n + 1)} \left[ \lambda (\text{div} \, u)^2 + \lambda \Delta \mathbf{D} \cdot \mathbf{D} + 4 \mu_r \left( \frac{1}{2} \text{rot} u - w \right)^2 + c_0 (\text{div} w)^2 \\
+ (c_a + c_d) \nabla \, w : \nabla \, w^T + (c_d - c_a) \nabla \, w : \nabla \, w \right],$

with
\[
(n, u, w, m)(x, 0) = (n_0, u_0, w_0, m_0)(x). (4.2)
\]

4.1. Global existence. In this subsection, we will show that there exists a unique global-in-time solution to the system (1.1)-(1.2). We first obtain the uniform a priori estimates.

The following lemma in [20] is useful for the forthcoming estimates.

**Lemma 4.1.** There exists a constant $C > 0$ such that for any $f, g, h \in H^s (\mathbb{R}^3)$ and any multi-index $\alpha$ with $1 \leq |\alpha| \leq N$
\[
\left\| fgh \right\| L^1(\mathbb{R}^3) \leq C \left\| f \right\| L^2(\mathbb{R}^3) \left\| g \right\| L^2(\mathbb{R}^3) \left\| h \right\| L^2(\mathbb{R}^3), \\
\left\| f \right\| L^\infty(\mathbb{R}^3) \leq C \left\| f \right\| H^1(\mathbb{R}^3), \\
\left\| f \right\| L^\infty(\mathbb{R}^3) \leq C \left\| \nabla f \right\| L^2(\mathbb{R}^3), \\
\left\| \partial^\alpha_x (fg) \right\| L^2(\mathbb{R}^3) \leq C \left\| \partial^\alpha_x f \right\| H^{\alpha - 1}(\mathbb{R}^3) \left\| \partial^\alpha_x g \right\| H^{\alpha - 1}(\mathbb{R}^3).
\]

**Lemma 4.2** (Moser-type inequality). For functions $f, g \in H^s \cap L^\infty$, and $|\alpha| \leq N$, we have
\[
\left\| \nabla^\alpha_x (fg) - f \nabla^\alpha_x g \right\| L^2 \leq C \left( \left\| \nabla^\alpha_x f \right\| L^\infty \left\| \nabla^{N-1} g \right\| L^2 + \left\| \nabla^N f \right\| L^2 \right),
\]
with the constant depends only on $N$.

Then, the global existence of the reformulated Cauchy problem (2.1)-(2.2) with small smooth initial data can be stated as follows.
Theorem 4.3. For any real number $s \geq 4$. There exists a small enough $\delta_1 > 0$, such that
\[ \|U_0\|_{H^s} \leq \delta_1, \]
then the Cauchy problem (1.1)-(1.3) admits a unique global solution $U = [n, u, w, m]$ satisfying
\[ U \in C \left( [0, \infty); H^s (\mathbb{R}^3) \right), \quad \nabla n \in L^2 \left( [0, \infty); H^{s-1} (\mathbb{R}^3) \right), \quad \nabla u \in L^2 \left( [0, \infty); H^s (\mathbb{R}^3) \right), \]
\[ \nabla w \in L^2 \left( [0, \infty); H^s (\mathbb{R}^3) \right), \quad \nabla m \in L^2 \left( [0, \infty); H^s (\mathbb{R}^3) \right), \]
and
\[ \|U(t)\|_{H^s}^2 + \int_0^t \left( \|\nabla n(\tau)\|_{H^{s-1}}^2 + \|\nabla u(\tau)\|_{H^s}^2 + \|\nabla w(\tau)\|_{H^s}^2 + \|\nabla m(\tau)\|_{H^s}^2 \right) d\tau \leq C\|U_0\|_{H^s}^2. \]

To prove Theorem 4.3, it suffices to prove the following uniform-in-time a priori estimate.

Define
\[ \mathcal{E}_N(U(t)) = \sum_{|\alpha| \leq N} \int_{\mathbb{R}^3} |\partial^{(|\alpha|)} [n, u, w, m]|^2 dx + \eta \sum_{|\alpha| \leq N-1} \left\langle \partial^{(|\alpha|)} u, \partial^{(|\alpha|)} \nabla n \right\rangle \]
where $\eta$ is a constant to be properly chosen later. Notice that, since the constant $\eta$ is small enough, one has
\[ \mathcal{E}_N(U(t)) \sim \| [n, u, w, m] \|_{H^s}, \]
and the corresponding dissipation rate given by
\[ D_N(U(t)) = \|\nabla n\|_{H^{s-1}}^2 + \|\nabla u, w, m\|_{H^s}^2. \]
Here, $0 < \eta \ll 1$. Then, for any smooth solution $U$ to the Cauchy problem (4.1) – (4.2) over $0 \leq t \leq T$ with $T > 0$, one has
\[ \frac{d}{dt} \mathcal{E}_N(U(t)) + cD_N(U(t)) \leq CD_N(t) \sum_{k=1}^{N+4} \mathcal{E}_k^N(t). \quad (4.3) \]
As long as the above estimate is proved, Theorem 4.3 follows in the standard way by combining it with the local-in-time existence and uniqueness as well as the continuity argument. Therefore, in what follows we prove $(4.3)$ only, and other details are omitted for simplicity.

Proof. We will establish zero-order energy estimates. Multiplying (4.1)-(4.2) by $n, u, w, m$ respectively, then taking integration and summation, we finally get
\[ \frac{1}{2} \frac{d}{dt} \left\{ \|n\|^2 + \|u\|^2 + \|w\|^2 + \|m\|^2 \right\} + 4\mu_r \int \frac{|w|^2}{n+1} dx + (\mu + \mu_r) \int \frac{|\nabla u|^2}{n+1} dx + (\mu + \lambda - \mu_r) \int \frac{|\text{div} u|^2}{n+1} dx \]
\[ + (c_0 + c_d) \int \frac{|\nabla w|^2}{n+1} dx + (c_0 + c_d - c_a) \int \frac{|\text{div} w|^2}{n+1} dx + \int \frac{|\nabla m|^2}{n+1} dx \]
\[ = - \int (n+1) \text{div} u \ n dx - \int u \cdot \nabla n \ n dx + 2\mu_r \int \frac{\nabla \times w \cdot u}{n+1} dx \]
\[ + \int \left( \frac{m+1}{n+1} \nabla n + \nabla m \right) \cdot u dx - \int \text{div} u \cdot \nabla \left( \frac{1}{1+n} \right) dx \]
\[ - \int u \cdot \nabla u \cdot u dx + 2\mu_r \int \frac{w \cdot \nabla u}{n+1} dx - \int u \cdot \nabla w \ w dx - \int u \cdot \nabla m \ m dx \]
\[- \int \frac{\gamma - 1}{1 + n} \text{div} u \, m \, dx \leq \int \text{div} \left( -u \cdot \nabla n \right) \, m \, dx \leq \mu_r \| \nabla u \|^2 + \mu_r \| w \|^2 + C \| n, u, w, m \|_{H^1} \| \nabla [n, u, w, m] \|^2 + \| \nabla n \| (\| \nabla^2 u \|_{H^1} + \| \nabla^2 m \| \| m \|_{H^1}) + \| m \|_{H^1} (\| \nabla^2 u \| + \| \nabla^2 w \|) \| \nabla m \| .\]

Here, we follow the similar argument in [12]. Let \( 1 \leq |\alpha| \leq N \). Applying differentiation \( \partial^\alpha \) to (4.1) yields

\[
\begin{aligned}
\partial^\alpha n_t + (n + 1)\partial^\alpha \text{div} u &= -u \cdot \nabla n + f_1, \\
\partial^\alpha u_t + \partial^\alpha \nabla n + \partial^\alpha \nabla m - \frac{\mu + \mu_r}{n + 1} \Delta \partial^\alpha u &= \frac{\mu + \lambda - \mu_r}{n + 1} \nabla \text{div} \partial^\alpha u \\
&\quad + \frac{m - n}{n + 1} \partial^\alpha \nabla n - \frac{2\mu_r}{n + 1} \nabla \times \partial^\alpha w = -u \cdot \partial^\alpha \nabla u + f_2, \\
\partial^\alpha w_t + \frac{4\mu_r}{n + 1} \partial^\alpha w - \frac{c_0 + c_d}{n + 1} \Delta \partial^\alpha w - \frac{c_0 + c_d - c_a}{n + 1} \nabla \text{div} \partial^\alpha w &= -\frac{(\gamma - 1)}{n + 1} \Delta \partial^\alpha m + \frac{\gamma - 1}{n + 1} \text{div} \partial^\alpha u = -u \cdot \partial^\alpha \nabla m + \partial^\alpha h + f_4,
\end{aligned}
\]

where

\[
\begin{aligned}
f_1 &= -\left[ \partial^\alpha u \cdot \nabla n - \partial^\alpha (n + 1) \text{div} \right] u, \\
f_2 &= -\left[ \partial^\alpha u \cdot \nabla u - \partial^\alpha m + \frac{\mu + \mu_r}{n + 1} \nabla \right] u + \left[ \partial^\alpha \frac{2\mu_r}{n + 1} \nabla \times \frac{\partial^\alpha w}{\partial^\alpha w} \right] u, \\
f_3 &= -\left[ \partial^\alpha \frac{4\mu_r}{n + 1} \right] w + \left[ \partial^\alpha \frac{c_0 + c_d}{n + 1} \Delta \right] w \\
&\quad + \left[ \partial^\alpha \frac{c_0 + c_d - c_a}{n + 1} \nabla \right] w + \left[ \partial^\alpha \frac{2\mu_r}{n + 1} \nabla \times \right] u, \\
f_4 &= \left[ \partial^\alpha \frac{\gamma - 1}{n + 1} \Delta \right] w + \left[ \partial^\alpha \frac{\gamma - 1}{n + 1} \text{div} \right] w - \left[ \partial^\alpha u \cdot \nabla \right] w,
\end{aligned}
\]

and \([A, B]\) denotes the commutator \((AB - BA)\) for two operators \(A\) and \(B\). Multiplying the first equation of (4.4) by \(\partial^\alpha n\), and taking integration in \(x\) give

\[
\frac{1}{2} \frac{d}{dt} \int \left| \partial^\alpha n \right|^2 \, dx + \int (n + 1) \text{div} \partial^\alpha u \partial^\alpha n \, dx = \int f_1 \partial^\alpha n \, dx - \int u \cdot \nabla \partial^\alpha n \partial^\alpha n \, dx.
\]

Multiplying the second equation of (4.4) by \(\partial^\alpha u\), and taking integration in \(x\) give

\[
\frac{1}{2} \frac{d}{dt} \int \left| \partial^\alpha u \right|^2 \, dx + \int (\nabla \partial^\alpha \nabla n + \nabla \partial^\alpha \nabla m) \partial^\alpha u \, dx \\
&\quad + (\mu + \mu_r) \int \frac{\nabla \partial^\alpha u}{n + 1} \, dx + (\mu + \lambda - \mu_r) \int \left| \text{div} \partial^\alpha u \right|^2 \, dx \\
&= - (\mu + \mu_r) \int \nabla \left( \frac{1}{n + 1} \right) \cdot \partial^\alpha \nabla u \cdot \partial^\alpha u \, dx \\
&\quad - (\mu + \lambda - \mu_r) \int \nabla \left( \frac{1}{n + 1} \right) \cdot \partial^\alpha \text{div} u \cdot \partial^\alpha u \, dx
\]
\[ + 2\mu_r \int \nabla \times \partial^{[\alpha]} w \cdot \partial^{[\alpha]} u \, dx - \int u \partial^{[\alpha]} \nabla u \cdot \partial^{[\alpha]} u \, dx + \int f_2 \partial^{[\alpha]} u \, dx. \tag{4.6} \]

Multiplying the third equation of (4.4) by \( \partial^{[\alpha]} w \), and taking integration in \( x \) give
\[
\frac{1}{2} \frac{d}{dt} \int \left| \partial^{[\alpha]} w \right|^2 \, dx + 4\mu_r \int \frac{\left| \partial^{[\alpha]} w \right|^2}{n+1} \, dx \\
+ (c_0 + c_d) \int \frac{\left| \nabla \partial^{[\alpha]} w \right|^2}{n+1} \, dx + (c_0 + c_d - c_a) \int \left| \text{div} \partial^{[\alpha]} w \right|^2 \, dx \\
= - (\gamma - 1) \int \nabla \left( \frac{1}{(n+1)} \right) \cdot \partial^{[\alpha]} \nabla w \cdot \partial^{[\alpha]} w \, dx \\
- (c_0 + c_d - c_a) \int \nabla \left( \frac{1}{(n+1)} \right) \cdot \partial^{[\alpha]} \text{div} w \cdot \partial^{[\alpha]} w \, dx \\
+ 2\mu_r \int \nabla \times \partial^{[\alpha]} w \cdot \partial^{[\alpha]} u \, dx - \int u \partial^{[\alpha]} \nabla w \cdot \partial^{[\alpha]} w \, dx + \int f_3 \partial^{[\alpha]} w \, dx. \tag{4.7} \]

Multiplying the fourth equation of (4.4) by \( \partial^{[\alpha]} m \), and taking integration in \( x \) give
\[
\frac{1}{2} \frac{d}{dt} \int \left| \partial^{[\alpha]} m \right|^2 \, dx + (\gamma - 1) \int \frac{\left| \nabla \partial^{[\alpha]} m \right|^2}{n+1} \, dx + (\gamma - 1) \int \left| \text{div} \partial^{[\alpha]} m \right|^2 \, dx \\
= - (\gamma - 1) \int \nabla \left( \frac{1}{(n+1)} \right) \cdot \partial^{[\alpha]} \nabla m \cdot \partial^{[\alpha]} m \, dx \\
- (\gamma - 1) \int \nabla \left( \frac{1}{(n+1)} \right) \cdot \partial^{[\alpha]} \text{div} m \cdot \partial^{[\alpha]} m \, dx \\
- \int u \partial^{[\alpha]} \nabla m \cdot \partial^{[\alpha]} m \, dx + \int f_4 \partial^{[\alpha]} m \, dx + \int \partial^{[\alpha]} h \partial^{[\alpha]} m \, dx. \tag{4.8} \]

Taking the summation of (4.5) – (4.8), we get
\[
\frac{1}{2} \frac{d}{dt} \left\{ \int \left| \partial^{[\alpha]} n \right|^2 \, dx + \int \left| \partial^{[\alpha]} u \right|^2 \, dx + \int \left| \partial^{[\alpha]} w \right|^2 \, dx + \int \left| \partial^{[\alpha]} m \right|^2 \, dx \right\} \\
+ (\mu + \mu_r) \int \frac{\left| \nabla \partial^{[\alpha]} u \right|^2}{n+1} \, dx + (\mu + \lambda - \mu_r) \int \frac{\left| \text{div} \partial^{[\alpha]} u \right|^2}{n+1} \, dx \\
+ (c_0 + c_d) \int \frac{\left| \nabla \partial^{[\alpha]} u \right|^2}{n+1} \, dx + (c_0 + c_d - c_a) \int \frac{\left| \text{div} \partial^{[\alpha]} u \right|^2}{n+1} \, dx \\
+ (\gamma - 1) \int \frac{\left| \nabla \partial^{[\alpha]} m \right|^2}{n+1} \, dx + (\gamma - 1) \int \frac{\left| \text{div} \partial^{[\alpha]} m \right|^2}{n+1} \, dx \\
+ 4\mu_r \int \frac{\left| \partial^{[\alpha]} w \right|^2}{n+1} \, dx \\
= \int f_1 \partial^{[\alpha]} n \, dx - \int u \cdot \nabla \partial^{[\alpha]} n \partial^{[\alpha]} n \, dx - \int (n+1) \text{div} \partial^{[\alpha]} u \partial^{[\alpha]} n \, dx \\
- (\mu + \mu_r) \int \nabla \left( \frac{1}{(n+1)} \right) \cdot \partial^{[\alpha]} \nabla u \cdot \partial^{[\alpha]} u \, dx \\
- (\mu + \lambda - \mu_r) \int \nabla \left( \frac{1}{(n+1)} \right) \cdot \partial^{[\alpha]} \text{div} u \cdot \partial^{[\alpha]} u \, dx \\
+ 2\mu_r \int \frac{\nabla \times \partial^{[\alpha]} w \cdot \partial^{[\alpha]} u}{n+1} \, dx - \int u \partial^{[\alpha]} \nabla u \cdot \partial^{[\alpha]} u \, dx + \int f_2 \partial^{[\alpha]} u \, dx \]
using Leibniz formula, Lemma 4.1 and Lemma 4.2, we can get

\(- (\gamma - 1) \int \nabla \left( \frac{1}{(n+1)} \right) \cdot \partial^{[\alpha]} \nabla w \cdot \partial^{[\alpha]} w \, dx\)

\(- (c_0 + c_d - c_0) \int \nabla \left( \frac{1}{(n+1)} \right) \cdot \partial^{[\alpha]} \text{div} w \cdot \partial^{[\alpha]} w \, dx\)

\(+ 2\mu_r \int \frac{\nabla \times \partial^{[\alpha]} u \cdot \partial^{[\alpha]} w}{n+1} \, dx - \int u \partial^{[\alpha]} \nabla w \cdot \partial^{[\alpha]} w \, dx + \int f_3 \partial^{[\alpha]} w \, dx\)

\(- (\gamma - 1) \int \nabla \left( \frac{1}{(n+1)} \right) \cdot \partial^{[\alpha]} \nabla \delta \cdot \partial^{[\alpha]} m \, dx\)

\(- (\gamma - 1) \int \nabla \left( \frac{1}{(n+1)} \right) \cdot \partial^{[\alpha]} \text{div} \delta \cdot \partial^{[\alpha]} m \, dx\)

\(- \int u \partial^{[\alpha]} \nabla \delta \cdot \partial^{[\alpha]} m \, dx + \int f_4 \partial^{[\alpha]} m \, dx + \int \partial^{[\alpha]} \partial \partial^{[\alpha]} m \, dx\)

\(:= \sum_{j=1}^{17} I_j.\)

Noticing the similarity of the quadratically nonlinear terms in \(f_1, f_2, f_3,\) and \(f_4,\) we first give useful estimates which will play an important role in our calculation. By using Leibniz formula, Lemma 4.1 and Lemma 4.2, we can get

\[\left\| \partial^{[\alpha]}, \frac{\gamma - 1}{n+1} \text{div} |w| \right\| \leq C \sum_{|\alpha_1| + |\alpha_2| = |\alpha|} \left\| \partial^{\alpha_1} \left( \frac{\gamma - 1}{n+1} \right) \partial^{\alpha_2} \text{div} w \right\|\]

\[\leq C \| \nabla w \|_{H^s} \sum_{j=1}^{N} \left\| \nabla n \right\|_{H^{s-1}}^j,\]

\[\left\| \partial^{[\alpha]}, (n + 1) \text{div} |u| \right\| \leq C \| \nabla n \|_{L^\infty} \left\| \partial^{[\alpha]} \left( \frac{1}{n+1} \right) \nabla u \right\| + \left\| \text{div} u \right\|_{L^\infty} \left\| \partial^{[\alpha]} \left( \frac{1}{n+1} \right) \nabla n \right\|,\]

\[\left\| \partial^{[\alpha]}, \frac{\mu + \mu_p}{n + 1} \Delta |u| \right\| \leq C \sum_{|\alpha_1| + |\alpha_2| = |\alpha|} \left\| \partial^{\alpha_1} \left( \frac{\mu + \mu_p}{n + 1} \right) \partial^{\alpha_2} \Delta u \right\|\]

\[\leq C \| \nabla u \|_{H^s} \sum_{j=1}^{N} \left\| \nabla n \right\|_{H^{s-1}}^j,\]

\[\left\| \partial^{[\alpha]}, \frac{m + 1}{n + 1} \nabla |n| \right\| \leq C \sum_{|\alpha_1| + |\alpha_2| = |\alpha|} \left\| \partial^{\alpha_1} \left( \frac{1}{n+1} \right) \partial^{\alpha_2} \left( \frac{m + 1}{n+1} \nabla n \right) \right\|\]

\[\leq C \| \nabla n \|_{H^s} \| m \|_{H^s} \sum_{j=1}^{N} \left\| \nabla n \right\|_{H^{s-1}}^j,\]

\[\left\| \partial^{[\alpha]}, \frac{4\mu_p}{n + 1} \right\|_{H^s} \leq C \| \nabla w \|_{H^s} \sum_{j=1}^{N} \left\| \nabla n \right\|_{H^{s-1}}^j,\]

\[\left\| \partial^{[\alpha]}, \frac{2\mu_p}{n + 1} \nabla \times |w| \right\| \leq C \sum_{|\alpha_1| + |\alpha_2| = |\alpha|} \left\| \partial^{\alpha_1} \left( \frac{2\mu_p}{n + 1} \right) \partial^{\alpha_2} \nabla \times w \right\|\]
\[ \leq C \| \nabla w \|_{H^r} \sum_{j=1}^{N} \| \nabla n \|_{H^{r-1}}^2, \]
\[ \left\| \partial^{[\alpha]} \left( \frac{1}{n+1} (\text{div} u)^2 \right) \right\| \leq C \sum_{|\alpha_1| + |\alpha_2| = |\alpha|} \left\| \partial^{[\alpha_1]} \left( \frac{1}{n+1} \right) \partial^{[\alpha_2]} (\text{div} u)^2 \right\| + C \| \partial^{[\alpha]} (\text{div} u)^2 \| \]
\[ \leq C \| \nabla u \|_{H^r}^2 \sum_{j=1}^{N} \| \nabla n \|_{H^{r-1}}^j + C \| \partial^{[\alpha]} (\nabla w : \nabla w) \|, \]
\[ \left\| \partial^{[\alpha]} \left( \frac{1}{n+1} (\nabla w : \nabla w) \right) \right\| \leq C \sum_{|\alpha_1| + |\alpha_2| = |\alpha|} \left\| \partial^{[\alpha_1]} \left( \frac{1}{n+1} \right) \partial^{[\alpha_2]} (\nabla w : \nabla w) \right\| + C \| \partial^{[\alpha]} (\nabla w : \nabla w) \| \]
\[ \leq C \| \nabla w \|_{H^r}^2 \sum_{j=1}^{N} \| \nabla n \|_{H^{r-1}}^j + C \| \partial^{[\alpha]} (\nabla w : \nabla w) \|, \]
\[ \left\| \partial^{[\alpha]} \left( \frac{1}{n+1} (2 \text{rot} u - w)^2 \right) \right\| \leq C \sum_{|\alpha_1| + |\alpha_2| = |\alpha|} \left\| \partial^{[\alpha_1]} \left( \frac{1}{n+1} \right) \partial^{[\alpha_2]} (2 \text{rot} u - w)^2 \right\| + C \| \partial^{[\alpha]} (2 \text{rot} u - w)^2 \| \]
\[ \leq C (\| w \|_{H^r}^2 + \| \nabla u \|_{H^r}^2) \sum_{j=1}^{N} \| \nabla n \|_{H^{r-1}}^j + C \| \partial^{[\alpha]} (2 \text{rot} u - w)^2 \|. \]

Now, we will estimate \( I_j \) respectively, the following is a long and standard procedure to verify by using the Leibniz formula, Hölder and Sobolev inequalities [19, 20],

\[ I_1 = \int \left( - \left[ \partial^{[\alpha]}(n \cdot \nabla) u - \left[ \partial^{[\alpha]}(n + 1) \text{div} \right] u \right] \right. \partial^{[\alpha]} n \text{d}x \]
\[ \leq C \| \nabla n \|_{H^{r-1}}^2 \| u \|_{H^r}, \]
\[ I_2 = \int u \cdot \nabla \partial^{[\alpha]} n \partial^{[\alpha]} n \text{d}x - \int (n + 1) \text{div} \partial^{[\alpha]} u \partial^{[\alpha]} n \text{d}x \]
\[ \leq C \| \nabla n \|_{H^{r-1}}^2 \| u \|_{H^r}, \]
\[ I_3 + I_4 = - (\mu + \mu_r) \int \nabla \left( \frac{1}{(n+1)} \right) \cdot \partial^{[\alpha]} n \nabla u \cdot \partial^{[\alpha]} n \text{d}x \]
\[ - (\mu + \lambda - \mu_r) \int \nabla \left( \frac{1}{(n+1)} \right) \cdot \partial^{[\alpha]} \text{div} u \cdot \partial^{[\alpha]} n \text{d}x \]
\[ \leq C \| \nabla n \|_{H^{r-1}} \| \nabla u \|_{H^r}^2, \]
\[ I_5 + I_6 = 2 \mu_r \int \nabla \times \partial^{[\alpha]} n \nabla u + \partial^{[\alpha]} n \text{d}x - \int u \partial^{[\alpha]} n \nabla u \cdot \partial^{[\alpha]} n \text{d}x \]
\[ \leq C \| \nabla n \|_{H^{r-1}} \| \nabla u \|_{H^r} \| \nabla w \|_{H^r} + \| \nabla u \|_{H^r} \| u \|_{H^r}^2, \]
\[ I_7 = \int f_2 \partial^{(\alpha)} u^d \, dx \]
\[ \leq C \left( \| \nabla u \|^2_{H^s} + \| \nabla w \|^2_{H^s} \right) \| u \|_{H^s} \]
\[ + C \| u \|_{H^s} \left( \| \nabla w \|_{H^s} + \| \nabla u \|^2_{H^s} + \| \nabla n \|_{H^s} \| m \|_{H^s} \right) \sum_{j=1}^N \| \nabla n \|^2_{H^{s-1}}, \]

\[ I_8 + I_9 = - (\gamma - 1) \int \nabla \left( \frac{1}{n+1} \right) \cdot \partial^{(\alpha)} \nabla w \cdot \partial^{(\alpha)} w \, dx \]
\[ - (c_0 + c_1 - c_3) \int \nabla \left( \frac{1}{n+1} \right) \cdot \partial^{(\alpha)} \text{div} w \cdot \partial^{(\alpha)} w \, dx \]
\[ \leq C \| \nabla n \|_{H^{s-1}} \| \nabla w \|^2_{H^s}, \]

\[ I_{10} + I_{11} = 2 \mu \int \nabla \times \partial^{(\alpha)} u \cdot \partial^{(\alpha)} w \, dx - \int u \partial^{(\alpha)} \nabla w \cdot \partial^{(\alpha)} w \, dx \]
\[ \leq C \| \nabla n \|_{H^{s-1}} \| \nabla u \|_{H^s} \| \nabla w \|_{H^s} + C \| \nabla w \|_{H^s} \| u \|_{H^s} \| w \|_{H^s}, \]

\[ I_{12} = \int f_3 \partial^{(\alpha)} w \, dx \]
\[ \leq C \left( \| \nabla u \|^2_{H^s} + \| \nabla w \|^2_{H^s} \right) \| u \|_{H^s} \]
\[ + C \| u \|_{H^s} \left( \| \nabla w \|_{H^s} + \| \nabla u \|^2_{H^s} \right) \sum_{j=1}^N \| \nabla n \|^2_{H^{s-1}}, \]

\[ I_{13} + I_{14} + I_{15} = - (\gamma - 1) \int \nabla \left( \frac{1}{n+1} \right) \cdot \partial^{(\alpha)} \nabla m \cdot \partial^{(\alpha)} m \, dx \]
\[ - (\gamma - 1) \int \nabla \left( \frac{1}{n+1} \right) \cdot \partial^{(\alpha)} \text{div} m \cdot \partial^{(\alpha)} m \, dx \]
\[ - \int u \partial^{(\alpha)} \nabla m \cdot \partial^{(\alpha)} m \, dx \]
\[ \leq C \left( \| \nabla n \|_{H^{s-1}} + \| u \|_{H^s} \right) \| \nabla m \|_{H^s} \| m \|_{H^s}, \]

\[ I_{16} = \int f_4 \partial^{(\alpha)} m \, dx \]
\[ \leq C \| m \|_{H^s} \| \nabla w \|_{H^s} \sum_{j=1}^N \| \nabla n \|^2_{H^{s-1}} + C \| \nabla w \|_{H^s} \| m \|_{H^s} \| \nabla u \|_{H^{s-1}}, \]

\[ I_{17} = \int \partial^{(\alpha)} h \partial^{(\alpha)} m \, dx \]
\[ = \int \frac{1}{1+n} \partial^{(\alpha)} ((1+n)h) \partial^{(\alpha)} m \, dx \]
\[ + \sum_{|\alpha_1| + |\alpha_2| = |\alpha|} \partial^{(\alpha_1)} \left( \frac{1}{1+n} \right) \partial^{(\alpha_2)} ((1+n)h) \partial^{(\alpha)} m \, dx \]
\[ \leq C \| m \|_{H^s} \| \nabla u \|^2_{H^s} \left( 1 + \sum_{j=1}^N \| \nabla n \|^2_{H^{s-1}} \right). \]
Then
\[
\frac{1}{2} \frac{d}{dt} \left\{ \int |\partial_x^{\alpha} n|^2 \, dx + \int |\partial_x^{\alpha} u|^2 \, dx + \int |\partial_x^{\alpha} w|^2 \, dx + \int |\partial_x^{\alpha} m|^2 \, dx \right\} \\
+ (\mu + \mu_r) \int \frac{1}{n+1} |\nabla \partial_x^{\alpha} u|^2 \, dx + (\mu + \lambda - \mu_r) \int \frac{1}{n+1} |\text{div} \partial_x^{\alpha} u|^2 \, dx \\
+ (c_0 + c_d) \int \frac{1}{n+1} |\nabla \partial_x^{\alpha} w|^2 \, dx + (c_0 + c_d - c_0) \int \frac{1}{n+1} |\text{div} \partial_x^{\alpha} w|^2 \, dx \\
+ (\gamma - 1) \int \frac{1}{n+1} |\nabla \partial_x^{\alpha} m|^2 \, dx + (\gamma - 1) \int \frac{1}{n+1} |\text{div} \partial_x^{\alpha} m|^2 \, dx + 4\mu_r \int \frac{1}{n+1} |\partial_x^{\alpha} w|^2 \, dx \leq C\|\nabla n\|_{L^2}^2 \|u\|_{H^s} + C\|\nabla n\|_{L^2} \|\nabla u\|_{H^s}^2 + C\|\nabla n\|_{H^{s-1}} \|\nabla u\|_{H^s} \|\nabla w\|_{H^s} \\
+ \|\nabla u\|_{H^s} \|u\|_{H^s}^2 + C \left( \|\nabla u\|_{H^s}^2 + \|\nabla w\|_{H^s}^2 \right) \sum_{j=1}^N \|\nabla n\|_{H^{s-1}}^j \\
+ C\|\nabla n\|_{H^{s-1}} \|\nabla u\|_{H^s}^2 + C\|\nabla n\|_{H^{s-1}} \|\nabla u\|_{H^s} \|\nabla w\|_{H^s} \\
+ C\|\nabla w\|_{H^s} \|u\|_{H^s} \|w\|_{H^s} + C(\|\nabla n\|_{H^{s-1}} + \|u\|_{H^s}) \|\nabla m\|_{H^s} \|m\|_{H^s} \\
+ C\|m\|_{H^s} \|\nabla w\|_{H^s} \sum_{j=1}^N \|\nabla n\|_{H^{s-1}}^j + C\|\nabla w\|_{H^s} \|m\|_{H^s} \|\nabla u\|_{H^{s-1}} \\
+ C\|m\|_{H^s} \|\nabla u\|_{H^s}^2 (1 + \sum_{j=1}^N \|\nabla n\|_{H^{s-1}}^j).
\]

In the same way, we can obtain
\[
\frac{d}{dt} \sum_{|\ell| \leq N-1} \int \partial_x^{\ell} u \cdot \partial_x^{\ell} \nabla n \, dx + c\|\nabla n\|_{H^{s-1}}^2 \\
\leq C\|\nabla u, w, m\|_{H^s} + C\|\nabla u, w, m\|_{H^s} \left( \|\nabla n\|_{H^{s-1}}^2 + \|\nabla u, w, m\|_{H^s}^2 \right).
\]

Let $0 \leq \ell \leq N-1$, applying $\partial_x^{\ell} \nabla n$ to equation of $(2.1)_2$ and then multiplying the resultant equation by $\partial_x^{\ell} \nabla n$, taking integrations in $x$, using integration by parts and replacing $\partial_t n$ from $(2.1)_1$, we have
\[
\frac{d}{dt} \langle \partial_x^{\ell} u, \partial_x^{\ell} \nabla n \rangle + \gamma \|\partial_x^{\ell} \nabla n\|^2 \\
= - \int \partial_x^{\ell} \nabla m \cdot \partial_x^{\ell} \nabla n \, dx + \|\nabla \cdot \partial_x^{\ell} u\|^2 - \int \nabla \cdot \partial_x^{\ell} u \partial_x^{\ell} S_1 \, dx \\
+ (\mu + \mu_r) \int \Delta \partial_x^{\ell} u \cdot \partial_x^{\ell} \nabla n \, dx + (\mu + \lambda - \mu_r) \int \nabla \text{div} \partial_x^{\ell} u \cdot \partial_x^{\ell} \nabla n \, dx \\
+ 2\mu_r \int \partial_x^{\ell} \nabla \times w \cdot \partial_x^{\ell} \nabla n \, dx + \int \partial_x^{\ell} S_2 \cdot \partial_x^{\ell} \nabla n \, dx.
\]

Then, it follows from the Cauchy-Schwarz inequality that
\[
\frac{d}{dt} \langle \partial_x^{\ell} u, \partial_x^{\ell} \nabla n \rangle + \frac{\gamma}{2} \|\partial_x^{\ell} \nabla n\|^2 \\
\leq C \left( \|\partial_x^{\ell+1} [u, w, m]\|^2 + \|\partial_x^{\ell+2} u\|^2 \right) \|\nabla n\|_{H^s} + C \left( \|\partial_x^{\ell+1} S_1\|^2 + \|\partial_x^{\ell+2} S_2\|^2 \right) \|\nabla n\|_{H^s}.
\]
Direct calculation implies,

\[ S_2 = -u \cdot \nabla u - f(n)(\mu + \mu_r)\Delta u + (\mu + \lambda - \mu_r)\nabla \text{div} u + 2\mu_r \nabla \times w + h(n)\nabla n. \]

Hence

\[
\| \partial_x S_1 \|^2 + \| \partial_x S_2 \|^2 \leq C \| [n, u, m] \|_{L^\infty}^2 \left( \| \partial_x^{t+1} [n, u, w] \|^2 + \| \partial_x^{t+2} u \|^2 \right) \\
+ \| \partial_x^t [n, u, m] \|^2 \left( \| \nabla [n, u, w] \|_{L^\infty}^2 + \| \Delta u \|_{L^\infty}^2 \right) \\
+ \sum_{|\alpha_1| + |\alpha_2| = |\alpha|} \| \partial_x^{\alpha_1} \left( \frac{1}{1 + n} \right) \partial_x^{\alpha_2} (\Delta u + \nabla \text{div} u + \nabla \times w) \| \\
+ \sum_{|\alpha_1| + |\alpha_2| = |\alpha|} \| \partial_x^{\alpha_1} \left( \frac{1}{1 + n} \right) \partial_x^{\alpha_2} (m \nabla n) \| \\
\leq C \| [n, u, w, m] \|_{H^1} (\| \nabla n \|_{H^{-1}}^2 + \| \nabla [u, w, m] \|_{H^2}^2) \\
+ C (\| \nabla u \|_{H^s} + \| \nabla w \|_{H^{s-1}} + \| m \|_{H^{s-1}} \| \nabla n \|_{H^{s-1}}^s) \sum_{j=1}^{N} \| \nabla n \|_{H^{s-1}}^j .
\]

Then, by putting the above estimates together for properly chosen constants \( 0 < \eta \ll 1 \), one has

\[
\frac{d}{dt} \mathcal{E}_N(U(t)) + cD_N(U(t)) \leq CD_N(t) \sum_{k=1}^{N+4} \mathcal{E}_N^k(t).
\]

The proof of Theorem 4.3 is complete. \( \square \)

4.2. Large-time behavior. From now, we suppose that all conditions in Theorem 4.3 hold and \( U = [n, u, w, m] \) is the obtained solution to the Cauchy problem (4.1) and (4.2). In this subsection we devote ourselves to proving the time decay rate of the full energy \( \| U(t) \|_{H^s}^2 \) or equivalently \( \mathcal{E}_N(U(t)) \). For that purpose, define

\[
X(t) = \sup_{0 < \tau < t} (1 + s)^{\frac{3}{2}} \mathcal{E}_N(U(\tau)), \quad t \geq 0
\]

4.2.1. Time decay rate for the full instant energy functional.

**Lemma 4.4.** If \( \| U_0 \|_{L^1 \cap H^s} \) is sufficiently small, then

\[
\sup_{t \geq 0} X(t) \leq C \| U_0 \|_{L^1 \cap H^s}^2 . \tag{4.9}
\]

**Proof.** Under the smallness of \( \| U_0 \|_{H^s} \), (4.3) implies that

\[
\frac{d}{dt} \mathcal{E}_N(U(t)) + cD_N(U(t)) \leq 0, \tag{4.10}
\]

for any \( t \geq 0 \). We now apply the time-weighted energy estimate and iteration to the Lyapunov inequality (4.10). Let \( |\alpha| \geq 0 \). Multiplying (4.10) by \( (1 + t)^{\frac{3}{2} + \epsilon} \) and taking integration over \([0, t]\) give

\[
(1 + t)^{\frac{3}{2} + \epsilon} \mathcal{E}_N(U(t)) + c \int_0^t (1 + s)^{\frac{3}{2} + \epsilon} D_N(U(\tau)) d\tau \\
\leq \mathcal{E}_N(U_0) + \left( \frac{3}{2} + \epsilon \right) \int_0^t (1 + s)^{\epsilon + \frac{1}{2}} \mathcal{E}_N(U(\tau)) d\tau. \tag{4.11}
\]
Considering that
\[ \mathcal{E}_N(U(t)) \leq C \left( D_N(U(t)) + \|n, u, w, m\|^2 \right), \]
and
\[ \int_0^t D_N(U(\tau)) d\tau \leq C \mathcal{E}_N(U_0), \]
it follows that
\[
(1 + t)^{\frac{3}{2} + c} \mathcal{E}_N(U(t)) + c \int_0^t (1 + s)^{\frac{3}{2} + c} D_N(U(\tau)) d\tau \\
\leq C \mathcal{E}_N(U_0) + C \left( \frac{3}{2} + \epsilon \right) \int_0^t (1 + s)^{r + \frac{2}{3}} \left( \|n, u, w, m\|^2 \right) d\tau.
\]
It is straightforward to verify
\[ \| [S_1, S_2, S_3, S_4] \|_{L^4 \cap L^2} \leq C \mathcal{E}_N(U). \]
Due to Corollary 1 one has
\[
\| u(t) \| \leq C (1 + t)^{-\frac{2}{3}} \| [n_0, u_0, w_0, m_0] \|_{L^1 \cap L^2} \\
+ C \int_0^t (1 + t - s)^{-\frac{2}{3}} \| [S_1(\tau), S_2(\tau)] \|_{L^1 \cap L^2} d\tau \\
\leq C (1 + t)^{-\frac{2}{3}} \| [n_0, u_0] \|_{L^1 \cap L^2} + C \int_0^t (1 + t - s)^{-\frac{2}{3}} \mathcal{E}_N(U) d\tau \\
\leq C (1 + t)^{-\frac{2}{3}} \| [n_0, u_0, w_0, m_0] \|_{L^1 \cap L^2} + C \int_0^t (1 + t - s)^{-\frac{2}{3}} (1 + s)^{-\frac{1}{3}} d\tau X(t),
\]
\[
\| w(t) \| \leq C (1 + t)^{-\frac{2}{3}} \| [n_0, u_0, w_0] \|_{L^1 \cap L^2} \\
+ C \int_0^t (1 + t - s)^{-\frac{2}{3}} \| [S_1(\tau), S_2(\tau), S_3(\tau)] \|_{L^1 \cap L^2} d\tau \\
\leq C (1 + t)^{-\frac{2}{3}} \| [n_0, u_0, w_0] \|_{L^1 \cap L^2} + C \int_0^t (1 + t - s)^{-\frac{2}{3}} \mathcal{E}_N(U) d\tau \\
\leq C (1 + t)^{-\frac{2}{3}} \| [n_0, u_0, w_0] \|_{L^1 \cap L^2} + C (1 + t)^{-\frac{2}{3}} X(t), \quad (4.12)
\]
\[
\| m(t) \| \leq C (1 + t)^{-\frac{2}{3}} \| [n_0, u_0, w_0, m_0] \|_{L^1 \cap L^2} \\
+ C \int_0^t (1 + t - s)^{-\frac{2}{3}} \| [S_1(\tau), S_2(\tau), S_3(\tau), S_4(\tau)] \|_{L^1 \cap L^2} d\tau \\
\leq C (1 + t)^{-\frac{2}{3}} \| [n_0, u_0, w_0, m_0] \|_{L^1 \cap L^2} + C \int_0^t (1 + t - s)^{-\frac{2}{3}} \mathcal{E}_N(U) d\tau
\]
\[ \leq C(1 + t)^{-\frac{3}{4}} \| [n_0, u_0, w_0, m_0] \|_{L^1 \cap L^2} + C(1 + t)^{-\frac{3}{4}} X(t). \]

Therefore, one has
\[
\int_0^t (1 + s)^{\frac{1}{2} + \epsilon} \| [n, u, w, m] \|^2 \, d\tau \\
\leq C \int_0^t (1 + s)^{\frac{1}{2} + \epsilon} (1 + s)^{-\frac{3}{4}} \, d\tau \left( \| U_0 \|_{L^1 \cap L^2}^2 + X(t)^2 \right) \\
\leq C(1 + t)^{\epsilon} \left( \| U_0 \|_{L^1 \cap L^2}^2 + X(t)^2 \right).
\]

Substituting it into (4.11) gives
\[
(1 + t)^{\frac{1}{2} + \epsilon} \mathcal{E}_N(U(t)) + c \int_0^t (1 + s)^{\frac{1}{2} + \epsilon} D_N(U(\tau)) \, d\tau \\
\leq C \mathcal{E}_N(U_0) + C(1 + t)^{\epsilon} \left( \| [n_0, u_0, w_0, m_0] \|^2_{L^1 \cap L^2} + X(t)^2 \right),
\]
which imply
\[
X(t) \leq C \left( \| U_0 \|^2_{L^1 \cap L^2} + X(t)^2 \right).
\]

Since \( \| U_0 \|^2_{L^1 \cap L^2} \) is sufficiently small, \( X(t) \) is bounded uniformly in time and also (4.9) holds true. This completes the proof to Lemma 4.4.

**Corollary 2.** If \( \| U_0 \|_{L^1 \cap H^s} \) is sufficiently small, then
\[
\| w \| \leq C(1 + t)^{-\frac{3}{4}} \| U_0 \|_{L^1 \cap H^s}.
\]

**Proof.** This can be proved by using Lemma 4.4 and (4.12). We omit the details.

### 4.3. Proof of Theorem 1.1.
Combining Theorem 2.1, Theorem 4.3 and Corollary 2 give Theorem 1.1.

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