Singular Limit for the Compressible Navier–Stokes Equations with the Hard Sphere Pressure Law on Expanding Domains

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Abstract. The article is devoted to the asymptotic limit of the compressible Navier–Stokes system with a pressure obeying a hard–sphere equation of state on a domain expanding to the whole physical space $\mathbb{R}^3$. Under the assumptions that acoustic waves generated in the case of ill-prepared data do not reach the boundary of the expanding domain in the given time interval and a certain relation between the Reynolds and Mach numbers and the radius of the expanding domain we prove that the target system is the incompressible Euler system on $\mathbb{R}^3$. We also provide an estimate of the rate of convergence expressed in terms of characteristic numbers and the radius of domains.

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1. Introduction

The main goal of many works devoted to the study of the asymptotic analysis of various physical system is to derive “a simplified system of equations” which can be solved analytically or numerically. For such studies we can refer e.g. to work of Zeytonian [42]. They obtain the simplified system formally via a scaling argument. The mathematical justification of the derivation of such models goes back to the seminal work of Klainerman and Majda [25]. Studying various types of singular limits allows to eliminate unimportant or unwanted modes of motion as a consequence of scaling and asymptotic analysis, cf. the works of Masmoudi [31,32], of Wang and Jiang [41], and of Feireisl and Novotný [18] among others.

Motivated by Feireisl et al. [17] and Kelliher, Lopes and Nussenweig-Lopes [24], where authors consider the inviscid limit for the Navier–Stokes system on an expanding domain in the isentropic regime for the compressible, incompressible case respectively, we will consider the compressible Navier–Stokes equations with the hard pressure.

Let us begin with the introduction of the problem. For $T > 0$ and $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ a bounded domain we consider the compressible Navier–Stokes system in the time–space cylinder $Q_T = (0,T) \times \Omega$

$$\partial_t \rho + \text{div}(\rho u) = 0 \quad \text{in } Q_T,$$

$$\partial_t (\rho u) + \text{div}(\rho u \otimes u - S(\nabla u)) + \nabla p(\rho) = \rho f \quad \text{in } Q_T,$$

$$\rho(0, \cdot) = \rho_0, \quad u(0, \cdot) = u_0 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } (0, T) \times \partial \Omega$$

The article was inspired by several discussions with our colleague and friend Antonín Novotný. We never forget him.

This article is part of the topical collection “In memory of Antonín Novotný” edited by Eduard Feireisl, Paolo Galdi, and Milan Pokorný.
for the unknown density $\rho : Q_T \to \mathbb{R}$ and the velocity $u : Q_T \to \mathbb{R}^d$. The structural relation between the pressure $p$ and the density $\rho$ is discussed later. The external forces are denoted by $f$. Here, $\mathcal{S}(\nabla u)$ denotes the Newtonian stress tensor defined as
\[
\mathcal{S}(\nabla u) = \mu^S \left( \frac{1}{2} (\nabla u + (\nabla u)^	op) - \frac{1}{d} \text{div} \mathbb{I}_d \right) + \mu^B \text{div} \mathbb{I}_d,
\]
where $\mu^S > 0$ and $\mu^B \geq 0$ are shear and bulk viscosity coefficients and $\mathbb{I}_d$ stands for the $d \times d$ identity matrix. Furthermore, the velocity gradient $\nabla u$ and the divergence of a $d \times d$ matrix valued function $\text{div} \mathbb{A}$ are defined as
\[
\nabla u = (\partial_{x_j} u_i)_{i,j=1}^d, \quad (\text{div} \mathbb{A})_i = \sum_{j=1}^d \partial_{x_j} \mathbb{A}_{i,j}, \quad i = 1, \ldots, d.
\]

Before we give the precise definition of a weak solution to (1) we introduce hypotheses on the pressure. The relation between the pressure $p$ and the density $\rho$ of the fluid is so-called hard–sphere equation of state in the interval $[0, \bar{\rho}]$.

\[
p \in C^1([0, \bar{\rho}]), \quad p(0) = 0, \quad p' > 0 \quad \text{on } (0, \bar{\rho}), \quad \lim_{s \to \rho_+} p(s) = +\infty.
\]

We also define the pressure potential $P \in C^1([0, \bar{\rho}])$ as
\[
P(s) = s \int_{\frac{\rho}{2}}^s \frac{p(z)}{z^2}
\]
and note that
\[
P'(s) - p(s) = p(s), \quad P''(s) = \frac{p'(s)}{s} \quad \text{for } s \in [0, \bar{\rho}).
\]

We are interested in the well-accepted Carnahan-Starling equation of the state characterized by the properties in (4) and (15). As explained in e.g. [38], it is a suitable approximate equation of state for the fluid phase of the hard–sphere model. The derivation of such model was performed from a quadratic relation between the integer portions of the virial coefficients and their orders. This model is used for the study of the behavior of dense gases and liquids. The interested reader can find more details regarding the model or its corrections (Percus-Yevick equation, Kolafa correction, Liu correction) in [8,21,22,26]. Singular pressure laws of similar type appeared in modeling of various phenomena, including the collective motion, see [11,12,30], and the traffic flow, see [1,2].

Asymptotic limits for problems involving a singular pressure law were studied e.g in [3–6,35].

\[1\] Several works are devoted to the study of macroscopic models for heterogeneous media like mixtures, suspensions or crowds, in dense regimes. These regimes show interesting behaviors such as transition phases with congestion (also called jamming for granular flows) and non-local (in time and/or in space) effects which are both due to a physical packing constraint, that is the finite size of the microscopic components. At the macroscopic scale this limitation corresponds to a maximal density constraint $\rho \leq \rho^*$. A very challenging question in physics and mathematics is to model and analyze the behavior in congested domains $\rho = \rho^*$ and close to a transition phase $\rho^* - \varepsilon < \rho < \rho^*$. Two different ways are generally considered in the literature to model congestion phenomena at the macroscopic level. The first approach, called hard approach, consists in coupling compressible dynamics in the free domain $\{ \rho < \rho^* \}$, with incompressible dynamics in the congested domain $\{ \rho = \rho^* \}$. Associated to the incompressibility constraint on the velocity field, an additional potential (seen as the Lagrange multiplier) is activated in the congested regions. The second approach is called soft approach, prevents the apparition of congested phases by introducing in the compressible dynamics repulsive forces which become singular as $\rho$ approaches $\rho^*$. A first generic hard congestion model is derived in [7] by Bouchut et al. from one-dimensional two-phase gas/liquid flows for more details on the model see e.g. [6, Section 2].

The system proposed by Bouchut was studied theoretically by Berthelin in [1,12]. Degond et al. applied numerical discretization, see [11,12], to solve the soft congestion system. Let us also mention on the subject the study [5] which addresses the issue of the creation of congested zones in 1D and highlights the multi-scale nature of the problem.

\[2\] In paper [6] authors study the compressible Brinkman equations, where maximal packing is encoded in a singular pressure and a singular bulk viscosity. It was shown that the global weak solutions converge (up to a subsequence) to global weak solutions of the two-phase compressible/incompressible Brinkman equations with respect to a parameter $\varepsilon$ which measures effects close to the maximal packing value. Depending on the importance of the bulk viscosity with respect to the pressure in the dense regimes, memory effects are activated or not at the limit in the congested (incompressible) domain. At the limit...
The study of existence of weak solutions to the compressible Navier–Stokes equations in the isentropic setting on a bounded domain goes back to the seminal work by Lions [27] and the later improvement by Feireisl et al. [20]. Concerning systems with a hard pressure law in a bounded domain with no–slip boundary conditions, the existence of weak solutions was shown by Feireisl et al. [15] and Feireisl and Zhang [19, Section 3]. Recently, the existence of weak solutions to compressible Navier–Stokes equations with the hard–sphere pressure was investigated by Choe et al. [9] for the case with a general inflow/outflow and in an exterior domain by Neˇcasov´a et al. [33]. Weak–strong uniqueness for the compressible Navier–Stokes equations with the hard pressure in periodic spatial domains was shown by Feireisl et al. [16].

This paper is motivated by the result in [17] concerning the assymptotic limits for the compressible Navier–Stokes system in the isentropic setting on an expanding domain with ill–prepared initial data. We point out that the presence of the hard sphere pressure requires a more involved form of the relative entropy inequality than the one used in [17]. Our second motivation comes from [16], where the modification of the relative entropy inequality involving the operator \( \nabla \Delta^{-1} \), origanally derived for the isentropic regime by Feireisl et al. [14], was derived in the case of periodic boundary conditions. We emphasize that our version of the relative energy inequality involves the Bogovskii operator. Since for the singular limit problem we consider a negative power of a scaling parameter in the relative entropy, see (49), we obtain terms with positive power of the scaling parameter during handling of the reminder terms of the relative entropy inequality. These terms then vanish in the limit. This fact avoids the necessity of using the adjoint operator to the Bogovskii operator, which causes technical difficulties also mentioned in [16]. The aim of this paper is twofold. First, we want to derive the relative entropy inequality for the Navier–Stokes problem in the setting with the hard–sphere pressure and no–slip boundary conditions. Second, we want to study the asymptotic limit of the compressible Navier–Stokes system with the pressure obeying a hard–sphere equation of state on a domain expanding to the whole physical space \( \mathbb{R}^3 \).

The outline of the paper is as follows. Section 2 deals with the description of the problem, the meaning of weak solution to the problem, the statement of the main result of the paper and the derivation of the relative entropy inequality. Section 3 is devoted to the study of the assymptotic limit in the primitive compressible Navier–Stokes problem (1) yielding the Euler incompressible equations in the whole physical space \( \mathbb{R}^3 \) as the target system. Finally, in the “Appendix” we deal with renormalized solutions of the continuity equation adopted for a function \( b \) satisfying (11) and (16).

2. Definition of Weak Solutions and Preliminaries

We introduce the definition of weak solutions as was done in [16].

**Definition 2.1.** Let the following hypotheses be imposed on the initial data

\[
\rho_0 \in [0, \bar{\rho}) \text{ a.e. in } \Omega, \quad \int_{\Omega} P(\rho_0) < \infty, \quad \int_{\Omega} \rho_0|u_0|^2 < \infty. \tag{7}
\]

A pair \((\rho, u)\) is said to be a finite–energy weak solution to (1) if

- \( \rho \in [0, \bar{\rho}) \text{ a.e. in } Q_T, \rho \in C_w([0, T]; L^\gamma(\Omega)) \) for any \( \gamma > 1, p(\rho) \in L^1(Q_T), \)
- \( u \in L^2(0, T; W_{0, \text{div}}^{1,2}(\Omega)^d), \rho u \in C_w([0, T]; L^2(\Omega)^d), \rho|u|^2 \in L^\infty(0, T; L^1(\Omega)). \)
- The continuity equation

\[
\int_0^T \int_{\Omega} \rho \partial_t \phi + \rho u \cdot \nabla \phi = \int_{\Omega} \rho(\tau, \cdot) \phi(\tau, \cdot) - \int_{\Omega} \rho_0 \phi(0, \cdot) \tag{8}
\]

Footnote 2 continued

on the hard congestion system, authors cover in particular the two cases introduced in [35] and [36] where pressure effects or memory effects are activated. The first justification of the link between a soft congestion system and a hard congestion system is given in [4] for the one-dimensional case. More details concerning soft and hard congestion problem can be found in [6], Sect. 2. In the paper [3] the singular limits of Euler equations with hard-sphere pressure towards the constrained Euler equations for smooth solutions are investigated.
is satisfied for any $\tau \in (0, T)$ and any test function $\varphi \in C^1([0, T]; C^1(\overline{\Omega}))$.

- The momentum equation
\[
\int_0^T \int_\Omega \rho u \partial_t \varphi + (\rho u \otimes u - \mathcal{S}(\nabla u)) \cdot \nabla \varphi + p(\rho) \text{div} \varphi \, d\tau = -\int_0^T \int_\Omega \rho f \cdot \varphi + \int_\Omega \rho(\tau, \cdot)\varphi(\tau, \cdot) - \int_\Omega \rho_0 \varphi(0, \cdot)
\]  
\tag{9}
\]
is satisfied for any $\tau \in (0, T)$ and any test function $\varphi \in C^1([0, T] \times \Omega)^d$.

- The continuity equation holds in the sense of renormalized solutions
\[
\int_0^T \int_\Omega b(\rho) \partial_t \psi + b(\rho) u \cdot \nabla \psi + (b'(\rho) \rho - b(\rho)) \text{div} \psi = 0
\]  
\tag{10}
\]
for any test function $\psi \in C^\infty_c(Q_T)$ and any function $b \in C^1([0, \bar{\rho}))$ satisfying
\[
|b'(s)|^2 + |b(s)|^2 \leq c(1 + p(s)) \text{ for some } c > 0 \text{ and any } s \in [0, \bar{\rho}). \tag{11}
\]

- The energy inequality holds for a.a. $\tau \in (0, T)$:
\[
\int_\Omega \left(\frac{1}{2} \rho |u|^2 + P(\rho)\right)(\tau) + \int_0^\tau \int_\Omega \mathcal{S}(\nabla u) \cdot \nabla u \, d\tau \leq \int_\Omega \left(\frac{1}{2} \rho_0 |u_0|^2 + P(\rho_0)\right) + \int_0^\tau \int_\Omega \rho f \cdot u \, d\tau. \tag{12}
\]

**Remark 2.1.** If the class of admissible test functions in (9) is reduced to $C^\infty_c(Q_T)^3$ one can conclude that $\nabla p(\rho) \in X^*$, where $X = L^2(0, T; W^{1,2}_0(\Omega)) \cap W^{1,2}(0, T; L^3(\Omega))$. Indeed, the regularity of $\rho, u$ and $\rho u$ specified in Definition 2.1 and
\[
\rho u \otimes u \in L^\infty(0, T; L^1(\Omega)) \cap L^1(0, T; L^3(\Omega)) \subset L^2(Q_T).
\]
where also the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)^3$ was applied, imply the regularity of the distribution $\nabla p$ provided that (9) is used for the expression of the duality $\int_0^T \langle \nabla p, \varphi \rangle$. Using the regularity of $\rho, u, \rho u, \rho_0$ and $\nabla p(\rho)$ and a density argument one can alternatively formulate the momentum equation
\[
\int_0^T \int_\Omega \rho u \partial_t \varphi + (\rho u \otimes u - \mathcal{S}(\nabla u)) \cdot \nabla \varphi - \int_0^T \langle \nabla p(\rho), \varphi \rangle \, d\tau = -\int_0^T \int_\Omega \rho f \cdot \varphi + \int_\Omega \rho(\tau, \cdot)\varphi(\tau, \cdot) - \int_\Omega \rho_0 \varphi(0, \cdot)
\]  
\tag{13}
\]
for any $\tau \in (0, T)$ and any test function $\varphi \in X$.

For the purposes of this paper we define the relative entropy functional as
\[
\mathcal{E}(\rho, u|r, U)(t) = \int_\Omega \left(\frac{1}{2} \rho |u - U|^2 + P(\rho) - P(r) - P'(r)(\rho - r)\right)(t, x) \, dx. \tag{14}
\]

The ensuing theorem deals with the global in time existence of a finite–energy weak solution to (1) that satisfies a version of so called relative entropy inequality.

**Theorem 2.1.** Suppose $T > 0$ and $\Omega \subset \mathbb{R}^d$ $d = 2, 3$ be a bounded domain with $C^{2, \nu}$–boundary for some $\nu > 0$. Let the pressure functional satisfy besides (4) also the constraint
\[
\lim_{\rho \to \bar{\rho}^-} p(\rho)(\bar{\rho} - \rho)^3 > 0 \text{ for some } \beta > \frac{5}{2}. \tag{15}
\]

\footnote{In fact for $\Omega \subset \mathbb{R}^2$, $W^{1,2}(\Omega)$ is embedded into $L^q(\Omega)$ for any $q \in [1, \infty)$ but the better integrability will not bring any benefits in further analysis. For the sake of clarity we will not distinguish between the 2d and 3d case.}
and the initial data satisfy \((7)\). Let \(b \in C^1([0,\bar{\rho}))\) be a nonnegative function such that
\[
|b'|^{\frac{1}{2}} + |b|^\frac{1}{2} \leq c(1 + p) \text{ on } [0,\bar{\rho}) \text{ for some } c > 0.
\]

(16)

Then there exists a finite–energy weak solution \((\rho, u)\) to (1) in the sense of Definition 2.1. Moreover, if \((r, U) \in C^1(Q_T) \times C^1([0, T]; C^2(\Omega)^d)\) with \(U = 0\) on \((0, T) \times \partial \Omega\) satisfies
\[
0 < \inf_{Q_T} r \leq \sup_{Q_T} r < \bar{\rho}
\]
then the relative entropy inequality
\[
\mathcal{E}(\rho, u|r, U) + \int_0^T \int_\Omega \left( \mathcal{S}(\nabla u) - \mathcal{S}(\nabla U) \right) \cdot \nabla (u - U) + \int_0^T \int_\Omega \rho p(b) \leq \mathcal{E}(\rho_0, u_0|r(0, \cdot), U(0, \cdot)) + \int_0^T \mathcal{R}_1(t) + \mathcal{R}_2(t) \, dt + \mathcal{R}_3(\tau)
\]
holds for a.a. \(\tau \in (0, T)\). The remainder terms \(\mathcal{R}_i, i = 1, 2, 3\) read
\[
\mathcal{R}_1(t) = \int_\Omega \rho (\partial_t U + (u \cdot \nabla) U) \cdot (U - u) + \int_\Omega \nabla \mathcal{S}(\nabla U) \cdot (U - u) + \int_\Omega \rho f \cdot (u - U)
\]
\[
+ \int_\Omega \left( (r - \rho) \partial_t P'(r) + (ru - \rho u) \cdot \nabla P'(r) \right) + \int_\Omega \text{div}_B(p(r) - p(r)))
\]
\[
\mathcal{R}_2(t) = \int_\Omega \rho p(b) - \int_\Omega \rho u \cdot \nabla B \left( b - \langle b \rangle \right) + \int_\Omega \nabla \mathcal{S}(\nabla u) \cdot \nabla B \left( b - \langle b \rangle \right)
\]
\[
- \int_\Omega \rho f \cdot \mathcal{B} \left( b - \langle b \rangle \right) + \int_\Omega \text{div} \mathcal{B} \left( \text{div} (b u) - \langle \text{div} (b u) \rangle \right)
\]
\[
+ \int_\Omega \rho u \cdot \mathcal{B} \left( \left( b' \rho - b \rho \right) \text{div} u - \langle \left( b' \rho - b \rho \right) \text{div} u \rangle \right)
\]
\[
\mathcal{R}_3(\tau) = \int_\Omega \rho u \cdot \mathcal{B} \left( b - \langle b \rangle \right) \left( \tau, \cdot \right) - \int_\Omega \rho_0 u_0 \cdot \mathcal{B} \left( b_0 - \langle b_0 \rangle \right)
\]

We note that \(\mathcal{B}\) stands for the Bogovskii operator and the notation \(\langle g \rangle = \langle g, 1 \rangle\) is used whenever \(g\) belongs to a dual space to a Banach space containing the element 1. In particular, if \(g \in L^p(\Omega)\) then \(\langle g \rangle = \dfrac{1}{|\Omega|} \int_\Omega g\). Next, we comment on the assumption concerning the monotone behavior in the neighborhood of the possible singular point \(\bar{\rho}\) of the renormalizing function and its derivative in (16). This rather technical assumption allows us to prove the existence of a renormalized solution to the continuity equation via the renormalizing procedure developed by DiPerna and Lions in [13] for a certain class of renormalizing functions, see the proof of Lemma 4.1. It is necessitated by a particular choice of the renormalizing function in the proof of Theorem 3.1, see (76), which is not covered by the known results on the existence of renormalized solutions, cf. [34, Lemma 6.9].

Proof of Theorem 2.1. The existence of a global in time weak solution to (1) in the sense of Definition 2.1 can be shown by employing the standard approximation scheme for the compressible Navier–Stokes system. One adopts the regularization of the pressure from [19]. The existence proof in the latter reference relies on a constraint (15) for \(\beta \geq 3\). Later it turned out that the exponent \(\beta > \frac{5}{2}\) in (15) is sufficient for the existence proof, cf. [15].

Therefore we now concentrate on proving the relative entropy inequality. Following the arguments employed in [14] we obtain that
\[
\mathcal{E}(\rho, u|r, U)(\tau) + \int_0^T \int_\Omega \mathcal{S}(\nabla (u - U)) \cdot \nabla (u - U) \leq \mathcal{E}(\rho_0, u_0|r(0, \cdot), U(0, \cdot)) + \int_0^T \mathcal{R}_1.
\]

(17)

Moreover, it was derived in [16] that
\[
|b(s)| + |b'(s)| \leq c(1 + p(s))^\frac{1}{2} \leq c(1 + P(s))^{\frac{1}{2}},
\]

(18)
provided that (5) and (15) are taken into account. The latter inequalities in combination with (7) and (16) yield

\begin{align}
&b(\rho_0), b'(\rho_0) \in L^2(\Omega), \\
&b(\rho), b'(\rho) \in L^\infty(0, T; L^{\frac{2}{3}}(\Omega)) \cap L^{\frac{2}{3}}(Q_T).
\end{align}

Having (17) at hand we are left with the proof of

\begin{equation}
\int_0^T \int_\Omega p(\rho)b(\rho) = \int_0^T \mathcal{R}_2(t) \, dt + \mathcal{R}_3(\tau).
\end{equation}

The idea of proving this identity is to employ \(\mathcal{B}(b(\rho) - \langle b(\rho) \rangle)\) as a test function in the momentum equation. Unfortunately, the low regularity of the time derivative of the latter function excludes this possibility. Indeed, expressing the time derivative of \(\mathcal{B}(b(\rho) - \langle b(\rho) \rangle)\) in terms of the renormalized continuity equation with the function \(b\) we obtain

\[
\partial_t \mathcal{B}(b(\rho) - \langle b(\rho) \rangle) = \mathcal{B}(\partial_t (b(\rho) - \langle b(\rho) \rangle)) = \mathcal{B}(\text{div}(b(\rho)u) - \langle \text{div}(b(\rho)u) \rangle)
\]

\[
+ \mathcal{B}((b'(\rho)\rho - b(\rho))\text{div}u - \langle (b'(\rho)\rho - b(\rho))\text{div}u \rangle).
\]

Taking into consideration \(b'(\rho) \in L^{\frac{2}{3}}(Q_T)\), \(\text{div}u \in L^2(Q_T)\) and the continuity of \(\mathcal{B}\) from \(L^{\frac{10}{3}}(\Omega)\) to \(W^{1, \frac{5}{2}}(\Omega)^d\) it follows that the last term in the latter identity belongs to \(L^{\frac{10}{3}}(0, T; W^{1, \frac{5}{2}}(\Omega)^d)\), which does not imply the regularity required for the time derivative of a test function in (9). In order to circumvent this obstacle, we consider a suitable regularization of the function \(b\). Namely, we define for \(\alpha \in (0, \alpha_0)\) with \(\alpha_0\) from (16) the regularization \(b_\alpha\) of \(b\) as

\[
b_\alpha(s) = \begin{cases} b(s) & s \in [0, \bar{\rho} - \alpha], \\ b(\bar{\rho} - \alpha) & s \in (\bar{\rho} - \alpha, \bar{\rho}) \end{cases}
\]

Next, considering the function \(\varphi = \mathcal{B}(b_\alpha(\rho) - \langle b_\alpha(\rho) \rangle)\) we immediately deduce that \(\varphi \in L^\infty(0, T; W^{1, q}_0(\Omega)^d)\) for any \(q \in [1, \infty)\), cf. Lemma 4.2. Moreover, using the linearity of \(\mathcal{B}\) we get

\[
\partial_t \varphi = \mathcal{B}(\partial_t b_\alpha(\rho) - \langle \partial_t b_\alpha(\rho) \rangle).
\]

Employing (112) we have

\[
\partial_t \varphi = \mathcal{B}(\text{div}(b_\alpha(\rho)u) + (b'_\alpha(\rho)\rho - b_\alpha(\rho))\text{div}u - \langle (b'_\alpha(\rho)\rho - b_\alpha(\rho))\text{div}u \rangle).
\]

We notice that \(\langle \text{div}(b_\alpha(\rho)u) \rangle = 0\) provided \(\text{div}(b_\alpha(\rho)u)\) is understood as an element of \(L^2(0, T; (W^{1, \frac{5}{2}}(\Omega))^d)\).

To show this fact we consider a sequence \(\{S_\varepsilon(b_\alpha(\rho)u)\}\), where \(S_\varepsilon\) is a mollifier with respect to the space variables. Applying properties of mollifiers, the facts that \(b_\alpha(\rho) \in L^\infty(Q_T)\), \(u \in L^2(0, T; L^6(\Omega))\) and the Lebesgue dominated convergence theorem it follows that

\[
S_\varepsilon(b_\alpha(\rho)u) \rightarrow b_\alpha(\rho)u \text{ in } L^2(0, T; L^6(\Omega)).
\]

Moreover, as \(S_\varepsilon(b_\alpha(\rho)u)\) possesses the vanishing trace on \(\partial \Omega\), which is not clear for \(b_\alpha(\rho)u\), there is a representation for the duality

\[
\int_0^T \langle \text{div}(S_\varepsilon(b_\alpha(\rho)u)), \phi \rangle_{(W^{1, \frac{5}{2}}(\Omega))', W^{1, \frac{5}{2}}(\Omega)} = - \int_0^T \int_\Omega S_\varepsilon(b_\alpha(\rho)u) \cdot \nabla \phi
\]

for any \(\phi \in L^2(0, T; W^{1, \frac{5}{2}}(\Omega))\).

Accordingly, by (23) we conclude that \(\{\text{div}(S_\varepsilon(b_\alpha(\rho)u))\}\) is a Cauchy sequence in \(L^2(0, T; (W^{1, \frac{5}{2}}(\Omega))^d)\) implying \(\text{div}(b_\alpha(\rho)u) \in L^2(0, T; (W^{1, \frac{5}{2}}(\Omega))^d)\) and the representation

\[
\int_0^T \langle \text{div}(b_\alpha(\rho)u), \phi \rangle_{(W^{1, \frac{5}{2}}(\Omega))', W^{1, \frac{5}{2}}(\Omega)} = - \int_0^T \int_\Omega b_\alpha(\rho)u \cdot \nabla \phi \text{ for any } \phi \in L^2(0, T; W^{1, \frac{5}{2}}(\Omega)).
\]

The suitable choice of an arbitrary \(\phi\) independent of the space variable yields \(\langle \text{div}(b_\alpha(\rho)u) \rangle = 0\) a.e. in \((0, T)\).
Taking into account \( b_\alpha(\rho), b'_\alpha(\rho), \rho \in L^\infty(Q_T) \) and \( u \in L^2(0, T; L^6(\Omega)) \) it follows that
\[
\text{div}(b_\alpha(\rho)u) + (b'_\alpha(\rho) - b_\alpha(\rho)) \text{div}u - ((b'_\alpha(\rho) - b_\alpha(\rho)) \text{div}u) \in L^2(0, T; (W^{1, \frac{6}{3}}(\Omega))').
\]
Hence using Lemma 4.2 it follows that \( \partial_t \varphi \in L^2(0, T; L^6(\Omega)^d) \). Consequently, we obtain that \( \varphi \) is an admissible test function in (13). Employing \( \varphi \) as a test function in (13) we infer
\[
\int_0^T \int_\Omega p(\rho) b_\alpha(\rho) - I_1^\alpha = - \int_0^T \langle \nabla p(\rho), B(b_\alpha(\rho) - \langle b_\alpha(\rho) \rangle) \rangle = \sum_{j=2}^6 I_j^\alpha + J^\alpha(\tau),
\] (25)
where
\[
I_1^\alpha = \int_0^T \int_\Omega p(\rho) \langle b_\alpha(\rho) \rangle,
I_2^\alpha = - \int_0^T \int_\Omega \rho \otimes u \cdot \nabla B(b_\alpha(\rho) - \langle b_\alpha(\rho) \rangle),
I_3^\alpha = \int_0^T \int_\Omega \mathbb{S}(\nabla u) \cdot \nabla B(b_\alpha(\rho) - \langle b_\alpha(\rho) \rangle),
I_4^\alpha = - \int_0^T \int_\Omega \rho f \cdot B(b_\alpha(\rho) - \langle b_\alpha(\rho) \rangle),
I_5^\alpha = \int_0^T \int_\Omega \rho u \cdot B(\text{div}(b_\alpha(\rho)u)),
I_6^\alpha = \int_0^T \int_\Omega \rho u \cdot B((b'_\alpha(\rho) - b_\alpha(\rho)) \text{div}u - ((b'_\alpha(\rho) - b_\alpha(\rho)) \text{div}u)),
J^\alpha(\tau) = \int_\Omega \rho u \cdot B(b_\alpha(\rho) - \langle b_\alpha(\rho) \rangle)(\tau, \cdot) - \int_\Omega \rho_0 u_0 \cdot B(b(\rho_0) - \langle b(\rho_0) \rangle).
\]
(26)
The next task is the limit passage \( \alpha \to 0_+ \) in (25). To this end, we use the following convergences as \( \alpha \to 0_+ \)
\[
\begin{align*}
b_\alpha(\rho) & \to b(\rho) \quad \text{in } L^\frac{5}{3}(Q_T) \text{ and a.e. in } Q_T, \\
b'_\alpha(\rho) & \to b'(\rho) \quad \text{in } L^\frac{5}{3}(Q_T), \\
b_\alpha(\rho_0) & \to b(\rho) \quad \text{in } L^\frac{5}{3}(\Omega), \\
b'_\alpha(\rho_0) & \to b'(\rho) \quad \text{in } L^\frac{5}{3}(\Omega).
\end{align*}
\] (27)
We note that the latter convergences follow by the definition of \( b_\alpha \) in (21), the assumption that \( b \in C^1([0, \bar{\rho}]), (19) \) and the Lebesgue dominated convergence theorem. Next, taking into account that \( u \in L^2(0, T; L^6(\Omega)^d) \) we get as \( \alpha \to 0_+ \)
\[
\begin{align*}
b_\alpha(\rho)u & \to b(\rho)u \quad \text{in } L^{\frac{10}{3}}(0, T; L^{\frac{6}{3}}(\Omega)^d), \\
\text{div}(b_\alpha(\rho)u) & \to \text{div}(b(\rho)u) \quad \text{in } L^{10}(0, T; (W^{1, \frac{6}{3}}(\Omega))'), \\
(b'_\alpha(\rho) - b_\alpha(\rho)) \text{div}u & \to (b'(\rho) - b(\rho)) \text{div}u \quad \text{in } L^{\frac{10}{3}}(Q_T).
\end{align*}
\] (28)
Let us point out that (28)2 follows from (28)1 by repeating the procedure, which leads to (24). Moreover, by Lemma 4.2 we conclude
\[
\begin{align*}
\mathcal{B}([b_\alpha(\rho)]_0) & \to \mathcal{B}([b(\rho)]_0) \quad \text{in } L^\frac{5}{3}(0, T; W^{1, \frac{5}{3}}(\Omega)^d), \\
\mathcal{B}(\text{div}(b_\alpha(\rho)u)) & \to \mathcal{B}(\text{div}(b(\rho)u)) \quad \text{in } L^{10}(0, T; (W^{1, \frac{6}{3}}(\Omega))'), \\
\mathcal{B} \left( ([b'_\alpha(\rho) - b_\alpha(\rho)] \text{div}u)_0 \right) & \to \mathcal{B} \left( ([b'(\rho) - b(\rho)] \text{div}u)_0 \right) \quad \text{in } L^{\frac{10}{3}}(0, T; W^{1, \frac{6}{3}}(\Omega)^d),
\end{align*}
\] (29)
where the notation \( [g]_0 = g - \langle g \rangle \) was used. By (27)1 and the properties of \( b \) that allow to consider \( |\Omega|^{-1} p(\rho)\|b(\rho)\|_{L^\infty(0,T;L^1(\Omega))} \) as an integrable majorant for \( p(\rho)\langle b(\rho) \rangle \) we employ the Lebesgue dominated
convergence theorem for the limit passage $\alpha \to 0_+$ in $I_6^\alpha$. We use the convergence (29) for the passage to the limit $\alpha \to 0_+$ in $I_6^\alpha$, $I_6^3$, $I_6^4$. Moreover, this convergence implies $B(b_\alpha(\rho) - (b_\alpha(\rho)))(\tau) \in W_{0, T}^{1, 2}(\Omega)^d$ for a.a. $\tau \in (0, T)$ allowing for the passage $\alpha \to 0_+$ in $J^\alpha(\tau)$. One also applies (27)$_{3, 4}$ in this limit passage. As $\rho u \in L^{10}(0, T; L^{20}(\Omega)^d) \subset L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; L^6(\Omega)^d)$ it follows that (29)$_2$ allows us to pass to the limit $\alpha \to 0_+$ in $I_6^\alpha$. Applying (29)$_3$, the Sobolev embedding, the fact that $\rho \in L^\infty(Q_T)$, $u \in L^\infty(0, T; L^2(\Omega)^d)$ we pass to the limit $\alpha \to 0_+$ in $I_6^\alpha$. Therefore (25) yields

$$
\int_0^\tau \int_\Omega p(\rho)b(\rho) = \lim_{\alpha \to 0_+} \int_0^\tau \int_\Omega p(\rho)\langle b(\rho) \rangle = \lim_{\alpha \to 0_+} \left( \sum_{j=1}^6 I_j^\alpha + J^\alpha(\tau) \right)
$$

$$
= \int_0^\tau \int_\Omega p(\rho)\langle b(\rho) \rangle - \int_0^\tau \int_\Omega \rho u \otimes u \cdot \nabla B(b(\rho) - \langle b(\rho) \rangle)
$$

$$
+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla u) \cdot \nabla B(b(\rho) - \langle b(\rho) \rangle) - \int_0^\tau \int_\Omega \rho f \cdot B(b(\rho) - \langle b(\rho) \rangle)
$$

$$
+ \int_0^\tau \int_\Omega \rho u \cdot B(\operatorname{div}(b(\rho)u))
$$

$$
+ \int_0^\tau \int_\Omega \rho u \cdot B((b'(\rho) - b(\rho))\operatorname{div}u - (\langle b'(\rho) - b(\rho) \rangle)\operatorname{div}u)
$$

$$
+ \int_\Omega \rho u \cdot B(b(\rho) - \langle b(\rho) \rangle)(\tau, \cdot) - \int_\Omega \rho_0 u_0 \cdot B(b(\rho_0) - \langle b(\rho_0) \rangle).
$$

The first equality follows by the Lebesgue monotone convergence theorem provided we take account the definition of $b_\alpha$ implying $b_{\alpha 1} \leq b_{\alpha 2}$ for $\alpha_1 \geq \alpha_2$ and the pointwise convergence $p(\rho)b_\alpha(\rho) \to p(\rho)b(\rho)$.

By the application of the following lemma we obtain further estimates from the relative entropy inequality. It states properties of a quantity related to the pressure potential. It is a version of [16, Lemma 4.1 and (4.15)].

**Lemma 2.1.** Let the function be defined via (5) where the function $p$ satisfies (4) and (15). Let $\rho \in [0, \bar{\rho}]$ and $r \in (0, \bar{\rho})$ be such that $0 < \alpha_0 \leq r \leq \bar{\rho} - \alpha_0 < \bar{\rho}$ for some $\alpha_0 \in (0, \bar{\rho})$. Then there exist $\alpha_1 \in (0, \alpha_0)$ and a constant $c > 0$ such that

$$
P(\rho) - P(r) - P'(r)(\rho - r) \geq \begin{cases} 
c(\rho - r)^2, & \text{if } \rho \in (\alpha_1, \bar{\rho} - \alpha_1), 
\frac{p(r)}{2}, & \text{if } \rho \in [0, \alpha_1], 
\frac{P(\rho)}{2} > 1, & \text{if } \rho \in [\bar{\rho} - \alpha_1, \bar{\rho}). 
\end{cases} \tag{30}
$$

Additionally, we have

$$
p(\rho) - p(r) - p'(r)(\rho - r) \leq \begin{cases} 
c(\rho - r)^2, & \text{if } \rho \in (\alpha_1, \bar{\rho} - \alpha_1), 
1 + p'(r)r - p(r), & \text{if } \rho \in [0, \alpha_1], 
2p(r), & \text{if } \rho \in [\bar{\rho} - \alpha_1, \bar{\rho}). 
\end{cases} \tag{31}
$$

**Lemma 2.2.** Let the function $r : Q_T \to \mathbb{R}$ satisfy $0 < \alpha_0 \leq r \leq \bar{\rho} - \alpha_0$ and the function $p : Q_T \to \mathbb{R}$ take values in $[0, \bar{\rho})$. Let the function $P \in C^1([0, \bar{\rho})$ be defined in (5), where the function $p$ satisfies (4) and (15). Then there is a constant $C > 0$ such that for a.a. $t \in (0, T)$

$$
\frac{\|\rho - r(t)\|^2}{L^2(\Omega)} \leq C \int_\Omega (P(\rho) - P(r) - P'(r)(\rho - r))(t, x) \, dx. \tag{32}
$$

**Proof.** We observe that for a fixed $t \in (0, T)$

$$
\int_\Omega |\rho - r|^2(t, x) \, dx = \sum_{i=1}^3 \int_{\Omega_i} |\rho - r|^2(t, x) \, dx, \tag{33}
$$
We consider the scaled system with parameters $\nu > 3$. Singular Limit

where

there is a constant $D > 0$, pressure potential be defined via

$D \in \mathbb{R}$ constant and $\alpha$ is increasing was taken into account.

\[ \int_\Omega |\rho - r|^2(t, x) \, dx \leq \frac{2(\bar{\rho})^2}{p(\alpha_0)} \int_{\Omega_1} \frac{P(r)}{2} (t, x) \, dx + c^{-1} \int_{\Omega_2} c|\rho - r|^2(t, x) \, dx + 4(\bar{\rho})^2 \int_{\Omega_3} \frac{P(\rho)}{2} (t, x) \, dx \]

\[ \leq C \int_\Omega (P(\rho) - P(r) - P'(r)(\rho - r)) (t, x) \, dx, \]

where also the assumption that $p$ is increasing was taken into account.

3. Singular Limit

We consider the scaled system with parameters $\nu > 0$, $\varepsilon > 0$ and $R > 0$ satisfying

\[ \partial_t \rho + \text{div}(\rho u) = 0 \quad \text{in} \quad (0, T) \times \Omega_R, \]

\[ \partial_t (\rho u) + \text{div}(\rho u \otimes u) - \nu \text{div}\mathcal{S}(\nabla u) + \varepsilon^{-2} \nabla p(\rho) = \rho f \quad \text{in} \quad (0, T) \times \Omega_R, \]

\[ \rho(0, \cdot) = \rho_0, \quad u(0, \cdot) = u_0 \quad \text{in} \quad \Omega_R, \]

\[ u = 0 \quad \text{on} \quad (0, T) \times \partial \Omega_R, \]

where the behavior of a domain $\Omega_R \subset \mathbb{R}^3$ will be specified later. We assume that

\[ \mathcal{S}(\nabla u) = \mu \left( \nabla u + (\nabla u)^\top - \frac{2}{3} \text{div} u I_3 \right). \]

\[ (\partial_t v + v \cdot \nabla v + \nabla \Pi) = 0, \quad \text{div} v = 0, \]

\[ v(0) = v_0 = H(u_0), \]

whose properties are summarized in Lemma 4.3, and where $H$ denotes the standard Helmholtz projection.

Assuming that the dependence of the pressure on the density is given by a hard–sphere equation of state and the initial data are ill-prepared the goal of this section is the rigorous proof of the above described formal process.

Before the precise formulation of the main theorem we describe the geometry of the physical space. We consider a family of expanding domains $\{\Omega_R\}$ with the following properties

\[ \Omega_R \subset \mathbb{R}^d \text{ is simply connected, bounded } C^{2,\nu} \text{ domain uniformly for } R \to \infty, \]

\[ \Omega_R \text{ is star-shaped with respect to the ball } B(0, R) = \{ x \in \mathbb{R}^d : |x| < R \} \]

there is a constant $D > 0$ such that

\[ \partial \Omega_R \subset \{ x \in \mathbb{R}^d : R < |x| < R + D \}. \]

**Theorem 3.1.** Let the pressure function $p$ satisfy assumption (4) and additionaly $p \in C^2((0, \bar{p}))$ and the pressure potential be defined via (5). Let $\{\Omega_R\}_{R \geq 1}$ be a family of uniformly $C^{2,\nu}$ domains for which (37), (38) and (39) hold. Let the positive constants $D$, $\bar{\rho}$ be given and $\varepsilon_0 > 0$ be such that

\[ D^{-1} < \bar{\rho} - \varepsilon_0 D, \bar{\rho} + \varepsilon_0 D < \bar{\rho}. \]
Let $\varepsilon \in (0, \varepsilon_0)$ be fixed and $(\rho, u)$ be a finite energy weak solution of system (34) emanating from the initial data

$$\rho(0, \cdot) = \rho_{0, \varepsilon} = \varrho + \varepsilon \rho^{(1)}_{0, \varepsilon}, \quad u(0, \cdot) = u_{0, \varepsilon}$$

with

$$\|u_{0, \varepsilon}\|_{L^2(\mathbb{R}^3)} + \|\rho^{(1)}_{0, \varepsilon}\|_{L^2(\mathbb{R}^3)} + \|\rho^{(1)}_{0, \varepsilon}\|_{L^\infty(\mathbb{R}^3)} \leq D.$$  \hspace{1cm} (42)

In addition, let

$$R > D + \frac{\sqrt{\rho'(\varrho)}}{\varepsilon} T.$$  \hspace{1cm} (43)

Furthermore, assume that there are functions $u_0 \in C^m(\mathbb{R}^3; \mathbb{R}^3)$, $\rho^{(1)}_0 \in C^m(\mathbb{R}^d)$, $m \geq 4$ supported in $B(0, D)$ such that

$$\|u_0\|_{C^m(\mathbb{R}^3)} + \|\rho^{(1)}_0\|_{C^m(\mathbb{R}^3)} \leq D.$$  \hspace{1cm} (44)

Let $v$ be a strong solution to the incompressible Euler system (36) in $(0, T_{\text{max}}) \times \mathbb{R}^3$ and $T \in (0, T_{\text{max}})$. Let $(s, \Psi)$ be the solution of the acoustic system

$$\varepsilon \partial_t s + \varrho \Delta \Psi = 0,$$

$$\varepsilon \partial_t \nabla \Psi + \frac{\rho'(\varrho)}{\varrho} \nabla s = 0$$  \hspace{1cm} (45)

in $(0, T) \times \mathbb{R}^d$ and supplemented with the initial data

$$s(0, \cdot) = \rho^{(1)}_0, \quad \nabla \Psi(0, \cdot) = \nabla \Psi_0 = u_0 - H(u_0).$$  \hspace{1cm} (46)

Then there is $\varepsilon_1 > 0$ such that

$$\int_{\Omega_R} \rho |u - \nabla \Psi - v|^2 (\tau, \cdot) + \left\| \frac{\rho(\tau, \cdot) - \varrho}{\varepsilon} - s(\tau, \cdot) \right\|_{L^2(\Omega_R)}^2 \leq \left( c(D, T)(\varepsilon^\alpha + R^{-1} + \varepsilon^2 (1 + R^{-2}) + \varepsilon \nu^{-1}) + c_2 \left( \|u_{0, \varepsilon} - u_0\|_{L^2(\Omega_R)}^2 + \|\rho^{(1)}_{0, \varepsilon} - \rho^{(1)}_0\|_{L^2(\Omega_R)}^2 \right) \right) \times \exp \left( c(D, T) \left( 1 + \varepsilon^2 \nu + \varepsilon^2 \nu^{-1} (1 + R^{-4}) + \varepsilon^2 + R^{-2} + \varepsilon^2 R^{-2} \right) \right) + \frac{c \varepsilon^2}{\varrho - \varrho - \varepsilon_0 D} \left( \nu^{-1} R^{-1} + c \nu^2 R^{-\frac{2}{3}} + 1 \right).$$  \hspace{1cm} (47)

for any $\varepsilon \in (0, \min\{\varepsilon_0, \varepsilon_1\})$, any $\tau \in [0, T]$ and any $\alpha \in (0, 1)$.

**Corollary 3.1.** Assuming that $R, \nu$ are dependent on $\varepsilon$ and are such that $R(\varepsilon) \to \infty$, $\varepsilon R(\varepsilon) \to \infty$, $\nu(\varepsilon) \to 0$, $\varepsilon \nu^{-1}(\varepsilon) \to 0$, $u_{0, \varepsilon} \to u_0$ in $L^2(\mathbb{R}^3)$ and $\rho^{(1)}_{0, \varepsilon} \to \rho^{(1)}_0$ in $L^2(\mathbb{R}^3)$ as $\varepsilon \to 0_+$ estimate (47) yields the uniform in time convergence of $u$ towards the solution $v$ corrected by the oscillatory component $\nabla \Psi$ and the convergence of the difference $\rho - \varrho$ scaled by the factor $\varepsilon^{-1}$ towards the oscillatory component $s$.

We point out that the relative energy inequality from the previous section holds also for the scaled system in (34) with the factor $\varepsilon^{-2}$ in front of the pressure and the pressure potential. Therefore for the relative entropy

$$\mathcal{E}(\rho, u|\tau, U)(\tau) = \frac{1}{2} \int_{\Omega_R} (\rho|u - U|^2)(\tau, \cdot) + \varepsilon^{-2} (P(\rho) - P(r) - P'(r)(\varrho - r))(t, \cdot) \, dx$$  \hspace{1cm} (48)
we obtain
\[
\mathcal{E}(\rho, u|r, U)(\tau) + \nu \int_0^\tau \int_{\Omega_R} (S(\nabla u) - S(\nabla U)) \cdot \nabla (u - U) + \varepsilon^{-2} \int_0^\tau \int_{\Omega_R} p(\rho)b(\rho)
\leq \mathcal{E}(\rho_0, u_0|r(0, \cdot), U(0, \cdot)) + \int_0^\tau \mathcal{R}_1(t) + \mathcal{R}_2(t) \, dt + \mathcal{R}_3(\tau),
\]
(49)
where
\[
\mathcal{R}_1(t) = \int_\Omega \rho (\partial_t U + (u \cdot \nabla) U) \cdot (U - u) + \nu \int_\Omega S(\nabla U) \cdot \nabla (U - u)
+ \varepsilon^{-2} \int_\Omega (r - \rho) \partial_t P'(r) + \varepsilon^{-2} \int_{\Omega_R} (rU - \rho u) \cdot \nabla P'(r) +
+ \varepsilon^{-2} \int_\Omega \text{div} U \left( p(r) - p(\rho) \right),
\]
\[
\mathcal{R}_2(t) = \varepsilon^{-2} \int_\Omega p(\rho) \langle b(\rho) \rangle - \int_\Omega \rho u \otimes u \cdot \nabla \mathcal{B}(b(\rho) - \langle b(\rho) \rangle) + \int_\Omega S(\nabla u) \cdot \nabla \mathcal{B}(b(\rho) - \langle b(\rho) \rangle)
+ \int_\Omega \rho u \cdot \mathcal{B}(\text{div}(b(\rho)u)) + \int_\Omega \rho u \cdot ((b'(\rho)\rho - b(\rho))\text{div}u - \langle (b'(\rho)\rho - b(\rho))\text{div}u \rangle)
\]
\[
\mathcal{R}_3(\tau) = \int_\Omega \rho u \cdot \mathcal{B}(b(\rho) - \langle b(\rho) \rangle)(\tau, \cdot) - \int_\Omega \rho_0 u_0 \cdot \mathcal{B}(b(\rho_0) - \langle b(\rho_0) \rangle).
\]

The rest of the section is devoted to the proof of Theorem 2.1. It consists of three steps. First, taking into account the fact that we are in the situation with ill-prepared data, we choose properly test functions in the relative entropy inequality. Second, the relation between the values of the relative entropy functional at the time from the given interval and the initial value is deduced. At last, the Gronwall type argument is employed for the evaluation of the distance between the solutions of the primitive and target systems by means of the relative entropy functional. Moreover, the estimate of the rate of convergence is expressed in terms of characteristic numbers and the radius of the expanding domains. We note that the convergence result is path dependent, i.e., there is a specific fashion in which the characteristic numbers and the radius of the expanding domain are interrelated.

The following two subsections deal with preparatory work that will justify our choice of test functions in the relative entropy inequality and also helps us in further estimates. The third subsection contains a collection of estimates that are independent of parameters \( \varepsilon, \nu \) and \( R \).

Remark 3.1.  

- Let us explain the choice of initial data. There are two possibilities the well-prepared initial data or the ill-prepared initial data. A proper choice of initial data can eliminate the effect of acoustic wave since the acoustic equation conserves the norm in the associated energy space. For more details see [18, Chapters 4 and 10].

- The limit system (target system) - Euler system is defined on \( \mathbb{R}^3 \) and the primitive system (original system) is defined on \( \Omega_R \). This implies that solution \( v \) is not an admissible test function in the relative entropy inequality. Similarly for the solutions of the associated acoustic system.

---

4 Let us mention that the incompressible inviscid limit was investigated by Lions and Masmoudi [28] in the case of well-prepared initial data. A different approach by using Strichartz’s estimates for the linear wave equation has been developed by Schochet in [39]. For the application of Strichartz’s estimates in the case of low Mach number limit the reader can consult the work of Desjardins and Grenier [10]. The incompressible inviscid limit in the case of ill-prepared initial data was studied by Masmoudi [31] on the whole space and also on the torus.
3.1. Acoustic system

An oscillatory motion with small amplitude in a compressible fluid is called an acoustic wave. Hence, the compressibility of the fluid allows the propagation of acoustic waves. Consequently, acoustic waves should definitely disappear in the incompressible limit regime.5

This subsection is devoted to some properties of a solution to (45) endowed with initial data (46). Following the steps in [18, Section 8.6] it is possible to show that the solution of (45) admits the finite speed of propagation $\sqrt{p'(\varrho)}/\varepsilon$. Therefore the solution of (45) satisfies

$$\nabla \Psi(t, x) = \nabla \Psi_0(x), \quad \Delta \Psi(t, x) = s(t, x) = 0 \text{ for } t \geq 0, \quad |x| > D + \frac{\sqrt{p'(\varrho)}}{\varepsilon} t.$$  (50)

The physical assumption that the acoustic waves do not reach the boundary $\partial \Omega_R$ in the time lap $[0, T]$ is expressed by condition (43).

The conservation of energy of system (45) is expressed in the form

$$\frac{d}{dt} \left( \nabla \Psi(\tau) \right)_{L^2(\mathbb{R}^3)}^2 + \varrho^2 \| \nabla \Psi(\tau) \|_{L^2(\mathbb{R}^3)}^2 = 0.$$  (51)

Furthermore, the solution to (45) obeys the higher energy estimates

$$\| s(\tau) \|_{W^{k,2}(\mathbb{R}^3)}^2 + \| \nabla \Psi(\tau) \|_{W^{k,2}(\mathbb{R}^3)}^2 \leq C \left( \| p_0(\varrho) \|_{W^{k+2,2}(\mathbb{R}^3)}^2 + \| \nabla \Psi_0 \|_{W^{k+2,2}(\mathbb{R}^3)}^2 \right)$$  (52)

for any $\tau > 0$ and $k = 0, 1, 2, \ldots$ and also the following estimates of Strichartz type

$$\| s(\tau) \|_{W^{k,\eta}(\mathbb{R}^3)} + \| \nabla \Psi(\tau) \|_{W^{k,\eta}(\mathbb{R}^3)} \leq C(p, q) \left( 1 + \frac{\tau}{\varepsilon} \right)^{\frac{1}{2} - \frac{1}{p}} \left( \| p_0(\varrho) \|_{W^{k+4,\eta}(\mathbb{R}^3)} + \| \nabla \Psi_0 \|_{W^{k+4,\eta}(\mathbb{R}^3)} \right)$$  (53)

for any $\tau > 0$, $k = 0, 1, 2, \ldots$, and $\frac{1}{p} + \frac{1}{q} = 1$, $p \in (1, 2]$ that follow from results in [40, Section 3], cf. [37, Section 1.1], by a suitable rescaling in the time variable.

3.2. Correctors

The proof of Theorem 3.1 is based on the relative entropy inequality which involves test function $U$ possessing zero trace on the boundary of the physical domain $\Omega_R$. This excludes the possibility to employ $v + \nabla \Psi$ as the test function in the relative entropy inequality. To overcome this obstacle we introduce a corrector that will have certain properties which is later incorporated for the definition of an admissible test function in the relative entropy inequality that is suitable for our purposes. For fixed $R > 0$ we define the corrector $w_R$ as

$$w_R = -\chi_R(v + \nabla \Psi_0)$$  (54)

where

$$\chi_R \in C^\infty(\mathbb{R}^3), \quad 0 \leq \chi_R \leq 1, \quad \chi_R(x) = 1$$

if

$$\text{dist}(x, \partial \Omega_R) \leq \frac{1}{2} \min\{ \text{dist}(B_R, \partial \Omega), \text{dist}(B_{R+D}, \partial \Omega) \}$$

and the support of $\chi_R$ is contained in $B_{R+D} \setminus B_R$. Before stating estimates involving the corrector $w_R$ we focus on properties of functions $v$ and $\nabla \Psi_0$. Namely, we shortly discuss the fact that $\Psi_0$ and $\text{curl} \, v$ are harmonic functions in the exterior of a ball with enough large radius. Employing the Biot-Savart law, cf. [29, Section 2.4.1], for the expression of $v$ we get

$$v = -\text{curl} \, \Delta^{-1} \text{curl} \, \varrho$$

5 see [10, 31].
where $\Delta^{-1} \text{curl} \, v$ is obtained as the convolution of the Newtonian potential with $\text{curl} \, v$. As the support of $u_0$ is assumed to be compact in $B(0, D)$, it follows that the support of $\text{curl} \, v_0$, $\Delta \Psi_0$ respectively, is also compact in $B(0, D)$. Moreover, as $v$ is a smooth solution of the Euler system, the quantity $\text{curl} \, v$ obeys a transport equation with a compactly supported initial datum. The latter implies that $\Delta^{-1} \text{curl} \, v$ is a harmonic function in the exterior of the ball $B(0, R)$ for $R > D + T \|v\|_{L^\infty((0,T) \times \mathbb{R}^3)}$. Taking into account the fact that both $v$ and $\nabla \Psi_0$ are derivatives of functions that are harmonic outside a ball $B(0, R)$ it follows that

$$\|\nabla^k \nabla \Psi_0(x), |\nabla^k v(x)| \| \leq c|x|^{-2-k} \text{ for } x \in \mathbb{R}^3 \setminus B(0, R) \text{ and } k = 0, 1, 2.$$  

Hence we conclude

$$\|\partial_t w_R(t)\|_{L^p(\mathbb{R}^3)} + \|w_R(t)\|_{W^{2,p}(\mathbb{R}^3)} \leq cR^{2\left(\frac{1}{p} - 1\right)}$$

as $|B_{R+D} \setminus B_R| \leq cDR^2$ provided that $R > 1$.

### 3.3. Uniform Estimates

We observe that setting $(r, U) = (\rho, 0)$ in relative entropy inequality (49) we conclude

$$\|\sqrt{\rho} u\|_{L^\infty(0,T;L^2(\Omega_R))} \leq c,$$

$$\|\rho - \rho\|_{L^\infty(0,T;L^2(\Omega_R))} \leq c,$$

$$\nu \frac{1}{2} \|\nabla u\|_{L^2(0,T;L^2(\Omega_R))} \leq c$$

where the constant $c$ is independent of $\varepsilon, \nu, R$. We point out that the latter bound follows by the Korn inequality, see [18, Theorem 11.22], provided that the extension of the function $u$ by zero in $(0, T) \times (\mathbb{R}^3 \setminus \Omega_R)$ is considered. Moreover, taking into account Lemma 2.1 it follows from (49) that there is $\alpha_1 \in (0, \bar{\rho})$ such that

$$\text{esssup}_{t \in (0,T)} \left( \|\chi(\rho(t,\cdot) < 0, \alpha_1)\|_{L^1(\Omega_R)} = \|P(\rho(t,\cdot) \chi(\rho(t,\cdot) > \rho - \alpha_1)\|_{L^1(\Omega_R)} \right) \leq \varepsilon^2. \quad (57)$$

### 3.4. Convergence

Let us begin the proof of inequality (47) by specifying of the value of $\varepsilon_1$. Since our intention is to set $r = \bar{\rho} + \varepsilon s$ in the relative entropy inequality (49), we need

$$0 \leq \bar{\rho} + \varepsilon s < \bar{\rho} \quad (58)$$

to have $P(r)$ well defined. To this end we get by the Sobolev embedding, (52) and (44)

$$\|s\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))} \leq c\|s\|_{L^\infty(0,T;W^{3,2}(\mathbb{R}^3))} \leq cD(\|\text{supp } \rho_0^{(1)}\| + |\text{supp } u_0|).$$

Therefore taking

$$\varepsilon_1 = \min\{\bar{\rho} - \bar{\rho}, \varepsilon\} \left( cD(\|\text{supp } \rho_0^{(1)}\| + |\text{supp } u_0|) \right)^{-1}, \quad (59)$$

we conclude the validity of (58) for any $\varepsilon \in (0, \varepsilon_1)$. We set $(r, U) = (\bar{\rho} + \varepsilon s, v + \nabla \Psi + w_R)$ in the relative entropy inequality. Such a pair is admissible in (49) as the boundary condition $U = 0$ on $\partial \Omega_R$ is satisfied due to the definition of the corrector $w_R$ in (54).

Moreover, taking into account the fact that $P''(z) = \frac{p''(\varepsilon)}{\varepsilon}$ we infer for the initial data given in (41) that

$$\mathcal{E}(\rho, u|r, U)(0) = \int_{\Omega_R} \frac{1}{2} \rho_{0,\varepsilon}|u_{0,\varepsilon} - H(u_0) - \nabla \Psi_0 - w_R(0, \cdot)|^2$$

$$\quad + \varepsilon^{-2} \int_{\Omega_R} \left( P(\rho + \varepsilon \rho_0^{(1)}) - P(\rho + \varepsilon \rho_0^{(1)}) - \varepsilon P'(\rho + \varepsilon \rho_0^{(1)})(\rho_0^{(1)} - \rho_0^{(1)}) \right)$$
We proceed by estimating \( I_1 \) by the Sobolev embedding and (52). Using the Euler system for \( \rho, U \), independent of \( \varepsilon \in (0, \varepsilon_0) \). From now on we use the following notation for the integrals involved in terms \( R_1, R_2, R_3 \).

\[
\int_0^t R_1 = \sum_{j=1}^5 I_j, \int_0^t R_2 = \sum_{j=9}^{10} I_j, R_3 = I_{11} + I_{12}
\]

We proceed by estimating \( I_j \)'s in terms of the relative entropy functional and terms involving powers of quantities \( \varepsilon, \nu, \frac{1}{\rho} \). First, we rewrite

\[
I_1 = - \int_0^t \int_{\Omega_R} \rho ((u - U) \cdot \nabla) U \cdot (U - u) + \int_0^t \int_{\Omega_R} \rho (\partial_t U + (U \cdot \nabla) U) \cdot (U - u)
\]

\[
= - \int_0^t \int_{\Omega_R} \rho ((u - U) \cdot \nabla) U \cdot (U - u) + \int_0^t \int_{\Omega_R} \rho (U - u) \cdot (\partial_t v + (v \cdot \nabla) v) + \int_0^t \int_{\Omega_R} (U - u) \cdot \partial_t w_R + \int_0^t \int_{\Omega_R} \rho (U - u) \cdot \partial_t \Psi + \int_0^t \int_{\Omega_R} \rho ((v + \nabla \Psi + w_R) \cdot \nabla) (\nabla \Psi + w_R) \cdot (U - u) + \int_0^t \int_{\Omega_R} \rho (\nabla \Psi + w_R) \cdot \nabla v \cdot (U - u) = \sum_{k=1}^6 J_k.
\]

We immediately see that

\[
|J_1| \leq \int_0^t \| v + \nabla \Psi + w_R \|_{L^{\infty}(\Omega_R)} \mathcal{E}(\rho, u|r, U) \leq c \int_0^t (\| v \|_{W^{3,2}(\mathbb{R}^3)} + \| \Psi \|_{W^{4,2}(\mathbb{R}^3)} + R^{-2}) \mathcal{E}(\rho, u|r, U) \leq c(D, T) \int_0^t \mathcal{E}(\rho, u|r, U)
\]

by the Sobolev embedding and (52). Using the Euler system for \( v \), the weak formulation of the continuity equation for \( \rho, U = 0 \) on \( \partial \Omega_R \) and \( \text{div} v = 0 \) in \( (0, T) \times \mathbb{R}^3 \) it follows that

\[
J_2 = \int_0^t \int_{\Omega_R} \rho (U - u) \cdot \nabla \Pi = -\varepsilon \int_0^T \int_{\Omega_R} \frac{\rho - \Theta}{\varepsilon} \partial_t \Pi + \varepsilon \left[ \int_{\Omega_R} \frac{\rho - \Theta}{\varepsilon} \Pi \right]_{t=0}^{t=T} - \varepsilon \int_0^T \int_{\Omega_R} \frac{\rho - \Theta}{\varepsilon} U \cdot \nabla \Pi + \Theta \int_0^T \int_{\Omega_R} \left( \text{div} w_R + \Delta \Psi \right) \Pi = J_{2,a} + J_{2,b} + J_{2,c} + J_{2,d}.
\]

Employing (56)_2 it follows that

\[
|J_{2,a}| + |J_{2,b}| + |J_{2,c}| \leq c \varepsilon \left( \| \partial_t \Pi \|_{L^1(0,T;L^2(\mathbb{R}^3))} + \| \Pi \|_{L^{\infty}(0,T;L^2(\mathbb{R}^3))} + \| U \|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))} \right) \leq c \varepsilon.
\]
where the last inequality is obtained with help of Lemma 4.3. Using this lemma in combination with (54) and (53) yield

\[ |J_{2,d}| \leq \varepsilon \left( \| w_R \|_{L^\infty(0,T;W^{1,p}(\Omega_R))} + \| \Delta \Psi \|_{L^\infty(0,T;L^p(\Omega_R))} \right) \| \Pi \|_{L^1(0,T;L^p(\Omega_R))} \leq c \left( R^{-1} + \varepsilon^{1-\frac{2}{q_1}} \right) \]

for any \( p > 2 \). Obviously, by (51), (56)_1 and (55) we conclude

\[ |J_3| \leq \left( \bar{p} \| U \|_{L^\infty(0,T;L^2(\mathbb{R}^3))} + \sqrt{\bar{p}} \| \sqrt{\rho} u \|_{L^\infty(0,T;L^2(\Omega_R))} \right) \| \partial_t w_R \|_{L^1(0,T;L^2(\Omega_R))} \leq cR^{-1}. \]

Next, we rewrite using (43) and (50)

\[ J_4 = \int_0^T \int_{\Omega_R} (\rho - \varrho)v \cdot \partial_t \nabla \Psi + \frac{\rho}{2} \int_{0}^{t} \int_{\Omega_R} \left| \nabla \Psi \right|^2_{L^2} \mathrm{d}t + \int_0^T \int_{\Omega_R} \rho \varrho \cdot \partial_t \nabla \Psi - \int_0^T \int_{\Omega_R} \rho u \cdot \partial_t \nabla \Psi \]

Employing equation (45)_2, the regularity of \( v \) and estimate (53) we obtain

\[ |J_{4,a}| + |J_{4,c}| \leq \frac{p'(\varrho)}{\varrho} \varepsilon^{-1} \| \rho - \varrho \|_{L^\infty(0,T;L^2(\Omega_R))} \left( \| v \|_{L^\infty(0,T;L^{q_1}(\Omega_R))} + \| \nabla \Psi \|_{L^\infty(0,T;L^{q_2}(\Omega_R))} \right) \times \| \nabla s \|_{L^1(0,T;L^{q_2}(\Omega_R))} \]

\[ \leq c \varepsilon^{1-\frac{2}{q_2}} \]

for any \( q_1, q_2 > 2 \) such that \( \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2} \). As (43) and (50) imply \( \Psi = \Psi_0 \) in \((0, T) \times (\mathbb{R}^3 \setminus B(0, R))\) and \( \Psi_0 \) is independent of time, it follows that \( \mathbb{R}^3 \) can be taken as the domain of integration in \( J_{4,d} \) and that

\[ J_{4,b} = \int_0^T \int_{\partial \Omega_R} v \cdot n \partial_t \Psi_0 = 0, \]

where \( \text{div} v = 0 \) in \((0, T) \times \mathbb{R}^3 \) was also applied. Using (55) and (45)_2 we conclude

\[ |J_{4,e}| \leq \bar{p} \| w_R \|_{L^2(0,T;L^2(\Omega_R))} \varepsilon^{-1} \| \nabla s \|_{L^2(0,T;L^2(\mathbb{R}^3))} \leq \frac{c}{\varepsilon R}. \]

Next, using bound (56)_1, the fact that \( \| v \|_{L^\infty(0,T;W^{1,\infty}(\mathbb{R}^3))} \) is finite, see Lemma 4.3 and (55) it follows that

\[ |J_5| + |J_6| \leq \left( \| \rho u \|_{L^\infty(0,T;L^2(\Omega_R))} + \bar{p} \| U \|_{L^\infty(0,T;L^2(\Omega_R))} \right) \]

\[ \times \left( \| v \|_{L^\infty(0,T;W^{1,q_1}(\Omega_R))} + \| \nabla \Psi \|_{L^\infty(0,T;L^{q_2}(\Omega_R))} + \| w_R \|_{L^\infty(0,T;L^{q_1}(\Omega_R))} \right) \times \left( \| \Psi \|_{L^1(0,T;W^{2,q_2}(\Omega_R))} + \| w_R \|_{L^1(0,T;W^{1,q_2}(\Omega_R))} \right) \]

\[ \leq c \left( \varepsilon^{1-\frac{2}{q_2}} + R^{-2(1-\frac{1}{q_2})} \right) \]

for any \( q_1, q_2 > 2 \) such that \( \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2} \).

Before estimating the term \( I_2 \), we note that by the Korn inequality, cf. [18, Theorem 11.22] and the structure of the tensor \( S \) defined in (35) it follows that

\[ \int_{\Omega_R} |\nabla (U - u)|^2 \mathrm{d}x \leq c \int_{\Omega_R} (S(\nabla u) - S(\nabla U)) \cdot \nabla (u - U) \mathrm{d}x. \]

We notice that the difference \((u - U)(t)\) can be understood as an element of \( W^{1,2}(\mathbb{R}^3) \) for a.e. \( t \in (0, T) \) after the extension by zero. Therefore [18, Theorem 11.22 (i)] implies that the constant \( c \) in the latter
inequality is independent of $R$. Moreover, one deduces by the Hölder and Young inequalities and (68) similarly as in [17]

$$\nu \int_{\Omega_R} S(\nabla U) \cdot \nabla (U - u) dx \leq \frac{\nu}{16} \int_{\Omega_R} (S(\nabla U) - S(\nabla u)) \cdot (\nabla U - \nabla u) dx + c\nu \int_{\Omega_R} |S(\nabla U)|^2 dx.$$  

Having the latter inequality at hand, we obtain

$$|I_2| \leq \nu \int_0^T \left( c_1 \|S(\nabla U)\|_{L^2(\Omega_R)}^2 + c_2 \|\nabla (U - u)\|_{L^2(\Omega_R)}^2 \right)$$

$$\leq c\nu \|U\|_{L^2(0,T;W^{2,2}(\Omega_R))}^2 + \frac{\nu}{2} \int_0^T \int_{\Omega_R} (S(\nabla U) - S(\nabla u)) \cdot \nabla (U - u) dx$$

$$\leq c(D,T)\nu(1 + R^{-2}) + \frac{\nu}{2} \int_0^T \int_{\Omega_R} (S(\nabla U) - S(\nabla u)) \cdot \nabla (U - u) dx$$

for suitably chosen $c_2$ by Lemma 4.3, (52) and (55). We proceed with treating the term $I_3$. Expanding derivatives of $P'(r)$ one has

$$I_3 = \varepsilon^{-1} \int_0^T \int_{\Omega_R} (r - \rho)P''(r)\partial_t s = \int_0^T \int_{\Omega_R} sP''(r)\partial_t s + \int_0^T \int_{\Omega_R} \varepsilon^{-1}(r - \rho)P''(r)\partial_t s$$

$$= \int_0^T \int_{\Omega_R} s(P''(r) - P''(\rho))\partial_t s + \frac{p'(\rho)}{2q} \left[ \int_0^T \int_{\Omega_R} s^2 \right]_{t=0}^{t=T} + \int_0^T \int_{\Omega_R} \varepsilon^{-1}(r - \rho)(P''(r) - P''(\rho))\partial_t s$$

$$+ \int_0^T \int_{\Omega_R} \varepsilon^{-1}(r - \rho)P''(\rho)\partial_t s$$

$$= I_{3,a} + I_{3,b} + I_{3,c} + I_{3,d}.$$  

Notice that $I_{3,b}$ cancels out $J_{4,d}$ from (64) because of (51), (43) and (50). We estimate using (45)

$$|I_{3,a}| \leq \int_0^T \int_{\Omega} \varepsilon s^2 |\partial_t s| P = \varepsilon \int_0^T \int_{\Omega} s^2 |\Delta \Psi|$$

$$\leq \varepsilon \|s\|_{L^2(0,T;L^2(\Omega))} \|s\|_{L^{q_1}(0,T;L^{q_1}(\Omega))} \|\Delta \Psi\|_{L^{q_2}(0,T;L^{q_2}(\Omega))}$$

where we denoted $P = \max_{z \in [-\varepsilon_1, \|s\|_{L^\infty(\Omega)}} |P''(z)|$, where $\varepsilon_1$ is specified in (59). The exponents $q_1, q_2$ satisfy $\frac{q_1}{q_1} + \frac{1}{q_1} + \frac{1}{q_2} = 1$. Hence using (52) and (53) with any $q_1, q_2 > 2$ it follows that

$$|I_{3,a}| \leq c(D,T)\varepsilon^{2(1 - \frac{1}{q_1} - \frac{1}{q_2})} \leq c(D,T)\varepsilon.$$  

Similarly, we obtain

$$|I_{3,c}| \leq \varepsilon \|s\|_{L^{q_1}(0,T;L^{q_1}(\Omega))} \|\Delta \Psi\|_{L^{q_2}(0,T;L^{q_2}(\Omega))} \leq c(D,T)\varepsilon^{2(1 - \frac{1}{q_1} - \frac{1}{q_2}).}$$  

Taking into account that

$$\int_{\Omega_R} \nabla P'(r) \cdot r U = \int_{\Omega_R} p'(r) \nabla r \cdot U = - \int_{\Omega_R} p(r) \text{div} U$$
by (6) we realize that
\[ I_4 + I_5 = -\varepsilon^{-2} \int_0^T \int_{\Omega_R} \nabla P'(r) \cdot \rho u - \varepsilon^{-2} \int_0^T \int_{\Omega_R} p(\rho) \text{div}U = \tilde{I}_4 + \tilde{I}_5. \] (73)

Applying (6) and equation (45) we get
\begin{align*}
\tilde{I}_4 &= -\varepsilon^{-1} \int_0^T \int_{\Omega_R} P''(r) \nabla s \cdot \rho u \\
&= -\varepsilon^{-1} \int_0^T \int_{\Omega_R} (P''(r) - P''(\rho)) \nabla s \cdot \rho u - \varepsilon^{-1} \int_0^T \int_{\Omega_R} \frac{p'(\rho)}{\rho} \nabla s \cdot \rho u \\
&= -\varepsilon^{-1} \int_0^T \int_{\Omega_R} (P''(r) - P''(\rho)) \nabla s \cdot \rho u + \int_0^T \int_{\Omega_R} \rho u \cdot \partial_t \nabla \Psi = \tilde{I}_{4,a} + \tilde{I}_{4,b}.
\end{align*}

We proceed with estimates \( \tilde{I}_{4,a} \). We note that the term \( \tilde{I}_{4,b} \) cancels out its counterpart \( J_{4,f} \) from (64). Similarly to estimate (71), we obtain
\[ |\tilde{I}_{4,a}| \leq g^2 \mathcal{P} \int_0^T \int_{\Omega_R} |s| |\nabla s| |\rho u| \leq c ||s||_{L^{\gamma_1}(0,T;L^{\gamma_1}(\Omega))} ||\nabla s||_{L^{\gamma_2}(0,T;L^{\gamma_2}(\Omega))} ||\rho u||_{L^2(0,T;L^2(\Omega))} \leq c(D,T)\varepsilon^{2(1-\frac{1}{\gamma_1} - \frac{1}{\gamma_2})}. \] (74)

We continue with the estimate of \( \tilde{I}_5 \) from (73). We first rewrite it as
\[ \tilde{I}_5 = \varepsilon^{-2} \int_0^T \int_{\Omega_R} (p(\rho) - p(\rho) - p'(\rho)(\rho - \rho)) \text{div}U + \varepsilon^{-2} \int_0^T \int_{\Omega_R} (p(\rho) + p'(\rho)(\rho - \rho)) \text{div}U = \tilde{I}_{5,a} + \tilde{I}_{5,b}. \]

By the definition of \( U \) we obtain \( \text{div}U = \text{div}(w_R + \nabla \Psi) \). Next we observe that thanks to estimates (55), (52) and the Sobolev embedding there is \( \alpha_2 \) such that
\[ -\log(\overline{\rho} - s) \geq 8 ||\text{div}(w_R + \nabla \Psi)||_{L^\infty((0,T) \times \mathbb{R}^d)} \text{ if } s \in (\overline{\rho} - \alpha_2, \overline{\rho}). \] (75)

For the purposes of this subsection we define \( b \in C^1([0,\overline{\rho}) \) with \( b' \geq 0 \) in the following way
\[ b(s) = \begin{cases} 0 & \text{if } s \leq \overline{\rho} - \alpha_1 \\ -\log(\overline{\rho} - s) & \text{if } \overline{\rho} - \alpha_2 \leq s < \overline{\rho} \end{cases} \] (76)

and \( b'(s) > 0 \) for \( s \in (\overline{\rho} - \alpha_1, \overline{\rho} - \alpha_2) \) with \( \alpha_1 \) from Lemma 2.1. We point out that such a function \( b \) is admissible in Theorem 2.1 as assumption (15) implies that the conditions in (16) are satisfied. Then we have
\[ \tilde{I}_{5,a} = \varepsilon^{-2} \int_{\rho \leq \overline{\rho} - \alpha_1} (p(\rho) - p(\rho) - p'(\rho)(\rho - \rho)) (\text{div}w_R + \Delta \Psi) \\
+ \varepsilon^{-2} \int_{\rho \in (\overline{\rho} - \alpha_1, \overline{\rho} - \alpha_2)} (p(\rho) - p(\rho) - p'(\rho)(\rho - \rho)) (\text{div}w_R + \Delta \Psi) \\
+ \varepsilon^{-2} \int_{\rho \geq \overline{\rho} - \alpha_2} (p(\rho) - p(\rho) - p'(\rho)(\rho - \rho)) (\text{div}w_R + \Delta \Psi). \]
Combining (31) with (30) we obtain, using also the definition of the function \(b(\rho)\) in (76) and the definition of \(\alpha_2\) in (75),
\[
|\tilde{I}_{5,a}| \leq \varepsilon^{-2} \int_{\{\rho \leq \rho_{-\alpha_1}\}} \|\text{div}(w_R + \nabla \Psi)\|_{L^\infty(\Omega_R)} (P(\rho) - P(\rho) - P'(\rho)(\rho - \rho)) \\
+ \varepsilon^{-2} \int_{\{\rho \in (\rho_{-\alpha_1}, \rho_{-\alpha_2})\}} \|\text{div}(w_R + \nabla \Psi)\|_{L^\infty(\Omega_R)} \max_{z \in [\rho_{-\alpha_1}, \rho_{-\alpha_2}]} p''(z)(\rho - \rho)^2 \\
+ 2\varepsilon^{-2} \int_{\{\rho \geq \rho_{-\alpha_2}\}} p(\rho)\|\text{div}(w_R + \nabla \Psi)\|_{L^\infty((0,T) \times \Omega_R)} \\
\leq c\varepsilon^{-2} \left( \int_{\{\rho \leq \rho_{-\alpha_1}\}} (P(\rho) - P(\rho) - P'(\rho)(\rho - \rho)) \\
+ \int_{\{\rho \in (\rho_{-\alpha_1}, \rho_{-\alpha_2})\}} (P(\rho) - P(\rho) - P'(\rho)(\rho - \rho)) \\
+ \frac{1}{4\varepsilon^2} \int_{\{\rho \geq \rho_{-\alpha_2}\}} p(\rho)b(\rho) \right)
\leq c(D,T)(1 + R^{-2}) \int_0^T \mathcal{E}(\rho, u|r, U)(t) \, dt + \frac{1}{4\varepsilon^2} \int_0^T \int_{\Omega_R} p(\rho)b(\rho).
\tag{77}
\]

Let us handle the term \(\tilde{I}_{5,b}\). Using the divergence theorem and the fact that \(U\) possesses zero trace on \(\partial \Omega_R\) we get
\[
\tilde{I}_{5,b} = -2p'(\rho) \int_0^T \int_{\Omega_R} (\rho - \rho)\text{div}w_R + \varepsilon^{-2}p'(\rho) \int_0^T \int_{\Omega_R} (\rho - \rho)\Delta \Psi = \tilde{J}_{5,a} + \tilde{J}_{5,b}.
\]

By (56) and (55) we deduce
\[
|\tilde{J}_{5,a}| \leq \varepsilon^{-1}\|\rho - \rho\|_{L^\infty(0,T;L^2(\Omega_R))} \varepsilon^{-1}\|w_R\|_{L^1(0,T;W^{1,2}(\Omega_R))} \leq \frac{c(D,T)}{\varepsilon R}.
\tag{78}
\]

Using (45) we deduce that \(\tilde{J}_{5,b}\) cancels out \(I_{3,d}\) from (70). Collecting estimates (60), (61), (62), (65), (67), (69), (71), (72), (74), (77) and (78) we obtain
\[
\int_0^T \mathcal{R}_1(t) \, dt \leq c(D,T)(1 + R^{-2}) \int_0^T \mathcal{E}(\rho, u|r, U) + \frac{1}{4}\varepsilon^{-2} \int_0^T \int_{\Omega_R} p(\rho)b(\rho) \\
+ c(D,T) (\varepsilon^\alpha + R^{-1} + (\varepsilon R)^{-1} + \nu) \\
+ \frac{\nu}{2} \int_0^T \int_{\Omega_R} S(\nabla(u - U)) \cdot \nabla(u - U)
\tag{79}
\]
for any \(\alpha \in (0, 1)\). For estimates of the terms in \(\mathcal{R}_2\) we need some preparations. First, it follows from assumption (15) and (5) that for any \(\gamma > 0\)
\[
\lim_{s \to \rho^-} \frac{p(s)}{(b(s))^\gamma} = \lim_{s \to \rho^-} \frac{P(s)}{(b(s))^\gamma} = \lim_{s \to \rho^-} \frac{p(s)}{(b'(s))^\beta} = \lim_{s \to \rho^-} \frac{P(s)}{(b(s))^\beta - 1} = +\infty.
\tag{80}
\]

The latter results imply that for \(\gamma \geq 1\)
\[
\int_{\Omega_R} |b(\rho)|^\gamma = \int_{\{\rho > \rho_{-\alpha_1}\}} |b(\rho)|^\gamma \leq c \int_{\{\rho > \rho_{-\alpha_1}\}} P(\rho) \leq c \int_{\{\rho > \rho_{-\alpha_1}\}} (P(\rho) - P(r) - P'(r)(\rho - r)),
\tag{81}
\]
with \(\alpha_1\) from Lemma 2.1. Moreover, for any \(\beta_0 \in [2, \beta]\) we have
\[
\int_{\Omega_R} |b'(\rho)|^{\beta_0 - 1} \leq c \int_{\{\rho > \rho_{-\alpha_1}\}} P(\rho) \leq c \int_{\{\rho > \rho_{-\alpha_1}\}} (P(\rho) - P(r) - P'(r)(\rho - r)),
\tag{82}
\]
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Using (81) with $\gamma = 1$ we conclude
\[
I_6 \leq c \int_0^r \frac{1}{|\Omega_R|} \int_{\Omega_R} p(\rho) \varepsilon^{-2} \int_{\Omega_R} (P(\rho) - P(r) - P'(r)(\rho - r)) \, d\rho \\
\leq c \int_0^r \frac{1}{|\Omega_R|} \int_{\Omega_R} p(\rho) \mathcal{E}(\rho, u|\tau, U)(t)
\]
In order to treat the second term in $\mathcal{R}_2$ we adopt the computations from [16, Section 4.5] in the following way. First, we write
\[
I_7 = -\int_0^r \int_{\Omega_R} \rho (u - U) \otimes (u - U) \cdot \nabla \mathcal{B} \left( b(\rho) - \frac{1}{|\Omega_R|} \int_{\Omega_R} b(\rho) \right) \\
- \int_0^r \int_{\Omega_R} \rho U \otimes (u - U) \cdot \nabla \mathcal{B} \left( b(\rho) - \frac{1}{|\Omega_R|} \int_{\Omega_R} b(\rho) \right) \\
- \int_0^r \int_{\Omega_R} \rho (u - U) \otimes U \cdot \nabla \mathcal{B} \left( b(\rho) - \frac{1}{|\Omega_R|} \int_{\Omega_R} b(\rho) \right) \\
- \int_0^r \int_{\Omega_R} (\rho - r) U \otimes U \cdot \nabla \mathcal{B} \left( b(\rho) - \frac{1}{|\Omega_R|} \int_{\Omega_R} b(\rho) \right) \\
- \int_0^r \int_{\Omega_R} r U \otimes U \cdot \nabla \mathcal{B} \left( b(\rho) - \frac{1}{|\Omega_R|} \int_{\Omega_R} b(\rho) \right)
\]
\[= I_{7,a} + I_{7,b} + I_{7,c} + I_{7,d} + I_{7,e}.
\]
Using the Hölder and Young inequalities, the Sobolev embedding, (68) and Lemma 4.2 we obtain
\[
|I_{7,a}| \leq \int_0^r \|\rho(u - U)\|_{L^2(\Omega_R)} \|u - U\|_{L^4(\Omega_R)} \left\| \nabla \mathcal{B} \left( b(\rho) - \frac{1}{|\Omega_R|} \int_{\Omega_R} b(\rho) \right) \right\|_{L^2(\Omega_R)} \\
\leq c \varepsilon^{-1} \int_0^r \|\sqrt{\rho}(u - U)\|_{L^2(\Omega_R)}^2 \|b(\rho)\|_{L^2(\Omega_R)} + \frac{\nu}{16} \int_0^r \int_{\Omega_R} \mathcal{S}(\nabla(u - U)) \cdot \nabla(u - U).
\]
Hence using (81) and (57) we conclude
\[
|I_{7,a}| \leq c \varepsilon^{-1} \int_0^r \left( \int_{\Omega_R} P(\rho) \right)^{\frac{1}{2}} \mathcal{E}(\rho, u|\tau, U) + \frac{\nu}{16} \int_0^r \int_{\Omega_R} \mathcal{S}(\nabla(u - U)) \cdot \nabla(u - U)
\]
\[\leq c \varepsilon^{\frac{3}{2}} \varepsilon^{-1} \int_0^r \mathcal{E}(\rho, u|\tau, U) + \frac{\nu}{16} \int_0^r \int_{\Omega_R} \mathcal{S}(\nabla(u - U)) \cdot \nabla(u - U).
\]
Next, (81), (57) and Lemma 4.2 imply
\[
|I_{7,b}| + |I_{7,c}| \leq \bar{p} \int_0^r \|U\|_{L^\infty(\Omega_R)} \|\sqrt{\rho}(u - U)\|_{L^2(\Omega_R)}^2 \\
+ \int_0^r \|U\|_{L^\infty(\Omega_R)} \left\| \nabla \mathcal{B} \left( b(\rho) - \frac{1}{|\Omega_R|} \int_{\Omega_R} b(\rho) \right) \right\|_{L^2(\Omega_R)}^2 \\
\leq c \int_0^r \|U\|_{L^\infty(\Omega_R)} \left( \|\sqrt{\rho}(u - U)\|_{L^2(\Omega_R)}^2 + \|b(\rho)\|_{L^2(\Omega_R)}^2 \right) \\
\leq c \int_0^r \|U\|_{L^\infty(\Omega_R)} \left( \|\sqrt{\rho}(u - U)\|_{L^2(\Omega_R)}^2 + \int_{\Omega_R} (P(\rho) - P(r) - P'(r)(\rho - r)) \right) \\
\leq c(D, T)(1 + \varepsilon^2)(1 + R^{-2}) \int_0^r \mathcal{E}(\rho, u|\tau, U).
\]
Applying the Young inequality and (30) it follows that
\[
|I_{7,c}| \leq c \int_0^\tau \||U||_{L^\infty(\Omega)}^2 (\|\rho - r\|_{L^2(\Omega_R)}^2 + \|\nabla B (b(\rho) - \frac{1}{|\Omega_R|} \int_{\Omega_R} b(\rho))\|_{L^2(\Omega_R)})^2 \leq c \int_0^\tau \|U||_{L^\infty(\Omega_R)}^2 \left( \int_{\Omega_R} (P(\rho) - P(r) - P'(r)(\rho - r) + \|b(\rho)||_{L^2(\Omega_R)}^2 \right) \
\leq c(D,T)\varepsilon^2 (1 + R^{-2}) \int_0^\tau \mathcal{E}(\rho, u|r, U).
\]

Then we estimate by the Sobolev embedding and Lemma 4.2
\[
|I_{7,c}| \leq c \int_0^\tau \|r||_{L^\infty(\Omega_R)} \|U||_{L^4(\Omega_R)} ||b(\rho)||_{L^2(\Omega_R)} \leq c\varepsilon^2 \int_0^\tau (\varepsilon + \varepsilon_0 s||_{L^\infty(\Omega_R)})^2 (\|v||_{L^4(\Omega_R)} + \|\nabla \Psi||_{L^4(\Omega)})^4 \|w_R||_{L^4(\Omega_R)}^4 \leq c(D,T,\varepsilon_0)\varepsilon^2 + c \int_0^\tau \mathcal{E}(\rho, u|r, U).
\]

In order to handle the next term of $\mathcal{R}_2$ we write
\[
I_8 = \nu \int_0^\tau \int_{\Omega_R} (S(\nabla u) - S(\nabla U)) \cdot \nabla B \left( b(\rho) - \frac{1}{|\Omega_R|} \int_{\Omega_R} b(\rho) \right) + \nu \int_0^\tau \int_{\Omega_R} S(\nabla u) \cdot \nabla B \left( b(\rho) - \frac{1}{|\Omega_R|} \int_{\Omega_R} b(\rho) \right) = I_{8,a} + I_{8,b}.
\]

Using the Hölder and Young inequalities, the structure of the tensor $S(\nabla u)$, Lemma 4.2 and (30) we deduce
\[
|I_{8,a}| \leq \nu \int_0^\tau \|S(\nabla u) - S(\nabla U)||_{L^2(\Omega_R)} \left( \|\nabla B \left( b(\rho) - \frac{1}{|\Omega_R|} \int_{\Omega_R} b(\rho) \right)\|_{L^2(\Omega_R)} \right) \leq \frac{\nu}{16} \int_0^\tau \int_{\Omega_R} (S(\nabla u) - S(\nabla U)) \cdot \nabla (u - U) + c\varepsilon \int_0^\tau \|b(\rho)||_{L^2(\Omega_R)}^2 \leq \frac{\nu}{16} \int_0^\tau \int_{\Omega_R} (S(\nabla u) - S(\nabla U)) \cdot \nabla (u - U) + c\varepsilon_0 \int_0^\tau \mathcal{E}(\rho, u|r, U).
\]

Folowing the arguments used in (87) we get
\[
|I_{8,b}| = \left| \nu \int_0^\tau \int_{\Omega_R} \text{div} S(\nabla u) \cdot B \left( b(\rho) - \frac{1}{|\Omega_R|} \int_{\Omega_R} b(\rho) \right) \right| \leq c\varepsilon \int_0^\tau \|U||_{W^{2,2}(\Omega_R)} ||b(\rho)||_{L^2(\Omega_R)} ||b(\rho)||_{L^2(\Omega_R)} \leq c(D,T)\varepsilon^2 \nu^2 (1 + R^{-2}) + c\varepsilon_0 \int_0^\tau \|b(\rho)||_{L^2(\Omega_R)}^2 \leq c(D,T)\varepsilon^2 \nu^2 (1 + R^{-2}) + c \int_0^\tau \mathcal{E}(\rho, u|r, U).
\]
Next, we write

\[ I_9 = \int_0^T \int_{\Omega_R} \rho u \cdot \mathcal{B}(\text{div}(\rho)(u - U)) \]
\[ + \int_0^T \int_{\Omega_R} \rho(u - U) \cdot \mathcal{B}(\text{div}(\rho)U) \]
\[ + \int_0^T \int_{\Omega_R} (\rho - r)U \cdot \mathcal{B}(\text{div}(\rho)U) \]
\[ + \int_0^T \int_{\Omega_R} rU \cdot \mathcal{B}(\text{div}(\rho)U) = I_{9,a} + I_{9,b} + I_{9,c} + I_{9,d}. \]

By the Hölder and Young inequalities, Lemma 4.2, the fact that \( \|\text{div}(\rho)(u - U)\|_{W^{1,2}(\Omega)} \leq \|\rho(u - U)\|_{L^{2}(\Omega)} \), \((81), (57)\), the Sobolev embedding and the Korn inequality it follows that

\[
|I_{9,a}| \leq c \int_0^T \|\rho u\|_{L^2(\Omega_R)} \|\text{div}(\rho)(u - U)\|_{L^2(\Omega_R)}
\leq c\nu^{-1} \|ho u\|_{L^2(0,T;L^2(\Omega_R))} \int_0^T \|\text{div}(\rho)\|_{L^4(\Omega_R)}^2 + \sigma \nu \int_0^T \|u - U\|_{L^4(\Omega_R)}^2
\leq c\nu^{-1} \int_0^T \left( \int_{\{\rho(t) > \rho^{-\alpha}\}} P(\rho) \right)^{\frac{1}{2}} + \frac{\nu}{16} \int_0^T \int_{\Omega_R} (\mathcal{S}(\nabla u) - \mathcal{S}(\nabla U)) \cdot \nabla (u - U)
\leq c\varepsilon \nu^{-1} + \frac{\nu}{16} \int_0^T \int_{\Omega_R} (\mathcal{S}(\nabla u) - \mathcal{S}(\nabla U)) \cdot \nabla (u - U). \quad (90)
\]

We similarly deduce

\[
|I_{9,b}| \leq \bar{p} \int_0^T \int_{\Omega_R} \rho |u - U|^2 + c \int_0^T \|b(\rho)U\|_{L^2(\Omega_R)}^2
\leq c \int_0^T \mathcal{E}(\rho, u|r, U) + c\varepsilon^2 \int_0^T \|U\|_{L^\infty(\Omega_R)}^2 \varepsilon^{-2} \int_{\Omega_R} (P(\rho) - P(r) - P'(r)(\rho - r))
\leq c(D, T) \int_0^T (1 + \varepsilon^2(1 + R^{-2})) \mathcal{E}(\rho, u|r, U), \quad (91)
\]

\[
|I_{9,c}| \leq \int_0^T \|U\|_{L^\infty(\Omega_R)}^2 |(\rho - r)^2 + \int_0^T \|U\|_{L^\infty(\Omega_R)}^2 \|b(\rho)\|_{L^2(\Omega_R)}^2 \leq c(1 + \varepsilon^2)(1 + R^{-2}) \int_0^T \mathcal{E}(\rho, u|r, U) \quad (92)
\]

and

\[
|I_{9,d}| \leq c \int_0^T \|rU\|_{L^2(\Omega_R)} \|b(\rho)U\|_{L^2(\Omega_R)} \leq c\varepsilon^2 \int_0^T \|r\|_{L^\infty(\Omega_R)}^2 \|U\|_{L^2(\Omega_R)}^2
\leq c(D, T) \varepsilon^2 (1 + R^{-4}) \int_0^T (\rho \varepsilon_0 \|s\|_{L^\infty(\Omega_R)})^2 + c(D, T)(1 + R^{-2}) \int_0^T \mathcal{E}(\rho, u|r, U). \quad (93)
\]
Next, we write
\[
I_{10} = \int_0^T \int_{\Omega_R} \rho u \cdot \mathbf{B} \left( (b'(\rho)\rho - b(\rho)) \text{div} U - \frac{1}{|\Omega_R|} \int_{\Omega_R} (b'(\rho)\rho - b(\rho)) \text{div} U \right) + \frac{\nu}{16} \int_0^T \int_{\Omega_R} (\mathbf{S}(\nabla u) - \mathbf{S}(\nabla U)) \cdot \nabla (u - U)\]
\[
\quad + \frac{\nu}{16} \int_0^T \int_{\Omega_R} (\mathbf{S}(\nabla u) - \mathbf{S}(\nabla U)) \cdot \nabla (u - U)\]
\[
= I_{10,a} + I_{10,b} + I_{10,c}.
\]

By Lemma 4.2, (56) and (81) it follows that
\[
|I_{10,a}| \leq c \int_0^T \|\rho u\|_{L^2(\Omega_R)} \|\nabla (b'(\rho)\rho - b(\rho))\|_{L^2(\Omega_R)} \cdot \text{div} U \|
\leq c\|\rho u\|_{L^\infty(0,T;L^2(\Omega_R))} \int_0^T \|\nabla (b'(\rho)\rho - b(\rho))\|_{L^2(\Omega_R)} \cdot \text{div} U \|
\leq c\varepsilon^2 \int_0^T \|\nabla (b'(\rho)\rho - b(\rho))\|_{L^2(\Omega_R)} \cdot \text{div} U \| + \epsilon^{-2} \int_0^T \|\nabla (b'(\rho)\rho - b(\rho))\|_{L^2(\Omega_R)} \cdot \text{div} U \|
\leq c(D,T)\varepsilon^2 (1 + R^{-4}) + c \int_0^T \mathcal{E}(\rho, u|\tau, U).
\]

The Sobolev embedding \(W^{1,\frac{8}{5}}(\Omega_R)\) into \(L^1(\Omega_R)\), Lemma 4.2, (81) and (57) yield
\[
|I_{10,b}| \leq c \int_0^T \|\nabla (b'(\rho)\rho - b(\rho))\|_{L^2(\Omega_R)} \cdot \text{div} U \|
\leq c \int_0^T \|\nabla (b'(\rho)\rho - b(\rho))\|_{L^2(\Omega_R)} \cdot \text{div} U \|
\leq c \varepsilon^2 \int_0^T \|\nabla (b'(\rho)\rho - b(\rho))\|_{L^2(\Omega_R)} \cdot \text{div} U \| + \epsilon^{-2} \int_0^T \|\nabla (b'(\rho)\rho - b(\rho))\|_{L^2(\Omega_R)} \cdot \text{div} U \|
\leq c(D,T)\varepsilon^2 (1 + R^{-4}) + c \int_0^T \mathcal{E}(\rho, u|\tau, U).
\]

as well as
\[
|I_{10,c}| \leq c \int_0^T \|\nabla (b'(\rho)\rho - b(\rho))\|_{L^2(\Omega_R)} \cdot \text{div} U \|
\leq c \int_0^T \|\nabla (b'(\rho)\rho - b(\rho))\|_{L^2(\Omega_R)} \cdot \text{div} U \|
\leq c \varepsilon^2 \int_0^T \|\nabla (b'(\rho)\rho - b(\rho))\|_{L^2(\Omega_R)} \cdot \text{div} U \| + \epsilon^{-2} \int_0^T \|\nabla (b'(\rho)\rho - b(\rho))\|_{L^2(\Omega_R)} \cdot \text{div} U \|
\leq c(D,T)\varepsilon^2 (1 + R^{-4}) + c \int_0^T \mathcal{E}(\rho, u|\tau, U).
\]
Collecting estimates (83)–(96) we conclude
\[
\int_0^T \mathcal{R}_2(t) \, dt \\
\leq c(D, T) \int_0^T \left( \frac{1}{|\Omega_R|} \int_{\Omega_R} p(\rho) + \varepsilon^\frac{4}{3} \nu^{-1}(1 + R^{-4}) + (1 + \varepsilon^2)(1 + R^{-2}) + \varepsilon^2 \nu + 1 \right) \\
\times \mathcal{E}(\rho, u|r, U)(t) \, dt \\
+ \frac{5\nu}{16} \int_0^T \int_{\Omega_R} (\mathcal{S}(\nabla u) - \mathcal{S}(\nabla U)) \cdot \nabla (u - U) + c(D, T)(\varepsilon^2 + \varepsilon^2 \nu^2)(1 + R^{-2}) + \varepsilon \nu^{-1})
\]
Similarly as before we deduce
\[
\mathcal{R}_3(t) \leq c\|\rho u\|_{L^\infty(0, T; L^2(\Omega_R))} \|b(\rho)(t, \cdot)\|_{L^2(\Omega_R)} + c\|\rho_0 u_0\|_{L^2(\Omega_R)} \|b(\rho_0)\|_{L^2(\Omega_R)} \\
\leq c(D, T)\varepsilon^2 + \sigma \varepsilon^{-2} \int_{\{\rho(t, \cdot) > \sigma^{-\alpha_1}\}} |b(\rho)|^2 + \sigma \varepsilon^{-2} \int_{\{\rho_0(\cdot) > \sigma^{-\alpha_1}\}} |b(\rho_0)|^2 \\
\leq c(D, T)\varepsilon^2 + \frac{1}{4} \mathcal{E}(\rho, u|r, U)(t) + \frac{1}{4} \mathcal{E}(\rho_0, u_0|r(0, \cdot), U(0, \cdot))
\]
where \(\sigma\) was suitably chosen and (56), along with assumption (42) were taken into account. Notice that the generic constants appearing in estimates (97) and (98) contain also positive powers of the expression \(\frac{\text{diam}(\Omega_R)}{R}\) that is bounded with respect to \(R\) due to assumption (39). Going back to (49), which we combine with estimates (79), (97) and (98), it follows by the Gronwall lemma that
\[
\mathcal{E}(\rho, u|r, U)(t) \leq \gamma \exp \left( \int_0^t \delta(s) \, ds \right) \quad t \in [0, T],
\]
where
\[
\gamma = c\mathcal{E}(\rho_0, u_0|r(0, \cdot), U(0, \cdot)) + c(D, T)(\varepsilon^n + R^{-1} + (\varepsilon R)^{-1} + \varepsilon^2 \nu^2(1 + R^{-2}) + \varepsilon \nu^{-1}) \\
\delta(\tau) = c(D, T) \left( \frac{1}{|\Omega_R|} \int_{\Omega_R} p(\rho) + \varepsilon^\frac{4}{3} \nu^{-1}(1 + R^{-4}) + (1 + \varepsilon^2)(1 + R^{-2}) + \varepsilon^2 \nu + 1 \right)
\]
In order to proceed we estimate the quantity \(|\Omega_R|^{-1} \int_0^T \int_{\Omega_R} p(\rho)\). We begin by repeating the part of the proof of Theorem 2.1, namely the proof of (20), as \(b(s) = s\) satisfies the assumptions of Theorem 2.1, to get
\[
\varepsilon^{-2} \int_0^T \int_{\Omega_R} p(\rho) \left( \rho - \frac{1}{|\Omega_R|} \int_{\Omega_R} \rho \right) = \sum_{i=1}^4 I_i
\]
where
\[
I_1 = -\int_0^T \int_{\Omega_R} \rho u \otimes u \cdot \nabla \mathcal{B} \left( \rho - \frac{1}{|\Omega_R|} \int_{\Omega_R} \rho \right), \\
I_2 = \int_0^T \int_{\Omega_R} \nu \mathcal{S}(\nabla u) \cdot \nabla \mathcal{B} \left( \rho - \frac{1}{|\Omega_R|} \int_{\Omega_R} \rho \right), \\
I_3 = \int_0^T \int_{\Omega_R} \rho u \cdot \mathcal{B} \left( \text{div}(\rho u) - \frac{1}{|\Omega_R|} \int_{\Omega_R} \text{div}(\rho u) \right), \\
I_4 = \int_{\Omega_R} \rho u \cdot \mathcal{B} \left( \rho - \frac{1}{|\Omega_R|} \int_{\Omega_R} \rho \right) - \int_{\Omega_R} \rho_0 u_0 \mathcal{B} \left( \rho_0 - \frac{1}{|\Omega_R|} \int_{\Omega_R} \rho_0 \right).
\]
By Lemma 4.2 we conclude
\[
\left\| \mathcal{B} \left( \rho - \frac{1}{|\Omega_R|} \int_{\Omega_R} \rho \right) \right\|_{L^p(0, T; W^{1, q}(\Omega_R))} \leq c\|\rho\|_{L^p(0, T; L^q(\Omega_R))} \quad \text{for } p \in [1, \infty], q \in (1, \infty)
\]
with the constant $c$ dependent also on the term of the form $\frac{\text{diam}(\Omega_R)}{R}$ that is bounded with respect to $R$ due to the assumption (39). Using the fact that $\rho < \bar{\rho}$, (56)$_{1,3}$, the Sobolev embedding, (101) and Lemma 4.2 we conclude

$$|I_1| \leq c\bar{\rho}
abla u\|_{L^2(0,T;L^6(\Omega_R))}^2 \|\rho\|_{L^\infty(0,T;L^{\frac{6}{5}}(\Omega_R))} \leq c\nu^{-1}|\Omega_R|^{\frac{3}{5}},$$

$$|I_2| \leq c\nu \|\mathcal{S}(\nabla u)\|_{L^2(0,T;L^2(\Omega_R))} \|\rho\|_{L^2(0,T;L^2(\Omega_R))} \leq c\nu^{\frac{1}{2}}T^{\frac{3}{2}}|\Omega_R|^{\frac{1}{2}},$$

$$|I_3| \leq c\epsilon\|\rho u\|_{L^2(0,T;L^2(\Omega_R))} \leq c,$$

$$|I_4| \leq c\epsilon\|\rho u\|_{L^\infty(0,T;L^2(\Omega_R))} \|\rho\|_{L^\infty(0,T;L^2(\Omega_R))} + c\|\rho_0u_0\|_{L^2(\Omega_R)} \|\rho_0\|_{L^2(\Omega_R)} \leq c|\Omega_R|^{\frac{3}{2}}.$$

Therefore, we get from (100) as $|\Omega_R| \geq cR^3$ by (39)

$$\frac{1}{|\Omega_R|} \int_0^T \int_{\Omega_R} p(\rho - \frac{1}{|\Omega_R|} \int_{\Omega_R} \rho) \leq c(D,T)\epsilon^2(\nu^{-1}R^{-1} + c\nu^{\frac{1}{2}}R^{-\frac{3}{2}} + 1).$$

(102)

Next, we denote

$$m_{\epsilon,R} = \frac{1}{|\Omega_R|} \int_{\Omega_R} \rho_0, \epsilon = \bar{\rho} + \frac{\epsilon}{|\Omega_R|} \int_{\Omega_R} \rho_0(1).$$

By (40) it follows that for any $\epsilon < \epsilon_0$

$$\frac{1}{|\Omega_R|} \int_{\Omega_R} \rho(t, x) \, dx = m_{\epsilon,R} < \bar{\rho}.$$  

(103)

Therefore we have

$$\frac{1}{|\Omega_R|} \int_0^T \int_{\Omega_R} p(\rho - \frac{1}{|\Omega_R|} \int_{\Omega_R} \rho) = \frac{1}{|\Omega_R|} \int_{\rho \leq \frac{\pi + m_{\epsilon,R}}{2}} p(\rho) \left( \rho - \frac{1}{|\Omega_R|} \int_{\Omega_R} \rho \right) + \frac{1}{|\Omega_R|} \int_{\rho > \frac{\pi + m_{\epsilon,R}}{2}} p(\rho) \left( \rho - \frac{1}{|\Omega_R|} \int_{\Omega_R} \rho \right) = J_1 + J_2.$$  

(104)

Next, we conclude

$$J_1 \leq \max_{s \in [0, \frac{1}{2}(\pi + \epsilon_0D)]} p(s) \frac{\bar{\rho} - m_{\epsilon,R}}{2}$$

(105)

by the assumed continuity of $p$. On the other hand we get

$$J_2 \geq \frac{\bar{\rho} - m_{\epsilon,R}}{2|\Omega_R|} \int_{\rho > \frac{\pi + m_{\epsilon,R}}{2}} p(\rho) \geq \frac{\bar{\rho} - m_{\epsilon,R}}{2|\Omega_R|} \int_{\rho > \frac{1}{2}(\pi + \epsilon_0D)} p(\rho).$$

(106)

Since $\inf_{\epsilon < \epsilon_0}(\bar{\rho} - m_{\epsilon,R}) \geq \bar{\rho} - \epsilon_0D > 0$ by (40), it follows that

$$\frac{1}{|\Omega_R|} \int_0^T \int_{\Omega_R} p(\rho) = \frac{1}{|\Omega_R|} \int_{\rho \leq \frac{\pi + \epsilon_0D}{2}} p(\rho) + \frac{1}{|\Omega_R|} \int_{\rho > \frac{\pi + \epsilon_0D}{2}} p(\rho) \leq \max_{s \in [0, \frac{\pi + \epsilon_0D}{2}]} p(s) + \frac{2}{\bar{\rho} - \epsilon_0D} J_2$$

$$\leq \max_{s \in [0, \frac{\pi + \epsilon_0D}{2}]} p(s) + \frac{2}{\bar{\rho} - \epsilon_0D} \left( \frac{1}{|\Omega_R|} \int_0^T \int_{\Omega_R} p(\rho) \left( \rho - \frac{1}{|\Omega_R|} \int_{\Omega_R} \rho \right) \right)$$

$$\leq \frac{c\epsilon^2}{\bar{\rho} - \epsilon_0D} (\nu^{-1}R^{-1} + c\nu^{\frac{1}{2}}R^{-\frac{3}{2}} + 1) + c \max_{s \in [0, \frac{\pi + \epsilon_0D}{2}]} p(s).$$

Employing the latter inequality in the $\delta$-term of (99) we conclude (47) by Lemma 2.2 and the proof is finished.

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Declarations

Conflict of interest On behalf of authors, the corresponding author states that there is no conflict of interest.

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4. Appendix

The ensuing lemma deals with renormalized solutions of the continuity equation. It collects versions of assertions [34, Lemmas 6.9 and 6.11] adopted for a function \( b \) considered in this paper.

Lemma 4.1. Let \( T > 0 \) and \( \Omega \subset \mathbb{R}^d, d \geq 2, \) be a bounded Lipschitz domain. Let \( \rho \in L^\infty(Q_T) \) be such that \( 0 \leq \rho < \bar{\rho} \) a.e. in \( Q_T \) and \( \rho \) together with \( u \in L^2(W^{1,2}_0(\Omega)^d) \) satisfy the continuity equation
\[
\partial_t \rho + \text{div}(\rho u) = 0 \quad \text{in} \quad \mathcal{D}'(Q_T)
\] (107)

Let \( b \in C^1([0, \bar{\rho}]) \) be a nonnegative function such that
\[
b(\rho) \in L^2(Q_T), b'(\rho) \in L^2(Q_T)
\] (108)

and

\[
b, b' \text{ are nondecreasing on } [\bar{\rho} - \alpha_0, \bar{\rho}] \text{ for some } \alpha_0 \in (0, \bar{\rho}).
\] (109)

Let \( \rho \) and \( u \) be extended by zero in \( (0, T) \times (\mathbb{R}^d \setminus \Omega) \).

1. Then the continuity equation (107) holds in the sense of renormalized solutions
\[
\partial_t b(\rho) + \text{div}(b(\rho)u) + (b'(\rho)\rho - b(\rho))\text{div} u = 0 \quad \text{in} \quad \mathcal{D}'((0, T) \times \mathbb{R}^d),
\] (110)

2. Moreover, for \( b_\alpha \) with \( \alpha \in (0, \alpha_0) \) defined as
\[
b_\alpha(s) = \begin{cases} 
b(s) & s \leq \bar{\rho} - \alpha, \\
b(\bar{\rho} - \alpha) & s > \bar{\rho} - \alpha
\end{cases}
\] (111)

the renormalized continuity equation holds in the form
\[
\partial_t b_\alpha(\rho) + \text{div}(b_\alpha(\rho)u) + (b'_\alpha(\rho)\rho - b_\alpha(\rho))\text{div} u = 0 \quad \text{in} \quad \mathcal{D}'((0, T) \times \mathbb{R}^d),
\] (112)

where we set \( b'_\alpha(\rho) = 0 \) in \( \{\rho = \bar{\rho} - \alpha\} \).

Proof. Applying the extension procedure from [34, Lemma 6.8] we get
\[
\partial_t \rho + \text{div}(\rho u) = 0 \quad \text{in} \quad \mathcal{D}'((0, T) \times \mathbb{R}^d)
\]

for the extensions \( \rho \in L^\infty((0, T) \times \mathbb{R}^d) \) and \( u \in L^2(0, T; W^{1,2}_{\text{loc}}(\mathbb{R}^d)) \) of \( \rho \) and \( u \) from assumptions of the lemma by zero in \( (0, T) \times (\mathbb{R}^d \setminus \Omega) \). Regularizing the latter equation over the spatial variables by the usual mollifier \( S_\varepsilon \) with \( \varepsilon > 0 \) yields
\[
\partial_t S_\varepsilon(\rho) + \text{div}(S_\varepsilon(\rho)u) = r_\varepsilon(\rho, u) \quad \text{a.e. in} \quad (0, T) \times \mathbb{R}^d,
\] (113)
implies the existence of a nonrelabeled subsequence \( S_\varepsilon(\rho) \rightarrow \rho \) a.e. in \((0, T) \times \mathbb{R}^d\). Next we multiply \((113)\) on \(b'_{(\delta)}(S_\varepsilon(\rho))\) and obtain

\[
\partial_t b_{(\delta)}(S_\varepsilon(\rho)) + \text{div}(b_{(\delta)}(S_\varepsilon(\rho))) + \left(b'_{(\delta)}(S_\varepsilon(\rho))S_\varepsilon(\rho) - b_{(\delta)}(S_\varepsilon(\rho))\right) \text{div} u = b'_{(\delta)}(S_\varepsilon(\rho))r_\varepsilon(\rho, u) \quad \text{a.e. in } (0, T) \times \mathbb{R}^d. \tag{116}
\]

The approximating property

\[
S_\varepsilon(\rho) \rightarrow \rho \quad \text{in } L^q(0, T; L^q_{\text{loc}}(\mathbb{R}^d)) \quad \text{for any } q \in [1, \infty) \tag{117}
\]

implies the existence of a nonrelabeled subsequence \( \{S_\varepsilon(\rho)\} \) such that \( S_\varepsilon(\rho) \rightarrow \rho \) a.e. in \((0, T) \times \mathbb{R}^d\). Taking into account the continuity of \(b^{(j)}\), where \( j \in \{0, 1\} \) denotes the order of the derivative, we have \(b^{(j)}_{(\delta)}(S_\varepsilon(\rho)) \rightarrow b^{(j)}_{(\delta)}(\rho)\) a.e. in \((0, T) \times \mathbb{R}^d\). Taking into consideration also \((120)\) and \((109)\) it follows that

\[
b_{(\delta)}(S_\varepsilon(\rho)) \rightarrow b_{(\delta)}(\rho) \quad \text{in } L^p(0, T; L^p_{\text{loc}}(\mathbb{R}^d)),
\]

\[
b'_{(\delta)}(S_\varepsilon(\rho)) \rightarrow b'_{(\delta)}(\rho) \quad \text{in } L^p(0, T; L^p_{\text{loc}}(\mathbb{R}^d)) \tag{119}
\]

for any \( p \in [1, \infty) \). The latter convergences and \((117)\) imply

\[
b'_{(\delta)}(S_\varepsilon(\rho))S_\varepsilon(\rho) - b_{(\delta)}(S_\varepsilon(\rho)) \rightarrow b'_{(\delta)}(\rho)\rho - b_{(\delta)}(\rho) \quad \text{in } L^p(0, T; L^p_{\text{loc}}(\mathbb{R}^d)).
\]

Combining \((116), (114), (119)\) and the latter convergence one arrives at \((110)\) for \( b = b_{(\delta)} \). Now we pass to the limit \( \delta \rightarrow 0_+ \). We have \(b^{(j)}_{(\delta)}(\rho) \rightarrow b^{(j)}(\rho)\) a.e. in \((0, T) \times \mathbb{R}^d, j \in \{0, 1\}\). Moreover, \((109)\) implies

\[
b_{(\delta)}(\rho) \leq \max\{\max_{|0, \tilde{\rho} - \alpha|} b, b(\rho)\},
\]

\[
|b'_{(\delta)}(\rho)| \leq \max\{\max_{|0, \tilde{\rho} - \alpha|} |b'|, |b'(\rho)|\}. \tag{120}
\]

Hence we infer by the Lebesgue dominated convergence theorem and \((108)\)

\[
b_{(\delta)}(\rho) \rightarrow b(\rho) \quad \text{in } L^2(0, T; L^2_{\text{loc}}(\mathbb{R}^d)),
\]

\[
b'_{(\delta)}(\rho) \rightarrow b'(\rho) \quad \text{in } L^2(0, T; L^2_{\text{loc}}(\mathbb{R}^d)).
\]

Having the latter convergences at hand we pass to the limit \( \delta \rightarrow 0_+ \) in \((110)\) with \( b = b_{(\delta)} \) and the first assertion of the lemma is proved.

In order to prove the second assertion, we begin with the proof of the following auxiliary identity

\[
(\tilde{\rho} - \alpha) \text{div} u = 0 \quad \text{a.e. in } \{\rho = \tilde{\rho} - \alpha\}. \tag{121}
\]
To this end we consider \( b \in C^1_c((0, \infty)) \) such that \( b(s) = s \) in \( \left[ \frac{2}{3}(\bar{\rho} - \alpha), \bar{\rho} - \frac{3}{4}\alpha \right] \) and \( b, b' \) are nondecreasing in \( [\bar{\rho} - \frac{3}{4}\alpha, \bar{\rho}] \). We define \( b_{+,\varepsilon}^+ = S^\varepsilon_2(b_{\alpha+\varepsilon}), b_{-,\varepsilon}^- = S^\varepsilon_2(b_{\alpha-\varepsilon}) \). Then we have as \( \varepsilon \to 0_+ \)

\[
\begin{align*}
  b_{+,\varepsilon}^+(s), b_{-,\varepsilon}^-(s) &\to b_{\alpha}(s) \quad s \in [0, \bar{\rho}], \\
  (b_{+,\varepsilon}^+)'(s), (b_{-,\varepsilon}^-)'(s) &\to b_{\alpha}'(s) \quad s \in [0, \bar{\rho}] \setminus \{ \bar{\rho} - \alpha \}, \\
  (b_{+,\varepsilon}^+)'(\bar{\rho} - \alpha) &\to 0, \\
  (b_{-,\varepsilon}^-)'(\bar{\rho} - \alpha) &\to 1.
\end{align*}
\]

By the first assertion of the lemma we have

\[
\begin{align*}
  \partial_t b_{+,\varepsilon}(\rho) + \text{div}(b_{+,\varepsilon}(\rho)u) + (\rho(b_{+,\varepsilon})'(\rho) - b_{+,\varepsilon}(\rho))\text{div}u &= 0 \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}^d), \\
  \partial_t b_{-,\varepsilon}(\rho) + \text{div}(b_{-,\varepsilon}(\rho)u) + (\rho(b_{-,\varepsilon})'(\rho) - b_{-,\varepsilon}(\rho))\text{div}u &= 0 \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}^d).
\end{align*}
\]

Letting \( \varepsilon \to 0_+ \) and employing the convergences from (122) we deduce by the Lebesgue dominated convergence theorem from the latter identities

\[
\begin{align*}
  \partial_t b_{\alpha}(\rho) + \text{div}(b_{\alpha}(\rho)u) + (\rho(b_{\alpha})'(\rho)\chi_{\{\rho \neq \bar{\rho} - \alpha\}} - b_{\alpha}(\rho))\text{div}u &= 0 \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}^d), \\
  \partial_t b_{\alpha}(\rho) + \text{div}(b_{\alpha}(\rho)u) + (\rho(b_{\alpha})'(\rho)\chi_{\{\rho \neq \bar{\rho} - \alpha\}} + (\bar{\rho} - \alpha)\chi_{\{\rho = \bar{\rho} - \alpha\}} - b_{\alpha}(\rho))\text{div}u &= 0 \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}^d).
\end{align*}
\]

Subtracting the latter equations we conclude (121).

Next, we consider for fixed \( \varepsilon < \alpha S_\varepsilon(b_{\alpha}) \), the mollification of \( b_{\alpha} \) extended by \( b(\bar{\rho} - \alpha) \) in \( (\bar{\rho}, \bar{\rho} + 1] \) and by \( 0 \) outside of \( [0, \bar{\rho} + 1] \). As \( S_\varepsilon(b_{\alpha}) \) is constant in a vicinity of \( \bar{\rho} \), it fulfills (109) and \( S_\varepsilon(b_{\alpha})(\rho) \) satisfies (19), it follows that

\[
\begin{align*}
  \partial_t S_\varepsilon(b_{\alpha})(\rho) + \text{div}(S_\varepsilon(b_{\alpha})(\rho)u) + ((S_\varepsilon(b_{\alpha}))'(\rho)\rho - S_\varepsilon(b_{\alpha}(\rho)))\text{div}u &= 0 \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}^d).
\end{align*}
\]

Furthermore, we have as \( \varepsilon \to 0_+ \)

\[
\begin{align*}
  S_\varepsilon(b_{\alpha})(s) &\to b_{\alpha}(s) \quad s \in [0, \bar{\rho}], \\
  (S_\varepsilon(b_{\alpha}))'(s) &\to b_{\alpha}'(s) \quad s \in [0, \bar{\rho} - \alpha) \cup (\bar{\rho} - \alpha, \bar{\rho}].
\end{align*}
\]

Hence we infer that as \( \varepsilon \to 0_+ \)

\[
\begin{align*}
  S_\varepsilon(b_{\alpha})(\rho) &\to b_{\alpha}(\rho) \quad \text{a.e. in } (0, T) \times \mathbb{R}^d, \\
  (S_\varepsilon(b_{\alpha}))'(\rho) &\to b_{\alpha}'(\rho) \quad \text{a.e. in } (0, T) \times \mathbb{R}^d \setminus \{ \rho = \bar{\rho} - \alpha \}.
\end{align*}
\]

Employing (121) we deduce that \( \rho(S_\varepsilon(b_{\alpha}))'(\rho)\text{div}u = 0 \) a.e. in \( \{ \rho = \bar{\rho} - \alpha \} \). Using the convergences from (124), the uniform bounds with respect to \( \varepsilon \) on \( S_\varepsilon(b_{\alpha}), S_\varepsilon(b_{\alpha})' \) and the Lebesgue dominated convergence theorem we pass to the limit \( \varepsilon \to 0_+ \) in (123) to conclude (122).

**Lemma 4.2.** Let \( \Omega \subset \mathbb{R}^d \) be a starshaped domain with respect to a ball \( B \) possessing the radius \( R \). There exists a linear operator \( B : C^\infty_c(\Omega) \to C^\infty_c(\Omega)^d \) such that \( \text{div}B(f) = f \) provided that \( \int_{\Omega} f = 0 \). Moreover, \( B \) can be extended in a unique way as a bounded linear operator

1. \( B : L^p(\Omega) \to W^{1,p}(\Omega)^d \) such that \( \|B(f)\|_{W^{1,p}(\Omega)} \leq c\|f\|_{L^p(\Omega)} \)
2. \( B : \{ f \in (W^{1,p}(\Omega))' : \langle f, 1 \rangle = 0 \} \to L^p(\Omega)^d \) such that \( \|B(f)\|_{L^p(\Omega)} \leq c\|f\|_{(W^{1,p}(\Omega))'} \)

for any \( p \in (1, \infty) \) where the constants \( c \) take the form

\[
c = c_0(p, d) \left( \frac{\text{diam}(\Omega)}{R} \right)^d \left( 1 + \frac{\text{diam}(\Omega)}{R} \right).
\]

Assertions in the following lemma are based on the results from [23]

**Lemma 4.3.** Let \( v_0 \in W^{m,2}(\mathbb{R}^3) \) with \( m > 4 \) be such that \( \text{div}v_0 = 0 \) in \( \mathbb{R}^3 \). Then there is \( T_{max} > 0 \) and a classical solution \( v \), unique in the class

\[
v \in C([0, T_{max}), W^{m,2}(\mathbb{R}^3)^3), \quad \partial_t v \in C([0, T_{max}); W^{m-1,2}(\mathbb{R}^3)^3)
\]
to the initial value problem
\[ \partial_t v + v \cdot \nabla v + \nabla \Pi = 0 \text{ in } (0, T_{max}) \times \mathbb{R}^3, \]
\[ v(0, \cdot) = v_0, \text{div} v_0 = 0 \text{ in } \mathbb{R}^3. \]

Furthermore, the associate pressure \( \Pi \) can be expressed as
\[ \Pi = (-\Delta)^{-1} \text{div} \text{div} (v \otimes v). \]

implying particularly that \( \Pi \in C^1([0, T]; C^1(\mathbb{R}^3) \cap W^{1,2}(\mathbb{R}^3)), T \in (0, T_{max}). \)

References

[1] Berthelin, F., Degond, P., Le Blanc, V., Moutari, S., Rascle, M., Royer, J.: A traffic-flow model with constraints for the modeling of traffic jams. Math. Models Methods Appl. Sci. 18, 1269–1298 (2008)
[2] Berthelin, F., Degond, P., Delitala, M., Rascle, M.: A model for the formation and evolution of traffic jams. Arch. Ration. Mech. Anal. 187(2), 185–220 (2008)
[3] Bianchini, R., Perrin, C.: Soft congestion approximation to the one-dimensional constrained Euler equations. Nonlinearity 34(10), 6901–6929 (2021)
[4] Bresch, D., Perrin, C., Zatorska, E.: Singular limit of a Navier–Stokes system leading to a free/congested zones two-phase model. C.R. Math. 352(9), 685–690 (2014)
[5] Bresch, D., Benard, M.: Development of congestion in compressible flow with singular pressure. Asymptot. Anal. 103(1–2), 95–101 (2017)
[6] Bresch, D., Nečasová, Š, Perrin, C.: Compression effects in heterogeneous media. J. Éc. Polytech. Math. 6, 433–467 (2019)
[7] Bouchut, F., Brenier, Y., Cortes, J., Ripoll, J.-F.: A hierarchy of models for two-phase flows. J. NonLinear Sci. 10(6), 639–660 (2000)
[8] Carnahan, N.F., Starling, K.E.: Equation of state for nonattracting rigid spheres. J. Chem. Phys. 51, 635–636 (1969)
[9] Choe, H.J., Novotný, A., Yang, M.: Compressible Navier-Stokes system with hard sphere pressure law and general inflow-outflow boundary conditions. J. Differ. Equ. 266(6), 3066–3099 (2019)
[10] Desjardins, B., Grenier, E.: Low Mach number limit of viscous compressible flows in the whole space. Proc. R. Soc. Lond. 455, 2271–2279 (1999)
[11] Degond, P., Hua, J.: Self-organized hydrodynamics and path formation in crowds. J. Comput. Phys. 237, 299–319 (2013)
[12] Degond, P., Hua, J., Navoret, L.: Numerical simulations of the Euler system with congestion constraint. J. Comput. Phys. 230(22), 8057–8088 (2011)
[13] DiPerna, R.J., Lions, P.-L.: Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math. 98(3), 511–547 (1989)
[14] Feireisl, E., Jin, B.J., Novotný, A.: Relative entropies, suitable weak solutions, and weak-strong uniqueness for the compressible Navier–Stokes system. J. Math. Fluid Mech. 14(4), 717–730 (2012)
[15] Feireisl, E., Lu, Y., Málek, J.: On PDE analysis of flows of quasi-incompressible fluids. ZAMM Z. Angew. Math. Mech. 96(4), 491–508 (2016)
[16] Feireisl, E., Lu, Y., Novotný, A.: Weak-strong uniqueness for the compressible Navier–Stokes equations with a hard-sphere pressure law. Sci China Math 61(11), 2003–2016 (2018)
[17] Feireisl, E., Nečasová, Š, Sun, Y.: Inviscid incompressible limits on expanding domains. Nonlinearity 27(10), 2465–2478 (2014)
[18] Feireisl, E., Novotný, A.: Singular limits in thermodynamics of viscous fluids, Advances in Mathe- matical Fluid Mechanics, 2nd edn. Birkhäuser, Basel (2017)
[19] Feireisl, E., Zhang, P.: Quasi-neutral limit for a model of viscous plasma. Arch. Ration. Mech. Anal. 197(1), 271–295 (2010)
[20] Feireisl, E., Novotný, A., Petzeltová, H.: On the existence of globally defined weak solutions to the Navier–Stokes equations. J. Math. Fluid Mech. 3(4), 358–392 (2001)
[21] Hongqin, L.: Carnahan-Starling type equations of state for stable hard disk and hard sphere fluids. Molecular Phys. 119(9) (2021)
[22] Kastler, A., Vichnievsky, R., Bruhat, G.: Cours de physique générale à l’usage de l’enseignement supérieur scientifique et technique: Thermodynamique, Masson et Cie (1962)
[23] Kato, T., Lai, C.Y.: Nonlinear evolution equations and the Euler flow. J. Funct. Anal. 56, 15–28 (1984)
[24] Kellihé, J.P., Lopes Filho, M.C., Nussenzveig-Lopes, H.: Vanishing viscosity limit for an expanding domain in space. Ann. Inst. H. Poincaré Anal. Non Linéaire 26(6), 2521–2537 (2009)
[25] Klarinerman, S., Majda, A.: Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. Commun. Pure Appl. Math. 34, 481–524 (1981)
[26] Kolafa, J., Labík, S., Malijevsky, A.: Accurate equation of state of the hard sphere fluid in stable and metastable regions. Phys. Chem. Chem. Phys. 6, 2335–2340 (2004)
[27] Lions, P.-L.: Mathematical topics in fluid mechanics Vol. 2, Oxford Lecture Series in Mathematics and its Applications, 10 (1998)
[28] Lions, P.-L., Masmoudi, N.: Incompressible limit for a viscous compressible fluid. J. Math. Pures Appl 77, 585–627 (1998)
[29] Majda, A.J., Bertozzi, A.L.: Vorticity and Incompressible Flow. Cambridge Texts in Applied Mathematics, 27. Cambridge University Press, Cambridge (2002)
[30] Maury, B.: Prise en compte de la congestion dans les modèles de mouvements de foules, Actes des colloques Caen (2012)
[31] Masmoudi, N.: Incompressible inviscid limit of the compressible Navier-Stokes system. Ann. Inst. H. Poincaré Anal. Non Linéaire 18, 199–224 (2001)
[32] Masmoudi, N.: Examples of singular limits in hydrodynamics. In: Dafermos, C., Feireisl, E. (ed.) Handbook of Differential Equations, III. Elsevier, Amsterdam (2006)
[33] Nečasová, S., Novotný, A., Roy, A.: Compressible Navier–Stokes system with the hard sphere pressure law in an exterior domain. Z. Angew. Math. Phys. 73(5) (2022)
[34] Novotný, A., Stráskraba, I.: Introduction to the Mathematical Theory of Compressible Flow. Oxford University Press, Oxford (2004)
[35] Perrin, C., Zatorska, E.: Free/congested two-phase model from weak solutions to multi-dimensional compressible Navier-Stokes equations. Commun. Partial Differ. Equ. 40(8), 1558–1589 (2015)
[36] Perrin, C.: Modelling of phase transitions in one-dimensional granular flows. ESAIM: Proc. Surv. 58, 78–97 (2017)
[37] Ruzhansky, M., Smith, J.: Dispersive and Strichartz estimates for hyperbolic equations with constant coefficients, MSJ Memoirs, 22. Mathematical Society of Japan, Tokyo (2010)
[38] Song, Y., Mason, E.A., Stratt, R.M.: Why does the Carnahan-Starling equation work so well? J. Phys. Chem. 95(19), 6916–6919 (1991)
[39] Schochet, S.: Fast singular limits of hyperbolic PDEs. J. Differ. Equ. 114(2), 476–512 (1994)
[40] Strichartz, R.: Convolutions with kernels having singularities on a sphere. Trans. Amer. Math. Soc. 148, 461–471 (1970)
[41] Wang, S., Jiang, S.: The convergence of the Navier–Stokes–Poisson system to the incompressible Euler equations. Commun. Partial Differ. Equ. 31(4–6), 571–591 (2006)
[42] Zeytounian, R.K.: Asymptotic Modeling of Atmospheric Flows. Springer, Berlin (1990)

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