A Note on $G$-intersecting Families

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Abstract

Consider a graph $G$ and a $k$-uniform hypergraph $H$ on common vertex set $[n]$. We say that $H$ is $G$-intersecting if for every pair of edges in $X, Y ∈ H$ there are vertices $x ∈ X$ and $y ∈ Y$ such that $x = y$ or $x$ and $y$ are joined by an edge in $G$. This notion was introduced by Bohman, Frieze, Ruszinkó and Thoma who proved a natural generalization of the Erdős-Ko-Rado Theorem for $G$-intersecting $k$-uniform hypergraphs for $G$ sparse and $k = O(n^{1/4})$. In this note, we extend this result to $k = O(√n)$.

1 Introduction

A hypergraph is said to be intersecting if every pair of edges has a nonempty intersection. The well-known theorem of Erdős, Ko and Rado \[2, 3\] details the extremal $k$-uniform intersecting hypergraph on $n$ vertices.

**Theorem 1** (Erdős-Ko-Rado). Let $k ≤ n/2$ and $H$ be a $k$-uniform, intersecting hypergraph on vertex set $[n]$. We have $|H| ≤ \binom{n-1}{k-1}$. Furthermore, $|H| = \binom{n-1}{k-1}$ if and only if there exists $v ∈ [n]$ such that $H = \{e ∈ \binom{n}{k} : v ∈ e\}$.

Of course, for $k > n/2$ the hypergraph consisting of all $k$-sets is intersecting. So, extremal $k$-intersecting hypergraphs come in one of two forms, depending on the value of $k$.

Bohman, Frieze, Ruszinkó and Thoma [1] introduced a generalization of the notion of an intersecting hypergraph. Let $G$ be a graph on a vertex set $[n]$ and $H$ be a hypergraph, also on vertex set $[n]$. We say $H$ is $G$-intersecting if for any $e, f ∈ H$, we have $e ∩ f ≠ ∅$ or there are vertices $v, w$ with $v ∈ e, w ∈ f$ and $v ∼_G w$. We are interested in the size and structure of maximum $G$-intersecting hypergraphs; in particular, we investigate

$$N(G, k) = \max \left\{|H| : H ⊆ \binom{[n]}{k} \text{ and } H \text{ is } G\text{-intersecting}\right\}.$$
Clearly, Erdős-Ko-Rado gives the value of $N(E_n, k)$ where $E_n$ is the empty graph on vertex set $[n]$. For a discussion of $N(G, k)$ for some other specific graphs see [1].

In this note we restrict our attention to sparse graphs: those graphs for which $n$ is large and the maximum degree of $G$, $\Delta(G)$, is a constant in $n$. What form can a maximum $G$-intersecting family take? If $K$ is a maximum clique in $G$ then a candidate for a maximum $G$-intersecting family is

$$\mathcal{H}_K := \left\{ X \in \binom{[n]}{k} : X \cap K \neq \emptyset \right\}.$$ 

Note that such a hypergraph can be viewed as a natural generalization of the maximum intersecting hypergraphs given by Erdős-Ko-Rado. However, for many graphs and maximum cliques $K$ one can add hyperedges to $\mathcal{H}_K$ to obtain a larger $G$-intersecting hypergraph.

Consider, for example, $C_n$, the cycle on vertex set $[n]$ (i.e. the graph on $[n]$ in which $u$ and $v$ are adjacent iff $u - v \in \{1, n - 1\}$ mod $n$). The set $\{2, 3\}$ is a maximum clique in $C_n$ and the set

$$\mathcal{H}_{\{2,3\}} \cup \left\{ X \in \binom{[n]}{k} : \{1, 4\} \subseteq X \right\}$$ 

is $G$ intersecting. Bohman, Frieze, Ruszinkó and Thoma showed that

$$N(C_n, k) = \binom{n}{k} - \binom{n - 2}{k} + \binom{n - 4}{k - 2}$$

(i.e. the hypergraph given in (1) is maximum) for $k$ less than a certain constant times $n^{1/4}$. In fact, they showed that for arbitrary sparse graphs and $k$ small, $N(G, k)$ is given by a hypergraph that consists of $\mathcal{H}_K$ for some clique $K$ together with a number of ‘extra’ hyperedges that cover the clique $K$ in $G$ (see Theorem 1 of [1]). In this note we extend this result to larger values of $k$.

**Theorem 2.** Let $G$ be a graph on $n$ vertices with maximum degree $\Delta$ and clique number $\omega$. There exists a constant $C$ (depending only on $\Delta$ and $\omega$) such that if $\mathcal{H}$ is a $G$-intersecting $k$-uniform hypergraph and $k < C n^{1/2}$ then

$$|\mathcal{H}| \leq \binom{n}{k} - \binom{n - \omega}{k} + \binom{\omega(\Delta - \omega + 1)}{2} \binom{n - \omega - 2}{k - 2}.$$ 

Furthermore, if $\mathcal{H}$ is a $G$-intersecting family of maximum cardinality then there exists a maximum clique $K$ in $G$ such that $\mathcal{H}$ contains all $k$-sets that intersect $K$.

An immediate corollary of this Theorem is that (2) holds for $k < C \sqrt{n}$.

Of course, a maximum $G$-intersecting hypergraph will not be of the form ‘$\mathcal{H}_K$ together with some extra hyperedges’ if $k$ is too large. Even for sparse graphs, when $k$ is large enough, there are hypergraphs that consist of nearly all of $\binom{[n]}{k}$ that are $G$-intersecting. In particular, Bohman, Frieze, Ruszinkó and Thoma showed that if $G$ is a sparse graph with
minimum degree \( \delta \), \( c \) is a constant such that \( c - (1 - c)^{\delta+1} > 0 \) and \( k > cn \), then the size of the largest \( G \)-intersecting, \( k \)-uniform hypergraph is at least \( (1 - e^{-\Omega(n)})(\binom{n}{k}) \) (see Theorem 7 of [1]). In some sense, this generalizes the trivial observation that \( \binom{n}{k} \) is intersecting for \( k > n/2 \).

There is a considerable gap between the values of \( k \) for which we have established these two types of behavior for maximum \( G \)-intersecting families. For example, for \( C_n \) we have \( k < C \sqrt{n} \) while we have \( N(C_n, k) > (1 - o(1))\binom{n}{k} \) for \( k \) greater than roughly \( .32n \). What happens for other values of \( k \)? Are there other forms that a maximum \( G \)-intersecting family can take? Bohman, Frieze, Ruszinkó and Thoma conjecture that this is not the case, at least for the cycle.

**Conjecture 1.** There exists a constant \( c \) such that for any fixed \( \epsilon > 0 \)

\[
    k \leq (c - \epsilon)n \Rightarrow N(C_n, k) = \binom{n}{k} - \binom{n - 2}{k} + \binom{n - 4}{k - 2} 
\]

\[
    k \geq (c + \epsilon)n \Rightarrow N(C_n, k) = (1 - o(1))\binom{n}{k} 
\]

The remainder of this note consists of the proof of Theorem 2.

**2 Utilizing \( \tau \)**

Let \( H \) be a hypergraph and \( G \) be a graph on vertex set \([n]\). For \( X \subseteq [n] \), we define

\[
    N(X) := \{ v \in V(G) : v \sim_G w \text{ for some } w \in X \} \cup X. 
\]

For \( x \in [n] \) we write \( N(x) \) for \( N(\{x\}) \). We will define the hypergraph \( F \) by setting \( f \in F \) if and only if \( f = N(h) \) for some \( h \in H \). Note that if \( H \) is \( G \)-intersecting, then

\[
    h \in H, f \in F \Rightarrow h \cap f \neq \emptyset. \quad (3) 
\]

The quantity \( \tau(F) \) is the cover number of \( F \).

The proof of Theorem 2 follows immediately from Lemma 1 which deals with the case where \( \tau(F) \geq 2 \) and Lemma 2 which deals with the case where \( \tau(F) = 1 \).

**Lemma 1.** Let \( G \) be a graph on \( n \) vertices with maximum degree \( \Delta \) and clique number \( \omega \), both constants. If \( k < \sqrt{\frac{\omega n}{2(\Delta+1)^2}} \), \( H \) is a \( k \)-uniform, \( G \)-intersecting hypergraph on \( n \) vertices and \( n \) is sufficiently large, then \( \tau(F) = 1 \) or

\[
    |H| < \binom{n}{k} - \binom{n - \omega}{k}. \quad (4) 
\]
Proof.

Suppose, by way of contradiction, that $\tau = \tau(\mathcal{F}) \geq 2$ and (4) does not hold. For $v \in [n]$ set $\mathcal{H}_v = \{ f \in \mathcal{F} : u \in f \}$, and for $Y \subseteq [n]$ set $\mathcal{H}_Y = \{ f \in \mathcal{F} : Y \subseteq f \}$. Let $\mathcal{F}_u$ and $\mathcal{F}_Y$ be defined analogously.

We first use $\tau > 1$ to get an upper bound $|H_u|$ for an arbitrary $u \in [n]$. First note that, since $\tau > 1$, there exists $X_1 \in \mathcal{F}$ such that $u \notin X_1$. It follows from (3) that each $f \in \mathcal{F}_u$ must intersect $X_1$. In other words, we have

$$\mathcal{F}_u = \bigcup_{u_1 \in X_1} \mathcal{F}_{\{u, u_1\}}.$$ 

This observation can be iterated: if $i < \tau$ and $Y = \{ u = u_0, u_1, \ldots, u_{i-1} \}$ then there exists $X_i \in \mathcal{F}$ such that $X_i \cap Y = \emptyset$, and we have

$$\mathcal{F}_Y = \bigcup_{u_i \in X_i} \mathcal{F}_{Y \cup \{u_i\}}.$$ 

Since $|f| \leq (\Delta + 1)k$ for all $f \in \mathcal{F}$, it follows that we have

$$|\mathcal{H}_u| \leq ((\Delta + 1)k)^{\tau-1} \binom{n-\tau}{k-\tau}. \quad (5)$$

On the other hand, by the definition of $\tau$, there exists $v \in [n]$ for which

$$\frac{1}{\tau} \left[ \binom{n}{k} - \binom{n - \omega(G)}{k} \right] \leq |\mathcal{F}_v|.$$ 

It follows that there exists $u \in [n]$ such that

$$\frac{1}{\tau(\Delta + 1)} \left[ \binom{n}{k} - \binom{n - \omega(G)}{k} \right] \leq |\mathcal{H}_u|.$$ 

Applying (5) to this vertex we have

$$\binom{n}{k} - \binom{n - \omega(G)}{k} \leq \tau(\Delta + 1)^{\tau-1} \binom{n-\tau}{k-\tau}.$$ 

In order to show that this is a contradiction, we first note that $\tau(\Delta + 1)^{\tau-1} \binom{n-\tau}{k-\tau}$ is a function that is decreasing in $\tau$. Indeed, for $\tau \geq 2$ we have

$$\frac{n - \tau}{k-\tau} \geq \frac{n - 2}{k-2} \geq \frac{3}{2}(\Delta + 1)k \geq \frac{\tau + 1}{\tau}(\Delta + 1)k$$

(note that the condition $k < \sqrt{\frac{\omega}{2(\Delta + 1)^2}}$ is used in the second inequality). It follows that we have

$$\binom{n}{k} - \binom{n - \omega(G)}{k} \leq 2(\Delta + 1)^2 k \binom{n-2}{k-2},$$

...
which is not true if $k < \sqrt{\frac{n\omega(G)}{2(\Delta+1)^2}}$ and $n$ is large enough.

\[ \square \]

**Lemma 2.** Let $G$ be a graph on $[n]$ with maximum degree $\Delta$, a constant. If $\mathcal{H}$ is a $k$-uniform, $G$-intersecting hypergraph on $[n]$, $k \leq \sqrt{\frac{n}{\Delta(\Delta+1)}}$, $\tau(\mathcal{F}) = 1$, $n$ is sufficiently large and $\mathcal{H}$ is of maximum size, then there exists a maximum-sized clique $K$ in $G$ such that $\mathcal{H}$ contains every $k$-set that intersects $K$.

**Proof.** Let us suppose $\mathcal{H}$ is of maximum size and let $u$ be a cover for $\mathcal{F}$, the hypergraph defined above.

For $v \in [n]$, let $\mathcal{H}_v$ denote the members of $\mathcal{H}$ that contain $v$. Since $\mathcal{H}$ is assumed to be extremal, we may assume that $|\mathcal{H}_u| = \binom{n-1}{k-1}$. Let $K$ be the set of $v \in [n]$ such that $|\mathcal{H}_v| = \binom{n-1}{k-1}$. If $n > (\Delta + 2)k$ then $K$ must be a clique in $G$; otherwise, we could find two sets that are not $G$-intersecting in $\mathcal{H}$.

We now show that the clique $K$ is maximal. Assume for the sake of contradiction that $v$ is adjacent to every element of $K$ but $v \notin K$ (i.e. $|\mathcal{H}_v| < \binom{n-1}{k-1}$). There exists $h \in \mathcal{H}$ that $h$ contains no member of $N(u)$. It follows from (3) that we have

$$|\mathcal{H}_v| < (\Delta + 1)k \binom{n-2}{k-2}.$$ 

Since this bounds holds for all vertices in $N(u) \setminus K$, if we have

$$\Delta(\Delta + 1)k \binom{n-2}{k-2} < \binom{n-|K|-1}{k-1}$$

then the number of $k$-sets that contain $v$ but do not intersect $K$ outnumber those edges in $\mathcal{H}$ that contain no member of $K$. In other words, if (6) holds then we get a contradiction to the maximality of $\mathcal{H}$. However, (6) holds for $n$ sufficiently large (here we use $k < \sqrt{\frac{n}{\Delta(\Delta+1)}}$).

It remains to show that $K$ is a maximum clique. Since $K$ is maximal, it must be that any member of $\mathcal{H}$ that does not contain a member of $K$ must contain at least 2 members of $N(K) \setminus K$. If

$$\binom{n}{k} - \binom{n-|K|}{k} + \binom{|K|(\Delta - |K| + 1)}{2} \binom{n-|K|-2}{k-2} < \binom{n}{k} - \binom{n-|K|-1}{k}$$

and there is some clique of size $|K| + 1$, then $\mathcal{H}$ cannot be maximum-sized. But (7) holds for $k = o(n)$. So the maximum-sized $G$ intersecting family must contain all members of $\bigcup_{v \in K} \mathcal{F}_v$ for some $K$ with $|K| = \omega(G)$. \[ \square \]
References

[1] T. Bohman, A. Frieze, M. Ruszinkó, L. Thoma, G-intersecting Families, Combinatorics, Probability and Computing 10, 376-384.

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[3] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. 2 12 (1961), 313-320.