\( \mathcal{W}_\infty \)-algebras in \( n \) complex dimensions and Kodaira-Spencer deformations: a symplectic approach

G. BANDELLONI \(^{a}\) and S. LAZZARINI \(^{b}\)

\(^a\) Dipartimento di Fisica dell’Università di Genova, 
Via Dodecaneso 33, I-16146 GENOVA, Italy 
and 
Istituto Nazionale di Fisica Nucleare, INFN, Sezione di Genova 
via Dodecaneso 33, I-16146 GENOVA, Italy

\(^b\) Centre de Physique Théorique, CNRS Luminy, Case 907, 
F-13288 MARSEILLE Cedex, France

Abstract

It is shown that the notion of \( \mathcal{W}_\infty \)-algebra originally carried out over a (compact) Riemann surface can be extended to \( n \) complex dimensional (compact) manifolds within a symplectic geometrical setup. The relationships with the Kodaira-Spencer deformation theory of complex structures are discussed. Subsequently, some field theoretical aspects at the classical level are briefly underlined.

1998 PACS Classification: 11.10.Gh - 11.25 hf - 03.70 
Keywords: Complex manifolds, Kodaira-Spencer deformation, symplectic geometry, \( W \)-algebras.
1 Introduction

It is fair to say that the concept of dimensionality plays an important role in Physics. In particular, the developments in quantum field theory as well as in statistical mechanics have greatly enlarged its importance. In renormalization theory, string field models, the concept of dimension is found to be not only a characterization of the background space were the physical phenomena are supposed to take place, but also a physical regularizing parameter. Indeed, a world with a given dimension very often shows merits and faults not found in some others of different dimensions. This led to the search for hyperspaces which could gather together the praises and avoid the imperfections of the theoretical models.

For instance, two dimensional models show the great relevance of complex structures in Quantum Field Theory. Moreover, this approach produces a dimensional halving, but, in spite of the low dimensionality, the conformal models are described by means of an infinite dimensional algebra.

So, the wide class of these “new” symmetries has been supporting the conjecture that life in two dimensions could be easier and more convenient. The so-called $W$-algebras were a byproduct of this feasibility in two dimensional spaces. For an extensive review on the various possibilities offered by these kinds of symmetries we refer to. Thus the question of extending this type of symmetries to higher dimensional spaces comes naturally. The extension required the use of the Kodaira-Spencer deformation theory. In particular, chiral symmetries have already been extended from 2D conformal models built on a Riemann surface to models to a $n$ complex dimensional complex manifold. Note that Kodaira-Spencer type deformation theories have been already used to describe $W_\infty$ in two (or more) dimensions in order to study holomorphic properties (chiral splitting) or mirror manifolds of arbitrary complex dimension. Their cohomologies have been investigated both in Lagrangian Field Theory models and in more general mathematical aspects.

Therefore we shall address in the present paper the extension to $n$ complex dimensions of our BRS treatment for $W_\infty$-algebra grounded on a symplectic approach. In the latter, the algebra emerges from a ghost realization geometrically constructed from the symplectic approach and as a byproduct the infinite number of chiral ghost fields $C^{(n)}$, $n = 1, 2, ..., turn out to be $(-n, 0)$-conformal fields and their infinitesimal variations have a well defined geometrical setting. To be more specific, let us remind how the chiral $W_\infty$-algebra is recovered in the bidimensional case over a Riemann surface. For any positive integer $n$, the local variations of the chiral ghosts are

$$SC^{(n)}(z, \bar{z}) = \sum_{m=1}^{n} m C^{(m)}(z, \bar{z}) \partial_{z} C^{(n-m+1)}(z, \bar{z}).$$

(1.1)

Introducing by duality to each ghost a local operator $T_{(n)}(z, \bar{z})$ in order to construct the anti-
commuting functional BRS operator \[23\],

\[
\delta = \sum_{n \geq 1} \int d^2 z \wedge d z \left( C^{(n)}(z, \bar{z}) T_n(z, \bar{z}) + SC^{(n)}(z, \bar{z}) \frac{\delta}{\delta C^{(n)}(z, \bar{z})} \right),
\] (1.2)

namely, \(\{\delta, \delta\} = 0\), leads to the following local commutation relations,

\[
\left[ T_n(z, \bar{z}), T_m(z', \bar{z}') \right] = n \partial_{z'} \delta^{(2)}(z' - z) T_{n+m-1}(z, \bar{z}) - m \partial_z \delta^{(2)}(z - z') T_{n+m-1}(z', \bar{z}) (1.3)
\]

which turn out to be a realization of the so-called \(W_\infty\)-algebra if one goes to the Fourier modes.

We stress that the well defined ghost realization allows one to write down the extension of the \(W_\infty\)-algebra to higher dimensions. Moreover we want to take advantage of the symplectic description for incompressible flows in order to extend to \(n\) dimensions the notion of \(W_\infty\)-algebra, which in two dimensions is related to area preserving diffeomorphisms, see for instance \[13\] and references therein.

The algebra will be described in our approach by means of the Kodaira-Spencer deformation theory of complex structures but reformulated in a symplectic framework. The physical motivation of investigating the subject relies is connected to the so-called \(W\)-gravity and also on the fact that quantizing a conformal gravitational theory would incorporate all the possible configurations of the gravitational fields. By the way, “well defined” gravitational conformal models are fully described by means of the complex structure of the surrounding space. Therefore a complete description just at the classical level of all its possible deformations might be relevant for a successful quantum improvement.

The paper is organized as follows. We shall first briefly introduce in a non-technical way, the Kodaira-Spencer deformations, referring the reader to the book by Kodaira \[6\] for a more complete survey, especially Chapters 2, 4 and 5. Then Section 3 will give a geometrical setting of symplectomorphisms in a generic \(n\) complex dimensional space in order to introduce the BRS formulation of the (infinitesimal) diffeomorphisms of a symplectic space. Furthermore in Section 4 the specific Kodaira-Spencer deformation of complex structures related to \(W_\infty\)-algebra will be presented through a symplectic approach by using a ghost representation. It is recalled that the symplectic treatment of the two-dimensional case for \(W\)-algebras \[20\] provides a well defined geometrical definition of the ghost fields and their BRS variations as well. In the present paper we avail ourselves of that symplectic approach, in order to address the problem of extending to arbitrary complex dimensions the notion of \(W_\infty\)-algebra and its consequences, in particular, for the study of Lagrangians subject to that type of symmetry to which a very brief Section will be devoted.
2 A short account on the Kodaira-Spencer deformation

Let \( M \) be a \( n \) dimensional (compact) complex manifold described in terms of background local complex coordinates:
\[
(z^k) := (z^1, z^2, \ldots, z^n), \quad k = 1 \cdots n
\] (2.4)
and the subordinated differentiable structure \((z^k, \bar{z}^k)\) turns \( M \) into a \( 2n \) real dimensional manifold.

Its complex structure is determined by the \( \overline{\partial} \equiv n \sum_{i=1}^{n} dz^i \partial_i \) operator. In order to control the deformation, usually a complex deformation parameter \( t = (t_1, \ldots, t_n) \) is introduced. Basically the physical implications of this mathematical field of interest, rises from the primitive idea that a complex manifold is composed of a set of coordinate neighborhoods patched together. Obviously the patching procedure sewing should be irrelevant to the manifold description. In this philosophy a deformation of \( M \) is considered to be the sewing of the same patches, through a fit of the parameters \( t \) via various identifications. Four our purpose the dimension of the parameter space will be exactly equal to that of \( M \). According to Chapter 5 of [6] one considers a complex family of compact complex manifolds as a complex manifold \( M \) and a holomorphic map \( \varpi : M \to B \) where \( B \) is a domain in \( \mathbb{C}^n \) such that \( \varpi^{-1}(t) = M_t \) is a compact complex manifold. For \( \Delta \subset B \) sufficiently small, \( M_\Delta := \varpi^{-1}(\Delta) \) can be identified as a complex manifold with the complex structure defined on the smooth manifold \( M \times \Delta \) since the subordinated smooth structure is always the same and does not depend on \( t \) ([6] Thm 2.3). Accordingly, local complex coordinates on \( M_\Delta \) will be given by the system of local complex coordinates \((\mathcal{Z}^\alpha((z, \bar{z}), t), t^\alpha)\), \( \alpha = 1, \ldots, n \) and for fixed \( t \), \( M_t \) is the complex structure of the differentiable manifold \( M \) defined by the system of local complex coordinates \((\mathcal{Z}^\alpha((z, \bar{z}), t))\), \( \alpha = 1, \ldots, n \) considered as a smooth change of local complex coordinates on \( M \) i.e. the Jacobian does not vanish.

On the other hand, the deformation of complex structure is thus described by the change of the \( \overline{\partial} \)-operator [6]
\[
\overline{\partial} \rightarrow \overline{\partial} - \sum_{\ell=1}^{n} \mu^\ell((z, \bar{z}), t) \partial_\ell, \quad (2.5)
\]
the \( \mu^\ell((z, \bar{z}), t) \) are unique smooth \((0, 1)\)-forms on \( M \times \Delta \). In this way, one can describe both infinitesimal and finite deformations. Indeed, by looking for, at fixed \( t \), the local solutions \( \mathcal{Z}^\alpha((z, \bar{z}), t) \) of this family of deformed \( \overline{\partial} \)-operators
\[
\left( \overline{\partial} - \sum_{\ell=1}^{n} \mu^\ell((z, \bar{z}), t) \partial_\ell \right) \mathcal{Z}^\alpha((z, \bar{z}), t) = 0 \quad (2.6)
\]
them they will patch together holomorphically with respect to the complex structure \( M_t \) and thus they will define a new complex structure parametrized by the \( \mu \) on \( M \).
To be consistent with the deformation philosophy discussed before, the previous equation (2.6) must be coupled (Newlander-Nirenberg integrability theorem) with the Kodaira-Spencer (integrability) equation

\[ \partial \mu((z, \overline{z}), t) - \frac{1}{2} [\mu((z, \overline{z}), t), \mu((z, \overline{z}), t)] = 0 \quad (2.7) \]

where \( \mu((z, \overline{z}), t) = \mu^\ell((z, \overline{z}), t) \partial_\ell \), is a smooth \((1, 0)\)-vector field valued \((0, 1)\)-form on \( M \) and the graded brackets \([,]\) means the commutator of two vector fields and wedging.

To sum up, two solutions of Eq(2.7) correspond to the same complex structure if they differ by an holomorphic diffeomorphism. Since for \( t = 0 \) both \((z^k)\) and \((Z^\alpha(z, 0))\) are local complex coordinates on the complex manifold \( M \), then \( Z^\alpha(z, 0) \) are holomorphic functions of \((z^k)\), showing that \( \mu(z, 0) = 0 \). The construction of the new local complex coordinates \( Z^\alpha((z, \overline{z}), t) \) for each fixed \( t \) will correspond to a smooth change of local complex coordinates \((z^k) \mapsto (Z^\alpha((z, \overline{z}), t))\). The construction holds in each holomorphic sector in \( t \). Embedding in a symplectic framework generates an infinite sequence of changes of local complex coordinates. It is the signature of their behavior under symplectomorphisms which gives rise to an algebra. The latter extends to higher dimensions the usual \( W_\infty \)-algebra.

For this reason, if we wish now to settle the Kodaira-Spencer deformation in a symplectic framework, we may consider the deformation parameters as the conjugate variables (by symplectic doubling, as it will be better specified later on) to those of the configuration space by identifying locally, as differentiable manifolds, the cotangent space \( T^*M \) with \( M_\Delta \) endowed with local smooth coordinates \((z^k, \overline{z}^\alpha; t, \overline{t})\). Then all the requirements to perform a Kodaira-Spencer deformation will be satisfied, so that this mathematical artillery will be at our disposal to investigate the possible extension of our symplectic approach to a \( n \) complex dimensional manifold and the consequences for physical models, in particular higher spin fields and their sources.

3 Symplectomorphisms in \( 2n \) complex dimensional complex symplectic space

Symplectomorphisms describe diffeomorphisms preserving a given symplectic structure on the cotangent bundle \( T^*M \). They can be respectively described in terms of local coordinates, namely,

\[
\mathcal{U}(z, y) = (z^1 \cdots z^n, \overline{z}^1 \cdots \overline{z}^n; y_1 \cdots y_n, \overline{y}_1 \cdots \overline{y}_n),
\]

\[
\mathcal{U}(Z, \mathcal{Y}) = (Z^1 \cdots Z^n, \overline{Z}^1 \cdots \overline{Z}^n; \mathcal{Y}_1 \cdots \mathcal{Y}_n, \overline{\mathcal{Y}}_1 \cdots \overline{\mathcal{Y}}_n)
\]

and respectively endowed with the symplectic fundamental 2-form which, in full generality, locally writes according to the system of local coordinates -not necessarily the Darboux’s ones,
\[ \Omega_{U(z,y)} = \sum_{i,j=1}^{n} \omega_{ij}^i dz^i \wedge dy_j + \text{c.c.} = d\theta_{U(z,y)} \]  
(3.2)

\[ \Omega_{U(Z,Y)} = \sum_{\alpha,\beta=1}^{n} \omega_{\alpha\beta}^\alpha dZ^\beta \wedge dY_\alpha + \text{c.c.} = d\theta_{U(Z,Y)} \]  
(3.3)

with the following local requirements
\[ \det |\omega_{(i,j)}| \neq 0; \quad \det |\omega_{(\beta,\alpha)}| \neq 0; \quad d_z \omega_{ij}^i = d_y \omega_{ij}^j = d_z \omega_{\alpha\beta}^\alpha = d_y \omega_{\alpha\beta}^\beta = 0. \]  
(3.4)

and the invariance of the fundamental 2-form is locally expressed by
\[ \Omega_{U(z,y)} = \Omega_{U(Z,Y)}. \]  
(3.5)

Locally, this implies on \( U(z,y) \cap U(Z,Y) \neq \emptyset \) that
\[ \theta_{U(z,y)} - \theta_{U(Z,Y)} = dF. \]  
(3.6)

From now on, we shall work locally in terms of the ‘mixed’ local independent coordinates \( (z,Y) \),
\[ U(z,Y) = (z^1 \cdots z^n, \bar{z}^1 \cdots \bar{z}^n; \, Y_1 \cdots Y_n, \bar{Y}_1 \cdots \bar{Y}_n) \]  
(3.7)

where we define the differential operators (from now on the Einstein’s convention for summation will be used throughout the paper):
\[ d = d_z + d_Y; \quad d_z = d_z^i \frac{\partial}{\partial z^i} + d_{\bar{z}} \frac{\partial}{\partial \bar{z}^i} = d_z^i \partial_i + d_{\bar{z}}^i \partial_i; \quad d_Y = d_Y_\alpha \frac{\partial}{\partial Y_\alpha} + d_{\bar{Y}}^\alpha \frac{\partial}{\partial \bar{Y}_\alpha} \]  
(3.8)

The corresponding generating function \( \Phi(z,Y) \) is obtained through the Legendre transformation
\[ d\Phi(z,Y) = d(F + \omega_{\alpha\beta}^\alpha Z^\beta Y_\alpha + \text{c.c.}) = \omega_{ij}^i dz^i + \omega_{\alpha\beta}^\alpha Z^\beta dY_\alpha + \text{c.c.} \]  
(3.9)

In the cotangent space \( T^*M \) endowed with this system of local coordinates, the mappings:
\[ y_i(z,Y) \equiv \omega_{ij}^j y_j = \frac{\partial \Phi(z,Y)}{\partial z^i} \equiv \partial_i \Phi(z,Y) \]  
(3.10)

\[ Z^\alpha(z,Y) \equiv \omega_{\alpha\beta}^\beta Z^\beta = \frac{\partial \Phi(z,Y)}{\partial Y_\alpha} \]  
(3.11)

are canonical and define new canonical variables via the \( \omega \) matrices.
Several ways can settle this canonical procedure: Poisson brackets (or something similar), flow analysis of hierarchical structures. We shall be concerned with the study of this aspect in a field theoretical language by using the BRS formulation. Moreover we can rewrite:

\[ \Omega_{\mathcal{U}(z,\mathcal{Y})} = \partial_i Z^\alpha(z, \mathcal{Y}) dz^i \wedge d\mathcal{Y}_\alpha + \partial_i \overline{Z}^\alpha(z, \mathcal{Y}) d\overline{\mathcal{Y}}^\alpha \wedge d\mathcal{Y}_\alpha \]

from which we get the relations of duality (with their complex conjugate expressions as well):

\[ \partial_i Z^\alpha(z, \mathcal{Y}) = \frac{\partial}{\partial \mathcal{Y}_\alpha} y_i(z, \mathcal{Y}), \quad \partial_\mathcal{Y} Z^\alpha(z, \mathcal{Y}) = \frac{\partial}{\partial \mathcal{Y}_\alpha} \overline{y}_i(z, \mathcal{Y}). \]  

(3.13)

In order to parametrize our space we define [20, 21, 22] the Hessian matrix elements by

\[ \partial_i \frac{\partial}{\partial \mathcal{Y}_\alpha} \Phi(z, \mathcal{Y}) \equiv \lambda^\alpha_i(z, \mathcal{Y}) \]

(3.14)

\[ \overline{\partial}_\mathcal{Y} \frac{\partial}{\partial \mathcal{Y}_\alpha} \Phi(z, \mathcal{Y}) \equiv \lambda^\alpha_i(z, \mathcal{Y}) \mu^i_j(z, \mathcal{Y}) \equiv \overline{\lambda}^i_j(z, \mathcal{Y}) \mu^{\overline{i}}_{\overline{j}}(z, \mathcal{Y}) \]

(3.15)

with \( \det |\lambda| \neq 0 \) for non singularity requirement and also for the complex conjugate expressions. From Eqs (3.14) (3.15) we get the following identities:

\[ \partial_j \lambda^\alpha_i(z, \mathcal{Y}) = \partial_i \lambda^\alpha_j(z, \mathcal{Y}) \quad; \quad \frac{\partial}{\partial \mathcal{Y}_\beta} \lambda^\alpha_i(z, \mathcal{Y}) = \frac{\partial}{\partial \mathcal{Y}_\alpha} \lambda^\beta_i(z, \mathcal{Y}) \]

(3.16)

\[ \overline{\partial}_\mathcal{Y} \lambda^\alpha_i(z, \mathcal{Y}) = \partial_i \left( \lambda^\alpha_i(z, \mathcal{Y}) \mu^i_j(z, \mathcal{Y}) \right) \quad; \quad \frac{\partial}{\partial \mathcal{Y}_\beta} \lambda^\alpha_i(z, \mathcal{Y}) = \frac{\partial}{\partial \mathcal{Y}_\alpha} \left( \overline{\lambda}^i_j(z, \mathcal{Y}) \overline{\mu}^{\overline{i}}_{\overline{j}}(z, \mathcal{Y}) \right) \]

(3.17)

\[ \overline{\partial}_\mathcal{Y} \lambda^\alpha_i(z, \mathcal{Y}) = \partial_i \left( \overline{\lambda}^\alpha_i(z, \mathcal{Y}) \overline{\mu}^{\overline{i}}_{\overline{j}}(z, \mathcal{Y}) \right) \quad; \quad \overline{\partial}_\mathcal{Y} \lambda^\alpha_i(z, \mathcal{Y}) = \frac{\partial}{\partial \mathcal{Y}_\alpha} \left( \lambda^\alpha_i(z, \mathcal{Y}) \mu^i_j(z, \mathcal{Y}) \right) \]

(3.18)

So from Eqs (3.13) (3.14) (3.15) we have the following two main identities which must be viewed within the Kodaira-Spencer spirit of Eq (2.6):

\[ \left( \overline{\partial}_\mathcal{Y} - \mu^i_j(z, \mathcal{Y}) \partial_i \right) Z^\alpha(z, \mathcal{Y}) \equiv L_\mathcal{Y}(z, \mathcal{Y}) Z^\alpha(z, \mathcal{Y}) = 0 \]

(3.19)

\[ \left( \frac{\partial}{\partial \mathcal{Y}_\overline{\alpha}} - \overline{\mu}^\overline{i}_{\overline{j}}(z, \mathcal{Y}) \frac{\partial}{\partial \mathcal{Y}_{\overline{i}}} \right) y_{\overline{i}}(z, \mathcal{Y}) \equiv L^{\overline{Y}}(z, \mathcal{Y}) y_{\overline{i}}(z, \mathcal{Y}) = 0 \]

(3.20)
where the role of the parameter $t$ of deformation is presently played by the covariant coordinates $(\mathcal{Y}, \bar{\mathcal{Y}})$ in the former or by the background complex coordinates $(z, \bar{z})$ in the latter. The first of the two equations tells that a local deformation of the complex structure on the base complex manifold $M$ can be implemented by using the symplectic structure on the cotangent bundle $T^* M$, while the second one governs the vertical deformation. This coincidence justifies our point of view of taking the conjugate variables as the deformation parameter. Hence, the complex family of complex manifolds $M_{\mathcal{Y}}$ is locally recasted as the symplectic cotangent bundle $T^* M$ when the differentiable structure is considered.

Let us write down the following Pfaff system

$$
dz Z^\alpha(z, \mathcal{Y}) = \lambda^\alpha(z, \mathcal{Y}) \left( dz^i + \mu^i(z, \mathcal{Y}) d\bar{z}^j \right) =: \left( dz + d\bar{z} \cdot \mu(z, \mathcal{Y}) \right) \cdot \lambda(z, \mathcal{Y})
$$

$$
dy y_i(z, \mathcal{Y}) = \lambda^\alpha_i(z, \mathcal{Y}) \left( dy^\alpha + \mu^\alpha_i(z, \mathcal{Y}) d\bar{y}^\tau \right)
$$

(3.21)

The system serves to define two types of Kodaira-Spencer differentials, namely, $\mu^i(z, \mathcal{Y})$ and $\mu^\alpha_i(z, \mathcal{Y})$ which parametrize the complex structures on the base space $M$ with background local complex coordinates $(z, \bar{z})$ and the fibers with local coordinates $(\mathcal{Y}, \bar{\mathcal{Y}})$, respectively. These complex structures are interlinked by the duality relations Eqs(3.13) (3.15)

$$
\mu^j_i(z, \mathcal{Y}) = D^j_i \partial_i \Phi(z, \mathcal{Y})
$$

(3.26)

where $\mathcal{Y} = \mathcal{Y}(z, y)$ has to be taken into account. The most relevant properties of the $D^i(z, \mathcal{Y})$ and $\partial_{\mathcal{Y}^i}$ operators can be summarized as

$$
\left[ D^i(z, \mathcal{Y}), D^j(z, \mathcal{Y}) \right] = 0, \quad \left[ \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right] = 0, \quad \left[ \frac{\partial}{\partial y_i}, \frac{\partial}{\partial \mathcal{Y}^\tau} \right] = 0.
$$

(3.27)
The third order derivatives of $\Phi$ yields The integrability conditions Eq(2.7) for the deformation of complex structures in the $(z, \bar{z})$ and $(Y, \bar{Y})$ spaces respectively write

\begin{equation}
L_ı(z, Y) \mu_ı(z, Y) = L_ı(z, Y) \mu_ı(z, Y)
\end{equation}

(3.28)

\begin{equation}
\lambda_ı^β(z, Y) \partial_j \mu_ı^β(z, Y) = \lambda_ı^β(z, Y) \partial_j \mu_ı^β(z, Y)
\end{equation}

(3.29)

Moreover in the $(Y, \bar{Y})$ space, the partner of the Kodaira-Spencer equations can be immediately recovered computing $\partial_j \partial_i \partial Y_α \partial Y_β \Phi(z, Y)$

\begin{equation}
L_α(z, Y) \mu_β(z, Y) = L_α(z, Y) \mu_β(z, Y)
\end{equation}

(3.30)

with the consistency conditions:

\begin{equation}
[L_ı(z, Y), L_ı(z, Y)] = 0 ; \quad [L_α(z, Y), L_β(z, Y)] = 0.
\end{equation}

(3.31)

3.1 BRS setting of symplectomorphisms in $2n$ complex dimensions

As said before the Kodaira-Spencer deformations reparametrize in a consistent way the space of complex structures. Furthermore, we shall study the action of reparametrizations on symplectic space (symplectomorphisms).

The BRS setting for symplectomorphisms can be performed along the lines developed in [20]. Let us define by $S$ the nilpotent BRS operation associated to the infinitesimal symplectomorphisms. Locally, $S$ will be represented in $(z, Y)$ coordinates by

\begin{equation}
S \Phi(z, Y) = \Lambda(z, Y), \quad S \Lambda(z, Y) = 0
\end{equation}

(3.32)

The infinitesimal BRS transformation of the deformed coordinate $Z_α(z, \bar{Y})$ can be calculated from its canonical definition Eq(3.10),(3.11)

\begin{equation}
SZ_α(z, Y) = \partial_i \Lambda(z, Y) = C_i(z, Y) \partial_i \Lambda(z, Y) = C_i(z, Y) \partial_i Z_α(z, Y)
\end{equation}

(3.33)

where the chiral ghost fields $C_i(z, Y)$ naturally emerge and are related to the ordinary diffeomorphism ghosts $c_i(z, Y)$, $\bar{c}_i(z, Y)$ on $T^*M$ within this symplectic framework by

\begin{equation}
C_i(z, Y) \equiv D_i \Lambda(z, Y) = \left( \frac{\partial \Lambda(z, Y)}{\partial Y_i} + \mu_ı^j(z, Y) \frac{\partial \Lambda(z, Y)}{\partial Y_ı} \right) = c_i(z, Y) + \mu_ı^j(z, Y) \bar{c}_i(z, Y)
\end{equation}

(3.34)

which explicitly corresponds to a change of generators for symplectomorphisms. Their BRS variations read

\begin{equation}
SC_i(z, Y) = C_i(z, Y) \partial_j C_i(z, Y)
\end{equation}

(3.35)

\begin{equation}
Sc_i(z, Y) = \left[ c_i(z, Y) \partial_j + \bar{c}_i(z, Y) \partial_ı \right] c_i(z, Y)
\end{equation}

(3.36)
These BRS transformations correspond to an infinitesimal reparametrization of $Z^\alpha(z,\mathcal{Y})$ due to an infinitesimal shift of the $(z,\bar{z})$ background, keeping $(\mathcal{Y},\bar{\mathcal{Y}})$ fixed.

We can easily derive:

$$S\lambda^\alpha_i(z,\mathcal{Y}) = \partial_i \left( \lambda^\alpha_j(z,\mathcal{Y}) C^j(z,\mathcal{Y}) \right)$$  \hspace{1cm} (3.37)

$$S\left( \lambda^\alpha_i(z,\mathcal{Y}) \mu^r_j(z,\mathcal{Y}) \right) = \partial_j \left( \lambda^\alpha_i(z,\mathcal{Y}) C^r(z,\mathcal{Y}) \right)$$  \hspace{1cm} (3.38)

so that

$$S\mu^r_j(z,\mathcal{Y}) = C^i(z,\mathcal{Y}) \partial_i \left( c^j(z,\mathcal{Y}) \right) C^r(z,\mathcal{Y}) \left( \partial_i \lambda^\alpha_j(z,\mathcal{Y}) C^r(z,\mathcal{Y}) \right)$$  \hspace{1cm} (3.39)

The non-chiral representation of this algebra can be easily given following the lines of \[9\], where we have stressed the relevance of the $(z,\bar{z})$ counterpart of the Kodaira-Spencer equation (3.30).

Moreover the ordinary ghosts $c^j(z,\mathcal{Y})$ transform as:

$$Sc^j(z,\mathcal{Y}) = \left( c^l(z,\mathcal{Y}) \partial_l + \bar{c}^l(z,\mathcal{Y}) \partial_l \right) c^j(z,\mathcal{Y})$$  \hspace{1cm} (3.40)

Finally, note the important commutators coming from the combination of the commutators $[S, \frac{\partial}{\partial \mathcal{Y}}] = 0 = [\partial_i, \frac{\partial}{\partial \mathcal{Y}}]$ with (3.11), (3.25):

$$\left[ S, D^i(z,\mathcal{Y}) \right] = -[\lambda(z,\mathcal{Y})^{-1}]_i^j \partial_j \left( \lambda^\alpha_i(z,\mathcal{Y}) C^j(z,\mathcal{Y}) \right) D^i(z,\mathcal{Y})$$

$$= -\partial_i C^j(z,\mathcal{Y}) D^r(z,\mathcal{Y}) + C^r(z,\mathcal{Y}) \left[ \partial_r, D^i(z,\mathcal{Y}) \right].$$  \hspace{1cm} (3.41)

Conversely, from Eqs (3.10), (3.11), we can derive the infinitesimal transformation of $y_i(z,\mathcal{Y})$ due to an infinitesimal reparametrization on $(\mathcal{Y},\bar{\mathcal{Y}})$ space, keeping the $(z,\bar{z})$ background fixed.

$$Sy_i(z,\mathcal{Y}) = \partial_i \Lambda(z,\mathcal{Y}) = \lambda^\alpha_i(z,\mathcal{Y}) \left( \partial_i + \mu^\alpha_i(z,\mathcal{Y}) \partial_i \Lambda(z,\mathcal{Y}) \right)$$

$$= \left( \omega^\alpha_i(z,\mathcal{Y}) + \mu^\alpha_i(z,\mathcal{Y}) \Lambda(z,\mathcal{Y}) \right) \frac{\partial}{\partial \mathcal{Y}} y_i(z,\mathcal{Y})$$

$$= \mathcal{O}_\alpha(z,\mathcal{Y}) \frac{\partial}{\partial \mathcal{Y}} y_i(z,\mathcal{Y})$$  \hspace{1cm} (3.42)

where it has been set

$$\omega^\alpha_i(z,\mathcal{Y}) = \partial_i \Lambda(z,\mathcal{Y})$$  \hspace{1cm} (3.43)

and:

$$SO_\alpha(z,\mathcal{Y}) = \mathcal{O}_\beta \frac{\partial}{\partial \mathcal{Y}} \mathcal{O}_\alpha(z,\mathcal{Y}).$$  \hspace{1cm} (3.44)
Now the generating function $\Phi(z, Y)$ for such canonical transformations will be so chosen in order to view the holomorphic deformation process in the $Y$ direction as being a canonical transformation.

For the purpose it will be convenient to use a multi-index notation. Let $A, B$ denote multi-indices on the fibers related to Greek indices and while and $I, J$ denote multi-indices on $M$ related to Latin indices. For $A = (a_1, \ldots, a_n)$ with positive integers $a_\alpha \geq 0$, $|A| = \sum a_k$ will be the order of $A$ and one sets $A + 1_\beta = (a_1, \ldots, a_{\beta-1}, a_\beta + 1, a_{\beta+1}, \ldots, a_n)$, $A! = \prod_{\alpha} a_\alpha!$. For the sake of notational completeness, on the base $M$ one will similarly use $I + 1_k = (i_1, \ldots, i_{k-1}, i_k + 1, i_{k+1}, \ldots, i_n)$. Now, one chooses a $Y$-holomorphically split generating function $\Phi(z, Y) = \sum_{|A| \geq 1} Z^{(A)}(z, \overline{z}) Y_A + \text{c.c.}$, \hspace{1cm} (3.45)

where for $|A| \geq 1$ and $Y_A := \prod_{\alpha} (Y_\alpha)^{a_\alpha}$, we have set

$$Z^{(A)}(z, \overline{z}) := \frac{1}{|A|!} \frac{\partial^{|A|} \Phi(z, Y)}{\partial Y_A} \bigg|_{Y=0} = \frac{1}{|A|!} \frac{\partial^{|A|} \Phi(z, Y)}{(\partial Y_1)^{a_1} \cdots (\partial Y_n)^{a_n}} \bigg|_{Y=0}$$

for the $\frac{|A| + n - 1)!}{|A|!(n-1)!}$ independent derivatives of order $|A|$. With such a generating function the symplectic two-form (3.12) is locally written as

$$\Omega = \sum_{|A| \geq 1} d_z Z^{(A)}(z, \overline{z}) \wedge d_{\overline{z}} Y_A + \text{c.c.}$$ \hspace{1cm} (3.47)

while the new coordinates defined in (3.10) and (3.11) are respectively given by

$$y_i(z, \mathcal{Y}) = \sum_{|A| \geq 0} \partial_i Z^{(A)}(z, \overline{z}) Y_A + \sum_{|B| \geq 0} \partial_i \overline{Z^{(B)}}(z, \overline{z}) \overline{Y_B}.$$ \hspace{1cm} (3.48)

$$Z^\alpha(z, \mathcal{Y}) = \sum_{|A| \geq 0} \sum_{\alpha=1}^n (a_\alpha + 1) Z^{(A+1_\alpha)}(z, \overline{z}) Y_A$$

$$= Z^\alpha(z, \overline{z}) + \sum_{|A| \geq 1} \sum_{\alpha=1}^n (a_\alpha + 1) Z^{(A+1_\alpha)}(z, \overline{z}) Y_A$$ \hspace{1cm} (3.49)

Note that $Z^\alpha(z, \mathcal{Y}) \big|_{Y=0} = Z^\alpha(z, \overline{z})$ showing that the complex structure given by the local complex coordinates $Z^\alpha$ is the one which is actually deformed. Recall that the latter are local complex coordinates solutions of (3.19) at $Y_\alpha = \overline{Y_\alpha} = 0$ and have already been treated in the context $n$ complex dimensional manifolds in [9].

As explicitly shown above, the local coefficients $Z^{(A)}(z, \overline{z}), |A| \geq 1$ thus describe the response to the deformation of the $Z^\alpha(z, \overline{z})$ complex coordinates. Combining the decomposition (3.47)
with the covariance requirement \( (3.3) \) leads to an infinite sequence of changes of local complex coordinates \((z^k) \rightarrow (\mathcal{Z}^{(A)}(z, \bar{z}))\) whose the algebra of infinitesimal transformations can be derived by means of BRS techniques.

Furthermore, the role of the complex structures involved in the present approach can be deepened. Indeed, the Kodaira-Spencer differentials \( \mu_{\mathcal{Y}}^{(A)}(z, \mathcal{Y}) \) reflect the general behavior (see Eq \((3.26)\)) of the generating function of the canonical transformations. Their infinitesimal behavior in the \((z, \bar{z})\) and \((\mathcal{Y}, \bar{\mathcal{Y}})\) spaces are constrained by both Eq \((3.29)\) and \((3.30)\). Now the explicit complex deformation will be chosen as a particular case of \( \mu_{\mathcal{Y}}^{(A)}(z, \mathcal{Y}) \), according to

\[
\mu_{\mathcal{Y}}^{(A)}(z, \mathcal{Y}) = \sum_{|A|>0} \mu_{\mathcal{Y}}^{(A)}(z, \bar{z}) \mathcal{Y}_A , \tag{3.50}
\]

with \( \mu_{\mathcal{Y}}^{(A)}(z, \bar{z}) = \frac{1}{|A|!} \frac{\partial^{|A|+1}}{\partial \mathcal{Y}_{A+1,\beta}} \left( \partial_\mathcal{Y} \Phi(z, \mathcal{Y}) \left[ \lambda(z, \mathcal{Y})^{-1} \right]^{i}_\beta \right) \bigg|_{\mathcal{Y}=0} \). This series converges in a Holder norm \( \| \) and represents a deformation of the integrable complex structure defined by \( \mu_{\mathcal{Y}}^{(0)} \) with the role of deformation parameters is played by \( \mathcal{Y} \) as already said before. Since the use of this space doubling is to introduce a symplectic structure in order that the smooth local changes of complex coordinates \((z^k) \rightarrow (\mathcal{Z}^{(A)}(z, \bar{z}))\) are interpreted as coming from a symplectomorphism symmetry. Recall that the generating function \((3.45)\) for the canonical transformations has been chosen to be compatible with the deformation \((3.50)\). The holomorphic character of the deformations will define, in a BRS framework, a series of infinitesimal symmetry transformations which will reproduce the \( n \) complex dimensional extension of the \( \mathcal{W}_{\infty} \)-algebra as will be shown in the next Section.

The link of the parametrization in Eq \((3.49)\) with the one of \((3.50)\) is given through \((3.15)\) by, for \( |A| \geq 0 \) and for each \( \alpha = 1, \ldots, n \) –no summation over \( \alpha \)–

\[
(a_\alpha + 1) \partial_\mathcal{Y} \mathcal{Z}^{(A+1,\alpha)}(z, \bar{z}) = \sum_{|B|, |C| \geq 0} \frac{1}{B + C = A} \partial_\mathcal{Y} \mathcal{Z}^{(B+1,\alpha)}(z, \bar{z}) \mu_{\mathcal{Y}}^{(C)}(z, \bar{z}) \tag{3.51}
\]

which, in the particular case of \( |A| = 0 \), reduces to the usual Beltrami equations

\[
\overline{\partial}_\mathcal{Y} \mathcal{Z}^{(A)}(z, \bar{z}) = \partial_\mathcal{Y} \mathcal{Z}^{(A)}(z, \bar{z}) \mu_{\mathcal{Y}}^{(0)}(z, \bar{z}) \tag{3.52}
\]

which were fully treated in \( \| \) in the two dimensional case. In this context the integrability condition \((3.28)\) is transfered on the jet coordinates \( \mu_{\mathcal{Y}}^{(A)}(z, \bar{z}) \) with \( |A| \geq 0 \), as follows

\[
\overline{\partial}_\mathcal{Y} \mu_{\mathcal{Y}}^{(A)}(z, \bar{z}) - \partial_\mathcal{Y} \mu_{\mathcal{Y}}^{(A)}(z, \bar{z}) = \sum_{|B|, |C| \geq 0} \partial_\mathcal{Y} \left( \mu_{\mathcal{Y}}^{(B)}(z, \bar{z}) \partial_\mathcal{Y} \mu_{\mathcal{Y}}^{(C)}(z, \bar{z}) - \mu_{\mathcal{Y}}^{(B)}(z, \bar{z}) \partial_\mathcal{Y} \mu_{\mathcal{Y}}^{(C)}(z, \bar{z}) \right) \tag{3.53}
\]

For \( |A| \geq 1 \),

\[
\overline{\partial}_\mathcal{Y} \mathcal{Z}^{(A)}(z, \bar{z}) = \frac{1}{A!} \frac{\partial^{|A|}}{\partial \mathcal{Y}_A} \overline{\partial}_\mathcal{Y} \Phi(z, \mathcal{Y}) \bigg|_{\mathcal{Y}=0} = \sum_{1 \leq |I| \leq |A|} \mathcal{G}^{(A)}_{(I)}(z, \bar{z}) \mu_{\mathcal{Y}}^{(I)}(z, \bar{z}) \tag{3.54}
\]
where we have set for $|I| \geq 1$,

$$
\mu^{(I)}_\partial(z, \bar{z}) := \frac{1}{I!} D^{(I)}(z, \mathcal{Y}) \partial_\partial^I \Phi(z, \mathcal{Y}) \bigg|_{y=0} = \prod_{k=1}^{n} \left( \frac{1}{ik!} \left( D^k(z, \mathcal{Y}) \right)^{i_k} \right) \partial_\partial^I \Phi(z, \mathcal{Y}) \bigg|_{y=0} \tag{3.55}
$$

as representing the $n$-dimensional version for the $\mathcal{W}$-extension of the Beltrami multipliers introduced by Bilal Fock and Kogan [24]. It is worthwhile to say that the coefficients $\mathcal{G}^{(A)}_{\partial(I)}(z, \bar{z})$ are very intricate non local expressions depending on the derivatives up to order $|A|$ of $\mathcal{Z}^{(B)}$, with $1 \leq |B| \leq |A|$. Writing (3.54) in more precise terms one has for $|A| \geq 1$,

$$
\bar{\partial}_\partial \mathcal{Z}^{(A)}(z, \bar{z}) = \mu^{(0)}_\partial(z, \bar{z}) \partial_\partial \mathcal{Z}^{(A)}(z, \bar{z}) + \cdots
$$

$$
+ \sum_{\alpha=1}^{n} \sum_{|I_{\alpha}|=\alpha} \frac{(I_1 + \cdots + I_n)!}{I_1! \cdots I_n!} \left( \prod_{\beta=1}^{n} \lambda^\partial_{\beta}(z, \bar{z}) \right) \mu^{(I_1 + \cdots + I_n)}_\partial(z, \bar{z}) \tag{3.56}
$$

where on the multi-indices $I^\alpha = (i_1^\alpha, \ldots, i_n^\alpha)$ the summand $I = \sum_{\alpha=1}^{n} I^\alpha$ is the linear addition on the monoid of positive integers $\mathbb{N}^n$ while $\lambda^\partial_{\beta}(z, \bar{z}) = \prod_{r=1}^{n} \left( \partial_r \mathcal{Z}^{\beta}(z, \bar{z}) \right)^{i_r^\beta}$. Moreover, in the above expansion $\mu^{(0)}$ must be identified with $\mu^{(0)}_\partial$ –see (3.51) and (3.52).

Furthermore, the symplectic structure of the space ought to provide by virtue of (3.50) a recursive construction for the coefficients $\mu^{(A)}_\partial$ defined in (3.50) for the complex structure in terms of those of Bilal-Fock-Kogan defined in (3.55). This certainly allows to write

$$
\mu^{(I)}_\partial(z, \mathcal{Y}) = \sum_{|I| \geq 1} \mathcal{F}^{(I)}_\partial(z, \mathcal{Y}) \mu^{(I)}_\partial(z, \bar{z}) \tag{3.57}
$$

where the very complicate coefficients $\mathcal{F}^{(I)}_\partial(z, \mathcal{Y})$ depending on the $\mathcal{G}$ carry a well defined geometrical meaning.

4 Classical $\mathcal{W}_\infty$-algebra in $n$-complex dimensions

Due to the holomorphically split expansion (3.45), the action of the BRS operator $\mathcal{S}$ on the theory can be parametrized by means of new ghost fields directly obtained from this expansion. These will be intimately related to the $\mathcal{W}_\infty$-algebra. Indeed, since $\mathcal{S}\mathcal{Y}_\alpha = 0$, by using (3.46), for $|A| \geq 1$, one gets the same combinatorial expansion as (3.54)

$$
\mathcal{S} \mathcal{Z}^{(A)}(z, \bar{z}) = \frac{1}{A!} \frac{\partial |A|}{\partial \mathcal{Y}_A} \mathcal{L}(z, \mathcal{Y}) \bigg|_{y=0} = \sum_{1 \leq |I| \leq |A|} \mathcal{G}^{(A)}_{\partial(I)}(z, \bar{z}) \mathcal{C}^{(I)}(z, \bar{z}) \tag{4.58}
$$

where we have introduced the independent ghost fields

$$
\mathcal{C}^{(I)}(z, \bar{z}) := \frac{1}{I!} D^{(I)}(z, \mathcal{Y}) \mathcal{L}(z, \mathcal{Y}) \bigg|_{y=0} = \prod_{k=1}^{n} \left( \frac{1}{ik!} \left( D^k(z, \mathcal{Y}) \right)^{i_k} \right) \mathcal{L}(z, \mathcal{Y}) \bigg|_{y=0}. \tag{4.59}
$$
Note that from the very definitions, the dependence on the generalized Bilal, Fock and Kogan parameters can be isolated and turns out to be coupled to the ghost \( \mathcal{C}^{(0, \tau)} := \mathcal{C}^{(0, \tau)} \),

\[
\mathcal{C}^{(I)}(z, \bar{z}) = \mu^{(I)}_\tau(z, \bar{z}) \mathcal{C}^{(0, \tau)}(z, \bar{z}) + \cdots ,
\]

the full detailed expression will be given down below –see (4.67).

Notably, after a tedious combinatorial calculation based upon the commutators (3.41), the BRS variations of the ghosts defined by (4.59) turn out to be local (in the sense that do not depend on the \( \lambda \)-fields), namely, for \( |I| \geq 1 \),

\[
\mathcal{S}C^{(I)}(z, \bar{z}) = \sum_{k=1}^{n} (1 - \delta_{0k}) \sum_{J(k) \leq I(k) - 1_k} \frac{(I - J^{(k)} - 1_k + 1_r)!}{(I - J^{(k)} - 1_k)!} \times
\]

\[
\mathcal{C}^{(I - J^{(k)} - 1_k + 1_r)}(z, \bar{z}) \partial_r \mathcal{C}^{(J^{(k)} + 1_k)}(z, \bar{z})
\]

where the notation \( J^{(k)} \) means \( J^{(k)} = (j_1, \ldots, j_k, 0, \ldots, 0) \) (and similarly for \( I^{(k)} \)), \( J \leq I \) is a shorthand for \( j_k \leq i_k, k = 1, \ldots, n \) and

\[
\frac{(I - J^{(k)} - 1_k + 1_r)!}{(I - J^{(k)} - 1_k)!} = \begin{cases} 
  i_r - j_r + 1 & \text{if } 1 \leq r \leq k - 1 \\
  i_k - j_k & \text{if } r = k \\
  i_r + 1 & \text{if } k + 1 \leq r \leq n
\end{cases}
\]

This formula represents the extended version to \( n \) complex dimensions of the chiral \( \mathcal{W}_\infty \)-algebra. Indeed, let us consider \( n = 1 \) a complex curve which represents a bidimensional theory built on a Riemann surface. In that case, the multi-index \( I \) reduces to a simple index and for \( |I| = \ell = m, J^{(1)} = (j_1) = j, \) \( r = k = 1 \) , the formula (4.61) reduces (with \( \ell = m - j \)) to that found in [20, 21]

\[
\mathcal{S}C^{(m)}(z, \bar{z}) = \sum_{\ell=1}^{m} \ell \mathcal{C}^{(\ell)}(z, \bar{z}) \partial \mathcal{C}^{(m-\ell+1)}(z, \bar{z}),
\]

and recalled in the introduction -see (1.1). Going back to the general case, at first order \( |I| = 1 \), we refine the usual BRS transformations for the chiral ghosts \( C^i \) under diffeomorphisms of \( M \)

\[
\mathcal{S}C^i(z, \bar{z}) = \mathcal{C}^{(1)}(z, \bar{z}) \partial \mathcal{C}^i(z, \bar{z}),
\]

showing that diffeomorphisms are actually captured by the \( \mathcal{W}_\infty \)-symmetry. In order to exemplify once more (4.61), at the second order \( |I| = 2 \), for \( 1 \leq i \leq j \leq n \), the multi-index \( I = (0, \cdots, 0, 1, 0, \cdots, 0, 1, 0, \cdots, 0) \), where 1 is at the both \( i \)-th and \( j \)-th places will be shorthandly written as \( I = (ij) \) in order to recover a tensorial notation. With this notation, one gets

\[
\mathcal{S}C^{(ij)}(z, \bar{z}) = \mathcal{C}^{(1)}(z, \bar{z}) \partial_r \mathcal{C}^{(1)}(z, \bar{z}) + 2\mathcal{C}^{(ii)}(z, \bar{z}) \partial \mathcal{C}^j(z, \bar{z}) + 2\mathcal{C}^{(ij)}(z, \bar{z}) \partial j \mathcal{C}^i(z, \bar{z}) + \sum_{r=1}^{n} \mathcal{C}^{(ir)}(z, \bar{z}) \partial r \mathcal{C}^j(z, \bar{z}) + \sum_{r=1}^{n} \mathcal{C}^{(jr)}(z, \bar{z}) \partial r \mathcal{C}^i(z, \bar{z}),
\]
By performing this construction for the chiral ghosts, one should directly get a generalization of the functional operator (1.2) from the BRS transformations (4.61) of the chiral ghost fields themselves.

\[ SC^{(ii)}(z, \bar{z}) = C^r(z, \bar{z}) \partial_r C^{(ii)}(z, \bar{z}) + 2C^{(ii)}(z, \bar{z}) \partial_r C^i(z, \bar{z}) + \sum_{r \neq i} C^{(ri)}(z, \bar{z}) \partial_r C^i(z, \bar{z}) \]

as in (4.62) with \( m = 2 \)

where \( I = (ii) \) means 2 at the \( i \)-th place, a shorthand notation saying that the multi-index entries are \( i_k = 2\delta_{kk} \). Of course, there is the complex conjugate expression to (4.61) as well.

Following the BRS method recalled in the introduction, the algebra of the \( \mathcal{W}_\infty \)-generators in the \( n \) complex dimensional case will be obtained by duality through the corresponding BRS functional operator (1.3) from the BRS transformations (4.61) of the chiral ghost fields themselves. By performing this construction for the chiral ghosts, one should directly get a generalization to \( n \) dimensions of the chiral \( \mathcal{W}_\infty \)-algebra (1.3).

Accordingly, the BRS variations of the generalized Bilal-Fock-Kogan parameters (3.52) can be directly computed from (4.61) by using a trick related to diffeomorphisms [26], namely, \( \hat{\{ S, \partial_r \} = \partial_r \} \), together with (4.60),

\[ S \mu_f^{(I)}(z, \bar{z}) = \overline{\partial_f} C^{(I)}(z, \bar{z}) + \sum_{k=1}^{n} (1 - \delta_{0k}) \sum_{j(k) \leq I(k) - 1_k} (I - J(k) - 1_k + 1_r)! \left( I - J(k) - 1_k \right)! \times \]

\[ \left( C^{(I - J(k) - 1_k + 1_r)}(z, \bar{z}) \partial_r \mu_f^{(j(k) + 1_k)}(z, \bar{z}) - \mu_f^{(I - J(k) - 1_k + 1_r)}(z, \bar{z}) \partial_r C^{(j(k) + 1_k)}(z, \bar{z}) \right) \]

(4.65)

By using once more the previous trick on (4.61) one ends up with the counterpart of the integrability condition (3.31) in terms of the external fields (3.57) for \( |I| \geq 1 \), and with the aforementioned notation

\[ \overline{\partial_f} \mu_f^{(I)}(z, \bar{z}) - \overline{\partial_f} \mu_f^{(I)}(z, \bar{z}) = \sum_{k=1}^{n} (1 - \delta_{0k}) \sum_{j(k) \leq I(k) - 1_k} (I - J(k) - 1_k + 1_r)! \left( I - J(k) - 1_k \right)! \times \]

\[ \left( \mu_f^{(I - J(k) - 1_k + 1_r)}(z, \bar{z}) \partial_r \mu_f^{(j(k) + 1_k)}(z, \bar{z}) - \mu_f^{(I - J(k) - 1_k + 1_r)}(z, \bar{z}) \partial_r C^{(j(k) + 1_k)}(z, \bar{z}) \right) \]

(4.66)

The chiral ghost fields \( C^{(I)} \) admit a local decomposition in terms of the fields (3.55) which generalizes the well known conformal one [22] in two dimensions. The latter has already been extended in [20] for Riemann surfaces. By definition the promised detailed expression for (4.60) writes

\[ C^{(I)}(z, \bar{z}) = \sum_{|P| + |Q| = 0} |P| \cdot |Q| \sum_{P + a_1J_1 + \cdots + a_{|I|}J_{|I|} = I} \overline{a_1} + \cdots + \overline{a_{|I|}} = |Q| \]
where in the second summand $a_k$ enumerates (the number of multi-indices identical to $J_k$, $J_k \neq J_{k'}$ for $k \neq k'$ and $k, k' = 1, \ldots, |I|$, $(0 \leq a_k \leq |I|)$, the sum goes with no repetition, and, for a given multi-index $\underline{s}_k = (s_1^{(k)}, \ldots, s_n^{(k)})$

$$
(\mu^{(J_k)}(z, \bar{z}))_{\underline{s}_k} = \left( \mu^{(J_k)}_{\Sigma}(z, \bar{z}) \right)^{\underline{s}_1^{(k)}} \cdots \left( \mu^{(J_k)}_{\Sigma}(z, \bar{z}) \right)^{\underline{s}_n^{(k)}}
$$

and where new independent ghost fields have been introduced by

$$
c^{(P, \bar{Q})}(z, \bar{z}) := \frac{1}{P! \bar{Q}!} \frac{\partial^{|P|}}{\partial y_P} \frac{\partial^{|\bar{Q}|}}{\partial \bar{y}_{\bar{Q}}} \Lambda(z, \bar{z}) \bigg|_{y = 0}.
$$

Remark that expression (4.67) which expresses a change of generators for the $\mathcal{W}$-symmetry is local. For instance, the case $|I| = 1$ gives

$$
C^{(i)}(z, \bar{z}) = c^{(i,0)}(z, \bar{z}) + \mu^{(i)}_{\Sigma}(z, \bar{z}) c^{0,\bar{\Sigma}}(z, \bar{z})
$$

which is the expression of the chiral ghost fields in terms of the true ghost fields $c^{(i,0)}$ and $c^{0,\bar{\Sigma}}$ for (infinitesimal) diffeomorphisms of $M$ \cite{ref}, while $|I| = 2$ yields respectively for $I = (ii)$ and $I = (ij)$, $i < j$,

$$
C^{(ii)}(z, \bar{z}) = c^{(i,0)}(z, \bar{z}) + \mu^{(i)}_{\Sigma}(z, \bar{z}) c^{(i,\bar{\Sigma})}(z, \bar{z}) + \left( \mu^{(i)}_{\Sigma}(z, \bar{z}) \right)^2 c^{(0,\bar{\Sigma})}(z, \bar{z}) + \mu^{(ii)}_{\Sigma}(z, \bar{z}) c^{0,\bar{\Sigma}}(z, \bar{z}) + \sum_{\tau < \Sigma} \mu^{(\tau)}_{\Sigma}(z, \bar{z}) \mu^{(\tau)}_{\Sigma}(z, \bar{z}) c^{0,\bar{\Sigma}}(z, \bar{z})
$$

$$
C^{(ij)}(z, \bar{z}) = c^{(i,0)}(z, \bar{z}) + \mu^{(i)}_{\Sigma}(z, \bar{z}) c^{(j,\bar{\Sigma})}(z, \bar{z}) + \mu^{(i)}_{\Sigma}(z, \bar{z}) c^{(i,\bar{\Sigma})}(z, \bar{z}) + \mu^{(ij)}_{\Sigma}(z, \bar{z}) c^{0,\bar{\Sigma}}(z, \bar{z}) + 2 \mu^{(i)}_{\Sigma}(z, \bar{z}) \mu^{(j)}_{\Sigma}(z, \bar{z}) c^{0,\bar{\Sigma}}(z, \bar{z}) + \sum_{\tau < \Sigma} \left( \mu^{(\tau)}_{\Sigma}(z, \bar{z}) \mu^{(i)}_{\Sigma}(z, \bar{z}) + \mu^{(\tau)}_{\Sigma}(z, \bar{z}) \mu^{(j)}_{\Sigma}(z, \bar{z}) \right) c^{0,\bar{\Sigma}}(z, \bar{z}).
$$

The ghosts $c^{(R,\bar{\Sigma})}(z, \bar{z})$ satisfy rather elaborate BRS transformations, which generalize formula (4.61) to the non chiral sectors. They can be obtained either from the very definition Eq (4.67) or from the combined action of the decomposition Eq (4.67) and the BRS variations (4.61) and (4.65). For $|P| + |\bar{Q}| \geq 1$ the variations look like

$$
\mathcal{S} c^{(P,\bar{Q})}(z, \bar{z}) = \sum_{\underline{s} \leq \bar{Q}} \sum_{k=1}^{n} (1 - \delta_{0p_k}) \sum_{R^{(k)} \leq P^{(k)} - 1_k} \frac{(P - R^{(k)} - 1_k + 1_r)!}{(P - R^{(k)} - 1_k)!} \times c^{(P - R^{(k)} - 1_k + 1_r, \bar{Q} - \bar{\Sigma})}(z, \bar{z}) \partial_r c^{(R^{(k)} + 1_k, \bar{\Sigma})}(z, \bar{z})
$$
\( + (1 - \delta_{0\mathcal{Q}}) \frac{(P+1_r)!}{P!} \sum_{|\mathcal{S}| \leq \mathcal{Q}, |\mathcal{S}| \geq 1} c^{(P+1_r, \mathcal{Q}-\mathcal{S})}(z, \bar{z}) \partial_r c^{(0, \mathcal{S})}(z, \bar{z}) \)

\( + \sum_{R \leq P} \sum_{k=1}^{n} (1 - \delta_{0\mathcal{Q}}) \sum_{|\mathcal{S}| \leq |\mathcal{Q}| - 1_r} \frac{(Q - \mathcal{S}^{(k)} - 1_r + 1_p)!}{(Q - \mathcal{S}^{(k)} - 1_p)!} \times \)

\( c^{(P-R, \mathcal{Q}-\mathcal{S}^{(k)} - 1_r + 1_p)}(z, \bar{z}) \partial_r c^{(R, \mathcal{S}^{(k)} + 1_r)}(z, \bar{z}) \)

\( + (1 - \delta_{0P}) \frac{(Q + 1_p)!}{Q!} \sum_{R \leq P, |R| \geq 1} c^{(P-R, \mathcal{Q} + 1_p)}(z, \bar{z}) \partial_r c^{(R, 0)}(z, \bar{z}), \)

and, according to the BRS technique briefly recalled in the introduction, give rise to the \( \mathcal{W}_\infty \)-structure at a non chiral level. These results provide (in the \( (z, \bar{z}) \)-space) an infinite \( \mathcal{W}_\infty \)-algebra of which the first step describes the reparametrization invariance \( (z, \bar{z}) \rightarrow (\mathcal{Z}^\alpha(z, \bar{z}), \overline{\mathcal{Z}^\alpha}(z, \bar{z})) \) studied in [1].

Here, what is left over is the relic of the deformation process for \( \mathcal{Y}_\alpha, \overline{\mathcal{Y}_\pi} \neq 0 \) given by an infinite hierarchy of smooth changes of local complex coordinates \( (z, \bar{z}) \rightarrow (\mathcal{Z}^A(z, \bar{z}), \overline{\mathcal{Z}^A}(z, \bar{z})) \) on the base (the \( (z, \bar{z}) \)-space) of the symplectic space. The new \( \mathcal{W}_\infty \)-algebra really encodes the behavior under symplectomorphisms of this hierarchy.

### 5 Towards a Lagrangian formulation

If we wish to construct now a Lagrangian field theory whose classical limit is invariant under this \( n \)-dimensional extension of a \( \mathcal{W}_\infty \)-algebra, it would retain the imprinting of the infinite expansion from which the algebra is extracted, by reproducing a theory which is badly packed in the \( (z, \bar{z}) \) space and makes attempt to get away in the full symplectic space. The ‘classical’ fields whose dynamics serve to probe the \( \mathcal{Y}_\alpha, \overline{\mathcal{Y}_\pi} \neq 0 \) sector are the generalized Bilal-Fock-Kogan parameters \( \mu^{(I)}_J(z, \bar{z}) \) defined in (3.55). Indeed they are the only “true” local fields from which the pure gravitational theory would depend on. They are sources related to higher spin fields as in the unidimensional complex case, see e.g. [22].

So from the BRS approach an infinite set of Ward operators \( \mathcal{W}_{(I)}(z, \bar{z}) \) can be obtained and from which a classical action \( \Gamma^{CI} \) may be defined in the vacuum sector as follows. For \( |I| \geq 1 \),

\[ \mathcal{W}_{(I)}(z, \bar{z}) \Gamma^{CI} = - \overline{\partial}_r \frac{\delta \Gamma^{CI}}{\mu^{(I)}_J(z, \bar{z})} + \sum_{|L| \geq 1} \left( (i_r + \ell_r \partial_r \mu^{(L)}_J(z, \bar{z})) + \ell_r \mu^{(L)}_J(z, \bar{z}) \partial_r \right) \frac{\delta \Gamma^{CI}}{\mu^{(I+L-1_r)}_J(z, \bar{z})} = 0. \] (5.1)

These encapsulate the first order case \( |I| = |L| = 1 \) already treated in [3]. Remark also that in order to now the \( \overline{\partial} \) divergence of the higher spin current, dual to \( \mu^{(I)}_J \), the infinite collection of higher spin fields must be known.
Using the usual techniques, which in two dimensional limit, lead from Ward identities to O.P.E. expansion \[24\] we can derive a generalization of the “O.P.E.” algebra which would promote the present symplectic approach, since a \(n\) complex dimensional short-distance product could generally be a difficult task to manage.

For both \(|I|\) and \(|J|\) greater than 0, one obtains

\[
\begin{align*}
\pi \sum_{j=1}^{n} \int_{\mathbb{C}^{n-1}} \left( \prod_{\ell=1}^{n} \frac{d\ell^\tau \wedge d\ell^\bar{\tau}}{2i(w^\ell - z^{\ell'})} \right) & \frac{\delta^2 \Gamma_{CI}}{\delta \mu^{(J)}(w^1, \ldots, w^n, \overline{\tau}^1, \ldots, \overline{\tau}^j, \ldots, \overline{\tau}^n) \delta \mu^{(J)}(z, \overline{\tau})} \bigg|_{\mu=0} \\
+ \sum_{r=1}^{n} \left( \frac{i_r + j_r}{(z^r - z^{r'})^2} - \frac{i_r}{z^r - z^{r'}} \right) & \left( \prod_{\ell=1}^{n} \frac{1}{z^{\ell} - z^{\ell'}} \right) \frac{\delta \Gamma_{CI}}{\delta \mu^{(I+J-1,r)}(z, \overline{\tau})} \bigg|_{\mu=0} = 0. \quad (5.2)
\end{align*}
\]

leading to a convolution algebra, where the directional properties of the (short) distance limit is taken into account, for the Green functions generated by the generalized Bilal-Fock-Kogan fields. Its two dimensional limit gives the usual classical O.P.E. expansion.

Anyhow, due to the anomalous character of the diffeomorphism symmetry at the quantum level, we must foresee whether this defect would be transmitted to the residual part of the algebra. Hence quantum corrections would be required to give a meaning to the theoretical model and are still under investigation.

6 Conclusions

It has been shown how a symplectic approach gives a strong geometrical way of extending the notion of \(\mathcal{W}_\infty\)-algebra as a symmetry arising in the one complex dimensional case to a generic \(n\) complex dimensional (compact) manifold.

This symmetry appears from consistent deformations of integrable complex structures in the spirit of Kodaira-Spencer deformation. The decomposition in terms of local quantities as the Bilal-Fock-Kogan coefficients considered as generalized sources for higher spin fields naturally emerges from the construction. However, in this symplectic framework, a truncation process analogous to the one for Riemann surfaces \[22\] from \(\mathcal{W}_\infty\)-algebra to a finite \(\mathcal{W}\)-algebra is still lacking. In particular, the latter could be of some interest in both string and brane theories (see e.g. \[22\]) where higher spin fields appear in four real dimensions. In these theories, the fields seem to be related to some finite \(\mathcal{W}\)-algebra. However theoretical models with explicit higher spin fields still remain to be constructed.

More generally, even if the topic ought to seem, according to the physical context, rather technical and strongly grounded on mathematics, we would emphasize that the important problem of a metric or a complex structure for a physical theory embedded in a gravitational model is far of being understood. So any little step in that direction could give profit to discover the
real of Nature and the intricacies of geometrical implications within the formulation of Physical Theories.

References

[1] A.A. Belavin. V.G. Knizhnik. Phys.Lett. B168(1986) 201, Sov. Phys. JEPT 64 (1986) 214
[2] A.A. Belavin, A.M. Polyakov,A.B. Zamolodchikov. "Infinite conformal symmetry in two dimensional quantum field theory". Nucl.Phys. B 241 (1984) 333
[3] A.K. Dewdney. "Planiverse". Poseidon Press 1984.
[4] A.M. Zamolodchicov. Theor. Math. Phys. 65 (1985) 1205
[5] T.Tjin. "Finite and infinite $W$ algebras". PhD. Thesis hep-th/9308146
[6] K. Kodaira. Complex Manifolds and Deformation of Complex Structures. Comprehensive Studies in Mathematics. Spinger-Verlag, New-York, 1986.
[7] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa. "Kodaira-spencer theory of gravity and exact results for quantum string amplitudes". Comm. Math. Phys. 165, (1994) 311–428.
[8] A. Losev, G. Moore, N. Nekrasov, and S. Shatashvili. “Chiral Lagrangians, anomalies, supersymmetry and holomorphy”, hep-th/9606082 Nucl. Phys. B484, (1997) 196.
[9] G. Bandelloni and S. Lazzarini. "Kodaira-Spencer Deformation Of Complex Structures And Lagrangian Field Theory" J. Math. Phys. 39 (1998) 3619-3642. e-Print Archive: hep-th/9802086
[10] D. B. Fairlie and J. Nuyts. “Deformations And Renormalizations Of W(Infinity),” Commun. Math. Phys. 134 (1990) 413.
[11] R. Dijkgraaf."Chiral deformations of conformal field theories," Nucl. Phys. B 493 (1997) 588 hep-th/9609022.
[12] C. Castro. "$W$ Geometry from Fedosov’s deformation quantization”, J. Geom. Phys. 33 (2000) 173-190, hep-th/9802023
[13] X. Shen, “$W$ infinity and string theory”, Int. J. Mod. Phys. A7 (1992) 6953-6993.
[14] C. Vafa. "Mirror transform and string theory”. Talk given in the Geometry and Topology Conference, April ’93, Havard in honour of R. Bott, hep-th/940315, March 1994.
[15] M. Bershadsky and V. Sadov. “Theory of Kähler gravity”. Int. J. Mod. Phys. A11 (1996) 4689-4730, hep-th/9410011.
[16] S. Barannikov. “Extended Moduli Spaces And Mirror Symmetry In Dimension $n > 3$” Ph.D.Thesis math-AG/9903124

[17] J.M.F. Labastida and M. Marino (Santiago de Compostela U.). “Type B topological matter, Kodaira-Spencer theory, and mirror symmetry”. Phys. Lett. B333 (1994) 386–395.

[18] J.-L. Koszul. Lectures on fibre bundles and differential geometry. Reissued 1965. Tata Institute of Fundamental Research, Bombay, 1960.

[19] D.B. Fucks. Cohomology of infinite Dimensional Algebra. Consultant Bureau, New York, 1986.

[20] G. Bandelloni, S. Lazzarini. CPT-99-P-3923, "W-algebras from symplectomorphisms” J.Math.Phys.41:2233-2250,2000 hep-th/9912202

[21] G. Bandelloni,S. Lazzarini. "The role of complex structures in W symmetry” Nucl.Phys.B577:471-499,2000 e-Print Archive: hep-th/0003027

[22] G. Bandelloni, S. Lazzarini. "The geometry of W(3) algebra: a twofold way for the rebirth of a symmetry” Nucl.Phys.B594:477-500,2001 e-Print Archive: hep-th/0011208

[23] C. Becchi, A. Rouet and R. Stora, Annals Phys. 98, 287 (1976).

[24] A.A Bilal , V,V Fock and I.I Kogan. “On the origin of W-algebras”. Nucl.Phys B359 (1991) 635.

[25] C.Becchi. “On the covariant quantization of the free string: the conformal structure”. Nucl. Phys. B 304 (1988) 513

[26] G. Bandelloni, “Diffeomorphism cohomology in Quantum Field Theory models”, Phys. Rev. D38 (1988) 1156.