How does the electromagnetic field couple to gravity, in particular to metric, nonmetricity, torsion, and curvature?

Friedrich W. Hehl$^{1,2}$ and Yuri N. Obukhov$^3$

1 School of Natural Sciences  
Institute for Advanced Study  
Princeton, NJ 08540, USA  
2 Institute for Theoretical Physics  
University of Cologne  
50923 Köln, Germany  
3 Department of Theoretical Physics  
Moscow State University  
117234 Moscow, Russia

Abstract. The coupling of the electromagnetic field to gravity is an age-old problem. Presently, there is a resurgence of interest in it, mainly for two reasons: (i) Experimental investigations are under way with ever increasing precision, be it in the laboratory or by observing outer space. (ii) One desires to test out alternatives to Einstein’s gravitational theory, in particular those of a gauge-theoretical nature, like Einstein-Cartan theory or metric-affine gravity.— A clean discussion requires a reflection on the foundations of electrodynamics. If one bases electrodynamics on the conservation laws of electric charge and magnetic flux, one finds Maxwell’s equations expressed in terms of the excitation $H = (D, H)$ and the field strength $F = (E, B)$ without any intervention of the metric or the linear connection of spacetime. In other words, there is still no coupling to gravity. Only the constitutive law $H = \text{functional}(F)$ mediates such a coupling. We discuss the different ways of how metric, nonmetricity, torsion, and curvature can come into play here. Along the way, we touch on non-local laws (Mashhoon), non-linear ones (Born-Infeld, Heisenberg-Euler, Plebański), linear ones, including the Abelian axion (Ni), and find a method for deriving the metric from linear electrodynamics (Toupin, Schönberg). Finally, we discuss possible non-minimal coupling schemes.

1 Introduction

General relativity was proposed in 1915. One of its predictions was the bending of light rays of stars in the gravitational field of the Sun. This effect was verified observationally soon afterwards by Dyson et al. in 1920 and put, as a result, Einstein’s theory in the forefront of gravitational research.

Within the framework of general relativity, a light ray can be extracted from classical electrodynamics in its geometrical optics limit, i.e., for wavelengths much smaller than the local curvature radius of space. Accordingly, the bending of light can be understood as a result of a nontrivial refractive index of spacetime, see Skrotskii et al. [81,89], due to the coupling of the electromagnetic field $F$ to
the gravitational field \( g \). Classically, we have in nature just these two fundamental fields \( F \) and \( g \), the weak and the strong fields being confined to microphysical dimensions of \( 10^{-19} \text{m} \) or \( 10^{-15} \text{m} \), respectively. Therefore, the coupling of \( F \) and \( g \) is of foremost importance in classical physics.

The conventional way that coupling is achieved is to display the Maxwell-Lorentz equations of vacuum electrodynamics in the (flat) Minkowski world of special relativity theory in Cartesian coordinates. For this purpose, usually the formalism of tensor analysis (Ricci calculus) is used, see [78]:

\[
F^{ij,j} = 0, \quad F^{ij,k} + F^{jk,i} + F^{ki,j} = 0.
\]

(1)

Here \( F^{ij} = -F^{ji} = (F_{01}, F_{02}, F_{03}, F_{23}, F_{31}, F_{12}) = (E, B) \) is the electromagnetic field strength, \( I^i \) the electric 4-vector current, and

\[
F^{ij} := g^{ik} g^{jl} F_{kl},
\]

(2)

with \( g^{ij} \) as the contravariant components of the metric. The commas in (1) denote partial differentiation with respect to the local spacetime coordinates \( x^i \).

If we switch on gravity, the flat Minkowski world becomes curved, the spacetime geometry now being a Riemannian geometry with a variable metric \( g_{ij} \) of Minkowskian signature (+−−−). The coupling of the Maxwell-Lorentz set to gravity is now brought about by the comma goes to semicolon rule, (see [52]), where the semicolon represents the covariant derivative \( \nabla_i \equiv ;i \) with respect to the Riemannian connection (“Levi-Civita connection”):

\[
F^{ij,j} = I^i, \quad F^{ij;k} + F^{jk;i} + F^{ki;j} = 0.
\]

(3)

This translation rule from special to general relativity is also alluded to as minimal coupling with the additional understanding that the components of the metric in (2) become spacetime dependent fields.

The metric field \( g_{ij}(x) \), entering (3) via (2) and via the covariant derivatives, i.e., via the semicolons, has to fulfill the Einstein field equation,

\[
\text{Ricci}_ij - \frac{1}{2} g_{ij} \text{Ricci}_k^k + \kappa \left( \text{Max}_i^j + \text{mat}_i^j \right) = 0,
\]

(4)

with

\[
\text{Ricci}_ij := R_{ij}^k, \quad \text{Max}_i^j := \sqrt{\frac{\varepsilon_0}{\mu_0}} \left( -F_{ik} F^{jk} + \frac{1}{4} \delta^j_i F_{kl} F^{kl} \right).
\]

(5)

Here \( R_{ij}^k \) is the curvature and \( \text{Max}_i^j \) the material energy-momentum tensor. The coupled Einstein-Maxwell system describes correctly a wealth of experiments, in particular the gravitational bending of light, the gravitational redshift, the time delay of radar pulses in the gravitational field of the Sun, and the gravitational lensing and microlensing of starlight in the gravitational field of galaxies.

But in all these experiments, we study the propagation of light along nullgeodesics in a prescribed (and perhaps slowly varying) gravitational field which is
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a solution of the Einstein vacuum equation – and not of the electro-vacuum equation. We could call this the non self-consistent Einstein-Maxwell theory. In the solar system, e.g., the Schwarzschild metric is taken as solution of the Einstein vacuum equation and the motion of a “photon” is described by the null geodesic equation on this background. A true novel effect of the Einstein-Maxwell theory would be, e.g., the generation of electromagnetic waves by gravitational waves. Because of their smallness, no such effects were ever observed. Accordingly, the interaction of a classical electromagnetic field $F_{ij}$ in the form of a lightray with a prescribed gravitational field $g_{ij}(x)$ is well described by means of Eqs. (3). Nevertheless, further consequences of these equations need to be compared with experiment as soon as more sensitive measuring methods are available.

2 On the equivalence principle

According to Einstein’s equivalence principle, see [20], gravity can be locally simulated in a gravity-free region of spacetime by going over from the Cartesian coordinates, anchored in an inertial frame of reference (including an inertial clock) and used in (1), to arbitrary curvilinear coordinates yielding a non-inertial frame in general, as in (3). In this context, the metric $g_{ij}$, occurring in (3) and in the semicolons of (3), is understood as a flat metric in curvilinear coordinates. Thus, the minimal coupling can be interpreted, in a first step, just as a coordinate transformation from Cartesian to curvilinear coordinates. And, moreover, it identifies the metric as the gravitational potential.

On the other hand, let us assume that we are in a region with gravity and (2) and (3) are valid together with the Einstein equation for the metric. Then, also according to Einstein’s equivalence principle, we must be able to pick suitable coordinates such that locally the equations look like in special relativity in Cartesian coordinates. In Riemannian geometry, the local coordinates are called Riemannian normal (hence geodesic) coordinates at one point $P$, if the Christoffel symbols

$$\Gamma_{ij}^k := \frac{1}{2} g^{kl} \left( g_{ui,j} + g_{jt,i} - g_{ij,l} \right)$$

vanish at $P$ and and the metric becomes Minkowskian:

$$\Gamma_{ij}^k|_P \equiv 0 , \quad g_{ij}|_P \equiv \text{diag} \left( +1, -1, -1, -1 \right) .$$

Accordingly, the semicolon becomes a comma and the metric in (3), at one given point, looks flat.

Still, the curvature is non-vanishing, of course: $R_{ijkl}|_P \neq 0$. The equations look flat since they contain only first derivatives. If they contained second derivatives, then the semicolons goes to comma rule and its reverse would not work since on that level not only the Christoffels enter but potentially also the curvature which, in contrast to the Christoffels, is a tensor and cannot be nullified by means of a suitable choice of coordinates. For that reason, the minimal coupling
procedure, being in this context an expression of the equivalence principle, must be
applied only to first order differential equations. The safest thing is then to
apply it, as a rule, only on the level of a Lagrangian, since there ordinarily only
first-order expressions are allowed for. Non-minimal couplings of the gravita-
tional field to electromagnetism have also been investigated, see Prasanna [74],
Buchdahl [8], Goenner [23], and Müller-Hoissen [58,59,60], for example, or for
light rays in non-minimally coupled theories, see Drummond and Hathrell [17],
but the price one has to pay is to introduce a new constant of nature; and there
is no evidence for such a constant in nature – unless one takes the Planck length
itself. We will come back to these questions in Sec.7.

Therefore we can conclude that the equivalence principle and minimal cou-
pling work well for the Maxwell-Lorentz equations (1) and that they lead to
experimentally established equations.

Wave equation for the electromagnetic field strength We hasten to add
that, within the framework of the minimally coupled Maxwell-Lorentz equations, we
find 2nd derivatives if we derive the wave equation for the electromagnetic field strength
$F$ — and then also curvature terms are expected to emerge. This is exactly what
happens, as already found by Gordon [24] and Eddington [18].

In the framework of exterior calculus (see Frankel [21]), let us consider the elec-
tromagnetic field strength 2-form

$$
F_{ij} = \frac{1}{2} F_{ij} \, dx^i \wedge dx^j.
$$

In Maxwell-Lorentz vacuum electrodynamics, it satisfies $dF = 0$, $\varepsilon_0 \varepsilon^* F = \frac{1}{c} J$. We denote the codifferential by $\delta := * d^*$. Then we find, with the wave operator (d’Alembertian)

$$
\Box := \delta d + d \delta, \quad (8)
$$

and by using the Maxwell-Lorentz equations, the wave equation

$$
\Box F = \frac{1}{\varepsilon_0 c} d^* J, \quad (9)
$$

see [53]. The left hand side of this equation, in terms of components, can be determined
by substituting (8):

$$
\Box F = \frac{1}{2} \left( \nabla^k \nabla_k F_{ij} + 2 R_{[i} F_{j]k} - R^{k[i} F_{j]k} \right) dx^i \wedge dx^j. \quad (10)
$$

Accordingly, minimal coupling can lead to curvature terms of a prescribed form.\footnote{Of course, we could have non-minimal coupling as, e.g., in $\Box F + \gamma \left( e_a \epsilon_{ij} R^{a} \right) \wedge F = \frac{1}{\varepsilon_0 c} d^* J$, see Sec.7.}

3 A caveat

Soon after general relativity had been proposed, it became clear, see Einstein [19], that one can introduce as auxiliary variables the densities

$$
F^i := \sqrt{-g(x)} g^{ik}(x) g^{jl}(x) F_{kl}, \quad I^i := \sqrt{-g(x)} I^i, \quad (11)
$$
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with \( g(x) := \det g_{ij}(x) \), in terms of which the Maxwell-Lorentz equations (11) can be rewritten in a metric-free way as

\[
F_{ij,j} = \mathcal{I}^i, \quad F_{ij,k} + F_{jk,i} + F_{ki,j} = 0.  \tag{12}
\]

Similarly, the charge conservation law \( I^{i;i} = 0 \) can be put in the form

\[
\mathcal{I}^i = 0.  \tag{13}
\]

The metric enters only via the densities defined in (11). In fact, if we started from the set (12) in special relativity right away, then no comma goes to semicolon rule would have been necessary: These equations are generally covariant, they are valid in arbitrary curvilinear coordinates, be it in the framework of special or general relativity theory.

In the calculus of exterior differential forms (Cartan calculus), see Frankel [21], these equations can be formulated very succinctly. We introduce the electric current as odd 3-form,

\[
J := \rho - j \wedge dt = \frac{1}{3!} J_{ijk} dx^i \wedge dx^j \wedge dx^k,  \tag{14}
\]

the electromagnetic excitation as odd 2-form

\[
H = D - \mathcal{H} \wedge dt = \frac{1}{2!} H_{ij} dx^i \wedge dx^j,  \tag{15}
\]

and the electromagnetic field strength as even 2-form

\[
F = B + E \wedge dt = \frac{1}{2!} F_{ij} dx^i \wedge dx^j.  \tag{16}
\]

Then (12) reads

\[
dH = J, \quad dF = 0,  \tag{17}
\]

with

\[
dJ = 0.  \tag{18}
\]

The set (17) represents the Maxwell equations. They are independent of metric and connection. The constitutive relation for the vacuum reads

\[
H = \sqrt{\frac{\varepsilon_0}{\mu_0}} * F,  \tag{19}
\]

where the star \( * \) represents the metric-dependent and odd Hodge duality operator. Eq.(19) corresponds to (11) and \( \mathcal{I}^i \) can be related to the components of \( J \),

\[
\mathcal{F}^{ij} = \frac{1}{2!} \sqrt{\frac{\mu_0}{\varepsilon_0}} \epsilon^{ijkl} H_{kl}, \quad \mathcal{I}^i = \frac{1}{3!} \epsilon^{ijkl} J_{jkl},  \tag{20}
\]

with \( \epsilon^{ijkl} = \pm 1,0 \), the totally antisymmetric Levi-Civita tensor density.

Now the equivalence principle looks empty: Since the Maxwell equations (17) are formulated in a coordinate and frame independent way, they are valid in this
form in arbitrary coordinate systems and frames, be it in a flat or in a curved spacetime. Only the constitutive relation (19) “feels”, up to a conformal factor, the presence of a flat or a non-flat metric, i.e., the constitutive relation couples to the conformally invariant part of the metric. The coupling of electromagnetism to gravity becomes almost trivial. Is all this just a mathematical trick, which distracts from the physical content of Maxwell’s theory, or is it more?

One further observation hints also at the need for clarification. The Einstein-Cartan theory of gravity is a viable gravitational theory, see [30,51,86]. It is the simplest model of the metric-affine gauge theory of gravity, see [31,25]. In the Einstein-Cartan theory, spacetime is described by means of a Riemann-Cartan geometry with torsion and curvature. If we couple (3) to gravity, do we have to use the semicolons as covariant derivatives with respect to the Riemann-Cartan connection or still with respect to the Christoffels, see [14]? In the context of (17), this question cannot even be posed, since the exterior derivative $d$ is all what is needed. Are then the equations (17) misleading as a starting point for coupling to gauge gravity? What could be the appropriate starting point?

Provided one formulates Maxwell’s theory and its coupling to gravity in terms of a Lagrangian with the electromagnetic potential $A$ as variable, gauge covariance of the formalism results in (17) cum (19), as was pointed out by Benn, Dereli, and Tucker [5]. However, we would like to have some more immediate insight into the structure of electromagnetism as induced by experiment even without having a variational formulation at our disposal.

4 Electric charge and magnetic flux conservation

The metric is a quantity which allows to define lengths and angles in spacetime. There are, however, laws in physics which don’t require the knowledge of a metric. Take the conservation law of electric charge as an example. Mark a 3-dimensional simply connected submanifold $\Omega_3$. We know from experiment that a possible electric charge inside $\Omega_3$ is composed of charge “quanta”, i.e., there is an integer number of elementary charges in $\Omega_3$. Recent advances in technology made it possible, see [13,44], to trap and to count single electrons and protons. Thus, as soon as we have such quanta available, we can rely on counting procedures, see Post [73], the use of a meter stick or a chronometer is superfluous under such circumstances.

Electric charge conservation is experimentally well-established and is one of the pillars electromagnetism rests on. We formulate it, following Kottler-Cartan-van Dantzig, see [72,87] and also [32], most appropriately as an integral law. According to (17), we assume the existence of the odd electric current 3-form $J$. We take charge conservation as axiom 1, that is, $J$ integrated over a closed 3-dimensional hypersurface $\Omega_3$ has to vanish, if this hypersurface is the boundary

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2 A proper discussion of the equivalence principle in the context of Einstein-Cartan theory requires the introduction of local coframes, see [25,14,72]. Being concerned here only with electromagnetism, it is sufficient to use natural, i.e., holonomic coframes.
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of a connected 4-volume $\Omega_4$:

$$\int_{\partial \Omega_4} J = \int_{\Omega_4} dJ = 0, \quad \text{or} \quad dJ = 0.$$  \hspace{1cm} (21)

Here we applied the Stokes theorem.

If (21) is assumed to be valid $\int_{C_3} J = 0$ for all three-cycles $\partial C_3 = 0$, $C_3 \neq \partial \Omega_4$, then, according to a theorem of de Rham, $J$ is exact, see [73]. Thus the inhomogeneous Maxwell equation is a consequence,

$$J = dH,$$ \hspace{1cm} (22)

with the odd 2-form $H$ of the electromagnetic excitation, see [13]. The excitation is only determined up to an exact form. Nevertheless, the electric excitation $\mathcal{D}$ can be measured by means of Maxwellian double plates as charge per unit area, the magnetic excitation $\mathcal{H}$ by means of a small test coil, which compensates the $\mathcal{H}$-field to be measured, as current per unit length. This is possible since in these null experiments vanishing field strength $F$ implies vanishing excitation $H$, see [23]. In other words, the extensive quantities $\mathcal{D}$ and $\mathcal{H}$ – and thus the 4-dimensional excitation $H$ – have an operationally significance of their own, since they are related to charge at rest or in motion, respectively. Accordingly, the somewhat formalistic introduction of the densities in [11] has now been legitimized as a transition to operationally meaningful additive quantities. Note that up to now only the differential structure of the spacetime was needed, a metric has not been involved.

Let us choose a field of 4-frames $e_\alpha$ and consider the motion of a point particle with respect to the reference frame thus defined. As axiom 2 one can take an operational definition of the electromagnetic field strength $F$ via the Lorentz force density

$$f_\alpha = (e_\alpha \lrcorner F) \wedge J.$$ \hspace{1cm} (23)

The interior product (contraction) is denoted by $\lrcorner$. The force density $f_\alpha$ is a notion from classical mechanics. It is an odd covector-valued 4-form. Accordingly, Eq. (23) can be read as a definition of the even 2-form $F$, see [10]. Again, we don’t need a metric. And we know the recipe of how to proceed in the same manner.

That $\int_{\Omega_2} F$ can be interpreted as magnetic flux is obvious if we choose $\Omega_2$ as a ‘spacelike’ surface (strictly, at this point we don’t know what spacelike means; we will come back to this later). In superconductors under suitable circumstances we can count (in an Abrikosov flux line lattice) quantized magnetic flux lines. This suggests that magnetic flux is a conserved quantity (axiom 3):

$$\int_{\partial \Omega_3} F = \int_{\Omega_3} dF = 0 \quad \text{or} \quad dF = 0.$$ \hspace{1cm} (24)

In this way, by means of the axioms (21), (23), and (24), we recovered the fundamental structure of Maxwell’s theory: $dH = J$, $dF = 0$. This is what had
been called *metric-free electrodynamics*. What is missing so far is the relation between the excitation $H$ and the field strength $F$, and it is exactly there where the metric, i.e., the gravitational potential comes in.

5 No interaction of charge and flux “substrata” with gravity

We now understand that the inhomogeneous Maxwell equation $dH = J$, as an expression of electric charge conservation, cannot be influenced by gravity, i.e., by the metric tensor $g$, or, in the case of metric-affine gravity or its specific subcases, such as Einstein-Cartan theory, by the connection $\Gamma$ of spacetime. The electric charge substratum of spacetime has rules of its own. Spacetime can be deformed by the presence of metric and connection, but the charge substratum and the net electric charge to be attributed to a prescribed 3-dimensional (3D) volume won’t change. Thus the additivity 3D volume-wise of the charge lays at the foundation of the Maxwellian framework. And it translates into the 2D additivity of the integrated excitation $\int_{\Omega_2} H$ – this being the reason why one uses this integral for the operational interpretation of $H$.

Similar arguments can be advanced for the homogeneous Maxwell equation $dF = 0$. However, first of all it should be stressed that the axiomatics we are using strongly suggests the non-existence of magnetic charges. If there were magnetic charges, then we would have no reason to believe in electric charge conservation either; compare for this argument axiom 1, Eq.(21), with axiom 3, Eq.(24). Conventionally, the inhomogeneous equation $dH = J$ is seen in analogy to the homogeneous one $dF = 0$. But not so in the framework of our axiomatics which has a firm empirical basis. We put $dJ = 0$ in analogy to $dF = 0$. The whole historical development of electromagnetism, starting with Ørsted and Ampère, points to the elimination of the phenomenologically introduced magnetic charges. Most recent experiments, see [1,29], exclude magnetic charges with very good precision. Thus theoretical as well as experimental evidence speak against the existence of magnetic charges.

Having said this, we hasten to add that, nevertheless, there is some kind of magnetic substratum in spacetime, namely the magnetic flux $\int_{\Omega_2} F$. It is a substratum of its own right. The fluxoids, the quantized magnetic flux lines in superconductors, see [24], do convey a clear message. Besides electric charge, magnetic flux (and not magnetic charge) has an independent standing in ele-

3 Stachel [83] calls it *generalized* electrodynamics. We don’t follow this suggestion, since Maxwell’s equations were originally given in terms of $(D, H)$ and $(E, \mu H)$ in a form ‘isomorphic’ to the $(1 + 3)$-decomposition of $dH = J$ and $dF = 0$, see [24]. Therefore, the “generalized” Maxwell equations, $dH = J$ and $dF = 0$, correspond in actual fact, just to Maxwell’s equations (modulo the substitution $B \to \mu H$). And this is how we will name them.

4 SI-unit Coulomb, elementary charge $e = 1.60217733 \times 10^{-19} \, \text{C}$.

5 SI-unit Weber, elementary fluxoid $h/(2e) = 2.06783461 \times 10^{-15} \, \text{Wb}$. 

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Axiom 3 is again a conservation theorem. In contrast to axiom 1, which has a fermionic smell, axiom 3 is more of a bosonic nature. Moreover, magnetic flux adds up 2D area-wise. For this reason, magnetic flux is represented by a 2-form and not, like the charge, by a 3-form. Accordingly, there are essential differences between these two conservation laws which express the peculiarities of the electromagnetic phenomena. Electric and magnetic effects enter the Maxwellian framework in an asymmetric way; in spite of all that talk about a duality between electricity and magnetism. But there is also a similarity in that both axioms are formulated as integral conservation laws. The possibility to count the fluxoids assures us that axiom 3 has to be again a law free of metric and connection.

Incidentally, there is a nice visualization of the fundamental quantities entering electrodynamics. If one describes the quantum Hall effect for low lying Landau levels, then the concept of a composite fermion is very helpful: it consists of one electron and an even number of fluxoids is attached to it, see Jain [41,42] and [43]. Isn’t that a very clear indication of what the fundamental quantities are in electrodynamics? Namely, electric charge (see axiom 1) and magnetic flux (see axiom 3), see also Nambu [61] in this general context.

Our conclusion is then that, as long as we opt for electric charge and magnetic flux conservation, the Maxwell equations in gravity-free regions, i.e., in the Minkowski spacetime of special relativity, read $dH = J$ and $dF = 0$; they remain the same irrespective of the switching on of gravity, be it in Einstein’s theory, in metric-affine gravity (see [76]), or in any other geometrical theory of gravity.

6 Constitutive law of electrodynamics and its relation to gravity

After having discussed extensively that gravity does not influence the Maxwell equations, we eventually turn to the constitutive law via which gravity does influence electrodynamics. It is true, the charge substratum and the flux substratum themselves do not couple to gravity, as we have shown in the last section. However, the interrelationship between both substrata is affected by gravity. Metaphorically speaking, the “flow” of each of the substrata is ruled by a particular gravity-independent conservation theorem, but the flows of electric charge and magnetic flux are coupled via a gravity-dependent constitutive law since, in the end, magnetism has to be expressed in terms of electricity.

Let us choose arbitrary local spacetime coordinates $x^i$. Then we have,

$$H = \frac{1}{2} H_{ij} \, dx^i \wedge dx^j, \quad F = \frac{1}{2} F_{ij} \, dx^i \wedge dx^j.$$  \hspace{1cm} (25)$$

We will turn first to the electrodynamics of material media in order to develop some intuition on the concept involved, but eventually, it will be the vacuum, be it in inertial or non-inertial frames, which will occupy the center of our interest.
Moving macroscopic matter defines a \((1 + 3)\)-splitting of spacetime specified by a well defined average 4-vector velocity field \(u\) which describes the congruence of worldlines of the flow of the medium. Such a vector field can be defined operationally from the motion of matter as follows.

Let a 3–dimensional arithmetic space \(R^3\) be equipped with the coordinates \(\xi^a, a = 1, 2, 3\). We will use these coordinates (known as Lagrange coordinates in continuum mechanics) as labels which enumerate elements of a material medium. A smooth mapping \(x_{(0)} : R^3 \rightarrow X_4\) into the spacetime defines a 3–dimensional space domain (hypersurface) \(V\) which represents the initial distribution of matter. In local spacetime coordinates, this mapping (or labeling) is given by the four functions \(x^i_{(0)}(\xi^a)\). It should be preserved at any time, i.e. along any worldline of a particular element its labels \(\xi^a\) are constant.

Given the initial configuration \(V\) of matter, we parameterize dynamics of the medium by the “time” coordinate \(\tau\) which is defined as the proper time measured along an element’s worldline from the original hypersurface \(V\). The resulting local coordinates \((\tau, \xi^a)\) are usually called the normalized comoving coordinates. Thus finally, the motion of matter is described by the functions \(x^i(\tau, \xi^a)\). Subsequently, we define the 4–velocity vector field by

\[
\mathbf{u} := \partial_{\tau} = \left( \frac{dx^i}{d\tau} \right)_{\xi^a=\text{const}} \partial_i .
\]

Evidently, a family of observers comoving with the matter is characterized by the same timelike congruence \(x^i(\tau, \xi^a)\). They are making physical (in particular, electrodynamical) measurements in their local reference frames which drift with the material motion.

One says that a medium, moving in general, has dispersion properties when the electromagnetic fields produce non-instantaneous polarization and magnetization effects. The most general linear constitutive law is then given, in the comoving system, by means of the integral

\[
H_{ij}(\tau, \xi) = \frac{1}{2} \int d\tau' K_{ijkl}(\tau, \tau') F_{kl}(\tau', \xi) .
\]

The coefficients of the kernel \(K_{ijkl}(\tau, \tau')\) are called the response functions. We expect the metric to be involved in their set-up. Their form is defined by the internal properties of matter and by the motion of a medium.

Mashhoon [49] has proposed a physically very interesting example of such a non-local electrodynamics in which non-locality comes as a direct consequence of the non-inertial dynamics of observers. In this case, instead of (25), one should use the field expansions

\[
H = \frac{1}{2} H_{\alpha\beta} \vartheta^\alpha \wedge \vartheta^\beta, \quad F = \frac{1}{2} F_{\alpha\beta} \vartheta^\alpha \wedge \vartheta^\beta
\]
with respect to the coframe of a non-inertial observer $\vartheta^\alpha = e_i^\alpha \, dx^i$. The constitutive law is then replaced by

$$H_{\alpha\beta}(\tau, \xi) = \frac{1}{2} \int d\tau' K_{\alpha\beta}^{\gamma\delta}(\tau, \tau') \, F_{\gamma\delta}(\tau', \xi), \quad (29)$$

and the response kernel in (29) is now defined by the acceleration and rotation of the observer’s reference system. It is a constitutive law for the vacuum as viewed from a non-inertial frame of reference.

Mashhoon imposes an additional assumption that the kernel is of convolution type, i.e., $K_{\alpha\beta}^{\gamma\delta}(\tau, \tau') = K_{\alpha\beta}^{\gamma\delta}(\tau - \tau')$. Then the kernel can be uniquely determined by means of the Volterra technique, and often it is possible to use the Laplace transformation in order to simplify the computations. Unfortunately, although Mashhoon’s kernel is always calculable in principle, in actual practice one normally cannot obtain $K$ explicitly in terms of the observer’s acceleration and rotation.

Preserving the main ideas of Mashhoon’s approach, one can abandon the convolution condition. Then the general form of the kernel can be worked out explicitly ($u$ is the observer’s 4-velocity):

$$K_{\alpha\beta}^{\gamma\delta}(\tau, \tau') = \frac{1}{2} \epsilon_{\alpha\beta}^{\lambda\delta} \left( \delta_\chi^{\gamma} \delta(\tau - \tau') - u^\lambda \Gamma_\lambda^{\gamma}(\tau') \right). \quad (30)$$

The influence of non-inertiality is manifest in the presence of the connection 1-form. The kernel (30) coincides with the original Mashhoon kernel in the case of constant acceleration and rotation, but in general the two kernels are different [55]. Perhaps, only the direct observations would establish the true form of the non-local constitutive law. However, such a non-local effect has not been confirmed experimentally as yet.

### 6.2 Non-linear

But the constitutive law can also be non-linear (or non-local and non-linear at the same time). In the local and non-linear Born-Infeld electrodynamics [6], with the dimensionfull parameter $f_e$ as maximal attainable electric field strength, we have

$$H = -\frac{\partial V_{BI}}{\partial F} \sim \frac{\partial \sqrt{-\det|g_{kl} + \frac{1}{2} f_e^2 F_{kl}|}}{\partial F}. \quad (31)$$

The metric as symmetric second rank tensor enters here in a very natural way. It adds up with the antisymmetric electromagnetic field to an asymmetric tensor – much in the way Einstein had hoped to find for his unified field theories of gravity and electromagnetism. By differentiation, we find

$$H = \frac{\varepsilon_0}{\mu_0} \frac{\ast F + \frac{1}{2 f_e^2} \ast(F \wedge F) \, F}{\sqrt{\sqrt{1 - \frac{1}{f_e^2} \ast(F \wedge \ast F) - \frac{1}{f_e^4} \left[\ast(F \wedge F)\right]^2}}}. \quad (32)$$
now the metric being absorbed in the (odd) Hodge star operator, see [21]. For $f_o \to \infty$, we recover the conventional local and linear Maxwell-Lorentz theory for vacuum with $H = \sqrt{\mu_o \epsilon_0} F$. The Born-Infeld electrodynamics is presently used as a toy model in string theories, see [22]. The problem with Born-Infeld electrodynamics is that, in contrast to Maxwell’s theory, it defies quantization. It is an interesting model, but nothing like an established theory.

A similar example is the non-linear Heisenberg-Euler electrodynamics [34]. Quantum electrodynamical vacuum fluctuations yield corrections to Maxwell’s theory that can be accounted for by an effective constitutive law constructed by Heisenberg and Euler. To the first order in the fine structure constant $\alpha_f = \frac{e^2}{4 \pi \epsilon_0 \bar{h} c}$, it is given by (see also [39, 35])

$$H = \sqrt{\mu_o \epsilon_0} \left\{ 1 + \frac{8 \alpha_f}{45 B_k^2} (F \wedge *F) *F + \frac{14 \alpha_f}{45 B_k^2} * (F \wedge F) F \right\},$$

(33)

where $B_k^2 = \frac{m^2 c^2 \bar{h}}{e} \approx 4.4 \times 10^9 \text{T}$, with the mass of the electron $m$. The metric is again hidden in the Hodge star and the Maxwell-Lorentz limit results analogously for $m \to \infty$. The Casimir force between two uncharged electrically conducting plates, also an effect of vacuum fluctuations, has been experimentally verified as have been non-linear effects in the “superposition” of strong laser beams. Accordingly, the non-linear constitutive law (33) is a valid post-classical approximation of vacuum electrodynamics and as such experimentally confirmed.

Note that these variants of classical electrodynamics respect charge and flux conservation. This underlines the fact that our axiomatics clearly points to that structure of electrodynamics, namely the constitutive law, which can be changed without giving up the essentials of electrodynamics.

Both, Eqs. (32) and (33) are special cases of Plebański’s more general non-linear electrodynamics [71]. Let the quadratic invariants of the electromagnetic field strength be denoted by

$$I_1 := \frac{1}{2} * (F \wedge *F) = \frac{1}{2} (E^2 - B^2) \quad \text{and} \quad I_2 := \frac{1}{2} * (F \wedge F) = E \cdot B,$$

(34)

where $I_1$ is an even and $I_2$ is an odd scalar (the Hodge operator is odd). Then Plebański postulated a non-linear electrodynamics with the constitutive law

$$H = U(I_1, I_2) * F + V(I_1, I_2) F,$$

(35)

where $U$ and $V$ are functions of the two invariants. Note that in the Born-Infeld case $U$ and $V$ depend on both invariants whereas in the Heisenberg-Euler case we have $U_{\text{HE}}(I_1)$ and $V_{\text{HE}}(I_2)$. Nevertheless, in both cases $U$ is required as well as $V$. And in both cases, see [32] and [33]. $U$ is an even function and $V$ and odd one such as to preserve parity invariance.

\footnote{Strictly, Plebański assumed a Lagrangian which yields \((17)\) together with the structural relations $F = u(I_1, I_2) * H + v(I_1, I_2) H$. The latter law, apart from singular cases, is equivalent to \((35)\).}
If one chose $V$ to be an even function, e.g., then parity violating terms would emerge. Such terms were most recently discussed by Majumdar, Mukhopadhyaya, and SenGupta [48,56]; for the experimental situation (there seem no signatures for parity violations) compare Lue et al. [47].

**Singularity-free electro-gravitodynamics** Recently, Ayón-Beato & García [2], for earlier work see Shikin [79], have proposed a constitutive law

$$H = U(I_1) \ast F,$$

which, as subcases, does neither encompass (32) nor (33) and thus makes it appear as rather academic. The explicit form of $U$ is defined by the requirement of obtaining completely singularity-free solutions of the coupled system of the gravitational field (Einstein) and the electromagnetic field (non-linear Maxwell).

Examples of suitable functions $U(I_1)$ are given in [79,2].

In terms of the local time and space coordinates $(t, r, \theta, \phi)$, the general spherically symmetric ansatz for the coframe can be written as

$$\vartheta^0 = f(r) \, dt, \quad \vartheta^1 = \frac{1}{f(r)} \, dr, \quad \vartheta^2 = r \, d\theta, \quad \vartheta^3 = r \sin \theta \, d\phi,$$

whereas, for the electromagnetic field, we have

$$F = \varphi(r) \, \vartheta^0 \wedge \vartheta^1.$$

The exact solution of the coupled system of gravitational and electromagnetic field equations, i.e., of Einstein’s equation (4) and Maxwell’s equations $dF = 0$, $dH = 0$, reads

$$\varphi = \frac{q}{U(I_1) r^2}, \quad f^2 = 1 - \frac{2m}{r} + \frac{Q(r)}{r^2},$$

where $q, m$ are integration constants and (‘Tolman’s integral’)

$$Q(r) = \kappa r \int_r^\infty d' r' \mathcal{K}(r') r'^2, \quad \mathcal{K} = 2I_1 U(I_1) - \int I_1 U(I_1).$$

In the last function one should substitute the explicit form of the quadratic invariant $I_1$ computed on the spherically symmetric configuration (37) with (38).

It is shown in [79,2,3,4] that the constitutive function $U(I_1)$ can be chosen in such a way that the functions in (39) describe a completely regular, i.e., singularity-free configuration.

**6.3 Linear: Abelian axion, inter alia**

A very important case is that of a linear constitutive law between the components of the two-forms $H$ and $F$. It postulates the existence of the $6 \times 6 = 36$ constitutive functions $\kappa_{ij}^{kl}(t, x) = -\kappa_{ij}^{lk} = -\kappa_{ij}^{kl}$ such that

$$H_{ij} = \frac{1}{2} \kappa_{ij}^{kl} F_{kl}.$$
This kind of an ansatz we know from the physics of anisotropic crystals. The factor $1/2$ is chosen in order to have a smooth transition to the conventional $D = \varepsilon_0 E$ etc. relations, cf. [72] p.127.

The choice of the local coordinates is clearly unimportant. In a different coordinate system the linear constitutive law preserves its form due to the tensorial transformation properties of $\kappa_{ijkl}$. Alternatively, instead of the local coordinates, one may choose an anholonomic frame and may then decompose the two-forms $H$ and $F$ with respect to it.

Since $H$ is an odd and $F$ an even form, the constitutive functions $\kappa_{ijkl}(t,x)$ are odd. Taking the Levi-Civita symbol, we can split off the odd piece according to

$$\kappa_{ijkl} = \frac{1}{2} \epsilon_{ijmn} \chi^{mnkl} \quad \text{or} \quad \chi_{ijkl} = \frac{1}{2} \epsilon_{ijmn} \kappa_{mnkl}. \quad (42)$$

Because of the corresponding properties of the Levi-Civita symbol, the $\chi_{ijkl}$ are even scalar densities of weight +1. For the Levi-Civita symbols with upper and lower indices, we have $\epsilon_{ijkl} \epsilon_{mnpq} = \delta_{ij}^{mp} \delta_{kl}^{nq}$.

With the linear constitutive law (as with more general laws), we can set up a Lagrangian 4-form; here we call it $V_{\text{lin}}$. Because of $H = -\partial V_{\text{lin}}/\partial F$, the Lagrangian must be quadratic in $F$. Thus we find

$$V_{\text{lin}} = -\frac{1}{2} H \wedge F = -\frac{1}{8} H_{ij} F_{pq} \, dx^i \wedge dx^j \wedge dx^p \wedge dx^q \quad \begin{array}{l} = -\frac{1}{32} (\epsilon^{pqij} \epsilon_{ijmn} \chi^{mnkl}) F_{kl} F_{pq} \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \end{array} \quad (43)$$

The components of the field strength $F$ enter in a symmetric way. Therefore, without loss of generality, we can impose the symmetry condition $\chi_{ijkl} = \chi_{klij}$ on the constitutive functions reducing them to 21 independent functions at this stage.

The $\kappa_{ijkl}$ carry the dimension $[\kappa] = [\chi] = e^2/\hbar$. Therefore, still before introducing the metric, we can split off the totally antisymmetric part of $\chi_{ijkl}$ and define the dimensionless constitutive functions according to

$$\chi_{ijkl} = f \overset{\circ}{\chi}_{ijkl} + \alpha \epsilon_{ijkl}, \quad \text{with} \quad \overset{\circ}{\chi}_{ijkl} = 0. \quad (44)$$

Here $[f] = [\alpha] = \hbar/e^2$, and $f = f(t,x)$ and $\alpha = \alpha(t,x)$ represent one scalar and one pseudo-scalar constitutive function, respectively. Thus the linearity ansatz eventually reads

$$H_{ij} = \frac{1}{4} \epsilon_{ijmn} \chi^{mnkl} F_{kl} = \frac{f}{4} \epsilon_{ijmn} \overset{\circ}{\chi}^{mnkl} F_{kl} + \alpha F_{ij}, \quad (45)$$

with

$$\overset{\circ}{\chi}^{mnkl} = -\overset{\circ}{\chi}^{nmkl} = -\overset{\circ}{\chi}^{mnlk} = \overset{\circ}{\chi}^{klmn} \quad \text{and} \quad \overset{\circ}{\chi}^{[mnkl]} = 0, \quad (46)$$

i.e., besides $\alpha$, we have 20 independent constitutive functions. Thus $\overset{\circ}{\chi}^{mnkl}$ has the same algebraic symmetries and the same number of independent components as a curvature tensor in a Riemannian spacetime.
Pseudo-scalars are also called axial scalars. So far, our axial scalar $\alpha(x)$ is some kind of permittivity/permeability field. If one adds a kinetic term of the $\alpha$-field to the electromagnetic Lagrangian \((43)\), then $\alpha(x)$ becomes propagating and one can call it legitimately an Abelian$^7$ axion. Ni$^{[62]}$ was the first to introduce such an axion field $\alpha$ in the context of the coupling of electromagnetism to gravity, see also deSabbata & Sivaram$^{[10]}$ and the references given there.

The Abelian axion has the following properties:

- Pseudoscalar field, i.e., spin $= 0$, parity $= -1$.
- Couples to Maxwell’s field in the Lagrangian according to $\alpha F \wedge F = 2\alpha E \wedge B \wedge dt$, see \((45)\) and \((43)\). Here $E$ is the 3-dimensional electric field 1-form and $B$ the corresponding magnetic field 2-form. This term in the total Lagrangian can be written as
  $$\alpha F \wedge F = -d\alpha \wedge A \wedge F, \quad (47)$$
  dropping, as usual, the total derivative. This contributes to the excitation
  $$H = -\partial L/\partial F$$ a term
  $$\sim d\alpha \wedge A. \quad (48)$$
- Since it arises on the same level as the metric, see Eq.\((64)\) below, it is a field of a similar universality as the gravitational field.

As yet, the Abelian axion has not been found experimentally, see the discussion of Cooper & Stedman$^{[12]}$ on corresponding ring laser experiments.

### 6.4 Isotropic

The linearity ansatz \((45)\) can be further constrained in order to arrive eventually at an isotropic constitutive tensor. We will proceed here somewhat unconventional in that we don’t assume a metric of spacetime beforehand but rather derive it in the following way:

**Duality operator, electric and magnetic reciprocity** The constitutive tensor $\hat{\chi}^{klnn}$ of \((45)\) defines a new duality operator which acts on 2-forms on $X$. In components, an arbitrary 2-form $\Theta = \frac{1}{2}\Theta_{ij} \, dx^i \wedge dx^j$ is mapped into the 2-form $\# \Theta$ by
$$\# \Theta_{ij} := \frac{1}{4} \epsilon_{ijkl} \hat{\chi}^{klmn} \Theta_{mn}, \quad (49)$$
see \[(43)\]. No metric is involved in this process. Now the linear material law \((45)\) can be written as
$$H = (f \# + \alpha) F. \quad (50)$$

We postulate that the duality operator, applied twice, should, up to a sign, lead back to the identity. Such a closure relation or the “electric and magnetic

---

$^7$ In contrast to the axions related to non-Abelian gauge theories, see \[(11)\], \[(12)\], \[(54)\] and the reviews in \[(13)\] and \[(80)\].
reciprocity\[85\] reduces the number of independent components of $\chi$ to 9 (without using a metric). One can demonstrate that this is a sufficient condition for the nonexistence of birefringence in vacuum, see [62, 40, 63, 28]. Then the fourth-order general Fresnel equation degenerates to the second-order light cone equation. Therefore, we impose

$$
{\#\#} = -1. 
$$

(51)
The minus sign yields Minkowskian signature\[8\], whereas the condition $\#\# = +1$ would lead to Euclidean or to the mixed signature $(+, +, -, -)$.

Seemingly Toupin \[85\] and Schönberg \[77\] were the first to deduce the conformally invariant part of a spacetime metric from duality operators and relations like (49) and (51). This was later rediscovered by Jadczyk \[44\], whereas Wang \[90\] gave a revised presentation of Toupin’s results. A forerunner was Peres \[69\], see in this context also the more recent papers by Piron and Moore \[70\]. Brans \[7\] and subsequently numerous authors discussed such structures in the framework of general relativity theory, see, e.g., \[137, 26, 68\] and the references given there.

It is convenient to adopt a more compact bivector notation by defining the indices $I, J, \ldots = (01, 02, 03, 23, 31, 12)$. Then $\chi^{ijkl}$ becomes the $6 \times 6$ matrix $\chi^{IK}$ and (51) goes over into

$$
\chi^{IJ} \epsilon_{JK} \chi^{KL} \epsilon_{LM} = -\delta^I_M. 
$$

(52)
In terms of $3 \times 3$-constituents an arbitrary symmetric $\chi^{IK} = \chi^{KI}$ constitutive matrix reads

$$
\chi^{IJ} = \chi^{JI} = \begin{pmatrix} A & C \end{pmatrix}, \quad \epsilon^{IJ} = \epsilon^{JI} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
$$

(53)
where $A = A^T, B = B^T$, and the superscript $^T$ denotes transposition. The algebraic condition $\epsilon_{IJ} \chi^{IJ} \equiv 0$ is provided by $\text{tr} C = 0$.

The general non-trivial solution of the closure relation (52) can be written in the form

$$
\chi^{IJ} = \begin{pmatrix} pB^{-1} + qN B^{-1}K \\ -KB^{-1} \\ B \end{pmatrix}. 
$$

(54)
Here $B$ is a nondegenerate arbitrary symmetric $3 \times 3$ matrix (6 independent components $B_{ab}$), $K$ an arbitrary antisymmetric matrix (3 independent components $K_{ab} =: \epsilon_{abc} k^c$), $N$ the symmetric matrix with components $N_{ab} := k^a k^b$, and $q := -1/\det B$, $p := [\text{tr}(NB)/\det B] - 1$. Thus, Eq. (54) subsumes 9 independent components.

---

\[8\] One could define a different duality operator by $\hat{\Theta}_{ij} = {\frac{1}{4}} \epsilon_{ijkl} \chi^{klmn} \Theta_{mn}$ such that $\hat{\Theta} \hat{\Theta} = -f^2$. 
How does the electromagnetic field couple to gravity?

The duality operator \( \# \) induces a decomposition of the 6-dimensional space of 2-forms into two 3-dimensional invariant subspaces corresponding to the eigenvalues \( \pm i \). Writing the 2-form basis \( \Theta^I = dx^i \wedge dx^j \) in terms of the two 3-dimensional column vectors

\[
\Theta^I = \begin{pmatrix} \beta^a \\ \gamma_b \end{pmatrix}, \quad a, b, \ldots = 1, 2, 3,
\]

(55)

one can construct the corresponding self-dual basis \( (s) \Theta^I := \frac{1}{2} (\Theta^I - i \# \Theta^I) \). In the 3-vector representation,

\[
(s) \Theta^I = \begin{pmatrix} (s) \beta^a \\ (s) \gamma_b \end{pmatrix},
\]

(56)

one of the 3-dimensional invariant subspaces can be spanned either by the upper or by the lower components which are related to each other by a linear transformation. For example, \( (s) \beta \) can be expressed in terms of \( (s) \gamma \) according to

\[
(s) \beta = (i + B^{-1} K) B^{-1} (s) \gamma.
\]

(57)

Therefore \( (s) \gamma \) or, equivalently, the triplet of 2-forms

\[
S^{(a)} := - (B^{-1})^{ab} (s) \gamma_b
\]

subsume the properties of this invariant subspace. Each of the 2-forms carry 3 independent components, i.e., they add up to 9 components.

The information of the constitutive matrix \( X^I_{\, J} \) is now encoded into the triplet of 2-forms \( S^{(a)} \). One can verify that the latter satisfies the completeness relation

\[
S^{(a)} \wedge S^{(b)} = \frac{1}{3} (B^{-1})^{ab} (B)_{cd} S^{(c)} \wedge S^{(d)}.
\]

(58)

**Extracting the metric** Within the context of \( SU(2) \) Yang-Mills theory, Urbantke [88] was able to derive a 4-dimensional spacetime metric \( g_{ij} \) from a triplet of 2-forms satisfying a completeness condition of the type (58). Explicitly, the Urbantke formulas read

\[
\sqrt{\det g} \ g_{ij} = - \frac{2}{3} \sqrt{\det B} \epsilon_{abc} \epsilon^{klmn} S_{ik}^{(a)} S_{lm}^{(b)} S_{nj}^{(c)},
\]

(59)

\[
\sqrt{\det g} = - \frac{1}{6} \epsilon^{klmn} B_{cd} S_{kl}^{(c)} S_{mn}^{(d)}.
\]

(60)

The \( S_{ij}^{(a)} \) are the components of the 2-form triplet \( S^{(a)} = S_{ij}^{(a)} dx^i \wedge dx^j / 2. \) If we substitute the forms (57) into (59) and (60), we can display the metric explicitly in terms of the constitutive coefficients:

\[
g_{ij} = \frac{1}{\sqrt{\det B}} \begin{pmatrix} \det B & -k_a \\ -k_b & -B_{ab} + (\det B)^{-1} k_a k_b \end{pmatrix}.
\]

(61)
Here \( k_a := B_{ab} k^b = B_{ab} \varepsilon^{bcd} K_{cd}/2 \). One can verify that the metric in (61) has Minkowskian signature. Since the triplet \( S^{(a)} \) is defined up to an arbitrary scalar factor, we obtain a conformal class of metrics.

Given a metric, we can now define eventually the notion of local isotropy. Let \( T^{i_1 \cdots i_p} \) be the contravariant coordinate components of a tensor field and \( T^{\alpha_1 \cdots \alpha_p} := e_{i_1}^{\alpha_1} \cdots e_{i_p}^{\alpha_p} T^{i_1 \cdots i_p} \) its frame components with respect to an orthonormal frame \( e_{\alpha} = e_{i_\alpha} \partial_i \). A tensor is said to be locally isotropic at a given point, if its frame components are invariant under a Lorentz rotation of the orthonormal frame. Similar considerations extend to tensor densities.

There are only two geometrical objects which are numerically invariant under local Lorentz transformations: the Minkowski metric \( \delta_{\alpha\beta} = \text{diag}(+1, -1, -1, -1) \) and the Levi-Civita tensor density \( \varepsilon^{\alpha\beta\gamma\delta} \).

One can prove that the constitutive tensor in (45) with the closure property (51) is locally isotropic with respect to the metric (61), see also [63]. Accordingly, for the constitutive tensor, we finally have

\[
\hat{\chi}_{ijkl} = 2 \sqrt{-g} g^{k[i} g^{j]l}.
\]

Thus, the isotropic law reads

\[
H_{ij} = f_{ijmn} \sqrt{-g} g^{km} g^{ln} F_{kl} + \alpha F_{ij}
\]

or, if written with the help of the Hodge star operator belonging to the metric (61),

\[
H = (f + \alpha) F.
\]

### 6.5 Centrosymmetric

If we want the constitutive tensor to be reflection symmetric at each point of spacetime, i.e., if we require centrosymmetry, then we have to kill the Abelian axion and arrive, provided \( F \) is chosen in accordance with the SI-conventions, at the usual law for Maxwell-Lorentz vacuum electrodynamics

\[
H = f^* F = \sqrt{\frac{\varepsilon_0}{\mu_0}} f^* F.
\]

---

9 Remember that in Ricci calculus the excitation is defined according to \( H_{ij}^{[Ric]} = e^{ijkl} H_{kl}/2 \). Then \( H_{ij}^{[Ric]} = -f \sqrt{-g} F^{ij} \), see (11).
7 Non-minimal coupling involving curvature, nonmetricity and torsion?

7.1 Non-minimal coupling violating charge and/or flux conservation

The Maxwell equation $dH = J$ reflects (and comes from) the electric current conservation, $dJ = 0$, see Sec.4. By modifying the left hand side of $dH = J$, one can arrive at a model violating charge conservation. Such a modification can typically originate from a non-minimal coupling of the electromagnetic to the gravitational field. Given the torsion 2-form $T^\alpha$, one can consider, for example, the field equation

$$dH + \alpha (e_\alpha | T^\alpha) \wedge H + \beta ^*(\vartheta_\alpha \wedge T^\alpha) \wedge H = J,$$

or, with the nonmetricity 1-form $Q_{\alpha\beta} := -Dg_{\alpha\beta}$ and the Weyl 1-form $Q := Q_\alpha^\alpha/4$,

$$dH + \gamma Q \wedge H + \delta ^*(\vartheta^\alpha \wedge \vartheta^\beta \wedge Q_{\alpha\beta}) \wedge H = J.$$  \hspace{1cm} (67)

Similar non-minimal terms could emerge in $dF = 0$. However, curvature dependent terms cannot be accommodated at the level of the Maxwell equations, since the contraction of the indices produces always a form of even rank whereas the Maxwell equations are represented by 3-forms, i.e., by forms of odd rank. In any case, violating charge or flux conservation is not possible without giving up most of the experimentally established structure of the theory of electromagnetism. Therefore we will not follow this path.

Incidentally, there are some papers in the literature in which the conventional vacuum constitutive law $H \sim \star F$ is upheld, but the Maxwell equations are coupled to torsion in an inconsistent way. A closer inspection of the papers [75,15,82] shows that the proposed “non-minimal” coupling of torsion to the electromagnetic field is void of physical contents. In fact, torsion drops out if the algebra is done correctly.

Another procedure comes to mind if we talk about the violation of the conservation laws. Hojman et al. [36] introduced a new scalar field $\varphi(x)$, the *tlaplon*, see also Mukku & Sayed [57]. The gradient $d\varphi$ of the tlaplon was put proportional to the trace part $(^2T^\alpha) := \vartheta^\alpha \wedge T$ of the torsion $T^\alpha$, see [31]; here $T := e_\beta | T^\beta$. In fact, we have $T = \frac{1}{2} d\varphi$.

Superficially, the axial scalar $\alpha(x)$ (Abelian axion) and the scalar tlaplon $\varphi(x)$ may look similar. However, the axion already emerges from spacetime viewed as a differential manifold as soon as a linear constitutive law is assumed for electromagnetism, whereas the tlaplon can only be introduced if the differential manifold is equipped with a linear connection. In other words, the axion is a pre-metric and a pre-connection animal, the tlaplon, in contrast, needs to be ‘housed’ in a linear connection.

Moreover, as we saw, the axion respects the conservation laws (and pleases us thereby), whereas the tlaplon defies these rules and appears as an anti-electromagnetic creature. The electromagnetic field is not defined in the conventional way, namely by $F = dA$. Instead, Hojman et al. define it via a “covariant”
Thus the “electromagnetic” Lagrangian becomes
\[ \hat{F} \wedge \star \hat{F} = F \wedge \star F + d\varphi \wedge A \wedge \star F + \frac{1}{4} T \wedge A \wedge \star (T \wedge A) . \] (70)
The last term does not contribute to the excitation \[ H = -\partial L / \partial F , \] but the second term produces a contribution
\[ \sim \star (d\varphi \wedge A) . \] (71)

We can compare now the two contributions from the axion (48) and the tlaplon (71). They are reminiscent of each other since one is equal to the Hodge dual of the other. Therefore, in the tlaplon case, besides a connection, we need additionally a metric. But the most decisive difference is, as can be read off from (70), that the Maxwell equations get amended and axiom 1 and axiom 3 are no longer valid.

7.2 “Admissible” non-minimal coupling

The message is then that a change of the Maxwell equations \[ dH = J , \ dF = 0 \] is to be avoided, unless one allows for a violation of electric charge or magnetic flux conservation. By introducing the metric \( g^{\alpha} \) into the constitutive law, one gets a smooth and natural transition from special relativity to general relativity and to gauge theories of gravity. In the constitutive law for vacuum one could imagine, along with the contributions depending only on the metric, couplings like
\[ \chi_{ijkl} = A_1 R^{ijkl} + A_2 R^{ijkl} + A_3 R^{ijkl} + A_4 (Ric^{ijkl} - Ric^{ijkl} + A_5 R g^{ijkl} R + A_6 R \epsilon^{ijkl} (72) \]
without violating the conservation laws. Here \( R := \epsilon^{\alpha \beta} R_{\alpha \beta} \) is the curvature scalar, and we denote the right or Lie dual of an \( so(1,3) \)-valued form \( \psi_{\alpha \beta} \) by \( \psi^{\star \alpha \beta} := \frac{1}{2} \epsilon_{\alpha \beta \mu \nu} \psi^{\mu \nu} \).

Choose, for example, \( \chi_{ijkl} = 2 \sqrt{-g} f_0 (1 + \beta^2 R) g^{ijkl} \), then the inhomogeneous Maxwell equation would read
\[ d \star F + \beta^2 d \star (R \star F) = \frac{1}{f_0} J , \] (73)
i.e., a coupling of curvature and electromagnetic field strength would be possible. However, one had to introduce a new natural constant with the dimension of \( [\beta] = 1/\text{length} \). On the level of the Lagrangian, this coupling would be non-minimal,
\[ V_{\text{non-m}} = -\frac{f_0}{2} (1 + \beta^2 R) \star F \wedge F , \] (74)
but such an ansatz would not spoil the fundamental principles of electrodynamics; it would seem to be the most natural way of achieving an RF-coupling. Goenner [23], see also the literature given there, derived this non-minimal Lagrangian from some fundamental principles, like the existence of a decent Newtonian limit. However, in his view, such a model violates charge conservation. We disagree with him on this point.

We want to stress that one cannot achieve a similar non-minimal coupling to the torsion $T^\alpha$ of spacetime. First of all, the Maxwell equations are independent of torsion. By means of the constitutive law one maps the 2-form $F$ to the 2-form $H$. The curvature $R_{\alpha \beta}$ is a 2-form of type $(1,1)$, i.e., it carries two $GL(4,R)$ indices, whereas the torsion 2-form carries only one index. Therefore, by contraction, we cannot get a scalar out of the torsion. A coupling like $(e_\alpha | T^\alpha) \wedge \ast F$ is not possible, since this is a 3-form. However, higher powers in $T^\alpha$ would be possible such as

$$H = f_0 \left[ 1 + \gamma^2 \ast (T^\alpha \wedge T_\alpha) \right] \wedge \ast F$$

or

$$H = f_0 \left[ 1 + \delta^4 \ast (e_\alpha | T^\alpha) \wedge (e_\beta | T^\beta) \wedge (e_\gamma | T^\gamma) \wedge (e_\delta | T^\delta) \right] \wedge \ast F.$$  

Here $[\gamma] = [\delta] = 1/\text{length}$. Accordingly, it is not too difficult to introduce a coupling of torsion to electromagnetism. However, the price one has to pay is the introduction of new natural constants $\gamma$ and $\delta$. In other words, even if possible, we don’t take such models too seriously.

Also non-minimally coupled nonmetricity could be installed by additional quadratic pieces such as

$$H = f_0 \left[ 1 + \xi^2 (e_\alpha | Q^{\alpha \gamma})(e_\beta | Q^{\beta \gamma}) \right] \ast F.$$  

Therefore, there are quite a number of different “admissible” options available as soon as we allow non-minimal couplings to arise.

8 Outlook

Using astronomical observations on the propagation of light, the upper bounds for non-minimal coupling effects should be determined in a systematic way, as is done, for example, by Haugan and Lämmerzahl [28]. For such a purpose, we will develop the geometrical optics limit of the Maxwell equations $dH = J$, $dF = 0$ and will use particular constitutive laws, as, e.g., the linear law. Possible couplings to curvature, torsion, and nonmetricity should come under sharper focus in this way. Non-linear effect à la Ayón-Beato & García should be investigated in the context of, say, the metric-affine gauge theory of gravity.

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