Exponential improvement for quantum cooling through finite memory effects

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Practical implementations of quantum technologies require one to prepare physical states with a high degree of purity—or, in thermodynamic terms, very low temperatures. The ability to do so is restricted by the Third Law of thermodynamics, which prohibits the attainability of perfect cooling given finite resources. For a finite dimensional system and environment from which to draw energy, attainable upper bounds for the asymptotic ground state population of the system repeatedly interacting with quantum machines have recently been derived. These bounds apply within a memoryless (Markovian) setting, in which each step of the process proceeds independently of those previous. Here, we expand this framework to study the effects of memory on the task of quantum cooling and derive bounds that provide an exponential advantage over the memoryless case and can be achieved asymptotically. We do this by introducing a microscopic memory mechanism through a generalized collision model, which can be embedded as a Markovian dynamics on a larger system space. For qubits, our asymptotic bound coincides with that achievable through heat-bath algorithmic cooling, of which our framework provides a generalization to arbitrary dimensions. We lastly describe the step-wise optimal protocol, which requires implementing an adaptive strategy, that outperforms all standard (non-adaptive) procedures.

Introduction.—Cooling a physical system is a thermodynamic task of both fundamental and practical importance [1–6]. From the foundational perspective, the ability to cool a system is limited by the Third Law of thermodynamics, which posits the necessity of an infinite resource to be able to do so perfectly [7]. This resource is subject to trade-off relations: absolute zero is attainable in finite time given an infinitely large environment from which to draw energy; alternatively, given a finite energy source, one can only achieve perfect cooling asymptotically in the infinite time limit. From a practical perspective, one usually cannot utilize such an infinite resource, and so the concern naturally turns to the question: how cold can a system be prepared given certain resource constraints?

Formulating a theory that is meaningful with respect to the resource constraints is typically scenario-dependent; nonetheless, the aim is to develop such theories that are abstract enough to apply to a wide range of settings. For example, the resource theories of quantum thermodynamics typically allow one to unitarily interact the system with a thermal environment to which it is initially uncorrelated [8, 9]. Analyzing the transformations possible for various environments (e.g., finite dimensional) and dynamical structures (e.g., repeated application of energy-conserving unitaries) provides insight into the limitations of thermodynamic tasks.

Recent work has examined the details of such a setting applied to the task of cooling a quantum system [10, 11]; the main result posits a universal bound for the ground state population of the system in the limit of infinite cycles that is valid for arbitrary unitary dynamics and environment size. However, these results are derived under the assumption of memorylessness (Markovianity) of the protocol, which is often not well-justified in experimental platforms where system-environment correlations can lead to memory effects that affect performance goals; for instance, Landauer’s principle [12] has been demonstrated to break down in the non-Markovian regime [13, 14].

A natural extension to this work is to examine the role of memory effects in quantum cooling. Depending on the task at hand and the level of control enjoyed by the experimenter, memory effects can have a detrimental or advantageous impact on the figure of merit [15–22]. Many attempts to generalize thermodynamics to the non-Markovian paradigm have recently pursued, including trajectory-based unravelings of the dynamics [23, 24] and those based on the operational process tensor formalism [25–30], amongst others [31, 32]. However, such general approaches often come at the cost of obscuring insight regarding the important resources for particular tasks; for instance, it is often unclear whether reported “quantum advantages” are due to genuinely quantum effects (e.g., coherence) or those stemming from memory.

Here, we propose a microscopic mechanism for memory transfer through a physically-motivated generalized collision model [33–35], which—while not fully general—permits meaningful comparison between various memory structures. We first show that in the asymptotic limit of repeated cycles, the memory depth of the protocol plays an important role and can lead to exponential improvement over the Markovian case; in fact, perhaps surprisingly, the role of memory depth is more significant than that of the ability of the agent to implement multi-partite interactions between the system and machines at each step. Our result coincides with the asymptotic limit of heat-bath algorithmic cooling protocols [36–43] and provides a wide-ranging generalization of this setting that permits adaptive strategies and is applicable for all system and finite dimensional environment structures. We further show that adaptive strategies can out-perform non-adaptive ones and provide a finite time cooling advantage. This connection illustrates the practical advantages achievable with increasing levels of control.

Task: Cooling a quantum system.—A physical system is
never totally isolated from its environment, stipulating that one must work within the theory of open systems, in which the system of interest and its environment evolve together as a closed system, but the environmental degrees of freedom are finally disregarded. Allowing for arbitrary environments in open system dynamics permits perfect cooling with finite resources, as any physical transformation on quantum systems can be realized via a unitary interaction with a sufficiently large environment (including discarding the input system and preparing a pure ground state); thus, further restrictions on the environment are necessary.

We consider a system, $S$, whose state is described by a (positive semidefinite, unit trace) density operator $\varrho_S$ with an associated Hamiltonian $H_S$. The open system dynamics proceeds via interactions with its environment, $E$, which is a quantum system with an associated state $\varrho_E$ and Hamiltonian $H_E$. Initially, the system and environment begin uncorrelated and in equilibrium at inverse temperature $\beta := \frac{1}{k_B T}$ (we set $k_B = 1$). The joint system-environment evolves unitarily, with the system dynamics between the initial time and a later one $t$ completely described by the induced dynamical map, $\varrho_S^{(t)}(\beta) := \Lambda^{(t)}[\tau^{(t)}_S(\beta)]$, defined such that:

$$\varrho_S^{(t)}(\beta) = \text{tr}_E \left[ U^{(t)} \tau^{(t)}_S (\beta) \otimes \tau^{(t)}_E (\beta) U^{(t)*} \right],$$

where $\tau_X(\beta)$ denotes a thermal (Gibbs) state of system $X$ at inverse temperature $\beta$, i.e., $\tau_X(\beta) := \mathcal{Z}_X^{-1}(\beta) \exp(-\beta H_X)$ with partition function $\mathcal{Z}_X(\beta) := \text{tr} [\exp(-\beta H_X)]$.

The aim is to prepare $\varrho_S^{(t)}(\beta)$ in as cold a state as possible for a given initial temperature, i.e., we seek the optimal family $\{U^{(t)}\}$ of sequential transformations to the initial system that leads to the coolest output system. Cooling a system, however, can have several meanings: for one that remains in a thermal state, it could mean driving it to one of a lower temperature; otherwise, one could consider increasing its ground state population or its purity, or decreasing its entropy or energy. In general, such notions are nonequivalent and so any study concerning cooling is dependent on the choice of objective function [10]. We focus on achieving target states majorizing all other potential target states. This includes maximizing the ground state population, but is more general and directly translates to all Schur-convex functions of the eigenvalues as the optimal strategy (when arbitrary unitaries are allowed).

**Framework: Collision models with memory.**—We are interested in the effects of memory on cooling a quantum system. One of the main difficulties in examining non-Markovian dynamics is that memory effects can arise in various ways: they can be the manifestation of initial system-environment correlations, recurring system-environment or intra-environment interactions; or, in general, a culmination of all three. In any case, the dynamical map in Eq. (1) fails to completely describe the evolution of the system, in contrast to the Markovian setting. In general, one must track all degrees of freedom of the system and environment to predict the state of the system at any time, which becomes practically unfeasible. Thus, we seek a physically-motivated framework that permits a tractable amount of memory and allows meaningful comparison to the memoryless case and between different levels of memory. To this end, we propose a microscopic model for the environment structure and its interactions with the system. We consider the system to be of dimension $d_S$ with $H_S = \sum_{i=0}^{d_S-1} E_i |i\rangle \langle i|_S$ and assume the environment comprises a number of identical individual units—which we call machines—each of which is a quantum system of dimension $d_M$ with associated state $\varrho_E$ and Hamiltonian $H_M = \sum_{i=0}^{d_M-1} E_i |i\rangle \langle i|_M$. Without loss of generality, we order all Hamiltonians with respect to non-decreasing energies, and set $E_0 = E_0 = 0$ and $E_{\text{max}} = E_{d_M-1}$. Assuming that the dynamics proceeds via successive instantaneous unitary “collisions” between the system and subsets of machines yields a collision model with memory.

The memory effects that arise from endowing such collision models with various dynamical structures have recently been examined: approaches include considering initially correlated machines [44, 45], permitting inter-machine interactions [33, 46–49], allowing for repeated system-machine collisions [50, 51], or hybrid variations [34, 35, 52] (see Ref. [53] for an overview). In certain cases, the model has been shown to exhibit finite length memory behavior [54–56]. In the limit where the number of machines is large, the system is expected to interact with only mutually exclusive subsets of machines that begin uncorrelated with each other; since any machines that are interacted with once never play a role in later dynamics, one yields a microscopic picture of Markovian evolution, for which the (memoryless) collision model formalism was originally developed, and which reduces to a Lindbladian master equation in the continuous time limit [57–60].

Although not fully general, this setting allows for tractable non-Markovian dynamics. Moreover, with respect to the task at hand, the assumptions seem reasonable: they stipulate that the experimenter can control the system and part of the environment (i.e., a subset of machines) at each step of the protocol, with the ability to retain some of the machines in time. In this Letter, we will analyze the memory effects that arise from repeated system-machine interactions, as depicted in Fig. 1. Precisely, we consider $k$ machines to interact with the system between timesteps, with some $\ell \leq k$ of these carrying memory forward by interacting with the system again later; this reduces to a Markovian protocol involving $k$ machines for $\ell = 0$. Choices we can vary are the number of machines involved in each interaction, $k$, [61], the number of memory
carriers, $\ell$, the initial temperature, $\beta$, and the Hamiltonians.

This framework allows for various cooling protocols. For instance, one can compare adaptive strategies, where the experimenter can perform different unitaries at each step, versus non-adaptive ones, where a fixed dynamics is repeatedly applied. Additionally, one can impose restrictions on the types of unitaries possible, such as limiting the set from the fully general “coherent” ones (that require an external energy source) to “incoherent” transformations that are energy conserving [11, 62]. Lastly, one could, in principle, allow for the memory structure itself to be adaptive, where e.g., $k$, $\ell$ vary between timesteps; although this would fit within our framework, we do not consider such cases and focus instead on the optimal cooling limits for fixed structures of dynamics. A choice of $k$ and $\ell$, along with the dimension of the system $d_S$ and machines $d_M$, determines the control complexity afforded to the experimenter: intuitively, $k$ is related to the spatial complexity and $\ell$ to the temporal. We now compare the achievable ground state population of the system for a given initial inverse temperature $\beta$ for different memory structures. Our results highlight an interesting interplay between the advantages to be gained from spatial and temporal control.

Memory-enhanced Cooling.—The fundamental cooling bounds in the Markovian setting have recently been derived in Refs. [10, 11]. The ground state population of the system at any finite time generally depends upon the energy level structure between the system and machines and the level of control afforded to the experimenter (coherent vs. incoherent, adaptive vs. non-adaptive). However, in the asymptotic limit of Markovian operation (i.e., for any $k$ and $\ell = 0$), the vector of eigenvalues of the asymptotic state (in any of the aforementioned control paradigms) is majorized by that of the following state

$$
g'_S(\ell_{\text{max}}, \beta, k) = \sum_{n=0}^{d_S-1} e^{-\beta n k \ell_{\text{max}}} |n\rangle \langle n|_S, \tag{2}
$$

whenever the initial state $\tau^0_S(\beta)$ is majorized by $g'_S(\ell_{\text{max}}, \beta, k)$; here $Z_X(\beta, \varepsilon) := \sum_{n=0}^{d_X-1} e^{-\beta n \varepsilon}$ is a quasi-partition function (depending only on the maximum energy gap rather than the full spectrum) and $\ell_{\text{max}}$ is the largest energy eigenvalue of each of the (identical) machines. The asymptotic state in Eq. (2) is furthermore attainable in the coherent control setting, where arbitrary unitaries are permitted (as we mainly focus on throughout this work); its ground state population thus posits the ultimate cooling limit in the Markovian setting.

The intuition behind this result is that the optimal protocol performs an interaction that reordered the eigenvalues of the system and relevant machines in such a way that the maximum population is placed into the ground state subspace of the system. When such a cycle is repeated with fresh machines at each timestep, the net result is that the asymptotic state looks as if it has interacted with only the qubit subspace with maximum energy difference of each machine, and hence depends only upon $\ell_{\text{max}}$. However, the result cannot immediately be extended to the non-Markovian regime, as its derivation relies on an inductive argument based on the system state after each step; for non-Markovian dynamics, this is typically correlated with its environment and can therefore not be expressed as a dynamical map acting upon its previous state, posing a roadblock for the above logic.

Whenever $\ell > 0$ the generalized collision model that we propose is inherently non-Markovian. Nonetheless, an important result relevant to our work is the fact that such non-Markovian collision models can be embedded as Markovian dynamics on a larger (finite) dimensional state space [49]. For a system interacting at each step with $k$ machines, of which $\ell$ feed forward to take part in the next interaction, the dynamics can be embedded into a Markovian one by considering the system and the $\ell$ memory carriers as a unified target system, which interacts at each step with $k - \ell$ fresh machines; such a dynamics is therefore said to have memory depth $\ell$. In Appendix A we provide details of the Markovian embedding, which allows us to derive the following results.

Asymptotic cooling advantage.—Using this embedding we can now present the universal cooling bound in the non-Markovian regime of the generalized collision model in the limit of infinite cycles.

**Theorem 1.** For any $d_S$-dimensional system interacting at each step with $k$ identical $d_M$-dimensional machines, with $\ell$ of the machines (labeled $L$) used at each step carrying the memory forward, in the limit of infinitely many cycles:

i) The ground state population of $S$ is upper bounded by

$$
p^* (\ell_{\text{max}}, \beta, k, \ell) = \left( \sum_{n=0}^{d_S-1} e^{-\beta n d_M (k-\ell) \ell_{\text{max}}} \right)^{-1}. \tag{3}
$$

ii) The vector of eigenvalues of the output state system is majorized by that of the following attainable state

$$
g^*_S(\ell_{\text{max}}, \beta, k, \ell) = \sum_{n=0}^{d_S-1} e^{-\beta n d_M (k-\ell) \ell_{\text{max}}} |n\rangle \langle n|_S. \tag{4}
$$

whenever the initial state $\tau_S(\beta) \otimes \tau_M(\beta) \otimes \ell$ is majorized by

$$
g^*_S(\ell_{\text{max}}, \beta, k, \ell) = \sum_{n=0}^{d_S-1} e^{-\beta n d_M (k-\ell) \ell_{\text{max}}} |n\rangle \langle n|_{S L}. \tag{5}
$$

The proof is provided in Appendix B.

Regarding the effects of memory, the optimal ground state population is enhanced by a factor of $d_M^\ell$ compared to the Markovian case, highlighting that the dimension and number of memory carriers play a dramatic role in the asymptotic cooling limit; in particular, one achieves an exponential improvement in $\ell$. This factor arises from the additional space that the memory carriers offer to compress populations into. The bound is achievable and the protocol to do so in the case of adaptive, coherent control is to perform the operation that reordered the eigenvalues of the system and machines in such a way that the largest are in the ground state of the system, and subject this constraint, the next largest are placed into ground state subspaces of the memory carriers. Intuitively, at each step the system and the memory carriers are optimally cooled, thereby dissipating as much heat as possible into the machines that are never interacted with again.
Finally, the analytic form of Eq. (4) allows us to compare cooling limits for various of $k, \ell, \beta$ and $E_{\text{max}}$. We have the following corollary:

**Corollary 1.** The asymptotic hierarchy is determined via:

$$d_k^\beta(E_{\text{max}}, \beta, k, \ell) < \langle \rangle \langle \rangle d_{k'}^\beta(E_{\text{max}}, \beta', k', \ell')$$

if

$$(\beta(k - \ell))d_M^\ell E_{\text{max}} \leq (\beta'(k' - \ell'))d_M^\ell E_{\text{max}}.$$

(6)

In particular, for a fixed $\beta$, $E_{\text{max}}$, it is the memory depth, $\ell$, that plays a more prominent role in the asymptotic limit than the number of parties interacting, $k$, as exhibited in Fig. 2.

**Relation to heat-bath algorithmic cooling.**—Above we have derived the optimally cool system state in the asymptotic limit when an experimenter can control a subset of machines that act as memory carriers. For qubit targets, this result aligns with the achievable limit in the heat-bath algorithmic cooling (HBAC) protocol [36–43]. Here, the experimenter cools a quantum system (called the target system) by cooling a larger ensemble of machines (referred to as compression systems) via interactions with a number of “reset” systems that rethermalize between steps. This permits better cooling than simply cooling the system alone; indeed, such a protocol is non-Markovian and is a special case of the collision model dynamics considered here, as we detail in Appendix C.

In particular, our approach encompasses various HBAC-like scenarios and offers scope for generalization in the following ways: i) each of the target/memory carrier/reset systems that we consider can be themselves composed of multiple systems of arbitrary dimension, with arbitrary energy spectra and initial temperatures, which allows us to clarify the determining features of the asymptotic hierarchy via comparison between various subsystem structures; in contrast, many studies of HBAC focus only on target and reset qubits [39, 40]; ii) it provides an opportunity to obtain the the asymptotic state, instead of simply the ground state population, thereby our results apply to more general notions of cooling [37]; and iii) it permits variation of the controlled interactions between system and machines at each timestep, which is often not considered in the standard HBAC framework. Indeed, most HBAC protocols (such as the partner pairing algorithm [37]) consider the experimenter to perform a non-adaptive strategy, in which the same unitary is performed at each step. One such protocol that maximizes asymptotic ground state population of the system has been established in Ref. [42]; Theorem 2 shows that any adaptive strategy, the asymptotic limit coincides with that achievable in the standard (non-adaptive) HBAC scenario. However, although the asymptotic limits behave identically, we now show that adaptive strategies can lead to finite time advantages.

**Finite time cooling advantage.**—To reiterate, although non-adaptive protocols that achieve the optimal asymptotic limit have been presented [42], such protocols do not necessarily maximize the ground state population of the system at each step. Intuitively, the asymptotic advantage over the Markovian case arises from cooling the memory carriers, which can then act to cool the target more efficiently. However, a non-adaptive strategy achieving this uses the additional degrees of freedom of the memory carriers, but does not make optimal use of the memory per se, which also manifests itself as $SL$ correlations. Although no advantage is offered in the asymptotic limit, controlling such correlations is pivotal to the step-wise optimal protocol.

In fact, the strategy outlined below Theorem 1 provides this protocol; in Appendix D we prove its optimality. The logic is that by performing the unitary that optimally cools the system and memory carriers at each step, one is able to prepare the coolest $SL$ state given the entire history of the protocol at any time, and from this the optimal S state at each time can be attained. In general, such a step-wise optimal unitary depends upon the microscopic relation between the eigenspectrum of the current $SL$ state and the fresh machines, and as such differs at each step. In Appendix D, we further examine the finite time behavior of this step-wise optimal protocol for various values of $k$ and $\ell$, showing that for a given number of total machines, it can be advantageous to make use of a greater number of memory carriers as opposed to more multi-partite interactions.

**Conclusions.**—In this Letter, we have put forward a framework for consistently dealing with memory effects when cooling quantum systems. In doing so, we have revealed the potential for exponential improvements in reachable ground state populations and more general notions of cooling. This yields drastic improvements already for modest memory depths. Furthermore, through a Markovian embedding of our framework, we could show an operational equivalence to heat-bath algorithmic cooling for qubit targets. As our results apply to any number of systems of arbitrary dimensions, from this perspective, they can also be read as a generalization of HBAC applicable to arbitrary target and compression systems and arbitrary bath spectra. By permitting adaptive strategies, we have outlined the step-wise optimal protocol that outperforms standard (non-adaptive) HBAC procedures at any finite time. This advantage comes at the cost of needing to pre-optimize the sequence of operations in order to modify them at each step optimally. Together with those of Ref. [10], our
results encompass a unification of HBAC with the resource theory of thermodynamics, as all of our results can either be reached by coherently controlled interactions or equivalently with energy-conserving unitaries on enlarged systems.

The results presented here consolidate the ultimate limits for quantum refrigeration in a setting with perfect control over a few constituents and high-quality isolation of the target. However, in most experimental settings, further challenges arise: we have—just as in HBAC—assumed that the uncontrolled interactions between the target and its environment are negligible compared to the interaction strength of the controlled refrigeration step. For finite times, this makes our improved results reliable, and due to the exponential scaling, it is sufficient to run the protocol for a finite time to achieve cooling close to the asymptotic limit. From a practical perspective, it would be interesting to study the impact of uncontrolled environment interactions: either to model less perfect isolation of the target or to better understand the realistic asymptotic limit, as for infinite steps, the natural rethermalization of the target cannot always safely be neglected. Another assumption worth analyzing in this context is that of perfect control: to implement a unitary perfectly, one requires both precise clocks [63], which comes with its own thermodynamic limitations and costs [64, 65], and a high degree of control over the interaction terms. While this paradigm is plausible for quantum computing devices, other systems are more challenging to control, in particular those with multi-partite interaction terms that are only perturbatively accessible [66]. Our proposed framework naturally lends itself to such a more realistic study of cooling, deriving similar bounds in the setting of imprecise control or further restrictions of interactions shows the potential to elevate our results beyond fundamental limits and towards practical guidelines for a larger class of quantum experiments.

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SUPPLEMENTAL MATERIAL

Appendix A: Markovian embedding of collision models with memory

We are interested in exploring analytically the effects of memory regarding the task of cooling a quantum system. We do not wish to allow for arbitrary non-Markovianity, as this would lead to an infinite resource in a sense that allows us to cool the system to the ground state perfectly. Rather we would like to obtain a cooling bound in the limit of infinite cycles for a generalized collision model endowed with memory. Such collision models with memory are quite general, simply assuming that between each step of the dynamics, the system interacts unitarily (blue outline) with a subset of these; the cardinality of this set is labelled by $k$ throughout this work. A further subset of these previously-interacted-with machines of cardinality $\ell < k$ take part in the following interaction, becoming memory carriers (red). At each timestep, $k - \ell$ fresh machines are incorporated into the dynamics (yellow). Here we have shown $k = 2, \ell = 1$.

FIG. 3. Generalized collision model with memory. In the collision model picture, the environment that the system (green) interacts with is assumed to be comprised of individual sub-units, which we call machines. Between each step of dynamics, the system interacts unitarily (blue outline) with a subset of these; the cardinality of this set is labelled by $k$ throughout this work. A further subset of these previously-interacted-with machines of cardinality $\ell < k$ take part in the following interaction, becoming memory carriers (red). At each timestep, $k - \ell$ fresh machines are incorporated into the dynamics (yellow). Here we have shown $k = 2, \ell = 1$. 
We consider a target system of dimension $d_S$ and local Hamiltonian $H_S = \sum_{i=0}^{d_S-1} E_i |i\rangle\langle i|_S$, where $\{E_i\}$ are sorted in non-decreasing order and a machine comprising of a number of constituent identical subsystems of finite size $d_M$, each of which has the local Hamiltonian $H_M = \sum_{i=0}^{d_M-1} E_i |i\rangle\langle i|_M$, where $\{E_i\}$ are also sorted in non-decreasing order. The system and machines begin uncorrelated in a thermal state with the same inverse temperature $\beta$:

$$g_S^{(\beta)}(\beta) = \tau_S(\beta) \quad \text{and} \quad g_M^{(\beta)}(\beta) = \bigotimes_{j=1}^{m} \tau_{M_j}(\beta) \quad (A1)$$

where $\tau_X(\beta) := Z_X^{-1} e^{-\beta H_X}$ with the partition function $Z_X = \text{tr}[e^{-\beta H_X}]$.

Fixing $k$ and $\ell$ provides a particular dynamical structure of the non-Markovian process: it stipulates that at each timestep there are $k$ machines interacting with the system at each step and let $\ell$ of them through to perpetuate the memory. For example, after $n$ steps, the system state is

$$g_S^{(n)}(\beta, k, \ell) = \text{tr}_M[U_{k,\ell}^{(n)} \cdots U_{k,\ell}^{(1)} (\tau_S(\beta) \bigotimes \tau_{M_j}(\beta)) U_{k,\ell}^{(1)\dagger} \cdots U_{k,\ell}^{(n)\dagger}], \quad (A2)$$

where $U_{k,\ell}^{(n)}$ is an arbitrary unitary transformation between the target system and the $k$ machines labeled by $\{(n-1)(k-\ell)+1, \ldots, n(k-\ell)+\ell\}$ (an identity map is implied on the other machines) and $m := k + (n-1)(k-\ell)$ is the total number of machines used by the protocol up until timestep $n$, which will be important in making finite time comparisons (as we do in Appendix D; see Fig. 1 in the main text for a graphical depiction in terms of a circuit diagram).

Importantly, the state of the system at any time is a function of the full microscopic energy structure $\{E_i\}$ and $\{E_i\}$ (which we do not explicitly label for ease of notation), $\beta$, $k$ and $\ell$; the latter two numbers specify a particular dynamical structure in terms of which systems the unitaries act upon between steps. If $\ell = 0$, the dynamics of the system is Markovian, since at each step, the system interacts with a fresh machine that can contain no memory of the past dynamics of the system. Otherwise, each of the machines interacts more than once with the target and at each step, and only $k - \ell$ fresh machines are added into the interaction at each step.

Eq. (A2) highlights the restriction imposed by the assumption of generalized collision model dynamics from the fully general case of Eq. (1). However, it is important to note that on the level of the system dynamics, memory effects still play a role. We first show that for $\ell > 0$ the dynamics considered is indeed non-Markovian in general on the level of the system.

To analyze the proposed setting, we need to look at the evolution of the entire joint system and machines to consider the effect of the memory in the protocol. For instance, consider rewriting Eq. (A2) as a dynamical map taking the initial system state $g_S^{(0)}(\beta)$ to the later one under a generic dynamical structure determined by the choice of $k$ and $\ell$, i.e., define $\Lambda_{k,\ell}^{(n,0)}(\beta)[X_S] := \text{tr}_M[U_{k,\ell}^{(n)} \cdots U_{k,\ell}^{(1)} (X_S \bigotimes_{j=1}^{m} \tau_{M_j}(\beta)) U_{k,\ell}^{(1)\dagger} \cdots U_{k,\ell}^{(n)\dagger}]$ such that

$$g_S^{(n)}(\beta, k, \ell) = \Lambda_{k,\ell}^{(n,0)}(\beta)[g_S^{(0)}(\beta)]. \quad (A3)$$

Linearity, complete positivity and trace preservation of $\Lambda_{k,\ell}^{(n,0)}(\beta)$ is guaranteed for any $\beta$, $k$, $\ell$ and most importantly $n$ by the fact that $S$ and $M$ begin initially uncorrelated and the dynamics evolves unitarily on the global level, before a final partial trace is taken over the machine degrees of freedom. Complete positivity is particularly important to ensure that the map takes valid quantum states to valid quantum states. In general, the global state $g_{SM}^{(n)}(\beta, k, \ell)$ at some time involves correlations between $S$ and $M$; taking the final partial trace over $M$ destroys all such correlations. Thus, it is clear to see that one cannot, in general, describe the evolution of the system between multiple times as a divisible concatenation of completely positive and trace preserving (CPTP) maps, i.e.,

$$g_S^{(n)}(\beta, k, \ell) = \Lambda_{k,\ell}^{(n,0)}(\beta)[g_S^{(0)}(\beta)] \neq \Lambda_{k,\ell}^{(n,\ell)}(\beta) \circ \Lambda_{k,\ell}^{(\ell,0)}(\beta)[g_S^{(0)}(\beta)]. \quad (A4)$$

Here, we have defined $\Lambda_{k,\ell}^{(n,\ell)}(\beta)$ as the map that would be tomographically constructed if one were to discard the system at time $\ell$ (which is generally correlated to $M$) and perform a quantum channel tomography by preparing a fresh basis of input states (see Fig. 4); since these are uncorrelated to $M$ by construction, the map $\Lambda_{k,\ell}^{(n,\ell)}(\beta)$ is guaranteed to be CPTP for any choice of parameters [67]. Testing for equality in Eq. (A4) then corresponds to the operational notion of CP-divisibility proposed in Ref. [68]; importantly, its breakdown acts as a valid witness for non-Markovianity that is stricter than other notions of CP-divisibility proposed throughout the literature (in particular, it is stronger than that based on invertible CP-divisibility in any case where the dynamical maps are invertible). Of course, the fact that Eq. (A4) is generally an inequality for generic dynamics does not imply that it is so for the particular optimal cooling dynamics described throughout this article; however, it is simple to show that the optimal cooling protocol indeed generates correlations between the system and machine that lead to a breakdown of (operational) CP-divisibility, and hence the particular dynamics considered are non-Markovian.
FIG. 4. **Operational CP-divisibility.** A breakdown of operational CP-divisibility is a witness for non-Markovianity. The test consists of tomographically constructing a set of maps describing the dynamics and checking the validity of Eq. (A4). We illustrate the scenario for a subset of times \( t = 2, n = 4, \) with \( k = 2, \ell = 1 \): the left panel depicts the map \( \Lambda_{2,1}^{(2:0)} \), which comprises everything within the purple border, i.e., the initial states of all machines, all joint unitary interactions, and the final partial trace. Note that the final memory carrier \( M_2 \) should continue forward, but the map \( \Lambda_{2,1}^{(2:0)} \) does not capture this and traces out that subsystem. The middle panel depicts the map \( \Lambda_{2,1}^{(2:0)} \). Both maps can be tomographically reconstructed by preparing a basis of input states at the initial time and measuring the outputs at time \( n = 4 \) and \( t = 2 \), respectively. As the system begins initially uncorrelated with the machines, the unitary dilation guarantees that the maps constructed are CPTP. The final map needed, \( \Lambda_{2,1}^{(2:2)} \), is shown in the right panel. In general, at time \( t = 2 \), the system is correlated to the machines, thus breaking the assumption of no initial correlations. An operational circumvention is to discard the system state at \( t = 2 \) and reprepare a fresh one, thereby erasing all system-machine correlations. This has the effect of rendering the memory carriers into a fixed quantum state, which can be included in the description of \( \Lambda_{2,1}^{(2:2)} \) to ensure that it is CP. When memory is present, tracing out the system at the intermediary timestep generally conditions the state of the memory carriers into a state that generally differs from \( \tau(\beta) \), labeled here \( \sigma(\beta) \) with the altered part of the evolution depicted in green; thus, the full dynamics generically differs from the concatenation.

Nonetheless, the collision model memory structure that we have introduced crucially allows for a Markovian embedding that permits a significant simplification in the analysis [49]. In general, one would need to track the total joint evolution throughout the entire protocol, which quickly becomes computationally exhaustive as \( k \) grows. However, the choices of \( k \) and \( \ell \) group the system \( S \) and \( \ell \) of the machines into a larger joint target system, which we label \( S_L \), which interacts with \( k - \ell \) fresh machines at each timestep; we label all of the fresh machines with \( R \) as they model rethermalization of some of the machines with the environment. On the level of \( S_L \), the dynamics is Markovian, as the degrees of freedom carrying the memory have been included in the description of the target system. One can obtain the state of the overall target via tracing out the machines at the final step. We therefore have

\[
\rho_{SL}^{(n)}(\beta, k, \ell) = \text{tr}_R[\tilde{U}_{k,\ell}^{(n)} \cdots \tilde{U}_{k,\ell}^{(1)} (\rho_{SL}^{(0)}(\beta, k, \ell) \otimes \rho_{R}^{(0)}(\beta, k, \ell)) \tilde{U}_{k,\ell}^{(1)\dagger} \cdots \tilde{U}_{k,\ell}^{(n)\dagger}],
\]

where \( \rho_{SL}^{(0)}(\beta, k, \ell) := \tau_{S}(\beta) \otimes \prod_{j=1}^{\ell} \tau_{M_j}(\beta) \), \( \rho_{R}^{(0)}(\beta, k, \ell) := \bigotimes_{k=\ell+1}^{m} \tau_{M_k} \), and \( \tilde{U}_{k,\ell}^{(n)} \) is an arbitrary unitary interaction between \( S_L \) and \( k - \ell \) fresh machines occurring immediately prior to timestep \( n \) (see Fig. 5). Due to the fact that no memory transportation occurs on the \( S_L \) level throughout the protocol, the full dynamics of the system and memory carriers is captured by the following concatenation of dynamical maps:

\[
\rho_{SL}^{(n:0)}(\beta, k, \ell) = \tilde{\Lambda}_{k,\ell}^{(n:t)} \circ \tilde{\Lambda}_{k,\ell}^{(t:0)}[\rho_{SL}^{(0)}(\beta, \ell)],
\]

where \( \tilde{\Lambda}_{k,\ell}^{(n:t)} \) is a CPTP map that acts only upon \( S_L \) and depends on the unitary operators \( \tilde{U}_{k,\ell}^{(n)} \), \( \tilde{U}_{k,\ell}^{(t)} \) and the initial state of the \( k - \ell \) fresh machines taking part in each interaction. Thus the dynamics is (operationally) CP-divisible on the level of \( S_L \), and it is easy to see that it is even Markovian in the stronger sense provided in Refs. [69, 70].

This “Markovian embedding” of the non-Markovian dynamics provides an opportunity to investigate the problem at hand with a simplified Markovian dynamics on the larger \( S_L \) system instead of complicated non-Markovian dynamics that occurs on the level of \( S \). In the sense of Ref. [49], the number of memory carriers \( \ell \) corresponds to the memory depth of the dynamics; intuitively, this is the number of additional subsystems that need to be included in the description of the system so that the dynamics is rendered Markovian.

**Appendix B: Proof of Theorem 1**

Here we prove Theorem 1. The proof makes use of the main result of Refs. [10, 11], which derive the ultimate cooling bounds for a Markovian protocol. We first embed the non-Markovian dynamics of \( S \) as a Markovian one by considering the target system \( S_L \), before finding the optimally cool \( S_L \) state in the asymptotic limit, which we denote \( \varrho_{SL}^{*} \). We then combine this result with the fact that there always exists a unitary that can finally be implemented on just \( S_L \) such that the reduced state of \( S \) majorizes all the reduced state of any possible state of the system \( S_L \), as long as \( \varrho_{SL}^{*} \) is majorized by \( \varrho_{SL}^{0} \). This implies that
the state $\varrho^*_S$ calculated from the reduced state of $\varrho_{SL}$ which has the same eigenvalue spectrum of the asymptotically optimal $SL$ state.

Before we begin with the proof, we provide a definition of majorization for completeness:

**Definition 1.** Given a vector of real numbers $\mathbf{a} \in \mathbb{R}^d$, we denote by $\mathbf{a}^\uparrow$ the vector with the same components but sorted in non-decreasing order. Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, we say that $\mathbf{a} \succ \mathbf{b}$ (a majorizes $\mathbf{b}$) iff

$$\sum_{i=1}^k a_i^k \geq \sum_{i=1}^k b_i^k \quad \forall k \in \{1, \ldots, d\}. \quad (B1)$$

**Proof of Theorem 1.** We first perform a Markovian embedding of the non-Markovian collision model dynamics by considering the evolution of the larger target system $SL$ (with dimension $d_S d_M$); for a given number $\ell$ of memory carriers, the Markovian embedding corresponds to a memory depth of $\ell$ in the sense of Ref. [49], which is to say that by including the description of the $\ell$ memory carrying systems with that of the original target system $S$, the dynamics is rendered Markovian. This is because, at each step of the protocol, $SL$ interacts with $k - \ell$ fresh machine systems (with total dimension $d_M^{k-\ell}$), which are subsequently discarded and play no further role in the dynamics.

In the Markovian regime, we can use the theorem of universal cooling bound presented in Ref. [10] for an arbitrary target system interacting with an arbitrary machine, which are initially in a thermal state with inverse temperature $\beta$ in the limit of infinite cycles.

**Lemma 1** (Markovian asymptotic cooling limit). (Theorem 1 in Ref. [10]): For any $d_S$-dimensional system with Hamiltonian $H_S = \sum_{i=0}^{d_S-1} E_i \ket{i} \bra{i}_S$ interacting with a $d_M$ dimensional machine with Hamiltonian $H_M = \sum_{i=1}^{d_M} \mathcal{E}_i \ket{i} \bra{i}_M$ with $\{E_i\}$, $\{\mathcal{E}_i\}$ sorted in non-decreasing order, in the limit of infinite cycles,

- The ground state population of the target system $\tilde{S}$ is upper bounded by

$$\tilde{p}^*(\beta) = \left( \sum_{n=0}^{d_S-1} e^{-\beta n \tilde{E}_{\text{max}}} \right)^{-1} \quad (B2)$$

where $\tilde{E}_{\text{max}}$ is the largest energy gap of the machine.

- In both coherent and incoherent control scenarios, the vectorized form of eigenvalues of the final state is majorized by that of the following state,

$$\varrho^*_S(\beta) = \sum_{n=0}^{d_S-1} e^{-\beta n \tilde{E}_{\text{max}}} \frac{\varrho_S(\beta, \tilde{E}_{\text{max}})}{\varrho_S(\beta, \tilde{E}_{\text{max}})} \ket{n} \bra{n}_S, \quad (B3)$$

if the initial state $\varrho_S^{(0)}(\beta)$ is majorized by $\varrho^*_S(\beta)$. 

---

**FIG. 5.** Markovian embedding of generalized collision model. The generalized collision models considered in this work can be embedded as a Markovian process by grouping together the system and memory carriers into a larger target system. On the left, we show that by considering the original circuit shown in Fig. 1, plus allowing for a swap interaction between relevant machines, the dynamics can be transformed into the circuit in the right panel. Here, we identify the memory carrier systems as $L$; the entire $SL$ system now interacts with fresh machines between each timestep, which we label $R$. The full dynamics of $SL$ can be described by a Markovian sequence of CPTP maps: here we show only the first two, $\tilde{\Lambda}_{2:1}^{(1)}$, $\tilde{\Lambda}_{2:1}^{(2)}$ within the purple borders, although the dynamics between any steps can be described similarly. Note that, in contrast to the dynamics of the system itself, the dynamical maps on the level of $SL$ contain the complete description and Eq. (A6) always holds; intuitively, this is because none of the systems carrying memory are artificially “cut” by the description of the dynamical map (see the partial traces on the red lines in Fig. 4 for comparison).
• In the optimal coherent operation, the asymptotically optimal state, which is also achievable, is given by \( \rho_{\text{opt}}^{AB} \).

In view of the fact that the final state \( \rho_{\text{opt}}^{AB}(\beta) \) has a unique eigenvalue distribution and is achievable in the infinite cycle limit, it is possible to investigate this bound on the population of a particular subspace of dimension \( d \), rather than just the ground state population. It is straightforward to show that its population is upper bounded by the \( d \) largest eigenvalues

\[
\tilde{p}_{H_d}(\beta) \leq \sum_{n=0}^{d-1} e^{-\beta n \tilde{E}_{\text{max}}}.
\]  

(B4)

With this knowledge, we are in the position to study the optimal cooling of the non-Markovian collision model protocol in the limit of infinite cycles by employing Lemma 1. In our case, an arbitrary target system \( SL \) interacts with \( k \) machines at each step and \( \ell \) of these carries memory forward to be involved in the next interaction. Thus, the system \( SL \) corresponds to the target system \( \tilde{S} \) here, which undergoes Markovian dynamics with respect to the \( k - \ell \) fresh machines added at each step, which comprise \( \tilde{M} \); hence, \( \tilde{E}_{\text{max}} \) is equal to \((k - \ell)\tilde{E}_{\text{max}}\). It turns out that the maximum energy gap of the fresh machines and the total dimension and number of the memory carriers play an important role in the ultimate cooling bound.

Using our Markovian embedding of the dynamics and Lemma 1, we see that in the limit of infinite cycles for any control paradigm, the vector of the eigenvalues of the asymptotic state is majorized by

\[
\tilde{\sigma}_{\text{SL}}(\tilde{E}_{\text{max}}, \beta, k, \ell) = \sum_{n=0}^{\tilde{d}-1} e^{-\beta n \tilde{E}_{\text{max}}} |n\rangle\langle n|_{\text{SL}}
\]  

(B5)

if \( \tilde{\rho}_{\text{SL}}^{(0)}(\tilde{E}_{\text{max}}, k, \ell) \propto \tilde{\rho}_{\text{SL}}^{(0)}(\tilde{E}_{\text{max}}, \beta, k, \ell) \) and \( \{ |n\rangle_{\text{SL}} \} \) is the energy eigenbasis with respect to which the eigenvalues are sorted in increasing order. So far, we have found the achievable passive state that majorizes all other reachable states of \( SL \) via unitary operations on \( SLR \). However, this state is not unique as the characterization is based solely on its eigenstate distribution: one can indeed find a whole set of reachable states, \( \tilde{\lambda}[\tilde{\rho}_{\text{SL}}^{(0)}(\tilde{E}_{\text{max}}, \beta, k, \ell)] = \tilde{\lambda}[U_{\text{SL}} \tilde{\rho}_{\text{SL}}^{(0)}(\tilde{E}_{\text{max}}, \beta, k, \ell) U_{\text{SL}}^\dagger] \forall U_{\text{SL}} \) where \( \tilde{\lambda}[\rho] \) indicates the vectorized form of the eigenvalues of \( \rho \). We now present another lemma which says that from any such state of \( SL \), one can reach the optimally cool state of \( S \), \( \rho_{\text{opt}}^{S} \), helping us complete the proof.

**Lemma 2** (Reduced state majorization). For any pair states \( \varrho_{AB} \) and \( \sigma_{AB} \), if \( \sigma_{AB} \prec \varrho_{AB} \), there exist a unitary \( U_{AB}^{\text{opt}} \) on \( H_{AB} \) such that

\[
\text{tr}_B \left[ U_{AB}^{\text{opt}} \sigma_{AB} U_{AB}^{\text{opt}} \dagger \right] < \text{tr}_B \left[ U_{AB}^{\text{opt}} \varrho_{AB} U_{AB}^{\text{opt}} \dagger \right] \quad \forall U_{AB}.
\]  

(B6)

**Proof.** Without loss of generality, we assume that the eigenvalues of both states \( \varrho_{AB} \) and \( \sigma_{AB} \) are sorted in non-increasing order as the following

\[
P_{AB} = \{ p_{\alpha} \}_{\alpha=0}^{d_{A} d_{B} - 1} \quad \text{and} \quad Q_{AB} = \{ q_{\alpha} \}_{\alpha=0}^{d_{A} d_{B} - 1}.
\]  

(B7)

Based on the sorted eigenvalues, \( \sigma_{AB} \prec \varrho_{AB} \) if and only if

\[
\sum_{\alpha=0}^{k} q_{\alpha} \leq \sum_{\alpha=0}^{k} p_{\alpha} \quad \forall k \in \{0, 1, \ldots, d_{A} d_{B} - 1\}.
\]  

(B8)

Now we aim to find the reduced state \( \sigma_{A}^{\text{opt}} \) majorizing all of the achievable reduced states possible to generate by a unitary transformation of \( \sigma_{AB} \), which we assume to be diagonal in the orthonormal basis \( \{ |ij\rangle_{AB} \} \) without loss of generality:

\[
\sigma_{AB} = \sum_{i=0}^{d_{A} - 1} \sum_{j=0}^{d_{B} - 1} \rho_{ij} |ij\rangle_{AB} \langle ij|_{AB}
\]  

(B9)

One can show that it is possible to obtain \( \sigma_{A}^{\text{opt}} \) from a bipartite state that is diagonal in the same basis. Then we have,

\[
\tilde{\sigma}_{AB} = U_{AB}^{\text{opt}} \sigma_{AB} U_{AB}^{\text{opt}} \dagger = \sum_{i=0}^{d_{A} - 1} \sum_{j=0}^{d_{B} - 1} \tilde{q}_{ij} |ij\rangle_{AB} \langle ij|_{AB},
\]  

(B10)

where \( U_{AB}^{\text{opt}} \) is simply a permutation matrix that reorders the eigenvalues appropriately. The final reduced state is then given by

\[
\tilde{\sigma}_{A} = \sum_{i=0}^{d_{A} - 1} \left( \sum_{j=0}^{d_{B} - 1} \tilde{q}_{ij} \right) |i\rangle_{A} \langle i|_{A}.
\]  

(B11)
We now need to maximize the eigenvalues of the reduced state with respect to eigenvalues of $\sigma_{AB}$. If we rearrange the eigenvalues in such a way that $\tilde{g}_{ij} = q_i^\dagger$, where $\alpha$ is given by $\alpha = i d_B + j$, we obtain $\sigma_A^{opt}$ as the following

$$
\sigma_A^{opt} = \sum_{i=0}^{d_A-1} \eta_i^\dagger |i\rangle_A \langle i|_A = \sum_{i=0}^{d_A-1} \left( \sum_{j=0}^{d_B-1} \tilde{g}_{i,j}^\dagger \right) |i\rangle_A \langle i|_A, 
$$

(B12)

where, due to the sorting of $\{q_i^\dagger\}$, the eigenvalues of $\sigma_A^{opt}$ are sorted in non-decreasing order. The final reduced state satisfies the following condition

$$
\text{tr}_B \left[ U_{AB} \sigma_{AB} U_{AB}^\dagger \right] \propto \text{tr}_B \left[ U_{AB}^{opt} \sigma_{AB} U_{AB}^{opt \dagger} \right] = \sigma_A^{opt} \quad \forall U_{AB}. 
$$

(B13)

Similarly one can find $\rho_A^{opt}$ by applying a unitary $V_{AB}^{opt}$.

$$
\rho_A^{opt} = \sum_{i=0}^{d_A-1} \xi_i^\dagger |i\rangle_A \langle i|_A = \sum_{i=0}^{d_A-1} \left( \sum_{j=0}^{d_B-1} p_{i,j}^\dagger \right) |i\rangle_A \langle i|_A, 
$$

(B14)

whose eigenvalues are also in non-decreasing order by construction. The final step of the proof is to show that $\sigma_A^{opt} \prec \rho_A^{opt}$ whenever $\sigma_{AB} \prec \rho_{AB}$. This majorization condition can be recast in the form of

$$
\sum_{i=0}^{k} \eta_i \leq \sum_{i=0}^{k} \xi_i \Rightarrow \sum_{\alpha=0}^{(k+1)d_B-1} \tilde{q}_{\alpha} \leq \sum_{\alpha=0}^{(k+1)d_B-1} \tilde{p}_{\alpha} \quad \forall k \in \{0, 1, \ldots, d_A-1\}. 
$$

(B15)

Using inequality (B8), one can easily show that inequality (B15) always holds, i.e., $\sigma_A^{opt} \prec \rho_A^{opt}$, completing the proof.  

In the next step, we aim to maximize the population of the system ground state, i.e., the maximum population of the specific subspace of the $SL$ target given in Eq. (B4), one must find the target state $\rho_{SL}^{opt}(\beta, k, \ell)$ that can be achieved from the states $\rho_{SL}$ with the same eigenvalues as $\rho_{SL}(\epsilon_{max}, \beta, k, \ell)$, since, from Lemma 2 we know that this state majorizes the largest set of states in $S$. In order to do so, we maximize the eigenvalues of $\rho_{SL}^{opt}(\epsilon_{max}, \beta, k, \ell)$ with respect to those of $\rho_{SL}^{opt}(\epsilon_{max}, \beta, k, \ell)$. One can appropriately sort the eigenvalues of the system $S$ and the memory carrier machines $L$ through the following unitary:

$$
U_{SL}^{opt} g_{SL}^\star(\epsilon_{max}, \beta, k, \ell) U_{SL}^{opt \dagger} = \sum_{n=0}^{d_S-1} \sum_{j=0}^{d_L-1} e^{-\beta(n d_S^\ell + j)(k-\ell)\epsilon_{max}} |nj\rangle_\star \langle nj|_\star, 
$$

(B16)

Thus, beginning with the optimally cool $SL$ state in Eq. (B5), we can reorder the eigenvalues via $U_{SL}^{opt}$ in Eq. (B16) such that the subsystem $S$ is optimally cool; finally applying Lemma 2 implies that

$$
\text{tr}_L \left[ U_{SL}^{opt} g_{SL}^\star(\epsilon_{max}, \beta, k, \ell) U_{SL}^{opt \dagger} \right] \propto g_S^\star(\epsilon_{max}, \beta, k, \ell) \quad \forall U_{SL}, 
$$

(B17)

where $g_S^\star(\epsilon_{max}, \beta, k, \ell)$ is indeed given by taking the partial trace over $L$ of $g_{SL}^\star(\epsilon_{max}, \beta, k, \ell)$:

$$
g_S^\star(\epsilon_{max}, \beta, k, \ell) = \sum_{n=0}^{d_S-1} \sum_{n'=0}^{d_L-1} e^{-\beta(n d_S^\ell + j)(k-\ell)\epsilon_{max}} |n\rangle_\star \langle n|_\star, 
$$

(B18)

thus establishing $g_S^\star(\epsilon_{max}, \beta, k, \ell)$ as the optimal system state in the asymptotic limit.
In conclusion, from Lemma 1, we know that the final state of the system $SL$, for any control paradigm in the infinite cycle limit, is majorized by $g^*_S(E_{\text{max}}, \beta, k, \ell)$. Consequently, the final state of $S$ is also majorized by $g^*_S(E_{\text{max}}, \beta, k, \ell)$. Then, the population of the ground state of the system is upper bounded by the sum of the $d_{M}^\ell$ largest eigenvalues of $g^*_S(E_{\text{max}}, \beta, k, \ell)$, i.e., $p^\ell \left(E_{\text{max}}, \beta, k, \ell \right) = \left( \sum_{i=0}^{d_{M}^\ell-1} e^{-\beta \mu_i d_{M}^\ell E_{\text{max}}} \right)^{-1}$.

We finally prove that in the coherent scenario, the state $g^*_S(E_{\text{max}}, \beta, k, \ell)$ is achievable in the limit of infinite cycles. Using Lemma 1, one can easily show that the final state of $SL$ under optimal coherent operations converges to $g^*_S(E_{\text{max}}, \beta, k, \ell)$. To do so, we use the fact that in the coherent scenario, one can apply any unitary operation on the system $SL$ at the final step. We then achieve the desired target state $g^*_S(E_{\text{max}}, \beta, k, \ell)$ via employing the unitary $U_{SL}^\text{opt}$, completing the proof.

Appendix C: Relation to heat-bath algorithmic cooling

Here, we propose a general heat-bath algorithm cooling (generalized collision model with memory) technique to optimally cool down a target system in the limit of infinite cycles. To obtain the cooling limit, in general we need to change the operations based on the state of $SL$ output by the dynamics at the most recent step. However, using this technique, we can show that not only it is possible to cool down the system by a state-independent, fixed sequence of operations, but also that the protocol converges to the optimally cool state in the asymptotic limit. The result hence draws attention to the fact that in the limit of infinitely many repeated cycles, the dimension of the memory carriers of the protocol (not necessarily knowledge about the state at intermediate times) plays an important role and can lead to exponential improvement over the Markovian case; in fact, perhaps surprisingly, the role of memory depth is more significant than that of the ability of the agent to implement multi-partite interactions between the system and machines at each step (although, of course, the number of memory carriers is upper bound by how multi-partite the interactions are allowed to be).

Here we will consider the effect of adding compression systems (in the terminology of the HBAC community) or a number of machines that carry memory forward (in the language of our generalized collision model) for a non-adaptive cooling protocol in which a fixed interaction between the target system and a subset of machines at each timestep is repeated infinitely many times. As we have previously, we assume that $k$ machines interact with the system at each step and $\ell$ of them carry memory forward to the next step. This means that $(k-\ell)$ fresh machines and $\ell$ memory carriers participate in the interaction at each timestep. We fix at the outset, for any given choice of these parameters, the dimension of the machines $d_M$, which (along with $k$) fixes the control complexity in each of the many cases we will look at, and also we fix the temperature at which everything begins, $\beta$.

In Appendix A, we showed how the dynamics of the system $S$ in the non-Markovian collision model can be described by Markovian dynamics on the larger system $SL$ (with total dimension $d_{SL}d_{M}^\ell$); in HBAC community, the larger target system of such an embedding is known as the computation system. In this case, the system $SL$ interact with $(k-\ell)$ fresh machines (with total dimension $d_{M}^{(k-\ell)}$) with maximum energy gap $(k-\ell)E_{\text{max}}$; this is known as the reset system, since they are the machines that are discarded after each interaction step, modelling a rethermalization with the external environment. One can decompose the total Hilbert space into the computation part and the reset part $R$, i.e., $H_{SLR} = H_{SL} \otimes H_R$. At any timestep, the dynamics of the system $SL$, which comes arises from unitary evolution on the system $SLR$, is given by

$$g_{SL}^{(n)}(\beta, k, \ell) = \Lambda_{k,\ell}^{(n)}(\beta)[g_{SL}^{(n-1)}(\beta, k, \ell)] := \text{tr}_R \left[ U^{(n)}_{k,\ell} \left( g_{SL}^{(n-1)}(\beta, k, \ell) \otimes g_{R}^{(0)}(\beta, k, \ell) \right) U^{(n)\dagger}_{k,\ell} \right].$$

(C1)

Note that $g_{R}^{(0)}(\beta, k, \ell) = \bigotimes_{j=\ell+1}^{k} \tau_{M_j}$ is fixed and the same at each step of the protocol as it refers to the $k-\ell$ fresh machines taken from a thermal bath. In Fig. 6, we depict the equivalence between the standard HBAC protocol and the generalized collision model formalism.

We now consider a non-adaptive protocol, in which the agent is only allowed to repeatedly apply a fixed unitary operation (non-adaptive scenario), i.e., $U^{(n)}_{k,\ell} = U_{k,\ell} \forall \ n$. The dynamics can then be simplified to

$$g_{SL}^{(n)}(\beta, k, \ell) = \varphi^n \Lambda_{k,\ell}(\beta)[g_{SL}^{(0)}(\beta, \ell)],$$

(C2)

where $g_{SL}^{(0)}(\beta, \ell) = \tau_S(\beta) \bigotimes_{j=1}^{\ell} \tau_{M_j}(\beta)$ and $\varphi^n \Lambda_{k,\ell}(\beta)$ is an $n$-fold concatenation of the dynamical map induced between any pair of timesteps, with $\Lambda_{k,\ell}(\beta)$ defined such that $\Lambda_{k,\ell}(\beta)[X_{SL}] := \text{tr}_R \left[ U_{k,\ell}(X_{SL} \otimes g_{R}^{(0)}(\beta, k, \ell) U^{(\dagger)}_{k,\ell} \right]$. This dynamical map is thus independent of the timestep and fully determined by the unitary $U_{k,\ell}$ and the initial state of the fresh machines. In the following, we will show that it is possible to asymptotically reach the ultimate cooling limit via this non-adaptive protocol.

**Theorem 2.** In the non-adaptive scenario, for a given $d_S$-dimensional system interacting at each step with $k$ $d_M$-dimensional identical machines, with $\ell$ of the machines used at each step carrying the memory forward, in the limit of infinite cycles, it is possible to reach the state $g_{SL}^{(0)}(\beta, k, \ell)$ if the initial state $g_{SL}^{(0)}(\beta, \ell) = \tau_S(\beta) \bigotimes_{j=1}^{\ell} \tau_{M_j}(\beta)$ is majorized by $g_{SL}(\beta, k, \ell)$.
FIG. 6. Equivalence between HBAC and generalized collision model. The circuit for a HBAC protocol applied to a quantum system $S$ with one compression system (labeled $L$ to be consistent with our notation) and one reset system $R$. The compression systems store memory of previous interactions (red), whereas the reset ones are assumed to rethermalize with a bath at inverse temperature $\beta$ between each step of the protocol (orange). Noting that the “reset” step has the effect of tracing out the system and preparing a fresh one in the thermal state with the same temperature as the bath, it is clear that HBAC is equivalent to generalized collision model we consider (here, $k = 2, \ell = 1$). Further comparison with Fig. 5 highlights that HBAC need not require the agent to control the compression systems with high fidelity for the entire duration of the protocol: by making appropriate swaps, one only needs to track the $\ell$ compression systems / memory carriers for $\ell$ timesteps.

**Proof.** Due to our definition of cooling being based upon majorization, only the eigenvalues of the asymptotic state play a role in determining the fundamental cooling limit. We can therefore restrict our analysis to a specific orthonormal basis, e.g., energy eigenbasis (it is straightforward to generalise the obtained result to an arbitrary orthonormal basis). Here we focus on group of unitary operations that map diagonal density operators of the system $SL$ to diagonal ones. This restriction hence provides us an opportunity to describe the dynamics via stochastic maps that act upon the vector constructed with the eigenvalues of the system and memory carriers.

Here we employ a specific unitary $U$ on the entire $SLR$ system, which can be decomposed as follows

$$ U = V \otimes \tilde{1} \quad \text{(C3)} $$

where $V$ acts unitarily on the Hilbert space $\mathcal{H}_{SL} \otimes \mathcal{H}_G$, in which $\mathcal{H}_G \subset \mathcal{H}_R$ is a subspace spanned by the two eigenstates of the reset systems (fresh machines) that have the maximum energy gap of $(k - \ell)\varepsilon_{\text{max}}$, i.e., $|0\rangle_R$ and $|d_m^{k-\ell} - 1\rangle_R$, and $\tilde{1}$ represents the identity on the subspace $\mathcal{H}_{\overline{G}} = \mathcal{H}_R \setminus \mathcal{H}_G$. In the energy eigenbasis, $V$ can also be written in the form of

$$ V = \begin{bmatrix} 1 & \sigma_x & \cdots & \sigma_x \\ \sigma_x & 1 & \cdots & \sigma_x \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_x & \cdots & \cdots & 1 \end{bmatrix}, \quad \text{(C4)} $$

where $\sigma_x$ is the Pauli X operator. The energy eigenvectors of the Hilbert space $\mathcal{H}_{SL} \otimes \mathcal{H}_G$ are sorted as

$$ |2q\rangle_{SLG} = |2q\rangle_{SLR} = |q\rangle_{SL} \otimes |0\rangle_R $$

$$ |2q + 1\rangle_{SLG} = |2q + 1\rangle_{SLR} = |q\rangle_{SL} \otimes |d_m^{k-\ell} - 1\rangle_R \quad \text{(C5)} $$

with corresponding eigenvalues of $\xi_{2q} = p_{q}^{(0)} / Z_R$ and $\xi_{2q + 1} = p_{q}^{(0)} e^{-\beta(k-\ell)\varepsilon_{\text{max}}} / Z_R$ for $q \in \{0, 1, \ldots, d_sd_M^{k-\ell}\}$, respectively, where $Z_R = Z_M(\beta)^{(k-\ell)}$ is the partition function of the reset system and $\{p_{q}^{(0)}\}$ are the eigenvalues of the initial state of $SL$.

The unitary $U$ acts to swap every neighboring element on the diagonal part of the global density matrix in the subspace $\mathcal{H}_{SL} \otimes \mathcal{H}_G$ and leave the other elements untouched. We now focus on the transformation of the diagonal elements on the global space $SLR$ under such dynamics. We write the initial state as

$$ \varrho_{SLR}^{(0)} = \sum_{r=0}^{2d_sd_M^{k-\ell} - 1} \xi_{r}^{(0)} |r\rangle_{SLR} \otimes |\tilde{r}\rangle_{\overline{G}} + \sum_{r=2d_sd_M^{k-\ell}}^{d_sd_M^{k-\ell}} \xi_{r}^{(0)} |r\rangle_{SLR} $$

$$ = \alpha_k \varrho_{SLG}^{(0)} \oplus (1 - \alpha_k) \varrho_{SL\overline{G}}^{(0)} \quad \text{(C6)} $$
where $\rho^{(0)}_{SLG}$ and $\varrho^{(0)}_{SLG}$ are normalized density matrices and $\alpha_{kt} = (1 + e^{-\beta(k-t)\varepsilon_{\text{max}}})/Z_R$. After applying the unitary $U$, we have

$$
\rho^{(1)}_{SLR} = U^\dagger \rho^{(0)}_{SLR} U^\dagger = \alpha_{kt} V \rho^{(0)}_{SLG} V^\dagger \oplus (1 - \alpha_{kt}) \varrho^{(0)}_{SLG}
$$

$$
= \alpha_{kt} \left[ \frac{(p_0^{(0)} + p_1^{(0)})}{1 + e^{-\beta(k\varepsilon_{\text{max}})}} |0\rangle\langle 0|_{SL} + \frac{p_{dSD_M^2-2}^{(0)} + p_{dSD_M^2-1}^{(0)}}{1 + e^{-\beta(k\varepsilon_{\text{max}})}} |dSD_M^2 - 1\rangle\langle dSD_M^2 - 1|_{SL} + \right.
$$

$$
\sum_{dSD_M^2-2}^{dSD_M^2-1} \left( \frac{p_{r}^{(0)} - e^{-\beta(k\varepsilon_{\text{max}})}}{1 + e^{-\beta(k\varepsilon_{\text{max}})}} |r\rangle\langle r|_{SL} + (1 - \alpha_{kt}) \sum_{r=0}^{dSD_M^2-1} p_r^{(0)} |r\rangle\langle r|_{SL} \right) \right) \oplus (1 - \alpha_{kt}) \varrho^{(0)}_{SLG} \quad (C7)
$$

It is clear that the output state is also diagonal in the energy eigenbasis. One can easily obtain the reduced state of the system $SL$ after one timestep from Eq. (C7) by taking a partial trace over $R$:

$$
\rho^{(1)}_{SL} = \text{tr}_R \rho^{(1)}_{SLR}
$$

$$
= \alpha_{kt} \left[ \frac{(p_0^{(0)} + p_1^{(0)})}{1 + e^{-\beta(k\varepsilon_{\text{max}})}} |0\rangle\langle 0|_{SL} + \frac{p_{dSD_M^2-2}^{(0)} + p_{dSD_M^2-1}^{(0)}}{1 + e^{-\beta(k\varepsilon_{\text{max}})}} |dSD_M^2 - 1\rangle\langle dSD_M^2 - 1|_{SL} + \right.
$$

$$
\sum_{r=0}^{dSD_M^2-2} \left( \frac{p_{r}^{(0)} - e^{-\beta(k\varepsilon_{\text{max}})}}{1 + e^{-\beta(k\varepsilon_{\text{max}})}} |r\rangle\langle r|_{SL} + (1 - \alpha_{kt}) \sum_{r=0}^{dSD_M^2-1} p_r^{(0)} |r\rangle\langle r|_{SL} \right) \right) \oplus (1 - \alpha_{kt}) \varrho^{(0)}_{SLG} \quad (C8)
$$

Since the output state on $H_{SLR}$ has a block-diagonal structure with respect to this subspace decomposition, it is locally classical, i.e., has diagonal marginals with respect to the local energy eigenbasis. Therefore, the dynamics of the relevant part of the reduced state can be described in terms of a classical stochastic matrix acting on $SL$ (instead of a CPTP map as would be required if coherences were relevant). In addition, this stochastic matrix is independent of the timestep (since the protocol is non-adaptive) and the state transformation of the system $SL$ can be described in terms of a classical stochastic matrix acting on $SL$, with the remaining eigenvalues given by

$$
\lambda_q = \alpha_{kt} \nu_q + (1 - \alpha_{kt}), \quad \nu_q = \frac{2e^{-\frac{q}{2} (k\varepsilon_{\text{max}}) \cos \left( \frac{\pi q}{dSD_M^2} \right)}}{1 + e^{-\beta(k\varepsilon_{\text{max}})}} \quad \forall q \in \{1, \ldots, dSD_M^2 - 1\}. \quad (C12)
$$

Since $\mathbb{I}$ is diagonal with respect to any orthonormal basis and has uniform eigenvalues, it is straightforward to show that the eigenvalues of $\mathbb{T}$ are obtained by

$$
\lambda_q = \alpha_{kt} \nu_q + (1 - \alpha_{kt}). \quad (C13)
where \( \lambda \) second largest in \( \varrho \) of the system \( SL \). Thus, \( T \) system and the environment \[42\]:

This result provides an estimate for the number of iterations of the protocol to reach the optimally cool system.

For the protocol considered, the spectral gap can be explicitly calculated

\[
\Delta = \lambda_0 - \lambda_1 = 1 - \alpha_{k \ell} \mu_1 - (1 - \alpha_{k \ell})
\]

\[
= \alpha_{k \ell} \left( 1 - \frac{2e^{-\frac{z}{2}(k-\ell)\varepsilon_{\text{max}}} \cos \left( \frac{\pi}{\delta d_{M}^{\ell}} \right)}{1 + e^{-(k-\ell)\varepsilon_{\text{max}}}} \right)
\]

\[
\geq \frac{1 + e^{-(k-\ell)\varepsilon_{\text{max}}}}{(Z_M(\beta))^{k-\ell}} \left( \frac{(1 - e^{-\frac{z}{2}(k-\ell)\varepsilon_{\text{max}}}^2)}{1 + e^{-(k-\ell)\varepsilon_{\text{max}}}} \right)
\]

\[
= \frac{(1 - e^{-\frac{z}{2}(k-\ell)\varepsilon_{\text{max}}}^2)}{(Z_M(\beta))^{k-\ell}}.
\]

Then we have

\[
t_{\text{mix}}(\eta) \leq \frac{(Z_M(\beta))^{k-\ell}}{(1 - e^{-\frac{z}{2}(k-\ell)\varepsilon_{\text{max}}}^2)} \log \left( \frac{1}{\eta p_0 e^{-(k-\ell)\varepsilon_{\text{max}}^{\ell}}} \right).
\]

This result provides an estimate for the number of iterations of the protocol to reach the optimally cool system.

**Appendix D: Step-wise optimal protocol and finite time comparisons**

Here we provide some analysis on the finite time behavior for the cooling strategies discussed throughout the main text. It is important to note that the finite time properties in general depend upon the details of the full complex energy spectrum of the machines; nonetheless, we have the following observations.

We first detail the step-wise optimal protocol, briefly described in the main text and prove its optimality.

**Definition 2** (Step-wise optimal cooling unitary). Given a joint state \( \varrho_{SLR} \), let \( V^\text{opt}_{SLR} \) be the unitary that reorders the eigenvalues of \( \varrho_{SLM} \) within each block partitioned by \( R \) such that the largest is in the subspace \( (000)_{SLR}(000) \), second largest in \( (001)_{SLR}(001) \), third largest in \( (002)_{SLR}(002) \), and so on until the smallest eigenvalue is in \( |d_S - 1, d_M - 1, d_{M-\ell} - 1\rangle_{SLR}|d_S - 1, d_M - 1, d_{M-\ell} - 1\rangle \), i.e., perform

\[
V^\text{opt}_{SLR} \varrho_{SLR} V^\text{opt}_{SLR} = \sum_{\mu=0}^{d_{S}^{\ell}-1} \sum_{\nu=0}^{d_{M}^{\ell}-1} \sum_{\omega=0}^{d_{M-\ell}^{\ell}-1} \lambda^{\uparrow}_\mu \delta_\mu \delta_\nu \delta_\omega \langle \mu | V(\omega) | \nu \rangle_{SLR} (\mu | V(\omega) | \nu),
\]

where \( \lambda^{\uparrow} \) denotes the vector of eigenvalues of \( \varrho_{SLR} \) labeled in non-increasing order.
**Theorem 3** (Step-wise optimal cooling protocol). By applying the unitary defined in Eq. (D1) at each step, the cooling protocol is step-wise optimal.

In the Markovian case, the step-wise optimal protocol simply considers all of the eigenvalues of the joint system-machine at each timestep and optimally reorders them such that the system is as cool as possible. However, such a protocol does not ensure step-wise optimality when memory is present: here, not only must we optimally cool the system by rearranging the eigenvalues of the total accessible state at each step, but we must also ensure that this accessible state at each step is as cool as possible given its history. As the only information pertaining to the history is transmitted by the system SL, this means that the optimal protocol must at each step optimally cool $S$, and then subject to this constraint, optimally cool the memory carriers $L$ which go on to further cool the system at later times.

**Proof.** We first need to show that $q_{SL}$ obtained from Eq. (D1) majorizes all the of the reachable marginal states of $S$; this problem reduces to a constrained rearrangement of the eigenvalues of the entire system, i.e., the eigenvalues are to be arranged optimally with respect to certain eigenspaces. Since majorization theory is independent of the eigenbasis, we choose the energy eigenbasis for simplicity. To obtain the eigenspectrum of the system $S$ that majorizes all of the reachable states under unitary transformations on $SLR$, the final state of the entire system can be written in the form of

$$
\tilde{V}_{SLR}^{\text{opt}} V_{SLR}^{\text{opt}} = \sum_{\mu=0}^{d_S^2-1} \sum_{\eta=0}^{d_L-1} \lambda_{\mu,\nu}^{\mu+\nu d_M} |\mu\eta\rangle_{SLR} \langle \mu\eta|,
$$

(D2)

where $|\mu\eta\rangle_{SLR} = |\mu\rangle_S \otimes |\eta\rangle_{LR}$.

Second, we show that the state of the memory carriers after applying this unitary, i.e., $q_{\tilde{L}} = \text{tr}_S \left[ V_{SLR}^{\text{opt}} V_{SLR}^{\text{opt}} \right]$, also majorizes all of the reachable states of $L$ given the mentioned majorization condition. We must therefore rearrange the eigenvalues of $q_{SLR}$ within each block corresponding to a fixed $\mu$, i.e., sort $\{\lambda_{\mu,\nu}^{\mu+\nu d_M} d_M\}_{\nu=0}^{d_M^2}$ in such a way that $\nu$-th largest $d_M^{k-\ell}$ eigenvalues are placed in the $\nu$-th eigenspace of the system $L$, which gives the state $q_{\tilde{L}}$ that majorizes all of those reachable via unitary transformations on $SLR$. To do so, we rearrange the eigenvalues of the joint $SLR$ system as $\lambda_{\mu,\nu}^{\mu+\nu d_M} \cdot d_M^{k-\ell}$, via the unitary transformation defined in Eq. (D1), where $|\mu\nu\omega\rangle_{SLR} = |\mu\rangle_S \otimes |\nu\rangle_L \otimes |\omega\rangle_R$. It is clear that the reduced state satisfies the required majorization condition for $q_{\tilde{L}}$, i.e., for all $\mu$, we have

$$
\lambda_{\mu,\nu}^{\mu+\nu d_M} \cdot d_M^{k-\ell} + \omega > \lambda_{\mu\prime,\nu\prime}^{\mu\prime+\nu\prime d_M} \cdot d_M^{k-\ell} + \omega \quad \text{if} \quad \nu > \nu\prime \quad \forall \omega, \omega' \in \{0, 1, \ldots, d_M^{k-\ell} - 1\},
$$

(D3)

where this inequality holds due to the eigenvalue ordering of joint state of $SLR$.

Finally, we say that the output state of the system $SL$ from Eq. (D1) majorizes all of those reachable states of $SL$. To do so, must show that $n$-th largest $d_M^{k-\ell}$ eigenvalues of $SLR$ only contribute to the $n$-th eigenvalue of $SL$. This statement follows from

$$
\lambda_{\mu,\nu}^{\mu+\nu d_M} \cdot d_M^{k-\ell} + \omega > \lambda_{\mu\prime,\nu\prime}^{\mu\prime+\nu\prime d_M} \cdot d_M^{k-\ell} + \omega \quad \text{if} \quad \mu \cdot d_M^{k-\ell} + \nu \cdot d_M^{k-\ell} > \mu' \cdot d_M^{k-\ell} + \nu' \cdot d_M^{k-\ell} \quad \forall \omega, \omega' \in \{0, 1, \ldots, d_M^{k-\ell} - 1\}. \quad \text{(D4)}
$$

Eq. (D4) states that under the protocol considered, one achieves the $SL$ state that majorizes all other reachable states via unitaries on $SLR$. We now need to show that if this is achieved at some timestep $n$, this leads to the optimal cooling at any later time. By the stability of majorization under tensor products [71], we know that $q_{SLR}^{(n)opt} \otimes \tau_R$, where $q_{SLR}^{(n)opt} = \text{tr}_R \left[ V_{SLR}^{\text{opt}} V_{SLR}^{\text{opt}} \right]$, majorizes all of the states of $q_{SLR}^{(n)} \otimes \tau_R$, where $q_{SLR}^{(n)}$ is generated by any other protocol and $\tau_R$ are the thermal bath machines to be added at said timestep. This majorization relation cannot be changed by performing the optimal $SLR$ unitary on $q_{SLR}^{(n)opt} \otimes \tau_R$ and any other unitary on $q_{SLR}^{(n)} \otimes \tau_R$. Lastly, invoking the subspace majorization result of Lemma 2, it follows that $q_{SLR}^{(n+1)opt} \gg q_{SLR}^{(n+1)}$.

Thus, we have shown that at each step of the protocol, we have reached the optimal $SL$ state possible given the history; it is important to note that at this level, the process is Markovian, allowing for an inductive extension of the above argumentation to hold. By further invoking Lemma 2 on the level of $SL$ at the final timestep, we yield the optimally cool state of the system $S$, thereby completing the proof.

We now compare the cooling performance between adaptive and non-adaptive strategies for a given choice of memory structure. In the non-adaptive strategy, the rate of cooling is determined completely by the spectral gap $\Delta$ in Eq. (C16), as the same dynamics is repeated at each step. In the adaptive scenario, this is no longer the case and a single parameter does not dictate the rate of convergence to the asymptotic state. Instead, in general, the cooling rate depends upon the entire energy structure of the system and all machines, making a closed form expression difficult to derive. Nonetheless, we can describe the solution to the problem of reaching a step-wise provably optimal system state at finite times as a protocol, as we do in the main text. This protocol converges to the same asymptotic value as the non-adaptive case, but offers an advantage at finite times, as we show in Fig. 7.
FIG. 7. **Finite time advantage with adaptive protocols.** Here we compare the adaptive and non-adaptive protocols for a qubit system (with $E_{\text{max}} = 1$) interacting with qubit machines (with $E_{\text{max}} = 2$) with initial temperature $\beta = 0.2$ and memory structure given by $k = 3, \ell = 2$. In the adaptive scenario, we make use of the step-wise optimal protocol described in the main text; in the non-adaptive, the unitary in Eq. (C4) is repeatedly implemented. We see that, although both scenarios asymptotically converge to the same ground state population, the adaptive protocol outperforms the non-adaptive one at finite times. This behavior is more pronounced for larger dimensions. Note that, in this case, every second step of the non-adaptive protocol does not act to cool the system.

FIG. 8. **Finite time behavior of optimal protocol.** Here we plot the ground state population after $m$ machines have been completely exhausted, for various values of $k, \ell$ as described in the text (we use a qubit system and qubit machines, with $\beta = 0.2, E_{\text{max}} = 1$ and $E_{\text{max}} = 2$). The finite time behavior generally depends upon the full structure of the energy spectra, but already here we see some interesting effects exhibited. For instance, note that at $m = 7$ (first point shown), the $k = 7, \ell = 0$ case provides the best possible cooling, as it allows for a full 8-partite unitary between the system and seven machines. However, for repeated applications of such a unitary with in the memoryless scenario, the performance can be worse than other cases with more local interactions (smaller $k$) and longer memory (larger $\ell$), e.g., see the hierarchy at $m = 28$, which shows the $k = 7, \ell = 0$ scenario to be outperformed by most other memory structures.

We lastly make comparison between various memory structures (i.e., values of $k, \ell$) with respect to the optimal adaptive protocol. In order to do so in a meaningful way, we compute the ground state population of the system for after a fixed number $m = k + (n - 1)(k - \ell)$ machines have been exhausted. If one were to compare the ground state populations after $n$ unitaries had been implemented, for various values of $k, \ell$ one would be making an unfair comparison with respect to the total resources at hand; e.g., after three unitaries with $k = 4, \ell = 3$ the experimenter has made use of six machines, whereas for $k = 3, \ell = 0$, they have used nine. Comparing various scenarios at fixed values of $m$ provides insight into how cool the system can be prepared after all constituents of a finite sized environment are used up for the given memory structure. This change of perspective comes at the cost of the fact that the number of physical unitaries $n$ needed to be implemented in order to exhaust the resources (quantified by $m$) now varies; e.g., to use six machines with $k = 4, \ell = 3$ takes three unitaries, whereas with $k = 1, \ell = 0$ it takes six. Lastly note that not all values of $k, \ell$ are valid for a given $m$, due to the restriction that $n$ must be an integer. The behavior is displayed in Fig. 8.