On Symmetric Extendible States

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Studies on symmetric extendibility of quantum states become especially important in a context of analysis of one-way quantum measures of entanglement, distillability and security of quantum protocols. In this paper we analyse composite systems containing a symmetric extendible part with a particular attention devoted to one-way security of such systems. Further, we introduce a new one-way monotone based on the best symmetric approximation of quantum state and differentiate symmetric extendibility into weak and strong type. We underpin those results with geometric observations on structures of multi-party settings which possess in sub-spaces substantial symmetric extendible components. Finally, we state a very important conjecture linking symmetric-extendibility with one-way distillability and security of all quantum states analyzing behavior of private key in neighborhood of symmetric extendible states.

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I. INTRODUCTION

Recent years have proved a great interest of symmetric extendibility concept showing its usability in quantum communication theory, especially in domain of one-way communication. A natural relation between monogamy of entanglement and symmetric extendibility concept was established [14] with an important application to analysis of Bell inequalities for multipartite settings where some of the parties possess the same sets of measurement settings. Further, the concept is central for studies on one-way quantum channel capacities, entanglement distillability and private key analysis deriving new upper bounds on these communication rates [17] [18] [19]. It seems also that symmetric extendibility is fundamental for studies on recovery and entanglement breaking channels including its neighborhood [10] as well as for such measures like squashed entanglement and quantum discord [22] or analysis of directed communication in 1D/2D spin chains [17]. The aforementioned applications sufficiently prove importance of the notion for quantum communication theory. The challenge for the present quantum information theory in domain of one-way communication is to better understand behavior of all quantum states in the region of non-symmetric extendibility and in particular in a region of non-positive coherent information [8] putting an important question about their distillability or existence of one-way bound entanglement. We also give a formalized structure to some natural intuitions about nature of composite systems and its reference to k-extendible states.

II. SYMMETRIC K-EXTENDIBLE STATES

Particularly symmetric extendibility [13] of a given bipartite state \( \rho_{AB} \in B(\mathcal{H}_A \otimes \mathcal{H}_B) \) denotes that there exists a tripartite state \( \rho_{ABE} \in B(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E) \) invariant due to permutation of B and E part, namely, if:

\[
P = \sum_{ijk} |ijk\rangle\langle ikj|
\]
then \( P \rho_{ABE} P^\dagger = \rho_{ABE} \).

We introduce a general notion of k-extendibility of bipartite states differentiating those with full symmetry on Bobs’ side in their extensions and those with broken symmetry:

Definition II.1. A state \( \rho_{AB_1} \in B(\mathcal{H}_A \otimes \mathcal{H}_B) \) is **strong k-extendible** if there exists such a state \( \rho_{AB_1...B_k B_{k+1}} \in B(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E) \) that \( \text{Tr} \rho_{B_2...B_{k+1}} \rho_{AB_1...B_{k+1}} = \rho_{AB_1} \) and \( \rho_{AB_1...B_{k+1}} = \rho_{AB_1(e_1)...B_{k+1}(e_{k+1})} \) where \( e \) runs over all permutations of \( k+1 \) indices. The state \( \rho_{AB_1...B_k B_{k+1}} \) is called a **strong k-rank symmetric extension** of \( \rho_{AB_1} \).

Definition II.2. A state \( \rho_{AB_1} \in B(\mathcal{H}_A \otimes \mathcal{H}_B) \) (\( \dim \mathcal{H}_A \leq \dim \mathcal{H}_B \)) is **weak k-extendible** if there exists such a state \( \rho_{AB_1...B_k B_{k+1}} \in B(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E) \) that \( \forall 1 \leq i \leq k+1 \rho_{AB_1} = \rho_{AB_i} \) where \( B_i \) is arbitrary chosen part
from B-part $B_1 \ldots B_k B_{k+1}$. The state $\rho_{AB_1 \ldots B_k B_{k+1}}$ is called a weak k-rank symmetric extension of $\rho_{AB_1}$.

Since there are considered only strong symmetric extendible states or invariance of symmetric extensions under permutations in the literature, it is important to show that they do not cover the whole family of states with extensions characterized by high symmetrization having important communication features, e.g. zero one-way distillable entanglement or zero secret key.

Remark. Noteworthy, in case of weak extensions it is necessary to assume $\dim H_A \leq \dim H_B$. Conversely, if we take the composition of symmetric extendible state $\rho_{AB}$ and a singlet state $\Phi AB = |\Phi + \rangle \langle \Phi +| AB$, then we achieve naturally a non-extendible state $\rho_{AB} \otimes \Phi_{AB}$. However, the three parties can construct a symmetrized version $\Gamma_{ABE} = \Phi_{AB} \otimes \rho_{ABE} \otimes \Phi_{AE}$ which is an extension of a state $\Gamma_{AB} = \Phi_{AB} \otimes \rho_{AB} \otimes id_A / n$ (where $\Gamma_{AB} = \Gamma_{AE}$ and naturally two pairs (Alice-Bob, Alice-Eve) can still distill a singlet in their pairs, i.e. $\Phi_{AB} \otimes \rho_{AB}$ and $\Phi_{AE} \otimes \rho_{AE}$.

The aforementioned definitions are equivalent locally for a particular $AB_i$ system, yet inequivalent globally as the extensions can vary depending on local phase factors. One can simply derive a strong symmetric extension $\overline{\rho}_{AB_1 B_2 \ldots B_k}$ from any weak extension $\rho_{AB_1 B_2 \ldots B_k}$ applying permutations on B-part:

$$\overline{\rho}_{AB_1 B_2 \ldots B_k} = \frac{1}{k!} \sum_\pi \rho_{AB_{\pi(1)} B_{\pi(2)} \ldots B_{\pi(k)}}$$

Example 1. The first class will represent strong k-rank symmetric extensions on $k + 2$ qudits:

$$|\Psi_{AB_1 \ldots B_{k+1}}\rangle = \frac{1}{d^{k+1}} \sum_{j=0}^{d-1} |j\rangle_A |\phi_j\rangle$$

where $|\phi_j\rangle = \alpha_j \sum_{\sigma} |0\rangle_{\sigma(1)} \ldots |0\rangle_{\sigma(j)} |1\rangle_{\sigma(j+1)} \ldots |1\rangle_{\sigma(k+1)}$ are states from symmetric space, $\alpha_j$ are normalization factors and $\sigma$ runs over all permutations of indexes without repetitions. This state possess full symmetry on Bobs’ side and for reduced states there holds: $\forall 1 \leq \ell \leq k \; \rho_{AB_\ell} = \rho_{AB_1}$.

The second class will represent weak k-rank symmetric extensions:

$$|\Psi_{AB_1 \ldots B_{k+1}}\rangle = \frac{1}{d^{k+1}} \sum_{j=0}^{d-1} |j\rangle_A |\phi_j\rangle$$

where $|\phi_j\rangle = \alpha_j \sum_{\sigma} e^{i\varphi_{ij}} |0\rangle_{\sigma(1)} \ldots |0\rangle_{\sigma(j)} |1\rangle_{\sigma(j+1)} \ldots |1\rangle_{\sigma(k+1)}$ where phase $\varphi_{ij}$ is arbitrary chosen for each summand in $|\phi_j\rangle$. This state possess broken symmetry on Bobs’ side due to phase factors introduced in $|\phi_j\rangle$ but for reduced states there still holds: $\forall 1 \leq \ell \leq k \; \rho_{AB_\ell} = \rho_{AB_1}$.

We will denote the set of all strong symmetric extendible states by $\mathcal{F}$ and the set of weak symmetric extendible states by $\mathcal{W}$. There holds the natural inclusion relation for the sets $\mathcal{F} \subseteq \mathcal{W}$. Although the discussion about differences between such extensions seems to be purely mathematical, in case of weak extensions one can think about Bobs differentiated by phase factors which might be potentially utilized by more global protocols on Bobs’s side e.g. applying optical devices for phase coding.

By 0-extendible states we will denote those that are not symmetrically extendible at all. One could note that it might be useful to partition the set of all symmetric extendible states $\mathcal{S}$ by relation of $k$-extendibility. If $\mathcal{S}_k$ denotes a convex set $[18]$ of all states being $k$-extendible, there holds the natural inclusion relation $[\text{Fig. 1}]:$

$$\mathcal{S}_1 \supset \mathcal{S}_2 \supset \ldots \supset \mathcal{S}_k$$

Of a great importance is the fact that for a given $\rho_{AB} \in \mathcal{S}$ there may exist different $k$-rank symmetric extensions so that the property is not unique and one could represent the set of appropriate symmetric extensions by means of equivalence classes given by the relation $B(H_A \otimes H_B) \ni \rho_{AB} \sim \rho \in B(H_A \otimes H_B^{(k+1)})$ if and only if $\rho$ is a k-rank symmetric extension of state $\rho_{AB}$. As the trivial example note that for $\rho_{AB} = \frac{1}{2} |\langle 00 | + |11\rangle\langle 11 ||$ at least the following are extensions of rank one: $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)$ and $\rho = \frac{1}{2} (|000\rangle\langle 000 | + |111\rangle\langle 111 |)$.

For $k$-extendible states it might be useful to introduce an operator swapping $k + 1$ particles:

$$P_\pi = \sum_{i_1 i_2 \ldots i_{k+1}} |i_1 i_2 \ldots i_{k+1}\rangle \langle \pi(i_1) \pi(i_2) \ldots \pi(i_{k+1}) |$$

where swapping is performed for an arbitrary permutation $\pi$. Hence, there holds a general relation for $k$-extendibility that explicitly derives set $\mathcal{S}_k$: $\forall \pi \; P_\pi \rho_{AB_1 \ldots B_k B_{k+1}} P_\pi^\dagger = \rho_{AB_1 \ldots B_k B_{k+1}}$.

Example 2. As a 1-extendible state we present $\rho_{AB} = \frac{1}{3} |\langle 00 | + |01\rangle\langle 01 | + |10\rangle\langle 10 |$ that obviously possess rank-1 symmetric purification to W-state $|W\rangle = \frac{1}{\sqrt{3}} (|001\rangle + |010 \rangle + |100\rangle)$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{The space of quantum states can be decomposed by the relation of k-extendibility. $\mathcal{S}_0$ denotes the set of all non-extendible states whereas $\mathcal{S}_k$ the set of states having $n$-rank symmetric extensions.}
\end{figure}
We could derive for this example a general form of n-extendible states in W-like n-partite states:

\[ T_{AB}(n) = \frac{n}{n+2} |00\rangle\langle 00| + \frac{2}{n+2} |\Phi_+\rangle\langle \Phi_+| \]  

(7)

where \(|\Phi_+\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle)\). Interestingly one can simply show that for e.g. GHZ-like n-partite states being a maximal extension of \(\rho_{AB} = \frac{1}{3} (|00\rangle\langle 00| + |11\rangle\langle 11|)\) there holds \(\rho_{AB} = \lim_{n \to \infty} T_{AB}(n)\) that is in agreement with theorems [3] stating implicitly that \(\rho\) is separable if and only if is \(\infty\)-extendible (where \(T_{AB}(n)\) is derived from n-partite GHZ state).

Following we present two different approaches to the problem of representation of weak symmetric extensions in extended space. The first approach is widely used in previous papers (see [1,3] on extendibility of quantum states. Every bipartite state \(\rho_{AB} \in B(\mathcal{H}_A \otimes \mathcal{H}_B)\) where \(\mathcal{H}_A = \mathbb{C}^n\) and \(\mathcal{H}_B = \mathbb{C}^n\) can be represented in the basis of generators of group \(SU(m) \otimes SU(n)\) as follows:

\[ \rho_{AB} = \Gamma \sigma_A^0 \otimes \sigma_B^0 + \sum_{i,j} \alpha_{ij} \sigma_A^i \otimes \sigma_B^j \]  

(8)

where \(\sigma_i^0\) are basis elements of \(SU(n)\) and respectively \(\sigma_A^i\) for \(SU(m)\). Elements of the basis satisfy relations:

\[ Tr[\sigma_i^0 \sigma_j^0] = \eta_i \delta_{ij} \text{ and } Tr[\sigma_i^0] = \delta_{i1} \text{ with } S = \{A,B\}. \]

Therefore, one could derive a general representation of all 1-rank symmetric extensions:

\[ \rho_{AB_{1}B_{2}} = \sum_{i,j \neq 0} \alpha_{ij} \sigma_A^i \otimes \sigma_B^i \otimes \sigma_A^j \]  

(9)

\[ + \sum_{i,j,k} \beta_{ijk} (\gamma \sigma_A^i \otimes \sigma_B^i \otimes \sigma_B^j + \sigma_A^i \otimes \sigma_B^i \otimes \sigma_B^j) \]

and further, for general \(k\)-extendible:

\[ \rho_{AB_1...B_{k+1}} = \sum_{i,j \neq 0} \alpha_{ij} \sigma_A^i \otimes \sigma_B^i \otimes ... \otimes \sigma_B^j_{B_{k+1}} \]  

(10)

\[ + \sum_{i_1 < i_2 < ... < i_{k+1}} \beta_{i_1i_2...i_{k+1}} \sigma_A^i \otimes \sigma_{B_1}^{i} \otimes ... \otimes \sigma_{B_{k+1}}^{i} \]

The latter approach that we will utilize in this paper is based on partitioning a space on which Bobs’ states operate into symmetric and antisymmetric subspace. Following we will prove some lemmas about Schmidt decomposition of k-rank pure symmetric states that supports in course of the paper more powerful theorem about properties of symmetric extendible states.

**Lemma II.3.** Let \(\rho_{AB_1} \in B(\mathcal{H}_A \otimes \mathcal{H}_{B_1})\) be symmetrically extendible to a k-rank pure extension \(\Psi_{AB_1...B_{k+1}} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}^{k+1}_{B_1}\) then there exists a Schmidt decomposition:

\[ \Psi_{AB_1...B_{k+1}} = \sum_i \alpha_i \phi_i^{AB_1} |\psi_i^{B_{2...k+1}}\rangle \]  

(11)

where \(\{|\phi_i^{AB_1}\rangle\}, \{|\psi_i^{B_{2...k+1}}\rangle\}\) are orthonormal and \(\{\phi_i^{AB_1}\}\) are symmetric sets of pure states so that \(\Psi_{AB_1...B_{k+1}}\) is symmetric extension.

**Proof.** Since:

\[ \forall_\pi I_{AB_1} \otimes P_\pi \Psi_{AB_1...B_{k+1}} = \pm \Psi_{AB_1...B_{k+1}} \]

where \(P_\pi\) operates only on \(B_2...B_{k+1}\) of the system, then \(\sum_i \alpha_i |\phi_i^{AB_1}\rangle \otimes |\psi_i^{B_{2...k+1}}\rangle = \pm \sum_i \alpha_i |\phi_i^{AB_1}\rangle |\psi_i^{B_{2...k+1}}\rangle\). However, the above Schmidt decomposition cannot be changed by any permutation due to symmetry of the pure state and \(|\phi_i^{AB_1}\rangle\) indexes uniquely the \(|\psi_i^{B_{2...k+1}}\rangle\) states so \(P_\pi\) transforms \(\Psi_{AB_1...B_{k+1}}\) onto itself. Therefore, the second multiplicands of Schmidt decomposition represent either symmetric or antisymmetric orthonormal states.

Basing on the above observation we put a general statement about structure of any k-rank extension, not necessarily possessing pure extensions.

**Lemma II.4.** For every k-rank symmetric state \(\rho_{AB_1...B_{k+1}} \in B(\mathcal{H}_A \otimes \mathcal{H}^{k+1}_{B_1})\) the following conditions are equivalent:

1. \(\forall_\pi S_{k+1} \{ I \otimes P_\pi \} |\rho_{AB_1...B_{k+1}}\rangle = \rho_{AB_1...B_{k+1}}\)

2. \(\exists \Lambda_A : \rho_{AB_1...B_{k+1}} = |\Lambda_A \otimes I_{B_1...B_{k+1}}\rangle \langle \Lambda_A|\Phi\langle \Phi|\)

where \(\Phi = \Phi_{AB_1...B_{k+1}}\) is a symmetric orthonormal state and \(\rho_{AB_1...B_{k+1}} = \sum_{i=1}^{N_S} \alpha_i |\phi_i\rangle \rangle \otimes \sum_{j=1}^{N_A} \beta_j |f_j\rangle \langle f_j|\)

where \(N_S = \binom{k+d}{d-1}\), \(N_A = \binom{d}{k+1}\) denote dimensions of symmetric and antisymmetric subspaces respectively. Further, \(\Lambda_A : B\langle\mathcal{H}_A\rangle \to B\langle\mathcal{H}_A\rangle\) denotes a quantum channel. The state fulfilling the conditions is k-rank symmetric extension of \(\rho_{AB_1...B_{k+1}}\).

**Proof.** The system in state \(\rho_{AB_1...B_{k+1}}\) can by viewed as a bipartite system consisting A-part and B = \(B_1...B_{k+1}\)-part. By Jamiołkowski isomorphism [12] there always exists such a quantum channel \(\Lambda_A : B\langle\mathcal{H}_A\rangle \to B\langle\mathcal{H}_A\rangle\) acting on A-part that \(\rho_{AB} = |\Lambda_A \otimes I\rangle \langle \Phi|\rangle\) where \(\Phi\) is a unique pure state. Since \(\Lambda_A\) acts only on A-part and the resulting state \(\rho_{AB} = \rho_{AB_1...B_{k+1}}\) is permutation invariant for B-part, the protocol is inherited by \(\Phi\) state. However, by virtue of (11) one observes that any rank-one operator \(\rho = |\Phi\rangle \langle \Phi|\) is \(\pi\)-invariant under \(\pi \in S_n\) being invariant under \(\pi \in S_{n}\) is supported either on symmetric or antisymmetric subspace of \(\mathcal{H}^n\) for B-part. Therefore, the following implication holds (1.) \(\Rightarrow\) (2.), for (2.) \(\Rightarrow\) (1.) the proof is obvious by symmetry of second multiplicands in Schmidt decomposition, namely, action of any permutation cannot change the structure of
the decomposition of the purification at most introducing change of sign and the state of the reduced system $B_1\ldots B_kB_{k+1}$ is invariant under $\pi$.

When in $\mathbb{H}$ spectral conditions for 1-rank symmetric extensions were stated, following we derive general statement about spectral conditions for k-extendible states basing on the observation about decomposition of symmetric states.

**Observation II.5.** Every pure normalized state $|\Psi\rangle \in \text{Sym}^{k+1}\bigoplus_\pi \text{Asym}^{k+1}(\mathcal{H}_{B_i})$ of k+1-partite system can be decomposed to the following Schmidt form:

$$\forall 1< i < k |\Psi\rangle = \sum_i |\phi_i^{B_1\ldots B_i}\rangle |\phi_i^{B_{i+1}\ldots B_{k+1}}\rangle$$

where the multiplicands form respectively symmetric or antisymmetric orthonormal sets.

**Proof.** One can conduct the proof similarly to (II.3). Since $\forall i \; P_{\pi}|\Psi\rangle < |\Psi\rangle$, then for all possible permutations the operation cannot change Schmidt decomposition of $\sum_i |\phi_i^{B_1\ldots B_i}\rangle |\phi_i^{B_{i+1}\ldots B_{k+1}}\rangle$. Furthermore, due to assumed symmetry property of $|\Psi\rangle$, a state of any subsystem $B_1\ldots B_i$ represented by the first multiplicand is permutationally invariant and the same is applied to the second multiplicand.

This observation with application of theorem II.4 can be effectively used to generate k-extendible states.

**Observation II.6.** Let $\rho_{AB_i}$ be k-extendable to a pure symmetric state $|\Psi\rangle_{AB_1\ldots B_{k+1}}$ then for ordered vectors of eigenvalues of $\rho_{AB_i}$ and $\rho_{B_2\ldots B_{k+1}}$ there holds:

$$\lambda_i^k(\rho_{AB_i}) = \lambda_i^k(\rho_{B_2\ldots B_{k+1}})$$

**Proof.** The proof is immediate applying Schmidt decomposition and results of (II.3).

### III. SYMMETRIC EXTENDIBILITY OF COMPOSITE SYSTEMS

In this section we explore symmetric extendibility of complex systems consisting of n pairs. All following statements can be applied both to weak and strong symmetric extendible states and the results are vital for protocols acting on multiple pairs of states.

For further results of the following section we will present a generalized version of a lemma [18] up to k-extendible maps stating that no matter what operation Alice and Bob can perform, the symmetric state shared between Alice and Bob will keep its symmetric extendibility. The following lemma indicates a natural fact that one cannot produce k-extendible state from n-extendible state $(n > k)$ by means of 1-LOCC $\Lambda_\alpha(\cdot)$ even if acts on any number of pairs:

**Lemma III.1.** Let $\Lambda_\alpha$ be a 1-LOCC quantum operation (not necessarily trace-preserving):

$$\Lambda_\alpha(\rho) = \sum_{ij} (I \otimes B_{ij})(A_i \otimes I)\rho(A_i \otimes I)^\dagger(I \otimes B_{ij})^\dagger$$

where $\sum_i A_iA_i^\dagger \leq I$ and $\sum_j B_jB_j^\dagger = I$ for all i since Bob cannot communicate the outcome of a probabilistic operation back to Alice. If $\rho$ is k-extendable state then $\Lambda_\alpha(\rho)$ is n-extendible and $n \geq k$.

One may state a non-trivial question if it is feasible to achieve symmetric extendibility of a composition of quantum states when at least one of them is not-symmetric extendible. The result of this question is crucial both for quantum security applications and measuring quantum entanglement. The following theorem casts some light on this field:

**Lemma III.2.** If $\rho_{AB} \in B(\mathcal{H}_A^N \otimes \mathcal{H}_B^M)$ is not symmetrically extendible state then there does not exist any such a state $\rho_{A'}B'$ in $B(\mathcal{H}_A^K \otimes \mathcal{H}_B^L)$ that $\rho_{AB}$ would be symmetrically extendible in respect to $BB'$ subsystem.

**Proof.** Conversely, let $\rho_{A'B'B'} = \rho_{AB} \otimes \rho_{A'B'}$ be a symmetrically extendible state acting on $B(\mathcal{H}_A^K \otimes \mathcal{H}_B^L \otimes \mathcal{H}_A^N \otimes \mathcal{H}_B^M)$. Therefore, one notes that $\rho_{A'B'B'}$ after swapping to $\rho_{AA'B'B}$ can be represented by method [8] in an appropriate basis including generators of group $SU(N) \otimes SU(K) \otimes SU(M) \otimes SU(L)$ and further, can be extended to a 1-rank symmetric extension $\rho_{AA'B'B'B}$ where we extend $BB'$ part as follows:

$$\rho_{AA'B'B'B} = \sum_{ijkl} \alpha_{ijkl}T_{ijklkl} + \sum_{ijklmn} \beta_{ijklmn}T_{ijklmn} + T_{ijklkl}$$

with tensors $T_{ijklmn} = \sigma^i \otimes \sigma^j \otimes \sigma^k \otimes \sigma^m \otimes \sigma^n$. Following we derive the state $\rho_{A'B'B}$ of system $AB$ tracing out that of $A'B'B'$. For the fact that $Tr[\sigma^i \otimes \sigma^j \otimes \sigma^k] = Tr(\sigma^i)Tr(\sigma^j)Tr(\sigma^k)$ and $Tr[\sigma^0] = \delta_{11}$ after tracing out only elements with $\sigma^0 = I$ remain, namely, one obtains:

$$\rho_{AB} = \sum_{ik} \alpha_{ik} T_{ikikkl} + \sum_{iklm} \beta_{iklm}T_{iklmkl} + T_{iklmkl}$$

Hence, $\rho_{AB}$ is 1-rank symmetric extension of $\rho_{AB}$ that is in contradiction with the assumption that the latter is not symmetrically extendible.

**Corollary III.3.** If $\rho_{AB} \in B(\mathcal{H}_A^N \otimes \mathcal{H}_B^M)$ is at most k-extendible state then there does not exist any such a state $\rho_{A'B'} \in B(\mathcal{H}_A^K \otimes \mathcal{H}_B^L)$ that $\rho_{AB} \otimes \rho_{A'B'}$ would be k+1-extendible in respect to $BB'$ subsystem.
Lemma III.4. Assume that $\rho_{AB} \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ is not symmetric extendible and there exists a local operation $\tilde{\sigma}$ acting on $A$-part such that $\tilde{\sigma} = (F \otimes id)\rho_{AB}(F^\dagger \otimes id)/Tr[(F \otimes id)\rho_{AB}(F^\dagger \otimes id)]$ is a symmetric extendible state.

Then for any local operations $\mathcal{A}$ and $\mathcal{B}$ acting on $A$ and $B$ part of the system:

$$\mathcal{A} = U \begin{pmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_i \\ \end{pmatrix} U^\dagger$$

$$\Lambda(\rho_{AB}) = \frac{\mathcal{A} \otimes \mathcal{B}\rho_{AB}\mathcal{A}^\dagger \otimes \mathcal{B}^\dagger}{Tr(\mathcal{A} \otimes \mathcal{B}\rho_{AB}\mathcal{A}^\dagger \otimes \mathcal{B}^\dagger)}$$

where for all $i \geq 0$, $\alpha_i \leq 1$ and $U$ denotes an unitary operation ($\mathcal{B}$ has a corresponding structure), there exists a local operation $\tilde{F}$ such that $\tilde{\sigma} = (\tilde{F} \otimes id)\Lambda(\rho_{AB})(\tilde{F}^\dagger \otimes id)/Tr[(\tilde{F} \otimes id)\Lambda(\rho_{AB})(\tilde{F}^\dagger \otimes id)]$ is symmetric extendible and $\dim \tilde{F} = \dim F$.

Proof. To prove this lemma, it suffices to note that $\mathcal{A} = UD\mathcal{U}^\dagger$ with a diagonal $D$. Further, we observe that $\tilde{F} = F \circ U D\mathcal{U}^\dagger$ where $D' = id$. The latter is possible due to the condition that for all $i$ there holds: $0 < \alpha_i \leq 1$ and we easily observe that $F = \tilde{F} \circ \mathcal{A}$. This brings us to conclusion that $\tilde{F}\Lambda(\rho_{AB})\tilde{F}^\dagger$ is symmetric extendible operator (after normalization becoming a physical state). For $B$-part the proof can be conducted in a similar manner as in particular, the local operation $\mathcal{B}$ is also reversible.

Remark. It naturally holds for both weak and strong case of symmetric extensions since symmetry on Bobs side does not influence local operations of Bobs separately. It also casts some light on a fact that local operations actually does not change the amount of symmetric extendibility embedded in a state.

This lemma is of a great importance for private security and entanglement distillation studies, as we can always build a symmetric extension $\Gamma_{ABE}$ of a state $\sigma_{AB}$ which means that Eve potentially has a state $\rho_E = \rho_B$ and operates on such a space. To support this statement one can further derive the corollary about extendibility of any quantum state with a proposal of new extendible number of a quantum state:

Definition III.5. For any $\rho_{AB}$, $\eta_{SE}(\rho_{AB}) = \max_{\mathcal{F}} \dim \mathcal{F}$ (where $(\mathcal{F} \otimes id)\rho_{AB}(\mathcal{F}^\dagger \otimes id)$ is a symmetric extendible operator) is called the extendible number of a state $\rho_{AB}$.

Corollary III.6. Any state $\rho_{AB} \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ with extendible number $\eta_{SE}$ can be extended to a state $\rho_{ABE} \in B(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)$ ($\dim \mathcal{H}_B = \dim \mathcal{H}_E$) where exists filtering operation $\mathcal{F}$ on $A$ so that $(\mathcal{F} \otimes id)\rho_{ABE}(\mathcal{F}^\dagger \otimes id)$ is invariant due to permutation of $B$ and $E$.

Naturally, there holds: if $\eta_{SE}(\rho_{AB}) = \text{rank}(\rho_A)$, then the state is symmetric extendible.

One may raise further a very important question how to create the property of symmetric non-extendibility both in case of single states and collective systems using only local operations or additionally one-way communication that naturally will have implications for distillability and capacities of corresponding states and channels.

Lemma III.7. Let $\rho_{AB} \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a state possessing at most $k$-rank symmetric extension where $k < \infty$ then there does not exist any 1-LOCC protocol represented by $\Lambda_{A \rightarrow BC} : B(\mathcal{H}_{ABC}) \rightarrow (\mathcal{H}_{ABC})$ (not necessarily trace-preserving):

$$\Lambda_{A \rightarrow BC}(\rho_{AB} \otimes \sigma_C) = \tilde{\rho}_{ABC}$$

so that $\tilde{\rho}_{ABC}$ is a symmetric extension of $\rho_{AB}$ and $\sigma_C \in B(\mathcal{H}_C)$ is an additional resource on Bob’s side.

Proof. Since $\rho_{AB}$ is k-extendible, one can assume that its symmetric extension is realized to $\rho_{ABB_1...B_k}$ but $B_1\ldots B_K$-part is possessed by Eve. Obviously no communication between Eve and Bob in such a scenario is allowed so that Bob cannot detect locally Eve and further, since the set of symmetric extendible states is closed under $1 – \text{LOCC}$ operations [6] even if Alice and Bob had engaged one-way communication they cannot break symmetric extendability of $\rho_{AB}$ and so cannot eliminate Eve if the symmetric extension had been realized.

Therefore, assuming that on the contrary $\Lambda_{A \rightarrow BC}$ enables creation of a symmetric extension:

$$\Lambda_{A \rightarrow BC} \otimes \text{id}_{B_1...B_k}(\rho_{ABB_1...B_k} \otimes \sigma_C) = \Omega$$

resulting state $\Omega$ would be k+1-symmetric extension of $\rho_{AB}$ that contradicts the lemma’s assumption about extendibility of this state and completes the proof.

Remark. The aforementioned statements holds as well in asymptotic regime due to results of [11] that can be extended for an infinite case.

As a result of the above lemmas we can conclude that in general for creation of any symmetric extension one needs to engage two-way communication.

IV. LOCKING NON-SYMMETRIC EXTENDIBILITY

The general idea of locking a property of a quantum state relates to the loss or decrease of this property subjected to a measurement or discarding of one qubit. It has been shown [14, 25] that entanglement of formation, entanglement cost and logarithmic negativity are lockable measures which manifests as an arbitrary decrease of those measures after measuring one qubit.

Herewith, we analyze in fact locking of non-symmetric extendibility in sense that discarding one qubit from the
quantum state that is not symmetric extendible leads to the loss of this property. Further, we derive implications for quantum security applying one-way communication between engaged parties Alice and Bob.

We shall show now that the property of non-symmetric extendibility of an arbitrary state $\rho_{AB}$ can be destroyed by measurement of one qubit and in result it presents how easily a quantum state can be removed of one-way distillability and security.

Let us consider bipartite quantum state shared between Alice and Bob on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \cong \mathbb{C}^{d+2} \otimes \mathbb{C}^{d+2}$

$$\rho_{AB} = \frac{1}{2d-1} \begin{bmatrix} dP_+ & 0 & 0 & A \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A^\dagger & 0 & 0 & \sigma \end{bmatrix}$$

(19)

where $P_+$ is a maximally entangled state on $\mathbb{C}^d \otimes \mathbb{C}^d$, $\sigma = \sum_{i=0}^{d-1} |i \ 0\rangle \langle i \ 0|$ and $A$ is an arbitrary chosen operator so that $\rho_{AB}$ represents a correct quantum state. This matrix is represented in the computational basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ held by Alice and Bob and possess a singlet-like structure. Whenever one party (Alice or Bob) measures the state in the local computational basis, the state decoheres and off-diagonal elements vanish which leads to a symmetric extendible state [15]:

$$\mathcal{Y}_{AB} = \frac{d}{2d-1} P_+ + \frac{1}{2d-1} \sum_{i=1}^{d-1} |i \ 0\rangle \langle i \ 0|$$

(20)

from which no entanglement nor secret key can be distilled by means of one-way communication and local operations. Clearly this example shows that from a non-symmetric extendible state possessing large entanglement cost and non-zero secret key one can easily obtain a symmetric structure by discarding small part of the whole system destroying possibility of entanglement distillation and secret key generation by means of 1-LOCC.

Thus, it is interesting to consider how much of symmetric extendibility is locked in a given state $\rho_{AB}$ as it can be expected that the more symmetric extendibility is hidden in a state, the less vulnerable for losses of one-way distillability and security it is. Besides analysis of symmetric structures in projected subspaces, we will also propose to perform this task by means of the best symmetric extendible approximation [8][20] that decomposes the state into a symmetric extendible component $\sigma_{\text{ext}}$ and non-symmetric extendible component $\sigma_{\text{next}}$:

$$\rho_{AB} = \max_{\lambda} \lambda \sigma_{\text{ext}} + (1-\lambda) \sigma_{\text{next}}$$

(21)

We denote by $\lambda_{\text{max}}(\rho)$ the maximum weight of extendibility [8] of $\rho_{AB}$ where $0 \leq \lambda_{\text{max}}(\rho) \leq 1$, thus, all symmetric extendible states have the weight $\lambda_{\text{max}} = 1$. It is proved in [5][8] that in case of one-way protocols only the non-symmetric extendible component can be effectively utilized for generation of a secret key and it confirms that the notion of symmetric extendibility is crucial for consideration of one-way entanglement and key distillation.

However, we show that there exist states which do not possess any symmetric extendible component in the aforementioned decomposition but there can be a large symmetric extendible component embedded in them. An example of such a state is given above [19] and one can derive the following statement about general structure of such states:

**Lemma IV.1.** Consider a state $\gamma$ on $\mathcal{H}_{AA'BB'}' = \mathcal{H}_A \otimes \mathcal{H}_A' \otimes \mathcal{H}_B \otimes \mathcal{H}_B' \sim \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$:

$$\gamma = \rho \otimes \sigma$$

(22)

being a composition of an arbitrary chosen state $\sigma \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ and a non-symmetric extendible state $\rho \in B(\mathcal{H}_A' \otimes \mathcal{H}_B')$ with no symmetric extendible component $\lambda_{\text{max}}(\rho) = 0$. Then for the best extendible approximation of $\gamma$ there holds $\lambda_{\text{max}}(\gamma) = 0$, i.e. there is no symmetric extendible component in $\gamma \in B(\mathcal{H}_{AA'BB'}')$.

**Proof.** Conversely, assume that there exists decomposition of $\gamma_{AA'BB'}'$ with non-zero symmetric extendible component, i.e. $\lambda \neq 0$:

$$\gamma_{AA'BB'}' = \lambda \sigma_{\text{ext}} + (1-\lambda) \rho_{\text{ne}}$$

(23)

then both components would be supported on $\mathcal{H}_{AA'BB'}'$ and one can search for a decomposition of $\gamma_{AA'BB'}'$ after tracing out $A'B'$-part. Due to additivity of a partial trace operation $\Gamma_X(\cdot) = Tr_X(\cdot)$ we obtain:

$$\Gamma_{A'B'}(\gamma_{AA'BB'}') = \lambda \Gamma_{A'B'}(\sigma_{\text{ext}}) + (1-\lambda) \Gamma_{A'B'}(\rho_{\text{ne}})$$

(24)

and, further, basing on a symmetric extendibility property of composite systems [13] one derives that tracing out $A'B'$ from $\sigma_{\text{ext}}$ does not destroy its symmetric extendibility and produces symmetric extendible state $\tilde{\sigma}_{\text{ext}}$:

$$\rho = \lambda \tilde{\sigma}_{\text{ext}} + (1-\lambda) \rho_{\text{ne}}$$

(25)

Thus, the initial assumption would imply existence of a non-zero symmetric extendible component of the state $\rho$ that contradicts the aforementioned decomposition. □

Following one can make an immediate observation about any private quantum state [13]:

**Corollary IV.2.** Any private quantum state $\gamma_{ABA'B'} \in B(\mathcal{H}_{ABA'B'})$:

$$\gamma_{ABA'B'} = \frac{1}{2} \sum_{i,j=0}^{1} |ii\rangle \langle jj| \otimes U_{i} \rho_{ABA'B'} U^\dagger_{i}$$

(26)

does not possess symmetric extendible component, i.e. $\lambda_{\text{max}} = 0$. 
Remark. The proof is conducted in analogy to the proof of [IV.1] but this state represents a twisted composition of singlet and an arbitrary chosen state \( \sigma \) where AB-part is the key part of the state and is not symmetric extendible due to the observation that secure states cannot be symmetric extendible [6].

Basing on previous studies of entanglement measures and importance of symmetric extendible states, we introduce the following best symmetric approximated entanglement monotone (as a counterpart of BSA in [20]):

**Proposition IV.3.** For any \( \rho \in B(\mathcal{H}_A \otimes \mathcal{H}_B) \) having best symmetric decomposition \( \rho_{AB} = \max_\lambda \lambda \sigma_{ext} + (1 - \lambda) \sigma_{next} \), best symmetric approximated entanglement monotone is defined as:

\[
E^{ss}(\rho) = 1 - \lambda_{\max}(\rho)
\] (27)

**Proof.** (We will prove that the quantity meets necessary conditions to be an entanglement monotone.)

1. If \( \rho \) is separable, i.e. also symmetric extendible, then \( \lambda_{\max} = 1 \) and \( E^{ss}(\rho) = 0 \).
2. \( E^{ss}(\rho) \) is invariant under local unitary operations since application of local operations \( U_A \) and \( U_B \) on \( \sigma_{ext} \) is extendible to the third part \( B' \).
3. For any local POVM \( V_i \), there holds:

\[
1 - \lambda_{\max}(\rho) \geq \sum_i (1 - \lambda_i^{\max} (\rho_i) Tr(V_i \rho V_i^\dagger))
\]

\[
\geq \sum_i E^{ss}(\rho_i) Tr(V_i \rho V_i^\dagger))
\]

and \( \rho_i = V_i \rho V_i^\dagger / Tr(V_i \rho V_i^\dagger) \). \( \square \)

It is interesting to notice that for two-qubit states on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) there holds a non-trivial observation about best symmetric approximated decomposition:

\[
\rho = \lambda \sigma_{ext} + (1 - \lambda) |\Psi\rangle \langle \Psi|
\] (28)

with \( \sigma_{ext} \) being a symmetric extendible component that appears in \( \rho \) with highest probability. The proof of this observation can be based on BSA with separable components [20] where \( \rho = \alpha \sigma_{sep} + (1 - \alpha) |\Psi\rangle \langle \Psi| \). As set of separable states is a subset of the convex set of symmetric extendible states, then for any dimension \( \alpha \leq \lambda \). Further, due to the fact that any two-qubit state has best separable decomposition into a separable and projective entangled component, we conclude that \( \lambda \sigma_{ext} = \alpha \sigma_{sep} + \beta |\Psi\rangle \langle \Psi| \) for arbitrary chosen \( \beta \).

These propositions can simplify potentially many research problems like analysis of CHSH regions vs. symmetric extendibility of states [21] represented in the steering ellipsoid formalism or just further analysis on security and distillability of all \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) states.

Following the results of [23], one can immediately propose max-relative entropy monotone based on this decomposition, i.e. \( D_{max}(\sigma \| \rho) \equiv \log \min \{ \lambda : \sigma \leq \lambda \rho \} \) and supp \( \rho \subset \text{supp} \rho \) with max-relative entropy being interpreted as a probability of finding \( \sigma \) in decompositions of \( \rho \). This leads immediately to \( \lambda = \max(2^{-D_{max}(\sigma_{ext} \| \rho)}) \).

An open question is: whether for one-way distillable entanglement we can state that \( D_{\rightarrow}(\rho) \leq (1 - \lambda_{\max}(\rho)) D_{\rightarrow}(\sigma_{next}) \)?

**V. IMPLICATIONS FOR ONE-WAY ENTANGLEMENT DISTILLABILITY AND PRIVATE KEY**

Studies on symmetric extendibility in a context of measures of entanglement like squashed entanglement [23, 26], security of quantum protocols [6] and quantum maps gain a substantial interest. Recently a great attention has been paid to so called k-extendible maps [22, 27, 28] and recovery maps [29, 30] where it is proved that small value of squashed entanglement implies closeness to highly extendible states. These results show importance of symmetric extendibility notion for analysis of one-way quantum communication rates. Inspired by these findings, we propose further an important conjecture about distillability of all non-symmetric extendible states and analyze behavior of a secret key rate in a neighborhood of symmetric extendible states.

Basing on theory of entanglement distillability we state the following conjecture in domain of one-way communication linking it directly with symmetric extendibility of quantum states:

**Conjecture V.1.** Any state \( \rho_{AB} \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \) is one-way distillable if and only if there exists a two-dimensional projector \( P : \mathcal{H}_A \to \mathbb{C}^2 \) such that for some \( n \geq 1 \) the state:

\[
\tilde{\rho}_{AB} = (P \otimes \text{id} ) \rho_{AB} (P \otimes \text{id})^\dagger
\] (29)

is not symmetrically extendible.

For a potential proof, it is an immediate observation that one-way distillable quantum states cannot be symmetric extendible [15], yet it is an open question if there exists a two-qubit state that is not at the same time symmetric extendible nor one-way distillable. Since we know conditions for symmetric extendibility of two-qubit states [4, 7], this conjecture if true would simplify analysis of entanglement of two-qubit states and capacity of channels acting on such spaces substantially. On the contrary, if there exist two-qubit states that are neither symmetric extendible nor one-way distillable then they would be one-way counterparts of bound entangled states for two-way distillability in higher dimensions. An analysis of this subject seems to be of a great importance for further studies on quantum secure protocols and structure of entanglement.

As an example, it is worth mentioning Werner states [31] and the hypothesis about NPT bound entangled states [32, 33]. The structure of the Werner states is as follows:

\[
\rho_W(\alpha) = \frac{id + \alpha P}{d^2 + \alpha d}
\] (30)
where $\mathbb{P} = \sum_{i,j=0}^{d-1} |ij\rangle\langle ij|$. The state is separable for $1 \geq \alpha \geq -\frac{1}{d}$, NPT for $-\frac{1}{d} > \alpha \geq 1$ and two-way 1-distillable for $-\frac{1}{2} > \alpha \geq -1$. Applying the conditions for symmetric extendibility \cite{2}, we found that for $d = 2$, the state is non-symmetric extendible for $-0.8 \geq \alpha \geq -1$. We analyzed potential one-way distillability of the state for the region of non-symmetric extendible Werner states with non-positive coherent information, namely for $-0.8 \geq \alpha > \alpha_{E} = -0.85559$. The latter condition excludes all those states being distilled by well-known one-way hashing protocol. The analysis was performed for two-copies of the state and over $10^{8}$ random filtering operations on Alice’ side and random unitary operations on Bob’s side. However, the protocol was not able to distill states with positive coherent information which suffices to distill entanglement with the hashing protocol. Therefore, it is an open question if the state is one-way distillable in the region $-0.8 \geq \alpha > \alpha_{E} = -0.85559$ or it is one-way ‘bound entangled’ which would be a counterpart of bound entanglement concept in two-way communication domain.

As all symmetric extendible state do not posses any private key, we can expect that in close neighborhood to the set of such states all other states can have only a small amount of distillable private key. That would have to be true assuming at least local continuity of private key $K_{\alpha}(\cdot)$ in such a neighborhood. To analyze this subject, we start reminding an important theorem about entropic inequalities for conditional entropies of sufficiently close states in terms of a trace norm:

**Theorem V.2.** \cite{3} For any two states $\rho_{AB}$ and $\bar{\rho}_{AB}$ on $\mathcal{H}_{AB} = \mathcal{H}_{A} \otimes \mathcal{H}_{B}$, let $\epsilon = \|\rho_{AB} - \bar{\rho}_{AB}\|_{1}$ and let $d_{A}$ be the dimension of $\mathcal{H}_{A}$, then the following estimate holds:

$$|S(\rho_{AB}) - S(\bar{\rho}_{AB})| \leq 4\epsilon \log d_{A} + 2\eta(1 - \epsilon) + 2\eta(\epsilon)$$

In particular, the right hand side of \cite{3} does not explicitly depend on the dimension of $\mathcal{H}_{B}$.

Further, to generate a secret key between Alice and Bob one can use \cite{8,9} a general tripartite pure state $\rho_{ABE}$. Alice engages a particular strategy to perform a quantum measurement (POVM) described by $Q = (Q_{x})_{x \in X}$ which leads to: $\rho_{ABE} = \sum_{x} |x\rangle \langle x|_{A} \otimes Tr_{A}(\rho_{ABE}(Q_{x}) \otimes I_{BE})$. Therefore, starting from many copies of $\rho_{ABE}$ we obtain many copies of cqg-states $\bar{\rho}_{ABE}$ and we restate the theorem defining one-way secret key $K_{\rightarrow}$:

**Theorem I.** \cite{3} For every state $\rho_{ABE}$, $K_{\rightarrow}(\rho) = \lim_{n \rightarrow \infty} K_{\rightarrow}^{(n)}(\rho)$, with $K_{\rightarrow}^{(n)}(\rho) = \max_{Q,T,I} I(X : B|T) - I(X : E|T)$ where the maximization is over all POVMs $Q = (Q_{x})_{x \in X}$ and channels $R$ such that $T = R(X)$ and the information quantities refer to the state: $\omega_{T,R,B} = \sum_{t,x} P(t|x)T(t|t)T(t|x)_{A} \otimes Tr_{A}(\rho_{ABE}(Q_{x}) \otimes I_{BE})$. The range of the measurement $Q$ and the random variable $T$ may be assumed to be bounded as follows: $|T| \leq d_{B}^{2}$ and $|X| \leq d_{A}^{2}$ where $T$ can be taken a (deterministic) function of $X$.

Basing one the above results we will prove continuity of the quantity $K_{\rightarrow}^{(1)}(\rho)$ for one copy of a state $\rho$ and further, consider behavior of the measure in the asymptotic regime.

**Lemma V.3.** For any two states $\rho$ and $\bar{\rho}$ on $\mathcal{H}_{AB} = \mathcal{H}_{A} \otimes \mathcal{H}_{B}$, let $\epsilon = \|\rho - \bar{\rho}\|_{1}$, and let $d_{A}$ be the dimension of $\mathcal{H}_{A}$, then the following estimate holds:

$$|K_{\rightarrow}^{(1)}(\rho) - K_{\rightarrow}^{(1)}(\bar{\rho})| \leq 8\epsilon \log d_{A} + 4\eta(1 - \epsilon) + 4\eta(\epsilon)$$

**Proof.** One can put for the quantity $K_{\rightarrow}^{(1)}(\rho) = S(\rho_{AB}) - S(\bar{\rho}_{AB}) - S(\rho_{AC}) + S(\bar{\rho}_{AC}) = -S(\rho_{AB}) + S(\bar{\rho}_{AB})$ and respectively for $\bar{\rho}$ there holds $K_{\rightarrow}^{(1)}(\bar{\rho}) = -S(\rho_{AB}) + S(\bar{\rho}_{AB})$. Further, engaging the results of \cite{31} it is easy to conduct the following implications for a chain of inequalities:

$$|K_{\rightarrow}^{(1)}(\rho) - K_{\rightarrow}^{(1)}(\bar{\rho})| =$$

$$\leq \|S(\rho_{AB}) - S(\bar{\rho}_{AB})\|_{1} + \|S(\rho_{AC}) - S(\bar{\rho}_{AC})\|_{1}$$

$$\leq 2\epsilon \log d_{A} + 2\eta(1 - \epsilon) + 2\eta(\epsilon)$$

Since it is not possible to distill any secret key by means of one-way communication and local operations from all symmetric extendible states, one can easily derive the following:

**Corollary V.4.** For any state $\rho$ on $\mathcal{H}_{AB} = \mathcal{H}_{A} \otimes \mathcal{H}_{B}$ being in distance $\epsilon$ to the nearest symmetric extendible state $\bar{\sigma}$ in sense of a trace norm: $\epsilon = \inf_{\bar{\sigma} \in \Omega} \|\rho - \bar{\sigma}\|_{1}$ where $\Omega$ denotes a convex set of symmetric extendible states on $\mathcal{H}_{AB}$, there holds:

$$K_{\rightarrow}^{(1)}(\rho) \leq 8\epsilon \log d_{A} + 4\eta(1 - \epsilon) + 4\eta(\epsilon)$$

**Example 3.** As an example of application of the above corollary we will consider two states very close to one another in sense of a trace norm $\|\cdot\|_{1}$ from which one is symmetric extendible and the another is non-symmetric extendible. This shows that for one-copy applications the theorem can be used operationally to estimate one-way secret key rate of quantum states. Following results of \cite{12}, let us consider two arbitrary instances of a state on $\mathcal{H}_{AB} \cong \mathbb{C}^{d} \otimes \mathbb{C}^{d}$:

$$\Upsilon(\epsilon) = \frac{d}{2d - 1} + \epsilon/2 + \frac{1}{2d - 1} - \frac{\epsilon}{2d - 1} \sum_{i = 1}^{d - 1} |i\rangle\langle i|$$

(34)

which is non-symmetric extendible for $\epsilon > 0$. Namely, one can put into the inequality \cite{32} two states $\Upsilon(\epsilon = 0)$ and $\Upsilon(\epsilon > 0)$. Since for all symmetric extendible states $\rho$ there holds: $K_{\rightarrow}^{(1)}(\rho) = 0$, then:

$$K_{\rightarrow}^{(1)}(\Upsilon(\epsilon > 0)) \leq 8\epsilon \log d_{A} + 4\eta(1 - \epsilon) + 4\eta(\epsilon).$$
where $\epsilon \leq \frac{2(d_A-1)}{2d_A-1}$.

It is proved [31] that in any open set of distillable states, all asymptotic entanglement measures $E(\rho)$ are continuous as a function of a single copy of $\rho$, even though they quantify the entanglement properties of $\rho^{\otimes N}$ in the large $N$ limit. However, the aforementioned theorem does not cast any light on the behavior of function $K_\rightarrow(\cdot)$ on the boundary of a set of all one-way distillable states adjacent to symmetric extendible states just due to the open conjecture V.1. Motivated by this insight we put an open question 

**Conjecture V.5.** For any state $\rho$ on $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ being in distance $\epsilon$ to the nearest symmetric extendible state $\tilde{\rho}$ in sense of a trace norm: $\epsilon = \inf_{\rho \in \Omega} \| \rho - \tilde{\rho} \|_1$ where $\Omega$ denotes a convex set of symmetric extendible states on $\mathcal{H}_{AB}$, there holds:

$$K_\rightarrow(\rho) \leq 8\epsilon \log d_A + 4\eta(1-\epsilon) + 4\eta(\epsilon) \quad (35)$$

**VI. CONCLUSIONS**

The theory of symmetric extendible states being crucial for analysis of one-way distillability and security of quantum states has still many unsolved problems. In this paper we introduced some new concepts related to classification of all symmetric extendible states and analyzed mainly composite systems including also a symmetric extendible part. As showed in the paper, beside analysis of best symmetric extendible decompositions it might be very useful to analyze a maximal symmetric extendible state that can be achieved by filtering on Alice’ side. We studied also behavior of private key in neighborhood of symmetric extendible states showing that for one-copy a quantum state close to symmetric extendible state can possess only a small amount of private key. One of the most intriguing open question relates to the conjecture about one-way distillability of all two-qubit states which are not symmetric extendible. In consequence, that would simplify substantially full characterization of two-qubit states in terms of their privacy and distillability. Finally, in relation to this question we analyzed Werner states in the domain of non-positive coherent information which would indicate one-way NPT bound entangled features in case the conjecture is not true.

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