Structured UH model considering induced anisotropy

Enyang Zhu*, Shaokun Wang
School of Civil Engineering, North China University of Technology, Beijing 100144, China
*zhuenyang@ncut.edu.cn

Abstract: Based on the structured unified hardening model (structured UH model), introducing rotational hardening, a structured UH model considering induced anisotropy is presented. Applying the transformed method, the presented model is extended in three-dimensional stress space. Comparing the test data with the model simulations, the structured UH model considering induced anisotropy is qualified to smoothly and continuously describe the behaviors of natural soils in cyclic loading.

1. Introduction
Both of soil structure and induced anisotropy are widely existed in natural soils. And soil structure was regarded as a core problem of soil mechanics in the 21st century[1]. In the 1990s, an elasto-plastic damage model for cemented clays was presented, and the model was simplified later[2][3]. In 1944, soil was regarded as anisotropic, and the anisotropy was divided into two types: inherent anisotropy and induced anisotropy[4]. The induced anisotropy is caused by the unequal stress of soil, which play a major role in soft clay[5]. Both soil structure and induced anisotropy have a great influence on the behaviors of natural soil, so it is necessary to build a constitutive model that is qualified to reflect both of soil structure and induced anisotropy.

Based on the Camclay model, a unified hardening model (UH model)[6][7] was developed to describe the over-consolidation behaviors of clay. Based on the UH model, several constitutive models have been developed, such as: UH model for unsaturated soils[8], UH model considering temperature effects[9], elastic-viscous-plastic model for over-consolidated clays[10], UH model considering expansion of soils[11]. Among the models developed from the UH model, the structured UH model[12] describes the soil structure, and the dynamic UH model[13] describes the induced anisotropy. Thus in this paper, a model considering both soil structure and induced anisotropy is established.

2. Model establishment

2.1. Review of the structured UH model
In the structured UH model, the normal compression line (NCL) was extended to a moving normal compression line (MNCL). An internal variable structure potential $\Delta e$, the vertical distance of MNCL and NCL, is applied to reflect soil structure. With loading, MNCL moves down and tends to NCL finally, as is shown in Figure 1. Compared with the UH model, two model parameters are added in structured UH model, one is initial structure potential $\Delta e_0$ and the other is structure decay rate $\zeta$. 

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In the structured UH model, the current yield function $f$ and reference yield function $\tilde{f}$ are:

$$f = c_p \left[ \ln \frac{p}{p_{x_0}} + \ln \left( 1 + \frac{\eta^2}{M^2} \right) - c_p \ln \frac{p_x}{p_{x_0}} \right] = 0 \quad (1)$$

$$\tilde{f} = c_p \left[ \ln \frac{p}{p_{x_0}} + \ln \left( 1 + \frac{\eta^2}{M^2} \right) - c_p \ln \frac{p_{\tilde{x}}}{p_{x_0}} \right] = 0 \quad (2)$$

where $c_p = (\lambda - \kappa)/(1 + e_0)$; $\lambda$ and $\kappa$ are respectively isotropic compression and swelling indexes; $e_0$ is initial void ratio; stress ratio $\eta = q/p$; $p$ is mean principal stress; $q$ is shear stress; $M$ is critical stress ratio; $p_x$ and $\tilde{p}_x$ are respectively the intercepts of the right end of yielding surface on $p$-axe; With the initial values $p_{x_0}$ and $\tilde{p}_{x_0}$, $p_x$ and $\tilde{p}_x$ evolve as:

$$p_x = p_{x_0} \exp \left( f \frac{d\varepsilon}{c_p} \right) \quad (3)$$

$$\tilde{p}_x = \tilde{p}_{x_0} \exp \left( f \frac{d\varepsilon}{c_p} \right) \quad (4)$$

where $H$ is the hardening parameter, and is related with the plastic volumetric strain $\varepsilon^p$.

$$H = \int \frac{d\varepsilon^p}{\Omega} \quad (5)$$

The evolutions of the internal variables are:

$$\frac{1}{\Omega} = R \Delta \varepsilon_0 (\eta - M) . M^4 - \eta^4 \quad (6)$$

$$M_f = 6 \left[ \sqrt{\frac{\chi}{R}} \left( \frac{1 + \chi}{R} - \frac{\chi}{R} \right) \right] \quad (7)$$

$$\chi = M^2 / \left[ 12 (3 - M) \right] \quad (8)$$

$$R = \frac{p_x}{p_{x_0}} = \frac{p \left( 1 + \frac{\eta^2}{M^2} \right)}{p_{x_0} \exp \left( - f \frac{d\varepsilon}{c_p} - f \frac{d(\Delta\varepsilon)}{\lambda - \kappa} \right)} \quad (9)$$

$$d(\Delta\varepsilon) = -\zeta R \Delta \varepsilon (c_p d(\ln p_x)) \quad (10)$$

In equation (6) to (10): $R$ is the ratio of the size of the current yield surface to the reference yield surface; $\Delta \varepsilon_0$ is the initial value of $\Delta \varepsilon$, $d(\Delta\varepsilon)$ is the increment of $\Delta \varepsilon$; $M_f$ is the potential failure stress ratio; $\zeta$ is the structure decay rate; the Macaulay-bracket $\langle \rangle$ means:
\[ \langle x \rangle = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases} \] (11)

Applying the associated flow rule, the plastic potential function \( g = f \):
\[ g = c_p \left[ \ln \frac{P}{P_{x_0}} + \ln \left( 1 + \frac{\eta^2}{M^2 - \beta^2} \right) \right] - H = 0 \] (12)

2.2. Rotational hardening
Based on the UH model, Yao et al. established the dynamic UH model by introducing rotational hardening\(^{[13]}\). Correspondingly, as is shown in stress space in Figure 2, the yield surface changes from symmetry of the \( p \) axis to symmetry of a rotating axis in stress space.

![Figure 2. Rotational hardening surface](image)

Then the yield function is rewritten as:
\[ f = c_p \left[ \ln \frac{P}{P_{x_0}} + \ln \left( 1 + \frac{\eta^2}{M^2 - \beta^2} \right) \right] - H = 0 \] (13)

In equation (13), \( \beta \) represents the deviation between the rotating axis and the \( p \) axis, which can be calculated by rotating axis tensor \( \beta_{ij} \):
\[ \beta = \sqrt{3} \hat{\beta}_{ij} \hat{\beta}_{ij} \] (14)

\( \eta^* \) is the relative stress ratio, which is calculated by relative stress ratio tensor \( \hat{\eta}_{ij} \):
\[ \eta^* = \sqrt{3} \hat{\eta}_{ij} \hat{\eta}_{ij} \] (15)

\( \hat{\eta}_{ij} \) is the difference between the stress ratio tensor \( \eta_{ij} \) and the rotating axis tensor \( \beta_{ij} \):
\[ \hat{\eta}_{ij} = \eta_{ij} - \beta_{ij} \] (16)

And the stress ratio tensor \( \eta_{ij} \) is described by stress tensor \( \sigma_{ij} \).
\[ \eta_{ij} = \frac{\sigma_{ij} - p\delta_{ij}}{p} \] (17)

The rotating axis tensor \( \beta_{ij} \) evolves with plastic shear strain \( \varepsilon^p_d \).
\[ d\beta_{ij} = \sqrt{3} \hat{b}_M \hat{b}_M (\hat{b}_M - \beta) \frac{\hat{\eta}_{ij}}{\eta} \frac{\partial^p_{ij}}{d^p} \] (18)

In equation (18): \( b_0 \) is the rotation rate parameter and \( b_1 \) is the rotation limit parameter.

2.3. Establishment of the structured UH model considering induced anisotropy
Adding parameter \( \beta \) in yield function (13) modifies the evolution of internal variable \( R \).
\[ R = \frac{P_s}{P_s} = \frac{p \left( 1 + \frac{\eta^2}{M^2 - \beta^2} \right)}{P_{x_0}} \exp \left( - \int \frac{d\varepsilon^P}{c_p} - \int \frac{d(\Delta \varepsilon)}{\lambda - \kappa} \right) \] (19)
Applying associated flow rule, the current yield function $f$ and the plastic potential function $g$ are:

$$g = c_p \left[ \ln \frac{p}{p_{xo}} + \ln \left( 1 + \frac{\eta^2}{M^2 - \beta^2} \right) \right] - H = 0$$  \hspace{1cm} (20)

The differential form of the yield function (20) is:

$$\frac{\partial f}{\partial p} dp + \frac{\partial f}{\partial \eta} d\eta^* + \frac{\partial f}{\partial \beta} d\beta = dH$$  \hspace{1cm} (21)

And it can also be generalized as:

$$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{\partial \beta_{ij}} d\beta_{ij} = dH$$  \hspace{1cm} (22)

Referring to reference [13], $\Pi = \eta/M$, the new hardening parameter is expressed as:

$$H = f \left( \frac{dp}{\Pi c_p} + \frac{2}{M} \frac{df}{d\beta_{ij}} \right)$$  \hspace{1cm} (23)

Substituting equation (18) into equation (23), the hardening parameter can be rewritten as:

$$H = \frac{\sqrt{3} (3b^2 \beta_{ij} - 2M^2)(b(M - \beta)b_0 \eta^*)}{(M^2 - \beta^2)(M^2 - \beta^2 + \eta^2)}$$  \hspace{1cm} (24)

In equation (24), $A = \frac{\sqrt{3} (3 \beta_{ij} \beta_{ij} - 2M^2)(b(M - \beta)b_0 \eta^*)}{(M^2 - \beta^2)(M^2 - \beta^2 + \eta^2)}$.

2.4. Stress transformation

In order to describe behaviors in three-dimensional stress space, stress transformation is applied[14][15]. The relationship between real stress state $(p, q, \theta)$ and transformed stress state $(\tilde{p}, \tilde{q}, \tilde{\theta})$ is:

$$\begin{align*}
\tilde{p} &= p \\
\tilde{q} &= q \\
\tilde{\theta} &= \theta
\end{align*}$$  \hspace{1cm} (25)

$q_c$ is the shear stress in the SMP curve under triaxial compression.

$$q_c = \frac{21}{2} \frac{I_1}{I_2 I_3 - I_1^2 - I_3^2}$$  \hspace{1cm} (26)

where: $I_1$, $I_2$, and $I_3$ are the first, the second and the third stress invariants, which can be obtained from the principal stress respectively:

$$\begin{align*}
I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\
I_2 &= \sigma_1 \sigma_2 + \sigma_1 \sigma_3 + \sigma_2 \sigma_3 \\
I_3 &= \sigma_1 \sigma_2 \sigma_3
\end{align*}$$  \hspace{1cm} (27)

According to equations (25)~(27), the transformed stress tensor $\tilde{\sigma}_{ij}$ can be described as:

$$\begin{align*}
\tilde{\sigma}_{ij} &= p \delta_{ij} + \frac{q_c}{q} (\sigma_{ij} - p \delta_{ij}) & q \neq 0 \\
\tilde{\sigma}_{ij} &= \sigma_{ij} & q = 0
\end{align*}$$  \hspace{1cm} (28)

Then the current yield function $f$ and plastic potential function surface $g$ are rewritten as:

$$\begin{align*}
f &= c_p \left[ \ln \frac{p}{p_{xo}} + \ln \left( 1 + \frac{\tilde{\eta}^2}{M^2 - \beta^2} \right) \right] - H = 0 \\
g &= c_p \left[ \ln \frac{p}{p_{xo}} + \ln \left( 1 + \frac{\tilde{\eta}^2}{M^2 - \beta^2} \right) \right] - H = 0
\end{align*}$$  \hspace{1cm} (29) \hspace{1cm} (30)

where: $\tilde{\eta}^* = \sqrt{\frac{3}{2} \tilde{\eta} \tilde{\eta}^*}$; $\tilde{\eta} = \tilde{\eta}_{ij} - \beta_{ij}$; $\tilde{\eta}_{ij} = \tilde{\eta}_{ij} - \tilde{\beta}_{ij}$. And the parameter $A$ is rewritten as:

$$A = \frac{\sqrt{3} (3 b_0 \beta_{ij} - 2M^2)(b(M - \beta)b_0 \eta^*)}{\sqrt{2} (M^2 - \beta^2)(M^2 - \beta^2 + \eta^2)}$$  \hspace{1cm} (31)

Consequently, the stiffness matrix $C_{ijkl}$ of the presented model can be obtained:
\[ d\sigma_{ij} = C_{ijkl} d\varepsilon_{kl} = \left( C_{ijkl}^0 \right) d\varepsilon_{kl} \]  

(32)

where:

\[ \frac{\partial g}{\partial \sigma_{ij}} = \frac{\partial f}{\partial \sigma_{ij}} \]

\[ \frac{\partial g}{\partial \sigma_{kl}} = \frac{\partial f}{\partial \sigma_{kl}} \]

\[ \frac{\partial \bar{g}}{\partial \sigma_{ij}} = \frac{\partial f}{\partial \sigma_{ij}} \]

\[ \frac{\partial \bar{g}}{\partial \sigma_{kl}} = \frac{\partial f}{\partial \sigma_{kl}} \]

\[ H' = \int \frac{dE}{\Delta p} + \left( \frac{\bar{g}}{M} - 1 \right) \frac{\partial f}{\partial \beta_i} d\beta_i \]

\[ \frac{\partial H'}{\partial \sigma_{ij}} = \frac{\partial H'}{\partial \sigma_{kl}} = 3 \frac{\partial g}{\partial \sigma_{ij}} \frac{\partial \bar{g}}{\partial \sigma_{kl}} + \beta_i \bar{g} \frac{\partial g}{\partial \sigma_{ij}} \frac{\partial \bar{g}}{\partial \sigma_{kl}} \]

In equation (32), \( C_{ijkl}^0 \) is the elastic stiffness matrix.

2.5. Model parameters

There are 9 basic parameters in the structured UH model considering induced anisotropy: \( \lambda, \kappa, M, N \) (The void ratio \( e \) corresponding to \( p = 1 \)kPa on NCL), \( n, \Delta e_0, \zeta, b_r, \) and \( b_l \). The first five parameters are the same as those in the Modified Camclay model, which can be determined by conventional tests on reconstituted soil. Parameter \( b_r \) and \( b_l \) respectively control the speed and limit of the rotation of the rotating axis, and the determination methods were discussed in reference [16]. Parameter \( \Delta e_0 \) and \( \zeta \) are the structural parameters. Due to the consideration of the rotating axis, the determination method of structural parameters may be different from that in reference [12]: If the initial rotating axis is not on the \( p \) axis, then the structural parameters \( \Delta e_0 \) and \( \zeta \) should be obtained by fitting the test data of isotropic strain compression test.

3. Model evolution

In order to illustrate the coupling effects of soil structure and induced anisotropy, shear tests under constant \( p \) value are simulated. The Camclay parameters applied are listed in table 1.

| \( \lambda \) | 0.2 | 0.03 | 2.0 | 1.5 | 0.25 |
| --- | --- | --- | --- | --- | --- |
| \( \kappa \) |  \( \Delta e_0 \) |  \( \zeta \) |  \( b_r \) |  \( b_l \) |  \( n \) |  \( M \) |  \( \nu \) |

(a) Reconstituted soil  
(b) Structured soil
Four different soil samples are applied to be simulated in this section, their common initial states are: $e_0 = 1.2$, $p_0 = 100\text{kPa}$, and no initial induced anisotropy. Sample (a): Reconstituted soil; Sample (b): Structured soil; Sample (c): Reconstituted soil considering induced anisotropy; Sample (d): Structured soil considering induced anisotropy. Structure parameters and rotation parameters are: $\Delta e_0 = 0.4$, $\zeta = 10$, $b_c = 1.0$, $b_l = 0.8$. The samples are loaded by cyclic triaxial compression and the simulations are shown in Figure 3. Taking soil sample (a) as the reference, soil structure effects are obvious in sample (b) and induced anisotropy effects are obvious in sample (c).

For more intuitively analyzing the combined action of soil structure and induced anisotropy in the cyclic compression, peak point of shear stress ($\sigma_a - \sigma_r$) for every cyclic loading of every sample are connected and plot in Figure 4. In the early loading stage, modulus of structured soil is significantly greater than that of reconstituted soil no matter whether or not the induced anisotropy being considered. As loading processing, soil structure decay, plastic strains develop, and induced anisotropy effects gradually emerge. Finally, after fully decay of soil structure, simulations of triaxial compressions considering induced anisotropy attain higher shear stress.

4. Model prediction
To verify the rationality of the structured UH model considering induced anisotropy, the test data of 2 kinds of natural soils are compared with the model predictions. In the following figures, the points represent the test data, broken lines represent the model simulations without considering induced anisotropy and full lines represent the model simulations considering induced anisotropy.
4.1. Bothkennar clay
Bothkennar clay is a marine from Scotland\cite{17}. The clay was deposited 6000 to 8500 years ago, and the soil layer is very uniform. Figure 5 and Figure 6 indicate the comparisons between test data and model predictions of undrained shear tests of Bothkennar clay. The initial structure potential of isotropic stress state is \( \Delta e_0 = 0.254 \) and the rotating axis initial state is \( \beta_1 = \beta_2 = \beta_3 = 0.0 \); the initial structure potential of \( K_0 \) compression state is \( \Delta e_0 = 0.188 \), and the rotating axis initial state is \( \beta_1 = 0.47 \). The parameters used for model predictions are shown in Table 2:

| \( \lambda \) | \( \kappa \) | \( N \) | \( M \) | \( \nu \) | \( \zeta \) | \( b_r \) | \( b_l \) |
|---|---|---|---|---|---|---|---|
| 0.25 | 0.025 | 2.45 | 1.37 | 0.3 | 10 | 1.0 | 0.8 |

Figure 5 shows the model predictions of the undrained triaxial shearing from isotropic stress state and the rotational hardening is not obvious. However, for the predictions shown in Figure 8 whose initial stress states are \( K_0 \) compression states, model simulations considering rotational hardening are significantly better.
4.2. Shanghai soft clay
Shanghai soft clay is a kind of muddy soft clay\(^{[18]}\). For that some model parameters cannot be determined according to the original literature, the model simulations and the test data are compared qualitatively in Figure 7 and Figure 8, and the model parameters are shown in Table 3:

Table 3. Model parameters for Shanghai soft clay

| \( \lambda \) | \( \kappa \) | \( M \) | \( v \) | \( N \) | \( \Delta e_0 \) | \( \zeta \) | \( b_r \) | \( b_l \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0.173 | 0.034 | 1.21 | 0.2 | 1.4 | 0.2 | 10 | 1.0 | 0.8 |

Compared to the model without considering induced anisotropy, the presented structured UH model considering induced anisotropy simulates greater shear stiffness and higher shear stress, which is more consistent to the test data.

![Figure 7. Stress path of undrained shear test](image1)

![Figure 8. Stress-strain relationship of undrained shear test](image2)

5. Conclusion
Based on the structured UH model and referring to rotational hardening, a structured UH model considering induced anisotropy is established and the model is then applied in three-dimensional stress space. By comparing the test data with the model predictions, the established model is qualified to reasonably simulate the behaviors of natural soil.

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