Abstract
Quantum mechanical time operator is introduced following the parametric formulation of classical mechanics in the extended phase space. Quantum constraint on the extended quantum system is defined in analogy to the constraint of the classical extended system, and is interpreted as the condition defining the space of physical events. It is seen that the peculiar properties of the time observable, otherwise obtained in the models of time measurement, are of the classical origin, i.e., due to the quantized classical constraint of the parametric Hamiltonian dynamics.

1 Introduction
In non-relativistic quantum mechanics, as well as in classical mechanics, time is considered as a parameter of the dynamical orbits, with an arbitrary initial value and an irrelevant global scale. Consequently, there is no phase space function in the classical theory and no self-adjoint operator in quantum mechanics that would correspond to the time as a physical measurable quantity. Nevertheless, one does measure the duration of various processes and one does obtain and records information about time of occurrence of various events. The need to formulate a consistent theory of time measurement in quantum mechanics on one hand and the relativistic covariance on the other demand that the same mathematical objects should be associated with time and space observables.

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The main obstacle to define an appropriate self-adjoint operator to represent adequately a measurement of quantum mechanical time are known since long ago [1, 2]. If the Hamiltonian of the system, representing its energy, has semi-bounded, bounded or discrete spectrum, then its canonically conjugated operator can not posses spectral representation in terms of projector measure on $R$ (PM) like the operators representing other usual observables. On the other hand, Galileian covariance demands that the operators representing energy and time must be conjugated. The way out is to associate the time observable not with PM but with a measure on $R$ in terms of positive operators that need not be orthogonal on non-overlapping domains (POVM)[2, 3]. This approach has been used by many to model time observables for different physical systems [2] or to model particular measurement schemes corresponding to different notions of time, such as the time of arrival, the tunneling time and the time of a quantum clock associated with the phase variable [4].

Well known parametric formulation of classical Hamiltonian systems, based on an extended phase space, is often used to transform an explicitly time-dependent system into an equivalent autonomous one [9, 10, 11]. In this formalism the time parameter is transformed into a coordinate of the extended phase space, and treated (almost) analogously as other canonical coordinates. The time-coordinate of the extended phase space is sometimes called the ideal time, or external time, because it is meant to be universal and not related to particular type of time measurement. However, the extended system has an additional constraint, which introduces ambiguity into its quantization. This theme is usually treated under the name of quantization of parameterized non-relativistic or relativistic particle a comprehensive review is presented in [6]), which are understood as simplified models with the typical problems that appear in the quantization of general relativity (a recent review of ”the problem of time” in general relativity can be found in [7],[8]).

On the other hand, and independently of the quantization of the parametric time, different types of quantum time observables have been introduced as mathematical models of measurements of occurrence times of particular events. As we have already stated, within this research theme an important common property of different time observables has been established: the time observables are mathematically represented by POVM’s on time values domains, and not by PM’s like most dynamical observables.

The goal of our paper is to show that the main mathematical property of different time observables, i.e. the POVM representation, can be seen as a property of the parameter time variable and its quantized version. Essential property, common to the different quantum time observables is thus seen
as implied already by the classical constraint on the extended Hamiltonian system. This indicates deep relation and unity of these different concepts of the time observables.

In the next section we shall review the parametric formulation of a Hamiltonian dynamical system. This includes the definition of the classical time variable in the extended phase space, and the constrained classical evolution equation. In Section 3 we introduce the extended quantum system. It should be stressed right at the beginning that we shall not quantize the constrained classical extended system. Instead the analog of the classical constraint is introduced as a condition on physically admissible states of the extended Hilbert space. We shall see that this constraint implies properties of the quantum time observable. Summary of the main argument is presented in Section IV.

2 Extended phase space and representation of time

Well known parametric formulation of Hamiltonian systems via the extended symplectic phase space \[9,10,11\] suggests a way to associate time with an operator defined on an extended Hilbert space. The parametric formulation is usually used to transform a non-autonomous system into an autonomous one. However, here it is used in order to introduce the time variable for an autonomous system. Consider a Hamiltonian system \((R^{2n},\Omega,H)\), where \(\Omega\) is the standard symplectic structure

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
\]

where 0 and 1 are the \(n\) dimensional zero and unit matrices, and \(H\) is the Hamilton’s function. Associated with this system is an extended system on \((R^{2n+2},\Omega_{ex},H_{ex})\) defined as follows. The canonical coordinates of the extended phase space \(R^{2n+2}\) are \(q_1,q_2,\ldots,q_{n+1},p_1,p_2,\ldots,p_n,p_{n+1}\) where \(q_1,q_2,\ldots,q_n,p_1,p_2,\ldots,p_n\) are the canonical coordinates of the original system and \((T \equiv q_{n+1}, S \equiv p_{n+1})\) are canonical coordinates in the added symplectic plane. The extended symplectic form \(\Omega_{ex}\) is of the form \((1)\) with 0 and 1 being the \(n+1\) dimensional zero and unit matrices. The Poisson brackets between the canonical coordinates are determined by the extended \(\Omega_{ex}\). In particular:

\[
\{T,S\} = 1, \{T,q\} = \{T,p\} = \{S,q\} = \{S,p\} = 0.
\]
Let us mention that we do not gain anything by treating more general symplectic manifold instead of $R^{2n}$ since in order to quantize the classical system, which is to be done in the next section, canonical coordinates have to be specified in advance.

In order that the two systems $(R^{2n}, \Omega, H)$ and $(R^{2n+2}, \Omega_{ex}, H_{ex})$ describe the same dynamics the Hamiltonians $H$ and $H_{ex}$ are related as follows: Hamilton variational principle for the original and extended systems are equivalent if

$$H_{ex}d\theta = (H + S)dt,$$

where $t$ and $\theta$ are evolution parameters in the original and in the extended systems. The original Hamiltonian defines invariant hypersurfaces in $R^{2n}$ by $H(q, p) = const$. The equivalent requirement on the extended $H_{ex}$, which need to be determined only up to an additive constant, suggests to define $H_{ex}$ as an implicit function $H_{ex} = 0$. This now defines a $2n + 1$ dimensional hypersurface in $R^{2n+2}$. In view of the equation (3) this choice relates the value of the canonical coordinate $S$ to the value of the original Hamiltonian $H$

$$S(\theta) = -H(q(\theta), p(\theta)) \equiv -h(\theta).$$

This choice uniquely determines the new Hamiltonian $H_{ex}$ as

$$H_{ex}d\theta = (H - h)dt$$

so that the Hamilton’s variational principle with the original Hamiltonian $H$ is equivalent to the extended formulation with the extended Hamiltonian and the constraint

$$H_{ex} = k[H - h], \quad k = dt/d\theta,$$

$$H_{ex} = 0.$$  

The appearance of the scaling factor $k = dt/d\theta$ ensures that the extended system is covariant with respect to the canonical transformations which might act nontrivially on all extended canonical coordinates including $T$. The most common and traditional choice for the value of the scaling factor $k$ is $k = 1$. In order to simplify the presentation of the main arguments, in what follows we shall also always take $k = 1$.

The dynamical equations of the extended system in terms of the new parameter $\theta$ are of the usual canonical form:

$$\frac{dq_i}{d\theta} = \frac{\partial H_{ex}}{\partial p_i}, \quad \frac{dp_i}{d\theta} = -\frac{\partial H_{ex}}{\partial q_i}, \quad i = 1, 2, \ldots n + 1,$$
and are equivalent to the original equations in terms of the parameter $t$:

\[
\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1, 2, \ldots n,
\]

\[
\frac{dt}{d\theta} = 1, \quad \frac{dh}{d\theta} = \frac{\partial H}{\partial t} = 0
\]

with the assumed relation between the parameters.

The advantage of the extended formulation is that the parameter time of the original formulation is treated as a canonical coordinate $T$ on an equal footing as the other canonical variables that correspond to the spatial degrees of freedom. The extended phase space and its submanifold given by the constraint (6) could perhaps be called the space of events and the manifold of physical events respectively.

Remarks

The following three remarks are not crucial for our main argument.

1° It is obviously equivalent to the presented formulas to define the extended Hamilton’s function with $-p_{n+1}$ in the equation (3) instead of $+p_{n+1}$, and to accompany this by the same change in the corresponding canonical Poisson bracket.

2° Going in the direction opposite to the construction of the extended system one can attempt to introduce an intrinsic time variable as a suitable function of the $2n$ canonical variables (see for example [12]). Such intrinsic time behaves as the phase variable of the dynamical system. However, the possibility to define a valuable intrinsic time is tightly related to a very difficult problem of integrability of the dynamical system.

3° It is well known that a quantum systems with the Hilbert space $\mathcal{H}$ can be considered as a Hamiltonian dynamical system with the projective Hilbert space $P\mathcal{H}$ as the symplectic phase space [13, 14, 15]. It might be tempting to try to introduce a quantum mechanical time observable using the parametric formulation of the Hamiltonian representation of quantum mechanics. In the Hamiltonian formulation the real and the imaginary parts of the Hermitian scalar product, reduced on $P\mathcal{H}$, generate the Riemannian and the simplectic structure on the phase space respectively. The quantum states are represented by points of the phase space and the observables $\hat{A}$ by functions $<\hat{A}>$ whose Hamiltonian vector fields generate isometries. The Schroedinger evolution equation is reproduced by the Hamilton dynamical equations with the Hamilton’s function $H = <\hat{H}>$ where $\hat{H}$ is the Hamiltonian. The function representing the commutator between two observables is given by the Poisson bracket of the corresponding functions. The geometric formulation of quantum mechanics has been used to study also the constrained quantum dynamics [16, 17, 18]. One could now introduce the
parametric formulation on the extended phase space of the Hamiltonian system corresponding to a quantum system. However, thus introduced canonical time coordinate does not define an observable. Only a small subset of functions on the phase space $P\mathcal{H}_{ex}$ correspond to quantum mechanical observables. In particular the canonical coordinates $(q, p, T, S)$ of a point do not represent observables. The canonical coordinates represent components of the quantum state in some basis. Thus, the coordinate corresponding to the time in the extended parametric formulation does not correspond to an observable. Several extensions of quantum mechanics [14] that generalize the geometric formulation to include the coordinates $(q, p)$ as legitimate observables are seen as theories with hidden variables enabling one to uniquely measure and determine the quantum state.

3 Quantum mechanical time observable

Extended quantum system

The extended phase space of a classical system is formed by treating time as an additional independent degree of freedom, and the parametric dynamics represents a constrained system on the extended phase space. Quantization of the system (6) is often discussed as a simple example of constrained parametric system quantization [19], [20]. We shall follow one of the possible quantization procedures in which the extended phase space is canonically quantized as if there are no constraints. The constraints are then included in the form of conditions imposed on the space of physical states.

The classical approach suggests an analogous treatment of a quantum system in which the system’s Hilbert space $\mathcal{H}$ is considered as the tensor product of the Hilbert spaces $\mathcal{H}_s$ corresponding to the spatial degrees of freedom and the Hilbert space $\mathcal{H}_T$ that corresponds to the time treated as an additional degree of freedom. The extended system will be defined by: a) the commutation relations between the time and its conjugate, b) the extended Hamiltonian and c) the constraint. There is an ambiguity in the choices of the sign in the commutation relations between the time and its conjugate. However the alternatives are equivalent in the sense that the properties of the physical time observable constructed in the two alternative ways are the same. We shall present both alternatives in parallel.

The extended Hilbert space is defined as the direct product $\mathcal{H}_{ex} = \mathcal{H}_s \otimes \mathcal{H}_T$. The structure of $\mathcal{H}_T$ is dictated by the desired algebraic properties of the time variable represented by an operator $I \otimes \hat{T}$ acting on $\mathcal{H}_T$. Pursuing the analogy with the classical extended phase space with the Poisson-Lie bracket $\{T, S\} = 1$ the Hilbert space $\mathcal{H}_T$ should carry an irreducible representation
of the same Lie algebra
\[ [\hat{T}, \pm \hat{S}] = i. \] (9)

Thus, \(\hat{T}\) and \(\hat{S}\) are represented by multiplication and differentiation operators acting on functions from the corresponding domains in \(\mathcal{H}_T\). Of course, \(\hat{T}\) and its conjugate \(\hat{S}\) commute with operators acting in \(\mathcal{H}_s\). In particular, \(\hat{T}\) commutes with Hamiltonian operators \(\hat{H}_s\). The \(\pm\) sign will be fixed to correspond to the sign of the operator \(\hat{S}\) in the extend Hamiltonian.

Operators \(\hat{T}\) and \(\hat{S}\) have continuous spectra on \(\mathcal{H}_{ex}\). Nevertheless, we shall often use the terminology ”eigenvalue” and ”eigenvectors” for those operators indicating by the quotation marks that these should be understood in the generalized sense. For example if the common ”eigenvectors” of the coordinate variables are denoted by \(|q>\) and if the ”eigenvector” of the time operator \(\hat{T}\) is denoted by \(|T>\) then the operator \(\int_V |q><q|d\mathbf{q} \otimes \int_{\Delta T} |T><T|dT\) is interpreted as the property that the system is in the volume \(V\) during the time interval \(\Delta T\). Unhated letters like \(H_s, T, S, H_{ex}\) denote the eigenvalues or ”eigenvalues” of the corresponding operators, which are, of course, to be distinguished from the corresponding classical functions denoted by the same symbols in the previous section.

At this stage we can propose that general mixed state of the extended system are represented by statistical operators \(\rho_{ex}\) in \(\mathcal{H}_{ex}\), and pure states are linear combinations of separable states:
\[
\sum_{i,j} c_{ij} |\psi_i>_{s} \otimes |\phi_j>_{T}.
\] (10)

Of course, vectors corresponding to pure states that have different norm or phase are all supposed to represent the same state. In what follows we shall not explicitly take care about this gauge invariance of the states.

Consider a system with the Hamiltonian \(\hat{H}_s = \hat{H}_s(\hat{q}, \hat{p})\). In order that the original system on \(\mathcal{H}_s\) and the extended one on \(\mathcal{H}_{ex}\) describe the same quantum evolution the Hamiltonian of the extended system is defined as
\[
\hat{H}_{ex} = (\hat{H} \pm \hat{S}).
\] (11)

Notice that \(\pm\) signs in the Hamiltonian (11) correspond to \(\pm\) sign in the commutation relation (9), analogously to the situation in the classical case, eq. (2) and (3).

Indeed, consider the Schroedinger evolution of an extended separable pure state \(|\psi> = |\psi >_s \otimes |\psi >_T\)
\[
\frac{\partial |\psi>}{\partial \theta} = \hat{H}_{ex} |\psi> = (\hat{H}_s |\psi >_s) \otimes |\psi >_T \pm |\psi >_s \hat{S}|\psi >_T.
\] (12)
On the other hand
\[ \frac{i}{\partial \theta} |\psi > = \frac{i}{\partial t} \otimes |\psi >_T + i |\psi >_s \otimes \frac{\partial |\psi >_T}{\partial t}. \] (13)

Thus
\[ \frac{i}{\partial t} |\psi >_s = \hat{H}_s |\psi >_s \] (14)
\[ \frac{i}{\partial t} |\psi >_T = \pm \hat{S} |\psi >_T \] (15)

The Schroedinger equation for \( \hat{H}_s \) (14) is reproduced, and the equation (15) determines the operator \( \hat{S} \). We see that the choice of \( \pm \) in the commutation relations (9) has to be performed together with the corresponding choice in the extended Hamiltonian (11).

In either case (10) the Hamiltonian \( \hat{H}_{ex} \) and the ideal time operator \( \hat{T} \) satisfy
\[ \Delta H_{ex} \Delta T \geq <i[\hat{H}_{ex}, \hat{T}]/2> = 1/2. \] (16)

Of course, the physical interpretation of (15) is not that of the time-energy uncertainty relation, since the system’s energy is represented by the Hamiltonian \( \hat{H}_s \) and commutes with \( \hat{T} \).

**Constraint**

The classical constraint \( H_{ex} = 0 \) is introduced into quantum mechanics by a constraint which must be satisfied by the states of the extended system as follows: It is declared that not all vectors from \( H_{ex} \) should be considered as representing states of the physical system but only those that satisfy the following condition, analogous to the classical equation of the constraint
\[ \hat{H}_{ex} |\psi >= (\hat{H}_s \pm \hat{S}) |\psi >= 0, \] (17)
which is equivalent to
\[ <\psi |(\hat{H}_s \pm \hat{S})^2 |\psi >= 0. \] (18)

Since \( \hat{H}_s \) and \( \hat{S} \) are linear, the set of physical states is a linear subspace of \( H_{ex} \). We denote the space of physical states, *i.e.*, those that satisfy (17) by \( H_{phys} \subset H_{ex} \). The space \( H_{phys} \) will be called the space of physical events to emphasize the fact that \( H_{phys} \) is a subspace of \( H_{ex} \) and not of \( H_s \).

In summary, the evolution equation on \( H_{ex} \) is given by the Schroedinger equation with the extended Hamiltonian \( H_{ex} \), but the physical pure states are only those vectors in \( H_{ex} \) that satisfy the condition of the (quantum) constraint (17). The quantum constraint (17) is obviously consistent with the extended dynamical equation.
The constraint (17) implies that the action of \( \hat{H_s} \) and that of \( \mp \hat{S} \) on the vectors from the subspace \( \mathcal{H}_{\text{phys}} \) coincide. Consider the vector \( |\psi > = |E_i > \otimes |S > \) where \( |E_i > \) is an eigenvector of \( \hat{H_s} \) and \( |S > \) is a "eigenvector" of \( \mp \hat{S} \). Due to the constraint (17) the spectra and the eigenstates of \( \mp \hat{S} \) restricted on the space of physical events \( \mathcal{H}_{\text{phys}} \) are equal to the spectra and the eigenstates of \( \hat{H_s} \) restricted on \( \mathcal{H}_{\text{phys}} \). The general physical state \( |\psi >_{\text{phys}} \in \mathcal{H}_{\text{phys}} \) is a linear combination of the states on the diagonal of \( \mathcal{H}_{\text{ex}} \):

\[
|\psi >_{\text{phys}} = \sum_i c_i |E_i > \otimes |S_i = E_i >, \tag{19}
\]

where \( |S_i = E_i > \) denotes the "eigenstate" of \( \hat{S} \) with the "eigenvalue" numerically equal to an energy "eigenvalue" \( E_i \). The formula (18) for a physical state is to be compared with the representation (10) of a general vector in \( \mathcal{H}_{\text{ex}} \).

Notice that the restriction of \( \hat{S} \) has the spectra of the restriction of \( -\hat{H_s} \) if the commutator \( [\hat{T}, \hat{S}] = i \) and the spectra of \( +\hat{H_s} \) if \( [\hat{T}, \hat{S}] = -i \). Thus, the relation between the algebra of the operators \( \hat{T} \) and \( \hat{S} \) and the physical consequences of the quantum constraint (17), is the same as the relation between the Poisson algebra of the classical quantities \( T, S \) and the physical consequences of the classical constraint given by equation (4).

The restriction of \( \hat{T} \) onto the \( \mathcal{H}_{\text{phys}} \) is denoted by \( \hat{T}_{\text{phys}} \), and could be called the physical time. Due to the constraint (17) and the consequent properties of \( \hat{S}_{\text{phys}} \), we can conclude that \( \hat{T}_{\text{phys}} \) does not generate an orthogonal resolution of unity. To this end we recall the operational treatment of the time observable\([2],[3]\). In this approach the canonical commutation relation between an energy operator and a formal operator representing time is interpreted in mathematically rigorous way as existence of a covariant POVM associated to the time observable. The restrictions of \( \pm \hat{S} \) and \( \hat{T} \) on the subspace of physical events are in the same relation as the energy operator and the formal time operator in the operational approach. Thus, we conclude that \( \hat{T}_{\text{phys}} \), i.e., the restriction of \( \hat{T} \) onto the space of physical events \( \mathcal{H}_{\text{phys}} \), generates the corresponding POVM from \( R \) onto the subspaces of \( \mathcal{H}_{\text{phys}} \). Let us stress that the properties of the physical subspace \( \mathcal{H}_{\text{phys}} \) are determined by the Hamiltonian of the system \( \hat{H_s} \), but the fact that \( \hat{T}_{\text{phys}} \) generates a POVM and not a PM is universal for all physically plausible Hamiltonian operators.

In conclusion, we see that, due to the quantized classical constraint (17), there is no physical state where the formal time observable \( \hat{T} \) has definite sharp value. "Eigenstates" of the restriction of \( \hat{T} \) on \( \mathcal{H}_{\text{phys}} \) are not only non-normalizable, but are more importantly non-orthogonal. Thus, despite the
fact that the operators representing the system’s energy $\hat{H}_s$ and the formal time $\hat{T}$ do commute, there is no physical event where the formal time $\hat{T}$, or its restriction $\hat{T}_{\text{phys}}$, has definite value and could be measured with an infinite precision. Our construction indicates that the main reason for this fact is of the classical origin, namely the classical constraint (6), with the quantized form (17) that determines the space of physical events.

4 Summary and discussion

We have explored the consequences of the possibility to introduce a time observable into quantum mechanics by following the procedure of parametric Hamiltonian mechanics on the extended phase space. The time appears in the extended classical system as an additional coordinate, but this sets a constraint that must be satisfied by the extended system. Quantization of the extended system results with a pair of canonically conjugate self-adjoint operators on the extended Hilbert space, which can be considered as an ideal formal time $\hat{T}$ and its conjugate. In order that the time operator corresponds to the classical external time the analogy of the classical constraint had to be introduced. The constraint in quantum mechanics assumes the form of a condition on the states that are considered as physical, and this constraint is trivially compatible with the Schroedinger evolution. Thus, the time observable is represented by the restriction $\hat{T}_{\text{phys}}$ of the formal time operator $\hat{T}$ from the extended Hilbert space $\mathcal{H}_{\text{ex}}$ onto the space of physical events $\mathcal{H}_{\text{phys}} \subset \mathcal{H}_{\text{ex}}$. It is then argued that the observable time $\hat{T}_{\text{phys}}$ must have non-orthogonal generalized eigenstates, and must be represented by a POVM. It is thus seen that the need to represent the time observable by a POVM can be traced to the constrained on the extended state space, and is thus of an essentially classical origin. This is the main claim of our paper.

It would be interesting to use the time observable $\hat{T}_{\text{phys}}$, as introduced here, to study the process of time measurement. Such an analyzes should establish relations between $\hat{T}_{\text{phys}}$ and the POVM’s related to particular measurements of different time observables.

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