A characterization of elementary abelian 2-groups

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Abstract

In this note we give a characterization of elementary abelian 2-groups in terms of their maximal sum-free subsets.

MSC (2010): Primary 11B75; Secondary 20D60, 20K01.

Key words: elementary abelian 2-groups, sum-free subsets, maximal sum-free subsets, maximal subgroups.

1 Introduction

Throughout this paper $G$ will denote an additive, but not necessarily abelian, group. A subset $A$ of $G$ is said to be sum-free if $(A + A) \cap A = \emptyset$, that is the equation $x + y = z$ has no solutions $x, y, z \in A$. Sum-free sets were introduced by Schur [12] whose celebrated result states that the set of positive integers cannot be partitioned into finitely many sum-free sets. Another famous result on sum-free sets is a conjecture of Cameron and Erdős [3, 4], which asserts that the number of sum-free subsets of $\{1, 2, ..., n\}$ is $O(2^{n/2})$. It has been recently proved by Green [7].

Sum-free subsets of abelian groups are central objects of interest in Additive Combinatorics, and have been studied intensively in the last years (see e.g. [1, 2], [5, 6] and [8–11]). Probably the most important problems in this direction are: How large a sum-free subset of $G$ can be? and How many sum-free sets of $G$ are there?

In the following we will focus on maximal sum-free subsets of $G$. Our main theorem characterizes elementary abelian 2-groups by connecting these subsets with the complements of maximal subgroups.
**Theorem 1.1.** A finite group $G$ is an elementary abelian $2$-group if and only if the set of maximal sum-free subsets coincides with the set of complements of maximal subgroups.

The following corollary is an immediate consequence of Theorem 1.1.

**Corollary 1.2.** The number of maximal sum-free subsets of $\mathbb{Z}_2^n$ is $2^n - 1$, and each of them has $2^{n-1}$ elements.

Theorem 1.1 can be also used to count the sum-free subsets of $\mathbb{Z}_2^n$ since these are all subsets of maximal sum-free subsets. Let $k = 2^n - 1$ and denote by $M_1, M_2, ..., M_k$ the maximal subgroups of $\mathbb{Z}_2^n$. Then, by applying the well-known Inclusion-Exclusion Principle, we infer that the total number of sum-free subsets of $\mathbb{Z}_2^n$ is

$$x_n = \left| \bigcup_{i=1}^{k} P(\mathbb{Z}_2^n \setminus M_i) \right| = \sum_{r=1}^{k} (-1)^{r-1} \sum_{1 \leq i_1 < ... < i_r \leq k} |P(\mathbb{Z}_2^n \setminus M_{i_1}) \cap ... \cap P(\mathbb{Z}_2^n \setminus M_{i_r})|$$

$$= \sum_{r=1}^{k} (-1)^{r-1} \sum_{1 \leq i_1 < ... < i_r \leq k} |P(\mathbb{Z}_2^n \setminus M_{i_1} \cup ... \cup M_{i_r})| = \sum_{r=1}^{k} (-1)^{r-1} \sum_{1 \leq i_1 < ... < i_r \leq k} 2^{n-|M_{i_1} \cup ... \cup M_{i_r}|}.$$  

We remark that this depends only on the cardinality of any union of $M_i$’s. Note that its first values are: $x_1 = 2$, $x_2 = 7$, $x_3 = 64$, $x_4 = 3049$, ... and so on. Thus, a natural problem is the following.

**Open problem.** Give an explicit formula for $x_n$.

Most of our notation is standard and will usually not be repeated here. Given a subset $A$ of $G$, we write $A + A = \{a_1 + a_2 | a_1, a_2 \in A\}$, $A - A = \{a_1 - a_2 | a_1, a_2 \in A\}$, $\frac{1}{2} A = \{a \in G | 2a \in A\}$ and $G \setminus A = \{x \in G | x \notin A\}$.

## 2 Proofs of the main results

We start with two easy but important lemmas.

**Lemma 2.1.** Let $G$ be a group and $H$ be a subgroup of $G$. Then $G \setminus H$ is a non-empty maximal sum-free subset of $G$ if and only if $[G : H] = 2$.  


Proof. First of all, we remark that if $A$ is a sum-free of $G$, then $|A| \leq \frac{|G|}{2}$. Indeed, given $x \in A$ the set $x + A = \{x + y \mid y \in A\}$ has the same number of elements as $A$ and $(x + A) \cap A = \emptyset$. In other words, $G$ contains at least $2|A|$ elements.

Assume first that $G \setminus H$ is a non-empty maximal sum-free subset of $G$. By the above remark we have $|G \setminus H| \leq \frac{|G|}{2}$, or equivalently $[G : H] \leq 2$. This leads to $[G : H] = 2$ because $G \setminus H$ is non-empty.

Conversely, assume that $[G : H] = 2$ and let $x, y \in G \setminus H$. Since $x + H$ and $-y + H$ are non-trivial cosets, it follows that $x + H = -y + H$, that is $x + y \in H$. Therefore $G \setminus H$ is a sum-free subset of $G$. Now, let $x \in G \setminus H$ and $z \in H$. Then $x + z \in G \setminus H$. This shows that $(G \setminus H) \cup \{z\}$ is not a sum-free subset of $G$, and consequently $G \setminus H$ is a non-empty maximal sum-free subset of $G$, as desired.

Lemma 2.2. Let $G$ be a group and $A$ be a subset of $G$. Then $A$ is a maximal sum-free subset of $G$ if and only if $G \setminus A = (A + A) \cup (A - A) \cup \frac{1}{2}A$.

Proof. Assume first that $A$ is a maximal sum-free subset of $G$. Then it is a routine to check that $(A + A) \cup (A - A) \cup \frac{1}{2}A \subseteq G \setminus A$. Let $x \in G \setminus A$. By our hypothesis, we infer that $A \cup \{x\}$ is not a maximal sum-free subset of $G$ and so there are $a, b \in A \cup \{x\}$ such that $a + b \in A \cup \{x\}$. If $a = b = x$ we have either $x + x \in A$, i.e. $x \in \frac{1}{2}A$, or $x + x = x$, i.e. $x = 0 \in A - A$.

If $a \in A$ and $b = x$, then $a + x \neq x$ and so $a + x \in A$, i.e. $x \in A - A$. Note that a similar conclusion follows for $b \in A$ and $a = x$. Finally, if $a, b \in A$, then we must have $a + b = x$, i.e. $x \in A + A$. These show that $G \setminus A \subseteq (A + A) \cup (A - A) \cup \frac{1}{2}A$.

Conversely, assume that $G \setminus A = (A + A) \cup (A - A) \cup \frac{1}{2}A$. Since $A + A \subseteq G \setminus A$, we infer that $A$ is a sum-free subset of $G$. It remains to prove that this is maximal, that is $A \cup \{x\}$ is not a sum-free subset of $G$ for any $x \in G \setminus A$. By our hypothesis, we distinguish the following three cases. If $x \in A + A$, then $x = a + b$ for some $a, b \in A$; thus $a, b \in A \cup \{x\}$ and $a + b \in A \cup \{x\}$. If $x \in A - A$, then $x = a - b$ for some $a, b \in A$; thus $x, b \in A \cup \{x\}$ and $x + b = a \in A \cup \{x\}$. Finally, if $x \in \frac{1}{2}A$, then $x \in A \cup \{x\}$ and $x + x \in A \subseteq A \cup \{x\}$. This completes the proof.
We are now able to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( G = \mathbb{Z}_2^n \) for some \( n \geq 1 \). Then all maximal subgroups of \( G \) are of index 2, and therefore their complements are maximal sum-free subsets by Lemma 2.1. Suppose now that \( A \) is a maximal sum-free subset of \( G \). Then \( x+x = 0 \notin A \forall x \in G \), that is \( \frac{1}{2}A = \emptyset \), and \( A+A = A - A \). These lead to 
\[
G \setminus A = A + A
\]
in view of Lemma 2.2. Let \( \{e_1, e_2, ..., e_m\} \) be a maximal linearly independent subset of \( A \) (over \( \mathbb{Z}_2 \)). We infer that \( A \) consists of all sums of an odd number of \( e_i \)'s. If \( m < n \) we can choose \( e_{m+1} \in G \) such that the vectors \( e_1, e_2, ..., e_{m+1} \) are linearly independent. One obtains that
\[
A' = \{e_1, e_2, ..., e_{m+1}, e_1 + e_2 + e_3, e_1 + e_2 + e_4, ..., e_{m-1} + e_m + e_{m+1}, ... \}
\]
is a sum-free subset of \( G \) and \( A \subsetneq A' \), contradicting the maximality of \( A \). Thus \( m = n \) and 
\[
|A| = \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{2r + 1} + \cdots = 2^{n-1},
\]
which implies that
\[
|G \setminus A| = |A + A| = 2^{n-1}.
\]
On the other hand, \( G \setminus A \) is a subgroup of \( G \) because it consists of all sums of an even number of \( e_i \)'s, and so it is a maximal subgroup.

Conversely, let \( M_1, M_2, ..., M_k \) be the maximal subgroups of \( G \) and suppose that \( G \setminus M_i, i = 1, 2, ..., k \), are the maximal sum-free subsets of \( G \). Then \( [G : M_i] = 2 \), for any \( i = 1, 2, ..., k \), by Lemma 2.1. We infer that \( G \) is a nilpotent group, more precisely a 2-group. Since every non-trivial element of \( G \) is contained in a maximal sum-free subset of \( G \), we have
\[
G \setminus 1 = \bigcup_{i=1}^{k} G \setminus M_i = G \setminus \bigcap_{i=1}^{k} M_i = G \setminus \Phi(G),
\]
that is \( \Phi(G) = 1 \). Hence \( G \) is an elementary abelian 2-group. \( \blacksquare \)

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A result on the maximal sum-free subsets of finite groups

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Abstract

In this note we give a characterization of the elementary abelian 2-groups \( \mathbb{Z}_2^n \), \( n = 1, 2, 3 \), in terms of their maximal sum-free subsets.

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1 Introduction

Throughout this paper \( G \) will denote an additive, but not necessarily abelian, group. A subset \( A \) of \( G \) is said to be sum-free if \((A + A) \cap A = \emptyset\), that is the equation \( x + y = z \) has no solutions \( x, y, z \in A \). Sum-free sets were introduced by Schur [12] whose celebrated result states that the set of positive integers cannot be partitioned into finitely many sum-free sets. Another famous result on sum-free sets is a conjecture of Cameron and Erdös [3, 4], which asserts that the number of sum-free subsets of \( \{1, 2, ..., n\} \) is \( O(2^{n/2}) \). It has been recently proved by Green [7].

Sum-free subsets of abelian groups are central objects of interest in Additive Combinatorics, and have been studied intensively in the last years (see e.g. [1, 2], [5, 6] and [8]-[11]). Probably the most important problems in this direction are: How large a sum-free subset of \( G \) can be? and How many sum-free sets of \( G \) are there?

In the following we will focus on the above second question. We remark that it is enough to study the maximal sum-free subsets of \( G \) because they
contain all sum-free subsets of $G$. We can easily check that for $G \cong \mathbb{Z}_2^n$, $n \leq 3$, the maximal sum-free subsets coincide with the complements of maximal subgroups. Our main theorem shows that $\mathbb{Z}_2^n$, $n = 1, 2, 3$, exhaust all finite groups with this property.

**Theorem 1.1.** Let $G$ be a finite group. Then the set of maximal sum-free subsets coincides with the set of complements of maximal subgroups if and only if $G \cong \mathbb{Z}_2^n$ for some $n = 1, 2, 3$.

The next corollary follows immediately from the proof of Theorem 1.1.

**Corollary 1.2.** The elementary abelian 2-group $\mathbb{Z}_2^n$ has $2^n - 1$ maximal sum-free subsets for $n = 1, 2, 3$, and at least $2^n - 1$ maximal sum-free subsets for $n \geq 4$.

A lower bound for the number of sum-free subsets of $\mathbb{Z}_2^n$ can be also given from the above results. Let $k = 2^n - 1$ and denote by $M_1, M_2, ..., M_k$ the maximal subgroups of $\mathbb{Z}_2^n$. Then, by applying the well-known Inclusion-Exclusion Principle, we infer that the total number of sum-free subsets of $\mathbb{Z}_2^n$ is at least

$$x_n = | \bigcup_{i=1}^{k} \mathcal{P}(\mathbb{Z}_2^n \setminus M_i)| = \sum_{r=1}^{k} (-1)^{r-1} \sum_{1 \leq i_1 < \cdots < i_r \leq k} |\mathcal{P}(\mathbb{Z}_2^n \setminus M_{i_1}) \cap \cdots \cap \mathcal{P}(\mathbb{Z}_2^n \setminus M_{i_r})|$$

$$= \sum_{r=1}^{k} (-1)^{r-1} \sum_{1 \leq i_1 < \cdots < i_r \leq k} |\mathcal{P}(\mathbb{Z}_2^n \setminus M_{i_1} \cup \cdots \cup M_{i_r})| = \sum_{r=1}^{k} (-1)^{r-1} \sum_{1 \leq i_1 < \cdots < i_r \leq k} 2^{2^n - |M_{i_1} \cup \cdots \cup M_{i_r}|},$$

and we have equality if and only if $n \leq 3$. We remark that $x_n$ depends only on the cardinality of any union of $M_i$’s. Its first values are: $x_1 = 2$, $x_2 = 7$, $x_3 = 64$, $x_4 = 3049$, ... and so on.

Finally, we indicate two natural open problems concerning the above study.

**Problem 1.** Describe all maximal sum-free subsets of $\mathbb{Z}_2^n$ for $n \geq 4$.

**Problem 2.** Give a precise lower bound for the total number of sum-free subsets of $\mathbb{Z}_2^n$ by computing explicitly $x_n$.

Most of our notation is standard and will usually not be repeated here. Given a subset $A$ of $G$, we write $A + A = \{a_1 + a_2 \mid a_1, a_2 \in A\}$ and $G \setminus A = \{x \in G \mid x \notin A\}$.
2 Proof of the main result

We start with the following easy but important lemma.

**Lemma 2.1.** Let $G$ be a group and $H$ be a subgroup of $G$. Then $G \setminus H$ is a maximal sum-free subset of $G$ if and only if $[G : H] = 2$.

**Proof.** Assume first that $G \setminus H$ is a maximal sum-free subset of $G$ and let $x + H, y + H$ be two non-trivial cosets. Then $x, -y \in G \setminus H$ and so $x - y \in H$, or equivalently $x + H = y + H$. In other words, $G$ has a unique non-trivial coset with respect to $H$, i.e. $[G : H] = 2$.

Conversely, assume that $[G : H] = 2$ and let $x, y \in G \setminus H$. Since $x + H$ and $-y + H$ are non-trivial cosets, it follows that $x + H = -y + H$, that is $x + y \in H$. Therefore $G \setminus H$ is a sum-free subset of $G$. Now, let $x, y \in G \setminus H$ and $z \in H$. Then $x + z \in G \setminus H$. This shows that $(G \setminus H) \cup \{z\}$ is not a sum-free subset of $G$, and consequently $G \setminus H$ is a maximal sum-free subset of $G$, as desired.

We are now able to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $M_1, M_2, ..., M_k$ be the maximal subgroups of $G$ and assume that $G \setminus M_i$, $i = 1, 2, ..., k$, are the maximal sum-free subsets of $G$. Then $[G : M_i] = 2$, for any $i = 1, 2, ..., k$, by Lemma 2.1. We infer that $G$ is a nilpotent group, more precisely a 2-group. Since every non-trivial element of $G$ is contained in a maximal sum-free subset of $G$, we have

$$G \setminus 1 = \bigcup_{i=1}^{k} G \setminus M_i = G \setminus \bigcap_{i=1}^{k} M_i = G \setminus \Phi(G),$$

that is $\Phi(G) = 1$. Consequently, $G$ is an elementary abelian 2-group, say $G \cong \mathbb{Z}_2^n$.

Next we will prove that $n \in \{1, 2, 3\}$. Assume that $n \geq 4$ and denote by $e_1, e_2, ..., e_n$ the canonical basis of $G$ over $\mathbb{Z}_2$. It is clear that $A = \{e_1, e_2, e_3, e_4, e_1 + e_2 + e_3 + e_4\}$ is a sum-free subset of $G$. If there is a maximal subgroup $M$ such that $A \subseteq G \setminus M$, then $M$ must be of index 2, which implies that the sum of any two elements of $A$ belongs to $M$. Therefore $e_1 + e_2, e_3 + e_4 \in M$ and so $e_1 + e_2 + e_3 + e_4 \in M$, a contradiction. This completes the proof.
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