Duplication self vertex switchings in self centered graphs

C. Jayasekaran¹ and V. Prabavathy²

Abstract
A vertex \( v \in V(G) \) is said to be a self vertex switching of \( G \) if \( G \) is isomorphic to \( G' \), where \( G' \) is the graph obtained from \( G \) by deleting all edges of \( G \) incident to \( v \) and adding all edges incident to \( v' \) which are not in \( G \). Duplication of a vertex \( v \) of graph \( G \) produces a new graph \( G' \) by adding a new vertex \( v' \) such that \( N(v') = N(v) \). In other words, a vertex \( v' \) is said to be duplication of \( v \) if all the vertices which are adjacent to \( v \) in \( G \) are also adjacent to \( v' \) in \( G' \). A vertex \( v \) is called a duplication self vertex switching of a graph \( G \) if the resultant graph obtained after duplication of \( v \) has \( v \) as a self vertex switching. In this paper, we give the existence of self centered graphs with duplication self vertex switching.

Keywords
Switching, self vertex switching, duplication self vertex switching, dss₁(\( G \)).

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1. Introduction

For a finite undirected graph \( G(V,E) \) with \( |V| = p \) and a set \( \sigma \subseteq V \), Seidel [9] defined the switching of \( G \) by \( \sigma \) as the graph \( G^\sigma(V,E) \), which is obtained from \( G \) by removing all edges between \( \sigma \) and its compliment \( V - \sigma \) and adding as edges all non edges between \( \sigma \) and \( V - \sigma \). When \( \sigma = \{v\} \subseteq V \), the corresponding switching \( G^\{v\} \) is called a vertex switching and is denoted by \( G^v \) [10]. Switching is an equivalence relation and the associated equivalence classes are called switching classes. For a survey of switching classes of graphs we refer to Seidel [9]. A subset \( \sigma \) of \( V(G) \) to be a self switching of \( G \) if \( G \cong G^\sigma \). The set of all self switchings of \( G \) with cardinality \( k \) is denoted by \( SS_k(G) \) and its cardinality by \( ss_k(G) \). If \( k = 1 \), then we call the corresponding self switching as self vertex switching [11]. Duplication of a vertex \( v \) of graph \( G \) produces a new graph \( G' \) by adding a new vertex \( v' \) such that \( N(v') = N(v) \). In other words, a vertex \( v' \) is said to be duplication of \( v \) if all the vertices which are adjacent to \( v \) in \( G \) are also adjacent to \( v' \) in \( G' \). The concept of duplication self vertex switching was introduced by C. Jayasekaran and V. Prabavathy [5]. A vertex \( v \) is called a duplication self vertex switching of a graph \( G \) if the resultant graph obtained after duplication of \( v \) has \( v \) as a self vertex switching. The number of duplication self vertex switching is denoted by \( dss_1(G) \). For any vertex \( v \in V(G) \), the open neighbourhood \( N(v) \) of \( v \) is the set of all vertices adjacent to \( v \). That is \( N(v) = \{u \in V(G)/ uv \in E(G)\} \). The closed neighbourhood of \( v \) is defined by \( N[v] = N(v) \cup \{v\} \). The existence of graphs with given number of self vertex switchings were discussed in [1]. The trees [3] and unicyclic graphs [2], [4] are characterized for self vertex switchings. We consider simple graphs only.

In this paper, we discuss the possibility of the presence of duplication self vertex switching in some graphs.

We consider the following results which are required in the subsequent section.

Theorem 1.1. [5] If \( v \) is a duplication self vertex switching of a graph \( G \) of order \( p \) then \( p \) is even and \( d_G(v) = p/2 \).

Theorem 1.2. [5] Let \( G \) be a graph and let \( v \) be any vertex of
G. Then v is a duplication self vertex switching of G iff there exists an automorphism on G which maps elements of N(v) onto elements of [N(v)]^c.

### 2. Main Results

**Theorem 2.1.** Every complete graph of order \( n \geq 1 \) is an induced subgraph of a graph G of order \( 2n \) with \( dss_1(G) = 2n \).

**Proof.** Let H be a complete graph of order \( n \geq 1 \) and let \( H' \) be another copy of H with \( V(H) = \{v_i/1 \leq i \leq n\} \), \( V(H') = \{v'_i/1 \leq i \leq n\} \) and \( v_i \) corresponds to \( v'_i(1 \leq i \leq n) \). Construct a graph G with \( V(G) = V(H) \cup V(H') \) and \( E(G) = E(H) \cup E(H') \cup \{v_iv'_i/1 \leq i \leq n\} \). The resulting graph G contains H as an induced subgraph. Moreover, the graph G is of order \( 2n \) and for \( 1 \leq i \leq n \), \( d_G(v_i) = d_G(v'_i) = n \). For \( 1 \leq i, k \leq n \), define \( \alpha_k : V(G) \rightarrow V(G) \) such that \( \alpha_k(v_i) = v'_i \) and \( \alpha_k(v'_i) = v_i \). Clearly \( \alpha_k \) is an automorphism of G which maps elements of \( N(v_i) \) onto \( [N(v_i)]^c \) and vice versa. By Theorem 1.2, each \( v_i \) is a duplication self vertex switching of G. Similarly each \( v'_i \) is a duplication self vertex switching of G. Hence each vertex in G is a duplication self vertex switching of G. For example, the construction of the graph G for \( n = 5 \) is shown in Fig. 2.1.

![Fig. 2.1](image1)

**Theorem 2.2.** For every \( n \geq 2 \), there exists a self centered graph of order \( 2n + 2 \) with a duplication self vertex switching.

**Proof.** Consider a path \( P_n \), where \( n \geq 2 \). Let \( P'_n \) be another copy of \( P_n \). Let \( V(P_n) = \{v_i/1 \leq i \leq n\} \) and \( V(P'_n) = \{v'_i/1 \leq i \leq n\} \), \( v_i \) corresponds to \( v'_i(1 \leq i \leq n) \). Construct a graph G with \( V(G) = V(P_n) \cup V(P'_n) \cup \{u, v\} \) and \( E(G) = E(P_n) \cup E(P'_n) \cup \{v_iv'_i, v'_iv_i, v_iv, v'_iv/vuv, u/vv_i/1 \leq i \leq n\} \). The resulting graph G is of order \( 2n + 2 \) and the eccentricity of each vertex in G is atmost two and hence \( \text{diam}(G) \leq 2 \). Also G does not contain any vertex of degree \( 2n + 1 \) and hence the radius of G is atleast two. This implies that the eccentricity of each vertex is two and so each vertex of G is a central vertex. Hence G is a self centered graph of radius 2. Define \( \alpha : V(G) \rightarrow V(G) \) such that \( \alpha(v_i) = u, \alpha(u) = v, \alpha(v_i) = v'_i \) and \( \alpha(v'_i) = v_i \), where \( 1 \leq i \leq n \). Clearly \( \alpha \) is an automorphism of G which maps elements of \( N(v) \) onto \( N(v)^c \) and vice versa. By Theorem 1.2, \( v \) is a duplication self vertex switching of G. Since \( \alpha(v) = u, u \) is also a duplication self vertex switching of G. The graph when \( n = 4 \) is illustrated in Fig. 2.2.

![Fig. 2.2](image2)

**Theorem 2.3.** Every connected graph is an induced subgraph of a self centered graph with a duplication self vertex switching.

**Proof.** Consider a graph G with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \). Now we construct a graph H with vertex set \( V(H) = \{v, u, u_1, u_2, \ldots, u_n, w_1, w_2, \ldots, w_n\} \) and the edge set \( E(H) = \{uv, u_v, u_w_i/1 \leq i \leq n\} \cup \{u_1u_j, w_jw_i/v_iw_j/1 \leq i, j \leq n\} \). Clearly G is an induced subgraph of H.

**Claim(i)**. H is a self centered graph.

From our construction, it is easy to verify that the eccentricity of every vertex is almost two and hence \( \text{diam}(H) \leq 2 \). Also the graph H is of order \( 2n + 2 \) and there is no vertex in H is of degree \( 2n + 1 \). This implies that \( r(H) \geq 2 \). Hence we conclude that H is a self centered graph of radius 2.

**Claim(ii)**. \( v \) is a duplication self vertex switching of H.

By our construction, \( N(v) = \{u, u_1, u_2, \ldots, u_n\} \) and \( N(v)^c = \{v, w_1, w_2, \ldots, w_n\} \). Define a map \( \alpha : V(H) \rightarrow V(H) \) such that \( \alpha(v) = u, \alpha(u) = v, \alpha(u_i) = w_i \) and \( \alpha(w_i) = u_i \), for \( 1 \leq i \leq n \). Clearly \( \alpha \) is an automorphism of G which maps elements of \( N(v) \) onto \( N(v)^c \) and vice versa. By Theorem 1.2, \( v \) is a duplication self vertex switching of H. For example, a graph G and the corresponding self centered graph H are shown in Fig. 2.3 and Fig. 2.4 respectively.

![Fig. 2.3](image3)
Theorem 2.4. Every connected graph is an induced subgraph of a graph $H$ with duplication self vertex switching $u$ and $v$ such that $H - \{u, v\}$ is self complementary.

**Proof.** Let $G$ be a connected graph of order $n$ and let $G'$ be another copy of $G$ with $V(G) = \{v_i/1 \leq i \leq n\}$, $V(G') = \{v'_i/1 \leq i \leq n\}$ and $v_i$ corresponds to $v_i(1 \leq i \leq n).$ Construct a graph $H$ with $V(H) = V(G) \cup V(G') \cup \{u, v, u_i, u'_i/1 \leq i \leq n\}$ and $E(H) = E(G) \cup E(G') \cup \{u_i u_j, u'_i u'_j, v_i v_j \notin E(G); 1 \leq i, j \leq n; i \neq j\} \cup \{v_i u'_i, u'_i v_i/1 \leq i \leq n\} \cup \{u, v, u_i, u'_i, v_i, v'_i/1 \leq i \leq n\}$. The resulting graph $H$ contains $G$ as an induced subgraph. Moreover, the graph $H$ has $4n + 2$ vertices and $d_H(v) = d_H(u) = 2n + 1$. Define $\alpha : V(H) \rightarrow V(H)$ such that $\alpha(v) = u; \alpha(u) = v; \alpha(v_i) = v'_i; \alpha(u_i) = u'_i$ and $\alpha(u'_i) = u_i$ where $1 \leq i \leq n$. Clearly $\alpha$ is an automorphism of $H$ which maps elements of $N(v)$ onto $[N(v)]'$ and vice versa. By Theorem 1.2, $v$ is a duplication self vertex switching of $G$. Since $\alpha(v) = u$, $u$ is also a duplication self vertex switching of $G$. For $1 \leq i \leq n$, we define a map $f : V(H - \{u, v\}) \rightarrow V(H - \{u, v\})'$ such that $f(v_i') = u_i'$; $f(u_i') = v_i'$; $f(v_i) = u_i'$; $f(u_i) = v_i$. Clearly $f$ is an isomorphism between $H - \{u, v\}$ onto $[H - \{u, v\}]'$. Therefore, $H - \{u, v\}$ is a self complementary graph of order $4n$. For example, a graph $G$ and the corresponding graph $H$ are shown in Fig. 2.5 and Fig. 2.6 respectively. \qed

3. Conclusion

In this paper we have discussed the possibility of the presence of duplication self vertex switching in some graphs. Also we have proved the existence of self centered graphs with duplication self vertex switching and we have constructed a graph $H$ with duplication self vertex switching $u$ and $v$ such that $H - \{u, v\}$ is self complementary.

References

[1] C. Jayasekaran, Graphs with a given number of self vertex switchings, *International Journal of Algorithms, Computing and Mathematics*, 3(3)(2010), 27-36.
[2] C. Jayasekaran, Self vertex switchings of connected unicyclic graphs, *Journal of Discrete Mathematical Sciences and Cryptography*, 15(6)(2012), 377-388.
[3] C. Jayasekaran, Self vertex switchings of trees, *Ars Combinatoria*, 127(2016), 33-43.
[4] C. Jayasekaran, Self vertex switchings of disconnected unicyclic graphs, *Ars Combinatoria*, 129(2016), 51-63.
[5] C. Jayasekaran, V. Prabavathy, A characterisation of duplication self vertex switching in graphs, *International Journal of Pure and Applied Mathematics*, 118(2)(2017), 1-10.
[6] C. Jayasekaran, V. Prabavathy, Duplication self vertex switching in some graphs, (Communicated).
[7] C. Jayasekaran, V. Prabavathy, Some results on duplication self vertex switchings, *International Journal of Pure and Applied Mathematics*, 116(2)(2017), 427-435.
[8] J. Lauri, Pseudosimilarity in graphs - a survey, *Ars Combinatoria*, 46(1997), 77-95.
[9] J.J. Seidel, A survey of two graphs, in *Proceedings of the International Coll. Theorie combinatorie* (Rome 1973), Tomo I, Acca. Naz. Lincei, (1976), pp. 481-511.
[10] R. Stanley, Reconstruction from vertex switching, *J. Combi. Theory. Series B*, 38 (1985), 138-142.
[11] V. VilFred, C. Jayasekaran, Interchange similar self vertex switchings in self centered graphs — 558/559.
switchings in graphs, *Journal of Discrete Mathematical Sciences and Cryptography*, 12(4)(2009), 467-480.

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