The Hamilton–Jacobi analysis for higher-order Maxwell–Chern–Simons gauge theory

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Abstract By using the Hamilton–Jacobi [HJ] framework, the higher-order Maxwell–Chern–Simons theory is analyzed. The complete set of HJ Hamiltonians and a generalized HJ differential are reported, from which all symmetries of the theory are identified. In addition, we complete our study by performing the higher-order Gitman–Lyakhovich–Tyutin [GLT] framework and compare the results of both formalisms.

1 Introduction

It is well known that higher-order theories are of interest in theoretical physics. In 1850, Ostrogradski developed works concerning the Hamiltonian formalism for systems with higher derivatives [1]. Since then, the research of systems such as generalizations of electrodynamics [2–5], string theory [6], and dark energy physics [7,8] has led to interesting results about gravity, where higher-order Lagrangians, which include quadratic products of the curvature tensor, have ensured the renormalization of these theories [9,10]. Moreover, when it comes to a theory with gauge symmetries, the standard study of higher-order theories is done by using the so-called Ostrogradski–Dirac framework. The Ostrogradski–Dirac framework is based on the extension of the phase space and on the choice of the fields and their temporal derivatives as canonical variables; the identification of the constraints is then performed as usual [11]. Nonetheless, the classification of the constraints into first or second class is a difficult task. Alternative approaches can be employed, like the one developed by Güler [12] based on the identification of the constraints, called Hamiltonians, and the construction of a fundamental differential. These Hamiltonians can be either involutive or non-involutive and are used to obtain the characteristic equations, the gauge symmetries, and the generalized HJ brackets of the theory. Using this approach, the construction of the fundamental differential is straightforward and the identification of symmetries is, in general, more economical than other approaches [12–18].

There is also a generalization of Ostrogradski’s framework, the so-called GLT formalism [19,20]. Based on the introduction of the canonical momenta as Lagrange multipliers, this framework allows one to reformulate the problem to one with only first-order time derivatives.

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Then, through a proper gauge fixing and making use of the Dirac brackets, the unphysical degrees of freedom can be removed. At the end of the calculations, the identification of the constraints is less complicated than in Ostrogradski’s method.

In this paper, we will analyze the higher-order Maxwell–Chern–Simons gauge theory [21,22]. We start with the Güler-HJ approach; we will introduce additional fields to reduce the problem to a first-order time derivative one. Due to this, non-physical degrees of freedom will appear, which are associated with non-involutive Hamiltonians; however, these will be removed with the introduction of the generalized HJ brackets. In this manner, the identification of the Hamiltonians will be straightforward; we also extend the results reported in [21,22]. Incidentally, we report an alternative study beyond the Ostrogradski–Dirac framework in search of the best alternative for analyzing higher-order singular systems. Then, we will finish our work by performing a GLT analysis. In fact, we will analyze the higher-order Maxwell–Chern–Simons from two different perspectives and we will compare our results.

The paper is organized as follows. In the Sect. 2, we develop the HJ analysis for the higher-order Maxwell–Chern–Simons gauge theory. We construct a fundamental differential, where the characteristics equations and all symmetries of the theory are found. Then, we reproduce and extend the results reported in [21,22]. In Sect. 3, the GLT formalism is implemented; we reduce the higher-order theory to a first-order one; then, we identify all constraints of the theory and present a complete description of the Dirac algebra.

2 The Hamilton–Jacobi analysis

The action under consideration is given by [21]

\[ \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{\theta}{4} \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} + \frac{1}{4m} \epsilon^{\mu\nu\lambda} (\Box A_\mu) F_{\nu\lambda}, \]  

where \( A_\mu \) is the gauge potential, \( F_{\mu\nu} \) is the curvature tensor, and \( \epsilon^{\mu\nu\lambda} \) is the Levi-Civita antisymmetric tensor. Throughout this paper, we will use the following metric convention \( \eta_{\mu\nu} = (-1, 1, 1) \); spacetime indices will be represented by Greek alphabet \( \alpha, \beta = 0, 1, 2 \) and space indices by the Latin one \( i, j, k = 1, 2 \).

By performing the \( 2 + 1 \) decomposition, we can write the action as

\[ \mathcal{L} = \frac{1}{2} \dot{A}^i \dot{A}_i - \dot{A}_i \partial^i A_0 - \frac{1}{2} \partial^i A_0 \partial_i A_0 - \frac{1}{4} F^{ij} F_{ij} + \frac{\theta}{2} \epsilon^{ij} A_0 \partial_i A_j - \frac{\theta}{2} \epsilon^{ij} A_i \dot{A}_j + \frac{\theta}{2} \epsilon^{ij} A_j \dot{A}_i A_0 \]

\[ - \frac{1}{2m} \epsilon^{ij} \ddot{A}_0 \partial_i A_j + \frac{1}{2m} \epsilon^{ij} \nabla^2 A_0 \partial_i A_j + \frac{1}{2m} \epsilon^{ij} \ddot{A}_i A_j - \frac{1}{2m} \epsilon^{ij} \nabla^2 A_i A_j - \frac{1}{2m} \epsilon^{ij} \ddot{A}_j A_i A_0 \]

\[ + \frac{1}{2m} \epsilon^{ij} \nabla^2 A_i \partial_j A_0, \]

(2)

For the purpose of analysis, we will write the Lagrangian (2) in a new fashion by introducing the following variables \( A_\mu \rightarrow \xi_\mu, \dot{A}_\mu \rightarrow v_\mu \). By doing this, the following constraints, given by \( \dot{\xi}_\mu - v_\mu = 0 \), will be added to the Lagrangian by means of new unphysical variables \( \psi^\mu \).

Thus, the Lagrangian takes the form

\[ \mathcal{L} = \frac{1}{2} v^j v_i - v_i \partial^j \xi_0 - \frac{1}{2} \partial^j \xi_0 \partial_i \xi_0 - \frac{1}{4} F^{ij} F_{ij} + \frac{\theta}{2} \epsilon^{ij} \xi_0 \partial_i \xi_j - \frac{\theta}{2} \epsilon^{ij} \dot{\xi}_i v_j + \frac{\theta}{2} \epsilon^{ij} \dot{\xi}_j v_i + \frac{1}{2m} \epsilon^{ij} \left( -\ddot{v}_0 + \nabla^2 \xi_0 \right) \partial_i \xi_j - \frac{1}{2m} \epsilon^{ij} \left( -\ddot{v}_i + \nabla^2 \xi_i \right) \partial_j \xi_0 \]

\[ + \psi^0 \left( v_0 - \dot{\xi}_0 \right) + \psi^j \left( v_i - \dot{\xi}_i \right). \]

(3)
We can observe that the theory is now linear in the temporal derivatives and we can apply the HJ analysis. From the definition of the momenta

\[ P^\mu = \frac{\partial \mathcal{L}}{\partial \dot{Q}_\mu}, \]

where \( Q_\mu = (\xi_0, \xi_i, v_0, v_i, \psi_0, \psi_i) \) are the canonical variables and \( P^\mu = (\pi^0, \pi^i, \pi^0, \pi^i, p^0, p^i) \) their corresponding momenta, we find the following Hamiltonians [12–18]

\[
\begin{align*}
\Omega_1^0 &= \pi^0 + \psi^0 = 0, \\
\Omega_1^i &= \pi^i + \psi^i = 0, \\
\Omega_2^0 &= \pi^0 + \frac{1}{2m} \epsilon^{ij} \partial_i \xi_j = 0, \\
\Omega_2^i &= \pi^i - \frac{1}{2m} \epsilon^{ij} (v_j - \partial_j \xi_0) = 0, \\
\Omega_3^0 &= p^0 = 0, \\
\Omega_3^i &= p^i = 0;
\end{align*}
\]

and the canonical Hamiltonian, given by

\[
\mathcal{H} = \dot{\xi}_0 \pi^0 + \dot{\xi}_i \pi^i + \dot{\psi}_0 p^0 - \mathcal{L} = -\frac{1}{2} v^i v_i + v_i \partial^i \xi_0 + \frac{1}{2} \partial^i \epsilon^{0j} \partial_j \xi_0 + \frac{1}{4} F^{ij} F_{ij} - \frac{\theta}{2} \epsilon^{ij} \xi_0 \partial_i \xi_j + \frac{\theta}{2} \epsilon^{ij} \partial_i v_j - \frac{\theta}{2} \epsilon^{ij} \partial_i \xi_0
\]

\[ + \pi^0 \nabla^2 \xi_0 + \nabla^2 \xi_i + \pi^0 v_0 + \pi^i v_i. \]

Thus, with the Hamiltonians identified, we construct the fundamental differential, which describes the evolution of any function, say \( F \), on the phase space [12–18]

\[
dF = \int \left[ \{ F, \mathcal{H} \} dt^0 + \{ F, \Omega_1^0 \} dw^0_0 + \{ F, \Omega_1^i \} dw^0_i + \{ F, \Omega_2^0 \} dw^2_0 + \{ F, \Omega_2^i \} dw^2_i + \{ F, \Omega_3^0 \} dw^3_0 + \{ F, \Omega_3^i \} dw^3_i \right] d^2 y,
\]

where \( \omega^0_0, \omega^0_1, \omega^2_0, \omega^2_1, \omega^3_0, \omega^3_1 \) are parameters associated with the Hamiltonians. To this end, we separate the Hamiltonians into involutive and non-involutive. Involutive Hamiltonians are those whose Poisson brackets with all Hamiltonians, including themselves, vanish; otherwise, they are called non-involutive. These will be labeled by \( \Gamma \) and \( \Lambda \), respectively. The Poisson algebra between the Hamiltonians in (4) is given by

\[
\begin{align*}
\{ \Omega_1^0(x), \Omega_1^0(y) \} &= -\frac{1}{2m} \epsilon^{ij} \partial_i \delta^2(x-y), \quad \{ \Omega_1^0(x), \Omega_2^0(y) \} = -\delta^2(x-y) \\
\{ \Omega_1^i(x), \Omega_2^0(y) \} &= -\frac{1}{2m} \epsilon^{ij} \partial_i \delta^2(x-y), \quad \{ \Omega_1^i(x), \Omega_3^0(y) \} = \eta^{ij} \delta^2(x-y), \\
\{ \Omega_1^i(x), \Omega_2^i(y) \} &= -\frac{1}{m} \epsilon^{ij} \delta^2(x-y)
\end{align*}
\]

and hence, we observe that all the Hamiltonians are non-involutive, particularly those related to the unphysical fields \( \psi^\mu \) and their momenta \( p^\mu \). The matrix composed of these Poisson brackets, namely

\[
\Delta_{ab} = \begin{pmatrix}
0 & 0 & 0 & -\frac{1}{2m} \epsilon^{jk} \partial_k -1 & 0 \\
0 & 0 & 0 & 0 & \eta^{ij} & 0 \\
0 & -\frac{1}{2m} \epsilon^{jk} \partial_j & 0 & 0 & 0 & 0 \\
\frac{1}{2m} \epsilon^{ij} \partial_j & 0 & 0 & -\frac{1}{m} \epsilon^{ij} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\eta^{ij} & 0 & 0 & 0 & 0
\end{pmatrix} \delta^2(x-y),
\]
is not invertible, which means that the Hamiltonians are not independent. We will use the null vectors $\zeta$ of the matrix $\Delta_{ab}$ to identify the independent ones, such as it is done in a pure Dirac framework [23]

$$
\int d^2y \Delta^{\mu\nu} \zeta(y)_{\mu} = 0.
$$

Then, a null vector is found, $\zeta_{\mu} = (0, 0, w, 0, -\frac{1}{2m} \epsilon_{ij} \partial^j w)$, where $w$ is an arbitrary function. Contracting $\zeta_{\mu}$ with a vector composed of the non-involutive Hamiltonians $\Omega^\mu = (\Omega^0_1, \Omega^i_1, \Omega^0_2, \Omega^i_2, \Omega^0_3, \Omega^i_3)$ yields a new Hamiltonian, given by

$$
\zeta_{\mu} \Omega^\mu = 0,
$$

$$
\rightarrow \Gamma_1 : \pi^0 + \frac{1}{2m} \epsilon^{ij} \partial_i \xi_j - \frac{1}{2m} \epsilon^{ij} \partial_j p_l = 0.
$$

Since its Poisson brackets with all other Hamiltonians (4) vanishes, this new Hamiltonian is an involutive one. In this manner, the complete set of non-involutives Hamiltonians is given by

$$
\Lambda_1 = \pi^0 + \psi^0 = 0,
\Lambda_2 = \pi^i + \psi^i = 0,
\Lambda_3^i = \tilde{\pi}^i - \frac{1}{2m} \epsilon^{ij} (v_j - \partial_j \xi_0) = 0,
\Lambda_4 = p^0 = 0,
\Lambda_5^i = p^i = 0,.
$$

Thus, the new $\Delta_{ab}$ matrix, whose entries will be the Poisson brackets between the new non-involutives Hamiltonians (10), takes the form

$$
\Delta_{ab} =
\begin{pmatrix}
0 & 0 & -\frac{1}{2m} \epsilon^{ij} \partial_k -1 & 0 \\
0 & 0 & 0 & -\eta^{ij} \\
\frac{1}{2m} \epsilon^{ij} \partial_j & 0 & -\frac{1}{m} \epsilon^{ij} & 0 \\
1 & 0 & 0 & 0 \\
0 & -\eta^{ij} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \delta^2(x - y).
$$

which is found to not be singular; therefore, it has an inverse, given by

$$
\Delta^{-1}_{ab}(x, y) =
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -\eta_{ij} & 0 \\
0 & 0 & m \epsilon_{jl} \frac{1}{2} \partial_j & 0 \\
-1 & 0 & -\frac{1}{2} \partial_l & 0 & 0 \\
0 & \eta_{jl} & 0 & 0 & 0
\end{pmatrix} \delta^2(x - y).
$$

With this inverse matrix at hand, we introduce the generalized brackets, defined as

$$
\{A(x), B(x')\}^* = \{A(x), B(x')\} - \int \int \{A(x), \xi^a(y)\} \Delta^{-1}_{ab}(y, z) \{\xi^b(z), B(x')\} \, d^2y \, d^2z,
$$

where $\xi^\mu$ represent the non-involutives Hamiltonians and $\Delta^{-1}_{ab}$ is the inverse of the matrix $\Delta_{ab}$, whose entries are the Poisson brackets between the non-involutives Hamiltonians. Hence, by
using the generalized brackets (12) we can calculate the nontrivial ones between the phase space variables; these are
\[
\{\xi_{\mu} , \pi^\nu\}^* = \delta^\nu_{\mu} \delta^2(x - y), \quad \{\xi_{\mu} , \psi_\nu\}^* = -\eta_{\mu\nu} \delta^2(x - y),
\]
\[
\{\pi^0 , v_k\}^* = \frac{1}{2} \partial_k \delta^2(x - y), \quad \{\pi^0 , \tilde{\pi}^k\}^* = - \frac{1}{4m} \epsilon^{kij} \partial_j \delta^2(x - y),
\]
\[
\{v_0 , \tilde{\pi}^0\}^* = \delta^2(x - y), \quad \{v_i , v_k\}^* = m \epsilon_{ijk} \delta^2(x - y),
\]
\[
\{v_i , \tilde{\pi}^k\}^* = \frac{1}{2} \delta_i^k \delta^2(x - y), \quad \{v_i , \psi_0\}^* = \frac{1}{2} \partial_i \delta^2(x - y),
\]
\[
\{\tilde{\pi}^i , \tilde{\pi}^k\}^* = \frac{1}{4m} \epsilon^{ikj} \delta^2(x - y), \quad \{\tilde{\pi}^i , \psi_0\}^* = - \frac{1}{4m} \epsilon^{ij} \partial_j \delta^2(x - y).
\]

These generalized brackets will coincide with those of Dirac, which are calculated in the next section. In particular, we can observe that the generalized \(HJ\) bracket between the Hamiltonian (9) with itself gives
\[
\{\Gamma_1(x) , \Gamma_1(y)\}^* = 0,
\]
confirming that \(\Gamma_1\) is indeed involutive. In this manner, the introduction of the \(HJ\) brackets removes the non-involutive Hamiltonians and leaves us with a new fundamental differential, given by
\[
dF = \int \left[ \{F , \mathcal{H}(y)\}^* dt^0 + \{F , \Gamma_1(y)\}^* d\sigma^1 \right] d^2y.
\]

By using the fundamental differential, we have removed the unphysical degrees of freedom \(\psi^0\) and \(\psi^i\), making the results in this section match those of the next section. In this regard, once the generalized brackets are introduced, we could perform the substitution of the fields \(\psi^s\) by the momenta \(\pi^s\) in the action (3); the result would be that the \(HJ\) and GLT actions are equivalent. In other approaches, see ref. [22], the unphysical degrees of freedom are removed until the end of the calculations, because the separation of the constraints into first and second class allows the introduction of the Dirac brackets; in contrast, in the \(HJ\) framework, the elimination of unphysical degrees of freedom is more convenient. We also have to take into account the Frobenius integrability conditions, which ensure that system is integrable. Applying this conditions to the Hamiltonian \(\Gamma_1\), the following Hamiltonian arises
\[
d\Gamma_1(x) = \int \left[ \{\Gamma_1(x) , \mathcal{H}(y)\}^* dt^0 + \{\Gamma_1(x) , \Gamma_1(y)\}^* d\sigma^1 \right] d^2y = 0,
\]
\[
\rightarrow \Gamma_2 \equiv \pi^0 - \partial_i \tilde{\pi}^i = 0,
\]
we observe that since \(\{\Gamma_2(x) , \Gamma_2(y)\}^* = 0, \Gamma_2\) is an involutive Hamiltonian. We add this new involutive Hamiltonian to the fundamental differential and then calculate its integrability, obtaining a new involutive Hamiltonian
\[
d\Gamma_2(x) = \int \left[ \{\Gamma_2(x) , \mathcal{H}(y)\}^* dt^0 + \{\Gamma_2(x) , \Gamma_1(y)\}^* d\sigma^1 + \{\Gamma_2(x) , \Gamma_2(y)\}^* d\sigma^2 \right] d^2y = 0
\]
\[
\rightarrow \Gamma_3 \equiv \partial_i \pi^i + \frac{\theta}{2} \epsilon^{ij} \partial_i \xi_j + \frac{1}{2m} \epsilon^{ij} \nabla^2 \partial_i \xi_j = 0.
\]

No further Hamiltonians emerge from the integrability conditions of \(\Gamma_3\). As a result, the complete set of involutive Hamiltonians is given by
\[
\Gamma_1 = \tilde{\pi}^0 + \frac{1}{2m} \epsilon^{ij} \partial_i \xi_j = 0,
\]
\[
\Gamma_2 = \pi^0 - \partial_i \tilde{\pi}^i = 0,
\]
\[
\Gamma_3 = \partial_i \pi^i + \frac{\theta}{2} \epsilon^{ij} \partial_i \xi_j + \frac{1}{2m} \epsilon^{ij} \nabla^2 \partial_i \xi_j = 0.
\]
And the complete fundamental differential becomes

\[
dF = \int \left[ ( F, \mathcal{H}(y) )^* d\sigma^0 + ( F, \Gamma_1(y) )^* d\sigma^1 + ( F, \Gamma_2(y) )^* d\sigma^2 + ( F, \Gamma_3(y) )^* d\sigma^3 \right] d^2y,
\]

where \( \sigma^1, \sigma^2, \sigma^3 \) are parameters associated with the Hamiltonians. Therefore, we have presented an alternative for studying higher-order theories in the context of \( HJ \) theory, which is more economical than those previously reported in the literature. With the fundamental differential, we can obtain the characteristic equations and then identify the symmetries, which is done in appendix A.

### 3 The Gitman–Lyakhovich–Tyutin framework

In order to complete our analysis, the \( GLT \) formalism will be performed. Starting with the Lagrangian (2), and following the formalism, we introduce the following variables \[19,20\]

\[
v_\mu = \dot{A}_\mu, \quad \beta_\mu = \dot{\nu}_\mu,
\]

and their conjugated canonical momenta, satisfying

\[
\{ A_\mu, \pi^v \} = \delta_\mu^v \delta^2(x - y),
\]

\[
\{ v_\mu, \tilde{\pi}^v \} = \delta_\mu^v \delta^2(x - y).
\]

Thus, the Lagrangian (2) can be written as

\[
\tilde{\mathcal{L}} = \mathcal{L} + \pi^\mu \left( \dot{A}_\mu - v_\mu \right) + \tilde{\pi}^\mu \left( \dot{\nu}_\mu - \beta_\mu \right),
\]

that is,

\[
\tilde{\mathcal{L}} = \frac{1}{2} v^i v_i - v_i \delta^i A_0 - \frac{1}{2} \delta^i A^0 \partial_i A_0 - \frac{1}{4} F^{ij} F_{ij} + \frac{\theta}{2} \epsilon^{ij} A_0 \partial_i A_j - \frac{\theta}{2} \epsilon^{ij} A_i v_j + \frac{\theta}{2} \epsilon^{ij} A_i \partial_j A_0
\]

\[
- \frac{1}{2m} \epsilon^{ij} \beta_0 \partial_i A_j + \frac{1}{2m} \epsilon^{ij} \nabla^2 A_0 \partial_i A_j + \frac{1}{2m} \epsilon^{ij} \beta_i v_j - \frac{1}{2m} \epsilon^{ij} \nabla^2 A_i v_j - \frac{1}{2m} \epsilon^{ij} \beta_i \partial_j A_0
\]

\[
+ \frac{1}{2m} \epsilon^{ij} \nabla^2 A_i \partial_j A_0 + \pi^\mu \left( \dot{A}_\mu - v_\mu \right) + \tilde{\pi}^\mu \left( \dot{\nu}_\mu - \beta_\mu \right).
\]

We can observe that the theory is now first order in the time derivatives, as well as that the canonical momenta have been introduced from the beginning. It is worth mentioning that the introduction of the momenta allows us to more easily identify the constraints compared to Ostrogradski’s formalism. In fact, in the \( GLT \) framework it is not necessary to introduce a generalized canonical momenta for the higher-order time derivatives of the fields, as is done in Ostrogradski’s framework \[19–22\]. Subsequently, the canonical Hamiltonian is given as usual

\[
\mathcal{H} = \dot{A}_\mu \pi^\mu + \dot{\nu}_\mu \tilde{\pi}^\mu - \tilde{\mathcal{L}},
\]

\[
= v_0 \pi^0 + v_i \pi^i + \beta_0 \tilde{\pi}^0 + \beta_i \tilde{\pi}^i - \frac{1}{2} v^i v_i + v_i \partial^i A_0
\]

\[
+ \frac{1}{4} \delta^i A^0 \partial_i A_0 + \frac{1}{2} \delta^i F_{ij} + \frac{\theta}{2} \epsilon^{ij} A_0 \partial_i A_j
\]

\[
+ \frac{\theta}{2} \epsilon^{ij} A_i v_j - \frac{\theta}{2} \epsilon^{ij} A_i \partial_j A_0
\]
\[
+ \frac{1}{2m} \epsilon^{ij} \beta_0 \partial_i A_j - \frac{1}{2m} \epsilon^{ij} \nabla^2 A_0 \partial_i A_j - \frac{1}{2m} \epsilon^{ij} \beta_i v_j \\
+ \frac{1}{2m} \epsilon^{ij} \nabla^2 A_i v_j \\
+ \frac{1}{2m} \epsilon^{ij} \beta_i \partial_j A_0 - \frac{1}{2m} \epsilon^{ij} \nabla^2 A_i \partial_j A_0.
\] (21)

Thus, the primary constraints (called \( \phi \)) are given by [19,20]

\[
\phi^3 = \frac{\partial L}{\partial \beta_0} - \tilde{\pi}^0 = -\frac{1}{2m} \epsilon^{ij} \partial_i A_j - \tilde{\pi}^0 \approx 0, \tag{22}
\]

\[
\phi^i = \frac{\partial L}{\partial \beta_i} - \tilde{\pi}^i = \frac{1}{2m} \epsilon^{ij} v_j - \frac{1}{2m} \epsilon^{ij} \partial_j A_0 - \tilde{\pi}^i \approx 0. \tag{23}
\]

They coincide only with \( \Omega_{2}^{0} \), and \( \Omega_{2}^{i} \) from equation 4. Their algebra is given by

\[
\{ \phi^3, \phi^i \} = 0,
\]

\[
\{ \phi^i, \phi^j \} = -\frac{1}{m} \epsilon^{ij} \delta^2(x-y). \tag{24}
\]

At this point, it is important to comment the differences between the \( HJ \) formalism and the GLT formulation. On the one hand, in GLT’s formulation we must identify future constraints through consistency and then perform the classification of the constraints into first and second class; then, Dirac’s brackets are introduced and second class constraints can be taken strongly as zero. Only at the end of the calculations, we can compare both formalisms. On the other hand, in the \( HJ \) scheme the generalized brackets, which has an equivalent construction just as the Dirac ones, are introduced from the beginning. At the end of the calculations, one ends up only with involutive Hamiltonians, which will agree with the set of first-class constraints of the GLT formalism.

We continue with the classification of the constraints. By using the primary constraints, we introduce the primary Hamiltonian

\[
\mathcal{H}' = \mathcal{H} + \lambda_3 \phi^3 + \lambda_i \phi^i, \tag{25}
\]

where \( \lambda_3 \) and \( \lambda_i \) are Lagrange multipliers; thus, by using (24) and by requiring consistency of the primary constraints, we obtain a secondary constraint

\[
\chi^0 : \phi^3 = \{ \phi^3, \mathcal{H}' \} = \pi^0 - \frac{1}{2m} \epsilon^{ij} \partial_i v_j \approx 0, \tag{26}
\]

Consistency of \( \phi^i \) provides a relation between the Lagrange multipliers, that is,

\[
\dot{\phi}^i = \{ \phi^i, \mathcal{H}' \} = -\epsilon^{ij} \lambda_j + \epsilon^{ij} \beta_j - \frac{1}{2} \epsilon^{ij} \partial_j v_0 + m \pi^i - m v^i + m \partial^i A_0 - \frac{\partial m}{2} \epsilon^{ij} A_j - \frac{1}{2} \epsilon^{ij} \nabla^2 A_j \approx 0. \tag{27}
\]

Now, by demanding consistency of the secondary constraint \( \chi^0 \) we find

\[
\dot{\chi}^0 = \{ \chi^0, \mathcal{H}' \} = -\epsilon^{ij} \partial_i \lambda_j + \epsilon^{ij} \partial_i \beta_j - m \partial^i v_i + m \nabla^2 A_0 - \partial m \epsilon^{ij} \partial_i A_j - \epsilon^{ij} \nabla^2 \partial_i A_j \approx 0, \tag{28}
\]
which also contains relations between the Lagrange multipliers. Furthermore, from (27) and (28) we can eliminate the Lagrange multipliers to obtain yet another secondary constraint

$$\chi^1 = \partial_i \pi^i + \frac{\theta}{2} \epsilon^{ij} \partial_i A_j + \frac{1}{2m} \epsilon^{ij} \nabla^2 \partial_i A_j \approx 0,$$

(29)

From consistency of $\chi^1$, no more constraints are found. In this manner, the complete set of GLT constrains is given by

$$\phi^3 = \tilde{\pi}^0 + \frac{1}{2m} \epsilon^{ij} \partial_i A_j \approx 0,$$

$$\phi^i = \tilde{\pi}^i - \frac{1}{2m} \epsilon^{ij} v_j + \frac{1}{2m} \epsilon^{ij} \partial_j A_0 \approx 0,$$

$$\chi^0 = \pi^0 - \frac{1}{2m} \epsilon^{ij} \partial_i v_j \approx 0,$$

$$\chi^1 = \partial_i \pi^i + \frac{\theta}{2} \epsilon^{ij} \partial_i A_j + \frac{1}{2m} \epsilon^{ij} \nabla^2 \partial_i A_j \approx 0.$$

(30)

Notice that $\phi^i$ is actually two constraints, so there are five in total. To separate them into first and second class, we calculate the $5 \times 5$ matrix whose entries are the Poisson brackets between all constraints. This, in compact form, is

$$A = \begin{pmatrix}
-\frac{1}{m} \epsilon^{ij} & 0 & \frac{1}{m} \epsilon^{ij} \partial_j 0 \\
0 & 0 & 0 \\
-\frac{1}{m} \epsilon^{ij} \partial_i & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \delta^2(x - y),$$

(31)

we observe that this matrix has a rank=2 and 3 null vectors; this means that there will be two second class constraints and three first class ones [23]. The contraction of the null vectors with the constraints (30) allows us identify the following first class constraints

$$\gamma^1 = \tilde{\pi}^0 + \frac{1}{2m} \epsilon^{ij} \partial_i A_j,$$

$$\gamma^2 = \pi^0 - \partial_i \tilde{\pi}^i,$$

$$\gamma^3 = \partial_i \pi^i + \frac{\theta}{2} \epsilon^{ij} \partial_i A_j + \frac{1}{2m} \epsilon^{ij} \nabla^2 \partial_i A_j,$$

(32)

e.g., one null vector is given by $\tilde{v} = (0, \partial_i w, w, 0),$ and from the contraction with (30) we obtain $\gamma^2$. We observe that the constraints (32) coincide with the Hamiltonians (16) obtained with $HJ$ framework in the previous section. The two second class constraints are

$$\xi^i = \tilde{\pi}^i - \frac{1}{2m} \epsilon^{ij} v_j + \frac{1}{2m} \epsilon^{ij} \partial_j A_0,$$

(33)

even if these constraints are removed from the beginning in $HJ$ approach, in this sense the $HJ$ is more economical. It is worth commenting that the constraints have been obtained in consistent form by using the ideas presented in [23] and it is not necessary to fix them by hand such as has been done previously in the literature [22]. With the identification of the correct constraints, we can carry out the counting of physical degrees of freedom as follows: there are 12 canonical variables $\{A_\mu, \pi^\nu\}, \{v_\mu, \tilde{\pi}^\nu\}$, three first class constraints ($\gamma^1, \gamma^2, \gamma^3$) and two second class constraints ($\xi^i$); therefore, there are two physical degrees of freedom, as expected [21].
The second class constraints can be removed by means of the Dirac bracket
\[\{A(x), B(x')\}_D = \{A(x), B(x')\} - \int \int \{A(x), \xi^a(u)\} \Delta_{ab}^{-1}(y, z) (\xi^b(z), B(x')) \, d^2y \, d^2z,\]
where \(\Delta_{ab}^{-1}\) is the inverse of \(\Delta_{ab}\), which consists of Poisson brackets among the second class constraints: \(\Delta_{ab} = \{\xi^a, \xi^b\}\). This \(2 \times 2\) matrix is as follows
\[\Delta^{ij}(x, y) = me^{ij}\delta^2(x - y)\]
(35)

This results in the following non-trivial Dirac’s brackets
\[\{A^0, \pi^0\}_D = \delta^2(x - y), \quad \{A^i, \pi^j\}_D = \delta_{ij}\delta^2(x - y), \quad \{\pi^0, v_i\}_D = \frac{1}{2} \partial_i \delta^2(x - y), \quad \{\pi^0, \tilde{\pi}^i\}_D = -\frac{1}{4m} \epsilon^{ij} \partial_j \delta^2(x - y), \quad \{v_0, \pi^i\}_D = \delta^2(x - y), \quad \{v_i, \pi^j\}_D = \frac{1}{2} \delta_{ij}\delta^2(x - y), \quad \{v_i, v_j\}_D = me^{ij}\delta^2(x - y), \quad \{\tilde{\pi}^i, \tilde{\pi}^j\}_D = \frac{1}{4m} \epsilon^{ij} \partial_j \delta^2(x - y).\]
(36)

Using these, we see that the constraints \(\gamma^1\), \(\gamma^2\), and \(\gamma^3\) are still first class. We will now fix the gauge in order to remove all first class constraints, turning them into second class. It is important to comment that the gauge-fixing condition removes the redundant degrees of freedom [11]. Demanding consistency of the Coulomb gauge \(\gamma^4 = \partial_t A^i\) results in
\[\gamma^5 := \{\partial_t A^i, H\}_D = \partial_t v^i.\]
(37)

Demanding consistency of \(\gamma^5\) yields \(\gamma^6\)
\[\gamma^6 := \{\partial_t v^i, H\}_D = \frac{1}{2} \nabla^2 v_0 + me^{ij} \partial_i \pi_j - me^{ij} \partial_i v_j;\]
(38)

preservation in time of \(\gamma^6\) gives no new constraints. Below we present the nontrivial brackets among all constraints
\[\{\gamma^4, \gamma^3\}_D = -\nabla^2 \delta^2(x - y), \quad \{\gamma^5, \gamma^2\}_D = \nabla^2 \delta^2(x - y), \quad \{\gamma^6, \gamma^1\}_D = \nabla^2 \delta^2(x - y), \quad \{\gamma^6, \gamma^3\}_D = \frac{\theta m}{2} \nabla^2 \delta^2(x - y) + \frac{1}{2} \nabla^4 \delta^2(x - y), \quad \{\gamma^5, \gamma^6\}_D = m^2 \nabla^2 \delta^2(x - y).\]
(39)

Since, as can be easily seen, \(\gamma_1, ..., \gamma_6\) are all second class constraints, a new Dirac bracket can be introduced. In fact, by using (39) the new Dirac’s brackets, say \(\{,\}_D\), are given by
\[\{A^0, v_0\}_D = m^2 \frac{1}{2} \frac{\theta m}{m^2\nu^2} \delta^2(x - y), \quad \{A^0, \pi^i\}_D = me^{ij} \partial_j \delta^2(x - y), \quad \{A^i, \pi^j\}_D = \frac{\theta m}{2} \nabla^2 \delta^2(x - y), \quad \{A^i, \pi^0\}_D = -\frac{1}{2} \frac{\theta m}{m^2\nu^2} \delta^2(x - y), \quad \{A^0, \pi^0\}_D = -\frac{1}{2} \frac{\theta m}{m^2\nu^2} \delta^2(x - y), \quad \{A^i, v_i\}_D = -me^{ij} \partial_j \delta^2(x - y), \quad \{v_0, \pi^i\}_D = \frac{1}{2} \nabla^2 \delta^2(x - y), \quad \{v_0, \pi^0\}_D = -\frac{m}{2} \nabla^2 \delta^2(x - y), \quad \{v_i, \pi^0\}_D = -\frac{1}{2} \frac{\theta m}{m^2\nu^2} \delta^2(x - y), \quad \{v_i, \pi^i\}_D = -\frac{1}{4m} \epsilon^{ij} \partial_j \delta^2(x - y), \quad \{\pi^0, \pi_i\}_D = \frac{1}{4m} \epsilon^{ij} \partial_j \delta^2(x - y), \quad \{\pi^i, \pi^0\}_D = \frac{1}{4m} \epsilon^{ij} \partial_j \delta^2(x - y), \quad \{\pi^i, \pi^j\}_D = \frac{1}{4} \left( \delta^{ij} - \frac{\theta m}{m^2\nu^2} \right) \delta^2(x - y).\]
(40)
These brackets were not reported in [21,22]. They can be used for quantization of the theory by using the methods reported in [28], where a procedure of gauge fixing is developed in the path integral approach. In this manner, our results extend those reported in the literature.

4 Conclusions

A detailed HJ and GLT analysis for higher-order Maxwell–Chern–Simons theory was developed. Regarding the HJ study, with the introduction of auxiliary fields the theory was written as a first-order time derivative Lagrangian, and by means of the null vectors all Hamiltonians were identified. Then, with the introduction of the generalized HJ brackets, all unphysical fields were removed. We then constructed a fundamental differential given in terms of the generalized brackets and involutive Hamiltonians. This allowed us to identify the characteristic equations of the theory, where the equations of motion and the gauge transformations were reported. In this manner, we showed that the HJ is an excellent framework for analyzing higher-order systems.

On the other hand, from the GLT we report the complete structure of the constraints. We observed that the constraints were obtained in a consistent way, and there was no need to fix their structure by hand, as developed previously in the literature. Additionally, by fixing the gauge the complete structure of the Dirac brackets was presented. Therefore, our analysis extends those results presented in [21,22], where different approaches were used. Finally, the study developed in this paper can be extended to theories with a more extensive structure, such as gravity and string theory. However, all those results are in progress and will be the subject of forthcoming works [29].

5 Appendix: Gauge transformations

5.1 HJ formalism

We start by calculating the characteristic equations from the fundamental differential, which will reveal the symmetries of the theory. Using (17), we find them to be

\[ d\xi_0 = v_0 dt - d\sigma^2, \]

\[ d\xi_i = v_i dt + \partial_i d\sigma^3, \]

\[ d\pi^0 = \left[ \frac{1}{2} \partial_i v^i - \frac{1}{2} \nabla^2 \xi_0 + \frac{3\theta}{4} \epsilon^{ij} \partial_i \xi_j - \nabla^2 \tilde{\pi}^0 + \frac{1}{4m} \epsilon^{ij} \nabla^2 \partial_i \xi_j + \frac{1}{2} \partial_i \pi^i \right] dt, \]

\[ d\pi^i = \left[ -\partial_j F^{ij} - \frac{\theta}{2} \epsilon^{ij} v_j - \nabla^2 \tilde{\pi}^i \right] dt - \frac{1}{2m} \epsilon^{ij} \partial_j d\sigma^1 + \left[ \frac{\theta}{2} \epsilon^{ij} \partial_j + \frac{1}{2m} \epsilon^{ij} \nabla^2 \partial_j \right] d\sigma^3, \]

\[ dv_0 = \nabla^2 \xi_0 dt + d\sigma^1, \]

\[ dv_i = \left[ \frac{1}{2} \nabla^2 \xi_i + \frac{1}{2} \partial_i v_0 - m \epsilon_{ij} v^j + m \epsilon_{ij} \partial^i \xi_0 + \frac{\theta m}{2} \xi_i + m \epsilon_{ij} \pi^j \right] dt - \partial_i d\sigma^2, \]

\[ d\tilde{\pi}^0 = -\pi^0 dt, \]

\[ d\tilde{\pi}^i = \left[ \frac{1}{2} v^i - \frac{1}{2} \partial^i \xi_0 + \frac{\theta}{4} \epsilon^{ij} \xi_j - \frac{1}{4m} \epsilon^{ij} \partial_j v_0 + \frac{1}{4m} \epsilon^{ij} \nabla^2 \xi_j - \frac{1}{2} \pi^i \right] dt. \] (41)

The evolution of the dynamical variables with respect to our parameters \( \sigma^i \) is understood as canonical transformations, with the corresponding Hamiltonians \( \Gamma^i \) as generators [24,25].
Due to Frobenius’ theorem [25], the transformation with respect to one of these parameters is independent of the evolution along the others. To relate these canonical transformations to the gauge ones, we set $dt = 0$ [15], obtaining

$$
\delta \xi_0 = -\delta \sigma^2, \\
\delta \xi_i = \partial_i \delta \sigma^3, \\
\delta \pi^0 = 0, \\
\delta \pi^i = -\frac{1}{2m} \epsilon^{ij} \partial_j \delta \sigma^1 + \left[ \frac{\theta}{2} \epsilon^{ij} \partial_j + \frac{1}{2m} \epsilon^{ij} \nabla^2 \partial_j \right] \delta \sigma^3, \\
\delta \nu_0 = \delta \sigma^1, \\
\delta \nu_i = -\partial_i \delta \sigma^2, \\
\delta \tilde{\pi}^0 = 0, \\
\delta \tilde{\pi}^i = 0.
$$

(42)

In $\text{HJ}$, to find the gauge transformations it is necessary to see the specific conditions in which (42) acts into the Lagrangian. Thus, the Lagrangian (1) becomes invariant under these transformations if $\delta L = 0$. This will result in relations between the parameters $\sigma^2, \sigma^3$. The variation in the Lagrangian is

$$
\delta L = \int dt \, d^2 x \left[ \frac{\partial L}{\partial A_\mu} \delta A_\mu + \frac{\partial L}{\partial (\partial_\nu A_\mu)} \delta (\partial_\nu A_\mu) + \frac{\partial L}{\partial (\partial_\nu \partial_\mu A_\mu)} \delta (\partial_\nu \partial_\mu A_\mu) \right]:
$$

here we use $A_\mu$ instead of $\xi_\mu$ to more easily compare both formalisms. This, up to a total time derivative, is found to be

$$
\delta L = \int dt \, d^2 x \left[ \theta \epsilon^{\nu \lambda} \partial_\nu A_\lambda + \partial_\rho F^{\rho \sigma} - \frac{1}{2m} \epsilon^{\rho \sigma \mu} (\partial_0 \partial_\mu A_\mu) + \frac{1}{m} \epsilon^{\sigma \nu \lambda} \nabla^2 \partial_\nu A_\lambda \right] \delta A_\sigma = 0.
$$

(43)

We can combine the first and second equations in (42) to write the variation in $A_\sigma$ as

$$
\delta A_\sigma = -\delta_0^0 \delta \sigma^2 + \delta_\sigma^i \partial_i \delta \sigma^3;
$$

(44)

thus, by using (44) into (43) the variation in the action takes the form

$$
\delta L = -\int dt \, d^2 x \left( \theta \epsilon^{ij} \partial_i A_j + \partial_i F^{ij0} - \frac{1}{2m} \epsilon^{ij} \partial_i \tilde{A}_j + \frac{1}{m} \epsilon^{ij} \nabla^2 \partial_i A_j \right) \left( \delta \sigma^2 + \partial_0 \delta \sigma^3 \right) = 0.
$$

(45)

The theory will be invariant under (42) if the parameters $\sigma^i$ obey

$$
\delta \sigma^2 = -\partial_0 \delta \sigma^3;
$$

(46)

hence, from (44) the gauge transformations are given by

$$
\delta A_\mu = \partial_\mu \delta \sigma^3.
$$

(47)

Additionally, since $v_\mu = \dot{A}_\mu$, it can be seen that $\delta \sigma^1 = \partial_0 \partial_0 \delta \sigma^3$. 

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5.2 GLT formalism

In this section, we use Castellani’s procedure \cite{22,26,27} to obtain the gauge transformations. We start this calculation with the Hamiltonian (21), the constraints given in (32), and the Dirac brackets (36). First, we define the gauge generator as

$$G = \int \epsilon_a \gamma^a d^2 x,$$

(48)

where \(\epsilon_a\) are the gauge parameters and \(a = 1, 2, 3\). This generates infinitesimal gauge transformations on phase space variables, say \(F\), through

$$\delta F = \int \delta \epsilon_a(y) \left\{ F(x), \gamma^a(y) \right\}_D d^2 y.$$

(49)

In particular, the generator obeys the following equation, called the master equation,

$$\frac{\partial}{\partial t} G + \{G, \mathcal{H}_T\}_D = 0.$$

(50)

where \(\mathcal{H}_T = \mathcal{H} + u_a \gamma^a\) is the total Hamiltonian. From the algebra of the constraints and the canonical Hamiltonian \(\mathcal{H}\), we can obtain the structure functions \(V_{ab}, C^a_{bc}\), given by

$$\left\{ \mathcal{H}, \gamma^a(x) \right\}_D = \int d^2 y V^a_b(x, y) \gamma^b(y),$$

(51)

$$\left\{ \gamma^a(x), \gamma^b(y) \right\}_D = \int d^2 z C^a_{bc}(x, y, z) \gamma^c(z).$$

(52)

Using these, the master equation becomes

$$\frac{d \epsilon_a(x)}{dt} - \int d^2 y \epsilon_b(y) V^b_a(x, y) - \int d^2 y d^2 z \epsilon_b(y) \gamma_c(z) C^b_c(x, y, z) = 0.$$

(53)

Since the only nonzero structure functions are

$$V^1_2 = -\delta^2(x - y), \quad V^2_3 = -\delta^2(x - y),$$

with all the \(C^a_{bc} = 0\). We obtain the following relations between the generators.

$$\epsilon_1 = \dot{\epsilon}_3, \quad \epsilon_2 = -\dot{\epsilon}_3.$$

(54)

Therefore, the generator has only one parameter and can be written as

$$G = \int d^2 x \left( \delta \epsilon_3 \gamma^1 - \delta \epsilon_3 \gamma^2 + \delta \epsilon_3 \gamma^3 \right);$$

(55)

using (49) the gauge transformations of the variables are

$$\delta A_0 = \int \delta \epsilon_2(y) \left[ \delta^2(x - y) \right] d^2 y,$$

$$\delta A_i = \int \delta \epsilon_3(y) \left[ \frac{\partial}{\partial y^i} \delta^2(x - y) \right] d^2 y,$$

$$\delta \pi^0 = \int 0 d^2 y,$$

$$\delta \pi^i = \int \delta \epsilon_1(y) \left[ \frac{1}{2m} \epsilon^{ij} \frac{\partial}{\partial x^j} \delta^2(x - y) \right].$$
+\delta \epsilon_3(y) \left[ -\frac{\theta}{2} \epsilon^{ij} \frac{\partial}{\partial x^j} \delta^2(x - y) \right. \\
\left. - \frac{1}{2m} \epsilon^{ij} \nabla^2_y \frac{\partial}{\partial x^j} \delta^2(x - y) \right] d^2 y, \\
\delta v_0 = \int \delta \epsilon_1(y) \left[ -\delta^2(x - y) \right] d^2 y, \\
\delta v_i = \int \delta \epsilon_2(y) \left[ \frac{\partial}{\partial x^i} \delta^2(x - y) \right] d^2 y, \\
\delta \tilde{\pi}^0 = \int 0 d^2 y, \\
\delta \tilde{\pi}^i = \int 0 d^2 y, 
\end{aligned}
\tag{56}

and by using (54) the following gauge transformations are found
\begin{align}
\delta A_\mu &= -\partial_\mu \delta \epsilon_3, \\
\delta \pi^\mu &= \epsilon^{\mu ij} \left( -\frac{\theta}{2} + \frac{1}{2m} - \frac{1}{2m} \nabla^2 \right) \partial_j \delta \epsilon_3, \\
\delta v_\mu &= -\partial_\mu \delta \dot{\epsilon}_3, \\
\delta \tilde{\pi}^\mu &= 0. 
\end{align}
\tag{57}

By identifying \( \sigma^3 = -\epsilon_3 \), both formalisms agree (see equations (47) and (42)).

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