Tate Resolutions for Products of Projective Spaces
(joint work with David A. Cox)

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Background for Tate Resolutions

Basic Notation

- $V$ and $W$ are dual vector spaces over $k$: $V = W^*$
- $\dim(V) = \dim(W) = N + 1$
- $E = \wedge^\bullet V$ is a graded exterior algebra
- $E_{-i} = \wedge^i V$ are graded parts (we assume $\deg(V) = -1$)

The Dualizing Module of $E$

- $\hat{E} = \omega_E = \text{Hom}_k(E, k)$ is a left $E$-module
- $\hat{E}_i = \text{Hom}_k(E_{-i}, k) = \text{Hom}_k(\wedge^i V, k) = \wedge^i W$
- $\hat{E}(p)$ is a graded $E$-module with $\hat{E}(p)_q = \hat{E}_{p+q}$
- $\hat{E} \cong E(-N-1)$ (non-canonically)
Definition of Tate Resolution

**Tate Resolution**

- **V** and **W** are dual vector spaces over **k**: $V = W^*$ ($\dim V = N + 1$)
- **F** is a coherent sheaf on $\mathbb{P}^N = \mathbb{P}(W) = (W - \{0\})/k^*$
- **Tate resolution** is a bi-infinite exact sequence

$$T^\bullet(\mathcal{F}) : \cdots \to T^{-1}(\mathcal{F}) \to T^0(\mathcal{F}) \to T^1(\mathcal{F}) \to \cdots \to T^p(\mathcal{F}) \to \cdots$$

of free graded modules over exterior algebras $E = \wedge^\bullet V$.

**Terms (Eisenbud, Fløystad and Schreyer, 2003, [EFS 03, ES 03])**

$$T^p(\mathcal{F}) = \bigoplus_i \hat{E}(i - p) \otimes_k H^i(\mathbb{P}(W), \mathcal{F}(p - i)),$$

where $\hat{E} = \text{Hom}_k(E, k) = \wedge^\bullet W$ as an $E$-module.
Why should we study Tate resolutions?

Tate resolution keeps a LOT of information

1. Beilinson-Gelfand-Gelfand (BGG) correspondence:
   \[ D^b(\mathbb{P}(W)) = \text{Kom}^\bullet(E - \text{mod}) \]

2. Algebraic properties of coherent sheaves
   - Regularity of \( \mathcal{F} \)
   - Duality
   - Koszul cohomology (introduced by M. Green) \textbf{new}

3. Elimination theory
   - Resultants (A. Khetan, [Kh1, Kh2], D. Eisenbud, F.-O. Schreyer, [ES 03])
   - Hyperdeterminants
Known facts about the maps

1. If $i < j$, then the map

$$d_{i,j}^p : \hat{E}(i - p) \otimes H^i(F(p - i)) \to \hat{E}(j - p - 1) \otimes H^i(F(p + 1 - j))$$

in $d^p = \bigoplus_{i,j} d_{i,j}^p : T^p \to T^{p+1}$ is zero.

2. The $(i, i)$-components of the map $d^p : T^p \to T^{p+1}$ are known explicitly:

$$\hat{E}(i - p) \otimes H^i(F(p - i)) \to \hat{E}(i - p - 1) \otimes H^i(F(p + 1 - i))$$

$$f \otimes m \mapsto \sum_i f e_i^* \otimes e_i m$$

where $\{e_i\}_{i=1,N}$ is a basis of $V$; $\{e_i^*\}_{i=1,N}$ is a basis of $W$, and correspond to

$$W \otimes H^i(F(p - i)) \to H^i(F(p + 1 - i)),$$

i.e., are the Koszul-type maps.
Known facts about the maps

3. For each differential
   \[ d^p : T^p(F) \to T^{p+1}(F) \]
   - \( T^{\geq p}(F) \) is a minimal injective resolution of \( \ker(d^p) \)
   - \( T^{< p}(F) \) is a minimal projective resolution of \( \ker(d^p) \)

4. Recall that a coherent sheaf \( F \) is called \textit{m-regular} if
   \[ H^i(F(m-i)) = 0, \quad \text{for all } i > 0. \]
   If \( p \geq m = \text{reg}(F) \), then
   \[ \cdots \to T^{m-2}(F) \to T^{m-1}(F) \to \widehat{E}(-m) \otimes H^0(F(m)) \to \cdots. \]
• $X = \mathbb{P}^{\ell_1} \times \cdots \times \mathbb{P}^{\ell_r}$, $\dim(X) = \ell_1 + \cdots + \ell_r = \ell$

• $S = k[x^{(1)}, \ldots, x^{(r)}]$ – graded polynomial ring in $r$ groups of variables

• $x^{(i)} = (x_0^{(i)}, \ldots, x_{\ell_i}^{(i)})$, where for all $i = 1, \ldots, r$
  $\deg(x_0^{(i)}) = \cdots = \deg(x_{\ell_i}^{(i)}) = (0, \ldots, 1, \ldots, 0)$

• For $(n_1, \ldots, n_r) \in \mathbb{Z}^r$ denote the sheaf
  $$\mathcal{O}_X(n_1, \ldots, n_r) = p_1^*\mathcal{O}_{\mathbb{P}^{\ell_1}}(n_1) \otimes \cdots \otimes p_r^*\mathcal{O}_{\mathbb{P}^{\ell_r}}(n_r),$$
  where $p_j : \mathbb{P}^{\ell_1} \times \cdots \times \mathbb{P}^{\ell_r} \rightarrow \mathbb{P}^{\ell_j}$ is the projection.

• The subspace in $S$ of polynomials in $x^{(1)}, \ldots, x^{(r)}$ homogeneous of degrees $n_i \geq 0$ in each $x^{(i)}$ is
  $$S_{n_1,\ldots,n_r} = H^0(X,\mathcal{O}_X(n_1,\ldots,n_r))$$
Fix the degree vector $d = (d_1, \ldots, d_r) \in \mathbb{Z}^r_{>0}$.

Let $\nu_d$ be the embedding

$$\nu_d : X = \mathbb{P}^{\ell_1} \times \cdots \times \mathbb{P}^{\ell_r} \rightarrow \mathbb{P}(W), \quad W = S_{d_1, \ldots, d_r}$$

which is a combination of Veronese and Segre embeddings.

Consider the sheaf

$$\mathcal{F} = \nu_{d*} \mathcal{O}_X(m_1, \ldots, m_r)$$

Since

$$\mathcal{O}_{\mathbb{P}(W)}(1)|_{\nu_d(X)} = \nu_{d*} \mathcal{O}_X(d_1, \ldots, d_r),$$

we have:

$$H^i(\mathbb{P}(W), \mathcal{F}(j)) = H^i(X, \mathcal{O}_X(m_1 + jd_1, \ldots, m_r + jd_r))$$

Now the Tate resolution has the terms $T^p(\mathcal{F}) = \bigoplus_i T^p_i$:

$$T^p_i = \hat{E}(i - p) \otimes H^i(X, \mathcal{O}_X(m_1 + (p - i)d_1, \ldots, m_r + (p - i)d_r))$$
The Case of Veronese embedding of $\mathbb{P}^n$ (D. Cox)

- $X = \mathbb{P}^n$; $\mathcal{F} = \nu_\ell^* \mathcal{O}_{\mathbb{P}^n}(\ell)$ for any $\ell \in \mathbb{Z}$, where
- $\nu_d : \mathbb{P}^n \rightarrow \mathbb{P}(W)$ is the $d$-fold Veronese embedding
- $W = S_d \subset S = k[x_0, \ldots, x_n]$ polynomials of degree $d$
- Since $\mathcal{O}_{\mathbb{P}(W)}(1)|_{\nu_d(\mathbb{P}^n)} = \nu_\ell^* \mathcal{O}_{\mathbb{P}^n}(d)$, we have
  $$ T^p(\mathcal{F}) = \widehat{E}(-p) \otimes S_{\ell + pd} \oplus \widehat{E}(n - p) \otimes S^{*}_{-n-1-(\ell+(p-n)d)} $$
- The map $T^p(\mathcal{F}) \rightarrow T^{p+1}(\mathcal{F})$ has the following form:

\[
\begin{align*}
\widehat{E}(n - p) \otimes S^{*}_{\rho - a} & \xrightarrow{\alpha_p} \widehat{E}(n - p - 1) \otimes S^{*}_{\rho - a - d} \\
\oplus & \quad \delta_p \\
\widehat{E}(-p) \otimes S_{a - d} & \xrightarrow{\beta_p} \widehat{E}(-p - 1) \otimes S_{a},
\end{align*}
\]

where $a = \ell + (p + 1)d$, $\rho = (n + 1)(d - 1)$
The Case of Veronese embedding of $\mathbb{P}^n$ (D. Cox)

- For $f \in k[x_0, \ldots, x_n]$ and $0 \leq j \leq n$ of degree $d$, define
  \[ \Delta_j(f) = \frac{f(y_0, \ldots, y_{j-1}, x_j, x_{j+1}, \ldots, x_n) - f(y_0, \ldots, y_{j-1}, y_j, x_{j+1}, \ldots, x_n)}{x_j - y_j} \]

- The **Bezoutian** of homogeneous polynomials $f_0, \ldots, f_n \in k[x_0, \ldots, x_n]$ of degree $d$ is the determinant
  \[ \Delta = \det \Delta_j(f_i) = \sum_{\alpha} \Delta_\alpha(x) y^\alpha = \sum_{\alpha} \Delta_\alpha(x) \otimes x^\alpha \]

- The Bezoutian in degree $(\rho - a, a)$ gives a linear map
  \[ \bigwedge^{n+1} W = \bigwedge^{n+1} S_d \rightarrow S_{\rho - a} \otimes S_a, \]
  which corresponds to an $E$-module homomorphism
  \[ B_\rho : \widehat{E}(n - p) \otimes S_{\rho - a}^* \rightarrow \widehat{E}(-p - 1) \otimes S_a \]

- **Theorem.** (D. Cox, 2007, [Cox 07]) The map $\delta_p$ in $T^p(\mathcal{F}) \rightarrow T^{p+1}(\mathcal{F})$ is equal to $(-1)^p B_\rho$ defined by the Bezoutian.
The Case of Segre embedding of $\mathbb{P}^a \times \mathbb{P}^b$

- $X = \mathbb{P}^a \times \mathbb{P}^b$ – product of projective spaces
- $\mathcal{F} = \nu_* O_X(k, \ell)$ for any $k, \ell \in \mathbb{Z}$, where
- $\nu : X \to \mathbb{P}(W)$ is the Segre embedding
- $W$ is spanned by $x_iy_j$, $0 \leq i \leq a$, $0 \leq j \leq b$
- In particular,
  \[ \text{reg}(\mathcal{F}) = \max\{-\min\{k, \ell\}, \min\{b - k, a - \ell\}\} \]
- $S = k[x, y] = k[x_0, \ldots, x_a; y_0, \ldots, y_b]$ – polynomial ring
- Grading: $\deg(x_i) = (1, 0)$, $\deg(y_j) = (0, 1)$
- Bi-homogeneous part of $S_{m,n} \subset S$ is spanned by $x^\alpha y^\beta$, $|\alpha| = m$, $|\beta| = n$
The Shape of the Resolution

Terms

The terms of the Tate resolution are

\[ T^p(F) = \bigoplus_i \widetilde{E}(i - p) \otimes H^i(X, \mathcal{O}_X(k + p - i, \ell + p - i)), \]

where

\[ H^i(X, \mathcal{O}_X(k + p - i, \ell + p - i)) = 0 \text{ for } i \notin \{0, a, b, a + b\}. \]

Types of the resolution

There are three types of the resolution of \( F = \nu_* \mathcal{O}_X(k, \ell) \) on \( X = \mathbb{P}^a \times \mathbb{P}^b \):

I) \(-a \leq k - \ell \leq b\)
II) \(k - \ell > b\)
III) \(k - \ell < -a \) (similar to type II)
Terms of the Resolution of Type I

Terms corresponding to Koszul maps (for all types)

Define the numbers:

\[ p^+ = \max\{-\min\{k, \ell\}, \min\{b - k, a - \ell\}\} = \text{reg}(\mathcal{F}), \]
\[ p^- = \min\{-\min\{k, \ell\}, \min\{b - k, a - \ell\}\} - 1. \]

Then

\[ T^p(\mathcal{F}) = \begin{cases} \hat{E}(-p) \otimes S_{k+p, \ell+p} & p \geq p^+ \\ \hat{E}(a + b - p) \otimes S^*_{b - k - 1 - p, a - \ell - 1 - p} & p \leq p^- . \end{cases} \]

Terms corresponding to the non-Koszul maps of Type I resolution

Assume that \( \mathcal{F} \) has Type I \((-a \leq k - \ell \leq b)\).
Then \( p^- = -\min\{k, \ell\} - 1 \) and \( p^+ = \min\{b - k, a - \ell\} \).
Furthermore, if \( p^- < p < p^+ \), then

\[ T^p(\mathcal{F}) = \hat{E}(a + b - p) \otimes S^*_{b - k - 1 - p, a - \ell - 1 - p} \]

\[ T^p(\mathcal{F}) = \bigoplus \hat{E}(-p) \otimes S_{k+p, \ell+p}. \]
Maps in the resolution of Type I

Toric Jacobian

Given $f_0, \ldots, f_{a+b} \in W = S_{1,1}$, where $f_j(x, y) = \sum_{i,k} a_{i,j,k} x_i y_j$, the toric Jacobian is

$$J(f_0, \ldots, f_{a+b}) = \frac{1}{x_0 y_b} \det \begin{pmatrix}
\frac{\partial f_0}{\partial x_1} & \cdots & \frac{\partial f_{a+b}}{\partial x_1} \\
\frac{\partial f_0}{\partial x_1} & \cdots & \frac{\partial f_{a+b}}{\partial x_1} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_0}{\partial y_{b-1}} & \cdots & \frac{\partial f_{a+b}}{\partial y_{b-1}}
\end{pmatrix} \in S_{b,a}.$$
Maps in the resolution of Type I

The non-Koszul part of the differential $d^p : T^p(F) \rightarrow T^{p+1}(F)$ looks like

$$\hat{E}(a + b - p) \otimes S_{b - k - 1 - p, a - \ell - 1 - p} \rightarrow \hat{E}(-p - 1) \otimes S_{k + p + 1, \ell + p + 1}$$

and is induced by the map

$$\delta_1 : \wedge^{a+b+1} W \rightarrow S_{b - k - 1 - p, a - \ell - 1 - p} \otimes S_{k + p + 1, \ell + p + 1}.$$

The change of variables

$$J \mapsto J(X_i + x_i, Y_j + y_j) \in k[X, Y, x, y] \cong S \otimes S$$

in the toric Jacobian extends the map $J$ to

$$J = \bigoplus_{\alpha, \beta} J_{\alpha, \beta}, \quad J_{\alpha, \beta} : \wedge^{a+b+1} W \rightarrow S_{a - \alpha, a - \beta} \otimes S_{\alpha, \beta}.$$

**Theorem.** The map $\delta_1$ can be chosen to be $(-1)^p J_{k + p + 1, \ell + p + 1}$. 
Maps in the resolution of Type II

Terms corresponding to the non-Koszul maps of Type II resolution

Assume that $\mathcal{F}$ has Type 2 ($k - \ell > b$).
Then $p^- = b - k - 1$ and $p^+ = -\ell$. Furthermore, if $p^- < p < p^+$, then

$$T^p(\mathcal{F}) = \widehat{E}(b - p) \otimes S_{k+p-b,0} \otimes S_{0,-\ell-p-1}^*.$$

Differentials:

The differential in $T^{p^-}(\mathcal{F}) \rightarrow T^{p^-+1}(\mathcal{F})$ looks like

$$d^- : \widehat{E}(a + 1 + k) \otimes S_{0,a+k-\ell-b}^* \rightarrow \widehat{E}(k) \otimes S_{0,0} \otimes S_{0,k-\ell-b-1}^*.$$

The differential in $T^{p^+-1}(\mathcal{F}) \rightarrow T^{p^+}(\mathcal{F})$ looks like

$$d^+ : \widehat{E}(b + 1 + \ell) \otimes S_{k-\ell-b-1,0} \otimes S_{0,0}^* \rightarrow \widehat{E}(\ell) \otimes S_{k-\ell,0}.$$
For given $f_0, \ldots, f_{a+b} \in W = S_{1,1}$ write $f_i = \sum_j A_{ij} x_j, A_{ij} \in S_{0,1}$ and define the map

$$\gamma_\alpha : \bigwedge^{a+1} W \xrightarrow{M} S_{0,a+1} \longrightarrow S^*_{0,\alpha} \otimes S_{0,a+1+\alpha},$$

where $M(f_0, \ldots, f_a) = \det(A_{ij}) \in S_{0,a+1}$, and

$$S_{0,a+1} \longrightarrow S^*_{0,\alpha} \otimes S_{0,a+1+\alpha}$$

is the comultiplication map. This induces (by abuse of notation) the maps:

$$\gamma_\alpha : \tilde{E}(a+1+k) \otimes S_{0,\alpha} \longrightarrow \tilde{E}(k) \otimes S_{0,a+1+\alpha}$$

$$\gamma^*_\alpha : \tilde{E}(a+1+k) \otimes S^*_{0,a+1+\alpha} \longrightarrow \tilde{E}(k) \otimes S^*_{0,\alpha}.$$

**Theorem.** The non-Koszul differentials in the Tate resolution of Type II can be chosen to be $d^- = \gamma^*_{k-\ell-b-1}$ and $d^+ = \gamma_{k-\ell-b-1}$. 
Example (on maps of Type I)

Let \( \nu : X = \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}(W) = \mathbb{P}^5 \), \( \mathcal{F} = \nu_* \mathcal{O}_X(0, 1) \).

\[
\cdots \to \mathcal{E}(4) \otimes S_{1,1}^* \to \mathcal{E}(3) \otimes S_{0,0}^* \to \mathcal{E}(2) \otimes S_{0,1}^* \otimes S_{1,0} \to \mathcal{E}(1) \otimes S_{0,2}^* \otimes S_{1,1} \to \mathcal{E}(0) \otimes S_{0,3}^* \otimes S_{1,2} \to \mathcal{E}(-1) \otimes S_{0,4}^* \otimes S_{1,3} \to \cdots
\]
Example (on maps of Type II)

Let $\nu : X = \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}(W) = \mathbb{P}^5$, $\mathcal{F} = \nu_* \mathcal{O}_X(3, 0)$.

The only nonzero diagonal maps appear in $T^{-3}(\mathcal{F}) \to T^{-2}(\mathcal{F})$:

$$
\cdots \to \hat{E}(6) \otimes S^*_{0,4} \to \hat{E}(3) \otimes S_{0,0} \otimes S^*_{0,1} \to \cdots
$$

(at cohomological levels 3 and 1) and in $T^{-1}(\mathcal{F}) \to T^0(\mathcal{F})$:

$$
\cdots \to \hat{E}(2) \otimes S_{1,0} \otimes S^*_{0,0} \to \hat{E}(0) \otimes S_{3,0} \to \cdots
$$

(at cohomological levels 1 and 0).
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