ABOUT A CONJECTURE ON DIFFERENCE EQUATIONS IN QUASIANALYTIC CARLEMAN CLASSES

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This modest work is dedicated to the memory of our beloved master Ahmed Intissar (1951-2017), a distinguished professor, a brilliant mathematician, a man with a golden heart.

Abstract. In this paper we consider the difference equation \((E)\):
\[
\sum_{j=1}^{q} a_j(x) \varphi(x + \alpha_i) = \chi(x)
\]
where \(a_1 < ... < a_q (q \geq 3)\) are given real constants, \(a_j (j = 1, ..., q)\) are given holomorphic functions on some strip \(R_\delta (\delta > 0)\) such that \(a_1 \) and \(a_q\) are nowhere vanishing on \(R_\delta\), and \(\chi\) a function which belongs to a quasianalytic Carleman class \(C_M(\mathbb{R})\). We prove under a growth condition on the functions \(a_j\) that the equation \((E)\) is solvable in the class \(C_M(\mathbb{R})\).

1. Introduction

In the paper ([1]), G. Belitskii, E. M. Dyn'kin and V. Tkachenko have formulated the following conjecture:

"Let \(\chi, a_i, i = 1, ..., q,\) be functions in a Carleman class \(C_M(\mathbb{R})\) such that \(a_1\) and \(a_q\) are nowhere vanishing on \(\mathbb{R},\) and \(\alpha_1 < ... < \alpha_q\) some real numbers. Then the difference equation:
\[
(E) : \sum_{j=1}^{q} a_j(x) \varphi(x + \alpha_i) = \chi(x)
\]
is solvable in the Carleman class \(C_M(\mathbb{R})\)."

In that paper the authors, relying on a result of decomposition in Carleman classes, have proved the conjecture in the particular cases where the coefficients \(a_j\) are constants or when the coefficients are variables with \(q = 2\). They have also suggested that the same method could be used to show the solvability of the equation \((E)\) in a quasianalytic Carleman class \(C_M(\mathbb{R})\), if we assume that the functions \(\frac{1}{a_1}, \frac{1}{a_q}, \frac{a_1}{a_2}, ..., \frac{a_1}{a_q}, \frac{a_2}{a_3}, ..., \frac{a_q}{a_{q-1}} (q \geq 3)\) can be continued in some strip \(R_\delta := \{z \in \mathbb{C} : |\text{Im}(z)| < \delta\}\) as analytic functions increasing on \(R_\delta\) not too fast at infinity. As an example of such coefficients, they have mentioned the class of rational functions. In this paper, our purpose is to give a precise meaning to this assertion, by proving that the result is true even if the functions \(\frac{1}{a_1}, \frac{1}{a_q}, \frac{a_1}{a_2}, ..., \frac{a_1}{a_q}, \frac{a_2}{a_3}, ..., \frac{a_q}{a_{q-1}}\) have more rapide increase at infinity provided that it is of the form \(e^{C|\text{Re}(z)|}\) where \(C > 0\) is a constant.

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2. Notations, definitions and statement of the main result

We set for every $\rho > 0$, $a \geq 0$ :
\[
\begin{align*}
\mathbb{R}_\rho := \{ z \in \mathbb{C} : |\text{Im}(z)| < \rho \}, \quad \mathbb{R}_\rho^\pm := \{ z \in \mathbb{R}_\rho : \pm \text{Re}(z) > \rho \} \\
\mathbb{R}_{\rho,a} := \{ z \in \mathbb{R}_\rho : |\text{Re}(z)| \leq a \} \\
\Delta_\rho := \{ z \in \mathbb{C} : |z| < \rho \}, \quad \Delta_\rho^\pm := \{ z \in \Delta_\rho : \pm \text{Re}(z) \leq 0 \} \\
\Gamma_\rho := \{ z \in \mathbb{C} : |z| = \rho \}, \quad \Gamma_\rho^\pm := \{ z \in \Gamma_\rho : \pm \text{Re}(z) \leq 0 \}
\end{align*}
\]

For every non empty subset $V$ of $\mathbb{C}$ and every $z \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we set :
\[
\begin{align*}
\{ V^{(0)} := V, \quad V^{(n)} := \{ u_1 + ... + u_n : u_j \in V, \ j = 1, ..., n \}, \ n \geq 1 \\
z + V := \{ z + u : u \in V \}, \ z - V := \{ z - u : u \in V \}
\end{align*}
\]

dm(\zeta) denotes the Lebesgue measure on $\mathbb{C}$.

Let $S$ be a nonempty subset of $\mathbb{C}$. By $O(S)$ we denote the set of holomorphic functions on some neighborhood of $S$.

Let $F : U \subset \mathbb{C} \rightarrow \mathbb{C}$ be a function of class $C^1$ on an open subset $U$ of $\mathbb{C}$. We set for all $z \in U :$
\[
\overline{\partial} F(z) := \frac{1}{2} \left( \frac{\partial F}{\partial x}(z) + i \frac{\partial F}{\partial y}(z) \right)
\]

$\overline{\partial}$ is called the operator of Cauchy-Riemann.

Let $M := (M_n)_{n \in \mathbb{N}}$ be a sequence of strictly positive numbers. Let $M := (M_n)_{n \geq 0}$ be a sequence of strictly positive real numbers. The Carleman class $C_M\{\mathbb{R}\}$ is the set of all functions $f : W \rightarrow \mathbb{C}$ of class $C^\infty$ such that
\[
||f^{(n)}||_{\infty,I} \leq C_I \rho_I^n M_n, \ n \in \mathbb{N}
\]

for every compact interval $I$ contained in $\mathbb{R}$ with some constants $C_I, \rho_I > 0$.

The Carleman class $C_M\{\mathbb{R}\}$ is said to be quasianalytic if every function $f \in C_M\{\mathbb{R}\}$ such that $f^{(n)}(u) = 0$ for some $u \in \mathbb{R}$ and every $n \in \mathbb{N}$ is identically equal to 0.

The Carleman class $C_M\{\mathbb{R}\}$ is called regular if the following conditions hold :
\[
\begin{align*}
\left( \frac{M_{n+1}}{(n+1)!} \right)^2 \leq \frac{M_n}{n!} \frac{M_{n+2}}{(n+2)!}, \ n \in \mathbb{N} \\
\sup_{n \in \mathbb{N}} \left( \frac{M_{n+1}}{(n+1)!} M_n \right)^{\frac{1}{n}} < +\infty \\
\lim_{n \rightarrow +\infty} M_n^{\frac{1}{n}} = +\infty
\end{align*}
\]

To the Carleman class $C_M\{\mathbb{R}\}$ we associate its weight $H_M$ defined by the following relation :
\[
H_M(x) := \inf_{n \in \mathbb{N}} \left( \frac{M_n}{n!} x^n \right), \ x > 0
\]

In this paper the following result will play a crucial role.

**Theorem 1.** (\cite{2}) We assume that the Carleman class $C_M\{\mathbb{R}\}$ is regular. A function $f : \mathbb{R} \rightarrow \mathbb{C}$ belongs to $C_M\{\mathbb{R}\}$ if and only if there exists for every compact interval $I$ of $\mathbb{R}$ a compactly supported function $F_I : \mathbb{C} \rightarrow \mathbb{C}$ of class $C^1$ such that $F_I$ is an extension of $f$ and satisfies the following estimate :
\[
|\overline{\partial} F_I(z)| \leq A_I H_M(|B_I| \text{Im}(z)|), \ z \in \mathbb{C}
\]

where $A_I, B_I > 0$ are constants.
Throughout this paper, we assume that the Carleman class \( C_M(\mathbb{R}) \) is regular and quasianalytic.

Our main result in this paper is the following.

**Theorem 2.** Let \( q \in \mathbb{N} \setminus \{1, 2\} \), \( \delta > 0 \), \( \chi \in C_M(\mathbb{R}) \) and \( a_j \in O(\mathbb{R}_\delta) \) (\( j = 1, \ldots, q \)) such that \( a_1 \) and \( a_q \) are nowhere vanishing on \( \mathbb{R}_\delta \). We assume that the following growth condition holds:

\[
\sup_{z \in \mathbb{R}_\delta} \left( \left| \sum_{j=2}^{q} \frac{a_j(z)}{a_1(z)} \right| + \left| \sum_{j=1}^{q-1} \frac{a_j(z)}{a_q(z)} \right| + \frac{1}{|a_1(z)|} + \frac{1}{|a_q(z)|} \right) e^{-C|\Re z|} < +\infty
\]

for some constant \( C > 0 \). Then the difference equation:

\[
(E): \sum_{j=1}^{q} a_i(x) \varphi(x + a_i) = \chi(x)
\]

is solvable in the Carleman class \( C_M(\mathbb{R}) \).

### 3. Proof of the main result

Let us first prove the following lemma.

**Lemma 3.** Given \( f \in C_M(\mathbb{R}), C_0 > 0 \) and \( \rho \in \left[ 0, \frac{\pi}{2C_0} \right] \), there exist two functions \( f_{\pm}: (\mathbb{C}\setminus \Delta_\rho^\pm) \cup \mathbb{R} \rightarrow \mathbb{C} \) which are holomorphic on \( \mathbb{C}\setminus (\Gamma_\rho^+ \cup \Delta_\rho^\pm) \), whose restrictions to \( \mathbb{R} \) belong to \( C_M(\mathbb{R}) \), and such that the following conditions hold:

\[
\begin{align*}
|f_{\pm}(z)| & \leq D_0 e^{-\cos(\rho C_0) e^{C|\Re z|}}, \quad z \in \mathbb{R}_\rho^+ \\
|f_{\pm}(z)| & \leq D_0 e^{-\cos(\rho C_0) e^{C|\Re z|}}, \quad z \in \mathbb{R}_\rho^-
\end{align*}
\]

where \( D_0 > 0 \) is a constant.

**Proof.** Since \( f \) belongs to \( C_M(\mathbb{R}) \) there exists, according to Dyn’kin’s theorem (\( \square \)), a compactly supported function \( F: \mathbb{C} \rightarrow \mathbb{C} \) of class \( C^1 \) such that \( F \) is an extension of the restriction of \( f \) to the interval \([-\rho, \rho]\) and satisfies the following estimate:

\[
|\partial F(z)| \leq AH_M(B|\Im(z)|), \quad z \in \mathbb{C}
\]

where \( A, B > 0 \) are constants. Following the same approach as that of (\( \square \), pages 34, 35), but using the Cauchy-Pompeiu formula on the disk \( \Delta_\rho \), for the function \( e^{C^{-1}z^*}+e^{-C_0 z^*} f(z) \), we show that the functions:

\[
\begin{cases}
 f_{\pm}(z) = \frac{1}{2\pi i} e^{-e^{C_0 z^*}} f(z) \int_{\Gamma_{\rho}^\pm} e^{C_0 \zeta^*+e^{-C_0 z}} F(\zeta) \frac{d\zeta}{\zeta-z} \\
 -\frac{1}{2\pi i} e^{-e^{C z^*}} f(z) \int_{\Gamma_{\rho}^\pm} e^{C \zeta^*+e^{-C_0 z}} F(\zeta) \frac{d\zeta}{\zeta-z}
\end{cases}
\]

satisfy the required conditions. \( \square \)
Now we set:

\[
\begin{align*}
\beta_j := & \alpha_j - \alpha_1, \quad j = 2, \ldots, q \\
b_j(z) := & -\frac{a_j(z)}{a_1(z)}, \quad z \in \mathbb{R} \delta, \quad j = 2, \ldots, q \\
\gamma_j := & \alpha_q - \alpha_j, \quad j = 1, \ldots, q - 1 \\
c_j(z) := & -\frac{a_j(z)}{a_q(z)}, \quad z \in \mathbb{R} \delta, \quad j = 1, \ldots, q - 1
\end{align*}
\]

Let \( C_1 > C \) and \( \delta_0 \in [0, \min(\delta, \frac{\pi}{2 M})] \). Then according to the lemma above, there exist constant \( D_1 > 0 \) and two functions \( \chi_{\pm} : (\mathbb{C} \setminus \Delta_{\delta_0}^{\pm}) \cup \mathbb{R} \to \mathbb{C} \) which are holomorphic on \( \mathbb{C} \setminus (\Gamma_{\delta_0}^{\pm} \cup \Delta_{\delta_0}^{\pm}) \), whose restrictions to \( \mathbb{R} \) belong to \( C_M(\mathbb{R}) \), and such that the following conditions hold:

\[
\begin{align*}
\chi(x) = & \chi_{\pm}(x) + \chi_{\mp}(x), \quad x \in [-\delta_0, \delta_0] \\
|\chi_{\pm}(x)| \leq & D_1 e^{-\cos(C_1 \delta_0) e^{C_1 |\text{Re}(x)|}}, \quad z \in \mathbb{R} \delta_0^+ \\
|\chi_{\mp}(x)| \leq & D_1 e^{-\cos(C_1 \delta_0) e^{C_1 |\text{Re}(x)|}}, \quad z \in \mathbb{R} \delta_0^-
\end{align*}
\]

(3.1)

Let \((g_n)_{n \in \mathbb{N}}\) and \((h_n)_{n \in \mathbb{N}}\) be the sequences of complex valued functions defined on the strip \( \mathbb{R} \delta_0 \) by the formulas:

\[
\begin{align*}
g_0(z) := & \frac{\chi(z)}{a_1(z)}, \quad g_{n+1}(z) := \sum_{j=2}^{q} b_j(z) g_n(z + \beta_j) \\
h_0(z) := & \frac{\chi(z)}{a_q(z)}, \quad h_{n+1}(z) := \sum_{j=1}^{q-1} c_j(z) h_n(z - \gamma_j)
\end{align*}
\]

It is clear that all the functions \( g_n|\mathbb{R} \) and \( h_n|\mathbb{R} \) belong to \( C_M(\mathbb{R}) \). Let us set:

\[
\begin{align*}
K_1 := & \{ \beta_j : j = 2, \ldots, q \} \\
K_2 := & \{ \gamma_j : j = 1, \ldots, q - 1 \}
\end{align*}
\]

It follows from (2.1) that we have for every \( n \in \mathbb{N}, z \in \mathbb{R} \delta_0 \):

\[
\begin{align*}
|g_{n+1}(z)| \leq & e^{L e^{C |\text{Re}(z)|}} \max_{u \in z + K_j^{(n)}} |g_n(u)| \\
|h_{n+1}(z)| \leq & e^{L e^{C |\text{Re}(z)|}} \max_{u \in z - K_j^{(n)}} |h_n(u)|
\end{align*}
\]

where \( L > 1 \) is a constant. Hence we have for all \( n \in \mathbb{N}^*, z \in \mathbb{R} \delta_0^* \):

\[
\begin{align*}
g_n(z) := & e^{\sum_{j=0}^{n-1} L e^{C |\text{Re}(z)|} + j \beta_q}} \max_{u \in z + K_j^{(n)}} |g_0(u)| \\
\leq & e^n L e^{C |\text{Re}(z)| + n \beta_q}} \max_{u \in z + K_j^{(n)}} |\chi_{\pm}(u)| \\
|h_n(z)| := & e^{\sum_{j=0}^{n-1} L e^{C |\text{Re}(z)|} + j \gamma_1}} \max_{u \in z - K_j^{(n)}} |h_0(u)| \\
\leq & e^n L e^{C |\text{Re}(z)| + n \gamma_1}} \max_{u \in z - K_j^{(n)}} |\chi_{\mp}(u)|
\end{align*}
\]

Let \( a > 0 \). There exists \( N_a \in \mathbb{N}^* \) such that \( (\beta_q + \gamma_{q-1})N_a \geq a \) and:

\[
\begin{align*}
z + K_j^{(n)} \subset & \mathbb{R} \delta_0^+, \quad n \geq N_a, \quad z \in \mathbb{R} \delta_0, a \\
z - K_j^{(n)} \subset & \mathbb{R} \delta_0^-, \quad n \geq N_a, \quad z \in \mathbb{R} \delta_0, a
\end{align*}
\]
It follows then from (3.1) that we have for all \( n \geq N_a, \, z \in \mathbb{R}_{\delta_0, a} \):

\[
\begin{align*}
\max_{u \in z + K_1^{(n)}} |\chi_+(u)| & \leq D_1 e^{\min_{u \in z + K_1^{(n)}} |\text{Re}(u)|} e^{-C_1} \\
& \leq D_1 e^{\cos(C_1 \delta_0) e^{-C_1(-a + n \beta_2)}} e^{C_1} \\
\max_{u \in z - K_2^{(n)}} |\chi_-(u)| & \leq D_1 e^{\cos(C_1 \delta_0) e^{-C_1(-a + n \beta_2)}} e^{C_1} \\
& \leq D_1 e^{\cos(C_1 \delta_0) e^{C_1(-a + n \gamma_q - 1)}}
\end{align*}
\]

Consequently we have for all \( n \geq N_a, \, z \in \mathbb{R}_{\delta_0, a} \):

\[
\begin{align*}
|g_n(z)| & \leq D_1 e^{n \text{Le}^C (a + n \beta_q)} e^{\cos(C_1 \delta_0) e^{C_1(-a + n \beta_2)}} \\
|h_n(z)| & \leq D_1 e^{n \text{Le}^C (a + n \gamma_1)} e^{\cos(C_1 \delta_0) e^{C_1(-a + n \gamma_q - 1)}}
\end{align*}
\]

On the other hand we have :

\[
\begin{align*}
n \text{Le}^C (a + n \beta_q) & = 0 \rightarrow +\infty \left( \cos(C_1 \delta_0) e^{C_1(-a + n \beta_2)} \right) \\
n \text{Le}^C (a + n \gamma_1) & = 0 \rightarrow +\infty \left( \cos(C_1 \delta_0) e^{C_1(-a + n \gamma_q - 1)} \right)
\end{align*}
\]

Thence there exist real constants \( D_a > 0 \) and \( E_a > 0 \) and an integer \( P_a \geq N_a \) such that the following inequalities hold :

\[
\begin{align*}
|g_n(z)| & \leq D_a e^{-E_a e^{C_1(-a + n \beta_2)}} \, , \, z \in \mathbb{R}_{\delta_0, a} \, , \, n \geq P_a \\
|h_n(z)| & \leq D_a e^{-E_a e^{C_1(-a + n \gamma_q - 1)}} \, , \, z \in \mathbb{R}_{\delta_0, a} \, , \, n \geq P_a
\end{align*}
\]

It follows that the function series \( \sum g_n|_{\mathbb{R}_{\delta_0}} \) and \( \sum h_n|_{\mathbb{R}_{\delta_0}} \) are uniformly convergent on every compact subset of \( \mathbb{R}_{\delta_0} \) and that the functions \( \sum_{n=N_a}^{+\infty} g_n \) and \( \sum_{n=N_a}^{+\infty} h_n \) are holomorphic on \( \mathbb{R}_{\delta_0, a} \) for every \( a > 0 \). Let \( G_+ \) and \( G_- \) be respectively the sums of \( \sum g_n|_{\mathbb{R}_{\delta_0}} \) and \( \sum h_n|_{\mathbb{R}_{\delta_0}} \). Since all the functions \( g_n|_{\mathbb{R}} \) and \( h_n|_{\mathbb{R}} \) belong to \( C_M(\mathbb{R}) \), it follows then that the functions \( g_+ := G_+|_{\mathbb{R}} \) and \( g_- := G_-|_{\mathbb{R}} \) belong to \( C_M(\mathbb{R}) \). Elementary computations show that :

\[
\begin{align*}
\sum_{j=1}^{q} a_j(x)g_+(x + \alpha_j) & = \chi_+(x) \, , \, x \in \mathbb{R} \\
\sum_{j=1}^{q} a_j(x)g_-(x + \alpha_j) & = \chi_-(x) \, , \, x \in \mathbb{R}
\end{align*}
\]

Then it follows from (3.1) that the function \( g := g_+ + g_- \) is a solution on the interval \([-\delta_0, \delta_0]\) of the difference equation \( (E) \). But the function \( x \mapsto \sum_{j=1}^{q} a_j(x)g(x + \alpha_j) - \chi(x) \) belongs to the quasianalytic Carleman class \( C_M(\mathbb{R}) \). Consequently the function \( g \in C_M(\mathbb{R}) \) is a solution on \( \mathbb{R} \) of the difference equation \( (E) \). Thence the proof of the main result is complete.

\[\square\]

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