Nonstationary frames of translates and frames from the Weyl–Heisenberg group and the extended affine group

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Abstract

In this work, we analyze Gabor frames for the Weyl–Heisenberg group and wavelet frames for the extended affine group. Firstly, we give necessary and sufficient conditions for the existence of nonstationary frames of translates. Using these conditions, we prove the existence of Gabor frames from the Weyl–Heisenberg group and wavelet frames for the extended affine group. We present a representation of functions in the closure of the linear span of a Gabor frame sequence in terms of the Fourier transform of window functions. We show that the canonical dual of frames of translates has the same structure. An approximation of the inverse of the frame operator of nonstationary frames of translates is presented. It is shown that a nonstationary frame of translates is a Riesz basis if it is linearly independent and satisfies the approximation of the inverse frame operator. Finally, we give equivalent conditions for a nonstationary sequence of translates to be linearly independent.

Keywords: frames, Gabor frames, wavelet frames, Weyl–Heisenberg group, extended affine group

1. Introduction

The notion of frames was introduced by Duffin and Schaeffer [14] in connection with the study of non-harmonic Fourier series. The study of image processing is the first application of the frames described by Daubechies, Grossmann, and Meyer [9]. Frames have been extensively studied in the last three decades due to their link to physics [2, 15, 16, 32, 46], applied...
mathematics [17, 24], operator theory [27, 31, 33], etc. Wavelet analysis and Gabor analysis are part of this development. Wavelet analysis corresponds to unitary irreducible representations of the affine group, while Gabor analysis corresponds to unitary irreducible representations of the Weyl–Heisenberg group. Whereas Gabor analysis yields a time-frequency representation of signals, wavelet analysis provides a time-scale representation. There is huge literature on Gabor analysis and wavelet analysis and its applications in several branches of physics, pure mathematics and engineering science, see [4, 8, 16–21, 24, 28, 29, 31, 43, 45] and many references therein.

In 1998, Dai and Liang proved that if \( \psi_1 \) and \( \psi_2 \) are two affine orthonormal MRA wavelets, then there exists a continuous map \( A : [0, 1] \to L^2(\mathbb{R}, dx) \) such that \( A(0) = \psi_1, A(1) = \psi_2, \) and \( A(t) \) is an affine orthonormal MRA wavelet for all \( t \in [0, 1] \). The proof appeared in [41, theorem 4] and may also be found in the review article [40] by Weiss and Wilson. Years later, Dahlke, Fornasier, Rauhut, Steidl and Teschke in [7] and Torresani in [38, 39] established a connection between Gabor analysis and wavelet analysis by exhibiting both the affine group and the Weyl–Heisenberg group as subgroups of the four dimensional affine Weyl–Heisenberg group. They also contracted the corresponding coherent state families, resolutions of the identity, and tight frames. One of the key insights in this paper is the realization that the affine group and the Weyl–Heisenberg group have different dimensions. They overcame this problem by introducing the extended affine group and an additional parameter, \( \epsilon \). The frames thus obtained vary continuously with \( \epsilon, \epsilon = 0 \) yields a Gabor frame and \( \epsilon = 1 \) yields a wavelet frame.

Recently, the authors of [37] studied the relationship between Gabor and wavelet analyses by using ‘contraction’ between the affine group and the Weyl–Heisenberg group. They also presented unitary irreducible representations of the Weyl–Heisenberg group as a contraction of the extended affine group. Starting from a standard wavelet frame, they constructed a family of frequency-localized wavelet frames that contract to a nonstandard Gabor frame. To be exact, a deformation of Gabor frames to wavelet frames is presented in [37]. In [22], the authors studied the construction of frames with multiple generators for both the Weyl–Heisenberg group and the extended affine group. They also proved a Paley-Wiener type stability result for frames from the Weyl–Heisenberg group. Most recently, in [23], the authors studied frame properties of matrix-valued nonstationary Gabor systems and wavelet systems corresponding to the Weyl–Heisenberg group and the extended affine group, respectively. Inspired by the above works, we study nonstationary frames of translates or generalized shift-invariant frames and frames from the Weyl–Heisenberg group and the extended affine group. Notable contribution in this paper includes necessary and sufficient conditions for nonstationary frames of translates, Gabor frames, and wavelet frames with several generators associated with the Weyl–Heisenberg group and the extended affine group. Using the results of Casazza and Christensen [3] and Christensen and Hasannasab [5], we give an approximation of inverse frame operator and linear independence of nonstationary frames of the space \( L^2(\mathbb{R}, dx) \).

The work in this paper is structured as follows. In section 2, we briefly review Hilbert frames, Riesz basis, Gabor systems and wavelet systems. The main results start from section 3. Theorem 3.1 gives necessary and sufficient conditions for the existence of Gabor frames associated with the Weyl–Heisenberg group. Its proof is based on the existence of nonstationary frames of translates for \( L^2(\mathbb{R}, dx) \), see theorem 3.2. In theorem 3.7, we give an interplay between modulation and translation parameters in Gabor frames associated with the Weyl–Heisenberg group. Theorem 3.10 gives a representation of a function in the closure of the linear span of a Gabor frame sequence in terms of the Fourier transform of window functions. In section 4, we give necessary and sufficient conditions for wavelet frames associated with the extended affine group, see theorem 4.1. In section 5, we discuss the structure of the canonical dual of nonstationary frames of translates for \( L^2(\mathbb{R}, dx) \). In theorem 5.1, we show
that the canonical dual of nonstationary frames of translates has the same structure. By using a
technique given in [3], we give an approximation of the inverse of the frame operator of non-
stationary frames of translates in theorem 6.1 of section 6. Sufficient conditions for nonstation-
ary Riesz bases of the space \( L^2(\mathbb{R}, dx) \) are given in theorem 6.3. In section 7, we give equivalent
conditions for linear independence of a nonstationary sequence of translates in \( L^2(\mathbb{R}, dx) \).

2. Preliminaries

As usual \( \mathbb{N}, \mathbb{Z}, \mathbb{R}^+, \mathbb{R}^*, \mathbb{R} \) denote the set of positive natural numbers, integers, positive real
numbers, non-zero real numbers and real numbers, respectively. \( I \) denotes a countable (finite
or infinite) index set. The support of a function \( f \) defined on \( \mathbb{R} \) is the closure of the set \( \{ x \in \mathbb{R} : f(x) \neq 0 \} \). Throughout this paper, \( \mathcal{H} \neq \{ 0 \} \) denotes a separable Hilbert space with an inner
product \( \langle \cdot, \cdot \rangle \) and the standard norm on \( \mathcal{H} \) is given by \( \| f \| = \sqrt{\langle f, f \rangle}, f \in \mathcal{H} \).

2.1. Hilbert frames and Riesz bases

A collection of vectors \( \mathcal{F} := \{ \varphi_k \}_{k \in \Xi} \) in \( \mathcal{H} \) is called a frame (or Hilbert frame) of \( \mathcal{H} \) if for some
\( \alpha_o, \beta_o \in (0, \infty) \) the following inequality holds

\[
\alpha_o \| f \|^2 \leq \sum_{k \in \Xi} |\langle f, \varphi_k \rangle|^2 \leq \beta_o \| f \|^2 \tag{2.1}
\]

for all \( f \in \mathcal{H} \). Inequality (2.1) is called the frame inequality. Scalars \( \alpha_o \) and \( \beta_o \), obviously not unique, are known as lower frame bound and upper frame bound of \( \mathcal{F} \). The frame \( \mathcal{F} \) is tight
if \( \alpha_o = \beta_o \). Parseval, if \( \alpha_o = \beta_o = 1 \). If \( \mathcal{F} \) fulfills the upper inequality in (2.1), then we say
that \( \mathcal{F} \) is a Bessel sequence with Bessel bound \( \beta_o \). If \( \mathcal{F} \) is a Bessel sequence, then the map
\( S : \mathcal{H} \to \mathcal{H} \) given by \( Sf = \sum_{k \in \Xi} \langle f, \varphi_k \rangle \varphi_k \) is called the frame operator of the frame \( \mathcal{F} \). The
frame operator \( S \) is bounded and linear. It is invertible on \( \mathcal{H} \) if \( \mathcal{F} \) is a frame for \( \mathcal{H} \). In that
case \( \mathcal{F} \) gives the following reconstruction formula: \( f = SS^{-1}f = \sum_{k \in \Xi} \langle f, S^{-1} \varphi_k \rangle \varphi_k, f \in \mathcal{H} \).
The scalars \( \langle f, S^{-1} \varphi_k \rangle \) are known as frame coefficients. The supremum over all lower frame
bounds is called the optimal lower frame bound, and the infimum over all upper frame bounds
is called the optimal upper frame bound. The optimal bounds in terms of the frame operator
are given in the following result.

**Proposition 2.1** ([4, p 121]). The lower optimal frame bound \( \alpha_{opt} \) and the upper optimal frame
bound \( \beta_{opt} \) of a frame \( \mathcal{F} \) with frame operator \( S \) are given by \( \alpha_{opt} = \| S^{-1} \|^{-1}, \beta_{opt} = \| S \| \).

**Definition 2.2** ([4, definition 73.1]). A sequence \( \{ \varphi_k \}_{k \in \Xi} \) in \( \mathcal{H} \) is a Riesz frame of \( \mathcal{H} \) if every
subsequence of \( \{ \varphi_k \}_{k \in \Xi} \) is a frame for its closed linear span with the same frame bounds \( \alpha, \beta \).

**Definition 2.3** ([4, definition 3.6.1]). Let \( \Xi \) be a bounded, linear and bijective operator acting
on \( \mathcal{H} \) and \( \{ \chi_k \}_{k \in \Xi} \) be an orthonormal basis of \( \mathcal{H} \). A sequence of the form \( \{ \Xi \chi_k \}_{k \in \Xi} \) is called
a Riesz basis for \( \mathcal{H} \).

**Theorem 2.4** ([4, theorem 71.1]). A frame \( \mathcal{F} = \{ \varphi_k \}_{k \in \Xi} \) of \( \mathcal{H} \) is a Riesz basis for \( \mathcal{H} \) if and
only if \( \mathcal{F} \) has a biorthogonal sequence \( \{ \psi_k \}_{k \in \Xi} \), that is,

\[
\langle \varphi_i, \psi_j \rangle = \delta_{ij} = \begin{cases} 
1, & \text{if } i = j; \\
0, & \text{elsewhere}. 
\end{cases}
\]

**Remark 2.5.** Every Riesz basis is a basis and hence linearly independent. For other types of
independence of frames we refer to chapter 7 of [4].
The frame conditions given in (2.1) is a powerful tool in the study of operator theory [1, 27, 31], iterated function systems [13, 42], quantum physics [2, 25, 26], distributed signal processing [11, 12]. Among the many available texts on frames, Christensen [4], Heil [19] are excellent for basic theory of frames and Han [17] for applications of frames in many directions.

2.2. Gabor system and wavelet system

As is standard, \( L^2(\mathbb{R}, dx) := \{ f : \int_{\mathbb{R}} |f|^2 dx < \infty \} \), the space of square integrable (in the sense of Lebesgue) functions over \( \mathbb{R} \). \( L^2(\mathbb{R}, dx) \) is a Hilbert space with respect to the standard inner product \( \langle f, g \rangle = \int_{\mathbb{R}} f(x)g(x)dx \). The Fourier transform of a function \( f \), denoted by \( \hat{f} \), is defined as

\[
\hat{f}(\gamma) = \int_{\mathbb{R}} f(x)e^{-2\pi i x \gamma} dx, \quad \gamma \in \mathbb{R}.
\]

Mathematically, a Gabor system and a wavelet system are defined by using the three classes of operators that act unitarily on \( L^2(\mathbb{R}, dx) \). For \( a, b \in \mathbb{R} \) and \( c \in \mathbb{R}^* \) define translation operator, modulation operator and dilation operator, respectively, on \( L^2(\mathbb{R}, dx) \) by

\[
\begin{align*}
T_a f(x) & \mapsto f(x - a), f \in L^2(\mathbb{R}, dx) \quad \text{(Translation by } a); \\
E_b f(x) & \mapsto e^{2\pi i bx} f(x), f \in L^2(\mathbb{R}, dx) \quad \text{(Modulation by } b); \\
D_c f(x) & \mapsto \frac{1}{\sqrt{c}} f\left(\frac{x}{c}\right), f \in L^2(\mathbb{R}, dx) \quad \text{(Dilation by } c).
\end{align*}
\]

Let \( \phi \in L^2(\mathbb{R}, dx) \) be a non-zero function. A collection of the form \( \mathcal{G}(a, b, \phi) := \{ e^{2\pi i m b t} \phi(x - ma) \}_{m,n \in \mathbb{Z}} \) is called the Gabor system; and the collection \( \mathcal{W}(c, b, \phi) := \{ e^{-2\pi i x} \phi(e^{-j}x - kb) \}_{j \in \mathbb{Z}} \) is called the wavelet system. A frame of the form \( \mathcal{G}(a, b, \phi) \) and \( \mathcal{W}(a, b, \phi) \) for \( L^2(\mathbb{R}, dx) \) is called the Gabor frame and the wavelet frame, respectively. Christensen [4] and Heil [19] are good texts for fundamental properties of Gabor frames and wavelet frames.

The following fundamental properties of translation, modulation, and dilation operators will be used throughout the paper.

**Lemma 2.6 ([4, p 65]).** For any \( 0 \neq a \in \mathbb{R} \) and \( f \in L^2(\mathbb{R}, dx) \), we have
\[
\begin{align*}
T_a f & = E_{-a} \hat{f}, \\
E_a f & = T_a \hat{f}, \\
D_a f & = D_{\frac{1}{a}} \hat{f}.
\end{align*}
\]

**Lemma 2.7 ([4, p 65]).** For \( f \in L^2(\mathbb{R}, dx) \), \( a, b \in \mathbb{R}, c \in \mathbb{R}^* \) and \( x \in \mathbb{R} \), the following commutator relations hold
\[
\begin{align*}
T_a E_b f(x) & = e^{-2\pi i b x} E_b T_a f(x), \\
T_a D_c f(x) & = D_c T_a f(x), \\
D_c E_b f(x) & = E_{b/c} D_c f(x).
\end{align*}
\]

The following result is a direct consequence of Parseval’s equation and can be found in any standard text on analysis, for instance [19, 46].

**Theorem 2.8.** Assume that for \( \mu > 0 \) and \( f \in L^2([0, \mu], dx) \subset L^2(\mathbb{R}, dx) \), we have
\[
\mu \| f \|^2_{L^2(\mathbb{R}, dx)} = \sum_{k \in \mathbb{Z}} |c_k|^2, \quad \text{where } c_k = \int_{\mathbb{R}} f(x) e^{-2\pi i k x} dx.
\]
2.3. Nonstationary frames of translates

Motivated by the concept of nonstationary wavelet system in [28, chapter 8], we consider nonstationary frames of translates. Let $X$ be a countably infinite index set and for each $j \in X$, let $\varphi_j \in L^2(\mathbb{R}, dx)$ and $a^{(j)} \in \mathbb{R}$. A collection of functions of the form $\{T_{ka^{(j)}j}\varphi_j\}_{j \in X}$ in $L^2(\mathbb{R}, dx)$ is called a nonstationary system of translates.

**Definition 2.9.** For each $j \in X$, let $a^{(j)} \in \mathbb{R}$ and $\varphi_j \in L^2(\mathbb{R}, dx)$. A frame of the form $\{T_{ka^{(j)}j}\varphi_j\}_{j \in X}$ for $L^2(\mathbb{R}, dx)$ is called a nonstationary frame of translates.

In the case of nonstationary frames of translates, we consider infinitely many window functions $\varphi_j$, whereas in stationary frames of translates, we consider finitely many window functions. For basic results on stationary Bessel sequences and stationary frames of translates (with one window function), we refer to [4, chapter 7]. Nonstationary frames with different structures have been studied by many authors, we refer to [28] for nonstationary wavelets and related applications. Most recently, Jindal, Jyoti, and Vashisht [23] studied matrix-valued nonstationary frames in the matrix-valued signal space $L^2(\mathbb{R}, C^{n \times n})$.

3. Frames for the Weyl–Heisenberg group

Let $\mathcal{G}$ be a locally compact group with left Haar measure $\mu$. A unitary representation of $\mathcal{G}$ is a homomorphism $\pi$ from $\mathcal{G}$ into the group $U(\mathcal{H})$ of unitary operators on $\mathcal{H}$ that is continuous with respect to the strong operator topology, that is, a map $\zeta: \mathcal{G} \rightarrow U(\mathcal{H})$ that satisfies $\zeta(xy) = \zeta(x)\zeta(y)$ and $\zeta(x^{-1}) = \zeta(x)^\dagger = \zeta(x)^*$, and for which $x \rightarrow \pi(x)u$ is continuous from $\mathcal{G}$ to $\mathcal{H}$ for any $u \in \mathcal{H}$. Also, $\zeta$ is said to be irreducible if $\pi$ admits only trivial (in other words, $= \{0\}$ or $\mathcal{H}$) invariant subspaces.

The Weyl–Heisenberg group $W$ is the outer semidirect product $\mathbb{R}^2 \rtimes_0 \mathbb{R}$, where $\theta: \mathbb{R} \rightarrow \text{Aut}(\mathbb{R}^2)$ given by $\theta_\zeta \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 + \zeta a_2 \\ a_2 \end{pmatrix}$ for every $\zeta \in \mathbb{R}$ and $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^2$. The product of $(s, \vec{u})$ and $(t, \vec{v})$ in $W$ is given by
$$
(s, \vec{u})(t, \vec{v}) = (s + t, \theta_\zeta(\vec{u} + \vec{v})�).
$$

For every $A \in \mathbb{R}^*$, $B \in \mathbb{R}$, the unitary irreducible representation of $W$ is given by
$$
\Theta^{A, B}: W \rightarrow L^2(\mathbb{R}, dx)
$$
$$
(\Theta^{A, B}(c, (y_1, y_2))f)(x) = e^{2\pi i[A(c+y_1)+By_2]}f(c+x).
$$

All unitary irreducible representations of this group $W$ are unitarily equivalent, see [30, 32, 33, 36] for technical details. We refer to the [22, 37] for technical details about applications of the Weyl–Heisenberg group in the construction of Gabor frames.

For $j \in \{1, 2, \ldots, N\}$, where $N$ is some strictly positive number, let $p_0^{(j)}$ and $q_0^{(j)}$ be real numbers such that $|p_0^{(j)}|q_0^{(j)}| < 1$, and consider the discrete subset of $W$ given by
$$
W_{q_0^{(j)}, p_0^{(j)}} = \left\{ \left( nq_0^{(j)}, \left( \frac{nq_0^{(j)}p_0^{(j)}}{2}, lp_0^{(j)} \right) \right) | n, l \in \mathbb{Z}\right\}.
$$

For each $j \in \{1, 2, \ldots, N\}$, let $\Phi_j$ be a non-zero function in $L^2(\mathbb{R}, dx)$ (also known as ‘window function’). For $n, l \in \mathbb{Z}$, define
$$
\Phi^{A, B}_{n, l}(j) = \Theta^{A, B}(nq_0^{(j)}, \left( \frac{nq_0^{(j)}p_0^{(j)}}{2}, lp_0^{(j)} \right)) \Phi_j.
$$
Then
\[
\left\{ \Phi_{(n,l,j)}^A (x) \right\}_{n,l,j \in \mathbb{Z}, j \in \{1,2,\ldots,N\}} = \left\{ e^{2\pi i ((1/2)A n b_l (q_l j + B p_l (q_l j - E A) x + n q_l \beta_l j)} \Phi_j (x + n q_l \beta_l j) \right\}_{n,l,j \in \mathbb{Z}, j \in \{1,2,\ldots,N\}}
\]

which is a Gabor system in \( L^2(\mathbb{R}, dx) \).

Now we are ready to give the necessary and sufficient conditions for the existence of Gabor frames for the Weyl–Heisenberg group.

**Theorem 3.1.** Let \( \{ \Phi_j \}_{j \in \{1,2,\ldots,N\}} \subset L^2(\mathbb{R}, dx) \) be a finite collection of non-zero functions such that support of each \( \Phi_j \) is contained in an interval of length \( \lambda \). For each \( j \in \{1,2,\ldots,N\} \), let \( p_{(j)}^0 \) and \( q_{(j)}^0 \) be real numbers such that \( |p_{(j)}^0 q_{(j)}^0| < 1 \) and \( q_{(j)}^0 = \frac{1}{\lambda} \). Then, \( \{ \Phi_{(n,l,j)}^A \}_{n,l \in \mathbb{Z}, j \in \{1,2,\ldots,N\}} \) is a Gabor frame for \( L^2(\mathbb{R}, dx) \) with frame bounds \( \alpha_\gamma, \beta_\gamma \) if and only if

\[
\alpha_\gamma \leq \lambda \sum_{j=1}^N \sum_{l \in \mathbb{Z}} |\Phi_j (\gamma - A n q_l \beta_l |)^2 \leq \beta_\gamma \text{ for almost all } \gamma \in \mathbb{R}.
\]

For the proof of theorem 3.1, first we state and prove the following result which gives necessary and sufficient conditions for nonstationary frames of translates in terms of a series of the Fourier transforms of window functions. Frames of translates (or frames for shift–invariant subspaces) were first considered in [35] by Ron and Shen.

**Theorem 3.2.** For each \( l \in \mathbb{Z} \), let \( \phi_l \) be a non-zero function in \( L^2(\mathbb{R}, dx) \) such that support of each \( \hat{\phi}_l \) is contained in an interval of length \( \lambda \) and \( q_l^0 = \frac{1}{\lambda} \). Then, \( \{ T_{n q_l^0} \phi_l \}_{n \in \mathbb{Z}} \) is a nonstationary frame of translates for \( L^2(\mathbb{R}, dx) \) with frame bounds \( \alpha, \beta \) if and only if

\[
\frac{\alpha}{\lambda} \leq \sum_{l \in \mathbb{Z}} |\hat{\phi}_l (\gamma)|^2 \leq \frac{\beta}{\lambda} \text{ for almost all } \gamma \in \mathbb{R}. \tag{3.1}
\]

**Proof.** Assume first that (3.1) holds. Then, for any \( f \in L^2(\mathbb{R}, dx) \), we have

\[
\sum_{n,l \in \mathbb{Z}} |\langle f, T_{n q_l^0} \phi_l \rangle|^2 = \sum_{n,l \in \mathbb{Z}} |\langle \hat{f}, T_{n q_l^0} \hat{\phi}_l \rangle|^2 = \sum_{n,l \in \mathbb{Z}} |\langle \hat{f}, E_{-n q_l^0} \hat{\phi}_l \rangle|^2 = \sum_{n,l \in \mathbb{Z}} \left| \int_{\mathbb{R}} \hat{f}(\gamma) \overline{E_{-n q_l^0} \phi_l (\gamma)} d\gamma \right|^2 = \sum_{n,l \in \mathbb{Z}} \left| \int_{\mathbb{R}} \hat{f}(\gamma) e^{2\pi inq_l^0} \overline{\phi_l (\gamma)} d\gamma \right|^2 = \sum_{n,l \in \mathbb{Z}} \left| \int_{\mathbb{R}} \hat{f}(\gamma) e^{2\pi inq_l^0} \phi_l (\gamma) d\gamma \right|^2. \tag{3.2}
\]
Using theorem 2.8 in (3.2), we arrive at
\[
\sum_{n,j \in \mathbb{Z}} |\langle f, T_{nq^{(0)}} \phi_l \rangle|^2 = \lambda \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}} |\hat{f}(\gamma)\overline{\phi_l(\gamma)}|^2 d\gamma
\]
\[
= \lambda \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}} |\hat{f}(\gamma)|^2 |\overline{\phi_l(\gamma)}|^2 d\gamma
\]
\[
= \lambda \int_{\mathbb{R}} |\hat{f}(\gamma)|^2 \sum_{l \in \mathbb{Z}} |\overline{\phi_l(\gamma)}|^2 d\gamma. \quad (3.3)
\]

By using (3.1), we have
\[
\alpha \int_{\mathbb{R}} |\hat{f}(\gamma)|^2 d\gamma \leq \sum_{n,j \in \mathbb{Z}} |\langle f, T_{nq^{(0)}} \phi_l \rangle|^2 \leq \beta \int_{\mathbb{R}} |\hat{f}(\gamma)|^2 d\gamma,
\]
or
\[
\alpha \|f\|^2 \leq \sum_{n,j \in \mathbb{Z}} |\langle f, T_{nq^{(0)}} \phi_l \rangle|^2 \leq \beta \|f\|^2.
\]
That is
\[
\alpha \|f\|^2 \leq \sum_{n,j \in \mathbb{Z}} |\langle f, T_{nq^{(0)}} \phi_l \rangle|^2 \leq \beta \|f\|^2 \text{ for all } f \in L^2(\mathbb{R}, dx).
\]

Hence, \( \{ T_{nq^{(0)}} \phi_l \}_{n,j \in \mathbb{Z}} \) is a nonstationary frame of translates for \( L^2(\mathbb{R}, dx) \) with the desired frame bounds.

To prove the opposite implication, let \( \{ T_{nq^{(0)}} \phi_l \}_{n,j \in \mathbb{Z}} \) be a frame of translates for \( L^2(\mathbb{R}, dx) \) with frame bounds \( \alpha, \beta \). Then, using (3.3), for all \( f \in L^2(\mathbb{R}, dx) \), we have
\[
\alpha \|f\|^2 \leq \lambda \int_{\mathbb{R}} |\hat{f}(\gamma)|^2 \sum_{l \in \mathbb{Z}} |\overline{\phi_l(\gamma)}|^2 d\gamma \leq \beta \|f\|^2,
\]
which entails
\[
\alpha \int_{\mathbb{R}} |\hat{f}(\gamma)|^2 d\gamma \leq \lambda \int_{\mathbb{R}} |\hat{f}(\gamma)|^2 \sum_{l \in \mathbb{Z}} |\overline{\phi_l(\gamma)}|^2 d\gamma \leq \beta \int_{\mathbb{R}} |\hat{f}(\gamma)|^2 d\gamma
\]
for all \( f \in L^2(\mathbb{R}, dx) \). This completes the proof.

\( \square \)

\textbf{Remark 3.3.} Lemma 10.1.1 of [4] is a particular case of theorem 3.2.

The following example justifies theorem 3.2.

\textbf{Example 3.4.} Let \( \lambda = \frac{1}{2} \). For each \( l \in \mathbb{Z} \), let \( q^{(0)} = 2 \) and define functions \( \phi_l \in L^2(\mathbb{R}, dx) \) as follows:
\[
\hat{\phi}_l(x) = \begin{cases} 
2, & \text{if } x \in \left( \frac{l}{2}, \frac{l+1}{2} \right], \\
0, & \text{elsewhere}.
\end{cases}
\]

Then, \( \sum_{l \in \mathbb{Z}} |\hat{\phi}_l(\gamma)|^2 = 2 \) for almost all \( \gamma \in \mathbb{R} \). Thus, by theorem (3.2), \( \{ T_{nq^{(0)}} \phi_l \}_{n,j \in \mathbb{Z}} \) is a nonstationary Parseval frame of translates for \( L^2(\mathbb{R}, dx) \).
Proof of theorem 3.1. For all $n, l \in \mathbb{Z}$ and $j \in \{1, 2, \ldots, N\}$, we have
\[
\Phi^{A,B}_{(n,j,l)}(x) = e^{2\pi i [(1/2)Ap_0(j,l) + Bp_0(j,l)]} E_{Ap_0} T_{-nq_0} \Phi_j(x),
\]
(3.4)
and
\[
E_{Ap_0} T_{-nq_0} \Phi_j = e^{-2\pi i [Ap_0(j,l) + Bp_0(j,l)]} E_{Ap_0} \Phi_j.
\]
(3.5)
Therefore, for any $f \in L^2(\mathbb{R}, dx)$, using (3.4) and (3.5), we have
\[
\sum_{j=1}^{N} \sum_{n,l \in \mathbb{Z}} |\langle f, \Phi^{A,B}_{(n,j,l)} \rangle|^2 = \sum_{j=1}^{N} \sum_{n,l \in \mathbb{Z}} |\langle f, E_{Ap_0} T_{-nq_0} \Phi_j \rangle|^2
\]
\[
= \sum_{j=1}^{N} |\langle f, E_{Ap_0} \Phi_j \rangle|^2.
\]
Thus, $\{ \Phi^{A,B}_{(n,j,l)} \}_{n,l \in \mathbb{Z}, j \in \{1, 2, \ldots, N\}}$ is a frame for $L^2(\mathbb{R}, dx)$ if and only if $\{ E_{Ap_0} \Phi_j \}_{j \in \{1, 2, \ldots, N\}}$ is a frame for $L^2(\mathbb{R}, dx)$. The result now follows directly from theorem 3.2. □

The following example illustrates theorem 3.1.

Example 3.5. Let $A = 1$, $B = 0$, $\lambda = 2$ and define functions $\Phi_1, \Phi_2 \in L^2(\mathbb{R}, dx)$ as follows:
\[
\tilde{\Phi}_1(\gamma) = \begin{cases} 
1 + \gamma, & \text{if } \gamma \in (0, 1]; \\
\gamma, & \text{if } \gamma \in (1, 2]; \\
0, & \text{otherwise},
\end{cases}
\]
and
\[
\tilde{\Phi}_2(\gamma) = \begin{cases} 
1 + \gamma, & \text{if } \gamma \in (0, 1]; \\
\frac{\gamma}{2}, & \text{if } \gamma \in (1, 2]; \\
0, & \text{otherwise}.
\end{cases}
\]
Choose $p_0^{(1)} = p_0^{(2)} = 1$. Then one can easily observe that
\[
2 \leq \sum_{j \in \mathbb{Z}} |\tilde{\Phi}_1(\gamma - Ap_0(j,l))|^2 \leq 8 \text{ for almost all } \gamma \in \mathbb{R};
\]
and
\[
\frac{5}{4} \leq \sum_{j \in \mathbb{Z}} |\tilde{\Phi}_2(\gamma - Ap_0(j,l))|^2 \leq 5 \text{ for almost all } \gamma \in \mathbb{R}.
\]
Therefore
\[
\lambda \sum_{j=(1,2)} \sum_{l \in \mathbb{Z}} |\tilde{\Phi}_j(\gamma - Ap_0(j,l))|^2 \leq 26 \text{ for almost all } \gamma \in \mathbb{R}.
\]

Hence, by theorem 3.1, $\{ \Phi^{1,0}_{(n,j,l)} \}_{n,j,l \in \{1,2\}}$ is a Gabor frame for $L^2(\mathbb{R}, dx)$ with frame bounds $\alpha_0 = \frac{13}{2}$ and $\beta_0 = 26$.

Next, we discuss an interplay between modulation and translation parameters in Gabor frames for the Weyl–Heisenberg group. First we observe that if for each $l \in \mathbb{Z}$, $\phi_j$ is a non-zero function in $L^2(\mathbb{R}, dx)$ such that support of each $\phi_j$ is contained in an interval of length $\lambda$ and $q^{(j)} = \frac{1}{\lambda}$. Then, using the same approach as in theorem 3.2, we arrive at
\[ \sum_{n \in \mathbb{Z}} |\langle f, T_{\frac{nq}{\lambda}} \phi_l \rangle|^2 = \frac{1}{\lambda} \int_{\mathbb{R}} |\hat{f}(\gamma)|^2 \sum_{l \in \mathbb{Z}} |\hat{\phi}(\gamma)|^2 \, d\gamma \]
\[ = \frac{1}{\lambda^2} \sum_{n, l \in \mathbb{Z}} |\langle f, T_{\frac{nlq}{\lambda}} \phi_l \rangle|^2. \]

Thus, we have the following result:

**Proposition 3.6.** For \( l \in \mathbb{Z} \), let \( \phi_l \) be a non-zero function in \( L^2(\mathbb{R}, dx) \) such that support of each \( \hat{\phi}_l \) is contained in an interval of length \( \lambda \) and \( q^{(l)}_0 = \frac{1}{\lambda} \). Then, \( \left\{ T_{\frac{nlq}{\lambda}} \phi_l \right\}_{n \in \mathbb{Z}} \) is a nonstationary frame of translates for \( L^2(\mathbb{R}, dx) \) with frame bounds \( \alpha, \beta \) if and only if \( \left\{ T_{\frac{nlq}{\lambda}} \phi_l \right\}_{n \in \mathbb{Z}} \) is a nonstationary frame of translates for \( L^2(\mathbb{R}, dx) \) with frame bounds \( \frac{\alpha}{\lambda^2}, \frac{\beta}{\lambda^2} \).

Now, we are ready to give an interplay between modulation and translation parameters for Gabor frames associated with the Weyl–Heisenberg group.

**Theorem 3.7.** For \( j \in \{1, 2, \ldots, N\} \), let \( \Phi_j \) be non-zero functions in \( L^2(\mathbb{R}, dx) \) such that support of each \( \Phi_j \) is contained in an interval of length \( \lambda \) and \( q^{(l)}_0 = \frac{1}{\lambda} \). Then, \( \left\{ \Phi_{j, k}^{A, B} \right\}_{n, l \in \mathbb{Z}, j \in \{1, 2, \ldots, N\}} \) is a Gabor frame for \( L^2(\mathbb{R}, dx) \) with frame bounds \( \alpha, \beta \) if and only if \( \left\{ E_{nhq^{(l)}_0} \Phi_{j, k}^{A, B} \right\}_{n, l \in \mathbb{Z}, j \in \{1, 2, \ldots, N\}} \) is a Gabor frame for \( L^2(\mathbb{R}, dx) \) with frame bounds \( \frac{\alpha}{\lambda^2}, \frac{\beta}{\lambda^2} \).

**Proof.** Since

\[ E_{nhq^{(l)}_0} T_{-mq^{(l)}_0} \Phi_j = e^{-2\pi i h q^{(l)}_0} \Phi_j T_{-mq^{(l)}_0} E_{nhq^{(l)}_0} \Phi_j \text{ for all } n, l \in \mathbb{Z} \text{ and } j \in \{1, 2, \ldots, N\}. \] (3.6)

Therefore, from equation (3.6) and proposition 3.6, we can say that

\[ \left\{ \Phi_{j, k}^{A, B} \right\}_{n, l \in \mathbb{Z}, j \in \{1, 2, \ldots, N\}} \text{ is a Gabor frame for } L^2(\mathbb{R}, dx) \text{ with frame bounds } \alpha, \beta. \]
\[ \iff \left\{ T_{-mq^{(l)}_0} \Phi_{j, k}^{A, B} \right\}_{n, l \in \mathbb{Z}, j \in \{1, 2, \ldots, N\}} \text{ is a nonstationary frame of translates for } L^2(\mathbb{R}, dx) \text{ with frame bounds } \alpha, \beta, \text{ where } \Phi_{j, k}^{A, B} = E_{nhq^{(l)}_0} \Phi_j. \]
\[ \iff \left\{ T_{-mq^{(l)}_0} \Phi_{j, k}^{A, B} \right\}_{n, l \in \mathbb{Z}, j \in \{1, 2, \ldots, N\}} \text{ is a nonstationary frame of translates for } L^2(\mathbb{R}, dx) \text{ with frame bounds } \frac{\alpha}{\lambda^2}, \frac{\beta}{\lambda^2}. \]
\[ \iff \left\{ E_{nhq^{(l)}_0} T_{-mq^{(l)}_0} \Phi_j \right\}_{n, l \in \mathbb{Z}, j \in \{1, 2, \ldots, N\}} \text{ is a Gabor frame for } L^2(\mathbb{R}, dx) \text{ with frame bounds } \frac{\alpha}{\lambda^2}, \frac{\beta}{\lambda^2}. \]

This concludes the proof. \( \square \)

**Lemma 3.8 ([4, lemma 5.3.3]).** If \( \{ \psi_k \}_{k \in \mathbb{Z}} \) is a frame sequence with frame bounds \( \alpha \) and \( \beta \) and \( U: \mathcal{H} \rightarrow \mathcal{H} \) is a unitary operator, then \( \{ U\psi_k \}_{k \in \mathbb{Z}} \) is a frame sequence with frame bounds \( \alpha \) and \( \beta \).

**Proposition 3.9** Assume that \( \{ \Phi_{j, k}^{A, B} \}_{n, l \in \mathbb{Z}, j \in \{1, 2, \ldots, N\}} \) is a frame sequence with frame bounds \( \alpha, \beta \). Given \( a > 0 \), let \( \phi_{a} := D_n \phi_j \). Then, \( \{ \Phi_{j, k}^{A, B} \}_{n, l \in \mathbb{Z}, j \in \{1, 2, \ldots, N\}} \) is a frame sequence with frame bounds \( \alpha \) and \( \beta \).
Proof. Using the commutator relations of lemma 2.7, we have
\[
\Phi_{(n, l, j)}^{A,B}(x) = e^{2\pi i ((1/2)\Delta p_0 |a|_0 + B^2 \mu_0 |a|_0^2) E_{B\mathcal{P}}(n \mu_0^l) T \eta_{n\delta_0^l} D_n \Phi_j(x)}
\]
\[
= e^{2\pi i ((1/2)\Delta p_0 |a|_0 + B^2 \mu_0 |a|_0^2) E_{B\mathcal{P}}(n \mu_0^l) T \eta_{n\delta_0^l} D_n \Phi_j(x)}
\]
\[
= e^{2\pi i ((1/2)\Delta p_0 |a|_0 + B^2 \mu_0 |a|_0^2) E_{B\mathcal{P}}(n \mu_0^l) T \eta_{n\delta_0^l} D_n \Phi_j(x)}
\]
\[
= D_n e^{2\pi i B_2 \mu_0^l (1-a) \Phi_{(n,l,j)}^{A,B}(x)}.
\]
The proof now follows from lemma 3.8.

It is proved in [4, lemma 9.3.2] that a function in \( L^2(\mathbb{R}, dx) \) belongs to the closure of the span of a frame sequence of translates if and only if its Fourier transform can be expressed in terms of the Fourier transform of the window functions. The next result generalizes [4, lemma 9.3.2] for frame sequences with the Gabor structure associated with the Weyl–Heisenberg group.

**Theorem 3.10.** Assume that \{\( \Phi_{(n,l,j)}^{A,B} \)\}_{n,l,j \in \{1,2,\ldots,N\}} is a frame sequence in \( L^2(\mathbb{R}, dx) \). Then, a function \( f \in L^2(\mathbb{R}, dx) \) belongs to the space \( \text{span}\{\Phi_{(n,l,j)}^{A,B}\}_{n,l,j \in \{1,2,\ldots,N\}} \) if and only if there exists a sequence \{\( F_{ij} \)\}_{n,l,j \in \{1,2,\ldots,N\}} of 1-periodic functions such that
\[
\widehat{f}(\gamma) = \sum_{j=1}^{N} \sum_{l \in \mathbb{Z}} F_{ij}(\gamma - Alp_0^{(j)}),
\]
where restriction of each \( F_{ij} \) to \([0,1)\) belongs to \( L^2([0,1), dx) \).

**Proof.** A function \( f \in L^2(\mathbb{R}, dx) \) belongs to \( \text{span}\{\Phi_{(n,l,j)}^{A,B}\}_{n,l,j \in \{1,2,\ldots,N\}} \) if and only if there exists a sequence \{\( c_{n,l,j} \)\}_{n,l,j \in \{1,2,\ldots,N\}} \in \ell^2(\mathbb{Z} \times \mathbb{Z} \times \{1,2,\ldots,N\}) \) such that
\[
f = \sum_{j=1}^{N} \sum_{l \in \mathbb{Z}} c_{n,l,j} \Phi_{(n,l,j)}^{A,B}.
\]
Therefore
\[
\widehat{f}(\gamma) = \sum_{j=1}^{N} \sum_{l \in \mathbb{Z}} c_{n,l,j} e^{2\pi i ((1/2)\Delta p_0 |a|_0 + B^2 \mu_0 |a|_0^2) E_{B\mathcal{P}}(n \mu_0^l) T \eta_{n\delta_0^l} \Phi_j(\gamma)}
\]
\[
= \sum_{j=1}^{N} \sum_{l \in \mathbb{Z}} c_{n,l,j} e^{2\pi i ((1/2)\Delta p_0 |a|_0 + B^2 \mu_0 |a|_0^2) E_{B\mathcal{P}}(n \mu_0^l) T \eta_{n\delta_0^l} \Phi_j(\gamma - Alp_0^{(j)}),
\]
\[
= \sum_{j=1}^{N} \sum_{l \in \mathbb{Z}} F_{ij} \Phi_j(\gamma - Alp_0^{(j)}),
\]
where \( F_{ij} = \sum_{n \in \mathbb{Z}} c_{n,l,j} e^{2\pi i ((1/2)\Delta p_0 |a|_0 + B^2 \mu_0 |a|_0^2) e2\pi i n\mu_0^l \gamma} \). This concludes the proof.

4. Frames for the extended affine group

The affine group \( A \) realized as the upper half plane
\[
A = \{ (\alpha, \beta) : \alpha \in \mathbb{R}^+, \beta \in \mathbb{R} \}.
\]
The multiplication law in $A$ is given by

$$(\alpha, \beta)(x, y) = (\alpha x, \alpha y + \beta)$$

for all points $(\alpha, \beta), (x, y) \in A$.

Let $V$ denote the subspace of $L^2(\mathbb{R}, dx)$ consisting of all the functions whose Fourier transform vanishes on the negative half of $\mathbb{R}$. Recall that for any $d \in \mathbb{R}^*$, the unitary irreducible representation of the affine group on the subspace $V$ of $L^2(\mathbb{R}, dx)$ is the map given by

$$\rho^d: A \to U(V)$$

$$(\rho^d(\alpha, \beta)f)(x) = \frac{1}{\sqrt{\alpha}}e^{\frac{d\beta}{\alpha}}f(x), \quad x \in \mathbb{R}.$$ 

The extended affine group, denoted by $EA$, is the direct product of the affine group $A$ with the additive group of real numbers. That is, $EA = A \oplus \mathbb{R}$. The product of $(\alpha, \beta, \gamma)$ and $(x, y, z)$ in $EA$ is given by

$$(\alpha, \beta, \gamma)(x, y, z) = (\alpha x, \alpha y + \beta, \gamma + z).$$

For $c \in \mathbb{R}$, let $\chi_c(x) = e^{icx}$ be a unitary character of $\mathbb{R}$. For every $d \in \mathbb{R}^*, c \in \mathbb{R}$, the unitary irreducible representation of $EA$ acting on $V$ is the map

$$\Xi_c^d = \rho^d \otimes \chi_c: EA \to U(V)$$

$$(\Xi_c^d(\alpha, \beta, \gamma)f)(x) = e^{ic\gamma}e^{\frac{d\beta}{\alpha}}f(x).$$

We now discuss wavelet frames associated with the extended affine group. We fix a representation $\Theta^d_{\gamma}^{\lambda}$ of the Weyl–Heisenberg group on $L^2(\mathbb{R}, dx)$. For $j \in \{1, 2, \ldots, N\}$, where $N$ is some strictly positive number, consider the discrete subset of $EA$ that is defined by $EA_{\alpha, \beta, \gamma} = \{(\alpha, \beta, \gamma) : n, l \in \mathbb{Z}\}$, where

$$\alpha_l = e^{-ib_l^0},$$

$$\beta_{ld} = -\frac{nl(q_0^{(j)})^2}{\text{dim}(\alpha_l)} = \frac{nq_0^{(j)}}{d},$$

$$\gamma_{ld} = \beta_{ld} \frac{\ln(\alpha_l)}{\alpha_l - 1}.$$  

For $\Phi_j \in L^2(\mathbb{R}, dx) \setminus \{0\}$ and $j \in \{1, 2, \ldots, N\}$, define

$$\Phi_{c, l}^{\alpha, \beta, \gamma} = \Xi_c^d(\alpha_l, \beta_{ld}, \gamma_{ld})\Phi_j,$$

where $n, l \in \mathbb{Z}$. Then,

$$\Phi_{c, l}^{\alpha, \beta, \gamma}(x) = e^{-\frac{\alpha_l x}{\sqrt{e^{-b_l^0} - 1}}} \frac{1}{\sqrt{e^{-b_l^0} - 1}} \Phi_j(x + nq_0^{(j)})$$

$$\quad = e^{-\frac{\alpha_l x}{\sqrt{e^{-b_l^0} - 1}}} D_{-\frac{\alpha_l x}{\sqrt{e^{-b_l^0} - 1}}} \Phi_j(x), \quad \text{for } n, l \in \mathbb{Z}, j \in \{1, 2, \ldots, N\},$$

which is equivalent to a wavelet system. For more technical details about frames associated with extended affine group, we refer to [37].

The following result gives the existence of wavelet frames from the extended affine group.
Theorem 4.1. Under the assumptions of theorem 3.1, the sequence \( \{ \Phi_{(n,l)}^{d} \}_{n,l \in \mathbb{Z}} \) is a wavelet frame for \( L^2(\mathbb{R}, dx) \) with bounds \( \alpha_n, \beta_n \) if and only if
\[
\alpha_n \leq \lambda \sum_{j=1}^{N} \sum_{l \in \mathbb{Z}} \left| \hat{\Phi}_j \left( \frac{x}{2^{l}} \right) \right|^2 \leq \beta_n \text{ for almost all } \gamma \in \mathbb{R}.
\]

Proof. For all \( n, l \in \mathbb{Z} \) and \( j \in \{1, 2, \ldots, N\} \), we have
\[
D_{e^{-q}(n)} T_{-m_{0}}(l) \Phi_j = T_{-m_{0}}(l) D_{e^{-q}(n)} \Phi_j.
\]

Using (4.1) and (4.2), for \( f \in L^2(\mathbb{R}, dx) \), we have
\[
\sum_{j=1}^{N} \sum_{n,l \in \mathbb{Z}} |(f, \Phi_{(n,l)}^{d})|^2 = \sum_{j=1}^{N} \sum_{n,l \in \mathbb{Z}} |(f, D_{e^{-q}(n)} T_{-m_{0}}(l) \Phi_j)|^2
\]
\[
= \sum_{j=1}^{N} \sum_{n,l \in \mathbb{Z}} |(f, T_{-m_{0}}(l) D_{e^{-q}(n)} \Phi_j)|^2.
\]

Thus, \( \{ \Phi_{(n,l)}^{d} \}_{n,l \in \mathbb{Z}} \) is a frame for \( L^2(\mathbb{R}, dx) \) if and only if the \( \{ T_{-m_{0}}(l) D_{e^{-q}(n)} \Phi_j \}_{n,l \in \mathbb{Z}} \) is a frame for \( L^2(\mathbb{R}, dx) \). The result now directly follows from theorem 3.2. \( \square \)

5. The canonical dual of nonstationary frames of translates

Recently, the structure of the canonical dual of different types of frames has been studied by many authors, see [6, 10, 34] and references therein. Chui and Shi showed in [6] that the canonical dual of a wavelet frame for \( L^2(\mathbb{R}, dx) \) need not have a wavelet structure. The authors of [10] proved that the canonical dual of a discrete wavelet frame for the unitary space \( \mathbb{C}^N \) has the same structure. Further, the canonical dual of a discrete Gabor frame has the same structure, see proposition 6.1 of [34]. In this section, we show that the canonical dual of nonstationary frames of translates has the same structure.

Theorem 5.1. Let \( \{ T_{am}(l) \phi_l \}_{n,l \in \mathbb{Z}} \) be a nonstationary frame of translates for \( L^2(\mathbb{R}, dx) \) with frame operator \( S \). Then, the canonical dual frame of \( \{ T_{am}(l) \phi_l \}_{n,l \in \mathbb{Z}} \) is \( \{ T_{am}(l) S^{-1} \phi_l \}_{n,l \in \mathbb{Z}} \).

Proof. First, we show that the frame operator \( S \) commutes with the translation operator. For any \( n', l' \in \mathbb{Z} \) and \( \psi_{l'} \in L^2(\mathbb{R}, dx) \), we compute
\[
T_{a(m',l')} S \psi_{l'} = T_{a(m',l')} \sum_{n,l \in \mathbb{Z}} \langle \psi_{l'}, T_{am}(l) \phi_l \rangle T_{am}(l) \phi_l
\]
\[
= \sum_{n,l \in \mathbb{Z}} \langle \psi_{l'}, T_{am}(l) \phi_l \rangle T_{a(m',l')} T_{am}(l) \phi_l
\]
\[
= \sum_{n,l \in \mathbb{Z}} \langle \psi_{l'}, T_{am}(l) \phi_l \rangle T_{a(m'+l')}(l) \phi_l
\]
\[
= \sum_{n,l \in \mathbb{Z}} \langle \psi_{l'}, T_{am}(l) \phi_l \rangle T_{am}(l) \phi_l
\]
\[
= \sum_{n,l \in \mathbb{Z}} \langle \psi_{l'}, T_{-a(m',l')} \phi_l \rangle T_{am}(l) \phi_l.
\]
Therefore, the frame operator $S$ commutes with the translation operator. Now

$$S^{-1}T_{nq} \phi_l = (T_{nq}^{-1} S)^{-1} \phi_l = (T_{nq}^{-1} S) \phi_l = (ST_{nq})^{-1} \phi_l = T_{nq}^{-1} S^{-1} \phi_l, \quad n, l \in \mathbb{Z}.$$ 

Hence, the canonical dual frame of $\{T_{nq} \phi_l\}_{n,l \in \mathbb{Z}}$ is $\{T_{nq} S^{-1} \phi_l\}_{n,l \in \mathbb{Z}}$. This completes the proof. 

6. Nonstationary frames and Riesz bases of translates

In [3], Casazza and Christensen introduced a new method to approximate the inverse of the frame operator using finite subsets of the frame. In their study, they also consider Gabor frames and frames consisting of translates of a single function. In this section, we consider the technique given in [3] for nonstationary frames of translates. Consider a nonstationary frame of translates $\{T_{nq} \phi_l\}_{n,l \in \mathbb{Z}}$ of $L^2(\mathbb{R}, dx)$ with frame operator $S$, where $q(l) \in \mathbb{R}$ for each $l \in \mathbb{Z}$. We recall that every finite collection of vectors in a Hilbert space $\mathcal{H}$ is a frame for its span, see [4]. Therefore, $\{T_{nq} \phi_l\}_{n \in \mathbb{F}_1, l \in \mathbb{F}_2}$ is a frame for $\mathcal{V} = \text{span}\{T_{nq} \phi_l\}_{n \in \mathbb{F}_1, l \in \mathbb{F}_2}$, where $\mathbb{F}_1 = \{-s, \ldots, s\}$, $\mathbb{F}_2 = \{-t, \ldots, t\}$ for $s, t \in \mathbb{N}$. Denote its frame operator by $S_{s,t}$. Then, $S_{s,t} : \mathcal{V} \rightarrow \mathcal{V}$ is given by

$$S_{s,t} f = \sum_{n \in \mathbb{F}_1} \sum_{l \in \mathbb{F}_2} \langle f, T_{nq} \phi_l \rangle T_{nq} \phi_l,$$

and its frame decomposition is given by $f = \sum_{n \in \mathbb{F}_1} \sum_{l \in \mathbb{F}_2} \langle f, S_{s,t}^{-1} T_{nq} \phi_l \rangle T_{nq} \phi_l, f \in \mathcal{V}$. Since nonstationary frames of translates are multivariate frames, it would be interesting to approximate the inverse of the frame operator (of a multivariate frame) in some sense. In this direction, the following result gives a necessary and sufficient condition for approximation of the inverse of the frame operator of nonstationary frames of translates in the weak sense.

**Theorem 6.1.** For each $l \in \mathbb{Z}$, let $q(l) \in \mathbb{R}$. Suppose $\{T_{nq} \phi_l\}_{n,l \in \mathbb{Z}}$ is a nonstationary frame of translates for $L^2(\mathbb{R}, dx)$ with upper bound $\beta$. Then, for all $f \in L^2(\mathbb{R}, dx)$ and for all $n, l \in \mathbb{Z}$,

$$\langle f, S_{s,t}^{-1} T_{nq} \phi_l \rangle \rightarrow \langle f, S^{-1} T_{nq} \phi_l \rangle \quad \text{as} \quad s, t \rightarrow \infty$$

(6.1)

if and only if, for all $i, j \in \mathbb{N}$, there exists $c_{i,j} \in \mathbb{R}$ such that

$$\|T_{nq} S_{s,t}^{-1} \phi_l\| = \|S_{s,t}^{-1} \phi_l\| \leq c_{i,j} \quad \text{for all} \quad s \geq i, t \geq j.$$ 

(6.2)

**Proof.** Suppose first that (6.2) holds. For $i, j \in \mathbb{N}$ define functions $\psi_{s,t}$ as follows:

$$\psi_{s,t} = S_{s,t}^{-1} T_{nq} \phi_l - S^{-1} T_{nq} \phi_l.$$

We need to prove that for all $f \in L^2(\mathbb{R}, dx)$, $\langle f, \psi_{s,t} \rangle \rightarrow 0$ as $s, t \rightarrow \infty$.

Note that for all $f \in L^2(\mathbb{R}, dx)$, we have

$$S f = \sum_{n,l \in \mathbb{Z}} \langle f, T_{nq} \phi_l \rangle T_{nq} \phi_l = S_0 f + \sum_{n \in \mathbb{F}_1} \sum_{l \in \mathbb{F}_2} \langle f, T_{nq} \phi_l \rangle T_{nq} \phi_l + \sum_{n \in \mathbb{Z} \setminus \mathbb{F}_1} \sum_{l \in \mathbb{Z} \setminus \mathbb{F}_2} \langle f, T_{nq} \phi_l \rangle T_{nq} \phi_l.$$ 

(6.3)
We will use this to obtain an alternative formula for \( \psi_{s,t} \).

Since \( S\psi_{s,t} = SS_{s,t}^{-1}T_{q^0}(\phi_j) - T_{q^0}(\phi_j) \). Therefore, using equation (6.3) on \( S_{s,t}^{-1}T_{q^0}(\phi_j) \), we obtain

\[
S\psi_{s,t} = S_{s,t}SS_{s,t}^{-1}T_{q^0}(\phi_j) + \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle S_{s,t}^{-1}T_{q^0}(\phi_j), T_{q^0}(\phi_l) \rangle T_{q^0}(\phi_l) - T_{q^0}(\phi_j)
\]

\[
= \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle S_{s,t}^{-1}T_{q^0}(\phi_j), T_{q^0}(\phi_l) \rangle T_{q^0}(\phi_l) + \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle S_{s,t}^{-1}T_{q^0}(\phi_j), T_{q^0}(\phi_l) \rangle T_{q^0}(\phi_l) - T_{q^0}(\phi_j).
\]

Thus, for \( s \geq i \) and \( t \geq j \), we have

\[
\psi_{s,t} = \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle S_{s,t}^{-1}T_{q^0}(\phi_j), T_{q^0}(\phi_l) \rangle S^{-1}T_{q^0}(\phi_l) + \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle S_{s,t}^{-1}T_{q^0}(\phi_j), T_{q^0}(\phi_l) \rangle S^{-1}T_{q^0}(\phi_l).
\]

Therefore, for all \( f \in L^2(\mathbb{R}, dx) \), we have

\[
|\langle f, \psi_{s,t} \rangle|^2 = \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle S_{s,t}^{-1}T_{q^0}(\phi_j), T_{q^0}(\phi_l) \rangle \langle f, S^{-1}T_{q^0}(\phi_l) \rangle
\]

\[+ \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle S_{s,t}^{-1}T_{q^0}(\phi_j), T_{q^0}(\phi_l) \rangle \langle f, S^{-1}T_{q^0}(\phi_l) \rangle^2 \]

\[\leq 2 \left( \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle S_{s,t}^{-1}T_{q^0}(\phi_j), T_{q^0}(\phi_l) \rangle \langle f, S^{-1}T_{q^0}(\phi_l) \rangle \right)^2 + \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle f, S^{-1}T_{q^0}(\phi_l) \rangle^2.
\]

Using Cauchy-Schwartz' inequality in (6.4), we compute

\[
|\langle f, \psi_{s,t} \rangle|^2 \leq 2 \left( \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle S_{s,t}^{-1}T_{q^0}(\phi_j), T_{q^0}(\phi_l) \rangle \right)^2 + \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle f, S^{-1}T_{q^0}(\phi_l) \rangle^2
\]

\[\leq 2\beta \| S_{s,t}^{-1}T_{q^0}(\phi_j) \| \left( \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle f, S^{-1}T_{q^0}(\phi_l) \rangle \right)^2 \leq 2\beta \| S_{s,t}^{-1}T_{q^0}(\phi_j) \| \left( \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle f, S^{-1}T_{q^0}(\phi_l) \rangle \right)^2 \]

\[= 2\beta \| T_{q^0}S_{s,t}^{-1} \phi_j \| \left( \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle f, S^{-1}T_{q^0}(\phi_l) \rangle \right)^2 \leq 2\beta \| T_{q^0}S_{s,t}^{-1} \phi_j \| \left( \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle f, S^{-1}T_{q^0}(\phi_l) \rangle \right)^2 \]

\[\leq 2\beta \| T_{q^0}S_{s,t}^{-1} \phi_j \| \left( \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle f, S^{-1}T_{q^0}(\phi_l) \rangle \right)^2 + \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle f, S^{-1}T_{q^0}(\phi_l) \rangle^2.
\]
Since \( \{T_{mq^l_0}\phi_l\}_{n,l \in \mathbb{Z}} \) is a frame, we have
\[
\sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} |\langle S^{-1}f, T_{mq^l_0}\phi_l \rangle|^2 \to 0 \quad \text{and} \quad \sum_{n \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} |\langle S^{-1}f, T_{mq^l_0}\phi_l \rangle|^2 \to 0 \quad \text{as} \quad s,t \to \infty.
\]  
(6.6)

From (6.5) and (6.6), we conclude that
\[
\langle f, \psi_{s,t} \rangle \to 0 \quad \text{as} \quad s,t \to \infty
\]
for all \( f \in L^2(\mathbb{R}, dx) \).

To prove the converse, assume that (6.1) holds. For any \( i,j \in \mathbb{N} \), consider the functionals
\[
F_{s,t}^i : L^2(\mathbb{R}, dx) \to \mathbb{C},
\]
\[
F_{s,t} = \langle f, S_{s,t}^{-1} T_{mq^l_0}\phi_l \rangle, \quad s \geq i, \ t \geq j.
\]

One may observe that each \( F_{s,t} \) is bounded and linear. By condition (6.1), the family of functionals \( \{F_{s,t}\}_{s \geq i, t \geq j} \) is pointwise convergent. By invoking the Principal of Uniform Boundedness, the sequence \( \{||F_{s,t}||\}_{s \geq i, t \geq j} \) is bounded. Therefore, there is constant \( c_{i,j} > 0 \) such that
\[
c_{i,j} \geq ||F_{s,t}|| = ||S_{s,t}^{-1} T_{mq^l_0}\phi_l|| = ||T_{mq^l_0} S_{s,t}^{-1} \phi_l|| = ||T_{mq^l_0} S_{s,t}^{-1} \phi_l|| \quad \text{for all} \quad s \geq i, \ t \geq j.
\]

This completes the proof. \( \square \)

Next, we illustrate theorem 6.1 with an example.

**Example 6.2.** Consider the nonstationary Parseval frame \( \{T_{mq^l_0}\phi_l\}_{n,l \in \mathbb{Z}} \) given in example 3.4. Then, \( \alpha_{opt} = \beta_{opt} = 1 \) are optimal frame bounds for \( \{T_{mq^l_0}\phi_l\}_{n,l \in \mathbb{Z}} \). Observe that \( ||\phi_l|| = 1 \) for every \( l \in \mathbb{Z} \). Hence, \( ||T_{mq^l_0}\phi_l|| = 1; n,l \in \mathbb{Z} \). For all \( s \geq i, t \geq j \),
\[
||S_{s,t}^{-1}|| = \sup_{||T_{mq^l_0}\phi_l|| = 1} ||S_{s,t}^{-1} T_{mq^l_0}\phi_l|| \geq ||S_{s,t}^{-1} T_{mq^l_0}\phi_l|| = ||T_{mq^l_0} S_{s,t}^{-1} \phi_l||.
\]

By proposition 2.1, \( ||S_{s,t}|| = ||S_{s,t}^{-1}|| = 1 \) for all \( s \geq i, t \geq j \). Thus, \( ||T_{mq^l_0} S_{s,t}^{-1} \phi_l|| \leq 1 \) for all \( s \geq i, t \geq j \). Hence, by theorem 6.1, we can say that (6.1) holds.

We conclude this section with sufficient conditions for a nonstationary frame of translates to be a Riesz basis.

**Theorem 6.3.** For each \( l \in \mathbb{Z} \), let \( q^{(l)} \in \mathbb{R} \). A nonstationary frame of translates \( \{T_{mq^l_0}\phi_l\}_{n,l \in \mathbb{Z}} \) for \( L^2(\mathbb{R}, dx) \) is a Riesz basis of the space \( L^2(\mathbb{R}, dx) \) if \( \{T_{mq^l_0}\phi_l\}_{n,l \in \mathbb{Z}} \) is linearly independent and (6.1) holds.

**Proof.** Assume that \( \{T_{mq^l_0}\phi_l\}_{n,l \in \mathbb{Z}} \) is linearly independent and (6.1) holds. Let \( s,t \in \mathbb{N} \). By using linear independence of \( \{T_{mq^l_0}\phi_l\}_{n,l \in \mathbb{Z}} \), we can say that \( \{T_{mq^l_0}\phi_l\}_{n \in \mathbb{Z}, l \in \mathbb{Z}} \) is (Riesz) basis for the space \( \mathcal{V} = \text{span}\{T_{mq^l_0}\phi_l\}_{n \in \mathbb{Z}, l \in \mathbb{Z}} \). Let \( S_{s,t} \) be its associated frame operator. Then, dual basis of \( \{T_{mq^l_0}\phi_l\}_{n \in \mathbb{Z}, l \in \mathbb{Z}} \) is of the form \( \{S_{s,t}^{-1} T_{mq^l_0}\phi_l\}_{n \in \mathbb{Z}, l \in \mathbb{Z}} \). Thus, every \( f \in \mathcal{V} \) can be expressed (uniquely) as
\[
f = \sum_{n \in \mathbb{Z}, l \in \mathbb{Z}} \langle T_{mq^l_0}\phi_l, S_{s,t}^{-1} T_{mq^l_0}\phi_l \rangle T_{mq^l_0}\phi_l,
\]
and
\[
\langle T_{mq^l_0}\phi_l, S_{s,t}^{-1} T_{mq^l_0}\phi_l \rangle = \delta_{ml} \delta_{ij} = \begin{cases} 1, & \text{if} \ n = i, \ l = j; \\ 0, & \text{otherwise}, \end{cases}
\]
Consider any sequence \(f\). There exists a linear operator \(L\) such that
\[
\langle T_{nq}^0 \phi_l, S^{-1} T_{nq}^0 \phi_j \rangle = \delta_{n,l} \delta_{ij},
\]
for \(n, i, j \in \mathbb{F}_1\) and \(l, j \in \mathbb{F}_2\). Letting \(s, t \to \infty\), and using (6.1), for \(n, l, i, j \in \mathbb{Z}\), we obtain
\[
\langle T_{nq}^0 \phi_l, S^{-1} T_{nq}^0 \phi_j \rangle = \delta_{n,l} \delta_{ij} - \phi_{nq}(i) \phi_{nq}(j). \]
Thus, \(\{T_{nq}^0 \phi_l\}_{n \in \mathbb{Z}}\) has a biorthogonal sequence \(\{S^{-1} T_{nq}^0 \phi_l\}_{l \in \mathbb{Z}}\). Hence, by theorem 2.4, the sequence \(\{T_{nq}^0 \phi_l\}_{n \in \mathbb{Z}}\) is a Riesz basis for \(L^2(\mathbb{R}, dx)\).

## 7. Linear independence of nonstationary sequences of translates

In this section, we discuss linear independence of a nonstationary sequence of translates \(\{T_{nq}^0 \phi_l\}_{n \in \mathbb{Z}}\) in \(L^2(\mathbb{R}, dx)\), where \(q^0 \in \mathbb{R}\) for each \(l \in \mathbb{Z}\). Christensen and Hasannasab in [5] proved an equivalent criteria for a sequence \(\{f_k\}_{k \in \mathbb{N}}\) in a separable Hilbert space to be linearly independent in terms of a linear operator on \(\text{span}\{f_k\}_{k \in \mathbb{N}}\) such that iterated action of that operator on an element \(f_j\) in \(\mathcal{H}\) gives the sequence \(\{f_k\}_{k \in \mathbb{N}}\). They proved the following result.

**Proposition 7.1 ([5, proposition 2.3]).** Consider any sequence \(\{f_k\}_{k \in \mathbb{N}}\) in \(\mathcal{H}\) for which \(\text{span}\{f_k\}_{k \in \mathbb{N}}\) is infinite dimensional. Then, the following are equivalent:

(i) \(\{f_k\}_{k \in \mathbb{N}}\) is linearly independent.

(ii) There exists a linear operator \(L: \text{span}\{f_k\}_{k \in \mathbb{N}} \to \mathcal{H}\) such that \(\{f_k\}_{k \in \mathbb{N}} = \{L^n f_1\}_{n=0}^\infty\).

Now we give an equivalent criteria for a nonstationary sequence \(\{T_{nq}^0 \phi_l\}_{n \in \mathbb{Z}}\) to be linearly independent in terms of a linear operator defined on \(\text{span}\{T_{nq}^0 \phi_l\}_{n \in \mathbb{Z}}\) such that iterated action of that operator on \(T_{nq}^0 \phi_l\) gives the whole sequence \(T_{nq}^0 \phi_l\) when span \(\{T_{nq}^0 \phi_l\}_{n \in \mathbb{N}}\) is infinite dimensional. This extends proposition 7.1 to nonstationary sequences of translates in the space \(L^2(\mathbb{R}, dx)\).

**Theorem 7.2.** Suppose \(\{T_{nq}^0 \phi_l\}_{n \in \mathbb{Z}}\) is a nonstationary sequence of translates in \(L^2(\mathbb{R}, dx)\) such that \(\text{span}\{T_{nq}^0 \phi_l\}_{n \in \mathbb{N}}\) is infinite dimensional. Then, the following conditions are equivalent.

(i) \(\{T_{nq}^0 \phi_l\}_{n \in \mathbb{Z}}\) is linearly independent.

(ii) There exists a linear operator \(L: \text{span}\{T_{nq}^0 \phi_l\}_{n \in \mathbb{Z}} \to L^2(\mathbb{R}, dx)\) such that

\[
\{L^m T_{nq}^0 \phi_l\}_{m, k \in \mathbb{Z}} = \{T_{(m+1)q^0} \phi_{l+1}\}_{m, k \in \mathbb{Z}}.
\]

**Proof.** For convenience, write \(\phi_{n,l} = T_{nq}^0 \phi_{l}, n, l \in \mathbb{Z}\).

(i) \(\implies\) (ii): Assume that \(\{\phi_{n,l}\}_{n \in \mathbb{Z}}\) is linearly independent. Then, every subset of \(\{\phi_{n,l}\}_{n \in \mathbb{Z}}\) is also linearly independent. For a fixed \(l \in \mathbb{Z}\), define

\[
L^0 \phi_{n,l} = \phi_{n+1,l}, \ n \in \mathbb{Z}.
\]

Extend \(L^0\) to an operator on \(\text{span}\{\phi_{n,l}\}_{n \in \mathbb{N}}\) linearly. Then, for a fixed \(l \in \mathbb{Z}\), we have

\[
\{L^m \phi_{1,l}\}_{m \in \mathbb{Z}} = \{\phi_{m+1,l}\}_{m \in \mathbb{Z}}. \quad (7.1)
\]

Similarly, for fixed \(n \in \mathbb{Z}\), we have

\[
\{L^0 \phi_{n,1}\}_{k \in \mathbb{Z}} = \{\phi_{n,k+1}\}_{k \in \mathbb{Z}}. \quad (7.2)
\]
Using (7.1) and (7.2), there exists a linear operator on \( \text{span}\{\phi_{n,l}\}_{n,l}\in\mathbb{N} \) such that
\[
\{L^{m,k}\phi_{1,1}\}_{m,k}\in\mathbb{Z} = \{\phi_{m+1,k+1}\}_{m,k}\in\mathbb{Z}.
\]

Thus, (i) implies (ii).

(ii) \(\Rightarrow\) (i): By (ii) there exists a linear operator \( L : \text{span}\{\phi_{n,l}\}_{n,l}\in\mathbb{N} \to L^2(\mathbb{R}, dx) \) such that
\[
\{L^{m,k}\phi_{1,1}\}_{m,k}\in\mathbb{Z} = \{\phi_{m+1,k+1}\}_{m,k}\in\mathbb{Z}.
\]

Assume on the contrary that \( \{\phi_{n,l}\}_{n,l}\in\mathbb{Z} \) is linearly dependent. Then, there exist \( R, S > 0 \) such that
\[
\sum_{n=-R}^{R} \sum_{l=-S}^{S} c_{n,l} \phi_{n,l} = 0
\]
for some coefficients \( \{c_{n,l}\}_{n\in\{-R,...,R\}, l\in\{-S,...,S\}} \). Without loss of generality, assume that \( c_{R,S} \neq 0 \). Then, we can write
\[
\phi_{R,S} = \sum_{n=-R}^{R-1} \sum_{l=-S}^{S-1} c'_{n,l} \phi_{n,l} + \sum_{n=-R}^{R-1} \sum_{l=-S}^{S-1} c''_{n,l} \phi_{n,l} + \sum_{l=-S}^{S-1} c'_{R,l} \phi_{R,l}
\]
for some coefficients \( c'_{n,l} \).

Write
\[
V := \text{span}\{\phi_{n,l}\}_{n\in\{-R,...,R-1\}, l\in\{-S,...,S-1\}} \cup \{\phi_{n,S}\}_{n\in\{-R,...,R-1\}} \cup \{\phi_{R,l}\}_{l\in\{-S,...,S-1\}}.
\]

Then, \( \phi_{R,S} \in V \). Now, for any \( v \in V \), we have
\[
L^{1,0}v = L^{1,0} \left( \sum_{n=-R}^{R-1} \sum_{l=-S}^{S-1} d_{n,l} \phi_{n,l} + \sum_{n=-R}^{R-1} \sum_{l=-S}^{S-1} d'_{n,l} \phi_{n,l} + \sum_{l=-S}^{S-1} d''_{R,l} \phi_{R,l} \right)
\]
for some coefficients \( d_{n,l}, d'_{n,l} \) and \( d''_{R,l} \). Therefore, for any \( v \in V \), we have
\[
L^{1,0}v = \sum_{n=-R+1}^{R} \sum_{l=-S}^{S-1} d_{n,l} \phi_{n,l} + \sum_{n=-R+1}^{R} \sum_{l=-S}^{S-1} d'_{n,l} \phi_{n,l} + \sum_{l=-S}^{S-1} d''_{R,l} \phi_{R,l}.
\]

Thus, \( L^{1,0}v \in V \). This implies that \( V \) is invariant under \( L^{1,0} \). Therefore, \( L^{m,0}v \in V \) for \( m \in \mathbb{N} \cup \{0\} \). Similarly, \( L^{0,k}v \in V \) for \( k \in \mathbb{N} \cup \{0\} \). Now for any \( v \in V \) and for any \( m,k \in \mathbb{N} \cup \{0\} \), we have
\[
L^{m,k}v = L^{m,0}(L^{0,k}v)
\]
\[
= L^{m,0}(v') \in V \quad (\text{where } v' = L^{0,k}v \in V).
\]

Thus, (ii) implies that \( V = \text{span}\{\phi_{n,l}\}_{n,l\in\mathbb{N}} \), which is a contradiction. Hence, \( \{\phi_{n,l}\}_{n,l\in\mathbb{Z}} \) is linearly independent. This completes the proof. \( \square \)

**Remark 7.3.** Proposition 7.1 is a particular case of theorem 7.2. Indeed, for \( n = 1 \) and \( l \in \mathbb{N} \), take \( q^{(l)} = a = \text{constant} \). Then, \( \{T_n^{(l)}\phi\}_{n,l\in\mathbb{Z}} := \{T_n\phi\}_{l\in\mathbb{N}} = \{f_l\}_{l\in\mathbb{N}} \) (say). For \( k = 0 \) and \( m \in \mathbb{N} \cup \{0\} \), we have \( \{L^{m,k}T_n^{(l)}\phi\}_l\in\{0\},m\in\mathbb{N}\cup\{0\} = \{L^mT_n\phi\}_l\in\{0\},m\in\mathbb{N}\cup\{0\} = \{L^mf_l\}_l\in\{0\},m=0 = \{f_l\}_l\in\{0\} \) which is same as in proposition 7.1.
8. Discussion and conclusion

It is well known that shifts of finitely many functions in $L^2(\mathbb{R}, dx)$ never generate a frame of the space $L^2(\mathbb{R}, dx)$ [4]. In this work, we give new necessary and sufficient conditions in terms of the Fourier transform under which a nonstationary system of translates in $L^2(\mathbb{R}, dx)$ turns out to be a frame $L^2(\mathbb{R}, dx)$. As an application of this result, we derive new frame conditions for the space $L^2(\mathbb{R}, dx)$ from the Weyl–Heisenberg group and extended affine group in theorems 3.1 and 4.1, respectively. In theorem 5.1, we show that the canonical dual of nonstationary frames of translates in $L^2(\mathbb{R}, dx)$ has the same structure. This is not true for all types of frames, see [4, 10, 34] and many references therein. Sufficient conditions in terms of linear independence and convergence in the weak sense for a nonstationary frame of translates to be a Riesz basis are given in theorem 6.3. Finally, we give an equivalent criterion for a nonstationary frame of translates to be linearly independent in terms of bounded linear operators acting on a space that is generated by a given nonstationary sequence of translates in the space $L^2(\mathbb{R}, dx)$.

The ability to represent a signal in terms of frames lies at the heart of many applications in physics such as quantum physics [2, 16], quantum channels [44], signal processing [8, 19] and distributed signal processing [11, 12, 42]. In nonstationary frames of translates for the space $L^2(\mathbb{R}, dx)$, each frame vector is obtained by the action of a bounded linear operator (translation operator) on a single function $\phi$ in the space $L^2(\mathbb{R}, dx)$. It is useful in the sense that it simplifies manipulations on the frame conditions, and makes it easier to store information about the frame. Using frame conditions for a sequence of nonstationary translates, we can obtain frame conditions for frames (Gabor) from the Weyl–Heisenberg group. We believe that the frame conditions for nonstationary frames of translates in this paper will be useful in quantum channel, quantum physics, signal processing, and other branches of physics related to time-frequency analysis. The study of nonstationary frames of translates is a good area of research, both in theory and applications. We also plan to study sums of frames of nonstationary translates and their applications to the frame algorithm. It would be interesting to investigate the stability of nonstationary frames of translates and frames from the Weyl–Heisenberg group.

Data availability statement

No Data is used in this study. No new data were created or analyzed in this study.

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