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\textbf{n-ARY LIE AND ASSOCIATIVE ALGEBRAS}

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To Wlodek Tulczyjew, on the occasion of his 65th birthday.

\textbf{Abstract.} With the help of the multigraded Nijenhuis–Richardson bracket and the multigraded Gerstenhaber bracket from \cite{7} for every $n \geq 2$ we define $n$-ary associative algebras and their modules and also $n$-ary Lie algebras and their modules, and we give the relevant formulas for Hochschild and Chevalley cohomology.

\textbf{1. Introduction}

In 1985 V. Filipov \cite{3} proposed a generalization of the concept of a Lie algebra by replacing the binary operation by an $n$-ary one. He defined an $n$-ary Lie algebra structure on a vector space $V$ as an operation which associates with each $n$-tuple $(u_1, \ldots, u_n)$ of elements in $V$ another element $[u_1, \ldots, u_n]$ which is $n$-linear, skew symmetric, and satisfies the $n$-Jacobi identity:

\begin{equation}
[u_1, \ldots, u_{n-1}, [v_1, \ldots, v_n]] = \sum [v_1, \ldots, v_{i-1}, [u_1, \ldots, u_{n-1}, v_i], \ldots, v_n].
\end{equation}

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Apparently Filippov was motivated by the fact that with this definition one can develop a meaningful structure theory, in accordance with the aim of Malcev’s school: To look for algebraic structures that manifest good properties.

On the other hand, in 1973 Y. Nambu [13] proposed an \( n \)-ary generalization of Hamiltonian dynamics by means of the \( n \)-ary ‘Poisson bracket’

\[
\{f_1, \ldots, f_n\} = \det \left( \frac{\partial f_i}{\partial x_j} \right).
\]

Apparently he looked for a simple model which explains the unseparability of quarks. Much later, in the early 90’s, it was noticed by M. Flato, C. Fronsdal, and others, that the \( n \)-bracket (2) satisfies (1). On this basis L. Takhtajan [17] developed systematically the foundations of of the theory of \( n \)-Poisson or Nambu-Poisson manifolds. It seems that the work of Filippov was unknown then; in particular Takhtajan reproduces some results from [3] without referring to it.

Recently Alekseevsky and Guha [1] and later Marmo, Vilasi, and Vinogradov [9] proved that \( n \)-Poisson structures of the kind above are extremely rigid: Locally they are given by \( n \) commuting vector fields of rank \( n \), if \( n > 2 \); in other words, \( n \)-Poisson structures are locally given by (2). This rigidity suggests that one should look for alternative \( n \)-ary analogs of the concept of a Lie algebra. One of them is proposed below in this paper. It is based on the completely skew symmetrized version of Filippov’s Jacobi identity (2). It is shown in [20] that this approach leads to richer and more diverse structures which seem to be more useful for purposes of dynamics. In fact, we were lead in 1990-92 to the constructions of this paper by some expectations about \( n \)-body mechanics and the naturality of the machinery developed in [7]. So, our motives were quite different from that by Filippov, Nambu and Takhtajan. This paper is essentially based on our unpublished notes from 1990-92. In view of the recent developments we decided to publish them now. In this paper we consider \( G \)-graded \( n \)-ary generalizations of the concept of associative algebras, of Lie algebras, their modules, and their cohomologies; all this is produced by the algebraic machinery of [7]. Related (but not graded) concepts are discussed in [4] in terms of operads and their Koszul duality. The recent preprints [2] and [5] propose dynamical models which correspond to the not graded case with even \( n \) in our construction.

2. Review of binary algebras and bimodules

In this section we review the results from the paper [7] in a slightly different point of view.

2.1. Conventions and definitions. By a grading group \((G, +)\) together with a \( \mathbb{Z} \)-bilinear symmetric mapping (bicharacter) \( \langle \cdot, \cdot \rangle : G \times G \to \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} \). Elements of \( G \) will be called degrees, or \( G \)-degrees if more precision is necessary. A standard example of a grading group is \( \mathbb{Z}^m \) with \( \langle x, y \rangle = \sum_{i=1}^{m} x_i y_i \pmod{2} \). If \( G \) is a grading group we will consider the grading group \( \mathbb{Z} \times G \) with \( \langle (k, x), (l, y) \rangle = kl \pmod{2} + \langle x, y \rangle \).

A \( G \)-graded vector space is just a direct sum \( V = \bigoplus_{x \in G} V^x \), where the elements of \( V^x \) are said to be homogeneous of \( G \)-degree \( x \). We assume that vector spaces are defined over a field \( \mathbb{K} \) of characteristic 0. In the following \( X, Y, \) etc will always denote homogeneous elements of some \( G \)-graded vector space of \( G \)-degrees \( x, y, \) etc.
By an $G$-graded algebra $A = \bigoplus_{x \in G} A^x$ we mean an $G$-graded vector space which is also a $K$ algebra such that $A^x \cdot A^y \subseteq A^{x+y}$.

1. The $G$-graded algebra $(A, \cdot)$ is said to be $G$-graded commutative if for homogeneous elements $X, Y \in A$ of $G$-degree $x, y$, respectively, we have $X \cdot Y = (-1)^{\langle x, y \rangle} Y \cdot X$.
2. If $X \cdot Y = -(-1)^{\langle x, y \rangle} Y \cdot X$ holds it is called $G$-graded anticommutative.
3. By an $G$-graded Lie algebra we mean a $G$-graded anticommutative algebra $(\mathfrak{g}, [\ ,\ ]_G)$ for which the $G$-graded Jacobi identity holds:

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{\langle x, y \rangle} [Y, [X, Z]]$$

Obviously the space $\text{End}(V) = \bigoplus_{\delta \in G} \text{End}^\delta(V)$ of all endomorphisms of a $G$-graded vector space $V$ is a $G$-graded algebra under composition, where $\text{End}^\delta(V)$ is the space of linear endomorphisms $D$ of $V$ of $G$-degree $\delta$, i.e. $D(V^z) \subseteq V^{z+\delta}$. Clearly $\text{End}(V)$ is a $G$-graded Lie algebra under the $G$-graded commutator

$$[D_1, D_2] := D_1 \circ D_2 - (-1)^{\langle \delta_1, \delta_2 \rangle} D_2 \circ D_1.$$ 

If $A$ is a $G$-graded algebra, an endomorphism $D : A \rightarrow A$ of $G$-degree $\delta$ is called a $G$-graded derivation, if for $X, Y \in A$ we have

$$D(X \cdot Y) = D(X) \cdot Y + (-1)^{\langle \delta, x \rangle} X \cdot D(Y).$$

Let us write $\text{Der}^\delta(A)$ for the space of all $G$-graded derivations of degree $\delta$ of the algebra $A$, and we put

$$\text{Der}(A) = \bigoplus_{\delta \in G} \text{Der}^\delta(A).$$

The following lemma is standard:

**Lemma.** If $A$ is an $G$-graded algebra, then the space $\text{Der}(A)$ of $G$-graded derivations is an $G$-graded Lie algebra under the $G$-graded commutator.

### 2.2 Graded associative algebras

Let $V = \bigoplus_{x \in G} V^x$ be an $G$-graded vector space. We define

$$M(V) := \bigoplus_{(k, \kappa) \in \mathbb{Z} \times G} M^{(k, \kappa)}(V),$$

where $M^{(k, \kappa)}(V)$ is the space of all $k+1$-linear mappings $K : V \times \ldots \times V \rightarrow V$ such that $K(V^{x_0} \times \ldots \times V^{x_k}) \subseteq V^{x_0+\ldots+x_k+\kappa}$. We call $k$ the form degree and $\kappa$ the weight degree of $K$. We define for $K_i \in M^{(k_i, \kappa_i)}(V)$ and $X_j \in V^{x_j}$

$$(j(K_1)K_2)(X_0, \ldots, X_{k_1+k_2}) := \sum_{i=0}^{k_2} (-1)^{k_1i+(\kappa_1+\kappa_2+x_0+\ldots+x_{i-1})} K_2(X_0, \ldots, K_1(X_i, \ldots, X_{i+k_1}), \ldots, X_{k_1+k_2}),$$

$$[K_1, K_2] = j(K_1)K_2 - (-1)^{k_1+k_2+(\kappa_1+\kappa_2)} j(K_2)K_1.$$
Theorem. Let $V$ be an $G$-graded vector space. Then we have:

1. $(M(V), [\cdot, \cdot]_{\Delta})$ is a $(\mathbb{Z} \times G)$-graded Lie algebra.
2. If $\mu \in M^{(1,0)}(V)$, so $\mu : V \times V \to V$ is bilinear of weight $0 \in G$, then $\mu$ is an associative $G$-graded multiplication if and only if $[\mu, \mu]_{\Delta} = 0$.
3. If $\nu \in M^{(1,n)}(V)$, so $\nu : V \times V \to V$ is bilinear of weight $n \in G$, then $j(\nu)\nu = 0$ is equivalent to

$$\nu(\nu(X_0, X_1), X_2) - (-1)^{(n,n)}\nu(X_0, \nu(X_1, X_2)) = 0$$

which is the natural notion of an associative multiplication of weight $n \in G$.

Proof. The first assertion is from [7]. The second and third assertion follows by writing out the definitions. \qed

In [7] the formulation was as follows: $\mu \in M^{(1,0)}(V)$ is an associative $G$-graded algebra structure if and only if $[\mu, \mu]_{\Delta} = 2j(\mu)\mu = 0$. For $\nu \in M^{(1,n)}(V)$ we have $[\nu, \nu]_{\Delta} = (1 + (-1)^{(n,n)})j(\nu)\nu$.

2.3. Multigraded bimodules. Let $V$ and $W$ be $G$-graded vector spaces and $\mu : V \times V \to V$ a $G$-graded algebra structure. A $G$-graded bimodule $M = (W, \lambda, \rho)$ over $A = (V, \mu)$ is given by $\lambda, \rho : V \to \text{End}(W)$ of weight $0$ such that

1. $j(\mu)\mu = 0$ so $A$ is associative
2. $\lambda(\mu(X_1, X_2)) = \lambda(X_1) \circ \lambda(X_2)$
3. $\rho(\mu(X_1, X_2)) = (-1)^{|x_1\cdot x_2|}\rho(X_2) \circ \rho(X_1)$
4. $\lambda(X_1) \circ \rho(X_2) = (-1)^{|x_1\cdot x_2|}\rho(X_2) \circ \lambda(X_1)$

where $X_i \in V^{x_i}$ and $\circ$ denotes the composition in $\text{End}(W)$.

2.4. Theorem. Let $E$ be the $(\mathbb{Z} \times G)$-graded vector space defined by

$$E^{(k, \ast)} = \begin{cases} V & \text{if } k = 0 \\ W & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $P \in M^{(1,0)}(E)$ defines a bimodule structure on $W$ if and only if $j(P)P = 0$.

Proof. We define

$$\mu(X_1, X_2) := P(X_1, X_2)$$
$$\lambda(X)Y := P(X, Y)$$
$$\rho(X)Y := (-1)^{|x\cdot y|}P(Y, X)$$

where we suppose the $X_i$’s $\in V$ and $Y \in W$ to be embedded in $E$. Then if $Z_i \in E$ is arbitrary we get

$$(j(P)P)(Z_0, Z_1, Z_2) = P((Z_0, Z_1), Z_2) - P(Z_0, (Z_1, Z_2)).$$

Now specify $Z_i \in V$ resp. $W$ to get eight independent equations. Four of them vanish identically because of their degree of homogeneity, the others recover the defining equations for the $G$-graded bimodules. \qed
2.5 Corollary. In the above situation we have the following decomposition of the \((\mathbb{Z}^2 \times G)\)-graded space \(M(E)\):

\[
M^{(k,q,\ast)}(E) = \begin{cases} 
0 & \text{for } q > 1 \\
L^{(k+1,\ast)}(V,W) & \text{for } q = 1 \\
M^{(k,\ast)}(V) \bigoplus (L^{(k,\ast)}(V,\operatorname{End}(W))) & \text{for } q = 0 
\end{cases}
\]

where \(L^{(k,\ast)}(V,W)\) denotes the space of \(k\)-linear mappings \(V \times \ldots \times V \rightarrow W\). If \(P\) is as above, then \(P = \mu + \lambda + \rho\) corresponds exactly to this decomposition. 

2.6. Hochschild cohomology and multiplicative structures. Let \(V, W\) and \(P\) be as in Theorem 2.4 and let \(\nu : W \times W \rightarrow W\) be a \(G\)-graded algebra structure, so \(\nu \in M^{(1,-1,0)}(E)\). Then for \(C_i \in L^{(k_i,c_i)}(V,W)\) we define

\[C_1 \bullet C_2 := [C_1,[C_2,\nu]^\Delta] = \pm \nu(C_1,C_2).\]

Since \([C_1,C_2]^\Delta = 0\) it follows that \((L(V,W),\bullet)\) is \((\mathbb{Z} \times G)\)-graded commutative.

Theorem.

1. The mapping \([P,\ ]^\Delta : M(E) \rightarrow M(E)\) is a differential. Its restriction \(\delta_P\) to \(L(V,W)\) is a generalization of the Hochschild coboundary operator to the \(G\)-graded case: If \(C \in L^{(k,c)}(V,W)\), then we have for \(X_i \in V^x,\)

\[
(\delta_P C)(X_0, \ldots, X_k) = \lambda(X_0)C(X_1, \ldots, X_k)
\]

\[
- \sum_{i=0}^{k-1} (-1)^i C(X_0, \ldots, \mu(X_i, X_{i+1}), \ldots, X_k)
\]

\[
+ (-1)^{k+1+c} \rho(X_k)C(X_0, \ldots, X_{k-1})
\]

The corresponding \((\mathbb{Z} \times G)\)-graded cohomology will be denoted by \(H(A, M)\).

2. If \([P,\nu]^{\Delta} = 0\), then \(\delta_P\) is a derivation of \(L(V,W)\) of \((\mathbb{Z} \times G)\)-degree \((1,0)\). In this case the product \(\bullet\) carries over to a \((\mathbb{Z} \times G)\)-graded (cup) product on \(H(A, M)\).

3. \(n\)-ary \(G\)-graded associative algebras and \(n\)-ary modules

3.1. Definition. Let \(V\) be a \(G\)-graded vector space. Let \(\mu \in M^{(n-1,0)}(V)\), so \(\mu : V^{
abla n} \rightarrow V\) is \(n\)-linear of weight 0 \(\in G\).

We call \(\mu\) an \(n\)-ary associative \(G\)-graded multiplication of weight 0 \(\in G\) if \(j(\mu)\mu = 0 \in M^{(2n-2,0)}(V)\).

Remark. We are forced to use \(j(\mu)\mu = 0\) instead of \([\mu,\mu]^\Delta = 0\) since the latter condition is automatically satisfied for odd \(n\).

3.2. Example. If \(V\) is 0-graded, then a ternary associative multiplication \(\mu : V \times V \times V \rightarrow V\) satisfies

\[
(j(\mu)\mu)(X_0, \ldots, X_5) = \mu(\mu(X_0, X_1, X_2), X_3, X_4) + \\
+ \mu(X_0, \mu(X_1, X_2, X_3), X_4) + \mu(X_0, X_1, \mu(X_2, X_3, X_4)) = 0.
\]
3.3. Definition. Let \( V \) and \( W \) be \( G \)-graded vector spaces. We consider the \( (\mathbb{Z} \times G) \)-graded vector space \( E \) defined by

\[
E^{(k,*)} = \begin{cases} 
    V & \text{if } k = 0 \\
    W & \text{if } k = 1 \\
    0 & \text{otherwise.}
\end{cases}
\]

Then \( P \in M^{(n-1,0,0)}(E) \) is called an \( n \)-ary \( G \)-graded module structure on \( W \) over an \( n \)-ary algebra structure on \( V \) if \( j(P)P = 0 \). Let us denote the resulting \( n \)-ary algebra by \( A \), and the \( n \)-ary module by \( W \).

The mapping \( P \) is the sum of partial mappings

\[
\begin{align*}
    \mu &= P : V \times \ldots \times V \to V \quad \text{the } n \text{-ary algebra structure} \\
    P &= W \times V \times \ldots \times V \to W \quad \text{the rightmost } n \text{-ary module structure} \\
    P &= V \times W \times V \times \ldots \times V \to W \\
    \ldots \\
    P &= V \times \ldots \times V \times W \times V \to W \\
    P &= V \times \ldots \times V \times W \to W \quad \text{the leftmost } n \text{-ary module structure}
\end{align*}
\]

This decomposition of \( P \) corresponds exactly to the last line in the decomposition of \( M^{(n-1,0,*)} \) of 2.5.

The above definition is easily generalized by changing the form degree of \( W \) or/and by augmenting the number of \( W \)'s. For simplicity we don’t discuss this possibility here.

3.4. Example. If \( V \) and \( W \) are 0-graded then a ternary module satisfies the following conditions besides the one from 3.2 describing the ternary algebra structure on \( V \):

\[
\begin{align*}
    P(P(w_0, v_1, v_2), v_3, v_4) + P(w_0, \mu(v_1, v_2, v_3), v_4) + P(w_0, v_1, \mu(v_2, v_3, v_4)) &= 0 \\
    P(P(v_0, w_1, v_2), v_3, v_4) + P(v_0, P(w_1, v_2, v_3), v_4) + P(v_0, w_1, \mu(v_2, v_3, v_4)) &= 0 \\
    P(P(v_0, v_1, w_2), v_3, v_4) + P(v_0, P(v_1, w_2, v_3), v_4) + P(v_0, v_1, P(w_2, v_3, v_4)) &= 0 \\
    P(\mu(v_0, v_1, v_2), w_3, v_4) + P(v_0, P(v_1, v_2, w_3), v_4) + P(v_0, v_1, P(v_2, w_3, v_4)) &= 0 \\
    P(\mu(v_0, v_1, v_2), v_3, w_4) + P(v_0, \mu(v_1, v_2, v_3), w_4) + P(v_0, v_1, P(v_2, v_3, w_4)) &= 0
\end{align*}
\]

3.5. Hochschild cohomology for even \( n \). Let \( V \) and \( W \) be \( G \)-graded vector spaces, and let \( P \in M^{(n-1,0,0)}(E) \) be an \( n \)-ary module structure on \( W \) over an \( n \)-ary \( G \)-graded algebra structure on \( V \) as in definition 3.3.

Theorem. Let \( n = 2k \) be even. Then we have:

The mapping \([ P, \Delta ] : M(E) \to M(E)\) is a differential. Its restriction \( \delta_p \) to \( L(V, W) \) is called the Hochschild coboundary operator. For a cochain \( C \in M^{(k,1,c)} = L^{(k+1,c)}(V, W) \) and with \( p = n - 1 \) we have for \( X_i \in V^x \),

\[
(\delta_p C)(X_0, \ldots, X_{k+p}) = \sum_{i=0}^{k} (-1)^{pi} C(X_0, \ldots, P(X_i, \ldots, X_{i+p}), \ldots, X_{k+p}) \\
- \sum_{j=0}^{p} (-1)^{k(j+p)+(x_0+\cdots+x_{j-1},c)} P(X_0, \ldots, C(X_j, \ldots, X_{j+k}), \ldots, X_{k+p}).
\]
The corresponding \((\mathbb{Z} \times G)\)-graded cohomology will be denoted by \(H(A,M)\).

**Proof.** We have by the \((\mathbb{Z}^2 \times G)\)-graded Jacobi identity

\[
[P, [P, Q]^\Delta] = [[P, P]^\Delta, Q]^\Delta + (-1)^{(n-1)^2} [P, [P, Q]^\Delta]^\Delta
\]

which implies that \([P, P]\) is a differential since \(n - 1\) is odd and \([P, P]^\Delta = j(P)P - (-1)^{(n-1)^2} j(P)P - 2j(P)P = 0\). The rest follows from a computation. □

3.6. **Remark.** We get an easy extension of the Hochschild coboundary operator for \(n\)-ary algebra structures for odd \(n\) if we choose the weight accordingly. Let \(P \in M^n(E)\) be an \(n\)-ary module structure of weight \(p\) on \(W\) over an \(n\)-ary \(G\)-algebra structure of weight \(p\) on \(V\), similarly as in definition 3.3: We require that \(j(P)P = 0\). Let us suppose that \(\|(n-1,0,p)\|^2 = (n-1)^2 + (p,p)\) is odd. Then by 2.2 we have

\[
[P, P]^\Delta = \left(1 - (-1)^{(n-1)^2 + (p,p)}\right) j(P)P = 2j(P)P = 0,
\]

so that we get a differential. A dual version of this can be seen in 7.2.(3) below.

3.7. **Ideals.** Let \((V, \mu)\) be an \(n\)-ary \(G\)-graded associative algebra. An ideal \(I\) in \((V, \mu)\) is a linear subspace \(I \subset V\) such that \(\mu(X_1, \ldots, X_n) \in I\) whenever one of the \(X_i \in I\). Then \(\mu\) factors to an \(n\)-ary associative multiplication on the quotient space \(V/I\). This quotient space is again \(G\)-graded, if \(I\) is a \(G\)-graded subspace in the sense that \(I = \bigoplus_{x \in G} (I \cap V^x)\).

Of course any ideal \(I\) is an \(n\)-ary module over \((V, \mu)\) which is \(G\)-graded if and only if \(I\) is \(G\)-graded. Conversely, any \(n\)-ary module \(W\) over \((V, \mu)\) is an ideal in the \(n\)-ary algebra \(V \oplus W = E\) with the multiplication \(P\) from 3.3. Here \(P(X_1, \ldots, X_n) = 0\) if any two elements \(X_i\) lie in \(W\), so that \(E\) may be regarded as an \(G\)-graded or as a \((\mathbb{Z} \times G)\)-graded algebra. It could be called also the semidirect product of \(V\) and \(W\).

3.8. **Homomorphisms.** A linear mapping \(f : V \to W\) of degree 0 between two \(G\)-graded algebras \((V, \mu)\) and \((W, \nu)\) is called a homomorphism of \(G\)-graded algebras if it is compatible with the two \(n\)-ary multiplications:

\[
f(\mu(X_1, \ldots, X_n)) = \nu(f(X_1), \ldots, f(X_n))
\]

Then the kernel of \(f\) is an \(n\)-ary ideal in \((V, \mu)\) and the image of \(f\) is an \(n\)-ary subalgebra of \((W, \nu)\) which is isomorphic to \(V/\ker(f)\).

Similarly we can define the notion of an \(n\)-ary \(V\)-module homomorphism between two \(V\)-modules \(W_0\) and \(W_1\). Then the category of all \((G\)-graded\) \(n\)-ary \(V\)-modules and of their homomorphisms is an abelian category. We did not investigate the relation to the embedding theorem of Freyd and Mitchell.

4. **Review of \(G\)-graded Lie algebras and modules**

In this section we sketch the theory from [7] for \(G\)-graded Lie algebras from a slightly different angle. In this section section we need that the ground field \(\mathbb{K}\) has characteristic 0.
4.1. Multigraded signs of permutations. Let \( x = (x_1, \ldots, x_k) \in G^k \) be a multi index of \( G \)-degrees \( x_i \in G \) and let \( \sigma \in S_k \) be a permutation of \( k \) symbols. Then we define the \( G \)-graded sign \( \text{sign}(\sigma, x) \) as follows: For a transposition \( \sigma = (i, i+1) \) we put \( \text{sign}(\sigma, x) = -(-1)^{(x_i, x_{i+1})} \); it can be checked by combinatorics that this gives a well defined mapping \( \text{sign}(\sigma, x) : S_k \to \{-1, +1\} \).

Let us write \( \sigma x = (x_{\sigma 1}, \ldots, x_{\sigma k}) \), then we have the following

**Lemma.** \( \text{sign}(\sigma \circ \tau, x) = \text{sign}(\sigma, x), \text{sign}(\tau, \sigma x) \). □

4.2 Multigraded Nijenhuis-Richardson algebra. We define the \( G \)-graded alternator \( \alpha : M(V) \to M(V) \) by

\[
(\alpha K)(X_0, \ldots, X_k) = \frac{1}{(k+1)!} \sum_{\sigma \in S_{k+1}} \text{sign}(\sigma, x)K(X_{\sigma 0}, \ldots, X_{\sigma k})
\]

for \( K \in M^{(k,\kappa)}(V) \) and \( X_i \in V^{x_i} \). By lemma 4.1 we have \( \alpha^2 = \alpha \) so \( \alpha \) is a projection on \( M(V) \), homogeneous of \((\mathbb{Z} \times G)\)-degree 0, and we set

\[
A(V) = \bigoplus_{(k,\kappa) \in \mathbb{Z} \times G} A^{(k,\kappa)}(V) = \bigoplus_{(k,\kappa) \in \mathbb{Z} \times G} \alpha(M^{(k,\kappa)}(V)).
\]

A long but straightforward computation shows that for \( K_i \in M^{(k_i,\kappa_i)}(V) \)

\[
\alpha(j(\alpha K_1)\alpha K_2) = \alpha(j(K_1)K_2),
\]

so the following operator and bracket is well defined:

\[
i(K_1)K_2 := \frac{(k_1 + k_2 + 1)!}{(k_1 + 1)!(k_2 + 1)!} \alpha(j(K_1)K_2)
\]

\[
[K_1, K_2]^\wedge = \frac{(k_1 + k_2 + 1)!}{(k_1 + 1)!(k_2 + 1)!} \alpha([K_1, K_2]^\Delta)
\]

\[
= i(K_1)K_2 - (-1)^{(k_1,\kappa_1),(k_2,\kappa_2)} i(K_2)K_1
\]

The combinatorial factor is explained in [7], 3.4.

4.3. Theorem. 1. If \( K_i \) are as above, then

\[
i(K_1)K_2(X_0, \ldots, X_{k_1+k_2}) = \frac{1}{(k_1 + 1)k_2!} \sum_{\sigma \in S_{k_1+k_2+1}} \text{sign}(\sigma, x)(-1)^{(\kappa_1,\kappa_2)} \cdot K_2((X_{\sigma 0}, \ldots, X_{\sigma k_1}), \ldots, X_{\sigma (k_1+k_2)}).
\]

2. \( A(V), [\ , \ ]^\wedge \) is a \((\mathbb{Z} \times G)\)-graded Lie algebra.

3. If \( \mu \in A^{(1,0)}(V), \) so \( \mu : V \times V \to V \) is bilinear \( G \)-graded anticommutative mapping of weight 0 \( \in G \), then \( i(\mu)\mu = 0 \) if and only if \((V,\mu)\) is a \( G \)-graded Lie algebra.

**Proof.** For 1 and 2 see [7].

3. Let \( \mu \in A^{(1,0)}(V), \) then from 1 we see that

\[
i(\mu)\mu(X_0, X_1, X_2) = \frac{1}{1!} \sum_{\sigma \in S_3} \text{sign}(\sigma, x) \cdot \mu(X_{\sigma 0}, X_{\sigma 1}, X_{\sigma 2})
\]

which is equivalent to the \( G \)-graded Jacobi expression of \((V,\mu)\). □

\((A(V), [\ , \ ]^\wedge)\) is called the \((\mathbb{Z} \times G)\)-graded Nijenhuis-Richardson algebra, since \( A(V) \) coincides for \( G = 0 \) with \( \text{Alt}(V) \) of [14].
4.4. Theorem. Let $V$ and $W$ be $G$-graded vector spaces. Let $E$ be the $(\mathbb{Z} \times G)$-graded vector space defined by

$$E^{(k,*)} = \begin{cases} V & \text{if } k = 0 \\ W & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $P \in A^{(1,0,0)}(E)$ then $i(P)P = 0$ if and only if

(a) $i(\mu)\mu = 0$

so $(V,\mu) = g$ is a $G$-graded Lie algebra, and

(b) $\rho(\mu(X_1, X_2))Y = [\rho(X_1), \rho(X_2)]Y$

where $\mu(X_1, X_2) = P(X_1, X_2) \in V$ and $\rho(X)Y = P(X, Y) \in W$ for $X, X_i \in V$ and $Y \in W$, and where $[\ , \ ]$ denotes the $G$-graded commutator in $\text{End}(W)$. So $i(P)P = 0$ is by definition equivalent to the fact that $M := (W, \rho)$ is a $G$-graded Lie-$g$ module.

If $P$ is as above the mapping $\partial P := [P, \ ]^\wedge : A(E) \to A(E)$ is a differential and its restriction to

$$\bigoplus_{k \in \mathbb{Z}} H^{(k,*)}(g, M) := \bigoplus_{k \in \mathbb{Z}} A^{(k,1,*)}(E)$$

generalizes the Chevalley-Eilenberg coboundary operator to the $G$-graded case:

$$(\partial P)C(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^{\alpha_i(x)} + \langle x_i, c \rangle \rho(X_i)C(X_0, \ldots, \hat{X_i}, \ldots, X_k)$$

$$+ \sum_{i<j} (-1)^{\alpha_{ij}(x)} C(\mu(X_i, X_j), \ldots, \hat{X_i}, \ldots, \hat{X_j}, \ldots)$$

where

$$\left\{ \begin{array}{l} \alpha_i(x) = \langle x_i, x_0 + \cdots + x_{i-1} \rangle + i \\ \alpha_{ij}(x) = \alpha_i(x) + \alpha_i(x) + \langle x_i, x_j \rangle \end{array} \right.$$ 

We denote the corresponding $(\mathbb{Z} \times G)$-graded cohomology space by $H(g, M)$.

If $\nu : W \times W \to W$ is $G$-graded symmetric (so $\nu \in A^{(1,-1,*)}(E)$) and $[P, \nu]^\wedge = 0$ then $\partial P$ acts as derivation of $G$-degree $(1,0)$ on the $(\mathbb{Z} \times G)$-graded commutative algebra $(\Lambda(g, M), \bullet)$, where

$$C_1 \bullet C_2 := [C_1, [C_2, \nu]^\wedge]^\wedge C_i \in \Lambda^{(k_i,c_i)}(g, M).$$

In this situation the product $\bullet$ carries over to a $(\mathbb{Z} \times G)$-graded symmetric (cup) product on $H(g, M)$.

Proof. Apply the $G$-graded alternator $\alpha$ to the results of 2.3, 2.4, 2.5, and 2.6. □
5. \textit{n-ary G-graded Lie algebras and their modules}

5.1. \textbf{Definition.} Let \( V \) be a \( G \)-graded vector space. Let \( \mu \in A^{(n-1,0)}(V) \), so \( \mu : V^n \rightarrow V \) is a \( G \)-graded skew symmetric \( n \)-linear mapping.

We call \( \mu \) an \( n \)-ary \( G \)-graded Lie algebra structure on \( V \) if \( i(\mu) = 0 \).

5.2. \textbf{Example.} If \( V \) is 0-graded, then a ternary Lie algebra structure on \( V \) is a skew symmetric trilinear mapping \( \mu : V \times V \times V \rightarrow V \) satisfying

\[
0 = (i(\mu) \mu)(X_0, \ldots, X_4) = \frac{1}{3!2!} \sum_{\sigma \in S_3} \text{sign}(\sigma) \mu(\mu(X_{\sigma_0}, X_{\sigma_1}, X_{\sigma_2}), X_{\sigma_3}, X_{\sigma_4})
\]

\[
= +\mu(\mu(X_0, X_1, X_2), X_3, X_4) - \mu(\mu(X_0, X_1, X_3), X_2, X_4)
\]

\[
+ \mu(\mu(X_0, X_1, X_4), X_2, X_3) + \mu(\mu(X_0, X_2, X_3), X_1, X_4)
\]

\[
- \mu(\mu(X_0, X_3, X_4), X_1, X_2) - \mu(\mu(X_1, X_2, X_3), X_0, X_4) + \mu(\mu(X_1, X_2, X_4), X_0, X_3)
\]

\[
- \mu(\mu(X_1, X_3, X_4), X_0, X_2) + \mu(\mu(X_2, X_3, X_4), X_0, X_1)
\]

5.3. \textbf{Definition.} Let \( V \) and \( W \) be \( G \)-graded vector spaces. We consider the \((\mathbb{Z} \times G)\)-graded vector space \( E \) defined by

\[
E^{(k,*)} = \begin{cases} 
V & \text{if } k = 0 \\
W & \text{if } k = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( P \in A^{(n-1,0)}(E) \) is called an \( n \)-ary \( G \)-graded Lie module structure on \( W \) over an \( n \)-ary Lie algebra structure on \( V \) if \( i(P)P = 0 \). Let us denote the resulting \( n \)-ary Lie algebra by \( g \), and the \( n \)-ary module by \( W \).

Ordering by degree and using the \( G \)-graded skew symmetry we see that \( P \) is now the sum of only two partial \( n \)-linear mappings

\[
\mu = P : V \times \ldots \times V \rightarrow V \quad \text{the } n \text{-ary Lie algebra structure}
\]

\[
\rho = P : V \times \ldots \times V \times W \rightarrow W \quad \text{the } n \text{-ary Lie module structure}
\]

5.4. \textbf{Example.} If \( V \) and \( W \) are 0-graded, then a ternary Lie module satisfies the following condition besides the one from 5.2 describing the ternary Lie algebra structure on \( V \):

\[
0 = \rho(\mu(v_0, v_1, v_2), v_3, w) - \rho(\mu(v_0, v_1, v_3), v_2, w) + \rho(v_2, v_3, \rho(v_0, v_1, w))
\]

\[
+ \rho(\mu(v_0, v_2, v_3), v_1, w) - \rho(v_1, v_3, \rho(v_0, v_2, w)) + \rho(v_1, v_2, \rho(v_0, v_3, w))
\]

\[
- \rho(\mu(v_1, v_2, v_3), v_0, w) + \rho(v_0, v_3, \rho(v_1, v_2, w)) - \rho(v_0, v_2, \rho(v_1, v_3, w))
\]

\[
+ \rho(v_1, v_0, \rho(v_2, v_3, w)).
\]

5.5. \textbf{Theorem.} If \( P \) is as in 5.3 above and if \( n \) is even then the mapping \( \partial_P := [P, \ ] : A(E) \rightarrow A(E) \) is a differential. Its restriction to

\[
\bigoplus_{k \in \mathbb{Z}} A^{(k,*)}(V, W) := \bigoplus_{k \in \mathbb{Z}} A^{(k,1,*)}(E)
\]
graded algebras \((V, \mu)\) if it is compatible with the two 

\[ \text{two} \]

\[ V \]

\[ (f) \]

Then the kernel of 

\[ 5.7. \text{Homomorphisms}. \]

A linear mapping \(W \rightarrow W\) is an \(n\)-ary \(G\)-graded subalgebra of \((W, \nu)\) which is isomorphic to \(V/\ker(f)\).

Similarly, we can define the notion of an \(n\)-ary \(V\)-module homomorphism between two \(V\)-modules \(W_0\) and \(W_1\).
6. Relations between $n$-ary algebras and Lie algebras

6.1. The $n$-ary commutator. Let $\mu \in M^{(n-1,0)}(V)$, so $\mu : V \times \ldots \times V \to V$ is an $n$-ary multiplication. The $G$-graded alternator $\alpha$ from 4.2 transforms $\mu$ into an element

$$\gamma \mu := n! \alpha \mu \in A^{(n,0)}(V),$$

which we call the $n$-ary commutator of $\mu$. From 4.2 we also have:

If $\mu$ is $n$-ary associative, then $\gamma \mu$ is an $n$-ary Lie algebra structure on $V$.

Definition. An $n$-ary $(\mathbb{Z} \times G)$-graded multiplication $\mu \in M^{(n-1,0)}(V)$ is called $n$-ary Lie admissible if $\gamma \mu$ is an $n$-ary $(\mathbb{Z} \times G)$-graded Lie algebra structure. By 5.1 this is the case if and only if $i(\gamma \mu)(\gamma \mu) = \frac{(2n-1)!}{(n!)^2} \alpha(j(\mu)\mu) = 0$; i.e. the alternation of the $n$-ary associator $j(\mu)(\mu)$ vanishes. For the binary version of this notion see [12] and [11].

An $n$-ary multiplication $\mu$ is called $n$-ary commutative if $\gamma \mu = 0$.

6.2. Induced mapping in cohomology. Let $V$ and $W$ be $G$-graded vector spaces and let $E$ be the $(\mathbb{Z} \times G)$-graded vector space

$$E^{(k,*)} = \begin{cases} V & \text{if } k = 0 \\ W & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

as in 3.3. Let $P \in M^{(n-1,0)}(E)$ be an $n$-ary $G$-graded module structure on $W$ over an $n$-ary algebra structure on $V$, i.e. $j(P)P = 0$.

Then $\gamma P = n! \alpha P \in A^{(n-1,0)}(E)$ is an $n$-ary $G$-graded Lie module structure on $W$ over $V$ and some multiple of $\alpha$ defines a homomorphism from the Hochschild cohomology of $(V, \mu)$ with values in $W$ into the Chevalley cohomology of $(V, \gamma \mu)$ with values in the Lie module $V$.

7. Hochschild operations and noncommutative differential calculus

7.1. Let $V$ be a $G$-graded vector space. We consider the tensor algebra $V^\otimes = \bigoplus_{k=0}^{\infty} V^\otimes k$ which is now $(\mathbb{Z} \times G)$-graded such that the degree of $X_1 \otimes \ldots \otimes X_i$ is $(i, x_1 + \ldots + x_i)$. Put also $V^\otimes n = \bigoplus_{k \geq n} V^\otimes k$. Obviously, $V^\otimes 0 = V^\otimes$.

The Hochschild operator $\delta_K$ associated with $K \in M^{(k,\kappa)}(V)$ (as in 2.2) is a map $\delta_K : V^\otimes k \to V_1^\otimes$ given by

$$\delta_K = 0 \text{ on } V \otimes k$$

and

$$\delta_K(X_0 \otimes \ldots \otimes X_l) := \sum_{i=0}^{l-k} (-1)^{ki + \langle x_0, x_0 + \ldots + x_{i-1} \rangle}X_0 \otimes \ldots \otimes X_{i-1} \otimes K(X_i \otimes \ldots \otimes X_{i+k}) \otimes \ldots \otimes X_l$$

In the natural $(\mathbb{Z} \times G)$-grading of $L(V^\otimes, V^\otimes)$ the operator $\delta_K$ has degree $(-k, \kappa)$. The mapping $\delta$ is called the Hochschild operation since for an associative multiplication $\mu : V \times V \to V$ the operator $\delta_\mu$ is the differential of the Hochschild homology.

For $K_i \in M^{(k_i,\kappa)}(V)$ with $k_i > 0$ the composition $\delta_{K_1} \circ \delta_{K_2}$ is well-defined as a map from $V_1^\otimes$ to $V_1^\otimes$. 

7.2. Proposition. For \( K_i \in M^{(k_i, \kappa_i)}(V) \) we have

1. In general \( \delta_{K_i} \circ \delta_{K_j} \neq \delta_{[K_i, K_j]} \).
2. \([\delta_{K_i}, \delta_{K_j}] = \delta_{K_i} \circ \delta_{K_j} - (-1)^{k_i k_j + \langle \kappa_i, \kappa_j \rangle} \delta_{K_j} \circ \delta_{K_i} = \delta_{[K_i, K_j]}.\)
3. \([\delta_{K_i}, \delta_{K_j}] = 2 \delta_K \circ \delta_K \) if and only if \( \| \delta_K ||^2 = k^2 + \langle \kappa, \kappa \rangle \equiv 1 \) mod 2.

Proof. We get

\[
\delta_{K_i} \circ \delta_{K_j}(X_1 \otimes \cdots \otimes X_s) = \\
= \sum_{j + k_2 < i} (-1)^{k_1 i + \langle \kappa_1, x_0 + \cdots + x_{i-1} \rangle + k_2 j + \langle \kappa_2, x_0 + \cdots + x_{i-1} \rangle} X_0 \otimes \cdots \otimes K_2(X_j \otimes \cdots \otimes X_{j+k_2}) \otimes \cdots \otimes K_1(X_1 \otimes \cdots \otimes X_{i+k_1}) \otimes \cdots \otimes X_s \\
+ \sum_{i-k_2 \leq j \leq i} (-1)^{k_1 i + \langle \kappa_1, x_0 + \cdots + x_{i-1} \rangle + k_2 j + \langle \kappa_2, x_0 + \cdots + x_{i-1} \rangle} X_0 \otimes \cdots \otimes K_2(X_j \otimes \cdots \otimes X_{j+k_2}) \otimes \cdots \otimes X_s \\
+ \sum_j X_0 \otimes \cdots \otimes K_1(X_1 \otimes \cdots \otimes X_{i+k_1}) \otimes \cdots \otimes K_2(X_j \otimes \cdots \otimes X_{j+k_2}) \otimes \cdots \otimes X_s.
\]

From this all assertions follow. \( \square \)

7.3. Rudiments of a non commutative differential calculus. An intrinsic characterization of the Hochschild operators can be given as follows. For \( X \in V^x \) we consider the left and right multiplication operators \( X^l, X^r \in L(V^{\otimes}, V^{\otimes})^{(1, x)} \) which are given by

\[
X^l(X_1 \otimes \cdots \otimes X_k) := X \otimes X_1 \otimes \cdots \otimes X_k, \\
X^r(X_1 \otimes \cdots \otimes X_k) := (-1)^{k+x_i+\cdots+x_k} X_1 \otimes \cdots \otimes X_k \otimes X.
\]

Then we have \([X^l, Y^r] = 0 \) in \( L(V^{\otimes}, V^{\otimes}) \) for all \( X, Y \in V \).

Proposition. An operator \( A \in L(V^\otimes, V^\otimes) \) is of the form \( A = \delta_K \) for an uniquely defined \( K \in M(V)^{(k, \kappa)} \) if and only if \( A[V^{\otimes}, V^{\otimes}] = O \) and \([X_0, [X_1, A]] = 0 \) in \( L(V^\otimes, V^\otimes) \) for all \( X_i \in V \).

Proof. A computation. \( \square \)

In view of the theory developed in [18] (see also [6], [19]) the Hochschild operators \( \delta_K \) can be naturally interpreted as the first order differential operators in the current non–commutative context.

7.4. Example. An element \( e \in V \) is the left (resp., right) unit of a binary multiplication \( \mu \) on \( V \) if and only if \([\delta_{\mu}, e^l] = id \) (on \( V^{\otimes} \)) (resp., \([\delta_{\mu}, e^r] = id \)). Differential calculus touched in 7.3 can be put in the following general cadre.

7.5. Definition. Let \( A \) be a \( G \)-graded associative (binary) algebra. For \( A, B \in A \) let \( A^l, B^r : A \rightarrow A \) be the left and (signed) right multiplications, \( A^l(B) = (-1)^{(a,b)} B^r(A) = AB \). Then we have

\[
[A^l, B^r] = A^l \circ B^r - (-1)^{(a,b)} B^r \circ A^l = 0.
\]
A differential operator $A \to A$ of order $(p, q)$ is an element $\Delta \in L(A, A)$ such that

$$[X_1, \ldots, [X_p, [Y_1, \ldots, [Y_q, \Delta], \ldots]] = 0 \quad \text{for all } X_i, Y_j \in A,$$

which we also denote by the shorthand $[p, p]_q \Delta = 0$. Obviously this definition also makes sense for mappings $M \to N$ between $G$-graded $A$-bimodules, where now $A^i$ is left multiplication of $A \in A$ on any $G$-graded $A$-bimodule, etc.

### 7.6. Example

Let $V$ be a finite dimensional vector space, ungraded for simplicity’s sake, and let us consider the associative algebra $A = L(V, V)$.

**Proposition.** If $\Delta : L(V, V) \to L(V, V)$ is a differential operator of order $(p, q)$ with $(p, q > 0)$, then

$$\Delta = \begin{cases} P^r, & \text{if } \delta^r_p = 0 \\ Q^l, & \text{if } \delta^q_r = 0 \\ P^r + Q^l, & \text{if } \delta^r_p \delta^q_r = 0 \end{cases}$$

where $P$ and $Q$ are in $L(V, V)$.

**Proof.** We shall use the notation $l_Y \Delta := [Y^i, \Delta]$ and similarly $r_Y \Delta = [Y^r, \Delta]$, for $Y \in L(V, V)$. We start with the following

**Claim.** If $l_Y \Delta = P^r_Y + Q^l_Y$ for each $Y \in L(V, V)$ and suitable $P = P_Y, Q = Q_Y : L(V, V) \to L(V, V)$, then we have $\Delta = A^l + B^r$ where $A = 0$ if $P = 0$. If on the other hand $r_Y \Delta = P^r_Y + Q^r_Y$ for each $Y$ then we have $\Delta = A^l + B^r$ where $B = 0$ if $Q = 0$.

Let us assume that $l_Y \Delta = P^r_Y + Q^l_Y$ for each $Y$. By replacing $\Delta$ by $\Delta - (\Delta(1))^r$ we may assume without loss that $\Delta(1) = 0$. We have $(l_Y \Delta)(X) = PX + QX = (P + Q)X - [Q, X] = [R, X] + SX$; if we assume that $R$ is traceless then $R = -Q$ and $S = P + Q$ are uniquely determined, thus linear in $Y$. Thus

$$Y \Delta(X) - \Delta(YX) = [R_Y, X] + SYX$$

Insert $X = 1$ and use $\Delta(1) = 0$ to obtain $\Delta(Y) = -S_Y$, hence

(1) $$[R_Y, X] = Y \Delta(X) + \Delta(YX) - \Delta(YX)$$

Replacing $Y$ by $YZ$ and applying the equation (1) repeatedly we obtain

$$[R_{YZ}, X] = YZ \Delta(X) + \Delta(YZ)X - \Delta(YZX)$$

$$= YZ \Delta(X) + Y \Delta(Z)X + \Delta(Y)ZX - [R_Y, Z]X$$

$$- Y \Delta(ZX) - \Delta(Y)ZX + [R_Y, ZX]$$

$$= YZ \Delta(X) + Y \Delta(Z)X - YZ \Delta(X) - Y \Delta(Z)X + Y[R_Z, X] + Z[R_Y, X]$$

$$= Y[R_{Z}, X] + Z[R_Y, X].$$

The right hand side is symmetric in $Y$ and $Z$, thus $[R_{YZ}, X] = 0$; inserting $Y = Z = 1$ we get also $[R_1, X] = 0$, hence $R = 0$. From (1) we see that $\Delta : L(V, V) \to L(V, V)$ is a derivation, thus of the form $\Delta(X) = [A, X] = (A^l - A^r)(X)$. If $P = 0$ then $\Delta = -S = R - P = 0$. So the first part of the claim follows since we already substracted $\Delta(1)^r$ from the original $\Delta$. 

The second part of the claim follows by mirroring the above proof.
Now we prove the proposition itself. If \( lp \Delta = 0 \) then by induction using the first part of the claim with \( P = 0 \) we have \( \Delta = B^t \). Similarly for \( r^q \Delta = 0 \) we get \( \Delta = A^l \).

If \( lp \Delta = 0 \) with \( p, q > 0 \), by induction on \( p + q \geq 2 \), using the claim, the result follows. □

The obtained result is parallel to the obvious fact that differential operators over 0–dimensional manifolds are of zero order.

8. Remarks on Filipov’s \( n \)-ary Lie algebras

Here we show how Filipov’s concept of an \( n \)–Lie algabra is related with that of 5.1 and sketch a similar framework for it. For simplicity’s sake no grading on the vector space is assumed.

8.1. Let \( V \) be a vector space. According to [3], an \( n \)-linear skew symmetric mapping \( \mu : V \times \ldots \times V \rightarrow V \) is called an F-Lie algebra structure if we have

\[
\mu(\mu(Y_1, \ldots, Y_n), X_2, \ldots, X_n) = \sum_{i=1}^n \mu(Y_1, \ldots, Y_{i-1}, \mu(Y_i, X_2, \ldots, X_n), Y_{i+1}, \ldots, Y_n)
\]

The idea is that \( \mu(Y_1, X_2, \ldots, X_n) \) should act as derivation with respect to the ‘multiplication’ \( \mu(Y_1, \ldots, Y_n) \).

8.2. The dot product. For \( P \in L^p(\mathbb{V}; L(V,V)) \) and \( Q \in L^q(\mathbb{V}; L(V,V)) \) let us consider the first entry as the distinguished one (belonging to \( L(V,V) \)), so that \( P(Y_1, \ldots, X_p) \in L(V,V) \) and then let us define \( P \cdot Q \in L^{p+q}(\mathbb{V}; L(V,V)) \) by

\[
(P \cdot Q)(Z, Y_1, \ldots, Y_q, X_1, \ldots, X_p) := P(Q(Z, Y_1, \ldots, Y_q), X_1, \ldots, X_p) - Q(P(Z, X_1, \ldots, X_p), Y_1, \ldots, Y_q) - \sum_{i=1}^q Q(Z, Y_1, \ldots, P(Y_i, X_1, \ldots, X_p), \ldots, Y_q)
\]

Then \( \mu \in L^{n-1}(\mathbb{V}; L(V,V)) \) which is skew symmetric in all arguments, is an F-Lie algebra structure if and only if \( \mu \cdot \mu = 0 \).

8.3. Lemma. We have

\[
\text{Alt}(P \cdot Q) = (p + 1)!(q + 1)!(\frac{1}{p!} \text{Alt} Q \text{ Alt} P - (-1)^{pq} i_{\text{Alt} P} \text{ Alt} Q),
\]

where \( \text{Alt} : L^p(V, L(V,V)) \rightarrow L^{p+1}_{\text{skew}}(V; V) = A^p(V) \) is the alternator in all appearing variables.

In particular, if \( \mu \) is an \( n \)-ary F-Lie algebra structure, then \( \text{Alt} \mu \) is a Lie algebra structure in the sense of 5.1.

Proof. An easy computation. □
8.4. **The grading operator.** For a permutation \( \sigma \in S_p \) and \( a = (a_1, \ldots, a_p) \in \mathbb{N}_0^p \) let the grading operator or (generalized) sign operator be given by

\[
S^a_\sigma : L^{a_1+\cdots+a_p}(V; W) \to L^{a_1+\cdots+a_p}(V; W),
\]

\[
(S^a_\sigma P)(X_1^{\sigma_1}, \ldots, X_{a_1}^{\sigma_1}, \ldots, X_{a_p}^{\sigma_p}) = P(X_1^{\sigma_1}, \ldots, X_{a_1}^{\sigma_1}, \ldots, X_{a_p}^{\sigma_p}),
\]

which obviously satisfies

\[
S^a_{\mu \sigma} = S^\sigma(a) \circ S^\mu_{\sigma}.
\]

We shall use the simplified version \( S^{a_1,a_2} = S^{(12);a_2,a_1} \) for the permutation of the first two blocks of arguments of length \( a_1 \) and \( a_2 \). Note that also \( S^{a,b}(\alpha \otimes \beta \otimes \gamma) = \beta \otimes \alpha \otimes \gamma \).

If \( P \) is skew symmetric on \( V \), then \( S^a_\sigma P = \text{sign}(\sigma, a)P \), the sign from [7] or 4.1.

8.5. **Lemma.** For \( P \in L^p(V; L(V, V)) \) and \( \psi \in L^q(V, W) \) let

\[
(\rho(P)\psi)(X_1, \ldots, X_p, Y_1, \ldots, Y_q) := -\sum_{i=1}^q \psi(Y_1, \ldots, P(Y_i, X_1, \ldots, X_p), \ldots, Y_q)
\]

then we have for \( \omega \in L^*(V; \mathbb{R}) \)

\[
\rho(P)(\psi \otimes \omega) = (\rho(P)\psi) \otimes \omega + S^{q,p}\psi \otimes \rho(P)\omega.
\]

*Proof.* A straightforward computation. \( \square \)

8.6. **Lemma 8.5 suggests that** \( \rho(P) \) behaves like a derivation with coefficients in a trivial representation of \( \mathfrak{gl}(V) \) with respect to the sign operators from 8.4. The corresponding derivation with coefficients in the adjoint representation of \( \mathfrak{gl}(V) \) then is given by the formula which follows directly from the definitions:

\[
P \cdot Q = [P, Q]_{\mathfrak{gl}(V)} + \rho(P)Q,
\]

where \([P, Q]_{\mathfrak{gl}(V)}\) is the pointwise bracket

\[
[P, Q]_{\mathfrak{gl}(V)}(X_1, \ldots) = [P(X_1, \ldots), Q(X_{p+1}, \ldots)].
\]

Moreover we have the following result

8.7. **Proposition.** For \( P \in L^p(V; L(V, V)) \) and \( Q \in L^q(V; L(V, V)) \) we have

\[
P \cdot (Q \cdot R) - S^{q,p}(Q \cdot (P \cdot R)) = [P, Q] \cdot R,
\]

where

\[
[P, Q]^S = [P, Q]_{\mathfrak{gl}(V)} + \rho(P)Q - S^{q,p}\rho(Q)P
\]

is a graded Lie bracket in the sense that

\[
[P, Q] = -S^{q,p}[Q, P]^S,
\]

\[
[P, [Q, R]^S]^S = [[P, Q]^S, R]^S + S^{q,p}[Q, [P, R]^S]^S.
\]

Also the derivation \( \rho \) is well behaved with respect to this bracket,

\[
\rho(P)\rho(Q) - S^{q,p}\rho(Q)\rho(P) = \rho([P, Q]^S).
\]

*Proof.* For decomposable elements like in the proof of lemma 8.5 this is a long but straightforward computation. \( \square \)
It is natural to expect an eventual dynamical realization of algebraic constructions discussed above when the underlying vector space $V$ is the algebra of observables of a mechanical or physical system. In the classical approach it should be an algebra of the form $V = C^\infty(M)$ with $M$ being the space–time, configuration or phase space of a system, etc. The localizability principle forces us to limit the considerations to $n$–ary operations which are given by means of multi–ferential operators. The following list of definitions is in conformity with these remarks.

**9.1 Definition.** An $n$–Lie algebra structure $\mu(f_1,\ldots,f_n)$ on $C^\infty(M)$ is called

1. *Local*, if $\mu$ is a multi–differential operator
2. *$n$–Jacobi*, if $\mu$ is a first–order differential operator with respect to any its argument
3. *$n$–Poisson* if $\mu$ is an $n$–derivation.

$(M,\mu)$ is called an $n$–Jacobi or $n$–Poisson manifold if $\mu$ is an $n$–Jacobi or, respectively, $n$–Poisson structure on $C^\infty(M)$.

It seems plausible that Kirillov's theorem is still valid for the proposed $n$–ary generalization. It so, $n$–Jacobi structures exhaust all local ones.

**9.2 Examples.** Any $k$–derivation $\mu$ on a manifold $M$ is of the form

$$\mu(f_1,\ldots,f_k) = P(df_1,\ldots,df_k)$$

where $P = P_\mu$ is a $k$–vector field on $M$ and vice versa. If $k$ is even, then $\mu$ is an $n$–Poisson structure on $M$ iff $[P_\mu,P_\mu]_{\text{Schouten}} = 0$. In particular, $\mu$ is a $k$–Poisson structure in each of below listed cases:

1. $P_\mu$ is of constant coefficients on $M = \mathbb{R}^m$
2. $P_\mu = X \wedge Q$ where $X$ is a vector field on $M$ such that $L_X(Q) = 0$
3. $P_\mu = Q_1 \wedge \cdots \wedge Q_r$ where all multi–vector fields $Q_i$'s are of even degree and such that $[Q_i,Q_j]_{\text{Schouten}} = 0, \quad \forall i,j$.

These examples are taken from [20] where the reader will find a systematical exposition and further structural results.

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