Bilateral tail estimate for distribution of self normalize s sum of independent centered random variables under natural norming

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Abstract

We derive in this article the exact non-asymptotical exponential and power estimates for self-normalized sums of centered independent random variables (r.v.) under natural norming.

We will use also the theory of the so-called Grand Lebesgue Spaces (GLS) of random variables.

Key words and phrases: Random variables, independence, self-normalizes sums, Rosenthal’s inequality, Cramer’s condition, Lebesgue-Riesz and Grand Lebesgue Spaces (GLS), exponential and power tail of distribution, Young-Fenchel transform, rearrangement invariant space, exponential Orlicz spaces, natural norming.

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1 Definitions. Notations. Previous results. Statement of problem.

Let \( \{\xi(i)\}, i = 1, 2, \ldots, n; \xi := \xi(1) \) be a sequence of centered: \( \mathbb{E}\xi(i) = 0 \) independent identically distributed (i., i.d.) random variables (r.v.) defined on certain probability space, having a finite non-zero variance \( \sigma^2 := \mathbb{E}\xi^2(i) \in (0, \infty) \).

Introduce the following self-normalized sequence of sums under natural norming

\[
T(n) = \sqrt{n} \cdot \frac{\sum_i \xi(i)}{\sum_i \xi^2(i)},
\]

here and in what follow

\[
\sum = \sum_i = \sum_{i=1}^n,
\]
and define the correspondent tail probabilities

\[ Q_n = Q_n(B) := P\left(T(n) > B\right), \quad B = \text{const} > 0; \quad Q = Q(B) := \sup_n Q_n(B). \quad (1.2) \]

**Our purpose in this preprint is obtaining non-asymptotical exponential and power bounds for introduced in (1.2) tail probabilities.**

This problem with another self norming sequence was considered in many works, see e.g. [3], [7], [8], [10], [16], [17], [18]-[19], [21]. Note that in these works was considered as a rule only asymptotical approach, i.e. when \( n \to \infty \); for instance, was investigated the classical Central Limit Theorem (CLT), Law of Iterated Logarithm (LIL) and Large Deviations (LD) for these variables. Several interest applications in the non-parametrical statistics are described in [3], [7], [17], [18]-[19] etc.

We must introduce now some needed notions and notations. \( \eta(i) = \eta(i; n, B) := \)

\[ \sqrt{n} \xi(i) + B(\sigma^2 - \xi^2(i)), \quad \eta = \eta(n; B) = \eta(1) = \sqrt{n} \xi + B (\sigma^2 - \xi^2); \]

\[ \gamma(i) = \gamma(i; n, B) := \eta(i)/\sqrt{n} = \xi(i) + B(\sigma^2 - \xi^2(i))/\sqrt{n}, \quad \gamma = \gamma(1); \]

\[ S = S(n) = \sum \xi(i), \quad V = V(n) = \sum (\xi^2(i) - \sigma^2), \]

so that

\[ \mathbf{E}S(n) = \mathbf{E}V(n) = 0 \]

and

\[ T(n) = \frac{\sqrt{n} \cdot S(n)}{V(n) + n \sigma^2}. \quad (1.4) \]

Further, we introduce the functions of two variables

\[ \phi(\lambda_1, \lambda_2) \overset{def}{=} \ln \mathbf{E} \exp(\lambda_1 \xi + \lambda_2 (\sigma^2 - \xi^2)), \]

so that

\[ \mathbf{E} \exp\left(\mu_1 \sqrt{n} \xi + \mu_2 B (\sigma^2 - \xi^2)\right) = \exp\phi(\mu_1 \sqrt{n}, \mu_2 B)). \quad (1.6) \]

Denote as ordinary for any r.v. \( \zeta \) its classical Lebesgue-Riesz \( L(p) \) norm

\[ |\zeta|_p := \left[ \mathbf{E}|\zeta|^p \right]^{1/p}, \quad p \geq 1, \]

and introduce the variables
\[ m(p) = m(p; B, \sigma, n) := |n^{-1/2} \xi + B (\sigma^2 - \xi^2)|_p = n^{-1/2}|\eta|_p, \] (1.7)

if of course \( m(p) \) is finite for certain value \( p, p \geq 1; \)

\[ w = w(\sigma) = w(\sigma; \xi) := \mathbf{E}(\sigma^2 - \xi^2); \quad z = z(\sigma) = z(\sigma; \xi) := \mathbf{E}(\sigma\xi - \xi^3), \] (1.8)

so that

\[ D^2(\sigma, n, B; \xi) := \operatorname{Var}(\eta) = n \sigma^2 + 2 B \sqrt{n} z + B^2 w \] (1.9)

and

\[ \mathbf{E}e^{\theta \eta(i;n)} = e^{\nu(\theta)} = e^{\nu(\theta;n,B)}, \] (1.10)

where

\[ \nu(\theta) = \nu(\theta; n, B) = \phi(\theta \sqrt{n}, B \theta). \] (1.11)

## 2 Grand Lebesgue Spaces (GLS).

Let \( Z = (Z, M, \mu) \) be probability space with non-trivial measure \( \mu \). Let also \( \psi = \psi(p), \ p \in [1, b), \ b = \text{const} \in (1, \infty] \) be certain bounded from below: \( \inf \psi(p) > 0 \) continuous inside the semi-open interval \( p \in [1, b) \) numerical function. We can and will suppose \( b = \sup \{p, \ \psi(p) < \infty\} \), so that \( \text{supp} \psi = [1, b) \) or \( \text{supp} \psi = [1, b] \). The set of all such a functions will be denoted by \( \Psi(b) = \{\psi(\cdot)\}; \ \Psi := \Psi(\infty). \)

By definition, the (Banach) Grand Lebesgue Space (GLS) space \( G\psi = G\psi(b) \) consists on all the real (or complex) numerical valued measurable functions (random variables, r.v.) \( \zeta \) defined on our probability space and having a finite norm

\[ ||\zeta|| = ||\zeta||_{G\psi} \overset{def}{=} \sup_{p \in [1,b]} \left[ \frac{|\zeta|_p}{\psi(p)} \right]. \] (2.1)

Here \( |\zeta|_p = |\zeta|_{L_p(Z)}. \)

These spaces are Banach functional space, are complete, and rearrangement invariant in the classical sense, see [1], chapters 1, 2; and were investigated in particular in many works, see e.g. [2], [5], [6], [9], [11], [12]-[15] etc. We refer here some used in the sequel facts about these spaces and supplement more.

It is known that if \( \zeta \neq 0 \), then

\[ \mathbf{P}(|\zeta| > y) \leq \exp \left(-v_{\psi}^*(\ln(u/||\zeta||))\right). \] (2.2)

Here and in the sequel the operator \( f \to f^* \) will denote the Young-Fenchel transform.
\[ f^*(u) \overset{\text{def}}{=} \sup_{x \in \text{Dom}(f)} (xu - f(x)). \]

Conversely, the last inequality may be reversed in the following version: if

\[ P(\zeta > u) \leq \exp \left(-v_\psi^*(\ln(u/K))\right), \quad u \geq e \cdot K, \]

and if the auxiliary function

\[ v(p) = v_\psi(p) \overset{\text{def}}{=} p \ln(p), \quad p \in [1, b) \]

is positive, continuous, convex and such that

\[ \lim_{p \to \infty} \psi(p) = \infty, \]

then \( \zeta \in \mathcal{G}(\psi) \) and besides \( \|\zeta\| \leq C(\psi) \cdot K. \)

Let us consider the so-called exponential Orlicz space \( L(M) \) builded over source probability space with correspondent Young-Orlicz function

\[ M(y) = \exp \left(v_\psi^*(\ln y)\right), \quad y \geq e; \quad M(y) = Cy^2, \quad |y| < e. \]

The Orlicz \( \|\cdot\| \) and \( G\psi \) norms are quite equivalent:

\[ \|\zeta\|_{G\psi} \leq C_1\|\zeta\|_{L(M)} \leq C_2\|\zeta\|_{G\psi}, \quad 0 < C_1 < C_2 < \infty. \quad (2.3) \]

Let us consider also the so-called degenerate \( \Psi \) – function \( \psi_{(r)}(p) \), where \( r = \text{const} \in [1, \infty) \):

\[ \psi_{(r)}(p) \overset{\text{def}}{=} 1, \quad p \in [1, r]; \]

so that the correspondent value \( b = b(r) \) is equal to \( r \). One can extrapolate formally this function onto the whole semi-axis \( R^1_+ \):

\[ \psi_{(r)}(p) := \infty, \quad p > r. \]

The classical Lebesgue-Riesz \( L_r \) norm for the r.v. \( \eta \) is quite equal to the GLS norm \( \|\eta\|_{G\psi_{(r)}} \):

\[ |\eta|_r = \|\eta\|_{G\psi_{(r)}}. \]

Thus, the ordinary Lebesgue-Riesz spaces are particular, more precisely, extremal cases of the Grand-Lebesgue ones.

Further, let \( \phi = \phi(\lambda), \quad |\lambda| < \lambda_0 = \text{const} \in (0, \infty] \) be numerical twice continuous differentiable positive even convex function such that \( \phi(0) = 0; \phi(\cdot) \) is monotonically increasing in the positive semi-interval \([0, \lambda_0)\) and such that \( \phi(\lambda) \sim C(\phi) \cdot \lambda^2, \lambda \to 0. \) The set of all such a functions will be denoted by \( \Phi = \{ \phi \} \).
**Definition.** The random variable $\zeta$ belongs to the space $\mathcal{B}(\phi)$, for certain fixed function $\phi \in \Phi$, if there exists a non-negative constant $\tau$ such that

$$\forall \lambda : |\lambda| < \lambda_0 \Rightarrow E \exp(\lambda \zeta) \leq \exp(\phi(\lambda \tau)). \quad (2.4)$$

The minimal value of the constant $\tau$ which satisfies the inequality (2.4) is said to be the $\mathcal{B}(\phi)$ norm of the r.v. $\zeta$:

$$||\zeta||_{\mathcal{B}(\phi)} \overset{def}{=} \max_{\pm} \sup_{\lambda \in (0, \lambda_0)} \phi^{-1}\{\ln E \exp(\pm \lambda \zeta)\}/|\lambda|, \quad (2.5)$$

so that

$$\forall \lambda : |\lambda| < \lambda_0 \Rightarrow E \exp(\lambda \zeta) \leq \exp(\phi(\lambda ||\zeta||_{\mathcal{B}(\phi)})). \quad (2.6)$$

We suppose in fact that the r.v. $\zeta$ is mean zero and satisfies the well-known Cramer’s condition. In this case the generated function $\phi(\cdot)$ may be introduced naturally:

$$\phi_{\zeta}(\lambda) := \max_{\pm} \ln E \exp(\pm \lambda \zeta).$$

See for example the equalities (1.10), (1.11).

This natural function $\phi_{\zeta}(\lambda)$ play a very important role in the theory of Large Deviations (L.D.) Namely, it is well known that

$$\lim_{n \to \infty} \left[ P(S(n)/n > x) \right]^{1/n} = -\phi^*_x(x), \ x > 0.$$  

Analogous result for the self-normalized sums was obtained by Qi-Man Shao in an article [18]:

$$\lim_{n \to \infty} \left[ P(S(n)/(V(n)\sqrt{n}) > x) \right]^{1/n} = \sup_{c > 0} \inf_{t > 0} E \exp \left[ t(c\zeta - x(\zeta^2 + c^2)/2) \right].$$

These spaces are complete Banach functional and rearrangement invariant, as well as considered before Grand Lebesgue Spaces. They were introduced at first in the article [11]; the detail investigation of these spaces may be found in the monographs [2] and [12], chapters 1,2.

It is known that if $\lambda_0 = \infty$ and $0 \neq \zeta \in \mathcal{B}(\phi)$ if and only if $E\zeta = 0$ and

$$\exists K = \text{const} \in (0, \infty) \Rightarrow \max [P(\zeta \geq u), P(\zeta \leq -u)] \leq \exp \{-\phi^*(u/K)\}, \ u > 0,$$

and herewith

$$||\zeta||_{\mathcal{B}(\phi)} \leq C_1(\phi)K \leq C_2(\phi)||\zeta||_{\mathcal{B}(\phi)}.$$  

More exactly, if $0 < ||\zeta||_{\mathcal{B}(\phi)} = ||\zeta|| < \infty$, then

$$\max [P(\zeta \geq u), P(\zeta \leq -u)] \leq \exp (-\phi^*(u/||\zeta||)). \quad (2.7)$$
If the r.v. $\zeta$ belongs to some $B(\phi)$ space, then it belongs also to certain $G\psi$ space with

$$\psi = \psi_\phi(p) = \frac{\phi^{-1}(p)}{p}, \quad p \geq 1. \quad (2.8)$$

The inverse conclusion in not true. Namely, the mean zero r.v. $\zeta$ can has finite
all the moments $|\zeta|^p < \infty, \quad p \geq 1$, but may not satisfy the Cramer’s condition.

A very popular class of these spaces form the subgaussian random variables, i.e. for which $\phi(\lambda) = \lambda^2$ and $\lambda_0 = \infty$. The correspondent $\psi$ function has a form $\psi(p) = \psi_2(p) = \sqrt{p}$.

More generally, suppose

$$\phi(\lambda) = \phi_m(\lambda) = |\lambda|^m/m, \quad |\lambda| \geq 1, \quad \lambda_0 = \infty, \quad m = \text{const} > 0. \quad (2.9)$$

The correspondent $\psi$ function has a form

$$\psi(p) = \psi_m(p) = p^{1/m}$$

and the correspondent tail estimate is follow:

$$\max \left\{ P(\zeta \geq u), \ P(\zeta \leq -u) \right\} \leq \exp \left\{ -(u/K)^m \right\}, \quad u > 0. \quad (2.10)$$

These space are used for obtaining of exponential estimates for sums of independent random variables, see e.g. [11]; [12], sections 1.6, 2.1-2.5. Indeed, introduce for any function $\phi(\cdot)$ from the set $\Phi$ a new function $\bar{\phi}(\cdot)$ which belongs also at the same set:

$$\bar{\phi}(\lambda) \overset{\text{def}}{=} \sup_{n=1,2,...} n \left[ \phi\{\lambda/\sqrt{n}\} \right]. \quad (2.11)$$

It is easily to see that

$$\sup_n \mathbb{E} \exp(\lambda S(n)/\sqrt{n}) \leq \exp[\bar{\phi}(\lambda)] \quad (2.12)$$

with correspondent uniform relative the variable $n$ exponential tail estimate.

For instance, if for some value $m = \text{const} > 0$

$$\max \left\{ P(\xi \geq u), \ P(\xi \leq -u) \right\} \leq \exp \left\{ -u^m \right\}, \quad u > 0, \quad (2.13)$$

then

$$\max \left\{ P(S(n)/\sqrt{n} \geq u), \ P(S(n)/\sqrt{n} \leq -u) \right\} \leq \exp \left\{ -(C(m)u^\min(m,2)) \right\}, \quad u > 0, \quad C(m) \in (0, \infty), \quad (2.14)$$

and the last estimate is essentially non-improvable.
3 Main results. Exponential level.

Denote

\[ \beta_n(\theta; B) = \beta(\theta; n, B) := n\phi(\theta/\sqrt{n}, B \theta/n); \]

\[ \beta^*(z; n, B) := \sup_{\theta > 0} (\theta z - \beta(\theta; n, B)); \]

Theorem 3.1.

\[ Q_n(B) \leq \exp \left( -\beta^*(B; n, B) \right). \] (3.0)

Corollary 3.1.

\[ Q(B) \leq \sup_n \exp \left( -\beta^*(B; n, B) \right). \] (3.0a)

Proof. We have:

\[ Q_n(B) = P \left( \sqrt{n} \cdot \frac{\sum \xi(i)}{\sum \xi^2(i)} \right) = P \left( \sqrt{n} \sum \xi(i) - B \sum \xi^2(i) > 0 \right) = \]

\[ P \left( \sum (\sqrt{n}\xi(i) + B((\sigma^2 - \xi^2(i))) > nB\sigma^2 \right) = \]

\[ P \left( n^{-1} \sum \eta(i, n) > B\sigma^2 \right). \] (3.1)

Since the random variables \( \eta(i) = \eta(i, n), \ i = 1,2,...,n \) are centered and common independent,

\[ E \exp \left( \theta \cdot n^{-1} \sum \eta(i, n) \right) = [E \exp(\theta \eta/n)]^n = \]

\[ \exp \left( n\phi(\theta/\sqrt{n}, B \theta/n) \right) = \exp \beta(\theta; n, B). \] (3.2)

It remains to use the estimate (2.7).

4 Main results. Power level

Let us introduce the natural \( G^\psi \) function for the r.v., more exactly for the sequence of r.v. \( \gamma(i) = \gamma(i, n) : \)

\[ \Delta(p) = \Delta(p; n) := |\gamma(i, n)|_p = |\xi + n^{-1/2} B(\sigma^2 - \xi^2)|_p, \] (4.0)
if of course it is finite for some value \( p, \ p \geq 1 \). The sufficient condition for this conclusion is following:

\[ \xi \in \bigcup_{s=2}^{\infty} L_s(\Omega, \mathcal{P}) \]

and does not dependent on the variable \( n \).

We deduce

\[ Q_n(B) = P \left( \sum \eta(i) > nB\sigma^2 \right) = P \left( n^{-1/2} \sum \gamma(i) > B\sigma^2 \right). \tag{4.1} \]

We need to use the famous Rosenthal’s inequality, see [20]:

\[ |n^{-1/2} \sum \gamma(i)|_p \leq K_R \cdot \frac{p}{\ln p} \cdot |\eta|_p, \ p \in \text{supp } \Delta(\cdot). \tag{4.2} \]

where the exact value of the Rosenthal’s constant may be found in [14], [15]: \( K(R) \approx 0.6379 \ldots \)

Introduce for such a function \( \Delta(p) \) its Rosenthal transform \( K_R[\Delta](p) \) as follows

\[ K_R[\Delta](p) = K_R[\Delta; n](p) \overset{def}{=} K_R \cdot \frac{p}{\ln p} \cdot \Delta(p; n). \tag{4.3} \]

We obtain by virtue of Rosenthal’s inequality

\[ ||n^{-1/2} \sum \gamma(i)||GK_R[\Delta] \leq 1. \tag{4.4} \]

It follows immediately from the estimate(2.2) the next proposition.

**Theorem 4.1.**

\[ Q_n(B) \leq \exp \left\{ -v_{K_R; \Delta, n}^*(\ln B) \right\}, \ B \geq e, \tag{4.5} \]

and therefore

\[ Q(B) \leq \sup_{n} \exp \left\{ -v_{K_R; \Delta, n}^*(\ln B) \right\}, \ B \geq e. \tag{4.6} \]

## 5 Lower bounds for introduces tail probabilities.

Suppose in this section for simplicity \( \sigma = 1 \). It follows immediately from the classical CLT and LLN that

\[ Q(B) \geq \lim_{n \to \infty} Q_n(B) \geq \exp \left( -B^2/2 \right), \ B \geq 1. \tag{5.1} \]

a first trivial estimate. A second one is follows:

\[ Q(B) \geq Q_1(B) = P(\xi \geq B, \xi^2) = P(0 \leq \xi \leq 1/B), \ B > 1; \tag{5.2} \]
so that

\[ Q(B) \geq Q_1(B) = \int_{0^+}^{(1/B)+} dF_\xi(x), \]

and in the case when the r.v \( \xi \) has a density of distribution \( f_\xi(x) \),

\[ Q(B) \geq Q_1(B) = \int_{0^+}^{1/B+} f_\xi(x) \, dx. \tag{5.3} \]

If in addition the density function \( f_\xi(x) \) is bounded from below in some positive neighborhood of origin, say

\[ f_\xi(x) \geq c_1 = \text{const} > 0, \; x \in (0, 1), \]

then

\[ Q(B) \geq Q_1(B) \geq c_1/B, \; B \geq 1. \tag{5.4} \]

If the function \( f_\xi(x) \) is right continuous at the point \( x = 0^+ \) and \( f_\xi(0) = c_2 = \text{const} > 0 \), then obviously

\[ Q_1(B) \sim c_2/B, \; B \to +\infty. \tag{5.5} \]

On the other words,

\[ Q_1(B) \asymp c_2/B, \; B \geq 1. \tag{5.5a} \]

This state is very different from the other norming function statement, see [16]-[20]; as well as from the alike estimate for one for classical estimates for i., i.d. r.v., see (2.14).

### 6 Examples.

**Example 6.0.** Let the r.v. \( \xi(i) \) be the Rademacher sequence, i.e. they are independent and

\[ P(\xi(i) = 1) = P(\xi(i) = -1) = 1/2. \]

Then \( \xi^2(i) = 1 \) and hence \( V(n) = n \). We return to the classical Khinchine theorem

\[ |\ln Q(B)| \asymp B^2/2, \; B \in (1, \infty). \tag{6.1} \]

**Example 6.1.** Let now \( \{\xi(i)\} \) be independent standard Gaussian distributed r.v. We deduce after some computations by virtue of theorem 3.1

\[ Q(B) \leq c_3/B, \; B \geq 1; \tag{6.2} \]
and the right-hand side of the last inequality coincides up to multiplicative constant with the low estimate (5.5).

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