Equivariant Relative Submajorization

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Abstract—We study a generalization of relative submajorization that compares pairs of positive operators on representation spaces of some fixed group. A pair equivariantly relatively submajorizes another if there is an equivariant subnormalized channel that takes the components of the first pair to a pair satisfying similar positivity constraints as in the definition of relative submajorization. In the context of the resource theory approach to thermodynamics, this generalization allows one to study transformations by Gibbs-preserving maps that are in addition time-translation symmetric. We find a sufficient condition for the existence of catalytic transformations and a characterization of an asymptotic relaxation of the relation. For classical and certain quantum pairs the characterization is in terms of explicit monotone quantities related to the sandwiched quantum Rényi divergences. In the general quantum case the relevant quantities are given only implicitly. Nevertheless, we find a large collection of monotones that provide necessary conditions for asymptotic or catalytic transformations. When applied to time-translation symmetric maps, these give rise to second laws that constrain catalytic transformations. When applied to time-translation symmetric maps, these give rise to second laws that constrain catalytic transformations. When applied to time-translation symmetric maps, these give rise to second laws that constrain catalytic transformations.

Index Terms—Asymptotic and catalytic transformations, equivariant channels, group symmetric hypothesis testing, relative submajorization, thermal processes.

I. INTRODUCTION

A PAIR of positive vectors \( (p, q) \in \mathbb{R}_+^d \times \mathbb{R}_+^d \) is said to relatively submajorize another pair \( (p', q') \) if there exists a substochastic map \( T \) such that \( T(p) \geq p' \) and \( T(q) \leq q' \) componentwise [1]. This relation can be used to characterize probabilistic and work-assisted thermal operations between incoherent states, as well as error probabilities in hypothesis testing. However, conditions based on relative (sub)majorization (or thermo-majorization) are insufficient to characterize thermal transformations in the presence of quantum coherence [2].

Quantum majorization is a relation between bipartite quantum states sharing a marginal. A state \( \rho_{AB} \) quantum majorizes \( \rho'_{AB} \) if there is a quantum channel \( T : \mathcal{B}(\mathcal{H}_B) \rightarrow \mathcal{B}(\mathcal{H}_{B'}) \) such that \( \rho'_{AB} = (\text{id}_A \otimes T)(\rho_{AB}) \). In [3] it was shown that this relation, as well as a \( G \)-covariant version (for some compact group \( G \)) can be characterized using an infinite family of monotones defined in terms of the conditional min-entropy. For specific classical-quantum states (with \( A \) a classical bit), quantum majorization with covariance encodes time-translation symmetric Gibbs-preserving transformations which, like thermal operations, puts constraints on the evolution of states with coherence between energy eigenstates.

In this paper, focusing on the classical-quantum case, we study transformations between pairs of positive operators by equivariant maps in a sense similar to relative submajorization: given representations \( \pi : G \rightarrow U(\mathcal{H}) \) and \( \pi' : G \rightarrow U(\mathcal{H}') \), and pairs of positive operators \( (\rho, \sigma) \) on \( \mathcal{H} \) and \( (\rho', \sigma') \) on \( \mathcal{H}' \), we say that \((\pi, \rho, \sigma)\) equivariantly relatively submajorizes \((\pi', \rho', \sigma')\) if there is a completely positive trace-nonincreasing map \( T \) that is equivariant, i.e. satisfies \( T(\pi(g)A\pi(g)^*) = \pi'(g)T(A)\pi'(g)^* \) for all \( g \in G \) and operator \( A \), in addition to the inequalities \( T(\rho) \geq \rho' \) and \( T(\sigma) \leq \sigma' \).

An averaging argument shows that this relation can equivalently be understood as transformations between families of positive operators parametrized by two copies of \( G \). Somewhat more generally, we will consider pairs of continuous families of positive operators \( \rho : X \rightarrow \mathcal{B}(\mathcal{H})_{++} \), \( \sigma : Y \rightarrow \mathcal{B}(\mathcal{H})_{++} \), where \( X \) and \( Y \) are fixed nonempty compact topological spaces (when studying \( G \)-equivariant transformations for a compact group \( G \), one would use \( X = Y = G \)). In this case we say that \((\sigma, \rho)\) relatively submajorizes \((\sigma', \rho')\) (notation: \((\sigma, \rho) \succ (\sigma', \rho')\)) if there is a completely positive trace-nonincreasing map \( T \) such that \( T(\rho(x)) \geq \rho'(x) \) and \( T(\sigma(y)) \leq \sigma'(y) \) for all \( x \in X \) and \( y \in Y \).

Our main result is a characterization of an asymptotic relaxation of this relation and a sufficient condition for the possibility of a catalytic transformation. We say that \((\rho, \sigma)\) asymptotically relatively submajorizes \((\rho', \sigma')\) if \( (2^{\sigma(\alpha)} \rho^{\otimes n} , \sigma^{\otimes n}) \succ (\rho'^{\otimes n} , \sigma'^{\otimes n}) \). Assuming that the image of \( \sigma \) and \( \sigma' \) consist of commuting operators, the characterization is in terms of explicitly given monotones: \((\rho, \sigma)\) asymptotically relatively submajorizes \((\rho', \sigma')\) iff the inequalities

\[
\tilde{D}_\alpha \left( \rho(x) \| \exp \int_Y \ln \sigma \ d\gamma \right) \geq \tilde{D}_\alpha \left( \rho'(x) \| \exp \int_Y \ln \sigma' \ d\gamma \right)
\]
hold for every $\alpha \geq 1$, $x \in X$ and probability measure $\gamma$ on $Y$, where $\tilde{D}_\alpha$ is

$$
\tilde{D}_\alpha (\rho||\sigma) = \frac{1}{\alpha - 1} \log \tilde{Q}_\alpha (\rho||\sigma) = \frac{1}{\alpha - 1} \log \text{Tr} \left( \sigma^{\frac{1}{2\alpha}} \rho \sigma^{\frac{1}{2\alpha}} \right)^\alpha,
$$

the minimal (or sandwiched) Rényi divergence [4], [5]. If the inequalities are strict and $\text{Tr} \rho(x) > \text{Tr} \rho'(x)$ for all $X$, then relative submajorization holds after tensoring both pairs with a suitable catalyst. Without the commutativity assumption, we find generalizations of the conditions (1) that are necessary for asymptotic or catalytic ordering. In these the second argument is replaced with a suitable non-commutative geometric mean. For example,

$$
\tilde{D}_\alpha (\rho(x)||\sigma(y_1)\#\sigma(y_2))
$$

is one of these monotones, where $x \in X$, $y_1, y_2 \in Y$, and $\sigma(y_1)\#\sigma(y_2)$ is the matrix geometric mean [6].

To prove our results, we use recent results from the theory of preordered semirings to find conditions in terms of monotone quantities that are additive under direct sums and multiplicative under the tensor product, following some of the ideas of [7], [8]. While in general these monotones are defined only implicitly, under the additional assumption that the image of $\sigma$ consists of commuting operators, we obtain a complete classification, identifying them as exponentiated preorder on the semiring of pairs of families satisfies the preorder is defined by $\rho \succcurlyeq \sigma$ if there is a sequence $\{\xi_k\}_{k \in \mathbb{N}}$ that is in the closure (with respect to the vague topology) of the set of positive linear combinations of Dirac measures and neutral elements (with the multiplication sign often omitted as usual). As the operations and the preorder is usually clear from the context, we will simply denote the preordered semiring with the symbol of the underlying set.

Two preordered semirings will play a distinguished role: the first is the set $\mathbb{R}_{\geq 0}$ of nonnegative real numbers with its usual addition, multiplication and total order; the second is the tropical semiring. In the multiplicative picture, as a set, the tropical real semiring is the set of nonnegative real numbers $\mathbb{R}_+ = \mathbb{R}_{\geq 0}$, the sum of $x$ and $y$ is defined as $\max\{x, y\}$, while is the usual multiplication. Equipped with the usual total order of the real numbers this set is a preordered semiring.

A pair of additional conditions must hold true for the preordered semirings considered here. First the $\mathbb{N} \to S$ canonical map which sends $n$ to the $n$-term sum $1 + 1 + \cdots + 1$ should be an order embedding (i.e. injective and $m \leq n$ as natural numbers iff their images, also denoted by $m$ and $n$, satisfy $m \leq n$). We require polynomial growth [11]. A semiring is of polynomial growth if there exist a $u \in S$ power universal element such that $u \geq 1$ and for every nonzero $x \in S$ there is a $k \in \mathbb{N}$ such that $x \preceq u^k$ and $1 \preceq u^k x$. The power universal element is not necessarily unique, but it can be shown that the subsequent definitions do not depend on a particular choice.

**Definition 2.2:** Let $S$ be a preordered semiring of polynomial growth and $u \in S$ power universal. The asymptotic preorder is defined by $x \succeq y$ if there is a sequence $(k_n)_{n \in \mathbb{N}}$
of natural numbers such that \( \lim_{n \to \infty} k_n/n = 0 \) and for all \( n \) the inequality \( u^{k_n} x^n \succ y^n \) holds.

**Definition 2.3:** Let \( S \) be a preordered semiring and let \( x, y \in S \). If \( \exists a \in S \) such that \( ax \succ ay \) we say that \( x \) is catalytically larger than \( y \), in notation \( x \succ_{c} y \).

**Proposition 2.4:** \( x \succeq y \implies x \succ_{c} y \implies x \succ_{c} y \).

**Proof:** The first implication is obvious. For the second implication consider \( ax \succ ay \). Then there exist \( k_1 \in \mathbb{N} \) such that \( u^{k_1} \succ a \) and \( k_2 \in \mathbb{N} \) such that \( u^{k_2} a \succ 1 \). Thus \( u^{k_1 + k_2} x \succ u^{k_1} ax \succ u^{k_1} ay \succ y \). In fact there is a constant power realizing the asymptotic ordering. \( \square \)

**Definition 2.5:** A \( \varphi : S_1 \to S_2 \) map is homomorphism between the semirings \((S_1, \preceq_1)\) and \((S_2, \preceq_2)\) if \( \varphi(0) = 0 \), \( \varphi(1) = 1 \), \( \varphi(x + y) = \varphi(x) + \varphi(y) \) and \( \varphi(xy) = \varphi(x) \varphi(y) \), for every \( x, y \in S_1 \). If furthermore \( x \preceq_1 y \implies \varphi(x) \preceq_2 \varphi(y) \) for \( x, y \in S_1 \), then we say that \( \varphi \) is a monotone semiring homomorphism.

We will be particularly interested in monotone homomorphisms into the real and tropical real semirings. For these we introduce the following notations: given a preordered semiring of polynomial growth \((S, \preceq)\) with power universal \( u \) we let \( T_{\text{Sp}1}(S, \preceq) = \text{Hom}(S, \mathbb{R}_{\geq 0}) \cup \{ f \in \text{Hom}(S, \mathbb{TR}) | f(u) = 2 \} \) and call it the 1-test spectrum. Note that in [12] monotone decreasing maps are also part of the 1-test spectrum. In our case these parts will be empty, since relative submajorization defined in Section III-A will assure that \( 0 \preceq 1 \). While there is a natural normalization condition in the definition of a homomorphism into the nonnegative reals, in the tropical case homomorphisms can always be rescaled in a multiplicative sense by replacing \( f(x) \) with \( f^c(x) \) for some \( c > 0 \) (see also [12, Section 13.1]). This is the reason for requiring that \( f(u) = 2 \) in our definition and the number 2 itself is arbitrary, but will be convenient relative to our choice of the power universal element \( u \) later.

Our strategy will be to use the elements of the spectrum to characterize the catalytic preorder. The main tool will be the following result from [12].

**Theorem 2.6 (Special Case of [12, second part of 1.4. Theorem]):** Let \( S \) be a preordered semiring of polynomial growth with \( 0 \preceq 1 \). Suppose that \( x, y \in S \setminus \{0\} \) such that for all \( f \in T_{\text{Sp}1}(S, \preceq) \) the strict inequality \( f(x) > f(y) \) holds. Then also the following hold:
1) there is a \( k \in \mathbb{N} \) such that \( u^{k}x^{n} \succ u^{k}y^{n} \) for every sufficiently large \( n \)
2) if in addition \( x \) is power universal then \( x^{n} \succ y^{n} \) for every sufficiently large \( n \)
3) there is a nonzero \( a \in S \) such that \( ax \succeq ay \).

In [12] a catalyst is given explicitly in terms of the \( k \) above. We note that any of the listed conditions implies the non-strict inequalities \( f(x) \geq f(y) \) for the monotone homomorphisms.

The following proposition will be useful when dealing with the logarithm of elements of the spectrum.

**Proposition 2.7:** Let \( S \) be a preordered semiring of polynomial growth and \( u \in S \) power universal. Then for every monotone homomorphism \( f \) from \( S \) into either \( \mathbb{R} \) or \( \mathbb{TR} \) and for every nonzero \( x \in S \) we have that \( f(x) > 0 \).

**Proof:** Since \( u \) is power universal, \( u \geq 1 \) and thus \( f(u) \geq f(1) = 1 \). Then for every nonzero \( x \in S \) there is a \( k \in \mathbb{N} \) such that \( 1 < u^{kx} \). This yields \( f(u)^{k}f(x) \geq f(1) = 1 \) and \( f(x) \geq f(1)f(u)^{-k} > 0 \). \( \square \)

Observe that the inequality \( f(u) \geq 1 \) can be strengthened as follows. On the one hand, by the chosen normalization of tropical monotones we have \( f(u) \geq 2 > 1 \). On the other hand, by power universality, there is a \( k \in \mathbb{N} \) such that \( u^{k} \geq 2 \). Apply the real monotone homomorphism \( f \) and rearrange to get \( f(u) \geq 2^{1/k} > 1 \). This allows us to show that the asymptotic relaxation of the preorder holds even if we only have non-strict inequalities on the spectrum.

**Corollary 2.8:** Let \( S \) be a preordered semiring of polynomial growth and \( u \in S \) power universal. Then \( x \succeq y \) iff for all \( f \) in the 1-test spectrum the inequality \( f(x) \geq f(y) \) holds.

**Proof:** The only direction is clear: \( u^{n}x^{n} \succeq y^{n} \) implies \( f(u)^{kn}f(x) \geq f(y) \), and by taking the limit as \( n \to \infty \), also \( f(x) \geq f(y) \). For the if direction, recall that \( f(u) > 1 \) for all \( f \) in the spectrum. Assuming \( f(x) \geq f(y) \) for all \( f \), this implies that for all \( n \in \mathbb{N} \) we have the strict inequalities \( f(u^{n}x) > f(y^{n}) \). By Theorem 2.6, there exists nonzero \( a \in S \) (which may depend on \( n \), such that \( u^{n}x \succeq ay^{n} \). [13, Lemma 2 (iv)], this implies \( ux^{n} \succeq y^{n} \) for all \( n \), which in turn by [13, Lemma 3] implies \( x \succeq y \). \( \square \)

### III. Relative Submajorization of State Families

#### A. The Preordered Semiring of Pairs of Families

Let \( X, Y \) be nonempty compact Hausdorff topological spaces. These will be the index sets for the families, and can be considered fixed throughout this section. We will consider pairs of continuous maps \((\rho, \sigma)\), where \( \rho : X \to B(\mathcal{H})_{++} \) and \( \sigma : Y \to B(\mathcal{H})_{++} \) for some finite dimensional Hilbert space \( \mathcal{H} \). Two pairs \((\rho, \sigma)\) and \((\rho', \sigma')\) are equivalent if there is a unitary \( U : \mathcal{H} \to \mathcal{H}' \) such that \( \forall x \in X : U\rho(x)U^{*} = \rho'(x) \) and \( \forall y \in Y : U\sigma(y)U^{*} = \sigma'(y) \). We let \( S_{X,Y} \) denote the set of equivalence classes of pairs of such families. The pointwise direct sum and tensor product operations are well-defined on equivalence classes, and turn \( S_{X,Y} \) into a commutative semiring. The zero element is represented by the unique pair over a zero dimensional Hilbert space, while 1 is represented by the pair consisting of constant functions with value \( I \) over \( \mathbb{C} \).

We adopt the convention that if any map \( \rho : X \to B(\mathcal{H})_{++} \) and \( \sigma : Y \to B(\mathcal{H})_{++} \) appear outside of brackets, then any operations or relations they appear in, are to be understood pointwise, including sum, product, direct sum, tensor product or image under a linear super operator, usually notated by \( T \). More precisely, given any map \( B(\mathcal{H})_{++} \to B(\mathcal{H}')_{++} \) we understand the composition \( T(\rho) : x \mapsto T(\rho(x)) \) and \( T(\sigma) : y \mapsto T(\sigma(y)) \).

**Definition 3.1:** \((\rho, \sigma)\) relatively submajorizes \((\rho', \sigma')\), in notation \((\rho, \sigma) \preceq (\rho', \sigma')\), if there exists a completely positive trace-nonincreasing map \( T : B(\mathcal{H}) \to B(\mathcal{H}')\) such that \( T(\rho) \geq \rho' \) and \( T(\sigma) \leq \sigma' \).

**Example 3.2:** Let \( X \) and \( Y \) be one-point spaces and let \( \rho, \sigma \) be states of full support over some Hilbert space \( \mathcal{H} \) and let \( \rho', \sigma' \) be states of full support over some Hilbert space \( \mathcal{K} \). Then \((\rho, \sigma) \preceq (\rho', \sigma')\) translates to relative majorization between the pairs of states.
1) If we further specify $\mathcal{H} = \mathcal{K}$ and $\sigma = \sigma' = \frac{Tr}{\dim \mathcal{H}}$, then $(\rho, \sigma) \preceq (\rho', \sigma')$ translates to the so-called majorization preorder between $\rho$ and $\rho'$, meaning there is a unital channel mapping $\rho$ to $\rho'$.

2) If we specify instead $\sigma = \frac{e^{-\beta H}}{Tr e^{-\beta H}}$ and $\sigma' = \frac{e^{-\beta H'}}{Tr e^{-\beta H'}}$ to be Gibbs states, then $(\rho, \sigma) \succ (\rho', \sigma')$ translates to the so-called thermomajorization preorder [14] between $\rho$ and $\rho'$, meaning there is a Gibbs preserving channel mapping $\rho$ to $\rho'$.

**Proposition 3.5:** $S_{X,Y}$ is a preordered semiring with relative submajorization.

**Proof:** We need to verify that the preorder is compatible with the semiring operations. Suppose that $(\rho, \sigma) \succ (\rho', \sigma')$ and let $T$ be a completely positive trace non-increasing map as in Definition 3.1. Let $(\omega, \tau) \in S_{X,Y}$ be a pair of families on $\mathcal{K}$, i.e. $\omega : X \to B(\mathcal{K})_{++}$ and $\tau : Y \to B(\mathcal{K})_{++}$. Then

$$
(T \otimes \text{id}_{B(\mathcal{K})})(\rho \otimes \omega) = T(\rho) \otimes \omega \geq \rho' \otimes \omega
$$

and

$$
(T \otimes \text{id}_{B(\mathcal{K})})(\sigma \otimes \tau) = T(\sigma) \otimes \tau \leq \sigma' \otimes \tau,
$$

therefore $(\rho, \sigma)(\omega, \tau) \succ (\rho', \sigma')(\omega, \tau)$. The map $\bar{T} : \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{B}(\mathcal{H}' \otimes \mathcal{K})$ defined as

$$
\bar{T} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} T(A) & 0 \\ 0 & D \end{pmatrix}
$$

is also completely positive and trace non-increasing, and satisfies

$$
\bar{T}(\rho \otimes \omega) = T(\rho) \otimes \omega \geq \rho' \otimes \omega
$$

and

$$
\bar{T}(\sigma \otimes \tau) = T(\sigma) \otimes \tau \leq \sigma' \otimes \tau,
$$

therefore $(\rho, \sigma) + (\omega, \tau) \succ (\rho', \sigma')(\omega, \tau)$. \hfill \Box

**Proposition 3.6:** $S_{X,Y}$ is of polynomial growth and $u = (2, 1)$ is a power universal.

**Proof:** For the pair of families $(\rho, \sigma)$ let us choose the substochastic map $T(.) := c_1 Tr(.)$, with $c_1 = \min\{1, [\max_{y \in Y} Tr \sigma(y)]^{-1}\}$. Then we have $c_1 Tr \rho \leq 1$ and $T(u^k(\rho, \sigma)) = (T(2^k \rho), T(\sigma)) = (c_2 2^k Tr \rho, c_1 Tr \sigma)$. Choosing a large enough $k$ will satisfy $c_1 2^k Tr \rho \geq 1$, since $Tr \rho$ is bounded on $X$ and so $\exists k \in \mathbb{N} : u^k(\rho, \sigma) \succ (1, 1)$. Let us choose now $T(.) := c_2(\cdot) \otimes \frac{\sigma}{\rho}$, with $c_2 = \min\{1, d[\min_{y \in Y} \min\{\text{spec}(\sigma(y))\}]\}$. Then we have $\sigma \succ \frac{\rho}{\sigma} \otimes 1_d$ and $T(u^k) = (T(2^k), T(1)) = (\frac{2^k}{c_2}, \frac{\sigma}{\rho} \otimes 1_d)$. Choosing a large enough $k$ will satisfy $\frac{2^k}{c_2} \geq \rho$, since $\max\{\text{spec}(\sigma(y))\}$ is bounded on $X$ and so $\exists k \in \mathbb{N} : u^k \succ (\rho, \sigma)$. \hfill \Box

$S_{X,Y}$ is then a semiring of polynomial growth and in $S_{X,Y}$ we have $0 \prec 1$ and thus Theorem 2.6 and Corollary 2.8 are applicable.

### B. Classical Families

**Definition 3.5:** The subsemiring generated by the one-dimensional elements is called the subsemiring of classical families, in notation $S^c_{X,Y}$. That is, $(\rho, \sigma) \in S^c_{X,Y}$ if and only if $[\rho(x), \rho(x')] = [\rho(x), \sigma(y)] = [\sigma(y), \sigma(y')] = 0$, $\forall x, x' \in X$, $\forall y, y' \in Y$.

We turn to the classification of real and tropical real valued monotone homomorphisms on the subsemiring of classical families. By definition every element in $S^c_{X,Y}$ is a sum of one-dimensional elements. A one-dimensional element of the semiring on the other hand can be identified by a pair of strictly positive continuous functions on $X$ and $Y$. Suppose that $f$ is a multiplicative map from the one-dimensional pairs into either the real or the tropical numbers. Then the extension of $f$ to multi-dimensional pairs via additivity also enjoys multiplicativity. Since $S^c_{X,Y}$ is generated by the one-dimensional pairs, the value of every $f \in TS_{X,Y}$ is determined by its behaviour on one-dimensional pairs.

**Proposition 3.7:** If $f \in TS_{X,Y}$ then there exists unique, non-negative Radon measures $\mu$ and $\nu$ on $X$ and $Y$ such that for every multidimensional classical pair $(\sum_{i=1}^d p_i, \sum_{i=1}^d q_i)$ ($p_i \in C(X), q_i \in C(Y)$), if $f$ is real valued, it admits the form

$$
\sum_{i=1}^d \exp \left( \int_X \ln p_i \, d\mu - \int_Y \ln q_i \, d\nu \right), \quad \mu(X) = \nu(Y) = 1,
$$

while if $f$ is tropical valued it admits the form

$$
\max_{i \in [d]} \exp \left( \int_X \ln p_i \, d\mu - \int_Y \ln q_i \, d\nu \right), \quad \mu(X) = \nu(Y) = 0
$$

and functions of these form are monotone under relative majorization if and only if they satisfy the data processing inequality.

**Proof:** Let $f \in TS_{X,Y}$ be an element of the spectrum. For every $\xi, \eta > 0$ one has $(e^\epsilon, 1_Y) \succ (1_X, e^{-\epsilon})$ and $(1_X, e^{-\epsilon}) \succ (1_X, 1_Y)$, thus the maps $\xi \mapsto \ln f(e^\epsilon, 1_Y)$, from $C(X)$ to $\mathbb{R}$ and $\eta \mapsto \ln f(1_X, e^{-\epsilon})$, from $C(Y)$ to $\mathbb{R}$ are well defined positive linear functionals on $C(X)$ and $C(Y)$ (note that we can take the logarithm by Proposition 2.7). Thus by Theorem 2.1, $\ln f(e^\epsilon, 1_Y) = \int_X \xi(x) d\mu(x)$ and $\ln f(1_X, e^{-\epsilon}) = \int_Y \eta(y) d\nu(y)$ for some unique $\mu, \nu$ Radon measures on $X$ and $Y$. Since $f$ is multiplicative and $\ln f(e^\epsilon, 1_Y) = \ln f(e^\epsilon, 1_Y) + \ln f(1_X, e^{-\epsilon}) = \int_X \xi d\mu + \int_Y \eta d\nu$. From this $f(p, q) = \exp(\int_X \ln p \, d\mu - \int_Y \ln q \, d\nu)$. We used only the multiplicative property of $f$ but not the additive property, thus this part of the proof works for either real or tropical valued elements of the spectrum. Consider now $f(1_X + 1_Y, 1_Y + 1_Y) = f(1_X, 1_Y) + f(1_X, 1_Y)$. In the real case this translates to $f(1_X + 1_Y, 1_Y + 1_Y) = 2$, in the tropical case to $f(1_X + 1_Y, 1_Y + 1_Y) = 1$. Then $t \mapsto \ln f(e^t, e^{1-Y})$ is additive, normalized and monotone, therefore satisfies Cauchy’s functional equation and admits the form $f(e^t, e^{1-Y}) = t$ in the real case and the form $f(e^t, e^{1-Y}) = 0$ in the tropical case. This leads to $1 = \ln f(e^t, e^{1-Y}) = \mu(X) - \nu(Y)$ in the real case and $0 = \ln f(e^t, e^{1-Y}) = \mu(X) - \nu(Y)$ in the tropical case. This further shows that elements of the spectrum are homogeneous of degree 1 in the real case and homogeneous of degree 0 in the tropical case.

Now from additivity any real or tropical valued element of the spectrum admits the forms (9) and (10). We fully exploited additivity and multiplicativity, we further need to impose monotonicity under relative submajorization on multidimensional pairs of families. Elements of the spectrum
need to be monotone under relative submajorization, so in particular relative majorization. Functions of the form (9) and (10) are monotone decreasing under increase of any of the \( q_i \) or decrease of any of the \( p_i \). So these functions are elements of the spectrum if and only if they are monotone decreasing under relative majorization, i.e. under stochastic maps, classical channels, that is we further require (9) and (10) to satisfy the data-processing inequality.

Lemma 3.7: Let \( f \) be an additive function from \( S_{X,Y} \) into either the real or tropical numbers. Then

1) if \( f \) is homogeneous of degree 1 and \( f \) goes into the real numbers, then it satisfies the data-processing inequality if and only if it is jointly convex;
2) if \( f \) is homogeneous of degree 0 and \( f \) goes into the tropical numbers, then it satisfies the data-processing inequality if and only if it is jointly quasi-convex.

Proof: Let \( f \) be an additive function from \( S_{X,Y} \) into either the real or tropical numbers and let \( f \) be homogeneous of degree \( k \). Whenever the sum symbol is outside of \( f \) let it stand as summing with respect to the semiring: usual summing in the real case and maximum in the tropical case. Suppose \( f \) is monotone under quantum channels. Applying monotonicity to \( \tilde{\rho} := \sum_i p_i |i\rangle\langle i|_E \otimes \rho_i \) and \( \tilde{\sigma} := \sum_i p_i |i\rangle\langle i|_E \otimes \sigma_i \) under the partial trace \( \text{Tr}_E \), where \( \langle |i\rangle \rangle = 1 \) is an ONS in \( \mathcal{H}_E \) yields

\[
\sum \frac{k}{i} f(\rho_i, \sigma_i) = f\left( \sum p_i |i\rangle\langle i|_E \otimes \rho_i, \sum p_i |i\rangle\langle i|_E \otimes \sigma_i \right)
\geq f\left( \sum p_i \rho_i, \sum p_i \sigma_i \right),
\]

which translates to joint convexity when \( k = 1 \), i.e. in the real case and joint quasi-convexity, when \( k = 0 \), i.e. in the tropical case. Suppose now that \( f \) is jointly convex in the real case or jointly quasi-convex in the tropical case. Using Stinespring dilation \( \Phi(\cdot) = \text{Tr}_E V(\cdot) V^* \) with an isometry \( V : \mathcal{H} \rightarrow \mathcal{K} \otimes \mathcal{H}_E \), and writing the partial trace multiplied by the maximally mixed state as a convex combination of unitary conjugations (e.g., by the discrete Weyl unitaries):

\[
f(\Phi(\rho), \Phi(\sigma)) = \sum_{i=1}^{d_E} \frac{1}{d_E} f(\Phi(\rho), \Phi(\sigma))
= f\left( \sum \frac{1}{d_E} I_E \otimes \text{Tr}_E V(\rho) V^* \otimes I_E \otimes \text{Tr}_E V(\sigma) V^* \right)
= f\left( \sum_n \frac{1}{n} U_i V \rho V^* U_i^* \otimes \sum_n \frac{1}{n} U_i V \sigma V^* U_i^* \right)
\leq \frac{1}{n^k} \sum_{i=1}^{n^k} f(\rho, \sigma) = f(\rho, \sigma).
\]

Proposition 3.8: If functions of the form (9) and (10) satisfy the data processing inequality then the measure \( \mu \) is concentrated on one point.

Proof: By Lemma 3.7 we require functions of the form (9) and (10) to be jointly convex and jointly quasi-convex. In particular both family of functions needs to be jointly quasiconvex in the one dimensional special case. These functions are totally differentiable and if we restrict \( f \) to a line segment then having a zero directional derivative and negative second derivative would mean strict local maximum and would contradict quasi-convexity. Consider the general directional derivative of \( f \) at 1. The forms of \( f \) in (9) and (10) are differentiable and the derivative of the integrands are continuous on \( X \) and \( Y \) and thus bounded. Then by [9, Theorem 2.27] the differentiation and the integration commute and we get

\[
\frac{d}{ds} f(1 + s, 1_Y) \bigg|_{s=0} = \frac{d}{ds} \left[ \exp \int_X \ln(1 + s, \xi) \, d\mu \right] \bigg|_{s=0} = f(1 + s, 1_Y) \int_X \frac{\xi}{1 + s, \xi} \, d\mu \bigg|_{s=0} = \int_X \xi \, d\mu
\]

and

\[
\frac{d^2}{ds^2} f(1 + s, 1_Y) \bigg|_{s=0} = \frac{d^2}{ds^2} \left[ \exp \int_X \ln(1 + s, \xi) \, d\mu \right] \bigg|_{s=0} = f(1 + s, 1_Y) \int_X \left( \frac{\xi}{1 + s, \xi} \right)^2 \, d\mu \bigg|_{s=0} = \int_X \xi^2 \, d\mu.
\]

To get a contradiction with quasi-convexity, we need to find a continuous function \( \xi \) whose integral is zero and such that \( \xi^2 \) has nonzero integral, suppose that \( \mu \) is not concentrated on a point. To this end let \( x_1, x_2 \) be distinct points in the support of \( \mu \). Choose disjoint closed neighbourhoods \( A_1 \) and \( A_2 \) (possible since \( X \) is a compact Hausdorff space). By Urysohn’s lemma, there exist continuous functions \( \xi_1, \xi_2 : X \rightarrow [0, 1] \) such that \( \xi_1 \) is 1 on \( A_1 \) and 0 on \( A_2 \), and \( \xi_2 \) is 0 on \( A_1 \) and 1 on \( A_2 \). Let

\[
\xi = \left( \int_X \xi_2 \, d\mu \right) \xi_1 - \left( \int_X \xi_1 \, d\mu \right) \xi_2.
\]

By construction, the integral of \( \xi \) vanishes, while

\[
\int_X \xi^2 \, d\mu \geq \mu(A_1) \left( \int_X \xi_2 \, d\mu \right)^2 > 0.
\]

□

Let \( \alpha \) := \( \mu(X) \) and \( \gamma := \frac{d}{d\mu(X)} \). Taking Propositions 3.6 and 3.8 into account an element of the 1-test spectrum must have one of the following forms.

\[
f \left( \bigoplus_{i=1}^d p_i, \bigoplus_{i=1}^d q_i \right) = \sum_{i=1}^d p_i(x)^\alpha \exp \left[ (1 - \alpha) \int_X \ln q_i \, d\gamma \right]
\]

(17)
if $f$ goes into the reals and
\[
f \left( \bigoplus_{i=1}^{d} p_i, \bigoplus_{i=1}^{d} q_i \right) = \max_{i \in [d]} p_i(x) \exp \left[ -\int_Y \ln q_i \, d\gamma \right], \tag{18}
\]
if $f$ goes into the tropicals, where we further took into account that $\int f = 2$ is required for elements of the spectrum going into the tropical numbers. (17) is characterized by the point $x$, the weight $\alpha \geq 1$ and the probability measure $\gamma$, while (18) is characterized by the point $x$ and the probability measure $\gamma$.

**Proposition 3.9:** (17) and (18) satisfy the data processing inequality.

**Proof:** By Lemma 3.7 the functions (17) are monotone decreasing under stochastic inequality, they are monotone decreasing under stochastic convexity in the 1-dimensional case. Joint convexity of (17) is characterized by the point $\gamma$. Theorem 3.10: (17) and (18) satisfy the data processing inequality.

Now for a function $f$ of the form (17) consider
\[
g_\alpha \left( \bigoplus_{i=1}^{d} p_i, \bigoplus_{i=1}^{d} q_i \right) := \left( \sum_{i=1}^{d} f(p_i, q_i) \right)^{\frac{1}{\alpha}} = \left( \sum_{i=1}^{d} p_i(x) \exp \left[ \left(1 - \frac{1}{\alpha}\right) \int_Y \ln q_i \, d\gamma \right] \right)^{\frac{1}{\alpha}} \tag{21}
\]
g_\alpha then satisfies the data processing inequality and preserves this property in the $\alpha \rightarrow \infty$ limit. However showing that functions of the form (18) satisfy the data processing inequality too.

Note that functions of the form (17) can be viewed as the $\alpha$-Rényi quasi-divergences of a positive vector $p$ and some pointwise geometric mean of positive vectors $q_i$. What we used in the last proof is that the max divergence, i.e. functions in (18), can be given as a $\alpha \rightarrow \infty$ limit of Rényi divergences.

**Theorem 3.10:** $\text{TS}_\text{per}(S_{X,Y})$ consists of the functions
\[
f_{\alpha,x,\gamma} \left( \bigoplus_{i=1}^{d} p_i, \bigoplus_{i=1}^{d} q_i \right) = \sum_{i=1}^{d} p_i(x)^\alpha \exp \left[ \left(1 - \frac{1}{\alpha}\right) \int_Y \ln q_i \, d\gamma \right], \tag{23}
\]
where $\alpha \geq 1$, $x \in X$ and $\gamma$ is a probability measure on $Y$, if $f$ is real-valued and
\[
f_{x,\gamma} \left( \bigoplus_{i=1}^{d} p_i, \bigoplus_{i=1}^{d} q_i \right) = \max_{i \in [d]} p_i(x) \exp \left[ -\int_Y \ln q_i \, d\gamma \right], \tag{24}
\]
where, $x \in X$ and $\gamma$ is a probability measure on $Y$, if $f$ is tropical real-valued.

**Proof:** Follows from Propositions 3.6, 3.8 and 3.9.

We will be refering to elements of $\text{TS}_\text{per}(S_{X,Y})$ as $f_{(\alpha),x,\gamma}$ or $g_{(\alpha),x,\gamma}$, if we want distinct multiple elements of the spectrum, signifying the characterizing quantities $x, \gamma$ and possibly $\alpha$ as well, the same time.

If $X$ and $Y$ are one-point spaces, then the elements of the spectrum are exponentiated Rényi divergences. With normalized arguments, these are bounded from below by 1, with equality iff the two arguments are the same. We conclude this section with an analogous statement for the spectral points on the classical semiring for general spaces.

**Proposition 3.11:** Let $\alpha > 1$, $x \in X$ and $\gamma$ a probability measure on $Y$. Suppose that $(p, q)$ is normalized, i.e. $\sum_{i=1}^{d} p_i(x') = 1$ and $\sum_{i=1}^{d} q_i(y') = 1$ for all $x' \in X$ and $y' \in Y$. Then $f_{\alpha,\gamma}(p, q) \geq 1$ with equality iff $p(x) = q(y)$ for all $y \in \text{supp} \gamma$. Similarly, $f_{\alpha,\gamma}(p, q) \geq 1$ with equality iff $p(x) = q(y)$ for all $y \in \text{supp} \gamma$. 

Proof: The inequality follows from monotonicity under the stochastic map \( p \mapsto \sum_{i=1}^{d} p_i \) and the normalization \( f(1_X, 1_Y) = 1 \). Let 
\[
\tilde{q}_i = \exp \int_Y \ln q_i \, d\gamma,
\]
so that
\[
f_{\alpha,x,\gamma}(p,q) = \sum_{i=1}^{d} p_i(x)^{\alpha} \tilde{q}_i^{1-\alpha} = 2^{(\alpha-1) D_{\gamma}(p||\tilde{q})}.
\]
By the Jensen inequality,
\[
\sum_{i=1}^{d} \tilde{q}_i \leq \sum_{i=1}^{d} \int_Y q_i \, d\gamma = \int_Y \sum_{i=1}^{d} q_i \, d\gamma = 1.
\]
Since the Rényi divergence is anti-monotone in the second argument and strictly positive when the arguments are distinct probability distributions, the equality \( f_{\alpha,x,\gamma}(p,q) = 1 \) is equivalent to \( p(x) = \tilde{q} \). This means that the Jensen inequality holds with equality which, by strict concavity of the logarithm, implies \( q(y) = \tilde{q} = p \) for all \( y \in \text{supp}\gamma \).

Similarly, if \( f_{x,\gamma}(p,q) = 1 \) then for all \( i \) we have \( p_i(x) \leq \tilde{q}_i \) but \( \sum_{i=1}^{d} p_i(x) = 1 \geq \sum_{i=1}^{d} \tilde{q}_i \), which implies \( p(x) = \tilde{q} \). \qed

C. Quantum Extensions

Proposition 3.12: Suppose that \( \tilde{f} \) is an element of the spectrum. Let \( f_{\alpha,x,\gamma} \) be \( f \) constrained on the classical semiring according to (23) or (24) in Theorem 3.10. Then for any \( \rho, \rho' : X \rightarrow B(H)_{++} \) such that \( \rho(x) = \rho'(x) \) and for any \( \sigma, \sigma' : Y \rightarrow B(H)_{++} \) such that \( \sigma(y) = \sigma'(y) \) for all \( y \in \text{supp}\gamma \), it follows that \( \tilde{f}(\rho, \sigma) = f(\rho', \sigma') \).

Proof: Let
\[
\begin{align*}
c_1(x) &= \min \{ t : \rho'(x) \leq t \rho(x) \} \\
d_1(x) &= \max \{ t : \sigma'(y) \geq t \sigma(y) \} \\
c_2(x) &= \min \{ t : \rho(x) \leq t \rho'(x) \} \\
d_2(x) &= \max \{ t : \sigma(y) \geq t \sigma'(y) \}.
\end{align*}
\]
Then \( c_1, c_2, d_1, d_2 \) are strictly positive continuous functions on \( X \) and \( Y \), respectively, and
\[
(\rho', \sigma') \not\leq (c_1, d_1)(\rho, \sigma) \not\leq (c_2, d_2)(\rho', \sigma).
\]
\[
c_1(x) = c_2(x) = 1 \text{ and } d_1(y) = d_2(y) = 1 \text{ for all } y \in \text{supp}\gamma,
\]
\[
(c_1, d_1) \text{ and } (c_2, d_2) \text{ are classical pairs and thus } f(c_1, d_1) = f_{\alpha,x,\gamma}(c_1, d_1) = 1 \text{ and } f = f_{\alpha,x,\gamma}(c_2, d_2) = 1.
\]
Applying now \( \tilde{f} \) to all three parts of the above inequality yields
\[
\tilde{f}(\rho', \sigma) \leq \tilde{f}(\rho, \sigma) \leq \tilde{f}(\rho', \sigma).
\]
\qed

Proposition 3.13: Let \( (\rho, \sigma) \in S_{X,Y} \) and let \( \tilde{f} \) be a real element of the spectrum and let \( f_{\alpha,x,\gamma} \) be its restriction on the classical subsemiring. Let \( \tilde{g} \) be a tropical element of the spectrum and let \( g_{x,\gamma} \) be its restriction onto the classical subsemiring. If \( [\sigma(y), \sigma(y')] = 0 \) \( \forall y, y' \in Y \) then
\[
\tilde{f}(\rho, \sigma) = \tilde{Q}_\alpha \left( \rho(x) \left( \exp \int_Y \ln \sigma \, d\gamma \right) \right).
\]

and
\[
\tilde{g}(\rho, \sigma) = \left\| \rho \left( \exp \int_Y \ln \sigma \, d\gamma \right)^{-1} \rho \right\|_\infty (35)
\]
Proof: There is a positive definite operator \( \bar{\sigma} \) such that the eigenbasis of \( \bar{\sigma} \) simultaneously diagonalizes all \( \sigma(y) \). Let \( \mathcal{P}_{\bar{\sigma}_n} \) denote the pinching by \( \bar{\sigma}^{\otimes n} \), then \( \mathcal{P}_{\bar{\sigma}_n} \) leaves \( \sigma^{\otimes n}(y) \) invariant for all \( y \in Y \). It follows that
\[
(\rho, \sigma)^n \cong (\mathcal{P}_{\bar{\sigma}_n} (\rho^{\otimes n}), \mathcal{P}_{\bar{\sigma}_n} (\sigma^{\otimes n}))
\]
\[
\cong (\mathcal{P}_{\bar{\sigma}_n} (\rho^{\otimes n}), \sigma^{\otimes n})
\]
\[
\cong \left( \frac{1}{\text{poly}(n)}, 1 \right) (\rho^{\otimes n}, \sigma^{\otimes n}), (36)
\]
where \( \text{poly}(n) \) is a polynomial of \( n \) and we used that any pinching is a completely positive trace preserving map and the pinching inequality:
\[
\rho^{\otimes n} \leq [\text{spec}(\bar{\sigma}^{\otimes n})] \mathcal{P}_{\bar{\sigma}_n} (\rho^{\otimes n}) = \text{poly}(n) \mathcal{P}_{\bar{\sigma}_n} (\rho^{\otimes n}). (37)
\]
We have shown in Proposition 3.12 that \( \tilde{f} \) and \( \tilde{g} \) only depend on one point of \( \rho \) apart from \( \sigma \), but after the pinching all these operators commute and thus we are evaluating \( \tilde{f} \) and \( \tilde{g} \) on the classical subsemiring, where \( \tilde{f} \) and \( \tilde{g} \) are determined by Theorem 3.10. Applying \( \tilde{f} \) to all three parts and taking the \( n \)-th root yields
\[
\tilde{f}(\rho, \sigma) \geq \sqrt[n]{f_{\alpha,x,\gamma}(\mathcal{P}_{\bar{\sigma}_n} (\rho^{\otimes n}(x)), \sigma^{\otimes n})}
\]
\[
= \sqrt[n]{\text{Tr}(\mathcal{P}_{\bar{\sigma}_n} (\rho^{\otimes n})))^\alpha \left( \exp \int_Y \ln \sigma^{\otimes n} \, d\gamma \right)^{1-\alpha}}
\]
\[
= \sqrt[n]{\text{Tr}(\mathcal{P}_{\bar{\sigma}_n} (\rho^{\otimes n}(x))))^\alpha \left( \exp \int_Y \ln \sigma \, d\gamma \right)^{1-\alpha}}
\]
\[
\geq \sqrt[n]{\frac{1}{\text{poly}(n)} \tilde{f}(\rho, \sigma)}. (38)
\]
Taking the limit \( n \rightarrow +\infty \) gives us
\[
\tilde{f}(\rho(x), \sigma) \geq \tilde{Q}_\alpha \left( \rho \left( \exp \int_Y \ln \sigma \, d\gamma \right) \right) \geq \tilde{f}(\rho, \sigma), (39)
\]
where we refer to [15, Proposition 4.12.] (see also [7, Theorem 4.4.]) in taking the limit of the middle term.

Now applying \( \tilde{g} \) to all three parts yields
\[
\tilde{g}(\rho, \sigma) \geq \sqrt[n]{g_{x,\gamma}(\mathcal{P}_{\bar{\sigma}_n} (\rho^{\otimes n}(x)), \sigma^{\otimes n})}
\]
\[
= \sqrt[n]{\left( \mathcal{P}_{\bar{\sigma}_n} (\rho^{\otimes n}(x))) \left( \exp \int_Y \ln \sigma^{\otimes n} \, d\gamma \right)^{-1} \right)^{-1}}
\]
\[
= \sqrt[n]{\left( \mathcal{P}_{\bar{\sigma}_n} (\rho^{\otimes n}(x))) \left( \exp \int_Y \ln \sigma \, d\gamma \right)^{-1} \right)^{-1}}
\]
\[
\geq \sqrt[n]{\frac{1}{\text{poly}(n)} \tilde{g}(\rho, \sigma)}. (40)
\]
Taking the limit \( n \rightarrow +\infty \) gives us
\[
\tilde{g}(\rho, \sigma) \geq \left\| \rho \left( \exp \int_Y \ln \sigma \, d\gamma \right)^{-1} \rho \right\|_\infty \tilde{g}(\rho, \sigma), (41)
\]
where we refer to [16] and [15, Section 4.2.4] in taking the limit of the middle term.

**Corollary 3.14:** Suppose that \( \hat{f} \) is an element of the spectrum and let \( f_{\alpha, x, \gamma} \) be its restriction to the classical semiring. If \( \sigma(y) = \sigma_0 \) for all \( y \in \text{supp} \gamma \), \( \hat{f}(\rho, \sigma) = Q_\alpha(\rho(x)||\sigma_0) \).

**Proof:** By Proposition 3.12, \( \hat{f}(\rho, \sigma) = \epsilon \int f(\rho, \sigma) \) where \( \sigma(y) := \sigma_0 \) for all \( y \in Y \) is constant, hence commuting. The statement follows by Proposition 3.13. \( \square \)

We prove an analogue of Proposition 3.11 for the quantum extensions:

**Proposition 3.15:** Suppose that \( \tilde{f} \) is an element of the spectrum and let \( f_{\alpha, x, \gamma} \) be its restriction to the classical semiring. Suppose that \( (\rho, \sigma) \) is normalized, i.e. \( \text{Tr} \rho(x') = 1 \) and \( \text{Tr} \sigma(y') \) for all \( x' \in X \) and \( y' \in Y \). Then \( f_{\alpha, x, \gamma}(\rho, \sigma) \geq 1 \) with equality iff \( \rho(x) = \sigma(y) \) for all \( y \in \text{supp} \gamma \). Similarly, if \( \tilde{f} \) is tropical and its restriction is \( f_{\alpha, x, \gamma} \), then \( \hat{f}(\rho, \sigma) \geq 1 \) with equality iff \( \rho(x) = \sigma(y) \) for all \( y \in \text{supp} \gamma \).

**Proof:** As in Proposition 3.11, the inequality follows by applying monotonicity under the trace map. The “if” part is a consequence of Corollary 3.14. For the “only if” direction, suppose that \( \rho(x) \neq \sigma(y) \) for some \( y \in \text{supp} \gamma \). Let \( F \) be a measurement channel (i.e. a completely positive trace preserving map with classical output) such that \( F(\rho(x)) \neq F(\sigma(y)) \). Then, \( \rho(x) \succ (F(\rho), F(\sigma)) \), so by monotonicity and Proposition 3.11 we have the strict inequality. \( \square \)

The expression \( \exp \int_Y \ln \sigma \, d\gamma \) can be viewed as a continuous analogue of a weighted geometric mean of positive numbers. The form of the spectrum elements in the case of commuting \( \sigma \) suggests looking for fully quantum generalizations of the form \( f(\rho, \sigma) = Q_\alpha(\rho(x)||M(\sigma)) \), where \( \alpha \geq 1 \), \( x \in X \) and \( M \) is some noncommutative version of the weighted geometric mean. We note that multi-state Rényi divergences constructed using Kubo–Ando means have been studied in [17].

In the following definition we make the requirements more precise and also more flexible by allowing the result to be also a continuous family of positive operators. The advantages of this formulation are that a simple composition property conveniently allows for the construction of many examples, and that these objects also give rise to monotone homomorphisms between different semirings \( S_{X,Y} \to S_{X',Y'} \). We equip \( C(Y, B(\mathcal{H}_{++})) \) with the pointwise semidefinite partial order.

**Definition 3.16:** Let \( Y \) and \( Y' \) be nonempty compact spaces. A family of continuous geometric means indexed by \( Y' \) is a collection of maps \( M : C(Y, B(\mathcal{H}_{++})) \to C(Y', B(\mathcal{H}_{++})) \) which are unitary equivariant (i.e. if \( \sigma' = U\sigma U^* \) for some unitary \( U : \mathcal{H} \to \mathcal{H}' \), then \( M(\sigma') = UM(\sigma)U^* \) satisfying the following properties for all \( \sigma \in C(Y, B(\mathcal{H}_{++})) \), \( \sigma' \in C(Y', B(\mathcal{H}'_{++})) \) and \( \lambda \in \mathbb{R}_{\geq 0} \):

(i) \( M(\sigma \oplus \sigma') = M(\sigma) \oplus M(\sigma') \),
(ii) \( M(\sigma \otimes \sigma') = M(\sigma) \otimes M(\sigma') \),
(iii) \( M(\lambda\sigma) = \lambda M(\sigma) \),
(iv) if \( \sigma \leq \sigma' \), then \( M(\sigma) \leq M(\sigma') \),
(v) \( M \) is concave.

The set of families of geometric means is denoted by \( G(Y, Y') \). When \( Y' \) is a one-point space, we identify \( C(Y', B(\mathcal{H}_{++})) \) with \( B(\mathcal{H}_{++}) \) and write \( G(Y) \) instead of \( G(Y, Y') \).

Because of unitary equivariance, it is sufficient to specify a family of means for families of operators on \( \mathbb{C}^d \) for all \( d \).

We note that the properties of families of geometric means that we consider imply that they can be extended to positive semidefinite operators (by \( \lim_{n \to 0} M(\sigma + c_1I \otimes I_{\mathcal{H}}) \)), and that they are increasing under completely positive trace-preserving maps in the sense that if \( M \in G(Y, Y') \), \( \sigma \in C(Y, B(\mathcal{H}_{++})) \) and \( T : B(\mathcal{H}) \to B(\mathcal{H}') \) is a completely positive trace-preserving map, then \( M(T(\sigma)) \geq T(M(\sigma)) \). To see this, consider the Stinespring dilation of \( T \), and write the partial trace over the environment, followed by tensoring with the maximally mixed state, as a convex combination of unitary conjugations. On numbers (\( \mathcal{H} = \mathbb{C} \)) every \( M \in G(Y) \) has the form \( M(\sigma) = \exp \int_Y \ln \sigma \, d\gamma \) for some probability measure \( \gamma \).

An example of an element of \( G(\{1, 2\}) \) is given by

\[
\sigma(1)\sigma(2) = \sigma(1)^{1/2}(\sigma(1)^{-1/2}\sigma(2)\sigma(1)^{-1/2})^{1/2}\sigma(1)^{1/2},
\]

the (unweighted) geometric mean of a pair of matrices, introduced in [6] and put in a general context by Kubo and Ando [18]. Extensions to several variables have been constructed by building on the bivariate geometric mean or generalizing characterizations thereof (see e.g. [19], [20], [21], [22], [23]), and also studied from an axiomatic point of view [24]. Examples of elements of \( G(\{1, 2, \ldots, n\}) \) are the Karcher means [22], [23]. Since our axioms in Definition 3.16 are directly motivated by their use in constructing monotone homomorphisms, they differ from the ones considered in the literature on geometric means of matrices, in particular in their emphasis on relating the means of matrices of different sizes (by the tensor product or the direct sum). In addition, we need to consider every possible weighting of the arguments, therefore symmetry is not a relevant property in our problem.

The following proposition lists basic constructions that allow one to exhibit many elements of the form \( G(Y) \). Geometric means that can be obtained in this way include the Ando–Li–Mathias mean [24] and the Bini–Meini–Polini means [25].

**Proposition 3.17:** Let \( Y, Y', Y'' \) be nonempty compact spaces.

(i) If \( M \in G(Y, Y') \) and \( N \in G(Y', Y'') \), then \( N \circ M \in G(Y, Y'') \) (here the composition \( N \circ M \) is understood separately for all \( \mathcal{H} \)).

(ii) If \( f : Y' \to Y \) is a continuous map, then \( M(\sigma) = \sigma \circ f \) defines an element of \( G(Y, Y') \).

(iii) \( M(\sigma) = \sigma' \) with

\[
\sigma'(y_1, y_2, \gamma) := \sigma(y_1)\#\sigma(y_2) = \sigma(y_1)^{1/2} \left( \sigma(y_1)^{-1/2}\sigma(y_2)\sigma(y_1)^{-1/2} \right)^\gamma \sigma(y_1)^{1/2}
\]

defines an element of \( G(Y, Y \times Y \times [0, 1]) \).

(iv) \( G(Y) \) is compact with respect to the pointwise convergence (i.e. convergence of \( i \mapsto M_i(\sigma) \) for all \( \sigma \)).

**Proof:** (i): The composition is clearly additive, multiplicative, homogeneous and monotone. For concavity, we apply \( N \) to the inequality \( M(\lambda\sigma + (1 - \lambda)\sigma') \geq \lambda M(\sigma) + (1 - \lambda)M(\sigma') \), which expresses the concavity of \( M \), using that \( N \)
is monotone:
\[(N \circ M)(\lambda\sigma + (1 - \lambda)\sigma')
= N(M(\lambda\sigma + (1 - \lambda)\sigma'))
\geq N(\lambda M(\sigma) + (1 - \lambda)M(\sigma'))
\geq \lambda N(M(\sigma)) + (1 - \lambda)N(M(\sigma')).\] (44)

(ii): \(M\) is clearly additive, multiplicative, homogeneous, monotone and affine (hence concave).

(iii): \(\sigma'\) is clearly continuous for every continuous \(\sigma\).
The geometric mean is clearly additive, multiplicative and homogeneous. For concavity and monotonicity see [18] and [26, Theorem 37.1].

(iv): If \(\sigma \in C(Y, B(\mathcal{H})_{++})\), then there exist constants \(c_1, c_2 > 0\) such that for all \(y \in Y\) the inequalities \(c_1 I \leq \sigma(y) \leq c_2 I\) hold. By the direct sum and monotonicity properties, it follows that \(c_1 I \leq M(\sigma) \leq c_2 I\) for every \(M \in \mathcal{G}(Y)\). The interval \([c_1 I, c_2 I] = \{A \in B(Cd)_{++} | c_1 I \leq A \leq c_2 I\}\) is compact for every \(d\), therefore the evaluations embed \(\mathcal{G}(Y)\) into the compact space \(\prod_{d \in \mathbb{N}} \prod_{y \in C(Y, B(Cd)_{++})} [c_1 I, c_2 I]\). The conditions defining \(\mathcal{G}\) are closed (equalities and non-strict inequalities with respect to the semidefinite partial order), therefore the image under the embedding is closed. \(\square\)

**Proposition 3.18:** Let \(X, Y, X', Y'\) be nonempty compact spaces, \(M \in \mathcal{G}(Y, Y')\) and \(f : X' \to X\) continuous. Then the map \((\rho, \sigma) \mapsto (\rho \circ f, M(\sigma))\) is a monotone semiring homomorphism.

**Proof:** This map is by definition additive and multiplicative. We have yet to show monotonicity. Suppose that the completely positive trace-nonincreasing map \(T\) realizes \((\rho, \sigma) \geq (\rho', \sigma')\). Then \(T(\rho) \geq \rho'\) and \(T(\sigma) \leq \sigma'\). From monotonicity of \(M\) in its variables under completely positive trace-nonincreasing maps:
\[T(M(\sigma)) \leq M(T(\sigma)) \leq M(\sigma').\] (45)
This yields \((\rho, M(\sigma)) \geq (\rho', M(\sigma'))\) by the same map \(T\). \(\square\)

**Theorem 3.19:** Let \(X, Y\) be nonempty compact spaces. For all \(\alpha \geq 1, x \in X\) and \(M \in \mathcal{G}(Y)\) the functional
\[f(\rho, \sigma) = \tilde{Q}_\alpha(\rho(x)||M(\sigma))\] (46)
is an element of the real part of the spectrum, and
\[f(\rho, \sigma) = \left\|M(\sigma)^{-1/2}\rho(x)M(\sigma)^{-1/2}\right\|_\infty\] (47)
is an element of the tropical part.

**Proof:** By Proposition 3.18, the map \((\rho, \sigma) \mapsto (\rho(x), M(\sigma))\) determines a monotone semiring homomorphism from \(S_{X,Y}\) to \(S_{1,1}\), where 1 is a one-point space. On \(S_{1,1}\) the functionals \(f_\alpha(\rho, \sigma) = \tilde{Q}_\alpha(\rho||\sigma)\) are in the real spectrum and \((\rho, \sigma) \mapsto \left\|\sigma^{-1/2}\rho\sigma^{-1/2}\right\|_\infty\) is in the tropical spectrum, as follows from Proposition 3.13 (see also [7], [8]). Therefore (46) and (47) are compositions of monotone semiring homomorphisms, which implies that they are points in the real (respectively tropical) part of the spectrum. \(\square\)

In the special case when \(X\) is a one-point space and \(Y\) has two elements, the \(\alpha \to 1\) limit has recently found application in composite binary state discrimination [27].

While Theorem 3.19 identifies a vast collection of elements of \(\text{TSpers}(S_{X,Y})\), it still provides an incomplete picture of the spectrum. Our results highlight several open problems:
- Is every real spectral point of the form \(f(\rho, \sigma) = \tilde{Q}_\alpha(\rho(x)||P(\sigma))\) for some map \(P : C(Y, B(\mathcal{H})_{++}) \to B(\mathcal{H})_{++}\) (defined in a consistent way for all \(\mathcal{H}\))?
- Assuming that a real spectral point does have the form \(f(\rho, \sigma) = \tilde{Q}_\alpha(\rho(x)||P(\sigma))\), is \(P\) necessarily an element of \(\mathcal{G}(Y)\)?
- Classify the elements of \(\mathcal{G}(Y)\).

The analogous questions for tropical points are also interesting.

**IV. APPLICATIONS**

A. Equivariant Relative Submajorization

In this section we consider pairs of operators on a representation space of some fixed group, and a variant of relative sub-majorization that takes into account the group actions. Let \(G\) be a topological group. Let \(\pi, \pi'\) be finite dimensional unitary representations of \(G\) on \(\mathcal{H}\) and \(\mathcal{H}'\), respectively. Suppose that \((\rho_0, \sigma_0) \in B(\mathcal{H})_{++}\) and \((\rho_0', \sigma_0') \in B(\mathcal{H}')_{++}\). We say that \((\pi, \rho_0, \sigma_0)\) equivariantly relatively submajorizes \((\pi', \rho_0', \sigma_0')\) if there exists a completely positive trace-nonincreasing map \(T : B(\mathcal{H}) \to B(\mathcal{H}')\) such that
\[T(\rho_0) \geq \rho_0'\] \[T(\sigma_0) \leq \sigma_0'\] \[\forall g \in G \forall \pi \in B(\mathcal{H}) : T(\pi(g)A\pi(g)^*\pi'(g)^*) = \pi'(g)T(A)\pi'(g)^*\] (50)
On these triples the direct sum and tensor product (of representations and of operators) give binary operations that are compatible with equivariant relative submajorization.

It will be convenient to restrict to compact groups \(G\), and it can be done without loss of generality for the following reason. Consider the closure \(K\) of \(\{(\pi(g), \pi'(g))\}g \in G\) \(\subseteq \mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H}')\). This is a compact group (in fact, a Lie group), the map \(g \mapsto (\pi(g), \pi'(g))\) is a homomorphism and the representations \(\pi, \pi'\) of \(G\) extend to representations of \(K\) (namely the first and second projections provide the required homomorphisms). By continuity, a map \(T : B(\mathcal{H}) \to B(\mathcal{H}')\) is \(K\)-equivariant iff it is \(K\)-equivariant. Therefore the condition for \((\pi, \rho_0, \sigma_0) \geq (\pi', \rho_0', \sigma_0')\) can be formulated in terms of the compact group \(K\) instead of \(G\). Note that in this case the compact group in general depends on the specific pair of triples to be compared (through the representations), which may not always be desirable. Alternatively, one may construct \(K\) in a universal way, by taking the Bohr compactification of \(G\).

Recall that the Bohr compactification of a topological group \(G\) is a compact Hausdorff topological group \(b(G)\) together with a continuous homomorphism \(b : G \to b(G)\) that is universal in the sense that every continuous homomorphism from \(G\) into a compact group factors through \(b\) in a unique way. Every topological group has an essentially unique Bohr compactification. We can apply the universal property to the homomorphisms \(\pi : G \to \mathcal{U}(\mathcal{H})\) to get a representation \(b(\pi) : b(G) \to \mathcal{U}(\mathcal{H})\). Thus, instead of each triple \((\pi, \rho_0, \sigma_0)\) we may consider the modified triple \((b(\pi), \rho_0, \sigma_0)\). For notational
simplicity, from now on we will assume that $G$ itself is a compact Hausdorff group.

We now show how to map the triples $(\pi, \rho_0, \sigma_0)$ to pairs of families in such a way that the operations are preserved and equivariant relative submajorization translates to the relative submajorization of the families. In this way a triple $(\pi, \rho_0, \sigma_0)$ gives rise to the following pair of families, parametrized by $G$:

$$\rho(g) = \pi(g)\rho_0\pi(g)^*, \quad \sigma(g) = \pi(g)\sigma_0\pi(g)^*.$$  \hfill (51)

$(\rho, \sigma)$ determines an element of $S_{G,G}$, and this element remains the same if we replace the triple $(\pi, \rho_0, \sigma_0)$ by a unitary equivalent one. This map clearly respects the sum and product operations.

**Example 4.1:**

(i) Let $G = \{0,1\}$ with addition modulo 2 as the group operation. A representation $\pi$ of $G$ on $\mathcal{H}$ is determined by the image of 1, which is a unitary $Z$ satisfying $Z^2 = I$. The triple $(\pi, \rho_0, \sigma_0)$ then gives rise to the element $((\rho, Z\rho Z), (\sigma, Z\sigma Z))$.

(ii) Let $G = U(1)$. A representation $\pi$ of $G$ is given by the infinitesimal generator $A \in B_2(\mathcal{H})$ as $e^{itA}$, which is well-defined if $A$ has integer spectrum. The element corresponding to $(\pi, \rho_0, \sigma_0)$ may be identified with $((e^{itA}\rho Z), (e^{itA}\sigma Z))$, where $0 \leq t \leq 2\pi$ and both represent the identity element of $U(1)$.

Suppose that $(\pi, \rho_0, \sigma_0)$ equivariantly relatively submajorizes $(\pi', \rho_0', \sigma_0')$, and let $T$ be an equivariantly relatively submajorizing triple $T(\rho_0) \geq \rho_0'$ and $T(\sigma_0) \leq \sigma_0'$. Consider the corresponding elements $(\sigma, \rho)$ and $(\sigma', \rho')$ of $S_{G,G}$. Then for all $g \in G$ the inequality

$$T(\rho(g)) = T(\pi(g)\rho_0\pi(g)^*) = \pi(g)T(\rho_0) \pi'(g)^* \geq T(\sigma_0) \leq \sigma_0' = \sigma(g)' \pi'(g)^* = \rho'(g)^* T(\sigma(g)) \leq \sigma'(g).$$  \hfill (53)

holds and similarly $T(\sigma(g)) \leq \sigma'(g)$. This means that $(\rho, \sigma)$ $\geq (\rho', \sigma')$ holds.

Conversely, suppose that $(\rho, \sigma) \geq (\rho', \sigma')$ is true in $S_{G,G}$ for the families defined above. This means that there exists a (not necessarily equivariant) completely positive trace-nonincreasing map $T_0$ such that for all $g \in G$ the inequalities $T_0(\rho(g)) \geq \rho'(g)$ and $T_0(\sigma(g)) \leq \sigma'(g)$ hold. We construct an equivariant map $T$ by averaging:

$$T(X) = \int_G \pi'(g)^* T_0(\pi(g)X\pi(g)^*) \pi'(g) \, d\mu(g),$$  \hfill (54)

where $\mu$ is the Haar probability measure on $G$. Then $T$ is $G$-equivariant and in addition

$$T(\rho_0) = \int_G \pi'(g)^* T_0(\pi(g)\rho_0\pi(g)^*) \pi'(g) \, d\mu(g) \geq \int_G \pi'(g)^* \rho'(g) \pi'(g) \, d\mu(g) = \rho_0' \hfill (55)$$

and similarly $T(\sigma_0) \leq \sigma_0'$

Note that even though the map $(\pi, \rho_0, \sigma_0) \mapsto (\rho, \sigma)$ is order-preserving and order-reflecting, it is in general not injective (on equivalence classes). Now we can apply our result on general pairs of families to the question of asymptotic equivariant relative submajorization.

**Theorem 4.2:** Let $G$ be a topological group and consider the triples $(\pi, \rho_0, \sigma_0)$ and $(\pi', \rho_0', \sigma_0')$, where $\pi$ is a unitary representation of $G$ on $\mathcal{H}$, $\rho_0, \sigma_0$ are positive definite operators on $\mathcal{H}$ and similarly for $\pi', \rho_0', \sigma_0'$ on $\mathcal{H}'$. The following are equivalent:

(i) there exists a sequence of $G$-equivariant completely positive trace-nonincreasing operations $T_n : B(\mathcal{H} \otimes \mathcal{H}) \to B(\mathcal{H}' \otimes \mathcal{H})$ such that for all $x \in X$ the inequalities

$$T_n(\rho_0^{\otimes n}) \geq 2^{-\alpha(n)} \rho_0^{\otimes n} \hfill (56)$$

$$T_n(\sigma_0^{\otimes n}) \leq \sigma_0^{\otimes n} \hfill (57)$$

hold, with the $\alpha(n)$ uniform in $x$,

(ii) the following inequality holds for all $f \in TSper_{1}(S_{G,G})$

$$f((\pi(g)\rho_0\pi(g)^*)_{g \in G}, (\pi(g)\sigma_0\pi(g)^*)_{g \in G}) \geq f((\pi(g)\rho_0'\pi(g)^*)_{g \in G}, (\pi(g)\sigma_0'\pi(g)^*)_{g \in G}). \quad (58)$$

**Remark 4.3:** Compact groups include many familiar groups, in particular finite groups and compact Lie groups such as $U(1)$ or $SU(2)$. Theorem 4.2 can be applied to any of these, but due to our incomplete knowledge of the test-spectrum, in practice it only gives necessary conditions for catalytic or asymptotic transformations (it would become sufficient if we could evaluate all elements of the real and tropical spectrum). Under the additional assumption that the orbit of $\sigma$ consists of commuting operators, Corollary 4.5 below gives an explicit necessary and sufficient condition.

We note that in general many elements of $TSper_{1}(S_{G,G})$ collapse to the same function when restricted to pairs of the form $((\pi(g)\rho_0\pi(g)^*)_{g \in G}, (\pi(g)\sigma_0\pi(g)^*)_{g \in G})$. The reason is that left translations of $G$ give rise to automorphisms of $S_{G,G}$ of the form $(\rho, \sigma) \mapsto (\rho \circ L_{h}, \sigma \circ L_{h})$ (where $h \in G$ and $L_{h} : G \to G$ is the map $L_{h}(g) = hg$), which in turn induce nontrivial automorphisms of $TSper_{1}(S_{G,G})$, while the equivalence class of $((\pi(g)\rho_0\pi(g)^*)_{g \in G}, (\pi(g)\sigma_0\pi(g)^*)_{g \in G})$ remains unchanged by these transformations. This can be seen explicitly on the subemerging of pairs with commuting $\sigma$, where the precise form of spectral points is known: if $\rho(g) = \pi(g)\rho_0\pi(g)^*$ and $\sigma(g) = \pi(g)\sigma_0\pi(g)^*$ such that $\sigma(g)|\sigma(e) = \sigma(e)\sigma(g)^*$ for all $g \in G$, then

$$f_{\alpha,\gamma}(\rho \circ L_{h}, \sigma \circ L_{h}) \hfill (59)$$

The first line of this calculation also shows that

$$f_{\alpha,\gamma}(\rho \circ L_{h}, \sigma \circ L_{h}) = f_{\alpha,\gamma}(\rho, \sigma), \hfill (60)$$

In particular, $f_{\alpha,\gamma}$ and $f_{\alpha,\gamma}^{-1}(\rho, \sigma)$ coincide on these elements.
Example 4.4:

(i) Let $G = \{0, 1\}$ with addition modulo 2. $\text{TSper}_1(S_{G,G})$ contains (at least) the maps $(\rho_0, \rho_1), (\sigma_0, \sigma_1)$ and $(\rho_1, \rho_1), (\sigma_0, \sigma_1)$.

(ii) Let $G = U(1)$, and consider the representation $\pi(e^{\i tA}) = e^{\i tA}$. An example of a geometric mean in $G(U(1))$ is the map $M_{t_1, t_2, t_3}(\sigma) = (\sigma_{t_1})\#(\sigma_{t_2})\#(\sigma_{t_3})$ for some $t_1, t_2, t_3 \in [0, 2\pi]$ (for notational simplicity, we identify $[0, 2\pi)$ with $U(1)$, see 4.1). For any $t$ the maps $(\rho, \sigma) \mapsto Q_\alpha(\rho(t)|M_{t_1, t_2, t_3}(\sigma))$ are in $\text{TSper}_1(S_{U(1), U(1)})$. When evaluated on an element of the form $\rho(t) = e^{\i tA}\rho(0)e^{-\i tA}$, $\sigma(t) = e^{\i tA}\sigma(0)e^{-\i tA}$, it gives

$$Q_\alpha(\rho(t)|M_{t_1, t_2, t_3}(\sigma)) = \text{Tr}(\rho(t)|M_{t_1, t_2, t_3}(\sigma)) = \frac{\text{Tr}(\rho(t)|M_{t_1, t_2, t_3}(\sigma))}{\text{Tr}(\rho(0)|M_{t_1, t_2, t_3}(\sigma))}.$$

Corollary 4.5: Under the conditions of Theorem 4.2, suppose that $[\sigma_0, \pi(g)]\sigma_0 \pi(g)^* = 0$ and $[\sigma_0, \pi(g)]\sigma_0 \pi(g)^* = 0$ for all $g \in G$. Then $(\sigma_0, \rho_0, \sigma_0) \simeq (\pi(\sigma_0, \rho_0, \sigma_0)$ (in the sense of asymptotic equivariant relative submajorization) iff for all $\alpha \geq 0$ and Radon probability measure $\gamma$ on $G$ the inequality

$$D_\alpha(\rho_0 \otimes \text{exp} \int \pi(g)\sigma_0 \pi(g)^* \text{d} \gamma(g)) \geq D_\alpha(\rho_0 \otimes \text{exp} \int \pi(g)\sigma_0 \pi(g)^* \text{d} \gamma(g))$$

holds.

Remark 4.6: The assumption that the orbit of $\sigma$ consists of commuting operators is a strong one due to the following rigidity property: if the orbit of $\sigma$ under the action of a connected group contains only operators that commute with $\sigma$, then $\sigma$ is a fixed point of the action. To see this, note that $\pi(\sigma) = \sigma(G)$ is a connected Lie subgroup of $U(H)$, therefore the exponential map is surjective. If $iA$ is an element of the Lie algebra, then $e^{\i tA}\sigma e^{-\i tA}, \sigma = 0$ for all $t \in \mathbb{R}$, which implies by differentiation that $[A, \sigma], [A, \sigma] = 0$. Since $\sigma$ is diagonalizable, so is the map $X \mapsto [X, \sigma]$, therefore its kernel is equal to the kernel of its square. It follows that $[A, \sigma] = 0$, i.e. $e^{\i tA}\sigma e^{-\i tA} = \sigma$ for all $t$.

1) Asymptotic Transformations by Thermal Processes:

Thermal operations are central to the resource theoretic approach to quantum thermodynamics. This is the class of quantum channels that can be obtained by preparing Gibbs states at a fixed temperature $T$, performing energy-preserving unitaries and tracing out subsystems [14], [28], [29]. This characterization does not suggest a simple way to decide whether a given channel is a thermal operation or whether a transformation between given states is feasible by a thermal operation, which motivates the study of channels and transformations admitting a simpler description at the cost of satisfying only some of the constraints governing thermal operations.

In the absence of coherence between energy eigenspaces, Gibbs-preserving maps provide an especially useful relaxation, which turns out to allow the same transitions as thermal operations. This is no longer true if coherence is present [30], and in addition to being Gibbs-preserving, the condition of time-translation symmetry has been identified as another key property of thermal operations [2]. Adding this requirement leads to the notion of thermal processes [3].

Transformations by such processes are an instance of equivariant real majorization: the group is that of time-translations, isomorphic to $\mathbb{R}$, and to a system with Hilbert space $\mathcal{H}$, Hamiltonian $H \in \mathcal{B}(\mathcal{H})$ and state $\rho$ we associate the triple $(\pi_H, \rho, e^{-\beta H})$, where $\pi_H : \mathcal{R} \rightarrow U(H)$ is the representation $t \mapsto e^{-\i tH}$ and $\beta$ is the inverse temperature. By definition, $(\pi_H, \rho, e^{-\beta H}) \simeq (\pi_H, \sigma, e^{-\beta H})$ if there is a completely positive trace-preserving map $T : B(H) \rightarrow B(H)$ such that $T(\rho) = \sigma$, $T(e^{-\beta H}) = e^{-\beta H}$ (by linearity, this amounts to preserving the Gibbs state $e^{-\beta H}/\text{Tr} e^{-\beta H}$), and $T(e^{-\beta H} \omega e^{\i tH}) = e^{-\beta H}T(\omega)e^{\i tH}$ for all states $\omega$ and $t \in \mathbb{R}$.

If we relax these transformations to equivariant relative submajorization and consider the asymptotic limit, then Theorem 4.2 provides a characterization of the resulting preorder in terms of the spectrum $\text{TSper}_1(S_{\mathcal{B}(\mathcal{H})}, \mathcal{B}(\mathcal{H}))$. Moreover, since $e^{-\beta H}e^{-\beta H}e^{\i tH} = e^{-\beta H}$, the orbit of $e^{-\beta H}$ has only one element, the simpler characterization of Corollary 4.5 can be applied.

Proposition 4.7: Let $H \in \mathcal{B}(\mathcal{H})$ be a Hamiltonian on a Hilbert space $\mathcal{H}$, and $\rho, \sigma \in \mathcal{S}(\mathcal{H})$. Then the following are equivalent:

(i) there exists a sequence of trace-nonincreasing thermal processes $T_n : \mathcal{B}(\mathcal{H}^{\otimes n}) \rightarrow \mathcal{B}(\mathcal{H}^{\otimes n})$ such that $T_n(\rho^{\otimes n}) \geq 2^{-\alpha(n)}\sigma^{\otimes n}$,

(ii) for all $\alpha \geq 1$ the inequality $D_\alpha(\rho\|e^{-\beta H}) \geq D_\alpha(\sigma\|e^{-\beta H})$ holds.

Proof: As explained in Section IV-A, we replace the non-compact group $\mathcal{R}$ with its Bohr compactification $\mathcal{B}(\mathcal{R})$ and consider the unique representation $\pi : b(\mathcal{R}) \rightarrow U(\mathcal{H})$ such that $\pi(b(t)) = e^{-\i tH}$, where $b : \mathcal{R} \rightarrow b(\mathcal{R})$ is the universal map. We claim that the image of $\pi$ is the closure of $\{e^{-\i tH} | t \in \mathbb{R}\}$, which we denote by $K$. Since $\pi(b(\mathcal{R}))$ is compact (hence closed) and contains $e^{-\i tH}$ for all $t$, it must contain $K$ as well. $K$ is a closed subgroup of $U(\mathcal{H})$, therefore compact. By the universal property, there is map $\varphi : b(\mathcal{R}) \rightarrow K$ such that $\varphi(b(t)) = e^{-\i tH}$ for all $t \in \mathbb{R}$. Composing with the inclusion
\( \iota : K \to U(\mathcal{H}) \) we get \( \pi(b(t)) \) for all \( t \in \mathbb{R} \), so by uniqueness we have \( \iota \circ \varphi = \pi \).

By continuity, we have \( \pi(g)e^{-\beta H} \pi(g)^* = e^{-\beta H} \) for every \( g \in b(\mathbb{R}) \), therefore
\[
\exp \int_{b(\mathbb{R})} \ln \pi(g)e^{-\beta H} \pi(g)^* \, d\gamma(g) = e^{-\beta H}
\]
for every probability measure \( \gamma \) on \( b(\mathbb{R}) \). This means that the condition \( D_n(\rho\|e^{-\beta H}) \geq D_\alpha(\sigma\|e^{-\beta H}) \) for all \( \alpha \geq 1 \) is equivalent to the condition in Proposition 4.5.

To see the equivalence as stated, note that for any trace-nonincreasing equivariant Gibbs sub-preserving completely positive map \( T_n \), there exists a trace-preserving thermal process \( T_n' \) such that \( T_n \leq T_n' \) (in the completely positive partial order), for example the map
\[
T_n'(X) = T_n(X) + (\text{Tr} X - \text{Tr} T_n(X)) \text{Tr} (e^{-\beta H})^{\otimes n} - T_n((e^{-\beta H})^{\otimes n})
\]
for some \( T_n' \) preserves the Gibbs state \( (e^{-\beta H})^{\otimes n} / \text{Tr}(e^{-\beta H})^{\otimes n} \) by construction, it is the sum of \( T_n \) and a completely positive equivariant map (since \( (e^{-\beta H})^{\otimes n} \) is invariant and therefore its image under the equivariant map \( T_n \) is invariant as well). Finally, we have \( T_n'(\rho^{\otimes n}) \geq T_n(\rho^{\otimes n}) \geq 2^{-o(n)}\sigma^{\otimes n} \).

We note that this characterization is exactly the same as the one obtained in [7] for Gibbs-preserving maps without the equivariance condition. This means that in this asymptotic limit, Gibbs-preserving maps are no more powerful than thermal processes.

2) **Hypothesis Testing With Group Symmetry**: The task in asymptotic binary state discrimination is to decide, based on measurements on many copies, if the state of a system is \( \rho \) or \( \sigma \), with the promise that it is one of the two. A type I error occurs if \( \rho \) is accepted but the state in fact was \( \sigma \), the opposite case is called the type II error. In the asymptotic setting, one is interested in the restrictions on the limiting behaviours of the two kinds of errors.

In the group-symmetric variant of this problem, the measurements are restricted to be invariant with respect to a group representation \( \pi : G \to U(\mathcal{H}) \). This problem was considered in [31] in three asymptotic regimes: when both error probabilities decay exponentially with the same exponent (Chernoff); when the exponential decays are different (Hoeffding); and when the type I error probability approaches zero arbitrarily, and the type II error decays exponentially (Stein).

We now focus on yet another limiting behavior, the strong converse domain. In this problem the type II error is required to decay faster than what is allowed by the Stein lemma, so that the type I error approaches one exponentially fast, and the task is to find the smallest exponent (i.e. slowest possible convergence). In the unrestricted (i.e. without group symmetry) case, this was characterized in [32], providing an operational interpretation of the sandwiched Rényi divergences with orders \( \alpha > 1 \). Following the strategy of [7], we show that the strong converse error exponent can be characterized in terms of the asymptotic preorder associated with equivariant relative submajorization.

We begin with single-copy measurements. Given a representation \( \pi : G \to U(\mathcal{H}) \), by a group-symmetric POVM (or invariant measurement) we mean a POVM \( \{F_i\}_{i=1}^r \) with measurement operators \( F_i \) satisfying \( \pi(g)F_i \pi(g)^* = F_i \) for all \( g \in G \). Since we are interested in discriminating between two hypotheses, the measurement will be a test, i.e. a two-outcome POVM with measurement operators \( (\Pi, I - \Pi) \). We interpret the outcome associated with \( \Pi \) as accepting (not rejecting) the null hypothesis \( \rho \), and the other outcome corresponds to accepting \( \sigma \). The probability of a type I error is therefore \( \text{Tr}(I - \Pi)\rho = 1 - \text{Tr} \Pi \rho \), while that of the type II error is \( \text{Tr} \Pi \sigma \).

***Lemma 4.8***: Let \( \pi : G \to U(\mathcal{H}) \) be a representation and \( \rho, \sigma \in S(\mathcal{H}) \). Then the following are equivalent for \( a, b \in \mathbb{R}_{>0} \):

(i) there exists a two-outcome group-symmetric POVM \( (\Pi, I - \Pi) \) on \( \mathcal{H} \) such that the type I and type II errors satisfy
\[
\text{Tr}(I - \Pi)\rho \leq 1 - a,
\]
\[
\text{Tr} \Pi \sigma \leq b.
\]

(ii) \( (\pi, \rho, \sigma) \succeq (1, a, b) \), where 1 denotes the trivial representation of \( G \) on \( \mathbb{C} \), and we identify \( B(\mathbb{C}) \simeq \mathbb{C} \).

**Proof**: An invariant measurement on \( \mathcal{H} \) can be identified with an equivariant (completely) positive trace-nonincreasing map \( T : B(\mathcal{H}) \to B(\mathbb{C}) \simeq \mathbb{C} \), where \( \mathbb{C} \) carries the trivial representation 1: the POVM \( (\Pi, I - \Pi) \) corresponds to the map \( T(X) = \text{Tr}(X\Pi) \). This is indeed equivariant (which in this special case means invariant), since
\[
T(\pi(g)X\pi(g)^*) = \text{Tr}(\pi(g)X\pi(g)^*)\Pi = \text{Tr}(X\pi(g^{-1})\Pi\pi(g^{-1})^*) = \text{Tr}(X\Pi) = \text{Tr}(X).
\]

Conversely, any equivariant linear map \( T : B(\mathcal{H}) \to \mathbb{C} \) is of the form \( T(X) = \text{Tr}(X\Pi) \) for a unique \( \Pi \) that is necessarily invariant, and \( T \) is completely positive and trace-nonincreasing if \( 0 \leq \Pi \leq I \). It follows that \( (\pi, \rho, \sigma) \succeq (1, a, b) \) for some \( a, b > 0 \) iff there is a POVM \( (\Pi, I - \Pi) \) such that the type I error satisfies \( 1 - \text{Tr} \Pi \rho \leq 1 - a \) and the type II error satisfies \( \text{Tr} \Pi \sigma \leq b \).

The following is an immediate consequence of Lemma 4.8, the definition of the asymptotic preorder, Theorem 4.2, and Corollary 4.5:

***Corollary 4.9***: Let \( G \) be a compact group and \( \pi : G \to U(\mathcal{H}) \) a representation and \( \rho, \sigma \in S(\mathcal{H}) \). Then the following are equivalent for \( r, R \geq 0 \):

(i) there exists a sequence of \( \pi^{\otimes n} \)-invariant measurements \( (\Pi_n, I_n - \Pi_n) \) with \( \Pi_n \in B(\mathcal{H}^{\otimes n}) \) such that the type I error is at most \( 1 - 2^{-Rn+o(n)} \) and the type II error is at most \( 2^{-r n + o(n)} \);

(ii) \( (\pi, \rho, \sigma) \succeq (1, 2^{-R}, 2^{-r}) \);

(iii) \( \forall f \in \text{TSeq}_{1}(S(\mathcal{G})): \frac{f((\pi(g)\rho\pi(g)^*)_{g\in G}(\pi(g)\pi(g)^*)_{g\in G})}{f(2^{-R}, 2^{-r})} \geq 1 \).
Moreover, if the orbit of $\sigma$ consists of operators that commute with $\sigma$, then the condition is equivalent to

$$R \geq \sup_{\alpha > 1} \max_{\gamma \in [0,1]} \frac{\alpha - 1}{\alpha} \left[ r - \tilde{D}_\alpha (\rho \| \sigma \#_\gamma (Z\sigma Z)) \right].$$

where the maximum is over Radon probability measures $\gamma$ on $G$.

In the following, $R^*(r)$ denotes the smallest possible error exponent for a given $r$.

**Example 4.10:** Consider the states and representation as in [31, Example 6.1]: The group is $G = \{0, 1\}$ with addition modulo 2, $\mathcal{H} = \mathbb{C}^2$, and the nontrivial element acts as the Pauli $Z$ matrix. Let the states be

$$\rho = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (70)$$

$$\sigma = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (71)$$

for some $\lambda, \mu \in (0, 1)$. Then $Z\sigma Z$ is also of the same form with $\mu$ replaced with $1 - \mu$, and these states commute with each other. From (69) we obtain

$$R^*(r) = \sup_{\alpha > 1} \max_{\gamma \in [0,1]} \frac{\alpha - 1}{\alpha} \left[ r - \frac{1}{\alpha - 1} \log \left( \frac{\lambda^\alpha \mu^{1-\alpha} \gamma(\alpha-\beta)}{1 - \lambda^\alpha \mu^{1-\alpha} \gamma(\alpha-\beta)} \right) \right].$$

This can be strictly larger than the maximum of the strong converse exponents for testing $\rho$ against $\sigma$ or $Z\sigma Z$, which correspond to $\gamma = 0$ and $\gamma = 1$. Note that a similar strict inequality was shown for unrestricted hypothesis testing with a composite alternative hypothesis in [27].

**Example 4.11:** Let $G = \{0, 1\}$, and $\pi : G \to U(\mathcal{H})$, $\rho, \sigma \in S(\mathcal{H})$ arbitrary full-rank, and let $Z = \pi(1)$. Without the commuting orbit assumption, we can only give a lower bound on the strong converse exponent, corresponding to the inner approximation of the test-spectrum given by Theorem 3.19 and Lemma 3.17:

$$R^*(r) \geq \sup_{\alpha > 1} \max_{\gamma \in [0,1]} \frac{\alpha - 1}{\alpha} \left[ r - \tilde{D}_\alpha (\rho \| \sigma \#_\gamma (Z\sigma Z)) \right].$$

We do not know whether this inequality can be strict for some $Z, \rho, \sigma$.

**Example 4.12:** Let $G$ be an arbitrary compact group, $\pi : G \to U(\mathcal{H})$, $\rho \in S(\mathcal{H})$ arbitrary as supposed that $\sigma \in S(\mathcal{H})$ is $G$-invariant, i.e. $\pi(g)\sigma\pi(g)^* = \sigma$ for all $g \in G$ (as always, both $\rho$ and $\sigma$ are assumed to be of full rank). Then the orbit of $\sigma$ is a single point, therefore (69) may be used and simplified because the exponentiated integral is equal to $\sigma$ independently of the measure $\gamma$. Therefore the bound reduces to

$$R^*(r) = \sup_{\alpha > 1} \max_{\gamma \in [0,1]} \frac{\alpha - 1}{\alpha} \left[ r - \tilde{D}_\alpha (\rho \| \sigma) \right],$$

which is the same as with unrestricted measurements [32]. This phenomenon is similar to the equality of the Stein exponents in the group-symmetric and unrestricted problems, and is in contrast with the Chernoff and Hoeffding exponents, which can be strictly worse with group-symmetric tests, even if $\sigma$ is $G$-invariant [31, Example 6.2].

**3) Reference Frames in Hypothesis Testing:** When the dynamics of a system is constrained by symmetries, an additional supply of asymmetric states (imperfect reference frames) becomes a resource, which allows to partially overcome the limitations of symmetric evolutions [33]. Suppose that $\pi_{\text{ref}} : G \to U(\mathcal{K})$ is a representation and $\Omega \in S(\mathcal{K})$ is a state with full support and trivial stabilizer. In the setting of group-symmetric hypothesis testing as modeled above in terms of equivariant relative submajorization, the reference frame corresponds to the triple $(\pi_{\text{ref}}, W, \Omega)$. The two states in this triple are equal, therefore they cannot be distinguished even with unrestricted measurements. The mathematical property that makes it a useful resource in group-symmetric hypothesis testing is the following:

**Proposition 4.13:** Let $\pi_{\text{ref}} : G \to U(\mathcal{K})$ be a representation and $\Omega \in S(\mathcal{K})$ a state with full support and trivial stabilizer. Let $f \in \mathcal{T}(\pi_{\text{ref}}, W, \Omega)$ and let its restriction to $S_{\mathcal{K}} G$ be characterized by $\alpha > 1$ (for real points, $x \in G$, and the probability measure $\gamma$. Then there are two possibilities: either $\gamma$ is the Dirac measure at $x$, or

$$f((\pi_{\text{ref}}(g)\Omega\pi_{\text{ref}}(g)^*)_{g \in G}, (\pi_{\text{ref}}(g)\Omega_{\pi_{\text{ref}}(g)^*})_{g \in G}) > 1. \quad (75)$$

**Proof:** Since the stabilizer of $\Omega$ is trivial, for all $g, g' \in G, g \neq g'$ we have $\pi_{\text{ref}}(g)\Omega_{\pi_{\text{ref}}(g)^*} \neq \pi_{\text{ref}}(g')\Omega_{\pi_{\text{ref}}(g')^*}$. By assumption, $\text{Tr} \pi_{\text{ref}}(g)\Omega_{\pi_{\text{ref}}(g)^*} = \text{Tr} \Omega = 1$, so by Proposition 3.15, the condition for

$$f((\pi_{\text{ref}}(g)\Omega_{\pi_{\text{ref}}(g)^*})_{g \in G}, (\pi_{\text{ref}}(g)\Omega_{\pi_{\text{ref}}(g)^*})_{g \in G}) = 1 \text{ is that } \gamma \text{ is concentrated at } x. \quad \square$$

**Example 4.14:**

(i) Let $G = \{0, 1\}$, and let the nontrivial element act on $\mathbb{C}^2$ by the Pauli $X$ operator. A biased coin $\Omega = q|0\rangle\langle 0| + (1-q)\langle 1|\langle 1|$ with $q \in (0, 1/2)$ is an example of a reference frame. Since $[\Omega, X\Omega X]$ = 0, we can use the explicit form of the spectrum of the classical semiring $S_{\mathcal{K}} G$. Identifying the probability measure $\gamma$ with the value $\gamma(\{0\}) \in [0, 1]$, for $\alpha > 1$ we have

$$f_{0,0,\gamma}((\Omega, X\Omega X), (\Omega, X\Omega X)) = q^\alpha q^{1-\alpha}(1-q)^{1-\alpha}(1-\gamma) + (1-q)^\alpha q^{1-\alpha}(1-q)^{1-\alpha}(1-\gamma) = q^{1-\alpha}(1-\gamma) \left( (1-q)^{1-\alpha}(1-\gamma) + (1-q)^{1-\alpha}(1-\gamma) \right), \quad (76)$$

which is equal to 1 iff $\gamma = 1$.

(ii) Let $G = U(1)$, which we can think of as the group of complex numbers with modulus 1. To find a reference frame, we consider the representation

$$\pi(z) = \left[ \begin{array}{c} 1 \\
0 \\
z \end{array} \right], \quad (77)$$

and let $\Omega$ be any full-rank state with nonzero off-diagonal entries, such as $\Omega = (1 - \epsilon) |+\rangle \langle + | + \epsilon |\rangle \langle \rangle | \rangle - | \rangle \langle \rangle | \rangle$, where $\epsilon \in (0, 1/2)$ and $| \rangle \langle \rangle | \rangle = \frac{1}{\sqrt{2}} \left( | \rangle \langle 0 | + | \rangle \langle 1 | \right)$. Conjugation by
\( \pi(z) \) means that the off-diagonal elements pick up the phase factor \( z \) and \( \overline{z} \), therefore the stabilizer is trivial. We can find a lower bound on the values of the spectral points by applying a suitable measurement. Let us measure in the basis \( (|+\rangle,|-\rangle) \). The probability of obtaining the plus outcome in the state \( \pi(z)\Omega\pi(z)^* \) is

\[
\frac{1}{4} \text{Tr} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & (1-2\epsilon)z \\ (1-2\epsilon)z & 1 \end{bmatrix} = \frac{1}{2} + \frac{1-2\epsilon}{2} \text{Re} z,
\]

(78)

which has a strict maximum at \( z = 1 \), therefore this measurement gives a nontrivial bound for any probability measure \( \gamma \neq \delta_1 \). More precisely, if \( f \in \text{TSper}_1(S_{U(1)}), \) and its restriction to the classical subsemiring is characterized by \( \alpha > 1, x = 1 \in U(1) \) and the probability measure \( \gamma \), then

\[
f((\pi(z)\Omega\pi(z)^*)_{z\in U(1)}, (\pi(z)\Omega\pi(z)^*)_{z\in U(1)}) \geq f_{\alpha,1,\gamma} \begin{bmatrix} (1/2 + 1-2\epsilon \text{Re} z) & 1 \\ 1 & (1/2 + 1-2\epsilon \text{Re} z) \end{bmatrix} \]

\[
= (1-\epsilon)^n \exp \left[ (1-\alpha) \int_0^{2\pi} \ln \left( \frac{1}{2} + \frac{1-2\epsilon}{2} \cos \varphi \right) d\gamma(\varphi) \right] + \epsilon^n \exp \left[ (1-\alpha) \int_0^{2\pi} \ln \left( \frac{1}{2} - \frac{1-2\epsilon}{2} \cos \varphi \right) d\gamma(\varphi) \right].
\]

(79)

Testing \((\pi,\rho,\sigma)\) aided by the reference frame corresponds to comparing \((\pi \otimes \pi_{\text{ref}}, \rho \otimes \sigma \otimes \Omega)\) with a one-dimensional triple with trivial representation. In an asymptotic setting, more generally, we may use \( \kappa \) copies of the reference frame per sample of the state to be discriminated. In this case the exponent pair \((R, r)\) is achievable iff

\[
\forall f \in \text{TSper}_1(S_G, G), \quad f((\pi(g)\rho_0\pi(g)^*)_{g\in G}, (\pi(g)\sigma_0\pi(g)^*)_{g\in G}) \cdot f((\pi_{\text{ref}}(g)\Omega\pi_{\text{ref}}(g)^*)_{g\in G}, (\pi_{\text{ref}}(g)\Omega\pi_{\text{ref}}(g)^*)_{g\in G})^\kappa \geq f(2^{-R}, 2^{-r}),
\]

(80)

therefore

\[
R^*(r, \kappa) = \sup_{\alpha > 1} \sup_{f \in \text{TSper}_1(S_G, G)} \frac{\alpha - 1}{\log f(2,1)=\alpha} \left[ r - \frac{1}{\alpha} \text{sup} \left( f((\pi(g)\rho_0\pi(g)^*)_{g\in G}, (\pi(g)\sigma_0\pi(g)^*)_{g\in G}) \right) - \kappa \frac{1}{\alpha} \text{sup} \left( f((\pi(g)\Omega\pi(g)^*)_{g\in G}, (\pi(g)\Omega\pi(g)^*)_{g\in G}) \right) \right] \]

(81)

where we may as well restrict to those elements of the spectrum that depend on the first family through its value at the identity. As \( \kappa \to \infty \) (i.e. in the limit of unlimited supply of the reference frame), the supremum is achieved for the \( \gamma \) (which, as before, is determined by the restriction of \( f \) to the classical subsemiring) that is concentrated at the identity by Proposition 4.13, since this is the only point where the last term vanishes. This means that we recover the unrestricted strong converse exponent \([32]\), which is potentially much smaller than the group-symmetric one. In an extreme example, \( \rho_0 \) and \( \sigma_0 \) might be in the same \( G \)-orbit, in which case \( R^*(r,0) = r \), i.e. a group-symmetric measurement cannot offer any advantage over guessing.

When the orbits of \( \sigma_0 \) and \( \Omega \) consist of commuting operators, we can use Corollary 4.5 to obtain an explicit form of the smallest type I strong converse exponent \( R^* \) for a given decay rate \( r \) of the type II error:

\[
R^*(r,\kappa) = \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} \left[ r - \frac{D_\alpha}{\Omega} \left( \rho \right) \exp \left( \int_G \ln \pi(g)\sigma_0\pi(g)^* \text{d}\gamma(g) \right) - \frac{1}{\kappa} D_\alpha \left( \Omega \right) \exp \left( \int_G \ln \pi_{\text{ref}}(g)\Omega\pi_{\text{ref}}(g)^* \text{d}\gamma(g) \right) \right].
\]

(82)

B. Approximate Joint Transformations

In this section we specialize our results and derive a characterization of approximate joint transformations with respect to the symmetrized max-divergence, in the asymptotic limit. Recall that the max-divergence between a pair of states \( \rho, \sigma \) is \( D_{\max}(\rho||\sigma) = \log \|\sigma^{-1/2}\rho\sigma^{-1/2}\|_\infty = \min \{ \lambda \in \mathbb{R} | 2^\lambda \sigma \geq \rho \} \). We will use the max-divergence as a measure of dissimilarity between states. It vanishes iff the two states are equal, but for subnormalized states this is no longer true, and it is not symmetric. However, the closely related quantity \( d_T(\rho, \sigma) := \max \{ D_{\max}(\rho||\sigma), D_{\max}(\sigma||\rho) \} \) is a metric on the set of positive definite operators (the Thompson metric [34] associated with the semidefinite cone). This metric is unbounded even on a fixed Hilbert space and satisfies \( d_T(\rho^{\otimes n}, \sigma^{\otimes n}) = nd_T(\rho, \sigma) \). The notion of approximate transformations that we consider will be that the distance increases sublinearly as the number of copies grows.

Proposition 4.15: Let \( X \) be a compact space, \( \rho \in C(X, \mathcal{B}(\mathcal{H}^{\otimes n})), \) and \( \rho' \in C(X, \mathcal{B}(\mathcal{H}^{'\otimes n})). \) The following are equivalent:

(i) there is a sequence of completely positive trace-nonincreasing maps \( T_n : B(\mathcal{H}^{\otimes n}) \to B(\mathcal{H}^{'\otimes n}) \) such that for all \( n \)

\[
2^{-o(n)} \rho^{\otimes n} \leq T_n(\rho^{\otimes n}) \leq \rho^{\otimes n},
\]

(83)

i.e. for all \( x \in X \) we have

\[
\lim_{n \to \infty} \frac{1}{n} d_T(T_n(\rho(x)^{\otimes n}), \rho'(x)^{\otimes n}) = 0,
\]

(84)

uniformly in \( x \);

(ii) \((\rho, \rho') \preceq (\rho', \rho')\) in the preorder of the semiring \( S_{X,X} \);

(iii) for all \( f \in \text{TSper}_1(S_{X,X}) \) the inequality \( f(\rho, \rho) \geq f(\rho', \rho') \) holds.

Proof: The equivalence of ii and iii is a special case of Corollary 2.8. Using the definition of the asymptotic preorder
with the power universal element \( u = (2 \cdot 1_X, 1_X) \), the condition for \( (\rho, \rho) \succeq (\rho', \rho') \) is that there is a sequence of completely positive trace-nonincreasing maps \( T_n : \mathcal{B}(\mathcal{H}^\otimes n) \to \mathcal{B}(\mathcal{H}^\otimes n) \) and a sequence \( k_n \) of natural numbers such that \( k_n/n \to 0 \) and for all \( n \in \mathbb{N} \) and \( x \in X \) the inequalities \( T_n(2^{k_n} \rho(x)) \geq \rho'(x) \) and \( T_n(\rho(x)) \leq \rho'(x) \) hold, which is the same as i. □

**Remark 4.16:** The (non-asymptotic) relative submajorization preorder between the pairs \( (\rho, \rho) \) and \( (\rho', \rho') \) means that \( T(\rho(x)) \leq \rho'(x) \) and \( T(\rho(x)) \geq \rho'(x) \) for some subchannel \( T \) and all \( x \), i.e. that \( T(\rho(x)) = \rho'(x) \).

Specializing to classical families and using the explicit form of the 1-test spectrum of the classical semiring (Theorem 3.10), we have the following characterization of asymptotic joint transformations in the above sense.

**Theorem 4.17:** Let \( p : X \to \mathcal{P}([d]), p' : X \to \mathcal{P}([d']) \) where \( X \) is a compact Hausdorff space and \( d, d' \in \mathbb{N}_{>0} \) are finite sets. The following are equivalent:
1) there exists a sequence of substochastic maps \( T_n \) from \((\mathbb{R}^d)^\otimes n\) to \((\mathbb{R}^{d'})^\otimes n\) such that for all \( x \in X \)
   \[
   \lim_{n \to \infty} \frac{1}{n} d_T(T_n(p(x)^\otimes n), p'(x)^\otimes n) = 0, \tag{85}
   
   
   uniformly in \( x \);  
2) for all \( x \in X \), \( \alpha \geq 1 \) and probability measure \( \gamma \) on \( X \) the inequality
   \[
   \sum_{i=1}^d p_i(x)^\alpha \exp(1 - \alpha) \int_X \ln p_i \, d\gamma \geq \sum_{i=1}^d p'_i(x)^\alpha \exp(1 - \alpha) \int_X \ln p'_i \, d\gamma \tag{86}
   
   

   holds.

The ideas in this section can be combined with our considerations on equi-variant transformations:

**Proposition 4.18:** Let \( X_0 \) be a compact space, \( G \) a compact group, \( \pi : G \to U(\mathcal{H}), \pi' : G \to U(\mathcal{H}') \) unitary representations, \( \rho_0 \in C(X_0, \mathcal{B}(\mathcal{H}_{++})) \) and \( \rho_0' \in C(X_0, \mathcal{B}(\mathcal{H}'_{++})) \). Let \( X = X_0 \times G \) and consider \( \rho \in C(X, \mathcal{B}(\mathcal{H}_{++})) \) defined as \( \rho(x, g) = \pi(g)\rho(x)\pi(g)^* \), and similarly \( \rho' \), which determine the elements \( (\rho, \rho) \) and \( (\rho', \rho') \) in \( S_{X \times X} \). Then the following are equivalent:
1) there exists an equi-variant completely positive trace-nonincreasing map \( T \) such that \( T(\rho_0(x)) = \rho'_0(x) \) for all \( x \in X_0 \);
2) \( (\rho, \rho) \succeq (\rho', \rho') \) in \( S_{X \times X} \).

**Proof:** 1 \( \Rightarrow \) 2: Let \( \mathcal{T} \) be an equi-variant subchannel such that \( T(\rho_0(x)) = \rho'_0(x) \) for all \( x \in X_0 \). Then for all \( x \in X_0 \) and \( g \in G \) we have
   \[
   T(\rho(x, g)) = T(\pi(g)\rho_0(x)\pi(g)^*) = \pi(g)T(\rho_0(x))\pi(g)^* = \pi(g)\rho'_0(x)\pi(g)^* = \rho'(x, g). \tag{87}
   
   Therefore \( (\rho, \rho) \succeq (\rho', \rho') \) (see Remark 4.16).

2 \( \Rightarrow \) 1: Suppose that \( (\rho, \rho) \succeq (\rho', \rho') \) in \( S_{X \times X} \). Then there exists a subchannel \( T_0 \) such that \( T_0(\rho(x, g)) = \rho'(x, g) \) for all \( x \in X_0 \) and \( g \in G \). Let us define the completely positive trace-nonincreasing map \( T \) as
   \[
   T(\sigma) = \int_G \pi'(g)^*T_0(\pi(g)\sigma(\pi(g)^*)\pi'(g)) \, d\mu(g). \tag{88}
   
   
   where \( \mu \) is the Haar probability measure on \( G \). This is equi-variant by construction, and satisfies
   \[
   T(\rho_0(x)) = \int_G \pi'(g)^*T_0(\pi(g)\rho_0(x)\pi(g)^*)\pi'(g) \, d\mu(g) = \int_G \pi'(g)^*T_0(\rho(x, g))\pi'(g) \, d\mu(g) = \int_G \pi'(g)^*\rho'(x, g)\pi'(g) \, d\mu(g) = \rho'_0(x, g). \tag{89}
   
   

While a complete classification of \( TS_{\text{per}}(S_{X \times X}) \) is required to obtain an explicit characterization of the asymptotic preorder, any subset of the spectrum gives necessary conditions for asymptotic, single or multiple copy, and catalytic transformations. In particular, in the setting of Section IV-A.1, the spectrum gives rise to many “second laws” of thermodynamics in the sense of [35]. For example, if a transformation is possible by a thermal process under the Hamiltonian \( H \) and at temperature \( \beta^{-1} \), then the value of
   \[
   \tilde{D}_\alpha \left( \rho \big|\| (e^{-itH}\rho e^{itH}) \big\|_\# e^{-\beta H} \right) \tag{90}
   
   

cannot increase under the process, for every \( \alpha \geq 1, \gamma \in [0, 1] \) and \( t \in \mathbb{R} \). Here the second argument may be replaced with any weighted geometric mean of the Gibbs state and arbitrary time-translated versions of \( \rho \), and in addition the first argument may be replaced with the Gibbs state. To ensure that the quantity is finite, one generally needs to restrict to full-rank states \( \rho \).

**Example 4.19:** As a concrete example, consider a transition studied in [30], perturbed slightly to get full-rank states. With the Hamiltonian \( H = |1|1| + |1|1| \) on \( \mathbb{C}^2 \), the transition \( |1|1| \to |+|+| \) was shown to be possible by Gibbs-preserving maps, but not possible with thermal processes. Let \( \tau = e^{-\beta H}/Tr e^{-\beta H} \) be the Gibbs state at temperature \( \beta^{-1} \). Then the transition \( (1 - \epsilon)|1|1| + \epsilon \tau \to (1 - \epsilon)|+|+| + \epsilon \tau \) is still possible by a Gibbs-preserving map. However, (90) with \( \alpha = 2, \gamma = 1/2 \) and \( t = \pi \) evaluates to \( c \log(1 + e^{\beta \epsilon} + O(\epsilon)) \) on the initial state while it diverges logarithmically on the target state as \( \epsilon \to 0 \). This implies that, for sufficiently small \( \epsilon \), the transition is not possible under a thermal process, even in the presence of a catalyst or assuming multi-copy transformations. In fact, a numerical comparison suggests that this holds for all \( \epsilon \in (0, 1) \).

**C. A Two-Parameter Family of Quantum Rényi Divergences**

A defining property of the monotone quantities in \( TS_{\text{per}}(S_{X \times X}) \) is that they are increasing in the first argument and decreasing in the second one. From the point of view of relative majorization, this is a severe and unnecessary restriction, and it is reasonable to expect that by dropping this...
requirement one gets more constraints on joint transformations that are violated by relative submajorization.

In this section we point out that it is possible to derive some of these additional constraints by specialization, thanks to the possibility of relative submajorization to express relative majorization as a special case. We also used this in Section IV-B for classical families, but now, with a different viewpoint, we consider quantum pairs instead, and introduce a two-parameter family of monotone quantum Rényi divergences. We note that $\alpha$-$\gamma$-divergences, another two-parameter quantum extension of the Rényi divergences introduced in [36], do not seem to have any obvious relation to ours.

**Definition 4.20:** For $\gamma \in [0, 1)$ and $\alpha > 1$, let us define the $\gamma$-weighted geometric sandwiched Rényi divergence for positive definite arguments $\rho, \sigma$ as

\[
\tilde{D}^\#_{\alpha, \gamma} (\rho\|\sigma) := \frac{1}{1-\gamma} \tilde{D}^\#_{\alpha, \gamma} (\rho - \gamma \rho\|\sigma - \gamma \sigma)
\]

\[
= \frac{1}{\alpha - 1} \log \text{Tr} \left( \sqrt{\sigma} \left( \sigma^{\frac{1}{2}} \rho \sigma^{\frac{1}{2}} \right)^\gamma \sigma \right)^{\frac{1}{1-\gamma}}.
\]

(91)

The $\alpha \to 1$ limit is studied in a more general context in [37]. We summarize some properties in the following proposition:

**Proposition 4.21:** Let $\gamma \in [0, 1)$ and $\alpha > 1$. Then for all $\rho, \sigma \in \mathcal{B}(\mathcal{H})_{++}$ and $\rho', \sigma' \in \mathcal{B}(\mathcal{H}')_{++}$ we have

(i) if $[\rho, \sigma] = 0$, then $\tilde{D}^\#_{\alpha, \gamma} (\rho\|\sigma) = \frac{1}{\alpha - 1} \text{Tr} \rho^{\alpha\sigma^{1-\alpha}}$ (extension of classical Rényi divergence);

(ii) $\tilde{D}^\#_{\alpha, \gamma} (\rho \otimes \rho'\|\sigma \otimes \sigma') = \tilde{D}^\#_{\alpha, \gamma} (\rho\|\sigma) + \tilde{D}^\#_{\alpha, \gamma} (\rho'\|\sigma')$ (additivity);

(iii) $2(\alpha - 1) \tilde{D}^\#_{\alpha, \gamma} (\rho \otimes \rho'\|\sigma \otimes \sigma') = 2(\alpha - 1) \tilde{D}^\#_{\alpha, \gamma} (\rho\|\sigma) + 2(\alpha - 1) \tilde{D}^\#_{\alpha, \gamma} (\rho'\|\sigma')$ (block additivity);

(iv) if there is a channel $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}')$ such that $T(\rho) = \rho'$ and $T(\sigma) = \sigma'$, then $\tilde{D}^\#_{\alpha, \gamma} (\rho\|\sigma) \geq \tilde{D}^\#_{\alpha, \gamma} (\rho'\|\sigma')$ (data processing inequality);

(v) if $\text{Tr} \rho = \text{Tr} \sigma = 1$, then $\tilde{D}^\#_{\alpha, \gamma} (\rho\|\sigma) \geq 0$ with equality iff $\rho = \sigma$ (positive definiteness);

(vi) if $\mathcal{H} = \mathcal{H}'$ and $\sigma \leq \sigma'$ in the semidefinite partial order, then $\tilde{D}^\#_{\alpha, \gamma} (\rho\|\sigma) \geq \tilde{D}^\#_{\alpha, \gamma} (\rho\|\sigma')$ (anti-monotonicity in the second argument).

**Proof:** Although the properties can be proved directly, in order to illustrate the general idea, we show them by a reduction to Theorem 3.19 and Proposition 3.17 that give us a partial description of the test-spectrum $\text{TS}_{\mathcal{P}_{\mathcal{R}_{\gamma}}} (S_{X,Y})$ for any spaces $X$ and $Y$. Let $X = \{1\}$ and $Y = \{1, 2\}$, so that the elements of $S_{X,Y}$ may be written as $(\rho, (\sigma_1, \sigma_2))$. Then for $\alpha > 1$, $\alpha_0 := \frac{1}{\alpha - 1}$ and $\gamma \in [0, 1)$ the functional

\[
f(\rho, (\sigma_1, \sigma_2)) = \tilde{Q}_{\alpha_0} (\rho\|\sigma_2\#\gamma \sigma_1)
\]

(92)

belongs to $\text{TS}_{\mathcal{P}_{\mathcal{R}_{\gamma}}} (S_{\{1\}, \{1,2\}})$ by Theorem 3.19 and Proposition 3.17. It is related to the $\gamma$-weighted geometric sandwiched Rényi divergence as

\[
\tilde{D}^\#_{\alpha, \gamma} (\rho\|\sigma) = \frac{1}{\alpha - 1} \log f(\rho, (\rho, \rho)).
\]

(93)

Note that the map $\varphi : (\rho, \sigma) \mapsto (\rho, (\rho, \rho))$ is clearly a semiring-homomorphism, which implies properties i and iii. If $\mathcal{P}$ is a channel such that $T(\rho) = \rho'$ and $T(\sigma) \leq \sigma'$, then $(\rho, (\rho, \rho)) \succ (\rho', (\rho', \rho'))$, which implies iv and vi (the equality $T(\rho) = \rho'$ is important: $\varphi$ is not monotone with respect to relative submajorization). Property i can be seen by a direct calculation, and vi is a consequence of i and that the classical Rényi divergence is positive definite. □

**Remark 4.22:** For completeness, we give a direct proof of properties iv and vi (the remaining ones either follow easily from the definition or by a reduction to the classical case as in the proof of Proposition 4.21).

If $T$ is a completely positive trace-preserving map, then

\[
\tilde{D}^\#_{\alpha, \gamma} (T(\rho)\|T(\sigma)) = \frac{1}{1 - \gamma} \tilde{D}^\#_{\alpha, \gamma} (T(\rho)\|T(\sigma)\#_{\gamma} T(\rho))
\]

\[
\geq \frac{1}{1 - \gamma} \tilde{D}^\#_{\alpha, \gamma} (T(\rho)\|T(\sigma)\#_{\gamma} T(\rho))
\]

\[
\geq \frac{1 - \gamma}{1 - \gamma} \tilde{D}^\#_{\alpha, \gamma} (\rho\|\sigma\#_{\gamma} \rho)
\]

\[
= \tilde{D}^\#_{\alpha, \gamma} (\rho\|\sigma),
\]

(94)

where the first inequality uses that $T(\sigma)\#_{\gamma} T(\rho) \geq T(\sigma\#_{\gamma} \rho)$ and that the sandwiched Rényi divergence is anti-monotone in its second argument, and the second inequality uses the data processing inequality for the sandwiched Rényi divergence. This proves iv. Property vi is true since the matrix geometric mean is monotone in both arguments and the sandwiched Rényi divergence is anti-monotone in its second argument.

Finally, we note that when $\gamma = 0$, (91) agrees with the minimal Rényi divergence, and when $\alpha = 1$, the limit $\gamma \to 1$ is the Belavkin–Staszewski relative entropy [37], but for $\alpha > 1$ we do not know what the limit $\gamma \to 1$ is. We leave the detailed study of these divergences for future work.

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