Dimensional crossover in dipolar magnetic layers

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Abstract. We investigate the static critical behaviour of a uniaxial magnetic layer, with finite thickness $L$ in one direction, yet infinitely extended in the remaining $d$ dimensions. The magnetic dipole-dipole interaction is taken into account. We apply a variant of Wilson's momentum shell renormalisation group approach to describe the crossover between the critical behaviour of the 3-D Ising, 2-d Ising, 3-D uniaxial dipolar, and the 2-d uniaxial dipolar universality classes. The corresponding renormalisation group fixed points are in addition to different effective dimensionalities characterised by distinct analytic structures of the propagator, and are consequently associated with varying upper critical dimensions. While the limiting cases can be discussed by means of dimensional $\epsilon$ expansions with respect to the appropriate upper critical dimensions, respectively, the crossover features must be addressed in terms of the renormalisation group flow trajectories at fixed dimensionality $d$.

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1. Introduction

Layered magnetic systems are of obvious technological importance, e.g., in magnetic storage devices. Presently, magneto-optic storage devices based on amorphous Gd TbFe and TbFeCo alloys are in commercial use, allowing for storage densities of up to $10^8$ bits/cm$^2$. In the future, however, Pt/Co, Pd/Co, and Co/Au multilayers of a few Angstrom thickness are expected to lead to four times as high storage capacities. On the purely theoretical side, they furthermore represent an intriguing model system with the competing exchange- and dipolar-dominated universality classes on the one hand, and the dimensional crossover between three and two effective dimensions on the other hand. The mathematical description of the associated crossover features within the framework of the renormalisation group theory is technically challenging, as it involves both varying analytic structures of the propagators, and different upper critical dimensions for the associated renormalisation group fixed points. The aim of the present paper is to demonstrate that such a unifying crossover description is feasible, even with the aforementioned difficulties at hand. For a layered uniaxial magnetic system

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of finite width $L$ in one direction and with dipolar interactions taken into account, the
renormalisation program is explicitly performed to one-loop order for the entire crossover
regime by means of a variant of Wilson’s momentum shell method at fixed transverse
dimension $d$. Moreover, it is shown that the scheme described here encompasses all the
relevant universality classes involved, and the critical exponents at each renormalisation
group fixed point may be computed within an $\epsilon$ expansion about the respective upper
critical dimensions.

In the theory of phase transitions there has been long-standing interest in the
critical properties of finite systems as well as in the influence of the magnetic dipole-
dipole interaction on static and dynamic critical behaviour. The first investigations for
the crossover behaviour between the different critical regimes in dipolar systems date
back thirty years: In their pioneering work, Larkin and Khmel’nikitii [1] showed that the
upper critical dimension of a uniaxial magnet with magnetic dipole-dipole interaction
is reduced by one as compared to the pure Ising system. The mathematical description
of the crossover in a three-dimensional system with easy axis and dipolar forces
therefore necessitated a renormalisation scheme capable of dealing with varying upper
critical dimensions associated with the different non-trivial fixed points. Modifying
the generalised dimensional regularisation method by Amit and Goldschmidt [2], such
a description was developed in Ref. [3] for the crossover between the Ising and 3-
dimensional dipolar fixed points of a uniaxial dipolar magnet, and in Ref. [4] for the
crossover from isotropic to directed percolation. This approach was then applied to an
even more complex scenario by Ried et al. [5], who considered the crossovers between
two possible fixed points of a uniaxial dipolar system, taking the finiteness of the
anisotropy into account. It should be noted, though, that all these calculations employed
the harmonic dipolar propagator originally derived in Ref. [6]. Yet this propagator is
adequate only in the case of a three-dimensional infinite system, or, more generally,
for any system of the same dimensionality as the one used for computing the effective
magnetic dipole-dipole interaction. It is therefore not applicable for 3-d electrodynamics,
if the system itself is two-dimensional, for example, or is finite in one or more directions.

Static critical behaviour in finite systems with purely short-range interactions was
investigated in Refs. [7, 8, 9, 10]. Here, the asymptotic critical behaviour was addressed,
and no crossover scaling functions were calculated. Dimensional crossovers were studied
in the context of quantum phase transitions by Chakravarty, Halperin, and Nelson for
quantum antiferromagnets [11], and by O’Connor and Stephens for an Ising system
between four and three dimensions [12]. Yet we are not aware of any work concerning
the critical behaviour in a finite system that takes into account the magnetic dipole-
dipole interaction.

In this paper we go beyond the cited literature in describing the crossover scenario
for a uniaxial magnetic system with dipolar interactions that is infinitely extended in
$d$ dimensions, but has a finite thickness $L$ in one additional direction. For simplicity,
we assume that the anisotropy energy is larger than all other relevant energy scales,
and hence only take into account the Ising-type up-down symmetry. In the case
of $d = 2$, such a system is supposed to exhibit four different regimes of non-trivial critical behaviour, in addition to the classical one described by mean-field critical exponents; these are described by the 3-D Ising, 2-d Ising, 3-D uniaxial dipolar, and 2-d uniaxial dipolar universality classes, respectively. As will be shown, these regimes are characterised in addition to varying effective system dimensions by different analytic forms of the propagator, which originate in the dipole-dipole interaction. Consequently, three different upper critical dimensions $d_c$ emerge: For the Ising systems, $D_c = 4$ and $d_c = 4$, respectively, for the 3-D uniaxial dipolar system $D_c = 3$, and finally the upper critical dimension for the truly asymptotic critical behaviour of a 2-d uniaxial dipolar system with an easy axis in the system’s plane will be demonstrated to be $d_c = \frac{7}{2}$.

This paper is organised as follows: In section 2 we introduce the effective hamiltonian (free energy functional) and specifically the propagator (two-point correlation function in Gaussian approximation) that is to be used for our calculations. In section 3 we explain our variant of the momentum shell renormalisation method devised to mathematically describe the crossover scenarios. In section 4 the resulting renormalisation group flow equations are derived to one-loop order and discussed. Section 5 contains the resulting limiting cases within the $\epsilon$ expansions in the vicinities of the appropriate upper critical dimensions. In section 6 the crossovers between the critical fixed points are discussed in terms of the renormalisation group trajectories. We conclude with a summary and brief discussion in section 7.

2. Effective hamiltonian

At first we need to deduce an effective free energy functional that is capable of capturing the crossover features we are aiming at. However, at this point we encounter the difficulty that the analytic forms of the propagators (Gaussian two-point correlation functions) describing uniaxial dipolar magnetic systems are quite different depending on the dimensionality $D = d + 1$ of the system. We employ periodic boundary conditions to avoid surface effects, in which we are not specifically interested here. Rather we are concerned with the effect of the system’s limited extent in one space direction as such.

Denoting the part of the wave vector $k$ in the direction of the easy axis of the magnetisation as $q$, and the perpendicular component as $p$, $k = q + p$, the propagators look as follows: In the case of a three-dimensional system, the relevant part of the inverse propagator is of the form $G_0(q, p)^{-1} = r + c k^2 + g q^2 / p^2$ [13], while for a two-dimensional system containing the easy axis, $G_0(q, p)^{-1} = r + c k^2 + g q^2 / k$, with $k = |k|$ [14]. On the other hand, if the easy axis is aligned perpendicular to the 2-d plane, the result is of the form $G_0(q, p)^{-1} = r + c k^2 - g k$ [14]. Here $r = T - T_0$, $T_0$ denotes the temperature distance to the mean-field transition temperature $T_0$, $c$ is a constant describing the stiffness of the system with respect to inhomogeneous order parameter configurations, and $g$ measures the strength of the dipole-dipole interaction as compared to the exchange interaction.

As our task is the description of a system with finite thickness, we have to apply some care in the derivation of the appropriate form of the propagator. The contribution
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to the free energy stemming from the dipole-dipole interaction between the magnetic moments connected to the spin variables $S_R$ at lattice sites $R$ reads

$$W_D = g \sum_{R_i \neq R_2} \sum_{\alpha \beta} A^{\alpha \beta}(R_1, R_2) S_{R_1}^{\alpha} S_{R_2}^{\beta},$$

(1)

with the dipole tensor in real space being defined as

$$A^{\alpha \beta}(R_1, R_2) = \frac{\delta^{\alpha \beta}}{|R_1 - R_2|^3} - \frac{3(R_1 - R_2)^\alpha (R_1 - R_2)^\beta}{|R_1 - R_2|^5}.$$  

(2)

Here, the indices $\alpha$ and $\beta$ denote real space vector components. In Fourier space, the dipole sum for a cubic lattice becomes

$$A^{\alpha \beta}(k) = A^{\alpha \beta}_{2d}(p) + \frac{2\pi}{a_0} \sum_{\rho \neq 0} e^{i q \rho} \sum_{g} \frac{(p - g)^{\alpha} (p - g)^{\beta}}{|p - g|} e^{-|\rho||p-g|},$$

(3)

if the spins are oriented in the plane, and

$$A^{\alpha \beta}(k) = A^{\alpha \beta}_{2d}(p) - \frac{2\pi}{a_0} \sum_{\rho \neq 0} e^{i q \rho} \sum_{g} |p - g| e^{-|\rho||p-g|},$$

if the spins are oriented perpendicular to the plane \(^{[14]} \). $A^{\alpha \beta}_{2d}(p)$ represents the dipole tensor for the two-dimensional system, with wave vectors $p$ and associated 2-d reciprocal lattice vectors $g$. $\rho$ is the discrete coordinate of a lattice plane along the finite direction, and $a_0$ and $q$ indicate the corresponding lattice constant and wave numbers, respectively. For the later treatment by means of the renormalisation group technique, we need only retain the relevant long-wavelength contributions, and may thus simplify the above expressions by restricting the reciprocal wave vector sums to $g = 0$ only. This part yields the correct limits for the propagators of both the three- and the two-dimensional system.

Finally, we expand the resulting expressions to obtain the long-wavelength limits. For the easy axis lying in the plane we find the following form:

$$A^{\alpha \beta}(q, p) = (1 + f_1) \frac{p^\alpha p^\beta}{p} + f_2 \frac{p^\alpha p^\beta}{p} q^2 + f_3 p^\alpha p^\beta + f_4 p^\alpha p^\beta + f_5 p^\alpha p^\beta q^2 + \mathcal{O}(p^3 q^2, p^4, p^2 q^4).$$

(5)

Denoting with $2N + 1$ the number of lattice planes in the finite direction, $L = 2Na_0$, the $N$-dependent functions $f_i$ read:

$$f_1 = 2N,$$  

(6)

$$f_2 = -\frac{a_0^2}{6} N \left(1 + 3N + 2N^2\right),$$  

(7)

$$f_3 = -a_0 N(1 + N),$$  

(8)

$$f_4 = \frac{a_0^2}{6} N \left(1 + 3N + 2N^2\right),$$  

(9)

$$f_5 = \frac{a_0^3}{4} N^2(1 + N)^2.$$  

(10)

If the easy axis is aligned perpendicular to the plane, we arrive at

$$A^{\alpha \beta}(q, p) = -(1 + \tilde{f}_1) p - \tilde{f}_2 p^2 - \tilde{f}_3 p^2 q^2 + \mathcal{O}(p^3 q^2, p^4).$$

(11)
instead, with the \( N \)-dependent functions

\[
\tilde{f}_1 = 2N, \\
\tilde{f}_2 = -\frac{a_0^2}{6} N \left(1 + 3N + 2N^2\right), \\
\tilde{f}_3 = -a_0 N(1 + N).
\]

We note that the long-wavelength expansion does \textit{not} commute with taking the limit of infinite system thickness, \( N \to \infty \). Consequently, the above formulas are strictly valid only for finite system thickness \( L \). Nonetheless we shall see that the method to be described below will yield the correct results even for the three-dimensional system.

We are now in the position to write down the effective hamiltonians that serve as starting points for our renormalisation group treatment. Dealing with a finite system in one direction, we symbolically write the sum over all Fourier modes as

\[
\int \ldots = \frac{1}{(2\pi)^d} \int_{0 < |p| < \Lambda} d^d p \, \frac{1}{L} \sum_{q = \frac{\pi}{L} n} \ldots, \quad n \in \mathbb{Z}.
\]

In the limit of the fully infinite system, \( L \to \infty \), this definition is to be replaced with

\[
\int \ldots = \frac{1}{(2\pi)^D} \int_{0 < |p| < \Lambda} d^d p \int dq, \quad \text{where} \quad D = d + 1
\]

denotes the dimension of the entire system. In both situations, \( \Lambda \) represents a cutoff whose numerical value is not relevant for the critical behaviour. Therefore we shall use \( \Lambda = 1 \) throughout the paper (in appropriate dimensionless units). Notice that we only apply the cutoff for the wave vector components in the plane, and \textit{not} for the perpendicular component.

Thus the final effective free energy for a \textit{layer with easy axis in the plane} assumes the form

\[
\mathcal{H}_{2d,||} = \frac{1}{2} \int_k \left[ r_0 + c_0 k^2 + g(1 + f_1) \frac{p_i^2}{p} + g f_2 \frac{p_i^2}{p} q_i^2 \\
+ g f_3 p_i^2 + g f_4 p_i^2 p + g f_5 p_i^2 q_i^2 \right] S_0(k) S_0(-k) \\
+ \frac{u_0}{4!} \int_{k_1} \int_{k_2} \int_{k_3} \int_{k_4} S_0(k_1) S_0(k_2) S_0(k_3) S_0(k_4) \delta \left( \sum_{i=1}^4 k_i \right),
\]

where \( q \) again is the wave vector component perpendicular to the system, and \( p \) the in-plane part, with \( p_i \) denoting the component pointing along the direction of the easy axis. The hamiltonian for a \textit{layer with easy axis perpendicular to the plane} reads

\[
\mathcal{H}_{2d,\perp} = \frac{1}{2} \int_k \left[ r_0 + c_0 k^2 - g(1 + \tilde{f}_1) p - g \tilde{f}_3 p^2 - g \tilde{f}_2 q^2 p \right] S_0(k) S_0(-k) \\
+ \frac{u_0}{4!} \int_{k_1} \int_{k_2} \int_{k_3} \int_{k_4} S_0(k_1) S_0(k_2) S_0(k_3) S_0(k_4) \delta \left( \sum_{i=1}^4 k_i \right).
\]

For a \textit{three-dimensional system} we find instead

\[
\mathcal{H}_{3d} = \frac{1}{2} \int_k \left[ r_0 + c_0 k^2 + g \frac{q^2}{p^2} \right] S_0(k) S_0(-k) \\
+ \frac{u_0}{4!} \int_{k_1} \int_{k_2} \int_{k_3} \int_{k_4} S_0(k_1) S_0(k_2) S_0(k_3) S_0(k_4) \delta \left( \sum_{i=1}^4 k_i \right),
\]
where \( q \) denotes the component of \( k \) along the direction of the easy axis, and \( p \) is the part of \( k \) perpendicular to the easy axis.

Thus, henceforth we are concerned with the description of critical phenomena with those three different hamiltonians. The free energy of the three-dimensional system cannot be recovered any more from the layered hamiltonian (18) by taking the limit of infinite layer thickness \( N \to \infty \). Nevertheless we will show that all the limiting cases, including the three-dimensional uniaxial dipolar one, are described correctly within our scheme.

3. Renormalisation scheme

As already mentioned, the expected critical regimes are clearly characterised by different system dimensions, as well as different analytical structures of the propagators, leading to different upper critical dimensions. Hence we need a renormalisation scheme capable of dealing with these different fixed points in a unified manner, and sufficiently flexible as to describe the crossovers between them. In addition, we aim to capture the crossover features with the correct transition temperature \( T_c(L, g) \) (see appendix A) of the finite system, and not with that of the infinite system as in Refs. [7, 8, 9, 10], where furthermore the studied systems were infinite in at most one direction (and consequently do not display a proper phase transition). As our system asymptotically becomes two-dimensional, we have to circumvent problems with infrared divergences appearing in field-theoretic approaches, e.g., dimensional regularisation, for such low-dimensional systems [2, 3, 12, 15]. One more aspect is that we must include the full gradient terms in order to describe the crossover between Ising and uniaxial dipolar behaviour, and cannot restrict ourselves to merely the most relevant parts in one of the limits as in Ref. [16].

In order to accommodate all these requirements, we apply a variant of Wilson’s momentum shell renormalisation method, similar to the one used previously by Chakravarty, Halperin, and Nelson for the crossover theory of a quantum non-linear sigma model [11]. (Certainly to one-loop order, the ensuing wave vector shell integrals over \( p \), followed by the summations over \( q \), are technically considerably easier to perform in this approach as compared to the field theory version.) One renormalisation transformation now consists of two steps: Starting from the effective hamiltonian functional \( \mathcal{H} \), we first integrate out the short-wavelength order parameter fluctuations in the partition function \( Z(\mathcal{H}) \) in the \( d \) directions of the plane, and then sum over the wave vector components perpendicular to the plane (without any cutoff). In wave vector space, the modes to be integrated out are depicted in figure 1. In the second step, the wave vectors \( k \), spin variables \( S_k \), and system thickness \( L \) are rescaled with an as yet arbitrary scale factor \( b > 1 \), and a spin rescaling factor \( \zeta(b) \) which eventually has to be chosen appropriately in terms of \( b \):

\[
\begin{align*}
  k' &= b \ k, \\
  S_{k'} &= \zeta(b)^{-1} S_k,
\end{align*}
\]
After this transformation, we arrive at a new free energy functional $\mathcal{H}'$ of the same form as before, but with renormalised coefficients fulfilling $Z(\mathcal{H}) = Z(\mathcal{H}')$. By successively evaluating partial partition functions, and subsequently applying scale transformations, we approach the interesting infrared behaviour of our theory, and in effect map the perturbationally inaccessible critical regime onto a region in parameter space where perturbation theory may be applied. As usual the critical regions themselves, characterised by algebraic singularities and scale-invariance, are represented by fixed points of the renormalisation (semi-)group transformation.

We apply this scheme in the framework of the one-loop approximation, however refraining from any $\epsilon$ expansion, for in our problem there is no unique and common upper critical dimension. We choose our rescaling factors isotropic in contrast to, e.g., Ref. [16]; this induces apparently diverging parameters under the renormalisation flow [13]. Yet these may be readily absorbed into appropriately defined effective couplings, by means of which we can describe the associated crossover features [2, 3, 4, 5]. In contrast to Ref. [17] we define the spin rescaling factor $\zeta(b)$ such that the gradient term of the hamiltonian functional remains unchanged during the renormalisation procedure. Again, this helps us capturing the entire crossover regime, while the advantages of the scheme in Ref. [17], where the term with the lowest wave vector power is held fixed, are gathered through the construction and discussion of suitable effective couplings.

Figure 1. The modes being integrated out during one renormalisation step.
4. Renormalisation group flow equations

In this section, we compute the differential renormalisation group flow equations for a magnetic layer with the easy axis in the plane, a layer with easy axis perpendicular to the plane, and a three-dimensional system. Our starting points will be the different hamiltonians (17), (18), and (19), respectively, for these systems. Following through the renormalisation scheme as described in the previous section above, with \( b^\ell = e^{\delta \ell} \), \( \delta \to 0 \), for the finite systems we arrive at the following differential flow equations for the system thickness \( L \), the coefficient \( c \) of the gradient term, and the strength \( g \) of the dipole-dipole interaction:

\[
\frac{dL}{d\ell} = -L, \quad (23)
\]
\[
\frac{dc}{d\ell} = 0, \quad (24)
\]
\[
\frac{dg}{d\ell} = g. \quad (25)
\]

Before moving on to the flow equations specific to the different systems, we discuss these differential equations common to both finite systems. The flow equation (23) for the layer thickness \( L \) is exact, and is solved by \( L = L_0 e^{-\ell} \) with \( \ell \to +\infty \) in the infrared limit. Thus, the fixed points for the layer thickness are \( L = \infty \) and \( L = 0 \), with \( L = 0 \) being infrared-stable. Hence the asymptotic critical behaviour is that of the two-dimensional system: As the correlation length diverges, the system thickness becomes irrelevant in the renormalisation group sense (as indicated by the negative sign on the r.h.s. in equation (23).) The coefficient \( c \) of the gradient term in the free energy functional is held fixed during the renormalisation procedure through the choice of the spin rescaling factor \( \zeta \), whence equation (24) follows from definition. The strength of the non-analytic dipole-dipole interaction \( g \) cannot be renormalised perturbationally, and consequently equation (25) simply follows from dimensional analysis and holds to all orders in perturbation theory. Its solution reads \( g = g_0 e^{\ell} \), and the fixed points for the dipole strength are \( g = 0 \) and \( g = \infty \), and the latter is infrared-stable. As to be expected, the critical behaviour of the system is eventually dominated by the dipole interaction. This is a consequence of the long-range character of the dipolar forces in contrast to the short-ranged exchange interactions. For the derivation of the remaining flow equations we have to distinguish between the different systems under consideration.

First we focus on the layer with the easy axis in the plane. The remaining differential flow equations read

\[
\frac{df_1}{d\ell} = 0, \quad (26)
\]
\[
\frac{df_2}{d\ell} = -2 f_2, \quad (27)
\]
\[
\frac{df_3}{d\ell} = -f_3, \quad (28)
\]
\[
\frac{df_4}{d\ell} = -2 f_4, \quad (29)
\]
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Figure 2. Definitions of \( p_l \) and the angle \( \theta \).

\[
\frac{df_5}{dl} = -3 f_5 , \tag{30}
\]

following, to one-loop order, simply from dimensional analysis, and

\[
\frac{dr}{dl} = 2r + \frac{u}{2} \frac{\Lambda^2}{L} \int_0^{2\pi} d\theta \frac{L/2}{\sqrt{a(\theta)b(\theta)}} \coth\left(\frac{L}{2}\sqrt{\frac{a(\theta)}{b(\theta)}}\right) , \tag{31}
\]

\[
\frac{du}{dl} = (3 - d) u - \frac{3}{2} u^2 \frac{\Lambda^2}{L} \int_0^{2\pi} d\theta \frac{L/4}{\sqrt{b(\theta)^{1/2} a(\theta)^{3/2}}} \frac{1}{\sinh\left(\frac{L}{2}\sqrt{\frac{a(\theta)}{b(\theta)}}\right)} \times \left[ \cosh\left(\frac{L}{2}\sqrt{\frac{a(\theta)}{b(\theta)}}\right) + \frac{L}{2} \frac{a(\theta)}{b(\theta)} \frac{1}{\sinh\left(\frac{L}{2}\sqrt{\frac{a(\theta)}{b(\theta)}}\right)} \right] , \tag{32}
\]

to one-loop order. Here \( \theta \) indicates the angle between the wave vector component in the plane \( p \) and the direction of the easy axis \( p_l \) (see figure 2). The \( \theta \)-dependent functions \( a \) and \( b \) are defined as follows:

\[
a(\theta) = r + c \Lambda^2 + g(1 + f_1) \Lambda \cos^2 \theta + gf_3 \Lambda^2 \cos^2 \theta + gf_4 \Lambda^3 \cos^2 \theta , \tag{33}
\]

\[
b(\theta) = c + gf_2 \Lambda \cos^2 \theta + gf_5 \Lambda^2 \cos^2 \theta . \tag{34}
\]

While the coefficient \( f_1 \) remains invariant under the renormalisation group flow, the irrelevant parameters \( f_2, f_3, f_4 \) trivially flow towards their stable fixed points \( f_2 = f_3 = f_4 = 0 \). Notice that there exist no fixed points with \( f_i = \infty \), because the functions \( f_i \) were only defined for a finite system thickness. The more intricate flow equations for the temperature variable \( r \) and the non-linear coupling \( u \) will be discussed in the following sections.

For a layer with easy axis perpendicular to the plane, on the other hand, we find the following flow equations

\[
\frac{d\tilde{f}_1}{dl} = 0 , \tag{35}
\]

\[
\frac{d\tilde{f}_2}{dl} = -2 \tilde{f}_2 , \tag{36}
\]
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\[ \frac{d\tilde{f}_3}{d\ell} = -\tilde{f}_3, \quad (37) \]

from power counting, and

\[ \frac{dr}{d\ell} = 2r + \frac{u}{2} K \sum_q \frac{1}{a + bq^2}, \quad (38) \]
\[ \frac{du}{d\ell} = (3 - d) u - \frac{3}{2} u^2 K \sum_q \frac{1}{(a + bq^2)^2}. \quad (39) \]

Here the functions \( a \) and \( b \) are defined as follows

\[ a = r + (c - g\tilde{f}_3) \Lambda^2 - g(1 + \tilde{f}_1) \Lambda, \quad (40) \]
\[ b = c - g\tilde{f}_2 \Lambda - g\tilde{f}_4 \Lambda^2, \quad (41) \]

and we have introduced the used the dimension-dependent factor

\[ K = \frac{\Lambda^d}{(2\pi)^{d-1}}. \quad (42) \]

Closer inspection reveals that in the case of an easy axis perpendicular to the layer and with infinite anisotropy, the propagator is not positive definite. Inevitably therefore, the terms in the sums in equations (38) and (39) will become negative under the renormalisation group flow. This indicates an instability at non-zero wave vector, and the ground state of the system becomes inhomogeneous, with the characteristic wavelength of the emerging spatial structures given by that instability wave vector (the typical size of the structures appearing in the low-temperature phase is estimated in appendix B). At any rate, it is impossible to analyse the asymptotic critical behaviour with the effective hamiltonian (18) which was obtained via an expansion about a presumed homogeneous ground state. Hence, in the remainder of this paper we shall concentrate on the study of the critical behaviour of a layer with in-plane easy axis, and on the three-dimensional limit.

For a three-dimensional system, finally we obtain the renormalisation group flow equations

\[ \frac{dc}{d\ell} = 0 , \quad (43) \]
\[ \frac{dg}{d\ell} = 2g , \quad (44) \]
\[ \frac{dr}{d\ell} = 2r + \frac{u}{4} \frac{K}{\sqrt{(c + \frac{u^2}{4})(r + c\Lambda^2)}}, \quad (45) \]
\[ \frac{du}{d\ell} = (4 - D) u - \frac{3}{8} u^2 \frac{K}{\sqrt{(c + \frac{u^2}{4})(r + c\Lambda^2)^3}}. \quad (46) \]

These equations do not appear as the limit \( L \to \infty \) of the flow equations of the layer (31) and (32), where the effective free energy has already been adjusted to the specific geometries. In order to arrive at the correct 3-D expression, one has to start from the correct, full hamiltonian (19), as the long-wavelength limit and the limit \( N \to \infty \) do
not commute, as already mentioned in section 2. Again, \( c \) is held fixed by construction, and the renormalisation group flow for the relevant dipole-dipole interaction has the unstable fixed point \( g = 0 \) and the infrared-stable fixed point \( g = \infty \).

Before discussing the different critical regions and the crossover between them within our scheme, we focus on the obvious limiting cases of the general flow equations, and compute the critical exponents within dimensional \( \epsilon \) expansions about the appropriate upper critical dimensions, respectively.

5. Limiting cases in \( \epsilon \) expansions

5.1. Three-dimensional Ising limit

We arrive at the limit of the flow equations at the unstable fixed point \( g = 0 \) and \( L = \infty \) by expanding the hyperbolic functions in the differential flow equations (31) and (32) for the layer in the limit of large argument. Assuringly, we can alternatively take the flow equations for the three-dimensional system (45) and (46) with \( g = 0 \). With either approach, we obtain

\[
\frac{dr}{d\ell} = 2r + \frac{u}{4c(r + c\Lambda^2)} K, \tag{47}
\]

\[
\frac{du}{d\ell} = \epsilon u - \frac{3}{8} u^2 \frac{K}{\sqrt{c(r + c\Lambda^2)^3}}, \tag{48}
\]

where \( \epsilon = 4 - D \) gives the distance to the upper critical dimension \( D_c = 4 \) of the system. From these equations we get the critical exponents at the non-trivial fixed point to first order in \( \epsilon \),

\[
r^* = -\frac{\epsilon}{3} c \Lambda^2 + \mathcal{O}(\epsilon^2), \tag{49}
\]

\[
K u^* = \frac{8}{3} \epsilon \frac{1}{\sqrt{c(c\Lambda^2)^3}} + \mathcal{O}(\epsilon^2), \tag{50}
\]

as usual via the eigenvalues of the linearised flow equations. The results, consistently expanded to first order in \( \epsilon = 4 - D \) are

\[
\nu = \frac{1}{2} \left( 1 + \frac{\epsilon}{6} + \mathcal{O}(\epsilon^2) \right), \tag{51}
\]

\[
\eta = 0 + \mathcal{O}(\epsilon^2), \tag{52}
\]

\[
\gamma = 1 + \frac{\epsilon}{6} + \mathcal{O}(\epsilon^2), \tag{53}
\]

in accordance with the well-known results for the single-component \( \phi^4 \) model (see, e.g., Ref. [15]), with \( \epsilon = 1 \) in \( D = d + 1 = 3 \) dimensions.

5.2. Three-dimensional uniaxial dipolar limit

Next, we discuss the second limiting case in three dimensions, namely the uniaxial dipolar fixed point at \( g = \infty \) and \( L = \infty \). We have to start from equations (45) and
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Upon introducing the effective expansion parameter $v_{\text{eff}} = u/\sqrt{g}$, we arrive at the flow equations

$$
\frac{dr}{d\ell} = 2r + \frac{v_{\text{eff}}}{4} \frac{K\Lambda}{\sqrt{r + c\Lambda^2}},
$$

$$
\frac{dv_{\text{eff}}}{d\ell} = \epsilon v_{\text{eff}} - \frac{3}{8} v_{\text{eff}}^2 \frac{K\Lambda}{(r + c\Lambda^2)^{3/2}},
$$

where now $\epsilon = 3 - D$ was used for the deviation from the upper critical dimension $D_c = 3$. In $D = d + 1 = 3$ dimensions, we thus expect the mean-field critical exponents with logarithmic corrections. This is in agreement with previous work addressing the critical behaviour of a three-dimensional uniaxial dipolar magnet.

5.3. Two-dimensional Ising limit

In the limit of absent dipole-dipole interaction $g = 0$ and zero thickness $L = 0$ we have to expand the hyperbolic functions in the flow equations for vanishing arguments. With the effective coupling $v_{\text{eff}} = u/L$, this procedure gives the flow equations

$$
\frac{dr}{d\ell} = 2r + \frac{v_{\text{eff}}}{2} \frac{K}{(r + c\Lambda^2)},
$$

$$
\frac{dv_{\text{eff}}}{d\ell} = \epsilon v_{\text{eff}} - \frac{3}{2} v_{\text{eff}}^2 \frac{K}{(r + c\Lambda^2)^2};
$$

here, however, $\epsilon = 4 - d (= 2$ in $D = d + 1 = 3$ dimensions). Linearising in the vicinity of the non-trivial fixed point

$$
r^* = -\frac{\epsilon}{6} c\Lambda^2 + \mathcal{O}(\epsilon^2),
$$

$$
K v_{\text{eff}}^* = \frac{2}{3} \epsilon c^2 \Lambda^4 + \mathcal{O}(\epsilon^2),
$$

we find the critical exponents to order $\epsilon = 4 - d (= 2$ in two dimensions):

$$
\nu = \frac{1}{2} \left( 1 + \frac{\epsilon}{6} + \mathcal{O}(\epsilon^2) \right),
$$

$$
\eta = 0 + \mathcal{O}(\epsilon^2)
$$

$$
\gamma = 1 + \frac{\epsilon}{6} + \mathcal{O}(\epsilon^2).
$$

Thus, this low-dimensional limit is also correctly included in the general flow equations. Note that in terms of the expansion parameter $\epsilon$, these exponents of course coincide with the three-dimensional Ising case above. Remember, however, that there $\epsilon = 4 - D = 1$ and here $\epsilon = 4 - d = 2$ in the relevant physical situation $D = d + 1 = 3$. 
5.4. Two-dimensional uniaxial dipolar limit, in-plane easy axis

This limit is especially interesting for two reasons: First, it represents the asymptotic critical behaviour of the system under consideration; and second it is intriguing on its own right because we are aware of only one other reference which addresses the critical behaviour of a two-dimensional uniaxial dipolar system with the easy axis in the plane [18]. Yet, in Ref. [18] the critical exponents are not presented explicitly. Using the same approximation for the hyperbolic functions as before and introducing the effective expansion parameter

$$v_{\text{eff}} = \frac{u}{L^\sqrt{g}} ,$$

we obtain the asymptotic flow equations

$$\frac{dr}{d\ell} = 2r + \frac{v_{\text{eff}}}{\sqrt{2d\pi^{d-1}}(r + c \Lambda^2)(1 + f_1) \Lambda},$$

$$\frac{dv_{\text{eff}}}{d\ell} = \left(\frac{7}{2} - d\right) v_{\text{eff}} - \frac{3}{2} \frac{v_{\text{eff}}^2}{\sqrt{2d\pi^{d-1}}(r + c \Lambda^2)^3(1 + f_1) \Lambda} .$$

Consequently, we identify the upper critical dimension here as $d_c = 7/2$. The non-trivial fixed point to order $\epsilon = \frac{7}{2} - d$ is

$$r^* = -\frac{\epsilon}{3} c \Lambda^2 + O(\epsilon^2) ,$$

$$v_{\text{eff}}^* = \frac{2}{3} \frac{2d\pi^{d-1}}{\Lambda^2} [(1 + f_1) \Lambda]^{1/2} (c \Lambda^2)^{3/2} + O(\epsilon^2) .$$

Linearisation near this fixed point leads to the following asymptotic critical exponents to first order in $\epsilon = \frac{7}{2} - d$

$$\nu = \frac{1}{2} \left(1 + \frac{\epsilon}{6} + O(\epsilon^2)\right) ,$$

$$\eta = 0 + O(\epsilon^2) ,$$

$$\gamma = 1 + \frac{\epsilon}{6} + O(\epsilon^2) .$$

Once again, these exponents look superficially identical to those of the short-range Ising model; yet the critical dimension and the value of $\epsilon$ is different: in $d = 2$, $\epsilon = 1/2$. In summary, all the asymptotic limits are correctly included in the general flow equations.

6. Crossover analysis

In the previous section, we have demonstrated the consistency of our general approach with the $\epsilon$ expansions near the respective critical dimensions of the limiting universality classes. We now proceed to the application of our scheme to the description of the actual crossovers between the different critical regions.
6.1. General procedure

At first we want to describe the general procedure to be used. We use the calculated renormalisation group flow equations to investigate the scale dependence of the effective interaction parameters. Specifically, we map the critical region onto a region in parameter space that is perturbationally accessible. In the infrared region, we can, e.g., evaluate the static susceptibility perturbationally, and thus obtain the static susceptibility $\chi$ in the critical region via the solution of the renormalisation group equation that takes the form (the Fisher exponent $\eta$ vanishes in the one-loop approximation)

$$\chi(r_0, g_0, u_0, L_0, k) = e^{2\ell} \chi(r(\ell), g(\ell), u(\ell), L(\ell), k(\ell)).$$  \hspace{1cm} (73)

Therefrom we can infer the effective exponent

$$\gamma_{\text{eff}}(\tau) = \frac{d \ln \chi^{-1}}{d \ln \tau}$$ \hspace{1cm} (74)

as function of reduced temperature $\tau = \frac{T-T_c}{T_c}$. This method was used first in [19] in the framework of an $\epsilon$ expansion; but here we make no use of any dimensional expansion, but perform all our calculations numerically at a fixed dimension. The one-loop recursion relations are then solved without recourse to any additional approximation.

It is vital here to introduce an appropriate effective expansion parameter that yields the correct 3-D and 2-d limits as given in the previous section. This effective coupling has to have a finite limit at the asymptotic fixed point below the upper critical dimension of the system. As suggested by the flow equations in section [4], we use the effective couplings

$$v_{\text{eff}} = u \int_0^{2\pi} d\theta \frac{1}{\sqrt{a(\theta)b(\theta)}} \coth \left( \frac{L}{2} \sqrt{\frac{a(\theta)}{b(\theta)}} \right)$$ \hspace{1cm} (75)

for the layer with in-plane easy axis, and

$$v_{\text{eff}} = \frac{2\pi}{\sqrt{r+c\Lambda^2}} \frac{u}{\sqrt{g}}$$ \hspace{1cm} (76)

for the three-dimensional case. Then within our methodical framework, the critical regions are again represented by fixed points of the renormalisation group transformation, and we may obtain the critical exponents from the eigenvalues of the linearised transformation in the vicinity of the fixed points (yet without applying any further approximation like dimensional expansions).

6.2. Limiting cases at fixed dimension

The appearance of renormalisation group fixed points to first order in an expansion does not necessarily guarantee the existence of fixed points if we refrain from an $\epsilon$ expansion. At first, therefore, we have to re-calculate the non-trivial fixed points and the associated critical exponents at fixed dimensionality $D = d + 1$. The actual fixed points thus obtained are listed in table [4]. The eigenvalues of the linearised flow equations can be
Table 1. Renormalisation group fixed points at fixed dimension to one-loop order.

| Fixed point       | $r^*$ | $v_{eff}^*$ |
|-------------------|-------|-------------|
| 3-D Ising         | $-\frac{2}{3}(4-D)c\Lambda^d\left[2 + \frac{2}{3}(4-D)\right]^{-1}$ | $\frac{16\pi}{3K_d}(4-D)\sqrt{(r^* + c\Lambda^2)}$ |
| 2-d Ising         | $-(4-d)c\Lambda^2/(4-d+6)$ | $\frac{16\pi}{K_d}(4-d)c\Lambda^2/(4-d+6)$ |
| 3-D uniaxial dipolar | 0     | 0           |
| 2-d uniaxial dipolar | $-\left(\frac{7}{2} - d\right)c\Lambda^2/\left(\frac{7}{2} - d + 3\right)$ | $\frac{16\pi}{K_d}\left(\frac{7}{2} - d\right)c\Lambda^2/\left(\frac{7}{2} - d + 3\right)$ |

Table 2. Numerically determined fixed point values and critical exponents.

| Fixed point       | $r^*$ | $v_{eff}^*$ | $L^*$ | $\gamma$ | $\nu$ |
|-------------------|-------|-------------|-------|----------|-------|
| 3-D Ising         | -0.25 | 78.95       | $\infty$ | 1        | 1/2   |
| 2-d Ising         | -0.25 | 78.95       | 0     | 1        | 1/2   |
| 3-D uniaxial dipolar | 0     | 0           | $\infty$ | 1        | 1/2   |
| 2-d uniaxial dipolar | -0.33 | 105.27      | 0     | 0.94     | 0.47  |

easily calculated analytically, and are not presented explicitly here. In table 2 we give the numerical values of the fixed points as well as the critical exponents $\nu$ and $\gamma$ for the system with two infinite dimensions ($D = d + 1 = 3$). As is readily seen, the numerical results for the critical exponents cannot be judged as good approximations to their realistic values. The reason is very likely the inadequacy of the one-loop approximation. An unfortunate artefact of the system under consideration is that the actual fixed point values as well as the critical exponents for the 3-D Ising and the 2-d Ising limit turn out identical to this order. This happens to be the case only in this dimension $D = 3$, as can be seen from figure 3, where we plot the numerical values of the critical exponent $\nu$ at both the higher and lower dimensional fixed points, respectively, vs the dimension. This artificial coincidence only appears at the dimensions $d = \frac{5}{2}$ and $d = 2$, the physical dimension in which we are interested. Nevertheless we shall argue for the usefulness of our scheme in describing different crossover features.

6.3. Dimensional crossover

The first crossover that we want to analyse within our scheme is the dimensional crossover for an Ising layer (dipolar strength $g = 0$). In order to circumvent the unfortunate exponent coincidence mentioned above, we evaluate the crossover features between 3.2 and 2.2 dimensions. In this situation, we are not only able to calculate the flow equations describing the crossover, but also the effective critical exponent $\gamma_{eff}$ as a function of the reduced temperature $\tau$. The corresponding renormalisation group fixed point values are listed in table 3. The effective exponent $\gamma_{eff}$ describing the crossover is calculated numerically for different layer thicknesses $L$, and the results are plotted in figure 4. It can be seen that at a temperature sufficiently far from the
transition temperature, the susceptibility exponent takes on the value of the higher-dimensional fixed point, then with decreasing temperature goes through a minimum which is obviously independent of the layer thickness, and finally reaches the asymptotic value of the lower-dimensional fixed point. The thicker the layer, the system remains the longer in the higher-dimensional critical region as the temperature is lowered, and reaches the later the asymptotic critical region. This is fully to be expected based on straightforward considerations.

6.4. Crossover exchange-dominated / dipole-dominated

Next we apply our scheme to the fully three-dimensional system, where the crossover occurs between the region with dominating exchange interactions and the asymptotic region with dominating dipole-dipole interaction. Because the numerical values of the critical exponents happen to be identical at both these fixed points within our scheme at fixed dimension $D = 3$, see table 3, the crossover cannot be noticed at all in the temperature dependence of the effective critical exponent $\gamma_{\text{eff}}$. However, we can discuss the crossover by means of the flow diagrams in the $r - v_{\text{eff}}$ plane. In figure 3, the flow diagram is depicted for different initial temperatures (which cannot be distinguished in this graph, however). The arrow indicates the starting point of the trajectories. These first approach the unstable Ising fixed point (filled circle on the right), before finally approaching the asymptotically stable fixed point that represents the uniaxial dipolar

| fixed point | $r^*$ | $v_{\text{eff}}^*$ | $L^*$ | $\nu$ |
|-------------|-------|-------------------|-------|-------|
| 3.2-D Ising | $-0.2105$ | $96.0267$ | $\infty$ | $0.5090$ |
| 2.2-d Ising | $-0.2308$ | $105.2601$ | $0$ | $0.5070$ |
Figure 4. Layer with vanishing dipole-dipole interaction: Effective critical exponent $\gamma_{\text{eff}}$ vs reduced temperature $\tau \sim r - r_c$ for systems of different thicknesses: $N = 20$ (full line), $N = 50$ (dashed), $N = 100$ (dotted), $N = 500$ (long-dashed).

regime (filled circle on the left). Eventually the trajectories diverge in the temperature variable $r$ due to the numerically inevitably finite distance from the critical point (even small inaccuracies in determining the true critical temperature will eventually blow up under renormalisation). In figure 4 we depict the same scenario for a stronger dipole-dipole interaction. In this situation, the unstable fixed point is not approached as closely as before, and the stable fixed point is approached more closely before the trajectories eventually diverge to $r \to \infty$ due to the finite distance from the critical temperature.

6.5. Crossover features of the dipolar layer

At last, we address the description of the crossover scenario for a dipolar magnetic layer with in-plane easy axis. As the starting point is the hamiltonian (17), which is only defined for finite system thickness, we cannot reach the fixed point corresponding to the three-dimensional uniaxial dipolar region. The crossover will thus occur between the critical regions of the three- and two-dimensional Ising systems, and the two-dimensional uniaxial dipolar regime, which represents the asymptotic universality class. As the numerical fixed point values of the effective coupling, the temperature variable, and the effective critical exponent of the static susceptibility $\gamma_{\text{eff}}$ of the two- and the three-dimensional Ising fixed point are identical, we illustrate the flow trajectories in three-
Figure 5. Infinite three-dimensional dipolar system: Flow of the effective coupling $v_{\text{eff}}$ and temperature variable $r$ for different initial temperature values, with $g_0 = 10^{-5}$ and $v_0 = 5$: $r \approx r_c$ (full line), $r - r_c = 6.42 \cdot 10^{-11}$ (dotted), $r - r_c = 6.66 \cdot 10^{-9}$ (dashed), $r - r_c = 1.67 \cdot 10^{-8}$ (long-dashed).

dimensional parameter space, consisting of $r$, $v_{\text{eff}}$, and the thickness-dependent variable $\frac{N}{N+1}$ as third coordinate. In terms of this quantity, the three- and two-dimensional Ising fixed points assume the value 1 and 0, respectively, and thus become distinguishable. In figures 6, 7, and 8 we show the trajectories with initial values in the vicinity of the critical temperature for increasing values of the dipole strength. At first the trajectories approach the three-dimensional Ising fixed point, then the two-dimensional Ising fixed point, and finally the asymptotic two-dimensional uniaxial dipolar fixed point. Eventually even the tiny numerical deviation from the critical temperature gives rise to a divergence in $r$ again. For stronger dipole interaction, the Ising fixed points are not approached as closely. In figure 10 we furthermore depict the temperature dependence of the trajectories: The higher the temperature the sooner the trajectories start to diverge.
7. Summary

In this paper we have analysed crossover scenarios for a uniaxial dipolar magnetic layer. We have derived the appropriate effective free energy functional in the long-wavelength limit for a finite system with thickness $L$ in one spatial direction; the resulting dipolar hamiltonian turned out to be considerably different from the one for the infinite three-dimensional system. In the long-wavelength limit, which does not commute with the limit $L \to \infty$, the propagator of the three-dimensional system cannot be recovered from the propagator of the layer in the limit of infinite layer thickness.

We have presented a unified renormalisation scheme, within which the critical behaviour of the layer with finite thickness can nevertheless be treated as well as that of the infinite system. This renormalisation procedure is required to be capable of capturing the crossovers between critical regions with different system dimensions, different analytical structures of the propagator due to the dipole-dipole interaction,
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Figure 7. Dipolar layer with easy in-plane axis: Flow diagram for $g_0 = 10^{-7}, u_0 = 10, r_0 \sim r_c, N_0 = 100$.

Figure 8. Dipolar layer with easy in-plane axis: Flow diagram for $g_0 = 10^{-6}, u_0 = 10, r_0 \sim r_c, N_0 = 100$. 
Figure 9. Dipolar layer with easy in-plane axis: Flow diagram for $g_0 = 10^{-5}, u_0 = 10, r_0 \sim r_c, N_0 = 100$.

Figure 10. Dipolar layer with easy in-plane axis: Flow diagram for $g_0 = 10^{-6}, u_0 = 10, r_0 - r_c = 8.33 \cdot 10^{-10}, 2.33 \cdot 10^{-10}, 0.53 \cdot 10^{-10}, r_0 \sim r_c$, (from left to right), $N_0 = 100$. 
and consequently different upper critical dimensions. Hence we had to refrain from any dimensional $\epsilon$ expansion near any of the fixed points, and have derived the renormalisation group flow equations in the one-loop approximation at fixed dimension. Furthermore, we have introduced a common effective expansion parameter, as suggested by the flow equations, which leads to finite non-trivial fixed points characterising the different critical regions. As a test of the consistency of our approach, the critical exponents in the limiting cases of the 3-d Ising, 2-d Ising, 3-d uniaxial dipolar, and 2-d uniaxial (in-plane easy axis) dipolar universality classes were calculated within $\epsilon$ expansions with respect to the appropriate upper critical dimensions, respectively, and reproduced correctly known results from the literature. The critical exponents for the asymptotic limit of a two-dimensional uniaxial dipolar system with in-plane easy axis were, to our knowledge, computed explicitly for the first time by means of an expansion near the upper critical dimension $d_c = 7/2$.

Performing the calculations without the $\epsilon$ expansion, however, reveals an unfortunate insufficiency of the one-loop approximation: The numerical values of the temperature variable, the effective coupling, and the critical exponents at the three-dimensional and the two-dimensional Ising fixed points happen to be identical in $D = 3$ dimensions. In order to test our scheme, we have first evaluated the dimensional crossover of the effective critical exponent $\gamma_{\text{eff}}$ of an Ising system from $D = 3.2$ to $D = 2.2$ dimensions. Second, the crossover features of the three-dimensional uniaxial dipolar system as well as the uniaxial dipolar layer were analysed and discussed in form of renormalisation group flow diagrams. Thus, we hope to have demonstrated the feasibility of the description of such very complex crossover features within the framework of our renormalisation group approach, which should be further applicable to a diverse range of interesting physical systems.

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Appendix A. Critical temperature

A number of the above calculations depended on the knowledge of the phase transition temperature which, due to thermal fluctuations, is shifted downwards as compared to the mean-field transition temperature. The correct location of the critical point (to one-loop order) was determined numerically through identifying that starting point of the temperature variable $r$, which separates those renormalisation group flow trajectories that eventually flow to $r = +\infty$ from the ones that eventually flow to $r = -\infty$.

Figures A1 and A2 show the dependence of the numerically obtained phase transition temperature $r_c$ upon the strength of the dipole-dipole interaction $g$ and the
layer thickness $N$. As one would expect physically, $r_c$ grows both with increasing dipole strength and increasing layer thickness.

**Appendix B. Layer with out-of-plane easy axis**

As already mentioned, the propagator for the dipolar layer with out-of-plane easy axis,

$$G_0(q, p)^{-1} = r_0 + c_0 k^2 - g(1 + \tilde{f}_1) p - g\tilde{f}_3 p^2 - g\tilde{f}_2 q^2 p,$$

is not positive definite under the renormalisation group flow. This indicates an instability of the homogeneous ground state at some finite wave vector. Therefore this propagator cannot be used in a straightforward manner for the calculation of the critical behaviour of the system. Nevertheless we can draw some information regarding the instability from it. Thus, from the equation

$$\left. \frac{\partial}{\partial p} \right|_{q=0} \left( r + c k^2 - g(1 + \tilde{f}_1) p - g\tilde{f}_3 p^2 - g\tilde{f}_2 q^2 p \right) = 0 \quad \text{(B.2)}$$

we may infer the typical size of the structures in the low-temperature phase, as function of the dipole strength $g$ and the layer thickness $L$. The result is

$$D_{\text{domain}} = \frac{2\pi \left[ c + g \frac{k}{2} \left( 1 + \frac{L}{a_0} \right) \right]}{g \left( 1 + \frac{L}{a_0} \right)}. \quad \text{(B.3)}$$

I.e., starting from a constant domain size at vanishing layer thickness, the domain size decreases with increasing thickness, runs through a minimum, and finally grows
Figure A2. Phase transition temperature $r_c(N)$ vs layer thickness ($g_0 = 1 \cdot 10^{-5}, u_0 = 0.1$).

again linearly with $L$. The result that a layer with out-of-plane easy axis should condense inhomogeneously is in accordance with Refs. [20, 21, 22]; but the increase of the domain size is predicted to be proportional to the square root of the layer thickness in Refs. [20, 22], and $\sim L^{1/3}$ in Ref. [21].

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