Improved bound on vertex degree version of Erdős matching conjecture

Mingyang Guo | Hongliang Lu | Yaolin Jiang

School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an, Shaanxi, China

Correspondence
Hongliang Lu, School of Mathematics and Statistics, Xi’an Jiaotong University, 710049 Xi’an, Shaanxi, China. Email: luhongliang215@sina.com

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Abstract
For a $k$-uniform hypergraph $H$, let $\delta_1(H)$ denote the minimum vertex degree of $H$, and $\nu(H)$ denote the size of the largest matching in $H$. In this paper, we show that for any $k \geq 3$ and $\beta > 0$, there exists an integer $n_0(\beta, k)$ such that for positive integers $n \geq n_0$ and $m \leq (\frac{k}{2(k-1)} - \beta)n$, if $H$ is an $n$-vertex $k$-graph with $\delta_1(H) > \binom{n-1}{k-1} - \binom{n-m}{k-1}$, then $\nu(H) \geq m$. This improves upon earlier results of Bollobás, Daykin, and Erdős for the range $n > 2k^3(m+1)$ and Huang and Zhao for the range $n \geq 3k^2m$.

KEYWORDS
fractional matching, hypergraph, matching, minimum degree

MATHEMATICAL SUBJECT CLASSIFICATION
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1 | INTRODUCTION

Let $k$ be a positive integer. For a set $S$, let $\binom{S}{k} := \{T \subseteq S : |T| = k\}$. A hypergraph $H$ consists of a vertex set $V(H)$ and an edge set $E(H)$ whose members are subsets of $V(H)$. A hypergraph $H$ is $k$-uniform if $E(H) \subseteq \binom{V(H)}{k}$, and a $k$-uniform hypergraph is also called a $k$-graph. We use $e(H)$ to denote the number of edges of $H$.

Let $H$ be a $k$-graph and $T \subseteq V(H)$. The degree of $T$ in $H$, denoted by $d_H(T)$, is the number of edges of $H$ containing $T$. Let $\ell$ be a nonnegative integer; then $\delta_{\ell}(H) := \min\{d_H(T) : T \in \binom{V(H)}{\ell}\}$ denotes the minimum $\ell$-degree of $H$. Hence, $\delta_0(H)$ is the number of edges in $H$. Note that $\delta_1(H)$ is often called the minimum vertex degree of $H$, and $\delta_{k-1}(H)$ is also known as the minimum codegree of $H$. A matching in $H$ is a set of pairwise disjoint edges of $H$, and it is called a perfect matching if the union of all edges of the matching is $V(H)$. We use
\(\nu(H)\) to denote the size of the largest matching in \(H\). Let \(K^k_r\) denote a complete \(k\)-graph with \(r\) vertices and edge set \(\binom{\nu(k^k)}{k}\).

Erdős and Gallai [5] determined the threshold of \(\delta_0(G)\) for a 2-graph \(G\) to contain a matching of given size, and Erdős [4] conjectured the following generalization to \(k\)-graphs for \(k \geq 3\): The threshold of \(\delta_0(H)\) for a \(k\)-graph \(H\) on \(n\) vertices to contain a matching of size \(m\) is

\[
\max \left\{ \left( \frac{km - 1}{k} \right), \left( \frac{n}{k} - \frac{n - m + 1}{k} \right) \right\} + 1.
\]

For recent results on this conjecture, we refer the reader to [1, 6, 7, 10].

Rödl, Ruciński, and Szemerédi [19] determined the minimum codegree threshold for the existence of perfect matchings in \(k\)-graphs. It is conjectured in [9, 15] that the \(\ell\)-degree threshold for the existence of a perfect matching in a \(k\)-graph \(H\) is

\[
\delta_\ell(H) > \left( \max \left\{ \frac{1}{2}, 1 - \left( 1 - \frac{1}{k} \right)^{\ell-1} \right\} + o(1) \right) \left( \frac{n - \ell}{k - \ell} \right)
\]

for \(k \geq 3\) and \(1 \leq \ell < k\). The first term \((1/2 + o(1))\left(\frac{n - \ell}{k - \ell}\right)\) comes from a parity construction: Take disjoint nonempty sets \(A\) and \(B\) with \(|A| - |B| \leq 2\) and \(|A| \equiv 1 \pmod{2}\), and form a hypergraph \(H\) by taking all \(k\)-subsets of \(A \cup B\) with \(|f \cap A| \equiv 0 \pmod{2}\). The second term is given by the hypergraph obtained from \(K^k_n\) (the complete \(k\)-graph on \(n\) vertices) by deleting all edges from a subgraph isomorphic to \(K^k_{n-k+1}\). Treglown and Zhao [20, 21] determined the minimum \(\ell\)-degree threshold for the appearance of perfect matchings in \(k\)-graphs, for \(k/2 \leq \ell \leq k - 2\). For a 3-graph \(H\), Hán, Person, and Schacht [9] showed that \(\delta_1(H) > (5/9 + o(1))\left(\frac{\nu(H)}{2}\right)\) is sufficient for the appearance of a perfect matching of \(H\). Kühn, Osthus, and Townsend [16] proved a stronger result: There exists a positive integer \(n_0\) such that if \(H\) is a 3-graph with \(|\nu(H)| = n \geq n_0\), \(m\) is an integer with \(1 \leq m \leq n/3\), and \(\delta_1(H) > \left(\frac{n-1}{2}\right) - \left(\frac{n-m}{2}\right)\), then \(\nu(H) \geq m\). For \(k \in \{3, 4\}\), Khan [12, 13] showed that there exists a positive integer \(n_0\) such that if \(H\) is a \(k\)-graph with \(|\nu(H)| = n \geq n_0\) and \(\delta_1(H) > \left(\frac{n-1}{k-1}\right) - \left(\frac{n-m}{k-1}\right)\), then \(H\) has a perfect matching, where \(n \equiv 0 \pmod{k}\).

Bollobás, Daykin, and Erdős [3] proved that for integers \(k \geq 2\) and \(m \geq 1\), if \(H\) is a \(k\)-graph with \(|\nu(H)| = n > 2k^3(m + 1)\) and \(\delta_1(H) > \left(\frac{n-1}{k-1}\right) - \left(\frac{n-m}{k-1}\right)\), then \(\nu(H) \geq m\). The bound for \(\delta_1(H)\) is tight. To see it, we define an \(n\)-vertex \(k\)-graph \(H_k(U, W) = (V, E)\), where \(V\) is partitioned into \(U \cup W\), and \(E = \{e \in \binom{V}{k}: 1 \leq |e \cap W| \leq k - 1\}\). When \(|W| = m - 1\), it is easy to see that \(\delta_1(H_k(U, W)) = \left(\frac{n-1}{k-1}\right) - \left(\frac{n-m}{k-1}\right)\) and \(\nu(H_k(U, W)) = m - 1\). We denote \(H_k(U, W)\) with \(|U| = n - m + 1\) and \(|W| = m - 1\) by \(H_k(n, m)\).

This result was improved by Huang and Zhao [11], who proved that for \(n > 3k^2m\), if \(\delta_1(H) > \left(\frac{n-1}{k-1}\right) - \left(\frac{n-m}{k-1}\right)\), then \(\nu(H) \geq m\). Huang and Zhao [11] also proposed the following conjecture.
**Conjecture 1** (Huang and Zhao [11]). Given positive integers \( m, k, n \) such that \( m < n/k \), let \( H \) be a \( k \)-graph on \( n \) vertices. If \( \delta_1(H) > \left( \frac{n-1}{k-1} \right) - \left( \frac{n-m}{k-1} \right) \), then \( \nu(H) \geq m \).

Kupavskii [14] proved an \( \ell \)-degree version result for \( 1 \leq \ell \leq k - 1 \), which confirmed Conjecture 1 for \( n \geq 2k^2 \) and \( k \geq 3(m - 1) \). When \( n \geq 2(k - 1)m \), an asymptotic version of this conjecture was proved by Kühn, Osthus, and Treglown [17]. For fractional matchings, Huang and Zhao [11] proved that when \( n \geq (2m - 1)(k - 1) - m + 2 \), every \( k \)-graph \( H \) on \( n \) vertices with \( \delta_1(H) > \left( \frac{n-1}{k-1} \right) - \left( \frac{n-m}{k-1} \right) \) contains a fractional matching of size \( m \). Frankl and Kupavskii [7] improved the result of Kühn, Osthus, and Treglown [17] and the fractional version result of Huang and Zhao [11] by extending the range of \( n \) to \( n \geq \frac{5}{3}(k - 1)m - \frac{2}{3}m + 1 \).

In the paper, we obtain the following result.

**Theorem 2.** Let \( k \geq 3 \) be an integer. For any \( \beta > 0 \), there exists an integer \( n_0 = n_0(\beta, k) \) such that the following holds. Let \( n, m \) be integers such that \( n \geq n_0 \) and \( 1 \leq m \leq \left( \frac{k}{2(k-1)} - \beta \right)^n \). Let \( H \) be a \( k \)-graph on \( n \) vertices. If \( \delta_1(H) > \left( \frac{n-1}{k-1} \right) - \left( \frac{n-m}{k-1} \right) \), then \( \nu(H) \geq m \).

Given two \( k \)-graphs \( H_1, H_2 \) and a real number \( \varepsilon > 0 \), we say that \( H_2 \) \( \varepsilon \)-contains \( H_1 \) if \( V(H_1) = V(H_2) \) and \( |E(H_1) \setminus E(H_2)| \leq \varepsilon |V(H_1)|^k \). Let \( H \) be an \( n \)-vertex \( k \)-graph. We say that \( H \) \( \varepsilon \)-contains \( H_k(n, m) \) if there exists a partition \( U, W \) of \( V(H) \) with \( |W| = m - 1 \) and \( H \) \( \varepsilon \)-contains \( H_k(U, W) \). Otherwise, we say that \( H \) does not \( \varepsilon \)-contain \( H_k(n, m) \). Our proof of Theorem 2 consists of two parts by considering whether or not \( H \) \( \varepsilon \)-contains \( H_k(n, m) \), which is similar to the arguments in [19].

When \( H \) \( \varepsilon \)-contains \( H_k(n, m) \), Lu, Yu, and Yuan [18] use the structure of \( H_k(n, m) \) to greedily find the desired matching (Lemma 2.3 in [18]).

**Lemma 3** (Lu, Yu, and Yuan [18]). Let \( n, m, k \) be integers and \( 0 < \varepsilon < (8k^{-1}k^{-5}(k-1)!)^{-3} \), such that \( k \geq 3 \), \( n \geq 8k^2/(1 - 5k^2\varepsilon) \), and \( n/(2k^4) + 1 < m \leq n/k \). Let \( H \) be a \( k \)-graph on \( n \) vertices such that \( \delta_1(H) > \left( \frac{n-1}{k-1} \right) - \left( \frac{n-m}{k-1} \right) \) and \( H \) \( \varepsilon \)-contains \( H_k(n, m) \), then \( \nu(H) \geq m \).

Thus for completing the proof of Theorem 2, it suffices to prove the following lemma. By \( x \ll y \) we mean that \( x \) is sufficiently small compared with \( y \) such that \( x, y \) satisfy finitely many inequalities in the proof.

**Lemma 4.** Let \( k \geq 3 \) be an integer and let \( \varepsilon, \rho, \beta \) be constants such that \( 0 < \varepsilon < (3^{k-2}k^3)^{-1} \) and \( 0 < \beta \ll \rho < (\varepsilon^2)/(18k)^2 \). Let \( n, m \) be two positive integers such that \( n \) is sufficiently large, and \( n/\varepsilon^4 \leq m \leq \left( \frac{k}{2(k-1)} - \beta \right)^n \). Let \( H \) be a \( k \)-graph on \( n \) vertices. If \( H \) does not \( \varepsilon \)-contain \( H_k(n, m) \) and \( \delta_1(H) > \left( \frac{n-1}{k-1} \right) - \left( \frac{n-m}{k-1} \right) - \rho n^k \), then \( \nu(H) \geq m \).
To prove Lemma 4, we first construct a $k$-graph $H^k_r$ from $H$ such that $v(H) \geq m$ if and only if $H^k_r$ has an almost perfect matching. In Section 2, we follow some ideas from [1, 7, 17, 18] to prove that $H^k_r$ has a perfect fractional matching by using a stability result proved by Lu, Yu, and Yuan [18]. In Section 3, we use the two-round randomization method from [1] to show that $H^k_r$ has a nearly regular spanning subhypergraph in which all 2-degrees are much smaller than the vertex degrees. Then a result of Frankl and Rödl [8] implies that $H^k_r$ has an almost perfect matching.

In section 2, we apply the stability result proved by Lu, Yu, and Yuan [18] to an $(n - 1)$-vertex $(k - 1)$-graph. Since their result holds for $m \leq n/(2k)$, we are only able to prove Lemma 7 for $m \leq \frac{n-1}{2(k-1)}$. In section 3, we use randomization method and Lemma 7 to prove Lemma 4. To prove a random subgraph has a perfect fractional matching by using Lemma 7, we need $m \leq \left(\frac{k}{2(k-1)} - \beta\right)^n$, where $\beta$ is a small positive.

We end this section with additional notations. For any positive integer $n$, let $[n] := \{1, ..., n\}$. For a $k$-graph $H$ and $S \subseteq V(H)$, we use $H - S$ to denote the hypergraph obtained from $H$ by deleting $S$ and all edges of $H$ intersecting set $S$, and we use $H[S]$ to denote the subhypergraph with vertex set $S$ and edge set $\{e \in E(H) : e \subseteq S\}$. For a $k$-graph $H$ and a vertex $v \in V(H)$, let $N_H(v) := \{e \in \binom{V(H)}{k-1} : e \cup \{v\} \in E(H)\}$. We omit the floor and ceiling functions when they do not affect the proof.

2 | FRACTIONAL MATCHING

A fractional matching in a $k$-graph $H$ is a function $\varphi : E(H) \rightarrow [0, 1]$ such that for any $v \in V(H)$, $\sum_{e \in E: v \in e} \varphi(e) \leq 1$. A fractional matching is called perfect if $\sum_{e \in E} \varphi(e) = \frac{|V(H)|}{k}$. For a hypergraph $H$, let

$$v'(H) = \max \left\{ \sum_{e \in E(H)} \varphi(e) : \varphi \text{ is a fractional matching in } H \right\}.$$  

A fractional vertex cover of $H$ is a function $w : V(H) \rightarrow [0, 1]$ such that for each $e \in E$, $\sum_{v \in e} w(v) \geq 1$. Let

$$\tau'(H) = \min \left\{ \sum_{v \in V(H)} w(v) : w \text{ is a fractional vertex cover of } H \right\}.$$  

Then the strong duality theorem of linear programming gives

$$v'(H) = \tau'(H).$$  

For a complete $k$-graph, we have the following observation.

**Observation 5.** Let $n, k$ be two integers such that $n > k \geq 2$. Let $H$ be a $k$-graph on $n$ vertex with edge set $\binom{V(H)}{k}$, then $H$ has a perfect fractional matching.
Proof. Let $V(H) = [n]$ and let $e_i = \{i, ..., i + k - 1\}$ for $i = 1, ..., n$, where the addition is on modular $n$ (except we write $n$ instead of 0). Write $E = \{e_1, ..., e_n\}$. Note that $e_i \in E(H)$. Let $\varphi : E(H) \rightarrow [0, 1]$ such that

$$\varphi(e) = \begin{cases} 1/k, & e \in E, \\ 0, & \text{otherwise}. \end{cases}$$

It is not difficult to see that $\varphi$ is a perfect fractional matching in $H$. \hfill \Box

Recall that $K_r^k$ is a complete $k$-graph with $r$ vertices and edge set $\binom{V(K_r^k)}{k}$. Let $H_r^k = H + K_r^k$ denote a $k$-graph with vertex set $V(H) \cup V(K_r^k)$ and edge set

$$E(H_r^k) = E(H) \cup \left\{ e \in \binom{V(H) \cup V(K_r^k)}{k} : e \cap V(K_r^k) \neq \emptyset \right\}.$$ 

In this section, we prove that $H_r^k$ satisfying the conditions in Lemma 7 has a perfect fractional matching. To prove $H_r^k$ has a perfect fractional matching, we need the following lemma for stable $k$-graphs (Lemma 4.2 in [18]). Let $H$ be a $k$-graph with vertex set $[n]$. For any $\{u_1, ..., u_k\}, \{v_1, ..., v_k\} \subseteq \binom{[n]}{k}$ with $u_i < u_{i+1}$ and $v_i < v_{i+1}$ for $1 \leq i \leq k - 1$, we write $\{u_1, ..., u_k\} \leq \{v_1, ..., v_k\}$ if $u_i \leq v_i$ for $1 \leq i \leq k$. $H$ is called stable if for $e, f \in \binom{[n]}{k}$ with $e \leq f$, $f \in E(H)$ implies that $e \in E(H)$.

Lemma 6 (Lu, Yu, and Yuan [18]). Let $k$ be a positive integer, and let $c$ and $\rho$ be constants such that $0 < c < 1/(2k)$ and $0 < \rho \leq (1 + 18(k - 1)/c)^{-2}$. Let $n, m$ be positive integers such that $n$ is sufficiently large and $cn \leq m \leq n/(2k)$. Let $H$ be a stable $k$-graph with vertex set $[n]$. If $e(H) > \binom{n}{k} - \binom{n-m}{k} - \rho n^k$ and $\nu(H) \leq m$, then $H$ $\sqrt{\rho}$-contains $H_k([n] \setminus [m], [m])$.\textsuperscript{1}

Let $H$ be a $k$-graph on $n$ vertices and $U, W$ be a partition of $V(H)$. Given $0 < \theta < 1$, a vertex $v \in V(H)$ is $\theta$-good with respect to $H_k(U, W)$ if $|N_{H_k(U, W)}(v) \setminus N_{H}(v)| \leq \theta n^{k-1}$. Otherwise $v$ is $\theta$-bad. A set $I \subseteq V(H)$ that contains no edge of $H$ is called an independent set in $H$. We use $\alpha(H)$ to denote the size of the largest independent set in the hypergraph $H$.

Lemma 7. Let $k \geq 3$ be an integer and let $\rho, \varepsilon$ be constants such that $0 < \varepsilon \leq (3k^2 - 2k)k^{-1}$ and $0 < \rho < \varepsilon^4/(2k^8)$. Let $n, m, r$ be integers such that $n$ is sufficiently large, $\frac{n-1}{2k^4} \leq m \leq \frac{n-1}{2(k-1)}$ and $(r-k)(k-1) \geq n - km$. Let $H$ be a $k$-graph on $n$ vertices such that $\alpha(H) < n - m - \varepsilon n$. If $\delta_1(H) > \binom{n-1}{k-1} - \binom{n-m}{k-1} - \rho n^{k-1}$, then $H_r^k$ has a perfect fractional matching.

\textsuperscript{1}In [18], the conclusion is $H$ $\sqrt{\rho}$-contains $H_k([n] \setminus [m], [m])$, where $H_k([n] \setminus [m], [m])$ is the $k$-graph with vertex set $[n]$ and edge set $\{e \in \binom{[n]}{k} : e \cap [m] \neq \emptyset\}$. Since $H_k([n] \setminus [m], [m])$ is a subgraph of $H_k([n] \setminus [m], [m])$, this conclusion implies $H$ $\sqrt{\rho}$-contains $H_k([n] \setminus [m], [m])$. 


Proof. Let \( V(H) = [n] \) and let \( Q = V(H^k_r) \setminus V(H) = \{ n + 1, \ldots, n + r \} \). Let \( w : V(H^k_r) \to [0, 1] \) be a minimum fractional vertex cover of \( H^k_r \). Rename the vertices in \([n]\) such that
\[
w(1) \geq w(2) \geq \cdots \geq w(n).
\]
(1)

Let \( H' \) be a \( k \)-graph with vertex set \( V(H^k_r) \) and edge set
\[
E(H') = \left\{ e \in \left( V(H^k_r) \right)_k : \sum_{x \in e} w(x) \geq 1 \right\}.
\]

Since \( \sum_{x \in e} w(x) \geq 1 \) for every \( e \in E(H^k_r) \), \( H' \) is a superhypergraph of \( H^k_r \). Let \( G = H' - Q \). Thus \( G \) is a superhypergraph of \( H \). Let \( G' \) be the \((k-1)\)-graph with vertex set \([n-1]\) and edge set \( N_G(n) \).

Claim 1.

(i) \( G' \) is stable.
(ii) Let \( S := [m + \varepsilon n] \), then \( G[S] \) is a complete \( k \)-graph.
(iii) For any \( e \in N_G(n) \), if \( i \in [n] \setminus e \), then \( e \in N_G(i) \).

Proof. For two sets \( \{x_1, \ldots, x_{k-1}\}, \{y_1, \ldots, y_{k-1}\} \subseteq \binom{[n-1]}{k-1} \) such that \( x_i \leq y_i \) for \( 1 \leq i \leq k-1 \), if \( \{y_1, \ldots, y_{k-1}\} \subseteq E(G') \), then \( \{y_1, \ldots, y_{k-1}, n\} \subseteq E(G) \). Thus \( \sum_{i=1}^{k-1} w(x_i) + w(n) \geq 1 \). By (1), we can derive that \( \sum_{i=1}^{k-1} w(x_i) + w(n) \geq 1 \). That is, \( \{x_1, \ldots, x_{k-1}\} \subseteq E(G') \). Thus \( G' \) is stable, and (i) follows. To prove (ii), suppose that \( G[S] \) is not a complete \( k \)-graph. Then there exists a set \( \{v_1, \ldots, v_k\} \subseteq \binom{S}{k} \) such that \( \{v_1, \ldots, v_k\} \notin E(H') \). Thus \( \sum_{i=1}^{k} w(v_i) < 1 \). Let \( S' := \{m + \varepsilon n - 1, \ldots, n\} \). By (1), for every \( e \in \binom{S}{k} \), we have \( \sum_{x \in e} w(x) \leq \sum_{i=1}^{k} w(v_i) < 1 \). Thus \( S' \) is an independent set in \( G \) with \( |S'| \geq n - m - \varepsilon n \), contradicting the fact that \( \alpha(G) \leq \alpha(H) < n - m - \varepsilon n \). To prove (iii), let \( e \in N_G(n) \). Since \( E(G) \subseteq E(H') \), we have \( \sum_{x \in e} w(x) + w(n) \geq 1 \). Thus by (1), \( \sum_{x \in e} w(x) + w(i) \geq 1 \) for any \( i \in [n] \setminus e \). That is, \( e \in N_G(i) \) for any \( i \in [n] \setminus e \).

One can see that \( w \) is also a fractional vertex cover of \( H' \). Thus \( w \) is also a minimum fractional vertex cover of \( H' \). By Linear Programming Duality Theory, we have \( \nu'(H^k_r) = \tau'(H^k_r) = \tau'(H') = \nu'(H') \). Thus it suffices to prove that \( H' \) has a perfect fractional matching. Suppose that \( n + r \equiv s \pmod{k} \), where \( 0 \leq s \leq k - 1 \). Let \( Q' = \{n + 1, \ldots, n + s\} \). We first find a perfect matching in \( H' - Q' \), then use it to construct a perfect fractional matching in \( H' \).

Claim 2. \( \nu(G) \geq m \).
Proof. Let \( W := [m], \, U := [n - 1] \setminus [m] \). Assume \( G' \) does not \( \sqrt{2\rho} \)-contain \( H_{k-1}(U, W) \). Let \( c = 1/(2k^4) \) be a constant as in Lemma 6. Note that \( 2\rho < c^h/k^8 < (k!k^5)^{-4} \leq (1 + 18(k - 2)!/c)^{-2}, \, \frac{n - 1}{2k^4} \leq m \leq \frac{n - 1}{2(k - 1)}, \) and \( e(G') = |N_G(n)| > \left( \frac{n - 1}{k - 1} \right) - \left( \frac{n - 1 - m}{k - 1} \right) - 2\rho(n - 1)^{k-1} \). Recall that \( G' \) is stable by Claim 1. We thus derive that \( \nu(G') > m \) by the contrapositive of Lemma 6. Let \( M_i = \{ f_i, \ldots, f_m \} \) be a matching of size \( m \) in \( G' \) and let \( \{ v_1, \ldots, v_m \} \subseteq [n] \setminus \bigcup_{i=1}^m f_i \). By Claim 1(iii), we have \( M_i \subseteq N_G(v) \) for any \( v \in [n] \setminus \bigcup_{i=1}^m f_i \). Thus \( M'_i = \{ f_i \cup \{ v_i \} : i \in [m] \} \) is a matching of size \( m \) in \( G \).

So we may assume that \( G' \sqrt{2\rho} \)-contains \( H_{k-1}(U, W) \). Then \( G' \) contains less than \( (k - 1)(2\rho)^{1/4}n \) \( (2\rho)^{1/4} \)-bad vertices with respect to \( H_{k-1}(U, W) \). Otherwise,

\[
|E(H_{k-1}(U, W)) \setminus E(G')| \geq \frac{1}{k - 1} \sum_{v \in V(G')} |N_{H_{k-1}(U, W)}(v) \setminus N_G(v)|
\]

\[
> \frac{1}{k - 1} \cdot (k - 1)(2\rho)^{1/4}n \cdot (2\rho)^{1/4}(n - 1)^{k-2}
\]

\[
> \sqrt{2\rho}(n - 1)^{k-1},
\]

a contradiction. Let \( B \) denote the set of \( (2\rho)^{1/4} \)-bad vertices in \( W \). Write \( b := |B|. \) So \( b < (k - 1)(2\rho)^{1/4}n. \)

First we find a matching \( M_{21} \) of size \( b \) in \( G[U] \). Let \( S_i := \{ m, \ldots, m + \varepsilon n \}. \) By Claim 1(ii), we know that \( G[S_i] \) is a complete \( k \)-graph. Since \( b < (k - 1)(2\rho)^{1/4}n \) and \( 2\rho < c^h/k^8 \), we have \( bk \leq \varepsilon n. \) Thus we can find pairwise disjoint edges \( f_1, \ldots, f_b \) in \( G[S_i]. \) Thus \( M_{21} = \{ f_1, \ldots, f_b \} \) is a matching in \( G[U]. \)

Let \( U_1 := U \setminus V(M_{21}), \, W_1 := W \setminus B \), and \( G'' := G' - (V(M_{21}) \cup B). \) Thus \( V(G'') = U_1 \cup W_1 \) and \( |W_1| = m - b. \) For every \( x \in W_1 \), since \( x \) is \( (2\rho)^{1/4} \)-good in \( G' \) with respect to \( H_{k-1}(U, W) \) and \( N_{H_{k-1}(U, W)}(x) \setminus N_G(x) \subseteq N_{H_{k-1}(U, W)}(x) \setminus N_G'(x), \) we have

\[
|N_{H_{k-1}(U, W)}(x) \setminus N_G(x)| \leq |N_{H_{k-1}(U, W)}(x) \setminus N_G'(x)| \leq (2\rho)^{1/4}(n - 1)^{k-2} < \left( \frac{n/3}{k - 2} \right).
\]

It follows that

\[
N_G'(x) \cap \left( \begin{array}{l}
U_1 \\
k - 2
\end{array} \right) \geq \left| N_{H_{k-1}(U, W)}(x) \cap \left( \begin{array}{l}
U_1 \\
k - 2
\end{array} \right) \right| - \left| N_{H_{k-1}(U, W)}(x) \setminus N_G'(x) \right|
\]

\[
> \left( \begin{array}{l}
|U_1| \\
k - 2
\end{array} \right) - \left( \frac{n/3}{k - 2} \right).
\]

That is, every vertex \( x \in W_1 \) has large degree in \( G''. \)

Now we use vertices in \( W_1 \) to construct a matching \( M_{22} = \{ e_1, \ldots, e_{m-b} \} \) in \( G'' \) such that \( |e_i \cap W_1| = 1 \) for \( i = 1, \ldots, m - b. \) Suppose for some integer \( 0 < t \leq m - b - 1, \) we have found a matching \( \{ e_1, \ldots, e_t \} \) in \( G'' \) such that \( |e_i \cap W_1| = 1 \) for \( 1 \leq i \leq t. \) Note that
Let $x \in W_1 \setminus \bigcup_{i=1}^{t} e_i$, by inequalities (2) and (3), we have

$$\left| N_{G'}(x) \cap \left( U_i \setminus \bigcup_{i=1}^{t} e_i \right) \right| \geq \left| N_{G'}(x) \cap \left( U_i \setminus k - 2 \right) \right| - \left| \left| U_i \setminus \bigcup_{i=1}^{t} e_i \right| \right| > 0.$$ 

Thus there exists an edge $e_{+i+1} \subseteq V(G') \setminus \bigcup_{i=1}^{t} e_i$ such that $e_{+i+1} \cap W_i = \{x\}$. Continue this process until $t = m - b - 1$. Then $M_{22} = \{e_1, ..., e_{m-b}\}$ is the desired matching.

Recall that $n \geq km$. Let $v_1, ..., v_{m-b}$ be $m - b$ vertices of $G - V(M_{21} \cup M_{22})$. By Claim 1(iii), we have $M_{22} \subseteq N_{G}(v_i)$ for $1 \leq i \leq m - b$. Thus $M_{22}' = \{e_i \cup \{v_i\} : i \in [m - b]\}$ is a matching of size $m - b$ in $G - V(M_{21})$. Then $M_{21} \cup M_{22}'$ is a matching of size $m$ in $G$ and thus $\nu(G) \geq m$. \hfill \Box

Let $M$ be a matching of size $m$ of $G$. Note that $N_{H'}(n+i) = \binom{[n+r]-i}{k-1}$ for $1 \leq i \leq r$ and $r - s \geq (n - km)/(k - 1)$.

Thus $H' - Q' - V(M)$ has a perfect matching, say $M'$. For the case $s = 0$, $M \cup M'$ is a perfect matching in $H'$. For the case $s \neq 0$, let $f \in M'$. By Observation 5, $H'[f \cup Q']$ has a perfect fractional matching $\varphi$. Let $\varphi' : E(H') \to [0, 1]$ such that

$$\varphi'(f) = \begin{cases} 1, & e \in M \cup (M' - f), \\ \varphi(e), & e \in E(H'[f \cup Q']), \\ 0, & \text{otherwise}. \end{cases}$$

Recall that $M \cup (M' - f)$ is a perfect matching in $H' - (Q' \cup f)$. So $\varphi'$ is a perfect fractional matching in $H'$. This completes the proof.

### 3 ALMOST PERFECT MATCHING

The following lemma asserts that the existence of an almost perfect matching in any nearly regular $k$-graph in which all 2-degrees are much smaller than the vertex degrees (see Theorem 1.1 in [8] or Lemma 4.2 in [1]). For any positive integer $\ell$, we use $\Delta_\ell(H)$ to denote the maximum $\ell$-degree of a hypergraph $H$. 
Lemma 8 (Frankl and Rödl [8]). For every integer \( k \geq 2 \) and any real \( \sigma > 0 \), there exist \( \tau = \tau(k, \sigma) \), \( d_0 = d_0(k, \sigma) \) such that for every \( n \geq D \geq d_0 \) the following holds: Every \( k \)-graph \( H \) on \( n \) vertices with \( (1 - \tau)D < d_H(v) < (1 + \tau)D \) and \( \Delta_2(H) < \tau D \) contains a matching covering all but at most \( \sigma n \) vertices.

Let \( Bi(n, p) \) be the binomial distribution with parameters \( n \) and \( p \). The following lemma on Chernoff bound can be found in Alon and Spencer [2, p. 313].

Lemma 9 (Chernoff). Suppose \( X_1, \ldots, X_n \) are independent random variables taking values in \( \{0, 1\} \). Let \( X = \sum_{i=1}^{n} X_i \) and \( \mu = \mathbb{E}[X] \). Then, for any \( 0 < \delta \leq 1 \),

\[
P[X \geq (1 + \delta)\mu] \leq e^{-\delta^2\mu/3} \quad \text{and} \quad P[X \leq (1 - \delta)\mu] \leq e^{-\delta^2\mu/2}.
\]

In particular, when \( X \sim Bi(n, p) \) and \( \lambda \leq \frac{3}{2} np \), then

\[
P(|X - np| \geq \lambda) \leq e^{-\Omega(\lambda^2/np)}.
\]

To find a spanning subgraph in a hypergraph satisfying conditions in Lemma 8, we use the same two-round randomization technique as in [1]. The only difference is that between the two rounds, we also need to bound the independence number of the subgraph. The following lemma (Lemma 5.4 in [18]) was proved by Lu, Yu, and Yuan using hypergraph container method.

Lemma 10 (Lu, Yu, and Yuan [18]). Let \( 0 < \gamma < c, \vartheta, \zeta \) be positive reals and let \( k, n \) be positive integers such that \( n \) is sufficiently large. Let \( H \) be an \( n \)-vertex \( k \)-graph such that \( e(H) \geq cn^k \) and \( e(H[S]) \geq \vartheta e(H) \) for all \( S \subseteq V(H) \) with \( |S| \geq \zeta n \). Let \( R \subseteq V(H) \) be obtained by taking each vertex of \( H \) uniformly at random with probability \( n^{-0.9} \). Then \( \alpha(H[R]) \leq (\zeta + \gamma)n^{0.1} \) with probability at least \( 1 - e^{-\Omega(n^{0.1})} \).

Lemma 11. Let \( n, k, m \) be integers such that \( k \geq 3, n \) \( n/2k^4 \leq m < n/k \). Let \( 0 < \varepsilon < 1/k \) and \( 0 < \varphi < \varepsilon/12 \). Let \( H \) be a \( k \)-graph on \( n \) vertices. If \( \delta_1(H) \geq \left( \frac{n-1}{k-1} \right) - \left( \frac{n-m}{k-1} \right) - \varphi n^{k-1} \) and \( H \) does not \( \varepsilon \)-contain \( H_k(n, m) \), then \( e(H[S]) \geq \frac{cn^k}{2k^2} \) for any set \( S \subseteq V(H) \) with \( |S| \geq \left( 1 - \frac{m}{n} - \frac{\varepsilon}{7} \right)n \).

Proof. Suppose that the result does not hold. Then \( H \) has a set \( A \) such that \( |A| \geq \left( 1 - \frac{m}{n} - \frac{\varepsilon}{7} \right)n \) and \( e(H[A]) < \frac{cn^k}{2k^2} \). After removing vertices from \( A \) if necessary, we may assume that \( |A| \leq n - m \). Let \( W \subseteq V(H) \setminus A \) such that \( |W| = m - 1 \). Let \( U = V(H) \setminus W \) and \( B = U \setminus A \). One can see that \( |B| \leq \varepsilon n/7 + 1 \). Since \( e(H[A]) < \frac{cn^k}{2k^2} \), the number of edges belonging to \( H[U] \) is at most

\[
\sum_{i=1}^{k} \binom{|B|}{i} \binom{|U \setminus B|}{k-i} + e(H[A]) < \sum_{i=1}^{k} \frac{\left( \frac{cn}{7} + 1 \right)^i n^{k-i}}{i! (k-i)!} + \frac{\varepsilon n^k}{2k^2} \leq \sum_{i=1}^{k} \frac{\varepsilon^i n^k}{i! (k-i)!} + \frac{\varepsilon n^k}{2k^2} \leq \frac{k \varepsilon n^k}{6(k-1)!} + \frac{\varepsilon n^k}{2k^2}.
\]
for sufficiently large $n$. So we have

$$\sum_{x \in U} |N_{H[U]}(x)| = k \cdot e(H[U]) \leq k \left( \frac{k \epsilon n^k}{6(k-1)!} + \frac{\epsilon n^k}{2k^2} \right).$$

Since every edge of $H_k(U, W)$ intersects $U$, we may infer that

$$|E(H_k(U, W)) \backslash E(H)| \leq \sum_{x \in U} |N_{H_k(U,W)}(x) \backslash N_H(x)|$$

$$= \sum_{x \in U} \left( |N_{H_k(U,W)}(x)| - |N_H(x) \backslash N_{H[U]}(x)| \right)$$

$$\leq \sum_{x \in U} \left( \binom{n-1}{k-1} - \binom{n-m}{k-1} - \binom{n-1}{k-1} - \binom{n-m}{k-1} \right)$$

$$\leq \varrho n^{k-1} - |N_{H[U]}(x)|$$

$$= |U| \left( \binom{n-1}{k-1} - \binom{n-m}{k-1} - \binom{n-1}{k-1} - \binom{n-m}{k-1} \right)$$

$$\leq |U| \varrho n^{k-1} + k \left( \frac{k \epsilon n^k}{6(k-1)!} + \frac{\epsilon n^k}{2k^2} \right)$$

$$\leq |U| \varrho n^{k-1} + 3\epsilon n^k/4 + \epsilon n^k/2k \quad \text{(since } k \geq 3)$$

$$\leq \epsilon n^k,$$

where the second equality follows from $N_{H_k(U,W)}(x) \cap N_H(x) = N_H(x) \backslash N_{H[U]}(x)$ for each $x \in U$. Hence $H$ $\epsilon$-contains $H_k(U, W)$, a contradiction. \(\square\)

The following lemma can be found in [1, 18] (Lemma 5.5 in [18]), which is the first round of randomization.

**Lemma 12.** Let $k \geq 3$ be an integer. Let $H$ be a $k$-graph on $n$ vertices. Take $n^{1.1}$ independent copies of $R$ and denote them by $R^i$, $1 \leq i \leq n^{1.1}$, where $R$ is chosen from $V(H)$ by taking each vertex uniformly at random with probability $n^{-0.9}$ and then deleting less than $k$ vertices uniformly at random so that $|R| \in k\mathbb{Z}$. For each $S \subseteq V(H)$, let $Y_S := |\{i : S \subseteq R^i\}|$. Then with probability at least $1 - o(1)$, all of the following statements hold:

(i) for every $v \in V$, $Y_{[v]} = (1 + o(1))n^{0.2}$,

(ii) every pair $\{u, v\} \subseteq V$ is contained in at most two copies $R^i$,

(iii) every edge $e \in E(H)$ is contained in at most one copy $R^i$,

(iv) for all $i = 1, \ldots, n^{1.1}$, we have $||R^i| - n^{0.1}|| \leq n^{0.95}$, and

(v) if $\mu, \rho$ are constants with $0 < \mu < \rho$, $n/k - \mu n \leq m \leq n/k$, and $\delta_i(H) \geq \binom{n-1}{k-1} - \binom{n-m}{k-1} - \rho n^{k-1}$, then for all $i = 1, \ldots, n^{1.1}$ and any positive real $\rho' \geq 2\rho$, we have
\[ \delta_1(H[R_i]) > \left(\frac{|R_i| - 1}{k - 1}\right) - \left(\frac{|R_i| - |R_i|/k}{k - 1}\right) - \rho'|R_i|^{k-1}. \]

We summarize the second round of randomization in [1] as the following lemma (see the proof of Claim 4.1 in [1]).

**Lemma 13.** Assume \( R^i, i = 1, 2, \ldots, n^{1.1} \) satisfy (i)-(v) in Lemma 12, and each \( H[R^i] \) has a perfect fractional matching \( \phi^i \). Then there exists a spanning subgraph \( H' \) of \( H \) such that \( d_{H'}(v) = (1 + o(1))n^{0.2} \) for each \( v \in V \), and \( \Delta_2(H') \leq n^{0.1} \).

**Proof of Lemma 4.** Let \( V(H) = [n] \) and \( \eta = \beta/3 \). We choose an integer \( r \) such that
\[ r = \left\lfloor n - km - \eta n \right\rfloor/k - 1. \]

Let \( Q := V(K_r^k) = \{n + 1, \ldots, n + r\} \). Recall that \( H_r^k = H + K_r^k \) and let \( n_i := n + r \). By \( \delta_1(H) > \left(\frac{n_i - 1}{k - 1}\right) - \left(\frac{n_i - m - \rho n^k}{k - 1}\right) - \rho n^k \), we can derive that
\[ \delta_1(H_r^k) > \left(\frac{n_i - 1}{k - 1}\right) - \left(\frac{n_i - m - r}{k - 1}\right) - \rho n^k. \]

It suffices to show that \( \nu(H_r^k) \geq m + r \). Indeed, if there is a matching \( M \) of size \( m + r \) in \( H_r^k \), then there are at most \( r \) edges in \( M \) intersecting \( Q \) and thus \( \nu(H) \geq m \).

Let \( R \subseteq V(H_r^k) \) be obtained by taking each vertex of \( H_r^k \) uniformly at random with probability \( n_i^{-0.9} \). Take \( n_i^{1.1} \) independent copies of \( R \) and denote them by \( R^i, 1 \leq i \leq n_i^{1.1} \).

By Lemma 12(iv), we have
\[ n_i^{0.1} - n_i^{-0.095} \leq |R^i| \leq n_i^{0.1} + n_i^{-0.095} \quad \text{for all } i = 1, \ldots, n_i^{1.1} \]
with probability \( 1 - o(1) \). One can see that \( |V(H)| \geq \frac{k-1}{k}|V\left(H_r^k\right)| \geq \frac{2}{3}|V\left(H_r^k\right)| \) since \( r(k - 1) < n \) and \( k \geq 3 \). For each \( i \), \( |R^i \cap V(H)| \) is a binomial random variable with expectation \( nn_i^{-0.9} \). Applying Lemma 9 with \( \lambda = n_i^{-0.095} \), we have
\[ P\left(|R^i \cap V(H)| - nn_i^{-0.9} \right) \geq n_i^{0.095}\right) \leq e^{-\Omega(n_i^{0.09})}. \]

Thus by the union bound, we have
\[ nn_i^{-0.9} - n_i^{-0.095} \leq |R^i \cap V(H)| \leq nn_i^{-0.9} + n_i^{-0.095} \quad \text{for all } i = 1, \ldots, n_i^{1.1} \]
with probability at least \( 1 - n_i^{1.1}e^{-\Omega(n_i^{0.09})} \). Write \( r_i := |R^i \cap Q| \). With similar discussion, one can see that
\[ rn_i^{-0.9} - n_i^{-0.095} \leq r_i \leq rn_i^{-0.9} + n_i^{-0.095} \quad \text{for all } i = 1, \ldots, n_i^{1.1} \]
with probability at least $1 - n_1^{-1}e^{-\Omega(n_1^{0.09})}$. Thus by (4), (7), and (8), we have

$$ (n - k)(k - 1) \geq \left( rn_1^{-0.9} - n_1^{-0.095} - k \right)(k - 1) $$
$$ \geq (n - km - \eta n)n_1^{-0.9} - \left( n_1^{-0.095} + k \right)(k - 1) $$
$$ \geq |R^i \cap V(H)| - kmn_1^{-0.9} - 2\eta mn_1^{-0.9} \quad \text{for all } i = 1, ..., n_1^{1.1} \tag{9} $$

with probability $1 - o(1)$.

Since $H$ does not $\varepsilon$-contain $H_k(n, m)$, then by Lemma 11, $e(H[S]) \geq \varepsilon n^k/2k^2$ for all $S \subseteq V(H)$ with $|S| \geq \alpha n$, where $\alpha = 1 - m/n - \varepsilon/7$. Since each vertex in $Q$ has degree $\left( n_1^{-1} \right)$, replacing a vertex in $S$ by a vertex in $Q$ will not decrease the number of edges in $H_k^n[S]$. Thus for every $S \subseteq V(H_k^n)$ with $|S| \geq \alpha n$, we have $e(H_k^n[S]) \geq \varepsilon n^k/2k^2 \geq (\frac{k-1}{k})^k \varepsilon n_1^k/2k^2$. Then by Lemma 10, with probability $1 - o(1)$, for each $i$, $H^k_r[R^i]$ has independence number $\alpha(H^k_r[R^i]) \leq (n_1 - m - r - \varepsilon n/8)n_1^{-0.9} = (n - m - \varepsilon n/8)n_1^{-0.9}$. Note that $\alpha(H[R^i \cap V(H)]) = \alpha(H^k_r[R^i])$. So we have

$$ \alpha(H[R^i \cap V(H)]) \leq (1 - m/n - 2\eta/k - \varepsilon/9)|R^i \cap V(H)| \quad \text{for all } i = 1, ..., n_1^{1.1} \tag{10} $$

with probability $1 - o(1)$.

Note that $n_1/k - 2\eta m_1 \leq m + r \leq n_1/k$, where $2\eta \ll \rho$. By Lemma 12(v) and inequality (5), with probability $1 - o(1)$, for all $i = 1, ..., n_1^{1.1}$, we have

$$ \delta_1(H^k_r[R^i]) > \frac{|R^i| - 1}{k - 1} - \frac{|R^i| - |R^i|/k}{k - 1} - 3\rho |R^i \cap V(H)|^{k-1}. \tag{11} $$

So by inequalities (6), (8), (11) and $n_1/k - 2\eta m_1 \leq m + r \leq n_1/k$, with probability $1 - o(1)$, for all $i = 1, ..., n_1^{1.1}$, we have

$$ \delta_1(H[R^i \cap V(H)]) = \delta_1(H^k_r[R^i]) - \left( \frac{|R^i| - 1}{k - 1} - \frac{|R^i \cap V(H)| - 1}{k - 1} \right) $$
$$ > \left( \frac{|R^i \cap V(H)| - 1}{k - 1} - \frac{|R^i| - |R^i|/k}{k - 1} - 3\rho |R^i \cap V(H)|^{k-1} \right) $$
$$ = \left( \frac{|R^i \cap V(H)| - 1}{k - 1} - \frac{|R^i \cap V(H)| - (|R^i| - kn_i)/k}{k - 1} - 3\rho |R^i \cap V(H)|^{k-1} \right) $$
$$ > \left( \frac{|R^i \cap V(H)| - 1}{k - 1} - \frac{|R^i \cap V(H)| - (m + 2\eta n/k)n_1^{-0.9}}{k - 1} - 4\rho |R^i \cap V(H)|^{k-1} \right). \tag{12} $$

By $\frac{n_1^{-1}}{m} \leq \frac{k}{2(k - 1)} - \beta \frac{n_1^{-1}}{k}$ and inequality (6), we have $\frac{|R^i \cap V(H)| - 1}{k^2} \leq (m + 2\eta n/k)n_1^{-0.9} \leq \frac{|R^i \cap V(H)| - 1}{2(k - 1)}$, with probability $1 - o(1)$, for all $i = 1, ..., n_1^{1.1}$. Hence by Lemma 7 and by (9), (10), and (12), with probability $1 - o(1)$, for all $i = 1, ..., n_1^{1.1}$, $H^k_r[R^i]$ has a perfect fractional matching.
Now for the $k$-graph $H^k_r$, we have chosen $n_1^{1.1}$ subgraphs $R^1, \ldots, R^{n_1^{1.1}}$ such that (i)–(v) in Lemma 12 hold and $H^k_r[R^i]$ has a perfect fractional matching for $1 \leq i \leq n_1^{1.1}$. Then by Lemma 13, there is a spanning subgraph $H'$ of $H^k_r$ such that $d_{H'}(v) = (1 + o(1))n_1^{0.2}$ for any vertex $v$, and $\Delta_2(H') \leq n_1^{0.1}$.

Thus we may apply Lemma 8 to find a matching covering all but at most $\sigma n_1$ vertices in $H^k_r$, where $\sigma < 2\eta/3$ is a positive constant. Hence we have $\nu(H^k_r) \geq (n + r - \sigma n_1)/k > m + r$ by (4). This completes the proof.

4 | PROOF OF THEOREM 2

Since the case $n \geq 3k^2m$ was proved by Huang and Zhao [11], we may assume $m \geq n/3k^2$ in the proof. Let $\rho, \varepsilon$ be constants such that $0 < \varepsilon < (k^8 - 1)k^{5(k-1)k!} - 3 < (3k^2 - 2k!k^3)^{-1}$ and $0 < \rho < \varepsilon^4/(18k^2)^4$. For the case $H$ does not $\varepsilon$-contain $H_k(n, m)$, let $\beta_0 \leq \beta$ be a constant such that $0 < \beta_0 \ll \rho$. By Lemma 4, $\nu(H) \geq m$ for $n/3k^2 \leq m \leq (\frac{k}{2(k-1)} - \beta_0)n/k$ and sufficiently large $n$. For the case $H$ $\varepsilon$-contains $H_k(n, m)$, $\nu(H) \geq m$ for $n/3k^2 \leq m \leq n/k$ and sufficiently large $n$ by Theorem 3.

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ORCID
Hongliang Lu © http://orcid.org/0000-0003-4447-2416

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