On Top-k Weighted Sum Aggregate Nearest and Farthest Neighbors in the $L_1$ Plane

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Abstract. In this paper, we study top-$k$ aggregate (or group) nearest neighbor queries using the weighted Sum operator under the $L_1$ metric in the plane. Given a set $P$ of $n$ points, for any query consisting of a set $Q$ of $m$ weighted points and an integer $k$, $1 \leq k \leq n$, the top-$k$ aggregate nearest neighbor query asks for the $k$ points of $P$ whose aggregate distances to $Q$ are the smallest, where the aggregate distance of each point $p$ of $P$ to $Q$ is the sum of the weighted distances from $p$ to all points of $Q$. We build an $O(n \log n \log \log n)$-size data structure in $O(n \log n \log \log n)$ time, such that each top-$k$ query can be answered in $O(m \log m + (k + m) \log^2 n)$ time. We also obtain other results with trade-off between preprocessing and query. Even for the special case where $k = 1$, our results are better than the previously best method (in PODS 2012), which requires $O(n \log^2 n)$ preprocessing time, $O(n \log^2 n)$ space, and $O(m^2 \log^2 n)$ query time. In addition, for the one-dimensional version of this problem, our approach can build an $O(n)$-size data structure in $O(n \log n)$ time that can support $O(\min\{k, \log m\} \cdot m + k + \log n)$ time queries. Further, we extend our techniques to the top-$k$ aggregate farthest neighbor queries, with the same bounds.

1 Introduction

Top-$k$ nearest neighbor searching has been well-studied, e.g., see [19] for a survey. For a set $P$ of points in the $d$-D space $\mathbb{R}^d$, the problem asks for a data structure to quickly report the $k$ nearest neighbors in $P$ for any query point. Aggregate nearest neighbor (ANN) searching, also known as group nearest neighbor searching [2,16,17,18,19,21,22,23,24,26], is a generalization of the basic problem, where each query consists of a set of (weighted) points and the result of the query is based on applying aggregate operators, such as (weighted) Sum and Max, on all the points in the query. In this paper, we study top-$k$ ANN queries using the weighted Sum operator under the $L_1$ metric in the plane.

1.1 Problem Statement, Previous Work, and Our Results

For any two points $p$ and $q$ in the plane, denote by $d(p,q)$ the distance of $p$ and $q$. Let $Q$ be a set of points and each point $q \in Q$ has a weight $w(q) > 0$. Throughout the paper, we use $m$ to denote the size of $Q$ (note that $m$ is not a fixed value). For any point $p$ in the plane, the aggregate distance from $p$ to $Q$, denoted by $Ad(p,Q)$, is defined to be

$$Ad(p,Q) = \sum_{q \in Q} w(q)d(p,q).$$

Let $P$ be a set of $n$ points in the plane. Given a query consisting of a set $Q$ of weighted points and an integer $k$, $1 \leq k \leq n$, the top-$k$ aggregate nearest neighbors (top-$k$ ANNs) of $Q$ in $P$ are the
points of $P$ whose aggregate distances to $Q$ are the smallest among all points in $P$; we denote by $S_k(P, Q)$ the set of the top-$k$ ANNs. Our goal is to design a data structure to quickly report the set $S_k(P, Q)$ for any query set $Q$ and $k$.

In this paper, we consider the $L_1$ metric. Specifically, for any two points $p = (x(p), y(p))$ and $q = (x(q), y(q))$ in the plane, their distance is defined to be $d(p, q) = |x(p) - x(q)| + |y(p) - y(q)|$. We build an $O(n \log n \log \log n)$-size data structure in $O(n \log n \log \log n)$ time that can support each query in $O(m \log m + (k + m) \log^2 n)$ time. Note that we also return the aggregate distance of each point in $S_k(P, Q)$ to $Q$ and the points of $S_k(P, Q)$ are actually reported in sorted order by their aggregate distances to $Q$. With trade-off between preprocessing and query time, we also build two other data structures: the first one has $O(n \log n)$ preprocessing time and space with $O(m \log m + (k + m) \log^2 n \log \log n)$ query time; the second one has $O(n \log n \log^* n)$ preprocessing time and space with $O(m \log m + (k + m) \log^2 n \log^* n)$ query time.

For the 1-D version of this problem, our approach can build an $O(n)$-size data structure in $O(n \log n)$ time with $O(\min\{k, \log m\} \cdot m + k + \log n)$ query time, and the query time can be reduced to $O(k + m + \log n)$ time if the points of $Q$ are given in sorted order.

Further, we extend our techniques to solve the top-$k$ aggregate farthest neighbor (AFN) searching problem, with the same bounds as above.

### 1.2 Related Work

Previously, only approximation and heuristic results were given for the top-$k$ ANN query problem [20]. For the special case where $k = 1$, Agarwal et al. [2] built an $O(n \log^2 n)$-size data structure in $O(n \log^2 n)$ time that can answer each top-1 ANN query in $O(m^2 \log^3 n)$ time. Hence, even for the special case where $k = 1$, our results are better than that in [2] in all three aspects: preprocessing time, space, and query time. Recently, Ahn et al. [3] studied the unweighted version of the problem, where they gave two data structures under the assumption that the maximum value of $|Q|$ is known in advance as $m$ for all queries, with the following time bounds: the first one is built in $O(m^2 n \log^2 n)$ time and space with $O(m^2 \log n + k(\log \log n + \log m))$ query time; the second one is built in $O(m^2 \log n)$ time and $O(m^2 n)$ space with $O(m^2 \log n + (k + m) \log^2 n)$ query time. Clearly, our results, albeit on the weighted version and do not require the assumption, are generally better than the results in [3] for most cases (e.g., if $m = O(1)$, their second result is better than ours on the unweighted version with the assumption).

For the $L_2$ metric (i.e., the Euclidean distance), only heuristic and approximation algorithms were known previously for answering even top-1 ANN queries [17,18,21,22,23,24,26]; the best known heuristic method for the top-1 ANN queries is based on R-tree [23], and Li et al. [18] gave a data structure with 3-approximation query performance for the top-1 ANN. Agarwal et al. [2] gave a data structure with a polynomial-time approximation scheme for the top-1 ANN queries.

If the $\text{Max}$ operator is used to define the aggregate distance, i.e., $\text{Ad}(p, Q) = \max_{q \in Q} w(q)d(p, q)$, we refer the problem as top-$k$ ANN-MAX queries. To the best of our knowledge, we are not aware of any previous work on the general weighted top-$k$ ANN-MAX queries, even for $k = 1$. Below is some previous work on the unweighted versions. For top-1 ANN-MAX queries, Papadias et al. [23] presented a heuristic Minimum Bounding Method with worst case query time $O(n + m)$ for $L_2$ metric. Recently, Li et al. [16] gave more results on the $L_2$ top-1 ANN-MAX queries (the queries were called group enclosing queries): by using R-tree [15], they [16] gave an exact algorithm that is very fast in practice although theoretically the worst case query time is $O(n + m)$; they [16] also gave a $\sqrt{2}$-approximation algorithm with query time $O(m + \log n)$ for any fixed dimensions and
they further extended the algorithm to obtain a $(1 + \epsilon)$-approximation result. Wang \cite{25} gave an exact algorithm that can answer each $L_2$ top-1 ANN-MAX query in $O(m\sqrt{n}\log O(1)n)$ time. For the $L_1$ metric, Wang \cite{25} constructed a data structure of $O(n)$ size in $O(n \log n)$ time that can answer each (unweighted) $L_1$ top-$k$ ANN-MAX query in $O(m + k \log n)$ time.

In addition, Li et al. \cite{18} proposed the flexible top-$k$ ANN queries, which extend the classical ANN queries, and they provided constant ratio approximation algorithms that work for both SUM and MAX operators in any metric space and any fixed dimension.

We should point out that the weighted ANN queries studied in this paper can be used to solve the expected nearest neighbor (ENN) queries for uncertain query points under $L_1$ metric. In each ENN query, an uncertain point $Q$ is given with $m$ different locations and each location $q$ is associated with a probability $w(q)$ of being the true location of $Q$, and the query asks for the point in $P$ that has the smallest expected distance to $Q$. Agarwal et al. \cite{2} gave the first nontrivial methods for answering exact or approximate ENN queries under $L_1$, $L_2$, and the squared Euclidean distance, with provable performance guarantees. We have mentioned their exact top-1 query algorithm on $L_1$ metric earlier. Other formulations on nearest neighbor queries over uncertain data have also been studied in \cite{2} and elsewhere, e.g., \cite{15,7,8,20,27}.

For the top-$k$ AFN queries, to the best of our knowledge, we are not aware of any previous work on the weighted queries. For unweighted queries, Gao et al. \cite{13} gave heuristic algorithms using R-trees for the $L_2$ metric. For the $L_1$ metric, Ahn et al. \cite{3} also extended their techniques to top-$k$ AFN queries with the same time bounds, assuming that the maximum value $m$ is known for all queries. For $k = 1$, farthest Voronoi diagrams \cite{4} can be used for answering top-1 AFN queries.

The rest of the paper is organized as follows. In Section 2 we give our results in the 1-D space, which are generalized to the 2-D space in Section 3. One may view Section 2 as a "warm-up" for Section 3. Section 4 extends our techniques to solve the AFN queries. Section 5 concludes the paper.

For simplicity of discussion, we make a general position assumption that no two points in $P \cup Q$ have the same $x$- or $y$-coordinate for any query $Q$; we also assume no two points of $P$ have the same aggregate distance to $Q$. Our techniques can be extended to the general case without these assumptions, although the discussion would be more tedious.

Throughout the paper, we use $Q$ to denote the uncertain query point and assume $k < n$. To simplify the notation, we will write $\text{Ad}(p)$ for $\text{Ad}(p, Q)$, and $S_k(P)$ for $S_k(P, Q)$. When we say "the ANN", we mean the top-1 ANN. For any subset $P' \subseteq P$, denote by $S_k(P')$ the set of the top-$k$ ANNs of $Q$ in $P'$. Let $W = \sum_{q \in Q} w(q)$. We assume $m < n$ always holds since otherwise we could compute $S_k(P)$ in $O((m + n)\log m) = O(m \log m)$ time by directly computing the aggregate distances for all points in $P$, and we omit the details.

## 2 Top-k ANN Searching in the 1-D Space

In the 1-D space, all points of $P$ lie on a real line $L$. We assume $L$ is the $x$-axis. For any point $p$ on $L$, denote by $x(p)$ the coordinate of $p$ on $L$. Consider any query set $Q = \{q_1, \ldots, q_m\}$ on $L$. For any point $p$ on $L$, the aggregate distance from $p$ to $Q$ is $\text{Ad}(p) = \sum_{q \in Q} w(q)d(p, q)$, where $d(p, q) = |x(p) - x(q)|$. Given any $Q$ and any $k$, our goal is to compute $S_k(P)$, i.e., the set of the top-$k$ ANNs of $Q$ in $P$.

For a fixed query set $Q$, a point $p$ on $L$ is called a global minimum point if it minimizes the aggregate distance $\text{Ad}(p)$ among all points on $L$. Such a global minimum point on $L$ may not be unique. The global minimum point is also known as weighted Fermat-Weber point \cite{12}, and as shown below, it is very easy to compute in our problem setting.
To find $S_k(P)$, we will use the following strategy. First, we find a global minimum point $q^*$ on $L$. Second, the point $q^*$ partitions $P$ into two subsets $P_l$ and $P_r$, for which we compute $S_k(P_l)$ and $S_k(P_r)$. Finally, $S_k(P)$ is obtained by taking the first $k$ points after merging $S_k(P_l)$ and $S_k(P_r)$.

Note that the points in $Q$ may not be given sorted on $L$. Recall that $W = \sum_{q \in Q} w(q)$. Let $q^*$ be the point in $Q$ such that

$$\sum_{x(q) < x(q^*), q \in Q} w(q) < W/2 \text{ and } w(q^*) + \sum_{x(q) < x(q^*), q \in Q} w(q) \geq W/2.$$ 

If we view $w(q)$ as the weight of $x(q)$, then $q^*$ is the weighted median of the set $\{x(q) \mid q \in Q\}$. We claim that $q^*$ is a global minimum point on $L$. To prove the claim, we first present Lemma 1. We say a function $f(x)$ is monotonically increasing (resp., decreasing) if $f(x_1) \leq f(x_2)$ for any $x_1 \leq x_2$ (resp., $x_1 \geq x_2$).

**Lemma 1.** For any point $p$ on $L$ and $p \neq q^*$, if we move $p$ on $L$ towards $q^*$, the aggregate distance $\text{Ad}(p)$ is monotonically decreasing.

**Proof:** Without loss of generality, assume $p$ is on the left side of $q^*$ and we move $p$ on $L$ to the right towards $q^*$. The case where $p$ is on the right side of $q^*$ can be analyzed similarly. At any moment during the movement of $p$, let $Q_L = \{q \in Q \mid x(q) \leq x(p)\}$ and $Q_R = Q \setminus Q_L$. According to the definition of $\text{Ad}(p)$, we have

$$\text{Ad}(p) = \sum_{q \in Q} |x(p) - x(q)| = \sum_{q \in Q_L} w(q) \cdot |x(p) - x(q)| + \sum_{q \in Q_R} w(q) \cdot |x(q) - x(p)|$$

$$= \left[ \sum_{q \in Q_L} w(q) - \sum_{q \in Q_R} w(q) \right] \cdot x(p) - \sum_{q \in Q_L} w(q) \cdot x(q) + \sum_{q \in Q_R} w(q) \cdot x(q). \quad (1)$$

Because $p$ is to the left of $q^*$, according to the definition of $q^*$, $\sum_{q \in Q_L} w(q) \leq W/2 \leq \sum_{q \in Q_R} w(q)$ holds. Further, as $p$ moves to the right towards $q^*$, the value $x(p)$ is monotonically increasing.

Suppose $p$ is between two points $q_i$ and $q_j$ of $Q$ such that $x(q_i) \leq x(p) < x(q_j)$ and there are no other points of $Q$ between $q_i$ and $q_j$. Note that it is possible that such a point $q_i$ does not exist (i.e., no point of $Q$ is on the left side of $p$), in which case we let $x(q_i) = -\infty$.

If $p$ moves in the interval $[x(q_i), x(q_j)]$ to the right, then both sets $Q_L$ and $Q_R$ stay the same, and thus, the value $\sum_{q \in Q_L} w(q) - \sum_{q \in Q_R} w(q)$ of $x(p)$ is monotonically decreasing and neither $\sum_{q \in Q_L} w(q) \cdot x(q)$ nor $\sum_{q \in Q_R} w(q) \cdot x(q)$ changes. Therefore, if $p$ moves in the interval $[x(q_i), x(q_j)]$ to the right, $\text{Ad}(p)$ is monotonically decreasing. We claim that for any $p$ in $[x(q_i), x(q_j)]$, it always holds that $\text{Ad}(p) \geq \text{Ad}(q_j)$, which leads to the lemma. Indeed, it can be verified that $\text{Ad}(q_j) - \text{Ad}(p) = (x(q_j) - x(p)) \cdot \sum_{q \in Q_L} w(q) - \sum_{q \in Q_R} w(q)].$ Since $x(q_j) - x(p) > 0$ and $|\sum_{q \in Q_L} w(q) - \sum_{q \in Q_R} w(q)| \leq 0$, we obtain that $\text{Ad}(q_j) - \text{Ad}(p) \leq 0$.

The lemma thus follows. \hfill \Box

Lemma 1 implies that $\text{Ad}(p)$ attains a global minimum at $p = q^*$. Hence, the point $q^*$ is a global minimum point on $L$.

Next, we find the set $S_k(P)$ with the help of $q^*$ and Lemma 1. Let $P_l = \{p \in P \mid x(p) \leq x(q^*)\}$ and $P_r = P \setminus P_l$. We find the set $S_k(P_r)$ of top-$k$ ANNs of $Q$ in $P_r$, by scanning the sorted list of $P_r$ from left to right and reporting the first $k$ scanned points. $S_k(P_l)$ can be obtained similarly. Among the $2k$ points obtained above, we report the set of $k$ points with the smallest aggregate distances to $Q$ as $S_k(P)$. We deduce the following theorem.
In this section, we present our results in two-dimensional space, where the input point set $P$ and the query point $Q$ are given in the plane.

We generalize the techniques in Section 2. For any query $Q$, we first find a global minimum $q^*$ in the plane. Then, for each quadrant $R$ of the four quadrants with respect to $q^*$ (i.e., the four
quadrants partitioned by the vertical line and the horizontal line through \( q^* \), we find the top-\( k \) \( Q \)-ANNs of \( Q \) in \( P \cap R \) (i.e., \( S_k(P \cap R) \)) and compute the aggregate distance values \( \text{Ad}(p) \) for all \( p \in S_k(P \cap R) \); among the found \( 4k \) points, we report the set of \( k \) points with smallest aggregate distances to \( Q \) as \( S_k(P) \). Note that we view each quadrant as a closed region including its two bounding half-lines (with the common endpoint \( q^* \)).

We describe our algorithm for the first quadrant, and the other three quadrants can be treated in a similar manner. Let \( P^1 \subseteq P \) be the set of points lying in the first quadrant, i.e., \( P^1 = \{ p \in P \mid x(p) \geq x(q^*), y(p) \geq y(q^*) \} \). Our goal is to find \( S_k(P^1) \), the set of top-\( k \) \( Q \)-ANNs of \( Q \) in \( P^1 \). Let \( z_i \) denote the \( i \)-th ANN of \( Q \) in \( P^1 \). Our algorithm computes \( S_k(P^1) \) in the order of \( z_1, z_2, \ldots, z_k \).

The problem here is more difficult than that in the 1-D case. For example, in the 1-D case, \( S_1(x) \) is found, we determine a new skyline without considering \( z_1, \ldots, z_i \), and then find \( z_{i+1} \) by searching the new skyline. This procedure continues until \( z_k \) is obtained. Advanced data structures (e.g., compact interval trees [14] and segment-dragging query data structure [6]) are also used for efficient implementations.

Consider any query set \( Q = \{ q_1, q_2, \ldots, q_m \} \) and any \( k \). For any point \( p \) in the plane, denote by \( x(p) \) the \( x \)-coordinate of \( p \) and by \( y(p) \) the \( y \)-coordinate of \( p \). The aggregate distance of \( p \) to \( Q \) is \( \text{Ad}(p) = \sum_{q \in Q} w(q)d(p, q) \), where \( d(p, q) = |x(p) - x(q)| + |y(p) - y(q)| \). Our goal is to find the top-\( k \) ANN set \( S_k(P^1) \) in the first quadrant.

### 3.1 The Global Minimum Point and the Monotonicity Property

A point \( p \) in the plane is called a **global minimum point** if it minimizes the aggregate distance \( \text{Ad}(p) \) among all points in the plane. Below, we first find a global minimum point and prove a monotonicity property. Recall that \( W = \sum_{q \in Q} w(q) \). Let \( q_x^* \in Q \) be the point such that

\[
\sum_{x(q) < x(q_x^*), q \in Q} w(q) < W/2 \quad \text{and} \quad \sum_{x(q) < x(q_x^*), q \in Q} w(q) + \sum_{x(q) > x(q_x^*), q \in Q} w(q) \geq W/2.
\]

If we view \( w(q) \) as the weight of \( x(q) \) for each \( q \in Q \), then \( x(q_x^*) \) is the weighted median of the set \( \{ x(q) \mid q \in Q \} \) [10]. Similarly, let \( q_y^* \) be the point in \( Q \) such that

\[
\sum_{y(q) < y(q_y^*), q \in Q} w(q) < W/2 \quad \text{and} \quad \sum_{y(q) < y(q_y^*), q \in Q} w(q) + \sum_{y(q) > y(q_y^*), q \in Q} w(q) \geq W/2.
\]

We claim that \( q^* = (x(q_x^*), y(q_y^*)) \) is a global minimum point. To prove the claim, we first present Lemma 2 which generalizes Lemma 1. A path in the plane is **monotone** if we move from one endpoint of it to the other, the \( x \)-coordinate (resp. \( y \)-coordinate) is monotonically changing (either increasing or decreasing).

**Lemma 2.** For any point \( p \) in the plane with \( p \neq q^* \), if we move \( p \) towards \( q^* \) along a monotone path, the aggregate distance \( \text{Ad}(p) \) is monotonically decreasing.

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Fig. 1. The four (red) points connected by the dashed lines are minimal points, and the dashed line connecting them is a skyline. $p_1$ dominates $p_2$ and the dotted curve connecting $q^*$ and $p_2$ is a monotone path.

**Proof:** According to the definition of $A_d(p)$, we have

$$A_d(p) = \sum_{q\in Q} w(q) \cdot d(p, q) = \sum_{q\in Q} w(q) \cdot \left( |x(p) - x(q)| + |y(p) - y(q)| \right)$$

$$= \sum_{q\in Q} w(q) \cdot |x(p) - x(q)| + \sum_{q\in Q} w(q) \cdot |y(p) - y(q)|$$

$$= A_{dx}(x(p)) + A_{dy}(y(p)),$$

where

$$A_{dx}(x(p)) = \sum_{q\in Q} w(q) \cdot |x(p) - x(q)| \quad \text{and} \quad A_{dy}(y(p)) = \sum_{q\in Q} w(q) \cdot |y(p) - y(q)|.$$

If we move $p$ towards $q^*$ along a monotone path, on the $x$-projection, we are moving $x(p)$ towards $x(q^*)$. By Lemma 1, $A_{dx}(x(p))$ is monotonically decreasing, so is $A_{dy}(y(p))$. The lemma thus follows.

Lemma 2 implies that $A_d(p)$ attains a global minimum at $p = q^*$. Hence, the point $q^*$ is a global minimum point in the plane (note that $q^*$ is not necessarily in $Q$). Next, based on the point $q^*$ and Lemma 2, we introduce the minimal points and the skyline, and present some observations.

### 3.2 The Minimal Points and the Skyline

We first show how to find $z_1$ (i.e., the ANN of $Q$ in $P^1$). For any two different points $p_1$ and $p_2$ in $P^1$, we say that $p_1$ dominates $p_2$ if and only if $x(p_1) \leq x(p_2)$ and $y(p_1) \leq y(p_2)$. A point $p$ in $P^1$ is called a minimal point if no point in $P^1$ dominates $p$ (note that the “minimal” here is different from the “global minimum” defined earlier). If $p_1 \in P^1$ dominates $p_2 \in P^1$, then there exists a monotone path $\pi$ connecting $p_2$ and $q^*$ such that $p_1 \in \pi$ (see Fig. 1). By Lemma 2, $A_d(p_1) \leq A_d(p_2)$. Therefore, to compute $z_1$, we only need to consider the set of minimal points in $P^1$, denoted by $M$. Our discussion above leads to the following lemma.

**Lemma 3.** $z_1 = \arg\min_{p\in M} A_d(p)$.

One tempting approach is to first find the set $M$ and then find $z_1$. Unfortunately, here $M$ may have $\Theta(n)$ points and we cannot afford to check every point of $M$. Below, we give a better approach.

For each $q \in Q$, we induce a horizontal line and a vertical line through $q$, respectively; let $A$ be the arrangement of the resulting $2m$ lines. Each cell of $A$ is a (possibly unbounded) rectangle. Each point in $Q$ is a vertex of $A$. Note that our algorithm does not explicitly compute $A$. 

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We will show below that \( \text{Ad}(p) \) is a linear function of \( x(p) \) and \( y(p) \) inside any cell \( C \) of \( A \), implying that the ANN (i.e., the top-1 ANN) of \( Q \) in \( P \cap C \) is on the convex hull of \( P \cap C \), as discussed in [2].

For any cell \( C \), suppose \( C = [x_l, x_r] \times [y_b, y_t] \). Set \( Q_L = \{ q \in Q \mid x(q) \leq x_l \} \), \( Q_R = \{ q \in Q \mid x(q) \geq x_r \} \), \( Q_B = \{ q \in Q \mid y(q) \leq y_b \} \) and \( Q_T = \{ q \in Q \mid y(q) \geq y_t \} \). According to the construction of \( A \), no point of \( Q \) lies strictly inside \( C \) and \( Q = Q_L \cup Q_R = Q_B \cup Q_T \).

We have the following lemma.

**Lemma 4.** For any point \( p \) in the cell \( C \), \( \text{Ad}(p) = C_a \cdot x(p) + C_b \cdot y(p) + C_c \), where

\[
C_a = \sum_{q \in Q_L} w(q) - \sum_{q \in Q_R} w(q), \quad C_b = \sum_{q \in Q_B} w(q) - \sum_{q \in Q_T} w(q), \quad \text{and}
\]

\[
C_c = \sum_{q \in Q_R} w(q)x(q) - \sum_{q \in Q_L} w(q)x(q) + \sum_{q \in Q_T} w(q)y(q) - \sum_{q \in Q_B} w(q)y(q).
\]

Further, with \( O(m \log m) \) time preprocessing on \( Q \), given any cell \( C \) of \( A \), we can compute \( C_a, C_b, \) and \( C_c \) in \( O(\log m) \) time.

**Proof:** The first part (i.e., computing the values of \( C_a, C_b, \) and \( C_c \)) has been discussed in [2] and it can also be easily verified by our analysis in Lemma [2]. Hence, we omit the proof for it.

For the second part, given any cell \( C \), our goal is to compute the three values \( C_a, C_b, \) and \( C_c \). Generally speaking, if, as preprocessing, we compute the prefix sums of the values \( w(q) \) and \( w(q)x(q) \) in the sorted list of the points of \( Q \) by their \( x \)-coordinates, and compute the prefix sum of \( w(q)y(q) \) in the sorted list of the points of \( Q \) by their \( y \)-coordinates, then \( C_a, C_b, \) and \( C_c \) can be computed in \( O(\log m) \) time. The details are given below.

To compute \( C_a \), we need to know the value \( \sum_{q \in Q_L} w(q) \) and the value \( \sum_{q \in Q_R} w(q) \). Note that \( \sum_{q \in Q_R} w(q) = \sum_{q \in Q_L} w(q) \). We can do the following preprocessing. We sort all points in \( Q \) by their \( x \)-coordinates. Suppose the sorted list is \( q_1, q_2, \ldots, q_m \) from left to right. For each \( 1 \leq j \leq m \), we compute the value \( W_1(q_j) = \sum_{i=1}^j w(q_i) \). For any given cell \( C \), let \( x_l \) be the \( x \)-coordinate of the vertical line containing the left side of \( C \). By binary search on the sorted list \( q_1, q_2, \ldots, q_m \), in \( O(\log m) \) time, we can find the rightmost point \( q' \) in \( Q \) such that \( x(q') \leq x_l \). It is easy to see that \( \sum_{q \in Q_L} w(q) = W_1(q') \). Note that the above preprocessing takes \( O(m \log m) \) time, and \( C_a \) can be computed in \( O(\log m) \) time.

In similar ways, we can compute \( C_b \) and \( C_c \) in \( O(\log m) \) time, with \( O(m \log m) \) preprocessing time. Hence, the second part of the lemma follows. \( \square \)

As discussed in [2], Lemma 4 implies that \( S_1(P \cap C) \) (i.e., the ANN of \( Q \) in \( P \cap C \)) is on the convex hull of \( P \cap C \). More specifically, \( S_1(P \cap C) \) is an extreme point of \( P \cap C \) along a certain direction that is determined by \( C_a \) and \( C_b \), and thus we can do binary search on the convex hull to find it.

To compute \( z_1 \), the algorithm in [2] checks every cell \( C \) of \( A \) in the first quadrant, and it finds \( S_1(P \cap C) \) by doing binary search on the convex hull of the points in \( P \cap C \). The number of cells checked in [2] is \( O(m^2) \). In contrast, based on Lemma 3 we show below that we only need to check \( O(m) \) cells. Although the number of minimal points in \( M \) can be \( \Theta(n) \), we show that the number of cells of \( A \) that contain these minimal points is \( O(m) \), and further, we can find these cells efficiently.

If we order the points in \( M \) by their \( x \)-coordinates and connect every pair of adjacent points by a line segment, then we can obtain a path \( \pi_1 \), which we call a *skysline* (see Fig. 1). The points
of $M$ are also considered as the vertices of $\pi_1$. If we move on $\pi_1$ from its left endpoint to its right endpoint, then the $x$-coordinate is monotonically increasing and the $y$-coordinate is monotonically decreasing. Hence, $\pi_1$ is a monotone path.

Denote by $C_1$ the set of cells of $A$ that contain the minimal points in $M$.

**Lemma 5.** $|C_1| = O(m)$.

**Proof:** Due to our general position assumption that no two points in $P \cup Q$ have the same $x$-coordinate or $y$-coordinate. Each edge of $\pi_1$ is neither horizontal nor vertical. Because $\pi_1$ is a monotone path, each line of $A$ can intersect $\pi_1$ at most once. Hence, the number of intersections between $\pi_1$ and $A$ is $O(m)$, which implies that the number of cells that intersect $\pi_1$ is $O(m)$. Since all points in $M$ are on $\pi_1$, the lemma follows. $\Box$

Due to Lemma 5 the following lemma is obvious.

**Lemma 6.** The point $z_1$ is in one of the cells of $C_1$.

### 3.3 Computing the Set $C_1$

Next, we show how to compute $C_1$. A straightforward way is to first compute $A$ and then traverse $A$ by following the skyline $\pi_1$. But this approach is not efficient due to: (1) computing $A$ takes $\Theta(m^2)$ time; (2) the size of $\pi_1$ may be $\Theta(n)$ due to $|M| = \Theta(n)$ in the worst case. Below in Lemma 7 we propose to compute $C_1$ in $O(m \log n + m \log m)$ time.

First of all, we sort all points in $Q$ by their $x$-coordinates and $y$-coordinates, respectively; accordingly, we obtain a sorted list for the horizontal lines of $A$ and a sorted list for the vertical lines of $A$. With these two sorted lists, given any point $p$, we can determine the cell of $A$ that contains $p$ in $O(\log m)$ time by doing binary search on the two sorted lists. We should point out that there might be other ways to compute $C_1$, but the algorithm we propose for Lemma 7 is particularly useful later when we compute other points in $S_k(P^1)$ than $z_1$.

**Lemma 7.** $P$ can be preprocessed in $O(n \log n)$ time using $O(n)$ space, such that given any $Q$, we can compute the set $C_1$ in $O(m \log n + m \log m)$ time.

**Proof:** One operation frequently used for computing $C_1$ is the following segment-dragging queries. Given any horizontal or vertical line segment $s$, we move $s$ along a given direction perpendicular to $s$; the query asks for the first point of $P$ hit by $s$ or reports no such point exists. Chazelle [6] constructed an $O(n)$-size data structure in $O(n \log n)$ time such that each segment-dragging query can be answered in $O(\log n)$ time. As preprocessing, we build such a data structure on $P$. Hence, the preprocessing takes $O(n \log n)$ time and $O(n)$ space.

For each cell $C$ of $C_1$, we call the leftmost point of $M \cap C$ the skyline-left point of $C$ and call the bottommost point of $M \cap C$ the skyline-bottom point of $C$. In other words, if we move along the skyline $\pi_1$ from its left endpoint to its right endpoint, then the skyline-left point of $C$ is the first vertex of $\pi_1$ we meet in $C$ and the skyline-bottom point of $C$ is the last vertex of $\pi_1$ we meet in $C$. Note that if $C$ has only one minimal point of $M$, then the only minimum point is both the skyline-left point and the skyline-bottom point of $C$.

We will find the skyline-left point and the skyline-bottom point for each cell $C \in C_1$. Each such point $p$ is determined by a segment-dragging query on a segment $s$ and we call $s$ the generating segment of $p$; $s$ will be associated with $p$ for later use (for computing other points in $S_k(P^1)$ than
Further, we will classify these generating segments into four types, and again, they will be useful later in Lemma 10 for computing $S_k(P_1)$.

All the vertical lines passing through points in $Q$ partition the space into $O(m)$ regions, which we refer to as columns (including bounding lines). Let $D_M$ denote the set of columns of $A$ each of which contains at least one cell of $C_1$. We search the columns of $D_M$ from left to right. For each column $D \in D_M$, we will first find the topmost cell and the bottommost cell of $C_1$ in $D$; then, from the bottommost cell to the topmost cell, we search all other cells of $C_1$ in $D$ in a bottom-up fashion. After the searching on $D$ is done, we proceed to the next column of $D_M$. The details are given below.

Note that due to the general position assumption that no two points in $P \cup Q$ have the same $x$- or $y$-coordinate, each point of $P$ lies strictly inside a cell of $A$.

We first determine the leftmost column of $D_M$, denoted by $D$, which is the one containing the leftmost point $p_0$ of $M$ (see Fig. 2). $p_0$ can be found by the following segment-dragging query. Let $y_{\text{max}} = \max_{p \in P_1} y(p)$. Consider a vertical segment $s_0 = q^*b$ where $b = (x(q^*), y_{\text{max}})$. If we drag $s_0$ rightwards (i.e., horizontally to the right), $p_0$ will be the first point of $P_1$ hit by $s_0$. By using the segment-dragging query data structure on $P$, $p_0$ can be found in $O(\log n)$ time. After having $p_0$, $D$ can be determined in $O(\log m)$ time using binary search on the sorted list of the vertical lines of $A$.

Notice that the cell of $A$ that contains $p_0$ is the topmost cell in $D \cap C_1$, which we denote by $C_t$, and that $p_0$ is the skyline-left point of $C_t$ (see Fig. 2). The segment $s_0$ is the generating segment of $p_0$ and we classify $s_0$ as an $s_0$-type generating segment. In general, the $s_0$-type generating segments are used to find the skyline-left points of the topmost cells of the columns of $D_M$.

Next, we determine the bottommost cell of $D \cap C_t$, denoted by $C_b$. We first determine the skyline-bottom point $p_1$ of $C_b$ by a segment-dragging query as follows. Let $\ell$ denote the horizontal line $y = y(q^*)$. Set $s_1 = D \cap \ell$. If we drag $s_1$ upwards, $p_1$ will be the first point of $P_1$ hit by $s_1$ (see Fig. 2). After $p_1$ is found in $O(\log n)$ time, $C_b$ can be determined in additional $O(\log m)$ time. $s_1$ is the generating segment of $p_1$ and we classify $s_1$ as the $s_1$-type generating segment. In general, $s_1$-type generating segments are used to find the skyline-bottom points of the bottommost cells of the columns of $D_M$.

If $C_b = C_t$, then the column $D$ contains only one cell of $C_1$, and our searching on $D$ is done. Below, we assume $C_b \neq C_t$. 

![Fig. 2. Illustrating the algorithm in Lemma 7: the dashed grid is $A$.](image)
In the sequel, from the bottommost cell $C_b$, we search the cells of $C_1$ in $D$ in a bottom-up manner until we meet the topmost cell $C_t$. We first show how to determine the second lowest cell of $C_1 \cap D$ (i.e., the one of $C_1 \cap D$ right above $C_b$), denoted by $C_s$.

To determine $C_s$, we first find the skyline-left point $p_2$ of $C_b$ using a segment-dragging query, as follows. Let $s_2$ be the left side of $C_b$, $p_2$ is the first point in $P^1$ hit by dragging $s_2$ rightwards (see Fig. 2). $s_2$ is the generating segment of $p_2$ and we classify $s_2$ as the $s_2$-type generating segment. In general, each $s_2$-type generating segment is used to find the skyline-left point of a cell whose skyline-bottom point has just been found. Next, we determine $C_s$ by using $p_2$.

We first determine the skyline-bottom point $p_3$ of $C_s$. Since $y(p_3) > y(p_2)$, $x(p_3) < x(p_2)$ (otherwise $p_2$ would dominate $p_3$). An easy observation is that $p_3$ is the lowest point among all points of $P^1 \cap D$ whose $x$-coordinates are less than $x(p_2)$ (see Fig. 2). We can determine $p_3$ by the following segment-dragging query. Let $s_3$ be the horizontal line segment on the top side of the cell $C_b$ such that the left endpoint of $s_3$ is the upper left vertex of $C_b$ and the right endpoint has $x$-coordinate $x(p_2)$ (see Fig. 2). Due to our general position assumption that no two points in $P \cup Q$ have the same $x$- or $y$-coordinate, $p_3$ is the point of $P^1$ hit first by dragging $s_3$ upwards. After $p_3$ is found, $C_s$ can be determined. Therefore, we can determine $C_s$ in $O(\log n + \log m)$ time. $s_3$ is the generating segment of $p_3$ and we classify $s_3$ as the $s_3$-type generating segment. In general, $s_3$-type generating segments are used to find the skyline-bottom points for non-bottommost cells of the columns of $D_M$.

If $C_s = C_t$, we are done searching on $D$. Otherwise, we continue the above procedure to search other cells of $C_1 \cap D$ until we meet the topmost cell $C_t$.

Now we proceed to the next column $D' \in D_M$, in the following way. We first determine $D'$ by a segment-dragging query as follows. Recall that $p_1$ is the lowest point in $P^1 \cap D$. Let $s_4$ be the vertical line segment on the right bounding line of $D$ such that the lower endpoint of $s_4$ has $y$-coordinate $y(q^*)$ and the upper endpoint has $y$-coordinate $y(p_1)$ (see Fig. 2). We drag the segment $s_4$ rightwards, and let $p_4$ be the first point of $P^1$ hit by $s_4$ (see Fig. 2). It is not difficult to see that $p_4$ is a minimal point and the column of $A$ containing $p_4$ is $D'$. Further, $p_4$ is the skyline-left point of the topmost cell of $C_1 \cap D'$. Hence, after $p_4$ is found, $D'$ and the topmost cell of $C_1 \cap D'$ can be determined in $O(\log m)$ time. $s_4$ is the generating segment of $p_4$; note that $s_4$ is an $s_0$-type generating segment.

Note that if the above segment-dragging query on $s_4$ fails to find any point (i.e., such a point $p_4$ does not exist), then all cells of $C_1$ have been found, and we terminate. Otherwise, we proceed to search all cells in $C_1 \cap D'$ in the same way as in the column $D$, and then search other columns of $D_M$ similarly.

For the running time, as shown above, the algorithm spends $O(\log n + \log m)$ time finding each cell of $C_1$. Due to $|C_1| = O(m)$ (by Lemma 5), computing $C_1$ takes $O(m \log n + m \log m)$ time. The lemma thus follows. \hfill \&

3.4 Computing the Top-$k$ ANN Set $S_k(P^1)$

In this section, we compute $S_k(P^1)$ in the order of $z_1, z_2, \ldots, z_k$.

Since $z_1$ is in one of the cells of $C_1$, once we have $C_1$, we compute the ANN of $Q$ in $C \cap P$ in each cell $C \in C_1$; among the $|C_1|$ candidate points, $z_1$ is the one with the smallest aggregate distance to $Q$. Once $z_1$ is obtained, we use a similar approach to compute $z_2$. Let $\pi_2$ be the skyline of $P^1 \setminus \{z_1\}$, and let $C_2$ be the set of cells of $A$ that contain the vertices of $\pi_2$. Again, $z_2$ must be in one of the cells of $C_2$, and we find $z_2$ by searching the cells of $C_2$. In general, let $\pi_i$ be the skyline
of $P^1 \setminus \{z_1, \ldots, z_{i-1}\}$, and let $C_i$ be the set of cells of $A$ that contain the vertices of $\pi_i$. The point $z_i$ must be in one of the cells of $C_i$, and we find $z_i$ by searching the cells of $C_i$. We repeat this till $z_k$ is found.

For each $1 \leq i \leq k$, since $\pi_i$ is a skyline, $|C_i| = O(m)$. A straightforward implementation to compute $S_k(P^1)$ will need to search $O(km)$ cells. We will show that we only need to search $O(k+m)$ cells in total, and more importantly, we can find all these cells efficiently. Specifically, we propose an algorithm that can efficiently determine the set $C_i$ by updating the set $C_{i-1}$, for all $2 \leq i \leq k$.

In the sequel, we first present an algorithm that can quickly compute the ANN of $Q$ in $C \cap P$ for any cell $C$ of $A$. An $O(n \log^2 n)$-size data structure was given in [2] that can be built in $O(n \log^2 n)$ time and can compute the ANN in any cell $C$ of $A$ in $O(\log^3 n)$ time. By using compact interval trees [14], we have the following improved result in Lemma 8.

**Lemma 8.** For a set $P$ of $n$ points in the plane, an $O(n \log n \log \log n)$-size data structure can be built in $O(n \log n \log \log n)$ time, such that given any axis-parallel rectangle $C$ (e.g., any cell of $A$), the ANN of $Q$ in $P \cap C$ can be computed in $O(\log^2 n)$ time. With trade-off between preprocessing and query time, we can build two other data structures: the first one has $O(n \log n)$ preprocessing time and space with $O(n \log^2 n \log \log n)$ query time; the second one has $O(n \log n \log^* n)$ preprocessing time and space with $O(n \log^2 n \log^* n)$ query time.

**Proof:** Our data structure uses the compact interval tree [14], which was for solving the following sub-path hull queries in [14]. Let $\pi$ be a simple path of $n$ vertices in the plane and suppose the vertices are $v_1, v_2, \ldots, v_n$ ordered along $\pi$. Given two vertex indices $i$ and $j$ with $i < j$, the sub-path hull query asks for the convex hull of all vertices $v_i, v_{i+1}, \ldots, v_j$. A compact interval tree data structure was given in [14], and for each sub-path hull query, it can report in $O(\log n)$ time a data structure that represents the convex hull such that any standard binary-search based operation on the convex hull can be implemented in $O(\log n)$ time (e.g., finding an extreme point on the convex hull along any given direction). Assume the vertices of $\pi$ are sorted by their $x$- or $y$-coordinates. The compact interval tree is of $O(n \log n \log \log n)$ size and can be built in the same time. With trade-off between preprocessing and query time, two other compact interval trees can be built for the sub-path hull queries: the first one has $O(n \log n)$ preprocessing time and space with $O(\log n \log \log n)$ query time; the second one has $O(n \log^* n)$ preprocessing time and space with $O(\log n \log^* n)$ query time.

Our data structure for the lemma is constructed as follows. At the high-level, it is similar to the two-dimensional orthogonal range tree [11]. A balanced binary search tree $T$ is built based on the $x$-coordinates of the points in $P$. The leaves of $T$ store the points of $P$ in sorted order from left to right, and the internal nodes store splitting values to guide the search on $T$. For each node $v$ of $T$, it also stores the subset $P(v) \subseteq P$ of points in the subtree of $T$ rooted at $v$, and $P(v)$ is called the canonical subset of $v$. For each canonical subset $P(v)$, we build a compact interval tree in the following way. If we sort the points of $P(v)$ by their $y$-coordinates and connect each pair of adjacent points in the sorted list by a line segment, we obtain a path $\pi(v)$. The points in $P(v)$ are vertices of $\pi(v)$. Note that $\pi(v)$ is a simple path and each horizontal line intersects $\pi(v)$ at most once. We build a compact interval tree data structure on $\pi(v)$ using the approaches in [14]. This finishes the construction of our data structure.

For each canonical subset $P(v)$, depending on which of the three compact interval trees is used, constructing the compact interval tree on $\pi(v)$ takes $O(\mu \log \log \mu)$ (or $O(\mu)$ or $O(\mu \log^* \mu)$) time and space, where $\mu = |P(v)|$. Note that the $y$-sorted list of $P(v)$ can be built during the construction of $T$ in a bottom-up manner. Hence, the preprocessing time and space is $O(n \log n \log \log n)$ (or $O(n \log n)$ or $O(n \log n \log^* n)$), the same as claimed in the lemma statement.
Given any axis-parallel rectangle $C$, our goal is to find the ANN of $Q$ in $C \cap P$. Essentially, we are looking for an extreme point in $C \cap P$ along a certain direction, denoted by $\sigma$. As discussed in \cite{2}, $\sigma$ is determined by the two factors $C_a$ and $C_b$ defined in Lemma \[4\] and can be computed in $O(\log m)$ time by Lemma \[4\]. Recall that we have assumed that $m < n$ holds; hence $\log m = O(\log n)$.

Suppose $C = [x_1, x_r] \times [y_1, y_r]$. Using the range $[x_1, x_r]$, we first find the $O(\log n)$ canonical subsets whose union is the set of points in $P$ lying between the two vertical lines $x = x_1$ and $x = x_r$. For each such canonical subset $P(v)$, we use the range $[y_1, y_r]$ to determine the sub-path of $\pi(v)$ inside $C$, which can be done by binary search on the $y$-sorted list of $P(v)$; subsequently, we use the compact interval tree data structure on $\pi(v)$ to (implicitly) report the convex hull of the sub-path in $O(\log n)$ time (or $O(\log n \log \log n)$ or $O(\log n \log^* n)$ time), after which we search the extreme point on the convex hull along the direction $\sigma$ in $O(\log n)$ time. In this way, we obtain $O(\log n)$ extreme points for these $O(\log n)$ canonical subsets, and the one minimizing the aggregate distance to $Q$ is the ANN of $Q$ in $C \cap P$. Assuming that we have computed the three factors $C_a$, $C_b$, and $C_c$ as defined in Lemma \[4\], for each extreme point found above, its aggregate distance to $Q$ can be computed in constant time.

Therefore, the ANN of $Q$ in $C \cap P$ can be found in $O(\log^2 n)$ time (or $O(\log^2 n \log \log n)$ or $O(\log^2 n \log^* n)$ time). The lemma thus follows. \hfill $\square$

In the following, to avoid tedious discussions, unless otherwise stated, when we refer to Lemma \[8\], we will always use the data structure with $O(\log^2 n)$ query time, with the understanding that using different data structures will give different performances (i.e., preprocessing and query time) accordingly.

Let $m_i = |C_i|$ for each $1 \leq i \leq k$. By Lemma \[8\], we can determine $z_1$ in $O(\log m_1)$ time. Next we continue to compute $z_2$. To this end, we need to find the set $C_2$ first. Instead of computing $C_2$ from scratch as we did for $C_1$, we obtain $C_2$ by updating $C_1$. Specifically, if some cells are both in $C_1$ and $C_2$, we do not need to compute them again. In other words, we only need to compute the cells in $C_2 \setminus C_1$. Let $C(z_1)$ denote the cell containing $z_1$. In fact, we will show that all the cells of $C_1$ except $C(z_1)$ must be in $C_2$. The cell $C(z_1)$ may or may not be in $C_2$. If $C(z_1) \in C_2$, then special care needs to be taken when searching $C(z_1)$ because we are looking for $z_2$ and the point $z_1$ should not be considered any more. The details are given below.

For each $2 \leq i \leq k$, let $C_i' = C_i \setminus C_{i-1}$ and $m_i' = |C_i'|$. We first show that $C_2$ can be obtained in $O(m_2' (\log^2 n + \log m))$ time, and specifically, we compute the cells of $C_2'$ and determine whether $C(z_1) \in C_2$, which is done in Lemma \[10\].

The algorithm in Lemma \[10\] needs a dynamic version of the segment-dragging query data structure that can support point deletions and insertions for $P$. Later after we finish the query, we also need to insert those points that have been deleted back to $P$, and we call them special insertions, i.e., whenever we insert a point $p$ to $P$ for the segment-dragging query data structure, $p$ has already been deleted from $P$ before. In the following Lemma \[9\], we present such a data structure by using the range trees \[11\]. Note that the performance of the data structure in Lemma \[9\] may not be the best: Since other parts of our algorithm for computing $S_k(P^1)$ dominate the overall running time, we choose to present a data structure that is simple and does not affect the overall performance.

**Lemma 9.** For a set $P$ of $n$ points in the plane, we can build a data structure in $O(n \log n)$ time and $O(n \log n)$ space that can answer each segment-dragging query in $O(\log^2 n)$ time and support each point deletion and special insertion for $P$ in $O(\log^2 n)$ time.
**Proof:** Our data structure consists of two range trees, one for horizontal segment-dragging queries and the other for vertical segment-dragging queries. Below, we only present the one for horizontal segment-dragging queries and the other one can be obtained similarly.

We first sort the points in $P$ by their $x$-coordinates and $y$-coordinates, respectively. We build a balanced binary search tree $T$ based on the $x$-coordinates of the points in $P$. The leaves of $T$ store the points of $P$ in sorted order from left to right. Each node $v$ of $T$ also stores the subset $P(v)$ of points stored in the leaves of the subtree rooted at $v$; $P(v)$ is called the canonical subset of $v$. For each node $v$, we use another balanced binary search tree $T(v)$ to store the points in $P(v)$ based on the $y$-coordinates of the points. It is commonly known that $T$ can be constructed in $O(n \log n)$ time using $O(n \log n)$ space \[11\].

Consider any segment-dragging query. Without loss of generality, assume we drag upwards a horizontal segment $s = [x_1(s), x_2(s)] \times \{y(s)\}$ (i.e., its $y$-coordinate is $y(s)$ and its $x$-coordinate spans the interval $[x_1(s), x_2(s)]$). We first determine the $O(\log n)$ canonical subsets of $T$ whose union is the subset of points of $P$ with $x$-coordinates lying in $[x_1(s), x_2(s)]$. For each canonical subset $P(v)$, we use the tree $T(v)$ to determine in $O(\log n)$ time the lowest point of $P(v)$ whose $y$-coordinate is no less than $y(s)$ and that point will be the first point hit by dragging $s$ upwards. After we find such a point in each canonical subset, we report the point with smallest $y$-coordinate as the answer to the segment-dragging query for $s$. The total query time is $O(\log^2 n)$ time.

Now consider deleting a point $p$ from $P$. We first find the leaf $v_p$ of $T$ storing $p$. Then, for each node $v$ in the path of $T$ from the root to $v_p$, we delete $p$ from the tree $T(v)$, which can be done in $O(\log n)$ time. Hence, it takes $O(\log^2 n)$ time for each point deletion. Consider a special insertion that inserts a point $p$ to $P$. Since it is a special insertion, $p$ was in $P$ before but has been deleted. We first find the leaf $v_p$ of $T$ that stored $p$ before. Then, for each node $v$ in the path of $T$ from the root to $v_p$, we insert $p$ to the tree $T(v)$, which can be done in $O(\log n)$ time. Hence, it takes $O(\log^2 n)$ time for each special insertion.

The lemma thus follows. \[ \square \]

Next in Lemma \[10\] we compute $C_2$ based on $C_1$, by using the data structure in Lemma \[9\]. The algorithm for Lemma \[10\] essentially follows the behavior of the algorithm for Lemma \[7\] but only focuses on searching the cells of $C'_2$. The efficiency of the algorithm for Lemma \[10\] also hinges on the observation that the cells of $C'_2$ form at most two subsets (separated by $C(z_1)$ if $C(z_1) \in C_2$) of consecutive cells of $C_2$ if we order the cells of $C_2$ from “northwest” to “southeast”.

**Lemma 10.** We can determine the set $C_2$ in $O((1 + m'_2)(\log^2 n + \log m))$ time, where $m'_2 = |C'_2|$, and more specifically, our algorithm will compute the cells of $C'_2$ and determine whether $C(z_1) \in C_2$.

**Proof:** We call the order of the cells of $C_1$ by which the skyline $\pi_1$ crosses them from left to right the canonical order of $C_1$. In other words, the canonical order of $C_1$ follows the northwest-to-southeast order. We define the canonical order of $C_2$ similarly.

Suppose the canonical order of the cells of $C_1$ is: $C_1, C_2, \cdots, C_m$. Note that we can obtain this ordered list during computing $C_1$ in Lemma \[7\] within the same running time. Recall that when computing $C_1$ we also computed a skyline-left point and a skyline-bottom point for each cell of $C_1$ as well as their generating segments. Let $C_i = C(z_i)$, i.e., the cell that contains $z_i$. We assume $i \neq 1$ and $i \neq m_1$ (otherwise the algorithm is similar and much simpler).

In order to better understand the algorithm we will present below, we first discuss a question: which cells are possibly in $C'_2$? Imagine that we partition the plane into four quadrants with respect to $z_1$ by the vertical line through $z_1$ and the horizontal line through $z_1$; an easy observation is that
only the cells intersecting the first quadrant can possibly be in $C'_2$ because only points in the first quadrant are dominated by $z_1$. Further, for each cell $C_j$ with $j \neq i$, none of the vertices of the skyline $\pi_1$ in $C_j$ is dominated by $z_1$, and thus $C_j$ is still in $C_2$. In other words, all cells of $C_1 \setminus \{C_i\}$ are still in $C_2$. The cell $C_i$ may or may not be in $C_2$. Also note that if we remove $z_1$ from $P$, then the skyline-bottom point of $C_{i-1}$ may be changed (see Fig. 3), but the skyline-left point of $C_{i-1}$ does not change; for each cell $C_j$ with $1 \leq j \leq i - 2$, neither its skyline-left point nor its skyline-bottom point changes. Similarly, due to the removal of $z_1$, the skyline-left point of $C_{i+1}$ may be changed, but its skyline-bottom point does not change; for each cell $C_j$ with $i + 2 \leq j \leq m_1$, neither its skyline-left point nor its skyline-bottom point changes.

The above implies that to determine $C_2$, we need to do the following. (1) Find all cells in $C'_2$, and as in Lemma 7, for each cell of $C'_2$, compute its skyline-left point and skyline-bottom point as well as their generating segments. (2) Determine whether $C_i$ is still in $C_2$, and if yes, compute its new skyline-left point and skyline-bottom point as well as their generating segments, if any of them changes. (3) Compute the new skyline-bottom point (and its generating segment) for $C_{i-1}$ if it changes. (4) Compute the new skyline-left point (and its generating segment) for $C_{i+1}$ if it changes.

Let $D_P$ be the dynamic segment-dragging data structure in Lemma 9 we built on $P$. Below, we give an algorithm that can determine $C_2$ in $O((1 + m'_2)(\log^2 n + \log m))$ time, and in particular, we need to find the cells of $C'_2$. Intuitively, if $C_i \notin C_2$, then $C'_2$ consists of all cells of $C_2$ between $C_{i-1}$ and $C_{i+1}$ in the canonical order; otherwise, $C'_2$ consists of all cells of $C_2$ between $C_{i-1}$ and $C_i$ and all cells between $C_i$ and $C_{i+1}$. Our algorithm essentially follows the behavior of the algorithm in Lemma 7 but only focuses on the cells in $C'_2 \cup \{C_{i-1}, C_i, C_{i+1}\}$. Recall that $C_i$ is the cell of $C_1$ that contains $z_1$.

First of all, we delete the point $z_1$ from the data structure $D_P$. The point $z_1$ can be the skyline-left point of $C_i$, or the skyline-bottom point of $C_i$, or both of them, or neither of them. Our algorithm works differently for these cases, as follows. Recall that according to our algorithm in Lemma 7, if $z_1$ is either the skyline-left point or the skyline-bottom point of $C_i$, then $z_1$ has a generating segment, denoted by $s(z_1)$. In other words, $z_1$ is identified by a segment-dragging query on $s(z_1)$ in our algorithm in Lemma 7.
1. If \( z_1 \) is neither the skyline-left nor the skyline-bottom point of \( C_i \), then \( C_i \) is still in \( C_2 \) and \( C'_2 = \emptyset \). In fact, \( C_2 = C_1 \). Further, the skyline-left and skyline-bottom points of any cell of \( C_1 \) do not change. Hence, we are done for this case.

2. If \( z_1 \) is the skyline-left point but not the skyline-bottom point, then according to our algorithm in Lemma \( \text{[Lemma]} \), the generating segment \( s(z_1) \) is either an \( s_2 \)-type or an \( s_0 \)-type. Note that since \( z_1 \) is not the skyline-bottom point of \( C_i \), the skyline-bottom point of \( C_i \) is still in the skyline \( \pi_2 \), which implies that \( C_i \) is still in \( C_2 \) and no cell of \( C'_2 \) is between \( C_i \) and \( C_{i+1} \) in the canonical order of \( C_2 \). In other words, all cells of \( C'_2 \) are between \( C_{i-1} \) and \( C_i \) in the canonical order of \( C_2 \). Denote by \( D \) the column of \( A \) that contains \( C_i \).

(a) If \( s(z_1) \) is an \( s_2 \)-type, then \( C_i \) is not the topmost cell of \( C_1 \) in the column \( D \), which implies that \( C_{i-1} \) is in \( D \). According to the algorithm in Lemma \( \text{[Lemma]} \), \( s(z_1) \) is the left side of \( C_i \) (i.e., \( z_1 \) is the first point of \( P \) hit by dragging \( s(z_1) \) rightwards). By using the data structure \( D_P \) (after deleting \( z_1 \)), we do a segment-dragging query by dragging \( s(z_1) \) rightwards to find the first point of \( P \setminus \{z_1\} \) hit by \( s(z_1) \), and we denote the point by \( p \). Then, \( p \) is the new skyline-left point of \( C_i \) (without considering \( z_1 \)). Note that \( s(z_1) \) is still an \( s_2 \)-type generating segment for \( p \).

Next, from \( C_i \), we continue to find the cells of \( C'_2 \) in a bottom-up manner in the same way as the algorithm in Lemma \( \text{[Lemma]} \) until we meet the cell \( C_{i-1} \). Note that it is possible that \( C'_2 = \emptyset \). Again, it takes two segment-dragging queries (using \( D_P \) on each cell of \( C'_2 \)) to find its skyline-left and skyline-bottom point as well as their generating segments. Also, the algorithm will find the new skyline-bottom point of \( C_{i-1} \) if it changes in \( \pi_2 \). Recall that given any point \( p \), we can determine the cell of \( A \) that contains \( p \) in \( O(\log m) \) time (by binary search on the sorted vertical lines of \( A \) and on the sorted horizontal lines of \( A \)). Therefore, in this case, the total running time to determine \( C_2 \) is \( O((1 + m'_2)(\log^2 n + \log m)) \) time.

(b) If \( s(z_1) \) is an \( s_0 \)-type, then \( C_i \) is the topmost cell of \( C_1 \) in the column \( D \), which implies that \( C_{i-1} \) is in a column to the left of \( D \). Denote by \( D' \) the column of \( A \) containing \( C_{i-1} \) and let \( p \) be the skyline-bottom point of \( C_{i-1} \). According to the algorithm in Lemma \( \text{[Lemma]} \), \( s(z_1) \) is the vertical line segment on the right side of \( D' \) where the lower endpoint of \( s(z_1) \) is on the horizontal line \( y = y(q^*) \) and the upper endpoint has the same \( y \)-coordinate as \( p \), and \( z_1 \) is the first point of \( P \) hit by dragging \( s(z_1) \) rightwards.

By using the data structure \( D_P \) (after deleting \( z_1 \)), we do a segment-dragging query by dragging \( s(z_1) \) rightwards; let \( p' \) be the point returned by the query (i.e., \( p' \) is the first point of \( P \setminus \{z_1\} \) hit by dragging \( s(z_1) \) rightwards). Note that \( s(z_1) \) is still an \( s_0 \)-type generating segment for \( p' \).

i. If \( p' \) is in \( C_i \), then \( p' \) is the new skyline-left point of \( C_i \), and \( C_i \) is still the topmost cell of \( C_2 \) in \( D \), which implies \( C'_2 = \emptyset \).

ii. If \( p' \) is not in \( C_i \), then let \( C' \) be the cell containing \( p' \) and \( p' \) is the skyline-left point of \( C' \). Since \( z_1 \) is not the skyline-bottom point of \( C_i \), the cell \( C' \) is still in the column \( D \) and is higher than \( C_i \) (see Fig. \( \text{[Fig]} \)). Then, from the cell \( C_i \) to \( C' \), we use the bottom-up procedure as in the algorithm in Lemma \( \text{[Lemma]} \) to find the cells of \( C_2 \) between \( C_i \) and \( C' \) in the column \( D \) and these cells (expect \( C'_2 \)) constitute the set \( C'_2 \). Again, it takes two segment-dragging queries (using \( D_P \)) for each cell of \( C'_2 \) to find its skyline-left and skyline-bottom point as well as their generating segments.

The total running time is \( O((1 + m'_2)(\log^2 n + \log m)) \) time.

3. If \( z_1 \) is the skyline-bottom point but not the skyline-left point, then according to our algorithm in Lemma \( \text{[Lemma]} \), \( s(z_1) \) is either an \( s_1 \)-type or an \( s_3 \)-type. Note that since \( z_1 \) is not the skyline-left
point of $C_i$, the skyline-left point of $C_i$ is still in the skyline $\pi_2$, which implies that $C_i$ is still in $\mathcal{C}_2$ and no cell of $\mathcal{C}_2$ is between $C_{i-1}$ and $C_i$ in the canonical order of $\mathcal{C}_2$. In other words, all cells of $\mathcal{C}_2$ are between $C_i$ and $C_{i+1}$ in the canonical order of $\mathcal{C}_2$. Denote by $D$ the column of $A$ that contains $C_i$.

(a) If $s(z_1)$ is an $s_1$-type, then $C_i$ is the bottommost cell of $C_1$ in $D$. According to the algorithm in Lemma 7, $s(z_1)$ is the intersection of $D$ and the horizontal line $y = y(q^*)$. By using the data structure $D_P$ (after deleting $z_1$), we do a segment-dragging query by dragging $s(z_1)$ upwards and let $p$ be the point returned by the query. Then $p$ is the new skyline-bottom point of $C_i$. Next, we find the cells in $\mathcal{C}_2$.

Let $s'$ be the vertical segment on the right side of $D$ where the lower endpoint of $s'$ is on the horizontal line $y = y(q^*)$ and the upper endpoint of $s'$ has the same $y$-coordinate as $p$. We do a segment-dragging query by dragging $s'$ rightwards and let $p'$ be the point given by the query. The segment $s'$ is the generating segment of $p'$, and in fact, $s'$ is an $s_0$-type generating segment based on our definition in the proof of Lemma 7. Denote by $C(p')$ the cell of $A$ that contains $p'$. Let $D'$ be the column that contains $C_{i+1}$.

i. If $C(p')$ is in $D'$, then there are further two cases. If $C(p')$ is $C_{i+1}$, then $p'$ is the new skyline-left point of $C_{i+1}$, and $\mathcal{C}_2 = \emptyset$. Otherwise, from $C_{i+1}$ to $C(p')$, we use the same bottom-up procedure as in the algorithm in Lemma 7 to find all cells of $\mathcal{C}_2$ between $C_{i+1}$ and $C(p')$, and these cells (expect $C_{i+1}$) constitute the set $\mathcal{C}_2'$.

ii. If $C(p')$ is not in $D'$, then it must be in a column to the left of $D'$. From the cell $C(p')$, we proceed in the same way as in the algorithm in Lemma 7 until the first time we find a cell in the column $D'$. Then, we use the same algorithm as the above case where $C(p')$ is in $D'$.

(b) If $s(z_1)$ is an $s_3$-type, then $C_i$ is the not bottommost cell of $C_1$ in $D$, which implies that $C_{i+1}$ is in $D$. We show below that $\mathcal{C}_2 = \emptyset$; further, we will find a new skyline-bottom point in $C_i$ (without considering $z_1$).

Based on our algorithm in Lemma 7, the generating segment $s(z_1)$ of $z_1$ is the horizontal line segment on the top side of $C_{i+1}$ whose left endpoint is the upper left vertex of $C_{i+1}$ and right endpoint has the same $x$-coordinate as the skyline-left point of $C_{i+1}$. By using the data structure $D_P$ (after deleting $z_1$), we do a segment-dragging query by dragging $s(z_1)$ upwards, and let $p$ be the point returned by the query.
Note that \( z_1 \) is the lowest point of \( P \) that will be hit by dragging \( s(z_1) \) upwards and \( z_1 \) is in \( C_i \). The point \( p \) is the lowest point of \( P \setminus \{ z_1 \} \) that will be hit by dragging \( s(z_1) \) upwards (see Fig. 4(b)). Clearly, \( p \) cannot be any cell of \( D \) lower than \( C_i \). On the other hand, since \( z_1 \) is not the skyline-left point of \( C_i \), the skyline-left point of \( C_i \) is still in \( C_i \). Note that when we drag \( s(z_1) \) upwards, the skyline-left point of \( C_i \) will be hit by \( s(z_1) \) (but not necessarily the first point hit by \( s(z_1) \)), and this implies that the point \( p \) must be in \( C_i \). In other words, \( p \) is the skyline-bottom point of \( C_i \) in the new skyline \( \pi_2 \), and further \( C_i' = \emptyset \).

In any case above, the total running time is \( O((1 + m_2')(\log^2 n + \log m)) \) time.

4. It remains to discuss the case where \( z_1 \) is both the skyline-left point and the skyline-bottom point of \( C_i \). In this case, \( z_1 \) is the first time identified as either the skyline-left point or the skyline-bottom point. In general, unlike the second and the third cases where the cells of \( C_i' \) are either between \( C_{i-1} \) and \( C_i \) or between \( C_i \) and \( C_{i+1} \) in the canonical order of \( C_2 \), in this case the cells of \( C_i' \) may lie both between \( C_{i-1} \) and \( C_i \) and between \( C_i \) and \( C_{i+1} \). Hence, our algorithm may need to search on both “directions”. In addition, in the previous three cases, the cell \( C_i \) must be in \( C_2 \); in this case, however, it is possible that \( C_i \) is not in \( C_2 \).

(a) If \( z_1 \) is the first time identified as the skyline-left point of \( C_i \), then \( C_i \) must be the topmost cell of \( C_1 \) in \( D \) where \( D \) is the column of \( A \) that contains \( C_i \), which implies that its generating segment \( s(z_1) \) must be an \( s_0 \)-type.

Let \( p \) be the skyline-bottom point of the cell \( C_{i-1} \). Let \( D' \) be the column of \( A \) that contains \( C_{i-1} \). According to the algorithm in Lemma 4, \( s(z_1) \) is the vertical line segment on the right side of \( D' \) where the lower endpoint of \( s(z_1) \) is on the horizontal line \( y = y(q') \) and the upper endpoint has the same \( y \)-coordinate as \( p \). By using the data structure \( \mathcal{D}_P \) (after deleting \( z_1 \)), we do a segment-dragging query by dragging \( s(z_1) \) rightwards, and let \( p' \) be the point given by the query. Let \( C(p') \) be the cell that contains \( p' \). Note that \( p' \) is the skyline-left point of \( C(p') \). Let \( D'' \) be the column that contains the cell \( C_{i+1} \). Note that it is possible that \( D'' = D \).

i. If \( C(p') \) is also in \( D'' \), then there are further two cases.

If \( C(p') = C_{i+1} \), then \( C_2' = \emptyset \) and \( C_i \notin C_2 \).

Otherwise, \( C(p') \) must be higher than \( C_{i+1} \) in \( D'' \). Then, from the cell \( C_{i+1} \) to \( C(p') \), we use the same bottom-up procedure as in the algorithm in Lemma 4 to find all cells of \( C_2 \) between \( C_{i+1} \) and \( C(p') \), and these cells (except \( C_{i+1} \) and possibly \( C_i \)) constitute the set \( C_2' \). Note that the cell \( C_i \) may or may not be identified as in \( C_2 \) in the above procedure.

ii. If \( C(p') \) is not in \( D'' \), then \( C(p') \) must be in a column to the left of \( D'' \). We proceed from \( C(p') \) in the same way as in the algorithm in Lemma 4 until the first time we find a cell in \( D'' \). Then, we use the same algorithm as in the above case (i.) to determine \( C_2' \).

In any case, the total running time is \( O((1 + m_2')(\log^2 n + \log m)) \) time.

(b) If \( z_1 \) is the first time identified as the skyline-bottom point, then its generating segment \( s(z_1) \) can be either an \( s_1 \)-type or an \( s_3 \)-type segment. Let \( D \) be the column that contains \( C_i \). In this case, \( C_i \) is not the topmost cell of \( C_2 \) in \( D \) since otherwise \( z_1 \) would be the first time identified as the skyline-left point of \( C_i \). This means that \( C_{i-1} \) is also in \( D \).

i. If \( s(z_1) \) is an \( s_1 \)-type segment, then \( C_i \) must be the bottommost cell of \( C_1 \) in the column \( D \).

According to the algorithm in Lemma 4, \( s(z_1) \) is the intersection of \( D \) and the horizontal line \( y = y(q') \). By using the data structure \( \mathcal{D}_P \) (after deleting \( z_1 \)), we do a segment-dragging query by dragging \( s(z_1) \) upwards and let \( p \) be the point returned by the query. Let \( C(p) \) be the cell that contains \( p \). Clearly, \( C(p) \) is in \( C_2 \). Let \( C_{21} \) be the subset of cells.
in $C'_i$ that are between $C_{i-1}$ and $C(p)$ in the canonical order of $C_2$, and let $C'_{22} = C'_2 \setminus C'_{21}$; in other words, $C'_{22}$ is the subset of cells in $C'_2$ that are between $C(p)$ and $C_{i+1}$ in the canonical order of $C_2$. Below, we will find $C'_{21}$ and $C'_{22}$ separately, by searching from $C(p)$ towards two “directions”: one towards $C_{i-1}$ and the other towards $C_{i+1}$.

Since $C_{i-1}$ is also in $D$, from $C(p)$ to $C_{i-1}$, we use the bottom-up procedure to find the cells of $C_2$ between $C(p)$ and $C_{i-1}$, and these cells (except $C_{i-1}$ and possibly $C_i$) constitute the set $C'_{21}$. Note that the cell $C_i$ may also be identified in $C_2$.

Next, we find the set $C'_{22}$, which can be done by the same algorithm as in Case 3a. We omit the details.

ii. If $s(z_1)$ is an $s_3$-type, then $C_i$ is the not bottommost cell of $C_1$ in $D$, which implies that $C_{i+1}$ is in $D$.

Based on our algorithm in Lemma 7, the generating segment $s(z_1)$ of $z_1$ is the horizontal line segment on the top side of $C_{i+1}$ whose left endpoint is the upper left vertex of $C_{i+1}$ and right endpoint has the same $x$-coordinate as the skyline-left point of $C_{i+1}$. By using the data structure $D_P$ (after deleting $z_1$), we do a segment-dragging query by dragging $s(z_1)$ upwards, and let $p$ be the point returned by the query. Let $C(p)$ be the cell that contains $p$.

Note that $z_1$ is the lowest point of $P$ that will be hit by dragging $s(z_1)$ upwards and $z_1$ is in $C_i$. The point $p$ is the lowest point of $P \setminus \{z_1\}$ that will be hit by dragging $s(z_1)$ upwards. Since $C_{i-1}$ is also in $D$, $C_{i-1}$ is higher than $C_i$. Hence, the cell $C(p)$ is one of the cells of $D$ between (and including) $C_{i-1}$ and $C_i$ (this is because the vertices of $\pi_1$ in $C_{i-1}$ are all to the left of the right endpoint of $s(z_1)$ and to the right of the left endpoint of $s(z_1)$). Hence, from $C(p)$ to $C_{i-1}$, we can use the bottom-up procedure as before to find the cells of $C_2$, these cells (except $C_{i-1}$ and possibly $C_i$) constitute the set $C'_{22}$.

In any case, the total running time is $O((1 + m'_2)(\log^2 n + \log m))$ time.

In summary, we can determine the set $C_2$ in $O((1 + m'_2)(\log^2 n + \log m))$ time. More specifically, for each cell in $C'_2$, we have computed its skyline-left point and its skyline-bottom point as well as their generating segments. We have also determined whether $C_i$ is in $C_2$, and if yes, its new skyline-bottom point and skyline-left point are computed if any of them changes. The new skyline-bottom point of $C_{i-1}$ has been found if it changes, and the new skyline-left point of $C_{i+1}$ has been found if it changes. In addition, in the above algorithm, we can also order all cells of $C'_2$ (with $C_i$ if $C_i \in C_2$) from northwest to southeast with the same running time, and therefore, along with the ordered cells from $C_1$ to $C_{i-1}$ and the ordered cells from $C_{i+1}$ to $C_{m_1}$, we have obtained a canonical order for $C_2$.

By Lemma 10, we can determine the set $C_2$, and in particular, we have the set $C'_2$ explicitly, and we know whether the cell $C(z_1) \in C_2$. Similarly to Lemma 8, the second ANN $z_2$ is in one of the cells of $C_2$. Denote by $P_1 = P \setminus \{z_1\}$.

To find $z_2$, as in the case for finding $z_1$, a straightforward approach is to compute the ANN of $Q$ in $P_1 \cap C$ for each $C \in C_2$, and then among the $|C_2|$ candidate points, report the one with the smallest aggregate distance to $Q$ as $z_2$. This approach will lead to an $O(km)$ time query algorithm for finding $S_k(P_1)$. Below, we present a better method.

Note that when computing $z_1$, we have computed the ANN $S_1(P \cap C)$ for each $C \in C_1$. Also, for each cell $C \in C_1$, if $C \neq C(z_1)$, then $C \in C_2$ and $P \cap C = P_1 \cap C$. Therefore, if we maintain the ANNs for all cells of $C_1 \setminus C(z_1)$, we do not have to compute them again. In other words, when computing $z_2$, we only need to compute the ANNs in the cells of $C'_2$. In addition, if $C(z_1) \in C_2$, we
will use a special approach to compute \( S_1(P_1 \cap C(z_1)) \). To maintain the ANNs in the involved cells mentioned above, we use a min-heap \( H \), as follows.

When searching \( z_1 \), for each \( C \in C_1 \), after the ANN \( S_1(P \cap C) \) is computed, we insert it into \( H \) with its aggregate distance to \( Q \) as the “key”. After the ANNs for all cells of \( C_1 \) are computed and inserted into \( H \), the point in \( H \) with the smallest key is \( z_1 \). Note that \( H \) has \( m_1 = |C_1| \) points. To compute the second ANN \( z_2 \), we first determine \( C_2' \) by Lemma 10. By the “Extract-Min” operation of min-heaps [10], we remove \( z_1 \) from \( H \). We compute the ANNs of the cells in \( C_2' \) and insert them into \( H \). If \( C(z_1) \notin C_2 \), then the point of \( H \) with the smallest key is \( z_2 \). Otherwise, we use the following special approach to determine \( S_1(P_1 \cap C(z_1)) \).

One tempting approach is to have a dynamic version of the data structure in Lemma 8 to support point deletions from \( P \). Unfortunately, due to the “static” nature of compact interval trees, it is not clear to us how to design such a dynamic data structure without deteriorating the performance. Instead, we present another method to “mimic” point deletions, as follows.

We divide the cell \( C(z_1) \) into two sub-cells \( C_1(z_1) \) and \( C_2(z_1) \) using the horizontal line through \( z_1 \). Hence, \( z_1 \) is on the common edge of the two sub-cells. Note that due to our general position assumption, no point of \( P \) is on the boundary of \( C(z_1) \). Hence, no point of \( P_1 = P \setminus \{z_1\} \) is on the boundary of \( C_1(z_1) \) (or \( C_2(z_1) \)). Below, we use \( C_1(z_1) \) (resp., \( C_2(z_1) \)) to refer to only its interior. Instead of computing the ANN \( S_1(P_1 \cap C(z_1)) \) and insert it into \( H \), we compute the ANNs \( S_1(P \cap C_1(z_1)) \) and \( S_1(P \cap C_2(z_1)) \) and insert them into \( H \); note that one of them is \( S_1(P_1 \cap C(z_1)) \). The reason we divide \( C(z_1) \) into two sub-cells is because that we can now simply use the data structure in Lemma 8 to compute \( S_1(P \cap C_1(z_1)) \) and \( S_1(P \cap C_2(z_1)) \); in other words, \( z_1 \) appears to be “deleted” from the data structure of Lemma 8. Clearly, now, the point of \( H \) with smallest key is \( z_2 \).

To analyze the running time for computing \( z_2 \), \( C_2 \) can be determined in \( O((1 + m_2')(\log^2 n + \log m)) \) time, after which, we compute the ANNs of the cells of \( C_2' \) possibly for the two sub-cells of \( C(z_1) \) in \( O((2 + m_2') \log^2 n) \) time by Lemma 8. Then, one “Extract-Min” operation and at most \( m_2' + 2 \) insertions on \( H \) together take \( O((m_2' + 3) \log(|H|)) \) time; note that \(|H| \leq m_1 + m_2' + 2 \) (here “2” corresponds to the number of possible sub-cells).

It should be noted that we need to explicitly maintain the two sub-cells \( C_1(z_1) \) and \( C_2(z_1) \) because later they may be further divided into smaller sub-cells (e.g., if \( z_2 \in C_1(z_1) \) and \( C(z_1) \in C_3 \), then \( C_1(z_1) \) will be divided for computing \( z_3 \)). Also note that these sub-cells are only maintained for computing ANNs and they will not be considered when we determine the sets \( C_i \)’s (in Lemma 10). After \( z_2 \) is found, we proceed to search the third ANN \( z_3 \) similarly.

In general, suppose we have computed \( C_i \) and \( z_i \), and we are about to find \( z_{i+1} \). We first determine \( C_{i+1} \) by computing \( C_{i+1}' \) and determining whether \( C(z_i) \in C_{i+1} \), where \( C(z_i) \) is the cell of \( C_i \) that contains \( z_i \); this can be done in \( O((1 + m_{i+1}')(\log^2 n + \log m)) \) time similarly as in Lemma 10.

Note that for any cell \( C \in C_{i+1}' \), it never appears in \( C_j \) for any \( 1 \leq j \leq i \). Next, we determine the ANNs in the cells of \( C_{i+1}' \) by Lemma 8 and insert them into the heap \( H \). We also need to remove \( z_i \) from \( H \). If \( C(z_i) \notin C_{i+1} \), then the point of \( H \) with smallest key is \( z_{i+1} \). Otherwise, as before, we divide \( C(z_i) \) into two sub-cells and compute their ANNs and insert them into \( H \). Note that \( C(z_i) \) may have already been divided into many sub-cells before. If so, they are explicitly maintained, and we can find the sub-cell that contains \( z_i \) in \( O(\log k) \) time by binary search since \( C(z_i) \) has at most \( k - 1 \) sub-cells ordered by \( y \) values. Then, we divide the sub-cell into two smaller sub-cells by
the horizontal line through $z_i$ and compute the ANNs in the two smaller sub-cells by Lemma $8$ and insert them into $H$. Now, the point of $H$ with smallest key is $z_{i+1}$.

To analyze the running time for computing $z_{i+1}$, $C_{i+1}$ can be determined in $O((1+m_{i+1}'')(\log^2 n + \log m))$ time. The time for computing the ANNs for the cells in $C_{i+1}'$ and possibly two sub-cells is bounded by $O((2+m_{i+1}'') \log^2 n)$. There are $O(2+m_{i+1}'')$ insertions and one “Extract-Min” operation on $H$, which together take $O((m_{i+1}''+3) \log(|H|))$ time. Note that $|H| \leq m_i + m_{i+1}'' + 2$.

We repeat the above procedure until $z_k$ is found. We have the following lemma (a crucial observation is that $m_1 + \sum_{i=2}^{k} m_i' = O(m + k)$).

**Lemma 11.** The overall running time of our query algorithm for finding $S_k(P^1) = \{z_1, z_2, \ldots, z_k\}$ is $O(m \log m + (k + m) \log^2 n)$.

**Proof:** Let $\lambda = m_1 + \sum_{i=2}^{k} m_i'$ denote the total number of cells in $C_1 \cup \bigcup_{i=2}^{k} C_i'$.

By Lemma $7$, we compute $C_1$ in $O(m \log n + m \log m)$ time. By Lemma $10$, the total time for finding all cells of $\bigcup_{i=2}^{k} C_i'$ is $O((k + \lambda) \log^2 n + \log m)$.

In the entire algorithm, the total number of operations for finding the ANNs in the cells of $A$ (not including the sub-cells) is $O(\lambda)$ because the above cells are those in $C_1 \cup \bigcup_{i=2}^{k} C_i'$. After finding $z_i$ for each $1 \leq i \leq k$, we have at most two more sub-cells, and thus the total number of operations for finding the ANNs in the sub-cells is $O(k)$. Hence, by Lemma $8$ the total time for finding the ANNs in the cells and sub-cells is $O((\lambda + k) \log^2 n)$. Also, we only need to explicitly maintain at most $O(k)$ sub-cells in the entire algorithm.

Similarly, the total number of operations on the heap $H$ is $O(\lambda+k)$, and the size of $H$ in the entire algorithm is always bounded by $O(\lambda+k)$. Hence, the total operations on $H$ take $O((\lambda+k) \log(\lambda+k))$ time.

In summary, the overall running time is $O((k + \lambda) \log^2 n + \log m + \log(\lambda + k)))$. To prove the lemma, we prove an important **claim:** $\lambda = O(m + k)$.

The proof for the claim is based on the fact that $|C_i| = O(m)$ for each $1 \leq i \leq k$, since each skyline $\pi_i$ intersects $O(m)$ cells. In particular, $|C_k| = O(m)$. For each $1 \leq i \leq k-1$, all the cells of $C_i$ except $C(z_i)$ are in $C_{i+1}$ and the cell $C(z_i)$ may or may not be in $C_{i+1}$. Hence, $|C_{i+1}| \geq |C_i| - |C_i'| + 1$, i.e., $m_{i+1} \geq m_i - 1 + m_{i+1}'$. Therefore, $m_k \geq m_1 + \sum_{i=2}^{k} m_i' - (k - 1)$. Due to $m_k = O(m)$, we have $\lambda = m_1 + \sum_{i=2}^{k} m_i' \leq m_k + k - 1 = O(m + k)$. The above claim thus follows.

Due to the above claim, the overall running time for finding $S_k(P^1)$ is $O((k + m) \log^2 n + \log(k + m))$, which is $O(m \log m + (k + m) \log^2 n)$ (to see this, note that if $k > m$, then since $n \geq k$, $(k + m) \log(k + m) = O((k + m) \log^2 n)$ holds).

Note that after obtaining $S_k(P^1)$, we also need to insert the points of $S_k(P^1)$ back to the data structure in Lemma $4$ for answering other top-$k$ ANN queries in future.

### 3.5 Wrapping Things Up

We summarize our methods for the top-$k$ ANN queries in the $L_1$ metric.

Our preprocessing on $P$ includes the following steps. (1) Sort all points in $P$ by their $x$-coordinates and $y$-coordinates, respectively. (2) Build the dynamic segment-dragging query data structure in Lemma $9$ on $P$. (3) Construct the data structure in Lemma $8$. The total time and space are dominated by Step (3), regardless of which data structure of Lemma $8$ is used.

Given any query set $Q$ and any $k$, we compute $S_k(P)$ in the following steps. (1) Sort all points in $Q$ by their their $x$-coordinates and $y$-coordinates, respectively. (2) Process $Q$ as in Lemma $4$. (3) Construct the data structure in Lemma $8$. (4) Process the query as in Lemma $8$. (5) Output the results.
(1) Compute a global minimum point \(q^*\). (2) Divide the plane into four quadrants with respect to \(q^*\). In each quadrant \(R\), we find the top-\(k\) ANNs of \(Q\) in \(P \cap R\) as follows. Suppose \(R\) is the first quadrant. (2.1) Find the set \(C_1\) by Lemma 2 and for each cell \(C \in C_1\), find the ANN of \(Q\) in \(P \cap C\) by Lemma 3 and insert the point into a min-heap \(H\); the point of \(H\) with smallest aggregate distance to \(Q\) is the ANN of \(Q\) in \(P \cap R\). (2.2) Based on \(C_1\) and \(z_1\), determine \(C_2\) and find \(z_2\). (2.3) The above procedure continues until we find \(z_k\). (3) Among the found \(4k\) points from all four quadrants of \(q^*\) (their aggregate distances to \(Q\) have also been computed), we report the \(k\) points with smallest aggregate distances to \(Q\) as \(S_k(P)\). (4) Insert the above \(4k\) points back to the data structure in Lemma 3 (for answering other top-\(k\) ANN queries in future).

For the running time of the query algorithm, the first three steps can be done in \(O(m \log m)\) time; Step (4) can be done in \(O(m \log m + (k + m) \log^2 n)\) time. Step (5) takes \(O(k)\) time. Step (6) needs \(O(k \log^2 n)\) time. Hence, the total query time is bounded by \(O(m \log m + (k + m) \log^2 n)\). If we use the other two data structures in Lemma 3, then we have the query times of \(O(m \log m + (k + m) \log^2 n \log \log n)\) and \(O(m \log m + (k + m) \log^2 n \log^* n)\), respectively.

**Theorem 2.** Given a set \(P\) of \(n\) points in the plane, a data structure of \(O(n \log n \log \log n)\) size can be built in \(O(n \log n \log \log n)\) time, such that for any weighted set \(Q\) and any \(k\), the top-\(k\) ANNs can be found in \(O(m \log m + (k + m) \log^2 n)\) time. With trade-off between preprocessing and query time, we also build two other data structures: the first one has \(O(n \log n)\) preprocessing time and space with \(O(m \log m + (k + m) \log^2 n \log \log n)\) query time; the second one has \(O(n \log n \log^* n)\) preprocessing time and space with \(O(m \log m + (k + m) \log^2 n \log^* n)\) query time.

4 Top-\(k\) Aggregate Farthest Neighbor (AFN) Searching in the \(L_1\) Metric

Our techniques can be extended to solve the top-\(k\) AFN searching, with the same time bounds as in Theorems 1 and 2.

For the 1-D case, given a query set \(Q\) and \(k\), recall that for computing the top-\(k\) ANNs, we first compute the global minimum point \(q^*\) and then search in \(P\) beginning from \(q^*\) simultaneously towards left and right. To compute the top-\(k\) AFNs, due to Lemma 1, we search in \(P\) simultaneously beginning from the leftmost and rightmost points of \(P\) and towards the middle (e.g., either the leftmost or the rightmost point of \(P\) is the top-1 AFN of \(Q\) by Lemma 1). The rest of the algorithm is similar as that for Theorem 1 and we omit the details. Hence, we can obtain the following result.

**Theorem 3.** Given a set \(P\) of \(n\) points on the real line \(L\), with \(O(n \log n)\) preprocessing time and \(O(n)\) space, the top-\(k\) AFNs can be found in \(O(\min\{k, \log m\} \cdot m + k + \log n)\) time for any query set \(Q\) and any \(k\); if the points of \(Q\) are given sorted on \(L\), then the query time is \(O(k + m + \log n)\).

For the 2-D case, consider any query set \(Q\) and \(k\). As in the ANN case, we first compute a global minimum point \(q^*\) and then compute the top-\(k\) AFNs in each quadrant of \(q^*\). Suppose \(R\) is the first quadrant with respect to \(q^*\). We find the top-\(k\) AFNs in \(R\) as follows. Recall that in the ANN case we search the top-\(k\) ANNs in a direction from \(q^*\) towards northeast. In the AFN case, due to Lemma 2 we search the top-\(k\) AFNs along the opposite direction, i.e., from northeast towards \(q^*\). Specifically, here we re-define the “dominate” relationship in the opposite way as before: a point \(p_1\) dominates \(p_2\) if and only if \(x(p_1) \geq x(p_2)\) and \(y(p_1) \geq y(p_2)\). A point \(p\) in \(P \cap R\) is called a maximal point if no other point in \(P \cap R\) dominates \(p\). Similarly, we re-define the skyline as the path connecting all maximal points of \(P \cap R\). According to Lemma 2, the AFN of \(Q\) in \(P \cap R\) must be in
the skyline. Then, we can use a similar algorithm as in the ANN case to compute all top-$k$ AFNs. More specifically, we first compute the AFN $p$ on the skyline and then search the second AFN on the next skyline (without considering $p$); we continue this procedure until we find the $k$-th AFN. The algorithm is similar (or symmetric) as the ANN case and we omit the details. Hence, we can obtain the following result.

**Theorem 4.** Given a set $P$ of $n$ points in the plane, a data structure of $O(n \log n \log \log n)$ size can be built in $O(n \log n \log \log n)$ time, such that for any weighted set $Q$ and integer $k$, the top-$k$ AFNs can be found in $O(m \log m + (k + m) \log^2 n)$ time. With trade-off between preprocessing and query time, we also build two other data structures: the first one has $O(n \log n)$ preprocessing time and space with $O(m \log m + (k + m) \log n \log n)$ query time; the second one has $O(n \log n \log^* n)$ preprocessing time and space with $O(m \log m + (k + m) \log^2 n \log^* n)$ query time.

5 Conclusions

We presented efficient methods for the top-$k$ aggregate nearest and farthest neighbor searching in the plane under the $L_1$ metric. Our results are the first-known solutions for the general top-$k$ queries on the weighted query points. Even for the special case where $k = 1$ or the unweighted query points, our results are generally better than the previous work. While it would be interesting to investigate whether any further improvements are possible, another open problem is whether and how the techniques proposed in this paper can be extended to higher dimensional spaces.

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