THE GLOBAL RENORMALIZATION GROUP TRAJECTORY IN A CRITICAL
SUPERSYMMETRIC FIELD THEORY ON THE LATTICE $\mathbb{Z}^3$

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Abstract: We consider an Euclidean supersymmetric field theory in $\mathbb{Z}^3$ given by a supersymmetric $\Phi^4$ perturbation of an underlying massless Gaussian measure on scalar bosonic and Grassmann fields with covariance the Green’s function of a (stable) Lévy random walk in $\mathbb{Z}^3$. The Green’s function depends on the Lévy-Khintchine parameter $\alpha = \frac{3+\varepsilon}{2}$ with $0 < \alpha < 2$. For $\alpha = \frac{3}{2}$ the $\Phi^4$ interaction is marginal. We prove for $\alpha - \frac{3}{2} = \frac{\varepsilon}{2} > 0$ sufficiently small and initial parameters held in an appropriate domain the existence of a global renormalization group trajectory uniformly bounded on all renormalization group scales and therefore on lattices which become arbitrarily fine. At the same time we establish the existence of the critical (stable) manifold. The interactions are uniformly bounded away from zero on all scales and therefore we are constructing a non-Gaussian supersymmetric field theory on all scales. The interest of this theory comes from the easily established fact that the Green’s function of a (weakly) self-avoiding Lévy walk in $\mathbb{Z}^3$ is a second moment (two point correlation function) of the supersymmetric measure governing this model. The rigorous control of the critical renormalization group trajectory is a preparation for the study of the critical exponents of the (weakly) self-avoiding Lévy walk in $\mathbb{Z}^3$.

0. Introduction

It was observed long ago, [PS, McK], that the Green’s function of weakly self avoiding simple random walks (SAW) on a lattice $\mathbb{Z}^d$ can be expressed as a correlation function in a supersymmetric field theory. This can be shown rigorously by the same derivation as in [BEI, BI1, BI2] for SAWs on hierarchical lattices. Consider instead of simple random walks the more general case of continuous time (stable) Lévy walks whose scaling limits are stable Lévy distributions, [KG, F]. Such walks can be realized as jump processes with probability distributions permitting long range jumps, [F]. Their characteristic functions are given by the Lévy-Khintchine formula with characteristic exponent $\alpha$, $0 < \alpha \leq 2$, [F], $\alpha = 2$ corresponding to simple random walks. The Green’s function of continuous time weakly self avoiding Lévy walks (SALW) can also be realized as a two point correlation function in a supersymmetric field theory by the same derivation as in [BEI, BI1, BI2]. This paper is concerned with proving the existence of a critical uniformly bounded renormalization group (RG) trajectory for the interactions in the underlying supersymmetric field theory corresponding to the class of SALWs where $\alpha = \frac{3+\varepsilon}{2}$ with $0 < \varepsilon < 1$ and $\varepsilon$ held small. The case $\alpha = \frac{3}{2}$ corresponds to mean field theory. Uniformity is with respect to the lattice scale which changes with each step of the renormalization group map. We find that the interactions are non-vanishing at all renormalization group scales, which is the lattice version of a non-Gaussian fixed point. This gives the foundation for the study of the Green’s function of SALWs in the scaling limit which is postponed to the sequel. Ultimately
one would like to be able to extract from this the end-to-end distance behaviour for SALWs.

The supersymmetric field theory in question is a lattice supersymmetric generalization of the model considered in [BMS]. We describe it informally here and leave the details for the next section. Let $\Delta$ be the standard Laplacian in $\mathbb{Z}^3$. Then for $x, y \in \mathbb{Z}^3$, and $0 < \alpha < 2$, $C(x - y) = (-\Delta)^{-\alpha/2}(x - y)$ is the Green’s function of a stable Lévy walk. Let $\varphi_1, \varphi_2$ be independent identically distributed Gaussian random fields in $\mathbb{Z}^3$ with covariance $\frac{1}{2}C$. Let $\varphi = \varphi_1 + i\varphi_2$ and $\bar{\varphi}$ its complex conjugate. Introduce a pair of Grassmann fields $\psi, \bar{\psi}$ of degree 1 and $-1$ respectively. Let $\Phi = (\varphi, \psi)$ and $\bar{\Phi} = (\bar{\varphi}, \bar{\psi})$. The inner product is given by $\langle \Phi, \Phi \rangle = \Phi \bar{\Phi} = \varphi \bar{\varphi} + \psi \bar{\psi}$. Let $\Lambda \subset \mathbb{Z}^3$ be a finite subset. Define

$$V_0(\Lambda, \Phi) = g_0 \int_{\Lambda} dx (\Phi \bar{\Phi})^2(x) + \tilde{\mu}_0 \int_{\Lambda} dx \Phi \bar{\Phi}(x) \quad (0.0)$$

where the coupling constant $g_0 > 0$ and $dx$ is the counting measure in $\mathbb{Z}^3$. Then our model in finite volume $\Lambda$ is defined by the supermeasure

$$d\mu_\Lambda(\Phi) = d\mu_{C_\Lambda}(\Phi) e^{-V_0(\Lambda, \Phi)} \quad (0.1)$$

where $C_\Lambda$ is the restriction of $C$ to the points of $\Lambda$ and $d\mu_{C_\Lambda}(\Phi)$ is the Gaussian supermeasure

$$d\mu_{C_\Lambda}(\Phi) = \prod_{x \in \Lambda} d\varphi_1(x) d\varphi_2(x) d\psi(x) d\bar{\psi}(x) e^{-(\Phi, C^{-1}_\Lambda \bar{\Phi})} \quad (0.2)$$

Integration over the Grassmann fields is Berezin integration and $d\mu_\Lambda(\Phi)$ is interpreted as a linear functional on the Grassman algebra (generated by the $\psi, \bar{\psi}$ with coefficients which are functionals of the $\varphi, \bar{\varphi}$). An important fact is that the potential $V_0(\Lambda, \Phi)$ is supersymmetric ($\text{supersymmetry in this context and some of its consequences are given in the Section 1.1}$). As a consequence we have that the supermeasure $d\mu_\Lambda(\Phi)$ is normalized:

$$\int d\mu_\Lambda(\Phi) 1 = 1 \quad (0.3)$$

The parameters of the supermeasure $\mu_\Lambda$ defined in (0.1) correspond to those of SALWs. Thus $g_0$ measures the strength of self-repulsion and $\tilde{\mu}_0$ the killing rate of a weakly self-avoiding Lévy walk. The reader will get a full dictionary in [BEI, BI1, BI2] where the end-to-end distance behaviour was studied for SAWs in a four dimensional hierarchical lattice with the help of supersymmetry.

We give an informal description of the results of this paper. We will choose $\alpha = 3 + 2\varepsilon$ with $0 < \varepsilon < 1$, in particular we hold $\varepsilon > 0$ very small. We will take $\Lambda$ to be a very large cube. By successive RG transformations we will get a sequence of measures (the RG trajectory of measures) living in smaller and smaller cubes in finer and finer lattices till we arrive at a fixed small cube in a very fine lattice. This will take $\log \Lambda$ steps. At every step the measure is a new Gaussian measure times a new supersymmetric density. The Gaussian measure is characterized by a covariance and the sequence of covariances converge to a smooth continuum covariance. The supersymmetric density incorporates the interactions. The principal information is in the local interactions incorporated in local potentials of the above type albeit with new parameters (coupling constants) and on a finer lattice. The other interactions are contracting (irrelevant) in an appropriate sense and are expressed in the form of polymer activities. The coupling constants and polymer activities give coordinates of the RG trajectory. These coordinates provide Banach spaces of interactions which permit a rigorous study of the Wilson RG [WK] avoiding real space renormalization group pathologies, [GP1], [GP2], related to the Griffiths singularity problem in disordered systems, [G]. See [BKL] for a review of these pathologies. The goal of this paper is to study the RG trajectory of these coordinates in the infinite volume limit which makes sense for these coordinates. The true infinite volume limit and the scaling limit will be taken at the level of correlation functions.
In section 1 we define the model, introduce supersymmetry and develop some of its consequences. The RG analysis of this paper is based on the finite range multiscale expansion of covariances of [BGM]. We summarize the basic results of [BGM] pertinent to this paper in Theorem 1.1. This is an alternative to the Kadanoff- Wilson block spin RG developed extensively by Gawedzki and Kupiainen [GK 1,2], and Balaban [Bal 1,2,3]. A crucial simplification arises due to the finite range of the fluctuation covariances: Cluster expansions are no longer needed in the control of the fluctuation integration which is an essential step of RG transformations. As a result all estimates are local in character. In this section we also define lattice polymers and polymer activities.

In section 2 we introduce norms which will measure the size of polymer activities. These norms are suggested by those in the continuum analysis of [BMS] but now take account of the presence of Grassmann fields. The choice of these norms was inspired by discussions with David Brydges. They are closely related to norms which will appear in the forthcoming study of self-avoiding simple random walks in four dimensions by Brydges and Slade [BS].

In section 3 we define the RG map as we will use it and in section 4 apply it to our model. In particular we develop second order perturbation theory. The task is to control the contributions from the remainder and this is taken up in the next section.

Section 5 gives the basic estimates that we will need for the control of the RG trajectory. These estimates are extensions of those in section 5 of [BMS]. The latter paper studied a critical bosonic theory in the continuum. Our present estimates take account of the presence of Grassmann variables as well as the lattice which has led to a considerable number of new details. The upshot is Theorem 5.1.

Section 6 is devoted to the proof of existence of the stable manifold: there exists an initial critical mass \( \hat{\mu}_0 \) which is a Lipshitz continuous function of the coupling constant \( g_0 \) such that RG trajectory is bounded uniformly on all scales. The proof is established by a combination of three theorems, namely Theorems 6.2, 6.4, and 6.6.

Finally we observe that the coupling constant \( g_n \) is uniformly bounded away from 0 at all scales \( n \geq 0 \). As a result the global RG trajectory gives rise to a non-Gaussian field theory. We remark that in a continuum version of this model with a cutoff modelled on that of [BMS] one can prove more: the continuum RG trajectory ends at a non-trivial fixed point. But the notion of a fixed point is devoid of meaning for lattice field theories because the RG map even in infinite volume does not give an autonomous action on a fixed Banach space.

### 1.1 Definitions, model, supersymmetry

Let \( e_1, e_2, e_3 \) be the standard basis of unit vectors specifying the orientation of \( \mathbb{Z}^3 \). We let \( \partial_\mu \) denote the forward lattice derivative in direction \( e_\mu \) and \( \partial^*_\mu \) its \( L^2(\mathbb{Z}^3) \) adjoint. The latter is the backward derivative. Then the lattice Laplacian \( \Delta \) in \( \mathbb{Z}^3 \) is defined by

\[
-\Delta = \sum_{\mu=1}^{3} \partial^*_\mu \partial_\mu
\]

Let \( \hat{\Delta}(p) \) be the Fourier transform of the integral kernel of \( \Delta \) in \( \mathbb{Z}^3 \), namely

\[
\hat{\Delta}(p) = 2 \sum_{\mu=1}^{3} (\cos(p_\mu) - 1) \tag{1.2}
\]

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\( ^1 \) A. Abdesselam in [A] corrected an error which occurs in [BDH-eps, BMS] where it is wrongly asserted that certain normed spaces are complete. This problem was resolved in [A] by some changes in definitions which fortunately are such that, as noted in [A], the estimates, theorems and proofs of [BDH-eps, BMS] remain true without change. On the lattice however the function space subtleties encountered in [A] disappear.
Let $\alpha = \frac{3\pm \sqrt{5}}{2}$ be a real number with $0 < \alpha < 2$. Then the Green’s function of a (stable) Lévy walk in $\mathbb{Z}^3$ is given by

$$C(x - y) = (-\Delta)^{-\alpha/2}(x - y) = \int_{[\pi, \pi]^3} \frac{d^3p}{(2\pi)^3} e^{ip(x-y)} (-\tilde{\Delta}(p))^{-\alpha/2}$$

(1.3)

$C$ is positive-definite and therefore qualifies as the covariance of a Gaussian random field in $\mathbb{Z}^3$. We introduce a pair of independent identically distributed Gaussian random fields $\varphi_1, \varphi_2$ with mean 0 and covariance

$$E(\varphi_j(x)\varphi_j(y)) = \frac{1}{2} C(x - y)$$

(1.4)

for $j = 1, 2$.

Let $\varphi(x) = \varphi_1(x) + i\varphi_2(x)$ be a complex scalar field and $\bar{\varphi}(x)$ its complex conjugate. On the space of functionals of $\varphi, \bar{\varphi}$ we have the Gaussian probability measure

$$d\mu_C(\phi) = d\mu_{\frac{1}{2}C}(\phi_1)d\mu_{\frac{1}{2}C}(\phi_2)$$

(1.5)

Then each of $\varphi, \bar{\varphi}$ has zero covariance and

$$E(\bar{\varphi}(x)\varphi(y)) = C(x - y)$$

(1.6)

**Grassmann algebra and integration**

Let $\Lambda \subset \mathbb{Z}^3$ be a bounded subset. $\mathcal{F}_\Lambda$ represents the algebra of $\mathbb{C}$ valued functionals of the fields $\{\phi, \bar{\phi}: \Lambda \to \mathbb{C}\}$. Let $\psi(x), \bar{\psi}(x)$ for all $x \in \Lambda$ be the (anti-commuting) Grassman elements of degree 1, $-1$ respectively. Following standard usage we will refer to them as (scalar) fermions. We denote by $\Omega_\Lambda$ the Grassman algebra generated by the $\psi(x), \bar{\psi}(y)$ by multiplication and linear sums for all $x, y \in \Lambda$ with coefficients in $\mathcal{F}_\Lambda$. The Grassmann algebra is naturally graded $\Omega_\Lambda = \oplus_{p} \Omega^p_\Lambda$ where the integer $p$ is the degree and each $\Omega^p_\Lambda$ is a $\mathcal{F}_\Lambda$ module. $\Omega^0_\Lambda$ is an algebra. Because of the anticommuting property of the generators, and because $\Lambda$ is a finite lattice an element of $\Omega^p_\Lambda$ is a finite sum of degree $p$ elements with coefficients in $\mathcal{F}_\Lambda$. For example, an element $F_\Lambda$ of $\Omega^0_\Lambda$ can be uniquely represented as

$$F_\Lambda(\varphi, \psi) = \sum_{p \geq 0} \int_{[\Lambda^3]} \prod_{j=1}^p dx_j dy_j F_{\Lambda, 2p}(\varphi; x_1, \ldots, x_p, y_1, \ldots, y_p) \prod_{j=1}^p \psi(x_j)\bar{\psi}(y_j)$$

(1.7)

where $dx$ is the counting measure in $\mathbb{Z}^3$. The coefficients, $F_{\Lambda, 2p}(\varphi; x_1, \ldots, x_p, y_1, \ldots, y_p)$, are antisymmetric in $(x_1, \ldots, x_p)$ and in $(y_1, \ldots, y_p)$. When $\Lambda$ is a finite subset the above multiple sum is finite. In the following we will often refer to the coefficients $F_{\Lambda, 2k}$ above as bosonic coefficients. Here and in the following we suppress indicating the dependence of the bosonic coefficients on $\bar{\varphi}$.

These considerations are of course valid for a lattice $(\delta \mathbb{Z})^3$ for any lattice spacing $\delta$ with the corresponding notations $\Lambda_\delta, \mathcal{F}_{\Lambda_\delta}, \Omega_{\Lambda_\delta}, dx$ being $\delta^3$ times the counting measure in $(\delta \mathbb{Z})^3$.

Now we define fermionic expectation (integration) using Berezin integration which we review briefly and set up our conventions. Berezin integration is a linear map $\Omega_\Lambda \to \Omega_\Lambda$ which satisfies

$$\int d\psi(x) F_\Lambda(\psi, \bar{\psi}, \phi, \bar{\phi}) = \pi^{-1/2} \frac{\partial}{\partial\psi(x)} F_\Lambda$$

(1.8)

where $F_\Lambda \in \Omega_\Lambda$ and the fermionic derivative $\frac{\partial}{\partial\psi(x)}$ is an antiderivation: If $f \in \Omega^p_\Lambda$ and $g \in \Omega^q_\Lambda$ then

$$\frac{\partial}{\partial\psi(x)} (fg) = \frac{\partial}{\partial\psi} fg + (-1)^p f \frac{\partial}{\partial\psi} g$$
Integration with respect to $d\bar{\psi}(x)$ is given by the same formula with $\frac{\partial}{\partial \bar{\psi}(x)}$ on the right hand side. Multiple integration is repeated integration using the above rule, keeping in mind that fermionic derivatives anticommute.

Define $C_\Lambda(x - y) = C(x - y): x,y \in \Lambda$. We consider this as a $|\Lambda| \times |\Lambda|$ dimensional positive definite symmetric matrix with $x,y$ labelling the entries. Then we define the fermionic expectation $E_{f,\Lambda}$ as a linear map $\Omega_\Lambda \to F_\Lambda$ as follows: Let $F_\Lambda \in \Omega_\Lambda$. We adopt the convention $F_\Lambda(\psi, \phi) \equiv F_\Lambda(\psi, \bar{\phi}, \phi)$. Then

$$E_{f,\Lambda}(F) = \int d\mu_{C_\Lambda}(\psi) F(\psi, \phi)$$

where

$$\int d\mu_{C_\Lambda}(\psi) F(\psi, \phi) = (\det \pi C_\Lambda)^{|\Lambda|} \int \prod_{x \in \Lambda} (d\psi(x)d\bar{\psi}(x)) e^{-(\psi, C_\Lambda^{-1}\bar{\psi})_{L^2(\Lambda)}} F(\psi, \phi)$$

We call $d\mu_{C_\Lambda}(\psi)$ a fermionic Gaussian measure and use the terminology measure and expectation interchangeably. It is not difficult to show that we have a fermionic counterpart of the bosonic gaussian formula, namely

$$\int d\mu_{C_\Lambda}(\psi) F(\psi, \phi) = e^{\int_{\Lambda \times \Lambda} dxdy C_\Lambda(x - y) \frac{\partial}{\partial \psi(x)} \frac{\partial}{\partial \psi(y)} F(\psi, \phi)} |_{\psi = \bar{\psi} = 0}$$

More generally, if $x_j, y_j \in \Lambda$, $j = 1, 2, .., n$

$$E_{f,\Lambda}(\prod_{j=1}^n \bar{\psi}(x_j)\psi(y_j)) = \det (C(x_j - y_k))_{j,k}^{n}$$

We define the field $\Phi(x)$ (called superfield in anticipation) as the pair

$$\Phi(x) = (\phi(x), \bar{\psi}(x))$$

with the scalar product

$$(\Phi(x), \Phi(y)) = \Phi(x)\bar{\Phi}(y) = \phi(x)\bar{\phi}(y) + \psi(x)\bar{\psi}(y)$$

More generally if $A(x, y)$ is a matrix for $x, y \in \Lambda$ we define

$$(\Phi, A\Phi)_{L^2(\Lambda)} = \int_{\Lambda \times \Lambda} dxdy \Phi(x)A(x, y)\bar{\Phi}(y) = \int_{\Lambda \times \Lambda} dxdy (\phi(x)A(x, y)\bar{\phi}(y) + \psi(x)A(x, y)\bar{\psi}(y))$$

Let $F_\Lambda(\Phi)$ belong to $\Omega_\Lambda$. $F_\Lambda$ also depends on $\Phi$ but here and in the following this is not explicitly indicated. Since $F_\Lambda(\Phi) \in \Omega_\Lambda$ it has the representation (1.7). We define the expectation $E_\Lambda$ as a linear map $\Omega_\Lambda \to C$ obtained by combining the bosonic and fermionic expectations: If $F_\Lambda \in \Omega_\Lambda$ with $\mu_C$ integrable bosonic coefficients then

$$E_\Lambda(F_\Lambda(\Phi)) = \int d\mu_{C_\Lambda}(\Phi) F_\Lambda(\Phi) = \int d\mu_{C_\Lambda}(\phi) d\mu_{C_\Lambda}(\psi) F_\Lambda(\Phi)$$
Thus
\[
E_\Lambda(F_\Lambda(\Phi)) = \int \prod_{x \in \Lambda} d\varphi(x) d\bar{\varphi}(x) \prod_{x \in \Lambda} d\psi(x) \prod_{x \in \Lambda} d\bar{\psi}(x) \ e^{-(\Phi, C^\Lambda_\Lambda \bar{\Phi}) L^2(\Lambda)} F_\Lambda(\Phi)
\] (1.18)

Notice that the determinant in the fermionic integration formula (1.10) has cancelled out with the inverse of the same determinant which appears in the bosonic integration measure.

The expectation defined above is normalized. In other words if \(1_{\Lambda}(\Phi)\) is the indicator function of \(\Omega_\Lambda\) then
\[
E_\Lambda(1_{\Lambda}(\Phi)) = 1
\] (1.19)

We have the natural order relation \(\Omega_\Lambda \subset \Omega_{\Lambda'}\) if \(\Lambda \subset \Lambda'\). Moreover if \(\Lambda \subset \Lambda'\) and \(F_\Lambda \in \Omega_\Lambda\) then \(E_{\Lambda'}(F_\Lambda) = E_\Lambda(F_\Lambda)\) as is not difficult to show. We define \(\Omega\) as the inductive limit of the \(\Omega_\Lambda\) as \(\Lambda \subset \mathbb{Z}^3\) varies over increasing subsets tending to \(\mathbb{Z}^3\) respecting the order relation above. The \(\{E_\Lambda, \Omega_\Lambda\}\) constitute a projective family. We denote by \(E\) the projective limit: Let \(F \in \Omega\) with \(\mu_C\) integrable bosonic coefficients. We have
\[
E(F) = \int d\mu_C(\Phi) F(\Phi) = \lim_{\Lambda' \uparrow \mathbb{Z}^3} E_{\Lambda'}(F)
\] (1.20)

and this limit exists since \(F \in \Omega_\Lambda\) for some finite set \(\Lambda\) and therefore \(E(F) = E_\Lambda(F)\) which exists.

Remark: The above construction is motivated by analogous considerations in [BEI].

Lattice integration: In the following and throughout this paper we will represent lattice sums as integrals where for the \((\delta \mathbb{Z})^3\) lattice the integration measure is the counting measure in \((\delta \mathbb{Z})^3\) times a factor \(\delta^3\). Thus if \(f\) is a function on \((\delta \mathbb{Z})^3\) we define
\[
\int_{(\delta \mathbb{Z})^3} dxf(x) = \delta^3 \sum_{x \in (\delta \mathbb{Z})^3} f(x)
\] (1.21)

We now define a Laplacian acting on functionals in \(\Omega\)
\[
\Delta_C = \int_{\mathbb{Z}^3 \times \mathbb{Z}^3} dxdy \ C(x-y) \left( \frac{\partial}{\partial \Phi(x)} \cdot \frac{\partial}{\partial \Phi(y)} \right) = \int_{\mathbb{Z}^3 \times \mathbb{Z}^3} dxdy \ C(x-y) \left[ \frac{\partial}{\partial \phi(x)} \frac{\partial}{\partial \phi(y)} + \frac{\partial}{\partial \psi(x)} \frac{\partial}{\partial \bar{\psi}(y)} \right]
\] (1.22)

These integrals on \(\mathbb{Z}^3\) automatically restricts to \(\Lambda \times \Lambda\) when applied to functionals of \(\Phi\) which live in a bounded subset \(\Lambda\) of \(\mathbb{Z}^3\). It follows from (1.11) and its bosonic counterpart that if \(F_\Lambda(\Phi) \in \Omega_\Lambda^0\) with \(\mu_C\) integrable bosonic coefficients then
\[
E(F_\Lambda(\Phi)) = e^{\Delta_C} F_\Lambda(\Phi)|_{\varphi=\bar{\varphi}=\psi=\bar{\psi}=0}
\] (1.23)

Note that the action of \(e^{\Delta_C}\) is well defined. In fact since \(F_\Lambda(\Phi)\) is in \(\Omega_\Lambda^0\) and \(\Lambda\) is a finite lattice, it can be expressed as a finite sum of Grassmann elements with coefficients in \(F_\Lambda\). \(e^{\Delta_C}\) factorises into bosonic and Grassmann exponentials. The expansion of the Grassman exponential acting on \(F_\Lambda(\Phi)\) evaluated at \(\psi = \bar{\psi} = 0\) thus terminates and we are left with the expectation of the bosonic coefficients which is well defined since they are \(\mu_C\) integrable by assumption.

We have in particular
\[
E(\Phi(x) \Phi(y)) = 0
\] (1.24)

and more generally
\[ E(\prod_{j=1}^{n} \Phi(x_j)\bar{\Phi}(y_j)) = 0 \quad (1.25) \]

This can be proved by computation or more simply using supersymmetry (introduced later). The integrand is supersymmetric and Lemma 1.1 below gives the result.

Wick polynomials \( P(\Phi) \) are defined by the formula

\[ :P(\Phi) :_C = e^{-\Delta C} P(\Phi) \quad (1.26) \]

This implies in particular that

\[ :\Phi(x)\bar{\Phi}(y) :_C = \Phi(x)\bar{\Phi}(y) \quad (1.27) \]

and

\[ :(\Phi\bar{\Phi})^2 :_C (x) = (\Phi\bar{\Phi})^2(x) - 2C(0)(\Phi \cdot \bar{\Phi})(x) \quad (1.28) \]

For future reference we note that for \( \alpha = 1, 2 \)

\[ :(\Phi\bar{\Phi})\Phi_{\alpha} :_C (x) = (\Phi\bar{\Phi})\Phi_{\alpha}(x) - C(0)\Phi_{\alpha}(x) \quad (1.29) \]

where \( \Phi_1 = \varphi, \Phi_2 = \psi. \)

**Remark:** The considerations from (1.8) to (1.28) remain valid on a lattice \((\delta\mathbb{Z})^3\) if we replace in the above \( \Lambda \) by a bounded subset \( \Lambda_\delta \subset (\delta\mathbb{Z})^3 \) and the positive definite matrix \( C \) by an arbitrary positive definite matrix \( C_\delta(x,y) \) with \( x, y \in (\delta\mathbb{Z})^3 \). The functional Laplacian \( \Delta C \) in (1.22) is replaced by \( \Delta_{C_\delta} \) with the integration over \((\delta\mathbb{Z})^3 \times (\delta\mathbb{Z})^3. \)

**The model:**

Let \( L \) be a triadic integer, \( L = 3^p \) with integer \( p \geq 2 \). Let \( \Lambda_N = (-L_N/2, L_N/2)^3 \subset \mathbb{R}^3 \), with \( N \) large be a large open cube in \( \mathbb{R}^3 \). Distances in \( \mathbb{R}^3 \) and lattices \((\delta\mathbb{Z})^3\) will be measured in the norm

\[ |x - y| = \max_{1 \leq j \leq 3} |x_j - y_j| \quad (1.30) \]

Define \( \Lambda_{N,0} = \Lambda_N \cap \mathbb{Z}^3 \). This is a (large) cube in \( \mathbb{Z}^3 \) of edge length \( L^N \). The second index 0 in \( \Lambda_{N,0} \) emphasizes that this is a cube in \( \mathbb{Z}^3 \). The local potential \((0,0)\) will be written in a \( C \)-Wick ordered form by using (1.28) and (1.27). This gives

\[ V_0(\Lambda_{N,0}, \Phi) = \int_{\Lambda_{N,0}} dx \ g_0 \ : (\Phi\bar{\Phi})^2 :_C (x) + \mu_0 \int_{\Lambda_{N,0}} dx \ : \Phi\bar{\Phi} :_C (x) \quad (1.31) \]

where \( \mu_0 = \tilde{\mu}_0 + 2C(0)g_0. \)

Define

\[ Z_0(\Lambda_{N,0}, \Phi) = e^{-V(\Lambda_{N,0}, \Phi)} \quad (1.32) \]

We define the measure

\[ d\mu_{N,0}(\Phi) = d\mu_C(\Phi) Z_0(\Lambda_{N,0}, \Phi) \quad (1.33) \]

Note that the measure is normalized

\[ \int d\mu_{N,0}(\Phi) = 1 \quad (1.34) \]
This follows from Lemma 1.1 below which exploits supersymmetry introduced later. However heuristically this is evident if we formally expand the exponential, integrate term by term and use (1.25). This measure defines our model.

**Supersymmetry**

The density of the measure $d\mu_{N,0}(\Phi)$ as well as its RG evolution have the important property of being *supersymmetric*. This will restrict considerably the form of the evolved density.

A *supersymmetry* transformation $Q : \Omega_\Lambda \to \Omega_\Lambda$ is a derivation on the bosonic fields and an antiderivation on the grassman fields which acts on the fields as follows:

\[
\begin{align*}
Q\phi &= \psi \\
Q\bar{\phi} &= -\bar{\psi} \\
Q\psi &= \varphi \\
Q\bar{\psi} &= \bar{\phi}
\end{align*}
\tag{1.35}
\]

Let $F_\Lambda(\Phi) = F_\Lambda(\varphi, \bar{\varphi}, \psi, \bar{\psi})$ belong to $\Omega_\Lambda$ with bosonic coefficients differentiable in the bosonic fields $\varphi(x), x \in \Lambda$. Then the action of $Q$ on $F_\Lambda$ is given by a super vector field denoted by the same symbol $Q$

\[
QF_\Lambda = \int_\Lambda dx \left( \psi(x) \frac{\partial}{\partial \varphi(x)} - \bar{\psi}(x) \frac{\partial}{\partial \bar{\varphi}(x)} + \varphi(x) \frac{\partial}{\partial \psi(x)} + \bar{\varphi}(x) \frac{\partial}{\partial \bar{\psi}(x)} \right) F_\Lambda \tag{1.36}
\]

We say that a functional $F_\Lambda$ is supersymmetric if $QF = 0$.

*Remark*: A super vector field is not a vector field because fermionic derivatives are antiderivations.

An (infinitesimal) *gauge transformation* $G : \Omega_\Lambda \to \Omega_\Lambda$ is a derivation whose action is given by

\[
\begin{align*}
G\varphi &= i\varphi \\
G\bar{\varphi} &= -i\bar{\varphi} \\
G\psi &= i\psi \\
G\bar{\psi} &= -i\bar{\psi}
\end{align*}
\tag{1.37}
\]

This induces on an $\Omega_\Lambda$ function $F_\Lambda$ the action of a vector field denoted by the same symbol $G$

\[
GF_\Lambda = i \int_\Lambda dx \left( \varphi(x) \frac{\partial}{\partial \varphi(x)} - \bar{\varphi}(x) \frac{\partial}{\partial \bar{\varphi}(x)} + \psi(x) \frac{\partial}{\partial \psi(x)} + \bar{\psi}(x) \frac{\partial}{\partial \bar{\psi}(x)} \right) F_\Lambda \tag{1.38}
\]

We say that a functional $F_\Lambda$ is gauge invariant if $GF_\Lambda = 0$.

From (1.35) we see that $Q^2$ engenders an infinitesimal gauge transformation (1.37). Thus acting on gauge invariant functionals

\[
Q^2 = 0 \tag{1.39}
\]

An important property of the super vector field $Q$ which we will exploit later is that it commutes with the super Laplacian $\Delta_C$ defined in (1.22):

\[
[ Q, \Delta_C ] = 0
\]

as is easy to verify.
It is easy to verify that any polynomial in \( \Phi \bar{\Phi} \) and their (lattice) derivatives is supersymmetric. As a consequence we have \( QV(\Lambda, \Phi) = 0 \) where \( V \) is given in (1.31) and thus the starting interaction potential is supersymmetric.

Let \( \Gamma(x, y) \) be any positive definite symmetric matrix. Let \( \Delta_\Gamma \) be a super Laplacian given by (1.22) with \( C \) replaced by \( \Gamma \). Let \( F_\Lambda(\Phi) \) be an \( \Omega_\Lambda \) functional with \( \mu_C \) integrable bosonic coefficients. Let \( \xi = (\zeta, \eta) \) be another superfield. Define the convolution
\[
\mu_\Gamma \ast F_\Lambda(\Phi) = \int d\mu_\Gamma(\xi) F_\Lambda(\Phi + \xi) = e^{\Delta_\Gamma} F_\Lambda(\Phi) \tag{1.40}
\]
Since \( Q \) commutes with \( \Delta_\Gamma \), \( Q \) also commutes with convolution with the measure \( \mu_\Gamma \):
\[
\mu_\Gamma \ast QF_\Lambda(\Phi) = Q\mu_\Gamma \ast F_\Lambda(\Phi) \tag{1.41}
\]
Therefore if \( F_\Lambda \) is supersymmetric so is \( \mu_\Gamma \ast F_\Lambda \). This observation prefigures the supersymmetry invariance of the renormalization group map which we will introduce later.

It follows by evaluating (1.41) at \( \Phi = 0 \) that
\[
\int d\mu_\Gamma(\Phi) QF_\Lambda(\Phi) = 0 \tag{1.42}
\]
since the left hand side is given by \( (\mu_\Gamma \ast QF_\Lambda(\Phi)) \big|_{\Phi=0} \) and this vanishes by virtue of (1.41) since the coefficients of the super vector field \( Q \) vanish when the fields vanish. ■

**Lemma 1.1**: Let \( F_\Lambda(\Phi) \) be a supersymmetric \( \Omega_\Lambda \) functional with differentiable bosonic coefficients which are \( \mu_\Gamma \) integrable. Then
\[
\int d\mu_\Gamma(\Phi) F_\Lambda(\Phi) = F_\Lambda(0) \tag{1.43}
\]
**Proof**: \( \lambda \) be a real parameter. Define
\[
f(\lambda) = \int d\mu_\Gamma(\Phi) F_\Lambda(\lambda \Phi) \tag{1.44}
\]
We will prove
\[
\frac{d}{d\lambda} f(\lambda) = 0 \tag{1.45}
\]
This implies that \( f(\lambda) \) is a constant and hence evaluating at \( \lambda = 0 \) gives (1.43).

Taking the \( \lambda \) derivative in (1.44) we get
\[
\frac{d}{d\lambda} f(\lambda) = \int d\mu_\Gamma(\Phi) (DF_\Lambda)(\lambda \Phi) \tag{1.46}
\]
where
\[
D = \int_\Lambda dx \left( \phi(x) \frac{\partial}{\partial \phi(x)} + \bar{\phi}(x) \frac{\partial}{\partial \bar{\phi}(x)} + \psi(x) \frac{\partial}{\partial \psi(x)} + \bar{\psi}(x) \frac{\partial}{\partial \bar{\psi}(x)} \right) \tag{1.47}
\]
Note that the four coefficients of \( D \) can also be written as \( (Q\psi(x), Q\bar{\psi}(x), Q\phi(x), -Q\bar{\phi}(x)) \) which we have taken in the same order as above. This suggests that we consider the operator
\[
L = \int_\Lambda dx \left( \psi(x) \frac{\partial}{\partial \phi(x)} + \bar{\psi}(x) \frac{\partial}{\partial \bar{\phi}(x)} + \phi(x) \frac{\partial}{\partial \psi(x)} - \bar{\phi}(x) \frac{\partial}{\partial \bar{\psi}(x)} \right) \tag{1.48}
\]
and act with $Q$ on it. We also consider the action of $L$ on $Q$. A straightforward computation gives the nice formula

$$D = \frac{1}{2}(QL + LQ)$$

We substitute for $D$ in (1.46) the right hand side of (1.49). The contribution of the first term vanishes by (1.42). The contribution of the second term vanishes because $F_\lambda$ is supersymmetric by hypothesis. This proves (1.45) and we are done. ■

Remark: The special case of Lemma 1.1 for a hierarchical lattice is Lemma 2.1 of [BI]. This Lemma has the important consequence that no field independent relevant parts (defined later) will arise in the renormalization group analysis to follow.

1.2 Lattice renormalization group transformations

We say that a function $f(x,y)$ has finite range $L$ if $f(x,y) = 0 : |x-y| \geq L$. Lattice renormalization group transformations will be based on the finite range multiscale expansion of the covariance $C$ established in [BGM].

Let $L$ be a large triadic integer, $L = 3^n$, $p \geq 2$. Define $\delta_n = L^{-n}$. We have a sequence of compatible lattices $(\delta_n\mathbb{Z})^3 \subset \mathbb{R}^3$, $(\delta_n\mathbb{Z})^3 \subset (\delta_{n+1}\mathbb{Z})^3$, with $n = 0, 1, 2, \ldots$. $B_{\delta_n} = [-\frac{\delta_n}{2}, \frac{\delta_n}{2}]^3$ denotes the first Brillouin zone of the dual of the $\delta_n$ lattice. We have the following theorem which gives the multiscale expansion of the covariance $C$ on $\mathbb{Z}^3$ as a sum of finite range fluctuation covariances living on increasingly finer lattices, together with their properties which we will need later:

**Theorem 1.1 (finite range multiscale expansion):** For $0 < \alpha < 2$, $d_\alpha = \frac{(3-\alpha)}{2}$ and $n = 0, 1, 2, \ldots$ there exist positive definite functions $\Gamma_n(x)$ defined for $x \in (\delta_n\mathbb{Z})^3$ and a smooth positive definite function $\Gamma_{c,\ast}$ in $\mathbb{R}^3$ such that for all $k \geq 0$, constants $c_{k,L}$, $c_{L,m}$ independent of $n$ and $q = \frac{1}{2}$,

1. $C(x-y) = \sum_{n \geq 0} L^{-2nd_\alpha} \Gamma_n \left( \frac{x-y}{L^n} \right)$ and the series converges in $L^\infty(\mathbb{Z}^3)$
2. $\Gamma_n(x) = 0$ for $|x| \geq \frac{L}{2}$
3. $|\hat{\Gamma}_n(p)| \leq c_{k,L}(1 + p^2)^{-2k}$ for $p \in B_{\delta_n}$, $\forall k \geq 0$
4. $\hat{\Gamma}_{c,\ast}(p) = \lim_{n \to \infty} \hat{\Gamma}_n(p)$ exists pointwise in $p$
5. $|\hat{\Gamma}_n(p) - \hat{\Gamma}_{c,\ast}(p)| \leq c_{k,L}(1 + p^2)^{-2k}(1 + \frac{1}{p^2})L^{-qn}$, $\forall n \geq 3$, $\forall k \geq 0$, $p \in B_{\delta_n} \setminus 0$
6. $\|\partial_{\delta_n}^m \Gamma_n\|_{L^\infty((\delta_n\mathbb{Z})^3)} \leq c_{L,m}, \forall m \geq 0$
7. $\partial_{c,\ast}^m \Gamma_{c,\ast} = \lim_{n \to \infty} \partial_{\delta_n}^m \Gamma_n$ exists in $L^\infty((\delta\mathbb{Z})^3)$
8. $\|\partial_{\delta_n}^m \Gamma_n - \partial_{c,\ast}^m \Gamma_{c,\ast}\|_{L^\infty((\delta\mathbb{Z})^3)} \leq c_{L,m}L^{-qn}$, $\forall n \geq 3$, $\forall m \geq 0$

where $\partial_{c}$ is a continuum partial derivative, $\partial_{\delta_n}$ is a forward lattice partial derivative in $(\delta_n\mathbb{Z})^3$ and the dependence on the direction vectors have been suppressed. For $\partial_{\delta_n}^m$ and $\partial_{c}^m$ a multi-index convention is implicit.

Remark: The theorem is for the most part a combination of results obtained in various theorems in [BGM]. Before we outline the proof note that in [BGM], $L$ was a large dyadic integer whereas we have chosen here $L$ to be triadic. The results of [BGM] remain unaffected provided we define the continuum cube $U_\alpha(R) \subset \mathbb{R}^3$ in section 1 of [BGM] to be $(-\frac{R}{3}, \frac{R}{3})^3$. This guarantees in particular that if $R = R_m = L^{-(m-1)}$, $0 \leq m \leq n$, 

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and $U_{\delta_n}(R_m) = U_c(R_m) \cap (\delta_n \mathbb{Z})^3$ then the important property $\partial U_{\delta_n}(R_m) \subset \partial U_c(R_m)$ remains true. This last property is invoked in section 6, page 439 of [BGM], in preparation for the convergence proof therein.

**Proof**: The multiscale expansion in part (1) and the finite range property of part (2) were given in Section 4, [BGM]. The factor 6 in the range 6$L$ of $\Gamma_n$ is an artifact. By scaling down $R_m$ in the cube $U_{\delta_n}(R_m)$ by a factor of $3^{-4}$ and the range of the function $g$ in section 1 to $3^{-6}L$ we get $\Gamma_n$ to have range $L/2$. Convergence of (1) in $L^\infty((\delta Z)^3)$ follows on using $d_s > 0$, Corollary 5.6 and the Sobolev embedding inequality for lattice $L_k^2 = H_k$ spaces with $k$ in the Corollary sufficiently large. Part (3) follows from (5.10) of Theorem 5.5 by integration on $a$ with the measure $da a^{-\alpha/2}$ (see (4.3) of section 4). Corollary 5.6 and lattice Sobolev embedding gives (5a). Corollary 6.2 gives parts (4a) and (5b). The convergence rate estimates of parts (4b) and (5c) which were not given in [BGM] also follow from the results therein. The proof is given elsewhere, [BM]. $\blacksquare$

**Remark**: (4b) is not necessarily the best possible estimate. The left hand side has no singularity at $p = 0$ whereas the right hand side does. However it suffices for our purposes because (5c) above follows from (4b) and it is (5c) which will be put to use later. In fact (4b) implies that for fixed $l \geq 3$ and all $n \geq l$, $k \geq 0$,

$$||\Gamma_n - \Gamma_{c,*}||_{L_k^1((\delta \mathbb{Z})^3)} \leq c_{k,L}L^{-\eta}$$

where $L_k^1((\delta \mathbb{Z})^3)$ is a lattice Sobolev space. The finite range of $\Gamma_n$, $\Gamma_{c,*}$ and lattice Sobolev embedding for $k \geq 3 + m$ implies (5c). The singularity at $p = 0$ in the right hand side of (4b) is integrable in $B_{\delta_n}$. It thus turns out to be harmless.

Define for all $n \geq 0$ the positive definite functions $C_n$, $C_{c,*}$ on $(\delta \mathbb{Z})^3$ and $\mathbb{R}^3$ respectively by the recursion relations

$$C_n(x) = \Gamma_n(x) + L^{-2d_s}C_{n+1}\left(\frac{x}{L}\right) \quad (1.50)$$

$$C_{c,*}(x) = \Gamma_{c,*}(x) + L^{-2d_s}C_{c,*}\left(\frac{x}{L}\right) \quad (1.51)$$

Solving these relations by iteration gives

$$C_n(x) = \sum_{j=0}^{\infty} L^{-2jd_s} \Gamma_{n+j}\left(\frac{x}{L^j}\right) \quad (1.52)$$

$$C_{c,*}(x) = \sum_{j=0}^{\infty} L^{-2jd_s} \Gamma_{c,*}\left(\frac{x}{L^j}\right) \quad (1.53)$$

Note that $C_0 = C$ as follows from (1) of Theorem 1.1.

**Corollary 1.2**: The series (1.52) for $C_n$ together with that for its multiple lattice derivatives in $(\delta \mathbb{Z})^3$ converge in $L^\infty((\delta \mathbb{Z})^3)$. For every integer $m \geq 0$ we have a constant $c_{L,m}$ such that

$$||\partial_{\delta_n}^m C_n||_{L^\infty((\delta \mathbb{Z})^3)} \leq c_{L,m} \quad (1.54)$$

The series (1.52) defining $C_{c,*}$ and its multiple continuum derivatives of arbitrary order converge in $L^\infty(\mathbb{R}^3)$ so that $C_{c,*}$ is a smooth continuum function. For all $m \geq 0$ and $\partial_c$ the continuum partial derivative

$$\sup_{x \in \mathbb{R}^3} |\partial_c^m C_{c,*}(x)| \leq c_{L,m} \quad (1.55)$$

Moreover for $n \geq l \geq 3$ with $l$ fixed and $\forall m \geq 0$, there exists a constant $c_{L,m}$ such that
\[ \| \partial^n \partial C_n - \partial^n C_{c, \ast} \|_{L^\infty((\delta_n \mathbb{Z})^3)} \leq c_{L, m} L^{-qn} \]  

(1.56)

**Proof:** The first part together with the bound (1.54) follow from (5a) of Theorem 1.1. In fact from (1.52) we have

\[ \partial^n C_n = \sum_{j=0}^{\infty} L^{-2jd_s} L^{-mj} (\partial^n_{\delta_{n+j}} \Gamma_{n+j}) \left( \frac{x}{L^j} \right) \]

where we have used repeatedly (m-times) the identity \( \partial^n \Gamma_{n+j} \left( \frac{x}{L^j} \right) = L^{-j} (\partial^n_{\delta_{n+j}} \Gamma_{n+j}) \left( \frac{x}{L^j} \right) \) as is easy to show. Therefore

\[
\| \partial^n C_n \|_{L^\infty((\delta_n \mathbb{Z})^3)} \leq \sum_{j=0}^{\infty} L^{-2jd_s} L^{-mj} \sup_{x \in (\delta_n \mathbb{Z})^3} \left| (\partial^n_{\delta_{n+j}} \Gamma_{n+j}) \left( \frac{x}{L^j} \right) \right|
\]

where we have used repeatedly (m-times) the identity \( \partial^n \Gamma_{n+j} \left( \frac{x}{L^j} \right) = L^{-j} (\partial^n_{\delta_{n+j}} \Gamma_{n+j}) \left( \frac{x}{L^j} \right) \) as is easy to show. Therefore

\[
\| \partial^n C_n \|_{L^\infty((\delta_n \mathbb{Z})^3)} \leq \sum_{j=0}^{\infty} L^{-2jd_s} L^{-mj} \sup_{y \in (\delta_n \mathbb{Z})^3} \left| (\partial^n_{\delta_{n+j}} \Gamma_{n+j}) (y) \right|
\]

Now use the bound in (5a) together with \( d_s \geq \frac{1}{2} \) to get (1.54). To prove the next statement observe that the first part together with the bound (1.54) follow from (5a) of Theorem 1.1. In fact from (1.52) the considerations in Section 1.1 for fields in \( \mathbb{Z}^3 \) remain valid for every lattice \( (\delta_n \mathbb{Z})^3 \) provided for the expectations we replace the covariance \( C \) by \( C_n \). Let the fields \( \varphi, \psi, \bar{\psi} \) be defined in \( (\delta_N \mathbb{Z})^3 \). These fields restrict to the coarser lattices \( (\delta_n \mathbb{Z})^3 \) for every \( n \) with \( 0 \leq n \leq N \).

We consider the finite sequence of compatible lattices \( \{ (\delta_n \mathbb{Z})^3 \} \) for \( 0 \leq n \leq N \). The considerations in Section 1.1 for fields in \( \mathbb{Z}^3 \) remain valid for every lattice \( (\delta_n \mathbb{Z})^3 \) provided for the expectations we replace the covariance \( C \) by \( C_n \). Let the fields \( \varphi, \psi, \bar{\psi} \) be defined in \( (\delta_N \mathbb{Z})^3 \). These fields restrict to the coarser lattices \( (\delta_n \mathbb{Z})^3 \) for every \( n \) with \( 0 \leq n \leq N \).

We introduce a parameter \( \varepsilon \) with \( 0 < \varepsilon \leq 1 \) and define

\[ \alpha = \frac{3 + \varepsilon}{2} \]  

(1.57)

Let \( x \in (\delta_n \mathbb{Z})^3 \). For every \( n \leq N - 1 \) we define the scale transformation \( S_L \) by

\[ S_L \Phi(x) = \Phi_{L^{-1}}(x) = L^{-d_s} \Phi \left( \frac{x}{L} \right) \]  

(1.58)

where

\[ d_s = \frac{(3 - \alpha)}{2} = \frac{3 - \varepsilon}{4} \]  

(1.59)
is the *dimension* of the field \( \Phi \). The fields \( \varphi, \hat{\varphi}, \psi, \hat{\psi} \) are thus assigned the same dimension \( d_s \) and the same transformation law (1.58). Note that the scale transformed fields now live in \((\delta_n \mathbb{Z})^3\). Let \( \Lambda \subset \mathbb{R}^3 \) and \( \Lambda_{\delta_n} = \Lambda \cap (\delta_n \mathbb{Z})^3 \). We define the scale transformation on functionals of fields by

\[
(S_L F)(L^{-1} \Lambda_{\delta_{n+1}}, \Phi) = F(\Lambda_{\delta_n}, S_L \Phi)
\]  

(1.60)

The \( C_n \) and \( \Gamma_n \) are positive definite and therefore qualify as covariances of Gaussian measures. For \( x, y \in (\delta_n \mathbb{Z})^3 \) we define the scale transformation of the covariance \( C_{n+1} \) by

\[
S_L C_{n+1}(x - y) = L^{-2d_s} C_{n+1}\left(\frac{x - y}{L}\right)
\]  

(1.61)

which permits us to write (1.50) as

\[
C_n(x - y) = \Gamma_n(x - y) + S_L C_{n+1}(x - y)
\]  

(1.62)

Let \( \Lambda_{\delta_n} \subset (\delta_n \mathbb{Z})^3 \) be a bounded subset. Then (1.62) implies upon using (1.23) (with \( C \) replaced by \( C_n \)) that

\[
\int d\mu_{C_n}(\Phi) F(\Lambda_{\delta_n}, \Phi) = \int d\mu_{S_L C_{n+1}}(\Phi) \int d\mu_{\Gamma_n}(\xi) F(\Lambda_{\delta_n}, \xi + \Phi)
\]  

(1.63)

Let \( L = 3^p \) with integer \( p \geq 2 \) and let \( \Lambda_m = (-\frac{L^m}{2}, \frac{L^m}{2})^3 \subset \mathbb{R}^3 \) be an open cube in \( \mathbb{R}^3 \) centered at the origin. We denote by

\[
\Lambda_{m,n} = \Lambda_m \cap (\delta_n \mathbb{Z})^3
\]  

(1.64)

the induced cube of side length \( L^m \) in \((\delta_n \mathbb{Z})^3\) centered at the origin. Let \( F_0(\Lambda_{N,0}, \Phi) \) be a functional of \( \Phi \) and \((\tilde{\Phi})\) belonging to \( \Omega^0(\Lambda_{N,0}) \). By virtue of (1.63) we have for \( n = 0 \)

\[
\int d\mu_{C_0}(\Phi) F_0(\Lambda_{N,0}, \Phi) = \int d\mu_{C_1}(\Phi) F_1(\Lambda_{N-1,1}, \Phi)
\]  

(1.65)

where

\[
F_1(\Lambda_{N-1,1}, \Phi) := (S_L \mu \gamma_0 \ast F_0)(\Lambda_{N-1,1}, \Phi) = \int d\mu_{\Gamma_0}(\xi) F_0(\Lambda_{N,0}, \xi + S_L \Phi)
\]  

(1.66)

The final scale transformation takes us to a finer lattice as well as scaling down the size of the cube. The iteration of (1.66) using (1.65) gives after \( n \) steps

\[
\int d\mu_{C_0} F_0(\Lambda_{N,0}, \Phi) = \int d\mu_{C_n} F_n(\Lambda_{N-n,n}, \Phi)
\]  

(1.67)

where

\[
F_n(\Lambda_{N-n,n}, \Phi) := \mu \Gamma_{n-1} \ast F_{n-1}(\Lambda_{N-n+1,n-1}, S_L \Phi)
\]  

(1.68)

(1.68) defines for \( N > 0 \) fixed and \( 1 \leq n \leq N - 1 \) a sequence of maps

\[
T_{N-n,n} : \Omega^0(\Lambda_{N-n+1,n-1}) \to \Omega^0(\Lambda_{N-n,n})
\]  

(1.69)

any member of which we call a *renormalization group (RG) transformation*. The map is clearly not autonomous. The first index refers to the cube whose size has gotten reduced because of the rescaling. The second index refers to the lattice spacing which has gotten finer because of the rescaling. In the following we will apply the RG transformation iteratively to the (interaction) density \( Z_0(\Lambda_{N,0}, \Phi) \) of the measure
the total number of 1-blocks in $X$ of the open continuum unit cubes defined above. Therefore, as is easy to verify, we denote by $|\Delta|$. If the 1-blocks are not connected we say that they are disconnected. If ∆ is a (connected) union of a finite subset of disjoint 1-blocks $\Delta$, we say that $\Delta$ is strictly disjoint. A continuum (connected) 1-polymer $X$ is a (connected) union of a finite subset of unit cubes chosen from the paving and is thus open. Henceforth, unless otherwise mentioned, a polymer is connected by default.

We will measure distances in $\mathbb{R}^3$ and in all embedded lattices in the norm

$$|x - y| = \max_{1 \leq j \leq 3} |x_j - y_j|$$

(1.71)

If $\Delta_1$ and $\Delta_2$ are two unit cubes from the paving then the distance between them is

$$d(\Delta_1, \Delta_2) = \inf_{x \in \Delta_1, y \in \Delta_2} |x - y|$$

(1.72)

If $\Delta_1$ and $\Delta_2$ are strictly disjoint than $d(\Delta_1, \Delta_2) \geq 1$.

Let $\delta_n = L^{-n}$ where $L = 3^p$ is a triadic integer. Let $\delta$ be any member of the sequence $\{\delta_n\}_{n \geq 0}$. Define the unit block or 1-block in $(\delta \mathbb{Z})^3$ by

$$\Delta_\delta = \Delta \cap (\delta \mathbb{Z})^3$$

(1.73)

and the lattice 1-polymer $X_\delta$ by

$$X_\delta = X \cap (\delta \mathbb{Z})^3$$

(1.74)

where $X$ is a continuum 1-polymer. Note that as point sets $X_{\delta_n} \subset X_{\delta_{n+1}}$.

We denote by $|X_\delta|$ the volume of $X_\delta$ measured in accordance with (1.21). The 1-blocks are lattice restrictions of the open continuum unit cubes defined above. Therefore, as is easy to verify, $|\Delta_\delta_n| = 1$ and

$$|X_{\delta_n}| = \sharp \{ \Delta_\delta_n : \Delta_\delta_n \subset X_\delta \}$$

(1.75)

the total number of 1-blocks in $X_{\delta_n}$. This is equal to $|X|$ the total number of 1-cubes in $X$ by our construction.

As a consequence we have $|X_{\delta_n}| = |X_{\delta_{n+1}}|$.

We say two 1-blocks in $X_\delta$ are connected if the continuum 1-cubes of which they are the lattice restrictions are connected (see above). If the 1-blocks are not connected we say that they are strictly disjoint. The distance between two strictly disjoint 1-blocks is $\geq 1$. The lattice (connected) polymer $X_\delta$ is a (connected) union of a finite subset of disjoint 1-blocks $\Delta_\delta$. Let $X_\delta$ and $Y_\delta$ be each a connected polymer. We say that $X_\delta, Y_\delta$ are strictly disjoint if they are mutually disconnected i.e. if every 1-block from $X_\delta$ is strictly disjoint from every 1-block from $Y_\delta$. Then the distance $d(X_\delta, Y_\delta) \geq 1$. 

1.3. Polymer gas representation.

In order to analyze the RG evolution we will write the densities $Z_n$ in a polymer gas representation whose form is preserved under RG transformations.

**Polymers:** We pave $\mathbb{R}^3$ with a disjoint union of open cubes $\Delta \subset \mathbb{R}^3$ of edge length 1 called unit cubes or 1-cubes defined by

$$\Delta = (-\frac{1}{2} + m_1, \frac{1}{2} + m_1) \times (-\frac{1}{2} + m_2, \frac{1}{2} + m_2) \times (-\frac{1}{2} + m_3, \frac{1}{2} + m_3)$$

(1.70)

where $(m_1, m_2, m_3) \in \mathbb{Z}^3$. We say two unit cubes from the paving are connected if their closures share at least a vertex in common. If they are not connected (i.e. their closures are disjoint) we say that they are strictly disjoint. A continuum (connected) 1-polymer $X$ is a (connected) union of a finite subset of unit cubes chosen from the paving and is thus open. Henceforth, unless otherwise mentioned, a polymer is connected by default.

The densities $Z_n$ are indefinitely large in the infinite volume limit (as explained at the end of section 3).

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Given an integer \( n \geq 1 \) we define the \( n \)-collar of \( X_\delta \), denoted \( \partial_n X_\delta \) by

\[
\partial_n X_\delta = \{ y \notin X_\delta : |x - y| \leq n\delta, \text{some } x \in X_\delta \}
\] (1.76)

where \(| \cdot |\) is the distance function inherited from \( \mathbb{R}^3 \). We define

\[
\tilde{X}_\delta^{(n)} = X_\delta \cup \partial_n X_\delta
\] (1.77)

Let \( f : (\delta \mathbb{Z})^3 \rightarrow \mathbb{C} \). We define the forward lattice partial derivative \( \partial_{\delta, \mu} \) and the backward lattice derivative \( \partial_{\delta, -\mu} \) by

\[
\partial_{\delta, \mu} f(x) = \delta^{-1}(f(x + \delta e_\mu) - f(x))
\] (1.78)

\[
\partial_{\delta, -\mu} f(x) = \partial^*_\delta f(x) = \delta^{-1}(f(x - \delta e_\mu) - f(x))
\] (1.79)

where \( e_1, e_2, e_3 \) is the standard basis of unit vectors which provides the orientation of \( \mathbb{R}^3 \) and thus of all the embedded lattices we will encounter. \( \partial^*_\delta \) is the \( L^2((\delta \mathbb{Z})^3) \) adjoint of \( \partial_{\delta, \mu} \).

**Polymer activity:**

A polymer activity \( K(X_\delta, \Phi) = \tilde{K}(X_\delta, \varphi, \psi) \), where it is henceforth understood that it also depends on \( \varphi, \tilde{\psi} \), is a map \( X_\delta, \Phi \rightarrow \Omega^0_{X_\delta} \) where the fields \( \Phi \) depend only on the points of \( \tilde{X}_\delta^{(2)} \).

The polymer activities of this paper are of degree 0, gauge invariant and supersymmetric, and invariant under translations, reflections and rotations which leave the lattice invariant. In addition they satisfy the condition \( K(X_\delta, \Phi) = K(X_\delta, -\Phi) \) together with the support condition : \( K(X_\delta, \Phi) = 0 \) if \( X \) is not connected. Furthermore \( K(X_\delta, 0) = 0 \).

We write the generic density \( Z(\Lambda_\delta)(\Phi) \) in the form

\[
Z(\Lambda_\delta) = \sum_{N=0}^\infty \frac{1}{N!} e^{-V(X_\delta^{(c)})} \prod_{X_\delta, \ldots, X_N, j=1}^N K(X_\delta, j)
\] (1.80)

where the connected polymers \( X_{\delta, j} \subset \Lambda_\delta \) are strictly disjoint, \( X_\delta = \bigcup_j X_{\delta, j} \), \( X_\delta^{(c)} = \Lambda_\delta \setminus X_\delta \) and \( V(Y_\delta) = V(Y_\delta, \Phi, C, g, \mu) \) is given by (1.31) with parameters \( g, \mu \) and integration over \( Y_\delta \) with measure \( dx \) defined as the counting measure in \( (\delta \mathbb{Z})^3 \) times \( \delta^3 \). The Wick ordering covariance \( C = C_n \) (see (1.52)) if \( \delta = \delta_n \). We have suppressed the field dependence in (1.80). Initially the activities \( K \) vanish but they do arise under RG transformations. The representation (1.80) remains stable under RG transformations as we will see in Section 3.

Polymer activities \( K(X_\delta, \Phi) = K(X_\delta, \varphi, \psi) \in \Omega^0_{X_\delta} \) can be represented uniquely as a (finite) series in the fermionic fields \( \psi, \tilde{\psi} \) with coefficients which are functionals of the bosonic fields \( \varphi \):

\[
K(X_\delta, \Phi) = K(X_\delta, \varphi, \psi) = \sum_{p \geq 0} \frac{1}{(p!)^2} \int_{X_\delta^p \times X_\delta^p} dxdy \ (D^2_{F} K)(X_\delta, \varphi, x, y) \prod_{j=1}^p \psi(x_j)\tilde{\psi}(y_j)
\] (1.81)

where:

\( x = (x_1, \ldots, x_p) \), \( y = (y_1, \ldots, y_p) \) and \( dx = \prod_{i=1}^p dx_i \) where \( dx_i \) is the counting measure multiplied by \( \delta^3 \) on \( (\delta \mathbb{Z})^3 \). \( y \) and \( dy \) are similarly defined. The coefficient \( (D^2_{F} K)(X_\delta, \varphi, x, y) \) is defined by

\[
(D^2_{F} K)(X_\delta, \varphi, x, y) = \prod_{j=0}^{p-1} \frac{\partial}{\partial \psi(y_{p-j})} \frac{\partial}{\partial \psi(x_{p-j})} K(X_\delta, \varphi, \psi)|_{\psi = \tilde{\psi} = 0}
\] (1.82)
This defines a lattice analogue of a distributional kernel which is henceforth restricted so as to contain at most (lattice) delta functions and their first and second (lattice) derivatives. It is clearly antisymmetric in \((x_1, \ldots, x_p)\) and in \((y_1, \ldots, y_p)\). It is gauge invariant as is the Grassmann monomial of degree 0. The polymer activities in question also satisfy

\[ K(X_\delta, 0) = 0 \]  

(1.83)

Remarks: We will see that the representations (1.80), (1.81) are preserved by renormalization group transformations. The RG transformations are gauge invariant, preserve supersymmetry by virtue of (1.41), as well as the vanishing condition (1.83) by virtue of Lemma 1.1. The RG transformations preserve invariance of the polymer activities under translations, reflections and rotations which leave the lattice invariant.

2. REGULATORS, DERIVATIVES AND NORMS

In this section we will introduce Banach spaces of polymer activities. These are lattice analogues of the continuum constructions in [BDH-est, BMS, A] albeit with changes because of the presence of Grassman variable. The Banach space norms that we will presently introduce measure differentiability properties of the activities with respect to fields \(\phi, \psi\), as well as the behaviour with respect to large fields \(\partial \phi\) and large sets. The behaviour for large \(\phi\) itself will be controlled with the help of lattice Sobolev inequalities and the local potential.

2.1 Regulators

Let \(\partial_{\delta,\mu}\) and \(\partial_{\delta,-\mu}\) be respectively the forward and backward lattice derivatives in \((\delta\mathbb{Z})^3\) along the unit vector \(e_\mu\) defined in (1.78) and (1.79). Here as before \(\delta\) is any member of the sequence \(\{\delta_n\}\) where \(\delta_n = L^{-n}\) and \(L = 3^p\) with integer \(p \geq 2\). Define

\[ \partial^j_{\mu_1,\mu_2,\ldots,\mu_j} = \partial_{\delta,\mu_1}\partial_{\delta,\mu_2}\ldots\partial_{\delta,\mu_j} \]

Let \(X\) be a connected polymer in \(\mathbb{R}^3\) and \(X_\delta = X \cap (\delta\mathbb{Z})^3\). Let \(\tilde{X}_\delta^{(n)} = X_\delta \cup \partial_n X_\delta\) as defined earlier ((1.76) and (1.77)). Let \(\varphi: \tilde{X}_\delta^{(5)} \to \mathbb{C}\). We define a norm \(\| \cdot \|_{X_\delta, 1, 5}:\)

\[ \| \varphi \|^2_{X_\delta, 1, 5} = \sum_{j=1}^{5} 2^{-j} \sum_{\mu_j \in \delta, \nu_j} \int_{X_\delta} |\partial^j_{\mu_1,\mu_2,\ldots,\mu_j} \varphi(x)|^2 \]

(2.1)

where \(S = \{1, -1, 2, -2, 3, -3\}\). This is a lattice Sobolev norm of the type introduced in Section 5, page 421 of [BGM] but now without the \(L^2\) piece.

We define now the large field regulator

\[ G_\kappa: X_\delta \times \mathcal{F}_{\tilde{X}_\delta^{(5)}} \to \mathbb{R} \]

(2.2)

where \(\mathcal{F}_{\tilde{X}_\delta^{(5)}}\) is the algebra of \(\mathbb{C}\) valued functions on \(\tilde{X}_\delta^{(5)}\) by

\[ G_\kappa(X_\delta, \varphi) = e^{\kappa \| \varphi \|^2_{X_\delta, 1, 5}} \]

(2.3)

\(G_\kappa\) satisfies the multiplicative property: If \(X_\delta, Y_\delta\) are disjoint sets then

\[ G_\kappa(X_\delta \cup Y_\delta, \varphi) = G_\kappa(X_\delta, \varphi)G_\kappa(Y_\delta, \varphi) \]

(2.4)

\(G_\kappa^{-1}\) will be a weight function in polymer activity norms. The norm \(\| \cdot \|_{X_\delta, 1, 5}\) can be used in lattice Sobolev inequalities, in conjunction with the stability provided by the local potential, to control \(\varphi\) and its first two
lattice derivatives pointwise. The parameter $\kappa = \kappa(L) > 0$ is chosen so that for all $L \geq 2$ the large field regulator satisfies the stability property given in the following Lemma:

**Lemma 2.1 (stability property):** There exists a constant $\kappa_0 = \kappa_0(L) > 0$ independent of $n$ such that for all $\kappa$ with $0 < \kappa \leq \kappa_0$

$$\int d\mu_{\Gamma_n}(\zeta) G_{\kappa}(X_{\delta_n}, \zeta + \varphi) \leq 2^{\|X_{\delta_n}\|} G_{2\kappa}(X_{\delta_n}, \varphi)$$

(2.5)

where $|X_{\delta_n}|$ is the number of unit blocks in $X_{\delta_n}$.

**Proof:** (2.5) is proved in exactly the same way as in the proof of the stability property of the continuum large field regulator in Lemma 3 of [BDH-est]. The proof uses a flow equation for the measure convolution with interpolated covariance which remains true for the lattice. Another ingredient is Young’s convolution inequality for functions which is also true on the lattice. In the cited proof we replace the covariance $C$ by $\Gamma_n$ and continuum derivatives by lattice derivatives. From the proof of Lemma 3 of [BDH-est] we see that two conditions have to be satisfied by $\kappa_0$, namely: 1) $\kappa_0 \max_{2 \leq m \leq 10} \|\partial^{m}_n \Gamma_n\|_{L^\infty((\delta_n, Z)^3)}$ is sufficiently small and 2) $\kappa_0 \|\Gamma_n\|_{L^1((\delta_n, Z)^3)}$ is sufficiently small. Parts (5a) of Theorem 1.1 shows that that 1) and 2) above can be assured by a $\kappa_0$ independent of $n$. From (5a) we have

$$\kappa_0 \max_{2 \leq m \leq 10} \|\partial^{m}_n \Gamma_n\|_{L^\infty((\delta_n, Z)^3)} \leq \kappa_0 c_L$$

and from (5a) and the finite range property

$$\kappa_0 \|\Gamma_n\|_{L^1((\delta_n, Z)^3)} \leq \kappa_0 L^3 c'_L$$

It is sufficient to choose $\kappa_0$ so that the right hand side of both inequalities are sufficiently small. This is achieved independent of $n$. □

Now hold $L = 3^p$ sufficiently large by taking $p$ large. Recall that $\alpha = \frac{3+\varepsilon}{2}$ where $0 < \varepsilon < 1$ so that $\alpha < 2$. Then we get after rescaling

$$\int d\mu_{\Gamma}(\zeta) G_{\kappa}(X_{\delta_n}, \zeta + S_L \varphi) \leq 2^{\|X_{\delta_n}\|} G_{\kappa}(L^{-1} X_{\delta_n+1}, \varphi)$$

(2.6)

because from the scaling property of the fields $\varphi$, see (1.58), (1.59) we have

$$\|S_L \varphi\|_{L^2(X_{\delta_n}, 1.5)} \leq L^{-(2-n)} \|\varphi\|_{L^2(X_{\delta_n+1}, 1.5)}$$

(2.7)

Next we introduce a large set regulator. Let $X_3$ be a connected $1$-polymer in $(\delta Z)^3$. This is a connected union of $1$-blocks defined earlier. We define

$$A_p(X_3) = 2^{p \|X_3\|} L^{(D+2) \|X_3\|}$$

(2.8)

where for us the dimension of space $D = 3$, and $p$ is an integer.

**Small sets:** We call a connected polymer $X_3$ small if $|X_3| \leq 2^D$. A connected polymer which is not small is called large.

**$L$-polymers and $L$-closure:** Pave $\mathbb{R}^3$ by a disjoint union of open cubes $L \Delta$ of edge length $L$, called $L$-cubes:

$$L \Delta = (\frac{L}{2} + m_1 L, \frac{L}{2} + m_1 L) \times (\frac{L}{2} + m_2 L, \frac{L}{2} + m_2 L) \times (\frac{L}{2} + m_3 L, \frac{L}{2} + m_3 L)$$

(2.9)

where $(m_1, m_2, m_3) \in \mathbb{Z}^3$. Each $L$-cube is a union of $1$-cubes. Let $\delta$ be any member of the sequence $\{\delta_n\}_{n \geq 0}$ where $\delta_n = L^{-n}, L = 3^p$ and $p \geq 2$. Take the restriction of these $L$-cubes to $(\delta Z)^3$ and call the latter cubes...
$L$-blocks. Each $L$-block is a union of 1-blocks. The paving of $\mathbb{R}^3$ by $L$-cubes induces a paving of $(\delta \mathbb{Z})^3$ by $L$-blocks. An $L$-polymer is a union of $L$-blocks. We define the $L$-closure of the 1-polymer $X_\delta$, denoted $\bar{X}_\delta^{(L)}$, as the $L$-polymer given by the smallest union of $L$-blocks containing $X_\delta$. The notions of connectedness and strict disjointness carry over from the case of 1-blocks and 1-polymers. Thus we say two $L$-blocks from the $L$-paving are connected if the closures of the corresponding continuum $L$-cubes are connected (i.e. share at least a vertex in common). If they are not connected we say that they are strictly disjoint. Strictly disjoint $L$-blocks are separated by a distance $\geq L$. A connected $L$-polymer is a connected union of $L$-blocks. If two connected $L$-polymers are not connected to each other we say they are strictly disjoint. Strictly disjoint $L$-polymers are separated by a distance $\geq L$.

**Lemma 2.2**: Fix any integer $p \geq 0$ and let $L$ be sufficiently large depending on $p$. Then for any connected 1-polymer $X_\delta$

$$\mathcal{A}(L^{-1}\bar{X}_\delta^{(L)}) \leq c_p \mathcal{A}_p(X_\delta)$$  \hspace{1cm} (2.10)

For $X_\delta$ a large connected 1-polymer,

$$\mathcal{A}(L^{-1}\bar{X}_\delta^{(L)}) \leq c_p L^{-D-1} \mathcal{A}_p(X_\delta)$$  \hspace{1cm} (2.11)

Here $c_p = O(1)$ is a constant independent of $L$ and $\delta$.

**Remark**: This is the lattice version of Lemma 1 of [BDH-est]. It is purely geometrical and proved in the same way.

### 2.2 Field derivatives and norms

The polymer activities in question are degree 0 gauge invariant supersymmetric functionals of the complex bosonic fields $\varphi, \bar{\varphi}$ and the fermionic fields $\psi, \bar{\psi}$. Lattice field derivatives are partial derivatives with respect to the fields at different points of the lattice. The fermionic derivative is an antiderivation. However in order to measure the size of the lattice field derivatives it turns out to be useful to generalize the notion of field derivatives as directional derivatives (directional in field space). For the bosonic coefficient this is the lattice transcription of that given in [BDH]. For the fermionic part there is no clear sense of direction and the definition we give below suggested to us by David Brydges is both natural and useful.

Let $X_\delta \subset (\delta \mathbb{Z})^3$ be a connected polymer. Let $f_j$ for $j = 1, \ldots, m$ be $\mathbb{C}$ valued functions on $\bar{X}_\delta^{(2)}$. Let $g_{2p}(x,y) =: g_{2p}(x_1, \ldots, x_p, y_1, \ldots, y_p)$ be a $\mathbb{C}$ valued function on $(\bar{X}_\delta^{(2)})^p \times (\bar{X}_\delta^{(2)})^p$, antisymmetric in the $x_j$ and in the $y_j$. A polymer activity $K(X_\delta, \Phi)$ has the representation (1.81) with the coefficients defined in (1.82).

We consider it as a function of $\varphi, \bar{\varphi}, \psi, \bar{\psi}$ denoted as $K(X_\delta, \varphi, \psi)$ where we have suppressed the dependence on $\bar{\varphi}, \bar{\psi}$. We define using the notations of (1.81), (1.82) for the coefficients,

$$D^{2p,m}_B K_p(X_\delta, \varphi, \psi, f^{x,m}, g_{2p}) =: \int_{X_p^p \times X_p^p} dxdy D_B^m D_{\bar{p}}^2 (X_\delta, \varphi, x, y; f^{x,m}) g_{2p}(x_1, \ldots, x_p, y_1, \ldots, y_p)$$  \hspace{1cm} (2.12)

where $f^{x,m} = (f_1, \ldots, f_m)$ and

$$D_B^m D_{\bar{p}}^2 K_{2p}(X_\delta, \varphi, \psi, f^{x,m}) = \partial_{s_1} \ldots \partial_{s_m} D_{\bar{p}}^2 K(X_\delta, \varphi + s_1 f_1, \ldots, \varphi + s_m f_m, x, y)|_{s_1 = \ldots = s_m = 0}$$  \hspace{1cm} (2.13)

and the $s_j$ are real parameters.

Let $\partial_{\delta, \mu}$, $\partial_{\delta, -\mu}$ be the forward and backward lattice derivative in the direction $\epsilon_\mu$. Let the index set $S$ be defined as after (2.1). We endow the linear space of $\mathbb{C}$ valued functions $f$ as above with the norm

$$\|f\|_{C^2(X_\delta)} = \sup_{\mu, \nu \in S} \left( \|f\|_{L^\infty(X_\delta)}, \|\partial_{\delta, \mu} f\|_{L^\infty(X_\delta)}, \|\partial_{\delta, \mu} \partial_{\delta, \nu} f\|_{L^\infty(X_\delta)} \right)$$  \hspace{1cm} (2.14)
and call the resulting normed space $C^2(\mathcal{X}_\delta)$. Let \( \partial_{\delta,\mu_j} \), \( \mu_j \in S \), acting on \( g_{2p}(x,y) \) denote the forward or backward lattice derivative with respect to \( x_j \) or \( y_j \) in the direction \( e_{\mu_j} \). We endow the linear space of \( C \) valued functions \( g_{2p}(x,y) \) on \( (\mathcal{X}_\delta^{(2)})^p \times (\mathcal{X}_\delta^{(2)})^p \), antisymmetric in the \( x_j \), and in the \( y_j \), with the norm

\[
\|g_{2p}\|_{C^2(\mathcal{X}_\delta^{(2)^p})} = \sup_{\mu_j, \mu_k \in \delta \atop 1 \leq j, k \leq 2p} (\|\partial_{\delta,\mu_j}g_{2p}\|_{L^\infty(\mathcal{X}_\delta^{(2)^p})}, \|\partial_{\delta,\mu_j}g_{2p}\|_{L^\infty(\mathcal{X}_\delta^{(2)^p})}, \|\partial_{\delta,\mu_j}g_{2p}\|_{L^\infty(\mathcal{X}_\delta^{(2)^p})})
\]

(2.15)

and call the resulting normed space $C^2(\mathcal{X}_\delta^{(2)^p})$. The above norms always exist for lattice functions since \( X_\delta \) is a finite set.

(2.12) then defines a \( C \) valued multilinear functional on \( C^2(\mathcal{X}_\delta)^m \times C^2(\mathcal{X}_\delta^{(2)^p}) \) whose norm is defined to be

\[
\|D^{2p,m}K(\mathcal{X}_\delta, \varphi, 0)\| = \sup_{\|f_j\|_{C^2(\mathcal{X}_\delta)^{m}}, \|g_{2p}\|_{C^2(\mathcal{X}_\delta^{(2)^p})} \leq 1, \forall 1 \leq j \leq m} \left| D^{2p,m}K(\mathcal{X}_\delta, \varphi, 0; f^{x,m}, g_{2p}) \right|
\]

(2.16)

The space of \( C \) valued multilinear functionals defined in (2.12) which are bounded in the norm (2.16) is complete and thus a Banach space.

Remarks: It is well known that the space of bounded \( C \) valued multilinear functionals on a normed space is complete (even if the normed space is not). The completeness follows on using the completeness of the number field \( C \) by a standard argument.

Let \( h = (h_F, h_B) \) where \( h_F, h_B > 0 \) are strictly positive real numbers. We define the following set of norms. The \( h \) norm is defined by

\[
\|K(\mathcal{X}_\delta, \varphi, 0)\|_h = \sum_{p=0}^{\infty} \sum_{m=0}^{m_0} \frac{h_F^{2p}}{(p!)^2} \frac{h_B^m}{m!} \|D^{2p,m}K(\mathcal{X}_\delta, \varphi, 0)\|
\]

(2.17)

In addition we define a kernel norm with \( h_* = (h_F, h_B*) \)

\[
|K(\mathcal{X}_\delta)|_{h_*} = \sum_{p=0}^{\infty} \sum_{m=0}^{m_0} \frac{h_F^{2p}}{(p!)^2} \frac{h_B^m}{m!} \|D^{2p,m}K(\mathcal{X}_\delta, 0, 0)\|
\]

(2.18)

\( h, h_* \) will be chosen later in Section 5. We now define the \( h, G_\kappa \) norm by

\[
\|K(\mathcal{X}_\delta)\|_{h,G_\kappa} = \sup_{\varphi \in \mathcal{F}_\kappa^{(5)}} \|K(\mathcal{X}_\delta, \varphi, 0)\|_{h} \|G_\kappa^{(5)}(\mathcal{X}_\delta, \varphi)\|
\]

(2.19)

Let \( \mathcal{A}(\mathcal{X}_\delta) \) be the large set regulator defined earlier. We then have our final set of norms

\[
\|K\|_{h,G_\kappa,\mathcal{A},\delta} = \sup_{\Delta(\mathcal{X}_\delta) \in \Delta} \sum_{\Delta(\mathcal{X}_\delta) \supset \Delta_\delta} \|(K(\mathcal{X}_\delta))\|_{h,G_\kappa,\mathcal{A}(\mathcal{X}_\delta)}
\]

(2.20)

where \( \Delta_\delta = \Delta \cap (\delta\mathbb{Z})^3 \) and \( \Delta \) is a unit cube in \( \mathbb{R}^3 \) as defined earlier, and

\[
|K|_{h,\mathcal{A},\delta} = \sup_{\Delta(\mathcal{X}_\delta) \in \Delta} \sum_{\Delta(\mathcal{X}_\delta) \supset \Delta_\delta} |K(\mathcal{X}_\delta)|_{h,\mathcal{A}(\mathcal{X}_\delta)}
\]

(2.21)

The index \( \delta \) in our final norms (2.20) and (2.21) indicate that the large set norm is being taken over polymers in \((\delta\mathbb{Z})^3\). Under each of above norms we have Banach spaces. Moreover it is easy to verify that the multiplicative (Banach algebra) property holds for the polymer activities \( K(\mathcal{X}_\delta) \) under the \( h \)-norm (2.17), the kernel norm (2.18), and, for activities supported on disjoint polymers , under the \( h, G_\kappa \) norm. The
multiplicative property plays a very important role in the estimates in the rest of the paper. We therefore state it as Proposition 2.3 below and supply a proof.

**Proposition 2.3**: Let \( X_{\delta,1}, X_{\delta,2} \) denote two connected polymers. Let \( X_{\delta,1} = X_{\delta,2} \) or \( X_{\delta,1} \cap X_{\delta,2} = \emptyset \). \( K_j(X_{\delta,j}, \varphi, \psi) \), \( j = 1, 2 \) are polymer activities of degree 0. Define a new polymer activity

\[
K(X_{\delta,1} \cup X_{\delta,2}, \varphi, \psi) = K_1(X_{\delta,1}, \varphi, \psi)K_2(X_{\delta,2}, \varphi, \psi)
\]

Then

\[
\|K(X_{\delta,1} \cup X_{\delta,2}, \varphi, 0)\|_h \leq \|K_1(X_{\delta,1}, \varphi, 0)\|_h \|K_2(X_{\delta,2}, \varphi, 0)\|_h
\]

The same inequality holds for the \( h \) norm. If \( X_{\delta,1} \) and \( X_{\delta,2} \) are disjoint we have

\[
\|K(X_{\delta,1} \cup X_{\delta,2})\|_{h, g_*} \leq \|K_1(X_{\delta,1})\|_{h, g_*} \|K_2(X_{\delta,2})\|_{h, g_*}
\]

**Proof**

Let \( f_j \), \( j = 1, \ldots, m \) be functions on \( \tilde{X}_{\delta,1}^{(2)} \cup \tilde{X}_{\delta,2}^{(2)} \) and \( g_{2p}(x, y) = g_{2p}(x_1, \ldots, x_p, y_1, \ldots, y_p) \) be a function on \( (\tilde{X}_{\delta,1}^{(2)} \cup \tilde{X}_{\delta,2}^{(2)})^p \times (\tilde{X}_{\delta,1}^{(2)} \cup \tilde{X}_{\delta,2}^{(2)})^p = (\tilde{X}_{\delta,1}^{(2)} \cup \tilde{X}_{\delta,2}^{(2)})^{2p} \). \( g_{2p} \) is antisymmetric in the \( x_j \) and in the \( y_j \). By definition

\[
\|D^{2p, m}K(X_{\delta,1} \cup X_{\delta,2}, \varphi, 0)\| = \sup_{\|f_j\|_{C^2(\tilde{X}_{\delta,1}^{(2)} \cup \tilde{X}_{\delta,2}^{(2)})^p} \leq 1} |D^{2p, m}K(X_{\delta,1} \cup X_{\delta,2}, \varphi, f^m; g_{2p})|
\]

where \( f^m = (f_1, \ldots, f_m) \), \( f^M = \{f_j\}_{j \in M} \) and \( M \subset \{1, 2, \ldots, m\} \). We extend the coefficients of \( K_j(X_{\delta,j}, \varphi, \psi) \), \( j = 1, 2 \) to \( X_{\delta,1} \cup X_{\delta,2} \) by declaring that they have support in \( X_{\delta,j} \). Now \( D^p_{F} \) is a partial (anti) derivation of order \( 2p \). \( D^m_B \) a derivation of order \( m \). Distributing \( D^p_B \) and \( D^m_B \) on the product of polymer activities gives

\[
D^p_B D^m_B K(X_{\delta,1} \cup X_{\delta,2}, \varphi, x_1, \ldots, x_p, y_1, \ldots, y_p; f^m) = \sum_{p_1 + p_2 = p, m_1 + m_2 = m} \sum_{\substack{M_1 \cup M_2 = \{1, \ldots, m\} \cap M_1 \cap M_2 = \emptyset \cr |M_1| = m_1, |M_2| = m_2}} \sum_{I \subset \{1, \ldots, p\}} \sum_{J \subset \{1, \ldots, p\}} \sum_{|I| = |J| = p_1} \sum_{|I^c| = |J^c| = p_2} K_1(X_{\delta,1}, \varphi, x_I, y_J; f^M_1)D^m_{M_1} D^p_{M_2} K_2(X_{\delta,2}, \varphi, x_{I^c}, y_{J^c}; f^M_2) \times g_{2p}(x_I, x_{I^c}, y_J, y_{J^c}) \times (-1)^J
\]

where \((-1)^J\) is a sign factor which plays no role in the norm bounds to follow, \( I^c, J^c \) are respectively the complements of \( I, J \) in \( \{1, \ldots, p\} \). We have \( |I^c| = |J^c| = p_2 \). We now integrate this with respect to \( x_1, \ldots, x_p, y_1, \ldots, y_p \) in \( (X_{\delta,1} \cup X_{\delta,2})^p \times (X_{\delta,1} \cup X_{\delta,2})^p \). Because of the support properties of the coefficients the integral splits over the products on the right hand side. We get

\[
D^{2p, m}K(X_{\delta,1} \cup X_{\delta,2}, \varphi, 0; f^m, g_{2p}) = \sum_{p_1 + p_2 = p, m_1 + m_2 = m} \sum_{\substack{M_1 \cup M_2 = \{1, \ldots, m\} \cap M_1 \cap M_2 = \emptyset \cr |M_1| = m_1, |M_2| = m_2}} \sum_{I \subset \{1, \ldots, p\}} \sum_{J \subset \{1, \ldots, p\}} \sum_{|I| = |J| = p_1} \sum_{|I^c| = |J^c| = p_2} \int_{X_{\delta,1}^{(2)} \times X_{\delta,2}^{(2)}} d\tau_1 d\tau_2 \int_{X_{\delta,1}^{(2)} \times X_{\delta,2}^{(2)}} d\xi_1 d\xi_2 D^m_{M_1} D^p_{M_2} K_1(X_{\delta,1}, \varphi, x_I, y_J; f^M_1)D^m_{M_1} D^p_{M_2} K_2(X_{\delta,2}, \varphi, x_{I^c}, y_{J^c}; f^M_2) \times g_{2p}(x_I, x_{I^c}, y_J, y_{J^c}) \times (-1)^J \]

where \( X_{\delta,j} = X_{\delta,j}^{p_j} \times X_{\delta,j}^{p_j} \). Define
\[ \tilde{g}_{2p_1}(x_I, y_J) = \int_{X_{\delta,2}^{2p_1}} dx_I \cdot dy_J \cdot D_B^{m_2} D_{M_2}^{2p_2} K_2(x_{\delta,2}, \varphi, x_{f^c \cdot y_J}; f \times M_2) g_{2p}(x_I, x_{f^c \cdot y_J}, y_J, y_{J^c}) \\
= D^{2p_2, m_2} K_2(x_{\delta,2}, \varphi, 0; g_{2p}(x_I, \cdot, y_J, \cdot), f \times M_2) \] 

(2.24)

where the dependence of \( \tilde{g}_{2p_1}(x_I, y_J) \) on \( K_2, x_{\delta,2}, p_2, f \times M_2 \) has been suppressed. Note that \( \tilde{g}_{2p_1}(x_I, y_J) \) is antisymmetric in the \( \{x_i : i \in I\} \) and in the \( \{y_j : j \in J\} \) and therefore qualifies as a test function.

From (2.22) and (2.24) we have

\[ D^{2p, m_1} K_1(x_{\delta,1}, \varphi, 0; f^{M_1}, \tilde{g}_{2p_1}) \times (-1)^d \] 

(2.25)

Therefore

\[ |D^{2p, m_1} K_1(x_{\delta,1}, \varphi, 0; f^{M_1}, \tilde{g}_{2p_1})| \leq \sum_{p_1+p_2=p} \sum_{m_1+m_2=m} \sum_{M_1 \cup M_2 = \{1, \ldots, m\}} \sum_{M_1 \cap M_2 = \emptyset} \sum_{I \subset \{1, \ldots, p\}} \sum_{J \subset \{1, \ldots, p\}} \prod_{i \in M_1} \|f_i\|_{C^2(X_{\delta,1}, \varphi)} \|\tilde{g}_{2p_1}\|_{C^2(X_{\delta,1}^{2p_1})} \] 

(2.26)

From (2.24) we have for \( 0 \leq k \leq 2 \)

\[ \partial^k \tilde{g}_{2p_1}(x_I, y_J) = D^{2p_2, m_2} K_2(x_{\delta,2}, \varphi, 0; \partial^k \tilde{g}_{2p_1}(x_I, \cdot, y_J, \cdot), f \times M_2) \]

where \( \partial^k \) is the partial derivative of degree \( k \) with respect to \( x_I, y_J \) in multi-index notation. Whence

\[ |\partial^k \tilde{g}_{2p_1}(x_I, y_J)| \leq \|D^{p_2, m_2} K_2(x_{\delta,2}, \varphi, 0)\| \prod_{j \in M_2} \|f_j\|_{C^2(X_{\delta,2}, \varphi)} \|\partial^k \tilde{g}_{2p_1}(x_I, \cdot, y_J, \cdot)\|_{C^2(X_{\delta,2}^{2p_2})} \]

and therefore

\[ \|\tilde{g}_{2p_1}\|_{C^2(X_{\delta,1}^{2p_1})} \leq \|D^{p_2, m_2} K_2(x_{\delta,2}, \varphi, 0)\| \prod_{j \in M_2} \|f_j\|_{C^2(X_{\delta,2}, \varphi)} \|\tilde{g}_{2p_1}\|_{C^2(X_{\delta,2}^{2p_2} \times X_{\delta,1}^{2p_1})} \] 

(2.27)

Now \( X_{\delta,2}^{2p_2} \times X_{\delta,1}^{2p_1} \subset (X_{\delta,2} \times X_{\delta,1})^{2p_2} \) where \( p = p_1 + p_2 \). Therefore from (2.26) and (2.27) we get

\[ \|D^{2p, m_1} K_1(x_{\delta,1}, \varphi, 0)\| \|D^{p_2, m_2} K_2(x_{\delta,2}, \varphi, 0)\| \]

Now

\[ \sum_{M_1 \cup M_2 = \{1, \ldots, m\}} \sum_{M_1 \cap M_2 = \emptyset} \frac{1}{m! m_1 m_2! (p_1!)^2 (p_2!)^2} \]

Therefore
\[ \|D^{2p,m}K(X_{\delta,1} \cup X_{\delta,2}, \varphi, 0)\| \leq \sum_{p_1+p_2=p} \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \frac{(p!)^2}{(p_1!)^2(p_2!)^2} \|D^{2p_1,m_1}K_1(X_{1,\delta}, \varphi, 0)\| \times \]

\[ \|D^{2p_2,m_2}K_2(X_{2,\delta}, \varphi, 0)\| \]  

(2.28)

Multiply both sides of the previous inequality by \(h_{2p}/m!\) and \(h_{2p}^2/(p!)^2\). Sum over integers \(m, 0 \leq m \leq m_0\), and over all integers \(p \geq 0\) to obtain

\[ \|K(X_{\delta,1} \cup X_{\delta,2}, \varphi, 0)\|_h \leq \|K_1(X_{\delta,1}, \varphi, 0)\|_h \|K_2(X_{\delta,2}, \varphi, 0)\|_h \]

This proves the first inequality of Proposition 2.3. The second inequality follows from the first because for union of disjoint sets

\[ G_\kappa(X_\delta \cup Y_\delta, \varphi) = G_\kappa(X_\delta, \varphi)G_\kappa(Y_\delta, \varphi) \]

3. THE RG MAP

In this section we describe the RG map applied to the generic density in the polymer representation given in (1.80). This is a lattice transcription of the continuum RG map described in [BMS], (see also [M]). This goes in several steps. First we must perform the fluctuation integration and rescaling (see (1.68))

\[ Z'(L^{-1}\Lambda_{L^{-1}\delta}, \varphi) = S_L \mu \ast Z(\Lambda_\delta, \Phi) \]  

(3.1)

where \(\Lambda_\delta \subset (\delta \mathbb{Z})^3\) is the volume arrived at after a certain number of previous RG steps and \(\mu\) is the fluctuation covariance for the next step. \(\Gamma\) is one of the covariances \(\Gamma_n\) of Theorem 1.1 and has the finite range property stated in that theorem. Thus after \(n\) RG steps (see (1.64)-(1.69)) \(\delta = \delta_n, \Gamma = \Gamma_n, \Lambda_\delta = \Lambda_{N-n, n}\) and \(L^{-1}\Lambda_{L^{-1}\delta} = \Lambda_{N-n-1, n+1}\).

The polymer representation (1.80) for \(Z(\Lambda_\delta)\) is parametrized by the coordinates \((V, K)\) on the scale \(\delta\) where \(V\) is a local functional (potential):

\[ V(X_\delta) = \sum_{\Delta_\delta \subset X_\delta} V(\Delta_\delta) \]

(3.2)

Let \(\tilde{V}(X_\delta, \Phi)\) be an arbitrary local supersymmetric functional with \(\tilde{V}(X_\delta, 0) = 0\). We will see that the polymer representation is preserved under the RG transformation (3.1) with new coordinates \(\tilde{V}_L, F(K)\) on the next scale \(L^{-1}\delta\). \(F\) depends on \(\tilde{V}\). The finite range property of \(\mu\) leads to a simple description of this map:

\[ V \to \tilde{V}_L, \quad \tilde{V}_L(\Delta_{L^{-1}\delta}, \Phi) = (S_L V)(\Delta_{L^{-1}\delta}, \Phi) = \tilde{V}(L\Delta_\delta, S_L \Phi) \]

\[ K \to F(K) : F(K)(X_{L^{-1}\delta}, \Phi) = \int d\mu_\Gamma(\xi) \mathcal{B}K(LX_\delta, \xi, S_L \Phi) \]

(3.3)

where \(\mathcal{B}K\) is a \(\tilde{V}\) dependent nonlinear functional of \(K\) to be presently described. We call this map the fluctuation map.

We can take advantage of the arbitrariness of the local potential \(\tilde{V}\) in the above map so as to remove the expanding (relevant) parts \(F\) in the polymer activity \(F(K)\) and compensate by a change \(\tilde{V}_L(F)\) in the local potential \(\tilde{V}_L\) in such a way that the evolved density \(Z'(L^{-1}\Lambda_{L^{-1}\delta})\) on the left hand side of (3.1) remains unchanged. This operation gives rise to the extraction map, [BDH-est]
\[
\tilde{V}_L \to V'(F) = \tilde{V}_L - \tilde{V}_L(F), \quad \mathcal{F}(K) \to K' = \mathcal{E}(\mathcal{F}(K), F)
\]

where the image is on the same scale \(L^{-1}\delta\). \(V'(F)\) and the nonlinear map \(\mathcal{E}\) have simple expressions which are lattice transcriptions of those given in [BDH-est]. The composition of the fluctuation map (3.3) and the extraction map (3.4) gives the \(\text{RG map}\)

\[
f : \quad f(V, K) = (f_V(V, K), f_K(V, K))
\]

where

\[
f_V : V \to \tilde{V}_L \to V'(F)
\]

\[
f_K : K \to \mathcal{F}(K) \to K' = \mathcal{E}(\mathcal{F}(K), F)
\]

The operation of extraction leads in particular to a discrete flow of the coupling constants in \(V\) on scale \(L^{-1}\delta\) provided we choose \(F, \tilde{V}_L(F)\) appropriately. The expanding functionals will be gathered in the local potential \(V'(F)\) whereas the polymer activity \(\mathcal{E}(\mathcal{F}(K), F)\) will be a contracting (irrelevant) error term.

3.1 The Fluctuation Map

We now construct the map (3.3) starting from (3.1) with the density in the polymer representation (1.80). In performing the fluctuation integration

\[
\mu_\Gamma * Z(\Lambda_{\delta}, \Phi) = \int d\mu_\Gamma(\xi) \sum_{N} \frac{1}{N!} e^{-V(X_{\delta}^{(N)}, \Phi + \xi)} \sum_{X_{\delta,1}, \ldots, X_{\delta,N}} \prod_{j=1}^{N} K(X_{\delta,j}, \Phi + \xi)
\]

we will exploit the independence of \(\xi(x)\) and \(\xi(y)\) when \(|x-y| \geq L\). To this end we construct an \(L\)-paving of \(\Lambda_{\delta}\) and the \(L\)-closure of \(X_{\delta}^{(L)}\) of a connected 1-polymer \(X_{\delta}\) as in the paragraph preceding Lemma 2.1. The 1-polymers will be combined into larger connected \(L\)-polymers which by definition are connected unions of \(L\)-blocks, (for the relevant definitions intervening here and in the following see the paragraph on \(L\)-polymers and \(L\)-closures before Lemma 2.1). The combination is performed in such a way that the new polymers are associated to independent functionals of \(\xi\). This is the lattice adaptation of Section 3.1 of [BMS]. Define the polymer activity \(P\), supported on unit blocks, by:

\[ P(\Delta_{\delta}, \xi, \Phi) = e^{-V(\Delta_{\delta}, \xi + \Phi)} - e^{-V(\Delta_{\delta}, \Phi)} \quad (3.7) \]

with \(\tilde{V}\), to be chosen. \(\tilde{V}(\Delta_{\delta}, \Phi)\) is required to satisfy \(\tilde{V}(\Delta_{\delta}, 0) = 0\). In the following \(V, K\) has field argument \(\xi + \Phi\) whereas \(\tilde{V}\) depends only on \(\Phi\). The dependence of \(P\) on \(\xi, \Phi\) is as defined above.

\[ X_{\delta}^c = \Lambda_{\delta} \cup_{j=1}^{N_{\delta}} X_{\delta,j} \quad \text{is a union of disjoint 1-blocks} \quad \Delta_{\delta}. \]

Therefore

\[ e^{-V(X_{\delta}^c)} = \prod_{\Delta_{\delta} \subset X_{\delta}} [e^{-V(\Delta_{\delta})} + P(\Delta_{\delta})] \]

Expand the product and insert the expansion into the integrand of in (3.6) which gives

\[ \text{integrand} = \sum_{N} \frac{1}{N!} \sum_{(X_{\delta,j}, (\Delta_{\delta,i}))} e^{-V(X_{\delta,0})} \prod_{j=1}^{N} K(X_{\delta,j}) \prod_{i=1}^{M} P(\Delta_{\delta,i}) \quad (3.8) \]

where \(X_{\delta,0} = \Lambda_{\delta} \setminus (\cup X_{\delta,j}) \cup (\cup \Delta_{\delta,i})\). Let \(Y_{\delta}\) be the \(L\)-closure of \((\cup X_{\delta,j}) \cup (\cup \Delta_{\delta,i})\) and let \(Y_{\delta,1}, \ldots, Y_{\delta,P}\) be the connected components of \(Y_{\delta}\). These are \(L\)-polymers. Let \(f\) be the function that maps \(\pi := (X_{\delta,j}, (\Delta_{\delta,i}))\) into \(\{Y_{\delta,1}, \ldots, Y_{\delta,P}\}\). Now we perform the sum over \((X_{\delta,j}, (\Delta_{\delta,i}))\) in (3.8) by summing over \(\pi \in f^{-1}(\{Y_{\delta,1}, \ldots, Y_{\delta,P}\})\) and then \(\{Y_{\delta,1}, \ldots, Y_{\delta,P}\}\). The result is:
\[
\text{integrand} = \sum_N \frac{1}{N!} \sum_{(Y_{\delta,j})} e^{-\bar{\mathcal{V}}(\bar{Y}_{\delta,j})} \prod_{j=1}^{N} BK(Y_{\delta,j})
\]
where the sum is over strictly disjoint connected \(L\) polymers and
\[
(BK)(Y_{\delta}) = \sum_{N+M \geq 1} \frac{1}{N! M!} \sum_{(X_{\delta,j}), (\Delta_{\delta,i}) - (Y)} e^{-\bar{\mathcal{V}}(X_{\delta,0})} \prod_{j=1}^{N} K(X_{\delta,j}) \prod_{i=1}^{M} P(\Delta_{\delta,i})
\]
where \(X_{\delta,0} = Y \setminus (\cup X_{\delta,j}) \cup (\cup \Delta_{\delta,i})\) and the \(\rightarrow\) is the map \(f\). In other words the sum in (3.10) is over distinct \(\Delta_{\delta,j}\) and disjoint 1-polymers \(X_{\delta,j}\) such that their \(L\)-closure is the connected \(L\)-polymer \(Y_{\delta}\).

We now perform the fluctuation integration of (3.9) over \(\xi\) followed by rescaling. Since \(\bar{\mathcal{V}}(Y_{\delta}^{(2)})\) is independent of \(\xi\) the \(\xi\) integration factors through and acts on the product of polymer activities \(\prod_{j}(BK)(Y_{\delta,j})\). A polymer activity \((BK)(Y_{\delta,j})\) belongs to \(\Omega^0(\bar{Y}_{\delta,j}^{(2)})\). The \(Y_{\delta,j}\) are strictly disjoint connected \(L\)-polymers and thus necessarily separated from each other by a distance \(\geq L\). The 2-collar attached \(L\)-polymers \(\Omega^0(\bar{Y}_{\delta,j}^{(2)})\) are therefore separated from each other by a distance \(\geq L - 4\). The fluctuation covariance \(\Gamma\) has finite range \(L/2\) and for \(L\) sufficiently large \(L - 4 \geq L/2\). Therefore the fluctuation integration over the product of polymer activities factorizes. We now follow this up by applying the rescaling operator to both sides. This has the effect of bringing us back to 1-polymers but on the scale \(L^{-1}\). Therefore we obtain
\[
(S_L \mu_{\Gamma} * Z)(L^{-1} \Lambda_{L^{-1} \delta}, \Phi) = \sum_{N} \frac{1}{N!} \sum_{(X_{L^{-1} \delta,j})} e^{-\bar{\mathcal{V}}(X_{L^{-1} \delta,j})} \prod_{j=1}^{N} \int d\mu_{\Gamma}(\xi) BK(LX_{\delta,j}, S_L \Phi, \xi)
\]
where \(X_{L^{-1} \delta,j} = X_j \cap (L^{-1} \delta Z)^3\) (as well as \(X_{\delta,j} = X_j \cap (\delta Z)^3\)) are disjoint 1-polymers, \(X_{L^{-1} \delta,j} = L^{-1} \Lambda_{L^{-1} \delta} \setminus \cup_j X_{L^{-1} \delta,j}\), and \(\bar{V}_L(\Delta_{L^{-1} \delta}) = S_L \bar{V}(L \Delta_{\delta})\). This gives the fluctuation map (3.3): \(V \rightarrow \bar{V}_L, K \rightarrow F(K)\) with \(BK\) defined as above. At the same time we have shown that the polymer representation is stable with respect to the RG transformation.

Consider
\[
F(K)(X_{L^{-1} \delta,j}, \Phi) = \int d\mu_{\Gamma}(\xi) BK(LX_{\delta,j}, \xi, S_L \Phi)
\]
By construction \(BK\) is supersymmetric. Therefore since the supersymmetry operator commutes with the measure \(F(K)\) is also supersymmetric. Now since \(P(\Delta_{\delta}, \xi, 0)\) and \(K(X_{\delta}, \xi)\) vanish for \(\xi = 0\) (the latter by hypothesis, see (1.83)) it follows that \(BK(LX_{\delta}, \xi, 0)\) also vanishes for \(\xi = 0\). Therefore by Lemma 1.1
\[
F(K)(X_{L^{-1} \delta, 0}) = \int d\mu_{\Gamma}(\xi) BK(LX_{\delta}, \xi, 0) = BK(LX_{\delta}, 0, 0) = 0
\]
Thus the condition (1.83) is satisfied by the new polymer activities. This implies in particular that no field independent relevant parts are generated by the fluctuation integration as a consequence of supersymmetry.

3.2 Extraction

Let \(\delta' = L^{-1} \delta\) and let \(\Lambda' = L^{-1} \Lambda_{\delta'}\). The fluctuation map gave us \(\bar{V}_L, F(K)\) as the coordinates of the evolved density \(Z(\Lambda')\). We want to change the local potential \(\bar{V}_L\) and the polymer activity \(F(K)\) simultaneously such that \(Z(\Lambda')\) remains invariant. To this end let \(P(\Phi(x))\) be a local polynomial, which means that it is a polynomial in \(\Phi(x)\) for \(x \in \Lambda'\). Furthermore we require that \(P(0) = 0\), i.e. \(P\) has no field independent part. Given \(\Delta_{\delta'}\) a unit block in \(\Lambda'\) we consider a change in \(\bar{V}_L(\Delta_{\delta'})\) of the form
\[
\bar{V}_L(F)(\Delta_{\delta'}) = \sum_P \int_{\Delta_{\delta'}} dx \alpha_P(x) P(\Phi(x))
\]
where the sum ranges over finitely many local polynomials and, for each such \( P \), \( \alpha_P(x) \) has the form

\[
\alpha_P(x) = \sum_{X_{\delta'} \ni x} \alpha_P(X_{\delta'}, x)
\]

(3.14)
such that \( \alpha_P(X_{\delta'}, x) = 0 \) if \( x \notin X_{\delta'} \), \( \alpha_P(X_{\delta'}, x) = 0 \) if \( X_{\delta'} \notin \Lambda' \) and \( \alpha_P(X_{\delta'}, x) = 0 \) if \( X_{\delta'} \) is not a small set (see definition after (2.8)). The corresponding change in \( \mathcal{F}(K) \) is given in terms of the relevant parts

\[
F(X_{\delta'}, \Phi) = \sum_p \int_{X_{\delta'}} dx \, \alpha_p(X_{\delta'}, x) P(\Phi(x)), \quad F(X_{\delta'}, \Delta_{\delta'}) = \sum_p \int_{\Delta_{\delta'}} dx \, \alpha_p(X_{\delta'}, x) P(\Phi(x))
\]

(3.15)

Note that \( F(X_{\delta'}, 0) = 0 \).

**Extraction Map:**

**Theorem 3.1 (after Brydges, Dimock and Hurd, [BDH-est]):** Given \( F, \tilde{V}_L(F) \) as above there exists a polymer activity which is a non-linear functional \( \mathcal{E}(\mathcal{F}(K), F) \) of \( \mathcal{F}(K) \), \( F \) such that

\[
V_L \to V'(F) = \tilde{V}_L - \tilde{V}_L(F), \quad \mathcal{F}(K) \to K' = \mathcal{E}(\mathcal{F}(K), F)
\]

(3.16)
preserves the polymer representation for the density \( Z(\Lambda') \) with new coordinates \( (V', K') \) satisfying \( V'(F)(\Delta_{\delta'}, 0) = K'(X_{\delta'}, 0) = 0 \). Let \( \mathcal{E}_1 \) denote the linearization of \( \mathcal{E} \). Then the linearization of the extraction map is given by

\[
\mathcal{E}_1(\mathcal{F}(K), F) = \mathcal{F}(K) - F e^{-\tilde{V}_L}, \quad V'(F) = \tilde{V}_L - \tilde{V}_L(F)
\]

(3.17)
We say that \( \tilde{V}_L \) is stable with respect to perturbation \( F \) if there are positive numbers \( f(X) \) such that

\[
\|e^{-\tilde{V}_L(\Delta_{\delta'})} - \sum_{X_{\delta'} \ni \Delta_{\delta'}} z(X) P(X_{\delta'}, \Delta_{\delta'})\|_{h, G_n} \leq 2
\]

(3.18)
for all complex numbers \( z(X_{\delta'}) \) with \( |z(X_{\delta'})| f(X_{\delta'}) \leq 2 \). Assume that \( \tilde{V}_L \) is stable. Then \( \mathcal{E}(\mathcal{F}(K), F) \) is norm analytic and satisfies the bounds

\[
\|\mathcal{E}(\mathcal{F}(K), F)\|_{h, G_n, A, \delta'} \leq O(1) (\|\mathcal{F}(K)\|_{h, G_n, A, \delta'} + \|f\|_{A, \delta'})
\]

(3.19)
\[
|\mathcal{E}(\mathcal{F}(K), F)|_{h, A, \delta'} \leq O(1) (\|\mathcal{F}(K)\|_{h, A, \delta'} + \|f\|_{A, \delta'})
\]

(3.20)

**Proof:** This is a restatement of Theorem 5 in Sec. 4.2 of [BDH-est] with the substitution \((\tilde{V}_L, \mathcal{F}(K))\) for \((V, K)\), adapted to the lattice. The proof of Theorem 5 exploited Lemmas 10, 11, 12, 13 the last of them providing the extraction formula in equation (121), page 781 of [BDH-est]. In [BDH-est] the continuum unit blocks are open. Our lattice unit blocks are lattice restrictions of continuum open unit cubes. Overlap connectedness is replaced by connectedness. With this in mind the proofs of Lemmas 10, 11, 12, 13 go through intact on the lattice providing the extraction map above. The estimates in Theorem 5 on the norms of \( \mathcal{E}(K, F) \) together with norm analyticity remain valid on the lattice.

**Remark:** The stability property (3.18) is proved in Section 5 once we have chosen \( \tilde{V} \) appropriately. The estimate (3.19) on the extraction operator \( \mathcal{E} \) plays an essential role and is exploited in Section 5.

**Formal infinite volume limit:** We reestablish the notations leading to (1.69). Choose \( \delta = \delta_n, \Gamma = \Gamma_n, \delta' = L^{-1} \delta = \delta_{n+1} \), \( \Lambda_n = \Lambda_{N-n,n} \), \( \Lambda_{n, n'} = \Lambda_{N-n-1, n+1} \) and \( \mathcal{F} = \mathcal{F}_{n+1} \) in (3.3). The RG transformation \( T_{n+1} \) of (1.69) induces the RG map \( \mathcal{F}_{n+1}(V, K) \) of (3.5) for the coordinates of the density \( Z_{n+1}(\Lambda_{N-n, n}) \) in the polymer representation. \( \alpha_P(X_{\delta_{n+1}}, x) \) in (3.14) is chosen later in Section 4. This choice will be local, in the sense that it is determined by \( \tilde{V}_L(\Delta_{\delta_{n+1}}) \), \( \Delta_{\delta_{n+1}} \subset X_{\delta_{n+1}} \) and by
\( \mathcal{F}_{n+1}(K)(X_{\delta_{n+1}}) \). Lemma 13 and equation (112) of [BDH-est] imply that \( \mathcal{E}(\mathcal{F}_{n+1}(K), F))(X_{\delta_{n+1}}) \) also is local: it is determined by \( \mathcal{F}_{n+1}(K)(Y_{\delta_{n+1}}) \), \( Y_{\delta_{n+1}} \subset X_{\delta_{n+1}} \) and \( \tilde{V}_L(\Delta_{\delta_{n+1}}) \), \( \Delta_{\delta_{n+1}} \subset X_{\delta_{n+1}} \), where \( X_{\delta_{n+1}} \) is a neighbourhood of \( X_{\delta_{n+1}} \), namely the union of \( X_{\delta_{n+1}} \) with all small sets that intersect \( X_{\delta_{n}} \). Therefore the \( K \) component of the map \( f_{N-n-1,n+1} \) representing the action of the \( n+1 \)th step of RG, namely \( f_{N-n-1,n+1,K}(K,V)(X_{\delta_{n+1}}, \Phi) \) is independent of \( N \) for all \( N \) large enough so that \( \Lambda_{N-n-1,n+1} \) contains \( X_{\delta_{n+1}} \). Thus \( \lim_{N \to \infty} f_{N-n-1,n+1,K}(K,V)(X_{\delta_{n+1}}, \Phi) \) exists pointwise in \( X_{\delta_{n+1}} \). In this paper we are studying the action of this pointwise infinite volume limit called the *formal infinite volume limit*.

3.3 Appendix:
We record here some definitions which have either already been used or will be used later. The object is to be able to move scaling past fluctuation integration.

We define for \( x, y \in (L^{-1}\delta \mathbb{Z})^3 \) and any covariance \( u \) on \((\delta \mathbb{Z})^3\)
\[
u_{L}(x-y) = S_{L^{-1}}u(x-y) = L^{2d_s}u(L(x-y))
\]
(3.21)
Since the fluctuation covariance \( \Gamma \) defined on \((\delta \mathbb{Z})^3\) has finite range \( L/2 \) we have that \( \Gamma_L \) defined on \((L^{-1}\delta \mathbb{Z})^3\) has finite range \( 1/2 \). We recall from section 1.3 that a polymer \( X_\delta \) is defined by \( X_\delta = X \cap (\delta \mathbb{Z})^3 \) where \( X \) is a continuum polymer. We define the rescaling of polymer activities by
\[
S_{L,K}(X_{L^{-1}\delta}, \Phi) = K_L(X_{L^{-1}\delta}, \Phi) = K(LX_\delta, S_L \Phi)
\]
(3.22)
We write the fluctuation integration of the polymer activity \( K(X_\delta, \Phi, \xi) \) with respect to \( \mu_G \) as
\[
K^2(X_\delta, \Phi) = \int d\mu_G(\xi)K(X_\delta, \Phi, \xi)
\]
(3.23)
We write the fluctuation integration of the polymer activity \( K(X_{L^{-1}\delta}, \Phi, \xi) \) with respect to \( \mu_{G_L} \) as
\[
K^2(X_{L^{-1}\delta}, \Phi) = \int d\mu_{G_L}(\xi)K(X_{L^{-1}\delta}, \Phi, \xi)
\]
(3.24)
We define
\[
S_L = S_L \mathcal{B}
\]
(3.25)
With these notations it is easy to see that the fluctuation map can be written as
\[
\mathcal{F}(K)(X_{L^{-1}\delta}, \Phi) = (BK)^2(LX_\delta, S_L \Phi) = (S_L K)^2(X_{L^{-1}\delta}, \Phi)
\]
(3.26)

4. THE RENORMALIZATION GROUP MAP APPLIED

In this section we specify the RG map of Section 3 by making choices for the local potential \( \tilde{V} \), and relevant parts \( F, \tilde{V} \) is chosen via first order perturbation theory. \( F \) is chosen so as to remove the expanding part of the fluctuation map. This is the extraction step. This will be done in second order perturbation theory as well as in the error term. We will follow closely the strategy in Section 4 of [BMS]. We will use the notations established in the Appendix to section 3.3, (3.21)-(3.26). We take \( \delta = \delta_n, \Gamma = \Gamma_n \). Recall that, see (1.57),
\[
\alpha = \frac{3\varepsilon}{2}, \quad \text{where we take } 0 < \varepsilon < 1.
\]
The field scaling dimension is \( d_s = \frac{3\varepsilon}{4} \), see (1.58),(1.59).
We assume that starting from the unit lattice where only the local potential (1.31) is present \( n \) steps of the renormalization group map has been carried out. This produces a new local potential
\[ V_n(\Delta_{\delta_n}, \Phi) = V(\Delta_{\delta_n}, \Phi, C_n, g_n, \mu_n) = g_n \int_{\Delta_{\delta_n}} dx : (\Phi \bar{\Phi})^2(x) : C_n + \mu_n \int_{\Delta_{\delta_n}} dx (\Phi \bar{\Phi}) (x) \quad (4.1) \]

together with a polymer activity \( K_n \) supported on connected polymers in \((\delta_n \mathbb{Z})^3\). Note that \((\Phi \bar{\Phi})(x) =: \Phi \bar{\Phi} : C_n (x)\) by virtue of (1.27). We write \( K_n \) in the form

\[ K_n = Q_n e^{-V_n} + R_n \quad (4.2) \]

where \( Q_n \) is a polymer activity which is given by second order perturbation theory in \( g \) assuming that \( \mu \) is \( O(g^2) \). \( Q_n \) is specified below. \( R_n \) is the remainder which is formally of \( O(g^3) \). \( Q_n, R_n \) vanish when \( \Phi = 0 \) by hypothesis. The RG map will preserve this property.

In order to carry through the next step of the RG map as described in Section 3 we must also specify \( \tilde{V}(\Delta_{\delta_n}, \Phi) \). We define

\[ \tilde{V}_n(\Delta_{\delta_n}, \Phi) = V(\Delta_{\delta_n}, \Phi, C_{n+1, L^{-1}}, g_n, \mu_n) = g_n \int_{\Delta_{\delta_n}} dx : (\Phi \bar{\Phi})^2(x) : C_{n+1, L^{-1}} + \mu_n \int_{\Delta_{\delta_n}} d^3x (\Phi \bar{\Phi})(x) \quad (4.3) \]

where we have used the notation \( C_{n+1, L^{-1}} = S_L C_{n+1} \). Here and in what follows we adopt the notations introduced in the Appendix of Section 3.3. Thus \( \bar{\Phi} \) denotes fluctuation integration with respect to the measure \( d\mu_n (\xi) \) and \( \bar{\Phi} \) denotes fluctuation integration with respect to the measure \( d\mu_{n, L} (\xi) \), with \( \Gamma_{n, L} = S_L^{-1} \Gamma_n \).

We recall (see section 3.1) that when we perform the fluctuation integration the fluctuation field \( \xi \) enters \( V \) through \( V(\Delta_{\delta_n}, \Phi + \xi) \) but \( \tilde{V} \) will remain independent of \( \xi \).

We now define \( Q_n \): \( Q_n \) is supported on connected polymers \( X_{\delta_n} \) such that \( |X_{\delta_n}| \leq 2 \). We assume it can be written in the form

\[ Q_n(X_{\delta_n}, \Phi) = Q(X_{\delta_n}, \Phi, C_n, w_n, g_n) = g_n^2 \sum_{j=1}^3 Q^{(j, j)}(X_{\delta_n}, \Phi; C_n, w_n^{(4-j)}) \quad (4.4) \]

where \( w_n = (w_n^{(1)}, w_n^{(2)}, w_n^{(3)}) \) is a triple of integral kernels to be obtained inductively and

\[ \tilde{X}_{\delta_n} = \begin{cases} \Delta_{\delta_n} \times \Delta_{\delta_n} & \text{if } X_{\delta_n} = \Delta_{\delta_n} \\ (\Delta_{\delta_n, 1} \times \Delta_{\delta_n, 2}) \cup (\Delta_{\delta_n, 2} \times \Delta_{\delta_n, 1}) & \text{if } X_{\delta_n} = \Delta_{\delta_n, 1} \cup \Delta_{\delta_n, 2} \\ \emptyset & \text{otherwise} \end{cases} \quad (4.5) \]

\[ Q^{(1, 1)}(\tilde{X}_{\delta_n}, \Phi; C_n, w_n^{(3)}) = -2 \int_{\tilde{X}_{\delta_n}} dx dy (\Phi(x) - \Phi(y)) (\bar{\Phi}(x) - \bar{\Phi}(y)) w_n^{(3)}(x - y) \]

\[ Q^{(2, 2)}(\tilde{X}_{\delta_n}, \Phi; C_n, w_n^{(2)}) = - \int_{\tilde{X}_{\delta_n}} dx dy \left[ (\Phi(x) - \Phi(y)) (\bar{\Phi}(x) - \bar{\Phi}(y)) (\Phi(x) + \Phi(y)) (\bar{\Phi}(x) + \bar{\Phi}(y)) : C_n + \right. \\
+3 : [(\Phi \bar{\Phi})(x) - (\Phi \bar{\Phi})(y)]^2 : C_n \left. \right] w_n^{(2)}(x - y) \]

\[ Q^{(3, 3)}(\tilde{X}_{\delta_n}, \Phi; C_n, w_n^{(1)}) = 4 \int_{\tilde{X}_{\delta_n}} dx dy \left[ \Phi(x) \Phi(x) \Phi(x) \Phi(y) \bar{\Phi}(y) \bar{\Phi}(y) : C_n \right. \\
\left. \right] w_n^{(1)}(x - y) \quad (4.6) \]

Note that in the expression for \( Q^{(1, 1)} \) is equal to its \( C_n \) Wick ordered form because of (1.27). Next we define the second order approximation to the RG map. Let \( p_n \) be the activity supported on unit blocks defined by

\[ p_n(\Delta_{\delta_n}, \xi, \Phi) = V_n(\Delta_{\delta_n}, \xi + \Phi) - \tilde{V}_n(\Delta_{\delta_n}, \Phi) = p_{n, g} + p_{n, \mu} \quad (4.7) \]

where

\[ p_{n, g} = g \int_{\Delta_{\delta_n}} dx : (\xi \bar{\xi})^2 : \gamma_n (x) + 2 \sum_{\alpha} \left[ \Phi_\alpha (x) : \bar{\xi}_\alpha (\xi \bar{\xi}) : \gamma_n (x) + : (\xi \bar{\xi}) \xi_\alpha : \gamma_n (x) \Phi_\alpha (x) \right] \]
+2\langle \Phi \bar{\Phi} \rangle(x) \xi(x) + \langle \Phi \bar{\Phi} \rangle^2(x) + \langle \xi \bar{\Phi} \rangle^2(x) + 2 \sum_{\alpha, \beta} \langle \xi_\alpha \bar{\xi}_\beta \rangle \cdot \gamma_n(x) \cdot \langle \Phi_\alpha \Phi_\beta \rangle(x) :C_{n+1-L-1} +
\nonumber
\nonumber
+2 \sum_\alpha \left[ \xi_\alpha(x) \cdot \Phi_\alpha(\Phi \bar{\Phi}) :C_{n+1-L-1} + \langle \Phi \bar{\Phi} \rangle \cdot \Phi_\alpha \cdot C_{n+1-L-1} (x) \langle \bar{\xi}_\alpha(x) \rangle \right]
\nonumber
\nonumber
p_{n,\mu} = \mu \int_{\Delta_{n,\mu}} dx \left( (\xi \bar{\Phi})(x) + (\Phi \bar{\xi})(x) + (\xi \bar{\xi})(x) \right) \quad (4.8)

In (4.8) we have used a component notation. Thus \Phi_1 = \varphi, \Phi_2 = \psi. Similarly for the fluctuation field \xi; \xi_1 = \zeta, \xi_2 = \eta. \zeta is bosonic (degree 0) and \eta fermionic (degree 1). In deriving (4.8) from (4.7) have used \Gamma_n = \Gamma_n + C_{n+1-L-1} (see (1.50)), the independence of \Phi, \xi in the sense that their components are independent and distributed with covariances \Gamma_n, \Gamma_n respectively. The unordered objects in (4.8) are equal to their Wick ordered form.

We will effectuate the RG map of Section 3 following closely the strategy in [BMS]. Namely, we insert a complex parameter \lambda into our previous definitions in such a way that (i) at \lambda = 1 our \lambda dependent objects correspond with the previous definitions. (ii) The expansion through order \lambda^2 is second order perturbation theory in \lambda n counting \mu_n = O(g_n^2). (iii) Powers of \lambda are determined so as to correspond with leading powers of \lambda n, buried inside polymer activities. (iv) All functions will turn out to be norm analytic in \lambda and this will enable us in section 5 to profit from Cauchy estimates.

We define

\begin{align*}
P_n(\lambda) &= e^{-V_n} \left( -\lambda p_{n,g} - \lambda^2 p_{n,\mu} + \frac{1}{2} \lambda^2 p_{n,\gamma}^2 \right) + \lambda^3 r_{n,1}
\end{align*}
(4.9)

where \lambda r_{n,1} is defined by the condition \lambda P_n(\lambda = 1) = P_n = e^{-V_n} - e^{-V_n}. Similarly, we define

\begin{align*}
K_n(\lambda) &= \lambda^2 e^{-V_n} Q_n + \lambda^3 \left( [e^{-V_n} - e^{-V_n}] Q_n + R_n \right)
\end{align*}
(4.10)

which, for \lambda = 1 coincides with \lambda K_n = e^{-V_n} Q_n + R_n. Corresponding to (3.10) we define

\begin{align*}
\mathcal{B}(\lambda, K_n)(Y_{\delta_n}) &= \sum_{N,M \geq 1} \frac{1}{N! M!} \sum_{(X_{\delta_n,j}(\Delta_{\delta_n,i} - \delta_{\delta_n,j}))} e^{-V_n(X_{\delta_n,0})} \prod_{j=1}^N K_n(\lambda, X_{\delta_n,j}) \prod_{i=1}^M p_n(\lambda, \Delta_{\delta_n,i})
\end{align*}
(4.11)

where \lambda \delta_{\delta_n,j} = \lambda Y_{\delta_n} \setminus (\cup X_{\delta_n,j}) \cup (\cup \Delta_{\delta_n,i}). Let \lambda S(\lambda, K_n) = S_L \lambda \mathcal{B}(\lambda, K_n), where \lambda S_L is the rescaling defined in the last section.

The RG map (see section 3) for \lambda K_n with parameter \lambda is \lambda K_n \mapsto \lambda f_{n+1,K}(\lambda, K_n) = \mathcal{E}(S(\lambda, K_n)^3, F_n(\lambda)), where the superscript \sharp denotes integration over the fluctuation field \xi = (\zeta, \eta) with the measure \lambda \mu_{\Gamma_{n,L}} and \lambda \Gamma_{n,L} is the rescaled covariance \lambda S_L^{-1} \lambda \Gamma_n as in the Appendix to Section 3. The relevant part \lambda F_n(\lambda)) is defined on polymers in (\delta_{n+1,Z})^3 and will be written as

\begin{align*}
F_n(\lambda) &= \lambda^2 F_{Q_n} + \lambda^3 F_{R_n}
\end{align*}
(4.12)

and \lambda F_n(\lambda) = F_n, when \lambda = 1.

**Perturbative contribution to \lambda f_{n+1}**

Given a function \lambda f(\lambda) let

\begin{align*}
T_{\lambda} f = f(0) + f'(0) + \frac{1}{2} f''(0)
\end{align*}
(4.13)

be the Taylor expansion to second order evaluated at \lambda = 1. Then the second order approximation to the RG map is \lambda f_{n+1}^{(2)} = (\lambda f_{n+1,K}^{(2)}, \lambda f_{n+1,V}^{(2)}) with
\[ f^{(\leq 2)}_{n+1,K}(K_n, V_n) = T_\lambda \mathcal{E}(S(\lambda, K_n)^2, F_n(\lambda)) = \mathcal{E}_1(T_\lambda S(\lambda, K_n)^2, F_{Q_n}), \quad f^{(\leq 2)}_{n+1,V}(K_n, V_n) = V^{(\leq 2)}_{n+1} \] (4.14)

where

\[ V^{(\leq 2)}_{n+1} = \hat{V}_{n,L} - \hat{V}_{n,L}(F_{Q_n}) \]

Note also that only the linearized \( \mathcal{E}_1 \) intervenes, because it will turn out that the nonlinear part of extraction generates terms only at order \( \lambda^3 \) or higher.

**Proposition 4.1:** There is a choice of \( F_Q \) such that the form of \( Q \) remains invariant under the RG evolution at second order. In more detail, \( f^{(\leq 2)}_{n+1}(V_n, Q_n e^{-V_n}) = (V^{(\leq 2)}_{n+1}, Q^{(\leq 2)}_{n+1} e^{-\hat{V}_{n,L}}) \) where the parameters in \( Q^{(\leq 2)}_{n+1} = Q(C_{n+1}, \mathbf{w}_{n+1}, g_{n,L}) \), where \( g_{n,L} = L^\varepsilon g_n \), evolved according to

\[ g_{n+1,(\leq 2)} = L^\varepsilon g_n (1 - L^\varepsilon a_n g_n) \quad \mu'_{n+1,(\leq 2)} = L^{2\varepsilon} \mu_n - L^\varepsilon b_n g_n^2 \] (4.15)

The parameters in \( Q^{(\leq 2)}_{n+1} = Q(C_{n+1}, \mathbf{w}_{n+1}, g_{n,L}) \), where \( g_{n,L} = L^\varepsilon g_n \), evolved according to

\[ \mathbf{w}_{n+1} = \mathbf{v}_{n+1} + \mathbf{w}_{n,L} \quad \nu^{(1)}_{n+1} = \gamma_{n,L} \quad \nu^{(p)}_{n+1} = (C_{n,L})^p - (C_{n+1})^p \quad p = 2, 3 \] (4.16)

The constants \( a_n, b_n \) are given by

\[ a_n = 4 \int_{(\delta_{n+1})^3} dy v^{(2)}_{n+1}(y), \quad b_n = 2 \int_{(\delta_{n+1})^3} dy v^{(3)}_{n+1}(y) \] (4.17)

**Proof:** We define a polymer activity \( \hat{Q}_{n,L} \) supported on connected polymers \( X_{\delta_{n+1}} \) with \( |X_{\delta_{n+1}}| \leq 2 \) as follows: if \( |X_{\delta_{n+1}}| = 1 \), say \( X_{\delta_{n+1}} = \Delta_{\delta_{n+1}} \), then

\[ \hat{Q}_{n,L}(\Delta_{\delta_{n+1}}, \xi, \Phi) = \frac{1}{2} (p_{n,L}(\Delta_{\delta_{n+1}}, \xi, \Phi))^2 \]

If \( |X_{\delta_{n+1}}| = 2 \) then

\[ \hat{Q}_{n,L}(X_{\delta_{n+1}}, \xi, \Phi) = \frac{1}{2} \sum_{\Delta_{\delta_{n+1}} \cup \Delta_{\delta_{n+1}} = X_{\delta_{n+1}}} p_{n,L,g}(\Delta_{n+1,1}, \xi, \Phi) p_{n,L,g}(\Delta_{n+1,2}, \xi, \Phi) \] (4.18)

where \( p_{n,L,g} \) is defined by replacing in (4.7) and (4.8) \( (g_n, \mu_n, \Gamma_n, C_{n+1,1}) \) by \( (g_{n,L}, \mu_{n,L}, \Gamma_{n,L}, C_{n+1}) \) with \( g_{n,L} = L^\varepsilon g_n \) and \( \mu_{n,L} = L^{2\varepsilon} \mu_n \).

It is easy to check that

\[ T_\lambda S(K_n, \lambda) = -p_{n,L}e^{-\hat{V}_{n,L}} + (e^{-\hat{V}_{n,L} + \hat{Q}_{n,L}} + e^{-\hat{V}_{n,L} + \hat{Q}_{n,L}}) \] (4.19)

where

\[ Q_{n,L}(X_{\delta_{n+1}}, \xi + \Phi) = Q(X_{\delta_{n+1}}, \xi + \Phi, C_{n,L}, \mathbf{w}_{n,L}, g_{n,L}) \]

Using \( C_{n,L} = \Gamma_{n,L} + C_{n+1} \) and remembering that \( \hat{V} \) depends only on \( \Phi \) we get

\[ (e^{-\hat{V}_{n,L} Q_{n,L}})^2 = e^{-\hat{V}_{n,L} Q(X_{\delta_{n+1}}, \Phi, C_{n+1}, \mathbf{w}_{n,L}, g_{n,L})} \]

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Therefore

\[ T_\lambda S(K_n, \lambda)^\sharp(X_{k_n+1}, \Phi) = e^{-\tilde{V}_{n,L}} \left( Q(X_{k_n+1}, \Phi, C_{n+1}, w_{n,L}, g_{n,L}) + \tilde{Q}(X_{k_n+1}, \Phi, v_{n+1}, C_{n+1}, g_{n,L}) \right) \]  

(4.20)

where \( \tilde{Q} = \tilde{Q}_{n,L}^2 \) is given after a straightforward but lengthy computation by

\[ \tilde{Q}(X_{k_n+1}, \Phi, v_{n+1}, C_{n+1}, g_{n,L}) = 2 \int_{X_{k_n+1}} dxdy \Phi(x)\bar{\Phi}(y) + \Phi(y)\bar{\Phi}(x) \]

(4.21)

where

\[ \tilde{Q}^{(1,1)}(X_{k_n+1}, \Phi, C_{n+1}, u) = 2 \int_{X_{k_n+1}} dxdy \Phi(x)\bar{\Phi}(y) + \Phi(y)\bar{\Phi}(x) \]

(4.22)

\[ \tilde{Q}^{(2,2)}(X_{k_n+1}, \Phi, C_{n+1}, u) = \int_{X_{k_n+1}} dxdy \left\{ \Phi(x)\bar{\Phi}(y) + \Phi(y)\bar{\Phi}(x) \right\} :C_{n+1} \]

\[ + 4 : (\Phi(x)\bar{\Phi}(x))(\Phi(y)\bar{\Phi}(y)) :C_{n+1} \] \( u(x - y) \)

(4.22)

\[ \tilde{Q}^{(3,3)}(X_{k_n+1}, \Phi, C_{n+1}, u) = 4 \int_{X_{k_n+1}} dxdy :\Phi(x)\bar{\Phi}(x)\Phi(x)\bar{\Phi}(y)\Phi(y)\bar{\Phi}(y) :C_{n+1} \] \( u(x - y) \)

Define

\[ F Q_n = \tilde{Q}(C_{n+1}, v_{n+1}, g_{n,L}) - Q(C_{n+1}, v_{n+1}, g_{n,L}) \]  

(4.23)

evaluated on \( X_{k_n+1}, \Phi \).

Then we have from (4.20) and (4.23)

\[ \mathcal{E}_1 (T_\lambda S(\lambda, K_n)^\sharp, F_n) = T_\lambda S(\lambda, K_n)^\sharp - F_n e^{-\tilde{V}_{n,L}} = e^{-\tilde{V}_{n,L}} Q(C_{n+1}, w_{n+1}, g_{n,L}) \]  

(4.24)

which shows that \( Q \) is stable under RG evolution and verifies (4.16). It remains to show that the chosen perturbative relevant part \( F Q_n \) given by (4.23) is of the form (3.15) and thus suitable for extraction.

To compute the difference in (4.23) we will make use of the following localization formulae

\[ \Phi(x)\bar{\Phi}(y) + \Phi(y)\bar{\Phi}(x) = \Phi(x)\bar{\Phi}(x) + \Phi(y)\bar{\Phi}(y) - (\Phi(x) - \Phi(y))(\bar{\Phi}(x) - \bar{\Phi}(y)) \]  

(4.25)

\[ (\Phi(x)\bar{\Phi}(y) + \Phi(y)\bar{\Phi}(x))^2 + 4(\Phi(x)\bar{\Phi}(x))(\Phi(y)\bar{\Phi}(y)) = 4[(\Phi\bar{\Phi})^2(x) + (\Phi\bar{\Phi})^2(y)] - \]

\[ -(\Phi(x) - \Phi(y))(\bar{\Phi}(x) - \bar{\Phi}(y))(\Phi(x) + \Phi(y))(\bar{\Phi}(x) + \bar{\Phi}(y)) - 3[(\Phi\bar{\Phi})(x) - (\Phi\bar{\Phi})(y)]^2 \]  

(4.26)

that are immediate to check. We get
\[ F_{Q_n}(X_{\delta_{n+1}}) = 2g_{n,L}^2 \int_{X_{\delta_{n+1}}} dx dy \left[ (\Phi \Phi)(x) + (\Phi \Phi)(y) \right] v^{(3)}(x-y) + 4g_{n,L}^2 \int_{X_{\delta_{n+1}}} dx dy \left[ (\Phi \Phi)^2 : C_{n+1} (x) + (\Phi \Phi)^2 : C_{n+1} (y) \right] v^{(2)}(x-y) \] 

(4.27)

Note that due to supersymmetry there is no field independent part in \( F_{Q_n} \). We can write \( F_{Q_n}(X_{\delta_{n+1}}) \) as:

\[ F_{Q_n}(X_{\delta_{n+1}}) = \sum_{\Delta_{\delta_{n+1}} \subset X_{\delta_{n+1}}} F_{Q_n}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) \] 

(4.28)

where

\[ F_{Q_n}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) = 4g_{n,L}^2 F^{(2)}_{Q_n}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) + 2g_{n,L}^2 F^{(1)}_{Q_n}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) \] 

(4.29)

and

\[ F^{(m)}_{Q_n}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) = \int_{\Delta_{\delta_{n+1}}} dx : (\Phi \Phi)^{m}(x) : C_{n+1} f_{Q_n}^{(m)}(x, X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) \] 

(4.30)

with

\[
f_{Q_n}^{(m)}(x, X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) = \begin{cases} \int_{\Delta_{\delta_{n+1}}} dy v^{(m)}(x-y) & X_{\delta_{n+1}} = \Delta_{\delta_{n+1}} \\ \int_{\Delta'_{\delta_{n+1}}} dy v^{(m)}(x-y) & X_{\delta_{n+1}} = \Delta_{\delta_{n+1}} \cup \Delta'_{\delta_{n+1}} \text{ connected} \end{cases}
\] 

(4.31)

and \( m' = 4 - m \).

\[ V(F_{Q_n}, \Delta_{\delta_{n+1}}) = \sum_{X_{\delta_{n+1}} \supset \Delta_{\delta_{n+1}}} F_{Q_n}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) = 4g_{n,L}^2 \sum_{X_{\delta_{n+1}} \supset \Delta_{\delta_{n+1}}} F^{(2)}_{Q_n}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) + 2g_{n,L}^2 \sum_{X_{\delta_{n+1}} \supset \Delta_{\delta_{n+1}}} F^{(1)}_{Q_n}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) \] 

(4.32)

where we have used (3.13), (3.14) and (3.15) for the first equality.

In the following we will use the fact that the \( v_{n+1}^{(j)}(x-y) \), \( 1 \leq j \leq 3 \) vanish for \(|x-y| \geq 1\). This follows from the fact that \( \Gamma_{n,L}(x-y) \) appears as a factor in the expression (4.16) for \( v_{\delta_{n+1}}^{(j)}(x-y) \) and \( \Gamma_{n,L} \) has range 1. Returning to (4.32) we have

\[ \sum_{X_{\delta_{n+1}} \supset \Delta_{\delta_{n+1}}} F^{(m)}_{Q_n}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) = \int_{\Delta_{\delta_{n+1}}} dx : (\Phi \Phi)^{m}(x) : C_{n+1} \left[ \int_{\Delta_{\delta_{n+1}}} dy v^{(m')}_{n+1}(x-y) + \right. \]

\[ + \sum_{\Delta'_{\delta_{n+1}} \subset \Delta_{\delta_{n+1}}} \int_{\Delta'_{\delta_{n+1}}} dy v^{(m')}_{n+1}(x-y) \] 

On the r.h.s. use \( v^{(m')}_{n+1}(x-y) = 0 \) for \(|x-y| \geq 1/2\) to extend the sum on \( \Delta'_{\delta_{n+1}} \) to all the \( \Delta'_{\delta_{n+1}} \neq \Delta_{\delta_{n+1}} \). We then get

\[ \sum_{X_{\delta_{n+1}} \supset \Delta_{\delta_{n+1}}} F^{(m)}_{Q_n}(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}) = \int_{\Delta_{\delta_{n+1}}} dx : (\Phi \Phi)^{m}(x) : C_{n+1} \int dy v^{(m')}(x-y) \]
Hence from (4.32) and above we get

\[ V(F_{Q_n}, \Delta_{\delta_{n+1}}) = a_n g_n^2 \int_{\Delta_{\delta_{n+1}}} dx : (\Phi \Phi)^2(x) : c_n + b_n g_n^3 \int_{\Delta_{\delta_{n+1}}} dx (\Phi \Phi)(x) \]  

(4.33)

where

\[ a_n = 4 \int_{(\delta_{n+1})^3} dy v_{n+1}^{(2)}(y) \quad \quad b = 2 \int_{(\delta_{n+1})^3} dy v_{n+1}^{(3)}(y) \]  

(4.34)

\[ \text{Remark: } a_n \text{ and } b_n \text{ are well defined since the } v_{n+1}^j \text{ have compact support. They are positive and their properties are discussed in Lemma 5.12 of Section 5.} \]

The exact RG map \( f_{n+1} \) for \( K_n = Q_n e^{-V_n} + R_n \).

\[ K_n \mapsto K_{n+1} = f_{n+1,K}(\lambda, K_n, V_n) |_{\lambda = 1} = \mathcal{E}(S(\lambda, K_n)^2, F_n(\lambda)) |_{\lambda = 1} \]  

(4.35)

induces an evolution of the remainder \( R_n \) which is studied by Taylor series around \( \lambda = 0 \) with remainder written using the Cauchy formula:

\[ f_{n+1,K}(\lambda = 1) = \sum_{j=0}^{\infty} \frac{f^{(j)}_{n+1,K}(0)}{j!} + \left. \frac{1}{2\pi i} \oint_{\gamma} \frac{d\lambda}{\lambda^j(\lambda - 1)^2} f_{n+1,K}(\lambda) \right|_{\lambda = 1} \]  

The terms \( j = 0, 1, 2 \) are the second order part \( f^{(\leq 2)}_{n+1,K} \). In the \( j = 3 \) term there are no terms mixing \( R_n \) with \( Q_n, P_n \) because of the \( \lambda^3 \) in front of \( R_n \). Therefore it splits \( f^{(3)}_{n+1,K}(0) = R_{n+1,1} + R_{n+1,2} \) into the third order derivative at \( R_n = 0 \), which we write using the Cauchy formula as

\[ R_{n+1,1} \equiv R_{n+1,\text{main}} = \frac{1}{2\pi i} \oint_{\gamma} \frac{d\lambda}{\lambda^2} \mathcal{E} \left( S(\lambda, Q_n e^{-V_n})^2, F_n(\lambda) \right) \]  

(4.36)

and terms linear in \( R_n \):

\[ R_{n+1,2} \equiv R_{n+1,\text{linear}} = (S_1 R_n)^{\frac{3}{2}} - F_{R_n} e^{-V_{\delta_n}} \]

\[ S_1 R_n(Z_{\delta_{n+1}}) = \sum_{X_{\delta_{n+1}}: L^{-1} X_{\delta_{n+1}} = Z_{\delta_{n+1}}} e^{-V_{\delta_n,L}} g_{L^{-1} X_{\delta_{n+1}}} R_{n,L}(L^{-1} X_{\delta_{n+1}}) \]  

(4.37)

The remainder term in the Taylor expansion is

\[ R_{n+1,3} = \frac{1}{2\pi i} \oint_{\gamma} \frac{d\lambda}{\lambda^4(\lambda - 1)^2} \mathcal{E}(S(\lambda, K_n)^2, F_n(\lambda)) \]  

(4.38)

In Proposition 4.1 the coupling constant in \( g^{(\leq 2)}_{n+1} \) is not the same as the coupling constant in \( V^{(\leq 2)}_{n+1} \). Furthermore, the coupling constant in \( V^{(\leq 2)}_{n+1} \) will further change because of the contribution from \( F_n \). To take this into account we introduce

\[ V_{n+1}(\Delta_{\delta_{n+1}}) = V(\Delta_{\delta_{n+1}}, g_{n+1}, \mu_{n+1}) \]

\[ g_{n+1} = L^\varepsilon g_n (1 - L^\varepsilon a_n) + \xi_n(u_n) \]  

(4.39)

\[ \mu_{n+1} = L^\frac{3}{2} \mu_n - L^2 b_n g_n^2 + \rho_n(u_n) \]
where \( u_n = (g_n, \mu_n, R_n) \) and the remainders \( \xi_n(u_n), \rho_n(u_n) \) anticipate the effects of a yet-to-be-specified \( F_{R_n} \). Then we set

\[
R_{n+1,4} = e^{-V_{n+1}} Q(C_{n+1}, \mathbf{w}_{n+1}, g_{n+1}) - e^{-\bar{V}_{n, L}} Q(C_{n+1}, \mathbf{w}_{n+1}, g_{n, L})
\]

(4.40)

and define

\[
\begin{align*}
Q_{n+1} &= Q(C_{n+1}, \mathbf{w}_{n+1}, g_{n+1}) \\
R_{n+1} &= R_{n+1, \text{main}} + R_{n+1, \text{linear}} + R_{n+1, 3} + R_{n+1, 4} \\
K_{n+1} &= Q_{n+1} e^{-V_{n+1}} + R_{n+1}
\end{align*}
\]

(4.41)

With these definitions we have obtained the RG map

\[
f_{n+1,V}(V_n, K_n) = V_{n+1}, \quad f_{n+1,K}(V_n, K_n) = K_{n+1}
\]

(4.42)

**Definition of \( F_{R_n} \)**

To complete the RG step we must specify the relevant part \( F_{R_n} \) from the remainder \( R_n \). The goal is to choose \( F_{R_n} \) so that the map \( R_n \to R_{n+1, \text{linear}} \) will be contractive in the following sense. \( R_n \) is measured in the norm \((2.20)\), and the kernel norm \((2.21)\), with \( \delta = \delta_n \) with a choice of \( h \) and \( h' \) (to be made in section 5). \( R_{n+1} \) is measured in the same norms but on the lattice scale \( \delta_{n+1} \). We will say that the map is contractive if the size of \( R_{n+1, \text{linear}} \) is less than the size of \( R_n \).

\( F_{R_n} \) will have the form given in \((3.15)\) with \( P \) a supersymmetric polynomial vanishing at \( \Phi = 0 \). The coefficients \( \alpha_P \) will be identified via normalization conditions on the small set part of \( R_{n+1, \text{linear}} \). This means that certain derivatives with respect to \( \Phi = (\phi, \psi) \) vanish when \( \Phi = 0 \). That the map in question is contractive when \( R_{n+1, \text{linear}} \) is suitably normalized is proven in Section 5.

For given coefficients \( \tilde{\alpha}_{n,P}(X) \), we define

\[
\tilde{F}_{R_n}(X_{\delta_n}, \Phi) = \sum_P \int_{X_{\delta_n}} dx \tilde{\alpha}_{n,P}(X_{\delta_n}) P(\Phi(x), \partial_{\delta_n} \Phi(x))
\]

(4.43)

\[
\tilde{F}_{R_n}(X_{\delta_n}, \Phi) = 0 : X_{\delta_n} \text{is not a small set}
\]

(4.44)

\( P \) runs over the relevant monomials which in this model are \( P = \Phi \bar{\Phi}, (\Phi \bar{\Phi})^2, \Phi \partial_{\delta_n, \mu} \bar{\Phi}, \partial_{\delta_n, \mu} \Phi \bar{\Phi}, \mu \in S \), with the corresponding coefficients \( \tilde{\alpha}_P(X_{\delta_n}) = \tilde{\alpha}_{n,2,0}(X_{\delta_n}), \tilde{\alpha}_{n,4}(X_{\delta_n}), \tilde{\alpha}_{n,2,1}(X_{\delta_n}, \mu), \tilde{\alpha}_{n,2,1}(X_{\delta_n}, \mu) \). The index set \( S \) was defined in section 2.1 after \((2.1)\). Note that \( P = 1 \) is not a relevant monomial in this model: \( R_n \) vanishes when \( \Phi = 0 \) vanishes by hypothesis. Then \( R_{n}^2(X_{\delta_n}, \Phi) \) vanishes when \( \Phi = 0 \) by supersymmetry, (Lemma 1.1) so that no subtraction is necessary at \( \Phi = 0 \).

Choose the coefficients \( \tilde{\alpha}_{n,P} \) so that

\[
J_n = R_{n}^2 - \tilde{F}_{R_n} e^{-V_n}
\]

(4.45)

is normalized (details are given below). Note that \( J(X_{\delta_n}, 0) = 0 \). We define the relevant part, supported on small sets, by

\[
F_{R_n}(Z_{\delta_{n+1}}, \Phi) = \sum_{\substack{X_{\delta_n+1} \text{small sets} \quad L^{-1} X_{\delta_n+1} L = Z_{\delta_{n+1}}}} \tilde{F}_{R_n,L}(L^{-1} X_{\delta_n+1}, \Phi) = \sum_{\substack{X_{\delta_n} \text{small sets} \quad L^{-1} X_{\delta_n} L = Z_{\delta_n}} \tilde{F}_{R_n}(X_{\delta_n}, S_L \Phi)
\]

(4.46)

\( F_{R_n} \) is supported on small sets by construction. From the definition of \( R_{n+1, \text{linear}} \) in \((4.37)\) we get
\[ R_{n+1, \text{linear}}(Z_{\delta_{n+1}}) = \sum_{X_{\delta_{n+1}} \text{small sets}} e^{-\tilde{V}_L(Z_{\delta_{n+1}} \setminus L^{-1}X_{\delta_{n+1}})} J_{n,L}(L^{-1}X_{\delta_{n+1}}) + \sum_{X_{\delta_{n+1}} \text{large sets}} e^{-\tilde{V}_L(Z_{\delta_{n+1}} \setminus L^{-1}X_{\delta_{n+1}})} J_{n,L}(L^{-1}X_{\delta_{n+1}}) \]

(4.47)

Therefore the first sum in \( R_{\text{linear}} \) is also normalized because normalization as defined below is preserved under multiplication by smooth functionals of \( \Phi \) and rescaling.

Substitution of (4.43) in (4.46) shows that \( F_{R_n} \) is of the form required in (3.15). We have

\[ F_{R_n}(Z_{\delta_{n+1}}, \Phi) = \sum_P \int_{Z_{\delta_{n+1}}} dx \frac{\alpha_n, P(Z_{\delta_{n+1}}, x) P(\Phi(x), \partial_{\delta_{n+1}} \Phi(x))}{\alpha_n, P(Z_{\delta_{n+1}}, x)} \]

(4.48)

where

\[ \alpha_n, P(Z_{\delta_{n+1}}, x) = \sum_{X_{\delta_{n+1}} \text{small set}} \tilde{\alpha}_n, P(X_{\delta_{n+1}}) L^{-[P]+3} 1_{L^{-1}X_{\delta_{n+1}}}(x) \]

(4.49)

In (4.49) \( 1_X \) is the characteristic function of the set \( X \). Note that \( X_{\delta_{n+1}} \) fixes \( X_{\delta_n} \) by restriction by our construction of polymers in section 1.3. \([P]\) is the dimension of the monomial \( P \), \((2kd_s + 2d_n + 1) \) for \( \Phi \partial_{\delta_{n+1}} \Phi \), \( \alpha_n, P(Z_{\delta_{n+1}}, x) \) is supported on small sets \( Z_{\delta_{n+1}} \) and vanishes if \( x \notin Z_{\delta_{n+1}} \).

We now compute \( V_{F_n} \) following (3.13). Define

\[ \alpha_{n, P} := \sum_{Z_{\delta_{n+1}} \supset x} \alpha_n, P(Z_{\delta_{n+1}}, x) \]

(4.50)

This is independent of \( x \) by translation invariance. In fact given an \( x \) it belongs uniquely to a block \( \Delta_{\delta_{n+1}} \), since our blocks which are restrictions of half open continuum cubes are always disjoint (see section 1.3).

The sum over all polymers containing a block \( \Delta_{\delta_{n+1}} \) is independent of \( \Delta_{\delta_{n+1}} \) by translation invariance.

From (4.49) and (4.50) we get

\[ \alpha_{n,P} = L^{-[P]+3} \sum_{X_{\delta_{n+1}} \text{small set}: L^{-1}X_{\delta_{n+1}} \supset x} \tilde{\alpha}_n, P(X_{\delta_{n+1}}) \]

(4.51)

\( \alpha_{n, P} = 0 \) for \( P = \Phi \partial_{\delta_{n+1}} \Phi \) or \( \partial_{\delta_{n+1}} \Phi \Phi \) by reflection invariance of polymer activities.

Therefore

\[ \tilde{V}_L(F_{R_n}, \Delta_{\delta_{n+1}}) = \int_{\Delta_{\delta_{n+1}}} dx \left\{ \alpha_{n,2,0} \Phi \Phi + \alpha_{n,4,0} (\Phi \Phi)^2 \right\} \]

(4.52)

\[ = \int_{\Delta_{\delta_{n+1}}} dx \left\{ \rho_n(u_n) : \Phi \Phi : C_{\delta_{n+1}} + \xi_n(u_n) : (\Phi \Phi)^2 : C_{\delta_{n+1}} \right\} \]

where \( u_n = (g_n, \mu_n, R_n) \) and

\[ \rho_n = \alpha_{n,2,0} + 2C_{\delta_{n+1}}(0) \alpha_{n,4,0} \quad \xi_n = \alpha_{4,0} \]

(4.53)

which are formulas for the error terms in (4.39).

**Normalization conditions**

By an abuse of notation let 1 denote the constant function in \( C^2(X_{\delta_n}) \) equal to 1. Similarly let \( 1^{2p} \) denote the constant function in \( C^2(X_{\delta_n}^{2p}) \) equal to 1. We will identify the \( C^2(X_{\delta_n}) \) function \( f(x) = x_\mu \) with \( x_\mu \)
Note that \( x_\mu \) is defined with respect to an origin which belongs to \( X_{\xi_n} \). Similarly we will identify \( C^2(X^2_{\xi_n}) \) functions \( g_2(x_1, x_2) = x_{1,\mu}, g_2(x_1, x_2) = x_{2,\mu} \) with \( x_{1,\mu}, x_{2,\mu} \) respectively.

Suppose the polymer activity \( J(X_{\xi_n}, \Phi) = J(X_{\xi_n}, \varphi, \psi) \) is of degree 0, gauge invariant and supersymmetric. We have the following identities:

\[
D^{2.0} J(X_{\xi_n}, 0, 0; 1^2) = D^{0.2} J(X_{\xi_n}, 0, 0; 1, 1) \tag{4.54}
\]

\[
D^{2.0} J(X_{\xi_n}, 0, 0; x_{1,\mu}) = D^{0.2} J(X_{\xi_n}, 0, 0; x_{2,\mu}) \tag{4.55}
\]

\[
D^{2.0} J(X_{\xi_n}, 0, 0; x_{2,\mu}) = D^{0.2} J(X_{\xi_n}, 0, 0; 1, x_{\mu}) \tag{4.56}
\]

\[
D^{2.2} J(X_{\xi_n}, 0, 0; 1, 1, 1^2) = 2D^{0.4} J(X_{\xi_n}, 0, 0; 1, 1, 1, 1) \tag{4.57}
\]

\[
D^{4.0} J(X_{\xi_n}, 0, 0; 1^4) = 0 \tag{4.58}
\]

where the field derivatives are taken according to (2.12). The identities (4.54)-(4.57) follow by expanding \( J(X_{\xi_n}, \Phi) \) in the fields, retaining a degree 4 supersymmetric polynomial in \( \Phi \) and \( \partial\xi_n \Phi \) which is all that enters into the computation. Then express it in the Grassmann representation (1.81). (4.58) is trivial. Because \( J \) is of degree 0 the only term that survives for the computation of (4.58) is of the form \( \int_\mathcal{X}_{\xi_n} dx a(x_1, x_2, x_3, x_4) \psi(x_1) \bar{\psi}(x_2) \psi(x_3) \bar{\psi}(x_4) \) where the kernel \( a \) is antisymmetric in \( x_1, x_3 \) and in \( x_2, x_4 \). The integral vanishes if we replace the grassmann piece by \( 1^4 \). Derivatives on the grassmann fields annihilate \( 1^4 \).

We say that a degree 0, gauge invariant, supersymmetric polymer activity \( J(X_{\xi_n}, \Phi) = J(X_{\xi_n}, \varphi, \psi) \) with \( J(X_{\xi_n}, 0) = 0 \) is \textit{normalized} if, for all small sets \( X_{\xi_n} \),

\[
D^{2.0} J(X_{\xi_n}, 0, 0; 1^2) = D^{0.2} J(X_{\xi_n}, 0, 0; 1, 1) = 0
\]

\[
D^{2.0} J(X_{\xi_n}, 0, 0; x_{1,\mu}) = D^{2.0} J(X_{\xi_n}, 0, 0; x_{2,\mu}) = 0
\]

\[
D^{0.2} J(X_{\xi_n}, 0, 0; 1, x_{\mu}) = D^{0.2} J(X_{\xi_n}, 0, 0; x_{\mu}, 1) = 0
\]

\[
2D^{0.4} J(X_{\xi_n}, 0, 0; 1, 1, 1, 1) = D^{2.2} J(X_{\xi_n}, 0, 0; 1, 1, 1^2) = 0 \tag{4.59}
\]

\textit{Determining coefficients from (4.59)}

We will apply the normalization conditions to \( J = J_n \) defined in (4.45). This will determine the dependence of the error terms \( \xi_n, \rho_n \) on \( R_n \). Lemma 5.17 will show that these terms are bounded by the kernel norm of \( R_n \).

In doing the following computations note that \( J_n(X_{\xi_n}, 0, 0) = 0 \) as shown earlier. Moreover the odd derivatives \( D^{0.2} J_n(X_{\xi_n}, 0; f^{x_1}) \), \( j \)-odd integer, vanish identically by gauge invariance. It is enough to take derivatives with respect to the bosonic fields \( \varphi \) because of the identities stated above, (4.54) et seq. Taking derivatives of (4.45) and remembering that \( \tilde{F}_{R_n}(X_{\xi_n}, 0) = 0, \tilde{V}_{n}(X_{\xi_n}, 0) = 0 \) we get

\[
D^{0.2} J_n(X_{\xi_n}, 0, 0; f, \bar{f}) = D^{0.2} R^2_n(X_{\xi_n}, 0, 0; f, \bar{f}) - D^{0.2} \tilde{F}_{R_n}(X_{\xi_n}, 0, 0; f, \bar{f})
\]

\[
D^{0.4} J_n(X_{\xi_n}, 0, 0; f_1, \bar{f}_1, f_2, \bar{f}_2) = D^{0.4} R^4_n(X_{\xi_n}, 0, 0; f_1, \bar{f}_1, f_2, \bar{f}_2) + D^{0.4} \tilde{F}_{R_n}(X_{\xi_n}, 0, 0; f_1, \bar{f}_1, f_2, \bar{f}_2) +
\]

\[\quad + 4D^{0.2} \tilde{F}_{R_n}(X_{\xi_n}, 0, 0; f, \bar{f}) D^{0.2} \tilde{V}_{n}(X_{\xi_n}, 0, 0; f, \bar{f}) \] \tag{4.60}
where the \( f \) are complex valued functions in \( C^2(\Delta_n) \). A variation of \( \varphi \) along \( f \) implies that we vary \( \tilde{\varphi} \) along \( \tilde{f} \). Note that from (4.43)

\[
D^{0.2}F_{\alpha_n}(\Delta_n, 0, 0; 1, 1) = |X_{\Delta_n}||\tilde{\alpha}_{n, 2, 0}(X_{\Delta_n})
\]

\[
D^{0.2}F_{\alpha_n}(\Delta_n, 0, 0; 1, x_\mu) = |X_{\Delta_n}|\tilde{\alpha}_{n, 2, 1}(X_{\Delta_n}, \mu) + \tilde{\alpha}_{n, 2, 0}(X_{\Delta_n}) \int_{X_{\Delta_n}} dx x_\mu
\]

\[
D^{0.2}F_{\alpha_n}(\Delta_n, 0, 0; 0, 1) = |X_{\Delta_n}|\tilde{\alpha}_{n, 2, 1}(X_{\Delta_n}, \mu) + \tilde{\alpha}_{n, 2, 0}(X_{\Delta_n}) \int_{X_{\Delta_n}} dx x_\mu
\]

\[
D^{0.4}F_{\alpha_n}(\Delta_n, 0, 0, 0; 1, 1, 1, 1) = 4|X_{\Delta_n}|\tilde{\alpha}_{n, 4}(X_{\Delta_n})
\]

Now imposing successively the conditions (4.59) we get

\[
\tilde{\alpha}_{n, 2, 0}(X_{\Delta_n}) = \frac{1}{|X_{\Delta_n}|}D^{0.2}R_{\alpha_n}(X_{\Delta_n}, 0, 0; 1, 1, 1)
\]

\[
\tilde{\alpha}_{n, 2, 1}(X_{\Delta_n}, \mu) = \frac{1}{|X_{\Delta_n}|}D^{0.2}R_{\alpha_n}(X_{\Delta_n}, 0, 0; 1, x_\mu) - \frac{1}{|X_{\Delta_n}|}\tilde{\alpha}_{2, 0}(X_{\Delta_n}) \int_{X_{\Delta_n}} dx x_\mu
\]

\[
\tilde{\alpha}_{n, 2, 1}(X_{\Delta_n}, \mu) = \frac{1}{|X_{\Delta_n}|}D^{0.2}R_{\alpha_n}(X_{\Delta_n}, 0, 0; x_\mu, 1) - \frac{1}{|X_{\Delta_n}|}\tilde{\alpha}_{n, 2, 0}(X_{\Delta_n}) \int_{X_{\Delta_n}} dx x_\mu
\]

\[
\tilde{\alpha}_{n, 4}(X_{\Delta_n}) = \frac{1}{4|X_{\Delta_n}|} \left( D^{0.4}R_{\alpha_n}(X_{\Delta_n}, 0, 0; 1, 1, 1, 1) + D^{0.2}V_{\alpha_n}(X_{\Delta_n}, 0, 0; 1, 1, 1, 1) D^{0.2}R_{\alpha_n}(X_{\Delta_n}, 0, 0; 1, 1) \right)
\]

We remind the reader that RG transformations preserve the invariance of polymer activities under translations, reflections, and rotations which leave the lattice invariant.

5. ESTIMATES

Let \( u_n = (g_n, \mu_n, R_n) \). Then \( (w_n, u_n) \) are the coordinates of the measure density in the polymer representation after \( n \) successive applications of the RG map \( f_j \), \( 1 \leq j \leq n \), of Section 4. The \( w_n \) evolve according to \( w_{n+1} = f_{n+1}(w_n) = v_{n+1} + w_{n,L} \) as given in (4.16). This evolution is independent of \( u_n \) and is solved in Lemma 5.9 below. The sequence \( \{w_n, u_n\} \) with \( u_{n+1} = f_{n+1}(u_n) \), where the solution for \( w_n \) is incorporated in the map \( f_n \), is the RG trajectory. The index \( n \) in \( R_n \) also indicates that \( R_n \) is supported on polymers in \( (\Delta_n\mathbb{Z})^3 \). Correspondingly the norms for Banach spaces of polymer activities given in Section 2 are indexed by the lattice spacing \( \Delta_n \). In this section we first set up a uniformly bounded domain \( D_n \) for \( u_n \). The rest of this section is then devoted to the proof of Theorem 5.1 below. This theorem controls the remainders \( (\xi_n, \rho_n) \) in the flow equations (4.39) together with \( R_{n+1} \) in (4.41) when \( u_n \) belongs \( D_n \). It also gives bounds on \( g_{n+1} \) and \( \mu_{n+1} \). Theorem 5.1 will provide essential ingredients for the proof (in Section 6) of existence of an initial choice of the mass parameter such that there is a uniformly bounded RG trajectory at all scales labelled by \( n \).

The aforementioned domain will be a ball defined with Banach space norms with the center of the ball fixed i.e. independent of \( n \). To this end we first obtain an approximate discrete flow of the coupling constant \( g_n \) from the first equation in (4.39) by ignoring the remainder \( \xi_n(g_n, \mu_n, R_n) \). The approximate flow equation has \( n \)-dependent coefficients. However we show below (Lemma 5.12), with no assumption about the domain \( D_n \) given below, that the the positive coefficients \( a_n \) converge geometrically as \( n \to \infty \) to a constant \( a_{c,s} > 0 \). This leads us to set up a reference approximate discrete flow of the coupling constant

\[
g_{c,n+1} = L^c g_{c,n} (1 - L^c a_{c,s} g_{c,n})
\]
This may be thought of as an approximate flow in an underlying continuum theory. This approximate flow has a nontrivial fixed point, namely

\[ \bar{g} = \frac{L^\varepsilon - 1}{L^{2\varepsilon}a_{c,*}} > 0 \]  

(5.2)

The constant \( a_{c,*} = a_{c,*}(L, \varepsilon) \) depends on \( L, \varepsilon \) in such a way that when \( \varepsilon \to 0 \) with \( L \) fixed \( a_{c,*}(L, \varepsilon) \to a_{c,*}(L) \) which depends only on \( L \). We will assume \( L \) large but fixed for the rest of the paper. We then choose \( \varepsilon \) sufficiently small depending on \( L \).

We have

\[ 0 < \bar{g} < C_L \varepsilon \]  

(5.3)

where \( C_L \) is a constant which depends only on \( L \). \( \varepsilon \) is then a measure of smallness of \( \bar{g} \).

In the following \( O(1) \) denotes a constant independent of \( L, \varepsilon \) and \( n \). Constants \( C \) are independent of \( \varepsilon \) and \( n \) but may depend on \( L \). These constants may change from line to line. It will not be necessary to keep track of these changes.

**The Domain \( \mathcal{D}_n \):**

We will say that \( u_n = (g_n, \mu_n, R_n) \) belongs to the domain \( \mathcal{D}_n \) if

\[ |g_n - \bar{g}| < \nu \bar{g}, \quad |\mu_n| < \bar{g}^{-\delta} \]  

(5.4)

\[ ||| R_n |||_n < \bar{g}^{11/4 - \eta} \]  

(5.5)

where the constant \( \nu \) is held in the range \( 0 < \nu < \frac{1}{2} \), and

\[ ||| R_n |||_n = \max\{ |R_n|_{h_{\cdot},A,\delta_n}, \bar{g}^2 \| R_n \|_{h_{G_{\cdot}},A,\delta_n} \} \]  

(5.6)

We choose \( \kappa = \kappa(L) \) as in Lemma 2.1 and \( \rho = \rho(L) \) as in Lemma 5.3 (independent of the domain hypothesis). \( \kappa, \rho \) will be held fixed after \( L \) has been chosen sufficiently large. \( \delta, \eta = O(1) > 0 \) are very small fixed numbers, say \( 1/64 \), and \( h_B = c_{\bar{g}}^{-1/4} \) with \( c = O(1) > 0 \) a very small number. Furthermore we take \( h_{B*} = \rho^{-1/2} + \kappa^{-1/2} \) and choose \( m_0 = 9 \). \( h_F = h_F(L) \) is an \( \varepsilon \) independent constant which depends on \( L \) and is taken to be sufficiently large. (The dependence of \( h_F \) on \( L \) enters in the proofs of Lemma 5.15 and Lemma 5.16 below).

We recall that \( h = (h_B, h_F) \), \( h_* = (h_{B*}, h_F) \).

**Remark 1.** Note that condition (5.5) is equivalent to having both

\[ ||| R_n |||_{h_{G_{\cdot}},A,\delta_n} < \bar{g}^{3/4 - \eta} \]  

(5.7)

\[ |R_n|_{h_{\cdot},A,\delta_n} < \bar{g}^{11/4 - \eta} \]  

(5.8)

2. In [BMS], see equations (5.1)-(5.3) therein, the domain was specified using \( \varepsilon \). In contrast here we specify the domain as in [A] by using \( \bar{g} \) instead of \( \varepsilon \) and moreover we enlarge the domain of \( g_n \) slightly for technical reasons.

Recall the definitions of \( \rho_n(g_n, \mu_n, R_n) \) and \( \xi_n(g_n, \mu_n, R_n) \) from (4.53). We will prove in this section

**Theorem 5.1**

Let \( u_n = (g_n, \mu_n, R_n) \in \mathcal{D}_n \). Let \( L \) be large but fixed followed by \( \varepsilon \) sufficiently small depending on \( L \). \( \bar{g} \) is thus sufficiently small. Let \( u_{n+1} = f_{n+1}(u_n) \) where \( f_{n+1} \) is the RG map of section 4. Then there exist constants \( C_L \) independent of \( n \) and \( \varepsilon \) such that

\[ |\xi_n| \leq C_L \bar{g}^{11/4 - \eta} \]  

(5.9)
\[ |\rho_n| \leq C_L \tilde{g}^{11/4 - \eta} \quad (5.10) \]

\[ |||R_{n+1}|||_{m+1} \leq L^{-1/4} \tilde{g}^{11/4 - \eta} \quad (5.11) \]

\[ |g_{n+1} - \bar{g}| < 2\nu \tilde{g}^{3/2}, \quad |\mu_{n+1}| < O(1) L^{2\delta} \tilde{g}^{2 - \delta} \quad (5.12) \]

**Remark**: The lemmas which follow will serve to prove Theorem 5.1. They are organized as in Section 5 of [BMS]. We remark that Lemmas 5.1-5.4, 5.9, and Lemma 5.12 are independent of the domain hypothesis.

All the other lemmas are under the assumption that \((g_0, \mu, R_n)\) belong to the domain \(D_n\). Lemmas 5.21, 5.22, 5.23 and 5.27 are the major parts of the program. \(R_{n+1, \text{main}}\) is bounded in Lemma 5.21 and this result determines the qualitative form of the bound on the remainder. \(R_{n+1,1}\) and \(R_{n+1,4}\) are seen, in Lemmas 5.22, 5.23 to be negligible in comparison. \(R_{n+1, \text{linear}}\) is the crux of the program and it is bounded in Lemma 5.27.

The remaining Lemmas are auxiliary results on the way to these Lemmas. These auxiliary lemmas implement some of the following principles: Wick constants \(C_n(0)\) are uniformly bounded by constants \(C = C_L\). In bounds by \(G_n, h, \wedge\) norms, a fluctuation field \(\zeta\) contributes a constant \(C = O(1)(\rho(L)\kappa(L))^{-1/2}\) and a field \(\varphi\) contributes a constant \(O(1)\tilde{g}^{-1/4}\). The contributions of these fields as well as of the Grassmann fields \(\psi, \eta\) are controlled by the structure of the norms defined in section 2 (with above choice of \(h, h_1\)) and later in this section ((5.20) et seq). Integration over the Grassman fluctuation fields \(\eta\) is controlled by the Gramm inequality. In bounds by the \(h, \wedge\) norms, fluctuation fields \(\zeta\) contribute a constant \(C = O(1)(\rho(L)\kappa(L))^{-1/2}\) and fields \(\varphi\) contribute \(O(1)\). \(h_{B_n}\) has been adjusted to take care of the constant \(C\) above in the contribution of the fluctuation field.

**Lattice Taylor expansions**

In the following we will have occasion to estimate the difference of lattice fields at two different points of a hypercubic lattice \((\delta_n \mathbb{Z})^d\). Let \(f\) be a lattice function and \(x, y\) be two points in the lattice. We write \(y - x = \sum_{j=1}^{d} \delta_n \varepsilon_j h_j e_j\) where \(h_j \in \mathbb{Z}_+\), \(\varepsilon_j = \text{sign}(y_j - x_j)\) and the \(e_j\) are the unit vectors of the lattice. We will express the difference \(f(y) - f(x)\) as a sum of forward and backward lattice derivatives of \(f\) along segments of a specified lattice path. As usual a forward derivative in the direction \(e_j\) is denoted \(\partial_{\delta_n, e_j}\) and the backward derivative is denoted \(\partial_{\delta_n, -e_j}\). Given \(j \in \{1, 2, \ldots, d\}\), \(s \in \mathbb{Z}_+\) we define \(p_j(x - y, s) \in (\delta_n \mathbb{Z})^d\) by

\[ p_j(y - x, s) = \sum_{i=1}^{j-1} (y - x, e_i) e_i + \delta_n \varepsilon_j s e_j \quad (5.13) \]

with the convention that if \(j = 1\) the sum is empty. Then it is a simple matter to verify that

\[ f(y) - f(x) = \delta_n \sum_{j=1}^{d} \sum_{s_j=0}^{h_j-1} \partial_{\delta_n, e_j} f(x + p_j(y - x, s_j)) \quad (5.14) \]

By iterating (5.14) we get the second order lattice Taylor expansion

\[ f(y) - f(x) = \sum_{j=1}^{d} ((y-x), e_j) \partial_{\delta_n, e_j} f(x) + \delta_n^2 \sum_{j,k=1}^{d} \sum_{s_j=0}^{h_j-1} \sum_{s_k=0}^{h_k-1} \partial_{\delta_n, e_j} \partial_{\delta_n, e_k} f(x + p_j(y-x, s_j), s_k) \quad (5.15) \]

**Lemma 5.1**

Let \(Z_{\delta_n}, X_{\delta_n}\) be connected 1-polymers in \((\delta_n \mathbb{Z})^3\). Let \(Y_{\delta_n} = \emptyset\) or \(Y_{\delta_n} = L^{-1} X_{\delta_n} \subset Z_{\delta_n}\) such that \(\text{vol}(Z_{\delta_n} \setminus Y_{\delta_n})\)
Choose any $\gamma = O(1) > 0$ and $\kappa = \kappa(L) > 0$ as in Lemma 2.1. Let $\bar{g}$ be sufficiently small so that $0 \leq \bar{g} \leq \kappa^2$. Let $\varphi : \bar{Z}_n^{(5)} \to \mathbb{C}$ where $\bar{Z}_n^{(5)}$ is $Z_{\delta_n}$ with 5-collar attached (see (1.76), (1.77)). Then there exists an $O(1)$ constant which depends on $j$ such that

$$
\|\varphi\|_{C^j(Z_{\delta_n})} \leq O(1)^{j} \bar{g}^{-\frac{j}{4}} \int_{Z_{\delta_n} \setminus Y_{\delta_n}} dy |\varphi(y)|^j G_\kappa(Z_{\delta_n}, \varphi)
$$

(5.16)

For $Y_{\delta_n} = \emptyset$ the above bound holds without the factor $2^{[Z]}$.

Proof: This is the lattice analogue of Lemma 5.1 in [BMS]. The proof reposes on the Hölder inequality and the lattice Sobolev inequality (see [BGM], Appendix B for an elementary proof). Let $x \in Z_{\delta_n}$. Write

$$
\varphi(x) = \frac{1}{\text{vol}(Z_{\delta_n} \setminus Y_{\delta_n})} \int_{Z_{\delta_n} \setminus Y_{\delta_n}} dy (\varphi(y) + \varphi(x) - \varphi(y))
$$

and bound

$$
|\varphi(x)| \leq \frac{1}{\text{vol}(Z_{\delta_n} \setminus Y_{\delta_n})} \int_{Z_{\delta_n} \setminus Y_{\delta_n}} dy |\varphi(y)| + \frac{1}{\text{vol}(Z_{\delta_n} \setminus Y_{\delta_n})} \int_{Z_{\delta_n} \setminus Y_{\delta_n}} dy |\varphi(x) - \varphi(y)|
$$

We bound the first term on the right hand side by $O(1)\|\varphi\|_{L^4(Z_{\delta_n} \setminus Y_{\delta_n})}$. To bound the second term we write the difference $\varphi(y) - \varphi(x)$ as a sum of lattice derivatives along the segments of the path as in (5.14). Any connected polymer $Z_{\delta_n}$ as defined in section 1.3 can be represented as $Z_{\delta_n} = Z \cap (\delta_n \mathbb{Z})^3$ where $Z$ is a connected continuum polymer. If $Z_{\delta_n}$ is a block (unit cube) then the path $p_j(y - x, s_j)$ in (5.14) lies entirely in $Z_{\delta_n}$. If $Z_{\delta_n}$ is not a block then it decomposes as a connected union of blocks. If $x, y$ are not in the same block then it suffices to consider the case when they are in adjacent components. We pick a point $z_0$ in the intersection of the closures, write $f(x) - f(y) = (f(x) - f(z_0)) + (f(z_0) - f(y))$ and use the above first order taylor expansion for each summand. The estimates below remain valid. Therefore it is sufficient to consider the case $Z_{\delta_n}$ is a block. From

$$
\varphi(y) - \varphi(x) = \delta_n \sum_{j=1}^3 \sum_{s_j=0}^{h_j-1} \partial_{\delta_n, \epsilon j} \varphi(x + p_j(y - x, s_j))
$$

(5.17)

we get the bound

$$
|\varphi(y) - \varphi(x)| \leq \sum_{j=1}^3 \delta_n |h_j| \sup_{z \in Z_{\delta_n}} |\partial_{\delta_n, \epsilon j} \varphi(z)| \leq 3\delta_n (\max_j |h_j|) \max_{z \in Z_{\delta_n}} |\partial_{\delta_n, \epsilon j} \varphi(z)| \leq O(1) |y - x| \|\varphi\|_{Z_{\delta_n}, 1.5}
$$

(5.18)

where in the last step we have used the lattice Sobolev embedding theorem. We have $|y - x| \leq |Z|$. Putting the above bounds together we get

$$
|\varphi(x)| \leq O(1) |Z| (\|\varphi\|_{L^4(Z_{\delta_n} \setminus Y_{\delta_n})} + \|\varphi\|_{Z_{\delta_n}, 1.5})
$$

We also have for $k = 1, 2$, $|\partial_{\delta_n, \mu_1, \ldots, \mu_k} \varphi(x)| \leq \|\varphi\|_{Z_{\delta_n}, 1.5}$ by Sobolev embedding. Therefore combining with the previous inequality we get

$$
\|\varphi\|_{C^k(Z_{\delta_n})} \leq O(1) |Z| (\|\varphi\|_{L^4(Z_{\delta_n} \setminus Y_{\delta_n})} + \|\varphi\|_{Z_{\delta_n}, 1.5})
$$

Hence

$$
\|\varphi\|_{C^j(Z_{\delta_n})} \leq O(1)^j |Z|^j (\|\varphi\|_{L^4(Z_{\delta_n} \setminus Y_{\delta_n})} + \|\varphi\|_{Z_{\delta_n}, 1.5}) \leq C(j) 2^j |Z| \bar{g}^{-j/4} \int_{Z_{\delta_n} \setminus Y_{\delta_n}} dy |\varphi(y)|^j G_\kappa(Z_{\delta_n}, \varphi)
$$

for some $C(j) > 0$. Therefore

$$
\|\varphi\|_{C^j(Z_{\delta_n})} \leq O(1)^j |Z|^j (\|\varphi\|_{L^4(Z_{\delta_n} \setminus Y_{\delta_n})} + \|\varphi\|_{Z_{\delta_n}, 1.5}) \leq C(j) 2^j |Z| \bar{g}^{-j/4} \int_{Z_{\delta_n} \setminus Y_{\delta_n}} dy |\varphi(y)|^j G_\kappa(Z_{\delta_n}, \varphi)
$$

for some $C(j) > 0$. Therefore
where \( C(j) \) is an \( O(1) \) constant that depends on \( j \). We have used the hypothesis that \( \bar{g} \) is sufficiently small so that \( 0 \leq \bar{g} \leq \kappa^2 \). This proves the bound \( (5.16) \). We now prove the statement following \( (5.16) \). Let \( Y_{\delta_n} = \emptyset \). For \( x \in Z_{\delta_n} \) pick the unit block \( \Delta_{\delta_n} \subset Z_{\delta_n}, \Delta_{\delta_n} \ni x \). We have

\[
|\varphi(x)| \leq \int_{\Delta_{\delta_n}} dy \ |f(y)| + \int_{\Delta_{\delta_n}} dy \ |\varphi(x) - \varphi(y)|
\]

Proceeding as before the first term is bounded by the \( L^4(\Delta_{\delta_n}) \) norm which is less than the \( L^4(Z_{\delta_n}) \) norm. The second term is bounded as before except that since \( x, y \in \Delta_{\delta_n} \) we have \( |x - y| \leq O(1) \). The rest is as before. ■

In effecting the fluctuation map in Section 3.1 we created polymer activities which depended separately on \( \Phi \) and the fluctuation field \( \xi \). The following lemmas will enable us to estimate the contributions of the bosonic fluctuation field \( \zeta \) at various steps. Define a large field regulator for the bosonic fluctuation field \( \zeta : \tilde{X}_{\delta_n}^3 \rightarrow \mathbb{C} \)

\[
\tilde{G}_{\kappa,\rho}(X_{\delta_n}, \zeta) = e^{\rho ||\zeta||^2_{L^2(X_{\delta_n})}} G_{\kappa}(X_{\delta_n}, \zeta), \quad \rho, \kappa > 0
\]  
(5.19)

\( \kappa \) is chosen as in Lemma 2.1 and is held sufficiently small. The choice of \( \rho > 0 \) is dictated by Lemma 5.3 below.

**Lemma 5.2**

*For any \( x \in X_{\delta_n} \)*

\[
|\zeta(x)|^3 \leq C_{\rho,j,\kappa} \tilde{G}_{\kappa,\rho}(X_{\delta_n}, \zeta)
\]  
(5.20)

where \( C_{\rho,j} = (\rho^{-1/2} + \kappa^{-1/2}) O(1) \) and \( O(1) \) depends on \( j \). We have isolated out the \( \rho, \kappa \) dependence in the bound.

**Proof:** The proof follows the lines of the proof of Lemma 5.1 for the case \( Y_{\delta_n} = \emptyset \). Take the unit block \( \Delta_{\delta_n} \subset X_{\delta_n} \) such that \( \Delta_{\delta_n} \ni x \). We replace the \( L^4 \) norm by the \( L^2 \) norm in the appropriate place and estimate \( |\zeta(x) - \zeta(y)| \) with \( x, y \in \Delta_{\delta_n} \) as before now using the regulator \( \tilde{G}_{\kappa,\rho} \). ■

The parameter \( \rho > 0 \) is chosen such that the following Lemma 5.3 holds. \( \rho \) depends on \( L \).

**Lemma 5.3:** Let \( \kappa > 0 \) be chosen as in Lemma 2.1. Then there exists \( \rho_0 = \rho_0(L) > 0 \) independent of \( n \) such that for all \( \rho, 0 < \rho \leq \rho_0 \)

\[
\int d\mu_{\Gamma_n}(\zeta) \tilde{G}_{\kappa,\rho}(X_{\delta_n}, \zeta) \leq 2^{\bar{j}|X_{\delta_n}|}
\]  
(5.21)

Lemma 5.3 is proved in the same way as Lemma 2.1.

We introduce a new intermediate large field regulator which combines the ones introduced earlier

\[
\hat{G}_{\kappa,\rho}(X_{\delta_n}, \zeta, \varphi) = G_{\kappa}(X_{\delta_n}, \zeta + \varphi) G_{\kappa}(X_{\delta_n}, \varphi) \tilde{G}_{\kappa,\rho}(X_{\delta_n}, \zeta)
\]  
(5.22)

**Lemma 5.4:** Let \( \kappa, \rho \) be chosen as in Lemma 2.1 and Lemma 5.3 respectively. Then we have

\[
\int d\mu_{\Gamma_n}(\zeta) \hat{G}_{\kappa,\rho}(X_{\delta_n}, \zeta, \varphi) \leq 2^{\bar{j}|X_{\delta_n}|} G_{3c}(X_{\delta_n}, \varphi)
\]  
(5.23)

**Proof:** The proof follows from an application of the Hölder inequality and Lemmas 2.1, 5.3.

**Intermediate norms:**

We will set up some additional norms to help us control intermediate steps where we encounter polymer activities which are functions of the four separate fields \( \varphi, \zeta, \psi, \eta \). These norms supplement the basic norms
We define the norms

In the beginning of this section we specified $\Omega^0(X_{\delta_n})$ denotes the subalgebra of degree 0 elements. Note that $\Omega^0(X_{\delta_n}) \subset \Omega^0(X_{\delta_n})$. Consider any polymer activity $\tilde{K}(X_{\delta_n}, \varphi, \zeta, \psi, \eta) \in \Omega^0(X_{\delta_n})$. Let $I \subset \{1, 2, \ldots, p\}$ and $J \subset \{1, 2, \ldots, p\}$. Define $I_\ast = \{1, 2, \ldots, p\} \setminus I$. We introduce the abbreviated notation $x_I := (x_{i_1}, \ldots, x_{i_I})$ where for $i_j \in I$ for $j = 1, \ldots, |I|$. We will refer to the $x_{i_j}$ as the members of $x_I$. Define $\psi(x_I) := \psi(x_{i_1}, \ldots, x_{i_I})$, and $\frac{\partial}{\partial \psi(x_I)} := \prod_{j=0}^{|I|-1} \frac{\partial}{\partial \psi(x_{i_{|I|-j}})}$. We now define

$$D^{2p,I,J}_F \tilde{K}(X_{\delta_n}, \varphi, \zeta, x_I, x_{I'}, y_J, y_{J'}) := \frac{\partial}{\partial \tilde{K}(y_{J'})} \frac{\partial}{\partial \psi(x_{I'})} \tilde{K}(X_{\delta_n}, \varphi, \zeta, \psi, \eta) \bigg|_{\psi = \eta = 0}$$

(54.24)

Note that the left hand side is antisymmetric respectively in the members of $x_I, x_{I'}$ and $y_J, y_{J'}$. Let $x := (x_I, x_{I'})$ and $y := (y_J, y_{J'})$. Then $\tilde{K}(X_{\delta_n}, \varphi, \zeta, \psi, \eta)$ can be represented uniquely as

$$\tilde{K}(X_{\delta_n}, \varphi, \zeta, \psi, \eta) = \sum_{p \geq 0, J \subset \{1, \ldots, p\}} \frac{1}{J!} \frac{\partial}{\partial x_{|J|}} \frac{\partial}{\partial y_{|J'|}} \int_{X_{\delta_n}^2} \frac{dxdy}{D^{2p,I,J}_F} \tilde{K}(X_{\delta_n}, \varphi, \zeta, x_I, x_{I'}, y_J, y_{J'}) \times$$

(54.25)

$$\psi(x_I) \psi(y_J) \eta(x_{I'}) \eta(y_{J'})$$

Let $g_{IJ} : (\tilde{X}_{\delta_n}^{(2)})^{|I|} \times (\tilde{X}_{\delta_n}^{(2)})^{|J|} \to C$ such that $g_{IJ}(x_I, y_J)$ is antisymmetric respectively in the members of $x_I$ and those of $y_J$. Let $h_{I', J'} : (\tilde{X}_{\delta_n}^{(2)})^{|I'|} \times (\tilde{X}_{\delta_n}^{(2)})^{|J'|} \to C$ such that $h_{I', J'}(x_{I'}, y_{J'})$ is antisymmetric respectively in the members of $x_{I'}$ and those of $y_{J'}$. The tensor product $g_{IJ} \otimes h_{I', J'}$ maps $(\tilde{X}_{\delta_n}^{(2)})^{|I|} \times (\tilde{X}_{\delta_n}^{(2)})^{|J|} \times (\tilde{X}_{\delta_n}^{(2)})^{|I'|} \times (\tilde{X}_{\delta_n}^{(2)})^{|J'|} = (\tilde{X}_{\delta_n}^{(2)})^{2p} \to C$. By definition $(g_{IJ} \otimes h_{I', J'})(x_I, x_{I'}, y_J, y_{J'}) = g_{IJ}(x_I, y_J)h_{I', J'}(x_{I'}, y_{J'})$. We consider the space of functions $g_{IJ}$ endowed with the $C^2((X_{\delta_n})^{|I|} \times (X_{\delta_n})^{|J|})$ norm. Similarly we consider the space of functions $h_{I', J'}$ endowed with the $C^2((X_{\delta_n})^{|I'|} \times (X_{\delta_n})^{|J'|})$ norm. Define

$$D^{2p,I,J,m}_F \tilde{K}(X_{\delta_n}, \varphi, \zeta, 0, 0; f^m, g_{IJ} \otimes h_{I', J'} = \int_{X_{\delta_n}^2} \frac{dxdy}{D^m_B D^{2p,I,J}_F} \tilde{K}(X_{\delta_n}, \varphi, \zeta, x_I, x_{I'}, y_J, y_{J'}; f^m)$$

(54.26)

where the derivative $D^m_B$ of the bosonic coefficient is with respect to the field $\varphi$ (and not the fluctuation field $\zeta$). This defines a multilinear functional on the normed subspace of antisymmetric functions in $C^2((X_{\delta_n})^{|I|} \times (X_{\delta_n})^{|J|}) \times C^2((X_{\delta_n})^{|I'|} \times (X_{\delta_n})^{|J'|})$.

The norm of the multilinear functional (54.26) is defined analogously to (1.26), namely

$$\|D^{2p,I,J,m}_F \tilde{K}(X_{\delta_n}, \varphi, \zeta, 0, 0)\| = \sup_{\|f\|_{L^2(X_{\delta_n})} \leq 1, \|g_{IJ}\|_{C^2(X_{\delta_n}^{|I|} \times (X_{\delta_n}^{|J|})} \leq 1} \left| \int_{X_{\delta_n}^2} \frac{dxdy}{D^m_B D^{2p,I,J}_F} \tilde{K}(X_{\delta_n}, \varphi, \zeta, x_I, x_{I'}, y_J, y_{J'}; f^m) (g_{IJ} \otimes h_{I', J'})(x_I, x_{I'}, y_J, y_{J'}) \right|$$

(54.27)

In the beginning of this section we specified $h = (h_F, h_B)$ and $h_\ast = (h_F, h_B, h_\ast)$. We define the norms

$$\|\tilde{K}(X_{\delta_n}, \varphi, \zeta, 0, 0)\|_h = \sum_{p=0}^{m_0} \sum_{m=0}^{m} \sum_{J \subset \{1, \ldots, p\}} \sum_{J \subset \{1, \ldots, p\}} \frac{h^m_B}{m!} \frac{h^{2p}_F}{|I|! |J|! |I'|! |J'|!} \|D^{2p,I,J,m}_F \tilde{K}(X_{\delta_n}, \varphi, \zeta, 0, 0)\|_h$$

(54.28)
\[ \| \tilde{K}(X_{\delta_{n}}) \|_{h, \tilde{G}_{n, \rho}} = \sup_{\varphi, \zeta \in \mathcal{F}_{\delta_{n}}^{(5)}} \| \tilde{K}(X_{\delta_{n}}, \varphi, \zeta, 0, 0) \|_{h, \tilde{G}_{n, \rho}^{-1}}(X_{\delta_{n}}, \varphi, \zeta) \]  
\hspace{1cm} (5.29)

\[ \| \tilde{K}(X_{\delta_{n}}, 0, \zeta, 0, 0) \|_{h} = \sum_{p=0}^{\infty} \sum_{m=0}^{m_{p}} \sum_{J \subset \{1, \ldots, p\}} \frac{h_{m, p}^{2p}}{m!} \frac{h_{2p}^{2p}}{|I|! |J|! |\left| J \right| !} \| D^{2p, I, J, m} \tilde{K}(X_{\delta_{n}}, 0, \zeta, 0, 0) \| \]  
\hspace{1cm} (5.30)

\[ \| \tilde{K}(X_{\delta_{n}}) \|_{h, \tilde{G}_{n, \rho}} = \sup_{\zeta \in \mathcal{F}_{\delta_{n}}^{(5)}} \| \tilde{K}(X_{\delta_{n}}, 0, \zeta, 0, 0) \|_{h, \tilde{G}_{n, \rho}^{-1}}(X_{\delta_{n}}, \zeta) \]  
\hspace{1cm} (5.31)

\[ |\tilde{K}(X_{\delta_{n}})|_{h_{*}} = \sum_{p=0}^{\infty} \sum_{m=0}^{m_{p}} \sum_{J \subset \{1, \ldots, p\}} \frac{h_{m, p}^{m_{p}}}{m!} \frac{h_{2p}^{2p}}{|I|! |J|! |\left| J \right| !} \| D^{2p, I, J, m} \tilde{K}(X_{\delta_{n}}, 0, 0, 0, 0) \| \]  
\hspace{1cm} (5.32)

It is straightforward to prove that the above \( h \) and \( h_{*} \) norms satisfy the multiplicative property of Proposition 2.3.

**Special case:** Consider the map \( \hat{\Omega}^{0}(X_{\delta_{n}}) \rightarrow \Omega^{0}(X_{\delta_{n}}) \) given by

\[ \hat{K}(X_{\delta_{n}}, \varphi, \zeta, \psi, \eta) = K(X_{\delta_{n}}, \varphi + \zeta, \psi + \eta) \]  
\hspace{1cm} (5.33)

Norms for \( K \) were defined earlier earlier in section 2, see (2.12)-(2.18). On the other hand the \( h \) and \( h_{*} \) norms of \( \hat{K} \) are defined in (5.28), (5.30) above. We have

**Lemma 5.4A:**

Define \( h := (\frac{\partial}{\partial t}, h_{B}) \) and \( h_{*} := (\frac{\partial}{\partial t}, h_{B*}) \). Then we have for the polymer activity defined in (5.33)

\[ \| \hat{K}(X_{\delta_{n}}, \varphi, \zeta, 0, 0) \|_{h} \leq \| K(X_{\delta_{n}}, \varphi + \zeta, 0, 0) \|_{h} \]  
\hspace{1cm} (5.34)

\[ \| \hat{K}(X_{\delta_{n}}, 0, \zeta, 0, 0) \|_{h_{*}} \leq \| K(X_{\delta_{n}}, \zeta, 0, 0) \|_{h_{*}} \]  
\hspace{1cm} (5.35)

**Proof:** Let \( I \subset \{1, \ldots, p\} \) and \( J \subset \{1, \ldots, p\} \). From the definitions (5.24) and (1.82) we have upto a sign factor

\[ D_{B}^{m} D_{F}^{2p, I, J} \hat{K}(X_{\delta_{n}}, \varphi, \zeta, x_{I}, x_{I'}, x_{J}, y_{J}, y_{J'}, f^{x}, f^{y}) = (-1)^{I} D_{B}^{m} D_{F}^{2p} K(X_{\delta_{n}}, \varphi + \zeta, x_{I}, x_{I'}, y_{J}, y_{J'}, f^{x}, f^{y}) \]  
\hspace{1cm} (5.36)

Let \( (g_{I, J} \otimes h_{I'}, J')(x_{I}, x_{I'}, y_{J}, y_{J'}) \) be a test function as defined after (5.25). It is antisymmetric respectively in the members of \( x_{I}, x_{I'}, y_{J}, y_{J'} \). From (5.26) and (5.36) we get

\[ D_{B}^{m} D_{F}^{2p, I, J, m} \hat{K}(X_{\delta_{n}}, \varphi, \zeta, 0, 0, f^{x}, g_{I, J}) = (-1)^{I} \int_{X_{\delta_{n}}^{2p}} dx_{I_{1}} \cdots dx_{I_{p}} D_{B}^{m} D_{F}^{2p} K(X_{\delta_{n}}, \varphi + \zeta, x_{I}, x_{I'}, y_{J}, y_{J'}, f^{x}, f^{y}) \times \]

\[ (g_{I, J} \otimes h_{I', J'})(x_{I}, x_{I'}, y_{J}, y_{J'}) \]  
\hspace{1cm} (5.37)

Let \( (x_{I}, x_{I'}) = (x_{1}, \ldots, x_{p}) \) and \( (y_{J}, y_{J'}) = (y_{1}, \ldots, y_{p}) \). Write \( (g_{I, J} \otimes h_{I', J'})(x_{I}, x_{I'}, y_{J}, y_{J'}) = (g_{I, J} \otimes h_{I', J'})(x_{1}, \ldots, x_{p}, y_{1}, \ldots, y_{p}) \). Let \( S_{p} \) be the permutation group of \( \{1, \ldots, p\} \). Define

\[ A(g_{I, J} \otimes h_{I', J'})(x_{1}, \ldots, x_{p}, y_{1}, \ldots, y_{p}) = \frac{1}{(p!)^{2}} \sum_{\sigma, \sigma' \in S_{p}} (g_{I, J} \otimes h_{I', J'})(x_{\sigma(1)}, \ldots, x_{\sigma(p)}, y_{\sigma'(1)}, \ldots, y_{\sigma'(p)}) \]
Now the coefficient function $D_{ij}^1 D_{k}^2 K(X_{\delta_n}, \varphi + \zeta, x_I, x_I; y_I, y_J; f^{x,m})$ is antisymmetric in $(x_I, x_{I'} = (x_1, ..., x_p)$ and in $(y_I, y_{J'}) = (y_1, ..., y_p)$. Therefore we can replace $g_{ij} \otimes h_{I'j}$ in (5.37) by $A(g_{ij} \otimes h_{I'j})$ and hence

$$
\left| D_{ij}^1 D_{k}^2 \tilde{K}(X_{\delta_n}, \varphi, \zeta, 0; f^{x,m}, g_{p,IJ}) \right| \leq \| D_{ij}^1 D_{k}^2 K(X_{\delta_n}, \varphi + \zeta, 0) \| \prod_{j=1}^m \| f_j \|_{C^2(X_{\delta_n})} \| A(g_{ij} \otimes h_{I'j}) \|_{C^2(X_{\delta_n}^2)}
$$

Now $\| A(g_{ij} \otimes h_{I'j}) \|_{C^2(X_{\delta_n}^2)} \leq \| g_{ij} \|_{C^2((X_{\delta_n})^{ij} \times (X_{\delta_n})^{ij})} \| h_{I'j} \|_{C^2((X_{\delta_n})^{ij} \times (X_{\delta_n})^{ij})}$. Using this in the previous inequality gives

$$
\| D_{ij}^1 D_{k}^2 \tilde{K}(X_{\delta_n}, \varphi, \zeta, 0, 0) \| \leq \| D_{ij}^1 D_{k}^2 K(X_{\delta_n}, \varphi + \zeta, 0) \| (5.38)
$$

Therefore

$$
\| \tilde{K}(X_{\delta_n}, \varphi, \zeta, 0, 0) \|_h \leq \sum_{p=0}^\infty \sum_{m=0}^{m_0} \sum_{I \subset \{1, ..., p \}} \sum_{J \subset \{1, ..., p \}} \left( \left\| \frac{2^{-2p} h_{ij}^2 p^2}{m! |I||J||J'!|} \right\| \| D_{ij}^1 D_{k}^2 K(X_{\delta_n}, \varphi, 0) \| \right) (5.39)
$$

Now

$$
\sum_{I \subset \{1, ..., p \}} \frac{1}{I!|J||J'|!} = \frac{1}{(p!)^2} 2^{2p}
$$

Substituting this in the previous inequality and using the definition (2.17) gives

$$
\| \tilde{K}(X_{\delta_n}, \varphi, \zeta, 0, 0) \|_h \leq \| K(X_{\delta_n}, \varphi + \zeta, 0) \|_h
$$

which proves (5.34). The proof of (5.35) is the same. ■

**Lemma 5.5**

Let $g_n, \mu_n$ belong to $D_n$. Let $Y_{\delta_n}$ be a 1-polymer. Then for $V_n(Y_{\delta_n}, \Phi, \xi) = V(Y_{\delta_n}, \Phi + \zeta, C_n, g_n, \mu_n)$ or $V(Y_{\delta_n}, \Phi, S_L C_{n+1}, g_n, \mu_n)$, we have

$$
\| e^{-V_n(Y_{\delta_n}, \Phi, \zeta)} \|_h \leq 2 |Y_{\delta_n}| e^{-\widetilde{\Phi} \int_{Y_{\delta_n}} dx \varphi(x)} (5.40)
$$

$$
\| e^{-V_n(Y_{\delta_n}, \Phi, \zeta)} \|_{h_*} \leq 2 |Y_{\delta_n}| (5.41)
$$

for $\varepsilon > 0$ sufficiently small depending on $L$. In the above norms $h = (h_B, h_F)$ and $h_* = (h_{Bs}, h_F)$ are chosen as in the hypothesis for the domain $D_{\delta_n}$. Thus $h_F = h_F(L)$, $h_B = cY^{-\frac{3}{4}}$ with $c = O(1)$ sufficiently small, and $h_{Bs} = \rho^{-1/2} + \kappa^{-1/2}$. Note that $h_{Bs}$ depends on $L$ via $\rho$ and $\kappa$.

**Proof**: It is sufficient to prove this when $Y_{\delta_n}$ is a 1-block $\Delta_{\delta_n}$. Because otherwise we can write $Y_{\Delta_n}$ as a disjoint union of 1-blocks and write the left hand side as a product over 1-block contributions. Then the multiplicative property of the $h$ norm (Proposition 2.3) gives the lemma.

From the definition of $V$ in (4.1) we get on undoing the Wick ordering, (see (1.28)),

$$
V_n(\Delta_{\delta_n}, \Phi) = V_{n,u}(\Delta_{\delta_n}, \varphi) + 2g_n \int_{\Delta_{\delta_n}} dx \varphi(x) \bar{\psi}(x) + \bar{\mu}_n \int_{\Delta_{\delta_n}} dx \bar{\psi}(x) (5.42)
$$

where
\[ V_{u,n}(\Delta_{s_n}, \varphi) = \int_{\Delta_{s_n}} dx [g_n(\varphi \bar{\varphi}(x))^2 + \bar{\mu}_n \varphi \bar{\varphi}(x)] \]  

(5.43)

and \( \bar{\mu}_n = \mu_n - 2g_n C_n(0) \).

By the multiplicative property of the \( h \) norm,

\[ \| e^{-V(\Delta_{s_n}, \Phi)} \|_h \leq \| e^{-V_{u,n}(\Delta_{s_n}, \varphi)} \|_h \| e^{-2g_n \int_{\Delta_{s_n}} dx \varphi \bar{\varphi}(x) \psi \bar{\psi}(x)} \|_h \| e^{-\bar{\mu}_n \int_{\Delta_{s_n}} dx \psi \bar{\psi}(x)} \|_h \]  

(5.44)

We estimate each of the factors on the right hand side in turn. We observe that by taking \( \varepsilon \) sufficiently small we can make \( \bar{g} \) as small as necessary since \( 0 < \bar{g} \leq C \varepsilon \). Since \( g_n, \mu_n \) belong to \( D_n \) and \( 0 < \nu < \frac{1}{2} \), we have \( \frac{\varepsilon}{2} \leq g_n \leq \frac{\bar{g}}{2} \) and \( \mu = O(\bar{g}^2 - \delta) \). Moreover from from (5a) of Theorem 1.1 and (1.52) we have the uniform bound \( |C_n(0)| \leq C_L \). Therefore \( |\bar{\mu}_n| \leq C_L \bar{g} \) with a new constant \( C_L \).

For the first factor on the right hand side of (5.44) we have the bound

\[ \| e^{-V_{u,n}(\Delta_{s_n}, \varphi)} \|_h \leq 2^{\frac{\varepsilon}{2}} |\Delta_{s_n}| e^{-\frac{\varepsilon}{4} \int_{\Delta_{s_n}} dx |\varphi|^4} \]  

(5.45)

where \( |\Delta_{s_n}| = 1 \). This can be proved on the lines of the proof of Lemma 5.5 of [BMS] by substituting there \( \bar{g} \) for \( \varepsilon \) and taking account of the previous observations. Thus

\[ V_{u,n}(\Delta_{s_n}, \varphi) - \frac{g_n}{2} \int_{\Delta_n} dx |\varphi|^4 \geq \frac{\bar{g}}{4} \int_{\Delta_n} dx (|\varphi|^4 - C_L |\varphi|^2) \]

Now \( C_L |\varphi|^2 \leq \frac{1}{2} (|\varphi|^4 + C_L^2) \). Let \( \bar{g} \) be sufficiently small so that \( \bar{g} C_L^2 \leq \frac{\bar{g}}{2} \). Using these two observations we get from the previous inequality

\[ e^{-V_{u,n}(\Delta_{s_n}, \varphi)} \leq (1 + O(\bar{g}^2))(e^{-\frac{\varepsilon}{4} \int_{\Delta_n} dx |\varphi|^4} \]

The rest of the proof which consists of estimating, for \( k \geq 1 \), \( \frac{\varepsilon}{4} \| D^k e^{-V_{u,n}} \| \) goes through as in the proof of Lemma 5.5 of [BMS] on replacing \( \varepsilon \) by \( \bar{g} \).

Now consider the second factor in the right hand side of (5.44). From the multiplicative property of the \( h \) norm applied to the series expansion of the exponential we get

\[ \| e^{-2g_n \int_{\Delta_{s_n}} dx \varphi \bar{\varphi}(x) \psi \bar{\psi}(x)} \|_h \leq e^{2g_n \frac{\bar{h}_B^2}{2} \int_{\Delta_{s_n}} dx \varphi \bar{\varphi}(x)} \]  

Now \( g_n \leq \frac{\bar{g}}{2} \) and \( \bar{g}^2 = O(1) \bar{h}_B^2 \). Let \( t = \frac{|\varphi(x)|}{\bar{h}_B} \). Then

\[ 2g_n \frac{\bar{h}_B^2}{2} \| \varphi \bar{\varphi}(x) \|_h \leq O(1) \frac{\bar{h}_B^2}{t^2} (t^2 + t + 1) \leq O(1) \frac{\bar{h}_B^2}{t^2} (t^4 + 1) \]

which can be proved by two applications of Hölder’s inequality. and therefore for the second factor we have the bound

\[ \| e^{-2g_n \int_{\Delta_{s_n}} dx \varphi \bar{\varphi}(x) \psi \bar{\psi}(x)} \|_h \leq 2 \frac{\bar{h}_B^2}{t^2} (t^4 + 1) \]  

(5.46)

Finally for the third factor we have straightforwardly the bound

\[ \| e^{-\bar{\mu}_n \int_{\Delta_{s_n}} dx \psi \bar{\psi}(x)} \|_h \leq 2 \bar{h}_B^3 C_L |\Delta_{s_n}| \]  

(5.47)

where we have used the bound \( |\bar{\mu}_n| \leq C_L \bar{g} \) (see above).

Put together the bounds for the three factors. In the bound (5.45) use \( g_n \geq \frac{\bar{g}}{2} \). Let \( \bar{g} \) be sufficiently small (thus making \( \bar{h}_B \) sufficiently large) so that \( \max(O(1), C_L)(\bar{h}_B/h_B)^2 \leq \frac{1}{16} \) where \( O(1) \) is the constant in
(5.46) and $C_L$ is the constant encountered above. This ensures that in the bound (5.46) the exponent $O(1)(h_F/h_B)^2 \leq \frac{1}{16}$. It also ensures that in the bound (5.47) the exponent $h_F^2 C_L \bar{g} \leq \frac{1}{16}$. Thus we have obtained for $\bar{g}$ sufficiently small

$$\| e^{-V(\Delta_{\delta_n},\Phi)} \|_{\bar{g}} \leq 2^{(\Delta_{\delta_n})} e^{-\bar{g}/8 \int_{\Delta_{\delta_n}} dx (\varphi \bar{\varphi})^2(x)}$$

and the first part of the lemma now follows on invoking the argument in the beginning of the proof. The proof of the second part follows the same lines. 

\textbf{Lemma 5.6}

Let $p_{n,g}(\Delta_{\delta_n},\xi,\Phi)$ and $p_{n,\mu}(\Delta_{\delta_n},\xi,\Phi)$ be as given in (4.8). Let $g_n,\mu_n$ belong to $D_n$. Let $h_B = e^{-1/4}$ and $h_B$, be as in the definition of $D_n$. Recall that $h = (h_B,h_F)$, and $h_0 = (h_B,h_F), \text{where} h_F = h_F(L)$. Let $\kappa = \kappa(L) > 0$ and $\rho = \rho(L) > 0$, be as specified in Lemmas 2.1 and 5.3. Then for any $\gamma = O(1) > 0, 0 \leq s < 1$ we have constants $C_L$ independent of $n$ and $\varepsilon$ but depending on $L$ such that

$$\| p_{n,g}(\Delta_{\delta_n},\varphi,\zeta,0,0) \|_{\bar{g}} \leq C_L g^{-1/4}(1-s)^{-3/4} \bar{G}_{\kappa,\rho}(\Delta_{\delta_n},\zeta) G_{\kappa}(\Delta_{\delta_n},\varphi) e^{\bar{g} \gamma(1-s)} \int_{\Delta_{\delta_n}} dx (\varphi \bar{\varphi})^2(x)$$

(5.48)

$$\| p_{n,\mu}(\Delta_{\delta_n},\varphi,\zeta,0,0) \|_{\bar{g}} \leq C_L g^{7/4-\delta}(1-s)^{-1/2} \bar{G}_{\kappa,\rho}(\Delta_{\delta_n},\zeta) G_{\kappa}(\Delta_{\delta_n},\varphi) e^{\bar{g} \gamma(1-s)} \int_{\Delta_{\delta_n}} dx (\varphi \bar{\varphi})^2(x)$$

(5.49)

$$\| p_{n,g}(\Delta_{\delta_n},0,\zeta,0,0) \|_{\bar{g}} \leq C_L \bar{g} \bar{G}_{\kappa,\rho}(\Delta_{\delta_n},\zeta)$$

(5.50)

$$\| p_{n,\mu}(\Delta_{\delta_n},0,\zeta,0,0) \|_{\bar{g}} \leq C_L \bar{g}^{2-\delta} \bar{G}_{\kappa,\rho}(\Delta_{\delta_n},\zeta)$$

(5.51)

\textbf{Proof :} $p_{n,g}(\Delta_{\delta_n},\xi,\Phi)$ is given in (4.7). We undo the Wick ordering which produces constants $C_n'(0)$ uniformly bounded by constant $C_L$ from Corollary 1.2. We can then write it in the form (5.25) by expanding out in the Grassmann fields. Since it is a local polynomial of degree four we get,

$$p_{n,g}(\Delta_{\delta_n},\varphi,\zeta,\psi,\eta) = \sum_{p=0}^{2} \sum_{a} \sum_{l \subset \{1,\ldots,2p\}} \int_{\Delta_{\delta_n}} dx p_{\Delta_{\delta_n},2p}^{0,a,l}(\Delta_{\delta_n},\varphi,\zeta,\psi,\eta) \prod_{i \in l} \psi_{a_i}(x) \prod_{i \in l^c} \eta_{a_i}(x)$$

(5.52)

where 0 means that $l_i = 0 \forall i$. We have following the definition of the norm in (5.28) with $\hat{h}$ replaced by $h$

$$\| p_{n,g}(\Delta_{\delta_n},\varphi,\zeta,0,0) \|_{\bar{g}} \leq \sum_{p=0}^{2} \sum_{a} \sup_{\|g_{2p}\| \leq 1} \sum_{l \subset \{1,\ldots,2p\}} \int_{\Delta_{\delta_n}} dx \| p_{\Delta_{\delta_n},2p}^{0,a,l}(\Delta_{\delta_n},\varphi,\zeta,\psi,\eta) \|_{h_B} \| g_{2p}(x) \|$$

$$\leq \sum_{p=0}^{2} \sum_{a} \sum_{l \subset \{1,\ldots,2p\}} \int_{\Delta_{\delta_n}} dx \| p_{\Delta_{\delta_n},2p}^{0,a,l}(\Delta_{\delta_n},\varphi,\zeta,\psi,\eta) \|_{h_B}$$

(5.53)

where $g_{2p}(x) = g_{2p}(x,\ldots,x)$ and $\|g_{2p}\|$ is the $C^2(\Delta_{\delta_n})$ norm of $g_{2p}$. $p_{\Delta_{\delta_n},2p}^{0,a,l}(\Delta_{\delta_n},\varphi,\zeta,\psi,\eta)$ is a polynomial in $\varphi,\zeta$ and every term in the $h_B$ norm of $p_{\Delta_{\delta_n},2p}$ can be estimated as in the proof of Lemma 5.6 of [BMS]. Each term carries a factor $g_n = O(\bar{g})$. The fluctuation fields $\zeta$ are estimated via Lemma 5.2 and the fields $\varphi$ via Lemma 5.1. For each field $\varphi$ we lose $\bar{g}^{-4}$. In the $p = 0$ term the maximum power of $\varphi$ in $p_{\Delta_{\delta_n},p}$ is 3, for $p = 1$
the maximum power is 2, and for \( p = 2 \) it is 0. The bound (5.48) now follows as in Lemma 5.6, [BMS]. The bound (5.49) for \( p_{n,\mu}(\Delta_{\delta_n}, \xi, \Phi) \) is proved in the same way. We just have to remember that \( \mu_n = O(\tilde{g}^{2-\delta}) \) from the domain hypothesis, and that the maximum power of the field \( \varphi \) in the \( \tilde{p}_{n,\mu,p} \) is 1. The remaining parts are proved in the same way. 

Define \( p_n(s) = p_n(s, \Delta_{\delta_n}, \Phi, \xi) \) by \( p_n(s) = sp_{n,g} + s^2p_{n,\mu} \). Then \( r_{n,1} = r_{n,1}(\Delta_{\delta_n}, \Phi, \xi) \) defined by (4.9) is given by

\[
r_{n,1} = \frac{1}{2} \int_0^1 ds(1-s)^2 e^{-p_n(s)}V_n(-p_n(s)^3 + 6p_n(s)p_{n,\mu})
\]

(5.54)

with \( p_n'(s) = \frac{d}{ds}p_n(s) = p_{n,g} + 2sp_{n,\mu} \) and \( p_{n,\mu}'(s) = 2p_{n,\mu} \).

Lemma 5.7

Under the conditions of the domain \( D_n \) there exists a constant \( C_L \) independent of \( n \) and \( \varepsilon \) but dependent on \( L \) such that

\[
\|r_{n,1}(\Delta_{\delta_n})\|_{h,\tilde{G}_{\varepsilon,\rho}} \leq C_L\tilde{g}^{3/4}
\]

(5.55)

\[
\|r_{n,1}(\Delta_{\delta_n})\|_{h,\tilde{G}_{\varepsilon,\rho}} \leq C_L\tilde{g}^{3-\delta}
\]

(5.56)

Proof: Follow the proof of the corresponding Lemma 5.7 of [BMS]. Write \( \tilde{V}_n + p_n(s) = V_{n,1}(s) + V_{n,2}(s) \) where

\[
V_{n,1}(s) = V(\Delta_{\delta_n}, \Phi + \xi, C_n, sg_n, s^2\mu_n), \quad V_{n,2}(s) = V(\Delta_{\delta_n}, \Phi, SLC_n, (1-s)g_n, (1-s^2)\mu_n)
\]

We have

\[
\|r_{n,1}(\Delta_{\delta_n}, \varphi, \zeta, 0, 0)\|_h \leq \frac{1}{2} \int_0^1 ds(1-s)^2\|e^{-V_{n,1}(s)}\|_h\|e^{-V_{n,2}(s)}\|_h\left(\|p'(s)\|_h + 6\|p(s)\|_h\|p\|_h\right)
\]

(5.57)

\[
\|e^{-V_{n,1}(s)}\|_h \leq 2 \quad \text{and} \quad \|e^{-V_{n,2}(s)}\|_h \leq 2e^{-(1-s)\frac{s}{2}\int_{\Delta_{\delta_n}} dx|\varphi|^4(x)} \cdot \text{We bound the remaining factor (using Lemma 5.6) by} \quad C_L\tilde{g}^{3}\tilde{G}_{\varepsilon,\rho}(\Delta_{\delta_n}, \varphi, \zeta)e^{-(1-s)\frac{s}{2}\int_{\Delta_{\delta_n}} dx|\varphi|^4(x)} \cdot \text{We put the three bounds together and choose} \quad 0 < \gamma < \frac{1}{17}. \text{This gives the bound (5.55). The proof of (5.56) is similar.} \]

Lemma 5.8

Under the conditions for the domain \( D_{\delta_n} \) there exists a constant \( C_L \), independent of \( n \) and \( \varepsilon \) but dependent on \( L \) such that

\[
\|P_n(\lambda)\|_{h,\tilde{G}_{\varepsilon,\rho}, A, \delta_n} \leq C_L|\lambda\tilde{g}^{1/4}| \quad \text{for} \quad |\lambda\tilde{g}^{1/4}| \leq 1
\]

(5.58)

\[
\|P_n(\lambda)\|_{h,\tilde{G}_{\varepsilon,\rho}, A, \delta_n} \leq C_L|\lambda\tilde{g}^{1-\delta/2}| \quad \text{for} \quad |\lambda\tilde{g}^{1-\delta/2}| \leq 1
\]

Proof: This follows on applying Lemmas 5.5, 5.6 and 5.7 to \( P_n(\lambda) \) defined in (4.9). 

Estimates for \( Q_n e^{-V_n} \)

We now turn to the estimate of \( Q_n e^{-V_n} \). From (4.4)
\[ Q_n(X_{\delta_n}, \Phi) = Q(X_{\delta_n}, \Phi; C_n, w_n, g) = g_n^2 \sum_{j=1}^{3} Q^{(j,j)}(X_{\delta_n}, \Phi; C_n, w^{(j,j)}_n) \] (5.59)

where the \( Q^{(m,m)} \) are given in (4.6). Under an iteration, see Proposition 4.1, we have

\[ w^{(p)}_n \rightarrow w^{(p)}_{n+1} = v^{(p)}_{n+1} + w^{p}_{n,L} \]

, where \( p = 1, 2, 3 \) and the \( v^{(p)}_n \) are given in Proposition 4.1. Starting with \( w^{(p)}_0 = 0 \) we get by iterating

\[ w^{(p)}_n = \sum_{j=0}^{n-1} v^{(p)}_{n-j,L,j} \] (5.60)

For every integer \( n \geq 0 \) we consider the Banach spaces \( W_{p, \delta_n} \) of functions \( f : (\delta_n \mathbb{Z})^3 \rightarrow \mathbb{R} \) with norms \( || \cdot ||_{p,n}, p = 1, 2, 3 \):

\[ ||f||_{p,n} = \sup_{x \in (\delta_n \mathbb{Z})^3} (|x| + \delta_n)^{6p+1} |f(x)| \] (5.61)

We define the Banach space \( W_n = W_{1,n} \times W_{2,n} \times W_{3,n} \) consisting of vectors \( f = (f^{(1)}, f^{(2)}, f^{(3)}), f^{(p)} : (\delta_n \mathbb{Z})^3 \rightarrow \mathbb{R} \) with the norm

\[ ||f||_n = \max_p ||f^{(p)}||_n \] (5.62)

Let \( w_n = (w^{(1)}_n, w^{(2)}_n, w^{(3)}_n) \) as above.

**Lemma 5.9**

1. For \( L \) sufficiently large and \( \varepsilon > 0 \) sufficiently small there exists a constant \( k_L \) independent of \( n \) and \( \varepsilon \) such that for all \( n \geq 1 \),

\[ ||w_n||_n \leq k_L/2 \] (5.63)

If we start the sequence \( \{w_n\}_{n \geq 0} \) with \( w_0 \neq 0 \), with \( ||w_n||_{\delta_0} \leq k_L/2 \), then

\[ ||w_n||_n \leq k_L, \forall n \geq 0 \] (5.64)

2. There exists a function \( w_* \) defined on \( \cup_{n \geq 0}(\delta_n \mathbb{Z})^3 \subset \mathbb{Q}^3 \) such that for every integer \( l \geq 0 \) held fixed , the sequence \( \{w_n\}_{l \leq n} \) converges to \( w_* \) in the norm \( || \cdot ||_l \) as \( n \rightarrow \infty \). The convergence rate is given by

\[ ||w_n - w_*||_l \leq \delta_L L^{-qn} \] (5.65)

where \( q > 0 \) is the constant in Theorem 1.1 and Corollary 1.2.. We have \( w_* = v_{c*} + w_{*,L} \) in \( W_l \) for every \( l \geq 0 \).

**Proof:**

1. Let \( m = n - j \) with \( 0 \leq j \leq n - 1 \). By definition \( v^{(p)}_m = C^p_{m,L} - C^p_{m+1} \) with pointwise multiplication.

Since \( C_{m,L} = \Gamma_{m,L} + C_{m+1} \), it follows that \( v^{(p)}_m \) has \( \Gamma_{m,L} \) as factor. From the finite range property of \( \Gamma_{m,L} \) it follows that

\[ v^{(p)}_m(x) = 0 : |x| \geq 1 \]
Theorem 1.1, part (5a), and Corollary 1.2 give uniform bounds on the $\Gamma_m$ and $C_m$. Therefore there exists a constant $c_{L,p}$ independent of $n$ such that

$$\|v_{m}^{(p)}\|_{L^\infty((\delta_n\mathbb{Z})^3)} \leq c_{L,p}$$

2. By definition

$$\|v_{n-j,L}^{(p)}\|_{p,n} = \sup_{x \in (\delta_n\mathbb{Z})^3} \left( L^{2d_j} \left( |x| + \delta_n \right)^{\frac{p+1}{2}} |v_{n-j}(L^j x)| \right)$$

$$= L^{-j \left( \frac{p+1}{2} - 2d_j \right)} \sup_{y \in (\delta_n \mathbb{Z})^3} \left( (|y| + \delta_{n-j})^{\frac{p+1}{2}} |v_{n-j}(y)| \right)$$

Because of the finite range property of $v_{n-j}$ of paragraph 1, we can bound $|y| \leq 1$ in the weight factor on the right. Because $n-j \geq 1$ we can bound in the weight factor $\delta_{n-j} \leq \delta_1 = L^{-1}$. Therefore on using the bound on $v_{n-j}$ of paragraph 1 we get

$$\|v_{n-j,L}^{(p)}\|_{p,n} \leq L^{-j \left( \frac{p+1}{2} - 2d_j \right)} (1 + L^{-1})^{\frac{p+1}{2}} c_{L,p}$$

We bound the first geometric factor by taking $p = 1$ and $\varepsilon > 0$ very small in $d_s = (3 - \varepsilon)/4$. This gives $L^{-j/5}$. This gives the bound

$$\|v_{n-j,L}^{(p)}\|_{p,n} \leq L^{-j/5} c_{L,p}$$

with a new constant $c_{L,p}$ independent of $n$. Using the above bound we get from (5.60) the bound

$$\|u_{n}^{(p)}\|_{p,n} \leq c_{L,p} \sum_{j=0}^{\infty} L^{-j/5} \leq 2c_{L,p}$$

for $L$ sufficiently large. Therefore setting $k_L = 4 \max_p c_{L,p}$ we get

$$\|w_n\|_{n} \leq k_L/2$$

which proves (5.63). (5.64) is a trivial consequence of the above. This proves the first part of the lemma.

3. Let $v_{cs}^{(p)} = C_{cs,L}^{p} - C_{cs}^{p}$, with pointwise multiplication, where $C_{cs}$ is the smooth continuum covariance in $\mathbb{R}^3$ of Corollary 1.2. By factoring out $\Gamma_{cs,L}$, Theorem 1.1 and Corollary 1.2 we have that $v_{cs}^{(p)}$ exists in $L^\infty(\mathbb{R}^3)$ and has finite range: $v_{cs}^{(p)}(x) = 0 : |x| \geq 1$. Moreover by Theorem 1.1 and Corollary 1.2 we have, see the proof of Lemma 5.12 for the detailed argument,

$$\|v_{m}^{(p)} - v_{cs}^{(p)}\|_{L^\infty((\delta_m\mathbb{Z})^3)} \leq c_{L,p} L^{-\eta(m-1)}$$

Define

$$u_{s}^{(p)} = \sum_{j=0}^{n-1} v_{cs,L}^{(p)}$$

Fix any integer $l \geq 0$. Then for $n \geq l$ the $(p,l)$ norm is dominated by the $(p,n)$ norm. Then proceeding as in the first part and using the previous inequality we get

$$\|u_{n}^{p} - u_{s}^{p}\|_{p,l} \leq \sum_{j=0}^{n-1} \|v_{n-j,L}^{p} - v_{cs,L}^{p}\|_{p,n} \leq c_{L,p} \sum_{j=0}^{n-1} L^{-j \left( \frac{p+1}{2} - 2d_j \right)} \|v_{n-j}^{p} - v_{cs}^{p}\|_{L^\infty((\delta_{n-j}\mathbb{Z})^3)}$$

$$\leq c_{L,p} \sum_{j=0}^{n-1} L^{-j/5} L^{-\eta(n-j-1)} \leq c_{L,p} L^{-\eta n}$$
Now take the maximum over $p$. This proves (5.61) and at the same time the convergence statement of part 2 of the lemma. The last statement of part 2 is trivial to prove. This completes the proof of the second part of the lemma. ■

**Lemma 5.10**

*Under the conditions of the domain $\mathcal{D}_n$, there exists constants $C_{p,L}$ independent of $n$ such that*

\[
\|Q_n e^{-V_n}\|_{h,G_n,A_p,\delta_n} \leq C_{p,L} \bar{g}^{1/2}
\]

\[
|Q_n e^{-V_n}|_{h,A_p,\delta_n} \leq C_{p,L} \bar{g}^2
\]

**Proof**

\[
Q_n(X_{\delta_n}) e^{-V_n(X_{\delta_n})} = g_n^2 \sum_{m=1}^{3} Q^{(m,m)}(\hat{X}_{\delta_n}, \Phi; C_n, w_n^{(4-m)}) e^{-V_n(X_{\delta_n})}
\]

\[
\left\| Q_n(X_{\delta_n}, \varphi) e^{-V_n(X_{\delta_n}, \varphi)} \right\|_h \leq g_n^2 \sum_{m=1}^{3} \left\| Q^{(m,m)}(\hat{X}_{\delta_n}, \Phi; C_n, w_n^{(4-m)}) \right\|_h \left\| e^{-V_n(X_{\delta_n})} \right\|_h
\]

Here $X_{\delta_n}$ is a small set because of the support properties of $Q_n$. The last factor will be estimated by Lemma 5.5. From (4.6) we have

\[
Q^{(3,3)}(\hat{X}_{\delta_n}, \Phi; w_n^{(1)}) = 4 \int_{X_{\delta_n}} dx dy : \Phi(x) \bar{\Phi}(x) \Phi(x) \Phi(y) \bar{\Phi}(y) :_{C_n} w_n^{(1)}(x - y)
\]

We exhibit (5.70) as an element of the Grassmann algebra:

\[
Q^{(3,3)}(\hat{X}_{\delta_n}, \Phi; w_n^{(1)}) = Q_0^{(3,3)}(\hat{X}_{\delta_n}, \varphi; w_n^{(1)}) + \int_{X_{\delta_n}} dx dy Q_1^{(3,3)}(\hat{X}_{\delta_n}, \varphi, x, y; w_n^{(1)}) : \psi(x) \bar{\psi}(y) :_{C_n} +
\]

\[
+ \int_{X_{\delta_n}} dx Q_2^{(3,3)}(\hat{X}_{\delta_n}, \varphi, x; w_n^{(1)}) : \psi(x) \bar{\psi}(x) :_{C_n} +
\]

\[
+ \int_{X_{\delta_n}} dx dy Q_3^{(3,3)}(\hat{X}_{\delta_n}, \varphi, x, y; w_n^{(1)}) : \psi(x) \bar{\psi}(x) \psi(y) \bar{\psi}(y) :_{C_n}
\]

where

\[
Q_0^{(3,3)}(\hat{X}_{\delta_n}, \varphi; w_n^{(1)}) = 4 \int_{X_{\delta_n}} dx dy : \varphi(x) \bar{\varphi}(x) \varphi(x) \bar{\varphi}(y) \varphi(y) :_{C_n} w_n^{(1)}(x - y)
\]

\[
Q_1^{(3,3)}(\hat{X}_{\delta_n}, \varphi, x, y; w_n^{(1)}) = 4 : \varphi(x) \bar{\varphi}(x) \varphi(y) \bar{\varphi}(y) :_{C_n} w_n^{(1)}(x - y)
\]

\[
Q_2^{(3,3)}(\hat{X}_{\delta_n}, \varphi, x; w_n^{(1)}) = 4 \int_{X_{\delta_n}} dy : (\varphi(x) \bar{\varphi}(y) + \varphi(y) \bar{\varphi}(x)) \varphi(y) \bar{\varphi}(y) :_{C_n} w_n^{(1)}(x - y)
\]

\[
Q_3^{(3,3)}(\hat{X}_{\delta_n}, \varphi, x, y; w_n^{(1)}) = 4 : \varphi(x) \bar{\varphi}(y) :_{C_n} w_n^{(1)}(x - y)
\]

where, denoting with $\Delta(x)$ the block $\Delta$ such that $x \in \Delta$, we have $X'_{\delta_n} = \left\{ \Delta \mid X_{\delta_n} \setminus \Delta(x) \right\}$ if $X_{\delta_n} = \Delta$. Undo the Wick ordering, which produces lower order terms with coefficients which are uniformly bounded independent of $n$ by Corollary 1.2. It is therefore enough to estimate with the Wick ordering taken off. We get
\[ \|Q^{(3,3)}(\tilde{X}_{\delta_n}, \Phi; w_n^{(1)})\|_h \leq \|Q_0^{(3,3)}(\tilde{X}_{\delta_n}, \varphi; w_n^{(1)})\|_h + \]
\[ + h_F^2 \sup_{\|g\|_{C^{2}(X_{\delta_n}^2)}} \int_{X_{\delta_n}} dx \|Q_1^{(3,3)}(\tilde{X}_{\delta_n}, \varphi, x, y; w_n^{(1)})\|_h \|g_2(x, y)\| + \]
\[ + h_F^2 \sup_{\|g\|_{C^{2}(X_{\delta_n}^2)}} \int_{X_{\delta_n}} dx \|Q_2^{(3,3)}(\tilde{X}_{\delta_n}, \varphi, x; w_n^{(1)})\|_h \|g_2(x, x)\| + \]
\[ + h_F^4 \sup_{\|g\|_{C^{2}(X_{\delta_n}^2)}} \int_{X_{\delta_n}} dxdy \|Q_3^{(3,3)}(\tilde{X}_{\delta_n}, \varphi, x; w_n^{(1)})\|_h \|g_4(x, x, y, y)\| \]

To estimate the \(h_B\) norm of the \(Q_3^{(3)}\) we apply to (5.72) \(h_B^k D^k\), with \(D\) the bosonic field derivative, \(h_B = c\bar{g}^{-1/4}\) and use Lemma 5.1. Contributions for \(k > 4\) vanish. We use \(g_n = O(\bar{g})\) from the domain hypothesis of Theorem 5.1. We estimate the kernel \(w_n^{(1)}(x - y)\) using Lemma 5.9. As a result we get

\[ \|Q^{(3,3)}(\tilde{X}_{\delta_n}, \Phi; w_n^{(1)})\|_h \leq C \bar{L} \bar{g}^{-3/2} \int_{X_{\delta_n}} dxdy \frac{1}{(|x - y| + \delta_n)^{7/4}} \bar{g}^{3/4} \int_{X_{\delta_n}} dx |\varphi(x)|^2 G_n(X_{\delta_n}, \varphi) \tag{5.73} \]

The integral over \(\tilde{X}_{\delta_n}\) exists and is of \(O(1)\) since \(X_{\delta_n}\) is a small set in \((\delta_n \mathbb{Z})^3\). Therefore

\[ \|Q^{(3,3)}(\tilde{X}_{\delta_n}, \Phi; w_n^{(1)})\|_h \leq C \bar{L} \bar{g}^{-3/2} e^{\bar{g}/4} \int_{X_{\delta_n}} dx |\varphi(x)|^2 G_n(X_{\delta_n}, \varphi) \tag{5.74} \]

Next turn to \(Q^{(m,m)}, m = 1, 2\). From (4.6)

\[ Q^{(2,2)}(\tilde{X}_{\delta_n}, \Phi; C, w_n^{(2)}) = - \int_{X_{\delta_n}} dxdy [:(\Phi(x) - \Phi(y))(\Phi(x) - \Phi(y))(\Phi(x) + \Phi(y))(\Phi(x) + \Phi(y)) :c_n + \]
\[ + :[(\Phi\Phi)(x) - (\Phi\Phi)(y)]^2 :c_n] w_n^{(2)}(x - y) \]

We exhibit this as an element of the Grassmann algebra. This gives

\[ Q^{(2,2)}(\tilde{X}_{\delta_n}, \Phi; C, w_n^{(2)}) = -Q_0^{(2,2)}(\tilde{X}_{\delta_n}, \varphi; C_n, w_n^{(2)}) - \int_{X_{\delta_n}} dxdy \]
\[ \left[ Q_1^{(2,2)}(\tilde{X}_{\delta_n}, \varphi, x, y; C_n, w_n^{(2)}) : (\psi(x) + \bar{\psi}(y))(\bar{\psi}(x) + \bar{\psi}(y)) :c_n + \]
\[ Q_2^{(2,2)}(\tilde{X}_{\delta_n}, \varphi, x, y; C_n, w_n^{(2)}) : (\psi(x) - \bar{\psi}(y))(\bar{\psi}(x) - \bar{\psi}(y)) :c_n + \]
\[ + Q_3^{(2,2)}(\tilde{X}_{\delta_n}, \varphi, x, y; C_n, w_n^{(2)}) : (\psi(x) - \psi(y))(\bar{\psi}(x) - \bar{\psi}(y)) :c_n + \]
\[ + Q_4^{(2,2)}(\tilde{X}_{\delta_n}, x, y, w_n^{(2)}) \{ : (\psi(x) - \psi(y))(\bar{\psi}(x) - \bar{\psi}(y))(\psi(x) + \psi(y))(\bar{\psi}(x) + \bar{\psi}(y)) :c_n + \]
\[ + : (\psi\bar{\psi} - \psi\bar{\psi})^2 :c_n \} \right] \tag{5.75} \]

where

\[ Q_0^{(2,2)}(\tilde{X}_{\delta_n}, \varphi; C_n, w_n^{(2)}) = \int_{X_{\delta_n}} dxdy w_n^{(2)}(x - y)(|\varphi(x) - \varphi(y)|^2 :c_n + \]
\[ + :(|\varphi|^2(x) - |\varphi|^2(y))^2 :c_n) \]
\[ Q_1^{(2,2)}(\tilde{X}_{\delta_n}, \varphi, x, y; C_n, w_n^{(2)}) = w_n^{(2)}(x - y) :|\varphi(x) - \varphi(y)|^2 :c_n \tag{5.76} \]
\[ Q_2^{(2,2)}(\tilde{X}_{\delta_n}, \varphi, x, y; C_n, w_n^{(2)}) = w_n^{(2)}(x - y) :|\varphi(x) + \varphi(y)|^2 :c_n \]
\[ Q_3^{(2,2)}(\tilde{X}_{\delta_n}, \varphi, x, y; C_n, w_n^{(2)}) = 2w_n^{(2)}(x - y) :(|\varphi|^2(x) - |\varphi|^2(y)) :c_n \]
\[ Q_4^{(2,2)}(\tilde{X}_{\delta_n}, x, y, w_n^{(2)}) = w_n^{(2)}(x - y) \]
The \( \| \cdot \|_{h,G_n,A_p} \) norm estimate for \( Q^{(2,2)}(\bar{X}_{\delta_n}, \Phi; C, w_n^{(2)}) \) reposes on the following principles:

1. Undoing the Wick ordering produces lower order terms with Wick coefficients which together with their derivatives are uniformly bounded independent of \( n \) by Corollary 1.2. Moreover by the domain hypothesis \( g_n = O(\bar{g}) \).

2. By Lemma 5.9, the kernel \( w_n^{(2)} \) has the bound \( |w_n^{(2)}(x-y)| \leq k_L(|x-y| + \delta_n)^{-13/4} \) where the constant \( k_L \) is independent of \( n \).

3. The fields \( \varphi(x) \) are estimated by Lemma 5.1. Differences of fields \( |\varphi(x) - \varphi(y)| \) are estimated by (5.18). This produces a factor \( |x|\) which we retain, and majorise the Sobolev factor by the large field regulator. Differences of fields \( \varphi\varphi(x) - \varphi\varphi(y) \) can also be expressed as in (5.17), substituting \( \varphi\varphi \) for \( \varphi \). This requires estimating \( (\partial_{\delta_n,e_j}\varphi\varphi)(x+\cdot) \). We apply the lattice Leibnitz which modifies the continuum rule by producing an extra term \( \delta_n |\partial_{\delta_n,e_j}\varphi(x+\cdot)|^2 \) (see equation (5.2), page 432 of [BGM]). We estimate the \( \varphi \) by Lemma 5.1, with \( \kappa/2 \) in the large field regulator. We estimate the gradient pieces by the Sobolev inequality as in (5.18), and then by the large field regulator with \( \kappa/2 \). We have also produced a factor \( |x|\) as in (5.18).

Invoking the above principles we get the following bounds for the bosonic coefficients:

\[
\frac{h^k}{k!} \| D^k Q_0^{(2,2)}(\bar{X}_{\delta_n}, \varphi; C_n, w_n^{(2)}) \| \leq c L g^{-1/2} \int_{X_{\delta_n}} dx dy (|x-y| + \delta_n)^{-4} \ddbar{\varphi}^2 G_n(\bar{X}_{\delta_n}, \varphi)
\]

and for \( j = 1, 2, 3 \)

\[
\frac{h^k}{k!} \| D^k Q_j^{(2,2)}(\bar{X}_{\delta_n}, \varphi; x, y, C_n, w_n^{(2)}) \| \leq c L g^{-1/2} \delta_n^{-13/4} f_j(x-y) \ddbar{\varphi}^2 G_n(\bar{X}_{\delta_n}, \varphi)
\]

where the maximum value of \( k \) which gives a nonvanising contribution is 4 and

\[
f_1(x-y) = |x-y|^2, \quad f_2(x-y) = 1, \quad f_3(x-y) = |x-y|, \quad f_4(x-y) = 1
\]

4. We must estimate the contribution of the fermionic pieces to the \( h \) norm. To this end denote by \( F_j(\psi) \) the fermionic factor multiplying \( Q_j^{(2,2)} \) in (5.75). Express the differences \( \psi(x) - \psi(y) \) by the fermionic analogue of (5.14). We do the same also for \( \psi\psi(x) - \psi\psi(y) \) and then apply the lattice Leibnitz rule to \( \partial_{\delta_n} \partial_{\delta_n} \psi\psi(x+\cdot) \). We replace the fermionic pieces by the functions \( g_{2_p} \) on \( \bar{X}_{\delta_n} \cup \Omega_2 \bar{X}_{\delta_n} \) and their lattice derivatives. Corresponeding to \( F_j(\psi) \) we get the contribution \( G_j \) which is a linear form on \( g_{2_p} \), where \( p_j = 1 \) for \( j = 1, 2, 3 \) and \( p_4 = 2 \). Let \( \delta_n h_j, h_j \in \mathbb{Z} \) be the component of \( y-x \) along the unit vector \( e_j \). We have

\[
G_1 = g_2(x, x) + g_2(x, y) + g_2(y, x) + g_2(y, y)
\]

\[
G_2 = \delta_n^2 \sum_{i_1, i_2 = 1, 0 \leq i_1, i_2 \leq h_j - 1} \partial_{\delta_n, e_{i_1}}^{(1)} \partial_{\delta_n, e_{i_2}}^{(2)} g_2(x + p_{i_1}(y - x, s_{i_1}), x + p_{i_2}(y - x, s_{i_2}))
\]

\[
G_3 = \delta_n \sum_{i=1}^{h_n-1} \sum_{t=0}^{h_n-1} \partial_{\delta_n, e_t}^{(1)} \partial_{\delta_n, e_{t+1}}^{(2)} g_2(x + p_t(y - x, s_t), x + p_{t+1}(y - x, s_{t+1})) + \delta_n \partial_{\delta_n, e_t}^{(1)} \partial_{\delta_n, e_{t+1}}^{(2)} g_2(x + p_t(y - x, s_t), x + p_t(y - x, s_{t+1}))
\]

\[
G_4 = \delta_n^2 \sum_{i_1, i_2 = 1, 0 \leq i_1, i_2 \leq h_j - 1} \partial_{\delta_n, e_{i_1}}^{(1)} \partial_{\delta_n, e_{i_2}}^{(2)} (g_4(x + p_{i_1}(y - x, s_{i_1}), x + p_{i_2}(y - x, s_{i_2}), x, x) +
\]

\[
9/\text{settembre}/2008 [51] 5:51
\]
where the superscript on the lattice derivative denotes the argument on which it acts. The omitted terms \( \cdots \) in \( G_4 \) comes from the square of the (first order) lattice Taylor expansion of \( \psi \bar{\psi}(x) - \psi \bar{\psi}(y) \) and then replacing the product of 4 Grassmann fields by the test function \( g_4 \). For \( j = 1, 2, 3 \) we have the bounds

\[
|G_j| \leq O(1)\tilde{f}_j(x-y)\|g_2\|_{C^2(\hat{X}^2_n)} \tag{5.80}
\]

and for \( j = 4 \) we have

\[
|G_4| \leq O(1)\tilde{f}_4(x-y)\|g_4\|_{C^2(\hat{X}^4_n)} \tag{5.81}
\]

where

\[
\tilde{f}_1 = 1, \quad \tilde{f}_2 = |x-y|^2, \quad \tilde{f}_3 = |x-y|, \quad \tilde{f}_4 = |x-y|^2 \tag{5.82}
\]

On using the bounds (5.77)-(5.82) we get for \( 0 \leq k \leq 4 \) and \( 0 \leq p \leq 2 \)

\[
h_E^{2p} h_n^p (D^{2p,n} Q^{(2,2)}(\hat{X}_\delta, \varphi, 0; f^{\times k}, g_2)) \leq c_L g_\delta^{-1/2} \int_{\hat{X}_n} dxdy (|x-y| + \delta_n)^{-\left(\frac{d+2}{2}\right)} 
\]

\[
\times e^{\frac{\tilde{h}}{4} \int_{\hat{X}_n} dx|\varphi(x)|^2} G_n(\hat{X}_\delta, \varphi) \|f\|_{C^2(\hat{X}_n)} \|g_2\|_{C^2(\hat{X}^2_n)} \tag{5.83}
\]

For \( k > 4 \) or \( p > 2 \) we have vanishing contribution. The integral over \( \hat{X}_\delta \) exists and gives a contribution of \( O(1) \) since \( X_\delta \) is a small set in \( (\delta_n \mathbb{Z})^3 \). Therefore we obtain from the previous inequality

\[
\|Q^{(2,2)}(\hat{X}_\delta, \varphi, 0)\|_{\mathfrak{h}} \leq c_L g_\delta^{-1/2} e^{\frac{\tilde{h}}{4} \int_{\hat{X}_n} dx|\varphi(x)|^2} G_n(\hat{X}_\delta, \varphi) \tag{5.84}
\]

We can estimate in the same way the case \( m = 1 \). We have

\[
\|Q^{(1,1)}(\hat{X}_\delta, \Phi; C, w^{(3)})\|_{\mathfrak{h}} \leq c_L g_\delta^{-1/2} e^{\frac{\tilde{h}}{4} \int_{\hat{X}_n} dx|\varphi(x)|^2} G_n(\hat{X}_\delta, \varphi) \tag{5.85}
\]

The integral over \( \hat{X}_\delta \) exists and gives a contribution of \( O(1) \). Therefore

\[
\|Q^{(1,1)}(\hat{X}_\delta, \Phi; C, w^{(3)})\|_{\mathfrak{h}} \leq c_L g_\delta^{-1/2} e^{\frac{\tilde{h}}{4} \int_{\hat{X}_n} dx|\varphi(x)|^2} G_n(\hat{X}_\delta, \varphi) \tag{5.86}
\]

Therefore from (5.69), Lemma 5.5 and the bounds (5.74), (5.84),(5.86) we get

\[
\left\| Q_n(X_\delta) e^{-V_n(X_\delta)} \right\|_{\mathfrak{h}, G_n} \leq c_L g_\delta^{1/2}
\]

and since \( Q_n \) is supported on small sets we get

\[
\left\| Q_n e^{-V_n} \right\|_{\mathfrak{h}, G_n, \mathcal{A}_p} \leq C_L \bar{g}^{1/2} \tag{5.87}
\]

which is (5.66). To prove (5.67) we estimate the r.h.s of (5.69) at \( \Phi = 0 \) after undoing the Wick ordering, set \( \mathfrak{h} = \mathfrak{h}_* \), and use Lemma 5.5. ✷

In the following lemma we consider \( Q_n(\Phi + \xi)e^{-V_n(\Phi + \xi)} \) as a function of \( \varphi, \zeta, \psi, \eta \).

**Lemma 5.11**
Under the conditions of the domain \( D_n \) there exists constants \( C_{L,p} \) independent of \( n \) such that

\[
\|Q_n e^{-V_n}\|_{h,\bar{G}_{\alpha_n},\bar{A}_p,\delta_n} \leq C_{L,p} g^{1/2} \quad (5.88)
\]

\[
\|Q_n e^{-V_n}\|_{h,\bar{G}_{\alpha_n},\bar{A}_p,\delta_n} \leq C_{L,p} g^2 \quad (5.89)
\]

**Proof**

The bound (5.88) follows from (5.66) since \( \hat{G}_{\alpha_p} > G_{\alpha} \). To prove (5.89) we first express \( Q_n^{m,m}(X_{\delta_n},\varphi + \zeta,\psi + \eta) \) in the Grassmann representation as in the proof of Lemma 5.10, substituting in the expressions there \( \varphi \to \varphi + \zeta, \psi \to \psi + \eta \). Field derivatives are defined as in (5.26). For the bosonic coefficients we take derivatives at \( \varphi = 0 \). The resulting dependence on \( \zeta \) is estimated by Lemma 5.2. The rest of the proof follows that of Lemma 5.10. We use Lemma 5.5 which implies that \( \|e^{-V_n(X_{\delta_n},\cdot)}\|_{h,\bar{G}_{\alpha_n}} \leq 2|X_{\delta_n}| \), and \( X_{\delta_n} \) is a small set. \( \blacksquare \)

We now prove a lemma to control the perturbative flow coefficients \( a_n, b_n \) given in (4.15) and (4.17). This lemma is independent of the domain \( D_n \).

**Lemma 5.12**

Let \( v_{c_s}^{(p)} = C_{c_s,L} - C_{c_s} \), with pointwise multiplication, where \( C_{c_s} \) is the smooth continuum covariance in \( \mathbb{R}^3 \) of Corollary 1.2. Define

\[
a_{c_s} = 2 \int_{\mathbb{R}^3} dy v_{c_s}^{(2)}(y), \quad b_{c_s} = 4 \int_{\mathbb{R}^3} dy v_{c_s}^{(3)}(y)
\]

We have that \( a_n, b_n, a_{c_s}, b_{c_s} \) are strictly positive. Moreover there exist constants \( c_L \) independent of \( n \) such that

\[
|a_n| \leq c_L, \quad |b_n| \leq c_L, \quad |a_{c_s}| \leq c_L, \quad |b_{c_s}| \leq c_L \quad (5.90)
\]

and

\[
|a_n - a_{c_s}| \leq c_L L^{-q_n}, \quad |b_n - b_{c_s}| \leq c_L L^{-q_n} \quad (5.91)
\]

where \( q > 0 \) is as in Theorem 1.1.

**Remark**: The convergence rate estimates (5.91) play no role in the estimates of the present section. They are used in Section 6 for the existence proof of a global renormalization group trajectory.

**Proof**

From (4.16), for \( p = 2, 3 \), using \( C_{n,L} = \Gamma_{n,L} + C_{n+1} \)

\[
v_{n+1}^{(p)} = C_{n,L}^{p} - C_{n+1}^{p} = \Gamma_{n,L}(\Gamma_{n,L}^{p-1} + p\Gamma_{n,L}^{p-2}C_{n+1} + \delta_{p,3} C_{n+1}^2)
\]

(5.92)

with pointwise multiplication. The positivity in Fourier space of the integral kernels on the right hand side implies that \( a_n > 0, b_n > 0 \) as claimed. The common factor of \( \Gamma_{n,L}(x) \) which has finite range 1 implies that \( v_{n+1}^{(p)}(x) \) has support in the unit ball in \( (\delta_{n+1}Z)^3 \). From Theorem 1.1 and Corollary 1.2 we have that \( v_{n+1}^{(p)} \) above are uniformly bounded in \( L^\infty((\delta_{n+1}Z)^3) \) by constants \( c_L \). By the same arguments \( v_{c_s} \) has finite range and belongs to \( L^\infty(\mathbb{R}^3) \). The uniform bounds in the first part of the lemma now follow.

By the same arguments using \( C_{c_s,L} = C_{c_s} + C_{c_s} \) we have that \( a_{c_s} > 0, b_{c_s} > 0 \) and \( v_{c_s}^{(p)}(x) \) has support in the unit ball in \( \mathbb{R}^3 \). Moreover using Corollary 1.2 we have \( \|v_{c_s}^{(p)}\|_{\bar{C}_\kappa(\mathbb{R}^3)} \leq c_{k,L} \) for all \( \kappa \geq 0 \).

Define
\[
a_n^{(p)} = \int_{(\delta_n + 1)Z} dy \, v_{n+1}^{(p)}(y), \quad a_{cs}^{(p)} = \int_{\mathbb{R}^3} dy \, v_{cs}^{(p)}(y)
\] (5.93)

Then using the compact support property of \(v_{n+1}^{(p)}\) and \(v_{cs}^{(p)}\) we get

\[
|a_n^{(p)} - a_{cs}^{(p)}| \leq \|v_{n+1}^{(p)} - v_{cs}^{(p)}\|_{L^\infty(\delta_n + 1Z)} + \left| \int_{(\delta_n + 1)Z} dy \, v_{cs}^{(p)}(y) - \int_{\mathbb{R}^3} dy \, v_{cs}^{(p)}(y) \right|
\] (5.94)

We estimate the first term on the right hand side of (5.94). We have

\[
\|v_{n+1}^{(p)} - v_{cs}^{(p)}\|_{L^\infty((\delta_n + 1Z))} \leq \|C_{c, L}^p - C_{n, L}^p\|_{L^\infty((\delta_n + 1Z))} + \|C_{cs}^p - C_{n+1}^p\|_{L^\infty((\delta_n + 1Z))}
\] (5.95)

In the first term on the right in (5.95) we factor out \(C_{c, L} - C_{n, L}\) and in the second term we factor out \(C_{cs} - C_{n+1}\). Then use of the bounds in Corollary 1.2 gives

\[
\|v_{n+1}^{(p)} - v_{cs}^{(p)}\|_{L^\infty((\delta_n + 1Z))} \leq c_{L, p}L^{-qn}
\] (5.96)

We estimate the second term on the right in (5.94) using Lemma 6.6 of [BGM] and the compact support of \(v_{cs}\). This gives

\[
\left| \int_{(\delta_n + 1)Z} dy \, v_{cs}^{(p)}(y) - \int_{\mathbb{R}^3} dy \, v_{cs}^{(p)}(y) \right| \leq O(1)\delta_n^{1+1}\|v_{cs}^{(p)}\|_{C^1(\mathbb{R}^3)} \leq c_{L, p}L^{-(n+1)}
\] (5.97)

From (5.94), (5.96) and (5.97) we get with \(q\) that of Theorem 1.1

\[
|a_n^{(p)} - a_{cs}^{(p)}| \leq c_{L, p}L^{-qn}
\] (5.98)

which completes the proof of the lemma. ■

Lemma 5.13

Under the conditions of the domain \(D_n\) there exist constants \(C_{p, L}\) independent of \(n\) such that

\[
\|Q_n(e^{-V_n} - e^{-\tilde{V}_n})\|_{h, G_{n, p, A_p, \delta_n}} \leq C_{L, p}\tilde{g}^{3/4}
\] (5.99)

\[
\|Q_n(e^{-V_n} - e^{-\tilde{V}_n})\|_{h, G_{n, p, A_p, \delta_n}} \leq C_{L, p}\tilde{g}^{3}
\] (5.100)

Proof: The proof is on the same lines as that of the corresponding Lemma 5.13 in [BMS]. It follows from Lemmas 5.11, 5.6, 5.10, and 5.5 which are lattice equivalents of the corresponding lemmas in [BMS]. ■

Lemma 5.14

Under the conditions of the domain \(D_n\) there exists constants \(C_L\) independent of \(n\) such that \(K_n(\lambda)\) given by (4.10) satisfies the bounds

\[
\|K_n(\lambda)\|_{h, G_{n, p, A, \delta_n}} \leq C_L|\lambda\tilde{g}^{1/4 - \eta/3}|^2 \quad \text{for } |\lambda\tilde{g}^{1/4 - \eta/3}| < 1
\] (5.101)

\[
\|K(\lambda)\|_{h, G_{n, p, A, \delta_n}} \leq C_L|\lambda\tilde{g}^{11/12 - \eta/3}|^2 \quad \text{for } |\lambda\tilde{g}^{11/12 - \eta/3}| < 1
\] (5.102)
The following proposition shows how fluctuation integration of polymer activities passes through \( h \) and \( h_* \) norms. It will be put to use in the subsequent lemmas.

**Lemma 5.14A**

1. Let \( \tilde{K}(X_{\delta_n}, \varphi, \zeta, \psi, \eta) \) be a polymer activity (see (5.24) and (5.25)) with norms defined as in (5.26)-(5.32). Let \( h = (h_B, h_F) \) and \( h_* = (h_B, h_F) \). Let \( \tilde{K}^\dagger(X_{\delta_n}, \varphi, \psi) = \int d\mu_{\Gamma_n}(\zeta) \frac{1}{|I|!|I'|!|J|!|J'|!} \times \int_{X_{\delta_n}^{2p}} dxdy D_{E}^{2p, I, J} \tilde{K}(X_{\delta_n}, \varphi, \zeta, x_I, x_{IJ}, y_J, y_{IJ}, y_{J'}; f^x, f^y) \psi(x_I) \tilde{\psi}(y_J) \det_{\Gamma_n, I, J, I', J'} (x_I, y_J, y_{J'}) \times (-1)^2 \] (5.107)

Note that \(|I'| = |I|\) since \(|I| = |J|\). The matrix \( \det_{\Gamma_n, I, J, I', J'} \) is an \(|I'| \times |I'|\) square matrix whose entry \( (\Gamma_n, I, J, I', J')_{rs} \) is given by

\[
(\Gamma_n, I, J, I', J')_{rs} = \Gamma_n(x_r - y_s) \] (5.108)

for \( r \in I', s \in J' \). \((-1)^2\) is a sign factor which it is not necessary to specify. It may change from line to line. We have

\[
\int_{X_{\delta_n}^{2p}} dxdy D_{E}^{2p, I, J} \tilde{K}(X_{\delta_n}, \varphi, \zeta, x_I, x_{IJ}, y_J, y_{IJ}, y_{J'}; f^x, f^y) \psi(x_I) \tilde{\psi}(y_J) \det_{\Gamma_n, I, J, I', J'} (x_I, y_J, y_{J'}) \times (-1)^2 \]

By definition \( g_{2j}(x_I, y_J) \), (note that \(|I| = |J| = j|\), is antisymmetric in the members of \( x_I \) and in the members of \( y_J \). The determinant is antisymmetric in the members of \( x_{I'} \) and in the members of \( y_{J'} \). Therefore the function

\[
(g_{2j} \otimes \det_{\Gamma_n, I, J, I', J'})(x_I, x_{I'}, y_J, y_{J'}) = g_{2j}(x_I, y_J) \det_{\Gamma_n, I, J, I', J'} (x_I, y_J, y_{J'}) \] (5.110)
on $X_{\delta_n}^{(I)} \times X_{\delta_n}^{(I')} \times X_{\delta_n}^{(I)} \times X_{\delta_n}^{(I')} = X_{\delta_n}^{2p}$ is an admissible test function for the norm defined in (5.26), (5.27). Hence we get from (5.109) and (5.110)

$$\frac{h_p^{2j}}{(J!)^2} \left| D^{2j,m} \tilde{K}^2(X_{\delta_n}, \varphi, 0, f^{\times m}, g_{2j}) \right| \leq \int d\mu_{\gamma_n}(\zeta) \left[ \sum_{p \geq 0} \sum_{\{I \subset \{1, \ldots, p\}\} \cap \{J \subset \{1, \ldots, p\}\}} \delta_{|I|, |J|, |I'|, |J'|} \right] \frac{h_p^{2j}}{(J!)^2} \times$$

$$\left| D^{2p,1,I,m} \tilde{K}(X_{\delta_n}, \varphi, \zeta, 0, 0) \right| \prod_{j=1}^{m} \left\| f_j \right\|_{C^2(X_{\delta_n}^{2p})} \left\| g_{2j} \otimes \det \Gamma_{n, I, J} \right\|_{C^2(X_{\delta_n}^{2p})}$$

(5.111)

We have

$$\left\| g_{2j} \otimes \det \Gamma_{n, I, J} \right\|_{C^2(X_{\delta_n}^{2p})} \leq \left(\left\| g_{2j} \right\|_{C^2(X_{\delta_n}^{2p})} \right) \left(\left\| \det \Gamma_{n, I, J} \right\|_{C^2(X_{\delta_n}^{2p})} \right)$$

(5.112)

where $X_{\delta_n}^{2j} = X_{\delta_n}^{(I)} \times X_{\delta_n}^{(I')}$ and $X_{\delta_n}^{2(p-j)} = X_{\delta_n}^{(I')} \times X_{\delta_n}^{(I)}$, since $|I| = |J| = j$ and $|I'| = |J'| = p - j$. Let $\partial_{\delta_n}^k$ be the lattice forward derivative of order $k$ in multi-index notation. By part (5a) of Theorem 1.1 we have

$$\max_{0 \leq k \leq 4} \left\| \partial_{\delta_n}^k \Gamma_{n} \right\|_{L^\infty((\delta_n Z)^3)} \leq C_L$$

(5.113)

where $C_L$ is a constant independent of $n$. Relabel $(\xi_1, \ldots, \xi_{2(p-j)}) = (x_1, \ldots, x_{|I'|}, y_1, \ldots, y_{|J'|})$ where the $x_i \in I'$ and the $y_j \in J'$. Let $\partial_{\delta_n}^k$ be the forward lattice derivative of order $k$, $0 \leq k \leq 2$ with respect to the points $\xi_r$. Let $k = (k_1, \ldots, k_2(p-j))$ and define $\partial_{\delta_n}^k = \prod_{r=1}^{2(p-j)} \partial_{\delta_n}^{k_r}$. Let $\partial_{\delta_n}^k$ act on the determinant. This produces another determinant with derivatives acting on the matrix elements $\Gamma_{n}(x_r - y_s)$. Since $\Gamma_{n}$ is positive definite these matrices can be written as Gram matrices by a standard argument. Thus the matrix $a_{rs} = \partial_{\delta_n}^k \partial_{\delta_n}^k \Gamma_{n}(x_r - y_s)$ with $(x_r, y_s) \in X_{\delta_n} \times X_{\delta_n}$ can be written as

$$a_{rs} = (f_r, g_s)_{L^2((\delta_n Z)^3)}$$

where $f_r(\cdot) = \partial_{\delta_n}^k \Gamma_{n}(x_r - \cdot)$ and $g_s(\cdot) = \partial_{\delta_n}^k \Gamma_{n}(y_s - \cdot)$. Gram's inequality says

$$\left| \det a_{rs} \right| \leq \prod_{r=1}^{2(p-j)} \left\| f_r \right\|_{L^2((\delta_n Z)^3)} \prod_{s=1}^{2(p-j)} \left\| g_s \right\|_{L^2((\delta_n Z)^3)}$$

We have $\left\| f_r \right\|_{L^2((\delta_n Z)^3)} = \partial_{\delta_n}^k \partial_{\delta_n}^k \Gamma_{n}(x_r - x)_{r=s}$. Similarly, $\left\| g_s \right\|_{L^2((\delta_n Z)^3)} = \partial_{\delta_n}^k \partial_{\delta_n}^k \Gamma_{n}(y_r - y)_{r=s}$. Since $2k_r \leq 4$, we have by (5.113) the bound $\left\| f_r \right\|_{L^2((\delta_n Z)^3)} \leq C_L$. Similarly $\left\| g_s \right\|_{L^2((\delta_n Z)^3)} \leq C_L$. We therefore get

$$\left| \partial_{\delta_n}^k \det \Gamma_{n, I, J}(x_{I'}, y_{J'}) \right| \leq C_L^{p-j}$$

(5.114)

Hence

$$\left| \det \Gamma_{n, I, J} \right|_{C^2(X_{\delta_n}^{2(p-j)})} \leq C_L^{p-j}$$

(5.115)

From (5.114), (5.112) and (5.111) we get

$$\frac{h_p^{2j}}{(J!)^2} \left| D^{2j,m} \tilde{K}^2(X_{\delta_n}, \varphi, 0) \right| \leq \int d\mu_{\gamma_n}(\zeta) \left[ \sum_{p \geq 0} \sum_{\{I \subset \{1, \ldots, p\}\} \cap \{J \subset \{1, \ldots, p\}\}} \delta_{|I|, |J|, |I'|, |J'|} \left(\frac{C_L}{h_p} \right)^{p-j} \left| \frac{h_p^{2j}}{(J!)^2} \times \right| \right]$$

$$\left| D^{2p,1,I,m} \tilde{K}(X_{\delta_n}, \varphi, \zeta, 0, 0) \right|$$

(5.116)

Now choose $h_p$ sufficiently large depending on $L$ such that $h_p^2 \geq C_L$ which implies that $(\frac{C_L}{h_p})^{p-j} \leq 1$. Putting in this bound and then summing over $j$ we get

$$\sum_{j \geq 0} \frac{h_p^{2j}}{(J!)^2} \left| D^{2j,m} \tilde{K}^2(X_{\delta_n}, \varphi, 0) \right| \leq \int d\mu_{\gamma_n}(\zeta) \left[ \sum_{p \geq 0} \sum_{\{I \subset \{1, \ldots, p\}\} \cap \{J \subset \{1, \ldots, p\}\}} \delta_{|I|, |J|, |I'|, |J'|} \frac{h_p^{2j}}{(J!)^2} \times \right]$$

5:56
Set $\varphi = 0$ in (5.117). Multiply both sides by $\frac{h_m}{m!}$ and sum over $m$, $0 \leq m \leq m_0$. Majorize by dropping the constraint $\delta_{|l_1|,|l_1|}$ on the right hand side. This gives

$$|\tilde{K}^2(X_{\delta_0})|_{\mathbf{h}_*} \leq \int d\mu_{\Gamma_n}(\zeta) \| \tilde{K}(X_{\delta_0}, 0, \zeta, 0, 0) \|_{\mathbf{h}}.$$  

On the other hand multiplying both sides of (5.117) by $\frac{h_m}{m!}$ and summing over $m$, $0 \leq m \leq m_0$ gives

$$\| \tilde{K}^2(X_{\delta_0}, \varphi, 0) \|_{\mathbf{h}_*} \leq \int d\mu_{\Gamma_n}(\zeta) \| \tilde{K}(X_{\delta_0}, \varphi, \zeta, 0, 0) \|_{\mathbf{h}}$$

This proves the first part of the Lemma.

We next turn to the second part of the Lemma. This is a consequence of the first part and Lemma 5.4A. Define $\tilde{K}(X_{\delta_0}, \varphi, \zeta, \psi, \eta) = K(X_{\delta_0}, \varphi + \zeta, \psi + \eta)$. Then from (5.104) of the first part and (5.34) of Lemma 5.4A we have

$$\| \tilde{K}^2(X_{\delta_0}, \varphi, 0) \|_{\mathbf{h}} = \| \tilde{K}^2(X_{\delta_0}, \varphi, 0) \|_{\mathbf{h}} \leq \int d\mu_{\Gamma_n}(\zeta) \| \tilde{K}(X_{\delta_0}, \varphi, \zeta, 0, 0) \|_{\mathbf{h}} \leq \int d\mu_{\Gamma_n}(\zeta) \| K(X_{\delta_0}, \varphi + \zeta, 0) \|_{\mathbf{h}}$$

which proves (5.106). (5.105) follows similarly by using (5.103) of the first part followed by (5.35) of Lemma 5.4A.

The following lemma generalizes Lemma 5.15 of [BMS] to the lattice and the additional presence of Grassmann fields. It will play a key role later in obtaining contractive estimates.

**Lemma 5.15**

For any polymer activity $\tilde{K}(X_{\delta_0}, \phi + \zeta, \psi + \eta)$:

$$\| \tilde{K}(X_{\delta_0}, \phi + \zeta, \psi + \eta) \|_{\mathbf{h}_*} \leq O(1) \tilde{G}_{\rho, \kappa}(X_{\delta_0}, \zeta) \left[ \| \tilde{K}(X_{\delta_0}) \|_{\mathbf{h}_*} \right] + h_B^{-m_0} h_{B*} \| \tilde{K}(X_{\delta_0}) \|_{\mathbf{h}, \mathbf{G}_*}$$  

(5.117)

$$\| \tilde{K}(Y_{\delta_0}, \phi, 0) \|_{\mathbf{h}_*} \leq O(1) e^{\gamma \theta} \int_{\mathbb{Z}^d} \int_{\mathbb{Z}^d} d\phi(d\phi(y)) G_{\kappa}(Z_{\delta_0}, \phi) \left[ \| \tilde{K}(Y_{\delta_0}) \|_{\mathbf{h}_*} \right] + L^{-m_{0d}} \| \tilde{K}(Y_{\delta_0}) \|_{\mathbf{h}, L^d, h_B, \mathbf{G}_*}$$  

(5.119)

$$\| \tilde{K}^2(X_{\delta_0}) \|_{\mathbf{h}_*} \leq O(1) 2^{d X_{\delta_0}} \left[ \| \tilde{K}(X_{\delta_0}) \|_{\mathbf{h}_*} \right] + h_B^{-m_0} h_{B*} \| \tilde{K}(X_{\delta_0}) \|_{\mathbf{h}, \mathbf{G}_*}$$  

(5.120)

$$\| \tilde{K}^4(X_{\delta_0}) \|_{\mathbf{h}, \mathbf{G}_*} \leq 2^{d X_{\delta_0}} \| \tilde{K}(X_{\delta_0}) \|_{\mathbf{h}, \mathbf{G}_*}$$  

(5.121)

where $\tilde{G}_{\rho, \kappa}$ is as defined in (5.19), and $m_0 = 9$ is the maximum number of derivatives appearing in the definition of Kernel and $h$ norms. In (5.119), $Y_{\delta_0}, Z_{\delta_0}, \gamma$ are as described in Lemma 5.1. Moreover in the above norms $\mathbf{h}_* = (h_F, h_B)$, and $\mathbf{h}_* = (\frac{h_B}{h_F}, h_B)$ where $h_B \ast = (\rho \kappa)^{-1/2}$, $h = (h_F, h_B)$, $\tilde{h} = (\frac{h_B}{h_F}, h_B)$, $h_B = c \gamma^{-1/4}$, $\gamma = O(1)$ very small, and $h_F$ is taken to be sufficiently large depending on $L$.

The superscript $\sharp$ stands for $d\mu_{\Gamma_n}(\zeta)$ integration. $\rho$ is chosen as in Lemma 5.3, and $\kappa$ as in Lemma 2.1. Note that we have that the constant $C(\rho, \kappa, j)$ appearing in Lemma 5.2 (this bounds $\zeta^j$) satisfies

$$C(\rho, \kappa, j) = h_B^j, O(1)^j$$

(5.122)
Proof
We will first prove (5.118) following the lines of the proof of lemma 5.15 of [BMS] where the Grassman fields were absent. Recall from the definition in (5.30) that

$$\| \tilde{K}(X_{\delta_0}, 0, 0, \zeta, 0) \|_{\mathfrak{n}} = \sum_{m=0}^{m_0} \frac{h_B^m}{m!} A_m$$  \quad (5.123)$$

where

$$A_m = \sum_{p \geq 0} h_B^p \frac{h_B^m}{m!} \|D^{2p,m} \tilde{K}(X_{\delta_0}, 0, 0, \zeta, 0)\|$$  \quad (5.124)

First consider the case \(m = m_0\). Then

$$A_{m_0} \leq h_B^{m_0} h_B^{-m_0} \|\tilde{K}(X_{\delta_0})\|_{\mathfrak{n}, G_\kappa} \tilde{G}_{\rho, \kappa}$$  \quad (5.125)

since \(G_\kappa \leq \tilde{G}_{\rho, \kappa}\). Now let \(m < m_0\).

We expand in \(\zeta\) in Taylor series with remainder

$$(D^{2p,m} \tilde{K})(X_{\delta_0}, \zeta, 0; f^x m, g_{2n}) = \sum_{j=0}^{m_0-m-1} \frac{1}{j!} (D^{2p,j+m} \tilde{K})(X_{\delta_0}, 0, 0; f^x m, \zeta^j, g_{2p}) +$$

$$+ \frac{1}{(m_0 - m - 1)!} \int_0^1 ds (1 - s)^{m_0-m-1} (D^{2p,m} \tilde{K})(X_{\delta_0}, s\zeta, 0; f^x m, \zeta^{m_0-m}, g_{2p})$$  \quad (5.126)

Therefore

$$\|(D^{2p,m} \tilde{K})(X_{\delta_0}, \zeta, 0)\| \leq \sum_{j=0}^{m_0-m-1} \frac{1}{j!} (D^{2p,j+m} \tilde{K})(X_{\delta_0}, 0, 0) \|\zeta\|_{C^2(X_{\delta_0})}^j +$$

$$+ \frac{1}{(m_0 - m - 1)!} \int_0^1 ds (1 - s)^{m_0-m-1} \|\zeta\|_{C^2(X_{\delta_0})}^{m_0-m} (D^{2p,m} \tilde{K})(X_{\delta_0}, s\zeta, 0)\|$$

Hence

$$A_m \leq \sum_{j=0}^{m_0-m-1} \frac{(j+m)!}{j! m!} h_B^{-j} \|\zeta\|_{C^2(X_{\delta_0})}^j \|\tilde{K}(X_{\delta_0})\|_{\mathfrak{n}} +$$

$$+ \frac{m_0! h_B^{m_0} h_B^{-m_0}}{m! (m_0 - m - 1)!} \int_0^1 ds (1 - s)^{m_0-m-1} h_B^{-m_0} \|\zeta\|_{C^2(X_{\delta_0})}^{m_0-m} \|\tilde{K}(X_{\delta_0})\|_{\mathfrak{n}, G_\kappa} \tilde{G}_{\rho, \kappa}(X_{\delta_0}, s\zeta)$$

By Lemma 5.2, with \(\zeta\) replaced by \(\sqrt{1 - s^2} \zeta\), and (5.122),

$$h_B^{-j} \|\zeta\|_{C^2(X_{\delta_0})}^j \leq O(1) \tilde{G}_{\rho, \kappa}(X_{\delta_0}, \sqrt{1 - s^2} \zeta) \frac{1}{(1 - s^2)^{j/2}}$$  \quad (5.127)

where \(0 \leq s < 1\). With \(s = 0\) this bound is applied to the terms in the sum over \(j\). For the Taylor remainder term take \(j = m_0 - m\) and note that \((1 - s)^{(m_0-m)/2-1}\) is integrable since \(m_0 > m\). Hence:
\[ A_m \leq O(1)\tilde{G}_{\rho,n}(X,\zeta) \left[ \sum_{j=0}^{m_0-1} \frac{(j+m)!}{j!\mu!} \left\| \tilde{K}(X_{\delta_j}) \right\|_{h_*} + \frac{m_0!h_{B_*}m_0h_B^{m_0}}{m!(m_0-m-1)!} \left\| \tilde{K}(X_{\delta_j}) \right\|_{h,G_*} \right] \]  

(5.128)

Summing (5.128) over \( 0 \leq m \leq m_0 - 1 \) and adding (5.125) proves (5.118).

Inequality (5.119) is also proved in the same way as (5.118). We are estimating the \( h \) norm which is given by (5.34) with \( h_* \) replaced by \( h, h_B \) by \( h_B \) and \( \zeta \) by \( \varphi \). We replace \( \tilde{G}_{\rho,n} \) by \( G_\kappa \). Then (5.125) remains true with \( \zeta \) replaced by \( \varphi \) and \( \tilde{G}_{\rho,n} \) replaced by \( G_\kappa \). Subsequently for \( m < m_0 \) we expand in Taylor series as above but now in \( \varphi \). We do the norm estimate as above but now using Lemma 5.1 in place of Lemma 5.2. For \( \varepsilon \) sufficiently small depending on \( L \) we have \( \varrho \) sufficiently small and therefore \( h_B^{-1} \) is sufficiently small. Hence \( h_B^{-1}C \leq O(1) \) where \( C = \kappa^{-1/2}O(1) \) is the constant appearing in Lemma 5.1. In the Taylor remainder term we replace \( h_B \) by \( L^d h_B \). which leads to the factor \( L^{-mad_*} \).

Finally note that the inequality (5.120) now follows on using (5.105) of Lemma 5.14A, followed by (5.118) and then Lemma 5.3. (5.121) follows from (5.106) of Lemma 5.14A on using the stability of the large field regulator \( G_\kappa \).

The next lemma extends lemma 5.16 of [BMS] to the case when Grassmann fields are also present.

**Lemma 5.16**

For any \( q > 0 \), there exists constants \( c_L \) independent of \( n \) such that for \( L \) sufficiently large, \( \varepsilon \) sufficiently small and \( h_F \) sufficiently large depending on \( L \),

\[ \|S(\lambda, K_n)^\sharp\|_{h,G_\kappa,A_\rho,\delta_{n+1}} \leq q \quad \text{when} \quad |\lambda g^{1/4-\eta/3}| \leq c_L \]  

(5.129)

\[ \|S(\lambda, K_n)^\sharp\|_{h_*,A_\rho,\delta_{n+1}} \leq q \quad \text{when} \quad |\lambda g^{11/12-\eta/3}| \leq c_L \]  

(5.130)

where \( h_* = (h_*, h_F) \) and \( \sharp \) denotes integration with respect to \( d\mu_{\Gamma_{\rho,n}}(\xi) \), \( \Gamma_{n,L} = S_{L,\rho+1}\Gamma_n \) being the rescaled fluctuation covariance.

When \( R_n = 0 \) we may set \( \eta = 0 \) in (5.129) and replace \( \lambda g^{1/4-\eta/3} \) by \( \lambda g^{-1/2} \) in (5.130)

**Proof**

We suppress the dependence on \( \lambda \) which plays a passive role in most of the following and make the dependence explicit towards the end when necessary. We will apply the first part of Lemma 5.14A to the the reblocked polymer activity \( BK_n(LZ_{\delta_n}, S_L\Phi, \xi) = BK_n(LZ_{\delta_n}, S_L\varphi, \zeta, S_L\psi, \eta) \) which is a functional of \( \tilde{K}_n \) and \( P_n \) where \( \tilde{K}_n(X_{\delta_n}, \varphi, \zeta, \psi, \eta) = K_n(X_{\delta_n}, \varphi + \zeta, \psi + \eta) \) ( see the definition of reblocking in section 3.1 ). Recall the definition of rescaled polymer activities and rescaled covariances given by (3.22) and (3.21) in the Appendix to section 3. The rescaled, reblocked activities are defined by (3.25). We get by virtue of (5.104) and (5.103)

\[ \|S(K_n)^\sharp(Z_{\delta_{n+1}}, \varphi, 0)\|_h \leq \int d\mu_{\Gamma_{\rho,n}}(\zeta)(||S_L BK_n)(Z_{\delta_n}, \varphi, \zeta, 0)||_h \]  

(5.131)

\[ \|S(K_n)^\sharp(Z_{\delta_{n+1}})\|_h \leq \int d\mu_{\Gamma_{\rho,n}}(\zeta)(||S_L BK_n)(Z_{\delta_n}, 0, \zeta, 0)||_h. \]  

(5.132)

We now prove (5.129) starting from (5.131). This follows the lines of the proof of Lemma 5.16, [BMS]. Unfortunately, in the latter proof a minor error crept in \(^{(1)}\) and we take this opportunity to correct it. Inserting the definition (4.11) in (5.131) and using the multiplicative property of the \( h \) norm we get

\(^{(1)}\) A. Abdesselam, private communication
\[\| (S(K_n)^2)(Z_{\delta_{n+1}}, \varphi, 0) \|_h \leq \sum_{N+M \geq 1} \frac{1}{N!M!} \sum_{(x_{\delta_{n+1}}), (\Delta_{\delta_{n+1}}) \rightarrow LZ\delta_{n+1}} \| e^{-\nu_n L}(Z_{\delta_{n+1}} \cup L^{-1}(X_{\delta_{n+1}} \cup \Delta_{\delta_{n+1}}), \varphi) \|_h \times \]
\[\times \int d\mu_{n,L}(\zeta) \prod_{j=1}^N K_{n,L}(L^{-1} X_{\delta_{n+1}, j}, \varphi, \zeta, 0) \| h \prod_{i=1}^M \| P_n, L(\delta_{n+1}, i, \varphi, \zeta, 0) \|_h \]
whence
\[\| (S(K_n)^2)(Z_{\delta_{n+1}}, \varphi, 0) \|_h \leq 2|Z_{\delta_{n+1}}| \sum_{N+M \geq 1} \frac{1}{N!M!} \sum_{(x_{\delta_{n+1}}), (\Delta_{\delta_{n+1}}) \rightarrow LZ\delta_{n+1}} \int d\mu_{n,L}(\zeta) \hat{G}_{n, \rho}(X_{\delta_{n+1}} \cup \Delta_{\delta_{n+1}}, S_L \varphi, \zeta) \times \]
\[\prod_{j=1}^N \| K_n(X_j) \|_{h_L, \hat{G}_{n, \rho}} \prod_{i=1}^M \| P_n(\Delta_{\delta_{n+1}}) \|_{h_L, \hat{G}_{n, \rho}} \]
(5.133)
where \( h_L = (L^{-d_1} h_B, L^{-d_2} h_F) \), \( X_{\delta_n} = \cup X_{\delta_{n,j}} \), \( \Delta_{\delta_n} = \cup \Delta_{\delta_{n,i}} \) and we have bounded \( e^{-\nu_n L} \) using lemma 5.5 which continues to apply.

Lemma 5.4 bounds the \( \zeta \) integral by
\[2 |X_{\delta_{n+1}} \cup \Delta_{\delta_{n+1}}| G_{3, L}(X_{\delta_{n+1}} \cup \Delta_{\delta_{n+1}}, S_L \varphi) \leq 2 |X_{\delta_{n+1}} \cup \Delta_{\delta_{n+1}}| G_n(L^{-1}(X_{\delta_{n+1}} \cup \Delta_{\delta_{n+1}}), \varphi) \]
\[\leq \prod_{j=1}^M 2 |X_{\delta_{n+1}, j}| \prod_{i=1}^N 2 |\Delta_{\delta_{n+1}, i}| G_{n}(Z_{\delta_{n+1}}, \varphi) \]
since \( L^{-1}(X_{\delta_{n+1}} \cup \Delta_{\delta_{n+1}}) \subset Z_{\delta_{n+1}} \). Moreover for \( L \) sufficiently large
\[\| K_n(X_j) \|_{h_L, \hat{G}_{n, \rho}} \leq \| K_n(X_j) \|_{h, \hat{G}_{n, \rho}} \leq \| K_n(X_j) \|_{h, \hat{G}_{n, \rho}} \]
where we have used Lemma 5.4A, (5.34) in the last step. Therefore
\[\| (S(K_n)^2)(Z_{\delta_{n+1}}) \|_{h, \hat{G}_{n}} \leq 2 |Z_{\delta_{n+1}}| \sum_{N+M \geq 1} \frac{1}{N!M!} \sum_{(x_{\delta_{n+1}}), (\Delta_{\delta_{n+1}}) \rightarrow LZ\delta_{n+1}} \prod_{j=1}^M 2 |X_{\delta_{n+1}, j}| \| K_n(X_j) \|_{h, \hat{G}_{n, \rho}} \times \]
\[\prod_{i=1}^N 2 |\Delta_{\delta_{n+1}, i}| \| P_n(\Delta_{\delta_{n+1}}) \|_{h, \hat{G}_{n, \rho}} \]
From this point on we proceed as in the proof of Lemma 5.16, [BMS], the only difference being is that now we are on a lattice. The condition on the sum over the polymers above implies that \( Z_{\delta_n} = (\cup L^{-1} X_{\delta_{n+1}, j}) \) \( \cup (\cup L^{-1} \Delta_{\delta_{n+1}, i}) \). This also implies that \( Z_{\delta_{n+1}} = (\cup L^{-1} X_{\delta_{n+1}, j}) \) \( \cup (\cup L^{-1} \Delta_{\delta_{n+1}, i}) \).

Multiply both sides by \( A_p(Z_{\delta_{n+1}}) \) and observe on the right hand side
\[A_{p+1}(Z_{\delta_{n+1}}) \leq \prod_{j=1}^M A_{p+1}(L^{-1} X_{\delta_{n+1}, j}) \prod_{i=1}^N A_{p+1}(L^{-1} \Delta_{\delta_{n+1}, i}) \]
\[\leq O(1)^{N+M} \prod_{j=1}^M A_{-2}(X_{\delta_{n+1}, j}) \prod_{i=1}^N A_{-2}(\Delta_{\delta_{n+1}, i}) \]
where we have first used the fact that the \( L \)-closures of the polymers are connected by definition of the reblocking operation, then Lemma 2.2 together with \( |X_{\delta_{n+1}}| = |X_{\delta_n}| \) and \( |\Delta_{\delta_{n+1}, i}| = |\Delta_{\delta_n, i}| \). The last observation follows from our definition of polymers in section 1.3 and (1.75). Therefore
\[\| (S(K_n)^2)(Z_{\delta_{n+1}}) \|_{h, \hat{G}_{n}} A_p(Z_{\delta_{n+1}}) \leq 2 |Z_{\delta_{n+1}}| \sum_{N+M \geq 1} \frac{1}{N!M!} O(1)^{N+M} \sum_{(x_{\delta_{n+1}}), (\Delta_{\delta_{n+1}}) \rightarrow LZ\delta_{n+1}} \]
5:60
Fix any $\Delta_{\delta_{n+1}}$ and sum over $Z_{\delta_{n+1}} \ni \Delta_{\delta_{n}}$. This fixes on the right hand side the sum over $Z_{\delta_{n}} \ni \Delta_{\delta_{n}}$ with $\Delta_{\delta_{n}}$ fixed by restriction. The spanning tree argument of Lemma 7.1 of [BY] controls the sums over $N,M,Z_{\delta_{n}},(X_{j,\delta_{n}}),(\Delta_{i,\delta_{n}}) \to LZ_{\delta_{n}}$ with the result (we have now made explicit the dependence on $\lambda$)

$$
\prod_{j=1}^{M} \|K_{n}(X_{j,\delta_{n}})\|_{h_{\gamma},A_{-1}(X_{j,\delta_{n}})} \prod_{i=1}^{N} \|P_{n}(\Delta_{\delta_{n},i})\|_{h_{\gamma},A_{-1}(\Delta_{\delta_{n},i})}
$$

The proof of (5.129) is completed by Lemmas 5.8 and 5.14. When $R = 0$ we can use Lemma 5.8 and replace Lemma 5.14 by Lemma 5.11.

To prove (5.130) we start from (5.132) and proceed as before. We replace $\hat{\gamma}$ by $\tilde{\gamma}$ and then use Lemma 5.3 to estimate the $\zeta$ integral. We use (5.35) of Lemma 5.4A. Proceeding as before now leads to

$$
\|\langle S(\lambda, K_{n}) \rangle_{h, G_{\lambda}, A_{-1}, \delta_{n+1}} \leq O(1) \sum_{N \geq 1} O(1)^{N} L^{3N} \left( \|K_{\lambda}(\lambda)\|_{h_{\gamma}, G_{\lambda}, A_{-1}, \delta_{n}} + \|P_{\lambda}(\lambda)\|_{h_{\gamma}, G_{\lambda}, A_{-1}, \delta_{n}} \right)^{N}
$$

Now use Lemmas 5.8 and 5.14 to complete the proof of (5.130). Finally when $R = 0$ use Lemmas 5.8 and 5.11 as before. 

**Estimates on relevant parts and flow coefficients from the remainder**

Let $(\hat{\alpha}_{n,p})$ be the coefficients $(\hat{\alpha}_{n,2,0}, \hat{\alpha}_{n,2,1}, \hat{\alpha}_{n,2,1})\gamma_{n,4})$ defined in (4.43) and (4.61). The flow coefficients $\xi_{n,p}$, $\rho_{n,p}$ are given in (4.53).

**Lemma 5.17:** Under the conditions of the domain $D_{n}$ we have

$$
\|R^{2}_{n}\|_{h, G_{\lambda}, A_{-1}, \delta_{n}} \leq \tilde{g}^{3/4-\eta}
$$

$$
\|R^{2}_{n}\|_{\hat{h}_{n}, A_{-1}, \delta_{n}} \leq O(1) \tilde{g}^{11/4-\eta}
$$

$$
|\hat{\alpha}_{n,p}|_{A_{-1}, \delta_{n}} \leq O(1) \tilde{g}^{11/4-\eta}
$$

$$
|\xi_{n}| \leq C_{L} \tilde{g}^{11/4-\eta}
$$

$$
|\rho_{n}| \leq C_{L} \tilde{g}^{11/4-\eta}
$$

where the constants $C_{L}$ are independent of $n$ and $\varepsilon$

**Proof**

(5.134) follows from (5.7) and Lemma 5.15, (5.121). (5.135) follow from (5.8) and lemma 5.15 with $m_{0} = 9$ and $\varepsilon$ sufficiently small depending on $L$ so that $\tilde{g}$ is sufficiently small. In fact in lemma 5.15 (with $K = R_{n}$) the first term has the desired bound by (5.8). By (5.7) together with $h_{B_{\varepsilon}}^{-1} = e\tilde{g}^{1/2}$ and $h_{B_{\varepsilon}} = h_{B_{\varepsilon}}(L) \geq 1$ we see that the second term is bounded by $O(1) \tilde{g}^{11/4-\eta} \leq \tilde{g}^{11/4-\eta}$ for $\tilde{g}$ sufficiently small.

Recall that $\hat{\alpha}_{n,p}(X_{\delta_{n}})$ are supported on small spans. Then (5.136) follows from (4.61) and (5.135). In fact the dominant contribution comes by setting $\hat{V}_{n} = 0$ because the difference gives additional powers of $\tilde{g}$. Then we have

$$
\|\hat{\alpha}_{n,p}|_{A_{-1}, \delta_{n}} \leq O(1)n(P)(\tilde{V}_{n})^{n(P)}|1S| \tilde{R}^{2}_{n}\|_{h_{\lambda}, A_{-1}, \delta_{n}}
$$

5:61
where \( n(P) \) is the number of fields in the monomial \( P \), we have used the shorthand notation \( \hat{h}^{-n(P)} = \max_{n(P),B,n(P)B=n(P)}(\hat{h}^{-n(P)} B \hat{h}^{-n(P)F}) \) and \( 1_S \) is the indicator function on small sets. Now use (5.135) to get (5.136). (5.137), (5.138) follow from (5.136), the definitions (4.53), (4.51) and Wick coefficients \( C (0) \) are uniformly bounded by a \( L \) dependent constant by Corollary 1.2. ■

**Lemma 5.18**

*Under the conditions of \( D_n \) and \( \varepsilon \) sufficiently small depending on \( L \), there exists a constant \( C_L \) independent of \( \varepsilon \) and \( n \) such that*

\[
|g_{n+1} - \bar{g}| < 2\nu \hat{g}^{3/2}, \quad |\mu_{n+1}| < C_L \hat{g}^{2-\delta} \tag{5.139}
\]

**Proof:**

It is convenient to define

\[
\tilde{g}_n = g_n - \bar{g}
\]

Then from the flow equation (4.39) for \( g_n \) and the definition of \( \bar{g} \) in (5.2) we get

\[
\tilde{g}_{n+1} = (2 - L^2)\tilde{g}_n + \tilde{\xi}_n \tag{5.140}
\]

where

\[
\tilde{\xi}_n = -L^2 a_{\bar{g}} \hat{g}_n^2 - L^2 e(a_n - a_{\bar{g}})g_n^2 + \xi_n \tag{5.141}
\]

From Lemma 5.12, Lemma 5.17 and \( g_n \in D_n \) we get for \( \varepsilon \) sufficiently small depending on \( L \) the bound

\[
|\tilde{\xi}_n| \leq C_L \hat{g}^2 \tag{5.142}
\]

Therefore

\[
|\tilde{g}_{n+1}| < \nu \hat{g}((2 - L^2) + \frac{C_L}{\nu} \hat{g}) \tag{5.143}
\]

For \( \varepsilon \) sufficiently small depending on \( L \) we get

\[
|(1 - L^2) + \frac{C_L}{\nu} \hat{g}| \leq 1 \tag{5.144}
\]

Therefore \( |\tilde{g}_{n+1}| \leq 2\nu \hat{g} \) which proves the first inequality of (5.139).

The bound on \( \mu_{n+1} \) follows from the second of the flow equations (4.15), on using \( \mu_n \), \( g_n \) belong to \( D_n \), Lemma 5.12 and the bound (5.138) on \( \rho_n \). ■

As stated in Theorem 3.1 borrowed from [BDH] the assumption of stability of the local potential with respect to perturbation by relevant parts (see (3.18) ensures the extraction estimate of (3.19). The following lemma proves the stability for the case at hand, namely that of \( V_{n,L}(\Delta_{n+1}) \) with respect to the relevant part \( F_n \) defined in section 4.

Recall from (4.12) that \( F_n(\lambda) = \lambda^2 F_{Q_n} + \lambda^3 F_{R_n} \) and from (3.15) that (each part of) \( F_n \) decomposes:

\[
F_n(X_{\delta_{n+1}}) = \sum_{\Delta_{\delta_{n+1}}} F_n(X_{\delta_{n+1}}, \Delta_{\delta_{n+1}}).
\]

**Lemma 5.19**

*For any \( R > 0 \) and \( \xi := R \max(|\lambda^2|\bar{g},|\lambda^3|\hat{g}^{7/4-\eta}) \) sufficiently small,

\[
\|e^{-\tilde{V}_{n,L}(\Delta_{n+1}) - \sum_{X_{\delta_{n+1}}} z(X_{\delta_{n+1}}) F(\lambda, X_{\delta_{n+1}}, \Delta_{\delta_{n+1}})}\|_{h,G_n} \leq 2^2 \tag{5.145}
\]
where \( z(X_{\delta_{n+1}}) \) are complex parameters with \(|z(X_{\delta_{n+1}})| \leq R \).

**Proof**

It is easy to see that Lemma 5.5 still holds if we replace \( \tilde{V}_n \) by \( \tilde{V}_{n,L} \) provided \( \varepsilon \) is sufficiently small. This implies that \( \tilde{g} \) is sufficiently small. We then have

\[
\| e^{-\tilde{V}_{n,L}(\Delta_{n+1}) - \sum_{\delta_{n+1} \supset \Delta_{n+1}} z(X_{\delta_{n+1}}) F(\lambda, X_{\delta_{n+1}}, \Delta_{n+1})} \|_h \leq 2e^{-\tilde{g}/4} \int_{\Delta_{n+1}} dx (|\varphi(x)|^2)^2 + \sum_{\delta_{n+1} \supset \Delta_{n+1}} R \| F(\lambda, X_{\delta_{n+1}}, \Delta_{n+1}) \|_h 
\]

(5.146)

Recall that the relevant parts \( F(X_{\delta_{n+1}}, \Delta_{n+1}) \) are supported on small sets \( X_{\delta_{n+1}} \). The proof now follows easily from the following

**Claim:** For \( \varepsilon \) sufficiently small

\[
|z(X_{\delta_{n+1}})||F(\lambda, X_{\delta_{n+1}}, \Delta_{n+1})| \leq C_L \xi \left( \bar{g} \int_{\Delta_{n+1}} d^3 x (|\varphi(x)|^2)^2 + \frac{\bar{g}}{2} \| |\varphi|^2 \|_{\Delta_{n+1}, 1.5} + 1 \right) 
\]

(5.147)

where \( \| \phi \|_{\Delta_{n+1}, 1.5} \) is the square of the lattice Sobolev norm defined in (2.1).

**Proof of the Claim:**

We have \( \| F(\lambda, X_{\delta_{n+1}}, \Delta_{n+1}) \|_h \leq |\lambda|^2 \| F_Q(X_{\delta_{n+1}}, \Delta_{n+1}) \|_h + |\lambda|^3 \| F_R(X_{\delta_{n+1}}, \Delta_{n+1}) \|_h \).

Consider (4.27)-(4.31). Undo the Wick ordering on the superfield field and note that, by virtue of supersymmetry, no field independent terms arise. The \( m = 1 \) term in (4.30) remains unchanged and in the \( m = 2 \) case there results an additional contribution \( -2C_{n+1}(0) \Phi \Phi \). The Wick constant \( C_{n+1}(0) \) has a uniform bound \( C \) which depends only on \( L \) by Corollary 1.2. We write this in the Grassmann representation and notice that for the \( m = 2 \) case the \( \psi \bar{\psi}(x)^2 \) contribution vanishes by statistics. From the definition of the \( h \) norm with \( h_B = c \bar{g}^{-1/4} \) and \( h_F = h_F(L) \) we get the bound

\[
\| F_Q(X_{\delta_{n+1}}, \Delta_{n+1}) \|_h \leq C_L \bar{g}^2 \left( \bar{g} \int_{\Delta_{n+1}} d^3 x (|\varphi|^2)^2(x) \|_{h_B} \sup_{x \in \Delta_{n+1}} |f_Q^{(2)}(X_{\delta_{n+1}}, x, \Delta_{n+1})| \right) + \sum_{m=1}^{2} \sup_{x \in \Delta_{n+1}} |f_Q^{(m)}(X_{\delta_{n+1}}, x, \Delta_{n+1})| \left( \bar{g} \int_{\Delta_{n+1}} d^3 x (|\varphi|^2)^m(x) \|_{h_B} \right)
\]

Now for \( m = 1, 2 \)

\[
\bar{g} \int_{\Delta_{n+1}} d^3 x (|\varphi|^2)^m(x) \|_{h_B} \leq O(1) \left( \bar{g} \int_{\Delta_{n+1}} d^3 x (|\varphi|^2)^2(x) + 1 \right)
\]

From the definition (4.31) and the estimates obtained in the course of proving Lemma 5.12, we have

\[
\sup_{x \in \Delta_{n+1}} |f_Q^{(m)}(X_{\delta_{n+1}}, x, \Delta_{n+1})| \leq C_L
\]

Therefore

\[
|\lambda|^2 |z(X_{\delta_{n+1}})| \| F_Q(X_{\delta_{n+1}}, \Delta_{n+1}) \|_h \leq C_L R |\lambda|^2 \bar{g} \left( \bar{g} \int_{\Delta_{n+1}} d x (|\varphi|^2)^2(x) + 1 \right) 
\]

(5.148)

Next consider \( F_R \), supported on small sets, defined in (4.43), (4.46). Recall (4.48),

\[
F_R(X_{\delta_{n+1}}, \Phi) = \sum_p \int_{\Delta_{n+1}} d x \alpha_p(X_{\delta_{n+1}}, x) P(\Phi(x), \partial_{\delta_{n+1}} \Phi(x))
\]

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By Lemma 5.17 and (4.49) we have $|\alpha P(X_{\delta_{n+1}}, x)| \leq C_Lg^{31/4-\eta}$, so that

$$|\lambda|^3|z(X_{\delta_{n+1}})|\|F_R(X_{\delta_{n+1}}, \Delta_{n+1})\|_h \leq C_L|R|\lambda|^3g^{31/4-\eta}\sum_{p} \int_{\Delta_{n+1}} dx \|P(\Phi(x), \partial_{\delta_{n+1}+1}\Phi(x))\|_h$$

$$\leq C_L|R|\lambda|^3g^{7/4-\eta}\left(\bar{g} \int_{\Delta_{n+1}} dx (|\varphi|^2)^2(x) + \bar{g}^{1/2}||\varphi||^2_{\Delta_{n+1}+1, 1, 5} + 1\right)$$

The claim follows by combining this with (5.148). In the above inequality the Sobolev norm when estimating the term giving arise to $\phi \partial_{\delta_{n+1}}\phi$. We bound $|\phi \partial_{\delta_{n+1}}\phi| \leq 1/2(||\varphi||^2 + ||\partial_{\delta_{n+1}}\phi||^2)$ and then use the lattice Sobolev embedding inequality. ■

**Lemma 5.20**

For any $R > 0$ and $\xi := R \max(|\lambda^2|\bar{g}^2, |\lambda^3|\bar{g}^{31/4-\eta})$ sufficiently small,

$$|e^{-\tilde{V}_{n,\epsilon}(\Delta_{n+1})-\sum_{\lambda_{\delta_{n+1}} \supset \Delta_{n+1}} z(X_{\delta_{n+1}})F(\lambda, X_{\delta_{n+1}}, \Delta_{n+1})}|_{h_*} \leq 2^{|z(X)|}$$

(5.149)

where $z(X)$ are complex parameters with $|z(X)| \leq R$.

**Proof**

The proof is similar to the previous one except that we can use the estimate $|F(\lambda, X, \Delta)|_{h_*} \leq C_L\xi$ in place of (5.147) since the $h_*$ norm is computed with field derivatives at $\Phi = 0$. ■

We will now bound the remainder $R_{n+1}$ given in (4.41). It consist of a sum of four contributions, namely $R_{n+1, \text{main}}, R_{n+1, \text{linear}}, R_{n+1, \text{3}}$ and $R_{n+1, \text{4}}$ which we will estimate in turn. These estimates parallel those obtained for the continuum bosonic theory in [BMS].

Recall from (4.36) that

$$R_{n+1, \text{main}} = \frac{1}{2\pi i} \int_{\gamma} d\lambda \frac{\lambda^4}{\lambda^4} \mathcal{E}\left(S(\lambda, Q_{\epsilon}e^{-V_{\epsilon}})^2, F_{Q_{\epsilon}}(\lambda)\right)$$

(5.150)

**Lemma 5.21**

$$\|R_{n+1, \text{main}}\|_{h, G, A, \delta_{n+1}} \leq C_L\bar{g}^{3/4}$$

(5.151)

$$|R_{n+1, \text{main}}|_{h, A, \delta_{n+1}} \leq C_L\bar{g}^{3-3\delta/2}$$

(5.152)

**Proof**

The proof is identical to that of Lemma 5.21 of [BMS] except that we replace $\epsilon$ by $\bar{g}$, put in lattice subscript $n$ where appropriate, and note that the field independent piece $F_0$ is now absent. We apply Theorem 3.1 (which is a restatement of Theorem 5 in section 4.2 of [BDH-est]) instead of Theorem 6 of [BDH-est]. ■

Recall from (4.38) that

$$R_{n+1, \text{3}} = \frac{1}{2\pi i} \int_{\gamma} d\lambda \frac{\lambda^4}{\lambda^4(\lambda - 1)} \mathcal{E}\left(S(\lambda, K_{\epsilon})^2, F_{\epsilon}(\lambda)\right)$$

(5.153)

**Lemma 5.22**

$$\|R_{n+1, \text{3}}\|_{h, G, A, \delta_{n+1}} \leq C_L\bar{g}^{-4\gamma/3}$$

(5.154)

$$|R_{n+1, \text{3}}|_{h, A, \delta_{n+1}} \leq C_L\bar{g}^{10/3-4\gamma/3}$$

(5.155)
Proof
The proof follows the lines of that of Lemma 5.21. To prove (5.154) we take the contour $\gamma$ to of radius $|\lambda| = c_L\tilde{g}^{-(1/4-n/3)}$. This ensures that the hypothesis of Lemma 5.16, (5.130) is satisfied and $\xi$ of Lemma 5.19 is sufficiently small so that stability holds. (5.154) now follows from the extraction estimate (3.19) and the Cauchy bound as before. To prove (5.154) we take $\gamma$ to be of radius $|\lambda| = c_L\tilde{g}^{-(5/6-n/3)}$. Then for $\tilde{g}$ sufficiently small the hypothesis for (5.130) of Lemma 5.16 is satisfied. Moreover then $\xi$ of Lemma 5.20 is sufficiently small and Lemma 5.20 holds. (5.154) now follows from the extraction estimate (3.20) and the Cauchy bound as before.

From the definition of $R_{n+1,4}$ in (4.40) we have

$$R_{n+1,4} = \left( e^{-V_{n+1}} - e^{-\tilde{V}_{n,L}} \right) Q(C_{n+1}, \omega_{n+1}, g_{n+1}) + e^{-\tilde{V}_{n,L}} Q(C_{n+1}, \omega_{n+1}, (g_{n+1}^2 - g_{n,L}^2))$$

**Lemma 5.23**

$$\|R_{n+1,4}\|_{h,A,\delta_{n+1}} \leq C_L\tilde{g}^{3/2} \quad |R_{n+1,4}|_{h,,A,\delta_{n+1}} \leq C_L\tilde{g}^3$$

**Proof**
The proof is the same as that of Lemma 5.23 of [BMS] except that we replace $\varepsilon$ by $\tilde{g}$, insert lattice subscript $n$ as appropriate and use the lattice counterparts (that we have already established) of the lemmas exploited in [BMS] for the proof.

**Lemma 5.24**

Let $X_{\delta_n}$ be a small set and let $J(X_{\delta_n}, \Phi)$ be normalized as in (4.59). Recall that the rescaled activity $J_L$ is defined by $J_L(L^{-1}X_{\delta_n+1}, \varphi, \psi) = J(X_{\delta_n}, \varphi_{L^{-1}}, \psi_{L^{-1}})$ where $\varphi_{L^{-1}} = S_L\varphi$ and $\psi_{L^{-1}} = S_L\psi$. Then we have

1. For $2p + m = 2$ so that $(p, m) = (1, 0), (0, 2)$

$$\|D^{2p,m}J_L(L^{-1}X_{\delta_{n+1}}, 0, 0)\| \leq O(1)L^{-(7-\varepsilon)/2}\|D^{2p,m}J(X_{\delta_n}, 0, 0)\|$$  \hspace{1cm} (5.156)

2. For $2p + m = 4$ so that $(p, m) = (2, 0), (1, 2), (0, 4)$

$$\|D^{2p,m}J_L(L^{-1}X_{\delta_{n+1}}, 0, 0)\| \leq O(1)L^{-(4-\varepsilon)}\|D^{2p,m}J(X_{\delta_n}, 0, 0)\|$$  \hspace{1cm} (5.157)

**Proof**: In the proof of this lemma we will need to use the lattice Taylor expansion introduced in (5.13),(5.14) and (5.15) with a particular choice of a lattice path joining two points. The polymer $X_{\delta_n}$ being a small set is connected. It can be represented as $X_{\delta_n} = X \cap (\delta_n \mathbb{Z})$ where $X$ is a continuum connected polymer which is a small set. By the argument in the proof of Lemma 5.1 it suffices to consider the case when $X_{\delta_n}$ is a block. Then the lattice path lies entirely in $X_{\delta_n}$.

Let $u_k$ be a function defined on $(\tilde{X}_{\delta_n}^{(2)})^k$ for $k \geq 2$. In the following $u$ is one of the test functions of subsection 2.2. Thus $u$ will represent either one of the functions $f_j$ defined on $\tilde{X}_{\delta_n}^{(2)}$ giving a direction for a bosonic derivative or a function $g_{2p}$ defined on $(\tilde{X}_{\delta_n}^{(2)})^{2p} = (\tilde{X}_{\delta_n}^{(2)})^p \times (\tilde{X}_{\delta_n}^{(2)})^p$ associated with a fermionic derivative of order $2p$. Note that $g_{2p}$ is restricted to be antisymmetric in the sense explained in the lines preceding equation (2.12). The $C^2(X_{\delta_n}^k)$ norms of these functions for $k = 1, 2p$ are defined as in (2.14) and (2.15) of subsection 2.2.

We recall the definition of the rescaled function

$$S_Lu_k(x) = u_{k,L^{-1}}(x) = L^{-kd}u_k(\frac{x}{L})$$
Observe that

\[ \|u_{k,L}^{-1}\|_{C^2(X_{\delta_n^k}^k)} \leq L^{-kd}\epsilon \|u_k\|_{C^2(L^{-1}X_{\delta_n^k+1})} \]  

(5.158)

Let \( e_1, e_2, \ldots, e_{3k} \) be the basis vectors of \((\delta_n\mathbb{Z})^{3k}\). Let \( x = \sum_{i=1}^{3k} (x, e_i) e_i \) denote a point in \( X_{\delta_n^k}^k \). Fix a point \( x_0 \in X_{\delta_n}^k \). We write \( x - x_0 = \delta_n \sum_{i=1}^{3k} h_i e_i e_i \) where \( h_i \) are non-negative integers and \( \epsilon_i = \text{sign}(x - x_0) \).

From (5.15) we have

\[ u_{k,L}^{-1}(x) = u_{k,L}^{-1}(x_0) + \sum_{i=1}^{3k} ((x - x_0), e_i) \partial_{\delta_n,e_i,e_i} u_{k,L}^{-1}(x_0) + \delta_n^2 \sum_{i,j=1}^{3k} \sum_{s_i=0}^{h_i-1} \sum_{s_j=0}^{h_j-1} \partial_{\delta_n,e_i,e_i} u_{k,L}^{-1}(x_0 + p_j (p_i (x - x_0, s_i), s_j)) \]

The argument of \( u_{k,L}^{-1} \) in the last term lies entirely in \( X_{\delta_n}^k \) since \( X_{\delta_n} \) is a block. Define

\[ \delta u_{k,L}^{-1}(x) = u_{k,L}^{-1}(x) - u_{k,L}^{-1}(x_0) = \delta_n \sum_{j=1}^{3k} \partial_{\delta_n,e_j,e_j} u_{k,L}^{-1}(x_0 + p_j (p_i (x - x_0, s_i), s_j)) \]

(5.159)

and

\[ \delta^2 u_{k,L}^{-1}(x) = h_{k,L}^{-1}(x) - u_{k,L}^{-1}(x_0) - \sum_{i=1}^{3k} ((x - x_0), e_i) \partial_{\delta_n,e_i,e_i} u_{k,L}^{-1}(x_0) \]

= \[ \delta_n^2 \sum_{i,j=1}^{3k} \sum_{s_i=0}^{h_i-1} \sum_{s_j=0}^{h_j-1} \partial_{\delta_n,e_i,e_i} \partial_{\delta_n,e_j,e_j} u_{k,L}^{-1}(x_0 + p_j (p_i (x - x_0, s_i), s_j)) \]

(5.160)

Now from (5.159), (5.160) we have using the definition of the rescaled function \( u_{k,L}^{-1} \)

\[ \delta u_{k,L}^{-1}(x) = L^{-(1+k\frac{3k}{4})} \delta_n \sum_{j=1}^{3k} \partial_{\delta_n,e_j,e_j} u_{k,L}^{-1}(L^{-1}(x_0 + p_j (p_i (x - x_0, s_i), s_j))) \]

and for \( l \geq 1 \)

\[ \partial_{\delta_n^l} \delta u_{k,L}^{-1}(x) = L^{-(1+k\frac{3k}{4})} (\partial_{\delta_n^l} u_k)(L^{-1}x) \]

where for \( \partial_{\delta_n^l} \) a multi-index convention is implicit. This implies that

\[ \|\delta u_{k,L}^{-1}\|_{C^2(X_{\delta_n}^k)} \leq c_1 L^{-(1+k\frac{3k}{4})}\|u_k\|_{C^2(L^{-1}X_{\delta_n^k+1})} \]

(5.161)

where \( c_1 = O(1) \) since \( X \) is a small set. In the same way starting from (5.160) a little bit of work shows that

\[ \|\delta^2 u_{k,L}^{-1}\|_{C^2(X_{\delta_n}^k)} \leq c_2 L^{-(2+k\frac{3k}{4})}\|u_k\|_{C^2(L^{-1}X_{\delta_n^k+1})} \]

(5.162)

where \( c_2 = O(1) \).

We will first prove the bounds of (5.161).

Consider the case \((p,m) = (1, 0)\). We have

\[ D^{2,0} J_L(L^{-1}X_{\delta_n+1}, 0, 0; g_2) = D^{2,0} J(X_{\delta_n}, 0, 0; g_{2,L^{-1}}) = D^{2,0} J(X_{\delta_n}, 0, 0; \delta^2 g_{2,L^{-1}}) \]

(5.163)
where we have Taylor expanded the function \( g_{2,L^{-1}} \) as in (5.160) and then used the first and second normalization conditions in (4.59). Therefore

\[
|D^{2,0}_L(L^{-1}X_{\delta_n+1}, 0; g_2)| \leq \|D^{2,0}_L(X_{\delta_n}, 0; 0)\| \|\delta^2 g_{2,L^{-1}}\|_{C^2(X_{\delta_n}^2)} \\
\leq O(1)L^{-7-\varepsilon)/2} \|D^{2,0}_L(X_{\delta_n}, 0)\| \|g_2\|_{C^2(L^{-1}X_{\delta_n+1}^2)}
\]

where in the last step we have used the bound in (5.162) for \( k = 2 \) This proves the case \((p,m) = (1,0)\) of the lemma.

To prove the case \((p,m) = (0,2)\) we Taylor expand the function \( f_{L^{-1}}(x) \) to second order, then use the first and the third conditions in (4.59) to get

\[
D^{0,2}_L(X_{\delta_n}, 0; f_{L^{-1}}^{2^2}) = D^{0,2}_L(X_{\delta_n}, 0; f_{1,L^{-1}}(x_0), \delta^2 f_{2,L^{-1}}) + D^{0,2}_L(X_{\delta_n}, 0; \delta^2 f_{1,L^{-1}}, \delta^2 f_{2,L^{-1}})
\]

Therefore

\[
|D^{0,2}_L(X_{\delta_n}, 0; f_{L^{-1}}^{2^2})| \leq \|D^{0,2}_L(X_{\delta_n}, 0, 0)\| \left( \|f_{1,L^{-1}}\|_{C^2(X_{\delta_n})} \|\delta^2 f_{2,L^{-1}}\|_{C^2(X_{\delta_n})} \\
+ \|f_{2,L^{-1}}\|_{C^2(X_{\delta_n})} \|\delta^2 f_{1,L^{-1}}\|_{C^2(X_{\delta_n})} \\
+ \|\delta f_{1,L^{-1}}\|_{C^2(X_{\delta_n})} \|\delta f_{2,L^{-1}}\|_{C^2(X_{\delta_n})} \right) \\
\leq O(1)L^{-7-\varepsilon)/2} \|D^{0,2}_L(X_{\delta_n}, 0, 0)\| \prod_{j=1}^2 \|f_j\|_{C^2(L^{-1}X_{\delta_n+1}^1)}
\]

where we have used the bounds (5.158), (5.161) and (5.162) for the case \(k = 1\). This proves the case \((p,m) = (0,2)\).

Next we prove the bounds (5.157) . For this case \((p,m) = (2,0), (1,2), (0,4)\) so that \(2p + m = 4\). Taylor expand test functions around the fixed point \( x_0 \in X_{\delta_n} \) to first order with remainder. We get for \((p,m) = (2,0)\)

\[
D^{4,0}_L(L^{-1}X_{\delta_n+1}, 0; g_4) = D^{4,0}_L(X_{\delta_n}, 0; g_{4,L^{-1}}) = D^{4,0}_L(X_{\delta_n}, 0; \delta g_{4,L^{-1}})
\]

where we have Taylor expanded the function \( g_{4,L^{-1}} \) as in (5.159) and then used (4.58).

Therefore exploiting the bound (5.161) for \(k = 4\) we get

\[
|D^{4,0}_L(L^{-1}X_{\delta_n+1}, 0, 0; g_4)| \leq O(1)L^{-4-\varepsilon)} \|D^{4,0}_L(X_{\delta_n}, 0)\| \|g_4\|_{C^2(L^{-1}X_{\delta_n+1}^4)}
\]

which proves the case \((p,m) = (2,0)\).

Next we turn to the case \((p,m) = (1,2)\). We have

\[
D^{2,2}_L(L^{-1}X_{\delta_n+1}, 0; f_x^{2^2}, g_2) = D^{2,2}_L(X_{\delta_n}, 0; f_{L^{-1}}^{2^2}, g_{2,L^{-1}}) = D^{2,2}_L(X_{\delta_n}, 0; \delta f_{L^{-1}}^{(1)}, f_{L^{-1}}^{(2)}, g_{2,L^{-1}}) + \\
D^{2,2}_L(X_{\delta_n}, 0; f_{L^{-1}}^{(1)}(x_0), \delta f_{L^{-1}}^{(2)}, g_{2,L^{-1}}) + D^{2,2}_L(X_{\delta_n}, 0; f_{L^{-1}}^{(1)}(x_0), f_{L^{-1}}^{(2)}(x_0), \delta g_{2,L^{-1}})
\]

where we have used the fourth condition in (4.59). Therefore

\[
|D^{2,2}_L(L^{-1}X_{\delta_n+1}, 0; f_x^{2^2}, g_2)| \leq \|D^{2,2}_L(X_{\delta_n}, 0, 0)\| \left( \|\delta f_{L^{-1}}^{(1)}\|_{C^2(X_{\delta_n})} \|f_{L^{-1}}^{(2)}\|_{C^2(X_{\delta_n})} \|g_{2,L^{-1}}\|_{C^2(X_{\delta_n})} \\
+ \|\delta f_{L^{-1}}^{(2)}\|_{C^2(X_{\delta_n})} \|f_{L^{-1}}^{(1)}\|_{C^2(X_{\delta_n})} \|g_{2,L^{-1}}\|_{C^2(X_{\delta_n})} \\
+ \|\delta g_{2,L^{-1}}\|_{C^2(X_{\delta_n})} \|f_{L^{-1}}^{(1)}\|_{C^2(X_{\delta_n})} \|f_{L^{-1}}^{(2)}\|_{C^2(X_{\delta_n})} \right)
\]

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Then using the bounds (5.158), (5.161) for \(k = 1, 2\) and (5.162) for \(k = 1\) we get

\[
|D^{2,2} J_L(L^{-1} X_{\delta_{n+1}}, 0, f \times^2, g_2)| \leq O(1) L^{-((4-\epsilon)} \|D^{2,2} J(X_{\delta_n}, 0)\| \prod_{j=1}^{2} \|f^{[j]} \| C^2(L^{-1} X_{\delta_{n+1}}) \|g_2\| C^2(L^{-1} X_{\delta_{n+1}})
\]

which proves the case \((p, m) = (1, 2)\).

Finally we treat the case \((n, m) = (0, 4)\). Let \(N_4 = (1, 2, 3, 4)\). Then using the fourth condition of (4.59) we get

\[
D^{0,4} J_L(L^{-1} X_{\delta_{n+1}}, 0, f \times^4) = D^{0,4} J(X_{\delta_n}, 0, f \times^4) = \sum_{I \subseteq N_4, |I| \neq 4} D^{0,4} J(X_{\delta_n}, 0, f_{L^{-1}(x_0)} \times^{|I|}, \delta f_{L^{-1}}^{X_{\delta_n}}(I))
\]

where \(f \times^{|I|} = (f_j)_{j \in I}\).

Therefore

\[
|D^{0,4} J_L(L^{-1} X_{\delta_{n+1}}, 0, f \times^4)| \leq \|D^{0,4} J(X_{\delta_n}, 0, 0)\| \sum_{I \subseteq N_4, |I| \neq 4} \|f_{L^{-1}} \| C^2(X_{\delta_n}) \|\delta f_{L^{-1}}^{X_{\delta_n}}(I)\|
\]

Because of the condition on \(I\) in the above sum \(|I_c| \geq 1\). Therefore using the bounds (5.158) and (5.161) we get

\[
|D^{0,4} J_L(L^{-1} X_{\delta_{n+1}}, 0, 0, f \times^4)| \leq O(1) L^{-(4-\epsilon)} \|D^{0,4} J(X_{\delta_n}, 0, 0)\| \prod_{j=1}^{4} \|f_j\| C^2(L^{-1} X_{\delta_{n+1}})
\]

which proves the case \((p, m) = (0, 4)\) and thus completes the proof of Lemma 5.24.

**Corollary 5.25**

Let \(Y_{\delta_{n+1}} = L^{-1} X_{\delta_{n+1}}\) where \(X\) is a small set, \(Z_{\delta_{n+1}} = L^{-1} X_{\delta_{n+1}}^L\) and let \(J(X_{\delta_n}, \Phi)\) be normalized as in (4.59). By definition \(J_L(Y_{\delta_{n+1}}, \Phi) = J(X_{\delta_n}, S_L \Phi)\). Then

\[
|J_L(Y_{\delta_{n+1}})|_{\bar{h}} \leq O(1) L^{-(\gamma-\epsilon)/2} |J(X_{\delta_n})|_{\bar{h}}
\]

(5.167)

\[
\|J_L(Y_{\delta_{n+1}}) e^{-Y_L(Z_{\delta_{n+1}} \setminus Y_{\delta_{n+1}})} \|_{G_{\alpha}} \leq O(1) L^{-(\gamma-\epsilon)/2} \left[ |J(X_{\delta_n})|_{\bar{h}} + \|J(X_{\delta_n})\|_{\bar{h}, G_{\alpha}} \right]
\]

(5.168)

\[
|J_L(Y_{\delta_{n+1}})|_{\bar{h}*} \leq O(1) L^{-(\gamma-\epsilon)/2} |J(X_{\delta_n})|_{\bar{h}*}
\]

(5.169)

where (see Lemma 5.15) \(\bar{h} = (\bar{h}_F, h_B)\), \(\bar{h}_* = (\bar{h}_F, h_{B*})\) and \(\bar{h}_F = h_F/2\).

**Proof**

(5.167), (5.169) follow easily from Lemma 5.24 taking advantage of the scaling present to shift from \(h\) to \(\bar{h}\).

To see this observe that since \(h_F = 2\bar{h}_F\) we have from the definition of the \(h\) norm

\[
|J_L(Y_{\delta_{n+1}})|_h = \sum_{n=0}^{\infty} \sum_{m=0}^{m_0} \bar{h}_F^{2n} \frac{h_{B*}^{2n}}{m!} 2^{2n} \|D^{2n,m} J_L(Y_{\delta_{n+1}}, 0)\|
\]

(5.170)

Only terms with \(2n + m\) even contribute. For \(2n + m \leq 4\) use Lemma 5.24 and observe that \(2^{2n} \leq O(1)\). For \(2n + m \geq 6\), we have for \(L\) sufficiently large and \(\epsilon\) sufficiently small depending on \(L\)
Recall that \( m \) by scaling from where we used Lemmas 5.5 and Lemma 5.15. By (5.167), and rewriting the second term by moving the \( h \) for any \( \gamma \). First we prove (5.171). Proof

\[
\| J_L(Y_{\delta_{n+1}}, \Phi)e^{-\tilde{V}_L(Z_{\delta_{n+1}} \setminus Y_{\delta_{n+1}}, \Phi)} \|_h \leq \| J_L(Y_{\delta_{n+1}}, \Phi)\|_h \| e^{-\tilde{V}_L(Z_{\delta_{n+1}} \setminus Y_{\delta_{n+1}}, \Phi)} \|_h \\
\leq O(1)G_\kappa(Z_{\delta_{n+1}}, \phi) \left[ \| J_L(Y_{\delta_{n+1}}) \|_h + L^{-\text{modz}} \| J_L(Y_{\delta_{n+1}}) \|_{h,F,L} \right]
\]

where we used Lemmas 5.5 and Lemma 5.15. By (5.167), and rewriting the second term by moving the scaling from \( J \) to the norm,

\[
\| J_L(Y_{\delta_{n+1}}, \Phi)e^{-\tilde{V}_L(Z_{\delta_{n+1}} \setminus Y_{\delta_{n+1}}, \Phi)} \|_h \leq O(1)G_\kappa(Z_{\delta_{n+1}}, \varphi) \left[ L^{-(7-\varepsilon)/2} \| J(X_{\delta_n}) \|_h + L^{-\text{modz}} \| J(X_{\delta_n}) \|_{\hat{h},G_{\kappa}} \right]
\]

Recall that \( m_0 = 9 \) and the scaling dimension \( d_s = (3-\varepsilon)/4 \). (5.168) now follows by multiplying both sides by \( G_\kappa^{-1}(Z_{\delta_{n+1}}, \varphi) \), and taking the supremum over \( \varphi \). ■

**Lemma 5.26**

\[
\| \tilde{F}_{R_n} e^{-\tilde{V}_n} \|_{h, G_{\kappa}, A, \delta_n} \leq O(1)\varepsilon^{3/4 - \eta} \quad (5.171)
\]

\[
| \tilde{F}_{R_n} e^{-\tilde{V}_n} |_{\hat{h}, A, \delta_n} \leq O(1)\varepsilon^{11/4 - \eta} \quad (5.172)
\]

and \( J_n = R_n^2 - \tilde{F}_{R_n} e^{-\tilde{V}_n} \) satisfies on small sets the bounds

\[
\| J_n \|_{\hat{h}, G_{\kappa}, A, \delta_n} \leq O(1)\tilde{g}^{3\eta - \eta} \quad | J_n |_{\hat{h}, A, \delta_n} \leq O(1)\tilde{g}^{3\eta - \eta} \quad | J_n |_{\hat{h}, A, \delta_n} \leq O(1)\tilde{g}^{3\eta - \eta} \quad (5.173)
\]

**Proof**

First we prove (5.171). \( \tilde{F}_{R_n} \) is defined in (4.43) and (4.45), and is supported on small sets. We estimate its \( h \) norm as in the proof of Lemma 5.19.

\[
\| \tilde{F}_{R_n}(X_{\delta_n}, \Phi) \|_h \leq \sum_P | \tilde{a}_{n,P}(X_{\delta_n}) | \int_{X_{\delta_n}} dx \| P(\Phi(x), \partial \Phi(x)) \|_h \\
\leq C_L \sum_P | \tilde{a}_{n,P}(X_{\delta_n}) | \tilde{g}^{-1} \left( \tilde{g} \int_{X_{\delta_n}} dx (|\varphi|^2)^2 (x) + \tilde{g}^{1/2} \| \varphi \|_{X_{\delta_n,1},\sigma}^2 + 1 \right) \\
\leq C_L \sum_P | \tilde{a}_{n,P}(X_{\delta_n}) | \tilde{g}^{-1} G_\kappa(X_{\delta_n}, \varphi) \tilde{g} \int_{X_{\delta_n}} d\gamma (|\varphi|^2)^2 (y)
\]

for any \( \gamma = O(1) > 0 \). Hence, using Lemma 5.5

\[
\| \tilde{F}_{R_n}(X_{\delta_n}, \Phi)e^{-\tilde{V}_n(X_{\delta_n}, \Phi)} \|_h \leq \| \tilde{F}_{R_n}(X_{\delta_n}, \Phi) \|_h \| e^{-\tilde{V}_n(X_{\delta_n}, \Phi)} \|_h \leq C_L \sum_P 2^{|X_{\delta_n}|} \tilde{a}_{n,P}(X_{\delta_n}) \tilde{g}^{-1} G_\kappa(X_{\delta_n}, \varphi)
\]
We thus obtain (remembering that $\tilde{\alpha}_{n,P}$ are supported on small sets) on using (5.136) of Lemma 5.17 for $\varepsilon$ sufficiently small depending on $L$, implying $\bar{g}$ sufficiently small,

$$\|\tilde{F}_{R_{n}}(X_{\delta_{n}}, \Phi)e^{-\tilde{V}_{n}(X_{\delta_{n}}, \Phi)}\|_{h,G_n, A, \delta_n} \leq C\bar{g}^{-1}\sum_{P}\|\tilde{\alpha}_{n,P}\|_{A, \delta_n} \leq O(1)\bar{g}^{3/4-\eta}$$

This proves (5.171). Now we turn to the proof of (5.172). As observed in the proof of Lemma 5.17, $\tilde{h}_{nP}^a|\tilde{\alpha}_{n,P}|_{A, \delta_n} \leq n(P)!\|sR_{n}^{d}|\tilde{h}_{n,A, \delta_n} \leq O(1)\bar{g}^{11/4-\eta}$. We have from the definition of $\tilde{F}_{R_{n}}$, given in (4.43) $|\tilde{F}_{R_{n}}(X_{\delta_{n}})|_{\tilde{h}_{n}} \leq O(1)\sum_{P}|\tilde{\alpha}_{n,P}(X_{\delta_{n}})|\tilde{h}_{n,P}^{a}$, whence

$$|\tilde{F}_{R_{n}}|_{\tilde{h}_{n,A, \delta_n}} \leq \sum_{P}|\tilde{\alpha}_{n,P}|_{A, \delta_n}\tilde{h}_{n,P}^{a} \leq O(1)\bar{g}^{11/4-\eta}$$

which proves (5.172).

To get these bounds for $J_{n} = R_{n}^{d} - \tilde{F}_{R_{n}}e^{-\tilde{V}_{n}}$ we use (5.171) and (5.172) to bound $\tilde{F}_{R_{n}}e^{-\tilde{V}_{n}}$ part. We can substitute $\tilde{h}$ for $h$ in (5.171) since the $\tilde{h}$ norm is smaller than the $h$ norm. We bound $R_{n}^{d}$ by Lemma 5.17.

We have also used the trivial bound $|J_{n}|_{\tilde{h}_{n,A, \delta_n}} \leq \|J\|_{\tilde{h},G_{3n}, A, \delta_n}$ to obtain the second inequality for $J$ from the first.

**Lemma 5.27**

$$\|R_{n+1, \text{linear}}\|_{h,G_n, A, \delta_{n+1}} \leq O(1)L^{-(1-\varepsilon)/2}\bar{g}^{3/4-\eta}$$

(5.174)

$$|R_{n+1, \text{linear}}|_{h_{n}, A, \delta_{n+1}} \leq O(1)L^{-(1-\varepsilon)/2}\bar{g}^{11/4-\eta}$$

(5.175)

**Remark:** This is a crucial lemma which also figures as Lemma 5.27 in [BMS] and its proof is the same. For the readers benefit we give the details below. The proof is based on the principle that the contribution to the linearized part of the remainder from large sets is very small. For small sets the expanding contributions have been subtracted out leading to normalized polymer activities and this is sufficient to provide a contracting factor.

**Proof**

Let $R_{n+1, \text{linear}}$, given in (4.47) is the sum of two terms which represent contributions from small/large sets respectively. Let $R_{n+1, \text{linear, s,s}}$ denote the first term:

$$R_{n+1, \text{linear, s,s}}(Z_{\delta_{n+1}}) = \sum_{X_{\delta_{n+1}} \text{small sets}} e^{-V_{L}(Z_{\delta_{n+1}} \setminus Y_{\delta_{n+1}})}J_{n,L}(Y_{\delta_{n+1}})$$

(5.176)

where $Y_{\delta_{n+1}} = L^{-1}X_{\delta_{n+1}}$ and $J_{n} = R_{n}^{d} - \tilde{F}_{R_{n}}e^{-\tilde{V}_{n}}$. By Corollary 5.25 we get

$$\|R_{n+1, \text{linear, s,s}}(Z_{\delta_{n+1}})\|_{h,G_{n}} \leq O(1)L^{-(7-\varepsilon)/2}\sum_{X_{\delta_{n+1}} \text{small sets}}|J_{n}(X_{\delta_{n}})|_{\tilde{h}_{n}} + \|J(X_{\delta_{n}})\|_{\tilde{h}, G_{3n}}$$

(5.177)

Note that $Z_{\delta_{n+1}}$ fixes $Z_{\delta_{n}}$ by restriction and the sum on the right hand side is the same as the sum over $X_{\delta_{n}}$ such that $L^{-1}X_{\delta_{n}} = Z_{\delta_{n}}$. We multiply both sides by $A(Z_{\delta_{n+1}})$. On the right hand side we have $A(Z_{\delta_{n+1}}) = A(Z_{\delta_{n}}) = A(X_{\delta_{n}}) \leq O(1)A(X_{\delta_{n}})$ by (2.10). We fix a unit block $\Delta_{n+1}$ and sum over $Z_{\delta_{n+1}} \supset \Delta_{n+1}$. This fixes by restriction to the over $Z_{\delta_{n}} \supset \Delta_{n}$ on the right hand side. The argument on p.790 of [BDH-est] controls the constrained sum on $X_{\delta_{n}}$ such that $L^{-1}X_{\delta_{n}} = Z_{\delta_{n}} \supset \Delta_{n}$ by $L^{3}$ times the sum over $X_{\delta_{n}} \supset \Delta_{n}$. Taking then the supremum over the fixed unit block gives
\[ \| R_{n+1,\text{linear},s}\|_{h,G_{\alpha},A,\delta_{n+1}} \leq O(1)L^{-\varepsilon_2/2}L^3 \left[ |J_n|_{h,A,\delta_n} + \| J_n \|_{h,G_{\delta_n},A,\delta_n} \right] \]

\[ \leq O(1)L^{-(1-\varepsilon)/2}g^{3/4-\eta} \]  

(5.178)

where for the second inequality we used Lemma 5.26.

The second term in (4.47) for \( R_{n+1, \text{linear}} \) which gets contributions only from large sets is

\[ R_{n+1, \text{linear}, l.s.}(Z_{\delta_{n+1}}) = \sum_{L^{-1}X_{\delta_{n+1}}=Z_{\delta_{n+1}}} \sum_{X_{\delta_{n+1}} \text{large sets}} e^{-\tilde{V}_L(Z_{\delta_{n+1}})} R_{n,L}^L(L^{-1}X_{\delta_{n+1}}) \]

(5.179)

where we have used \( J_n = R^L_n \) since the relevant part \( \tilde{F}_n \) is supported on small sets. We first bound in the \( h \) norm and observe that because of the rescaling involved and Lemma 5.17

\[ \| R_{n+1, \text{linear}, l.s.}(Z_{\delta_{n+1}}) \|_{h,G_{\alpha}} \leq O(1)g^{3/4-\eta} \sum_{X_{\delta_{n+1}} \text{large sets}} 2^{[X_{\delta_{n+1}}]} \]

so that on using Lemma 5.5 for \( e^{-\tilde{V}_L} \)

\[ \| R_{n+1, \text{linear}, l.s.}(Z_{\delta_{n+1}}) \|_{h,G_{\alpha}} \leq O(1)g^{3/4-\eta} \sum_{X_{\delta_{n+1}} \text{large sets}} 2^{[X_{\delta_{n+1}}]} \]

(5.180)

We estimate the \( A \) norm as before except that for large sets we use from (2.11)

\[ \mathcal{A}(L^{-1}X^L_{\delta_{n+1}}) = c_p L^{-4} \mathcal{A}_p(X_{\delta_{n+1}}) \] for any positive integer \( p \) with \( c_p = O(1) \). Choose \( p = 2 \). Therefore

\[ \| R_{n+1, \text{linear}, l.s.}\|_{h,G_{\alpha},A,\delta_{n+1}} \leq O(1)L^{-1}g^{3/4-\eta} \]

(5.180)

Adding the contributions (5.176) and (5.180) we get (5.174). (5.175) can be proved in the same way. For the small set contribution we use the kernel bounds in Corollary 5.25 and Lemma 5.26. For the large set contribution we first use the rescaling involved to shift on the right hand side the \( h \) norm to the \( \hat{h} \) norm followed by the kernel bound in Lemma 5.17. □

Proof of Theorem 5.1

From (4.41), \( R_{n+1} \) is the sum of \( R_{n+1, \text{main}}, R_{n+1, \text{linear}}, R_{n+1, 3}, \) and \( R_{n+1, 4} \). \( R_{n+1, \text{main}} \) satisfies the bound given in Lemmas 5.21. For \( L \) large and \( \varepsilon \) small depending on \( L \) implying \( \tilde{g} \) sufficiently small \( C_L g^{3/4} \leq L^{-1/2}g^{3/4-\eta} \) with \( \eta = 1/64 \). Similarly \( C_L g^{3-3/2} \leq L^{-1/2}g^{11/4-\eta} \) for \( \delta = \eta \). Therefore \( \| R_{n+1, \text{main}} \|_{n+1} \leq L^{-1/2}g^{11/4-\eta} \). Similarly from Lemmas 5.22 and 5.23 we get \( \| R_{n+1, j} \|_{n+1} \leq L^{-1/2}g^{11/4-\eta} \), for \( j = 3, 4 \). Adding these bounds to that provided for \( R_{n+1, \text{main}} \) by Lemma 5.27 we have that the sum satisfies the bound (5.11) for \( L \) sufficiently large. The bounds (5.9), (5.10) and (5.12) have been proved in Lemmas 5.17 and 5.18. □

6. EXISTENCE OF THE GLOBAL RENORMALIZATION GROUP TRAJECTORY AND THE STABLE MANIFOLD

This section is devoted to the proof of existence of the stable manifold starting from the unit lattice. Namely, there exists an initial critical mass \( \mu_0 \) which is a Lipshitz continuous function of the coupling constant \( g_0 \) such that RG trajectory is bounded uniformly on all scales. The proof is complicated because of the presence of lattice artifacts which become innocuous if we advance sufficiently on the RG trajectory. We therefore prove the result by a combination of three theorems, namely Theorems 6.2, 6.4, and 6.6. We first iterate the RG map a finite number \( n_0 \) of times and then restart the trajectory. Theorem 6.2 says that there
exists a critical mass such that the RG trajectory is uniformly bounded at all scales $n \geq n_0 \geq 0$. Theorem 6.4 says that if $n_0$ is sufficiently large then the critical mass $\mu_{n_0}$ at this scale is a Lipschitz continuous function of the contracting variables. The stable (critical) manifold at scale $n_0$, appropriately interpreted for a sequence of non-autonomous maps, is constructed. Finally Theorem 6.6 says that there exists an initial critical mass $\mu$ which is a $C^1$ function of $g_0$ such that after $n_0$ applications of the RG map we arrive at the critical mass $\mu_{n_0}$ of Theorem 6.4. Combining Theorem 6.4 with Theorem 6.6 proves the existence of the stable manifold starting from the unit lattice. One consequence is that the coupling constant $g_n$ is bounded away from $0$ uniformly in $n$.

6.1. Define $\tilde{g}_n = g_n - \bar{g}$ with $\bar{g}$ defined by (5.2) and $\tilde{w}_n = w_n - w_*$. Here $w_*$ is the function on $\cup_{n \geq 0}(\delta_n \mathbb{Z})^3$ defined in Lemma 5.9. According to Lemma 5.9, $\tilde{w}_n \to 0$ geometrically fast in $\mathcal{W}_l$ for all $l \geq 0$.

We will use as coordinates of the RG trajectory

$$v_n = (\tilde{g}_n, \mu_n, R_n, \tilde{w}_n)$$ \hspace{1cm} (6.1)

The RG map

$$v_{n+1} = f_{n+1}(v_n)$$ \hspace{1cm} (6.2)

can be written in components (see (4.39), (4.42), (4.16) and Lemma 5.9)

$$\tilde{g}_{n+1} = f_{n+1, g}(v_n) = \alpha(\varepsilon)\tilde{g}_n + \tilde{\xi}_n(v_n)$$ \hspace{1cm} (6.3)

$$\mu_{n+1} = f_{n+1, \mu}(v_n) = \tilde{L}^{2\varepsilon} \mu_n + \tilde{\rho}_n(v_n)$$ \hspace{1cm} (6.4)

$$R_{n+1} = f_{n+1, R}(v_n) =: U_{n+1}(v_n)$$ \hspace{1cm} (6.5)

$$\tilde{w}_{n+1} = f_{n+1, \tilde{w}}(v_n) = (v_{n+1} - v_{c,*}) + \tilde{w}_{n,L}$$ \hspace{1cm} (6.6)

with initial $w_0 = 0$ and $R_0 = 0$. This implies that $\tilde{w}_0 = -w_*$ and our initial condition is

$$v_0 = (\tilde{g}_0, \mu_0, 0, \tilde{w}_0)$$ \hspace{1cm} (6.7)

$\alpha(\varepsilon)$, $\xi_n$, $\rho_n$ are defined by

$$\alpha(\varepsilon) = 2 - L^\varepsilon = 1 - O(\log L)\varepsilon$$

for $\varepsilon$ sufficiently small depending on $L$, and

$$\tilde{\xi}_n(v_n) = -L^{2\varepsilon} a_{c,*} \tilde{g}_n^2 - L^{2\varepsilon}(a_n - a_{c,*})(\tilde{g}_n + \bar{g})^2 + \xi_n(v_n)$$ \hspace{1cm} (6.8)

$$\tilde{\rho}_n(v_n) = -L^{2\varepsilon} b_n (\tilde{g} + \bar{g})^2 + \rho_n(v_n)$$ \hspace{1cm} (6.9)

Let $E_n$ be the Banach space consisting of elements $v_n$ with the norm

$$\|v_n\|_n = \max((\nu \tilde{g})^{-1}\|\tilde{g}_n\|, \tilde{g}^{-(2-\delta)}\|\mu_n\|, \tilde{g}^{-(11/4-\eta)}\|R_n\|, \tilde{c}^{-1}_L\|\tilde{w}_n\|_n)$$ \hspace{1cm} (6.10)

where the norm $|||R|||_{\delta_n}$ of $R_n$ is as defined in (5.6). $\nu$ is the $O(1)$ constant which figures in the specification of the domain $D_n$, (5.4). We have $0 < \nu < 1$. We will take $\nu > 0$ sufficiently small depending on $L$ and this
will be specified in the proofs of Lemmas 6.3 and 6.4 below. The norm $\|\tilde{w}_n\|_{\delta_n}$ and the constant $\tilde{c}_L$ are as specified in Lemma 5.9.

6.2. Domains and bounds

Let $E_n(r) \subset E_n$ be the open ball of radius $r$, centered at the origin:

$$E_n(r) = \{v_n \in E_n : \|v_n\|_n < r\}$$

(6.11)

Let $D_n$ be the domain of $(g_n, \mu_n, R_n)$ defined in (5.4) and (5.5).

$$v_n \in E_n(1) \Rightarrow (g_n, \mu_n, R_n) \in D_n$$

(6.12)

and then Theorem 5.1 holds.

Let $v_n \in E_n(1)$. Let $\varepsilon > 0$ be sufficiently small (depending on $L$). Then from Theorem 5.1, Lemma 5.12, (6.8) and (6.9) we get the bounds

$$|\tilde{\xi}_n(v_n)| \leq C_L \left((\nu^2 + L^{-\eta})\tilde{g}^2 + \tilde{g}^{11/4-\eta}\right)$$

$$|\tilde{\rho}_n(v_n)| \leq C_L \tilde{g}^2$$

||U_{n+1}(v_n)||_{n+1} \leq L^{-1/4}\tilde{g}^{11/4-\eta}$$

(6.13)

We have the following Lipshitz bounds:

Lemma 6.1

Let $v_n, v_n' \in E_n(1/4)$. Then we have:

(i) $|\tilde{\xi}_n(v_n) - \tilde{\xi}_n(v_n')| \leq C_L \left((\nu^2 + L^{-\eta})\tilde{g}^2 + \tilde{g}^{11/4-\eta}\right)\|v_n - v_n'\|_n$

(ii) $|\tilde{\rho}_n(v_n) - \tilde{\rho}_n(v_n')| \leq C_L \tilde{g}^2\|v_n - v_n'\|_n$

(iii) $||U_{n+1}(v_n) - U_{n+1}(v_n')||_{n+1} \leq O(1)L^{-1/4}\tilde{g}^{11/4-\eta}\|v_n - v_n'\|_n$

(iv) $\tilde{c}_L^{-1}\|f_{n+1, w}(v_n) - f_{n+1, w}(v_n')\|_{n+1} \leq L^{-1/5}\|v_n - v_n'\|_n$

Proof:

$\tilde{\xi}_n, \tilde{\rho}_n, U_{n+1}$ are (norm) analytic functions in $D_n$ and thus in $E_n(1)$. The analyticity follows from the algebraic operations in Section 4, the norm analyticity of the reblocking map together with the norm analyticity of the extraction map (Theorem 5 [BDH-est]). Therefore we can use Cauchy estimates exactly as in the proof of Lemma 6.1 of [BMS] together with the bounds (6.13) to get (i), (ii) and (iii). To get (iv) note that from (6.6) and the definition of the norms in (5.61) we have

$$\|f_{n+1, w}(v_n) - f_{n+1, w}(v_n')\|_{n+1} \leq L^{2d_n} \max_{1 \leq p \leq 3} \sup_{x \in (\delta_n, z)^3} \left(\|z + \delta_n + \delta_{n+1}\|_{\delta_n}^{\delta_n + \delta_{n+1}}|\tilde{w}_{n+(p)}(Lx) - \tilde{w}_{n+(p)}(Lx)|\right)$$

$$\leq L^{2d_n} \max_{1 \leq p \leq 3} \sup_{y \in (\delta_n, z)^3} \left(\|y + \delta_n + \delta_{n+1}\|_{\delta_n}^{\delta_n + \delta_{n+1}}|\tilde{w}_{n+(p)}(y) - \tilde{w}_{n+(p)}(y)|\right)$$

$$\leq L^{2d_n - 7/4} \max_{1 \leq p \leq 3} \sup_{y \in (\delta_n, z)^3} \left(\|y + \delta_n + \delta_{n+1}\|_{\delta_n}^{\delta_n + \delta_{n+1}}|\tilde{w}_{n+(p)}(y) - \tilde{w}_{n+(p)}(y)|\right)$$

$$\leq L^{-1/5}\|\tilde{w}_n - \tilde{w}_n'\|_n$$
where we have used \( d_s = \frac{3-\varepsilon}{\mu} \) and then \( L \) large and \( \varepsilon \) sufficiently small. Now dividing both sides by \( c \), we get (iv).

6.3. Existence of the global RG trajectory

Let \( \mathcal{D}_n \) be the domain specified by (5.4),(5.5), and (5.6). Let \( (\tilde{g}_0, \mu_0, 0) \) belong to \( \tilde{D}_0 \subset \mathcal{D}_0 \) where \( \tilde{D}_0 \) is specified by

\[
|\tilde{g}_0| < 2^{-(n_0+5)} \nu \bar{g}, \quad |\mu_0| < 2^{-(n_0+5)} L^{-\frac{3+\varepsilon}{\mu} n_0} \bar{g}^{2-\delta}
\]

Let \( n_0 \) be a positive integer. By iterating the RG map \( n_0 \) times using Theorem 5.1 and the flow equation (4.39) recursively we obtain for \( \varepsilon \) sufficiently small depending on \( L \) and \( n_0 \), \( (g_{n_0}, \mu_{n_0}, R_{n_0}) \in \mathcal{D}_{n_0}(1/32) \) where \( \mathcal{D}_{n_0}(1/32) \) is specified by

\[
|\tilde{g}_{n_0}| < \frac{1}{32} \nu \bar{g}, \quad |\mu_{n_0}| < \frac{1}{32} \bar{g}^{2-\delta}
\]

\[
\|\|R_{n_0}\|\|_{n_0} < \frac{1}{32} \bar{g}^{11/4-\eta}
\]

We will now prove the existence of a global solution to the discrete flow map (6.2):

\[
v_{n+1} = f_{n+1}(v_n), \quad \forall \ n \geq n_0
\]

with initial condition

\[
v_{n_0} = (\tilde{g}_{n_0}, \mu_{n_0}, \tilde{w}_{n_0}, R_{n_0})
\]

in a bounded domain. We will say that \( \{v_n: v_{n+1} = f_{n+1}(v_n), n \geq n_0 \} \) is the RG trajectory restarted at scale \( n_0 \). To this end we consider the Banach space \( \mathbf{E}_{n_0} \) of sequences \( s_{n_0} = \{v_n\}_{n \geq n_0} \), each \( v_n \in \mathbf{E}_n \), with the norm

\[
\|s_{n_0}\| = \sup_{n \geq n_0} \|v_n\|_n \tag{6.15}
\]

and the open ball \( \mathbf{E}_{n_0}(r) \subset \mathbf{E} \)

\[
\mathbf{E}_{n_0}(r) = \{s_{n_0}: \|s_{n_0}\| < r\} \tag{6.16}
\]

We will derive on the space of sequences \( \mathbf{E}_{n_0} \) an equation that a global RG trajectory must solve and then prove for \( v_{n_0} \in \mathbf{E}_{n_0}(1/32) \) the existence of a unique solution in the ball \( \mathbf{E}_{n_0}(1/4) \), for a suitable choice of \( r \), by the contraction mapping principle. This adapts a standard method from the theory of hyperbolic dynamical systems in Banach spaces due to Irwin in [I]. Irwin’s analysis is explained by Shub in Appendix 2, Chapter 5 of [S]. For earlier applications see section 5 of [BDH-eps] and section 6 of [BMS].

**Theorem 6.2**

Let \( L \) be large, \( \nu \) be sufficiently small depending on \( L \), then \( \varepsilon \) sufficiently small depending on \( L \). Let \( v_{n_0} \in \mathbf{E}_{n_0}(1/32) \) for any integer \( n_0 \geq 0 \). Let \( (\tilde{g}_{n_0}, R_{n_0}, \tilde{w}_{n_0}) \) be held fixed. Then there is a \( \mu_{n_0} \) such that there exists a sequence \( s_{n_0} = \{v_n\}_{n \geq n_0} \) in \( \mathbf{E}_{n_0}(1/4) \) satisfying \( v_{n+1} = f_{n+1}(v_n) \) for all \( n \geq n_0 \).

**Remark:** The \( \mu_{n_0} \) of the theorem is called a critical mass. We write \( \mu_{n_0} = \mu_{n_0,c} \).

**Proof**

Our initial data will be at scale \( n_0 \). Let \( n_0 \leq n \leq N - 1 \). We iterate the map (6.3) forwards \( N \) times. We iterate the map (6.4) backwards \( N - n_0 \) times starting from a given \( \mu_N \). We then easily derive
\[\hat{g}_{n+1} = \alpha(\epsilon)^{n+1-n_0} \tilde{g}_{n_0} + \sum_{j=n_0}^{n} \alpha(\epsilon)^{n-j} \tilde{\xi}_j(v_j), \quad n_0 \leq n \leq N - 1\]

\[\mu_{n+1} = L^{-\frac{3+\varepsilon}{2}(N-n-1)} \mu_N - \sum_{j=n+1}^{N-1} L^{-\frac{3+\varepsilon}{2}(j-n)} \tilde{\rho}_j(v_j), \quad n_0 - 1 \leq n \leq N - 2\]

Let us fix \(\mu_N = \mu_f\) and take \(N \to \infty\). In other words we assume the \(\mu_n\) flow is bounded and then must show that such a flow exists. We have

\[\hat{g}_{n+1} = \alpha(\epsilon)^{n+1-n_0} \tilde{g}_{n_0} + \sum_{j=n_0}^{n} \alpha(\epsilon)^{n-j} \tilde{\xi}_j(v_j), \quad n \geq n_0 \quad (6.17)\]

\[\mu_{n+1} = -\sum_{j=n+1}^{\infty} L^{-\frac{3+\varepsilon}{2}(j-n)} \tilde{\rho}_j(v_j), \quad n \geq n_0 - 1 \quad (6.18)\]

together with

\[R_{n+1} = U_{n+1}(v_n), \quad n \geq n_0 \quad (6.19)\]

\[\tilde{w}_{n+1} = (v_{n+1} - v_{cs}) + \tilde{w}_{n,L} \quad (6.20)\]

The flow \(\tilde{w}_n\) is independent of that of \(g_n, \mu_n, R_n\), is solved by (5.60) and satisfies the bounds of Lemma 5.9. This solution can be incorporated in the \(v_n\) and \(\tilde{w}_n\) is then no longer a flow variable.

For \(\varepsilon\) sufficiently small (depending on \(L\))

\[0 < \alpha(\epsilon) < 1 \quad (6.21)\]

Note that \(s_{n_0} \in E_{n_0}(1/4)\) implies \(v_j \in E_j(1/4)\) for all \(j \geq n_0\). Then the infinite sum of (6.18) converges by (6.21) and (6.13). So \(\mu_{n_0}\) has now been determined provided (6.17)-(6.19) has a solution in the aforementioned ball. It is easy to verify that any solution of (6.17)-(6.18), together with the \(\tilde{w}\) flow, is a solution of the RG flow \(v_{n+1} = f_{n+1}(v_n)\) for \(n \geq n_0\).

We write (6.17)-(6.19) in the form

\[v_{n+1} = F_{n+1}(s_{n_0}), \quad n \geq n_0 \quad (6.22)\]

where \(s_{n_0} = (v_{n_0}, v_{n_0+1}, v_{n_0+2}, \ldots)\) and \(F_{n+1}\) has components \(F_{n+1}^{(g)}, F_{n+1}^{(\mu)}, F_{n+1}^{(\tilde{w})}\) given by the r.h.s. of (6.17), (6.18), (6.19) respectively.

If we write

\[F(s_{n_0}) = (F_{n_0}(s_{n_0}), F_{n_0+1}(s_{n_0}), \ldots)\]

then (6.22) can be written as a fixed point equation

\[s_{n_0} = F(s_{n_0}) \quad (6.23)\]

We seek a solution of (6.23) in the open ball \(E_{n_0}(1/4)\) with initial data \(v_{n_0} = (\tilde{g}_{n_0}, \mu_{n_0}, R_{n_0}, \tilde{w}_{n_0})\) in \(E_{n_0}(1/32)\) with \((\tilde{g}_{n_0}, R_{n_0}, \tilde{w}_{n_0})\) held fixed. The existence of a unique solution follows by the standard contraction mapping principle and the next Lemma. ■

\textbf{Lemma 6.3}
\[ s_{n_0} \in E_{n_0}(1/32) \Rightarrow F(s_{n_0}) \in E_{n_0}(1/16) \]  
(6.24)

Moreover, for \( s_{n_0}, s'_{n_0} \in E_{n_0}(1/4) \)

\[ \|F(s_{n_0}) - F(s'_{n_0})\| \leq \frac{1}{2}\|s_{n_0} - s'_{n_0}\| \]  
(6.25)

**Proof**

First we prove (6.24), and thus take \( s_{n_0} \in E_{n_0}(1/32) \). Then \( v_n \in E_n(1/32) \) for every \( n \geq n_0 \) and we can use the estimates (6.13). From (6.17) and the estimates in (6.13) we have

\[ (\nu \bar{g})^{-1}|F^{(g)}_{n+1}(s_{n_0})| < \alpha(\epsilon) \frac{1}{32} + (\nu \bar{g})^{-1}\frac{n}{32} \alpha(\epsilon)^{n-j}C_L \left( (\nu^2 + L^{-jq})\bar{g}^2 + \bar{g}^{11/4-\eta} \right) \]

\[ \leq \frac{1}{32} + C_L \frac{\bar{g}^{7/4-\eta}}{\nu(1-\alpha(\epsilon))} + C_L \frac{\bar{g}^\nu}{1-\alpha(\epsilon)} + C_L \frac{\bar{g}}{\nu(1-\alpha(\epsilon))^{-1}L^{-\eta}} \]

\[ < \frac{1}{32} + \frac{C_L}{\nu \log L} \frac{\bar{g}^{3/4-\eta}}{\epsilon} + \frac{C_L}{\nu \log L} \frac{\bar{g}}{\nu(1-\epsilon)} < \frac{1}{16} \]

for \( L \) sufficiently large, \( \nu \) sufficiently small depending on \( L \) so that \( \frac{C_L}{\nu \log L} \nu \leq 1/96 \) and then \( \epsilon \) sufficiently small depending on \( L \) so that \( \bar{g} \leq C_L \bar{g} \) is sufficiently small, \( \frac{C_L}{\nu(1-\epsilon)} \leq 1/96 \) and \( \frac{C_L}{\nu \log L} \epsilon^{1/4-\eta} \leq 1/96 \)

Similarly from (6.18) and (6.13) we have

\[ \bar{g}^{-(2-\delta)}|F^{(1)}_{n+1}(s_{n_0})| \leq c_L \bar{g}^\delta \sum_{j=n+1}^{\infty} L^{-\frac{3\bar{g}}{2}(j-n)} \leq L^{-\frac{3\bar{g}}{2}(1-\epsilon)} \leq < \frac{1}{16} \]

since \( \delta = 1/64 \), \( L \) sufficiently large, and \( \epsilon \) sufficiently small depending on \( L \).

Finally from (6.19) and (6.13)

\[ \bar{g}^{-(11/4-\eta)}||F^{(R)}_{n+1}(s_{n_0})||_{n+1} \leq L^{-1/4} < \frac{1}{16} \]

for \( L \) sufficiently large. This proves (6.24).

To prove (6.25), take \( s_{n_0}, s'_{n_0} \in E_{n_0}(1/4) \). This implies that \( v_n, v'_n \in E_n(1/4) \) for every \( n \geq n_0 \) and we can use the Lipshitz estimates of lemma 6.1. Note that the initial coupling \( g_{n_0} \) is held fixed. Then we have

\[ (\nu \bar{g})^{-1}|F^{(g)}_{n+1}(s_{n_0}) - F^{(g)}_{n+1}(s'_{n_0})| \leq \sum_{j=n_0}^{n} \alpha(\epsilon)^{n-j}(\nu \bar{g})^{-1}\left| \tilde{\xi}_j(v_j) - \tilde{\xi}_j(v'_j) \right| \]

\[ \leq (\nu \bar{g})^{-1}\sum_{j=n_0}^{n} \alpha(\epsilon)^{n-j}C_L \left( (\nu^2 + L^{-jq})\bar{g}^2 + \bar{g}^{11/4-\eta} \right)\|s_{n_0} - s'_{n_0}\| \]

\[ \leq \left( \frac{C_L}{\nu \log L} \frac{\bar{g}^{3/4-\eta}}{\epsilon} + C_L \frac{\bar{g}^\nu}{1-\alpha(\epsilon)} + C_L \frac{\bar{g}}{\nu(1-\alpha(\epsilon))^{-1}L^{-\eta}} \right)\|s_{n_0} - s'_{n_0}\| \leq \frac{1}{2}\|s_{n_0} - s'_{n_0}\| \]

by estimating as above in the bound for \( F^{(g)}_{n+1}(s_{n_0}) \) with \( L \) sufficiently large, \( \nu \) sufficiently small depending on \( L \) and \( \epsilon \) sufficiently small depending on \( L \). Similarly,

\[ \bar{g}^{-(2-\delta)}|F^{(1)}_{n+1}(s_{n_0}) - F^{(1)}_{n+1}(s'_{n_0})| \leq \sum_{j=n+1}^{\infty} L^{-\frac{3\bar{g}}{2}(j-n)} \bar{g}^{-(2-\delta)}\left| \tilde{\xi}_j(v_j) - \tilde{\xi}_j(v'_j) \right| \]
\[
\leq L^{-\frac{3\lambda}{2}} (1 - L^{-\frac{3\lambda}{2}})^{-1} C L \bar{g}' ||s_{n_0} - s'_{n_0}| | \leq \frac{1}{2} ||s_{n_0} - s'_{n_0}| |
\]
for \( L \) sufficiently large and \( \varepsilon \) sufficiently small depending on \( L \). Finally
\[
\bar{g}^{-\left(11/4-\eta\right)||} F_{n+1}^{(R)}(s_{n_0}) - F_{n+1}^{(R)}(s'_{n_0})|| ||n+1 = \bar{g}^{-\left(11/4-\eta\right)||} ||U_{n+1}(v_n) - U_{n+1}(v'_n)|| ||n+1
\]
\[
\leq O(1) L^{-1/4} ||s_{n_0} - s'_{n_0}| | \leq \frac{1}{2} ||s_{n_0} - s'_{n_0}| |
\]
for \( L \) sufficiently large. Thus (6.25) has been proved. This completes the proof of Theorem 6.2. 

6.4. Theorem 6.2 says that if \( v_{n_0} = (\bar{g}_{n_0}, \mu_{n_0}, R_{n_0}, \hat{w}_{n_0}) \in E_{n_0}(1/32) \) for any \( n_0 \geq 0 \) then there is a critical mass \( \mu_{n_0} = \mu_{n_0,c} \) such that a uniformly bounded RG trajectory exists. The Theorem 6.4 below proves the uniqueness of \( \mu_{n_0,c} \) for \( n_0 \) sufficiently large: \( \mu_{n_0,c} \) is a Lipschitz continuous function of \( (\bar{g}_{n_0}, R_{n_0}, \hat{w}_{n_0}) \). In Theorem 6.6 below we prove that given \( \mu_{n_0} \) as above there is a \( \mu_0 \) given by a \( C^1 \) function of \( \bar{g}_0 \) such that after \( n_0 \) applications of the RG map we arrive at \( \mu_{n_0} \).

To this end we represent the Banach space \( E_n \) as a product of two Banach spaces \( E_n = E_{n,1} \times E_{n,2} \). We write \( v_n \in E_n \) as \( v_n = (v_{n,1}, v_{n,2}) \) where \( v_{n,1} = (\bar{g}_n, R_n, \hat{w}_n) \) and \( v_{n,2} = \mu_n \). \( v_{n,2} \) is the expanding (relevant) variable. Let \( p_i, i = 1, 2 \), denote the projector onto \( E_{n,i} \) and \( f_{n,i} = p_i \circ f_n \). The norm \( || \cdot ||_n \) on \( E_n \) being a box norm we have
\[
||v_n||_n = \max(||v_{n,1}||_n, ||v_{n,2}||_n)
\]
where \( ||v_{n,2}||_n = \bar{g}^{-(2-\delta)}||\mu_n|| \) and \( ||v_{n,1}||_n = \max(\nu||\bar{g}||^-1||\mu_n||, \bar{g}^{-\left(11/4-\eta\right)||} ||R_n|| ||\hat{w}_n|| ||\mu_n|| \).

In the following we continue to assume that \( L \) is sufficiently large, followed by \( \nu \) sufficiently small depending on \( L \), then \( \varepsilon \) sufficiently small depending on \( L \). The last condition also implies that \( \bar{g} \leq C_L \varepsilon \).

Theorem 6.4: Let \( s_{n_0} = \{v_n : v_{n+1} = f_{n+1}(v_n)\}_{n \geq n_0} \in E(1/4) \) be the global RG trajectory of Theorem 6.2. Then for \( n_0 \) sufficiently large there exists a Lipschitz continuous function \( h : E_{n_0,1} \rightarrow \mathbb{R} \) with Lipschitz constant 1 such that the stable manifold of the sequence of maps \( \{f_n\}_{n \geq n_0+1}, W^s_{n_0} = \{v_{n_0} \in E_{n_0}(1/32) : s_{n_0} \in E(1/4)\} \) is the graph
\[
W^s_{n_0} = \{v_{n_0,1}, h(v_{n_0,1})\}
\]

We will prove the theorem following the analysis of Shub in [S, Section 5]. The Schub analysis has been employed earlier in the context of continuum models, (see [Section 5.3, BDH-eps] and [Section 6, BMS]). Here we have to take account of additional features stemming from the lattice which results in \( n_0 \) having to be taken sufficiently large (sufficiently fine lattice) for the argument to work. This will be clear from the proof of the following lemma from which Theorem 6.4 follows.

Lemma 6.5
Let \( v_n, v'_n \in E_n(1/4) \). Then for \( n \geq n_0, n_0 \) sufficiently large depending on \( \nu \) and \( L \)
\[
||f_{n+1,1}(v_n) - f_{n+1,1}(v'_n)||_{n+1} \leq (1 - \varepsilon)||v_n - v'_n||_n
\]
and, if \( ||v_{n,2} - v'_{n,2}||_n \geq ||v_{n,1} - v'_{n,1}||_n \) then
\[
||f_{n+1,2}(v_n) - f_{n+1,2}(v'_n)||_{n+1} \geq (1 + \varepsilon)||v_n - v'_n||_n
\]

Proof
First we prove (6.26). \( f_{n+1,1} \) has components \( f_{n+1,g}, f_{n+1,R} \) and \( f_{n+1,w} \). From (6.3)
Since $v_n, v'_n \in E_n(1/4)$ we can use lemma 6.1. Therefore for $n \geq n_0$

\[
(n\bar{g})^{-1} |f_{n+1,g}(v_n) - f_{n+1,g}(v'_n)| \leq \alpha(\varepsilon) \|v_n - v'_n\|_n + (n\bar{g})^{-1} |\tilde{\xi}(v_n) - \tilde{\xi}(v'_n)|
\]

\[
\leq \left( 1 - \varepsilon \log(L) + C_L\varepsilon + \frac{1}{\nu}L^{-\nu q} + \frac{1}{\nu^{3/4}} \|v_n - v'_n\|_n
\]

Let $L$ be large. Let $\nu$ be sufficiently small and $n_0$ sufficiently large so that $C_L L^{-\nu q/2} \leq \nu \leq \frac{1}{\varepsilon L}$ and $\varepsilon$ sufficiently small so that $\nu \geq C_L \varepsilon^{3/8 - n}$. Then we have

\[
1 - \varepsilon \log(L) + \varepsilon C_L(\nu + \frac{1}{\nu}L^{-\nu q} + \frac{1}{\nu^{3/4}}) \leq 1 + \varepsilon(-\log(L) + 1 + L^{-\nu q/2} + \varepsilon^{3/8})
\]

\[
\leq 1 + \varepsilon(-\log(L) + 3) \leq 1 - \varepsilon
\]

Therefore

\[
(n\bar{g})^{-1} |f_{n+1,g}(v_n) - f_{n+1,g}(v'_n)| \leq (1 - \varepsilon)\|v_n - v'_n\|_n
\]

Since $f_{n+1,R}(v_n) = U_{n+1}(v_n)$, we have from lemma 6.1 for $L$ sufficiently large

\[
\bar{g}^{-(1/4 - n)} \|f_{n+1,R}(v_n) - f_{n+1,R}(v'_n)\|_{n+1} \leq (1 - \varepsilon)\|v_n - v'_n\|_n
\]

as well as

\[
\tilde{e}_L^{-1} \|f_{n+1,w}(v_n) - f_{n+1,w}(v'_n)\|_{n+1} \leq (1 - \varepsilon)\|v_n - v'_n\|_n
\]

These three inequalities prove (6.26).

Remark: $n_0$ had to be chosen sufficiently large because the $\bar{g}$ flow coefficient $a_n$, see (6.7), depends on the lattice scale $n$. $a_n$ converges geometrically (Lemma 5.12) to a constant $\alpha_{\infty}$. As a result we have to wait sufficiently long before $\bar{g}$ becomes irrelevant. A consequence of this is the presence of the $L^{-\nu q} \bar{g}^2$ term in the first inequality of Lemma 6.1. It has to be sufficiently small to ensure the validity of the first of three inequalities above.

Next we turn to (6.27). In this case by assumption \(\|v_{n,2} - v'_{n,2}\| \geq \|v_{n,1} - v'_{n,1}\|\) and hence, since our norms are box norms, we have

\[
\|v_n - v'_n\|_n = \|v_{n,2} - v'_{n,2}\|_n = \bar{g}^{-(2-\delta)}|\mu_n - \mu'_n|
\]

From (6.4)

\[
f_{n+1,\mu}(v_n) = L^{\frac{3\delta}{2}}\mu_n + \bar{\rho}_n(v_n)
\]

Then, using lemma 6.1, we have

\[
\bar{g}^{-(2-\delta)}|f_{\mu}(v_n) - f_{\mu}(v'_n)| \geq L^{\frac{3\delta}{2}} \|v_n - v'_n\|_n - \bar{g}^{-(2-\delta)}|\bar{\rho}(v_n) - \bar{\rho}(v'_n)|
\]

\[
\geq (L^{\frac{3\delta}{2}} - C_L \bar{g}^\delta)\|v_n - v'_n\|_n
\]

\[
\geq (1 + \varepsilon)\|v_n - v'_n\|
\]

for $\varepsilon$ sufficiently small depending on $L$. This proves (6.27).

Proof of Theorem 6.4
Given Lemma 6.5, the proof follows the Schub argument as in [BMS]. Namely, to prove that $W_n^s$ is given by a graph of a function $v_{n_0,2} = h(y_{n_0,1})$ it is enough to prove that if in $W_n^s$ we take two points $v_{n_0} = (v_{n_0,1}, v_{n_0,2})$ and $v_{n_0}' = (v_{n_0,1}', v_{n_0,2}')$ then

$$||v_{n_0,2} - v_{n_0,2}'||_{n_0} \leq ||v_{n_0,1} - v_{n_0,1}'||_{n_0}$$

(6.28)

because then for a given $v_{n_0,1}$ we would have at most one $v_{n_0,2}$, and by theorem 6.2 there exists such a $v_{n_0,2}$. This means that $W_n^s$ is the graph of a function $h$, $v_{n_0,2} = h(v_{n_0,1})$, and moreover

$$||h(v_{n_0,1}) - h(v_{n_0,1}')||_{n_0} \leq ||v_{n_0,1} - v_{n_0,1}'||_{n_0}$$

Suppose (6.28) is not true. Then

$$||v_{n_0,2} - v_{n_0,2}'||_{n_0} > ||v_{n_0,1} - v_{n_0,1}'||_{n_0}$$

(6.29) implies that (6.27) holds. The latter followed by (6.26) gives

$$||f_{n_0+1,2}(v_{n_0}) - f_{n_0+1,2}(v_{n_0}')||_{n_0+1} \geq (1 + \varepsilon)||v_{n_0} - v_{n_0}'||_{n_0} > (1 - \varepsilon)||v_{n_0} - v_{n_0}'||_{n_0}$$

$$\geq ||f_{n_0+1,1}(v_{n_0}) - f_{n_0+1,1}(v_{n_0}')||_{n_0+1}$$

(6.30)

and hence

$$||f_{n_0+1}(v_{n_0}) - f_{n_0+1}(v_{n_0}')||_{n_0+1} = ||f_{n_0+1,2}(v_{n_0}) - f_{n_0+1,2}(v_{n_0}')||_{n_0+1} \geq (1 + \varepsilon)||v_{n_0} - v_{n_0}'||_{n_0}$$

(6.31)

Define the composition of maps

$$\mathcal{P}^k_n = f_{n+k} \circ \ldots \circ f_{n+2} \circ f_{n+1}$$

Now

$$||\mathcal{P}^2_{n_0}(v_{n_0}) - \mathcal{P}^2_{n_0}(v_{n_0}')||_{n_0+2} = ||f_{n_0+2}(f_{n_0+1}(v_{n_0})) - f_{n_0+2}(f_{n_0+1}(v_{n_0}'))||_{n_0+2}$$

By (6.30) and the second part of Lemma 6.4 followed by (6.31) we get

$$||\mathcal{P}^2_{n_0}(v_{n_0}) - \mathcal{P}^2_{n_0}(v_{n_0}')||_{n_0+2} \geq (1 + \varepsilon)||f_{n_0+1}(v_{n_0}) - f_{n_0+1}(v_{n_0}')||_{n_0+1} \geq (1 + \varepsilon)^2||v_{n_0} - v_{n_0}'||_{n_0}$$

Repeating this $k$ times we get for all $k \geq 0$

$$||\mathcal{P}^k_{n_0}(v_{n_0}) - \mathcal{P}^k_{n_0}(v_{n_0}')||_{n_0+k} \geq (1 + \varepsilon)^k||v_{n_0} - v_{n_0}'||_{n_0}$$

(6.32)

Now $v_{n_0}, v_{n_0}'$ belong to $W_n^s$ and $\mathcal{P}^k_{n_0}(v_{n_0})$ is a member of the sequence $s_{n_0} \in E(1/4)$. Therefore $||\mathcal{P}^k_{n_0}(v_{n_0})||_{n_0+k} < 1/4$. Therefore we have from (6.32) the bound $\frac{1}{4} > (1 + \varepsilon)^k||v_{n_0} - v_{n_0}'||_{n_0}$. By making $k$ arbitrarily large we get a contradiction because $v_{n_0} \neq v_{n_0}'$ under (6.29). Hence (6.28) is true and the theorem 6.4 has been proved.

The next theorem establishes the uniqueness of the critical mass at the unit lattice scale.

**Theorem 6.6:** Let $n_0$ and $\mu_{n_0}$ be as in Theorem 6.4. Let $\tilde{g}_0, \mu_0, R_0, w_0$ belong to $\tilde{\mathcal{D}}_0$, defined in the beginning of subsection 6.2, with $R_0 = 0, w_0 = 0$. Let $U_1(1) = \{ \tilde{g}_0 : 2^{(n_0+5)(\nu g)^{-1}}|\tilde{g}_0| < 1 \}$. Let $\varepsilon$ be sufficiently small depending on $L$ and $n_0$. Then there exists an open ball $U_0 \subset U_1(1)$ and a $C^1$ function $h_0 : U_0 \rightarrow \mathbb{R}$ such that for $\mu_0 = h_0(\tilde{g}_0)$ the RG map applied $n_0$ times gives the effective critical mass $\mu_{n_0}$.
Remark: This is the first time in our estimates that $\varepsilon$ has been chosen to depend on $n_0$. Recall that in Lemma 6.5 $n_0$ was taken to be sufficiently large depending on $\nu$ and $L$.

Proof: Let $v_n$ be as in (6.1). Let $r_n = 2^{-(n_0-n+5)}$ and $\lambda_n = L^{-\frac{(2n+2)}{2}(n_0-n)}$. Let $\tilde{E}_n$, $1 \leq n \leq n_0$, be the Banach space consisting of $v_n$ with norm

$$||v_n||_n = \max(r_n^{-1}(\nu \tilde{g})^{-1}|\tilde{g}_n|, r_n^{-1}\lambda_n^{-1}\bar{\tilde{g}}^{-(2-\delta)}|\mu_n|, \bar{\tilde{g}}^{-(\frac{11}{4} - \varepsilon)}||E_n||, \bar{\tilde{E}}_n^{-1}||\tilde{w}_n||_n)$$

(6.33)

We have $\tilde{E}_n \subset E_n$. $\tilde{E}_n(1)$ is the open unit ball in $E_n$. Note that $\tilde{E}_0(1)$ coincides with $\tilde{D}_0$ as defined in the beginning of subsection 6.2. For $R_0 = 0$ and $w_0 = 0$, and $\tilde{E}_n(1)$ was taken to be sufficiently large depending on $\rho$. Then by Theorem 5.1 and Lemma 5.9 for $1 \leq n \leq n_0$ we have each RG map $f_n : \tilde{E}_n(1) \rightarrow \tilde{E}_{n+1}(1)$. Moreover each such map is (norm) analytic. Define the composition of maps

$$\mathcal{P}_0^{n_0} = f_{n_0} \circ f_{n_0-1} \circ \cdots f_1 : \tilde{E}_0(1) \rightarrow \tilde{E}_{n_0}(1)$$

(6.34)

$\mathcal{P}_0^{n_0}$ is the composition of a finite number of analytic maps and therefore analytic. We consider the equation $v_{n_0} = \mathcal{P}_0^{n_0}(v_0)$ in the direction $\mu$:

$$\mu_{n_0} = (\mathcal{P}_0^{n_0})_\mu(v_0)$$

(6.35)

with $v_0 = (\tilde{g}_0, \mu_0, 0, \tilde{w}_0)$ with $w_0 = 0$, ( recall that $\tilde{w}_0 = w_0 - w_*$).

We will solve (6.33) for $\mu_{n_0}$ for fixed $\mu_{n_0}$ using the (Banach space) implicit function theorem. Let $x = (\tilde{g}_0, \mu_{n_0})$ and $y = \mu_0$. We have set $w_0 = R_0 = 0$. Let $V_1$ be the Banach space of elements $x$ with norm

$$||x|| = \max(r_0^{-1}(\nu \tilde{g})^{-1}|\tilde{g}_0|, r_0^{-1}\lambda_0^{-1}\bar{\tilde{g}}^{-(2-\delta)}|\mu_{n_0}|)$$

Let $V_2$ be the Banach space of elements $y$ with norm

$$||y|| = r_0^{-1}\lambda_0^{-1}\bar{\tilde{g}}^{-(2-\delta)}|\mu_0|$$

Let $V_j(r)$ be the open ball in $V_j$ of radius $r$, centered at the origin. Define

$$F(x, y) = (\mathcal{P}_0^{n_0})_\mu(v_0) - \mu_{n_0}$$

(6.36)

Solving (6.35) is equivalent to solving $F(x, y) = 0$ for $y$.

Recall that $v_0 = (\tilde{g}_0, \mu_0, 0, \tilde{w}_0)$. We have $F(0, 0) = 0$ and $F(\cdot, \cdot) : V_1(1) \times V_2(1) \rightarrow \mathbb{R}$ is an analytic map and therefore $C^2$. Taking a $y$ derivative of $F(x, y)$ gives $D_y F(x, y) = D_{\mu_0}(\mathcal{P}_0^{n_0})_\mu(v_0)$. We will prove that the linear map

$$D_y F(0, 0) : V_2 \rightarrow \mathbb{R}$$

is injective. It is easy to see that

$$(\mathcal{P}_0^{n_0})_\mu(v_0) = L^{\frac{(2n+2)}{2} n_0} (\mu_0 + L^{-\frac{(3n+4)}{2}} \sum_{j=0}^{n_0-1} L^{-\frac{(3n+4)}{2} j} \hat{\rho}_j(v_j))$$

(6.37)

$\hat{\rho}_j$ was defined earlier in (6.9) and is analytic in $E_j(1)$. The map $\hat{\rho}_j \circ (\mathcal{P}_0^{j})_\mu$ is analytic since it is a composition of analytic maps. Let $\mu_0 \in V_2(\frac{1}{4})$. Let $\gamma$ be the closed contour $\gamma = \{ \mu : \mu - \mu_0 = Re^{i\theta} \}$ with $R = \frac{1}{4} r_0 \lambda_0 \bar{\tilde{g}}^{(2-\delta)}$.

We estimate the $\mu$ derivative of $\hat{\rho}_j \circ (\mathcal{P}_0^{j})_\mu(v_0)$ by using the Cauchy integral formula integrating along the contour $\gamma$ enclosing a pole at $\mu_0$ together with the estimate for $\hat{\rho}_j(v_j)$ given in (6.13) which is valid in $E_j(1)$. The latter is guaranteed by our choice of contour. We have
\[ D_{\mu_0} \tilde{f}_j \circ (P_0^j)_\mu(v_0) \big|_{\bar{g}_0=\mu_0=0} \leq \frac{cL\bar{g}^2}{R} \leq c_{L,n_0}\bar{g}^\delta \]

Using this estimate we get from (6.37)
\[ D_{\mu_0}(P_0^{n_0})_\mu(v_0) \big|_{\bar{g}_0=\mu_0=0} \geq L \frac{\alpha+\delta}{\alpha} \left( 1 - c'_{L,n_0}\bar{g}^\delta \right) \]

Taking \( \varepsilon \) sufficiently small depending on \( L \) and \( n_0 \) makes \( \bar{g} \) sufficiently small so as to ensure
\[ D_{\mu_0}(P_0^{n_0})_\mu(v_0) \big|_{\bar{g}_0=\mu_0=0} \geq \frac{1}{2} \]. Therefore the map \( D_y F(0,0) : V_2 \to \mathbb{R} \) is injective. Hence by the implicit function theorem there exists a ball \( \tilde{U}_0 \) containing \( x = 0 \) with \( \tilde{U}_0 \subset V_1(1) \), and a \( C^1 \) function \( \tilde{h}_0 \) in \( \tilde{U}_0 \) with \( \tilde{h}_0(x) \in V_2(\frac{1}{2}) \) such that \( F(x, \tilde{h}_0(x)) = 0 \). For \( \varepsilon \) sufficiently small depending on \( L \) and \( n_0 \) we have \( \mu_{n_0} \in \tilde{U}_0 \). This completes the proof of the theorem because for \( \mu_{n_0} \in \tilde{U}_0 \), \( \tilde{U}_0 \) restricts to the ball \( U_0 \) and correspondingly \( \tilde{h}_0 \) restricts to the desired function \( h_0 \).

Theorem 6.4 and Theorem 6.6 put together completes our construction of the stable manifold starting from the unit lattice.

Finally we remark that as a consequence of theorem 6.4 we have \( v_n \in E_n(1/4), \forall n \geq n_0 \). This implies that \( |\bar{g}_n| < \frac{1}{4} \nu \bar{g}, \forall n \geq n_0 \). By construction the same statement is also true for \( 0 \leq n \leq n_0 \). Whence for all \( n \geq 0 \)
\[ (1 - \frac{1}{4} \nu)\bar{g} < g_n < (1 + \frac{1}{4} \nu)\bar{g} \]  \( (6.38) \)

We have \( 0 < \nu < \frac{1}{2} \). Therefore the effective coupling constant generated by the discrete RG flow is uniformly bounded away from 0 at all RG scales.

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