Fibre bundle formulation of nonrelativistic quantum mechanics

II. Equations of motion and observables

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Abstract

We propose a new systematic fibre bundle formulation of nonrelativistic quantum mechanics. The new form of the theory is equivalent to the usual one but it is in harmony with the modern trends in theoretical physics and potentially admits new generalizations in different directions. In it a pure state of some quantum system is described by a state section (along paths) of a (Hilbert) fibre bundle. Its evolution is determined through the bundle (analogue of the) Schrödinger equation. Now the dynamical variables and the density operator are described via bundle morphisms (along paths). The mentioned quantities are connected by a number of relations derived in this work.

In the second part of this investigation we derive several forms of the bundle (analogue of the) Schrödinger equation governing the time evolution of state sections. We prove that up to a constant the matrix-bundle Hamiltonian, entering in the bundle analogue of the matrix form of the conventional Schrödinger equation, coincides with the matrix of coefficients of the evolution transport. This allows to interpret the Hamiltonian as a gauge field. Here we also apply the bundle approach to the description of observables. It is shown that to any observable there corresponds a unique Hermitian bundle morphism (along paths) and vice versa.
1. Introduction

This paper is a second part of our investigation devoted to the fibre bundle approach to nonrelativistic quantum mechanics. It is a straightforward continuation of [?].

The organization of the material is the following.

Sect. 2 is devoted to the bundle analogues of the Schrödinger equation which are fully equivalent to it. In particular, in it is introduced the matrix-bundle Hamiltonian which governs the quantum evolution through the matrix-bundle Schrödinger equation. The corresponding matrix of the bundle-evolution transport (operator) is found. It is proved that up to a constant the matrix of the coefficients of the bundle evolution transport coincides with the matrix-bundle Hamiltonian. On this basis is derived the (invariant) bundle-Schrödinger equation. Geometrically it simply means that the corresponding state sections are (parallelly, or, more precisely, linearly) transported by means of the bundle evolution transport (along paths).

In Sect. 3 is considered the question for the bundle description of observables. It turns out that to any observable there corresponds a unique Hermitian bundle morphism (along paths) and vice versa.

Sect. 4 closes the work.

The notation of the present work is the the same as the one in [?] and we are not going to recall it here.

The references to sections, equations, footnotes etc. from [?] are obtained from their sequential numbers in [?] by adding in front of them the Roman one (I) and a dot as a separator. For instance, Sect. I.4 and (I.5.13) mean respectively section 4 and equation (5.13) (equation 13 in Sect. 5) of [?].

Below, for reference purposes, we present a list of some essential equations of [?] which are used in this paper. Following the just given convention, we retain their original reference numbers.

\[ i\hbar \frac{d\psi(t)}{dt} = \mathcal{H}(t)\psi(t), \quad (I.2.6) \]

\[ i\hbar \frac{\partial U(t,t_0)}{\partial t} = \mathcal{H}(t) \circ U(t,t_0), \quad U(t_0,t_0) = \text{id}_\mathcal{F}, \quad (I.2.8) \]

\[ \mathcal{H}(t) = i\hbar \frac{\partial U(t,t_0)}{\partial t} \circ U^{-1}(t,t_0) = i\hbar \frac{\partial U(t,t_0)}{\partial t} \circ U(t_0,t), \quad (I.2.9) \]

\[ \langle A(t) \rangle_\psi^{\psi} := \frac{\langle \psi(t)|A(t)\psi(t)\rangle}{\langle \psi(t)|\psi(t)\rangle}, \quad (I.2.11) \]

\[ \psi(t) = l_\gamma(t)(\Psi_{\gamma}(t)) \in \mathcal{F}, \quad (I.4.1) \]

\[ \langle \cdot | \cdot \rangle_x = \langle l_x : l_x \cdot | \cdot \rangle, \quad x \in M, \quad (I.4.4) \]

\[ \langle A^\dagger \Phi_x | \Psi_x \rangle_x := \langle \Phi_x | A \Psi_x \rangle_x, \quad \Phi_x, \Psi_x \in F_x, \quad (I.4.19) \]

\[ \Psi_{\gamma}(t) = U_{\gamma}(t,s)\Psi_{\gamma}(s), \quad (I.5.7) \]

\[ U_{\gamma}(t,s) = l^{-1}_\gamma(t) \circ U(t,s) \circ l_\gamma(s), \quad s, t \in J. \quad (I.5.10) \]
2. The bundle equations of motion

If we substitute (I.5.11) into (I.2.6)–(I.2.10), we ‘get’ the ‘bundle’ analogues of (I.2.6)–(I.2.10). But they will be wrong! The reason for this being that they will contain partial derivatives like \( \partial l_\gamma(t) / \partial t \), \( \partial \Psi_\gamma(t) / \partial t \), and \( \partial U_\gamma(t, t_0) / \partial t \), which are not defined at all. For instance, for the first one we must have \( \partial l_\gamma(t) / \partial t \equiv \lim_{\varepsilon \to 0} \left( \frac{1}{\varepsilon} (l_\gamma(t+\varepsilon) - l_\gamma(t)) \right) \), but the ‘difference’ in this limit is not defined (for \( \varepsilon \neq 0 \)) because \( l_\gamma(t+\varepsilon) \) and \( l_\gamma(t) \) act on different spaces, viz. resp. on \( F_\gamma(t+\varepsilon) \) and \( F_\gamma(t) \). The same is the situation with \( \partial U_\gamma(t, t_0) / \partial t \). The most obvious is the contradiction in \( \partial \Psi_\gamma(t) / \partial t \equiv \lim_{\varepsilon \to 0} \left( \frac{1}{\varepsilon} (\Psi_\gamma(t+\varepsilon) - \Psi_\gamma(t)) \right) \), because \( \Psi_\gamma(t+\varepsilon) \) and \( \Psi_\gamma(t) \) belong to different (for \( \varepsilon \neq 0 \)) vector spaces.

One way to go through this difficulty is to define, e.g. \( \partial \Psi_\gamma(t) / \partial t \) like \( l_\gamma^{-1}(t) \partial \psi_\gamma(t) / \partial t \) (cf. (I.4.1)) but this does not lead to something important and new.

To overcome this problem, we are going to introduce local bases (or coordinates) and to work with the matrices of the corresponding operators and vectors in them.

Let \( \{ e_a(x) \}, \ a \in \Lambda \) be a basis in \( F_x = \pi^{-1}(x), \ x \in M \). The indices \( a, b, c, \ldots \in \Lambda \) may take discrete, or continuous, or both values. More precisely, the set \( \Lambda \) has a decomposition \( \Lambda = \Lambda_\text{d} \cup \Lambda_\text{c} \) where \( \Lambda_\text{d} \) is a union of (a finite or countable) subsets of \( \mathbb{N} \) (or, equivalently, of \( \mathbb{Z} \)) and \( \Lambda_\text{c} \) is union of subsets of \( \mathbb{R} \) (or, equivalently, of \( \mathbb{C} \)). Note that \( \Lambda_\text{d} \) or \( \Lambda_\text{c} \), but not both, can be empty. This is why sums like \( \lambda^a e_a(x) \) or \( \lambda_a \mu^a \) for \( a \in \Lambda \), \( \lambda^a, \mu^a \in \mathbb{C} \) must be understood as a sum over the discrete (enumerable) part(s) of \( \Lambda \), if any, plus the (Stieltjes or Lebesgue) integrals over the continuous part(s) of \( \Lambda \), if any. For instance: \( \lambda^a e_a(x) := \sum_{a \in \Lambda_\text{d}} \lambda^a e_a(x) + \int_{a \in \Lambda_\text{c}} \lambda^a e_a(x) \). By this reason it is better to write \( \sum_{a \in \Lambda} := \sum_{a \in \Lambda_\text{d}} + \int_{a \in \Lambda_\text{c}} da \) instead of \( \sum_{a \in \Lambda} \), but we shall avoid this complicated notation by using the assumed summation convention on indices repeated on different levels.\(^1\)

The matrices corresponding to vectors or operators in a given field of bases will be denoted with the same symbol but in \textbf{boldface}, for example: \( U_\gamma(t, s) := [(U_\gamma(t, s))^b_a] \) and \( \Psi_\gamma(s) := [\Psi_\gamma(s)] \), where \( U_\gamma(t, s) (e_b(\gamma(s))) =: (U_\gamma(t, s))^a_b e_a(\gamma(t)) \) and \( \Psi_\gamma(s) =: \Psi_\gamma(s)^a e_a(\gamma(s)) \).\(^2\)

Analogously, we suppose in \( \mathcal{F} \) to be fixed a basis \( \{ f_a(t), \ a \in \Lambda \} \) with respect to which we shall use the same bold-faced matrix notation, for instance: \( \mathcal{U}(t, s) = [\mathcal{U}_a^b(t, s)], \mathcal{U}(t, s) (f_a(s)) =: (\mathcal{U}(t, s))^b_a f_b(t), \psi(t) = \).

\(^1\)For details about infinite dimensional matrices see, for instance, [7] and [?, chapter VII, § 18]. A comprehensive presentation of the theory of infinite matrices is given in [7]; this book is mainly devoted to infinite discrete matrices but it contains also some results on continuous infinite matrices related to Hilbert spaces.

\(^2\)The matrices \( U(t, s) \) and \( \mathcal{U}(t, s) \) are closely related to propagator functions [7], but we will not need these explicit connections. For explicit calculations and construction of \( mape(t, s) \) see [7, § 21, §22].
[ψa(t)], ψ(t) =: ψa(t)f_a(t), and, at last, \( L_x(t) = [(l_x)_a^b(t)] \), \( l_x(e_a(x)) =: (l_x)_a^b(t)f_b(t) \). Generally \( L_x(t) \) depends on \( x \) and \( t \), but if \( x = γ(s) \) for some \( s \in J \), we put \( t = s \) as from physical reasons is clear that \( F_γ(t) \) corresponds to \( F \) at the ‘moment’ \( t \), i.e. the components of \( l_γ(s) \) are with respect to \( \{ e_a(γ(s)) \} \) and \( \{ f_a(s) \} \). The same remark concerns ‘two-point’ objects like \( U_γ(t, s) \) and \( U(t, s) \) whose components will be taken with respect to pairs of bases like \( \{ e_a(γ(t)) \}, \{ e_a(γ(s)) \} \) and \( \{ f_a(t) \}, \{ f_a(s) \} \) respectively.

Evidently, the equations (I.4.1), (I.5.7)–(I.5.10) remain valid {	extit{mutatis mutandis}} in the introduced matrix notation: the kernel letters have to be made bold-faced, the operator composition (product) must be replaced by matrix multiplication, and the identity map \( F_a \) has to be replaced by the unit matrix \( 1_{F_x} := [δ_a^b] : = [(id_{F_x})_a^b] \) of \( F_x \) in \( \{ e_a(x) \} \). Here \( δ^b_a = 1 \) for \( a = b \) and \( δ^b_a = 0 \) for \( a \neq b \), which means that \( e_a(x) = δ^b_a e_b(x) \). For instance, using the above definitions, one verifies that (I.5.10) is equivalent to

\[
U_γ(t, s) = l^{-1}(t)U(t, s)l_γ(s)(s).
\]

Let \( Ω(x) := [Ω_a^b(x)] \) and \( ω(t) := [ω_a^b(t)] \) be nondegenerate matrices. The changes

\[
\{ e_a(x) \} \rightarrow \{ e'_a(x) := Ω_a^b(x)e_b(x) \}, \quad \{ f_a(t) \} \rightarrow \{ f'_a(t) := ω_a^b(t)e_b(t) \}
\]

de the bases in \( F_x \) and \( F \), respectively, lead to the transformation of the matrices of the components of \( Φ_x \) \( F_x \) and \( φ \) \( F \), respectively, according to

\[
Φ_x \rightarrow Φ'_x = (Ω^T(x))^{-1}Φ_x, \quad φ \rightarrow φ' = (ω^T(t))^{-1}φ.
\]

Here the super script \( T \) means matrix transposition, for example \( Ω^T(x) := [(Ω^T(x))_a^b] \) with \( (Ω^T(x))_a^b := Ω_b^a(x) \). One easily verifies the transformation

\[
l_x(t) \rightarrow l'_x(t) = (ω^T(t))^{-1}l_x(t)Ω^T(x)
\]

of the components of the linear isomorphisms \( l_x : F_x \rightarrow F \) under the above changes of bases.

For any operator \( A(t) : F \rightarrow F \) we have

\[
A(t) \rightarrow A'(t) = (ω^T(t))^{-1}A(t)ω^T(t).
\]

Analogously, if \( A(t) \) is a morphism of \( (F, π, M) \), i.e. if \( A : F \rightarrow F \) and \( π \circ A = id_M \), and \( Φ_x := A(t)|_{F_x} \), then

\[
Φ_x(t) \rightarrow Φ'_x(t) = (Ω^T(t))^{-1}Φ_x(t)Ω^T(t).
\]
Note that the components of $\mathcal{U}(t, s)$, when referred to a pair of bases 
$\{e_a(t)\}$ and $\{e_a(s)\}$, transform according to

$$\mathcal{U}(t, s) \mapsto \mathcal{U}'(t, s) = \left(\omega^\top(t)\right)^{-1} \mathcal{U}(t, s) \omega^\top(s). \quad (2.5)$$

Analogously, the change $\{e_a(\gamma(t))\} \to \{e'_a(\gamma(t))\} := \Omega^b_a(t; \gamma) e_b(\gamma(t))$, with a nondegenerate matrix $\Omega(t; \gamma) := [\Omega^b_a(t; \gamma)]$ along $\gamma$, implies

$$\mathcal{U}_\gamma(t, s) \mapsto \mathcal{U}'_\gamma(t, s) = \left(\Omega^\top(t; \gamma)\right)^{-1} \mathcal{U}_\gamma(t, s) \Omega^\top(s; \gamma). \quad (2.6)$$

Substituting $\psi(t) = \psi^a(t)f_a(t)$ into (I.2.6), we get the *matrix Schrödinger equation*

$$\frac{d\psi(t)}{dt} = \mathcal{H}^m(t)\psi(t) \quad (2.7)$$

where

$$\mathcal{H}^m(t) := \mathcal{H}(t) - \hbar \mathbf{E}(t) \quad (2.8)$$

is the *matrix Hamiltonian* (in the Hilbert space description). Here $\mathbf{E}(t) = [E^b_a(t)]$ determines the expansion of $df_a(t)/dt$ over $\{f_a(t)\} \subset \mathcal{F}$, that is $df_a(t)/dt = E^b_a(t)f_b(t)$; if $f_a(t)$ are independent of $t$, which is the usual case, we have $\mathbf{E}(t) = 0$. In the last case $\mathcal{H}^m = \mathcal{H}$. It is important to be noted that $\mathcal{H}^m$ is independent of $\mathbf{E}(t)$. In fact, applying (I.2.9) to the basic vector $f_a(t)$, we get

$$\mathcal{H}(t)f_a(t) = i\hbar(\Omega(t)U(t, t_0))f_a(t_0)U_\gamma^b(t_0, t) = i\hbar[\Omega(t)\mathcal{U}(t_0, t)]U_\gamma^b(t_0, t),$$

that is

$$\mathcal{H}(t) = i\hbar \frac{\partial \mathcal{U}(t_0, t)}{\partial t} \mathcal{U}(t_0, t) + i\hbar \mathbf{E}(t) \quad (2.9)$$

which leads to

$$\mathcal{H}^m(t) = i\hbar \frac{\partial \mathcal{U}(t_0, t)}{\partial t} \mathcal{U}(t_0, t). \quad (2.10)$$

Substituting into (2.7) the matrix form of (I.4.1), we find the *matrix-bundle Schrödinger equation*

$$i\hbar \frac{d\mathbf{\Psi}_\gamma(t)}{dt} = \mathcal{H}^m(t)\mathbf{\Psi}_\gamma(t) \quad (2.11)$$

where the *matrix-bundle Hamiltonian* is

$$\mathcal{H}^m_\gamma(t) := \mathcal{H}(t) - \hbar \mathbf{E}(t) \quad (2.12)$$

and the notation $\mathbf{L}(t, s; \gamma) = H(t, s; \gamma) = \mathcal{U}_\gamma(s, t; \gamma)$ and $\mathbf{A}(t) = \Omega^\top(t; \gamma)$ is used.
Combining (2.8) and (2.12), we find the following connection between the conventional and bundle matrix Hamiltonians:

\[
H_m^\gamma(t) = l_{\gamma(t)}^{-1}(t)H_\gamma^m(t)l_{\gamma(t)}(t) - i\hbar l_{\gamma(t)}^{-1}(t)\frac{dl_{\gamma(t)}(t)}{dt}.
\]

(2.13)

Remark 2.1. Choosing \( e_a(x) = l_x^{-1}(f_a) \) for \( df_a(t)/dt = 0 \), we get \( l_x(t) = [\delta_a^x] \). Then \( H_\gamma(t) = H(t) \). So, as \( H^\dagger = H \), we have \( (H^m_\gamma(t))^\dagger = H(t) = H^m_\gamma(t) \) where we use the dagger (\( \dagger \)) to denote also matrix Hermitian conjugation. Here \( H^m_\gamma(t) \) is a Hermitian matrix in the chosen basis, but in other bases it may not be such (see (2.23) below). Analogously, choosing \( \{f_a(t)\} \) such that \( E(t) = 0 \), we see that \( H^m(t) = H(t) \) is a Hermitian matrix, otherwise it may not be such.

Remark 2.2. Note that, due to (2.13), the transition \( H^m_\gamma \rightarrow H^m_\gamma \) is very much alike a gauge (or connection) transformation [6] (see also below (2.21)–(2.23)).

Because of (2.11) and (I.5.7) there is 1:1 correspondence between \( U_\gamma \) and \( H^m_\gamma \) expressed through the initial-value problem (cf. (I.2.8))

\[
i\hbar \frac{\partial U_\gamma(t, t_0)}{\partial t} = H^m_\gamma(t)U_\gamma(t, t_0), \quad U_\gamma(t_0, t_0) = \mathbb{1}_{F_\gamma(t_0)},
\]

(2.14)

or via the equivalent to it integral equation

\[
U_\gamma(t, t_0) = \mathbb{1}_{F_\gamma(t_0)} + \frac{1}{i\hbar} \int_{t_0}^{t} H^m_\gamma(\tau)U_\gamma(\tau, t_0)d\tau.
\]

(2.15)

So, if \( H^m_\gamma \) is given, we have (cf. (I.2.10))

\[
U_\gamma(t, t_0) = \text{Texp} \left\{ \int_{t_0}^{t} \frac{1}{i\hbar} H^m_\gamma(\tau)d\tau \right\}
\]

(2.16)

and, conversely, if \( U_\gamma \) is given, then (cf. (I.2.9) and (2.10))

\[
H^m_\gamma(t) = i\hbar \frac{\partial U_\gamma(t, t_0)}{\partial t}U^{-1}_\gamma(t, t_0) = \frac{\partial U_\gamma(t, t_0)}{\partial t}U_\gamma(t_0, t).
\]

(2.17)

The next step is to write the above matrix equations into an invariant, i.e. basis-independent, form. For this purpose we shall use the introduce in [7], derivation along paths uniquely corresponding to any linear transport along paths in a vector bundle.
Let $D$ be the derivation along paths corresponding to the bundle evolution transport $U$, that is (cf. [?, definition 2.3] or [?, definition 4.1]) 

$D: \gamma \mapsto D\gamma$, where $D\gamma$, called derivation along $\gamma$ generated by $U$, is such that $D\gamma: s \mapsto D\gamma s$ and the derivation $D\gamma s : \text{Sec}^1 \left( (F, \pi, M) \mid_{\gamma(J)} \right) \rightarrow \pi^{-1}(\gamma(s))$

along $\gamma$ at $s$ generated by $U$ is defined by

$D\gamma s \chi := \lim_{\varepsilon \to 0} \left\{ \frac{1}{\varepsilon} [U\gamma(s, s + \varepsilon)\chi(\gamma(s + \varepsilon)) - \chi(\gamma(s))] \right\} \quad (2.18)$

for any $C^1$ section $\chi$ over $\gamma(J)$ in $(F, \pi, M)$.

By [?, equation (2.27)] or [?, proposition 4.2] the local explicit form of (2.18) is

$D\gamma s \chi = \left( \frac{d\chi^a(\gamma(s))}{ds} + \Gamma^a_b(s; \gamma)\chi^b(\gamma(s)) \right) e_a(\gamma(s)) \quad (2.19)$

where the coefficients $\Gamma^b_a(s; \gamma)$ of $U$ are defined by

$\Gamma^b_a(s; \gamma) := \frac{\partial (U\gamma(s, t))^b}{\partial t} \bigg|_{t=s} = -\frac{\partial (U\gamma(t, s))^b}{\partial t} \bigg|_{t=s}. \quad (2.20)$

Using (I.5.9) and (2.17), both for $t_0 = t$, we see that

$\Gamma(\gamma)(t) := \left[ \Gamma^b_a(t; \gamma) \right] = -\frac{1}{i\hbar} H^m_\gamma(t) \quad (2.21)$

which expresses a fundamental result: up to a constant the matrix-bundle Hamiltonian coincides with the matrix of coefficients of the bundle evolution transport (in a given field of bases). Let us recall that, using another arguments, analogous result was obtained in [?, sect. 5].

There are two invariant operators corresponding to the Hamiltonian $\mathcal{H}$ in $\mathcal{F}$: the bundle-evolution transport $U$ and the corresponding to it derivation along paths $D$. The equations (2.11)–(2.21), as well as the general results of [?, § 2] and [?, § 4], imply that these three operators, namely $\mathcal{H}$, $U$, and $D$, are equivalent in a sense that if one of them is given, then the remaining ones are uniquely determined.

**Example 2.1.** Let $\{e_a(x)\}$ be fixed by $e_a(x) = l^{-1}_x(f_a)$ for $df(t)/dt = 0$. Then $H^m_\gamma(t)$ is a Hermitian matrix (see remark 2.1). Consequently, in this case, $\Gamma(\gamma)$ is anti-Hermitian, i.e. $(\Gamma(\gamma))^\dagger = -\Gamma(\gamma)$. Note that for other choices of the bases this property may not hold.
Example 2.2. Let $\mathcal{H}$ be given and independent of $t$, i.e. $\partial \mathcal{H}(t)/\partial t = 0$, and $\{e_a(x)\}$ be fixed by $e_a(x) = t_x^{-1}(f_a)$ for $df(t)/dt = 0$. Then $L_x(t) = [\delta^a_0]$ with $\delta^a_0$ defined above. Equations (2.12) and (2.21) yield $H_m^\gamma(t) = \mathcal{H}(t)$ and $\Gamma_\gamma(t) = -\mathcal{H}(t)/i\hbar$. Finally, now the solution of (2.14) is $U_\gamma(t,t_0) = \exp (\mathcal{H}(t)(t-t_0)/i\hbar)$ (cf. (2.16)).

According to [?, equation (2.30)] (or [?, equation (4.11)]) and footnote I.17, if a basis $\{e_a(\gamma(t))\}$ is change to $\{e'_a(t;\gamma) = \Omega_a^b(t;\gamma)e_b(\gamma(t))\}$ with $\det \Omega(t;\gamma) \neq 0$, $\Omega(t;\gamma) := [\Omega_a^b(t;\gamma)]$, then $\Gamma_\gamma(t)$ transforms into\footnote{In [?, ?] the matrix $A(t) = \Omega^\top(t;\gamma)$ is used instead of $\Omega(t;\gamma)$.}

$$\Gamma'_\gamma(t) = (\Omega^\top(t;\gamma))^{-1}\Gamma_\gamma(t)\Omega^\top(t;\gamma) + (\Omega^\top(t;\gamma))^{-1}\frac{d\Omega^\top(t;\gamma)}{dt}. \quad (2.22)$$

This result is also a corollary of (2.5) and (2.20).

Hence (see (2.21)), the matrix-bundle Hamiltonian undergoes the change $H_m^\gamma(t) \mapsto 'H_m^\gamma(t)$ where

$$'H_m^\gamma(t) = (\Omega^\top(t;\gamma))^{-1}H_m^\gamma(t)\Omega^\top(t;\gamma) - i\hbar(\Omega^\top(t;\gamma))^{-1}\frac{d\Omega^\top(t;\gamma)}{dt}, \quad (2.23)$$

which can be deduced from (2.13) too.

Now we are able to write into an invariant form the matrix-bundle Schrödinger equation (2.11). Substituting (2.21) into (2.11) and using (2.19), we find that (2.11) is equivalent to

$$D^\gamma_\gamma \Psi_\gamma = 0. \quad (2.24)$$

This is the (invariant) bundle Schrödinger equation (for the state sections). Since it coincides with the linear transport equation along $\gamma$ [?, definition 5.2] for the bundle evolution transport, it has a very simple and fundamental geometrical meaning. By [?, proposition 5.4] this is equivalent to the statement that $\Psi_\gamma$ is a (linearly) transported along $\gamma$ section with respect to the bundle evolution transport (expressed in other terms via (I.5.7); see [?, definition 2.2]). Note that (2.24) and (I.5.7) are compatible as [?, equation (4.5)] is fulfilled (see also [?, equation (2.25)]): $D^\gamma_\gamma \circ U_\gamma(t,t_0) \equiv 0$, $t,t_0 \in J$ ($\gamma$ is not a summation index here!). Moreover, if $D$ is given (independently of $U$, e.g. through (2.19)), then from [?, proposition 5.4] follows that $U$ is the unique solution of the (invariant) initial-value problem

$$D^\gamma_\gamma \circ U_\gamma(t,t_0) = 0 \quad U_\gamma(t_0,t_0) = \text{id}_{F_\gamma(t_0)}. \quad (2.25)$$

This is the bundle Schrödinger equation for the evolution transport (operator) $U$. In fact it is the inversion of (2.18) with respect to $U$.

Let us summarize in conclusion. There are two equivalent ways of describing the unitary evolution of a quantum system: (i) through the evolution transport $\mathcal{U}$ (see (I.2.1)) or by the Hermitian Hamiltonian $\mathcal{H}$ (see (I.2.6))
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in the Hilbert space \( \mathcal{F} \) (which is the typical fibre in the bundle description) and (ii) through the bundle evolution transport \( U \) (see (I.5.7)), which is a Hermitian (and unitary) transport along paths, or the derivation along paths \( D \) (see (2.24)) in the Hilbert fibre bundle \( (F, \pi, M) \). In the bundle description \( U \) corresponds to \( U \) (see (I.5.10)) and \( D \) to \( H \) (see (2.19) and (2.21)).

Since now we have in our disposal the machinery required for analysis of [2], we, as promised in Sect. I.1, want to make some comments on it. In [2], p. 1455, left column, paragraph 4] is stated "that in the Heisenberg gauge (picture) the Hamiltonian is the null operator”. If so, all eigenvalues of the Hamiltonian vanish and, as they are picture-independent, they are null in any picture of quantum mechanics. Consequently, form here one deduces the absurd conclusion that the ‘energy levels of any system coincide and correspond to one and the same energy equal to zero’. Since the paper [2] is mathematically completely correct and rigorous, there is something wrong with the physical interpretation of the mathematical scheme developed in it. Without going into details, we describe below the solution of this puzzle which simultaneously throws a bridge between [?] and the present investigation.

In [2] the system’s Hilbert space \( \mathcal{H} \) is replaced by a differentiable Hilbert bundle \( \mathcal{E}(\mathbb{R}_+, \mathcal{H}) \) (in our terms \( (\mathcal{E}, \pi, \mathbb{R}_+) \) with a fibre \( \mathcal{H} \)), \( \mathbb{R}_+ := \{ t : t \in \mathbb{R}, t \geq 0 \} \), which is an associated Hilbert bundle of the principle fibre bundle \( \mathcal{P}(\mathbb{R}_+, \mathfrak{U}(\mathcal{H})) \) of orthonormal bases of \( \mathcal{H} \) where \( \mathfrak{U}(\mathcal{H}) \) is the unitary group of (linear) bounded invertible operators in \( \mathcal{H} \) with bounded inverse. Let \( p : \mathfrak{U}(\mathcal{H}) \rightarrow GL(\mathbb{C}, dim\mathcal{H}) \) be a (linear and continuous) representation of \( \mathfrak{U}(\mathcal{H}) \) into the general linear group of \( dim\mathcal{H} \)-dimensional matrices. An obvious observation is that [2, equation (4.6)] under \( p \) transforms, up to notation, to our equation (2.23) (in [2] is taken \( \hbar = 1 \)). Thus we see that what in [2] is called Hamiltonian is actually the (analogue of the) matrix-bundle Hamiltonian \( H_m^\gamma(t) \), not the Hamiltonian \( H \) itself (see also Sect. 3). This immediately removes the above-pointed conflict: as we shall see later in the third part of this series, along any \( \gamma \) (or, over \( \mathbb{R}_+ \) in the notation of [2] - see below), we can choose a field of frames (bases) in which \( H_m^\gamma(t) \) identically vanishes but, due to (2.12), this does not imply the vanishment of the Hamiltonian at all. This particular choice of the frames along \( \gamma \) corresponds to the ‘Heisenberg gauge’ in [2], normally known as Heisenberg picture.

Having in mind the above, we can describe [?] as follows. In it we have \( F = \mathcal{E}, M = \mathbb{R}_+, \mathcal{F} = \mathcal{H} \) (the conventional system’s Hilbert space), \( J = \mathbb{R}_+, \gamma = \text{id}_{\mathbb{R}_+} \) (other choices of \( \gamma \) correspond to reparametrization of the time), and \( \frac{\partial}{\partial t}, t \in \mathbb{R}_+ \) is the analog of \( D^+ \) in [2]. As we already pointed, the matrix-bundle Hamiltonian \( H_m^\gamma(t) \) represents the operator \( A(t) \) of [2], incorrectly identified there with the ‘Hamiltonian’ and the choice of a field of bases over \( \gamma(J) = \mathbb{R}_+ = M \) corresponds to an appropriate ‘choice of the gauge’ in [2]. Now, after its correspondence between [?] and the present
work is set, one can see that under the representation \( p \) the main results of [?], expressed by [?], equations (4.5), (4.6) and (4.8), correspond to our equations (2.24) (see also (2.19)), (2.23) and (2.4) respectively.

Ending with the comment on [?], we note two things. First, this paper uses a rigorous mathematical base, analogous to the one in [?], which is not a goal of our work. And, second, the ideas of [?] are a very good motivation for the present investigation and are helpful for its better understanding.

3. The bundle description of observables

In quantum mechanics is accepted that to any dynamical variable, say \( A \), there corresponds a unique observable, say \( \mathcal{A}(t) \), which is a Hermitian linear operator in the Hilbert space \( \mathcal{F} \), i.e. \( \mathcal{A}(t) : \mathcal{F} \to \mathcal{F} \), \( \mathcal{A}(t) \) is linear, and \( \mathcal{A}^\dagger = \mathcal{A} \). What is the analogue of \( \mathcal{A}(t) \) in the developed here bundle description? Below we prove that this is a suitable bundle morphism \( \mathcal{A} \) of the introduced in Sect. I.5 fibre bundle \( (\mathcal{F}, \pi, M) \).

Let \( \psi^{(\lambda)}(t) \in \mathcal{F} \) be an eigenvector of \( \mathcal{A}(t) \) with eigenvalue \( \lambda \) (\( \in \mathbb{R} \)), i.e. \( \mathcal{A}(t)\psi^{(\lambda)}(t) = \lambda\psi^{(\lambda)}(t) \). According to (I.4.1) to \( \psi^{(\lambda)}(t) \) corresponds the vector \( \Psi^{(\lambda)}(t) = l^{-1}_{\gamma(t)}\psi^{(\lambda)}(t) \in F_{\gamma(t)} \) in the bundle description. But the Hilbert space and Hilbert bundle descriptions of a quantum evolution are fully equivalent (see Sect. I.5). Hence to \( \mathcal{A}(t) \) in \( F_{\gamma(t)} \) must correspond certain operator which we denote by \( A_\gamma(t) \). We define this operator by demanding any \( \Psi^{(\lambda)}(t) \) to be its eigenvector with eigenvalue \( \lambda \), i.e. \( (A_\gamma(t))\Psi^{(\lambda)}(t) := \lambda\Psi^{(\lambda)}(t) \). Combining this equality with the preceding two, we easily verify that \( \mathcal{A}(t) \circ l^{-1}_{\gamma(t)} \psi^{(\lambda)}(t) = (l^{-1}_{\gamma(t)} \circ \mathcal{A}(t)) \psi^{(\lambda)}(t) \) where the linearity of \( l_x \) has been used. Admitting that \( \{\psi^{(\lambda)}(t)\} \) form a complete set of vectors, i.e a basis of \( \mathcal{F} \), we find

\[
A_\gamma(t) = l^{-1}_{\gamma(t)} \circ \mathcal{A}(t) \circ l_{\gamma(t)} : F_{\gamma(t)} \to F_{\gamma(t)}. \tag{3.1}
\]

More ‘physically’ the same result is derivable from (I.2.11) too. The mean value \( \langle A \rangle_\psi \) of \( \mathcal{A} \) at a state \( \psi(t) \) is given by (I.2.11) and the mean value of \( A_\gamma(t) \) at a state \( \Psi_{\gamma}(t) \) is

\[
\langle A_\gamma(t) \rangle_{\Psi_{\gamma}} = \frac{\langle \Psi_{\gamma}(t)|A_\gamma(t)|\Psi_{\gamma}(t) \rangle_{\gamma(t)}}{\langle \Psi_{\gamma}(t)|\Psi_{\gamma}(t) \rangle_{\gamma(t)}}, \tag{3.2}
\]

that is it is given via (I.2.11) in which the scalar product \( \langle \cdot | \cdot \rangle_x \), defined by (I.4.4), is used instead of \( \langle \cdot | \cdot \rangle \). We define \( A_\gamma(t) \) by demanding

\[
\langle A(t) \rangle_{\psi} = \langle A_\gamma(t) \rangle_{\Psi_{\gamma}}. \tag{3.3}
\]

Physically this condition is very natural as it means that the observed values of the dynamical variables are independent of the way we calculate them.
From this equality, (I.4.1), and (I.4.4), we get \( \langle \psi(t) | \mathcal{A}(t) | \psi(t) \rangle = \langle \psi(t) | \mathcal{A}^{-1}(t) \circ \mathcal{A}_\gamma(t) \circ l_{\gamma(t)}^{-1} \circ \mathcal{A}_\gamma(t) \circ l_{\gamma(t)} \rangle \) which, again, leads to (3.1). Thus we have also proved the equivalence of (3.1) and (3.3).

According to equation (3.1), along \( \gamma: J \to M \) to any operator \( \mathcal{A}(t): \mathcal{F} \to \mathcal{F} \), \( t \in J \), there corresponds a unique map \( A_{\gamma}(t): \mathcal{F}_{\gamma(t)} \to \mathcal{F}_{\gamma(t)} \) in any fibre \( \mathcal{F}_{\gamma(t)} \), \( t \in J \), in \( (\mathcal{F}, \pi, M) \). If \( J' \subseteq J \) is a subinterval on which \( \gamma \) is without self-intersections and \( A_{\gamma|J'}: \pi^{-1}(\gamma(J')) \to \pi^{-1}(\gamma(J')) \) is defined by \( A_{\gamma|J'}|_{\mathcal{F}_{\gamma(t')}} = A_{\gamma}(t'): \mathcal{F}_{\gamma(t')} \to \mathcal{F}_{\gamma(t')} \), \( t' \in J' \), then \( A_{\gamma|J'} \in \text{Morf}(\mathcal{F}, \pi, M) \), i.e. \( A_{\gamma|J'} \) is a morphism on the restricted on \( \gamma(J') \) fibre bundle \( (\mathcal{F}, \pi, M) \). In the general case we define the multiple-valued map \( A_{\gamma}: \mathcal{F} \to \mathcal{F} \) via \( A_{\gamma}|_{\mathcal{F}} := \{ A_{\gamma}(t): t \in J, \ \gamma(t) = x \} \) for every \( x \in M \). Evidently \( A_{\gamma}|_{\mathcal{F}_x} = \emptyset \) for \( x \not\in \gamma(J) \) and \( A_{\gamma}|_{\mathcal{F}_{\gamma(t)}} : \mathcal{F}_{\gamma(t)} \to \mathcal{F}_{\gamma(t)} \), \( t \in J \), the multiplicity of \( A_{\gamma}|_{\mathcal{F}_{\gamma(t)}} \) being equal to one plus the number of self-intersections of \( \gamma \) at the point \( \gamma(t) \). We call a \textit{(bundle) morphism along paths} any map \( A : \gamma \mapsto A_{\gamma} \), where \( A_{\gamma}: \mathcal{F} \to \mathcal{F} \) can be multiple-valued and such that \( \pi \circ (A_{\gamma}|_{\pi^{-1}(\gamma(J))}) = \text{id}_{\gamma(J)} \) and \( A_{\gamma}|_{\mathcal{F}_x} = \emptyset \) for \( x \not\in \gamma(J) \). We call the (possibly multiple-valued) map \( A_{\gamma} \) a \textit{(bundle) morphism along the path} \( \gamma \). Hence, the above-defined map \( A_{\gamma} \) is a morphism along \( \gamma \) which is single-valued (and consequently a morphism over \( \gamma(J) \)) iff \( \gamma \) is without self-intersections. Therefore the map \( A : \gamma \mapsto A_{\gamma} \) is morphism along paths. We call \( A \) \textit{Hermitian} and denote this by \( A^\dagger = A \), if \( A_{\gamma} \) are such, i.e. if (I.4.19) holds for \( A_{\gamma} \) instead of \( A \). The morphism along paths \( A \) is Hermitian if \( \mathcal{A}(t) \) is a Hermitian operator, viz. we have

\[
A^\dagger = A \iff A_{\gamma}^\dagger = A_{\gamma}(t) \iff \mathcal{A}^\dagger = \mathcal{A}(t),
\]

which is a simple corollary of (3.1) and (I.4.20). Hence, if the morphism \( A_{\gamma}(t) \) along \( \gamma \) corresponds to an observable \( \mathcal{A} \), it is Hermitian because \( \mathcal{A}(t) \) is such by assumption [?, ?]

Consequently, to any observable \( \mathcal{A} \) there corresponds a unique Hermitian bundle morphism \( A \) along paths and vice versa. Explicitly this correspondence is given by (3.1) which will be assumed hereafter. Its consequence is the independence of the physically measurable quantities (and the eigenvalues of the corresponding operators) of the mathematical way we describe them, as it should be.

Generally to any operator \( \mathcal{A}: \mathcal{F} \to \mathcal{F} \) there corresponds a unique (global) morphism \( \overline{\mathcal{A}} \in \text{Morf}(\mathcal{F}, \pi, M) \) given by

\[
\overline{\mathcal{A}}|_{\mathcal{F}_x} = \overline{\mathcal{A}}|_{\mathcal{F}_x} = l_x^{-1} \circ \mathcal{A} \circ l_x, \quad x \in M, \quad \mathcal{A}: \mathcal{F} \to \mathcal{F}.
\]

Consequently to an observable \( \mathcal{A}(t) \) can be assigned the (global) morphism \( \overline{\mathcal{A}}(t) \), \( \overline{\mathcal{A}}|_{\mathcal{F}_x} = l_x^{-1} \circ \mathcal{A}(t) \circ l_x \). But this morphism \( \overline{\mathcal{A}}(t) \) is almost useless for

\footnote{Cf. the definition of a (bundle) morphism \( C \in \text{Morf}(\mathcal{F}, \pi, M) := \{ B : \mathcal{F} \to \mathcal{F}, \pi \circ B = \text{id}_M \} \) of \( (\mathcal{F}, \pi, M) \).}
our goals as it simply gives in any fibre \( F_x \) a linearly isomorphic image of the initial observable \( A(t) \) (see Sect. I.4).

Notice that \( A_\gamma(t) \) generally depends explicitly on \( t \) even if \( A \) does not. In fact, from (3.1) we get

\[
\frac{\partial A_\gamma(t)}{\partial t} = \left[ g_\gamma(t), A_\gamma(t) \right] + l_{\gamma(t)}^{-1}(t) \frac{\partial A(t)}{\partial t} l_{\gamma(t)}(t),
\]

where \( \left[ \cdot, \cdot \right] \) denotes the commutator of corresponding quantities and

\[
g_\gamma(t) := -l_{\gamma(t)}^{-1}(t) \frac{dl_{\gamma(t)}(t)}{dt}.
\]

In particular, to the Hamiltonian \( H \) in \( F \) there corresponds the bundle Hamiltonian (or the bundle-Hamiltonian morphism along paths) given by

\[
H_\gamma(t) := l_{\gamma(t)}^{-1}(t) \circ H(t) \circ l_{\gamma(t)}.
\]

It is a Hermitian bundle morphism along paths, \( H_\gamma \) is a Hermitian operator.

From (3.8), using (I.2.9) and (I.5.10), we find

\[
H_\gamma(t) = i\hbar l_{\gamma(t)}^{-1} \circ \frac{\partial U(t,t_0)}{\partial t} \circ l_{\gamma(t_0)} \circ U_\gamma(t_0, t).
\]

From here we can get a relationship between the matrix-bundle Hamiltonian and the bundle Hamiltonian. For this purpose we write (3.9) in a matrix form and using (2.17) and \( df_a(t)/dt = E_b^a f_b(t) \), we obtain:

\[
H_\gamma(t) = H^m_\gamma(t) + i\hbar l_{\gamma(t)}^{-1}(t) \left( \frac{dl_{\gamma(t)}(t)}{dt} + E(t) l_{\gamma(t)}(t) \right).
\]

Substituting here (2.12), we get

\[
H_\gamma(t) = l_{\gamma(t)}^{-1}(t) H(t) l_{\gamma(t)}(t)
\]

which is simply the matrix form of (3.8). Combining (3.10) with (2.13), we find the following connection between the matrix of the bundle Hamiltonian and the matrix Hamiltonian:

\[
H_\gamma(t) = l_{\gamma(t)}^{-1}(t) H^m(t) l_{\gamma(t)}(t) + i\hbar l_{\gamma(t)}^{-1}(t) E(t) l_{\gamma(t)}(t).
\]

We notice that, due to (3.5) as well as to (3.1), to the identity map of \( F \) there corresponds a morphism along paths equal to the identity map of \( F \):

\[
id_F \longleftrightarrow \id_F.
\]
of \((F, \pi, M)\), respectively. We will illustrate this in the case of, e.g., two variables. Let \(G: (A(t), B(t)) \mapsto G(A(t), B(t)) : F \to F\) be a function of the observables \(A(t), B(t): F \to F\). It is natural to define the bundle analogue \(G\) of \(G\) by

\[
G: (A, B) \mapsto G(A, B): \gamma \mapsto G_\gamma(A, B): \pi^{-1}(\gamma(J)) \to \pi^{-1}(\gamma(J)),
\]

where \(G_\gamma(A, B)|_{F_x} = \emptyset\) for \(x \notin \gamma(J)\) and

\[
G_\gamma(A, B)|_{F_{\gamma(t)}} := l_{\gamma(t)}^{-1} \circ G(A(t), B(t)) \circ l_{\gamma(t)} = l_{\gamma(t)}^{-1} \circ G(l_{\gamma(t)} \circ A_\gamma(t) \circ l_{\gamma(t)}^{-1}, l_{\gamma(t)} \circ B_\gamma(t) \circ l_{\gamma(t)}^{-1}) \circ l_{\gamma(t)}. \tag{3.14}
\]

Thus \(G(A, B)\) is a bundle morphism along paths. This definition becomes evident in the cases when \(G\) is a polynomial or if it is expressible as a convergent power series; in both cases the multiplication has to be understood as an operator composition. If we are dealing with one of these cases, the definition (3.14) follows from the fact that for any morphisms \(A_1, \ldots, A_k, k \in \mathbb{N}\) along paths of \((F, \pi, M)\) the equality

\[
A_{1, \gamma(t)} \circ A_{2, \gamma(t)} \circ \cdots \circ A_{k, \gamma(t)} = l_{\gamma(t)}^{-1} \circ (A_1(t) \circ A_2(t) \circ \cdots \circ A_k(t)) \circ l_{\gamma(t)} \tag{3.15}
\]

holds due to (3.1). In these cases \(G(A, B)\) depends only on \(A\) and \(B\) and it is explicitly independent on the isomorphisms \(l_x, x \in M\).

The commutator of two operators is an important operator function in quantum mechanics. In the Hilbert space and bundle descriptions it is defined by \([A, B]_\gamma := A \circ B - B \circ A\) and \([A, B]_\omega := A \circ B - B \circ A\) respectively. The relation

\[
[A_\gamma(t), B_\gamma(t)]_\omega = l_{\gamma(t)}^{-1} \circ [A, B]_\omega \circ l_{\gamma(t)} \tag{3.16}
\]

is an almost evident corollary of (3.1). It can also be considered as a special case of (3.14). In particular, to commuting observables (in \(F\)) there correspond commuting bundle morphisms (of \((F, \pi, M)\)):

\[
[A, B]_\gamma = 0 \iff [A, B]_\omega = 0. \tag{3.17}
\]

A little more general is the result, following from (3.16), that to observables whose commutator is a c-number there correspond bundle morphisms with the same c-number as a commutator:

\[
[A, B]_\omega = c(\text{id}_F) \iff [A, B]_\gamma = c(\text{id}_F). \tag{3.18}
\]

for some \(c \in \mathbb{C}\). In particular, the bundle analogue of the famous relation \([Q, P]_\gamma = i\hbar(\text{id}_F)\) between a coordinate \(Q\) and the conjugated to it momentum \(P\) is \([Q, P]_\omega = i\hbar(\text{id}_F)\).
A bit more complicated is the case for operators and morphisms along paths at different ‘moments’. Let $\gamma : \mathcal{J} \rightarrow \mathcal{M}$ and $r, s, t \in \mathcal{J}$. If $\mathcal{G}_{s,t} : (\mathcal{A}, \mathcal{B}) \mapsto \mathcal{G}(\mathcal{A}(s), \mathcal{B}(t))$, we define the bundle analogue $\mathcal{G}_{s,t}$ of $\mathcal{G}_{s,t}$ by

$$\mathcal{G}_{s,t} : (A, B) \mapsto \mathcal{G}_{s,t}(A, B) : \gamma \mapsto \mathcal{G}_{r,s,t}(A, B) : \pi^{-1}(\gamma(J)) \rightarrow \pi^{-1}(\gamma(J)),$$

where

$$\mathcal{G}_{r,s,t}(A, B) \bigg|_{\gamma(t)} := l_{\gamma(r)}^{-1} \circ \mathcal{G}(\mathcal{A}(s), \mathcal{B}(t)) \circ l_{\gamma(r)}$$

$$= l_{\gamma(r)}^{-1} \circ \mathcal{G}(l_{\gamma(r)} \circ A_{\gamma}(s) \circ l_{\gamma(r)}^{-1} l_{\gamma(r)} \circ B_{\gamma}(t) \circ l_{\gamma(r)}^{-1} l_{\gamma(r)} : F_{\gamma(r)} \rightarrow F_{\gamma(r)}.$$

(3.19)

Here

$$\mathcal{A}_{\gamma; r} := l_{\gamma(r)}^{-1} \circ A(t) \circ l_{\gamma(r)} = l_{\gamma(r)}^{-1} \mathcal{A}(t) \circ l_{\gamma(r)},$$

(3.20)

where (3.1) has been used and $l_{\gamma(r)}^\gamma : l_{\gamma(s)} \rightarrow l_{\gamma(t)}$ is the (flat) linear transport (along paths) from $\gamma(s)$ to $\gamma(t)$ assigned to the isomorphisms $l_{\gamma(r)}$, $x \in \mathcal{M}$ (see Sect. I.4, equation (I.4.16)).

Now the analogue of (3.15) is

$$\mathcal{A}_{1; \gamma; r} := l_{\gamma(r)}^{-1} \circ A(t_1) \circ l_{\gamma(r)} = l_{\gamma(r)}^{-1} \mathcal{A}_{1}(t_1) \circ l_{\gamma(r)}.$$

(3.21)

So, if $\mathcal{G}$ is a polynomial or a convergent power series, the morphism $\mathcal{G}_{\gamma; s,t}(A, B)$ along $\gamma$ depends only on $\mathcal{A}_{\gamma; r}(A)$ and $B_{\gamma; r}(A)$.

In particular for $\mathcal{G}(\cdot, \cdot) = [\cdot, \cdot]_\gamma$, have

$$\left[\mathcal{A}_{\gamma; s}(A), B_{\gamma; r}(A)\right]_\gamma = l_{\gamma(r)}^{-1} \circ [A(s), B(t)]_\gamma \circ l_{\gamma(r)}$$

(3.22)

which for $s = r = t$ reduces to (3.16). In this case the analogues of (3.17) and (3.18) are

$$[A(s), B(t)]_\gamma = 0 \iff [\mathcal{A}_{\gamma; s}(A), B_{\gamma; r}(A)]_\gamma = 0,$$

(3.23)

$$[A(s), B(t)]_\gamma = c(\mathrm{id}_F) \iff [\mathcal{A}_{\gamma; s}(A), B_{\gamma; r}(A)]_\gamma = c(\mathrm{id}_{F_{\gamma}(r)}),$$

(3.24)

respectively.

The above considerations can mutatis mutandis, e.g. by replacing $\gamma(t)$ with $x$, $\mathcal{A}(t)$ with $A$, $A$ with $A$, etc., be transferred to global morphisms of $(F, \pi, M)$, but this is not needed for the present investigation.

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7 According to [? sections 2 and 3] the morphism $\mathcal{A}_{\gamma; r}(A)$ along $\gamma$ is obtained by linear transportation of $A_{\gamma}(t)$ along $\gamma$ by means of the induced by $l_{\gamma(r)}^\gamma$ linear transport along paths in the fibre bundle morf $(F, \pi, M)$ of bundle morphisms of $(F, \pi, M)$. 
Further we will need a kind of ‘extension’ of the differentiation along paths $D^\gamma_t$ (see (2.18)) on some fibre morphisms that will be presented below.

Let $\text{Morf}^p(F, \pi, M), \ p \in \mathbb{N} \cup 0$, be the set of $C^p$ bundle morphisms of $(F, \pi, M)$. We define $\tilde{D}^\gamma_t$ to be the differentiation along $\gamma$ (at $t$) of bundle morphisms from $\text{Morf}^1(F, \pi, M)|_{\gamma(t)}$ and given by

$$\tilde{D}^\gamma_t : C \mapsto \tilde{D}^\gamma_t(C) := D^\gamma_t \circ C|_{\gamma(t)}$$

where $C \in \text{Morf}^1(F, \pi, M)$ acts on state vectors (only).

Applying (2.19), we can find the explicit (matrix) action of $\tilde{D}^\gamma_t$. Let $C_t := C|_{\gamma(t)}$ and $[X]$ be the matrix of a vector or an operator $X$ in $\{e_a\}$. Due to (2.19), we have $[\tilde{D}^\gamma_t(C)\Psi_\gamma(t)] = (\frac{d}{dt}C_t)\Psi_\gamma(t) + C_t(\frac{d}{dt}\Psi_\gamma(t)) + \Gamma_\gamma(t)C_t\Psi_\gamma(t)$. Substituting here $\frac{d}{dt}\Psi_\gamma(t)$ from (2.11) and using (2.21), we obtain the matrix equation

$$[\tilde{D}^\gamma_t(C)\Psi_\gamma(t)] = \left(\frac{d}{dt}C_t\right)\Psi_\gamma(t) + [\Gamma_\gamma(t), C_t]_\cdot \Psi_\gamma(t),$$

where $[\cdot, \cdot]$ is the commutator of the corresponding matrices, or

$$[\tilde{D}^\gamma_t(C)] = \frac{d}{dt}C_t + [\Gamma_\gamma(t), C_t]_\cdot.$$  

(3.27)

Nevertheless that the last equation is valid in any local basis it cannot be written in an invariant (operator) form as the action of $\frac{d}{dt}$ on bundle morphisms or sections is not defined, as well as to $\Gamma_\gamma(t)$ alone there does not correspond some invariant operator or morphism.

We derived (3.27) under the assumption that $\tilde{D}^\gamma_t$ acts on state vectors, i.e. on ones satisfying the matrix-bundle Schrödinger equation (2.11). Conversely, if we apply (3.27) to some vector $\Phi_\gamma(t) \in F_{\gamma(t)}$ and compare the result with the one for $\left(D^\gamma_t(C)\right)(\Phi_\gamma(t))$ obtained through (2.19) (see above), we see that $\Phi_\gamma(t)$ satisfies (2.11). Consequently, equation (3.27) is valid if and only if it is applied on vectors representing the evolution of a quantum system. Hence $\Psi_\gamma(t)$ is a state vector, i.e. it satisfies, for instance, the bundle Schrödinger equation (2.24), if and only if the equation (3.26) is valid for any bundle morphism $C$. In particular (3.26) is valid for the (Hermitian) morphisms (along paths) corresponding to observables and $\Psi_\gamma(t)$ satisfying the bundle Schrödinger equation (2.24).

The over-all above discussion shows the equivalence of (3.26) (for every morphisms $C$) with the Schrödinger equation (in anyone of its (equivalent) forms mentioned until now). That is why (3.26) can be called matrix-morphism Schrödinger equation.
4. Conclusion

Here we have continued to apply the fibre bundle formalism to nonrelativistic quantum mechanics. We derived different forms of the bundle Schrödinger equation which governs the time evolution of state sections along paths in the Hilbert bundle description.

In this description, as we have seen, the observables are described via Hermitian bundle morphisms along paths. We also have concerned some technical problems connected with functions of observables.

In the future continuation of the present series we plan to consider from a fibre bundle point of view the following items: pictures and integrals of motion, mixed states, evolution transport’s curvature, interpretation of the theory and its possible further developments.