Optimal Sampling and Remote Estimation of the Wiener Process over a Channel with Random Delay

Yin Sun, Yury Polyanskiy, and Elif Uysal-Biyikoglu
Dept. of ECE, the Ohio State University, Columbus, OH
Dept. of EECS, Massachusetts Institute of Technology, Cambridge, MA
Dept. of EEE, Middle East Technical University, Ankara, Turkey

Abstract—In this paper, we consider a sampling and remote estimation problem, where samples of a Wiener process are forwarded to a remote estimator via a channel with queueing and random delay. The estimator reconstructs an estimate of the real-time signal value from causally received samples. We obtain the jointly optimal sampling and estimation strategy that minimizes the mean-square estimation error subject to a maximum sampling rate constraint. We prove that a threshold-based sampler and a minimum mean-square error (MMSE) estimator are jointly optimal, and the optimal threshold is found exactly. Our jointly optimal solution exhibits an interesting coupling between the source and channel, which is different from the source-channel separation in many previous information theoretical studies. If the sampling times are independent of the observed Wiener process, the joint sampling and estimation optimization problem reduces to an age-of-information optimization problem that has been recently solved. Our theoretical and numerical comparisons show that the estimation error of the optimal sampling policy can be much smaller than those of age-optimal sampling, zero-wait sampling, and classic periodic sampling.

Index Terms—Sampling, remote estimation, age-of-information, Wiener process, queueing.

I. INTRODUCTION

In many networked control and monitoring systems (e.g., airplane/vehicular control, smart grid, stock trading, robotics, etc.), real-time updates of the system status state are critical for making decisions, and should be reported to the supervisor or control center in a timely fashion. Recently, the age-of-information, or simply the age, has been proposed as a metric for characterizing the timeliness of information updates. Suppose that the $i$-th update is generated at time $S_i$ and delivered at time $D_i$. At time $t$, the freshest update delivered to the destination was generated at time $\max\{S_i : D_i \leq t\}$. Then, the age at time $t$ is defined as

$$\Delta(t) = t - \max\{S_i : D_i \leq t\}. \quad (1)$$

Hence, the age is the time difference between the generation time of the freshest received update and the current time $t$.

The age-of-information, as well as the general non-linear age penalty models in [4]–[7], are appropriate for measuring the timeliness of message updates (e.g., news, fire alarm, email notifications, social updates, etc). However, the status state of many systems is in the form of a continuous-time signal (e.g., the location/speed/acceleration of a vehicle, the voltage and frequency in an electric power grid, stock price and chart, etc.). The age may not be a perfect metric for evaluating the timeliness of signal updates. For example, as illustrated in Fig. 1 a signal may change slow during a time interval and become more dynamic in a later time interval. Hence, the time difference in the age-of-information (1) is insufficient for precisely characterizing the variability of a signal over time.

In this paper, we investigate timely updates of signal measurements (i.e., signal samples), where the signal is modeled as a Wiener process. Consider a system with two terminals (see Fig. 2): An observer measuring a Wiener process $W_t$ and an estimator, whose goal is to provide the best-guess $\hat{W}_t$ for the real-time value of $W_t$ at every time $t \geq 0$. These two terminals are connected by a channel that transmits time-stamped samples of the form $(S_i, W_{S_i})$, where the sampling times $S_i$ satisfy $0 \leq S_1 \leq S_2 \leq \ldots$. The channel is modeled as a work-conserving FIFO queue with random i.i.d. delay $Y_i$, where $Y_i \geq 0$ is the channel delay (i.e., random transmission time) of sample $i$. The observer can choose the sampling times $S_i$ causally subject to an average sampling rate constraint

$$\liminf_{n \to \infty} \frac{1}{n} E[S_n] \geq \frac{1}{f_{\max}},$$

where $f_{\max}$ is the maximum allowed sampling rate.

Unless it arrives at an empty system, sample $i$ needs to wait in the queue until its transmission starts. Let $G_i$ be the transmission starting time of sample $i$ such that $S_i \leq G_i$. The delivery time of sample $i$ is $D_i = G_i + Y_i$. The initial value $W_0 = 0$ is known by the estimator for free, represented

By “work-conserving”, we meant that the channel is kept busy whenever there exist some generated samples that have not been delivered to the estimator.

Fig. 1: Signal variations during time intervals of length $T$. 

This paper was presented in part at IEEE ISIT 2017 [1].

This work was supported in part by the Center for Science of Information (CSoI), an NSF Science and Technology Center, under grant agreement CCF-09-39370, by ONR grant N00014-17-1-2417, and by TUBITAK.
by \( S_0 = D_0 = 0 \). At time \( t \), the estimator forms \( \hat{W}_t \) using causally received samples with \( D_i \leq t \). We measure the quality of remote estimation via the mean-square error (MSE) between \( W_t \) and \( \hat{W}_t \):

\[
\text{mse} = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T (W_t - \hat{W}_t)^2 dt \right].
\]

We study the joint optimization of causal sampling and estimation strategies that achieve the fundamental tradeoff between \( f_{\text{max}} \) and mse. One key challenge in this study, and some related work, e.g., \([8], [9]\), is how to optimally convey information to the estimator through samples: A sample \((S_t, W_S)\) naturally reveals the value of the Wiener process \( W_t \) at time \( t = S_t \). Interestingly, because the sampling time \( S_t \) is chosen based on the history observations of \( W_t \), a decision on when to generate a sample reveals additional information about \( W_t \). For example, if a sample is taken when \( W_t > 0 \), then not receiving a sample for a long time suggests that \( W_t \) is likely to be negative. The estimator can use this additional information to reduce the MSE. Accordingly, the sampler needs to carefully choose the sampling times \( S_t \) to make the best usage of the channel and help to reduce the MSE.

This, in spirit, is similar to the information theoretical studies in \([10], [11]\), where the timing of message arrival times at a queue is used to convey additional information to the receiver. Hence, the Shannon capacity of a queueing system is higher than its service rate \( \mu \). One difference is that the encoding schemes in \([10], [11]\) introduce some encoding delay, but we cares about real-time reconstruction of \( W_t \) in this study.

A. Contributions

The contributions of this paper are summarized as follows:

- We formulate the jointly optimal sampling and remote estimation problem as a constrained continuous-time Markov decision problem with a continuous state space, and solve it exactly. We prove that a threshold-based sampler and a minimum mean-square error (MMSE) estimator are jointly optimal, and the optimal threshold is found. Let \( Y \) be a random variable with the same distribution as \( Y_i \). The optimal threshold is determined by \( f_{\text{max}} \) and \( W_Y \), i.e., the amount of signal variation during the random channel delay. Notice that \( W_Y \) is a random variable that tightly couples the source process \( W_t \) and the channel delay \( Y \), which, on a high level, is different from the classic information theory where source coding and channel coding can usually be treated separately.
- Our threshold-based sampling policy has an important difference from the previous threshold-based sampling policies studied in, e.g., \([8], [9], [12]–[29]\): In our model, each sample waiting in the queue for its transmission opportunity unnecessarily becomes stale. We have proven that it is suboptimal to take a new sample when the channel is busy (using the strong Markov property of the Wiener process). Consequently, the threshold should be disabled when the channel is busy and reactivated once the channel becomes idle.

- An unexpected consequence of our study is that even in the absence of the sampling rate constraint (i.e., \( f_{\text{max}} = \infty \)), the optimal sampling strategy is not zero-wait sampling in which a new sample is generated once the previous sample is delivered; rather, it is optimal to wait a positive amount of time after the previous sample is delivered, and then take the next sample.

- If the sampling times are independent of the observed Wiener process (i.e., the sampling strategy is forced to depend only on the channel but not the source), the joint sampling and estimation optimization problem reduces to an age-of-information optimization problem that has been solved recently \([3], [5]\). The asymptotics of the MSE-optimal and age-optimal sampling policies at low/high channel delay or low/high sampling frequencies are studied.

- Our theoretical and numerical comparisons show that the MSE of the optimal sampling policy can be much smaller than those of age-optimal sampling, zero-wait sampling, and classic periodic sampling. In particular, periodic sampling is far from optimal if the sampling rate is low or high; age-optimal sampling is far from optimal if the sampling rate is low; periodic sampling, age-optimal sampling, and zero-wait sampling policies are all far from optimal if the channel transmission times are highly random.

B. Paper Structure

The rest of this paper is organized as follows. In Section II, we discuss some related work. In Section III we describe the system model and the formulation of the optimal sampling and remote estimation problem. In Section IV, we present the optimal sampling and estimation solution to this problem and compare it with some other sampling policies. In Section V we describe the proof of this optimal solution. Some simulation results are provided in Section VI. Finally, in Section VII we conclude the paper.

II. RELATED WORK

A. Sampling and Source Coding of the Wiener Process

In \([30]\), Berger calculated the rate-distortion function of the Wiener process. In source coding theory, rate-distortion function represents the optimal tradeoff between the bitrate of source coding and the distortion (i.e., MSE) in the recovery of the process from its coded version. To achieve the rate-distortion function, Berger used Karhunen-Loève (KL) transform to map each realization of the Wiener process over a finite time interval to a sequence of discrete coefficients, and then applied the optimal source coding to the KL coefficients.
Based on this, a lossy source coding theorem was established in [30]. In [31], Kipnis et al. considered a combined sampling and source coding problem, and derived the minimal distortion in recovering the continuous-time Wiener process from a coded version of its periodic samples. In this setting, the authors showed that the rate-distortion function is attained in three steps: First, obtain the MMSE estimate of the Wiener process from its periodic samples, which is given by linear interpolating between neighboring samples. The obtained estimate is a continuous-time process. Second, compute the KL coefficients of the continuous-time estimate process. Third, apply the optimal source coding to the KL coefficients.

The KL transform in [30], [31] requires using the realization of the Wiener process during a sufficiently long time interval. Hence, the source coding schemes therein have long encoding delay, and cannot be used to recover the real-time value of the Wiener process as in this paper.

B. Age-of-Information Optimization

The results in this paper are closely related to the recent age-of-information studies, e.g., [3]–[7], [32]–[40]. In [32], Yates and Kaul provided a simple example about an age-of-information studies, e.g., [3]–[7], [32]–[40]. In [32], [33], Bernhardsson considered an infinite-horizon state estimation of continuous-time state processes. In [8], Rabie et al. investigated the joint optimization problem of the Wiener process, and the optimal threshold is essentially the same with a joint sampling and estimation optimization problem with an indicator-type cost function and an infinite time horizon. They used majorization theory and Riesz’s rearrangement inequality to show that, if the state process is modeled as a symmetric or Gaussian random walk, a threshold-based sampler and a nearest distance estimator are jointly optimal. This is the first study pointing out that the sampler and estimator have different information patterns. In [15], [16], Lipsa and Martins considered remote state estimation in first-order linear time-invariant (LTI) discrete time systems with a quadratic cost function and finite time horizon. They showed that a time-dependent threshold-based sampler and Kalman-like estimator are jointly optimal. In [9], Nayyar et al. considered a remote estimation problem with an energy-harvesting sensor and a remote estimator, where the sampling decision at the sensor is constrained by the energy level of the battery. They proved that an energy-level dependent threshold-based sampler and a Kalman-like estimator are jointly optimal. The proof techniques in [9], [15], [16] are similar with those of [13], [14]. In [17], the authors provided an alternative way to prove the main results of [13], [16]. In these studies, dynamic programming based iterative algorithms were used to find the optimal threshold. Recently, more efficient approaches are provided in [18] to find the optimal threshold.

In [19], Imre and Başar studied the optimal sampling and remote estimation of an i.i.d. state process over a finite time horizon. In this study, the sampler has a hard upper bound on the number of allowed samples (i.e., a hard constraint). It was shown that there exists a unique optimal sampling policy within the class of threshold-based sampling policies. In [20], the authors derived the exact MMSE estimator for a class of threshold-based sampling policies. The jointly optimal design was not proven in [19], [20].

There exist a few studies on the sampling and remote estimation of continuous-time state processes. In [8], Rabi et al. considered a continuous-time version of [19], again with a hard constraint on the number of samples. In [21], Nar and Başar studied the optimal sampling of the Wiener process subject to a time-average sampling rate constraint (i.e., a soft constraint). In both [8] and [21], it was shown that a threshold-based sampling policy is optimal among all causal sampling policies of the Wiener process, and the optimal threshold was obtained exactly. Following [13], [14] for discrete-time random walks, the authors of [8] conjectured that, when the state is a continuous-time linear diffusion process, the MMSE estimator under deterministic sampling is the optimal sampling policy.

In the studies mentioned above, it was assumed that the samples are transmitted from the sampler to the estimator over a perfect channel that is error and noise free. There exists some recent studies with explicit channel models. In [22], [23], Gao et al. considered optimal communication scheduling and remote estimation over an additive noise channel. Because of the noise, the transmitter needs to encode its message before transmission. In [22], [23], it was shown that if the transmission scheduling policy is threshold-based, then the optimal encoder and decoder are piecewise affine. In [24], it
was shown that if (i) the encoder and decoder (i.e., estimator) satisfies some technical assumption, the optimal transmission scheduling policy is threshold-based. Some extensions of this research were reported in [25–27]. In [28, 29], Chakraborty and Mahajan considered optimal communication scheduling and remote estimation over the error channel and the Gilbert-Elliott channel, where it was proved that a threshold-based transmitter and a Kalman-like estimator are jointly optimal. In [41], Mahajan and Teneketzis provided a dynamic programming based numerical method to find the optimal transmission and estimation strategies over a channel with a constant transmission delay (without queueing).

This paper is distinguished from these previous studies on remote estimation in the following aspects:

- **Problem Setup**: We consider the jointly optimal sampling and remote estimation of a continuous-time state/signal process (i.e., the Wiener process) over a channel that consists of a queue with random delay. To the best of our knowledge, this queueing model has not been considered in the literature of remote estimation.

- **Optimal Solution**: We prove that a threshold-based sampler and an MMSE estimator are jointly optimal, and the optimal threshold is found exactly. As we will see later, the threshold is disabled when there is a packet in transmission and is reactivated when all packets are delivered. We have not seen such disabling period in the threshold-based policies in the previous studies.

- **Methodology**: A novel proof procedure (involving the calculus of variations, the probability theory of Wiener processes, sufficient statistics and Lagrangian duality theory for Markov decision problems, and optimal stopping theory [42]) is developed to establish our results.

To the best of our knowledge, [21] is the closest study to this paper. Because there is no queueing and random delay in [21] (i.e., \( Y_i = 0 \)), the problem analyzed therein is a special case of ours.

### III. SYSTEM MODEL AND PROBLEM FORMULATION

#### A. Sampling Policy Notations

Let \( I_t \in \{0, 1\} \) denote the idle/busy state of the channel at time \( t \). As shown in Fig. 2 the channel state \( I_t \) is known by the sampler through acknowledgements (ACKs). We assume that once a sample is delivered to the estimator, an ACK is fed back to the sampler with zero delay. Hence, the information that is available to the sampler at time \( t \) can be expressed as \( W_t, I_t : 0 \leq s \leq t \).

Each sampling time \( S_i \) is chosen causally using the information available at the sampler. To characterize this statement precisely, we define the \( \sigma \)-fields

\[
\mathcal{N}_t = \sigma(W_s, I_s : 0 \leq s \leq t), \quad \mathcal{N}_t^+ = \cap_{s > t} \mathcal{N}_s.
\]

Then, \( \{\mathcal{N}_t^+, t \geq 0\} \) is the filtration (i.e., a non-decreasing and right-continuous family of \( \sigma \)-fields) of the information available to the sampler. Each sampling time \( S_i \) is a stopping time with respect to the filtration \( \{\mathcal{N}_t^+, t \geq 0\} \), i.e.,

\[
\{ S_i \leq t \} \in \mathcal{N}_t^+.
\]

![Fig. 3: Illustration of the threshold-based sampling policy (7), where no sample is taken during \( [S_i, S_i + Y_i) \).](attachment:image.png)

Let \( \pi = (S_0, S_1, \ldots) \) denote a sampling policy and \( \pi \) be the set of causal sampling policies satisfying the following conditions: (i) Each sampling policy \( \pi \in \Pi \) satisfies [2] for all \( i = 0, 1, \ldots \) and \( t \geq 0 \). (ii) The inter-sampling times \( \{T_i = S_{i+1} - S_i, i = 0, 1, \ldots\} \) form a regenerative process

\[
\text{Section 6.1: There exist integers } 0 \leq k_1 < k_2 < \ldots \text{ such that the post-} k_j \text{ process } \{T_{k_j+i}, i = 0, 1, \ldots\} \text{ has the same distribution as the post-} k_1 \text{ process } \{T_{k_1+i}, i = 0, 1, \ldots\} \text{ and is independent of the pre-} k_j \text{ process } \{T_{i}, i = 0, 1, \ldots, k_j-1\}; \text{ in addition, } \lim_{j \to \infty} \mathbb{E}[S_{k_j}^2] < \infty \text{ and } 0 < \mathbb{E}[(S_{k_j+1} - S_{k_j})^2] < \infty \text{ for } j = 1, 2, \ldots.\]

By Condition (ii), we can obtain that, almost surely,

\[
\lim_{i \to \infty} S_i = \infty, \quad \lim_{i \to \infty} D_i = \infty. \tag{3}
\]

Some examples of the sampling policies in \( \Pi \) are:

1. **Periodic sampling** [43, 45]: The inter-sampling times are constant, such that for some \( \beta \geq 0 \),

\[
S_{i+1} = S_i + \beta. \tag{4}
\]

2. **Zero-wait sampling** [3, 4, 34]: A new sample is generated once the previous sample is delivered, i.e.,

\[
S_{i+1} = S_i + Y_i. \tag{5}
\]

3. **Channel delay based sampling** [4, 5, 34]: The sampling times are given by

\[
S_{i+1} = \inf \{t \geq S_i + Y_i : t - S_i \geq \beta\}. \tag{6}
\]

4. **Threshold-based sampling**: The sampling times are given by

\[
S_{i+1} = \inf \{t \geq S_i + Y_i : |W_t - W_{S_i}| \geq \sqrt{\beta}\}, \tag{7}
\]

which is illustrated in Fig. 3. If \( |W_{S_i + Y_i} - W_{S_i}| \geq \sqrt{\beta} \), sample \( i+1 \) is generated at the time \( S_{i+1} = S_i + Y_i \) when sample \( i \) is delivered; otherwise, if \( |W_{S_i + Y_i} - W_{S_i}| < \sqrt{\beta} \), sample \( i+1 \) is generated at the earliest time \( t \) such that \( t \geq S_i + Y_i \) and \( |W_t - W_{S_i}| \) reaches the threshold

\[3\text{Really, we assume that } T_i \text{ is a regenerative process because we analyze the time-average MMSE in (10), but operationally a nicer definition is } \lim_{n \to \infty} \mathbb{E}[\int_0^T (W_t - \bar{X}_t)^2 dt]/\mathbb{E}[D_n]. \text{ These two definitions are equivalent when } T_i \text{ is a regenerative process.}\]
Fig. 4: Illustration of the MMSE estimation policy (9).

(a) Wiener process $W_t$ and its samples.

(b) Estimate process $\hat{W}_t$ using causally received samples.

The optimal value of (10) is then given by

$$\text{mse}_{\text{opt}} = \frac{\mathbb{E}[\max(\beta, W_Y^2)]}{6\mathbb{E}[\max(\beta, W_Y^2)]} + \mathbb{E}[Y].$$

Proof. See Section V.

The sampling policy in (7) and (12) is called the “MSE-optimal” sampling policy. The equation (12) can be solved by using the bisection method or other one-dimensional search methods with quite low complexity. Therefore, this optimal policy does not suffer from the “curse of dimensionality” issue encountered in many Markov decision problems.

Moreover, according to (12), the threshold $\sqrt{\beta}$ is determined by the maximum sampling rate $f_{\text{max}}$ and the distribution of the signal variation $W_Y$ during the random channel delay $Y$. It is worth noting that $W_Y$ is a random variable that tightly couples the source process $W_t$ and the channel delay $Y$. This is different from the traditional wisdom of information theory where source coding and channel coding can usually be treated separately.

A. Discussions

We now fix $\xi$ as the MMSE estimation policy (9) that is optimal for Problem (10) and analyze the performance of different sampling policies.

1) Signal-Independent Sampling and Age-of-Information Optimization: Let $\Pi_{\text{sig-independent}} \subseteq \Pi$ denote the set of signal-independent sampling policies, defined as

$$\Pi_{\text{sig-independent}} = \{ \pi \in \Pi : \pi \text{ is independent of } W_t, t \geq 0 \}.$$

In these policies, the sampling decisions can only depend on the channel delay $Y_t$ but not the source process $W_t$. If $\pi \in \Pi_{\text{sig-independent}}$ and $\xi$ is fixed as the MMSE estimation policy

3If $\beta \to 0$, the last terms in (12) and (13) are determined by L’Hôpital’s rule.
where $\Delta(t) = t - S_t$, $t \in [D_i, D_{i+1})$, $i = 0, 1, 2, \ldots$ (15) is the age-of-information [3], that is, the time difference between the generation time of the freshest received sample and the current time $t$. In this case, (10) reduces to the following age-of-information optimization problem [4], [5]:

$$
\text{mse}_{\text{age-opt}} = \inf_{\pi \in \Pi_{\text{age-independent}}} \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T \Delta(t) \, dt \right],
$$

where $\text{mse}_{\text{age-opt}}$ is the optimal value of (16). Because $\Pi_{\text{age-independent}} \subseteq \Pi$ and the MMSE estimation policy (9) is optimal for Problem (10),

$$
\text{mse}_{\text{opt}} \leq \text{mse}_{\text{age-opt}}.
$$

Theorem 2. [4], [5] An optimal solution to (16) is given by the sampling policy (6), where the optimal $\beta \geq 0$ is determined by solving

$$
\mathbb{E}[\max(\beta, Y)] = \max\left( \frac{1}{f_{\text{max}}}, \frac{\mathbb{E}[\max(\beta^2, Y^2)]}{2\beta} \right).
$$

The optimal value $\text{mse}_{\text{age-opt}}$ of (16) is then given by

$$
\text{mse}_{\text{age-opt}} = \frac{\mathbb{E}[\max(\beta^2, Y^2)]}{2\mathbb{E}[\max(\beta, Y)]} + \mathbb{E}[Y].
$$

The sampling policy in (6) and (18) is referred to as the “age-optimal” sampling policy. In the following, the asymptotics of the MSE-optimal and age-optimal sampling policies at low/high channel delay or low/high sampling frequencies are studied.

2) Low Channel Delay or Low Sampling Rate: Let

$$
Y_i = dX_i
$$

represent the scaling of the channel delay $Y_i$ with $d$, where $d \geq 0$ and the $X_i$'s are i.i.d. positive random variables. If $d \to 0$ or $f_{\text{max}} \to 0$, we can obtain from (12) that (see Appendix B for its proof)

$$
\beta = \frac{1}{f_{\text{max}}} + o\left( \frac{1}{f_{\text{max}}} \right),
$$

where $f(x) = o(g(x))$ as $x \to a$ means that $\lim_{x \to a} f(x)/g(x) = 0$. Hence, the MSE-optimal sampling policy in (7) and (12) becomes

$$
S_{i+1} = \inf \left\{ t \geq S_i : |W_t - W_{S_i}| \geq \sqrt{\frac{1}{f_{\text{max}}}} \right\},
$$

and as shown in Appendix B the optimal value of (10) becomes

$$
\text{mse}_{\text{opt}} = \frac{1}{\beta_{\text{max}}} + o\left( \frac{1}{\beta_{\text{max}}} \right).
$$

The sampling policy (22) was also obtained in [21] for the case that $Y_i = 0$ for all $i$.

If $d \to 0$ or $f_{\text{max}} \to 0$, one can show that the age-optimal sampling policy in (6) and (18) becomes periodic sampling with $\beta = 1/f_{\text{max}} + o(1/f_{\text{max}})$, and the optimal value of (16) is $\text{mse}_{\text{age-opt}} = 1/(2f_{\text{max}}) + o(1/f_{\text{max}})$. Therefore,

$$
\lim d \to 0 \frac{\text{mse}_{\text{opt}}}{\text{mse}_{\text{age-opt}}} = \lim f_{\text{max}} \to 0 \frac{\text{mse}_{\text{opt}}}{\text{mse}_{\text{age-opt}}} = \frac{1}{3}.
$$

3) High Channel Delay or Unbounded Sampling Rate: If $d \to \infty$ or $f_{\text{max}} \to \infty$, as shown in Appendix C the MSE-optimal sampling policy for solving (10) is given by (7) where $\beta$ is determined by solving

$$
2\beta \mathbb{E}[\max(\beta, Y^2)] = \mathbb{E}[\max(\beta^2, W_Y^2)].
$$

Similarly, if $d \to \infty$ or $f_{\text{max}} \to \infty$, the age-optimal sampling policy for solving (16) is given by (6) where $\beta$ is determined by solving

$$
2\beta \mathbb{E}[\max(\beta, Y)] = \mathbb{E}[\max(\beta^2, Y^2)].
$$

In these limits, the ratio between $\text{mse}_{\text{opt}}$ and $\text{mse}_{\text{age-opt}}$ depends on the distribution of $Y$.

When the sampling rate is unbounded, i.e., $f_{\text{max}} = \infty$, one logically reasonable policy is the zero-wait sampling policy in [5] [3] [4]. This zero-wait sampling policy achieves the maximum throughput and the minimum queuing delay of the channel. Surprisingly, this zero-wait sampling policy does not always minimize the age-of-information in (16) and almost never minimizes the MSE in (10), as stated below:

Theorem 3. When $f_{\text{max}} = \infty$, the zero-wait sampling policy [5] and the MMSE estimation policy (9) are optimal for solving (10) if and only if $Y = 0$ with probability one.

Proof. See Appendix D.

Theorem 4. [5] When $f_{\text{max}} = \infty$, the zero-wait sampling policy (5) is optimal for solving (16) if and only if

$$
\mathbb{E}[Y^2] \leq 2 \text{ess sup} \mathbb{Y} \mathbb{E}[Y],
$$

where $\text{ess sup} \mathbb{Y} = \sup \{ y \in [0, \infty) : \text{Pr}[Y < y] = 0 \}$.

Proof. See Appendix D.

V. PROOF OF THE MAIN RESULT

We prove Theorem 1 in five steps: First, we show that given any sampling policy, the MMSE estimation policy (9) is optimal for minimizing the objective function in (10). The remaining task is to solve the optimal sampling problem given an MMSE estimator. Second, we show that no sample should be generated when the channel is busy, which simplifies the
optimal sampling problem. Third, we study the Lagrangian dual problem of the simplified problem, and decompose the Lagrangian dual problem into a series of mutually independent per-sample control problems. Each of these per-sample control problems is a continuous-time Markov decision problem. Fourth, we utilize optimal stopping theory \cite{42} to solve the per-sample control problems. Finally, we show that the Lagrangian duality gap of our Markov decision problem is zero. By this, Problem \cite{10} is solved. The details are as follows.

A. Optimal Estimation Policy

**Lemma 1.** For any given sampling policy $\pi \in \Pi$, the MMSE estimation policy $\bar{\pi}$ minimizes the objective function in \cite{10} among the estimation policies in $\Xi$.

Lemma 1 was proven in \cite{17} for first-order discrete-time LTI processes. For continuous-time processes, Lemma 1 was used in \cite{21} Eq. (5)] without a proof, and was stated in \cite{8, Eq. (2.3)] as a conjecture. Here, we provide a rigorous proof of this result by using the calculus of variations. In addition, there is no queueing and random delay in \cite{8, 17, 21}.

**Proof.** See Appendix E

After finding the optimal estimation policy $\bar{\pi}$, the remaining task is to solve the following optimal sampling problem:

$$\inf_{\pi \in \Pi} \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T (W_t - \hat{W}_t)^2 dt \right]$$ \hspace{1cm} (28)

$$\text{s.t.} \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[S_n] \geq \frac{1}{f_{\text{max}}},$$ \hspace{1cm} (29)

$\hat{W}_i$ is given by \cite{9}.

B. Simplification of Problem (28)

The following lemma is useful for simplifying (28).

**Lemma 2.** In the optimal sampling problem (28), it is suboptimal to take a new sample before the previous sample is delivered.

**Proof.** See Appendix E

In recent studies on age-of-information \cite{4, 5}, Lemma 2 was intuitive and hence was used without a proof: If a sample is taken when the channel is busy, it needs to wait in the queue until its transmission starts, and becomes stale while waiting. A better method is to wait until the channel becomes idle, and then generate a new sample, as stated in Lemma 2. However, this lemma is not intuitive in the MSE minimization problem (28): The proof of Lemma 2 relies on the strong Markov property of the Wiener process, which may not hold for other signal processes.

By Lemma 2 we only need to consider a sub-class of sampling policies $\Pi_1 \subset \Pi$ such that each sample is generated and submitted to the channel after the previous sample is delivered, i.e.,

$$\Pi_1 = \{ \pi \in \Pi : S_i = G_i \geq D_{i-1} \text{ for all } i \}. \hspace{1cm} (30)$$

This completely eliminates the waiting time wasted in the queue, and hence the queue is always kept empty. The information that is available for determining $S_i$ includes the history of signal values $(W_t : t \in [0, S_i])$ and the channel delay $(Y_1, \ldots, Y_{i-1})$ of the previous samples.\footnote{Note that the generation times $(S_1, \ldots, S_{i-1})$ of previous samples are also included in this information.}

To characterize this statement precisely, let us define the $\sigma$-fields $F_t = \sigma(W_s : s \in [0, t])$ and $F_t^+ = \cap_{s>t} F_s$. Then, $\{F_t^+, t \geq 0\}$ is the filtration (i.e., a non-decreasing and right-continuous family of $\sigma$-fields) of the Wiener process $W_t$. Given the transmission durations $(Y_1, \ldots, Y_{i-1})$ of previous samples, $S_i$ is a stopping time with respect to the filtration $\{F_t^+, t \geq 0\}$ of the Wiener process $W_t$, that is

$$\{(S_i \leq t) | Y_1, \ldots, Y_{i-1} \in F_t^+ \}.$$ \hspace{1cm} (31)

Then, the policy space $\Pi_1$ can be alternatively expressed as

$$\Pi_1 = \{ \pi : (S_i \leq t) | Y_1, \ldots, Y_{i-1} \in F_t^+, \text{ } S_i = G_i \geq D_{i-1} \text{ for all } i, \text{ } T_i = S_{i+1} - S_i \text{ is a regenerative process} \}.$$ \hspace{1cm} (32)

Let $Z_i = S_{i+1} - D_i \geq 0$ represent the waiting time between the delivery time $D_i$ of sample $i$ and the generation time $S_{i+1}$ of sample $i + 1$. Then, $S_i = Z_0 + \sum_{j=1}^{i-1} (Y_j + Z_j)$ and $D_i = \sum_{j=0}^{i-1} (Z_j + Y_{j+1})$. If $(Y_1, Y_2, \ldots)$ is given, $(S_0, S_1, \ldots)$ is uniquely determined by $(Z_0, Z_1, \ldots)$. Hence, one can also use $\pi = (Z_0, Z_1, \ldots)$ to represent a sampling policy.

Because $T_i$ is a regenerative process, by following the renewal theory in \cite{47} and \cite{45} Section 6.1, one can show that in Problem (28), $f \mathbb{E}[S_n]$ is a convergent sequence and

$$\limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T (W_t - \hat{W}_t)^2 dt \right] = \lim_{n \to \infty} \frac{\mathbb{E} \left[ \int_0^{D_{n-1}} (W_t - \hat{W}_t)^2 dt \right]}{\mathbb{E}[D_n]} = \lim_{n \to \infty} \frac{\sum_{i=0}^{n-1} \mathbb{E} \left[ \int_{D_{i+1}}^{D_{i+1}+1} (W_t - \hat{W}_t)^2 dt \right]}{\sum_{i=0}^{n-1} \mathbb{E}[Y_i + Z_i]},$$

where in the last step we have used $\mathbb{E}[D_n] = \mathbb{E} \left[ \sum_{i=0}^{n-1} (Z_i + Y_{i+1}) \right] = \mathbb{E} \left[ \sum_{i=0}^{n-1} (Y_i + Z_i) \right]$. Hence, (28) can be rewritten as the following Markov decision problem:

$$\text{mse}_{\text{opt}} \triangleq \inf_{\pi \in \Pi_1} \lim_{n \to \infty} \frac{\sum_{i=0}^{n-1} \mathbb{E} \left[ \int_{D_{i+1}}^{D_{i+1}+1} (W_t - \hat{W}_t)^2 dt \right]}{\sum_{i=0}^{n-1} \mathbb{E}[Y_i + Z_i]}$$ \hspace{1cm} (33)

$$\text{s.t.} \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}[Y_i + Z_i] \geq \frac{1}{f_{\text{max}}},$$ \hspace{1cm} (34)

where $\text{mse}_{\text{opt}}$ is the optimal value of (33).

In order to solve (33), let us consider the following Markov
decision problem with a parameter $c \geq 0$:

$$
p(c) \equiv \inf_{\pi \in \Pi} \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}\left[ \int_{D_i} (W_t - W_{S_i})^2 dt - c(Y_i + Z_i) \right]
$$

(35)

\[ \text{s.t. } \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}[Y_i + Z_i] \geq \frac{1}{f_{\max}}, \]

where $p(c)$ is the optimum value of (35).

**Lemma 3.** The following assertions are true:

(a) $\text{mse}_{\text{opt}} \geq c$ if and only if $p(c) \geq 0$.

(b) If $p(c) = 0$, the solutions to (33) and (35) are identical.

**Proof.** See Appendix [6] \[ \square \]

Hence, the solution to (33) can be obtained by solving (35) and seeking a $c_{\text{opt}} \geq 0$ such that

$$
p(c_{\text{opt}}) = 0.
$$

(36)

**C. Lagrangian Dual Problem of (35) when $c = c_{\text{opt}}$**

Although (35) is a continuous-time Markov decision problem with a continuous state space (not a convex optimization problem), it is possible to use the Lagrangian dual approach to solve (35) and show that it admits no duality gap.

When $c = c_{\text{opt}}$, define the following Lagrangian

$$
L(\pi; \lambda) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}\left[ \int_{D_i} (W_t - W_{S_i})^2 dt - (c_{\text{opt}} + \lambda)(Y_i + Z_i) \right]
+ \lambda \frac{1}{f_{\max}}.
$$

(37)

Let

$$
g(\lambda) \equiv \inf_{\pi \in \Pi} L(\pi; \lambda).
$$

(38)

Then, the Lagrangian dual problem of (35) is defined by

$$
d \equiv \max_{\lambda \geq 0} g(\lambda),
$$

(39)

where $d$ is the optimum value of (39). Weak duality [48], [49] implies that $d \leq p(c_{\text{opt}})$. In Section V-B, we will establish strong duality, i.e., $d = p(c_{\text{opt}}).

In the sequel, we solve (38). Using the stopping times and martingale theory of the Wiener process, we can obtain the following lemma:

**Lemma 4.** Let $\tau \geq 0$ be a stopping time of the Wiener process $W_t$ with $\mathbb{E}[\tau^2] < \infty$, then

$$
\mathbb{E}\left[ \int_{0}^{\tau} W_t^2 dt \right] = \frac{1}{6} \mathbb{E} \left[ W_\tau^4 \right].
$$

(40)

**Proof.** See Appendix [I] \[ \square \]

By using Lemma 4 and the sufficient statistics of (38), we can show that for every $i = 1, 2, \ldots$,

$$
\mathbb{E}\left[ \int_{D_i} (W_t - W_{S_i})^2 dt \right] \leq \frac{1}{6} \mathbb{E} \left[ (W_{S_i} + Y_i + Z_i - W_{S_i})^4 \right] + \mathbb{E}[Y_i + Z_i] \mathbb{E}[Y_i],
$$

(41)

which is proven in Appendix [I].

Define the $\sigma$-fields $\mathcal{F}_t = \sigma(W_{s\uparrow} - W_s : v \in [0, t])$ and $\mathcal{F}_t^+ = \cap_{v \geq t} \mathcal{F}_v$, as well as the filtration $\{\mathcal{F}_t^+, t \geq 0\}$ of the time-shifted Wiener process $\{W_{s+t} - W_s, t \in [0, \infty)\}$. Define $\mathfrak{M}_s$ as the set of square-integrable stopping times of $\{W_{s+t} - W_s, t \in [0, \infty)\}$, i.e.,

$$
\mathfrak{M}_s = \{\tau \geq 0 : \{\tau \leq t\} \in \mathcal{F}_t^+, \mathbb{E}[\tau^2] < \infty\}.
$$

By substituting (41) into (38) and using again the sufficient statistics of (35), we can obtain

**Theorem 5.** An optimal solution $(Z_0, Z_1, \ldots) \to (38)$ satisfies

$$
Z_t = \arg \inf_{\tau \in \mathfrak{M}_s + Y_t} \mathbb{E}\left[ \frac{1}{2} (W_{S_t + Y_t + \tau} - W_{S_t})^4 \right]
- \beta(Y_t + \tau) | W_{S_t + Y_t} - W_{S_t}, Y_t,
$$

(42)

where $\beta$ is given by

$$
\beta = 3(c_{\text{opt}} + \lambda - \mathbb{E}[Y]) \geq 0.
$$

(43)

**Proof.** See Appendix [I] \[ \square \]

Note that because the $Y_i$’s are i.i.d. and the strong Markov property of the Wiener process, the $Z_i$’s in (42) are i.i.d.

**D. Per-Sample Optimal Stopping Solution to (42)**

We use optimal stopping theory [52] to solve (42). Let us first pose (42) in the language of optimal stopping. A continuous-time two-dimensional Markov chain $X_t$ on a probability space $(\mathbb{R}^2, \mathcal{F}_t, \mathbb{P})$ is defined as follows: Given the initial state $X_0 = (s, b)$, the state $X_t$ at time $t$ is

$$
X_t = (s + t, b + W_t),
$$

(44)

where $\{W_t, t \geq 0\}$ is a standard Wiener process. Define $\mathbb{P}_x(A) = \mathbb{P}(A | X_0 = x)$ and $\mathbb{E}_x Z = \mathbb{E}(Z | X_0 = x)$, respectively, as the conditional probability of event $A$ and the conditional expectation of random variable $Z$ for given initial state $X_0 = x$. Define the $\sigma$-fields $\mathcal{F}_t^X = \sigma(X_v : v \in [0, t])$ and $\mathcal{F}_t^{X+} = \cap_{v \geq t} \mathcal{F}_v^X$, as well as the filtration $\{\mathcal{F}_t^{X+}, t \geq 0\}$ of the Markov chain $X_t$. A random variable $\tau: \mathbb{R}^2 \to [0, \infty)$ is said to be a stopping time of $X_t$ if $\{\tau \leq t\} \in \mathcal{F}_t^{X+}$ for all $t \geq 0$. Let $\mathfrak{N}$ be the set of square-integrable stopping times of $X_t$, i.e.,

$$
\mathfrak{N} = \{\tau \geq 0 : \{\tau \leq t\} \in \mathcal{F}_t^{X+}, \mathbb{E}[\tau^2] < \infty\}.
$$

Our goal is to solve the following optimal stopping problem:

$$
\sup_{\tau \in \mathfrak{N}} \mathbb{E}_x g(X_\tau),
$$

(45)
where the function \( g : \mathbb{R}^2 \to \mathbb{R} \) is defined as
\[
g(s, b) = \beta s - \frac{1}{2} b^4
\]  
(46)
with parameter \( \beta \geq 0 \), and \( x \) is the initial state of the Markov chain \( X(t) \). Notice that (45) becomes (42) if the initial state is \( x = (Y_i, W_{S_i + Y_i} - W_{S_i}) \), and \( W_t \) is replaced by \( W_{S_i + Y_i + t} - W_{S_i} \).

**Theorem 6.** For all initial state \((s, b) \in \mathbb{R}^2 \) and \( \beta \geq 0 \), an optimal stopping time for solving (45) is
\[
\tau^* = \inf \left\{ t \geq 0 : |b + W_t| \geq \sqrt{\beta} \right\}.
\]  
(47)
In order to prove Theorem 6 let us define the function \( u(x) = \mathbb{E}_x g(X_{\tau^*}) \) and establish some properties of \( u(x) \).

**Lemma 5.** \( u(x) \geq g(x) \) for all \( x \in \mathbb{R}^2 \), and
\[
u(s, b) = \begin{cases} 
\beta s - \frac{1}{2} b^4, & \text{if } b^2 \geq \beta; \\
\beta s + \frac{1}{2} \beta^2 - \beta b^2, & \text{if } b^2 < \beta.
\end{cases}
\]  
(48)

**Proof.** See Appendix K.

A function \( f(x) \) is said to be excessive for the process \( X_t \) if
\[
\mathbb{E}_x f(X_t) \leq f(x), \text{ for all } t \geq 0, \ x \in \mathbb{R}^2.
\]  
(49)
By using the Itô-Tanaka-Meyer formula in stochastic calculus, we can obtain

**Lemma 6.** The function \( u(x) \) is excessive for the process \( X_t \).

**Proof.** See Appendix L.

Now, we are ready to prove Theorem 6.

**Proof of Theorem 6** In Lemma 5 and Lemma 6, we have shown that \( u(x) = \mathbb{E}_x g(X_{\tau^*}) \) is an excessive function and \( u(x) \geq g(x) \). In addition, it is known that \( \mathbb{E}_x(\tau^* < \infty) = 1 \) for all \( x \in \mathbb{R}^2 \) [Theorem 8.5.3]. These conditions, together with the Corollary to Theorem 1 in [42] Section 3.3.1], imply that \( \tau^* \) is an optimal stopping time of (45). This completes the proof.

An immediate consequence of Theorem 6 is

**Corollary 1.** An optimal solution to (42) is
\[
Z_i = \inf \left\{ t \geq 0 : |W_{S_i + Y_i + t} - W_{S_i}| \geq \sqrt{\beta} \right\}.
\]  
(50)

E. Zero Duality Gap between (35) and (39)

Strong duality is established in the following theorem:

**Theorem 7.** The following assertions are true:

(a) If \( c = c_{opt} \), the duality gap between (35) and (39) is zero, i.e., \( d = p(c_{opt}) \).

(b) A common optimal solution to (28), (33), and (35) with \( c = c_{opt} \) is given by (7) and (12). The optimal value of (28) is given by (13).

**Proof Sketch of Theorem 7** If \( c = c_{opt} \), we first use Theorem 5 and Corollary 1 to find a geometric multiplier [48] for Problem (35). Hence, the duality gap between (35) and (39) is zero, because otherwise no geometric multiplier exists [48] Section 6.1-6.2]. By this, part (a) is proven. Next, we use Lemma 5 and the properties of geometric multiplier [48] Prop. 6.2.5] to prove part (b). See Appendix M for the details.

Hence, Theorem 1 follows from Lemma 1 and Theorem 7.

VI. NUMERICAL RESULTS

In this section, we evaluate the estimation performance achieved by the following four sampling policies:

1. **Periodic sampling:** The policy in (4) with \( \beta = f_{max} \).
2. **Zero-wait sampling** [3, 4, 14]: The sampling policy in (5), which is feasible when \( f_{max} \geq \mathbb{E}[X_i] \).
3. **Age-optimal sampling** [4]: The sampling policy in (6) and (13), which is the optimal solution to (16).
4. **MSE-optimal sampling:** The sampling policy in (7) and (12), which is the optimal solution to (10).

Let \( \text{mse}_{\text{periodic}}, \text{mse}_{\text{zero-wait}}, \text{mse}_{\text{age-opt}}, \text{and } \text{mse}_{\text{opt}}, \) be the MSES of periodic sampling, zero-wait sampling, age-optimal sampling, MSE-optimal sampling, respectively. According to (17), as well as the facts that periodic sampling is feasible for (16) and zero-wait sampling is feasible for (16) when \( f_{max} \geq \mathbb{E}[X_i] \), we can obtain
\[
\text{mse}_{\text{opt}} \leq \text{mse}_{\text{age-opt}} \leq \text{mse}_{\text{periodic}} ;
\]
\[
\text{mse}_{\text{opt}} \leq \text{mse}_{\text{age-opt}} \leq \text{mse}_{\text{zero-wait}}, \text{ when } f_{max} \geq \mathbb{E}[X_i],
\]
which fit with our numerical results below.

Figure 5 depicts the tradeoff between MSE and $f_{\text{max}}$ for i.i.d. exponential channel delay with mean $E[Y_i] = 1/\mu = 1$. Hence, the maximum throughput of the channel is $\mu = 1$. In this setting, $\text{mse}_{\text{periodic}}$ is characterized by eq. (25) of [3], which was obtained using a D/M/1 queueing model. For small values of $f_{\text{max}}$, age-optimal sampling is similar with periodic sampling, and hence $\text{mse}_{\text{age-opt}}$ and $\text{mse}_{\text{periodic}}$ are of similar values. However, as $f_{\text{max}}$ approaches the maximum throughput 1, $\text{mse}_{\text{periodic}}$ increases to infinite. This is because the queue length in periodic sampling is large at high sampling frequencies, and the samples become stale during their long waiting times in the queue. On the other hand, $\text{mse}_{\text{opt}}$ and $\text{mse}_{\text{age-opt}}$ decrease with respect to $f_{\text{max}}$. The reason is that the set of feasible policies satisfying the constraints in (10) and (16) becomes larger as $f_{\text{max}}$ grows, and hence the optimal values of (10) and (16) are decreasing in $f_{\text{max}}$. Moreover, the gap between $\text{mse}_{\text{opt}}$ and $\text{mse}_{\text{age-opt}}$ is large for small values of $f_{\text{max}}$. The ratio $\text{mse}_{\text{opt}}/\text{mse}_{\text{age-opt}}$ tends to 1/3 as $f_{\text{max}} \rightarrow 0$, which is in accordance with (23). As we expected, $\text{mse}_{\text{zero-wait}}$ is larger than $\text{mse}_{\text{opt}}$ and $\text{mse}_{\text{age-opt}}$ when $f_{\text{max}} \geq 1$.

Figure 6 and Figure 7 illustrate the MSE of i.i.d. log-normal channel delay for $f_{\text{max}} = 0.8$ and $f_{\text{max}} = 1.5$, respectively, where $Y_i = e^{X_i}/E[e^{X_i}]$, $\sigma > 0$ is the scale parameter of log-normal distribution, and $(X_1, X_2, \ldots)$ are i.i.d. Gaussian random variables with zero mean and unit variance. Because $E[Y_i] = 1$, the maximum throughput of the channel is 1. In Fig. 6 since $f_{\text{max}} < 1$, zero-wait sampling is not feasible and hence is not plotted. As the scale parameter $\sigma$ grows, the tail of the log-normal distribution becomes heavier and heavier. We observe that $\text{mse}_{\text{periodic}}$ grows quickly with respect to $\sigma$, much faster than $\text{mse}_{\text{opt}}$ and $\text{mse}_{\text{age-opt}}$. In addition, the gap between $\text{mse}_{\text{opt}}$ and $\text{mse}_{\text{age-opt}}$ increases as $\sigma$ grows. In Fig. 7 because $f_{\text{max}} > 1$, $\text{mse}_{\text{periodic}}$ is infinite and hence is not plotted. We can find that $\text{mse}_{\text{zero-wait}}$ grows quickly with respect to $\sigma$ and is much larger than $\text{mse}_{\text{opt}}$ and $\text{mse}_{\text{age-opt}}$.

In summary, periodic sampling is far from optimal if the sampling rate is low or high; age-optimal sampling is far from optimal if the sampling rate is low; periodic sampling, age-optimal sampling, and zero-wait sampling policies are all far from optimal if the channel transmission times are highly random.

VII. CONCLUSIONS

In this paper, we have investigated optimal sampling and remote estimation of the Wiener process over a channel with queuing and random delay. The jointly optimal sampling and estimation policies for minimizing the mean square estimation error subject to a sampling rate constraint has been obtained. We prove that a threshold-based sampler and an MMSE estimator are jointly optimal, and the optimal threshold is found exactly. Analytical and numerical comparisons with several important sampling policies, including age-optimal sampling, zero-wait sampling, and classic periodic sampling, have been provided. The results in this paper generalize recent research on age-of-information by adding a signal-based control model, and generalize existing studies on remote estimation by adding a queueing model.

ACKNOWLEDGEMENT

The first author is grateful to Ness B. Shroff and Roy D. Yates for their careful reading of a draft of this paper and their valuable suggestions.

APPENDIX A

PROOF OF (14)

If $\pi$ is independent of $\{W_t, t \in [0, \infty)\}$, the $S_i$’s and $D_i$’s are independent of $\{W_t, t \in [0, \infty)\}$. Define $x \wedge y = \inf\{x, y\}$. Consider any $T > 0$, we need to consider two cases:

Case 1: If $D_i \wedge T \geq S_i$, we can obtain

$$
\mathbb{E} \left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - \hat{W}_t)^2 dt \right\} 
\leq \mathbb{E} \left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - W_{S_i})^2 dt \right\} 
\leq \mathbb{E} \left\{ (W_t - W_{S_i})^2 | S_i, D_i, D_{i+1} \right\} dt
$$

Case 2: If $D_i \wedge T < S_i$, then the fact $D_i \geq S_i$ implies that $T < S_i \leq D_i \leq D_{i+1}$. Hence, $D_i \wedge T = D_{i+1} \wedge T$ and

$$
\mathbb{E} \left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - \hat{W}_t)^2 dt \right\} = \mathbb{E} \left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} \Delta(t) dt \right\} = 0.
$$
Therefore, (51) holds in both cases. In addition,
\[
\lim_{n \to \infty} \sum_{i=0}^{n} \mathbb{E} \left\{ \int_{D_i \cup T} (W_t - \hat{W}_t)^2 dt \right\} = \lim_{n \to \infty} \mathbb{E} \left\{ \int_{0}^{D_n \cup T} (W_t - \hat{W}_t)^2 dt \right\} \tag{52}
\]
\[
(a) \mathbb{E} \left\{ \lim_{n \to \infty} \int_{0}^{D_n \cup T} (W_t - \hat{W}_t)^2 dt \right\} = \mathbb{E} \left\{ \int_{0}^{T} (W_t - \hat{W}_t)^2 dt \right\},
\]
\[
(b) \mathbb{E} \left\{ \lim_{n \to \infty} \int_{0}^{D_n \cup T} \Delta(t) dt \right\} = \mathbb{E} \left\{ \int_{0}^{T} \Delta(t) dt \right\}. \tag{53}
\]

By combining (51), (53), (14) is proven.

\section*{Appendix B}

\textbf{Proofs of (21) and (23)}

If \(f_{\text{max}} \to 0\), (12) tells us that
\[
\mathbb{E}[\max(\beta, W_Y^2)] = \frac{1}{f_{\text{max}}},
\]
which implies
\[
\beta \leq \frac{1}{f_{\text{max}}} \leq \beta + \mathbb{E}[W_Y^2] = \beta + \mathbb{E}[Y].
\]
Hence,
\[
\frac{1}{f_{\text{max}}} - \mathbb{E}[Y] \leq \beta \leq \frac{1}{f_{\text{max}}}. \tag{54}
\]

If \(f_{\text{max}} \to 0\), (21) follows. Because \(Y\) is independent of the Wiener process, using the law of iterated expectations and the Gaussian distribution of the Wiener process, we can obtain \(\mathbb{E}[W_Y^2] = 3\mathbb{E}[Y^2]\) and \(\mathbb{E}[W_Y^4] = 3\mathbb{E}[Y^4]\). Hence,
\[
\beta \leq \mathbb{E}[\max(\beta, W_Y^2)] \leq \beta + \mathbb{E}[W_Y^2] = \beta + \mathbb{E}[Y],
\]
\[
\beta^2 \leq \mathbb{E}[\max(\beta^2, W_Y^4)] \leq \beta^2 + \mathbb{E}[W_Y^4] = \beta^2 + 3\mathbb{E}[Y^2].
\]
Therefore,
\[
\frac{\beta^2}{\beta + \mathbb{E}[Y]} \leq \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{\mathbb{E}[\max(\beta, W_Y^2)]} \leq \frac{\beta^2 + 3\mathbb{E}[Y^2]}{\beta}. \tag{54}
\]

By combining (13), (21), and (54), (23) follows in the case of \(f_{\text{max}} \to 0\).

If \(d \to 0\), then \(Y \to 0\) and \(W_Y \to 0\) with probability one. Hence, \(\mathbb{E}[\max(\beta, W_Y^2)] \to \beta\) and \(\mathbb{E}[\max(\beta^2, W_Y^4)] \to \beta^2\). Substituting these into (12) and (54), yields
\[
\lim_{d \to 0} \beta = \frac{1}{f_{\text{max}}} \lim_{d \to 0} \left\{ \frac{\mathbb{E}[\max(\beta, W_Y^2)]}{6\mathbb{E}[\max(\beta, W_Y^2)]} + \mathbb{E}[Y] \right\} = \frac{1}{6f_{\text{max}}}.
\]

By this, (21) and (23) are proven in the case of \(d \to 0\). This completes the proof.

\section*{Appendix C}

\textbf{Proof of (25)}

If \(f_{\text{max}} \to \infty\), the sampling rate constraint in (10) can be removed. By (12), the optimal \(\beta\) is determined by (25).

If \(d \to \infty\), let us consider the equation
\[
\mathbb{E}[\max(\beta, W_Y^2)] = \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{2\beta}. \tag{55}
\]

If \(Y\) grows by \(a\) times, then \(\beta\) and \(\mathbb{E}[\max(\beta, W_Y^2)]\) in (55) should grow by \(a\) times, and \(\mathbb{E}[\max(\beta^2, W_Y^4)]\) in (55) should grow by \(a^2\) times. Hence, if \(d \to \infty\), it holds in (12) that
\[
\frac{1}{f_{\text{max}}} \leq \frac{\mathbb{E}[\max(\beta^2, W_Y^4)]}{2\beta}
\]
and the solution to (12) is given by (25). This completes the proof.

\section*{Appendix D}

\textbf{Proofs of Theorems 3 and 4}

\textbf{Proof of Theorem 3} The zero-wait policy can be expressed as (7) with \(\beta = 0\). Because \(Y\) is independent of the Wiener process, using the law of iterated expectations and the Gaussian distribution of the Wiener process, we can obtain \(\mathbb{E}[W_Y^2] = 3\mathbb{E}[Y^2]\). According to (25), \(\beta = 0\) if and only if \(\mathbb{E}[W_Y^4] = 3\mathbb{E}[Y^4] = 0\) which is equivalent to \(Y = 0\) with probability one. This completes the proof.

\textbf{Proof of Theorem 4} In the one direction, the zero-wait policy can be expressed as (6) with \(\beta \leq \text{ess inf } Y\). If the zero-wait policy is optimal, then the solution to (26) must satisfy \(\beta \leq \text{ess inf } Y\), which further implies \(\beta \leq Y\) with probability one. From this, we can get
\[
2\text{ess inf } Y \mathbb{E}[Y] \geq 2\beta \mathbb{E}[Y] = \mathbb{E}[Y^2],
\]
By this, (27) follows.

In the other direction, if (27) holds, we will show that the zero-wait policy is age-optimal by considering the following two cases.

\text{Case 1: } \mathbb{E}[Y] > 0. By choosing
\[
\beta = \frac{\mathbb{E}[Y^2]}{2\mathbb{E}[Y]}, \tag{56}
\]
we can get \(\beta \leq \text{ess inf } Y\) and hence
\[
\beta \leq Y. \tag{57}
\]
with probability one. According to (56) and (57), such a \(\beta\) is the solution to (26). Hence, the zero-wait policy expressed by (6) with \(\beta \leq \text{ess inf } Y\) is the age-optimal policy.

\text{Case 2: } \mathbb{E}[Y] = 0\text{ and hence } Y = 0\text{ with probability one. In this case, } \beta = 0\text{ is the solution to (26). Hence, the zero-wait policy expressed by (6) with } \beta = 0\text{ is the age-optimal policy.}

Combining these two cases, the proof is completed.
APPENDIX E
PROOF OF LEMMA [1]

We use the calculus of variations to prove Lemma [1].

Motivated by (52), we consider a functional $h$ defined as

$$h(W_t) = E \left\{ \int_{D_{i+1} \wedge T} (\hat{W}_t - W_t)^2 dt \bigg| S_j, W_{S_j}, D_j, j \leq i \right\}.$$

Let $f_t$ and $g_t$ be two functions of the information available at the estimator \{s, W_{S}, D : D \leq t}\}. Similar to the one-sided sub-gradient in finite dimensional space, the one-sided Gâteaux derivative of the functional $h$ in the direction of $g$ at a point $f$ is given by

$$\delta h(f; g) = \lim_{\epsilon \to 0} \frac{h(f + \epsilon g) - h(f)}{\epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} E \left\{ \int_{D_{i+1} \wedge T} (f_t + \epsilon g_t - W_t)^2 - (f_t - W_t)^2 dt \bigg| S_j, W_{S_j}, D_j, j \leq i \right\}$$

$$= \lim_{\epsilon \to 0} E \left\{ \int_{D_i \wedge T} 2(f_t - W_t)g_t dt + \epsilon g_t^2 \right\} \bigg\{ S_j, W_{S_j}, D_j, j \leq i \bigg\} + \epsilon \int_{D_i \wedge T} (f_t - W_t)g_t dt$$

where the last step follows from the iterated law of expectations. According to [52, p. 710], $f_t$ is an optimal solution to

$$f_t = \arg \min_{W_t} h(W_t)$$

if and only if

$$\delta h(f; g) \geq 0, \ \forall g.$$

By $\delta h(f; g) = -\delta h(f; -g)$, we get

$$\delta h(f; g) = 0, \ \forall g.$$

Since $g_t$ is arbitrary, by (58), the optimal solution to (59) is

$$f_t = E[W_t | S_j, W_{S_j}, D_j, j \leq i]$$

for all $t \in [D_i \wedge T, D_{i+1} \wedge T].$

Note that under any sampling policy $\pi$, \{S, W_{S}, D, j \leq i\} are determined by (W, t \in [0, S_j]) and (Y_1, \ldots, Y_i). Because of the strong Markov property of the Wiener process [51, Theorem 2.16] and the Y_i's are independent of the Wiener process W, \{W_{S+t} - W_{S_t}, t \geq 0\} is a Wiener process independent of (W_t, t \in [0, S_j]) and (Y_1, \ldots, Y_i). Hence, \{W_{S+t} - W_{S_t}, t \geq 0\} is independent of \{S, W_{S}, D, j \leq i\}. By this, for each $t \in [D_i \wedge T, D_{i+1} \wedge T]$

$$E[W_t | S_j, W_{S_j}, D_j, j \leq i]$$

$$= E[W_t - W_{S_j} | S_j, W_{S_j}, D_j, j \leq i] + W_{S_j}$$

Recall that the channel is busy whenever there exist some samples that have been generated but not delivered to the estimator. Hence, during the time interval [S_i, D_{i+1}], the channel is busy sending the first (i + 1) samples. Because $E[Y_i^2] < \infty$, we can get $E[Y_i^2] < \infty$ and

$$E[D_{i+1} - S_i] \leq E \left[ \sum_{j=1}^{i+1} Y_j \right] < \infty.$$

By Wald's identity [51, Theorem 2.44], we have $E[W_t - W_{S_i}] = 0$ and hence

$$E[W_t | S_j, W_{S_j}, D_j, j \leq i] = W_{S_i}$$

for each $t \in [D_i \wedge T, D_{i+1} \wedge T]$. Let $T \to \infty$, we obtain that the optimal estimator is given by (9). By this, Lemma [1] is proven.

APPENDIX F
PROOF OF LEMMA [2]

Suppose that in the sampling policy $\pi$, sample $i$ is generated when the channel is busy sending another sample, and hence sample $i$ needs to wait for some time before submitted to the channel, i.e., $S_i < G_i$. Let us consider a virtual sampling policy $\pi' = \{S_0, \ldots, S_{i-1}, S'_i = G_i, S_{i+1}, \ldots\}$. We call policy $\pi'$ a virtual policy because the generation time of sample $i$ in policy $\pi'$ is $S'_i = G_i \geq S_i$ and hence it may happen that $S'_i > S_{i+1}$. However, this will not affect our proof below. We will show that the MSE of the sampling policy $\pi'$ is smaller than that of the sampling policy $\pi = \{S_0, \ldots, S_{i-1}, S_i, S_{i+1}, \ldots\}$.

Note that the Wiener process \{W_t : t \in [0, \infty)\} does not change according to the sampling policy, and the sample delivery times \{D_0, D_1, D_2, \ldots\} remain the same in policy $\pi$ and policy $\pi'$. Hence, the only difference between policies $\pi$ and $\pi'$ is that the generation time of sample $i$ is postponed from $S_i$ to $G_i$. The MSE estimator under policy $\pi$ is given by (9) and the MSE estimator under policy $\pi'$ is given by

$$\hat{W}_t = \begin{cases} 0, & t \in [0, D_i); \\ W_{G_i}, & t \in [D_i, D_{i+1}); \\ W_{S_j}, & t \in [D_j, D_{j+1}], \ j \neq i, j \geq 1. \end{cases}$$

For any $T > 0$, we consider two cases:

**Case 1:** If $D_i \wedge T \geq G_i$, we can obtain $S_i \leq G_i \leq D_i \wedge T \leq D_{i+1} \wedge T$. By the strong Markov property of the Wiener process [51, Theorem 2.16], \( \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - W_{G_t}) dt \)
and $W_G - W_S$ are mutually independent. Hence,

$$
\begin{align*}
\mathbb{E} \left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - W_{S_i})^2 dt \right\} &= \mathbb{E} \left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - W_{G_i})^2 dt \right\} \\
&\quad + \mathbb{E} \left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - W_{G_i})^2 dt \right\}.
\end{align*}
$$

Note that the channel is busy whenever there exist some samples that have been generated but not delivered to the estimator. Hence, during the time interval $[S_i, G_i]$, the channel is busy sending some samples generated before $S_i$ in policy $\pi$. Because $\mathbb{E}[Y_i^2] < \infty$, we can get $\mathbb{E}[Y_j^2] < \infty$ and

$$
\mathbb{E}[G_i - S_i] \leq \mathbb{E} \left\{ \sum_{j=1}^{i-1} Y_j \right\} < \infty.
$$

By Wald’s identity [51, Theorem 2.44], we have $\mathbb{E}[W_G - W_S] = 0$ and hence

$$
\begin{align*}
\mathbb{E} \left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - W_{S_i})^2 dt \right\} &\geq \mathbb{E} \left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - W_{G_i})^2 dt \right\}.
\end{align*}
$$

Case 2: If $D_i \wedge T < G_i$, then the fact $D_i \geq G_i$ implies that $T < G_i \leq D_i \leq D_{i+1}$. Hence, $D_i \wedge T = D_{i+1} \wedge T$ and

$$
\begin{align*}
\mathbb{E} \left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - W_{S_i})^2 dt \right\} = \mathbb{E} \left\{ \int_{D_i \wedge T}^{D_{i+1} \wedge T} (W_t - W_{G_i})^2 dt \right\} = 0.
\end{align*}
$$

Combining these two cases with (52), yields that the MSE of the sampling policy $\pi'$ is no greater than that of the sampling policy $\pi$.

By repeating the above arguments for all samples $i$ satisfying $S_i < G_i$, one can show that the sampling policy $\pi'' = \{S_0, G_1, \ldots, G_{i-1}, G_i, G_{i+1}, \ldots\}$ is better than the sampling policy $\pi = \{S_0, S_1, \ldots, S_{i-1}, S_i, S_{i+1}, \ldots\}$. This completes the proof.

If $\text{mse}_{\text{opt}} \leq c$, then there exists a policy $\pi = (Z_0, Z_1, \ldots) \in \Pi_1$ that is feasible for both (33) and (35), which satisfies

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E} \left[ f_{D_{i+1}}(W_t - W_{S_i})^2 dt \right] \leq c.
$$

Hence,

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E} \left[ f_{D_{i+1}}(W_t - W_{S_i})^2 dt - c(Y_t + Z_i) \right] \leq 0.
$$

Because the inter-sampling times $T_i = Y_i + Z_i$ are regeneration, the renewal theory [47] tells us that the limit $\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E} [Y_i + Z_i]$ exists and is positive. By this, we get

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E} \left[ \int_{D_i} (W_t - W_{S_i})^2 dt - c(Y_t + Z_i) \right] = 0.
$$

Therefore, $p(c) \leq 0$.

On the reverse direction, if $p(c) \leq 0$, then there exists a policy $\pi = (Z_0, Z_1, \ldots) \in \Pi_1$ that is feasible for both (33) and (35), which satisfies (62). From (62), we can derive (61) and (60). Hence, $\text{mse}_{\text{opt}} \leq c$. By this, we have proven that $\text{mse}_{\text{opt}} \leq c$ if and only if $p(c) \leq 0$.

Step 2: We need to prove that $\text{mse}_{\text{opt}} < c$ if and only if $p(c) < 0$. This statement can be proven by using the arguments in Step 1, in which “≤” should be replaced by “<”. Finally, from the statement of Step 1, it immediately follows that $\text{mse}_{\text{opt}} > c$ if and only if $p(c) > 0$. This completes the proof of part (a).

Part (b): We first show that each optimal solution to (33) is an optimal solution to (35). By the claim of part (a), $p(c) = 0$ is equivalent to $\text{mse}_{\text{opt}} = c$. Suppose that policy $\pi = (Z_0, Z_1, \ldots) \in \Pi_1$ is an optimal solution to (33). Then, $\text{mse}_{\pi} = \text{mse}_{\text{opt}} = c$. Applying this in the arguments of (60)-(62), we can show that policy $\pi$ satisfies

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E} \left[ \int_{D_i} (W_t - W_{S_i})^2 dt - c(Y_t + Z_i) \right] = 0.
$$

This and $p(c) = 0$ imply that policy $\pi$ is an optimal solution to (35).

Similarly, we can prove that each optimal solution to (35) is an optimal solution to (33). By this, part (b) is proven.

APPENDIX G

PROOF OF LEMMA 3

Part (a) is proven in two steps:

Step 1: We will prove that $\text{mse}_{\text{opt}} \leq c$ if and only if $p(c) \leq 0$.

APPENDIX H

PROOF OF LEMMA 4

According to Theorem 2.51 and Exercise 2.15 of [51], $W_t - 6 \int_0^t W_s^2 ds$ and $W_t - 6tW_t^2 + 3t^2$ are two martingales of the Wiener process \( \{W_t, t \in [0, \infty)\} \). Hence, $\int_0^t W_s^2 ds - tW_t^2 + t^2/2$ is also a martingale of the Wiener process.

Because the minimum of two stopping times is a stopping time and constant times are stopping times [50], it follows that $t \wedge \tau$ is a bounded stopping time for every $t \in [0, \infty)$, where
\(x \wedge y = \inf\{x, y\}\). Then, it follows from Theorem 8.5.1 of [50] that for every \(t \in [0, \infty)\)
\[
E \left[ \int_0^{t \wedge \tau} W_d^2 \, ds \right] = \frac{1}{6} E \left[ W_{t \wedge \tau}^4 \right]
\]
\[
= E \left[ (t \wedge \tau) W_{t \wedge \tau}^2 - \frac{1}{2} (t \wedge \tau)^2 \right].
\]
(63)

Notice that \(\int_0^{t \wedge \tau} W_d^2 \, ds\) is positive and increasing with respect to \(t\). By applying the monotone convergence theorem [50] Theorem 1.5.5], we can obtain
\[
\lim_{t \to \infty} E \left[ \int_0^{t \wedge \tau} W_d^2 \, ds \right] = E \left[ \int_0^\tau W_d^2 \, ds \right].
\]
The remaining task is to show that
\[
\lim_{t \to \infty} E \left[ W_{t \wedge \tau}^4 \right] = E \left[ W_\tau^4 \right].
\]
(65)

Towards this goal, we combine (63) and (64), and apply Cauchy-Schwarz inequality to get
\[
E \left[ W_{t \wedge \tau}^4 \right] \\
= E \left[ 6(t \wedge \tau) W_{t \wedge \tau}^2 - 3(t \wedge \tau)^2 \right] \\
\leq 6 \sqrt{E \left[ (t \wedge \tau)^2 \right] E \left[ W_{t \wedge \tau}^4 \right] - 3E \left[ (t \wedge \tau)^2 \right]}.
\]
Let \(x = \sqrt{E \left[ W_{t \wedge \tau}^4 \right] / E \left[ (t \wedge \tau)^2 \right]}\), then \(x^2 - 6x + 3 \leq 0\).
By the roots and properties of quadratic functions, we obtain
\[
E \left[ W_{t \wedge \tau}^4 \right] \leq (3 + 6\sqrt{6})^2 E \left[ (t \wedge \tau)^2 \right] \leq (3 + 6\sqrt{6})^2 E \left[ \tau^2 \right] < \infty.
\]
Then, we use Fatou’s lemma [50] Theorem 1.5.4] to derive
\[
E \left[ W_\tau^4 \right] \\
= E \left[ \lim_{t \to \infty} W_{t \wedge \tau}^4 \right] \\
\leq \liminf_{t \to \infty} E \left[ W_{t \wedge \tau}^4 \right] \\
\leq (3 + 6\sqrt{6})^2 E \left[ \tau^2 \right] < \infty.
\]
(66)

Further, by (66) and Doob’s maximal inequality [51] Theorem 12.30] and [50] Theorem 5.4.3],
\[
E \left[ \sup_{t \in [0, \infty)} W_{t \wedge \tau}^4 \right] = E \left[ \sup_{t \in [0, \tau]} W_{t \wedge \tau}^4 \right] \leq \left( \frac{4}{3} \right)^4 E \left[ W_\tau^4 \right] < \infty.
\]
Because \(W_{t \wedge \tau}^4 \leq \sup_{t \in [0, \infty)} W_{t \wedge \tau}^4\) and \(\sup_{t \in [0, \tau]} W_{t \wedge \tau}^4\) is integrable, (65) follows from dominated convergence theorem [50] Theorem 1.5.6]. This completes the proof.

**APPENDIX I**

**PROOF OF (41)**

The following lemma is needed in the proof of (41):

**Lemma 7.** For any \(\lambda \geq 0\), there exists an optimal solution \((Z_0, Z_1, \ldots)\) to (38) in which \(Z_i\) is independent of \((W_t, t \in [0, W_{S_i}])\) for all \(i = 1, 2, \ldots\)

**Proof.** Because the \(Y_i\)’s are i.i.d., \(Z_i\) is independent of \(Y_{i+1}, Y_{i+2}, \ldots\), and the strong Markov property of the Wiener process [51] Theorem 2.16], in the Lagrangian \(L(\pi; \lambda)\) the term related to \(Z_i\) is
\[
\int_{S_i + Y_i}^{S_i + Y_i + Z_i + Y_{i+1}} (W_t - W_{S_i})^2 dt - (\kappa_{opt} + \lambda)(Y_i + Z_i),
\]
which is determined by the control decision \(Z_i\) and the recent information of the system \(Z_i = (Y_i, W_{S_i + t} - W_{S_i}, t \geq 0)\).
According to [53] p. 252] and [54] Chapter 6], \(Z_i\) is a sufficient statistics for determining \(Z_i\) in (38). Therefore, there exists an optimal policy \((Z_0, Z_1, \ldots)\) in which \(Z_i\) is determined based on only \(Z_i\), which is independent of \((W_t : t \in [0, S_i])\). This completes the proof. □

**Proof of (41).** By using (12) and Lemma 7 we obtain that for given \(Y_i\) and \(Y_{i+1}\), \(Y_i\) and \(Y_i + Z_i + Y_{i+1}\) are stopping times of the time-shifted Wiener process \(\{W_{S_i + t} - W_{S_i}, t \geq 0\}\).

\[
E \left[ \int_{S_i + Y_i}^{S_i + Y_i + Z_i + Y_{i+1}} (W_t - W_{S_i})^2 dt \right] \\
= E \left[ \int_{Y_i}^{Y_i + Z_i + Y_{i+1}} (W_{S_i + t} - W_{S_i})^2 dt \right] \\
= \frac{1}{6} E \left[ \left( \int_{Y_i}^{Y_i + Z_i + Y_{i+1}} (W_{S_i + t} - W_{S_i})^2 dt \right) Y_i, Y_{i+1} \right] \\
= \frac{1}{6} E \left[ \int_{Y_i}^{Y_i + Z_i + Y_{i+1}} (W_{S_i + t} - W_{S_i})^2 dt \right] Y_i, Y_{i+1} \\
\leq \frac{1}{6} E \left[ \int_{Y_i}^{Y_i + Z_i + Y_{i+1}} (W_{S_i + t} - W_{S_i})^2 dt \right] Y_i, Y_{i+1} \\
\leq \frac{1}{6} E \left[ \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^4 \right] Y_i, Y_{i+1} \\
\leq \frac{1}{6} E \left[ \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^4 \right] Y_i, Y_{i+1}.
\]
(68)

where Step (a) and Step (c) are due to the law of iterated expectations, and Step (b) is due to Lemma 4 Because \(S_{i+1} = S_i + Y_i + Z_i\), we have
\[
E \left[ \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^4 \right] \\
= E \left[ \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^4 \right] \\
= E \left[ \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^4 \right] \\
+ 4E \left[ \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^2 \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^2 \right] \\
+ 6E \left[ \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^2 \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^2 \right] \\
+ 4E \left[ \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^2 \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^2 \right] \\
+ E \left[ \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^4 \right] \\
= E \left[ \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^4 \right] \\
+ 4E \left[ \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^2 \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^2 \right] \\
+ 4E \left[ \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^2 \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^2 \right] \\
+ E \left[ \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^4 \right] \\
+ E \left[ \left( W_{S_i + Y_i + Z_i + Y_{i+1}} - W_{S_i} \right)^4 \right],
\]
where in the last equation we have used the fact that \(Y_{i+1}\) is independent of \(Y_i\) and \(Z_i\), and the strong Markov property of
the Wiener process [51] Theorem 2.16], because
\[ \mathbb{E} \left[ (W_{S_{t+1}+Y_{t+1}} - W_{S_{t+1}})^2 \right] = 0, \]
by the law of iterated expectations, we have
\[ \mathbb{E} \left[ (W_{S_{t+1}+Y_{t+1}} - W_{S_{t+1}})^4 \right] = \mathbb{E} \left[ (W_{S_{t+1}+Y_{t+1}} - W_{S_{t+1}})^2 \right] = 0. \]
In addition, Wald’s identity tells us that \( \mathbb{E} [W_2^2] = \mathbb{E} [\tau] \) for any stopping time \( \tau \) with \( \mathbb{E} [\tau] < \infty \). Hence,
\[ \mathbb{E} \left[ (W_{S_{t+1}+Y_{t+1}} - W_{S_{t+1}})^4 \right] = \mathbb{E} \left[ (W_{S_{t+1}+Y_{t+1}} - W_{S_{t+1}})^2 \right] + \mathbb{E} [Y_{t+1}^2] \mathbb{E} [Y_{t+1}] + \mathbb{E} \left[ (W_{S_{t+1}+Y_{t+1}} - W_{S_{t+1}})^4 \right]. \]
Finally, because \((W_{S_{t+1}} - W_{S_{t}})\) and \((W_{S_{t+1}+Y_{t+1}} - W_{S_{t+1}})\) are both Wiener processes, and the \(Y_i\)’s are i.i.d.,
\[ \mathbb{E} \left[ (W_{S_{t+1}+Y_{t+1}} - W_{S_{t+1}})^4 \right] = \mathbb{E} \left[ (W_{S_{t+1}+Y_{t+1}} - W_{S_{t+1}})^2 \right] = 0. \]
Combining (68)–(70), yields (41).

**APPENDIX J**

**PROOF OF THEOREM 5**

By (41), (67) can be rewritten as
\[ \mathbb{E} \left[ \int_{Z_t}^{S_{t+1}+Y_{t+1}} (W_t - W_{S_{t+1}})^2 dt - (\text{opt} + \lambda)(Y_{t+1} + Z_{t}) \right] = \mathbb{E} \left[ \frac{1}{6} (W_{S_{t+1}+Y_{t+1}} - W_{S_{t+1}})^4 \right] (\text{opt} + \lambda - \mathbb{E}[Y_{t+1}]) (Y_{t+1} + Z_{t}) \]
\[ = \mathbb{E} \left[ \frac{1}{6} (W_{S_{t+1}+Y_{t+1}} - W_{S_{t+1}})^4 \right] \frac{\beta}{3} (Y_{t+1} + Z_{t}) \]
\[ = \mathbb{E} \left[ \frac{1}{6} (W_{S_{t+1}+Y_{t+1}} - W_{S_{t+1}})^4 \right] \frac{\beta}{3} (Y_{t+1} + Z_{t}). \]
(71)

Because the \(Y_i\)’s are i.i.d. and the strong Markov property of the Wiener process [51] Theorem 2.16], the expectation in (71) is determined by the control decision \(Z_i\) and the information \(Z'_i = (W_{S_{t+1}+Y_{t+1}} - W_{S_{t+1}}, Y_{t+1}, (W_{S_{t+1}+Y_{t+1}+t} - W_{S_{t+1}+Y_{t+1}}, t \geq 0))\) (According to [53] p. 252) and [54] Chapter 6, \(Z'_i\) is a sufficient statistics for determining the waiting time \(Z_i\) in (38). Therefore, there exists an optimal policy \((Z_0, Z_1, \ldots)\) in which \(Z_i\) is determined based on only \(Z'_i\). By this, (38) is decomposed into a sequence of per-control problem controls (42). Combining (33), (41), and Lemma 5 yields \(c_{\text{opt}} \geq \mathbb{E}[Y]\). Hence, \(\beta \geq 0\).

We note that, because the \(Y_i\)’s are i.i.d. and the strong Markov property of the Wiener process, the \(Z_i\)’s in this optimal policy are i.i.d. Similarly, the \((W_{S_{t+1}+Y_{t+1}+Z_{t} - W_{S_{t+1}}})\)’s in this optimal policy are i.i.d.

**APPENDIX K**

**PROOF OF Lemma 5**

Case 1: If \(b^2 \geq \beta\), then (47) tells us that \(\tau^* = 0\).

and
\[ u(x) = \mathbb{E}[g(X_0), X_0 = x] = g(x) = \beta s - \frac{1}{2} b^4. \]

Case 2: If \(b^2 < \beta\), then \(\tau^* > 0\) and \((b + W_{\tau^*})^2 = \beta\). Invoking Theorem 8.5.5 in [50], yields
\[ \mathbb{E}_x \tau^* = -(\sqrt{\beta} - b)(-\sqrt{\beta} - b) = \beta - b^2. \]
Using this, we can obtain
\[ u(x) = \mathbb{E}_x g(X(\tau^*)) = \beta(s + \mathbb{E}_x \tau^*) - \frac{1}{2} \mathbb{E}_x [(b + W_{\tau^*})^4] \]
\[ = \beta(s + \beta - b^2) - \frac{1}{2} \beta^2 \]
\[ = \beta s + \frac{1}{2} \beta^2 - b^2 \beta. \]
Hence, in Case 2,
\[ u(x) - g(x) = \frac{1}{2} \beta^2 - b^2 \beta + \frac{1}{2} b^4 = \frac{1}{2} \beta^2 \geq 0. \]

By combining these two cases, Lemma 5 is proven.

**APPENDIX L**

**PROOF OF Lemma 6**

The function \(u(s, b)\) is continuous differentiable in \((s, b)\). In addition, \(\frac{\partial^2}{\partial b^2} u(s, b)\) is continuous everywhere but at \(b = \pm \sqrt{\beta}\). By the Itô-Tanaka-Meyer formula [51] Theorem 7.14 and Corollary 7.35], we obtain that almost surely
\[ u(s + t, b + W_t) - u(s, b) \]
\[ = \int_0^t \frac{\partial}{\partial b} u(s + r, b + W_r) dW_r \]
\[ + \int_0^t \frac{\partial}{\partial s} u(s + r, b + W_r) dr \]
\[ + \frac{1}{2} \int_{-\infty}^\infty L^a(t) \frac{\partial^2}{\partial b^2} u(s + r, b + a) da, \]
(74)
where \(L^a(t)\) is the local time that the Wiener process spends at the level \(a\), i.e.,
\[ L^a(t) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{W_s - a \leq \epsilon\}} ds, \]
(75)
and \(1_A\) is the indicator function of event \(A\). By the property of local times of the Wiener process [51] Theorem 6.18], we obtain that almost surely
\[ u(s + t, b + W_t) - u(s, b) \]
\[ = \int_0^t \frac{\partial}{\partial b} u(s + r, b + W_r) dW_r \]
\[ + \int_0^t \frac{\partial}{\partial s} u(s + r, b + W_r) dr \]
\[ + \frac{1}{2} \int_0^t \frac{\partial^2}{\partial b^2} u(s + r, b + W_r) dr. \]
(76)
Because
\[ \frac{\partial}{\partial b} u(s, b) = \begin{cases} -2b^3, & \text{if } b^2 \geq \beta; \cr -2\beta b, & \text{if } b^2 < \beta, \end{cases} \]
we can obtain that for all \( t \geq 0 \) and all \( x = (s, b) \in \mathbb{R}^2 \)

\[
\mathbb{E}_x \left\{ \int_0^t \left[ \frac{\partial}{\partial b} u(s + r, b + W_r) \right]^2 dr \right\} < \infty.
\]

This and Theorem 7.11 of [51] imply that \( \int_0^t \frac{\partial}{\partial b} u(s + r, b + W_r) dW_r \) is a martingale and

\[
\mathbb{E}_x \left[ \int_0^t \frac{\partial}{\partial b} u(s + r, b + W_r) dW_r \right] = 0, \ \forall \ t \geq 0.
\]

By combining (44), (76), and (77), we get

\[
\mathbb{E}_x [u(X_t)] - u(x) = \mathbb{E}_x \left\{ \int_0^t \left[ \frac{\partial}{\partial s} u(X_r) + \frac{1}{2} \frac{\partial^2}{\partial b^2} u(X_r) \right] dr \right\}.
\]

(78)

It is easy to compute that if \( b^2 > \beta \),

\[
\frac{\partial}{\partial s} u(s, b) + \frac{1}{2} \frac{\partial^2}{\partial b^2} u(s, b) = \beta - 3b^2 \leq 0;
\]

and if \( b^2 < \beta \),

\[
\frac{\partial}{\partial s} u(s, b) + \frac{1}{2} \frac{\partial^2}{\partial b^2} u(s, b) = \beta - \beta = 0.
\]

Hence,

\[
\frac{\partial}{\partial s} u(s, b) + \frac{1}{2} \frac{\partial^2}{\partial b^2} u(s, b) \leq 0
\]

(79)

for all \((s, b) \in \mathbb{R}^2\) except for \( b = \pm \sqrt{\beta} \). Since the Lebesgue measure of those \( r \) for which \( b + W_r = \pm \sqrt{\beta} \) is zero, we get from (78) and (79) that \( \mathbb{E}_x [u(X_t)] \leq u(x) \) for all \( x \in \mathbb{R}^2 \) and \( t \geq 0 \). This completes the proof.

**APPENDIX M**

**PROOF OF THEOREM 7**

Theorem 7 is proven in three steps:

**Step 1:** If \( c = c_{\text{opt}} \), we will show that the duality gap between (35) and (39) is zero, i.e., \( d = p(c_{\text{opt}}) \).

To that end, we need to find \( \pi^* = (Z_0, Z_1, \ldots) \) and \( \lambda^* \) that satisfy the following conditions:

\[
\pi^* \in \Pi, \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E} [Y_i + Z_i] - \frac{1}{f_{\text{max}}} \geq 0,
\]

(80)

\[
\lambda^* \geq 0,
\]

(81)

\[
L(\pi^*; \lambda^*) = \inf_{\pi \in \Pi_{\lambda}} L(\pi; \lambda^*),
\]

(82)

\[
\lambda^* \left\{ \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E} [Y_i + Z_i] - \frac{1}{f_{\text{max}}} \right\} = 0.
\]

(83)

According to Theorem 5 and Corollary 1 the solution \( \pi^* \) to (82) is given by (50) where \( \beta = 3(c_{\text{opt}} + \lambda^* - \mathbb{E} [Y]) \). In addition, as shown in the proof of Theorem 5 the \( Z_i \)'s in policy \( \pi^* \) are i.i.d. From (80), (81), and (83), \( \lambda^* \) is determined by considering two cases: If \( \lambda^* > 0 \), because the \( Z_i \)'s are i.i.d., we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E} [Y_i + Z_i] = \mathbb{E} [Y_i + Z_i] = \frac{1}{\beta f_{\text{max}}}.
\]

(84)

If \( \lambda^* = 0 \), then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E} [Y_i + Z_i] = \mathbb{E} [Y_i + Z_i] = \frac{1}{f_{\text{max}}}.
\]

(85)

Hence, such \( \pi^* \) and \( \lambda^* \) satisfy (80)-(83). By [48] Prop. 6.2.5, \( \pi^* \) is an optimal solution to the primal problem (35) and \( \lambda^* \) is a geometric multiplier [49] for the primal problem (35).

The duality gap between (35) and (39) must be zero, because otherwise there exists no geometric multiplier [48] Section 6.1-6.2.

**Step 2:** We will show that a common optimal solution to (28), (33), and (35) is given by (7) where \( \beta \geq 0 \) is determined by solving

\[
\mathbb{E} [Y_i + Z_i] = \max \left( \frac{1}{f_{\text{max}}}, \frac{\mathbb{E} [(W_{S_i} + Y_i + Z_i - W_{S_i})^4]}{2\beta} \right).
\]

(86)

In Step 1, we have shown that policy \( \pi^* \) in (50) with \( \beta = 3(c_{\text{opt}} + \lambda^* - \mathbb{E} [Y]) \) satisfying (84) and (85) is an optimal solution to (35). By Lemma 3(b), this policy \( \pi^* \) is also an optimal solution to (33). Because (28) is equivalent to (33) and (7) is equivalent to (50), (7) with \( \beta \) satisfying (84) and (85) is a common optimal solution to (28), (33), and (35).

The remaining task is to find the value of \( \beta = 3(c_{\text{opt}} + \lambda^* - \mathbb{E} [Y]) \) satisfying (84) and (85). By (p(c_{\text{opt}}) = 0 and Lemma 3(a), we have \( \text{mse}_{\text{opt}} = c_{\text{opt}} \). Substituting policy \( \pi^* \) and (41) into (33), yields

\[
\text{mse}_{\text{opt}} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E} [(W_{S_i} + Y_i + Z_i - W_{S_i})^4]^4 + \mathbb{E} [Y_i + Z_i] \]

(87)

where in the last equation we have used that the \( Z_i \)'s are i.i.d. and the \( (W_{S_i} + Y_i + Z_i - W_{S_i}) \)'s are i.i.d., which were shown in the proof of Theorem 5 Using (87), the value of \( \beta = 3(c_{\text{opt}} + \lambda^* - \mathbb{E} [Y]) \) can be obtained by considering the following two cases:

**Case 1:** If \( \lambda^* > 0 \), then (87) and (84) imply that

\[
\mathbb{E} [Y_i + Z_i] = \frac{1}{f_{\text{max}}}.
\]

(88)

\[
\beta > 3(c_{\text{opt}} - \mathbb{E} [Y]) = \frac{\mathbb{E} [(W_{S_i} + Y_i + Z_i - W_{S_i})^4]}{2\mathbb{E} [Y_i + Z_i]}. \]

(89)

**Case 2:** If \( \lambda^* = 0 \), then (87) and (85) imply that

\[
\mathbb{E} [Y_i + Z_i] \geq \frac{1}{f_{\text{max}}}.
\]

(90)

\[
\beta = 3(c_{\text{opt}} - \mathbb{E} [Y]) = \frac{\mathbb{E} [(W_{S_i} + Y_i + Z_i - W_{S_i})^4]}{2\mathbb{E} [Y_i + Z_i]}. \]

(91)

Combining (89)-(91). (86) follows. By (87), the optimal value of (33) is given by

\[
\text{mse}_{\text{opt}} = \frac{\mathbb{E} [(W_{S_i} + Y_i + Z_i - W_{S_i})^4]}{6\mathbb{E} [Y_i + Z_i]} + \mathbb{E} [Y].
\]

(92)

**Step 3:** We will show that the expectations in (86) and (92)
are given by
\[ E[Y_i + Z_i] = E[\max(\beta, W_i^2)], \]
\[ E[(W_{i+1} + Y_i + Z_i - W_i)^4] = E[\max(\beta^2, W_i^2)]. \]  
\[ (93) \]
\[ (94) \]
According to (90) with $\beta \geq 0$, we have
\[ W_{i+1} + Y_i - W_i = \begin{cases} \sqrt{\beta}, & \text{if } |W_{i+1} + Y_i - W_i| \geq \sqrt{\beta} ; \\ \beta, & \text{if } |W_{i+1} + Y_i - W_i| < \sqrt{\beta}. \end{cases} \]

Hence,
\[ E[(W_{i+1} + Y_i - W_i)^4] = E[\max(\beta^2, (W_{i+1} + Y_i - W_i)^4)]. \]  
\[ (95) \]
In addition, from (72) and (73) we know that if $|W_{i+1} + Y_i - W_i| \geq \sqrt{\beta}$, then
\[ E[Z_i | Y_i] = 0; \]
on otherwise, if $|W_{i+1} + Y_i - W_i| < \sqrt{\beta}$, then
\[ E[Z_i | Y_i] = \beta - (W_{i+1} + Y_i - W_i)^2. \]

Hence,
\[ E[Z_i | Y_i] = \max(\beta - (W_{i+1} + Y_i - W_i)^2, 0). \]

Using the law of iterated expectations, the strong Markov property of the Wiener process, and Wald’s identity $E[(W_{i+1} + Y_i - W_i)^2] = E[Y_i]$, yields
\[ E[Z_i | Y_i] = E[E[Z_i | Y_i] + Y_i] \\
= E[\max(\beta - (W_{i+1} + Y_i - W_i)^2, 0) + Y_i] \\
= E[\max(\beta - (W_{i+1} + Y_i - W_i)^2, 0) + (W_{i+1} + Y_i - W_i)^2] \\
= E[\max(\beta, (W_{i+1} + Y_i - W_i)^2)]. \]  
\[ (96) \]
Finally, because $W_t$ and $W_{i+t} - W_i$ are of the same distribution, (95) and (96) follow from (90) and (95), respectively. Substituting (93) and (94) into (86) and (92), yields (12) and (13). This completes the proof.

**REFERENCES**

[1] Y. Sun, Y. Polyanskiy, and E. Uysal-Biyikoglu, “Remote estimation of the Wiener process over a channel with random delay,” in *IEEE ISIT*, 2017.

[2] X. Song and J. W. S. Liu, “Performance of multiversion concurrency control algorithms in maintaining temporal consistency,” in *Fourteenth Annual International Computer Software and Applications Conference*, Oct 1990, pp. 132–139.

[3] S. Kaul, R. D. Yates, and M. Gruteser, “Real-time status: How often should one update?” in *IEEE INFOCOM*, 2012.

[4] Y. Sun, E. Uysal-Biyikoglu, R. D. Yates, C. E. Koksal, and N. B. Shroff, “Update or wait: How to keep your data fresh,” in *IEEE INFOCOM*, 2016.

[5] ——, “Update or wait: How to keep your data fresh,” *IEEE Trans. Inf. Theory*, in press, 2017.

[6] A. M. Bedewy, Y. Sun, and N. B. Shroff, “Optimizing data freshness, throughput, and delay in multi-server information-update systems,” in *IEEE ISIT*, 2016.

[7] ——, “Age-optimal information updates in multihop networks,” in *IEEE ISIT*, 2017.

[8] M. Rabi, G. V. Moustakides, and J. S. Baras, “Adaptive sampling for linear state estimation,” *SIAM Journal on Control and Optimization*, vol. 50, no. 2, pp. 672–702, 2012.

[9] A. Nayyar, T. Başar, D. Teneketzis, and V. Veeravalli, “Optimal strategies for communication and remote estimation with an energy harvesting sensor,” *IEEE Trans. Auto. Control*, vol. 58, no. 9, 2013.

[10] R. Gallager, “Basic limits on protocol information in data communication networks,” *IEEE Trans. Inf. Theory*, vol. 22, no. 4, pp. 385–398, Jul 1976.

[11] V. Anantharam and S. Verdú, “Bits through queues,” *IEEE Trans. Inf. Theory*, vol. 42, no. 1, pp. 4–18, Jan 1996.

[12] K. J. Åström and B. M. Bernhardsson, “Comparison of Riemann and Lebesgue sampling for first order stochastic systems,” in *IEEE CDC*, 2002.

[13] B. Hajek, “Jointly optimal paging and registration for a symmetric random walk,” in *IEEE ITW*, Oct 2002, pp. 20–23.

[14] B. Hajek, K. Mitzel, and S. Yang, “Paging and registration in cellular networks: Jointly optimal policies and an iterative algorithm,” *IEEE Trans. Inf. Theory*, vol. 54, no. 2, pp. 608–622, Feb 2008.

[15] G. M. Lipsa and N. C. Martins, “Optimal state estimation in the presence of communication costs and packet drops,” in *47th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, Sept 2009, pp. 160–169.

[16] ——, “Remote state estimation with communication costs for first-order LTI systems,” *IEEE Trans. Auto. Control*, vol. 56, no. 9, pp. 2013–2025, Sept 2011.

[17] A. Molin and S. Hirche, “An iterative algorithm for optimal event-triggered estimation,” *IFAC Proceedings Volumes*, vol. 45, no. 9, pp. 64–69, 2012, 4th IFAC Conference on Analysis and Design of Hybrid Systems.

[18] J. Chakravorty and A. Mahajan, “Fundamental limits of remote estimation of autoregressive Markov processes under communication constraints,” *IEEE Trans. on Auto. Control*, vol. 62, no. 3, pp. 1109–1124, March 2017.

[19] O. C. Imer and T. Başar, “Optimal estimation with limited measurements,” *International Journal of Systems Control and Communications*, vol. 2, no. 1–3, pp. 5–29, 2010.

[20] J. Wu, Q. Jia, K. H. Johansson, and L. Shi, “Event-based sensor data scheduling: Tradeoff between communication rate and estimation quality,” *IEEE Trans. Auto. Control*, vol. 58, no. 4, 2013.

[21] K. Nar and T. Başar, “Sampling multidimensional Wiener processes,” in *IEEE CDC*, Dec 2014, pp. 3426–3431.

[22] X. Gao, E. Akyol, and T. Başar, “Optimal sensor scheduling and remote estimation over an additive noise channel,” in *American Control Conference (ACC)*, July 2015, pp. 2723–2728.

[23] ——, “Optimal estimation with limited measurements and noisy communication,” in *IEEE CDC*, 2015.

[24] ——, “Optimal communication scheduling and remote estimation over an additive noise channel,” 2016, https://arxiv.org/abs/1610.05471.

[25] ——, “On remote estimation with multiple communication channels,” in *American Control Conference (ACC)*, July 2016, pp. 5425–5430.

[26] ——, “Joint optimization of communication scheduling and online power allocation in remote estimation,” in *58th Asilomar Conference on Signals, Systems and Computers*, Nov 2016, pp. 714–718.

[27] ——, “On remote estimation with communication scheduling and power allocation,” in *IEEE 55th Conference on Decision and Control (CDC)*, Dec 2016, pp. 5900–5905.

[28] J. Chakravorty and A. Mahajan, “Remote-state estimation with packet drop,” 6th *IFAC Workshop on Distributed Estimation and Control in Networked Systems*, 2016.

[29] ——, “Structure of optimal strategies for remote estimation over Gilbert-Elliott channel with feedback,” in *IEEE ISIT*, 2017.

[30] T. Berger, “Information rates of Wiener processes,” *IEEE Trans. Inf. Theory*, vol. 16, no. 2, pp. 134–139, March 1970.

[31] A. Kipnis, A. Goldsmith, and Y. Eldar, “The distortion-rate function of sampled Wiener processes,” 2016, https://arxiv.org/abs/1608.04679.

[32] R. D. Yates and S. K. Kaul, “Real-time status updating: Multiple sources,” in *IEEE ISIT*, Jul 2012.

[33] ——, “The age of information: Real-time status updating by multiple sources,” *CoRR*, abs/1608.08622, submitted to *IEEE Trans. Inf. Theory*, 2016.

[34] R. D. Yates, “Lazy is timely: Status updates by an energy harvesting source,” in *IEEE ISIT*, 2015.

[35] C. Kam, S. Kompella, and A. Ephremides, “Age of information under random updates,” in *IEEE ISIT*, 2013.

[36] B. T. Bacinoğlu, E. T. Ceran, and E. Uysal-Biyikoglu, “Age of information under energy replenishment constraints,” in *Information Theory and Applications Workshop (ITA)*, 2015.
[37] M. Costa, M. Codreanu, and A. Ephremides, “On the age of information in status update systems with packet management,” IEEE Trans. Inf. Theory, vol. 62, no. 4, pp. 1897–1910, April 2016.

[38] C. Kam, S. Kompella, G. D. Nguyen, and A. Ephremides, “Effect of message transmission path diversity on status age,” IEEE Trans. Inf. Theory, vol. 62, no. 3, pp. 1360–1374, March 2016.

[39] I. Kadota, E. Uysal-Biyikoglu, R. Singh, and E. Modiano, “Minimizing the age of information in broadcast wireless networks,” in Allerton Conference, 2016.

[40] B. T. Bacinoglu and E. Uysal-Biyikoglu, “Scheduling status updates to minimize age of information with an energy harvesting sensor,” in IEEE ISIT, 2017.

[41] A. Mahajan and D. Teneketzis, “Optimal design of sequential real-time communication systems,” IEEE Trans. Inf. Theory, vol. 55, no. 11, pp. 5317–5338, Nov 2009.

[42] A. N. Shiryaev, Optimal Stopping Rules. New York: Springer-Verlag, 1978.

[43] P. J. Haas, Stochastic Petri Nets: Modelling, Stability, Simulation. New York, NY: Springer New York, 2002.

[44] H. Nyquist, “Certain topics in telegraph transmission theory,” Transactions of the American Institute of Electrical Engineers, vol. 47, no. 2, pp. 617–644, April 1928.

[45] C. E. Shannon, “Communication in the presence of noise,” Proceedings of the IRE, vol. 37, no. 1, pp. 10–21, Jan 1949.

[46] H. V. Poor, An Introduction to Signal Detection and Estimation, 2nd ed. New York, NY, USA: Springer-Verlag New York, Inc., 1994.

[47] S. M. Ross, Stochastic Processes, 2nd ed. John Wiley & Sons, 1996.

[48] D. P. Bertsekas, A. Nedić, and A. E. Ozdaglar, Convex Analysis and Optimization. Belmont, MA: Athena Scientific, 2003.

[49] S. Boyd and L. Vandenberghe, Convex Optimization. Cambridge, U.K.: Cambridge University Press, 2004.

[50] R. Durrett, Probability: Theory and Examples, 4th ed. Cambridge University Press, 2010.

[51] P. Morters and Y. Peres, Brownian Motion. Cambridge University Press, 2010.

[52] D. P. Bertsekas, Nonlinear Programming, 2nd ed. Belmont, MA: Athena Scientific, 1999.

[53] ——, Dynamic Programming and Optimal Control, 3rd ed. Belmont, MA: Athena Scientific, 2005, vol. 1.

[54] P. R. Kumar and P. Varaiya, Stochastic Systems: Estimation, Identification, and Adaptive Control. Englewood Cliffs, NY: Prentice-Hall, Inc., 1986.