Approximation of the Euclidean Ball by polytopes with restricted number of facets

Gil Kur
Weizmann Institute of Science
gil.kur@weizmann.ac.il

Abstract
We prove that there is a universal constant $C$ such that for every $n \geq 2$ and $N \geq 10^n$, there exists a polytope $P_{n,N}$ in $\mathbb{R}^n$ with $N$ facets that satisfies

$$\text{vol}_n(D_n \Delta P_{n,N}) \leq C N^{-\frac{2}{n-1}} \text{vol}_n(D_n)$$

and

$$\Delta_s(D_n, P_n) := \text{vol}_{n-1} (\partial (D_n \cup P_n)) - \text{vol}_{n-1} (\partial (D_n \cap P_n)) \leq 3CN^{-\frac{2}{n-1}} \text{vol}_{n-1}(\partial D_n),$$

where $D_n$ is the $n$-dimensional Euclidean ball, and $\text{vol}_n(D_n \Delta P_{n,N})$ is known as the symmetric volume difference metric.

This result closes gaps from several papers of Hoehner, Ludwig, Schütte and Werner [6, 7, 4] that are related to the approximation of a convex body by an arbitrary positioned polytope with restricted number of facets. These upper bounds are optimal up to an absolute constants.

1 Introduction
Let $K$ be a convex body in $\mathbb{R}^n$ with $C^2$ boundary $\partial K$, we assume that the boundary has a strictly positive curvature $\kappa$ everywhere. In [8], it was shown that for general polytopes the following holds:

$$\lim_{N \to \infty} \inf \{\text{vol}_n(K \Delta P) \mid P \text{ is a polytope with at most } N \text{ facets} \} =$$

$$\frac{1}{2} \text{ldiv}_{n-1} \left( \int_{\partial K} \kappa(x) \frac{1}{n-1} d\mu_{\partial K}(x) \right)^\frac{n+1}{n-1}$$

where $\mu_{\partial K}$ denotes the surface measure of $\partial K$ and $\text{ldiv}_{n-1}$ is a positive constant that depends only on the dimension $n$. In [9], it was shown that $\text{ldiv}_{n-1} \geq c$, They prove it, by the following way:

---

2010 Mathematics Subject Classification:Primary 52A22; Secondary 60D05.

Key words and phrases: Random polytopes, approximation, convex bodies.
The convex set $K = D_n$ was chosen, and they showed that when $N \geq 10^n$ all polytopes $P$ in $\mathbb{R}^n$ with $N$ facets, satisfy

$$\text{vol}_n(D_n \Delta P) \geq cN^{-\frac{2}{n+1}} \text{vol}_n(D_n),$$

for more details see Theorem 2 in [7].

In [3, 7] it was shown that

$$\lim_{N \to \infty} \inf \{\text{vol}_n(P \setminus K) \mid K \subset P \text{ and } P \text{ is a polytope with at most } N \text{ facets}\}$$

$$= \frac{1}{2} \text{div}_{n-1} \left( \int_{\partial K} \kappa(x) \frac{1}{n+1} d\mu_{K}(x) \right)^{\frac{n+1}{n}}.$$ 

where $c_1 n \leq \text{div}_{n-1} \leq c_2 n$, for some $c_1, c_2$ positive numbers.

Until now it was known that $c \leq \text{ldiv}_{n-1} \leq c_2 n$. In this paper we show that $\text{ldiv}_{n-1} \leq C$, where $C$ is a universal constant. We prove this by using a random construction, specifically we show that for $N \geq 10^n$, there is a polytope $P_{n,N}$ in $\mathbb{R}^n$ with $N$ facets, which is generated from a random construction, that satisfies

$$\text{vol}_n(D_n \Delta P_{n,N}) \leq C_n N^{-\frac{2}{n+1}} \text{vol}_n(D_n),$$

where $C_n \leq C$ for some universal constant $C > 0$. Moreover when $N$ is large enough, we can improve the upper bound to

$$\text{vol}_n(D_n \Delta P_{n,N}) \leq (1 + e^{-1} + O(n^{-0.5}))N^{-\frac{2}{n+1}} \text{vol}_n(D_n).$$

which implies that $\frac{1}{2} \text{ldiv}_{n-1} \leq 1 + e^{-1} + O(n^{-0.5})$. We also show that $\frac{1}{2} \text{ldiv}_{n-1} \geq \frac{1}{4} + O(n^{-0.5})$.

By Equation (2) we know that there exists a polytope with $f(\varepsilon) := (c \varepsilon)^{-\frac{n+1}{n}}$ facets that gives an $\varepsilon < \frac{1}{100}$ approximation to the $n$-dimensional Euclidean ball with respect to the symmetric volume difference, i.e. $\frac{|P_{n,N} \Delta D_n|}{|D_n|} \leq \varepsilon$. This result is optimal up to a universal constant by [7]. A classical result in asymptotic geometry states that if the Banach-Mazur distance between a polytope $P$ and the $n$-dimensional Euclidean Ball is less than $\varepsilon$, then the polytope $P$ must have at least $2e \varepsilon^{-\frac{n+1}{n}}$ facets and vertices (see for example Proposition 5.6 in [2]). Hence we can deduce that when $\varepsilon$ is small enough, it’s “harder” to approximate the Euclidean Ball in the Banach-Mazur distance than in the symmetric volume difference.

The author believes that equation (2) holds for every convex body $K$ in $\mathbb{R}^n$, i.e for every $N \geq 10^n$, $n \geq 2$ there exist a polytope $P_{n,N}$ with $N$ facets such that

$$\text{vol}_n(K \Delta P_{n,N}) \leq C_n N^{-\frac{2}{n+1}} \text{vol}_n(K),$$

where $C_n$ is the same constant from equation (2), for more details see subsection [2, 12].

In [4], they considered the surface area deviation, which is defined for every two sets $A, B$ with measurable boundary as follows:

$$\Delta_s(A, B) := \text{vol}_{n-1}(\partial (B \cup A)) - \text{vol}_{n-1}(\partial (A \cap B)).$$

It was shown that for all polytopes $Q$ in $\mathbb{R}^n$ with $N \geq M_n$ facets the following holds:

$$\Delta_s(Q, D_n) \geq c_1 N^{-\frac{2}{n+1}} \text{vol}_n(\partial D_n),$$

where $c_1$ is a positive constant. Finally, we prove that

$$\text{vol}_n(D_n \Delta P_{n,N}) \geq C_n N^{-\frac{2}{n+1}} \text{vol}_n(D_n),$$

where $C_n$ is a positive constant.
where \( M_n \) is a natural number that depends only on the dimension \( n \), and \( c_1 \) is a positive absolute constant. We show that this bound is optimal up to an absolute constant. We use the same random construction that was mentioned above and find a polytope \( Q_{n,N} \) in \( \mathbb{R}^n \) with \( N \geq 10^n \) facets that satisfies

\[
\Delta_s (Q_{n,N}, D_n) \leq 4C_n N^{-\frac{n}{n+1}} \text{vol}_{n-1} (\partial D_n) \leq 4C N^{-\frac{n}{n+1}} \text{vol}_{n-1} (\partial D_n),
\]

where \( C_n \) are the constants that were defined in Equation (2).

**Notations and Preliminary Results**

\(|A|\) is the volume of a set \( A \), and similarly \(|\partial A|\) is the surface area of the set \( A \). \( \text{conv}(A) \) denotes the the convex hull of the set \( A \). \( A^c \) denotes the complementary set of \( A \).

Throughout the paper \( c, c', C, C', c_1, c_2, C_1, C_2 \) denote positive absolute constants that may change from line to line.

We shall use the following auxiliary lemmas:

**Lemma 1.1** (Lemma 2.2 in [10]).

\[
\frac{\sqrt{2\pi}}{\sqrt{n+2}} \leq \frac{|D_n|}{|D_{n-1}|} \leq \frac{\sqrt{2\pi}}{\sqrt{n}}
\]

**Lemma 1.2** (Classical isoperimetric inequality, see [1, 2]). Let \( K \subset \mathbb{R}^n \) be a convex body, then the following holds

\[
|\partial K| \geq n |K|^\frac{n-1}{n} |D_n|^{\frac{n}{n-1}}.
\]

**Lemma 1.3** (Affine isoperimetric inequality [8]). Let \( K \subset \mathbb{R}^n \) be a convex body with \( |K| = |D_n| \) in \( \mathbb{R}^n \) and denote by \( \text{as}(K) = \int_{\partial K} \kappa(x)^{\frac{n+1}{n}} d\mu_{\partial K} (x) \), then the following holds

\[
\text{as}(K) \leq \text{as}(D_n)
\]

**Lemma 1.4** (Theorem 1 in [6]).

\[
\lim_{N \to \infty} \inf \{ \text{vol}_n (K \Delta P) \mid P \text{ is a polytope with at most } N \text{ facets} \} = \frac{1}{2 \text{div}_{n-1}} \left( \int_{\partial K} \kappa(x)^{\frac{n+1}{n}} d\mu_{\partial K} (x) \right)^{\frac{n}{n+1}}
\]

**Lemma 1.5** (Theorem 2 in [4]). Assume that \( N > 10^n \), let \( P \) be a polytope with \( N \) facets in \( \mathbb{R}^n \), then there exists \( c > 0 \) such that

\[
|P \Delta D_n| > cN^{-\frac{n}{n+1}} |D_n|
\]

**Lemma 1.6** (Equation 1.2 from section 1.2.1 in [5]). Let \( g(x) \in C^2([a, b]) \) and \( n > 0 \), assume \( g'(x) \neq 0 \) on \([a, b]\), then the following holds

\[
\int_a^b e^{ng(x)} dx = \frac{1}{n} \left[ \frac{1}{g'(b)} e^{ng(b)} - \frac{1}{g'(a)} e^{ng(a)} \right] = \frac{1}{n} \int_a^b \frac{d}{dx} \left( \frac{1}{g'(x)} \right) e^{ng(x)} dx
\]
2 Main results

**Theorem 2.1.** Let $P_{n,N}$ be the best approximating polytope for the $n$-dimensional Euclidean ball with $N$ facets with respect to the symmetric volume difference. Under the assumption that $N \geq n^n$, the following holds:

$$|D_n \Delta P_{n,N}| \leq (1 + e^{-1} + O(n^{-0.5})) N^{-\frac{1}{n^2}} |D_n|.$$  \hfill (3)

**Remark 2.2.** The bound on $N$, can be improved from $N \geq n^n$ to $N \geq 10^n$. This causes a slight change to the constant before $N^{-\frac{1}{n^2}}$. The proof of this remark is almost identical to the proof of Theorem 2.1, yet for the completeness of this paper we provide a detailed sketch of the proof in Section 4.

**Remark 2.3.** Under the assumptions of Theorem 2.1, every polytope $P$ with $N$ facets in $\mathbb{R}^n$ satisfies

$$|P \Delta D_n| \geq \left(\frac{1}{4} + O(n^{-0.5})\right) N^{-\frac{1}{n^2}} |D_n|,$$

hence $\text{ldiv}_{n-1} \geq \frac{1}{2} + O(n^{-0.5})$.

We give a detailed sketch to the proof of the last remark in the Section 4.

**Theorem 2.4.** Let $Q_{n,N}$ be the best approximating polytope for the $n$-dimensional Euclidean ball with $N$ facets with respect to the surface area deviation. Under the assumptions of Theorem 2.1 the following holds:

$$\Delta_s(Q_{n,N}, D_n) \leq (2.5 + 2e^{-1} + O(n^{-0.5})) N^{-\frac{1}{n^2}} |\partial D_n|.$$  \hfill (4)

**Remark 2.5.** Under the assumptions of Theorems 2.1 and 2.4, their proofs imply that there is a polytope $P$ in $\mathbb{R}^n$ with $N$ facets, that satisfies both

$$|D_n \Delta P| \leq (1 + e^{-1} + O(n^{-0.5})) N^{-\frac{1}{n^2}} |D_n|$$

and

$$\Delta_s(Q_{n,N}, D_n) \leq (2.5 + 2e^{-1} + O(n^{-0.5})) N^{-\frac{1}{n^2}} |\partial D_n|.$$  

**Remark 2.6.** Also, in Theorem 2.4 the bound of the number of facets can be improved from $N \geq n^n$ to $N \geq 10^n$. This causes a slight change to the constant before $N^{-\frac{1}{n^2}}$.

**Remark 2.7.** The constant before $N^{-\frac{1}{n^2}}$ in both Theorems 2.1 and 2.4 can be improved.

2.1 Asymptotic results

In this section we present some asymptotic results, denote by $P_{n,N}$ to be the best approximating polytope with respect to symmetric volume difference in $\mathbb{R}^n$ with $N$ facets. The following corollaries are consequences of Theorem 2.1, Lemma 1.5 and Remark 2.2.

**Corollary 2.8.** Let $A \geq 10$, and $n$ is large enough, then the following holds:

$$\frac{|D_n \Delta P_{n,A^n}|}{|D_n|} \in [cA^{-2}, CA^{-2}].$$

We conjecture that the limit $\lim_{n \to \infty} \frac{|D_n \Delta P_{n,A^n}|}{|D_n|}$ exists.
Corollary 2.9. Let \( f(n) \) be a sequence that satisfies \( \frac{n}{\log(f(n))} = o(1) \). Then the following holds:

\[
\lim_{n \to \infty} \frac{|D_n \Delta P_{n,f(n)}|}{|D_n|} = 0
\]

Remark 2.10. It can easily be proven that for \( f(n) = o(n^a) \) \( \forall a \in (1, \infty) \), the following holds:

\[
\lim_{n \to \infty} \frac{|D_n \Delta P_{n,f(n)}|}{|D_n|} = 1
\]

2.2 Conjectures

We present conjecture that is stronger than the result of Remark 2.10; the author believe that due to symmetry consideration that the following holds

Conjecture 2.11. If \( N \leq 2^n \) then

\[
\lim_{n \to \infty} \frac{|D_n \Delta P_{n,N}|}{|D_n|} = 1.
\]

In order to present the last conjecture, first we show by using the affine isoperimetric inequality (Lemma 1.3), that when the dimension is fixed and the number of facets tends to infinity, then the convex body which is the hardest to approximate is the Euclidean-ball.

For this purpose, let \( K \) be a convex body in \( \mathbb{R}^n \), and assume without loss of generality \( |K| = |D_n| \), and also denote by \( as(K) = \int_{\partial K} \kappa(x) \, d\mu_{\partial K}(x) \)

\[
\lim_{N \to \infty} \inf_{P \text{ has } N \text{ facets}} |P \Delta K| = \frac{1}{2} \operatorname{div}_{n-1} as(K) N^{n-1} - N^{n-1} N^{n-2} \leq \frac{1}{2} \operatorname{div}_{n-1} as(D_n) N^{n-1} - N^{n-1} \leq \lim_{N \to \infty} \inf_{P \text{ has } N \text{ facets}} |P \Delta D_n| = \frac{1}{2} \operatorname{div}_{n-1} as(D_n) N^{n-1} - N^{n-1}
\]

where the first and the last equality follow from Lemma 1.4, the second inequality follows from the affine isoperimetric inequality (Lemma 1.3). The author believes that the limit in equation (5) is unnecessary, i.e.

Conjecture 2.12. For every \( n \geq 0, N \geq n + 1 \) the following holds:

\[
\min_{P \text{ has at most } N \text{ facets}} \frac{|P \Delta K|}{|K|} \leq \min_{P \text{ has at most } N \text{ facets}} \frac{|P \Delta D_n|}{|D_n|}
\]

Observe that by Theorem 2.1 there is a polytope with \( f(\varepsilon) := (c\varepsilon)^{- \frac{n-1}{2}} \) facets that gives an approximation of the \( n \)-dimensional Euclidean ball up to an accuracy of \( \varepsilon \leq \frac{1}{100} \), i.e \( \frac{|P_{n,f(\varepsilon)} \Delta D_n|}{|D_n|} \leq \varepsilon \). This result is optimal by up to a universal constant by Lemma 1.5. This conjecture implies that this result holds for every convex set.

Remark 2.13. In [9] a similar result was proven, it was shown that for every \( n \geq 2, N \geq n + 1 \) the following holds:

\[
\min_{P \text{ has at most } N \text{ vertices, } P \subset K} \frac{|P \Delta K|}{|K|} \leq \min_{P \text{ has at most } N \text{ vertices, } P \subset D_n} \frac{|P \Delta D_n|}{|D_n|}
\]
3 Proofs

For the proofs of Theorems 2.1 and 2.4 we may assume that $N$ is even. We also denote by $\sigma$ the uniform probability measure on $S^{n-1}$. Also recall that $N \geq n^n$

3.1 Proof of Theorem 2.1

We define a random slab with width $t$ as follows: First choose a random $y \in S^{n-1}$ from the uniform distribution on the sphere, and the random slab is the set $\{x : |\langle x, y \rangle| \leq t\}$. The probability of a point $x \in \mathbb{R}^n$ to be outside of a random slab with width $t \in (0, 1)$ is the following:

$$
\sigma_{y \in S^{n-1}} (|\langle x, y \rangle| \geq t) = \sigma_{y \in S^{n-1}} \left( \left| \frac{x}{\|x\|_2}, y \right| \geq \frac{t}{\|x\|_2} \right)
= \frac{|\text{conv}(\widehat{y}, \{y \in S^{n-1} : |\langle x, y \rangle| \geq \frac{t}{\|x\|_2} \} \cap \partial D_n)|}{|D_n|}
+ \frac{|\{y \in \mathbb{R}^n : |\langle x, y \rangle| \geq \frac{t}{\|x\|_2} \} \cap D_n|}{|D_n|}
= \frac{2|D_{n-1}|}{|D_n|} \left( \int_{1-\frac{t^2}{n}}^1 (1-x^2)^{\frac{n-2}{2}} dx + \frac{t}{n\|x\|_2} \left(1 - \frac{t^2}{\|x\|_2^2}\right)^{\frac{n-1}{2}} \right)
$$

where the first term is the volume of the cap and the second is the volume of the cone whose common base is $\{y \in \mathbb{R}^n : \langle \frac{x}{\|x\|_2}, y \rangle = \frac{t}{\|x\|_2} \} \cap D_n$.

We denote by $r = \|x\|_2$ and the probability $\sigma_{y \in \mathbb{R}^n} \left( |\langle x, y \rangle| \geq t \right)$ by $\alpha_{n,r,t}$. Let $P$ be the random polytope that is generated by the intersection of $\frac{N}{2}$ independent random slabs with the same width $t$. Observe that with probability one $P$ is bounded and has $N$ facets.

The following expression is the probability of a point $x \in \mathbb{R}^n$ to be inside the random polytope $P$:

$$
\Pr(x \in P) = \Pr_{y_1,\ldots,y_N \in S^{n-1}} \left( \bigcap_{i=1}^N |\langle x, y_i \rangle| \leq t \right) = (1 - \alpha_{n,r,t})^{\frac{N}{2}}.
$$

We express the expectation of the volume of $P$ as follows

$$
E[|P|] = \int_{\mathbb{R}^n} \int_{S^{n-1}} \mathbb{1}_{\{x \in \cap_{i=1}^N \{\langle x, y_i \rangle \leq t\}} \, dxd\sigma(y_1) \ldots d\sigma(y_N)
= \int_{\mathbb{R}^n} \int_{S^{n-1}} \mathbb{1}_{\{x \in \cap_{i=1}^N \{\langle x, y_i \rangle \leq t\}} \, d\sigma(y_1) \ldots d\sigma(y_N) dx
= \int_{\mathbb{R}^n} \left(1 - \alpha_{n,\|x\|_2,t}\right)^{\frac{N}{2}} dx
= |\partial D_n| \int_0^\infty r^{n-1} \int_{S^{n-1}} \left(1 - \alpha_{n,r,t}\right)^{\frac{N}{2}} d\sigma dr
= |\partial D_n| \int_0^\infty r^{n-1} \left(1 - \alpha_{n,r,t}\right)^{\frac{N}{2}} dr,
$$

6
where we used Fubini and polar coordinates. Similarly $\mathbb{E}[|D_n \Delta P|]$ can be expressed

$$\mathbb{E}[|D_n \Delta P|] = \mathbb{E}[|D_n \setminus P|] + \mathbb{E}[|P \setminus D_n|]$$

$$= |\partial D_n| \left( \int_0^1 r^{n-1} \left( 1 - (1 - \alpha_{n,r,t}) \frac{\alpha}{\pi} \right) dr + \int_1^\infty r^{n-1} (1 - \alpha_{n,r,t}) \frac{\alpha}{\pi} dr \right).$$  (7)

Now we set $t$ with

$$t_0 = \sqrt{1 - \left( \frac{2|\partial D_n|}{N|D_{n-1}|} \right)^{\frac{n}{2-n}}}. $$

From now on we use the notation $\alpha_{n,r}$ instead of $\alpha_{n,r,t_0}$. The following lemma gives an upper bound for the first term in $(7)$.

**Lemma 3.1.**

$$\mathbb{E}[|D_n \setminus P|] = |\partial D_n| \int_0^1 r^{n-1} \left( 1 - (1 - \alpha_{n,r}) \frac{\alpha}{\pi} \right) dr$$

$$\leq \left( 1 + O\left(n^{-0.5}\right) \right) N^{-\frac{2}{2-n}} |D_n|. $$  (8)

**Proof.** Recall that $P = \bigcap_{i=1}^N \{ x \in \mathbb{R}^n : |\langle y_i, x \rangle| \leq t_0 \}$, the following bound holds for every realization of $P$.

$$|D_n \setminus P| = \bigcup_{i=1}^N \{ x \in \mathbb{R}^n : |\langle y_i, x \rangle| > t_0 \} \cap D_n$$

$$\leq N \left\{ x \in \mathbb{R}^n : \langle y, x \rangle \in (t_0, 1] \right\} \cap D_n, $$  (9)

where $y$ is an arbitrary vector in $\mathbb{S}^{n-1}$. Now we give an upper bound for the volume of a cap $C$, under the assumption that $N \geq n^n$

$$|C| = |D_{n-1}| \int_{t_0}^1 \left( 1 - x^2 \right)^{\frac{n-1}{2}} dx \leq |D_{n-1}| \int_1^{\frac{1}{2}} \left( \frac{|\partial D_n|}{|D_{n-1}|} \right)^{\frac{n}{2-n}} \left( 1 - x^2 \right)^{\frac{n-1}{2}} dx$$

$$= |D_{n-1}| \int_1^{\frac{1}{2}} \left( \frac{|\partial D_n|}{|D_{n-1}|} \right)^{\frac{n}{2-n}} \left( 1 - x \right) \left( 1 + x \right)^{\frac{n-1}{2}} dx$$

$$\leq |D_{n-1}| 2^{\frac{n-1}{2}} \int_1^{\frac{1}{2}} \left( \frac{|\partial D_n|}{|D_{n-1}|} \right)^{\frac{n}{2-n}} \left( 1 - x \right)^{\frac{n-1}{2}} dx$$

$$\leq \left( 1 + O\left( \frac{\ln(n)}{n} \right) \right) N^{-1} N^{-\frac{2}{2-n}} |D_n|. $$  (10)

Thus, by equations $(9)$ and $(10)$, when $N$ large enough the following holds for every realization of $P$

$$|D_n \setminus P| \leq \left( 1 + O\left(n^{-0.5}\right) \right) N^{-\frac{2}{2-n}} |D_n|,$$

and by taking expectation we get

$$\mathbb{E}[|D_n \setminus P|] \leq \left( 1 + O\left(n^{-0.5}\right) \right) N^{-\frac{2}{2-n}} |D_n|.$$
and the lemma follows.

**Lemma 3.2.**

\[ E[|P \setminus D_n|] = |\partial D_n| \int_1^{\infty} n^{-1} (1 - \alpha_{n,r})^N dr \left( e^{-1} + O \left( n^{-0.5} \right) \right) N^{-\frac{2}{N}} |D_n|. \]

we dedicate a whole subsection for the this Lemma due to its technicality. We denote by \( \delta = (n - 1)^{-0.5} N^{-\frac{2}{N}} \), also we shall use the following lemma which is proven in Section 4.

**Lemma 3.3.** Assume that \( r \in [1, 1 + \frac{N - \delta}{\sqrt{n - 1}}] \), then the following holds:

\[ \alpha_{n,r} = \frac{2 \left( 1 + O \left( n^{-1} \right) \right)}{N} e^{(n-1)(r-1)N\frac{2}{N} \left( 1 + O \left( n^{-0.5} \right) \right)} \]

**Proof of Lemma 3.2**

Let us split \( E[|P \setminus D_n|] \) to five parts:

\[
\begin{align*}
|\partial D_n| & \left[ \int_1^{1+\delta} r^{n-1} (1 - \alpha_{n,r})^N dr + \int_{1+\delta}^{1+2N^{-\frac{2}{N}}} r^{n-1} (1 - \alpha_{n,r})^N dr \\
& + \int_{1+2N^{-\frac{2}{N}}}^{1+\frac{N}{\sqrt{n}}} r^{n-1} (1 - \alpha_{n,r})^N dr + \int_{1+\frac{N}{\sqrt{n}}}^{n^2} r^{n-1} (1 - \alpha_{n,r})^N dr \\
& + \int_{n^2}^{\infty} r^{n-1} (1 - \alpha_{n,r})^N dr \right]
\end{align*}
\]

Observe that \( (1 - \alpha_{n,r}) \) is decreasing in \( r \), therefore we need to derive a lower bound for \( \alpha_{n,r} \).

**Lemma 3.4.**

\( \int_1^{1+\delta} r^{n-1} (1 - \alpha_{n,r})^N dr \leq \left( e^{-1} + O \left( n^{-0.5} \right) \right) N^{-\frac{2}{N}} |D_n|. \)

**Proof.** First by Lemma 3.3 we evaluate \( \alpha_{n,r} \) for \( r \in [1, 1 + \delta] \)

\[ \alpha_{n,r} = \frac{2 \left( 1 + O \left( n^{-1} \right) \right)}{N} e^{(n-1)(r-1)N\frac{2}{N} \left( 1 + O \left( n^{-0.5} \right) \right)}. \]

Hence,

\[
\begin{align*}
|\partial D_n| \int_1^{1+\delta} r^{n-1} (1 - \alpha_{n,r})^N dr & \leq \\
|\partial D_n| \int_1^{1+\delta} r^{n-1} \left( 1 - \frac{2 \left( 1 + O \left( n^{-1} \right) \right)}{N} e^{(n-1)(r-1)N\frac{2}{N} \left( 1 + O \left( n^{-0.5} \right) \right)} \right)^N dr & \leq \\
|\partial D_n| \int_1^{1+\delta} r^{n-1} e^{(1+O(n^{-1}))e^{(n-1)(r-1)N\frac{2}{N} \left( 1 + O \left( n^{-0.5} \right) \right)}} dr & \leq \\
(e^{-1} + O \left( n^{-1} \right)) |\partial D_n| \int_1^{1+\delta} r^{n-1} e^{(n-1)(r-1)N\frac{2}{N} \left( 1 + O \left( n^{-0.5} \right) \right)} dr & \leq \\
\end{align*}
\]
in the last inequality, we used the following inequality: $e^{-x} \leq \frac{1}{ex}$ on $[1, \infty)$ and we continue

\[
(e^{-1} + O(n^{-1})) |\partial D_n| \int_0^\delta (r + 1)^{n-1} e^{-(n-1)rN_{\pi^2 \pi}} dr \leq \\
(e^{-1} + O(n^{-1})) |\partial D_n| \int_0^{(n-1)^{-0.5}N_{\pi^2 \pi}} e^{-(n-1)rN_{\pi^2 \pi}} dr = \\
\frac{(e^{-1} + O(n^{-0.5})) |\partial D_n|}{(n-1)} N^{-\frac{n}{n-1}} \left(1 - e^{-(n-1)(n-1)^{-0.5}N_{\pi^2 \pi}^2 N_{\pi^2 \pi}} \right) = \\
(e^{-1} + O(n^{-0.5})) N^{-\frac{2}{n-1}} |D_n|.
\]

\[
\Box
\]

**Lemma 3.5.**

\[
|\partial D_n| \int_{1+\delta}^{1+2N_{\pi^2 \pi}} r^{n-1} \left(1 - \alpha_{n,r} \right)^\frac{n}{n-1} dr = |D_n| N^{-\frac{2}{n-1}} o(n^{-0.5}).
\]

**Proof.** First by Lemma 3.3 applied to $r = 1 + \delta = 1 + (n-1)^{-0.5} N_{\pi^2 \pi}$, we get that

\[
\alpha_{n,1+\delta} = \frac{2(1 + O(n^{-1}))}{N} e^{(n-1)(r-1)N_{\pi^2 \pi}^2 (1 + O(n^{-0.5}))} = \frac{2(1 + O(n^{-1}))}{N} e^{\frac{\sqrt{n-1}(1 + O(n^{-0.5}))}{N}}.
\]

Hence,

\[
|\partial D_n| \int_{1+\delta}^{1+2N_{\pi^2 \pi}} r^{n-1} \left(1 - \alpha_{n,r} \right)^\frac{n}{n-1} dr \leq \\
|\partial D_n| \int_{1+\delta}^{1+2N_{\pi^2 \pi}} r^{n-1} \left(1 - \alpha_{n,1+(n-1)^{-0.5}N_{\pi^2 \pi}} \right)^\frac{n}{n-1} \left(1 - \frac{2(1 + O(n^{-1}))}{N} e^{\frac{\sqrt{n-1}(1 + O(n^{-0.5}))}{N}} \right) \frac{n}{n-1} dr \leq \\
|\partial D_n| \int_{1+\delta}^{1+2N_{\pi^2 \pi}} r^{n-1} e^{-(1 + O(n^{-1})) e^{\frac{\sqrt{n-1}(1 + O(n^{-0.5}))}{N}}} dr \leq \\
(1 + o(n^{-1})) |\partial D_n| e^{-(1 + O(n^{-1})) e^{\frac{\sqrt{n-1}(1 + O(n^{-0.5}))}{N}}} |\partial D_n| \int_1^{1+2N_{\pi^2 \pi}} r^{n-1} dr = |D_n| N^{-\frac{2}{n-1}} o(n^{-0.5}).
\]

\[
\Box
\]

**Lemma 3.6.**

\[
|\partial D_n| \int_{1+2N_{\pi^2 \pi}}^{1+2N_{\pi^2 \pi}} r^{n-1} \left(1 - \alpha_{n,r} \right)^\frac{n}{n-1} dr = |D_n| N^{-\frac{2}{n-1}} o(n^{-0.5}).
\]
Proof. Using equation (1) we know that
\[
\alpha_{n,r} \geq \frac{2|D_{n-1}|}{|D_n|} \frac{t}{nr} \left(1 - \frac{t^2}{r^2}\right)^{\frac{n-1}{2}}
\]
\[
= (1 + O(n^{-1})) \frac{2|D_{n-1}|}{|D_n|} \left(1 - \frac{t^2}{r^2}\right)^{\frac{n-1}{2}}
\]
where \( t = \sqrt{1 - \left(\frac{2|D_{n-1}|}{|D_n|}\right)^{\frac{n-1}{2}}} \). Hence,
\[
|\partial D_n| \int_{1+2N}^{1+\frac{n}{2}N} e^{-n^{-1}(1 - \alpha_{n,r})^N} \, dr \leq
\]
e\^2|\partial D_n| \int_{1+2N}^{1+\frac{n}{2}N} (1 - \alpha_{n,r})^N \, dr \leq
\]
e\^2|\partial D_n| \int_{1+2N}^{1+\frac{n}{2}N} \left(1 - (1 + O(n^{-1})) \frac{2|D_{n-1}|}{|D_n|} \left(1 - \frac{t^2}{r^2}\right)^{\frac{n-1}{2}}\right)^N \, dr \leq
\]
e\^2|\partial D_n| \int_{1+2N}^{1+\frac{n}{2}N} \left(-N(1+O(n^{-1})) \frac{|D_{n-1}|}{|D_n|} (1 - \frac{t^2}{r^2})^{\frac{n-1}{2}}\right) \, dr \leq
\]
e\^2|\partial D_n| \int_{1+2N}^{1+\frac{n}{2}N} \left(-N(1+O(n^{-1})) \frac{|D_{n-1}|}{|D_n|} (1 - \frac{t^2}{r^2})^{\frac{n-1}{2}}\right) \, dr \leq
\]
we use again the fact that \( r \leq 1 + \frac{2}{n} \), and continue
\[
e\^2|\partial D_n| \int_{1+2N}^{1+\frac{n}{2}N} e^{-N(1+O(n^{-1})) \frac{|D_{n-1}|}{|D_n|} ((1+(r-1)^2-t^2)^{\frac{n-1}{2}}}) \, dr =
\]
e\^2|\partial D_n| \int_{1+2N}^{1+\frac{n}{2}N} e^{-N(1+O(n^{-1})) \frac{|D_{n-1}|}{|D_n|} (1-2(r-1))^{\frac{n-1}{2}}} \, dr =
\]
now we that \( t = t_0 = \sqrt{1 - \left(\frac{2|D_{n-1}|}{|D_n|}\right)^{\frac{n-1}{2}}} \)
\[
e\^2|\partial D_n| \int_{1+2N}^{1+\frac{n}{2}N} e^{-N(1+O(n^{-1})) \frac{|D_{n-1}|}{|D_n|} \left(\frac{|\partial D_n|}{2|D_n|}\right)^{\frac{n-1}{2}} + 2(r-1))^{\frac{n-1}{2}}} \, dr \leq
\]
e\^2|\partial D_n| \int_{1+2N}^{1+\frac{n}{2}N} e^{-2N(1+O(n^{-1})) \left(1+(r-1)^2-t^2\right)^{\frac{n-1}{2}}} \, dr \leq
\]
now we use the following \((1 + b)^n \geq 1 + nb \) on \([0, \infty)\) and continue
\[
e\^2|\partial D_n| \int_{1+2N}^{1+\frac{n}{2}N} e^{-2N(1+O(n^{-1})) \left(1+(r-1)^2-t^2\right)^{\frac{n-1}{2}}} \, dr \leq
\]
e\^2|\partial D_n| \int_{1+2N}^{1+\frac{n}{2}N} \left(1+O(\frac{ln(n)}{n})\right) \, dr =
\]
e\^2|\partial D_n| \int_{1+2N}^{1+\frac{n}{2}N} \left(1+O(\frac{ln(n)}{n})\right) \, dr =
\]
\[
|D_n|N^{-\frac{n-1}{2}} \, o(n^{-1}).
\]
Lemma 3.7.

\[ |\partial D_n| \int_{1 + \frac{2}{n}}^{n^2} r^{n-1} (1 - \alpha_{n,r})^\frac{n}{2} \ dr = |D_n| \mathcal{N}^{-\frac{n}{2n+2}} o (n^{-0.5}) \]

Proof. First recall that \( \alpha_{n,r} \) is decreasing in \( r \), hence

\[ |\partial D_n| \int_{1 + \frac{2}{n}}^{n^2} r^{n-1} (1 - \alpha_{n,r})^\frac{n}{2} \ dr \leq |\partial D_n| n^{2n} \int_{1 + \frac{2}{n}}^{n^2} (1 - \alpha_{n,1+\frac{2}{n}})^\frac{n}{2} \ dr. \]  

(16)

In order to continue we derive an upper bound for \( \alpha_{n,1+\frac{2}{n}} \). We use the fact that \( \alpha_{n,r} > \frac{2}{|\partial D_n|} \left( 1 - \frac{t^2}{r^2} \right)^{\frac{n-1}{2}} \), and since \( t = 1 - O\left( \frac{1}{n} \right) \) and \( r = 1 + \frac{2}{n} \), the following holds

\[ \alpha_{n,1+\frac{2}{n}} \geq (1 + O\left( n^{-1} \right)) \frac{2|D_{n-1}|}{|\partial D_n|} = (1 + O\left( n^{-1} \right)) \frac{2|D_{n-1}|}{|\partial D_n|} \left( 1 - \frac{t^2}{r^2} \right)^{\frac{n-1}{2}} \geq (1 + O\left( n^{-1} \right)) \frac{2|D_{n-1}|}{|\partial D_n|} \left( \frac{4}{n} + o\left( n^{-1} \right) \right)^{\frac{n-1}{2}} \geq 2c \left( \frac{4}{n} \right)^{\frac{n}{2}}. \]

Now we continue from the end of equation (16)

\[ |\partial D_n| n^{2n} \int_{1 + \frac{2}{n}}^{n^2} \left( 1 - 2c \left( \frac{4}{n} \right)^{\frac{n}{2}} \right)^\frac{n}{2} \ dr \leq |\partial D_n| n^{2n} \int_{1 + \frac{2}{n}}^{n^2} e^{-Nc\left( \frac{4}{n} \right)^{\frac{n}{2}}} \ dr \leq |\partial D_n| \mathcal{N}^{-n^{2n+2}e^{-Nn^{2n}} - \frac{n}{2} \mathcal{N}^{-n^{2n+2}N^{-4}} \leq |\partial D_n| \mathcal{N}^{-1} = |D_n| \mathcal{N}^{-\frac{2}{2n+2}} o (n^{-0.5}). \]  

(17)

In the last two inequality we used both that \( N \geq n^n \) and that when \( x \) is sufficiently large then \( e^{-x} \leq x^{-10} \).

Lemma 3.8.

\[ |\partial D_n| \int_{n^2}^{\infty} r^{n-1} (1 - \alpha_{n,r})^\frac{n}{2} \ dr = |D_n| \mathcal{N}^{-\frac{n}{2n+2}} o (n^{-0.5}) \]

Proof. In order to prove to lemma we use the following lemma which is proven in the Section 3.

Lemma 3.9. Assume that \( r \geq n^2 \), then the following holds:

\[ \alpha_{n,r} \geq 1 - \frac{C\sqrt{n}}{r} \]  

(18)
Now we prove the lemma,
\[ |\partial D_n| \int_{n^2}^{\infty} r^{n-1} \left( 1 - \alpha_{n,r} \right) \frac{n}{n+2} dr \leq |\partial D_n| \int_{n^2}^{\infty} r^{n-1} \left( C \sqrt{n} \right) \frac{n}{n+2} dr \leq |\partial D_n| \int_{n^2}^{\infty} r^{n-1} \left( 1 - \alpha_{n,r} \right) \frac{n}{n+2} N dr = \]
(19)
and the lemma follows.

Now Lemma 3.2 follows from all the lemmas that are proven in this subsection, which give us
the following upper bound for \( E[|P \setminus D_n|] \)
\[ E[|P \setminus D_n|] \leq (e^{-1} + O(n^{-0.5})) N^{-\frac{n}{n+2}} |D_n|. \]
(20)

**Proof of Theorem 2.1.** Using Lemmas 3.1 and 3.2 give the following upper bound
\[ E[|P \Delta D_n|] \leq (1 + e^{-1} + O(n^{-0.5})) N^{-\frac{n}{n+2}} |D_n| \]
The theorem follows from the fact that there is polytope \( P_{n,N} \), a realization of \( P \), which its
symmetric volume difference is less than \( E[|D_n \Delta P|] \).

### 3.2 Proof of Theorem 2.4

Recall that we want to find an upper bound for \( \Delta_s(Q_{n,N}, D_n) \), where \( Q_{n,N} \) is a polytope in \( \mathbb{R}^n \)
with at most \( N \) facets that minimizes the surface area deviation with the Euclidean ball.

For this purpose choose a polytope \( P \) from the random construction that was shown in Theorem
2.1 that satisfies
\[ |P \Delta D_n| \leq (1 + e^{-1} + O(n^{-0.5})) N^{-\frac{n}{n+2}} |D_n|. \]
Recall that the surface area deviation defined as follows
\[ \Delta_s(D_n, P) = |\partial (D_n \cup P) | - |\partial (D_n \cap P) |. \]
First we prove a lower bound for \( |\partial (D_n \cap P) |. \)

**Lemma 3.10.**
\[ |\partial (P \cap D_n) | \geq \left( 1 - N^{-\frac{n}{n+2}} (1 + e^{-1} + O(n^{-0.5})) \right) |\partial D_n|. \]
(21)

**Proof.** By the definition of the symmetric volume difference, \( P \) satisfies the following:
\[ |P \cap D_n| \geq \left( 1 - (1 + e^{-1} + O(n^{-0.5})) N^{-\frac{n}{n+2}} \right) |D_n|, \]
and by the isoperimetric inequality (Lemma 1.2)
\[ |\partial (P \cap D_n) | \geq n|P \cap D_n|^{-\frac{1}{n}} |D_n|^{\frac{1}{n}} \]
\[ \geq \left( 1 - (1 + e^{-1} + O(n^{-0.5})) N^{-\frac{n}{n+2}} \right) |\partial D_n|, \]
and the lemma follows.
Now we prove an upper bound for $|\partial (D_n \cup P)|$:

**Lemma 3.11.**

$$|\partial (D_n \cup P)| \leq \left( 1 + \left( 1 + \frac{1}{2} + e^{-1} + O \left( n^{-0.5} \right) \right) N^{-\frac{2}{n-1}} \right) |\partial D_n|.$$  \hspace{1cm} (22)

**Proof.** By the definition of the symmetric volume difference, $P$ satisfies the following:

$$|P \cup D_n| \leq \left( 1 + \left( 1 + e^{-1} + O \left( n^{-0.5} \right) \right) N^{-\frac{2}{n-1}} \right) |D_n|.$$  \hspace{1cm} (23)

First by using volume considerations, it’s easy to see that the origin is in the interior of $P$. Hence, the following holds:

$$|D_n \cup P| = |\text{conv}(\vec{0}, \partial P \cap D_n^c)| + |\text{conv}(\vec{0}, \partial D_n \cap P^c)|$$

in the last equation we used the fact all the facets have the same height $t_0$. Now we use both Equations (23) and (24) and derive that

$$t_0 |\partial (P \cup D_n)| \leq \left( 1 + \left( 1 + e^{-1} + O \left( n^{-0.5} \right) \right) N^{-\frac{2}{n-1}} \right) |D_n|,$$

and since $t_0 = 1 - \frac{1}{2} \left( 1 + O \left( n^{-0.5} \right) \right) N^{-\frac{2}{n-1}}$, the lemma follows. \hfill \Box

**Proof of Theorem 2.4.** The theorem follows by using Lemmas 3.10 and 3.11

$$\Delta_s (D_n, P) = |\partial (P \cup D_n)| - |\partial (D_n \cap P)|$$

$$\leq \left( 2 + \frac{1}{2} + 2e^{-1} + O \left( n^{-0.5} \right) \right) N^{-\frac{2}{n-1}} |\partial D_n|.$$  \hspace{1cm} (25)

\hfill \Box

**ACKNOWLEDGMENTS**

I would like to express my sincerest gratitude to Prof. Boaz Klartag for the inspiring discussions, and also to Prof. Gideon Schechtman and Dr. Ronen Eldan. Also I express my gratitude to my lovely girlfriend, Anna, for editing the content of this paper.

4 Technical Lemmas and loose ends

Recall that

$$\alpha_{n,r} = \frac{2|D_{n-1}|}{|D_n|} \left( \int_{\frac{1}{2}}^{1} \left( 1 - x^2 \right)^{\frac{n-1}{2}} dx + \frac{t}{nr} \left( 1 - \frac{t^2}{r^2} \right)^{\frac{n-1}{2}} \right)$$

where $t = t_0 = \sqrt{1 - \left( \frac{2|\partial D_n|}{N|D_{n-1}|} \right)^{\frac{n-1}{n}}}$.

Recall that the first term is the volume of the cap and the second is the volume of the cone whose common base is $\{ x \in \mathbb{R}^N : x_1 = \frac{1}{r} \} \cap D_n$. Under the assumptions of Theorem 2.1, $t$ is very close to 1. When $r$ is close to 1, the volume of the cone is significantly larger than the volume of the cap. The following lemma formalizes it.
Lemma 4.1. Assume that \( r \in [1, 1 + \sqrt{n}^{-1}] \) under the assumptions of Theorem 2.1, the following holds:

\[
\int_{\frac{1}{r}}^{1} (1 - x^2)^{\frac{n-1}{2}} \, dx \leq C \frac{n^2}{n^2} \left( \frac{t}{nr} \left( 1 - \frac{t^2}{r^2} \right)^{\frac{n-1}{2}} \right)
\]  

(26)

Proof. Observe that under the assumptions of Theorem 2.1 we know that 
\[ N - \frac{2}{n-1} = O(n^{-2}), \]
which implies that \( \frac{1}{r} - 1 = O(n^{-2}) \). Hence,

\[
\int_{\frac{1}{r}}^{1} (1 - x^2)^{\frac{n-1}{2}} \, dx = \int_{0}^{1} (2x - x^3)^{\frac{n-1}{2}} \, dx \\
\leq 2^{\frac{n-1}{2}} \int_{0}^{1} x^{\frac{n-1}{2}} \, dx \\
= 2^{\frac{n-1}{2}} \left( 1 - \frac{t}{r} \right)^{\frac{n-1}{2}} \\
= C n^2 \left( 1 - \frac{t}{r} \right)^{\frac{n-1}{2}} \\
\leq C n^2 \left( 1 - \frac{t}{r} \right)^{\frac{n-1}{2}}.
\]  

(27)

Now we can complete all the missing details from the proof of Theorem 2.1. First we prove Lemma 3.3:

Lemma. Assume that \( r \in [1, 1 + \sqrt{n}^{-1}] \), then the following holds:

\[
\alpha_{n,r} = \frac{2}{N} \left( 1 + O\left( n^{-1} \right) \right) e^{(n-1)(r-1)N \frac{2}{n^2}} (1+O(n^{-0.5}))
\]  

(28)
Proof. We use Lemma 4.1 and remind that both $t, r$ are $1 - O\left(\frac{1}{n^2}\right)$

\[
\alpha_{n,r} = (1 + O\left(\frac{1}{n^2}\right)) \frac{2|D_{n-1}|}{|D_n|} \left( \frac{t}{nr} \right) \left( 1 - \frac{t^2}{(1 + (r - 1))^2} \right)^{\frac{n-1}{2}}
\]

\[
= (1 + O\left(\frac{1}{n^2}\right)) \frac{2|D_{n-1}|}{|\partial D_n|} \frac{1}{(1 + (r - 1))^{n-1}} \left( (1 + (r - 1))^2 - t^2 \right)^{\frac{n-1}{2}}
\]

\[
= (1 + O\left(\frac{1}{n^2}\right)) \frac{2|D_{n-1}|}{|\partial D_n|} \left( 1 - t^2 + 2(r - 1) + (r - 1)^2 \right)^{\frac{n-1}{2}}
\]

\[
= (1 + O\left(\frac{1}{n^2}\right)) \frac{2|D_{n-1}|}{|\partial D_n|} \left( \frac{|\partial D_n|}{|D_{n-1}|N} \right)^{\frac{n}{2}} + 2 (r - 1) + (r - 1)^2 \right)^{\frac{n-1}{2}}
\]

\[
= 2 \left(1 + O\left(n^{-1}\right)\right) \frac{1}{N} \left( 1 + 2(r - 1)N \frac{n^{2/3} (1 + O\left(\frac{ln(n)}{n}\right))}{N^{1/3}} \right)^{\frac{n-1}{2}}
\]

\[
= 2 \left(1 + O\left(n^{-1}\right)\right) e^{(n-1)(r-1)N^{1/3}} \left(1 + O\left(n^{-0.5}\right)\right)
\]

The following is proof the of Lemma 3.9.

**Lemma.** Assume that $r \geq n^2$, then the following holds:

\[
\alpha_{n,r} \geq 1 - \frac{C \sqrt{n}}{r}
\]  

(29)

**Proof.**

\[
\alpha_{n,r} = \frac{2|D_{n-1}|}{|D_n|} \left( \int_{\frac{r}{2}}^{1} (1 - x^2)^{\frac{n-1}{2}} dx + \frac{t}{nr} \left( 1 - \frac{t^2}{r^2} \right)^{\frac{n-1}{2}} \right)
\]

\[
\geq \frac{2|D_{n-1}|}{|D_n|} \int_{\frac{r}{2}}^{1} (1 - x^2)^{\frac{n-1}{2}} dx
\]

\[
\geq \frac{2|D_{n-1}|}{|D_n|} \int_{\frac{r}{2}}^{1} (1 - x^2)^{\frac{n-1}{2}} dx = 1 - 2 \frac{|D_{n-1}|}{|D_n|} \int_{0}^{r} (1 - x^2)^{\frac{n-1}{2}} dx
\]
in the last equality we used the fact that |D_{n-1}| \int_0^1 (1 - x^2)^{\frac{n-1}{2}} \, dx = \frac{|D_n|}{2n}, and continue
\begin{align*}
&\geq 1 - 2\frac{|D_{n-1}|}{|D_n|} \int_0^1 (1 - x^2)^{\frac{n-1}{2}} \, dx \geq 1 - c\sqrt{n} \int_0^1 (1 - x^2)^{\frac{n-1}{2}} \, dx \\
&\geq 1 - c'\sqrt{n} \int_0^1 (1 - x)^{\frac{n-1}{2}} \, dx \\
&\geq 1 - \frac{c}{\sqrt{n}} \left(1 - \left(1 - \frac{1}{r}\right)\left(1 - (1 - \frac{n+1}{2r} + \ldots)\right)\right) \\
&\geq 1 - \frac{C}{r} \sqrt{n}.
\end{align*}

\[ \square \]

Sketch to the proof of Remark 2.2

We give a detailed sketch for the required modification to the proof of Theorem 2.1 in order that its result also holds when \( 10^n \leq N \leq n^n \). We show in this subsection how to modify the two main Lemmas 3.1 and 3.2 that will also be correct when \( 10^n \leq N \leq n^n \). For both the aforementioned lemmas, we need to estimate the volume of a spherical cap with height \( h < 1 \).

**Lemma 4.2.** Let \( a_n \in (\frac{2}{3}, 1) \), \( a_n \) be a number that may depend on the dimension \( n \), then the following holds:
\[ \int_{a_n}^1 (1 - x^2)^{\frac{n-1}{2}} \, dx = \left(1 - a_n^2\right)^{\frac{n+1}{2}} \frac{n+1}{n} + O\left(\frac{1}{a_n^2 \sqrt{n}}\right). \]

**Proof.** Let \( \varepsilon < \frac{1-a_n}{2} \), then the following holds:
\[ \int_{a_n}^{1-\varepsilon} (1 - x^2)^{\frac{n-1}{2}} \, dx = \int_{a_n}^{1-\varepsilon} e^{\frac{n+1}{2} \ln(1-x^2)} \, dx \]
\[ = \frac{2}{n-1} \left[ - \frac{1 - (1 - \varepsilon)^2}{2(1 - \varepsilon)} \left(1 - (1 - \varepsilon)^2\right)^{\frac{n-1}{2}} \right. \\
+ \frac{1 - a_n^2}{2a_n} \left(1 - a_n^2\right)^{\frac{n-1}{2}} \right] - \frac{2}{n-1} \int_{a_n}^{1-\varepsilon} \frac{1 - (1 - x^2)}{2x} \left(1 - x^2\right)^{\frac{n-1}{2}} \, dx \]
\[ \leq \frac{2}{n-1} \left[ - \frac{(1 - (1 - \varepsilon)^2)^{\frac{n+1}{2}}}{2(1 - \varepsilon)} \right. \\
+ 1 - a_n^2 \left. \left(1 - a_n^2\right)^{\frac{n+1}{2}} \right] + \frac{C}{n} \int_{a_n}^{1-\varepsilon} (1 - x^2)^{\frac{n-1}{2}} \, dx, \]
where the second equality follows from Lemma 1.6. By taking \( \varepsilon \to 0 \), the lemma follows. \[ \square \]
Lemma 4.2 gives an upper bound for the volume of the cap $C$ in Lemma 3.1 when $10^n \leq N \leq n^n$. Thus we can repeat the proof of Lemma 3.1 and get that

**Lemma 4.3. (Modification for Lemma 3.1)**

$$\mathbb{E}[|D_n \setminus P|] \leq \left(1 + \frac{1}{20}\right) + O\left(n^{-0.5}\right) N^{-\frac{n-1}{2}} |D_n|$$

(31)

Now we modify Lemma 3.2, this clearly requires more changes, first we give a lower bound for $\alpha_{n,r}$. Recall that $t = t_0 = \sqrt{1 - \frac{2|\partial D_n|}{N|D_{n-1}|}}$.

**Lemma 4.4. Assume that $r \in [1, 1 + \frac{N^{-\frac{n}{2}}}{\sqrt{n-1}}]$ and $10^n \leq N \leq n^n$ and when the dimension is sufficiently large, then the following holds:**

$$\int_{\frac{t}{r}}^{1} (1 - x^2)^{\frac{n-1}{2}} dx \leq \frac{1}{100 nr} \left(1 - \frac{t^2}{r^2}\right)^{\frac{n-1}{2}}.$$

**Proof.** Applying Lemma 4.2 with $a_n = \frac{t}{r}$, gives that

$$\int_{\frac{t}{r}}^{1} (1 - x^2)^{\frac{n-1}{2}} dx = \frac{1}{n-1} t \left(1 - \frac{t^2}{r^2}\right)^{\frac{n-1}{2}} + O\left(n^{-1} \int_{\frac{t}{r}}^{1} (1 - x^2)^{\frac{n-1}{2}} dx\right)$$

$$\leq \frac{1}{100 nr} \left(1 - \frac{t^2}{r^2}\right)^{\frac{n-1}{2}}.$$

Using the last Lemma one can repeat the proof of Lemma 3.3 and get that:

**Lemma 4.5. (Modification of Lemma 3.3)** Assume that $r \in [1, 1 + \frac{N^{-\frac{n}{2}}}{\sqrt{n-1}}]$ and $10^n \leq N \leq n^n$, then the following holds:

$$\frac{2}{N} \left(1 - \frac{1}{20} + O\left(n^{-1}\right)\right) e^{(n-1)(r-1)N^{-\frac{n}{2}}} (1 + O(n^{-0.5})) \leq \alpha_{n,r}.$$

Finally we show how to modify Lemma 3.2.

**Lemma 4.6. (Modification of Lemma 3.2)**

$$\mathbb{E}[|P \setminus D_n|] \leq \left(\frac{e^{-1}}{1 - \frac{1}{20}} + O\left(n^{-0.5}\right)\right) N^{-\frac{n-1}{2}} |D_n|.$$

**Proof.** We denote by $\delta = \min\{N^{-\frac{n}{2}}, (n-1)^{-0.5}, (100n)^{-1}\}$ and split $\mathbb{E}[|P \setminus D_n|]$ to three parts

$$|\partial D_n| \left(\int_{1}^{1+\delta} r^{n-1} (1 - \alpha_{n,r})^N dr + \int_{1+\delta}^{n^2} r^{n-1} (1 - \alpha_{n,r})^N dr + \int_{n^2}^{\infty} r^{n-1} (1 - \alpha_{n,r})^N dr\right).$$
We handle the last integral in the same way as in the original Lemma. Now we prove that the second integral is negligible.

\[
|\partial D_n| \int_{1+\delta}^{n^2} r^{n-1} (1 - \alpha_{n,r}) \frac{N}{2} \, dr \leq
\]

\[
|\partial D_n| \int_{1+\delta}^{n^2} r^{n-1} \left( 1 - \frac{2 \left( 1 - \frac{1}{25} + O\left( n^{-1} \right) \right) e^{\sqrt{n-1} (1+O(n^{-0.5}))} \right) \frac{N}{2} \, dr \leq
\]

\[
|\partial D_n| \int_{1+\delta}^{n^2} r^{n-1} e^{-\left( 1 - \frac{1}{25} + O\left( n^{-1} \right) \right) e^{\sqrt{n-1} (1+O(n^{-0.5}))}} \, dr \leq
\]

\[
C |D_n| n^2 e^{-\left( 1 - \frac{1}{25} + O\left( n^{-1} \right) \right) e^{\sqrt{n-1} (1+O(n^{-0.5}))}} = o(n^{-3}) |D_n| N^{-\frac{2}{n-1}}.
\]

Finally, by using the lower bound for \( \alpha_{n,r} \) that was proven in Lemma 4.5, we handle the first integral as in the original Lemma, and we derive the following

\[
\mathbb{E}[|P \setminus D_n|] \leq \left( \frac{e^{-1}}{1 - \frac{1}{25}} + O\left( n^{-0.5} \right) \right) N^{-\frac{2}{n-1}} |D_n|.
\]

\[\square\]

**Sketch to the proof of Remark 2.3**

Let \( P_{n,N} \) be the best approximating polytope in \( \mathbb{R}^h \) with at most \( N \) facets with respect to the symmetric volume difference.

In Theorem 2 at [7], it was shown that

\[
|D_n \setminus P_{n,N}| \geq \frac{1}{n} \sum_{i=1}^{N} |F_i \cap D_n| \sqrt{1 - (F_i \cap D_n)^2}, \tag{32}
\]

where \( F_i \) denotes a facet of \( P_{n,N} \). We use also Lemma 9 at [7]

\[
\sum_{i=1}^{N} |F_i \cap D_n| = \frac{1}{2} |\partial P| = \frac{1}{2} |\partial D_n| + f(N). \tag{33}
\]

where \( f(N) \to 0 \) as \( N \to \infty \). Thus we can write this as an optimization problem, which the target function is that is smaller than the right side of equation (32) and constraint is the last equation.

\[
\min \left\{ f(r_1, \ldots, r_N) : \sum_{i=1}^{N} r_i^{n-1} = \frac{1}{2} |\partial P|, 0 \leq r_i \leq 1, \forall i \in 1, \ldots, N \right\}.
\]

18
where
\[ f(r_1, \ldots, r_N) = n^{-1} |D_{n-1}| \sum_{i=1}^{N} r_i^{n-1} \left( 1 - \sqrt{1 - r_i^2} \right). \]

Now we use both Lagrange multipliers and the separability of both \( f \) and the constraints, and we get that the minimum is achieved on
\[ r_1^*, \ldots, r_N^* = \left( \frac{|\partial P|}{2|D_{n-1}|N} \right)^{\frac{1}{n-1}}. \]

Now we conclude the remark
\[ \frac{1}{2} \text{div}_{n-1} \geq f(r_1^*, \ldots, r_N^*) = \frac{|D_{n-1}|}{n} \sum_{i=1}^{N} \frac{|\partial P|}{2N|D_{n-1}|} \left( 1 - \sqrt{1 - \left( \frac{|\partial P|}{2|D_{n-1}|N} \right)^{\frac{2}{n-1}}} \right) \]
\[ = \frac{|\partial P|}{2n} \left( 1 - \sqrt{1 - \left( \frac{|\partial P|}{2|D_{n-1}|N} \right)^{\frac{n-1}{n}}} \right) \geq \left( \frac{1}{4} + O \left( n^{-0.5} \right) \right)^{\frac{1}{n}} |\partial D_n|, \quad (34) \]
in the last inequality we used the isoperimetric inequality (Lemma 1.2 and Theorem 2.1) which implies that \( |\partial P| \geq (1 - cN^{-\frac{2}{n-1}})|\partial D_n| \).

References

[1] Shiri Artstein-Avidan, Apostolos Giannopoulos, and Vitali D Milman. Asymptotic geometric analysis, part i. Mathematical Surveys and Monographs, 202, 2015.
[2] Guillaume Aubrun and Stanislaw J Szarek. Alice and Bob Meet Banach: The Interface of Asymptotic Geometric Analysis and Quantum Information Theory, volume 223. American Mathematical Soc., 2017.
[3] Peter M Gruber. Asymptotic estimates for best and stepwise approximation of convex bodies ii. In Forum Mathematicum, volume 5, pages 521–538, 1993.
[4] Steven D Hoehner, Carsten Schütz, and Elisabeth M Werner. The surface area deviation of the euclidean ball and a polytope. Journal of Theoretical Probability, pages 1–24, 2015.
[5] Daan Huybrechs and Sheehan Olver. Highly oscillatory quadrature. Highly oscillatory problems, (366):25–50, 2009.
[6] Monika Ludwig. Asymptotic approximation of smooth convex bodies by general polytopes. Mathematika, 46(01):103–125, 1999.
[7] Monika Ludwig, Carsten Schütt, and Elisabeth Werner. Approximation of the Euclidean ball by polytopes. Studia Math., 173(1):1–18, 2006. ISSN 0039-3223.
[8] Erwin Lutwak. The brunn–minkowski–firey theory ii: affine and geominimal surface areas. Advances in Mathematics, 118(2):244–294, 1996.
[9] Alexander Murray Macbeath. An extremal property of the hypersphere. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 47, pages 245–247. Cambridge University Press, 1951.

[10] Shlomo Reisner, Carsten Schütt, and Elisabeth Werner. Dropping a vertex or a facet from a convex polytope. In *Forum Mathematicum*, volume 13, pages 359–378. Berlin; New York: De Gruyter, c1989-, 2001.