On Carpi and Alessandro conjecture

Mikhail V. Berlinkov

Department of Algebra and Discrete Mathematics, Ural State University,
620083 Ekaterinburg, Russia
berlm@mail.ru

Abstract. The well known open Černý conjecture states that each synchronizing automaton with \(n\) states has a synchronizing word of length at most \((n-1)^2\). On the other hand, the best known upper bound is cubic of \(n\). Recently, in the paper [1] of Alessandro and Carpi, the authors introduced the new notion of strongly transitivity for automata and conjectured that this property with a help of Extension method allows to get a quadratic upper bound for the length of the shortest synchronizing words. They also confirmed this conjecture for circular automata. We disprove this conjecture and the long-standing Extension conjecture too. We also consider the widely used Extension method and its perspectives.

1 Strongly transitivity and the Černý conjecture

Let \(A = \langle Q, \Sigma, \delta \rangle\) be a complete deterministic finite automaton (DFA), where \(Q\) is the state set, \(\Sigma\) is the input alphabet, and \(\delta: Q \times \Sigma \rightarrow Q\) is the transition function. The function \(\delta\) extends uniquely to a function \(Q \times \Sigma^* \rightarrow Q\), where \(\Sigma^*\) stands for the free monoid over \(\Sigma\); the latter function is still denoted by \(\delta\) and \(\lambda\) denotes an empty word. Thus, each word in \(\Sigma^*\) acts on the set \(Q\) via \(\delta\). The DFA \(A\) is called synchronizing if there exists a word \(w \in \Sigma^*\) whose action resets \(A\), that is, leaves the automaton in one particular state no matter which state in \(Q\) it starts at: \(\delta(q, w) = \delta(q', w)\) for all \(q, q' \in Q\). Any such word \(w\) is called a synchronizing word for \(A\). The minimum length of synchronizing words for \(A\) is denoted by \(\min_{\text{synch}}(A)\).

Synchronizing automata serve as transparent and natural models of error-resistant systems in many applications (coding theory, robotics, testing of reactive systems) and also reveal interesting connections with symbolic dynamics and other parts of mathematics. For a brief introduction to the theory of synchronizing automata we refer the reader to the recent survey [12]. Here we discuss one of the main problem in this theory: the Černý conjecture and related problems.

In the paper [3] at 1964 Černý conjectured that each synchronizing automaton with \(n\) states has a synchronizing word of length at most
(n − 1)^2. He also presented the extremal series of the n-state circular automata with a shortest synchronizing word of length (n − 1)^2. Thus he proved the lower bound of the conjecture. The conjecture is still open and the best known upper bound for the length of the shortest synchronizing word is \( n^3 - n \). Pin proved this result at 1983 in [8] using combinatorial result of Frankl [5]. Since the lower bound is quadratic and the upper bound is cubic, it is of certain importance to prove a quadratic upper bound.

All existing methods for proving the upper bound of minimal length of synchronizing words can be divided to «compress» and «extension» methods. Methods of both types construct a finite ordered collection of words \( V = (v_1, v_2, \ldots, v_m) \), which concatenation is synchronizing. Let us say that \( m = |V| \) is the size of the collection \( V \) and \( L_V = \max_i |v_i| \) is the length of the collection \( V \). The difference between these types of methods is that the compress collection subsequently compresses the set of states \( Q \) to some state \( p \), i.e

\[
|Q| > |Q.v_1| > |Q.v_1v_2| > \ldots > |Q.v_1v_2\ldots v_m| = |\{p\}|
\]

while the extension collection subsequently extends some state \( p \) to the set of states \( Q \), i.e

\[
|\{p\}| < |p.v_1^{-1}| < |p.v_1^{-1}v_2^{-1}| < \ldots < |p.v_1^{-1}v_2^{-1}\ldots v_m^{-1}| = |Q|.
\]

Since the size \( m \) of the collections can not be more than \( n - 1 \), the proof of a quadratic upper bound can be reduced to the proof a linear upper bound for the length of the collection \( L_V \).

The compress method is used to prove the cubic upper bound \( n^3 - n \) in the general case mentioned above. It is also used to prove the Černý conjecture for few «small» classes of automata such as automata with zero, aperiodic automata [13] or interval automata. Since the Černý conjecture is proved for automata with zero, we assume automata is strongly connected in the rest of the paper, otherwise the considered problem can be reduced by using the construction of automaton with zero (see [11] for example).

The extension methods seem more productive to prove a quadratic upper bounds. In 1998 Dubuc [4] proved the Černý conjecture for circular automata, i.e. the automata with a letter, which acts as a cyclic substitution. He used an extension method combined with the skilful linear algebra techniques to prove this result. In 2003 Kari [7] proved the Černý conjecture for Euclidian automata using extension method. The quadratic upper bound was also confirmed for the one-cluster automata in the paper
Let us note that it is the largest class of synchronizing automata with proved quadratic upper bound.

In 2008 Arturo Carpi and Flavio D’Alessandro introduced the new ideas for constructing the extension collection $V$ of linear length. The ideas are based on the notion of the independent collection (or set) of words. The collection of words $W = (w_1, w_2, \ldots, w_n)$ of the $n$-state automaton $\mathcal{A}$ is called independent, if for any two given state $s$ and $t$ there exists an index $i$ such that $s.w_i = t$. The automaton $\mathcal{A}$ is called strongly transitive, if it admits some independent collection of words $W$. It is easy to check, that each synchronizing strongly connected automaton $\mathcal{A}$ is strongly transitive. Moreover, if $u$ is synchronizing, then $\mathcal{A}$ has an independent collection of length not more than $|u| + n - 1$. The authors also proved that this bound is tight and if the $n$-state automaton $\mathcal{A}$ is strongly transitive with some independent collection $W$, then it has a synchronizing word of length not more than $(n - 2)(n + L_W - 1) + 1$. Later they conjectured that each synchronizing automata has an independent collection of linear length. Formally, for some number $k > 0$ the following $kn$-Independent-Set conjecture holds true.

**Conjecture 1** Each strongly connected $n$-state synchronizing automaton has an independent collection $W = (w_1, w_2, \ldots, w_n)$ of length less than $kn$.

Since $k$ is a constant, this conjecture implies quadratic upper bound of the minimal length of synchronizing word for all synchronizing automata. If the automaton is circular and $a$ denotes the circular letter, then the independent collection $W$ can be chosen as $(\lambda, a, a^2, \ldots, a^{n-1})$. Hence, the $1 \ast n$-Independent-Set conjecture is true for circular automata. This implies the upper bound $2(n - 2)(n - 1) + 1$ for this class of automata.

Our paper is organized as follows. At first, in the section 2 we consider the Extension Algorithm in the universal form, introduce the $kn$-Extension and $kn$-Balanced conjecture and prove that the last one implies $kn$-Independent-Set conjecture. After this, in the section 3 we construct a series, which disproves introduced conjectures, in particular, the $kn$-Independent-Set conjecture of Carpi and Alessandro for each $k > 0$. Finally, in the section 4 we generalize the disproved conjectures to the «local» form and discuss the perspectives of the extension method.

## 2 Extension Method

Let us consider precisely the extension method, implicitly used in the papers [2,1,4,7,9]. In the rest of the paper we assume $\mathcal{A} = (Q, \Sigma, \delta)$ is an $n$-state strongly connected synchronizing automaton.
Suppose $C_s, C_e$ are some subsets of $Q$ and $v_s, v_e$ are some words such that $Q.v_e = C_e \supseteq C_s, |C_s.v_s| = 1$. Then the following algorithm returns a synchronizing word by constructing an extension collection of words.

**Expansion Algorithm (EA)**

**input** $\mathcal{A}, C_s, C_e, v_s, v_e$

**initialization** $v \leftarrow v_s$

$S \leftarrow C_s$

**while** $|S \cap C_e| < |C_e|$, find a word $u(S) = u \in \Sigma^*$ of minimum length with $|S.u^{-1} \cap C_e| > |S \cap C_e|$; if none exists, **return** Failure

$v \leftarrow uv$

$S \leftarrow S.u^{-1}$

**return** $v.e.v$

It is easy to show that the algorithm $EA$ works. At first, the cycle iterates not more than $|C_e| - |C_f|$ times, because each iteration expands the set $S \cap C_e$ to a one or more elements. Let us show $EA$ does not fail, i.e. the word $u = u(S)$ exists in each iteration. We now consider the one iteration of the cycle. Let $|S \cap C_e| < |C_e|$. Since $\mathcal{A}$ is synchronizing, there exists a synchronizing word $u$, i.e. $Q.u = \{p\}$. Moreover, since $\mathcal{A}$ is strongly connected, the word $u$ can be chosen to satisfy $p \in S$. Thus the following calculations hold true.

$$Q \supseteq S.u^{-1} \supseteq p.u^{-1} = Q \Rightarrow S.u^{-1} = Q$$

$$|S.u^{-1} \cap C_e| = |Q \cap C_e| = |C_e| > |S \cap C_e|$$

Hence $u$ satisfies desired condition and after the last iteration we have $C_e \subseteq S$ and $S = p.v^{-1}$ for some state $p \in Q$. Since $C_e.v.e^{-1} = Q$, then

$$p.(v.e.v)^{-1} = p.v^{-1}v.e^{-1} = S.v.e^{-1} = Q.$$ 

Hence the word $v.e.v$ is synchronizing. Furthermore, since the cycle of $EA$ iterates not more than $|C_e| - |C_f|$ times, the length of $v.e.v$ does not exceed

$$|v_e| + |v_s| + (|C_e| - |C_f|) \max_{S \subseteq C_e} |u(S)|$$

Thus, in order to prove the upper bound, we need to estimate the maximal possible length of the extension words $u(S)$ (the length of the extension collection). Let us introduce one of the basic definition in the paper.
Definition 1. The subset $S \subseteq Q$ of automaton $\mathcal{A}$ is called $m$-Extendable in the subset $C_e \subseteq Q$, if there exists some (extension) word $v$ of length not more than $m$ such that

$$|(S \cap C_e).v^{-1}| > |S \cap C_e|.$$  

We now formulate the Extension conjecture.

Conjecture 2 Each proper subset $S$ of $Q$ is $n$-Extendable.

Suppose the automaton $\mathcal{A}$ satisfies the Extension conjecture. Let us set $C_e = Q$ and $v_e = \lambda$. Since $\mathcal{A}$ is synchronizing, there exists a letter $v_s$ and a subset $C_s$ such that $|C_s.v_s| = 1 < |C_s|$. Applying $EA$ for this input data, we get a synchronizing word $v_e v_s$ as a result. Finally, we have that $\mathcal{A}$ satisfies the Černý conjecture

$$|v_e v| \leq |v_e| + |v_s| + (|C_e| - |C_s|) \max_{S \subseteq C_e} |u(S)| \leq 0 + 1 + (n - 2)n = (n - 1)^2$$

Thus the Černý conjecture follows from the Extension conjecture. This fact is used in the papers [4] and [7] to prove the Černý conjecture for circular and Euclidian automata respectively. The counterexample for Extension conjecture is presented in the paper of Kari [6]. However, the example is the 6-state automaton, so the conjecture is still open for $n > 6$. In the next section we present a series of counterexamples for $n > 3$. We now generalize this conjecture to the $kn$-Extension conjecture.

Conjecture 3 Each proper subset $S$ of $Q$ is $kn$-Extendable.

If the $kn$-Extension conjecture holds true for $\mathcal{A}$, then $EA$ returns a synchronizing word of length at most $(n - 2)kn + 1$. Since $k$ is a constant, this bound is also quadratic. This conjecture is often proved by using the following $kn$-Balanced conjecture.

Conjecture 4 Each proper subset $S$ of $Q$ admits a word collection $v_1, v_2, \ldots v_m$ such that $|v_i| < kn$ with the following property.

$$\sum_{i=1}^{m} [S.v_i^{-1}] = m \frac{|S|}{|Q|}[Q],$$

where $[T]$ denotes the characteristic vector of the set $T$ in the linear space $\mathbb{R}^n$. 
One can prove that the \((k - 1)n\)-Balanced conjecture implies the \(kn\)-Extension conjecture (for synchronizing automaton). Proofs of this or equivalent facts can be found in the papers [9,2,1]. Thus \(kn\)-Balanced conjecture also implies the quadratic upper bound. The following lemma shows that \(kn\)-Balanced conjecture implies \(kn\)-Independent-Set conjecture.

**Lemma 1.** If synchronizing \(n\)-state automaton \(A\) satisfies \(kn\)-Independent-Set conjecture then \(A\) satisfies \(kn\)-Balanced conjecture.

**Proof.** Suppose \(W = \{w_1, w_2, \ldots, w_n\}\) is an independent set in the automaton \(A\) of length less than \(kn\), i.e. \(|w_i| < kn\) and for any two given state \(s\) and \(t\) there exists an index \(i\) such that \(s.w_i = t\). Let us fix the arbitrary state \(t\). Then for each \(s \in Q\) there exists an index \(i\) such that \(s.w_i = t\) or equivalently \(\bigcup_{i=1}^n t.w_i^{-1} = Q\). In the linear form it can be written as \(\sum_{i=1}^n [t.w_i^{-1}] = |Q|\). Since the automaton is deterministic then for each subset \(S\) of \(Q\), we have the desired property of \(kn\)-Balanced conjecture

\[
\sum_{i=1}^n [S.w_i^{-1}] = \sum_{i=1}^n \sum_{q \in S} [q.w_i^{-1}] = \sum_{q \in S} \sum_{i=1}^n [q.w_i^{-1}] = \sum_{q \in S} |Q| = |S||Q| = n \frac{|S|}{|Q|}|Q|.
\]

3 Slow extended series

The 2-letter automaton \(A(m, k) = \langle Q, \Sigma, \delta \rangle\) is drawn at the Figure. If for some state \(q \in Q\) and some letter \(d \in \Sigma\) there is no output edge from the state \(q\) labeled by \(d\), then we assume the loop is drawn there. All such edges are omitted for the sake of simplicity.

Let us denote by \(C_b\) the set of all states unstable by \(b\), i.e. \(C_b = \{q_0, s_1, s_2, \ldots, s_k\}\).

The following remark is directly follows from the construction of the automaton and shows when the letter \(b\) can appear in the shortest expanding word.

**Remark 1.** Suppose \(S\) is a subset of \(Q\) unstable by \(b\), i.e. \(S.b^{-1} \neq S\); then \(C_b \cap S \neq \emptyset\) and \(C_b \subseteq S\).

We now formulate the main proposition about properties of the collection of automata \(A(m, k)\).
**Proposition 1.** 1. The series $B_n = \mathcal{A}(n-2,1)$ is a counterexample of the Extension conjecture for $n > 3$;
2. For each $c < 2$ the series $B_n$ is also a counterexample of the $cn$-Extension conjecture for $n > \frac{3}{2-c}$;
3. For each $k \in \mathbb{N}$ the series $C_n = \mathcal{A}(n-k,k)$ for $n > k^2$ is a counterexample of the $kn$-Balanced conjecture and $kn$-Independent-Set conjecture of Carpi and Alessandro, therefore.

**Proof.** Consider the subset $S = C_b$. Let $v$ be a shortest word such that $|S.v^{-1}| > |S|$, then it is easily proved by using Remark 1 that $v = a^mba^m$ and the length of $v$ is equal to $2m + 1$. Indeed, since $S = C_b$, then by Remark 1 we have $v(1) = a$ and $S_1 = S.v(1)^{-1} = S.a^{-1} = \{q_m\}$. Further, since $C_b \cap S_1 = \emptyset$, then $v(2) = a$. Applying these argumentations $m$ times, we have

$$S_m = S.v(1 \ldots m)^{-1} = S.(a^m)^{-1} = \{q_1, s_1, s_2 \ldots s_k\}.$$

Since $S_m.a^{-1} = \{q_0\} \subseteq S$, then $v(m + 1) = b$ and $S_m.b^{-1} = \{q_0, q_1\}$. If we repeat these arguments, we have that $v = a^mba^m$. 

---

**Fig. 1. Automaton $\mathcal{A}(m,k)$**
Thus $B_n = \mathcal{A}(n-2, 1)$ is the $n$-state automaton and the shortest extension word for the subset $S$ is $v$ and its length is $2m + 1 = 2n - 3$. Thus the first and the second items of the proposition are proved.

We now consider the third one. It is clear that $C_n$ is a synchronizing $n$-state automaton. Arguing by contradiction, suppose the $kn$-Balanced conjecture is true for $C_n$ within the subset $S$. Then there exists a word collection $v_1, v_2 \ldots v_m$ such that $|v_i| \leq kn$ with the following property.

$$\sum_{i=1}^{m} |S.v_i^{-1}| = m \frac{|S|}{|Q|}|Q|$$

Since $|S| = k + 1$, it is evident that there exists $j$ such that

$$|S.v_j^{-1} \cap \{q_0, q_1, q_2 \ldots q_m\}| \geq k + 1.$$  

Repeating the same argumentations as above for expanding $S$ in $\{q_0, q_1, q_2 \ldots q_m\}$, we have

$$|v_j| > |(a^m b)^k a^m| = k(m + 1) + m = k(n - k + 1) + (n - k) = (k + 1)n - k^2$$

Since $n > k^2$, then $|v_j| > (k + 1)n - k^2 > kn$ and we get the contradiction with the assumption. Hence, the $kn$-Balanced conjecture is false for the $C_n$ series. It completes the proof of the proposition.

4 Conclusions

In the previous section in the Proposition 1 we disproved the Extension and the $cn$-Extension conjecture for $c < 2$. Moreover, we disproved the $kn$-Balanced conjecture and the conjecture of Alessandro and Carpi, therefore. However, it does not mean that the Extension method can not be applied to prove the Černý conjecture or quadratic upper bound for the general case. Our results show that we can not use these ways directly only when $C_e = Q$. For instance, this method can be used to prove the upper bound $2n^2 - 7n + 7$ for automata with a connecting letter (one-cluster automaton). Furthermore, we can generalize the $kn$-Extension conjecture to the $kn$-Local-Extension conjecture as follows.

Conjecture 5 There are subsets $C_s, C_e$ and words $v_s, v_e$ such that

$$|C_s.v_s| = 1, C_s \subseteq C_e, |v_s| \leq k + kn(|C_s| - 2)$$

and $C_e.v_e^{-1} = Q, |v_e| \leq kn(n - |C_e|)$ with the following property. Each proper subset $S$ of $C_e$ is $kn$-Extendable in $C_e$, i.e. $|S.v^{-1} \cap C_e| > |S \cap C_e|$ and $|v| \leq kn$ for some word $v$. 
If $kn$-Local-Extension is true for the automaton $\mathcal{A}$, then it has a synchronizing word of length at most

$$kn(n - |C_e|) + k + kn(|C_s| - 2) + (|C_e| - |C_s|)kn = k(n - 1)^2$$

Particulary, if $k = 1$ the Černý conjecture holds true for the automaton $\mathcal{A}$. Note that the disproved $kn$-Balanced conjecture also can be generalized to the $kn$-Local-Balanced conjecture by the similar way for some subset $C_e$.

**Conjecture 6** Each proper subset $S$ of $C_e$ admits a word collection $v_1, v_2 \ldots v_m$ such that $|v_i| < kn$ with the following property.

$$\sum_{i=1}^{m} [S.v_i^{-1}] = m \frac{|S|}{|Q|}[Q],$$

where $[T]$ is a characteristic vector of the set $T$ in the linear space $\mathbb{R}^n$.

The $kn$-Local-Balanced conjecture implies the main property of $(k + 1)n$-Local-Extension, i.e. each proper subset $S$ of $C_e$ can be extended in $C_e$ by using the word $v$ of length at most $(k + 1)n$. One can easily prove this fact using ideas from the papers [9,2,1] again.

It is easy to see that the $1 \ast n$-Local-Extension conjecture is true for the automaton $\mathcal{A}(m, k)$ with $C_s = \{q_0, q_1\}, v_s = ba$ and $C_e = \{q_0, q_1, \ldots, q_m\}, v_e = a$. Moreover, by using $EA$ with this input we get the synchronizing word $a(ba^m)^{m-1}ba = v$ and $|v| = m^2 + 2 = (n - k - 1)^2 + 2$. By using Remark 1 one can easily prove that this word is a shortest synchronizing word for this automaton.

The following remark is also trivially proved.

**Remark 2.** The $cn$-Local-Balanced conjecture is true for the automaton $\mathcal{A}(m, n - m - 1)$ with the same input for $c = 1$ and this value is the minimal with this property.

In order to prove the quadratic bound for the one-cluster automata, Beal and Perrin in the paper [2] actually proved $2n$-Local-Extension conjecture, using the $1 \ast n$-Local-Balanced conjecture as an auxiliary statement. Remark 2 shows that this way, **directly** applied for this subclass of automata, gives the order $O(2n^2)$ for the upper bound. Hence, the upper bound $2n^2 - 7n + 7$ is the best polynomial upper bound for the one-cluster automata, one can achieve follow these techniques, because
\[ p(n) = 2n^2 - 7n + 7 \] is the least polynomial function with a first coefficient 2 such that
\[ p(2) = 1 = (2 - 1)^2, p(3) = 4 = (3 - 1)^2, \]
i.e. \( p(n) \) coincides with a lower bound for the one-cluster automata, which is not circular (see examples in [10]).

Nevertheless, the basic results of this paper are rejections of the conjectures, the author wants to emphasize that it also can be considered from the “positive” viewpoint, because it directs us to the probably correct way for the proof of the quadratic bounds for the length of the shortest synchronizing words for some subclass of automata. Finally, remark the conjectures of 2n-Extension, n-Local-Extension and n-Local-Balanced seem to be most interesting for proving or rejecting.

References

1. Alessandro, F., Carpi, A.: The Synchronization Problem for Strongly Transitive Automata. 12th Int. Conf. DLT, Kyoto, LNCS 5257, 240-251, 2008
2. Béal, M., Perrin D.: A quadratic upper bound on the size of a synchronizing word in one-cluster automata. In: Diekert, V. (ed.) Developments in Language Theory. LNCS, to appear. Springer, Heidelberg (2009)
3. Černý, J.: Poznámka k homogénnym eksperimentom s konečnými automatami. Matematicko-fyzikalny Časopis Slovensk. Akad. Vied 14(3) 208–216 (1964) (in Slovak)
4. Dubuc, L.: Sur les automates circulaires et la conjecture de Černý. RAIRO Inform. Theor. Appl. 32, 21–34 (1998) (in French)
5. Frankl, P.: An extremal problem for two families of sets. Eur. J. Comb. 3, 125–127 (1982)
6. Kari, J.: A counter example to a conjecture concerning synchronizing words in finite automata. Department of computer science, 15 MLH, University of Iowa, Iowa City, IA, 52242 USA, email: jjkari@cs.uiowa.edu
7. Kari, J.: Synchronizing finite automata on Eulerian digraphs. Theoret. Comput. Sci. 295, 223–232 (2003)
8. Pin, J.-E.: On two combinatorial problems arising from automata theory. Ann. Discrete Math. 17, 535–548 (1983)
9. Rystsov, I.: Quasioptimal bound for the length of reset words for regular automata, Acta Cybernetica 12 (1995), 145-152
10. Trahtman, A.: An efficient algorithm finds noticable trends and examples concerning the Černý conjecture. Lecture Notes in Computer Science, 4162(2006), 789-800
11. Volkov M. V. Synchronizing automata preserving a chain of partial orders Implementation and Application of Automata. Proc. 12th Int. Conf. CIAA 2007, Lect. Notes Comp. Sci., Springer-Verlag, Berlin-Heidelberg-New York. 2007. V.4783. P.27–37.
12. Volkov, M.V.: Synchronizing automata and the Černý conjecture. In: Martín-Vide, C.; Otto, F.; Fernau, H. (eds.) Languages and Automata: Theory and Applications. Lect. Notes Comput. Sci., vol. 5196, pp. 11–27. Springer, Heidelberg (2008)
13. Trahtman A. *The Černý conjecture for aperiodic automata* // Discrete Math. Theor. Comput. Sci. 2007. V.9. No.2. P.3–10.