Abstract. By introducing a structure of the balance sheets of the banks, which takes into account their bilateral exposures in terms of stocks or lendings, we get a structural model for default analysis. This model allows us to distinguish the exogenous and endogenous default dependence. We prove the existence and uniqueness of the liquidation equilibrium, we study the consequences of exogenous shocks on the banking system and we measure contagion phenomena. This approach is illustrated by an application to the French banking system.

Expositions bilatérales et risque systémique pour la solvabilité. En utilisant une structure des états financiers des banques qui tient compte de leurs expositions bilatérales en termes d’actions et de prêts, on développe un modèle structurel de faillite. Ce modèle permet de distinguer les facteurs exogènes et endogènes dont dépend la faillite. On prouve l’existence et l’unicité de l’équilibre de liquidation, on étudie les conséquences des chocs exogènes sur le système bancaire, et on mesure le phénomène de contagion. On illustre l’usage de cette approche en l’appliquant au système bancaire français.

1. Introduction

Following the crisis, the measurement of risk of financial institutions has become a critical question. How far will the value of a particular bank fall after an...
exogenous shock? How large would a shock on asset values have to be in order for a particular bank to go bankrupt? Consider a set of financial institutions. If these institutions are not linked, then measuring the exposure to a change in the prices of several assets would be straightforward, given information on the institution's balance sheet only. For instance, if a bank owned 100 millions in a specific corporate stock and if the market price of this stock dropped by 2%, then the bank's assets would drop from 100 to 98 millions.

The problem, as revealed emphatically in the financial crisis, is that financial institutions are linked: each bank has ownership of a set of exogenous assets as well as shares of other banks equity and loans. Therefore, measuring the risk of a financial institution needs to take into account the interconnections between banks' balance sheets, that is, to find a consistent set of balance sheet values prior to the shock and a consistent set of values afterwards.

A main deficiency of the regulations and practices before the recent financial crisis is the stand-alone computation of risk measures, that is, the evaluation of risk made independently for the different financial institutions, followed by a crude aggregation to deduce the magnitude of the global risk. This practice concerns the assets themselves: for instance, the ratings of sovereign bonds (resp. credits) are determined separately for the different countries (resp. borrowers) and are not really informative on the risk of a portfolio of such bonds (resp. credits). Loosely speaking, a portfolio of AAA bonds might be as risky as a portfolio of AA bonds if the AAA bonds were positively dependent. Similarly, the required capitals in Basel 2 were computed bank by bank, without taking into account the dependence between the risks of these institutions. Even if this practice were not the cause of the financial crisis, it participated in its development. Since they were jointly exposed to an exogenous adverse shock, the banks had to increase their required capital simultaneously, and, thus, they had an important demand for cash, or risk-free asset. As a consequence of this need for liquidity, they tried to sell quickly the stocks they possessed, which implied in two days a significant drop in market stock prices.

The post-crisis regulation (Basel 3, Financial Stability Board) highlights the importance of risk dependencies and considers financial institutions (banks and insurance companies) as parts of a system. They focus on the risk of the system (systemic risk) and the role of each institution in this systemic risk.

For a given system, like the set of European banks, say, there exist two reasons for a joint increase of risks for a large number of institutions, that is, common exogenous adverse shocks and contagion.

i) First, there can exist shocks on a factor exogenous to the system. For instance, the increase of a prime rate will have an effect on the monthly payment for adjustable rate mortgages, will imply default clustering for individual mortgages, and will diminish the results of all institutions having an important quantity of such mortgages, or associated mortgage-backed securities, in their balance sheets. The default on a sovereign bond is
another example of an exogenous shock with joint effects on the risk of the institutions.

ii) Contagion phenomena may arise in a second step and can amplify significantly the effect of exogenous shocks. They are due to the connection between the institutions through the structure of their balance sheets. For instance, a bank failure will have an impact on the institutions holding loans, bonds, stocks of this bank. In extreme cases, this may imply the failure of other institutions and so on. These contagion phenomena and chains of failures (the so-called domino effect of solvency) can result from an exogenous shock specific to an institution, such as a management error or a fraud, not necessarily from a shock on a common risk factor.

There exist two streams of literature on risk dependencies, depending on the kind of available data.

i) Some analyses are based on the values of the institutions. These values can be deduced from their balance sheets, possibly disaggregated by class of assets, or from their capitalizations if they are quoted on a stock market. However, these balance sheets give no information on the existing contagion channels. This explains why it is difficult with such reduced-form approaches to disentangle the exogenous and contagion effects. Systemic risk measures such as the CoVaR (Adrian and Brunnermeier 2008), the Marginal Expected Shortfall (MES) (Acharya et al. 2010; Brownlees and Engle 2011), the Euler allocations (see Gouriéroux and Monfort 2011 for a detailed discussion), are examples of reduced-form measures unable to identify the two components of risk dependencies. As usual, there exist two solutions to an identification problem. First, we can constrain the model by introducing identification restrictions. This approach is followed in a static framework by Rosch and Winterfeld (2008), who set ex ante to 20% the number of contaminating firms. Another identification method is used in a dynamic framework by Gagliardini and Gouriéroux (2012) and Darolles, Gagliardini, and Gouriéroux (2012). Intuitively, simultaneity effects can be disentangled from lagged exogenous factor effects, interpreted as contagion. However, such identification restrictions are always rather ad hoc.

ii) An alternative approach is based on more informative data sets. In our framework, we need balance sheets disaggregated by class of assets and counterparties, not by class of assets only. Equivalently, we need the exposures of each bank for each class of asset and each counterparty. This type of data might become available soon, owing to the reporting by banks and insurance companies required by the new regulations on financial stability. They were not available in the past, except for specific segments of bank interlending, corresponding to some payment systems. For instance, Humphrey (1986) uses data from the Clearinghouse Interbank Payments System and Furfine (2003) from Fedwire, the Federal Reserve’s large value
transfer system (see also McAndrews and Wasilyev 1995; Angelini, Maresca, and Russo 1996; Elsinger, Lehar, and Summer 2004, 2006). Other papers try to construct the missing data by using the knowledge on marginal exposures and by looking for the least favourable bilateral exposures (see, e.g., Maurer and Sheldon 1998; Upper and Worms 2004; Upper 2011; Moussa 2011). This methodology is largely used in the central banks (see, e.g., Wells 2004; Degryse and Nguyen 2007; Mistrulli 2007; Toivanen 2009). However, this methodology is based on a rather ad hoc statistical criterion, called an information criterion,\(^1\) to reconstruct the missing exposures and thus is without any financial or risk interpretation.

The aim of our paper is to provide a complete theoretical analysis that distinguishes different types of contagion channels. We extend the seminal paper by Eisenberg and Noe (2001) (see also Demange 2011) to channels involving stocks and lendings, instead of lendings only. Moreover, we allow for stakeholders, that is, shareholders and debtholders, outside the system. In section 2, we describe the system and the balance sheets of the institutions when all institutions are alive. Their interconnections can be summarized by matrices of exposures through stocks or lendings. Thus, the framework requires that the counterparties of any financial assets be identified. Note that, for a large set of assets during the financial crisis, such as credit derivatives, it was often impossible to know who the counterparties were. The regular collection of this information is a main innovation of the new European regulation. Examples of exposure matrices are given for the French banking sector. In section 3, we study the consequences on the system of an exogenous shock. This shock may imply defaults of some institutions and changes in the balance sheets of the surviving ones. We prove the existence and uniqueness of the equilibrium after the shock. We discuss how the equilibrium depends on the magnitude of the shock. In particular, we

\[\begin{align*}
\min_{\tilde{X}} & \sum_{k=1}^{n^2-n} \tilde{x}_k \ln \left( \frac{\tilde{x}_k}{\tilde{z}_k} \right) \\
\text{s.t.} & \tilde{X} \geq 0 \\
\text{s.t.} & A \tilde{X} = [d', l'].
\end{align*}\]

\(^1\) The usual method is the so-called entropy minimization method. The underlying principle is that each institution seeks for diversifying its interbank interconnections. Consider \(n\) banks whose total interbank assets and total interbank liabilities, respectively denoted \(a_i\) and \(l_i\), for \(i = 1, \ldots, n\), are known. The issue is to estimate the bilateral exposures \(x_{ij}, i = 1, \ldots, n, j = 1, \ldots, n\), considering that \(\sum_j x_{ij} = a_i\) and \(\sum_i x_{ij} = l_j\). Moreover, a usual assumption is to set \(x_{ii} = 0\) for \(i = 1, \ldots, n\). In practice, this assumption is required to avoid the situation that using the 'entropy minimization method,' which leads to almost only self-exposures (see Upper 2011). Technically, let us denote \(\tilde{X}\) the vector of size \(n^2 - n\) containing the off-diagonal elements of the bilateral exposure matrix to be estimated, \(Z\) the vector of size \(n^2 - n\) containing the off-diagonal elements of matrix \((a_i, l_j)_{i,j=1,\ldots,n}\), \(A\) a \(2n \times (n^2 - n)\) matrix such that

\[A\tilde{X} = \begin{bmatrix} a_1, & \cdots, & a_n \\ l_1, & \cdots, & l_n \end{bmatrix} \begin{bmatrix} x_{12}, & \cdots, & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1}, & \cdots, & x_{nn} \end{bmatrix} = \begin{bmatrix} d'_1, & \cdots, & d'_n \\ l'_1, & \cdots, & l'_n \end{bmatrix}.
\]
construct impulse response functions and we consider the case of stochastic shocks. In section 4, we provide a methodology to disentangle the direct and contagion systemic effects on the liquidation equilibrium. Section 5 concludes. The proofs are gathered in appendices.

2. Balance sheet and exposure

2.1. System and systemic risk
Before we discuss systemic risk and its exogenous or contagion components, it is necessary to precisely define the system. The perimeter of the system has to specify

- the type of institutions: banks/insurance-companies/hedge funds...
- the activity zone: France/Europe/World.
- the type of balance sheet, including or not the off, the intraday payments and settlements...
- the numeraire: Dollar/Euro...
- the assets which are the possible channels for contagion: loans/stocks/derivatives...
- the existing regulation: definition and management of failure, bankruptcy, unsolvency...

It is also necessary to say what are the changes of a given system considered as risky. It is possible to consider the structure of the system, for instance, the number of institutions (possibly weighted by their values), then to analyze the (weighted) number of failures following a shock and among these failures the part due to the initial shock and the part due to contagion.

However, in some situations, a defaulted bank can be merged with a safer one. This will modify the structure of this system, but not necessarily with a significant impact on the account of the system, obtained by consolidating the balance sheets of all institutions. Thus, it is necessary to choose between a global (consolidated) analysis of the system and an analysis of its structure.

2.2. Balance sheet
We consider below a simplified description of the balance sheet of the institutions $i = 1, \ldots, n$ in the system, and assume that the possible interconnections appear through either stocks, or debts. The structure of the balance sheet of institution $i$ is given in table 1.

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2 We do not distinguish in the paper among bonds, loans, and lendings. Thus, we assume a uniform debt structure.
TABLE 1
Balance sheet of Bank \( i \)

| Asset                  | Liability |
|------------------------|-----------|
| \( \pi_{i,1} Y_1 \)   | \( L_i \) |
| \( \vdots \)          | \( \vdots \) |
| \( \pi_{i,n} Y_n \)   | \( \gamma_{1,1} L_1 \) |
| \( \gamma_{1,n} L_n \) | \( \gamma_{n,1} L_1 \) |
| \( Ax_i \)            | \( L_i \) |

\( Y_i \) denotes the value of institution \( i \), \( L_i \) the total value of its debt. This debt value is equal to the nominal (contractual) value \( L_i^* \), if the institution is alive, but can be strictly smaller if there is default. The debt includes the issued bonds as well as the deposits, the interlending, and the over-the-counter loans. For expository purpose, we assume that the debt is homogeneous; that is, we do not distinguish the seniority levels, the maturities, and the different degrees of liquidity of the debt. In particular, we focus on solvency constraints, not on liquidity constraints, contrary to a large part of the theoretical literature. Institution \( i \) holds a proportion \( \pi_{i,j} \) of the total number of shares of institution \( j \), and a given proportion \( \gamma_{i,j} \) of its total debt. Thus, we assume a proportional sharing among counterparties of the debt of institution \( j \) in case of default. \( Ax_j \) gathers the asset components that are outside the system, that is, that correspond to Treasury, corporates, households, or even banks (which do not belong to this system). In general \( \sum \gamma_{i,j} \) is much smaller than 1, since a significant part of the debt is held by outsiders, such as depositors.

At this stage we do not explain how the asset and liability components are balanced. Indeed, the values of \( Y \) and \( L \) depend on the situations of the banks, that is, if they are in default, or alive.

When all institutions of the system are alive, the balance sheets are characterized by

i) the exogenous asset values: \( Ax = (Ax_1, \ldots, Ax_n)' \);
ii) the nominal values of the debt: \( L^* = (L^*_1, \ldots, L^*_n)' \);

3 Different terminologies are used in the literature, such as
- for institution \( i \): node...
- for value of the firm \( Y_i \): equity capital, equity, net equity...
- for debt \( L_i \): liability, debt obligation...
- for external asset \( Ax_i \): net worth, operating cash-flow...
4 See Gouriéroux, Héam, and Monfort (2012) for an extension to multiple seniority levels.
5 These assets may be held or correspond to an uncovered operation.
iii) the interconnections induced by stocks and debts, that is, the \((n, n)\) exposure matrices \(\Pi = (\pi_{i,j})\) and \(\Gamma = (\gamma_{i,j})\), respectively.

In this situation, we get the standard accounting relationships:

\[
\begin{align*}
L_i &= L_i^*, \\
Y_i &= A_i - L_i, \quad i = 1, \ldots, n, \\
&= \sum_{j=1}^{n} (\pi_{i,j}Y_j) + \sum_{j=1}^{n} (\gamma_{i,j}L_j^*) - L_i^* + Ax_i.
\end{align*}
\]

(1)

They provide the values of the firms when all institutions are alive:

\[
Y = (Id - \Pi)^{-1}[(\Gamma - Id)L^* + Ax],
\]

(2)

whenever \(Id - \Pi\) is invertible (see lemma A.1 in appendix B).

2.3. Exposure matrices

The banks and insurance companies regularly report detailed balance sheets intended to give shareholders, investors, and Supervisory Authorities information on their activities. The information on the structure of the balance sheets can be obtained by an appropriate treatment of the Financial Report database established by the European Banking Authority. An example of templates is provided in appendix A. The banks (and insurance companies) have to report their connections when the amount is larger than 300 MEuros, or 10\% if its total equity. The reports on balance sheets allow us to reallocate assets and liabilities by categories and counterparties. We can deduce for every quarter \(t\) since June 2007 the matrices of exposures \(\Pi_t, \Gamma_t\), as well as the vector of contractual debts \(L_t^*\) and the vector of exogenous asset components \(Ax_t\).

i) The exposure matrices depend on the selected perimeter. We provide in table 2 the exposure matrices at date 12/31/2010. They concern the system of French banks. We have kept five firms, which are large in terms of total assets to get exposure matrices of reasonable dimension. In fact, the number of financial institutions can reach several hundreds of firms. There are about 1000 banking institutions, reduced to about 200 consolidated groups for France. However, the first dozen consolidated groups represent about 95\% of the total asset value. The firms include banks quoted on the stock markets, which are banks C and D, and mutual saving banks, which are banks A and B. Bank E is mixed: E is originally a mutual saving bank with several regional mutual saving funds, but this bank has developed a publicly traded subsidiary, which represents approximately 60\% of bank E.
TABLE 2
Exposure matrices for the banking sector (at 12/31/2010)

|     | A    | B    | C    | D    | E    |
|-----|------|------|------|------|------|
| A   | B    | 0.00 | 0.00 | 0.23 | 0.14 | 0.23 |
| B   | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| C   | 0.00 | 0.00 | 1.65 | 0.21 | 0.34 | 0.34 |
| D   | 0.00 | 0.00 | 0.31 | 0.30 | 0.30 | 0.30 |
| E   | 0.00 | 0.00 | 0.71 | 0.42 | 0.39 | 0.39 |

|     | A    | B    | C    | D    | E    |
|-----|------|------|------|------|------|
| A   | 0.00 | 0.90 | 1.04 | 1.14 | 1.22 |
| B   | 0.43 | 0.00 | 0.35 | 0.38 | 0.41 |
| C   | 0.63 | 0.45 | 0.00 | 0.57 | 0.61 |
| D   | 0.57 | 0.40 | 0.46 | 0.00 | 0.54 |
| E   | 3.27 | 2.32 | 2.66 | 2.93 | 0.00 |

Let us first describe the exposure matrix for stocks $\Pi$. For pure mutual saving banks (A and B), the absence of stocks implies zero columns. Only a part of bank E is quoted, so that the corresponding column is much lower than the two columns for quoted banks, which are C and D. The diagonal reports the part of the total equity of a group held by itself.

The exposure matrix for loans $\Gamma$ has non-zero coefficients out of the diagonal: every bank is lending and borrowing from every other bank. This corresponds to a complete structure in Allen and Gale (2001) terminology. Since we consider consolidated groups, there is no self-lending and the diagonal elements of $\Gamma$ are equal to zero.

These exposure matrices can vary significantly over time. This arises, for instance after the supporting plans from governments and after new Basel 3 regulations introduced to reduce systemic risk and potential risk contagion.

ii) The knowledge of the exogenous asset components and their joint dynamics is also important, since it may be used to define the static/dynamic, deterministic/stochastic shocks of interest. We provide in figures 1–2 the evolutions of these exogenous components for A and C. The frequency is quarterly.

Two pricing methods coexist in the balance sheet reports, which are the market-to-market approach mainly for trading activities and the contractual values for credit activities. A given asset has to remain evaluated using the same method during its holding time, but many exceptions exist (see, e.g., WSJ 2011). These pricing methods differ from the liquidation values, which are implicit in Merton’s

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6 Rigorously, banks A and B also have a publicly traded subsidiary. But since they are very small, we neglected them. Moreover, it might happen that one commercial bank holds shares of a mutual saving banks following a merger. But this type of link is very uncommon.
model. For expository purpose, the consequences of these different valorizations are not considered here.

The exogenous assets fall into four main categories. The trading category gathers all elements that are marked-to-market or corresponding to short-term
perspectives. The second category corresponds to the retail activity; it mainly
consists in mortgages. Corporate loans form a third category. The last one,
sovereign, includes the assets whose counterparty is part of the public sector
(governments, states, local, etc).

We observe that banks A and C have a different structure of activity portfolios.
Bank A splits its assets similarly between retail and corporate activities. Bank C
seems to favour trading activities over other activities.

Despite the differences in term of structure, a similar trend and cycle drive the
evolution of the structure of balance sheets. We observe that the banks diminish
the part of asset dedicated to trading activity, especially for bank A.

3. Consequences of an exogenous shock

3.1. The liquidation equilibrium

Let us now consider an exogenous shock changing the initial exogenous asset
value $A^0$ into $A$. This shock implies a change of the value of the firms and
maybe the default of some institutions, which are no longer able to cover their
nominal debt and will have zero value.

The values of the firms and the values of their debts after this shock are
solutions of the system:

$$\begin{align*}
Y_i &= (A_i - L_i)^+,
L_i &= \min(A_i, L^*_i).
\end{align*}$$

The first equation takes into account the possibility of default (when $A_i < L_i$)
and the limited liability of shareholders (see Merton 1974). The second equation
shows the seniority of debtholders with respect to shareholders. This implies an
endogenous recovery rate equal to $A_i/L^*_i$ in case of default.\footnote{A part
of the literature assumes a constant exogenous recovery rate (see, e.g., Furfine 2003;
Upper and Worms 2004) possibly set to zero (Cont et al. 2010). In such a case,
the piecewise affine mapping defining the equilibrium is no longer continuous
and the existence and uniqueness of equilibrium is no longer ensured (Gouriéroux,
Laffont, and Monfort 1980). However, this assumption does not correspond to reality.
For instance, the estimation of the recovery rates by Moody’s are based on the
value of the zero-coupon bonds of the firm one month after failure. Indeed, the bond
market for a defaulted firm is still open and often shows a non-zero price
of these bonds.}

It is easily seen that system (3) reduces to the standard Merton model for a system
with a single firm and no self-holding of stocks or bonds (see appendix 3 for
the analysis of equilibrium in the standard Merton’s model).
System (3) can be written explicitly as

\[
Y_i = \left[ \sum_{j=1}^{n} (\pi_{ij} Y_j) + \sum_{j=1}^{n} (\gamma_{ij} L_j) + A x_i - L_i \right]^{+}, \quad i = 1, \ldots, n, \tag{4}
\]

\[
L_i = \min \left[ \sum_{j=1}^{n} (\pi_{i,j} Y_j) + \sum_{j=1}^{n} (\gamma_{i,j} L_j) + A x_i, L_i^{\ast} \right].
\]

We get a $2n$-dimensional piecewise linear system, which can be solved to find a consistent set of values $Y = (Y_1, \ldots, Y_n)^\prime$, $L = (L_1, \ldots, L_n)^\prime$. As a by-product, the resolution of the system provides the institutions, which are in default: $\{Y_i = 0, L_i < L_i^{\ast}\}$, the set of institutions, which are still alive $\{Y_i > 0, L_i = L_i^{\ast}\}$, and the values of each asset business line of the alive institutions. In some particular cases, it has been proved that the consistent set of values $Y$, $L$ can be interpreted as equilibrium values in an appropriate liquidation process managed by a centralized liquidator (see, e.g., Demange 2011). This justifies the terminology liquidation equilibrium values used later on in the text.

The following proposition is derived in appendix B:

**Proposition 1.** If $\pi_{i,j} \geq 0$, $\gamma_{i,j} \geq 0$, $\forall i, j$, $\sum_{i=1}^{n} \pi_{i,j} < 1$, $\forall j$, $\sum_{i=1}^{n} \gamma_{i,j} < 1$, $\forall j$, the liquidation equilibrium $Y$, $L$ exists and is unique for any choices of nonnegative $A x_i, L_i^{\ast}, i = 1, \ldots, n$.

This equilibrium concerns the values of the institutions $Y$ and the values of the debt $L$, and depends on the financial system $S = \{\Pi, \Gamma, L^{\ast}, A x\}$. Equivalently, if the numbers of shares are given and if there is a unique maturity of the debt, this is an equilibrium in the prices of stocks and digital credit default swap (CDS) written on the $n$ institutions.

The result in proposition 1 can be compared with the literature analyzing the existence and uniqueness of clearing repayment vector in the interlending market (see Eisenberg and Noe 2001; Demange 2011). In our notations, these papers assume no contagion by means of stocks, that is, $\Pi = 0$, and an exposure

8 As noted in Demange (2011), there can exist situations with negative exogenous net worth $A x_i$.

In this case, the second equation in system (4) has to be written with an additional zero threshold as (see Elsinger et al. 2006):

\[
L_i = \max \left( \min \left[ \sum_{j=1}^{n} (\pi_{i,j} Y_j) + \sum_{j=1}^{n} (\gamma_{i,j} L_j) + A x_i, L_i^{\ast} \right], 0 \right),
\]

and the regimes of default can now distinguish whether the recovery rate is equal to zero. However, this case arises if some debtors, such as depositors, are served before the banks in the system in case of default. This is an example of a model with different seniority levels (see Gouriéroux, Héam, and Monfort 2012).
matrix $\Gamma$ with all columns summing up to 1. This explains why their proofs of existence and uniqueness rely on the interpretation in terms of graph structure of stochastic matrices. Proposition 1 completes their analysis in two respects: by introducing interconnection by means of stocks and by considering creditors outside the system. The proof is based on a necessary and sufficient condition for the invertibility of a piecewise linear function.

Let us illustrate the liquidation equilibrium as a function of the exogenous asset components for a system of two banks. For expository purposes, it is more appropriate to write the liquidation equilibrium conditions in terms of variables $Y$ and $\Delta L = L^* - L$. Let us also denote $\Delta Ax = Ax - Ax^*$, where $Ax^* = (Id - \Gamma)L^*$. We get four possible regimes with the following liquidation equilibrium values:

**Regime 1:** No default. We get $Y = (Id - \Pi)^{-1}\Delta Ax$, $\Delta L = 0$, and this regime occurs iff

$$\Delta Ax \in (Id - \Pi)(IR^+)^2 \equiv C_1.$$

**Regime 2:** Joint default. We get $Y = 0$, $\Delta L = (\Gamma - Id)^{-1}\Delta Ax$, and this regime occurs iff

$$\Delta Ax \in (\Gamma - Id)[0; L^*_1] \times [0; L^*_2] \equiv C_2.$$

**Regime 3:** Default of bank 1 only. We get $Y_1 = 0$, $\Delta L_2 = 0$, and

$$\begin{pmatrix} \Delta L_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \gamma_{1,1} - 1 & -\pi_{1,2} \\ \gamma_{2,1} & 1 - \pi_{2,2} \end{pmatrix}^{-1} \Delta Ax.$$

The regime occurs if

$$\Delta Ax \in \begin{pmatrix} \gamma_{1,1} - 1 & -\pi_{1,2} \\ \gamma_{2,1} & 1 - \pi_{2,2} \end{pmatrix}[0; L^*_1] \times IR^+ \equiv C_3.$$

**Regime 4:** Default of bank 2 only. We get $\Delta L_1 = 0$, $Y_2 = 0$, and

$$\begin{pmatrix} Y_1 \\ \Delta L_2 \end{pmatrix} = \begin{pmatrix} 1 - \pi_{1,1} & \gamma_{1,2} \\ -\pi_{2,1} & \gamma_{2,2} - 1 \end{pmatrix}^{-1} \Delta Ax.$$

The regime occurs iff

$$\Delta Ax \in \begin{pmatrix} 1 - \pi_{1,1} & \gamma_{1,2} \\ -\pi_{2,1} & \gamma_{2,2} - 1 \end{pmatrix} IR^+ \times [0; L^*_2] \equiv C_4.$$
The sets $C_j, j = 1, \ldots, 4$, are truncated positive cones. They are generated by the following pairs of vectors: $C_1$ is generated by $(u_1, u_2)$, $C_4$ is generated by $(u_4, u_1)$, $C_2$ is generated by $(u_3, u_4)$, and $C_3$ is generated by $(u_2, u_3)$, where

$$u_1 = \begin{pmatrix} 1 - \pi_{1,1} \\ -\pi_{2,1} \end{pmatrix}, \quad u_2 = \begin{pmatrix} -\pi_{1,2} \\ 1 - \pi_{2,2} \end{pmatrix}, \quad u_3 = \begin{pmatrix} \gamma_{1,1} - 1 \\ \gamma_{2,1} \end{pmatrix}, \quad u_4 = \begin{pmatrix} \gamma_{1,2} \\ \gamma_{2,2} - 1 \end{pmatrix}. \quad (5)$$

The conditions in proposition 1 ensure that these truncated cones do not overlap. The regimes are represented in figure 3.

There is no default for both banks if the exogenous asset components $Ax_1$ and $Ax_2$ are sufficiently large. We observe that there exist thresholds

$$\overline{Ax}_1 = Ax_1^* + \frac{1 - \pi_{1,1}}{\pi_{2,1}} Ax_2^*, \quad \overline{Ax}_2 = Ax_2^* + \frac{1 - \pi_{2,2}}{\pi_{1,2}} Ax_1^*,$$

such that if $Ax_i > \overline{Ax}_i$, institution $i$ will not default whatever the exogenous asset component of the other institution.

The proof of the existence and uniqueness in the case of two banks is easy to understand. Indeed, there exists a unique equilibrium if the cones defining the regimes in figure 3 do not overlap. Since the sign of $det(u, v)$, where $u, v$ are
vectors of \( \mathbb{R}^2 \), gives the direction of rotation from \( u \) to \( v \), the condition is simply that the four determinants \( \det(u_1, u_2), \det(u_2, u_3), \det(u_3, u_4) \) and \( \det(u_4, u_1) \) – have the same sign. It is easily checked that all these determinants are strictly positive under the assumption on the exposure matrices given in proposition 1.

The conditions on exposure matrices given in proposition 1 are sufficient for the existence and uniqueness of the liquidation equilibrium. When they are not satisfied, we do not have necessarily a unique liquidation equilibrium. For instance, we have a multiplicity of liquidation equilibria in Regime 1 if \( \Pi_1 \) has a unitary eigenvalue, in particular if \( \sum_{i=1}^{n} \pi_{i,j} = 1, j = 1, \ldots, n \), since \( (I - \Pi) \) is not invertible. This is easily understood: if \( \sum_{i=1}^{n} \pi_{i,j} = 1, j = 1, \ldots, n \), the stock cross-holdings are so large, that we have, in fact, a unique group. The multiplicity of liquidation equilibria reveals that the consolidation step has not been well done.\(^9\)

### 3.2. Impulse response analysis and stochastic shock

#### 3.2.1. Comparative statics

We can now discuss how the equilibrium responds to shocks on the exogenous asset components. We have the following monotonicity property:

**Proposition 2.** If

\[
\pi_{i,j} \geq 0, \quad \gamma_{i,j} \geq 0, \quad \forall \ i, j, \quad \sum_{i=1}^{n} \pi_{i,j} < 1, \quad \forall \ j, \quad \sum_{i=1}^{n} \gamma_{i,j} < 1, \quad \forall \ j,
\]

the equilibrium values \( Y_i, L_i, i = 1, \ldots, n \) are non-decreasing functions of the asset components \( Ax_j, j = 1, \ldots, n \), for any given nominal debt and exposure matrices.

**Proof.** See the appendix on line.

This result was expected. An increase of the exogenous component decreases the default occurrence, increases the value of the firm, and also the recovery rate of any defaulting firm. It has been shown in Eisenberg and Noe (2001), that the debt level \( L \) is a componentwise concave function of \( Ax \), when \( \Pi = 0 \) and \( \sum_j Y_{i,j} = 1, \forall i \). This result is no longer valid when there is a feedback effect by means of stock cross-holdings.

#### 3.2.2. Impulse response analysis

Let us now consider an initial exogenous asset component \( Ax_0 \), a (multidimensional) direction of shocks \( \beta = (\beta_1, \ldots, \beta_n)' \) and the new exogenous asset

\(^9\) The counterexamples provided in Eisenberg and Noe (2001, appendix 2), and Demange (2011, 11) are of the same type.
components defined by

$$Ax(\delta) = Ax^0 + \delta\beta,$$

where $\delta$, $\delta > 0$, is the magnitude of the deterministic shock. The impulse response$^{10}$ explains how the equilibrium values $Y$ and $L$ depend on $\delta$, for given $\beta$ and initial conditions.

As is usual in the current regulation, the effects of the shocks are analyzed with crystallized, that is, fixed, bilateral exposure matrices. From an economic point of view, this might be interpreted as the effect of an immediate not anticipated shock. From a practical point of view, it is difficult for the regulator to make reasonable assumptions about the reaction of the financial institutions to the different types of shocks, or to get reliable information on the future strategies of the institutions under stress. However, these reactions are partly taken into account if these exercises are performed on a regular basis, monthly or quarterly, with updated bilateral exposure matrices.$^{11}$

A simple case is that of uniformly adverse shocks, when all components of the direction of the shocks are non-positive: $\beta_i \leq 0$, $\forall i$. Indeed, by proposition 2, we can apply the monotonicity property and deduce a minimal value of $\delta$ ($\delta^*_1$, say) for which we observe the first default, then a minimal value, $\delta^*_2$, say, for which we observe the first two defaults, and so on. By studying the thresholds of magnitude $\delta$ of the shock that trigger default, we build the inverse impulse response. Central bankers call this approach ‘reverse stress test’ (see, e.g., BIS 2009 or FSA 2009). This is illustrated in figure 4 for a system of two banks.

The initial situation corresponds to a banking system in a joint no default regime. A direction of shock $\beta = (\beta_1, \beta_2)'$, where $\beta_1 \leq 0$, $\beta_2 \leq 0$, defines a half-line with negative slope. This line can cross between 0 and 2 other regimes. For instance, the directions $\beta_1$ and $\beta_2$ displayed in figure 4 cross two other regimes, whereas direction $\beta_3$ crosses only one.

Figure 5 reports the impulse response functions with a direction $\beta^2$ for different characteristics of the equilibrium, which are the values of the exogenous asset components, the default indicators, the values of the firms and the values of the debts. The magnitude $\delta$ of the shock is on the x-axis. We set

$$\Pi = \begin{pmatrix} 0.05 & 0.37 \\ 0.46 & 0.07 \end{pmatrix}; \quad \Gamma = \begin{pmatrix} 0.07 & 0.13 \\ 0.15 & 0.00 \end{pmatrix}; \quad Ax^0 = \begin{pmatrix} 2.9 \\ 2.2 \end{pmatrix}; \quad L^* = \begin{pmatrix} 2.5 \\ 2 \end{pmatrix}$$

$$\beta^2 = \begin{pmatrix} -0.003 \\ -0.005 \end{pmatrix}.$$

$^{10}$ An impulse response describes the reaction of a system to a function of time or some other independent variable. The latter interpretation is used in our static framework.

$^{11}$ The histories of bilateral exposure matrices may be used to introduce a dynamic definition of shocks and to understand how the financial institutions adjust their strategies.
FIGURE 4 Directions of the shocks

The effects of the shocks on the external asset component are reported in the North-West panel. As the values of external asset decrease for both bank 1 and bank 2, the values of the banks plotted on the South-West panel decrease and stop at 0, triggering first the default of bank 2, then the default of bank 1. When in default, the value of the bank remains constant, equal to zero, but the value of its debt, that is, the recovery rate, starts decreasing (South-East panel). The status of a bank (North-East panel) is 0 when it is alive and 1 otherwise: bank 1 defaults first at $\delta_1^* \approx 125$, then bank 2 defaults at $\delta_2^* \approx 250$. We also observe the convexity of the value of the bank and the concavity of the value of its debt corresponding to the call and put interpretations, respectively (see appendix C, d). Some components of $Ax$ may become negative when $\delta$ is too large. For this reason, we increase $\delta$ up to the first zero component of $Ax$.

3.2.3. Stochastic shock

We can also consider a stochastic new situation $Ax$ and the associated stochastic shock $Ax - Ax^0$. This situation is characterized by the multivariate distribution of the vector of exogenous asset components. Then we can deduce explicitly the distribution of the equilibrium values $Y$ and $L$. More precisely, let us introduce the regime indicator, $z = (z_1, \ldots, z_n)$, where $z_i = 1$ if bank $i$ defaults, $z_i = 0$
otherwise. It is shown in appendix B that regime $z$ occurs iff

$$\Delta Ax = Ax - Ax^* \in C(z),$$

where $Ax^* = (Id - \Gamma)L^*$, $C(z)$ is a truncated cone, function of $\Pi$, $\Gamma$ and $L^*$, defined in appendix B. Therefore the probability to be in regime $z$ is:

$$\mathbb{P}(\text{regime } z) = \mathbb{P}[Ax - Ax^* \in C(z)].$$

Then it is easy to derive the conditional distribution of $(Y, L)' = X$, say, given the regime. Indeed, in regime $z$ we have

$$Y_i = 0, \quad \text{if } z_i = 1, \quad L_i = L_i^*, \quad \text{if } z_i = 0.$$ 

Let us denote by $X_z$ the $n$-dimensional vector obtained by stacking the value of $Y_i$ for the banks such that $z_i = 0$, and the value of $L_i$, for the banks such that $z_i = 1$. It is proved in appendix B that $X_z$ is an invertible linear function of $\Delta Ax$
in regime $z$:

$$X_z = B_z \Delta Ax,$$

where $B_z$ is a matrix function of $\Pi$ and $\Gamma$, whose expression is given in appendix B. Therefore, in regime $z$, the vector $X_z$ has an $n$-dimensional distribution with density

$$h(x_z) = \frac{1}{\text{det}(B_z)} f \left( B_z^{-1} x_z \right),$$

where $f$ denotes the density of $\Delta Ax$.

To summarize, the joint distribution of $(Y, L)$ is a mixture of $2n$-dimensional distributions, which are continuous for $n$ coordinates and discrete for the other ones.

We get a complicated uncertainty, which is well illustrated by considering, for instance, the probability of default ($PD$) of a given bank. The probability of default of bank 1, say, is given by

$$PD_1 = \mathbb{P}(z_1 = 1) = \sum_{z/z_1=1} \mathbb{P}($$ regime $z$)

$$= \mathbb{P}($$ regime $(1, 0, \ldots, 0)$) + \sum_{i \neq 1} \mathbb{P}($$ regime $z_1 = 1, z_i = 1, z_j = 0, j \neq 1, i)$

$$+ \sum_{i,j/i\neq j\neq 1} \mathbb{P}($$ regime $z_1 = 1, z_i = 1, z_j = 1, z_k = 0, k \neq 1, i,j)$

$$+ \cdots$$

Thus, the standard $PD$ can be decomposed to highlight the number of banks, which are in default jointly with bank 1

$$PD_1 = PD_1(1) + PD_1(2) + \cdots + PD_1(n), \quad \text{say.} \quad (7)$$

Similarly, we may compute the probability of a joint default of two banks, $PD_{1,2}$, say, if these banks are 1 and 2, and decompose it according to the total number of defaults in the system. Such a decomposition may be used to complete the standard analysis of default correlation.
4. Contagion measure

4.1. The standard analysis in a linear framework
Let us consider linear system (1), which can be rewritten:

\[ Y = \Pi Y + (\Gamma - Id)L^* + Ax^0, \text{ say,} \tag{8} \]

and let us introduce a deterministic shock on the exogenous asset component:

\[ Ax = Ax^0 + \delta \beta, \tag{9} \]

where \( \beta \) denotes the direction of the shock and \( \delta \) its magnitude. The effect on the equilibrium values of the firms is

\[ \Delta Y = \delta (Id - \Pi)^{-1} \beta. \tag{10} \]

This shock is linear in \( \delta \) and involves both a direct effect of the shock and a contagion effect. To disentangle these two components, we usually introduce a recursive version of model (8):

\[ Y_k = \Pi Y_{k-1} + (\Gamma - Id)L^* + Ax^0, \tag{11} \]

leading to the equilibrium solution (8), when \( k \) tends to infinity, assuming that matrix \( \Pi \) has eigenvalues with modulus strictly smaller than one. Then we compute the short-term effect of the shock, equal to \( \delta \beta \), and decompose the total effect as

\[ \Delta Y = \delta (Id - \Pi)^{-1} \beta = \delta \beta + \delta \left( \sum_{j=1}^{\infty} \Pi^j \right) \beta. \tag{12} \]

In this linear framework, both the direct and the contagion effects are linear in the direction \( \beta \) of the shock and its magnitude \( \delta \). In particular, they can easily be deduced from the shocks specific to each institution, since \( \beta = \beta_1(1, 0, \ldots, 0)' + \beta_2(0, 1, 0, \ldots, 0)' + \cdots + \beta_n(0, 0, \ldots, 0, 1)' \). Moreover, the two components in (12) can be obtained directly without specifying an underlying recursive process. Indeed, the direct effect is simply obtained by setting \( \Pi = 0 \) in formula (4), that is, by cancelling the contagion channel in terms of stocks. Note that the direct effect is independent of \( \Gamma \) and therefore can be also computed under \( \Pi = \Gamma = 0 \).
4.2. How to disentangle exogenous and contagion effects?

Let us consider an initial financial system with exogenous asset components $Ax^0$, in which all institutions are alive. As noted earlier, the equilibrium values are

$$Y^0 = (Id - \Pi)^{-1}[(\Gamma - Id)\Lambda^* + Ax^0].$$

(13)

The equilibrium values with contagion, when $Ax = Ax^0 + \delta \beta \geq 0$, are the solutions of system (4). They will be denoted by $Y(S^0; \delta, \beta)$ and $L(S^0; \delta, \beta)$, where $S^0 = \{\Pi, \Gamma, \Lambda^*, Ax^0\}$ characterizes the financial system.

It is easy to suppress the contagion channel in our framework, that is, to get $\Pi = \Gamma = 0$. Indeed, let us assume that, in the initial financial system $S^0$, the institutions cash their stocks and bonds of the other institutions. The balance sheet becomes

| Asset | Liability |
|-------|-----------|
| $\tilde{Ax}^0_i$ | $L_i$ |

where $\tilde{Ax}^0_i = \Pi_i Y^0_i + \Gamma_i \Lambda^*_i + Ax^0_i = Y^0_i + \Lambda^*_i$. We have eliminated the contagions by setting $\Pi = 0$ and $\Gamma = 0$ while keeping the same value of the firm. Let us now apply the exogenous shock to this new financial system $\tilde{S}^0 = \{0, 0, \Lambda^*, \tilde{Ax}^0\}$. We get another equilibrium $Y(\tilde{S}^0; \delta, \beta)$ and $L(\tilde{S}^0; \delta, \beta)$, such that

- institution $i$ is alive if and only if $Y^0_i + \delta \beta_i > 0$,
- $\tilde{Y}_i = (Y^0_i + \delta \beta_i)^+$,
- $\tilde{L}_i = \min(\tilde{Ax}^0_i + \delta \beta_i; \Lambda^*_i)$.

By comparing the two liquidation equilibria associated with $S^0$ and $\tilde{S}^0$, respectively, we get the effect of contagion. This approach can be applied to different aggregate measures of the final state of the system, such as

i) the number of non-defaulted banks:

$$N_0 = \sum_{i=1}^{n} 1_{Y_i > 0} = \sum_{i=1}^{n} 1_{L_i - \Lambda_i^* = 0},$$

(14)

where $1_A$ denotes the indicator function of $A$.

ii) the total value of the banks:

$$\tilde{Y} = \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} Y_i 1_{Y_i > 0},$$

(15)

which is a criterion appropriate for shareholders.

12 Implicitly, we assume liquid markets for stocks and bonds. This assumption is consistent with the conditions of $\sum_i \pi_{i,j} < 1$ and $\sum_i \gamma_{i,j} < 1$, which means that a part of stocks and bonds issued by institutions are held by external agents (households, corporations...).
iii) the total value of the debt:

\[ L = \sum_{i=1}^{n} L_i = \sum_{i=1}^{n} L_i^* I_{y_i > 0} + \sum_{i=1}^{n} L_i I_{L_i < L_i^*}, \]  

which is a criterion appropriate for bondholders

All these scalar measures of the global state of the banking system are non-decreasing functions of \( y_i, L_i, i = 1, \ldots, n \), and, therefore, also non-decreasing functions of the exogenous asset components \( Ax_i \), by the monotonicity property established in section 3.2. If we consider, for instance, the number of non-defaulted banks, we compute \( N_0(S_0; \delta, \beta) \), \( N_0(S^0; \delta, \beta) \), which are decreasing functions of \( \delta \) (if \( \beta_i < 0, \forall i \)), and decompose the total effect \( N_0(S_0; \delta, \beta) \) into the direct effect \( N_0(S_0; \delta, \beta) \) and the contagion effect equal to the difference \( N_0(S_0; \delta, \beta) - N_0(S^0; \delta, \beta) \). The (absolute and percent) contagion effects depend on the initial configuration \( S_0 \), but also on the direction and magnitude of the shock. This dependence is rather complex, and, as already noted, the contagion effect is not a linear function of \( \beta \). Therefore, we cannot deduce the effect of a global shock from the effects of the specific shocks. Because of this dependence of the shock, we have to consider with care:

- the ranking of Systematically Important Financial Institutions (the so-called SIFIs in the terminology of the Financial Stability Board),
- the distinction between ‘shock transmitters’ and ‘shock absorbers’ (Nier et al. 2007),
- a definition of contagion measure based on a unique type of shock, such as the so-called market shock (Cont, Moussa, and Santos 2010).

The previous approach can also be applied by partly cancelling contagion channels. For instance, we can set to zero, that is, cash, all cross-holdings between the bank and insurance sectors to evaluate the effect of bancassurance business model on systemic risk. We may also cancel all links that do not involve a given institution \( i \) to focus on the contagion channel passing by this institution.

Let us illustrate the contagion effect in the case of two banks with nominal debts \( L^* = (2, 3)' \) and values \( Y^0 = (1, 1)' \) in all experiments below. We consider the following set of exposure matrices:

- Set 1: \( \Pi = (0, 30\% \ 0)' \), \( \Gamma = (0 \ 0)' \)
- Set 2: \( \Pi = (0 \ 0)' \), \( \Gamma = (0, 30\% \ 0)' \)
- Set 3: \( \Pi = (0 \ 30\% \ 0)' \), \( \Gamma = (0 \ 0)' \)
- Set 4: \( \Pi = (0 \ 0)' \), \( \Gamma = (0 \ 30\% \ 0)' \)
• Set 5: $\Pi = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\Gamma = \begin{pmatrix} 0 & 0 \\ 30\% & 0 \end{pmatrix}$
• Set 6: $\Pi = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\Gamma = \begin{pmatrix} 0 & 0 \\ 0 & 30\% \end{pmatrix}$.

For each set, both banks are assumed alive and the exogenous asset components are deduced by the formula $A x^0 = (I_d - \Pi) Y^0 + (I_d - \Gamma) L^*$. We consider a shock specific to bank 1, that is: $\beta = (-1, 0)'$. We report in figures 6 and 7, the impulse response functions for the total debt, with their decompositions into direct and contagion effects.

The contagion effect depends on the system that is considered. It is always more pronounced on the total value of the debt. However, the effect of contagion is far from clear in a general framework. It depends on the form of the exposure matrices, but also on the type of exogenous shock, deterministic or stochastic, of its direction and magnitude (see, e.g., Dubecq and Gouriéroux 2012). As an illustration, let us consider a small stochastic shock such that $\delta = 1$ and $\beta$ is such that the system stays in the no joint default regime. We know that there is no effect of debt exposure and that the decomposition of the effect of the shock on the value of the firm corresponds to equation (12). The percent contagion effect on the expected total value of the firm is

$$1 - \frac{1'\mathbb{E}(\beta)}{1'(I_d - \Pi)^{-1}\mathbb{E}(\beta)},$$

where $\mathbb{E}(\beta)$ denotes the expected multivariate shock and $1 = (1, \ldots, 1)'$. In fact, the terminology percent contagion effect is misleading. Indeed, if this quantity is between 0 and 1 when the expected shocks are uniformly adverse $\mathbb{E}[\beta_i] \leq 0$, $i = 1, \ldots, n$, this is no longer the case in general. The contagion effect on the total value of the firm is more difficult to discuss for stochastic shock because of the dependence between direct and contagion effects (see also Darolles, Gagliardini, and Gouriéroux 2012). Indeed, we get from equation (12)

$$V(1' \Delta Y) = V(1' \beta) + V \left[ 1' \left( \sum_{j=1}^{\infty} \Pi^j \right) \beta \right] + 2 \times \text{Cov} \left( 1' \beta; 1' \left( \sum_{j=1}^{\infty} \Pi^j \right) \beta \right)$$

$$= 1' V(\beta) 1 + 1' \left( \sum_{j=1}^{\infty} \Pi^j \right) V(\beta) \left( \sum_{j=1}^{\infty} \Pi^j \right)' 1 + 2 \times 1' V(\beta) \left( \sum_{j=1}^{\infty} \Pi^j \right)' 1.$$
4.3. The effect of contagion on reverse stress-test
For a given bank, bank 1, say, and a shock specific to the exogenous asset component of this bank,

\[ Ax = Ax^0 - \delta(Ax_1, 0, \ldots, 0)^t, \]  

(17)
where $\delta \in (0, 1)$, there is a minimal value of $\delta$, which implies the first failure of a bank in the system. We might expect that this bank will be bank 1, but this is not always the case because of the stock interconnections. For instance, in a system of two banks with balance sheets such as

FIGURE 7 Decomposition of the impulse response functions for sets 4, 5, and 6
Bilateral exposures and systemic solvency risk

TABLE 3
Reverse stress-tests for the banking sector (at 12/31/2010); δ in percent

| Specific shock on | A  | B  | C  | D  | E  |
|-------------------|----|----|----|----|----|
| First bank to fail|    |    |    |    |    |
| δ (with contagion,%) | 4.349 | 5.495 | 4.359 | 4.161 | 3.085 |
| δ (without contagion,%) | 4.349 | 5.495 | 4.432 | 4.174 | 3.097 |
| 1 − L_i^* / Ax_i^0 (%) | −1.01 | 1.61 | 1.64 | 3.05 | −5.88 |

• Bank 1: π_{1,1} = π_{1,2} = γ_{1,1} = γ_{1,2} = 0, L_1^* = 100 and Ax_1 = 200
• Bank 2: π_{2,1} = 50%, π_{2,2} = γ_{2,1} = γ_{2,2} = 0, L_2^* = 100 and Ax_2 = 55,

bank 2 will first fail for a decrease of 5% of Ax_1.

To illustrate the effect of contagion, we consider the French banking system. For each bank \( i \), we consider: \( i \) the reverse stress-test of the initial system for a specific shock, \( ii \) the reverse stress-test when the contagion effects are canceled, \( iii \) the quantity \( 1 − L_i^* / Ax_i^0 \). The levels of exogenous asset components and nominal debts at 12/31/2010 (in trillions of Euros) are:

\[
Ax_0 = ((1 : 13; 0 : 60; 1 : 15; 2 : 06; 1 : 58)',
L^* = (1 : 14; 0 : 59; 1 : 13; 2 : 00; 1 : 67)'.
\]

The results of the reverse stress-tests are given in table 3.

When we compare the δs with and without contagion, two cases arise. First, for mutual saving banks, there is no significant effect of contagion, since the δs are equal. Second, the effect of contagion lowers the δ for the three other banks. The difference in δ corresponds to 826, 260, and 207 millions of euros, respectively, for these banks. These amounts represent 1.99%, 0.30%, and 0.46% of their prudential capital, respectively.

A positive value in the last row means that the initial exogenous assets are sufficient to cover any possible loss on the debt and stockholding of the other banks, whereas a negative value points out that interbank assets are needed to cover all the debt. Thus, the solvency of A and E seems to be more sensitive to the health of the banking sector, whereas B and C, and particularly bank D are solvable without it. This is consistent with the larger row in the exposure matrices of banks A and E. The small contagion effects observed above are contingent to the selected system. We have retained the five large banks in terms of total assets. It is often said in the literature that the size of the bank is a major determinant for systemic risk. This feature is likely not valid for the French banking system. Indeed, a smaller bank has stronger connections especially to banks A and B. This extension of the perimeter is out of the scope of the present paper.
The optimal \( \delta \) values in table 3 admit explicit forms. Indeed, given the initial configuration \( S^0 \) and a direction of shock with non-positive components, we have

\[
\delta = \max_{d} \left\{ \text{there exists } i \text{ such that} \right. \\
[(Id - \Pi)^{-1}((\Gamma - Id)\Gamma^* + Ax^0 + d\beta)]_i = 0 \\
[(Id - \Pi)^{-1}((\Gamma - Id)\Gamma^* + Ax^0 + d\beta)]_j > 0, \forall j \neq i \\
\left. \max \left\{ [(Id - \Pi)^{-1}((\Gamma - Id)\Gamma^* + Ax^0)]_i/[(Id - \Pi)^{-1}\beta]_i \right\} \right.
\]

4.4. Decomposition of a probability of default

Let us consider the financial system \( S^0 = \{\Pi, \Gamma, L^*, Ax^0\} \) and a stochastic shock, that is, a new exogenous asset component \( Ax \) that is stochastic. After the shock, we get the system \( S = \{\Pi, \Gamma, L^*, Ax\} \). The equilibrium values \( Y(S), L(S) \) are stochastic (see section 3.2.iii). In particular, we can compute the probability of default of bank \( i: PD_i \).

As noted in section 4.2, it is possible to cancel the contagion channel by considering the virtual initial financial system \( \tilde{S}^0 = \{0, 0, L^*, \tilde{Ax}^0\} \), where \( \tilde{Ax}^0 = \Pi Y^0 + \Gamma L^* + Ax^0 = Y^0 + L^* \). Then we can apply the same shock to system \( \tilde{S}^0 \) in order to get a virtual financial system after shock \( \tilde{S} = \{0, 0, L^*, Ax - Ax^0 + \tilde{Ax}^0\} = \{0, 0, L^*, Ax + \Pi Y^0 + \Gamma L^*\} \) and define the probability of default without contagion (or direct PD) as \( PD^d_i \). A measure of the effect of contagion on the probability of default is

\[
K_i = \frac{PD_i}{PD^d_i}.
\]

Contagion has an increasing effect (resp. decreasing effect), if \( K_i > 1 \) (resp. \( K_i < 1 \)).

As an illustration of the effect of stochastic shocks on the probability of default, we consider the same initial situation of the French banking system as in section 4.3. Then, the stochastic shocks are introduced on the exogenous asset components as in the standard Vasicek extension of the Value-of-the-Firm model (Vasicek 1987). We assume that

\[
\log(Ax_i) = \log(Ax^0_i) + u_i, \quad i = 1, \ldots, n,
\]

where the stochastic \( u_i \)'s are Gaussian. We set \( \mathbb{E}(u) = 0, \sigma^2 = 0.0141 \).

The probability of default with and without contagion can be easily derived by simulation and can be converted into ratings. For instance, for this type of
TABLE 4
Simulated probabilities of default for the banking sector (at 12/31/2010); 100,000 simulations

|       | PD (in %) Without connection | PD (in %) With connection | ΔPD |
|-------|-----------------------------|---------------------------|-----|
| A     | 0.058                       | 0.023                     | −0.035 |
| B     | 0.001                       | 0.000                     | −0.001 |
| C     | 0.040                       | 0.001                     | −0.039 |
| D     | 0.073                       | 0.002                     | −0.071 |
| E     | 1.017                       | 1.059                     | +0.042 |

TABLE 5
Simulated probabilities of joint default for the banking sector (at 12/31/2010); 100,000 simulations

|       | Without | A     | B     | C     | D     | E     |
|-------|---------|-------|-------|-------|-------|-------|
| A     | 0.058   | 0.    | 0.    | 0.    | 0.    | 0.    |
| B     | 0.001   | 0.    | 0.    | 0.    | 0.    | 0.    |
| C     | 0.04    | 0.04  | 0.072 | 0.001 | 1.016 |
| D     | 0.072   | 0.    | 0.001 | 0.    | 0.    |
| E     | 1.016   | 1.022 | 1.059 | 1.057 | 1.016 |

Considering the variation of individual probabilities of default in table 4, the effect of interconnection is not uniform across banks. Except for bank E, being interconnected lowers the probability of default. The interconnections can be seen as an efficient diversification of risk, since the stochastic shocks $u_i$, are independent. The probabilities of joint default are reported table 5, respectively without and with interconnections. Without interconnections, joint default are almost absent. With connections, individual defaults are no longer the rule. Pairwise defaults appear. Contrary to the situation without interconnections, regimes with at least three banks in default have strictly positive probability.
5. Concluding remarks

Until now very little has been known about the actual structure of bilateral exposures in the finance and insurance sectors. The new regulations for financial stability require a periodic reporting by banks and insurance companies about their counterparties by class of assets, possibly distinguished by maturities and seniorities. This information might be used to quantify the bilateral exposures in terms of stocks, lendings, or derivatives. In our paper, we considered a simplified framework, which does not distinguish seniorities, maturities, and more generally liquidity features, and focus on solvency risk. We saw how such a structural information on the balance sheet can be used to define the system of banks and its structure after an exogenous shock, the so-called liquidation equilibrium. We also saw that this information can be used to decompose the systemic effect of an exogenous deterministic or stochastic shock into a direct and a contagion effect, respectively. Such a decomposition is appealing for the interpretation of stress tests and reserve for systemic risk. It is also appealing in a perspective of controlling systemic risk. Indeed, an alternative to a control of systematic exogenous factors, such as the sovereign Greek debt for instance, is the control of the exposure matrices. Such a policy was followed by the Federal Reserve of New York to avoid the forced liquidation of LTCM. In order to avoid an uncontrolled transmission of losses from LTCM to its counterparties that could put the financial sector in distress, the FED asked several private banks to take control of this institution, that is, to change the matrix of bilateral stock exposures (see Greenspan 1998; McDonough 1998).

Appendix A: Balance sheet and large exposure European templates

In order to collect data, the European National Supervisory Authorities could use the templates of the Committee of European Banking Supervisors (CEBS). These templates are filled by banks and controlled by National Supervisory Authorities.

Figures A.1 and A.2 are extracts from the balance sheet. The first one is the general structure of the asset side with in column: the financial item, the accounting rules (IFRS for International Financial Reporting Standards and IAS for International Accounting Standards), the reference to a sub-table where the amount is decomposed, and, in the last column, the amount. Figure A.2 is the sub-table for Financial Assets Held for Trading. The decomposition mixes financial instruments (Equity, Debt...) and nature of counterparties (Central Bank, General governments...). Besides the general accounting rules and the total market-to-market amount, a decomposition between price and volume is given.
Figure A.1 is a table for large exposures report. In this template, the main counterparties (of the filling bank) are reported in column. The rows report the name of the counterparty, the total exposed amount, a breakdown of this amount across financial instruments, provision...
Appendix B: Proof of proposition 1

In the case of \( n \) banks, the \( 2^n \) regimes can be indexed by a sequence \( z = (z_1, \ldots, z_n) \) of 0 and 1, where \( z_i = 1 \) if bank \( i \) defaults, \( z_i = 0 \) otherwise. Let us define the matrix \( Q(z) \) as follows. The \( i \)th column of matrix \( Q(z) \) is the \( i \)th column of \( \Gamma \) when \( z_i = 1 \), of \( \Pi \) when \( z_i = 0 \).

B.1. A preliminary lemma

**Lemma A.1.** If \( \pi_{i,j} \geq 0, \gamma_{i,j} \geq 0, \forall i,j, \sum_{i=1}^{n} \pi_{i,j} < 1, \forall j, \sum_{i=1}^{n} \gamma_{i,j} < 1, \forall j, \text{ then } \det[Id - Q(z)] > 0, \forall z. \)

**Proof.** By the assumptions in proposition 1, the matrices \( Q'(z) \) have non-negative coefficients, which sum up to a value strictly smaller than 1 per row. By applying the Perron-Frobenius theorem, we deduce that the eigenvalues of \( Q'(z) \), which are also equal to the eigenvalues of \( Q(z) \), have a modulus strictly smaller than 1. Therefore, the eigenvalues of \( Id - Q(z) \) are either complex conjugates, or real positive, and their product equal to \( \det[Id - Q(z)] \) is strictly positive. QED

In particular, the matrices \( Id - \Pi \) and \( Id - \Gamma \) are invertible.
B.2. Existence and uniqueness

The first equation of system (4) can be rewritten as

\[
Y_i = \left[ \sum_{j=1}^{n} \pi_{i,j} Y_j - \sum_{j=1}^{n} \gamma_{i,j} \Delta L_j + Ax_i - L_i + \sum_{j=1}^{n} \gamma_{i,j} L_j^* \right]^{+}
\]

\[
= \left[ \sum_{j=1}^{n} \pi_{i,j} Y_j - \sum_{j=1}^{n} \gamma_{i,j} \Delta L_j + \Delta L_i + \Delta Ax_i \right]^{+},
\]

where \( \Delta L_i = L_i^* - L_i \), \( \Delta Ax_i = Ax_i - Ax_i^* \) and \( Ax_i^* = L_i^* - \sum_{j=1}^{n} \gamma_{i,j} L_j^* \). The second equation of system (4) can be rewritten as

\[
\Delta L_i = -\min \left[ \sum_{j=1}^{n} \pi_{i,j} Y_j + \sum_{j=1}^{n} \gamma_{i,j} L_j + Ax_i - L_i^*, 0 \right]
\]

\[
= \left[ -\sum_{j=1}^{n} \pi_{i,j} Y_j + \sum_{j=1}^{n} \gamma_{i,j} \Delta L_j - \Delta Ax_i \right]^{+}.
\]

Therefore, in regime \( z \), the equilibrium values are such that

\[
z_i Y_i + (1 - z_i) \Delta L_i = 0, \quad i = 1, \ldots, n,
\]

\[
\begin{bmatrix}
(1 - z_1) Y_1 - z_1 \Delta L_1 \\
\vdots \\
(1 - z_n) Y_n - z_n \Delta L_n
\end{bmatrix}
= [Id - Q(z)]^{-1} \Delta Ax.
\]

Equations (A.1) say that \( Y = 0 \) for a defaulted bank and \( \Delta L = 0 \) for a non-defaulted bank. Equations (A.1) and (A.2) provide the equilibrium values of \( \Delta L \) for the defaulted banks and the equilibrium values of \( Y \) for the non-defaulted banks.

We deduce that regime \( z \) occurs iff

\[
\Delta Ax \in [Id - Q(z)] \prod_{i=1}^{n} \left\{ (1 - z_i)(\mathbb{R}^+) + z_i[-L_i^*, 0] \right\} = C(z), \; \text{say.}
\]

The liquidation equilibrium exists for any admissible \( Ax \) iff the union of the truncated cones \( C(z) \), \( z \) varying, contains the set of admissible values of \( \Delta Ax \). This condition is

\[
\bigcup_{z} C(z) \supset -Ax^* + (\mathbb{R}^+)^n = -(Id - \Gamma)L^* + (\mathbb{R}^+)^n,
\]
since the exogenous asset components $A x_i$ are positive. Note that, since $L^* \in (\mathbb{R}^+)^n$, $A x^*$ cannot belong to $(\mathbb{R}^-)^n$, because, in this case, $L^* = (I d - \Gamma)^{-1} A x^* = (I d + \Gamma + \Gamma^2 + \ldots) A x^*$ would belong to $(\mathbb{R}^-)^n$.

When it exists the liquidation equilibrium is unique iff the truncated cones $C(z)$, $z$ varying, do not overlap.

To analyze the existence and uniqueness of this equilibrium let us consider the piecewise linear function from $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$g(x) = \sum_z [I d - Q(z)] x 1_{[x \in C^*(z)]}, \quad (A.3)$$

where $C^*(z) = \prod_{i=1}^n \{ (1 - z_i) \mathbb{R}^+ + z_i \mathbb{R}^- \}$ denotes the orthants of $\mathbb{R}^n$.

We have

$$g(C^*(z)) = \overline{C}(z), \quad (A.4)$$

where $\overline{C}(z)$ is the cone generated by the truncated cones $C(z)$.

The proof of proposition 1 is based on theorem 1 in Gouriéroux, Laffont, and Monfort (1981), given below in our framework.

**Theorem 1.** The following properties are equivalent:

- i) function $g$ is one-to-one from $\mathbb{R}^n$ to $\mathbb{R}^n$;
- ii) $\det [I d - Q(z)]$, $z$ varying, have the same sign;
- iii) $\bigcup_z C(z) = \mathbb{R}^n$;
- iv) the cones $\overline{C}(z)$, $z$ varying, do not overlap.

i) First, note that the truncated cones are non-degenerate, that is, reduced to $\{0\}$, and that they do not overlap iff their extensions $\overline{C}(z)$ do not overlap. By theorem 1, we deduce that a necessary and sufficient condition for the uniqueness of the liquidation equilibrium is ‘$\det [I d - Q(z)]$, $z$ varying, have the same sign.’

ii) Then, we have to check that the equivalent condition $\bigcup_z C(z) = \mathbb{R}^n$ implies the condition for existence:

$$\bigcup_z C(z) \supset -A x^* + (\mathbb{R}^+)^n,$$

where $A x^* = (I d - \Gamma)L^*$. Since $\bigcup_z \overline{C}(z) = \mathbb{R}^n$, the previous condition can be written:

$$\bigcup_z C(z) \supset (\bigcup_z \overline{C}(z)) \cap (-A x^* + (\mathbb{R}^+)^n)$$

$$\supset \bigcup_z (\overline{C}(z) \cap (-A x^* + (\mathbb{R}^+)^n)),$$
which is equivalent to

\[ C(z) \supset \overline{C}(z) \cap (-Ax^* + (IR^+)\mathbb{N}), \quad \forall z \]

(since \( C(z) \subset \overline{C}(z) \) and the \( \overline{C}(z) \) do not overlap). If we denote by \( M(z) \) the point in \( IR^n \) with \( i \)th coordinate 0 if \( z_i = 0 \) and \(-L_i^*\) if \( z_i = 1 \), we get:

\[ C(z) = \overline{C}(z) \cap ([Id - Q(z)][M(z) + (IR^+)\mathbb{N}]) \]

Moreover, \([Id - Q(z)](IR^+)\mathbb{N} \supset (IR^+)\mathbb{N}\), since any point \( y \) of \((IR^+)\mathbb{N}\) is the image by \( Id - Q(z) \) of \([Id - Q(z)]^{-1}y = y + Q(z)y + Q^2(z)y + \cdots\) which belongs to \((IR^+)\mathbb{N}\), and we get

\[ C(z) \supset \overline{C}(z) \cap ([Id - Q(z)]M(z) + (IR^+)\mathbb{N}) \]

We now have to check that \( \overline{C}(z) \cap ([Id - Q(z)]M(z) + (IR^+)\mathbb{N}) \supset \overline{C}(z) \cap (-Ax^* + (IR^+)\mathbb{N}) \) for any \( z \). For instance, for \( z = (1, \ldots, 1)' = e \), we have \([Id - Q(e)]M(e) = -Ax^*\) and the result holds.

iii) Finally, from lemma A.1 above, the determinants of \( Id - Q(z) \) have the same positive sign under the assumptions of proposition 1 and the results of this proposition follow.

**Appendix C: Merton’s model**

Merton (1974) presents a stylized approach for evaluating the credit risk of a single firm. Let us summarize and discuss the main features of this paper.

**C.1. Merton’s model**

The firm has the following simple balance sheet:

| Asset | Liability |
|-------|-----------|
| Ax    | L*        |

Ax includes all the assets of the firm, whereas its nominal debt is \( L^* \). Besides, Merton identifies two types of stakeholders: the shareholders and the debtors. The shareholders own the value of the firm \( Y \), while the debtors hold its debt \( L \) of nominal value \( L^* \). Based on these elements, Merton derives the pricing of those components with respect to the status of the firm: either default, or alive.
The status is triggered by the relative value of asset $Ax$ over nominal debt $L^*$. We get

$$\begin{align*}
  Y &= (Ax - L^*)^+ \\
  L &= \min(Ax, L^*).
\end{align*}$$

**C.2. Equilibrium conditions and solutions**

Let us now consider the system

$$\begin{align*}
  Y &= (Ax - L)^+ \\
  L &= \min(Ax, L^*),
\end{align*}$$

where $Y$ and $L$ are simultaneously defined. The equilibrium solution of this new system is exactly the quantity given in (A.8). Therefore, the standard Merton’s model is a special system with a single bank and no self-holding of both stocks and lendings.

**C.3. Interpretations in terms of options**

A standard interpretation of Merton’s model is to consider that shareholders buy a call on $Ax$ with strike $L^*$, whereas debtors sell a portfolio including a put on $Ax$ with strike $L^*$ and risk-free asset. This interpretation is summarized in the following table:

| Status  | Shareholder          | Debtor            |
|---------|----------------------|-------------------|
| $Ax > L^*$ | $Y = Ax - L^*$     | $L = L^*$        |
| $Ax \leq L^*$ | $Y = 0$               | $L = Ax$         |
|         | $Y = (Ax - L^*)^+$  | $L = \min(Ax, L^*)$ |
|         | buying a call       | selling a put     |

**C.4. Convexity property**

$L$ is concave in $Ax$. Since $Y$ is convex in $Ax$, $-Y$ is concave in $Ax$.

**C.5. Impulse response**

Consider an initial situation where $Ax = Ax^0$. In figure A.4 we plot the evolutions of $L$ and $Y$ as $Ax$ decreases through a factor $\delta$ to zero value. They show the convexity (resp. concavity) property of $Y$ (resp. $L$) as a function of $\delta$. 
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