REGULARITY CONDITIONS IN THE REALISABILITY PROBLEM WITH APPLICATIONS TO POINT PROCESSES AND RANDOM CLOSED SETS

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We study existence of random elements with partially specified distributions. The technique relies on the existence of a positive extension for linear functionals accompanied by additional conditions that ensure the regularity of the extension needed for interpreting it as a probability measure. It is shown in which case the extension can be chosen to possess some invariance properties.

The results are applied to the existence of point processes with given correlation measure and random closed sets with given two-point covering function or contact distribution function. It is shown that the regularity condition can be efficiently checked in many cases in order to ensure that the obtained point processes are indeed locally finite and random sets have closed realisations.

1. Introduction. Defining the distribution of a random element $\xi$ in a topological space $\mathcal{X}$ is equivalent to specialising the expected values for all bounded continuous functions $g(\xi)$. These expected values define a linear functional $\Phi(g) = \mathbb{E}g(\xi)$ on the space of bounded continuous functions $g : \mathcal{X} \rightarrow \mathbb{R}$. It is well known that a functional $\Phi$ indeed corresponds to a random element if and only if $\Phi$ is positive (i.e. $\Phi(g) \geq 0$ if $g$ is non-negative) and upper semicontinuous (i.e. $\Phi(g_n) \downarrow 0$ if $g_n \downarrow 0$), see e.g. [35].

Below we consider the case of functional $\Phi$ defined only on some functions on $\mathcal{X}$ and address the realisability of $\Phi$, i.e. the mere existence of a random element $\xi$ such that $\Phi(g) = \mathbb{E}g(\xi)$ for $g$ from the chosen family $G$ of functions. The uniqueness is not on the agenda, since typically the family $G$ will not
suffice to uniquely specify the distribution of $\xi$. A classical example of this setting is the existence of a probability distribution with given marginals, see [8]. The present paper focuses on some geometric instances of the problem. We will see that in most cases the answer to the existence problem consists of the two main steps.

1. **(Positivity)** Checking the positivity condition on $\Phi$ — in most cases this requires checking a system of inequalities, which is a serious (but unavoidable) computational burden.

2. **(Regularity)** Ensuring that the extended functional is regular (namely, upper semicontinuous) and so defines a $\sigma$-additive measure.

The first step ensures that it is possible to extend functional $\Phi$ positively from a certain family of functions to a wider family. In this work we put the emphasis on the latter step — checking the regularity condition, leaving aside the computational difficulties arising from validating the positivity assumption.

The use of positive extension techniques (that go back to L.V. Kantorovitch) in the framework of stochastic geometry was pioneered by T. Kuna, J. Lebowitz and E.R. Speer [12] in application to point processes, which greatly inspired the current work. In this paper we establish the general nature of an idea proposed in [12] and show how it leads to various further realisability results. The new idea is to introduce an additional function, what we call the regularity modulus, and to formulate sufficient and necessary conditions in terms of a positive extension of a functional onto the linear space containing the regularity modulus and requiring only a priori integrability of the regularity modulus.

We concentrate on two basic examples of the realisability problem: the existence of point processes with given correlation (factorial moment) measure and the existence of a random closed set with given two-point coverage probabilities or contact distribution functions. The introduction to the realisability issue for point processes is available in several papers by T. Kuna, J. Lebowitz and E.R. Speer [11, 12], see also Section 3 of this paper. The realisability problem for random closed sets has been widely studied in physics and material science literature, see [7, 15, 30, 32, 33] and in particular the comprehensive monograph by S. Torquato [31] and a recent survey by J. Quintanilla [22]. If $\xi$ is a random closed set (see Sec. 4 for formal definitions) in a locally compact Hausdorff second countable space $X$, its one-point covering functions is defined by

$$p_x = P\{x \in \xi\}, \quad x \in X.$$
It is easy to characterise all one-point covering functions of random closed sets as follows.

**Theorem 1.1.** A function $p_x$, $x \in \mathbb{X}$, with values in $[0,1]$ is the one-point covering function of a random closed set if and only if $p$ is upper semicontinuous.

The upper-semi continuity of the one-point covering function of a random closed set $\xi$ is a straightforward consequence of the upper semicontinuity property of the capacity functional of a random closed set, see [20, Sec. 1.1.2]. Conversely, the function $p$ from the theorem is realised (for instance) as the one-point covering function of the random set $\xi = \{ x : p_x \geq v \}$ where $v$ is a uniformly distributed variable (the details are left to the reader).

It is considerably more complicated to characterise two-point covering functions

$$p_{x,y} = P\{x, y \in \xi\}, \quad x, y \in \mathbb{X}.$$  

In view of applications to modelling of random media it is often assumed that $\xi$ is a stationary set in $\mathbb{R}^d$, so that the one-point covering function is constant and the two-point covering function $p_{x,y}$ depends only on $x - y$. Since a random closed set can be considered as an upper semicontinuous indicator function, the realisability problem for the two-point covering function can be rephrased as follows.

Characterise covariance functions of (stationary) upper semicontinuous random functions with values in $\{0,1\}$.

These covariances are obviously a subfamily of positive semi-definite functions. Without the upper semicontinuity requirement, this problem, of combinatorial nature, was solved by B. McMillan [19] and L. Shepp [26, 27] using the extension argument from [8]. More exactly, they normalised indicators by letting them take values $+1$ or $-1$ and assumed that the mean is zero. Their result does not rely on the topological structure of the underlying space and so does not necessarily lead to an upper semicontinuous indicator function.

**Example 1.2.** Let $p_{x,y} = \frac{1}{4}$ and let $p_x = \frac{1}{2}$ for all $x, y \in \mathbb{R}$. While this two-point covering function corresponds, e.g., to the indicator field with independent values, it cannot be obtained as the two-point covering function of a random closed set, see Proposition 4.4.

Even leaving aside the upper semicontinuity property, the McMillan–Shepp condition involves a family of corner-positive matrices, which is poorly
understood. As a result, its practical use to check the realisability for random media is rather limited. A number of authors have attempted to come up with simpler (but only necessary) conditions, see, e.g., [7, 17, 22, 32]. Another set of conditions for joint distributions of binary random variables is formulated in [25] in terms of the corresponding copulas.

The realisability problem can be also posed for point processes in terms of their moment measures. In case of moment measures of arbitrary order it has been solved by A. Lenard [13, 14]. The case of moment measures up to the second order has been studied by T. Kuna, J. Lebowitz and E.R. Speer [11], whose recent paper [12] contains (among other results) a complete solution of this realisability problem for point processes with finite third-order moments and hard-core type conditions with fixed exclusion distance. The results of [12] can be extended to higher order moment measures, as was explicitly indicated there. Again, the positivity condition of [12] is extremely difficult to verify, even more complicated than the original condition for point processes because of new polynomial functionals involved in the positivity condition.

The paper is organised as follows. Section 2 presents a series of general results on regular extensions and also invariant extensions (relevant for the existence of stationary random elements). These results form the theoretical backbone of our study, and are new even in the abstract setting of extending general positive linear functionals.

Section 3 presents a number of realisability conditions for correlation measures of point processes that considerably extend the results of [12] by relaxing the moment and hardcore conditions. One of our most important results is Theorem 3.3 that shows how to split the positivity and regularity conditions, so that the latter can be efficiently checked. The importance of the packing number in relation to realisability conditions for hard-core point processes is also explained.

Section 4 deals with the realisability problem for two-point covering probabilities of random sets. The closedness of the corresponding random set can be ensured by imposing appropriate regularity conditions. Section 5 addresses a further variant of the realisability problem that involves contact distribution functions of random sets.

The notational convention is that the carrier space is denoted as $X$ (e.g. $\mathbb{R}$, $\mathbb{R}^d$), points in the carrier space are $x, y$, subsets of carrier spaces are denoted by capitals $X, Y, F$ (while $Y$ is reserved for counting measures identified with corresponding support sets), the families of sets (or families of counting measures) as $\mathcal{X}, \mathcal{N}, \mathcal{F}$ (while in Section 2 $\mathcal{X}$ denotes also a rather general space), and random element in these spaces (random sets or point processes) as $\xi$, real functions acting on $\mathcal{X}, \mathcal{N}, \mathcal{F}$ are $g, v$ and families of such
functions are $G, E, V$, a functional on $G, E, V$ is denoted by $\Phi$, real numbers are denoted by $t, r, \lambda$, while $c$ denotes a generic constant and at the same time the corresponding constant function.

2. Extending positive functionals. Fundamental results about the extension of positive operators form the heart of our main results, and are necessary to understand the machinery of the proofs. Nevertheless, the results of the subsequent sections can be understood without Section 2, with the exception of Definition 2.5.

2.1. General extension theorems. Consider a vector lattice $E$, i.e. a linear space with a partial order and such that for any $v_1, v_2 \in E$ their maximum $v_1 \vee v_2$ also belongs to $E$. The absolute value $|v|$ of $v$ is defined as the sum of $v \vee 0$ and $(-v) \vee 0$.

Let $G$ be a vector subspace of $E$, which is not necessarily a lattice itself, i.e. $G$ may be not closed with respect to the maximum operation. We say that $G$ majorises $E$ if each $v \in E$ satisfies $|v| \leq g$ for some $g \in G$. A real-valued functional $\Phi$ defined on $E$ (resp. $G$) is said to be positive if $\Phi(v) \geq 0$ whenever $v \geq 0$ and $v \in E$ (resp. $v \in G$). A functional defined on $E$ is said to be an extension of $\Phi : G \mapsto \mathbb{R}$ if it coincides with $\Phi$ on $G$. The extended $\Phi$ is always denoted by the same letter. The following result about extension of positive functionals goes back to L.V. Kantorovich.

**Theorem 2.1** (see [1], Th. 8.12 and [34], Th. X.3.1). Assume that $G$ is a majorising vector subspace of a vector lattice $E$. Then each positive linear functional on $G$ admits a positive extension on the whole $E$.

If $G$ is a lattice itself, then it is possible to gain much more control over the extension of $\Phi$, e.g. a continuous functional admits a continuous extension, see [34, Sec. X.5]. On the contrary, very little is known about regularity properties of the extension if $G$ is not a lattice.

In the following we assume that $G$ and $E$ are families of functions $g$ on a certain space $X$. If $G$ contains constant functions, the positivity of $\Phi$ over $G$ can be equivalently formulated as

$$\Phi(g) \geq \inf_{X \in \chi} g(X). \quad (2.1)$$

This equivalence is a particular case of the following result for $\chi = 0$ (replace $g$ with $-g$ in (2.2)).

**Proposition 2.2.** Assume that vector space $G$ contains constant functions and denote by $G \setminus \mathbb{R}$ the family of non-constant functions from $G$. If $\chi$
is any non-negative function on $\mathcal{X}$, then a linear functional $\Phi$ on $G$ admits a positive extension on $G + \mathbb{R}\chi$ with $\Phi(\chi) = r$ if and only if

$$r = \sup_{g \in G, g \leq \chi} \Phi(g) = \sup_{g \in G} \inf_{X \in \mathcal{X}} [\chi(X) - g(X)] + \Phi(g) < \infty. \tag{2.2}$$

**Proof.** Since every element of $G$ can be written $c + g$ with $g \in G \setminus \mathbb{R}$ and $c \in \mathbb{R}$, the left-hand side of (2.2) equals

$$r = \sup_{g \in G \setminus \mathbb{R}} \sup_{c \in \mathbb{R}} c + \Phi(g) = \sup_{g \in G} c_g + \Phi(g),$$

where $c_g = \inf_{X \in \mathcal{X}} (\chi - g)(X)$ is the largest $c$ such that $c + g \leq \chi$, which yields the equality in (2.2).

The necessity of (2.2) is straightforward because $r \leq \Phi(\chi) < \infty$. For the sufficiency, assume that (2.2) holds. The proof consists in checking that assigning the value $\Phi(\chi) = r$ yields a positive extension on $G + \mathbb{R}\chi$. Let us first prove that $\Phi$ is positive on $G$. If some $g \leq 0$ satisfies $\Phi(g) > 0$, then $\Phi(tg) \uparrow \infty$ as $t \to \infty$ whereas $tg \leq \chi$, which contradicts (2.2).

Let $g + \lambda \chi \geq 0$ for $\lambda \neq 0$ and $g \in G$. If $\lambda > 0$, then $-\lambda^{-1}g \leq \chi$, whence $\Phi(-\lambda^{-1}g) \leq r$ and $\Phi(g + \lambda \chi) \geq -\lambda r + \lambda \Phi(\chi) = 0$. If $\lambda < 0$, $-\lambda^{-1}g \geq \chi$ whence $-\lambda^{-1}g$ is larger than any $g' \leq \chi$, and

$$\Phi(-\lambda^{-1}g) \geq \sup_{g' \in G, g' \leq \chi} \Phi(g') = r$$

by monotonicity of $\Phi$ on $G$. Hence $\Phi(g + \lambda \chi) \geq -\lambda r + \lambda \Phi(\chi) = 0$. \qed

The advantage of the latter condition in (2.2) consists in the explicit reference to the space $\mathcal{X}$ where random elements lie instead of checking the inequality $g \leq \chi$.

**2.2. Regularity conditions and distributions of random elements.** Let $\mathcal{E}$ be a certain family of functions $v : \mathcal{X} \mapsto \mathbb{R}$ defined on a space $\mathcal{X}$ with lattice operation being the pointwise maximum and the corresponding partial order.

**Theorem 2.3** (Daniell, see Sec. 4.5 [3] and Th. 14.1 [10]). Let a vector lattice $\mathcal{E}$ consist of real-valued functions on $\mathcal{X}$ and let $\mathcal{E}$ contain constants. If $\Phi$ is a positive functional on $\mathcal{E}$ such that $\Phi(v_n) \downarrow 0$ for each sequence $v_n \downarrow 0$ and $\Phi(1) = 1$, then there exists a unique random element $\xi$ in $\mathcal{X}$, measurable with respect to the $\sigma$-algebra generated by all functions from $\mathcal{E}$, such that $\Phi(v) = \mathbb{E}v(\xi)$ for all $v \in \mathcal{E}$.
In view of the positivity of $\Phi$, the condition imposed on $\Phi$ is equivalent to its upper semicontinuity on $E$. In this paper, we start with a functional $\Phi$ defined on a vector sub-space $G \subset E$ and discuss the existence of a random element $\xi \in X$ such that $\Phi(\mathbf{g}) = \mathbf{E}_g(\xi)$ for all $\mathbf{g} \in G$. In this case $\Phi$ is said to be realisable as a probability distribution on $X$.

Assumption 2.4. The vector space $G$ of functions on $X$ contains constants and, for each $\mathbf{g}_1, \mathbf{g}_2 \in G$, there exists a $\mathbf{g} \in G$ such that $(\mathbf{g}_1 \lor \mathbf{g}_2) \leq \mathbf{g}$.

From now on assume that $X$ is a topological space.

Definition 2.5. Given a vector space $G$ of functions on $X$, a regularity modulus on $X$ is a lower semicontinuous function $\chi : X \rightarrow [0, \infty]$ such that

$$\mathcal{H}_g = \{X \in X : \chi(X) \leq g(X)\}$$

is relatively compact for each $\mathbf{g} \in G$ (if all $\mathbf{g} \in G$ are bounded, $\chi$ is a regularity modulus if and only if it has compact level sets).

Examples of regularity moduli are given in Sections 3 and 4. A function $v : X \rightarrow \mathbb{R}$ is said to be $\chi$-regular if $v$ is continuous on $\mathcal{H}_g$ for each $\mathbf{g}$ in $G$. Each continuous function is trivially $\chi$-regular. The proof of the following central result is based on the ideas from the proof of [12, Th. 3.14]. It should be noted that our result entails not only the realisability, but also provides a bound for the expected value of the regularity modulus.

Theorem 2.6. Consider a vector space $G$ of functions on $X$ satisfying Assumption 2.4 and such that each $\mathbf{g}$ from $G$ is $\chi$-regular for a regularity modulus $\chi$. Let $\Phi$ be a linear functional on $G$ with $\Phi(1) = 1$. Then, for any given $r \geq 0$, there exists a Borel random element $\xi$ in $X$ such that

$$\begin{cases} 
\mathbf{E}_g(\xi) = \Phi(g) \text{ for all } g \in G, \\
\mathbf{E}_\chi(\xi) \leq r,
\end{cases}$$

if and only if

$$\sup_{g \in G, \mathbf{g} \leq \chi} \Phi(g) \leq r.$$  

Proof. Condition (2.5) is necessary because $g \leq \chi$ implies $\Phi(g) = \mathbf{E}_g(\xi) \leq \mathbf{E}_\chi(\xi) \leq r$.

Sufficiency. Let $E$ be the family of all $\chi$-regular functions $v$ that satisfy $v \leq g$ for some $\mathbf{g} \in G$. Each function $v \in E$ is Borel measurable. Note that $E$
contains all bounded continuous functions that generate the Baire $\sigma$-algebra on $X$ being in general a sub-$\sigma$-algebra of the Borel one. For each $v_1, v_2 \in E$, the function $v_1 \vee v_2$ is $\chi$-regular and is majorised by $g_1 \vee g_2$, where $g_1, g_2 \in G$ majorise $v_1$ and $v_2$ respectively. In view of Assumption 2.4, $E$ is a lattice.

Without loss of generality assume that the supremum in (2.5) equals $r$. By Proposition 2.2, $\Phi$ is positive on $G$ and can be positively extended onto $G + R\chi$ with $\Phi(\chi) = r$, and further on to $E + R\chi$ by Theorem 2.1. It remains to prove that the obtained extension satisfies conditions of Theorem 2.3. For that, we use an argument similar to that of [12]. First restrict the obtained functional $\Phi$ onto $E$. Assume that $\chi$ is strictly positive. Consider a sequence $\{v_n, n \geq 1\} \subset E$ such that $v_n \downarrow 0$. For each $n$, let $g_n$ be a function of $G$ such that $v_n \leq g_n$. Take $\varepsilon > 0$. Then $\mathcal{K}_n = \{X : v_n(X) \geq \varepsilon\chi(X)\}$ is a subset of relatively compact $H_{g_n/\varepsilon}$, since $\chi$ is a regularity modulus. Since $v_n$ is continuous on $H_{g_n/\varepsilon}$, the set $\mathcal{K}_n$ is closed and therefore compact. The pointwise convergence $v_n \downarrow 0$ yields that $\cap_n \mathcal{K}_n = \emptyset$ (recall that $\chi$ is strictly positive). Since $\{\mathcal{K}_n\}$ is a decreasing sequence of compact sets, $\mathcal{K}_{n_0} = \emptyset$ for some $n_0$, whence $v_n(X) < \varepsilon\chi(X)$ for sufficiently large $n$. The positivity of $\Phi$ on $E + R\chi$ implies $\Phi(v_n) \leq \varepsilon\Phi(\chi) = \varepsilon r$, whence $\Phi(v_n) \downarrow 0$. Theorem 2.3 yields the existence of a random element $\xi$ in $X$ such that $\Phi(v) = E v(\xi)$ for all $v \in E$.

Since $\chi$ is lower semicontinuous, it can be pointwisely approximated from below by a sequence $\{v_n\}$ of non-negative continuous functions. Then $\tilde{v}_n = \min(n, v_n)$ belongs to $E$ and also approximates $\chi$ from below, so that $E \tilde{v}_n(\xi) = \Phi(\tilde{v}_n) \leq \Phi(\chi) = r$, while the monotone convergence theorem yields

$$E \chi(\xi) = \lim_{n \to \infty} E \tilde{v}_n(\xi) \leq r.$$ 

If $\chi$ is not strictly positive, it suffices to apply the above argument to $\chi' = 1 + \chi$ and use the linearity of $\Phi$. 

Condition (2.5), equivalent to (2.2), is expressed solely in terms of the values taken by $\Phi$ on $G$, and therefore yields a self-contained solution of the realisability problem. It is not easy to check in general, but if $\chi$ can be approximated by functions $\chi_n \in G$, $n \geq 1$, then it is possible to “split” (2.5) into the positivity condition on $\Phi$ and the uniform boundedness of $\Phi(\chi_n)$, $n \geq 1$. This idea is used successfully in several different frameworks, which justify the abstract setting of Theorem 2.6: in Section 3 for point processes (see Theorem 3.1), in Section 4.4 for random closed sets (see Theorem 4.9), and in [5] in the framework of random measurable sets with the regularity modulus being the perimeter of a set.
The realizability problem is particularly simple if $\mathcal{X}$ is compact and $G$ consists of continuous functions. Then, for identically vanishing $\chi$, Theorem 2.6 yields the following result, which is similar to the Riesz–Markov theorem, see [10].

**Corollary 2.7.** Let $\mathcal{X}$ be a compact space with its Borel $\sigma$-algebra. Consider a vector space $G$ containing constants such that each $g \in G$ is continuous and a map $\Phi : G \mapsto \mathbb{R}$ such that $\Phi(1) = 1$. Then there exists a random element $\xi$ in $\mathcal{X}$ such that $E_g(\xi) = \Phi(g)$ for all $g \in G$ if and only if $\Phi$ is a linear positive functional on $G$.

2.3. **Passing to the limit.** The following result shows that the family of all random elements that realise $\Phi$ in the sense of (2.4) is weakly compact.

**Theorem 2.8.** Assume that $G$ satisfies Assumption 2.4 and consists of continuous functions on a Polish space $\mathcal{X}$ with regularity modulus $\chi$. Let $\Phi$ be a linear positive functional on $G$. Then the family $M$ of all Borel random elements $\xi$ that satisfy (2.4) for any given $r \geq 0$ is compact in the weak topology.

**Proof.** Since $\chi$ is a regularity modulus, the set $H_{r/\varepsilon}$ is compact. By Markov’s inequality,

$$P\{\xi \notin H_{r/\varepsilon}\} = P\{\chi(\xi) > r/\varepsilon\} \leq \varepsilon,$$

for all $\xi \in \mathcal{M}$, so that $\mathcal{M}$ is tight.

Let $\{\xi_n, n \geq 1\}$ be random elements from $\mathcal{M}$. Assume that $\xi_n$ converges weakly to some $\xi$. Without loss of generality assume that the $\xi_n$’s are defined on the same probability space and converge almost surely to $\xi$. Since $\chi$ is non-negative, Fatou’s lemma yields

$$r \geq \lim \inf E_{\chi}(\xi_n) \geq E \lim \inf \chi(\xi_n) \geq E \chi(\lim \xi_n) = E \chi(\xi),$$

where the lower semicontinuity of $\chi$ also has been used.

Take an arbitrary $g \in G$ and define $H_{\lambda g}$ as in (2.3). Let $g^+(X) = \max(g(X), 0)$ be the positive part of $g$. Then, for $\lambda > 0$,

$$E g^+(\xi_n) = E g^+(\xi_n) 1_{\xi_n \notin H_{\lambda g}} + E g^+(\xi_n) 1_{\xi_n \in H_{\lambda g}}.$$

Since $g$ is continuous, $H_{\lambda g}$ is closed (and compact), so that if $\xi_n \in H_{\lambda g}$ for infinitely many $n$, then also $\xi \in H_{\lambda g}$. Furthermore, $\lambda g$ and also $g$ itself, are continuous and bounded on $H_{\lambda g}$, so that Fatou’s lemma yields

$$\lim \sup E g^+(\xi_n) 1_{\xi_n \in H_{\lambda g}} \leq E \lim \sup (g^+(\xi_n) 1_{\xi_n \in H_{\lambda g}}) \leq E g^+(\xi) 1_{\xi \in H_{\lambda g}} \leq E g^+(\xi).$$
Thus
\[
\limsup E g^+(\xi_n) \leq \frac{\chi(\xi_n)}{\lambda} + E g^+(\xi) \leq \frac{r}{\lambda} + E g^+(\xi).
\]
Since \(\lambda\) is arbitrary,
\[
\limsup E g^+(\xi_n) \leq E g^+(\xi).
\]
Since \(g^+\) is non-negative, Fatou's lemma yields that
\[
E g^+(\xi_n) \to E g^+(\xi),
\]
so that \(E g(\xi) = \Phi(g)\) for all \(g \in G\). Therefore, \(\xi \in \mathcal{M}\).

The following result concerns realisability of pointwise limits of linear functionals. Special conditions of this type for correlation measures of point processes are given in [12, Sec. 3.4].

**Theorem 2.9.** Let \(\{\Phi_n, n \geq 1\}\) be a sequence of linear positive functionals on a space \(G\) that satisfies the assumptions of Theorem 2.8. Assume that
\[
(2.6) \quad \liminf_n \sup_{g \in G, g \leq \chi} \Phi_n(g) < \infty.
\]
If \(\Phi_n(g) \to \Phi(g)\) for all \(g \in G\), then \(\Phi\) is realisable as a random element \(\xi\) satisfying (2.4) and such that \(\xi\) is the weak limit of random elements realising \(\Phi_{n_k}\) for a subsequence \(n_k\).

**Proof.** By passing to a subsequence, it suffices to assume that (2.6) holds for the limit instead of the lower limit. Let \(\xi_n\) be a random element that realises \(\Phi_n\). If \(r\) is larger than the limit of (2.6), then \(P\{\xi_n \notin \mathcal{H}_{r/\epsilon}\} \leq \epsilon\), so that \(\{\xi_n\}\) is a tight sequence. Without loss of generality assume that \(\xi_n\) weakly converges to a random element \(\xi\).

The pointwise convergence of \(\Phi_n\) yields that \(E g(\xi_n) \to E g(\xi)\) for all \(g \in G\). Now the arguments from the proof of Theorem 2.8 can be used to show that \(E g(\xi_n) \to E g(\xi)\), so that \(E g(\xi) = \Phi(g)\) for all \(g \in G\), i.e. \(\xi\) indeed satisfies (2.4).

2.4. **Invariant extension.** Consider an abelian group \(\Theta\) of continuous transformations acting on \(\mathcal{X}\). For a function \(\nu\) on \(\mathcal{X}\), define
\[
(\theta \nu)(X) = \nu(\theta X), \quad \theta \in \Theta, X \in \mathcal{X}.
\]
A functional \(\Phi\) is said to be \(\Theta\)-invariant if, for each \(\theta \in \Theta\) and \(\nu\) from the domain of definition of \(\Phi\), \(\Phi(\theta \nu)\) is defined and equal to \(\Phi(\nu)\).
A Borel random element $\xi$ in $X$ is said to be $\Theta$-stationary if, for each $\theta \in \Theta$, $\theta \xi$ has the same distribution as $\xi$. A variant of the following result for correlation measures of point processes is given in [12, Th. 4.3].

**Theorem 2.10.** Assume that $G$ is a $\Theta$-invariant space satisfying Assumption 2.4 and consisting of $\chi$-regular functions. Furthermore, assume that at least one of the following conditions holds:

(i) $G$ consists of continuous functions and $\chi$ is pointwisely approximated from below by a monotone sequence of functions $g_n \in G$, $n \geq 1$.

(ii) $\chi$ is $\Theta$-invariant.

Let $\Phi$ be a $\Theta$-invariant functional on $G$. Then, for every given $r \geq 0$, there exists a $\Theta$-stationary random element $\xi$ in $X$ satisfying (2.4) if and only if (2.5) holds.

**Proof.** As in [12, Prop. 4.1], the proof consists in checking hypotheses of the Markov–Kakutani fixed point theorem. Let $M$ be the family of random elements $\xi$ that realise $\Phi$ on $G$, and satisfy $E\chi(\theta \xi) \leq r$ for every $\theta \in \Theta$. The family $M$ is easily seen to be convex with respect to addition of measures, it is compact by Theorem 2.8, and $\Theta$-invariant, since $\Phi$ is $\Theta$-invariant on $G$.

It remains to prove that $M$ is not empty.

(i) In view of (2.5), it is possible to extend $\Phi$ positively onto $G + \mathbb{R}_\chi$, so that $E\chi(\xi) \leq r$. The $\Theta$-invariance of $\Phi$ on $G$ together with the monotone convergence theorem imply that $E\chi(\theta \xi) = E\chi(\xi) \leq r$, whence $\xi \in M$.

(ii) By Proposition 2.2, we can extend $\Phi$ positively onto the $\Theta$-invariant vector space $V = G + \mathbb{R}_\chi$. Since $\Phi$ is $\Theta$-invariant on $G$, we have $\Phi(\theta(g + t\chi)) = \Phi(\theta g) + t\Phi(\theta \chi) = \Phi(g + t\chi)$ for $g + t\chi$ in $V$, whence $\Phi$ is $\Theta$-invariant on $V$.

According to [28, Th. 3], $\Phi$ admits a positive $\Theta$-invariant extension to the space $E + \mathbb{R}_\chi$, defined like in the proof of Theorem 2.6. The restriction of the obtained functional onto $E$ corresponds to a random element $\xi$ in $X$ that verifies (2.4) and satisfies $E(\theta \nu)(\xi) = E(\nu) = E\nu(\xi)$ for $\nu$ in $E$. Since $E$ contains all bounded continuous functions on $X$, $\theta \xi$ and $\xi$ are identically distributed for all $\theta \in \Theta$.

3. Correlation measures of point processes.

3.1. Framework and main results. Let $\mathcal{N}$ be the family of locally finite counting measures on a locally compact separable metric space $X$. We denote the support of $Y \in \mathcal{N}$ by the same letter $Y$, so that $x \in Y$ means $Y(\{x\}) \geq 1$.

Equip $\mathcal{N}$ with the vague topology, see [2]. A random element $\xi$ in $\mathcal{N}$ with the corresponding Borel $\sigma$-algebra is called a point process. Denote by $\mathcal{N}_0$
the family of simple counting measures, i.e. those which do not attach mass 2 or more to any given point. If $\xi$ is simple, i.e. $\xi \in \mathcal{N}_0$ a.s., then $\xi$ can be identified with a locally finite random set in $\mathbb{X}$, which is also denoted by $\xi$.

For a real function $h$ on $\mathbb{X} \times \mathbb{X}$ and counting measure $Y = \sum_i \delta_{x_i}$, given by the sum of Dirac measures, define

$$g_h(Y) = \sum_{x_i, x_j \in Y, i \neq j} h(x_i, x_j),$$

whenever the series absolutely converges, the empty sum being 0. Note that the sum in the right-hand side is taken over all pairs of distinct points from the support of $Y$, where multiple points appear several times according to their multiplicities. The value $g_h(Y)$ is necessarily finite if $h$ is bounded and has a bounded support. The value $g_h(Y)$ is termed in [12] the quadratic polynomial of $Y$, while polynomials of order $n \geq 1$ are sums of functions of $n$ points of the process, and are constants if $n = 0$.

Let $G$ be the vector space formed by constants and functions $g_h$ for $h$ from the space $\mathcal{C}_o$ of symmetric continuous functions with compact support. Note that $G$ satisfies Assumption 2.4, since

$$(c_1 + g_{h_1}) \vee (c_2 + g_{h_2}) \leq c_1 \vee c_2 + g_{h_1 \vee h_2} \in G$$

for all $c_1, c_2 \in \mathbb{R}$ and $h_1, h_2 \in \mathcal{C}_o$. Furthermore, each $g_h$ is continuous in the vague topology, and so is $\chi$-regular for any regularity modulus $\chi$.

Assume that $\xi$ has locally finite second moment, i.e. $E\xi(A)^2$ is finite for each bounded $A$. The correlation measure $\rho$ (also called the second factorial moment measure) of a point process $\xi$ is a measure on $\mathbb{X} \times \mathbb{X}$ that satisfies

$$(3.1) \quad \int_{\mathbb{X} \times \mathbb{X}} h(x,y)\rho(dx,dy) = E g_h(\xi)$$

for each $h \in \mathcal{C}_o$, see [2, Sec. 5.4] and [29, Sec. 4.3]. The left-hand side defines a linear functional $\Phi(g_h)$ on $g_h \in G$.

Let $\mathcal{X}$ be a subset of $\mathcal{N}$, which may be $\mathcal{N}$ itself. Given a symmetric locally finite measure $\rho$ on $\mathbb{X} \times \mathbb{X}$, the realisability problem amounts to the existence of a point process $\xi$ with realisations from $\mathcal{X}$ and with correlation measure $\rho$, so that $\Phi(g_h) = E g_h(\xi)$ for all $h \in \mathcal{C}_o$.

By (2.1), the positivity of $\Phi$ means

$$(3.2) \quad \Phi(g_h) \geq \inf_{Y \in \mathcal{X}} g_h(Y)$$

for all $h \in \mathcal{C}_o$. Then it is clear that the positivity of $\Phi$ is necessary for its realisability. If $\mathcal{X}$ is compact in the vague topology, then Corollary 2.7
applies and the positivity condition (3.2) is necessary and sufficient for the realisability of $\rho$.

However, in general the positivity condition alone is not sufficient for the realisability, see [11, Ex. 3.12]. In the following we find another condition that is not directly related to the positivity, but, together with the positivity, is necessary and sufficient for the realisability.

As an introduction, let us present our results for $X$ being a subset of the Euclidean space $\mathbb{R}^d$. For $\varepsilon \geq 0$, define

$$\chi_\varepsilon(Y) = \sum_{x,y \in Y, x \neq y} \|x - y\|^{-d-\varepsilon}, \quad Y \in \mathcal{N},$$

which is later acknowledged as being a regularity modulus (see Definition 2.5) if $\varepsilon \neq 0$. Note that $\chi_\varepsilon(Y)$ is infinite if $Y$ has multiple points. The tools developed in this paper enable us to resolve the original realisability problem with a supplementary regularity condition involving $\chi_\varepsilon$.

**Theorem 3.1.** (i) Let $X$ be a compact subset of $\mathbb{R}^d$ without isolated points. A symmetric finite measure $\rho(dx,dy)$ on $X \times X$ is the correlation measure of a simple point process $\xi \subset X$ such that $\mathbb{E}\chi_0(\xi) < \infty$ if and only if $\Phi$ given by the left-hand side of (3.1) is positive and

$$\int_{X \times X} \|x - y\|^{-d} \rho(dx,dy) < \infty.$$  

(ii) Let $\rho$ be a symmetric locally finite measure on $\mathbb{R}^d \times \mathbb{R}^d$ such that $\rho((A + x) \times (B + x)) = \rho(A \times B)$ for all $x \in \mathbb{R}^d$ and measurable sets $A$ and $B$. Then there exists a simple stationary point process $\xi$ with correlation measure $\rho$, such that

$$\mathbb{E}\chi_0(\xi \cap C) < \infty$$

for every compact $C \subset \mathbb{R}^d$, if and only if $\Phi$ defined by (3.1) is positive and

(3.3) \[ \int_{B \times B} \|x - y\|^{-d} \rho(dx,dy) < \infty \]

for some open set $B$.

**Proof.** The first statement follows from Theorem 3.5 using the fact that the packing number $P_t(X)$ of $X$ is bounded by $ct^{-d}$ for all sufficiently small $t$. For (ii), apply Theorem 3.9(ii) noticing that the imposed condition is equivalent to (3.24). \[ \square \]
In the remainder of this section, one can find a quantification of this result (i.e. how the left hand member of (3.3) controls the value of $E \chi_0(X \cap C)$) as well as generalisations for general metric spaces. The main argument used is a splitting method based on Theorem 2.6, the details are made clear in the proof of Theorem 3.3. Note that the packing number of the metric space appears as a crucial quantity to uncouple in this way the realisability problem, see Lemma 3.2.

3.2. Moment conditions. The family $\mathcal{X}_k$ of all counting measures with total mass at most $k$ on a compact space $X$ is compact. Thus, a measure $\rho$ on $X \times X$ is realisable as a point process with at most $k$ points if (3.2) holds with $\mathcal{X} = \mathcal{X}_k$.

Assume that $Y$ is a finite counting measure. For $\alpha > 2$ define

$$\chi_\alpha(Y) = Y(X)\alpha, \quad Y \in \mathcal{N}.$$ 

The finiteness of $E \chi_\alpha(\xi)$ amounts to the finiteness of the moment of order $\alpha$ for the total mass of $\xi$. Since $h \in C_\alpha$ is bounded by a constant $c'$ and $\alpha > 2$, the family

$$\{Y \in \mathcal{N} : \chi_\alpha(Y) \leq c + g_h(Y)\} \subset \{Y \in \mathcal{N} : Y(X)\alpha \leq c + c'Y(X)^2\}$$

consists of counting measures with total masses bounded by a certain constant and therefore is compact in the space $\mathcal{N}$. Hence $\chi_\alpha$ is a regularity modulus and so Theorem 2.6 yields the realisability condition

$$\sup_{g \in \mathcal{G}, g \leq \chi_\alpha} \Phi(g) < \infty$$

of $\rho$ by a point process $\xi$ whose total number of points has finite moment of order $\alpha$. Note that [12, Th. 3.14] provides a variant of this result assuming the existence of the third factorial moment of the cardinality of $\xi$ (i.e. with $\alpha = 3$) and for the joint realisability of the intensity and the correlation measures. The condition of [12, Th. 3.14] (reformulated for the correlation measure only) reads in our notation as $c + \Phi(g_h) + br \geq 0$ whenever $c + g_h + b\chi_3$ is non-negative on $\mathcal{N}$. Noticing that $b \geq 0$, this is equivalent to the fact that $c + \Phi(g_h) \leq r$ whenever $c + g_h \leq \chi_3$, being exactly (3.4). If $\Theta$ is a group of continuous transformations acting on $X$ and $\rho$ is $\Theta$-invariant, then the point process $\xi$ can be chosen $\Theta$-stationary by Theorem 2.10(ii).

In order to handle possibly non-finite point processes $\xi$ define

$$\chi_{\alpha,\beta}(Y) = \left(\sum_{x \in Y} \beta(x)\right)^\alpha, \quad Y \in \mathcal{N},$$

where $\beta$ is a non-negative measurable function on $X$. Note that $\chi_{\alpha,\beta}$ is a regularity modulus and so Theorem 2.6 yields the realisability condition

$$\sup_{g \in \mathcal{G}, g \leq \chi_{\alpha,\beta}} \Phi(g) < \infty$$

of $\rho$ by a point process $\xi$ whose total number of points has finite moment of order $\alpha$. Note that [12, Th. 3.14] provides a variant of this result assuming the existence of the third factorial moment of the cardinality of $\xi$ (i.e. with $\alpha = 3$) and for the joint realisability of the intensity and the correlation measures. The condition of [12, Th. 3.14] (reformulated for the correlation measure only) reads in our notation as $c + \Phi(g_h) + br \geq 0$ whenever $c + g_h + b\chi_3$ is non-negative on $\mathcal{N}$. Noticing that $b \geq 0$, this is equivalent to the fact that $c + \Phi(g_h) \leq r$ whenever $c + g_h \leq \chi_3$, being exactly (3.4). If $\Theta$ is a group of continuous transformations acting on $X$ and $\rho$ is $\Theta$-invariant, then the point process $\xi$ can be chosen $\Theta$-stationary by Theorem 2.10(ii).

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for a lower semicontinuous strictly positive function $\beta : X \to \mathbb{R}$ and $\alpha > 2$. By approximating $\beta$ from below with compactly supported functions, it is easy to see that $\chi_{\alpha,\beta}$ is a regularity modulus. By Theorem 2.6 and (2.2), for any given $r \geq 0$, there is a point process $\xi$ with correlation measure $\rho$ such that $\mathbb{E} \chi_{\alpha,\beta}(\xi) \leq r$ if and only if $\rho$ satisfies

$$\inf_{Y \in \mathcal{X}} [\chi_{\alpha,\beta}(Y) - g_h(Y)] + \int_{X \times X} h(x,y) \rho(dx)dy \leq r, \quad h \in \mathcal{C}_o.$$  

For $\alpha = 3$, condition (3.5) is a reformulation of [12, Th. 3.17] meaning the positivity of $\Phi$ on a family of positive polynomials that involve symmetric functions of the support points up to the third order. The realisability condition for $\Theta$-stationary random elements can be obtained by applying Theorem 2.10.

3.3. **Hardcore point processes on a compact space.** Assume that $X$ is a compact metric space with metric $d$. Let $\mathcal{N}_\varepsilon$ be the family of $\varepsilon$-hard-core point sets in $X$ (including the empty set), i.e. each $Y \in \mathcal{N}_\varepsilon$ attaches unit masses to distinct points with pairwise distances at least $\varepsilon$ with a fixed $\varepsilon > 0$. In this case no multiple points are allowed, i.e. $\mathcal{N}_\varepsilon \subset \mathcal{N}_0$.

According to [6, 9], a subset $\mathcal{X}$ of simple counting measures $\mathcal{N}_0$ is relatively compact if and only if $\sup\{Y(K) : Y \in \mathcal{X}\}$ is finite and the infimum over $Y \in \mathcal{X}$ of the minimal distance between two points in $Y \cap K$ is strictly positive for each compact set $K \subset X$. The hard-core condition yields that the number of points in any compact set is uniformly bounded, and so $\mathcal{N}_\varepsilon$ is indeed compact. By Corollary 2.7, $\rho$ is realisable as the correlation measure of an $\varepsilon$-hard-core point process with given $\varepsilon > 0$ if and only if

$$\Phi(g_h) \geq \inf_{Y \in \mathcal{N}_\varepsilon} g_h(Y)$$  

for all $h \in \mathcal{C}_o$. This result is formulated in [12, Th. 3.4], which essentially reduces to the positivity of $\Phi$ over the family $c + g_h$ (in our setting).

In this paper we assume that the hardcore distance is not predetermined and the point process takes realisations from $\cup_{\varepsilon > 0} \mathcal{N}_\varepsilon$, which coincides with $\mathcal{N}_0$ in case of compact $X$. Note that (3.6) is stronger than the positivity of $\Phi$ on functions $g_h$ defined on the whole family $\mathcal{N}_0$ and formulated as

$$\Phi(g_h) \geq \inf_{Y \in \mathcal{N}_0} g_h(Y), \quad h \in \mathcal{C}_o.$$  

If $X$ does not have isolated points, then the infimum in (3.7) can be taken over $\mathcal{N}$. This is seen by approximating a multiple atom with a sequence
of simple counting measures supported by points converging to the atom’s location.

In the following we use the (hard-core) regularity modulus of the form

\[ \chi^\text{hc}_\psi(Y) = \sum_{x_i, x_j \in Y, i \neq j} \psi(d(x_i, x_j)), \quad Y \in \mathcal{N}_0, \]

where \( \psi : (0, \infty) \to [0, \infty] \) is a monotone decreasing right-continuous function, such that \( \lim_{t \downarrow 0} \psi(t) = \infty \). The compactness of \( \mathbb{X} \) and the lower semicontinuity of \( \psi \) imply that \( \chi^\text{hc}_\psi \) is lower semicontinuous on \( \mathcal{N}_0 \). As shown below \( \chi^\text{hc}_\psi \) is a regularity modulus if \( \psi \) grows sufficiently fast at zero.

Let \( P_t(\mathbb{X}) \) be the packing number of \( \mathbb{X} \), i.e. the maximum number of points in \( \mathbb{X} \) with pairwise distances exceeding \( t \), see [18, p. 78]. It is convenient to define the packing number at \( t = 0 \) as \( P_0(\mathbb{X}) = \infty \) if \( \mathbb{X} \) is infinite and otherwise let \( P_0(\mathbb{X}) \) be the cardinality of \( \mathbb{X} \).

**Lemma 3.2.** Function \( \chi^\text{hc}_\psi \) is a regularity modulus on \( \mathcal{N}_0 \) if

\[ \psi(t)/P_t(\mathbb{X}) \to \infty \quad \text{as} \quad t \downarrow 0. \]

**Proof.** In view of the compactness of \( \mathbb{X} \), it is possible to bound \( h \in \mathcal{C}_\infty \) by a constant \( \lambda \), so that \( \chi^\text{hc}_\psi \) is a regularity modulus if

\[ \mathcal{H}_\lambda = \{ Y \in \mathcal{N}_0 : \chi^\text{hc}_\psi(Y) \leq \lambda Y(\mathbb{X})^2 \} \]

is compact in \( \mathcal{N}_0 \) for each \( \lambda > 0 \). For this, it suffices to show that the total mass of all \( Y \in \mathcal{H}_\lambda \) is bounded by a fixed number and \( \mathcal{H}_\lambda \subset \mathcal{N}_\varepsilon \) for some \( \varepsilon > 0 \).

Let \( \gamma_t(n) \) be the minimal number of pairs \( (x_i, x_j) \) with \( i \neq j \), such that \( x_i, x_j \in Y \) and \( d(x_i, x_j) \leq t \) over all counting measures \( Y \) of total mass \( n \).

Take \( t \) such that \( \psi(t)/P_t(\mathbb{X}) > \lambda \). If \( Y(\mathbb{X}) \geq n \), then

\[ \chi^\text{hc}_\psi(Y) \geq \sum_{x_i, x_j \in Y, i \neq j} \psi(t) I_{d(x_i, x_j) \leq t} \geq \gamma_t(n) \psi(t). \]

Therefore,

\[ \mathcal{H}_\lambda \subset \{ Y : n^{-2} \gamma_t(n) \psi(t) \leq \lambda \} \]

consists of \( Y \) with total mass uniformly bounded by fixed number \( n_\lambda \). Indeed, by Lemma A.1,

\[ \lim_{n \to \infty} n^{-2} \gamma_t(n) \geq \lim_{n \to \infty} n^{-2} n \left( \frac{n}{P_t(\mathbb{X})} - 1 \right) = P_t(\mathbb{X})^{-1}. \]
Choose $\varepsilon > 0$ so that $\psi(t) \geq \lambda n_2^2$ for $t \leq \varepsilon$. For $Y \in \mathcal{H}_\lambda$ and any $x_i, x_j \in Y$, $\psi(d(x_i, x_j)) \leq \chi_{\psi}^{hc}(Y) \leq \lambda n_2^2$, whence $d(x_i, x_j) \geq \varepsilon$. Thus $\mathcal{H}_\lambda \subset \mathcal{N}_\varepsilon$, so $\mathcal{H}_\lambda$ is relatively compact.

The following theorem shows that the realizability condition can be split into the positivity condition (3.7) on the linear functional $\Phi$ and the regularity condition (3.9) on the correlation measure, so that the latter can be easily checked. Such a split is possible because the regularity modulus $\chi_{\psi}^{hc}$ can be approximated by functions from $\mathcal{G}$.

**Theorem 3.3.** A locally finite measure $\rho$ on $\mathbb{X} \times \mathbb{X}$ is the correlation measure of a simple point process $\xi$ such that $E_{\chi_{\psi}^{hc}}(\xi) \leq r$ for some $r \geq 0$ with $\psi$ satisfying (3.8) if and only if (3.7) holds and

\[(3.9) \quad \int_{\mathbb{X} \times \mathbb{X}} \psi(d(x, y)) \rho(dx dy) \leq r.\]

**Proof.** Necessity. The definition of the correlation measure implies that

\[\int_{\mathbb{X} \times \mathbb{X}} \psi(d(x, y)) \rho(dx dy) = E_{\chi_{\psi}^{hc}}(\xi) \leq r.\]

Sufficiency. First assume that $\psi$ only takes finite values. The proof consists in checking (2.2), which is equivalent to (2.5).

For each family of positive numbers $\{t_g, g \in \mathcal{G}\}$,

\[(3.10) \quad \sup_{g \in \mathcal{G}} \inf_{Y \in \mathcal{N}_0} [\chi(Y) - g(Y)] + \Phi(g) \leq \sup_{g \in \mathcal{G}} \inf_{Y \in \mathcal{N}_t} [\chi(Y) - g(Y)] + \Phi(g).\]

The crucial step of the proof consists in the careful choice of $t_g > 0$.

Fix $g \in \mathcal{G}$. For $t > 0$, define $\psi_t(s) = \psi(\max(t, s))$, $s \geq 0$. Since any $Y \in \mathcal{N}_t$ does not contain any two points at distance less than $t$, $\chi(Y) = g_{\psi_t}(X)$. Therefore,

\[(3.11) \quad \inf_{Y \in \mathcal{N}_t} [\chi(Y) - g(Y)] = \inf_{Y \in \mathcal{N}_t} (g_{\psi_t} - g)(Y).\]

Our aim is to prove that

\[(3.12) \quad \inf_{Y \in \mathcal{N}_t} (g_{\psi_t} - g)(Y) = \inf_{Y \in \mathcal{N}_0} (g_{\psi_t} - g)(Y),\]

because then, since $g_{\psi_t} \in \mathcal{G}$, the positivity of $\Phi$ on $\mathcal{G}$ yields that (3.11) is not greater than $\Phi(g_{\psi_t} - g)$. Thus, (3.10) is bounded above by

\[\sup_{g \in \mathcal{G}} \Phi(g_{\psi_t} - g) + \Phi(g) \leq \sup_{t} \Phi(g_{\psi_t}) = \int_{\mathbb{X} \times \mathbb{X}} \psi(d(x, y)) \rho(dx dy)\]
by the monotone convergence theorem.

The proof of (3.12) relies on the proper choice for \( t \) (depending on \( g \)). Assume without loss of generality \( g = g_h \) for \( h \in C_0 \) with absolute value bounded by \( \lambda > 0 \). By (3.8), there exists \( t_0 \) such that \( \psi(t_0)/P_{t_0}(X) \geq \lambda + 1 \). By Lemma A.1, there is \( n_0 \) such that for all \( Y \) with mass \( n \geq n_0 \), the number of pairs of points of \( Y \) at distance at most \( t_0 \) satisfies

\[
\gamma_{t_0}(n) \geq n^2 \frac{1}{P_{t_0}(X)}.
\]

Choose \( t \leq t_0 \) so that \( \psi(t) > \lambda n_0^2 \) and consider any \( Y \in N_t \setminus N_t \). If \( Y(X) \leq n_0 \), then

\[
g_{\psi_0}(Y) \geq \psi(t) > \lambda n_0^2 \geq g_h(Y),
\]

while if \( Y(X) > n_0 \), then

\[
g_{\psi_0}(Y) - g_h(Y) \geq g_{\psi_0}(Y) - g_h(Y) \geq n^2 \psi(t_0) \gamma_{t_0}(Y) - \lambda n^2 > 0.
\]

Thus for \( Y \notin N_t \), we have \( g_{\psi_0}(Y) - g_h(Y) > 0 \). Therefore, the infimum of \( g_{\psi_0} - g_h \), which is non-positive because zero is obtained for \( Y = \emptyset \), is reached on \( N_t \), and (3.12) is proved.

Now assume that \( \psi(t) \) is infinite for \( t \in [0, \delta) \) and finite on \((\delta, \infty)\) with \( \delta > 0 \). If \( \psi(t) \to \infty \) as \( t \downarrow \delta \), then the above arguments apply with \( t_0 > \delta \) chosen such that \( \psi(t_0)/P_\delta(X) > \lambda \).

Assume that \( \psi(\delta) \) is finite. Let \( \psi_0(t) \) be a function satisfying (3.8) and finite for all \( t > 0 \), e.g. \( \psi_0(t) = t^{-1} P_t(X) \). Define \( \psi^*(t) = \psi(t) \) for \( t \geq \delta \) and let \( \psi^*(t) = \psi_0(t) + a \) for \( t \in (0, \delta) \) with a sufficiently large \( a \), so that \( \psi^* \) is monotone right-continuous, and \( \chi_{\psi^*}^h \) is a regularity modulus. Applying the previous arguments to \( \psi^* \) yields that there exists a point process \( \xi \) such that \( E \chi_{\psi^*}^h(\xi) \leq r \). Since \( r < \infty \), \( \rho \) vanishes on \( \{(x, y) : d(x, y) < \delta\} \), and so \( E \chi_{\psi}^h(\xi) = E \chi_{\psi^*}^h(\xi) \leq r \).

The following result is obtained by letting \( \psi \) be infinite on \([0, \varepsilon)\) and otherwise setting it to zero.

**Corollary 3.4.** A measure \( \rho \) on \( \mathbb{X} \times \mathbb{X} \) is the correlation measure of a point process \( \xi \) with \( \xi \in N_\varepsilon \) a.s. if and only if (3.7) holds and \( \rho(\{(x, y) : d(x, y) < \delta\}) = 0 \).

The following result yields a direct realisability condition for \( \rho \) without mentioning a regularity modulus.
Theorem 3.5. Let $\rho$ be a locally finite measure on $\mathbb{X} \times \mathbb{X}$, and fix any $r \geq 0$. Then there exists, for every $r' > r$, a simple point process $\xi$ with correlation measure $\rho$, such that

\begin{equation}
E \sum_{x_i, x_j \in \xi, i \neq j} P_{d(x_i, x_j)}(\mathbb{X}) \leq r',
\end{equation}

if and only if (3.7) holds and

\begin{equation}
\int_{\mathbb{X} \times \mathbb{X}} P_{d(x, y)}(\mathbb{X}) \rho(dx dy) \leq r.
\end{equation}

Proof. Necessity. Call $h_t(x, y) = \min(t, P_{d(x, y)}(\mathbb{X}))$ for $x \neq y \in \mathbb{X}$ and $t > 0$. Assume that $\xi$ realises $\rho$ and satisfies (3.13). The monotone convergence theorem yields that

\[ \int_{\mathbb{X} \times \mathbb{X}} P_{d(x, y)}(\mathbb{X}) \rho(dx dy) = \lim_{t \to \infty} E g_{h_t}(\xi) \leq r' \]

for every $r' > r$, whence (3.14) holds.

Sufficiency. Define a measure on $\mathbb{R}_+$ by

\[ \rho'([a, b)) = \rho(\{(x, y) \in \mathbb{X} \times \mathbb{X} : a \leq d(x, y) < b\}). \]

Fubini’s theorem yields that

\[ r = \int_{\mathbb{R}_+} P_t(\mathbb{X}) \rho'(dt). \]

Let $\{t_k, k \geq 1\}$ be a strictly decreasing sequence of numbers such that

\[ \int_{[0, t_k)} P_t(\mathbb{X}) \rho'(dt) \leq 2^{-k}. \]

For $m \geq 1$, the function

\[ \psi_m(t) = \begin{cases} kP_t(\mathbb{X}) & \text{if } t_{k+1} \leq t < t_k, k \geq 1, \\ P_t(\mathbb{X}) & \text{if } t \geq t_m \end{cases} \]

is monotone right-continuous and satisfies $\psi_m(t)/P_t(\mathbb{X}) \to \infty$ as $t \to 0$. Then

\[ \int_{\mathbb{X} \times \mathbb{X}} \psi_m(d(x, y)) \rho(dx dy) = \int_{\mathbb{R}_+} \psi_m(t) \rho'(dt) \]

\[ \leq \int_{\mathbb{R}_+} P_t(\mathbb{X}) \rho'(dt) + \sum_{k \geq m} k2^{-k} \leq r + \sum_{k \geq m} k2^{-k}. \]
By Theorem 3.3, choosing $m$ sufficiently large yields the realisability of $\rho$ by a point process $\xi$ satisfying

$$
E \sum_{x_i, x_j \in \xi, i \neq j} P_d(x_i, x_j)(\mathbb{X}) \leq E \chi_{\psi/m}^h(\xi) \leq r + \sum_{k \geq m} k 2^{-k} < r'.
$$

\[\square\]

**Remark 3.6.** Let $\Theta$ be a group of continuous transformations on $\mathbb{X}$ that leave $\rho$ invariant, i.e. $\rho(\theta A \times \theta B) = \rho(A \times B)$ for all $\theta \in \Theta$ and Borel $A, B$. Since the regularity modulus $\chi_{\psi}^h$ can be approximated from below by a sequence of functions from $\mathcal{G}$, Theorem 2.10(i) is applicable and so the corresponding point process $\xi$ in Theorems 3.3, 3.5 and Corollary 3.4 can be chosen $\Theta$-stationary. If $\Theta$ consists of isometric transformations, then Theorem 2.10(ii) is also applicable.

### 3.4. Non-compact case and stationarity

Assume that $X = \mathbb{R}^d$ and $d(x, y) = \|x - y\|$ is the Euclidean metric. Let $\psi$ be a positive right-continuous monotone function on $\mathbb{R}_+$ such that $\psi(t)t^d \to \infty$ as $t \to 0$. Denote by $B_n$ the open ball of radius $n$ centred at 0. Given a known bound for the packing number in the Euclidean space [18, p. 78], Lemma 3.2 implies that $\chi_{\psi}^h$ is a regularity modulus on every $B_n, n \geq 1$. Define

$$
(3.15) \quad \chi_{\beta\psi}^h(Y) = \sum_{x_i, x_j \in Y, i \neq j} \beta(x_i, x_j) \psi(\|x_i - x_j\|)
$$

for a bounded lower semicontinuous strictly positive on $\mathbb{R}^d \times \mathbb{R}^d$ function $\beta$.

**Theorem 3.7.** Let $\rho$ be a locally finite measure on $\mathbb{R}^d \times \mathbb{R}^d$.

(i) The measure $\rho$ is realisable as the correlation measure of a point process $\xi$ that satisfies $E \chi_{\beta\psi}^h(\xi) \leq r$ if and only if (3.7) holds and

$$
(3.16) \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} \beta(x, y) \psi(\|x - y\|) \rho(dx dy) \leq r.
$$

(ii) Fix $r \geq 0$, let

$$
(3.17) \quad r_n = \int_{B_n \times B_n} \|x - y\|^{-d} \rho(dx dy), \quad n \geq 1,
$$

and let $\{\beta_n, n \geq 1\}$ be a sequence of non-increasing numbers converging to 0. Then the following assertions are equivalent.
REALISABILITY PROBLEM

(a) \((3.7)\) holds and

\[\sum_{n \geq 1} \beta_n(r_{n+1} - r_n) \leq r < \infty,\]

in particular every \(r_n, n \geq 1\), is finite.

(b) For every \(r' > r\) there exists \(\xi\) with correlation measure \(\rho\) and such that

\[\sum_{n \geq 1} (\beta_n - \beta_{n+1}) \mathbb{E} \sum_{x_i, x_j \in B_n, i \neq j} \|x_i - x_j\|^{-d} \leq r'.\]

Proof. Sufficiency. (i) The function \(\chi^{hc}_{\beta\psi}\) is a regularity modulus on \(\mathcal{N}_0\), since

\[\mathcal{H}_{c,h} = \{Y \in \mathcal{X} : \chi^{hc}_{\beta\psi}(Y) \leq c + g_h(Y)\}, \quad c \in \mathbb{R}, \ h \in \mathcal{C}_o,\]

is compact in \(\mathcal{N}_0\). This follows from Lemma 3.2, which yields the compactness of the restriction of \(Y\) from \(\mathcal{H}_{c,h}\) onto any compact set \(C\). Indeed, this family of restricted counting measures coincides with the family of simple counting measures supported by \(C\) such that \(\chi^{hc}_{\beta\psi}(Y) \leq c/m + g_{h/m}(Y)\), where \(m > 0\) is a lower bound of \(\beta(x,y)\) for \(x,y \in C\).

In order to apply Theorem 2.6 with the regularity modulus \(3.15\) and in view of \(2.5\) it suffices to show that

\[\inf_{Y \in \mathcal{N}_0} \left[\chi^{hc}_{\beta\psi}(Y) - g_h(Y)\right] + \int_{\mathbb{R}^d \times \mathbb{R}^d} h(x,y) \rho(dxdy) \leq r\]

for all \(h \in \mathcal{C}_o\). Assume that \(h\) is supported by a subset of \(B_n \times B_n\) for some \(n \geq 1\). Then \(3.20\) holds by the same reasoning as in the proof of Theorem 3.3 applied to the compact space \(B_n\) (One might first consider only \(Y \subset B_n\), and then note that the infimum over all \(Y \in \mathcal{N}_0\) is necessarily smaller). By Theorem 2.6, \(3.16\) implies the existence of a point process \(\xi\) with correlation measure \(\rho\) that satisfies \(\mathbb{E}X^{hc}_{\beta\psi}(\xi) \leq r\).

(ii) Define \(\mathcal{Y}_n = (B_n \times B_n) \setminus (B_{n-1} \times B_{n-1})\), \(n \geq 1\) (with \(B_0 = \emptyset\)). For every \(n \geq 1\), define the measure

\[\rho'_n((a,b)) = \rho((x,y) \in B_n \times B_n : a \leq \|x - y\| < b)),\]

and let

\[t^n_k = \sup \left\{ t > 0 : \int_{[0,t]} s^{-d} \rho'_n(ds) \leq 2^{-k} \right\}, \quad k \geq 1.\]
Since \( \rho_{n+1} \geq \rho'_n \) for every \( n \geq 1 \), for every \( k, n \geq 1 \) we have \( t_{k+1}^n \leq t_k^n \). Let \( \{m_n, n \geq 1\} \) be a non-decreasing sequence of positive integers so that

\[
\sum_{n \geq 1} \beta_n (r_n - r_{n-1} + \sum_{k \geq m_n} k2^{-k}) \leq r'.
\]

(3.21)

Now define

\[
\psi_n(t) = \begin{cases} 
kt^{-d} & \text{if } t_{k+1}^n \leq t < t_k^n , \\
t^{-d} & \text{if } t \geq t_k^n .
\end{cases}
\]

Since \( m_n \leq m_{n+1} \), \( \psi_{n+1} \leq \psi_n \) for every \( n \geq 1 \). Function \( \psi_n \) satisfies \( \psi_n(t)t^d \to \infty \) as \( t \to 0 \), whence, for every \( n \geq 1 \), \( \psi_n \) is a regularity modulus on counting measures supported by \( B_n \) and

\[
\int_{\mathbb{Y}_n} \psi_n(||x - y||)\rho(\text{d}xdy) \leq \int_{\mathbb{R}^+} \psi_n(t)\rho_n''(\text{d}t) ,
\]

where

\[
\rho_n''([a, b]) = \rho(\{(x, y) \in \mathbb{Y}_n : a \leq ||x - y|| < b\}) \leq \rho_n'([a, b]) .
\]

Then

\[
\int_{\mathbb{Y}_n} \psi_n(||x - y||)\rho(\text{d}xdy) \leq \int_{\mathbb{R}^+} t^{-d}\rho_n''(\text{d}t) + \int_{t \leq t_{m_n}} \psi_n(t)\rho_n''(\text{d}t) .
\]

We have

\[
\int_{\mathbb{R}^+} t^{-d}\rho_n''(\text{d}t) = \int_{(B_n \times B_n) \setminus (B_{n-1} \times B_{n-1})} ||x - y||^{-d}\rho(\text{d}xdy) = r_n - r_{n-1}
\]

and

\[
\int_{t \leq t_{m_n}} \psi_n(t)\rho_n''(\text{d}t) \leq \int_{t \leq t_{m_n}} \psi_n(t)\rho_n'(\text{d}t) \leq \sum_{k \geq m_n} k2^{-k} ,
\]

whence

\[
\int_{\mathbb{Y}_n} \psi_n(||x - y||)\rho(\text{d}xdy) \leq r_n - r_{n-1} + \sum_{k \geq m_n} k2^{-k} .
\]

(3.22)

Define \( \psi(x, y) = \psi_n(||x - y||) \) for \( x, y \in \mathbb{Y}_n \). Since \( \psi_{n+1} \leq \psi_n \) and functions \( \psi_n, n \geq 1 \), are lower semicontinuous, the function \( \psi \) is lower semicontinuous on \( \mathbb{R}^d \times \mathbb{R}^d \). Define \( \beta(x, y) = \beta_n \) on \( \mathbb{Y}_n \). Since \( \beta_n, n \geq 1 \), decrease, \( \beta \) is a lower semicontinuous function on \( \mathbb{R}^d \times \mathbb{R}^d \). Since \( \psi_n \leq \psi_k \) for every \( k \leq n \), the restriction of \( \psi_{\beta_n} \) onto sets \( Y \subset B_n \) is larger than \( \psi_{\beta_n} \), whence \( \psi_{\beta_n} \)
is a regularity modulus on \( N_0 \). By Theorem 2.6, \( \Phi \) is realised by a point process \( \xi \) satisfying

\[
\mathbf{E}\chi_{\beta\psi}^{hc}(\xi) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \beta(x,y)\psi(x,y)\rho(dx\,dy) = \sum_{n \geq 1} \beta_n \int_{Y_n} \psi(||x-y||)\rho(dx\,dy).
\]

Since \( t^{-d} \leq \psi_n(t) \) for each \( n \) and \( t > 0 \),

\[
\mathbf{E}\chi_{\beta\psi}^{hc}(\xi) = \lim_{m \to \infty} \mathbf{E} \sum_{n=1}^m \beta_n (\chi_{\psi_n}^{hc}(\xi \cap B_n) - \chi_{\psi_n}^{hc}(\xi \cap B_{n-1})) \\
\geq \lim_{m \to \infty} \mathbf{E} \sum_{n=1}^m (\beta_n - \beta_{n+1})\chi_{\psi_n}^{hc}(\xi \cap B_n)) \\
\geq \sum_{n \geq 1} (\beta_n - \beta_{n+1}) \mathbf{E} \sum_{i \neq j, x_i, x_j \in B_n} ||x_i - x_j||^{-d},
\]

Using successively (3.22) and (3.21)

\[
\sum_{n \geq 1} \beta_n \int_{Y_n} \psi_n(||x-y||)\rho(dx\,dy) \leq \sum_{n \geq 1} \beta_n (r_n - r_{n-1} + \sum_{k \geq m_n} k2^{-k}) \leq r',
\]

we arrive at (3.19).

**Necessity.** For (ii) remark first that for \( x \neq y \in \mathbb{R}^d \)

\[
\beta(x,y)||x-y||^{-d} = \sum_{n \geq 1} (\beta_n - \beta_{n+1}) \mathbf{1}_{x,y \in B_n} ||x-y||^{-d},
\]

where \( \beta \) is defined in the sufficiency part of the proof. The function \( \beta\psi \) in (i) and the function \( (x,y) \mapsto \beta(x,y)||x-y||^{-d} \) in (ii) are lower semi-continuous and can therefore be approximated from below by compactly supported continuous functions. The rest follows from the monotone convergence theorem similarly to the proof of necessity in Theorem 3.5. \( \square \)

**Remark 3.8.** Remark for point (ii) that if each \( r_n, n \geq 1 \), is finite, there always exists a sequence \( \{\beta_n\} \) of sufficiently small numbers such that the right-hand side of (3.18) is finite.

If the distribution of point process \( \xi \) is invariant with respect to the group \( \Theta \) of translations of \( \mathbb{R}^d \), then \( \xi \) is called *stationary*. Its correlation measure \( \rho \) is translation invariant, i.e. \( \rho((A+x) \times (B+x)) = \rho(A \times B) \) for all \( x \in \mathbb{R}^d \) and so

\[
(3.23) \quad \rho(A \times B) = \lambda^2 \int_A \int_{\mathbb{R}^d} \mathbf{1}_{x+y \in B} \tilde{\rho}(dy)dx,
\]
where \( \lambda \) is the intensity of \( \xi \) and \( \bar{\rho} \) is a measure on \( \mathbb{R}^d \) called the reduced correlation measure, see [24, p. 76].

**Theorem 3.9.** Let \( \bar{\rho} \) be a locally finite measure on \( \mathbb{R}^d \), let \( \beta \) be a bounded lower semicontinuous strictly positive function on \( \mathbb{R}^d \) satisfying

\[
\beta(y) = \int_{\mathbb{R}^d} \beta(x, x + y) dx < \infty, \quad y \in \mathbb{R}^d,
\]

and let \( \psi \) be a monotone decreasing non-negative function such that \( t^d \psi(t) \to \infty \).

(i) \( \bar{\rho} \) is the reduced correlation measure of a stationary point process \( \xi \) that satisfies \( E \chi_{B_{\beta \psi}}(\xi) \leq r \) if and only if (3.7) holds and

\[
\int_{\mathbb{R}^d} \bar{\beta}(y) \psi(\|y\|) \bar{\rho}(dy) \leq r.
\]

(ii) \( \bar{\rho} \) is realisable as the reduced correlation measure of a stationary point process \( \xi \) that satisfies (3.19) for some sequence \( \{\beta_n, n \geq 1\} \) if and only if

\[
(3.24) \quad \int_B \|y\|^{-d} \bar{\rho}(dy) < \infty
\]

for some open ball \( B \) containing the origin. If \( \int_{\mathbb{R}^d} \|y\|^{-d} \bar{\rho}(dy) \) is finite, it is possible to let \( \beta_n = n^{-d-\delta}, n \geq 1 \), for any fixed \( \delta > 0 \).

**Proof.** It suffices to use (3.23) to confirm the conditions imposed in Theorem 3.7, see also Remark 3.8. In order to show that \( \xi \) can be chosen stationary, note that \( \chi_{B_{\beta \psi}} \) can be pointwisely approximated from below by a monotone sequence of functions from \( G \), so Theorem 2.10(i) applies.

3.5. Joint realisability of the intensity and correlation. Recall that the intensity measure \( \rho_1 \) of a point process \( \xi \) is defined from

\[
E \sum_{x_i \in \xi} h(x_i) = \int h(x) \rho_1(dx), \quad h \in C_{0,1},
\]

where \( C_{0,1} \) is the family of continuous functions on \( \mathbb{R} \) with compact support. A pair \( (\rho_1, \rho) \) of locally finite non-negative measures on \( \mathbb{R} \) and \( \mathbb{R} \times \mathbb{R} \) respectively is said to be jointly realisable if there exists a point process \( \xi \) with intensity measure \( \rho_1 \) and correlation measure \( \rho \).
Let $G_1$ be the vector space formed by constants and functions

$$g_{h_1,h}(Y) = \sum_{x \in Y} h(x) + g_h(Y), \quad Y \in \mathcal{N},$$

for $h_1 \in \mathcal{C}_{o,1}$ and $h \in \mathcal{C}_o$. It is easy to see that Assumption 2.4 is verified in this case. The pair $(\rho_1, \rho)$ yields a linear functional

$$\Phi(g_{h_1,h}) = \int_X h_1(x)\rho_1(dx) + \int_{X \times X} h(x,y)\rho(dx dy).$$

The realisability of $\Phi$ by a point process $\xi$ means that $\Phi(g_{h_1,h}) = E g_{h_1,h}(\xi)$. Functional $\Phi$ is positive on $G_1$ if and only if

$$\Phi(g_{h_1,h}) \geq \inf_{Y \in X} g_{h_1,h}(Y), \quad h_1 \in \mathcal{C}_{o,1}, \ h \in \mathcal{C}_o.$$

Similar arguments as before apply and yield the joint realisability conditions. Consider the special case of stationary processes in $X = \mathbb{R}^d$ with the reduced correlation measure $\bar{\rho}$ (see (3.23)) and intensity $\rho_1(dx) = \lambda dx$ proportional to the Lebesgue measure.

**Theorem 3.10.** Let $\lambda$ be a constant, and let $\bar{\rho}$ be a locally finite measure of $\mathbb{R}^d$. Then there is a stationary point process $\xi$ with intensity $\rho_1(dx) = \lambda dx$ and reduced correlation measure $\bar{\rho}$ if $\Phi$ given by (3.25) satisfies (3.26) with $X = \mathcal{N}_0$ and

$$\int_B \|z\|^{-d}\bar{\rho}(dz) < \infty$$

for some open set $B$ containing the origin.

**Proof.** It suffices to note that $g_{h_1,h}$ is dominated by $c g_h$ for a constant $c$ and follow the proof of (ii) in Theorem 3.7. The condition on $\bar{\rho}$ follows from (3.17) and (3.23). 

**4. Realisability of covering probabilities for random sets.** The nature of realisability problem changes with the choice of the family of subsets of a carrier space $X$ taken as possible values for a random set. We start with the case when a random set is allowed to be any subset of $X$, where realisability results are available under minimal conditions, while the obtained random set lacks properties and might even not be measurable. In the remainder of this section we treat the case of random closed sets, a classical setting in stochastic geometry. Some examples of possible regularity moduli are presented, along with the corresponding realisability results that resemble those of [11] in the point processes setting. The framework of random
measurable sets with finite perimeter (in the variational sense), developed in the forthcoming paper [5], provides a compromise between regularity of the random set and the explicitness of conditions.

4.1. Random binary functions. Let \( \mathcal{X} \) be the family of all subsets of \( \mathbb{X} \) identified with their indicator functions. Endow \( \mathcal{X} \) with the topology of pointwise convergence and the corresponding \( \sigma \)-algebra. Since \( \mathcal{X} \) is compact, Corollary 2.7 yields the following result.

**Theorem 4.1.** Let \( G \) be a vector space that consists of continuous functions on \( \mathcal{X} \) and includes constants, and let \( \Phi \) be a map from \( G \) to \( \mathbb{R} \). Then there exists a random indicator function \( \xi \), such that \( \Phi(g) = \mathbb{E}g(\xi) \) for all \( g \in G \) if and only if \( \Phi \) is a linear positive functional on \( G \) and \( \Phi(1) = 1 \).

The key issue in applying Theorem 4.1 is the choice of the space \( G \).

**Example 4.2 (One-point covering function).** Let \( G \) be generated by constants \( c \) and one-point indicator functions \( g_x(F) = \mathbb{I}_{x \in F}, F \in \mathcal{X} \), for \( x \in \mathbb{X} \). The positivity of a linear functional \( \Phi : G \mapsto \mathbb{R} \) together with \( \Phi(1) = 1 \) means that \( \Phi(g_x) \in [0,1] \) for all \( x \in \mathbb{X} \). Thus, a function \( p_x = \Phi(g_x) \) is a one-point covering function \( P\{x \in \xi\} \) for a random set \( \xi \) if and only if \( p_x \) takes values in \([0,1] \). Compare with Theorem 1.1, where the extra upper semicontinuity condition ensures that the corresponding random binary function is upper semicontinuous and so \( \xi \) is a random closed set.

**Example 4.3 (Covariances of random sets).** Consider vector space \( G \) generated by constants and functions \( g_{x,y}(F) = \mathbb{I}_{x,y \in F} \) for \( x,y \in \mathbb{X} \). The values of a linear functional \( \Phi \) on \( G \) are determined by \( p_{x,y} = \Phi(g_{x,y}) \), \( x,y \in \mathbb{X} \). By (2.1), \( \Phi \) is positive on \( G \) if and only if

\[
\sum_{ij=1}^n a_{ij}p_{x_i,x_j} \geq \inf_{F \in \mathcal{X}} \sum_{ij=1}^n a_{ij} \mathbb{I}_{x_i,x_j \in F}
\]

for all \( n \geq 1 \) and all matrices \((a_{ij})_{ij=1}^n \). In particular, if \( a_{ij} = a_i a_j \), then (4.1) implies the non-negative definiteness of \( p_{x,y} \), \( x,y \in \mathbb{X} \). Note that the one-point covering probabilities are specified if \( p_{x,y} \) are given.

4.2. The closedness condition. A random closed set \( \xi \) in a locally compact Hausdorff second countable space \( \mathbb{X} \) is a random element that takes values in the family \( \mathcal{X} = \mathcal{F} \) of closed subsets of \( \mathbb{X} \) equipped with the \( \sigma \)-algebra (called the Effros \( \sigma \)-algebra) generated by families \( \{ F \in \mathcal{F} : F \cap K \neq \emptyset \} \) for...
all compact sets $K$. The distribution of a random closed set $\xi$ is uniquely determined by its capacity functional

$$T(K) = \mathbb{P}\{\xi \cap K \neq \emptyset\}$$

for all $K$ from the family of all compact sets in $\mathbb{X}$, see [16] and [20, Th. 1.1.13].

Theorem 4.1 ensures only the existence of a binary stochastic process with given marginal distributions up to a certain order. However, it is not guaranteed that the constructed stochastic process has upper semicontinuous realisations, which should be the case if this process is the indicator of a random closed set in a topological space $\mathbb{X}$. If the carrier space $\mathbb{X}$ is finite (more generally, has a discrete topology), then this problem is avoided, since each random set is closed. Furthermore, the closedness issue can be settled in the following special case of two-point probabilities in the product form (and can be generalised for multi-point covering probabilities). The following result implies, in particular, that the random indicator function from Example 1.2 does not correspond to a random closed set. It also illustrates regularity problems arising in realisability problems for random closed sets.

**Theorem 4.4.** Assume that $\mathbb{X}$ is a separable space. A function

$$p_{x,y} = \begin{cases} p_x p_y & \text{if } x \neq y, \\ p_x & \text{if } x = y \end{cases}$$

is a two-point covering function of a random closed set if and only if $p_x$, $x \in \mathbb{X}$, is an upper semicontinuous function with values in $[0, 1]$ such that each point $x$ with $p_x \in (0, 1)$ has an open neighbourhood $U$ such that $p_y > 0$ only for at most a countable number of $y \in U$ and the sum of $p_y$ for $y \in U$ is finite.

**Proof.** Sufficiency. Note that the set $L = \{x : p_x = 1\}$ is closed by the upper semicontinuity of $p_x$. The separability of $\mathbb{X}$ and the condition of theorem imply that the set $M = \{x : p_x \in (0, 1)\}$ is at most countable. The sufficiency is obtained by a direct construction of a random subset $Z$ of $M$ that contains each point $x$ with probability $p_x$ independently of all other points. It remains to show that the random set $\xi = Z \cup L$ is closed. Consider $x \in M$ and its neighbourhood from the condition of theorem. Since $\sum p_y < \infty$, only a finite number of $y$ belong to $Z$ and so they do not converge to $x$. Thus, $x$ with probability zero appears as a limit of other points from $\xi$ unless $x \in L$ and so belongs to $\xi$ almost surely.

Necessity. The function $p_x = \mathbb{P}\{x \in \xi\}$ is upper semicontinuous, since $\xi$ is a random closed set. The product form of the two-point covering function
implies that the capacity functional on two-point set is given by
\[ T(\{x, y\}) = p_x + p_y - p_x p_y. \]
The upper semicontinuity property of the capacity functional yields that
\[ \limsup_{y \to x} T(\{x, y\}) \leq p_x, \]
while the monotonicity implies that \( T(\{x, y\}) \to p_x \) as \( y \to x \) for all \( x \). Unless \( p_x = 1 \), we have \( p_y \to 0 \).

Assume that \( p_x > 0 \) and \( p_{x_n} > 0 \), where \( x_n \to x \) and \( x_n \neq x \) with \( \sum p_{x_n} = \infty \). A variant of the lemma of Borel–Cantelli for pairwise independent events (see [4, Lemma 6.2.5]) implies that almost surely infinitely many points \( x_n \) belong to \( \xi \), so that \( x \in \xi \) a.s. by the closedness of \( \xi \) and so \( p_x = 1 \). Thus, the sum of \( p_{x_n} \) for each sequence \( \{x_n\} \) in a neighbourhood of \( x \) is finite. This rules out the existence of uncountably many \( y \) with \( p_y > 0 \) in any neighbourhood of \( x \). Indeed, then \( \{y : p_y \geq 1/n\} \) is finite for all \( n \), and so the union of such sets is countable. \( \square \)

It is known that \( \mathcal{F} \) is compact in the Fell topology that generates the Effros \( \sigma \)-algebra, see [20]. However, Corollary 2.7 is not applicable, since functions \( I_{x, y \in F}, F \in \mathcal{F} \), generating the vector space \( G \), do not generate the Effros \( \sigma \)-algebra on \( \mathcal{F} \).

It is known [20, Th. 1.2.6] that the \( \sigma \)-algebra generated by \( G \) on the family of regular closed sets (that coincide with closures of their interiors) coincides with the trace of the Effros \( \sigma \)-algebra on the family of regular closed sets. However, the family of regular closed sets is no longer compact in the Fell topology. Furthermore, indicator functions are not continuous in the Fell topology, so it is again not possible to appeal to Corollary 2.7 or explicitly check the upper semicontinuity condition required in Daniell’s theorem.

In order to describe a useful family \( G \) of functionals acting on sets, consider a \( \sigma \)-finite measure \( \nu \) on \( X \) and define
\[
(4.2) \quad g_h(F) = \int_{F \times F} h(x, y) \nu(dx) \nu(dy)
\]
for all measurable \( F \subset X \) and \( h \) from the family \( \mathcal{C}_o \) of symmetric continuous functions with compact support in \( X \times X \). A function \( p_{x,y}, x, y \in X \), generates a functional acting on \( g_h \) as
\[
(4.3) \quad \phi(g_h) = \int_{X \times X} p_{x,y} h(x, y) \nu(dx) \nu(dy). 
\]
The function \( p_{x,y} \) is said to be weakly realisable as the two-point covering function if there exists a random set \( \xi \) (or the corresponding random indicator function) such that \( \xi \) is almost surely measurable and \( \mathbb{E}g_h(\xi) = \Phi(g_h) \) for all \( h \in \mathcal{C}_o \). By approximating the atomic masses at two points with continuous functions, it is easy to see that the weak realisability is equivalent to \( \Phi(g_{x,y}) = p_{x,y} \) for \( \nu \otimes \nu \)-almost all \((x,y)\), in contrast to the strong realisability requiring this equality everywhere. The strong and weak realisability do not coincide in general. For instance, a non-positive function which vanishes almost everywhere, but takes some negative values is weakly realisable by the empty set, but not strongly realisable. Nevertheless, in the case of a stationary random regular closed set \( \xi \) in \( \mathbb{R}^d \) and the Lebesgue measure \( \nu \), the strong and weak realisability properties coincide, see Theorem 4.7.

In view of the required continuity property of functions from \( \mathcal{G} \) it is essential to ensure that \( g_h(F), F \in \mathcal{F} \), defined in (4.2) is continuous in the Fell topology. Note that it is not the case for most non-trivial measures \( \nu \), even for the Lebesgue measure. The continuity holds only on some subfamilies of \( \mathcal{F} \) considered in the following sections.

4.3. Neighbourhoods of closed sets. For simplicity, in the following consider random sets in the Euclidean space, i.e. assume that \( X = \mathbb{R}^d \) with Euclidean metric \( d \).

Let \( \mathcal{F}^\varepsilon \) be the family of \( \varepsilon \)-neighbourhoods of closed sets in \( \mathbb{R}^d \), i.e. \( \mathcal{F}^\varepsilon \) consists of \( F^\varepsilon = \{ x : d(x,F) \leq \varepsilon \} \) for all \( F \in \mathcal{F} \) and also contains the empty set. The vector space \( \mathcal{G} \) is generated by constants and the functions \( g_h \) defined by (4.2) with the Lebesgue measure \( \nu \).

**Lemma 4.5.** The space \( \mathcal{F}^\varepsilon \) with the Fell topology is compact and, for each \( h \in \mathcal{C}_o \), the function \( g_h \) is continuous on \( \mathcal{F}^\varepsilon \).

**Proof.** Recall that the upper limit of a sequence of sets \( \{F_n\} \) is the set of all limits for sequences \( \{x_{n_k}\} \) such that \( x_{n_k} \in F_{n_k} \) for all \( k \), while the lower limit is the set of all limits for convergent sequences \( \{x_n\} \) such that \( x_n \in F_n \) for all \( n \). The sequence of closed sets converges in the Fell topology if its upper and lower limits coincide.

If \( F_n = F^\varepsilon_{n,0} \in \mathcal{F}^\varepsilon \) converges to \( F \) in the Fell topology, then we can assume without loss of generality (by passing to subsequences) that \( F_{n,0} \) converges to \( F_0 \), so that \( F = F_0 \) and \( F \in \mathcal{F}^\varepsilon \). Thus, \( \mathcal{F}^\varepsilon \) is a closed subset of \( \mathcal{F} \) and so is compact, since \( \mathcal{F} \) is compact itself.

Consider a non-negative \( h \in \mathcal{C}_o \) supported by a ball \( B_R \) centred at the origin with sufficiently large radius \( R \). If \( F_n \to F \) in the Fell topology, then the upper limit of \( (F_n \cap B_R) \) is a subset of \( (F \cap B_R) \). Thus, \( g_h(F) = \)
\( \Phi( (4.4) \inf \Phi( x, y )) \)

where

\( \Phi( (4.4) \inf \Phi( x, y )) \)

is given by (4.3).

**Proof.** In view of the continuity of \( \Phi( (4.4) \inf \Phi( x, y )) \), it suffices to refer to Corollary 2.7.

In order to handle random sets with realisations from the space \( \mathcal{F}^{\infty} = \bigcup_{\varepsilon > 0} \mathcal{F}^{\varepsilon} \), we need the regularity modulus \( \chi(F) \) defined as the infimum of \( \varepsilon > 0 \) such that \( F \in \mathcal{F}^{1/\varepsilon} \) and \( \chi(F) = \infty \) if \( F \notin \mathcal{F}^{0} \).

**Theorem 4.7.** For any given \( r > 0 \), a function \( p_{x,y}, x, y \in \mathbb{R}^{d} \), is weakly realisable by a random closed set \( \xi \) such that \( \mathbb{E} \chi(\xi) \leq r \) if and only if

\[ \inf_{F \in \mathcal{F}^{0}} [\chi(F) - \Phi( (4.4) \inf \Phi( x, y ))) + \Phi( (4.4) \inf \Phi( x, y ))) \leq r, \quad h \in \mathcal{C}_0, \]

where \( \Phi( (4.4) \inf \Phi( x, y ))) \) is given by (4.3). If, additionally, \( p_{x,y} \) is an even continuous function of \( x - y \), then \( p_{x,y} \) is strongly realisable by a stationary random closed set \( \xi \).
Proof. Function $\chi$ is lower semicontinuous, since $\{F \in \mathcal{F} : \chi(F) \leq c\} = \mathcal{F}^{1/c}$ is closed for all $c > 0$. Furthermore,

$$\{F \in \mathcal{F} : \chi(F) \leq g_h(F)\} \subset \{F \in \mathcal{F} : \chi(F) \leq c\},$$

where $c = \int |h(x,y)|\nu(dx)\nu(dy)$ is a finite upper bound for $g_h(F)$. The left-hand side of (4.5) is compact, since $g_h$ is continuous on $\mathcal{F}^{1/c}$ by Lemma 4.5 and the right-hand side of (4.5) is compact. Thus, $\chi$ is a regularity modulus and the result follows from Theorem 2.6 and (2.2).

Note that the regularity modulus $\chi$ is invariant for the group $\Theta$ of translations of $\mathbb{R}^d$. By Theorem 2.10(ii), $\xi$ can be chosen to be stationary. In order to confirm the strong realisability, it remains to show that the covariance function of a stationary regular closed random set is continuous.

Since $\chi(\xi)$ is integrable, $\xi \in \mathcal{F}^0$, so that $\xi$ is almost surely regular closed and its boundary $\partial \xi$ has a.s. vanishing Lebesgue measure. By Fubini’s theorem, almost every point $x$ belongs to the boundary of $\xi$ with probability zero, and so $P\{x \in \partial \xi\} = 0$ for all $x$ in view of the stationarity property.

Let $P\{x,y \in \xi\}$ be the covariance function of $\xi$. Take $x,y \in \mathbb{R}^d$, and $(x_n, y_n)$ that converges to $(x,y)$. Since with probability 1, $x$ does not belong to $\partial \xi$, $\mathbb{I}_{x \in \xi}$ is almost surely equal to $\mathbb{I}_{x \in \text{Int}(\xi)}$ for the interior $\text{Int}(\xi)$ of $\xi$ and so $\mathbb{I}_{x_n \in \xi}$ almost surely converges to $\mathbb{I}_{x \in \xi}$. Similarly, $\mathbb{I}_{y_n \in \xi} \to \mathbb{I}_{y \in \xi}$ a.s., whence the product converges too $\mathbb{I}_{x_n \in \xi, y_n \in \xi} \to \mathbb{I}_{x \in \xi, y \in \xi}$. The Lebesgue theorem yields that $P\{x_n, y_n \in \xi\} \to P\{x, y \in \xi\}$. Since $p_{x,y}$ and $P\{x, y \in \xi\}$ are both continuous and coincide almost surely, they are equal everywhere. The continuity of $P\{x, y \in \xi\}$ can be also obtained by referring to a result of [21] saying that the capacity functional of each stationary regular closed random set is continuous in the Hausdorff metric.

4.4. Convexity restrictions. The family $\mathcal{C}$ of convex closed sets in $\mathbb{R}^d$ (including the empty set) is closed in the Fell topology and it is easy to see that the function $g_h$ given by (4.2) is continuous on $\mathcal{C}$. Corollary 2.7 yields that $p_{x,y}$ is weakly realisable for a convex random closed set if and only if

$$\Phi(g_h) \geq \inf_{F \in \mathcal{C}} g_h(F)$$

for the functional $\Phi(g_h)$ given by (4.3).

Let $\mathcal{P}$ be the convex ring in $\mathbb{R}^d$, i.e. the family of finite unions of compact convex subsets of $\mathbb{R}^d$. For $F \in \mathcal{P}$ let $\chi(F)$ be the smallest number $k$, such that $F$ can be represented as the union of $k$ convex compact sets.

Theorem 4.8. Let $\Phi$ be linear functional defined by (4.3). Fix any $r > 0$. Then there is a random closed set $\xi$ with realisations in $\mathcal{P}$ such that
\[ E_{g_h}(\xi) = \Phi(g_h) \text{ for all } h \in C_0 \text{ and } E\chi(\xi) \leq r \text{ if and only if } \]
\[ \inf_{F \in P} \left[ \chi(F) - g_h(F) \right] + \Phi(g_h) \leq r, \quad h \in C_0. \]

**Proof.** The family \( P_k \) of unions of at most \( k \) convex compact sets is closed in \( F \) and so is compact, whence \( \chi \) is lower semicontinuous. It is easily seen that \( g_h \) is continuous on convex compact sets, and so is also continuous on \( P_k \). Thus, \( g_h \) is \( \chi \)-regular and Theorem 2.6 applies.

If \( X = [0,1] \), then \( P \) is the family of finite unions of segments in \([0,1]\). The number of convex components of \( F \subset [0,1] \) is the variation of its indicator function,
\[ \chi(F) = \sup \sum_{i=0}^{n-1} |\mathbb{I}_{t_i \in F} - \mathbb{I}_{t_{i+1} \in F}| \]
where the supremum is taken over partitions \( 0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1 \).

The quantity
\[ \nu(F) = \sup_{\varphi \in C^1, 0 \leq \varphi \leq 1} \int_F \varphi'(x)dx, \]
where \( C^1 \) is the family of differentiable functions on \([0,1]\), captures the number of components of \( F \) with non-empty interiors, in particular \( \nu(F) \leq \chi(F) \). Remark that \( \nu \) is not a regularity modulus, because a set \( F \) with small \( \nu(F) \) can contain an arbitrarily large number of isolated singletons.

**Theorem 4.9.** If \( p_{x,y} \) is a function of \( x,y \in [0,1] \) such that
\[ \sup_{\varphi \in C^1, 0 \leq \varphi \leq 1} \int_{[0,1] \times [0,1]} p_{x,y} \varphi'(x)\varphi'(y)dxdy = \infty, \]
then there is no random closed set \( \xi \) satisfying \( E\chi(\xi)^2 < \infty \) having \( p_{x,y} \) as its two-point covering function.

**Proof.** Let \( H \) be the family of functions \( h(x,y) = \varphi'(x)\varphi'(y) \) for \( \varphi \in C^1 \) with \( 0 \leq \varphi \leq 1 \). Then
\[ \nu(F)^2 = \sup_{h \in H} \int_{[0,1] \times [0,1]} \mathbb{I}_{x,y \in F} h(x,y)dxdy = \sup_{h \in H} g_h(F). \]

Theorem 2.6 implies that \( \Phi \) is realisable by a random closed set \( \xi \) with \( E\chi(\xi)^2 < \infty \) if and only if
\[ \sup_{h \in C_0} \left[ \inf_{F \in F} \left[ \chi(F)^2 - g_h(F) \right] + \Phi(g_h) \right] < \infty. \]
It implies in particular

\[ \sup_{h \in H} \left[ \inf_{F \in \mathcal{X}} [\chi(F)^2 - g_h(F)] + \Phi(g_h) \right] < \infty. \]

Since \( \chi(F)^2 \geq \nu(F)^2 \geq g_h(F) \) for \( h \in H \), this condition would imply that

\[ \sup_{h \in H} \Phi(g_h) < \infty, \]

contradicting (4.6). Thus \( \Phi \) is not realisable.

Further results on realisability of random sets can be found in [5], where it is shown that by relaxing the closedness assumption it is possible to split the positivity and regularity conditions as it was the case in Section 3.3.

5. Contact distribution functions for random sets. Results from Section 4 concern realisability of the two-point covering probabilities, which are closely related to the values of the capacity functional (hitting probabilities) on two-point sets. Here we consider the realisability problem for a capacity functional defined on the family of balls in \( \mathbb{R}^d \). If \( T \) is the capacity functional of a random closed set \( \xi \), then

\[ T(B_R(x)) = P\{\xi \cap B_R(x) \neq \emptyset\} \]

is closely related to the spherical contact distribution function \( P\{d(x, \xi) \leq R | x \notin \xi\}, R \geq 0 \), which is the cumulative distribution function of the distance between \( \xi \) and \( x \) given that \( x \notin \xi \).

**Theorem 5.1.** A function \( \tau_x(R), R \geq 0, x \in A \subset \mathbb{R}^d \), is realisable as \( T(B_R(x)) \) for a random closed set \( \xi \) if and only if

\[ \Phi(g) = \sum_{i=1}^{m} a_i \tau_{x_i}(R_i) \geq 0 \tag{5.1} \]

for all \( m \geq 1, x_1, \ldots, x_m \in A \) and \( R_1, \ldots, R_m \geq 0 \), such that the function

\[ g(F) = \sum_{i=1}^{m} a_i \mathbb{1}_{B_{R_i}(x_i) \cap F \neq \emptyset} \geq 0, \quad F \in \mathcal{F} \tag{5.2} \]

is non-negative.
PROOF. The necessity is evident.

Sufficiency. Let $G$ be the vector space generated by constants and functions $g_{h,x}(F) = h(d(x,F))$, $F \in \mathcal{F}$, where $d(x,F)$ is the distance from $x \in \mathbb{R}^d$ to the nearest point of $F$, and $h$ is a continuous function on $\mathbb{R}$ with bounded support. The functions $g_{h,x}$ are all continuous in the Fell topology, since the Fell topology in $\mathbb{R}^d$ coincides with the topology of pointwise convergence of distance functions $d(x,F)$ for $x \in \mathbb{R}^d$, see [20, Th. B.12].

It suffices to show that $\Phi$ is positive on $G$. Let $g(F) = \sum_{i=1}^m a_i h_i(d(x_i,F))$. Uniform approximation of $h_1,\ldots,h_m$ by step functions on their supports yields a function $\hat{g}$ of the form (5.2) so that $\hat{g}(F) \geq -\varepsilon$ for some $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ and using (5.1) yield that

$$
\Phi(g) = \sum_{i=1}^m a_i \int h_i(t) d\tau_{x_i}(t) \geq 0.
$$

\[ \square \]

If $\tau_x(R) = \tau(R)$ does not depend on $x$, it may be possible to realise it as the contact distribution function of a stationary random closed set. If the argument $x$ of $\tau_x(R)$ takes only a single value, then the necessary and sufficient condition on $\tau_x(\cdot)$ is that it is a non-decreasing right-continuous function with values in $[0,1]$, i.e. the cumulative distribution function of a sub-probability measure on $\mathbb{R}_+$. The following result concerns the case of $x$ taking two possible values.

**Theorem 5.2.** Let $x_1, x_2 \in \mathbb{R}^d$, with $l = \|x_1 - x_2\|$, and let $\tau_{x_1}$ and $\tau_{x_2}$ be cumulative distribution functions of two sub-probability measures on $\mathbb{R}_+$. Then there exists a random closed set $\xi$ such that $\tau_{x_i}(R) = T(B_R(x_i))$ for $r \geq 0$ and $i = 1, 2$ if and only if for all $r \geq 0$

$$
\tau_{x_1}(\max(R-l,0)) \leq \tau_{x_2}(R) \leq \tau_{x_1}(R+l).
$$

**Proof.** Necessity: Let $\xi$ be a random closed set with $\tau_{x_1}(R) = T(B_R(x_i))$. Let $a_1$ and $a_2$ be random points such that $a_1, a_2 \in \xi$ a.s. and $R_i = d(x_i,a_i) = d(x_i,\xi)$, $i = 1, 2$, have cumulative distribution functions $\tau_{x_1}$ and $\tau_{x_2}$ respectively. Then $|R_1 - R_2| \leq l$. Indeed, if, for instance, $R_1 > R_2 + l$, then $a_2$ is nearer to $x_1$ than $a_1$ contrary to the assumption. Thus $R_1 \leq R$ implies $R_2 \leq R + l$, so that $\tau_{x_1}(R) \leq \tau_{x_2}(R+l)$. The symmetry argument with $x_1$ and $x_2$ interchanged yields (5.3).

Sufficiency. Define two random variables $R_1$ and $R_2$ as inverse functions to $\tau_{x_1}$ and $\tau_{x_2}$ applied to a single uniform random variable, so that (5.3)
yields that $|R_1 - R_2| \leq l$ a.s. This means that none of the balls $B_{R_1}(x_1)$ and $B_{R_2}(x_2)$ lies in the interior of the other one. Now construct random closed set $\xi$ consisting of two points: $a_1$ on the boundary of $B_{R_1}(x_1)$ but outside of the interior of $B_{R_2}(x_2)$ and $a_2$ on the boundary of $B_{R_2}(x_2)$ but outside of the interior of $B_{R_1}(x_1)$. Then $a_1$ is nearest to $x_1$ and $a_2$ is nearest to $x_2$ with given distributions of the distance.

APPENDIX: A COMBINATORIAL LEMMA

Recall that $P_t(\mathbb{X})$ denotes the packing number of $\mathbb{X}$ with metric $d$, i.e. the maximum number of points in the space $\mathbb{X}$ with pairwise distance exceeding $t$, see [18, p. 78].

**Lemma A.1.** If $Y = \sum \delta_{x_i}$ is a counting measure of total mass $n$, then for all $t > 0$, 
$$ \sum_{i \neq j} \mathbb{I}_{d(x_i, x_j) \leq t} \geq n \left( \frac{n}{P_t(\mathbb{X})} - 1 \right) . $$

**Proof.** Denote 
$$ n(Y, x_i) = Y(B_t(x_i)) - 1 , $$
where $B_t(x_i)$ is the closed ball of radius $t$ centred at $x_i$. Furthermore, denote 
$$ g_{h_t}(Y) = \sum_{i \neq j} \mathbb{I}_{d(x_i, x_j) \leq t} . $$

Then 
$$ g_{h_t}(Y - \delta_{x_i}) = g_{h_t}(Y) - 2n(Y, x_i) , $$
$$ g_{h_t}(Y + \delta_{x_i}) = g_{h_t}(Y) + 2n(Y, x_i) + 2 . $$

Let $x_i$ and $x_j$ be two distinct points from the support of $Y$ with $d(x_i, x_j) \leq t$. Assume that $n(Y, x_i) < n(Y, x_j)$ or $n(Y, x_i) = n(Y, x_j)$ with $i < j$ and define 
$$ Y' = Y - \delta_{x_j} + \delta_{x_i} $$
obtained from $Y$ by transferring a mass 1 from $x_j$ to $x_i$. Call $Y'' = Y - \delta_{x_j}$. Remark that $n(Y'', x_i) = n(Y, x_i) - 1$ because $d(x_i, x_j) \leq t$. Since $n(Y, x_j) \geq n(Y, x_i)$, 
$$ g_{h_t}(Y') = g_{h_t}(Y'') + 2n(Y'', x_i) + 2 $$
$$ = g_{h_t}(Y) - 2n(Y, x_j) + 2n(Y'', x_i) + 2 $$
$$ = g_{h_t}(Y) - 2n(Y, x_j) + 2n(Y, x_i) - 2 + 2 $$
$$ \leq g_{h_t}(Y) . $$
Furthermore, \( n(Y', x_i) = n(Y, x_i) \) because the transferred mass remains in the ball with centre \( x_i \) and radius \( t \), and \( n(Y', x_j) = n(Y, x_j) \) as well. Thus \( n(Y', x_i) \leq n(Y', x_j) \). Repeat the mass transfer from \( x_j \) to \( x_i \) until the mass at \( x_j \) disappears. Call the resulting counting measure \( Y_1 \).

Apply the same construction to \( Y_1 \) and repeat it until there are no more distinct points at distance at most \( t \). This happens in a finite time because the cardinality of the support of \( Y \) strictly decreases at each step.

The obtained counting measure \( \hat{Y} \) is supported by a set of points \( \{y_1, \ldots, y_q\} \) with pairwise distances exceeding \( t \). Thus,

\[
g_{ht}(Y) \geq g_{ht}(\hat{Y}) = \sum_{i=1}^{q} m_i(m_i - 1),
\]

where \( m_i = \hat{Y}(\{y_i\}) \). Under the restriction \( \sum_{i=1}^{q} m_i = n \), the minimal value \( \sum_{i=1}^{q} m_i(m_i - 1) \) is reached for \( m_i = n/q \), whence

\[
g_{ht}(Y) \geq n \left( \frac{n}{q} - 1 \right).
\]

It remains to note that \( q \leq P_t(\mathbb{X}) \).

It is also possible to define a counting measure by placing masses from the interval \( [n/q, n/q + 1) \) at the points forming the packing net of \( \mathbb{X} \). Thus, there exists a counting measure \( Y \) such that

\[
g_{ht}(Y) \leq n \left( \frac{n}{P_t(\mathbb{X})} + 1 \right).
\]

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