Polar Duality, John Ellipsoid, and Generalized Gaussians

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Abstract

We apply the notion of polar duality from convex geometry to the study quantum covariance ellipsoids in symplectic phase space. We consider in particular the case of “quantum blobs” introduced in previous work; quantum blobs are the smallest symplectic invariant regions of the phase space compatible with the uncertainty principle in its strong Robertson–Schrödinger form. We show that they can be characterized by a simple condition using polar duality, thus improving previous results. We apply these geometric results to the characterization of pure Gaussian states in terms of partial information on the covariance ellipsoid.

Keywords: polar duality; Lagrangian plane; polar duality; symplectic capacity; John and Löwner ellipsoids; uncertainty principle
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1 Introduction

In a recent paper [13] we pointed out the usefulness of the geometric notion of polar duality in expressing the uncertainty principle of quantum mechanics. In our discussion of polar duality we suggested that a quantum system localized in the position representation in a set $X$ cannot be localized in the
momentum representation in a set smaller than its polar dual $X^{\hbar}$, the latter being defined as the set of all $p$ in momentum space such that $px \leq \hbar$ for all $x \in X$. In the present work we go several steps further by studying the product sets $X \times X^{\hbar}$. In particular we find that when $X$ is an ellipsoid, then the John ellipsoid $X \times X^{\hbar}$ is a “quantum blob” (as defined in previous work [9, 11, 15]) to which one canonically associates a squeezed coherent state.

The two main results of this paper are

- **Theorem 9**: we prove that a centered phase space ellipsoid $\Omega$ is a quantum blob (i.e. a symplectic ball with radius $\sqrt{\hbar}$) if and only if the polar dual of the projection of $\Omega$ on the position space is the intese ction of $\Omega$ with the momentum space; this considerably strengthens a previous result obtained in [13];

- **Theorem 11**: it is an analytical version of Theorem 9 which we use to give a simple characterization of pure Gaussian states in terms of partial information on the covariance ellipsoid of a Gaussian state. This result is related to the so-called “Pauli problem”.

**Notation 1** The configuration space of a system with $n$ degrees of freedom will in general be written $\mathbb{R}^n_x$, and its dual (the momentum space) $\mathbb{R}^n_p$. The position variables will be written $x = (x_1, ..., x_n)$ and the momentum variables $p = (p_1, ..., p_n)$. The duality form (identified with the usual inner product) is $p \cdot x = p_1 x_1 + \cdots + p_n x_n$. The product $\mathbb{R}^n_x \times \mathbb{R}^n_p$ is identified with $\mathbb{R}^{2n}$ and is equipped with the standard symplectic form $\sigma$ defined by $\omega(z, z') = p \cdot x' - p' \cdot x$ if $z = (x, p)$, $z' = (x', p')$. The corresponding symplectic group is denoted $\text{Sp}(n)$: $S \in \text{Sp}(n)$ if and only $\omega(Sz, Sz') = \omega(z, z')$ for all $z, z'$. We denote by $\text{Sym}^{++}(n, \mathbb{R})$ the cone of real positive definite symmetric $n \times n$ matrices, and by $\text{GL}(n, \mathbb{R})$ the general (real) linear group (the invertible real $n \times n$ matrices).

## 2 A Geometric Quantum Phase Space

### 2.1 Polar duality and quantum states

Let $X \subset \mathbb{R}^n_x$ be a convex body: $X$ is compact and convex and has non-empty interior $\text{int}(X)$. If $0 \in \text{int}(X)$ we define the $\hbar$-polar dual $X^{\hbar} \subset \mathbb{R}^n_p$ of $X$ by

$$X^{\hbar} = \{ p \in \mathbb{R}^m : \sup_{x \in X} (p \cdot x) \leq \hbar \} \quad (1)$$
where \(\hbar\) is a positive constant (we have \(X^h = \hbar X^o\) where \(X^o\) is the traditional polar dual dual from convex geometry). The following properties of polar duality are obvious \[23]\:

- \((X^h)^h = X\) (reflexivity) and \(X \subset Y \implies Y^h \subset X^h\) (anti-monotonicity),
- For all \(L \in GL(n, \mathbb{R})\):
  \[ (LX)^h = (L^T)^{-1}X^h \]
  (scaling property). In particular \((\lambda X)^h = \lambda^{-1}X^h\) for all \(\lambda \in \mathbb{R}, \lambda \neq 0\).

We can view \(X\) and \(X^h\) as subsets of phase space by the identifications \(\mathbb{R}^n_x \equiv \mathbb{R}^n_x \times 0\) and \(\mathbb{R}^n_p \equiv 0 \times \mathbb{R}^n_p\). Writing \((LX)^h = (L^T)^{-1}X^h\) for all \(L \in GL(n, \mathbb{R})\) the transformation \(X \rightarrow X^h\) is a mapping \(\ell_X \rightarrow \ell_P\). With this interpretation formula (2) can be rewritten in symplectic form as

\[ (M_{L^{-1}}X)^h = M_{L^T}X^h \]  (3)

where \(M_{L^{-1}} = \begin{pmatrix} L^{-1} & 0 \\ 0 & L^T \end{pmatrix}\) is in \(Sp(n)\). Notice that \(M_{L^{-1}} : \ell_X \rightarrow \ell_X\) and \(M_{L^T} : \ell_P \rightarrow \ell_P\).

Suppose now that \(X\) is an ellipsoid centered at the origin:

\[ X = \{ x \in \mathbb{R}^n_x : Ax \cdot x \leq \hbar \} \]  (4)

where \(A \in \text{Sym}_{++}(n, \mathbb{R})\). The polar dual \(X^h\) is the ellipsoid

\[ X^h = \{ p \in \mathbb{R}^n_p : A^{-1}p \cdot p \leq \hbar \} \]  (5)

In particular the ball \(B^n_X(\sqrt{\hbar}) = \{ x : |x| \leq \sqrt{\hbar} \}\) is \((B^n_X(\sqrt{\hbar}))^h = B^n_P(\sqrt{\hbar})\).

Let \(\Omega\) be a convex body in \(\mathbb{R}^{2n}\). Recall \[4\] that the John ellipsoid \(\Omega_{\text{John}}\) is the unique ellipsoid in \(\mathbb{R}^{2n}\) with maximum volume contained in \(\Omega\). If \(M \in GL(2n, \mathbb{R})\) then

\[ (M(\Omega))_{\text{John}} = M(\Omega_{\text{John}}). \]  (6)

In previous work \[11\ [15\] we called the image of the phase space ball \(B^{2n}(\sqrt{\hbar})\) by some \(S \in \text{Sp}(n)\) a “quantum blob”. Quantum blobs are minimum quantum uncertainty phase space units. The product \(X \times X^h\) contains a unique quantum blob:
**Proposition 2** Let \( X = \{ x : Ax \cdot x \leq h \} \). The John ellipsoid of the quantum state \( X \times X^h \) is a a quantum blob, namely

\[
(X \times X^h)_{\text{John}} = M_{A^{1/2}}(B^{2n}(\sqrt{h}))
\]

where \( M_{A^{1/2}} = \begin{pmatrix} A^{-1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \in \text{Sp}(n) \).

**Proof.** That \( S_{A^{1/2}} \in \text{Sp}(n) \) is clear. Let \( B^n_X(\sqrt{h}) \) and \( B^n_P(\sqrt{h}) \) be the balls with radius \( \sqrt{h} \) in \( \mathbb{R}^n_x \) and \( \mathbb{R}^n_p \), respectively. We have, by (4), (5), and (6),

\[
(X \times X^h)_{\text{John}} = (A^{-1/2}B^n_X(\sqrt{h}) \times A^{1/2}B^n_P(\sqrt{h}))_{\text{John}}
= M_{A^{1/2}}(B^n_X(\sqrt{h}) \times B^n_P(\sqrt{h}))_{\text{John}}
\]

Let us show that

\[
(B^n_X(\sqrt{h}) \times B^n_P(\sqrt{h}))_{\text{John}} = B^{2n}(\sqrt{h});
\]

this will prove our assertion. The inclusion \( B^{2n}(\sqrt{h}) \subset B^n_X(\sqrt{h}) \times B^n_P(\sqrt{h}) \) is obvious, and we cannot have \( B^{2n}(R) \subset B^n_X(\sqrt{h}) \times B^n_P(\sqrt{h}) \) if \( R > 1 \). Assume now that the John ellipsoid \( \Omega_{\text{John}} \) of \( \Omega = B^n_X(\sqrt{h}) \times B^n_P(\sqrt{h}) \) is defined by

\[
Ax \cdot x + Bx \cdot p + Cp \cdot p \leq h
\]

where \( A, C > 0 \) and \( B \) are real symmetric \( n \times n \) matrices. Since \( \Omega \) is invariant by the transformation \( (x, p) \mapsto (p, x) \) so is \( \Omega_{\text{John}} \) and we must thus have \( A = C \) and \( B = B^T \). Similarly, \( \Omega \) being invariant by the partial reflection \( (x, p) \mapsto (-x, p) \) we get \( B = 0 \) so \( \Omega_{\text{John}} \) is defined by \( Ax \cdot x + Ap \cdot p \leq 1 \). The next step is to observe that \( \Omega \), and hence \( \Omega_{\text{John}} \), are invariant under all symplectic rotations \( (x, p) \mapsto (Hx, HP) \) where \( H \in O(n, \mathbb{R}) \) so we must have \( AH = HA \) for all \( H \in O(n, \mathbb{R}) \), but this is only possible if \( A = \lambda I_n \) for some \( \lambda \in \mathbb{R} \). The John ellipsoid is thus of the type \( B^{2n}(\lambda^{-1/2}) \) for some \( \lambda \geq 1 \) and this concludes the proof in view of the inclusion \( B^{2n}(\sqrt{h}) \subset B^n_X(\sqrt{h}) \times B^n_P(\sqrt{h}) \) since we cannot have \( \lambda > 1 \).

**Remark 3** The John ellipsoid \( (X \times X^h)_{\text{John}} \) is the set of all \( (x, p) \in \mathbb{R}^{2n} \) such that \( Ax \cdot x + A^{-1}p \cdot p \leq h \). The orthogonal projections of \( (X \times X^h)_{\text{John}} \) on the coordinate planes \( \ell_X = \mathbb{R}^n_x \times 0 \) and \( \ell_P = 0 \times \mathbb{R}^n_p \) are therefore \( \Pi_X(X \times X^h)_{\text{John}} = X \) and \( \Pi_P(X \times X^h)_{\text{John}} = X^h \).
The construction above shows that we have a canonical identification between the ellipsoids $X = \{x : Ax \cdot x \leq h\}$ and the squeezed coherent states

$$\phi_A(x) = (\pi h)^{-n/4} (\det A)^{1/4} e^{-Ax \cdot x/2h}.$$  

(8)

In fact, the covariance ellipsoid of $\phi_A$ is precisely the John ellipsoid of the product $X \times X^h$ as can be seen calculating the Wigner transform of $\phi_A$

$$W\phi_A(z) = (\pi h)^{-n} (\det A)^{1/4} \exp \left[ -\frac{1}{h} (Ax \cdot x + A^{-1}p \cdot p) \right]$$  

(9)

which corresponds to the canonical bijection

$$X \mapsto (X \times X^h)_{\text{John}}$$

between (centered) configuration space ellipsoids $X$ and John ellipsoids of $X \times X^h$ (we will have more to say about this correspondence in the forthcoming sections).

### 2.2 Polar duality and the symplectic camel

Symplectic capacities (see [4, 15] for reviews) are numerical invariants that serve as a fundamental tool in the study of various symplectic and Hamiltonian rigidity phenomena; they are closely related to Gromov’s symplectic non-squeezing theorem [16].

We denote $\text{Symp}(n)$ the group of all symplectomorphisms $(\mathbb{R}^{2n}, \omega) \rightarrow (\mathbb{R}^{2n}, \omega)$. That is, $f \in \text{Symp}(n)$ if and only if $f$ is a diffeomorphism of $\mathbb{R}^{2n}$ whose Jacobian matrix $Df(z)$ is in $\text{Sp}(n)$ for every $z \in \mathbb{R}^{2n}$.

A (normalized) symplectic capacity on $(\mathbb{R}^{2n}, \sigma)$ associates to every subset $\Omega \subset \mathbb{R}^{2n}$ a number $c(\Omega) \in [0, +\infty]$ such that the following properties hold:

**SC1 Monotonicity:** If $\Omega \subset \Omega'$ then $c(\Omega) \leq c(\Omega')$;

**SC2 Conformality:** For every $\lambda \in \mathbb{R}$ we have $c(\lambda \Omega) = \lambda^2 c(\Omega)$;

**SC3 Symplectic invariance:** $c(f(\Omega)) = c(\Omega)$ for every $f \in \text{Symp}(n)$;

**SC4 Normalization:** For $1 \leq j \leq n$ we have $c(B^{2n}(r)) = \pi r = c(Z^{2n}_j(r))$

where $Z^{2n}_j(r)$ is the cylinder with radius $r$ based on the $x_j, p_j$ plane.

There exists a symplectic capacity, denoted by $c_{\text{max}}$, such that $c \leq c_{\text{max}}$ for every symplectic capacity. It is defined by

$$c_{\text{max}}(\Omega) = \inf_{f \in \text{Symp}(n)} \{ \pi r^2 : f(\Omega) \subset Z^{2n}_j(r) \}$$  

(10)
where $Z_j^{2n}(r)$ is the phase space cylinder defined by $x_j^2 + p_j^2 \leq r^2$ and $\text{Symp}(n)$ the group of all symplectomorphisms of $\mathbb{R}^{2n}$ equipped with the standard symplectic structure. Similarly, there exists a smallest symplectic capacity $c_{\min}$, it is defined by

$$c_{\min}(\Omega) = \sup_{f \in \text{Symp}(n)} \{ \pi r^2 : f(B^{2n}(r)) \subset \Omega \}.$$  

One shows [1, 2] that if $X \subset \mathbb{R}^n$ and $P \subset \mathbb{R}^n$ are centrally symmetric convex bodies then we have

$$c_{\max}(X \times P) = 4\hbar \sup\{ \lambda > 0 : \lambda X \subset P \}.$$  

In particular,

$$c_{\max}(X \times X) = 4\hbar.$$  

One also has the weaker notion of linear symplectic capacity, obtained by replacing condition (SC3) with

**SC3lin** Linear symplectic invariance: $c(S(\Omega)) = c(\Omega)$ for every $S \in \text{Sp}(n)$ and $c(\Omega + z)) = c(\Omega)$ for every $z \in \mathbb{R}^{2n}$.

One then defines the corresponding minimal and maximal linear symplectic capacities $c_{\min}^{\text{lin}}$ and $c_{\max}^{\text{lin}}$

$$c_{\min}^{\text{lin}}(\Omega) = \sup_{S \in \text{Sp}(n)} \{ \pi R^2 : S(B^{2n}(z, R)), z \in \mathbb{R}^{2n} \}$$

$$c_{\max}^{\text{lin}}(\Omega) = \inf_{f \in \text{Sp}(n)} \{ \pi r^2 : S(\Omega) \subset Z_j^{2n}(z, r), z \in \mathbb{R}^{2n} \}.$$  

It turns out that all symplectic capacities agree on ellipsoids. They are calculated as follows: assume that

$$\Omega = \{ z \in \mathbb{R}^{2n} : Mz \cdot z \leq r^2 \}$$

where $M \in \text{Sym}^+(2n, \mathbb{R})$, and let $\lambda_1^\sigma, \lambda_2^\sigma, \ldots, \lambda_n^\sigma$ be the symplectic eigenvalue of $M$, i.e. the numbers $\lambda_j^\sigma > 0$ ($1 \leq j \leq n$) such that the $\pm i\lambda_j^\sigma$ are the eigenvalues of the antisymmetric matrix $M^{1/2}JM^{1/2}$. Then

$$c(\Omega) = \pi r^2 / \lambda_{\max}^\sigma$$  

where $\lambda_{\max}^\sigma = \max\{ \lambda_1^\sigma, \lambda_2^\sigma, \ldots, \lambda_n^\sigma \}$ (see [10, 15]). The following technical Lemma will allows us to prove a refinement of formula (12).
Lemma 4 Let $\Omega \subset \mathbb{R}^{2n}$ be a centrally symmetric body. We have

$$c_{\text{lin}}^{\min}(\Omega) = \sup_{S \in \text{Sp}(n)} \{ \pi R^2 : S(B^{2n}(R)) \subset \Omega \}. \quad (16)$$

Proof. Since $\Omega$ is centrally symmetric we have $S(B^{2n}(z_0, R)) \subset \Omega$ if and only if $S(B^{2n}(-z_0, R)) \subset \Omega$. The ellipsoid $S(B^{2n}(R))$ is interpolated between $S(B^{2n}(z_0, R))$ and $S(B^{2n}(-z_0, R))$ using the mapping $t \mapsto z(t) = z - 2tz_0$ where $z \in S(B^{2n}(z_0, R))$, and is hence contained in $\Omega$ by convexity. □

Proposition 5 Let $c_{\text{lin}}^{\min}$ be the smallest linear symplectic capacity and $X \subset \mathbb{R}^n_x$ a centered ellipsoid. We have

$$c_{\text{lin}}^{\min}(X \times X^h) = 4h. \quad (17)$$

Proof. In view of Lemma 4, $c_{\text{lin}}^{\min}(X \times X^h)$ is the greatest number $\pi R^2$ such that $X \times X^h$ contains a symplectic ball $S(B^{2n}(R))$, $S \in \text{Sp}(n)$. In view of Proposition 2, $M_{A_1/2}(B^{2n}(\sqrt{\hbar}))$ is such a symplectic ball; since it is also the largest ellipsoid contained in $X \times X^h$ we must have

$$c_{\text{lin}}^{\min}(X \times X^h) = c_{\text{lin}}^{\min}(M_{A_1/2}(B^{2n}(\sqrt{\hbar}))) = \pi h. \quad (18)$$

3 Projections of Quantum Blobs

In this section we generalize the observation made in Remark 3. Projecting Phase Space Ellipsoids

3.1 Block matrix notation

For $M \in \text{Sym}_{++}(2n, \mathbb{R})$ we define the phase space ellipsoid

$$\Omega = \{ z \in \mathbb{R}^{2n} : Mz \cdot z \leq \hbar \}. \quad (18)$$

Setting $M = \frac{1}{2} h \Sigma^{-1}$ we can visualize $\Omega$ as the covariance matrix of a (classical or quantum) state:

$$\Omega = \{ z \in \mathbb{R}^{2n} : \frac{1}{2} \Sigma^{-1} z \cdot z \leq 1 \}. \quad (19)$$

Writing $M$ and $\Sigma$ in block-matrix form Let us write $M$ in block form

$$M = \begin{pmatrix} M_{XX} & M_{XP} \\ M_{PX} & M_{PP} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XP} \\ \Sigma_{PX} & \Sigma_{PP} \end{pmatrix} \quad (20)$$

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where the blocks are \( n \times n \) matrices. The condition \( M > 0 \) ensures us that \( M_{XX} > 0, M_{PP} > 0, \) and \( M_{PX} = M_{XP}^T \) (resp. \( \Sigma_{XX} > 0, \Sigma_{PP} > 0, \) and \( \Sigma_{PX} = \Sigma_{XP}^T \); see [24]). Using classical formulas for the inversion of block matrices [22] we have

\[
M^{-1} = \left( \begin{array}{cc}
(M/M_{PP})^{-1} & -(M/M_{PP})^{-1}M_{XP}M_{PP}^{-1} \\
-M_{PP}^{-1}M_{PX}M_{PP}(M/M_{XX})^{-1} & (M/M_{XX})^{-1}
\end{array} \right) \tag{21}
\]

where \( M/M_{PP} \) and \( M/M_{XX} \) are the Schur complements:

\[
M/M_{PP} = M_{XX} - M_{XP}M_{PP}^{-1}M_{PX} \tag{22}
\]

\[
M/M_{XX} = M_{PP} - M_{PX}M_{XX}^{-1}M_{XP} \tag{23}
\]

Similarly,

\[
\Sigma^{-1} = \left( \begin{array}{cc}
(\Sigma/\Sigma_{PP})^{-1} & -(\Sigma/\Sigma_{PP})^{-1}\Sigma_{XP}\Sigma_{PP}^{-1} \\
-\Sigma_{PP}^{-1}\Sigma_{PX}(\Sigma/\Sigma_{BB})^{-1} & (\Sigma/\Sigma_{XX})^{-1}
\end{array} \right) \tag{24}
\]

Notice that these formulas imply

\[
\Sigma_{XX} = \frac{\hbar}{2}(M/M_{PP})^{-1}, \Sigma_{PP} = \frac{\hbar}{2}(M/M_{XX})^{-1} \tag{25}
\]

\[
\Sigma_{XP} = -\frac{\hbar}{2}(M/M_{PP})^{-1}M_{XP}M_{PP}^{-1}. \tag{26}
\]

**Lemma 6** The ellipsoid \( \Omega \) is a quantum blob \( S(B^{2n}(\sqrt{\hbar})) \), \( S \in \text{Sp}(n) \) if and only if the block entries of \( M = (SS^T)^{-1} \) satisfy

\[
M_{XX}M_{PP} - M_{XP}^2 = I_{n \times n}, M_{PX}M_{PP} = M_{PP}M_{XP}. \tag{27}
\]

**Proof.** The ellipsoid \( \Omega \) is the set of all \( z \in \mathbb{R}^{2n} \) such that \((SS^T)^{-1}z \cdot z \leq \hbar\). The positive definite matrix \( M = (S^TS)^{-1} \) is thus symplectic. This condition is equivalent to the matrix relation \( MJM = J \), hence [28].

Notice that the conditions above can be written, in terms of the covariance matrix,

\[
\Sigma_{XX}\Sigma_{PP} - \Sigma_{XP}^2 = \frac{1}{4}\hbar^2 I_{n \times n} \text{ and } \Sigma_{PX}\Sigma_{PP} = \Sigma_{PP} \Sigma_{XP}. \tag{28}
\]

This is a matrix form of the saturated Robertson–Schrödinger uncertainty principle [10] [15].

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3.2 Orthogonal projections and intersections

Let $M$ be the symmetric positive definite matrix $(20)$. The following results is well-known (see for instance [13]):

**Lemma 7** The orthogonal projections $\Pi_\ell X$ and $P = \Pi_\ell P \Omega$ on the coordinate subspaces $\ell_X = \mathbb{R}^n_x \times 0$ and $\ell_P = 0 \times \mathbb{R}^n_p$ of $\Omega$ are the ellipsoids

\[
\Pi_\ell X \Omega = \{ x \in \mathbb{R}^n_x : (M/M_{PP})x^2 \leq h \} \\
\Pi_\ell P \Omega = \{ p \in \mathbb{R}^n_p : (M/M_{XX})p^2 \leq h \}.
\]

In terms of the covariance matrix $\Sigma$ and the formulas $(25)$ this is

\[
\Pi_\ell X \Omega = \{ x \in \mathbb{R}^n_x : \frac{1}{2} \Sigma^{-1}_{XX} x^2 \leq 1 \} \\
\Pi_\ell P \Omega = \{ p \in \mathbb{R}^n_p : \frac{1}{2} \Sigma^{-1}_{PP} p^2 \leq 1 \}.
\]

Orthogonal projections and intersections are exchanged by polar duality:

**Lemma 8** For every linear subspace $\ell$ of $\mathbb{R}^n$ we have

\[
(X \cap \ell)^h = \Pi_\ell(X^h) \quad \text{and} \quad (\Pi_\ell X)^h = X^h \cap \ell
\]

where $\Pi_\ell$ is the orthogonal projection $\mathbb{R}^n_x \rightarrow \ell$. (In both equalities, the operation of taking the polar set in the left hand side is made inside $\ell$.)

**Proof.** (See Vershynin [23].) Let us first show that $\Pi_\ell(X^h) \subset (X \cap \ell)^h$. Let $p \in X^h$. We have, for every $x \in X \cap \ell$,

\[
x \cdot \Pi_\ell p = \Pi_\ell x \cdot p = x \cdot p \leq h
\]

hence $\Pi_\ell p \in (X \cap \ell)^h$. To prove the reverse inclusion we note that it is sufficient, by the anti-monotonicity property of polar duality, to prove that $(\Pi_\ell(X^h))^h \subset X \cap \ell$. Let $x \in (\Pi_\ell(X^h))^h$; we have $x \cdot \Pi_\ell p \leq h$ for every $p \in X^h$. Since $x \in \ell$ (because the dual of a subset of $\ell$ is in $\ell$) we also have

\[
h \geq x \cdot \Pi_\ell p = \Pi_\ell x \cdot p = x \cdot p
\]

from which follows that $x \in (X^h)^h = X$, which shows that $x \in X \cap \ell$. This completes the proof of the first formula in $(33)$. The second formula in $(33)$ follows by duality, noting that in view of the reflexivity of polar duality we have

\[
(X^h \cap \ell)^h = \Pi_\ell(X^h)^h = \Pi_\ell X
\]

and hence $X^h \cap \ell = (\Pi_\ell X)^h$. □
3.3 Quantum blobs from projections and intersections

In [13] we proved that if $\Omega$ is a quantum blob then $\Pi_{\ell_X}\Omega$ and $\Pi_{\ell_P}\Omega$ are polar dual of each other. The following result considerably improves this statement:

**Theorem 9** A centered phase space ellipsoid

$$\Omega = \{z \in \mathbb{R}^{2n}_z : Mz \cdot z \leq \hbar\}$$

$(M \in \text{Sym}_{++}(2n, \mathbb{R}))$ is a quantum blob $S(B^{2n}(\sqrt{\hbar}))$, $S \in \text{Sp}(n)$ if and only if the equivalent conditions

$$(\Pi_{\ell_X}\Omega)^h = \Omega \cap \ell_P , \quad \Pi_{\ell_X}\Omega = (\Omega \cap \ell_P)^h. \quad (34)$$

are satisfied. In terms of the matrix $M$ these conditions are equivalent to the identity

$$M_{PP}(M/M_{PP}) = I_{n \times n}. \quad (35)$$

**Proof.** That the conditions $(\Pi_{\ell_X}\Omega)^h = \Omega \cap \ell_P$ and $(\Pi_{\ell_X}\Omega)^h = \Omega^h \cap \ell_X$ are equivalent is clear by the reflexivity of polar duality. Writing $M$ in block matrix form, the condition $z = (x,p) \in \Omega$ means that

$$M_{XX}x^2 + 2M_{XP}xp + M_{PP}p^2 \leq \hbar$$

(we are using again the abbreviations $M_{XX}x \cdot x = M_{XX}x^2$, etc.) and the intersection $\Omega \cap \ell_P$ is therefore

$$\Omega \cap \ell_P = \{p : M_{PP}p^2 \leq \hbar\}.$$

On the other hand, in view of Lemma [7]

$$\Pi_{\ell_X}\Omega = \{x : (M/M_{PP})x^2 \leq \hbar\}$$

and the polar dual $(\Pi_{\ell_X}\Omega)^h$ is

$$(\Pi_{\ell_X}\Omega)^h = \{p : (M/M_{PP})^{-1}p^2 \leq \hbar\}$$

so we have to prove that $\Omega$ is a quantum blob if and only if (35) holds. Using the explicit expression (22) of the Schur complement this is equivalent to the condition

$$(M_{XX} - M_{XP}M_{PP}^{-1}M_{PX})M_{PP} = I_{n \times n}. \quad (36)$$

Assume now that $\Omega$ is a quantum blob; then $\Omega = S(B^{2n}(\sqrt{\hbar}))$ for some $S \in \text{Sp}(n)$; then $z \in \Omega$ if and only if $Mz \cdot z \leq \hbar$ where $M = (S^T)^{-1}S^{-1}$. 

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Since $M \in \text{Sp}(n) \cap \text{Sym}^+(2n, \mathbb{R})$ we have $M_{PP}M_{XP} = M_{PX}M_{PP}$ (second formula (27) in Lemma 6) and hence

$$(M_{XX} - M_{XP}M_{PP}^{-1}M_{PX})M_{PP} = M_{XX}M_{PP} - M_{XP}M_{PP}^{-1}(M_{PX}M_{PP})$$

$$= M_{XX}M_{PP} - (M_{XP})^2.$$ 

Using the first formula (27) in Lemma 6 we thus have

$$(M_{XX} - M_{XP}M_{PP}^{-1}M_{PX})M_{PP} = I_{n \times n} \quad (37)$$

which implies that $(\Pi_{\ell_X}\Omega)^\hbar = \Omega \cap \ell_P$, so we have proven the necessity of the condition (34). Let us prove that this condition is sufficient as well. Let us perform a Williamson diagonalization [9] of the matrix $M$: there exists $S \in \text{Sp}(n)$ such that

$$M = S^T_0 D S_0 \quad , \quad D = \begin{pmatrix} \Lambda^\omega & 0_{n \times n} \\ 0_{n \times n} & \Lambda^\omega \end{pmatrix} \quad (38)$$

where $\Lambda^\omega = \text{diag}(\lambda_1^\omega, \ldots, \lambda_n^\omega)$; here $\lambda_1^\omega, \ldots, \lambda_n^\omega$ the symplectic eigenvalues of $M$ (i.e. the moduli of the usual eigenvalues of the matrix $JM$; they are the same as those of the antisymmetric matrix $M^{1/2}JM^{1/2}$ and hence of the type $\pm i\lambda, \lambda > 0$). Since a symplectic automorphism transforms a quantum blob into another quantum blob, we can reduce the proof of the sufficiency of (34) to the case where $\Omega$ is the ellipsoid

$$\Omega_0 = \{z \in \mathbb{R}^{2n} : \Lambda^\omega x^2 + \Lambda^\omega p^2 \leq \hbar\}.$$ 

We have here $\Pi_{\ell_X}\Omega_0 = \{x : \Lambda^\omega x^2 \leq \hbar\}$ hence $(\Pi_{\ell_X}\Omega_0)^\hbar = \{x : (\Lambda^\omega)^{-1}x^2 \leq \hbar\}$ and $\Omega \cap \ell_P = \{x : \Lambda^\omega x^2 \leq \hbar\}$. The equality $(\Pi_{\ell_X}\Omega_0)^\hbar = \Omega \cap \ell_P$ thus implies that $\Lambda^\omega = I_{n \times n}$ hence $M = S^T_0 S_0 \in \text{Sp}(n)$. □

4 Gaussian Quantum Phase Space

4.1 The Wigner transform

Recall that the Wigner transform (or function) of a square integrable function $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$ is the function $W \psi : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ defined by the absolutely convergent integral

$$W \psi(x, p) = \left(\frac{1}{2\pi\hbar}\right)^n \int e^{-\frac{i}{\hbar}p y} \psi(x + \frac{1}{2}y)\psi^*(x - \frac{1}{2}y) d^n y \quad . \quad (39)$$
It satisfies the Moyal identity

\[(W\psi|W\phi)_{L^2(\mathbb{R}^{2n})} = (2\pi\hbar)^{-n} |(\psi|\phi)|^2_{L^2(\mathbb{R}^n)} \quad (40)\]

which implies that \[||W\psi||_{L^2(\mathbb{R}^{2n})} = (2\pi\hbar)^{-n/2} ||\psi||^2_{L^2(\mathbb{R}^n)}.\]

An important property satisfied by the Wigner transform is its symplectic covariance: for every \(S \in \text{Sp}(n)\) and \(\psi \in L^2(\mathbb{R}^n)\) we have

\[W\psi(S^{-1}z) = W(\hat{S}\psi)(z) \quad (41)\]

where \(\hat{S} \in \text{Mp}(n)\) is one of the two metaplectic operators projecting onto \(S\) (recall [9] that \(\text{Mp}(n)\), the metaplectic group, is a unitary representation in \(L^2(\mathbb{R}^n)\) of the double cover of \(\text{Sp}(n)\)). The covering projection \(\pi_{\text{Mp}} : \text{Mp}(n) \rightarrow \text{Sp}(n)\) is uniquely determined by its action of the generators of \(\text{Mp}(n)\).

Here is a basic example. Let \(X \in \text{Sym}_{++}(n, \mathbb{R})\) and \(Y \in \text{Sym}(n, \mathbb{R})\). The associated generalized Gaussian \(\psi_{X,Y}\) is defined by

\[\psi_{X,Y}(x) = (\pi\hbar)^{-n/4}(\det X)^{1/4} e^{-\frac{1}{\pi}(X+iY)x^2}. \quad (42)\]

Its Wigner transform is given by \([5, 9, 12]\)

\[W\psi_{X,Y}(z) = (\pi\hbar)^{-n} e^{-\frac{1}{\pi}Gz^2} \quad (43)\]

where

\[G = \begin{pmatrix} X + YX^{-1}Y & YX^{-1} \\ X^{-1}Y & X^{-1} \end{pmatrix}. \quad (44)\]

It is essential to observe that \(G = G^T \in \text{Sp}(n)\); this is most easily seen using the factorization where

\[G = S^T S, \quad S = \begin{pmatrix} X^{1/2} & 0 \\ X^{-1/2}Y & X^{-1/2} \end{pmatrix} \in \text{Sp}(n). \quad (45)\]

### 4.2 Gaussian density operators

Let \(\hat{\rho} \in \mathcal{L}^1(L^2(\mathbb{R}^n))\) be a trace class operator on \(L^2(\mathbb{R}^n)\). If \(\text{Tr}(\hat{\rho}) = 1\) and \(\hat{\rho}\) is positive semidefinite \((\hat{\rho} \geq 0)\) one says that \(\hat{\rho}\) is a density operator (it represents the mixed states in quantum mechanics). One shows, using the spectral theorem for compact operators, that the Weyl symbol of \(\hat{\rho}\) can be written as \((2\pi\hbar)^n \rho\) where \(\rho\) (the “Wigner distribution of \(\hat{\rho}\)”) is a convex sum

\[\rho = \sum_j \lambda_j W\psi_j, \quad \lambda_j \geq 0, \quad \sum_j \lambda_j = 1\]

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where \((\psi_j)_j\) is an orthonormal set of vectors in \(L^2(\mathbb{R}^n)\) (the series is absolutely convergent in \(L^2(\mathbb{R}^n)\)). Of particular interest are Gaussian density operators, by definition these are the density operators whose Wigner distribution can be written

\[
\rho(z) = \frac{1}{(2\pi)^n \sqrt{\det \Sigma}} e^{-\frac{1}{2} \Sigma^{-1}(z-z_0)(z-z_0)}
\]

(46)

where \(z_0 \in \mathbb{R}^n\) and the covariance matrix \(\Sigma \in \text{Sym}_{++}(n, \mathbb{R})\) (we will from now on choose \(z_0 = 0\), but all the statements on the covariance matrix and ellipsoid that follow are not influenced by this assumption). While the operator \(\hat{\rho}\) with Weyl symbol \((2\pi\hbar)^n \rho\) automatically has trace one, the condition \(\hat{\rho} \geq 0\) is equivalent to \([6, 8, 9]\)

\[
\Sigma + i\hbar^2 J \geq 0.
\]

(47)

(that is, the eigenvalues of the Hermitian matrix \(\Sigma + i\hbar^2 J\) are \(\geq 0\)).

By definition the purity of a density operator \(\rho\) is the number \(\mu(\hat{\rho}) = \text{Tr}(\hat{\rho}^2)\). We have \(0 < \mu(\hat{\rho}) \leq 1\) and \(\mu(\hat{\rho}) = 1\) if and only the Wigner distribution \(\rho\) of \(\hat{\rho}\) consists of a single term: \(\rho = W \psi\) for some \(\psi \in L^2(\mathbb{R}^n)\).

**Proposition 10** Let \(\hat{\rho}\) be a Gaussian density operator with covariance matrix \(\Sigma\). (i) The condition \(\Sigma + i\hbar^2 J \geq 0\) holds if and only if the covariance ellipsoid \(\Omega\) associated with \(\Sigma\) contains a quantum blob. (ii) We have \(\mu(\hat{\rho}) = 1\) if and only \(\Omega\) is a quantum blob and we have in this case \(\rho = W \psi_{X,Y}\) for some pair of matrices \((X,Y)\).

**Proof.** We have proven part (i) in \([9, 10]\) (also see \([15]\)). To prove (ii) we note that the purity of a Gaussian state \(\hat{\rho}\) is \([9]\)

\[
\mu(\hat{\rho}) = \left(\frac{\hbar}{2}\right)^n (\det \Sigma)^{-1/2}
\]

hence \(\mu(\hat{\rho}) = 1\) if and only if \(\det \Sigma = (\hbar/2)^{2n}\). Let \(\lambda_1^\omega, \ldots, \lambda_n^\omega\) be the symplectic eigenvalues of \(\Sigma\) as in the proof of Theorem \([8]\) in view of Williamson’s symplectic diagonalization theorem there exists \(S \in \text{Sp}(n)\) such that \(\Sigma = S^{-1} D(S^T)^{-1}\) where \(D = \begin{pmatrix} \Lambda^\omega & 0 \\ 0 & \Lambda^\omega \end{pmatrix}\) with \(\Lambda^\omega = \text{diag}(\lambda_1^\omega, \ldots, \lambda_n^\omega)\). The quantum condition \([47]\) is equivalent to \(\lambda_j^\omega \geq \hbar/2\) for all \(j\) hence

\[
\det \Sigma = (\lambda_1^\omega)^2 \cdots (\lambda_n^\omega)^2 = 1
\]

if and only if \(\lambda_j^\omega = \hbar/2\) for all \(j\), hence \(\Sigma = \frac{\hbar}{2} S^{-1} (S^T)^{-1}\) and \(\Omega = S(B^{2n}(\sqrt{\hbar}))\) is a quantum blob. ■
4.3 A characterization of Gaussian density operators

We are going to apply Theorem 9 to characterize pure Gaussian density operators without prior knowledge of the full covariance matrix. This is related to the so-called “Pauli reconstruction problem” [21] we have discussed in [14]. The latter can be reformulated in terms of the Wigner transform as follows: given a function \( \psi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) whose Fourier transform is also in \( L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) the question is whether we reconstruct \( \psi \) from the knowledge of the marginal distributions

\[
\int W\psi(x,p)d^n p = |\psi(x)|^2, \quad \int W\psi(x,p)d^n x = |\widehat{\psi}(p)|^2
\]

where the Fourier transform \( \widehat{\psi} \) of \( \psi \) is given by

\[
\widehat{\psi}(p) = \left( \frac{1}{2\pi\hbar} \right)^{n/2} e^{-\frac{i}{\hbar} \Sigma^{-1} x \cdot p} \int e^{-i\Sigma x \cdot \xi} \psi(x) d^n x.
\]

The answer to Pauli’s question is negative; the study of this problem has led to many developments, one of them being the theory of symplectic quantum tomography (see e. g. [19]). The following result is essentially an analytic restatement of Theorem 9.

**Theorem 11** Let \( \hat{\rho} \in \mathcal{L}(L^2(\mathbb{R}^n)) \) be a density operator with Gaussian Wigner distribution

\[
\rho(z) = \frac{1}{(2\pi)^n \sqrt{\det \Sigma}} e^{-\frac{1}{2} \Sigma^{-1} z \cdot z}.
\]

Then \( \hat{\rho} \) is a pure density operator if and only if

\[
\Phi(x) = 2^n \int \rho(x,p)d^n p
\]

where \( \Phi \) is the Fourier transform of the function \( p \mapsto \rho(0,p/2) \).

**Proof.** We begin by noting that by the well-known formula about marginals in probability theory we have

\[
\int \rho(x,p)d^n p = \frac{1}{(2\pi)^{n/2} \sqrt{\det \Sigma_{XX}}} e^{-\frac{1}{2} \Sigma_{XX}^{-1} x \cdot x}.
\]

Returning to the notation \( M = \frac{\hbar}{2} \Sigma^{-1} \) we have

\[
\rho(z) = (\pi\hbar)^{-n} (\det M)^{1/2} e^{-\frac{1}{\hbar} M z \cdot z}
\]
and the margin formula (51) reads

\[ \int \rho(x,p) d^n p = (\pi \hbar)^{-n/2} (\det M/M_{PP})^{1/2} e^{-\frac{1}{\hbar} (M/M_{PP}) x \cdot x}. \]  

(52)

Assume now that \( \hat{\rho} \) is a pure density operator and let us show that (50) holds (also see Remark 12 below). In view of Proposition 10 we then have \( \rho = W \psi_{X,Y} \) for some Gaussian (42) and thus \( \rho(z) = (\pi \hbar)^{-n/2} e^{-\frac{1}{\hbar} G z \cdot z} \), where \( G \) is the symmetric symplectic matrix (44). Using the first marginal property (48) and the definition of \( \psi_{X,Y} \) it follows that

\[ \int \rho(x,p) d^n p = |\psi_{X,Y}(x)|^2 = (\pi \hbar)^{-n/2} (\det X)^{1/2} e^{-\frac{1}{\hbar} X x \cdot x}. \]

On the other hand

\[ W \psi_{X,Y}(0,p/2) = (\pi \hbar)^{-n} e^{-\frac{1}{\hbar} X^{-1} p \cdot p} \]

and its Fourier transform is

\[ \Phi(p) = \left( \frac{2}{\pi \hbar} \right)^n (\det M)^{1/2} e^{-\frac{1}{\hbar} M^{-1} p \cdot p}. \]

hence the equality (50). Assume now that, conversely, (50) holds. We have

\[ \rho(0,p/2) = (\pi \hbar)^{-n} (\det M)^{1/2} e^{-\frac{1}{\hbar} M_{PP} p \cdot p}. \]

and the Fourier transform \( \Phi \) of the function \( p \mapsto \rho(0,p/2) \) is given by

\[ \Phi(p) = \left( \frac{2}{\pi \hbar} \right)^n (\det M)^{1/2} (\det M_{PP})^{-1/2} e^{-\frac{1}{\hbar} M_{PP}^{-1} p \cdot p}. \]

The equality (50) requires that

\[ (\det M)^{1/2} (\det M_{PP})^{-1/2} e^{-\frac{1}{\hbar} M_{PP}^{-1} x \cdot x} = (\det M/M_{PP})^{1/2} e^{-\frac{1}{\hbar} (M/M_{PP}) x \cdot x} \]

that is, equivalently,

\[ M_{PP}^{-1} = (M/M_{PP}) \]

\[ (\det M)^{1/2} (\det M_{PP})^{-1/2} = (\det M/M_{PP})^{1/2}. \]

The first of these two conditions implies that the covariance ellipsoid \( \Omega \) is a quantum blob (formula (35)) in Theorem 9; the second condition is then automatically satisfied since \( \det M = 1 \) in this case.
Remark 12 Condition (50) is actually satisfied by all even Wigner transformations (and hence by all pure density operators corresponding to an even function $\psi$). Suppose indeed that $\rho = W\psi$ for some suitable even function $\psi \in L^2(\mathbb{R}^n)$. Then

$$W\psi(0,p/2) = (\pi \hbar)^{-n} \int e^{\frac{\pi}{\hbar} p \cdot y} |\psi(y)|^2 d^n y;$$

Taking the Fourier transform of both sides and using the first marginal property (48) yields the identity (50).

5 Perspectives and Comments

Among all states (classical, or quantum) the Gaussians are those which are entirely characterized by their covariance matrices. The notion of polar duality thus appears informally as being a generalization of the uncertainty principle of quantum mechanics as expressed in terms of variances and covariances. Polar duality actually is a more general concept than the usual uncertainty principle, expressed in terms of covariances and variances of position and momentum variables (and the derived notion of quantum blob). As was already in the work of Uffink and Hilgevoord [17, 18], variances and covariances are satisfactory measures of uncertainties only for Gaussian (or almost Gaussian) distribution. For more general distributions having nonvanishing “tails” they can lead to gross errors and misinterpretation. Another advantage of the notion of polar duality is that it might precisely be extended to study uncertainties when non-Gaussianity appears. Instead of considering ellipsoids $X$ in configuration space $\mathbb{R}^n_x$ one might want to consider sets $X$ which are only convex. In this case the polar dual $X^\circ$ is still well-defined and one might envisage, using the machinery of the Minkowski functional to generalize the results presented here to general non-centrally symmetric convex bodies in $\mathbb{R}^n_x$. The difficulty comes from the fact that we then need to choose the correct center with respect to which the polar duality is defined since there is no privileged “center”; different choices may lead to polar duals with very different sizes and volumes. These are difficult questions, but they may lead to a better understanding of very general uncertainty principles for the density operators of quantum mechanics.

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