Analysis of the long and short-period terms due the nonsphericity of the central body: applications for Mercury

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Abstract.
In this work, we present an approach taking into account the single-averaged equations and unaveraged equations to investigate the dynamics of artificial satellites on the effect due to the non-spherical shape of the planet Mercury. An analysis considering the long-period terms and another taking into account the short-period terms is presented. The numerical integrations of the equations developed are performed using the Maple software. We consider the numerical values of the most updated spherical harmonic coefficients in the literature. Emphasis is given to analyze the effect of the $C_{22}$ term in the dynamics of the spacecraft. We show that the two techniques are in agreement (average or not average). We found orbits that librates around an equilibrium point with small variation of the orbital elements, in particular the eccentricity and argument of the pericenter. We also note that the $C_{22}$ term contributes to reduce the growth of the orbital eccentricity.

1. Introduction
Several authors have analyzed the dynamics of artificial satellites around Mercury, taking into account the non-spherical shape of the planet, see for example, [1, 2, 3, 4, 5]. These authors have considered the values of the harmonic coefficients obtained by the Mariner probe (1974). But, these coefficients are now available in the literature in an updated version, see for example, [6, 7]. In [6] it is realized an analyze 3 years of radio tracking data from the MESSENGER spacecraft in orbit around Mercury, where the gravity field of the planet was determined. Here, in this research, we using the updated values of the spherical harmonics coefficients to describe the gravity field of Mercury, obtained from [6]. In this work, an approach that takes into account the effect due to the non-spherical shape of the planet Mercury on the dynamics of artificial satellites is presented, where an analysis considering the terms of long-period and another taking into account the terms of short-period is considered.

2. Nonsphericity of the central body
In this section, are presented the long-period equations, where the averaged model is applied. In the next section, we show the development of the short-period equations. Consider the satellite orbit around the central body with semimajor axis $a$, eccentricity $e$, inclination $i$, right ascension of the ascending node $h$, argument of the pericentre $g$ and mean motion $n$. 
To develop the disturbing potential, that represents the non-uniform distribution of mass of the planet, we use Eq. (1)(see [8])

\[
U_M = -\frac{\mu}{r} \left[ \sum_{n=2}^{\infty} \left( \frac{R_e}{r} \right)^n J_n P_n(\sin \phi) - \left( \frac{R_e}{r} \right)^2 C_{22} P_{22}(\sin \phi) \cos(2\lambda) \right],
\]

(1)

where \( \mu \) and \( R_e \) are the gravitational parameter and equatorial radius of the planet, respectively. Here \( P_n \) are the Legendre polynomials, \( P_{nm} \) the associated Legendre polynomials, the angle \( \phi \) is the latitude of the orbit with respect to the equator of the planet, the angle \( \lambda \) is the longitude measured from the direction of the longest axis of the planet. However, \( \lambda \) contains the time explicitly (see [8]). Where \( r \) is the position of the satellite with respect to the Mercury center.

The disturbing potential due to the non-uniform distribution of mass of the central body up to the third order for the zonal terms and the \( C_{22} \) term (equatorial ellipticity) it is presented in [9, 10, 11]. We get,

\[
<R_{J2}> = -\frac{1}{4} \frac{J_2 R_e^2}{(1-e^2)^{3/2}} n^2 (3s^2 - 2)
\]

\[
<R_{J3}> = -\frac{3}{8} \frac{J_3 R_e^4}{(1-e^2)^{5/2}} n^2 s \sin(g) (5s^2 - 4)
\]

\[
<R_{C22}> = -\frac{3}{8} \frac{C_{22} R_e^2}{(1-e^2)^{5/2}} n^2 (c^2 - 1) \cos(2wt - 2h)
\]

(2) (3) (4)

where \( s = \sin i \) and \( c = \cos i \). Here \( w = 6.138506839^\circ/day \) is the rotation rate of Mercury ([12]) and \( t \) is the time.

Thus, the disturbing potential due to the effect of the non-spherical shape of the central body can be written as follows

\[
R = <R_{J2}> + <R_{J3}> + <R_{C22}>
\]

(5)

Considering the equations that represent the non-sphericity of the central body, the artificial satellite is studied under the single-averaged analytical model. The coefficients of the gravity field is given in table 1. Note that in Eq. (4) the time appears explicitly on the right side. Replacing the potential given by Eq.(5) in the Lagrange planetary equations we obtain a system of non-linear and non-autonomous differential equations. This system of differential equations is numerically integrated using the Maple software.

| Coefficients | Value          |
|--------------|----------------|
| \( J_2 \)   | \( 2.25045 \times 10^{-5} \) |
| \( J_3 \)   | \( 4.76589 \times 10^{-6} \) |
| \( C_{22} \) | \( 1.24538 \times 10^{-5} \) |

Table 1. Normalized harmonics coefficients of Mercury ([6]).
2.1. Short-period equations

Now, the development of the short-period equations is presented, were the averaged model is not applied. The Legendre polynomials for the $J_2$ and $J_3$ zonal terms and the Legendre associated functions for the $C_{22}$ sectorial term can be written in the form [8]

$$P_2(\sin \phi) = \frac{1}{2}(3s^2\sin^2(f + g) - 1),$$
$$P_3(\sin \phi) = \frac{3}{2}(s^3\sin^3(f + g)) - \frac{3}{2} s \sin(f + g),$$
$$P_{22}(\sin \phi) \cos 2\lambda = 6(\xi^2 \cos^2 f + \chi^2 \sin^2 f + \xi \chi \sin 2f) - 3(1 - s^2 \sin^2(f + g)),$$

where we used the shortcut $\xi = \cos g \cos h - c \sin g \sin h$ and $\chi = - \sin g \cos h - c \cos g \sin h$.

To represent the flattening due to $J_2$ term, Eqs. (1) and (6) are used. We get,

$$U_{20} = -\frac{1}{2} \frac{a^3}{pi} \epsilon n^2 (3s^2(\sin^2(f + g)) - 1)$$

where $\epsilon = J_2 R_c^2$.

We used known equations of celestial mechanics to put the equation in function of the orbital elements (see [13]), we get

$$a/r = (1 + e \cos(f))/(1 - e^2)$$
$$\cos(f) = \cos(l) + e(\cos(2l) - 1) + e^2 \left( \frac{3}{8} \cos(3l) - \frac{3}{8} \cos(l) \right)$$
$$\sin(f) = \sin(l) + e \sin(2l) + e^2 \left( \frac{3}{6} \sin(3l) - \frac{3}{6} \sin(l) \right)$$

where $l$ is the mean anomaly and $f$ is true anomaly of the satellite. The expression $\mu = n^2 a^3$ is also used.

Using Eqs. (7), (8), (9) and (10) and after some algebraic manipulations, the potential due to $J_2$ term is written as follows

$$R_{J_2} = -\frac{15}{8} e ((e^2 - 2/5)(s^2) \cos(2g + 2l) - 7/5 e(s^2) \cos(2g + 3l) -$$
$$\frac{17}{5} e^2(s^2) \cos(2g + 4l) + 1/5 e(s^2) \cos(l + 2g) +$$
$$9/5((s^2 - 2/3) (e^2 \cos(2l) + 1/3 e^2 + 2/3 e \cos(l + 2/9)) n^2$$

To represent the flattening due to $J_3$ term, Eqs. (1) and (6) are used. We get,

$$U_{30} = -\frac{1}{2} \frac{a^4}{pi} \epsilon_1 (5s^3(\sin(f + g))^3 - 3s \sin(f + g))$$

$\epsilon_1 = J_3 R_c^3$.

Using Eqs. (12), (8), (9) and (10) and after some algebraic manipulations, the potential due to $J_3$ term is written as follows

$$R_{J_3} = -\frac{165}{64} n^2 \epsilon_1 s^2 ((e^2 - 1/6) (s^2) \sin(3g + 3l) + \frac{8}{33} (s^2) \sin(2l + 3g) e -$$
$$\frac{40}{33} (s^2) \sin(4l + 3g) e - \frac{127}{33} (s^2) \sin(5l + 3g) e^2 + e^2 ((s^2 - 4/5) \sin(g + l) +$$
$$\frac{53}{11} e^2 ((s^2 - 4/5) \sin(g + l) + (\frac{24}{11} e(s^2 - \frac{96}{33} e) \sin(2l + g) -$$
$$1/33 (s^2) \sin(l + 3g) e^2 + \frac{16}{11} ((e^2 + 1/2) \sin(g + l) + 1/2 \sin(g) e)((s^2 - 4/5)) a^{-1}$$

To describe the equatorial ellipticity, where we represent by the $C_{22}$ term, we use Eq.(5) of the paper from De Saedeleer ([14]) given by Eq. (14)
\[ U_{22} = \frac{3}{4} \delta \dot{e}^2 \left( 2 s^2 \cos(2h) + (c + 1)^2 \cos(2f + 2g + 2h) + (c - 1)^2 \cos(2f + 2g - 2h) \right) \] 

but here \( \delta = C_{22} R_c^2 \).

Using Eqs. (14), (8), (9) and (10) and after some algebraic manipulations, the potential due to \( C_{22} \) term is written as follows

\[
R_{C_{22}} = \frac{9}{4} \delta n^2 (-5/6(c - 1)^2(e^2 - 2/5) \cos(2g + 2l - 2h) - 5/6(e^2 - 2/5)(c + 1)^2 \cos(2g + 2l + 2h) + 7/6e(c - 1)^2 \cos(2g + 3l - 2h) + 7/6e(c + 1)^2 \cos(2g + 3l + 2h) + \frac{17}{6}e^2(c - 1)^2 \cos(2g + 4l - 2h) + \frac{17}{6}e^2(c + 1)^2 \cos(2g + 4l + 2h) - 1/6e(c - 1)^2 \cos(2g + l - 2h) + 1/6e(c + 1)^2 \cos(2g + l + 2h) + e^2 \cos(2h) + 3/2e^2 \cos(-2l + 2h) + 3/2e^2 \cos(2l + 2h) + e \cos(-l + 2h) + e \cos(l + 2h) + 2/3 \cos(2l))(s^2) \]

For the time to appear explicitly in Eq. (15), we use the expression \( h = h_0 - \omega t \), given by Sehnel ([15]), see also [8, 16]. Therefore, the longitude of the ascending node \( h \) in Eq. (15) should be replaced by \( h = h_0 - \omega t \). For more details see the References [10, 16].

Thus, the short-period disturbing potential can be written in the form

\[ K = R_{J_2} + R_{J_3} + R_{C_{22}} \]

3. Results

Now, the short-period perturbing potential (Eq. (16)) is replaced in the Lagrange planetary equations and numerically integrated using the Maple software. This section presents the results obtained from the numerical simulations. The numerical integration used the routine "dsolve" of the software Maple (version 17), with the options "numeric" and "method=rkf45". It finds a numerical solution using a Fehlberg fourth-fifth order Runge-Kutta method with degree four. Figures 1 and 2 show a comparison when the \( C_{22} \) term is considered in the dynamics (see Fig. 2) and when this term is neglected (see Fig. 1). Note that the \( C_{22} \) term contributes to reduce eccentricity growth, in particular, for the eccentricity 0.02.

For the single-averaged potential, given by Eq. (5), we obtain the following results. The diagram \( e \times g \), see Fig. 3, shows orbits that librates around an equilibrium point, see in particular the orbit with \( e = 0.02 \), and orbits that circulates \( e = 0.05, e = 0.06 \). We call here the orbit with \( e = 0.02 \) as an almost frozen orbit, which presents a smaller variation of the orbital elements. Note that Fig. 2, considering the short-period terms, shows exactly this value of the eccentricity as the orbit with smaller growth. Figure 4 shows the behavior of the eccentricity for different initial values, as expected, the orbit with \( e = 0.02 \) has smaller growth when compared to the other trajectories. The orbit that circulates in Fig. 3 \( (e = 0.05) \), also as expected, shows a high rate of variation of the orbital eccentricity (see Fig. 4). The orbits shown in Figs. 3 and 4 are polar \( (i = 90 \text{ degrees}) \).

Figures 5 and 6 show orbits with initial inclination of 65 degrees and 125 degrees, respectively. Note that more orbits circulate, and the orbits that librates around an equilibrium point, now show lager variation of the orbital elements when compared to Fig. 3. This characteristic of the polar orbit is important, as scientific missions often require polar orbits. For example, the BepiColombo is Europe’s first mission to Mercury. It will set off in 2018. In this mission are planned two spacecraft orbiting Mercury both with inclination of 90 degrees.
4. Conclusions

An analytical approach is presented considering the single-averaged equations, and equations without averaged. These equations are numerically integrated to analyze the dynamics of artificial satellites around Mercury. The perturbation considered in this research is the non-uniform distribution of mass of the planet considering zonal terms and the $C_{22}$ sectoral term. We show that the single-averaged model and the model without average are in agreement. In both models we found the orbit with smaller variation of the orbital elements, in particular the eccentricity. In previous work we have already shown that the $C_{22}$ term coupled with the zonal
Figure 5. Diagram $e$ versus $g$. Initial conditions: $a = 2700$ km, $i = 65^\circ$. $g = 270^\circ$ and $h = 90^\circ$. Disturbing Potential: $R_{J_2} + R_{J_3} + R_{C_{22}}$.

Figure 6. Diagram $e$ versus $g$. Initial conditions: $a = 2700$ km, $i = 125^\circ$. $g = 270^\circ$ and $h = 90^\circ$. Disturbing Potential: $R_{J_2} + R_{J_3} + R_{C_{22}}$.

terms helps to control the growth of the eccentricity, here we also show that the same happens with the equations without average.

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