ISOMETRIC EMBEDDINGS OF BANACH SPACES UNDER OPTIMAL PROJECTION CONSTANTS

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Abstract. Let $X$ be a Banach space with a shrinking FDD $(E_n)$ and $\mathcal{D}$ its strong bimonotonicity projection constant. We prove that for every $\epsilon \in (0,1)$, $X$ embeds isometrically into a Banach space with a shrinking basis which is $(1 + \epsilon)$-monotone and has strong bimonotonicity projection constant not exceeding $\mathcal{D}(1 + \epsilon)$. The proof uses renorming and skipped blocking decomposition techniques. As an application, we prove that if $X$ has a shrinking $\mathcal{D}$-unconditional basis with $\mathcal{D} < \sqrt{6} - 1$, then $X$ has the weak fixed point property.

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A classical result of Banach and Mazur [B, p. 185] states that every separable Banach space embeds isometrically into $C[0, 1]$, that is, $C[0, 1]$ is isometrically universal for the class of all separable spaces. This remarkable theorem raised an important question in Banach space theory: Does a Banach space satisfying some property can be embedded into a Banach space with a basis having some related and desirable property. In 1988, Zippin [Z] answering a question originally posed by Pełczyński in 1964 [Pel, Problem I, p. 133], proved the following fundamental embedding results.

**Theorem I.** Every separable reflexive Banach space embeds into a reflexive Banach space with a Schauder basis.

**Theorem II.** Every Banach space with a separable dual embeds into a space with a shrinking Schauder basis.

Remarkably Theorem II also solved an important question posed by Lindenstrauss and Tzafriri [LT]. There are nowadays an awful lot of embedding theorems. Among others, we mention Johnson and Zheng [JoZ], Odell and Schlumprecht [OS], Ghoussoub, Maurey and Schachermayer [GMS], Dodos and Ferenczi [DF]. In [S] Schlumprecht used coordinate system methods and obtained new proofs for Theorems I and II with more accurate information on the target space. Notably, he proved that every separable dual Banach space $X$ embeds into a space $W$ with a shrinking basis whose Szlenk index of $X$ and the Szlenk index of $W$ are the same. Recent developments in embedding theory suggest quite natural questions. For example, is it possible to get isometric embeddings into spaces with optimal projection constants? Some known results [Ku, MV] suggest that at least the monotonicity projection constant can become as optimal as we wish. In 2016, e.g., Kurka [Ku] proved (among other results) that separable dual Banach spaces embed isometrically into a Banach space with a shrinking monotone basis, and every separable reflexive space embeds isometrically into a reflexive space with a shrinking monotone basis. His proof relies heavily on interpolation methods and methods from descriptive set theory. The main points can be roughly summarized as follows. Let $1$ be the constant function on the Cantor set $2^N$ and take $g_0 \in C(2^N)$ a function that separates points in $2^N$. If $X$ is separable and reflexive, then $E_X := \text{span}(X \cup \{1, g_0\})$ is a separable dual reflexive subspace of $C(2^N)$. From descriptive set theory one gets a shrinking monotone Schauder basis $(e^X_i)$ for $E_X$ (cf. [D, Theorem 5.17]). Finally, applying the interpolation method of Davis, Figiel, Johnson and Pełczyński [DFJP], a
new norm on $E_X$ is build in which the basis $(e^X_i)$ becomes shrinking and monotone, and yet $E_X$ contains $X$ isometrically ([Ku, Lemma 4]).

The goal of the present work is to study another related question about embedding theorems. Precisely, assuming that $X$ is a Banach space having a Schauder basis, does $X$ embed isometrically into a Banach space with a quasi-monotone basis and strong bimonotonicity of the same magnitude as that of the basis of $X$? Our main result covers the class of shrinking basis and can be seen as a monotonicity reduction theorem. As we shall see, it is useful in metric fixed point theory.

2. Results

Our main result is the following.

Main Theorem. Let $X$ be a Banach space with a shrinking FDD $(E_n)$ and $D > 1$ its strong bimonotonicity projection constant. Then for every $\varepsilon > 0$, $X$ embeds isometrically into a Banach space $W$ with a shrinking Schauder basis which is $(1 + \varepsilon)$-monotone and has strong bimonotonicity projection constant at most $D(1 + \varepsilon)$.

As far as we know, no mention of this result has been made in the literature. Let us point out that, in general, it may happen that a Banach space may have a FDD but it may not have a basis (cf. [SZ, Theorem 1.1]). Recall the strong bimonotonicity projection constant of an FDD $(E_n)$ is the number

$$\sup_{m<n} \max \left( \|P_{E_{[m,n]}}^E\|, \|I - P_{E_{[m,n]}}^E\| \right),$$

where for $A \subset \mathbb{N}$, $P_A^E$ denotes the basis projection onto span$(E_i: i \in A)$ and $I$ is the identity operator on $X$. Here $[m,n]$ denotes the interval $\{m, m+1, \ldots, n\}$ in $\mathbb{N}$. Recall the FDD $(E_n)$ is called $K$-monotone if $\|P_n^E\| \leq K$ for all $n \in \mathbb{N}$ ($P_n^E := P_{[1,n]}^E$).

The Main Theorem will directly follow from the following two results.

Theorem A. Let $X$ be a Banach space with a shrinking FDD $(E_n)$ and $D > 1$ its strong bimonotonicity projection constant. Then for every $\varepsilon > 0$, $X$ embeds isometrically into a Banach space $Z$ with a shrinking FDD $(Z_j)$ which is $(1 + \varepsilon)$-monotone and whose strong bimonotonicity projection constant is not larger than $D(1 + \varepsilon)$.

Theorem B. Let $V$ be a Banach space with a shrinking $(1 + \varepsilon)$-monotone FDD $(V_j)$. Assume that $(V_j)$ has strong bimonotonicity projection constant at most $D(1 + \varepsilon)$. Then $V$ embeds isometrically into a Banach space $W$ with a shrinking Schauder basis which is $(1 + \varepsilon)^4$-monotone and has strong bimonotonicity projection constant at most $D(1 + \varepsilon)^4$. 
These results are deeply linked to Theorems A and B of $\mathbb{S}$, where $M$-basis and skipped blocking decomposition techniques were instrumental in their proofs. In fact, as will be seen later on, Schlumprecht’s approach can be suitably adapted to prove our results.

Theorem A is in turn a consequence of the following

**Theorem C.** Let $X$ be Banach space with a shrinking FDD $(E_n)$ and $D > 1$ its strong bimonotonicity projection constant. Then for every $\varepsilon > 0$ there exist a Banach space $Z$ with a shrinking FDD $(Z_j)$ and a bounded linear mapping $T: X \to Z$ satisfying the following properties:

(i) $(1 - \varepsilon)\|x\| \leq \|Tx\| \leq \|x\|$ for all $x \in X$.

(ii) $(Z_j)$ is $(1 + \varepsilon)$-monotone and has strong bimonotonicity projection constant not larger than $D$.

Theorem B is also a consequence of an almost-isometric embedding type result. To be precise, it will follow from the following.

**Theorem D.** Let $\varepsilon > 0$ and $D \geq 1$. Assume that $V$ is a Banach space with a shrinking $(1 + \varepsilon)$-monotone FDD $(V_j)$ whose strong bimonotonicity projection constant is not larger than $D(1 + \varepsilon)$. Then there exist a Banach space $W$ with a shrinking monotone Schauder basis $(e_\gamma)_{\gamma \in \Gamma}$ and a linear embedding $J: V \to W$ satisfying the following properties:

(i) $(1 - \varepsilon/2)(1 + \varepsilon)^{-2}\|v\| \leq \|J(v)\| \leq \|v\|$ for all $v \in V$.

(ii) $(e_\gamma)_{\gamma \in \Gamma}$ has strong bimonotonicity projection constant not larger than $D(1 + \varepsilon)$.

As an application of our Main Theorem we obtain the following improvement of a known fixed point result by P.-K. Lin $\left[\text{Lin}\right]$ stating that every Banach space with a $D$-unconditional basis satisfying $D < (\sqrt{33} - 3)/5$ has the weak fixed point property.

**Theorem E.** Let $X$ be a Banach space with a shrinking $D$-unconditional basis. Assume that $D < \sqrt{6} - 1$. Then $X$ has the weak fixed point property.

Recall that a Banach space $X$ is said to have the (weak) fixed point property if whenever a (weakly compact) closed bounded convex subset $C$ of $X$ is given, then every nonexpansive (i.e., 1-Lipschitz) mapping $f: C \to C$ has a fixed point. It should be noted that a Banach space with an unconditional FDD is isomorphic with a Banach space with an unconditional basis (cf. $\left[\text{LT}\right]$ Theorem 1.g.5). Theorem E immediately provides the following.
Theorem F. Every reflexive Banach space with a $\mathcal{D}$-unconditional basis satisfying $\mathcal{D} < \sqrt{6} - 1$ has the fixed point property.

In the light of the above results, it would be interesting to remove the assumption of shrinkingness in Theorem E, but this seems to require another methodology since the proof of our main result strongly uses this technical condition to get skipped blocking decompositions. It however can be replaced by a more weaker one, as our last result shows.

Theorem G. Every Banach space with a $\mathcal{D}$-unconditional spreading basis satisfying $\mathcal{D} < \sqrt{6} - 1$ has the weak fixed point property.

This paper is organized as follows. In Section 3, we recall some basic definitions and set up the main ingredients used throughout the paper. In Section 4, we present four technical lemmas. The first is a basic renorming result. The second one correspond to [S, Lemma 2.4] and the third is a refinement of [S, Lemma 2.3]. The construction of the spaces $Z$ and $W$ are performed in Sections 5 and 8, respectively. The proofs of theorems are delivered in Sections 6, 7, 8 and 9, respectively. In Section 10 we present the proofs of Theorems E and G.

3. Background

We will use standard Banach space notation and terminology, mostly contained e.g. in [AK, HJ, S]. We below recall some material used in the proof of Main Theorem.

3.1. Finite Dimensional Decompositions. Recall from [JJ] that a sequence of finite dimensional subspaces $(Z_j)$ of a Banach space $(Z, \| \cdot \|)$ is called a finite dimensional decomposition (FDD) of $Z$ if each $z \in Z$ can be uniquely written in the form $z = \sum_{n=1}^{\infty} z_n$ with $z_n \in Z_n$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, the $n$-th projection $P_n^Z$ on $Z$ associated to an FDD $(Z_j)$ is defined by $P_n^Z(z) = z_n$. It is well known that the sequence of projections $(P_n^Z)$ satisfies $\sup_n \| P_n^Z \| < \infty$. For a finite set $A \subset \mathbb{N}$, define $P_A^Z = \sum_{n \in A} P_n^Z$. The projection constant of $(Z_j)$ is defined by $K_b^Z((Z_j)) = \sup_{m \leq n} \| P_{[m,n]}^Z \|$. Here $[m, n]$ stands for the interval $\{ i \in \mathbb{N} : m \leq i \leq n \}$. An FDD $(Z_j)$ of $Z$ is called $\mathcal{D}$-bimonotone ($\mathcal{D} \geq 1$) if $K_b^Z((Z_j)) \leq \mathcal{D}$. If $K_n^Z((Z_j)) = \sup_{n \geq 1} \| P_{[1,n]}^Z \| \leq \mathcal{D}$, then it is called $\mathcal{D}$-monotone. We say that $(Z_j)$ is $\mathcal{D}$-strongly bimonotone if $K_{sh}^Z((Z_j)) := \max \{ \| P_A^Z \|, \| I - P_A^Z \| \} \leq \mathcal{D}$ for every interval $A = [m, n]$. In the special case when $\mathcal{D} = 1$, we simply say that $(Z_j)$ is bimonotone, monotone or strongly bimonotone, respectively.
An FDD \((Z_j)\) is called **shrinking** if its biorthogonal sequence \((F_n)\) given by

\[
F_n = \text{span}(Z_j : j \in \mathbb{N}\setminus\{n\})^\perp,
\]

spans a dense subspace of \(Z^*\), and \((Z_j)\) is said to be **boundedly complete** if for every block sequence \((x_j)\) (i.e., a sequence with \(x_j \in Z_j\) for every \(j \in \mathbb{N}\)) with \(\sup_{n \in \mathbb{N}} \| \sum_{j=1}^{n} x_j \| < \infty\), then the series \(\sum_{j=1}^{\infty} x_j\) converges.

**Remark 3.1.** It is known that James’ characterization of reflexivity passes to FDD’s, that is, a Banach space with an FDD is reflexive if and only if the FDD is both shrinking and boundedly complete.

If \((E_j)\) is an FDD and \((F_j)\) is its biorthogonal sequence, then \((G_k)\) is a **blocking of** \((E_j)\) if \(G_k = \text{span}(E_j : n_{k-1} < j \leq n_k)\), for all \(k \in \mathbb{N}\), and some natural numbers \(0 = n_0 < n_1 < n_2 < \ldots\) \((G_k)\) is also, then, an FDD of \(X\) and its biorthogonal sequence is \((H_k)\) with \(H_k = \text{span}(F_j : n_{k-1} < j \leq n_k)\), and \((H_k)\) is c-norming if \((F_j)\) was c-norming. For \(x^* \in X^*\) we also define the **support of** \(x^*\) **with respect to** \((E_j)\) by

\[
\text{supp}_{E}(x^*) = \{j \in \mathbb{N} : x^*|_{E_j} \neq 0\}.
\]

The **range of** \(x \in X\) or \(x^* \in X^*\) is the smallest interval in \(\mathbb{N}\) containing \(\text{supp}_E(x)\), or \(\text{supp}_E(x^*)\), and is denoted by \(\text{rg}_E(x)\), or \(\text{rg}_E(x^*)\). A **block sequence with respect to** \((E_j)\) in \(X\) or in \(X^*\) is a finite or infinite sequence \((x_n)\) in \(X\), or a sequence \((x^*_n)\) in \(X^*\) for which max \(\text{rg}_E(x_n) < \min \text{rg}_E(x_{n+1})\) or max \(\text{rg}_E(x^*_n) < \min \text{rg}_E(x^*_{n+1})\), respectively, for all \(n \in \mathbb{N}\) for which \(x_{n+1}\), or \(x^*_{n+1}\) are defined. In the case that max \(\text{rg}_E(x_n) < \min \text{rg}_E(x_{n+1}) - 1\) or max \(\text{rg}_E(x^*_n) < \min \text{rg}_E(x^*_{n+1}) - 1\), respectively, we call the sequence a **skipped block sequence with respect to** \((E_j)\) in \(X\) or in \(X^*\).

We close this section with a basic definition of monotonicity which is quite natural for Schauder basis.

**Definition 3.2.** Let \(X\) be a separable Banach space. An FDD \((E_j)\) of \(X\) is called **\(D\)-strongly bimonotone** if

\[
\left\| \sum_{j=1}^{m} x_j + \sum_{j=n}^{\infty} x_j \right\| \leq D \left\| \sum_{j=1}^{\infty} x_j \right\|
\]

for all \(x = \sum_{j=1}^{\infty} x_j \in \text{span}(E_j : j \in \mathbb{N})\) and for all \(m < n \in \mathbb{N}\).

**Remark 3.3.** Clearly, if an FDD \(E = (E_j)\) of \(X\) is \(D\)-strongly bimonotone then its projection constant satisfies \(K_b((E_j)) \leq D\). It is known that \(D\)-unconditional bases are
Theorem 7

It follows that every shrinking \( D \)-unconditional basis gives rise to a shrinking \( (D + 1)/2 \)-strongly bimonotone FDD.

4. Technical lemmas

In this section we gather some necessary tools which will help us to prove the Main Theorem. The first lemma we shall use here is a well known folklore fact concerning isometric renormings, for a proof we refer to ([Cow, Lemma 4.0.10]). The second and third lemmas are Lemmas 4.2 and 4.4 that were proved in [S, Lemmas 2.3 and 2.4] in the slightly more general context of FMD’s. The last one is Lemma 4.6, the first two parts of which are only restatements of Lemma 4.4 and the third part follows directly from strong bimonotonicity; we include its proof here for completeness’ sake.

Lemma 4.1. Suppose that \( X \) and \( Y \) are normed spaces and that \( T : X \rightarrow Y \) is an isomorphic embedding from \( X \) into \( Y \). Assume that for some \( M > 0 \) one has

\[
\|x\|_X \leq M\|Tx\|_Y \quad \text{for all } x \in X.
\]

Then the quantity

\[
|y| = \inf \left\{ \|x\|_X + M\|y - Tx\|_Y : x \in X \right\}
\]

defines an equivalent norm on \( Y \) with respect to which \( T \) is an isometry. Furthermore, for all \( y \in Y \), one has

\[
\frac{1}{\|T\|}\|y\|_Y \leq |y| \leq M\|y\|_Y.
\]

Lemma 4.2. Let \( X \) be a Banach space and \( Y' \) be a (not necessarily closed) subspace of \( X^* \) for which \( B_{Y'} \) is \( w^* \)-dense in \( B_{X^*} \). If \( E \subseteq X \) is a finite dimensional subspace, then for every \( \varepsilon > 0 \) there is a finite dimensional subspace \( F \subset Y' \) so that each functional \( e^* \in E^* \) can be extended to an element \( x^* \in F \) with \( \|x^*\| \leq (1 + \varepsilon)\|e^*\| \).

Remark 4.3. It is worth to point out that the extension \( x^* \) obtained above belongs to \( X^* \), that is, \( x^* \) is globally defined on \( X \).

Lemma 4.4. Let \( X \) be a Banach space. Assume that \((E'_n)\) is a 1-norming FDD of \( X \) and \((F'_j)\) is its biorthogonal sequence. Then for every \( \varepsilon > 0 \), \((E'_j)\) can be blocked to an FDD \((E_n)\) satisfying with its biorthogonal sequence \((F_n)\) the following conditions:

\[
\begin{align*}
\forall n \in \mathbb{N}, & \exists e^* \in (E_1 + E_2 + \cdots + E_n)^*, \\
\exists x^* \in F_1 + F_2 + \cdots + F_n + F_{n+1} & \text{ such that } x^*|_{E_1 + \cdots + E_n} = e^* \text{ and } \|x^*\| \leq (1 + \varepsilon)\|e^*\|,
\end{align*}
\]
\[
\begin{align*}
\forall n \in \mathbb{N}, \forall f^* \in (F_1 + F_2 + \cdots + F_n)^*, \\
\exists z \in E_1 + E_2 + \cdots + E_n + E_{n+1} \text{ such that } z|_{F_1+\cdots+F_n} = f^* \quad \text{and} \quad \|z\| \leq (1 + \varepsilon)\|f^*\|,
\end{align*}
\]
and
\[
\begin{align*}
\forall m < n \in \mathbb{N}, \forall e^* \in (E_m + E_{m+1} + \cdots + E_n)^*, \\
\exists x^* \in F_{m-1} + F_m + \cdots + F_n + F_{n+1} \text{ such that } x^*|_{E_m+\cdots+E_n} = e^* \quad \text{and} \quad \|x^*\| \leq (2 + \varepsilon)\|e^*\|.
\end{align*}
\]

**Remark 4.5.** It is worth noting that the argument used to prove Lemma 4.4 involves an appropriate use of Lemma 4.2. In particular, the proofs of (3)–(5) are respectively contained in the proofs of [S, Lemma 2.3, (1)-(2)]. For instance, to get the factor $1 + \varepsilon$ as well as the factor $2 + \varepsilon$, it suffices to proceed as in [S, Lemma 2.3] with $\rho = p - \varepsilon$.

**Lemma 4.6.** Let $X$ be a Banach space and $(E'_j)$ be a 1-norming FDD of $X$ with biorthogonal sequence $(F'_j)$. Assume that $(E'_j)$ is $\mathcal{D}$-strongly bimonotone. Then $(E'_j)$ can be blocked to an FDD $(E_n)$ satisfying with its biorthogonal sequence $(F_n)$ the properties (3)–(5) in Lemma 4.4, and the following ones:

\[
\begin{align*}
\forall n \in \mathbb{N}, & \forall x^* \in F_1 + \cdots + F_n, \\
\|x^*\| & \leq (1 + \varepsilon) \sup_{x \in E_1 + \cdots + E_n + E_{n+1}, |x| \leq 1} |x^*(x)|. \\
(6)
\end{align*}
\]

\[
\begin{align*}
\forall m < n \in \mathbb{N}, & \forall x^* \in \text{span}(F_1, F_2, \ldots, F_m, F_j; j \geq n), \\
\|x^*\| & \leq D \sup_{x \in \text{span}(E_1, \ldots, E_m, E_j; j \geq m-1), |x| \leq 1} |x^*(x)|. \\
(7)
\end{align*}
\]

**Proof.** Let $(E_n)$ be the FDD given in Lemma 4.4. Assertion (6) can be proved in a similar way as the proof of [S, (3), pp. 842-843] but, in our context, we only have to use that $m = 1$. Let us proceed to check (7). Fix any $x^* \in \text{span}(F_1, \ldots, F_m, F_j; j \geq n)$, $m < n$, $x^* \neq 0$, and let

\[G := \text{span}(E_1, E_2, \ldots, E_m, E_j; j \geq n-1).\]

Thus, assertion (7) is equivalent to the assertion that

\[\|x^*\| \leq D \sup_{u \in G, |u| \leq 1} |x^*(u)|.\]
This in turn is a direct consequence of the fact that \((E_j)\) is \(D\)-strongly bimonotone. Indeed let \(\theta > 0\) be arbitrary and choose a vector \(x \in X\) such that \(\|x\| \leq 1\) and \(\|x^*\| - \theta \leq |x^*(x)|\). We may uniquely write \(x = \sum_{i=1}^{\infty} x_i\) with each \(x_i \in E_i\) for all \(i \in \mathbb{N}\). Thus, since \(x^* \in \text{span}(F_1, \ldots, F_m, F_j : j \geq n)\), we have

\[
x^*(x) = x^* \left( \sum_{i=1}^{m} x_i + \sum_{i=n}^{\infty} x_i \right).
\]

It follows that

\[
\|x^*\| - \theta \leq \left| x^* \left( \sum_{i=1}^{m} x_i + \sum_{i=n}^{\infty} x_i \right) \right| \leq \left\| \sum_{i=1}^{m} x_i + \sum_{i=n}^{\infty} x_i \right\| \sup_{u \in G, \|u\| \leq 1} |x^*(u)|.
\]

Since \((E_j)\) is \(D\)-strongly bimonotone, \(\| \sum_{i=1}^{m} x_i + \sum_{i=n}^{\infty} x_i \| \leq D\|x\| \leq D\). Since \(\theta\) was arbitrary, this proves assertion (7). The proof of lemma is complete. \(\square\)

5. Construction of the space \(Z\)

Throughout this section \(X\) stands for a Banach space with a shrinking \(D\)-strongly bimonotone FDD \((E_n)\) with biorthogonal sequence \((F_n)\). Let \(\varepsilon > 0\) be fixed. We also assume that we have chosen a shrinking FDD \((E_n)\) of \(X\) which, together with its biorthogonal sequence \((F_n)\), satisfies the conclusions of Lemma 4.6.

5.1. Construction of \(Z\). Following [S] we will construct a Banach space \(Z\) with a shrinking \(D\)-strongly bimonotone FDD \((E_n)\) with biorthogonal sequence \((F_n)\). Let \(\varepsilon > 0\) be fixed. We also assume that we have chosen a shrinking FDD \((E_n)\) of \(X\) which, together with its biorthogonal sequence \((F_n)\), satisfies the conclusions of Lemma 4.6.

**Lemma 5.1.** Let \((\varepsilon_k) \subset (0, 1)\) be given. Then there exists an increasing sequence \((n_k) \subset \mathbb{N}\), so that \(n_{k+1} > n_k + 2\) for all \(k \in \mathbb{N}\) and, moreover, for each \(x^* \in B_{X^*}\) and each \(k \in \mathbb{N}\) there is a \(j_k \in [n_k, n_{k+1}]\) so that \(\|x^*|_{E_{j_k}}\| \leq \varepsilon_k\).

The statement of this lemma is slightly different from the statement of [S] Lemma 3.1, but the proof is the same and strongly uses the fact that \((E_n)\) is shrinking. Fix now a decreasing sequence \((\varepsilon_k) \subset (0, 1)\) such that

\[
\sum_{k=1}^{\infty} \varepsilon_k \ll \frac{\varepsilon}{2(2 + \varepsilon)}
\]

and take \((n_k) \subset \mathbb{N}\) so as to satisfy together with \((\varepsilon_k)\) the conclusion of Lemma 5.1. Next define

\[
B^* = D^* \cap B_{X^*},
\]
where

\[ D^* = \{ x^* \in X^*: \forall k \in \mathbb{N} \exists j \in [n_k, n_{k+1}] \text{ s.t. } x^*|_{E_j} = 0 \}. \]

**Lemma 5.2.** \( B^* \) is \( \varepsilon \)-dense in \( B_{X^*} \).

**Proof.** The proof is an adaptation to our situation of the proof of [S, Lemma 3.2]. Let \( x^* \in B_{X^*} \) be fixed and choose, according to Lemma 5.1, \( j_k \in [n_k, n_{k+1}] \), for each \( k \in \mathbb{N} \), so that \( \| x^*|_{E_{j_k}} \| \leq \varepsilon_k \). Let

\[ K = \{ k \in \mathbb{N}: j_k \neq j_{k-1} \}. \]

Note that, since \( n_{k+1} > n_k + 2 \), for all \( k \in \mathbb{N} \), \((j_k : k \in K)\) is a skipped sequence in \( K \), and thus we still have \( j_k \in [n_k, n_{k+1}] \) and \( \| x^*|_{E_{j_k}} \| \leq \varepsilon_k \) for all \( k \in K \) (cf. details in [S, p. 844]). Applying for each \( k \in K \), part (5) of Lemma 4.4 to \( e_k^* := x^*|_{E_{j_k}} \), we obtain an extension \( f_k \in F_{j_k-1} + F_{j_k} + F_{j_k+1} \) of \( e_k^* \) which satisfies \( \| f_k \| \leq (2 + \varepsilon) \varepsilon_k \). Define

\[ y^* = x^* - \sum_{k \in K} f_k \quad \text{and} \quad z^* = \frac{1}{1 + (2 + \varepsilon) \sum_k \varepsilon_k} y^*. \]

Then

\[ \| y^* \| \leq \| x^* \| + (2 + \varepsilon) \sum_k \varepsilon_k \leq 1 + (2 + \varepsilon) \sum_k \varepsilon_k, \]

so \( \| z^* \| \leq 1 \) and hence \( z^* \in B^* \). On the other hand,

\[
\| x^* - z^* \| \leq \| x^* - y^* \| + \| y^* - z^* \|
\leq (2 + \varepsilon) \sum_k \varepsilon_k + \frac{(2 + \varepsilon) \sum_k \varepsilon_k}{1 + (2 + \varepsilon) \sum_k \varepsilon_k} \| y^* \|
\leq 2(2 + \varepsilon) \sum_k \varepsilon_k < \varepsilon.
\]

The proof is complete. \( \square \)

From now on, we will consider the following norm on \( X \)

\[ \| x \| = \sup_{x^* \in B^*} | x^*(x) | \quad \text{for all } x \in X. \quad (8) \]

The fact that \( \| \cdot \| \) defines a norm follows from the following immediate consequence of Lemma 5.2.

**Proposition 5.3.** Let \( B^* \) be as above. Then

\[ (1 - \varepsilon) \| x \| \leq \| x \| \leq \| x \| \quad \text{for all } x \in X. \]
Let us continue with the construction of the space $Z$. For $x^* \in D^*$ pick $J = (j_k) \in \Pi_{k=1}^{\infty} [n_k, n_{k+1}]$ so that $x^*|_{E_{j_k}} \equiv 0$ for all $k \in \mathbb{N}$, and put

$$x^*_k := P^F_{(j_{k-1}, j_k)}(x^*) \quad \text{for } k \in \mathbb{N} \ (\text{with } j_0 = 0).$$

**Proposition 5.4.** For every $x^* \in D^*$, $(x^*_k)_{k=1}^{\infty}$ fulfills the following properties:

(i) $x^* = \sum_{k=1}^{\infty} x^*_k$ (convergence in norm).

(ii) $\| \sum_{k=1}^{m} x^*_k \| \leq (1 + \varepsilon) \| \sum_{k=1}^{\infty} x^*_k \|$ for all $n \in \mathbb{N}$.

(iii) $\| \sum_{k=m}^{\infty} x^*_k \| \leq D \| \sum_{k=1}^{\infty} x^*_k \|$ for all $m < n$.

(iv) $\| \sum_{k=1}^{n} x^*_k + \sum_{k=n}^{\infty} x^*_k \| \leq D \| \sum_{k=1}^{\infty} x^*_k \|$ for all $m < n$ in $\mathbb{N}$.

(v) If $\| x^* \| \leq 1$ then

$$\left\| \sum_{k=n}^{\infty} x^*_k (z_k) \right\| \leq D \left\| \sum_{k=1}^{\infty} z_k \right\|$$

for all $z = (z_k) \in c_{00} \left( \bigoplus_{k=1}^{\infty} \text{span}(E_j : n_{k-1} < j < n_{k+1}) \right)$ and $n \in \mathbb{N}$.

**Proof.** (i) Using (7)-Lemma 4.6 together with the fact that $(E_j)$ is shrinking, we get

$$\left\| x^* - \sum_{k=1}^{m} x^*_k \right\| \leq D \sup_{x \in \text{span}(E_j : j \geq j_m), \|x\| \leq 1} \left\| (x^* - \sum_{k=1}^{m} x^*_k) (x) \right\| = D \| x^*|_{\text{span}(E_j : j \geq j_m)} \| \to_{m \to \infty} 0.$$

Let us prove (ii). Combining (i) and part (ii) of Lemma 4.6 we conclude that

$$\left\| \sum_{k=1}^{n} x^*_k \right\| \leq (1 + \varepsilon) \sup_{x \in E_1 + E_2 + \cdots + E_{j_m}, \|x\| \leq 1} \left\| \sum_{k=1}^{n} x^*_k (x) \right\| = (1 + \varepsilon) \sup_{x \in E_1 + E_2 + \cdots + E_{j_m}, \|x\| \leq 1} \left\| \sum_{k=1}^{\infty} x^*_k (x) \right\| \leq (1 + \varepsilon) \left\| \sum_{k=1}^{\infty} x^*_k \right\|.$$

Assertions (iii) and (iv) follow directly from (7). Finally, assertion (v) easily follows from (iii).

Following the notation in [3] we now define sets $\mathbb{D}^*$ and $\mathbb{B}^*$ as follows.
\[
\mathbb{D}^* = \left\{ (x_k^*) \subset X^* : \exists (j_k) \in \prod_{k=1}^{\infty} [n_k, n_{k+1}] \text{ so that } \right.
\]
\[
\text{rg}_{E}(x_k^*) \subset (j_{k-1}, j_{k}), \text{ for } k \in \mathbb{N}, \left\| \sum_{k=1}^{\infty} x_k^* \right\| < \infty \right\} \tag{9}
\]
(Note that in the definition of \(\mathbb{D}^*\) it is possible that \(j_{k-1} = j_k = n_k\) or \(j_k = j_{k-1} + 1\), and that in either case \(x_k^* \equiv 0\) and)
\[
\mathbb{B}^* = \mathbb{D}^* \cap \left\{ (x_k^*) \subset X^* : \left\| \sum_{k=1}^{\infty} x_k^* \right\| \leq 1 \right\}. \tag{10}
\]

It thus follows that the sets \(D^*\) and \(B^*\) can respectively be rewritten as
\[
D^* = \left\{ \sum_{k=1}^{\infty} x_k^* : (x_k^*) \in \mathbb{D}^* \right\},
\]
and
\[
B^* = D^* \cap B_X = \left\{ \sum_{k=1}^{\infty} x_k^* : (x_k^*) \in \mathbb{B}^* \right\}.
\]

According to the machinery of \([Z]\), the construction of the desired space \(Z\) depends on a new blocking of \((E_j)\). This will be witnessed in what follows. For \(k \in \mathbb{N}\), let
\[
Z_k := \text{span}(E_j : n_{k-1} < j < n_{k+1}), \quad \text{for } k \in \mathbb{N} \ (n_0 = 0),
\]
and note for \((x_k^*) \in \mathbb{B}^*\) that
\[
\text{rg}_{E}(x_k^*) \subset (n_{k-1}, n_{k+1}), \quad \forall k \in \mathbb{N}.
\]

This shows that each \(x_k^*\) can be seen as a functional acting on \(Z_k\). We then define the space \(Z\) as the completion of \(c_{00} \left( \bigoplus_{k=1}^{\infty} Z_k \right)\) with respect to the norm
\[
\left\| z \right\|_Z := \sup \left\{ \left\| \sum_{k=1}^{\infty} x_k^* (z_k) \right\| : (x_k^*) \in \mathbb{B}^* \right\}
\]
where \(z = (z_k) \in c_{00} \left( \bigoplus_{k=1}^{\infty} Z_k \right)\).

The construction of \(Z\) is complete.
6. Proof of Theorem \( \text{C} \)

Theorem \( \text{C} \) is implied by the following result.

**Lemma 6.1.** Let \( X \) be a Banach space with a shrinking \( \mathcal{D} \)-strongly bimonotone FDD \((E_n)\), and let \( \varepsilon > 0 \), \( Z \) and \((Z_j)\) be as in previous section. Then

(i) The map
\[
I : c_0(\bigoplus_{j=1}^{\infty} E_j) \to Z, \quad x \mapsto \left( P_{(n_{k-1}, n_k)}^E(x) : k \in \mathbb{N} \right)
\]
extends to an embedding \( T \) on \( X \) that satisfies
\[
(1 - \varepsilon)\|x\| \leq T(x)\|Z\| \leq \|x\| \quad \forall x \in X.
\]

(ii) \((Z_j)\) is a \((1 + \varepsilon)\)-monotone and \( \mathcal{D} \)-strongly bimonotone FDD of \( Z \).

(iii) \((Z_j)\) is shrinking.

**Proof.** To verify (i), let \( x \in c_0(\bigoplus_{j=1}^{\infty} E_j) \) be arbitrary and note that
\[
\|I(x)\|_Z = \sup_{(x_k^*) \in B*} \left| \sum_{k=1}^{\infty} x_k^* \left( P_{(n_{k-1}, n_k)}^E(x) \right) \right|
= \sup_{(x_k^*) \in B*} \left| \sum_{k=1}^{\infty} x_k^*(x) \right|
= \sup_{x^* \in B*} |x^*(x)| = \|x\|.
\]
Thus by Proposition 5.3 we have
\[
(1 - \varepsilon)\|x\| \leq \|I(x)\|_Z \leq \|x\| \quad \forall x \in c_0(\bigoplus_{j=1}^{\infty} E_j).
\]
Clearly, the same inequality also holds for the extension \( T \) of \( I \) to \( X \).

Let us proceed to the proof of (ii). The proof is essentially the same as that in \( \text{S}, \) except that we have to use our modified Lemma 4.6. As already observed in \( \text{S}, \) p. 845] we can always consider the elements of \( \mathbb{D}^* \) to be elements of \( Z^* \). This fact is also implicitly used here. Let \( m < n \) and \( z = (z_k) \in c_0(\bigoplus_{k=1}^{\infty} Z_k) \). Then
\[
\| P_{[m, n]}^Z(z) \|_Z = \sup_{(x_k^*) \in B*} \left| \sum_{k=m}^{n} x_k^*(z_k) \right|
= \sup_{(x_k^*) \in B*} \left| \sum_{k=m}^{n} x_k^* \right| \|z\|_Z.
\]
Claim: For every \((x_k^*) \in \mathbb{B}^*\) we have
\[
\left\| \sum_{k=m}^{n} x_k^* \right\|_{Z^*} \leq \hat{D} \quad \forall m < n,
\]
where \(\hat{D} = 1 + \varepsilon\) if \(m = 1\) and \(\hat{D} = D\) if \(1 < m\). Indeed, given \((x_k^*) \in \mathbb{B}^*\) we have \(\left\| \sum_{k=1}^{\infty} x_k^* \right\| \leq 1\). By Proposition 5.4, assertions (ii) and (iii), we get
\[
\left\| \sum_{k=m}^{n} x_k^* \right\| \leq \hat{D}.
\]
Define an element \((y_j^*)\) in \(Z^*\) by putting \(y_j^* = 0\) if \(j \notin [m, n]\) and
\[
y_j^* = \frac{x_j^*}{D} \quad \text{for } m \leq j \leq n.
\]
It is clear that \((y_j^*) \in \mathbb{B}^*\). Then
\[
\left\| \sum_{j=1}^{\infty} y_j^*(z_j) \right\| \leq 1, \quad \forall z = (z_j) \in B_Z,
\]
where \(B_Z\) stands for the closed unit ball of \((Z, \| \cdot \|_Z)\). Thus \(\left\| \sum_{j=m}^{n} y_j^* \right\|_{Z^*} \leq 1\), which certainly proves the claim. This shows in particular that
\[
\| P_{[1, n]}^Z(z) \|_Z \leq (1 + \varepsilon) \quad \text{and} \quad K_b^Z \leq D.
\]
Similarly, using now assertions (iv), (v)–Proposition 5.4 one can prove for all \(m < n\) that
\[
\| (I - P_{[1, n]}^Z)(z) \|_Z \leq D \| z \|_Z \quad \text{and} \quad \| (I - P_{[m, n]}^Z)(z) \|_Z \leq D \| z \|_Z.
\]
These inequalities show that \((Z_j)\) is \(D\)-strongly bimonotone.

(iii) The proof that \((Z_j)\) is shrinking is exactly the same as that in [S, Lemma 3.5]. \(\square\)

7. Proof of Theorem \(\Delta\)

Let \((E'_j)\) be any shrinking \(D\)-strongly bimonotone FDD of \(X\). Let \(\varepsilon > 0\) be fixed and set \(\tilde{\varepsilon} = \varepsilon/(2 + \varepsilon)\). Now take \(E = (E_n)\) to be the blocking FDD which, together with its biorthogonal sequence \((F_n)\), is associated to \(\tilde{\varepsilon}\) by Lemma 4.6. By Theorem \(\mathbb{C}\) there exists a Banach space \(Z = Z^E\) with a shrinking FDD \((Z_j)\) which is \((1 + \tilde{\varepsilon})\)-monotone and \(D\)-strongly bimonotone and, moreover, there exists a bounded linear map \(T: X \to Z\) such that
\[
\| x \|_X \leq \| Tx \|_Z \leq \frac{1}{1 - \tilde{\varepsilon}} \| x \|_X \quad \text{for all } x \in X.
\]
Let $|\cdot|_{Z}$ denote the norm given in Lemma 4.1. Then $|Tx|_Z = \|x\|_X$ for all $x \in X$, where $\|\cdot\|_X$ denotes the norm of $X$. Furthermore, inequality (2) implies

$$(1 - \tilde{\epsilon}) \|z\|_Z \leq |z|_Z \leq \|z\|_Z$$

for all $z \in Z$.

Note that if $m < n$ then, from the last inequality and the fact that the projection constant (relative to the norm $\|\cdot\|_Z$) is larger than $\mathcal{D}$, we get for all $z \in Z$ that

$$|P_{[m,n]}^Z(z)|_Z \leq \|P_{[m,n]}^Z(z)\|_Z \leq \mathcal{D} \|z\|_Z \leq \mathcal{D} \frac{1}{1 - \tilde{\epsilon}} |z|_Z \leq \mathcal{D}(1 + \epsilon) |z|_Z.$$ 

Similarly one shows that

$$\left|I - P_{[m,n]}^Z(z)\right|_Z \leq (1 + \epsilon)$$

and

$$\left|\pi_j - P_{[m,n]}^Z(z)\right|_Z \leq \mathcal{D}(1 + \epsilon) |z|_Z$$

for all $z \in Z$. The proof is over. \(\square\)

8. Construction of the space $W$ and Proof of Theorem $D$

In this section we shall adapt the construction of the space $W$ obtained in [S] to our present context. Then we will use it to prove Theorem $D$. The space $W$ in [S, p. 852] depends upon a set $A$ which is $2/3$-norming the space $V$. As it turns out however, this norming property is not sufficient to our purpose and it thus also needs to be adapted to our context. Proposition 8.1 below will play a fundamental role in this regard.

8.1. Construction of $W$. Let $\mathcal{D} > 1$ and $0 < \epsilon < 1$ be fixed. Assume that $V$ is a Banach space with a shrinking $(1 + \epsilon)$-monotone FDD $(V_j)$ which is $\mathcal{D}(1 + \epsilon)$-strongly bimonotone. Denote by $\|\cdot\|$ the norm of $V$. For $j \in \mathbb{N}$, take $\pi_j := (P_j^V)^*$ to be the adjoint operator of $P_j^V$, where

$$P_j^V : V \to V, \quad v := \sum_{j=1}^{\infty} v_j \mapsto P_j^V v = v_j.$$ 

It follows that $\pi_j(f)(v) = \langle f, P_j^V v \rangle$ for all $f \in V^*$ and $v \in V$. It is easy to see that the canonical embedding from $V_j$ to $V$ has operator norm at most $\mathcal{D}(1 + \epsilon)$. So, $\pi_j$ can be viewed as a linear mapping from $V^*$ into itself with

$$\|\pi_j\| \leq \mathcal{D}(1 + \epsilon) \quad \text{for all} \ j \in \mathbb{N}. \quad (11)$$
For \( j \in \mathbb{N} \), set \( \pi_{j,\varepsilon} := \frac{\pi_j}{(1 + \varepsilon)^2} \). Now fix a sequence \( (\varepsilon_n) \subset (0, 1) \) so that
\[
\sum_{n=1}^{\infty} \varepsilon_n \leq \frac{\varepsilon/2}{(1 + \varepsilon)^2}
\]
and choose, for each \( n \in \mathbb{N} \), a finite \( \varepsilon_n \)-net \( \{x_{(n,i)} : i = 1, 2, \ldots, \ell_n\} \) in the closed ball \( B_{V^\varepsilon}(D) \) of radius \( D \). At this point, we also note that each \( x_{(n,i)}^* \) can be viewed as a functional on \( V \) having norm at most \( D^2(1 + \varepsilon) \).

Then, for every choice of numbers \( (x_{n}) \in \prod_{n=1}^{\infty} \{1, 2, \ldots, \ell_n\} \), we may formally define a functional linear on \( V \) by
\[
\langle \sum_{n=1}^{\infty} a_n x_{(n,i_n)}^* , v \rangle = \sum_{n=1}^{\infty} a_n \langle x_{(n,i_n)}^* , v_n \rangle , \quad \text{for} \quad v = \sum_{n=1}^{\infty} v_n \in V.
\]

Notice that it may well happen for some scalars \( (a_n) \) that the series \( \sum_{n=1}^{\infty} a_n x_{(n,i_n)}^* \) does not converge. However, the next proposition shows in particular that the following set is nonempty
\[
A = \left\{ \sum_{n=1}^{\infty} a_n x_{(n,i_n)}^* : (x_{n}) \in \prod_{n=1}^{\infty} \{1, 2, \ldots, \ell_n\} , \sup_{\ell \geq 1} \left\| \sum_{n=1}^{\ell} a_n x_{(n,i_n)}^* \right\| \leq 1 \right\}.
\]
This set is a modification of the corresponding set \( A \) considered in [S, p. 851].

**Proposition 8.1.** Let \( \varepsilon \) and \( V \) be as above. Then for every \( v \in V \), the following inequality holds.
\[
\frac{1 - \varepsilon/2}{(1 + \varepsilon)^2} \| v \| \leq \sup_{v^* \in A} |\langle v^* , v \rangle| \leq \| v \|.
\]

**Proof.** It suffices to prove the left-hand side of (12). For \( v \in B_V \) pick a functional \( \varphi \in B_{V^*} \) so that \( \| v \| = \langle \varphi , v \rangle \). Since \( (V_j) \) is shrinking, \( \{\pi_j\}_{j=1}^{\infty} \) determines an FDD for \( V^* \). Thus we may write
\[
\varphi = \sum_{k=1}^{\infty} \pi_k(\varphi).
\]

From (11) we deduce \( \pi_{k,\varepsilon}(\varphi|_{V_k}) \in B_{V_k^*}(D) \). So there is \( \varepsilon(k) \in \{1, 2, \ldots, \ell_k\} \) with
\[
\| \pi_{k,\varepsilon}(\varphi|_{V_k}) - x_{(k,i(k))}^* \| \leq \varepsilon_k \quad \text{for all} \quad k \in \mathbb{N}.
\]
Fix any integer \( \ell \) in \( \mathbb{N} \). Using that \( (V_j) \) is \((1+\varepsilon)\)-monotone and applying triangle inequality we have

\[
\left\| \sum_{k=1}^{\ell} x^*_j(k) \right\| \leq \sum_{k=1}^{\ell} \left\| \pi_{k,\varepsilon}(\varphi|V_k) - x^*_j(k) \right\| + \sup_{y \in B_V} \left\| \sum_{k=1}^{\ell} \pi_{k,\varepsilon}(\varphi|V_k), y \right\| \\
\leq \sum_{k=1}^{\ell} \varepsilon_k + \frac{1}{(1+\varepsilon)^2} \sup_{y \in B_V} \left\langle \varphi, \sum_{k=1}^{\ell} P_k^V(y) \right\rangle \\
\leq \sum_{k=1}^{\ell} \varepsilon_k + \frac{\|\varphi\|}{1+\varepsilon} \leq 1.
\]

This shows in particular that \( \sum_{k=1}^{\ell} x^*_j(k) \in A \). Hence,

\[
\|v\| = \langle \varphi, v \rangle = \left\langle \sum_{k=1}^{\ell} \pi_k(\varphi), v \right\rangle = \left\langle \sum_{k=1}^{\ell} \pi_k(\varphi|V_k), v \right\rangle \\
= (1+\varepsilon)^2 \left\langle \sum_{k=1}^{\ell} (\pi_{k,\varepsilon}(\varphi|V_k) - x^*_j(k)), v \right\rangle + (1+\varepsilon)^2 \left\langle \sum_{k=1}^{\ell} x^*_j(k), v \right\rangle \\
\leq (1+\varepsilon)^2 \|v\| \sum_{k=1}^{\ell} \varepsilon_k + (1+\varepsilon)^2 \sup_{v^* \in A} |v^*(v)| \\
\leq \frac{\varepsilon}{2} \|v\| + (1+\varepsilon)^2 \sup_{v^* \in A} |v^*(v)|,
\]

which implies \((1-\varepsilon/2)\|v\| \leq (1+\varepsilon)^2 \sup_{v^* \in A} |v^*(v)|. \)

Let us define a new norm on \( V \) by

\[
\|v\|_V = \sup_{v^* \in A} |v^*(v)| \quad \forall v \in V.
\]

**Remark 8.2.** Let us note in passing that (14) shows that \( A \) is a \( 1 \)-norming set in \( B_{V^*} \) for the pair \((V, \| \cdot \|_V)\). As in [S, p. 854], this information will be crucial in order to prove shrinkingness property.

Let us begin the construction of \( W \). Let

\[
\Gamma := \{(n, i) : n \in \mathbb{N}, \text{ and } i = 1, 2, \ldots, \ell \}.
\]
Denote $\preceq$ the lexicographical order on $\Gamma$. Also, denote the unit vector basis of $c_{00}(\Gamma)$ by $(e_{\gamma}: \gamma \in \Gamma)$ and its coordinate functionals by $(e_{\gamma}^*: \gamma \in \Gamma)$. Define

$$B = \left\{ \sum_{n=1}^{\infty} a_n e_{(n,i_n)}^* : \sum_{n=1}^{\infty} a_n x_{(n,i_n)} \in A \right\}$$

and note that it can be proved using the right hand side of (12) that $B$ satisfies

$$B = \left\{ \sum_{n=1}^{\infty} a_n e_{(n,i_n)}^*: (i_n) \in \prod_{n=1}^{\infty} \{1, 2, \ldots, \ell_n\}, \sup_{(v, \|\|_V)} \left\| \sum_{n=1}^{\infty} a_n x_{(n,i_n)} \right\| \leq 1 \right\}.$$ 

We then define a norm $\|\cdot\|_W$ on $c_{00}(\Gamma)$ by

$$\|x\|_W := \sup_{w^* \in B} w^*(x), \quad x \in c_{00}(\Gamma).$$

Next define $W$ to be the completion of $c_{00}(\Gamma)$ with respect to $\|\cdot\|_W$.

After these considerations, we get the following result whose proof follows the same lines as the proof of [3, Theorem 3.9].

**Theorem 8.3.** Let $\Gamma$, $V$, $\|\|_V$, and $(W, \|\|_W)$ be as above. Then $(e_{\gamma}: \gamma \in \Gamma)$ is a shrinking monotone basis of $W$ which is $D(1+\varepsilon)$-strongly bimonotone. In addition, the map $J: V \to W$ given by

$$\sum v_n \mapsto \sum_{n=1}^{\infty} \sum_{i=1}^{\ell_n} x_{(n,i)}^*(v_n) e_{(n,i)}$$

is an isometric embedding of $(V, \|\|_V)$ into $(W, \|\|_W)$. Moreover, for every $v \in V$, one has

$$\frac{1-\varepsilon/2}{(1+\varepsilon)^2} \|v\| \leq \|J(v)\|_W \leq \|v\|.$$

**Proof.** The proof that $(e_{\gamma}: \gamma \in \Gamma)$ is monotone follows the same lines as the proof of bimonotonicity given in [3, p. 853], we include it here for completeness’ sake. Set $\gamma_1 = (1,1)$ and fix any $\gamma_+ = (n, j_+) \in \Gamma$ with $\gamma_1 \preceq \gamma_+$. We want to prove that $\|P_{[\gamma_1, \gamma_+]}\| \leq 1$. Let $w^* = \sum_{k=1}^{\infty} a_k e_{(k,i_k)}^* \in B$ be fixed. Then

$$P_{[\gamma_1, \gamma_+]}^*(w^*) = \sum_{\gamma_1 \preceq (k,i_k) \preceq \gamma_+} a_k e_{(k,i_k)}^* = \begin{cases} \sum_{k=1}^{n} a_k e_{(k,i_k)}^* & \text{if } i_n \leq j_+ \\ \sum_{k=1}^{n-1} a_k e_{(k,i_k)}^* & \text{if } i_n > j_+ \end{cases}.$$
Since $A$ is closed under projections of the form $P^V_{[1,j]}$ we get that $P^*_{[\gamma_1,\gamma_2]}(w^*) \in B$. Consequently, for every $w = \sum_{\gamma \in \Gamma} \xi_\gamma e_\gamma \in c_0(\Gamma)$ we have

$$
\|P_{[\gamma_1,\gamma_2]}(w)\|_W = \left\| \sum_{\gamma_1 \leq \gamma \leq \gamma_2} \xi_\gamma e_\gamma \right\|_W = \sup_{w^* \in B} P^*_{[\gamma_1,\gamma_2]}(w^*)(w) \leq \|w\|_W.
$$

Let us prove that $(e_\gamma : \gamma \in \Gamma)$ is $D(1+\varepsilon)$-strongly bimonotone. Take any $\gamma_- = (m,j_-)$ and $\gamma_+ = (n,j_+)$ in $\Gamma$, with $m \leq n$ and $j_- < j_+$, if $m = n$, and let $w^*$ be as above. Then

$$
P^*_{[\gamma_-:\gamma_+]}(w^*) = \sum_{\gamma_- \leq (k,i_k) \leq \gamma_+} a_{k}e^*_{(k,i_k)} = \begin{cases} 
\sum_{k=m}^{n} a_{k}e^*_{(k,i_k)} & \text{if } j_- \leq i_m \text{ and } i_n \leq j_+,
\sum_{k=m+1}^{n} a_{k}e^*_{(k,i_k)} & \text{if } j_- < i_m \text{ and } i_n \leq j_+,
\sum_{k=m}^{n-1} a_{k}e^*_{(k,i_k)} & \text{if } j_- \leq i_m \text{ and } i_n > j_+,
\sum_{k=m+1}^{n-1} a_{k}e^*_{(k,i_k)} & \text{if } j_- < i_m \text{ and } i_n > j_+.
\end{cases}
$$

Now using that $(V_j)$ has projection constant less than or equal to $D(1+\varepsilon)$, one can prove that $(1/D(1+\varepsilon)) \sum_{k=m}^{n} a_{k}x^*_{(k,i_k)} \in A$ for all $m \leq n \in \mathbb{N}$. Consequently, we have that $(1/D(1+\varepsilon))P^*_{[\gamma_-:\gamma_+]}(w^*) \in B$ for all $\gamma_- \leq \gamma_+ \text{ in } \Gamma$. It follows that

$$
\frac{1}{D(1+\varepsilon)}\|P_{[\gamma_-:\gamma_+]}(w)\|_W = \sup_{w^* \in B} \left(1/D(1+\varepsilon)\right) P^*_{[\gamma_-:\gamma_+]}(w^*)(w) \leq \|w\|_W,
$$

and hence $\|P_{[\gamma_-:\gamma_+]}(w)\|_W \leq D(1+\varepsilon)\|w\|_W$. Similarly, one can prove that

$$
\|(I - P_{[\gamma_-:\gamma_+]})(w)\|_W \leq D(1+\varepsilon)\|w\|_W.
$$

It follows that $(e_{\gamma} : \gamma \in \Gamma)$ is $D(1+\varepsilon)$-strongly bimonotone. Let us prove that $J$ is an isometry. For $v = \sum_{n=1}^{\infty} v_n \in V$, with $v_n \in V_n$, for $n \in \mathbb{N}$, we have

$$
\|J(\sum_{n=1}^{\infty} v_n)\|_W = \sup \left\{ \sum_{n=1}^{\infty} a_n x^*_n(v_n) : (i_n) \in \prod_{n=1}^{\infty} \{1,2,\ldots,\ell_n\}, \sup_{\ell \geq 1} \left\| \sum_{n=1}^{\ell} a_n x^*_n \right\| \leq 1 \right\}
= \sup \left\{ \left( \sum_{n=1}^{\infty} a_n x^*_n \right) \left( \sum_{n=1}^{\infty} v_n \right) : \sum_{n=1}^{\infty} a_n x^*_n \in A \right\}
= \|v\|_V.
$$
Hence $J: (V, \| \cdot \|_V) \to (W, \| \cdot \|_W)$ is an isometric embedding. It follows therefore from Proposition 8.3 that

$$\frac{1 - \varepsilon/2}{(1 + \varepsilon)^2} \| v \| \leq \| J(v) \|_W \leq \| v \|$$

for all $v \in V$.

This proves assertion (i). The proof that $(e_{\gamma}: \gamma \in \Gamma)$ is shrinking follows the same lines as the proof given in [S] p. 854. □

8.2. **Proof of Theorem D**. This is a direct consequence of Theorem 8.3. □

9. **Proof of Theorem B**

Let $\| \cdot \|_V$ denote the norm of $V$. Fix $\varepsilon > 0$ as in the statement of Theorem B. Let $J$ and $(W, \| \cdot \|_W)$ be as in the Section 8. Then $J$ embeds $(V, \| \cdot \|_V)$ into $(W, \| \cdot \|_W)$ according to the inequality

$$\frac{1 - \varepsilon/2}{(1 + \varepsilon)^2} \| v \|_V \leq \| Jv \|_W \leq \| v \|_V$$

for all $v \in V$.

Define now on $W$ the norm $| w |_W = \inf \{ \| v \|_V + \| w - J(v) \|_W : v \in V \}$.

By Lemma 11, $J: (V, \| \cdot \|_V) \to (W, | \cdot |_W)$ is an isometric embedding with $| \cdot |_W$ satisfying

$$\| w \|_W \leq | w |_W \leq \frac{1 + \varepsilon}{1 - \varepsilon/2} \| w \|_W$$

for all $w \in W$. This inequality certainly implies that the basis $(e_{\gamma}: \gamma \in \Gamma)$ is $(1 + \varepsilon)^4$-monotone and $D(1 + \varepsilon)^4$-strongly bimonotone. □

10. **Proofs of Theorem E and G**

Before starting the proofs we fix some notation. Let $\mathcal{U}$ stand for a free ultrafilter on $\mathbb{N}$. Denote by $\lim_{\mathcal{U} \to} \sup_{n \in \mathbb{N}} \| v_i \| < \infty$. The ultrapower $[X]$ of $X$ is the quotient space $\ell_\infty(X)/\mathcal{N}$ equipped with the norm $\| [v_i] \| = \lim_{\mathcal{U} \to} \| v_i \|$, where $\mathcal{N} = \{(v_i) \in \ell_\infty(X): \lim_{\mathcal{U} \to} \| v_i \| = 0 \}$ and $[v_i]$ denotes the element of $[X]$ associated to the sequence $(v_i)$. Clearly, $X$ embeds isometrically into $[X]$ through the map $x \to [x, x, \ldots] \in [X]$, so henceforth we will not distinguish between $x$ and $[x, x, \ldots]$. If $C$ is a weakly compact convex subset of $X$ and $T: C \to C$ is a nonexpansive fixed-point free mapping, then by Zorn’s lemma there is a weakly compact convex set
Let $T: K \to K$ be a nonexpansive fixed-point free mapping on a minimal weakly compact convex set. Assume that $(y_n)$ is an approximate fixed point sequence of $T$. Then

$$\lim_{n \to \infty} \|x - y_n\| = \text{diam} K \quad \text{for all } x \in K.$$ 

Recall that a sequence $(y_n) \subset K$ is called an approximate fixed point sequence of $T$ provided that $\|y_n - Ty_n\| \to 0$. In the ultra-product language, both $K$ and $T$ are described as follows

$$[K] = \{ [v_i] \in [X] : v_i \in K \forall i \in \mathbb{N} \}$$

and $[T]([v_i]) := [Tv_i]$ for all $[v_i] \in [K]$. The mapping $[T]$ is called the ultrapower mapping induced by $T$. It is easy to see that $[T]$ is nonexpansive and maps $[K]$ into itself. Moreover, the set of fixed points of $[T]$ is nonempty and consists of all points $[x_i] \in [K]$ for which $(x_i)$ is an approximate fixed point sequence of $T$. The ultrapower counterpart of Goebel-Karlovitz’s lemma (see also [AKK, Corollary 3.2]) is the following result due to P.-K. Lin [Lin].

**Lemma 10.2.** Let $K$ be a minimal weakly compact convex set for a nonexpansive fixed-point free mapping $T$. Assume that $[M]$ is a convex nonempty subset of $[K]$ which is invariant under $[T]$. Then for all $x \in K$,

$$\sup\{\|x - [v_i]\| : [v_i] \in [M]\} = \text{diam} K.$$ 

10.1. **Proof of Theorem** [Ein]. Let $X$ be as in the statement of the theorem and let $(e_i)$ denote its $D$-unconditional basis in which $D < \sqrt{6} - 1$. Towards a contradiction, assume that $X$ fails the weak fixed point property. Then there is a weakly compact convex subset $K$ of $X$ which is minimally invariant under a nonexpansive fixed-point free mapping $T$. Fix $(y_n)$ an approximate fixed point sequence of $T$. By Goebel-Karlovitz’s lemma, we have

$$\lim_{n \to \infty} \|x - y_n\| = \text{diam} K \quad \text{for all } x \in K.$$ 

By passing to a subsequence and considering a translation followed by an appropriate scaling of $K$, we may assume that diam $K = 1$ and $(y_n)$ is weakly convergent to 0. So, $(y_n)$ is a seminormalized weakly null sequence. Fix $\varepsilon > 0$ small enough so that

$$\frac{\varepsilon^2}{2} \cdot \left[ \frac{(D + 1)(D + 2 + \varepsilon)}{2} + 1 \right] < \frac{3}{2} - \frac{(D + 1)^2}{4}. \quad (15)$$
This is possible because $D < \sqrt{6} - 1$. Since $(e_i)$ is $D$-unconditional, every block basis of $(e_i)$ is $D$-unconditional, too. Thus, using a standard gliding hump argument, we can select a subsequence $(x_n)$ of $(y_n)$ which is $D + \varepsilon$-unconditional. It follows that $\|x_m + x_n\| \leq (D + \varepsilon)\|x_m - x_n\|$ for all $m < n$ in $\mathbb{N}$. As $\text{diam} K = 1$, we get

$$\|x_m + x_n\| \leq D + \varepsilon \quad \text{for all } m < n.$$  \hfill (16)

Note that $(e_i)$ is $(D + 1)/2$-suppression unconditional. In particular, it is $(D + 1)/2$-strongly bimonotone. By the Main Theorem, there is an isometric linear embedding $J$ from $X$ into a Banach space $W$ with a shrinking $(1 + \varepsilon)$-monotone basis $(w_i)$ which is $(D + 1)(1 + \varepsilon)/2$-strongly bimonotone. Denote by $\|\cdot\|$ the norm of $W$. It follows from (16) that

$$\|Jx_m + Jx_n\| \leq D + \varepsilon \quad \text{for all } m < n.$$  \hfill (17)

Set $K^J = J(K)$ and define $T^J: K^J \to K^J$ by $T^J(Jx) = J(Tx)$. Then $K^J$ is a minimal weakly compact convex set for the nonexpansive fixed-point free mapping $T^J$. Further, one also has $\text{diam} K^J = 1$. The next step of the proof concerns a standard procedure in metric fixed point theory. Let $(P_i)$ denote the natural projections associated to the basis $(w_i)$. For $n \in \mathbb{N}$, set $R_n = I - P_n$. Let $[W]$ denote the ultrapower of $W$. Since $(Jx_n)$ is weakly null, we can use the gliding hump argument to find a subsequence $(x_{m_i})$ of $(x_n)$ and a sequence of non-overlapping intervals $(F_i)$ in $[\mathbb{N}]^{< \infty}$ satisfying the following properties:

(P1) $\|[Jx_{m_i}] - [P_{F_i}Jx_{m_i}]\| = 0$.

(P2) $\|[Jx_{m_i + 2}] - [R_{\max F_i}Jx_{m_i + 2}]\| = 0$.

(P3) $\|[P_{F_i}Jx]\| = \|[R_{\max F_i}Jx]\| = 0$ for all $x \in X$.

Set $\hat{x} = [Jx_{m_i}]$ and $\hat{y} = [Jx_{m_i + 2}]$. Now consider the set

$$[M] = \left\{ [v_i] \in [K^J]: \exists u \in K \text{ such that } \|[v_i] - Ju\| \leq (D + \varepsilon)/2, \text{ and } \max (\|[v_i] - \hat{x}\|, \|[v_i] - \hat{y}\|) \leq 1/2 \right\}.$$  

Let $[T^J]: [K^J] \to [K^J]$ be the ultrapower mapping induced by $T^J$. Direct calculation shows that $[T^J]$ is nonexpansive and leaves $[M]$ invariant. Then Lemma 10.2 implies

$$\sup \{ \|[v_i]\|: [v_i] \in [M] \} = 1.$$  \hfill (18)

As it turns out, however, this cannot be true. Indeed, first, since $\text{diam} K^J = 1$, we deduce from (17) that $(\hat{x} + \hat{y})/2 \in [M]$. Let now $\tilde{P} = [P_{F_i}]$ and $\tilde{Q} = [R_{\max F_i}]$. Our
choice for the operator $\tilde{Q}$ was inspired by the work of Khamsi [K, p. 29]. Fix any $[v_i] \in [M]$ and choose $u \in K$ so that $\|[v_i] - Ju\| \leq (D + \varepsilon)/2$. Then

$$\|[v_i]\| \leq \frac{1}{2} \left( \left\| \tilde{P} + \tilde{Q} \right\| [v_i] - Ju \right\| + \left\| [I] - \tilde{P} \right\| [v_i] - \tilde{x} \right\| + \left\| [I] - \tilde{Q} \right\| [v_i] - \tilde{y} \right\|.$$  

Using now that $(w_i)$ is $(1+\varepsilon)$-monotone and is also $(D+1)(1+\varepsilon)/2$-strongly bimonotone, we conclude that

$$\max \left( \left\| \tilde{P} + \tilde{Q} \right\|, \left\| [I] - \tilde{P} \right\| \right) \leq \frac{(D+1)(1+\varepsilon)}{2} \quad \text{and} \quad \left\| [I] - \tilde{Q} \right\| \leq 1 + \varepsilon$$

It follows that

$$\|[v_i]\| \leq \frac{1}{2} \left( \frac{(D+1)(1+\varepsilon)}{2} \cdot (D+\varepsilon) + \frac{(D+1)(1+\varepsilon)}{2} \cdot \frac{1}{2} + \frac{1+\varepsilon}{2} \right),$$

and hence, taking the supremum over all $[v_i] \in [M]$, we obtain

$$\sup \left\{ \|[v_i]\| : [v_i] \in [M] \right\} \leq \frac{1}{2} \left( \frac{(D+1)(1+\varepsilon)}{2} \cdot (D+\varepsilon) + \frac{(D+1)(1+\varepsilon)}{2} \cdot \frac{1}{2} + \frac{1+\varepsilon}{2} \right).$$

From our choice of $\varepsilon$ in (15) we deduce that $\sup \left\{ \|[v_i]\| : [v_i] \in [M] \right\} < 1$, contradicting (18). This therefore completes the proof of theorem. \hfill \Box

10.2. **Proof of Theorem** Let $X$ be a Banach space with $D$-unconditional spreading basis $(e_i)$ satisfying $D < \sqrt{6} - 1$. Assume by way of contradiction that $X$ fails the weak fixed point property. In particular, $X$ does not have Schur’s property. Since $(e_i)$ is spreading, it follows that $(e_i)$ is a seminormalized weakly null sequence in $X$ (see [HJ, p. 199, Fact 37]). Fix $\varepsilon > 0$ as in (15). By Bessaga-Pelczyński’s selection principle, we can find a subsequence $(e_{n_i})$ of $(e_i)$ which is $1 + \varepsilon$-basic. Denote $(P_n)$ the natural projections associated with $(e_i)$. Since $(e_i)$ is spreading, we deduce that $\sup_{n \geq 1} \|P_n\| \leq 1 + \varepsilon$. Therefore, we can proceed in the same way as in the proof of Theorem [E] and arrive at a contradiction after applying Lin’s Lemma [10.2]. The proof is over. \hfill \Box

**Remark 10.3.** Recall that a basis $(e_i)$ is said to be spreading if it is 1-equivalent to all of its subsequences.
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