FOURIER NONUNIQUENESS SETS FOR THE HYPERBOLA AND THE PERRON-FROBENIUS OPERATORS

DEB KUMAR GIRI

ABSTRACT. Let $\Gamma$ be a smooth curve or finite disjoint union of smooth curves in the plane and $\Lambda$ be any subset of the plane. Let $X(\Gamma)$ be the space of all finite complex-valued Borel measures in the plane which are supported on $\Gamma$ and are absolutely continuous with respect to the arc length measure on $\Gamma$. Let $\mathcal{AC}(\Gamma, \Lambda) = \{\mu \in X(\Gamma) : \hat{\mu}|_{\Lambda} = 0\}$, then we say that $\Lambda$ is a Fourier uniqueness set for $\Gamma$ or $(\Gamma, \Lambda)$ is a Heisenberg uniqueness pair, if $\mathcal{AC}(\Gamma, \Lambda) = \{0\}$. In particular, let $\Gamma$ be the hyperbola $\{(x, y) \in \mathbb{R}^2 : xy = 1\}$ and $\Lambda_\beta$ be the lattice-cross in $\mathbb{R}^2$ is defined by $\Lambda_\beta = (\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta \mathbb{Z})$, where $\beta$ is a positive real. Then Canto-Martín, Hedenmalm and Montes-Rodríguez has shown that the space $\mathcal{AC}(\Gamma, \Lambda_\beta)$ is infinite-dimensional for $\beta > 1$. Further, they considered the branch $\Gamma_+ = \{(x, y) \in \mathbb{R}^2 : xy = 1, x > 0\}$ of the hyperbola $xy = 1$ and the lattice-cross $\Lambda_\gamma = (2\mathbb{Z} \times \{0\}) \cup (\{0\} \times 2\gamma \mathbb{Z})$, where $\gamma$ is a positive real, and prove that $\mathcal{AC}(\Gamma_+, \Lambda_\gamma)$ is infinite-dimensional for $\gamma > 1$. In this paper, we prove the following results:

(a) For a rational perturbation of $\Lambda_\beta$ namely, $\Lambda_\theta^\beta = ((\mathbb{Z} + \{\theta\}) \times \{0\}) \cup (\{0\} \times \beta \mathbb{Z})$, where $\theta = 1/p$, for some $p \in \mathbb{N}$, and $\beta$ is a positive real, $\mathcal{AC}(\Gamma, \Lambda_\theta^\beta)$ is infinite-dimensional whenever $\beta > p$.

(b) For a rational perturbation of $\Lambda_\gamma$ namely, $\Lambda_\theta^\gamma = ((2\mathbb{Z} + \{2\theta\}) \times \{0\}) \cup (\{0\} \times 2\gamma \mathbb{Z})$, where $\theta = 1/q$, for some $q \in \mathbb{N}$, and $\gamma$ is a positive real, $\mathcal{AC}(\Gamma_+, \Lambda_\theta^\gamma)$ is infinite-dimensional whenever $\gamma > q$.

1. INTRODUCTION

1.1. Heisenberg uniqueness pairs. The uncertainty principle for Fourier transform states that a nonzero function and its Fourier transform both cannot be too concentrated at the same time (for the details see [14, 18]). The notion of Heisenberg uniqueness pair introduced recently by Hedenmalm and Montes-Rodríguez as a version of this uncertainty principle. Further, the concept of Heisenberg uniqueness pair has significant similarity with mutually annihilating pairs of Borel measurable sets of positive measures. To describe this, consider a pair of Borel measurable sets $\mathcal{S}, \Sigma \subseteq \mathbb{R}$. Then $(\mathcal{S}, \Sigma)$ forms a mutually annihilating pair if for any $\varphi \in L^2(\mathbb{R})$ such that $\text{supp} \varphi \subseteq \mathcal{S}$ and whose Fourier transform $\hat{\varphi}$ supported on $\Sigma$, implies $\varphi$ is identically zero (for more details see [14]).

Heisenberg uniqueness pair. In [15], Hedenmalm and Montes-Rodríguez proposes the following: Let $\Gamma$ be a smooth curve in $\mathbb{R}^2$ and $\Lambda$ be a subset of $\mathbb{R}^2$. Let $X(\Gamma)$ be
of all finite complex-valued Borel measures $\mu$ in $\mathbb{R}^2$ which are supported on $\Gamma$ and are absolutely continuous with respect to the arc length measure on $\Gamma$. For $(\xi, \eta) \in \mathbb{R}^2$, the Fourier transform of $\mu$ is defined by

$$\hat{\mu}(\xi, \eta) = \int_{\Gamma} e^{\pi i (x\xi + y\eta)} d\mu(x, y).$$

Let $\mathcal{AC}(\Gamma, \Lambda) = \{\mu \in \mathcal{X}(\Gamma) : \hat{\mu}|_{\Lambda} = 0\}$, then following [15], $(\Gamma, \Lambda)$ is said to be a Heisenberg uniqueness pair (HUP) if $\mathcal{AC}(\Gamma, \Lambda) = \{0\}$. In this case, since $\Lambda$ determine the measures $\mu \in \mathcal{X}(\Gamma)$, we say that $\Lambda$ is a Fourier uniqueness set for $\Gamma$. The definition of Heisenberg uniqueness pair can be extended for more general measures but here we restrict our attention to those which are only absolutely continuous. Heisenberg uniqueness pairs satisfy the following invariance properties:

1. For any points $u_0, v_0 \in \mathbb{R}^2$, $(\Gamma + \{u_0\}, \Lambda + \{v_0\})$ is a HUP if and only if $(\Gamma, \Lambda)$ is a HUP.
2. $(\Gamma, \Lambda)$ is a HUP if and only if $(T^{-1}(\Gamma), T^*(\Lambda))$ is a HUP, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible linear transform with adjoint $T^*$.

The dual formulation. $(\Gamma, \Lambda)$ is a HUP if and only if the subspace of all linear span of the functions $\{e^{\pi i (x\xi + y\eta)} : (\xi, \eta) \in \Lambda\}$ is weak-star dense in $L^\infty(\Gamma)$.

Many examples of Heisenberg uniqueness pair have been obtained in the plane as well as in the higher dimensional Euclidean spaces. The Heisenberg uniqueness pairs $(\Gamma, \Lambda)$ in which $\Gamma$ is the union of any two parallel lines in the plane was investigated in [15]. In [2], Babot studied the HUP in which $\Gamma$ is the certain system of three parallel lines. Later, the author has studied the HUP corresponding to a certain system of four parallel lines together with some algebraic curves (see [10]). Further, the cases in which $\Gamma$ is the union of a certain system of finitely many parallel lines were studied in [3, 11]. The cases in which $\Gamma$ is the unit circle were independently investigated in [22, 23]. Later, González Vieli (see [12]) generalized the cases for the circle to the higher dimension using the properties of the Bessel functions $J_{(n+2k-2)/2}$, $k \in \mathbb{Z}_+$. In [25], Srivastava studied the cases for the sphere in which $\Lambda$ is the cone as well as the cone does not contain in the zero sets of any homogeneous harmonic polynomial on $\mathbb{R}^n$. In [24], Sjölin investigated the HUP corresponding to the parabola while the author (see [7]) studied certain exponential surfaces and connect the notion of HUP to the Euclidean motion groups. The dynamical system approach was used in [20] to study the cases for hyperbola, polygon, ellipse, and graph of $\varphi(t) = |t|^\alpha$, whenever $\alpha > 0$. Later, Gröchenig and Jaming [13] solved the cases corresponding to the quadratic surface. In a recent article [8], the authors had extended the notion of Heisenberg uniqueness pair to the Heisenberg group. As a major development along this direction, in [6, 15, 16, 17], the dynamics of Gauss-type maps, Ergodic theory, Klein-Gordon equation, and the Perron-Frobenius operators were used to studying HUP which advanced the theory.

In this paper, the problems (Theorem 1.7 and Theorem 1.13) we consider is inspired by the articles [6, 9, 15], and closely follows the methods of [6] with
For bounded and continuous function $\varphi$, in particular, for $(\xi,\eta)$, the lattice-cross $\Lambda$, the notation from [6] as much as possible. Moderate modifications at appropriate places. We will follow definitions and notation from [6] as much as possible.

1.2. Nonuniqueness sets for the hyperbola. Let $\Gamma = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$ be the hyperbola and $\mu \in \mathcal{X}(\Gamma)$, then there exists $g \in L^1(\mathbb{R}, \sqrt{1 + 1/t^4} \, dt)$ such that for bounded and continuous function $\varphi$ on $\mathbb{R}^2$,

$$\int_{\Gamma} \varphi(x, y) d\mu(x, y) = \int_{\mathbb{R}\setminus\{0\}} \varphi(t, 1/t) g(t) \sqrt{1 + 1/t^4} dt.$$  

In particular, for $(\xi, \eta) \in \mathbb{R}^2$, the Fourier transform of $\mu$ can be expressed as

$$\hat{\mu}(\xi, \eta) = \int_{\mathbb{R}\setminus\{0\}} e^{\pi i \xi t + \eta / t} f(t) dt,$$

where $f(t) := g(t) \sqrt{1 + 1/t^4} \in L^1(\mathbb{R})$.

As a first known result on HUP, in [15], Hedenmalm and Montes-Rodríguez have studied that some lattice-cross in the plane is a Fourier uniqueness set for the hyperbola and have proved the following result.

**Theorem 1.1.** [15] Let $\Gamma = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$ be the hyperbola and $\Lambda_\beta$ be the lattice-cross $\Lambda_\beta := (\mathbb{Z} \times \{0\}) \cup \{0\} \times \beta \mathbb{Z}$, where $\beta$ is a positive real. Then $\mathcal{AC}(\Gamma, \Lambda_\beta) = \{0\}$ if and only if $\beta \leq 1$.

In [6], Canto-Martín, Hedenmalm and Montes-Rodríguez have studied that some lattice-cross in $\mathbb{R}^2$, is a Fourier nonuniqueness set for the hyperbola and have proved the following result.

**Theorem 1.2.** [6] Let $\Gamma$ be the hyperbola $xy = 1$ and $\Lambda_\beta$ be the lattice-cross $\Lambda_\beta := (\mathbb{Z} \times \{0\}) \cup \{0\} \times \beta \mathbb{Z}$, where $\beta$ is a positive real. Then $\mathcal{AC}(\Gamma, \Lambda_\beta)$ is infinite-dimensional for $\beta > 1$.

Let $\mathcal{M}_\beta$ be the subspace of all linear span of the functions $\{e_n(x) := e^{\pi i n x}; n \in \mathbb{Z}\} \cup \{e_n^0(x) := e^{\pi i n \beta x}; n \in \mathbb{Z}\}$ in $L^\infty(\mathbb{R})$, where $\beta$ is a positive real. The codimension of the weak-star closure of $\mathcal{M}_\beta$ in $L^\infty(\mathbb{R})$ is the dimension of its pre-annihilator space

$$\mathcal{M}^\perp_\beta := \left\{ f \in L^1(\mathbb{R}) : \int_{\mathbb{R}} f(x) e_n(x) dx = \int_{\mathbb{R}} f(x) e_n^\beta(x) dx = 0 \text{ for all } n \in \mathbb{Z}\right\}.$$  

By dual formulation, Theorem 1.1 is equivalent to the following density result.

**Theorem 1.3.** [15] The space $\mathcal{M}_\beta$ is weak-star dense in $L^\infty(\mathbb{R})$ if and only if $0 < \beta \leq 1$.

Similarly, by dual formulation, Theorem 1.2 is equivalent to the following density result.

**Theorem 1.4.** [6] $\mathcal{M}^\perp_\beta$ is an infinite-dimensional subspace of $L^1(\mathbb{R})$ for $1 < \beta < \infty$.

Next, we consider a rational perturbation of the lattice-cross $\Lambda_\beta$, namely that

$$\Lambda_\beta^\theta := ((\mathbb{Z} + \{\theta\}) \times \{0\}) \cup \{0\} \times \beta \mathbb{Z},$$

(1.2)
where $\theta = 1/p$, for some $p \in \mathbb{N}$, and $\beta$ is a positive real. Let $\Gamma$ be the hyperbola $xy = 1$, then the following result shows that $(\Gamma, \Lambda^\theta_\beta)$ is a Heisenberg uniqueness pair for $0 < \beta \leq p$. In other words, $\Lambda := \Lambda^\theta_\beta$ is a Fourier uniqueness set for the hyperbola $\Gamma$ whenever $\beta \leq p$.

**Theorem 1.5.** [9] $\mathcal{AC} (\Gamma, \Lambda^\theta_\beta) = \{0\}$ if and only if $0 < \beta \leq p$.

**Remark 1.6.** (a) It is rather surprising that the condition on $\beta$ depends on $\theta$. The proof of Theorem 1.5 works along the same lines as in [15] but with moderate modifications at appropriate places. The question is remains open for irrational values of $\theta$.

(b) The notion of Heisenberg uniqueness pair may be extended for more general finite complex-valued Borel measures $\mu$ in $\mathbb{R}^2$ which are supported on $\Gamma$ without assuming absolute continuity with respect to the arc length measure on $\Gamma$. But for Theorem 1.5, the measures $\mu$ must be absolutely continuous with respect to the arc length measure on the hyperbola, without this assumption, Theorem 1.5 is not true.

Next, we state a result of this paper which is a variant of Theorem 1.2.

**Theorem 1.7.** Let $\Gamma$ be the hyperbola $xy = 1$ and $\Lambda^\theta_\beta$ be the lattice-cross defined in (1.2). Then $\mathcal{AC} (\Gamma, \Lambda^\theta_\beta)$ is infinite-dimensional for $\beta > p$.

**Remark 1.8.** (a) Theorem 1.7 asserts that $\Lambda := \Lambda^\theta_\beta$ is a Fourier nonuniqueness set for the hyperbola $xy = 1$ whenever $\beta > p$. The presence of $\theta$ showing up in the condition of $\beta$ which is somewhat unexpected. The proof of Theorem 1.7 works along the same lines as in [6] but with modifications at appropriate places.

(b) As a corollary to Theorem 1.7 let $\Gamma$ be the hyperbola $xy = 1$ and $\Lambda^\theta_\beta$ be the set $\Lambda^\theta_\beta := ((\mathbb{Z} + \{\zeta\}) \times \{0\}) \cup (\{0\} \times \beta \mathbb{Z})$, where $\zeta = r/p$, for some $p \in \mathbb{N}$ and $r \in \mathbb{Z}$ with gcd($p, r$) = 1 and $\beta$ is a positive real. Then $\mathcal{AC} (\Gamma, \Lambda^\theta_\beta)$ is infinite-dimensional for $\beta > p$.

(c) Let $\Gamma$ be the hyperbola $xy = 1$, then any $\mu \in \mathcal{AC} (\Gamma, \Lambda^\theta_\beta)$, $u := \hat{\mu}$ is a solution of the one-dimensional Klein-Gordon equation: $(\partial_\xi \partial_\eta + \pi^2) u(\xi, \eta) = 0$ in the sense of distributions. Theorem 1.7 says that for $\beta > p$, the solution space of the above partial differential equation is infinite-dimensional.

Let $\mathcal{F}_\beta$ be the subspace of all linear span of the functions $e_n^\theta(x), e_n^\beta(x); \ n \in \mathbb{Z}$ in $L^\infty(\mathbb{R})$, where $e_n^\theta(x) := e^{\pi i (n+1/p)x}$ and $e_n^\beta(x) := e^{\pi i n \beta x}$ with $p \in \mathbb{N}$ and $\beta$ is a positive real. The codimension of the weak-star closure of $\mathcal{F}_\beta$ in $L^\infty(\mathbb{R})$ is the dimension of its pre-annihilator space $\mathcal{F}_\beta^\perp$. By dual formulation, Theorem 1.5 is equivalent to the following result.

**Theorem 1.9.** [9] The space $\mathcal{F}_\beta$ is weak-star dense in $L^\infty(\mathbb{R})$ if and only if $0 < \beta \leq p$.

Similarly, Theorem 1.7 is equivalent to the following density result.

**Theorem 1.10.** $\mathcal{F}_\beta^\perp$ is an infinite-dimensional subspace of $L^1(\mathbb{R})$ for $p < \beta < \infty$. 

1.3. Nonuniqueness sets for the branch of the hyperbola. Let $\Gamma_+ = \{(x, y) \in \mathbb{R}^2 : xy = 1, \ x > 0\}$ be the branch of the hyperbola and $\Lambda_{\gamma}$ be the lattice-cross in $\mathbb{R}^2$, then there exists $g \in L^1(\mathbb{R}_+, \sqrt{1 + 1/t^4} \, dt)$ such that for bounded and continuous function $\varphi$ on $\mathbb{R}^2$,
\[
\int_{\Gamma_+} \varphi(x, y) d\mu(x, y) = \int_{\mathbb{R}_+\setminus\{0\}} \varphi(t, 1/t) g(t) \sqrt{1 + 1/t^4} dt.
\]
In particular, for $(\xi, \eta) \in \mathbb{R}^2$, the Fourier transform of $\mu$ can be expressed as
\[
\hat{\mu}(\xi, \eta) = \int_{\mathbb{R}_+\setminus\{0\}} e^{\pi i (\xi t + \eta/t)} f(t) dt,
\]
where $f(t) := g(t) \sqrt{1 + 1/t^4} \in L^1(\mathbb{R}_+)$. In [6], Canto-Martín, Hedenmalm and Montes-Rodríguez have studied that some lattice-cross in $\mathbb{R}^2$, is a Fourier nonuniqueness set for $\Gamma_+$ and have proved the following result.

Theorem 1.11. [6] Let $\Gamma_+ = \{(x, y) \in \mathbb{R}^2 : xy = 1, \ x > 0\}$ be the branch of the hyperbola and $\Lambda_{\gamma}$ be the lattice-cross $\Lambda_{\gamma} := (2\mathbb{Z} \times \{0\}) \cup \{0\} \times 2\gamma \mathbb{Z}$, where $\gamma$ is a positive real. Then $\mathcal{A}(\Gamma_+, \Lambda_{\gamma})$ is infinite-dimensional for $\gamma > 1$.

By dual formulation, Theorem 1.11 is equivalent to the following result.

Theorem 1.12. [6] Let $\mathcal{N}_{\gamma}$ be the subspace of all linear span of the functions $\{e_n(x) := e^{2\pi i n x}, \ n \in \mathbb{Z}\} \cup \{e_n^+ (x) := e^{2\pi i n x/\gamma}, \ n \in \mathbb{Z}\}$ in $L^\infty(\mathbb{R}_+)$, where $\gamma$ is a positive real. Then the pre-annihilator space $\mathcal{N}_{\gamma}^\perp$ is infinite-dimensional for $\gamma > 1$.

Next, we state a result of this paper which is a variant of Theorem 1.11.

Theorem 1.13. Let $\Gamma_+ = \{(x, y) \in \mathbb{R}^2 : xy = 1, \ x > 0\}$ be the branch of the hyperbola and $\Lambda_{\gamma}^\theta$ be the lattice-cross $\Lambda_{\gamma}^\theta := ((2\mathbb{Z} + \{2\theta\}) \times \{0\}) \cup \{0\} \times 2\gamma \mathbb{Z}$, where $\theta = 1/q, \ q \in \mathbb{N}$, and $\gamma$ is a positive real. Then $\mathcal{A}(\Gamma_+, \Lambda_{\gamma}^\theta)$ is infinite-dimensional for $\gamma > q$.

Remark 1.14. (a) Theorem 1.13 asserts that $\Lambda := \Lambda_{\gamma}^\theta$ is a Fourier nonuniqueness set for the branch $\Gamma_+$ whenever $\gamma > q$. The presence of $\theta$ showing up in the condition of $\gamma$ which is somewhat unexpected. The proof of Theorem 1.13 works along the same lines as in [6] but with modifications at appropriate places. The question is still open when $\theta$ is irrational.

(b) Let $\Lambda_{\gamma}^\theta$ be the lattice-cross $((2\mathbb{Z} + \{2\theta\}) \times \{0\}) \cup \{0\} \times 2\gamma \mathbb{Z}$, where $\theta = 1/q, \ q \in \mathbb{N}$, and $\gamma$ is a positive real. It seems likely, that $\mathcal{A}(\Gamma_+, \Lambda_{\gamma}^\theta) = \{0\}$ if and only if $\gamma < q$, and for the critical case $\gamma = q$, $\mathcal{A}(\Gamma_+, \Lambda_{\gamma}^\theta)$ is one-dimensional in analogy with the results in [10]. The question is still open.

By duality, Theorem 1.13 is equivalent to the following density result.

Theorem 1.15. Let $\mathcal{K}_{\gamma}$ be the subspace of all linear span of the functions $\{e_n(x) := e^{2\pi i n(1+1/q)x}, \ n \in \mathbb{Z}\} \cup \{e_n^+ (x) := e^{2\pi i n x/\gamma}, \ n \in \mathbb{Z}\}$ in $L^\infty(\mathbb{R}_+)$, where $q \in \mathbb{N}$ and $\gamma$ is a positive real. Then the pre-annihilator space $\mathcal{K}_{\gamma}^\perp$ is infinite-dimensional for $\gamma > q$. 
1.4. The Perron-Frobenius operators. In this section, we recall the definitions and notation related to the Perron-Frobenius operators associated with a $C^2$-smooth piecewise monotonic transform from ([6], Section 3) as far as possible. The spectral property of the Perron-Frobenius operators has played a significant role in the HUP.

1.4.1. Perron-Frobenius operators on bounded intervals.
Let $I \subset \mathbb{R}$ be a closed and bounded interval and $m$ be the Lebesgue measure defined on the $\sigma$-algebra of $I$. Following ([6], Definition 3.2), a measurable map $\tau : I \to I$ is said to be a "partially filling $C^2$-smooth piecewise monotonic transform" if there exists a countable collection of pairwise disjoint open intervals say, $\{I_u\}_{u \in U}$, where $U$ is the index set, such that the following holds:

(i) $m(I \setminus \bigcup\{I_u : u \in U\}) = 0$,
(ii) for any $u \in U$, the map $\tau_u := \tau|_{I_u}$ is strictly monotone and can be extended to a $C^2$-smooth function on $I_u$ with $\tau_u' \neq 0$ on $I_u$,
(iii) there exists a positive number say, $\delta$ such that $m(\tau(I_u)) \geq \delta$ for all $u \in U$.

Following ([6], Definition 3.1), $\tau$ is said to be a "filling $C^2$-smooth piecewise monotonic transform", if the above conditions (i),(ii) holds for $\tau$ along with (iii)' for every $u \in U$, the map $\tau_u : \overline{I_u} \to I$ is onto.

Observe that, condition (iii)' is much stronger than condition (iii). In the above context, each $I_u$ is called a "fundamental interval" and $\tau_u$ is the corresponding "branch".

The Koopman operator $C_\tau : L^\infty(I) \to L^\infty(I)$ corresponding to a measurable map $\tau : I \to I$ is defined by letting

$$C_\tau[\varphi] = \varphi \circ \tau.$$  

The Perron-Frobenius operator $P_\tau : L^1(I) \to L^1(I)$ is the pre-dual adjoint (Banach space dual) of $C_\tau$ is given by

$$\langle P_\tau[\psi], \varphi \rangle_I = \langle \psi, C_\tau[\varphi] \rangle_I,$$

where $\psi \in L^1(I)$ and $\varphi \in L^\infty(I)$.

The operator $P_\tau$ is linear and a norm contraction on $L^1(I)$, therefore, its spectrum $\sigma(P_\tau)$ is contained in the closed unit disk $\overline{D} = \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \}$.

1.4.2. The spectral decomposition of Perron-Frobenius operators.
Functions of bounded variation in one variable. Let $I \subset \mathbb{R}$ be a closed bounded interval. For any function $h : I \to \mathbb{C}$, the pointwise variation of $h$ in $I$ is defined by letting

$$pV(h, I) := \sup_{t_1, \ldots, t_n \in I, n \geq 2} \left\{ \sum_{i=1}^{n-1} |h(t_{i+1}) - h(t_i)| \right\}.$$  

Then $h$ is said to be a function of bounded variation provided $pV(h, I) < \infty$. Let $BV(I)$ denote the subspace of all functions in $L^1(I)$ such that the pointwise variation is finite. The essential variation $eV(h, I)$ of $h$ in which the pointwise variation is minimized in the equivalence class is defined by

$$eV(h, I) := \inf \left\{ pV(\tilde{h}, I) : \tilde{h} = h \text{ except on a set of Lebesgue measure zero} \right\}.$$
For any \( h \in \text{BV}(I) \), from (9), Theorem 3.27 we get that the infimum in the expression of \( eV(h, I) \) is achieved. Hence the space \( \text{BV}(I) \) equipped with the norm
\[
\|h\|_{\text{BV}} = \|h\|_{L^1(I)} + eV(h, I), \; h \in \text{BV}(I),
\]
becomes a Banach space.

Next, we state the spectral decomposition of Perron-Frobenius operators \( \mathcal{P}_\tau \) associated to \( \tau \) on \( I \). Let \( \partial \bar{D} \) denote the boundary of the closed unit disk \( \bar{D} \) and the point spectrum of \( \mathcal{P}_\tau \) is denoted by \( \sigma_{\text{point}}(\mathcal{P}_\tau) \). Then the following spectral decomposition for \( \mathcal{P}_\tau \) is stated in (9), Theorem C which is a consequence of the Ionescu-Tulcea and Marinescu theorem (for details see [5, 19]). Recall the definition of \( \mathcal{U}_m \); \( m \geq 1 \) from (9), p. 39. In particular, for \( m = 1 \), we have \( \mathcal{U}_1^m = \mathcal{U} \).

**Theorem A.** Let \( \tau : I \to I \) is a partially filling \( C^2 \)-smooth piecewise monotonic transform such that

(i) [uniform expansiveness] there exists an integer \( m \geq 1 \) and \( \epsilon > 0 \) such that
\[
|\tau'(x)| \geq 1 + \epsilon \quad \text{for all} \quad x \in \bigcup \{ I_u : u \in \mathcal{U}_m \},
\]

(ii) [second derivative condition] there exists \( M > 0 \) such that \( |\tau''(x)| \leq M |\tau'(x)|^2 \)
\[
\text{for all} \quad x \in \bigcup \{ I_u : u \in \mathcal{U} \},
\]

then \( \Lambda_\tau := \sigma_{\text{point}}(\mathcal{P}_\tau) \cap \partial \bar{D} \) is a finite set namely, \( \Lambda_\tau = \{ \alpha_1, \ldots, \alpha_s \} \) and one of the eigenvalues is 1, say \( \alpha_1 = 1 \). Let \( E_i \) denotes the eigenspace of \( \mathcal{P}_\tau \) for the eigenvalue \( \alpha_i \), then \( E_i \) is finite-dimensional and \( E_i \) is contained in \( \text{BV}(I) \). In addition,
\[
\mathcal{P}_\tau^n[h] = \sum_{i=1}^s \alpha_i^n \mathcal{P}_{\tau,i}[h] + \mathcal{Z}_\tau^n[h], \; h \in L^1(I), \; n = 1, 2, \ldots,
\]

where the operators \( \mathcal{P}_{\tau,i} \) are projections onto \( E_i \), and the operator \( \mathcal{Z}_\tau \) acts boundedly on \( L^1(I) \) as well as on \( \text{BV}(I) \). Moreover, \( \mathcal{Z}_\tau \) acting on \( \text{BV}(I) \) has spectral radius \( < 1 \).

## 2. Proof of Theorem 1.10

### 2.1. Dynamics of a Gauss-type map.

**A Gauss-type map.** For \( t \in \mathbb{R} \), the expression \( \{t\}_2 \) represent the unique number in \((-1, 1)\) such that \( t - \{t\}_2 \in 2\mathbb{Z} \). We consider a Gauss-type map \( U \) on the interval \((-p, p); \; p \in \mathbb{N} \) which is defined by letting
\[
U(x) := \begin{cases} 
p(\frac{-x}{2})_2, & x \neq 0, \\
0, & \text{for } x = 0.
\end{cases}
\]

For \( u \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\} \), the map \( U \) can be explicitly written as \( U(x) = p(2u - p/x) \) whenever \( \frac{p}{2u+1} < x \leq \frac{p}{2u-1} \), and hence \( U : \left( \frac{p}{2u+1}, \frac{p}{2u-1} \right) \to (-p, p) \) is one-to-one and for \( x \in (-p, p) \setminus \frac{p}{2z+1} \), the derivative of \( U \) is \( U'(x) = \frac{p^2}{x^2} \). For a continuous \( 2p \)-periodic function \( \varphi \) on \( \mathbb{R} \) and a finite complex-valued Borel measure \( \nu \) on \((-p, p)\), the integral \( \int_{(-p,p]} \varphi(x) d\nu(x) \) is well-defined. The above integral makes sense for all pseudo-continuous functions on \((-p, p)\).
Note that for a pseudo-continuous function \( \varphi \) on \((-p, p)\), \( \varphi \circ U \) is pseudo-continuous. Given \( \lambda \in \mathbb{C} \), a finite complex Borel measure \( \nu \) on \((-p, p)\) is \((U, \lambda)\)-invariant provided that

\[
\int_{(-p, p)} \varphi(U(x)) \, d\nu(x) = \lambda \int_{(-p, p)} \varphi(x) \, d\nu(x)
\]

holds for all pseudo-continuous functions \( \varphi \), that is, \( \lambda \nu = \nu(\{0\}) \delta_0 + \sum_{u \in \mathbb{Z}^*} \nu_u \), where \( \delta_t \) denote the point mass at \( t \), and \( d\nu_u(x) = d\nu \left( \frac{p^2}{2pu-x} \right) \). It is easy to see that, for \( |\lambda| > 1 \), there are no \((U, \lambda)\)-invariant measures except zero measure.

In this work, we mainly study the properties the following map which is associated to the parameter \( \beta \). For \( 0 < \beta < \infty \), the Gauss-type map \( U_\beta : (-p, p) \to (-p, p) \) is defined by letting

\[
U_\beta(x) := \begin{cases} 
  p \left\{ -\frac{\beta}{x} \right\}_2, & x \neq 0, \\
  0, & \text{for } x = 0.
\end{cases}
\]

For \( u \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\} \), on the interval \( \left( \frac{\beta}{2u+1}, \frac{\beta}{2u-1} \right] \), the map \( U_\beta \) can be expressed as \( U_\beta(x) = p \left( 2u - \beta / x \right) \). In particular, \( U_p = U \). For \( \lambda \in \partial \mathbb{D} \), a finite complex Borel measure \( \nu \) on \((-p, p)\) is \((U_\beta, \lambda)\)-invariant provided that \( \lambda \nu = \nu(\{0\}) \delta_0 + \sum_{u \in \mathbb{Z}^*} \nu_u \), where \( d\nu_u(x) = d\nu \left( \frac{p^2}{2pu-x} \right) \).

2.1.2. Dynamical properties of \( U_\beta \) for \( \beta > p \).

Denote \( \beta_0 := \beta / p > 1 \). In this section, we observe that \( U_\beta \) is a partially filling \( C^2 \)-smooth piecewise monotonic transform for \( \beta_0 > 1 \). We need to find a unique absolutely continuous invariant probability measure for \( U_\beta \) having positive density.

Let \( \mathcal{U} := U_{\beta_0} \) denote the index set which contain the points \( u \in \mathbb{Z}^* \) so that the associate fundamental interval is nonempty, that is,

\[
I_u := \left( \frac{-\beta}{2u+1}, \frac{\beta}{2u-1} \right) \cap (-p, p) \neq \emptyset.
\]

If \( \beta_0 \) is an odd integer, then \( \{\beta_0\}_2 = 1 \). In this case, it is easy to see that for all \( u \in \mathcal{U} \), where \( \mathcal{U} \) be the set of all nonzero integers with \( |u| \geq \frac{1}{2}(\beta_0 + 1) \), the fundamental intervals are given by

\[
I_u := \left( \frac{-\beta}{2u+1}, \frac{\beta}{2u-1} \right)
\]

so that \( U_\beta(I_u) = (-p, p) \) for all \( u \in \mathcal{U} \). Thus in this particular case, \( U_\beta \) fulfill the "filling" condition for all \( u \in \mathcal{U} \).

If \( \beta_0 \) is not an odd integer, then \(-1 < \{\beta_0\}_2 < 1 \). Denote \( u_0 := \frac{1}{2}(\beta_0 - \{\beta_0\}_2) \), then \( u_0 \in \mathbb{Z} \) with \( u_0 \geq 1 \). A simple calculation shows that for all \( u \in \mathcal{U} \setminus \{\pm u_0\} \),
where $\mathcal{U}$ be the set of all nonzero integers with $|u| \geq u_0$, the fundamental intervals are given by

\[
I_u := \left( \frac{\beta}{2u + 1}, \frac{\beta}{2u - 1} \right)
\]

so that $U_\beta(I_u) = (-p, p)$ for all $u \in \mathcal{U} \setminus \{ \pm u_0 \}$. Thus in this case, $U_\beta$ fulfill the "filing" condition for all $u \in \mathcal{U}$ except two branches corresponding to $\{ \pm u_0 \}$. The edge fundamental intervals corresponding to $\pm u_0$ are explicitly given by

\[
I_{u_0} := \left( \frac{\beta}{2u_0 + 1}, p \right), \quad I_{-u_0} := \left( -p, -\frac{\beta}{2u_0 - 1} \right).
\]

In view of the above facts, we conclude that $U_\beta$ is a partially filling $C^2$-smooth piecewise monotonic transform for $\beta_0 > 1$.

The map $U_\beta$ satisfy the "uniform expansiveness" condition with $m = 1$, because of the derivative $|U'_\beta(x)| = \frac{\beta^2}{x^2} \geq \beta_0 > 1$ for all $x \in \{ I_u : u \in \mathcal{U} \}$. Also, we have "second derivative condition" due to $|U''_\beta(x)| \leq \frac{2}{\beta^2}|U'_\beta(x)|^2$ for all $x \in \{ I_u : u \in \mathcal{U} \}$. Our aim is to find a unique absolutely continuous invariant measure for $U_\beta$ which has a positive density. Since $U_\beta$ is a partially filling $C^2$-smooth piecewise monotonic transform, (see [6], p. 45, Remark 5.1(b)), there exists a $U_\beta$-invariant absolutely continuous probability measure, but it may not be unique. Although, form ([6], p. 45, Remark 5.2) we can get the uniqueness for filling $C^2$-smooth piecewise monotonic transform. Also, $U_\beta$ may not always be a Markov map, therefore, to get the uniqueness of the absolutely continuous $U_\beta$-invariant measure, we have to prove the condition $(iii)$ of Adler’s theorem which is stated in ([6], p. 46, Theorem D), for the details see [5, 21].

2.1.3. The iterates of an interval.

The following result will help to get the unique ergodic $U_\beta$-invariant absolutely continuous probability measure with positive density.

**Lemma 2.1.** $(p < \beta < \infty)$ Let $J_0 \subset [-p, p]$ be any nonempty open interval, then for sufficiently large positive integers, namely that $n \geq n_0$, we have $C_p := (-p, p) \subset U^n_\beta(J_0)$.

**Proof.** We want to show that $U^n_\beta(J_0)$ will cover the open interval $C_p$ for sufficiently large positive integers namely, $n \geq n_0$. The proof will be carried out in the following cases.

1. The case $\beta_0$ is an odd integer.

In this case $\beta_0 \geq 3$ and hence $\beta_0/2 > 1$. The fundamental intervals $I_u$ are given by (2.2) for all $u \in \mathcal{U}$, where $\mathcal{U}$ be the set of all nonzero integers with $|u| \geq \frac{1}{2}(\beta_0 + 1)$. Here, we want to show that $C_p \subset U^n_\beta(J_0)$ for sufficiently large $n$. There are two possibilities:

(i). Suppose $J_0$ contains one of the fundamental intervals say, $I_u$ for some $u \in \mathcal{U}$, then it follows that $C_p = U_\beta(I_u) \subset U_\beta(J_0)$.

(ii). If none of the fundamental intervals are contained in $J_0$, then we have the following possibilities:
(a). Suppose $J_0$ is contained in one of the fundamental intervals namely, $I_u$, $u \in \mathcal{U}$, then by uniform expansiveness condition of $U_\beta$, we have $m(J_1) \geq \beta_0 m(J_0) \geq \frac{\beta_0}{2} m(J_0)$, where $J_1 := U_\beta(J_0)$.

(b). Suppose $J_0$ has nonempty intersection with two neighbouring fundamental intervals say $I_u, I_{u'}$ and $J_0$ is contained in the closure of $I_u \cup I_{u'}$. In this case, one of the sets $J_0 \cap I_u$, $J_0 \cap I_{u'}$ say, $J_0 \cap I_u$ has length at least $\frac{1}{2} m(J_0)$. It follows that for $J_1 := U_\beta(J_0 \cap I_u)$ with $J_1 \subset U_\beta(J_0)$, we have

$$m(J_1) = m(U_\beta(J_0 \cap I_u)) \geq \beta_0 m(J_0 \cap I_u) \geq \frac{\beta_0}{2} m(J_0).$$

For both the cases (a) and (b), there exist an interval $J_1$ which is contained in $U_\beta(J_0)$ and $m(J_1) \geq \frac{\beta_0}{2} m(J_0)$. Next, we consider $J_1$ in place of $J_0$, and therefore we get an interval namely, $J_2$ such that $m(J_2) \geq \frac{\beta_0}{2} m(J_1) \geq \left(\frac{\beta_0}{2}\right)^2 m(J_0)$. We repeat this process, and hence we get an increasing sequence of intervals namely, $J_0, J_1, J_2, \ldots$ such that $m(J_l) \geq \left(\frac{\beta_0}{2}\right)^l m(J_0)$ with $J_l \subseteq U_\beta^l(J_0)$. Since the length of $J_l$ depends on $l$, after finitely many steps say, $l = l_0$ we must stop this process, because $J_{l_0}$ will contain one of the fundamental intervals, and hence $\mathcal{C}_p \subset U_\beta^l(J_0) \subseteq U_\beta^{l+1}(J_0)$.

2. **The case $\beta_0$ is not an odd integer.**

Since $\{\beta_0\}_2 \in (-1, 1)$, it follows that $2u_0 - 1 < \beta_0 < 2u_0 + 1$. The fundamental intervals $I_u$ are given by (2.3) for all $u \in \mathcal{U} \setminus \{\pm u_0\}$, where $\mathcal{U}$ be the set of all nonzero integers with $|u| \geq u_0$. The edge fundamental intervals $I_{-u_0}$ are given by (2.4). For every $u \in \mathcal{U}$, the map $U_\beta$ is given by $U_\beta(x) = p(2u - \beta/x)$ for all $x \in I_u$.

2(A). **The case $J_0$ is an edge fundamental interval.** Assume that $J_0 = I_{-u_0}$, and the case $J_0 = I_{u_0}$ is similar. In this case, we want to show that $\mathcal{C}_p \subset U_\beta^l(J_0)$ for sufficiently large $n$. We first observe that

$$U_\beta(J_0) = U_\beta(I_{-u_0}) = \left(p(\beta_0 - 2u_0), p\right) \supset I_{u_0} := \left(p(\beta_0 - 2u_0), \frac{\beta}{2u_0 + 1}\right).$$

Then there are two possibilities:

(i). If $(\beta_0 - 2u_0) \leq \beta_0/(2u_0 + 3)$, then we have $I_{u_0+1} \subset I_{u_0}$. Therefore, $\mathcal{C}_p = U_\beta(I_{u_0+1}) \subset U_\beta(I_{-u_0}) \subset U_\beta^2(I_{-u_0}) = U_\beta^2(J_0)$.

(ii). If $(\beta_0 - 2u_0) > \beta_0/(2u_0 + 3)$, then $I_{u_0} \subset I_{u_0+1}$ and the point $p(\beta_0 - 2u_0) \in I_{u_0+1}$. Next, we claim that for $y \in I_{u_0} := \left(\frac{\beta}{2u_0 + 3}, p(\beta_0 - 2u_0)\right) \subset I_{u_0+1}$, there exists a positive number $\beta_0'$ with $\beta_0 \geq \beta_0' > 1$, which depends only on $\beta_0$ and closest to the point 1 such that

$$m(U_\beta(I_y)) \geq \beta_0' m(I_y),$$

where $I_y := \left(y, \frac{\beta}{2u_0 + 1}\right)$. Since $U_\beta(I_y) = \left(p(2u_0 + 2 - \beta/y), 1\right)$, it is enough to show that for $y \in I_{u_0}$,

$$\beta_0 y + p\beta/y \geq p(2u_0 + 1) + p\beta_0' .$$
By the change of variables \( y := py' \), it is equivalent to show that for \( y' \in I''_{w_0} := \left( \frac{\beta_0}{2u_0+3}, (\beta_0 - 2u_0) \right) \), we have

\[
\beta_0 y' + \beta_0/y' \geq 2u_0 + 1 + \beta'_0 .
\]

Now, \((2.7)\) is same as \((2.6)\) at \( y_0 := p(\beta_0 - 2u_0) \). Note that \( \beta_0 > 2 \) because \( y_0 > \beta_0/(2u_0 + 3) \). In particular, we have

\[
m\left( (U_\beta(y_0), p) \right) \geq \beta'_0 m\left( (y_0, p) \right).
\]

If we consider \( J_1 := U_\beta(J_0) = (y_0, p) \) and \( J_2 := U_\beta(I'_{w_0}) = (U_\beta(y_0), p) \), then \( J_2 \subset U^2_\beta(J_0) = U_\beta(J_1) \) with \( m(J_2) \geq \beta'_0 m(J_1) \). If \( U_\beta(y_0) \leq \beta/(2u_0 + 3) \), then \( I_{w_0+1} \subset J_2 \), and hence we are done, because \( C_p = U_\beta(I_{w_0+1}) \subset U_\beta(J_2) = U^2_\beta(I'_{w_0}) \subset U^2_\beta(1_{-w_0}) = U^2_\beta(J_0) \). If \( U_\beta(y_0) > \beta/(2u_0 + 3) \), then repeat the same argument to get a bigger interval say, \( J_3 \) with right end point 1 so that \( I_{w_0+1} \) is contained in \( J_3 \). This completes proof of the case \( J_0 = I_{-w_0} \).

2(B). The case \( J_0 \subset [-p, p] \) is an arbitrary nonempty open interval. Recall that \( 2u_0 - 1 < \beta_0 < 2u_0 + 1 \) and the edge fundamental intervals are given by \((2.4)\). Let the point \( x_0 \in I_{w_0} \) is given by

\[
x_0 := \frac{(2u_0 + 1)\beta}{2u_0(2u_0 + 1) + \beta_0}.
\]

A simple calculation gives \( p\beta/x_0^2 > 2 \). Since on the fundamental interval \( I_u \) the Gauss-type map is given by \( U_\beta(x) = 2u - \beta/x \), the point \( x_0 \in I_{w_0} \) has the property that

\[
U_\beta(I_{x_0}) = I_{-w_0} \text{ and } U_\beta(I_{-x_0}) = I_{w_0},
\]

where

\[
I_{x_0} := \left( \frac{\beta}{2u_0 + 1}, x_0 \right) \text{ and } I_{-x_0} := \left( -x_0, -\frac{\beta}{2u_0 + 1} \right).
\]

Moreover, for \( x \in [-x_0, x_0] \cap \bigcup \{ I_u : u \in U \} \), we have \( U'_\beta(x) \geq p\beta/x_0^2 > 2 \). Therefore, if we write \( \beta''_0 := \min\{\beta_0, \frac{p\beta}{x_0^2} \} \), then \( \beta''_0 > 1 \). In this case, we show that \( C_p \subset U''_\beta(J_0) \) for sufficiently large \( n \). There are two possibilities:

(i). Suppose \( J_0 \) contains one of the fundamental intervals say, \( I_u \) for some \( u \in U \setminus \{ \pm u_0 \} \), then it directly follows that \( C_p = U_\beta(I_u) \subset U_\beta(J_0) \). If \( J_0 \) contains the edge fundamental intervals, then also we are done by the case 2(A) above.

(ii). If none of the fundamental intervals are contained in \( J_0 \), then we have the following possibilities:

(a). Suppose \( J_0 \subset I_u \) for some \( u \in U \), then by uniform expansiveness condition of \( U_\beta \), we get that \( m(J_1) \geq \beta_0 m(J_0) \geq \beta''_0 m(J_0) \), where \( J_1 := U_\beta(J_0) \).

(b). Suppose \( J_0 \) has nonempty intersection with two neighbouring fundamental intervals say \( I_u, I_{u'} \) and \( J_0 \) is contained in the closure of \( I_u \cup I_{u'} \). There are two possibilities:
(b1). Assume that \( J_0 \subset [-x_0, x_0] \). In this case, one of the sets \( J_0 \cap I_u, J_0 \cap I_u' \) say, \( J_0 \cap I_u \) has length at least \( \frac{1}{2} m(J_0) \). It follows that for \( J_1 := U_\beta(J_0 \cap I_u) \) with \( J_1 \subset U_\beta(J_0) \),

\[
m(J_1) = m\left(U_\beta(J_0 \cap I_u)\right) \geq \frac{p\beta}{x_0^2} m(J_0 \cap I_u) \geq \frac{p\beta}{2x_0^2} m(J_0) \geq \beta''_0 m(J_0).
\]

For both the cases (a) and (b1), there exist an interval \( J_1 \) which is contained in \( U_\beta(J_0) \) with \( m(J_1) \geq \beta''_0 m(J_0) \). Next, we consider \( J_1 \) in place of \( J_0 \), and therefore we get a bigger interval namely, \( J_2 \) such that \( m(J_2) \geq \beta''_0 m(J_1) \geq \beta''_0^2 m(J_0) \).

We repeat this process and hence we get an increasing sequence of intervals namely, \( J_0, J_1, J_2, \ldots \) such that \( m(J_l) \geq \beta''_0^l m(J_0) \) with \( J_l \subset U_\beta(J_0) \). Since the length of \( J_l \) depends on \( l \), after finitely many steps say, \( l = l_0 \) we must stop this process, because \( J_{l_0} \) will contain one of the fundamental intervals, and hence \( \mathcal{C}_p \subset U_\beta(J_{l_0}) \subset U_\beta^{l+1}(J_0) \).

(b2). It only remains the case when \( J_0 \) is not contained in \( [-x_0, x_0] \). Then \( I_{x_0} \subset J_0 \cap I_{u_0} \) or \( I_{-x_0} \subset J_0 \cap I_{-u_0} \), and hence we are done, because in view of (2.19) we have one of the following:

(b21). \( \mathcal{C}_p = U_\beta(I_{-u_0}) = U_{\beta}^2(J_{0}) \subset U_{\beta}^2(J_0) \subset U_{\beta}^3(J_0) \),

(b22). \( \mathcal{C}_p = U_\beta(I_{u_0}) = U_{\beta}^2(J_{-x_0}) \subset U_{\beta}^2(J_0) \subset U_{\beta}^3(J_0) \).

This completes the proof of Lemma 2.1. \( \square \)

2.2. Characterization of the pre-annihilator space \( \mathcal{F}^+_{\beta} \).

2.2.1. Periodic and inverted periodic functions.

Let \( L_p^\infty(\mathbb{R}) \) denote the space of all functions \( f \in L^\infty(\mathbb{R}) \) such that the map \( x \mapsto e^{-\pi x/p} f(x) \) is 2-periodic. Then the weak-star closure in \( L^\infty(\mathbb{R}) \) of the linear span of the functions \( \{e^{\beta x/n}(x) := e^{\pi x/(n+1/p)}; \ n \in \mathbb{Z}\} \) equals to \( L_p^\infty(\mathbb{R}) \).

Let \( L_p^\infty(\mathbb{R}) \) denote the space of all functions \( f \in L^\infty(\mathbb{R}) \) such that the map \( x \mapsto f(\beta/x) \) is 2-periodic. Then the weak-star closure in \( L^\infty(\mathbb{R}) \) of the linear span of the functions \( \{e^{\beta x/n}(x) := e^{\pi x/(n+1/p)}; \ n \in \mathbb{Z}\} \) equals to \( L_p^\infty(\mathbb{R}) \).

Observe that the functions in \( L_p^\infty(\mathbb{R}) \) are defined freely on \( [-p, p] \), and due to periodicity they are uniquely determined on \( \mathbb{R} \). Similarly, the functions in \( L_p^\infty(\mathbb{R}) \) are defined freely on \( \mathbb{R} \setminus [-p, p] \), and due to periodicity they are uniquely determined on \( [-\beta, \beta] \). For \( E \subseteq \mathbb{R} \), the function \( \chi_E \) denote the characteristic function of \( E \) on \( \mathbb{R} \). This observations motivate to define the operators \( S_p, T_\beta \) as follows.

The operator \( S_p : L^\infty([-p, p]) \rightarrow L^\infty(\mathbb{R} \setminus [-p, p]) \) is defined by

\[
S_p[\varphi](x) = \varphi \left( \left\{ x/p \right\}_2 \right) \chi_{\mathbb{R}\setminus[-p, p]}(x),
\]

where \( \varphi \in L^\infty([-p, p]) \). The operator \( T_\beta : L^\infty(\mathbb{R} \setminus [-\beta, \beta]) \rightarrow L^\infty([-\beta, \beta]) \) is defined by

\[
T_\beta[\psi](x) = \psi \left( \frac{\beta}{\beta/x}_2 \right) \chi_{[-\beta, \beta] \setminus \{0\}}(x),
\]
where $\psi \in L^\infty(\mathbb{R} \setminus [-\beta, \beta])$. Now, in terms of the operators $S_p, T_\beta$ the functions space $L^\infty_p(\mathbb{R})$ and $L^\infty_\beta(\mathbb{R})$ are given by

$$
\begin{align*}
L^\infty_p(\mathbb{R}) &= \{ \varphi + S_p[\varphi] : \varphi \in L^\infty([-p, p]) \}, \\
L^\infty_\beta(\mathbb{R}) &= \{ \psi + T_\beta[\psi] : \psi \in L^\infty(\mathbb{R} \setminus [-\beta, \beta]) \}.
\end{align*}
$$

2.2.2. The Perron-Frobenius operator. For $p < \beta < \infty$, the Koopman operator $C_\beta : L^\infty([-p, p]) \to L^\infty([-p, p])$ associated to $U_\beta$ be the map

$$
C_\beta[\varphi](x) = \varphi \circ U_\beta(x), \ x \in [-p, p].
$$

The predual adjoint of $C_\beta$ is the Perron-Frobenius operator $P_\beta : L^1([-p, p]) \to L^1([-p, p])$ associated to $U_\beta$ is given by

$$
P_\beta[h](x) = \sum_{u \in \mathbb{Z}} \frac{p^\beta}{(2pu - x)^2} h \left( \frac{p^\beta}{2pu - x} \right), \ x \in [-p, p].
$$

The operator $P_\beta$ is linear and a norm contraction on $L^1([-p, p])$. Thus the point spectrum $\sigma_{\text{point}}(P_\beta)$ of $P_\beta$ is contained in the closed unit disk $\overline{D}$. Here, we use the notation $C_\beta$ in place of $C_{U_\beta}$ and $P_\beta$ for $P_{U_\beta}$.

Next, we build up a connection between the operators $S_p, T_\beta$ and $C_\beta$, so that we can study the pre-annihilator space $F_\beta^\perp$ via the properties of the Perron-Frobenius operators for $p < \beta < \infty$. To do this, we need to define some restriction operators. For a measurable set $E \subseteq \mathbb{R}$ with $m(E) > 0$, we denote $L^s(E)$ the closed subspace of $L^s(\mathbb{R})$ by extending the functions vanish on $\mathbb{R} \setminus E$, where $s = 1, \infty$. For $p < \beta < \infty$, consider the following restriction operators:

$$
\begin{align*}
R_1 &= L^\infty(\mathbb{R} \setminus [-p, p]) \to L^\infty(\mathbb{R} \setminus [-\beta, \beta]), \\
R_2 &= L^\infty([-\beta, \beta]) \to L^\infty([-p, p]), \\
R_3 &= L^\infty([-\beta, \beta]) \to L^\infty([-\beta, \beta] \setminus [-p, p]), \\
R_4 &= L^\infty(\mathbb{R} \setminus [-p, p]) \to L^\infty([-\beta, \beta] \setminus [-p, p]).
\end{align*}
$$

Then the pre-dual adjoints (Banach space dual) are the maps $R_1^*, R_2^*, R_3^*$ and $R_4^*$ defined on the corresponding $L^1$-spaces. A simple calculation shows that $C_\beta = R_2^* T_\beta R_1 S_p$, whenever $\beta > p$, and hence $P_\beta^2 = S_p^* R_4^* T_\beta^* R_2^* R_4$.

**Proposition 2.2.** For $p < \beta < \infty$, suppose $f \in L^1(\mathbb{R})$ such that $f = f_1 + f_2 + f_3$, where $f_1 \in L^1([-p, p])$, $f_2 \in L^1([-\beta, \beta] \setminus [-p, p])$, and $f_3 \in L^1(\mathbb{R} \setminus [-\beta, \beta])$. Then $f \in F_\beta^\perp$ if and only if (i) $(I - P_\beta^2) f_1 = S_p^* (-R_4^* + R_4^* T_\beta^* R_3^*) f_2$, where $I$ is the identity operator on $L^1([-p, p])$, and (ii) $f_3 = -T_\beta^* R_2^* f_1 - T_\beta^* R_3^* f_2$.

**Proof.** The proof of Proposition 2.2 works along the same lines as in the proof of ([6], p. 52, Proposition 6.1), hence omitted. \(\square\)

**Remark 2.3** (0 < $\beta$ ≤ $p$). The composition operator $T_\beta S_p : L^\infty([-p, p]) \to L^\infty([-p, p])$ is given by

$$
T_\beta S_p[\varphi](x) = \varphi \left( p \left\{ \frac{\beta_0}{(\beta/x)_2} \right\} \right) \chi_{E_p}(x), \ \varphi \in L^\infty([-p, p]),
$$
where $\beta_0 = \beta/p$ and $E_\beta = \left\{ x \in (-\beta, \beta) \setminus \{0\} : \frac{\beta_0}{|x|} \in \mathbb{R} \setminus (-1, 1) \right\}$, and the weighted Koopman operator $C_\beta : L^\infty([-p, p]) \to L^\infty([-\beta, \beta])$ associated to $U_\beta$ be the map $C_\beta[\varphi](x) = \varphi \circ U_\beta(x) \chi_{[-\beta, \beta]}(x)$, where $x \in \mathbb{R}$. The pre-dual adjoint of $C_\beta$ is the Perron-Frobenius operator $P_\beta : L^1([-p, p]) \to L^1([-\beta, \beta])$ is given by $P_\beta[h](x) = \sum_{u \in \mathbb{Z}} \frac{\beta}{2pu - x} h\left(\frac{\beta}{2pu - x}\right)$. As the operator $P_\beta$ is linear and a norm contraction, the point spectrum $\sigma_{\text{point}}(P_\beta)$ of $P_\beta$ is contained in $\mathbb{D}$. Observe that $T_\beta S_p = C_\beta^2$.

In [9], it has shown that $\lambda \in \partial \mathbb{D}$ is not an eigenvalue of $P_\beta$ whenever $0 < \beta \leq p$, which in turn implies that $\mathcal{A}(\Gamma, \Lambda^0_\beta) = \{0\}$ for $0 < \beta_0 \leq 1$.

### 2.3. Exterior spectrum of $P_\beta$, $\beta > p$ and the proof of Theorem 1.10

Next, we study the exterior spectrum of the Perron-Frobenius operator $P_\beta$ for $p < \beta < \infty$. We know from Theorem A that 1 is an eigenvalue of $P_\beta$ and the associated eigenfunction is in $\mathbb{B}(\mathbb{R})$. The proof of Theorem 2.4 gives us that one of the eigenfunctions for the eigenvalue 1 must be positive, and it can be normalized by a suitable constant so that we get the positive density of an ergodic $U_\beta$-invariant absolutely continuous probability measure.

**Theorem 2.4.** Let $p < \beta < \infty$, then $\alpha_1 = 1$ is a simple eigenvalue of $P_\beta$, and is the only eigenvalue of $P_\beta$ contained in $\partial \mathbb{D}$. Moreover, the eigenfunctions for $\alpha_1 = 1$ are nonzero scalar multiple of $\varrho_0$, where $\varrho_0 dm$ is the unique ergodic $U_\beta$-invariant absolutely continuous probability measure with $\varrho_0 > 0$ almost everywhere.

**Proof.** Here, Lemma 2.1 will help to show that $\varrho_0 > 0$ almost everywhere. The proof of Theorem 2.4 works along the same lines as in the proof of ([8], p. 55, Theorem 7.2), hence omitted. \hfill $\Box$

For $p < \beta < \infty$, the Gauss-type map $\tilde{U}_\beta : [-\beta, \beta] \to [-\beta, \beta]$ is defined by letting

$$\tilde{U}_\beta(x) := \begin{cases} \rho \left\{ -\frac{\beta}{2}, \frac{\beta}{2} \right\}_2, & x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

The Koopman operator $\tilde{C}_\beta : L^\infty([-\beta, \beta]) \to L^\infty([-\beta, \beta])$ associated to $\tilde{U}_\beta$ be the map $\tilde{C}_\beta[\varphi](x) = \varphi \circ \tilde{U}_\beta(x)$, $x \in [-\beta, \beta]$. Then the pre-dual adjoint of $\tilde{C}_\beta$ is the Perron-Frobenius operator $\tilde{P}_\beta : L^1([-\beta, \beta]) \to L^1([-\beta, \beta])$ associated to $\tilde{U}_\beta$. A simple calculation shows that $\tilde{C}_\beta^2 = T_\beta S_p R_1 S_m R_2$ whenever $\beta > p$, and hence $\tilde{P}_\beta^2 = R^*_{\tilde{C}_\beta^2} R_{\tilde{C}_\beta^2} S_m$. It is easy to see that $\tilde{U}_\beta$ is a partially filling $C^2$-smooth piecewise monotonic transform for $\beta > p$. Also, $\tilde{U}_\beta$ satisfy the uniform expansiveness condition with $m = 2$, and second derivative condition. Therefore, it follows from Theorem A that $\tilde{P}_\beta$ maps $\mathbb{B}([-\beta, \beta])$ into $\mathbb{B}([-\beta, \beta])$. Now, following the proof of ([8], Lemma 8.1), for any $f \in \mathbb{B}([-\beta, \beta] \setminus [-p, p])$, we have

$$-S_p^* R_1^* f + S_p^* R_1^* T_\beta^* R_2^* f \in \mathbb{B}([-p, p]).$$

To complete the proof of Theorem 1.10 we prove the following result. Although the proof of Theorem 2.5 follows the same lines as in ([8], Theorem 8.2), we write it here for the sake of completeness.
Theorem 2.5. For $p < \beta < \infty$, there exists a bounded linear operator $B : BV([-\beta, \beta]\setminus [-p, p]) \to L^1(\mathbb{R})$ such that the range of $B$ is infinite-dimensional, and contained in $F_\beta^1$. Moreover, the range of $B$ is contained in the weighted $L^2$-space $L^2(\mathbb{R}, \omega)$, where $\omega(x) = 1 + x^2$.

Proof. We actually prove more precise statement, namely that, there exits a bounded linear operator $B : BV([-\beta, \beta]\setminus [-p, p]) \to L^1(\mathbb{R})$ such that $Bf(x) = f(x)$ a.e. $x \in [-\beta, \beta]\setminus [-p, p]$, and for all $f \in BV([-\beta, \beta]\setminus [-p, p])$. Moreover, $B$ has infinite-dimensional range which is contained in $F_\beta^1$.

In view of Theorem A and Theorem 2.4, we have the following spectral decomposition for the Perron-Frobenius operator $P_\beta$ associated to $U_\beta$:

\[ P_\beta^n[h] = \{\langle h, \phi_0\rangle\}_{[-p,p]} + Z_\beta^n[h], \ n = 1, 2, \ldots, \]

where $h \in L^1([-p, p])$, and $\phi_0 \in L^\infty([-p, p])$ such that $\langle \phi_0, \phi_0\rangle_{[-p,p]} = 1$. Also, $\phi_0$ is the positive density of the ergodic $U_\beta$-invariant absolutely continuous probability measure on $[-p, p]$, and we have $\phi_0 \in BV([-p, p])$, in addition, we can normalized by a suitable constant so that $\langle \phi_0, 1 \rangle_{[-p,p]} = 1$. Moreover, $Z_\beta$ acts on $BV([-p, p])$ and its spectral radius smaller than 1. In particular, we have $Z_\beta[\phi_0] = 0$, because $\phi_0$ is invariant under $P_\beta$. Observe that, $Z_\beta^n[h] \to 0$ exponentially as $n \to \infty$.

Next, we claim that $\phi_0$ is the constant function 1 almost everywhere on $[-p, p]$. To see this, let $h \in BV([-p,p])$, then by (2.11) we infer that

\[ \langle h, 1 \rangle_{[-p,p]} = \langle h, c_\beta[1] \rangle_{[-p,p]} = \langle P_\beta^n[h], 1 \rangle_{[-p,p]} \]

\[ = \langle h, \phi_0 \rangle_{[-p,p]} \langle \phi_0, 1 \rangle_{[-p,p]} + \langle Z_\beta^n[h], 1 \rangle_{[-p,p]} \]

\[ = \langle h, \phi_0 \rangle_{[-p,p]} + \langle Z_\beta^n[h], 1 \rangle_{[-p,p]} \to \langle h, \phi_0 \rangle_{[-p,p]}, \ \text{as} \ n \to \infty. \]

It is well known that $BV([-p, p])$ is dense in $L^1([-p, p])$. Thus from (2.12) get the claim. Further, as soon as $\phi_0 = 1$, (2.11) can be rewrite as

\[ P_\beta^n[h] = \{\langle h, 1 \rangle_{[-p,p]}\}_n \phi_0 + Z_\beta^n[h], \ n = 1, 2, \ldots, \]

where $h \in L^1([-p, p])$, and from (2.12) we get that

\[ \langle Z_\beta^n[h], 1 \rangle_{[-p,p]} = 0; \ h \in L^1([-p, p]), \ n = 1, 2, \ldots. \]

Now, we are in a position to construct an extension operator $B$ from $BV([-\beta, \beta]\setminus [-p, p])$ onto $L^1(\mathbb{R})$. To do so, pick an arbitrary $f_2 \in BV([-\beta, \beta]\setminus [-p, p])$, then from (2.10), we know that $-S_p^*R_4^*f_2 + S_p^*R_2^*T_\beta^*R_3^*f_2 \in BV([-p, p])$. Since $Z_\beta$ acts on $BV([-p, p])$ has spectral radius smaller than 1, $I - Z_\beta^2$ is invertible and hence, we define the operator $B_1$ by letting

\[ B_1f_2 := (I - Z_\beta^2)^{-1}(-S_p^*R_4^*f_2 + S_p^*R_2^*T_\beta^*R_3^*f_2) \in BV([-p, p]). \]

A simple calculation gives $\langle -S_p^*R_4^*f_2 + S_p^*R_2^*T_\beta^*R_3^*f_2, 1 \rangle_{[-p,p]} = 0$. If we denote $f_1 := B_1f_2$, then by (2.14) and (2.15),

\[ \langle f_1, 1 \rangle_{[-p,p]} = \langle (I - Z_\beta^2)[f_1], 1 \rangle_{[-p,p]} = 0. \]
Next, we define the operator $B_3$ on $BV([\beta, \beta] \setminus [-p, p])$ by letting
\begin{equation}
B_3 f_2 := -T_{\beta}^* R_2 f_1 - T_{\beta}^* R_3 f_2 \in L^1(\mathbb{R} \setminus [-\beta, \beta]).
\end{equation}
We write $f_3 := B_3 f_2$. Now, we define the operator $B : BV([-\beta, \beta] \setminus [-p, p]) \to L^1(\mathbb{R})$ by letting
\[
B f_2 := f_1 + f_2 + f_3 \in L^1(\mathbb{R}),
\]
with the understanding that each $f_k; \ k = 1, 2, 3$ can be extended to $\mathbb{R}$ by considering zero outside of their domain of definition. Then the bounded and linear operator $B$ is clearly an extension operator, in the sense that for all $f_2 \in BV([-\beta, \beta] \setminus [-p, p]),$
\[
B f_2(x) = f_2(x); \ a.e. \ x \in [-\beta, \beta] \setminus [-p, p].
\]
Observe that the range of $B$ is infinite-dimensional. Next, we claim that the range of $B$ is contained in $F^{2}_{\beta}$. Actually, we have to verify the conditions (i) and (ii) of the Proposition 2.2 for the functions $f_k; \ k = 1, 2, 3$. In view of (2.16), from (2.13) we get that $P^n_\beta[f_1] = Z^n_\beta[f_1]; \ n = 1, 2, \ldots$. Thus, from (2.15) and (2.17) we have the conditions (i) and (ii) of the Proposition 2.2.

It follows from the proof of (Proposition 8.3, [6]) that the range of $B$ is contained in the weighted $L^2$-space $L^2(\mathbb{R}, \omega)$, where the weight $\omega(x) = 1 + x^2$. This completes the proof of Theorem 2.5.

3. Proof of Theorem 1.15

3.1. Dynamics of a Gauss-type map.

3.1.1. A Gauss-type map. For $t \in \mathbb{R}$, the expression $\{t\}$ is the unique number in $[0, 1)$ such that $t - \{t\} \in \mathbb{Z}$. For $0 < \gamma < \infty$, consider the Gauss-type map $V_\gamma$ on the interval $[0, q)$. The map $V_\gamma : [0, q) \to [0, q)$ is defined by letting
\[
V_\gamma(x) = \begin{cases} 
q \{\frac{x}{q}\} & x \neq 0 \\
0 & x = 0.
\end{cases}
\]

Note that, for $v \in \mathbb{N}$, the map $V_\gamma$ can be expressed as $V_\gamma(x) = q (\frac{x}{q} - v)$ whenever $\frac{x}{v+1} < x \leq \frac{x}{v}$, and hence $V_\gamma : \left(\frac{1}{v+1}, \frac{1}{v}\right] \to [0, q)$ is one-to-one.

3.1.2. Dynamical properties of Gauss-type map for $\gamma > q$.

Denote $\gamma_0 := \gamma/q$ which is $> 1$. In this section, we observe that $V_\gamma$ is a partially filling $C^2$-smooth piecewise monotonic transform for $\gamma_0 > 1$. We need to find a unique absolutely continuous invariant probability measure for $V_\gamma$ having positive density.

Let $\mathcal{V} := \mathcal{V}_\gamma$ denote the index set which contain the points $v \in \mathbb{N}$ so that the associate fundamental interval is nonempty, that is,
\[
J_v := \left(\frac{\gamma}{v+1}, \frac{\gamma}{v}\right) \cap (0, q) \neq \emptyset.
\]
If $\gamma_0$ is an integer, then it is easy to see that for all $v \in \mathcal{V}$, where $\mathcal{V}$ be the set of all nonzero positive integers with $v \geq \gamma_0$, the fundamental intervals are given by $J_v := \left(\frac{\gamma}{v+1}, \frac{\gamma}{v}\right)$, $v \in \mathcal{V}$ so that $V_\gamma(J_v) = (0, q)$ for all $v \in \mathcal{V}$. Thus in
this particular case, $V_\gamma$ fulfill the "filing" condition for all $v \in \mathcal{V}$. If $\gamma_0$ is not an integer, write $v_0 := \gamma_0 - \{\gamma_0\} \geq 1$, then a simple calculation shows that $J_v := \left( \frac{\gamma}{\pi i + 1}, \frac{\gamma}{\pi i} \right)$, $v \in \mathcal{V} \setminus \{v_0\}$, where $\mathcal{V}$ be the set of all positive integers with $v \geq v_0$. Observe that $V_\gamma(J_v) = (0, q)$ for all $v \in \mathcal{V} \setminus \{v_0\}$, that is, in this case, $V_\gamma$ fulfill the "filing" condition for all $v \in \mathcal{V}$ except one branch corresponding to $\{v_0\}$. In view of the above facts, we conclude that $V_\gamma$ is a partially filling $C^2$-smooth piecewise monotonic transform for $\gamma > 1$.

The map $V_\gamma$ satisfy the "uniform expansiveness" condition with $m = 1$, because of the derivative $|V'_\gamma(x)| = \frac{\beta_0}{2} \geq \gamma_0 > 1$ for all $x \in \{J_v : v \in \mathcal{V}\}$. Also, we have "second derivative condition" due to $|V''_\gamma(x)| \leq \frac{\beta_0}{q}|V'_\gamma(x)|^2$ for all $x \in \{J_v : v \in \mathcal{V}\}$. We aim to find a unique absolutely continuous invariant probability measure corresponding to $V_\gamma$ which has a positive density. Since $V_\gamma$ is a partially filling $C^2$-smooth piecewise monotonic transform, there exists a $V_\gamma$-invariant absolutely continuous probability measure, but it may not be unique. Although, form [6], p. 45, Remark 5.2 we can get the uniqueness for filling $C^2$-smooth piecewise monotonic transform. Also, $V_\gamma$ may not always be a Markov map, therefore, to get the uniqueness of the absolutely continuous $V_\gamma$-invariant measure, we have to prove the condition (iii) of Adler’s theorem which is stated in [6], p. 46, Theorem D) for the details see [5, 21].

3.2. Characterization of the pre-annihilator space $\mathcal{K}^\gamma_{+}$.

3.2.1. Periodic and inverted periodic functions.

Let $L_q^\infty(\mathbb{R}_+)$ denote the space of all functions $f \in L^\infty(\mathbb{R}_+)$ such that the map $x \mapsto e^{-2\pi i x/q} f(x)$ is 1-periodic. Then the weak-star closure in $L^\infty(\mathbb{R}_+)$ of the linear span of the functions $\{e^x_n(x) := e^{2\pi i n x/q}, n \in \mathbb{Z}\}$ equals to $L_q^\infty(\mathbb{R}_+)$. Let $L_q^\infty(\mathcal{R}_+)$ denote the space of all functions $f \in L^\infty(\mathbb{R}_+)$ such that the map $x \mapsto f(\gamma/x)$ is 1-periodic. Then the weak-star closure in $L^\infty(\mathbb{R}_+)$ of the linear span of the functions $\{e^x_n(x) := e^{2\pi i n x/q}, n \in \mathbb{Z}\}$ equals to $L_q^\infty(\mathbb{R}_+)$. Observe that the functions in $L_q^\infty(\mathbb{R}_+)$ are defined freely on $[0, q]$, and because of periodicity they are uniquely determined on $\mathbb{R}_+ \setminus [0, q]$. Similarly, the functions in $L_q^\infty(\mathbb{R}_+)$ are defined freely on $\mathbb{R}_+ \setminus [0, \gamma]$, and due to periodicity they are uniquely determined on $[0, \gamma]$.

The operator $O_q : L^\infty([0, q]) \to L^\infty(\mathbb{R}_+ \setminus [0, q])$ is defined by

$$\begin{equation}
O_q[\varphi]\left(x\right) = \varphi \left(q \left\{ x/q \right\}_1 \right) \chi_{\mathbb{R}_+ \setminus [0, q]}(x), \quad \text{where } \varphi \in L^\infty([0, q]).
\end{equation}
$$

The operator $T_\gamma : L^\infty(\mathbb{R}_+ \setminus [0, \gamma]) \to L^\infty([0, \gamma])$ is defined by

$$\begin{equation}
T_\gamma[\psi](x) = \psi \left( \frac{\gamma}{\{\gamma/x\}_1} \right) \chi_{[\gamma, \infty) \setminus \{0\}}(x), \quad \text{where } \psi \in L^\infty(\mathbb{R}_+ \setminus [0, \gamma]).
\end{equation}
$$

In view of the above facts, we get that $L_q^\infty(\mathbb{R}_+) = \{ \varphi + O_q[\varphi] : \varphi \in L^\infty([0, q]) \}$ and $L_q^\infty(\mathbb{R}_+) = \{ \psi + T_\gamma[\psi] : \psi \in L^\infty(\mathbb{R}_+ \setminus [0, \gamma]) \}$. 

3.2.2. The Perron-Frobenius operators. For \( q \leq \gamma < \infty \), the Koopman operator \( C_{\gamma} : L^{\infty}([0, q]) \rightarrow L^{\infty}([0, q]) \) associated to \( U_{\gamma} \) be the map \( C_{\gamma}[\varphi](x) = \varphi \circ U_{\gamma}(x) \), where \( \varphi \in L^{\infty}([0, q]) \). The predual adjoint of \( C_{\gamma} \) is the Perron-Frobenius operator \( P_{\gamma} : L^{1}([0, q]) \rightarrow L^{1}([0, q]) \) given by

\[
P_{\gamma}[h](x) = \sum_{v=1}^{\infty} \frac{q^{\gamma}}{(qv + x)^{2}} h\left(\frac{q^{\gamma}}{qv + x}\right).
\]

The operator \( P_{\gamma} \) is linear and a norm contraction on \( L^{1}([0, q]) \). Thus the point spectrum \( \sigma_{\text{point}}(P_{\gamma}) \) of \( P_{\gamma} \) is contained in \( \mathbb{D} \). For \( q < \gamma < \infty \), consider the following restriction operators:

\[
\begin{align*}
R_{5} : L^{\infty}(\mathbb{R}_{+} \setminus [0, q]) &\rightarrow L^{\infty}(\mathbb{R}_{+} \setminus [0, \gamma]) \\
R_{6} : L^{\infty}([0, \gamma]) &\rightarrow L^{\infty}([0, q]) \\
R_{7} : L^{\infty}([0, \gamma]) &\rightarrow L^{\infty}([0, \gamma] \setminus [0, q]) \\
R_{8} : L^{\infty}(\mathbb{R}_{+} \setminus [0, q]) &\rightarrow L^{\infty}([0, \gamma] \setminus [0, q])
\end{align*}
\]

The corresponding pre-dual adjoints are the maps \( R_{5}^{*}, R_{6}^{*}, R_{7}^{*} \) and \( R_{8}^{*} \) respectively. As \( \gamma > q \), a simple calculation shows that \( C_{\gamma}^{2} = R_{6}T_{\gamma}R_{5}O_{p} \) and hence \( P_{\gamma}^{2} = O_{q}R_{5}^{*}T_{\gamma}R_{8}^{*} \).

**Proposition 3.1.** For \( q < \gamma < \infty \), suppose \( f \in L^{1}(\mathbb{R}_{+}) \) such that \( f = f_{1} + f_{2} + f_{3} \), where \( f_{1} \in L^{1}([0, q]), f_{2} \in L^{1}([0, \gamma] \setminus [0, q]), \) and \( f_{3} \in L^{1}(\mathbb{R}_{+} \setminus [0, \gamma]) \). Then \( f \in \mathcal{K}_{\gamma} \) if and only if (i) \( (I - P_{\gamma}^{2})f_{1} = O_{q}(-R_{8}^{*} + R_{8}^{*}T_{\gamma}R_{7}^{*})f_{2} \), where \( I \) is the identity operator on \( L^{1}([0, q]) \), and (ii) \( f_{3} = -T_{\gamma}R_{6}^{*}f_{1} - T_{\gamma}R_{7}^{*}f_{2} \).

**Proof.** The proof of Proposition 3.1 works along the same lines as in the proof of [6], p. 52, Proposition 6.1, hence omitted.

**Remark 3.2.** For \( 0 < \gamma < q \), in analogy with the results in [16] it seems likely that, the Perron-Frobenius operator \( P_{\gamma} \) has no eigenfunction corresponding to the eigenvalue 1. For the critical case \( \gamma = q \), \( P_{\gamma} \) has one-dimensional eigenspace corresponding to the eigenvalue 1. This in turn implies that \( \mathcal{A}C(\Gamma_{+}, \Lambda_{+}) = \{0\} \) for \( 0 < \gamma < q \), and for \( \gamma = q \), \( \mathcal{A}C(\Gamma_{+}, \Lambda_{+}^{0}) \) is one-dimensional. This problem is open.

3.3. **Proof of Theorem 1.15** The proof of Theorem 1.15 directly follows from the proof of Theorem 3.3.

**Theorem 3.3.** For \( q < \gamma < \infty \), there exists a bounded linear operator \( B_{+} : BV([0, \gamma] \setminus [0, q]) \rightarrow L^{1}(\mathbb{R}_{+}) \) such that the range of \( B_{+} \) is infinite-dimensional, and contained in \( \mathcal{K}_{\gamma}^{+} \).

**Proof.** We actually prove more precise statement, namely that, there exists a bounded linear operator \( B_{+} : BV([0, \gamma] \setminus [0, q]) \rightarrow L^{1}(\mathbb{R}_{+}) \) such that \( B_{+}f(x) = f(x) \) a.e. \( x \in [0, \gamma] \setminus [0, q] \), and for all \( f \in BV([0, \gamma] \setminus [0, q]) \). Moreover, \( B_{+} \) has infinite-dimensional range which is contained in \( \mathcal{K}_{\gamma}^{+} \). The proof of Theorem 3.3 works along a similar path as in Theorem 2.5, hence omitted.
Acknowledgements. The author wishes to thank E. K. Narayanan, Rama Rawat, R. K. Srivastava, and Sundaram Thangavelu for several valuable suggestions during the preparation of this manuscript. The author gratefully acknowledges the support provided by NBHM post-doctoral fellowship from the Department of Atomic Energy (DAE), Government of India. The author was supported by the Department of Mathematics, IISc Bangalore, India.

References

[1] L. Ambrosio, N. Fusco, and D. Pallara Functions of bounded variation and free discontinuity problems, Oxford Mathematical Monographs, Oxford University Press, New York, 2000. MR1857292
[2] D. B. Babot, Heisenberg uniqueness pairs in the plane, Three parallel lines, Proc. Amer. Math. Soc. 141 (2013), no. 11, 3899-3904.
[3] S. Bagchi, Heisenberg uniqueness pairs corresponding to a finite number of parallel lines, Adv. Math. 325 (2018), 814-823.
[4] M. Benedicks, On Fourier transforms of functions supported on sets of finite Lebesgue measure, J. Math. Anal. Appl. 106 (1985), no. 1, 180-183.
[5] A. Boyarsky and P. Gora, Laws of Chaos. Invariant Measures and Dynamical System in One Dimension, Probab. Appl., Birkhäuser Boston (1997). MR 1461536
[6] F. Canto-Martín, H. Hedenmalm, and A. Montes-Rodríguez, Perron-Frobenius operators and the Klein-Gordon equation, J. Eur. Math. Soc. (JEMS) 16 (2014), no. 1, 31-66.
[7] A. Chattopadhyay, S. Ghosh, D. K. Giri, and R. K. Srivastava, Heisenberg uniqueness pairs on the Euclidean spaces and the motion group, C. R. Math. Acad. Sci. Paris 358 (2020), no. 3, 365-377.
[8] S. Ghosh and R. K. Srivastava, Heisenberg uniqueness pairs for the Fourier transform on the Heisenberg group, (2020). [arXiv:1810.06390]
[9] D. K. Giri and R. Rawat, Heisenberg uniqueness pairs for the hyperbola, Bull. Lond. Math. Soc., doi: 10.1112.blms.12391 (to appear)
[10] D. K. Giri and R. K. Srivastava, Heisenberg uniqueness pairs for some algebraic curves in the plane, Adv. Math. 310 (2017), 993-1016.
[11] D. K. Giri and R. K. Srivastava, Heisenberg uniqueness pairs for the finitely many parallel lines with an irregular gap, (submitted).
[12] F. J. González Vieli, A uniqueness result for the Fourier transform of measures on the sphere, Bull. Aust. Math. Soc. 86 (2012), 78-82.
[13] K. Gröchenig and P. Jaming, The Cramér-Wold theorem on quadratic surfaces and Heisenberg uniqueness pairs, J. Inst. Math. Jussieu 19 (2020), no. 1, 117-135.
[14] V. Havin and B. Jöricke, The Uncertainty Principle in Harmonic Analysis, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 28. Springer-Verlag, Berlin, 1994.
[15] H. Hedenmalm and A. Montes-Rodriguez, Heisenberg uniqueness pairs and the Klein-Gordon equation, Ann. of Math. (2) 173 (2011), no. 3, 1507-1527.
[16] H. Hedenmalm and A. Montes-Rodriguez, The Klein-Gordon equation, the Hilbert transform, and dynamics of Gauss-type maps, J. Eur. Math. Soc. (JEMS) 22 (2020), no. 6, 1703-1757.
[17] H. Hedenmalm and A. Montes-Rodriguez, The Klein-Gordon equation, the Hilbert transform, and Gauss-type maps: $H^\infty$ approximation, J. Anal. Math. (2020) (to appear).
[18] W. Heisenberg, Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, Z. Physik 43 (1927), 172-198. JFM 53.0853.05.
[19] M. Iosifescu and S. Grigorescu, Dependence with Complete Connections and its Applications, Cambridge Tracts in math, 96, Cambridge Univ. Press, Cambridge (1990). MR 1070097

[20] P. Jaming and K. Kellay, A dynamical system approach to Heisenberg uniqueness pairs, J. Anal. Math. 134 (2018), no. 1, 273-301.

[21] A. Lasota and J. A. Yorke Existence of invariant measures for piecewise monotonic transformations, Trans. Amer. Math. Soc. 186 (1973), 481-488.

[22] N. Lev, Uniqueness theorem for Fourier transform, Bull. Sci. Math. 135 (2011), 134-140.

[23] P. Sjölin, Heisenberg uniqueness pairs and a theorem of Beurling and Malliavin, Bull. Sci. Math. 135 (2011), 125-133.

[24] P. Sjölin, Heisenberg uniqueness pairs for the parabola, J. Fourier Anal. Appl. 19 (2013), 410-416.

[25] R. K. Srivastava, Non-harmonic cones are Heisenberg uniqueness pairs for the Fourier transform on $\mathbb{R}^n$, J. Fourier Anal. Appl. 24 (2018), no. 6, 1425-1437.

Deb Kumar Giri, Department of Mathematics, Indian Institute of Science, Bangalore-560012, India.
E-mail address: debkumarg@iisc.ac.in