NOTES ON W-DIRECTION CURVES IN EUCLIDEAN 3-SPACE

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Abstract
In this paper, we study the spherical indicatrices of W-direction curves in three dimensional Euclidean space which were defined by using the unit Darboux vector field $W$ of a Frenet curve, in [11]. We obtain the Frenet apparatus of these spherical indicatrix curves and the characterizations of being general helix and slant helix. Moreover we give some properties between the spherical indicatrix curves and their associated curves.

Then we investigate two special ruled surface that are normal and binormal surface by using W-direction curves. We give useful results involving the characterizations of these ruled surfaces. From those applications, we make use of such a work to interpret the Gaussian, mean curvatures of these surfaces and geodesic, normal curvature and geodesic torsion of the base curves with respect to these surfaces.

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1. Introduction
The theory of curves is a subbranch of geometry which deals with curves in Euclidean space or other spaces by using differential and integral calculus. One of the most studied topic in curve theory is associated curves like involute-evolute pairs, Bertrand curve pairs, Mannheim partner curves and W-direction curves. Working with these associated curves is nice aspect that these curves can be characterized by the properties and behavior of the main curves of them.
The most commonly used ones to characterize curves are general or cylindrical helix and slant helix. A general helix in $\mathbb{E}^3$ is defined as; its tangent vector field makes a constant angle with a fixed direction. If the principal normal vector field makes a constant angle with a fixed direction, it is called slant helix. Izumiya and Takeuchi founded that a curve is a slant helix if and only if the geodesic curvature of the principal image of the principal normal indicatrix which is

$$\delta(s) = \left(\frac{\kappa^2}{(\kappa^2 + \tau^2)^{\frac{3}{2}}} \cdot \left(\frac{\tau}{\kappa}\right)^{\frac{\prime}{\prime}}\right)(s)$$

(1.1)

is a constant function [6].

Choi and Kim created the principal(binormal)-direction curves and principal(binormal)-donor curve of a Frenet curve in $\mathbb{E}^3$. They gave the relation of curvature and torsion between the principal-direction curve and its mate curve. They also defined a new curve called PD-rectifying curve and gave a new characterization of a Bertrand curve by means of the PD-rectifying curve. They made an application of associated curves and studied a general helix and slant helix as principal-donor and second principal-donor curve of a plane, respectively [2]. Then Choi et al. worked on the principal(binormal)-direction curve and principal(binormal)-donor curve of a Frenet non-lightlike curves in $\mathbb{E}^3_1$ [3].

After that Körpınar et al. introduced associated curves according to Bishop frame in $\mathbb{E}^3$ [7].

Recently, Macit and Dülüşlü defined W-direction curve, W-rectifying curve and V-direction curve of a Frenet curve in $\mathbb{E}^3$ and also principal-direction curve, $B_1$-direction curve, $B_2$-direction curve and $B_2$-rectifying curve in $\mathbb{E}^4$. These curves were given as the integral curves of vector fields taken from the Frenet frame or Darboux frame along a curve. They gave the relationship of the Frenet vector fields, the curvature and the torsion between the associated curves and their main curve [11].

The spherical indicatrix of a curve in three dimensional Euclidean space is described by moving any of the unit Frenet vectors onto the unit sphere $S^2$. If any of the Frenet vectors is $X$ of a curve given with the arc-length parameter $s$, then the equation of the spherical indicatrix curve is given by

$$\beta(s^*) = X(s).$$

(1.2)

where $s^*$ is the arc-length parameter of spherical indicatrix.

There are many works on spherical indicatrix curves. Kula and Yaylı in [8], investigated spherical images of tangent and binormal indicatrix of a slant helix. They found that the spherical images are spherical helices.

Kula et al. gave some characterizations for a unit speed curve in $\mathbb{R}^3$ to being a slant helix by using its tangent, principal normal and binormal indicatrix [9].

Tunçer and Ünal studied spherical indicatrixes of a Bertrand curve and its mate curve. They obtained relations between spherical images and new representations of spherical indicatrixes [14].

A ruled surface in $\mathbb{R}^3$ is a surface which can be described as the set of points swept out by moving a straight line in surface. It therefore has a parametrization of the form

$$\Phi(s, v) = \alpha(s) + v\delta(s)$$

(1.3)
where $\alpha$ is a curve lying on the surface called base curve and $\delta$ is called director curve. The straight lines are called rulings. In using the equation of ruled surface we assume that $\alpha'$ is never zero and $\delta$ is not identically zero. The rulings of ruled surface are asymptotic curves. Furthermore, the Gaussian curvature of ruled surface is everywhere nonpositive. The ruled surface is developable if and only if the distribution parameter vanishes and it is minimal if and only if its mean curvature vanishes [4]. Recently, the ruled surfaces were studied on Euclidean space and Minkowski space, see, e.g., [1, 5, 6, 15, 16, 17]. Also Nurkan et al. investigated two ruled surfaces such as normal and binormal surface by using any base curve and adjoint curve of it [12].

In this paper, we study the spherical indicatrixes of W-direction curves. We obtain the Frenet apparatus of tangent indicatrix and binormal indicatrix via Frenet vector fields, curvature and torsion of the main curves. We give the characterizations of being general helix and slant helix of these image curves. We also investigate the normal surface and the binormal surface by taking the base curve as W-direction curve. We give useful results about developability, minimality of the surfaces and being asymptotic line, geodesic curve, principal line of the base curves.

2. Preliminaries

Let $\beta : I \longrightarrow \mathbb{E}^3$ be a curve and $\{T, N, B\}$ denote the Frenet frame of $\beta$. $T(s) = \beta'(s)$ is called the unit tangent vector of $\beta$ at $s$. $\beta$ is a unit speed curve (or parametrized by arc-length $s$) if and only if $\|\beta'(s)\| = 1$. The curvature of $\beta$ is given by $\kappa(s) = \|\beta''(s)\|$. The unit principal normal vector $N(s)$ of $\beta$ at $s$ is given by $\beta''(s) = \kappa(s).N(s)$. Also the unit vector $B(s) = T(s) \times N(s)$ is called the unit binormal vector of $\beta$ at $s$. Then the famous Frenet formula holds as:

$$
T'(s) = \kappa(s).N(s) \\
N'(s) = -\kappa(s).T(s) + \tau(s)B(s) \\
B'(s) = -\tau(s)N(s)
$$

where $\tau(s)$ is the torsion of $\beta$ at $s$ and calculated as $\tau(s) = \langle N'(s), B(s) \rangle$ or $\tau(s) = \|B'(s)\|$. Also the Frenet vectors of a curve $\beta$, which is given by arc-length parameter $s$, can be calculated as;

$$
T(s) = \beta'(s) \\
N(s) = \frac{\beta''(s)}{\|\beta''(s)\|} \quad (2.1) \\
B(s) = T(s) \times N(s).
$$

For the unit speed curve $\beta : I \longrightarrow \mathbb{E}^3$, the vector

$$
W(s) = \tau(s)T(s) + \kappa(s)B(s)
$$

is called the Darboux vector of $\beta$ which is the rotation vector of trihedron of the curve with curvature $\kappa \neq 0$ when a point moves along the curve $\beta$.

A unit speed curve $\beta : I \longrightarrow \mathbb{E}^n$ is a Frenet curve if $\beta''(s) \neq 0$ and it has the non-zero curvature.
Definition 1. Let $\beta$ be a Frenet curve in $\mathbb{E}^3$ and $W$ be the unit Darboux vector field of $\beta$. The integral curve of $W(s)$ is called $W$-direction curve of $\beta$. Namely, if $\beta$ is the $W$-direction curve of $\beta$, then $W(s) = \beta'(s)$, where $W = \frac{1}{\sqrt{\kappa^2 + \tau^2}}(\tau T + \kappa B)$ \[11\].

Let the Frenet apparatus of a Frenet curve $\beta$ and its $W$-direction curve be $\{T, N, B, \kappa, \tau\}$ and $\{\overline{T}, \overline{N}, \overline{B}, \overline{\kappa}, \overline{\tau}\}$ respectively. The relations of Frenet apparatus between the main curve and $W$-direction curve are given in [11] as;

\begin{align*}
T &= \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} B \\
\overline{N} &= -\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} B \\
\overline{B} &= -N \\
\overline{\kappa} &= \frac{|\tau \kappa' - \tau' \kappa|}{\kappa^2 + \tau^2}, \quad \overline{\tau} = \sqrt{\kappa^2 + \tau^2}
\end{align*}

(2.3)

Remark 1. In this paper we take the signs of absolute value positive. If the sign will be taken negative, the expressions similarly have the other signs.

Theorem 1. Let $\overline{\beta}$ be the $W$-direction curve of $\beta$ which is not a general helix. Then $\overline{\beta}$ is a general helix if and only if $\beta$ is a slant helix [11].

Theorem 2. A curve is a general helix if and only if $\frac{\tau}{\kappa}$ = constant [13].

A ruled surface given with the equation (1.3) is in the general form. The normal and the binormal surfaces which are also ruled surfaces are defined by

\begin{align*}
\Phi(s, v) &= \alpha(s) + vN(s) \\
\Phi(s, v) &= \alpha(s) + vB(s)
\end{align*}

(2.4)

where $\alpha$ is a curve with arc-length parameter $s$ and $\{T, N, B\}$ are the Frenet vector fields of $\alpha$ [5, 6].

The distribution parameter $\lambda$ of a ruled surface $\Phi$ given in equation (1.3) is dedicated as;

\[\lambda = \frac{\text{det}(\frac{d\alpha}{ds}, \delta, \frac{d\delta}{ds})}{\left\| \frac{d\delta}{ds} \right\|^2}.\]

(2.6)

The standard unit normal vector field $U$ on the ruled surface $\Phi$ is defined by

\[U = \frac{\Phi_s \times \Phi_v}{\left\| \Phi_s \times \Phi_v \right\|}.\]

(2.7)

where $\Phi_s = \frac{d\Phi}{ds}$ and $\Phi_v = \frac{d\Phi}{dv}$.

The Gaussian curvature and mean curvature of a ruled surface given in equation (1.3) are given respectively by

\[K = \frac{eg - f^2}{EG - F^2},\]

and

\[H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}\]

(2.8)

(2.9)
where \( E = \langle \Phi_s, \Phi_s \rangle \), \( F = \langle \Phi_s, \Phi_v \rangle \), \( G = \langle \Phi_v, \Phi_v \rangle \), \( e = \langle \Phi_{ss}, U \rangle \), \( f = \langle \Phi_{sv}, U \rangle \) and \( g = \langle \Phi_{vv}, U \rangle \) (see \[13\]).

**Definition 2.** A surface is developable (flat) provided its Gaussian curvature is zero and minimal provided its mean curvature is zero \[13\].

The equation between the distribution parameter and Gaussian curvature which is called Lamarle formula is defined as
\[
K = -\frac{\lambda^2}{(\lambda^2 + v^2)^2}
\]
where \( K \) is the Gaussian curvature and \( \lambda \) is the distribution parameter (see \[10\]).

Namely, if the distribution parameter is zero, then the surface is developable.

Let \( \alpha \) be any curve of arc-length parameter \( s \) and has the Frenet frame \( \{ T, N, B \} \) along \( \alpha \). If \( \alpha \) is on a surface, the frame \( \{ T, V, U \} \) along the curve \( \alpha \) is called the Darboux frame where \( T \) is the unit tangent vector of \( \alpha \), \( U \) is the unit normal of the surface and \( V \) is the unit vector given by \( V = U \times T \). The relations between these vectors and their derivatives are \[13\];
\[
\begin{bmatrix}
  T' \\
  V' \\
  U'
\end{bmatrix} =
\begin{bmatrix}
  0 & \kappa_g & \kappa_n \\
  -\kappa_g & 0 & \tau_g \\
  -\kappa_n & -\tau_g & 0
\end{bmatrix}
\begin{bmatrix}
  T \\
  V \\
  U
\end{bmatrix}
\]

where \( \kappa_g \) is the geodesic curvature, \( \kappa_n \) is the normal curvature and \( \tau_g \) is the geodesic torsion.

If the curve \( \alpha \) is the base curve of the ruled surface given in (1.3), then the geodesic curvature, normal curvature and geodesic torsion with respect to the ruled surface are also given respectively by \[13\];
\[
\kappa_g = \langle U \times T, T' \rangle \quad (2.10)
\]
\[
\kappa_n = \langle \alpha'', U \rangle \quad (2.11)
\]

and
\[
\tau_g = \langle U \times U', T' \rangle . \quad (2.12)
\]

For a curve \( \alpha \) which is lying on a surface, the following statements are satisfied \[13\];

1) \( \alpha \) is a geodesic curve if and only if the geodesic curvature of the curve with respect to the surface vanishes.

2) \( \alpha \) is a asymptotic line if and only if the normal curvature of the curve with respect to the surface vanishes.

3) \( \alpha \) is a principal line if and only if the geodesic torsion of the curve with respect to the surface vanishes.

### 3. Spherical Images of W-direction Curves

In this section, we will introduce tangent indicatrix and binormal indicatrix curves which are spherical indicatrices of W-direction curves. We find their Frenet apparatus and give some results of being general helix and slant helix.

Let \( \beta \) be a curve with arc-length parameter \( s \) and \( \bar{\beta} \) be the W-direction curve of \( \beta \). The arc-length parameter \( \bar{s} \) of \( \bar{\beta} \) which is an integral curve of \( \beta \), can be taken as \( \bar{s} = s \) (see \[2\]). The Frenet apparatus of \( \beta \) and \( \bar{\beta} \) are \( \{ T, N, B, \kappa, \tau \} \) and \( \{ T, \bar{N}, \bar{B}, \bar{k}, \bar{\tau} \} \) respectively. Here
also δ is the geodesic curvature of the principal image of the principal normal indicatrix given with the equation (1.1).

Using the equation (1.2), the tangent indicatrix and binormal indicatrix curves of W-direction curve \( \beta \) are given with the equations

\[
\alpha(s_\alpha) = \overline{T}(s) \\
\gamma(s_\gamma) = \overline{B}(s)
\]  

where \( s_\alpha \) and \( s_\gamma \) are arc-length parameters of tangent and binormal indicatrix curves, respectively.

**Theorem 3.** Let \( \beta \) be a curve with arc-length parameter \( s \) and \( \overline{\beta} \) be the W-direction curve of \( \beta \). The Frenet vector fields, curvature and torsion of the tangent indicatrix curve \( \alpha \) of W-direction curve are given by

\[
T_\alpha = -\frac{1}{\sqrt{1+f^2}}T + \frac{f}{\sqrt{1+f^2}}B
\]

\[
N_\alpha = -\frac{\kappa'(f-g)}{\sqrt{(\kappa')^2(f-g)^2 + \kappa^4(1+f^2)^2}} \left( fT + \frac{\kappa^2(1+f^2)^2}{\kappa'(f-g)}N + B \right)
\]

\[
B_\alpha = \frac{\kappa'(f-g)}{\sqrt{(\kappa')^2(f-g)^2 + \kappa^4(1+f^2)^2}} \left( \tau T - \frac{\kappa'(f-g)}{\kappa(1+f^2)}N + \kappa B \right)
\]

\[
\kappa_\alpha = \sqrt{1 + \frac{\kappa^4(1+f^2)^3}{(\kappa')^2(f-g)^2}}
\]

\[
\tau_\alpha = \frac{\kappa^2 \sqrt{(1+f^2)^3}}{\kappa'(f-g) ( (\kappa')^2(f-g)^2 + \kappa^4(1+f^2)^2 )} \left( 3(\kappa')^2(f-g)(f-g) - \kappa\kappa''(f-h)(1+f^2) \right)
\]

where \( f = \frac{\tau}{\kappa}, \quad g = \frac{\kappa'}{\kappa} \) and \( h = \frac{\kappa''}{\kappa} \).

**Proof.** The equation of tangent indicatrix curve \( \alpha \) is given in equation (3.1) with the arc-length parameter \( s_\alpha \). By differentiating equation (3.1) and using Frenet formulas we get

\[
\frac{d s_\alpha}{d s} = \kappa.
\]

If we use the equations in (2.1) and the relation \( \frac{d s_\alpha}{d s} = \kappa \), we find the tangent, principal and binormal vector fields respectively as;

\[
T_\alpha = N \\
N_\alpha = -\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}T + \frac{\tau}{\sqrt{\kappa^2 + \tau^2}}B \\
B_\alpha = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}}T + \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}B.
\]
By writing the relations (2.3) in the last equations, we have:

\[ T_\alpha = -\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} B \]

\[ N_\alpha = -\frac{\tau\kappa' - \tau'\kappa}{\sqrt{\tau\kappa' - \tau'\kappa}^2 + (\kappa^2 + \tau^2)^3} \left( \frac{\tau}{\sqrt{\tau\kappa' - \tau'\kappa}} T + \frac{\sqrt{(\kappa^2 + \tau^2)^3}}{\tau\kappa' - \tau'\kappa} N + \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} B \right) \]

\[ B_\alpha = \frac{\kappa^2 + \tau^2}{\sqrt{\tau\kappa' - \tau'\kappa}^2 + (\kappa^2 + \tau^2)^3} \left( \tau T - \frac{\tau\kappa' - \tau'\kappa}{\kappa^2 + \tau^2} N + \kappa B \right). \]

In the last equations assuming \( f = \frac{\tau}{\kappa}, \ g = \frac{\tau'}{\kappa'} \) and \( h = \frac{\tau''}{\kappa''} \) and arranging the expressions, we obtain the tangent, principal and binormal vector fields of the tangent indicatrix curve \( \alpha \) of the W-direction curve \( \beta \), with respect to the main curve \( \beta \).

Also the curvature and torsion of the tangent indicatrix curve \( \alpha \) are found as:

\[ \kappa_\alpha = \frac{\sqrt{\kappa^2 + \tau^2}}{\kappa} \quad (3.3) \]

\[ \tau_\alpha = \frac{\sqrt{(\tau\kappa' - \tau'\kappa)^2 + (\kappa^2 + \tau^2)^3}}{(\kappa^2 + \tau^2)^3} (\tau \kappa' - \tau' \kappa) \quad (3.4) \]

Again by using the relations (2.3) in equations (3.3) and (3.4), we get:

\[ \kappa_\alpha = \frac{\sqrt{(\tau\kappa' - \tau'\kappa)^2 + (\kappa^2 + \tau^2)^3}}{\tau\kappa' - \tau'\kappa} \]

\[ \tau_\alpha = \frac{\sqrt{(\kappa^2 + \tau^2)^3}}{(\tau\kappa' - \tau'\kappa)(\tau\kappa' - \tau'\kappa)^2 + (\kappa^2 + \tau^2)^3} \left( \frac{3(\kappa\kappa' + \tau\tau')(\tau'\kappa' - \kappa'\tau)}{-\tau\kappa'' + \tau'\kappa''(\kappa^2 + \tau^2)} \right) \]

Taking \( f, g \) and \( h \) in the last equations, we reach the result. \( \square \)

**Theorem 4.** If any curve \( \beta \) with arc-length parameter \( s \) is slant helix, then the tangent indicatrix of W-direction curve of \( \beta \) is a planar curve.

**Proof.** By Theorem 6 in [11], if the geodesic curvature of the principal image of the principal normal indicatrix of the curve \( \beta \) is \( \delta \), then \( \frac{\kappa}{\tau} = \delta \) where \( \kappa \) and \( \tau \) are curvature and torsion of the W-direction curve of \( \beta \).

By using the equation (3.4) and \( \frac{\kappa}{\tau} = \delta \), the torsion of the tangent indicatrix curve \( \alpha \) is found as:

\[ \tau_\alpha = \frac{\delta'}{\frac{\tau}{\delta}(1 + \delta^2)} \]

If the curve \( \beta \) is slant helix, then \( \delta' = 0 \). So by the last equation \( \tau_\alpha = 0 \) which means the tangent indicatrix curve \( \alpha \) is planar. \( \square \)

**Remark 2.** From now on, the equations \( f = \frac{\tau}{\kappa}, \ g = \frac{\tau'}{\kappa'}, \ h = \frac{\tau''}{\kappa''}, \ \delta = \frac{\kappa}{\tau} \) and \( \kappa'(f - g) = -\delta\kappa^2(1 + f^2)^{3/2} \) will be used in the calculations.
Theorem 5. The tangent indicatrix curve $\alpha$ of the W-direction curve is a general helix if and only if the following equation is satisfied;

$$A' - A \left( \frac{3\kappa'}{\kappa} + \frac{4ff'}{1 + f^2} + \frac{3\delta\delta'}{1 + \delta^2} \right) = 0$$

where \( A = \kappa''(f - h) + 3\kappa\kappa'\sqrt{1 + f^2(1 + fg)}\delta. \)

Proof. If we take ratio of the torsion and curvature of the tangent indicatrix which are in Theorem 3, use the relation $\kappa'(f - g) = -\delta\kappa^2(1 + f^2)^{3/2}$ and make some appropriate calculations, we have;

$$\tau_{\alpha} = A$$

(3.5)

By differentiating the equation (3.5), we find that;

$$\left( \frac{\tau_{\alpha}}{\kappa_{\alpha}} \right)' = -\frac{A' - A \left( \frac{3\kappa'}{\kappa} + \frac{4ff'}{1 + f^2} + \frac{3\delta\delta'}{1 + \delta^2} \right)}{\kappa^3(1 + f^2)^2(1 + \delta^2)^{3/2}}.$$

If the numerator of the last fraction is zero, then \( \left( \frac{\tau_{\alpha}}{\kappa_{\alpha}} \right)' = 0. \) Since the harmonic curvature of the curve $\alpha$ is constant and it is a general helix. \( \square \)

Corollary 1. If the curve $\beta$ is a general helix and and the equation $A' - 3A\kappa^2 = 0$ is satisfied, then the tangent indicatrix of the W-direction curve $\beta$ is a general helix.

Proof. If $\beta$ is a general helix, then $f$ is constant and also $f' = 0$. Since $f' = 0$, then $\delta = 0$. For the derivative, we find;

$$\left( \frac{\tau_{\alpha}}{\kappa_{\alpha}} \right)' = -\frac{A' - A\kappa^2}{\kappa^3(1 + f^2)^2(1 + \delta^2)^{3/2}}.$$

If the numerator of this fraction is zero, then we reach the result clearly. \( \square \)

Theorem 6. The tangent indicatrix curve $\alpha$ of the W-direction curve is a slant helix if and only if the following equation is satisfied;

$$\delta'(1 + 4\delta^2)\frac{X}{Y} + \delta(1 + \delta^2) \left( \frac{X}{Y} \right)' = 0$$

where $\delta = \frac{\kappa'}{\kappa}$, $X = (1 + \delta^2)\tau(\delta^2\tau - \delta'\tau') - 3\delta(\delta')^2\tau^2$ and $Y = (\tau^2(1 + \delta^2)^3 + (\delta')^2)^{3/2}$.

Proof. From the equation (1.1), the geodesic curvature of the principal image of the principal normal indicatrix of the tangent indicatrix curve $\alpha$ is given by

$$\delta_{\alpha} = \frac{\kappa_{\alpha}^2}{(\kappa_{\alpha}^2 + \tau_{\alpha}^2)^{3/2}} \left( \frac{\tau_{\alpha}}{\kappa_{\alpha}} \right)'.$$

(3.6)

We take into account that $\tau\kappa - \tau\kappa' = -\tau^2\delta'$, $\kappa'\frac{\kappa}{\tau} - \tau\kappa' = \tau^2\delta'$ and $\kappa^2 + \tau^2 = \tau^2(1 + \delta^2)$, then put the equations (3.3) and (3.4) in (3.6), we clearly find

$$\delta_{\alpha} = \frac{1 + \delta^2}{\delta^2} \left( \frac{\delta^2(1 + \delta^2)^2\tau^2}{(1 + \delta^2)^3 + (\delta')^2} \right)^{3/2} \left( \frac{\delta'}{(\tau^2(1 + \delta^2)^{3/2})} \right)'.$$
After some calculations we have
\[ \delta_\alpha = \frac{\delta(1 + \delta^2)^{3/2} X}{Y}. \] (3.7)
where \( X = (1 + \delta^2)\tau(\delta''\tau - \delta'\varphi') - 3\delta(1 + \delta^2)^2 \) and \( Y = (\tau^2(1 + \delta^2)^3 + (\delta')^2)^{3/2} \).

By differentiating the equation (3.7), we obtain
\[ (\delta_\alpha)' = (1 + \delta^2)^{1/2} \left( \delta'(1 + 4\delta^2)\frac{X}{Y} + \delta(1 + \delta^2) \left( \frac{X}{Y} \right)' \right). \]

\( (\delta_\alpha)' = 0 \) if and only if \( \delta'(1 + 4\delta^2)\frac{X}{Y} + \delta(1 + \delta^2) \left( \frac{X}{Y} \right)' = 0 \).

So by taking into consideration the equation (1.1), the proof is completed. \( \square \)

**Theorem 7.** Let \( \beta \) be a curve with arc-length parameter \( s \) and \( \overline{\beta} \) be the W-direction curve of \( \beta \). The Frenet vector fields, curvature and torsion of the binormal indicatrix curve \( \gamma \) of W-direction curve are given by

\[
\begin{align*}
T_\gamma &= \frac{1}{\sqrt{1 + f^2}} T - \frac{f}{\sqrt{1 + f^2}} B \\
N_\gamma &= \frac{\kappa'(f - g)}{\sqrt{(\kappa')^2(f - g)^2(1 + f^2) + \kappa^4(1 + f^2)^4}} \left( fT + \frac{\kappa^2(1 + f^2)^2}{\kappa'(f - g)} N + B \right) \\
B_\gamma &= \frac{\kappa(1 + f^2)}{\sqrt{(\kappa')^2(f - g)^2 + \kappa^4(1 + f^2)^3}} \left( \tau T - \frac{\kappa'(f - g)}{\kappa(1 + f^2)} N + \kappa B \right) \\
\kappa_\gamma &= \sqrt{1 + (\kappa')^2(f - g)^2 + \kappa^4(1 + f^2)^3} \\
\tau_\gamma &= \frac{1}{(\kappa')^2(f - g)^2 + \kappa^4(1 + f^2)^3} \left( 3(\kappa')^2(1 + f)g(f - g) - \kappa\kappa''(f - h)(1 + f^2) \right)
\end{align*}
\]

where \( f = \frac{\dot{\tau}}{\kappa}, \; g = \frac{\dot{\tau}}{\kappa'} \) and \( h = \frac{\ddot{\tau}}{\kappa''} \).

**Proof.** The equation of binormal indicatrix curve \( \gamma \) is given in equation (3.2) with the arc-length parameter \( s_\gamma \). By differentiating equation (3.2) and using Frenet formulas we get

\[ \frac{ds_\gamma}{ds} = |\tau| \]

Here we assume \( \tau \parallel 0 \). If \( \tau \parallel 0 \), the Frenet vector fields have other signs.

If we use the equations in (2.1) and the relation \( \frac{ds_\gamma}{ds} = \tau \), we find the tangent, principal and binormal vector fields respectively as;

\[
\begin{align*}
T_\gamma &= -N \\
N_\gamma &= \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} T - \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} B \\
B_\gamma &= \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} B.
\end{align*}
\]
By writing the relations (2.3) in the last equations, we have:

\[ T_\gamma = \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} T - \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} B \]

\[ N_\gamma = \frac{\tau K' - \tau' K}{\sqrt{(\tau K' - \tau' K)^2 + (K^2 + \tau^2)^2}^3} \left( \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\sqrt{(\kappa^2 + \tau^2)^3}}{\tau K' - \tau' K} N + \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} B \right) \]

\[ B_\gamma = \frac{K^2 + \tau^2}{\sqrt{(\tau K' - \tau' K)^2 + (K^2 + \tau^2)^3}} \left( \frac{\tau T - \tau K' - \tau' K}{K^2 + \tau^2} N + \kappa B \right). \]

In the last equations assuming \( f = \frac{\tau}{K}, \quad g = \frac{\tau'}{K} \quad \text{and} \quad h = \frac{\tau''}{K} \), arranging the expressions, we obtain the tangent, principal and binormal vector fields of the binormal indicatrix curve \( \gamma \) of the W-direction curve \( \beta \), with respect to the main curve \( \beta \).

Also the curvature and torsion of the binormal indicatrix curve \( \gamma \) by taking into account that \( \frac{ds}{ds} = \tau \) are found as:

\[ \kappa_\gamma = \frac{\sqrt{K^2 + \tau^2}}{\tau} \tag{3.8} \]

\[ \tau_\gamma = \sqrt{\frac{(\tau K' - \tau' K)^2 + (\tau K' - \tau' K)^2}{(\tau^2 + \tau^2)^3}} \tag{3.9} \]

Again by using the relations (2.3) in equations (3.8) and (3.9), we get;

\[ \kappa_\gamma = \frac{\sqrt{(\tau K' - \tau' K)^2 + (K^2 + \tau^2)^3}}{(\kappa^2 + \tau^2)^3} \]

\[ \tau_\gamma = \frac{1}{(\tau K' - \tau' K)^2 + (K^2 + \tau^2)^3} \left( \frac{3(\kappa K' + \tau \tau') (\tau K' - \tau' K)}{-(\tau K' - \tau' K)(\kappa^2 + \tau^2)} \right). \]

Taking \( f, g \) and \( h \) in the last equations, we reach the result. □

**Corollary 2.** The tangent indicatrix curve and the binormal indicatrix curve of a W-direction curve are Bertrand mate curves.

**Proof.** Since the relation between the principal normal vector fields of the tangent indicatrix and binormal indicatrix is;

\[ N_\alpha = -N_\gamma, \]

they are linearly dependent. □

**Theorem 8.** If any curve \( \beta \) with arc-length parameter \( s \) is slant helix, then the binormal indicatrix of W-direction curve of \( \beta \) is a planar curve.

**Proof.** By using the equation (3.9) and \( \frac{\Delta}{\Delta} = \delta \), the torsion of the binormal indicatrix curve \( \gamma \) is found as;

\[ \tau_\gamma = \frac{\delta'}{\tau (1 + \delta^2)}. \]

If the curve \( \beta \) is slant helix, then \( \delta' = 0 \). So by the last equation \( \tau_\gamma = 0 \) which means the binormal indicatrix curve \( \gamma \) is planar. □
Theorem 9. The binormal indicatrix curve $\gamma$ of the W-direction curve is a general helix if and only if the following equation is satisfied;

$$A' - A \left( \frac{3\kappa'}{\kappa} + \frac{4ff'}{1+f^2} + \frac{3\delta\delta'}{1+\delta^2} \right) = 0$$

where $A = \kappa''(f - h) + 3\kappa\kappa'(h') + \sqrt{1+f^2(1 + fg)}\delta$.

Proof. If we take ratio of the torsion and curvature of the binormal indicatrix which are in Theorem 7, use the relation $\kappa'((f - g) = -\delta\kappa^2(1 + f^2)^{3/2}$ and make some appropriate calculations, we have;

$$\frac{\tau_\gamma}{\kappa_\gamma} = -\frac{A}{\kappa^3(1 + f^2)^2(1 + \delta^2)^{3/2}}. \quad (3.10)$$

By differentiating the equation (3.10), we find that;

$$\left( \frac{\tau_\gamma}{\kappa_\gamma} \right)' = -\frac{A' - A \left( \frac{3\kappa'}{\kappa} + \frac{4ff'}{1+f^2} + \frac{3\delta\delta'}{1+\delta^2} \right)}{\kappa^3(1 + f^2)^2(1 + \delta^2)^{3/2}}. \quad (3.11)$$

If the numerator of the last fraction is zero, then $\left( \frac{\tau_\gamma}{\kappa_\gamma} \right)' = 0$. Since the harmonic curvature of the curve $\gamma$ is constant, it is a general helix. □

Corollary 3. Let $\beta$ be a curve with arc-length parameter $s$ and $\bar{\beta}$ be the W-direction curve of $\beta$. The tangent indicatrix curve of $\beta$ is a general helix if and only if the binormal indicatrix curve of $\bar{\beta}$ is general helix.

Proof. By the equations (3.5) and (3.10), the result is clear. □

Theorem 10. The binormal indicatrix curve $\gamma$ of the W-direction curve is a slant helix if and only if the following equation is satisfied;

$$3\delta\delta'X' + (1 + \delta^2)(\frac{X}{Y})' = 0$$

where $\delta = \frac{\tau}{\kappa}$, $X = (1 + \delta^2)(\overline{\tau}(\delta'\overline{\tau} - \delta\overline{\tau'}) - 3\delta(\delta')^2\overline{\tau'}$ and $Y = (\overline{\tau}^2(1 + \delta^2)^3 + (\delta')^2)^{3/2}$.

Proof. From the equation (1.1), the geodesic curvature of the principal image of the principal normal indicatrix of the binormal indicatrix curve $\gamma$ is given by

$$\delta_\gamma = \frac{\kappa'_\gamma}{(\kappa_\gamma^2 + \tau_\gamma^4)^{3/2}} \left( \frac{\tau_\gamma}{\kappa_\gamma} \right)' \quad (3.11)$$

We take into account that $\overline{\tau}\overline{\kappa} - \overline{\tau}\overline{\kappa'} = -\overline{\tau}^2\delta'$, $\kappa\overline{\tau}^2 = \overline{\tau}\overline{\tau}^2 = \overline{\tau}^2\delta'$ and $\kappa^2 + \overline{\tau}^2 = \overline{\tau}^2(1 + \delta^2)$, then put the equations (3.8) and (3.9) in (3.11), we clearly find

$$\delta_\gamma = \frac{\overline{\tau}^2(1 + \delta^2)^4}{((1 + \delta^2)^3\overline{\tau}^2 + (\delta')^2)^{3/2}} \left( \frac{\delta'}{\overline{\tau}(1 + \delta^2)^{3/2}} \right)'.$$

After some calculations we have

$$\delta_\gamma = \frac{(1 + \delta^2)^{3/2}X}{Y} \quad (3.12)$$
where $X = (1 + \delta^2)\tau(\delta''\tau - \delta'\tau') - 3\delta(\delta')^2\tau^2$ and $Y = (\tau^2(1 + \delta^2)^3 + (\delta')^2)^{3/2}$.

By differentiating the equation (3.12), we obtain

\[
(\delta_\gamma)' = (1 + \delta^2)^{1/2} \left(3\delta(\delta')X^2 + (1 + \delta^2)(\delta')^2 \left(\frac{X}{Y}\right)\right)
\]

$(\delta_\gamma)' = 0$ if and only if $3\delta(\delta')X^2 + (1 + \delta^2)(\delta')^2 (\frac{X}{Y})' = 0$.

So by taking into consideration the equation (1.1), the proof is completed.

**Corollary 4.** If any unit speed curve $\beta$ is slant helix, then the tangent indicatrix curve of $W$-direction curve of $\beta$ is slant helix if and only if the binormal indicatrix curve of $W$-direction curve of $\beta$ is slant helix.

**Proof.** By the equations (3.7) and (3.12), we have

\[
\delta_\alpha = \delta \cdot \delta_\gamma
\]

where $\delta = \frac{\pi}{\kappa}$.

If any unit speed curve $\beta$ is slant helix, then $\delta$ is constant by Theorem 1. If $\delta$ is constant, the result is apparent.

**Example:** Let a curve which is a slant helix be

\[
\beta(s) = \left(-\frac{3}{2}\cos\left(s^2\right) - \frac{1}{6}\cos\left(\frac{3s}{2}\right), -\frac{3}{2}\sin\left(s^2\right) - \frac{1}{6}\sin\left(\frac{3s}{2}\right), \sqrt{3}\cos\left(\frac{s}{2}\right)\right).
\]

The tangent, binormal vectors, the curvature, the torsion and the Darboux vector were found in [11].

\[
T(s) = \left(\frac{3}{4}\sin\left(s^2\right) + \frac{1}{4}\sin\left(\frac{3s}{2}\right), -\frac{3}{4}\cos\left(s^2\right) - \frac{1}{4}\cos\left(\frac{3s}{2}\right), -\frac{\sqrt{3}}{2}\sin\left(\frac{s}{2}\right)\right)
\]

\[
B(s) = \left(-\frac{1}{2}\cos\left(s^2\right)\left(2\cos^2\left(s^2\right) - 3\right), \sin^3\left(s^2\right), \frac{\sqrt{3}}{2}\cos\left(s^2\right)\right)
\]

and

\[
\kappa(s) = \frac{\sqrt{3}}{2}\cos\left(\frac{s}{2}\right), \quad \tau(s) = -\frac{\sqrt{3}}{2}\sin\left(\frac{s}{2}\right).
\]

Also the $W$-direction curve of $\beta$ was given as;

\[
\overline{\beta}(s) = \left(-\frac{9s}{8} - 6\sin\left(\frac{s}{2}\right) - \frac{3}{4}\sin(\frac{s}{2}) - \frac{1}{16}\sin(2s), -\frac{1}{2}\cos(s), \frac{\sqrt{3}s}{2}\right) + (c_1, c_2, c_3)
\]

where $c_1, c_2, c_3$ are constants.

Now lets find the tangent and binormal indicatrix curves of this $W$-direction curve $\overline{\beta}$. By using these expressions above and the equations (2.3), (3.1) and (3.2), the tangent indicatrix
and binormal indicatrix are obtained respectively:

$$\alpha(s_\alpha) = \left(-\frac{3}{4} \sin^2 \left(\frac{s}{2}\right) - \frac{1}{4} \sin \left(\frac{s}{2}\right) \sin \left(\frac{3s}{2}\right) - \frac{1}{2} \cos^2 \left(\frac{s}{2}\right) \left(2 \cos^2 \left(\frac{s}{2}\right) - 3\right) \right),$$

$$\sin \left(\frac{s}{2}\right) \left[\frac{3}{4} \cos \left(\frac{s}{2}\right) + \frac{1}{2} \cos \left(\frac{3s}{2}\right) + \cos \left(\frac{s}{2}\right) \sin^2 \left(\frac{s}{2}\right) \right], \sqrt{\frac{3}{2}}$$

and

$$\gamma(s_\gamma) = \left(\frac{\sqrt{3}}{2} \left[\frac{3}{4} \cos^2 \left(\frac{s}{2}\right) + \frac{1}{4} \cos \left(\frac{3s}{2}\right) \cos \left(\frac{s}{2}\right) - \sin^4 \left(\frac{s}{2}\right) \right] \right),$$

$$\frac{3}{4} \cos(s) - \frac{1}{4} \left[\sin^3 \left(\frac{s}{2}\right) \sin \left(\frac{3s}{2}\right) - \cos^3 \left(\frac{s}{2}\right) \cos \left(\frac{3s}{2}\right) \right]$$

$$+ \frac{3}{8} \cos \left(\frac{s}{2}\right) \left[3 \cos \left(\frac{s}{2}\right) + \cos \left(\frac{3s}{2}\right) \right].$$

4. Some Ruled Sufaces Related To W-direction Curves

In this section, we will identify some special ruled surfaces which are formed by using the base curve as the W-direction curve.

Let $\beta$ be a curve with the arc-length parameter $s$ and $\overline{\beta}$ be the W-direction curve of $\beta$. From the equations (2.4) and (2.5), the normal and binormal surfaces of $\overline{\beta}$ are given by:

$$\Phi_1(s, v) = \overline{\beta}(s) + v\overline{N}(s) \quad (4.1)$$

$$\Phi_2(s, v) = \overline{\beta}(s) + v\overline{B}(s). \quad (4.2)$$

**Theorem 11.** The normal surface and binormal surface of the W-direction curve are not developable.

**Proof.** By using equation (2.6), the distribution parameters of the normal and the binormal surfaces of W-direction curve given in (4.1) and (4.2) are;

$$\lambda_{\Phi_1} = \frac{\det(\frac{d\overline{N}}{ds}, \overline{N}, \frac{d\overline{N}}{ds})}{\left\|\frac{d\overline{N}}{ds}\right\|^2}$$

$$\lambda_{\Phi_2} = \frac{\det(\frac{d\overline{B}}{ds}, \overline{B}, \frac{d\overline{B}}{ds})}{\left\|\frac{d\overline{B}}{ds}\right\|^2}.$$

Taking into account that the equations given in (2.3) and

$$\frac{d\overline{N}}{ds} = -\left(\frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}\right)'T - \sqrt{\kappa^2 + \tau^2}N + \left(\frac{\tau}{\sqrt{\kappa^2 + \tau^2}}\right)'B$$

$$\frac{d\overline{B}}{ds} = \kappa T - \tau B$$
we have;

\[
\lambda_{\Phi_1} = \frac{(\kappa^2 + \tau^2)^{5/2}}{(\tau \kappa' - \kappa' \tau)^2 + (\kappa^2 + \tau^2)^3}
\]
\[
\lambda_{\Phi_2} = \frac{1}{\sqrt{\kappa^2 + \tau^2}}
\]

Since the distribution parameters cannot be zero, the normal and binormal surfaces of W-direction curve are not developable. 

**Theorem 12.** If any curve \( \beta \) with arc-length parameter \( s \) is general helix and the equation \( \kappa \kappa' + \tau \tau' = 0 \) is satisfied then the normal surface and the binormal surface of W-direction curve \( \overline{\beta} \) of \( \beta \) are minimal.

**Proof.** Let's find the mean curvatures of the normal and binormal surfaces in (4.1) and (4.2) for minimallity.

For the normal surfaces given in (4.1), by taking into consideration the equations (2.3), the following equations are obtained as;

\[
E_1 = \langle (\Phi_1)_s, (\Phi_1)_s \rangle = A^2 + C^2 + D^2
\]
\[
F_1 = \langle (\Phi_1)_s, (\Phi_1)_v \rangle = -AX + DY = 0
\]
\[
G_1 = \langle (\Phi_1)_v, (\Phi_1)_v \rangle = 1
\]
\[
e_1 = \langle (\Phi_1)_{ss}, U_1 \rangle = \frac{1}{Z_1} \left( -\left( A' + C \kappa \right) CY - \left( AY + DX \right) (A \kappa - C' - D \tau) \right) - (D' - C \tau) CX
\]
\[
f_1 = \langle (\Phi_1)_{sv}, U_1 \rangle = \frac{1}{Z_1} \left( C(X'Y - Y'X) + \sqrt{\kappa^2 + \tau^2} (AY + DX) \right)
\]
\[
g_1 = \langle (\Phi_1)_{vv}, U_1 \rangle = 0
\]

where \( X = \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \), \( Y = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \), \( Z_1 = \| (\Phi_1)_s \times (\Phi_1)_v \| \), \( U_1 = \frac{1}{Z_1} \left( (C\kappa) \mathbf{T} - (AY + DX) \mathbf{N} - (CX) \mathbf{B} \right) \), \( A = Y - vX', C = v \sqrt{\kappa^2 + \tau^2} \) and \( D = X + vY' \).

For the binormal surfaces given in (4.2);

\[
E_2 = \langle (\Phi_2)_s, (\Phi_2)_s \rangle = 1 + v^2 (\kappa^2 + \tau^2)
\]
\[
F_2 = \langle (\Phi_2)_s, (\Phi_2)_v \rangle = 0
\]
\[
G_2 = \langle (\Phi_2)_v, (\Phi_2)_v \rangle = 1
\]
\[
e_2 = \langle (\Phi_2)_{ss}, U_2 \rangle = \frac{1}{Z_2} \left( \left( \frac{\tau'}{\kappa} \right) \frac{1}{1 + (\frac{\tau}{\kappa})^2} + v \frac{\kappa \kappa' + \tau \tau'}{\sqrt{\kappa^2 + \tau^2}} + v^2 \kappa^2 \left( \frac{\tau'}{\kappa} \right) \right)
\]
\[
f_2 = \langle (\Phi_2)_{sv}, U_2 \rangle = \frac{\sqrt{\kappa^2 + \tau^2}}{Z_2}
\]
\[
g_2 = \langle (\Phi_2)_{vv}, U_2 \rangle = 0
\]

where \( Z_2 = \sqrt{1 + v^2 (\kappa^2 + \tau^2)} \) and \( U_2 = \frac{1}{Z_2} \left( (X - v \tau) \mathbf{T} - (Y + v \kappa) \mathbf{B} \right) \).
By using the equation (2.9), we find the mean curvatures as:

\[
H_1 = \frac{e_1}{2E_1}, \quad H_2 = \frac{e_2}{2E_2}.
\]

\(e_2\) is apparent from the above equation, let’s find \(e_1\). We can calculate simply that

\[
AY + DX = 1 + v\left(\frac{\tau'}{\kappa} - \frac{1}{1 + (\frac{\tau}{\kappa})^2}\right)
\]

\[
A\kappa - C' - D\tau = -v\frac{\kappa\kappa' + \tau\tau'}{\sqrt{\kappa^2 + \tau^2}}
\]

\[
-A'CY - C^2\kappa Y - D'C X + C^2\tau X = v^2 \cdot \frac{2(\tau'\kappa - \tau\kappa')(\kappa\kappa' + \tau\tau') - ((\tau'\kappa)' - (\tau\kappa')')(\kappa^2 + \tau^2)}{(\kappa^2 + \tau^2)^{3/2}}.
\]

If we put these expressions in the equation of \(e_1\), we have finally:

\[
e_1 = \frac{1}{Z_1} \left( \frac{v^2 \cdot \frac{2(\tau'\kappa - \tau\kappa')(\kappa\kappa' + \tau\tau') - ((\tau'\kappa)' - (\tau\kappa')')(\kappa^2 + \tau^2)}{(\kappa^2 + \tau^2)^{3/2}}}{1 + \left(\frac{\tau}{\kappa}\right)^2} \right) + v \left(1 + v\left(\frac{\tau'}{\kappa} - \frac{1}{1 + (\frac{\tau}{\kappa})^2}\right)\right) \frac{\kappa\kappa' + \tau\tau'}{\sqrt{\kappa^2 + \tau^2}}.
\]

Since the curve \(\beta\) is a general helix, then \((\frac{\tau}{\kappa})' = 0\) and so we have \((\tau'\kappa)' - (\tau\kappa')' = 0\). Also by the condition \(\kappa\kappa' + \tau\tau' = 0\) given in the theorem, we obtain \(e_1 = 0\) and \(e_2 = 0\).

So the proof is completed. \(\square\)

**Theorem 13.** Let \(\beta\) be any curve with arc-length parameter \(s\). If \(\beta\) is a general helix, then the \(W\)-direction curve \(\tilde{\beta}\) which is on the normal surface of the \(W\)-direction curve is a geodesic curve. Also the \(W\)-direction curve \(\tilde{\beta}\) which is on the binormal surface is geodesic curve.

**Proof.** Let’s find the geodesic curvatures of the \(W\)-direction curve \(\tilde{\beta}\) with respect to the normal and binormal surfaces of \(\tilde{\beta}\) given in (4.1) and (4.2). By using the equation (2.10), the geodesic curvatures are

\[
\kappa_{g_1} = \left\langle U_1 \times \overline{T}, \overline{T}' \right\rangle
\]

\[
\kappa_{g_2} = \left\langle U_2 \times \overline{T}, \overline{T}' \right\rangle.
\]

By using the same terminology with the previous proof and \(\overline{T} = YT + XB\), we have

\[
U_1 \times \overline{T} = \frac{1}{Z_1}(AY + DX)(YB - XT)
\]

\[
U_2 \times \overline{T} = \frac{-1}{Z_2}N.
\]

Deciding the equation \(\overline{T}' = Y'T + X'B\), we get

\[
\kappa_{g_1} = -\frac{1}{Z_1}(AY + DX)(Y'X - XY')
\]

\[
\kappa_{g_2} = 0.
\]
By using appropriate expressions and \( Y'X - YX' = \frac{\tau'\kappa - \tau\kappa'}{\kappa' + \tau'} \), we get finally that:

\[
\kappa_{g1} = -\frac{1}{Z_1} \left( 1 + v\left( \frac{\tau'}{\kappa} \right)' \frac{1}{1 + (\frac{\tau}{\kappa})^2} \right) \left( \frac{\tau'}{\kappa} \right)' \frac{1}{1 + (\frac{\tau}{\kappa})^2}. \]

If \( \beta \) is general helix, then \( (\frac{\tau}{\kappa})' = 0 \) and \( \kappa_{g1} = 0 \).

\[ \square \]

**Theorem 14.** Let \( \beta \) be any curve with arc-length parameter \( s \). The W-direction curve \( \bar{\beta} \) which is on the normal surface of the W-direction curve is an asymptotic line. \( \beta \) is general helix if and only if the W-direction curve \( \bar{\beta} \) which is on the binormal surface is an asymptotic line.

**Proof.** The normal curvatures of \( \bar{\beta} \) with respect to the normal and binormal surfaces in (4.1) and (4.2) are computed by the equation (2.11) as:

\[
\kappa_{n1} = \left\langle \bar{\beta}', U_1 \right\rangle \\
\kappa_{n2} = \left\langle \bar{\beta}', U_2 \right\rangle.
\]

We can write that; \( \bar{\beta}' = \bar{T} = Y''T + X'B \) and also by making the dot product of \( \bar{\beta}' \) and \( U_1, U_2 \), we get

\[
\kappa_{n1} = -\frac{C}{Z_1} (XX' + YY') \\
\kappa_{n2} = \frac{1}{Z_2} (Y'X - YX').
\]

Since \( XX' + YY' = 0 \) and \( Y'X - YX' = \frac{\tau'\kappa - \tau\kappa'}{\kappa' + \tau'} \), we simply obtain that;

\[
\kappa_{n1} = 0 \\
\kappa_{n2} = \frac{1}{Z_2} \left( \frac{\tau'}{\kappa} \right)' \frac{1}{1 + (\frac{\tau}{\kappa})^2}.
\]

So \( \bar{\beta} \) which is on the normal surface of the W-direction curve is an asymptotic line. \( \beta \) is general helix if and only if \( (\frac{\tau}{\kappa})' = 0 \) which means that \( \kappa_{n2} = 0 \) and it is an asymptotic line.

\[ \square \]

**Theorem 15.** Let \( \beta \) be any curve with arc-length parameter \( s \). If \( \beta \) is general helix, then the W-direction curve \( \bar{\beta} \) which is on the normal surface and the binormal surface of the W-direction curve is a principal line.

**Proof.** Lets find the geodesic torsions of the W-direction curve \( \bar{\beta} \) with respect to the normal and binormal surfaces given in (4.1) and (4.2). By using the equation (2.12), the geodesic torsions are:

\[
\tau_{g1} = \left\langle U_1 \times U_1', \bar{T}' \right\rangle \\
\tau_{g2} = \left\langle U_2 \times U_2', \bar{T}' \right\rangle.
\]
After some calculations and using the abbreviations as
\[ r = -\frac{CY}{Z_1}, \quad q = -\frac{AY + DX}{Z_1}, \quad t = -\frac{CX}{Z_1} \]
and
\[ K = \left( \frac{X+Y+\tau(\kappa-\tau)}{Z_2} \right)' \]
\[ L = \frac{X+Y+\tau(\kappa-\tau)}{Z_2}, \quad M = \left( \frac{Y+\nu\kappa}{Z_2} \right)' \]
we find;
\[ U_{1} \times U_{1}' = \left( \frac{CX}{Z_1} (r\kappa + q' - t\tau) - \frac{AY + DX}{Z_1} (q\tau + t') \right) T \]
\[ + \left( \frac{CY}{Z_1} (q\tau + t') - \frac{CX}{Z_1} (r' - q\kappa) \right) N \]
\[ + \left( \frac{AY + DX}{Z_1} (r' - q\kappa) - \frac{CY}{Z_1} (r\kappa + q' - t\tau) \right) B \]
and
\[ U_{2} \times U_{2}' = \left( \frac{Y + v\kappa}{Z_2} \right) L.T + \left( \left( \frac{X - v\tau}{Z_2} \right) M - \frac{Y + v\kappa}{Z_2} K \right) N \]
\[ + \left( \frac{X - v\tau}{Z_2} \right) L.B \]
By taking into account that \( T' = Y'T + X'B \), we obtain lastly;
\[ \tau_{g_1} = \frac{1}{Z_1} \left( \frac{C'}{Z_1} \right)' (\frac{\tau'}{\kappa})' \frac{1}{1 + (\frac{\tau}{\kappa})^2} \left( 1 + v (\frac{\tau'}{\kappa})' \frac{1}{1 + (\frac{\tau}{\kappa})^2} \right) \]
\[ + \frac{C(r\kappa + q' - t\tau)}{Z_1} (\frac{\tau'}{\kappa})' \frac{1}{1 + (\frac{\tau}{\kappa})^2} \]
\[ \tau_{g_2} = \frac{v\kappa^2(\kappa + \tau + v(\kappa - \tau)\sqrt{\kappa^2 + \tau^2})}{(\kappa^2 + \tau^2)(1 + v^2(\kappa^2 + \tau^2))} (\frac{\tau'}{\kappa})' \]
If \( \beta \) is general helix, then \( (\frac{\tau}{\kappa})' = 0 \) and so \( \tau_{g_1} = 0, \quad \tau_{g_2} = 0. \)

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