MATHAI-QUILEN FORMULATION OF TWISTED N = 4 SUPERSYMMETRIC GAUGE THEORIES IN FOUR DIMENSIONS

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ABSTRACT

We present a detailed description of the three inequivalent twists of \( N = 4 \) supersymmetric gauge theories. The resulting topological quantum field theories are reobtained in the framework of the Mathai-Quillen formalism and the corresponding moduli spaces are analyzed. We study their geometric features in each case. In one of the twists we make contact with the theory of non-abelian monopoles in the adjoint representation of the gauge group. In another twist we obtain a topological quantum field theory which is orientation reversal invariant. For this theory we show how the functional integral contributions to the vacuum expectation values leading to topological invariants notably simplify.

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1. Introduction

Topological quantum field theory [1] constitutes a very fruitful framework to apply and test different ideas emerged in the context of duality as a symmetry of extended supersymmetric gauge theories. Two salient examples are the introduction of Seiberg-Witten invariants in [2,3,4], and the strong coupling test of $S$-duality carried out by Vafa and Witten in [5] from the analysis of a twisted $N = 4$ supersymmetric gauge theory. Subsequent generalizations in the framework of Seiberg-Witten invariants have been presented in [6-14]. However, no further progress has been made on the role played by duality in twisted $N = 4$ supersymmetric gauge theories. The main goal of this paper is to construct a sound framework to pursue further developments on this issue.

The first analysis of twisted $N = 4$ supersymmetric gauge theories was carried out by Yamron in [15] where he presented the structure of two of the possible non-equivalent twists of these theories and pointed out the existence of a third one. This third twist was first addressed by Marcus in [16]. These twists have not been fully presented in these works. In the second section of this paper we will describe in full detail the twisting procedure in each of the cases and we will present complete off-shell topological actions for all the three cases. For the twist treated by Vafa and Witten the construction completes the action presented in [5] while for the twist treated by Marcus it provides an off-shell formulation which is equivalent to the one recently obtained in [17]. In the case of the other twist we make contact with the topological quantum field theory of non-abelian monopoles introduced in [6] for the case in which matter fields are in the adjoint representation.

It is well known that topological quantum field theories obtained after twisting $N = 2$ supersymmetric gauge theories can be formulated in the framework of the Mathai-Quillen formalism. One would expect that a similar formulation should exist for the $N = 4$ case. Though it turns out that this is so, there is an important issue that has to be addressed to clarify what it is meant by a Mathai-Quillen formulation in the latter case. Twisted $N = 2$ supersymmetric gauge theories have
an off-shell formulation such that the topological quantum field theory action can be expressed as a $Q$-exact expression, being $Q$ the part of the $N = 2$ supersymmetry which remains after the twisting and is valid on curved space. Actually, this is true only up to a $\theta$-term. However, due to the chiral anomaly inherent to the $R$-symmetry of $N = 2$ supersymmetric gauge theories, observables are independent of $\theta$-terms up to a rescaling. This allows to disregard these terms and to just consider the $Q$-exact part of the action which is precisely the one obtained in the Mathai-Quillen formalism.

In $N = 4$ supersymmetric gauge theories $\theta$-terms are observable. There is no chiral anomaly and these terms can not be shifted away as in the $N = 2$ case. This means that in the twisted theories one might have a dependence on the coupling constants (in fact, this was one of the key observations in [5] to make a strong coupling test of $S$-duality). This being so we first have to clarify what one expect to be the form of the twisted theories in the framework of the Mathai-Quillen formalism. To do this let us concentrate our attention on the part of the action of a twisted theory (originated from any gauge theory with extended supersymmetry) involving the gauge field strength,

$$S_X = -\frac{1}{4e^2} \int_X \sqrt{g} d^4 x \, \text{Tr}(F^{\mu\nu}F_{\mu\nu}) - \frac{i\theta}{16\pi^2} \int_X \text{Tr}(F \wedge F) + \ldots,$$  \hspace{1cm} (1.1)

where $X$ is an oriented Riemannian four-manifold and $g_{\mu\nu}$ a Riemannian metric on it. We are using conventions such that,

$$k = \frac{1}{16\pi^2} \int_X \text{Tr}(F \wedge F) = \frac{1}{32\pi^2} \int_X \sqrt{g} \text{Tr}(*F_{\mu\nu} F^{\mu\nu}) = \frac{1}{32\pi^2} \int_X \sqrt{g} \text{Tr}\{ (F^+)^2 - (F^-)^2 \},$$  \hspace{1cm} (1.2)

gives the instanton number. We also take the path integral to be $Z \sim \int e^{S}$. Using the decomposition of the field strength $F$ into its self-dual and anti-selfdual parts,

$$F_{\mu\nu}^\pm = \frac{1}{2} (F_{\mu\nu} \pm \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}),$$ \hspace{1cm} (1.3)
(1.1) can be written in the following two forms:

\[
S_X = -\frac{1}{2e^2} \int_X \sqrt{g} d^4 x \left[ \text{Tr}(F^{+\mu\nu} F^{\mu\nu}_+) - 2\pi i \tau \frac{1}{16\pi^2} \int_X \text{Tr}(F \wedge F) + \ldots \right]
\]

\[
= -\frac{1}{2e^2} \int_X \sqrt{g} d^4 x \left[ \text{Tr}(F^{-\mu\nu} F^{-\mu\nu}_+) - 2\pi i \bar{\tau} \frac{1}{16\pi^2} \int_X \text{Tr}(F \wedge F) + \ldots \right],
\]

being,

\[
\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}.
\]

The twist consists of considering as the new rotation group an exotic subgroup of the global group corresponding to the extended supersymmetry under consideration. The global group of extended supersymmetry has the form \( SU(2)_L \otimes SU(2)_R \otimes H \) where \( SU(2)_L \otimes SU(2)_R \) constitutes the rotation group and \( H \) the internal or isospin group. For \( N = 2 \), \( H = U(2) \), while for \( N = 4 \), \( H = SU(4) \). In the twisting procedure one first selects one of the two components of the rotation group and then replaces it by the diagonal sum of that component with a \( SU(2) \) subgroup of the internal group. In the case of \( N = 2 \) this can be done in only one way while for \( N = 4 \) there are three possibilities. These will be fully described in the next section. What we intend to discuss here is the difference between the two possible choices which are present when picking up one of the components of the rotation group. It turns out that choosing one of them, say, the left or twist \( T \), one must consider the first form of the action in (1.4) since then, after working out its off-shell formulation, it can be written as

\[
S^T_X = \frac{1}{2e^2} \int_X \sqrt{g} d^4 x \left\{ Q, \Lambda \right\} - 2\pi i \tau \frac{1}{16\pi^2} \int_X \text{Tr}(F \wedge F),
\]

for some \( \Lambda \), while it one chooses the other one, the right or twist \( \tilde{T} \), one finds,

\[
S^{\tilde{T}}_X = \frac{1}{2e^2} \int_X \sqrt{g} d^4 x \left\{ \tilde{Q}, \tilde{\Lambda} \right\} - 2\pi i \bar{\tau} \frac{1}{16\pi^2} \int_X \text{Tr}(F \wedge F),
\]

for some \( \tilde{\Lambda} \) and some \( \tilde{Q} \). These actions correspond to an orientable four-manifold \( X \).
with a given orientation. The actions of the two twists are related in the following way:

$$S_X^T = S_{\tilde{X}}^T \bigg|_{\tau \to -\bar{\tau}},$$

(1.8)

where the four-manifolds $X$ and $\tilde{X}$ are related by a reversal of orientation.

For twisted theories originated from $N = 2$ supersymmetric gauge theories, observables do not depend on $e$ because it appears only in a term which is $Q$-exact. They do not depend either on $\tau$, up to a rescaling, due to the chiral anomaly. In the case of twisted theories originated from $N = 4$ supersymmetric gauge theories, however, observables are independent of $e$ but possess a dependence on $\tau$. In both cases one needs to consider only one of the types of twist, say $T$, since, according to (1.8), the other just leads to the observables that one would obtain considering $\tilde{X}$ instead of $X$. In the first case this statement is exact and in the second case one must supplement it with the replacement $\tau \to -\bar{\tau}$. Therefore one can say that up to a reversal of orientation there is only one possible twist from $N = 2$ supersymmetric gauge theories and three, as stated in [15] and described in detail in the next section, from $N = 4$ theories.

After these remarks on the twisting procedure we will state what is meant by a Mathai-Quillen formulation of topological quantum field theories resulting after twisting $N = 4$ supersymmetric gauge theories. The Mathai-Quillen formulation builds out of a moduli problem a representative of the Thom class associated to the corresponding vector bundle. This representative can always be written as an integral of the exponential of a $Q$-exact expression. The three twists of $N = 4$, after working out their off-shell formulation, can be written as in (1.6). We will present for each case the moduli problem which in the context of the Mathai-Quillen approach leads to the $Q$-exact part of the action. In other words, we will find out the geometrical content which is behind each of the three twists.

One of the three twists, the one first considered by Marcus in [16], possesses special features. It turns out that the topological quantum field theories resulting
from the twist $T$ and from the twist $\tilde{T}$ are the same:

$$S_X^T = S_{\tilde{X}}^{\tilde{T}}. \quad (1.9)$$

We will call theories satisfying this property *amphicheiral* topological quantum field theories. The reason for this name is that for these theories, after using (1.8),

$$S_X^T = S_{\tilde{X}}^{\tilde{T}}|_{\tau \rightarrow -\bar{\tau}}, \quad (1.10)$$

in other words, for a fixed twist the observables of the theory on $X$ and on $\tilde{X}$ are related after reversing the sign of their dependence on the real part of $\tau$, i.e., of their dependence on the $\theta$-angle. Amphicheiral topological quantum field theories seem to possess very special properties which make them rather simple. An example of this type of theories, the one resulting from the third twist, will be analyzed in sect. 4.

The paper is organized as follows. In sect. 2 we present a detailed description of the three twists of $N = 4$ supersymmetric gauge theories, obtaining their off-shell formulation and their canonical form (1.6). In sect. 3 the three moduli problems associated to each of the twists are presented and the construction of the corresponding Thom class representatives is carried out making contact with the actions obtained in sect. 2. In sect. 4 we discuss the observables of these theories and the special features of the amphicheiral topological quantum field theory which results in the third the twist. Finally, in sect. 5 we state our conclusions. An appendix describes our conventions and collects a set of useful formulas used throughout the paper.
2. Twisting of $N=4$ Supersymmetric Gauge Theory

In this chapter we will obtain the actions and BRST-like symmetries which result after twisting $N=4$ supersymmetric gauge theories. We will first introduce the $N=4$ physical theory and then we will carry out in detail its three possible twists.

2.1. $N=4$ Supersymmetric Gauge Theory

We begin with the standard $N=4$ supersymmetric gauge theory on flat $\mathbb{R}^4$. Our conventions regarding spinor notation are almost as in Wess and Bagger [18], with some differences that we conveniently compile in the appendix. The field content of the model is the following: a gauge field $A_{a\dot{a}}$, gauginos $\lambda_u^\alpha$ and $\bar{\lambda}^u_{\dot{\alpha}}$, transforming respectively in the representations $4$ and $\bar{4}$ of $SU(4)_I$ ($SU(4)_I$ is the global isospin group of the theory, and indices $(u,v,w,\ldots)$ label its fundamental representation), and scalars $\phi_{uv}$ in the $6$ of $SU(4)_I$. All the fields above take values in the adjoint representation of some compact Lie group $G$. Being in the representation $6$, the scalars $\phi_{uv}$ satisfy antisymmetry and self-conjugacy constraints:

\begin{equation}
\phi_{uv} = -\phi_{vu}, \quad \phi^{uv} = (\phi_{uv})^\dagger = \phi_{vu} = -\frac{1}{2} \epsilon^{uvwz} \phi_{wz}; \quad \epsilon_{1234} = \epsilon^{1234} = +1.
\end{equation}

The action for the model in Euclidean space is:

\begin{equation}
S = \frac{1}{e^2} \int d^4x \text{Tr} \{ -\frac{1}{8} \nabla_{a\dot{a}} \phi_{uv} \nabla^{a\dot{a}} \phi^{uv} - i \lambda_u^\alpha \nabla_{a\dot{a}} \bar{\lambda}^{\dot{\alpha}} \lambda_{\dot{\alpha}}^v + \frac{1}{4} F_{mn} F^{mn} \\
- \frac{i}{\sqrt{2}} \lambda_u^\alpha [\lambda_{\alpha v}, \phi^{uv}] + \frac{i}{\sqrt{2}} \bar{\lambda}^u_{\dot{\alpha}} [\bar{\lambda}^{\dot{\alpha}} v, \phi_{uv}] + \frac{1}{16} [\phi_{uv}, \phi_{wz}] [\phi^{vw}, \phi^{wz}] \\
- \frac{i \theta}{32\pi^2} \int d^4x \text{Tr} \{ * F_{mn} F^{mn} \} \}
\end{equation}

We have introduced the covariant derivative $\nabla_{a\dot{a}} = \sigma^m_{a\dot{a}} (\partial_m + i [A_m, \ ])$ (together with its corresponding field strength $F_{mn} = \partial_m A_n - \partial_n A_m + i [A_m, A_n]$) and the
trace $\text{Tr}$ in the adjoint representation, which we normalize as follows: $\text{Tr}(T^a T^b) = \delta^{ab}$, being $T^a$, $a = 1, \ldots, \dim(G)$, the hermitian generators of the gauge group in the adjoint representation. The action (2.2) is invariant under the following four supersymmetries (in $SU(4)_I$ covariant notation):

$$
\begin{align*}
\delta A_{\alpha \dot{\alpha}} &= -2i \bar{\xi}^u_{\dot{\alpha}} \lambda_u \alpha + 2i \bar{\lambda}^u_{\dot{\alpha}} \xi_u \alpha, \\
\delta \lambda_u &= -i F^+_{\alpha \beta} \xi_u \beta + i \sqrt{2} \xi^v \nabla_{\alpha \dot{\alpha}} \phi_{vu} - i \xi_u \alpha [\phi_{uv}, \phi^{vw}], \\
\delta \phi_{uv} &= \sqrt{2} \{ \xi^u_{\alpha} \lambda_v \alpha - \xi^v_{\alpha} \lambda_u \alpha + \epsilon_{uvwz} \bar{\xi}^w_{\dot{\alpha}} \bar{\lambda}^z_{\dot{\alpha}} \},
\end{align*}
$$

(2.3)

where $F^+_{\alpha \beta} = \sigma^{mn}_{\alpha \beta} F_{mn}$. Notice that there are no auxiliary fields in the action (2.2). Correspondingly, the transformations (2.3) close the supersymmetry algebra on-shell.

As already discussed in the introduction, in $\mathbb{R}^4$, the global symmetry group of $N = 4$ supersymmetric theories is $\mathcal{H} = SU(2)_L \otimes SU(2)_R \otimes SU(4)_I$, where $\mathcal{K} = SU(2)_L \otimes SU(2)_R$ is the rotation group $SO(4)$. The supersymmetry generators responsible for the transformations (2.3) are $Q^u_{\alpha}$ and $\bar{Q}_{u\dot{\alpha}}$. They transform as $(2, 1, \bar{4}) \oplus (1, 2, 4)$ under $\mathcal{H}$.

From the point of view of $N = 1$ superspace, the theory contains one $N = 1$ vector multiplet and three $N = 1$ chiral multiplets. These supermultiplets are represented in $N = 1$ superspace by superfields $V$ and $\Phi_s$ ($s = 1, 2, 3$), which satisfy the constraints $V = V^\dagger$ and $\bar{D}_{\dot{\alpha}} \Phi_s = 0$, being $\bar{D}_{\dot{\alpha}}$ a superspace covariant derivative. The physical component fields of these superfields are:

$$
\begin{align*}
V &\longrightarrow A_{\alpha \dot{\alpha}}, \lambda_4_{\alpha}, \bar{\lambda}^4_{\dot{\alpha}}, \\
\Phi_s, \Phi^\dagger_s &\longrightarrow B_{s}, \lambda_{s\alpha}, \bar{B}^\dagger_{s\dot{\alpha}}.
\end{align*}
$$

(2.4)

In terms of these fields, the $SU(4)_I$ tensors that we introduced above are defined as follows:

$$
\begin{align*}
\{4\} &\rightarrow \lambda_u = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}, \\
\{6\} &\rightarrow \Phi_{uv} \sim \{B_s, B^\dagger_s\}, \\
\{\bar{4}\} &\rightarrow \bar{\lambda}^u = \{\bar{\lambda}^1, \bar{\lambda}^2, \bar{\lambda}^3, \bar{\lambda}^4\},
\end{align*}
$$

(2.5)
where by \sim we mean precisely:

\[
\begin{pmatrix}
0 & -B^{\dagger} & B^{\dagger} & -B \\
B^{\dagger} & 0 & -B^{\dagger} & -B \\
-B^{\dagger} & B^{\dagger} & 0 & -B \\
B & B & B & 0
\end{pmatrix},
\begin{pmatrix}
0 & B & -B & B^{\dagger} \\
-B & 0 & B & B^{\dagger} \\
B^{\dagger} & -B & 0 & B^{\dagger} \\
-B^{\dagger} & B^{\dagger} & -B^{\dagger} & 0
\end{pmatrix}.
\]

The action (2.2) takes the following form in \( N = 1 \) superspace:

\[
S = -\frac{i}{4\pi} \int d^4x d^2\theta \text{Tr}(W^2) + \frac{i}{4\pi} \int d^4x d^2\theta \text{Tr}(W^{\dagger 2})
\]

\[
+ \frac{1}{e^2} \sum_{s=1}^{3} \int d^4x d^2\theta d^2\bar{\theta} \text{Tr}(\Phi^{\dagger s} e^V \Phi_s)
\]

\[
+ \frac{i\sqrt{2}}{e^2} \int d^4x d^2\theta \text{Tr}\{\Phi_1[\Phi_2, \Phi_3]\} + \frac{i\sqrt{2}}{e^2} \int d^4x d^2\bar{\theta} \text{Tr}\{\Phi^{\dagger 1}[\Phi^{\dagger 2}, \Phi^{\dagger 3}]\},
\]

where \( W_\alpha = -\frac{1}{16} \bar{D}^2 e^{-V} D_\alpha e^V \).

2.2. Twisting \( N = 4 \) Supersymmetry Gauge Theory

The purpose of this section is to analyze in detail the twists of \( N = 4 \) supersymmetric gauge theory. We assume that the reader is familiar with the analogous (yet simpler) procedure in \( N = 2 \) theories [1,20,19,21,22]. The aim of the twist is to extract from the supersymmetries of the theory under consideration one (or several) scalar BRST-like symmetries which can be readily generalized to any arbitrary four manifold. To create a scalar supercharge out of spinor supercharges one has to modify somehow the rotation group. The idea, as discussed in the introduction, is to replace one of the \( SU(2) \) components of the rotation group \( K \) by its diagonal sum with an \( SU(2) \) subgroup of the isospin group \( SU(4)_I \). Depending on how we choose this subgroup, we will obtain different theories after the twisting. The possible choices are found just analyzing how the 4 of \( SU(4)_I \) splits in terms of representations of the rotation group \( K \). There are just three possibilities for a given choice of the \( SU(2) \) component of \( K \): (1) 4 \( \rightarrow \) (2, 1) \( \oplus \) (2, 1), (2)
$4 \rightarrow (2, 1) \oplus (1, 1) \oplus (1, 1)$ and (3) $4 \rightarrow (2, 2)$, each of which leads to a different topological quantum field theory. Choosing the other $SU(2)$ component of $K$ one would obtain the other three $\tilde{T}$ twists: $4 \rightarrow (1, 2) \oplus (1, 2)$, $4 \rightarrow (1, 2) \oplus (1, 1) \oplus (1, 1)$ and $4 \rightarrow (2, 2)$. As described in the introduction all these twists are related to the other ones after a reversal of orientation of the four-manifold $X$. Notice that in the third case both twists, $T$ and $\tilde{T}$, involve the same splitting of the 4 of $SU(4)_I$, anticipating its amphicheiral character.

(1) $4 \rightarrow (2, 1) \oplus (2, 1)$ Vafa-Witten Theory

This is the twist that has been considered by Vafa and Witten in [5]. After the twisting, the symmetry group of the theory becomes $\mathcal{H}' = SU(2)_L' \otimes SU(2)_R \otimes SU(2)_F$, where $SU(2)_F$ is a subgroup of $SU(4)_I$ that commutes with the defining identification $4 \rightarrow (2, 1) \oplus (2, 1)$ and remains in the theory as a residual isospin group. Under $\mathcal{H}'$, the supercharges split up as,

$$Q^v_\alpha \rightarrow Q^i_\alpha, \quad Q^i_{\alpha\beta}, \quad \bar{Q}^{\dot{v}}_\dot{\alpha} \rightarrow \bar{Q}^{\dot{i}}_\dot{\alpha},$$

(2.8)

where the index $i$ labels the fundamental representation of $SU(2)_F$. The twist has produced a scalar supercharge, the $SU(2)_F$ doublet $Q^i$. This scalar charge is defined in terms of the original supercharges as follows:

$$Q^i_{v=1} \equiv Q^v_{\alpha=1} + Q^v_{\alpha=2},$$

$$Q^i_{v=2} \equiv Q^v_{\alpha=3} + Q^v_{\alpha=4},$$

(2.9)

The fields of the $N = 4$ multiplet decompose under $\mathcal{H}'$ in the following manner:

$$A_{\alpha\dot{\alpha}} \rightarrow A_{\alpha\dot{\alpha}},$$

$$\lambda_{v\alpha} \rightarrow \chi_{ij\alpha}, \quad \eta_i,$$

$$\bar{\lambda}^{v}_{\dot{\alpha}} \rightarrow \psi^{i\alpha}_{\dot{\alpha}},$$

$$\phi_{uv} \rightarrow \varphi_{ij}, \quad G_{\alpha\beta}.$$  

(2.10)

Notice that the fields $\chi^i_{\alpha\beta}$ and $G_{\alpha\beta}$ are symmetric in their spinor indices and therefore can be regarded as components of self-dual two-forms. $\varphi_{ij}$ is also symmetric.
in its isospin indices and thus transforms in the representation $3$ of $SU(2)_F$. Some of the definitions in (2.10) need clarification. Our choices for the anticommuting fields are,

$$
\chi_{i=1(\alpha\beta)} = \begin{cases} 
\chi_{i=1(11)} = \lambda_{v=1,\alpha=1}, \\
\chi_{i=1(12)} = \frac{1}{2}(\lambda_{v=1,\alpha=2} + \lambda_{v=2,\alpha=1}), \\
\vdots 
\end{cases} 
$$

(2.11)

while for the scalars $\phi_{uv}$:

$$
\varphi_{ij} = \begin{pmatrix} 
\phi_{12} & \frac{1}{2}(\phi_{14} - \phi_{23}) \\
\frac{1}{2}(\phi_{14} - \phi_{23}) & \phi_{34}
\end{pmatrix},
$$

$$
G_{\alpha\beta} = \begin{pmatrix} 
\phi_{13} & \frac{1}{2}(\phi_{14} + \phi_{23}) \\
\frac{1}{2}(\phi_{14} + \phi_{23}) & \phi_{24}
\end{pmatrix}.
$$

(2.12)

In terms of the twisted fields, the $N = 4$ action (2.2) takes the form (remember that we are still on flat $\mathbb{R}^4$):

$$
S^{(0)} = \frac{1}{e^2} \int d^4x \text{Tr} \left\{ \frac{1}{4} \nabla_{\dot{\alpha}\dot{\beta}} \varphi_{ij} \nabla^{\dot{\alpha}\dot{\beta}} \varphi^{ij} - \frac{1}{4} \nabla_{\alpha\dot{\beta}} G_{\beta\gamma} \nabla^{\dot{\alpha}\alpha} G^{\beta\gamma} - i \psi_{\dot{\alpha}}^{i\alpha} \nabla^{\dot{\alpha} \alpha} \chi_{j\alpha\beta} \\
- \frac{i}{2} \psi_{\dot{\alpha}}^{i\alpha} \nabla^{\dot{\alpha}} \eta_{j} - \frac{1}{4} F_{mn} F^{mn} - \frac{i}{\sqrt{2}} \chi^{i\alpha\dot{\beta}} [\chi_{j\alpha\beta}, \varphi^{ij}] + \frac{i}{\sqrt{2}} \chi^{i\alpha \dot{\beta}} [\chi_{i\alpha\gamma}, G^{\beta\gamma}] \\
- \frac{i}{\sqrt{2}} \chi^{i\alpha \beta} [\eta_{j}, G^{\alpha\beta}] - \frac{i}{2\sqrt{2}} \eta_{j} [\eta_{j}, \varphi^{ij}] + \frac{i}{\sqrt{2}} \psi_{\dot{\alpha}}^{i\alpha} [\psi_{\dot{\alpha}}^{j\alpha}, G_{\alpha\beta}] \\
- \frac{i}{\sqrt{2}} \psi_{\dot{\alpha}}^{i\alpha} [\psi_{\dot{\alpha}}^{j\alpha}, \varphi_{ij}] + \frac{1}{4} [\varphi_{ij}, \varphi_{kl}][\varphi^{ij}, \varphi^{kl}] - \frac{1}{2} [\varphi_{ij}, G_{\alpha\beta}][\varphi^{ij}, G^{\alpha\beta}] \\
+ \frac{1}{4} [G_{\alpha\beta}, G_{\gamma\delta}][G^{\alpha\beta}, G^{\gamma\delta}] \right\} - \frac{i\theta}{32\pi^2} \int d^4x \text{Tr} \left\{ *F_{mn} F^{mn} \right\}.
$$

(2.13)

The $Q^{j}$-transformations of the twisted theory can be readily obtained from the corresponding $N = 4$ supersymmetry transformations. These last transformations are generated by $\xi_{\alpha}^{v} Q_{\alpha}^{v} + \xi_{\dot{\alpha}}^{v} \bar{Q}_{\dot{\alpha}}^{v}$. According to our conventions, to obtain the
where $C_{\beta\alpha}$ (or $C_{\dot{\beta}\dot{\alpha}}$, $C_{ij}$) is the antisymmetric (invariant) tensor of $SU(2)$ with the convention $C_{21} = C^{12} = +1$. The resulting transformations are:

\[
\begin{align*}
\delta A_{\alpha \dot{\alpha}} &= 2i\epsilon_j \psi^j_{\alpha \dot{\alpha}}, \\
\delta F^+_{\alpha \beta} &= 2\epsilon_i \nabla_\alpha \dot{\psi}^\beta_{\dot{\alpha}}, \\
\delta \psi^{i\alpha}_{\dot{\alpha}} &= -i\sqrt{2}\epsilon_j \nabla_\alpha \dot{\varphi}^j_{\dot{\alpha}} + i\sqrt{2}\epsilon_i \nabla_\beta \dot{G}^{\beta\alpha}, \\
\delta \chi_{i\alpha\dot{\beta}} &= -i\epsilon_i F^+_{\alpha \beta} - i\epsilon_i [G_{\gamma \alpha}, G^\gamma_{\dot{\beta}}] - 2i\epsilon_j [G_{\alpha \beta}, \varphi^j_i], \\
\delta \psi_i &= 2i\epsilon_k [\varphi_{ij}, \varphi^{jk}], \\
\delta \varphi_{ij} &= \sqrt{2}\epsilon_i \eta_j, \\
\delta G_{\alpha \beta} &= \sqrt{2}\epsilon_i \chi_{i\alpha\dot{\beta}},
\end{align*}
\]

where, for example, $\epsilon_i (\eta_{ij}) = \frac{1}{2}(\epsilon_i \eta_j + \epsilon_j \eta_i)$. The transformations (2.15) satisfy the on-shell algebra $[\delta_1, \delta_2] = 0$ modulo a non-abelian gauge transformation generated by the scalars $\varphi_{ij}$. For example, $[\delta_1, \delta_2] G_{\alpha \beta} = -4\sqrt{2}\epsilon_i \epsilon^j [\varphi_{ij}, G_{\alpha \beta}]$. In checking the algebra, use has to be made of the equations of motion for the anticommuting fields $\psi^i_{\alpha \dot{\alpha}}$ and $\chi^i_{\alpha\dot{\beta}}$. In terms of the generators $Q^i$, the algebra takes the form:

\[
\begin{align*}
\{Q^1, Q^1\} &= \delta_g(\varphi_{22}), \\
\{Q^1, Q^2\} &= \delta_g(\varphi_{12}), \\
\{Q^2, Q^2\} &= \delta_g(\varphi_{11}),
\end{align*}
\]

where by $\delta_g(\Phi)$ we denote the non-abelian gauge transformation generated by, say, $\Phi$. As explained in [15], it is possible to realize the algebra off-shell by inserting the auxiliary fields $N_{\alpha \beta}$ (symmetric in its spinor indices) and $M_{\alpha \dot{\alpha}}$ in the transformations of $\psi^i_{\alpha \dot{\alpha}}$ and $\chi^i_{\alpha\dot{\beta}}$. This is the opposite to the situation one encounters in the associated physical $N = 4$ theory, where an off-shell formulation in terms
of unconstrained fields is not possible. After some suitable manipulations [23], the off-shell formulation of the twisted theory takes the form:

\[
S^{(1)} = \frac{1}{e^2} \int d^4 x \operatorname{Tr} \left\{ \frac{1}{4} \nabla_{a\dot{a}} \psi_{ij} \nabla^{\dot{a}a} \varphi^{ij} + \frac{i}{\sqrt{2}} M^{a\dot{a}} \nabla_{\alpha\dot{a}} G^\beta_{\alpha} - i \psi^{j\beta}_{\dot{a}} \nabla^{\dot{a}a} \chi_{j\alpha\beta} \\
- \frac{i}{2} \psi_{a\dot{a}} \nabla^{\dot{a}a} \eta_j + \frac{i}{2} N^{a\dot{a}} \nabla^{\dot{a}a} F^+_{a\beta} - \frac{i}{\sqrt{2}} \chi^i_{\alpha} \nabla^{\dot{a}a} [\chi_{j\alpha\beta}, \varphi^{ij}] + \frac{i}{\sqrt{2}} \chi^i_{\alpha} \nabla^{\dot{a}a} \chi_{j\alpha\gamma} G^\beta_{\gamma} \\
- \frac{i}{\sqrt{2}} \psi^{j\alpha}_{a\dot{a}} \nabla^{\dot{a}a} \eta_j + \frac{i}{2} \eta_j [\nabla^{\dot{a}a} \varphi^{ij}] + \frac{i}{\sqrt{2}} \psi^{i}_{a\dot{a}} \nabla^{\dot{a}a} [\psi^{j\beta}_{\dot{a}}, G^\alpha_{\beta}] - \frac{1}{4} M_{a\dot{a}} M^{a\dot{a}} \\
- \frac{i}{\sqrt{2}} \psi^{j\alpha}_{a\dot{a}} \nabla^{\dot{a}a} \varphi_{ij} + \frac{i}{4} [\varphi_{ij}, \varphi_{kl}] [\varphi^{ij}, \varphi^{kl}] - \frac{1}{2} [\varphi_{ij}, G_{a\beta}] [\varphi^{ij}, G^\alpha_{\beta}] \\
+ \frac{i}{4} N^{a\dot{a}} N^{a\dot{a}} + \frac{i}{2} N^{a\dot{a}} [G^\alpha_{\gamma} G^\beta_{\gamma}] \right\} - 2\pi i \tau \frac{1}{32\pi^2} \int d^4 x \operatorname{Tr} \left\{ F_{mn}^{*mn} \right\}.
\]

(2.17)

The corresponding off-shell transformations are:

\[
\begin{align*}
\delta A_{a\dot{a}} &= 2i \epsilon_j \psi^{j}_{a\dot{a}}, \\
\delta F^+_{a\beta} &= 2\epsilon_i \nabla^{(a} \psi^{b)}_{\beta}, \\
\delta \psi_{a\dot{a}} &= -i \sqrt{2} \epsilon_j \nabla^{a\dot{a}} \varphi^{ji} + \epsilon_i M'_{a\dot{a}}, \\
\delta \chi_{i\alpha\beta} &= -2i \epsilon_j [G^\alpha_{\beta}, \varphi^{ji}] + \epsilon_i N'_{a\beta}, \\
\delta \eta_i &= 2i \epsilon_j [\varphi_{ik}, \varphi^{jk}], \\
\delta \varphi_{ij} &= \sqrt{2} \epsilon (i \eta_j), \\
\delta G_{a\beta} &= \sqrt{2} \epsilon \chi_{j\alpha\beta}, \\
\delta M'_{a\dot{a}} &= \epsilon_i \left\{ -i \nabla_{a\dot{a}} \eta_i + 2\sqrt{2} i [\psi^{j}_{a\dot{a}}, \varphi_{ij}] \right\}, \\
\delta N'_{a\beta} &= \epsilon_i \left\{ \sqrt{2} i [\eta_i, G_{a\beta}] - 2\sqrt{2} i [\chi_{j\alpha\beta}, \varphi^{ji}] \right\}.
\end{align*}
\]

(2.18)

With the aid of the transformations (2.18) it is easy (but rather lengthy) to show that the action (2.17) can be written as a double \(Q\)-commutator plus a \(\tau\)-dependent term, that is,

\[
\epsilon^2 S^{(1)} = \delta^2 \Lambda - \epsilon^2 2\pi i k \tau = -\frac{1}{2} \epsilon^2 \{Q^i, [Q_i, \Lambda]\} - \epsilon^2 2\pi i k \tau,
\]

(2.19)
(here $\delta \equiv e^i[Q_i,]$), with

$$\Lambda = \frac{1}{e^2} \int d^4x \left\{ \frac{i}{2\sqrt{2}} F_{\alpha\beta}^{\alpha\beta} + \frac{1}{4\sqrt{2}} N_{\alpha\beta} G_{\alpha\beta} + \frac{1}{8} \psi_{j\alpha\dot{\alpha}} \psi^{j\dot{\alpha}} + \frac{i}{6\sqrt{2}} G_{\alpha\beta} [G_{\beta\gamma}, G_{\gamma\alpha}] - \frac{i}{12\sqrt{2}} \varphi_{ij} [\varphi^j_k, \varphi^k_i] \right\}. \quad (2.20)$$

The next step is to couple the theory to an arbitrary background metric $g_{\mu\nu}$ of Euclidean signature. This can be done as follows: first, covariantize the expression (2.20) and the transformations (2.18); second, define the new action to be $\delta^2 \Lambda_{cov}$. The resulting action is:

$$S_c^{(1)} = \frac{1}{e^2} \int d^4x \sqrt{g} \text{Tr} \left\{ \frac{1}{4} \nabla_{\alpha\dot{\alpha}} \varphi_{ij} \nabla^\dot{\alpha} \psi^{ij} + \frac{i}{\sqrt{2}} M^{\dot{\alpha} \alpha} D_{\beta \dot{\alpha}} G_{\beta \alpha} - i \psi^{ij}_\alpha D^{\dot{\alpha} \alpha} \chi_{j\alpha\beta} - \frac{i}{2} \psi_{j\alpha\dot{\alpha}} \nabla^{\dot{\alpha}} \eta_j + \frac{i}{2} N^{\alpha\beta} F_{\alpha\beta} + \frac{i}{\sqrt{2}} \chi_i^{\alpha\beta} [\chi_{j\alpha\beta}, \varphi^{ij}] + \frac{i}{\sqrt{2}} \chi^{ij}_\alpha \chi_{j\alpha\beta} - \frac{i}{\sqrt{2}} \chi^{j\alpha}_\beta [\eta_j, G^{\alpha\beta}] - \frac{i}{\sqrt{2}} \chi^{ij}_\alpha [\psi^{j\dot{\alpha}}, \varphi_{ij}] + \frac{1}{4} [\varphi_{ij}, \varphi_{kl}] [\varphi^{ij}, \varphi^{kl}] - \frac{i}{4} [\varphi_{ij}, G^{\alpha\beta}] \right\} - 2\pi i \tau \frac{1}{32\pi^2} \int d^4x \sqrt{g} \text{Tr} \left\{ *F_{\mu\nu} F^{\mu\nu} \right\}, \quad (2.21)$$

where we have introduced the full covariant derivative $D_{\alpha\dot{\alpha}}$. The action (2.21) is, by construction, invariant under the appropriate covariantized version of the transformations (2.18).

A sensible action for a so-called cohomological topological quantum field theory is expected to meet two basic requirements. First of all, it should be real, since we will eventually interpret it as a real differential form defined on a certain moduli space. Likewise, it must display a non-trivial ghost number symmetry which, from the geometrical viewpoint, corresponds to the de Rham grading on the moduli space. These requirements are not fulfilled by the action (2.21). First of all, it is not manifestly real because it contains fields in the fundamental representation of
$SU(2)_F$, which are complex. Second, it is not possible to assign a non-trivial ghost number to the fields in (2.21).

We solve these problems by breaking the $SU(2)_F$ internal symmetry group of the theory down to its $T_3$ subgroup. This allows to introduce a non-anomalous ghost number in the theory (basically twice the corresponding charge under $T_3$). With respect to this ghost number, the field content of the theory can be reorganized as follows (in the notation of reference [5]): with ghost number $+2$, we have the scalar field $\phi \equiv \varphi_{11}$; with ghost number $+1$, the anticommuting fields $\psi_{a\dot{a}} \equiv i\psi_{1a\dot{a}}$, $\tilde{\psi}_{a\dot{a}} \equiv \chi_{1a\dot{a}}$ and $\zeta \equiv i\eta_1$; with ghost number 0, the gauge connection $A_{a\dot{a}}$, the scalar field $C \equiv i\varphi_{12}$, the self-dual two-form $B_{a\beta} \equiv G_{a\beta}$ and the auxiliary fields $H_{a\beta} \equiv iN_{a\beta}$ and $\tilde{H}_{a\dot{a}} \equiv M_{a\dot{a}}$; with ghost number $-1$, the anticommuting fields $\chi_{a\beta} \equiv i\chi_{2a\beta}$, $\tilde{\chi}_{a\dot{a}} \equiv \psi_{2a\dot{a}}$ and $\eta \equiv \eta_2$; and finally, with ghost number $-2$, the scalar field $\bar{\phi} \equiv \varphi_{22}$. Notice that now we can consistently assume that all the fields above are real, in order to guarantee the reality of the topological action.

In terms of these new fields, and after making the shifts:

\begin{equation}
\begin{align*}
\tilde{H}'_{a\dot{a}} &= \tilde{H}_{a\dot{a}} + \sqrt{2} \nabla_{a\dot{a}} C, \\
H'_{a\beta} &= H_{a\beta} + 2i[B_{a\beta}, C],
\end{align*}
\end{equation}

the action (2.21) takes the form:
\[ S^{(2)}_c = \frac{1}{e^2} \int d^4 x \sqrt{g} \text{Tr}\{ \frac{1}{2} D_{\alpha\dot{\alpha}} \bar{\phi} D^{\alpha\dot{\alpha}} \phi - \frac{1}{4} \bar{H}^{\alpha\dot{\alpha}} (\bar{H}'_{\alpha\dot{\alpha}} - 2\sqrt{2} D_{\alpha\dot{\alpha}} C - 2\sqrt{2} i D_{\beta\dot{\alpha}} B^\beta_{\alpha}) \]

\[ - \frac{1}{4} H^{\alpha\beta} (H'_{\alpha\beta} - 2 F^{\alpha\beta}_{\alpha\beta} - 2 \{B_{\gamma\alpha}, B_\beta^{\gamma}\} - 4i \{B_{\alpha\beta}, C\}) - i \bar{\psi}^\beta_{\dot{\alpha}} D^{\dot{\alpha}\alpha} \chi_{\alpha\beta} \]

\[ - i \bar{\chi}^\beta_{\dot{\alpha}} D^{\dot{\alpha}\alpha} \bar{\psi}_{\alpha\beta} - \frac{1}{2} \bar{\chi}_{\alpha\dot{\alpha}} D^{\dot{\alpha}\alpha} \zeta + \frac{1}{2} \bar{\psi}_{\alpha\dot{\alpha}} D^{\dot{\alpha}\alpha} \eta - \frac{i}{\sqrt{2}} \bar{\psi}^\beta_{\alpha\beta} [\bar{\psi}_{\alpha\beta}, \bar{\phi}] \]

\[ + \frac{i}{\sqrt{2}} \chi^{\alpha\beta} [\chi_{\alpha\beta}, \phi] - i \sqrt{2} \bar{\psi}^\alpha_{\alpha\beta} [\chi_{\alpha\beta}, C] - \sqrt{2} \bar{\psi}^\alpha_{\alpha\beta} [\chi_{\alpha\gamma}, B_\beta^{\gamma}] \]

\[ + \frac{i}{\sqrt{2}} \chi_{\alpha\beta} [\zeta, B^\alpha_{\beta}] + \frac{i}{\sqrt{2}} \bar{\psi}^\alpha_{\alpha\beta} [\eta, B^\alpha_{\beta}] + \frac{i}{2\sqrt{2}} \zeta [\zeta, \bar{\phi}] - \frac{i}{2\sqrt{2}} \eta [\eta, \phi] \]

\[ - \frac{i}{\sqrt{2}} [\eta, C] + \sqrt{2} \bar{\psi}^\alpha_{\alpha\beta} [\bar{\chi}^\dot{\alpha}_{\dot{\gamma}} B^\alpha_{\beta}] - \frac{i}{\sqrt{2}} \bar{\chi}_{\alpha\dot{\alpha}} [\bar{\chi}^\alpha_{\dot{\alpha}}, \phi] \]

\[ + \frac{i}{\sqrt{2}} \bar{\psi}^\alpha_{\alpha\beta} [\bar{\psi}^\alpha_{\beta\dot{\alpha}}, \phi] - i \sqrt{2} \bar{\psi}^\alpha_{\alpha\beta} [\bar{\chi}_{\alpha\dot{\alpha}}, C] - \frac{1}{2} [\phi, \bar{\phi}]^2 + 2 [\phi, C][\bar{\phi}, C] \]

\[- [\phi, B_{\alpha\beta}][\bar{\phi}, B^{\alpha\beta}] \} - 2\pi i \tau \frac{1}{32\pi^2} \int_X d^4 x \sqrt{g} \text{Tr}\{ *F_{\mu\nu} F^{\mu\nu} \}. \]

The analysis of the bosonic part of a topological action is of great importance. Apart from comparing to the corresponding theory on flat \( \mathbb{R}^4 \) (and possibly unveiling some non-minimal curvature couplings which are exclusive of the theory on general four-manifolds), it enables us to search for vanishing theorems that can be used to constrain the space on which the path integral localizes when passing to the weak coupling limit. After integrating out the auxiliary fields in (2.23) we find for the bosonic part of the action not involving the scalars \( \phi \) and \( \bar{\phi} \) the following expression:

\[ \int_X d^4 x \sqrt{g} \text{Tr}\{ \frac{1}{2} (D_{\alpha\alpha} C + i D_{\beta\alpha} B^\beta_{\alpha})^2 \]

\[ + \frac{1}{4} (F^{\alpha\beta}_{\alpha\beta} + [B_{\gamma\alpha}, B_\beta^{\gamma}] + 2i \{B_{\alpha\beta}, C\})^2 \}. \]

Expanding the squares in this expression one obtains,
\[ \int d^4 x \sqrt{g} \text{Tr} \left\{ -\frac{1}{2} D_\mu C D^\mu C - \frac{1}{2} (D_\beta \alpha B_\beta^\alpha D_\gamma \alpha B_\gamma^\alpha - F^{+\alpha\beta} [B_\gamma^\alpha, B_\beta^\gamma]) \right\} \]
\[ - \frac{1}{2} F^+_{\mu\nu} F^{+\mu\nu} + [B^+_{\mu\nu}, B^+_{\tau\lambda}] [B^{+\mu\nu}, B^{+\tau\lambda}] + 2 [B^+_{\mu\nu}, C] [B^{+\mu\nu}, C] \}
\]
\[ (2.25) \]

where we have used $D_\alpha \dot{\alpha} = \sigma^m_{\alpha\dot{\alpha}} D_m$ and $F^+_{\alpha\beta} \equiv \sigma_{\alpha\beta} F^+_{\mu\nu}$, $B_{\alpha\beta} \equiv \sigma_{\alpha\beta} B^+_{\mu\nu}$. Let us now focus on the expression inside the parenthesis. Further expansion leads to the identity:
\[ \int d^4 x \sqrt{g} \text{Tr} \left\{ -\frac{1}{2} D_\beta \alpha B_\beta^\alpha D_\gamma \alpha B_\gamma^\alpha + \frac{1}{2} F^{+\alpha\beta} [B_\gamma^\alpha, B_\beta^\gamma] \right\} = \]
\[ \int d^4 x \sqrt{g} \text{Tr} \left\{ -D_\mu B^+_{\nu\lambda} D^\mu B^{+\nu\lambda} - \frac{1}{2} R B^+_{\mu\nu} B^{+\mu\nu} + R_{\mu\nu\tau\lambda} B^{+\mu\nu} B^{+\tau\lambda} \right\}, \]

(2.26)

(using again $B_{\alpha\beta} \equiv \sigma_{\alpha\beta} B^+_{\mu\nu}$). If we now express the Riemann tensor in (2.26) in terms of its irreducible components \[24\],
\[ R_{\mu\nu\tau\lambda} = \frac{1}{2} (g_{\mu\tau} R_{\nu\lambda} - g_{\mu\lambda} R_{\nu\tau} - g_{\nu\tau} R_{\mu\lambda} + g_{\nu\lambda} R_{\mu\tau}) \]
\[ - \frac{R}{6} (g_{\mu\tau} g_{\nu\lambda} - g_{\nu\tau} g_{\mu\lambda}) + C_{\mu\nu\tau\lambda}, \]

with $C_{\mu\nu\tau\lambda}$ the Weyl tensor, we finally obtain,
\[ \int d^4 x \sqrt{g} \text{Tr} \left\{ -\frac{1}{2} D_\beta \alpha B_\beta^\alpha D_\gamma \alpha B_\gamma^\alpha + \frac{1}{2} F^{+\alpha\beta} [B_\gamma^\alpha, B_\beta^\gamma] \right\} = \]
\[ \int d^4 x \sqrt{g} \text{Tr} \left\{ -D_\mu B^+_{\nu\lambda} D^\mu B^{+\nu\lambda} - B^{+\mu\nu} \left( \frac{1}{6} R (g_{\mu\tau} g_{\nu\lambda} - g_{\nu\tau} g_{\mu\lambda}) - C_{\mu\nu\tau\lambda} \right) B^{+\tau\lambda} \right\}. \]

(2.28)

Notice that all the terms in (2.24) are negative definite except the terms contained in this last equation which involve the scalar curvature and the Weyl tensor.

The associated fermionic symmetry splits up as well into BRST ($Q^+ \equiv Q^1$)
and anti-BRST \((Q^{-} \equiv iQ^{2})\) parts. The explicit formulas are:

\[
[Q^{+}, A_{a\dot{a}}] = -2\psi_{a\dot{a}}, \quad [Q^{-}, A_{a\dot{a}}] = -2\chi_{a\dot{a}},
\]
\[
\{Q^{+}, \psi_{a\dot{a}}\} = -\sqrt{2}D_{a\dot{a}}\phi, \quad \{Q^{-}, \chi_{a\dot{a}}\} = \sqrt{2}D_{a\dot{a}}\bar{\phi},
\]
\[
[Q^{+}, \phi] = 0, \quad [Q^{-}, \bar{\phi}] = 0,
\]
\[
[Q^{+}, B_{\alpha\beta}] = \sqrt{2}\bar{\psi}_{\alpha\beta}, \quad [Q^{-}, B_{\alpha\beta}] = -\sqrt{2}\chi_{\alpha\beta},
\]
\[
\{Q^{+}, \bar{\psi}_{\alpha\beta}\} = 2i[B_{\alpha\beta}, \phi], \quad \{Q^{-}, \bar{\chi}_{\alpha\beta}\} = 2i[B_{\alpha\beta}, \bar{\phi}],
\]
\[
[Q^{+}, C] = \frac{1}{\sqrt{2}}\zeta, \quad [Q^{-}, C] = -\frac{1}{\sqrt{2}}\eta,
\]
\[
\{Q^{+}, \zeta\} = 4i[C, \phi], \quad \{Q^{-}, \eta\} = 4i[C, \bar{\phi}],
\]
\[
[Q^{+}, \bar{\phi}] = \sqrt{2}\eta, \quad [Q^{-}, \bar{\phi}] = \sqrt{2}\zeta,
\]
\[
\{Q^{+}, \eta\} = 2i[\bar{\phi}, \phi], \quad \{Q^{-}, \zeta\} = -2i[\phi, \bar{\phi}],
\]
\[
\{Q^{+}, \bar{\chi}_{a\dot{a}}\} = \bar{H}'_{a\dot{a}}, \quad \{Q^{-}, \psi_{a\dot{a}}\} = -\bar{H}'_{a\dot{a}} + 2\sqrt{2}D_{a\dot{a}}C,
\]
\[
[Q^{+}, \bar{H}'_{a\dot{a}}] = 2\sqrt{2}i[\bar{\chi}_{a\dot{a}}, \phi], \quad [Q^{-}, \bar{H}'_{a\dot{a}}] = -2D_{a\dot{a}}\eta + 2\sqrt{2}i[\psi_{a\dot{a}}, \bar{\phi}] - 4\sqrt{2}i[\bar{\chi}_{a\dot{a}}, C],
\]
\[
\{Q^{+}, \chi_{\alpha\beta}\} = H'_{\alpha\beta}, \quad \{Q^{-}, \bar{\psi}_{\alpha\beta}\} = H'_{\alpha\beta} - 4i[B_{\alpha\beta}, C],
\]
\[
[Q^{+}, H'_{\alpha\beta}] = 2\sqrt{2}i[\chi_{\alpha\beta}, \phi], \quad [Q^{-}, H'_{\alpha\beta}] = -2\sqrt{2}i[\bar{\psi}_{\alpha\beta}, \bar{\phi}] - 4\sqrt{2}i[\chi_{\alpha\beta}, C] - 2\sqrt{2}i[B_{\alpha\beta}, \eta],
\]

satisfying the algebra,

\[
\{Q^{+}, Q^{+}\} = \delta_{g}(\phi),
\]
\[
\{Q^{-}, Q^{-}\} = \delta_{g}(-\bar{\phi}),
\]
\[
\{Q^{+}, Q^{-}\} = \delta_{g}(C). \tag{2.30}
\]

The \(\tau\)-independent part of the action (2.23) can be written either as a BRST \((Q^{+})\) commutator or as an anti-BRST \((Q^{-})\) commutator. Let us focus on the
former possibility. The appropriate “gauge” fermion turns out to be:

\[
\Psi = \frac{1}{e^2} \int d^4 x \sqrt{g} \text{Tr}\left\{ -\frac{1}{4} \chi^{\dot{\alpha} \alpha} \left( \tilde{H}'_{\alpha \dot{\alpha}} - 2\sqrt{2} D_{\alpha \dot{\alpha}} C - 2\sqrt{2} i \mathcal{D}_{\dot{\alpha} \dot{\alpha}} B^\alpha \right) \right. \\
- \frac{1}{4} \chi^{\alpha \beta} \left( H_{\alpha \beta}^\prime - 2 F^+_{\alpha \beta} - 2 [B_{\gamma \alpha}, B_{\beta \gamma}] - 4 i [B_{\alpha \beta}, C] \right) \left. \right\} \\
+ \frac{1}{e^2} \int d^4 x \sqrt{g} \text{Tr}\left\{ \frac{1}{2\sqrt{2}} \tilde{\phi} \left( D_{\alpha \dot{\alpha}} \psi^{\dot{\alpha} \alpha} + i \sqrt{2} \left[ \psi_{\alpha \beta} B^{\alpha \beta} \right] - i \sqrt{2} [\zeta, C] \right) \right\} \\
- \frac{1}{e^2} \int d^4 x \sqrt{g} \text{Tr}\left\{ i \frac{\eta}{4} [\phi, \bar{\phi}] \right\}.
\]

(2.31)

For reasons of future convenience we will rewrite (2.31) in vector indices. With the definitions, \( X_{\alpha \dot{\alpha}} \equiv \sigma^\mu_{\alpha \dot{\alpha}} X_\mu \), and, \( Y_{\alpha \beta} \equiv \sigma_{\alpha \beta}^\mu Y_\mu \), for any two given fields \( X \) and \( Y \), (2.31) takes the form:

\[
\Psi = \frac{1}{e^2} \int d^4 x \sqrt{g} \text{Tr}\left\{ \frac{1}{2} \bar{\psi}^\mu \left( \tilde{H}'_\mu - 2\sqrt{2} \mathcal{D}_\mu C + 4\sqrt{2} D^\nu B^\nu_{\mu \nu} \right) \right. \\
+ \frac{1}{2} \chi^{\mu \nu} \left( H^\prime_{\mu \nu} - 2 F^+_{\mu \nu} - 4 i [B^+_{\mu \tau}, B^{+ \tau \nu}] - 4 i [B^+_{\mu \nu}, C] \right) \left. \right\} \\
- \frac{1}{e^2} \int d^4 x \sqrt{g} \text{Tr}\left\{ \frac{1}{2\sqrt{2}} \tilde{\phi} \left( 2 \mathcal{D}_\mu \bar{\psi}^\mu + 2\sqrt{2} i \left[ \bar{\psi}^+_{\mu \nu}, B^{+ \mu \nu} \right] + \sqrt{2} i [\zeta, C] \right) \right\} \\
- \frac{1}{e^2} \int d^4 x \sqrt{g} \text{Tr}\left\{ i \frac{\eta}{4} [\phi, \bar{\phi}] \right\}.
\]

(2.32)

The gauge fermion, in turn, can itself be written as an anti-BRST commutator (2.19):

\[
\Psi = \{ Q^-, \frac{1}{e^2} \int d^4 x \sqrt{g} \text{Tr}\left\{ -\frac{1}{2\sqrt{2}} B^\alpha \left( F^+_{\alpha \beta} - \frac{1}{2} H_{\alpha \beta}^\prime + \frac{1}{3} [B_{\alpha \gamma}, B_{\beta \gamma}] \right) \right. \\
+ \frac{i}{2\sqrt{2}} C [\phi, \bar{\phi}] + \frac{1}{4} \psi_{\alpha \dot{\alpha}} \tilde{\chi}^{\dot{\alpha} \alpha} \left. \right\}.
\]

(2.33)
As explained in [15], this amounts to a breaking of the $SU(4)_I$ isospin group down to a subgroup $SU(2)_A \otimes SU(2)_F \otimes U(1)$ and then a replacement of the $SU(2)_L$ factor of the rotation group by the diagonal sum $SU(2)'_L$ of $SU(2)_L$ and $SU(2)_A$. The subgroup $SU(2)_F \otimes U(1)$ remains in the theory as an internal symmetry group. Hence, we observe that, as a by-product of the twisting procedure, it remains in the theory a $U(1)$ symmetry which was not present in the original $N = 4$ theory, and which becomes, as we shall see in a moment, the ghost number symmetry associated to the topological theory. With respect to the new symmetry group $H' = SU(2)'_L \otimes SU(2)_R \otimes SU(2)_F \otimes U(1)$ the supercharges $Q^v_{\alpha}$ split up into three supercharges: $Q_{(\beta \alpha)} + Q \oplus Q^i_{\alpha}$, where the index $i$ labels the representation $2$ of $SU(2)_F$. In more detail,

\begin{align}
Q^v_{\alpha} = & 1, 2, 3, 4 \rightarrow \\
Q^v_{\alpha} = 1, 2 \rightarrow & Q^\beta \rightarrow \\
Q^v_{\alpha} = 3 \rightarrow & Q^i_{\alpha} = 1, \\
Q^v_{\alpha} = 4 \rightarrow & Q^i_{\alpha} = 2.
\end{align}

The conjugate supercharges $\bar{Q}_{\dot{\alpha} v}$ split up accordingly into a vector isosinglet and a right-handed spinor isodoublet supercharge, $\bar{Q}_{\dot{\alpha} v} \otimes \bar{Q}_{i \dot{\alpha}}$.

The fields of the $N = 4$ multiplet give rise, after the twisting, to the following topological multiplet (in the notation of reference [15]):

\begin{align}
A_{\alpha \dot{\alpha}} \rightarrow & A^{(0)}_{\alpha \dot{\alpha}}, \\
\lambda_{v \alpha} \rightarrow & \lambda^{(-1)}_{\beta \dot{\alpha}}, \eta^{(-1)}, \lambda^{(+1)}_{\dot{\alpha} \alpha}, \\
\bar{\lambda}^v_{\dot{\alpha}} \rightarrow & \psi^{(+1)}_{\dot{\alpha} \dot{\alpha}}, \zeta^{(-1)}_{\dot{\alpha} \dot{\alpha}}, \\
\phi_{uv} \rightarrow & B^{(-2)}, C^{(+2)}, G^{(0)}_{i \dot{\alpha}},
\end{align}

where we have indicated the ghost number carried by the fields after the twisting by a superscript. Some of the definitions in (2.35) need clarification. Our choices
for the anticommuting fields are:

\[
\lambda_{v\alpha} = \begin{cases} 
\lambda_{(v=1,2)\alpha} \rightarrow \lambda_{\beta\alpha} \\
\lambda_{(v=3,4)\alpha} \rightarrow \lambda_{i\alpha},
\end{cases} \quad \eta = 2\lambda_{[\beta\alpha]},
\]

\[
\bar{\lambda}_{\dot{v}\dot{\alpha}} = \begin{cases} 
\bar{\lambda}_{(v=1,2)\alpha} \rightarrow \psi_{\alpha\dot{\alpha}}, \\
\bar{\lambda}_{(v=3,4)\dot{\alpha}} \rightarrow \zeta_{i\dot{\alpha}}.
\end{cases}
\] (2.36)

whereas for the commuting ones:

\[
B = \phi_{12}, \quad C = \phi_{34},
\]

\[
G_{i\alpha} = \begin{cases} 
G_{(i=1)1} = \phi_{13}, \\
G_{(i=1)2} = \phi_{23},
\end{cases} \quad G_{(i=2)1} = \phi_{14}, \quad G_{(i=2)2} = \phi_{24}. \quad (2.37)
\]

In terms of the twisted fields, the action for the theory (on flat \(\mathbb{R}^4\)) takes the form:

\[
S^{(0)} = \frac{1}{e^2} \int d^4x \text{Tr} \left\{ \frac{1}{2} \nabla_{\dot{a}\dot{\alpha}} B \nabla^{\dot{a}\dot{\alpha}} C - \frac{1}{4} \nabla_{a\dot{a}} G_{i\beta} \nabla^{a\dot{\alpha}} G^{i\beta} - i\psi_{\alpha} \nabla^{\dot{a}\dot{\alpha}} \chi_{\alpha\beta} \\
- \frac{i}{2} \psi_{\alpha\dot{\alpha}} \nabla^{\dot{a}\dot{\alpha}} \eta - i\zeta_{i\dot{\alpha}} \nabla^{\dot{a}\dot{\alpha}} \lambda_{i\alpha} - \frac{1}{4} F_{mn} F^{mn} - \frac{i}{\sqrt{2}} \chi_{\alpha\beta} [\chi_{\alpha\beta}, C] \\
- \frac{i}{\sqrt{2}} \lambda^{i\alpha} [\lambda_{i\alpha}, B] + i\sqrt{2} \chi^{b\beta} [\lambda_{i\alpha}, G^{i\beta}] + \frac{i}{\sqrt{2}} \eta [\lambda_{i\alpha}, G^{i\alpha}] - \frac{i}{2\sqrt{2}} \eta [\eta, C] \\
+ \frac{i}{\sqrt{2}} \psi_{\alpha\dot{a}} [\psi^{\dot{a}\dot{\alpha}}, B] + i\sqrt{2} \psi_{\alpha\dot{a}} [\zeta^{i\dot{a}}, G_{i\alpha}] + \frac{i}{\sqrt{2}} \zeta_{i\dot{a}} [\zeta^{i\dot{a}}, C] - \frac{1}{2} [B, C]^2 \\
- [B, G_{i\alpha}] [C, G^{i\alpha}] + \frac{1}{4} [G_{i\alpha}, G_{j\beta}] [G^{i\alpha}, G^{j\beta}] \right\} - \frac{i\theta}{32\pi^2} \int d^4x \text{Tr} \left\{ *F_{mn} F^{mn} \right\}.
\] (2.38)

To obtain the corresponding topological symmetry we proceed as follows. First of all, we recall that the \(N = 4\) supersymmetry transformations (2.3) are generated by \(\xi_v^{\alpha} Q_{\beta\alpha} + \bar{\xi}_{\dot{v}\dot{\alpha}} \bar{Q}_{\dot{v}\dot{\alpha}}\). According to our conventions, to obtain the \(Q\)-transformations we must set \(\bar{\xi}_{\dot{v}\dot{\alpha}} = 0\) and make the replacement:

\[
\xi_{v\alpha} = \begin{cases} 
\xi_{(v=1,2)\alpha} \rightarrow \epsilon C_{\beta\alpha}, \\
\xi_{(v=3,4)\alpha} \rightarrow 0.
\end{cases}
\] (2.39)
The resulting transformations turn out to be:

\[
\begin{align*}
\delta A_{\alpha \dot{\alpha}} &= 2i\varepsilon \psi_{\alpha \dot{\alpha}}, \\
\delta \psi_{\alpha \dot{\alpha}} &= -i\sqrt{2}\varepsilon \nabla_{\alpha \dot{\alpha}} C, \\
\delta C &= 0, \\
\delta \chi_{\alpha \beta} &= -i\varepsilon F^+_{\alpha \beta} - i\varepsilon[G_{\alpha \dot{\alpha}}, G^\dot{\alpha}_{\beta}], \\
\delta \zeta_{\dot{\alpha}} &= -i\sqrt{2}\varepsilon \nabla_{\alpha \dot{\alpha}} G^{\dot{\alpha}}, \\
\delta G_{\alpha &} &= -\sqrt{2}\varepsilon \lambda_{\alpha}, \\
\delta \lambda_{\alpha} &= -2i\varepsilon [G_{\alpha \dot{\alpha}}, C], \\
\delta \eta &= 2i\varepsilon [B, C].
\end{align*}
\]

The BRST generator \(Q\) associated to the transformations (2.40) satisfies the on-shell algebra \(\{Q, Q\} = \delta g(C)\) where by \(\delta g(C)\) we mean a non-abelian gauge transformation generated by \(C\). It is possible to realize this algebra off-shell, i.e., without the input of the equations of motion for some of the fields in the theory. A minimal off-shell formulation can be constructed by introducing in the theory the auxiliary fields \(N_{\alpha \beta}\) (symmetric in its spinor indices) and \(P^i_{\alpha}\) (both with ghost number 0). The off-shell BRST transformations which correspond to the enlarged topological multiplet can be cast in the form:

\[
\begin{align*}
[Q, A_{\alpha \dot{\alpha}}] &= 2i\psi_{\alpha \dot{\alpha}}, & \{Q, \psi_{\alpha \dot{\alpha}}\} &= -i\sqrt{2}\nabla_{\alpha \dot{\alpha}} C, \\
[Q, F^+_{\alpha \beta}] &= 2\nabla_{(\alpha} \hat{\psi}_{\beta)\dot{\alpha}}, & [Q, C] &= 0, \\
[Q, G_{\alpha \dot{\alpha}}] &= -\sqrt{2}\varepsilon \lambda_{\alpha}, & \{Q, \lambda_{\alpha}\} &= -2i[G_{\alpha \dot{\alpha}}, C], \\
\{Q, \chi_{\alpha \beta}\} &= N_{\alpha \beta}, & [Q, N_{\alpha \beta}] &= 2\sqrt{2}\varepsilon [\chi_{\alpha \beta}, C], \\
\{Q, \zeta^j_{\dot{\alpha}}\} &= P^j_{\dot{\alpha}}, & [Q, P^j_{\dot{\alpha}}] &= 2\sqrt{2}\varepsilon [\zeta^j_{\dot{\alpha}}, C], \\
[Q, B] &= \sqrt{2}\varepsilon, & \{Q, \eta\} &= 2i[B, C].
\end{align*}
\]

After some suitable manipulations [23], the off-shell action which corresponds
to the topological symmetry (2.41) is:

$$S^{(1)} = \frac{1}{e^2} \int d^4 x \text{Tr} \left\{ \frac{1}{2} \nabla_{\alpha} B \nabla_{\dot{\alpha}} C + \frac{1}{4} P^i_{i\dot{\alpha}} \left( P^i_{i\dot{\alpha}} + 2\sqrt{2} \nabla_{\alpha} G^{i\alpha} \right) - i\psi_{\dot{\alpha}} \nabla_{\dot{\alpha}} \chi_{\alpha} \right\}$$

$$- \frac{i}{2} \psi_{\alpha \dot{\alpha}} \nabla_{\dot{\alpha}} \eta - i\chi^j \nabla_{\dot{\alpha}} \lambda^j \alpha + \frac{1}{4} N_{\alpha \beta} \left( N^{\alpha \beta} + 2i F^{\alpha \beta} + 2i [G^i_{\alpha}, G^{i\beta}] \right)$$

$$- \frac{i}{\sqrt{2}} \chi^{\alpha \beta} \left[ \chi_{\alpha \beta}, C \right] - \frac{i}{\sqrt{2}} \chi^{\alpha \beta} \left[ \lambda_{\alpha \beta}, B \right] + i\sqrt{2} \chi^{\alpha \beta} \left[ \lambda_{\alpha \beta}, G^{i\beta} \right] + \frac{i}{\sqrt{2}} \zeta_{i \dot{\alpha}} \left[ \zeta_{i \dot{\alpha}}, C \right]$$

$$+ \frac{i}{\sqrt{2}} \eta \left[ \lambda^{i \alpha}, G^i_{\alpha} \right] - \frac{i}{2\sqrt{2}} \eta \left[ \eta, C \right] + \frac{i}{\sqrt{2}} \psi_{\alpha \dot{\alpha}} \left[ \psi_{\dot{\alpha}} \alpha, B \right] + i\sqrt{2} \psi_{\dot{\alpha}} \left[ \psi_{\dot{\alpha}} \alpha, G^{i\alpha} \right]$$

$$- \frac{1}{2} [B, C]^2 - [B, G_{i\alpha}] [C, G^{i\alpha}] \right\} - 2\pi i \tau \frac{1}{32\pi^2} \int d^4 x \text{Tr} \left\{ * F_{mn} F^{mn} \right\}. \quad (2.42)$$

The $\tau$-independent part of the topological action above is, as it could be expected, BRST-exact, that is, it can be written as $\{Q, \Psi\}$. The appropriate gauge fermion is easily seen to be:

$$\Psi = \frac{1}{e^2} \int d^4 x \text{Tr} \left\{ \frac{1}{4} \psi_{\dot{\alpha}} \left( P^i_{i\dot{\alpha}} + 2\sqrt{2} \nabla_{\alpha} G^{i\alpha} \right) \right.$$

$$+ \frac{1}{4} \chi_{\alpha \beta} \left( N^{\alpha \beta} + 2i F^{\alpha \beta} + 2i [G^i_{\alpha}, G^{i\beta}] \right) \right\}$$

$$- \frac{1}{e^2} \int d^4 x \text{Tr} \left\{ \frac{i}{2\sqrt{2}} B \left( \nabla_{\alpha} \psi_{\dot{\alpha}} - \sqrt{2} [G_{i\alpha}, \lambda^{i\alpha}] \right) \right\}$$

$$- \frac{1}{e^2} \int d^4 x \text{Tr} \left\{ \frac{i}{4} B [\eta, C] \right\}. \quad (2.43)$$

The next step will consist of the coupling the theory to an arbitrary background metric $g_{\mu \nu}$ of Euclidean signature. To achieve this goal we make use of the covariantized version of the topological symmetry (2.41) (which is trivial to obtain), and of the gauge fermion $\Psi$, and then define the topological action to be $S^{(1)} = \{Q, \Psi\}_{\text{cov}} - 2\pi i k \tau$. The resulting action is:
\[ S_c^{(1)} = \frac{1}{\epsilon^2} \int d^4 x \sqrt{g} \text{Tr} \left\{ \frac{1}{2} \mathcal{D}_{\alpha\dot{\alpha}} B \mathcal{D}^{\dot{\alpha}\alpha} C + \frac{1}{4} P_i^\alpha \left( P_i^\alpha + 2\sqrt{2} i \mathcal{D}_{\alpha\dot{\alpha}} G^{i\dot{\alpha}} \right) - i\psi^\alpha_\beta \mathcal{D}^{\dot{\alpha}\alpha} \chi_{\alpha\beta} \right. \\
- \frac{i}{2} \psi_{\alpha\dot{\alpha}} \mathcal{D}^{\dot{\alpha}} \eta - i\zeta^j_\dot{\alpha} \mathcal{D}^{\dot{\alpha}} \lambda_{j\alpha} + \frac{1}{4} N_{\alpha\beta} \left( N^{\alpha\beta} + 2i F^{+\alpha\beta} + 2i [G_i^\alpha, G^i_\beta] \right) \right. \\
- \frac{i}{\sqrt{2}} \chi^{\alpha\beta} [\chi_{\alpha\beta}, C] - \frac{i}{\sqrt{2}} \chi^{\alpha\beta} [\lambda_{i\alpha}, B] + i\sqrt{2} \chi^{\alpha\beta} [\lambda_{ia}, G^i_\beta] + \frac{i}{\sqrt{2}} \zeta_{ia} [\zeta_i^{\alpha\dot{\alpha}}, C] \\
+ \frac{i}{\sqrt{2}} \eta [\lambda_i^{\alpha}, G_i^\alpha] - \frac{i}{2\sqrt{2}} \eta [\eta, C] + \frac{i}{\sqrt{2}} \psi_{\alpha\dot{\alpha}} [\psi^{\dot{\alpha}\alpha}, B] + i\sqrt{2} \psi^\alpha_\alpha [\psi^{\alpha\dot{\alpha}}, G_{i\alpha}] \\
- \frac{1}{2} [B, C]^2 - [B, G_{i\alpha}][C, G^{i\alpha}] \bigg\} - 2\pi i \tau \frac{1}{32\pi^2} \int d^4 x \text{Tr} \left\{ *F_{\mu\nu}F^{\mu\nu} \right\}. \tag{2.44} \]

Up to now we have carefully studied the “standard” formulation of the second twist, and we have been able to reproduce faithfully previously known results [15]. However, we think there are several subtleties that demand clarification. Since the twisted theory contains several spinor fields taking values in the fundamental representation of the internal SU(2)_F symmetry group, and these fields are necessarily complex, as they live in complex representations of the rotation group and of the isospin group, it can be seen that the action (2.44) is not real. Moreover, there are more fields in the twisted theory than in the physical theory. To see this, pick for example the scalar fields \( \phi_{uv} \) in the physical \( N = 4 \) theory. They are 6 real fields that after the twisting become the scalar fields \( B \) and \( C \) (which can be safely taken to be real, thus making a total of 2 real fields) and the isospin doublet bosonic spinor field \( G_{i\alpha} \), which is necessarily complex and thus is built out of \( 2 \times 2 \times 2 = 8 \) real fields. Thus we see that 6 real fields in the \( N = 4 \) theory give rise to 10 real fields in the twisted theory. With the anticommuting fields this overcounting is even worse. In what follows we will break \( SU(2)_F \) explicitly and rearrange the resulting fields wisely so as to avoid these problems. The outcome of this reformulation is that we will make contact with the non-abelian monopole theory formulated in [6,7,9]. For a thoroughful and self-contained review of these theories see [10].

We start with the fields \( G_{i\alpha} \), which we rearrange in a complex commuting two-
component Weyl spinor $M_\alpha \equiv G_{2\alpha}$ and its complex conjugate $\overline{M}_\alpha \equiv G_{1\alpha}$. The constraint $G_{1\alpha} = (G_{2\alpha})^*$ looks rather natural when considered from the viewpoint of the physical $N = 4$ theory, in terms of which –recall eqn. (2.6) and (2.37)–

$$G_{1\alpha} = \left(\phi_{13} \atop \phi_{23}\right) = \left(\frac{B^\dagger_2}{-B^\dagger_1}\right), \quad G_{2\alpha} = \left(\phi_{14} \atop \phi_{24}\right) = \left(-\frac{B_1}{-B^2}\right). \quad (2.45)$$

Similarly, for the other isodoublets in the theory we make the rearrangements:

$$\lambda_{1\alpha} = \overline{\mu}_\alpha, \quad \zeta_{1\dot{\alpha}} = \overline{\nu}_{\dot{\alpha}},$$
$$\lambda_{2\alpha} = \mu_\alpha, \quad \zeta_{2\dot{\alpha}} = \nu_{\dot{\alpha}}, \quad \lambda_1 = \overline{\mu}, \quad \zeta_1 = \overline{\nu},$$

$$P_{1\dot{\alpha}} = \overline{h}_{\dot{\alpha}}, \quad P_{2\dot{\alpha}} = h_{\dot{\alpha}}. \quad (2.46)$$

Finally, after redefining $\psi \to -i\psi, \ C \to \phi, \ B \to \lambda$ and $N_{\alpha\beta} \to H_{\alpha\beta}$ ($A$ and $\eta$ remain the same), the action (2.44) becomes:

$$S_c^{(1)} = \frac{1}{e^2} \int \frac{d^4x}{\lambda} \sqrt{g} \text{Tr} \left\{ \frac{1}{2} D_{\alpha\dot{\beta}} \phi D^{\dot{\alpha}\alpha} \lambda + \frac{1}{4} \overline{h}_{\dot{\alpha}} (h_{\dot{\alpha}} + 2\sqrt{2}i D_{\alpha\dot{\beta}} M_\alpha) \right. $$
$$- \frac{1}{4} h^{\dot{\alpha}} (h_{\dot{\alpha}} + 2\sqrt{2}i D_{\alpha\dot{\beta}} M_\alpha) - \psi^{\dot{\alpha}} \bar{\alpha} D^{\dot{\alpha}\alpha} \chi_{\alpha\beta} - \frac{1}{2} \psi_{\alpha\dot{\beta}} D^{\dot{\alpha}\alpha} \eta$$
$$- i\sqrt{2} \chi^{\alpha\beta} \eta[\chi_{\alpha\beta}, \phi] + i\sqrt{2} \bar{\mu}^{\alpha}[\mu_\alpha, \lambda]$$
$$+ i\sqrt{2} \chi^{\alpha\beta} \eta[\mu_\alpha, \phi] - i\sqrt{2} \bar{\nu}^{\alpha}[\nu_\alpha, \phi]$$
$$- \frac{i}{\sqrt{2}} \eta[\mu_\alpha, M_\alpha] + \frac{i}{\sqrt{2}} \eta[\mu_\alpha, M_\alpha] - \frac{i}{2\sqrt{2}} \eta[\eta, \phi] - \frac{i}{\sqrt{2}} \psi_{\alpha\dot{\alpha}}[\psi^{\dot{\alpha}\alpha}, \lambda]$$
$$+ \sqrt{2} \psi_{\alpha\dot{\alpha}}[\phi, M_\alpha] - \sqrt{2} \psi_{\alpha\dot{\alpha}}[\phi, M_\alpha] - \frac{1}{2}[\phi, \lambda]^2$$
$$- \frac{i}{\sqrt{2}} \eta[\mu_\alpha, M_\alpha] + \frac{i}{\sqrt{2}} \eta[\mu_\alpha, M_\alpha]$$
$$= 2\pi i \tau \frac{1}{32\pi^2} \int d^4x \text{Tr} \sqrt{g} \{ *F_{\mu\nu} F^{\mu\nu} \}. \quad (2.47)$$

Let us focus now on the bosonic part of the action not containing the scalar
fields $\phi$ and $\lambda$. After integrating out the auxiliary fields, this part reads:

$$\int d^4x \sqrt{g} Tr \left\{ -\mathcal{D}_{a\dot{\alpha}} \overline{M}^\beta \mathcal{D}_\beta M^\beta + \frac{1}{4} ( F^{+\alpha\beta} + 2 \overline{M} \gamma M) \right\}.$$  \hfill (2.48)

Expanding the squares we obtain the contributions:

$$\int d^4x \sqrt{g} Tr \left\{ -g^{\mu\nu} \mathcal{D}_\mu \overline{M}^\alpha \mathcal{D}_\nu M^\alpha - \frac{1}{4} R \overline{M}^\alpha M^\alpha - \frac{1}{2} F^{+\mu\nu} \right.$$

$$\left. + \overline{M} (\gamma M) \right\}.$$

\hfill (2.49)

In the derivation of (2.49) we have used the Weitzenböck formula,

$$\mathcal{D}_{a\dot{\alpha}} \mathcal{D}^{\dot{\alpha} \beta} = \frac{1}{2} \delta^\beta_{\dot{\alpha}} \mathcal{D}_{\gamma \dot{\alpha}} \mathcal{D}^{\gamma \dot{\alpha}} + \frac{1}{4} \delta^\beta_{\dot{\alpha}} R + F^{+\alpha\beta} T^a$$

\hfill (2.50)

being $R$ the scalar curvature and $T^a, a = 1, \ldots, \dim(G)$, the generators of the gauge group in the appropriate representation.

The corresponding BRST symmetry is readily obtained from (2.41):

$$[Q, A_{\alpha \dot{\alpha}}] = 2 \psi_{\alpha \dot{\alpha}}, \quad \{Q, \psi_{\alpha \dot{\alpha}}\} = \sqrt{2} \mathcal{D}_{\alpha \dot{\alpha}} \phi,$$

$$[Q, F_{\alpha \beta}^+] = 2 \mathcal{D}_{(\alpha \dot{\alpha})} \psi_{(\beta \dot{\alpha})}, \quad [Q, \phi] = 0,$$

$$[Q, M^\alpha] = -\sqrt{2} \mu^\alpha, \quad \{Q, \mu^\alpha\} = -2i [M^\alpha, \phi],$$

$$\{Q, \chi_{\alpha \beta}\} = H_{\alpha \beta}, \quad [Q, H_{\alpha \beta}] = 2\sqrt{2}i [\chi_{\alpha \beta}, \phi],$$

$$\{Q, \nu_{\dot{\alpha}}\} = h_{\dot{\alpha}}, \quad [Q, h_{\dot{\alpha}}] = 2\sqrt{2}i [\nu_{\dot{\alpha}}, \phi],$$

$$[Q, \lambda] = \sqrt{2} \eta, \quad \{Q, \eta\} = 2i [\lambda, \phi].$$

\hfill (2.51)
The covariantized gauge fermion (2.43) takes now the form:

\[
\Psi = \frac{1}{e^2} \int_X d^4x \sqrt{g} \text{Tr} \left\{ \frac{1}{4} \bar{\nu}^\dot{\alpha} (h_{\dot{\alpha}} + 2\sqrt{2i} D_{\dot{\alpha}} M^\alpha) - \frac{1}{4} \nu^\dot{\alpha} (\bar{h}_{\dot{\alpha}} + 2\sqrt{2i} D_{\dot{\alpha}} \bar{M}^\alpha) \right. \\
+ \frac{1}{4} \chi_{\alpha\beta} \left( H^{\alpha\beta} + 2i (F^{+\alpha\beta} + 2[M^\alpha, M^\beta]) \right) \bigg\} \\
- \frac{1}{e^2} \int_X d^4x \text{Tr} \left\{ \frac{1}{2\sqrt{2}} \lambda \left( D_{\dot{\alpha}} \psi^{\dot{\alpha}} + i\sqrt{2} [\bar{M}^\alpha, \mu\alpha] - i\sqrt{2} [\bar{\mu}^\alpha, M\alpha] \right) \right. \\
- \frac{1}{e^2} \int_X d^4x \text{Tr} \left\{ \frac{i}{4} \lambda [\eta, \phi] \right\}.
\]

(2.52)

The resulting theory is equivalent to the theory of non-abelian monopoles discussed at length in [6,7,10], but with the monopole multiplet in the adjoint representation of the gauge group. That theory in turn is a generalization of the abelian monopole equations proposed in [4]. The reason for this equivalence can be explained as follows. First recall that from the viewpoint of \(N = 1\) superspace both, \(N = 2\) supersymmetric gauge theory coupled to an \(N = 2\) hypermultiplet in the adjoint of the gauge group, and \(N = 4\) supersymmetric gauge theory, are built out of the same set of \(N = 1\) superfields, namely a vector superfield and three chiral superfields. In the case of \(N = 4\) supersymmetric gauge theory we have a quadruplet of gauginos in the 4 of \(SU(4)_I\), which correspond to a \(SU(2)_I\) doublet of gauginos and an \(SU(2)_I\) singlet Dirac spinor (i.e., two \(SU(2)_I\) singlet Weyl spinors) in the case of the \(N = 2\) theory. Notice that in the transition the decomposition \(4 \rightarrow (2, 2)\) has to be done, which is equivalent to the decomposition defining the second twist of \(N = 4\). In this framework, the \(T_3\) subgroup of the former \(SU(2)_F\) symmetry remains in the theory as an \(U(1)\) symmetry which involves the monopole sector only and which corresponds to the \(N = 2\) central charge (trivial in this case) that remains after the twisting [9].

(3) \(4 \rightarrow (2, 2)\) Amphicheiral Theory

The last theory we will consider was briefly introduced at the end of reference
[15], and afterwards it was considered in detail in [16,17]. It corresponds to the decomposition $4 \rightarrow (2,2)$, but it is easier (and equivalent) to start from the second twisted theory and replace $SU(2)_R$ with the diagonal sum $SU(2)'_R$ of $SU(2)_R$ itself and the remaining isospin group $SU(2)_F$ (this is very much alike to a conventional $N = 2$ twisting). This introduces in the theory a second BRST-like symmetry, which comes from the $N = 4$ spinor supercharges $\bar{Q}_{v^\alpha}$. As we pointed out at the end of the introduction, there are several unusual features in this theory that we think deserve a detailed analysis. We begin by recalling the fundamentals of the second twist. The symmetry group $\mathcal{H} = SU(2)_L \otimes SU(2)_R \otimes SU(4)_I$ of the original $N = 4$ supersymmetric gauge theory is twisted to give the symmetry group $\mathcal{H}' = SU(2)'_L \otimes SU(2)_R \otimes SU(2)_F \otimes U(1)$ (we will refer to this as the $L$ twist) of the half-twisted theory. The supersymmetry charges $Q^v_\alpha$ and $\bar{Q}_{v^\alpha}$ decompose under $\mathcal{H}'$ as:

$$Q^v_\alpha \oplus \bar{Q}_{v^\alpha} \rightarrow Q^{(+1)} \oplus Q^{(+1)}_{(\alpha\beta)} \oplus Q^{(-1)}_{i\alpha} \oplus \bar{Q}^{(-1)}_{i\alpha} \oplus \bar{Q}^{(+1)}_{i\dot{\alpha}}. \quad (2.53)$$

But one can also twist with $SU(2)_R$ thus obtaining its corresponding $\bar{T}$ twist with symmetry group $\bar{\mathcal{H}}' = SU(2)_L \otimes SU(2)'_R \otimes SU(2)_F \otimes U(1)$ ($R$ twist). Both formulations are related (1.8) through an orientation reversal and a change of sign in $\theta$. Now we can twist $SU(2)_F$ away in four different ways. Two of these ($LL$ and $RR$) take us back to the Vafa-Witten twists $T$ and $\bar{T}$. The other two ($LR$ and $RL$) should lead us to the twist considered in [16,17] and its corresponding $\bar{T}$ twist. The non-trivial result is that either of these two different choices leads to the same topological theory. This can be seen as follows. Pick one of the possibilities, say, $LR$. After the first twist we have the half-twisted theory with symmetry group $\mathcal{H}'$ and supersymmetry charges (2.53). If we now twist $SU(2)_F$ with $SU(2)_R$ we obtain, from the last charge in (2.53), a second scalar charge $\tilde{Q}$ given by:

$$\tilde{Q}_{i\dot{\alpha}} \rightarrow \tilde{Q}_{\dot{\beta}\dot{\alpha}} \rightarrow \tilde{Q} = C^{\dot{\beta}\dot{\alpha}} \bar{Q}_{\dot{\beta}\dot{\alpha}}. \quad (2.54)$$

Notice that both the anticommuting symmetries, $Q$ and $\tilde{Q}$, have the same ghost number, so they are both to be considered either as BRST or anti-BRST...
operators. This is in contrast with the situation we found in the first twist where, after explicitly breaking the isospin group $SU(2)_{F}$ down to its $T_{3}$ subgroup, we were left with two scalar charges $Q^{(+)}$ and $Q^{(-)}$ with opposite ghost numbers, which were then interpreted as a BRST-antiBRST system.

The fields of the new theory can be obtained from those in the half-twisted theory as follows:

\begin{align}
A_{\alpha\dot{\alpha}} & \rightarrow A_{\alpha\dot{\alpha}}^{(0)} \rightarrow A_{\alpha\dot{\alpha}}^{(0)}, \\
\lambda_{v\alpha} & \rightarrow \chi_{\beta\bar{\alpha}}^{(-1)}, \eta^{(-1)}, \lambda^{(+1)}_{i\alpha} \rightarrow \chi_{\beta\alpha}^{(-1)}, \eta^{(-1)}, \tilde{\psi}_{\alpha\dot{\alpha}}^{(+1)}, \\
\tilde{\lambda}_{\alpha} & \rightarrow \psi_{\alpha\dot{\alpha}}^{(+1)}, \tilde{\zeta}_{i\dot{\alpha}}^{(-1)} \rightarrow \tilde{\psi}_{\alpha\dot{\alpha}}^{(+1)}, \tilde{\eta}^{(-1)}, \tilde{\chi}_{\dot{\alpha}\dot{\beta}}^{(-1)}, \\
\phi_{uv} & \rightarrow B^{(-2)}, C^{(+2)}, G_{i\alpha}^{(0)} \rightarrow B^{(-2)}, C^{(+2)}, V_{\alpha\dot{\alpha}}^{(0)},
\end{align}

(2.55)

where we have included also the corresponding fields of the $N = 4$ theory and the ghost number carried by the twisted fields. The notation is similar to that in ref. [16]. Notice that if we exchange $SU(2)_{L}$ by $SU(2)_{R}$ the field content in (2.55) does not change. This in turn implies that the $LR$ and $RL$ twists are in fact the same,

\begin{align}
S_{X}^{LR} = S_{X}^{RL},
\end{align}

(2.56)

or, in other words, the third twist leads to an amphicheiral topological quantum field theory (see (1.9)). Since it is known that the two twists are related by $S_{X}^{LR} = S_{X}^{RL}|_{\tau \rightarrow -\tau}$ ($\bar{X}$ denotes the manifold $X$ with the opposite orientation), it follows that by reversing the sign of the $\theta$-angle one can jump from $X$ to $\bar{X}$:

\begin{align}
S_{X} = S_{\bar{X}}|_{\tau \rightarrow -\tau}.
\end{align}

(2.57)

We will see in a moment that this information is encoded in the conjugation discrete symmetry introduced in [16].
The definitions in (2.55) are almost self-evident. The only ones which need clarification are those corresponding to \( \tilde{\eta} \) and \( \tilde{\chi}_\dot{\alpha}\dot{\beta} \). Our conventions are:

\[
\zeta^i \dot{\alpha} \rightarrow \zeta^j \dot{\beta} \rightarrow \begin{cases} 
\tilde{\eta} = -\zeta^{\dot{\alpha} \dot{\beta}}, \\
\tilde{\chi}_\dot{\alpha}\dot{\beta} = -C_{\gamma(\dot{\beta} \zeta^{\dot{\gamma} \dot{\alpha}})}.
\end{cases}
\]  

(2.58)

In terms of the fields in (2.55), the on-shell action (2.38) takes the form:

\[
S^{(0)} = \frac{1}{e^2} \int d^4 x \text{Tr} \left\{ \frac{1}{2} \nabla_{\dot{\alpha}} B \nabla^{\dot{\alpha}} C - \frac{1}{4} \nabla_{\dot{\beta}} V_{\dot{\alpha}\dot{\beta}} \nabla^{\dot{\beta}} V^{\dot{\alpha}} C - i \psi_{\dot{\alpha}} \nabla^{\dot{\alpha}} \chi_{\dot{\alpha}\dot{\beta}} \\
- \frac{i}{2} \psi_{\dot{\alpha}\dot{\beta}} \nabla^{\dot{\alpha}} \psi_{\dot{\beta}} \eta + \frac{i}{2} \tilde{\eta} \nabla^{\dot{\alpha}} \psi_{\dot{\alpha}\dot{\beta}} + i \tilde{\chi}_{\dot{\alpha}\dot{\beta}} \nabla^{\dot{\alpha}} \tilde{\psi}^{\dot{\beta}} \eta - \frac{1}{4} F_{mn} F^{mn} \\
- \frac{i}{\sqrt{2}} \chi_{\dot{\alpha}\dot{\beta}} [\chi_{\dot{\alpha}\dot{\beta}}, C] - \frac{i}{\sqrt{2}} \tilde{\psi}^{\dot{\alpha}} \tilde{\psi}_{\dot{\alpha}\dot{\beta}} B + \frac{i}{\sqrt{2}} \tilde{\psi}^{\dot{\alpha}} \tilde{\psi}_{\dot{\alpha}\dot{\beta}} \tilde{\psi}_{\dot{\beta} C} \eta^{\dot{\alpha}} + \frac{i}{\sqrt{2}} \eta \tilde{\psi}_{\dot{\alpha}} \tilde{\psi}^{\dot{\alpha}} [\psi_{\dot{\alpha}\dot{\beta}}, V_{\dot{\alpha}}] \\
- \frac{i}{2 \sqrt{2}} \eta [\eta, C] + \frac{i}{\sqrt{2}} \psi_{\dot{\alpha}\dot{\beta}} \psi_{\dot{\alpha}} [\psi_{\dot{\alpha}\dot{\beta}}, B] - \frac{i}{\sqrt{2}} \tilde{\eta} \tilde{\psi}^{\dot{\alpha}} [\psi_{\dot{\alpha} \dot{\beta}}, V_{\dot{\alpha}}] - i \sqrt{2} \tilde{\chi}_{\dot{\alpha}\dot{\beta}} [\psi_{\dot{\alpha}\dot{\beta}}, V_{\dot{\alpha}}] \\
+ \frac{i}{\sqrt{2}} \tilde{\chi}_{\dot{\alpha}\dot{\beta}} [\tilde{\chi}_{\dot{\alpha}\dot{\beta}}, C] + \frac{i}{2 \sqrt{2}} \tilde{\eta} \tilde{\eta}, C] - \frac{i}{4} [B, C]^2 - [B, V_{\dot{\alpha}}] \tilde{C}, V^{\dot{\alpha} C} \tilde{C} \\
+ \frac{i}{4} [V_{\dot{\alpha}\dot{\beta}}, V_{\dot{\beta} C}] [V^{\dot{\alpha} C}, V^{\dot{\beta} C}] \right\} - \frac{i \theta}{32 \pi^2} \int d^4 x \text{Tr} \left\{ F_{mn} F^{mn} \right\}.
\]

(2.59)

The next thing to do is to obtain the symmetry transformations which correspond to the new model. Recall that we have now two fermionic charges \( Q \) and \( \tilde{Q} \). The transformations generated by \( Q \) are easily obtained from those in the previous twist (2.40). To obtain the transformations generated by \( \tilde{Q} \) we must return to the \( N = 4 \) theory. Let us recall that the \( N = 4 \) supersymmetry transformations are generated by \( \xi_v^{\alpha} Q_{\alpha}^v + \xi_v^{\dot{\alpha}} \tilde{Q}^v_{\dot{\alpha}} \). The transformations corresponding to \( \tilde{Q} \) are readily extracted by setting \( \xi^1 = \xi^2 = 0 \) and making the replacement

\[
\tilde{\xi}^{3,4} \tilde{\dot{\alpha}} \rightarrow \tilde{\xi}^i \tilde{\dot{\alpha}} \rightarrow \tilde{\xi}^{\dot{\beta}} \tilde{\dot{\alpha}} \rightarrow \tilde{\xi}^{\dot{\delta}} \tilde{\dot{\alpha}}.
\]  

(2.60)
In this way one gets the following set of transformations:

\[ \delta A_{\alpha\dot{\alpha}} = 2i\epsilon \psi_{\alpha\dot{\alpha}}, \]
\[ \delta F_{\alpha\beta}^+ = 2\epsilon \nabla_{(\alpha} \psi_{\beta)\dot{\alpha}}, \]
\[ \delta \psi_{\alpha\dot{\alpha}} = -i\sqrt{2}\epsilon \nabla_{\alpha\dot{\alpha}} C, \]
\[ \delta \eta = i\sqrt{2}\epsilon \nabla_{\alpha\dot{\alpha}} V^{\dot{\alpha}\alpha}, \]
\[ \delta \tilde{\chi}_{\dot{\alpha}\dot{\beta}} = -i\sqrt{2}\epsilon \nabla_{\dot{\alpha}} (V_{\beta})^{\dot{\alpha}}, \]
\[ \delta \chi_{\alpha\beta} = -i\epsilon \tilde{F}_{\alpha\dot{\beta}} - i\epsilon [V_{\alpha\dot{\beta}}, V_{\beta}], \]
\[ \delta \eta = 2i\epsilon [B, C], \]
\[ \delta \psi_{\alpha\dot{\alpha}} = -2i\epsilon [V_{\alpha\dot{\alpha}}, C], \]
\[ \delta B = \sqrt{2}\epsilon \eta, \]
\[ \delta C = 0, \]
\[ \delta V_{\alpha\dot{\alpha}} = -\sqrt{2}\epsilon \tilde{\psi}_{\alpha\dot{\alpha}}, \]
\[ \delta \tilde{A}_{\alpha\dot{\alpha}} = -2i\epsilon \tilde{\psi}_{\alpha\dot{\alpha}}, \]
\[ \delta \tilde{F}_{\alpha\beta}^+ = -2\epsilon \nabla_{(\alpha} \tilde{\psi}_{\beta)\dot{\alpha}}, \]
\[ \delta \tilde{\psi}_{\alpha\dot{\alpha}} = -2i\epsilon [V_{\alpha\dot{\alpha}}, C], \]
\[ \delta \tilde{\eta} = 2i\epsilon [B, C], \]
\[ \delta \tilde{\chi}_{\dot{\alpha}\dot{\beta}} = i\epsilon \tilde{F}_{\alpha\dot{\beta}}^+ - i\epsilon [V_{\alpha\dot{\beta}}, V_{\beta}], \]
\[ \delta \tilde{\chi}_{\dot{\alpha}\dot{\beta}} = i\sqrt{2}\epsilon \nabla_{\dot{\alpha}} (V_{\beta})^\dot{\alpha}, \]
\[ \delta \tilde{\eta} = i\sqrt{2}\epsilon \nabla_{\alpha\dot{\alpha}} V^{\dot{\alpha}\alpha}, \]
\[ \delta \tilde{\psi}_{\alpha\dot{\alpha}} = -i\sqrt{2}\epsilon \nabla_{\dot{\alpha}} C, \]
\[ \delta \tilde{B} = -\sqrt{2}\epsilon \tilde{\eta}, \]
\[ \delta \tilde{C} = 0, \]
\[ \delta \tilde{V}_{\alpha\dot{\alpha}} = \sqrt{2}\epsilon \tilde{\psi}_{\alpha\dot{\alpha}}. \]

Since there are no half-integer spin fields in the theory it is preferable to convert the spinor indices into vector indices. To do this we make the following definitions:

\[
\begin{pmatrix}
V \\
\psi \\
\tilde{\psi}
\end{pmatrix}_{\alpha\dot{\alpha}} \equiv \sigma^{m}_{\alpha\dot{\alpha}}
\begin{pmatrix}
V \\
\psi \\
\tilde{\psi}
\end{pmatrix}_{m}, \quad \chi_{\alpha\beta} = \sigma^{mn}_{\alpha\beta} \chi^{+}_{mn}, \quad \tilde{\chi}_{\dot{\alpha}\dot{\beta}} = \tilde{\sigma}^{mn}_{\dot{\alpha}\dot{\beta}} \chi^{-}_{mn} \tag{2.61}
\]

where \( \chi^{\pm}_{mn} = (1/2)\{\chi_{mn} \pm (1/2)\epsilon_{mpq}\chi^{pq}\} \). In order to extract a manifestly real action we also make the replacements \( \psi \to -i\psi, \chi^{+} \to i\chi^{+}, \tilde{\eta} \to i\tilde{\eta} \) and \( \tilde{Q} \to i\tilde{Q} \). The resulting action is:
\[ S^{(0)} = \frac{1}{\epsilon^2} \int d^4 x \text{Tr} \left\{ -\nabla_mB^{mn}C - \nabla_m V_n \nabla^n V^m + 4\psi^m \nabla^n \chi_{mn}^+ + \psi^m \nabla_m \eta \right\} \\
+ \bar{\psi}^m \nabla_m \bar{\eta} + 4\bar{\psi}^m \nabla^n \chi_{mn}^- - \frac{1}{4} F_{mn} F^{mn} - i \sqrt{2} \chi_{mn}^+ F_{mn} + i \sqrt{2} \eta \left[ \psi_m, V^n \right] - \frac{i}{2\sqrt{2}} \eta \left[ \psi_m, C \right] \\
+ i \sqrt{2} \bar{\psi}^m \left[ \psi_m, B \right] - 4i \sqrt{2} \bar{\eta} \left[ \psi_m, V^n \right] + 4 \sqrt{2} i \chi_{mn}^- \left[ \psi^m, V^n \right] - i \sqrt{2} \chi_{mn}^- \left[ \chi_{mn}^- , C \right] \\
- \frac{i}{2\sqrt{2}} \bar{\eta} \left[ \psi_m, C \right] - \frac{1}{2} \left[ B, C \right]^2 + 2 \left[ B, V_m \right] \left[ C, V^m \right] + \left[ V_m, V_n \right] \left[ V^m, V^n \right] \right\} \\
- \frac{i\theta}{32\pi^2} \int d^4 x \text{Tr} \left\{ *F_{mn} F^{mn} \right\}, \tag{2.63} \]

and the corresponding transformations become:

\[ \delta A_m = 2\epsilon \psi_m, \quad \delta A_m = -2\bar{\epsilon} \bar{\psi}_m, \]
\[ \delta \psi_m = \sqrt{2} \epsilon \nabla_m C, \quad \tilde{\delta} \psi_m = -2i\bar{\epsilon} \left[ V_m, C \right] \]
\[ \delta \bar{\eta} = -2\sqrt{2} \epsilon \nabla_m V^m, \quad \tilde{\delta} \bar{\eta} = -2i\bar{\epsilon} \left[ B, C \right], \]
\[ \delta \chi_{mn}^- = 2\sqrt{2} \epsilon (\nabla_m V_n)^-, \quad \tilde{\delta} \chi_{mn}^- = \bar{\epsilon} F_{mn}^- - 2i\bar{\epsilon} \left[ V_m, V_n \right]^-, \]
\[ \delta \chi_{mn}^+ = -\epsilon F_{mn}^+ + 2i\epsilon \left[ V_m, V_n \right]^+, \quad \tilde{\delta} \chi_{mn}^+ = 2\sqrt{2} \bar{\epsilon} (\nabla_m V_n)^+, \]
\[ \delta \eta = 2i\epsilon \left[ B, C \right], \quad \tilde{\delta} \eta = -2\sqrt{2} \bar{\epsilon} \nabla_m V^m, \]
\[ \delta \bar{\psi}_m = -2i\bar{\epsilon} \left[ V_m, C \right], \quad \tilde{\delta} \bar{\psi}_m = -\sqrt{2} \bar{\epsilon} \nabla_m C, \]
\[ \delta B = \sqrt{2} \epsilon \bar{\eta}, \quad \tilde{\delta} B = -\sqrt{2} \bar{\epsilon} \bar{\eta}, \]
\[ \delta C = 0, \quad \tilde{\delta} C = 0, \]
\[ \delta V_m = -\sqrt{2} \epsilon \bar{\psi}_m, \quad \tilde{\delta} V_m = -\sqrt{2} \bar{\epsilon} \psi_m, \tag{2.64} \]

where \((X_{mn})^\pm \equiv \frac{1}{2} (X_{mn} \pm *X_{mn})\), and \(X_{mn} \equiv \frac{1}{2} (X_{mn} - X_{nm})\). The generators \(Q\) and \(\tilde{Q}\) satisfy the on-shell algebra:

\[ \{ Q, Q \} = \delta_g (C), \]
\[ \{ \tilde{Q}, \tilde{Q} \} = \delta_g (C), \tag{2.65} \]
\[ \{ Q, \tilde{Q} \} = 0. \]

Now consider the following discrete transformations acting on the fields of the
theory:

\[
\begin{align*}
B & \rightarrow B, & C & \rightarrow C, \\
A & \rightarrow A, & V & \rightarrow -V, \\
\eta & \rightarrow -\bar{\eta}, & \psi & \rightarrow -\bar{\psi}, \\
\bar{\eta} & \rightarrow -\eta, & \bar{\psi} & \rightarrow -\psi,
\end{align*}
\]

(2.66)

\[
\begin{align*}
\chi^+ & \leftrightarrow -\chi^- \Rightarrow \begin{cases} 
\chi \rightarrow -\chi, \\
*\chi \rightarrow -*\chi,
\end{cases} \\
F^+ & \leftrightarrow F^- \Rightarrow \begin{cases} 
F \rightarrow F, \\
*F \rightarrow -*F.
\end{cases}
\end{align*}
\]

Notice that these transformations involve the simultaneous replacement \( \epsilon_{mnpq} \rightarrow -\epsilon_{mnpq} \), which is equivalent to a reversal of the orientation of the four-manifold \( X \). Because of this orientation reversal, the sign of the \( \theta \)-term in (2.63) is also reversed. Thus the \( \mathbb{Z}_2 \)-like transformations (2.66) map the action on \( X \) to the same action on \( \bar{X} \) but with \(-\theta\). This is precisely the information encoded in (2.57).

It is also noteworthy that the transformations (2.66) exchange the BRST generators \( Q \) and \( \bar{Q} \) (one can realize this by looking at (2.64)). Indeed, had we not known about the existence of one of the topological symmetries, say \( \bar{Q} \), we would have discovered it immediately with the aid of the symmetry (2.66). In addition to this, one can readily see that the replacements dictated by (2.66) preserve the ghost number assignments of the fields. In what follows, we will usually refer to the transformations (2.66) by \( \mathbb{Z}_2 \), but the reader must be aware of this abuse of notation.

Several things remain to be done. It would be desirable to obtain an off-shell formulation of the model. Besides, it would be interesting to find out whether the off-shell action (provided that it exists) can be written as a \( Q \)- (or \( \bar{Q} \), or both) commutator, and write down the explicit expression for the corresponding gauge fermion. And finally, the theory should be generalized to any arbitrary four-manifold of euclidean signature.
We have found a complete off-shell formulation involving both BRST symmetries simultaneously such that the action (2.63) is (up to appropriate theta-terms) \(Q\) and \(\tilde{Q}\)-exact. Let us examine these results in more detail. The on-shell algebra (2.65) can be extended off-shell by introducing the auxiliary fields \(N^+_{mn}, N^-_{mn}\) and \(P\), which have zero ghost numbers and are taken to transform under \(Z_2\) as \(N^+ \leftrightarrow -N^-, P \rightarrow -P\). In terms of these fields, the transformations (2.64) are modified as follows:

\[
\begin{align*}
\delta \tilde{\eta} &= -2\sqrt{2}e \nabla_m V^m + \epsilon_P, \\
\delta P &= -4\epsilon \nabla_m \tilde{\psi}^m + 4\sqrt{2}i\epsilon[\tilde{\psi}^m, V_m] + 2\sqrt{2}i\epsilon[\tilde{\eta}, C], \\
\delta \chi^-_{mn} &= 2\sqrt{2}e(\nabla_m V_n)^- + \epsilon N^-_{mn}, \\
\delta N^-_{mn} &= 4\epsilon \nabla_m \tilde{\psi}^-_n - 4\sqrt{2}i\epsilon[\tilde{\psi}^-_m, V_n]^-- + 2\sqrt{2}i\epsilon[\chi^-_{mn}, C], \\
\delta \chi^+_{mn} &= -\epsilon F^+_{mn} + 2i\epsilon([V_m, V_n])^+ + \epsilon N^+_{mn}, \\
\delta N^+_{mn} &= 4\epsilon \nabla_m \tilde{\psi}^+_n + 4\sqrt{2}i\epsilon[\tilde{\psi}^+_m, V_n]^+ + 2\sqrt{2}i\epsilon[\chi^+_{mn}, C].
\end{align*}
\] (2.67)

The other transformations in (2.64) remain the same. Equivalent formulas hold for \(\tilde{Q}\) and are related to those in (2.67) through the \(Z_2\) transformation. In this off-shell realization the auxiliary fields appear in the action only quadratically, that is,

\[
S^{(1)} = S^{(0)} + \int \text{Tr}\{\frac{1}{2}(N^+)^2 + \frac{1}{2}(N^-)^2 + \frac{1}{8}P^2\}.
\] (2.68)

The action \(S^{(1)}\) can be written either as a \(Q\) commutator or as a \(\tilde{Q}\) commutator and is invariant under both, \(Q\) and \(\tilde{Q}\), that is,

\[
S^{(1)} = \{Q, \hat{\Psi}^+\} = -2\pi ik\tau = \{\tilde{Q}, \hat{\Psi}^-\} = -2\pi ik\bar{\tau}; \quad [Q, S^{(1)}] = [\tilde{Q}, S^{(1)}] = 0 \quad (2.69)
\]

where the gauge fermions \(\hat{\Psi}^\pm\) are not equal but are formally interchanged by the \(Z_2\) transformation and \(k\) is the instanton number (1.2). It is possible to redefine the auxiliary fields to cast either the \(Q\) or the \(\tilde{Q}\) transformations (but not both
simultaneously) in the standard form,

\[
\{Q, \text{antighost}\} = \text{auxiliary field},
\]

\[
[Q, \text{auxiliary field}] = \delta^{\text{gauge}}\text{antighost},
\]

which is essential to make contact with the Mathai-Quillen interpretation. Performing the shifts,

\[
\begin{align*}
P & \rightarrow P + 2\sqrt{2} \nabla_m V^m, \\
N_{mn}^- & \rightarrow N_{mn}^- - 2\sqrt{2}(\nabla_{[m}V_{n]}^-), \quad (2.70) \\
N_{mn}^+ & \rightarrow N_{mn}^+ + F_{mn}^+ - 2i([V_m, V_n])^+,
\end{align*}
\]

which can be guessed from (2.67), the Q transformations take the simple form:

\[
\begin{align*}
\delta A_m &= 2\epsilon\psi_m, \\
\delta V_m &= -\sqrt{2}\epsilon\tilde{\psi}_m, \\
\delta C &= 0, \\
\delta B &= \sqrt{2}\epsilon\eta, \\
\delta \eta &= 2i\epsilon[B, C], \\
\delta \chi_{mn}^+ &= \epsilon N_{mn}^+, \\
\delta \chi_{mn}^- &= \epsilon N_{mn}^-.
\end{align*}
\]

(2.71)

The point is that if instead of (2.70) we make the Z\(_2\) conjugate shifts,

\[
\begin{align*}
P & \rightarrow P + 2\sqrt{2} \nabla_m V^m, \\
N_{mn}^+ & \rightarrow N_{mn}^+ - 2\sqrt{2}(\nabla_{[m}V_{n]}^+), \\
N_{mn}^- & \rightarrow N_{mn}^- - F_{mn}^- + 2i([V_m, V_n])^-,
\end{align*}
\]

(2.72)
it is \( \tilde{\delta} \equiv \tilde{\epsilon} \tilde{Q} \) the one which can be cast in the simple form:

\[
\begin{align*}
\tilde{\delta}A_m &= -2\tilde{\epsilon}\tilde{\psi}_m, \\
\tilde{\delta}V_m &= -\sqrt{2}\tilde{\epsilon}\psi_m, \\
\tilde{\delta}C &= 0, \\
\tilde{\delta}B &= -\sqrt{2}\tilde{\epsilon}\tilde{\eta}, \\
\tilde{\delta}\tilde{\eta} &= \tilde{\epsilon}P, \\
\tilde{\delta}\chi^+_{mn} &= \tilde{\epsilon}N^+_{mn}, \\
\tilde{\delta}\chi^-_{mn} &= \tilde{\epsilon}N^-_{mn}, \\
\tilde{\delta}\psi_m &= -\sqrt{2}\tilde{\epsilon}\nabla_mC, \\
\tilde{\delta}\psi &= -2i\tilde{\epsilon}[V_m, C], \\
\tilde{\delta}\eta &= -2i\tilde{\epsilon}[\chi^+_{mn}, \tilde{\psi}_m], \\
\tilde{\delta}\beta &= -2i\tilde{\epsilon}[\chi^-_{mn}, \tilde{\psi}_m].
\end{align*}
\]

Notice that since the appropriate shifts are in each case different, the one which simplifies the \( Q \) transformations makes the corresponding \( \tilde{Q} \) transformations (not shown) much more complicated and conversely, the shift which simplifies the \( \tilde{Q} \) transformations makes the corresponding \( Q \) transformations (not shown) much more complicated.

Keeping these results in mind from now on we will focus on the \( Q \) formulation, that is, on the off-shell formulation in which the \( Q \) transformations take the form (2.71). The off-shell action which corresponds to this formulation is:

\[
S^{(2)} = \frac{1}{e^2} \int d^4x \text{Tr} \left\{ -\nabla_mB\nabla^mC + \frac{1}{2}N^+_{mn}(N^{+mn} + 2F^{+mn} - 4i[V_m, V^n]^+) \\
+ \frac{1}{2}N^-_{mn}(N^{-mn} - 4\sqrt{2}(\nabla^m[V^n])^-) + \frac{1}{8}P(P + 4\sqrt{2}\nabla_mV^m) \\
+ 4\psi^m\psi^{n\chi^+_{mn}} + \psi^m\nabla_m\eta + \tilde{\psi}^m\nabla_m\tilde{\eta} + 4\tilde{\psi}^m\nabla_m\chi^-_{mn} \\
- i\sqrt{2}\chi^{+mn}[\chi^+_{mn}, C] + i\sqrt{2}\tilde{\psi}^m[\tilde{\psi}_m, B] - 4\sqrt{2}i\chi^{-mn}[\tilde{\psi}_m, V^n] \\
+ i\sqrt{2}\eta[\tilde{\psi}_m, V^m] - \frac{i}{2\sqrt{2}}\eta[\eta, C] + i\sqrt{2}\psi_m[\psi^m, B] - i\sqrt{2}\tilde{\eta}[\psi_m, V^m] \\
+ 4\sqrt{2}i\chi^-_{mn}[\psi^m, V^n] - i\sqrt{2}\chi^-_{mn}[\chi^{-mn}, C] - \frac{i}{2\sqrt{2}}\tilde{\eta}[\tilde{\eta}, C] - \frac{1}{2}[B, C]^2 \\
+ 2[B, V_m][C, V^m]\right\} - 2\pi i\tau \frac{1}{32\pi^2} \int d^4x \text{Tr} \left\{ *F_{mn}F^{mn} \right\},
\]

(2.74)

and reverts to (2.63) after integrating out the auxiliary fields. The \( \tau \)-independent part of the action (2.74) is \( Q \)-exact, that is, it can be written as a \( Q \)-commutator.
The appropriate gauge fermion is:

\[
\Psi^+ = \frac{1}{e^2} \int d^4 x \text{Tr} \left\{ \frac{1}{2} \chi_{+}^{mn} \left( N^{+mn} + 2F^{+mn} - 4i[V^m, V^n]^+ \right) 
+ \frac{1}{2} \chi_{-}^{mn} \left( N^{-mn} - 4\sqrt{2}(\nabla[mV^n])^- \right) + \frac{1}{8} \tilde{\eta} \left( P + 4\sqrt{2} \nabla_m V^m \right) \right\} 
+ \frac{1}{e^2} \int d^4 x \text{Tr} \left\{ \frac{1}{\sqrt{2}} B \left( \nabla_m \psi^m + i\sqrt{2}[\psi^m, V^m] \right) \right\} 
+ \frac{1}{e^2} \int d^4 x \text{Tr} \left\{ \frac{i}{4} \eta [B, C] \right\}.
\]

(2.75)

notice that \(\Psi^-\) would correspond to the \(\mathbb{Z}_2\)-transformed of \(\Psi^+\). The gauge fermions \(\hat{\Psi}^+\) and \(\hat{\Psi}^-\) in (2.69) are easily obtained after undoing the shifts (2.70) and (2.72), respectively.

Now we switch on an arbitrary background metric \(g_{\mu\nu}\) of euclidean signature. This is straightforward once we have expressed the model in the form of eqs. (2.71) and (2.75). The covariantized transformations are the following:

\[
\begin{align*}
\delta A_\mu &= 2\epsilon \psi_\mu, \\
\delta V_\mu &= -\sqrt{2}\epsilon \tilde{\psi}_\mu, \\
\delta C &= 0, \\
\delta B &= \sqrt{2}\epsilon \eta, \\
\delta \tilde{\eta} &= \epsilon P, \\
\delta \chi^+_{\mu\nu} &= \epsilon N^+_{\mu\nu}, \\
\delta N^+_{\mu\nu} &= 2\sqrt{2}\epsilon [\chi^+_{\mu\nu}, C], \\
\delta \chi^-_{\mu\nu} &= \epsilon N^-_{\mu\nu}, \\
\delta N^-_{\mu\nu} &= 2\sqrt{2}\epsilon [\chi^-_{\mu\nu}, C],
\end{align*}
\]

(2.76)

and the action for the model is defined to be \(S^{(2)}_c = \{Q, \Psi^+_c\} - 2\pi ik\tau\), with the
gauge fermion (appropriately covariantized):

\[
\Psi_c^+ = \frac{1}{e^2} \int_X d^4x \sqrt{g} \text{Tr} \left\{ \frac{1}{2} \chi^+_{\mu\nu} \left( N^{+\mu\nu} + 2F^{+\mu\nu} - 4i[V^\mu, V^\nu]^+ \right) + \frac{1}{2} \chi^-_{\mu\nu} \left( N^{-\mu\nu} - 4\sqrt{2}(D[^{\mu}V])^\nu \right) + \frac{1}{8} \bar{\eta} \left( P + 4\sqrt{2} D_\mu V^\mu \right) \right\}
\]

\[
+ \frac{1}{e^2} \int_X d^4x \sqrt{g} \text{Tr} \left\{ \frac{1}{\sqrt{2}} B \left( D_\mu \psi^\mu + i\sqrt{2}[\bar{\psi}_\mu, V^\mu] \right) \right\}
\]

\[
+ \frac{1}{e^2} \int_X d^4x \sqrt{g} \text{Tr} \left\{ \frac{i}{4} \eta[B, C] \right\}.
\]

The resulting action reads:

\[
S_c^{(2)} = \frac{1}{e^2} \int_X d^4x \sqrt{g} \text{Tr} \left\{ -D_\mu B D^\mu C + \frac{1}{2} N^+_{\mu\nu} \left( N^{+\mu\nu} + 2F^{+\mu\nu} - 4i[V^\mu, V^\nu]^+ \right) + \frac{1}{2} N^-_{\mu\nu} \left( N^{-\mu\nu} - 4\sqrt{2}(D[^{\mu}V])^\nu \right) + \frac{1}{8} P \left( P + 4\sqrt{2} D_\mu V^\mu \right) + 4\psi^\mu D^\nu \chi^+_{\mu\nu} + \psi^\mu D_\mu \eta + \bar{\psi}_\mu D_\mu \bar{\eta} + 4\bar{\psi}_\mu D^\nu \chi^-_{\mu\nu} - i\sqrt{2} \chi^+_{\mu\nu}[\chi^+_{\mu\nu}, C] + i\sqrt{2} \bar{\psi}_\mu[\bar{\psi}_\mu, B] - 4\sqrt{2} i\chi^+_{\mu\nu}[\bar{\psi}_\mu, V^\nu] + i\sqrt{2} \eta[\bar{\psi}_\mu, V^\mu] - \frac{i}{2\sqrt{2}} \eta[\eta, C] + i\sqrt{2} \psi^\mu[\psi^\mu, B] - i\sqrt{2} \bar{\eta}[\bar{\psi}_\mu, V^\mu] + 4\sqrt{2} i\chi^-_{\mu\nu}[\psi^\mu, V^\nu] - i\sqrt{2} \chi^-_{\mu\nu}[\chi^-_{\mu\nu}, C] - \frac{i}{2\sqrt{2}} \bar{\bar{\eta}}[\bar{\eta}, C] - \frac{1}{2} [B, C]^2 + 2[B, V_\mu][C, V^\mu] \right\}
\]

\[
- 2\pi i \tau \frac{1}{32\pi^2} \int_X d^4x \sqrt{g} \text{Tr} \left\{ *F_{\mu\nu} F^{\mu\nu} \right\}.
\]

If we integrate out the auxiliary fields in (2.78) we recover the action (2.63). Some important issues relative to this theory will be addressed in sect. 4.
3. The Topological Actions in the Mathai-Quillen Approach

In the first part of this paper we have reviewed in great detail the four dimensional topological field theories that can be obtained by twisting the symmetry group of the $N = 4$ supersymmetric gauge theory. The twisting procedure has been repeatedly shown to be a very powerful technique for the construction of topological quantum field theories. However, it suffers from serious drawbacks, the main one being that it is not possible to identify from the very beginning the underlying geometrical structure that is involved. Rather, in most of the cases the underlying geometrical scenario is unveiled only after a careful analysis with techniques borrowed from conventional quantum field theory is carried out [1]. In what follows, we will change our scope and try to concentrate on the geometrical formulation of these theories. We will make use of the Mathai-Quillen formalism (see [25-29] and references therein), which is very well suited for our purposes. Let us recall briefly the fundamentals of this approach. In the framework of topological quantum field theories of cohomological type [29], one deals with a certain set of fields (the field space, $\mathcal{M}$), on which the action of a symmetry group, $\mathcal{G}$, which is usually a local symmetry group, is defined. An appropriate set of basic equations imposed on the fields single out a certain subset (the moduli space) of $\mathcal{M}/\mathcal{G}$. The topological quantum field theory associated to this moduli problem studies intersection theory on the corresponding moduli space. In this context, the Mathai-Quillen formalism involves the following steps. Given the field space $\mathcal{M}$, the basic equations of the problem are introduced as sections of an appropriate vector bundle $\mathcal{V} \to \mathcal{M}$, in such a way that the zero locus of these sections is precisely the relevant moduli space. The Mathai-Quillen formalism allows the computation of a certain representative of the Thom class of the vector bundle $\mathcal{V}$, which turns out to be the exponential of the action of the topological field theory under consideration. The integration on $\mathcal{M}$ of the pullback under the sections of the Thom class of $\mathcal{V}$ gives the Euler characteristic of the bundle, which is the basic topological invariant associated to the moduli problem.
3.1. The Vafa-Witten Problem

In [5], Vafa and Witten studied the partition function of the first of the twisted $N = 4$ supersymmetric gauge theories we have considered, namely that corresponding to the defining embedding $4 \to (2, 1) \oplus (2, 1)$ (see section 2). They were able to show that, in favourable conditions, the partition function is the Euler characteristic of instanton moduli space, and then computed it on several 4-manifolds in order to make some non-trivial tests of S-duality. The analysis starts from two basic equations involving the self-dual part of the gauge connection $F^+$, a certain scalar field $C$ and a bosonic self-dual two-form $B^+$, all taking values in the adjoint representation of some compact finite dimensional Lie group $G$. These equations are:

$$\begin{cases}
  D_\mu C + \sqrt{2} D^\nu B^+_{\nu\mu} = 0, \\
  F^+_{\mu\nu} - \frac{i}{2} [B^+_{\mu\tau}, B^+_{\nu\sigma}] - \frac{i}{\sqrt{2}} [B^+_{\mu\nu}, C] = 0.
\end{cases} \tag{3.1}$$

One can consider the equations above as defining a certain moduli problem, and our aim is to construct the topological quantum field theory which corresponds to it within the framework of the Mathai-Quillen formalism. Our analysis will follow closely that in [6,10,30]. Recently, this formalism has been applied to the twist under consideration in [31]. The construction presented in that work differs from ours in the role assigned to the field $C$.

The topological framework

The geometrical setting is a certain oriented, compact Riemannian four-manifold $X$, and the field space is $\mathcal{M} = \mathcal{A} \times \Omega^0(X, \text{ad} P) \times \Omega^{2,+}(X, \text{ad} P)$, where $\mathcal{A}$ is the space of connections on a principal $G$-bundle $P \to X$, and the second and third factors denote, respectively, the 0-forms and self-dual differential forms of degree two on $X$ taking values in the Lie algebra of $G$. $\text{ad} P$ denotes the adjoint bundle of $P$, $P \times_{\text{ad} \mathfrak{g}}$. The space of sections of this bundle, $\Omega^0(X, \text{ad} P)$, is the Lie algebra of the group $G$ of gauge transformations (vertical automorphisms) of the
bundle \( P \), whose action on the field space is given locally by:

\[
\begin{align*}
g^*(A) &= i(dg)g^{-1} + gAg^{-1}, \\
g^*(C) &= gCg^{-1}, \\
g^*(B^+) &= gB^+g^{-1},
\end{align*}
\]  

(3.2)

where \( C \in \Omega^0(X, \text{ad}P) \) and \( B^+ \in \Omega^{2,+}(X, \text{ad}P) \). In terms of the covariant derivative, \( d_A = d + i[A, \cdot] \), the infinitesimal form of the transformations (3.2), with \( g = \exp(-i\phi) \) and \( \phi \in \Omega^0(X, \text{ad}P) \), takes the form:

\[
\begin{align*}
\delta_g(\phi) A &= dA + i[A, \phi], \\
\delta_g(\phi) C &= i[C, \phi], \\
\delta_g(\phi) B^+ &= i[B^+, \phi].
\end{align*}
\]  

(3.3)

The tangent space to the field space at the point \((A, C, B^+)\) is the vector space \( T_{(A,C,B^+)} \mathcal{M} = \Omega^1(X, \text{ad}P) \oplus \Omega^0(X, \text{ad}P) \oplus \Omega^{2,+}(X, \text{ad}P) \). On \( T_{(A,C,B^+)} \mathcal{M} \) we can define a gauge-invariant Riemannian metric (inherited from that on \( X \)) as follows:

\[
\langle (\psi, \zeta, \tilde{\psi}^+), (\theta, \xi, \tilde{\omega}^+) \rangle = \int_X \text{Tr}(\psi \wedge \ast \theta) + \int_X \text{Tr}(\zeta \wedge \ast \xi) + \int_X \text{Tr}(\tilde{\psi}^+ \wedge \ast \tilde{\omega}^+) 
\]  

(3.4)

where \( \psi, \theta \in \Omega^1(X, \text{ad}P), \zeta, \xi \in \Omega^0(X, \text{ad}P) \) and \( \tilde{\psi}^+, \tilde{\omega}^+ \in \Omega^{2,+}(X, \text{ad}P) \).

To introduce the basic equations (3.1) in this framework we proceed as follows. On the field space \( \mathcal{M} \) we build a trivial vector bundle \( \mathcal{V} = \mathcal{M} \times \mathcal{F} \), where the fibre is in this case \( \mathcal{F} = \Omega^1(X, \text{ad}P) \oplus \Omega^{2,+}(X, \text{ad}P) \). The basic equations (3.1) can then be identified to be a section \( s : \mathcal{M} \to \mathcal{V} \) of the vector bundle \( \mathcal{V} \). In our case, the section reads, with a certain choice of normalization that makes easier the comparison with the results in sect. 2:

\[
s(A, C, B^+) = (\sqrt{2}(D_\mu C + \sqrt{2}D^\nu B^+_{\nu\mu}), -2(F^+_{\mu\nu} - i[B^+_{\mu\tau}, B^+_{\nu\tau}] - \frac{i}{\sqrt{2}}[B^+_{\mu\nu}, C])).
\]  

(3.5)

Notice that this section is gauge ad-equivariant, and the zero locus of the associated section \( \tilde{s} : \mathcal{M}/\mathcal{G} \to \mathcal{V}/\mathcal{G} \) gives precisely the moduli space of the topological theory.
It would be desirable to compute the dimension of this moduli space. The best we can do is to build the corresponding deformation complex whose index is known to compute, under certain assumptions, the dimension of the tangent space to the moduli space. This index provides what is called the virtual dimension of the moduli space. The deformation complex that corresponds to our moduli space is the following:

\[ 0 \to \Omega^0(X, \text{ad}P) \xrightarrow{\mathcal{C}} \Omega^1(X, \text{ad}P) \oplus \Omega^0(X, \text{ad}P) \oplus \Omega^{2,+}(X, \text{ad}P) \xrightarrow{\text{ds}} \Omega^1(X, \text{ad}P) \oplus \Omega^{2,+}(X, \text{ad}P) \to 0. \] (3.6)

The map \( \mathcal{C} : \Omega^0(X, \text{ad}P) \to TM \), given by (recall (3.3)):

\[ \mathcal{C}(\phi) = (d_A\phi, i[C, \phi], i[B^+, \phi]), \quad \phi \in \Omega^0(X, \text{ad}P), \] (3.7)

defines the vertical tangent space (gauge orbits) to the principal \( G \)-bundle. The map \( ds : T_{(A,C,B^+)}M \to F \) is given by the linearization of the basic equations (3.1):

\[ ds(\psi, \zeta, \tilde{\psi}^+) = \left( \sqrt{2}(D_\mu \zeta + i[\psi_\mu, C] + \sqrt{2}D^\nu \tilde{\psi}^+_{\nu\mu} + i\sqrt{2}[\psi^\nu, B^+_{\nu\mu}]), \right. \]
\[ -2(2(D_\nu \psi_\mu)^+ + i[\tilde{\psi}^+_{\mu\nu}, B^{++}_{\nu\mu}] - \frac{i}{\sqrt{2}}[\tilde{\psi}^+_{\mu\nu}, C] - \frac{i}{\sqrt{2}}[\tilde{B}^+_{\mu\nu}, \zeta]) \). (3.8)

Under suitable conditions (which happen to be the same vanishing theorems discussed in [5]), the index of the complex (3.6) computes the dimension of \( \text{Ker}(ds)/\text{Im}(\mathcal{C}) \), that is, the dimension of the moduli space under consideration. To calculate its index, the complex (3.6) can be split up into the Atiyah-Hitchin-Singer instanton deformation complex [32] for anti-self-dual (ASD) connections,

\[ (1) \quad 0 \to \Omega^0(X, \text{ad}P) \xrightarrow{d_A} \Omega^1(X, \text{ad}P) \xrightarrow{p^+d_A^*} \Omega^{2,+}(X, \text{ad}P) \to 0, \] (3.9)

and the complex associated to the operator,

\[ (2) \quad D = p^+d_A^* + d_A : \Omega^0(X, \text{ad}P) \oplus \Omega^{2,+}(X, \text{ad}P) \to \Omega^1(X, \text{ad}P), \] (3.10)

which is also the ASD instanton deformation complex. They contribute with opposite signs and therefore the net contribution to the index is zero, leaving us with
the result that the virtual dimension of the moduli space is zero.

The topological action

We now proceed to construct the topological action associated to this moduli problem, and we will do it within the Mathai-Quillen formalism. The Mathai-Quillen form gives a representative of the Thom class of the bundle $\mathcal{E} = \mathcal{M} \times_G \mathcal{F}$, and the integration over $\mathcal{M}/\mathcal{G}$ of the pullback of this Thom class under the section $\tilde{s} : \mathcal{M}/\mathcal{G} \to \mathcal{E} = \mathcal{M} \times_G \mathcal{F}$ gives the (generalized) Euler characteristic of $\mathcal{E}$, which is to be identified, from the field-theory point of view, with the partition function of the associated topological quantum field theory.

As a first step to construct the topological theory which corresponds to the moduli problem defined by the basic equations (3.1), we have to give explicitly the field content and the BRST symmetry of the theory. This will help to clarify the structure of the topological multiplet we introduced in sect. 2. In the field space $\mathcal{M} = \mathcal{A} \times \Omega^0(X, \text{ad} P) \times \Omega^2(X, \text{ad} P)$ we have the gauge connection $A_\mu$, the scalar field $C$ and the self-dual two-form $B^+_{\mu\nu}$, all with ghost number 0. In the (co)tangent space $T(\mathcal{A}, C, B^+)\mathcal{M} = \Omega^1(X, \text{ad} P) \oplus \Omega^0(X, \text{ad} P) \oplus \Omega^2(X, \text{ad} P)$ we have the anticommuting fields $\psi_\mu$, $\zeta$ and $\tilde{\psi}^+_{\mu\nu}$, all with ghost number 1 and which are to be interpreted as differential forms on the moduli space. In the fibre $\mathcal{F} = \Omega^1(X, \text{ad} P) \oplus \Omega^2(X, \text{ad} P)$ we have anticommuting fields with the quantum numbers of the equations, namely a one-form $\tilde{\chi}_\mu$ and a self-dual two-form $\chi^+_{\mu\nu}$, both with ghost number $-1$, and their superpartners, a commuting one-form $\tilde{H}_\nu$ and a commuting self-dual two-form $H^+_{\mu\nu}$, both with ghost number 0 and which appear as auxiliary fields in the associated field theory. And finally, associated to the gauge symmetry, we have a commuting scalar field $\phi \in \Omega^0(X, \text{ad} P)$ with ghost number $+2$ [29], and a multiplet of scalar fields $\tilde{\phi}$ (commuting and with ghost number $-2$) and $\eta$ (anticommuting and with ghost number $-1$), both also in $\Omega^0(X, \text{ad} P)$ and which enforce the horizontal projection $\mathcal{M} \to \mathcal{M}/\mathcal{G}$ [28]. The BRST symmetry of
the model is given by:

\[
\begin{align*}
\{Q, A_\mu \} &= \psi_\mu, \\
\{Q, C \} &= \zeta, \\
\{Q, B^{\mu +} \} &= \tilde{\psi}^{\mu +}, \\
\{Q, \phi \} &= 0, \\
\{Q, \tilde{\chi}_\mu \} &= \tilde{H}_\mu, \\
\{Q, \chi^{\mu +}_\mu \} &= H^{\mu +}_\mu, \\
\{Q, \tilde{\phi} \} &= \eta,
\end{align*}
\]

(3.11)

The BRST generator \(Q\) satisfies the algebra \(\{Q, Q\} = \delta_g(\phi)\), and can be seen to correspond to the Cartan model for the \(G\)-equivariant cohomology of \(\mathcal{M}\).

We are now ready to write the action for the topological field theory under consideration. Instead of writing the full expression for the Mathai-Quillen form, we define the action to be \(\{Q, \Psi\}\) for some appropriate gauge invariant gauge fermion \(\Psi\) [28]. The use of gauge fermions was introduced in the context of topological quantum field theory in [33] (see [20] for a review). As it is explained in detail in [28], the gauge fermion consists of two basic pieces, a localization gauge fermion, which essentially involves the equations defining the moduli problem and which in our case takes the form:

\[
\Psi_{\text{loc}} = \langle (\check{\chi}, \chi^{+}_\mu), s(A, C, B^+) \rangle + \langle (\check{\chi}, \chi^{+}_\mu), (\tilde{H}, H^+) \rangle = \\
\int_X \sqrt{g} \text{Tr} \left\{ \frac{1}{2} \chi^{+}_\mu \left( H^{\mu +}_\mu - 2(F^{\mu +}_\mu - i[B^{\mu +}_\mu, B^{\mu +}_\mu] - \frac{i}{\sqrt{2}}[B^{\mu +}_\mu, C]) \right) \right. \\
+ \check{\chi}_\mu \left( \tilde{H}_\mu + \sqrt{2} (D^\mu C + \sqrt{2} D_\mu B^{\mu +}_\mu) \right) \right\},
\]

(3.12)
and a projection gauge fermion, which enforces the horizontal projection, and which can be written as:

\[
\Psi_{\text{proj}} = \langle \tilde{\phi}, C^\dagger (\psi, \zeta, \tilde{\psi}) \rangle_g.
\]

(3.13)

where \(\langle , \rangle_g\) denotes the gauge invariant metric in \(\Omega^0(X, \text{ad}P)\), and the map \(C^\dagger : T\mathcal{M} \to \Omega^0(X, \text{ad}P)\) is the adjoint of the map \(C\) (3.7) with respect to the
Riemannian metrics (3.4) in $T\mathcal{M}$ and $\Omega^0(X, \text{ad}P)$. Since $\mathcal{C}(\phi), \phi \in \Omega^0(X, \text{ad}P)$, is given by (3.7), its adjoint is readily computed to be:

$$C^+(\psi, \zeta, \tilde{\psi}) = -D_\mu \psi^\mu + \frac{i}{2} [\tilde{\psi}^+\mu\nu, B^{+\mu\nu}] + i[\zeta, C], \quad (3.14)$$

where $(\psi, \zeta, \tilde{\psi}) \in T_{(A,C,B^+)}\mathcal{M}$. This leaves for the projection fermion (3.13) the expression:

$$\Psi_{\text{proj}} = \int_X \sqrt{g} \text{Tr} \{ \bar{\phi}(-D_\mu \psi^\mu + \frac{i}{2} [\tilde{\psi}^+\mu\nu, B^{+\mu\nu}] + i[\zeta, C]) \}. \quad (3.15)$$

In the Mathai-Quillen formalism the action is built out of the terms (3.12) and (3.15). However, as in the case of the Mathai-Quillen formulation of Donaldson-Witten theory [25], one must add another piece to the gauge fermion to make full contact with the corresponding twisted supersymmetric theory. In our case, this extra term is:

$$\Psi_{\text{extra}} = -\int_X \sqrt{g} \text{Tr} \{ \frac{i}{2} \bar{\eta}[\phi, \bar{\phi}] \}. \quad (3.16)$$

It is now straightforward to see that after the rescalings

$$A' = A, \quad \psi' = -\frac{1}{2} \psi, \quad C' = -\frac{1}{\sqrt{2}} C, \quad \tilde{\chi}' = \sqrt{2} \tilde{\chi}, \quad \bar{\phi}' = \frac{1}{2\sqrt{2}} \bar{\phi}, \quad \zeta' = -\zeta, \quad \bar{H}' = \sqrt{2} \bar{H}, \quad B^{+'} = \frac{1}{2} B^{+}, \quad \chi^{+'} = \chi^{+}, \quad \tilde{\psi}' = \frac{1}{2\sqrt{2}} \tilde{\psi}, \quad H^{+'} = H^{+}, \quad (3.17)$$

one recovers, in terms of the primed fields, the twisted model we analyzed in sect. 2 and that is encoded in (2.29) and (2.32).
3.2. Adjoint Non-Abelian Monopoles

As we saw before, the model arising from the second twist is equivalent to the theory of non-abelian monopoles discussed at length in [6-9]. The relevant basic equations for this model involve the self-dual part of the gauge connection $F^+$ and a certain complex spinor field $M$ taking values in the adjoint representation of some compact finite dimensional Lie group $G$:

\[
\begin{align*}
F^+_{\alpha\beta} + [\overline{M}(\alpha, M_{\beta})] &= 0, \\
D_{\alpha\beta} M^\alpha &= 0,
\end{align*}
\]

(3.18)

where $\overline{M}$ is the complex conjugate of $M$.

The topological framework

The geometrical setting is a certain oriented, closed Riemannian four-manifold $X$, that we will also assume to be spin. We will denote the positive and negative chirality spin bundles by $S^+$ and $S^-$ respectively. The field space is $M = \mathcal{A} \times \Gamma(X, S^+ \otimes \text{ad}P)$, where $\mathcal{A}$ is the space of connections on a principal $G$-bundle $P \to X$, and the second factor denotes the space of sections of the product bundle $S^+ \otimes \text{ad}P$, that is, positive chirality spinors taking values in the Lie algebra of the gauge group. The group $\mathcal{G}$ of gauge transformations of the bundle $P$ has an action on the field space which is given locally by:

\[
\begin{align*}
g^*(A) &= i(dg)g^{-1} + gAg^{-1}, \\
g^*(M) &= gMg^{-1},
\end{align*}
\]

(3.19)

where $M \in \Gamma(X, S^+ \otimes \text{ad}P)$ and $A$ is the gauge connection. In terms of the covariant derivative $d_A = d + i[A, ]$, the infinitesimal form of the transformations (3.19), with $g = \exp(-i\phi)$ and $\phi \in \Omega^0(X, \text{ad}P)$, takes the form:

\[
\begin{align*}
\delta_g(\phi)A &= d_A\phi, \\
\delta_g(\phi)M &= i[M, \phi].
\end{align*}
\]

(3.20)

The tangent space to the field space at the point $(A, M)$ is the vector space
\[ T_{(A,M)} \mathcal{M} = \Omega^1(X, \text{ad}P) \oplus \Gamma(X, S^+ \otimes \text{ad}P). \]

On \( T_{(A,M)} \mathcal{M} \) we can define a gauge-invariant Riemannian metric given by:

\[
\langle(\psi, \mu), (\theta, \omega)\rangle = \int_X \text{Tr} (\psi \wedge * \theta) + \frac{1}{2} \int_X \text{Tr} (\bar{\mu}^\alpha \omega_\alpha + \bar{\omega}^\alpha \mu_\alpha),
\]

where \( \psi, \theta \in \Omega^1(X, \text{ad}P) \) and \( \mu, \omega \in \Gamma(X, S^+ \otimes \text{ad}P) \).

The basic equations (3.18) are introduced in this framework as sections of the trivial vector bundle \( V = \mathcal{M} \times \mathcal{F} \), where the fibre is in this case \( \mathcal{F} = \Omega^{2,+}(X, \text{ad}P) \oplus \Gamma(X, S^- \otimes \text{ad}P) \). Taking into account the form of the basic equations, the section reads, up to some harmless normalization factors that we introduce for reasons that will become apparent soon:

\[
s(A, M) = \left( -2(F_{\alpha\beta}^* + [\mathcal{M}(\alpha, M_\beta)]) \right), \sqrt{2D}_{\alpha\dot{\alpha}} M^\alpha. \tag{3.22}
\]

The section (3.22) can be alternatively seen as a gauge ad-equivariant map from the principal \( G \)-bundle \( \mathcal{M} \rightarrow \mathcal{M} / G \) to the vector space \( \mathcal{F} \), and in this way it descends naturally to a section \( \tilde{s} \) of the associated vector bundle \( \mathcal{M} \times_G \mathcal{F} \), whose zero locus gives precisely the moduli space of the topological theory. It would be desirable to compute the dimension of this moduli space. The relevant deformation complex (which allows one to compute, in a general situation, the virtual dimension of the moduli space) is the following:

\[
0 \rightarrow \Omega^0(X, \text{ad}P) \xrightarrow{C} \Omega^1(X, \text{ad}P) \oplus \Gamma(X, S^+ \otimes \text{ad}P) \xrightarrow{ds} \Omega^{2,+}(X, \text{ad}P) \oplus \Gamma(X, S^- \otimes \text{ad}P). \tag{3.23}
\]

The map \( C : \Omega^0(X, \text{ad}P) \rightarrow T\mathcal{M} \) is given by:

\[
C(\phi) = (d_A \phi, i[M, \phi]), \quad \phi \in \Omega^0(X, \text{ad}P), \tag{3.24}
\]

while the map \( ds : T_{(A,M)} \mathcal{M} \rightarrow \mathcal{F} \) is provided by the linearization of the basic
equations (3.18):

\[
ds(\psi, \mu) = (-4\sigma_{\alpha\beta}^{\mu\nu} D_\mu \psi_\nu - 2[\bar{\mu}_{(\alpha}, M_{\beta)}] - 2[M_{(\alpha, \mu}] \sqrt{2}D_{\alpha\beta}^\mu + \sqrt{2}[\psi_{\alpha\beta}, M^\alpha])
\]  

(3.25)

Under suitable conditions, the index of the complex (3.23) computes the dimension of \(\text{Ker}(ds)/\text{Im}(C)\). To calculate the index, the complex (3.23) can be split up into the ASD-instanton deformation complex:

\[
(1) \quad 0 \longrightarrow \Omega^0(X, \text{ad}P) \xrightarrow{d_4} \Omega^1(X, \text{ad}P) \xrightarrow{p_{+}} \Omega^2(X, \text{ad}P) \longrightarrow 0,
\]

(3.26)

whose index is \(p_1(\text{ad}P) + \dim(G)(\chi + \sigma)/2\), being \(p_1(\text{ad}P)\) the first Pontryagin class of the adjoint bundle \(\text{ad}P\), and the complex associated to the twisted Dirac operator

\[
(2) \quad D : \Gamma(X, S^+ \otimes \text{ad}P) \longrightarrow \Gamma(X, S^- \otimes \text{ad}P),
\]

(3.27)

whose index is \(p_1(\text{ad}P)/2 - \dim(G)\sigma/8\). Thus, the index of the total complex (which gives minus the virtual dimension of the moduli space) is:

\[
-\dim(M) = \text{ind}(1) - 2 \times \text{ind}(2) = \dim(G)\frac{(2\chi + 3\sigma)}{4}
\]

(3.28)

where \(\chi\) is the Euler characteristic of the 4-manifold \(X\) and \(\sigma\) is its signature. The factor of two appears in (3.28) since we want to compute the real dimension of the moduli space.

### The topological action

We now proceed as in the previous case. To build a topological theory out of the moduli problem defined by the equations (3.18) we need the following multiplet of fields. For the field space \(\mathcal{M} = \mathcal{A} \times \Gamma(X, S^+ \otimes \text{ad}P)\) we introduce commuting fields \((A, M)\), both with ghost number 0, and their corresponding superpartners, the anticommuting fields \(\psi\) and \(\mu\), both with ghost number 1. For the fibre
\[ \mathcal{F} = \Omega^{2,+}(X, \text{ad}P) \oplus \Gamma(X, S^- \otimes \text{ad}P) \] we introduce anticommuting fields \( \chi^+ \) and \( \nu \) respectively, both with ghost number \(-1\), and their superpartners, a commuting self-dual two-form \( H^+ \) and a commuting negative chirality spinor \( h \), both with ghost number \( 0 \) and which appear as auxiliary fields in the associated field theory. And finally, associated to the gauge symmetry, we have a commuting scalar field \( \phi \in \Omega^0(X, \text{ad}P) \) with ghost number \(+2\), and a multiplet of scalar fields \( \lambda \) (commuting and with ghost number \(-2\)) and \( \eta \) (anticommuting and with ghost number \(-1\)), both also in \( \Omega^0(X, \text{ad}P) \) and which enforce the horizontal projection \( \mathcal{M} \rightarrow \mathcal{M}/\mathcal{G} \).

The BRST symmetry of the model is given by:

\[
\begin{align*}
[Q, A_\mu] &= \psi_\mu, & \{Q, \psi_\mu\} &= D_\mu \phi, \\
[Q, M_\alpha] &= \mu_\alpha, & \{Q, \mu_\alpha\} &= i [M_\alpha, \phi], \\
[Q, \phi] &= 0, & \{Q, \chi_+^{\alpha\beta}\} &= H_+^{\alpha\beta}, \\
\{Q, \chi^+_\alpha\} &= H_+^\alpha, & [Q, H^+_{\alpha\beta}] &= i [\chi_+^{\alpha\beta}, \phi], \\
\{Q, \nu_\dot{\alpha}\} &= h_\dot{\alpha}, & [Q, h_\dot{\alpha}] &= i [\nu_\dot{\alpha}, \phi], \\
[Q, \lambda] &= \eta, & \{Q, \eta\} &= i [\lambda, \phi].
\end{align*}
\] (3.29)

This BRST algebra closes up to a gauge transformation generated by \( \phi \).

We have to give now the expressions for the different pieces of the gauge fermion. For the localization gauge fermion we have:

\[
\Psi_{\text{loc}} = i \langle (\chi^+, \nu), s(A, M) \rangle - \langle (\chi^+, \nu), (H^+, h) \rangle = \\
\int_X \sqrt{g} \text{Tr} \left\{ \frac{1}{4} \chi_+^{\alpha\beta} \left( H^{+\alpha\beta} + 2i (F^{+\alpha\beta} + [\overline{M}^\alpha, M^\beta]) \right) \right\} \\
+ \frac{1}{2} \bar{\nu}^{\dot{\alpha}} \left( h_\dot{\alpha} - i \sqrt{2} D_{\alpha\dot{\alpha}} M^\alpha \right) - \frac{1}{2} \bar{\nu}^{\dot{\alpha}} \left( \bar{h}_{\dot{\alpha}} - i \sqrt{2} D_{\alpha\dot{\alpha}} \bar{M}^\alpha \right),
\] (3.30)

and for the projection gauge fermion, which enforces the horizontal projection,

\[
\Psi_{\text{proj}} = \langle \lambda, \bar{c}^\dagger(\psi, \mu) \rangle g = \\
\int_X \sqrt{g} \text{Tr} \left\{ \lambda \left( -D_\mu \psi^\mu + i \frac{1}{2} [\bar{\mu}^\alpha, M_\alpha] - i \frac{1}{2} [\overline{M}^\alpha, \mu_\alpha] \right) \right\}.
\] (3.31)

As in the previous case, it is necessary to add an extra piece to the gauge
fermion to make full contact with the corresponding twisted supersymmetric theory. In this case, this extra term is:

\[ \Psi_{\text{extra}} = - \int_X \sqrt{g} \text{Tr} \left\{ \frac{i}{2} \lambda [\eta, \phi] \right\}. \]  

(3.32)

It is now straightforward to see that, after making the following redefinitions,

\[
\begin{align*}
A' &= A, & M' &= M, & H'^+ &= H^+, \\
\psi' &= \frac{1}{2} \psi, & M' &= \frac{1}{2} M, & \nu' &= -2\nu, \\
\phi' &= \frac{1}{2\sqrt{2}} \phi, & \mu' &= -\frac{1}{\sqrt{2}} \mu, & \bar{\nu}' &= -\bar{\nu}, \\
\chi' &= -2\sqrt{2} \lambda, & \bar{\mu}' &= -\frac{1}{2\sqrt{2}} \bar{\mu}, & h' &= 2h, \\
\eta' &= -2\eta, & \chi'^+ &= \chi^+, & \bar{h}' &= -\bar{h},
\end{align*}
\]

(3.33)

one recovers, in terms of the primed fields, the twisted model summarized in (2.47) and (2.51).

3.3. The Amphicheiral Theory

The relevant basic equations for this model involve the self-dual part of the gauge connection \( F^+ \) and a certain real vector field \( V_\mu \) taking values in the adjoint representation of some finite dimensional compact Lie group \( G \):

\[
\begin{align*}
F^+_{\mu\nu} - i[V_\mu, V_\nu]^+ &= 0, \\
(D_{[\mu} V_{\nu]})^- &= 0, \\
D_\mu V^\mu &= 0.
\end{align*}
\]

(3.34)

The topological framework

The geometrical setting is a certain compact, oriented Riemannian four-manifold \( X \), and the field space is \( \mathcal{M} = \mathcal{A} \times \Omega^1(X, \text{adP}) \), where \( \mathcal{A} \) is the space of
connections on a principal $G$-bundle $P \to X$, and the second factor denotes, as we have already seen before, 1-forms on $X$ taking values in the Lie algebra of $G$. The group $G$ of gauge transformations of the bundle $P$ has an action on the field space which is given locally by:

$$g^*(A) = i(dg)g^{-1} + gAg^{-1},$$
$$g^*(V) = gVg^{-1},$$  \hspace{1cm} (3.35)

where $V \in \Omega^1(X, \text{ad}P)$ and $A$ is the gauge connection. In terms of the covariant derivative $d_A = d + i[A, ]$, the infinitesimal form of the transformations (3.35), with $g = \exp(-iC)$ and $C \in \Omega^0(X, \text{ad}P)$, takes the form:

$$\delta_g(C)A = d_AC,$$
$$\delta_g(C)V = i[V, C].$$  \hspace{1cm} (3.36)

The tangent space to the field space at the point $(A, V)$ is the vector space $T_{(A, V)}\mathcal{M} = \Omega^1_{(A)}(X, \text{ad}P) \oplus \Omega^1_{(V)}(X, \text{ad}P)$, where $\Omega^1_{(A)}(X, \text{ad}P)$ denotes the tangent space to $\mathcal{A}$ at $A$, and $\Omega^1_{(V)}(X, \text{ad}P)$ denotes the tangent space to $\Omega^1(X, \text{ad}P)$ at $V$. On $T_{(A, V)}\mathcal{M}$, the gauge-invariant Riemannian metric (inherited from that on $X$) is defined as:

$$\langle (\psi, \tilde{\psi}), (\theta, \tilde{\omega}) \rangle = \int_X \text{Tr}(\psi \wedge *\theta) + \int_X \text{Tr}(\tilde{\psi} \wedge *\tilde{\omega})$$  \hspace{1cm} (3.37)

where $\psi, \theta \in \Omega^1_{(A)}(X, \text{ad}P)$ and $\tilde{\psi}, \tilde{\omega} \in \Omega^1_{(V)}(X, \text{ad}P)$.

The basic equations (3.34) are introduced in this framework as sections of the trivial vector bundle $\mathcal{V} = \mathcal{M} \times \mathcal{F}$, where the fibre is in this case $\mathcal{F} = \Omega^{2, +}(X, \text{ad}P) \oplus \Omega^{2, -}(X, \text{ad}P) \oplus \Omega^0(X, \text{ad}P)$. Taking into account the form of the basic equations, the section reads:

$$s(A, V) = (-2(F^+_{\mu} - i[V_\mu, V_\nu]^+), 4(D_{\mu}V_\nu)^-, \sqrt{2}D_\mu V^\mu).$$  \hspace{1cm} (3.38)

The section (3.38), being gauge ad-equivariant, descends to a section $\tilde{s}$ of the associated vector bundle $\mathcal{M} \times_G \mathcal{F}$ whose zero locus gives precisely the moduli
space of the topological theory. It would be desirable to compute the dimension of this moduli space. The relevant deformation complex is the following:

$$0 \rightarrow \Omega^0(X, \text{ad}P) \xrightarrow{C} \Omega^1_{(A)}(X, \text{ad}P) \oplus \Omega^1_{(V)}(X, \text{ad}P)$$

$$\xrightarrow{ds} \Omega^2_{(A)}(X, \text{ad}P) \oplus \Omega^2_{(V)}(X, \text{ad}P) \oplus \Omega^0(X, \text{ad}P) \rightarrow 0.$$ (3.39)

The map $C : \Omega^0(X, \text{ad}P) \rightarrow T\mathcal{M}$ is given by:

$$C(C) = (d_A C, i[V, C]), \quad C \in \Omega^0(X, \text{ad}P),$$ (3.40)

while the map $ds : T_{(A,V)}\mathcal{M} \rightarrow \mathcal{F}$ is given by the linearization of the basic equations (3.34):

$$ds(\psi, \tilde{\psi}) = (-4(D_{[\mu} \psi_{\nu]})^+ + 4i[\tilde{\psi}_{[\mu}, V_{\nu]])^+$$

$$4(D_{[\mu} \tilde{\psi}_{\nu]})^- + 4i[\psi_{[\mu}, V_{\nu]]^-)$$

$$+ \sqrt{2}D_\mu \tilde{\psi}^\mu + \sqrt{2}i[\psi_{[\mu}, V^\mu]^-).$$ (3.41)

Under suitable conditions, the index of the complex (3.39) computes the dimension of Ker($ds$)/Im($C$). To calculate its index, the complex (3.6) can be split up into the ASD-instanton deformation complex:

$$1) 0 \rightarrow \Omega^0(X, \text{ad}P) \xrightarrow{d_A} \Omega^1(X, \text{ad}P) \xrightarrow{p} \Omega^2(X, \text{ad}P) \rightarrow 0,$$ (3.42)

and the complex associated to the operator

$$2) D = p^-d_A + d_A^* : \Omega^1(X, \text{ad}P) \rightarrow \Omega^0(X, \text{ad}P) \oplus \Omega^2(X, \text{ad}P),$$ (3.43)

which is easily seen to correspond to the instanton deformation complex for self-dual (SD) connections. Thus, the index of the total complex (which gives minus the virtual dimension of the moduli space) is:

$$-\dim(\mathcal{M}) = \text{ind}(1) - \text{ind}(2) = \text{ind}(\text{ASD}) + \text{ind}(\text{SD}) =$$

$$= p_1(\text{ad}P) + \frac{1}{2} \dim(G)(\chi + \sigma) - p_1(\text{ad}P) + \frac{1}{2} \dim(G)(\chi - \sigma) = \dim(G)\chi,$$ (3.44)

where $p_1(\text{ad}P)$ is the first Pontryagin class of the adjoint bundle $\text{ad}P$, $\chi$ is the Euler characteristic of the 4-manifold $X$ and $\sigma$ is its signature.
The topological action

We now proceed as in the previous cases. To build a topological theory out of the moduli problem defined by the equations (3.34), we need the following multiplet of fields. For the field space \( \mathcal{M} = \mathcal{A} \times \Omega^1(X, \text{ad} P) \) we introduce the gauge connection \( A_\mu \) and the one-form \( V_\mu \), both commuting and with ghost number 0. For the (co)tangent space \( T(A, V)_\mathcal{M} = \Omega^1(A) \times \Omega^1(X, \text{ad} P) \oplus \Omega^1(V) \times \Omega^0(X, \text{ad} P) \) we introduce the anticommuting fields \( \psi_\mu \) and \( \tilde{\psi}_\mu \), both with ghost number 1 and which can be seen as differential forms on the moduli space. For the fibre \( \mathcal{F} = \Omega^2^+ (X, \text{ad} P) \oplus \Omega^2^- (X, \text{ad} P) \oplus \Omega^0 (X, \text{ad} P) \) we have anticommuting fields with the quantum numbers of the equations, namely a self-dual two-form \( \chi^+_{\mu\nu} \), an anti-self-dual two-form \( \chi^-_{\mu\nu} \) and a 0-form \( \tilde{\eta} \), all with ghost number \(-1\), and their superpartners, a commuting self-dual two-form \( N^+_{\mu\nu} \), a commuting anti-self-dual two-form \( N^-_{\mu\nu} \) and a commuting 0-form \( P \), all with ghost number 0 and which appear as auxiliary fields in the associated field theory. And finally, associated to the gauge symmetry, we introduce a commuting scalar field \( C \in \Omega^0(X, \text{ad} P) \) with ghost number +2, and a multiplet of scalar fields \( B \) (commuting and with ghost number \(-2\)) and \( \eta \) (anticommuting and with ghost number \(-1\)), both also in \( \Omega^0(X, \text{ad} P) \) and which enforce the horizontal projection \( \mathcal{M} \to \mathcal{M}/\mathcal{G} \) [28]. The BRST symmetry of the model is given by:

\[
\begin{align*}
[Q, A_\mu] &= \psi_\mu, & \{Q, \psi_\mu\} &= D_\mu C, \\
[Q, V_\mu] &= \tilde{\psi}_\mu, & \{Q, \tilde{\psi}_\mu\} &= i [V_\mu, C], \\
[Q, C] &= 0, \\
\{Q, \chi^+_{\mu\nu}\} &= N^+_{\mu\nu}, & [Q, N^+_{\mu\nu}] &= i [\chi^+_{\mu\nu}, C], \\
\{Q, \chi^-_{\mu\nu}\} &= N^-_{\mu\nu}, & [Q, N^-_{\mu\nu}] &= i [\chi^-_{\mu\nu}, C], \\
\{Q, \tilde{\eta}\} &= P, & [Q, P] &= i [\tilde{\eta}, C], \\
[Q, B] &= \eta, & \{Q, \eta\} &= i [B, C].
\end{align*}
\]

(3.45)

This BRST algebra closes up to a gauge transformation generated by \( C \).
We have to give now the expressions for the different pieces of the gauge fermion. For the localization gauge fermion we have:

\[
\Psi_{\text{loc}} = \langle (\chi^+, \chi^-, \tilde{\eta}), s(A, V) \rangle + \langle (\chi^+, \chi^-, \tilde{\eta}), (N^+, N^-, P) \rangle = \int_X \sqrt{g} \text{Tr} \left\{ \frac{1}{2} \chi^+_{\mu\nu} (N^+_{\mu\nu} - 2F^{+\mu\nu} + 2i[V^\mu, V^{\nu}]^+) 
+ \frac{1}{2} \chi^-_{\mu\nu} (N^-_{\mu\nu} + 4(D[V^\mu]^{-}) + \tilde{\eta}(P + \sqrt{2}D_V V^\mu) \right\},
\]

while for the projection gauge fermion, which enforces the horizontal projection, we have:

\[
\Psi_{\text{proj}} = \langle B, C^\dagger (\psi, \tilde{\psi}) \rangle g = \int_X \sqrt{g} \text{Tr} \left\{ B(-D^\mu \psi^\mu + i[\tilde{\psi}^\mu, V^\mu]) \right\}.
\]

As in the other cases we have studied, it is necessary to add an extra piece to the gauge fermion to make full contact with the corresponding twisted supersymmetric theory. In this case, this extra term is:

\[
\Psi_{\text{extra}} = \int_X \sqrt{g} \text{Tr} \left\{ \frac{i}{2} \eta[B, C] \right\}.
\]

It is now straightforward to see that, with the redefinitions

\[
A' = A, \\
\psi' = \frac{1}{2} \psi, \\
C' = \frac{1}{2\sqrt{2}} C, \\
B' = -2\sqrt{2} B, \\
\eta' = -2\eta,
\]

\[
V' = -\frac{1}{\sqrt{2}} V, \\
\tilde{\psi}' = \frac{1}{2} \tilde{\psi}, \\
\tilde{\eta}' = -2\sqrt{2} \tilde{\eta}, \\
P' = -2\sqrt{2} P, \\
\chi'^+ = -\chi^+, \\
\chi'^- = \chi^-, \\
N'^+ = -N^+, \\
N'^- = N^-,
\]

one recovers, in terms of the primed fields, the twisted model summarized in (2.76) and (2.77), which corresponds to the topological symmetry \(Q\).
It is worth to remark that one could also consider the “dual” problem built out of the basic equations:

\[
\begin{align*}
F_{\mu\nu}^- - i[V_\mu, V_\nu]^- &= 0, \\
(D_{\mu}V_{\nu})^+ &= 0, \\
D_{\mu}V^\mu &= 0.
\end{align*}
\]

The resulting theory corresponds precisely to the second type of theory obtained in the previous section in our discussion of the third twist. The corresponding action has the form \(\{\tilde{Q}, \psi^-\}\) where \(\tilde{Q}\) is given in (2.73) and \(\Psi^-\) is the result of performing a \(\mathbb{Z}_2\)-transformation (see (2.66)) on the gauge fermion \(\Psi^+\) in (2.75).
4. Observables

In this section we will analyze the structure of the observables for each of the three twists. Observables are operators which are $Q$-invariant but are not $Q$-exact. A quick look at the $Q$-transformations which hold in each twist shows that the observables are basically the same as in ordinary Donaldson-Witten theory. Indeed, from (2.29) or (3.11) one finds that the trio, $A_\mu$, $\psi_\mu$ and $\phi$, which is present in the first twist transform adequately so that the operators,

$$
W_0 = \text{Tr}(\phi^2), \quad W_1 = -\sqrt{2}\text{Tr}(\phi \wedge \psi), \\
W_2 = \text{Tr}\left(\frac{1}{2} \psi \wedge \psi + \frac{1}{\sqrt{2}} \phi \wedge F\right), \quad W_3 = -\frac{1}{2}\text{Tr}(\psi \wedge F),
$$

satisfy the descent equations,

$$
\delta W_i = dW_{i-1},
$$

which imply that,

$$
\mathcal{O}(\gamma_j) = \int_{\gamma_j} W_j,
$$

being $\gamma_j$ homology cycles of $X$, are observables. Of course, as usual, this set can be enlarged for gauge groups possessing other independent Casimirs besides the quadratic one. The transformations (2.51) or (3.29) for the second twist, and (2.76) or (3.45) for the third (after replacing $C$ by $\phi$) show that these other twists possess a similar set of observables.

Topological invariants are obtained considering the vacuum expectation value of arbitrary products of observables:

$$
\langle \prod_{\gamma_j} \mathcal{O}(\gamma_j) \rangle.
$$

The general form of this vacuum expectation value is,

$$
\langle \prod_{\gamma_j} \mathcal{O}(\gamma_j) \rangle = \sum_k \langle \prod_{\gamma_j} \mathcal{O}(\gamma_j) \rangle_k e^{-2\pi i k \tau},
$$

55
where $k$ is the instanton number and $\langle \prod_{\hat{\gamma}_j} \mathcal{O}^{(\gamma_j)} \rangle_k$ is the vacuum expectation value computed at a fixed value of $k$ with an action which is $Q$-exact,

$$\langle \prod_{\hat{\gamma}_j} \mathcal{O}^{(\gamma_j)} \rangle_k = \int [df]_k e^{\{Q, \Psi\}} \prod_{\hat{\gamma}_j} \mathcal{O}^{(\gamma_j)}.$$  \hspace{1cm} (4.6)

In this equation $[df]_k$ denotes collectively the measure indicating that only gauge configurations of instanton number $k$ enter in the functional integral. These quantities are independent of the coupling constant $e$. When analyzed in the weak coupling limit the contributions to the functional integral come from field configurations which are solutions to the equations which define the moduli problems which we have associated to each twist in the previous section. All the dependence of the observables on $\tau$ is contained in the sum (4.5).

The $Q$-symmetry of the theory imposes a selection rule for the products entering (4.4) which could lead to a possibly non-vanishing result: the ghost number of (4.4) must be equal to the virtual dimension of the corresponding moduli space. For the first twist this implies that the only observable is the partition function of the theory. In fact, this is the quantity computed by Vafa and Witten in [5] for some specific situations to obtain a test of duality. The resulting partition functions $Z(\tau)$ turn out to transform as modular forms under $Sl(2, \mathbb{Z})$ transformations.

For the other two twists the virtual dimension is not zero but it is independent of the instanton number $k$. This means that, as in the case of the first twist, one could obtain contributions from many values of $k$. Possibly non-trivial topological invariants for these cases correspond to products of operators (4.4) such that their ghost number matches the virtual dimension $\dim(G)(2\chi + 3\sigma)/4$ for the case of the second twist, or $\dim(G)\chi$ for the case of the third. One important question is whether or not the vacuum expectation values of these observables have good modular properties under $Sl(2, \mathbb{Z})$ transformations. We will show in the rest of this section that in the case of the third twist the vacuum expectation values are actually independent of $\tau$. Thus, further non-trivial duality tests can be addressed only in the second twist. We will not consider this issue in this paper.
As indicated in the introduction and proved in sect. 2, the third twist leads to a topological quantum field theory which is amphicheiral. We will show now that in addition this theory possesses the property that the vacuum expectation values of products of its observables are independent of $\tau$. Thus, in some sense the invariance under $\text{SL}(2, \mathbb{Z})$ is trivially realized in this case.

In order to study the vacuum expectation values of products of observables in the third twist we are going to consider the action (2.69) (in its covariantized form) in which the auxiliary fields appear quadratically. The bosonic part of this action involving only the field strength $F_{\mu\nu}$ and the vector field $V_\mu$ can be written in three equivalent forms. The form of the action, $S = \{Q, \hat{\Psi}^+\} - 2\pi ik\tau$, leads to,

$$
- \int_X d^4x \sqrt{g} \text{Tr} \left\{ \frac{1}{2e^2} (F^{+\mu\nu} - 2i[V^\mu, V^\nu]^+) \right. \\
+ \left. \frac{1}{e^2} (D_\mu V^\mu)^2 \right\} - 2\pi i \tau \frac{1}{32\pi^2} \int_X d^4x \sqrt{g} \text{Tr} \left\{ *F_{\mu\nu}F^{\mu\nu} \right\},
$$

the form, $S = \{\tilde{Q}, \hat{\Psi}^-\} - 2\pi ik\bar{\tau}$, to,

$$
- \int_X d^4x \sqrt{g} \text{Tr} \left\{ \frac{1}{2e^2} (F^{-\mu\nu} - 2i[V^\mu, V^\nu]^-) \right. \\
+ \left. \frac{1}{e^2} (D_\mu V^\mu)^2 \right\} - 2\pi i \bar{\tau} \frac{1}{32\pi^2} \int_X d^4x \sqrt{g} \text{Tr} \left\{ *F_{\mu\nu}F^{\mu\nu} \right\},
$$

and, finally, the form, $S = \frac{1}{2} \{Q, \hat{\Psi}^+\} + \frac{1}{2} \{\tilde{Q}, \hat{\Psi}^-\} - 2\pi i k \text{Re}(\tau)$, to,

$$
- \int_X \text{Tr} \left\{ \frac{1}{4} (F^{\mu\nu} - 2i[V^\mu, V^\nu])^2 \\
+ 2(D_\mu V^\mu)^2 + (D_\mu V^\mu)^2 \right\} \\
- 2\pi i \text{Re}(\tau) \frac{1}{32\pi^2} \int_X d^4x \sqrt{g} \text{Tr} \left\{ *F_{\mu\nu}F^{\mu\nu} \right\}.
$$

Standard arguments in topological quantum field theory show that the weak coupling limit is exact. In the first case this limit implies that the contributions to
the functional integral correspond to the moduli space defined by the equations (3.34). Notice that the normalization factor for $V_\mu$ in (3.49) has to be taken into account since (4.7) correspond to the action resulting after twisting. Similarly, in the second case the weak coupling limit contributions correspond to the moduli space defined by the equations (3.50). In the third case, however, the contributions correspond to the solution of the following set of equations:

$$\begin{cases} F_{\mu\nu} - 2i[V_\mu, V_\nu] = 0, \\ D_{[\mu} V_{\nu]} = 0, \\ D_\mu V^\mu = 0, \end{cases}$$

which define a moduli space which is the intersection of the other two. This is the moduli space which appears in the formulation of the third twist presented in [16,17]. Notice that the three points of view lead to three different types of dependence on $\tau$. The first one implies that vacuum expectation values are holomorphic in $\tau$, the second that they are antiholomorphic, and the third that they depend only on the real part of $\tau$. We will solve this puzzle showing that actually the vacuum expectation values are just real numbers and not functions of $\tau$.

We first prove that any solution of (4.10) must involve a gauge connection whose instanton number is zero. Indeed, from the identity,

$$\int_X d^4x \sqrt{g} \text{Tr} \left\{ *F_{\mu\nu} \left( F^{\mu\nu} - 2i[V_\mu, V_\nu] \right) - 4 * D^{[\mu} V^{\nu]} D_{[\mu} V_{\nu]} \right\} = \int_X d^4x \sqrt{g} \text{Tr} \left\{ *F_{\mu\nu} F^{\mu\nu} \right\},$$

follows that any solution of (4.10) must have $k = 0$. This implies that only configurations with vanishing instanton number contribute and therefore:

$$\langle \prod_{\gamma_j} O^{(\gamma_j)} \rangle = \langle \prod_{\gamma_j} O^{(\gamma_j)} \rangle_{k=0},$$

which is clearly independent of $\tau$. From (4.7) and (4.8) follows that for $k = 0$ a solution to the equations of the first moduli space (3.34) is also a solution to the
ones of the second (3.50) and therefore also to the ones of the third (4.10). For
$k \neq 0$, however, one can have solutions to the equations of the first moduli space
which are not solutions to the equations of the second and therefore neither to the
ones of the third. For $k \neq 0$ the quantities $\langle \prod_{\gamma_j} \mathcal{O}^{(\gamma_j)} \rangle_k$ are different in each point
of view. They clearly vanish in the third case. On the other hand, there is no
reason why they should also vanish in the other two cases. Our results, however,
suggest that they do vanish.

We will end this section discussing a vanishing theorem which tells us when
the third moduli space (4.10) reduces to the moduli space of flat connections. The
equations (4.10) have the immediate solution $V = 0$, $F = 0$, that is, the moduli
space of flat connections is contained in the moduli space defined by the equations
(4.10). We will show that under certain conditions both moduli spaces are in fact
the same. To see this note that since,

$$
\int_X d^4x \sqrt{g} \text{Tr} \left\{ \frac{1}{4} \left( F^{\mu\nu} - 2i[V^\mu, V^\nu] \right)^2 + 2(D[\mu V^{\nu}])^2 + (D_\mu V^\mu)^2 \right\} =
$$

$$
= \int_X d^4x \sqrt{g} \text{Tr} \left\{ D_\mu V_\nu D^{\mu\nu} + R_{\mu\nu} V^\mu V^\nu + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - ([V^\mu, V^\nu])^2 \right\}, \quad (4.13)
$$

it follows that if the Ricci tensor is such that

$$
R_{\mu\nu} V^\mu V^\nu > 0 \quad \text{for } V \neq 0, \quad (4.14)
$$

the solutions to the equations (4.10) are necessarily of the form $V = 0$, $F = 0$, and
thus the moduli space is the space of flat gauge connections on $X$. 59
5. Concluding Remarks

In this paper we have analyzed in full detail the three non-equivalent twists of $N = 4$ supersymmetric gauge theory. The first twist leads to a topological quantum field theory whose observables transform as modular forms under $Sl(2, \mathbb{Z})$ transformations [5]. The second twist leads to the theory of non-abelian monopoles in the adjoint representation of the gauge group. In this theory, as in the previous one, there is a non-trivial dependence on $\tau$ and one expects that its observables have good transformation properties under $Sl(2, \mathbb{Z})$. This is an important issue that certainly should be addressed. The third twist leads to a topological quantum field theory which is amphicheiral. We have shown that in this theory the vacuum expectation values of products of its observables do not depend on $k$. Hence, barring possible anomalous dependences in $\tau$ like the ones explicitly unveiled in [5], the theory is trivially invariant under $Sl(2, \mathbb{Z})$ transformations.

The moduli spaces which, from the point of view of the Mathai-Quillen formalism, correspond to each twisted theory have been identified. In the third twist, due to the amphicheiral character of the topological quantum field theory one finds three different moduli spaces defined by the equations (3.34), (3.50) and (4.10). These moduli spaces coincide when the integral of the Chern class of the gauge field vanishes. We have shown that only the $k = 0$ sector contributes to the functional integral, leading to topological invariants which, therefore, do not depend on $\tau$.

Of the three topological quantum field theories, the one corresponding to the second twist is not valid on arbitrary oriented four-manifolds. This theory contains spinors and therefore it only exists for spin manifolds. The generalization of this theory to arbitrary oriented four-manifolds can be easily done introducing a Spin$^c$ structure following the construction recently presented in [34]. In this construction the baryon number symmetry of the original physical theory is gauged introducing a connection which in the twisted theory is identified with the Spin$^c$ connection.

We finish by making the remark that the topological quantum field theories
originated from $N = 4$ supersymmetric gauge theories are not the only ones which can lead to a theory with a non-trivial dependence on $\tau$. Any conformally invariant $N = 2$ supersymmetric gauge theory would have the same property. This is for instance the case for an $N = 2$ supersymmetric gauge theory with gauge group $SU(N_c)$ and $2N_c$ hypermultiplets in the fundamental representation. These theories should be studied along the lines of this paper and the duality properties of the resulting topological quantum field theory should be analyzed.

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APPENDIX

We will now summarize the conventions used in this paper. Basically we will describe the elements of the positive and negative chirality spin bundles $S^+$ and $S^-$ on a four-dimensional spin manifold $X$ endowed with a vierbein $e^{m\mu}$ and a spin connection $\omega_{\mu}^{mn}$. The spaces of sections of the spin bundles $S^+$ and $S^-$ correspond, from the field-theory point of view, to the set of two-component Weyl spinors defined on the manifold $X$. These are the simplest irreducible representations of the holonomy group $SO(4)$. We will denote positive-chirality (or negative-chirality) spinors by indices $\alpha, \beta, \ldots = 1, 2$ (or $\dot{\alpha}, \dot{\beta}, \ldots = 1, 2$). Spinor indices are raised and lowered with the $SU(2)$ invariant tensor $C_{\alpha\beta}$ (or $\bar{C}_{\dot{\alpha}\dot{\beta}}$) and its inverse $C^{\alpha\beta}$ (or $\bar{C}^{\dot{\alpha}\dot{\beta}}$), with the conventions $C_{21} = C^{12} = +1$, so that,

$$
C_{\alpha\beta}C^{\beta\gamma} = \delta_\alpha^\gamma, \quad C_{\alpha\beta}C^{\gamma\delta} = \delta_\alpha^\delta \delta_\beta^\gamma - \delta_\alpha^\gamma \delta_\beta^\delta, \\
C_{\dot{\alpha}\dot{\beta}}C^{\dot{\beta}\dot{\gamma}} = \delta_{\dot{\alpha}}^\dot{\gamma}, \quad C_{\dot{\alpha}\dot{\beta}}C^{\dot{\gamma}\dot{\delta}} = \delta_{\dot{\alpha}}^\dot{\delta} \delta_{\dot{\beta}}^\dot{\gamma} - \delta_{\dot{\alpha}}^\dot{\gamma} \delta_{\dot{\beta}}^\dot{\delta}.
$$

(A.1)

The spinor representations and the vector representation associated to $S^+ \times S^-$ are related by the Clebsch-Gordan $\sigma^m_{\alpha\dot{\alpha}} = (i1, \bar{\tau})$ and $\bar{\sigma}^{m\dot{\alpha}} = (i1, -\bar{\tau})$, where $1$
is the $2 \times 2$ unit matrix and $\vec{\tau} = (\tau^1, \tau^2, \tau^3)$ are the Pauli matrices,

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hspace{1cm} (A.2)

The Pauli matrices satisfy:

$$\tau_a \tau_b = i \epsilon_{abc} \tau_c + \delta_{ab} 1,$$  \hspace{1cm} (A.3)

where $\epsilon_{abc}$ is the totally antisymmetric tensor with $\epsilon_{123} = 1$.

Under an infinitesimal $SO(4)$ rotation a Weyl spinor $M_\alpha$, $\alpha = 1, 2$, associated to $S^+$, transforms as:

$$\delta M_\alpha = \frac{i}{2} \epsilon_{mn} (\sigma^{mn})_\alpha^\beta M_\beta,$$  \hspace{1cm} (A.4)

where $\epsilon_{mn} = -\epsilon_{nm}$ are the infinitesimal parameters of the transformation. On the other hand, a Weyl spinor $N^{\dot{\alpha}}$, $\dot{\alpha} = 1, 2$, associated to $S^-$, transforms as,

$$\delta N^{\dot{\alpha}} = \frac{i}{2} \epsilon_{mn} (\bar{\sigma}^{mn})_{\dot{\alpha}}^{\dot{\beta}} N^{\dot{\beta}}.$$  \hspace{1cm} (A.5)

The matrices $\sigma^{mn}$ and $\bar{\sigma}^{mn}$ are antisymmetric in $m$ and $n$ and are defined as follows:

$$\sigma^{mn}_\alpha^\beta = \frac{i}{2} \sigma^{[m}_{\alpha \dot{\alpha}}^{n]}_{\dot{\alpha}}^\beta, \quad \sigma^{mn\dot{\alpha}}_{\dot{\beta}} = \frac{i}{2} \sigma^{[m\dot{\alpha}_{\alpha}^{n]}_{\alpha}^{\dot{\beta}}.}$$  \hspace{1cm} (A.6)

They satisfy the self-duality properties,

$$\sigma^{mn} = \frac{1}{2} \epsilon^{mnpq} \sigma_{pq}, \quad \bar{\sigma}^{mn} = -\frac{1}{2} \epsilon^{mnpq} \bar{\sigma}_{pq},$$  \hspace{1cm} (A.7)

and the $SO(4)$ algebra,

$$[\sigma_{mn}, \sigma_{pq}] = i (\delta_{mp} \sigma_{nq} - \delta_{mq} \sigma_{np} - \delta_{np} \sigma_{mq} + \delta_{nq} \sigma_{mp}).$$  \hspace{1cm} (A.8)

The same algebra is fulfilled by $\bar{\sigma}^{mn}$.
Let us consider the covariant derivative $D_\mu$ on the manifold $X$. Acting on an element of $\Gamma(X, S^+)$ it has the form:

\[ D_\mu M_\alpha = \partial_\mu M_\alpha + \frac{i}{2} \omega^{mn}_\mu (\sigma_{mn})_\alpha^\beta M_\beta, \quad (A.9) \]

where $\omega^{mn}_\mu$ is the spin connection. Defining $D_\alpha \hat{a}$ as,

\[ D_\alpha \hat{a} = (\sigma_n)_{\alpha\hat{a}} e^{n\mu} D_\mu, \quad (A.10) \]

where $e^{n\mu}$ is the vierbein on $X$, the Dirac equation for $M \in \Gamma(X, S^+)$ and $N \in \Gamma(X, S^-)$ can be simply written as,

\[ D_\alpha \hat{a} M^\alpha = 0, \quad D_\alpha \hat{a} N^{\hat{a}} = 0. \quad (A.11) \]

Let us now introduce a principal $G$-bundle $P \to X$ with its associated connection one-form $A$, and let us consider that the Weyl spinors $M_\alpha$ realize locally an element of $\Gamma(S^+ \otimes \text{ad} P)$, i.e., they transform under a $G$ gauge transformation in the adjoint representation --indeed, $\text{ad} P$ is the vector bundle associated to $P$ through the adjoint representation of the gauge group on its Lie algebra:

\[ \delta M^a_\alpha = i[M_\alpha, \phi]^a = -i(T^c)^{ab} M^b_\alpha \phi^c, \quad (A.12) \]

where $(T^a)^{bc} = -if^{abc}$, $a = 1, \cdots, \dim(G)$, are the generators of $G$ in the adjoint representation, which are traceless and chosen to be hermitian and are normalized as follows: $\text{Tr} (T^a T^b) = \delta^{ab}$. In (A.12) $\phi^a$, $a = 1, \cdots, \dim(G)$, denote the infinitesimal parameters of the gauge transformation.

In terms of the gauge connection $A$, the covariant derivative (A.9) can be
promoted to a full covariant derivative acting on sections in $\Gamma(X, S^+ \otimes \text{ad} P)$,

$$
D_\mu M_\alpha = \partial_\mu M_\alpha + \frac{i}{2} \omega_{\mu}^{mn}(\sigma_{mn})_{\alpha \beta} M_\beta + i[A_\mu, M_\alpha],
$$

(A.13)

and its analogue in (A.10):

$$
\mathcal{D}_{\alpha \dot{\alpha}} = (\sigma_n)_{\alpha \dot{\alpha}} e^{n\mu} D_\mu.
$$

(A.14)

In terms of the full covariant derivative the Dirac equations (A.11) become:

$$
\mathcal{D}_{\alpha \dot{\alpha}} M^\alpha = 0, \quad D_{\alpha \dot{\alpha}} N^\dot{\alpha} = 0.
$$

(A.15)

Given an element of $\Gamma(X, S^+ \otimes \text{ad} P)$, $M_\alpha = (a, b)$ we define $\overline{M}^\alpha = (a^*, b^*)$. In this way, given $M, N \in \Gamma(X, S^+ \otimes \text{ad} P)$, the gauge-invariant quantity entering the metric

$$
\frac{1}{2} \text{Tr} \left( \overline{M}^\alpha N_\alpha + \overline{N}^\alpha M_\alpha \right),
$$

(A.16)

is positive definite. With similar arguments the corresponding gauge invariant metric in the fibre $\Gamma(X, S^- \otimes \text{ad} P)$, which we define as

$$
\frac{1}{2} \text{Tr} \left( \overline{M}_\alpha N^\dot{\alpha} + \overline{N}_\dot{\alpha} M^\alpha \right),
$$

(A.17)

for $M, N \in \Gamma(X, S^- \otimes \text{ad} P)$, can be seen to be positive definite, too. For self-dual two-forms $Y, Z \in \Gamma(X, \Lambda^{2,+} T^* X \otimes \text{ad} P) \equiv \Omega^{2,+}(X, \text{ad} P)$ our definition of the metric is the following:

$$
\langle Y, Z \rangle = \int_X \text{Tr} \left( Y \wedge * Z \right) = \frac{1}{2} \int_X \text{Tr} \left( Y_{\mu \nu} Z^{\mu \nu} \right) = -\frac{1}{4} \int_X \text{Tr} \left( Y_{\alpha \beta} Z^{\alpha \beta} \right),
$$

(A.18)

where $(Y, Z)_{\alpha \beta} = \sigma^{\mu \nu}_{\alpha \beta}(Y, Z)_{\mu \nu}$ and we have used the identity $(Y, Z)_{\alpha \beta}(Y, Z)^{\alpha \beta} = -2(Y, Z)_{\mu \nu}^+(Y, Z)^{+ \mu \nu}$. 

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Acting on an element of $\Gamma(X, S^+ \otimes \text{ad}P)$ the covariant derivatives satisfy:

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]M_\alpha = i[F_{\mu\nu}, M_\alpha] + \frac{i}{2}R_{\mu\nu}^{\ mn}(\sigma_{mn})^\alpha_\beta M_\beta;$$

(A.19)

where $F_{\mu\nu}$ are the components of the two-form field strength:

$$F = dA + iA \wedge A,$$

(A.20)

and $R_{\mu\nu}^{\ mn}$ the components of the curvature two-form,

$$R^{\ mn} = d\omega^{\ mn} + \omega^{\ mp} \wedge \omega^{\ pn},$$

(A.21)

being $\omega^{\ mn}$ the spin connection one-form. The scalar curvature is defined as:

$$R = e^\mu_m e^\nu_n R_{\mu\nu}^{\ mn},$$

(A.22)

and the Ricci tensor as:

$$R_{\kappa\lambda} = e^\mu_m e_{\nu\lambda} R_{\mu\nu}^{\ mn}.$$  

(A.23)

The components of the curvature two-form (A.21) are related to the components of the Riemann tensor as follows:

$$R_{\mu\nu\kappa\lambda} = e_{\kappa m} e_{\lambda n} R_{\mu\nu}^{\ mn}. $$

(A.24)

The Riemann tensor satisfies the following algebraic properties:

(a) Symmetry:

$$R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu},$$

(A.25)

(b) Antisymmetry:

$$R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu} = +R_{\mu\lambda\kappa\nu},$$

(A.26)
(c) Cyclicity:

\[ R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0. \]  \hspace{1cm} (A.27)

Notice that (A.27) implies that

\[ \epsilon^{\mu\nu\kappa\sigma} R_{\lambda\mu\nu\kappa} = 0. \]  \hspace{1cm} (A.28)

This result is essential in the verification of the identity (4.11).
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