Semileptonic decays $B_c \to (\eta_c, J/\psi) l \bar{\nu}_l$ in the “PQCD + Lattice” approach

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In this paper, we studied the semileptonic decays $B_c^- \to (\eta_c, J/\psi) l^- \bar{\nu}_l$ by employing the PQCD factorization approach, using the newly defined distribution amplitudes of the $B_c$ meson and also taking into account the Lattice NRQCD results about the relevant form factors. We found the following main results: (a) the PQCD predictions for the branching ratios will become smaller by about $(10-50)\%$ when the Lattice NRQCD results are taken into account in the extrapolation of the relevant form factors; (b) the PQCD predictions for the ratios $R_{\eta_c}, R_{J/\psi}$ and the longitudinal polarization $P_{\tau}$ are $R_{\eta_c} = 0.373^{+0.003}_{-0.012}, R_{J/\psi} = 0.300^{+0.005}_{-0.004}, P_{\tau}^{\eta_c} = 0.356^{+0.003}_{-0.005}$ and $P_{\tau}^{J/\psi} = -0.557 \pm 0.002$; and (c) after the inclusion of the Lattice NRQCD results the theoretical predictions changed moderately: $R_{\eta_c} = 0.300^{+0.033}_{-0.031}, R_{J/\psi} = 0.230^{+0.041}_{-0.035}, P_{\tau}^{\eta_c} = 0.345 \pm 0.010$ and $P_{\tau}^{J/\psi} = -0.427^{+0.127}_{-0.093}$. The theoretical predictions for $R_{J/\psi}$ agree with the measured one within errors, and other predictions could be tested in the future LHCb experiments.

I. INTRODUCTION

In the standard model (SM), all electroweak gauge bosons ($Z, \gamma$ and $W^\pm$) have equivalent couplings to three generation leptons, and the only differences arise from the mass hierarchy: $m_e < m_\mu \ll m_\tau$: this is the so-called Lepton flavor universality (LFU) in the SM. The $B_c$ meson can only decay through weak interactions because $m_{B_c}$ is below the B-D threshold, it is therefore an ideal system to study the weak decays of heavy quarks. Since the rare semileptonic decays governed by the flavor-changing neutral currents (FCNC) are forbidden at the tree level in the SM, the precise measurements for such semileptonic $B_c$ meson decays can play an important role in testing the SM and in searching for the signal and/or evidence of NP beyond the SM. In recent years, the measured values of $R(D)$ and $R(D^*)$ [1, 2], defined as the ratios of the branching fractions $B(B \to D^{(*)}\tau \nu_\tau)$ and $B(B \to D^{(*)}\mu \nu_\mu)$, are larger than the SM predictions [1, 2]. The semileptonic decays $B \to D^{(*)} l \bar{\nu}_l$ with $l = (e, \mu, \tau)$ are therefore studied intensively by many authors in the framework of the SM [2–8], or in various new physics (NP) models beyond the SM for example in Refs. [5, 9–13].

If the above mentioned anomalies are indeed the first signal of the LFU violation (i.e. an indication of the new physics) in $B_{u,d}$ sector, it must appear in the similar semileptonic decays of $B_s$ and $B_c$ mesons. The $B_c$ ($bc$) meson, as a bound state of two heavy bottom and charm quarks, was firstly observed by the CDF Collaboration [14] and then by the Large Hadron Collider

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(LHC) experiments in recent years [15]. The properties of $B_c$ meson and the dynamics involved in $B_c$ decays could be fully exploited through the precise measurements at the LHC experiments, especially the measurements carried on by the LHCb Collaboration. Very recently, some hadronic and semileptonic $B_c$ meson decays have been measured by LHCb experiments [16, 17]. Analogous to the cases for $B$ decays, the generalization of the $R(D^{(*)})$ for the semileptonic $B_c$ decays are the ratio $R_{\eta_c}$ and $R_{J/\psi}$:

$$R_{X} = \frac{\mathcal{B}(B_c^- \rightarrow X\tau^-\bar{\nu}_\tau)}{\mathcal{B}(B_c^- \rightarrow X\mu^-\bar{\nu}_\mu)}$$

for $X = (\eta_c, J/\psi)$. \hfill (1)

But only the ratio $R_{J/\psi}$ has been measured by the LHCb Collaboration very recently [17],

$$R_{J/\psi}^{\exp} = 0.71 \pm 0.17 \text{(stat.)} \pm 0.18 \text{(syst.)},$$ \hfill (2)

which is consistent with currently available SM predictions [18–30] within $2\sigma$ errors.

During the past two decades, the semileptonic $B_c \rightarrow (\eta_c, J/\psi)l\bar{\nu}_l$ decays have been studied by many authors in rather different theories or models, for example, in the QCD sum rule (QCD SR) and light-cone sum rules (LCSR) [21, 28, 29, 31], the relativistic or non-relativistic quark models [26, 32], the light-front quark model (LFQM) [22, 33], the covariant confining quark model (CCQM) [34], the nonrelativistic QCD (NRQCD) [23, 35–38], the model-independent investigations (MII) [39–42], the lattice QCD (LQCD) [43–45], and the perturbative QCD (PQCD) factorization approach [19, 46, 47].

In a previous work [19], we calculated the ratio $R_{J/\psi}$ and $R_{\eta_c}$ by employing the PQCD approach [48, 49] and found the PQCD predictions [19]:

$$R_{J/\psi} \approx 0.29, \quad R_{\eta_c} \approx 0.31,$$ \hfill (3)

which also agree well with the ones from the QCDSR or other different approaches in the framework of the SM. In this paper, we will present a new systematic evaluation for the ratio $R_{J/\psi}$ and $R_{\eta_c}$ by using the PQCD factorization approach but with the following further improvements:

1. We here will use a newly developed distribution amplitude (DA) $\phi_{B_c}(x, b)$ as proposed very recently in Ref. [50]:

$$\phi_{B_c}(x, b) = \frac{f_{B_c}}{2\sqrt{6}} N_{B_c} x (1 - x) \exp \left[ -\frac{(1 - x)m_c^2 + x m_{B_c}^2}{8\beta_{B_c}^2 x(1 - x)} \right] \exp \left[-2\beta_{B_c}^2 x(1 - x)b^2 \right] ,$$ \hfill (4)

instead of the simple $\delta$-function as being used in previous works for example in Refs. [18, 19]:

$$\phi_{B_c}(x) = \frac{f_{B_c}}{2\sqrt{6}} \delta \left( x - \frac{m_c}{m_{B_c}} \right) .$$ \hfill (5)

2. We will take into account the Lattice QCD results for the semileptonic form factors of the decays $B_c \rightarrow (J/\psi, \eta_c)l\bar{\nu}_l$, as reported by the HPQCD Collaboration [43–45], in the extrapolation of the relevant form factors from the low-$q^2$ region to the high-$q^2$ region. We will calculate the ratios $R_{J/\psi}$ and $R_{\eta_c}$ by using both the PQCD approach and the ”PQCD+Lattice” method, and compare the resultant predictions.
(3) Besides the ratios $R_{\eta_c, J/\psi}$ of the branching ratios, we here will calculate the longitudinal polarization $P_{\tau}(\eta_c)$ and $P_{\tau}(J/\psi)$ of the final state tau lepton, which was absent in Ref. [19]. Just like the polarization $P^{D^*}_{\tau}$ firstly measured at Belle [51], both $P_{\tau}(\eta_c)$ and $P_{\tau}(J/\psi)$ might be measured in the future LHCb experiment.

The paper is organized as follows: after this introduction, we give the distribution amplitudes of the $B_c$ meson and the final state $\eta_c$ and $J/\psi$ mesons in Section 2. Based on the $k_T$ factorization formalism, we calculate and present the expressions for the $B_c \rightarrow (\eta_c, J/\psi)$ transition form factors in the large recoil regions in Section 3. The numerical results of the branching ratios, the ratios $R_M$ and the longitudinal polarization $P(\tau)$ are given in Section 4. A short summary will be given in final section.

![Feynman diagrams](image)

**FIG. 1.** The charged current tree Feynman diagrams for the semileptonic decays $B_c^- \rightarrow Xl^-\bar{\nu}_l$ with $X = (\eta_c, J/\psi)$ and $l = (e, \mu, \tau)$ in the PQCD approach.

## II. KINEMATICS AND THE WAVE FUNCTIONS

The lowest order Feynman diagrams for $B_c \rightarrow XL\nu$ are displayed in Fig. 1. The kinematics of these decays are discussed in the large-recoil (low $q^2$) region, where the PQCD factorization approach is applicable to the considered semileptonic decays involving $\eta_c$ or $J/\psi$ as the final state meson [52]. In the $B_c$ meson rest frame, we define the $B_c$ meson momentum $P_1$, and the final state meson momentum $P_2$ in the light-cone coordinates as[19, 53]

$$P_1 = \frac{m_{B_c}}{\sqrt{2}}(1, 1, 0_\perp), \quad P_2 = r\frac{m_{B_c}}{\sqrt{2}}(\eta^+, \eta^-, 0_\perp),$$

with

$$\eta^\pm = \eta \pm \sqrt{\eta^2 - 1}, \quad \eta = \frac{1}{2r} \left[ 1 + r^2 - \frac{q^2}{m_{B_c}^2} \right],$$

where $r = m_M/m_{B_c}$ is the mass ratio, and $q = p_1 - p_2$ is the momentum of the lepton pair. The longitudinal polarization vector $\epsilon_L$ and transverse polarization vector $\epsilon_T$ of the vector meson are defined in the same way as in Ref. [19]:

$$\epsilon_L = \frac{1}{\sqrt{2}}(\eta^+, -\eta^-, 0_\perp), \quad \epsilon_T = (0, 0, 1),$$

The momentum $k_1$ and $k_2$ of the spectator quark in $B_c$ or in final state $(J/\psi, \eta_c)$ are parameterized in the same way as in Ref. [19].
For the $B_c$ meson wave function, we make use of the same one as being used for example in Ref.[18, 19],

$$\Phi_{B_c}(x, b) = \frac{i}{\sqrt{6}}(f_1 + m_{B_c})\gamma_5\phi_{B_c}(x, b).$$

(9)

As for the distribution amplitude (DA) $\phi_{B_c}(x, b)$, we here will use a new $\phi_{B_c}(x, b)$ [50] as shown in Eq. (4) instead of the simple $\delta$-function as given in Eq. (5). As usual, the normalization constant $N_{B_c}$ in Eq. (4) is fixed by the relation

$$\int_0^1 \phi_{B_c}(x, b = 0)dx = \int_0^1 \phi_{B_c}(x)dx = \frac{f_{B_c}}{2\sqrt{6}}$$

(10)

where the decay constant $f_{B_c} = 0.489 \pm 0.005$ GeV has been obtained in lattice QCD by the TWQCD Collaboration [54]. We will set the factor $\beta_{B_c}$ in Eq. (4) in the value of $\beta_{B_c} = 1.0 \pm 0.1$ GeV in order to estimate the uncertainty.

For the pseudoscalar meson $\eta_c$ and the vector one $J/\Psi$, we use the same wave function as those in Ref. [19, 20]:

$$\Phi_{\eta_c}(x) = \frac{i}{\sqrt{6}}\gamma_5[\phi\phi^\nu(x) + m_{\eta_c}\phi^s(x)]$$

(11)

$$\Phi^T_{J/\Psi}(x) = \frac{1}{\sqrt{6}}[m_{J/\Psi}\phi_L(x) + \phi_L\phi^T(x)]$$

(12)

$$\Phi^V_{J/\Psi}(x) = \frac{1}{\sqrt{6}}[m_{J/\Psi}\phi_T(x) + \phi_T\phi^V(x)]$$

(13)

where the twist-2 asymptotic DAs $(\phi^\nu(x), \phi_L(x), \phi^T(x))$ and the twist-3 ones $(\phi^s(x), \phi^T(x), \phi^V(x))$ are the same as those being used in Refs. [19, 20].

### III. THE FORM FACTORS AND DIFFERENTIAL DECAY WIDTHS

For the considered charged current $B_c \rightarrow (\eta_c, J/\psi)L^-\bar{\nu}_L$ decays, the quark-level transition is the $b \rightarrow c l^-\bar{\nu}_L$ decay with the effective Hamiltonian

$$\mathcal{H}_{eff}(b \rightarrow c l^-\bar{\nu}_L) = \frac{G_F}{\sqrt{2}}V_{cb}\bar{c}\gamma_\mu(1 - \gamma_5)b\cdot\bar{\nu}_L(1 - \gamma_5)\nu_L.$$  

(14)

where $G_F = 1.16637 \times 10^{-5}$ GeV$^{-2}$ is the Fermi-coupling constant and $V_{cb}$ is the CKM matrix element. The differential decay widths of the semi-leptonic decays $B_c^- \rightarrow \eta_c l^-\bar{\nu}_L$ can be written [19, 22] in the following form:

$$\frac{d\Gamma(B_c^- \rightarrow \eta_c l^-\bar{\nu}_L)}{dq^2} = \frac{G_F^2|V_{cb}|^2}{192\pi^3m_{B_c}^3}\left(1 - \frac{m_{l}^2}{q^2}\right)^2 \frac{\lambda^{1/2}(q^2)}{2q^2}\left\{3\lambda(\frac{m_{l}^2}{q^2})^2|F_0(q^2)|^2 + (\frac{m_{l}^2}{q^2})^2 \lambda(q^2)|F_+(q^2)|^2\right\},$$

(15)

where $m_l$ is the mass of the charged leptons, $0 \leq q^2 \leq m_{B_c}^2 - m_{\eta_c}^2$ and $\lambda(q^2) = (m_{B_c}^2 + m_{\eta_c}^2 - q^2)^2 - 4m_{B_c}^2m_{\eta_c}^2$ is the phase space factor. In the PQCD factorization approach, the form factor
$F_0(q^2)$ and $F_+(q^2)$ in Eq. (15) defined through the matrix element $<\eta_c(p_2)|\bar{c}(0)\gamma_\mu b(0)|B_c(p_1)>$ [19, 22] can be calculated and written as a summation of the auxiliary form factor $f_{1,2}(q^2)$:

$$F_0(q^2) = F_+(q^2) + \frac{q^2}{2(m_{B_c}^2 - m_{\eta_c}^2)} [f_1(q^2) - f_2(q^2)].$$

After making the analytical calculations in PQCD approach, one found the function $f_{1,2}(q^2)$:

$$f_1(q^2) = 8\pi m_{B_c}^2 C_F \int dx_1 dx_2 \int b_1 db_1 b_2 db_2 \phi_{B_c}(x_1, b_1) \times \left\{ \begin{array}{l} [-2r^2x_2 \phi^v(x_2) + 2r(2 - r_b) \phi^s(x_2)] \cdot H_1(t_1) \\
\left( -2r^2 + \frac{r x_1 \eta^+ \eta^+}{\sqrt{\eta^2 - 1}} \right) \phi^v(x_2) + \left( 4rr_c - \frac{2r x_1 \eta^+}{\sqrt{\eta^2 - 1}} \right) \phi^s(x_2) \end{array} \right\} \cdot H_2(t_2),$$

$$f_2(q^2) = 8\pi m_{B_c}^2 C_F \int dx_1 dx_2 \int b_1 db_1 b_2 db_2 \phi_{B_c}(x_1, b_1) \times \left\{ \begin{array}{l} [(4r_b - 2 + 4r x_2 \eta) \phi^v(x_2) + (-4r x_2) \phi^s(x_2)] \cdot H_1(t_1) \\
\left( -2r^2 - \frac{x_1 \eta^+}{\sqrt{\eta^2 - 1}} \right) \phi^v(x_2) + \left( 4r + \frac{2r x_1}{\sqrt{\eta^2 - 1}} \right) \phi^s(x_2) \end{array} \right\} \cdot H_2(t_2),$$

with the functions $H_i(t_i)$ in the following form

$$H_i(t_i) = h_i(x_1, x_2, b_1, b_2) \cdot \alpha_s(t_i) \exp [-S_{ab}(t_i)], \quad \text{for} \quad i = (1, 2),$$

where $C_F = 4/3$ is a color factor, $r_c = m_c/m_{B_c}$, $r_b = m_b/m_{B_c}$, $r = m_{\eta_c}/m_{B_c}$. The explicit expressions of the hard functions $h_i(x_1, x_2, b_1, b_2)$ and the Sudakov functions $\exp [-S_{ab}(t_i)]$ will be given in Appendix.

For $B_C \rightarrow J/\psi l^- \bar{\nu}_l$ decays, the differential decay widths can be written in the following form [19, 22]:

$$\frac{d\Gamma_L}{dq^2} = \frac{G_F^2 |V_{cb}|^2}{192\pi^3 m_{B_c}^3} \left( 1 - \frac{m_1^2}{q^2} \right)^2 \frac{\lambda^{1/2}(q^2)}{2q^2} \cdot \left\{ 3m_1^2 \lambda(q^2) A_0^2(q^2) + \frac{m_1^2 + 2q^2}{4m_{J/\psi}^2} \cdot \left[ (m_{B_c}^2 - m_{J/\psi}^2 - q^2)(m_{B_c} + m_{J/\psi}) A_1(q^2) - \frac{\lambda(q^2)}{m_{B_c} + m_{J/\psi}} A_2(q^2) \right]^2 \right\},$$

$$\frac{d\Gamma_\pm}{dq^2} = \frac{G_F^2 |V_{cb}|^2}{192\pi^3 m_{B_c}^3} \left( 1 - \frac{m_1^2}{q^2} \right)^2 \frac{\lambda^{3/2}(q^2)}{2} \cdot \left\{ (m_1^2 + 2q^2) \left[ \frac{V(q^2)}{m_{B_c} + m_{J/\psi}} + \frac{(m_{B_c} + m_{J/\psi}) A_1(q^2)}{\sqrt{\lambda(q^2)}} \right]^2 \right\},$$
where \( 0 \leq q^2 \leq (m_{B_c} - m_{J/\psi})^2 \) and \( \lambda(q^2) = (m_{B_c}^2 + m_{J/\psi}^2 - q^2)^2 - 4m_{B_c}^2 m_{J/\psi}^2 \). The total differential decay widths is defined as

\[
\frac{d\Gamma}{dq^2} = \frac{d\Gamma_L}{dq^2} + \frac{d\Gamma_+}{dq^2} + \frac{d\Gamma_-}{dq^2}.
\]

The form factors \( V(q^2) \) and \( A_{0,1,2}(q^2) \) can also be calculated in the framework of the PQCD factorization approach:

\[
V(q^2) = 8\pi m_{B_c}^2 C_F \int dx_1 dx_2 \int b_1 db_1 b_2 db_2 \phi_{B_c}(x_1, b_1) \cdot (1 + r) \\
\times \left\{ [(2 - r_b) \phi^T(x_2) - r x_2 \phi^\prime(x_2)] \cdot H_1(t_1) + \left[ r + \frac{x_1}{2\sqrt{\eta^2} - 1} \right] \phi^V(x_2) \right\} \cdot H_2(t_2),
\]

\[
A_0(q^2) = 8\pi m_{B_c}^2 C_F \int dx_1 dx_2 \int b_1 db_1 b_2 db_2 \phi_{B_c}(x_1, b_1) \\
\times \left\{ [(2r_b - 1 - r^2 x_2 + 2 r x_2 \eta) \phi^L(x_2) + r (2 - r_b - 2 x_2) \phi^\prime(x_2)] \cdot H_1(t_1) + \left[ r^2 + r_c + \frac{x_1}{2} - r x_1 \eta + \frac{x_1 (\eta + r (1 - 2 \eta^2))}{2\sqrt{\eta^2} - 1} \right] \phi^L(x_2) \right\} \cdot H_2(t_2),
\]

\[
A_1(q^2) = 8\pi m_{B_c}^2 C_F \int dx_1 dx_2 \int b_1 db_1 b_2 db_2 \phi_{B_c}(x_1, b_1) \cdot \frac{r}{1 + r} \\
\times \left\{ [2(2r_b - 1 + rx_2) \phi^V(x_2) - 2(2rx_2 - (2 - r_b) \eta) \phi^T(x_2)] \cdot H_1(t_1) + [(2r_c - x_1 + 2r \eta) \phi^V(x_2)] \right\} \cdot H_2(t_2),
\]

\[
A_2(q^2) = \frac{(1 + r)^2 (\eta - r)}{2r (\eta^2 - 1)} \cdot A_1(q^2) - 8\pi m_{B_c}^2 C_F \int dx_1 dx_2 \int b_1 db_1 b_2 db_2 \phi_{B_c}(x_1, b_1) \cdot \frac{1 + r}{\eta^2 - 1} \\
\times \left\{ [2rx_2 \eta - x_1 (2 - r_b)(1 - r \eta)] \phi^T(x_2) + [1 - 2r_b] \eta - r x_2 + 2r x_2 \eta^2 - 2x_2 \eta^2 \phi^L(x_2) \right\} \cdot H_1(t_1) \\
+ \left[ x_1 \left( r \eta - \frac{1}{2} \right) \sqrt{\eta^2 - 1} + (r_c - r^2 - \frac{x_1}{2}) \eta + r \left( 1 - r_c - \frac{x_1}{2} + x_1 \eta^2 \right) \right] \\
\phi^L(x_2) \cdot H_2(t_2),
\]

where \( r = m_{J/\psi}/m_{B_c} \), and the functions \( H_i(t_i) \) are the same ones as those defined in Eq. (19).

### IV. NUMERICAL RESULTS

In the numerical calculations we use the following input parameters (here masses and decay constants are in units of GeV)[1, 15]:

\[
m_{B_c} = 6.275, \quad m_{J/\psi} = 3.097, \quad m_r = 1.777, \quad m_c = 1.27 \pm 0.03, \quad m_{B_c} = 2.983,
\]

\[
\tau_{B_c} = 0.507 \text{ ps}, \quad f_{B_c} = 0.489, \quad f_{B_c} = 0.45, \quad f_{J/\psi} = 0.405 \pm 0.014,
\]

\[
|V_{cb}| = (40.5 \pm 1.5) \times 10^{-3}, \quad \Lambda_{\overline{\text{MS}}}^{(\text{m} = 4)} = 0.287.
\]
TABLE I. The PQCD predictions for the form factors $F_{0, +}(q^2), V(q^2)$ and $A_{0,1,2}(q^2)$ at $q^2 = 0$, and the parametrization constants “a” and “b”.

| Form Factor | $F(0)$       | $a$       | $b$       |
|------------|--------------|-----------|-----------|
| $F_{0c}^{B_c \to \eta_c}$ | $0.56^{+0.07}_{-0.06}$ | $0.052^{+0.002}_{-0.001}$ | $0.0021^{+0.0002}_{-0.0004}$ |
| $F_{+}^{B_c \to \eta_c}$ | $0.56^{+0.07}_{-0.06}$ | $0.072^{+0.003}_{-0.001}$ | $0.0025^{+0.0002}_{-0.0005}$ |
| $V_{B_c \to J/\psi}$ | $0.75^{+0.09}_{-0.08}$ | $0.087^{+0.003}_{-0.004}$ | $0.0026^{+0.0001}_{-0.0002}$ |
| $V_{B_c \to J/\psi}$ | $0.40^{+0.04}_{-0.05}$ | $0.083^{+0.001}_{-0.001}$ | $0.0031^{+0.0006}_{-0.0007}$ |
| $A_{0c}^{B_c \to J/\psi}$ | $0.47^{+0.05}_{-0.05}$ | $0.058^{+0.002}_{-0.002}$ | $0.0025^{+0.0004}_{-0.0002}$ |
| $A_{1c}^{B_c \to J/\psi}$ | $0.62^{+0.06}_{-0.06}$ | $0.105^{+0.002}_{-0.003}$ | $0.0066^{+0.0003}_{-0.0002}$ |

TABLE II. The theoretical predictions for the form factors $F_{0, +}, V$ and $A_{0,1,2}$ at $q^2 = 0$, obtained by using different approaches [21, 22, 38, 57].

| Form Factor | PQCD LFQM [22] | BSW [57] | NRQCD [38] | LCSR [21] |
|------------|----------------|----------|------------|-----------|
| $F_0(0)$  | 0.56           | 0.61     | 0.58       | 1.67      | 0.87      |
| $F_+(0)$  | 0.56           | 0.61     | 0.58       | 1.67      | 0.87      |
| $V(0)$    | 0.75           | 0.74     | 0.91       | 2.24      | 1.69      |
| $A_0(0)$  | 0.40           | 0.53     | 0.58       | 1.43      | 0.27      |
| $A_1(0)$  | 0.47           | 0.50     | 0.63       | 1.57      | 0.75      |
| $A_2(0)$  | 0.62           | 0.44     | 0.74       | 1.73      | 1.69      |

A. Branching ratios

For the considered semileptonic $B_c$ meson decays, it is easy to see that the theoretical predictions for the differential decay rates and other physical observables strongly depend on the form factors $F_{0, +}(q^2)$ for $B_c \to \eta_c \ell \nu_\ell$ decays , and the form factors $V(q^2)$ and $A_{0,1,2}(q^2)$ for $B_c \to J/\psi \ell \nu_\ell$ decays [19, 22]. The value of these form factors at $q^2 = 0$ and their $q^2$ dependence in the whole range of $0 \leq q^2 \leq q^2_{\text{max}}$ contain a lot of information of the physical process. Up to now, these form factors have been calculated in many rather different methods, for example, in Refs. [21, 25, 26, 28, 29, 31, 32].

In Refs. [3, 4, 53, 55], the authors examined the applicability of the PQCD approach to $(B \to D^{(*)})$ transitions, and have shown that the PQCD approach with the inclusion of the Sudakov effects is applicable to the semileptonic decays $B \to D^{(*)} \ell \nu_\ell$ at the low $q^2$ region [3, 4]. Since the PQCD predictions for the considered form factors are reliable only at the low $q^2$ region, we first calculate explicitly the values of the relevant form factors at the sixteen points in the lower region $0 \leq q^2 \leq m_\tau^2$ by using the expressions as given in Eqs. (17,18,23-26) and the definitions in Eq. (16).

In the large $q^2$ region $m_\tau^2 \leq q^2 \leq q^2_{\text{max}}$ with $q^2_{\text{max}} = (m_{B_c} - m_x)^2$ and $x = (\eta_c, J/\psi)$, however, one has to make an extrapolation for all relevant form factors from the lower $q^2$ region to larger $q^2$ region. In this work we will make the extrapolation by using two different methods.

1. The first method is the same one as being used in Ref. [19]: using the parametrization
formula [22, 56]

\[ F_i(q^2) = F_i(0) \cdot \exp \left[ a \cdot q^2 + b \cdot (q^2)^2 \right]. \tag{28} \]

where \( F_i(q^2) \) stands for the relevant form factors \( F_{0,+}(q^2), V(q^2), A_{0,1,2}(q^2) \), and \( a, b \) are the parameters to be determined by the fitting procedure. In Table I, we list the PQCD predictions for all relevant form factors for \((B_c \rightarrow \eta_c, J/\psi)\) transitions at the points \( q^2 = 0 \), and the values of parameters \( a, b \). The errors of the PQCD predictions is the combination of the major errors from the uncertainty of \( \beta_{B_c} = 1.0 \pm 0.1 \) GeV, \( m_c = 1.27 \pm 0.03 \) GeV and the decay constant \( f_{\eta_c} \) or \( f_{J/\psi} \).

In Table II, as a comparison, we show the central values of all form factors \( F_i(0) \) in the PQCD approach and some other typical approaches, such as the BSW [57], the NRQCD [38] and the LCSR [21]. One can see easily that the theoretical predictions from different approaches are indeed rather different in values.

(2) The second one is the ”PQCD+Lattice” method, similar with what we did in Ref. [58]. The authors in HPQCD Collaboration [43, 44] calculated the form factors \( f_{0,+}(q^2) \) for \( B_c \rightarrow \eta_c \) transition, and \( V(q^2) \) and \( A_1(q^2) \) for \( B_c \rightarrow J/\psi \) transition by using the ”Lattice NRQCD” at \( q^2 = 0 \) and several other points of \( q^2 \). In order to reduce the theoretical uncertainty in the extrapolation of \( F_i(q^2) \), we use their results at point \( q^2 = 5.44, 8.72 \) GeV [43, 44],

\[ f_0(8.72) = 0.823 \pm 0.005, \quad f_+(8.72) = 0.995 \pm 0.005, \]
\[ V(5.44) = 1.06 \pm 0.03, \quad A_1(5.44) = 0.587 \pm 0.015, \tag{29} \]

as the Lattice QCD input in the fitting process. At present no Lattice QCD results are available for other two \( B_c \rightarrow J/\psi \) form factors \( A_{0,2}(q^2) \).

In Fig. 2 and 3, we show the theoretical predictions for the \( q^2 \)-dependence of the six relevant form factors for \( B_c \rightarrow (\eta_c, J/\psi) \) transitions, obtained by using the PQCD approach and the “PQCD+Lattice” approach. The theoretical predictions based on the LFQM [22] are also shown in these two figures as a comparison. In these two figures, the blue dashed curves show the theoretical predictions for the \( q^2 \)-dependence of \( f_{0,+}(q^2) \) and \( (V(q^2), A_{0,1,2}(q^2)) \) in the pQCD approach only [19], the green dot-dashed curves denote the form factors evaluated by using the LFQM as given for example in Ref. [22]. In Fig. 2 and 3 (a,c), the red solid curves show the four form factors \( (f_{0,+}(q^2), V(q^2), A_1(q^2)) \) obtained by using the ”PQCD+Lattice” approach.

One can see from the theoretical predictions as illustrated in Figs. 2 and 3 that (a) the theoretical predictions for the form factors from three different methods are very similar with each other at the low \( q^2 \) region, but become rather different at large \( q^2 \) region for the results from the PQCD and the LFQM method; (b) the difference between the theoretical predictions from the ”PQCD+Lattice” method and the LFQM remain small in the large region of \( q^2 \).

From the formulae of the differential decay rates as given in Eqs. (15,22), it is straightforward to make the integration over the range of \( m_t^2 \leq q^2 \leq (m_{B_c}^2 - m_x^2) \) with \( x = (\eta_c, J/\psi) \). The theoretical predictions (in unit of \( 10^{-3} \)) for the branching ratios of the considered semileptonic
FIG. 2. (Color online) The theoretical predictions for $B_c \to \eta_c$ form factors $f_+(q^2)$ and $f_0(q^2)$.

FIG. 3. (Color online) The theoretical predictions for the form factors $V(q^2)$ and $A_{0,1,2}(q^2)$ for $B_c \to J/\psi$ transition.

decays are the following:

\[
\mathcal{B}(B_c \to \eta_c \tau \bar{\nu}_\tau) = \begin{cases} 
2.93^{+0.87}_{-0.71}(\beta_{B_c}) \pm 0.22(V_{cb}) \pm 0.08(m_c), & \text{PQCD,} \\
2.16^{+0.25}_{-0.19}(\beta_{B_c}) \pm 0.16(V_{cb}) \pm 0.03(m_c), & \text{PQCD + Lattice,}
\end{cases}
\]  

(30)

\[
\mathcal{B}(B_c \to \eta_c \mu \bar{\nu}_\mu) = \begin{cases} 
7.87^{+2.12}_{-1.73}(\beta_{B_c}) \pm 0.53(V_{cb}) \pm 0.27(m_c), & \text{PQCD,} \\
7.20^{+1.71}_{-1.28}(\beta_{B_c}) \pm 0.53(V_{cb}) \pm 0.23(m_c), & \text{PQCD + Lattice,}
\end{cases}
\]  

(31)
\( \mathcal{B}(B_c \rightarrow J/\psi \tau \bar{\nu}_\tau) = \{ 4.98^{+1.59}_{-1.13}(\beta_{B_c}) \pm 0.37(V_{cb}) \pm 0.19(m_c), \text{ PQCD}, \\
2.30^{+0.57}_{-0.33}(\beta_{B_c}) \pm 0.17(V_{cb}) \pm 0.07(m_c), \text{ PQCD + Lattice} \}, \) \hspace{1cm} (32)

\( \mathcal{B}(B_c \rightarrow J/\psi \mu \bar{\nu}_\mu) = \{ 16.6^{+5.0}_{-3.6}(\beta_{B_c}) \pm 0.12(V_{cb}) \pm 0.06(m_c), \text{ PQCD}, \\
9.98^{+0.02}_{-0.03}(\beta_{B_c}) \pm 0.17(V_{cb}) \pm 0.05(m_c), \text{ PQCD + Lattice} \}, \) \hspace{1cm} (33)

where the major theoretical errors come from the uncertainties of the input parameters \( \beta_{B_c} = 1.0 \pm 0.1 \text{ GeV}, |V_{cb}| = (40.5 \pm 1.5) \times 10^{-3} \) and \( m_c = 1.27 \pm 0.03 \text{ GeV} \).

TABLE III. The central values of the theoretical predictions (in unit of \( 10^{-3} \)) for the branching ratios \( B_c \rightarrow (\eta_c, J/\psi)l \bar{\nu}_l \) calculated in this paper, or a previous PQCD work [19], and other three approaches [21, 22, 42].

| BR | PQCD | PQCD+Lattice | PQCD [19] | LFQM [22] | Z-Series [42] | LCSR [21] |
|----|------|--------------|-----------|-----------|--------------|-----------|
| \( B(B_c \rightarrow \eta_c \bar{\nu}_\mu) \) | 7.87^{+2.20}_{-1.82} | 7.20^{+1.90}_{-1.40} | 4.4^{+1.2}_{-1.1} | 6.7 | 6.6 | 16.7 |
| \( B(B_c \rightarrow \eta_c \bar{\nu}_\tau) \) | 2.93^{+0.90}_{-0.75} | 2.16^{+0.30}_{-0.25} | 1.4^{+1.2}_{-1.1} | 1.9 | 2.0 | 4.9 |
| \( B(B_c \rightarrow J/\psi \mu \bar{\nu}_\mu) \) | 16.6^{+5.0}_{-3.5} | 9.98^{+0.05}_{-0.18} | 10.0^{+1.3}_{-1.2} | 14.9 | 14.5 | 23.7 |
| \( B(B_c \rightarrow J/\psi \tau \bar{\nu}_\tau) \) | 4.98^{+1.64}_{-1.20} | 2.30^{+0.60}_{-0.38} | 2.9^{+0.4}_{-0.3} | 3.7 | 3.6 | 6.5 |

In Table III, we list the central values of the theoretical predictions (in unit of \( 10^{-3} \)) for the branching ratios of the decays \( B_c \rightarrow (\eta_c, J/\psi)l \bar{\nu}_l \) with \( l = (\mu, \tau) \), obtained in this paper, or from the previous PQCD work [19], and from other different models or approaches [21, 22, 38, 42]. One can see that the difference between different theoretical predictions can be as large as a factor of two for the same decay mode. In Table IV, we show the PQCD predictions for the ratios \( R_{\eta_c} \) and \( R_{J/\psi} \), as defined in Eq. (1) and evaluated in this paper, and the theoretical prediction as given in Ref. [19] and other papers [21, 22, 38, 42]. We also list the measured value of \( R_{J/\psi} \) as given by LHCb Collaboration [17] in last column of Table IV.

From the theoretical predictions for the branching ratios and the ratios \( R_{\eta_c, J/\psi} \) as listed in Eqs. (30-33) and in Table III and IV, we find the following points:

1. Because of the phase space suppression, the branching ratios of the decay modes with a final \( \tau \) lepton are smaller than those decay modes with a final \( \mu^- \) or \( e^- \) lepton. The PQCD predictions for the branching ratios become smaller by a degree of \((10 - 50)\%\) when the Lattice NRQCD results are taken into account in the extrapolation of the relevant form factors.

TABLE IV. The PQCD predictions for \( R_{\eta_c} \) and \( R_{J/\psi} \), and the theoretical predictions as given in Refs. [19, 21, 22, 42]. As a comparison, we also list the LHCb measured value of \( R_{J/\psi} \) [17] at last column.

| Ratios | PQCD | PQCD+Lattice | PQCD [19] | LFQM [22] | Z-series [42] | LCSR [21] | LHCb [17] |
|--------|------|--------------|-----------|-----------|--------------|-----------|-----------|
| \( R_{\eta_c} \) | 0.373^{+0.003}_{-0.012} | 0.300^{+0.034}_{-0.031} | 0.31 | 0.28 | 0.31 | 0.30 | – |
| \( R_{J/\psi} \) | 0.300^{+0.005}_{-0.004} | 0.230^{+0.041}_{-0.035} | 0.29 | 0.25 | 0.25 | 0.27 | 0.71 \pm 0.25 |
(2) The central values of the ratio \( R_{\eta_c} \) and \( R_{J/\psi} \) are around 0.23–0.37 in all considered models or approaches, while \( R_{J/\psi} \) is a little smaller than \( R_{\eta_c} \) due to the mass suppression of the phase space. The theoretical predictions for \( R_{J/\psi} \) in both PQCD and PQCD+Lattice methods are smaller than the measured value as given in Eq. (2), but still agree with it because of the large errors of the experimental measurements. These ratios could be measured in high precision at the future LHCb experiment and can help us to test the theoretical models or approaches.

(3) The dominant theoretical error comes from the uncertainty of the factor \( \beta_{B_c} \) in the \( B_c \) meson distribution amplitude \( \phi_{B_c}(x, b) \). The theoretical errors of the PQCD or PQCD+Lattice predictions for the branching ratios are still large, around (10–30)% in magnitude, as can be seen from the numerical results in Eqs. (30-33). For the ratios \( R_{\eta_c} \) and \( R_{J/\psi} \), however, the theoretical errors are largely cancelled in these ratios of the branching ratios. One can see from the numerical results as listed in Table IV that the theoretical error of \( R_{\eta_c} \) and \( R_{J/\psi} \) is now about (3–10)% in size.

B. Longitudinal polarizations

For both kinds of the semileptonic decays \( B \to D^{(*)} \ell^- \bar{\nu}_\ell \) and \( B_c^- \to (\eta_c, J/\psi) \ell^- \bar{\nu}_\ell \), their quark level weak decays are indeed the same charged current tree transitions: \( b \to c \ell^- \bar{\nu}_\ell \) with \( \ell = (e, \mu, \tau) \). The only difference between them is the spectator quark: one is the charm quark, another is the up or down quark. Consequently, it is reasonable to assume that the dynamics for these two kinds of semileptonic decays are similar in nature, we therefore can use similar method to study these two kinds of semileptonic decays.

For \( B \to D^{(*)} \tau \bar{\nu}_\tau \) decays, besides the decay rates and the ratios \( R(D^{(*)}) \), the longitudinal polarization \( P_\tau(D^{(*)}) \) of the tau lepton and the fraction of \( D^* \) longitudinal polarization \( F_\ell^D \) are also the additional physical observables and sensitive to some kinds of new physics [59–62]. The first measurement of \( P_\tau(D^*) \) and \( F_\ell^D \) have been reported very recently by Belle Collaboration [51, 63, 64]:

\[
P_\tau(D^*) = -0.38 \pm 0.51(\text{stat.})^{+0.21}_{-0.16}(\text{syst.}),
\]

\[
F_\ell^D(D^*) = 0.60 \pm 0.08(\text{stat.}) \pm 0.04(\text{syst.}).
\]

They are compatible with the SM predictions: \( P_\tau(D^*) = -0.497 \pm 0.013 \) for \( \bar{B} \to D^* \tau^- \bar{\nu}_\tau \) [60, 62], and \( F_\ell^D(D^*) = 0.441 \pm 0.006 [65] \) or 0.457 ± 0.010 [66].

For \( B_c \to (\eta_c, J/\psi) \tau \bar{\nu}_\tau \) decays, we consider the relevant longitudinal polarizations \( P_\tau(\eta_c) \) and \( P_\tau(J/\psi) \), and define them in the same way as the one for \( P_\tau(D^{(*)}) \) in Refs. [59–62]:

\[
P_\tau(X) = \frac{\Gamma^+(X) - \Gamma^-(X)}{\Gamma^+(X) + \Gamma^-(X)}, \quad \text{for} \quad X = (\eta_c, J/\psi),
\]

where \( \Gamma^\pm(X) \) denotes the decay rates of \( B_c^+ \to X \tau^+ \nu_\tau \) with a \( \tau \) lepton helicity \( \pm 1/2 \). Following Ref. [61], the explicit expressions of \( dT^\pm/dq^2 \) for the considered semileptonic \( B_c \) decays here can
be written in the following form:

\[
\frac{d\Gamma^+}{dq^2}(B_c \to \eta_c \tau \bar{\nu}_\tau) = \frac{G_F^2 |V_{cb}|^2}{192\pi^3 m_{B_c}^3} q^2 \sqrt{\lambda(q^2)} \left(1 - \frac{m_\tau^2}{q^2}\right)^2 \frac{m_\tau^2}{2q^2} (H_{V,0}^\tau + 3H_{V,t}^\tau),
\]

\[
\frac{d\Gamma^-}{dq^2}(B_c \to \eta_c \tau \bar{\nu}_\tau) = \frac{G_F^2 |V_{cb}|^2}{192\pi^3 m_{B_c}^3} q^2 \sqrt{\lambda(q^2)} \left(1 - \frac{m_\tau^2}{q^2}\right)^2 \frac{m_\tau^2}{2q^2} (H_{V,0}^\tau),
\]

\[
\frac{d\Gamma^+}{dq^2}(B_c \to J/\psi \tau \bar{\nu}_\tau) = \frac{G_F^2 |V_{cb}|^2}{192\pi^3 m_{B_c}^3} q^2 \sqrt{\lambda(q^2)} \left(1 - \frac{m_\tau^2}{q^2}\right)^2 \frac{m_\tau^2}{2q^2} (\Gamma_{V,+} + \Gamma_{V,0}^\tau + 3\Gamma_{V,t}^\tau),
\]

\[
\frac{d\Gamma^-}{dq^2}(B_c \to J/\psi \tau \bar{\nu}_\tau) = \frac{G_F^2 |V_{cb}|^2}{192\pi^3 m_{B_c}^3} q^2 \sqrt{\lambda(q^2)} \left(1 - \frac{m_\tau^2}{q^2}\right)^2 (\Gamma_{V,+}^\tau + \Gamma_{V,-}^\tau + \Gamma_{V,0}^\tau),
\]

with the functions \( H_i(q^2) \)

\[
H_{V,0}^\tau(q^2) = \sqrt{\frac{\lambda(q^2)}{q^2}} f_+(q^2),
\]

\[
H_{V,t}^\tau(q^2) = \frac{m_{B_c}^2 - m_{\eta_c}^2}{\sqrt{q^2}} f_0(q^2),
\]

\[
H_{V,\pm}(q^2) = (m_{B_c} + m_{J/\psi})A_1(q^2) \mp \sqrt{\frac{\lambda(q^2)}{q^2}} V(q^2),
\]

\[
H_{V,0}(q^2) = \frac{m_{B_c} + m_{J/\psi}}{2m_{J/\psi} \sqrt{q^2}} \left[ -\frac{(m_{B_c}^2 - m_{J/\psi}^2 - q^2)A_1(q^2) + \lambda(q^2) A_2(q^2)}{(m_{B_c} + m_{J/\psi})^2} \right],
\]

\[
H_{V,t}(q^2) = -\sqrt{\frac{\lambda(q^2)}{q^2}} A_0(q^2),
\]

where \( 0 \leq q^2 \leq (m_{B_c} - m_X)^2 \) and \( \lambda(q^2) = (m_{B_c}^2 + m_X^2 - q^2)^2 - 4m_{B_c}^2 m_X^2 \) with \( X = (\eta_c, J/\psi) \), and the form factors \( f_+(q^2), V(q^2) \) and \( A_{0,1,2}(q^2) \) in PQCD approach have been given in Eqs. (16,23-26).

After making the proper integrations over \( q^2 \), we found the theoretical predictions for the longitudinal polarization \( P_\tau \) for the considered semileptonic \( B_c \) decays:

\[
P_\tau(\eta_c) = 0.356^{+0.003}_{-0.005}, \quad P_\tau(J/\psi) = -0.557 \pm 0.002,
\]

in the PQCD approach, and

\[
P_\tau(\eta_c) = 0.345 \pm 0.010, \quad P_\tau(J/\psi) = -0.427^{+0.127}_{-0.063},
\]

in the "PQCD + Lattice" approach. The dominant errors come from the uncertainty of \( \beta_{B_c} \) and \( m_c \).

It is easy to see that the theoretical uncertainties of polarization \( P_\tau(\eta_c, J/\psi) \) as given in Eq. (42) are much larger than those in Eq. (41). The two reasons are the following:

1. For \( B_c \to \eta_c \) transition, we know both Lattice NRQCD input \( f_+(q^2) \) and \( f_0(q^2) \) simultaneously, the two relevant form factors \( f_+(q^2) \) and \( f_0(q^2) \) have similar \( q^2 \)-dependence in the large \( q^2 \) region after the inclusion of the Lattice NRQCD input. The PQCD predictions for both \( P_\tau(\eta_c) \) and \( P_\tau(J/\psi) \) therefore have a very small theoretical error due to the consistent cancelation of the errors.
For the four relevant form factors of $B_c \rightarrow J/\psi$ transition, however, we know only the two Lattice NRQCD input $V(5.44)$ and $A_1(5.44)$, the Lattice NRQCD results for other two form factor $A_0(q^2)$ and $A_2(q^2)$ are not available now. The partial Lattice NRQCD inputs, consequently, lead to the different high $q^2$ behaviour of the four form factors $F_i(q^2)$, as illustrated clearly by the curves in Fig. (3), which results in the inconsistency in some degree and a relatively large theoretical error. We do wish the Lattice NRQCD results for all the four form factors become available as soon as possible.

As listed in Eq. (34), the longitudinal polarization $P_\tau(D^*)$ for $B \rightarrow D^*\tau \nu_\tau$ decays has been measured by Belle Collaboration [51]. The similar longitudinal polarization $P_\tau(\eta_c)$ and $P_\tau(J/\psi)$ of $B_c \rightarrow (\eta_c, J/\psi)\tau \nu_\tau$ decays could be measured in the future LHCb experiment when enough amount of $B_c$ decay events are collected.

5. SUMMARY

In this paper, we studied the semileptonic decays $B_c \rightarrow (\eta_c, J/\psi)\bar{\nu}\nu$ by employing the pQCD factorization approach, using the newly defined distribution amplitudes of the $B_c$ meson and also taking into account the Lattice NRQCD results about the relevant form factors. We calculate the form factors $f_{0,1}(q^2)$, $V(q^2)$ and $A_{0,1,2}(q^2)$ of the $B_c \rightarrow (\eta_c, J/\psi)$ transitions, present the predictions for the branching ratios $B(B_c \rightarrow (\eta_c, J/\psi)\bar{\nu}\nu)$, the ratios $R_{\eta_c, J/\psi}$ and the longitudinal polarization $P_\tau(\eta_c, J/\psi)$ of the final $\tau$ lepton.

From the numerical calculations and phenomenological analysis we found the following points:

1. The PQCD predictions for the branching ratios of $B_c \rightarrow (\eta_c, J/\psi)\bar{\nu}\nu$ decays agree well with other SM predictions obtained by using rather different approaches or models. The theoretical predictions for the branching ratios will become smaller by about $(10 - 50)\%$ when the Lattice NRQCD results about the form factors are taken into account in the extrapolation of the relevant form factors.

2. The theoretical predictions for the ratios $R_{\eta_c}$ and $R_{J/\psi}$ are the following:

$$
\begin{align*}
R_{\eta_c} &= 0.373^{+0.003}_{-0.012}, \\
R_{J/\psi} &= 0.300^{+0.005}_{-0.004}, \quad \text{PQCD}, \\
R_{\eta_c} &= 0.300^{+0.033}_{-0.031}, \\
R_{J/\psi} &= 0.230^{+0.041}_{-0.035}, \quad \text{PQCD + Lattice},
\end{align*}
$$

The central values of above theoretical predictions for $R_{J/\psi}$ are smaller than the measured one as shown in Eq. (2), but still agree with it within the errors.

3. The theoretical predictions for the longitudinal polarization $P(\tau)$ of the tau lepton are the following:

$$
\begin{align*}
P_{\eta_c}^{\tau} &= 0.356^{+0.003}_{-0.005}, \\
P_{J/\psi}^{\tau} &= -0.557 \pm 0.002, \quad \text{PQCD}, \\
P_{\eta_c}^{\tau} &= 0.345 \pm 0.010, \\
P_{J/\psi}^{\tau} &= -0.427^{+0.127}_{-0.093}, \quad \text{PQCD + Lattice},
\end{align*}
$$

These predictions could be tested in the future LHCb experiments.
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Appendix A: Relevant functions

In this appendix, we present the explicit expressions for some functions appeared in the previous sections. The hard functions $h_1$ and $h_2$ appeared in Eq. (19) can be written as

$$h_1 = K_0(\beta_1 b_1) [\theta(b_1 - b_2) I_0(\alpha_1 b_2) K_0(\alpha_1 b_1) + \theta(b_2 - b_1) I_0(\alpha_1 b_1) K_0(\alpha_1 b_2)] ,$$

$$h_2 = K_0(\beta_2 b_2) [\theta(b_1 - b_2) I_0(\alpha_2 b_2) K_0(\alpha_2 b_1) + \theta(b_2 - b_1) I_0(\alpha_2 b_1) K_0(\alpha_2 b_2)] ,$$

with

$$\alpha_1 = m_{B_c} \sqrt{2r_s x_2 \eta + r_s^2 - 1 - r_s^2 x_2^2} ,$$

$$\alpha_2 = m_{B_c} \sqrt{r_s x_1 \eta^+ + r_s^2 - 2} ,$$

$$\beta_1 = \beta_2 = m_{B_c} \sqrt{x_1 x_2 r_s \eta^+ - r_s^2 x_2^2} ,$$

(A2)

where $r_s = m_\eta/m_{B_c}$ with $q = (c, b)$, $r = m_\eta/m_{B_c}$ ( $r = m_{J/\psi}/m_{B_c}$) when it appears in the form factors $f_{+,0}(q^2)$ ($V(q^2)$ and $A_{0,1,2}(q^2)$). $\eta$ and $\eta^+$ are defined in Eq. (7). The functions $K_0$ and $I_0$ in Eq. (A1) are the modified Bessel functions. The term inside the square-root symbol of $\alpha_{(1,2)}$ and $\beta_{(1,2)}$ may be positive or negative. When such term is negative, the argument of the functions $K_0$ and $I_0$ becomes imaginary, and the associated Bessel functions $K_0$ and $I_0$ will consequently transform in the following way

$$K_0(\sqrt{y})|_{y<0} = K_0(i\sqrt{|y|}) = \frac{i\pi}{2} [J_0(\sqrt{|y|}) + iY_0(\sqrt{|y|})]$$

$$I_0(\sqrt{y})|_{y<0} = J_0(\sqrt{|y|}) ,$$

(A3)

The factor $\exp[-S_{ab}(t)]$ in Eq. (19) contains the Sudakov logarithmic corrections and the renormalization group evolution effects of both the wave functions and the hard scattering amplitude with $S_{ab}(t) = S_{B_c}(t) + S_X(t)$ as given in Ref. [50]

$$S_{B_c} = s_c \left( \frac{x_1}{\sqrt{2}} m_{B_c}, b_1 \right) + \frac{5}{3} \int_{m_c}^{t} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(\alpha_s(\bar{\mu})) ,$$

$$S_{\eta_c} = s_c \left( \frac{x_2}{\sqrt{2}} m_{\eta_c}, \eta^+ , b_2 \right) + s_c \left( \frac{1 - x_2}{\sqrt{2}} m_{\eta_c}, \eta^+ , b_2 \right) + 2 \int_{m_c}^{t} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(\alpha_s(\bar{\mu})) ,$$

$$S_{J/\psi} = s_c \left( \frac{x_2}{\sqrt{2}} m_{J/\psi}, \eta^+ , b_2 \right) + s_c \left( \frac{1 - x_2}{\sqrt{2}} m_{J/\psi}, \eta^+ , b_2 \right) + 2 \int_{m_c}^{t} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(\alpha_s(\bar{\mu})) ,$$

(A5)

1 One can find the expression of $J_0(x)$ and $Y_0(x)$ in Sec.8.411 and 8.415 of Ref. [67].
where $\eta^+$ is defined in Eq. (7), while the hard scale $t$ and the quark anomalous dimension $\gamma_q = -\alpha_s/\pi$, which governs the aforementioned renormalization group evolution. The Sudakov exponent $s_c(Q, b)$ for an energetic charm quark is expressed [50] as the difference

$$s_c(Q, b) = s(Q, b) - s(m_c, b)$$

$$= \int_{m_c}^{Q} \frac{d\mu}{\mu} \left[ \int_{1/b}^{\mu} \frac{d\bar{\mu}}{\bar{\mu}} A(\alpha_s(\bar{\mu})) + B(\alpha_s(\mu)) \right].$$

(A6)

The hard scales $t_i$ are chosen as the largest scale of the virtuality of the internal particles in the hard $b$-quark decay diagram,

$$t_1 = \max\{\alpha_1, 1/b_1, 1/b_2\},$$

$$t_2 = \max\{\alpha_2, 1/b_1, 1/b_2\}.$$  

(A7)

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