Markov and Lagrange spectra for Laurent series in $1/T$
with rational coefficients

by

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1. Introduction. A binary quadratic form $q$ with real coefficients $a, b, c$
given by

$$q = q(x, y) := ax^2 + bxy + cy^2$$

has discriminant $d(q) = b^2 - 4ac$ and arithmetic minimum

$$M(q) = \inf_{x, y \in \mathbb{Z}, (x, y) \neq (0, 0)} |q(x, y)|.$$

Considering an appropriate normalisation of these minima yields a set of real
numbers, called the Markov spectrum:

$$M = \left\{ \frac{\sqrt{d(q)}}{M(q)} \mid q \text{ a real binary quadratic form with positive discriminant} \right\}.$$

We should remark that we exclude real binary quadratic forms that realise 0
for integers $x, y$ not both zero. In [M79], Markov exhaustively studied the
part of the spectrum below 3, showing that it is a discrete set. His methods
involved giving an alternative definition using doubly infinite sequences of
positive integers. Later, Perron pointed out that these objects can also be
used to describe the Lagrange spectrum, a set characterising the approxima-
tion of irrational numbers by rational numbers. For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we define the
Lagrange constant, $L(\alpha)$, to be the supremum of the real numbers $L$ such
that the inequality

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{Lq^2}$$

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is satisfied by infinitely many rational numbers \( p/q \). Running through all real irrationals, we obtain the Lagrange spectrum:

\[
L = \{ L(\alpha) \mid \alpha \in \mathbb{R} \setminus \mathbb{Q} \}.
\]

Using techniques of Markov, it has been shown that \( L \cap [0, 3] = M \cap [0, 3] \). However, if we consider the two spectra for numbers greater than 3, we have \( L \subset M \) \cite{F68, VDF58}. In \cite{VDF58} and \cite{DV59}, Delone together with Fuks and Vinogradov showed that there exists some \( \mu > 3 \) such that

\[
L \cap [\mu, \infty) = M \cap [\mu, \infty) = [\mu, \infty).
\]

The parts of the Markov and Lagrange spectrum in the interval \([3, \mu]\) have a more complex structure: they are closed sets with an infinite number of adjoining intervals, that is, they exhibit gaps. For example in the interval \((\sqrt{12}, \sqrt{13})\) there are no points of either the Lagrange or Markov spectrum.

For a more detailed survey of results over the reals, see \cite{CF89, M77}.

In this paper we add a twist to the study of these classical problems by changing the setting to formal Laurent series in \( 1/T \) with rational coefficients. The field \( \mathbb{Q}((1/T)) \) is the completion of \( \mathbb{Q}(T) \) under the valuation \(-\deg\), where for \( \alpha = \sum_{i=-\infty}^{m} a_i T^i \), \( a_m \neq 0 \), we define \( \deg \alpha := m \). We study binary quadratic forms \( Q \) with coefficients in \( \mathbb{Q}((1/T)) \) given by

\[
Q = Q(X, Y) = AX^2 + BXY + CY^2 \quad \text{with} \quad D(Q) = B^2 - 4AC,
\]

and the corresponding minima

\[
m(Q) = \inf_{X,Y \in \mathbb{Q}[T] \atop (X,Y) \neq (0,0)} \deg Q(X, Y).
\]

We define the Markov spectrum over \( \mathbb{Q}((1/T)) \) to be

\[
M = \{ \deg \sqrt{D(Q)} - m(Q) \mid Q \text{ an indefinite binary quadratic form} \}.
\]

Furthermore, given \( \alpha \in \mathbb{Q}((1/T)) \), we let \( l(\alpha) \) be the supremum of the integers \( k \) such that the inequality

\[
\deg(\alpha - p/q) \leq -2 \deg q - k
\]

is satisfied by infinitely many rational polynomials \( p, q \) with arbitrarily high degree of \( q \). We then define the Lagrange spectrum over \( \mathbb{Q}((1/T)) \) to be

\[
\mathcal{L} = \{ l(\alpha) \mid \alpha \in \mathbb{Q}((1/T)) \text{ not rational} \}.
\]

We explicitly compute the Lagrange and Markov spectra, showing that they exhibit no gaps. In a similar fashion to Markov, we show that the two sets can be alternatively realised via doubly infinite sequences of non-constant polynomials, and furthermore coincide:

**Main Theorem 1.1.** The Lagrange spectrum for \( \mathbb{Q}((1/T)) \) is equivalent to the Markov spectrum for \( \mathbb{Q}((1/T)) \), and is equal to \( \mathbb{N} \cup \{ \infty \} \).
Markov and Lagrange spectra

This statement is a combination of Theorems 5.3, 6.10 and Corollary 6.11.

A detailed survey of results on Diophantine approximation in fields of power series, some of which we recall in Section 3, is given in [L00]. Lasjaunias’ article [L00] then studies the approximation spectrum of an irrational element of $k((1/T))$, for a finite field $k$, first defined by Schmidt [S00].

Recently, some work has been done on the Lagrange spectrum in the setting of formal Laurent series over finite fields, by Parkkonen and Paulin [PP19] and Bugeaud [B19]. They define and study the non-archimedean quadratic Lagrange spectrum, whose elements are approximations by the orbit of a given quadratic irrational in $\mathbb{F}_q((T^{-1}))$. In particular, they give analogies to the well-known results over the reals about the closedness and boundedness of the spectrum, as well as computations of its maximum.

Organisation of the paper. In Section 2 we set up the scene by giving preliminary results and defining the continued fractions algorithm and convergents over $\mathbb{Q}((1/T))$. In Section 3 we develop the theory of indefinite binary quadratic forms in the setting of formal Laurent series in $1/T$ with rational coefficients and show results analogous to those for real indefinite binary quadratic forms. In Section 4 we prove results on the representation of formal Laurent series by indefinite binary quadratic forms and give a function field analogue to the classical definition of the Markov spectrum. In Section 5 we show that the Markov spectrum is equal to a set whose elements are quantities attached to doubly infinite sequences of non-constant polynomials, and use that result to explicitly compute it. Finally in Section 6 we explicitly compute the Lagrange constant for several sets of examples of quadratic irrationals of even degree polynomials and describe the Lagrange spectrum, exploring its connection with the Markov spectrum.

The results in Section 3 and the theorems in Section 4 regarding the representation of Laurent series by indefinite quadratic forms follow the approach of Dickson [D57], but there are essential differences in the details.

2. The field of formal Laurent series with rational coefficients.

Let $\mathbb{Q}[T]$ be the ring of polynomials with rational coefficients, and $\mathbb{Q}(T) = \{A/B \mid A, B \in \mathbb{Q}[T], B \neq 0\}$ be its field of fractions. Furthermore,

$$\mathbb{Q}((1/T)) = \left\{ \sum_{i=-\infty}^{m} a_i T^i \mid m \in \mathbb{Z}, a_i \in \mathbb{Q}, \forall i, a_m \neq 0 \right\}$$

will denote the set of formal Laurent series in $1/T$ with rational coefficients.

We can extend the usual definition of degree to $\mathbb{Q}((1/T))$ in the following way.
Definition 2.1. For \( \alpha = \sum_{i=-\infty}^{m} a_i T^i \), \( a_m \neq 0 \), define
\[
\deg : \mathbb{Q}(1/T) \rightarrow \mathbb{Z}, \quad \alpha \mapsto m.
\]
Furthermore, by convention, \( \deg 0 = -\infty \).

This map is well defined on rational functions and it agrees with the usual definition of degree on polynomials:

Lemma 2.2. For \( A, B \in \mathbb{Q}[T] \) with \( B \neq 0 \),
(1) \( \deg \frac{A}{B} = \deg A - \deg B \).
(2) \( \deg A = \deg A \).

Remark 2.3. For \( \alpha \in \mathbb{Q}((1/T)) \), the formula \( \text{ord}(\alpha) := -\deg \alpha \) defines a valuation. Furthermore, \( \mathbb{Q}((1/T)) \) is the completion of \( \mathbb{Q}(T) \) under it.

Definition 2.4. The polynomial part of \( \alpha = \sum_{i=-\infty}^{m} a_i T^i \in \mathbb{Q}((1/T)) \) is given by
\[
\lfloor \alpha \rfloor := \begin{cases} 
0 & \text{if } \deg \alpha < 0, \\
\sum_{i=0}^{m} a_i T^i & \text{if } \deg \alpha = m \geq 0.
\end{cases}
\]
The fractional part of \( \alpha \in \mathbb{Q}((1/T)) \) is defined as \( \{\alpha\} := \alpha - \lfloor \alpha \rfloor \).

With all the basics defined, we can describe the continued fraction algorithm for elements of \( \mathbb{Q}((1/T)) \).

2.1. Continued fraction expansion of elements in \( \mathbb{Q}((1/T)) \). Let \( \alpha \in \mathbb{Q}((1/T)) \). The continued fraction algorithm over function fields works in a similar fashion to the one over the reals. For all \( i \geq 0 \), we define
\[
a_i(T) := \lfloor \alpha_i(T) \rfloor \quad \text{and} \quad \alpha_{i+1}(T) := \{\alpha_i(T)\}^{-1}
\]
so
\[
\alpha_i = a_i + \frac{1}{\alpha_{i+1}}.
\]
Hence
\[
\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ddots + \frac{1}{\alpha_{i+1}}}}.
\]
written as
\[
\alpha = [a_0, a_1, \ldots, a_i, \alpha_{i+1}].
\]
The algorithm terminates if the fractional part \( \{\alpha_i(T)\} \) is 0.

The polynomials \( a_i \in \mathbb{Q}[T] \) are called the partial quotients of \( \alpha \). They are all of positive degree, except perhaps for \( i = 0 \). The partial quotient \( a_0(T) \) can be a constant, but the other partial quotients must have at least a linear term, since \( \deg a_i(T) = \deg \lfloor \alpha_i(T) \rfloor = -\deg \{\alpha_{i-1}(T)\} > 0 \).
The continued fraction of $\alpha$ is infinite for most $\alpha \in \mathbb{Q}((1/T))$. In fact we have

**Proposition 2.5.** The continued fraction of $\alpha \in \mathbb{Q}((1/T))$ has a finite number of terms if and only if $\alpha \in \mathbb{Q}(T)$.

Since the Euclidean algorithm works in $\mathbb{Q}[T]$, the proof of the proposition is identical to the one over the reals. When dealing with binary quadratic forms, we usually refer to a square root of a polynomial, but not all such can have a Laurent expansion.

**Lemma 2.6.** Let $D \in \mathbb{Q}[T]$ be a non-square, monic polynomial of even degree. Then $D$ is a square in $\mathbb{Q}((1/T))$, i.e. $\sqrt{D}$ has a Laurent series expansion in $1/T$ with rational coefficients.

To see this, factorise the leading term and use the expansion of $(1+x)^{1/2}$. Notice that in fact $D$ need not be monic: as long as the leading coefficient of $D$ is a square in $\mathbb{Q}$, the above lemma still holds. The proof actually gives us a method of computing the continued fraction expansion of $\sqrt{D}$. Furthermore, since $D$ is not a perfect square, $\sqrt{D} \notin \mathbb{Q}(T)$ and its continued fraction is infinite. However, unlike in the case of reals, the continued fraction of $\sqrt{D}$ will not always be periodic.

**Example 2.7.** For $D = t^4 + t^3 \in \mathbb{Q}[t]$, we have

$$\sqrt{D} = \left[ t^2 + \frac{t}{2} - \frac{1}{8}, 16t + 10, -\frac{4t}{3} - \frac{13}{18}, \frac{27t}{2} + \frac{225}{32}, -\frac{512}{405} + \frac{1312}{2025}, \ldots \right].$$

This can be shown to have a non-periodic continued fraction by exploiting the connection between periodicity of the continued fraction of $\sqrt{D(T)}$ and non-trivial solutions to Pell’s equation for $D(t)$, together with the ABC theorem for polynomials. For more details see [DS04].

**2.2. Convergents.** Given an infinite continued fraction expansion, we can truncate it at any point, say $[a_0, a_1, \ldots, a_h]$, and the resulting expression will be a rational function of the form $p_h/q_h(T)$. Similarly to the classical case, we can obtain a sequence of continuants $(p_h)_{h \geq 0}$ and $(q_h)_{h \geq 0}$ and thus convergents $p_h/q_h$ via

$$p_h = a_h p_{h-1} + p_{h-2}, \quad q_h = a_h q_{h-1} + q_{h-2},$$

with $p_{-1} = 1$, $q_{-1} = 0$.

From the matrix identity

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_h & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_h & p_{h-1} \\ q_h & q_{h-1} \end{pmatrix}$$

(2.1)
a correspondence between products of matrices and the convergents of continued fractions can be obtained (see for example [PS92]):

\[
\begin{pmatrix}
    p_h & p_{h-1} \\
    q_h & q_{h-1}
\end{pmatrix} = \begin{pmatrix}
    a_0 & 1 \\
    1 & 0
\end{pmatrix} \begin{pmatrix}
    a_1 & 1 \\
    1 & 0
\end{pmatrix} \cdots \begin{pmatrix}
    a_h & 1 \\
    1 & 0
\end{pmatrix} \iff [a_0, a_1, \ldots, a_h] = \frac{p_h}{q_h}.
\]

Since we can write \(\alpha = [a_0, a_1, \ldots, a_h, \alpha_{h+1}]\) we have the convergents correspondence

\[
\alpha \iff \begin{pmatrix}
    p_h & p_{h-1} \\
    q_h & q_{h-1}
\end{pmatrix} \begin{pmatrix}
    \alpha_{h+1} & 1 \\
    1 & 0
\end{pmatrix} \iff \frac{p_h\alpha_{h+1} + p_{h-1}}{q_h\alpha_{h+1} + q_{h-1}}.
\]

Therefore

\[
(2.2) \quad \alpha = \frac{p_h\alpha_{h+1} + p_{h-1}}{q_h\alpha_{h+1} + q_{h-1}}.
\]

Furthermore, if we take the determinants of the matrices above, we show

**Proposition 2.8.** Given a continued fraction expansion of a formal Laurent series \(\alpha = [a_0, a_1, \ldots]\), its continuants \(p_h\) and \(q_h\) satisfy

\((-1)^{h+1} = p_h q_{h-1} - p_{h-1} q_h.\)

**Proposition 2.9.** The continuants satisfy

\[\deg p_{h-1} < \deg p_h \quad \text{and} \quad \deg q_{h-1} < \deg q_h,\]

for \(h > 0.\)

Notice that from Propositions 2.8 and 2.9 we can deduce

**Proposition 2.10.** The pair of continuants \((q_{h-1}, p_{h-1})\) gives the unique solution, in polynomials with coefficients in \(\mathbb{Q}\), to \(p_h x - q_h y = 1\) such that \(\deg x < \deg q_h\) and \(\deg y < \deg p_h\), provided \(h\) is an odd integer. And if \(h\) is even then \((-q_{h-1}, -p_{h-1})\) gives the unique solution to \(p_h x - q_h y = 1\) such that \(\deg x < \deg q_h\) and \(\deg y < \deg p_h\).

A final algebraic property of the convergents, to be used later, is

**Proposition 2.11.** Given

\[
\begin{pmatrix}
    p_h \\
    q_h
\end{pmatrix} = [a_0, a_1, \ldots, a_h] \quad \text{and} \quad \begin{pmatrix}
    p_{h-1} \\
    q_{h-1}
\end{pmatrix} = [a_0, \ldots, a_{h-1}],
\]

we have

\[
\begin{pmatrix}
    p_h \\
    p_{h-1}
\end{pmatrix} = [a_{h-1}, a_{h-2}, \ldots, a_0] \quad \text{and} \quad \begin{pmatrix}
    q_h \\
    q_{h-1}
\end{pmatrix} = [a_{h-1}, a_{h-2}, \ldots, a_1].
\]

The proof is a direct computation using the recurrence relations connecting the \(p_h\)’s and \(q_h\)’s.

All the results up until now are well known and analogous to those over the reals and can be found in [O11], for example.
3. Binary quadratic forms over $\mathbb{Q}((1/T))$. The Markov spectrum over the reals concerns the real numbers represented by indefinite binary quadratic forms with real coefficients. So in order to examine the Markov spectrum over $\mathbb{Q}((1/T))$, we need to first develop the theory of indefinite binary quadratic forms in the setting of formal Laurent series in $T^{-1}$ with rational coefficients.

**Definition 3.1.** A binary quadratic form over $\mathbb{Q}((1/T))$ is defined to be an expression

$$Q = Q(X,Y) = (A,B,C) := AX^2 + BXY + CY^2,$$

where $A, B, C \in \mathbb{Q}((1/T))$. We define the discriminant to be $D := B^2 - 4AC$, also an element of $\mathbb{Q}((1/T))$.

**Definition 3.2.** We call a binary quadratic form $(A, B, C)$ indefinite if the discriminant $D$ is a square in $\mathbb{Q}((1/T))$. From Lemma 2.6, this is precisely when $D$ is a polynomial of even degree and with leading coefficient a rational square. Otherwise, we call the binary quadratic form definite.

For the rest of this paper we will solely consider indefinite binary quadratic forms. Analogously to the classical case, they are the central objects in the study of the Markov spectrum.

For an indefinite binary quadratic form $Q(X,Y)$, we see that $X - \omega Y$ is a factor, where $\omega$ is a root of

$$A\omega^2 + B\omega + C = 0.$$

We define the first and second roots to be respectively

$$f := \frac{\sqrt{D} - B}{2A}, \quad s := \frac{-\sqrt{D} - B}{2A}. \tag{3.1}$$

Furthermore, assuming $A \neq 0$ and $f, s \notin \mathbb{Q}(T)$, the Laurent series for $f, s$ and $\sqrt{D}$ uniquely determine $A, B, C$. Observe that $f$ and $s$ are both in $\mathbb{Q}(T)$ if and only if $A, B$ and $C$ are all rational functions in $T$, and $D$ is a perfect square in $\mathbb{Q}(T)$.

Suppose we substitute, in $q(x, y)$,

$$x = \alpha X + \beta Y, \quad y = \gamma X + \delta Y, \tag{3.2}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{Q}[T]$ not all 0. This takes the binary quadratic form $q(x, y)$ to the binary quadratic form $Q(X, Y)$. We can also use the matrix form

$$H = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

with the convention that applying the matrix to a binary quadratic form is the same as applying the linear transformation (3.2) to it.
DEFINITION 3.3. We say that two forms $q$ and $Q$ are equivalent if such a matrix $H$ exists and $\det H = \pm 1$. Furthermore, we say that $q$ and $Q$ are properly equivalent if $\det H = 1$, and improperly equivalent if $\det H = -1$.

PROPOSITION 3.4. The form $q = (a, b, c)$ is transformed into the form $Q = (A, B, C)$ via $H = (\alpha \beta \gamma \delta) \in \text{GL}_2(\mathbb{Q}[T])$ if and only if their first roots $f$ and $F$ and their second roots $s$ and $S$, respectively, are connected by the relations

$$f = \frac{\alpha F + \beta}{\gamma F + \delta} \quad \text{and} \quad s = \frac{\alpha S + \beta}{\gamma S + \delta}.$$

The proof is a computation analogous to the one over the reals [D57].

3.1. Reduced indefinite binary quadratic forms

DEFINITION 3.5. An indefinite binary quadratic form $Q = (A, B, C)$ is called reduced if $\deg f < 0 < \deg s$ and $f \neq 0$.

From (3.1), this is equivalent to

$$\deg(\sqrt{D} - B) < \deg A < \deg(\sqrt{D} + B) \quad \text{and} \quad \sqrt{D} \neq B.$$

PROPOSITION 3.6. If $q = (A, B, C)$ is reduced, then so is $Q = (C, B, A)$.

Proof. Consider the transformation $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$ taking $q$ to $Q$, and in particular the roots $(f, s)$ to $(F, S) = (1/s, 1/f)$. Since $q$ is reduced, we have $\deg f < 0 < \deg s$. Hence $\deg F = -\deg s < 0$ and $\deg S = -\deg f > 0$. ■

THEOREM 3.7. Every indefinite binary quadratic form is properly equivalent to a reduced one.

Proof. Let $q = (a, b, c) = ax^2 + bxy + cy^2$ with $a, b, c \in \mathbb{Q}((1/T))$ be an indefinite binary quadratic form of discriminant $D \neq 0$. It has the first and second root $f = (\sqrt{D} - b)/(2a)$ and $s = (-\sqrt{D} - b)/(2a)$, respectively. First we will show that $q$ is either a reduced form or is properly equivalent to a binary quadratic form with a first root of non-negative degree. Suppose the degree of $f$ is negative. Then either $q$ is already reduced or $\deg s \leq 0$. In the latter case, we apply the transformation $(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$. Then $q$ is properly equivalent to an indefinite binary quadratic form with roots $-1/f$ and $-1/s$, both of positive degree.

Hence $q$ is properly equivalent to a binary quadratic form with roots $(\varphi, \sigma)$ such that $\deg \varphi \geq 0$. Then we apply the transformation $(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix})$ with $h = [\varphi] \in \mathbb{Q}[T]$. This takes the roots $(\varphi, \sigma)$ to $(F, S)$, where $F = \{\varphi\}$ and $S = \sigma - h$. Now if $[\varphi] \neq [\sigma]$, then $\deg F < 0$ and $\deg S > 0$, hence $q$ is properly equivalent to a reduced form.

If $[\varphi] = [\sigma]$, then consider the continued fraction expansion $\varphi = [a_0, a_1, \ldots]$ and $\sigma = [b_0, b_1, \ldots]$. Pick the smallest $m$ such that $a_m \neq b_m$,
$m > 0$. Then
\[
\varphi = [a_0, a_1, \ldots, a_{m-1}, f_m] \quad \text{and} \quad \sigma = [a_0, a_1, \ldots, a_{m-1}, s_m].
\]

Since $a_m \neq b_m$, we have $f_m \neq s_m$, and in particular $|f_m| \neq |s_m|$. Observe that the convergents for $\varphi$ and $\sigma$ are the same up to and including the $(m - 1)$st term. Then the transformation $\begin{pmatrix} p_{m-1} & q_{m-2} \\ q_{m-1} & p_{m-2} \end{pmatrix}$ takes $(\varphi, \sigma)$ to $(f_m, s_m)$. Furthermore, this matrix has polynomial entries and determinant $(-1)^{m-2}$. We apply
\[
\begin{pmatrix} (-1)^m & h \\ 0 & 1 \end{pmatrix}
\]
with $h = |f_m|$.

This takes $(f_m, s_m)$ to $(F, S)$, where $F = (-1)^m f_m$ has negative degree and $S = (-1)^m (s_m - h)$ has non-negative degree. Since $|s_m| \neq |f_m|$, $\deg S$ is positive, and the new quadratic form is reduced and properly equivalent to $q$. $\blacksquare$

The same result holds over the reals, but a different reduction algorithm is used.

### 3.2. Chain of reduced forms

All the results in this section are the direct analogues to classical results over the reals displayed in \textsuperscript{[D57]} Even though morally we take the same approach as Dickson, our proofs differ in some details, in particular in the use of Lemma \textsuperscript{3.13}

\textbf{Definition 3.8.} Two reduced binary quadratic forms with coefficients in $\mathbb{Q}((\frac{1}{T}))$, $Q = (A, B, C)$ and $q = (C, b, c)$, are called \textit{neighbours} if they are properly equivalent and $B + b = 2PA$ for some polynomial $P \in \mathbb{Q}[T]$. Further, we then call $q$ a \textit{right neighbouring form} for $Q$, and $Q$ a \textit{left neighbouring form} for $q$.

\textbf{Theorem 3.9.} Each reduced indefinite binary quadratic form has a unique right neighbouring form.

\textit{Proof.} Let $Q = (A, B, A_1)$ be a reduced indefinite binary quadratic form of discriminant $D$. The transformation $\Delta = \begin{pmatrix} 0 & 1 \\ -1 & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Q}[T])$ takes $Q$ to the equivalent form $Q_1 = (A_1, B_1, A_2)$ such that $B_1 = -B - 2\delta A_1$ and $A_2$ is obtained from the discriminant $D$. Furthermore,
\[
\begin{align*}
F & \xrightarrow{\Delta} \delta - \frac{1}{f}, \\
S & \xrightarrow{\Delta} \delta - \frac{1}{s}.
\end{align*}
\]
Since $Q$ is reduced, $\deg F < 0 < \deg S$. Take $\delta = [1/f] \in \mathbb{Q}[T]$, which has positive degree. Then $\deg F = \deg \{1/f\} < 0$ and $\deg S = \deg (\delta - 1/s) = \deg \delta > 0$, i.e. $Q_1$ is reduced. Observe that if $\delta \neq [1/f]$, then $\deg F > 0$. Hence $Q_1$ is reduced only if $\delta$ is chosen to be $[1/f]$. $\blacksquare$


**Corollary 3.10.** Every reduced form has a unique reduced left neighbouring form.

**Proof.** If \((A, B, A_1)\) is reduced, then so is \((A_1, B, A)\), by Proposition 3.6. From the theorem above, there is a unique reduced right neighbouring form \((A, B_1, A_2)\). Then by Proposition 3.6 \((A_2, B_1, A)\) is also reduced. Furthermore, it has \((A, B, A_1)\) as its unique right neighbouring form. \(\blacksquare\)

Therefore, given an indefinite binary quadratic form of discriminant \(D \neq 0\) we can construct a chain of equivalent reduced indefinite binary quadratic forms of the same discriminant, say

\[
\ldots, \Phi_{-1}, \Phi_0, \Phi_1, \ldots,
\]

where \(\Phi_i = ((-1)^i A_i, B_i, (-1)^{i+1} A_{i+1})\). The transformation \(\Delta_i = \begin{pmatrix} 0 & 1 \\ -1 & \delta_i \end{pmatrix}\) takes \(\Phi_i\) to \(\Phi_{i+1}\). Furthermore, \(B_i + B_{i+1} = 2g_i A_{i+1}\), where \(g_i = (-1)^i \delta_i\).

Let \(f_i = \frac{\sqrt{D} - B_i}{(-1)^{i+1} 2A_{i}^{2}}\) and \(s_i = \frac{\sqrt{D} + B_i}{(-1)^{i+1} 2A_{i}}\) be the first and second roots of \(\Phi_i\), and define \(F_i := (-1)^i / f_i\) and \(S_i := (-1)^{i+1} / s_i\). Then

\[
F_i = \frac{\sqrt{D} + B_i}{2A_{i+1}} \quad \text{and} \quad S_i = \frac{\sqrt{D} - B_i}{2A_{i+1}},
\]

with \(\deg F_i > 0 > \deg S_i\), since \(\Phi_i\) is reduced. Furthermore, from the fact that \(\Delta_i\) takes \(\Phi_i\) to \(\Phi_{i+1}\) we know that their roots are related by

\[
f_{i+1} = \delta_i - \frac{1}{f_i} \quad \text{and} \quad s_{i+1} = \delta_i - \frac{1}{s_i}
\]

Multiplying both by \((-1)^{i+1}\) and using the definition of \(F_i\), \(S_i\) and \(g_i\) we get

\[
F_i = g_i + \frac{1}{F_{i+1}} \quad \text{and} \quad S_{i+1} = \frac{1}{g_i + S_i}.
\]

Hence

\[
F_i = [g_i, g_{i+1}, \ldots] \quad \text{and} \quad S_i = [0, g_{i-1}, g_{i-2}, \ldots].
\]

Furthermore, using the properties of continued fractions we obtain

\[
1/f_0 = F_0 = [g_0, g_1, \ldots, g_i, F_{i+1}],
\]

\[
(-1)^{i+1} s_i = 1/S_i = [g_{i-1}, g_{i-2}, \ldots, g_0, 1/S_0].
\]

**Remark 3.11.** Observe that

\[
(3.6) \quad F_i + S_i = \sqrt{D}/A_{i+1} = [g_i, g_{i+1}, \ldots] + [0, g_{i-1}, g_{i-2}, \ldots].
\]

**Theorem 3.12.** Two properly equivalent reduced indefinite binary quadratic forms belong to the same chain.

**Proof.** Let \(q\) and \(Q\) be reduced indefinite binary quadratic forms with coefficients in \(\mathbb{Q}((1/T))\) and discriminant \(D \neq 0\). Suppose the transformation \(H = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Q}[T])\) makes them properly equivalent. Then the entries of \(H\) satisfy \(\deg \alpha < \deg \beta\), \(\deg \gamma < \deg \delta\) and \(\deg \beta < \deg \delta\), as will be
shown in Lemma 3.13, after the proof. Moreover, from the determinant of $H$, we have $\alpha \delta - \beta \gamma = 1$ and thus $(\alpha, \gamma)$ is a solution to $\delta x - \beta y = 1$ such that $\deg x < \deg \beta$ and $\deg y < \deg \delta$. But also for $\delta/\beta = [a_0, a_1, \ldots, a_i]$, Proposition 2.10 implies that the pair $(q_{i-1}, p_{i-1})$ for $i$ odd, is the unique non-zero solution over $\mathbb{Q}[T]$ to $\delta x - \beta y = 1$ such that $\deg x < \deg \beta$ and $\deg y < \deg \delta$. Hence $\gamma/\alpha = [a_0, a_1, \ldots, a_{i-1}]$.

Let $F \in \mathbb{Q}((1/T))$ be the first root of $Q$, and consider the continued fraction expansion $[a_0, a_1, \ldots, a_i, 1/F]$. Using the convergents correspondence (2.2), together with the fact that $\delta/\beta = [a_0, a_1, \ldots, a_i]$, we get

$$[a_0, a_1, \ldots, a_i, 1/F] = \frac{\delta/F + \gamma}{\beta/F + \alpha}.$$ 

Furthermore, since $H$ sends $q$ to $Q$, by Proposition 3.4 it also connects their corresponding first roots $f$ and $F$ in the following way:

$$(3.7) \quad \frac{1}{f} = \frac{\delta/F + \gamma}{\beta/F + \alpha} = [a_0, a_1, \ldots, a_i, 1/F].$$

Observe that since $Q$ is reduced, $\deg 1/F > 0$, and since $\deg \delta > \deg \beta$, all other partial quotients $a_j$ are of positive degree. Thus (3.7) uniquely determines $1/f$ up to the $i$th partial quotient in its continued fraction expansion.

On the other hand for any chain $(\Phi_j)_{j \in \mathbb{Z}}$ of reduced forms, we have shown (3.4):

$$1/f_0 = F_0 = [g_0, g_1, \ldots, g_i, F_{i+1}],$$

where $f_0$ is the first root of the form $\Phi_0$, and $F_i = (-1)^{i+1}/f_{i+1}$, where $f_{i+1}$ is the first root of the form $\Phi_{i+1}$. Consider the chain of forms where $q$ is $\Phi_0$. This implies that $f = f_0$, and from the uniqueness of the expansion of $1/f$, we must have $g_j = a_j$ for all $0 \leq j \leq i$ and $F = 1/F_{i+1} = (-1)^{i+1}f_{i+1} = f_{i+1}$, since $i$ is odd. In particular this proves that $F$, the first root of $Q$, is also the first root of the form $\Phi_{i+1}$ in the chain where $\Phi_0 = q$.

It remains to show that the second root $s_{i+1}$ of $\Phi_{i+1}$ is equal to $S$ (the second root of $Q$), given the second root $s_0$ of $\Phi_0$ is equal to $s$ (the second root of $q$). The relations for the second roots $s_{i+1}$ of the chain forms given in (3.12) state

$$(-1)^{i+2}s_{i+1} = 1/S_{i+1} = [g_i, g_{i-1}, \ldots, g_0, 1/S_0],$$

so

$$-s_{i+1} = [a_i, a_{i-1}, \ldots, a_0, -s],$$

since $i$ is odd, $s = s_0$ and $g_j = a_j$ for all $0 \leq j \leq i$. Now, $\deg s$ is positive, so this expansion is unique up to the term $a_0$. Furthermore, from Proposition 2.11 applied to the continued fraction of $\delta/\beta$, we know that

$$\delta/\gamma = [a_i, a_{i-1}, \ldots, a_0] \quad \text{and} \quad \beta/\alpha = [a_i, a_{i-1}, \ldots, a_1].$$
Hence from the convergents correspondence, we have
\[-s_{i+1} = [a_i, a_{i-1}, \ldots, a_0, -s] = \frac{-s\delta + \beta}{-s\gamma + \alpha} = -S.\]

The final equality follows from \(s\) and \(S\) being connected via \(H\). Therefore, \(S\) is equal to the second root of the form \(\Phi_{i+1}\) in the chain with \(\Phi_0 = q\). Hence, \(q\) and \(Q\) are in the same chain. The case when \(i\) is even is identical, but instead we use the fact that \((-q_{i-1}, -p_{i-1}) = (\alpha, \gamma)\) provides the unique solution to the equation \(\delta x - \beta y = 1.\]

**Lemma 3.13.** If two distinct reduced indefinite binary quadratic forms of the same discriminant \(D \neq 0\) are properly equivalent via the transformation \((\alpha \gamma \beta \delta)\), then
\[
\deg \alpha \leq \deg \beta, \quad \deg \gamma < \deg \delta, \quad \text{and} \quad \deg \beta < \deg \delta.
\]

The inequalities (3.8) also hold in the classical case, but the proof then follows quickly from the properties of the roots. In the setting of this paper, it is a laborious case by case analysis.

**Proof of Lemma 3.13.** Since \(q\) and \(Q\) are properly equivalent, \(\alpha \delta = \beta \gamma + 1\). We consider several cases.

**Case (i).** Suppose \(\deg \alpha \delta < 0\). Since \(\alpha, \delta \in \mathbb{Q}[T]\), \(H\) is
\[
\begin{pmatrix}
0 & \pm 1 \\
\mp 1 & \delta
\end{pmatrix} \text{ or } \begin{pmatrix}
\alpha & \pm 1 \\
\mp 1 & 0
\end{pmatrix}.
\]

In the latter case, consider \(H^{-1} = \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & \alpha \end{pmatrix}\), taking \(Q\) to \(q\). The matrix \(H\) connects the roots by \(-\delta = 1/f + F\), hence \(\deg \delta = \deg 1/f > 0\); and since \(\deg \alpha < 0\) and \(\deg \beta = \deg \gamma = 0\), the conditions are satisfied. If \(H = \begin{pmatrix} \alpha & \pm 1 \\ \mp 1 & 0 \end{pmatrix}\), the conditions are thus satisfied for \(H^{-1}\).

If \(\beta \gamma = 0\), then \(H\) is
\[
\begin{pmatrix}
\pm 1 & \beta \\
0 & \pm 1
\end{pmatrix} \text{ or } \begin{pmatrix}
\pm 1 & 0 \\
0 & \pm 1
\end{pmatrix}.
\]

The first transformation connects the first roots \(f\) and \(F\), by \(f - F = \beta\), and since the degrees of both \(f\) and \(F\) are negative, \(\beta = 0\), i.e. \(H\) is the identity. For the latter matrix, consider the second roots \(s\) and \(S\). Then \(1/s = \gamma + 1/S\), and since \(s\) and \(S\) are of positive degree, we must have \(\gamma = 0\), and \(H\) is the identity matrix. However, \(q \neq Q\), so we can assume that \(\beta \gamma \neq 0\).

**Case (ii).** Suppose \(\deg \alpha \delta \geq 0\) and \(\beta \gamma \neq \pm 1\). Then \(\deg(\beta \gamma + 1) = \deg \beta \gamma \geq 0\). Hence
\[
\deg \beta + \deg \gamma = \deg \alpha + \deg \delta,
\]
and so
\[(3.10) \quad \deg \alpha < \deg \beta \iff \deg \gamma < \deg \delta.\]

(a) Suppose \(\deg \alpha < \deg \delta\). Then (3.9) implies \(\deg \beta + \deg \gamma < 2 \deg \delta\).

- If \(\deg \beta = \deg \gamma\), then \(\deg \beta < \deg \delta\) and \(\deg \gamma < \deg \delta\). Then from (3.10) \(\deg \alpha < \deg \beta\).
- If \(\deg \gamma < \deg \beta\), then (3.10) implies \(\deg \gamma < \deg \delta\) and \(\deg \alpha < \deg \beta\).

Furthermore, under \(H\), the first roots satisfy
\[
\frac{1}{f} = \frac{\gamma + \delta/F}{\alpha + \beta/F}
\]
and since \(\deg f < 0\), we must have \(\deg(\gamma + \delta/F) > \deg(\alpha + \beta/F)\). Moreover, \(\deg \gamma < \deg \delta + \deg 1/F\), since \(\deg 1/F > 0\). Hence
\[
\deg(\gamma + \delta/F) = \deg \delta + \deg \frac{1}{F} > \deg(\alpha + \beta/F) \geq \deg \frac{\beta}{F}.
\]
The latter inequality follows from \(\deg \frac{\beta}{F} > \deg \beta > \deg \alpha\). Therefore \(\deg \delta > \deg \beta\).

- If \(\deg \beta < \deg \gamma\), then (3.9) implies \(\deg \beta < \deg \delta\) and \(\deg \alpha < \deg \gamma\).

We use the relation of the second roots under the transformation \(H\), namely
\[
\frac{1}{s} = \frac{\gamma + \delta/S}{\alpha + \beta/S}, \quad \text{so} \quad 1 = \left(\frac{\alpha}{s} - \gamma\right)(\alpha S + \beta).
\]
Hence \(\deg(\alpha/s - \gamma) = -\deg(\alpha S + \beta)\). In addition, \(\deg 1/s < 0\) so
\[
\deg(\alpha/s - \gamma) = \deg \gamma = -\deg(\alpha S + \beta).
\]
Furthermore, \(\deg \gamma > \deg \alpha \geq 0\), i.e. \(\deg(\alpha S + \beta) < 0\). Since \(\alpha, \beta \in \mathbb{Q}[T]\) and \(\deg S > 0\), this can only happen if \(\deg \alpha S = \deg \beta\). Hence \(\deg \alpha < \deg \beta\).

(b) Suppose \(\deg \delta < \deg \alpha\). Consider \(H^{-1}\), taking \(Q\) to \(q\). Then
\[
H^{-1} = \begin{pmatrix} A & B \\ \Gamma & \Delta \end{pmatrix} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix},
\]
hence \(\deg A < \deg \Delta\), and the same analysis as in the above cases works.

(c) Suppose \(\deg \alpha = \deg \delta\). Then \(2 \deg \alpha = 2 \deg \delta = \deg \beta + \deg \gamma\).

- If \(\deg \beta = \deg \gamma\), then \(\deg \alpha = \deg \beta = \deg \gamma = \deg \delta\). Moreover, consider
\[(3.11) \quad 1 = (\alpha/f - \gamma)(\alpha F + \beta).
\]
Since \(\deg 1/f > 0\) and \(\deg F < 0\), we have
\[
-\deg \beta = -\deg(\alpha F + \beta) = \deg(\alpha/f - \gamma) > \deg \alpha,
\]
a contradiction.

- If \(\deg \beta > \deg \gamma\), then \(\deg \alpha < \deg \beta\) and \(\deg \delta < \deg \beta\). From (3.10), we have \(\deg \gamma < \deg \delta\) and \(\deg \gamma < \deg \alpha\). Furthermore, taking the degree of
\[ (3.11) \] we get
\[ \partial \deg (\alpha F + \beta) = -\partial \deg (\alpha/f - \gamma) \]
and since \( \deg 1/f > 0 \) and \( \deg F < 0 \) we have
\[ -\deg \beta = -\partial \deg (\alpha F + \beta) = \partial \deg (\alpha/f - \gamma) > \deg \alpha. \]
But also \( \deg \beta > \deg \alpha \), hence \( \deg \alpha < 0 \), i.e. \( \alpha = 0 = \delta \), and \( \beta = \pm 1 = \gamma \); but by assumption \( \deg \beta > \deg \gamma \), a contradiction.

- If \( \deg \beta < \deg \gamma \), then \( \deg \beta < \deg \alpha < \deg \gamma \) and \( \deg \beta < \deg \delta < \deg \gamma \). We next consider
\[ (3.12) \]
\[ 1 = (\alpha/s - \gamma)(\alpha S + \beta). \]
Taking degree and using \( \partial \deg 1/s < 0 < \partial \deg S \), we have
\[ \deg \alpha < \deg \gamma = -\partial \deg (\alpha/s - \gamma) = -\deg \alpha - \partial \deg S, \]
i.e. \( \partial \deg S < -2 \deg \alpha \) and \( \deg \alpha < 0 \). Thus \( \alpha = 0 \), and the same analysis as above gives us a contradiction.

4. Representation by indefinite binary quadratic forms and the Markov spectrum

 DEFINITION 4.1. We say that \( A \in \mathbb{Q}((1/T)) \) is represented by an indefinite binary quadratic form \( Q \) if there exist rational polynomials \( X, Y \), not both zero, such that
\[ A = Q(X, Y). \]

PROPOSITION 4.2. Properly equivalent binary quadratic forms represent the same elements of \( \mathbb{Q}((1/T)) \).

Proof. Let \( q \) and \( Q \) be two binary quadratic forms which are properly equivalent via the transformation \( H = (\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}) \in SL(\mathbb{Q}[T]) \), and let \( M \in \mathbb{Q}((1/T)) \) be represented by \( q \). That is, there are rational polynomials \( x, y \), not both 0, such that \( q(x, y) = M \). Let \( X = \delta x - \beta y \) and \( Y = -\gamma x + \alpha y \), also rational polynomials; then \( Q(X, Y) = M \). Finally, \( X \) and \( Y \) cannot both be zero, since \( x \) and \( y \) are not both zero and the determinant of \( H \) is 1. Therefore \( M \) is also represented by \( Q \).

The following theorem is the analogue of Theorem 85 in Dickson’s book [D57], which he attributes to Lagrange.

THEOREM 4.3. If the forms \( ((-1)^iA_i, B_i, (-1)^{i+1}A_{i+1}) \), for \( i \) an integer, constitute a chain of reduced forms of discriminant \( D \neq 0 \), a square in \( \mathbb{Q}((1/T)) \), then the \( A_i \)'s include all elements of \( \mathbb{Q}((1/T)) \) of degree less than \( \partial \deg \sqrt{D} \), which are represented by a form in the chain.

Proof. Let \( A \in \mathbb{Q}((1/T)) \) with \( \deg A < \deg \sqrt{D} \) be represented by a reduced form \( Q = ((-1)^iA_i, B_i, (-1)^{i+1}A_{i+1}) \) of discriminant \( D \) in the chain. That is, there exist rational polynomials \( x, y \), not both zero, such that \( (-1)^iA_ix^2 + B_ixy + (-1)^{i+1}A_{i+1}y^2 = A \). If we take \( \alpha = x \) and \( \gamma = y \),
where $x, y$ are co-prime, then there exist $\beta, \delta \in \mathbb{Q}[T]$ such that $\alpha \delta - \gamma \beta = 1$. Then the transformation $H = \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right)$ takes $Q$ to a properly equivalent form $(A, B, C)$ of the same discriminant $D$, which also represents $A$. However, this form is not necessarily reduced. Consider its first and second roots $f = (\sqrt{D} - B)/(2A)$ and $s = (-\sqrt{D} - B)/(2A)$. Observe that $\partial \deg(f - s) = \partial \deg \sqrt{D} - \partial \deg A > 0$. Therefore, we cannot have both degrees of $f$ and $s$ negative, and we can assume that $\partial \deg f \geq 0$, otherwise $Q$ is reduced. Furthermore, $[f] \neq [s]$, so we apply $\left( \begin{array}{cc} 1 & h \\ 0 & 1 \end{array} \right)$ with $h = [f]$. This transformation sends $(A, B, C)$ to $(A, B_1, C_1)$, which is reduced and represents $A$. From Theorem 3.12, $(A, B_1, C_1)$ must be one of the forms in the chain, i.e. $A$ must appear amongst the $A_i$’s.

**Theorem 4.4.** Suppose $Q$ is an indefinite binary quadratic form determining a chain of equivalent reduced forms $\Phi_i = ((-1)^i A_i, B_i, (-1)^{i+1} A_{i+1})$ for $i \in \mathbb{Z}$. Then

$$\inf_{X,Y \in \mathbb{Q}[T]} \partial \deg Q(X,Y) = \inf_{i \in \mathbb{Z}} \partial \deg A_i.$$ 

**Proof.** If $Q$ is not a reduced form, then we can use the algorithm of Theorem 3.7 to find a properly equivalent form $\hat{Q}$ which is reduced; moreover, since they are properly equivalent, they will represent the same elements by Proposition 4.2. Thus we can assume that $Q = (A, B, C)$ is a reduced form of discriminant $D$. Then

$$\partial \deg \left( \frac{\sqrt{D}}{A} \right) = \partial \deg \left( \frac{\sqrt{D} + B}{2A} + \frac{\sqrt{D} - B}{2A} \right) > 0.$$ 

In particular, $A \in \mathbb{Q}((1/T))$ is such that $\partial \deg A < \partial \deg \sqrt{D}$. Hence $Q$ represents an element of $\mathbb{Q}((1/T))$ of degree smaller than $\partial \deg \sqrt{D}$. Therefore by Theorem 4.3, it is represented by some $A_i$ in the chain of reduced forms equivalent to $Q$.

**Definition 4.5.** Let $Q$ be an indefinite binary quadratic form of discriminant $D \neq 0$. Let

$$m(Q) := \inf_{X,Y \in \mathbb{Q}[T]} \partial \deg Q(X,Y).$$

Then the **Markov spectrum** is defined to be

$$\mathcal{M} := \{ \partial \deg \sqrt{D(Q)} - m(Q) \mid Q \text{ an indefinite binary quadratic form} \}.$$ 

And from Theorem 4.4 we can conclude that

$$\mathcal{M} = \left\{ \partial \deg \sqrt{D(\Phi_i)} - \inf_{i \in \mathbb{Z}} \partial \deg A_i \mid \Phi_i = ((-1)^i A_i, B_i, (-1)^{i+1} A_{i+1}) \text{ a chain} \right\}.$$
5. Alternative realisation of the Markov spectrum. In the real case the Markov spectrum can alternatively be defined in terms of doubly infinite sequences of positive integers. In our setting we work with the analogous object, doubly infinite sequences of polynomials of positive degree, $G = (\ldots, g_{-1}, g_0, g_1, \ldots)$. We will use this new form of the spectrum to completely determine it.

To give some intuition on how the Markov spectrum is realised via doubly infinite sequences, we re-examine a few identities from Section 3. Suppose we are given an indefinite binary quadratic form $Q$ of discriminant $D$, which determines a chain of equivalent forms $\Phi_i = (\ldots, A_i, B_i, (-1)^{i+1} A_{i+1}).$

Just as in the discussion after Corollary 3.10, we can define $F_i := (-1)^i/A_i$ and $S_i := (-1)^{i+1}/s_i$, where $f_i$ and $s_i$ are the first and second roots of $\Phi_i$. Then from Remark 3.11 we have

$$F_i + S_i = \sqrt{D}/A_{i+1} = [g_i, g_{i+1}, \ldots] + [0, g_{i-1}, g_{i-2}, \ldots],$$

where the $g_i$ are rational polynomials of positive degree. Furthermore, according to Theorem 4.4 the elements of the Markov spectrum are given by $\deg \sqrt{D} - \inf_{i \in \mathbb{Z}} \deg A_i$. Hence it is natural to make the following definition.

**Definition 5.1.** For a doubly infinite sequence $G = (\ldots, g_{-1}, g_0, g_1, \ldots)$ of rational polynomials of positive degree, we define $\lambda_i(G) := [g_i, g_{i+1}, \ldots] + [0, g_{i-1}, g_{i-2}, \ldots]$.

**Theorem 5.2.** The Markov spectrum $\mathcal{M}$ can be realised as the set

$$\mathcal{M} = \{ M(G) \mid G = (\ldots, g_{-1}, g_0, g_1, \ldots) \text{ with } g_i \in \mathbb{Q}[T], \deg(g_i) > 0 \},$$

where $M(G) := \sup_{i \in \mathbb{Z}} \deg \lambda_i(G)$.

**Proof.** From the discussion above and Remark 3.11 given an indefinite binary quadratic form $Q$ of discriminant $D \neq 0$ we obtain a doubly infinite sequence $G$ of non-constant polynomials such that $M(G) = \sqrt{D} - m(Q)$. Hence $\mathcal{M} \subseteq \mathcal{M}$.

On the other hand, given a doubly infinite sequence $G = (\ldots, g_{-1}, g_0, g_1, \ldots)$ of rational polynomials of positive degree, we consider

$$\lambda_i(G) = [g_i, g_{i+1}, \ldots] + [0, g_{i-1}, g_{i-2}, \ldots] \in \mathbb{Q}((1/T)).$$

Thus we can find an element of $\mathbb{Q}((1/T))$, say $A_{i+1}$, of degree $-\deg g_i < 0$, such that $\lambda_i(G) = 1/A_{i+1}$. Let $F_i = [g_i, g_{i+1}, \ldots]$ and $S_i = [0, g_{i-1}, g_{i-2}, \ldots]$, then $F_i + S_i = 1/A_{i+1}$. Choose $B_i \in \mathbb{Q}((1/T))$ such that

$$F_i = \frac{1 + B_i}{2A_{i+1}} \quad \text{and} \quad S_i = \frac{1 - B_i}{2A_{i+1}}.$$
Then we consider $f_i = (-1)^i/F_i$ and $s_i = (-1)^i/S_i$, i.e.

$$f_i = \frac{1 - B_i}{2(-1)^i a_i} \quad \text{and} \quad s_i = \frac{1 + B_i}{2(-1)^i a_i},$$

where $4A_{i+1}a_i = 1 - B_i^2$. Furthermore, $\deg S_i < 0 < \deg F_i$ and thus $\deg f_i < 0 < \deg s_i$. Therefore, $f_i$ and $s_i$ are the roots of the reduced indefinite binary quadratic form $Q_i = ((-1)^i a_i, B_i, (-1)^{i+1} A_{i+1})$ of discriminant 1. From the continued fraction expansion of $F_i$ and $S_i$ we have

$$F_i = g_i + \frac{1}{F_{i+1}} \quad \text{and} \quad \frac{1}{S_i} = g_{i-1} + S_{i-1},$$

so

$$f_{i+1} = \delta_i - \frac{1}{f_i} \quad \text{and} \quad s_{i+1} = \delta_i - \frac{1}{s_i},$$

where $\delta_i = (-1)^i g_i$. Then the transformation $\Delta_i = (0 \ 1, -1 \ \delta_i)$ sends $Q_i$ to $Q_{i+1}$, and in particular $a_{i+1} = A_{i+1}$. Hence the forms $Q_i = ((-1)^i A_i, B_i, (-1)^{i+1} A_{i+1})$ are reduced, of discriminant 1 and in a chain. From Theorem 4.4 we know $\inf_{i \in \mathbb{Z}} A_i = m(Q)$, where $Q$ is an indefinite quadratic form of discriminant 1 properly equivalent to $Q_i$. Then

$$M(G) = \sup_{i \in \mathbb{Z}} \deg \lambda_i(G) = \sup_{i \in \mathbb{Z}} \deg \left(\frac{1}{A_{i+1}}\right) = -\inf_{i \in \mathbb{Z}} \deg A_{i+1} = -m(Q).$$

Hence $\mathbb{M} \subseteq \mathcal{M}$. \(\blacksquare\)

**Theorem 5.3.** The Markov spectrum $\mathcal{M}$ equals $\mathbb{N} \cup \{\infty\}$.

**Proof.** From the above theorem

$$\mathcal{M} = \mathbb{M} = \{M(G) \mid G = (\ldots, g_{-1}, g_0, g_1, \ldots) \text{ with } g_i \in \mathbb{Q}[T], \deg(g_i) > 0\}.$$ 

Furthermore, $M(G) = \sup_{i \in \mathbb{N}} \deg \lambda_i(G)$, and $\deg \lambda_i(G) = \deg g_i$, where $g_i \in \mathbb{Q}[T]$ has positive degree. The result follows. \(\blacksquare\)

Suppose that in Definition 5.1 instead of taking the supremum we consider $L(G) := \limsup_{i \in \mathbb{N}} \deg \lambda_i(G)$ and let $G$ range over all doubly infinite sequences of rational polynomials of positive degree; we will obtain a new set which we will show to be of the same size. The analogy of this over the integers produces an alternative realisation of the Lagrange spectrum. Moreover, Markov introduced doubly infinite sequences of integers in [M79] precisely to prove that the Markov spectrum, for numbers below 3, coincides with the Lagrange spectrum.

**6. Rational approximation and the Lagrange spectrum.** Over the reals, the classical definition of Lagrange spectrum is closely related to the rational approximation of algebraic real irrationals. We will first study the analogue to this over $\mathbb{Q}((1/T))$ and later prove that its elements are also given by $L(G)$ for a doubly infinite sequence $G$ of rational polynomials.
6.1. Classical definition. We begin by outlining some well-known results on rational approximations of algebraic elements of $\mathbb{Q}((1/T))$. Propositions 6.1 and 6.3 and Theorem 6.2 appear in slightly different formulations in this article, but in spirit they are well known in the literature: see for example [PT00]. Similarly to the case over the reals, the convergents of $\alpha$ provide a very good rational approximation.

**Proposition 6.1.** Suppose $\alpha \in \mathbb{Q}((1/T))$ and $p, q \in \mathbb{Q}[T]$ with $q \neq 0$. Then

$$\partial \deg(\alpha - p/q) < -2 \deg q$$

if and only if $p/q$ is a convergent for $\alpha$.

Furthermore, an explicit formula of how well, in terms of degree, $\alpha$ is approximated by its convergents, is known.

**Theorem 6.2.** Suppose $\alpha \in \mathbb{Q}((1/T))$ and let $p_h/q_h$ be its $h$th convergent. Then

$$\partial \deg(\alpha - p_h/q_h) = -2 \deg q_h - \deg a_{h+1}. \quad (6.1)$$

There is no result similar to the identity (6.1), expressing $|\alpha - p_h/q_h|$, for $\alpha \in \mathbb{R}$ and $p_h/q_h \in \mathbb{Q}$, as a linear combination of the partial quotients $a_h$ and continuants $q_h$.

Theorem 6.2 significantly simplifies the computation of the Lagrange spectrum, to be defined shortly, and the proofs of various results in Diophantine approximation.

One example is the following proposition, whose statement is a precise analogue to a corollary to Dirichlet’s approximation theorem (1), and whose proof over $\mathbb{Q}((1/T))$ follows almost immediately from Theorem 6.2.

**Proposition 6.3.** Given $\alpha \in \mathbb{Q}((1/T))$ not a rational function, there exist infinitely many pairs of rational polynomials $p, q$ with $q \neq 0$ such that

$$\partial \deg(\alpha - p/q) \leq -2 \deg q - 1.$$

**Proof.** Since the degree of $a_{n+1}$ is always greater than or equal to 1, Theorem 6.2 implies that the convergents $p_n/q_n$ satisfy the inequality. Furthermore, since $\alpha$ is not a rational function, Proposition 2.5 implies that there are infinitely many of those.

If we consider all non-rational $\alpha \in \mathbb{Q}((1/T))$ then we cannot improve the inequality in Proposition 6.3.

(1) The corollary to Dirichlet’s approximation theorem states that for any $\alpha \in \mathbb{R}$, there exist infinitely many $p/q \in \mathbb{Q}$ such that $|\alpha - p_n/q_n| < 1/q^2$. 
Example 6.4. Suppose $D \in \mathbb{Q}[T]$ is a quadratic polynomial, not a perfect square, say $D = (aT + b)^2 + c$ with $a, b, c \in \mathbb{Q}$ and $ac \neq 0$. Then
\[
\sqrt{D} = \sqrt{(aT + b)^2 + c} = \left[ aT + b, \frac{2}{c}(aT + b), 2(aT + b) \right].
\]
Notice that all partial quotients have degree 1 and therefore by Theorem 6.2
\[
\partial \deg(\alpha - p/q) = -2 \deg q - 1
\]
for all convergents $p/q$.

However, given a specific non-rational Laurent series $\alpha$, we might be able to sharpen the bound. This leads us to the following definition:

Definition 6.5. Given $\alpha \in \mathbb{Q}((1/T))$, we define the approximation (Lagrange) constant $l(\alpha)$ to be the supremum of the integers $k$ such that the inequality
\[
\partial \deg(\alpha - p/q) \leq -2 \deg q - k
\]
is satisfied by infinitely many rational polynomials $p, q$ with $q$ of arbitrarily large degree. We then define the Lagrange spectrum over $\mathbb{Q}((1/T))$ to be
\[
\mathcal{L} := \{ l(\alpha) \mid \alpha \in \mathbb{Q}((1/T)) \text{ not rational} \}.
\]

Remark 6.6. Theorem 6.2 simplifies the definition of Lagrange’s constant in this setting. Since from Proposition 6.1 the inequality
\[
\partial \deg(\alpha - p/q) < -2 \deg q
\]
is satisfied only by the convergents of $\alpha$, we have
\[
l(\alpha) = \lim \sup_{h \to \infty} \deg a_h,
\]
where $a_h$ denotes the partial quotients of $\alpha$.

6.2. Alternative definition. Observe that for $\alpha \in \mathbb{Q}((1/T))$ not a rational function, with continued fraction expansion $[a_0, a_1, \ldots, \alpha_{h+1}]$,
\[
(6.2) \quad \alpha - \frac{p_h}{q_h} = \frac{(-1)^h}{q_h^2(\alpha_{h+1} + q_{h-1}/q_h)}.
\]
This identity is in perfect correspondence with (1.15) from Aigner’s book [A13].

In particular, from (6.2) we obtain
\[
l(\alpha) = \lim \sup_{h \to \infty} \partial \deg(\alpha_{h+1} + q_{h-1}/q_h).
\]
Observe further that
\[
\alpha_{h+1} = [a_{h+1}, a_{h+2}, \ldots] \quad \text{and} \quad q_{h-1}/q_h = [0, a_h, a_{h-1}, \ldots, a_1],
\]
where each $a_i$ is a rational polynomial of positive degree. Recall that given a doubly infinite sequence $G = (\ldots, g_{-1}, g_0, g_1, \ldots)$ of non-constant rational
polynomials, we have defined
\[ \lambda_i(G) := [g_i, g_{i+1}, \ldots] + [0, g_{i-1}, g_i-2, \ldots]. \]
Hence we get

**Theorem 6.7.** The Lagrange spectrum \( \mathcal{L} \) can also be defined as
\[ \mathcal{L} = \{ L(G) \mid G = (\ldots, g_{-1}, g_0, g_1, \ldots) \text{ with } g_i \in \mathbb{Q}[T], \deg(g_i) > 0 \}, \]
where \( L(G) = \limsup_{i \to \infty} \deg \lambda_i(G). \)

**Proof.** For \( \alpha \in \mathbb{Q}((1/T))/\mathbb{Q}(T) \) with continued fraction expansion \( \alpha = [a_0, a_1, \ldots] \), let
\[ G = (\ldots, a_1, a_0, a_1, \ldots). \]
Then \( L(G) = \limsup_{i \to \infty} \deg \lambda_i(G) = \limsup_{i \to \infty} \deg a_i. \) Furthermore, from Theorem 6.2 we know that \( l(\alpha) = \limsup_{i \to \infty} \deg a_i. \) Therefore, \( l(\alpha) \in \{ L(G) \mid G \text{ as above} \}. \)

For the converse, let \( G \) be a doubly infinite sequence as in the definition. Then \( L(G) \) is either \( \limsup_{i \to +\infty} \deg \lambda_i(G) \) or \( \limsup_{i \to -\infty} \deg \lambda_i(G). \) In the former case we take \( \alpha = [g_0, g_1, \ldots] \) and in the latter \( \alpha = [g_0, g_{-1}, \ldots]. \) Then \( L(A) \in \mathbb{L}. \)

**6.3. Examples and computation.** An easy consequence of Example 6.4 is that for any square-free quadratic polynomial \( D \) with rational coefficients, \( l(\sqrt{D}) = 1. \) For more interesting examples of Lagrange constants we need to find \( \alpha \in \mathbb{Q}((1/T)) \) with infinitely many partial quotients of degree greater than 1.

**Theorem 6.8.** For \( a, b, c \in \mathbb{Q}[T] \), we have
(1) \( \sqrt{a^2 + 1} = [a, 2a]; \)
(2) \( \sqrt{a^2 + c} = [a, 2b, 2a] \) if \( a = bc. \)

The proof is an explicit computation following the techniques used in the real quadratic irrationals case. Furthermore, as a special case we have

**Example 6.9.** Let \( d > 1 \) be a positive integer. Then
(1) \( \sqrt{T^{2d} + T^d} = [T^d, 2T^d]; \)
(2) \( \sqrt{T^{2d} + T^l} = [T^d, 2T^{d-l}, 2T^{d}] \) for \( 0 \leq l < d. \)

And in particular, \( l(\sqrt{D}) = d \) in both cases. This is clear in Example 6.9(1), since \( \deg a_h = d \) for all \( h. \) The assertion for 6.9(2) follows from the observation that
\[ \deg a_h = \begin{cases} d & \text{if } h \text{ even,} \\ d - l & \text{if } h \text{ is odd.} \end{cases} \]
Since \( d - l < d, \) we have \( l(\sqrt{D}) = \limsup_{h \to \infty} \deg a_h = d. \)

**Theorem 6.10.** The Lagrange spectrum of \( \mathbb{Q}((1/T)) \) is equal to \( \mathbb{N} \cup \{ \infty \}. \)
Proof. For each positive integer \( k \), there exists \( \alpha \in \mathbb{Q}((T^{-1})) \) such that \( l(\alpha) = k \); just take \( \alpha \) to be one of the square roots described in Example 6.9.

Corollary 6.11. The Lagrange spectrum is equal to the Markov spectrum.

We have already computed the Markov spectrum in Theorem 5.3, and from Theorem 6.10, it is equal to the Lagrange spectrum.

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