DEPTH AND REGULARITY OF MONOMIAL IDEALS VIA POLARIZATION AND COMBINATORIAL OPTIMIZATION

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ABSTRACT. In this paper we use polarization to study the behavior of the depth and regularity of a monomial ideal $I$, locally at a variable $x_i$, when we lower the degree of all the highest powers of the variable $x_i$ occurring in the minimal generating set of $I$, and examine the depth and regularity of powers of edge ideals of clutters using combinatorial optimization techniques. If $I$ is the edge ideal of an unmixed clutter with the max-flow min-cut property, we show that the powers of $I$ have non-increasing depth and non-decreasing regularity. In particular edge ideals of unmixed bipartite graphs have non-decreasing regularity. We are able to show that the symbolic powers of the ideal of covers of the clique clutter of a strongly perfect graph have non-increasing depth. A similar result holds for the ideal of covers of a uniform ideal clutter.

1. Introduction

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field $K$, let $f$ be a monomial of $R$, and let $I \subset R$ be a monomial ideal. The following two inequalities were shown in [3, Theorem 3.1]:

(A) $\text{depth}(R/(I : f)) \geq \text{depth}(R/I)$,
(B) $\text{reg}(R/I) \geq \text{reg}(R/(I : f))$,

where depth($R/I$) and reg($R/I$) are the depth and regularity of the quotient ring $R/I$ and $(I : f) = \{g \in R | gf \in I\}$ is referred to as a colon ideal. If $I$ and $f$ are squarefree, we show that (A) and (B) are equivalent using a duality theorem of Terai [58] (Theorem 2.7) and some duality formulas for edge ideals of clutters (Lemma 2.6), that is, (A) and (B) are dual statements in the squarefree case (Proposition 2.8).

We introduce a formula expressing $\text{depth}(R/(I, f)) - \text{depth}(R/I)$, $\text{reg}(R/I)$, and $\text{reg}(R/(f, I))$ in terms of the depth and regularity of polarizations (Proposition 2.11). Then, as an application, we give an alternate proof of (A) and (B), and show some other known inequalities about depth and regularity (Corollary 2.12). If $\text{in}_>(I + f) = I + \text{in}_>(f)$ for some monomial order $<$ and some homogeneous polynomial $f$, we show that (A) and (B) hold (Corollary 2.13).

The aim of this paper is to use these results to study the behavior of the depth and regularity of $R/I$, locally at a variable $x_i$, when we lower the degree of all the highest powers of the variable $x_i$ occurring in the minimal generating set of $I$ and, furthermore, to examine the depth and regularity of powers and symbolic powers of edge ideals of clutters and graphs, and their ideals of covers, using combinatorial optimization techniques.

Fix a variable $x_i$ that occurs in the minimal generating set $G(I)$ of $I$. Let $q$ be the maximum of the degrees in $x_i$ of the monomials of $G(I)$, let $B_i$ be the set of all monomials of $G(I)$ of
degree $q$ in $x_i$, let $p$ be the maximum of the degrees in $x_i$ of the monomials of $\mathcal{A}_i = G(I) \setminus \mathcal{B}_i$, and consider the ideal $L = (\{x^a/x_i \mid x^a \in \mathcal{B}_i \cup \mathcal{A}_i\})$.

One of our main results shows that the depth is locally non-decreasing at each variable $x_i$ when lowering the top degree. Note that if $p = 0$, that is, if all generators of $I$ that are divisible by $x_i$ have degree $q$ in $x_i$, then $L = (I : x_i)$. Thus when $p = 0$ we have from (A) that $\text{depth}(R/L) = \text{depth}(R/(I : x_i)) \geq \text{depth}(R/I)$. This theorem allows control over the depth when the degrees in $x_i$ of the generators varies.

**Theorem 3.1**

(a) If $p \geq 1$ and $q - p \geq 2$, then $\text{depth}(R/I) = \text{depth}(R/L)$.

(b) If $p \geq 0$ and $q - p = 1$, then $\text{depth}(R/L) \geq \text{depth}(R/I)$.

(c) If $p = 0$ and $q \geq 2$, then $\text{depth}(R/I) = \text{depth}(R/(\{x^a/x_i^q \mid x^a \in \mathcal{B}_i \cup \mathcal{A}_i\}))$.

There are similar results for regularity (Theorem 3.7). As a consequence one recovers a result of Herzog, Takayama and Terai [32] showing that $\text{depth}(R/I)$ is equal to $\text{depth}(R/(I^{k+1} : f))$ for $k \geq 1$. This was exploited in [3, 13] and in [24, Corollary 3.11] in connection to normally torsion-free ideals.

There are some classes of monomial ideals whose powers have non-increasing depth and non-decreasing regularity [3, 5, 20, 27, 52, 54]. A natural way to show these properties for a monomial ideal is to prove the existence of a monomial $f$ such that $(I^{k+1} : f) = I^k$ for $k \geq 1$. This was exploited in [3, 13] and in [24, Corollary 3.11] in connection to normally torsion-free ideals.

Since any squarefree monomial ideal is the edge ideal $I(C)$ of a clutter $C$, we will study the depth and regularity of powers and symbolic powers of edge ideals of clutters and graphs—and their ideals of covers—that have nice combinatorial optimization properties (e.g., max-flow min-cut, ideal, uniform, and unmixed clutters, strongly perfect and very well-covered graphs). The $k$-th symbolic power of an ideal $I$ is denoted by $I^{(k)}$ (Definition 4.2). The ideal of covers of a clutter $C$, denoted $I(C)^\vee$, is the edge ideal of $C^\vee$, the clutter of minimal vertex covers of $C$.

If $I(C)$ is the edge ideal of a clutter $C$ which has a good leaf, then the powers of $I(C)$ have non-increasing depth and non-decreasing regularity [3, Theorem 5.1]. In particular edge ideals of forests or simplicial trees have these properties. Our next result gives a wide family of ideals with these properties.

**Theorem 4.9** If $I = I(C)$ is the edge ideal of an unmixed clutter $C$ with the max-flow min-cut property, then

(a) $\text{depth}(R/I^k) \geq \text{depth}(R/I^{k+1})$ for $k \geq 1$, and

(b) $\text{reg}(R/I^k) \leq \text{reg}(R/I^{k+1})$ for $k \geq 1$.

Let $G$ be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$. A result of T. N. Trung [61] shows that for $k \gg 0$ one has

$$\text{depth}(R/I(G)^k) = |\text{isol}(G)| + c_0(G),$$

where isol($G$) is the set of isolated vertices of $G$ and $c_0(G)$ is the number of non-trivial bipartite components of $G$. We complement this fact by observing that $\dim(R) - \ell(I(G))$ is equal to $|\text{isol}(G)| + c_0(G)$, where $\ell(I(G))$ is the analytic spread of $I(G)$, and by showing the inequality

$$\text{depth}(R/((I(G)^k : x_i)^k, N_G(x_i))) \leq \text{depth}(R/(I(G \setminus N_G(x_i))^k, N_G(x_i)))$$

for $k \geq 1$ and $i = 1, \ldots, n$ (Proposition 5.1), where $N_G(x_i)$ is the neighbor set of $x_i$. For $k = 1$ this inequality follows from the fact that $(I(G) : x_i)$ is equal to $(I(G \setminus N_G(x_i)), N_G(x_i))$. [60] p.
293] and using the inequality depth(R/(I(G): x_i)) ≥ depth(R/I(G)). The general case follows by successively applying Theorem 3.1 locally at each variable.

It is an open problem whether or not the powers of the edge ideal of a graph have non-increasing depth. To the best of our knowledge this is open even for bipartite graphs. Our next application extends the fact that the powers of $I(G)^\vee$, the ideal of covers of $G$, have non-increasing depth if $G$ is bipartite [5, 26, 27].

**Corollary 5.3** Let $G$ be a bipartite graph. The following hold.

(a) [39, Corollary 5.3] If $G$ is unmixed, then $I(G)$ has non-increasing depth.
(b) ([5, Theorem 3.2], [26, 27, Corollary 2.4]) $I(G)^\vee$ has non-increasing depth.
(c) $I(G)^\vee$ has non-decreasing regularity.

An interesting example due to Kaiser, Stehlík, and Škrekovski [38] shows that the powers of the ideal of covers of a graph does not always have non-increasing depth (Example 5.4), that is, part (b) of Corollary 5.3 fails for non-bipartite graphs. A nice result of L. T. Hoa, K. Kimura, N. Terai and T. N. Trung [33, Theorem 3.2] shows that the symbolic powers of the ideal of covers of a graph have non-increasing depth. A similar result holds for the ideal of covers of a uniform ideal clutter (Corollary 4.10).

If $G$ is a very well-covered graph, then the depths of symbolic powers of $I(G)^\vee$ form a non-increasing sequence [52] (cf. [33, Theorem 3.2]) and also the depths of symbolic powers of $I(G)$ form a non-increasing sequence [39, Theorem 5.2]. In this case we show that the symbolic powers of $I(G)$ have non-decreasing regularity (Proposition 5.6).

We will give another family of squarefree monomial ideals whose symbolic powers have non-increasing depth and non-decreasing regularity. A **clique** of a graph $G$ is a set of vertices inducing a complete subgraph. The **clique clutter** of $G$, denoted by $cl(G)$, is the clutter on $V(G)$ whose edges are the maximal cliques of $G$.

**Proposition 5.8** Let $G$ be a strongly perfect graph and let $cl(G)$ be its clique clutter. If $J$ is the ideal of covers of $cl(G)$, then

(a) $\text{depth}(R/J^{(k)}) \geq \text{depth}(R/J^{(k+1)})$ for $k \geq 1$,

(b) $\text{reg}(R/J^{(k)}) \leq \text{depth}(R/J^{(k+1)})$ for $k \geq 1$.

Bipartite graphs, chordal graphs, comparability graphs, and Meyniel graphs are strongly perfect (see [47] and the references therein). Thus this result generalizes Corollary 5.3(b) because if $G$ is a bipartite graph, then $cl(G) = G$ and $I(G)^\vee^{(k)} = I(G)^\vee^{(k)}$ for $k \geq 1$ [18].

For edge ideals of clutters the Cohen–Macaulay property of its $k$-th ordinary or symbolic power is well understood if $k \geq 3$. By a result of N. Terai and N. V. Trung [59], if $I(\mathcal{C})$ is the edge ideal of a clutter $\mathcal{C}$, then $I(\mathcal{C})^k$ (resp. $I(\mathcal{C})^{(k)}$) is Cohen–Macaulay for some $k \geq 3$ if and only if $I(\mathcal{C})$ is a complete intersection (resp. the independence complex $\Delta_{\mathcal{C}}$ of $\mathcal{C}$ is a matroid).

The case when $G$ is a graph and $k = 2$ is treated in [7, 35, 36, 60]. The Cohen–Macaulay property of the square of an edge ideal can be expressed in terms of its connected components [25, 48] (Corollary 5.10). Edge ideals of graphs whose square is Cohen–Macaulay have a rich combinatorial structure and have been classified combinatorially by D. T. Hoang, N. C. Minh and T. N. Trung [35, 36]. The Cohen-Macaulay property of $I(G)^2$ is also studied in [60] in terms of simplicial complexes.

As an application we recover the following fact.

**Corollary 5.14** ([7, Theorem 2.7], [35, Proposition 4.2]) Let $G$ be a bipartite graph without isolated vertices. Then $I(G)^2$ is Cohen-Macaulay if and only if $G$ is a disjoint union of edges.
For all unexplained terminology and additional information we refer to [22] (for commutative algebra), [6, 50, 51] (for combinatorial optimization), [28] (for graph theory), and [15, 21, 31, 62, 66] (for the theory of powers of edge ideals of clutters and monomial ideals).

2. Depth and regularity of monomial ideals via polarization

Let \( R = K[x_1, \ldots, x_n] \) be a polynomial ring over a field \( K \) and let \( I \) be a monomial ideal. The unique minimal set of generators of \( I \) consisting of monomials is denoted by \( G(I) \). The goal of this section is to use polarization to control the depth and regularity of \( R/I \) when the powers of a variable appearing in \( G(I) \) are reduced. To do so, we first recall some known results, then show a series of equivalent conditions that will allow us to study the behavior of the depth and the regularity of \( R/I \).

In [3] Lemma 5.1 it was shown that \( \text{depth}(R/(I: x_i)) \geq \text{depth}(R/I) \) for all \( i \). By noting that a generating set for \( (I: x_i) \) can be found from \( G(I) \) by reducing all powers of \( x_i \) by one, this can be viewed as the first step in reaching the goal. The result was recently generalized in [3, Theorem 3.1] to any monomial ideal. We provide an alternate proof using polarization. We begin by treating the squarefree case using Stanley-Reisner complexes.

Recall that if \( \Delta \) is a simplicial complex with vertices \( x_1, \ldots, x_n \), the Stanley-Reisner ideal of \( \Delta \), denoted by \( I_{\Delta} \), is the ideal of \( R \) whose squarefree monomial generators correspond to non-faces of \( \Delta \). That is,

\[
I_{\Delta} = (x_{i_1} \cdots x_{i_t} | \{i_1, \ldots, i_t\} \not\subseteq \Delta).
\]

The following result shows how the structure of the simplicial complex can be used to find the depth of the associated ideal.

**Theorem 2.1.** [50] Let \( \Delta \) be a simplicial complex with vertex set \( V = \{x_1, \ldots, x_n\} \), let \( I_{\Delta} \) be its Stanley-Reisner ideal, and \( K[\Delta] = R/I_{\Delta} \). Then

\[
\text{depth}(R/I_{\Delta}) = 1 + \max\{i | K[\Delta^i] \text{ is Cohen–Macaulay}\},
\]

where \( \Delta^i = \{F \in \Delta | \dim(F) \leq i\} \) is the \( i \)-skeleton and \(-1 \leq i \leq \dim(\Delta)\).

The star of a face \( \sigma \) in a simplicial complex \( \Delta \), denoted \( \text{star}_{\Delta}(\sigma) \), is defined to be the subcomplex of \( \Delta \) generated by all facets of \( \Delta \) that contain \( \sigma \).

**Lemma 2.2.** [3, Theorem 3.1] Let \( I \subset R \) be a squarefree monomial ideal and let \( f \) be a squarefree monomial. Then \( \text{depth}(R/(I: f)) \geq \text{depth}(R/I) \).

**Proof.** Let \( \sigma = \text{supp}(f) \) be the set of all variables that occur in \( f \). We may assume that \( f \) is a zero divisor of \( R/I \) because otherwise \( (I: f) = I \) and there is nothing to prove. We may also assume that \( f \) is not in all minimal primes of \( I \) because in this case \( (I: f) = R \) and \( \text{depth}(0) = \infty \). Let \( \Delta \) and \( \Delta' \) be the Stanley–Reisner complexes of \( I \) and \( (I: f) \), respectively. Setting \( d = \dim(\Delta) \), \( d' = \dim(\Delta') \), one has \( d' \leq d \). Assume that \( \Delta^i \) is Cohen–Macaulay for some \( i \leq d \). We claim that \( i \leq d' \). If \( i > d' \), take a facet \( F \) of \( \Delta' \) of dimension \( d' \), that is, \( F \) is a facet of \( \Delta \) of dimension \( d' \) containing \( \sigma \). As \( F \) is a face of \( \Delta^i \) and this complex is pure, we get that \( F \) is properly contained in a face of \( \Delta \) of dimension \( i \), a contradiction. Hence \( i \leq d' \). The simplicial complex \( \Delta' \) is equal to \( \text{star}_{\Delta}(\sigma) \). Therefore, from the equalities

\[
(\Delta')^i = (\text{star}_{\Delta}(\sigma))^i = \text{star}_{\Delta^i}(\sigma),
\]

and using that the star of a face of a Cohen–Macaulay complex is again Cohen–Macaulay [66, p. 224], we get that \( (\Delta')^i \) is Cohen–Macaulay. Hence, by Theorem 2.1 it follows that the depth of \( R/(I: f) \) is greater than or equal to \( \text{depth}(R/I) \). \( \square \)
A common technique in commutative algebra is to start with a short exact sequence of the form
\[ 0 \to R/(I : f)[-k] \xrightarrow{f} R/I \to R/(I, f) \to 0, \]
where \( I \subseteq R \) is a graded ideal and \( f \) is a homogeneous polynomial of degree \( k \), and use information about two of the terms to glean desired information about the third. Both depth and regularity are known to behave well relative to short exact sequences. There are several versions of the depth lemma that appear in the literature. The following lemmas provide the information relating the depths and regularity of the terms of a short exact sequence in a format that will be particularly useful in the remainder of this paper.

**Lemma 2.3.** Let \( 0 \to N \to M \to L \to 0 \) be a short exact sequence of modules over a local ring \( R \). The following conditions are equivalent.

(a) \( \text{depth}(N) \geq \text{depth}(M) \).
(b) \( \text{depth}(M) = \text{depth}(N) \) or \( \text{depth}(M) = \text{depth}(L) \).
(c) \( \text{depth}(L) \geq \text{depth}(M) - 1 \).

**Proof.** It follows from the depth lemma [66, Lemma 2.3.9]. \( \Box \)

There is a similar statement for the regularity.

**Lemma 2.4.** Let \( 0 \to N \to M \to L \to 0 \) be a short exact sequence of graded finitely generated \( R \)-modules. The following conditions are equivalent.

(a) \( \text{reg}(M) \geq \text{reg}(N) - 1 \).
(b) \( \text{reg}(M) = \text{reg}(N) \) or \( \text{reg}(M) = \text{reg}(L) \).
(c) \( \text{reg}(M) \geq \text{reg}(L) \).

**Proof.** It follows from [11 Corollary 20.19]. \( \Box \)

**Lemma 2.5.** Let \( 0 \to N \to M \to L \to 0 \) be an exact sequence of graded finitely generated \( R \)-modules with homomorphisms of degree 0 and \( k \geq 1 \) an integer. The following are equivalent.

(a) \( \text{reg}(N) \leq \text{reg}(M) + k \).
(b) \( \text{reg}(L) \leq \text{reg}(M) + k - 1 \).

**Proof.** (a) \( \Rightarrow \) (b): We may assume \( \text{reg}(M) \leq \text{reg}(L) - 1 \), otherwise there is nothing to prove. Hence, by [11 Corollary 20.19], we get
\[ \text{reg}(L) \leq \max(\text{reg}(N) - 1, \text{reg}(M)) \leq \text{reg}(M) + k - 1. \]

(b) \( \Rightarrow \) (a): As \( \text{reg}(L) + 1 \leq \text{reg}(M) + k \), by [11 Corollary 20.19], we get
\[ \text{reg}(N) \leq \max(\text{reg}(M), \text{reg}(L) + 1) \leq \text{reg}(M) + k. \] \( \Box \)

Let \( \mathcal{C} \) be a clutter with vertex set \( X = \{x_1, \ldots, x_n\} \), that is, \( \mathcal{C} \) consists of a family of subsets of \( X \), called edges, none of which is included in another. The sets of vertices and edges of \( \mathcal{C} \) are denoted by \( V(\mathcal{C}) \) and \( E(\mathcal{C}) \), respectively. If \( V \subset X \), the clutter obtained from \( \mathcal{C} \) by deleting all edges of \( \mathcal{C} \) that intersect \( V \) will be denoted by \( \mathcal{C} \setminus V \). The edge ideal of \( \mathcal{C} \), denoted \( I(\mathcal{C}) \), is the ideal of \( R \) generated by all squarefree monomials \( x_e = \prod_{i \in e} x_i \) such that \( e \in E(\mathcal{C}) \). The ideal of covers \( I(\mathcal{C})^\vee \) of \( \mathcal{C} \) is the edge ideal of \( C^\vee \), the clutter of minimal vertex covers of \( \mathcal{C} \) [66, p. 221]. The ideal \( I(\mathcal{C})^\vee \) is also called the Alexander dual of \( I(\mathcal{C}) \) or simply the cover ideal of \( \mathcal{C} \).

**Lemma 2.6.** Let \( I(\mathcal{C}) \subseteq R \) be the edge ideal of a clutter \( \mathcal{C} \) and let \( f = x_{i_1} \cdots x_{i_k} \) be a squarefree monomial of \( R \). The following hold.
Proof. (i): Let $E(C)$ be the set of edges of $C$. We set $V = \{x_{i_1}, \ldots, x_{i_k}\}$ and $I = I(C)$. Then

$$
(I^\lor : f)^\lor = (I(C \setminus \{x_{i_1}, \ldots, x_{i_k}\}))^\lor = (I(C^\lor) \setminus \{x_{i_1}, \ldots, x_{i_k}\})^\lor = (I(C \setminus V))^\lor = I(C \setminus V).
$$

(ii): Notice the equalities $I(C^\lor)^\lor = (I(C)^\lor)^\lor = I(C)$. Thus this part follows from (i) by replacing $C$ with $C^\lor$.

(iii): Setting $L = (I(C), x_i)$ and $J = I(C \setminus \{x_i\})$, it follows readily that

$$
L = (I(C \setminus \{x_i\}), x_i) = (J, x_i) = \bigcap_{p \in \text{Ass}(R/J)} (x_i, p).
$$

Hence, by duality [65, Theorem 6.3.39], one has $(I(C), x_i)^\lor = x_i I(C \setminus \{x_i\})^\lor$. 

Our interest in the duality results above is partially motivated by the following result relating regularity and projective dimension, and thus depth, when passing to the dual.

**Theorem 2.7.** (Terai [58]) If $I \subset R$ is a squarefree monomial ideal, then

$$
\text{reg}(I) = 1 + \text{reg}(R/I) = \text{pd}(R/I^\lor).
$$

In [3, Theorem 3.1] it is shown that conditions (ii) and (iv) of the next result hold (cf. [9, Lemmas 5.1 and 2.10]). For squarefree monomial ideals—using the above duality theorem of Terai [58]—we show that these conditions are in fact equivalent (cf. Remark 2.9). Roughly speaking the inequalities of (ii) and (iv) are dual of each other via the duality theorem of Terai.

**Proposition 2.8.** Let $I \subset R$ be a squarefree monomial ideal and let $f = x_{i_1} \cdots x_{i_k}$ be a squarefree monomial of $R$ of degree $k$. Then any of the following equivalent conditions hold.

(i) $\text{depth}(R/(f, I)) \geq \text{depth}(R/I) - 1$.

(ii) [3, Theorem 3.1] $\text{depth}(R/I) \leq \text{depth}(R/(I : f))$.

(iii) $\text{depth}(R/(x_{i_1}, \ldots, x_{i_k}, I)) \geq \text{depth}(R/I) - k$.

(iv) [3, Theorem 3.1] $\text{reg}(R/I) \geq \text{reg}(R/(I : f))$.

(v) $\text{reg}(R/(f, I)) \leq \text{reg}(R/I) + k - 1$.

Proof. By Lemma 2.2 condition (ii) holds for any squarefree monomial ideal $I$ and for any squarefree monomial $f$. Thus it suffices to show that (i) and (ii) are equivalent and that (i) and (iii)–(v) are equivalent conditions. Since $I$ is squarefree, there is a clutter $C$ such that $I = I(C)$.

(i) $\iff$ (ii): This follows from applying Lemma 2.3 to the short exact sequence

$$
(2.1)\ 0 \rightarrow R/(I : f)[-k] \xrightarrow{f} R/I \rightarrow R/(I, f) \rightarrow 0.
$$

(i) $\implies$ (iii): This follows directly by induction on $k$.

(iii) $\implies$ (iv): As (iii) holds for squarefree monomials, applying (iii) to $I(C')$, we get

$$
k + \text{depth}(R/(x_{i_1}, \ldots, x_{i_k}, I(C'))) \geq \text{depth}(R/I(C')).
$$

Therefore, setting $V = \{x_{i_1}, \ldots, x_{i_k}\}$ and $X = \{x_1, \ldots, x_n\}$, we get

$$
\text{depth}(R/I(C' \setminus V)) = k + \text{depth}(K[X \setminus V]/I(C' \setminus V)) \\
= k + \text{depth}(R/(V, I(C'))) \geq \text{depth}(R/I(C')), 
$$

where $K[X]$ is the polynomial ring in $X$.
that is, \( \text{depth}(R/I(C^\vee \setminus V)) \geq \text{depth}(R/I(C^\vee)) \), where \( I(C^\vee) = I(C)^\vee \). Hence, applying the Auslander–Buchsbaum formula \([66, \text{Theorem 3.5.13}]\) to both sides of this inequality and then using Terai’s formula of Theorem 2.7 we get

\[
\text{reg}(R/I(C)) \geq \text{reg}(R/I(C^\vee \setminus V)^\vee).
\]

By Lemma 2.6(ii) one has \( I(C^\vee \setminus V) = (I(C): f)^\vee \). Thus, by duality, \( I(C^\vee \setminus V)^\vee = (I(C): f) \), and the required inequality follows.

(iv) \(\Rightarrow\) (iii): As (iv) holds for squarefree monomials, applying (iv) to \( I(C^\vee) \), we get

\[
\text{reg}(R/I(C^\vee)) \geq \text{reg}(R/(I(C)^\vee : f)).
\]

Therefore, applying Terai’s formula of Theorem 2.7 and Lemma 2.6(i), we get

\[
\text{pd}_R(R/I(C)) \geq \text{pd}_R(R/I(C \setminus V)).
\]

Hence, applying the Auslander–Buchsbaum formula \([66, \text{Theorem 3.5.13}]\) to both sides of this inequality and using depth properties, we obtain

\[
k + \text{depth}(R/(V, I(C))) = k + \text{depth}(K[X \setminus V]/I(C \setminus V)) = \text{depth}(R/I(C \setminus V)) \geq \text{depth}(R/I(C)).
\]

(iv) \(\Leftrightarrow\) (v): Since \( \text{reg}((R/I : f)[-k]) = k + \text{reg}(R/I : f) \), the equivalence between (iv) and (v) follows applying Lemma 2.6 to the exact sequence of Eq. (2.1).

In \([3, \text{Corollary 3.3}]\) it is shown that condition (vii) below holds (cf. \([9, \text{Lemma 2.10}]\)).

**Remark 2.9.** (A) Conditions (i)–(v) are equivalent to

(vi) \( \text{depth}(R/I) = \text{depth}(R/(I : f)) \) or \( \text{depth}(R/I) = \text{depth}(R/(f, I)) \).

(B) For \( k = \deg(f) = 1 \) conditions (i)–(vi) are equivalent to:

(vii) \( \text{reg}(R/I) = \text{reg}(R/(I : f)) + 1 \) or \( \text{reg}(R/I) = \text{reg}(R/(f, I)) \).

This follows applying Lemmas 2.3 and 2.4 to the exact sequence given in Eq. (2.1).

**Depth and regularity via polarization.** In what follows we will use the polarization technique due to Fröberg that we briefly recall now (see \([66, \text{p. 203}]\) and the references therein). Note that alternate labelings of polarizations and partial polarizations exist in the literature (see, for example, \([14, 31, 45]\)); however, the notation used here will prove beneficial in Section 3.

Let \( J \subset R \) be a monomial ideal minimally generated by \( G(J) = \{g_1, \ldots, g_s\} \). We set \( \gamma_i \) equal to \( \max\{\deg_x(g) \mid g \in G(J)\} \). To polarize \( J \) we use the set of new variables

\[
X_J = \bigcup_{i=1}^{n} \{x_{i,2}, \ldots, x_{i,\gamma_i}\},
\]

where \( \{x_{i,2}, \ldots, x_{i,\gamma_i}\} \) is empty if \( \gamma_i = 0 \) or \( \gamma_i = 1 \). It is convenient to identify the variable \( x_i \) with \( x_{i,1} \) for all \( i \). Recall that a power \( x_i^c \) of a variable \( x_i \), \( 1 \leq c \leq \gamma_i \), polarizes to \( (x_i^c)^{\text{pol}} = x_i \) if \( \gamma_i = 1 \), to \( (x_i^c)^{\text{pol}} = x_{i,2} \cdots x_{i,c+1} \) if \( c < \gamma_i \), and to \( (x_i^c)^{\text{pol}} = x_{i,2} \cdots x_{i,\gamma_i}x_i \) if \( c = \gamma_i \). This induces a polarization \( g_i^{\text{pol}} \) of \( g_i \) for \( i = 1, \ldots, s \). The full polarization \( J^{\text{pol}} \) of \( J \) is the ideal of \( R[X_J] \) generated by \( g_1^{\text{pol}}, \ldots, g_s^{\text{pol}} \). The next lemma is well known.

**Lemma 2.10.** Let \( J \) be a monomial ideal of \( R \). Then

(a) (Fröberg \([15]\)) \( \text{depth}(R[X_J]/J^{\text{pol}}) = |X_J| + \text{depth}(R/J) = \text{depth}(R[X_J]/J) \).

(b) \( \text{pd}(R/J) = \text{pd}(R[X_J]/J^{\text{pol}}) \).

(c) \( \text{pd}(R/J) = \text{reg}(R[X_J]/(J^{\text{pol}})^\vee) + 1 \).

(d) \([31, \text{Corollary 1.6.3}]\) \( \text{reg}(R/J) = \text{reg}(R[X_J]/J^{\text{pol}}) \).
Proof. Part (b) follows applying the Auslander–Buchsbaum formula [66 Theorem 3.5.13] to part (a). Part (c) follows from Theorem 2.7 and part (b). □

Let $I \subset R$ be a monomial ideal and let $f$ be a monomial. Using polarization, one can extend Proposition 2.8 and Remark 2.9 to general monomial ideals. The following result will be needed when relating the depth and the regularity of a monomial ring $R/I$ with those of the ring $R[X_L]/I^\text{pol}$, where $L$ is the ideal $(f, I)$ and $I^\text{pol}$ is the polarization of $I$ with respect to $R[X_L]$ (cf. Lemma 2.10).

**Proposition 2.11.** Let $I \subset R$ be a monomial ideal and let $f$ be a monomial. If $L = (f, I)$ and $X_L$ is the set of new variables that are needed to polarize $L$, then

(i) $\text{depth}(R[X_L]/I^\text{pol}) = \text{depth}((f^\text{pol}, \ldots, f^\text{pol})) = \text{depth}(R/L) - \text{depth}(R/I),$

(ii) $\text{reg}(R[X_L]/I^\text{pol}) = \text{reg}(R/L)$ and $\text{reg}(R[X_L]/(f^1, \ldots, f^r)) = \text{reg}(R/I),$

where $G(I) = \{f_1, \ldots, f_r\}$, and $f^\text{pol}_i$ is the polarization of $f_i$ in $R[X_L]$.

**Proof.** (i): We may assume $f$ is not in $I$, otherwise there is nothing to prove. Let $L^\text{pol} \subset R[X_L]$ be the full polarization of $L$. For use below we set $\delta_i = \max\{\deg_i(g)\mid g \in G(I)\}$ and $f = x_1^{a_1} \cdots x_n^{a_n}$. The set of variables of $R$ is denoted by $X = \{x_1, \ldots, x_n\}$.

Subcase (i.a): $a_i > \delta_i$ for some $i$. Then $G(L) = \{f, f_1, \ldots, f_r\}$. For simplicity of notation we assume there is an integer $k$ such that $a_1 > \delta_1, \ldots, a_k > \delta_k$, and $a_i \leq \delta_i$ for $i > k$. If $\delta_i = 0$ for some $i > k$, then the variable $x_i$ does not occur in any element of $G(L)$ because $a_i = 0$. Hence we can replace $R$ by $K[X \setminus \{x_i\}]$. Thus we may assume that $\delta_i \geq 1$ for $i > k$. To polarize $L$ we use the set of variables

$$X_L = \left(\bigcup_{i=1}^k \{x_{i,1}, \ldots, x_{i,\delta}, x_{i,\delta+1}, \ldots, x_{i,a_i}\}\right) \cup \left(\bigcup_{i=k+1}^n \{x_{i,2}, \ldots, x_{i,\delta}\}\right),$$

where $\{x_{i,2}, \ldots, x_{i,c}\}$ is the empty set if $c = 0$ or $c = 1$. It is convenient to identify $x_i$ with $x_{i,1}$ for all $i$. In this setting the monomial $x_i^{a_i}$ polarizes to $(x_i^{a_i})^\text{pol} = x_{i,2} \cdots x_{i,a_i} x_i$ for $i = 1, \ldots, k$ and the monomial $x_i^{\delta_i}$ polarizes to $(x_i^{\delta_i})^\text{pol} = x_{i,2} \cdots x_{i,\delta} x_i$ for $i > k$. Let $f^\text{pol}$ and $f^\text{pol}_i$ be the polarizations in $R[X_L]$ of $f$ and $f_i$ (see Example 2.14). By Lemma 2.10 one has

$$\text{depth}(R[X_L]/I^\text{pol}) = |X_L| + \text{depth}(R/L) = \sum_{i=1}^{k} (a_i - 1) + \sum_{i=k+1}^{n} (\delta_i - 1) + \text{depth}(R/I).$$

Next we relate the depth of $R[X_L]/(f^1, \ldots, f^r)$ to the depth of $R/I$. For this consider the polynomial ring $R' = K[X']$, where $X' = (X \setminus \{x_i\}) \cup \{x_{i,\delta_i+1}\}$ and let $f'_i$ be the polynomial of $R'$ obtained from $f_i$ by replacing $x_i$ with $x_{i,\delta_i+1}$ for $i = 1, \ldots, k$. If $I'$ is the ideal of $R'$ generated by $f'_1, \ldots, f'_r$, then $K[X'/I' = K[X'/I'$ are isomorphic and have the same depth. By polarizing $f'_i$ with respect to

$$X'_I = \bigcup_{i=1}^n \{x_{i,2}, \ldots, x_{i,\delta}\},$$

we obtain that $(f'_i)^\text{pol}$ is equal to $f^\text{pol}_i$, the polarization of $f_i$ with respect to $X_L$. The full polarization of $I'$ is $(I')^\text{pol} = ((f'_1)^\text{pol}, \ldots, (f'_r)^\text{pol})$. Therefore, by Lemma 2.10 one has

$$\text{depth}(R/X'_I)/(f^1, \ldots, f^r) = \text{depth}(R[X_L]/(f^1, \ldots, f^r))^\text{pol},$$

(2.3) $\text{depth}(R/X'_I)/(f^1, \ldots, f^r) = |X'_I| + \text{depth}(R'/I') = |X_I| + \text{depth}(R/I).$
As $|X \cup X_L| = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{n} \delta_i$ and $|X' \cup X_{L'}| = \sum_{i=1}^{n} \delta_i$, we get

$$|(X \cup X_L) \setminus (X' \cup X_{L'})| = \sum_{i=1}^{k} (a_i - \delta_i),$$

that is, the number of variables of $R[X_L]$ that do not occur in $R'[X_{L'}]$ is $\sum_{i=1}^{k} (a_i - \delta_i)$. Therefore from Eqs. (2.3) and (2.4), and using that $|X_{L'}| = \sum_{i=1}^{n} (\delta_i - 1)$, we get

$$\text{depth}(R[X_L]/(f_1^{\text{pol}}, \ldots, f_r^{\text{pol}})) = \sum_{i=1}^{k} (a_i - \delta_i) + \text{depth}(R'[X_{L'}]/((f'_1)^{\text{pol}}, \ldots, (f'_r)^{\text{pol}}))$$

(2.5)

$$= \sum_{i=1}^{k} (a_i - 1) + \sum_{i=k+1}^{n} (\delta_i - 1) + \text{depth}(R/I).$$

Using Eqs. (2.2) and (2.5) the required equality follows.

Subcase (i.b): $a_i \leq \delta_i$ for all $i$. This case follows adapting the arguments of Subcase (i.a), noting that $k = 0$ in this case.

(ii): To prove this part we keep the notation of part (i).

Subcase (i.a): Assume that $a_i > \delta_i$ for some $i$. The first equality follows at once from Lemma 2.10. As $R'[X_{L'}]$ is a subring of $R[X_L]$, the regularity of $(I')^{\text{pol}}R'[X_{L'}]$ is equal to that of $(I')^{\text{pol}}R[X_L]$. Hence, by Lemma 2.10, we get

$$\text{reg}(R[X_L]/(I')^{\text{pol}}) = \text{reg}(R'[X_{L'}]/(I')^{\text{pol}}) = \text{reg}(R'/I') = \text{reg}(R/I).$$

Subcase (ii.b): $a_i \leq \delta_i$ for all $i$. This case follows adapting the arguments of Subcase (ii.a). □

The following corollary extends Proposition 2.8 and Remark 2.9 from squarefree monomial ideals to arbitrary monomial ideals using polarization. It will be used throughout the paper (e.g., Lemma 3.6 Theorem 4.9 Proposition 5.6). This result is later extended using Gröbner bases (Corollary 2.13).

**Corollary 2.12.** Let $I \subset R$ be a monomial ideal, let $f$ be a monomial of degree $k$, and let $x_{i_1}, \ldots, x_{i_k}$ be a set of distinct variables of $R$. The following hold.

(i) depth$(R/(f, I)) \geq \text{depth}(R/I) - 1$.

(ii) [3] Theorem 3.1 depth$(R/I) \leq \text{depth}(R/(I: f))$.

(iii) depth$(R/(x_{i_1}, \ldots, x_{i_k}, I)) \geq \text{depth}(R/I) - k$.

(iv) [3] Theorem 3.1 reg$(R/I) \geq \text{reg}(R/(I: f))$.

(v) reg$(R/(f, I)) \leq \text{reg}(R/I) + k - 1$.

(vi) depth$(R/I) = \text{depth}(R/(I: f))$ or depth$(R/I) = \text{depth}(R/(f, I))$.

(vii) [3] [9] If $k = 1$, then \(\text{reg}(R/I) = \text{reg}(R/(I: f)) + 1\) or \(\text{reg}(R/I) = \text{reg}(R/(f, I))\).

**Proof.** If $I$ and $f$ are squarefree, the result holds true. Indeed, by Lemma 2.2 one has the inequality $\text{depth}(R/(I: f)) \geq \text{depth}(R/I)$. Then by Proposition 2.8 and Remark 2.9 the statements all hold. To show the general case we will use the polarization technique.

(i) One may assume that $f \not\in I$. We set $G(I) = \{f_1, \ldots, f_r\}$ and $L = (f, I)$. Let $X_L$ be the set of new variables needed to polarize $L$ and let $f_1^{\text{pol}}, f_i^{\text{pol}}$ be the polarizations in $R[X_L]$ of $f, f_i$, respectively. As these polarizations are squarefree, by Proposition 2.8 one has

$$\text{depth}(R[X_L]/(f_1^{\text{pol}}, f_2^{\text{pol}}, \ldots, f_r^{\text{pol}})) \geq \text{depth}(R[X_L]/(f_1^{\text{pol}}, f_2^{\text{pol}}, \ldots, f_r^{\text{pol}})) - 1,$$

where $L^{\text{pol}} = (f_1^{\text{pol}}, f_2^{\text{pol}}, \ldots, f_r^{\text{pol}})$. Hence, by Proposition 2.11 $\text{depth}(R/L) \geq \text{depth}(R/I) - 1.$
(ii): According to Lemma 2.3 parts (ii) and (i) are equivalent.

(iii): It follows from part (i) using induction on $k$.

(iv)–(v): Setting $N = (R/(I : f))[-k]$, $M = R/I$ and $L = R/(I, f)$, and noticing that $\text{reg}(N) = k + \text{reg}(R/(I : f))$, from Lemma 2.3 it follows that (iv) and (v) are equivalent. Since $f^\text{pol}, f_1^\text{pol}, \ldots, f_r^\text{pol}$ are squarefree, by Proposition 2.3 one has

$$\text{reg}(R[X_L]/L^\text{pol}) - \text{reg}(R[X_L]/(f_1^\text{pol}, \ldots, f_r^\text{pol})) \leq k - 1.$$ 

Hence, by Proposition 2.11 one has $\text{reg}(R/L) - \text{reg}(R/I) \leq k - 1$. Thus (v) and (iv) hold.

(vi): This condition is equivalent to (i). This follows applying Lemma 2.3 to the exact sequence

$$0 \rightarrow R/(I : f)[-k] \xrightarrow{f} R/I \rightarrow R/(I, f) \rightarrow 0.$$

(vii): Recall that $\text{reg}(R/(I : f))[-k] = k + \text{reg}(R/(I : f))$. If $k = 1$, using Lemma 2.4 it follows that conditions (vii) and (iv) are equivalent. □

**Corollary 2.13.** Let $I \subset R$ be a monomial ideal and let $f$ be a homogeneous polynomial of degree $k$. If there exists a monomial order $<$ on $R$ such that $\text{in}_<(I, f) = I + (\text{in}_<(f))$, then

(a) $\text{depth}(R/(I : f)) \geq \text{depth}(R/I)$,
(b) $\text{reg}(R/(I, f)) \leq \text{reg}(R/I) + k - 1$, and
(c) $\text{reg}(R/I) \geq \text{reg}(R/(I : f))$.

**Proof.** (a): We proceed by contradiction assuming that $\text{depth}(R/I) > \text{depth}(R/(I : f))$. From the exact sequence

$$0 \rightarrow R/(I : f)[-k] \xrightarrow{f} R/I \rightarrow R/(I, f) \rightarrow 0,$$

using the depth lemma [66] Lemma 2.3.9 and the fact that the depth of $R/(I, f)$ is greater than or equal to the depth of $R/\text{in}_<(I, f)$ [31] Theorem 3.3.4(d), we get

$$\text{depth}(R/(I : f)) = \text{depth}(R/(I, f)) + 1 \geq \text{depth}(R/(I + \text{in}_<(f))) + 1.$$

By Corollary 2.12(i), we have $\text{depth}(R/(I + \text{in}_<(f))) \geq \text{depth}(R/I) - 1$. Hence we obtain $\text{depth}(R/(I : f)) \geq \text{depth}(R/I)$, a contradiction.

(b): Using that the regularity of $R/(I, f)$ is less than or equal to the regularity of $R/\text{in}_<(I, f)$ [31] Theorem 3.3.4(c) and Corollary 2.12(v), we get

$$\text{reg}(R/(I, f)) \leq \text{reg}(R/\text{in}_<(I, f)) = \text{reg}(R/(I + \text{in}_<(f))) \leq \text{reg}(R/I) + k - 1.$$

(c): Setting $N = R/(I : f)[-k]$, $M = R/I$, and $L = R/(I, f)$, we proceed by contradiction assuming $\text{reg}(R/(I : f)) > \text{reg}(R/I)$, that is, $\text{reg}(N) \geq \text{reg}(M) + k + 1$. On the other hand, by part (b), one has $\text{reg}(L) \leq \text{reg}(N) - 2$. According to [11] Corollary 20.19(a), one has either $\text{reg}(N) \leq \text{reg}(M)$ or $\text{reg}(N) \leq \text{reg}(L) + 1$, a contradiction. □

The next example illustrates the polarizations used in the proof of Proposition 2.11. For convenience we use the notation of that proof.

**Example 2.14.** Let $f = x_1^3x_2^3, f_1 = x_2^2x_3, f_2 = x_1x_3^3, f_3 = x_2^3x_3$ be monomials in the polynomial ring $R = K[x_1, x_2, x_3]$ and set $I = (f_1, f_2, f_3)$ and $L = (f, I)$. Setting

$$f^\text{pol} = x_1x_2^3x_3, f_1^\text{pol} = x_2^3x_3, f_2^\text{pol} = x_1x_3^2, f_3^\text{pol} = x_2^3x_3^2,$$

and $X_L = \{x_1, x_2, x_3\} \cup \{x_2, x_3\} \cup \{x_3, x_2\}$, the full polarization of $L$ is

$$L^\text{pol} = (f^\text{pol}, f_1^\text{pol}, f_2^\text{pol}, f_3^\text{pol}) \subset R[X_L].$$
Making the change of variables \(x_1 \rightarrow x_{1,3}, x_2 \rightarrow x_{2,3}\) in \(I\) and setting
\[
f'_1 = x_{1,3}^2 x_3, \quad f'_2 = x_{1,3} x_3^2, \quad f'_3 = x_2^2 x_3, \quad I' = (f'_1, f'_2, f'_3),
\]
\(X' = \{x_{1,2}\} \cup \{x_{2,2}\} \cup \{x_{3,2}\}, R' = K[x_{1,3}, x_{2,3}, x_3]\), the full polarization of \(I'\) is
\[
(I')_{\text{pol}} = ((f'_1)_{\text{pol}}, (f'_2)_{\text{pol}}, (f'_3)_{\text{pol}}) \subset R'[X'],
\]
where \((f'_1)_{\text{pol}} = x_{1,2} x_{1,3} x_3, (f'_2)_{\text{pol}} = x_{1,2} x_{3,2} x_3, (f'_3)_{\text{pol}} = x_{2,2} x_{2,3} x_3\). Thus \(f'_{i\text{pol}}\) is equal to \((f'_i)_{\text{pol}}\) for \(i = 1, 2, 3\). Setting \(X_I = \{x_{1,2}, x_{2,2}, x_{3,2}\}\) the full polarization of \(I\) is generated by the monomials \(x_{1,2} x_{1,3} x_3, x_{1,2} x_{3,2} x_3, x_{2,2} x_{2,3} x_3\).

3. DEPTH AND REGULARITY LOCALLY AT EACH VARIABLE

In this section we use polarization to study the behavior of the depth and regularity of a monomial ideal locally at each variable when lowering the top degree.

Let \(R = K[x_1, \ldots, x_n]\) be a polynomial ring over a field \(K\), let \(I\) be a monomial ideal of \(R\) and let \(x_i\) be a fixed variable that occurs in \(G(I)\). Given a monomial \(x^a = x_1^{a_1} \cdots x_n^{a_n}\), we set \(\deg_{x_i}(x^a) = a_i\). Consider the integer

\[
q := \max\{\deg_{x_i}(x^a) | x^a \in G(I)\},
\]
and the corresponding set \(B_i := \{x^a | \deg_{x_i}(x^a) = q\} \cap G(I)\). That is, \(B_i\) is the set of all monomial of \(G(I)\) of highest degree in \(x_i\). Setting

\[
A_i := \{x^a | \deg_{x_i}(x^a) < q\} \cap G(I) = G(I) \setminus B_i,
\]
\(p := \max\{\deg_{x_i}(x^a) | x^a \in A_i\}\) and \(L := (\{x^a/x_i | x^a \in B_i \cup A_i\})\), we are interested in comparing the depth (resp. regularity) of \(R/I\) with the depth (resp. regularity) of \(R/L\).

One of the main results of this section shows that the depth is locally non-decreasing at each variable \(x_i\) when lowering the top degree:

**Theorem 3.1.** Let \(I\) be a monomial ideal of \(R\) and let \(x_i\) be a variable. The following hold.

(a) If \(p \geq 1\) and \(q - p \geq 2\), then \(\text{depth}(R/I) = \text{depth}(R/L)\).

(b) If \(p \geq 0\) and \(q - p = 1\), then \(\text{depth}(R/L) \geq \text{depth}(R/I)\).

(c) If \(p = 0\) and \(q \geq 2\), then \(\text{depth}(R/I) = \text{depth}(R/(\{x_i^{q-1} | x^a \in B_i \cup A_i\}))\).

**Proof.** (a): To simplify notation we set \(i = 1\). We may assume that \(G(I) = \{f_1, \ldots, f_r\}\), where \(\{f_1, \ldots, f_m\}\) is the set of all elements of \(G(I)\) that contain \(x_1^q\) and \(\{f_{m+1}, \ldots, f_r\}\) is the set of all elements of \(G(I)\) that contain some positive power \(x_1^\ell\) of \(x_1\) for some \(1 \leq \ell < q\). Making a partial polarization of \(x_1^q\) with respect to the new variables \(x_1, \ldots, x_{1,q-1}\) [66 p. 203], gives that \(f_j\) polarizes to \(f_{j\text{pol}} = x_1 x_{1,q-1} x_1^2 f_j\) for \(j = 1, \ldots, m\), where \(f_1', \ldots, f_m'\) are monomials that do not contain \(x_1\) and \(f_j = x_1^q f_j'\) for \(j = 1, \ldots, m\). Hence, using that \(q - p \geq 2\), one has the partial polarization

\[
I_{\text{pol}} = (x_1 x_{1,q-1} x_1^2 f_1', \ldots, x_1 x_{1,q-1} x_1^2 f_m', f_{m+1\text{pol}}, \ldots, f_{s\text{pol}}, f_{s+1}, \ldots, f_r),
\]

where \(f_{m+1\text{pol}}, \ldots, f_{s\text{pol}}\) do not contain \(x_1\) and \(I_{\text{pol}}\) is an ideal of \(R_{\text{pol}} = R[x_1, \ldots, x_{1,q-1}]\). On the other hand, from the equality

\[
G(L) = \{f_1/x_1, \ldots, f_m/x_1, f_{m+1}, \ldots, f_r\},
\]

one has the partial polarization

\[
L_{\text{pol}} = (x_1 x_{1,q-1} x_1 f_1', \ldots, x_1 x_{1,q-1} x_1 f_m', f_{m+1\text{pol}}, \ldots, f_{s\text{pol}}, f_{s+1}, \ldots, f_r).
\]
By making the substitution $x_1^q \to x_1$ in each element of $G(I^{pol})$ this will not affect the depth of $R^{pol}/I^{pol}$ (see [44, Lemmas 3.3 and 3.5]). Thus
\[
q - 2 + \text{depth}(R/I) = \text{depth}(R^{pol}/I^{pol}) = \text{depth}(R^{pol}/L^{pol}) = q - 2 + \text{depth}(R/L),
\]
and consequently $\text{depth}(R/I) = \text{depth}(R/L)$.

(b): To simplify notation we set $i = 1$. Assume $p = 0$, then $q = 1$. Note that the ring $R/L$ is equal to $R/(I: x_1)$. Hence, by Corollary 2.12, its depth is greater than or equal to $\text{depth}(R/I)$. Thus we may assume that $p \geq 1$. We may also assume that $G(I) = \{f_1, \ldots, f_r\}$, where $f_1, \ldots, f_m$ is the set of all elements of $G(I)$ that contain $x_1^q$, and $f_{m+1}, \ldots, f_t$ is the set of all elements of $G(I)$ that contain $x_1^{q-1}$ but not $x_1^q$, and $f_{t+1}, \ldots, f_s$ is the set of all elements of $G(I)$ that contain some power $x_1^q$, with $1 \leq \ell < q - 1$, but not $x_1^{q+1}$. Let $R'$ be the polynomial ring $K[x_1, q, x_2, \ldots, x_n]$, with $x_1, q$ a new variable, and let $L'$ be the ideal of $R'$ obtained from $L$ by making the change of variable $x_1 \to x_1, q$ in each element of $G(L)$. Clearly
\[
\text{depth}(R/L) = \text{depth}(R'/L') = \text{depth}(R[x_1]/L') - 1.
\]

The partial polarization of $I$ with respect to $x_1$ using the variables $x_1, 2, \ldots, x_1, q$ is given by
\[
I^{pol} = (x_1, 2 \cdots x_1, q f_1', \ldots, x_1, 2 \cdots x_1, q f_m', \ldots, x_1, 2 \cdots x_1, q f_t', \ldots, x_1, 2 \cdots x_1, q f_{t+1}', \ldots, x_1, 2 \cdots x_1, q f_{s+1}', \ldots, f_r'),
\]
where $f_1', \ldots, f_t', f_{t+1}', \ldots, f_{s+1}', \ldots, f_r'$ do not contain $x_1$ and $I^{pol}$ is an ideal of the ring $R^{pol} = R[x_1, 2, \ldots, x_1, q]$. Therefore
\[
(I^{pol}: x_1) = (x_1, 2 \cdots x_1, q f_1', \ldots, x_1, 2 \cdots x_1, q f_m', \ldots, x_1, 2 \cdots x_1, q f_t', \ldots, x_1, 2 \cdots x_1, q f_{t+1}', \ldots, x_1, 2 \cdots x_1, q f_{s+1}', \ldots, f_r').
\]

The following is a generating set for $L'$, which is not necessarily minimal:
\[
L' = (x_1, q^{-1} f_1', \ldots, x_1, q^{-1} f_m', x_1, q^{-1} f_{m+1}', \ldots, x_1, q^{-1} f_t', \ldots, x_1, q^{-1} f_{t+1}', \ldots, x_1, q^{-1} f_{s+1}', \ldots, f_r'),
\]
where $1 \leq a_i < q - 1$ for $i = t + 1, \ldots, s$. Hence, it is seen that $(I^{pol}: x_1)$ is equal to $(L')^{pol}$, the polarization of $L'$ with respect to the variable $x_1, q$ using the variables $x_1, 2, \ldots, x_1, q - 1$. Therefore, using Lemma 2.2 we get
\[
(q - 1) + \text{depth}(R/L) = 1 + ((q - 2) + \text{depth}(R'/L')) = 1 + \text{depth}((R^{pol}/L^{pol}) = \text{depth}((R^{pol}/L^{pol})(I^{pol}: x_1)) - 1 + \text{depth}(R/I).
\]

Thus $\text{depth}(R/L) \geq \text{depth}(R/I)$.

(c): It suffices to notice that by making the substitution $x_1^q \to x_i$ in each element of $G(I)$ this will not affect the depth of $R/I$ (see [44, Lemmas 3.3 and 3.5]).

Let $D$ be a vertex-weighted digraph, that is, $D$ consists of a finite set $V(D) = \{x_1, \ldots, x_n\}$ of vertices, a prescribed collection $E(D)$ of ordered pairs of distinct points called edges or arrows,
and $D$ is endowed with a function $d: V(D) \to \mathbb{N}_+$, where $\mathbb{N}_+ := \{1, 2, \ldots\}$. The weight $d(x_i)$ of $x_i$ is denoted simply by $d_i$. The edge ideal of $D$, denoted $I(D)$, is the ideal of $R$ given by

$$I(D) := \langle x_i^{d_j} | (x_i, x_j) \in E(D) \rangle.$$

Edge ideals of vertex-weighted digraphs occur in the theory of Reed-Muller-type codes as initial ideals of vanishing ideals of projective spaces over a finite field [23, 41, 57].

**Corollary 3.2.** [17] Corollary 6] Let $I = I(D)$ be the edge ideal of a vertex-weighted digraph with vertices $x_1, \ldots, x_n$ and let $d_i$ be the weight of $x_i$. If $U$ is the digraph obtained from $D$ by assigning weight 2 to every vertex $x_i$ with $d_i \geq 2$, then $I$ is Cohen–Macaulay if and only if $I(U)$ is Cohen–Macaulay.

**Proof.** By applying Theorem 3.1 to each vertex $x_i$ of $D$ of weight at least 3, we obtain that the depth of $R/I(D)$ is equal to the depth of $R/I(U)$. Since $I(D)$ and $I(U)$ have the same height, then $I(D)$ is Cohen–Macaulay if and only if $I(U)$ is Cohen–Macaulay. □

**Corollary 3.3.** [32] If $I$ is a monomial ideal, then $\text{depth}(R/\text{rad}(I)) \geq \text{depth}(R/I)$. In particular if $I$ is Cohen–Macaulay, then $\text{rad}(I)$ is Cohen–Macaulay.

**Proof.** It follows by applying Theorem 3.1 to every vertex $x_i$ as many times as necessary. □

As a consequence if $I$ is squarefree, then $\text{depth}(R/I) \geq \text{depth}(R/I^k)$ for all $k \geq 1$.

**Remark 3.4.** Let $L \subset R$ be a monomial ideal. If $x_i^k$ is in $G(L)$ for some $k \geq 1$, $1 \leq i \leq n$ and $L'$ is the ideal of $R$ generated by all elements of $G(I)$ that do not contain $x_i$, then $(L, x_i) = (L', x_i)$ and by a repeated application of Theorem 3.1 one has

$$\text{depth}(R/L) \leq \text{depth}(R/(L', x_i)) = \text{depth}(R/L') - 1.$$ 

Before proving an analog of Theorem 3.1 for regularity, we first provide a basic fact regarding the effect of a change of variables on the resolution of an ideal.

**Lemma 3.5.** Let $I$ be a homogeneous ideal of $R$, let $d_1$ be a positive integer, and define $\phi: R \to R$ by $\phi(x_1) = x_1^{d_1}$ and $\phi(x_i) = x_i$ for $2 \leq i \leq n$. If $\phi(I)$ is homogeneous, then a minimal resolution of $\phi(I)$ over $R$ can be obtained by applying $\phi$ to a minimal resolution of $I$. Moreover, the (non-graded) Betti numbers of $I$ and $\phi(I)$ will be equal and $\text{reg}(\phi(I)) \geq \text{reg}(I)$.

**Proof.** Define $S = K[x_1, \ldots, x_n]$ to be a polynomial ring with the non-standard grading $d(x_1) = d_1$ and $d(x_i) = 1$ for $2 \leq i \leq n$. Note that the map $\phi$ factors through $S$. Write $\phi = \psi \sigma$, where $\sigma: R \to S$ is given by $\sigma(x_i) = x_i$ for all $i$ and $\psi: S \to R$ is given by $\psi(x_1) = x_1^{d_1}$ and $\psi(x_i) = x_i$ for $2 \leq i \leq n$. Then, by assumption, $I$ is a homogeneous ideal of $R$ and $\sigma(I)$ is again homogeneous in $S$. Applying $\sigma$ to a minimal resolution of $I$ yields a minimal resolution of $\sigma(I)$, where the modules and maps are unchanged except that the degrees of some of the maps, and thus the shifts in the resolution, may have increased, showing $\text{reg}(\sigma(I)) \geq \text{reg}(I)$. Now the map $\psi$ is precisely the map used in [44, Lemma 3.5 and Theorem 3.6(b)]. The result follows from combining these results. □

**Lemma 3.6.** Let $I$ and $J$ be monomial ideals of $R$ and let $x_i$ be a variable. If $(I: x_i) = J$ and $(I, x_i) = (J, x_i)$, then

(i) $\text{reg}(R/J) \leq \text{reg}(R/I) \leq \text{reg}(R/J) + 1$, and

(ii) $\text{depth}(R/J) - 1 \leq \text{depth}(R/I) \leq \text{depth}(R/J)$.
Proof. (i): By Corollary 2.12(v), we have \( \text{reg}(R/(I, x_1)) \leq \text{reg}(R/I) \) and \( \text{reg}(R/(J, x_1)) \leq \text{reg}(R/J) \), and by Corollary 2.12(vii), we have either \( \text{reg}(R/I) = \text{reg}(R/(I : x_1)) + 1 = \text{reg}(R/J) + 1 \) or \( \text{reg}(R/I) = \text{reg}(R/(I, x_1)) = \text{reg}(R/(J, x_1)) \leq \text{reg}(R/J) \). In the latter case one has \( \text{reg}(R/I) = \text{reg}(R/J) \) because by Corollary 2.12(iv), one has \( \text{reg}(R/J) \leq \text{reg}(R/I) \). Combining these facts yields \( \text{reg}(R/J) \leq \text{reg}(R/I) \leq \text{reg}(R/J) + 1 \).

(ii): By Corollary 2.12(vi), we have either \( \text{depth}(R/I) = \text{depth}(R/J) \) or \( \text{depth}(R/I) = \text{depth}(R/(I, x_1)) \). In the latter case one has

\[
\text{depth}(R/J) \geq \text{depth}(R/I) = \text{depth}(R/(I, x_1)) = \text{depth}(R/(J, x_1)) \geq \text{depth}(R/J) - 1
\]

because by parts (ii) and (i) of Corollary 2.12 one has the inequalities \( \text{depth}(R/J) \geq \text{depth}(R/I) \) and \( \text{depth}(R/(J, x_1)) \geq \text{depth}(R/J) - 1 \), respectively. \( \square \)

Using the notation introduced for Theorem 3.1, we are now able to control regularity when lowering the degrees of the generators of a monomial ideal.

**Theorem 3.7.** Let \( I \) be a monomial ideal and let \( L' \) be the ideal \( \{x^a/x_1^{q-1} | x_1 \in B_i \cup A_i \} \), where \( x_1 \) is a variable. The following hold.

(a) If \( p \geq 1 \) and \( q - p \geq 2 \), then \( \text{reg}(R/L) \leq \text{reg}(R/I) \leq \text{reg}(R/L) + 1 \).

(b) If \( p \geq 0 \) and \( q - p = 1 \), then \( \text{reg}(R/L) \leq \text{reg}(R/I) \).

(c) If \( p = 0 \) and \( q \geq 2 \), then \( \text{reg}(R/L') \leq \text{reg}(R/I) \leq \text{reg}(R/L') + q - 1 \).

**Proof.** (a): As in Theorem 3.1, we assume \( i = 1 \). Forming a partial polarization of \( x_1^q \) with respect to new variables \( x_{1,2}, \ldots, x_{1,q-1} \) will not change the regularity by Lemma 2.10(d). By the same argument, forming a full polarization of \( x_2, \ldots, x_n \) will also not change the regularity. Thus we may assume that \( I = (x_1^q h_1, \ldots, x_1^q h_m, h_{m+1}, \ldots, h_r) \) and \( L = (x_1 h_1, \ldots, x_1 h_m, h_{m+1}, \ldots, h_r) \) where \( h_j \) are squarefree monomials and \( x_1 \) does not divide \( h_j \) for all \( j \). Note that \( (I, x_1) = (L, x_1) \) and \( (I : x_1) = L \). Thus, by Lemma 3.6 we have \( \text{reg}(R/I) = \text{reg}(R/L) + 1 \) or \( \text{reg}(R/I) = \text{reg}(R/L) \) as claimed.

(b): This part follows from the proof of Theorem 3.1(b) and Lemma 2.10(d).

(c): We proceed by induction on \( q \geq 2 \). There are monomials \( h_1, \ldots, h_r \) not containing \( x_1 \) such that

\[
I = (x_1^q h_1, \ldots, x_1^q h_m, h_{m+1}, \ldots, h_r) \quad \text{and} \quad L = (x_1^{q-1} h_1, \ldots, x_1^{q-1} h_m, h_{m+1}, \ldots, h_r).
\]

Note that \( (I, x_1) = (L, x_1) \) and \( (I : x_1) = L \). Then, applying Lemma 3.6 to \( I \) and \( L \), one has \( \text{reg}(R/L) \leq \text{reg}(R/I) \leq \text{reg}(R/L) + 1 \). In particular the required inequality holds for \( q = 2 \). If \( q > 3 \), applying induction to \( L \), the inequality follows. \( \square \)

**Corollary 3.8.** Let \( I \) be a monomial ideal of \( R \) and let \( J \) be its radical. The following hold.

(i) \( \text{reg}(R/J) \leq \text{reg}(R/I) \).

(ii) If \( I \) is Cohen–Macaulay, then \( a(R/J) \leq a(R/I) \), where \( a(\cdot) \) is the \( a \)-invariant.

**Proof.** (i): It follows by applying Theorem 3.7 to every vertex \( x_i \) as many times as necessary.

(ii): By Corollary 3.3 \( J \) is Cohen–Macaulay. Hence, by [46, Corollary B.4.1], one has \( a(M) = \text{reg}(M) - \text{depth}(M) \) for \( M = R/I \) and \( M = R/J \). As \( \dim(R/I) = \dim(R/J) = \text{depth}(R/I) = \text{depth}(R/J) \), the inequality follows from part (i). \( \square \)

**Remark 3.9.** Let \( I \subset R \) be a monomial ideal and let \( f \) be a monomial which is a non-zero divisor of \( R/I \). Then \( \text{reg}(R/fI) = \text{reg}(R/I) + \deg(f) \) and \( \text{reg}(R/(I, f)) = \text{reg}(R/I) + \deg(f) - 1 \). This follows from Proposition 5.9. Thus the upper bound of Theorem 3.7(c) is tight.
Example 3.10. The ideals $I = (x_1^2 x_2 x_3^2, x_1^2 x_4, x_1^2 x_5)$ and $J = (x_1 x_2 x_3^2, x_1^2 x_4, x_1^2 x_5)$ have regularity 5. Thus the lower bound of Theorem 3.7(c) is also tight.

Example 3.11. The ideals $I = (x_1^2 x_2 x_3^2, x_1^2 x_4, x_1^2 x_5), L = (x_1^6 x_2 x_3^2, x_1^6 x_5, x_1^5 x_2 x_4, x_2 x_5^7)$ have regularity 16 and 13, respectively. Thus in Theorem 3.7(b), $\text{reg}(R/L) + 1$ is not an upper bound for $\text{reg}(R/I)$.

4. Edge ideals of clutters with non-increasing depth

Let $C$ be a clutter with vertex set $X = \{x_1, \ldots, x_n\}$ and let $\{x^{v_1}, \ldots, x^{v_r}\}$ be the minimal generating set of $I(C)$. The matrix $A$ whose column vectors are $v_1^\top, \ldots, v_r^\top$ is called the incidence matrix of $C$. The set covering polyhedron of $C$ is given by:

$$Q(A) := \{x \in \mathbb{R}^n | x \geq 0; xA \geq 1\},$$

where $1 = (1, \ldots, 1)$. The rational polyhedron $Q(A)$ is called integral if it has only integral vertices. A clutter is called uniform (resp. unmixed) if all its edges (resp. minimal vertex covers) have the same cardinality. A clutter is ideal if its set covering polyhedron is integral [6].

Definition 4.1. A clutter $C$, with incidence matrix $A$, has the max-flow min-cut (MFMC) property if both sides of the LP-duality equation

$$(4.1) \quad \min\{\langle \alpha, x \rangle | x \geq 0; xA \geq 1\} = \max\{\langle y, 1 \rangle | y \geq 0; Ay \leq \alpha\}$$

have integral optimum solutions $x, y$ for each nonnegative integral vector $\alpha$.

Definition 4.2. Let $I$ be a squarefree monomial ideal of $R$ and let $p_1, \ldots, p_r$ be the associated primes of $I$. Given an integer $k \geq 1$, we define the $k$-th symbolic power of $I$ to be the ideal

$$I^{(k)} := \bigcap_{i=1}^r (I^k R_{p_i} \cap R) = p_1^k \cap \cdots \cap p_r^k.$$

An ideal $I$ of $R$ is called normally torsion-free if $\text{Ass}(R/I^k)$ is contained in $\text{Ass}(R/I)$ for all $k \geq 1$. Notice that if $I$ is a squarefree monomial ideal, then $I$ is normally torsion-free if and only if $I^k = I^{(k)}$ for all $k \geq 1$. A major result of [18, 20] shows that a clutter $C$ has the max-flow min-cut property if and only if $I(C)$ is normally torsion-free (cf. [13, Proposition 3.4]).

Lemma 4.3. ([18 Lemma 5.6], [10 Lemma 2.1]) If $C$ is a uniform clutter and $Q(A)$ is integral, then there exists a minimal vertex cover of $C$ intersecting every edge of $C$ in exactly one vertex.

Theorem 4.4. ([6 Theorem 1.17]) If $Q(A)$ is integral and $B$ is the incidence matrix of the clutter $C^\lor$ of minimal vertex covers of $C$, then $Q(B)$ is integral.

Lemma 4.5. If $C$ is an unmixed clutter and $Q(A)$ is integral, then there exists an edge of $C$ intersecting every minimal vertex cover of $C$ in exactly one vertex.

Proof. By duality [66 Theorem 6.3.39] the minimal vertex covers of $C^\lor$ (resp. edges of $C^\lor$) are the edges of $C$ (resp. minimal vertex covers of $C$). Let $B$ be the incidence matrix of $C^\lor$. As $Q(A)$ is integral and $C$ is unmixed, by Lemma 4.4 $Q(B)$ is also integral and $C^\lor$ is uniform. Thus applying Lemma 4.3 to $C^\lor$, there exists a minimal vertex cover of $C^\lor$ intersecting every edge of $C^\lor$ in exactly one vertex. Hence by duality the result follows.

Let $I \subset R$ be a homogeneous ideal and let $m = (x_1, \ldots, x_n)$ be the maximal irrelevant ideal of $R$. Recall that the analytic spread of $I$, denoted by $\ell(I)$, is given by

$$\ell(I) = \dim R[I]/mR[I].$$
This number satisfies $\text{ht}(I) \leq \ell(I) \leq \dim(R)$ [63 Corollary 5.1.4].

**Theorem 4.6.** [12] \( \inf_i \{\text{depth}(R/I^i)\} \leq \dim(R) - \ell(I) \), with equality if the associated graded ring \( \text{gr}_I(R) \) is Cohen–Macaulay.

Brodmann [1] improved this inequality by showing that \( \text{depth}(R/I^k) \) is constant for \( k \gg 0 \) and that this constant value is bounded from above by \( \dim(R) - \ell(I) \). For a generalization of these results to other ideal filtrations see [30 Theorem 1.1]. The constant value of \( \text{depth}(R/I^k) \) for \( k \gg 0 \) is called the limit depth of \( I \) and is denoted by \( \lim_{k \to \infty} \text{depth}(R/I^k) \).

**Definition 4.7.** A homogeneous ideal \( I \subset R \) has non-increasing depth if

\[ \text{depth}(R/I^k) \geq \text{depth}(R/I^{k+1}) \quad \forall \ k \geq 1, \]

and \( I \) has non-decreasing regularity if \( \text{reg}(R/I^k) \leq \text{reg}(R/I^{k+1}) \) for all \( k \geq 1 \). The ideal \( I \) has the persistence property if \( \text{Ass}(R/I^k) \subset \text{Ass}(R/I^{k+1}) \) for \( k \geq 1 \).

There are some classes of monomial ideals with non-increasing depth and non-decreasing regularity. A natural way to show these properties for a monomial ideal \( I \) is to prove the existence of a monomial \( x \) such that \( (I^{k+1}: x) = I^k \) for \( k \geq 1 \). This was exploited in [33, 43] and in [24 Corollary 3.11] in connection to normally torsion-free ideals.

**Theorem 4.8.** [3] Theorem 5.1] If \( I(C) \) is the edge ideal of a clutter \( C \) which has a good leaf, then \( I(C) \) has non-increasing depth and non-decreasing regularity.

In particular edge ideals of forests or simplicial trees have non-increasing depth and non-decreasing regularity. Our next result gives another wide family of ideals with these properties.

**Theorem 4.9.** Let \( C \) be a clutter and let \( I = I(C) \) be its edge ideal. If \( C \) is unmixed and satisfies the max-flow min-cut property, then

(a) \( \text{depth}(R/I^k) \geq \text{depth}(R/I^{k+1}) \) for \( k \geq 1 \), and

(b) \( \text{reg}(R/I^k) \leq \text{reg}(R/I^{k+1}) \) for \( k \geq 1 \).

**Proof.** Let \( C_1, \ldots, C_s \) be the minimal vertex covers of \( C \). If \( p_i \) is the ideal of \( R \) generated by \( C_i \) for \( i = 1, \ldots, s \), then \( p_1, \ldots, p_s \) are the minimal primes of \( I \) [64 Theorem 6.3.39]. As \( C \) has the max-flow min-cut property, by [50 Corollary 22.1c], \( Q(A) \) is integral. Therefore, by Lemma 4.5, there exists an edge \( e \) of \( C \) intersecting every \( C_i \) in exactly one vertex. Thus \(|e \cap p_i| = 1\) for \( i = 1, \ldots, s \). We claim that \( (I^{k+1}: x_e) = I^k \) for \( k \geq 1 \), where \( x_e = \prod_{i \in e} x_i \). The \( k \)-th symbolic power of \( I \) is given by

\[ I^{(k)} = p_1^k \cap \cdots \cap p_s^k, \]

and by [20 Corollary 3.14], \( I^k = I^{(k)} \) for \( k \geq 1 \). Clearly \( I^k \) is contained in \( (I^{k+1}: x_e) \) because \( x_e \) is in \( I \). To show the other inclusion take \( x^a \) in \( (I^{k+1}: x_e) \). Fix any \( 1 \leq i \leq s \). Then \( x^a x_e \) is in \( I^{k+1} \subset p_i^{k+1} \). Thus there are \( x_{j_1}, \ldots, x_{j_{k+1}} \) in \( p_i \) with \( j_1 \leq \cdots \leq j_{k+1} \) such that

\[ x^a x_e = x_{j_1} \cdots x_{j_{k+1}} x^b, \]

for some \( x^b \). Since \(|e \cap p_i| = 1\) from this equality, we get that with one possible exception all variables that occur in \( x_e \) divide \( x^b \). Thus \( x^a \in p_i^k \). As \( 1 \leq i \leq s \) was an arbitrary fixed integer, using Eq. (1.2), we get \( x^a \in I^{(k)} = I^k \). Thus \( (I^{k+1}: x_e) = I^k \), as claimed. To prove parts (a) and (b) note that, by Corollary 2.12(ii), one has \( \text{depth}(R/I^k) = \text{depth}(R/(I^{k+1}: x_e)) \geq \text{depth}(R/I^{k+1}) \), and by Corollary 2.12(iv), one has \( \text{reg}(R/I^k) = \text{reg}(R/(I^{k+1}: x_e)) \leq \text{reg}(R/I^{k+1}) \). \( \square \)
Corollary 4.10. Let $C$ be a clutter and let $J = I(C)^\vee$ be its ideal of covers. If $C$ is uniform and its set covering polyhedron is integral, then

(a) $\text{depth}(R/J^{(k)}) \geq \text{depth}(R/J^{(k+1)})$ for $k \geq 1$, and

(b) $\text{reg}(R/J^{(k)}) \leq \text{reg}(R/J^{(k+1)})$ for $k \geq 1$.

Proof. This follows using duality and adapting the proof of Theorem 4.9.

5. Edge ideals of graphs

Let $G$ be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$. A connected component of $G$ with at least two vertices is called non-trivial. We denote the set of isolated vertices of $G$ by $\text{isol}(G)$ and the number of non-trivial bipartite components of $G$ by $c_0(G)$. The neighbor set of $x_i$, denoted $N_G(x_i)$, is the set of all $x_j \in V(G)$ such that $\{x_i, x_j\}$ is an edge of $G$.

Proposition 5.1. Let $I(G)$ be the edge ideal of $G$. The following hold for $k \geq 1$ and $i = 1, \ldots, n$.

(a) $\text{depth}(R/(I(G)^k : x_i^k)) \leq \text{depth}(R/(I(G \setminus N_G(x_i))^k, N_G(x_i)))$.

(b) $\text{dim}(R) - \ell(I(G)) = |\text{isol}(G)| + c_0(G)$.

(c) $\lim_{k \to \infty} \text{depth}(R/I(G)^k) = |\text{isol}(G)| + c_0(G)$.

(d) If $H = G \setminus N_G(x_i)$, then $\lim_{k \to \infty} \text{depth}(R/(I(H)^k, N_G(x_i))) = |\text{isol}(H)| + c_0(H)$.

Proof. (a): Clearly $x_i^k \in (I(G)^k : x_i^k)$ for $x_j \in N_G(x_i)$. Setting $H = G \setminus N_G(x_i)$, it is not hard to see that $x_i^k$ is a minimal generator of the ideal $(I(G)^k : x_i^k)$ for $x_j \in N_G(x_i)$ and that any minimal generator of $I(H)^k$ is a minimal generator of $(I(G)^k : x_i^k)$. The colon ideal $(I(G)^k : x_i^k)$ is minimally generated by

\[ \{x_i^k, x_j \in N_G(x_i)\} \cup H(I(H)^k) \cup \{x_{\alpha_1}, \ldots, x_{\alpha_r}\}, \]

for some monomials $x_{\alpha_1}, \ldots, x_{\alpha_r}$ such that each $x_{\alpha_i}$ contains at least one variable in $N_G(x_i)$. One has the equality

\[ (N_G(x_i), (I(G)^k : x_i^k)) = (N_G(x_i), I(H)^k). \]

Therefore, starting with the ideal $(I(G)^k : x_i^k)$ and any variable $x_j$ in $N_G(x_i)$, and successively applying Theorem 3.1 the required inequality follows.

(c): Let $G_1, \ldots, G_m$ be the non-trivial connected components of $G$. The analytic spread of $I(G_i)$ is equal to $|V(G_i)|$ if $G_i$ is non-bipartite and is equal to $|V(G_i)| - 1$ otherwise (see [66, Corollary 10.1.21 and Proposition 14.2.12]). Hence the equality follows from the fact that the analytic spread is additive in the sense of [40, Lemma 3.4].

(e): This follows at once from part (d).

Lemma 5.2. Let $G$ be a bipartite graph with vertices $x_1, \ldots, x_n$, let $I(G)$ be its edge ideal, and let $k \geq 1$, $1 \leq i \leq n$ be integers. The following hold.

(a) $(I(G) : x_i^k) = (I(G) : x_i)^{(k)}$.

(b) $(I(G)^k : x_i) = (I(G) : x_i)^k$.

Proof. (a): The graph $G \setminus N_G(x_i)$ is bipartite. Hence, according to [55, Theorem 5.9], the ideal $I(G \setminus N_G(x_i))$ is normally torsion-free and so is the ideal $(N_G(x_i))$ generated by $N_G(x_i)$. Therefore, by [53, Corollary 5.6], the ideal $(I(G \setminus N_G(x_i)), N_G(x_i))$ is normally torsion-free. Thus it suffices to observe that $(I(G) : x_i)$ is equal to $(I(G \setminus N_G(x_i)), N_G(x_i))$ (see [66, p. 293]).
Let $p_1, \ldots, p_s$ be the associated primes of $I(G)$. Since $G$ is bipartite, its edge ideal is normally torsion-free [53 Theorem 5.9]. Therefore, using part (a) and noticing that the primary decomposition of $(I(G) : x_i)$ is $\cap_{j \not\in p_i} p_j$, we get

$$(I(G)^k : x_i^k) = \left( \left( \bigcap_{j=1}^{s} p_j \right)^k : x_i^k \right) = \left( \left( \bigcap_{j=1}^{s} p_j^k \right) : x_i^k \right)_{x_i \not\in p_j^k} = (I(G) : x_i)^{(k)} = (I(G) : x_i)^k.$$

The regularity of powers of the cover ideal of a bipartite graph was studied in [37] and the depth of symbolic powers of cover ideals of graphs was examined in [33, 53].

**Corollary 5.3.** Let $G$ be a bipartite graph. The following hold.

(a) [39 Corollary 5.3] If $G$ is unmixed, then $I(G)$ has non-increasing depth.

(b) ([3 Theorem 3.2], [25, 27 Corollary 2.4]) $I(G)^\lor$ has non-increasing depth.

(c) $I(G)^\lor$ has non-decreasing regularity.

**Proof.**

(a): By [18 Theorem 4.6 and Proposition 4.27], the graph $G$ has the max-flow min-cut property and since $G$ is unmixed the result follows at once from Theorem 4.9.

(b)–(c): By [18 Theorem 4.6 and Corollary 4.28], $G^\lor$ has the max-flow min-cut property, and $G^\lor$ is unmixed because its minimal vertex covers are the edges of $G$. Thus by Theorem 4.9 the ideal $I(G^\lor) = I(G)^\lor$ has non-increasing depth and non-decreasing regularity.

The next interesting example is due to Kaiser, Stehlík, and Škrekovski [38]. It shows that the Alexander dual of a graph does not always have the persistence property for associated primes. This example also shows that part (b) of Corollary 5.3 fails for non-bipartite graphs.

**Example 5.4.** [38] Let $J = I^\lor$ be the Alexander dual of the edge ideal

$$I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7, x_7x_8, x_8x_9, x_9x_{10}, x_{10}x_{11}, x_{11}x_{12}, x_{12}x_1, x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7, x_7x_8, x_8x_9, x_9x_{10}, x_{10}x_{11}, x_{11}x_{12}, x_{12}x_1).$$

Using Macaulay2 [22], it is seen that the values of depth$(R/J^i)$, for $i = 1, \ldots, 4$ are 8, 5, 0, 4, respectively.

**Definition 5.5.** Let $I \subset R$ be a squarefree monomial ideal. The symbolic powers of $I$ have non-increasing depth if

$$\text{depth}(R/I^{(k)}) \geq \text{depth}(R/I^{(k+1)}) \ \forall \ k \geq 1,$$

and have non-decreasing regularity if $\text{reg}(R/I^{(k)}) \leq \text{reg}(R/I^{(k+1)})$ for all $k \geq 1$.

If $G$ is a very well-covered graph (i.e., $G$ is unmixed, has no isolated vertices and $|V(G)|$ is equal to $2\text{ht}(I(G)))$, then the symbolic powers of $I(G)^\lor$ have non-increasing depth [52 (cf. [33 Theorem 3.2]) and the symbolic powers of $I(G)$ have non-increasing depth [39 Theorem 5.2]. The next result complements these facts.

**Proposition 5.6.** If $G$ is a very well-covered graph, then the symbolic powers of $I(G)$ have non-decreasing regularity.

**Proof.** The graph $G$ has a perfect matching by [19 Corollary 3.7(ii)]. Pick an edge $e$ in a perfect matching of $G$ and set $x_e = \prod_{x_i \in e} x_i$. Note that any minimal vertex cover of $G$ intersects $e$ in
Corollary 3.3), the result follows from Proposition 5.9. □

Proof. Since the radical of a Cohen–Macaulay monomial ideal is Cohen–Macaulay [32] (see I, then we get Corollary 5.10. For additional results on the depth of powers of Proposition 5.8. Let \( p_1, \ldots, p_s \) be the set of all ideals \( (e) \) such that \( e \in E(\text{cl}(G)) \). From the equality

\[
J = I(\text{cl}(G)^\vee) = I(\text{cl}(G))^\vee = \bigcap_{e \in E(\text{cl}(G))} (e) = \bigcap_{i=1}^s p_i,
\]

we get \( J^{(k)} = \bigcap_{i=1}^s p_i^k \) for \( k \geq 1 \). As \( G \) is strongly perfect, \( G \) has a maximal independent set of vertices \( C \) such that \( |C \cap e| = 1 \) for any \( e \in \text{cl}(G) \), that is, \( |C \cap p_i| = 1 \) for \( i = 1, \ldots, s \). Hence, setting \( f = \prod_{x_i \in C} x_i \), one has the equalities

\[
(J^{(k+1)} : f) = \left( \bigcap_{i=1}^s p_i^{k+1} : f \right) = \left( \bigcap_{i=1}^s p_i^{k+1} : f \right) = \bigcap_{i=1}^s p_i^k = J^{(k)} \quad \text{for} \quad k \geq 1.
\]

Therefore, by parts (ii) and (iv) of Corollary 2.12 one has

\[
\text{depth}(R/J^{(k)}) = \text{depth}(R/(J^{(k+1)} : f)) \geq \text{depth}(R/J^{(k+1)}),
\]

and \( \text{reg}(R/J^{(k)}) = \text{reg}(R/(J^{(k+1)} : f)) \leq \text{reg}(R/J^{(k+1)}) \). □

Proposition 5.9. Let \( A = K[X] \) and \( B = K[Y] \) be polynomial rings over a field \( K \) in disjoint sets of variables, let \( I \) and \( J \) be nonzero homogeneous proper ideals of \( A \) and \( B \) respectively, and let \( R = K[X, Y] \). The following hold.

(a) [25 Proposition 3.7] \( R/(I + J)^i \) is Cohen–Macaulay for all \( i \leq k \) if and only if \( A/I^i \) and \( B/J^i \) are Cohen–Macaulay for all \( i \leq k \).

(b) [34 Lemma 3.2] \( \text{reg}(R/(I + J)) = \text{reg}(A/I) + \text{reg}(B/J) \).

(c) [34 Lemma 3.2] \( \text{reg}(R/IJ) = \text{reg}(A/I) + \text{reg}(B/J) + 1 \).

The Cohen–Macaulay property of the square of an edge ideal can be expressed in terms of its connected components (cf. [65 Lemma 4.1]). For additional results on the depth of powers of sums of ideals see [25] and the references therein.

Corollary 5.10. [48 Corollary 4.9] Let \( G \) be a graph with connected components \( G_1, \ldots, G_m \). Then \( I(G)^2 \) is Cohen–Macaulay if and only if \( I(G_i)^2 \) is Cohen–Macaulay for \( i = 1, \ldots, m \).

Proof. Since the radical of a Cohen–Macaulay monomial ideal is Cohen–Macaulay [32] (see Corollary 5.9), the results follows from Proposition 5.9. □
Example 5.11. Let $A = K[x_1, x_2, x_3]$ and $B = K[y_1, y_2, y_3]$ be polynomial rings over a field $K$, let $I = (x_1 x_2, x_2 x_3, x_1 x_3)$ and $J = (y_1 y_2, y_2 y_3, y_1 y_3)$ be ideals of $A$ and $B$ respectively, and let $R = K[X, Y]$. Then $A/I^2$ and $B/I^2$ have depth $0$ but $R/(I + J)^2$ has depth $1$, that is, the depth of squares of monomial ideals is not additive on disjoint sets of variables.

Lemma 5.12. Let $G$ be a graph without isolated vertices. The following hold.

(a) If $R/I(G)^2$ is Cohen–Macaulay, then $R/(I(G \setminus N_G(x_i))^2$ is Cohen–Macaulay for any $x_i$.

(b) $\text{depth}(R/I(G)^2) = 0$ if and only if $G$ has a triangle $C_3$ that intersects $N_G(x_i)$ for any $x_i$ outside $C_3$. In particular if the depth of $R/I(G)^2$ is $0$, then $G$ is connected.

Proof. (a): Using Proposition 5.1(a) and Corollary 2.12(ii), we get

$$\text{depth}(R/(I(G \setminus N_G(x_i))^2, N_G(x_i))) \geq \text{depth}(R/(I(G)^2 : x_i^2)) \geq \text{depth}(R/I(G)^2)$$

for all $i$. Thus $R/(I(G \setminus N_G(x_i))^2$ is Cohen–Macaulay for all $i$.

(b) ($\Rightarrow$): As $m = (x_1, \ldots, x_n)$ is an associated prime of $I(G)^2$, there is $x^a = x_1^{a_1} \cdots x_n^{a_n}$ such that $(I(G)^2 : x^a) = m$. Thus $x_i x^a \in I(G)^2$ for all $i$ and $x^a \notin I(G)^2$. Note that $x^a$ is squarefree. Indeed if $a_k \geq 2$ for some $k$, then $x_k x^a = x^b f_j$ for some monomial $x^b$ and some minimal generators $f_i, f_j$ of $I(G)$, which is impossible because $f_i, f_j$ are squarefree monomials of degree $2$ and $x^a \notin I(G)^2$. Thus we may assume that $x^a = x_1 \cdots x_r$, for some $r \geq 3$, and $x_1 x_2 \in I(G)$. Then $x_3 x^a = x^b f_j f_j$ for some $x^b$ and some minimal generators $f_i, f_j$ of $I(G)$. One can write $f_i = x_3 x_k$ and $f_j = x_3 x_\ell$, $k \neq \ell$, $k \neq 3$, $\ell \neq 3$. Clearly either $x_k = x_1$ or $x_k = x_2$ and either $x_\ell = x_1$ or $x_\ell = x_2$ because $x^a$ is not in $I(G)^2$. Thus $x_1, x_2, x_3$ are the vertices of a triangle of $G$ that we denote by $C_3$. Since $x_r x^a \in I(G)^2$, it follows that $r = 3$. Take any vertex $x_k$ not in $C_3$. As $x_k x^a = x_k (x_1 x_2 x_3)$ and $x_k x^a$ is in $I(G)^2$, we get that $x_k$ is adjacent to some vertex of $C_3$.

(b) ($\Leftarrow$): Pick a triangle $C_3$ of $G$ such that any vertex outside $C_3$ is adjacent to a vertex of $C_3$. Setting $x^a = \prod x_i \in V(C_3)$, we get that $(I(G)^2 : x^a)$ is the maximal ideal $m = (x_1, \ldots, x_n)$. Thus $m$ is an associated prime of $I(G)^2$, that is, $\text{depth}(R/I(G)^2) = 0$. This part could also follow from a general construction of [4].

In [35, 36] the Cohen–Macaulay property of the square of the edge ideal of a graph is classified.

Theorem 5.13. [36] Theorem 4.4 Let $G$ be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and without isolated vertices. Then $I(G)^2$ is Cohen–Macaulay if and only if $G$ is a triangle-free unmixed graph and $G \setminus \{x_i\}$ is unmixed for all $i$.

As an application we recover the following facts.

Corollary 5.14. ([71 Theorem 2.7], [35] Proposition 4.2) Let $G$ be a bipartite graph without isolated vertices. Then $I(G)^2$ is Cohen–Macaulay if and only if $I(G)$ is a complete intersection, i.e., $G$ is a disjoint union of edges.

Proof. $\Rightarrow$: Since $I(G)$ is the radical of $I(G)^2$, by Corollary 3.3, the ideal $I(G)$ is Cohen–Macaulay. Hence, according to a structure theorem for Cohen–Macaulay bipartite graphs [29, Theorem 3.4], there is a bipartition $V_1 = \{x_1, \ldots, x_y\}, V_2 = \{y_1, \ldots, y_y\}$ of $G$ such that:

(i) $\{x_i, y_i\} \in E(G)$ for all $i$,
(ii) if $\{x_i, y_j\} \in E(G)$, then $i \leq j$, and
(iii) if $\{x_i, y_j\}, \{x_j, y_k\}$ are in $E(G)$ and $i < j < k$, then $\{x_i, y_k\} \in E(G)$.

We proceed by induction on $g$. If $g = 1$, $I(G)$ is clearly a complete intersection. Using the connected components of $G$ together with Corollaries 3.3 and 5.10 and Proposition 5.9, we may
assume that \( I(G)^2 \) is Cohen–Macaulay and that \( G \) is a Cohen–Macaulay connected bipartite graph. Consider the graph \( H = G \setminus N_G(y_1) \). We set \( R = K[V_1 \cup V_2] \). Note that \( N_G(y_1) = \{ x_1 \} \). Hence, by Lemma \( 5.12(a) \), \( I(G \setminus \{ x_1 \})^2 \) is Cohen–Macaulay and so is \( I(G \setminus \{ x_1 \}) \). Therefore by induction \( I(G \setminus \{ x_1 \}) \) is generated by \( x_2y_2, \ldots, x_gy_g \). As \( G \) is connected, using (i)–(iii), it is seen that the edges of \( G \) are the edges of the perfect matching and all edges of the form \( \{ x_1, y_i \} \), \( i \geq 1 \). It is not hard to see (by a separate induction procedure) that the square of \( I(G) \) is not Cohen–Macaulay if \( g \geq 2 \). Thus \( g = 1 \).

\[ \Rightarrow \]: If \( I(G) \) is a complete intersection, it is well known that all powers of \( I(G) \) are Cohen–Macaulay \([42, 17.4, \text{p. 139}]\). □

Let \( G \) be a graph. The next corollary follows from the result that “\( I(G)^2 = I(G)^{(2)} \) if and only if \( G \) has no triangles”. This result originated in \([55]\) implicitly, written explicitly in \([49, \text{Lemma 3.1}]\). Fix an integer \( t \geq 2 \). This lemma shows that \( I(G)^t = I(G)^{(t)} \) if and only if \( G \) contains no odd cycles of length \( 2s - 1 \) for any \( 2 \leq s \leq t \) (cf. \([8, \text{Theorem 4.13}]\)).

**Corollary 5.15.** Let \( G \) be a graph without isolated vertices. If \( I(G)^2 \) is Cohen–Macaulay, then \( G \) has no triangles.

**Proof.** Let \( V(G) = \{ x_1, \ldots, x_n \} \) be the vertex set of \( G \) and let \( R \) be the polynomial ring \( K[V(G)] \). We proceed by induction on \( n \). The result is clear for \( n = 1, 2, 3 \). Assume \( n \geq 4 \). We proceed by contradiction assuming that \( G \) has a triangle \( C_3 \). Using the connected components of \( G \) together with Corollaries \( 3.3 \) and \( 5.10 \) and Proposition \( 5.9 \) we may assume that \( I(G)^2 \) is Cohen–Macaulay and \( G \) is connected. Thus, by Lemma \( 5.12(a) \), \( I(G \setminus N_G(x_i))^2 \) is Cohen–Macaulay for all \( i \). If \( G \) has a vertex \( x_i \) not in \( C_3 \) such that \( N_G(x_i) \) do not intersect the vertex set \( V(C_3) \) of \( C_3 \), then \( C_3 \) is a triangle of \( G \setminus N_G(x_i) \), a contradiction. Thus any vertex outside \( C_3 \) is adjacent to a vertex of \( C_3 \). Hence, by Lemma \( 5.12(b) \), we get \( \text{depth}(R/I(G)^2) = 0 \), a contradiction. This part could also follow from a general construction of \([4]\). □

**Example 5.16.** \([36, 48]\) The square of the edge ideal of the graph \( G \) of Fig. 1 is Cohen–Macaulay and \( I(G) \) is Gorenstein. This can be verified using Macaulay2 \([22]\). This example appears as a special case of \([48, \text{Conjecture 5.7}]\). A result of Hoang and Trung \([36, \text{Theorem 4.4}]\) shows that for a graph \( G \) without isolated vertices \( I(G)^2 \) is Cohen–Macaulay if and only if \( G \) is triangle-free and Gorenstein. The Cohen-Macaulay property of \( I(G)^2 \) is also studied in \([60] \) in terms of simplicial complexes.

![Figure 1. Gorenstein Graph G.](image-url)
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