Recent theoretical advances in decentralized distributed convex optimization

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Abstract In the last few years, the theory of decentralized distributed convex optimization has made significant progress. The lower bounds on communications rounds and oracle calls have appeared, as well as methods that reach both of these bounds. In this paper, we focus on how these results can be explained based on optimal algorithms for the non-distributed setup. In particular, we provide our recent results that have not been published yet and that could be found in details only in arXiv preprints.

1 Introduction

In this work, we focus on the following convex optimization problem

\[
\min_{x \in Q \subseteq \mathbb{R}^n} f(x) := \frac{1}{m} \sum_{i=1}^{m} f_i(x),
\]

where functions \(\{f_i\}_{i=1}^{m}\) are convex and \(Q\) is a convex set. Such kind of problems arise in many machine learning applications \[107\] (e.g., empirical risk minimization) and statistical applications \[112\] (e.g., maximum likelihood estimation). To solve these problems, decentralized distributed methods are widely used (see \[84\] and reference therein). This direction has gained popularity with the release of the book \[10\]. Many researchers (among which we especially note Angelia Nedich) have productively promoted distributed algorithms in the last 30 years. Due to the emergence of big data and the rapid growth of problems sizes, decentralized distributed methods have gained increased interest in the last decade. In this paper, we mainly focus on the last five years of theoretical advances, starting with the remarkable paper \[6\]. The authors of \[6\] introduce the lower complexity bounds for communication rounds required to achieve \(\varepsilon\)-accuracy solution \(x_N\) of (1) in function value, i.e., \(f(x_N) - \min_{x \in Q} f(x) \leq \varepsilon\), and at worst in Euclidean distance (more accurate definition in Sections 2, 3).

Let us formulate the result of \[6\] (see also \[111\] \[103\] \[105\] \[70\] \[123\]) formally. Assume that we have some connected undirected graph (network) with \(m\) nodes. For each node \(i\) of this graph, we privately assign function \(f_i\) and suppose that the node \(i\) can calculate \(\nabla f_i\) for some point \(x\). At each communication round the nodes can communicate with their neighbors, i.e., send and receive a message with no more than \(O(n)\) numbers. In the \(O(R)\) neighborhood of a solution \(x^*\) of (1) \((R = \|x^0 - x^*\|_2\) is the Euclidean distance between starting point \(x^0\) and the solution \(x^*\) that corresponds to the minimum of this norm), we suppose that functions \(f_i\)’s are \(M\)-Lipschitz continuous \((i.e., \|\nabla f_i(x)\|_2 \leq M)\) and \(L\)-Lipschitz smooth \(i.e., \|\nabla f_i(y) - \nabla f_i(x)\|_2 \leq L\|y - x\|_2\). The optimal bounds on the number of communications and the number of oracle calls per node are summarized in Table \[1\]. Here and below \(\tilde{O}()\) means the same as \(O()\) up to a \(\log(1/\varepsilon)\) factor, and \(\tilde{O}(\sqrt{\chi})\) corresponds to the consensus time, that is the number of communication rounds required to reach the consensus in the considered network (more accurate definition of \(\tilde{O}(\sqrt{\chi})\) is given in Sections 2, 3).

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In the last few years, algorithms have been developed that reach the lower bounds from Table 1. In Section 2, we consider one of such algorithms [102] for the case when the functions \( f_i \)'s are smooth. This algorithm has the simplest nature among all known alternatives: this is a direct consensus-projection generalization of Nesterov’s fast gradient method.

When communication networks vary from time to time (time-varying communication networks, see Section 2), we replace \( \sqrt{\frac{\chi}{\mu}} \) by \( \chi \) and we suppose that different \( f_i \)'s may have different constants of smoothness \( L_i \).

The non-smooth case (when functions \( f_i \)'s are Lipschitz continuous) is studied in Section 3 where the results from [23, 39] are summarized. The approach is based on reformulation of the distributed decentralized problem as non-distributed convex optimization problem with affine constraints, which are further brought into the target function as a composite quadratic penalty. To solve this problem, Lan’s sliding algorithm [69] can be used.

### Table 1 Optimal bounds for communication rounds and deterministic oracle calls of \( \nabla f_i \) per node

| \( f_i \) is \( \mu \)-strongly convex and \( L \)-smooth | \( f_i \) is \( L \)-smooth | \( f_i \) is \( \mu \)-strongly convex |
|---------------------------------|-----------------|-----------------|
| # communication rounds | \( \tilde{O} \left( \sqrt{\frac{\chi}{\mu}} \right) \) | \( \tilde{O} \left( \frac{L^2}{\mu^2} \chi \right) \) | \( O \left( \sqrt{\frac{\mu^2}{\mu^2}} \right) \) |
| # oracle calls of \( \nabla f_i \) per node \( i \) | \( \tilde{O} \left( \sqrt{\frac{\chi}{\mu}} \right) \) | \( O \left( \frac{L^2}{\mu^2} \chi \right) \) | \( O \left( \frac{\mu^2}{\mu^2} \right) \) |

The same construction and estimates hold (see Table 2) in the non-smooth stochastic case, when instead of subgradients \( \nabla f_i(x) \)'s we have an access only to their unbiased estimates \( \nabla f_i(x, \xi) \)'s. We assume here that \( E \| \nabla f(x, \xi) \|^2 \leq M^2 \) on an \( O(R) \) neighborhood of \( x^* \).

### Table 2 Optimal bounds for communication rounds and stochastic oracle calls of \( \nabla f_i(x, \xi) \) per node

| \( f_i \) is \( \mu \)-strongly convex and \( L \)-smooth | \( f_i \) is \( L \)-smooth | \( f_i \) is \( \mu \)-strongly convex |
|---------------------------------|-----------------|-----------------|
| # communication rounds | \( \tilde{O} \left( \sqrt{\frac{\chi}{\mu}} \right) \) | \( \tilde{O} \left( \frac{L^2}{\mu^2} \chi \right) \) | \( O \left( \frac{\mu^2}{\mu^2} \chi \right) \) |
| # oracle calls of \( \nabla f_i(x, \xi) \) per node \( i \) | \( \tilde{O} \left( \frac{\sigma^2}{\mu^2 \epsilon} \sqrt{\frac{\mu}{\sqrt{\mu}}} \right) \) | \( \tilde{O} \left( \max \left\{ \frac{\sigma^2}{\mu^2 \epsilon}, \left( \frac{L^2}{\mu^2} \right) \right\} \right) \) | \( O \left( \frac{\mu^2}{\mu^2} \right) \) |

The smooth part of Table 2 describes known lower bounds. There exist methods that are optimal only in one of two mentioned criteria [23]: either in communication rounds or in oracle calls per node. But, we expect that the technique from [102] (also described in Section 2) combined with proper batch-size policy [26] allows to reach these lower bounds.

Section 3 also contains analogues of the results mentioned in Tables 1 and 2 for dual (stochastic) gradient type oracle. That is instead of an access at each node to \( f_i \) (also described in Section 2) combined with proper batch-size policy [26] allows to reach these lower bounds. This algorithm has the simplest nature among all known alternatives: this is a direct consensus-projection generalization of Nesterov’s fast gradient method.

### Table 3 Optimal bounds for communication rounds and deterministic oracle calls of \( \nabla f_i \) per node

| \( f_i \) is \( \mu \)-strongly convex and \( L \)-smooth | \( f_i \) is \( L \)-smooth | \( f_i \) is \( \mu \)-strongly convex |
|---------------------------------|-----------------|-----------------|
| # communication rounds | \( \tilde{O} \left( \sqrt{\frac{\chi}{\mu}} \right) \) | \( \tilde{O} \left( \frac{L^2}{\mu^2} \chi \right) \) | \( O \left( \sqrt{\frac{\mu^2}{\mu^2}} \right) \) |
| # oracle calls of \( \nabla f_i \) per node \( i \) | \( \tilde{O} \left( \sqrt{\frac{\chi}{\mu}} \right) \) | \( O \left( \frac{L^2}{\mu^2} \chi \right) \) | \( O \left( \frac{\mu^2}{\mu^2} \right) \) |

The same construction and estimates hold (see Table 2) in the non-smooth stochastic case, when instead of subgradients \( \nabla f_i(x) \)'s we have an access only to their unbiased estimates \( \nabla f_i(x, \xi) \)'s. We assume here that \( E \| \nabla f(x, \xi) \|^2 \leq M^2 \) on an \( O(R) \) neighborhood of \( x^* \).

The smooth part of Table 2 describes known lower bounds. There exist methods that are optimal only in one of two mentioned criteria [23]: either in communication rounds or in oracle calls per node. But, we expect that the technique from [102] (also described in Section 2) combined with proper batch-size policy [26] allows to reach these lower bounds.

In Section 4, we transfer the results mentioned above to gradient-free oracle assuming that we have an access only to \( f_i \) instead of \( \nabla f_i \). In this case, a trivial solution comes to mind: to restore the gradient from finite differences. Based on optimal gradient-type methods in smooth case, it is possible to build optimal gradient-free methods. But what is about non-smooth case? To the best of our knowledge, until recently, it was an open question. Based on [11], we provide an answer for this question (Section 4). To say more precisely, we transfer optimal gradient-free algorithms for non-smooth (stochastic two-points) convex optimization problems [103, 123] from non-distributed set up to decentralized distributed one. Here, as in Section 3, we also mainly use the penalty trick and Lan’s sliding.

It is worth to add several results to the list of recent advances collected in Tables 1, 2. The first result describes the case when \( f_i(x) = \frac{1}{r} \sum_{j=1}^{r} f_i^j(x) \) in [1]. \( Q = \mathbb{R}^n \). All \( f_i^j \) are \( L \)-smooth and \( \mu \)-strongly convex. In this case, lower bounds were obtained in [47]. Optimal algorithms were proposed in [47, 77]. These algorithms require \( \tilde{O} \left( \frac{1}{\sqrt{\mu}} \chi \right) \) communication rounds and \( \tilde{O} \left( r + \sqrt{\frac{\chi}{\mu}} \right) \) oracle calls (\( \nabla f_i^j \) calculations) per node. This is valuable result since in real machine learning applications the sum-type representation of \( f_i \) is typical.

Another way to use this representation is statistical similarity of \( f_i \). The lower bound for communication rounds in deterministic smooth case was also obtained [6]. Roughly speaking, if the Hessians of the \( f_i \)'s are \( \beta \)-close in 2-norm, then the lower bound for communication rounds will be \( \tilde{O} \left( \sqrt{\frac{\beta}{\mu}} \chi \right) \). For example, if \( f_i(x) = \frac{1}{r} \sum_{j=1}^{r} f_i^j(x) \) and all \( f_i^j \) are...
μ-strongly convex we have \( \beta \simeq \mu + \text{const} \frac{\sqrt{r}}{\epsilon} \). In the decentralized distributed setup there is a gap between this lower bound and the optimal (non-accelerated) bound \( \widetilde{O} \left( \frac{\mu}{\epsilon^2} \right) \) that can be achieved at the moment \([118]\). But for centralized distributed architectures with additional assumptions on \( f_i \)’s, partial acceleration is possible \([48]\).

Other group of results relate to very specific (but rather popular) centralized federated learning architectures \([55]\). According to mentioned above estimates, heterogeneous federated learning can be considered as a partial case (with \( \chi = 1 \)) \([57, 126, 60, 41]\). In the paper \([60]\), this was explained based on the analysis of unified decentralized SGD. Paper \([60]\] also summarizes a lot of different distributed setups in one general approach. We partially try to use the generality from \([60]\) in Section 2. To the best of our knowledge, it is an open question to accelerate all the results of \([60]\). Section 2 contains such an acceleration only in deterministic case. Note also, that for homogeneous federated learning, the described above bounds are far from being optimal \([127]\).

2 Decentralized Optimization of Smooth Convex Functions

Consider problem \([1]\) and rewrite it in the following form:

\[
\min_{X \in \mathbb{R}^{m \times n}} F(X) = \sum_{i=1}^{m} f_i(x_i) \quad \text{s.t.} \quad x_1 = \ldots = x_m
\]  

(2)

where \( X = (x_1 \ldots x_m)^T \). Decentralized optimization problem is now reformulated as an optimization problem with linear constraints. The constraint set writes as \( C = \{x_1 = \ldots = x_m\} \).

Functions \( f_i \) are stored on the nodes across the network, which is represented as an undirected graph \( \mathcal{G} = (V, E) \). Every node has an access to the function and its first-order characteristics. Decentralized first-order methods use two types of steps – computational steps, i.e. performing local computations and communication steps, which is exchanging the information with neighbors. Alternating these two types of steps results in minimizing the objective while maintaining agents’ vectors approximately equal.

We begin with an overview of how communication procedures are developed and analyzed. The iterative information exchange is referred to as consensus of gossip \([15, 120, 128, 83]\) algorithms in the literature.

2.1 Consensus algorithms

Let each agent in the network initially hold a vector \( x_i^0 \) and let the communication network be represented by a connected graph \( \mathcal{G} = (V, E) \). The agents seek to find the average vector across the network, but their communication is restricted to sending and receiving information from their direct neighbors. In one communication round, every two nodes linked by an edge exchange their vector values. After that, agent \( i \) sums the received values with predefined coefficients \( m_{ij} \), where \( j \) is the number of the corresponding neighbor. In other words, every node runs an update

\[
x_i^{k+1} = m_{ii}x_i^k + \sum_{(i,j) \in E} m_{ij}x_j^k.
\]

Introduce a mixing matrix \( M = [m_{ij}]_{i,j=1}^{m} \). Then the update at one communication step takes the form

\[
X^{k+1} = MX^k.
\]

(3)

Under additional assumptions this iterative scheme converges to the average of initial vectors over network, i.e. to

\[
X^0 = \frac{1}{m} 11^T X^0 = \frac{1}{m} \sum_{i=1}^{m} x_i^0.
\]

Assumption 1 Mixing matrix \( M \) satisfies the following properties.

- (Decentralized property) If \((i,j) \notin E \) and \( i \neq j \), then \( |M|_{ij} = 0 \). Otherwise \( |M|_{ij} > 0 \).
- (Symmetry and double stochasticity) \( M1 = 1 \) and \( M = M^\top \).
The choice of weights for mixing matrix is an interesting problem we do not address here (see [15] for details). A mixing matrix with Metropolis weights satisfies Assumption 1:

\[
[M]_{ij} = \begin{cases} 
1/(1 + \max\{d_i, d_j\}) & \text{if } (i, j) \in E, \\
0 & \text{if } (i, j) \notin E, \\
1 - \sum_{m(i, m) \in E} [M]_{im} & \text{if } i = j,
\end{cases}
\]

where \(d_i\) denotes the degree of node \(i\).

Several variations of Assumption 1 can be found in literature. In particular, in [80] the mixing matrix is not needed to be symmetric. Instead, it is assumed to be doubly stochastic and have a real spectrum. Moreover, the spectrum property in Assumption 1 implies that 1 is the only (up to a scaling factor) eigenvector corresponding to eigenvalue 1, i.e. \(\ker (I - M) = \text{span}(1)\).

**Lemma 1.** For iterative consensus procedure (3) it holds

\[
\left\|X^k - \bar{x}^0\right\|_2 \leq (\lambda_2(M))^k \left\|X^0 - \bar{x}^0\right\|_2.
\]

**Proof.** Let \(x \in \mathbb{R}^n\) and \(\bar{x} = \frac{1}{m}11^\top x\). First, note that \(M\bar{x} = M \cdot \frac{1}{m}11^\top x = \frac{1}{m}11^\top x = \bar{x}\). It can be easily seen that \(x - \bar{x} \in (\text{span}(1))^\perp\) and \(Mx - x \in (\text{span}(1))^\perp\):

\[
\langle x - \bar{x}, 1 \rangle = \left\langle \left(I - \frac{1}{m}11^\top \right)x, 1 \right\rangle = \left\langle x, \left(I - \frac{1}{m}11^\top \right)1 \right\rangle = 0,
\]

\[
\langle Mx - \bar{x}, 1 \rangle = \left\langle \left(M - \frac{1}{m}11^\top \right)x, 1 \right\rangle = \left\langle x, \left(M - \frac{1}{m}11^\top \right)1 \right\rangle = 0.
\]

On subspace \((\text{span}(1))^\top\) the largest eigenvalue of \(M\) is \(\lambda_2(M)\). We have

\[
\|Mx - \bar{x}\|_2 = \|M(x - \bar{x})\|_2 \leq \lambda_2(M) \|x - \bar{x}\|_2.
\]

Applying the derived fact to every column of \(M\) we get \(\bar{x}^k = \bar{x}^0\) and \(\left\|X^{k+1} - \bar{x}^0\right\|_2 \leq \lambda_2(M) \left\|X^k - \bar{x}^0\right\|_2\), for every \(k \geq 0\), which concludes the proof.

By Lemma 1 consensus scheme (3) requires \(O\left(\frac{1}{\lambda_2(M)} \log \left(\frac{1}{\varepsilon}\right)\right)\) iterations to achieve accuracy \(\varepsilon\), i.e. to find arithmetic mean of vectors over the network with precision \(\varepsilon\): \(\left\|X^k - \bar{x}^0\right\|_2 \leq \varepsilon\).

### 2.1.1 Quadratic optimization point of view

For a given undirected graph \(\mathcal{G} = (V, E)\) introduce its Laplacian matrix

\[
[W]_{ij} = \begin{cases} 
-1, & \text{if } (i, j) \in E, \\
\deg(i), & \text{if } i = j, \\
0 & \text{otherwise}.
\end{cases}
\]

Laplacian matrix is positive semi-definite and for \(X = (x_1 \ldots x_m)^\top\) it holds \(WX = 0 \iff x_1 = \ldots = x_m\). A more detailed discussion of Laplacian matrix and its applications is provided in Section 3.3. The consensus problem can be reformulated as

\[
\min_{X \in \mathbb{R}^{m \times n}} \ g(X) := \frac{1}{2} \langle X, WX \rangle.
\]

Any matrix \(X^*\) with equal rows is a solution of Problem 4, and therefore the set of minimizers of Problem 4 is a linear subspace of form \(\mathcal{X}^* = \{1x^\top : x \in \mathbb{R}^n\}\). Denote \(\lambda_{\max}(W)\) and \(\lambda_{\min}(W)\) the largest and the smallest non-zero eigenvalues...
Analogously to non-accelerated scheme (5), the trajectory of accelerated Nesterov method lies in $X^0 + (\ker W)^\perp$. Therefore, gradient descent on $g(X)$ with constant step-size $\frac{1}{\lambda_{\max}(W)}$ is equivalent to non-accelerated consensus algorithm (3). Moreover, the iteration complexities coincide, since $\lambda_2(M) = 1 - \frac{\lambda_{\min}(W)}{\lambda_{\max}(W)}$ and therefore $O\left(\frac{\lambda_{\max}(W)}{\lambda_{\min}(W)} \log(\frac{1}{\epsilon})\right) = O\left(\frac{1}{1-\lambda_2(M)} \log(\frac{1}{\epsilon})\right)$.

In order to obtain a better dependence on $\frac{\lambda_{\max}(W)}{\lambda_{\min}(W)}$, Nesterov acceleration [88] may be employed. Consider Nesterov accelerated method for strongly convex objectives

$$\hat{\beta} = \sqrt{\lambda_{\max}(W)} - \sqrt{\lambda_{\min}(W)},$$

$$Y^k = X^k + \hat{\beta}(X^k - X^{k-1}),$$

$$X^{k+1} = Y^k - \frac{W}{\lambda_{\max}(W)} Y^k.$$

Analogously to non-accelerated scheme (5), the trajectory of accelerated Nesterov method lies in $X^0 + (\ker W)^\perp$. This can be easily seen by induction:

$$Y^k - X^0 = (X^k - X^0) + \hat{\beta}((X^k - X^0) - (X^{k-1} - X^0)) \in (\ker W)^\perp,$$

$$X^{k+1} - X^0 = Y^k - \frac{1}{\lambda_{\max}(W)} \frac{W}{\ker W} Y^k \in (\ker W)^\perp.$$

Therefore, accelerated scheme converges to the projection of $X^0$ onto $\ker W$, which is $X^0$, i.e. a matrix whose rows are arithmetic averages of $X^0$.

### 2.2 Main assumptions on objective functions

In this section, we introduce basic assumptions on the functions locally held by computational entities in the network.

**Assumption 2** For every $i = 1, \ldots, m$, function $f_i$ is differentiable, convex and $L_i$-smooth ($L_i > 0$).

**Assumption 3** For every $i = 1, \ldots, m$, function $f_i$ is $\mu_i$-strongly convex ($\mu_i > 0$).
Under Assumptions 2 and 3 for any $x_i, y_i \in \mathbb{R}^m$ for $i = 1, \ldots, n$ it holds
\[
\frac{\mu_i}{2} \|y_i - x_i\|_2^2 \leq f_i(y_i) - f_i(x_i) - \langle \nabla f_i(x_i), y_i - x_i \rangle \leq \frac{L_i}{2} \|y_i - x_i\|_2^2.
\]

Summing the above inequality on $i = 1, \ldots, n$ we obtain
\[
\frac{\min \mu_i}{2} \|Y - X\|_2^2 \leq \sum_{i=1}^m \frac{\mu_i}{2} \|y_i - x_i\|_2^2 \leq F(Y) - F(X) - \langle \nabla F(X), Y - X \rangle \leq \sum_{i=1}^m \frac{L_i}{2} \|y_i - x_i\|_2^2 \leq \frac{\max \mu_i}{2} \|Y - X\|_2^2.
\]

On the other hand, given that $X, Y \in C$, i.e. $x_1 = \ldots = x_m$, $y_1 = \ldots = y_m$, we have
\[
\frac{1}{2m} \sum_{i=1}^m \mu_i \|Y - X\|_2^2 \leq F(Y) - F(X) - \langle \nabla F(X), Y - X \rangle \leq \frac{1}{2m} \sum_{i=1}^m L_i \|Y - X\|_2^2.
\]

Therefore, $F(X)$ has different strong convexity and smoothness constants on $\mathbb{R}^{m \times n}$ and $C$. Following the definitions in [103], we introduce

- (local constants) $F(X)$ is $\mu_l$-strongly convex and $L_l$-smooth on $\mathbb{R}^{m \times d}$, where $\mu_l = \min \mu_i$, $L_l = \max L_i$.
- (global constants) $F(X)$ is $\mu_g$-strongly convex and $L_g$-smooth on $C$, where $\mu_g = \frac{1}{m} \sum_{i=1}^m \mu_i$, $L_g = \frac{1}{m} \sum_{i=1}^m L_i$.

Note that local smoothness and convexity constants may be significantly worse than global, i.e. $L_l \gg L_g$, $\mu_l \ll \mu_g$ (see [103] for details). We denote
\[
\kappa_l = \frac{L_l}{\mu_l}, \quad \kappa_g = \frac{L_g}{\mu_g}.
\]

the local and global condition numbers, respectively.

### 2.3 Distributed gradient descent

Distributed gradient methods alternate taking optimization updates and information exchange steps. One (synchronized) communication round can be represented as a multiplication by a mixing matrix compatible with the graph topology. One of the first distributed gradient dynamics studied in the literature [86, 133] uses a time-static mixing matrix and writes as
\[
x_{t+1}^k = \sum_{j=1}^m [M]_{ij} x_j^k - \alpha^k \nabla f_i(x_j^k).
\]

Using the notion of $X = (x_1 \ldots x_m)\top$ the above update rule takes the form
\[
X^{k+1} = MX^k - \alpha \nabla F(X^k),
\]

which is a combination of two step types: gradient step with constant step-size $\alpha$ and communication round with mixing matrix $M$. In [133] the authors showed that function residual $f(X^k) - f(x^*)$ in iterative scheme (8) decreases at $O(1/k)$ rate until reaching $O(\alpha)$-neighborhood of solution.

Method (6) does not find an exact solution in the general case. We follow the arguments in [111] to illustrate this fact. First, note that $\tilde{X}$ is a solution of (6) if and only if two following conditions hold.

1. (Consensus) $\tilde{X} = MX\tilde{X}$.
2. (Optimality) $1\top \nabla F(\tilde{X}) = 0$.

Let $X^\infty$ be a limit point of (8). Then
\[
X^\infty = MX^\infty - \alpha \nabla F(X^\infty).
\]

Consensus condition yields $X^\infty = MX^\infty$, i.e. $X^\infty$ has identical rows $[X]^\infty = x^\infty$. Therefore, $\nabla F(X^\infty) = 0$, which means $\nabla f_1(x^\infty) = \ldots = \nabla f_m(x^\infty) = 0$. Consequently, $x^\infty$ is a common minimizer of every $f_i$, which is not a realistic case.
2.4 EXTRA

Distributed gradient descent [8] is unable to converge to the exact minimum of (2), which is the major drawback of the method. An exact decentralized first-order algorithm EXTRA was proposed in [111]. The approach of [111] is based on using two different mixing matrices. Namely, consider two consequent updates of type (8).

\[ X^{k+2} = MX^{k+1} - \alpha \nabla F(X^{k+1}), \]
\[ X^{k+1} = \tilde{M}X^k - \alpha \nabla F(X^k) \]

where \( \tilde{M} \) is mixing matrix, i.e. \( \tilde{M} = (M + I)/2 \) as proposed in [111]. Subtracting (10) from (9) yields

\[ X^{k+2} - X^{k+1} = MX^{k+1} - \tilde{M}X^k - \alpha \left[ \nabla F(X^{k+1}) - \nabla F(X^k) \right] \]

thus leading to an algorithm

**Algorithm 1 EXTRA**

Input: Step-size \( \alpha > 0 \),
\( X^0 = MX^0 - \alpha \nabla F(X^0) \)
for \( k = 0, 1, \ldots \) do
\( X^{k+2} = (I + M)X^{k+1} - \tilde{M}X^k - \alpha \left[ \nabla F(X^{k+1}) - \nabla F(X^k) \right] \)
end for

Let \( X^\infty \) be a limit point of iterate sequence \( \{X^k\}_{k=0}^\infty \) generated by (9), (10). Then

\[ X^\infty - X^\infty = MX^\infty - \tilde{M}X^\infty - \alpha \left[ \nabla F(X^\infty) - \nabla F(X^\infty) \right], \]
\[ (M - \tilde{M})X^\infty = \frac{1}{2}(MX^\infty - X^\infty) = 0. \]

The last equality means that \( X^\infty \) is consensual, i.e. its rows are equal. On the other hand, rearranging the terms in (11) and taking into account that \( X^1 = MX^0 - \alpha \nabla F(X^0) \) gives

\[ X^{k+2} = \tilde{M}X^{k+1} - \alpha \nabla F(X^{k+1}) + \sum_{i=0}^{k+1} (M - \tilde{M})X^i. \]

Multiplying by \( 1^T \) from the left yields

\[ 1^T X^{k+2} = 1^T \tilde{M}X^{k+1} - 1^T \nabla F(X^{k+1}) + \sum_{i=0}^{k+1} 1^T (M - \tilde{M})X^i \]
\[ = 1^T \tilde{M}X^{k+1} - 1^T \nabla F(X^{k+1}) \]

and taking the limit over \( k \to \infty \) we obtain

\[ 1^T \nabla F(X^\infty) = 0 \]

which is the optimality condition for point \( X^\infty \). Therefore, a limit point of \( \{X^k\}_{k=0}^\infty \) generated by Algorithm 1 is both consensual and optimal, i.e. is a solution of (2).

In the original paper [111] Algorithm 1 was proved to converge at a \( O(1/k) \) rate for \( L \)-smooth objectives and achieve a geometric rate \( O(C^{-k}) \) (where \( C < 1 \) is some constant) for strongly convex smooth objectives. In [76] explicit dependencies on graph topology were established. Namely, EXTRA requires

Method [8] is a basic distributed first-order method. Its different variations include feasible point algorithms [74] and sub-gradient methods [86] (actually, the latter work initially proposed scheme [8]). Extensions to stochastic objectives and stochastic mixing matrices have been addressed in [60][1].
Communication complexities have been studied in [103]. On the other hand, lower complexity bound for deterministic methods over strongly convex smooth objectives is \( \Omega(\sqrt{\kappa \chi}) \) bounds, i.e. \( \sqrt{\kappa \chi} \) is an important direction of research in distributed optimization. This can be done by applying direct Nesterov acceleration [88] or by employing meta-acceleration techniques such as Catalyst [79]. The two major approaches studied in the literature are primal and dual algorithms.

Dual methods are based on a reformulation of problem (1) using a Laplacian matrix induced by the communication network. This reformulation is discussed in Section 3.3 in more details. The basic idea behind dual approach is to run first-order methods on a dual problem to (2). Every gradient step on the dual is equivalent to one communication round and one local gradient step taken by every node in the network. Accelerated algorithms achieving first-order methods on a dual problem to (2). Every gradient step on the dual is equivalent to one communication round and one local gradient step taken by every node in the network. Accelerated algorithms achieving \( O(\sqrt{\kappa \chi}) \) communication complexities have been studied in [103]. On the other hand, lower complexity bound for deterministic methods over strongly convex smooth objectives is \( \Omega(\sqrt{\kappa \chi \log(1/\epsilon)}) \), as shown in [103].

In dual approach, one may run non-distributed accelerated schemes on dual problem and obtain accelerated complexity bounds, i.e. \( \sqrt{\kappa \chi \log(1/\epsilon)} \). For primal-only methods this is not the case, and primal algorithms have to alternate optimization and communication steps to achieve accuracy \( \epsilon \), which does not match optimal bounds. EXTRA acceleration via Catalyst envelope [76] requires \( O((\sqrt{\kappa \chi \log(1/\epsilon)}) \) iterations for smooth strongly convex objectives. Recently a new method Mudag which unifies gradient tracking, Nesterov acceleration and multi-step consensus procedures was proposed in [130]. It enjoys

\[
O\left(\frac{L_i}{\mu_i} + \chi\right) \log \frac{\chi}{\epsilon}
\]

iterations for strongly convex smooth objectives,

\[
O\left(\frac{L_i}{\epsilon} + \chi\right) \log \chi
\]

iterations for strongly convex smooth objectives,

where

\[
\chi = \frac{1}{1 - \lambda_2(M)}
\]

(12)

and \( \lambda_2(M) \) denotes the second largest eigenvalue of mixing matrix \( M \). The term \( \chi \) characterizes graph connectivity. A similar term, also referred to as graph condition number, is used in Section 3.3 for graph Laplacian matrix. Graph condition numbers based on mixing matrix and Laplacian have the same meaning, as discussed in Section 2.1.

2.5 Accelerated decentralized algorithms

Performance of decentralized gradient methods typically depends on function (local or global) condition number \( \kappa \) and graph condition number \( \chi \) defined in (12). For non-accelerated dynamics [111] complexity bounds include \( \kappa \) and \( \chi \). Improving dependencies to \( \sqrt{\kappa} \) and \( \sqrt{\chi} \) is an important direction of research in distributed optimization. Double acceleration via Catalyst [79] requires \( \kappa \chi \sqrt{\log(1/\epsilon)} \) for smooth strongly convex objectives. Recently a new method Mudag which unifies gradient tracking, Nesterov acceleration and multi-step consensus procedures was proposed in [130].

Algorithm 2 Accelerated Distributed Nesterov Method

Input: Starting points \( x^0 = y^0 = v^0, S^0 = \nabla F(x^0) \), step-size \( \eta > 0 \), momentum term \( \alpha = \sqrt{\mu_i \eta} \)

for \( k = 0, 1, \ldots \), do

\[ x^{k+1} = \nabla y^k - \eta S^k \]

\[ v^{k+1} = (1 - \alpha |M|v^k + \alpha M y^k - \frac{\eta}{\mu_i} S^k \]

\[ y^{k+1} = \frac{x^{k+1} + \alpha v^{k+1}}{1 + \alpha} \]

\[ S^{k+1} = MS' + \nabla F(y^{k+1}) - \nabla F(y^{k}) \]

end for

In Algorithm 2 quantity \( S^{k+1} \) stands for a gradient estimator. The information about the gradients held by different agents is diffused through the network via consensus steps, i.e. \( MS' \) multiplication. Every node stores one row of \( S^k \) which approximates the average gradient over the nodes in network:

\[ s_k^i \approx \frac{1}{m} \sum_{k=1}^{m} \nabla f_i(y_i^k). \]

This technique is referred to as gradient tracking and is employed in several primal decentralized methods [96, 85, 130].

Algorithm 2 requires \( O((\sqrt{\kappa \chi \log(1/\epsilon)}) \) computation and communication steps to achieve accuracy \( \epsilon \), which does not match optimal bounds. EXTRA acceleration via Catalyst envelope [76] requires \( O((\sqrt{\kappa \chi \log(1/\epsilon)}) \) iterations for smooth strongly convex objectives. Recently a new method Mudag which unifies gradient tracking, Nesterov acceleration and multi-step consensus procedures was proposed in [130].
Recent theoretical advances in decentralized distributed convex optimization

Mudag reaches optimal computation complexity and optimal communication complexity up to \( \log \left( \frac{Lg}{\kappa g} \right) \) term. A valuable feature of the method is that it has dependencies on global condition number \( \kappa_l \) instead of local \( \kappa_g \). In the general case, global condition number may be significantly better. A proximal version of Mudag method for composite optimization is studied in [131]. The method in [131] requires an optimal \( O\left( \sqrt{\kappa g} \log \left( \frac{1}{\epsilon} \right) \right) \) number of computations and matches the lower communication complexity bound up to a logarithmic factor. Global condition number is also utilized in paper [102] where an inexact oracle framework [20, 21] for decentralized optimization is studied. The latter work is discussed in more details in Section 2.7. Finally, in [61] authors proposed a primal-only method which reaches both optimal computation and communication complexities (up to replacing \( \kappa_g \) with \( \kappa_l \)).

2.6 Time-varying networks

In the time-varying case, the communication network changes from time to time. In practice this changes are typically caused by malfunctions such as loss of connection between to agents. The network is represented as a sequence of undirected communication graphs \( \{ \mathcal{G}^k = (V, E^k) \}_{k=0}^{\infty} \) and every graph \( \mathcal{G}^k \) is associated with a mixing matrix \( M^k \). The algorithms capable of working over time-varying graphs must be robust to sudden network changes. A linearly convergent method DIGing was proposed in [85].

**Algorithm 3 DIYing**

**Input:** Step-size \( \alpha > 0 \), starting iterate \( X^0, Y^0 = \nabla F(X^0) \)

for \( k = 0, 1, \ldots \) do

\[
X^{k+1} = M^k X^k - \alpha Y^k
\]

\[
Y^{k+1} = M^k Y^k + \nabla F(X^{k+1}) - \nabla F(X^k)
\]

end for

DIGing incorporates a gradient-tracking scheme and achieves linear convergence under realistic assumptions such as \( B \)-connectivity (i.e. a union of any \( B \) consequent graphs is connected). In [117] authors propose an algorithm which utilizes specific convex surrogates of local functions and local functions similarity in order to enhance convergence speed.

2.7 Inexact oracle point of view

In [102] the authors study an algorithm which alternates making gradient updates and running multi-step communication procedures. Introduce

\[
\bar{X} = \frac{1}{m} \mathbf{1} \bar{X} = \Pi_C(X) = (\bar{x} \ldots \bar{x})^\top, \text{ where } \bar{x} = \frac{1}{m} \sum_{i=1}^{m} x_i \text{ and } \mathbf{1} = (1 \ldots 1)^\top.
\]

Also define an average gradient over nodes \( \nabla F(X) = 1/m \sum_{i=1}^{m} \nabla f_i(x_i) \). Consider a projection gradient method with trajectory lying in \( \mathbf{C} \)

\[
\bar{X}_{k+1} = \bar{X}_k - \beta \nabla F(\bar{X}_k).
\]

In a centralized scenario, the computational network is endowed with a master agent, which communicates with all agents in the network. The master node is able to collect vectors \( x_i \) from every node in the network and compute a precise average \( \bar{x} \). In decentralized case the master agent is not available, and therefore nodes are only able to compute an approximate average using consensus procedures.

**Assumption 4** Mixing matrix sequence \( \{ M^k \}_{k=0}^{\infty} \) satisfies the following properties.
(Decentralized property) If \((i, j) \notin E_k\), then \(M^k_{ij} = 0\).

(Double stochasticity) \(M^k \mathbf{1} = \mathbf{1}, \mathbf{1}^\top M^k = \mathbf{1}^\top\).

(Contraction property) There exist \(\tau \in \mathbb{Z}_{++}\) and \(\lambda \in (0, 1)\) such that for every \(k \geq \tau - 1\) it holds

\[
\left\| M^k X - \bar{X} \right\|_2 \leq (1 - \lambda) \left\| X - \bar{X} \right\|_2 ,
\]

where \(M^k = M^k \cdots M^{k-\tau+1}\).

The contraction property in the above assumption allows to ensure that consensus procedure converges to the average of agent’s vectors over the network.

---

**Algorithm 4 Consensus**

**Input:** Initial \(X^0 \in \mathcal{C}\), number of iterations \(T\).

```plaintext
for \(t = 1, \ldots, T\) do
    \(X^{t+1} = M^t X^t\)
end for
```

---

**Algorithm 5 Decentralized AGD with consensus subroutine**

**Input:** Initial guess \(X^0 \in \mathcal{C}\), constants \(L, \mu > 0\), \(U^0 = X^0\), \(\alpha^0 = M^0 = 0\)

1: for \(k = 0, 1, 2, \ldots\) do
2:   Find \(\alpha^{k+1}\) as the greatest root of \((A^k + \alpha^{k+1})(1 + A^k \mu) = L(\alpha^{k+1})^2\)
3:   \(A^{k+1} = A^k + \alpha^{k+1}\)
4:   \(Y^{k+1} = \alpha^{k+1} U^k + A^k X^k\)
5:   \(V^{k+1} = \mu Y^{k+1} + (1 + A^k \mu) U^k - \frac{\alpha^{k+1}}{1 + A^k \mu + \mu} \nabla F(Y^{k+1})\)
6:   \(U^{k+1} = \text{Consensus}(V^{k+1}, T^k)\)
7:   \(X^{k+1} = \frac{\alpha^{k+1} U^{k+1} + A^k X^k}{A^{k+1}}\)
8: end for

---

### 2.7.1 Inexact oracle construction

Trajectory of Algorithm 5 lies in the neighborhood of constraint set \(\mathcal{C}\). It is analyzed in [102] using the notation of inexact oracle [21, 20]. Algorithms of this type have been analyzed in time-static case [51] using inexact oracle notation, as well. Let \(h(x)\) be a convex function defined on a convex set \(Q \subseteq \mathbb{R}^m\). For \(\delta > 0\), \(L > \mu > 0\), a pair \((h_{\delta, L, \mu}(x), x_{\delta, L, \mu}(x))\) is called a \((\delta, L, \mu)\)-model of \(h(x)\) at point \(x \in Q\) i.e. if for all \(y \in Q\) it holds

\[
\frac{\mu}{2} \left\| y - x \right\|_2^2 \leq h(y) - \left(h_{\delta, L, \mu}(x) + \left\langle x_{\delta, L, \mu}(x), y - x \right\rangle \right) \leq \frac{L}{2} \left\| y - x \right\|_2^2 + \delta .
\]  

(13)

The inexactness originates from computation of gradient at a point in neighborhood of \(\mathcal{C}\). The next lemma identifies the size of neighborhood and describes the inexact oracle construction.

**Lemma 2.** Define

\[
\delta = \frac{1}{2n} \left( \frac{L_f^2}{L_g} + \frac{2L_f^2}{\mu_g} + L_f - \mu_f \right) s',
\]

\[
f_{\delta, L, \mu}(\bar{x}, X) = \frac{1}{n} \left[ F(X) + \left\langle \nabla F(X), \bar{X} - X \right\rangle + \frac{1}{2} \left( \mu_f - \frac{2L_f^2}{\mu_g} \right) \left\| \bar{X} - X \right\|_2^2 \right],
\]

\[
x_{\delta, L, \mu}(\bar{x}, X) = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(x_i).
\]

Then \((f_{\delta, L, \mu}(\bar{x}, X), x_{\delta, L, \mu}(\bar{x}, X))\) is a \((\delta, 2L_f, \mu_f / 2)\)-model of \(f\) at point \(\bar{x}\), i.e.
In order to achieve this, provided that consensus accuracy is maintained at level $\delta'$, i.e. $\|U^j - \overline{U}\|^2 \leq \delta' \text{ for } j = 1, \ldots, k$ and let Assumption 4 hold. Define

$$\sqrt{D} := \left( \frac{2L_i}{\sqrt{L_i \mu}} + 1 \right) \sqrt{\delta' L_i \mu} \sqrt{n} \left( \frac{8\delta'}{\sqrt{L_i \mu}} \right)^{1/2} + \frac{2 \|\nabla F(x^*)\|_2}{\sqrt{L_i \mu}}.$$

Then it is sufficient to make $T_k = T = \frac{\tau}{2 \lambda} \log \frac{D}{\delta}$ consensus iterations in order to obtain consensus with $\delta'$-accuracy on step $k + 1$, i.e. $\|U^{k+1} - \overline{U}^{k+1}\|^2 \leq \delta'$.

A basis for the proof of Lemma 3 is a contraction property of mixing matrix sequence $\{M^k\}_{k=0}^\infty$ (see Assumption 4).

Second, provided that projection accuracy on every step of Algorithm 5 is sustained at level $\delta'$, the algorithm turns into an accelerated scheme with inexactness. Its convergence is given by the following

**Lemma 4.** Provided that consensus accuracy is $\delta'$, i.e. $\|U^j - \overline{U}^j\|^2 \leq \delta' \text{ for } j = 1, \ldots, k$, we have

$$f(\overline{x}^j) - f(x^*) \leq \frac{\|\overline{x}^j - x^*\|^2}{2A^6} + \frac{2 \sum_{j=1}^k A^j \delta}{A^6}$$

$$\|\overline{x}^j - x^*\|^2 \leq \frac{\|\overline{x}^j - x^*\|^2}{1 + A^j \mu} + \frac{4 \sum_{j=1}^k A^j \delta}{1 + A^j \mu}$$

where $\delta$ is given in (13).

Finally, putting Lemmas 3 and 4 together yields a convergence result for Algorithm 5.

**Theorem 5.** Choose some $\varepsilon > 0$ and set

$$T_k = T = \frac{\tau}{2 \lambda} \log \frac{D}{\delta'}, \delta' = \frac{nL_i \mu_g}{32 L_i L_g^{1/2}}.$$

Also define

$$D_1 = \frac{L_i}{L_g^{1/2} \mu_g} \left[ 8\sqrt{2L_i} \|\overline{x}^j - x^*\|^2 \left( \frac{L_g}{\mu_g} \right)^{3/4} + \frac{4 \sqrt{2} \|\nabla F(x^*)\|_2}{\sqrt{n}} \left( \frac{L_g}{\mu_g} \right)^{1/4} \right]$$

$$D_2 = \frac{L_i}{L_g^{1/2} \mu_g} \left[ 3\sqrt{2} + 4 \sqrt{2n} \left( \frac{L_g}{\mu_g} \right)^{1/4} \right]$$

Then Algorithm 5 requires

$$N = 2 \sqrt{\frac{L_g}{\mu_g} \log \left( \frac{\|\overline{x}^j - x^*\|^2}{2\varepsilon L_g} \right)} \leq \frac{D}{\delta'}$$

for gradient computations at each node and
such that

$$f(X^N) - f(x^*) \leq \epsilon, \quad \|X^N - \bar{X}^N\|_2 \leq \delta'.$$

In the time-static case, contraction term $\tau/\lambda$ turns into $\chi(M)$, and an accelerated consensus procedure of type $O\left(\frac{L_g}{\mu_g} \sqrt{\chi(M)} \log^2\left(\frac{1}{\epsilon}\right)\right)$ which is optimal up to a logarithmic term. Similar results are attained in works which use penalty-based methods [25, 101, 39] (see Appendix B in [39]) for details.

3 Convex Problems with Affine Constraints

In this section[1] we consider convex optimization problem with affine constraints

$$\min_{Ax=0, x \in Q} f(x),$$

where $A \succeq 0$, $\text{Ker}A \neq \{0\}$ and $Q$ is a closed convex subset of $\mathbb{R}^n$. Up to a sign, the dual problem is defined as follows:

$$\min_y \psi(y), \quad \text{where}$$

$$\varphi(y) = \max_{x \in Q} \{y, x - f(x)\},$$

$$\psi(y) = \varphi(A^\top y) = \max_{x \in Q} \{y, Ax - f(x)\} = \langle A^\top y, x(A^\top y)\rangle - f(x(A^\top y)),$$

where $x(y) \overset{\text{def}}{=} \arg\max_{x \in Q} \{y, x - f(x)\}$. Since $\text{Ker}A \neq \{0\}$ the solution of the dual problem [19] is not unique. We use $y^*$ to denote the solution of [19] with the smallest $\ell_2$-norm $R_y \overset{\text{def}}{=} \|y^*\|_2$.

3.1 Primal Approach

In this section, we focus on primal approaches to solve (18) and, in particular, the main goal of this section is to present first-order methods that are optimal both in terms of $\nabla f(x)$ and $A^\top Ax$ calculations. One can apply the following trick [23, 39, 39] to solve problem (18): instead of (18) one can solve penalized problem

$$\min_{x \in Q} \left\{ F(x) = f(x) + \frac{R_y^2}{\epsilon} \|Ax\|_2^2 \right\},$$

where $\epsilon > 0$ is the desired accuracy of the solution in terms of $f(x)$ that we want to achieve (see the details in [39]).

Next, we assume that $f$ is $\mu$-strongly convex, but possibly non-smooth function with bounded (sub) gradients: $\|\nabla f(x)\|_2 \leq M$ for all $x \in Q$. In this setting, one can apply Sliding algorithm from [68, 67] to get optimal rates of convergence. The method is presented as Algorithm [5] and it is aimed to solve the following problem:

$$\min_{x \in Q} \left\{ \Psi(x) = h(x) + f(x) \right\},$$

where $h(x)$ is convex and $L$-smooth, $f(x)$ is convex, but can be non-smooth, and $x^*$ is an arbitrary solution of the problem. In this case, it is additionally assumed that $f(x)$ has uniformly bounded subgradients: there exists non-negative constant $M$ such that $\|\nabla f(x)\|_2 \leq M$ for all $x \in Q$ and all subgradients at this point $\nabla f(x) \in \partial f(x)$.

---

[1] The narrative in this section follows [39].

[2] For the sake of simplicity, we slightly abuse the notation and denote gradients and subgradients similarly.
Algorithm 6 Sliding Algorithm [68, 67]

**Input:** Initial point $x_0 \in \mathcal{O}$ and iteration limit $N$.
Let $\beta_k, \gamma_k \in \mathbb{R}^+$, and $T_k \in \mathbb{N}$, $k = 1, 2, \ldots$, be given and set $\tau_0 = x_0$.

for $k = 1, 2, \ldots, N$ do

1. Set $x_k = (1 - \gamma_k)x_k - 1 + \gamma_kx_{k-1}$, and let $h_k(\cdot) \equiv h_k(x_k, \cdot)$, where $h_k(x, y) = h(x) + (\nabla h(x), y - x)$.

2. Set $(x_k, 0) = \text{PS}(h_k, x_k, \beta_k, T_k)$.

3. Set $\tau_k = (1 - \gamma_k)x_k - 1 + \gamma_kx_k$.

end for

The **PS** (prox-sliding) procedure.

procedure: $(x^*, \hat{x}^*) = \text{PS}(g, x, \beta, T)$

Let the parameters $p_t \in \mathbb{R}^+$ and $\theta_t \in [0, 1], t = 1, \ldots,$ be given. Set $u_0 = \bar{u}_0 = x$.

for $t = 1, 2, \ldots, T$ do

$$u_t = \text{argmin}_{w \in \mathcal{Q}} \left\{ g(w) + l_f(w_{t-1}, u) + \frac{\beta}{2} \| w - x \|^2 + \frac{\beta p_t}{2} \| u - u_{t-1} \|^2 \right\},$$

$$u_t = (1 - \theta_t)u_{t-1} + \theta_t u_t,$$

where $l_f(x, y) = f(x) + (\nabla f(x), y - x)$.

end for

Set $x^* = u_T$ and $\hat{x}^* = \bar{u}_T$.

end procedure:

The key property of Algorithm 6 is its ability to separate oracle complexities for smooth and non-smooth parts of the objective. That is, to find such $\hat{x}$ that $\Psi(\hat{x}) - \Psi(x^*) \leq \varepsilon$. Sliding requires

$$O\left(\sqrt{\frac{LR^2}{\varepsilon}}\right)$$
calculations of $\nabla h(x)$

and

$$O\left(\frac{M^2R^2}{\varepsilon^2} + \sqrt{\frac{LR^2}{\varepsilon}}\right)$$
calculations of $\nabla f(x)$,

where $R = \|x^0 - x^*\|_2$.

Now, we go back to the problem (22) and consider the case when $\mu = 0$. In these settings, to find $\hat{x}$ such that

$$F(\hat{x}) - F(x^*) \leq \varepsilon$$

one can run Algorithm 6 considering $f(x)$ as the non-smooth term and $\frac{R^2}{\varepsilon} \| Ax \|^2_2$ as the smooth one. In this case, Sliding requires

$$O\left(\sqrt{\frac{\lambda_{\max}(A^TA)R^2R^2}{\varepsilon^2}}\right)$$
calculations of $A^T Ax$, \hspace{1cm} (25)

and

$$O\left(\frac{M^2R^2}{\varepsilon^2}\right)$$
calculations of $\nabla f(x)$. \hspace{1cm} (26)

Next, we consider the situation when $Q$ is a compact set, $\nabla f(x)$ is not available, and unbiased stochastic gradient $\nabla f(x, \xi)$ is used instead:

$$\| E[\nabla f(x, \xi)] - \nabla f(x) \|_2 \leq \delta,$$

$$\left| E[\exp\left(\frac{\| \nabla f(x, \xi) - E[\nabla f(x, \xi)] \|^2}{\sigma^2}\right)\right| \leq \exp(1),$$

where $\delta \geq 0$ and $\sigma \geq 0$. When $\delta = 0$, i.e., stochastic gradients are unbiased, one can show [68, 67] that Stochastic Sliding (S-Sliding) method can achieve (24) with probability at least $1 - \beta$, $\beta \in (0, 1)$ requiring the same number of calculations of $A^T Ax$ as in (25) up to logarithmic factors and
calculations of $\nabla f(x, \xi)$. \hfill (29)

When $\mu > 0$ one can apply restarts technique for $\text{S-Sliding}$ and get the method ($\text{RS-Sliding}$) \cite{23, 122} that guarantees \cite{24} with probability at least $1 - \beta$, $\beta \in (0, 1)$ using

$$
\tilde{O} \left( \frac{(M^2 + \sigma^2)R^2}{\epsilon^2} \right) \text{calculations of } A^\top Ax,
$$

\hfill (30)

$$
\tilde{O} \left( \frac{M^2 + \sigma^2}{\mu \epsilon} \right) \text{calculations of } \nabla f(x, \xi).
$$
\hfill (31)

We notice that bounds presented above for the non-smooth case are proved when $Q$ is bounded. For the case of unbounded $Q$ the convergence results with such rates were established only in expectation. Moreover, it would be interesting to study $\text{S-Sliding}$ and $\text{RS-Sliding}$ in the case when $\delta > 0$, i.e., stochastic gradient is biased.

### 3.2 Dual Approach

In this section, we assume that one can construct a dual problem for \cite{15}. If $f$ is $\mu$-strongly convex in $\ell_2$-norm, then $\psi$ and $\varphi$ have $L_\psi$-Lipschitz continuous and $L_\varphi$-Lipschitz continuous in $\ell_2$-norm gradients respectively \cite{56, 99}, where $L_\psi = \lambda_{\max}(A^\top A)/\mu$ and $L_\varphi = 1/\mu$. In our proofs, we often use Demyanov–Danskin theorem \cite{99} which states that

$$
\nabla \psi(y) = A\nabla x(y), \quad \nabla \varphi(y) = x(y).
$$
\hfill (32)

Moreover, we do not assume that $A$ is symmetric or positive semidefinite.

Below we propose a primal-dual method for the case when $f$ is additionally Lipschitz continuous on some ball and two methods for the problems when the primal function is also $L$-smooth and Lipschitz continuous on some ball. In the subsections below, we assume that $Q = \mathbb{R}^n$. The formal proofs of the presented results are given in \cite{39}.

#### 3.2.1 Convex Dual Function

In this section, we assume that the dual function $\varphi(y)$ could be rewritten as an expectation, i.e., $\varphi(y) = \mathbb{E}_\xi [\varphi(y, \xi)]$, where stochastic realizations $\varphi(y, \xi)$ are differentiable in $y$ functions almost surely in $\xi$. Then, we can also represent $\psi(y)$ as an expectation: $\psi(y) = \mathbb{E}_\xi [\psi(y, \xi)]$. Consider the stochastic function $f(x, \xi)$ which is defined implicitly as follows:

$$
\varphi(y, \xi) = \max_{x \in \mathbb{R}^n} \{ \langle y, x \rangle - f(x, \xi) \}.
$$
\hfill (33)

Similarly to the deterministic case, we introduce $x(y, \xi) \overset{\text{def}}{=} \arg \max_{x \in \mathbb{R}^n} \{ \langle y, x \rangle - f(x, \xi) \}$ which satisfies $\nabla \varphi(y, \xi) = x(y, \xi)$ due to Demyanov–Danskin theorem, where the gradient is taken w.r.t. $y$. As a simple corollary, we get $\nabla \psi(y, \xi) = \nabla x(y, \xi)$. Finally, introduced notations and obtained relations imply that $x(y) = \mathbb{E}_\xi [x(y, \xi)]$ and $\nabla \psi(y) = \mathbb{E}_\xi [\nabla \psi(y, \xi)]$.

Consider the situation when $x(y, \xi)$ is known only through the noisy observations $\tilde{x}(y, \xi) = x(y, \xi) + \delta(y, \xi)$ and assume that the noise is bounded in expectation, i.e., there exists non-negative deterministic constant $\delta_y \geq 0$, such that

$$
\| \mathbb{E}_\xi [\delta(y, \xi)] \|_2 \leq \delta_y, \quad \forall y \in \mathbb{R}^n.
$$
\hfill (34)

Assume additionally that $\tilde{x}(y, \xi)$ satisfies so-called “light-tails” inequality:

$$
\mathbb{E}_\xi \left[ \exp \left( \frac{\| \tilde{x}(y, \xi) - \mathbb{E}_\xi [\tilde{x}(y, \xi)] \|^2}{\sigma^2_\xi} \right) \right] \leq \exp(1), \quad \forall y \in \mathbb{R}^n,
$$
\hfill (35)

where $\sigma_\xi$ is some positive constant. It implies that we have an access to the biased stochastic gradient $\tilde{\nabla} \psi(y, \xi) \overset{\text{def}}{=} A\tilde{x}(y, \xi)$ which satisfies following relations:
Algorithm 7 SPDSTM

**Input:** $\tilde{y}^0 = y^0 = 0$, number of iterations $N$, $\alpha_0 = A_0 = 0$

1. for $k = 0, \ldots, N$ do
2. Set $L = 2L$
3. Set $A_{k+1} = A_k + A_{k+1}$, where $2L\alpha_k^2 = A_k + A_{k+1}$
4. $F_k = (A_k + \alpha_k^2)/A_{k+1}$
5. $x_k = x_{k-1} - \alpha_k \tilde{y}(x_{k-1}, x_k)$
6. $\tilde{x}_k = (A_k + \alpha_k^2)/A_{k+1}$
7. end for

**Output:** $\tilde{y}^N, \tilde{x}^N = \frac{1}{N} \sum_{k=0}^{N} \alpha_k \tilde{x}(A^T \tilde{y}^k, \xi^k)$.

Below we present the main convergence result of this section.

**Theorem 6 (Theorem 5.1 from [39]).** Assume that $f$ is $\mu$-strongly convex and $\|\nabla f(x^*)\|_2 = M_f$. Let $\varepsilon > 0$ be a desired accuracy. Next, assume that $f$ is $L_f$-Lipschitz continuous on the ball $B_{R_f}(0)$ with

$$R_f = \tilde{\Omega} \left( \max \left\{ \frac{R_y}{A_N \sqrt{\lambda_{\text{max}}(A^T A)}}, \frac{\sqrt{\lambda_{\text{max}}(A^T A)} R_y}{\mu}, R_A \right\} \right),$$

where $R_A$ is such that $\|y^N\|_2 \leq R_A$, $y^N$ is the solution of the dual problem (19), and $R_y = \|x(A^T y^*)\|_2$. Assume that at iteration $k$ of Algorithm 2 batch size is chosen according to the formula $r_k \geq \max \left\{ 1, \frac{\alpha_k^2 \alpha_k \ln(N/\beta)}{\varepsilon \mu} \right\}$, where $\alpha_k = \frac{k+1}{2L}$, $0 < \varepsilon \leq \frac{HL\alpha_k^2}{N}$, $0 \leq \delta \leq \frac{G_0 R_0}{(N+1)}$, and $N \geq 1$ for some numeric constant $H > 0$, $G > 0$ and $\hat{C} > 0$. Then with probability $\geq 1 - 4\beta$, where $\beta \in (0, 1/8)$, after $N = \tilde{O} \left( \frac{M_f}{\mu \varepsilon \chi(A^T A)} \right)$ iterations where $\chi(A^T A) = \frac{\lambda_{\text{max}}(A^T A)}{\lambda_{\text{min}}(A^T A)}$ the outputs $\tilde{x}^N$ and $y^N$ of Algorithm 7 satisfy the following condition

$$f(x^N) - f(x^\dagger) \leq \psi(y^N) \leq \varepsilon, \quad \|Ax^N\|_2 \leq \frac{\varepsilon}{R_y}$$

with probability at least $1 - 4\beta$. What is more, to guarantee (41) with probability at least $1 - 4\beta$ Algorithm 7 requires
3.2.2 Strongly Convex Dual Functions and Restarts Technique

In this section, we assume that primal functional $f$ is additionally $L$-smooth. It implies that the dual function $\psi$ in (19) is additionally $\mu_\psi$-strongly convex in $y^0 + (\text{Ker} A^\top)^\perp$ where $\mu_\psi = \lambda_{\text{min}}(A^\top A)/L$.\cite{5,99} and $\lambda_{\text{min}}(A^\top A)$ is the minimal positive eigenvalue of $A^\top A$.

From weak duality $-f(x^*) \leq \psi(y^*)$ and (21) we get the key relation of this section (see also \cite{4,5,89})

$$f(x(A^\top y)) - f(x^*) \leq \langle \nabla \psi(y), y \rangle = \langle A x(A^\top y), y \rangle.$$  \hspace{1cm} (43)

This inequality implies the following theorem.

**Theorem 7 (Theorem 5.2 from \cite{39}).** Consider function $f$ and its dual function $\psi$ defined in (21) such that problems (18) and (19) have solutions. Assume that $y^N$ is such that $\|\nabla \psi(y^N)\|_2 \leq \epsilon/R_y$ and $y^N \leq 2R_y$, where $\epsilon > 0$ is some positive number and $R_y = \|y^*\|_2$ where $y^*$ is any minimizer of $\psi$. Then for $x^N = x(A^\top y^N)$ following relations hold:

$$f(x^N) - f(x^*) \leq 2\epsilon, \quad \|Ax^N\|_2 \leq \frac{\epsilon}{R_y},$$  \hspace{1cm} (44)

where $x^*$ is any minimizer of $f$.

That is why, in this section we mainly focus on the methods that provide optimal convergence rates for the gradient norm. In particular, we consider Recursive Regularization Meta-Algorithm from (see Algorithm\cite{32}) with AC-SA\(^2\) (see Algorithm\cite{10}) as a subroutine (i.e. RRMA-AC-SA\(^2\)) which is based on AC-SA algorithm (see Algorithm\cite{9}) from \cite{37}. We notice that RRMA-AC-SA\(^2\) is applied for a regularized dual function

$$\tilde{\psi}(y) = \psi(y) + \frac{\lambda}{2} ||y - y^0||_2^2,$$  \hspace{1cm} (45)

where $\lambda > 0$ is some positive number which will be defined further. Function $\tilde{\psi}$ is $\lambda$-strongly convex and $\bar{L}_\psi$-smooth in $\mathbb{R}^n$ where $\bar{L}_\psi = L_{\text{opt}} + \lambda$. For now, we just assume w.l.o.g. that $\tilde{\psi}$ is $(\mu_\psi + \lambda)$-strongly convex in $\mathbb{R}^n$, but we will go back to this question further.

In this section we consider the same oracle as in Section 3.2.1 but we additionally assume that $\delta = 0$, i.e., stochastic first-order oracle is unbiased. To define batched version of the stochastic gradient we will use the following notation:

$$\nabla \Phi(y, \xi^t, r_t) = \frac{1}{r_t} \sum_{i=1}^{r_t} \nabla \psi(y, \xi^i), \quad x(y, \xi^t, r_t) = \frac{1}{r_t} \sum_{i=1}^{r_t} x(y, \xi^i).$$  \hspace{1cm} (46)

As before, in the cases when the batch-size $r_t$ can be restored from the context, we will use simplified notation $\nabla \Phi(y, \xi^t)$ and $x(y, \xi^t)$.

**Algorithm 8 RRMA-AC-SA\(^2\)\cite{32}**

**Input:** $y^0$ — starting point, $m$ — total number of iterations

1: $\psi_0 \leftarrow \tilde{\psi}, \ y^0 \leftarrow y^0, \ T \leftarrow \lceil \log_2 \frac{\epsilon R_y}{\mu_\psi} \rceil$
2: for $k = 1, \ldots, T$ do
3: Run AC-SA\(^2\) for $m/r$ iterations to optimize $\psi_{k-1}$ with $y^{k-1}$ as a starting point and get the output $\hat{y}^k$
4: $\psi_k(y) \leftarrow \psi(y) + \lambda \sum_{l=1}^{k-1} 2^{l-1} ||y - y^l||_2^2$
5: end for
**Output:** $\hat{y}^T$.

In the AC-SA algorithm we use batched stochastic gradients of functions $\psi_k$ which are defined as follows:
Corollary 1 (Corollary 5.5 from [39]).
Assume that Algorithm 8 is run for the objective

\[ y \overset{\geq}{\leftarrow} \frac{1}{r} \sum_{l=1}^{r_i} \nabla \psi_k(y, \xi_l^k), \]

and

\[ \nabla \psi_k(y, \xi) = \nabla \psi(y, \xi) + \lambda(y - y^0) + \lambda \sum_{l=1}^{k} \gamma^l(y - y^l). \]

Theorem 9 (Theorem 5.4 from [39]).

Algorithm 9 AC-SA [37]

Input: \( y^0 \) — starting point, \( m \) — number of iterations, \( \psi_k \) — objective function

1: \( y^0_{ag} \leftarrow y^0, y^0_{ad} \leftarrow y^0 \)
2: for \( t = 1, \ldots, m \) do
3: \( \alpha_t \leftarrow \frac{2}{r+1}, y \leftarrow \frac{4y_{ag}}{r+1} \)
4: \( y^t_{ad} \leftarrow \frac{1-\alpha_t(y+y)}{y+(1-\alpha_t^2)y} y^t_{ad} - \alpha_t \nabla \psi_k(y^t_{ad}, \xi^t_l) \)
5: \( \hat{c} \leftarrow \alpha_t \hat{c} + \frac{1-\alpha_t}{y+y} \psi_k(y^t_{ad}, \xi^t_l) \)
6: \( y^t_{ag} \leftarrow \alpha_t \hat{c} + \frac{1-\alpha_t}{y+y} y^t_{ad} \)
7: end for
Output: \( y^m_{ag} \)

Algorithm 10 AC-SA\(^2\) [32]

Input: \( y^0 \) — starting point, \( m \) — number of iterations, \( \psi_k \) — objective function

1: Run AC-SA for \( m/2 \) iterations to optimize \( \psi \) with \( y^0 \) as a starting point and get the output \( y^1 \)
2: Run AC-SA for \( m/2 \) iterations to optimize \( \psi \) with \( y^1 \) as a starting point and get the output \( y^2 \)
Output: \( y^2 \)

The following theorem states the main result for RRMA-AC-SA\(^2\) that we need in the section.

Theorem 8 (Corollary 1 from [32]). Let \( \psi \) be \( L_{\psi} \)-smooth and \( \mu_{\psi} \)-strongly convex function and \( \lambda = \Theta \left( \frac{l_{\psi} \ln^2 N}{N^2} \right) \) for some \( N > 1 \). If the Algorithm [3] performs \( N \) iterations in total\(^3\) with batch size \( r \) for all iterations, then it will provide such a point \( \hat{y} \) that

\[ \mathbb{E} \left[ \| \nabla \psi(y) \|^2 | y^0, r \right] \leq C \left( \frac{L_{\psi}^2 \| y^0 - y^* \|^2 \ln^4 N}{N^4} + \frac{\sigma_{\psi}^2 \ln^6 N}{rN} \right), \] \[ (48) \]

where \( C > 0 \) is some positive constant and \( y^* \) is a solution of the dual problem [19].

The following result shows that w.l.o.g. we can assume that function \( \psi \) defined in (21) is \( \mu_{\psi} \)-strongly convex everywhere with \( \mu_{\psi} = \lambda_{\text{min}}(A^T A)/L \). In fact, from \( L \)-smoothness of \( f \) we have only that \( \psi \) is \( \mu_{\psi} \)-strongly convex in \( y^0 + (\text{Ker}(A^T))^\perp \) (see [66,99] for the details). However, the structure of the considered here methods is such that all points generated by the RRMA-AC-SA\(^2\) and, in particular, AC-SA lie in \( y^0 + (\text{Ker}(A^T))^\perp \).

Theorem 9 (Theorem 5.4 from [39]). Assume that Algorithm [3] is run for the objective \( \psi_k(y) = \tilde{\psi}(y) + \lambda \sum_{l=1}^{k} \gamma^l \| y - y^l \|^2 \) with \( y^0 \) as a starting point, where \( y^0, y^1, \ldots, y^k \) are some points from \( y^0 + (\text{Ker}(A^T))^\perp \) and \( y^0 \in \mathbb{R}^n \). Then for all \( t \geq 0 \) we have \( y^t_{ad}, y^t_{ag} \in y^0 + (\text{Ker}(A^T))^\perp \).

Corollary 1 (Corollary 5.5 from [39]). Assume that Algorithm [3] is run for the objective \( \psi_k(y) = \tilde{\psi}(y) + \lambda \sum_{l=1}^{k} \gamma^l \| y - y^l \|^2 \) with \( y^0 \) as a starting point. Then for all \( k \geq 0 \) we have \( y^k \in y^0 + (\text{Ker}(A^T))^\perp \).

Now we are ready to present our approach\(^4\) of constructing an accelerated method for the strongly convex dual problem using restarts of RRMA-AC-SA\(^2\). To explain the main idea we start with the simplest case: \( \sigma_{\psi}^2 = 0, r = 0 \). It means that there is no stochasticity in the method and the bound [48] can be rewritten in the following form:

\[ \frac{L_{\psi}^2 \| y^0 - y^* \|^2 \ln^4 N}{N^4} + \frac{\sigma_{\psi}^2 \ln^6 N}{rN} \]
\[ \| \nabla \psi(\hat{y}) \|_2 \leq \frac{\sqrt{CL_\psi} \| y^0 - y^* \|_2 \ln^2 N}{\mu_\psi N^2} \leq \frac{\sqrt{CL_\psi} \| \nabla \psi(y^0) \|_2 \ln^2 N}{\mu_\psi N^2}, \]  

where we used inequality \( \| \nabla \psi(y^0) \| \geq \mu_\psi \| y^0 - y^* \| \) which follows from the \( \mu_\psi \)-strong convexity of \( \psi \). It implies that after \( N = O(\sqrt{L_\psi/\mu_\psi}) \) iterations of \( \text{RRMA-AC-SA}^2 \) the method returns such \( \hat{y}^1 = \hat{y} \) that \( \| \nabla \psi(\hat{y}^1) \|_2 \leq \frac{1}{2} \| \nabla \psi(y^0) \|_2 \). Next, applying \( \text{RRMA-AC-SA}^2 \) with \( \hat{y}^1 \) as a starting point for the same number of iterations we will get new point \( \hat{y}^2 \) such that \( \| \nabla \psi(\hat{y}^2) \|_2 \leq \frac{1}{2} \| \nabla \psi(\hat{y}^1) \|_2 \leq \frac{1}{4} \| \nabla \psi(y^0) \|_2 \). Then, after \( l = O(\ln(\bar{r}_l/\|\nabla \psi(y^0)\|_2/\varepsilon)) \) of such restarts we can get the point \( \hat{y}^l \) such that \( \| \nabla \psi(\hat{y}^l) \|_2 \leq \varepsilon/\bar{r}_l \) with total number of gradient computations \( \bar{N}l = \tilde{O} \left( \frac{\sqrt{L_\psi/\mu_\psi} \ln(\bar{r}_l/\|\nabla \psi(y^0)\|_2/\varepsilon)}{\varepsilon} \right) \).

When \( \sigma^2_\psi \neq 0 \) we need to modify this approach. The first ingredient to handle the stochasticity is enough batch size for the \( l \)-th restart: \( r_l \) should be \( \Omega \left( \sigma^2_\psi/\|\nabla \psi(y^l-1)\|_2^2 \right) \). However, in the stochastic case we do not have an access to the \( \nabla \psi(y^l-1) \), so, such batch size is impractical. One possible way to fix this issue is to independently sample large enough number \( \tilde{r}_l \sim \mathcal{N}(\bar{r}_l/\varepsilon^2) \) of stochastic gradients additionally, which is the second ingredient of our approach, in order to get good enough approximation \( \nabla \psi(y^l-1, \xi^l-1, \tilde{r}_l) \) of \( \nabla \psi(y^l-1) \) and use the norm of such an approximation which is close to the norm of the true gradient with big enough probability in order to estimate needed batch size \( r_l \) for the optimization procedure. Using this, we can get the bound of the following form:

\[ \mathbb{E} \left[ \| \nabla \psi(\hat{y}^l) \|_2^2 \mid y^{l-1}, r_l, \tilde{r}_l \right] \leq A_l \quad \text{def} \quad \frac{\| \nabla \psi(y^{l-1}) \|_2^2}{8} + \frac{\| \nabla \psi(y^{l-1}, \xi^{l-1}, \tilde{r}_l) - \nabla \psi(y^{l-1}) \|_2^2}{32}, \]

The third ingredient is the amplification trick: we run \( p_l = \Omega \left( \ln(1/\beta) \right) \) independent trajectories of \( \text{RRMA-AC-SA}^2 \), get points \( y^{l,1}, \ldots, y^{l,p_l} \) and choose such \( y^{l,p(l)} \) among of them that \( \| \nabla \psi(y^{l,p(l)}) \|_2 \) is close enough to \( \min_{p=1,\ldots,p_l} \| \nabla \psi(y^{l,p}) \|_2 \) with high probability, i.e., \( \| \nabla \psi(y^{l,p(l)}) \|_2 \leq 2 \min_{p=1,\ldots,p_l} \| \nabla \psi(y^{l,p}) \|_2 + \varepsilon^2/8R_y^2 \) with probability at least \( 1 - \beta \) for fixed \( \nabla \psi(y^{l-1}, \xi^{l-1}, \tilde{r}_l) \). We achieve it due to additional sampling of \( \tilde{r}_l \sim \mathcal{N}(\bar{r}_l/\varepsilon^2) \) stochastic gradients at \( y^{l,p} \) for each trajectory and choosing such \( p(l) \) corresponding to the smallest norm of the obtained batched stochastic gradient. By Markov’s inequality for all \( p = 1, \ldots, p_l \)

\[ \mathbb{P} \left\{ \| \nabla \psi(y^{l,p}) \|_2^2 \geq 2A_l \mid y^{l-1}, r_l, \tilde{r}_l \right\} \leq \frac{1}{2}, \]

hence

\[ \mathbb{P} \left\{ \min_{p=1,\ldots,p_l} \| \nabla \psi(y^{l,p}) \|_2^2 \geq 2A_l \mid y^{l-1}, r_l, \tilde{r}_l \right\} \leq \frac{1}{2p_l}. \]

That is, for \( p_l = \log_2(1/\beta) \) we have that with probability at least \( 1 - 2\beta \)

\[ \| \nabla \psi(y^{l,p(l)}) \|_2^2 \leq \frac{\| \nabla \psi(y^{l-1}) \|_2^2}{2} + \frac{\| \nabla \psi(y^{l-1}, \xi^{l-1}, \tilde{r}_l) - \nabla \psi(y^{l-1}) \|_2^2}{8} + \frac{\varepsilon^2}{8R_y^2}, \]

for fixed \( \nabla \psi(y^{l-1}, \xi^{l-1}, \tilde{r}_l) \) which means that

\[ \| \nabla \psi(y^{l,p(l)}) \|_2^2 \leq \frac{\| \nabla \psi(y^{l-1}) \|_2^2}{2} + \frac{\varepsilon^2}{4R_y^2}, \]

with probability at least \( 1 - 3\beta \). Therefore, after \( l = \log_2(2R_y \|\nabla \psi(y^0)\|_2/\varepsilon^2) \) of such restarts our method provides the point \( y^{l,p(l)} \) such that with probability at least \( 1 - 3\beta \)

\[ \| \nabla \psi(y^{l,p(l)}) \|_2^2 \leq \frac{\| \nabla \psi(y^0) \|_2^2}{2^l} + \frac{\varepsilon^2}{4R_y^2} \sum_{k=0}^{l-1} 2^{-k} \leq \frac{\varepsilon^2}{2R_y^2} + \frac{\varepsilon^2}{4R_y^2} \cdot 2 = \frac{\varepsilon^2}{R_y^2}. \]

The approach informally described above is stated as Algorithm[11].

**Theorem 10 (Theorem 5.6 from [39]).** Assume that \( \psi \) is \( \mu_\psi \)-strongly convex and \( L_\psi \)-smooth. If Algorithm[11] is run with
Algorithm 11 Restarted-RRMA-AC-SA^2

**Input:** $y^0$ — starting point, $l$ — number of restarts, $\{\hat{r}_k\}_{k=1}^l$, $\{\tilde{r}_k\}_{k=1}^l$ — batch-sizes, $\{p_k\}_{k=1}^l$ — amplification parameters

1. Choose the smallest integer $\bar{N} > 1$ such that $\frac{C L^2 \ln^4 N}{\mu y N^4} \leq \frac{1}{32}$

2. $y^{0, p(0)} \leftarrow y^0$

3. for $k = 1, \ldots , l$ do

4. Compute $\nabla \Psi(y^{k-1, p(k-1)}, \xi^{k-1, p(k-1)}, \bar{r}_k)$

5. $r_k \leftarrow \max \left\{ 1, \frac{64 C \sigma_y^2 \ln \bar{N}}{\nabla \Psi(y^{k-1, p(k-1)}, \xi^{k-1, p(k-1)}, \bar{r}_k)} \right\}$

6. Run $p_k$ independent trajectories of RRMA-AC-SA^2 for $\bar{N}$ iterations with batch-size $r_k$ with $y^{k-1, p(k-1)}$ as a starting point and get outputs $y^{k, p_k}$

7. Compute $\nabla \Psi(y^{k, p_k}, \xi^{k, p_k}, \bar{r}_k), \ldots , \nabla \Psi(y^{k, p_k}, \xi^{k, p_k}, \bar{r}_k)$

8. $p(k) \leftarrow \arg\min_{p=1, \ldots , p_k} \|\nabla \Psi(y^{k, p, \xi^{k, p}}, \bar{r}_k)\|_2$

9. end for

**Output:** $y^{l, p(l)}$

\[ l = \max \left\{ 1, \log_2 \frac{2 R^2 \|\nabla \Psi(y^0)\|_2^2}{\varepsilon^2} \right\} \]

\[ \hat{r}_k = \max \left\{ 1, \frac{4 \sigma_y^2 \left( 1 + \sqrt{3 \ln \frac{1}{\beta}} \right)^2 R^2}{\varepsilon^2} \right\}, \quad r_k = \max \left\{ 1, \frac{64 C \sigma_y^2 \ln \bar{N}}{\nabla \Psi(y^{k-1, p(k-1)}, \xi^{k-1, p(k-1)}, \bar{r}_k)} \right\}, \]

\[ p_k = \max \left\{ 1, \log_2 \frac{l}{\beta} \right\}, \quad \tilde{r}_k = \max \left\{ 1, \frac{128 \sigma_y^2 \left( 1 + \sqrt{3 \ln \frac{1}{\beta}} \right)^2 R^2}{\varepsilon^2} \right\} \]

\[ \text{for all } k = 1, \ldots , l \text{ where } \bar{N} > 1 \text{ is such that } \frac{C L^2 \ln^4 \bar{N}}{\mu y N^4} \leq \frac{1}{32}, \beta \in (0, 1/3) \text{ and } \varepsilon > 0, \text{ then with probability at least } 1 - 3\beta \]

\[ \|\nabla \Psi(y^{l, p(l)})\|_2 \leq \frac{\varepsilon}{R_y} \]  \hspace{1cm} (51)

\[ \text{and the total number of the oracle calls equals} \]

\[ \sum_{k=1}^l (\hat{r}_k + \bar{N} p_k r_k + p_k \tilde{r}_k) = \tilde{O} \left( \max \left\{ \frac{L_y}{\mu y}, \frac{\sigma_y^2 R^2}{\varepsilon^2} \right\} \right). \]  \hspace{1cm} (52)

**Corollary 2 (Corollary 5.7 from [39]).** Under assumptions of Theorem [10] we get that with probability at least $1 - 3\beta$

\[ \|y^{l, p(l)} - y^*\|_2 \leq \frac{\varepsilon}{\mu y R_y}, \]  \hspace{1cm} (53)

where $\beta \in (0, 1/3)$ the total number of the oracle calls is defined in (52).

Now we are ready to present convergence guarantees for the primal function and variables.

**Corollary 3 (Corollary 5.8 from [39]).** Let the assumptions of Theorem [10] hold. Assume that $f$ is $L_f$-Lipschitz continuous on $B_{R_f}(0)$ where

\[ R_f = \left( \frac{\mu y}{8 \sqrt{\lambda_{\max}(A^\top A)}} + \sqrt{\frac{\lambda_{\max}(A^\top A)}{\mu}} + \frac{R_y}{R_y} \right) R_y \]

and $R_x = \|x(A^\top y^*)\|_2$. Then, with probability at least $1 - 4\beta$

\[ f(x^l) - f(x^*) \leq \left( 2 + \frac{L_f}{8 R_y \sqrt{\lambda_{\max}(A^\top A)}} \right) \varepsilon, \quad \|Ax^l\| \leq \frac{9\varepsilon}{8 R_y}, \]  \hspace{1cm} (54)
where \( \beta \in (0, 1/4) \), \( \epsilon \in (0, \mu_\psi R^2) \), \( x' \stackrel{\text{def}}{=} x(A^T \bar{y}^{l,p(l)}, \bar{z}^{l,p(l)}, \bar{r}_l) \) and to achieve it we need the following number of oracle calls:

\[
\sum_{k=1}^t (\hat{r}_k + \bar{R}_k r_k + p_k \bar{r}_k) = \tilde{O} \left( \max \left\{ \frac{L}{\mu}, \frac{\sigma_2^2 M^2}{\epsilon^2} \chi(A^T A) \right\} \right)
\]

(55)

where \( M = \|\nabla f(x^*)\|_2 \).

### 3.2.3 Direct Acceleration for Strongly Convex Dual Function

First of all, we consider the following minimization problem:

\[
\min_{y \in \mathbb{R}^d} \psi(y),
\]

(56)

where \( \psi(y) \) is \( \mu_\psi \)-strongly convex and \( L_\psi \)-smooth. We use the same notation to define the objective in (56) as for the dual function from (19) because later in the section we apply the algorithm introduced below to the (19), but for now it is not important that \( \psi \) is a dual function for (18) and we prefer to consider more general situation. As in Section 3.2.1 we do not assume that we have an access to the exact gradient of \( \psi(y) \) and consider instead of it biased stochastic gradient \( \tilde{\nabla}_i \psi(y, \xi) \) satisfying inequalities (56) and (57) with \( \delta \geq 0 \) and \( \sigma_\psi \geq 0 \). In the main method of this section batched version of the stochastic gradient is used:

\[
\tilde{\nabla}_i \psi(y, \xi_k) = \frac{1}{r_k} \sum_{l=1}^{r_k} \tilde{\nabla}_i \psi(y, \xi_k),
\]

(57)

where \( r_k \) is the batch-size that we leave unspecified for now. Note that \( \tilde{\nabla}_i \psi(y, \xi_k) \) satisfies inequalities (39) and (40).

We use Stochastic Similar Triangles Method which is stated in this section as Algorithm 12 to solve problem (56). To define the iterate \( \hat{z}^{k+1} \) we use the following sequence of functions:

\[
\hat{g}_0(z) \stackrel{\text{def}}{=} \frac{1}{2} \|z - z^0\|_2^2 + \alpha_0 \left( \psi(y^0) + \langle \tilde{\nabla}_i \psi(y, \xi_0), z - y^0 \rangle + \frac{\mu_\psi}{2} \|z - y^0\|_2^2 \right),
\]

(58)

\[
\hat{g}_{k+1}(z) = \hat{g}_k(z) + \alpha_{k+1} \left( \psi(y^{k+1}) + \langle \tilde{\nabla}_i \psi(y, \xi), z - y^{k+1} \rangle + \frac{\mu_\psi}{2} \|z - y^{k+1}\|_2^2 \right)
\]

\[
= \frac{1}{2} \|z - z^0\|_2^2 + \sum_{l=0}^{k+1} \alpha_l \left( \psi(y^l) + \langle \tilde{\nabla}_i \psi(y, \xi^l), z - y^l \rangle + \frac{\mu_\psi}{2} \|z - y^l\|_2^2 \right)
\]

(58)

We notice that \( \hat{g}_k(z) \) is \((1 + A_k \mu_\psi)\)-strongly convex.

#### Algorithm 12 Stochastic Similar Triangles Methods for strongly convex problems (SSTM\(_{\psi} \circ \xi\))

**Input:** \( y^0 = z^0 = y^0 \) — starting point, \( N \) — number of iterations

1. Set \( \alpha_0 = A_0 = 1/\psi \)
2. Get \( \tilde{\nabla}_i \psi(y^0, \xi^0) \) to define \( \hat{g}_0(z) \)
3. for \( k = 0, 1, \ldots, N - 1 \) do
4. Choose \( \alpha_{k+1} \) such that \( A_{k+1} = A_k + \alpha_{k+1}, A_{k+1}(1 + A_k \mu_\psi) = \alpha_{k+1}^2 L_\psi \)
5. \( z^{k+1} = (A_0 + \alpha_{k+1} \delta^2)/A_{k+1} \)
6. \( z^{k+1} = \arg\min_{z \in \mathbb{R}^d} \hat{g}_{k+1}(z) \), where \( \hat{g}_{k+1}(z) \) is defined in (58)
7. \( z^{k+1} = (A_0 + \alpha_{k+1} \delta^2)/A_{k+1} \)
8. end for

**Output:** \( x^k \)

For this algorithm we have the following convergence result.

**Theorem 11 (Theorem 5.11 from (39)).** Assume that the function \( \psi \) is \( \mu_\psi \)-strongly convex and \( L_\psi \)-smooth,

\[
r_k = \Theta \left( \max \left\{ 1, \left( \frac{\mu_\psi}{L_\psi} \right)^{3/2} N^2 \sigma_\psi^2 \ln \frac{1}{\epsilon} \right\} \right),
\]

where \( \sigma_\psi \) is defined in (58).
Recent theoretical advances in decentralized distributed convex optimization

for all \( k \) and \( \varepsilon \) as in Section 3.2.1 but we additionally assume that the primal functional \( \psi \) is strongly convex and \( \nabla \) the equation

\[
\psi(y) = \psi(y^0) + \sum_{l=0}^{k+1} \alpha_l \nabla \psi(y^l, \xi^l)
\]

where \( \beta \in (0, 1/3) \).

Next, we apply the SSTM\(_{\infty}\)c to the problem \((19)\) when the objective of the primal problem \((18)\) is \( L\)-smooth, \( \mu \)-strongly convex and \( L\)-Lipschitz continuous on some ball which will be specified next, i.e., we consider the same setup as in Section 3.2.1 but we additionally assume that the primal functional \( f \) has \( L \)-Lipschitz continuous gradient. As in Section 3.2.1 we also consider the case when the gradient of the dual functional is known only through biased stochastic estimators, see \((33), (40)\) and the paragraphs containing these formulas.

In Section 3.2.1 and 3.2.2 we mentioned that in the considered case dual function \( \psi \) is \( L\)-smooth on \( \mathbb{R}^n \) and \( \mu \)-strongly convex on \( y^0 + (\text{Ker} A^\top) \) where \( L\psi = \lambda_{\max}(A^\top A) / \mu \) and \( \mu \psi = \lambda_{\min}(A^\top A) / \mu \). Using the same technique as in the proof of Theorem 9 we show next that w.l.o.g. one can assume that \( \psi \) is \( \mu \)-strongly convex on \( \mathbb{R}^n \) since \( \text{Im} A = (\text{Ker} A^\top) \) by definition of \( \nabla \psi(y, \xi^k) \). For this purposes we need the explicit formula for \( \zeta^{k+1} \) which follows from the equation \( \nabla \bar{g}_{k+1}(\zeta^{k+1}) = 0 \):

\[
\zeta^{k+1} = \frac{\zeta^0}{1 + A_k \mu} + \sum_{l=0}^{k+1} \frac{\alpha_l \mu}{1 + A_{k+1} \mu} y^l - \frac{1}{1 + A_{k+1} \mu} \sum_{l=0}^{k+1} \alpha_l \nabla \psi(y^l, \xi^l).
\]

**Theorem 12 (Theorem 5.12 from [39])**. For all \( k \geq 0 \) we have that the iterates of Algorithm \((22)\) \( \bar{g}^k, \zeta^k, y^k \) lie in \( y^0 + (\text{Ker} A^\top) \). This theorem makes it possible to apply the result from Theorem 11 for SSTM\(_{\infty}\)c which is run on the problem \((19)\).

**Corollary 4 (Corollary 5.13 from [39])**. Under assumptions of Theorem 11 we get that after \( N = \bar{O}\left( \sqrt{\frac{L}{\mu}} \ln \frac{1}{\varepsilon} \right) \) iterations of Algorithm \((22)\) which is run on the problem \((19)\) with probability at least \( 1 - 3\beta \)
\[\|\nabla \psi(y^N)\|_2 \leq \frac{\varepsilon}{R_y},\]  
where \(\beta \in (0, 1/3)\) and the total number of oracles calls equals
\[O\left(\max \left\{ \frac{L_{\psi}}{\mu_{\psi}} \frac{\sigma_{\psi}^2 R_y^2}{\varepsilon^2} \right\} \right).\]  

If additionally \(\varepsilon \leq \mu_{\psi} R_y^2\), then with probability at least \(1 - 3\beta\)
\[\|y^N - y^*\|_2 \leq \frac{\varepsilon}{\mu_{\psi} R_y},\]
\[\|y_N\|_2 \leq 2R_y\]  

Corollary 5 (Corollary 5.14 from [39]). Let the assumptions of Theorem 11 hold. Assume that \(f\) is \(L_f\)-Lipschitz continuous on \(B_{R_f}(0)\) where
\[R_f = \sqrt{2C} + G_1 + \frac{\lambda_{\max}(A^\top A)}{\mu} \varepsilon,\]
\[R_x = \|x(A^\top y^*)\|_2 \leq \mu_{\psi} R_y^2\] and \(\delta \leq \frac{G_1 \varepsilon}{R_x}\) for some positive constant \(G_1\). Assume additionally that the last batch-size \(r_N\) is slightly bigger than other batch-sizes, i.e.
\[r_N \geq \frac{1}{C} \max \left\{ 1, \left( \frac{\mu_{\psi}}{L_{\psi}} \right)^{3/2} N^2 \sigma_{\psi}^2 \left( 1 + \frac{2C}{\lambda_{\max}(A^\top A)} \right) + \frac{\sigma_{\psi}^2}{\varepsilon^2} R_y^2 \right\}.\]  
Then, with probability at least \(1 - 4\beta\)
\[f(x^N) - f(x^*) \leq 2 + \left( \sqrt{2C} + G_1 \frac{\lambda_{\max}(A^\top A)}{\mu} \varepsilon,\right)\]
\[\|A x^N\|_2 \leq \left( 1 + \sqrt{2C} + G_1 \frac{\lambda_{\max}(A^\top A)}{\mu} \varepsilon,\right)\]  
where \(\beta \in (0, 1/4)\), \(x^N \equiv \bar{x}(A^\top y^N, \xi^N, r_N)\) and to achieve it we need the total number of oracle calls including the cost of computing \(\bar{x}^N\) equals
\[O\left(\max \left\{ \frac{L}{\mu} \chi(A^\top A), \frac{\sigma_{\psi}^2 M^2}{\varepsilon^2} \right\} \right)\]  
where \(M = \|\nabla f(x^*)\|_2\).

3.3 Applications to Decentralized Distributed Optimization

In this section, we apply our results to the decentralized optimization problems. First of all, we want to add additional motivation to the problem we are focusing on. As it was stated in the introductory part of this work, we are interested in the convex optimization problem
\[\min_{x \in Q \subseteq \mathbb{R}^n} f(x),\]  
where \(f\) is a convex function and \(Q\) is closed and convex subset of \(\mathbb{R}^n\). More precisely, we study particular case of \(\bar{f}\) when the objective function \(f\) could be represented as a mathematical expectation
\[f(x) = \mathbb{E}_\xi [f(x, \xi)],\]
where $\xi$ is a random variable. Typically $x$ represents the feature vector defining the model, only samples of $\xi$ are available and the distribution of $\xi$ is unknown. One possible way to minimize generalization error (70) is to solve empirical risk minimization or finite-sum minimization problem instead, i.e., solve (69) with the objective

$$
\hat{f}(x) = \frac{1}{m} \sum_{i=1}^{m} f(x, \xi_i),
$$

(71)

where $m$ should be sufficiently large to approximate the initial problem. Indeed, if $f(x, \xi)$ is convex and $M$-Lipschitz continuous for all $\xi_i$, $Q$ has finite diameter $D$ and $\hat{x} = \text{argmin}_{x \in Q} \hat{f}(x)$, then (see [16, 108]) with probability at least $1 - \beta$

$$
f(\hat{x}) - \min_{x \in Q} f(x) \leq O \left( \sqrt{\frac{M^2 D^2 n \ln(m) \ln(n/\beta)}{m}} \right),
$$

(72)

and if additionally $f(x, \xi)$ is $\mu$-strongly convex for all $\xi$, then (see [31]) with probability at least $1 - \beta$

$$
f(\hat{x}) - \min_{x \in Q} f(x) = O \left( \frac{M^2 D^2 \ln(m) \ln(\mu/\beta)}{\mu m} + \frac{M^2 D^2 \ln(1/\beta)}{m} \right).
$$

(73)

In other words, to solve (69)+(70) with $\varepsilon$ functional accuracy via minimization of empirical risk (71) it is needed to have $m = O \left( \frac{M^2 D^2 n \varepsilon^2}{\mu \varepsilon} \right)$ in the convex case and $m = O \left( \max \left\{ \frac{M^2 D^2 \mu}{\varepsilon^2}, \frac{M^2 D^2}{\varepsilon^2} \right\} \right)$ in the $\mu$-strongly convex case where $O(\cdot)$ hides a constant factor, a logarithmic factor of $1/\beta$ and a polylogarithmic factor of $1/\varepsilon$.

Stochastic first-order methods such as Stochastic Gradient Descent (SGD) [45, 87, 92, 98, 124] or its accelerated variants like AC-SA [66] or Similar Triangles Method (STM) [28, 36, 90] are very popular choice to solve either (69)+(70) or (69)+(71). In contrast with their cheap iterations in terms of computational cost, these methods converge only to the neighbourhood of the solution, i.e., to the ball centered at the optimality and radius proportional to the standard deviation of the stochastic estimator. For the particular case of finite-sum minimization problem one can solve this issue via variance-reduction trick [18, 42, 52, 106] and its accelerated variants [2, 13, 14, 135]. Unfortunately, this technique is not applicable in general for the problems of type (69)+(70). Another possible way to reduce the variance is mini-batching. When the objective function is $L$-smooth one can accelerate the computations of batches using parallelization [19, 28, 36, 38], and it is one of the examples where centralized distributed optimization appears naturally [10].

In other words, in some situations, e.g., when the number of samples $m$ is too big, it is preferable in practice to split the data into $q$ blocks, assign each block to the separate worker, e.g., processor, and organize computation of the gradient or stochastic gradient in the parallel or distributed manner. Moreover, in view of (72)-(73) sometimes to solve an expectation minimization problem it is needed to have such a big number of samples that corresponding information (e.g. some objects like images, videos and etc.) cannot be stored on 1 machine because of the memory limitations (see Section 3.5 for the detailed example of such a situation). Then, we can rewrite the objective function in the following form

$$
f(x) = \frac{1}{q} \sum_{i=1}^{q} f_i(x), \quad f_i(x) = \mathbb{E}_{\xi_i} [f(x, \xi_i)] \quad \text{or} \quad f_i(x) = \frac{1}{s_i} \sum_{j=1}^{s_i} f(x, \xi_{ij}).
$$

(74)

Here $f_i$ corresponds to the loss on the $i$-th data block and could be also represented as an expectation or a finite sum. So, the general idea for parallel optimization is to compute gradients or stochastic gradients by each worker, then aggregate the results by the master node and broadcast new iterate or needed information to obtain the new iterate back to the workers.

The visual simplicity of the parallel scheme hides synchronization drawback and high requirement to master node [103]. The big line of works is aimed to solve this issue via periodical synchronization [9, 113, 132, 127, 126, 60, 41], error-compensation [58, 114, 12, 43], quantization [2, 49, 50, 82, 125] or combination of these techniques [2, 81]. However, in this work we mainly focus on another approach to deal with aforementioned drawbacks — decentralized distributed optimization [10, 59]. It is based on two basic principles: every node communicates only with its neighbours and communications are performed simultaneously. Moreover, this architecture is more robust, e.g., it can be applied to time-varying (wireless) communication networks [100].

But let us consider first the centralized or parallel architecture. As we mentioned in the introduction, when the objective function is $L$-smooth one can compute batches in parallel [19, 28, 36, 38] in order to accelerate the work of the method and get the method (see Section 3 from [39] for the details) using

$$
O \left( \frac{\sigma^2 R^2}{\mu \varepsilon} \right) \quad \text{or} \quad O \left( \frac{\sigma^2}{\mu \varepsilon} \right)
$$

(75)
workers and having the working time proportional to the number of iterations of an accelerated first-order method. However, the number of workers defined in (75) could be too big in order to use such an approach in practice. But still computing the batches in parallel even with much smaller number of workers could reduce the working time of the method if the communication is fast enough.

Besides the computation of batches in parallel for the general type of problem (69)+(70), parallel optimization is often applied to the finite-sum minimization problems (69)+(71) or (69)+(74) that we rewrite here in the following form:

$$\min_{x \in Q \subseteq \mathbb{R}^n} f(x) = \frac{1}{m} \sum_{k=1}^{m} f_k(x).$$  \hspace{1cm} (76)

We notice that in this section $m$ is a number of workers and $f_k(x)$ is known only for the $k$-th worker. Consider the situation when workers are connected in a network and one can construct a spanning tree for this network. Assume that the diameter of the obtained graph equals $d$, i.e., the height of the tree — maximal distance (in terms of connections) between the root and a leaf [103]. If we run Similar Triangles Methods (STM, [34]) on such a spanning tree then we will get that the number of communication rounds will be

$$O\left(dN + d \min \left\{ \frac{\sigma^2 R^2}{\epsilon^2} \ln \left( \frac{\sqrt{LR^2/\epsilon}}{\beta} \right), \frac{\sigma^2}{\mu \epsilon} \ln \left( \frac{LR^2}{\epsilon} \right) \ln \left( \frac{\sqrt{L/\mu}}{\beta} \right) \right\} \right),$$

where

$$N = O \left( \min \left\{ \sqrt{\frac{LR^2}{\epsilon}}, \frac{L}{\mu} \ln \left( \frac{LR^2}{\epsilon} \right) \right\} \right).$$

Now let us consider the decentralized case when workers can communicate only with their neighbours. Next, we describe the method of how to reflect this restriction in the problem (76). Consider the Laplacian matrix $\mathcal{W} \in \mathbb{R}^{m\times m}$ of the network with vertices $V$ and edges $E$ which is defined as follows:

$$\mathcal{W}_{ij} = \begin{cases} -1, & \text{if } (i, j) \in E, \\ \deg(i), & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \hspace{1cm} (77)$$

where $\deg(i)$ is degree of $i$-th node, i.e. number of neighbours of the $i$-th worker. Since we consider only connected networks the matrix $\mathcal{W}$ has unique eigenvector $1_m \overset{\text{def}}{=} (1, \ldots, 1) \top \in \mathbb{R}^m$ corresponding to the eigenvalue 0. It implies that for all vectors $a = (a_1, \ldots, a_m) \top \in \mathbb{R}^m$ the following equivalence holds:

$$a_1 = \ldots = a_m \iff \mathcal{W}a = 0. \hspace{1cm} (78)$$

Now let us think about $a_i$ as a number that $i$-th node stores. Then, using (78) we can use Laplacian matrix to express in the short matrix form the fact that all nodes of the network store the same number. In order to generalize it for the case when $a_i$ are vectors from $\mathbb{R}^n$ we should consider the matrix $W \overset{\text{def}}{=} \mathcal{W} \otimes I_n$ where $\otimes$ represents the Kronecker product. Indeed, if we consider vectors $x_1, \ldots, x_m \in \mathbb{R}^n$ and $x = (x_1 \top, \ldots, x_m \top) \in \mathbb{R}^{mn}$, then (78) implies

$$x_1 = \ldots = x_m \iff Wx = 0. \hspace{1cm} (79)$$

For simplicity, we also call $W$ as a Laplacian matrix and it does not lead to misunderstanding since everywhere below we use $W$ instead of $\mathcal{W}$. The key observation here that computation of $Wx$ requires one round of communications when the $k$-th worker sends $x_k$ to all its neighbours and receives $x_j$ for all $j$ such that $(k, j) \in E$, i.e. $k$-th worker gets vectors from all its neighbours. Note, that $W$ is symmetric and positive semidefinite [103] and, as a consequence, $\sqrt{W}$ exists. Moreover, we can replace $W$ by $\sqrt{W}$ in (79) and get the equivalent statement:

$$x_1 = \ldots = x_m \iff \sqrt{W}x = 0. \hspace{1cm} (80)$$

Using this we can rewrite the problem (76) in the following way:

$$\min_{\sqrt{W}x = 0, x_1, \ldots, x_m \in Q \subseteq \mathbb{R}^n} f(x) = \frac{1}{m} \sum_{k=1}^{m} f_k(x_k). \hspace{1cm} (81)$$
We are interested in the general case when \( f_k(x_k) = E_{\xi_k} [f_k(x_k, \xi_k)] \) where \( \{\xi_k\}_{k=1}^m \) are independent. This type of objective can be considered as a special case of (74). Then, as it was mentioned in the introduction it is natural to use stochastic gradients \( \nabla f_k(x_k, \xi_k) \) that satisfy

\[
\|E_{\xi_k} \nabla f_k(x_k, \xi_k) - \nabla f_k(x_k)\|_2 \leq \delta, \tag{82}
\]

\[
E_{\xi_k} \exp \left( \frac{\|\nabla f_k(x_k, \xi_k) - E_{\xi_k}[\nabla f_k(x_k, \xi_k)]\|_2^2}{\sigma^2} \right) \leq \exp(1). \tag{83}
\]

Then, the stochastic gradient

\[
\nabla f(x, \xi) \overset{\text{def}}{=} \nabla f(x, \{\xi_k\}_{k=1}^m) \overset{\text{def}}{=} \frac{1}{m} \sum_{k=1}^m \nabla f_k(x_k, \xi_k)
\]

satisfies (see also (40))

\[
E_{\xi} \exp \left( \frac{\|\nabla f(x, \xi) - E_{\xi}[\nabla f(x, \xi)]\|_2^2}{\sigma^2} \right) \leq \exp(1)
\]

with \( \sigma^2 = O(\sigma^2/m) \).

As always, we start with the smooth case with \( Q = \mathbb{R}^n \) and assume that each \( f_k \) is \( L \)-smooth, \( \mu \)-strongly convex and satisfies \( \|\nabla f_k(x_k)\|_2 \leq M \) on some ball \( B_{R_{\xi_k}(x^*)} \) where we use \( \nabla f(x_k) \) to emphasize that \( f_k \) depends only on the \( k \)-th \( n \)-dimensional block of \( x \). Since the functional \( f(x) \) in (81) has separable structure, it implies that \( f \) is \( L/m \)-smooth, \( \mu/m \)-strongly convex and satisfies \( \|\nabla f(x)\|_2 \leq M/\sqrt{m} \) on \( B_{\sqrt{m}L/M}(x^*) \). Indeed, for all \( x, y \in \mathbb{R}^n \)

\[
\|x - y\|_2^2 = \sum_{k=1}^m \|x_k - y_k\|_2^2,
\]

\[
\|\nabla f(x) - \nabla f(y)\|_2 \leq \sqrt{\frac{L^2}{m^2} \sum_{k=1}^m \|x_k - y_k\|_2^2} = \frac{L}{m} \|x - y\|_2.
\]

\[
f(x) = \frac{1}{m} \sum_{k=1}^m f_k(x_k) \geq \frac{1}{m} \sum_{k=1}^m \left( f(y_k) + \langle \nabla f_k(y_k), x_k - y_k \rangle + \frac{\mu}{2} \|x_k - y_k\|_2^2 \right)
\]

\[
= f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2m} \|x - y\|_2^2,
\]

\[
\|\nabla f(x)\|_2^2 = \frac{1}{m^2} \sum_{k=1}^m \|\nabla f_k(x_k)\|_2^2.
\]

Therefore, one can consider the problem (81) as (18) with \( A = \sqrt{W} \) and \( Q = \mathbb{R}^{nm} \). Next, if the starting point \( x^0 \) is such that \( x^0 = ((x^0)^1, \ldots, (x^0)^1) \) then

\[
R_x^2 \overset{\text{def}}{=} \|x^0 - x^*\|_2^2 = m \|x^0 - x^*\|_2^2 = mR^2, \quad R_y^2 \overset{\text{def}}{=} \|y^*\|_2^2 \leq \frac{\|\nabla f(x^*)\|_2^2}{\lambda^+_m(W)} \leq \frac{M^2}{m\lambda^+_m(W)}.
\]

Now it should become clear why in Section 3.1 we paid most of our attention on number of \( A^T A x \) calculations. In this particular scenario \( A^T A x = \sqrt{W}^T \sqrt{W} = W x \) which can be computed via one round of communications of each node with its neighbours as it was mentioned earlier in this section. That is, for the primal approach we can simply use the results discussed in Section 3.1. For convenience, we summarize them in Tables 3 and 4 which are obtained via plugging the parameters that we obtained above in the bounds from Section 3.1. Note that the results presented in this match the lower bounds obtained in (6) in terms of the number of communication rounds up to logarithmic factors and and there is a conjecture (23) that these bounds are also optimal in terms of number of oracle calls per node for the class of methods that require optimal number of communication rounds. Recently, the very similar result about the optimal balance between number of oracle calls per node and number of communication round was proved for the case when the primal functional is convex and \( L \)-smooth and deterministic first-order oracle is available [129].

Finally, consider the situation when \( Q = \mathbb{R}^n \) and each \( f_k \) from (81) is dual-friendly, i.e. one can construct dual problem for (81).
Table 3 Summary of the covered results in this paper for solving (81) using primal deterministic approach from Section 3.1. First column contains assumptions on $f_k$, $k = 1, \ldots, m$ in addition to the convexity, $\chi = \chi(W) = \lambda_{\text{max}}(W)/\lambda_{\text{min}}(W)$, where $\lambda_{\text{max}}(W)$ and $\lambda_{\text{min}}(W)$ are maximal and minimal positive eigenvalues of matrix $W$. All methods except $D$–MASG should be applied to solve (22).

| Assumptions on $f_k$ | Method | # of communication rounds | # of $V_{f_k}(x, \xi)$ oracle calls per node |
|----------------------|--------|---------------------------|-----------------------------------------------|
| $\mu$-strongly convex, $L$-smooth | $D$–MASG, $Q = \mathbb{R}^n$, $\chi$ with (82)–(83) | $O\left(\sqrt{\frac{L}{\mu}}\right)$ | $O\left(\frac{1}{\mu}\right)$ |
| $L$-smooth | SSTP/IPS with STP as a subroutine, $Q = \mathbb{R}^n$, conjecture, $\chi$ | $O\left(\sqrt{\frac{LR^2}{\epsilon}}\right)$ | $O\left(\frac{1}{\mu}\right)$ |
| $\mu$-strongly convex, $\|\nabla f_k(x)\|_2 \leq M$ | R-Sliding, | $O\left(\frac{M^2 \gamma^2}{\epsilon^2}\right)$ | $O\left(\frac{M^2 \gamma^2}{\epsilon^2}\right)$ |
| $\|\nabla f_k(x)\|_2 \leq M$ | Sliding, | $O\left(\frac{M^2 \gamma^2}{\epsilon^2}\right)$ | $O\left(\frac{M^2 \gamma^2}{\epsilon^2}\right)$ |

Table 4 Summary of the covered results in this paper for solving (81) using primal stochastic approach from Section 3.1 with the stochastic oracle satisfying (82)–(83) with $\delta = 0$. First column contains assumptions on $f_k$, $k = 1, \ldots, m$ in addition to the convexity, $\chi = \chi(W) = \lambda_{\text{max}}(W)/\lambda_{\text{min}}(W)$, where $\lambda_{\text{max}}(W)$ and $\lambda_{\text{min}}(W)$ are maximal and minimal positive eigenvalues of matrix $W$. All methods except $D$–MASG should be applied to solve (22). The bounds from the last two rows hold even in the case when $Q$ is unbounded, but in the expectation (see [72]).

$$
\min_{y \in \mathbb{R}^m} \Psi(y), \quad \text{where } y = (y_1^T, \ldots, y_m^T)^T \in \mathbb{R}^{mm}, y_1, \ldots, y_m \in \mathbb{R}^n,
$$

$$
\varphi_k(y_k) = \max_{x_k \in \mathbb{R}^n} \{\langle y_k, x_k \rangle - f_k(x_k)\},
$$

$$
\Phi(y) = \frac{1}{m} \sum_{k=1}^{m} \varphi_k(m[y_k]), \quad \Psi(y) = \Phi(\sqrt{W} y) = \frac{1}{m} \sum_{k=1}^{m} \varphi_k(m[\sqrt{W} x_k]),
$$

where $[\sqrt{W}]_k$ is the $k$-th $n$-dimensional block of $\sqrt{W}$. Note that

$$
\max_{x \in \mathbb{R}^m} \{\langle y, x \rangle - f(x)\} = \max_{x \in \mathbb{R}^m} \left\{\sum_{k=1}^{m} \langle y_k, x_k \rangle - \frac{1}{m} \sum_{k=1}^{m} f_k(x_k)\right\}
$$

$$
= \frac{1}{m} \sum_{k=1}^{m} \max_{x_k \in \mathbb{R}^n} \{\langle y_k, x_k \rangle - f_k(x_k)\} = \frac{1}{m} \sum_{k=1}^{m} \varphi_k(my_k) = \Phi(y),
$$

so, $\Phi(y)$ is a dual function for $f(x)$. As for the primal approach, we are interested in the general case when $\varphi_k(y_k) = E_{\xi_k} [\varphi_k(y_k, \xi_k)]$ where $\{\xi_k\}_{k=1}^{m}$ are independent and stochastic gradients $\nabla \varphi_k(x_k, \xi_k)$ satisfy

$$
||E_{\xi_k} \nabla \varphi_k(x_k, \xi_k) - \nabla \varphi_k(y_k)||_2 \leq \delta_\Phi,
$$

$$
E_{\xi_k} \left[\exp\left(\frac{||\nabla \varphi_k(y_k, \xi_k) - E_{\xi_k} \nabla \varphi_k(y_k, \xi_k)||_2^2}{\sigma^2}\right)\right] \leq \exp(1).
$$
Consider the stochastic function $f_k(x_k, \tilde{\xi}_k)$ which is defined implicitly as follows:
$$\varphi_k(y_k, \tilde{\xi}_k) = \max_{x_k \in \mathbb{R}^n} \left\{ \langle y_k, x_k \rangle - f_k(x_k) \right\}. \quad (89)$$

Since
$$\nabla \Phi(y) = \sum_{k=1}^{m} \nabla \varphi_k(m y_k) \triangleq \sum_{k=1}^{m} x_k(m y_k) \overset{\text{def}}{=} x(y), \quad x_k(y_k) \overset{\text{def}}{=} \arg\max_{x_k \in \mathbb{R}^n} \left\{ \langle y_k, x_k \rangle - f_k(x_k) \right\}$$

it is natural to define the stochastic gradient $\nabla \Phi(y, \xi)$ as follows:
$$\nabla \Phi(y, \xi) \overset{\text{def}}{=} \nabla \Phi(y, (\tilde{\xi}_k)_{k=1}^m) \overset{\text{def}}{=} \sum_{k=1}^{m} \nabla \varphi_k(m y_k, \tilde{\xi}_k) \overset{\text{def}}{=} \sum_{k=1}^{m} x_k(m y_k, \tilde{\xi}_k) \overset{\text{def}}{=} x(y, \xi), \quad x_k(y_k, \tilde{\xi}_k) \overset{\text{def}}{=} \arg\max_{x_k \in \mathbb{R}^n} \left\{ \langle y_k, x_k \rangle - f_k(x_k, \tilde{\xi}_k) \right\}.$$  

It satisfies (see also (40))
$$\mathbb{E}_{\xi} \left[ \exp \left( \frac{\| \mathbb{E}_{\xi} [\nabla \Phi(y, \xi)] - \nabla \Phi(y) \|_2^2}{\sigma_{\nabla \Phi}^2} \right) \right] \leq \exp(1)$$
with $\delta_\Phi = m \delta_\varphi$ and $\sigma_{\nabla \Phi}^2 = O(m \sigma_\varphi^2)$. Using this, we define the stochastic gradient of $\Psi(y)$ as $\nabla \Psi(y, \xi) \overset{\text{def}}{=} \sqrt{W} \nabla \Phi(\sqrt{W} y, \xi) = \sqrt{W} x(\sqrt{W} y, \xi)$ and, as a consequence, we get
$$\mathbb{E}_{\xi} \left[ \exp \left( \frac{\| \mathbb{E}_{\xi} [\nabla \Psi(y, \xi)] - \nabla \Psi(y) \|_2^2}{\sigma_{\nabla \Psi}^2} \right) \right] \leq \exp(1)$$
with $\delta_{\Psi} = \sqrt{\lambda_{\max}(W)} \delta_\varphi$ and $\sigma_{\nabla \Psi}^2 = \sqrt{\lambda_{\max}(W)} \sigma_{\nabla \Phi}$.

Taking all of this into account we conclude that problem (84) is a special case of (19) with $A = \sqrt{W}$. To make the algorithms from Section 3.2 distributed we should change the variables in those methods via multiplying them by $\sqrt{W}$ from the left $[23, 25, 122]$, e.g. for the iterates of SPDSTM we will get
$$\bar{y}_k^{k+1} := \sqrt{W} y_k^{k+1}, \quad \bar{\xi}_k^{k+1} := \sqrt{W} \xi_k^{k+1}, \quad y_k^{k+1} := \sqrt{W} y_k^{k+1},$$
which means that it is needed to multiply lines 4–6 of Algorithm 7 by $\sqrt{W}$ from the left. After such a change of variables all methods from Section 3.2 become suitable to run them in the distributed fashion. Besides that, it does not spoil the ability of recovering the primal variables since before the change of variables all of the methods mentioned in Section 3.2 used $x(\sqrt{W} y)$ or $x(\sqrt{W} y, \xi)$ where points $y$ were some dual iterates of those methods, so, after the change of variables we should use $x(y)$ or $x(y, \xi)$ respectively. Moreover, it is also possible to compute $\| \sqrt{W} x \|_2^2 = \langle x, W x \rangle$ in the distributed fashion using consensus type algorithms: one communication step is needed to compute $W x$, then each worker computes $\langle x_k, W x_k \rangle$ locally and after that it is needed to run consensus algorithm. We summarize the results for this case in Table 5. Note that the proposed bounds are optimal in terms of the number of communication rounds up to polylogarithmic factors $[6, 103, 104, 105]$. Note that the lower bounds from $[103, 104, 105]$ are presented for the convolution of two criteria: number of oracle calls per node and communication rounds. One can obtain lower bounds for the number of communication rounds itself using additional assumption that time needed for one communication is big enough and the term which corresponds to the number of oracle calls can be neglected. Regarding the number of oracle calls there is a conjecture $[23]$ that the bounds that we present in this paper are also optimal up to polylogarithmic factors for the class of methods that require optimal number of communication rounds.
| Assumptions on $f_k$ | Method | # of communication rounds | # of $\nabla \varphi_k(y, \xi)$ oracle calls per node |
|---------------------|--------|---------------------------|----------------------------------|
| $\mu$-strongly convex, $L$-smooth, $\|\nabla f_k(x)\|_2 \leq M$ | R-RRMA-AC-SA$^2$ (Algorithm 11), Corollary 3 | $\tilde{O}\left(\sqrt{\frac{\mu}{L}}\right)$ | $\tilde{O}\left(\max\left\{\sqrt{\frac{\mu}{L}}, \frac{\sigma^2 L}{m \mu \varepsilon} \chi\right\}\right)$ |
| $\mu$-strongly convex, $\|\nabla f_k(x)\|_2 \leq M$ | SPDSTM (Algorithm 7), Theorem 6 | $\tilde{O}\left(\sqrt{\frac{\mu}{L}}\right)$ | $\tilde{O}\left(\max\left\{\sqrt{\frac{\mu}{L}}, \frac{\sigma^2 L}{m \mu \varepsilon} \chi\right\}\right)$ |

Table 5 Summary of the covered results in this paper for solving (84) using dual stochastic approach from Section 3.2 with the stochastic oracle satisfying (82)-(83) with $\delta = 0$ for R-RRMA-AC-SA$^2$ and $\delta_\varphi = \tilde{O}(\varepsilon/(\mu \sqrt{m} \chi))$, for SSTM$^{sc}$ and SPDSTM. First column contains assumptions on $f_k$, $k = 1, \ldots, m$ in addition to the convexity, $\chi = \chi(W)$.

### 3.4 Discussion

In this section, we want to discuss some aspects of the proposed results that were not covered in the main part of this paper. First of all, we should say that in the smooth case for the primal approach our bounds for the number of communication steps coincide with the optimal bounds for the number of communication steps for parallel optimization if we substitute the diameter $d$ of the spanning tree in the bounds for parallel optimization by $\tilde{O}(\sqrt{\chi(W)})$.

However, we want to discuss another interesting difference between parallel and decentralized optimization in terms of the complexity results which was noticed in [23]. From the line of works [63, 64, 65, 71] it is known that for the problem (69)+(74) (here we use $m$ instead of $q$ and iterator $k$ instead of $i$ for consistency) with $L$-smooth and $\mu$-strongly convex $f_k$ for all $k = 1, \ldots, m$ the optimal number of oracle calls, i.e. calculations of of the stochastic gradients of $f_k$ with $\sigma^2$-subgaussian variance is

$$\tilde{O}\left(\frac{m}{\mu} + \sqrt{\frac{L}{\mu} + \frac{\sigma^2}{m \mu \varepsilon}}\right).$$

(90)

The bad news is that (90) does not work with full parallelization trick and the best possible way to parallelize it is described in [71]. However, standard accelerated scheme using mini-batched versions of the stochastic gradients without variance-reduction technique and incremental oracles which gives the bound

$$\tilde{O}\left(\frac{m}{\mu} \sqrt{\frac{L}{\mu} + \frac{\sigma^2}{m \mu \varepsilon}}\right)$$

(91)

for the number of oracle calls and it admits full parallelization. It means that in the parallel optimization setup when we have computational network with $m$ nodes and the spanning tree for it with diameter $d$ the number of oracle calls per node is

$$\tilde{O}\left(\sqrt{\frac{L}{\mu} + \frac{\sigma^2}{m \mu \varepsilon}}\right) = \tilde{O}\left(\max\left\{\sqrt{\frac{L}{\mu}}, \frac{\sigma^2}{m \mu \varepsilon}\right\}\right)$$

(92)

and the number of communication steps is

$$\tilde{O}\left(d \sqrt{\frac{L}{\mu}}\right).$$

(93)

However, for the decentralized setup the second row of Table 4 states that the number of communication rounds is the same as in (93) up to substitution of $d$ by $\sqrt{\chi(W)}$ and the number of oracle calls per node is

$$\tilde{O}\left(\max\left\{\sqrt{\frac{L}{\mu}}, \frac{\sigma^2}{m \mu \varepsilon}\right\}\right)$$

(94)

which has $m$ times bigger statistical term under the maximum than in (92). What is more, recently it was shown that there exists such a decentralized distributed method that requires

$$\tilde{O}\left(\frac{\sigma^2}{m \mu \varepsilon}\right)$$

(94)
stochastic gradient oracle calls per node \cite{93,94}, but it is not optimal in terms of the number of communications. Moreover, there is a hypothesis \cite{23} that in the smooth case the bounds from Tables 3 and 4 (rows 2 and 3) are not optimal in terms of the number of oracle calls per node and optimal ones can be found in Table 2.

3.5 Application for Population Wasserstein Barycenter Calculation

In this section we consider the problem of calculation of population Wasserstein barycenter since this example hides different interesting details connected with the theory discussed in this paper. In our presentation of this example we rely mostly on the recent work \cite{22}.

3.5.1 Definitions and Properties

We define the probability simplex in \( \mathbb{R}^n \) as \( S_n(1) = \{ x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i = 1 \} \). One can interpret the elements of \( S_n(1) \) as discrete probability measures with \( n \) shared atoms. For an arbitrary pair of measures \( p, q \in S_n(1) \) we introduce the set \( \Pi(p,q) = \{ \pi \in \mathbb{R}^{kn} \mid \pi 1 = p, \pi^\top 1 = q \} \) called transportation polytope. Optimal transportation (OT) problem between measures \( p, q \in S_n(1) \) is defined as follows

\[
\mathcal{W}(p,q) = \min_{\pi \in \Pi(p,q)} \langle C, \pi \rangle = \min_{\pi \in \Pi(p,q)} \sum_{i,j=1}^n C_{ij} \pi_{ij} \tag{95}
\]

where \( C \) is a transportation cost matrix. That is, \((i, j)\)-th component \( C_{ij} \) of \( C \) is a cost of transportation of the unit mass from point \( x_i \) to the point \( x_j \) where points are atoms of measures from \( S_n(1) \).

Next, we consider the entropic OT problem (see \cite{95,97}):

\[
\mathcal{W}_\mu(p,q) = \min_{\pi \in \Pi(p,q)} \sum_{i,j=1}^n (C_{ij} \pi_{ij} + \mu \pi_{ij} \ln \pi_{ij}) \tag{96}
\]

Consider some probability measure \( \mathbb{P} \) on \( S_n(1) \). Then one can define population barycenter of measures from \( S_n(1) \) as

\[
p^*_\mu = \arg\min_{p \in S_n(1)} \int_{q \in S_n(1)} \mathcal{W}_\mu(p,q) d\mathbb{P}(q) = \arg\min_{p \in S_n(1)} \mathbb{E}_q [\mathcal{W}_\mu(p,q)]. \tag{97}
\]

For a given set of samples \( q^1, \ldots, q^m \) we introduce empirical barycenter as

\[
\hat{p}^*_\mu = \arg\min_{p \in S_n(1)} \frac{1}{m} \sum_{i=1}^m \mathcal{W}_\mu(p,q^i). \tag{98}
\]

We consider the problem \cite{97} of finding population barycenter with some accuracy and discuss possible approaches to solve this problem in the following subsections.

However, before that, we need to mention some useful properties of \( \mathcal{W}_\mu(p,q) \). First of all, one can write explicitly the dual function of \( \mathcal{W}_\mu(p,q) \) for a fixed \( q \in S_n(1) \) (see \cite{17,22}):

\[
\mathcal{W}_\mu(p,q) = \max_{\lambda \in \mathbb{R}^n} \{ \langle \lambda, p \rangle - \mathcal{W}^{*\mu}_{q,\lambda}(\lambda) \} \tag{99}
\]

\[
\mathcal{W}^{*\mu}_{q,\lambda}(\lambda) = \mu \sum_{j=1}^n q_j \ln \left( \frac{1}{q_j} \sum_{i=1}^n \exp \left( \frac{-C_{ij} + \lambda_i}{\mu} \right) \right). \tag{100}
\]

Using this representation one can deduce the following theorem.
\textbf{Theorem 13 (22)}. For an arbitrary \( q \in S_n(1) \) the entropic Wasserstein distance \( \mathcal{W}_\mu (\cdot, q) : S_n(1) \rightarrow \mathbb{R} \) is \( \mu \)-strongly convex w.r.t. \( \ell_2 \)-norm and \( M \)-Lipschitz continuous w.r.t. \( \ell_\infty \)-norm. Moreover, \( M \leq \sqrt{n}\lambda_{\infty} \) where \( \lambda_{\infty} \) is Lipschitz constant of \( \mathcal{W}_\mu (\cdot, q) \) w.r.t. \( \ell_\infty \)-norm and \( \mathcal{W}_\mu (\cdot, q) \).  

We also want to notice that function \( \mathcal{W}_{q, \mu}^\ast (\lambda) \) is only strictly convex and the minimal eigenvalue of its hessian \( \gamma \defeq \lambda_{\min} (\nabla^2 \mathcal{W}_{q, \mu}^\ast (\lambda)) \) evaluated in the solution \( \lambda^\ast \defeq \arg \max_{\lambda \in \mathbb{R}^n} \{ (\lambda, p) - \mathcal{W}_{q, \mu}^\ast (\lambda) \} \) is very small and there exist only such bounds that are exponentially small in \( n \).

We will also use another useful relation (see [22]):

\[
\nabla \mathcal{W}_\mu (p, q) = \lambda^\ast, \quad \langle \lambda^\ast, 1 \rangle = 0
\]

where the gradient \( \nabla \mathcal{W}_\mu (p, q) \) is taken w.r.t. the first argument.

\subsection{3.5.2 SA Approach}

Assume that one can obtain and use fresh samples \( q^1, q^2, \ldots \) in online regime. This approach is called Stochastic Approximation (SA). It implies that at each iteration one can draw a fresh sample \( q^k \) and compute the gradient w.r.t. \( p \) of function \( \mathcal{W}_\mu (p, q^k) \) which is \( \mu \)-strongly convex and \( M \)-Lipschitz continuous with \( M = O(\sqrt{n}\|C\|_\infty) \). Optimal methods for this case are based on iterations of the following form

\[
p^{k+1} = \text{proj}_{S_n(1)} \left( p^k - \eta_k \nabla \mathcal{W}_\mu (p^k, q^k) \right)
\]

where \( \text{proj}_{S_n(1)} (x) \) is a projection of \( x \in \mathbb{R}^n \) on \( S_n(1) \) and the gradient \( \nabla \mathcal{W}_\mu (p^k, q^k) \) is taken w.r.t. the first argument. One can show that restarted-SGD (R-SGD) from [54] that using biased stochastic gradients (see also [53, 35, 22]) \( \tilde{\nabla} \mathcal{W}_\mu (p, q) \) such that

\[
\| \tilde{\nabla} \mathcal{W}_\mu (p, q) - \nabla \mathcal{W}_\mu (p, q) \|_2 \leq \delta
\]

for some \( \delta \geq 0 \) and for all \( p, q \in S_n(1) \) after \( N \) calls of this oracle produces such a point \( p^N \) that with probability at least \( 1 - \beta \) the following inequalities hold:

\[
\mathcal{W}_\mu (p^N) - \mathcal{W}_\mu (p^\ast) = O \left( \frac{n\|C\|_\infty^2 \ln(N/\alpha)}{\mu N} + \delta \right)
\]

and, as a consequence of \( \mu \)-strong convexity of \( \mathcal{W}_\mu (p, q) \) for all \( q \),

\[
\| p^N - p^\ast \|_2 = O \left( \frac{n\|C\|_\infty^2 \ln(N/\alpha)}{\mu^2 N} + \frac{\delta}{\mu} \right).
\]

That is, to guarantee

\[
\| p^N - p^\ast \|_2 \leq \epsilon
\]

with probability at least \( 1 - \beta \), R-SGD requires

\[
\tilde{O} \left( \frac{n\|C\|_\infty^2}{\mu^2 \epsilon^2} \right) \tilde{\nabla} \mathcal{W}_\mu (p, q) \text{ oracle calls}
\]

under additional assumption that \( \delta = O(\mu \epsilon^2) \).

However, it is computationally hard problem to find \( \nabla \mathcal{W}_\mu (p, q) \) with high-accuracy, i.e. find \( \tilde{\nabla} \mathcal{W}_\mu (p, q) \) satisfying (102) with \( \delta = O(\mu \epsilon^2) \). Taking into account the relation (101) we get that it is needed to solve the problem (99) with accuracy \( \delta = O(\mu \epsilon^2) \) in terms of the distance to the optimum. i.e. it is needed to find such \( \tilde{\lambda} \) that \( \| \tilde{\lambda} - \lambda^\ast \|_2 \leq \delta \) and set \( \tilde{\nabla} \mathcal{W}_\mu (p, q) = \tilde{\lambda} \). Using variants of Sinkhorn algorithm [62, 113, 46] one can show [22] that R-SGD finds point \( p^N \) such that (103) holds with probability at least \( 1 - \beta \) and it requires

\[
\tilde{O} \left( \frac{n^3\|C\|_\infty^2}{\mu^2 \epsilon^2} \min \left\{ \exp \left( \frac{\|C\|_\infty}{\mu} \right), \left( \frac{\|C\|_\infty}{\mu} + \ln \left( \frac{\|C\|_\infty}{\gamma \mu^2 \epsilon^4} \right) \right) \right\} \right)
\]

5 Under assumption that measures are separated from zero, see the details in [14] and the proof of Proposition 2.5 from [22].
arithmetic operations.

3.5.3 SAA Approach

Now let us assume that large enough collection of samples $q^1, \ldots, q^m$ is available. Our goal is to find such $p \in S_n(1)$ that $\|\hat{p} - p^*_\mu\|_2 \leq \epsilon$ with high probability, i.e. $\epsilon$-approximation of the population barycenter, via solving empirical barycenter problem \cite{107}. This approach is called Stochastic Average Approximation (SAA). Since $\mathcal{W}_\mu(p, q^i)$ is $\mu$-strongly convex and $M$-Lipschitz in $p$ with $M = \tilde{O}(\sqrt{n}\|C\|_\infty)$ for all $i = 1, \ldots, m$ we can conclude that with probability $\geq 1 - \beta$

$$\mathcal{W}_\mu(\hat{p}^*_\mu) - \mathcal{W}_\mu(p^*_\mu) \geq O \left( \frac{n\|C\|_\infty^2 \ln(m) \ln(m/\beta)}{\mu m} + \sqrt{\frac{n\|C\|_\infty^2 \ln(1/\beta)}{m}} \right) \quad (108)$$

where we use that the diameter of $S_n(1)$ is $O(1)$. Moreover, in \cite{108} it was shown that one can guarantee that with probability $\geq 1 - \beta$

$$\mathcal{W}_\mu(\hat{p}^*_\mu) - \mathcal{W}_\mu(p^*_\mu) \geq O \left( \frac{n\|C\|_\infty^2}{\mu \beta m} \right). \quad (109)$$

Taking advantages of both inequalities we get that if

$$m = \tilde{\Omega} \left( \min \left\{ \max \left\{ \frac{n\|C\|_\infty^2}{\mu \beta \epsilon^2}, \frac{n\|C\|_\infty^2}{\mu^2 \epsilon^4} \right\}, \frac{n\|C\|_\infty^2}{\beta \mu^2 \epsilon^2} \right\} \right) = \tilde{\Omega} \left( \min \left\{ \frac{\|C\|_\infty^2}{\mu \beta \epsilon^2}, \frac{\|C\|_\infty^2}{\mu^2 \epsilon^4}, \frac{\|C\|_\infty^2}{\beta \mu^2 \epsilon^2} \right\} \right) \quad (110)$$

then with probability at least $1 - \frac{\beta}{2}$

$$\|\hat{p}^*_\mu - p^*_\mu\|_2 \leq \sqrt{\frac{2}{\mu} \left( \mathcal{W}_\mu(\hat{p}^*_\mu) - \mathcal{W}_\mu(p^*_\mu) \right)} \leq \frac{\epsilon}{2}. \quad (111)$$

Assuming that we have such $\hat{p} \in S_n(1)$ that with probability at least $1 - \frac{\beta}{2}$ the inequality

$$\|\hat{p} - \hat{p}^*_\mu\|_2 \leq \frac{\epsilon}{2} \quad (112)$$

holds, we apply the union bound and get that with probability $\geq 1 - \beta$

$$\|\hat{p} - p^*_\mu\|_2 \leq \|\hat{p} - \hat{p}^*_\mu\|_2 + \|\hat{p}^*_\mu - p^*_\mu\|_2 \leq \epsilon. \quad (113)$$

It remains to describe the approach that finds such $\hat{p} \in S_n(1)$ that satisfies (113) with probability at least $1 - \beta$. Recall that in this subsection we consider the following problem

$$\mathcal{W}_\mu(p) = \frac{1}{m} \sum_{i=1}^m \mathcal{W}_\mu(p, q^i) \rightarrow \min_{p \in S_n(1)} \mathcal{W}_\mu(p, q^i). \quad (114)$$

For each summand $\mathcal{W}_\mu(p, q^i)$ in the sum above we have the explicit formula \cite{100} for the dual function $\mathcal{W}_{q^i, \mu}^*(\lambda)$. Note that one can compute the gradient of $\mathcal{W}_{q^i, \mu}^*(\lambda)$ via $O(n^2)$ arithmetic operations. What is more, $\mathcal{W}_{q^i, \mu}^*(\lambda)$ has a finite-sum structure, so, one can sample $j$-th component of $q^i$ with probability $q^i_j$ and get stochastic gradient

$$\nabla \mathcal{W}_{q^i, \mu}^*(\lambda, j) = \mu \nabla \left( \ln \left( \frac{1}{q^i_j} \sum_{j=1}^n \exp \left( -\frac{C_{ij} + \lambda j}{\mu} \right) \right) \right). \quad (115)$$

which requires $O(n)$ arithmetic operations to be computed.

We start with the simple situation. Assume that each measures $q^i$ are stored on $m$ separate machines that form some network with Laplacian matrix $\tilde{W} \in \mathbb{R}^{m \times m}$. For this scenario we can apply the dual approach described in Section 3.5.2 and apply bounds from Table 5 If for all $i = 1, \ldots, m$ the $i$-th node computes the full gradient of dual functions $\mathcal{W}_{q^i, \mu}$ at each iteration then in order to find such a point $\hat{p}$ that with probability at least $1 - \frac{\beta}{2}$
\[ \mathcal{W}_\mu (\hat{p}) - \mathcal{W}_\mu (\hat{p}_\mu) \leq \hat{\epsilon}, \]  
(116)

where \( W = \overline{W} \otimes I_n \), this approach requires \( \tilde{O}\left( \sqrt{\frac{n\|C\|_\infty}{\mu \epsilon}} \chi(W) \right) \) communication rounds and

\[ \tilde{O}\left( n^{2.5} \sqrt{\frac{\|C\|_\infty}{\mu \epsilon}} \chi(W) \right) \]

arithmetical operations per node to find gradients \( \nabla \mathcal{W}_\mu^* (\lambda) \). If instead of full gradients workers use stochastic gradients \( \nabla \mathcal{W}_\mu^* (\lambda, j) \) defined in (115) and these stochastic gradients have light-tailed distribution, i.e. satisfy the condition (88) with parameter \( \sigma > 0 \), then to guarantee (116) with probability \( \geq 1 - \frac{\beta}{q} \) the aforementioned approach needs the same number of communications rounds and \( \tilde{O}\left( n \max \left\{ \sqrt{\frac{n\|C\|_\infty}{\mu \epsilon}} \chi(W), \frac{m\sigma^2 n\|C\|_\infty^2}{\mu^2 \epsilon^4} \chi(W) \right\} \) arithmetical operations per node to find gradients \( \nabla \mathcal{W}_\mu^* (\lambda, j) \). Using \( \mu \)-strong convexity of \( \mathcal{W}_\mu(p, q) \) for all \( i = 1, \ldots, m \) and taking \( \hat{\epsilon} = \frac{\sqrt{c}}{8} \) we get that our approach finds such a point \( \hat{p} \) that satisfies (112) with probability at least \( 1 - \frac{\beta}{q} \) using

\[ \tilde{O}\left( \frac{\sqrt{n\|C\|_\infty}}{\mu \epsilon} \sqrt{\chi(W)} \right) \]

communication rounds

(117)

and

\[ \tilde{O}\left( n^{2.5} \frac{\|C\|_\infty}{\mu \epsilon} \sqrt{\chi(W)} \right) \]

(118)

arithmetical operations per node to find gradients in the deterministic case and

\[ \tilde{O}\left( n \max \left\{ \sqrt{\frac{n\|C\|_\infty}{\mu \epsilon}} \sqrt{\chi(W)}, \frac{m\sigma^2 n\|C\|_\infty^2}{\mu^2 \epsilon^4} \chi(W) \right\} \) \]

arithmetical operations per node to find stochastic gradients in the stochastic case. However, the state-of-the-art theory of learning states (see (110)) that \( m \) should so large that in the stochastic case the second term in the bound for arithmetical operations typically dominates the first term and the dimensional dependence reduction from \( n^{2.5} \) in the deterministic case to \( n^{1.5} \) in the stochastic case is typically negligible in comparison with how much \( \frac{m\sigma^2 \sqrt{n\|C\|_\infty^2}}{\mu^2 \epsilon^4} \chi(W) \) is larger than \( \frac{\|C\|_\infty}{\mu \epsilon} \sqrt{\chi(W)} \). That is, our theory says that it is better to use full gradients in the particular example considered in this section (see also Section 3.4). Therefore, further in the section we will assume that \( \sigma^2 = 0 \), i.e. workers use full gradients of dual functions \( \mathcal{W}_\mu^* (\lambda) \).

However, bounds (117)-(118) were obtained under very restrictive at the first sight assumption that we have \( m \) workers and each worker stores only one measure which is unrealistic. One can relax this assumption in the following way. Assume that we have \( \hat{l} < m \) machines connected in a network with Laplacian matrix \( \overline{W} \) and \( j \)-th machine stores \( \hat{m}_j \geq 1 \) measures for \( j = 1, \ldots, \hat{l} \) and \( \sum_{j=1}^{\hat{l}} \hat{m}_j = m \). Next, for \( j \)-th machine we introduce \( \hat{m}_j \) virtual workers also connected in some network that \( j \)-th machine can emulate along with communication between virtual workers and for every virtual worker we arrange one measure, e.g. it can be implemented as an array-like data structure with some formal rules for exchanging the data between cells that emulates communications. We also assume that inside the machine we can set the preferable network for the virtual nodes in such a way that each machine emulates communication between virtual nodes and computations inside them fast enough. Let us denote the Laplacian matrix of the obtained network of \( m \) virtual nodes as \( \overline{W} \). Then, our approach finds such a point \( \hat{p} \) that satisfies (112) with probability at least \( 1 - \frac{\beta}{q} \) using

\[ \tilde{O}\left( \frac{\max_{j=1, \ldots, \hat{l}} T_{cm,j}}{T_{cm,max}} \frac{\sqrt{n\|C\|_\infty}}{\mu \epsilon} \sqrt{\chi(W)} \right) \]

(119)

time to perform communications and

\[ \tilde{O}\left( \frac{\max_{j=1, \ldots, \hat{l}} T_{cp,j}}{T_{cp,max}} n^{2.5} \frac{\|C\|_\infty}{\mu \epsilon} \sqrt{\chi(W)} \right) \]

(120)
time for arithmetical operations per machine to find gradients where \( T_{cm,j} \) is time needed for \( j \)-th machine to emulate communication between corresponding virtual nodes at each iteration and \( T_{cp,j} \) is time required by \( j \)-th machine to perform 1 arithmetical operation for all corresponding virtual nodes in the gradients computation process at each iteration. For example, if we have only one machine and network of virtual nodes forms a complete graph than \( \chi(W) = 1 \), but \( T_{cm,max} \) and \( T_{cp,max} \) can be large and to reduce the running time one should use more powerful machine. In contrast, if we have \( m \) machines connected in a star-graph than \( \chi(W) \) will be much smaller, but \( \chi(W) \) will be of order \( m \) which is large. Therefore, it is very important to choose balanced architecture of the network at least for virtual nodes per machine if it is possible. This question requires a separate thorough study and lies out of scope of this paper.

3.5.4 SA vs SAA comparison

Recall that in SA approach we assume that it is possible to sample new measures in online regime which means that the computational process is performed on one machine, whereas in SAA approach we assume that large enough collection of measures is distributed among the network of machines that form some computational network. In practice measures from \( S_n(1) \) correspond to some images. As one can see from the complexity bounds, both SA and SAA approaches require large number of samples to learn the population barycenter defined in (97). If these samples are images, then they typically cannot be stored in RAM of one computer. Therefore, it is natural to use distributed systems to store the data.

Now let us compare complexity bounds for SA and SAA. We summarize them in Table 6. When the communication is fast enough and \( \mu \) is small we typically have that SAA approach significantly outperforms SA approach in terms of the complexity as well even for communication architectures with big \( \chi(W) \). Therefore, for balanced architecture one can expect that SAA approach will outperform SA even more.

To conclude, we state that population barycenter computation is a natural example when it is typically much more preferable to use distributed algorithms with dual oracle instead of SA approach in terms of memory and complexity bounds.

| Approach                | Complexity                                                                 |
|-------------------------|-----------------------------------------------------------------------------|
| SA                      | \( \tilde{O} \left( \frac{n|m|\epsilon^2}{\mu^2} \right) \) arithmetical operations |
| SAA, the 2-d term is smaller | \( \tilde{O} \left( \frac{n|m|\epsilon^2}{\mu^2} \right) \) arithmetical operations |
| SAA                     | \( \tilde{O} \left( T_{cm,max} \frac{\sqrt{m|\epsilon|}}{\mu} \sqrt{\chi(W)} \right) \) time to perform communications, \( \tilde{O} \left( T_{cp,max} \frac{n^2s|\epsilon|}{\mu^2} \sqrt{\chi(W)} \right) \) time for arithmetical operations per machine, where \( m = \Omega \left( n \min \left\{ \frac{|\epsilon|}{\mu^2 \epsilon^2}, \frac{|\epsilon|}{\mu^2} \right\} \right) \) |
| SAA, \( \chi(W) = \Omega(m) \), \( T_{cm,max} = O(1) \), \( T_{cp,max} = O(1) \), \( \sqrt{\beta} \geq \epsilon \) | \( \tilde{O} \left( \frac{n|m|\epsilon^2}{\sqrt{\mu^2 \epsilon^2}} \right) \) communication rounds, \( \tilde{O} \left( \frac{n|m|\epsilon^2}{\sqrt{\mu^2 \epsilon^2}} \right) \) arithmetical operations per machine |

Table 6 Complexity bounds for SA and SAA approaches for computation of population barycenter defined in (97) with accuracy \( \epsilon \). The third row states the complexity bound for SA approach when the second term under the minimum in (104) is dominated by the first one, e.g. when \( \mu \) is small enough. The last row corresponds to the case when \( T_{cm,max} = O(1) \), \( T_{cp,max} = O(1) \), \( \sqrt{\beta} \geq \epsilon \), e.g. \( \beta = 0.01 \) and \( \epsilon \leq 0.1 \), and the communication network is star-like, which implies \( \chi(W) = \Omega(m) \).

4 Derivative-Free Distributed Optimization

As mentioned above in Section 3 the decentralized optimization problem can be rewritten as a problem with affine constraints:

\[
\min_{\mathbf{x}} \{ f(\mathbf{x}) = \frac{1}{m} \sum_{i=1}^{m} f_i(x_i) \}.
\] (121)
where we use matrix $W \overset{\text{def}}{=} W \otimes I_n$ for Laplacian matrix $\mathcal{W} = \|W_{ij}\|_{i,j=1,1} \in \mathbb{R}^{m \times m}$ of the connection graph. In turn, the problem with affine constraints:

$$
\min_{Ax = 0, x \in Q} f(x),
$$

is rewritten in a penalized form as follows:

$$
\min_{x \in Q} F(x) = f(x) + \frac{R^2}{\varepsilon} \|Ax\|_2^2,
$$

with some positive constants $\varepsilon$ and $R$, for details see Section 3. As a result, we have a classical composite optimization problem, therefore this section will focus on this problem. In what follows, we will rely on work [11]. In particular, we will find out a method based on the Sliding Algorithm (see [67] and Section 3) for the convex composite optimization problem with smooth and non-smooth terms. One can find gradient-free methods for distributed optimization in the literature (see [78, 119]), but the method that will be discussed further is the first, which combines zeroth-order and first-order oracles.

4.1 Theoretical part

4.1.1 Convex Case

We consider the composite optimization problem

$$
\min_{x \in Q} \Psi_0(x) = f(x) + g(x).
$$

In this part of paper, we will work not in the Euclidean norm $\|\cdot\|_2$, but in a certain norm $\|\cdot\|$ (and the dual norm $\|\cdot\|_*$ for the norm $\|\cdot\|$). Also define the Bregman divergence associated with some function $\nu(x)$, which is 1-strongly convex w.r.t. $\|\cdot\|$-norm and differentiable on $Q$, as follows

$$
V(x, y) = \nu(y) - \nu(x) - \langle \nabla \nu(x), y - x \rangle, \quad \forall x, y \in Q.
$$

The use of Bregman divergence and special norms allows taking into account the geometric setup of the problem. For example, when we work with the problem in a probability simplex, it seems natural to use the $\|\cdot\|_1$-norm and the Kullback–Leibler divergence.

Next, we introduce some assumptions for problem (123): $Q \subseteq \mathbb{R}^n$ is a compact and convex set with diameter $D_Q$ in $\|\cdot\|$-norm, function $g$ is convex and $L$-smooth on $Q$ w.r.t. norm $\|\cdot\|$, i.e.

$$
\|\nabla g(x) - \nabla g(y)\|_* \leq L \|x - y\|, \quad \forall x, y \in Q,
$$

$f$ is convex differentiable function on $Q$.

Assume that we have an access to the first-order oracle for $g$, i.e. gradient $\nabla g(x)$ is available, and to the biased stochastic zeroth-order oracle for $f$ (see also [40, 13]) that for a given point $x$ returns noisy value $\tilde{f}(x, \xi)$ such that

$$
\tilde{f}(x, \xi) = f(x, \xi) + \Delta(x),
$$

where $\Delta(x)$ is a bounded noise of unknown nature

$$
|\Delta(x)| \leq \Delta
$$

and random variable $\xi$ is such that

$$
\mathbb{E}[f(x, \xi)] = f(x).
$$

Additionally, we assume that for all $x \in Q$, $s \leq D_Q$

$$
\|\nabla f(x, \xi)\|_2 \leq M(\xi), \quad \mathbb{E}[M^2(\xi)] = M^2.
$$

\footnote{The narrative in this section follows [11].}
It is important to note that for the function \( f(x) \) these assumptions are made only for theoretical estimates; we have no real access to \( \nabla f(x) \). The question is how to replace the gradient of the function \( f(x) \). The easiest way is to collect gradient completely using finite differences:

\[
f_{\text{full}} = \frac{1}{r} \sum_{i=1}^{n} (\tilde{f}(x + rh_i, \tilde{\xi}) - \tilde{f}(x - rh_i, \tilde{\xi})) h_i,
\]

(125)

Here we consider a standard orthogonal normalized basis \( \{h_1, \ldots, h_n\} \). This way we really get a vector close to the gradient. The obvious disadvantage of this method is that one need to call the oracle for \( \tilde{f}(x, \tilde{\xi}) \) 2n times. Another way is to use random direction \( e \) uniformly distributed on the Euclidean sphere (see [91, 110]):

\[
f_{\text{r}}(x, \tilde{\xi}, e) = \frac{r}{2r}(\tilde{f}(x + re, \tilde{\xi}) - \tilde{f}(x - re, \tilde{\xi}))e.
\]

(126)

In particular, the authors of [11] use this approximation.

Now another problem arises – we need to combine the zeroth-order and first-order oracles for different parts of the composite problem. It seems natural that the gradient-free oracle should be called more often than the gradient one. The authors of paper [11] solve this problem and propose to apply the algorithm based on Lan’s Sliding [67]. The basic idea is that we fix \( \nabla g \) and iterate through the inner loop (\( PS \) procedure), changing only the point \( x \) in \( \tilde{f}_{\text{r}}(x, \tilde{\xi}, e) \).

---

**Algorithm 13 Zeroth-Order Sliding Algorithm (z\(\cdot\)SA)**

**Input:** Initial point \( x_0 \in \mathcal{Q} \) and iteration limit \( N \).

Let \( \beta \in \mathbb{R}_{++}, \gamma \in \mathbb{R}_{++}, \) and \( T \in \mathbb{N}, k = 1, 2, \ldots \), be given and set \( x_0 = x_0 \).

for \( k = 1, 2, \ldots, N \) do

1. Set \( x_k = (1 - \gamma) x_{k-1} + \gamma x_{k-1} \), and let \( h_k(\cdot) \equiv I_{\beta}(x_k, \cdot) \) be defined in (127).

2. Set \( (x_k, \tilde{\xi}_k) = PS(h_k, x_{k-1}, \beta_k, T_k) \); \( \tilde{\xi}_k = \nabla_h \tilde{f}(x_k, \tilde{\xi}_k) \).

end for

**Output:** \( x_N \).

The \( PS \) (prox-sliding) procedure.

**procedure:** \( (x^+, \tilde{x}^+) = PS(h, x, \beta, T) \)

Let the parameters \( p_i \in \mathbb{R}_{++} \) and \( \theta_i \in [0, 1] \), \( t = 1, \ldots \), be given. Set \( u_0 = \tilde{u}_0 = x \).

for \( t = 1, 2, \ldots, T \) do

\[
u_t = \text{argmin}_{u_0 \in \mathcal{Q}} \left\{ h(u) + (\tilde{f}(u_{t-1}, \tilde{\xi}_{t-1}, e_{t-1}), u) + \beta V(x, u) + \beta p_t V(u_{t-1}, u) \right\},
\]

\[
\tilde{u}_t = (1 - \theta_t)u_{t-1} + \theta_t u_t.
\]

end for

Set \( x^+ = u_T \) and \( \tilde{x}^+ = \tilde{u}_T \).

end procedure:

In the Algorithm [13] we need the following function

\[
I_{\beta}(x, y) = \frac{\gamma}{2r} \sum_{i=1}^{n} (\tilde{f}(x + rh_i, \tilde{\xi}) - \tilde{f}(x - rh_i, \tilde{\xi})) h_i,
\]

(127)

It is important that the random variables \( \tilde{\xi} \) are independent, and also \( e_i \) is sampled independently from previous iterations.

We also note that \( z\cdot\)SA (in contrast to the basic version – Algorithm [6]) takes into account the geometric setting of the problem and uses Bregman divergence \( V(x, y) \) instead of the standard Euclidean distance in prox-sliding procedure.

Next, we will briefly talk about the convergence of this method (see the full version of the analysis in [11]). First of all, we note the universal technical lemmas that forms a general approach to working with gradient-free methods for non-smooth functions. But before that we introduce a new notation:

\[
F(x) = \mathbb{E}_e[f(x + re)],
\]

(128)

\( F(x) \) is called the smoothed function of \( f(x) \). It is important to note that the function \( F(x) \) is not calculated by the algorithm, this object is needed only for theoretical analysis. The first lemma states some properties of \( F(x) \):
Lemma 5. Assume that differentiable function $f$ defined on $Q$, satisfy $\|\nabla f(x)\|_2 \leq M$ with some constant $M > 0$. Then $F(x)$ defined in (128) is convex, differentiable and $F(x)$ satisfies

$$\sup_{x \in Q} |F(x) - f(x)| \leq rM, \quad \nabla F(x) = \mathbb{E}_r \left[ \nabla_f (x + re) \right], \quad \|\nabla F(x)\|_2 \leq c \bar{r} \sqrt{n}M,$$

where $\bar{c}$ is some positive constant independent of $n$ and $\bar{r}$ is determined by the following relation: $\sqrt{\mathbb{E}[\|g\|_2^2]} \leq \bar{r}$.

In other words, $F(x)$ provides a good approximation of $f(x)$ for small enough $r$.

Lemma 6. For $\tilde{f}_{z}(x, \xi, e)$ defined in (126), the following inequalities hold:

$$\mathbb{E}[\tilde{f}_{z}(x, \xi, e)] - \nabla F(x) \leq \frac{n\Delta p_s}{r}, \quad \mathbb{E}[\|\tilde{f}_{z}(x, \xi, e)\|_2^2] \leq 2p_s^2 \left( cnM^2 + \frac{n^2 \Delta^2}{r^2} \right),$$

where $c$ is some positive constant independent of $n$.

In other words, one can consider $\tilde{f}_{z}(x, \xi, e)$ as a biased stochastic gradient of $F(x)$ with bounded second moment. Therefore, instead of solving (123) directly one can focus on the problem

$$\min_{x \in \mathbb{R}^d} \Psi(x) = F(x) + g(x) \quad (129)$$

with small enough $r$. As mentioned earlier, this approach is universal. In particular, the analysis of gradient-free methods for non-smooth saddle-point problems can be carried out in a similar way [13].

Now we will give the main facts from [11] for SA algorithm itself. The following theorem states convergence guarantees:

Theorem 14. Suppose that \{\theta_i\}_{i \geq 1}, \{\beta_k\}_{i \geq 1} are

$$p_t = \frac{t}{2}, \quad \theta_t = \frac{2(t+1)}{t^3}, \quad \text{for all } t \geq 1,$$

$N$ is given, \{\beta_k\}, \{\gamma_k\}, \{T_k\} are

$$\beta_k = \frac{2L}{k}, \quad \gamma_k = \frac{1}{k+1}, \quad T_k = \frac{CN\rho_s^2 \left( nM^2 + \frac{n^2 \Delta^2}{r^2} \right)}{D^2} \quad (131)$$

with $D = 3p_s^2/4$, $D_{Q,V} = \max \{ \sqrt{2V(x,y)} \mid x, y \in Q \}$, $D_Q = \max \{ \|x - y\| \mid x, y \in Q \}$, with some positive constant $C$. Then for all $N \geq 1$

$$\mathbb{E}[\Psi(x_N) - \Psi(x^*)] \leq \frac{12LD_{Q,V}^2}{N(N+1)} + \frac{n\Delta D_{Q,V} p_s}{r}. \quad (132)$$

Finally, need to connect the result above to the initial problem (123).

Corollary 6. Under the assumptions of Theorem 14 we have that the following inequality holds for all $N \geq 1$:

$$\mathbb{E}[\Psi_0(x_N) - \Psi_0(x^*)] \leq 2rM + \frac{12LD_{Q,V}^2}{N(N+1)} + \frac{n\Delta D_{Q,V} p_s}{r}. \quad (132)$$

From (132) it follows that if

$$r = \Theta \left( \frac{\varepsilon}{M} \right), \quad \Delta = \Theta \left( \frac{\varepsilon^2}{nMD_{Q,V} \min\{p_s, 1\}} \right)$$

and $\varepsilon = O(\sqrt{nMD_{Q,V}})$, then the number of evaluations for $\nabla g$ and $\tilde{f}_{z}$, respectively, required by Algorithm 13 to find an $\varepsilon$-solution of (123), i.e. such $x_N$ that $\mathbb{E}[\Psi_0(x_N)] - \Psi_0(x^*) \leq \varepsilon$, can be bounded by

$$O \left( \sqrt{\frac{LD_{Q,V}^2}{\varepsilon}} \right) \quad \text{and} \quad O \left( \sqrt{\frac{LD_{Q,V}^2}{\varepsilon} + \frac{D_{Q,V}^2 p_s^2 nM^2}{\varepsilon^2}} \right).$$

(133)
It is interesting to analyze the obtained results depending on $p^*$, and these constants are determined depending on what geometry we have defined for our problem. For example, if we consider Euclidean proximal setup, i.e. $\| \cdot \| = \| \cdot \|_2$, $V(x,y) = \frac{1}{2} \| x - y \|_2^2, D_{Q,V} = D_Q$. In this case we have $p_s$ and bound (133) for the number of (124) oracle calls reduces to

$$O \left( \sqrt{\frac{LD_Q^2}{\varepsilon}} + \frac{D_Q^2 n M^2}{\varepsilon^2} \right)$$

and the number of $\nabla g(x)$ computations remains the same. It means that our result gives the same number of first-order oracle calls as in the original Gradient Sliding algorithm, while the number of the biased stochastic zeroth-order oracle calls is $n$ times larger in the leading term than in the analogous bound from the original first-order method. In the Euclidean case our bounds reflect the classical dimension dependence for the derivative-free optimization (see (73)).

But if we work on the probability simplex in $\mathbb{R}^n$ and the proximal setup is entropic: $V(x,y)$ is the Kullback–Leibler divergence, i.e. $V(x,y) = \sum_{i=1}^n x_i \ln \frac{x_i}{y_i}$. In this situation we have $D_{Q,V} = \sqrt{2 \log n}, D_Q = 2, p_s = O(\log(n)/n)$ (44). Then number of $\nabla g(x)$ calculations is bounded by $O \left( \sqrt{[\log n]/\varepsilon} \right)$. As for the number of $\tilde{f}_i^\varepsilon(x, \xi, \varepsilon)$ computations, we get the following bound:

$$O \left( \sqrt{\frac{L \log n}{\varepsilon}} + \frac{M^2 \log^2 n}{\varepsilon^2} \right). \quad (134)$$

### 4.1.2 Strongly Convex Case

In this section we additionally assume that $g$ is $\mu$-strongly convex w.r.t. Bregman divergence $V(x,y)$ (116), i.e. for all $x,y \in Q$

$$g(x) \geq g(y) + \langle \nabla g(y), x - y \rangle + \mu V(x,y).$$

The authors of (11) use restarts technique and get Algorithm (14)

**Algorithm 14** The Multi-phase Zeroth-Order Sliding Algorithm (M-zoSA)

**Input:** Initial point $y_0 \in Q$ and iteration limit $N_0$, initial estimate $\rho_0$ (s.t. $\Psi(y_0) - \Psi(y^*) \leq \rho_0$)

for $i = 1, 2, \ldots, I$

- Run zoSA with $x_0 = y_{i-1}, N = N_0, \{p_i\}$ and $\{\theta_i\}$ in (130), $\{\tilde{b}_i\}$ and $\{\tilde{y}_i\}, \{\tilde{t}_i\}$ in (131) with $\tilde{D} = \rho_0/\mu^2$, and $y_i$ is output.

end for

Output: $y_I$.

The following theorem states the main complexity results for $M$-zoSA.

**Theorem 15.** For $M$-zoSA with $N_0 = 2 \lfloor \sqrt{\mu/\rho_0} \rfloor$ we have

$$\mathbb{E}[\Psi(y_i) - \Psi(y^*)] \leq \frac{\rho_0}{2^i} + \frac{2nD_Qp_s}{r}.$$

Using this we derive the complexity bounds for $M$-zoSA.

**Corollary 7.** For all $N \geq 1$ the iterates of $M$-zoSA satisfy

$$\mathbb{E}[\Psi_0(y_i) - \Psi_0(y^*)] \leq 2rM + \frac{\rho_0}{2^i} + \frac{2nD_Qp_s}{r}. \quad (135)$$

From (135) it follows that if

$$r = O\left(\frac{\varepsilon}{M}\right), \quad \Delta = O\left(\frac{\varepsilon^2}{nMD_Q \min\{p_s, 1\}}\right)$$

and $\varepsilon = O(\sqrt{nMD_Q})$, then the number of evaluations for $\nabla g$ and $\tilde{f}_i^\varepsilon$, respectively, required by Algorithm (14) to find a $\varepsilon$-solution of (123) can be bounded by
the starting point is \( \parallel x \parallel \) that we reduce the original decentralized problem to the penalized problem. Next, we need to define parameters of

4.1.3 From Composite Optimization to Decentralized Distributed Optimization

Finally, we get an estimate for solving the decentralized optimization problem. With the help of (121) and (122), we reduce the original decentralized problem to the penalized problem. Next, we need to define parameters of

\[
O\left(\sqrt{\frac{L}{\mu}} \log_2 \max \{1, \rho_0/\varepsilon\}\right), \quad O\left(\sqrt{\frac{L}{\mu}} \log_2 \max \{1, \rho_0/\varepsilon\} + \frac{\rho_0^2 n M^2}{\mu \varepsilon}\right).
\]

At the same time, when we work on a simplex and use the Kullback-Leibler divergence, we get estimates similar to (134):

\[
O\left(\sqrt{\frac{\chi(W) M^2 D_Q^2}{\varepsilon^2}}\right) \text{ communication rounds and } O\left(\sqrt{\frac{\chi(W) M^2 D_Q^2}{\varepsilon^2} + \frac{n D_Q^2 M^2}{\varepsilon^2}}\right) \text{ calculations of } \tilde{f}(x, \xi) \text{ per node.}
\]

The bound for the communication rounds matches the lower bound from [105, 104] and one can note that under above assumptions the obtained bound for zeroth-order oracle calculations per node is optimal up to polylogarithmic factors in the class of methods with optimal number of communication rounds (see also [23, 39]). In particular, in the Euclidean case, we lose \( n \) times (which corresponds to the case if we were to restore the gradient in the way (125)), and in the case of a simplex, only in the log \( n \) times.

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References

1. A. Agahian and B. Touri. Distributed optimization over dependent random networks. *arXiv preprint arXiv:2010.01956*, 2020.
2. D. Alistarh, D. Grubic, J. Li, R. Tomioka, and M. Vojnovic. QSGD: Communication-efficient SGD via gradient quantization and encoding. In *Advances in Neural Information Processing Systems*, pages 1709–1720, 2017.
3. Z. Allen-Zhu. Katyusha: The first direct acceleration of stochastic gradient methods. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2017, pages 1200–1205, New York, NY, USA, 2017. ACM. *arXiv*:1603.05953.
4. Z. Allen-Zhu. How to make the gradients small stochastically: Even faster convex and nonconvex sgd. In *Advances in Neural Information Processing Systems*, pages 1157–1167, 2018.
5. A. S. Anikin, A. V. Gasnikov, P. E. Dvurechensky, A. I. Tyurin, and A. V. Chernov. Dual approaches to the minimization of strongly convex and smooth inequality constraints. *Computational Mathematics and Mathematical Physics*, 57(8):1262–1276, Aug 2017.
6. Y. Arjevani and O. Shamir. Communication complexity of distributed convex learning and optimization. In *Advances in neural information processing systems*, pages 1756–1764, 2015.
7. D. Basu, D. Data, C. Karakus, and S. Diggavi. Qsparse-local-sgd: Distributed sgd with quantization, sparsification, and local computations. *arXiv preprint arXiv:1906.02367*, 2019.
8. A. Bayandina, P. Dvurechensky, A. Gasnikov, F. Stonyakin, and A. Titov. Mirror descent and convex optimization problems with non-smooth inequality constraints. *arXiv:1710.06612*, 2017.
9. A. K. R. Bayoumi, K. Mishchenko, and P. Richtárik. Tighter theory for local sgd on identical and heterogeneous data. In *International Conference on Artificial Intelligence and Statistics*, pages 4519–4529, 2020.
10. D. P. Bertsekas and J. N. Tsitsiklis. *Parallel and distributed computation: numerical methods*, volume 23. Prentice hall Englewood Cliffs, NJ, 1989.

11. A. Beznosikov, E. Gorbunov, and A. Gasnikov. Derivative-free method for decentralized non-smooth optimization. *arXiv preprint arXiv:1911.10645*, 2019.

12. A. Beznosikov, S. Horváth, P. Richtárik, and M. Safaryan. On biased compression for distributed learning. *arXiv preprint arXiv:2002.12410*, 2020.

13. A. Beznosikov, A. Sadiev, and A. Gasnikov. Gradient-free methods with inexact oracle for convex-concave stochastic saddle-point problem. In *International Conference on Mathematical Optimization Theory and Operations Research*, pages 105–119. Springer, 2020.

14. J. Blanchet, A. Jambulapati, C. Kent, and A. Sidford. Towards optimal running times for optimal transport. *arXiv preprint arXiv:1810.07717*, 2018.

15. S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah. Randomized gossip algorithms. *IEEE transactions on information theory*, 52(6):2508–2530, 2006.

16. N. Cesa-bianchi, A. Conconi, and C. Gentile. On the generalization ability of on-line learning algorithms. In T. G. Dietterich, S. Becker, and Z. Ghahramani, editors, *Advances in Neural Information Processing Systems 14*, pages 359–366. MIT Press, 2002.

17. M. Cuturi and G. Peyré. A smoothed dual approach for variational wasserstein problems. *SIAM Journal on Imaging Sciences*, 9(1):320–343, 2016.

18. A. Defazio, F. Bach, and S. Lacoste-Julien. Saga: A fast incremental gradient method with support for non-strongly convex composite objectives. In *Proceedings of the 27th International Conference on Neural Information Processing Systems, NIPS’14*, pages 1646–1654, Cambridge, MA, USA, 2014. MIT Press.

19. O. Devolder. *Exactness, inexactness and stochasticity in first-order methods for large-scale convex optimization*. PhD thesis, PhD thesis, ICTEAM and CORE, Université Catholique de Louvain, 2013.

20. O. Devolder, F. Glineur, and Y. Nesterov. First-order methods with inexact oracle: the strongly convex case. *CORE Discussion Papers*, 2013016:47, 2013.

21. O. Devolder, F. Glineur, and Y. Nesterov. First-order methods of smooth convex optimization with inexact oracle. *Mathematical Programming*, 146(1):37–75, 2014.

22. D. Dvinskikh. Sa vs saa for population wasserstein barycenter calculation. *arXiv preprint arXiv:2001.07697*, 2020.

23. D. Dvinskikh and A. Gasnikov. Decentralized and parallelized primal and dual accelerated methods for stochastic convex programming problems. *arXiv preprint arXiv:1904.09015*, 2019.

24. D. Dvinskikh, A. Gasnikov, A. Rogozin, and A. Beznosikov. Parallel and distributed algorithms for ml problems. *arXiv preprint arXiv:2010.06585*, 2020.

25. D. Dvinskikh, E. Gorbunov, A. Gasnikov, P. Dvurechensky, and C. A. Uribe. On primal and dual approaches for distributed stochastic convex optimization over networks. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 7435–7440. IEEE, 2019.

26. D. M. Dvinskikh, A. I. Turin, A. V. Gasnikov, and S. S. Olemsenko. Accelerated and non accelerated stochastic gradient descent in model generality. *Matematicheskie Zametki*, 108(4):515–528, 2020.

27. P. Dvurechenskii, D. Dvinskikh, A. Gasnikov, C. Uribe, and A. Nedic. Decentralize and randomize: Faster algorithm for wasserstein barycenters. In *Advances in Neural Information Processing Systems*, pages 10760–10770, 2018.

28. P. Dvurechensky and A. Gasnikov. Stochastic intermediate gradient method for convex problems with stochastic inexact oracle. *Journal of Optimization Theory and Applications*, 171(1):121–145, 2016.

29. P. Dvurechensky, A. Gasnikov, and A. Tiurin. Randomized similar triangles method: A unifying framework for accelerated randomized optimization methods (coordinate descent, directional search, derivative-free method). *arXiv:1707.08486*, 2017.

30. A. Fallah, M. Gurbuzbalaban, A. Ozdaglar, U. Simsekli, and L. Zhu. Robust distributed accelerated stochastic gradient methods for multi-agent networks. *arXiv preprint arXiv:1910.08701*, 2019.

31. V. Feldman and J. Vondrak. High probability generalization bounds for uniformly stable algorithms with nearly optimal rate. *arXiv preprint arXiv:1902.10710*, 2019.

32. D. Foster, A. Sekhari, O. Shamir, N. Srebro, K. Srivastava, and B. Woodworth. The complexity of making the gradient small in stochastic convex optimization. *arXiv preprint arXiv:1902.04686*, 2019.

33. A. Gasnikov. Universal gradient descent. *arXiv preprint arXiv:1711.00394*, 2017.

34. A. Gasnikov and Y. Nesterov. Universal fast gradient method for stochastic compositional optimization problems. *arXiv:1604.05275*, 2016.

35. A. V. Gasnikov, A. A. Lagunovskaya, I. N. Usmanova, and F. A. Fedorenko. Gradient-free proximal methods with inexact oracle for convex stochastic nonsmooth optimization problems on the simplex. *Automation and Remote Control*, 77(11):2018–2034, Nov 2016. *arXiv:1412.3890*.

36. A. V. Gasnikov and Y. E. Nesterov. Universal method for stochastic compositional optimization problems. *Computational Mathematics and Mathematical Physics*, 58(1):48–64, 2018.

37. S. Ghadimi and G. Lan. Optimal stochastic approximation algorithms for strongly convex stochastic composite optimization i: A generic algorithmic framework. *SIAM Journal on Optimization*, 22(4):1469–1492, 2012.

38. S. Ghadimi and G. Lan. Stochastic first- and zeroth-order methods for nonconvex stochastic optimization. *SIAM Journal on Optimization*, 23(4):2341–2368, 2013. *arXiv:1309.5549*.

39. E. Gorbunov, D. Dvinskikh, and A. Gasnikov. Optimal decentralized distributed algorithms for stochastic convex optimization. *arXiv preprint arXiv:1911.07363*, 2019.

40. E. Gorbunov, P. Dvurechensky, and A. Gasnikov. An accelerated method for derivative-free smooth stochastic convex optimization. *arXiv preprint arXiv:1802.00622*, 2018.

41. E. Gorbunov, F. Hanzely, and P. Richtárik. Local sgld: Unified theory and new efficient methods. *arXiv preprint arXiv:2011.02828*, 2020.

42. E. Gorbunov, F. Hanzely, and P. Richtárik. A unified theory of sgld: Variance reduction, sampling, quantization and coordinate descent. In *International Conference on Artificial Intelligence and Statistics*, pages 680–690. PMLR, 2020.

43. E. Gorbunov, D. Kovalev, D. Makarenko, and P. Richtárik. Linearly converging error compensated sgld. *Advances in Neural Information Processing Systems*, 33, 2020.

44. E. Gorbunov, E. A. Vorontsova, and A. V. Gasnikov. On the upper bound for the expectation of the norm of a vector uniformly distributed on the sphere and the phenomenon of concentration of uniform measure on the sphere. *Mathematical Notes*, 106, 2019.
84. A. Nedic. Distributed gradient methods for convex machine learning problems in networks: Distributed optimization. *IEEE Signal Processing Magazine*, 37(3):92–101, 2020.

85. A. Nedic, A. Olshhevsky, and W. Shi. Achieving geometric convergence for distributed optimization over time-varying graphs. *SIAM Journal on Optimization*, 27(4):2597–2633, 2017.

86. A. Nedić and A. Ozdaglar. Distributed subgradient methods for multi-agent optimization. *IEEE Transactions on Automatic Control*, 54(1):48–61, 2009.

87. A. Nemirovski, A. Juditsky, G. Lan, and A. Shapiro. Robust stochastic approximation approach to stochastic programming. *SIAM Journal on Optimization*, 19(4):1574–1609, 2009.

88. Y. Nesterov. *Introductory Lectures on Convex Optimization: a basic course*. Kluwer Academic Publishers, Massachusetts, 2004.

89. Y. Nesterov. How to make the gradients small. *Optima*, 88:10–11, 2012.

90. Y. Nesterov. Lectures on convex optimization, volume 137. Springer, 2018.

91. Y. Nesterov and V. G. Spokoiny. Random gradient-free minimization of convex functions. *Foundations of Computational Mathematics*, 17(2):527–566, 2017.

92. L. M. Nguyen, P. H. Nguyen, M. van Dijk, P. Richtárik, K. Scheinberg, and M. Takač. Sgd and hogwild! convergence without the bounded gradients assumption. *arXiv preprint arXiv:1802.03801*, 2018.

93. A. Olshevsky, I. C. Paschalidis, and S. Pu. Asymptotic network independence in distributed optimization for machine learning. *arXiv preprint arXiv:1906.12345*, 2019.

94. A. Olshevsky, I. C. Paschalidis, and S. Pu. A non-asymptotic analysis of network independence for distributed stochastic gradient descent. *arXiv preprint arXiv:1906.02702*, 2019.

95. G. Peyré, M. Cuturi, et al. Computational optimal transport. *Foundations and Trends® in Machine Learning*, 11(5-6):355–607, 2019.

96. G. Qu and N. Li. Accelerated distributed nesterov gradient descent. *arXiv preprint arXiv:1906.12345*, 2019.

97. H. Robbins and S. Monro. A stochastic approximation method. *Annals of Mathematical Statistics*, 22:400–407, 1951.

98. R. T. Rockafellar. *Convex analysis*. Princeton university press, 2015.

99. A. Rogozin and A. Gasnikov. Projected gradient method for decentralized optimization over time-varying networks. *arXiv preprint arXiv:1911.08527*, 2019.

100. A. Rogozin and A. Gasnikov. Penalty-based method for decentralized optimization over time-varying graphs. In *International Conference on Optimization and Applications*, pages 239–256. Springer, 2020.

101. A. Rogozin, V. Lukoshkin, A. Gasnikov, D. Kovalov, and E. Shulgin. Towards accelerated rates for distributed optimization over time-varying networks. *arXiv preprint arXiv:2009.11069*, 2020.

102. K. Scaman, F. Bach, S. Bubeck, Y. T. Lee, and L. Massoulié. Optimal algorithms for smooth and strongly convex distributed optimization in networks. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 3027–3036. JMLR. org, 2017.

103. K. Scaman, F. Bach, S. Bubeck, Y. T. Lee, and L. Massoulié. Optimal convergence rates for convex distributed optimization in networks. *Journal of Machine Learning Research*, 20(159):1–31, 2019.

104. K. Scaman, F. Bach, S. Bubeck, L. Massoulié, and Y. T. Lee. Optimal algorithms for non-smooth distributed optimization in networks. In *Advances in Neural Information Processing Systems*, pages 2740–2749, 2018.

105. M. Schmidt, N. Le Roux, and F. Bach. Minimizing finite sums with the stochastic average gradient. *Mathematical Programming*, 162(1-2):83–112, 2017.

106. S. Shalev-Shwartz and S. Ben-David. *Understanding machine learning: From theory to algorithms*. Cambridge university press, 2014.

107. S. Shalev-Shwartz, O. Shamir, N. Srebro, and K. Sridharan. Stochastic convex optimization. In *COLT*, 2009.

108. O. Shamir. An optimal algorithm for bandit and zero-order convex optimization with two-point feedback. *Journal of Machine Learning Research*, 18:52:1–52:11, 2017. First appeared in arXiv:1507.08752.

109. O. Shamir. An optimal algorithm for bandit and zero-order convex optimization with two-point feedback. *Journal of Machine Learning Research*, 18(52):1–11, 2017.

110. W. Shi, Q. Ling, G. Wu, and W. Yin. Extra: An exact first-order algorithm for decentralized consensus optimization. *SIAM Journal on Optimization*, 25(2):944–966, 2015.

111. V. Spokoiny et al. Parametric estimation. finite sample theory. *The Annals of Statistics*, 40(6):2877–2909, 2012.

112. S. U. Stich. Local sgd converges fast and communicates little. *arXiv preprint arXiv:1805.09767*, 2018.

113. S. U. Stich, J.-B. Cordonnier, and M. Jaggi. Sparsified sgd with memory. In *Advances in Neural Information Processing Systems*, pages 4447–4458, 2018.

114. F. Stonyakin, D. Vdvinskikh, P. Dvurechensky, A. Kroshnin, O. Kuznetsova, A. Agafonov, A. Gasnikov, A. Tyurin, C. A. Uribe, D. Pasechnyuk, et al. Gradient methods for problems with inexact model of the objective. *arXiv preprint arXiv:1902.09001*, 2019.

115. F. Stonyakin, A. Tyurin, A. Gasnikov, P. Dvurechensky, A. Agafonov, D. Vdvinskikh, D. Pasechnyuk, S. Artamonov, and V. Piskunova. Inexact relative smoothness and strong convexity for optimization and variational inequalities by inexact model. *arXiv preprint arXiv:2001.09013*, 2020.

116. Y. Sun, A. Daneshmand, and G. Scutari. Convergence rate of distributed optimization algorithms based on gradient tracking. *arXiv preprint arXiv:1905.02637*, 2019.

117. Y. Sun, A. Daneshmand, and G. Scutari. Distributed optimization based on gradient-tracking revisited: Enhancing convergence rate via surrogation. *arXiv preprint arXiv:1905.02637*, 2020.

118. Y. Tang, J. Zhang, and N. Li. Distributed zero-order algorithms for nonconvex multi-agent optimization. *IEEE Transactions on Control of Network Systems*, 2020.

119. J. N. Tsitsiklis. Problems in decentralized decision making and computation. Technical report, Massachusetts Inst of Tech Cambridge Lab for Information and Decision Systems, 1984.

120. C. A. Uribe, D. Vdvinskikh, P. Dvurechensky, A. Gasnikov, and A. Nedić. Distributed computation of Wasserstein barycenters over networks. In *2018 IEEE 57th Annual Conference on Decision and Control (CDC)*, 2018. Accepted, arXiv:1803.02933.

121. C. A. Uribe, S. Lee, A. Gasnikov, and A. Nedić. Optimal algorithms for distributed optimization. *arXiv preprint arXiv:1712.00232*, 2017.

122. C. A. Uribe, S. Lee, A. Gasnikov, and A. Nedić. A dual approach for optimal algorithms in distributed optimization over networks. *Optimization Methods and Software*, pages 1–40, 2020.
124. S. Vaswani, F. Bach, and M. Schmidt. Fast and faster convergence of sgd for over-parameterized models and an accelerated perceptron. In *The 22nd International Conference on Artificial Intelligence and Statistics*, pages 1195–1204, 2019.

125. W. Wen, C. Xu, F. Yan, C. Wu, Y. Wang, Y. Chen, and H. Li. Terngrad: Ternary gradients to reduce communication in distributed deep learning. In *Advances in Neural Information Processing Systems*, pages 1509–1519, 2017.

126. B. Woodworth, K. K. Patel, and N. Srebro. Minibatch vs local sgd for heterogeneous distributed learning. *arXiv preprint arXiv:2006.04735*, 2020.

127. B. Woodworth, K. K. Patel, S. U. Stich, Z. Dai, B. Bullins, H. B. McMahan, O. Shamir, and N. Srebro. Is local sgd better than minibatch sgd? *arXiv preprint arXiv:2002.07839*, 2020.

128. L. Xiao and S. Boyd. Fast linear iterations for distributed averaging. *Systems & Control Letters*, 53(1):65–78, 2004.

129. J. Xu, Y. Tian, Y. Sun, and G. Scutari. Accelerated primal-dual algorithms for distributed smooth convex optimization over networks. *arXiv preprint arXiv:1910.10666*, 2019.

130. H. Ye, L. Luo, Z. Zhou, and T. Zhang. Multi-consensus decentralized accelerated gradient descent. *arXiv preprint arXiv:2005.00797*, 2020.

131. H. Ye, Z. Zhou, L. Luo, and T. Zhang. Decentralized accelerated proximal gradient descent. *Advances in Neural Information Processing Systems*, 33, 2020.

132. H. Yu, R. Jin, and S. Yang. On the linear speedup analysis of communication efficient momentum sgd for distributed non-convex optimization. *arXiv preprint arXiv:1905.03817*, 2019.

133. K. Yuan, Q. Ling, and W. Yin. On the convergence of decentralized gradient descent. *SIAM Journal on Optimization*, 26(3):1835–1854, 2016.

134. K. Zhou. Direct acceleration of saga using sampled negative momentum. *arXiv preprint arXiv:1806.11048*, 2018.

135. K. Zhou, F. Shang, and J. Cheng. A simple stochastic variance reduced algorithm with fast convergence rates. *arXiv preprint arXiv:1806.11027*, 2018.