Abstract—Rich interaction with the world requires extensive contact between robots and the objects in their environment. Most such contacts involve significant compliance between the contacting surfaces due to rubber pads or inflated grippers, soft objects to be manipulated, and soft surfaces for safe human-robot interaction. Accurate simulation of these contacts is critical for meaningful sim-to-real transfer. Compliant contact interactions generate contact surfaces of considerable extent, over which contact forces are distributed with varying pressure. Finite element methods can capture these effects but are too slow for most robotics applications. Consequently, in order to enable real-time simulation rates, most current simulation tools model contact as occurring between rigid bodies at a point or set of points using ad hoc methods to incorporate localized compliance. However, point contact is non-smooth, hard to extend to arbitrary geometry, and often introduces non-physical artifacts. Moreover, point contact misses important area-dependent phenomena critical for robust manipulation, such as net contact moment and slip control. Pressure Field Contact (PFC) was recently introduced as a method for detailed modeling of contact interface regions at rates much faster than elasticity-theory models, while at the same time predicting essential trends and capturing rich contact behavior. PFC was designed to work with coarsely-meshed objects while preserving continuity to permit use with error-controlled integrators. Here we introduce a discrete approximation of PFC suitable for use with velocity-level time steppers that enables execution at real-time rates. We evaluate the accuracy and performance gains of our approach and demonstrate its effectiveness in simulation of relevant manipulation tasks. The method is available in open source as part of Drake’s Hydroelastic Contact model.

Index Terms—Contact Modeling, Simulation and Animation, Grasping, Dynamics.

I. INTRODUCTION

There is a need for smooth, rich, artifact-free models of contact between arbitrary geometries as encountered in modern robotics applications such as grasping and manipulation, assistive and rehabilitative robotics, prosthetics, and unstructured environments. Most often these applications involve compliant surfaces such as padded grippers, deformable manipuland objects or soft surfaces for safe human-robot interaction. Moreover, with the emerging field of soft robotics, designers have begun to incorporate significant compliance in their robot designs; consider for instance the Soft-bubble gripper [1] in Fig. 1 for which the accurate prediction of contact patches is critical for meaningful sim-to-real transfer. Still, the rigid-body approximation of contact is at the core of many simulation engines enabling them to run at interactive rates.

Point contact is a useful and popular approximation of non-conforming contact (e.g. contact between a sphere and a half-space), but it does not extend well to conforming surfaces nor non-convex shapes. Localized compliance can be incorporated using spring-dampers [2], Hertz theory [3] and volumetric models [4], [5]. However, while point contact modeling approaches are fast, they are non-smooth, and extensions to arbitrary geometry often involve non-physical heuristics [6], [7] that heavily influence the correctness and accuracy of simulation results [8].

The Elastic Foundation Model [9] (EFM) computes rich contact patches providing an alternative to point contact that can solve many of its issues. However, current implementations [10] need highly refined meshes and can even miss contact interactions if coarse meshes are used. The work in [11] introduces the pressure field model; a modern rendition of EFM designed to work with coarse meshes at a computational cost suitable for real-time simulation. The pressure field model finds the contact surface as the manifold on which precalculated normal stresses from each object balance each other. The resultant wrench on each contacting body is then obtained as the surface integral of the normal stress. The model is completed with the Striebeck [12] model of friction leading to a system of ODEs that is advanced forward in time using error-controlled integration. An implementation of the pressure field model is available in open source as part of Drake’s [13] Hydroelastic Contact model, see Section II for details. We use hydroelastic contact model here to distinguish our work from the original pressure field model.
A drawback of the pressure field model is the large amount of stiffness introduced into the resulting system of ODEs to avoid viscous drift and to model near-rigid objects. With implicit integration, stability theory says we should be able to take large time steps even for very stiff ODEs. However, doing so in practice has proven difficult [14].

As an alternative to error-controlled integration, most current physics simulation engines use velocity-level time stepping. In this approach, time is advanced at discrete intervals of fixed size; contact impulses and the resulting velocities are found by solving a challenging Nonlinear Complementarity Problem (NCP), or some approximation of an NCP. This approach is taken by software targeted at engineering applications such as ODE [15], Dart [16], Vortex [17], MuJoCo [18] and Drake [13].

Given the success of velocity-level solvers in the community and guided by our own success with these methods [19], this work aims to incorporate the hydroelastic contact model [11] into a discrete formulation of contact. We strive to enable simulation of contact rich scenes, eliminate artifacts introduced by point contact, and capture area dependent phenomena otherwise missed by point contact while still performing at real-time rates.

The remainder of this paper is organized as follows. We briefly describe the hydroelastic contact model and introduce our notation in Section II. Section III describes the geometric computation of the contact surface. We present an equivalent tessellation in terms of polygonal face patches that leads to a drastic reduction in the number of contact constraints, and thus to a decrease in the size of the contact problems that needs to be solved at each time step. We introduce our discrete approximation of the hydroelastic contact model in Section IV. By drawing an analogy with point contact, we express the hydroelastic model as area-weighted point contact forces that we can incorporate within velocity-level discrete solvers. Finally, we demonstrate the effectiveness and accuracy of our approximation in a number of simulations relevant to robotics in Section V. We conclude with a summary of our findings in Section VI.

II. OVERVIEW OF THE HYDROELASTIC CONTACT MODEL

The hydroelastic contact model [11] combines two ideas: elastic foundation and hydrostatic pressure. Thus the model introduces an object-centric virtual pressure field $p_0$ to mimic the hydrostatic pressure field of a fluid. The pressure field is designed to be zero at the boundary of the object and to increase with depth towards the inside.

Given two overlapping objects $A$ and $B$ with pressure fields $p_{0A}$ and $p_{0B}$, respectively, the contact surface $S$ is modeled as the surface of equal pressure. The computation of the contact surface can be performed efficiently with a judicious choice of data structures [11]. We describe the original method and our specific contributions in Section III. The result of the contact surface computations is the surface $S$ along with the field of equal pressure $p_c = p_{0A} = p_{0B}$. Pressure $p_c$ only models the conservative, or elastic, forces. Dissipation is modeled using the Hunt and Crossley model [20] to define the total contact pressure

$$p(x) = p_c(x)(1 - \zeta v_n(x)), \quad (1)$$

where $v_n(x)$ is the normal component of the relative velocity between objects $A$ and $B$ at point $x \in S$, and $\zeta$ is the positive dissipation constant of the model. Friction is included using a regularized model of Coulomb friction [10]. Finally, the net forces and moments on each body is obtained as the surface integral of the normal and friction contributions on the contact surface, see [11] for details.

III. CONTACT SURFACE COMPUTATION

We represent the geometry of a compliant body with a tetrahedral volume mesh. Each vertex of this mesh stores a single scalar pressure value resulting in a piece-wise linear pressure field $p_0$ which can be used to interpolate pressure values at any point inside the volume.

A. Compliant-Compliant Contact

The contact surface between two compliant bodies $A$ and $B$ consists of a number of polygons [11]. We denote with $L_a : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $L_b : \mathbb{R}^3 \rightarrow \mathbb{R}$ the linear interpolation of the respective pressure fields within two tetrahedra $\tau_a \in A$ and $\tau_b \in B$ having a non-empty intersection. The surface on which $L_a$ equals $L_b$ defines an equilibrium plane $P_{ab}$. The contact surface is the intersection $P_{ab} \cap \tau_a \cap \tau_b$, a convex polygon with at most eight vertices. Fig. 2 illustrates this idea.

![Fig. 2: Steps to compute a contact polygon for compliant-compliant contact.](image)

b) Their equilibrium plane is clipped by the bottom tetrahedron into a square. c) The top tetrahedron clips the plane further into the final polygon, in this example, an octagon. d) Contact polygon with linearly interpolated equilibrium pressure.

B. Rigid-Compliant Contact

Rigid-Compliant contact is a feature introduced in Drake not previously considered in [11]. While a rigid object can be approximated as a compliant hydroelastic object with a very large modulus of elasticity, this approach can lead to numerical issues. Therefore, in Drake, we represent a rigid object solely with a surface mesh of triangles that tessellates its boundary. In this case, the contact surface corresponds to the surface of the rigid object clipped by the volume of the compliant object. Each polygon of the contact surface is the intersection of a triangle on the surface of the rigid object and a tetrahedron of the compliant object. The result is a convex polygon with at most seven vertices, Fig. 3.
The contact pressure \( p_0 \) is the linear interpolation of the compliant pressure field onto the contact surface.

![Fig. 3: The intersection of a rigid triangle and a compliant tetrahedron is a contact polygon with at most seven vertices. Pressure is linearly interpolated from the tetrahedron.](image)

**C. Triangulated vs. Polygonal Contact Surfaces**

A special treatment of these polygonal faces is presented in [11] to guarantee contact forces that are a continuous function of state as required for error-controlled integration. An \( n \) sided polygon is divided into a fan of \( n \) triangles that share a vertex at the polygon’s centroid, left in Fig. 4. This ensures that only zero area triangles are added/removed to the contact surface as objects move so that these topological changes do not introduce discontinuities in the contact forces. The result is a triangulation of the contact surface with a linear interpolation of the equilibrium pressure field defined on it.

This continuity, however, is not a requirement for discrete velocity-level methods used in this work and can therefore be relaxed. We propose to replace the fan of triangles by the original polygon, right in Fig. 4. In addition, we approximate the surface integral of the pressure field \( p_0 \) using a single quadrature point located at the polygon’s centroid. Since the pressure field is linear and the surface is planar, this approximation is exact. For non-linear quantities such as torque, this approximation is first-order in the size of the polygon.

Moreover, the polygonal representation leads to a significant reduction in the number of face elements representing the surface, a factor of seven in Fig. 4. As we will see in Section IV, this leads to the same reduction factor in the number of contact constraints and consequently to a much smaller contact problem.

**IV. POINT CONTACT APPROXIMATION**

Given the configuration of the system, the geometry engine used for point contact modeling reports a discrete set of contacts between pair of bodies. Each contact pair is described by the location of the contact point, a normal direction \( \hat{n} \in \mathbb{R}^3 \), and the signed distance or gap function \( \phi \in \mathbb{R} \). The relative velocity between the pair of bodies at the contact point is denoted with \( v_n \in \mathbb{R}^3 \). The normal velocity \( v_n = \hat{n} \cdot v_c = d\phi/dt \) is defined to be positive when bodies move apart.

A popular point contact model of compliance introduces a spring/damper at each contact point to model the normal force \( f_n \) as

\[
f_n = (-k\phi - d v_n)_+ ,
\]

where \( k > 0 \) is the point contact stiffness and \( d > 0 \) is a coefficient of linear dissipation. Since we take the positive part, the force is always repulsive. Software such as MSC Adams use similar state dependent laws and advance the acceleration-level dynamics of the system using numerical integration. Alternatively, physics engines such as ODE, Dart, and Algoryx use a velocity-level formulation in which Eq. (2) is cast as the complementarity condition

\[
0 \leq \phi + d c v_n + c f_n \perp f_n \geq 0 ,
\]

where \( c = k^{-1} \) is the compliance. In the limit \( c \to 0 \), Eq. (3) reduces to the rigid contact complementarity condition

\[
0 \leq \phi \perp f_n \geq 0 .
\]

Using the first order approximation \( \phi = \phi_0 + \delta t v_n \) where \( \phi_0 \) is the signed distance function at the previous time step and \( \delta t \) is step size, Eq. (3) becomes a linear complementarity condition between the velocities of the system and the contact forces

\[
0 \leq \phi_0 + (\delta t + d c) v_n + c f_n \perp f_n \geq 0 .
\]

The key idea introduced in this work is to approximate the force contribution from each of the polygons described in Section III-C using a first order expansion in time that resembles a point contact model. To this end, we write the elastic force contribution \( f_{n,e} \) in the normal direction from a polygon with area \( A \) as

\[
f_{n,e} = A p_e ,
\]

where \( p_e \) is the hydroelastic pressure field evaluated at the centroid of the polygon, as discussed in Section III-C. Using first order Taylor expansion, we approximate the pressure as

\[
f_{n,e} = A \left( p_{e,0} + \delta t \frac{dp_e}{dt} \right)_+ ,
\]

![Fig. 4: Triangular tessellation (left) as required in [11] and the proposed polygonal approximation (right). The pressure field is linear on the polygon as shown by the contour lines and the color shading. The white rectangle outline is a visual cue for the spanning plane of the contact polygon relative to the compliant tetrahedron drawn in orange outline.](image)
where $p_{e,0}$ is the hydroelastic pressure at the previous time step. The positive part must be taken to ensure repulsive forces. We will show in the next subsections that the time rate of the pressure at the surface can be approximated as

$$\frac{dp_e}{dt} = g v_n, \quad (7)$$

where $g$ is an effective pressure gradient, with units of Pa/m and $v_n$ is the normal velocity at the centroid. Using this approximation in Eq. (6), we can write

$$f_{n,e} = (-k \phi)_+, \quad (8)$$

with

$$k = -g A, \quad \phi_0 = \frac{p_{e,0}}{g}, \quad \phi = \phi_0 + \delta t v_n. \quad (9)$$

Using this surrogate signed distance $\phi_0$ and stiffness $k$, we can incorporate the hydroelastic model into a point contact solver framework.

A. One-Dimensional Analysis

We first analyze a much simpler one-dimensional configuration in which two bodies $A$ and $B$ with hydroelastic pressure fields $p_A(x)$ and $p_B(x)$, respectively, move relative to each other along the horizontal axis, see Fig. 5.

![Fig. 5: In one dimension, each shaded area represents a compliant hydroelastic body. The point at which pressure equalizes within the overlap region defines the contact surface, a point in one dimension.](image)

As the three-dimensional implementation of the model uses a linear representation of the pressure fields within each tetrahedron, we use linear pressure fields in our one-dimensional setup

$$p_A(x) = g_A(x - x_A(t)) + b_A, \quad (10)$$
$$p_B(x) = g_B(x - x_B(t)) + b_B, \quad (11)$$

where $g_A$ and $g_B$ are the pressure gradients in each body, $x_A(t)$ and $x_B(t)$ are points rigidly affixed to $A$ and $B$, respectively, and $b_A$ and $b_B$ are simply the pressure values at $x_A(t)$ and $x_B(t)$, respectively.

In one dimension, the contact surface is described by point $s(t)$ that moves along the horizontal axis. The surface point $s(t)$ is found by equating the hydroelastic pressures

$$p_e(s(t)) = g_A(s(t) - x_A(t)) + b_A = g_B(s(t) - x_B(t)) + b_B. \quad (12)$$

We find the velocity of the interface $\dot{s}(t)$ by taking the time derivative of Eq. (12)

$$g_A(\dot{s}(t) - v_A(t)) = g_B(\dot{s}(t) - v_B(t)), \quad (13)$$

where $v_A(t)$ and $v_B(t)$ are the velocities of body $A$ and $B$, respectively. Since the pressure fields are fixed in the body frames, $b_A = b_B = 0$. From this equation, we obtain the speed of the contact interface

$$\dot{s}(t) = \sigma_A v_A(t) + \sigma_B v_B(t), \quad (14)$$

with

$$\sigma_A = \frac{-g_A}{g_B - g_A}, \quad (15)$$
$$\sigma_B = \frac{g_B}{g_B - g_A}, \quad (16)$$

and $\sigma_A + \sigma_B = 1$. Notice that in one dimension $g_A < 0$, $g_B > 0$ and therefore $\sigma_A > 0$ and $\sigma_B > 0$.

We can obtain the rate of change of the pressure at the interface by taking the time derivative of Eq. (12) and substituting the interface velocity from Eq. (14). After some minimal algebraic manipulation, the result is

$$\frac{dp_e(s)}{dt} = \frac{g_A g_B}{g_B - g_A} v_n, \quad (17)$$

where we define the separation velocity as $v_n = v_B - v_A$. Since $g_A g_B < 0$, $dp/dt < 0$ for $v_n > 0$, the pressure decreases as the bodies move away from each other, as expected. Notice that Eq. (17) is also valid in the limit to one of the bodies being rigid, i.e., one of the gradients is infinity.

B. Three-Dimensional Extension

At each polygon in the contact surface, our approximation considers motions along its normal as depicted in Fig. 6. At the centroid $S$ of the polygon, we consider witness points $A_w$ and $B_w$ that are instantaneously coincident with $S$ but move as rigidly affixed to bodies $A$ and $B$, respectively. We denote with $S_{vA} = S_{vB} = \hat{S}_A = \hat{S}_B = \hat{S}_n$ the velocity of these witness points with respect to $S$. Along the normal direction, these witness points move with speed $S_{vA} = S_{vB} = S_{vA} = S_{vB} = \hat{S}_n$. The relative velocity of $B_w$ with respect to $A_w$ is the normal separation velocity $v_n = S_{vB} - S_{vA}$. Along the normal direction, body $A$ approaches $S$ with an effective gradient along the normal direction equal to $g_A = \nabla p_A \cdot \hat{n}$. Similarly, $B$ approaches the surface with an effective gradient $g_B = \nabla p_B \cdot \hat{n}$. We can now use these one-dimensional quantities along the normal direction with the results from the previous section to write

$$\frac{dp_e}{dt} = g v_n, \quad (18)$$
$$g = \frac{g_A g_B}{g_B - g_A}, \quad (19)$$
$$g_A = \nabla p_A \cdot \hat{n}, \quad (20)$$
$$g_B = \nabla p_B \cdot \hat{n}. \quad (21)$$

![Diagram](image)
as needed by the surrogate point contact quantities in Eq. (9).

It is important to notice that the approximation in Eq. (18) is exact only in the pedagogical case when the pressure gradients and the normal direction fall on the same line. In practice, this is generally not true. Moreover, special care must be taken when \( g_A > 0 \) and \( g_B < 0 \). Since the discrete approximation of point contact requires \( k > 0 \), we simply ignore polygons where the conditions \( g_A < 0 \) and \( g_B > 0 \) are not satisfied. We find that this is not a major problem in practice since this situation corresponds to corner cases of the hydroelastic contact model for which pushing into the object leads to a decrease of the contact forces instead of an increase as expected.

V. RESULTS AND DISCUSSION

We present a series of simulation cases to assess the robustness, accuracy, and performance of our method.

A. Sliding and Spinning Disk

To assess the accuracy of our method’s ability to capture the highly non-linear coupling between net force and torque, we study a sliding and spinning disk with a known analytical solution [21].

Based on the dimensions of a U.S. quarter dollar coin, we simulate a disk of radius \( R = 1.213 \) cm, thickness \( t = 1.75 \) mm, mass \( m = 5.67 \) g, friction coefficient \( \mu = 0.2 \), and elastic modulus \( E = 1.0 \) GPa lying flat on a horizontal plane set into motion with initial values of translational velocity \( v \) and angular velocity \( \omega \). The analytical result for this example establishes a dimensionless parameter \( \varepsilon = \frac{v}{R} \) that, regardless of initial conditions, converges to \( \varepsilon^* \approx 0.653 \) as the coin comes to rest. We set initial angular and translational velocities to span initial values \( \varepsilon_0 \) in the range \([0.1; 10]\).

A fan of 152 triangles discretizes the circular geometry of the coin. To estimate the error introduced by the discrete geometry, we first simulate our model using error controlled integration to a tight accuracy of \( 10^{-2}\% \). We find the numerical solution with discrete geometry converges to \( \varepsilon^*_{\text{disc}} = 0.64426 \), at only 1.3% error from \( \varepsilon^* \).

We now use our velocity-level discrete solver with a fixed time step of \( \delta t = 10^{-3} \) s to compute numerical approximations \( \varepsilon^*_{\text{num}} \) from various initial conditions. Theory [21] predicts a constant \( \varepsilon^* \) regardless of the initial conditions. The numerical results confirm this prediction within 0.01-0.5% of \( \varepsilon^*_{\text{disc}} \) and within 1.3% of the theoretical value \( \varepsilon^* = 0.653 \) (dashed line).

The hydroelastic model is able to resolve the strong coupling between force and torque on the disk at high accuracy compared to the analytical solution. This highly predictive level of results is not possible with point contact.

B. Pancake Flip

To demonstrate the effectiveness of our method, we compare results using both triangular and polygonal tessellations of the contact surfaces in a simulation of a relevant robotic task.

In this scenario, a Kinova JACO arm (6 DOF) is outfitted with a highly compliant Soft-bubble gripper [1]. The arm is anchored to a table which has a stand holding a spatula, a cylindrical stove top, and a pancake, modeled as a flat ellipsoid, on top. The Soft-bubble gripper and the pancake are modeled as compliant objects using the hydroelastic contact model, whereas point contact is used between the remaining rigid objects. We identify contact model parameters by matching the level of compliance observed in the real setup.

The controller process tracks a prescribed sequence of Cartesian end-effector keyframe poses. We use force feedback to gauge successful grasps and to know when the spatula makes contact with the stove top.

The robot is commanded to grab the spatula from the stand and subsequently scoop, raise, and flip the pancake over on the stove, see Fig. 8 and the accompanying supplemental video. Figure 9 shows the number of faces throughout the simulation using both triangular and polygonal tessellations. On average, the number of faces is 4.05 times smaller when using the polygonal tessellation. Still, the model is able to resolve the net torque on the spatula needed to achieve

Fig. 6: In three dimensions we approximate the motion of the surface by considering the one-dimensional motion along the normal direction \( \hat{n} \), taking into account directional gradients \( g_A = dp_A/dn = \nabla p_A \cdot \hat{n} \) and \( g_B = dp_B/dn = \nabla p_B \cdot \hat{n} \).
a secure grasp. Moreover, with the resulting reduction in the number of contact constraints, our solver performs 4.09 times faster. The computation of polygonal tessellations is only about 10% faster than the corresponding triangular tessellations.

In summary, our methodology enabled the simulation of this complex manipulation task at interactive real-time rates.

Figure 1 (left) shows a closeup of the contact geometry used for this model. Notice that while well resolved, we use a rather coarse tessellation of the compliant bubble surfaces of the gripper. The polygonal tessellation provides a rich representation of the contact patch exhibited by an elongated shape induced by the geometry of the handle, Fig. 1 (right).

This level of grasp control is achieved by properly resolving contact patch area changes; this degree of control would be very difficult, if not impossible, to emulate using point contact approaches.

![Figure 8: The scoop process of the pancake flip task. See associated video.](image)

![Figure 9: Number of faces generated as a function of time. Important events during the task are highlighted.](image)

**C. Spatula Slip Control**

We now demonstrate the effectiveness of our method to capture area-dependent phenomena such as the net torque required to successfully grasp an object. We simulate the aforementioned Soft-bubble gripper [1] anchored to the world holding a spatula by the handle horizontally. The grasp force is commanded to vary between 1 N and 16 N with square wave having a 6 second period and a 75% duty cycle, left on Fig. 10. This controller results in a periodic transition from a secure grasp with stiction to a loose grasp where the spatula is allowed to rotate within the grasp in a controlled manner, see Fig. 1 and the accompanying video.

![Figure 10: Commanded grasp force (left) and spatula pitch (right).](image)

**VI. CONCLUSIONS**

We presented a discrete approximation of the hydroelastic contact model to enable simulation of contact rich patches using velocity-level discrete solvers for simulation at real-time rates.

We demonstrated the highly predictive nature of this model in a test case with strong coupling between net force and torque, matching known analytical results to within 1.3% without parameter tuning beyond choosing a mesh that can reasonably represent the geometry and choosing a time step that can resolve the temporal dynamics of the problem. Even though the polygonal tessellations are coarser than the original triangular tessellations from [11], we demonstrated the effectiveness of the approach to predict area-dependent phenomena such as the net torque required for the successful completion of a manipulation task.

Our novel surface representation in terms of polygonal faces leads to a drastic reduction in the number of contact constraints, a significantly smaller contact problem at each time step, and consequently a substantial speedup enabling simulation at interactive rates.

The hydroelastic contact model and the discrete approximation presented in this work are made available in the open-source robotics toolbox Drake [13]. The new model has been used extensively for work conducted at the Toyota Research Institute on prototyping and validating controllers for dexterous manipulation of complex geometries [19].

**ACKNOWLEDGMENT**

We thank the reviewers for their feedback, Sean Curtis for his invaluable support on geometry and Drake implementation details, Michael Sherman for his insight and helpful discussions, and Naveen Kuppuswamy for advising with the pancake demo setup.
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