Concave-convex critical problems for the spectral fractional laplacian with mixed boundary conditions

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Abstract
In this work we study the existence of solutions to the following critical fractional problem with concave-convex nonlinearities,

\[
\begin{aligned}
(-\Delta)^s u &= \lambda u^q + u^{2s-1}, \quad u > 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Sigma_D, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Sigma_N,
\end{aligned}
\]

with \( \Omega \subset \mathbb{R}^N \), \( N > 2s \), a smooth bounded domain, \( \frac{1}{2} < s < 1, \ 0 < q < 2^*_s - 1, \ q \neq 1 \), being \( 2^*_s = \frac{2N}{N-2s} \) the critical fractional Sobolev exponent, \( \lambda > 0 \), \( \nu \) is the outwards normal to \( \partial \Omega \); \( \Sigma_D, \Sigma_N \) are smooth \((N-1)\)-dimensional submanifolds of \( \partial \Omega \) such that \( \Sigma_D \cup \Sigma_N = \partial \Omega, \Sigma_D \cap \Sigma_N = \emptyset, \) and \( \Sigma_D \cap \Sigma_N = \Gamma \) is a smooth \((N-2)\)-dimensional submanifold of \( \partial \Omega \). In particular, we will prove that, for the sublinear case \( 0 < q < 1 \), there exists at least two solutions for every \( 0 < \lambda < \Lambda \) for certain \( \Lambda \in \mathbb{R} \) while, for the superlinear case \( 1 < q < 2^*_s - 1 \), we will prove that there exists at least one solution for every \( \lambda > 0 \). We will also prove that solutions are bounded.

Keywords
Fractional Laplacian (Primary) · Critical problem · Concave-Convex nonlinearities · Mixed boundary conditions

Mathematics Subject Classification
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1 Introduction

In this work we study the existence of solutions to the following concave-convex critical problem involving the spectral fractional Laplacian,
\[
\begin{cases}
(-\Delta)^s u = \lambda u^q + u^{2^*_s - 1}, & \text{in } \Omega, \\
B(u) = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where \(\frac{1}{2} < s < 1\), \(0 < q < 2^*_s - 1\), \(q \neq 1\), being \(2^*_s = \frac{2N}{N-2s}\) the critical fractional Sobolev exponent, \(\lambda > 0\), \(\Omega \subset \mathbb{R}^N\) is a smooth bounded domain with mixed boundary conditions
\[
B(u) = u \chi_{\Sigma_D} + \frac{\partial u}{\partial \nu} \chi_{\Sigma_N},
\]

where \(\nu\) is the outward normal to \(\partial\Omega\), \(\chi_A\) denotes the characteristic function of a set \(A\), \(\Sigma_D\) and \(\Sigma_N\) are smooth \((N-1)\)-dimensional submanifolds of \(\partial\Omega\) such that \(\Sigma_D\) is a closed submanifold with measure \(|\Sigma_D| = \alpha\), \(\alpha \in (0, |\partial\Omega|)\); \(\Sigma_D \cap \Sigma_N = \emptyset\), \(\Sigma_D \cup \Sigma_N = \partial\Omega\) and \(\Sigma_D \cap \Sigma_N = \Gamma\) is a smooth \((N-2)\)-dimensional submanifold. The range \(\frac{1}{2} < s < 1\) is the appropriate range for mixed boundary problems due to the natural embedding of the associated functional space, see Remark 1.

Concave-convex critical problems are nowadays a well-known topic in the field of nonlinear PDE’s as they have been broadly studied since the works of Brezis and Nirenberg (cf. [9]) and Ambrosetti, Brezis and Cerami (cf. [2]). The seminal paper [9] deals with critical elliptic problems with Dirichlet boundary data for the classical Laplace operator \((s = 1)\) and the exponent \(q = 1\). The authors introduced the main ideas to prove existence of solutions to critical problems with lower order perturbation terms. In [2], the authors analyzed the main effects concave-convex nonlinearities, \(f_{\lambda}(u) = \lambda u^q + u^p\), \(0 < q < 1 < p \leq 2^* - 1 = \frac{N+2}{N-2}\), have on issues related to the existence and multiplicity of solutions. Problems similar to \((P_\lambda)\) have been also studied for the \(p\)-Laplace operator (cf. [4]) or fully nonlinear operators (cf. [15]) both considering Dirichlet boundary data.

Results in these lines also hold when one considers the classical Laplace operator endowed now with mixed Dirichlet-Neumann boundary data. Lions, Pacella and Tricarico (cf. [23]) analyzed the pure critical power problem \((\lambda = 0)\) and the attainability of the associated Sobolev constant. In [19], the corresponding mixed Brezis-Nirenberg problem was studied. Mixed concave-convex problems were addressed in [17] by considering subcritical nonlinearities and in [1] by considering critical problems involving Caffarelli-Kohn-Nirenberg weights.

Regarding the fractional setting, the aim of this work is to extend to the mixed boundary data framework the results of [5], where the Dirichlet problem \((|\Sigma_N| = 0)\) corresponding to \((P_\lambda)\) was studied. Results regarding multiplicity of solutions to concave-convex critical problems under Dirichlet conditions were recently proved in [18] for \(0 < q < 1\). Concave-convex critical fractional problems dealing with a different fractional operator, defined through a singular integral, were studied in [6].
Fractional problems or nonlocal problems involving more general kernels and critical nonlinearities were studied in [25, 27–31].

Coming back to problem (P_λ), using a generalized Pohozaev identity (cf. [26]) it can be seen that, under Dirichlet boundary conditions (|Σ_N| = 0), problem (P_λ) has no solution for λ = 0 and Ω a star-shaped domain (cf. [7]). Similar nonexistence results based on Pohozaev type identities for mixed problems can be found in [23] and [16]. Nevertheless, mixed boundary critical problems behave quite differently from critical Dirichlet problems and, taking the Dirichlet boundary part small enough, one can prove the existence of a positive solution for the pure critical problem corresponding to λ = 0, (cf. [23], [1, Theorem 2.1] and [16, Theorem 2.9]). Indeed, (P_λ) with λ ≥ 0 and the exponent q = 1 was analyzed in [16] where it was proved the following.

**Theorem 1** [16, Theorem 1.1] Assume that q = 1, \(\frac{1}{2} < s < 1\) and \(N \geq 4s\). Let \(λ_{1,s}\) be the first eigenvalue of the fractional operator \((−Δ)^s\) with mixed Dirichlet-Neumann boundary conditions (1.1). Then, the problem (P_λ)

1. has no solution for \(λ \geq λ_{1,s}\),
2. has at least one solution for \(0 < λ < λ_{1,s}\),
3. has at least one solution for \(λ = 0\) and \(|Σ_D|\) small enough.

Our aim is then to obtain existence results for (P_λ) for the whole range \(0 < q < 2_s^* - 1\), \(q \neq 1\). Precisely we will prove the following two main results.

**Theorem 2** Let \(0 < q < 1\), \(\frac{1}{2} < s < 1\) and \(N > 2s\). Then, there exists \(0 < Λ < ∞\) such that the problem (P_λ)

1. has no solution for \(λ > Λ\),
2. has a minimal solution for any \(0 < λ < Λ\). Moreover, the family of minimal solutions is increasing with respect to \(λ\),
3. has at least one solution for \(λ = Λ\),
4. has at least two solutions for \(0 < λ < Λ\).

**Theorem 3** Let \(1 < q < 2_s^* - 1\), \(\frac{1}{2} < s < 1\) and \(N > 2s \left(1 + \frac{1}{q}\right)\). Then, the problem (P_λ) has at least one solution provided that either

a) \(q + 1 > 2_s^* - 2\) and \(λ > 0\), or
b) \(q + 1 \leq 2_s^* - 2\) and \(λ\) is sufficiently large.

The proof of Theorem 2 follows from nowadays well-known arguments. The existence of a positive minimal solution follows by using sub and supersolution, comparison and iterative arguments. To prove the existence of a second positive solution we will need to use a recently proved Strong Maximum Principle for mixed fractional problems (cf. [24]), from which we will obtain a separation result (see Lemma 8 below) that implies that the minimal solution is indeed a minimum of the energy functional associated to (P_λ). This step is fundamental to prove (4) in Theorem 2 since it allows us to use a Mountain Pass type argument. Due to the lack of compactness of the Sobolev embedding at the critical exponent \(2_s^*\), we prove next that a local PS condition holds below a certain critical level \(c_{D−N}^*\). We conclude by constructing paths whose energy is below the critical level \(c_{D−N}^*\). At this point we have two options as the mixed pure
critical problem can have solution for Dirichlet boundary size small enough (Theorem 1 - (3) above). If the Sobolev constant associated to \((P_{\lambda})\) (see Definition 2 below) is attained we use the associated extremal functions to find paths with energy below the critical level \(c^*_{D-N}\). Otherwise, this step is accomplished by the use of appropriate truncations of the extremal functions of the fractional Sobolev inequality. Due to the presence of mixed boundary conditions a careful control on this truncation process is needed.

Most of the arguments of the concave case \(0 < q < 1\) also works for the convex case \(q > 1\) so we will only indicate the main steps to prove Theorem 3. Let us note that condition \(q + 1 > 2s - 2\) can be restated as \(N > \frac{2s(q+3)}{q+1}\) in the lines of [6, Theorem 1.2] (see also [4, Theorem 3.3]).

**Organization of the paper:** In Section 2 we introduce the appropriate functional setting and some results for a Sobolev-like constant associated to \((P_{\lambda})\) useful in the sequel. In Section 3 we will address the proof of Theorem 2. We finish with the proof of Theorem 3 in Section 4.

## 2 Functional setting and preliminaries

As far as the fractional Laplace operator is concerned, we recall its definition given through the spectral decomposition. Let \((\varphi_i, \lambda_i)\) be the eigenfunctions (normalized with respect to the \(L^2(\Omega)\)-norm) and the eigenvalues of \((-\Delta)\) under homogeneous mixed Dirichlet–Neumann boundary data, respectively. Then, \((\varphi_i, \lambda_i^s)\) are the eigenfunctions and eigenvalues of the fractional operator \((-\Delta)^s\), i.e., its action on a smooth function \(u = \sum_{j \geq 1} \langle u, \varphi_j \rangle \varphi_j\) is given by

\[
(-\Delta)^s u = \sum_{j \geq 1} \lambda_j^s \langle u, \varphi_j \rangle \varphi_j.
\]

Thus, the fractional Laplace operator \((-\Delta)^s\) is well defined through its spectral decomposition in the following space of functions that vanish on \(\Sigma_D\),

\[
H_{\Sigma_D}^s(\Omega) = \left\{ u = \sum_{j \geq 1} a_j \varphi_j \in L^2(\Omega) : u|_{\Sigma_D} = 0, \|u\|^2_{H_{\Sigma_D}^s(\Omega)} = \sum_{j \geq 1} a_j^2 \lambda_j^s < \infty \right\}.
\]

For \(u \in H_{\Sigma_D}^s(\Omega)\), it follows that \(\|u\|_{H_{\Sigma_D}^s(\Omega)} = \left\|(-\Delta)^{\frac{s}{2}} u\right\|_{L^2(\Omega)}\).

**Remark 1** By [21, Theorem 11.1], if \(0 < s \leq \frac{1}{2}\) then \(H_{0}^s(\Omega) = H^s(\Omega)\) and, therefore, also \(H_{\Sigma_D}^s(\Omega) = H^s(\Omega)\), while for \(\frac{1}{2} < s < 1\), \(H_{0}^s(\Omega) \subset H^s(\Omega)\). Hence, the range \(\frac{1}{2} < s < 1\) guarantees that \(H_{\Sigma_D}^s(\Omega) \subset H^s(\Omega)\) and it provides us with the appropriate functional space for the mixed boundary problem \((P_{\lambda})\).
This definition of fractional powers of the Laplace operator allows us to integrate by parts in the proper spaces, so that a natural definition of weak solution (cf. [5, 7, 10, 12]) to problem \((P_\lambda)\) is the following.

**Definition 1** We say that \(u \in H_{\Sigma_D}^s(\Omega)\) is a weak solution to \((P_\lambda)\) if, for all \(\psi \in H_{\Sigma_D}^s(\Omega)\),

\[
\int_{\Omega} (-\Delta)^{s/2} u (-\Delta)^{s/2} \psi \; dx = \int_{\Omega} \left( \lambda u^q + u^{2^*_s-1} \right) \psi \; dx.
\]  

(2.1)

Note that the right-hand side of (2.1) is well defined because of the embedding \(H_{\Sigma_D}^s(\Omega) \hookrightarrow L^{2^*_s}(\Omega)\), so \(u \in H_{\Sigma_D}^s(\Omega)\) then \(\lambda u^q + u^{2^*_s-1} \in L^{\frac{2N}{N+2}} \hookrightarrow (H_{\Sigma_D}^s(\Omega))'\).

The energy functional associated to problem \((P_\lambda)\) is

\[
I_\lambda(u) = \frac{1}{2} \int_{\Omega} |(-\Delta)^{s/2} u|^2 \; dx - \frac{\lambda}{q+1} \int_{\Omega} |u|^{q+1} \; dx - \frac{1}{2^*_s} \int_{\Omega} |u|^{2^*_s} \; dx.
\]  

(2.2)

\(I_\lambda\) is well defined in \(H_{\Sigma_D}^s(\Omega)\) and positive critical points of \(I_\lambda\) correspond to solutions of \((P_\lambda)\) (cf. [5, 7]).

Due to the nonlocal nature of the fractional operator \((-\Delta)^{s}\) some difficulties arise when one tries to obtain an explicit expression of the action of the fractional Laplacian on a given function. In order to overcome these difficulties, we use the ideas in [11] together with those of [7, 10, 12] to give an equivalent definition of \((-\Delta)^{s}\) by means of an auxiliary problem that we introduce next.

Given \(\Omega \subset \mathbb{R}^N\), we set the cylinder \(C_\Omega = \Omega \times (0, \infty) \subset \mathbb{R}^{N+1}_+\). We denote by \((x, y)\) those points that belong to \(C_\Omega\) and by \(\partial_L C_\Omega = \partial \Omega \times [0, \infty)\) the lateral boundary of the cylinder. Let us also denote by \(\Sigma_D^s = \Sigma_D \times [0, \infty)\) and \(\Sigma_N^s = \Sigma_N \times [0, \infty)\) as well as \(\Gamma^* = \Gamma \times (0, \infty)\). It is clear that, by construction,

\[
\Sigma_D^s \cap \Sigma_N^s = \emptyset, \quad \Sigma_D^s \cup \Sigma_N^s = \partial_L C_\Omega \quad \text{and} \quad \Sigma_D^s \cap \Sigma_N^s = \Gamma^*.
\]

Then, given a function \(u \in H_{\Sigma_D}^s(\Omega)\) we define its \(s\)-harmonic extension, \(w(x, y) = E_s[u(x)]\), as the solution to the problem

\[
\begin{cases}
-\text{div}(y^{1-2s} \nabla w) = 0 & \text{in } C_\Omega, \\
B(w) = 0 & \text{on } \partial_L C_\Omega, \\
w(x, 0) = u(x) & \text{on } \Omega \times \{y = 0\},
\end{cases}
\]

where \(B(w) = w \chi + \frac{\partial w}{\partial \nu} \chi\), being \(\nu\), with an abuse of notation, the outward normal to \(\partial_L C_\Omega\). The extension function belongs to the space

\[
X_{\Sigma_D}^s(C_\Omega) := \overline{C_0^\infty((\Omega \cup \Sigma_N) \times [0, \infty))}^{\|\cdot\|_{X_{\Sigma_D}^s(C_\Omega)}}.
\]
where we define
\[
\| \cdot \|_{X^s_{\Sigma} \Omega}^2 := \kappa_s \int_{\Omega} y^{1-2s} |\nabla (\cdot)|^2 \, dx \, dy,
\] (2.3)
with \( \kappa_s = 2^{2s-1} \frac{\Gamma(s)}{\Gamma(1-s)} \). The space \( X^s_{\Sigma} \Omega \) is a Hilbert space equipped with the norm \( \| \cdot \|_{X^s_{\Sigma} \Omega} \) which is induced by the scalar product
\[
\langle w, z \rangle_{X^s_{\Sigma} \Omega} = \kappa_s \int_{\Omega} y^{1-2s} \langle \nabla w, \nabla z \rangle \, dx \, dy.
\]
Moreover, the following inclusions are satisfied,
\[
X^s_0 \Omega \subset X^s_{\Sigma} \Omega \subset X^s \Omega,
\] (2.4)
being \( X^s_0 \Omega \) the space of functions in \( X^s \Omega \equiv H^1 \Omega, y^{1-2s} \, dx \, dy \) that vanish on the lateral boundary of \( \Omega \), denoted by \( \partial L \Omega \).

The key point of the extension function is that it is related to the fractional Laplacian of the original function through the formula
\[
\frac{\partial w}{\partial \nu^s} := -\kappa_s \lim_{y \to 0^+} y^{1-2s} \frac{\partial w}{\partial y} = (-\Delta)^s u(x).
\]

Then, we can reformulate \((P_{\lambda})\) in terms of the extension problem as follows
\[
\left\{ \begin{array}{ll}
-\text{div}(y^{1-2s} \nabla w) = 0 & \text{in } \Omega, \\
B(w) = 0 & \text{on } \partial L \Omega, \\
\frac{\partial w}{\partial \nu^s} = \lambda w^q + w^{2s-1} & \text{on } \Omega \times \{y = 0\}.
\end{array} \right.
\] \((P_{\lambda}^*)\)

A weak solution to \((P_{\lambda}^*)\) is a function \( w \in X^s_{\Sigma} \Omega \) such that
\[
\kappa_s \int_{\Omega} y^{1-2s} \langle \nabla w, \nabla \varphi \rangle \, dx \, dy = \int_{\Omega} \left( \lambda w^q(x, 0) + w^{2s-1}(x, 0) \right) \varphi(x, 0) \, dx,
\]
for all \( \varphi \in X^s_{\Sigma} \Omega \). Given \( w \in X^s_{\Sigma} \Omega \) a solution to \((P_{\lambda}^*)\), the function \( u(x) = Tr[w](x) = w(x, 0) \) belongs to \( H^s_{\Sigma} \Omega \) and it solves \((P_{\lambda})\) and vice-versa, if \( u \in H^s_{\Sigma} \Omega \) is a solution to \((P_{\lambda})\) then \( \tilde{w} = E_s[u] \in X^s_{\Sigma} \Omega \) solves \((P_{\lambda}^*)\). Thus, both formulations are equivalent and the extension operator
\[
E_s : H^s_{\Sigma} \Omega \to X^s_{\Sigma} \Omega,
\]
allows us to switch between each other. Moreover, due to the choice of the constant \( \kappa_s \), (cf. [7, 11]) the extension operator \( E_s \) is an isometry, i.e.,
\[
\| E_s[\varphi] \|_{X^s_{\Sigma} \Omega} = \| \varphi \|_{H^s_{\Sigma} \Omega} \quad \text{for all } \varphi \in H^s_{\Sigma} \Omega.
\] (2.5)
Finally, the energy functional associated to problem \((P^*_\lambda)\) is

\[
J_\lambda(w) = \frac{\kappa_s}{2} \int_{\Omega} y^{1-2s} |\nabla w|^2 \, dx \, dy - \frac{\lambda}{q+1} \int_{\Omega} |w|^{q+1} \, dx - \frac{1}{2s} \int_{\Omega} |w|^{2s} \, dx.
\]  

(2.6)

Plainly, (positive) critical points of \(J_\lambda\) in \(X^s_{\Sigma_D}(\mathcal{C}_\Omega)\) correspond to (positive) critical points of \(I_\lambda\) in \(H^s_{\Sigma_D}(\Omega)\). Moreover, minima of \(J_\lambda\) also correspond to minima of \(I_\lambda\). The proof of this fact is similar to the one of the Dirichlet case, (cf. [5, Proposition 3.1]).

When one considers Dirichlet boundary conditions the following trace inequality holds (cf. [7, Theorem 4.4]): there exists \(C = C(N, s, r, |\Omega|) > 0\) such that, for all \(z \in X^s_{0}(\mathcal{C}_\Omega)\),

\[
\kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s} |\nabla z(x, y)|^2 \, dx \, dy \geq C \left( \int_{\Omega} |z(x, 0)|^r \, dx \right)^{\frac{2}{r}}.
\]  

(2.7)

for \(1 \leq r \leq 2_s^*, \ N > 2s\). Because of (2.5) the trace inequality (2.7) is equivalent to the fractional Sobolev inequality,

\[
C \left( \int_{\Omega} |v|^r \, dx \right)^{\frac{2}{r}} \leq \int_{\Omega} |(-\Delta)^{\frac{s}{2}} v|^2 \, dx \quad \text{for all } v \in H^s_{0}(\Omega),
\]  

(2.8)

with \(1 \leq r \leq 2_s^*, \ N > 2s\). If \(r = 2_s^*\) the best constant in (2.8) (and, thanks to (2.5), in (2.7)), namely the fractional Sobolev constant, denoted by \(S(N, s)\), is independent of \(\Omega\) and \(S(N, s) = 2^{2s} \pi^s \frac{\Gamma \left( \frac{N+2s}{2} \right)}{\Gamma \left( \frac{N}{2} \right)} \left( \frac{\Gamma \left( \frac{N}{2} \right)}{\Gamma \left( \frac{N-2s}{2} \right)} \right)^2 \). Since it is not achieved in any bounded domain we have

\[
\kappa_s \int_{\mathbb{R}^{N+1}} y^{1-2s} |\nabla z(x, y)|^2 \, dx \, dy \geq S(N, s) \left( \int_{\mathbb{R}^{N}} |z(x, 0)|^{2s} \, dx \right)^{\frac{2}{2s}},
\]

for all \(z \in \mathcal{X}^s(\mathbb{R}^{N+1}_+),\) where \(\mathcal{X}^s(\mathbb{R}^{N+1}_+) = \mathcal{C}_0^\infty(\mathbb{R}^{N} \times [0, \infty)) || \cdot ||_{\mathcal{X}^s(\mathbb{R}^{N+1}_+)}\), with \(|| \cdot ||_{\mathcal{X}^s(\mathbb{R}^{N+1}_+)}\) defined as (2.3) replacing \(\mathcal{C}_\Omega\) by \(\mathbb{R}^{N+1}_+\). Indeed, in the whole space the latter inequality is achieved for the family \(u_{\varepsilon} = E_s[u_{\varepsilon}]\),

\[
u_{\varepsilon}(x) = \frac{\varepsilon^{-\frac{N-2s}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{N-2s}{2}}},
\]

(2.9)

with arbitrary \(\varepsilon > 0\), (cf. [7]).

When mixed boundary conditions are considered the situation is quite similar as Dirichlet condition is imposed on \(\Sigma_D \subset \partial \Omega\) with \(0 < |\Sigma_D| < |\partial \Omega|\).
Definition 2 The Sobolev constant relative to the Dirichlet boundary $\Sigma_D$ is defined by

$$\tilde{S}(\Sigma_D) = \inf_{\begin{subarray}{l} u \in H_{x,D}^s(\Omega) \\ u \not\equiv 0 \end{subarray}} \frac{\|u\|^2_{H_{x,D}^s(\Omega)}}{\|w\|^2_{L^s_0(\Omega)}} = \inf_{\begin{subarray}{l} w \in \mathcal{X}_{\Sigma_D}^s(\mathcal{C}\Omega) \\ w \not\equiv 0 \end{subarray}} \frac{\|w\|^2_{\mathcal{X}_{\Sigma_D}^s(\mathcal{C}\Omega)}}{\|w(\cdot, 0)\|^2_{L^s_0(\Omega)}}.$$

Note that the second equality in Definition 2 follows since the $s$-extension minimizes $\|\cdot\|_{\mathcal{X}_{\Sigma_D}^s(\mathcal{C}\Omega)}$ along all functions with the same trace on $\{y = 0\}$ (cf. [16, Lemma 2.4]) and the isometry (2.5). Moreover, since $0 < |\Sigma_D| < |\partial\Omega|$, by the inclusions (2.4), we have

$$0 < \tilde{S}(\Sigma_D) < \inf_{\begin{subarray}{l} u \in H_0^s(\Omega) \\ u \not\equiv 0 \end{subarray}} \frac{\|u\|^2_{H_0^s(\Omega)}}{\|u\|^2_{L^s_0(\Omega)}}. \quad (2.10)$$

The constant $\tilde{S}(\Sigma_D)$ plays a main role in existence issues of $(P_\lambda)$, i.e., $\tilde{S}(\Sigma_D)$ is to mixed problems what Sobolev constant $S(N, s)$ is to Dirichlet problems.

Remark 2 By the spectral definition of $(-\Delta)^s$ and Hölder’s inequality we get $\tilde{S}(\Sigma_D) \leq |\Omega|^{\frac{s}{2N}} \lambda_1^s(\alpha)$, with $\lambda_1(\alpha)$ the first eigenvalue of the Laplace operator under mixed boundary conditions on $\Sigma_D = \Sigma_D(\alpha)$ and $\Sigma_N = \Sigma_N(\alpha)$. Since $\lambda_1(\alpha) \to 0$ as $\alpha \to 0^+$, (cf. [17, Lemma 4.3]), we have $\tilde{S}(\Sigma_D) \to 0$ as $\alpha \to 0^+$.

Gathering together (2.10) and (2.5) it follows that, for all $\varphi \in \mathcal{X}_{\Sigma_D}^s(\mathcal{C}\Omega)$,

$$\tilde{S}(\Sigma_D) \|\varphi(x, 0)\|^2_{L^s_0(\Omega)} \leq \|\varphi(x, 0)\|^2_{H_{x,D}^s(\Omega)} = \|E_s[\varphi(x, 0)]\|^2_{\mathcal{X}_{\Sigma_D}^s(\mathcal{C}\Omega)}.$$ 

This Sobolev–type inequality provides us with a trace inequality adapted to the mixed boundary data framework.

Lemma 1 [16, Lemma 2.4] For all $\varphi \in \mathcal{X}_{\Sigma_D}^s(\mathcal{C}\Omega)$, we have

$$\tilde{S}(\Sigma_D) \left( \int_{\Omega} |\varphi(x, 0)|^{2s} \right)^{\frac{2}{2s}} \leq \kappa_s \int_{\mathcal{C}\Omega} y^{1-2s} |\nabla \varphi|^2 dxdy.$$ 

Let us collect some results for $\tilde{S}(\Sigma_D)$ proven in [16] useful in the sequel.

Proposition 1 [16, Proposition 3.6] If $\Omega \subset \mathbb{R}^N$, $N > 2s$, is a smooth bounded domain, then

$$\tilde{S}(\Sigma_D) \leq 2^{\frac{2s}{N}} S(N, s).$$

Theorem 4 [16, Theorem 2.9] If $\tilde{S}(\Sigma_D) < 2^{\frac{2s}{N}} S(N, s)$ then $\tilde{S}(\Sigma_D)$ is attained.
This result highlights the difference between Dirichlet problems and mixed Dirichlet-Neumann problems. If $|\Sigma_N| = 0$ (Dirichlet case) and $\lambda = 0$, we have the pure critical power problem which has no positive solution, under some geometrical assumptions on $\Omega$, for instance, under star-shapeness assumptions (cf. [7], [26]). Analogous non-existence results based on a Pohozaev–type identity and star-shapeness like assumptions hold for mixed problems (cf. [23], [16]). Nevertheless, in the mixed case, $\tilde{S}(\Sigma_D)$ can be achieved by Theorem 4. Even more, by Remark 2, the hypothesis of Theorem 4 holds by making $|\Sigma_D|$ small enough (cf. [16, Theorem 1.1-(3)]).

At last, we enunciate a concentration-compactness result adapted to our fractional setting with mixed boundary conditions useful in the sequel. First, we recall the concept of a tight sequence.

**Definition 3** We say that a sequence $\{y^{1-2s} |\nabla w_n|^2\}_{n\in\mathbb{N}} \subset L^1(\mathcal{C}_\Omega)$ is tight if for any $\eta > 0$ there exists $\rho > 0$ such that

$$\int_{\{|y| > \rho\}} \int_\Omega y^{1-2s} |\nabla w_n|^2 dx dy \leq \eta, \; \forall n \in \mathbb{N}.$$  

Roughly speaking, the tight condition avoids the 'evanescence to infinity' of the mass, i.e., it prevents the $L^1$-norm of the sequence from moving to infinity.

**Theorem 5** [16, Theorem 4.4] Let $\{w_n\} \subset X^s_\Sigma_D(\mathcal{C}_\Omega)$ be a weakly convergent sequence to $w$ in $X^s_\Sigma_D(\mathcal{C}_\Omega)$ such that $\{y^{1-2s} |\nabla w_n|^2\}_{n\in\mathbb{N}}$ is tight. Let us denote $u_n = Tr[w_n]$, $u = Tr[w]$ and let $\mu, \nu$ be two nonnegative measures such that, in the sense of measures,

$$\kappa s y^{1-2s} |\nabla w|^2 \to \mu \quad \text{and} \quad |u_n|^{2^*_s} \to \nu.$$  

Then there exist an at most countable set $\mathfrak{I}$ and points $\{x_i\}_{i\in\mathfrak{I}} \subset \overline{\Omega}$ such that

1. $\nu = |u|^{2^*_s} + \sum_{i\in\mathfrak{I}} v_i \delta_{x_i}$, $v_i > 0$,
2. $\mu \geq \kappa s y^{1-2s} |\nabla w|^2 + \sum_{i\in\mathfrak{I}} \mu_i \delta_{x_i}$, $\mu_i > 0$,
3. $\mu_i \geq \tilde{S}(\Sigma_D) v_i^{2^*_s}$.

Using Theorem 5 and the Brezis-Lieb Lemma (cf. [8]) it is proved the following.

**Theorem 6** [16, Theorem 4.5] Let $w_m$ be a minimizing sequence of $\tilde{S}(\Sigma_D)$. Then either $w_m$ is relatively compact or the weak limit, $w \equiv 0$. Even more, in the latter case there exist a subsequence $w_m$ and a point $x_0 \in \overline{\Sigma_N}$ such that

$$\kappa s y^{1-2s} |\nabla w|^2 \to \tilde{S}(\Sigma_D) \delta_{x_0} \quad \text{and} \quad |u_m|^{2^*_s} \to \delta_{x_0},$$  

with $u_m = Tr[w_m]$.
In this section we prove Theorem 2. The existence of a positive minimal solution and related results follow from nowadays well known arguments so we will be brief in details. The existence of a second positive solution follows from the following scheme: first we prove a separation result deduced from a recent work of the author (cf. [24]) which implies that the minimal solution is indeed a minimum of the energy functional \( I_\lambda \). Next, due to the lack of compactness of the Sobolev embedding at the critical exponent \( 2^*_s \), we prove a local PS condition below the critical level \( c_* \). Finally, we construct paths below the critical level \( c_* \). This is done either using the extremal functions of \( \tilde{S}(\Sigma_D) \) (in case this constant is attained) or, bearing in mind Theorems 5 and 6, by concentrating the extremal functions of the Sobolev inequality at points on the Neumann boundary \( \Sigma_N \).

Let us consider the fractional mixed problem with a general nonlinearity,

\[
\begin{aligned}
&(-\Delta)^s u = f(u), \quad u > 0 \text{ in } \Omega, \\
&B(u) = 0 \quad \text{on } \partial\Omega.
\end{aligned}
\]  

\[\text{(Pf)}\]

Note that \((P_\lambda)\) corresponds to \((P_f)\) for \( f(u) = f_\lambda(u) = \lambda|u|^{q-1}u + |u|^{2^*_s-2}u \). The associated energy functional, \( I_f : H^s_{\Sigma_D}(\Omega) \mapsto \mathbb{R} \), is given by

\[
I_f(u) = \frac{1}{2} \int_\Omega |(-\Delta)^{s/2}u|^2 dx - \int_\Omega F(u) dx,
\]

where \( F(u) = \int_0^u f(t) dt \) (resp. \( F_\lambda(u) = \int_0^u f_\lambda(t) dt = \frac{\lambda}{q+1} |u|^{q+1} + \frac{1}{2^*_s} |u|^{2^*_s} \)). The corresponding extension problem reads

\[
\begin{aligned}
&-\text{div}(y^{1-2s}\nabla w) = 0 \quad \text{in } C_\Omega, \\
&B(w) = 0 \quad \text{on } \partial C_\Omega, \\
&w > 0 \quad \text{on } \Omega \times \{y = 0\}, \\
&\frac{\partial w}{\partial \nu^s} = f(w(x, 0)) \quad \text{on } \Omega \times \{y = 0\}.
\end{aligned}
\]

\[\text{(P^*_f)}\]

The associated energy functional, \( J_f : \mathcal{X}_{\Sigma_D}(\mathbb{C}_\Omega) \mapsto \mathbb{R} \), is

\[
J_f(w) = \frac{\kappa_s}{2} \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla w|^2 dy dx - \int_\Omega F(w(x, 0)) dx.
\]

**Lemma 2** A function \( u_0 \in H^s_{\Sigma_D}(\Omega) \) is a local minimum of \( I_f \) if and only if \( w_0 = E_s[u_0] \in \mathcal{X}_{\Sigma_D}(\mathbb{C}_\Omega) \) is a local minimum of \( J_f \).

**Proof** The proof follows as in [5, Proposition 3.1], so we omit the details. \(\square\)

**Proposition 2** Let \( u \in H^s_{\Sigma_D}(\Omega) \) be a solution to \( (P_f) \) with

\[
0 \leq f(t) \leq C(1 + |t|^p) \quad \text{for all } t \in \mathbb{R} \text{ and some } 1 < p \leq 2^*_s - 1.
\]
Then \( u \in L^\infty(\Omega) \).

**Proof** Following verbatim the proof of [5, Proposition 5.1], by a truncation argument we deduce \( u^{\beta+1} \in L^{2\beta}(\Omega) \) for \( \beta \geq 0 \). Then, by an iteration argument, we get \( f(u(x)) \in L^r(\Omega) \), \( r > \frac{N}{2s} \) after a finite number of steps. We conclude \( u \in L^\infty(\Omega) \) by [13, Theorem 3.7]. \( \Box \)

The next result deals with the sub and supersolutions method.

**Lemma 3** [14, Lemma 5.2] Suppose that there exist a subsolution \( w_1 \) and a supersolution \( w_2 \) to \((P^*_f)\), i.e., \( w_1, w_2 \in X_{S_D}^s(\mathcal{E}_\Omega) \) such that, for any nonnegative \( \phi \in X_{S_D}^s(\mathcal{E}_\Omega) \),

\[
\kappa_s \int_{\mathcal{E}_\Omega} y^{1-2s} \nabla w_1 \nabla \phi dx dy \leq \int_{\Omega} f(w_1(x,0))\phi(x,0)dx ,
\]

\[
\kappa_s \int_{\mathcal{E}_\Omega} y^{1-2s} \nabla w_2 \nabla \phi dx dy \geq \int_{\Omega} f(w_2(x,0))\phi(x,0)dx .
\]

Assume moreover that \( w_1 \leq w_2 \) in \( \mathcal{E}_\Omega \). Then, there exists a solution \( w \) to problem \((P^*_f)\) verifying \( w_1 \leq w \leq w_2 \) in \( \mathcal{E}_\Omega \).

Finally we recall a comparison result.

**Lemma 4** [14, Lemma 5.3] Let \( w_1, w_2 \in X_{S_D}^s(\mathcal{E}_\Omega) \) be respectively a positive subsolution and a positive supersolution to \((P^*_f)\) and assume that \( f(t)/t \) is decreasing for \( t > 0 \). Then \( w_1 \leq w_2 \) in \( \mathcal{E}_\Omega \).

We address now the proof of Theorem 2.

**Lemma 5** Let \( \Lambda = \sup \{ \lambda > 0 : (P_\lambda) \text{has solution} \} \). Then, \( 0 < \Lambda < \infty \).

**Proof** Let \((\lambda_1, \varphi_1)\) be the first eigenvalue and a corresponding positive eigenfunction of \((-\Delta)^s\) in \( \Omega \). Using \( \varphi_1 \) as a test function in \((P_\lambda)\), we have

\[
\int_{\Omega} (\lambda u^q + u^{2s-1})\varphi_1 dx = \lambda_1^s \int_{\Omega} u \varphi_1 dx . \tag{3.1}
\]

Since there exists constants \( c = c(N, s, q) < 1 \) and \( \delta = \frac{2s-2}{2s-q} \) such that \( \lambda t^q + t^{2s-1} > c\lambda^\delta t \) for any \( t > 0 \), from (3.1) we deduce \( c\lambda^\delta < \lambda_1^s \) and hence \( \Lambda < \infty \). In particular, this also proves that there is no solution to \((P_\lambda)\) for \( \lambda > \Lambda \).

In order to prove that \( \Lambda > 0 \), we prove, by means of the sub and supersolution technique, the existence of solution to \((P^*_\lambda)\) for any small positive \( \lambda \). Indeed, for \( \varepsilon > 0 \) small enough, \( \hat{U} = \varepsilon E_s[\varphi_1] \) is a subsolution to \((P^*_\lambda)\). Because of Proposition 2, a supersolution can be constructed as an appropriate multiple of the function \( G \), the solution to

\[
\begin{cases}
-\text{div}(y^{1-2s} \nabla G) = 0 & \text{in } \mathcal{C}_\Omega , \\
B(G) = 0 & \text{on } \partial_L \mathcal{C}_\Omega , \\
\frac{\partial G}{\partial v^s} = 1 & \text{on } \Omega \times \{ y = 0 \} .
\end{cases}
\]

\( \Box \) Springer
Note that, as the trace function \( g(x) = G(x, 0) \) is a solution to
\[
\begin{cases}
(-\Delta)^s g = 1 & \text{in } \Omega, \\
B(g) = 0 & \text{on } \partial\Omega,
\end{cases}
\]
by [13, Theorem 3.7] we have \( \|g\|_{L^\infty(\Omega)} < +\infty \). Next, since \( 0 < q < 1 \) we can find \( \lambda_0 > 0 \) such that for all \( 0 < \lambda \leq \lambda_0 \) there exists \( M = M(\lambda) \) such that
\[
M \geq \lambda M^q \|g\|_{L^\infty(\Omega)} + M^{2s-1} \|g\|_{L^{2s-1}(\Omega)}^{2s-1}.
\]  
(3.2)

As a consequence, the function \( h = Mg \) satisfies \( M = (-\Delta)^s h \geq \lambda h^q + h^{2s-1} \) and, by the Maximum Principle (cf. [12, Lemma 2.3]), the extension function \( \overline{U} = Es[h] \) is a supersolution and \( \overline{U} \leq \overline{U} \). Applying Lemma 3 we conclude the existence of a solution \( U(x, y) \) to \((P^*_\lambda)\). Therefore, its trace \( u(x) = U(x, 0) \) is a solution to problem \((P_\lambda)\) with \( \lambda < \lambda_0 \).

**Lemma 6** Problem \((P_\lambda)\) has at least a positive minimal solution for every \( 0 < \lambda < \Lambda \). Moreover, the family \( \{u_\lambda\} \) of minimal solutions is increasing in \( \lambda \).

**Proof** By definition, for any \( 0 < \lambda < \Lambda \) there exists \( \mu \in (\lambda, \Lambda] \) such that \((P^*_\mu)\) has a solution \( W_\mu \). It is easy to see that \( W_\mu \) is a supersolution for \((P^*_\lambda)\).

On the other hand, let \( V_\lambda \) be the unique solution to \((P^*_f)\) with \( f(t) = \lambda t^q \) (the existence can be deduced by minimization, while uniqueness follows from Lemma 4). It is clear that \( V_\lambda \) is a subsolution to \((P^*_\lambda)\) and, by Lemma 4, we have \( V_\lambda \leq W_\mu \). Thus, by Lemma 3, we conclude that there is a solution to \((P^*_\lambda)\) and, as a consequence, for the whole open interval \((0, \Lambda)\).

Finally, we prove the existence of a minimal solution for all \( 0 < \lambda < \Lambda \). Given a solution \( u \) to \((P_\lambda)\) we take \( U = Es[u] \) with \( U \) solution to \((P^*_\lambda)\). By Lemma 4, we have \( V_\lambda \leq U \) with \( V_\lambda \) solution to \((P^*_f)\) with \( f(t) = \lambda t^q \). Then, \( u_\lambda(x) = V_\lambda(x, 0) \) is a subsolution to \((P_\lambda)\) and the monotone iteration
\[
(-\Delta)^s u_{n+1} = \lambda u_n^q + u_n^f, \quad u_n \in H^s_{\Sigma_D}(\Omega) \quad \text{with} \quad u_0 = v_\lambda,
\]
verifies \( u_n \leq U(x, 0) = u \) and \( u_n \not\leq u_\lambda \) with \( u_\lambda \) solution to problem \((P_\lambda)\). In particular, \( u_\lambda \leq u \) and we conclude that \( u_\lambda \) is a minimal solution.

The monotonicity follows directly from the first part of the proof, by taking \( U_\mu = Es[u_\mu] \) which leads to \( u_\lambda \leq u_\mu \) whenever \( 0 < \lambda < \mu \leq \Lambda \).

**Lemma 7** Problem \((P_\lambda)\) has at least one solution if \( \lambda = \Lambda \).

To prove Lemma 7 we need the following result which guarantees that the linearized equation of \((P_\lambda)\) has non-negative eigenvalues at the minimal solution.

**Proposition 3** Let \( u_\lambda \in H^s_{\Sigma_D}(\Omega) \) be the minimal solution to problem \((P_\lambda)\) and let us define \( a_\lambda = a_\lambda(x) = \lambda qu_\lambda^{q-1} + (2s - 1)u_\lambda^{2s-2} \). Then, the operator \( ((-\Delta)^s - a_\lambda(x)) \)

\(\square\)
with mixed boundary conditions has a first eigenvalue \( \nu_1 \geq 0 \). In particular, it follows that

\[
\int_{\Omega} \left( |(-\Delta)^{s/2} v|^2 - a_\lambda v^2 \right) \, dx \geq 0, \quad \text{for any } v \in H^s_{\Sigma D}(\Omega). \quad (3.3)
\]

**Proof** The proof follows verbatim that of [14, Proposition 5.1], so we omit the details. \(\square\)

**Proof of Lemma 7** Let \( \{\lambda_n\} \) be a sequence such that \( \lambda_n \nearrow \Lambda \) and let \( u_n = u_{\lambda_n} \) be the minimal solution to (\( P_{\lambda_n} \)). Let \( U_n = E_s[u_n] \), then

\[
I_{\lambda_n}(u_n) = \frac{1}{2} \int_{\Omega} |(-\Delta)^{s} u_n|^2 \, dx - \frac{\lambda_n}{q+1} \int_{\Omega} |u_n|^{q+1} \, dx - \frac{1}{2s} \int_{\Omega} |u_n|^{2s} \, dx.
\]

Moreover, since \( u_n \) is a solution to (\( P_{\lambda_n} \)), it also satisfies

\[
\int_{\Omega} |(-\Delta)^{s} u_n|^2 \, dx = \lambda_n \int_{\Omega} |u_n|^{q+1} \, dx + \int_{\Omega} |u_n|^{2s} \, dx.
\]

On the other hand, using (3.3) with \( v = u_n \),

\[
\int_{\Omega} |(-\Delta)^{s} u_n|^2 \, dx - \lambda_n q \int_{\Omega} |u_n|^{q+1} \, dx - (2s - 1) \int_{\Omega} |u_n|^{2s} \, dx \geq 0.
\]

As in [2, Theorem 2.1], we conclude \( I_{\lambda_n}(u_n) \) \( < \) 0. Moreover, as \( I_{\lambda_n}'(u_n) = 0 \), then \( \|u_n\|_{H^s_{\Sigma D}(\Omega)} \leq C \). Hence, there exists a weakly convergent subsequence \( u_n \rightarrow u \in H^s_{\Sigma D}(\Omega) \) and, thus, \( u \) is a weak solution of (\( P_\lambda \)) for \( \lambda = \Lambda \). \(\square\)

Having proved the first three items of Theorem 2, next we focus on proving the existence of a second solution. As commented above, first we show that the minimal solution is a local minimum of the energy functional \( I_\lambda \); so we can use the Mountain Pass Theorem, obtaining a minimax PS sequence. To find a second solution, we prove next a local PS\(_c\) condition for energy levels \( c \) under a critical level \( c^*_{D-N} \). Let us stress that, in order to find a second solution, it is fundamental that the minimal solution is a minimum of the energy functional \( I_\lambda \) or, equivalently, its \( s \)-harmonic extension \( w_\lambda = E_s[u_\lambda] \) is a minimum of \( J_\lambda \).

Following [17] we begin with a separation Lemma. Let \( v \) be the solution to

\[
\begin{aligned}
(-\Delta)^s v &= g & \text{in } \Omega, \\
B(v) &= 0 & \text{on } \partial\Omega,
\end{aligned}
\quad (3.4)
\]

with \( g \in L^p(\Omega), \quad p > \frac{N}{s} \).

The following result is proven in [24].

**Theorem 7** [24, Theorem 1.2] Let \( u \) be the solution to

\[
\begin{aligned}
(-\Delta)^s u &= f & \text{in } \Omega, \\
B(u) &= 0 & \text{on } \partial\Omega,
\end{aligned}
\quad (3.5)
\]

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with \( f \in L^\infty(\Omega) \), \( f \geq 0 \) and let \( v \) be the solution to (3.4). Then, there exists a constant \( C > 0 \) such that
\[
\left\| \frac{v}{u} \right\|_{L^\infty(\Omega)} \leq C \|g\|_{L^p(\Omega)},
\]
with the constant \( C \) depending on \( N, \ p, \ \Omega, \ \Sigma_D, \ \|u\|_{L^\infty(\Omega)}, \ \|f\|_{L^\infty(\Omega)} \) and \( 1/(\int_\Omega f(z)d(z)dz) \) where \( d(x) = dist(x, \partial\Omega) \).

Let us set \( v \) to be the solution to (3.4) with the particular choice \( g \equiv 1 \) and note that if \( u \) solves \((P_\lambda)\) then, by Proposition 2, \( f_\lambda(u(x)) \in L^\infty(\Omega) \), so it also satisfies (3.5). Then, by comparison, \( u(x) \leq C_1 v(x), \ x \in \Omega, \) by taking \( C_1 = \max_{x \in \Omega} f_\lambda(u(x)) \).
Precisely, Theorem 7 deals with the opposite inequality, namely, whether \( v(x) \leq C_2 u(x), \ x \in \Omega \) holds true for some constant \( C_2 > 0 \). Next, let us define the class
\[
\mathcal{C}_v(\Omega) = \left\{ \omega \in C^0(\overline{\Omega}) \cap H^1_{\Sigma_D}(\Omega) : \left\| \frac{\omega}{v} \right\|_{L^\infty(\Omega)} < \infty \right\}.
\]
Plainly, \( \mathcal{C}_v(\Omega) \) contains all \( \omega \in C^0(\overline{\Omega}) \cap H^1_{\Sigma_D}(\Omega) \) for which there exists some \( 0 < C < \infty \) such that \( \omega(x) \leq C v(x), \ x \in \Omega \). In particular, by Proposition 2 and [13, Theorem 1.2], if \( u \) solves \((P_\lambda)\) with \( 0 < \lambda < \Lambda \), then \( u \in \mathcal{C}_v(\Omega) \). It is worth to note that the function \( v \) is to mixed problems what the distance function \( d(x) = dist(x, \partial\Omega) \) is to Dirichlet problems (cf. [5, Lemma 3.2], [6, Lemma 2.3]). By application of Theorem 7 we obtain the following separation result in the \( \mathcal{C}_v(\Omega) \)-topology.

Lemma 8 Let \( 0 < \lambda_0 < \lambda_1 < \lambda_2 < \Lambda \) and \( u_{\lambda_0}, u_{\lambda_1} \) and \( u_{\lambda_2} \) be the minimal solutions to \((P_\lambda)\) with \( \lambda = \lambda_0, \lambda_1 \) and \( \lambda_2 \) respectively. Then, if \( X = \{ \omega \in \mathcal{C}_v(\Omega) : u_{\lambda_0} \leq \omega \leq u_{\lambda_2} \} \), there exists \( \varepsilon > 0 \) such that
\[
u_{\lambda_1} + \varepsilon B_1(0) \subset X,
\]
where \( B_1(0) = \{ \omega \in \mathcal{C}_v(\Omega) : \|\omega/v\|_{L^\infty(\Omega)} < 1 \} \).

Proof By the Maximum Principle (cf. [12, Lemma 2.3]), \( 0 < u_{\lambda_0} < u_{\lambda_1} < u_{\lambda_2} \). Next, the function \( \underline{u} := u_{\lambda_1} - u_{\lambda_0} \) solves \((-\Delta)\underline{u} = f_{\lambda_1}(u_{\lambda_1}) - f_{\lambda_0}(u_{\lambda_0}) \) and the function \( \overline{u} := u_{\lambda_2} - u_{\lambda_1} \) solves \((-\Delta)\overline{u} = f_{\lambda_2}(u_{\lambda_2}) - f_{\lambda_1}(u_{\lambda_1}) \), in both cases with the same boundary condition as in problem \((P_\lambda)\). Since \( f_{\lambda}(t) \) is increasing in \( t \), then both \( f_{\lambda_1}(u_{\lambda_1}) - f_{\lambda_0}(u_{\lambda_0}) \geq 0 \) and \( f_{\lambda_2}(u_{\lambda_2}) - f_{\lambda_1}(u_{\lambda_1}) \geq 0 \) so, by Proposition 2, they are also bounded. By Theorem 7, there exists \( \varepsilon > 0 \) such that
\[
u_{\lambda_0}(x) + \varepsilon v(x) \leq u_{\lambda_1}(x) \leq u_{\lambda_2}(x) - \varepsilon v(x), \quad \text{for all } x \in \overline{\Omega},
\]
and the result follows. \( \square \)

Let us define the functions
\[
\overline{F}_\lambda(u) = \int_0^u \overline{f}_\lambda(t)dt \quad \text{with} \quad \overline{f}_\lambda(t) = \begin{cases} \lambda t q^+ + t^{2s-1} & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases}
\]
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and the energy functional

\[ \overline{I}_\lambda(u) = \frac{1}{2} \|u\|^2_{H^s_{\Sigma D}(\Omega)} - \int_{\Omega} F_\lambda(u) \, dx. \]

Clearly \( \overline{I}_\lambda(u) = I_\lambda(u) \), if \( u > 0 \). The Euler–Lagrange equation of \( \overline{I}_\lambda(u) \) is

\[
\begin{cases}
(-\Delta)^s u = \lambda(u^+)^q + (u^+)^{2^*_s-1} & \text{in } \Omega, \\
B(u) = 0 & \text{on } \partial\Omega,
\end{cases}
\]

\((P^+_\lambda)\)

where \( u^+ = \max\{0, u\} \). If \( u \) is a nontrivial solution to \((P^+_\lambda)\), i.e., \( u \) is a nontrivial critical point of \( \overline{I}_\lambda \), by the Maximum Principle (cf. [12, Lemma 2.3]), \( u \) is strictly positive in \( \Omega \) and, therefore, it is also a solution to \((P_\lambda)\), i.e., a positive critical point of \( I_\lambda \); and vice-versa, if \( u \) is a solution to \((P_\lambda)\), then \( u > 0 \) in \( \Omega \) and \( u^+ = u \), so that \( u \) is a solution to \((P^+_\lambda)\).

**Proposition 4** For all \( \lambda \in (0, \Lambda) \) there exists \( u_0 \in H^s_{\Sigma D}(\Omega) \) a solution to \((P_\lambda)\) which is a local minimum of \( \overline{I}_\lambda \) in \( C_v(\Omega) \): there exists \( r > 0 \) such that

\[ \overline{I}_\lambda(u_0) \leq \overline{I}_\lambda(\varphi), \quad \text{for all } \varphi \in C_v(\Omega) \text{ with } \left\| u_0 - \varphi \right\|_{L^\infty(\Omega)} \leq r. \]

**Proof** Fixed \( 0 < \lambda_1 < \lambda < \lambda_2 < \Lambda \) let us consider the corresponding minimal solutions \( u_{\lambda_1} \) and \( u_{\lambda_2} \). Note that \( u_{\lambda_1} \leq u_{\lambda_2} \) and \( u_{\lambda_1}, u_{\lambda_2} \) are a subsolution and a supersolution to \((P_\lambda)\). Moreover, setting \( \varepsilon := u_{\lambda_2} - u_{\lambda_1} \), then

\[ \begin{cases}
(-\Delta)^s \varepsilon \geq 0 & \text{in } \Omega, \\
B(\varepsilon) = 0 & \text{on } \partial\Omega.
\end{cases} \]

Next, let us take the functions

\[ F^*_\lambda(u) = \int_0^u f^*_\lambda(t) \, dt \quad \text{with} \quad f^*_\lambda(t) = \begin{cases}
\overline{f}_\lambda(u_{\lambda_1}) & \text{if } t \leq u_{\lambda_1}, \\
\overline{f}_\lambda(t) & \text{if } u_{\lambda_1} < t < u_{\lambda_2}, \\
\overline{f}_\lambda(u_{\lambda_2}) & \text{if } t \geq u_{\lambda_2},
\end{cases} \]

and the energy functional

\[ I^*_\lambda(u) = \frac{1}{2} \|u\|^2_{H^s_{\Sigma D}(\Omega)} - \int_{\Omega} F^*_\lambda(u) \, dx. \]

It is clear that \( I^*_\lambda \) attains its global minimum at some \( u_0 \in H^s_{\Sigma D}(\Omega) \), that is

\[ I^*_\lambda(u_0) \leq I^*_\lambda(u), \quad \text{for all } u \in H^s_{\Sigma D}(\Omega). \]

(3.6)
Moreover,
\[
\begin{cases}
(-\Delta)^s u_0 = f_{\lambda}^*(u_0) & \text{in } \Omega, \\
B(u_0) = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Because of Theorem 7 we get, for some \(\varepsilon > 0\),
\[
\lambda_1 (x) + \varepsilon v(x) \leq u_0(x) \leq \lambda_2 (x) - \varepsilon v(x), \quad \text{for all } x \in \Omega,
\]
so that \(\lambda_1 < u_0 < \lambda_2\), for all \(x \in \Omega\). Then, taking \(\|\omega - u_0\|_{L^\infty(\Omega)} \leq \tau\) with \(0 < \tau < \varepsilon\), we get \(\lambda_1 \leq \omega \leq \lambda_2\) in \(\Omega\). Since \(I_\lambda^*(\omega) - T_\lambda(\omega)\) is zero for all \(\lambda_1 \leq \omega \leq \lambda_2\), by (3.6) we conclude
\[
T_\lambda(\omega) = I_\lambda^*(\omega) \geq I_\lambda^*(u_0) = T_\lambda(u_0),
\]
for all \(\omega \in \mathcal{C}_v(\Omega)\) with \(\|\omega - u_0\|_{L^\infty(\Omega)} \leq \tau\) and the result follows. \(\square\)

**Proposition 5** Let \(u_0 \in H^S_{\Sigma_D}(\Omega)\) be a local minimum of \(I_\lambda\) in \(\mathcal{C}_v(\Omega)\), i.e., there exists \(r > 0\) such that
\[
I_\lambda(u_0) \leq I_\lambda(u_0 + \omega), \quad \text{for all } \omega \in \mathcal{C}_v(\Omega) \text{ with } \|\omega\|_{L^\infty(\Omega)} \leq r.
\]

Then, \(u_0\) is a local minimum of \(I_\lambda\) in \(H^S_{\Sigma_D}(\Omega)\): there exists \(\delta > 0\) such that
\[
I_\lambda(u_0) \leq I_\lambda(u_0 + \varphi), \quad \text{for all } \varphi \in H^S_{\Sigma_D}(\Omega) \text{ with } \|\varphi\|_{H^S_{\Sigma_D}(\Omega)} \leq \delta.
\]

**Proof** Arguing by contradiction we assume that,
\[
\forall \varepsilon > 0, \exists u_\varepsilon \in B_\varepsilon(u_0) \text{ such that } I_\lambda(u_\varepsilon) < I_\lambda(u_0), \quad (3.7)
\]
where \(B_\varepsilon(u_0) = \{u \in H^S_{\Sigma_D}(\Omega) : \|u - u_0\|_{H^S_{\Sigma_D}(\Omega)} \leq \varepsilon\}\). For every \(j > 0\) let us consider the truncation map
\[
T_j(t) = \begin{cases}
\text{t} & \text{if } 0 < t < j, \\
\text{j} & \text{if } t \geq j,
\end{cases}
\]
let us set
\[
F_{\lambda,j}(u) = \int_0^u f_{\lambda,j}(t)dt \quad \text{where } f_{\lambda,j}(t) = f_{\lambda}(T_j(t)),
\]
with \(f_{\lambda}(t) = \lambda |t|^{q-1}t + |t|^{2_s-2}t\), and the energy functional
\[
I_{\lambda,j}(u) = \frac{1}{2}\|u\|_{H^S_{\Sigma_D}(\Omega)}^2 - \int_{\Omega} F_{\lambda,j}(u)dx.
\]
For each \( u \in H^s_{\Sigma D}(\Omega) \), we have \( I_{\lambda,j}(u) \to I_\lambda(u) \) as \( j \to \infty \). Hence, for any \( \varepsilon > 0 \) there exists \( j = j(\varepsilon) \) big enough such that \( I_{\lambda,j}(u_\varepsilon) < I_\lambda(u_0) \). Clearly, \( \min_{B_\varepsilon(u_0)} I_{\lambda,j}(\cdot) \) is attained at some point, say \( \omega_\varepsilon \). Therefore, we have

\[
I_{\lambda,j}(\omega_\varepsilon) \leq I_{\lambda,j}(u_\varepsilon) < I_\lambda(u_0).
\]

Now we want to prove that

\[
\omega_\varepsilon \to u_0 \quad \text{in} \quad C^v(\Omega) \quad \text{as} \quad \varepsilon \to 0, \quad (3.8)
\]

and we will arrive to a contradiction with (3.7). The Euler–Lagrange equation satisfied by \( \omega_\varepsilon \) involves a Lagrange multiplier \( \xi_\varepsilon \) such that

\[
\langle I_\lambda, j(\varepsilon)(\omega_\varepsilon), \psi \rangle_{H^{-s}(\Omega), H^s_{\Sigma D}(\Omega)} = \xi_\varepsilon \langle \omega_\varepsilon, \psi \rangle_{H^s_{\Sigma D}(\Omega)}, \quad \forall \psi \in H^s_{\Sigma D}(\Omega),
\]

that is,

\[
\int_{\Omega} (-\Delta)^{\frac{s}{2}} \omega_\varepsilon (-\Delta)^{\frac{s}{2}} \psi \, dx - \int_{\Omega} f_{\lambda,j}(\varepsilon)(\omega_\varepsilon) \psi \, dx = \xi_\varepsilon \int_{\Omega} (-\Delta)^{\frac{s}{2}} \omega_\varepsilon (-\Delta)^{\frac{s}{2}} \psi,
\]

for all \( \psi \in H^s_{\Sigma D}(\Omega) \). Since \( \omega_\varepsilon \) is a minimum of \( I_{\lambda,j}(\varepsilon) \), we have, for \( 0 < \varepsilon \ll 1 \),

\[
\xi_\varepsilon = \frac{\langle I_\lambda, j(\varepsilon)(\omega_\varepsilon), \omega_\varepsilon \rangle_{H^{-s}(\Omega), H^s_{\Sigma D}(\Omega)}}{\|\omega_\varepsilon\|_{H^s_{\Sigma D}(\Omega)}^2} \quad \text{and} \quad \xi_\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0. \quad (3.10)
\]

Let us observe that, by (3.9), the function \( \omega_\varepsilon \) satisfies the problem

\[
\begin{cases}
(-\Delta)^{\frac{s}{2}} \omega_\varepsilon = \frac{1}{1-\xi_\varepsilon} f_{\lambda,j}(\varepsilon)(\omega_\varepsilon) := f_\varepsilon(\omega_\varepsilon), & \text{in } \Omega, \\
B(\omega_\varepsilon) = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Clearly \( \|\omega_\varepsilon\|_{H^s_{\Sigma D}(\Omega)} \leq C \). Moreover, by Proposition 2, \( \|\omega_\varepsilon\|_{L^\infty(\Omega)} \leq C_1 \) for a constant \( C_1 > 0 \) independent of \( \varepsilon \). Thus, because of (3.10), we get \( \|f_\varepsilon(\omega_\varepsilon)\|_{L^\infty(\Omega)} \leq C_2 \) so that [13, Theorem 1.2] implies \( \|\omega_\varepsilon\|_{C^\gamma} \leq C_3 \) for some \( \gamma \in (0, \frac{1}{2}) \) and a constant \( C_3 > 0 \) independent of \( \varepsilon \). Here \( C^\gamma \) denotes the space of Hölder continuous functions with exponent \( \gamma \). Then, by the Ascoli–Arzelà Theorem, there exist a subsequence, still denoted by \( \omega_\varepsilon \), such that \( \omega_\varepsilon \to u_0 \) uniformly as \( \varepsilon \to 0 \). Since \( \omega_\varepsilon \to u_0 \) in \( H^s_{\Sigma D}(\Omega) \), we have \( \omega_0 = u_0 \) and, by the Maximum Principle and Theorem 7, we conclude (3.8) since

\[
\left\| \frac{\omega_\varepsilon - u_0}{v} \right\|_{L^\infty(\Omega)} \leq C \sup_{\Omega} |f_\varepsilon(\omega_\varepsilon) - f_\lambda^0(u_0)| \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

\( \square \)
Propositions 4 and 5 provide us with a local minimum of $I_\lambda$ in $H^s_{\Sigma_D}(\Omega)$ which will be denoted by $u_0$. Now, fixed $\lambda > 0$, we look for a second solution of the form $\tilde{u} = u_0 + \tilde{u}$ with $u_0$ the above solution and $\tilde{u} > 0$. The corresponding problem satisfied by $\tilde{u}$ is

$$\begin{cases} (-\Delta)^s \tilde{u} = \lambda (u_0 + \tilde{u})^q - \lambda u_0^q + (u_0 + \tilde{u})^{2^*_s-1} - u_0^{2^*_s-1} & \text{in } \Omega, \\ B(\tilde{u}) = 0 & \text{on } \partial \Omega, \end{cases}$$

Let us define the function $G_\lambda(u) = \int_0^u g_\lambda(t)dt$, with

$$g_\lambda(t) = \begin{cases} \lambda(u_0 + t)^q - \lambda u_0^q + (u_0 + t)^{2^*_s-1} - u_0^{2^*_s-1} & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases}$$

and the energy functional

$$\tilde{I}_\lambda(u) = \frac{1}{2} \|u\|^2_{H^s_{\Sigma_D}(\Omega)} - \int_{\Omega} G_\lambda(u)dx.$$ 

Since $u \in H^s_{\Sigma_D}(\Omega)$, then $\tilde{I}_\lambda$ is well defined. Finally let us set the moved problem

$$\begin{cases} (-\Delta)^s u = g_\lambda(u) & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial \Omega, \end{cases} \quad (\tilde{P}_\lambda)$$

If $\tilde{u} \neq 0$ is a critical point of $\tilde{I}_\lambda$ then it is a solution to $(\tilde{P}_\lambda)$. Moreover, by the Maximum Principle (cf. [12, Lemma 2.3]), we have $\tilde{u} > 0$ in $\Omega$. Hence $\tilde{u} = u_0 + \tilde{u}$ will be a second solution to $(P_\lambda)$ with $0 < q < 1$. Then, in order to prove the existence of a second solution we have to analyze the existence of nontrivial critical points of the functional $\tilde{I}_\lambda$. First we have the following.

**Lemma 9** $u = 0$ is a local minimum of $\tilde{I}_\lambda$ in $H^s_{\Sigma_D}(\Omega)$.

**Proof** Note that, because of Proposition 5, it is sufficient to prove that $u = 0$ is a local minimum of $\tilde{I}_\lambda$. Let $u^+$ be the positive part of $u$. Since

$$G_\lambda(u^+) = \tilde{F}_\lambda(u_0 + u^+) - \tilde{F}(u_0) - (\lambda u_0^q + u_0^{2^*_s-1}) u^+,$$

then

$$\tilde{I}_\lambda(u) = \frac{1}{2} \|u^+\|^2_{H^s_{\Sigma_D}(\Omega)} + \frac{1}{2} \|u^-\|^2_{H^s_{\Sigma_D}(\Omega)}$$

$$- \int_{\Omega} \tilde{F}_\lambda(u_0 + u^+) dx + \int_{\Omega} \tilde{F}(u_0) dx + \int_{\Omega} (\lambda u_0^q - u_0^{2^*_s-1}) u^+ dx.$$ 

$\square$ Springer
On the other hand
\[
\tilde{T}_\lambda (u_0 + u^+) = \frac{1}{2} \| u_0 + u^+ \|^2_{H^s(D)}(\Omega) - \int_{\Omega} F_\lambda (u_0 + u^+) \, dx
\]
\[
= \frac{1}{2} \| u_0 \|^2_{H^s(D)}(\Omega) + \frac{1}{2} \| u^+ \|^2_{H^s(D)}(\Omega)
\]
\[
+ \int_{\Omega} (-\Delta)^{\frac{s}{2}} u_0 (-\Delta)^{\frac{s}{2}} u^+ \, dx - \int_{\Omega} F_\lambda (u_0 + u^+) \, dx
\]
\[
= \frac{1}{2} \| u_0 \|^2_{H^s(D)}(\Omega) + \frac{1}{2} \| u^+ \|^2_{H^s(D)}(\Omega)
\]
\[
+ \int_{\Omega} \left( \lambda u_0^q + u_0^{2s-1} \right) u^+ \, dx - \int_{\Omega} F_\lambda (u_0 + u^+) \, dx.
\]

Hence
\[
\tilde{\gamma}_\lambda (u) = \frac{1}{2} \| u^- \|^2_{H^s(D)}(\Omega) + \tilde{T}_\lambda (u_0 + u) - \frac{1}{2} \| u_0 \|^2_{H^s(D)}(\Omega) + \int_{\Omega} F_\lambda (u_0) \, dx
\]
\[
= \frac{1}{2} \| u^- \|^2_{H^s(D)}(\Omega) + \tilde{T}_\lambda (u_0 + u) - \tilde{T}_\lambda (u_0).
\]

Using Proposition 4, it follows that \( \tilde{T}_\lambda (u) \geq \frac{1}{2} \| u^- \|^2_{H^s(D)}(\Omega) \geq 0 \), provided
\[
\left\| \frac{u}{v} \right\|_{L^\infty(\Omega)} \leq \varepsilon \quad \text{for some } \varepsilon > 0 \text{ small enough.} \tag*{\square}
\]

By Lemma 9, the following result holds for the energy functional associated to the extension problem corresponding to the moved problem \( \tilde{\mathcal{P}}_\lambda \), namely,
\[
\mathcal{J}_\lambda (w) = \frac{1}{2} \| w \|^2_{H^s(D)}(\mathcal{C}_\Omega) - \int_{\Omega} G_\lambda (w(x, 0)) \, dx.
\]

Corollary 1 \( w = 0 \) is a local minimum of \( \mathcal{J}_\lambda \) in \( \mathcal{X}^s_{\Sigma D}(\mathcal{C}_\Omega) \).

### 3.1 The Palais–Smale condition

Assuming \( w = 0 \) is the unique critical point of the energy functional \( \mathcal{J}_\lambda \), corresponding to the extension problem of \( \tilde{\mathcal{P}}_\lambda \), we prove that \( \mathcal{J}_\lambda \) satisfies a local PS condition for energy levels below the critical level \( c_{D-N}^* = \frac{S}{N} [\tilde{S}(\Sigma_D)]^{\frac{N}{N}} \). As commented above, the main tool for proving this fact is the extension to the fractional-mixed setting of the concentration-compactness principle by Lions (cf. [22]) provided by Theorem 5. We will also need some estimates for the action of the fractional Laplacian on the Sobolev extremal functions (2.9) in order to find paths with energy below \( c_{D-N}^* \).

Lemma 10 If \( w = 0 \) is the only critical point of \( \mathcal{J}_\lambda \) in \( \mathcal{X}^s_{\Sigma D}(\mathcal{C}_\Omega) \), then \( \mathcal{J}_\lambda \) satisfies a local Palis–Smale condition below the critical level
\[
c_{D-N}^* = \frac{S}{N} [\tilde{S}(\Sigma_D)]^{\frac{N}{N}}.
\]
Proof Let \( \{w_n\} \) be a PS sequence for \( \tilde{J}_\lambda \) verifying
\[
\tilde{J}_\lambda(w_n) \to c < c_\lambda^* - N \quad \text{and} \quad \tilde{J}_\lambda'(w_n) \to 0.
\]
(3.11)

Then, it is clear that the sequence \( \{w_n\} \) is uniformly bounded in \( X_{\Sigma D}(\mathcal{C}_\Omega) \), say \( \|w_n\|_{X_{\Sigma D}(\mathcal{C}_\Omega)}^2 \leq M \) and, since by hypothesis \( w = 0 \) is the unique critical point of \( \tilde{J}_\lambda \), it follows that
\[
w_n \rightharpoonup 0 \quad \text{weakly in} \quad X_{\Sigma D}(\mathcal{C}_\Omega),
\]
\[
w_n(x, 0) \to 0 \quad \text{strongly in} \quad L^r(\Omega), \quad 1 \leq r < 2^*_s,
\]
\[
w_n(x, 0) \to 0 \quad \text{a.e. in} \quad \Omega.
\]

Also, since \( w_0 = E_s[u_0] \) is a critical point of \( J_\lambda \), we get
\[
\tilde{J}_\lambda(w_n) + J_\lambda(w_0) \geq J_\lambda(z_n)
\]
(3.12)

where \( z_n = w_n + w_0 \) and, then,
\[
J_\lambda(z_n) \to c + J_\lambda(w_0) \quad \text{and} \quad J_\lambda'(z_n) \to 0.
\]
(3.13)

From (3.12) and (3.13) we get that the sequence \( \{z_m\} \) is uniformly bounded in \( X_{\Sigma D}(\mathcal{C}_\Omega) \), say \( \|z_n\|_{X_{\Sigma D}(\mathcal{C}_\Omega)}^2 \leq M_2 \). This, together with the fact that \( w = 0 \) is the unique critical point of \( \tilde{J}_\lambda \), implies that, up to a subsequence,
\[
z_n \rightharpoonup w_0 \quad \text{weakly in} \quad X_{\Sigma D}(\mathcal{C}_\Omega),
\]
\[
z_n(x, 0) \to w_0(x, 0) \quad \text{strongly in} \quad L^r(\Omega), \quad 1 \leq r < 2^*_s,
\]
\[
z_n(x, 0) \to w_0(x, 0) \quad \text{a.e. in} \quad \Omega.
\]
(3.14)

In order to apply the concentration-compactness result provided by Theorem 5 we claim the following:

Claim: The sequence \( \{y^{1-2s}|\nabla z_n|^2\}_{n \in \mathbb{N}} \) is tight.

The proof of the claim follows verbatim that of [5, Lemma 3.6] (see also the proof of [16, Theorem 4.5]) so we omit the details.

We can then apply Theorem 5. Hence, up to a subsequence, there exists an at most countable index set \( \mathcal{I} \), a sequence of points \( \{x_i\}_{i \in \mathcal{I}} \subset \mathcal{J}_\mathcal{D} \) and nonnegative real numbers \( \mu_i \) and \( v_i \) such that
\[
\kappa_s y^{1-2s}|\nabla z_n|^2 \to \mu \geq \kappa_s y^{1-2s}|\nabla w_0|^2 + \sum_{i \in \mathcal{I}} \mu_i \delta_{x_i},
\]
(3.15)

and
\[
z_n(x, 0) \to v = |w_0(x, 0)|^{2^*_s} + \sum_{i \in \mathcal{I}} v_i \delta_{x_i},
\]
(3.16)
in the sense of measures and satisfying the relation

$$\mu_i \geq \tilde{S}(\Sigma_{\mathcal{D}}) v_i^{\frac{2}{N}}, \quad \text{for } i \in \mathcal{I}.$$  \quad (3.17)

Next, we fix $i_0 \in \mathcal{I}$ and we let $\phi \in C^0_0(\mathbb{R}_{+}^{N+1})$ be a non-increasing smooth cut-off function verifying $\phi = 1$ in $B_i^+(x_{i_0})$ and $\phi = 0$ in $B_i^+(x_{i_0})^c$, with $B_i^+(x_{i_0}) \subset \mathbb{R}^N \times \{ y \geq 0 \}$ the $(N+1)$-dimensional semi-ball of radius $r > 0$ centered at $x_{i_0}$. Let now $\phi_\varepsilon(x, y) = \phi(x/\varepsilon, y/\varepsilon)$, clearly $|\nabla \phi_\varepsilon| \leq \frac{C}{\varepsilon}$ and let us denote by $\Gamma_\varepsilon = B_2^+(x_{i_0}) \cap \{ y = 0 \}$. Thus, taking the dual product in (3.13) with $\phi_\varepsilon \zeta_n$, we have

$$\lim_{n \to \infty} \kappa_s \int_{\mathcal{D}} y^{1-2s} \langle \nabla \zeta_n, \nabla \phi_\varepsilon \rangle \zeta_n \, dx 
= \lim_{n \to \infty} \lambda \int_{\Gamma_\varepsilon} |\zeta_n|^{q+1} \phi_\varepsilon \, dx + \int_{\Gamma_\varepsilon} |\zeta_n|^{2s} \phi_\varepsilon \, dx - \kappa_s \int_{B_2^+(x_{i_0})} y^{1-2s} \nabla \zeta_n \nabla \phi_\varepsilon \, dx.$$

Then, thanks to (3.14), (3.15) and (3.16), we find,

$$\lim_{n \to \infty} \kappa_s \int_{\mathcal{D}} y^{1-2s} \langle \nabla \zeta_n, \nabla \phi_\varepsilon \rangle \zeta_n \, dx 
= \lambda \int_{\Gamma_\varepsilon} |w_0|^{q+1} \phi_\varepsilon \, dx + \int_{\Gamma_\varepsilon} \phi_\varepsilon \, d\nu - \kappa_s \int_{B_2^+(x_{i_0})} y^{1-2s} \phi_\varepsilon \, d\mu. \quad (3.18)$$

Assume for the moment that the left hand side of (3.18) vanishes as $\varepsilon \to 0$. Thus,

$$0 = \lim_{\varepsilon \to 0} \lambda \int_{\Gamma_\varepsilon} |w_0|^{q+1} \phi_\varepsilon \, dx + \int_{\Gamma_\varepsilon} \phi_\varepsilon \, d\nu - \kappa_s \int_{B_2^+(x_{i_0})} y^{1-2s} \phi_\varepsilon \, d\mu \quad (3.19)$$

We have then two options, either the compactness of the PS sequence or concentration at the point $x_{i_0}$. In other words, either $\nu_{i_0} = 0$, so that $\mu_{i_0} = 0$ or, by (3.19) and (3.17),

$$\nu_{i_0} \geq \left( \tilde{S}(\Sigma_{\mathcal{D}}) \right)^{\frac{N}{2}}.$$

In case of having concentration, we find,

$$c + J_\lambda(w_0) = \lim_{n \to \infty} J_\lambda(\zeta_n) - \frac{1}{2} \langle J_\lambda'(\zeta_n), \zeta_n \rangle$$

$$\geq \lambda \left( \frac{1}{2} - \frac{1}{q+1} \right) \int_{\mathcal{D}} |w_0|^{q+1} \, dx + \frac{S}{N} \int_{\mathcal{D}} |w_0|^{2s} \, dx + \frac{S}{N} \nu_{i_0}$$

$$\geq J_\lambda(w_0) + \frac{S}{N} \left( \tilde{S}(\Sigma_{\mathcal{D}}) \right)^{\frac{N}{2}} = J_\lambda(w_0) + c^*_{D-N},$$

in contradiction with the hypotheses $c < c^*_{D-N}$. Since $i_0$ was chosen arbitrarily, $\nu_i = 0$ for all $i \in \mathcal{I}$. Thus, $u_n \to u_0$ in $L^{2s}(\Omega)$. Since convergence of $u_n$ in $L^{2s}(\Omega)$
implies convergence of $f_{\lambda}(u_n)$ in $L^{\frac{2N}{N+2\beta}}(\Omega)$ using the continuity of the inverse operator $(-\Delta)^{-s}$ we get the strong convergence of $u_n$ in $H^s_{\Sigma_D}(\Omega)$.

It remains to prove that the left hand side of (3.18) vanishes as $\varepsilon \to 0$. The PS sequence $\{z_m\}$ is uniformly bounded in $X^{s}_{\Sigma_D}(\mathcal{E}_\Omega)$ and, up to a subsequence, $z_n \to w_0 \in X^{s}_{\Sigma_D}(\mathcal{E}_\Omega)$ and $z_n \to w_0$ a.e. in $\mathcal{E}_\Omega$. Moreover, if $r < 2^* = \frac{2(N+1)}{N-1}$ we have the compact inclusion $X^{s}_{\Sigma_D}(\mathcal{E}_\Omega) \hookrightarrow L'(\mathcal{E}_\Omega, y^{1-2\beta}dx)$. Then, applying Hölder’s inequality with $p = \frac{N+1}{N-1}$ and $q = \frac{N+1}{2}$, we find,

\[
\int_{B_{2\varepsilon}(x_0)} y^{1-2s}|\nabla \phi_\varepsilon|^2 |z_n|^2 \, dx \, dy \leq \left( \int_{B_{2\varepsilon}(x_0)} y^{1-2s}|\nabla \phi_\varepsilon|^{N+1} \, dx \, dy \right)^{\frac{2}{N+1}} \left( \int_{B_{2\varepsilon}(x_0)} y^{1-2s}|z_n|^{2N+1} \, dx \, dy \right)^{\frac{N-1}{N+1}}
\]

\[
\leq \frac{1}{\varepsilon^2} \left( \int_{B_{2\varepsilon}(x_0)} \int_0^\varepsilon y^{1-2s} \, dx \, dy \right)^{\frac{2}{N+1}} \left( \int_{B_{2\varepsilon}(x_0)} y^{1-2s}|z_n|^{2N+1} \, dx \, dy \right)^{\frac{N-1}{N+1}}
\]

\[
\leq C_0 \varepsilon^{\frac{2(1-2\beta)}{N+1}} \left( \int_{B_{2\varepsilon}(x_0)} y^{1-2s}|z_n|^{2N+1} \, dx \, dy \right)^{\frac{N-1}{N+1}}
\]

\[
\leq C_0 \varepsilon^{\frac{2(1-2\beta)}{N+1}} \varepsilon^{\frac{(2+2\beta)(N-1)-2}{(N+1)}} \left( \int_{B_{2\varepsilon}(x_0)} y^{1-2s}|(\varepsilon x, \varepsilon y)|^{2N+1} \, dx \, dy \right)^{\frac{N-1}{N+1}}
\]

\[
\leq C_1 \varepsilon^{N-2s},
\]

for appropriate positive constants $c_0$ and $c_1$. Thus, we conclude,

\[
0 \leq \lim_{n \to \infty} \kappa_s \int_{\mathcal{E}_\Omega} y^{1-2s} \langle \nabla z_n, \nabla \phi_\varepsilon \rangle z_n \, dx \, dy
\]

\[
\leq \kappa_s \lim_{n \to \infty} \left( \int_{\mathcal{E}_\Omega} y^{1-2s} |\nabla z_n|^2 \, dx \, dy \right)^{1/2} \left( \int_{B_{2\varepsilon}(x_0)} y^{1-2s} |\nabla \phi_\varepsilon|^2 |z_n|^2 \, dx \, dy \right)^{1/2}
\]

\[
\leq C \varepsilon^{\frac{N-2s}{2}} \to 0,
\]

as $\varepsilon \to 0$ and the proof of the Lemma 10 is complete. \hfill \Box

In Lemma 10 we have proved that, if $w = 0$ is the only critical point of the functional $\tilde{J}_\lambda$, then $\tilde{J}_\lambda$ verifies the PS condition at any level $c < c^+_{D-N}$. Next, we have to show that we can obtain a local PS$\varepsilon$ sequence for the energy functional $\tilde{J}_\lambda$ with energy level $c < c^+_{D-N}$. To do that we will use the family of minimizers $u_\varepsilon$ of the fractional Sobolev inequality (2.8) given by (2.9) and its $s$-harmonic extension $w_\varepsilon = E_s[u_\varepsilon]$.

Consider a smooth non-increasing cut-off function $\phi_0(t) \in C^\infty(\mathbb{R}_+)$, such that $\phi_0(t) = 1$ for $0 \leq t \leq \frac{1}{2}$ and $\phi_0(t) = 0$ for $t \geq 1$, and $|\phi_0'(t)| \leq C$ for any $t \geq 0$. Assume, without loss of generality, that $0 \in \Omega$, and define, for some $\rho > 0$ small...
enough such that \( B_\rho^+((0,0)) \subseteq \mathcal{C}_\Omega \), the function \( \phi(x,y) = \phi_\rho(x,y) = \phi_0(\frac{r_{xy}}{\rho}) \) with \( r_{xy} = |(x,y)| = (|x|^2 + y^2)^{\frac{1}{2}} \).

**Lemma 11** [16, Lemma 3.3] The family \( \{\phi_\rho w_\epsilon\} \) and its trace on \( \{y = 0\} \), namely \( \{\phi_\rho u_\epsilon\} \), satisfy

\[
\|\phi_\rho w_\epsilon\|_{\mathcal{X}_s^2(\mathcal{C}_\Omega)}^2 = \|w_\epsilon\|_{\mathcal{X}_s^2(\mathcal{C}_\Omega)}^2 + O\left(\left(\frac{\epsilon}{\rho}\right)^{N-2s}\right)
\]

and

\[
\int_\Omega |\phi_\rho u_\epsilon|^2\,dx = \|u_\epsilon\|_{L_{s}^2(\Omega)}^2 + O\left(\left(\frac{\epsilon}{\rho}\right)^N\right).
\]

for \( \epsilon > 0 \) small enough.

We consider now a cut-off function centered at \( x_0 \in \Sigma_N \), namely, \( \psi(x,y) = \psi_\rho(x,y) = \phi_0(\frac{r_{xy}}{\rho}) \) with \( r_{xy} = |(x-x_0, y)| = (|x-x_0|^2 + y^2)^{\frac{1}{2}} \) and \( \phi_0(t) \) as the cut-off function of Lemma 11. Then, in the lines of [19, Lemma 3.2] the following holds.

**Lemma 12** The family \( \{\psi_\rho w_\epsilon\} \) and its trace on \( \{y = 0\} \), namely \( \{\psi_\rho u_\epsilon\} \), satisfy

\[
\|\psi_\rho w_\epsilon\|_{\mathcal{X}_s^2(\mathcal{C}_\Omega)}^2 = \frac{1}{2}\|w_\epsilon\|_{\mathcal{X}_s^2(\mathcal{C}_\Omega)}^2 + O\left(\left(\frac{\epsilon}{\rho}\right)^{N-2s}\right)
\]

and

\[
\|\psi_\rho u_\epsilon\|_{L_{s}^2(\Omega)}^2 = \frac{1}{2}\|u_\epsilon\|_{L_{s}^2(\mathbb{R}^N)}^2 + O\left(\left(\frac{\epsilon}{\rho}\right)^N\right).
\]

for \( \epsilon > 0 \) small enough.

**Proof** Take \( X_0 = (x_0, 0) \) and denote by \( \Omega_\rho \) the set \( \Omega \cap B_\rho(x_0) \) and by \( \mathcal{C}_\rho \) the set \( \mathcal{C}_\Omega \cap B_\rho^+(X_0) \). Let us estimate the norm of the truncated function centered at \( x_0 \in \Sigma_N \),

\[
\|\psi_\rho w_\epsilon\|_{\mathcal{X}_s^2(\mathcal{C}_\Omega)}^2 = \kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s} \psi_\rho^2 |\nabla w_\epsilon|^2\,dxdy + \kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s} \left(|w_\epsilon \nabla \psi_\rho|^2 + 2\langle w_\epsilon \nabla \psi_\rho, \psi_\rho \nabla w_\epsilon \rangle\right)\,dxdy.
\]

Following verbatim the proof of Lemma 11, the next estimate holds,

\[
\kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s} \left(|w_\epsilon \nabla \psi_\rho|^2 + 2\langle w_\epsilon \nabla \psi_\rho, \psi_\rho \nabla w_\epsilon \rangle\right)\,dxdy = O\left(\left(\frac{\epsilon}{\rho}\right)^{N-2s}\right).
\]
To estimate the remaining term in (3.22), we use that, by [5, Lemma 3.1],
\[ |\nabla w_{1,s}(x, y)| \leq C w_{1,s-\frac{1}{2}}(x, y), \quad \frac{1}{2} < s < 1, \quad (x, y) \in \mathbb{R}_{+}^{N+1}, \]
and, for \((x, y) \in B_{\rho}^{+}(X_0)\), we have \(w_1(x, y) = O\left(\left(\frac{\varepsilon}{\rho}\right)^{N-2s}\right)\), so that
\[
\int_{|r_{xy}| \leq \rho} y^{1-2s} |\nabla w_{\varepsilon}|^2 dxdy \leq C \int_{|r_{xy}| \leq \frac{\varepsilon}{\rho}} y^{1-2s} \left(\frac{\varepsilon}{\rho}\right)^{2(N-2s-1/2)} dxdy = O\left(\left(\frac{\varepsilon}{\rho}\right)^{N-2s}\right).
\]

Then, since \(\Omega\) is a smooth domain, by the above estimates we find
\[
\int_{\Omega} y^{1-2s} \psi_{\rho}^2 |\nabla w_{\varepsilon}|^2 dxdy = \int_{\varepsilon \rho} y^{1-2s} \psi_{\rho}^2 |\nabla w_{\varepsilon}|^2 dxdy = \int_{\varepsilon \rho} y^{1-2s} |\nabla w_{\varepsilon}|^2 dxdy + O\left(\left(\frac{\varepsilon}{\rho}\right)^{N-2s}\right) = \frac{1}{2} \int_{B_{\rho}(0)} y^{1-2s} |\nabla w_{\varepsilon}|^2 dxdy + O\left(\left(\frac{\varepsilon}{\rho}\right)^{N-2s}\right),
\]
Then, (3.20) follows by applying Lemma 11. On the other hand, since
\[
\int_{|x| < \rho} |u_{\varepsilon}|^{2s} dx = \int_{|x| < \rho} \frac{\varepsilon^N}{(\varepsilon^2 + |x|^2)^N} dx = O\left(\left(\frac{\varepsilon}{\rho}\right)^N\right),
\]
we have
\[
\int_{\Omega} |\psi_{\rho} u_{\varepsilon}|^{2s} dx = \int_{\Omega_{\rho}} |\psi_{\rho} u_{\varepsilon}|^{2s} dx = \int_{\Omega_{\rho}} |u_{\varepsilon}|^{2s} dx + O\left(\left(\frac{\varepsilon}{\rho}\right)^N\right) = \frac{1}{2} \int_{B_{\rho}(0)} |u_{\varepsilon}|^{2s} dx + O\left(\left(\frac{\varepsilon}{\rho}\right)^N\right) dx.
\]

Using Lemma 11 we get (3.21). \(\square\)

Next, let us consider the family of functions
\[
\eta_{\varepsilon} = \frac{\psi_{\rho} w_{\varepsilon}}{\|\psi_{\rho} u_{\varepsilon}\|_{L^{2s}_{\rho}(\Omega)}}. \tag{3.23}
\]
with $\rho = \varepsilon^\alpha$ and $0 < \alpha < 1$ to be chosen (see (3.27) below). By Lemma 12,

$$\|\eta_{\varepsilon}\|_{X_{\Sigma D}(\mathcal{C}\Omega)}^2 = \frac{1}{2} \|w_{\varepsilon}\|_{X_{\Sigma D}(\mathcal{C}\Omega)}^2 + O(\varepsilon^{1-\alpha}(N-2s))$$

$$\leq 2^{-\frac{2s}{N}} \|w_{\varepsilon}\|_{X_{\Sigma D}(\mathcal{C}\Omega)}^2 + O(\varepsilon^{1-\alpha}(N-2s))$$

by (3.24).

Lemma 13 [5, Lemma 3.8] The family $\{\phi_{\rho u_{\varepsilon}}\}$ satisfy

$$\|\phi_{\rho u_{\varepsilon}}\|_{L^2(\Omega)}^2 = \begin{cases} C\varepsilon^{2s} + O(\varepsilon^{N-2s}) & \text{if } N > 4s, \\ C\varepsilon^{2s} \log(1/\varepsilon) + O(\varepsilon^{2s}) & \text{if } N = 4s, \end{cases}$$

and

$$\|\phi_{\rho u_{\varepsilon}}\|_{L^2_{N-1}(\Omega)}^{2s-1} \geq C\varepsilon^{N-2s}, \quad \text{if } N > 2s,$$

for $\varepsilon > 0$ small enough.

Remark 3 In this case the dependence on $\rho > 0$ can be neglected as

$$\|\phi_{\rho u_{\varepsilon}}\|_{L^2(\Omega)}^2 \geq \int_{\{|x|<\frac{\rho}{2}\}} |u_{\varepsilon}|^2 \, dx = \varepsilon^{-(N-2s)} \int_{\{|x|<\frac{\varepsilon}{2}\}} |u_1(\frac{x}{\varepsilon})|^2 \, dx$$

so that, for $\rho = \varepsilon^\alpha$ with $0 < \alpha < 1$ as above, we have

$$\|\phi_{\rho u_{\varepsilon}}\|_{L^2(\Omega)}^2 \geq \varepsilon^{2s} \int_{\{|x|<\varepsilon^{1-\alpha}(1-\alpha)\}} |u_1(x)|^2 \, dx \geq \varepsilon^{2s} \int_{\{|x|<1\}} |u_1(x)|^2 \, dx,$$

for $\varepsilon < < 1$. Moreover, noticing that

$$\varepsilon^{2s} \int_{\{|x|>\frac{\varepsilon}{2}\}} |u_1|^2 \, dx = \varepsilon^{2s} \cdot O \left( \left( \frac{\varepsilon}{\rho} \right)^{N-4s} \right),$$

we conclude,

$$\|\phi_{\rho u_{\varepsilon}}\|_{L^2(\Omega)}^2 = \varepsilon^{2s} \left( \|u_1\|_{L^2(\mathbb{R}^N)}^2 + O(\varepsilon^{1-\alpha}(N-4s)) \right).$$

We finally prove that we can find paths with energy below the critical level $c^*_D - N$. As commented above, contrary to the fractional Sobolev constant (2.8), the constant $\tilde{S}(\Sigma D)$ is attained for $|\Sigma D|$ small enough. Indeed, as by Proposition 1 we have $\tilde{S}(\Sigma D) \leq 2^{-\frac{2s}{N}} S(N, s)$, we have two options:
1) $\tilde{S}(\Sigma_D) < 2\frac{s^2}{\lambda} S(N, s)$. In this case, the constant $\tilde{S}(\Sigma_D)$ is attained by Theorem 4 and no concentration occurs.

2) $\tilde{S}(\Sigma_D) = 2\frac{s^2}{\lambda} S(N, s)$. In this case, by Theorem 6, minimizing sequences for $\tilde{S}(\Sigma_D)$ concentrate at a boundary point $x_0 \in \Sigma_N$.

**Lemma 14** Assume $\tilde{S}(\Sigma_D) < 2\frac{s^2}{\lambda} S(N, s)$. Then, there exists $\tilde{w} \in X^s_{\Sigma_D}(\mathcal{C}_\Omega)$, $\tilde{w} > 0$ such that

$$\sup_{t \geq 0} \tilde{J}_\lambda(t\tilde{w}) < c^*_D_{-N}.$$  

**Proof** If $\tilde{S}(\Sigma_D) < 2\frac{s^2}{\lambda} S(N, s)$ then $\tilde{S}(\Sigma_D)$ is attained at some $\tilde{u} \in H^s_{\Sigma_D}(\Omega)$ that we can assume to be positive (cf. [16]). Hence, take $\tilde{w} = \frac{E_s[\tilde{u}]}{\|\tilde{u}\|_{2_s}^*}$ so that, by (2.5),

$$\|\tilde{w}\|_{X^s_{\Sigma_D}(\mathcal{C}_\Omega)}^2 = \tilde{S}(\Sigma_D).$$

Then, as $\lambda > 0$ and $\tilde{w} > 0$,

$$\tilde{J}_\lambda(t\tilde{w}) = \frac{t^2}{2} \tilde{S}(\Sigma_D) - \frac{\lambda}{q + 1} \|\tilde{w}(x, 0)\|_{L_{q+1}^s(\Omega)}^{q+1} - \frac{t^{2s}}{2s} < \frac{t^2}{2} \tilde{S}(\Sigma_D) - \frac{t^{2s}}{2s} = g(t).$$

Since $g(t)$ attains its maximum at $t_0 = [\tilde{S}(\Sigma_D)]^{\frac{1}{2s-2}}$ and $g(t_0) = \frac{s}{N}[\tilde{S}(\Sigma_D)]^{\frac{N}{2s-2}}$, we conclude

$$\sup_{t \geq 0} \tilde{J}_\lambda(t\tilde{w}) < \frac{s}{N}[\tilde{S}(\Sigma_D)]^{\frac{N}{2s-2}} = c^*_D_{-N}.$$  

\[\square\]

Next we address the case $\tilde{S}(\Sigma_D) = 2\frac{s^2}{\lambda} S(N, s)$. Since now $\tilde{S}(\Sigma_D)$ is not attained, we need to use the family of functions $\eta_\varepsilon$ defined in (3.23) to construct paths below the critical level $c^*_D_{-N} = \frac{s}{N}[\tilde{S}(\Sigma_D)]^{\frac{N}{2s-2}}$.

**Lemma 15** Assume $\tilde{S}(\Sigma_D) = 2\frac{s^2}{\lambda} S(N, s)$. Then, there exists $\varepsilon > 0$ small enough such that

$$\sup_{t \geq 0} \tilde{J}_\lambda(t\eta_\varepsilon) < c^*_D_{-N}.$$  

**Proof** Assume $N \geq 4s$. Then, using that, for $a, b \geq 0$ and $p > 1$, we have

$$(a + b)^p \geq a^p + b^p + \mu a^{p-1} b,$$  

(3.25)

for some $\mu = \mu(p) > 0$, it follows that $G_\lambda(w) \geq \frac{1}{2s} w_{2s}^{2s} + \frac{\mu}{2} w^2 w_0^{2s-2}$. Hence,

$$\tilde{J}_\lambda(t\eta_\varepsilon) \leq \frac{t^2}{2} \|\eta_\varepsilon\|_{X^s_{\Sigma_D}(\mathcal{C}_\Omega)}^2 - \frac{t^{2s}}{2s} - \frac{t^2}{2} \mu \int_\Omega w_0^{2s-2} \eta_\varepsilon^2 dx.$$  

\[\square\]
Since $w_0 \geq a_0 > 0$ in $supp(\eta_\varepsilon)$, we get
\[
\tilde{J}_\lambda(t\eta_\varepsilon) \leq \frac{t^2}{2} \|\eta_\varepsilon\|_{\mathcal{X}^s_{\Sigma_D}(\mathcal{C}_\Omega)}^2 - \frac{t^{2s}}{2s} - \frac{t^2}{2} \tilde{\mu} \|\eta_\varepsilon\|_{L^2(\Omega)}^2 = := g_\varepsilon(t).
\]
Clearly, $\lim_{t \to +\infty} g_\varepsilon(t) = -\infty$ and $\sup_{t \geq 0} g_\varepsilon(t)$ is attained at some $t_\varepsilon$. Indeed, since
\[
0 = g_\varepsilon'(t_\varepsilon) = t_\varepsilon \|\eta_\varepsilon\|_{\mathcal{X}^s_{\Sigma_D}(\mathcal{C}_\Omega)}^2 - t_\varepsilon^{2s-1} - t_\varepsilon \tilde{\mu} \|\eta_\varepsilon\|_{L^2(\Omega)},
\]
we have $t_\varepsilon \leq \|\eta_\varepsilon\|_{\mathcal{X}^s_{\Sigma_D}(\mathcal{C}_\Omega)}^{2s-2}$. Moreover, by Lemma 12, $t_\varepsilon \geq C > 0$. On the other hand, the function
\[
h_\varepsilon(t) = \frac{t^2}{2} \|\eta_\varepsilon\|_{\mathcal{X}^s_{\Sigma_D}(\mathcal{C}_\Omega)}^2 - \frac{t^{2s}}{2s},
\]
is increasing on $[0, \|\eta_\varepsilon\|_{\mathcal{X}^s_{\Sigma_D}(\mathcal{C}_\Omega)}^{2s-2}]$. Therefore,
\[
\sup_{t \geq 0} g_\varepsilon(t) = g_\varepsilon(t_\varepsilon) \leq \frac{s}{N} \|\eta_\varepsilon\|_{\mathcal{X}^s_{\Sigma_D}(\mathcal{C}_\Omega)}^2 - C \|\eta_\varepsilon\|_{L^2(\Omega)}^2.
\]
Since $\|u_\varepsilon\|_{L^{2s}(\Omega)}$ does not depend on $\varepsilon$, by (3.24) and Lemma 13, we have
\[
\|\eta_\varepsilon\|_{\mathcal{X}^s_{\Sigma_D}(\mathcal{C}_\Omega)}^2 \leq \tilde{S}(\Sigma_D) + O(\varepsilon^{(1-\alpha)(N-2s)}),
\]
\[
\|\eta_\varepsilon\|_{L^2(\Omega)}^2 = \begin{cases} O(\varepsilon^{2s}) & \text{if } N > 4s, \\ O(\varepsilon^{2s} \log(1/\varepsilon)) & \text{if } N = 4s. \end{cases}
\]
Thus, for $N > 4s$ we get
\[
g_\varepsilon(t_\varepsilon) \leq \frac{s}{N} [\tilde{S}(\Sigma_D)]^{\frac{N}{2s}} + C_1 \varepsilon^{(1-\alpha)(N-2s)} - C_2 \varepsilon^{2s} < c^*_D - N, \quad (3.26)
\]
for $\varepsilon > 0$ small enough and
\[
0 < \alpha < \frac{N - 4s}{N - 2s}. \quad (3.27)
\]
Note that a similar relation between the concentration parameter $\varepsilon > 0$ and the cut-off radius $\rho = \varepsilon^\alpha$ was obtained in [19, Lemma 3.2].

If $N = 4s$, the same conclusion follows. The case $2s < N < 4s$ follows using the inequality (3.25), which gives, for some $\mu' > 0$,
\[
G_\lambda(w) \geq \frac{1}{2s} w^{2s} + \mu' w_0 w^{2s-1}. \quad (3.28)
\]
The result then follows arguing in a similar way as above, using (3.28) together with the second estimate in Lemma 13.

**Proof of Theorem 2-(4)** Let us fix \( \lambda \in (0, \Lambda) \). If \( \tilde{S}(\Sigma_D) \leq 2^{\frac{2\rho}{\epsilon}} S(N, s) \), since \( \lim_{t \to +\infty} \tilde{J}_\lambda(t \tilde{w}) = -\infty \), there exists \( M \gg 1 \) such that \( \tilde{J}_\lambda(M \tilde{w}) < \tilde{J}_\lambda(0) \). If \( \tilde{S}(\Sigma_D) = 2^{-\frac{\rho}{\epsilon}} S(N, s) \), since \( \lim_{t \to +\infty} \tilde{J}_\lambda(t \eta_\epsilon) = -\infty \), there exists \( M_\epsilon \gg 1 \) such that \( \tilde{J}_\lambda(M_\epsilon \eta_\epsilon) < \tilde{J}_\lambda(0) \). By Corollary 1, there exists \( \rho > 0 \) such that, if \( \|w\| \mathcal{X}^s_{\Sigma_D}(\epsilon \Omega) = \rho \), then \( \tilde{J}_\lambda(w) \geq \tilde{J}_\lambda(0) \). Next, if \( \tilde{S}(\Sigma_D) < 2^{-\frac{\rho}{\epsilon}} S(N, s) \), let us define

\[
\Gamma = \{ \gamma \in \mathcal{C}([0, 1], \mathcal{X}^s_{\Sigma_D}(\epsilon \Omega)) : \gamma(0) = 0, \gamma(1) = M \tilde{w} \},
\]

and the minimax value \( c = \inf_{\gamma \in \Gamma} \sup_{0 \leq t \leq 1} \tilde{J}_\lambda(\gamma(t)) \). Due to the comments above, \( c \geq \tilde{J}_\lambda(0) \). Furthermore, \( c \leq \sup_{0 \leq t \leq 1} \tilde{J}_\lambda(t M \tilde{w}) = \sup_{t \geq 0} \tilde{J}_\lambda(t \tilde{w}) < c^*_D - \mathcal{N} \), because of Lemma 14. If \( \tilde{S}(\Sigma_D) = 2^{-\frac{\rho}{\epsilon}} S(N, s) \), let us define

\[
\tilde{\Gamma}_\epsilon = \{ \gamma \in \mathcal{C}([0, 1], \mathcal{X}^s_{\Sigma_D}(\epsilon \Omega)) : \gamma(0) = 0, \gamma(1) = M_\epsilon \eta_\epsilon \},
\]

and the minimax value \( c_\epsilon = \inf_{\gamma \in \tilde{\Gamma}_\epsilon} \sup_{0 \leq t \leq 1} \tilde{J}_\lambda(\gamma(t)) \). By the arguments above, \( c_\epsilon \geq \tilde{J}_\lambda(0) \). Moreover, \( c_\epsilon \leq \sup_{0 \leq t \leq 1} \tilde{J}_\lambda(t M_\epsilon \eta_\epsilon) = \sup_{t \geq 0} \tilde{J}_\lambda(t \eta_\epsilon) < c^*_D - \mathcal{N} \), for \( \epsilon \ll 1 \) small enough, by Lemma 15. Thus, by Lemma 10 and the Mountain Pass theorem (cf. [3]) if \( c > \tilde{J}_\lambda(0) \) (resp. \( c_\epsilon > \tilde{J}_\lambda(0) \)), or the refinement of the MPT (cf. [20]) if \( c > \tilde{J}_\lambda(0) \) (resp. \( c_\epsilon = \tilde{J}_\lambda(0) \)), there exists a non-trivial solution to (\( P_\lambda \)), provided \( u \equiv 0 \) is its unique solution.

Of course this is a contradiction. Hence, \( \tilde{J}_\lambda \) has a critical point \( \tilde{w} > 0 \) so that \( \tilde{J}_\lambda \) has a non-trivial critical point \( \tilde{\mu} = \tilde{w}(x, 0) > 0 \). Hence, \( \tilde{\mu} = u_0 + \tilde{\mu} \) is a solution, different of \( u_0 \), to (\( P_\lambda \)).

4 Convex case, \( 1 < q < 2^* - 1 \)

Since the arguments used for \( 0 < q < 1 \) work with minor modifications in this convex case we will only indicate the main differences. First, we have that the functional \( J_\lambda \) has the appropriate geometry.

**Proposition 6** Let \( \lambda > 0 \) and \( 1 < q < 2^* - 1 \). Then, the functional \( J_\lambda \) has the Mountain Pass geometry. That is, there exists \( \rho > 0 \) and \( \beta > 0 \) such that

1. \( J_\lambda(0) = 0 \),
2. for all \( w \in \mathcal{X}^s_{\Sigma_D}(\epsilon \Omega) \) with \( \|w\| \mathcal{X}^s_{\Sigma_D}(\epsilon \Omega) = \rho \) we have \( J_\lambda(w) \geq \beta \),
3. there exists a positive function \( h \in \mathcal{X}^s_{\Sigma_D}(\epsilon \Omega) \) such that \( \|h\| \mathcal{X}^s_{\Sigma_D}(\epsilon \Omega) > \rho \) and \( J_\lambda(h) < \beta \).
Therefore, the main point in order to prove Theorem 3 is then to show that we can find a local PS for any $\gamma > 0$ with energy level under the critical level $c_{D-\mathcal{N}}^*$. If $\tilde{S}(\Sigma_D) < 2^{\frac{2}{q^*}} S(N, s)$, we use the extremal functions of the constant $\tilde{S}(\Sigma_D)$. Otherwise, if $\tilde{S}(\Sigma_D) = 2^{\frac{2}{q^*}} S(N, s)$, we proceed as in Lemma 15 using now the following estimate (cf. [6, Lemma 3.4]) on the functions $\eta_\epsilon$ defined in (3.23),

$$
\|\eta_\epsilon\|_{L^{q+1}(\Omega)}^{q+1} \geq C \epsilon^{-\frac{N-2s}{2}}(q+1)^{q+1}, \quad \text{for } N > 2s \left(1 + \frac{1}{q}\right).
$$

(4.1)

Assume $q + 1 > 2s - 2$ so that, as $q > 1$, $N > 2s \left(1 + \frac{1}{q}\right)$. Then, in the lines of (3.26) and (3.27) we get

$$
g_\epsilon(t_\epsilon) \leq \frac{s}{N}[\tilde{S}(\Sigma_D)]^{\frac{N}{2}} + C_1 \epsilon^{1-\alpha(N-2s)} - C_2 \epsilon^{-\frac{N-2s}{2}}(q+1) < c_{D-\mathcal{N}}^*,
$$

for some $\alpha > 0$ such that $q + 1 > 2s - 2 + 2\alpha$ with is possible since $q + 1 > 2s - 2$. Note that in this case, there is no restriction on the size of $\lambda > 0$. The case $q + 1 \leq 2s - 2$ follows as in [6, Lemma 3.4] so we omit the details.

**Proof of Theorem 3** Assume $\tilde{S}(\Sigma_D) = 2^{\frac{2}{q^*}} S(N, s)$. Let us define

$$
\Gamma_\epsilon = \{\gamma \in \mathcal{C}([0, 1], \mathcal{X}_{\Sigma_D}^*(\mathcal{E}_\Omega)) : \gamma(0) = 0, \gamma(1) = M_\epsilon \eta_\epsilon\},
$$

for some $M_\epsilon \gg 1$ such that $J_\lambda(M_\epsilon \eta_\epsilon) < 0$ with $\eta_\epsilon$ defined as in (3.23). Note that, for any $\gamma \in \Gamma_\epsilon$ the function $t \mapsto \|\gamma(t)\|_{\mathcal{X}_{\Sigma_D}^*(\mathcal{E}_\Omega)}$ is continuous in $[0, 1]$. Then, since $\|\gamma(0)\|_{\mathcal{X}_{\Sigma_D}^*(\mathcal{E}_\Omega)} = 0$ and $\|\gamma(1)\|_{\mathcal{X}_{\Sigma_D}^*(\mathcal{E}_\Omega)} = \|M_\epsilon \eta_\epsilon\|_{\mathcal{X}_{\Sigma_D}^*(\mathcal{E}_\Omega)} > \rho$ for $M_\epsilon$ large enough, there exists $t_0 \in (0, 1)$ such that $\|\gamma(t_0)\|_{\mathcal{X}_{\Sigma_D}^*(\mathcal{E}_\Omega)} = \rho$ for $\rho$ given in Proposition 6. As a consequence,

$$
\sup_{0 \leq t \leq 1} J_\lambda(\gamma(t)) \geq J_\lambda(\gamma(t_0)) \geq \inf_{\|g\|_{\mathcal{X}_{\Sigma_D}^*(\mathcal{E}_\Omega)} = \rho} J_\lambda(g) \geq \beta > 0,
$$

with $\beta > 0$ given in Proposition 6. Thus, $c_\epsilon = \inf_{\gamma \in \Gamma_\epsilon} \sup_{0 \leq t \leq 1} J_\lambda(\gamma(t)) > 0$. By the Mountain Pass Theorem (cf. [3]), $J_\lambda$ has a critical point $w \in \mathcal{X}_{\Sigma_D}^*(\mathcal{E}_\Omega)$ if $N > 2s \left(1 + \frac{1}{q}\right)$. Moreover, as $J_\lambda(w) = c_\epsilon \geq \beta > 0$ and $J_\lambda(0) = 0$ then $w \neq 0$. Therefore, $u = w(x, 0)$ is a nontrivial solution to $(P_\lambda)$ for $q > 1, \lambda > 0$. The case $\tilde{S}(\Sigma_D) < 2^{\frac{2}{q^*}} S(N, s)$ follows analogously. 

\[ \square \]
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