ON LOCAL SOLVABILITY FOR A CLASS OF GENERALIZED MIZOHATA EQUATIONS

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Abstract. The image in $C^\infty$ for a class of complex vector fields, containing the Mizohata operator, was characterized.

1. Introduction. One of the basic problems in the theory of partial differential equations is to decide local solvability. In the second half of the 1950's two crucial studies on the subject were presented. The first result presented the famous example by Hans Lewy, see [12], which shows that the operator

$$L = -\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} + 2i(x_1 + ix_2)\frac{\partial}{\partial x_3}$$

is not locally solvable at any point of $\mathbb{R}^3$. In fact the result is even stronger: there exists $f \in C^\infty(\mathbb{R}^3)$ such that the equation $Lu = f$ has no solution for any open set (not empty) of $\mathbb{R}^3$. The second result is due to L. Hörmander, see [10], in which it is given a general class, containing Lewy’s operator, which is not locally solvable. In this work he presented a necessary condition for local solvability of a linear partial differential operator $P$, at a given point $x_0$, involving the commutator between $P$ and its conjugate $\overline{P}$:

$$C = [P, \overline{P}] = \overline{PP} - PP.$$  

Note that $C$ is a linear partial differential operator of order $\leq 2m - 1$, if $P$ has order $m$. The result is the following:

**Theorem 1.1.** Let $P$ be a solvable operator near $x_0 \in \mathbb{R}^n$, then

$$p_m(x_0, \xi) = 0, \quad \xi \in \mathbb{R}^n \Rightarrow \quad c_{2m-1}(x_0, \xi) = 0,$$

where $p_m$ and $c_{2m-1}$ represent the principal symbols of $P$ and $C$, respectively.
The nonsolvability of the Lewy operator is an immediate consequence of this theorem.

Finally, in the 1970’s, a necessary and sufficient condition for local solvability of a linear partial differential operator, for any smooth function, was proposed, with some deep evidence by L. Nirenberg and F. Treves. In the plane they considered

\[ P = A(x,t) \frac{\partial}{\partial x} + B(x,t) \frac{\partial}{\partial t}, \]

where \( A \) and \( B \) are smooth functions and complex valued, never vanishing simultaneously. Let \( p_m(x, \xi) = a + ib \) be the principal symbol of \( P \), with \( a \) and \( b \) real valued symbols, for the sake of generality we assume that the order of \( P \) is arbitrary. If \( \nabla \xi a \neq 0 \) in a neighborhood by a point \((x_0, \xi_0)\), the bicharacteristics of \( a \) are the oriented curves

\[ \frac{dx}{ds} = \nabla \xi a(x, \xi), \quad \frac{d\xi}{ds} = -\nabla x a(x, \xi). \]

Function \( a \) is constant on each such curve, and the curves on which \( a \) vanishes are called the null-bicharacteristics of \( a \). The condition is as follows:

\((P)\) : On every null-bicharacteristic \( \Gamma \) of \( \text{Re}(p) \) the function \( \text{Im}(p) \) does not change sign, i.e., we always have \( \text{Im}(p) \geq 0 \) or \( \text{Im}(p) \leq 0 \) on \( \Gamma \).

For more details see [15] and [16].

The condition \((P)\) implies that the \( \ell \)-Mizohata operators given by

\[ M_\ell = \frac{\partial}{\partial t} + i(\ell + 1)t^{\ell} \frac{\partial}{\partial x}, \quad \text{with } \ell \in \mathbb{N}, \]

are locally solvable if, and only if, \( \ell \) is even. Note that Theorem 1.1 gives some results regarding local solvability only for the case \( \ell = 1 \). Furthermore, if \( t \neq 0 \) then \( M_\ell \) is elliptic in \((x,t)\). Therefore, it is sufficient to solve \( M_\ell u = f \) near points of the form \((x,0)\).

In the early 1980s, F. Treves [19] and J. Sjöstrand [20] considered a class called of Mizohata type operators in \( \mathbb{R}^2 \), which are by definition smooth complex vector fields \( L \) defined near the origin \((0,0)\), such that

(i): \( L(0,0), \overline{L}(0,0) \) are linearly dependent, and
(ii): \( L(0,0), [L, \overline{L}] (0,0) \) are linearly independent.

Assuming that \( L \) never vanishes, by change of variables and multiplication of a non-zero function we can assume that near \((0,0)\) the operator \( L \) is of the form

\[ L = \frac{\partial}{\partial t} + i\lambda(x,t) \frac{\partial}{\partial x}, \quad (1.1) \]

where \( \lambda \) is smooth and real valued. In this case we have that conditions (i) and (ii) are equivalent to

(i)\': \( \lambda(0,0) = 0 \), and
(ii)\': \( \frac{\partial \lambda}{\partial x}(0,0) \neq 0 \).

Note that the Mizohata operator \( M_1 \) belongs to this class of operators.

We say that the operator \( L \), defined in a neighborhood \( U \) of the origin, is locally integrable at a given \( p_0 \in U \) if there is a neighborhood \( V_0 \subset U \) of \( p_0 \) and a smooth function \( Z \in C^\infty(V_0) \) such that

\[ \frac{\partial Z}{\partial x} \neq 0 \text{ in } V_0 \quad \text{and} \quad LZ = 0. \]

Suppose that \( L \) satisfies (i)' and (ii)'. The following results were obtained by F. Treves and J. Sjöstrand, respectively:
Theorem 1.2. The operator $L$ is locally integrable at the origin if, and only if, there exists a local coordinate change such that $L$ is a multiple, by a nonzero smooth function, of the 1-Mizohata operator.

Theorem 1.3. There are smooth functions $u^+$ and $u^-$ defined in $t \geq 0$ and $t \leq 0$, respectively, such that $u^\pm(x,0)$ are real, $\frac{\partial u^\pm}{\partial t}(x,0) > 0$ and $Lu^\pm = 0$. Also, $L$ is locally integrable at the origin if, and only if, the function $(u^+)^{-1} \circ u^-(x,0)$ is well defined and real analytic near the origin.

In [14], H. Ninomiya, considered an extension of this class of operators, containing the $\ell$-Mizohata operator $M_\ell$, for $\ell$ odd, which are the smooth complex vector fields $L$, defined in a neighborhood of origin of $\mathbb{R}^2$ satisfying:

(i): $L(x,0)$ and $C_n(x,0)$ are linearly dependent for $n = 0, 1, \ldots, \ell - 1$, and

(ii): $L(x,0)$ and $C_\ell(x,0)$ are linearly independent,

where

\[
C_0 = \overline{L}, C_1 = [L, \overline{L}], C_2 = [L, C_1], \ldots, C_n = [L, C_{n-1}], \quad n = 1, 2, \ldots, \ell. \quad (1.2)
\]

Hence he proved an extension the Theorem 1.2.

In this article, based on [18], we consider a class of operators which satisfies conditions (i) and (ii) of H. Ninomiya, near $t = 0$. This class will be called almost $\ell$-Mizohata operators in a 1-dimensional submanifold $\Sigma$ on $\Omega$.

Definition 1.4. Let $L$ be a smooth complex valued vector field defined in an open subset $\Omega$ of $\mathbb{R}^2$ and $\ell \in \mathbb{N}$, $\ell \geq 1$. We say that $L$ is an almost $\ell$-Mizohata operator in $\Sigma$, if:

(i): $L$ and $C_n$ are linearly dependent on $n = 0, 1, \ldots, \ell - 1$ in $\Sigma$;

(ii): $L$ and $C_\ell$ are linearly independent in $\Sigma$.

where $C_n$, $n = 0, 1, \ldots, \ell - 1$ is given by (1.2).

The class of almost $\ell$-Mizohata operators is invariant under change of variables and under multiplication by non vanishing smooth functions. Furthermore, the set where $L$ and $\overline{L}$ are linearly dependent is equal to the set where $L$ ceases to be elliptic, thus the submanifold $\Sigma$ is contained in this set.

For this class of operators we obtain a normal form and necessary and sufficient conditions for a smooth function to be in the range of these operators, which we now describe given by the following results. From now on, given $U$ a neighborhood of origin, we will consider

\[
U^- = \{(x,t) \in U : t < 0\}
\]

and

\[
U^+ = \{(x,t) \in U : t > 0\}.
\]

Theorem 1.5. Let $L$ be an almost $\ell$-Mizohata operator in $\Sigma$, $\ell$ is odd, then there exists a neighborhood of the origin $U$, such that $L$ can be transformed into

\[
L = \frac{\partial}{\partial t} + i(\ell + 1)t^\ell(1 + \rho(x,t))\frac{\partial}{\partial x}, \quad (1.3)
\]

where $\rho$ is smooth, real valued, $\rho \equiv 0$ in $U^-$ and $\Sigma = \Sigma_0$, with

\[
\Sigma_0 = \{(x,t) \in U : t = 0\}. \quad (1.4)
\]
Theorem 1.6. Let \( L \) be an operator having the form (1.3) in a neighborhood \( U \) of the origin and \( f \in C^\infty(U) \). There exists an integral operator \( K \), acting on \( f \), and a smooth function \( Z \) defined in the upper half plane, such that \( f \) is the image of \( L \) near \((0,0)\) if, and only if,

\[
(C): \text{ there exist two holomorphic functions, } A \text{ and } B, \text{ of tempered growth in the lower half plane such that}
\]

\[
(Kf)(x) = \lim_{t \to 0^+} A(Z(x,t)) + \lim_{t \to 0^-} B(x - it^{\ell+1}), \quad (1.5)
\]

where the limit is taken in the distribution sense.

In the second half of the 1980’s N. Hanges proved these results for the case \( \ell = 1 \), see [8], where he characterized the smooth functions which are in the image of the class of operators presented by Treves and Sjostrand, he called them Almost Mizohata Operators.

Observation 1.7. The image of \( Z(U^+) \) is contained in the lower half plane and the boundary points of the form \((x,0^+)\) are mapping in boundary points of the form \((x,0^-)\), see Proposition 2.6.

Observation 1.8. Theorem 1.6 is also true when \( \ell \) is even, but in this case there is no restriction on \( f \) for solvability. Here we say that a holomorphic function \( H \) is of tempered growth in the lower half-plane if there exists a neighborhood \( U \) of origin in the complex plane, a constant \( C > 0 \) and \( N \in \mathbb{N} \) such that

\[
|H(x + it)| \leq \frac{C}{|t|^N}, \quad x + it \in U^-.
\]

This paper is organized in the following way: in Section 2 we will prove Theorem 1.5, where we also present a first integral of an almost \( \ell \)-Mizohata operator in the upper and lower half-planes. In Section 3 we will characterize condition \( (C) \) by defining the integral operator \( K \) and prove Theorem 1.6.

We mention that, using this notion as proposed by us, G. Hoepfner and R. Medrado, in [9], proved propagation of singularities for this class of operators, in the Gevrey’s category.

2. Normal form. In this section we will prove Theorem 1.5, for this we will need some lemmas that will be stated and proved in the first subsection. In the second subsection we construct first integrals associated to the normal form. Finally, in the last subsection we prove the theorem.

2.1. Lemmas. The first lemma characterizes in a neighborhood of the origin properties (i) and (ii) of Definition 1.4. First observe that if \( L \) is an almost \( \ell \)-Mizohata operator in \( \Sigma \) then after change of variables \( L \) takes the form (1.1) and \( \Sigma \) is of the form (1.4).

**Lemma 2.1.** Let \( L \) of the form (1.1) be an almost \( \ell \)-Mizohata operator in \( \Sigma_0 \) then \( \lambda \) verifies:

(i)': \[ \frac{\partial^n \lambda}{\partial t^n} = 0 \text{ in } \Sigma_0, \text{ if } n = 0, 1, \ldots, \ell - 1; \]

(ii)': \[ \frac{\partial^\ell \lambda}{\partial t^\ell} \neq 0 \text{ in } \Sigma_0. \]
Proof. Note that if \( L \) has the form (1.1) then the bracket \( C_1 \) is given by

\[
C_1 = -2i \frac{\partial \lambda}{\partial t} \frac{\partial}{\partial x}.
\]

Thus \( L \) and \( C_1 \) are linearly dependent on \( \Sigma_0 \) if, and only if, \( \frac{\partial \lambda}{\partial t} = 0 \) in \( \Sigma_0 \). Analogously, we have

\[
C_2 = \left[ -2i \frac{\partial^2 \lambda}{\partial t^2} + 2 \lambda \frac{\partial^2 \lambda}{\partial t \partial x} - 2 \frac{\partial \lambda}{\partial t} \frac{\partial \lambda}{\partial x} \right] \frac{\partial}{\partial x}.
\]

Inductively, we have that brackets \( C_n \) are of the form

\[
C_n = \left[ -2i \frac{\partial^n \lambda}{\partial t^n} + P_n \left( \lambda, \frac{\partial \lambda}{\partial t}, \ldots, \frac{\partial^{n-1} \lambda}{\partial t^{n-1}} \right) \right] \frac{\partial}{\partial x},
\]

where \( P_n \) is a polynomial on the derivatives of \( \lambda \) up to order \( k \leq n - 1 \).

So it is easy to see that \( L \) and \( C_n \) are linearly dependent on \( \Sigma_0 \) if, and only if, \( \frac{\partial^n \lambda}{\partial t^n} = 0 \) in \( \Sigma_0 \).

**Example 2.2.** For each \( \ell \in \mathbb{N} \), operator \( M_\ell \) belongs to the class of almost \( \ell \)-Mizohata operators in the submanifold \( \Sigma_0 \).

By Lemma 2.1, Implicit Function Theorem and change of variables, we get the following lemma:

**Lemma 2.3.** Let \( L \) be an almost \( \ell \)-Mizohata operator in \( \Sigma \), then there exists a neighborhood of the origin \( U \), such that \( \Sigma \) takes the form of \( \Sigma_0 \) and \( L \) takes the form

\[
L = \frac{\partial}{\partial t} + i(\ell + 1)t^\ell \lambda(x,t) \frac{\partial}{\partial x},
\]

with \( \lambda \) smooth, real valued, \( \lambda(0,0) \neq 0 \).

Before stating the next lemma, we will need the following definition:

**Definition 2.4.** A smooth function defined in a neighborhood of the origin \( U \) is called flat at \( t = 0 \) if its derivatives of order \( \geq 0 \) vanish on the line \( \{ t = 0 \} \).

**Lemma 2.5.** Let \( L \) be of the form (2.1), then there exists a neighborhood of the origin \( U \) such that, after change of variables and multiplication by a smooth function which does not vanish, \( L \) takes the form

\[
L = \frac{\partial}{\partial t} + i(\ell + 1)t^\ell (1 + \rho(x,t)) \frac{\partial}{\partial x},
\]

where \( \rho \) is smooth, real and flat at \( \{ t = 0 \} \).

Proof. We can assume that \( \lambda \) in the form (2.1) is such that \( \lambda(0,0) > 0 \). We will formally solve the following initial value problem

\[
\begin{cases}
Lv = 0 \\
v|_{t=0} = x.
\end{cases}
\]

For this we consider formal Taylor series of \( \lambda \) and \( v \) with respect to \( t \) in \( t = 0 \), that is,

\[
\lambda(x,t) \approx \sum_{j=0}^{\infty} \lambda_j(x)t^j \quad \text{and} \quad v(x,t) \approx \sum_{j=0}^{\infty} v_j(x)t^j,
\]

here

\[
\lambda_j(x) = \frac{1}{j!} \frac{\partial^j \lambda}{\partial t^j}(x,0) \quad \text{and} \quad v_j(x) = \frac{1}{j!} \frac{\partial^j v}{\partial t^j}(x,0).
\]
Since we want to determine \( v_j \) so that \( v_{|t=0} = x \) then we take \( v_0(x) = x \). Now formally solving equation \( L v = 0 \) we have

\[
\frac{\partial v}{\partial t} + i(\ell + 1)t \lambda(x, t) \frac{\partial v}{\partial x} = 0,
\]

then

\[
\sum_{j=1}^{\infty} v_j(x)j^{\ell-1} + i(\ell + 1)t \sum_{j=0}^{\infty} \lambda_j(x)t^j \left( \sum_{j=0}^{\infty} \frac{\partial v_j}{\partial x} (x)t^j \right) = 0.
\]

Comparing the degree of the polynomials we get

\[
\begin{cases} v_j(x) = 0, & \text{if } 1 \leq j \leq \ell \\ v_{\ell+1}(x) = -i\lambda(x, 0). \end{cases}
\]

Note that we can find \( v_j, j \geq \ell + 1 \), recurrently by formula

\[
v_{\ell+j}(x) = \frac{-i(\ell + 1)}{\ell + j} \sum_{k=0}^{\ell - j} \lambda_k(x) \frac{\partial v_j - (k+1)}{\partial x}(x), \quad j = 1, 2, \ldots \tag{2.3}
\]

Using formula (2.3), we define

\[
u(x, t) = x - i\lambda(x, 0)t^{\ell+1} + \sum_{j=\ell+2}^{\infty} v_j(x)t^j \chi \left( \frac{t}{\varepsilon_j} \right),
\]

where the \( \varepsilon_j \)'s converging to zero are determined so that the series is convergent in \( C^\infty \) and \( \chi \in C^\infty_c(\mathbb{R}) \) is such that

\[
\chi(t) = \begin{cases} 1, & \text{if } |t| \leq 1 \\ 0, & \text{if } |t| \geq 2. \end{cases}
\]

We found a formal power series \( v \) in \( t \), with smooth coefficients in \( x \), so that \( L v = 0 \) formally at \( \{ t = 0 \} \). Now, since \( L \) is a linear first order differential operator and \( \chi \) is constant near \( \{ t = 0 \} \) we have that \( Lu \) is flat in \( \{ t = 0 \} \).

Note that

\[
u(x, t) = x + O(t^{\ell+2}) + i \left( -\lambda(x, 0)t^{\ell+1} + O(t^{\ell+2}) \right) = \text{Re}u(x, t) + i\text{Im}u(x, t). \tag{2.4}
\]

We consider the change of variables \( (x, t) \overset{\gamma}{\rightarrow} (y, s) \), where

\[
\begin{cases} y = \text{Re}u \\ s = t \left( \frac{-\text{Im}u}{t^{\ell+1}} \right)^{\frac{1}{\ell+1}}. \end{cases}
\]

A simple calculation shows that \( (\ell + 1)s^\ell Ls = -L(\text{Im}u) \). Thus, we have that the operator \( L \) has the form

\[
\frac{\partial}{\partial s} + \frac{Ly}{Ls} \frac{\partial}{\partial y} = \frac{\partial}{\partial s} + \frac{L(\text{Re}u)}{Ls} \frac{\partial}{\partial y} = \frac{\partial}{\partial s} - \frac{(\ell + 1)s^\ell L(\text{Re}u)}{L(\text{Im}u)} \frac{\partial}{\partial y} = \frac{\partial}{\partial s} - \frac{(\ell + 1)s^\ell (L\text{Re}u)}{L(\text{Im}u)} \frac{\partial}{\partial y}.
\]

Here we use that \( Lu = L(\text{Re}u) + iL(\text{Im}u) \) and consider \( \rho = \frac{iL\text{Re}u}{L(\text{Im}u)} \). We have that \( Lu \) vanishes to infinite order at \( t = 0 \) and by (2.4) we see that \( L(\text{Im}u) \) vanishes to order \( \ell \) at \( t = 0 \), thus we can conclude that \( \rho \) also vanishes to infinite order at \( t = 0 \) and therefore \( \rho \) is smooth.
As before, after an adequate change of variables we can assume that \( \rho \) is real.

2.2. First integral of \( L \) in \( U^+ \) and \( U^- \). From now on, we will restrict to the case when \( \ell \) is odd, since for \( \ell \) even by the condition (\( P \)), \( Lu = f \) is locally solvable for every \( f \in C_c(U) \).

**Proposition 2.6.** Let \( L \) be defined near the origin having the form (2.2). Then there exists a neighborhood of the origin \( U \) and functions \( Z^+ \in C^\infty(U^+) \) and \( Z^- \in C^\infty(U^-) \), satisfying:

1: \( LZ^\pm = 0 \) in \( U^\pm \);
2: \( \frac{\partial Z^\pm}{\partial x} \neq 0 \) in \( U^\pm \);
3: \( \text{Im} Z^\pm \leq 0 \) in \( U^\pm \);
4: \( Z^\pm|_{t=0} \) is real valued.

**Proof.** We will show the proposition only for case \( U^- \), in fact the case \( U^+ \) follows in an analogous fashion. We start by considering the following change of variables

\[
\begin{align*}
y &= x \\
s &= -t^{\ell+1}.
\end{align*}
\]

Note that, since \( \ell + 1 \) is even, \( (x, t) \xrightarrow{\varphi} (y, s) \) is a change of variables from \( U^- \) to \( \varphi(U^-) \subset \{(y, s) \in \mathbb{R}^2; s < 0\} \). Thus, in the new variables the operator \( L \) has the form

\[
L_s \frac{\partial}{\partial s} + L_y \frac{\partial}{\partial y} = - (\ell + 1)t^\ell \frac{\partial}{\partial s} + i(\ell + 1)t^\ell(1 + \rho(x, t)) \frac{\partial}{\partial y}
\]

\[
= - (\ell + 1)t^\ell \left( \frac{\partial}{\partial s} - i(1 + \rho(x, t)) \frac{\partial}{\partial y} \right)
\]

\[
= - (\ell + 1)(-s)^{\frac{1}{\ell+1}} \left( \frac{\partial}{\partial s} - i(1 + \tilde{\rho}(y, s)) \frac{\partial}{\partial y} \right),
\]

here, \( \tilde{\rho}(y, s) = \rho(x, t) \), that is, \( \tilde{\rho} = \rho \circ \varphi^{-1} \). Thus,

\[
L = -(\ell + 1)(-s)^{\frac{1}{\ell+1}} E,
\]

where

\[
E = \frac{\partial}{\partial s} - i(1 + \tilde{\rho}(y, s)) \frac{\partial}{\partial y},
\]

defined in \( \varphi(U^-) \). We consider \( \mathcal{L} = [E] \) the formally integrable structure generated by the vector field \( E \).

We have that \( \mathcal{L} \) is an elliptic structure for \( s < 0 \) small. Furthermore, \( \tilde{\rho} \) is smooth up to \( \{s = 0\} \) and flat there, see for example Corollary 1.1.2 of [11].

Now we define \( b \) by

\[
b(y, s) = \begin{cases} 
0, & \text{if } s \geq 0 \\
\tilde{\rho}(y, s), & \text{if } s < 0
\end{cases}
\]

and

\[
\tilde{E} = \frac{\partial}{\partial s} - i(1 + b(y, s)) \frac{\partial}{\partial y}.
\]

From the above observations, we have that \( \tilde{E} \) defines an elliptic structure in a neighborhood of the origin and has smooth coefficients. Thus, since every elliptic structure is locally integrable, there exists a smooth function \( \tilde{w} \) defined in a
neighborhood of the origin \( \tilde{U} \) such that
\[
\tilde{E}\tilde{w} = 0 \quad \text{and} \quad \frac{\partial\tilde{w}}{\partial x} \neq 0 \text{ in } \tilde{U}.
\]

We consider a neighborhood of the origin \( V \) so that, \( \varphi(V) \subset \tilde{U} \) and \( \varphi(V^-) \subset \tilde{U} \cap \varphi(U^-) \) and we define in \( V \)
\[
w(x,t) = \tilde{w}(x,-t^{\ell+1}).
\]
For \((x,t) \in V^-\) we have \( w(x,t) = \tilde{w} \circ \varphi(x,t) \).
Thus, given \((x,t) \in V^-\) we have that
\[
Lw(x,t) = L\tilde{w}(y,s) = -(\ell + 1)(-s)\tilde{E}\tilde{w}(y,s) = -(\ell + 1)(-s)\tilde{E}\tilde{w}(y,s) = 0.
\]
Thus \( Lw = 0 \) in \( V^- \). Furthermore, defining
\[
w(x,0) = \lim_{t \to 0^-} \tilde{w}(x,-t^{\ell+1}) = \tilde{w}(x,0),
\]
and using Corollary 1.1.2 of [11], we have that \( w \in C^\infty(\overline{V^-}) \). Moreover, given \((x,t) \in V^-\)
\[
\frac{\partial w}{\partial x}(x,t) = \frac{\partial \tilde{w}}{\partial x}(x,-t^{\ell+1}) \neq 0.
\]
Then, \( \frac{\partial w}{\partial x} \neq 0 \) in \( \overline{V^-} \).

Now write \( w = a + bi \) where \( a \) and \( b \) are real functions. Since \( \frac{\partial w}{\partial x} \neq 0 \), we can assume that \( \frac{\partial a}{\partial x} > 0 \). We will also assume that \( w(0,0) = 0 \). Thus, we define
\[
b_0(x) = b(x,0) = a_0(x) = a(x,0).
\]
We have that there exists \( a_0^{-1} \), a local inverse of \( a_0 \). For \( t \) near zero there is also a local inverse of \( a_t(x) = a(x,t) \). Now define, \( F(x) = b_0(a_t^{-1}(x))|_{t=0} \). From \( Lw = 0 \) in \( V^- \) we have
\[
\begin{align*}
\frac{\partial a}{\partial t}(x,t) &= (\ell + 1)t^\ell(1 + \rho(x,t))\frac{\partial b}{\partial x}(x,t), \\
\frac{\partial b}{\partial t}(x,t) &= -(\ell + 1)t^\ell(1 + \rho(x,t))\frac{\partial a}{\partial x}(x,t).
\end{align*}
\]
From the second equality follows that for each fixed \( x, t \mapsto b(x,t) \) is increasing. Then
\[
F(a(x,t)) = F(a_t(x)) = b_0(a_t^{-1}(a_t(x)))|_{t=0} = b_0(x) = b(x,0) \geq b(x,t).
\]
Therefore we have
\[
w : \overline{V^-} \mapsto \{ y + is \in \mathbb{C}; s \leq F(y) \} = \mathcal{O}.
\]
Now, since \( \mathcal{O} \) is simply connected, for \( V \) small enough, by the Riemann Mapping Theorem, there is a holomorphic function \( H \) defined in the interior of \( \mathcal{O} \), smooth up to \( \{ s = F(y) \} \), such that
\[
H(\text{int}\mathcal{O}) = \{ u + iv \in \mathbb{C}; |u^2 + v^2| < 1, v < 0 \}
\]
and \( H'(z) \neq 0 \), if \( z \in \text{int}\mathcal{O} \). Furthermore \( H \) applies the curve \( \{ s = F(y) \} \) on the real axis. From this we define
\[
Z^- (x,t) = H(w(x,t)).
\]
We easily see that \( Z^- \) has the desired properties. \( \Box \)
2.3. Proof of Theorem 1.5.

Proof. By Lemma 2.5 we have that there exists a neighborhood $U$ of the origin such that $L$ has the form (2.2). We consider the function $Z^- : \mathbb{C} \to \mathbb{C}$ given by Proposition 2.6, we write $Z^- = a + bi$ with $a, b$ real. Assume $\frac{\partial a}{\partial x} \neq 0$ and $Z^-(0, 0) = 0.$

Since $Z^-|_{t=0}$ is real we have $b(x, 0) \equiv 0$. This, together with the fact that $LZ^- \equiv 0$ in $U^-$ implies that

$$\frac{\partial^j b}{\partial y^j}(x, 0) = 0, \text{ if } j \leq \ell$$

so that one can use the Taylor formula of order $j = \ell + 1$ of $t \mapsto b(x, t)$ at $t = 0$ to obtain

$$\frac{\partial^{\ell+1} b}{\partial t^{\ell+1}}(x, 0) = -(\ell + 1)! (1 + \rho(x, 0)) \frac{\partial a}{\partial x}(x, 0) = -(\ell + 1)! \frac{\partial a}{\partial x}(x, 0) \neq 0.$$ 

Therefore

$$b(x, t) = -t^{\ell+1}v(x, t) \text{ in } U^-,$$

where $v(x, t) > 0$ in $U^-$. Now define,

$$s = \begin{cases} t(v(x, t))^{\frac{1}{\ell+1}}, & t > 0 \\ t(v(x, t))^{\frac{1}{\ell+1}}, & t \leq 0 \end{cases}$$

and

$$y = \begin{cases} a(x, -t), & t > 0 \\ a(x, t), & t \leq 0. \end{cases}$$

We consider the mapping $(x, t) \xrightarrow{\varphi} (y, s)$. Note that $\det J\varphi(0, 0) \neq 0$ and $\varphi$ is smooth mapping because the derivatives of odd order of $y$ and $s$ are zero at $t = 0$. We also have that

$$L(t(v(x, t))^{\frac{1}{\ell+1}})|_{(0,0)} = (v(0, 0))^{\frac{1}{\ell+1}} \neq 0.$$ 

Then in a neighborhood of the origin $L(t(v(x, t))^{\frac{1}{\ell+1}}) \neq 0$. Now applying this change of variables we have that $L$ has the form

$$\bar{L} = \begin{cases} \frac{\partial}{\partial s} + \frac{L(a(x, -t))}{L(t(v(x, -t))^{\frac{1}{\ell+1}})} \frac{\partial}{\partial y}, & t > 0 \\ \frac{\partial}{\partial s} + \frac{L(a(x, t))}{L(t(v(x, t))^{\frac{1}{\ell+1}})} \frac{\partial}{\partial y}, & t \leq 0. \end{cases}$$

We take

$$\lambda(x, t) = \frac{Ly(x, t)}{Ls(x, t)}.$$ 

Thus,

$$\bar{L} = \frac{\partial}{\partial s} + \lambda(x, t) \frac{\partial}{\partial y}.$$ 

If $t < 0$ we have

$$\frac{Ly(x, t)}{Ls(x, t)} = i(\ell + 1)s^\ell.$$ 

Therefore, if $t < 0$, $L$ has the following form

$$\bar{L} = \frac{\partial}{\partial s} + i(\ell + 1)s^\ell \frac{\partial}{\partial y}.$$
That is, \( L \) is an \( \ell \)-Mizohata operator. Now in the case \( t > 0 \) we define the following smooth function on \( U \)

\[
Z(x, t) = \begin{cases} 
Z^-(x, -t), & t > 0 \\
Z^-(x, t), & t \leq 0.
\end{cases}
\]

From \( LZ^- = 0 \) in \( U^- \) it follows that \( LZ = 0 \) in \( \overline{U^-} \) and in the new coordinates \( Z \) is written as \( \tilde{Z}(y, s) = y - is^{\ell+1} \). Thus, \( L\tilde{Z}(y, s) = -i(\ell + 1)s^\ell + \lambda(x, t) \). Therefore

\[
\lambda(x, t) = i(\ell + 1)s^{\ell+1} + \tilde{L}\tilde{Z}(y, s)
\]

and

\[
\tilde{L} = \frac{\partial}{\partial s} + \left( i(\ell + 1)s^\ell + \tilde{L}\tilde{Z}(y, s) \right) \frac{\partial}{\partial y} = \frac{\partial}{\partial s} + i(\ell + 1)s^\ell \left( 1 + \frac{\tilde{L}\tilde{Z}(y, s)}{i(\ell + 1)s^\ell} \right) \frac{\partial}{\partial y}.
\]

Therefore we must consider

\[
\tilde{\rho}(y, s) = \begin{cases} 
\frac{\tilde{L}\tilde{Z}(y, s)}{i(\ell + 1)s^\ell}, & \text{if } s > 0 \\
0, & \text{if } s \leq 0.
\end{cases}
\]

We have that \( \tilde{\rho} \) is smooth and after an adequate change of variables we can assume that \( \rho \) is real.

3. Characterization of the range. Let \( L \) be an almost \( \ell \)-Mizohata operator, \( \ell \) is odd, defined in an open neighborhood of the origin \( U \). Our objective in this section is to prove Theorem 1.6, that is, characterize all smooth \( f \) defined near the origin for which there exists a distribution \( u \) defined near the origin satisfying the equation

\[
Lu = f.
\]  

(3.1)

We will divide this section into two subsections, in the first subsection we will present a parametrix of the operator \( L \) in the half-planes and in the last subsection the integral operator \( K \) (as in (1.5)), which will be constructed to prove that condition (C) is necessary and sufficient.

We start by recalling the following Generalized Fourier Inversion Formula:

**Theorem 3.1.** Let \( g \in C^\infty_\ast(V) \), where \( V \) is an open set in \( \mathbb{R}^2 \). Suppose that \( Z : V \to \mathbb{C} \) is diffeomorphism (of class \( C^\infty \)) from \( V \) to \( Z(V) \) such that

\[
|\text{Im} \left( Z(x, t) - Z(y, t) \right)| \leq \frac{1}{2} |\text{Re} \left( Z(x, t) - Z(y, t) \right)|,
\]

(3.2)

for \( (x, t), (y, t) \in V \). Then we have

\[
g(x, t) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i[Z(x, t) - Z(y, t)]\xi - \varepsilon \xi^2} g(y, t) \, dZ(y, t) \, d\xi,
\]

(3.3)

where \( dZ(y, t) \) is the form \( Z_y(y, t) \, dy \). Here the convergence is valid in \( C^\infty(V) \).

For proof and more details see [1], page 344.
3.1. The parametrix of $L$ on $U^+$ and $U^-$. Let $Z^+$ be a function as in Proposition 2.6, we can also assume that $\frac{\partial Z^+}{\partial x}(0,0) = 1$. To simplify the notation we will write $Z^+ = Z$. Note that in $U^-$, since $L$ has the form (1.3) and $\rho \equiv 0$ in $U^-$, $L$ is an $t$-Mizohata operator in $U^-$. Thus in this case the function

\[ w(x,t) = x - it^{\ell+1}, \quad (3.4) \]

satisfies $Lw = 0$ in $U^-$ and, furthermore, it satisfies the remaining properties of Proposition 2.6 in $U^-$. Suppose $f$ is smooth, defined in a neighborhood of the origin and let $\chi \in C_c^\infty(U)$ be such that $\chi \equiv 1$ near the origin, with support contained in a neighborhood where $f$ is defined. We introduce

\[ u_{22}(x,t) = \frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \int_t^\infty \int_{-\infty}^\infty e^{i[Z(x,t) - Z(y,s)]\xi - \varepsilon\xi^2} \chi f(y,s) dZ(y,s) ds \quad (3.5) \]

and

\[ u_{21}(x,t) = \frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \int_0^t \int_{-\infty}^\infty e^{i[Z(x,t) - Z(y,s)]\xi - \varepsilon\xi^2} \chi f(y,s) dZ(y,s) ds \quad (3.6) \]

for $(x,t) \in U^-$. Thus in this case the function

\[ u_{11}(x,t) = \frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \int_0^t \int_{-\infty}^\infty e^{i(x-y+is^{\ell+1} - t^{\ell+1})\xi - \varepsilon\xi^2} \chi f(y,s) dy ds \quad (3.7) \]

and

\[ u_{12}(x,t) = \frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \int_{-\infty}^0 \int_{-\infty}^\infty e^{i(x-y+is^{\ell+1} - t^{\ell+1})\xi - \varepsilon\xi^2} \chi f(y,s) dy ds \quad (3.8) \]

for $(x,t) \in U^-$. 

**Lemma 3.2.** If $U$ is chosen small enough then $u_{22}, u_{21} \in C^\infty(\overline{U^+})$ and $u_{11}, u_{12} \in C^\infty(\overline{U^-})$.

**Proof.** We will prove for $u_{22}$, the regularity of $u_{21}, u_{11}$ and $u_{12}$ follows analogously. Write $Z = a + bi$, $a$ and $b$ real, from $LZ = 0$ in $U^+$ we have that

\[ \frac{\partial a}{\partial t} = (\ell + 1)t^\ell(1 + \rho)\frac{\partial b}{\partial x}, \quad (3.9) \]

\[ \frac{\partial b}{\partial t} = -(\ell + 1)t^\ell(1 + \rho)\frac{\partial a}{\partial x}. \quad (3.10) \]

If $U$ is chosen small then $\frac{\partial a}{\partial x}(x,t)$ is near 1, as well as $\frac{\partial b}{\partial x}(x,t)$ is near 0 and $(1 + \rho(x,t))$ is near 1. Therefore, there is a constant $C_1 > 1$ such that $(1 + \rho(x,t)) \leq \sqrt{C_1}$, $\frac{\partial a}{\partial x}(x,t) \leq \sqrt{C_1}$ and $|\frac{\partial b}{\partial x}(x,t)| \leq \frac{1}{8}$ for $(x,t)$ in $\overline{U^+}$. Hence, integrating (3.10) with respect to $t$ we obtain

\[ b(x,t) - b(x,s) \geq C_1(s^{\ell+1} - t^{\ell+1}), \quad (x,t),(x,s) \in \overline{U^+}. \quad (3.11) \]

Now, integrating (3.9) with respect to $t$ and using that $|\frac{\partial b}{\partial x}(x,t)| \leq \frac{1}{8} \leq \sqrt{C_1}$ we have that

\[ |a(x,t) - a(x,s)| \leq C_1 |t^{\ell+1} - s^{\ell+1}|, \quad (x,t),(x,s) \in \overline{U^+}. \quad (3.12) \]

As a consequence of the Mean Value Theorem we see that

\[ |b(x,t) - b(y,t)| \leq \frac{1}{8} |x - y|, \quad (x,t),(y,t) \in \overline{U^+}. \quad (3.13) \]
For $t$ fixed, we consider the Taylor Formula of $a(x, t)$ in the $x$ variable around the point $(y, t)$, that is,

$$a(x, t) = a(y, t) + \frac{\partial a}{\partial x}(y, t)(x - y) + \frac{1}{2!} \frac{\partial^2 a}{\partial x^2}(y, t)(x - y)^2.$$ 

Given $\varepsilon > 0$ we have $\left|\frac{\partial a}{\partial x}(y, t) - 1\right| < \varepsilon$. Thus

$$a(x, t) - a(y, t) = x - y + \left(\frac{\partial a}{\partial x}(y, t) - 1\right)(x - y) + \frac{1}{2!} \frac{\partial^2 a}{\partial x^2}(y, t)(x - y)^2 = x - y + A(x, y, t),$$

where $A(x, y, t)$ is such that

$$|A(x, y, t)| \leq \left|\left(\frac{\partial a}{\partial x}(y, t) - 1\right)(x - y)\right| + \frac{1}{2!} \left|\frac{\partial^2 a}{\partial x^2}(y, t)(x - y)\right| |x - y|.$$

Therefore, for sufficiently small $U$, taking $\varepsilon$ appropriate, we have

$$a(x, t) - a(y, t) = x - y + A(x, y, t), \quad (x, t), (y, t) \in U^+, \quad (3.14)$$

where $A$ is such that $|A(x, y, t)| \leq \frac{1}{4}|x - y|$.

Since the exponential term of the integral in (3.5) increases, it will be necessary to make a deformation of the path of integration. Given $R > 0$, we consider

$$f(\xi) = e^{i[Z(x, t) - Z(y, s)]\xi - \xi^2},$$

$$I_R = \int_0^R f(\xi) \, d\xi$$

and

$$\zeta = \zeta(R) = R \left(1 + \frac{i}{2} \frac{x - y}{|x - y|}\right). \quad (3.15)$$

Figure 1 shows the integration paths for the case that $\frac{x - y}{|x - y|} = 1$. Thus, we want to

![Figure 1. Integration domain complexification](image-url)

transform the integral from the $\gamma_1$ path to the $\gamma_3$ path. Applying Cauchy’s Theorem to the $f$ function, which is holomorphic in $C$, we have

$$lll = \int_\Delta f(z) \, dz = \int_{\gamma_1} f(z) \, dz + \int_{\gamma_2} f(z) \, dz - \int_{\gamma_3} f(z) \, dz \quad (3.16)$$

$$= \int_0^R f(\xi) \, d\xi + \int_0^1 f \left(R + \frac{i}{2} R \frac{x - y}{|x - y|}\right) \frac{R}{2} \, d\lambda - \sqrt{\frac{5}{4}} \int_0^R f \left(\xi + \frac{i}{2} \frac{x - y}{|x - y|}\right) d\xi.$$
But
\[
\int_{\gamma_2} f(z) \, dz = \int_0^1 f \left( R + \frac{i}{2} R \frac{x - y}{|x - y|} \lambda \right) \frac{R}{2} \, d\lambda \to 0 \text{ when } R \to +\infty. \tag{3.17}
\]
Then
\[
\int_0^\infty f(\xi) \, d\xi = C \int_0^\infty f(\zeta(\xi)) \, d\xi.
\]
To prove the limit given in (3.17) consider
\[
z = (Z(x, t) - Z(y, s)) \zeta + i \varepsilon \zeta^2,
\]
for \((x, t), (y, s) \in U^+\), supposing that \(0 \leq t \leq s\) and using (3.11)-(3.14) we have
\[
\text{Im}(z) = \text{Im} \left[ (a(x, t) + ib(x, t) - a(y, s) - ib(y, s)) \zeta + i \varepsilon \zeta^2 \right]
\]
\[
= (a(x, t) - a(x, s)) \frac{R}{2} \frac{x - y}{|x - y|} + (a(x, s) - a(y, s)) \frac{R}{2} \frac{x - y}{|x - y|}
\]
\[
+ (b(x, t) - b(x, s)) R + (b(x, s) - b(y, s)) R + \frac{3}{4} \varepsilon R^2
\]
\[
\geq -C_1 |t^{+1} - s^{+1}| \frac{R}{2} \frac{x - y}{|x - y|} + (x - y + A(x, y, t)) \frac{R}{2} \frac{x - y}{|x - y|}
\]
\[
+ C_1 (s^{+1} - t^{+1}) R - \frac{1}{8} |x - y| R + \varepsilon \frac{3}{4} R^2
\]
\[
\geq C_1 (s^{+1} - t^{+1}) \frac{R}{2} + |x - y| \frac{R}{4} + \varepsilon \frac{3}{4} R^2 \geq 0. \tag{3.18}
\]
Consequently,
\[
\text{Re}(iz) \leq -C_1 (s^{+1} - t^{+1}) \frac{R}{2} - |x - y| \frac{R}{4} - \varepsilon \frac{3}{4} R^2 \leq -\varepsilon \frac{3}{4} R^2. \tag{3.19}
\]
In addition, considering a parameter \(\lambda \in [0, 1]\), we have that
\[
\text{Im} \left[ (Z(x, t) - Z(y, s)) R \left( 1 + \frac{i}{2} \frac{x - y}{|x - y|} \lambda \right) + i \varepsilon R^2 \left( 1 + \frac{i}{2} \frac{x - y}{|x - y|} \lambda \right)^2 \right] \geq 0.
\]
Therefore, we can deform the path of integration. But before that, we will need to introduce a new operator, which will be very useful. Consider the operator
\[
L_0 = \frac{1}{Z_x} \frac{\partial}{\partial x}
\]
and its formal transpose \(L_0\), given by \(L_0(v) = \frac{\partial}{\partial x} \left( \frac{1}{Z_x} v \right)\). We have that operators \(L_0\) and \(L\) commute, that is, \([L_0, L] = 0\). Furthermore, by induction it is shown that for any integer \(N \geq 0\) we have
\[
\text{(i): } (1 - L_0)^N (e^{-iZ\xi}) = (1 + i\xi)^N (e^{-iZ\xi});
\]
\[
\text{(ii): } (1 - L_0)^N (Z_x v) = Z_x (1 + L_0)^N v, \text{ for any smooth } v.
\]
With this we can write \(u^{\alpha_2}\) as follows
\[
-\frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \int_1^\infty \int_{-\infty}^\infty \int_0^\infty e^{i[Z(x, t) - Z(y, s)]\xi - \varepsilon \xi^2}
\]
\[
\times (1 + i\xi)^{-N} d\xi (1 + L_0)^N (\chi f)(y, s) dZ(y, s) ds. \tag{3.21}
\]
We set
\[
h_\varepsilon(\xi) = e^{i[Z(x, t) - Z(y, s)]\xi - \varepsilon \xi^2} (1 + i\xi)^{-N}
\]
thus, by Cauchy’s Theorem we have
\[ u_x^{22}(x, t) = -\frac{1}{2\pi} \int_t^\infty \int_{-\infty}^\infty h_\varepsilon(\zeta) \, d\zeta (1 + L_0)^N(\chi f)(y, s) \, dZ(y, s) \, ds . \]

Note that
\[ h_\varepsilon(\zeta) \to e^{i[Z(x, t) - Z(y, s)]\zeta}(1 + i\zeta)^{-N} \pm h(\zeta) \quad \text{when } \varepsilon \to 0^+ \]

and
\[ |h_\varepsilon(\zeta)| = |e^{i[Z(x, t) - Z(y, s)]\zeta - \varepsilon\zeta^2}(1 + i\zeta)^{-N}| \]
\[ = e^{\text{Re}\{i[Z(x, t) - Z(y, s)]\zeta - \varepsilon\zeta^2\}}(1 + i\zeta)^{-N} \leq C(1 + i\zeta)^{-N} \in L^1. \]

(3.22)

Using the Dominated Convergence Theorem we have
\[ u_x^{22}(x, t) = \lim_{\varepsilon \to 0} u_x^{22}(x, t) \]
\[ = -\frac{1}{2\pi} \int_t^\infty \int_{-\infty}^\infty h(\zeta) \, d\zeta (1 + L_0)^N(\chi f)(y, s) \, dZ(y, s) \, ds . \]

(3.23)

Thus, since \( f \) is smooth in a neighborhood of the origin and \( Z \) is continuous in \( \overline{U^+} \), it follows that \( u_x^{22} \in C(\overline{U^+}) \). Choosing \( N \) as large as we want, differentiating under the integration sign, using the fact that \( Z \) is smooth in \( \overline{U^+} \) and applying the Dominated Convergence Theorem we conclude that \( u_x^{22} \in C^\infty(\overline{U^+}) \).

Now let \( H \) be the Heaviside function, that is
\[ H(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0. \end{cases} \]

Define \( u \) as follows
\[ u(x, t) = H(t)(u^{21}(x, t) + u_x^{22}(x, t)) + H(-t)(u^{11}(x, t) + u_x^{12}(x, t)). \]

(3.24)

Note that from (3.13) and (3.14) we have that \( Z \) verifies (3.2).

Hence, the Generalized Fourier Inversion Formula (3.3) is valid and applying it to \( \chi f \in C_c^\infty(U) \) we have
\[ (Lu)(x, t) = (\chi f)(x, t) - (Kf)(x) \otimes \delta(t), \]

(3.25)

for \((x, t) \in U\), where we define \( K = K^+ + K^- \) which
\[ (K^+ f)(x) = \frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \int_0^\infty \int_{-\infty}^\infty \int_0^\infty e^{i[Z(x, 0) - Z(y, s)]\xi - \varepsilon \xi^2} d\xi (\chi f)(y, s) \, dZ(y, s) \, ds \]
and
\[ (K^- f)(x) = \frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \int_{-\infty}^0 \int_{-\infty}^\infty \int_0^\infty e^{i[x - y + i\xi^2(t+1)]\xi - \varepsilon \xi^2} d\xi (\chi f)(y, s) \, dy \, ds . \]

3.2. Proof of Theorem 1.6.

Proof. First we will prove that the condition is sufficient. If we assume that \( f \) satisfies (C), we can define the distribution
\[ v(x, t) = H(t)A(Z(x, t)) - H(-t)B(x - it^{t+1}) \]

and conclude that
\[ (Lv)(x, t) = (Kf)(x) \otimes \delta(t). \]

In fact, writing \( w(x, t) = x - it^{t+1} \), we have
\[ Lv = L(H)(A \circ Z) + HL(A \circ Z) - L(H)(B \circ w) - \dot{H}L(B \circ w) . \]
Thus, given $\phi \in C_c^\infty (U)$, it follows
\[
\langle L v, \phi \rangle = \langle L (H(A \circ Z), \phi) + \langle H L (A \circ Z), \phi \rangle - \langle L (H(B \circ w), \phi) - \langle H L (B \circ w), \phi \rangle .
\]
Since $Z$ and $w$ are homogeneous solutions of $L$ in $U^+$ and $U^-$, respectively, and $A$ and $B$ are holomorphic functions then the second and fourth terms on the right side of the above equation are zero, thus
\[
\langle L v, \phi \rangle = \langle \delta (t)(A \circ Z)(x, t), \phi \rangle + \langle \delta (t)(B \circ w)(x, t), \phi \rangle
\]
\[
= \left\langle \delta (t) \left[ \lim_{t \to 0^+} (A \circ Z)(x, t) + \lim_{t \to 0^-} (B \circ w)(x, t) \right], \phi \right\rangle
\]
\[
= \langle K f \delta, \phi \rangle.
\]
The above limits exist and are distributions, since $A$ and $B$ are of tempered growth in $Z(x, t)$ and $x - i t^{\ell + 1}$, respectively (see Theorem 3.1.1 of [11]). Hence, we see that
\[
L(v + v) = Lu + L v = (\chi f) - (K f) \delta + (K f) \delta = (\chi f) = f,
\]
in a neighborhood of the origin where $\chi \equiv 1$. Therefore condition (C) is sufficient to solve (3.1).

Now we will prove that condition (C) is necessary. For the moment we will assume $u \in C^1$. Consider
\[
(K_\epsilon^+ f)(x) = \frac{1}{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i[Z(x, 0) - Z(y, s)]} e^{-\epsilon \xi^2} d\xi (\chi f)(y, s) dZ(y, s) ds,
\]
where $\chi \in C_c^\infty (\mathbb{R}^2)$, $\chi \equiv 1$ near the origin and support of $\chi$ is contained in a neighborhood of the origin where both $u$ and $f$ are defined. Using that $Lu = f$ and $LZ = 0$ in $U^+$ we have
\[
K_\epsilon^+ f = I_\epsilon^+ f + J_\epsilon^+ f,
\]
where
\[
(I_\epsilon^+ f)(x) = -\frac{1}{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i[Z(x, 0) - Z(y, s)]} e^{-\epsilon \xi^2} d\xi (L \chi)(y, s) u(y, s) dZ(y, s) ds
\]
(3.29)
and
\[
(J_\epsilon^+ f)(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i[Z(x, 0) - Z(y, 0)]} e^{-\epsilon \xi^2} d\xi \chi(y, 0) u(y, 0) dZ(y, 0).
\]
(3.30)
First we will analyze $I_\epsilon^+$. The idea is to show that $I_\epsilon^+$ extends to a holomorphic function, such that the limit as $\epsilon$ tends to $0^+$ exists and is a holomorphic function. For this, we will need to deform the integration path again, we will consider the same deformation as before, that is, we will consider $\epsilon$ given by (3.15). So we have,
\[
(I_\epsilon^+ f)(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i[Z(x, 0) - Z(y, s)]} \zeta (\xi) e^{-\epsilon (\zeta (\xi))^2} d\xi (L \chi)(y, s) u(y, s) dZ(y, s) ds.
\]
(3.19)
By (3.19) we have that
\[
\text{Re} \left\{ i[Z(x, 0) - Z(y, s)] \zeta (\xi) - \epsilon (\zeta (\xi))^2 \right\} \leq -C_1 s^{\ell + 1} \frac{R}{2} - |x - y| \frac{R}{4} - \epsilon \frac{3}{4} R^2,
\]
where $C_1 > 1$ is constant. Since $\chi \equiv 1$ in a neighborhood of the origin, then $L \chi \equiv 0$ in this neighborhood. So we need to estimate the exponential term only outside this neighborhood. Assuming that this neighborhood is a rectangle, the integral with respect to $s$ and $y$ is calculated out of this rectangle. If $(y, s)$ does not belong to
this rectangle, there are $C_2$ and $C_3$ such that $|s| > C_2$ or $|y| > C_3$. Consequently, for $x$ small enough, we have

$$\frac{C_1}{2}s^{\ell + 1} \geq C_4 \text{ or } \frac{|x - y|}{4} \geq C_5,$$

with $(C_j)_{j=2}^{5}$ positive constants, not depending on $s, y$ and $x$. Thus, for $x$ small enough, there exists a constant $C > 0$ not depending on $s, y$, and $x$ such that

$$\Re \left\{ i[Z(x, 0) - Z(y, s)]\zeta(\xi) - \varepsilon(\zeta(\xi))^2 \right\} \leq -CR - \frac{3}{4}\varepsilon R^2.$$

Thus,

$$|e^{i[Z(x, 0) - Z(y, s)]\zeta(\xi)} - e^{i[Z(x, 0) - Z(y, s)]\zeta(\xi) - \varepsilon(\zeta(\xi))^2}| \leq e^{-C\varepsilon - \frac{3}{4}\varepsilon^2}.$$  (3.31)

For $x$ small enough, let us look at $Z(x, 0)$ as a complex variable. And we consider

$$G_{\varepsilon}(Z(x, 0), y, s, \xi) = e^{i[Z(x, 0) - Z(y, s)]\zeta(\xi) - \varepsilon(\zeta(\xi))^2}(L\chi)(y, s)u(y, s)Z_y(y, s).$$

Then, from (3.31) we have

$$|G_{\varepsilon}(Z(x, 0), y, s, \xi)| \leq e^{-C\varepsilon - \frac{3}{4}\varepsilon^2}|(L\chi)(y, s)u(y, s)Z_y(y, s)| \leq \tilde{C}e^{-C\varepsilon - \frac{3}{4}\varepsilon^2} \in L^1.$$  (3.32)

Therefore, $I_{\varepsilon}^+ f$ is a holomorphic function of the variable $Z(x, 0)$, for $x$ small enough. By the Dominated Convergence Theorem we have that exists $\lim_{\varepsilon \to 0^+} I_{\varepsilon}^+ f(x)$ and

$$|(I_{\varepsilon}^+ f)(x)| \leq \frac{\tilde{C}}{2\pi} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-C\varepsilon - \frac{3}{4}\varepsilon^2} d\xi dy ds = \tilde{C}.$$  

Then, by Montel Theorem, following $\lim_{\varepsilon \to 0^+} I_{\varepsilon}^+ f(x)$ is a holomorphic function of the variable $Z(x, 0)$. Thus, there is a function $A^+$ holomorphic near the origin in $\mathbb{C}$ such that

$$\lim_{\varepsilon \to 0^+} (I_{\varepsilon}^+ f)(x) = A^+(Z(x, 0)).$$  (3.33)

In addition, $A^+$ is real analytic in the variable $Z(x, 0)$.

Let us now analyze $J_{\varepsilon}^+ f$. For this we start by applying the Generalized Fourier Inversion Formula to $\chi u \in C^2_0(\mathbb{R}^2)$, then

$$\chi u(y, 0) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i[Z(x, 0) - Z(y, 0)]\xi - \varepsilon \xi^2} d\xi \chi u(y, 0) dZ(y, 0).$$

Thus, for small $x$ we have that

$$\lim_{\varepsilon \to 0^+} J_{\varepsilon}^+ f(x) = -\chi u(y, 0) + \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{-\infty}^{0} \int_{-\infty}^{\infty} e^{i[Z(x, 0) - Z(y, 0)]\xi - \varepsilon \xi^2} d\xi \chi u(y, 0) dZ(y, 0)$$

$$= -u(y, 0) + \lim_{\varepsilon \to 0^+} F_{\varepsilon}(x),$$

where

$$F_{\varepsilon}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i[Z(x, 0) - Z(y, 0)]\xi - \varepsilon \xi^2} d\xi \chi u(y, 0) dZ(y, 0).$$

Consider the following neighborhood of the origin of the complex plane

$$V = \{z \in \mathbb{C}; z \in Z(U)\}$$

and write

$$f_{\varepsilon}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i[z - Z(y, 0)]\xi - \varepsilon \xi^2} d\xi \chi u(y, 0) dZ(y, 0).$$
We have that

1: \( f_\varepsilon \) is holomorphic in \( V^- \);
2: \( \exists \lim_{\varepsilon \to 0^+} f_\varepsilon, \) in \( V^- \);
3: \( \lim_{\varepsilon \to 0^+} f_\varepsilon \) is holomorphic in \( V^- \);
4: there are \( K > 0 \) and an integer \( N \geq 0 \) such that \( |\lim_{\varepsilon \to 0^+} f_\varepsilon(z)| \leq \frac{K}{|\text{Im} z|} \) in \( V^- \).

Note that in this case, we don’t need to apply deformation of the integration path as before.

Thus, \( B^+ \) is holomorphic of tempered growth in the lower half-plane. Hence, we can take the boundary limits, see Theorem 3.1.14 of [11], to obtain

\[
\lim_{t \to 0^+} B^+(Z(x, t)) = B^+(Z(x, 0)) = \lim_{\varepsilon \to 0^+} f_\varepsilon(Z(x, 0)) = \lim_{\varepsilon \to 0^+} f_\varepsilon(x).
\]

Then

\[
\lim_{t \to 0^+} J^+_t f(x) = -u(x, 0) + \lim_{t \to 0^+} B^+(Z(x, t)).
\]

Combining (3.28), (3.33) and (3.34) we see that there exists a function \( A \) holomorphic of slow growth in the lower half-plane such that

\[
K^+ f(x) = -u(x, 0) + \lim_{t \to 0^+} A(Z(x, t)).
\]

Using similar arguments it is also shown that there exists a function \( B \) holomorphic of tempered growth in the lower half-plane such that

\[
K^- f(x) = u(x, 0) + \lim_{t \to 0^-} B(x - it^{\ell+1}).
\]

Of (3.35) and (3.36) we see that

\[
K f(x) = \lim_{t \to 0^+} A(Z(x, t)) + \lim_{t \to 0^-} B(x - it^{\ell+1}).
\]

That is, (C) is necessary to solve (3.1), assuming \( u \in C^1 \).

Finally, suppose that \( u \) is a distribution. We take a neighborhood of the origin \( U \) and \( I_1 \) and \( I_2 \) open intervals in \( \mathbb{R} \) containing the origin, such that \( \bar{I}_1 \times \bar{I}_2 \subset U \).

Again, we consider the operator \( L_0 \) given by (3.20). Let us use here the following theorem, which is a particular case of a theorem due to Baouendi and Treves (see [2], pg 250).

**Theorem 3.3.** Suppose that \( L \) has the form (1.3) in \( U, u \in \mathcal{D}’(U) \) and \( f \in C^\infty(U) \) satisfying \( Lu = f \) in \( U \). Then there is an integer \( N \geq 0 \) and a continuous function \( v \) in \( I_1 \times I_2 \) such that

\[
u = L_0^N v.
\]

This result implies that there is \( v_2 \in C^0(\bar{I}_1 \times \bar{I}_2) \) such that

\[
u = L_0^N v_2 \quad \text{in} \quad I_1 \times I_2.
\]

Using that \( L \) and \( L_0 \) commute we have \( L(L_0^N v) = L_0^N (Lv) \). Thus, substituting \( L_0^N v \) for \( u \) in (3.27) we get

\[
(K^+_\varepsilon f)(x) = \frac{1}{2\pi i \varepsilon} \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{t(z(x,0) - z(y,s)} \xi - \varepsilon \xi^2 \, d\xi \chi(y, s) dZ(y, s).
\]

Hence, integrating by parts with respect to \( y \) we can proceed in a similar way.

Therefore, we conclude the proof of the Theorem.

**Observation 3.4.** The condition (C) is independent of the choice of the cut-off function \( \chi \).
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