Combining Experimental and Observational Data for Identification and Estimation of Long-Term Causal Effects

(Working Paper)

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Abstract

We consider the task of identifying and estimating the causal effect of a treatment variable on a long-term outcome variable using data from an observational domain and an experimental domain. The observational domain is subject to unobserved confounding. Furthermore, subjects in the experiment are only followed for a short period of time; hence, long-term effects of treatment are unobserved but short-term effects will be observed. Therefore, data from neither domain alone suffices for causal inference about the effect of the treatment on the long-term outcome, and must be pooled in a principled way, instead. [Athey et al. (2020)] proposed a method for systematically combining such data for identifying the downstream causal effect in view. Their approach is based on the assumptions of internal and external validity of the experimental data, and an extra novel assumption called latent unconfoundedness. In this paper, we first review their proposed approach, and then we propose three alternative approaches for data fusion for the purpose of identifying and estimating average treatment effect as well as the effect of treatment on the treated. Our first approach is based on assuming equi-confounding bias for the short-term and long-term outcomes. Our second approach is based on a relaxed version of the equi-confounding bias assumption, where we assume the existence of an observed confounder such that the short-term and long-term potential outcome variables have the same partial additive association with that confounder. Our third approach is based on the proximal causal inference framework, in which we assume the existence of an extra variable in the system which is a proxy of the latent confounder of the treatment-outcome relation. We propose influence function-based estimation strategies for each of our data fusion frameworks and study the robustness properties of the proposed estimators.

Keywords: Long-Term Causal Effects; Data Fusion; Equi-Confounding Bias; Bespoke Instrumental Variables; Proximal Causal Inference; Influence Functions
1 Introduction

The gold standard for estimating the causal effect of a treatment variable on an outcome variable of interest is to conduct a randomized experiment. This is due to the fact that randomization leads to the (conditional) exchangeability (also known as ignorability) property, which implies that the sub-populations in the arms of the experiment are comparable. However, running randomized experiments can be expensive or time consuming, and hence, usually only limited experimental data is available. On the other hand, in many applications, we have access to large amount of observational data from our system of interest, yet there may be unobserved confounders of the treatment-outcome relation in the system, rendering causal inference impossible without positing extra assumptions. Therefore, a natural question is how to combine the experimental and observational data for causal inference.

In some applications, experiments can be run on the target variable of interest, and hence, data fusion is merely for the sake of improving estimation efficiency (Kallus and Mao, 2020). However, in many settings, the experiment is run with another target variable different from the primary target. Especially, the practicalities of conducting an experiment with human subjects dictates that observations on subjects (and their compliance with the study protocol) only extends over a relatively short period of time from enrollment in the experiment. Hence, information is often missing on long-term outcomes (primary target) in randomized experiments. In this case, the experimental data alone cannot be used for estimating the causal effect of the treatment on the long-term outcome variable, and if the observational domain is confounded, the observational data alone cannot be used for causal inference either. This is the setup that we focus on in this work. An example of such setup, presented by Athey et al. (2020), is estimating the effect of class size on eighth grade test scores in New York schools, where we have access to observational data from New York schools, and Project STAR data is used as the experimental data which includes test scores only through the third grade.

Athey et al. (2020) proposed a method for combining data from an experimental domain which only includes short-term outcome variable with observational domain which contains confounded data from both the short-term and the long-term outcome variable. The authors showed that under the assumptions of internal and external validity of the experimental data, and an extra novel assumption termed latent unconfoundedness, the average treatment effect (ATE) on the long-term outcome in the observational data is identified. In this paper, we first review their proposed approach and discuss the latent unconfoundedness assumption. Then we propose three alternative approaches for data fusion for the purpose of estimating ATE as well as estimating the effect of treatment on the treated (ETT).

Our first proposed data fusion approach is based on assuming equi-confounding bias for the short-term and long-term outcomes, where we consider both additive and quantile-quantile equi-confounding bias. Roughly speaking, the equi-confounding bias assumption posits that the magnitude of the confounding bias for the short-term and the long-term outcome variables are the same. This approach is inspired by the literature of difference-in-differences (DiD) framework (Card, 1990), Augrist and Pischke (2008), change-in-changes framework (Athey and Imbens, 2006), and negative control-based causal inference (Lipsitch et al., 2010, Tchetgen Tchetgen, 2014, Sofer et al., 2016). Our second proposed data fusion approach is based on assuming the existence of an observed confounder in the system called the bespoke instrumental variable (BSIV), such that the short-term and long-term potential outcome variables have the same partial additive association with that confounder. The existence of such a variable allows us to relax the equi-confounding bias assumption in the first approach, as in the second approach, we do not assume restrictions on the selection bias for the short-term and the long-term outcomes. This approach is based on the recently proposed BSIV causal inference framework of Richardson and Tchetgen Tchetgen (2021). Our third proposed data fusion approach is based on the proximal causal inference framework (Miao et al., 2013, Tchetgen Tchetgen et al., 2020, Cui et al., 2020). For this approach, we require the existence of a treatment confounding proxy variable in the system which is associated with the latent confounder, and is independent of the outcome variables conditional on the treatment and all observed and unobserved confounder variables.
As we formally establish, unbiased information encoded in the experimental data effectively allows for relaxation of a key identifying condition of standard DiD methods, the original BSIV approach, and proximal causal inference methods. Standard DiD methods applied to our setting would require that the treatment cannot causally impact the short-term outcome: as the short-term outcome is a post-treatment outcome, one cannot rule out such a causal effect, however, as we demonstrate, experimental data still allows for identification of the causal effect of the treatment on the downstream outcome using DiD methods by essentially anchoring the short-term causal impact at that observed in the experimental sample. Similarly, in the original BSIV causal inference framework, it is assumed that there exists a reference domain in which the treatment is not applied and hence, the outcome in that domain in fact represents the potential outcome under no treatments. In our setup, we relax this assumption by utilizing the internal validity of the experimental domain. Likewise, in the proximal causal approach, the experimental sample relaxes a required exclusion restriction of no treatment effect on the short-term outcome, in this case standing for an outcome confounding proxy. This, together with availability of a valid treatment proxy, enables proximal causal identification anchored at the short-term causal impact learned from the experimental sample.

To the best of our knowledge, we are the first to propose non-parametric identification methods for long-term causal effect which allows for latent confounders in the system that influence the treatment and both the short-term and long-term outcomes. After the publication of the first draft of our work, Imbens et al. (2022) also proposed an approach for identification of long-term causal effects based on proximal causal inference framework. We have provided a discussion regarding that work in Subsection 6.2.2. The main identification result of that work is weaker than ours, as it is based on stronger assumptions. However, the authors have also provided an extension to their setup and results which is very similar to the proximal data fusion approach proposed in our work. Furthermore, access to three short-term outcome variables is required in that work such that each outcome variable can be directly affected only by outcomes immediately preceding them, which may not be feasible in many real-life applications.

The rest of the paper is organized as follows. We describe the model and parameters of interest in Section 2. In Section 3 we review the approach of Athey et al. (2020) and discuss their latent unconfoundedness assumption. Our proposed alternative methods, the equi-confounding bias-based approach, the BSIV data fusion approach, and the proximal causal data fusion approach, are presented in Sections 4, 5, and 6, respectively. In Section 7 we focus on the estimation aspect of the parameters of interest, and propose influence function-based estimation strategies for each of our data fusion frameworks and study the robustness properties of the proposed estimators. Our conclusion remarks are provided in Section 8. All the proofs are provided in the Appendix.

2 Problem Description

Let $A$ be a binary treatment variable, $X$ be the vector of pre-treatment covariates, $M$ be the short-term outcome variable, $Y$ be the long-term outcome variable, $U$ be all unobserved (latent) confounders of the treatment-outcome relation, and $G \in \{O, E\}$ be the indicator of the domain, where $G = O$ indicates the observational domain, and $G = E$ indicates the experimental domain. We denote the set of all observed variables by $V$. We denote the potential short-term and long-term outcome variables, had the treatment been set to value $a$ (possibly contrary to the fact) by $M^{(a)}$ and $Y^{(a)}$, respectively. Independent and identically distributed (iid) data from $\{A, X, M\}$ are available from the experimental domain, in which we have randomized the treatment. Also, iid data from $\{A, X, M, Y\}$ is available from the observational domain. That is, we only have data from the short-term outcome variable in the experimental domain, yet we have access to data from both short-term and long-term outcomes in the observational domain, however, the treatment outcome relationship in the latter is confounded by unobserved variables in $U$.

The goal is to identify the following two causal parameters of interest: The average treatment effect (ATE)
in the observational domain – i.e., in the observed study population
\[
\theta_{ATE} = \mathbb{E}[Y^{(1)} - Y^{(0)} \mid G = O],
\]
and the effect of treatment on the treated (ETT)
\[
\theta_{ETT} = \mathbb{E}[Y^{(1)} - Y^{(0)} \mid A = 1, G = O].
\]
Clearly, these parameters are not identifiable from the experimental data alone, as \(Y\) is missing in that dataset, and also the information regarding the treatment effect in the experimental domain may be not relevant to the observational domain. Furthermore, due to possible unobserved confounding, the parameters are not identifiable from the observational data alone, either, without further assumptions. We assume that the data generating process satisfies the following assumptions.

**Assumption 1.**
\[
A \perp \perp \{Y^{(a)}, M^{(a)}\} \mid \{X, U, G\} \quad \forall a \in \{0, 1\}.
\]
Note that Assumption 1 is a very mild version of exchangeability assumption because it is stated conditional on any unobserved confounder \(U\).
Assumption 2 (Internal validity of experimental data).
\[ A \perp \{Y^{(a)}, M^{(a)}\} \mid \{X, G = E\} \quad \forall a \in \{0, 1\}. \]

Note that Assumption 2 is a context-specific conditional independence assumption. That is, a conditional independence which is only realized conditional on a certain event.

Assumption 3 (External validity of experimental data).
\[ G \perp \{Y^{(a)}, M^{(a)}\} \mid X \quad \forall a \in \{0, 1\}. \]

We also posit the standard consistency and positivity assumptions. Figures 1-6 represent a graphical model that satisfies Assumptions 2 and 3. Variables in gray circle are unobserved. Figures 1 and 2 represent the pooled dataset, Figures 3 and 4 represent the experimental dataset, which is the pooled data set conditioned on \( G = E \), and Figures 5 and 6 represent the observational dataset, which is the pooled data set conditioned on \( G = O \). Figures 2, 4, and 6 represent single world intervention graphs (SWIGs), which are graphical models obtained from the original acyclic directed graphs or mixed graphs, which include potential outcome variables. See Richardson and Robins (2013) for the definition and details.

Assumptions 2 and 3 are not sufficient for identification of the causal parameter of interest. In the following, we first review an approach proposed by Athey et al. (2020) for identification based on an extra assumption called latent unconfoundedness, and then propose our alternative approaches.

3 Athey et al. (2020) Approach

Athey et al. (2020) introduced a method for combining experimental and observational data to identify the parameter \( \theta_{ATE} \) with the following extra assumptions. As we show here, this assumption also results in identification of the parameter \( \theta_{ETT} \).

Assumption 4 (Latent Unconfoundedness).
\[ A \perp Y^{(a)} \mid \{X, M^{(a)}, G = O\} \quad \forall a \in \{0, 1\}. \]

Theorem 1. Under Assumptions 2-4, parameters \( \theta_{ATE} \) and \( \theta_{ETT} \) are identified using the following formula
\[
\theta_{ATE} = E[E[E[Y \mid M, A = 1, X, G = O] \mid X, G = E] \mid G = O] - E[E[E[Y \mid M, A = 0, X, G = O] \mid X, G = E] \mid G = O],
\]
\[
\theta_{ETT} = E[Y \mid A = 1, G = O] - \frac{E[E[E[Y \mid M, A = 0, X, G = O] \mid X, G = E] \mid G = O]}{p(A = 1 \mid G = O)} + \frac{p(A = 0 \mid G = O)}{p(A = 1 \mid G = O)} E[Y \mid A = 0, G = O].
\]

Remark 1. We note that under Assumptions 2-4, the distribution \( p(Y^{(a)} \mid G = O) \) is, in fact, identified. Therefore, Assumption 4 in (Athey et al., 2020) is powerful enough to identify more general causal parameters than contrasts based on expected values. This is also the case for our proposed approach in Section 4.2 and the proximal data fusion approach in Section 6.

3.1 Discussion Regarding Assumption 4

Considering a non-parametric structural equation model as the data generating process, Assumption 4 simply implies that if all the conditional independence relations are reflected in the causal relations among
the variables (which we denote by edges in the graphical model), and conditional independence relations are not due to a specific alignment of the modules in the causal model, that is, if the distribution is *faithful* to the causal model (Spirtes et al., 2000), then the edge from latent variable \( U \) to the outcome variable \( Y \) should not exist. That is, there should not exist an unmeasured variable which is a direct cause of both \( A \) and \( Y \). Therefore, Assumption 4 may be too strong for many real-life setups, where we do not expect faithfulness violations.

If we allow for violation of faithfulness, then \( A \) and \( Y \) can still have a latent confounder. An example of this case is given in (Athey et al., 2020), which we discuss below.

**Example 1.** Suppose that variables \( M \) and \( Y \) are generated via the following linear structural equation models.

\[
M = \theta_M A + \gamma_M X + U, \\
Y = \theta_Y A + \gamma_Y X + \delta U + \epsilon,
\]

such that \( A \perp U \mid \{X, G = E\} \) and \( A \perp \epsilon \mid \{X, U, G = O\} \). A graphical model for this model is represented in Figure 7.

**Remark 2.** It is important to note that although we believe they are more practically plausible, the assumptions that we introduce in this paper are not logically weaker than Assumption 4. For instance, the model in Example 1, despite being pathological, does not necessarily satisfy our proposed assumptions, while it satisfies Assumption 4.

# 4 Approach 1: Equi-Confounding Bias Data Fusion

In this section, we provide an alternative identification assumption for the causal parameters \( \theta_{ATE} \) and \( \theta_{ETT} \). The assumption is inspired by the additive equi-confounding bias assumption in the identification approach using negative outcome controls (Lipsitch et al., 2010; Tchetgen Tchetgen, 2014; Sofer et al., 2016), and is similar in flavor to the parallel trends assumption in the difference-in-differences (DiD) framework (Card, 1990; Angrist and Pischke, 2008).

**Assumption 5** (Additive Equi-Confounding Bias).
Figure 8: Schematic representation of Assumption 5. The dashed lines, representing the difference between two unobserved parameters, are the assumptions.

(i)
\[ E[M(1) | A = 0, G = O] - E[M(1) | A = 1, G = O] = E[Y(1) | A = 0, G = O] - E[Y(1) | A = 1, G = O], \]

(ii)
\[ E[M(0) | A = 0, G = O] - E[M(0) | A = 1, G = O] = E[Y(0) | A = 0, G = O] - E[Y(0) | A = 1, G = O]. \]

See Figure 8 for a schematic representation of Assumption 5.

Example 2. Assumption 5 is satisfied if the data is generated from the following structural equations.

\[
\begin{align*}
M &= \tau A + f_M(X, U) + \epsilon_M, \\
Y &= \theta A + f_Y(X, M, U) + \epsilon_Y,
\end{align*}
\]

where \( \epsilon_M \) and \( \epsilon_Y \) are independent noise terms and we have \( E[f(X, M, U) | A = 1] = E[f(X, M, U) | A = 0] \), where \( f(X, M, U) = f_Y(X, M, Y) - f_M(X, U) \), and \( f_M \) and \( f_Y \) can be stochastic functions.

Remark 3. Unlike the exchangeability assumption, Assumption 5 allows for the presence of latent confounders in the system. Therefore, it does not assume that the treated and the control groups have the same potential outcomes. However, it posits that the difference of the expected value of the short-term potential outcome across these two groups (that is the bias due to confounding on an additive scale) is the same as that of the long-term potential outcome variable. Equivalently, it can be interpreted as the magnitude of change (in expectation) from the short-term potential outcome to long-term potential outcome is the same across the treated and control groups. Part (ii) of Assumption 5 is indeed the well-known parallel trends assumption in the DiD framework. However, in that framework, \( M \) is assumed to be a pre-treatment variable and hence, the treatment variable cannot have a causal effect on variable \( M \). Therefore, Assumption 5 can be considered as a generalization of the parallel trends assumption.
In order to identify the parameter $\theta_{\text{ETT}}$, we note that

$$
\theta_{\text{ETT}} = \mathbb{E}[Y^{(1)} - Y^{(0)} \mid A = 1, G = O]
\quad = \mathbb{E}[Y^{(1)} \mid A = 1, G = O] - \mathbb{E}[Y^{(0)} \mid A = 0, G = O]
\quad + \mathbb{E}[Y^{(0)} \mid A = 0, G = O] - \mathbb{E}[Y^{(0)} \mid A = 1, G = O]
\quad = \mathbb{E}[Y \mid A = 1, G = O] - \mathbb{E}[Y \mid A = 0, G = O]
\quad + \mathbb{E}[M \mid A = 0, G = O] - \mathbb{E}[M^{(0)} \mid A = 1, G = O].
$$

Therefore, it suffices to identify $\mathbb{E}[M^{(0)} \mid A = 1, G = O]$. One may reason that

$$
\mathbb{E}[M^{(0)} \mid A = 1, G = O] = \mathbb{E}[\mathbb{E}[M^{(0)} \mid A = 1, X, G = O] \mid A = 1, G = O]
\quad \overset{(a)}{=} \mathbb{E}[\mathbb{E}[M^{(0)} \mid A = 1, X, G = E] \mid A = 1, G = O]
\quad \overset{(b)}{=} \mathbb{E}[\mathbb{E}[M^{(0)} \mid A = 0, X, G = E] \mid A = 1, G = O]
\quad = \mathbb{E}[\mathbb{E}[M \mid A = 0, X, G = E] \mid A = 1, G = O],
$$

and conclude the identifiability. But step (a) is not necessarily correct. Note that Assumption 3 indicates that for $a \in \{0, 1\}$, $G \perp \perp \{Y^{(a)}, M^{(a)}\} \mid X$, but we do not have the conditional independence relation $G \perp \perp \{Y^{(a)}, M^{(a)}\} \mid X, A = 1$. This is due to the fact that variable $A$ is a collider on the path between $G$ and $Y^{(a)}$ or $M^{(a)}$. Therefore, in general $\mathbb{E}[M^{(0)} \mid X, A = 1, G = O] \neq \mathbb{E}[M^{(0)} \mid X, A = 1, G = E]$. In fact we can identify the difference between $\mathbb{E}[M^{(0)} \mid X, A = 1, G = O]$ and $\mathbb{E}[M^{(0)} \mid X, A = 1, G = E]$ using the following lemma.

**Lemma 1.** Under Assumptions 2 and 3 for $a \in \{0, 1\}$, the parameter $\mathbb{E}[M^{(a)} \mid X, A = 1 - a, G = O]$ is identified as

$$
\mathbb{E}[M^{(a)} \mid X, A = 1 - a, G = O] = \frac{\mathbb{E}[M \mid X, A = a, G = E] - \mathbb{E}[M \mid X, A = a, G = O]p(A = a \mid X, G = O)}{p(A = 1 - a \mid X, G = O)}.
$$

Hence, using Lemma 1

$$
\mathbb{E}[M^{(0)} \mid X, A = 1, G = O] - \mathbb{E}[M^{(0)} \mid X, A = 1, G = E]
\quad \overset{(a)}{=} \mathbb{E}[\mathbb{E}[M^{(0)} \mid X, A = 1, G = O] - \mathbb{E}[M^{(0)} \mid X, A = 0, G = E]
\quad = \mathbb{E}[M^{(0)} \mid X, A = 1, G = O] - \mathbb{E}[M \mid X, A = 0, G = E]
\quad = \frac{p(A = 0 \mid X, G = O)}{p(A = 1 \mid X, G = O)}\bigl\{\mathbb{E}[M \mid X, A = 0, G = E] - \mathbb{E}[M \mid X, A = 0, G = O]\bigr\}.
$$

Therefore, in order to identify the parameter $\theta_{\text{ETT}}$, we propose an approach based on the following Lemma 2.

**Lemma 2.** Under Assumptions 2 and 3 for $a \in \{0, 1\}$, the parameter $\mathbb{E}[M^{(a)} \mid A = 1 - a, G = O]$ is identified as

$$
\mathbb{E}[M^{(a)} \mid A = 1 - a, G = O] = \frac{\mathbb{E}[\mathbb{E}[M \mid X, A = a, G = E] \mid G = O] - \mathbb{E}[M \mid A = a, G = O]p(A = a \mid G = O)}{p(A = 1 - a \mid G = O)}.
$$
Theorem 2. Under Assumptions \(2\), \(3\), and \(5\), the parameter \(\theta_{ETT}\) is identified using the following formula

\[
\theta_{ETT} = E[Y | A = 1, G = O] - E[Y | A = 0, G = O] + E[M | A = 0, G = O] - E[M | G = O] p(A = 0 | G = O).
\]

Theorem 3. Under Assumptions \(2\), \(3\), and \(5\) for \(a \in \{0, 1\}\), the parameter \(E[Y^{(a)} | G = O]\) and hence the parameter \(\theta_{ATE}\) are identified. \(\theta_{ATE}\) is identified using the following formula

\[
\theta_{ATE} = E[Y | A = 1, G = O] - E[Y | A = 0, G = O] + E[M | X, A = 1, G = E] | G = O] - E[E[M | X, A = 0, G = E] | G = O] - E[M | A = 1, G = O] + E[M | A = 0, G = O].
\]

For the goal of estimating ATE, Assumption \(3\) is stronger than needed and can be replaced by the following weaker assumption.

Assumption 6. The average treatment effect of the treatment \(A\) on variable \(M\) in each stratum \(X\) is the same in the experimental and observational domains. That is,

\[
E[M^{(1)} - M^{(0)} | X, G = O] = E[M^{(1)} - M^{(0)} | X, G = E].
\]

Theorem 4. Under Assumptions \(2\), \(3\), and \(6\) the parameter \(\theta_{ATE}\) is identified using the formula in Theorem \(3\).

4.1 Conditional Additive Equi-Confounding Bias

In this section, we provide the conditional version of the additive equi-confounding bias assumption as the basis of identification, which is defined as follows.

Assumption 7 (Conditional Additive Equi-Confounding Bias).

(i) With probability one, we have

\[
E[M^{(1)} | X, A = 0, G = O] - E[M^{(1)} | X, A = 1, G = O] = E[Y^{(1)} | X, A = 0, G = O] - E[Y^{(1)} | X, A = 1, G = O],
\]

(ii) With probability one, we have

\[
E[M^{(0)} | X, A = 0, G = O] - E[M^{(0)} | X, A = 1, G = O] = E[Y^{(0)} | X, A = 0, G = O] - E[Y^{(0)} | X, A = 1, G = O].
\]

Example 3. Assumption \(7\) is satisfied if the data is generated from the following structural equations.

\[
M = g_M(A, X) + f_M(X, U) + \epsilon_M,
\]
\[
Y = g_Y(A, X) + f_Y(X, M, U) + \epsilon_Y,
\]

where \(\epsilon_M\) and \(\epsilon_Y\) are independent noise terms and we have \(E[f(X, M, U) | X, A = 1] = E[f(X, M, U) | X, A = 0]\), where \(f(X, M, U) = f_Y(X, M, Y) - f_M(X, U)\), and \(g_M\), \(f_M\), \(g_Y\), and \(f_Y\) can be stochastic functions.
Remark 4. Note that neither of Assumptions 5 and 7 imply the other. Yet the types of heterogeneity and interactions among the variables that they allow are different and can be the factor in choosing one assumption over the other. Assumption 5 posits that the magnitude of the confounding bias for the short-term and the long-term outcome variables are the same, whereas Assumption 7 implies the same marginal bias. In this case, Assumption 7 is more appropriate. In this sense, Assumption 7 can be considered as the generalization of Assumption 6. However, it might be the case that the researcher does not believe that the bias for the two outcomes are the same marginally, but it holds in each stratum of $X$. In this case, Assumption 6 is more appropriate. In this sense, Assumption 6 can be considered as the generalization of Assumption 5. But again, having the same conditional bias does not imply having the same marginal bias.

Theorem 5. Under Assumptions 2, 3, and 7 (ii), the parameter $\theta_{ETT}$ is identified using the following formula

$$
\theta_{ETT} = \mathbb{E}[Y \mid A = 1, G = 0] + \mathbb{E}\left[\frac{1}{p(A = 1 \mid X, G = 0)} \mathbb{E}[M \mid X, A = 0, G = 0] \mid A = 1, G = 0\right]
- \mathbb{E}\left[\frac{1}{p(A = 1 \mid X, G = 0)} \mathbb{E}[M \mid X, A = 0, G = E] + \mathbb{E}[Y \mid X, A = 0, G = 0] \mid A = 1, G = 0\right].
$$

Theorem 6. Under Assumptions 2, 3, and 7 for $a \in \{0, 1\}$, the parameter $\mathbb{E}[Y^{(a)} \mid G = O]$ is identified, and hence the parameter $\theta_{ATE}$ is identified using the following formula

$$
\theta_{ATE} = \mathbb{E}\left[\mathbb{E}[Y \mid X, A = 1, G = O] - \mathbb{E}[Y \mid X, A = 0, G = O]\right]
+ \mathbb{E}\left[\mathbb{E}[M \mid X, A = 1, G = E] - \mathbb{E}[M \mid X, A = 0, G = E]\right]
+ \mathbb{E}\left[\mathbb{E}[M \mid X, A = 0, G = O] - \mathbb{E}[M \mid X, A = 1, G = 0] \mid G = O\right].
$$

Similar to the case of marginal equi-confounding bias, for the goal of estimating ATE, Assumption 3 is stronger than needed and can be replaced by the weaker Assumption 6.

Theorem 7. Under Assumptions 2, 6, and 7, the parameter $\theta_{ATE}$ is identified using the formula in Theorem 6.

4.2 Quantile-Quantile Equi-Confounding Bias

We note that the (conditional) additive equi-confounding bias assumption may be restrictive as it requires the two outcomes are measures on the same scale. This is not an issue for the specific setup under study in this work, because in our setup, $M$ and $Y$ are short- and long-term versions of the same variable, yet it can be an issue in other setups. As a remedy, inspired by the changes-in-changes approach in the study of panel data (Athey and Imbens, 2006), and its counterpart in the negative control-based inference framework (Sofer et al., 2016), we consider the following generalization of the additive equi-confounding bias-based approach. Our approach here also enables us to estimate causal parameters of interest beyond average-based parameters such as ATE and ETT. We only provide the result for the treated sub-population; it can be extended to marginal parameters similar to our approach in the previous subsections.

To proceed, we first introduce the quantile-quantile association, as a measure of association between two variables, which we will use to encode confounding bias.

Definition 1. The quantile-quantile association between any variable $W$ and the binary treatment variable $A$ conditional on $X$ is defined as

$$
qw(v \mid x) := F_{W \mid A=0, X=x}^{-1} \circ F_{W \mid A=1, X=x}^{-1}(v), \quad v \in [0, 1],
$$

where for random variables $X_1$ and $X_2$, we denote the cumulative distribution function of $X_1$ conditioned on $X_2$ by $F_{X_1 \mid X_2}$, and the operator $\circ$ denotes function composition.
Assumption 8 (Quantile-Quantile Equi-Confounding Bias). For all $v \in [0, 1]$, with probability one, we have

$$q_{M(v)}(v \mid X, G = O) = q_{Y(v)}(v \mid X, G = O).$$

Example 4. Assumption 8 is satisfied if the data is generated from the following structural equations.

$$M = g_M(A, X, U),$$
$$Y = g_Y(A, X, U),$$

where $g_M$ and $g_Y$ are monotonically increasing functions of $U$ for any $A, X$.

To see this, we note that for $a \in \{0, 1\}$,

$$F_{Y(a)}|A=1,X|y = Pr(Y(a) \leq y \mid A = 1, X)$$
$$= Pr(g_Y(a, X, U) \leq y \mid A = 1, X)$$
$$= Pr(U \leq g_Y^{-1}(a, X, y) \mid A = 1, X)$$
$$= F_{U|A=1,X}(g_Y^{-1}(a, X, y)).$$

Also,

$$F_{Y(a)}|A=1,X|y = v \Rightarrow F_{Y(a)}|A=1,X = y,$$

and

$$F_{U|A=1,X}(g_Y^{-1}(a, X, y)) = v \Rightarrow g_Y(a, X, F_{U|A=1,X}^{-1}(v)) = y.$$

Therefore, we have

$$F_{Y(a)}|A=0,X \circ F_{Y(a)}|A=1,X^{-1}(v) = Pr(Y(a) \leq F_{Y(a)}|A=1,X^{-1}(v) \mid A = 0, X)$$
$$= Pr(g_Y(a, X, U) \leq F_{Y(a)}|A=1,X^{-1}(v) \mid A = 0, X)$$
$$= Pr(U \leq F_{U|A=1,X}^{-1}(F_{U|A=1,X}^{-1}(v)) \mid A = 0, X)$$
$$= F_{U|A=0,X} \circ F_{U|A=1,X}^{-1}(v).$$

Similarly,

$$F_{M(a)}|A=0,X \circ F_{M(a)}|A=1,X^{-1}(v) = F_{U|A=0,X} \circ F_{U|A=1,X}^{-1}(v).$$

Therefore,

$$q_{M(a)}(v \mid X, G = O) = F_{M(a)}|A=0,X \circ F_{M(a)}|A=1,X^{-1}(v)$$
$$= F_{Y(a)}|A=0,X \circ F_{Y(a)}|A=1,X^{-1}(v)$$
$$= q_{Y(a)}(v \mid X, G = O).$$

Theorem 8. Under Assumptions 2, 3, and 8, the conditional distribution $F_{Y(v)}|X,A=1,G=O$ is identified using the following formula

$$F_{Y(v)}|A=1,X,G=O(y) = \frac{F_{M|X,A=0,G=E \circ F_{M|A=0,X,G=O}^{-1} \circ F_{Y|A=0,X,G=O}(y)}}{p(A = 1 \mid X, G = O)}$$
$$- \frac{p(A = 0 \mid X, G = O)}{p(A = 1 \mid X, G = O)} F_{Y|A=0,X,G=O}(y).$$
5 Approach 2: Bespoke IV Data Fusion

In this section, we demonstrate that the (conditional) equi-confounding bias assumption can be relaxed if there exists a variable \( Z \) among the observed pre-treatment covariates which satisfies a certain condition regarding its association with the potential outcome variables. To make the notations similar to the previous section, we denote the rest of the observed covariates by \( X \). The approach is based on the recently proposed bespoke instrumental variable (BSIV) causal inference framework of [Richardson and Tchetgen Tchetgen (2021)]. Following that work, we refer to the variable \( Z \) as a BSIV, and the approach as the BSIV data fusion approach. We present the framework for binary \( Z \), but as discussed in Remark 5, the approach can be extended to non-binary BSIV. The requirements on the BSIV variable \( Z \) are the following.

**Assumption 9 (BSIV Relevance).**

\[ E[A \mid Z = 0, X, G = O] \neq E[A \mid Z = 1, X, G = O]. \]

**Assumption 10 (BSIV Partial Additive Equi-Association).**

(i) With probability one, we have

\[ E[M^{(1)} \mid X, Z = 1, G = O] - E[M^{(1)} \mid X, Z = 0, G = O] = E[Y^{(1)} \mid X, Z = 1, G = O] - E[Y^{(1)} \mid X, Z = 0, G = O], \]

(ii) With probability one, we have

\[ E[M^{(0)} \mid X, Z = 1, G = O] - E[M^{(0)} \mid X, Z = 0, G = O] = E[Y^{(0)} \mid X, Z = 1, G = O] - E[Y^{(0)} \mid X, Z = 0, G = O]. \]

Assumption 9 is the standard testable IV relevance assumption in all IV settings. Assumption 10 states that the short-term and long-term potential outcome variables have the same partial additive association with \( Z \). This assumption is a milder version of Assumption 7 in the sense that Assumption 7 in fact states that the selection bias is the same for variables \( M \) and \( Y \), but Assumption 10 does not restrict the selection bias and instead focuses on the partial additive association with some observed confounder \( Z \). Assumption 10 is closely related to the partial population exchangeability assumption in the framework of [Richardson and Tchetgen Tchetgen (2021)]. However, in that setup, it is assumed that in the reference population, the treatment is withheld by an external intervention, and hence the counterfactual outcome under \( A = 0 \) is the same as the observational outcome.

Besides Assumption 10, we require the variable \( Z \) to satisfy one of the following partial homogeneity conditions.

**Assumption 11 (Partial Homogeneity of Causal Effect Contrast).**

(i) With probability one, we have

\[ E[Y^{(1)} - Y^{(0)} \mid A = 0, Z = 1, X, G = O] - E[M^{(1)} - M^{(0)} \mid A = 0, Z = 1, X, G = O] = E[Y^{(1)} - Y^{(0)} \mid A = 0, Z = 0, X, G = O] - E[M^{(1)} - M^{(0)} \mid A = 0, Z = 0, X, G = O]. \]

(ii) With probability one, we have

\[ E[Y^{(1)} - Y^{(0)} \mid A = 1, Z = 1, X, G = O] - E[M^{(1)} - M^{(0)} \mid A = 1, Z = 1, X, G = O] = E[Y^{(1)} - Y^{(0)} \mid A = 1, Z = 0, X, G = O] - E[M^{(1)} - M^{(0)} \mid A = 1, Z = 0, X, G = O]. \]
Assumption 12 (Partial Homogeneity of Bias Contrast).

(i) With probability one, we have
\[
\begin{align*}
\{E[Y(1) \mid A = 1, Z = 1, X, G = O] - E[Y(1) \mid A = 0, Z = 1, X, G = O]\} \\
- \{E[M(1) \mid A = 1, Z = 1, X, G = O] - E[M(1) \mid A = 0, Z = 1, X, G = O]\}
\end{align*}
\]
\[= \{E[Y(1) \mid A = 1, Z = 0, X, G = O] - E[Y(1) \mid A = 0, Z = 0, X, G = O]\} \\
- \{E[M(1) \mid A = 1, Z = 0, X, G = O] - E[M(1) \mid A = 0, Z = 0, X, G = O]\}. \tag{1}
\]

(ii) With probability one, we have
\[
\begin{align*}
\{E[Y(0) \mid A = 1, Z = 1, X, G = O] - E[Y(0) \mid A = 0, Z = 1, X, G = O]\} \\
- \{E[M(0) \mid A = 1, Z = 1, X, G = O] - E[M(0) \mid A = 0, Z = 1, X, G = O]\}
\end{align*}
\]
\[= \{E[Y(0) \mid A = 1, Z = 0, X, G = O] - E[Y(0) \mid A = 0, Z = 0, X, G = O]\} \\
- \{E[M(0) \mid A = 1, Z = 0, X, G = O] - E[M(0) \mid A = 0, Z = 0, X, G = O]\}. \tag{2}
\]

Assumption 11 states that the conditional in-group causal effect with outcome \(M\) is not a function of \(Z\). As a special case, if the conditional in-group causal effect with neither the outcome \(Y\) nor \(M\) is a function of \(Z\), then Assumption 11 is satisfied. Note that \(Z\) can be a confounder of \(A, M, \) and \(Y\) and this assumption is still satisfied. In other words, this assumption does not restrict causal relations in the model.

Regarding Assumption 12, note that the conditional equi-confounding bias condition in Assumption 7 requires the curly brackets on the right hand side of equations (1) and (2) to be equal, and also the curly brackets on the left hand side of those equations to be equal. That is, Assumption 7 requires both sides of equations (1) and (2) to be zero. Therefore, Assumption 12 is strictly weaker than Assumption 7.

Theorem 9. Define \(\pi(Z, X) := p(A = 1 \mid Z, X, G = O)\), and for \(a, z \in \{0, 1\}\) define \(E_{a, z}(X) := E[Y - M \mid A = a, Z = z, X, G = O]\) and \(P_{a, z}(X) := p(A = a \mid Z = z, X, G = O)\).

(a) Under Assumptions 2, 3, 9, 10(ii), and 11(ii), the parameter \(\theta_{ETT}\) is identified by
\[
\theta_{ETT} = E\left[ \frac{E[Y - M \mid Z = 1, X, G = O] - E[Y - M \mid Z = 0, X, G = O]}{P_{11}(X) - P_{10}(X)} \right] \\
+ E[M \mid A = 1, Z, X, G = O] - \frac{E[M \mid A = 0, Z, X, G = E]}{\pi(Z, X)} \left[ A = 1, G = O \right].
\]

(b) Under Assumptions 2, 3, 9, 10(ii), and 12(ii), the parameter \(\theta_{ETT}\) is identified by
\[
\theta_{ETT} = E\left[ \frac{E[Y - M \mid Z = 1, X, G = O] - E[Y - M \mid Z = 0, X, G = O]}{P_{11}(X) - P_{10}(X)} \right] \\
- \frac{E_{01}(X) - E_{00}(X)}{P_{01}(X) - P_{00}(X)} + E[M \mid A = 1, Z, X, G = O] - \frac{E[M \mid A = 0, Z, X, G = E]}{\pi(Z, X)} \left[ A = 1, G = O \right] \\
+ \frac{E[M \mid A = 0, Z, X, G = O] - E[M \mid A = 1, Z, X, G = E]}{\pi(Z, X)} \left[ A = 1, G = O \right].
\]

(c) Under Assumptions 2, 3, 10, and 11 the parameter \(\theta_{ATE}\) is identified by
\[
\theta_{ATE} = E\left[ \frac{E[Y - M \mid Z = 1, X, G = O] - E[Y - M \mid Z = 0, X, G = O]}{P_{11}(X) - P_{10}(X)} \right] \\
+ E[M \mid A = 1, Z, X, G = E] - E[M \mid A = 0, Z, X, G = E] \left[ G = O \right].
\]
Under Assumptions 2, 3, 9, 10, and 12, the parameter $\theta_{ATE}$ is identified by

$$
\theta_{ATE} = \mathbb{E} \left[ \left( E_{11}(X) - E_{01}(X) - E_{10}(X) + E_{00}(X) \right) Z + E_{10}(X) - E_{00}(X) \right] \frac{P_01(X) - P_00(X)}{P_11(X) - P_{10}(X)}
$$

Remark 5. The bespoke instrumental variable $Z$ does not need to be a binary variable. In the following, we list the counterpart of the assumptions of our BSIV datafusion framework for non-binary $Z$. The identification proofs essentially remain unchanged.

- **BSIV Relevance.** For any $z \neq 0$,

$$
\mathbb{E}[A \mid Z = z, X, G = O] \neq \mathbb{E}[A \mid Z = 0, X, G = O].
$$

- **BSIV Partial Additive Equi-Association.**

  (i) With probability one, we have

$$
\mathbb{E}[M^{(1)} \mid X, Z, G = O] - \mathbb{E}[M^{(1)} \mid X, Z = 0, G = O] = \mathbb{E}[Y^{(1)} \mid X, Z, G = O] - \mathbb{E}[Y^{(1)} \mid X, Z = 0, G = O],
$$

  (ii) With probability one, we have

$$
\mathbb{E}[M^{(0)} \mid X, Z, G = O] - \mathbb{E}[M^{(0)} \mid X, Z = 0, G = O] = \mathbb{E}[Y^{(0)} \mid X, Z, G = O] - \mathbb{E}[Y^{(0)} \mid X, Z = 0, G = O].
$$

- **Partial Homogeneity of Causal Effect Contrast.**

  (i) The following quantity is not a function of $Z$.

$$
\mathbb{E}[Y^{(1)} - Y^{(0)} \mid A = 0, Z, X, G = O] - \mathbb{E}[M^{(1)} - M^{(0)} \mid A = 0, Z, X, G = O].
$$

  (ii) The following quantity is not a function of $Z$.

$$
\mathbb{E}[Y^{(1)} - Y^{(0)} \mid A = 1, Z, X, G = O] - \mathbb{E}[M^{(1)} - M^{(0)} \mid A = 1, Z, X, G = O].
$$

- **Partial Homogeneity of Bias Contrast.**

  (i) The following quantity is not a function of $Z$.

$$
\left\{ \mathbb{E}[Y^{(1)} \mid A = 1, Z, X, G = O] - \mathbb{E}[Y^{(1)} \mid A = 0, Z, X, G = O] \right\} - \left\{ \mathbb{E}[M^{(1)} \mid A = 1, Z, X, G = O] - \mathbb{E}[M^{(1)} \mid A = 0, Z, X, G = O] \right\}.
$$

  (ii) The following quantity is not a function of $Z$.

$$
\left\{ \mathbb{E}[Y^{(0)} \mid A = 1, Z, X, G = O] - \mathbb{E}[Y^{(0)} \mid A = 0, Z, X, G = O] \right\} - \left\{ \mathbb{E}[M^{(0)} \mid A = 1, Z, X, G = O] - \mathbb{E}[M^{(0)} \mid A = 0, Z, X, G = O] \right\}.
$$
5.1 The Case of the Unobserved Short-Term Outcome Variable in the Observational Domain

It could be shown that under stronger versions of the homogeneity assumptions, we in fact do not need to observe the short-term outcome variable $M$ in the observational domain for identification of the parameters $\theta_{ETT}$ and $\theta_{ATE}$. The modified assumptions are as follows.

**Assumption 13** (Partial Homogeneity of Causal Effect).

(i) With probability one, we have
\[ E[Y^{(1)} - Y^{(0)} | A = 1, Z = 1, X, G = O] = E[Y^{(1)} - Y^{(0)} | A = 0, Z = 0, X, G = O]. \]

(ii) With probability one, we have
\[ E[Y^{(1)} - Y^{(0)} | A = 1, Z = 1, X, G = O] = E[Y^{(1)} - Y^{(0)} | A = 1, Z = 0, X, G = O]. \]

**Assumption 14** (Partial Homogeneity of Bias).

(i) With probability one, we have
\[ \{ E[Y^{(1)} | A = 1, Z = 1, X, G = O] - E[Y^{(1)} | A = 0, Z = 1, X, G = O] \} = \{ E[Y^{(1)} | A = 1, Z = 0, X, G = O] - E[Y^{(1)} | A = 0, Z = 0, X, G = O] \}. \] (3)

(ii) With probability one, we have
\[ \{ E[Y^{(0)} | A = 1, Z = 1, X, G = O] - E[Y^{(0)} | A = 0, Z = 1, X, G = O] \} = \{ E[Y^{(0)} | A = 1, Z = 0, X, G = O] - E[Y^{(0)} | A = 0, Z = 0, X, G = O] \}. \] (4)

**Theorem 10.** Suppose the short-term outcome variable $M$ is unobserved in the observational domain $G = O$. Define $\pi(Z, X) := p(A = 1 | Z, X, G = O)$, for $a, z \in \{0, 1\}$ define $E^{a}_{1}(X) := E[Y | A = a, Z = z, X, G = O]$ and $P_{a}(X) := p(A = a | Z = z, X, G = O)$, and define $\omega_{0}^{1}(X) := E[M | A = 0, Z = 1, X, G = E] - E[M | A = 0, Z = 0, X, G = E]$ and $\omega_{1}^{1}(X) := E[M | A = 1, Z = 1, X, G = E] - E[M | A = 1, Z = 0, X, G = E]$.

(a) Under Assumptions 13(ii), and 13(ii), the parameter $\theta_{ETT}$ is identified by
\[ \theta_{ETT} = E \left[ \frac{E[Y | Z = 1, X, G = O] - E[Y | Z = 0, X, G = O] - \omega_{0}^{1}(X)}{P_{11}(X) - P_{10}(X)} \right] | A = 1, G = O. \]

(b) Under Assumptions 13(ii), and 13(ii), the parameter $\theta_{ETT}$ is identified by
\[ \theta_{ETT} = E \left[ \frac{E^{1}_{1}(X) - E^{1}_{01}(X) - E^{1}_{10}(X) + E^{1}_{00}(X))Z + E^{1}_{10}(X) - E^{1}_{00}(X)}{P_{01}(X) - P_{00}(X)} \right] | A = 1, G = O. \]

(c) Under Assumptions 13(ii), and 13(ii), the parameter $\theta_{ATE}$ is identified by
\[ \theta_{ATE} = E \left[ \frac{E[Y | Z = 1, X, G = O] - E[Y | Z = 0, X, G = O]}{P_{11}(X) - P_{10}(X)} \right. \]
\[ \left. - \frac{\omega_{0}^{1}(X)\pi(Z, X) + \omega_{1}^{1}(X)(1 - \pi(Z, X))}{P_{11}(X) - P_{10}(X)} \right] | G = O. \]
Remark 6. Note that Figures 9-14 only demonstrate one possible way for graphical models to satisfy the proxy variable conditions; the pooled data graph is represented in Figure 9. Figures 11 and 13 demonstrate an example of a graphical models for experimental and observational domains which can be satisfied if the edge from $Z$ to $A$ was flipped, i.e., $Z$ can be either a post-treatment variable or a pre-treatment variable. Other possibilities are that the edge between $Z$ and $A$ can be removed or these two variables can have an extra latent confounder and the requirement is still satisfied. This demonstrates the flexibility that the researcher possesses in terms of choosing a variable as the proxy variable $Z$.

In order to obtain identifiability, we require the following extra assumption.

Assumption 16.

(i) For square-integrable function $g$ and any $a$ and $x$, if $\mathbb{E}[g(U) | Z, a, x, G = O] = 0$ almost surely, then $g(U) = 0$ almost surely.

(ii) There exists an outcome bridge function $h(m, a, x)$ that solves the following integral equation

$$
\mathbb{E}[Y | Z, A, X, G = O] = \mathbb{E}[h(M, A, X) | Z, A, X, G = O].
$$

We have the following identification result.

Theorem 11. Under Assumptions \([\ref{assumption3}], \ref{assumption4}, \ref{assumption5}, \text{ and } \ref{assumption6}\) the parameter $\theta_{ATE}$ is identified by

$$
\theta_{ATE} = \mathbb{E}\left[\frac{(E_{11}^i(X) - E_{00}^{10}(X) - E_{10}^{11}(X))\pi(Z, X) + (E_{11}^i(X) - E_{10}^{11}(X) - \omega_0^i(X))(1 - \pi(Z, X))}{P_{01}(X) - P_{00}(X)} \bigg| G = O\right].
$$

6 Approach 3: Proximal Data Fusion

In this section, we introduce another identification assumption for the causal parameters $\theta_{ATE}$ and $\theta_{ETT}$, which uses ideas from the proximal causal inference framework \cite{Miao2018, TchetgenTchetgen2020, Cui2020}. For this approach, we require the existence of a proxy variable $Z$ in the system in the observational domain which satisfies the following.

Assumption 15 (Existence of Proxy Variable). There exists a proxy variable $Z$ of the latent confounder in the system in the observational domain satisfying

$$
Z \perp \{M, Y\} \mid \{U, X, A, G = O\}.
$$

No assumptions on $Z$ are needed in the experimental domain. In fact, $Z$ need not exist in that domain at all.

Figures \ref{figure9} and \ref{figure10} demonstrate an example of a graphical models for experimental and observational domains which satisfy the proxy variable conditions; the pooled data graph is represented in Figure 9.

Remark 6. Note that Figures 9, 14 only demonstrate one possible way for graphical models to satisfy the proxy variable conditions. The requirement here is $Z \perp \{M, Y\} \mid \{U, X, A, G = O\}$ which can be satisfied in several ways. For example, this requirement could also have been satisfied if the edge from $Z$ to $A$ was flipped, i.e., $Z$ can be either a post-treatment variable or a pre-treatment variable. Other possibilities are that the edge between $Z$ and $A$ can be removed or these two variables can have an extra latent confounder and the requirement is still satisfied. This demonstrates the flexibility that the researcher possesses in terms of choosing a variable as the proxy variable $Z$.

In order to obtain identifiability, we require the following extra assumption.

Assumption 16.

(i) For square-integrable function $g$ and any $a$ and $x$, if $\mathbb{E}[g(U) | Z, a, x, G = O] = 0$ almost surely, then $g(U) = 0$ almost surely.

(ii) There exists an outcome bridge function $h(m, a, x)$ that solves the following integral equation

$$
\mathbb{E}[Y | Z, A, X, G = O] = \mathbb{E}[h(M, A, X) | Z, A, X, G = O].
$$

We have the following identification result.

Theorem 11. Under Assumptions \([\ref{assumption3}], \ref{assumption4}, \ref{assumption5}, \text{ and } \ref{assumption6}\) the parameter $\theta_{ATE}$ is identified by

$$
\theta_{ATE} = \mathbb{E}\left[\frac{(E_{11}^i(X) - E_{00}^{10}(X) - E_{10}^{11}(X))\pi(Z, X) + (E_{11}^i(X) - E_{10}^{11}(X) - \omega_0^i(X))(1 - \pi(Z, X))}{P_{01}(X) - P_{00}(X)} \bigg| G = O\right].
$$

and the parameter $\theta_{ETT}$ is identified by

$$
\theta_{ETT} = \mathbb{E}[Y | A = 1, G = O] - \frac{\mathbb{E}[E[h(M, 0, X) | A = 0, X, G = E] | G = O]}{p(A = 1 | G = O)}
$$

$$
+ \frac{\mathbb{E}[Y | A = 0, G = O]p(A = 0 | G = O)}{p(A = 1 | G = O)}.
$$
6.1 An Alternative Proximal Identification Method

Next, we establish an alternative proximal identification result based on the following counterpart of Assumption 16.

Assumption 17.

(i) For square-integrable function $g$ and any $a$ and $x$, if $E[g(U) \mid M, a, x, G = O] = 0$ almost surely, then $g(U) = 0$ almost surely.

(ii) There exists an outcome bridge function $q(z, a, x)$ that solves the following integral equation

$$E[q(Z, A, X) \mid M, A, X, G = O] = \frac{p(M \mid A, X, G = E)}{p(M \mid A, X, G = O)p(A \mid X, G = O)}.$$  \tag{6}

We have the following identification result.

**Theorem 12.** Under Assumptions 15 and 17, the parameter $\theta_{ATE}$ is identified by

$$\theta_{ATE} = E[I(A = 1)Yq(Z, A, X) \mid G = O] - E[I(A = 0)Yq(Z, A, X) \mid G = O],$$
and the parameter $\theta_{ETT}$ is identified by

$$
\theta_{ETT} = \mathbb{E}[Y \mid A = 1, G = O] - \frac{\mathbb{E}[I(A = 0)Y q(Z, A, X) \mid G = O]}{p(A = 1 \mid G = O)} + \frac{\mathbb{E}[Y \mid A = 0, G = O]p(A = 0 \mid G = O)}{p(A = 1 \mid G = O)}.
$$

### 6.2 Comparison to Other Proximal Setups

#### 6.2.1 Comparison to the Original Proximal Causal Inference Setup

Our approach here can be viewed as an extension of the proximal causal inference framework (Miao et al., 2018; Tchetgen Tchetgen et al., 2020; Cui et al., 2020) to the data fusion setup. The original proximal causal inference framework only considers data from one domain and besides assuming existence of a proxy variable $Z$ which satisfies $Z \perp \perp Y \mid \{U, X, A\}$, it also assumes existence of a second proxy variable $M$ of the latent confounder in the system which satisfies $M \perp \perp \{A, Z\} \mid \{U, X\}$. Figures 15 demonstrates an example of a graphical model which satisfies the proxy variables conditions in the original setup. Importantly, that setup requires no treatment effect on the variable $M$. Our work shows that if this condition is violated, experimental data still allows for identification of the causal effect of the treatment on $Y$ by essentially anchoring the short-term causal impact at that observed in the experimental sample.

#### 6.2.2 Comparison to (Imbens et al., 2022)

After the publication of the first draft of our work, Imbens et al. (2022) also proposing an approach for identification of long-term causal effects based on proximal causal inference framework. The main identification result of that work is weaker than ours, as it is based on stronger assumptions. Specifically, as for the internal validity of the experimental domain, the authors assume that for $a \in \{0, 1\}$,

$$
\{Y^{(a)}, M^{(a)}, U, X\} \perp \perp A \mid G = E,
$$

for the external validity, the authors assume that for $a \in \{0, 1\}$,

$$
\{M^{(a)}, U, X\} \perp \perp G.
$$

Our internal validity assumption is weaker in the sense that we allow that the researcher assigns treatments in the trial based on observed covariates. Clearly choosing the treatment independent of the observed covariates is a special case. More importantly, our external validity assumption is much weaker as we allow the distribution of the covariates to be different in the experimental and observational datasets. The data in these two datasets may have been collected in completely geographically separated places, and hence it is important to have the capability of allowing different distributions for the covariates in the two domains.
However, we note that the authors have also provided an extension to their setup and results which is very similar to the proximal data fusion approach proposed in our work.

In the place of our Assumption 15 (i.e., existence of proximal variable) the authors of that work assume there exists three short-term outcomes \( S = (S_1, S_2, S_3) \) in the system, sorted in the temporal order, and posit the “sequential outcomes” assumption, which states that for \( a \in \{0, 1\} \),

\[
\{Y^{(a)}, S_3^{(a)}\} \perp\!\!\!\perp S_1^{(a)} \mid \{S_2^{(a)}, U, X, G = O\}.
\]

For example, this assumption is satisfied if the short-term and long-term outcomes can be directly affected by \textit{only} outcomes immediately preceding them. The assumption is designed in a way that if \( S_2 \) is in the conditioning set, \( S_1 \) becomes independent of \( S_3 \). Therefore, by including \( S_2 \) in the conditioning set, a setup similar to our requirement of conditional independence of \( Z \) and \( \{M, Y\} \) is created. However, it may be challenging in real-life setups to find three post-treatment variables that satisfy the specific sequential conditional independence required by the sequential outcomes assumption, whereas as clarified in Remark 9 our model allows for great flexibility in terms of choosing the proxy variable \( Z \).

7 Estimation Strategies

In this section, we focus on the estimation aspect of causal inference and propose influence function (IF)-based estimators for the parameters of interest in each of our data fusion frameworks. Since the estimators are based on the influence functions, the bias of the estimators will be of second order (Bickel et al., 1993; Newey, 1990; Robins et al., 2017), which is an important feature in the face of complex nuisance functions. We also analyze the robustness of the proposed estimators in each framework. Specifically, we show that consistent estimation of what subsets of the nuisance functions in the proposed estimators suffices for consistent estimation of the parameter of interest.

7.1 Equi-Confounding Bias Data Fusion

In this section, we consider the estimation aspect of the parameters of interest in the equi-confounding bias data fusion framework. We only focus on the case of conditional additive equi-confounding bias. Recall from Theorems 15 that under the assumptions of that framework, the task of inference for \( \theta_{ETT} \) and \( \theta_{ATE} \) reduces to the estimation of the following functionals of the observed data distribution respectively.

\[
\psi_{ETT} = E[Y \mid A = 1, G = O] + E[M \mid X, A = 0, G = O] - E[M \mid X, A = 0, G = O] + E[Y \mid X, A = 0, G = O] - E[M \mid X, A = 0, G = O],
\]

\[
\psi_{ATE} = E[M \mid X, A = 1, G = O] - E[Y \mid X, A = 0, G = O] + E[M \mid X, A = 1, G = E] - E[M \mid X, A = 0, G = E] + E[M \mid X, A = 0, G = O] - E[M \mid X, A = 1, G = O].
\]

We first derive the influence functions of the parameters \( \psi_{ETT} \) and \( \psi_{ATE} \). Based on the obtained influence functions, we propose new identification formulae for \( \theta_{ETT} \) and \( \theta_{ATE} \) as well a multiply robust estimation strategies for these parameters.

**Theorem 13.** Under the unrestricted non-parametric model, the influence function of the parameter \( \psi_{ETT} \)
is given by

\[
IF_{\psi_{ETT}}(V) = \frac{1}{p(A=1, G=O)} \left\{ \frac{I(G=O)I(A=0)}{1-p(A=1 | X, G=O)} (M - \mathbb{E}[M | X, A, G]) \right. \\
+ \frac{I(G=E)I(A=0)}{1-p(A=1 | X, G=E)} \left( \frac{1}{p(G=E | X)} - 1 \right) (M - \mathbb{E}[M | X, A, G]) \\
+ \frac{I(G=O)I(A=0)p(A=1 | X, G=O)}{1-p(A=1 | X, G=O)} (Y - \mathbb{E}[Y | A, X, G]) \\
+ I(G=O) \left\{ \mathbb{E}[M | X, A=1, G=O] + \mathbb{E}[M | X, A=0, G=E] \\
+ I(A=1) (Y + \mathbb{E}[Y | X, A=1, G=O]) - \psi_{ETT} \right\} \right\}.
\]

**Theorem 14.** Under the unrestricted non-parametric model, the influence function of the parameter \( \psi_{ATE} \) is given by

\[
IF_{\psi_{ATE}}(V) = \frac{(-1)^{1-A}}{p(A | X, G=E)} \cdot \frac{I(G=E)}{p(G=O)} (M - \mathbb{E}[M | A, X, G]) \cdot \left( \frac{1}{p(G=E | X)} - 1 \right) \\
+ \frac{I(G=O)}{p(G=O)} \cdot (-1)^{1-A} \cdot \left( \frac{1}{p(A | X, G=E)} - 1 \right) (M - \mathbb{E}[M | A, X, G]) \\
+ \mathbb{E}[Y | X, A=1, G=O] - \mathbb{E}[Y | X, A=0, G=O] \\
+ \mathbb{E}[M | X, A=1, G=E] - \mathbb{E}[M | X, A=0, G=E] \\
+ I(A=1) (Y + \mathbb{E}[Y | X, A=1, G=O]) - \psi_{ATE} \right\}.
\]

Based on the influence functions of the functionals \( \psi_{ETT} \) and \( \psi_{ATE} \), we propose the following identification result for parameters \( \theta_{ETT} \) and \( \theta_{ATE} \).

**Corollary 1.**

(a) Under Assumptions \#2 \#3 \#7 (ii), parameter \( \theta_{ETT} \) is identified by

\[
\theta_{ETT} = \mathbb{E} \left[ \frac{1}{p(A=1, G=O)} \left\{ \frac{I(G=O)I(A=0)}{1-p(A=1 | X, G=O)} (M - \mathbb{E}[M | X, A, G]) \right. \\
+ \frac{I(G=E)I(A=0)}{1-p(A=1 | X, G=E)} \left( \frac{1}{p(G=E | X)} - 1 \right) (M - \mathbb{E}[M | X, A, G]) \\
+ \frac{I(G=O)I(A=0)p(A=1 | X, G=O)}{1-p(A=1 | X, G=O)} (Y - \mathbb{E}[Y | A, X, G]) \\
+ I(G=O) \left\{ \mathbb{E}[M | X, A=1, G=O] + \mathbb{E}[M | X, A=0, G=E] \\
+ I(A=1) (Y + \mathbb{E}[Y | X, A=1, G=O]) \right\} \right\}.
\]

(b) Under Assumptions \#2 \#7 \#8 or \#2 parameter \( \theta_{ATE} \) is identified by

\[
\theta_{ATE} = \mathbb{E} \left[ \frac{(-1)^{1-A}}{p(A | X, G=E)} \cdot \frac{I(G=E)}{p(G=O)} (M - \mathbb{E}[M | A, X, G]) \cdot \left( \frac{1}{p(G=E | X)} - 1 \right) \\
+ \frac{I(G=O)}{p(G=O)} \cdot (-1)^{1-A} \cdot \left( \frac{1}{p(A | X, G=E)} - 1 \right) (M - \mathbb{E}[M | A, X, G]) \\
+ \mathbb{E}[Y | X, A=1, G=O] - \mathbb{E}[Y | X, A=0, G=O] \\
+ \mathbb{E}[M | X, A=1, G=E] - \mathbb{E}[M | X, A=0, G=E] \\
+ \mathbb{E}[M | X, A=1, G=O] \right\}.
\]
7.1.1 Estimation Strategy

Define outcome regression functions

\[ \begin{align*}
\mu_M^{G=E}(a, x) & := \mathbb{E}[M \mid X = x, A = a, G = E], \\
\mu_M^{G=O}(a, x) & := \mathbb{E}[M \mid X = x, A = a, G = O], \\
\mu_Y^{G=O}(a, x) & := \mathbb{E}[Y \mid X = x, A = a, G = O],
\end{align*} \]

and propensity score functions

\[ \begin{align*}
\pi^{G=E}(x) & := p(A = 1 \mid x, G = E), \\
\pi^{G=O}(x) & := p(A = 1 \mid x, G = O).
\end{align*} \]

Based on identification formulae in Corollary 11 we propose the following estimators for the parameters \( \theta_{ETT} \) and \( \theta_{ATE} \).

**IF-based Estimator for \( \theta_{ETT} \).** Given estimators for \( \mu_M^{G=E}, \mu_M^{G=O}, \mu_Y^{G=O}, \pi^{G=E}, \pi^{G=O} \), and \( p(G = E \mid x) \), we can estimate the parameter \( \theta_{ETT} \) as

\[
\mathbb{P}_n \left[ \frac{1}{p(A = 1, G = O)} \left\{ \frac{I(G = O)I(A = 0)}{1 - \pi^{G=O}(X)} \{ M - \hat{\mu}_M^{G=O}(0, X) \} + \frac{I(G = E)I(A = 0)}{1 - \pi^{G=E}(X)} \left\{ \frac{1}{p(G = E \mid X)} - 1 \right\} \{ M - \hat{\mu}_M^{G=E}(0, X) \} + \frac{I(G = O)I(A = 0)}{1 - \pi^{G=O}(X)} \{ Y - \hat{\mu}_Y^{G=O}(0, X) \} + I(G = O) \{ \hat{\mu}_M^{G=O}(0, X) + \hat{\mu}_M^{G=E}(0, X) + I(A = 1) \{ Y + \hat{\mu}_Y^{G=O}(0, X) \} \} \right\} \right].
\]

**IF-based Estimator for \( \theta_{ATE} \).** Given estimators for \( \mu_M^{G=E}, \mu_M^{G=O}, \mu_Y^{G=O}, \pi^{G=E}, \pi^{G=O} \), and \( p(G = E \mid x) \), we can estimate the parameter \( \theta_{ATE} \) as

\[
\mathbb{P}_n \left[ \frac{(-1)^{1-A}}{1 - A + (-1)^{1-A}\pi^{G=E}(X)} \cdot \frac{I(G = E)}{p(G = O)} \{ M - \hat{\mu}_M^{G=E}(A, X) \} \left\{ \frac{1}{p(G = E \mid X)} - 1 \right\} + \frac{I(G = O)}{p(G = O)} \left\{ \frac{(-1)^{1-A}}{1 - A + (-1)^{1-A}\pi^{G=O}(X)} \{ Y - \hat{\mu}_Y^{G=O}(A, X) - M + \hat{\mu}_M^{G=O}(A, X) \} + \hat{\mu}_Y^{G=O}(1, X) - \hat{\mu}_Y^{G=O}(0, X) + \hat{\mu}_M^{G=E}(1, X) - \hat{\mu}_M^{G=E}(0, X) + \hat{\mu}_M^{G=O}(0, X) - \hat{\mu}_M^{G=O}(1, X) \} \right\} \right].
\]

In the following result, we show that the proposed IF-based estimators are multiply robust.

**Proposition 1.** The IF-based estimators for \( \theta_{ETT} \) and \( \theta_{ATE} \) are multiply robust, in the sense that the estimators are consistent if at least one of the following sets of nuisance functions is correctly specified.

\[ \begin{align*}
&\{ \mu_M^{G=E}, \mu_M^{G=O}, \mu_Y^{G=O} \} \\
&\{ \pi^{G=E}, \pi^{G=O}, p(G = E \mid x) \} \\
&\{ \mu_M^{G=E}, \pi^{G=O} \} \\
&\{ \pi^{G=E}, \mu_M^{G=O}, \mu_Y^{G=O}, p(G = E \mid x) \}
\end{align*} \]

The multiple robustness property gives the user the chance of estimating the parameter of interest consistently even if the estimators of all the nuisance functions are not consistent.
7.2 Proximal Data Fusion

In this section, we focus on the estimation aspect of the parameters of interest in the proximal data fusion framework. We propose an influence function-based estimation method as our main focus, as well as other simpler yet less robust estimation strategies. The approach and derivations in this section are similar and in parallel to that of [Ghassami et al., 2021]. The proposal here can be viewed as an extension of the semi-parametric framework of Cui et al. (2020) to the data fusion setup.

For \( a \in \{0, 1\} \), we define the parameter \( \theta^{(a)} := \mathbb{E}[Y^{(a)} | G = 0] \), which with arguments similar to those in Theorems 11 and 12 can be identified using the functional

\[
\psi^{(a)} = \mathbb{E}\left[ \mathbb{E}[h(M, A, X) | A = a, X, G = E] | G = O \right] \\
= \mathbb{E}[I(A = a)Yq(Z, A, X) | G = O].
\]

(7)

From the identification formulae of the parameters \( \theta_{ATE} \) and \( \theta_{ETT} \), we note that \( \psi^{(a)} \) is the main piece which is needed to be estimated for obtaining an estimator for \( \theta_{ATE} \) and \( \theta_{ETT} \). Therefore, we focus on the estimation of the functional \( \psi^{(a)} \). We first derive the influence function of the parameter \( \psi^{(a)} \). Based on the obtained influence function, we propose a new identification formula for \( \theta^{(a)} \) as well as several estimation strategies for this parameter. In the following, we assume that the integral equations in Assumptions 11 (ii) and 17 (ii) have unique solutions.

**Theorem 15.** Under the unrestricted non-parametric model, for \( a \in \{0, 1\} \), the influence function of the parameter \( \psi^{(a)} \) is given by

\[
IF_{\psi^{(a)}}(V) = \frac{I(G = 0)}{p(G = O)}I(A = a)q(Z, A, X)\{Y - h(M, A, X)\} \\
+ \frac{I(G = E)}{p(G = O)} \cdot \frac{I(A = a)}{p(A = a | X, G = E)}\{h(M, A, X) - \eta(A, X)\}\{ \frac{1}{p(G = E | X)} - 1 \} \\
+ \frac{I(G = 0)}{p(G = O)}(\eta(a, X) - \psi^{(a)}),
\]

where

\[
\eta(a, x) := \mathbb{E}[h(M, A, X) | A = a, X = x, G = E] \\
= \mathbb{E}[I(A = a)h(M, A, X)q(Z, A, X) | X = x, G = E].
\]

Based on the influence function of the functional \( \psi^{(a)} \), we propose the following identification result for the parameter \( \theta^{(a)} \).

**Corollary 2.** Under Assumptions 11 and 17 for \( a \in \{0, 1\} \), parameter \( \theta^{(a)} \) is identified by

\[
\theta^{(a)} = \mathbb{E}\left[ \frac{I(G = 0)}{p(G = O)}I(A = a)q(Z, A, X)\{Y - h(M, A, X)\} \\
+ \frac{I(G = E)}{p(G = O)} \cdot \frac{I(A = a)}{p(A = a | X, G = E)}\{h(M, A, X) - \eta(A, X)\}\{ \frac{1}{p(G = E | X)} - 1 \} \\
+ \frac{I(G = 0)}{p(G = O)}(\eta(a, X)) \right],
\]

where

\[
\eta(a, x) := \mathbb{E}[h(M, A, X) | A = a, X = x, G = E] \\
= \mathbb{E}[I(A = a)h(M, A, X)q(Z, A, X) | X = x, G = E].
\]
7.2.1 Estimation Strategies

Based on identification formulae (7) and (8), and Corollary 2, we propose the following estimation strategies for the parameter $\theta^{(a)}$.

- **Estimation Strategy 1.** Given estimators for $h$ and $p(m \mid A, X, G = E)$, we can estimate the parameter $\theta^{(a)}$ as
  \[ P_n \left[ \sum_m \frac{I(G = O)}{p(G = O)} \hat{h}(m, a, X) \hat{p}(m \mid a, X, G = E) \right]. \]

- **Estimation Strategy 2.** Given estimators for $h, p(A = 1 \mid X, G = E)$, and $p(G = E \mid X)$, we can estimate the parameter $\theta^{(a)}$ as
  \[ P_n \left[ I(G = O) I(A = a) \frac{1}{p(G = O)} \hat{h}(M, a, X) \{ \frac{1}{\hat{p}(G = E \mid X)} - 1 \} \right]. \]

- **Estimation Strategy 3.** Given an estimator for $q$, we can estimate the parameter $\theta^{(a)}$ as
  \[ P_n \left[ I(G = O) I(A = a) \hat{q}(Z, a, X) \right]. \]

- **Estimation Strategy 4 (IF-based Strategy).** Given estimators for $h, q, p(m \mid A, X, G = E)$, $p(A = 1 \mid X, G = E)$, and $p(G = E \mid X)$, we can estimate the parameter $\theta^{(a)}$ as
  \[ P_n \left[ I(G = O) I(A = a) \hat{q}(Z, a, X) \right]. \]

In the following result, we show that the estimator in IF-based strategy is multiply robust.

**Proposition 2.** Estimation Strategy 4 is multiply robust, in the sense that the estimator is consistent if at least one of the following sets of nuisance functions is correctly specified.

- $\{h, p(m \mid a, x, G = E)\}$
- $\{h, p(A = 1 \mid x, G = E), p(G = E \mid x)\}$
- $\{q, p(A = 1 \mid x, G = E), p(G = E \mid x)\}$

The multiple robustness property gives the user the chance of estimating the parameter of interest consistently even if the estimators of all the nuisance functions are not consistent.
7.2.2 Estimating the Bridge Functions

We note that in all the proposed estimation strategies, an estimation of at least one of the nuisance functions \( h \) and \( q \) is needed for the estimation of the parameter of interest. However, as seen earlier, these nuisance functions are solutions to conditional moment equations, and hence they cannot be estimated by a simple standard regression. In a recent work, Dikkala et al. (2020) proposed a non-parametric estimation method based on an adversarial learning approach for solving such integral equations. The approach was adopted to the original semi-parametric proximal causal inference framework in Ghassami et al. (2022; Kallus et al., 2021). Here, we propose to use the same technique for estimating bridge functions \( h \) and \( q \) in our proximal data fusion setup.

In order to proceed, we note that the bridge function \( q \) satisfies a certain conditional moment equation provided in the following result, which we use for designing the estimator for the bridge function \( q \).

**Proposition 3.** The bridge function \( q \) satisfies the following conditional moment equation.

\[
E \left[ \frac{I(G = O)}{p(G = O \mid X)} q(Z, A, X) - \frac{I(G = E)}{p(A, G = E \mid X)} M, A, X \right] = 0.
\] (9)

Let \( \mathcal{H}, \mathcal{Q}, \) and \( \mathcal{F} \) be normed function spaces. Based on the conditional moment equations (5) and (9), we propose the following regularized optimization-based estimators for the bridge functions \( h \) and \( q \).

\[
\hat{h} = \arg \min_{h \in \mathcal{H}} \sup_{f \in \mathcal{F}} \mathbb{P}_n \left[ \frac{I(G = O)}{p(G = O \mid X)} h(W, A, X) - Y \right]^2 f(Z, A, X) - f^2(Z, A, X) - \lambda_h^h \| f \|_F^2 + \lambda_H^h \| h \|_H^2,
\]

\[
\hat{q} = \arg \min_{q \in \mathcal{Q}} \sup_{f \in \mathcal{F}} \mathbb{P}_n \left[ \frac{I(G = O)}{p(G = O \mid X)} q(Z, A, X) - \frac{I(G = E)}{p(A, G = E \mid X)} f(M, A, X) - f^2(M, A, X) \right] - \lambda_q^q \| f \|_F^2 + \lambda_Q^q \| q \|_Q^2.
\]

We refer the reader to Dikkala et al. (2020; Ghassami et al., 2022) for the convergence analysis of the proposed minimax estimators.

8 Conclusion

In many real-life systems, the available observational data is confounded by latent variables and hence cannot be used for identifying the causal effect of a treatment variable on an outcome variable of interest. However, we may have access to experimental data in which, due to limitations of performing experiments, the reported outcome is a short-term version of the original long-term outcome variable that we were interested in. Clearly, neither the observational dataset nor the experimental dataset alone can be used for identifying the causal parameter of interest. But the question is whether we can combine the information from these two datasets to gain identification. In this work, we demonstrated three frameworks under which identification of the long-term causal effect is possible using data fusion. Our proposed approaches were based on: 1. Assuming equi-confounding bias for the short-term and long-term outcomes. 2. A relaxed version of the equi-confounding bias assumption, where we assume the existence of an observed confounder such that the short-term and long-term potential outcome variables have the same partial additive association with that confounder. 3. The proximal causal inference framework, in which we assume the existence of an extra variable in the system which is a proxy of the latent confounder of the treatment-outcome relation. We proposed influence function-based estimation strategies for each of our data fusion frameworks and studied the robustness properties of the proposed estimators.

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Appendices

A Proofs

Proof of Theorem In [Athey et al., 2020], the authors only considered $\theta_{ATE}$ as the parameter of interest and provided the following proof.

\[
\mathbb{E}[Y^{(a)} \mid G = O] = \mathbb{E}[\mathbb{E}[Y^{(a)} \mid X, G = O] \mid G = O]
\]

\[
\mathbb{E}[\mathbb{E}[Y^{(a)} \mid X, G = E] \mid G = O]
\]

\[
= \mathbb{E}[\mathbb{E}[\mathbb{E}[Y^{(a)} \mid M^{(a)}, X, G = E] \mid X, G = E] \mid G = O]
\]

\[
\mathbb{E}[\mathbb{E}[\mathbb{E}[Y^{(a)} \mid M^{(a)}, X, G = O] \mid X, G = E] \mid G = O]
\]

\[
\mathbb{E}[\mathbb{E}[\mathbb{E}[Y^{(a)} \mid M^{(a)}, A = a, X, G = O] \mid X, G = E] \mid G = O]
\]

\[
= \mathbb{E}[\mathbb{E}[\mathbb{E}[Y \mid M, A = a, X, G = O] \mid X, G = E] \mid G = O].
\]

$f(M, X) = \mathbb{E}[Y \mid M, A = a, X, G = O]$ is identified from the observational data.

Note that the middle expectation is over $M^{(a)}$. By Assumption 2,

\[
\mathbb{E}[f(M^{(a)}, X) \mid X, G = E] = \mathbb{E}[f(M^{(a)}, X) \mid X, A = a, G = E] = \mathbb{E}[f(M, X) \mid X, A = a, G = E]
\]

is identified from the experimental data.

To see the back-and-forths between the two domains better, perhaps it is easier to look at the following alternative presentation of the proof:

\[
p(Y^{(a)} \mid G = O) = \sum_X p(Y^{(a)} \mid X, G = O)p(X \mid G = O)
\]

\[
= \sum_X p(Y^{(a)} \mid X, G = E)p(X \mid G = O)
\]

\[
= \sum_{X, M^{(a)}} p(Y^{(a)} \mid M^{(a)}, X, G = E)p(M^{(a)} \mid X, G = E)p(X \mid G = O)
\]

\[
= \sum_{X, M^{(a)}} p(Y^{(a)} \mid M^{(a)}, X, G = O)p(M^{(a)} \mid X, G = E)p(X \mid G = O)
\]

\[
= \sum_{X, M} p(Y \mid A = a, M, X, G = O)p(M \mid A = a, X, G = E)p(X \mid G = O).
\]

Therefore, $\mathbb{E}[Y^{(1)} \mid G = 0], \mathbb{E}[Y^{(0)} \mid G = 0]$, and hence ATE is identified.

Realizing that $\mathbb{E}[Y^{(0)} \mid G = O]$ is identified, it is easy to see that $\mathbb{E}[Y^{(0)} \mid A = 1, G = O]$ is also identified as

\[
\mathbb{E}[Y^{(0)} \mid G = O] = \mathbb{E}[Y^{(0)} \mid A = 1, G = 0]p(A = 1 \mid G = O) + \mathbb{E}[Y^{(0)} \mid A = 0, G = O]p(A = 0 \mid G = O)
\]

\[
\Rightarrow \mathbb{E}[Y^{(0)} \mid A = 1, G = O] = \frac{\mathbb{E}[Y^{(0)} \mid G = O] - \mathbb{E}[Y \mid A = 0, G = O]p(A = 0 \mid G = O)}{p(A = 1 \mid G = O)}.
\]

Therefore, $\theta_{ETT} = \mathbb{E}[Y \mid A = 1, G = O] - \mathbb{E}[Y^{(0)} \mid A = 1, G = O]$ is also identified.

\[\square\]
Therefore,
\[
\mathbb{E}[M^{(0)} | X, A = 1, G = O] = \mathbb{E}[M^{(0)} | X, A = 1, G = O]p(A = 1 | X, G = O) + \mathbb{E}[M^{(0)} | X, A = 0, G = O]p(A = 0 | X, G = O).
\]

Therefore,
\[
\mathbb{E}[M^{(0)} | X, A = 1, G = O] = \frac{\mathbb{E}[M^{(0)} | X, G = O] - \mathbb{E}[M | X, A = 0, G = O]p(A = 0 | X, G = O)}{p(A = 1 | X, G = O)} E[M^{(0)} | X, A = 0, G = E] - \mathbb{E}[M | X, A = 0, G = O]p(A = 0 | X, G = O) \]
\[
= \frac{\mathbb{E}[M | X, A = 0, G = E] - \mathbb{E}[M | X, A = 0, G = O]p(A = 0 | X, G = O)}{p(A = 1 | X, G = O)}.
\]

Proof of Lemma \[\square\] We only show that \(\mathbb{E}[M^{(0)} | A = 1, G = O]\) is identified.
\[
\mathbb{E}[M^{(0)} | A = 1, G = O] = \mathbb{E}[M^{(0)} | A = 1, G = O]p(A = 1 | G = O) + \mathbb{E}[M^{(0)} | A = 0, G = O]p(A = 0 | G = O).
\]

Therefore,
\[
\mathbb{E}[M^{(0)} | A = 1, G = O] = \frac{\mathbb{E}[M^{(0)} | G = O] - \mathbb{E}[M | A = 0, G = O]p(A = 0 | G = O)}{p(A = 1 | G = O)}.
\]

Moreover,
\[
\mathbb{E}[M^{(0)} | G = O] = \mathbb{E}[\mathbb{E}[M^{(0)} | X, G = O] | G = O] \quad \square
\]
\[
\mathbb{E}[M^{(0)} | X, A = 0, G = E] | G = O] = \mathbb{E}[\mathbb{E}[M^{(0)} | X, A = 0, G = E] | G = O].
\]

(10) and (11) imply that
\[
\mathbb{E}[M^{(0)} | A = 1, G = O] = \frac{\mathbb{E}[\mathbb{E}[M | X, A = 0, G = E] | G = O] - \mathbb{E}[M | A = 0, G = O]p(A = 0 | G = O)}{p(A = 1 | G = O)}.
\]

\(\square\)

In order to prove Theorems \[\square\] and \[\square\], we first state the following corollary of Lemma \[\square\]

Corollary 3. Under Assumptions \[\square\], \[\square\] and \[\square\] for \(a \in \{0, 1\}\), the parameter \(\mathbb{E}[Y^{(a)} | A = 1 - a, G = O]\) is identified.

Proof of Corollary \[\square\] We only show that \(\mathbb{E}[Y^{(0)} | A = 1, G = O]\) is identified. By Assumption \[\square(ii)\],
\[
\mathbb{E}[Y^{(0)} | A = 1, G = O] = \mathbb{E}[Y | A = 0, G = O] - \mathbb{E}[M | A = 0, G = O] + \mathbb{E}[M^{(0)} | A = 1, G = O].
\]
Therefore, by Lemma 2
\[
E[Y^{(0)} \mid A = 1, G = O] \\
= E[Y \mid A = 0, G = O] - E[M \mid A = 0, G = O] \\
+ \frac{E[E[M \mid X, A = 0, G = E] \mid G = O] - E[M \mid A = 0, G = O]p(A = 0 \mid G = O)}{p(A = 1 \mid G = O)}.
\]

Proof of Theorem 2. Using Corollary 3 we have
\[
\theta_{ETT} = E[Y^{(1)} \mid A = 1, G = O] - E[Y^{(0)} \mid A = 1, G = O] \\
= E[Y \mid A = 1, G = O] - E[Y \mid A = 0, G = O] + E[M \mid A = 0, G = O] \\
- \frac{E[E[M \mid X, A = 0, G = E] \mid G = O] - E[M \mid A = 0, G = O]p(A = 0 \mid G = O)}{p(A = 1 \mid G = O)}.
\]

Proof of Theorem 3. Using Corollary 3 we have
\[
E[Y^{(1)} \mid G = O] = E[Y^{(1)} \mid A = 1, G = O]p(A = 1 \mid G = O) \\
+ E[Y^{(1)} \mid A = 0, G = O]p(A = 0 \mid G = O) \\
= E[Y \mid A = 1, G = O]p(A = 1 \mid G = O) \\
+ E[Y \mid A = 1, G = O]p(A = 0 \mid G = O) \\
- E[M \mid A = 1, G = O]p(A = 1 \mid G = O) \\
+ E[M \mid X, A = 1, G = E] \mid G = O] \\
- E[M \mid A = 1, G = O]p(A = 1 \mid G = O) .
\]

Similarly,
\[
E[Y^{(0)} \mid G = O] = E[Y^{(0)} \mid A = 1, G = O]p(A = 1 \mid G = O) \\
+ E[Y^{(0)} \mid A = 0, G = O]p(A = 0 \mid G = O) \\
= E[Y \mid A = 0, G = O]p(A = 1 \mid G = O) \\
- E[M \mid A = 0, G = O]p(A = 1 \mid G = O) \\
+ E[M \mid X, A = 0, G = E] \mid G = O] \\
- E[M \mid A = 0, G = O]p(A = 0 \mid G = O) \\
+ E[Y \mid A = 0, G = O]p(A = 0 \mid G = O) .
\]

(12) and (13) conclude that
\[
\theta_{ATE} = E[Y^{(1)} \mid G = O] - E[Y^{(0)} \mid G = O] \\
= E[Y \mid A = 1, G = O] - E[Y \mid A = 0, G = O] \\
+ E[E[M \mid X, A = 1, G = E] \mid G = O] - E[E[M \mid X, A = 0, G = E] \mid G = O] \\
- E[M \mid A = 1, G = O] + E[M \mid A = 0, G = O].
\]

\[\square\]
Proof of Theorem 4. That $\theta_{ATE}$ is identified in Theorem 3 can also be proven directly as follows. We note that

$$E[Y^{(1)} - Y^{(0)} | G = O]$$

$$= E[Y^{(1)} | A = 1, G = O]p(A = 1 | G = O) + E[Y^{(1)} | A = 0, G = O]p(A = 0 | G = O) - E[Y^{(0)} | A = 1, G = O]p(A = 1 | G = O) - E[Y^{(0)} | A = 0, G = O]p(A = 0 | G = O)$$

Similarly,

$$E[M^{(1)} - M^{(0)} | G = O] = E[M | A = 1, G = O] - E[M | A = 0, G = O]$$

Therefore, by Assumption 3, we have

$$\theta_{ATE} = E[Y^{(1)} - Y^{(0)} | G = O]$$

$$= E[Y | A = 1, G = O] - E[Y | A = 0, G = O] + E[M^{(1)} - M^{(0)} | G = O] - E[M | A = 1, G = O] + E[M | A = 0, G = O].$$

By Assumptions 2 and 5, we have

$$E[M^{(1)} - M^{(0)} | G = O]$$

$$= E[E[M^{(1)} - M^{(0)} | X, G = O] | G = O]$$

$$= E[E[M^{(1)} - M^{(0)} | X, G = E] | G = O]$$

Therefore, the causal parameter of interest is identified.
Proof of Theorem 5. By Lemma 1 and Assumption 7 (ii),

\[-E[Y^{(0)} \mid X, A = 1, G = O] = E[M \mid X, A = 0, G = O] - E[Y \mid X, A = 0, G = O] - E[M^{(0)} \mid X, A = 1, G = O]
\]

\[= E[M \mid X, A = 0, G = O] - E[Y \mid X, A = 0, G = O] - \frac{E[M \mid X, A = 0, G = E] - E[M \mid X, A = 0, G = O]p(A = 0 \mid X, G = O)}{p(A = 1 \mid X, G = O)}\]

\[= \frac{1}{p(A = 1 \mid X, G = O)}E[M \mid X, A = 0, G = E] - \frac{1}{p(A = 1 \mid X, G = O)}E[M \mid X, A = 0, G = O].\]

Therefore,

\[\theta_{\text{ETT}} = E[Y^{(1)} \mid A = 1, G = O] - E[Y^{(0)} \mid A = 1, G = O]
\]

\[= E[Y \mid A = 1, G = O] + E[-E[Y^{(0)} \mid X, A = 1, G = O] \mid A = 1, G = O] - E[M \mid X, A = 0, G = O] \mid A = 1, G = O]
\]

\[= E[Y \mid A = 1, G = O] + E\left[\frac{1}{p(A = 1 \mid X, G = O)}E[M \mid X, A = 0, G = O] \mid A = 1, G = O\right]
\]

\[= E[Y \mid A = 1, G = O] + E[M \mid X, A = 0, G = E] + E[Y \mid X, A = 0, G = O] \mid A = 1, G = O].\]

Proof of Theorem 6. As seen in the proof of Theorem 5

\[E[Y^{(0)} \mid X, A = 1, G = O] = \frac{1}{p(A = 1 \mid X, G = O)}E[M \mid X, A = 0, G = O]
\]

\[= \frac{1}{p(A = 1 \mid X, G = O)}E[M \mid X, A = 0, G = E] + E[Y \mid X, A = 0, G = O].\]

Similarly,

\[E[Y^{(1)} \mid X, A = 0, G = O] = \frac{1}{p(A = 0 \mid X, G = O)}E[M \mid X, A = 1, G = O]
\]

\[= \frac{1}{p(A = 0 \mid X, G = O)}E[M \mid X, A = 1, G = E] + E[Y \mid X, A = 1, G = O].\]

Therefore,

\[E[Y^{(1)} \mid G = O] = E[E[Y^{(1)} \mid X, G = O] \mid G = O]
\]

\[= E[E[Y \mid X, A = 1, G = O]p(A = 1 \mid X, G = O) \mid G = O]
\]

\[+ E[E[Y^{(1)} \mid X, A = 0, G = O]p(A = 0 \mid X, G = O) \mid G = O]
\]

\[= E[Y \mid X, A = 1, G = O]p(A = 1 \mid X, G = O) \mid G = O]
\]

\[+ E[M \mid X, A = 1, G = E] - E[M \mid X, A = 1, G = O] + E[Y \mid X, A = 1, G = O]p(A = 0 \mid X, G = O) \mid G = O].\]
Proof of Theorem 7.

Similarly,
\[
\mathbb{E}[Y^{(0)} \mid G = O] = \mathbb{E}[\mathbb{E}[Y^{(0)} \mid X, G = O] \mid G = O]
\]
\[
= \mathbb{E}[\mathbb{E}[Y^{(0)} \mid X, A = 1, G = O]p(A = 1 \mid X, G = O) \mid G = O]
+ \mathbb{E}[\mathbb{E}[Y^{(0)} \mid X, A = 0, G = O]p(A = 0 \mid X, G = O) \mid G = O]
= \mathbb{E}[\mathbb{E}[M \mid X, A = 0, G = E] - \mathbb{E}[M \mid X, A = 0, G = O]
+ \mathbb{E}[Y^{(0)} \mid X, A = 0, G = O]p(A = 1 \mid X, G = O) \mid G = O]
+ \mathbb{E}[\mathbb{E}[Y^{(0)} \mid X, A = 0, G = O]p(A = 0 \mid X, G = O) \mid G = O].
\]

(14) \text{ and } (15) \text{ conclude that}

\[
\theta_{ATE} = \mathbb{E}[Y^{(1)} \mid G = O] - \mathbb{E}[Y^{(0)} \mid G = O]
= \mathbb{E}[\mathbb{E}[Y^{(1)} \mid X, A = 1, G = O] - \mathbb{E}[Y^{(0)} \mid X, A = 0, G = O]
+ \mathbb{E}[M \mid X, A = 1, G = E] - \mathbb{E}[M \mid X, A = 0, G = E]
+ \mathbb{E}[M \mid X, A = 0, G = O] - \mathbb{E}[M \mid X, A = 1, G = O] \mid G = O].
\]

Proof of Theorem 7.

\[
\mathbb{E}[Y^{(1)} - Y^{(0)} \mid G = O] = \mathbb{E}[\mathbb{E}[Y^{(1)} - Y^{(0)} \mid X, G = O] \mid G = O]
= \mathbb{E}[\mathbb{E}[Y^{(1)} \mid X, A = 1, G = O]p(A = 1 \mid X, G = O)
+ \mathbb{E}[Y^{(1)} \mid X, A = 0, G = O]p(A = 0 \mid X, G = O)
- \mathbb{E}[Y^{(0)} \mid X, A = 1, G = O]p(A = 1 \mid X, G = O)
- \mathbb{E}[Y^{(0)} \mid X, A = 0, G = O]p(A = 0 \mid X, G = O)
+ \mathbb{E}[Y^{(1)} \mid X, A = 1, G = O]p(A = 0 \mid X, G = O)
+ \mathbb{E}[Y^{(0)} \mid X, A = 0, G = O]p(A = 1 \mid X, G = O)
- \mathbb{E}[Y^{(0)} \mid X, A = 0, G = O]p(A = 1 \mid X, G = O) \mid G = O]
= \mathbb{E}[\mathbb{E}[Y^{(1)} \mid X, A = 1, G = O] - \mathbb{E}[Y^{(0)} \mid X, A = 0, G = O]
+ \{\mathbb{E}[Y^{(1)} \mid X, A = 0, G = O]p(A = 1 \mid X, G = O)
- \mathbb{E}[Y^{(1)} \mid X, A = 1, G = O] \mid G = O]
- \mathbb{E}[Y^{(0)} \mid X, A = 1, G = O]p(A = 1 \mid X, G = O) \mid G = O].
\]

Similarly,

\[
\mathbb{E}[M^{(1)} - M^{(0)} \mid G = O] = \mathbb{E}[\mathbb{E}[M \mid X, A = 1, G = O] - \mathbb{E}[M \mid X, A = 0, G = O]
+ \{\mathbb{E}[M^{(1)} \mid X, A = 0, G = O]
- \mathbb{E}[M^{(1)} \mid X, A = 1, G = O] \mid G = O]
+ \{\mathbb{E}[M^{(0)} \mid X, A = 0, G = O]
- \mathbb{E}[M^{(0)} \mid X, A = 1, G = O] \mid G = O].
\]

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Therefore, by Assumption 4
\[ \theta_{ATE} = E[Y^{(1)} - Y^{(0)} \mid G = O] \]
\[ = E[E[Y \mid X, A = 1, G = O] - E[Y \mid X, A = 0, G = O] \mid G = O] \]
\[ + E[M^{(1)} - M^{(0)} \mid G = O] + E[-E[M \mid X, A = 1, G = O] + E[M \mid X, A = 0, G = O] \mid G = O]. \]

By Assumptions 2 and 4 we have
\[ E[M^{(1)} - M^{(0)} \mid G = O] = E[E[M^{(1)} - M^{(0)} \mid X, G = O] \mid G = O] \]
\[ = E[E[M^{(1)} \mid X, A = 1, G = E] - E[M^{(0)} \mid X, A = 0, G = E] \mid G = O] \]
\[ = E[E[M \mid X, A = 1, G = E] \mid G = O] - E[E[M \mid X, A = 0, G = E] \mid G = O]. \]

Therefore, the causal parameter of interest is identified.

\[ \square \]

**Proof of Theorem 8** By Assumption 8
\[ F_{Y^{(0)} \mid A=0,X,G=O} F_{Y^{(0)} \mid A=1,X,G=O}^{-1}(v) = F_{M^{(0)} \mid A=0,X,G=O} \circ F_{M^{(0)} \mid A=1,X,G=O}^{-1}(v), \]
which implies that
\[ F_{Y^{(0)} \mid A=1,X,G=O}^{-1}(v) = F_{Y^{(0)} \mid A=0,X,G=O}^{-1}(v) \circ F_{M \mid A=0,X,G=O} \circ F_{M \mid A=1,X,G=O}^{-1}(v). \]

Note that
\[ F_{Y^{(0)} \mid A=1,X,G=O}^{-1}(v) = y \]
\[ \Rightarrow F_{Y^{(0)} \mid A=1,X,G=O}(y) = v, \]
and
\[ F_{Y^{(0)} \mid A=0,X,G=O} \circ F_{M \mid A=0,X,G=O} \circ F_{M \mid A=1,X,G=O}^{-1}(v) = y \]
\[ \Rightarrow F_{M \mid A=1,X,G=O} \circ F_{M \mid A=0,X,G=O} \circ F_{Y \mid A=0,X,G=O}(y) = v. \]

Therefore,
\[ F_{Y^{(0)} \mid A=1,X,G=O}(y) = F_{M \mid A=1,X,G=O} \circ F_{M \mid A=0,X,G=O} \circ F_{Y \mid A=0,X,G=O}(y). \]

Finally, we note that
\[ F_{M \mid X,G=O}(m) = F_{M \mid X,A=1,G=O}(m)p(A = 1 \mid X, G = O) \]
\[ + F_{M \mid X,A=0,G=O}(m)p(A = 0 \mid X, G = O), \]
and hence,
\[ F_{M \mid X,A=1,G=O}(m) = \frac{F_{M \mid X,A=0,G=O}(m) - F_{M \mid X,A=0,G=O}(m)p(A = 0 \mid X, G = O)}{p(A = 1 \mid X, G = O)} \]
\[ = \frac{F_{M \mid X,A=0,G=O}(m) - F_{M \mid X,A=0,G=O}(m)p(A = 0 \mid X, G = O)}{p(A = 1 \mid X, G = O)} \]
\[ = \frac{F_{M \mid X,A=0,G=O}(m) - F_{M \mid X,A=0,G=O}(m)p(A = 0 \mid X, G = O)}{p(A = 1 \mid X, G = O)}. \]
This concludes that
\[
F_{Y^{(0)}|A=1,X,G=O}(y) = \frac{F_{M|X,A=0,G=E \circ F^{-1}_{M|A=0,X,G=O} \circ F_{Y|A=0,X,G=O}(y)}}{p(A = 1 | X, G = O)}
- \frac{p(A = 0 | X, G = O)}{p(A = 1 | X, G = O)} F_{Y|A=0,X,G=O}(y).
\]

**Proof of Theorem 9** Defining

\[
\begin{align*}
  b_0(X) &:= \mathbb{E}[Y^{(0)} \mid Z = 0, X, G = O], \\
  b_1(X) &:= \mathbb{E}[Y^{(1)} \mid Z = 0, X, G = O], \\
  \omega_0(X) &:= \mathbb{E}[M \mid A = 0, Z = 0, X, G = E], \\
  \omega_1(X) &:= \mathbb{E}[M \mid A = 1, Z = 0, X, G = E], \\
  \beta_0(Z, X) &:= \mathbb{E}\{Y^{(1)} - M^{(1)}\} - \{Y^{(0)} - M^{(0)}\} \mid A = 0, Z, X, G = O, \\
  \beta_1(Z, X) &:= \mathbb{E}\{Y^{(1)} - M^{(1)}\} - \{Y^{(0)} - M^{(0)}\} \mid A = 1, Z, X, G = O, \\
  \gamma_0(Z, X) &:= \mathbb{E}[Y^{(0)} - M^{(0)} \mid A = 1, Z, X, G = O] - \mathbb{E}[Y^{(0)} - M^{(0)} \mid A = 0, Z, X, G = O], \\
  \gamma_1(Z, X) &:= \mathbb{E}[Y^{(1)} - M^{(1)} \mid A = 1, Z, X, G = O] - \mathbb{E}[Y^{(1)} - M^{(1)} \mid A = 0, Z, X, G = O], \\
  \pi(Z, X) &:= p(A = 1 \mid Z, X, G = O).
\end{align*}
\]

We note that by Assumptions 2, 3, and 10(ii), we have

\[
\mathbb{E}[Y^{(0)} - M^{(0)} \mid Z, X, G = O] = b_0(X) - \mathbb{E}[M^{(0)} \mid Z = 0, X, G = O] = \mathbb{E}[Y^{(0)} - M^{(0)} \mid Z = 0, X, G = E] = \mathbb{E}[Y^{(0)} - M^{(0)} \mid A = 0, Z = 0, X, G = E] = b_0(X) - \omega_0(X).
\]

Similarly, by Assumptions 2, 3, and 10(i), we have

\[
\mathbb{E}[Y^{(1)} - M^{(1)} \mid Z, X, G = O] = b_1(X) - \omega_1(X).
\]

Note that \(\omega_0(X)\) and \(\omega_1(X)\) are identified.

Using Robins' parametrization of the outcome conditional mean function [Robins et al., 2000], we have

\[
\mathbb{E}[Y - M \mid A = a, Z, X, G = O] = \mathbb{E}[Y^{(a)} - M^{(a)} \mid A = a, Z, X, G = O] - \mathbb{E}[Y^{(0)} - M^{(0)} \mid A = a, Z, X, G = O] + \mathbb{E}[Y^{(0)} - M^{(0)} \mid A = a, Z, X, G = O] - \mathbb{E}[Y^{(0)} - M^{(0)} \mid A = 0, Z, X, G = O] - \mathbb{E}[Y^{(0)} - M^{(0)} \mid A = 0, Z, X, G = O]\]
\[
- \{\mathbb{E}[Y^{(0)} - M^{(0)} \mid A = 1, Z, X, G = O] - \mathbb{E}[Y^{(0)} - M^{(0)} \mid A = 0, Z, X, G = O]\} p(A = 1 \mid Z, X, G = O) + \mathbb{E}[Y^{(0)} - M^{(0)} \mid Z, X, G = O] \]
\[
= \mathbb{E}\{Y^{(1)} - M^{(1)}\} - \{Y^{(0)} - M^{(0)}\} \mid A = 1, Z, X, G = O\}
\]
\[
+ \{\mathbb{E}[Y^{(0)} - M^{(0)} \mid A = 1, Z, X, G = O] - \mathbb{E}[Y^{(0)} - M^{(0)} \mid A = 0, Z, X, G = O]\}
\]
\[
\times \{a - p(A = 1 \mid Z, X, G = O)\} = \beta_1(Z, X)a + \gamma_0(Z, X)\{a - \pi(Z, X)\} + b_0(X) - \omega_0(X).
\]

(16)
Note that for every fixed $X$, the left hand side identifies 4 parameters. Under Assumption 11(ii), $\beta_1$ is not a function of $Z$. Therefore, we also have 4 unknown parameters on the right hand side: 1 corresponding to $\beta_1$, 2 corresponding to $\gamma_0$, and 1 corresponding to $b_0$. Similarly, under Assumption 12(ii), $\gamma_0$ is not a function of $Z$. Therefore, we also have 4 unknown parameters on the right hand side: 2 corresponding to $\beta_1$, 1 corresponding to $\gamma_1$, and 1 corresponding to $b_0$. Therefore, under either of these two assumptions, the parameter $\beta_1(Z, X)$ is identified.

Formally, define

$$E_{az}(X) := \mathbb{E}[Y - M \mid A = a, Z = z, X, G = O],$$
$$P_{az}(X) := p(A = a \mid Z = z, X, G = O).$$

From (16), we have

$$E_{00}(X) = -P_{01}(X)\gamma_0(0, X) + b_0(X) - \omega_0(X),$$
$$E_{01}(X) = -P_{11}(X)\gamma_0(1, X) + b_0(X) - \omega_0(X),$$
$$E_{10}(X) = \beta_1(0, X) + P_{00}(X)\gamma_0(0, X) + b_0(X) - \omega_0(X),$$
$$E_{11}(X) = \beta_1(1, X) + P_{01}(X)\gamma_0(1, X) + b_0(X) - \omega_0(X).$$

Under Assumption 11(ii), $\beta_1(X) := \beta_1(0, X) = \beta_1(1, X)$, which can be obtained as follows. Noting that

$$\mathbb{E}[Y - M \mid Z = 1, X, G = O] = E_{11}(X)P_{11}(X) + E_{01}(X)P_{01}(X) = P_{11}(X)\beta_1(X) + b_0(X) - \omega_0(X),$$

and

$$\mathbb{E}[Y - M \mid Z = 0, X, G = O] = E_{10}(X)P_{10}(X) + E_{00}(X)P_{00}(X) = P_{10}(X)\beta_1(X) + b_0(X) - \omega_0(X),$$

we have

$$\beta_1(X) = \frac{\mathbb{E}[Y - M \mid Z = 1, X, G = O] - \mathbb{E}[Y - M \mid Z = 0, X, G = O]}{P_{11}(X) - P_{10}(X)}.$$

Note that

$$\beta_1(Z, X) = \mathbb{E}[Y^{(1)} - Y^{(0)} \mid A = 1, Z, X, G = O]$$
$$- \mathbb{E}[M \mid A = 1, Z, X, G = O]$$
$$+ \mathbb{E}[M^{(0)} \mid A = 1, Z, X, G = O].$$

By Lemma 1, the last term is identified. Therefore, the first term on the right hand side is identified, and hence, the parameter $\theta_{ETT}$ is also identified as follows.

$$\theta_{ETT} = \mathbb{E} \left[ \frac{\mathbb{E}[Y - M \mid Z = 1, X, G = O] - \mathbb{E}[Y - M \mid Z = 0, X, G = O]}{P_{11}(X) - P_{10}(X)} \right.$$
$$+ \mathbb{E}[M \mid A = 1, Z, X, G = O] - \frac{\mathbb{E}[M \mid A = 0, Z, X, G = E]}{p(A = 1 \mid Z, X, G = O)}$$
$$+ \frac{\mathbb{E}[M \mid A = 0, Z, X, G = O)p(A = 0 \mid Z, X, G = O)}{p(A = 1 \mid Z, X, G = O)} \mid A = 1, G = O].$$

Under Assumption 12(ii), $\gamma_0(X) := \gamma_0(0, X) = \gamma_0(1, X)$, which can be obtained as

$$\gamma_0(X) = \frac{E_{01}(X) - E_{00}(X)}{P_{01}(X) - P_{00}(X)}.$$
We also note that
\[ b_0(X) - \omega_0(X) = E_{00}(X) + P_{10}(X)\gamma_0(X) \]
\[ = E_{01}(X) + P_{11}(X)\gamma_0(X), \]
by which we have
\[ \beta_1(0, X) = E_{10}(X) - P_{00}(X)\gamma_0(X) - E_{00}(X) - P_{10}(X)\gamma_0(X) \]
\[ = E_{10}(X) - E_{00}(X) - \gamma_0(X), \]
and
\[ \beta_1(1, X) = E_{11}(X) - P_{01}(X)\gamma_0(X) - E_{01}(X) - P_{11}(X)\gamma_0(X) \]
\[ = E_{11}(X) - E_{01}(X) - \gamma_0(X), \]
which implies that
\[ \beta_1(Z, X) = \{ E_{11}(X) - E_{01}(X) - E_{10}(X) + E_{00}(X) \} Z + E_{10}(X) - E_{00}(X) - \gamma_0(X). \]

Note that
\[ \beta_1(Z, X) = \mathbb{E}[Y^{(1)} - Y^{(0)} \mid A = 1, Z, X, G = O] \]
\[ - \mathbb{E}[M \mid A = 1, Z, X, G = O] \]
\[ + \mathbb{E}[M^{(0)} \mid A = 1, Z, X, G = O]. \]

By Lemma 1, the last term is identified. Therefore, the first term on the right hand side is identified, and hence, the parameter \( \theta_{ETT} \) is also identified as follows.

\[ \theta_{ETT} = \mathbb{E} \left[ \{ E_{11}(X) - E_{01}(X) - E_{10}(X) + E_{00}(X) \} Z + E_{10}(X) - E_{00}(X) \right] \]
\[ - \frac{E_{01}(X) - E_{00}(X)}{P_{01}(X) - P_{00}(X)} + \mathbb{E}[M \mid A = 1, Z, X, G = O] - \frac{E[M \mid A = 0, Z, X, G = E]}{p(A = 1 \mid Z, X, G = O)} \]
\[ + \frac{E[M \mid A = 0, Z, X, G = O]p(A = 0 \mid Z, X, G = O)}{p(A = 1 \mid Z, X, G = O)} \left| A = 1, G = O \right|. \]

In order to show the identifiability of \( \theta_{ATE} \), we note that using a similar parametrization, we have

\[ \mathbb{E}[Y - M \mid A = a, Z, X, G = O] \]
\[ = \beta_0(Z, X)(a - 1) + \gamma(X, X)(a - \pi(Z, X)) + b_1(X) - \omega_1(X). \]

A similar argument regarding counting the parameters as before shows that under either Assumption 11(i), or 12(i), the parameter \( \beta_0(Z, X) \) and consequently, the parameter \( \mathbb{E}[Y^{(1)} - Y^{(0)} \mid A = 0, Z, X, G = O] \) are identified.

Formally, we have
\[ E_{00}(X) = -\beta_0(0, X) - P_{10}(X)\gamma_1(0, X) + b_1(X) - \omega_1(X), \]
\[ E_{01}(X) = -\beta_0(1, X) - P_{11}(X)\gamma_1(1, X) + b_1(X) - \omega_1(X), \]
\[ E_{10}(X) = P_{00}(X)\gamma_1(0, X) + b_1(X) - \omega_1(X), \]
\[ E_{11}(X) = P_{01}(X)\gamma_1(1, X) + b_1(X) - \omega_1(X). \]

Under Assumption 11(i), \( \beta_0(X) := \beta_0(0, X) = \beta_0(1, X) \), which can be obtained similar to the previous case as

\[ \beta_0(X) = \frac{\mathbb{E}[Y - M \mid Z = 1, X, G = O] - \mathbb{E}[Y - M \mid Z = 0, X, G = O]}{P_{11}(X) - P_{10}(X)}. \]
Note that

$$
\beta_0(Z, X) = \mathbb{E}[Y^{(1)} - Y^{(0)} \mid A = 0, Z, X, G = O]
+ \mathbb{E}[M \mid A = 0, Z, X, G = O]
- \mathbb{E}[M^{(1)} \mid A = 0, Z, X, G = O].
$$

By Lemma 1, the last term is identified. Therefore, the first term on the right hand side is identified. The parameter \(\mathbb{E}[Y^{(1)} - Y^{(0)} \mid A = 0, Z, X, G = O]\) is identified as follows.

$$
\mathbb{E}[Y^{(1)} - Y^{(0)} \mid A = 0, Z, X, G = O] = \frac{\mathbb{E}[Y - M \mid Z = 1, X, G = O] - \mathbb{E}[Y - M \mid Z = 0, X, G = O]}{P_{11}(X) - P_{10}(X)}
- \mathbb{E}[M \mid A = 0, Z, X, G = O] + \frac{\mathbb{E}[M \mid A = 1, Z, X, G = E]}{p(A = 0 \mid Z, X, G = O)}
- \mathbb{E}[M \mid A = 1, Z, X, G = O]p(A = 1 \mid Z, X, G = O).
$$

Finally, note that

$$
\theta_{ATE} = \mathbb{E}[\mathbb{E}[Y^{(1)} - Y^{(0)} \mid Z, X, G = O] \mid G = O]
= \mathbb{E}[\mathbb{E}[Y^{(1)} - Y^{(0)} \mid A = 1, Z, X, G = O] \pi(Z, X)]
+ \mathbb{E}[Y^{(1)} - Y^{(0)} \mid A = 0, Z, X, G = O](1 - \pi(Z, X))]
= \mathbb{E}\left[\frac{\mathbb{E}[Y - M \mid Z = 1, X, G = O] - \mathbb{E}[Y - M \mid Z = 0, X, G = O]}{P_{11}(X) - P_{10}(X)}
+ \mathbb{E}[M \mid A = 1, Z, X, G = E] - \mathbb{E}[M \mid A = 0, Z, X, G = E] \bigg| G = O\right].
$$

Under Assumption 12(i), \(\gamma_1(X) := \gamma_1(0, X) = \gamma_1(1, X)\), which can be obtained as

$$
\gamma_1(X) = \frac{E_{11}(X) - E_{10}(X)}{P_{01}(X) - P_{00}(X)}.
$$

This by an approach similar to the previous case leads to

$$
\beta_0(Z, X) = \{E_{11}(X) - E_{01}(X) - E_{10}(X) + E_{00}(X)\}Z + E_{10}(X) - E_{00}(X) - \gamma_1(X).
$$

Note that

$$
\beta_0(Z, X) = \mathbb{E}[Y^{(1)} - Y^{(0)} \mid A = 0, Z, X, G = O]
+ \mathbb{E}[M \mid A = 0, Z, X, G = O]
- \mathbb{E}[M^{(1)} \mid A = 0, Z, X, G = O].
$$

By Lemma 1, the last term is identified. Therefore, the first term on the right hand side is identified. The parameter \(\mathbb{E}[Y^{(1)} - Y^{(0)} \mid A = 0, Z, X, G = O]\) is identified as follows.

$$
\mathbb{E}[Y^{(1)} - Y^{(0)} \mid A = 0, Z, X, G = O] = \frac{E_{11}(X) - E_{10}(X)}{P_{01}(X) - P_{00}(X)}
- \mathbb{E}[M \mid A = 0, Z, X, G = O] + \frac{\mathbb{E}[M \mid A = 1, Z, X, G = E]}{p(A = 0 \mid Z, X, G = O)}
- \mathbb{E}[M \mid A = 1, Z, X, G = O]p(A = 1 \mid Z, X, G = O).\]
Finally, note that
\[
\theta_{ATE} = \mathbb{E}[\mathbb{E}[Y(1) - Y(0) | Z, X, G = O | G = O]
 = \mathbb{E}[\mathbb{E}[Y(1) - Y(0) | A = 1, Z, X, G = O] \pi(Z, X)
+ \mathbb{E}[Y(1) - Y(0) | A = 0, Z, X, G = O|(1 - \pi(Z, X))]
 = \mathbb{E}\left[\{E_{11}(X) - E_{01}(X) - E_{10}(X) + E_{00}(X)\}Z + E_{10}(X) - E_{00}(X)
- \{E_{01}(X) - E_{00}(X)\} \pi(Z, X) + \{E_{11}(X) - E_{10}(X)\}(1 - \pi(Z, X))\right]
/ \mathbb{P}_{01}(X) - \mathbb{P}_{00}(X)
+ \mathbb{E}[M | A = 1, Z, X, G = E] - \mathbb{E}[M | A = 0, Z, X, G = E | G = O].
\]

\[\square\]

**Proof of Theorem 11** Define
\[
\begin{align*}
b_0(X) &:= \mathbb{E}[Y(0) | Z = 0, X, G = O], \\
b_1(X) &:= \mathbb{E}[Y(1) | Z = 0, X, G = O], \\
\omega_0^1(X) &:= \mathbb{E}[M | A = 0, Z = 1, X, G = E] - \mathbb{E}[M | A = 0, Z = 0, X, G = E], \\
\omega_1^1(X) &:= \mathbb{E}[M | A = 1, Z = 1, X, G = E] - \mathbb{E}[M | A = 1, Z = 0, X, G = E], \\
\beta_0^1(Z, X) &:= \mathbb{E}[Y(1) - Y(0) | A = 0, Z, X, G = O], \\
\beta_1^1(Z, X) &:= \mathbb{E}[Y(1) - Y(0) | A = 1, Z, X, G = O], \\
\gamma_0^1(Z, X) &:= \mathbb{E}[Y(0) | A = 1, Z, X, G = O] - \mathbb{E}[Y(0) | A = 0, Z, X, G = O], \\
\gamma_1^1(Z, X) &:= \mathbb{E}[Y(1) | A = 1, Z, X, G = O] - \mathbb{E}[Y(1) | A = 0, Z, X, G = O], \\
\pi(Z, X) &:= \pi(A = 1 | Z, X, G = O).
\end{align*}
\]

We note that by Assumptions 2, 3, and 10(ii), we have
\[
\begin{align*}
\mathbb{E}[Y(0) | Z, X, G = O] &= \mathbb{E}[Y(0) | Z, X, G = O] - \mathbb{E}[Y(0) | Z = 0, X, G = O] + \mathbb{E}[Y(0) | Z = 0, X, G = O] \\
&\overset{\text{Ass. 10(i) }}{=} \mathbb{E}[M(0) | Z, X, G = O] - \mathbb{E}[M(0) | Z = 0, X, G = O] + b_0(X) \\
&\overset{\text{Ass. 3 }}{=} \mathbb{E}[M(0) | Z, X, G = E] - \mathbb{E}[M(0) | Z = 0, X, G = E] + b_0(X) \\
&\overset{\text{Ass. 2 }}{=} \mathbb{E}[M(0) | A = 0, Z, X, G = E] - \mathbb{E}[M(0) | A = 0, Z = 0, X, G = E] + b_0(X) \\
&= \mathbb{E}[M | A = 0, Z, X, G = E] - \mathbb{E}[M | A = 0, Z = 0, X, G = E] + b_0(X) \\
&= \omega_0^1(X)Z + b_0(X).
\end{align*}
\]

Similarly, by Assumptions 2, 3, and 10(i), we have
\[
\mathbb{E}[Y(1) | Z, X, G = O] = \omega_1^1(X)Z + b_1(X).
\]

Note that \(\omega_0^1(X)\) and \(\omega_1^1(X)\) are identified from the experimental domain.

Using the same parametrization as the one in the proof of Theorem 10, we have
\[
\begin{align*}
\mathbb{E}[Y | A = a, Z, X, G = O] &= \beta_1^1(Z, X)a + \gamma_0^1(Z, X)a - \pi(Z, X) + \omega_0^1(X)Z + b_0(X).
\end{align*}
\]
A similar parameter counting argument as the one in the proof of Theorem 9 shows that under either Assumption 13(ii), or 14(ii), the parameter $\beta_1^1(Z, X)$, and hence, the parameter $\theta_{ETT}$ is identified.

Formally, define

$$E^\dagger_{az}(X) := \mathbb{E}[Y - M \mid A = a, Z = z, X, G = O],$$
$$P^\dagger_{az}(X) := p(A = a \mid Z = z, X, G = O).$$

From (17), we have

$$E^1_{00}(X) = -P_{10}(X)\gamma_0^1(0, X) + b_0(X),$$
$$E^1_{10}(X) = -P_{11}(X)\gamma_0^1(1, X) + b_0(X) + \omega_0^1(X),$$
$$E^1_{11}(X) = \beta_1^1(0, X) + P_{00}(X)\gamma_0^1(0, X) + b_0(X),$$
$$E^1_{11}(X) = \beta_1^1(1, X) + P_{01}(X)\gamma_0^1(1, X) + b_0(X) + \omega_0^1(X).$$

Under Assumption 13(ii), $\beta_1^1(X) := \beta_1^1(0, X) = \beta_1^1(1, X)$, which can be obtained as follows. Noting that

$$\mathbb{E}[Y \mid Z = 1, X, G = O] = E^1_{11}(X)P_{11}(X) + E^1_{01}(X)P_{01}(X)$$
$$\quad = P_{11}(X)\beta_1^1(X) + b_0(X) + \omega_0^1(X),$$

and

$$\mathbb{E}[Y \mid Z = 0, X, G = O] = E^1_{10}(X)P_{10}(X) + E^1_{00}(X)P_{00}(X)$$
$$\quad = P_{10}(X)\beta_1^1(X) + b_0(X),$$

we have

$$\beta_1^1(X) = \frac{\mathbb{E}[Y \mid Z = 1, X, G = O] - \mathbb{E}[Y \mid Z = 0, X, G = O] - \omega_0^1(X)}{P_{11}(X) - P_{10}(X)}.$$

Hence, the parameter $\theta_{ETT}$ is identified as follows.

$$\theta_{ETT} = \mathbb{E}\left[\frac{\mathbb{E}[Y \mid Z = 1, X, G = O] - \mathbb{E}[Y \mid Z = 0, X, G = O] - \omega_0^1(X)}{P_{11}(X) - P_{10}(X)} \mid A = 1, G = O\right].$$

Under Assumption 14(ii), $\gamma_0^1(X) := \gamma_0^1(0, X) = \gamma_0^1(1, X)$, which can be obtained as

$$\gamma_0^1(X) = \frac{E^1_{01}(X) - E^1_{00}(X) - \omega_0^1(X)}{P^\dagger_{01}(X) - P^\dagger_{00}(X)}.$$

We also note that

$$b_0(X) = E^\dagger_{00}(X) + P_{10}(X)\gamma_0^1(X),$$

and

$$b_0(X) + \omega_0^1(X) = E^\dagger_{01}(X) + P_{11}(X)\gamma_0^1(X),$$

by which we have

$$\beta_1^1(0, X) = E^\dagger_{10}(X) - P_{00}(X)\gamma_0^1(X) - E^\dagger_{00}(X) - P_{10}(X)\gamma_0^1(X)$$
$$\quad = E^\dagger_{10}(X) - E^\dagger_{00}(X) - \gamma_0^1(X),$$

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Under Assumption 13, as which implies that

\[ \beta_1(Z, X) = \{E_1^+(X) - E_0^+(X) - E_{10}^+(X) + E_{00}^+(X)\}Z + E_{10}^+(X) - E_{00}^+(X) - \gamma_0^+(X). \]

Hence, the parameter \( \theta_{ETT} \) is identified as follows.

\[
\theta_{ETT} = \mathbb{E}\left[\{E_1^+(X) - E_0^+(X) - E_{10}^+(X) + E_{00}^+(X)\}Z + E_{10}^+(X) - E_{00}^+(X) - \frac{\omega_0^+(X)}{P_{01}(X) - P_{00}(X)}|A = 1, G = O\right].
\]

In order to show the identifiability of \( \theta_{ATE} \), we note that using a similar parametrization, we have

\[
\mathbb{E}[Y | A = a, Z, X, G = O] = \beta_0^+(Z, X)(a - 1) + \gamma_1^+(Z, X)\{a - \pi(Z, X)\} + \omega_1^+(X)Z + b_1(X).
\]

A similar argument regarding counting the parameters as before shows that under either Assumption 13(i), or 14(i), the parameter \( \beta_0^+(Z, X) \), i.e., the parameter \( \mathbb{E}[Y^{(1)} - Y^{(0)} | A = 0, Z, X, G = O] \) is identified.

Formally, we have

\[
\begin{align*}
E_{00}^+(X) &= -\beta_0^+(0, X) - P_{10}(X)\gamma_1^+(0, X) + b_1(X), \\
E_{01}^+(X) &= -\beta_0^+(1, X) - P_{11}(X)\gamma_1^+(1, X) + b_1(X) + \omega_1^+(X), \\
E_{10}^+(X) &= P_{00}(X)\gamma_1^+(0, X) + b_1(X), \\
E_{11}^+(X) &= P_{01}(X)\gamma_1^+(1, X) + b_1(X) + \omega_1^+(X)
\end{align*}
\]

Under Assumption 13(i), \( \beta_0^+(X) := \beta_0^+(0, X) = \beta_0^+(1, X) \), which can be obtained similar to the previous case as

\[
\beta_0^+(X) = \frac{\mathbb{E}[Y | Z = 1, X, G = O] - \mathbb{E}[Y | Z = 0, X, G = O] - \omega_1^+(X)}{P_{11}(X) - P_{10}(X)}.
\]

Therefore,

\[
\theta_{ATE} = \mathbb{E}\left[\mathbb{E}[Y^{(1)} - Y^{(0)} | Z, X, G = O] | G = O\right] \\
= \mathbb{E}\left[\mathbb{E}[Y^{(1)} - Y^{(0)} | A = 1, Z, X, G = O | \pi(Z, X)] + \mathbb{E}[Y^{(1)} - Y^{(0)} | A = 0, Z, X, G = O | (1 - \pi(Z, X))]\right] \\
= \mathbb{E}\left[\frac{\mathbb{E}[Y | Z = 1, X, G = O] - \mathbb{E}[Y | Z = 0, X, G = O]}{P_{11}(X) - P_{10}(X)} - \frac{\omega_0^+(X)\pi(Z, X) + \omega_1^+(X)(1 - \pi(Z, X))}{P_{11}(X) - P_{10}(X)}| G = O\right].
\]

Under Assumption 14(i), \( \gamma_1^+(X) := \gamma_1^+(0, X) = \gamma_1^+(1, X) \), which can be obtained as

\[
\gamma_1^+(X) = \frac{E_{11}^+(X) - E_{10}^+(X) - \omega_1^+(X)}{P_{01}(X) - P_{00}(X)}.
\]
This by an approach similar to the previous case leads to

\[ \beta_0(Z, X) = \{E_{11}(X) - E_{01}(X) - E_{10}(X) + E_{00}(X)\}Z + E_{10}(X) - E_{00}(X) - \gamma_1(X). \]

Therefore,

\[ \theta_{ATE} = E[\{Y^{(1)} - Y^{(0)} \mid Z, X, G = O\} \mid G = O] \]

\[ = E(E[Y^{(1)} - Y^{(0)} \mid A = 1, Z, X, G = O] \mid G = O) \pi(Z, X) \]

\[ + E(E[Y^{(1)} - Y^{(0)} \mid A = 0, Z, X, G = O] \mid G = O)(1 - \pi(Z, X)) \]

\[ = E \left[ \frac{E_{11}(X) - E_{01}(X) - E_{10}(X) + E_{00}(X)}{P_{01}(X) - P_{00}(X)} \right] \pi(Z, X) + \frac{E_{11}(X) - E_{01}(X) - \gamma_1(X)}{P_{01}(X) - P_{00}(X)} \mid G = O \right]. \]

\[ \square \]

**Proof of Theorem** For identifying ATE, we show that the parameter \( E[Y^{(a)} \mid G = O] \) is identified. By Assumption (ii), for any choice of \( Z, A = a, X, \) we have

\[ E[Y \mid Z, A = a, X, G = O] = E[h(M, a, X) \mid Z, A = a, X, G = O] \]

\[ \Rightarrow E[E[Y \mid U, Z, A = a, X, G = O] \mid Z, A = a, X, G = O] \]

\[ = E[E[h(M, a, X) \mid U, Z, A = a, X, G = O] \mid Z, A = a, X, G = O] \]

\[ = E[E[Y \mid U, A = a, X, G = O] \mid Z, A = a, X, G = O] \]

\[ = E[E[h(M, a, X) \mid U, A = a, X, G = O] \mid Z, A = a, X, G = O] \]

\[ \Rightarrow E[Y^{(a)} \mid U, A = a, X, G = O] = E[E[h(M^{(a)}, a, X) \mid U, A = a, X, G = O] \mid G = O] \]

\[ \Rightarrow E[Y^{(a)} \mid U, X, G = O] = E[E[h(M^{(a)}, a, X) \mid U, X, G = O] \mid G = O] \]

\[ \Rightarrow E[Y^{(a)} \mid G = O] = E[E[h(M^{(a)}, a, X) \mid G = O]. \]

Therefore, we have

\[ E[Y^{(a)} \mid G = O] = E[E[h(M^{(a)}, a, X) \mid X, G = O] \mid G = O] \]

\[ = E[E[h(M^{(a)}, a, X) \mid X, G = E] \mid G = O] \]

\[ = E[E[h(M^{(a)}, a, X) \mid A = a, X, G = E] \mid G = O] \]

\[ = E[E[h(M, a, X) \mid A = a, X, G = E] \mid G = O], \]

which concludes the desired result.

Realizing that \( E[Y^{(0)} \mid G = O] \) is identified, it is easy to see that \( E[Y^{(0)} \mid A = 1, G = O] \) is also identified as

\[ E[Y^{(0)} \mid G = O] \]

\[ = E[Y^{(0)} \mid A = 1, G = 0] p(A = 1 | G = O) + E[Y^{(0)} \mid A = 0, G = O] p(A = 0 | G = O) \]

\[ \Rightarrow E[Y^{(0)} \mid A = 1, G = O] = \frac{E[Y^{(0)} \mid G = O] - E[Y^{(0)} \mid A = 0, G = O] p(A = 0 | G = O)}{p(A = 1 | G = O)}. \]

Therefore, \( \theta_{ETT} = E[Y \mid A = 1, G = O] - E[Y^{(0)} \mid A = 1, G = O] \) is also identified.

\[ \square \]
Proof of Theorem 12. For identifying ATE, we show that the parameter $\mathbb{E}[Y^{(a)} | G = O]$ is identified. By Assumption 17 (ii), for any choice of $M, A = a, X$, we have

$$
\mathbb{E}[q(Z, A, X) | M = m, A = a, X, G = O] = \frac{p(M = m | A = a, X, G = E)}{p(M = m | A = a, X, G = O)p(A = a | X, G = O)}
$$

where

$$
\frac{p(M^{(a)} = m | A = a, X, G = E)}{p(M^{(a)} = m | A = a, X, G = O)p(A = a | X, G = O)}
$$

and

$$
\frac{p(M^{(a)} = m | X, G = E)}{p(M^{(a)} = m | X, G = O)}
$$

Hence,

$$
\sum_z q(z, a, x)p(z | M = m, a, x, G = O) = \frac{1}{p(A = a | M^{(a)} = m, x, G = O)}
$$

where the last equality is due to the conditional independence $A \perp \perp M^{(a)} | \{X, U, G = O\}$. Note that here we are conditioning on $U$, all possible latent confounders.

Therefore, by Assumption 17 (i), we have

$$
\sum_z q(z, a, x)p(z | u, a, x, G = O) = \frac{1}{p(A = a | u, x, G = O)}.
$$
Therefore, we have
\[
\mathbb{E}[Y^{(a)} | G = O] = \sum_y yp(Y^{(a)}) = y | G = O)
\]
\[
= \sum_{y,u,x} yp(Y^{(a)}) = y | u, x, G = O)p(u, x | G = O)
\]
\[
= \sum_{y,u,x} yp(Y^{(a)}) = y | a, u, x, G = O)p(u, x | G = O)
\]
\[
= \sum_{y,u,x} yp(Y = y | a, u, x, G = O)p(a | u, x, G = O)p(u, x | G = O)
\]
\[
= \sum_{z,y,u,x} yq(z, a, x)p(z | a, u, x, G = O)p(y | a, u, x, G = O)p(a, u, x | G = O)
\]
\[
\sum_{z,y,u,x} yq(z, a, x)p(y, z | a, u, x, G = O)p(a, u, x | G = O)
\]
\[
= \sum_{\tilde{a}, z, y, u, x} I(\tilde{a} = a)yq(z, \tilde{a}, x)p(y, z, \tilde{a}, u, x | G = O)
\]
\[
= \sum_{\tilde{a}, z, y, u, x} I(\tilde{a} = a)yq(z, \tilde{a}, x)p(y, z, \tilde{a}, x | G = O)
\]
\[
= \mathbb{E}[I(A = a)Yq(Z, A, X) | G = O].
\]

which concludes the desired result.

Realizing that \(\mathbb{E}[Y^{(0)} | G = O]\) is identified, it is easy to see that \(\mathbb{E}[Y^{(0)} | A = 1, G = O]\) is also identified as

\[
\mathbb{E}[Y^{(0)} | G = O]
\]
\[
= \mathbb{E}[Y^{(0)} | A = 1, G = O]p(A = 1 | G = O) + \mathbb{E}[Y^{(0)} | A = 0, G = O]p(A = 0 | G = O)
\]
\[
\Rightarrow \mathbb{E}[Y^{(0)} | A = 1, G = O] = \frac{\mathbb{E}[Y^{(0)} | G = O] - \mathbb{E}[Y | A = 0, G = O]p(A = 0 | G = O)}{p(A = 1 | G = O)}.
\]

Therefore, \(\theta_{ETT} = \mathbb{E}[Y | A = 1, G = O] - \mathbb{E}[Y^{(0)} | A = 1, G = O]\) is also identified.

\[
\boxed{\text{Proof of Theorem 13}}
\]

Define

\[
\psi_1 = \mathbb{E}[Y | A = 1, G = O],
\]
\[
\psi_2 = \mathbb{E}\left[\frac{1}{p(A = 1 | X, G = O)} \mathbb{E}[M | X, A = 0, G = O | A = 1, G = O] \right],
\]
\[
\psi_3 = \mathbb{E}\left[\frac{1}{p(A = 1 | X, G = O)} \mathbb{E}[M | X, A = 0, G = E | A = 1, G = O] \right],
\]
\[
\psi_4 = \mathbb{E}\left[\mathbb{E}[Y | X, A = 0, G = O | A = 1, G = O] \right].
\]

We use the notation \(\partial_t f(t)\) to denote \(\frac{\partial f(t)}{\partial t} |_{t=0}\). For parameter \(\psi\), let \(\psi_t\) be the parameter under a regular parametric sub-model indexed by \(t\), that includes the ground-truth model at \(t = 0\). Let \(V\) be the set of all observed variables. In order to obtain an influence function, we need to find a random variable \(\Gamma\) with mean zero, that satisfies

\[
\partial_t \psi_t = \mathbb{E}[\Gamma S(V)],
\]

where \(S(V) = \partial_t \log p_t(V)\).
For $\psi_1$, note that
\[
\partial_t \psi_1 = \sum_y y \partial_t p_t(y \mid A = 1, G = O)
\]
\[
= \sum_y y S(y \mid A = 1, G = O) p(y \mid A = 1, G = O)
\]
\[
= \sum_{y,a,g} \frac{I(a = 1)I(g = O)}{p(A = 1, G = O)} y S(y \mid a,g) p(y,a,g)
\]
\[
= \mathbb{E} \left[ \frac{I(A = 1)I(G = O)}{p(A = 1, G = O)} Y S(Y \mid A,G) \right]
\]
\[
= \mathbb{E} \left[ \frac{I(A = 1)I(G = O)}{p(A = 1, G = O)} \{Y - \mathbb{E}[Y \mid A = 1, G = O]\} S(Y \mid A,G) \right]
\]
\[
= \mathbb{E} \left[ \frac{I(A = 1)I(G = O)}{p(A = 1, G = O)} \{Y - \psi_1\} S(Y \mid A,G) \right].
\]

Note that
\[
\mathbb{E} \left[ \frac{I(A = 1)I(G = O)}{p(A = 1, G = O)} \{Y - \psi_1\} S(A,G) \right] = 0.
\]

Therefore,
\[
\partial_t \psi_1 = \mathbb{E} \left[ \frac{I(A = 1)I(G = O)}{p(A = 1, G = O)} \{Y - \psi_1\} S(V) \right].
\]

This implies that
\[
\frac{I(A = 1)I(G = O)}{p(A = 1, G = O)} \{Y - \psi_1\}
\]
is the influence function of $\psi_1$.

For $\psi_2$, note that
\[
\partial_t \psi_2 = \partial_t \sum_{m,x} m \frac{1}{p_t(A = 1 \mid x, G = O)} p_t(m \mid x, A = 0, G = O) p_t(x \mid A = 1, G = O)
\]
\[
= \partial_t \sum_{m,x} m \frac{1}{p_t(A = 1, G = O)} p_t(m \mid x, A = 0, G = O) p_t(x, G = O)
\]
\[
= \sum_{m,x} m \partial_t \frac{1}{p_t(A = 1, G = O)} p(m \mid x, A = 0, G = O) p(x, G = O)
\]
\[
+ \sum_{m,x} m \frac{1}{p(A = 1, G = O)} \partial_t p_t(m \mid x, A = 0, G = O) p(x, G = O)
\]
\[
+ \sum_{m,x} m \frac{1}{p(A = 1, G = O)} p(m \mid x, A = 0, G = O) \partial_t p_t(x, G = O).
\]
For the first term in (19), we have

$$\sum_{m,x} \frac{1}{p(A = 1, G = O)} \partial p_t(m \mid x, A = 0, G = O)p(x, G = O)$$

$$= - \sum_{m,x} \frac{1}{p(A = 1, G = O)} p(m \mid x, A = 0, G = O)p(x, G = O)S(A = 1, G = O)$$

$$= -\psi_2 S(A = 1, G = O)$$

$$= -E \left[ \frac{I(A = 1)I(G = O)}{p(A = 1, G = O)} \psi_2 S(A, G) \right]$$

$$= -E \left[ \frac{I(A = 1)I(G = O)}{p(A = 1, G = O)} \psi_2 S(V) \right].$$

(20)

For the second term in (19), we have

$$\sum_{m,x} \frac{1}{p(A = 1, G = O)} \partial p_t(m \mid x, A = 0, G = O)p(x, G = O)$$

$$= \sum_{m,x} \frac{1}{p(A = 1, G = O)} S(m \mid x, A = 0, G = O)p(m \mid x, A = 0, G = O)p(x, G = O)$$

$$\sum_{m,x} \frac{1}{p(A = 1, G = O)} \cdot \frac{1}{p(A = 0 \mid x, G = O)} S(m \mid x, A = 0, G = O)p(m, x, A = 0, G = O)$$

$$\sum_{m,a,x,g} \frac{1}{p(A = 1, G = O)} \cdot \frac{I(a = 0)I(g = O)}{p(a = 0 \mid x, G = O)} S(m \mid x, a, g)p(m, x, a, g)$$

$$= E \left[ \frac{1}{p(A = 1, G = O)} \cdot \frac{I(A = 0)I(G = O)}{p(A = 0 \mid X, G = O)} MS(M \mid X, A, G) \right]$$

$$= E \left[ \frac{1}{p(A = 1, G = O)} \cdot \frac{I(A = 0)I(G = O)}{p(A = 0 \mid X, G = O)} (M - E[M \mid X, A, G])S(M \mid X, A, G) \right].$$

Note that

$$E \left[ \frac{1}{p(A = 1, G = O)} \cdot \frac{I(A = 0)I(G = O)}{p(A = 0 \mid X, G = O)} (M - E[M \mid X, A, G])S(X, A, G) \right] = 0.$$
For the third term in (19), we have

$$
\sum_{m,x} \frac{1}{p(A = 1, G = O)} p(m \mid x, A = 0, G = O) \partial_t p_t(x, G = O)
$$

$$
= \sum_x \frac{1}{p(A = 1, G = O)} E[M \mid X = x, A = 0, G = O] S(x, G = O) p(x, G = O)
$$

$$
= \sum_{x,g} \frac{I(g = O)}{p(A = 1, G = O)} E[M \mid X = x, A = 0, G = O] S(x, g) p(x, g)
$$

$$
= E \left[ \frac{I(G = O)}{p(A = 1, G = O)} E[M \mid X = x, A = 0, G = O] \right] .
$$

Combining (20)-(22) concludes that

$$
\partial_t \psi_2 = E \left[ \frac{I(G = O)}{p(A = 1, G = O)} \left\{ \frac{I(A = 0)}{p(A = 0 \mid X, G = O)} \{M - E[M \mid X, A, G]\} + E[M \mid X, A = 0, G = O] - I(A = 1) \psi_2 \right\} S(V) \right].
$$

The variable

$$
\frac{I(G = O)}{p(A = 1, G = O)} \left\{ \frac{I(A = 0)}{p(A = 0 \mid X, G = O)} \{M - E[M \mid X, A, G]\} + E[M \mid X, A = 0, G = O] - I(A = 1) \psi_2 \right\}
$$

is mean zero and hence is the influence function of $\psi_2$.

For $\psi_3$, note that

$$
\partial_t \psi_3 = \partial_t \sum_{m,x} \frac{1}{p_t(A = 1 \mid x, G = O)} p_t(m \mid x, A = 0, G = E) p_t(x \mid A = 1, G = O)
$$

$$
= \partial_t \sum_{m,x} \frac{1}{p_t(A = 1 \mid x, G = O)} p_t(m \mid x, A = 0, G = E) p_t(x, G = O)
$$

$$
= \sum_{m,x} m \partial_t \frac{1}{p_t(A = 1 \mid x, G = O)} p_t(m \mid x, A = 0, G = E) p_t(x, G = O)
$$

$$
+ \sum_{m,x} \frac{1}{p(A = 1, G = O)} \partial_t p_t(m \mid x, A = 0, G = E) p(x, G = O)
$$

$$
+ \sum_{m,x} \frac{1}{p(A = 1, G = O)} p(m \mid x, A = 0, G = E) \partial_t p_t(x, G = O).
$$
For the first term in (23), we have

\[
\sum_{m,x} m \frac{1}{p_t(A = 1, G = O)} p(m \mid x, A = 0, G = E)p(x, G = O)
\]

\[
= - \sum_{m,x} m \frac{1}{p(A = 1, G = O)} p(m \mid x, A = 0, G = E)p(x, G = O)S(A = 1, G = O)
\]

\[
= - \psi_3 S(A = 1, G = O)
\]

\[
= - \mathbb{E} \left[ \frac{I(A = 1)I(G = O)}{p(A = 1, G = O)} \psi_3 S(A, G) \right]
\]

For the second term in (23), we have

\[
\sum_{m,x} m \frac{1}{p(A = 1, G = O)} \partial_p p_t(m \mid x, A = 0, G = E)p(x, G = O)
\]

\[
= \sum_{m,x} m \frac{1}{p(A = 1, G = O)} S(m \mid x, A = 0, G = E)p(m \mid x, A = 0, G = E)\left( \frac{1}{p(G = E \mid x)} - 1 \right)p(x, G = E)
\]

\[
= \sum_{m,x} m \frac{1}{p(A = 1, G = O)} \cdot \frac{1}{p(A = 0 \mid x, G = E)} \left( \frac{1}{p(G = E \mid x)} - 1 \right)S(m \mid x, A = 0, G = E)p(m, x, A = 0, G = E)
\]

\[
= \mathbb{E} \left[ \frac{1}{p(A = 1, G = O)} \cdot \frac{I(a = 0)I(g = E)}{p(A = 0 \mid X, G = E)} \left( \frac{1}{p(G = E \mid X)} - 1 \right) MS(M \mid X, A, G) \right]
\]

\[
= \mathbb{E} \left[ \frac{1}{p(A = 1, G = O)} \cdot \frac{I(a = 0)I(g = E)}{p(A = 0 \mid X, G = E)} \frac{1}{p(G = E \mid X)} - 1 \right] \left\{ M - \mathbb{E}[M \mid X, A, G] \right\} S(M \mid X, A, G).
\]

Note that

\[
\mathbb{E} \left[ \frac{1}{p(A = 1, G = O)} \cdot \frac{I(a = 0)I(g = E)}{p(A = 0 \mid X, G = E)} \frac{1}{p(G = E \mid X)} - 1 \right] \left\{ M - \mathbb{E}[M \mid X, A, G] \right\} S(X, A, G) = 0.
\]

Therefore

\[
\sum_{m,x} m \frac{1}{p(A = 1, G = O)} \partial_p p_t(m \mid x, A = 0, G = E)p(x, G = O)
\]

\[
= \mathbb{E} \left[ \frac{1}{p(A = 1, G = O)} \cdot \frac{I(a = 0)I(g = E)}{p(A = 0 \mid X, G = E)} \frac{1}{p(G = E \mid X)} - 1 \right] \left\{ M - \mathbb{E}[M \mid X, A, G] \right\} S(V).
\]
For the first term in (27), we have
\[
\sum_{x} \frac{1}{p(A = 1, G = O)} p(m \mid x, A = 0, G = E) \partial_{t} p_{t}(x, G = O)
\]
\[
= \sum_{x} \frac{1}{p(A = 1, G = O)} \mathbb{E}[M \mid X = x, A = 0, G = E] S(x, G = O) p(x, G = O)
\]
\[
= \sum_{x,g} \frac{I(g = O)}{p(A = 1, G = O)} \mathbb{E}[M \mid X = x, A = 0, G = E] S(x, g) p(x, g)
\]
\[
= \mathbb{E} \left[ \frac{I(G = O)}{p(A = 1, G = O)} \mathbb{E}[M \mid X, A = 0, G = E] S(X, G) \right]
\]
\[
= \mathbb{E} \left[ \frac{I(G = O)}{p(A = 1, G = O)} \mathbb{E}[M \mid X, A = 0, G = E] S(V) \right].
\]

Combining (24) and (26) concludes that
\[
\partial_{t} \psi_{3r} = \mathbb{E} \left[ \frac{1}{p(A = 1, G = O)} \left\{ \frac{I(A = 0)I(G = E)}{p(A = 0 \mid X, G = E)} \left( \frac{1}{p(G = E \mid X)} - 1 \right) \{ M - \mathbb{E}[M \mid X, A, G] \} + I(G = O) \mathbb{E}[M \mid X, A = 0, G = E] - I(A = 1)I(G = O) \psi_{3} \right\} S(V) \right].
\]

The variable
\[
\frac{1}{p(A = 1, G = O)} \left\{ \frac{I(A = 0)I(G = E)}{p(A = 0 \mid X, G = E)} \left( \frac{1}{p(G = E \mid X)} - 1 \right) \{ M - \mathbb{E}[M \mid X, A, G] \} + I(G = O) \mathbb{E}[M \mid X, A = 0, G = E] - I(A = 1)I(G = O) \psi_{3} \right\}
\]
is mean zero and hence is the influence function of \( \psi_{3} \).

For \( \psi_{4} \), note that
\[
\partial_{t} \psi_{4r} = \partial_{t} \sum_{y,x} y p_{t}(y \mid x, A = 0, G = O)p_{t}(x \mid A = 1, G = O)
\]
\[
= \sum_{y,x} y \partial_{t} p_{t}(y \mid x, A = 0, G = O)p_{t}(x \mid A = 1, G = O)
\]
\[
+ \sum_{y,x} y p_{t}(y \mid x, A = 0, G = O) \partial_{t} p_{t}(x \mid A = 1, G = O).
\]

For the first term in (27), we have
\[
\sum_{y,x} y \partial_{t} p_{t}(y \mid x, A = 0, G = O)p_{t}(x \mid A = 1, G = O)
\]
\[
= \sum_{y,x} y S(y \mid x, A = 0, G = O)p(y \mid x, A = 0, G = O)p(x \mid A = 1, G = O)
\]
\[
= \sum_{y,x} \frac{1}{p(A = 1, G = O)} \cdot p(A = 1 \mid x, G = O) \cdot \frac{p(0)}{p(A = 0 \mid x, G = O)} y S(y \mid x, A = 0, G = O)p(y, x, A = 0, G = O)
\]
\[
= \sum_{y,a,x,g} \frac{I(a = 0)I(g = O)}{p(A = 1, G = O)} \cdot p(A = 1 \mid x, G = O) \cdot \frac{I(a = 0)}{p(A = 0 \mid x, G = O)} y S(y \mid x, a, g)p(y, x, a, g)
\]
\[
= \mathbb{E} \left[ \frac{I(A = 0)I(G = O)}{p(A = 1, G = O)} \cdot p(A = 1 \mid X, G = O) \cdot Y S(Y \mid A, X, G) \right]
\]
\[
= \mathbb{E} \left[ \frac{I(A = 0)I(G = O)}{p(A = 1, G = O)} \cdot p(A = 1 \mid X, G = O) \cdot [Y - \mathbb{E}[Y \mid A, X, G]] S(Y \mid A, X, G) \right].
\]
Note that
\[
\mathbb{E}\left[ \frac{I(A = 0)I(G = O)}{p(A = 1, G = O)} \cdot \frac{p(A = 1 \mid X, G = O)}{p(A = 0 \mid X, G = O)} \{Y - \mathbb{E}[Y \mid A, X, G]\}S(A, X, G) \right] = 0.
\]
Therefore,
\[
\sum_{y,x} y \partial_t p_t(y \mid x, A = 0, G = O)p(x \mid A = 1, G = O)
= \mathbb{E} \left[ \frac{I(A = 0)I(G = O)}{p(A = 1, G = O)} \cdot \frac{p(A = 1 \mid X, G = O)}{p(A = 0 \mid X, G = O)} \{Y - \mathbb{E}[Y \mid A, X, G]\}S(V) \right].
\quad (28)
\]
For the second term in (27), we have
\[
\sum_{y,x} yp(y \mid x, A = 0, G = O) \partial_t p_t(x \mid A = 1, G = O)
= \sum_x \mathbb{E}[Y \mid x, A = 0, G = O]S(x \mid A = 1, G = O)p(x \mid A = 1, G = O)
= \sum_{a,x,g} I(a = 1)I(g = O)p(A = 1, G = O)\mathbb{E}[Y \mid x, A = 0, G = O]S(x \mid a, g)p(x, a, g)
= \mathbb{E} \left[ \frac{I(A = 1)I(G = O)}{p(A = 1, G = O)} \mathbb{E}[Y \mid X, A = 0, G = O]S(X \mid A, G) \right]
= \mathbb{E} \left[ \frac{I(A = 1)I(G = O)}{p(A = 1, G = O)} \left\{ \mathbb{E}[Y \mid X, A = 0, G = O] - \mathbb{E}[Y \mid X, A = 0, G = O \mid A = 1, G = O] \right\}S(X \mid A, G) \right]
= \mathbb{E} \left[ \frac{I(A = 1)I(G = O)}{p(A = 1, G = O)} \left\{ \mathbb{E}[Y \mid X, A = 0, G = O] - \psi_4 \right\}S(X \mid A, G) \right].
\]
Note that
\[
\mathbb{E} \left[ \frac{I(A = 1)I(G = O)}{p(A = 1, G = O)} \{Y - \mathbb{E}[Y \mid A, X, G]\} \right] = 0.
\]
Therefore,
\[
\sum_{y,x} yp(y \mid x, A = 0, G = O) \partial_t p_t(x \mid A = 1, G = O)
= \mathbb{E} \left[ \frac{I(A = 1)I(G = O)}{p(A = 1, G = O)} \left\{ \mathbb{E}[Y \mid X, A = 0, G = O] - \psi_4 \right\}S(V) \right].
\quad (29)
\]
Combining (28) and (29) concludes that
\[
\partial_t \psi_4 = \mathbb{E} \left[ \frac{I(G = O)}{p(A = 1, G = O)} \left\{ \frac{I(A = 0)p(A = 1 \mid X, G = O)}{p(A = 0 \mid X, G = O)} \{Y - \mathbb{E}[Y \mid A, X, G]\} + I(A = 1)\{\mathbb{E}[Y \mid X, A = 0, G = O] - \psi_4\} \right\}S(V) \right].
\]
Therefore
\[
\frac{I(G = O)}{p(A = 1, G = O)} \left\{ \frac{I(A = 0)p(A = 1 \mid X, G = O)}{1 - p(A = 1 \mid X, G = O)} \{Y - \mathbb{E}[Y \mid A, X, G]\} + I(A = 1)\{\mathbb{E}[Y \mid X, A = 0, G = O] - \psi_4\} \right\}
\]
is the influence function of \( \psi_4 \).
For \( i \in \{1, 2, 3, 4\} \), denote the obtained influence functions by \( IF_{\psi_i} \). The influence function for \( \psi_{ETT} \) can be obtained as \( IF_{\psi_{ATT}} = \sum_{i=1}^{4} IF_{\psi_i} \). Therefore,

\[
\frac{1}{p(A = 1, G = O)} \left\{ \frac{I(G = O)I(A = 0)}{1 - p(A = 1 | X, G = O)} \{ M - E[M | X, A, G] \} + \frac{1}{1 - p(A = 1 | X, G = E)} \frac{1}{p(G = E | X)} - 1 \{ M - E[M | X, A, G] \} + I(G = O)I(A = 0) \{ Y - E[Y | A, X, G] \} + I(A = 1) \{ Y + E[Y | A, X, G = O] - \psi_{ETT} \} \right\} \]

is the influence function of \( \psi_{ETT} \).

\[\square\]

**Proof of Theorem 14.** For \( a \in \{0, 1\} \), define

\[
\psi^{(a)}_1 = E[E[M | A = a, X, G = E] | G = O], \quad \psi^{(a)}_2 = E[E[M | A = a, X, G = O] | G = O], \quad \psi^{(a)}_3 = E[E[Y | A = a, X, G = O] | G = O].
\]

We use the notation \( \partial_t f(t) \) to denote \( \frac{\partial f(t)}{\partial t} \). For parameter \( \psi^{(a)}_i \), let \( \psi^{(a)}_i \) be the parameter under a regular parametric sub-model indexed by \( t \), that includes the ground-truth model at \( t = 0 \). Let \( V \) be the set of all observed variables. In order to obtain an influence function, we need to find a random variable \( \Gamma \) with mean zero, that satisfies

\[
\partial_t \psi^{(a)}_i = E[\Gamma S(V)],
\]

where \( S(V) = \partial_t \log p_t(V) \).

For \( \psi^{(a)}_1 \), note that

\[
\partial_t \psi^{(a)}_1 = \partial_t \sum_{m,x} mp_t(m | a, x, G = E)p_t(x | G = O) = \sum_{m,x} m\partial_t p_t(m | a, x, G = E)p_t(x | G = O) + \sum_{m,x} mp(m | a, x, G = E)\partial_t p_t(x | G = O).
\]

(30)
For the first term in (30), we have

\[ \sum_{m,x} m \partial_1 p_t(m \mid a, x, G = E)p(x \mid G = O) \]

\[ = \sum_{m,x} m \partial_1 p_t(m \mid a, x, G = E)\left( \frac{1}{p(G = E \mid x)} - 1 \right) \frac{1}{p(G = O)}p(x, G = E) \]

\[ = \sum_{m,x} m S(m \mid a, x, G = E)\left( \frac{1}{p(G = E \mid x)} - 1 \right) \frac{1}{p(A = a \mid x, G = E)} \cdot \frac{1}{p(G = O)}p(m, a, x, G = E) \]

\[ = \sum_{m,a,x,g} m S(m \mid \tilde{a}, x, g)\left( \frac{1}{p(G = E \mid x)} - 1 \right) \frac{I(\tilde{a} = a)}{p(A = a \mid x, G = E)} \cdot \frac{I(g = E)}{p(G = O)}p(m, \tilde{a}, x, g) \]

\[ = \mathbb{E}\left[ \frac{I(A = a)}{p(A = a \mid X, G = E)} \cdot \frac{I(G = E)}{p(G = O)}M\left\{ \frac{1}{p(G = E \mid X)} - 1 \right\}S(M \mid A, X, G) \right] \]

\[ = \mathbb{E}\left[ \frac{I(A = a)}{p(A = a \mid X, G = E)} \cdot \frac{I(G = E)}{p(G = O)}\left[ M - \mathbb{E}[M \mid A, X, G] \right]\left\{ \frac{1}{p(G = E \mid X)} - 1 \right\}S(M \mid A, X, G) \right]. \]

Note that

\[ \mathbb{E}\left[ \frac{I(A = a)}{p(A = a \mid X, G = E)} \cdot \frac{I(G = E)}{p(G = O)}\left[ M - \mathbb{E}[M \mid A, X, G] \right]\left\{ \frac{1}{p(G = E \mid X)} - 1 \right\}S(A, X, G) \right] = 0. \]

Therefore,

\[ \sum_{m,x} m \partial_1 p_t(m \mid a, x, G = E)p(x \mid G = O) \]

\[ = \mathbb{E}\left[ \frac{I(A = a)}{p(A = a \mid X, G = E)} \cdot \frac{I(G = E)}{p(G = O)}\left[ M - \mathbb{E}[M \mid A, X, G] \right]\left\{ \frac{1}{p(G = E \mid X)} - 1 \right\}S(V) \right]. \] (31)

For the second term in (30), we have

\[ \sum_{m,x} mp(m \mid a, x, G = E)\partial_t p_t(x \mid G = O) \]

\[ = \sum_{x} \mathbb{E}[M \mid A = a, X = x, G = E]p(x \mid G = O) \]

\[ = \sum_{x} \mathbb{E}[M \mid A = a, X = x, G = E]S(x \mid G = O)p(x \mid G = O) \]

\[ = \sum_{x,g} \frac{I(g = O)}{p(G = O)}\mathbb{E}[M \mid A = a, X = x, G = E]S(x \mid g)p(x, g) \]

\[ = \mathbb{E}\left[ \frac{I(G = O)}{p(G = O)}\mathbb{E}[M \mid A = a, X, G = E]S(X \mid G) \right] \]

\[ = \mathbb{E}\left[ \frac{I(G = O)}{p(G = O)}\{ \mathbb{E}[M \mid A = a, X, G = E] - \mathbb{E}[\mathbb{E}[M \mid A = a, X, G = E] \mid G = O] \}S(X \mid G) \right] \]

\[ = \mathbb{E}\left[ \frac{I(G = O)}{p(G = O)}\{ \mathbb{E}[M \mid A = a, X, G = E] - \psi_1^{(a)} \}S(X \mid G) \right]. \]

Note that

\[ \mathbb{E}\left[ \frac{I(G = O)}{p(G = O)}\{ \mathbb{E}[M \mid A = a, X, G = E] - \psi_1^{(a)} \}S(G) \right] = 0. \]
Therefore,

\[
\sum_{m,x} mp(m \mid a, x, G = E) \partial_t p_t(x \mid G = O) = \mathbb{E} \left\{ \frac{I(G = O)}{p(G = O)} \left[ \mathbb{E}[M \mid A = a, X, G = E] - \psi_1^{(a)}(S(V)) \right] \right\}.
\] (32)

Combining (31) and (32) concludes that

\[
\partial_t \psi_1^{(a)} = \mathbb{E} \left\{ \frac{I(A = a)}{p(A = a \mid X, G = E)} \cdot \frac{I(G = E)}{p(G = O)} \{M - \mathbb{E}[M \mid A, X, G]\} \left\{ \frac{1}{p(G = E \mid X)} - 1 \right\} 
+ \frac{I(G = O)}{p(G = O)} \{\mathbb{E}[M \mid A = a, X, G = E] - \psi_1^{(a)}\} \right\} S(V).
\]

Therefore,

\[
\frac{I(A = a)}{p(A = a \mid X, G = E)} \cdot \frac{I(G = E)}{p(G = O)} \{M - \mathbb{E}[M \mid A, X, G]\} \left\{ \frac{1}{p(G = E \mid X)} - 1 \right\} 
+ \frac{I(G = O)}{p(G = O)} \{\mathbb{E}[M \mid A = a, X, G = E] - \psi_1^{(a)}\}
\]

is the influence function of \(\psi_1^{(a)}\).

For \(\psi_2^{(a)}\), note that

\[
\partial_t \psi_2^{(a)} = \partial_t \sum_{m,x} mp(m \mid a, x, G = O)p_t(x \mid G = O)
= \sum_{m,x} m \partial_t p_t(m \mid a, x, G = O)p(x \mid G = O)
+ \sum_{m,x} mp(m \mid a, x, G = O)\partial_t p_t(x \mid G = O).
\] (33)

For the first term in (33), we have

\[
\sum_{m,x} m \partial_t p_t(m \mid a, x, G = O)p(x \mid G = O)
= \sum_{m,x} mS(m \mid a, x, G = O) \frac{1}{p(A = a \mid x, G = O)} \cdot \frac{1}{p(G = O)} p(m, a, x, G = O)
= \sum_{m, \tilde{a}, x, g} mS(m \mid \tilde{a}, x, g) \frac{I(\tilde{a} = a)}{p(A = a \mid x, G = O)} \cdot \frac{I(g = O)}{p(G = O)} p(m, \tilde{a}, x, g)
= \mathbb{E} \left\{ \frac{I(A = a)}{p(A = a \mid X, G = O)} \cdot \frac{I(G = O)}{p(G = O)} MS(M \mid A, X, G) \right\}
= \mathbb{E} \left\{ \frac{I(A = a)}{p(A = a \mid X, G = O)} \cdot \frac{I(G = O)}{p(G = O)} \{M - \mathbb{E}[M \mid A, X, G]\} S(M \mid A, X, G) \right\}.
\]

Note that

\[
\mathbb{E} \left\{ \frac{I(A = a)}{p(A = a \mid X, G = O)} \cdot \frac{I(G = O)}{p(G = O)} \{M - \mathbb{E}[M \mid A, X, G]\} S(A, X, G) \right\} = 0.
\]

Therefore,

\[
\sum_{m,x} m \partial_t p_t(m \mid a, x, G = O)p(x \mid G = O)
= \mathbb{E} \left\{ \frac{I(A = a)}{p(A = a \mid X, G = O)} \cdot \frac{I(G = O)}{p(G = O)} \{M - \mathbb{E}[M \mid A, X, G]\} S(V) \right\}.
\] (34)
For the second term in (33), we have
\[
\sum_{m,x} mp(m \mid a, x, G = O) \partial_x p_t(x \mid G = O)
\]
\[
= \sum x \mathbb{E}[M \mid A = a, X = x, G = O] \partial_x p_t(x \mid G = O)
\]
\[
= \sum x \mathbb{E}[M \mid A = a, X = x, G = O] S(x \mid G = O) p(x \mid G = O)
\]
\[
= \sum_{x,g} I(g = O) \mathbb{E}[M \mid A = a, X = x, G = O] S(x \mid g) p(x, g)
\]
\[
= \mathbb{E} \left[ \frac{I(G = O)}{p(G = O)} \mathbb{E}[M \mid A = a, X, G = O] S(X \mid G) \right]
\]
\[
= \mathbb{E} \left[ \frac{I(G = O)}{p(G = O)} \{ \mathbb{E}[M \mid A = a, X, G = O] - \mathbb{E}[\mathbb{E}[M \mid A = a, X, G = O] \mid G = O] \} S(X \mid G) \right]
\]
\[
= \mathbb{E} \left[ \frac{I(G = O)}{p(G = O)} \{ \mathbb{E}[M \mid A = a, X, G = O] - \psi_2^{(a)} \} S(X \mid G) \right].
\]

Note that
\[
\mathbb{E} \left[ \frac{I(G = O)}{p(G = O)} \{ \mathbb{E}[M \mid A = a, X, G = O] - \psi_2^{(a)} \} S(G) \right] = 0.
\]

Therefore,
\[
\sum_{m,x} mp(m \mid a, x, G = O) \partial_x p_t(x \mid G = O) = \mathbb{E} \left[ \frac{I(G = O)}{p(G = O)} \{ \mathbb{E}[M \mid A = a, X, G = O] - \psi_2^{(a)} \} S(V) \right].
\]

Combining (34) and (35) concludes that
\[
\partial_t \psi_2^{(a)} = \mathbb{E} \left[ \left\{ \frac{I(A = a)}{p(A = a \mid X, G = O)} \cdot \frac{I(G = O)}{p(G = O)} \{ M - \mathbb{E}[M \mid A, X, G] \} + \frac{I(G = O)}{p(G = O)} \{ \mathbb{E}[M \mid A = a, X, G = O] - \psi_2^{(a)} \} \right\} S(V) \right].
\]

Therefore,
\[
\frac{I(G = O)}{p(G = O)} \left\{ \frac{I(A = a)}{p(A = a \mid X, G = O)} \{ M - \mathbb{E}[M \mid A, X, G] \} + \mathbb{E}[M \mid A = a, X, G = O] - \psi_2^{(a)} \right\}
\]
is the influence function of $\psi_2^{(a)}$.

Similarly,
\[
\frac{I(G = O)}{p(G = O)} \left\{ \frac{I(A = a)}{p(A = a \mid X, G = O)} \{ Y - \mathbb{E}[Y \mid A, X, G] \} + \mathbb{E}[Y \mid A = a, X, G = O] - \psi_3^{(a)} \right\}
\]
is the influence function of $\psi_3^{(a)}$.

For $i \in \{1, 2, 3\}$ and $a \in \{0, 1\}$, denote the obtained influence functions by $IF_{\psi_i^{(a)}}$. The influence function
for $\psi_{ATE}$ can be obtained as $IF_{\psi_{ATE}} = IF_{\psi_1^{(1)}} - IF_{\psi_1^{(0)}} + IF_{\psi_2^{(0)}} - IF_{\psi_2^{(1)}} + IF_{\psi_3^{(1)}} - IF_{\psi_3^{(0)}}$. Therefore,

\[
\frac{(-1)^{1-A}}{p(A \mid X, G = E)} \cdot \frac{I(G = E)}{p(G = O)} \left\{ M - E[M \mid A, X, G] \right\} \left\{ \frac{1}{p(G = E \mid X)} - 1 \right\} \\
+ \frac{I(G = O)}{p(G = O)} \left( \frac{(-1)^{1-A}}{p(A \mid X, G = O)} \left\{ Y - E[Y \mid A, X, G] - M + E[M \mid A, X, G] \right\} \\
+ E[Y \mid X, A = 1, G = O] - E[Y \mid X, A = 0, G = O] \\
+ E[M \mid X, A = 1, G = E] - E[M \mid X, A = 0, G = E] \\
+ E[M \mid X, A = 0, G = O] - E[M \mid X, A = 1, G = O] - \psi_{ATE} \right\}
\]

is the influence function of $\psi_{ATE}$.

**Proof of Proposition [2]** First, suppose the set $\{\mu_M^{G=E}, \mu_M^{G=O}, \mu_Y^{G=O}\}$ is correctly specified. We have

\[
E \left[ \frac{1}{p(A = 1, G = O)} \left\{ I(G = O)I(A = 0) \left\{ M - \hat{\mu}_M^{G=O}(0, X) \right\} \\
+ \frac{I(G = E)}{1 - \hat{\pi}_G^{G=O}(X)} \left\{ \frac{1}{p(G = E \mid X)} - 1 \right\} \left\{ M - \hat{\mu}_M^{G=E}(0, X) \right\} \\
+ \frac{I(G = O)}{1 - \hat{\pi}_G^{G=O}(X)} \left\{ Y - \hat{\mu}_Y^{G=O}(0, X) \right\} \right\} \\
+ I(G = O) \left\{ \hat{\mu}_M^{G=O}(0, X) + \hat{\mu}_M^{G=E}(0, X) + I(A = 1) \left\{ Y + \hat{\mu}_Y^{G=O}(0, X) \right\} \right\} \right]\}
\]

\[
= E \left[ \frac{1}{p(A = 1, G = O)} \left\{ I(G = O) \left\{ \hat{\mu}_M^{G=O}(0, X) + \hat{\mu}_M^{G=E}(0, X) + I(A = 1) \left\{ Y + \hat{\mu}_Y^{G=O}(0, X) \right\} \right\} \right\} \right]\}
\]

\[
= \theta_{ETT}.
\]

\[
E \left[ \frac{(-1)^{1-A}}{1 - A + (-1)^{1-A} \hat{\pi}_G^{G=O}(X)} \cdot \frac{I(G = E)}{p(G = O)} \left\{ M - \hat{\mu}_M^{G=E}(A, X) \right\} \left\{ \frac{1}{p(G = E \mid X)} - 1 \right\} \\
+ \frac{I(G = O)}{p(G = O)} \left( (-1)^{1-A} \right) \left\{ Y - \hat{\mu}_Y^{G=O}(A, X) - M + \hat{\mu}_M^{G=O}(A, X) \right\} \\
+ \hat{\mu}_Y^{G=O}(1, X) - \hat{\mu}_Y^{G=O}(0, X) + \hat{\mu}_M^{G=E}(1, X) - \hat{\mu}_M^{G=E}(0, X) + \hat{\mu}_Y^{G=O}(0, X) - \hat{\mu}_M^{G=O}(1, X) \right\} \right]
\]

\[
= E \left[ \frac{(-1)^{1-A}}{1 - A + (-1)^{1-A} \hat{\pi}_G^{G=O}(X)} \cdot \frac{I(G = E)}{p(G = O)} \left\{ E[M \mid X, A, G = E] - \hat{\mu}_M^{G=E}(A, X) \right\} \left\{ \frac{1}{p(G = E \mid X)} - 1 \right\} \\
+ \frac{I(G = O)}{p(G = O)} \left( (-1)^{1-A} \right) \left\{ E[Y \mid X, A, G = O] - \hat{\mu}_Y^{G=O}(A, X) - E[M \mid X, A, G = O] + \hat{\mu}_M^{G=O}(A, X) \right\} \right]
\]

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Second, suppose the set \( \{ \pi^{G=E}, \pi^{G=O}, p(G = E \mid x) \} \) is correctly specified. We have

\[
\begin{align*}
&\mathbb{E}\left[ \frac{1}{p(A = 1, G = O)} \left\{ I(G = O) I(A = 0) \{ M - \hat{\mu}_M^{G=O}(0, X) \} ight. \\
&\quad + \frac{I(G = E) I(A = 0)}{1 - \hat{\pi}^{G=E}(X)} \left( \frac{1}{\hat{p}(G = E \mid X)} - 1 \right) \{ M - \hat{\mu}_M^{G=E}(0, X) \} \\
&\quad + \frac{I(G = O) I(A = 0) \hat{\pi}^{G=O}(X)}{1 - \hat{\pi}^{G=O}(X)} \left\{ \hat{\mu}_Y^{G=O}(0, X) \right\} \\
&\quad + \frac{I(G = O) \{ \hat{\mu}_M^{G=O}(0, X) + \hat{\mu}_M^{G=E}(0, X) + I(A = 1) \left\{ Y + \hat{\mu}_Y^{G=O}(0, X) \right\} \} \right\} \right] \\
&= \mathbb{E}\left[ \frac{1}{p(A = 1, G = O)} \left\{ I(G = O) I(A = 0) \{ M - \hat{\mu}_M^{G=O}(0, X) \} ight. \\
&\quad + \frac{I(G = E) I(A = 0)}{1 - \hat{\pi}^{G=E}(X)} \left( \frac{1}{\hat{p}(G = E \mid X)} - 1 \right) \{ M - \hat{\mu}_M^{G=E}(0, X) \} \\
&\quad + \frac{I(G = O) I(A = 0) \hat{\pi}^{G=O}(X)}{1 - \hat{\pi}^{G=O}(X)} \left\{ \hat{\mu}_Y^{G=O}(0, X) \right\} \\
&\quad + \frac{I(G = O) \{ \hat{\mu}_M^{G=O}(0, X) + \hat{\mu}_M^{G=E}(0, X) + I(A = 1) \left\{ Y + \hat{\mu}_Y^{G=O}(0, X) \right\} \} \right\} \right] \\
&= \mathbb{E}\left[ \frac{1}{p(A = 1, G = O)} \left\{ I(G = O) I(A = 0) \{ M - \hat{\mu}_M^{G=O}(0, X) \} ight. \\
&\quad + \frac{I(G = E) I(A = 0)}{1 - \hat{\pi}^{G=E}(X)} \left( \frac{1}{\hat{p}(G = E \mid X)} - 1 \right) \{ M - \hat{\mu}_M^{G=E}(0, X) \} \\
&\quad + \frac{I(G = O) I(A = 0) \hat{\pi}^{G=O}(X)}{1 - \hat{\pi}^{G=O}(X)} \left\{ \hat{\mu}_Y^{G=O}(0, X) \right\} \\
&\quad + \frac{I(G = O) \{ \hat{\mu}_M^{G=O}(0, X) + \hat{\mu}_M^{G=E}(0, X) + I(A = 1) \left\{ Y + \hat{\mu}_Y^{G=O}(0, X) \right\} \} \right\} \right]
\end{align*}
\]

Second, suppose the set \( \{ \pi^{G=E}, \pi^{G=O}, p(G = E \mid x) \} \) is correctly specified. We have
\[ \begin{aligned}
+ I(G = O) I(A = 1)Y} \right] \\
= \theta_{\text{ETT}}.
\end{aligned} \]

\[
E \left[ \frac{(-1)^{1-A}}{1 - A + (-1)^{1-A} \hat{\theta} G = E(X)} \cdot \frac{I(G = E)}{p(G = O)} \left\{ M - \mu^G_M(A, X) \right\} \left\{ \frac{1}{p(G = E \mid X)} - 1 \right\} \\
+ \frac{I(G = O)}{p(G = O)} \left\{ (-1)^{1-A} Y - \mu^G_M(A, X) - M + \hat{\mu}^G_M(A, X) \right\} \\
+ \frac{\hat{\mu}^G_M(1, X) - \mu^G_M(0, X) + \hat{\mu}^G_M(1, X) - \mu^G_M(0, X) + \hat{\mu}^G_M(0, X) - \mu^G_M(1, X) \right\} \right] \\
= E \left[ \frac{(-1)^{1-A}}{1 - A + (-1)^{1-A} \hat{\theta} G = E(X)} \cdot \frac{I(G = E)}{p(G = O)} \left\{ M \right\} \left\{ \frac{1}{p(G = E \mid X)} - 1 \right\} \\
+ \frac{I(G = O)}{p(G = O)} \left\{ (-1)^{1-A} Y - M \right\} \\
+ \frac{I(G = E)}{p(G = O)} \left\{ (-1)^{1-A} \mu^G_M(1, X) + \mu^G_M(0, X) \right\} \left\{ \frac{1}{p(G = E \mid X)} - 1 \right\} \\
+ \frac{I(G = O)}{p(G = O)} \left\{ (-1)^{1-A} \hat{\mu}^G_M(1, X) + \mu^G_M(0, X) - \mu^G_M(0, X) - \mu^G_M(1, X) \right\} \right] \\
= E \left[ \frac{(-1)^{1-A}}{1 - A + (-1)^{1-A} \hat{\theta} G = E(X)} \cdot \frac{I(G = E)}{p(G = O)} \left\{ M \right\} \left\{ \frac{1}{p(G = E \mid X)} - 1 \right\} \\
+ \frac{I(G = O)}{p(G = O)} \left\{ (-1)^{1-A} Y - M \right\} \right]
= \theta_{\text{ATE}}.
\]

Parts 3 and 4 can be proven by combining the techniques used in parts 1 and 2, and thus we omit here.

\[
\square
\]

**Proof of Theorem 13** Recall the definition that for \( a \in \{0, 1\} \)
\[
\psi^{(a)} = E \left[ E[h(M, A, X) \mid A = a, X, G = E] \mid G = O \right].
\]

We use the notation \( \partial_t f(t) \) to denote \( \frac{\partial f(t)}{\partial t} \mid_{t=0} \). For parameter \( \psi^{(a)} \), let \( \psi^{(a)}_t \) be the parameter under a regular parametric sub-model indexed by \( t \), that includes the ground-truth model at \( t = 0 \). Let \( V \) be the set
Therefore, in order to obtain an influence function, we need to find a random variable \( \Gamma \) with mean zero, that satisfies
\[
\partial_t \psi_t^{(a)} = \mathbb{E}[S(V)],
\]
where \( S(V) = \partial_t \log p_t(V) \).

Note that
\[
\partial_t \psi_t^{(a)} = \partial_t \sum_{m,x} h_t(m, a, x)p_t(m | a, x, G = E)p_t(x | G = O)
\]
\[
= \sum_{m,x} \partial_t h_t(m, a, x)p_t(m | a, x, G = E)p_t(x | G = G = O)
\]
\[
+ \sum_{m,x} h(m, a, x)\partial_t p_t(m | a, x, G = E)p_t(x | G = O)
\]
\[
+ \sum_{m,x} h(m, a, x)p_t(m | a, x, G = E)\partial_t p_t(x | G = O).
\]

For the first term in (36), we have
\[
\sum_{m,x} \partial_t h_t(m, a, x)p_t(m | a, x, G = E)p_t(x | G = O)
\]
\[
= \sum_{m,x} \partial_t h_t(m, a, x)\frac{p_t(m | a, x, G = E)}{p_t(m | a, x, G = O)p_t(a | x, G = O)}p_t(m, a, x | G = O)
\]
\[
= \sum_{z,m,x} \partial_t h_t(m, a, x)q(z, a, x)p_t(z, m, a, x | G = O)
\]
\[
= \sum_{z,m,a,x} I(\tilde{a} = a)\partial_t h_t(m, \tilde{a}, x)q(z, \tilde{a}, x)p_t(z, m, \tilde{a}, x | G = O)
\]
\[
= \mathbb{E}[I(A = a)\partial_t h_t(M, A, X)q(Z, A, X | G = O]
\]
\[
= \mathbb{E}[I(A = a)\partial_t \mathbb{E}[h_t(M, A, X) | Z, A, X, G = O]q(Z, A, X | G = O] .
\]

Note that by Assumption 16 (ii)
\[
\mathbb{E}[Y - h_t(M, A, X) | Z, A, X, G = O] = 0
\]
\[
\Rightarrow \partial_t \mathbb{E}[Y - h_t(M, A, X) | Z, A, X, G = O] = 0
\]
\[
\Rightarrow \mathbb{E}[\partial_t (Y - h_t(M, A, X)) | Z, A, X, G = O]
\]
\[
+ \mathbb{E}[(Y - h_t(M, A, X))S(Y, M | Z, A, X, G = O) | Z, A, X, G = O] = 0
\]
\[
\Rightarrow \mathbb{E}[\partial_t h_t(M, A, X) | Z, A, X, G = O] = \mathbb{E}[(Y - h_t(M, A, X))S(Y, M | Z, A, X, G = O) | Z, A, X, G = O].
\]

Therefore,
\[
\sum_{m,x} \partial_t h_t(m, a, x)p_t(m | a, x, G = E)p_t(x | G = O)
\]
\[
= \mathbb{E}[I(A = a)q(Z, A, X)\mathbb{E}[(Y - h_t(M, A, X))S(Y, M | Z, A, X, G = O) | Z, A, X, G = O] | G = O]
\]
\[
= \mathbb{E}[I(A = a)q(Z, A, X)\{Y - h_t(M, A, X)\}S(Y, M | Z, A, X, G = O) | G = O]
\]
\[
= \mathbb{E}[I(A = a)\frac{I(G = O)}{p(G = O)}q(Z, A, X)\{Y - h_t(M, A, X)\}S(Y, M | Z, A, X, G)] .
\]

Also, note that
\[
\mathbb{E}[I(A = a)\frac{I(G = O)}{p(G = O)}q(Z, A, X)\{Y - h_t(M, A, X)\}S(Z, A, X, G)] = 0.
\]
Therefore,

\[
\sum_{m,x} \partial h_t(m, a, x)p(m \mid a, x, G = E)p(x \mid G = O) = E \left[ I(A = a) \frac{I(G = O)}{p(G = O)} q(Z, A, X) \{ Y - h(M, A, X) \} S(V) \right].
\]

(37)

For the second term in (36), we have

\[
\sum_{m,x} h(m, a, x) \partial p_t(m \mid a, x, G = E)p(x \mid G = O)
\]

\[
= \sum_{m,x} h(m, a, x) \partial p_t(m \mid a, x, G = E) \\left\{ \frac{1}{p(G = E \mid x)} - 1 \right\} \frac{1}{p(G = O)} p(x, G = E)
\]

\[
= \sum_{m,x} h(m, a, x) S(m \mid a, x, G = E) \\left\{ \frac{1}{p(G = E \mid x)} - 1 \right\} \frac{1}{p(A = a \mid x, G = E)} \cdot \frac{1}{p(G = O)} p(m, a, x, G = E)
\]

\[
= \sum_{m,\tilde{a},x,g} h(m, \tilde{a}, x) S(m \mid \tilde{a}, x, g) \\left\{ \frac{1}{p(G = E \mid x)} - 1 \right\} \frac{1}{p(A = a \mid x, G = E)} \cdot \frac{I(\tilde{a} = a)}{p(G = O)} p(m, a, x, G = E)
\]

\[
= E \left[ \frac{I(A = a)}{p(A = a \mid X, G = E)} \cdot \frac{I(G = E)}{p(G = O)} h(M, A, X) \{ \frac{1}{p(G = E \mid X)} - 1 \} S(M \mid A, X, G) \right]
\]

\[
= E \left[ \frac{I(A = a)}{p(A = a \mid X, G = E)} \cdot \frac{I(G = E)}{p(G = O)} \left\{ h(M, A, X) - \eta(A, X) \right\} \{ \frac{1}{p(G = E \mid X)} - 1 \} S(M \mid A, X, G) \right],
\]

where

\[
\eta(a, x) := E[h(M, A, X) \mid A = a, X = x, G = E]
\]

\[
= \sum_{m} h(m, a, x)p(m \mid a, x, G = E)
\]

\[
= \sum_{m,\tilde{a},x} I(\tilde{a} = a) h(m, \tilde{a}, x) q(z, \tilde{a}, x)p(z, m, \tilde{a} \mid x, G = E)
\]

\[
= \sum_{m,\tilde{a},x} \frac{I(\tilde{a} = a)}{p(A = a \mid X, G = E)} h(M, A, X) q(Z, A, X) \mid X = x, G = E.
\]

Note that

\[
E \left[ \frac{I(A = a)}{p(A = a \mid X, G = E)} \cdot \frac{I(G = E)}{p(G = O)} \left\{ h(M, A, X) - \eta(A, X) \right\} \{ \frac{1}{p(G = E \mid X)} - 1 \} S(A, X, G) \right] = 0.
\]

Therefore,

\[
\sum_{m,x} h(m, a, x) \partial p_t(m \mid a, x, G = E)p(x \mid G = O)
\]

\[
= E \left[ \frac{I(A = a)}{p(A = a \mid X, G = E)} \cdot \frac{I(G = E)}{p(G = O)} \left\{ h(M, A, X) - \eta(A, X) \right\} \{ \frac{1}{p(G = E \mid X)} - 1 \} S(V) \right].
\]

(38)
For the third term in (36), we have
\[
\sum_{m,x} h(m, a, x)p(m \mid a, x, G = E)\partial_t p_t(x \mid G = O)
\]
\[
= \sum_x \sum_m h(m, a, x)p(m \mid a, x, G = E)S(x \mid G = O)p(x \mid G = O)
\]
\[
= \sum_{x, g} \frac{I(g = O)}{p(G = O)}\eta(a, x)S(x \mid g)p(x, g)
\]
\[
= \mathbb{E}\left[\frac{I(G = O)}{p(G = O)}\eta(a, X)S(X \mid G)\right]
\]
\[
= \mathbb{E}\left[\frac{I(G = O)}{p(G = O)}\{\eta(a, X) - \mathbb{E}[\eta(a, X) \mid G = O]\}S(X \mid G)\right]
\]
\[
= \mathbb{E}\left[\frac{I(G = O)}{p(G = O)}\{\eta(a, X) - \psi^{(a)}\}S(X \mid G)\right].
\]

Note that
\[
\mathbb{E}\left[\frac{I(G = O)}{p(G = O)}\{\eta(a, X) - \psi^{(a)}\}S(G)\right] = 0.
\]

Therefore,
\[
\sum_{m,x} h(m, a, x)p(m \mid a, x, G = E)\partial_t p_t(x \mid G = O) = \mathbb{E}\left[\frac{I(G = O)}{p(G = O)}\{\eta(a, X) - \psi^{(a)}\}S(V)\right]. \tag{39}
\]

Combining (36) - (39) concludes that
\[
\partial_t \psi^{(a)} = \mathbb{E}\left[\left\{\frac{I(G = O)}{p(G = O)}I(A = a)q(Z, A, X)\{Y - h(M, A, X)\}
\right.ight.
\]
\[
+ \frac{I(G = E)}{p(G = O) p(A = a \mid X, G = E)}\{h(M, A, X) - \eta(A, X)\}\left\{\frac{1}{p(G = E \mid X)} - 1\right\}
\]
\[
+ \frac{I(G = O)}{p(G = O)}\{\eta(a, X) - \psi^{(a)}\}\right\}S(V)\right].
\]

Therefore,
\[
\frac{I(G = O)}{p(G = O)}I(A = a)q(Z, A, X)\{Y - h(M, A, X)\}
\]
\[
+ \frac{I(G = E)}{p(G = O) p(A = a \mid X, G = E)}\{h(M, A, X) - \eta(A, X)\}\left\{\frac{1}{p(G = E \mid X)} - 1\right\}
\]
\[
+ \frac{I(G = O)}{p(G = O)}\{\eta(a, X) - \psi^{(a)}\}
\]

is the influence function of $\psi^{(a)}$.

\[\Box\]

**Proof of Proposition** First, suppose the pair $\{h, p(m \mid a, x, G = E)\}$ is correctly specified. We have
\[
\mathbb{E}\left[\frac{I(G = O)}{p(G = O)}I(A = a)q(Z, a, X)\{Y - \hat{h}(M, a, X)\}
\]
Second, suppose the set \( \{h, p(A = 1 \mid x, G = E), p(G = E \mid x)\} \) is correctly specified. We have

\[
\begin{align*}
\mathbb{E} \left[ \frac{I(G = E)}{p(G = O)} I(A = a) \hat{q}(Z, a, X) \{Y - \hat{h}(M, a, X)\} \right] &+ \frac{I(G = E)}{p(G = O)} \frac{I(A = a)}{1 - a + (-1)^{1-a} \hat{p}(A = 1 \mid X, G = E)} \{\hat{h}(M, a, X) - \hat{q}(a, X)\} \{\frac{1}{\hat{p}(G = E \mid X)} - 1\} \\
&+ \frac{I(G = O)}{p(G = O)} \hat{q}(a, X) \\
= \mathbb{E} \left[ \frac{I(G = E)}{p(G = O)} I(A = a) \hat{q}(Z, a, X) \{Y - \hat{h}(M, a, X)\} \right] \\
&+ \frac{I(G = E)}{p(G = O)} \frac{I(A = a)}{1 - a + (-1)^{1-a} \hat{p}(A = 1 \mid X, G = E)} \{\hat{h}(M, a, X) - \hat{q}(a, X)\} \{\frac{1}{\hat{p}(G = E \mid X)} - 1\} \\
&+ \frac{I(G = O)}{p(G = O)} \hat{q}(a, X) \\
= \mathbb{E} \left[ \frac{I(G = O)}{p(G = O)} I(A = a) \hat{q}(Z, a, X) \{Y - \hat{h}(M, a, X)\} \right] \\
&+ \frac{I(G = E)}{p(G = O)} \frac{I(A = a)}{1 - a + (-1)^{1-a} \hat{p}(A = 1 \mid X, G = E)} \{\hat{h}(M, a, X) - \hat{q}(a, X)\} \{\frac{1}{\hat{p}(G = E \mid X)} - 1\} \\
&+ \frac{I(G = O)}{p(G = O)} \hat{q}(a, X) \\
= \mathbb{E} \left[ \frac{I(G = E)}{p(G = O)} I(A = a) \hat{q}(Z, a, X) \{Y - \hat{h}(M, a, X)\} \right] \\
&+ \frac{I(G = E)}{p(G = O)} \frac{I(A = a)}{1 - a + (-1)^{1-a} \hat{p}(A = 1 \mid X, G = E)} \{\hat{h}(M, a, X) - \hat{q}(a, X)\} \{\frac{1}{\hat{p}(G = E \mid X)} - 1\} \\
&+ \frac{I(G = O)}{p(G = O)} \hat{q}(a, X) \\
= \mathbb{E} \left[ \frac{I(G = E)}{p(G = O)} I(A = a) \hat{q}(Z, a, X) \{Y - \hat{h}(M, a, X)\} \right] \\
&+ \frac{I(G = E)}{p(G = O)} \frac{I(A = a)}{1 - a + (-1)^{1-a} \hat{p}(A = 1 \mid X, G = E)} \{\hat{h}(M, a, X) - \hat{q}(a, X)\} \{\frac{1}{\hat{p}(G = E \mid X)} - 1\} \\
&+ \frac{I(G = O)}{p(G = O)} \hat{q}(a, X) \\
= \mathbb{E} \left[ \frac{I(G = E)}{p(G = O)} I(A = a) \hat{q}(Z, a, X) \{Y - \hat{h}(M, a, X)\} \right] \\
&+ \frac{I(G = E)}{p(G = O)} \frac{I(A = a)}{1 - a + (-1)^{1-a} \hat{p}(A = 1 \mid X, G = E)} \{\hat{h}(M, a, X) - \hat{q}(a, X)\} \{\frac{1}{\hat{p}(G = E \mid X)} - 1\} \\
&+ \frac{I(G = O)}{p(G = O)} \hat{q}(a, X) \\
= \theta(a).
\end{align*}
\]
Third, suppose the set \( \{q, p(A = 1 \mid x, G = E), p(G = E \mid x)\} \) is correctly specified. We have

\[
\begin{align*}
\mathbb{E}\left[ \frac{I(G = O)}{p(G = O)} I(A = a) \hat{q}(Z, a, X) \{Y - \hat{h}(M, a, X)\} \right] \\
+ \frac{I(G = E)}{p(G = O)} \cdot \frac{I(A = a)}{1 - a + (-1)^{1-a} \tilde{p}(A = 1 \mid X, G = E)} \{\hat{h}(M, a, X) - \hat{q}(a, X)\} \{\frac{1}{\tilde{p}(G = E \mid X)} - 1\}
\end{align*}
\]

\[
= \mathbb{E}\left[ \frac{I(G = O)}{p(G = O)} \right] \mathbb{I}(A = a) \hat{q}(Z, a, X) Y \]

Note that

\[
\begin{align*}
\mathbb{E}\left[ \frac{I(G = O)}{p(G = O)} I(A = a) \hat{q}(Z, a, X) \hat{h}(M, a, X) \right] \\
= \mathbb{E}\left[ \frac{I(G = O)}{p(G = O)} I(A = a) \mathbb{E}[\hat{q}(Z, a, X) \mid M, A = a, X, G = O] \hat{h}(M, a, X) \right] \\
= \frac{1}{p(G = O)} \sum_{m, x} p(m \mid a, x, G = E) p(a \mid x, G = O) \hat{h}(m, a, x) p(m, a, x, G = O) \\
= \frac{1}{p(G = O)} \sum_{m, x} p(m \mid a, x, G = E) p(x, G = O) \hat{h}(m, a, x) p(m, a, x, G = E) \\
= \frac{1}{p(G = O)} \sum_{m, x} p(a \mid x, G = E) p(G = O \mid x) \hat{h}(m, a, x) p(m, a, x, G = E) \\
= \mathbb{E}\left[ \frac{I(G = E)}{p(G = O)} \right] \cdot \frac{I(A = a)}{1 - a + (-1)^{1-a} \tilde{p}(A = 1 \mid X, G = E)} \{\frac{1}{\tilde{p}(G = E \mid X)} - 1\} \hat{h}(M, a, X) \\
\end{align*}
\]

Therefore,

\[
\begin{align*}
\mathbb{E}\left[ \frac{I(G = O)}{p(G = O)} I(A = a) \hat{q}(Z, a, X) \{Y - \hat{h}(M, a, X)\} \right] \\
+ \frac{I(G = E)}{p(G = O)} \cdot \frac{I(A = a)}{1 - a + (-1)^{1-a} \tilde{p}(A = 1 \mid X, G = E)} \{\hat{h}(M, a, X) - \hat{q}(a, X)\} \{\frac{1}{\tilde{p}(G = E \mid X)} - 1\}
\end{align*}
\]

\[
= \mathbb{E}\left[ \frac{I(G = O)}{p(G = O)} I(A = a) \hat{q}(Z, a, X) Y \right] \\
= \theta(a).
\]
Proof of Proposition 3

\[ \mathbb{E}[q(Z, A, X) \mid M, A, X, G = O] = \frac{p(M \mid A, X, G = E)}{p(M \mid A, X, G = O)p(A \mid X, G = O)} \]
\[ = \sum_z q(z, a, x)p(z, m, a \mid x, G = O) = p(m \mid a, x, G = E) \]
\[ = \sum_z q(z, a, x)\frac{p(z, G = O \mid m, a, x)p(m, a \mid x)}{p(G = O \mid x)} = \frac{p(G = E \mid m, a, x)p(m, a \mid x)}{p(a, G = E \mid x)} \]
\[ = \sum_{z, g} I(g = O)q(z, a, x)\frac{p(z, g \mid m, a, x)}{p(G = O \mid x)} = \sum_g I(g = E)\frac{p(g \mid m, a, x)}{p(a, G = E \mid x)} \]
\[ \Rightarrow \mathbb{E} \left[ \frac{I(G = O)}{p(G = O \mid X)} q(Z, A, X) - \frac{I(G = E)}{p(A, G = E \mid X)} \right] = 0. \]