RATIONAL ERGODICITY OF STEP FUNCTION SKEW PRODUCTS

JON AARONSON, MICHAEL BROMBERG AND NISHANT CHANDGOTIA

Dedicated to the memory of Roy Adler

Abstract. We study rational step function skew products over certain rotations of the circle proving ergodicity and bounded rational ergodicity when the rotation number is a quadratic irrational. The latter arises from a consideration of the asymptotic temporal statistics of an orbit as modelled by an associated affine random walk.

1. Introduction

A rational step function is a right continuous, step function on the additive circle $T := \mathbb{R}/\mathbb{Z} \cong [0,1)$ taking values in $\mathbb{R}^d$, whose discontinuity points are rational.

Let $\varphi : T \to \mathbb{R}^d$ be a rational step function.

The skew products $T_{\alpha,\varphi} = T_\alpha : T \times \mathbb{R}^d \to T \times \mathbb{R}^d$ ($\alpha \in T$) defined by

$$T_{\alpha,\varphi}(x,y) := (x + \alpha, y + \varphi(x))$$

are conservative if and only if

$$\int_T \varphi(t) dt = 0.$$ 

Necessity follows from the ergodic theorem and sufficiency follows from the Denjoy-Koksma inequality (see below).

Consider the collections of badly approximable irrationals

$$\text{BAD} := \{ \alpha \in \mathbb{R} \setminus \mathbb{Q} : \exists \theta > 0, |\alpha - \frac{p}{q}| \geq \frac{\theta}{q^2} \}$$

and of quadratic irrationals

$$\text{QUAD} := \{ \alpha \in \mathbb{R} \setminus \mathbb{Q} : \alpha \text{ quadratic} \}.$$ 

It is known that $\text{QUAD} \subset \text{BAD}$ and that $\text{BAD}$ has Lebesgue measure zero (see, e.g., [8]).
Denominator of a rational step function. Fix \( d, Q \in \mathbb{N} \), \( Q \geq 2 \) and \( \Phi : \mathbb{Z}_Q \to \mathbb{R}^d \). The rational step function with denominator \( Q \) and values \( \Phi \) is the step function \( \varphi = \varphi^{(\Phi)} : \mathbb{T} \to \mathbb{R}^d \) defined by

\[
\varphi(x) = \Phi(\kappa(x)),
\]

where \( \kappa : [0,1) \to \mathbb{Z}_Q \) is defined by \( \kappa(x) := \lfloor Qx \rfloor \). Every rational step function is of this form for some \( Q \geq 2 \) and \( \Phi : \mathbb{Z}_Q \to \mathbb{R}^d \).

If \( \varphi : \mathbb{T} \to \mathbb{R}^d \) is a rational step function with denominator \( Q \), then

\[
\int_{\mathbb{T}} \varphi(x) dx = \frac{1}{Q} \sum_{k=0}^{Q-1} \Phi(k).
\]

We prove, for \( \varphi : \mathbb{T} \to \mathbb{R}^d \) a rational step function with denominator \( Q \) and values \( \Phi \) and which is centered in the sense that \( \int_{\mathbb{T}} \varphi(x) dx = 0 \):

**Theorem 1’ (Ergodicity).** There is a collection \( \text{SBAD} \subset \mathbb{R} \sim Q \) of full Lebesgue measure so that \( \text{SBAD} \supset \text{BAD} \) and so that if \( \alpha \in \text{SBAD} \), then

\[
(\mathbb{T} \times \Gamma, \mathcal{B}(\mathbb{T} \times \Gamma), m_{\mathbb{T}} \times m_{\Gamma}, T_{\alpha,\varphi})
\]

is a CEMPT where \( \Gamma := \langle \varphi(\mathbb{T}) \rangle \) is the closed subgroup of \( \mathbb{R}^d \) generated by \( \Phi(\mathbb{Z}_Q) \).

Here and throughout, CEMPT means conservative, ergodic, measure preserving transformation, \( m_{\mathbb{G}} \) denotes Haar measure on the locally compact, Polish, Abelian group \( \mathbb{G} \), normalized if \( \mathbb{G} \) is compact. Also, \( \varphi : \mathbb{T} \to \mathbb{R}^d \) is always going to mean a centered rational step function.

Theorem 1’ will follow from the stronger Theorem 1 (see below). The technique of the proof of Theorem 1 is not new. For older, related results, see [6], [16] and references therein.

**Theorem 2 (Temporal CLT).** If \( \alpha \in \text{QUAD} \) and \( \dim \text{span}_{\mathbb{R}} \varphi(\mathbb{T}) = d \), then \( \exists \ell_k \in \mathbb{N} \), \( \ell_k \uparrow \& \ell_k \propto \lambda^k \) for some \( \lambda > 1 \) and \( \mu^{(0)} \in \mathbb{R}^d \) so that for any box \( I \subset \mathbb{R}^d \),

\[
\frac{1}{\ell_k} \# \left\{ 1 \leq n \leq \ell_k : \frac{\varphi_n(0) - k\mu^{(0)}}{\sqrt{k}} \in I \right\} \rightarrow_{k \to \infty} \int_I f_Z(t) dt,
\]

where \( Z \) is a globally supported, centered, normal random variable on \( \mathbb{R}^d \) and \( f_Z \) is its probability density function.

Here and throughout \( \varphi_n(x) := \sum_{k=0}^{n-1} \varphi(x + k\alpha) \) and \( \# \) denotes counting measure.

For an introduction to temporal statistics in dynamics see [7]. Theorem 2 here is a generalization of a subsequence version of Theorem 1.1 in [3], which in turn has been recently strengthened in [4].

**Theorem 3 (Rational ergodicity).** Suppose that \( \alpha \in \text{QUAD} \) and that \( \langle \varphi(\mathbb{T}) \rangle = \mathbb{Z}^d \), then

\[
(\mathbb{T} \times \mathbb{Z}^d, \mathcal{B}(\mathbb{T} \times \mathbb{Z}^d), m_{\mathbb{T}} \times \#, T_{\alpha,\varphi})
\]

is boundedly rationally ergodic and \( a_n(T_{\alpha,\varphi}) \asymp \frac{n}{(\log n)^2} \).

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**Remark:**

The rational step functions in this paper are also called quasiperiodic step functions. References to these functions can be found in [11], [12].
See [2] for a definition of bounded rational ergodicity. Bounded rational ergodicity of $T_{\alpha, \varphi}$ for $\varphi = 1_{[0, \frac{1}{2})} - 1_{(\frac{1}{2}, 1)}$ was established in [2] for $\alpha \in \text{QUAD}$ and in [1] for $\alpha \in \text{BAD}$.

Notations. Here and throughout, for $a_n, b_n > 0$:
- $a_n \ll b_n$ means $\exists M > 0$ so that $a_n \leq Mb_n$ for each $n \geq 1$,
- $a_n \asymp b_n$ means $a_n \ll b_n$ and $b_n \ll a_n$,
- $a_n \sim b_n$ means $\frac{a_n}{b_n} \xrightarrow{n \to \infty} 1$.

Outline of the rest of the paper. In §2 we prove Theorem 1, a stronger version of Theorem 1'. The rest of the paper is devoted to the proofs of Theorems 2 and 3. As in [2], [1], proofs rely on recursive properties of the tuples $(\varphi_n(0) : 1 \leq n \leq \ell_k)$, for a suitably chosen sequence $\ell_k \uparrow \infty$.

To study the temporal statistics of these tuples, we consider the “temporal random variables" $x_k : \{1, \ldots, \ell_k\} \to \mathbb{R}^d$, defined by $x_k(n) = \varphi_n(0)$, where $n$ is a uniformly distributed random variable with values in $\{1, \ldots, \ell_k\}$. In other words,

$$\text{Prob}(x_k \in I) = \frac{1}{\ell_k} \# \{1 \leq n \leq \ell_k : \varphi_n(0) \in I\}.$$

The recursive properties of the tuples (see §2), allow us to construct an associated affine random walk (ARW) which models the distribution of the “temporal random variables” (see §3).

In §4 we show that when $\alpha$ is quadratic, the sequence of expectations $E(x_k)$ is asymptotically linear. This culminates in the approximation of the distribution of $x_k - E(x_k)$ by an affine random walk generated by a sequence of centered, independent, identically distributed affine transformations (see the ARW centering lemma).

This enables proof in §5 of Theorem 2 which is a central limit theorem for $(x_k : k \geq 1)$. The proof of Theorem 3 in §6 is based on a “weak, rough local limit theorem” for $(x_k : k \geq 1)$. Both proofs use a spectral theory of ARWs based on perturbation theory of stochastic matrices (as in [9]).

2. Ergodicity

Regular continued fractions. Recall that the regular continued fraction expansion of $\alpha \in (0, 1) \sim \mathbb{Q}$ is

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n + \ddots}}} = \frac{1}{|a_1| + \frac{1}{|a_2| + \frac{1}{\ddots + \frac{1}{|a_n| + \ddots}} = (a_1, a_2, \ldots)}}$$

where $a_n := a(G^{n-1} \alpha) \in \mathbb{N}$ with $a(\alpha) := \lfloor \frac{1}{|a|} \rfloor$ & $G(\alpha) := \lfloor \frac{1}{|a|} \rfloor = \alpha - \lfloor \frac{1}{|a|} \rfloor$ for $\alpha \in \mathbb{T} \sim \mathbb{Q}$. 

Recall that \( G((0,1) \sim Q) \subset (0,1) \sim Q \) and so every irrational in \((0,1)\) indeed has an infinite regular continued fraction expansion. On the other hand, if \( \alpha \in (0,1) \cap Q \) then \( \exists \ n \geq 1, \ G^n(\alpha) = 0, \) and \( \alpha \) has only a finite regular continued fraction expansion. In the sequel, we’ll consider modified continued fractions where the situation is different.

Fix \( \alpha = (a_1, a_2, \ldots) \in (0,1) \sim Q \) and \( n \geq 1 \) and define the principal convergents \( \frac{p_n}{q_n}, \ p_n, q_n \in \mathbb{N}_0, \ \gcd(p_n, q_n) = 1 \) by

\[
\frac{p_n}{q_n} := \frac{1}{|a_1|} + \frac{1}{|a_2|} + \cdots + \frac{1}{|a_n|}.
\]

Here, and throughout, for \( k \in \mathbb{Z}, \ k \geq 0, \) we denote

\[
\mathbb{N}_k := \{ n \in \mathbb{Z} : n \geq k \}.
\]

The principal denominators \( q_n \) of \( \alpha \) are given by

\[
q_0 = 1, \ q_1 = a_1, \ q_{n+1} = a_{n+1}q_n + q_{n-1};
\]

the numerators \( p_n \) are given by

\[
p_0 = 0, \ p_1 = 1, \ p_{n+1} = a_{n+1}p_n + p_{n-1},
\]

and the principal convergents \( \frac{p_n}{q_n} \) satisfy

\[
\frac{1}{q_n(q_n + q_{n+1})} < |\alpha - \frac{p_n}{q_n}| < \frac{1}{q_n q_{n+1}}.
\]

We’ll also need Theorems 16, 17 and 19 in [13]:

**Proposition.** For \( \alpha \in \mathbb{T} \sim Q, \)

- If for some \( 0 \leq a \leq b (b \in \mathbb{N}), \ |b\alpha - a| < |d\alpha - c| \) for all \( 0 < d < b, \) then \( b = q_k \) for some \( k \in \mathbb{N}. \)
- For \( 0 \leq a \leq b < q_k \) \( (b, k \in \mathbb{N}), \ |b\alpha - a| > |q_k\alpha - p_k|. \)
- If \( p, q \in \mathbb{N}, \ (p, q) = 1 \) and \( |\alpha - \frac{p}{q}| < \frac{1}{2q^2}, \) then \( q \in \{ q_k : k \geq 1 \}. \)

The following is also well known (see e.g. [8], [13]):

**Proposition.** Let \( \alpha = (a_1, a_2, \ldots) \in (0,1) \sim Q, \) then

(i) \( \alpha \in \text{QUAD} \ \text{iff} \ \exists \ K, \ L \geq 1 \ \text{so that} \ a_{k+L} = a_k \ \forall \ k \geq K; \)

(ii) \( \alpha \in \text{BAD} \ \text{iff} \ \sup_{k \geq 1} a_k < \infty. \)

For \( Q \geq 2, \) we’ll also need the collection

\[
\text{SBAD}_Q := \{ \alpha : \lim_{n \to \infty} \frac{a_n}{q_{n+1}} > 0 \ \text{where} \ n' := \max\{1 \leq m < n : \frac{q_m}{q_n} < \frac{1}{q}\} \}.
\]

Evidently, \( \text{BAD} \subset \text{SBAD} := \bigcap_{Q \geq 2} \text{SBAD}_Q \) and it is not hard to show that \( \text{SBAD} \) has full Lebesgue measure.

In the following, \( \varphi = \psi(\Phi) : \mathbb{T} \to \mathbb{R}^d \) is a rational step function with denominator \( Q \geq 2. \)

**Theorem 1.** Suppose that either (i) \( \alpha \in \text{SBAD}_Q, \) or (ii) \( \alpha \notin Q \) & \( Q \) is prime, then \( (\mathbb{T} \times \Gamma, \mathcal{B}(\mathbb{T} \times \Gamma), m_{\mathbb{T}} \times m_{\Gamma}, T_{\alpha, \varphi}) \) is a CEMPT.

The rest of this section is devoted to the proof of Theorem 1.
**Essential values and periods.** Let \((X, \mathcal{B}, m)\) be a standard probability space, and let \(T : X \to X\) be an invertible, ergodic, probability preserving transformation and \(\mathcal{B}^+ := \{A \in \mathcal{B} : m(A) > 0\}\).

Suppose that \(G\) is a locally compact, Polish, Abelian group equipped with the translation invariant metric \(\rho\) (e.g., \(\rho(x, y) = \|x - y\|\) if \(G \leq \mathbb{R}^d\)).

Let \(\varphi : X \to G\) be measurable and define \(\varphi_n : X \to G\) by
\[
\varphi_n := \sum_{k=0}^{n-1} \varphi \circ T^k.
\]

The collection of essential values of \(\varphi\) (as in [17]) is
\[
E(\varphi) := \{a \in G : \forall A \in \mathcal{B}^+, \epsilon > 0, \exists n \in \mathbb{Z}, m(A \cap T^{-n}A) \cap \{\rho(\varphi_n, a) < \epsilon\} > 0\}.
\]

The skew product \(T_\varphi : X \times G \to X \times G\) is defined by
\[
T_\varphi(x, y) := (Tx, y + \varphi(x))
\]
and preserves the measure \(m \times m_G\).

Define the collection of periods for \(T_\varphi\)-invariant functions:
\[
\text{Per}(\varphi) := \{a \in G : \tau_a A = A \mod m \times m_G, \forall A \in \mathcal{B}(X \times G), T_\varphi(A) = A\},
\]
where \(\tau_a(x, y) = (x, y + a)\).

It is not hard to see that \(T_\varphi\) is ergodic iff \(T\) is ergodic and \(\text{Per}(\varphi) = G\).

**Schmidt’s Theorem** ([17]). \(E(\varphi)\) is a closed subgroup of \(G\) and \(E(\varphi) = \text{Per}(\varphi)\).

In view of this, the conclusion of Theorem 1 is equivalent to
\[
(1) \quad \Gamma := \langle \Phi(\mathbb{Z}_Q) \rangle = E(\varphi).
\]

We prove this first in the case that \(\langle \Phi(\mathbb{Z}_Q) \rangle\) is countable and then deduce the uncountable case.

Let
\[
D(\alpha) := \{q \in \mathbb{N} : \exists p \in \mathbb{N}, |\alpha - \frac{p}{q}| < \frac{1}{q^2}\}.
\]

We’ll need

**Denjoy-Koksma Inequality** ([11, 10]).
\[
\|\varphi_q\|_\infty \leq \bigvee \varphi, \forall q \in D(\alpha),
\]
where \(\bigvee \varphi\) denotes the total variation of \(\varphi\).

**Remark 1.** Consequently, when \(\Gamma = \langle \Phi(\mathbb{Z}_Q) \rangle\) is countable, there is a finite set \(F \subset \Gamma\) such that \(\varphi_q(x) \in F\) for every \(x \in \mathbb{T}\) & \(q \in D(\alpha)\).

Given an Abelian group \(G\) and \(g_0 \in G\), let \(r_{g_0} : G \to G\) denote the group rotation on \(G\) given by \(r_{g_0}(g) := g + g_0\).
Proof of Theorem 1 in the countable case.

**Sublemma 1.** For Theorem 1 in the countable case, it suffices that

\[ \Phi(\epsilon + 1) - \Phi(\epsilon) \in \text{Per}(\varphi) \ \forall \ \epsilon \in \mathbb{Z}. \]  

*Proof.* Let \( \Gamma_0 \subset \Gamma \) be the group generated by \( \{ \Phi(\epsilon + 1) - \Phi(\epsilon) : \epsilon \in \mathbb{Z} \} \), that is,

\[ \Gamma_0 := \langle \{ \Phi(\epsilon + 1) - \Phi(\epsilon) : \epsilon \in \mathbb{Z} \} \rangle \leq \Gamma. \]

Evidently, \( \Phi(\epsilon) + \Gamma_0 = \Phi(0) + \Gamma_0 \ \forall \ \epsilon \in \mathbb{Z} \), whence \( \varphi + \Gamma_0 = \Phi(0) + \Gamma_0 \) and \( \Gamma/\Gamma_0 \) is cyclic.

We claim moreover that \( \#\Gamma/\Gamma_0 \leq Q \). To see this, using \( \sum_{\epsilon \in \mathbb{Z}} \Phi(\epsilon) = 0 \), we have

\[ \Gamma_0 \ni \sum_{\epsilon \in \mathbb{Z}} (\Phi(0) - \Phi(\epsilon)) = Q\Phi(0), \]

whence indeed \( \#\Gamma/\Gamma_0 \leq Q \). By (2), \( \Gamma_0 \subset \text{Per}(\varphi) \). By Schmidt’s Theorem, if \( h \in L^\infty(\mathbb{T} \times \Gamma) \) and \( h \circ T_{\alpha,\varphi} = h \), then \( h \circ \tau_a = h \) a.e. \( \forall \ a \in \Gamma_0 \) and \( \exists \ H \in L^\infty(\mathbb{T} \times \Gamma/\Gamma_0) \) so that

\[ h(x,\gamma) = H(x,\gamma + \Gamma_0) \text{ for a.e. } (x,\gamma) \in \mathbb{T} \times \Gamma. \]

Evidently

\[ H \circ T_{\alpha,\psi} = H \text{ a.e.}, \]

where \( \psi : \mathbb{T} \rightarrow \Gamma/\Gamma_0, \psi := \varphi + \Gamma_0 = \Phi(0) + \Gamma_0 \) (as before).

Defining \( T_{\alpha,\psi} : \mathbb{T} \times \Gamma/\Gamma_0 \rightarrow \mathbb{T} \times \Gamma/\Gamma_0 \) as usual, we have

\[ T_{\alpha,\psi} \equiv r_{\alpha} \times r_{\Phi(0)} \circ \Gamma_0 : \mathbb{T} \times \Gamma/\Gamma_0 \rightarrow \mathbb{T} \times \Gamma/\Gamma_0, \]

which is ergodic, being a product of two ergodic group rotations with disjoint spectra. Thus \( H \) is constant a.e., whence also \( h \), and \( T_{\alpha,\varphi} \) is ergodic. \( \square \)

**Sublemma 2.**

\[ \Phi(\epsilon + 1) - \Phi(\epsilon) \in \text{Per}(\varphi) \ \forall \ \epsilon \in \mathbb{Z}. \]

*Proof.* We’ll prove the sublemma using

**OREN’S LEMMA** ([16]). If there exist \( n_k \in \mathbb{N} \) and \( A_k \subset \mathbb{T} \) such that \( \varphi_{n_k} \) is constant on \( A_k \) and \( \varphi_{n_k}|_{A_k} \rightarrow a, \inf \mu(T(A_k)) > 0 \) and \( \lim_{k \rightarrow \infty} |||n_k a||| = 0 \), then \( a \in \text{Per}(\varphi) \).

Here and throughout, \( |||x||| := \min_{k \in \mathbb{Z}} |x - k| \).

Note that a version of Oren’s lemma is implicit in [5].

Next, we claim that for (2), it suffices to show

**Sublemma 3.** For any \( \epsilon \in \mathbb{Z} \) there are sequences of measurable sets \( (A_k), (B_k) \subset \mathbb{T} \) and positive integers \( n_k \in \mathbb{N} \) such that \( \varphi_{n_k} \) is constant on \( A_k \) and \( B_k \),

\[ \varphi_{n_k}|_{B_k} - \varphi_{n_k}|_{A_k} = \Phi(\epsilon + 1) - \Phi(\epsilon) \]

and \( m(A_k), m(B_k) > c > 0 \), where \( c \) does not depend on \( k \).
Indeed, by Remark 1, there is a finite set $F$ so that $\varphi_{q_n}(x) \in F$ $\forall \ k \geq 1, \ x \in T$. Thus, $\exists f \in F$ and $k_\ell \to \infty$ such that $\varphi_{q_{k_\ell}}|_{A_k} = f \ \forall \ \ell \geq 1$, whence $\varphi_{q_{nk_\ell}}|_{B_k} = f + \Phi(e + 1) - \Phi(e) \ \forall \ \ell \geq 1$.

By Oren's Lemma, $f, f + \Phi(e + 1) - \Phi(e) \in \text{Per}(q)$, whence, since $\text{Per}(q)$ is a group, $\Phi(e + 1) - \Phi(e) \in \text{Per}(q)$ and sufficiency of the statement in Sublemma 3 is established.

Finally, we construct the sequences of measurable sets $A_k, B_k \subset T$ as in Sublemma 3. To this end, we prove first that the discontinuities of $\varphi$ are "dynamically separated".

Let $q \in \mathbb{N}$. Since $\varphi$ is a step function with the set of discontinuities contained in $\left\{ \frac{\ell}{Q} : 0 \leq \ell \leq Q - 1 \right\}$ the set of discontinuities of $\varphi_q$ is contained in the set

$$\left\{ \frac{\ell}{Q} - j\alpha : 0 \leq \ell \leq Q - 1 \text{ and } 0 \leq j \leq q - 1 \right\} \subset T.$$

Hence the distance between the discontinuities is bounded below by

$$\text{disc}(q) = \min_{|\ell| \leq Q - 1, |j| \leq q - 1, (\ell, j) \neq (0, 0)} |||\frac{\ell}{Q} - j\alpha|||.$$

**Claim.**

(3) $\exists \ n_k \uparrow \infty$ and $\theta > 0$ s.t. $\text{disc}(q_{n_k}) \geq \frac{\theta}{q_{n_k}} \ \forall \ k \geq 1$.

**Proof of (3) when $\alpha \in \text{SBAD}$.** By definition of SBAD, there exist a sequence $(m_k) \in \mathbb{N}, v \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$\varepsilon < \frac{q_{m_k-v}}{q_{m_k+1}} < \frac{q_{m_k-v}}{q_{m_k}} < \frac{1}{Q},$$

whence for all $r \in \mathbb{Z}$ and $|l| \leq q_{m_k-v} < \frac{q_{m_k}}{Q}$, since $q_{m_k}$ is a principal denominator,

$$|l - Q\alpha| = \frac{1}{Q}|r - Q\alpha| > \frac{1}{Q}|r - q_{m_k}\alpha| > \frac{1}{Q}\frac{1}{q_{m_k-v}} > \frac{1}{Qq_{m_k+1}} > \frac{\varepsilon}{Qq_{m_k+v}}.$$

Relation (3) follows. \qed

**Proof of (3) when $Q$ is prime.** This further splits into two separate cases.

(i) There are only finitely many $n$’s such that $q_n = 0 \mod Q$: Choose $N$ large enough such that $q_n \neq 0 \mod Q$ for $n \geq N$. For $n > N$,

$$\text{disc}(q_n) \geq \frac{1}{Q} \min_{0 \leq j < q_n} |||jQ\alpha|||, |||j\alpha|||.\$$

As before, we have

$$\min_{0 \leq j < q_n} |||j\alpha||| > \frac{1}{2q_n}.$$

Since $q_h$ is prime to $Q$ for all $h \geq n$, for $0 < j < q_n$, $jQ$ is not a multiple of $q_{n+r}$ for $r \geq 0$. Thus by [13, Theorem 19] if

$$|||Qj\alpha||| \leq \frac{1}{2Qj},$$
then $Qj$ is a multiple of $q_r$ for some $r < n$; in this case

$$|||Qj\alpha||| \geq \frac{1}{2q_{r+1}} \geq \frac{1}{2q_n}.$$ 

Therefore

$$\min_{0 < j < q_n} |||Qj\alpha||| \geq \min \left( |||q_{n-1}\alpha|||, \frac{1}{2Qq_n} \right) = \frac{1}{2Qq_n}$$

implying

$$\text{disc}(q_n) \geq \frac{1}{2Q^2q_n}.$$ 

(ii) There are infinitely many $n$'s such that $q_n = 0 \mod Q$: Let $(n_k)$ be the subsequence such that $q_{n_k} = 0 \mod Q$. Let the $\nu$-th term of the continued fraction expansion of $\alpha$ be given by $a_\nu$; we know

$$\begin{pmatrix} a_n & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix}.$$ 

Since $\det \left( \begin{pmatrix} a_\nu & 1 \\ 1 & 0 \end{pmatrix} \right) = -1$ for all $\nu$, the determinant of the product is either $1$ or $-1$. Thus $q_{n_k+1} \neq 0 \mod Q$.

By the recursion formula for the principal denominators, we have that

$$q_{n_k+r+1} = M(r)q_{n_k} + S(r)q_{n_k+1}$$

for some $M(r), S(r) \in \mathbb{N}$. Again,

$$\min_{0 < j < q_{n_k+1}} |||j\alpha||| > \frac{1}{2q_{n_k+1}}.$$ 

If

$$|||j\alpha||| < \frac{1}{2Qq_k}$$

then by [13, Theorem 19], $jQ$ is a multiple of $q_v$ for some $v$. Since $q_{n_k+1} \neq 0 \mod Q$, if $jQ = iq_{n_k+1}$ for some $i \in \mathbb{N}$ then $j \geq q_{n_k+1}$; it follows that $jQ$ is not a multiple of $q_{n_k+1}$. Therefore $i \neq n_k + 1$. Since $q_{n_k} = 0 \mod Q$ if

$$jQ = iq_{n_k+r+1} = iM(r)q_{n_k} + iS(r)q_{n_k+1} = 0 \mod Q$$

for some $i$,

then $iS(r)$ is multiple of $Q$ implying $j \geq q_{n_k+1}$; thus the number $jQ$ cannot be a multiple of $q_{n_k+r+1}$ for any $r \in \mathbb{N}$ and it follows that $i \leq n_k$.

Hence by [13, Theorem 16] we have

$$\min_{0 < j < q_{n_k+1}} |||Qj\alpha||| \geq \min \left( |||q_{n_k}\alpha|||, \frac{1}{2Qq_{n_k+1}} \right) = \frac{1}{2Qq_{n_k+1}}$$

implying

$$\text{disc}(q_{n_k+1}) \geq \frac{1}{2Q^2q_{n_k+1}}.$$ 

This proves (3).
Construction of measurable sets as in Sublemma 3. By (3) there exist a subsequence \((n_k) \uparrow \infty\) and \(\theta > 0\) such that \(\text{disc}(q_{n_k}) > \frac{\theta}{q_{n_k}}\) and such that \(q_{n_k}\) is sufficiently large compared to \(|F|^2\), where \(F\) is the finite set of values taken by \(\phi_{q_{n_k}}\) as in Remark 1.

Fix \(0 \leq \epsilon \leq Q - 1\). To obtain the periodicity \(\Phi(\epsilon + 1) - \Phi(\epsilon)\), we build sequences of measurable sets \((A_k), (B_k) \subset T\) such that

- \(\phi_{q_{n_k}}\) is constant on \(A_k\) and \(B_k\),
- \(\phi_{q_{n_k}}|_{B_k} - \phi_{q_{n_k}}|_{A_k} = \Phi(\epsilon + 1) - \Phi(\epsilon)\), and
- \(m_T(A_k), m_T(B_k) > c > 0\).

Fix \(k\) and let \(\partial\) be the partition of \(T\) by the discontinuities \(\{\frac{\epsilon}{Q} - h\alpha : 0 \leq h \leq q_{n_k} - 1\}\) of the step function \(\phi_{q_{n_k}}\). For \(0 \leq h < q_{n_k}\), let \(I^-_h, I^+_h \in \partial\) be the interval with right endpoint \(\frac{\epsilon}{Q} - h\alpha\) and \(I^+_h \in \partial\) be the interval with left endpoint \(\frac{\epsilon}{Q} - h\alpha\).

We can choose \(0 < h_1, h_2, \ldots, h_{\left\lfloor \frac{q_{n_k}}{|F|^2}\right\rfloor} < q_{n_k}\) such that \(\phi_{q_{n_k}}\) is constant on

\[
A_k := \bigcup_{u=1}^{\left\lfloor \frac{q_{n_k}}{|F|^2}\right\rfloor} I^-_{h_u} \quad \text{and} \quad B_k := \bigcup_{u=1}^{\left\lfloor \frac{q_{n_k}}{|F|^2}\right\rfloor} I^+_{h_u}.
\]

Evidently,

\[
\phi_{q_{n_k}}|_{B_k} - \phi_{q_{n_k}}|_{A_k} = \Phi(\epsilon + 1) - \Phi(\epsilon)
\]

and by (3),

\[
m(A_k), m(B_k) \geq \text{disc}(q_{n_k}) \left| \frac{q_{n_k}}{|F|^2} \right| \geq \frac{\theta}{2|F|^2}.
\]

These sets are as in Sublemma 3 and the proof of Theorem 1 in the countable case is now complete.

Proof of Theorem 1 in the uncountable case. Let

\[
V := \text{Span}_Q \Phi(Z_Q) \subset \mathbb{R}^d,
\]

let \(K := \dim V\) and let \(\{e_k : 1 \leq k \leq K\}\) be a basis for \(V\) so that each

\[
\Phi(\epsilon) = \sum_{k=1}^{K} \phi_k(\epsilon) e_k \quad \text{with} \quad \phi_k(\epsilon) \in \mathbb{Z} \quad (1 \leq k \leq K, \epsilon \in \mathbb{Z}_Q).
\]

Consider the cocycle \(\Psi : T \to \mathbb{Z}^K\) defined by

\[
\Psi(x) := \phi((Qx)), \quad \text{where} \quad \phi(\epsilon) := (\phi_1(\epsilon), \ldots, \phi_K(\epsilon)) \quad (\epsilon \in \mathbb{Z}_Q).
\]

It follows that \(\langle \Psi(T) \rangle = \mathbb{Z}^K\). We claim that

\[
\int_T \Psi(x) dx = \frac{1}{\mathcal{O}} \sum_{\epsilon \in \mathbb{Z}_Q} \phi(\epsilon) = 0.
\]
To see this,

\[ 0 = \int_T \varphi(x) \, dx = \sum_{e \in \mathbb{Z}_Q} \Phi(e) = \sum_{k=1}^K \left( \sum_{e \in \mathbb{Z}_Q} \phi_k(e) \right) e_k. \]

By linear independence of \( \{e_k : 1 \leq k \leq K\} \), for each \( 1 \leq k \leq K \), \( \sum_{e \in \mathbb{Z}_Q} \phi_k(e) = 0 \) showing that, indeed, \( \int_T \Psi(x) \, dx = 0 \).

Thus, by (1) in the countable case, and Schmidt's Theorem,

\[ \langle \phi(Z_Q) \rangle \subset \Per(\Psi) = E(\Psi). \]

It follows that \( \Phi = L \circ \varphi \) (and \( \phi = L \circ \Psi \)) where \( L: \mathbb{Z}^K \to V \subset \mathbb{R}^d \) is given by

\[ L(z_1, \ldots, z_K) := \sum_{k=1}^K z_k e_k. \]

By linearity of \( L \),

\[ L(E(\Psi)) \subset E(L \circ \Psi) = E(\varphi) \]

and

\[ \Phi(Z_Q) = L(\phi(Z_Q)) \subset E(\varphi). \]

3. THE ORBIT SEQUENCE

Theorems 2 and 3 both depend on the modeling of the orbit sequence

\[ (\varphi_n(0) : n \geq 1) \]

by an associated affine random walk. To extract this affine random walk we first obtain a sequential substitution construction of the jump sequence

\[ (\varphi(n\alpha)) : n \geq 1 \] for \( \alpha \in (0, 1) \sim \mathbb{Q}. \)

To this end, let \( \beta = \lfloor Q\alpha \rfloor \) and \( P := \lfloor Q\alpha \rfloor \) so that \( \alpha = \frac{P + \beta}{Q} \).

Define the map \( \pi : [0, 1) \to [0, 1) \times \mathbb{Z}_Q \) by

\[ \pi(x) := ([Qx], [Qx]), \]

the transformation \( \tau : [0, 1) \times \mathbb{Z}_Q \to [0, 1) \times \mathbb{Z}_Q \) by \( \tau := \pi \circ r_\alpha \circ \pi^{-1} \) and \( k : [0, 1) \times \mathbb{Z}_Q \to \mathbb{Z}_Q \) by

\[ k(x, k) := \kappa \circ \pi^{-1}(x, k) = k. \]

Then

\[ \tau(y, k) = \pi(\lfloor \frac{y + k}{Q} + \alpha \rfloor) = \pi(\lfloor \frac{y + k + P + \beta}{Q} \rfloor) \]

\[ = (r_\beta(y), [k + P + y + \beta] \mod Q) \]

\[ = (r_\beta(y), k + P + 1_{[-\beta, 1]}(y) \mod Q). \]
Thus
\[
\kappa(n\alpha) = \kappa \circ r^n_a(0) = k \circ \pi \circ r^n_a(0) = k \circ \tau^n \pi(0)
\]
\[
= nP + \sum_{k=0}^{n-1} 1_{[1-\beta,1)}(\{k\beta\})
\]
\[
= nP + \sum_{k=1}^{n} \psi_k \mod Q,
\]
where \(\psi_k := 1_{[1-\beta,1)}(\{k-1\beta\})\). The sequence \((\psi_k : k \geq 1)\) is generated as follows.
The modified continued fraction expansion of \(\beta \in (0, 1)\) is
\[
\beta = \frac{1}{n_1 - \frac{1}{n_2 - \frac{1}{\ddots - \frac{1}{n_k - \ddots}}}} = \left[\frac{1}{n_1} - \frac{1}{n_2} - \cdots - \frac{1}{n_k} - \cdots = [n_1, n_2, \ldots],
\]
where \(n_k(\beta) := n(s^{k-1}(\beta))\) with \(n(\beta) := \left\lfloor \frac{1}{\beta} \right\rfloor\) and \(s(\beta) := 1 - \{\frac{1}{\beta}\} = n(\beta) - \frac{1}{\beta}\). See [12, 14].

**The quadratic case.** If \(\alpha \in \text{QUAD}\), then so does \(\beta = \{Q\alpha\}\) and there exist
\[
(n_1, n_2, \ldots, n_K) \in \mathbb{N}_2^K \text{ and } (m_1, \ldots, m_L) \in \mathbb{N}_2^L \sim [2]\]
such that
\[
\beta = [n_1, n_2, \ldots, n_K, m_1, \ldots, m_L].
\]
Here and throughout,
- \(\mathbf{1}\) denotes a vector all of whose coordinates are 1,
- \(b_0 \circ b_1\) denotes the concatenation of the finite sequences \(b_0\) and \(b_1\),
- \(b_0^{\otimes n}\) denotes the concatenation of \(n\) copies of \(b_0\).

**Theorem.** ([2, Theorem 2.1]) For \(\beta = [n_1, n_2, \ldots]\), let \(b_0(0) = 0, b_0(1) = 1\) and
\[
 b_{k+1}(0) = b_k(0)^{\otimes (n_{k+1}-1)} \circ b_k(1) \quad \& \quad b_{k+1}(1) = b_k(0)^{\otimes (n_{k+1}-2)} \circ b_k(1),
\]
then
\[
(\psi_1, \ldots, \psi_{\ell_k(i)}) = b_k(i) \quad (k \geq 1)
\]
if the last symbol in \(b_k(1)\) is changed from “1” to “0”.

Here \(\ell_k(i) = |b_k(i)|\) \((i = 0, 1)\) are the block lengths.

**Block lengths.** Let \(\ell_k^* := \left(\begin{array}{c} \ell_k(0) \\ \ell_k(1) \end{array}\right)\), then
\[
\ell_0 = \left(\begin{array}{c} 1 \\ 1 \end{array}\right) \quad \text{and} \quad \ell_{k+1} = \left(\begin{array}{c} n_{k+1} - 1 \\ n_{k+1} - 2 \end{array}\right) \ell_k^*.
\]
Parities, jumps, and orbits. Next, we compute the jump blocks. We call $\kappa(x) = [Qx] \in \mathbb{Z}_Q$ the parity of $x$ and we begin by calculating the parity blocks with a generalization of [2, Theorem 2.2].

For $\beta = (Q\alpha) = [n_1, n_2, \ldots], \alpha \in \mathbb{Z}_Q, i = 0, 1$, define

$$B_k(i, \epsilon) := (\epsilon + \kappa(([n - 1] \alpha)) : 1 \leq n \leq \ell_k(i)).$$

Then by (4) and [2, Theorem 2.1] respectively,

$$B_k(i, \epsilon) = (\epsilon + (n - 1)P + \sum_{v=1}^{n-1} 1_{[1-\beta, 1)}((v-1)\beta)) : 1 \leq n \leq \ell_k(i))$$

(5)

$$= (\epsilon + (n - 1)P + \sum_{v=1}^{n-1} b_k(i)_v : 1 \leq n \leq \ell_k(i)),$$

where the addition is $\mod Q$ and $\sum_{v \in \mathbb{Q}} := 0$. Note that $B_0(i, \epsilon) = (\epsilon)$.

**Theorem 3.1 (Parity recursions).**

(6) $B_{k+1}(i, \epsilon) = \bigotimes_{j=1}^{n_{k+1} - 1 - i} B_k(0, \epsilon + (j - 1)\epsilon_k) \otimes B_k(1, \epsilon + (n_{k+1} - 1 - i)\epsilon_k)$,

with $\epsilon_k := \sum_{j=1}^{\ell_k(0)} (b_k(0))_j + \ell_k(0)P \mod Q$ and $\bigotimes_{j \in \mathbb{Q}} H_j \otimes B := B$ for finite sequences $(H_j)$ and $B$. Here (as before) the addition is $\mod Q$.

It follows that

$$B_1(i, \epsilon) = (\epsilon, \epsilon + P, \ldots, \epsilon + (n_1 - 1 - i)P) \mod Q.$$

**Proof.** Fix $i = 0, 1, \epsilon \in \mathbb{Z}_Q, k \geq 1$ and $1 \leq n \leq \ell_{k+1}(i)$. Then $n = q\ell_k(0) + r$, where $0 \leq q \leq n_{k+1} - i - 1$ and $1 \leq r \leq \ell_k(j_q)$ with $j_{n_{k+1} - i - 1} = 1$ and $j_q = 0$ for $q < n_{k+1} - i - 1$.

Using [2, Theorem 2.1] and (5), we have $\mod Q$,

$$B_{k+1}(i, \epsilon)_n = \epsilon + (n - 1)P + \sum_{v=1}^{n-1} b_{k+1}(i)_v$$

$$= \epsilon + (q\ell_k(0) + r - 1)P + \sum_{v=1}^{q\ell_k(0) + r - 1} b_{k+1}(i)_v$$

$$= \epsilon + q\epsilon_k + (r - 1)P + \sum_{v=1}^{r-1} b_k(j_q)_v$$

$$= \bigg( \bigotimes_{j=1}^{n_{k+1} - 1 - i} B_k(0, \epsilon + (j - 1)\epsilon_k) \otimes B_k(1, \epsilon + (n_{k+1} - 1 - i)\epsilon_k) \bigg)_n.$$ 

**Parity states and transition algorithm.** The $k^{th}$ parity states are $\epsilon_k(i)$ ($i = 0, 1$), where

$$\epsilon_k(i) := \sum_{j=1}^{\ell_k(i)} (b_k(i))_j + \ell_k(i)P \mod Q.$$

In (6), $\epsilon_k = \epsilon_k(0)$.
The parity states are given by $\epsilon_0(i) = P + i \mod Q$ and
\begin{equation}
\epsilon_{k+1}(i) = (n_{k+1} - i - 1)\epsilon_k(0) + \epsilon_k(1) \mod Q \quad (i = 0, 1, \; k \geq 1).
\end{equation}

**Parity Proposition.** For every $k \geq 1$, $\langle \epsilon_k, \epsilon_{k+1} \rangle = \mathbb{Z}_Q$.

**Proof.** Define $\zeta_k = (\zeta_k(0), \zeta_k(1))$ by
\begin{align*}
\zeta_0(i) &= P + i \\
\zeta_{k+1}(i) &= (n_{k+1} - i - 1)\zeta_k(0) + \zeta_k(1),
\end{align*}
It follows that
\begin{itemize}
\item $\epsilon_k(i) \equiv \zeta_k(i) \mod Q$;
\item $\gcd(\zeta_k(0), \zeta_k(1)) = 1 \forall \; k \geq 1$;
\item $\gcd(\zeta_k(0), \zeta_{k+1}(0)) = 1 \forall \; k \geq 1$;
\item $\langle \zeta_k, \zeta_{k+1} \rangle = \mathbb{Z} \forall \; k \geq 1$;
\item $\langle \epsilon_k, \epsilon_{k+1} \rangle = \mathbb{Z}_Q$.
\end{itemize}

**Jump blocks.** Next, for $k \geq 1$, $\epsilon \in \mathbb{Z}_Q$ and $i = 0, 1$, define the auxiliary jump blocks
\begin{equation}
J_k(i, \epsilon) := \Phi(B_k(i, \epsilon)),
\end{equation}
where
\begin{equation}
\Phi((a_1, \ldots, a_n)) := (\Phi(a_1), \ldots, \Phi(a_n)).
\end{equation}
It follows from (6) that, for $i = 0, 1$,
\begin{equation}
J_{k+1}(i, \epsilon) = \bigoplus_{j=1}^{n_{k+1} - 1 - i} J_k(0, \epsilon + (j - 1)\epsilon_k) \odot J_k(1, \epsilon + (n_{k+1} - 1 - i)\epsilon_k),
\end{equation}
where addition is $\mod Q$, and that the jump block
\begin{equation}
(\varphi([j\alpha]))_{j=0}^{\ell_k(0)-1} = J_k(0, 0).
\end{equation}

**Orbit blocks.** Define the auxiliary orbit blocks
\begin{equation}
\Sigma_k(i, \epsilon) := (\sum_{v=1}^{j} J_k(i, \epsilon)_v : 1 \leq j \leq \ell_k(i)).
\end{equation}
In particular by (9)
\begin{equation}
\Sigma_k(0, 0) = (\varphi_1(0), \varphi_2(0), \ldots, \varphi_{\ell_k(0)}(0)).
\end{equation}
Our goal here is to obtain the transition between auxiliary orbit blocks.

**Orbit block transitions.** The simple displacement over the auxiliary jump block $J_k(i, \epsilon)$ is
\begin{equation}
\sigma_k(i, \epsilon) := \Sigma_k(i, \epsilon)_{\ell_k(i)} = \sum_{j=1}^{\ell_k(i)} J_k(i, \epsilon)_j.
\end{equation}
The cumulative displacements over the concatenation jump blocks
\begin{equation}
\bigoplus_{j=1}^{K} J_k(0, \epsilon + (j - 1)\epsilon_k \mod Q) \quad (K \geq 0)
\end{equation}
are
\[ s_k(K,\epsilon) := \sum_{j=1}^{K} \sigma_k(0,\epsilon + (j-1)\epsilon_k \mod Q). \]

By (8), for \( k \geq 1, \epsilon \in \mathbb{Z}/Q, i = 0, 1, \)
\[ \Sigma_{k+1}(i,\epsilon) = \bigcup_{j=1}^{n_{k+1}} (\Sigma_k(0,\epsilon + (j-1)\epsilon_k \mod Q) + s_k(j-1,\epsilon)\mathbb{1}) \]
\[ \cap (\Sigma_k(1,\epsilon + (n_{k+1} - i - 1)\epsilon_k \mod Q) + s_k(n_{k+1} - i - 1,\epsilon)\mathbb{1}). \]

**Generating functions of orbit blocks.** For \( k \geq 1, \) define the functions \( x_k(i,\epsilon) : \Omega_k(i) = [1, \ell_k(i)] \to \mathbb{T}^d \) by
\[ x_k(i,\epsilon)(\omega) := \Sigma_k(i,\epsilon)_\omega \quad (\omega \in \Omega_k(i)) \]
and their generating functions
\[ U_k(i,\epsilon,\theta) := \sum_{\omega \in \Omega_k(i)} e^{2\pi i (\theta, x_k(i,\epsilon)(\omega))} \quad (\theta \in \mathbb{T}^d). \]

Here and throughout, \( i := \sqrt{-1}. \)

**Transition matrices.** Noting that
\[ \Omega_{k+1}(i) = \bigcup_{j=1}^{n_{k+1} - i - 1} (\Omega_k(0) + (j-1)\ell_k(0)) \cup (\Omega_k(1) + (n_{k+1} - i - 1)\ell_k(0)), \]
we have
\[ U_{k+1}(i,\epsilon,\theta) = \sum_{\omega \in \Omega_{k+1}(i)} e^{2\pi i (\theta, x_{k+1}(i,\epsilon)(\omega))} \]
\[ = \left( \sum_{j=1}^{n_{k+1} - i - 1} \sum_{\omega \in \Omega_k(0) + (j-1)\ell_k(0)} + \sum_{\omega \in \Omega_k(1) + (n_{k+1} - i - 1)\ell_k(0)} \right) e^{2\pi i (\theta, x_{k+1}(i,\epsilon)(\omega))} \]
\[ = \sum_{j=1}^{n_{k+1} - i - 1} \sum_{\omega \in \Omega_k(0)} e^{2\pi i (\theta, x_k(0,\epsilon + (j-1)\epsilon_k(0)) + s_k(j-1,\epsilon))} \]
\[ + \sum_{\omega \in \Omega_k(1)} e^{2\pi i (\theta, x_k(1,\epsilon + (n_{k+1} - i - 1)\epsilon_k(0)) + s_k(n_{k+1} - i - 1,\epsilon))} \]
\[ = \sum_{j=1}^{n_{k+1} - i - 1} e^{2\pi i (\theta, s_k(j-1,\epsilon))} U_k(0,\epsilon + (j-1)\epsilon_k,\theta) \]
\[ + e^{2\pi i (\theta, v_k(n_{k+1} - i - 1,\epsilon))} U_k(1,\epsilon + (n_{k+1} - i - 1)\epsilon_k,\theta) \]
\[ = \sum_{\Delta \in \mathbb{Z}_Q} \sum_{j \in \mathbb{N}} e^{2\pi i (\theta, s_k(n_{k+1} - i - 1,\epsilon))} U_k(0,\epsilon + \Delta,\theta) \]
\[ + e^{2\pi i (\theta, v_k(n_{k+1} - i - 1,\epsilon))} U_k(1,\epsilon + (n_{k+1} - i - 1)\epsilon_k,\theta), \]
where for \( \epsilon, \Delta \in \mathbb{Z}/Q, \)
\[ m(\epsilon,\Delta) := \{ j \in \mathbb{N} : (j - 1)\epsilon = \Delta \mod Q \}. \]
(with $\sum_{\theta \in \mathcal{Q}} := 0$ as before). Equivalently, 
\[ U_{k+1}(\theta) = A^{(k+1)}(\theta)U_k(\theta), \]
where $S := \{0, 1\} \times \mathbb{Z}_Q$ and $U_k : \mathbb{T}^d \to \mathbb{C}^S$ is given by 
\[ U_k(\theta)_{(i,\epsilon)} := U_k(i,\epsilon,\theta) \text{ for } (i,\epsilon) \in S \]
and $A^{(k+1)} : \mathbb{T}^d \to M_{S \times S}(\mathbb{C}) := \{a : S \times S \to \mathbb{C}\}$ is given by:
\[ A^{(k+1)}_{(i,\epsilon),(0,\epsilon+\Delta)}(\theta) = \sum_{j \in \mathcal{m}(\epsilon,\Delta) \cap [1,n_{k+1}-i-1]} e^{2\pi i(\theta,s_k(j-1,\epsilon))} \quad \text{if } (n_{k+1},i) \neq (2,1), \]
\[ A^{(k+1)}_{(i,\epsilon),(0,\epsilon+\Delta)}(\theta) = 0 \quad \text{if } (n_{k+1},i) = (2,1), \]
\[ A^{(k+1)}_{(i,\epsilon),(1,\epsilon+\Delta)}(\theta) = e^{2\pi i(\theta,s_k(n_{k+1}-i-1,\epsilon))} 1_{[\Delta]}((n_{k+1}-i-1)\epsilon_k). \]

It follows that 
\[ A^{(k+1)}_{(i,\epsilon),(0,\epsilon+\Delta)}(0) = N_{k+1}(i,\Delta) \quad \text{and} \quad A^{(k+1)}_{(i,\epsilon),(1,\epsilon+\Delta)}(0) = 1_{[\Delta]}((n_{k+1}-i-1)\epsilon_k), \]
where $N_{k+1}(i,\Delta) := \# m(\epsilon_k,\Delta) \cap [1,n_{k+1}-i-1]$. 

4. The random affine model

Probabilities. Here, we consider the probabilities 
\[ p^{(i)}_k := \frac{\#}{\ell_k(i)} \in \mathcal{P}(\Omega_k(i)) \]
and each $x_k(i,\epsilon) \in \mathbb{R}^d$ as a random variable with sample space $(\Omega_k(i),F^{(i)}_k)$, and understand the transitions of the resulting stochastic processes 
\[ (x_k(i,\epsilon) : k \geq 1) \quad ((i,\epsilon) \in S) \]
in the RAT lemma.

Let 
\[ \Xi_k(i,\epsilon,\theta) := E(e^{2\pi i(\theta,x_k(i,\epsilon))}) = \frac{1}{\ell_k(i)} U_k(i,\epsilon,\theta), \]
then 
\[ (11) \quad \Xi_{k+1}(\theta) = \Pi^{(k+1)}(\theta)\Xi_k(\theta) \]
where $\Xi_k := (\Xi_k(i,\epsilon) : (i,\epsilon) \in S)$ and $\Pi^{(k+1)}(\theta) : S \times S \to \mathbb{C}$ is given by 
\[ \Pi^{(k+1)}_{(i,\epsilon),(j,\Delta)}(\theta) = \frac{\ell_k(j)}{\ell_{k+1}(i)} A^{(k+1)}_{(i,\epsilon),(j,\Delta)}(\theta). \]
Random variables. We denote by $\text{RV}(Z)$, for $Z$ a measurable space, the collection of $Z$-valued random variables.

Consider any sequence of independent, random vectors

$$
\left( \mathcal{L}^{(k+1)}_s, W^{(k+1)}_{s,t} : s, t \in S \right) \in \text{RV}(S^S \times (\mathbb{R}^d)^{S \times S}) \quad (k \geq 0)
$$

whose marginals satisfy

$$
P(\mathcal{L}^{(k+1)}_{i,\epsilon} = (0, \epsilon + \Delta)) = \frac{\ell_k(0) N_{k+1}(i, \Delta)}{\ell_{k+1}(i)},
$$

$$
P(\mathcal{L}^{(k+1)}_{i,\epsilon} = (1, \epsilon + \Delta)) = \frac{\ell_k(1) N_{k+1}(i, \Delta)}{\ell_{k+1}(i)},
$$

$$
P(\{W^{(k+1)}_{i,\epsilon} = 1\} \cap \{\mathcal{L}^{(k+1)}_{i,\epsilon} = (0, \epsilon + \Delta)\}) = \frac{\#\{j \in m(\epsilon, \Delta) \cap [1, n_{k+1} - i - 1] : s_k(j - 1, \epsilon) = s_k(j - 1, \epsilon)\}}{N_{k+1}(i, \Delta)}
$$

for $j \in m(\epsilon, \Delta) \cap [1, n_{k+1} - i - 1]$ and

Note that when $n_{k+1} = 2$, then $\mathcal{L}^{(k+1)}_{1,\epsilon} = (1, \epsilon + (n_{k+1} - i - 1)\epsilon_k)$ a.s. $\forall \epsilon \in \mathbb{Z}_Q$ and that $W^{(k+1)}_{s,t}$ is defined when and only when $P(\mathcal{L}^{(k+1)}_2 = t) > 0$.

Random affine transformations. Given a random vector

$$
(\mathcal{L}, W) \in \text{RV}(S^S \times (\mathbb{R}^d)^{S \times S}),
$$

the associated random affine transformation (RAT) $F \in \text{RV}(M_{S \times S}(\mathbb{R}) \times (\mathbb{R}^d)^S)$ defined by

$$
F(x)_s := x_{\mathcal{L}_s} + W_{s,\mathcal{L}_s} =: (a(F)x)_s + b(F)_s \text{ for } x \in (\mathbb{R}^d)^S.
$$

This RAT is of flip-type in the sense of [1].

Throughout this paper we’ll often denote a flip-type RAT

$$
F = (a(F), b(F)) \in \text{RV}\left(M_{S \times S}([0, 1]) \times (\mathbb{R}^d)^S\right)
$$

by

$$
F = (\mathcal{L}, W) = (\mathcal{L}(F), W(F)) \in \text{RV}(S^S \times (\mathbb{R}^d)^{S \times S}).
$$

Here

$$
a_{s,t} = \delta_{s,\mathcal{L}_t} \& b_s = W_{s,\mathcal{L}_t}.
$$

Given a sequence $\left( \mathcal{L}^{(k+1)}_s, W^{(k+1)}_{s,t} : s, t \in S \right) \quad (k \geq 0)$ of independent random vectors as before, consider the associated RAT sequence

$$
(F_k : k \geq 1) \in \text{RV}\left(M_{S \times S}([0, 1]) \times (\mathbb{R}^d)^S\right)^\mathbb{N}
$$

of independent RATs defined by (13).
RAT characteristic function. The characteristic function of the RAT \( F = (\mathcal{L}, W) \in \text{RV} (S^5 \times (\mathbb{R}^d)^{S \times S}) \) (RAT-CF) is \( \Pi_F : \mathbb{R}^d \to M_{S \times S}(\mathbb{C}) \) defined by
\[
\Pi_F(\theta)_{s,t} = P(\mathcal{L}_s = t) E(e^{2\pi i \langle \theta, W_{s,t} \rangle}) \quad (s, t \in S). \tag{14}
\]
Note that \( \Pi^{(k+1)} \) in (11) is the RAT-CF of the RAT \((\mathcal{L}^{(k+1)}, W^{(k+1)})\) where \( \mathcal{L}^{(k+1)} \) and \( W^{(k+1)} \) are as in (12).

RAT Lemma. For each \( k \geq 1, s \in S, \)
\[
\text{dist} F^k_{1}(0)_{s} = \text{dist} x_{k}(s),
\]
where \( x_{k}(s) \) is as in (10).

Here and throughout for \( K \leq L \) and RATs \((F_j : K \leq j \leq L)\)
\[
F^L_K := F_L \circ F_{L-1} \circ \cdots \circ F_{K+1} \circ F_K.
\]

Proof. For \( k \geq 1, \) define
\[
X^{(k)} := F_k \circ F_{k-1} \circ \cdots \circ F_1(0)
\]
and
\[
\hat{\Xi}_k := (E(e^{2\pi i \langle \theta, X^{(k)}(i,\epsilon) \rangle}) : (i,\epsilon) \in S).
\]
By construction,
\[
\hat{\Xi}_{k+1}(\theta) = \Pi^{(k+1)}(\theta) \hat{\Xi}_k(\theta).
\]
By (11),
\[
\Xi_{k+1}(\theta) = \Pi^{(k+1)}(\theta) \Xi_k(\theta).
\]
The result follows by induction since \( \Xi_0 = \hat{\Xi}_0 \equiv 1. \)

Associated affine random walks. We associate to a sequence
\[
(F_k : k \geq 1) \in \text{RV} \left( M_{S \times S}(\{0,1\}) \times (\mathbb{R}^d)^{S} \right)^N
\]
of independent RATs an affine random walk (ARW). This is the \((\mathbb{R}^d)^S\)-valued stochastic process
\[
(X^{(k)} = (X^{(k)}_s : s \in S) : k \geq 1)
\]
defined by
\[
X^{(k)} := F^k_1(0).
\]
Elementary presentation. We now split the random vectors \( (\mathcal{L}_s^{(k+1)}, W_s^{(k+1)} : s, t \in S) \) \((k \geq 0)\) into more elementary components. Write \[ \mathcal{L}_{(i,t)}^{(k+1)} = (\tau_{(i,t)}^{(k+1)}, s_{(i,t)}^{(k+1)}), \]
then \( \tau_{(i,t)}^{(k+1)} \) is a \([0,1]\)-valued random variable where \[ P(\tau_{(i,t)}^{(k+1)} = 0) = \frac{\ell_k(0)(n_{k+1} - i - 1)}{\ell_{k+1}(i)} \quad \text{and} \quad P(\tau_{(i,t)}^{(k+1)} = 1) = \frac{\ell_k(1)}{\ell_{k+1}(i)} \]
and \[ s_{(i,t)}^{(k+1)} = e + \epsilon_i^{(k+1)} \mod Q, \]
where \( s_{(i,t)}^{(k+1)} \) and \( \epsilon_i^{(k+1)} \) are \( \mathbb{Z}_Q \)-valued random variables; the latter is given by \[ P(\epsilon_i^{(k+1)} = \Delta \| \tau_i^{(k+1)} = 0) = \frac{N_{k+1}(i, \Delta)}{n_{k+1} - i - 1} \quad (\Delta \in \mathbb{Z}_Q) \]
and \[ P(\epsilon_i^{(k+1)} = (n_{k+1} - i - 1)\epsilon_k \| \tau_i^{(k+1)} = 1) = 1. \]
Next define random variables \( u^{(k+1)}(i) \) \((k \geq 1, i = 0, 1)\) by \[ u^{(k+1)}(i) \left\{ \begin{array}{l}
\text{uniform on} \ m(\epsilon_k, \epsilon_i^{(k+1)}) \cap [1, n_{k+1} - i - 1] \text{ if} \ \tau_i^{(k+1)} = 0 \\
\equiv n_{k+1} - i \quad \text{if} \ \tau_i^{(k+1)} = 1.
\end{array} \right. \]
Now we define random variables \( W_s^{(k+1)} \) \((k \geq 1, s \in S)\) by \[ W_s^{(k+1)} := s_k(u^{(k+1)}(i) - 1, \epsilon). \]
It is not hard to see that \[ P(W_s^{(k+1)} = j \| \mathcal{L}_{(i,t)}^{(k+1)} = (j, e + \Delta)) = P(W_s^{(k+1)} = j \| \tau_i^{(k+1)} = j, \epsilon_i^{(k+1)} = \Delta). \]
In the sequel, we'll have recourse to the elementary random vector sequence \( (\chi^{(k)} : k \geq 1) \in RV\left( ((0,1) \times \mathbb{Z}_Q \times \mathbb{N}_0)^{(0,1)} \right)^\mathbb{N} \) where \[ \chi^{(k)} := (\tau_i^{(k)}, \epsilon_i^{(k)}, u^{(k)}(i) : i = 0, 1). \]
The RAT \( F_k \) is constructed from \( \chi^{(k)} \) and the (deterministic) cumulative displacements \( s_{k-1} \).

5. The RAT Sequence in the Quadratic Case

We assume that \( \alpha \in \text{QUAD} \); thus \( \beta := [\alpha] \in \text{QUAD} \). These hold if and only if there exist \( (n_1, n_2, \ldots, n_K) \in \mathbb{N}_2^K \) and \( (m_1, \ldots, m_L) \in \mathbb{N}_2^L \) such that \[ \beta := [n_1, n_2, \ldots] = [n_1, \ldots, n_K, m_1, \ldots, m_L]. \]
We next establish that the centered RAT sequence (as in [1]) corresponding to a quadratic irrational and a rational step function is “asymptotically eventually periodic”. The proofs of Theorems 2 and 3 rely on this fact.
This asymptotic eventual periodicity is obtained via centering. We’ll see that elementary random vector sequence is always asymptotically eventually periodic, however, the cumulative displacements may have linear growth. The centering is needed to offset this possibility.

In this section, we’ll often “possibly extend the period in (16)” to demonstrate eventual periodicity of related sequences. This means that for some \( M \in \mathbb{N} \), we’ll modify (16) to

\[
\begin{bmatrix}
[ n_1, n_2, \ldots ] = [ n_1, \ldots, n_K, m_1, \ldots, m_L, \ldots, m_1, \ldots, m_L ]^{M \text{-times}}
\end{bmatrix}
\]

Recall (7) that the parity state transitions are given by

\[
\epsilon_{k+1}(i) = (n_{k+1} - i - 1)\epsilon_k(0) + \epsilon_k(1) \mod Q.
\]

In the quadratic case, these transitions form an eventually periodic sequence, whence

\[
((\epsilon_k(0), \epsilon_k(1)) : k \geq 1)
\]

is also eventually periodic.

These parity transitions only depend on \( \alpha \in \mathbb{T} \sim Q \) and \( Q \geq 2 \). If \( \alpha \in \text{QUAD} \), then by possibly extending the period in (16), we may assume that \((\epsilon_{k+L}(0), \epsilon_{k+L}(1)) = (\epsilon_k(0), \epsilon_k(1)) \forall k > K \).

**Simple displacement transitions.** Consider the simple displacement vectors

\[
\sigma_k := (\sigma(i, e) : (i, e) \in S) \in (\mathbb{R}^d)^S.
\]

By Theorem 3.1, for \( i = 0, 1 \) and \((n_{k+1}, i) \neq (2, 1)\),

\[
\sigma_{k+1}(i, e) = \sum_{j=1}^{n_{k+1} - 1 - i} \sigma_k(0, e + (j - 1)e_k) + \sigma_k(1, e + (n_{k+1} - 1 - i)e_k),
\]

where \( e_k = \epsilon_k(0) \) as before. Thus there exist matrices \( M^{(k+1)} : S \times S \to \mathbb{Z} \) such that

\[
\sigma_{k+1} = M^{(k+1)} \sigma_k
\]

for each \( k \geq 1 \).

The simple displacement transformations also only depend on \( \alpha \in \mathbb{T} \sim Q \) and \( Q \geq 2, \).

Seeing \( \sigma_k = (\sigma_k(s) : s \in S) \in (\mathbb{R}^d)^S \) as

\[
\sigma_k = ((\sigma_k^{(j)}(s) : s \in S) \mid 1 \leq j \leq d) \in (\mathbb{R}^S)^d,
\]

we note that each \( \sigma_k^{(j)} \in \mathbb{R}^S \) is a linear image of \( \Phi^{(j)}(\alpha) \in \mathbb{R}^Q \) (the \( j \)th coordinate of \( \Phi \)) and \( \sigma_{k+1}^{(j)} = M^{(k+1)} \sigma_k^{(j)} \) for each \( 1 \leq j \leq d \).

**Displacement Lemma.** Suppose that \( \alpha \in \text{QUAD} \), then there exist \( K, L \in \mathbb{N} \) and \( c, d \in (\mathbb{R}^d)^S \) so that

\[
\sigma_{K+Ln} = c + n \hat{d}.
\]
For $\alpha \in \text{QUAD}$, the simple displacement transitions are eventually periodic and the proof of the displacement lemma rests on the Denjoy-Koksma inequality and a spectral analysis of the simple displacement transformations on $C^S$ over a period (as in the “eigenvalue lemma” below).

**Subspace decomposition and eigenvalues.** For $\alpha \in \text{QUAD}$, the parity sequence $(\mathcal{e}_k : k \geq 1)$ is eventually periodic, whence the above sequence of matrices $(M^{(k)} : k \geq 1)$ giving the displacement transitions is also eventually periodic.

Suppose that

$$[n_1, n_2, \ldots] = [n_1, \ldots, n_K, m_1, \ldots, m_L];$$

$$\mathcal{e}_k : k \geq 1 = (e_1, \ldots, e_K, \eta_1, \ldots, \eta_L);$$

$$(M^{(k)} : k \geq 1) = (M^{(1)}, \ldots, M^{(K)}, E_1, \ldots, E_L).$$

Thus

$$\sigma_{K+nL} = B^n \sigma_K,$$ where $B = E_L \cdots E_1$.

Next, write $C^S = (C^Q)^{[0,1]}$ and $z \in C^S$ as $z = (z^{(0)}, z^{(1)}) \in (C^Q)^{[0,1]}$. The parity state transitions can now be rewritten as

$$\sigma_{k+1} = M^{(k+1)} \begin{pmatrix} \sigma(0)_k \\ \sigma(1)_k \end{pmatrix},$$

where

$$\sigma(i)_k(\mathcal{e}) = \sigma_k(i, \mathcal{e}) \text{ for } \mathcal{e} \in \mathbb{Z}_Q$$

and

$$M^{(k+1)} = \Psi^{(k+1)}(\rho_{\mathcal{e}_k}) = \begin{pmatrix} \Psi^{(k+1)}_{(0,0)}(\rho_{\mathcal{e}_k}) & \Psi^{(k+1)}_{(0,1)}(\rho_{\mathcal{e}_k}) \\ \Psi^{(k+1)}_{(1,0)}(\rho_{\mathcal{e}_k}) & \Psi^{(k+1)}_{(1,1)}(\rho_{\mathcal{e}_k}) \end{pmatrix}$$

(17)

$$= \begin{pmatrix} p_{n_k+1}(\rho_{\mathcal{e}_k}) & q_{n_k+1}(\rho_{\mathcal{e}_k}) \\ p_{n_k+1-1}(\rho_{\mathcal{e}_k}) & q_{n_k+1-1}(\rho_{\mathcal{e}_k}) \end{pmatrix}$$

with $\rho_{\mathcal{e}} \in M_{\mathbb{Z}_Q \times \mathbb{Z}_Q}(C)$ defined by

$$\rho_{\mathcal{e}} z(\delta) := z(\delta + \mathcal{e})$$

and

$$\Psi^{(k)}_{(0,0)}, \Psi^{(k)}_{(0,1)}, \Psi^{(k)}_{(1,0)}, \Psi^{(k)}_{(1,1)}, p_v, q_v$$

are polynomials given by

$$p_v(x) := \sum_{j=1}^{v-1} x^{j-1},$$

$$q_v(x) := x^{v-1} \text{ and}$$

$$\Psi^{(k+1)}(x) := \begin{pmatrix} \Psi^{(k+1)}_{(0,0)}(x) & \Psi^{(k+1)}_{(0,1)}(x) \\ \Psi^{(k+1)}_{(1,0)}(x) & \Psi^{(k+1)}_{(1,1)}(x) \end{pmatrix} = \begin{pmatrix} p_{n_k+1}(x) & q_{n_k+1}(x) \\ p_{n_k+1-1}(x) & q_{n_k+1-1}(x) \end{pmatrix}.$$

Set $\gamma_r = e^{2\pi r}$ and let $\mathcal{e}_r \in C^Q$ be given by

$$(\mathcal{e}_r)_s := \gamma_{rs}$$
for $0 \leq r \leq Q - 1$ and $1 \leq s \leq Q$.

Since $\vec{e}_s \perp \vec{e}_t \forall s, t \in \mathbb{Z}_Q, s \neq t$, we have that $(\vec{e}_r : 0 \leq r \leq Q - 1)$ form an orthogonal basis for $\mathbb{C}^Q$ and

$$
\text{Span} \{ \vec{e}_r : 1 \leq r \leq Q - 1 \} = \mathbb{C}^Q = \{ \vec{v} = (v_h) \in \mathbb{C}^Q : \sum_{h=0}^{Q-1} v_h = 0 \}.
$$

Moreover,

$$
\rho_r \vec{e}_r = \gamma_{r\epsilon} \vec{e}_r.
$$

Next, define the bracket $[\cdot, \cdot] : \mathbb{C}^{[0,1]} \times \mathbb{C}^Q \to \mathbb{C}^S = (\mathbb{C}^Q)^{[0,1]}$ by

$$
[\vec{c}, \vec{z}] := \begin{pmatrix} c_0 \vec{z} \\ c_1 \vec{z} \end{pmatrix},
$$

where $\vec{c} = (c_0, c_1)$. It follows from (5) that

$$
M^{(k+1)}[\vec{c}, \vec{e}_r] = \Psi^{(k+1)}(\rho_{\epsilon_1})[\vec{c}, \vec{e}_r] = [\Psi^{(k+1)}(\gamma_{\epsilon_1}) \vec{c}, \vec{e}_r].
$$

To summarize, letting for $0 \leq r \leq Q - 1$,

$$
V_r := \{ [\vec{c}, \vec{e}_r] : \vec{c} \in \mathbb{C}^{[0,1]} \},
$$

then

$$
\bigoplus_{r=0}^{Q-1} V_r = (\mathbb{C}^Q)^{[0,1]} \text{ and } B V_r = V_r \ (0 \leq r \leq Q - 1).
$$

**Eigenvalue Lemma.** For $1 \leq r \leq Q - 1$, all the eigenvalues of $B|_{V_r}$ are roots of unity.

**Proof.** We have that $B|_{V_0}$ is a product of integer matrices of the form $\begin{pmatrix} N & 1 \\ N-1 & 1 \end{pmatrix}$ with $N \in \mathbb{N}$; we have $N \geq 2$ for at least one of these matrices. Therefore $B|_{V_0}$ is a positive matrix with integer coefficients and unit determinant. It follows that the characteristic polynomial of $B|_{V_0}$ is an integer polynomial of form $z^2 - Jz + 1$ for some $J \in \mathbb{N}$ (and that $B|_{V_0}$ is hyperbolic).

For each $1 \leq r \leq Q - 1$,

$$
| \det B|_{V_r} | = | \det \Psi^{(k+1)}(\gamma_{\epsilon_1}) | = 1.
$$

We claim first that no $B|_{V_r}$ $(1 \leq r \leq Q - 1)$ is hyperbolic. If this were not the case for $1 \leq r \leq Q - 1$, there would be $\lambda > 1$ and a rational cocycle $\Phi(\neq 0) \perp \mathbb{C}$ with $\langle \Phi, \vec{e}_r \rangle \neq 0$ giving rise to either

- $||\sigma_{K+Ln}|| \gg \lambda^n$ which is impossible by the Denjoy-Koksma estimate, or
- $||\sigma_{K+Ln}|| \ll \frac{1}{\lambda^n}$ which is impossible by Theorem 1.

To continue, since $B$ is an integer matrix, $\det(B - z \mathbb{I})$ is a polynomial with integer coefficients. It follows that

$$
\det(B - z \mathbb{I})|_{V_0} = \frac{\det(B - z \mathbb{I})}{\det(B - z \mathbb{I})|_{V_0}}
$$

is also a polynomial with integer coefficients. As shown above, all its roots are of unit modulus. By Kronecker’s theorem ([15]), all these roots are roots of unity.

\[
\]

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Proof of the Displacement Lemma. Let \(|\gamma_j : j \in \mathcal{J}\) be the collection of eigenvalues of \(B_{V^1_j}\) counting multiplicity which are all roots of unity. Let \(V_j\) be the corresponding Jordan subspace, then by the above,
\[
\dim V_j = 2.
\]
We may extend the period in (16) so that \(\gamma_j = 1 \forall j \in \mathcal{J}\). For each \(j \in \mathcal{J}\) let \((e_j(j) : j = 1, 2)\) be the Jordan basis of \(V_j\). For \(j \in \mathcal{J}\), \(x = x_1 e_1(j) + x_2 e_2(j)\) and \(N \geq 1\), we have that
\[
B^N x = N x_1 e_1(j) + x_2 e_2(j).
\]
Thus for \(\Phi : \mathbb{Z}_Q \to \mathbb{R}^d\) and \(1 \leq k \leq d\),
\[
\sigma_{K + Ln}^{(k)} = B^N \sigma_K^{(k)} = \sum_{j \in \mathcal{J}} N \langle \sigma_K^{(k)}, e_1(j) \rangle e_1(j) + \langle \sigma_K^{(k)}, e_2(j) \rangle e_2(j) =: c^{(k)} + ND^{(k)}.
\]
This proves the lemma. \(\Box\)

In the sequel, we’ll also need the following.

**Positivity Proposition.** By possibly extending the period in (16), we may ensure that \(B_{s,t} > 0\) \(\forall s, t \in S\).

**Remark 2.** Evidently \(E(a(F_{K+L}^{K+L})_{s,t}) = \prod_{F_{K+L}^{K+L}}(0) > 0\) iff \(B_{s,t} > 0\). Recall the assumption as in the subsection on subspace decompositions and eigenvalues, that, the parity sequence \((\epsilon_k : k \geq 1)\) is given by
\[
(\epsilon_k : k \geq 1) = (\epsilon_1, \ldots, \epsilon_K, \bar{\eta_1}, \ldots, \bar{\eta_L}).
\]

**Proof.** It follows from (17) that
\[
E_{k+1} = \begin{pmatrix}
\sum_{j=1}^{m_{k+1}-1} \rho_{\eta_k}^{j-1} & \rho_{\eta_k}^{m_{k+1}-1} \\
\sum_{j=1}^{m_{k+1}-2} \rho_{\eta_k}^{j-1} & \rho_{\eta_k}^{m_{k+1}-2}
\end{pmatrix}.
\]
Choose \(1 \leq r \leq L\) such that \(m_r \neq 2\). A direct calculation shows
\[
B^2 > E_{r+1} E_r \geq \begin{pmatrix}
\rho_0 + \rho_{\eta_r} + \rho_{\eta_{r-1}} & D_1 \\
D_2 & D_3
\end{pmatrix}
\]
where \(D_1, D_2, D_3 \in M_{Q \times Q}(\mathbb{N}_0)\) are matrices where each row and column has at least one non-zero entry.

By the Parity Proposition, \(\eta_k\) and \(\eta_{k+1}\) generate the group \(\mathbb{Z}_Q\). Applying this to \(k = r - 1\), we get that there exists an \(N\) such that for all \(n \geq N\),
\[
(\rho_0 + \rho_{\eta_r} + \rho_{\eta_{r-1}})^n > 0,
\]
meaning all of its entries are positive. Thus \(B^{N+2} > 0\). This proves that \(B\) is aperiodic and irreducible and that by extending the period, we can ensure that \(B\) is a positive matrix. \(\Box\)
Asymptotic eventual periodicity and centering. Let $\alpha \in \mathbb{Q}\mathbb{D}$ and $\varphi$ be a step function with rational discontinuities with associated RAT sequence $(F_k : k \geq 1)$ and ARW $(X^{(k)} : k \geq 1)$.

By the Displacement Lemma, we may suppose that

\[ [n_1, n_2, \ldots] = [n_1, \ldots, n_K, m_1, \ldots, m_L]; \]

\[ (e_k : k \geq 1) = (e_1, \ldots, e_K, \eta_1, \ldots, \eta_L), \sigma_{k+1} = M^{(k+1)} \sigma_k; \]

\[ (M^{(k)} : k \geq 1) = (M^{(1)}, \ldots, M^{(K)}, E_1, \ldots, E_L) \text{ and } \sigma_{K+Ln} = e + n\varphi. \]

Next, we examine the asymptotic, distributional periodicity of the RAT sequence and, in particular, that of the elementary random vector sequence:

\[ \mathcal{X}^{(k)} = ((\xi_i^{(k)}, \eta_i^{(k)}, U_i^{(k)}(i) : i = 0, 1)) \]

as in (15).

Elementary Periodic Approximation Lemma. There are constants $\lambda$, $M > 1$ and, for each $1 \leq r \leq L$ there is a random vector

\[ \mathcal{X}^{(r)} := (\mathcal{Y}_i^{(r)}, \mathcal{E}_i^{(r)}, \mathcal{U}_i^{(r)}(i) : i = 0, 1) \in RV(([0, 1] \times \mathbb{Z}_Q \times \mathbb{N}_0)^{0, 1}) \]

so that

\[ \text{dist}(\xi_i^{(K+Ln+r)}, \xi_i^{(K+Ln+r)}(i) : i = 0, 1) \leq \epsilon, \] \[ \text{dist}(\mathcal{E}_i^{(r)}, \mathcal{U}_i^{(r)}(i) : i = 0, 1) \leq \epsilon \] \[ |P(\mathcal{X}^{(K+Ln+r)} = Z) - P(\mathcal{X}^{(r)} = Z)| \leq \frac{M}{\lambda^n} \] \forall \ n \geq 1, Z \in ([0, 1] \times \mathbb{Z}_Q \times \mathbb{N}_0)^{0, 1}.

Proof. We have that

\[ \mathcal{X}^{(K+Ln+r)} = B(m_r)B(m_{r-1}) \cdots B(m_1)C^\alpha \mathcal{X}, \]

where

\[ B(m) := \begin{pmatrix} m-1 & 1 \\ m-2 & 1 \end{pmatrix} \] \& \[ C := B(m_L)B(m_{L-1}) \cdots B(m_1). \]

Now $\det C = \prod_{i=1}^L \det B(m_i) = 1$ and each $C_{i,j} \in \mathbb{N}$, so $C$ is hyperbolic, with eigenvalues $\lambda > 1$ and $1/\lambda$. Moreover, there exists $c_r(i)$ (for $i = 0, 1$ and $0 \leq r \leq L$) with $c_L = \lambda c_0$ so that

\[ \ell^r_{K+Ln+r}(i) = c_r(i)\lambda^n + O\left(\frac{1}{\lambda^n}\right), \]

whence

\[ c_{r+1}(i) = \frac{\ell^r_{K+Ln+r+1}(i)}{\lambda^n} + O\left(\frac{1}{\lambda^n}\right) \]

\[ = \frac{1}{\lambda^n} (m_{r+1} - i - 1) \ell^r_{K+Ln+r}(0) + \ell^r_{K+Ln+r}(1) + O\left(\frac{1}{\lambda^n}\right) \]

\[ = (m_{r+1} - i - 1)c_r(0) + c_r(1) + O\left(\frac{1}{\lambda^n}\right). \]

Define random variables $\mathcal{Y}_i^{(r+1)}$ (for $i = 0, 1$, and $1 \leq r \leq L$) by

\[ P(\mathcal{Y}_i^{(r+1)} = 0) = \frac{(m_{r+1} - i - 1)c_r(0)}{c_{r+1}(i)} \] \text{ and } \[ P(\mathcal{Y}_i^{(r+1)} = 1) = \frac{c_r(1)}{c_{r+1}(i)} = 1 - P(\mathcal{Y}_i^{(r+1)} = 0). \]
It follows that for $i, j = 0, 1$ and $1 \leq r \leq L$,
$$P(\gamma_i^{(K+Ln+r)} = j) = P(\Omega_i^{(r)} = j) + O(\frac{1}{\lambda^n}).$$

Next, we observe that for $n \geq 1$, $1 \leq r \leq L$, $j = 0, 1$, the distribution of $\epsilon_i^{(K+Ln+r)}$ given $\gamma_i^{(K+Ln+r)}$ does not depend on $n \geq 1$ and define:
$$P((\epsilon_i^{(r+1)} = \Delta) || \Omega_i^{(r+1)} = 0)) = \frac{\#m(\eta_i, \Delta) \cap [1, m_{r+1} - i - 1]}{m_{r+1} - i - 1} \quad (\Delta \in \mathbb{Z}_Q)$$
and
$$P((\epsilon_i^{(r+1)} = (m_{r+1} - i - 1)\eta_i) || \Omega_i^{(r+1)} = 1)) = 1.$$  

Analogously, $u^{(L+Ln+r+1)}(i)$ has a conditional distribution independent of $n$ and we define
$$\Upsilon^{(r+1)}(i) := \begin{cases} 
\text{uniform on } m(\eta_i, \epsilon_i^{(r+1)}) \cap [1, m_{r+1} - i - 1] & \text{if } \Omega_i^{(r+1)} = 0 \\
\epsilon_i^{(r+1)} & \text{if } \Omega_i^{(r+1)} = 1.
\end{cases}$$

The random vectors $\mathcal{X}^{(r)} \in \text{RV}((\{0, 1\} \times \mathbb{Z}_Q \times \mathbb{N}_0)^{\{0,1\}})$, $1 \leq r \leq L$ where (18)
$$\mathcal{X}^{(r)} := (\Omega_i^{(r)}, \epsilon_i^{(r)}, \Upsilon^{(r)}(i) : i = 0, 1)$$
are as advertised by construction. $\square$

**RAT Periodic Approximation Lemma.** There are random variables
$$a \in \text{RV}((M_{S \times S}(\mathbb{Z})), \nu, \omega \in \text{RV}((\mathbb{R}^d)^S))$$
so that if
$$H^{(n)}(x) = ax + \nu + n\omega \text{ for } x \in (\mathbb{R}^d)^S,$$
then $\exists M > 0$ so that $\forall \ n \geq 1, f \in M_{S \times S}(\mathbb{Z}) \times (\mathbb{R}^d)^S$,
$$|P(\tilde{H}_n = f) - P(H^{(n)} = f)| \leq \frac{M}{\lambda^n},$$
$$P(\tilde{H}_n = f) > 0 \iff P(H^{(n)} = f) > 0,$$
where
$$\tilde{H}_n := \tilde{F}_n^{K+Ln+L}.$$  

**Proof.** Let $\mathcal{X}^{(r)}$ $(1 \leq r \leq L)$ be independent, each distributed as in (18). Define
$$\mathcal{E}^{(r+1)}_{(i,c)} := (\Omega_i^{(r+1)}, \epsilon_i^{(r+1)}).$$
Then, since
$$\mathcal{E}^{(K+Ln+r+1)}_{(i,c)} := (\gamma^{(K+Ln+r+1)}_i, \epsilon_i^{(K+Ln+r+1)}),$$
we have by the Elementary Periodic Approximation Lemma,
$$\sup_{s, t \in S} \frac{1}{S} \left| P(\mathcal{E}^{(K+Ln+r+1)}_s = t) - P(\mathcal{E}^{(K+Ln+r+1)}_s = t) \right| = O(\frac{1}{\lambda^n}).$$

To study the random variables $W^{(K+Ln+r)}_s$, we’ll need formulae for the cumulative displacements.
Using the Displacement Lemma, for $1 \leq r \leq L$

\[
s_{K+Ln+r}(K,\epsilon) = \sum_{v=1}^{K} \sigma_{K+Ln+r}(0,\epsilon + (v-1)\epsilon)_{K+Ln+r})
\]
\[
= \sum_{v=1}^{K} (c_{r} + n\alpha_{r})(0,\epsilon + (v-1)\epsilon_r)
\]
\[
= \mathcal{C}_{r}(K,\epsilon) + n\mathcal{D}_{r}(K,\epsilon),
\]
where

\[
\mathcal{C}_{r}(K,\epsilon) := \sum_{v=1}^{K} c_{r}(0,\epsilon + (v-1)\epsilon_r)
\]
\[
\mathcal{D}_{r}(K,\epsilon) := \sum_{v=1}^{K} \alpha_{r}(0,\epsilon + (v-1)\epsilon_r).
\]

It follows that

\[
\text{dist}(W_{(i,\epsilon)}^{(K+Ln+r+1)}(K+Ln+r+1)) = \text{dist}(s_{K+Ln+r+1}(U_{(K+Ln+r+1)}(i)-1,\epsilon))\|_{y}^{(K+Ln+r+1)}
\]
\[
= \text{dist}(s_{K+Ln+r+1}(U_{(r+1)}(i)-1,\epsilon))\|_{y}^{(r+1)}
\]
\[
= \text{dist}(\mathcal{C}_{r+1}(U_{(r)}(i)-1,\epsilon) + n\mathcal{D}_{r+1}(U_{(r)}(i)-1,\epsilon))\|_{y}^{(r+1)}.
\]

Now let $G_{r}^{(n)}$ ($1 \leq r \leq L$, $n \geq 1$) be the RATs defined by

\[
G_{r}^{(n)}(x)_{(i,\epsilon)} := x^{(r)}_{(i,\epsilon)} + \mathcal{C}_{r}(U_{(r)}(i)-1,\epsilon) + n\mathcal{D}_{r}(U_{(r)}(i)-1,\epsilon),
\]
for all $x \in (\mathbb{R}^{d})^{S}$. Then there is a constant $M > 0$ so that $\forall f \in M_{S \times S}(\mathbb{Z}) \times (\mathbb{R}^{d})^{S}$,

\[
(21) \quad |P(F_{K+Ln+r} = f) - P(G_{r}^{(n)} = f)| \leq \frac{M}{\lambda^{n}}.
\]

Finally, let

\[
H^{(n)} := G_{L}^{(n)} \circ G_{L-1}^{(n)} \cdots \circ G_{2}^{(n)} \circ G_{1}^{(n)}.
\]

This has the form

\[
H^{(n)}(x) = ax + v + nw
\]
for all $x \in (\mathbb{R}^{d})^{S}$, where $a \in \text{RV}(M_{S \times S}(\mathbb{Z}))$, $v, w \in \text{RV}((\mathbb{R}^{d})^{S})$.

It follows from (21) that $H^{(n)}$ satisfies (19) and (20).

**Coupling.** It follows that there exists a probability space $(\Omega,\mathcal{A},P)$ on which the independent random vectors $(H^{(n)},\tilde{F}_{n})$ ($n \geq 1$) can be defined so that

\[
P(H^{(n)} \neq \tilde{F}_{n}) \leq \frac{M}{\lambda^{n}}.
\]

Consider the ARW

\[
Y_{J}^{(n)} := H_{J+1}^{n}(X^{(K+Ln)}) \quad (n > J).
\]
\textbf{AR\textsuperscript{W} Periodic Approximation Lemma.} There is a constant \(M > 1\) so that for all \(n > J\),

\begin{equation}
|P(Y_J^{(n)} \neq X^{(K+Ln)})| \leq \frac{M}{\lambda^j};
\end{equation}

and

\begin{equation}
\sup_{n > j} |E(Y_J^{(n)v}) - E(X^{(K+Ln)v})| \xrightarrow{J \to \infty} 0 \ \forall \ v \geq 1.
\end{equation}

\textbf{Proof of (22).}

\[ P(Y_J^{(n)} \neq X^{(K+Ln)}) \leq P(\tilde{F}_n^J \neq H_n^J) \leq \sum_{j=J}^n P(H^{(t)} \neq \tilde{F}_j) \leq \sum_{j=J}^n \frac{M}{\lambda^j} = O\left( \frac{1}{\lambda^j} \right). \]

\textbf{Proof of (23).} For fixed \(n \geq 1\) and a measurable function \(g : \Omega \to \mathbb{R}^S\), for which \(|g|^v\) is integrable, let

\[ \|g\|^v := E(|g|^v)^\frac{1}{v}, \]

then \(\|\cdot\|^v\) is a norm. Next, it follows from the RAT periodic approximation lemma that

\[ \|b(\tilde{F}_n)^\infty\|, \|b(H\,(n)\|^\infty = O(n), \]

whence

\[ \|Y_J^{(n+1)}\|^\infty, \|X^{(K+L(n+1))}\|^\infty = O(n^2). \]

Thus, for some \(M' > 0\),

\[ \|Y_J^{(n+1)} - X^{(K+L(n+1))}\|^v = \|H^{(n+1)}(Y_J^{(n)}) - \tilde{F}_{n+1}(X^{(K+Ln)})\|^v \]

\[ \leq \|b(H^{(n+1)}) - b(\tilde{F}_{n+1})\|^v + \|(a(H^{(n+1)}) - a(\tilde{F}_{n+1}))X^{(K+Ln)}\|^v \]

\[ + \|a(H^{(n+1)})(Y_J^{(n)}) - X^{(K+Ln)}\|^v \]

\[ \leq \|Y_J^{(n)} - X^{(K+Ln)}\|^v + \frac{M'n^2}{\lambda^2}. \]

Thus possibly increasing \(M\),

\[ \|Y_J^{(n)} - X^{(K+Ln)}\|^v \leq \sum_{j=J}^n \frac{Mj^2}{\lambda^j} \xrightarrow{J \to \infty} 0 \]

and (23) follows. \(\square\)

\textbf{Corollary.} There are constant vectors \(\mu, \xi, \xi_J \in (\mathbb{R}^d)^S \ (J \geq 1)\) and \(0 < \rho < 1\) so that

\[ E(Y_J^{(n)}) = n\mu + \xi_J + O(\rho^n) \ \forall \ J \geq 1, \xi_J \xrightarrow{J \to \infty} \xi, \]

and

\[ E(X^{(K+Ln)}) = n\mu + \xi + O(\rho^n). \]
Proof. We have
\[ Y_f^{(n+1)} = H^{(n+1)}(Y_f^{(n)}), \]
where
\[ H^{(n)}(x) = a^{(n)}x + v^{(n)} + nw^{(n)} \]
and \((a^{(n)}, v^{(n)}, w^{(n)}: n \geq 1)\) are independent and identically distributed.

It follows as in [1] that
\[ E(Y_f^{(n)}) = E(a)^n E(X^{(K+L)}) + \sum_{k=1}^{n} E(a)^{n-k} E(v + kw). \]

By the Positivity Proposition, by possibly extending the period in (16), we may ensure that
\[ E(a(H^{(n)})) = \Pi_{H^{(n)}(0)} = 0 \]
is a simple, dominant eigenvalue with eigenvector \( \mathbb{1} \in C^S \).

Suppose that \( \pi \in \mathbb{R}_+^S \) satisfies \( \langle \pi, \mathbb{1} \rangle = 1 \) and \( E(a)^* \pi = \pi \) (where \( A^* \) is the transpose of the matrix \( A \)).

Let \( N: C^S \to \mathbb{C} \cdot 1, N(x) := \langle \pi, x \rangle 1, \) then
\[ E(a)^n x = N(x) 1 + R^n x, \]
where \( Rx := E(a)(x - N(x)) \) and \( \exists 0 < \rho < 1 \) so that \( \|R^n\| = O(\rho^n) \).

We claim next that \( \exists \mu, \xi_j \in (\mathbb{R}^d)^S \) so that
\[ E(Y_f^{(n)}) = \frac{n(n+1)}{2} N(E(v)) + n\mu + \xi_j + O(t\rho^n). \]

Indeed, by (24),
\[ E(Y_f^{(n)}) = E(a)^n E(X^{(K+L)}) + \sum_{k=1}^{n} E(a)^{n-k} E(v + kw) \]
\[ = N(E(X^{(K+L)})) + (n-1)N(E(v)) + \frac{n(n-1)}{2} N(E(v)) + \mathcal{E}(n), \]
where
\[ \mathcal{E}(n) := R^n E(X^{(K+L)}) + \sum_{k=1}^{n} R^{n-k} E(v + kw) \]
\[ = \sum_{n=1}^{n} R^{n-k} E(v + kw) + O(\rho^n) \]
\[ = \sum_{n=0}^{k-1} (n-k)R^k(v) + \sum_{k=0}^{n-1} R^k E(v) + O(\rho^n) \]
\[ = t \sum_{k=0}^{\infty} R^k(v) - \sum_{k=0}^{\infty} kR^k E(v) + \sum_{k=0}^{\infty} R^k E(v) + O(n\rho^n). \]

To obtain the expansion for \( E(Y_f^{(n)}) \) from (25) (with enlarged \( \rho \)), it suffices to show that \( N(E(v)) = 0 \). This will follow from the Denjoy-Koksma estimate. By (23), we have
\[ |E(X^{(K+Ln)}) - E(Y_f^{(n)})| = O(1). \]

Thus, if \( N(E(v)) \neq 0 \), then by (25), \( |E(X^{(K+Ln)})| \approx n^2 \) contradicting the Denjoy-Koksma estimate that \( |E(X^{(K+Ln)})| = O(n) \).
The expansion for $E(X^{(K+Ln)})$ follows. \hfill \Box

**Centering.** As in [1], set $(\hat{X}^{(n)} : n \geq 1)$ be the centered ARW defined by

$$\hat{X}^{(n)} := X^{(n)} - E(X^{(n)})$$

and let $(\mathcal{F}_n : n \geq 1)$ be the independent RAT sequence so that

$$\hat{X}^{(K+Ln)} = \mathcal{F}_1^n(\hat{X}^{(K)}).$$

**ARW Centering Lemma.** There is a centered, independent, identically distributed RAT sequence $(\mathcal{H}_n : n \geq 1)$ and $0 < \rho < r < 1$ so that if for $J \geq 1$, $(Z^{(n)}_J : n > J)$ is defined by

$$Z^{(n)}_J := \mathcal{H}_J^n(\hat{X}^{(K+Ln)}),$$

then

(i) \quad \lim_{J \to \infty} \sup_{n > J} |E(Z^{(n)}_J) - E(\hat{X}^{(K+Ln)})| = 0 \forall \ n \geq 1; \quad (ii) \quad P(\exists \ n \geq J, |Z^{(n)}_J - \hat{X}^{(K+Ln)}| \geq r^n) = O(\rho^J) \text{ as } J \to \infty.

**Proof.** Define

$$\hat{Y}^{(n)}_J := Y^{(n)}_J - E(Y^{(n)}_J) =: Y^{(n)}_J - c_n.$$

As in [1], $(\hat{Y}^{(n)}_J : n \geq 1)$ is given by the centered RAT sequence $(\mathcal{G}_n : n \geq 1)$, where $a(\mathcal{G}_n) = a^{(n)}$ and

$$b(\mathcal{G}_{n+1}) = a^{(n+1)}c_n - c_{n+1} + u^{(n+1)} + nw^{(n+1)} = (a^{(n+1)} - I)\xi_J + u^{(n+1)} - \mu + n[(a^{(n+1)} - I)\mu + w^{(n+1)}] + O(\rho^n) =: v^{(n+1)} + nw^{(n+1)} + O(\rho^n),$$

where $(a^{(n)}, u^{(n)}, w^{(n)})$ are independent, identically distributed random variables and $I$ is the identity matrix.

By Remark 2, $E(a)$ is irreducible and aperiodic. Thus, by the Variance Lemma in [1], for each $s \in S$,

$$E((\hat{Y}^{(n)}_J)^2) \geq \sum_{k=1}^n E(b(\mathcal{G}_k)^2)$$

$$\geq \sum_{k=1}^n E((u'_s + k\omega'_s + O(\rho^k))^2)$$

$$\geq nE(\omega'_s)^2 + n^2E(u'_s\omega'_s) + \frac{n^3}{3}E(\omega'_s^2).$$

By (23),

$$|E((\hat{Y}^{(n)}_J)^2) - E((\hat{X}^{(K+Ln)^2})_J)| = O(1)$$

whence, by the Denjoy-Koksma estimate, $E((\hat{Y}^{(n)}_J)^2) \ll n^2$ and $w^{(n+1)} \equiv 0$.

Thus

$$b(\mathcal{G}_{n+1}) = (a^{(n+1)} - I)\xi_J + u^{(n+1)} - \mu + O(\rho^n).$$
Accordingly, define \( \mathcal{H}_n \) by
\[
a(\mathcal{H}_n) := a^{(n)} \land b(\mathcal{H}_n) := (a^{(n)} - I)\xi + v^{(n)} - \mu.
\]
The lemma follows. \( \square \)

6. Spectral Theory and Theorem 2

By the Perron-Frobenius theorem, 1 is a simple, dominant eigenvalue of \( \Pi_{\mathcal{H}}(0) \) (where \( \mathcal{H} := \mathcal{H}_1 \) and \( \Pi_{\mathcal{H}} \) is the RAT-CF as defined by (14) with right eigenvector \( e \in \mathbb{R}^S_+ \) and left eigenvector \( \pi \in \mathbb{R}^S_+ \) satisfying \( \langle \pi, e \rangle = 1 \).

By the Implicit Function theorem, \( \exists r = r_{\mathcal{H}} > 0 \) and smooth functions
\[
\lambda : (-r, r)^d \to \mathbb{C}, \quad v : (-r, r)^d \to \mathbb{C}^S, \quad \pi : (-r, r)^d \to \mathbb{C}^S
\]
so that
\[
\begin{align*}
\langle \pi, 0 \rangle &= \langle \pi, v(0) \rangle = 1; \\
\lambda(0) &= 1, \quad v(0) = e \land \pi(0) = \pi;
\end{align*}
\]
\[
\text{for each } 1 \leq k \leq \infty, \quad \theta \in (-r, r)^d, \quad \lambda(\theta) \text{ is a simple, dominant eigenvalue of } \\
\Pi_{\mathcal{H}}(\theta) \text{ with eigenvector } v(\theta) \text{ and left eigenvector } \pi(\theta).
\]

As in [9], consider the principal projections \( N(\theta) : \mathbb{C}^S \to \mathbb{C}^S \) defined by
\[
N(\theta)x := \langle \pi(\theta), x \rangle v(\theta).
\]
Then, possibly reducing \( r_{\mathcal{H}} > 0 \), we ensure \( \exists 0 < \rho < 1 \) so that
\[
\Pi_{\mathcal{H}}(\theta)^n - \lambda(\theta)^n N(\theta) = R(\theta)^n = O(\rho^n) \text{ uniformly in } |\theta| \leq r_{\mathcal{H}},
\]
where \( R(\theta) := \Pi_{\mathcal{H}}(\theta)(I - \lambda(\theta))N(\theta) \).

The proofs of our limit theorems in the sequel use the following lemma.

**Lemma** (Taylor expansion of the eigenvalue). Under the assumptions of Theorem 2,
\[
\lambda(\theta) = 1 - \langle D\theta, \theta \rangle + o(\|\theta\|^2)
\]
as \( \theta \to 0 \), where \( D \in M_{d \times d}(\mathbb{C}) \) is positive definite.

**Proof.** We have
\[
\lambda(\theta) = 1 + \langle \nabla \lambda(0), \theta \rangle + \langle d^2 \lambda(0)\theta, \theta \rangle + o(\|\theta\|^2)
\]
where \( d^2 \lambda(\theta) \) is the matrix of second partial derivatives:
\[
d^2 \lambda(\theta)_{h, j} := \frac{\partial^2 \lambda}{\partial \theta_h \partial \theta_j}(\theta).
\]
and we must show that \( \nabla \lambda(0) = 0 \) and that \( D := -d^2 \lambda(0) \) is positive definite.

Fix \( \sigma \in \mathbb{R}^d, \|\sigma\| = 1 \) and write, for differentiable \( f : \mathbb{R}^d \to \mathbb{C}, \)
\[
D^k_\sigma f(\theta) := \frac{\partial^k f(\theta)}{\partial t^k} |_{t=0} = \frac{\partial^k f(\theta + t\sigma)}{\partial t^k} |_{t=0} \quad \text{and} \quad D^2_\sigma f(\theta) = \langle d^2 f(\theta)\sigma, \sigma \rangle.
\]
Accordingly, it suffices to show that for each \( \sigma \in \mathbb{R}^d, \|\sigma\| = 1, \)
whence

Thus

Differentiating

Proof of \( \Pi(\mathcal{H})(0) = 0 \).

Proof of (i). Since \( \langle \pi(0), v(\theta) \rangle = 1 \), we have that \( D_\sigma v(\theta) \perp \pi(0) \). Also

whence

and in particular

Thus

Proof. Differentiating \( \Pi(\mathcal{H})(\theta) v(\theta) = \lambda(\theta) v(\theta) \) at 0:

By \( \Pi(\mathcal{H})(0) D_\sigma v(0) = D_\sigma \lambda(0) \perp + D_\sigma v(0) \).

Thus \( D_\sigma v(0) = c \perp \) for some \( c \in \mathbb{C} \). But \( D_\sigma v(0) \perp \pi(0) \), and so

\( \overline{c} = \langle \pi(0), D_\sigma v(0) \rangle = 0 \).

Proof. Differentiating \( \Pi(\mathcal{H})(\theta) v(\theta) = \lambda(\theta) v(\theta) \) twice at 0 in direction \( \sigma \):

By (i) and \( \Pi(\mathcal{H})(0) D_\sigma v(0) = D_\sigma \lambda(0) \perp + D_\sigma v(0) \).

By (i) and \( \Pi(\mathcal{H})(0) D_\sigma v(0) = D_\sigma \lambda(0) \perp + D_\sigma v(0) \)
and in particular
\[ D_0^2(\lambda)(0) = \langle \pi(0), D_0^2(\lambda)(0) \rangle \]
\[ = \langle \pi(0), D_0^2(\Pi, \mathcal{H})(0) \rangle + \langle \pi(0), \Pi, \mathcal{H}(0) \rangle D_0^2 \nu(0) \]
\[ = \langle \pi(0), D_0^2(\Pi, \mathcal{H})(0) \rangle =: \Pi, \mathcal{H}(0) \rangle \pi(0) = \pi(0). \]

\[ \square \]

**Proof of (ii).** For fixed \( s \in S \):
\[(D_0^2(\Pi, \mathcal{H})(0) \rangle_s = - \sum_{i \in S} P_\mathcal{H}(\mathcal{H}) = t) E((\langle \sigma, b_s(\mathcal{H}) \rangle^2 || \mathcal{H} = t)
\[ = - E((\langle \sigma, b_s(\mathcal{H}) \rangle^2),\]
whence by \( \bullet \),
\[(D_0^2(\lambda)(0) = \langle \pi(0), D_0^2(\Pi, \mathcal{H})(0) \rangle \]
\[ = - \sum_{i \in S} \pi_i(0) E((\langle \sigma, b_s(\mathcal{H}) \rangle^2) \leq 0 \]
with equality iff \( \langle \sigma, b_s(\mathcal{H}) \rangle = 0 \forall \ s \in S \).
Next recall that \( Z^{(n)} := \mathcal{H}_1^{n}(0) = X^{(K+Ln)} - n\mu + O(1) \).
If \( \langle \sigma, b_s(\mathcal{H}) \rangle = 0 \forall \ s \in S \), then, taking \( s = (0,0) \) we have
\[ \sup_n |\langle \sigma, Z_s^{(n)} \rangle| < \infty \implies \sup_n |\langle \sigma, X_s^{(K+Ln)} - n\mu_s \rangle| =: W < \infty,\]
whence
\[ |\langle \sigma, \varphi_j(0) - n\mu_j \rangle| \leq W \forall \ n \geq 1, \ 1 \leq j \leq \ell_{K+Ln}(0). \]
It follows from this that \( \mu_s = 0 \) and that \( \langle \sigma, \varphi \rangle \) is a coboundary.
In view of the assumption that \( \text{span}_{\mathbb{R}} \varphi(\mathbb{T}) \) has full dimension, this contradicts Theorem 1 and completes the proof of the lemma.

\[ \square \]

**Proof of Theorem 2.** It suffices to prove that for fixed \( s \in S, \theta \in \mathbb{R}^d \)
\[ E(\exp[2\pi i (\theta, \frac{X^{(K+Ln)} - n\mu_s}{\sqrt{n}})]) = \exp[-\frac{(\theta, D\theta)}{2}], \]
By asymptotic, eventual periodicity \( \exists \rho \in (0,1) \) so that for any fixed \( J, s \in S \) and all \( n \geq 1, \)
\[ E(\exp[2\pi i (\theta, \frac{X^{(K+Ln)(+J+n\mu_s)}}{\sqrt{n}})]) = E(\exp[2\pi i (\theta, \frac{Z^{(n)}}{\sqrt{n}})]) + O_n(\rho^J) \]
\[ = \lambda(\frac{\theta}{\sqrt{n}})^n E(\exp[2\pi i (\theta, \frac{X^{(K+Ln)} - J\mu_s}{\sqrt{n}})]) + O_n(\rho^J), \]
where \( |O_n(\rho^J)| \leq M \rho^J \) for a constant \( M \) independent of \( n \). By the Taylor expansion of the eigenvalue,
\[ \lambda(\frac{\theta}{\sqrt{n}})^n E(\exp[2\pi i (\theta, \frac{X^{(K+Ln)} - J\mu_s}{\sqrt{n}})]) \xrightarrow{n \to \infty} \exp[-\frac{(\theta, D\theta)}{2}] \forall \ J \geq 1. \]
To deduce (26) from this, let \( \varepsilon > 0 \) and choose \( J = J_\varepsilon \geq 1 \) so that
\[ |E(\exp[2\pi i (\theta, \frac{X^{(K+Ln)} - (J+n\mu_s)}{\sqrt{n}})]) - \lambda(\frac{\theta}{\sqrt{n}})^n E(\exp[2\pi i (\theta, \frac{X^{(K+Ln)} - J\mu_s}{\sqrt{n}})])| < \frac{\varepsilon}{2} \forall \ n \geq J \]
and then choose \( N = N_{J,k} > J \) so that
\[
|\lambda \left( \frac{\partial}{\partial n} \right)^n E(\exp[2\pi i (\theta, x_{k+1}^{(k+1)} - J\mu)]) - \exp[-\left( \frac{\theta, D\theta}{2} \right)]| < \frac{\varepsilon}{2} \quad \forall \ n > N.
\]
This implies (26).

7. THE WRLT AND THEOREM 3

Visits to zero and RATs. Recall that we assume \( Q \geq 1, \alpha \in [0,1] - Q, \) with \( \{Q\alpha\} = [n_1, n_2, \ldots] \).

Let \( \Phi : Z_Q \to Z^d \) satisfy \( \sum_{k \in Z_Q} \Phi(k) = 0 \) & \( \langle \Phi(Z_Q) \rangle = Z^d \) and define \( \varphi : T \to Z^d \) by
\[
\varphi(x) := \Phi([Qx])
\]
and \( T = T_{\alpha,\Phi} : T \times Z^d \to T \times Z^d \) by
\[
T(x, z) := ([x + \alpha], z + \varphi(x)).
\]
Let \( \Psi^N(x) := \#\{1 \leq n \leq N : \varphi_n(x) = 0\} \).

**Visit Lemma.** Let \( (X^{(k)} : k \geq 1) \) be the associated ARW, then
(a) \[
\int_0^1 \Psi x_k(0)(x) \, dx \geq \frac{\ell_k(1)^2}{3\ell_k(0)} \min_{\psi \in Z_Q} \int_{T^d} |E(e^{2\pi i (\theta, X^{(k)}(1,\epsilon)))}|^2 \, d\theta - \frac{1}{2},
\]
(b) \[
\|\Psi x_k(1)\|_\infty \leq 2\ell_k(0) \max_{\psi \in Z_Q} \int_{T^d} |E(e^{2\pi i (\theta, X^{(k)}(0,\epsilon)))}| \, d\theta.
\]

**Visit sets.** The visit set to \( v \in Z^d \) is
\[
K_v := \{ n \geq 1 : \varphi_n(0) = v \}
\]
and the visit distributions are the discrete measures \( U^{(i)}_k \) on \( Z^d \) defined by
\[
U^{(i)}_k(v) := \#(K_v \cap [1, \ell_k(i)]) \quad (k \geq 1, i = 0, 1).
\]
The auxiliary visit sets to \( v \in Z^d \) are
\[
K_k(i,\epsilon, v) := \{ 1 \leq j \leq \ell_k(i) : \Sigma_k(i,\epsilon)_j = v \}
\]
and the auxiliary visit distributions are the discrete measures \( V^i_k(i,\epsilon) \) on \( Z^d \) defined by
\[
V^i_k(i,\epsilon)(v) := \#(K_k(i,\epsilon, v)) \quad (k \geq 1, i = 0, 1).
\]
As above,
\[
K_k(0,0,v) = K_v \cap [1, \ell_k(0)] \& U^{00}_k = V^0_k(0,0).
\]

**Visit Sublemma.**
(a') \[
\int_0^1 \Psi x_k(0)(x) \, dx \geq \frac{1}{3\ell_k(0)} \min_{\psi \in Z_Q} \sum_{v \in Z^d} V_k(1,\epsilon)(v)^2 - \frac{1}{2};
\]
(b') \[
\int_0^1 \Psi x_k(1)(x)^N \, dx \leq \frac{2N}{\ell_k(1)} \max_{\psi \in Z_Q} \sum_{v \in Z^d} V_k(0,\epsilon)(v)^{N+1} \quad \forall N \geq 1.
\]
Proof. Fix \( N, k \geq 1 \) & \( i = 0, 1 \), \( \Psi_{\ell_k(i)}^N \) is a step function on \( \mathbb{T} \), whence Riemann integrable. Using the unique ergodicity of \( x \mapsto x + \alpha \),

\[
\ell_{k+r}(0) \int_0^1 \Psi_{\ell_k(i)}(x)^N \, dx \sim \sum_{j=1}^{\ell_{k+r}(0)} \Psi_{\ell_k(i)}((j \alpha))^N
\]

\[= \sum_{j=1}^{\ell_{k+r}(0)} \#(1 \leq m \leq \ell_k(i) : \phi_m((j \alpha)) = 0)^N\]

\[= \sum_{j=1}^{\ell_{k+r}(0)} \#(1 \leq m \leq \ell_k(i) : \phi_{m+j}(0) = \phi_j(0))^N\]

(27)

\[= \sum_{j=1}^{\ell_{k+r}(0)} \#(1 \leq m \leq \ell_k(i) : \phi_{m+j}(0) = \phi_j(0))^N\]

\[= \sum_{j=1}^{\ell_{k+r}(0)} \sum_{j=1}^{\ell_{k+r}(0)} \#(K \cap [j+1, j+\ell_k(i)])^N.\]

By [2, Theorem 2.1] and the orbit block transitions, for \( r \geq 1 \), \( \exists J = J_{r,k} \geq 1 \), \( 0 = m_1 < \cdots < m_J = \ell_{k+r}(0) \) and \( \eta_1, \ldots, \eta_{J-1} \in \mathbb{Z}_Q \), \( i_1, \ldots, i_{J-1} = 0, 1 \) so that

\[m_{j+1} - m_j = \ell_k(i_j) \quad \forall \ j, \quad [1, \ell_{k+r}(0)] = \bigcup_{j=1}^{J-1} (m_j, m_{j+1}]\]

and

(28) \((\phi_{m_{j+1}}(0), \phi_{m_{j+2}}(0), \ldots, \phi_{m_{j+1}}(0)) = \phi_{m_j}(0) + \Sigma_k(i_j, \eta_j)\).

Since \( m_{j+1} - m_j = \ell_k(i_j) \), it follows that

(29) \( \ell_{k+r}(0) \in [J \ell_k(1), J \ell_k(0)] \).

Also by (28), for fixed \( v \in \mathbb{Z}^d \),

\[K \cap (m_j, m_{j+1}] = m_j + K(i_j, \eta_j, v - \phi_m(0)).\]

Proof of (a'). We have for fixed \( v \in \mathbb{Z}^d \)
\[
\sum_{h \in [1, \ell_{k+r}(0)] \cap K_v} \#(K_v \cap [h+1, h+\ell_k(0)]) \\
= \sum_{j=1}^{J-1} \sum_{h \in [m_j, m_{j+1}] \cap K_v} \#(K_v \cap [h+1, h+\ell_k(0)]) \\
\geq \sum_{j=1}^{J-1} \sum_{h \in [m_j, m_{j+1}] \cap K_v} \#(K_v \cap [h+1, m_{j+1}]) \quad \forall \ h, j : h + \ell_k(0) \geq h + \ell_k(i_j) \geq m_{j+1} \\
\geq \sum_{j=1}^{J-1} \sum_{k=1}^{L_j-1} k \quad \text{where} \quad L_j := \#((m_j, m_{j+1}] \cap K_v) = V_k(i_j, \eta_j)(\nu - \varphi_{m_j}(0)) \\
= \sum_{j=1}^{J-1} \sum_{k=1}^{L_j-1} t(L_j) \quad \text{where} \quad t(x) := \frac{x(x-1)}{2} \\
= \sum_{j=1}^{J-1} \sum_{k=1}^{L_j-1} t(V_k(i_j, \eta_j)(\nu - \varphi_{m_j}(0))). \\
\]

Thus

\[
\sum_{v \in Z^d} \sum_{j \in [1, \ell_{k+r}(0)] \cap K_v} \#(K_v \cap [j+1, j+\ell_k(0)]) \\
\geq \sum_{v \in Z^d} \sum_{j=1}^{J-1} t(V_k(i_j, \eta_j)(\nu - \varphi_{m_j}(0))) \\
\geq (J-1) \min_{\varepsilon \in \mathbb{Z}_Q} \sum_{v \in Z^d} t(V_k(1,\varepsilon)(\nu)) \quad \text{since} \ t \uparrow \text{ on} \ \mathbb{N}_0 \\
= \frac{J-1}{2} \min_{\varepsilon \in \mathbb{Z}_Q} \sum_{v \in Z^d} V_k(1,\varepsilon)(\nu)^2 - \frac{(J-1)\ell_k(1)}{2} \\
\]
and using (27) with \( N = 1 \) and \( i = 0 \),

\[
\int_0^1 \Psi_{\ell_k(0)}(x) dx \sim \frac{1}{\ell_k+r(0)} \sum_{j=1}^{\ell_k+r(0)} \Psi_{\ell_k(0)}([j\alpha]) \\
\geq \frac{1}{2\ell_k+r(0)} \min_{\varepsilon \in \mathbb{Z}_Q} \sum_{v \in Z^d} V_k(1,\varepsilon)(\nu)^2 - \frac{J\ell_k(1)}{2\ell_k+r(0)} \\
\geq \frac{1}{3\ell_k(0)} \min_{\varepsilon \in \mathbb{Z}_Q} \sum_{v \in Z^d} V_k(1,\varepsilon)(\nu)^2 - \frac{1}{2} \quad \text{by (29)}. \\
\]

**Proof of (b’).** Using (27) with \( k, N \geq 1 \) arbitrary and fixed, and \( i = 1 \) we have

\[
\ell_{k+r}(0) \int_0^1 \Psi_{\ell_k(1)}(x)^N dx \sim \sum_{v \in Z^d} \sum_{j \in [1, \ell_{k+r}(0)] \cap K_v} \#(K_v \cap [j+1, j+\ell_k(1)])^N. \\
\]

Similar to (28),

\[
(\varphi_{m_j+1}(0), \varphi_{m_j+2}(0), \ldots, \varphi_{m_{j+1}+\ell_k(1)}(0)) \\
= [\varphi_{m_j}(0) \uparrow + \Sigma_k(i_j, \eta_j)] \uparrow [\varphi_{m_{j+1}}(0) \uparrow + \Sigma_k(1, \Delta_j)]. \\
\]

for some \( \Delta_j \in \mathbb{Z}_Q \).
We have as before, for fixed \( \nu \in \mathbb{Z}^d \),

\[
\sum_{h \in [1, \ell_{k+1}(0)] \cap \mathcal{K}_\nu} \#(K_\nu \cap [h + 1, h + \ell_k(1)])^N
= \sum_{j=1}^{L-1} \sum_{h \in (m_j, m_{j+1}] \cap \mathcal{K}_\nu} \#(K_\nu \cap [h + 1, h + \ell_k(1)])^N
\leq \sum_{j=1}^{L-1} \sum_{h \in (m_j, m_{j+1}] \cap \mathcal{K}_\nu} \#(K_\nu \cap [h + 1, m_{j+1} + \ell_k(1)])^N.
\]

Fix \( j \). For fixed \( h \in (m_j, m_{j+1}] \),

\[
\#(K_\nu \cap [h + 1, m_{j+1} + \ell_k(1)])
= \#(K_\nu \cap [h + 1, m_{j+1}]) + \#(K_\nu \cap [m_{j+1} + 1, m_{j+1} + \ell_k(1)])
= \#(K_\nu \cap [h + 1, m_{j+1}]) + V_k(1, \Delta_j)(\nu - \varphi_{m_{j+1}}(0)).
\]

Thus

\[
\sum_{h \in (m_j, m_{j+1}] \cap \mathcal{K}_\nu} \#(K_\nu \cap [h + 1, m_{j+1} + \ell_k(1)])^N
= \sum_{h \in (m_j, m_{j+1}] \cap \mathcal{K}_\nu} \#(K_\nu \cap [h + 1, m_{j+1}]) + V_k(1, \Delta_j)(\nu - \varphi_{m_{j+1}}(0))^N
= \sum_{n=0}^{N} \binom{N}{n} \sum_{h \in (m_j, m_{j+1}] \cap \mathcal{K}_\nu} \#(K_\nu \cap [h + 1, m_{j+1}])^n V_k(1, \Delta_j)(\nu - \varphi_{m_{j+1}}(0))^{N-n}
\leq \sum_{n=0}^{N} \binom{N}{n} V_k(0, \eta_{j})(\nu - \varphi_{m_{j+1}}(0))^{n+1} V_k(0, \Delta_j)(\nu - \varphi_{m_{j+1}}(0))^{N-n}
\leq \sum_{n=0}^{N} \binom{N}{n} V_k(0, \eta_{j})(\nu - \varphi_{m_{j+1}}(0))^{n+1} V_k(0, \Delta_j)(\nu - \varphi_{m_{j+1}}(0))^{N-n}.
\]

Using this and Hölder’s inequality,

\[
\sum_{j \in \mathbb{Z}^d} \sum_{h \in [1, \ell_{k+1}(0)] \cap \mathcal{K}_\nu} \#(K_\nu \cap [j + 1, j + \ell_k(1)])^N
\leq \sum_{j=1}^{L} \sum_{n=0}^{N} \binom{N}{n} \sum_{j \in \mathbb{Z}^d} V_k(0, \eta_{j})(\nu - \varphi_{m_{j+1}}(0))^{n+1} V_k(0, \Delta_j)(\nu - \varphi_{m_{j+1}}(0))^{N-n}
\leq \sum_{j=1}^{L} \sum_{n=0}^{N} \binom{N}{n} \left( \sum_{j \in \mathbb{Z}^d} V_k(0, \eta_{j})(\nu - \varphi_{m_{j+1}}(0))^{N+1} \right)^{\frac{n+1}{N+1}} \left( \sum_{j \in \mathbb{Z}^d} V_k(0, \Delta_j)(\nu - \varphi_{m_{j+1}}(0))^{N+1} \right)^{\frac{N-n}{N+1}}
= 2^N \max_{\epsilon \in \mathbb{Z}_0} \sum_{j \in \mathbb{Z}^d} V_k(0, \epsilon)(\nu)^{N+1}.
\]
whence
\[
\int_0^1 \Psi_{\ell_k(1)}(x)^N dx \xrightarrow{r \to \infty} \frac{1}{\ell_{k+r}(0)} \sum_{v \in \mathbb{Z}^d} \sum_{j \in \mathbb{N}} \#(K_v \cap \{j + 1, j + \ell_k(1)\})^N \\
\leq \frac{2N \ell_{k+r}(0)}{\ell_k(1)} \max_{v \in \mathbb{Z}^d} \sum_{v \in \mathbb{Z}^d} V_k(0, \epsilon)(v)^{N+1} \\
\leq \frac{2N}{\ell_k(1)} \max_{v \in \mathbb{Z}^d} \sum_{v \in \mathbb{Z}^d} V_k(0, \epsilon)(v)^{N+1}.
\]

**Proof of the Visit Lemma.** Let
\[
\hat{V}_k(i, \epsilon)(\theta) := \sum_{v \in \mathbb{Z}^d} V_k(i, \epsilon)(v)e^{2\pi i (\theta, v)} \quad (\epsilon \in \mathbb{Z}^d, \ i = 0, 1, \ \theta \in \mathbb{T}^d),
\]
then
\[
\hat{V}_k(i, \epsilon)(\theta) = \ell_k(i) E(e^{2\pi i (\theta, X^{(k)}(i, \epsilon))}).
\]
Using (a') in the sublemma and the Riesz-Fischer theorem, we see that
\[
\int_0^1 \Psi_{\ell_k(0)}(x) dx \geq \frac{1}{3\ell_k(0)} \min_{v \in \mathbb{Z}} \int_{\mathbb{T}^d} |\hat{V}_k(1, \epsilon)(\theta)|^2 d\theta - \frac{1}{2} \\
= \frac{\ell_k(1)^2}{3\ell_k(0)} \min_{v \in \mathbb{Z}} \int_{\mathbb{T}^d} |E(e^{2\pi i (\theta, X^{(k)}(1, \epsilon))})|^2 d\theta - \frac{1}{2}.
\]
This is (a). To see (b),
\[
\|\Psi_{\ell_k(1)}\|_\infty \xrightarrow{N \to \infty} \left(\int_0^1 \Psi_{\ell_k(1)}(x)^N dx\right)^{\frac{1}{N}} \\
\leq \left(\frac{2N}{\ell_k(1)} \max_{v \in \mathbb{Z}^d} \sum_{v \in \mathbb{Z}^d} V_k(0, \epsilon)(v)^{N+1}\right)^{\frac{1}{N}} \text{ by (b')} \\
= \frac{2}{\ell_k(1)^{\frac{1}{N}}} \max_{v \in \mathbb{Z}^d} \left(\sum_{v \in \mathbb{Z}^d} V_k(0, \epsilon)(v)^{N+1}\right)^{\frac{1}{N}} \\
\leq \frac{2}{\ell_k(1)^{\frac{1}{N}}} \max_{v \in \mathbb{Z}^d} \int_{\mathbb{T}^d} |\hat{V}_k(0, \epsilon)(\theta)|^{1+\frac{2}{N}} d\theta \text{ by the Hausdorff-Young theorem} \\
\xrightarrow{N \to \infty} 2\max_{v \in \mathbb{Z}^d} \int_{\mathbb{T}^d} |\hat{V}_k(0, \epsilon)(\theta)| d\theta \\
= 2\ell_k(0) \max_{v \in \mathbb{Z}^d} \int_{\mathbb{T}^d} |E(e^{2\pi i (\theta, X^{(k)}(0, \epsilon))})| d\theta.
\]

**Adaptedness.** As in [1], the norm of a matrix \(A \in M_{S \times S}\) is given by
\[
\|A\| := \sup \{\|Ax\|_\infty : x \in \mathbb{R}^S, \ \|x\|_\infty = 1\},
\]
where \(\|(x_s : s \in S)\|_\infty := \sup_{s \in S} |x_s|\).
We'll call the RAT \(F \in RV(M_{S \times S}(\mathbb{R}) \times (\mathbb{R}^d)^S)\) adapted if \(\exists\) a discrete subgroup \(\Gamma = \Gamma_F \leq \mathbb{R}^d\) (called the *adaptivity group*) so that
\[
\theta \in \mathbb{R}^d & \ \|\Pi_F(\theta)\| = 1 \implies \theta \in \Gamma.
\]
Equivalently, for some \( r > 0 \),
\[
\| \Gamma_F(\theta) \| < 1 \quad \forall \theta \in B(0, r) \sim \{0\}.
\]
Now, writing \( F = (\mathcal{L}, W) \in RV(S^d, (\mathbb{R}^d)^{S\times S}) \), we have as \( \theta \to 0 \)
\[
\| \Gamma_F(\theta) \| = \max_{s \in S} \sum_{t \in S} P(\mathcal{L}_s = t) |E(e^{2\pi i(\theta, W_s, t)})|
= 1 - \min_{s \in S} \sum_{t \in S} P(\mathcal{L}_s = t) \langle \text{Cov}(W_s, t), \theta \rangle + o(\|\theta\|),
\]
where for \( V = (V_1, V_2, \ldots, V_d) \) a \( \mathbb{R}^d \)-valued \( L^2 \) random variable, the covariance matrix \( \text{Cov}(V) \in M_{d \times d}(\mathbb{R}) \) is defined by
\[
\text{Cov}(V)_{k,\ell} := E((V_k - E(V_k))(V_\ell - E(V_\ell))).
\]
A covariance matrix is non-negative definite in the sense that
\[
\langle \text{Cov}(V)\theta, \theta \rangle = E\left(\sum_{k=1}^{d} \theta_k (V_k - E(V_k))^2\right) \geq 0 \quad \forall \theta \in \mathbb{R}^d
\]
and is called positive definite if it is invertible. Equivalently, for some \( \epsilon > 0 \) (the minimum eigenvalue modulus)
\[
\langle \text{Cov}(V)\theta, \theta \rangle \geq \epsilon \|\theta\|^2 \quad \forall \theta \in \mathbb{R}^d.
\]
Thus, \( F \) is adapted iff
\[
\forall s \in S, \exists t \in S \text{ such that } P(\mathcal{L}_s = t) > 0 \quad \text{and} \quad \text{Cov}(W_s, t) \text{ is strictly positive definite.}
\]
It follows as in [1] that if \( F \) is adapted, then \( \forall \epsilon > 0 \) and \( M > 0 \), \( \exists \delta > 0 \) so that
\[
\| \Gamma_F(\theta) \| \leq 1 - \delta \quad \forall \quad n \geq 1 \quad \text{and} \quad \theta \in B(0, M) \sim B(\Gamma, \epsilon).
\]
The following lemma gives a sequence version of adaptedness similar to that in [1].

**Adaptedness Lemma.** For \( N, J, M \geq 1 \) large, \( \exists \) a discrete subgroup \( \Gamma \leq \mathbb{R}^d \) and \( \epsilon, b, c > 0 \), \( r \in (0,1) \) so that
\[
\begin{align*}
  \| \Pi_{E_n}(\theta + \gamma) \| &\leq 1 - c \|\theta\|^2 \quad \forall \quad \gamma \in \Gamma \cap B(0, M), \quad \theta \in B(0, r), \\
  \| \Pi_{E_n}(\theta) \| &\leq 1 - \epsilon \quad \forall \quad \theta \in B(0, M) \sim B(\Gamma, r), \\
  \langle \text{Cov}(W_{s,t}(E_n))\theta, \theta \rangle &\geq \epsilon \|\theta\|^2 \quad \forall \quad \theta \in \mathbb{R}^d,
\end{align*}
\]
where \( E_n := \mathcal{F}_{K+L+J+n+1}^{K+L+J+n+1} \) and \( \{\mathcal{F}_n : n \geq 1\} \) is the independent RAT sequence as on page 28.

**Proof.** The proof is in a series of steps, the first two of which are as in [18].
1. If \( \theta \in \mathbb{R}^d \), \( \xi \in \mathbb{C} \) and \( v \in C^S \) satisfy \( \Pi_{\mathcal{H}}(\theta)v = \xi v \), then \( |\xi| \leq 1 \) with equality iff
\[
\text{(a)} \quad v \in (S^1)^S \quad \text{and} \quad (\Pi_{\mathcal{H}}(\theta))_{s,t} = \xi v_s v_t (\Pi_{\mathcal{H}}(0))_{s,t} \quad \forall \quad s, t \in S
\]
where \( S^1 := \{ z \in \mathbb{C} : |z| = 1 \} \) is the multiplicative circle.
Proof. Write $\Pi := \Pi_{\mathcal{H}}(\theta)$ & $P := \Pi_{\mathcal{H}}(0)$. Evidently (a) $\implies \Pi v = \xi v$.

Now suppose that $\Pi v = \xi v$ with $J \in S$, $|v_J| = \|v\|_{\infty} = 1$, then

$$|\xi| = |\xi v_J| = \left| \sum_{t \in S} \Pi_{J,t} v_t \right| \leq \sum_{t \in S} |\Pi_{J,t}| |v_t| \leq \sum_{t \in S} P_{J,t} |v_t| \leq 1.$$  

If $|\xi| = 1$, then $v \in (S^1)^{S}$ and

$$\left| \sum_{t \in S} \Pi_{s,t} v_t \right| = 1 \ \forall \ s \in S$$

and $\exists z \in (S^1)^{S}$ so that

$$\Pi_{s,t} v_t = z_s P_{s,t} \ \forall \ s, t \in S.$$  

Thus, for $s \in S$,

$$\xi v_s = \sum_{t \in S} \Pi_{s,t} v_t = \sum_{t \in S} z_s P_{s,t} = z_s$$

which is (a).  

Next, for $\theta \in \mathbb{R}^d$, let

$$\mu(\theta) := \max \{|\xi| : \xi \in \mathbb{C} \& \exists v \in C^S, \Pi_{\mathcal{H}}(\theta) = \xi v \}.$$  

By ¶1, $\mu(\theta) \leq 1$. Set $\Gamma := \{ \gamma \in \mathbb{R}^d : \mu(\gamma) = 1 \}$.  

¶2 $\Gamma$ is a discrete subgroup of $\mathbb{R}^d$ and

(b) $\mu(\gamma + \theta) = |\lambda(\theta)| \ \forall \ \gamma \in \Gamma, \ |\theta\|_{\infty} \leq r_{\mathcal{H}}$.

Proof. Since $P(L_{s} = t) > 0 \ \forall \ s, t \in S$,

$$\Gamma = \{ \gamma \in \mathbb{R}^d : \exists x \in \mathbb{R}, \ (\gamma, W_{s,t}) + x \in 2\pi \mathbb{Z} \text{ a.s. } \forall \ s, t \in S \}$$

which is evidently a subgroup of $\mathbb{R}^d$.

Now suppose that $\gamma \in \Gamma$ with $\Pi_{\mathcal{H}}(\gamma) v = \xi v$ where $|v_s| = |\xi| = 1 \ \forall \ s \in S$. By ¶1,

$$(\Pi_{\mathcal{H}}(\theta))_{s,t} = \xi v_s \bar{v}_t (\Pi_{\mathcal{H}}(0))_{s,t} \ \forall \ s, t \in S.$$  

Equivalently $\forall \ s, t \in S$,

$$E(e^{2\pi i (\gamma, W_{s,t})}) = \xi v_s \bar{v}_t \implies E(e^{2\pi i (\gamma + \theta, W_{s,t})}) = \xi v_s \bar{v}_t E(e^{2\pi i (\theta, W_{s,t})}) \ \forall \ \theta \in \mathbb{R}^d,$$

whence

$$(\Pi_{\mathcal{H}}(\theta + \gamma))_{s,t} = \xi v_s \bar{v}_t (\Pi_{\mathcal{H}}(\theta))_{s,t} \ \forall \ s, t \in S, \ \theta \in \mathbb{R}^d.$$  

Statement (b) follows from this, whence ¶2 via the Taylor expansion of $\lambda$.  

¶3 For $N \geq 1$ sufficiently large, $\mathcal{H}_1^N$ is an adapted RAT with adaptivity group $\Gamma$.

Proof. Fix $0 < q < 1$ and $N \geq 1$ so that

$$\|Q(\theta)^N\| < q \text{ for } \theta \in B(0, r_{\mathcal{H}}) \& \text{ hence for } \theta \in B(\Gamma, r_{\mathcal{H}}); \text{ and}$$

$$\mu(\theta)^N < q \ \forall \ \theta \in \mathbb{R}^d \sim B(\Gamma, r_{\mathcal{H}}).$$  

It follows that $\mathcal{H}_1^N$ is adapted with adaptivity group $\Gamma$.  




To complete the proof of the lemma, fix $J \geq 1$ and let 
\[ \mathcal{E}_n := \mathcal{F}_{J+N(n+1)} : n \geq 1. \]

Statements (i) and (ii) follow because for each $M > 0$,
\[ \sup_{|\theta| \leq M} \| \Pi_{\mathcal{E}_n}(\theta) - \Pi_{\mathcal{H}_1}(\theta) \| \to 0 \quad \text{as} \quad n \to \infty \]
and statement (iii) follows from
\[ W_{s,t}(\mathcal{E}_n) \xrightarrow{n \to \infty} W_{s,t}(\mathcal{H}_1) \quad \forall \ s, t \in S. \]

To establish Theorem 3, we use the following.

**Weak, Rough Local Limit Theorem.** For each $s \in S$ and $1 \leq p \leq 2$,
\[ (\text{WRLLT}) \quad \int_{[-\pi, \pi]^d} |E(e^{2\pi i \langle \theta, x^{(J+Ln)}_s \rangle})|^p d\theta \geq \frac{1}{n^d}. \]

The proof is a multidimensional version of the proof of the WRLLT in [1].

**Proof of WRLLT.** By [1, Theorem 6.1], for each $1 \leq k \leq d$, we have
\[ \sum_{\nu=1}^n \min_{s \in S} E((b_s(\mathcal{E}_n)_k)^2) \leq E((b_s(\mathcal{H}_1)_k)^2) \leq \sum_{\nu=1}^n \max_{s \in S} E((b_s(\mathcal{E}_n)_k)^2). \]

Thus, by the Adaptedness Lemma, there exists $G > 0$ so that
\[ E(\|X_s^{(J+Ln)}\|^2) \leq Gn. \]

Next, fix $M = 2\sqrt{C}$, then by Chebyshev’s inequality,
\[ P(\|X_s^{(J+Ln)}\| \leq M\sqrt{n}) \geq \frac{3}{4}. \]

Now fix $\Delta > 0$ so that
\[ |1 - e^{2\pi i x}| < \frac{1}{4} \quad \forall \ |x| < \Delta. \]

We have
\[ n^{\frac{d}{2}} \int_{[-\pi, \pi]^d} |E(e^{2\pi i \langle \theta, x^{(J+Ln)}_s \rangle})|^2 d\theta = \int_{[-\pi, \pi]^d} |E(\exp[2\pi i \langle \theta, x^{(J+Ln)}_s \rangle])|^2 d\theta \]
\[ \geq \int_{[-\Delta, \Delta]^d} |E(\exp[2\pi i \langle \theta, x^{(J+Ln)}_s \rangle])|^2 d\theta. \]

For $\|\theta\| < \frac{\Delta}{4}$, we have
\[ |E(\exp[2\pi i \langle \theta, x^{(J+Ln)}_s \rangle])| \]
\[ \geq |E(\exp[2\pi i \langle \theta, x^{(J+Ln)}_s \rangle])| \cdot P([\|X_s^{(J+Ln)}\| < M\sqrt{n}]) - P([\|X_s^{(J+Ln)}\| \geq M\sqrt{n}]) \]
\[ \geq \frac{3}{4} \cdot \frac{1}{4} = \frac{5}{16}. \]
whence
\[ n^\frac{d}{2} \int_{[-\pi,\pi]^d} |E(e^{2\pi i (\theta, X_i^{(j+Ln)})})|^2 d\theta \geq \int_{[-\frac{\Delta}{M}, \frac{\Delta}{M}]^d} |E(\exp[2\pi i (\theta, X_i^{(j+Ln)})/\sqrt{n^n})])|^2 d\theta \]
\[ \geq \left(\frac{2\Delta}{M}\right)^d \cdot \frac{25}{256}. \]
\[ \square \]

**Proof of \( \ll \).** We have
\[ |E(e^{2\pi i (\theta, X_i^{(j+Ln)})})| = |(\Pi_{\theta_n}(\theta)\Pi_{\theta_{n-1}}(\theta)\cdots\Pi_{\theta_1}(\theta)\Xi_f(\theta))_s| \]
\[ \leq \prod_{k=1}^n \|\Pi_{\theta_k}(\theta)\|. \]

Fix \( M > 0 \) so that \( [-\pi, \pi]^d \subset B(0, M) \). By the Adaptedness Lemma, for \( n \geq 1, \gamma \in \Gamma \) we have
\[ \|\Pi_{\theta_n}(\gamma + \theta)\| \leq 1 - c\|\theta\|^2 \quad \forall \, |\theta| < r \]
and
\[ \|\Pi_{\theta_n}(\theta)\| \leq 1 - c \quad \forall \, \theta \in B(0, M) \sim B(\Gamma, r). \]

\[
\int_{[-\pi,\pi]^d} |E(e^{2\pi i (\theta, X_i^{(j+Ln)})})|d\theta \\
\leq \left( \int_{B(0,M) \cap B(\Gamma,r)} + \int_{B(0,M) \sim B(\Gamma,r)} \right) \prod_{k=1}^n \|\Pi_{\theta_k}(\theta)\| d\theta \\
\leq \sum_{\gamma \in B(0,M) \cap \Gamma} \int_{B(0,r)} \prod_{k=1}^n \|\Pi_{\theta_k}(\gamma + \theta)\| d\theta + \int_{B(0,M) \sim B(\Gamma,r)} \prod_{k=1}^n \|\Pi_{\theta_k}(\theta)\| d\theta \\
\leq \#(B(0,M) \cap \Gamma) \int_{B(0,r)} (1 - c\|\theta\|^2)^n d\theta + O((1 - c)^n) \\
\ll \frac{1}{n^\frac{d}{2}}. \]
\[ \square \]

**Proof of Theorem 3.** Set \( \nu_k(i) := \ell_{K+Lk}(i) \quad (i = 0, 1) \). The Visit Lemma and the \( \Psi RL LT \) show that
\[ \Psi_{\nu_k(i)}(x) \ll \nu_k(0) \int_{T^d} |E(e^{2\pi i (\theta, X_i^{(j+Ln)})})|d\theta \]
\[ \ll \frac{\nu_k(0)}{k^\frac{d}{2}} \ll \nu_k(0) \int_{T^d} |E(e^{2\pi i (\theta, X_i^{(j+Ln)})})|^2 d\theta \]
\[ = \int_{T^d} \Psi_{\nu_k(i)}(x) dx \]
\[ \ll \frac{\nu_k(0)}{k^\frac{d}{2}}. \]
Next, \( \exists \Lambda > 1 \) so that \( \nu_k(i) \propto \Lambda^k \) \((i = 0, 1)\), whence for \( \nu_k(0) \leq n \leq \nu_{k+1}(0) \),
\[
\int_{\mathbb{T}^d} \Psi_n(x) \, dx \geq \int_{\mathbb{T}^d} \Psi_{\nu_k(0)}(x) \, dx \geq \frac{\nu_k(1)}{k^d} \gg \frac{\nu_{k+1}(0)}{k^d} \gg \frac{n}{(\log n)^{\frac{d}{2}}}
\]
and for \( \nu_k(1) \leq n \leq \nu_{k+1}(1) \),
\[
\| \Psi_n(x) \|_{L^\infty(\mathbb{T}^d)} \ll \frac{\nu_{k+1}(1)}{k^d} \ll \frac{\nu_k(0)}{k^d} \ll \frac{n}{(\log n)^{\frac{d}{2}}}. \]

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JON AARONSON <aaro@post.tau.ac.il>: School of Mathematical Sciences, Tel Aviv University, 69978 Tel Aviv, Israel

MICHAEL BROMBERG <micbromberg@gmail.com>: School of Mathematics, Bristol University, Bristol BS8 1TW, UK

NISHANT CHANDGOTIA <nishant.chandgotia@gmail.com>: School of Mathematical Sciences, Tel Aviv University, 69978 Tel Aviv, Israel