Synchronization of P Systems with Simplex Channels

(Work in Progress)

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We solve the Firing Squad Synchronization Problem (FSSP), for P systems based on digraphs with simplex channels, where communication is restricted by the direction of structural arcs. Previous work on FSSP for P systems focused exclusively on P systems with duplex channels, where communication between parents and children is bidirectional. Our P solution, the first for simplex channels, requires cell IDs, strongly connected digraphs and some awareness of the local topology (such as each cell’s outdegree)—we argue that these requirements are necessary. Compared to the known solutions for cellular automata, our solution is substantially simpler and faster.

Keywords: P systems, digraphs, strongly connected, simplex channels, firing squad synchronization, cellular automata.

1 Introduction

The Firing Squad Synchronization Problem (FSSP), originally proposed by Myhill in 1957 [14], is one of the best studied problems for cellular automata. Essentially, the problem involves programming a network of cellular automata, so that, after the firing order is given by the general, after some finite time, all the cells in the system enter a designated firing state, simultaneously and for the first time.

Versions of FSSP have been proposed and studied for variety of network structures, from simple linear graphs to rings, trees, or general connected graphs; see, for example, [2, 4, 10, 11, 12, 13, 15, 17, 23, 25, 26]. However, most of these versions, require bidirectional communication (i.e. duplex channels): only a few notable exceptions consider the more restricted unidirectional communication (i.e. simplex channels), starting with Kobayashi [12]. Later, Even, Litman and Winkler [9] proposed improved solutions for arbitrary undirectional strongly-connected digraph, working in \(O(N^2)\) steps, where \(N\) is the digraph size (number of cells). Ostrovsky and Wilkerson [18] improved this further, to a solution which runs in \(O(ND)\) steps, where \(D\) is the digraph diameter (typically smaller than \(N\))— this still seems to be the best solution available.

Several FSSP solutions have recently been studied in the framework of P systems, although with somehow different formulations, stemming from their different computing capabilities. P solutions were proposed: for trees, by Bernardini et al. [3] and Alhazov et al. [1]; and for arbitrary connected graphs, by Dinneen at al. [5, 7, 8]. All these P solutions require duplex channels and follow the typical pattern of a wave algorithm [24], using three phases:
1. a **first broadcast**—which follows all shortest paths from the **general** and builds a **virtual BFS tree** (or **dag**);  
2. a **convergecast**—which helps determine the **general’s eccentricity**;  
3. a **second broadcast**—which carries the actual firing command (with a countdown counter).

The best P solutions need \(e_g + k\) steps for each of the three phases, for a total of \(3e_g + k\) steps, where \(e_g\) is the general’s eccentricity (height for trees, if the general is at the root).

Obviously, while the two broadcasts, of phases (1) and (3), would also work with simplex channels, duplex channels are essential for the convergecast of phase (2), where children need to talk back to their parents. At first sight, the convergecast seems impossible for simplex channels. However, children can still talk back to their parents, if the digraph is **strongly-connected**, albeit on a typically longer path. Moreover, if messages cannot be confused, all children can send messages to their parents, in parallel (overlapping in time without problems), achieving this way a **virtual convergecast**.

Based on ideas from Ostrovsky and Wilkerson [18], we propose a first FSSP solution for P systems with **simplex** channels, based on arbitrary **strongly-connected digraphs**. Our solution runs in \(O(e_gD)\) steps, specifically: (1) \(e_g\) steps for the first broadcast; (2) \(O(e_gD)\) steps for the virtual convergecast (maximally parallelized); (3) \(e_g\) steps for the second broadcast. Thus, in terms of execution time, our P solution compares favourably with the best known solution for cellular automata. Taking into account the different problem constraints and different computing capabilities of P systems vs. cellular automata, we were expecting a simpler and faster solution, but not necessarily such a substantial speed improvement. However, for the reasons mentioned, any performance comparison must be viewed with a grain of salt.

The actual design seems challenging and requires careful selection of the most adequate ingredients, some of which are available in cellular automata, but not typically available in P systems. Specifically, we argue that the P solution requires: (1) reified cell IDs; (2) reified local network information, such as the number of outgoing arcs (this information is available in cellular automata); and (3) high-level generic rules, if we want to have a fixed rule set, independent of the actual network size.

## 2 Preliminaries

We assume that the reader is familiar with the basic terminology and notations, such as relations, graphs, nodes (vertices), edges, directed graphs (digraphs), directed acyclic graphs (dags), arcs, alphabets, strings and multisets.

A **P system** is a parallel and distributed computational model, inspired by the structure and interactions of cell membranes. This model was introduced by Păun in 1998–2000 [19]. An in-depth overview of this model can be found in Păun et al. [22].

In this paper, we consider an ad-hoc definition of P systems, based on our definition of **simple P module** [5], which extends earlier versions of tissue and neural P systems [13, 20]. However, here we intentionally restrict rule transfer mode to broadcast to all children, \(\forall\).

**Definition 1** A P system of order \(n\) with simplex channels is a system \(\Pi = (O, K, \delta)\), where:  
1. \(O\) is a finite alphabet of elementary symbols; strings over \(O\) are interpreted as multisets;  
2. \(K = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}\) is a finite set of cells; where each cell is a system \(\sigma_i = (Q_i, R_i)\), with \(Q_i\) a finite set of states and \(R_i\) a finite set of rewriting rules over \(O\), further detailed below.
3. $\delta$ is an irreflexive binary relation on $K$, which represents a set of structural arcs between cells, with unidirectional communication capabilities, strictly from parents to children.

4. Each $R_i$ is a finite linearly ordered set of multiset rewriting rules with promoters, of the form: $S x \rightarrow \alpha S' x' (y)_\downarrow \cdots$, where $S, S' \in Q_i$, $x, x', y \in O^*$, $z \in O'$ is the promoter, $\alpha \in \{\min, \max\}$ is a rewriting operator and $\downarrow_y$ is a transfer operator, here restricted to send $y$ messages from a parent to all its children.

As usually, each cell, $\sigma_i \in K$, starts from an initial configuration $(S_0, w_0)$, where $S_0 \in Q_i$ is its initial state and $w_0 \in O^*$ is its initial content. A cell evolves by applying one or more rules, which can change its current configuration, i.e. its current state and current content, and send symbols to its children.

The application of a rule transforms the current state $S$ to the target state $S'$, rewrites multiset $x$ as $x'$ and sends multiset $y$ by replication to all its children. Note that, multisets $x'$ and $y$ will not be visible to further rules in this same step, but they will become visible after no more rules are applicable, i.e. they will be available since next step only. Promoters are symbols which enable rules, but are not consumed by the rules’ application.

When an applicable rule is applied, its rewriting operator $\alpha$ indicates how many times it is actually applied: once, if $\alpha = \min$; or as many times as possible, if $\alpha = \max$.

As used here, rules have priorities and are applied in weak priority order [21], with special attention to target state compatibility: (1) higher priority applicable rules are applied before lower priority applicable rules, and (2) a lower priority applicable rule is applied only if it indicates the same target state as the previously assigned rules (if any).

All cells evolve synchronously in one global step. An evolution of a $P$ system is a sequence of steps, where each cell starts from its initial configuration. An execution halts if no cell can evolve.

### 2.1 Further P systems extensions

We let each cell, $\sigma_i$, start with its own unique cell ID symbol, $i$. We thus reify the conceptual cell index, $i$, into an internal symbol, which is accessible to the rules, exclusively as an immutable promoter [16].

We enhance our vocabulary by recursive composition of elementary symbols from $O$ into a simple form of complex symbols [16]. Such complex symbols can be viewed as complex molecules, consisting of elementary atoms or other molecules.

Further, complex symbols let us process our multisets with high-level generic rules, using free variable matching. To explain these additional ingredients, consider this hypothetical rule (which uses an additional transfer mode, targeted to a specific child, not considered in the definition used here):

$$S a n_j \rightarrow_{\min, \min} S' b (c_i)_{\downarrow j} | t_i.$$  

This is a generic rule, which uses an extended rewriting mode, with complex symbols, $c_i$ and $n_j$, where $i$ and $j$ are free variables. In fact, $c_i$ and $n_j$ are just shorthands for tuples $(c, i)$ and $(n, j)$, or, equivalently, for compound terms $c(i)$ and $n(j)$. If needed, we can build more complex symbols by recursive composition; e.g., we could have complex symbols such as $d(e, i, f(j))$. Generally, a free variable could match anything, including another complex symbol. However, in this rule, $i$ and $j$ are constrained to match cell ID indices only:

1. $i$—because it also appears as the cell ID of the current cell, $t_i$;
2. $j$—because it also indicates the target of the transfer mode, $\downarrow j$. 

59
A generic rule is identified by using an extended version of the “classical” rewriting mode, in fact, by a combined instantiation and rewriting mode. Our sample rule uses the extended mode \( \min \min \), where the two \( \min \) operators have distinct semantics: the first \( \min \) operator is new and describes the generic instantiation; the second \( \min \) is the classical operator, which describes the rule application. Briefly:

1. according to the first \( \min \), this rule is instantiated once, for one of the existing \( n_j \) symbols (if any), while promoter, \( i \), constrains \( i \) to the cell ID index of the current cell, \( \sigma_i \);

2. according to the second \( \min \), the instantiated rule is applicable once, i.e. if applied, it consumes one \( a \) and one \( n_j \), produces one \( b \) and sends one \( c_i \) to child \( \sigma_j \) (if this exists).

As a further example, consider the scenario in which the current cell, \( \sigma_1 \), contains the multiset \( n_2 n_3 n_3 \). Here, our sample generic rule instantiates (randomly) one of the following two lower-level rules, which is then applied in the classical way (in the \( \min \) rewriting mode):

\[
S a n_2 \rightarrow \min S' b (c_1)_{i_2},
\]

\[
S a n_3 \rightarrow \min S' b (c_1)_{i_3}.
\]

We consider four basic combinations of the instantiation and rewriting modes, \( \min \min \), \( \min \max \), \( \max \min \), \( \max \max \); their semantics is:

- \( \min \min \) indicates that the generic rule is (randomly) instantiated once, if possible, and the instantiated rule is applied once, if possible.
- \( \min \max \) indicates that the generic rule is (randomly) instantiated once, if possible, and the instantiated rule is applied as many times as possible.
- \( \max \min \) indicates that the generic rule is instantiated as many times as possible, without superfluous instances (i.e. without duplicates or instances which are not applicable), and each one of the instantiated rules is applied once, if possible.
- \( \max \max \) indicates that the generic rule is instantiated as many times as possible, without superfluous instances (i.e. without duplicates or instances which are not applicable), and each one of the instantiated rules is applied as many times as possible.

All instantiations are ephemeral, created when rules are tested for applicability and disappearing at the end of the step.

### 3 FSSP problem for P systems with simplex channels

We are required to find:

1. an alphabet \( O \);
2. a cell prototype \( \sigma = (Q, R) \), where
   (a) \( R \) is a set of rules over \( O \);
   (b) \( Q \) contains two distinguished states:
      - \( S_0 \): a quiescent state, i.e. if \( \sigma \) is in state \( S_0 \) and empty, then there are no applicable rules;
      - \( S_f \): a final state, i.e. if \( \sigma \) is in state \( S_f \), then there are no applicable rules.

such that, given:

1. any finite set of \( \sigma \) copies, \( K = \{ \sigma_1, \sigma_2, \ldots, \sigma_n \} \), \( \sigma_i = \sigma \);
2. connected via any strongly-connected digraph $\delta$;

the P system $\Pi = (O, K, \delta)$ with simplex channels will evolve according to the following specification:

1. all cells start from quiescent state $S_0$: $S_{i0} = S_0$;
2. except a distinguished cell $\sigma_g$, called the general, all cells start with a restricted initial content, containing, the reified cell ID and a reified count of the cell’s outdegree: $w_{i0} \subseteq \{\iota_i, c_{\text{outdegree}}(\sigma_i)\}, \forall i \neq g$;
3. the evolution terminates and, during its last step: all cells enter state $s_f$ simultaneously and for the first time.

Remark 1 Our formulation has different constraints than the original problem for cellular automata. In the cellular automata formulation, there is a given fixed bound on the number of input and output connections of each cell (bounded indegree and outdegree). In our formulation, there is no bound on the number of input and output connections that a cell may have. However, this is compensated by the fact that in our formulation there are no size bounds on messages or cells’ internal memory. These trade-offs, as well as different computing capabilities, suggests that performance comparisons must be viewed with a grain of salt.

Remark 2 We argue that both the reified cell ID and the reified children count, or equivalent information, are necessary, definitely for our approach, and, likely, for any other approach. Note that children counts are implicit in the cellular automata version, where unconnected channels (out of the fixed sized pool) can be detected.

Remark 3 There is no constraint on the cells’ final contents. However, if needed, any left-over garbage could be collected in one extra step.

Remark 4 Practically, we are only required to design the rule set, $R$, because this implies the alphabet, $O$, and the state set, $Q$.

Remark 5 Note the rule set, $R$, must be fixed and applicable to any structural digraph. This is a strong requirement: we require a rule set which is independent of the size and structure of the actual system.

4 FSSP solution for P systems with simplex channels

Our solution runs in three phases (conceptually similar to the duplex case):

1. First phase: a first broadcast from the general. This phase builds the virtual-dag (the virtual BFS dag) and, for each cell: (a) records its virtual-dag-parent(s); and (b) successively computes its depth attribute, which represents this cell’s depth level in the virtual-dag (the same as this cell’s digraph distance from the general). A first phase broadcast message is a complex symbol, $x_{k,i}$, where $i$ is the sender’s ID and $k$ is the next depth level ($\sigma_i$’s own depth plus one).

2. Second phase: a virtual convergecast from the virtual-dag leaves. For each cell, this phase successively computes the max-depth attribute, which represents the maximum depth over all descendant cells in the virtual-dag. In the end, the general’s max-depth is its eccentricity. This virtual convergecast simulates impossible direct virtual-dag-child to virtual-dag-parent messages, by broadcasting them over the digraph (using ad-hoc BFS dags). A convergecast message is a complex symbol, $a_{j,i,k}$, where $i$ is the sender’s ID, $j$ is its virtual-dag-parent ID and $k$ is $\sigma_i$’s
own max-depth. In addition to virtual-dag leaves, a pseudo-convergecast message is also sent by a digraph cell to its digraph parents with larger depth attributes (if any). Because each message is uniquely identified by both its sender and its destination, any number of such convergecasts can run in parallel, without creating confusions.

Note that a cell needs to know its digraph outdegree—to detect when it receives its last outstanding convergecast message and to start its own convergecast. However, a cell does not know the identities of its digraph children, or how long a message from any one of them will take to reach it.

3. Third and last phase: a second broadcast from the general, with a countdown to firing. A last phase broadcast message is a complex symbol, \( f_k \), where \( k \) is the next countdown counter (\( \sigma_i \)'s own countdown minus one).

As possible extensions, not discussed here, we can extract from our solution a more general subprogram, to send any message from any cell to any other cell, which can run in parallel, without creating confusions. Also, we can consolidate the routing information, to speedup future messages with the same destination; however, this feature is not required here (each convergecast is performed exactly once).

Figures 1–8 show bird's eye views of the evolution of \( \Pi \): a sample P system, with simplex channels, based on a strongly connected digraph.

![Figure 1: Initial configuration of a sample P system with simplex channels, based on a strongly connected digraph. Normal arrows are arcs in the virtual dag, created by a BFS broadcast, started from the general, \( \sigma_1 \). The remaining digraph arcs are dotted arrows. Each cell knows its digraph outdegree, indicated by small circles at outgoing arrows' tails. Each cell blob shows three attributes, in order: (1) its cell ID; (2) its depth attribute (computed by the first broadcast); and (3) its max-depth attribute (computed by the virtual convergecast). At this stage, the depth and max-depth attributes are still indeterminate and all small circles are white, indicating still outstanding convergecast messages.](image-url)
4.1 Rule set \( R \)

0. Rules in state \( S_0 \):
   1. \( S_0 \ a \rightarrow_{\min \min} S_1 \ g \ n_0 \ m_0 \ (x_{1,1})_v \ | i \)
   2. \( S_0 \ x_{k,j} \rightarrow_{\max \min} S_1 \ l_k \ p_j \ (x_{k+1,1})_v \ | i \)
   3. \( S_0 \ x_{k,j} \rightarrow_{\max \max} S_1 \ p_j \ | i \)

1. Rules in state \( S_1 \):
   1. \( S_0 \ l_k \rightarrow_{\max \min} S_1 \ n_k \ m_k \)
   2. \( S_0 \ l_k \rightarrow_{\max \max} S_1 \)
   3. \( S_1 \ x_{k,j} \rightarrow_{\max \min} S_1 \ y_{j,0} \)
   4. \( S_1 \ x_{k,j} \rightarrow_{\max \max} S_1 \)
   5. \( S_1 \ s_k \rightarrow_{\max \min} \text{Max} \ w \ m_k \)
   6. \( S_1 \ y_{j,k} \rightarrow_{\max \min} S_1 \ v_{j,i} \ (a_{j,k,i})_v \ | i \)
   7. \( S_1 \ a_{j,k,l} \rightarrow_{\max \max} S_1 \ v_{j,k} \)
   8. \( S_1 \ a_{i,j,k} \rightarrow_{\max \min} S_1 \ v_{i,j} s_k \ | i \)
   9. \( S_1 \ a_{j,k,l} \rightarrow_{\max \min} S_1 \ v_{j,k} \ (a_{j,k,l})_v \)

2. Rules in state \( S_2 \):
   1. \( S_2 \ t \rightarrow_{\min} S_5 \ (t)_v \)
   2. \( S_2 \ s_k \rightarrow_{\max \max} \text{Max} \ w \ m_k \)
   3. \( S_2 \ y_{j,k} \rightarrow_{\max \min} S_2 \ v_{j,i} \ (a_{j,k,i})_v \ | i \)
   4. \( S_2 \ a_{j,k,l} \rightarrow_{\max \max} S_2 \ v_{j,k} \)
   5. \( S_2 \ a_{i,j,k} \rightarrow_{\max \min} S_2 \ v_{i,j} s_k \ | i \)
   6. \( S_2 \ a_{j,k,l} \rightarrow_{\max \min} S_2 \ v_{j,k} \ (a_{j,k,l})_v \)

3. Rules in state \( S_3 \):
   1. \( S_3 \ c \rightarrow_{\min} S_1 \ c \ e \)
   2. \( S_3 \ c \rightarrow_{\min} S_4 \ e \ b \)

4. Rules in state \( S_4 \):
   1. \( S_4 \rightarrow_{\max \min} S_2 \ t \ f_k (t)_v \ | g \ b \ m_k \)
   2. \( S_4 \rightarrow_{\max \min} S_2 \ y_{j,k} \ | p_j m_k \)

4.2 Alphabet \( O \), elementary and complex symbols

- \( i_v \): reified cell ID;
- \( a \): starts the process from the cell which next assume the general role;
- \( g \): marks the general;
- \( x_{j,k} \): complex symbol broadcasted in the first phase;
• \(n_k\): indicates the \textit{depth} attribute, \(k\);
• \(m_k\): indicates the \textit{max-depth} attribute, \(k\);
• \(l_k\): auxiliary symbol used to compute \textit{depth} attribute, \(n_k\);
• \(p_k\): pointer to a parent cell;
• \(y_{i,k}\): initiates the sending of message \(k\) to \(\sigma_i\);
• \(a_{i,j,k}\): complex symbol broadcasted in the convergecast phase, sent from \(\sigma_j\) to \(\sigma_i\) and carrying payload \(k\), representing \(\sigma_j\)'s \textit{max-depth} (if known, otherwise it is a pseudo-convergcast sent by a cell with lower \textit{depth});
• \(v_{i,j}\): records the passage of a message from \(\sigma_j\) to \(\sigma_i\);
• \(s_k\): auxiliary symbol used to compute \textit{max-depth} attribute, \(m_k\), via maximum;
• \(c\): used to count the children which have not yet sent their convergecast messages (initially cell’s outdegree);
• \(e\): used to count the children which have already sent their convergecast messages (initially zero);
• \(b\): marks a cell which has performed its convergecast;
• \(t\): auxiliary symbol used in the countdown to firing;
• \(f_k\): complex symbol broadcasted in the last phase, carrying the countdown to firing;

4.3 Brief description

1. State \(S_0\): initiates the first broadcast and computes \textit{depth} attributes.
2. State \(S_1\): continues the first broadcast and builds the virtual-dag.
3. State \(S_2\): performs the actual convergecast.
4. State \(S_3\): decides if a non-general cell is ready to start its convergecast, i.e. if it has received its all outstanding convergecast messages, from all its digraph children.
5. State \(S_4\): decides if the general is ready to start the second broadcast (countdown to firing), i.e. if it has received convergecast messages from all its children.
6. State \(\text{Max}\): prepares the convergecast, by determining the \textit{max-depth} attribute.
7. State \(S_5\): last broadcast, counts down to firing and erases unnecessary symbols.

5 Assessment

**Theorem 1** The synchronization time of the FSSP solution for digraph-based P systems with simplex channels is \(e_g + e_gD + e_g\), i.e. bounded by \(O(e_gD)\).

**Theorem 2** The digraph must be strongly connected, otherwise, there is no solution.

**Theorem 3** Cells must know the number of their outgoing arcs, otherwise, there is no solution.

**Theorem 4** Cell IDs must be reified, otherwise, there is no solution.
6 Experimental results

Besides our earlier example, we have empirically validated our solution in several test scenarios, with digraphs of different shapes and sizes, for example:

- A simple ring (Figure 9).
- A main ring linking a series of smaller rings of size two (Figure 10).
- A main ring linking a series of smaller rings of size three (Figure 11).
- A main ring linking a series of smaller rings of increasing size (Figure 12).
- A set of 11 random directed graphs, with up to 70 nodes each, generated by the standard networkx package.

The results support our claim for correctness and performance.

7 Conclusions

In this paper, we explicitly presented a first solution to the FSSP for synchronous digraph-based P systems with simplex channels. Our design suggests, but does not need, ways to consolidate routing information in such systems—this can be a topic for further study.

Our solution runs in $O(eD)$ steps and compares favourably, i.e. it is simpler and faster, than the best known solution for cellular automata [18], which runs in $O(ND)$ steps. Taking into account the different problem constraints and different computing capabilities of P systems vs cellular automata, we were expecting a simpler and a faster solution, but not necessarily such a substantial speed improvement. However, as noted before, any performance comparison must be viewed with a grain of salt.

Our solution used a fixed size high-level rule set, independent of the number of cells in the actual system and of its structure. This supports the case for reified cell IDs, complex symbols and generic rules and suggests that such ingredients could be useful or even essential in any distributed or just large system.

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References

[1] A. Alhazov, M. Margenstern, and S. Verlan. Fast Synchronization in P Systems. In D. W. Corne, P. Frisco, G. Păun, G. Rozenberg, and A. Salomaa, editors, Workshop on Membrane Computing, volume 5391 of Lecture Notes in Computer Science, pages 118–128. Springer, 2008.

[2] R. Balzer. An 8-state Minimal Time Solution to the Firing Squad Synchronization Problem. Information and Control, 10(1):22–42, 1967.

[3] F. Bernardini, M. Gheorghe, M. Margenstern, and S. Verlan. How to Synchronize the Activity of All Components of a P System? Int. J. Found. Comput. Sci., 19(5):1183–1198, 2008.
[4] A. Berthiaume, T. Bittner, L. Perkovic, A. Settle, and J. Simon. Bounding the Firing Synchronization Problem on a Ring. *Theor. Comput. Sci.*, 320(2-3):213–228, 2004.

[5] M. J. Dinneen, Y.-B. Kim, and R. Nicolescu. Edge- and node-disjoint paths in P systems. *Electronic Proceedings in Theoretical Computer Science*, 40:121–141, 2010.

[6] M. J. Dinneen, Y.-B. Kim, and R. Nicolescu. Synchronization in P Modules. In C. S. Calude, M. Hagiya, K. Morita, G. Rozenberg, and J. Timmis, editors, *Unconventional Computation*, volume 6079 of *Lecture Notes in Computer Science*, pages 32–44. Springer-Verlag, Berlin Heidelberg, 2010.

[7] M. J. Dinneen, Y.-B. Kim, and R. Nicolescu. An Adaptive Algorithm for P System Synchronization. In *Proceedings of the Twelfth International Workshop on Membrane Computing* (CMC11), Fontainebleau/Paris, France, pages 1–26, in press, 2011.

[8] M. J. Dinneen, Y.-B. Kim, and R. Nicolescu. Faster synchronization in P systems. *Journal of Natural Computing*, pages 1–17, in press, 2011.

[9] S. Even, A. Litman, and P. Winkler. Computing with Snakes in Directed Networks of Automata. *Journal of Algorithms*, 24:740–745, 1990.

[10] E. Goto. A Minimal Time Solution of the Firing Squad Problem. Course notes for Applied Mathematics 298, pages 52–59, Harvard University, 1962.

[11] J. J. Grefenstette. Network Structure and the Firing Squad Synchronization Problem. *J. Comput. Syst. Sci.*, 26(1):139–152, 1983.

[12] K. Kobayashi. The Firing Squad Synchronization Problem for a Class of Polyautomata Networks. *J. Comput. Syst. Sci.*, 17(3):300–318, 1978.

[13] C. Martín-Vide, G. Păun, J. Pazos, and A. Rodríguez-Patón. Tissue P systems. *Theor. Comput. Sci.*, 296(2):295–326, 2003.

[14] E. F. Moore. The Firing Squad Synchronization Problem. Moore, E.F. (ed.) *Sequential Machines, Selected Papers*, pages 213–214, 1964.

[15] F. R. Moore and G. G. Langdon. A Generalized Firing Squad Problem. *Information and Control*, 12(3):212–220, 1968.

[16] R. Nicolescu and H. Wu. BFS solution for disjoint paths in P systems. In C. Calude, J. Kari, I. Petre, and G. Rozenberg, editors, *Unconventional Computation*, volume 6714 of *Lecture Notes in Computer Science*, pages 164–176. Springer Berlin / Heidelberg, 2011.

[17] Y. Nishitani and N. Honda. The Firing Squad Synchronization Problem for Graphs. *Theor. Comput. Sci.*, 14:39–61, 1981.

[18] R. Ostrovsky and D. S. Wilkerson. Faster Computation On Directed Networks of Automata. In *Proceedings of the Fourteenth Annual ACM Symposium on Principles of Distributed Computing*, pages 38–46. ACM, 1995.

[19] G. Păun. Computing with Membranes. *Journal of Computer and System Sciences*, 61(1):108–143, 2000.

[20] G. Păun. *Membrane Computing: An Introduction*. Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2002.

[21] G. Păun. Introduction to Membrane Computing. In G. Ciobanu, M. J. Pérez-Jiménez, and G. Păun, editors, *Applications of Membrane Computing*, Natural Computing Series, pages 1–42. Springer-Verlag, 2006.

[22] G. Păun, G. Rozenberg, and A. Salomaa. *The Oxford Handbook of Membrane Computing*. Oxford University Press, Inc., New York, NY, USA, 2010.

[23] H. Schmid and T. Worsch. The Firing Squad Synchronization Problem with Many Generals For One-Dimensional CA. In J.-J. Lévy, E. W. Mayr, and J. C. Mitchell, editors, *IFIP TCS*, pages 111–124. Kluwer, 2004.

[24] G. Tel. *Introduction to Distributed Algorithms*. Cambridge University Press, 2000.

[25] H. Umeo, N. Kamikawa, K. Nishioka, and S. Akiyama. Generalized Firing Squad Synchronization Protocols for One-dimensional Cellular Automata — A Survey. *Acta Physica Polonica B Proceedings Supplement*, 3(2):267–289, 2010.
[26] A. Waksman. An optimum solution to the firing squad synchronization problem. *Information and Control*, 9(1):66–78, 1966.

Figure 2: $\Pi$: route from $\sigma_7$ to $\sigma_2$, indicated via thick arrows.

Figure 3: $\Pi$, step 1: The general, $\sigma_1$, starts the first phase, by broadcasting the complex symbol $a_{1,1}$. 
Figure 4: Π, step 9: Cell $\sigma_{10}$ has just learned that it is a virtual-dag leaf, by receiving cell’s $\sigma_4$’s pseudo-convergecast message, the complex symbol $a_{10,4}$. Cell $\sigma_{10}$ starts its virtual convergecast by broadcasting the complex symbol $a_{0,10,4}$ towards its virtual-dag-parent, $\sigma_9$. At this stage, each cell knows its virtual-dag-parent(s) and its own depth attribute. All other cells have already initiated their virtual convergecasts. All leaves, including $\sigma_{10}$, and some other cells already know their max-depth attribute: exactly, if they have received all their outstanding convergecast messages, or a lower bound, otherwise. Message $a_{0,10,4}$ will take three more steps: via paths $\sigma_{10}, \sigma_4, \sigma_6, \sigma_9$ and path $\sigma_{10}, \sigma_4, \sigma_7, \sigma_9$. Three other, earlier stared, virtual convergecasts run in parallel: $\sigma_2$ to $\sigma_1$, $\sigma_3$ to $\sigma_1$, $\sigma_8$ to $\sigma_6$. Smaller arrows near structural arcs indicate virtual convergecasts.

Figure 5: Π, step 12: After receiving its single expected convergecast message, cell $\sigma_9$ starts its own virtual convergecasts towards its parents, $\sigma_6$ and $\sigma_7$, by broadcasting complex symbols, $a_{6,9,4}$ and $a_{7,9,4}$, respectively. Each of these messages will take three more steps, via paths $\sigma_9, \sigma_{10}, \sigma_4, \sigma_6$ and via path $\sigma_9, \sigma_{10}, \sigma_4, \sigma_7$, respectively. These two convergecasts run in parallel with another virtual convergecast: $\sigma_9$ to $\sigma_6$. 

68
Figure 6: Π, step 18: The convergecast phase is almost completed. Cell $σ_4$ starts its convergecast towards its parent, $σ_1$, by broadcasting the complex symbol $a_{1,4,4}$. This convergecast will take four more steps, via paths $σ_4, σ_6, σ_3, σ_1$ and path $σ_4, σ_6, σ_8, σ_5, σ_1$. This is cell $σ_1$’s last outstanding convergecast.

Figure 7: Π, step 22: General $σ_1$ starts the last phase, by broadcasting the complex symbol $f_3$, carrying the countdown to firing. Small dots above a cell indicate the cell’s countdown counter, which is also broadcasted to all its digraph children.
Figure 8: $\Pi$, step 25: The last phase is almost complete. Cell $\sigma_9$ forwards the complex symbol $f_0$ to $\sigma_{10}$. This the last step before firing (not illustrated here).

Figure 9: Ring networks.
(a) Sample network, $N = 10$.

| $N$ | $e_g$ | $D$ | Steps |
|-----|-------|-----|-------|
| 2   | 1     | 1   | 18    |
| 4   | 3     | 3   | 40    |
| 6   | 5     | 5   | 62    |
| 8   | 7     | 7   | 90    |
| 10  | 9     | 9   | 122   |
| 12  | 11    | 11  | 158   |
| 14  | 13    | 13  | 198   |
| 16  | 15    | 15  | 242   |
| 18  | 17    | 17  | 290   |
| 20  | 19    | 19  | 342   |

(b) Results.

Figure 10: Ring networks of size 2 rings.

(a) Sample network, $N = 18$.

| $N$ | $e_g$ | $D$ | Steps |
|-----|-------|-----|-------|
| 5   | 4     | 4   | 57    |
| 10  | 5     | 5   | 67    |
| 15  | 6     | 6   | 78    |
| 20  | 7     | 7   | 90    |
| 25  | 8     | 8   | 101   |

(b) Results.

Figure 11: Ring networks of size 3 rings.

(a) Sample network, $N = 14$.

| $N$ | $e_g$ | $D$ | Steps |
|-----|-------|-----|-------|
| 2   | 1     | 1   | 18    |
| 5   | 3     | 3   | 40    |
| 9   | 5     | 5   | 63    |
| 14  | 7     | 7   | 90    |
| 20  | 9     | 9   | 121   |
| 27  | 11    | 11  | 156   |

(b) Results.

Figure 12: Ring networks of increasing size rings.