TRIANGULAR REDUCTIONS OF 2D TODA HIERARCHY

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Abstract. New reductions of the 2D Toda equations associated with low-triangular difference operators are proposed. Their explicit Hamiltonian description is obtained.

1. Introduction

A recent burst of interest to a theory of linear difference operators has been motivated by their connection with the theory of discrete-time integrable systems (pentagram map and its higher dimensional generalization), representation theory (Coxeter’s friezes) and the theory of cluster algebras.

The pentagram map is defined for $n$-gons in $\mathbb{RP}^2$ as follows: a vertex $v_i$ of $n$-gon $(v_1, \ldots, v_n)$ is mapped to a point which is the intersection of two diagonals $(v_{i-1}, v_{i+1})$ and $(v_{i}, v_{i+2})$. If $n$ and $k+1$ are co-prime, then, as shown in [13], the moduli space of $n$-gons in $\mathbb{RP}^k$ is isomorphic, as algebraic varieties, to the space $E_{k+1,n}$ of $n$-periodic linear difference equations

\begin{equation}
V_i = a_i^{(1)}V_{i-1} - a_i^{(2)}V_{i-2} + \cdots + (-1)^{k-1}a_i^{(k)}V_{i-k} + (-1)^kV_{i-k-1},
\end{equation}

whose all solutions are (anti)periodic

\begin{equation}
V_{i+n} = (-1)^kV_i.
\end{equation}

In [6] such equations were called superperiodic.

More generally, equations (1.1) without constraint (1.2) correspond to, the so-called, twisted $n$-gon in $\mathbb{RP}^k$, that is a sequence $v_j \in \mathbb{RP}^k$, $j \in \mathbb{Z}$, for which there is a projective linear transformation $M$ of $\mathbb{RP}^k$ such that $v_{j+n} = M v_j$.

In [13] it was shown that the pentagram map is a discrete complete integrable system, i.e. that the space of $n$-periodic lower-triangular operators (1.1) of order 3 is a Poisson manifold and a complete set of integrals in involution for the pentagram map was constructed. Algebraic-geometrical integrability of the pentagram map was proved in [14].

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In [12] an explicit construction of duality of the spaces $E_{k+1,n}$ and $E_{n-k-1,n}$ was proposed, which is a generalization of the classical Gale duality for $n$-gons. In [6] this duality was clarified and connected with the theory of commuting difference operators. Along the way of establishing this connection a spectral theory of strictly lower triangular difference operators

$$L = T^{-k-1} + \sum_{j=1}^{k} a_i^{(j)} T^{-j}, \quad a_i^{(j)} = a_i^{(j)}$$

was developed. Here $T$ is the shift operator, $T\psi_j = \psi_{j+1}$. Throughout the paper it is assumed that the leading coefficient of $L$ is non-zero:

$$a_i^{(1)} \neq 0.$$  

The spectral theory of the triangular difference operators is of its own interest. The starting point of this work was an observation: the spectral theory of triangular operators is naturally connected with a special reduction of the 2D Toda hierarchy.

**Remark 1.1.** For definiteness, in this paper we consider only the case of lower-triangular reductions, since the involution $L \rightarrow L^*$, where

$$L^* = T^{k+1} + \sum_{j=1}^{k} T^j a_i^{(j)} = T^{k+1} + \sum_{j=1}^{k} a_i^{(j)} T^j,$$

is the formal adjoint operator, establishes an equivalence of the cases of lower- and upper-triangular operators.

Recall, that the 2D Toda equations by themselves

$$\partial_{\xi \eta}^2 \varphi_i = e^{\varphi_{i-1}} - e^{\varphi_i - \varphi_{i+1}}$$

is the compatibility condition of two linear problems

$$\begin{cases}
\partial_{\xi} \Psi_i = v_i \Psi_i + \Psi_{i-1} \\
\partial_{\eta} \Psi_i = c_i \Psi_{i+1}, \quad c_i = e^{\varphi_i - \varphi_{i+1}}
\end{cases}$$

The 2D Toda hierarchy is a system of commuting flows on a space of functions $\varphi$ depending on one discrete variable $i$ and two sets of continuous variables $t^\pm_m$, i.e. $\varphi = \varphi_i(t^+_m, t^-_1, t^+_2, t^-_2, \ldots)$. The variables $t^+_m$ and $t^-_m$ are identified with $\xi$ and $\eta$. The hierarchy is the compatibility condition of a system of the linear problems

$$\partial_{t^+_m} \Psi = L^+_m \Psi$$
where $L_m^\pm$ are difference operators of the form:

\begin{equation}
L_m^\pm = \sum_{j=0}^{m} a_{i,m}^{(j,\pm)} T^{\pm j}
\end{equation}

with the leading coefficients

\begin{equation}
a_{i,m}^{(m,-)} = 1, \quad a_{i,m}^{(m,+)} = e^{\varphi_i - \varphi_{i-m}}
\end{equation}

It is easy to check that the compatibility of the second equation in (1.7) with (1.8) implies

\begin{equation}
a_{i,m}^{(0,-)} = \partial_{t_m} \varphi_i, \quad a_{i,m}^{(0,+)} = 0.
\end{equation}

**Remark.** It is necessary to emphasize that although the hierarchy of any soliton equation, as a linear space of commuting vector fields is well-defined, in general there is no canonical choice of the basic "times" of the hierarchy or equivalently a canonical basis of commuting vector fields. The condition above that the operators $L_m^\pm$ are upper (lower) triangular operators of order $m$ fixes that ambiguity only partially. By this constraint the times are defined up to a linear triangular transformation $t_m^\pm = t_m^\pm + \sum_{\mu<m} c_{\mu}^\pm t_m^\pm$. We will comment more on that in Sections 2 and 3 below.

Let us fix one of the times of the hierarchy: $t_{k+1}$ (or more generally a linear combination of the first $(k+1)$ times), and consider solutions the hierarchy that do not depend on it, i.e.

\begin{equation}
\partial_{t_{k+1}} \varphi_i = 0
\end{equation}

The space of such solutions can be identified with the space of the auxiliary operators $L_{k+1}^\pm$. Note, that from (1.11) it follows that under the constraint (1.12) the operator $L = L_{k+1}^\pm$ becomes strictly low triangular, i.e. it takes the form (1.3).

The restriction of each $t_m^\pm$ flow of the hierarchy onto the space of stationary with respect to $t_{k+1}$ solutions can be seen as a finite-dimensional system admitting Lax representation

\begin{equation}
\partial_{t_m} L = [L_m^\pm, L]
\end{equation}

For $\xi = t_1^+$ the auxiliary operator has the form $L_1^- = v_i + T^{-1}$ with $v_i = \partial_{t^+} \varphi_i$ and (1.13) is equivalent to the system equations for $a_i^{(1)} = e^{\varphi_i - \varphi_{i-1}}$ and $a_i^{(j)}, j = 2, \ldots, k$:

\begin{equation}
\begin{cases}
\partial_{\xi} a_i^{(j)} = a_{i-1}^{(j-1)} - a_i^{(j-1)} + a_i^{(j)} (v_i - v_{i-j}), & j = 2, \ldots, k \\
0 = a_{i-1}^{(k)} - a_i^{(k)} + (v_i - v_{i-k-1}), & v_i = \partial_{t} \varphi_i
\end{cases}
\end{equation}
Similarly, for $\eta = t_i^+$, we get the system
\begin{equation}
\partial_{\eta} a_i^{(j)} = c_i a_{i+1}^{(j+1)} - c_{i-j} a_i^{(j+1)}, j = 1, \ldots, k
\end{equation}
where $a_i^{(1)} = e^{\phi_i - \phi_{i-1}}$, $c_i = e^{\phi_i - \phi_{i+1}}$.

The main goal of this paper is to construct a bi-Hamiltonian theory of systems (1.14) and (1.15). We show that the space of strictly low-diagonal difference operators $L$ admits two different structures of the Poisson manifold and identify the corresponding Hamiltonians.

For $k = 1$ the systems (1.14) and (1.15) have the most simple and interesting form:
\begin{equation}
\partial_{\xi} \varphi_{i-1} - \partial_{\xi} \varphi_{i+1} = e^{\phi_i - \phi_{i-1}} - e^{\phi_{i+1} - \phi_i}
\end{equation}
\begin{equation}
\partial_{\eta} \varphi_i - \partial_{\eta} \varphi_{i-1} = e^{\phi_{i-1} - \phi_{i+1}} - e^{\phi_{i-2} - \phi_i}
\end{equation}

A posteriori, in these cases one of our main result can be directly verified. Namely, it is easy to check that the systems (1.16) and (1.17) are Hamiltonian with respect to the form $\omega = \sum_{i=1}^n d\phi_i \wedge d\phi_{i+1}$, $\varphi_i = \varphi_{i+n}$, with the Hamiltonians
\begin{equation}
H^- = \sum_{i=1}^n e^{\phi_i - \phi_{i-1}}, \quad H^+ = \sum_{i=1}^n e^{\phi_{i-2} - \phi_i}, \varphi_i = \varphi_{i+n},
\end{equation}
respectively. In this simple case the second Hamiltonian structure of equations (1.16) and (1.17) is not so obvious. In the last part we prove that under (one-to-one for odd $n$) change of variables $e^{\phi_i - \phi_{i-1}} = x_i - x_{i-2} + e_1$ equations (1.16) take the form of Hamiltonian equations with respect to the form $\tilde{\omega} = \sum_{i=1}^n dx_i \wedge dx_{i-1}$, $x_i = x_{i+n}$, with the Hamiltonian
\begin{equation}
\tilde{H}^- = \sum_{i=1}^n x_i^2(x_{i-1} - x_{i+1})
\end{equation}

2. Necessary facts

In this section we present necessary facts from the spectral theory of strictly lower-triangular operators and the construction of algebraic-geometrical solutions of the 2D Toda hierarchy.

2.1. The spectral theory of lower-triangular difference operators. In the modern approach to the spectral theory of periodic difference operators central is the notion of a spectral curve associated with each $n$-periodic difference operator $L$. By definition, points of the spectral curve parameterize Bloch solutions of the equation
\begin{equation}
L \psi = E \psi,
\end{equation}
i.e. the solutions of (2.1) that are eigenfunctions for the monodromy operator $T^{-n}$

\begin{equation}
T^{-n}\psi = w\psi.
\end{equation}

Let $\mathcal{L}(E)$ be a space of the solutions of equation (2.1). It is a linear space of dimension equal to the order of $L$. The monodromy operator preserves $\mathcal{L}(E)$ and, hence, defines on it a finite-dimensional operator $T^{-n}(E)$. The pairs of complex numbers $(w, E)$ for which there exists a common solution of equations (2.1) and (2.2) satisfy the characteristic equation:

\begin{equation}
R(w, E) = \det\left( w \cdot 1 - T^{-n}(E) \right) = 0
\end{equation}

Similarly, the same polynomial $R(w, E)$ can be obtained as the characteristic polynomial of the finite dimensional operator $L(w)$ that is a restriction of $L$ on the space $\mathcal{T}(w) := \{ \psi | w\psi_{i+n} = \psi_i \}$:

\begin{equation}
R(w, E) = \det\left( E \cdot 1 - L(w) \right) = 0, \quad L(w) := L|_{\mathcal{T}(w)}
\end{equation}

The family of algebraic curves that arises as spectral curves depends on a family of operators. In case of strictly lower-triangular difference operators in ([6]) it was noticed that the characteristic equation of such operator is of the form

\begin{equation}
R(w, E) = w^{k+1} - E^n + \sum_{i>0, j>0, ni+(k+1)j < n(k+1)} r_{ij} w^i E^j = 0,
\end{equation}

where $r_{1,0} = \prod_{i=1}^{n} a_i^{1} \neq 0$ (due to (1.4)).

If $n$ and $k+1$ are co-prime, then the affine curve defined in $\mathbb{C}^2$ by (2.4) is compactified by one point $p_-$, where the functions $w(p)$ and $E(p)$, naturally defined on $\Gamma$, have pole of order $n$ and $k+1$, respectively. In other words, if one chooses a local coordinate $z$ in the neighborhood of $p_-$ such that $w = z^{-n}$, then the Laurent expansion of $E$ has the form:

\begin{equation}
E = z^{-k-1} \left( 1 + \sum_{s=1}^{\infty} e_s z^s \right), \quad w = z^{-n}.
\end{equation}

As it was emphasized in [6], the specific form of equation (2.4) allows one to single out another marked point $p_+$ on $\Gamma$. Namely, it is the only preimage of $E = 0$ where $w = 0$. It turns out that at this point $E = E(p)$ has a simple zero, and the function $w = w(p)$ has zero of order $n$

\begin{equation}
w = \frac{1}{r_{1,0}} E^n \left( 1 + \sum_{s=1}^{\infty} w_s E^s \right)
\end{equation}
Analytic properties of the Bloch solutions in the neighborhoods of marked points are described by the following two statements:

**Lemma 2.1.** \((\text{[6]}\) Let \(L\) be an operator of form \((1.3)\) whose order and the period are co-prime. Then there is a unique formal series \(E(z)\) of the form \((2.5)\) such that the equation \(L\psi = E\psi\) has a unique formal solution of the form

\[
\psi_i(z) = z^i \left(1 + \sum_{s=1}^{\infty} \xi^-(i)z^s \right).
\]

with periodic coefficients \(\xi^-(i) = \xi^-(i+n)\) normalized by the condition \(\xi^- (0) = 0\).

For further use let us briefly outline the proof.

**Proof.** The substitution of \((2.7)\) and \((2.5)\) into the equation \(L\psi = E\psi\) gives a system of difference equations for unknown constants \(e_s\) and unknown functions \(\xi^-(i)\) of the discrete variable \(i\). The first of them is the equation

\[
e_1 + \xi^-(i) - \xi^-(i - k - 1) = a^{(k)}_i.
\]

The periodicity constraint for \(\xi^-\) uniquely defines

\[
e_1 = n^{-1} \sum_{i=1}^{n} a^{(k)}_i
\]

and reduces difference equation \((2.8)\) of order \(k+1\) to the difference equation of order 1:

\[
m e_s + \xi^- (i) - \xi^- (i - 1) = \sum_{j=0}^{m-1} a^{(k)}_{i-j(k+1)},
\]

where \(m\) is the integer \(1 \leq m < n\) such that \(m(k+1) = 1 \pmod{\text{n}}\). Equation \((2.10)\) and the initial condition \(\xi^- (0) = 0\) uniquely defines \(\xi^- (i)\).

For arbitrary \(s\) the defining equation for \(e_s\) and \(\xi^-\) has the form:

\[
e_s + \xi^- (i) - \xi^- (i - k - 1) = Q_s(e_1, \ldots, e_{s-1}; \xi_1, \ldots, \xi_{s-1}, a^{(j)}_i)
\]

where \(Q_s\) is an explicit function linear in \(e_s', \xi_{s'}\), \(s' < s\), and polynomial in \(a^{(j)}_i\). The same arguments as above show that it has a unique periodic solution. The lemma is proved.
Lemma 2.2. ([6]) Equation $L\psi = E\psi$ has an unique formal solution of the form

$$\psi_i(E) = e^{\varphi_i} E^{-i} \left( 1 + \sum_{s=1}^{\infty} \xi_+^s(i) E^s \right), \quad a_i^{(1)} = e^{\varphi_i - \varphi_{i-1}},$$

normalized by the condition $\xi_+^s(0) = 0$.

Proof. The substitution of (2.12) into (2.1) gives a system of non-homogeneous first order difference equations for unknown coefficients $\xi^-_s$. For $s = 1$ we have

$$\xi_1^+(i) - \xi_1^+(i-1) = e^{\varphi_i - 2 - \varphi_i} a_i^{(2)}$$

For any $s$ the equations have similar form

$$\xi_s^+(i) - \xi_s^+(i-1) = e^{-\varphi_i} q_s(\xi_1^+, \ldots, \xi_s^+, a_i^{(j)}),$$

which together with the initial conditions recurrently define $\xi_s^+(i)$ for all $i$.

The uniqueness of the formal solution (2.12) implies

Corollary 2.3. The formal series (2.12) is the Bloch solution, i.e. it satisfies (2.2) with

$$w(E) = \psi_n(E) = r_{i,0}^{-1} E^n \left( 1 + \sum_{s=1}^{\infty} w_s E^s \right)$$

From Lemma 2.1 it follows that the component $\psi_i(p), p := (w, E) \in \Gamma$, of the Bloch solution $\psi(p)$ considered as a function on the spectral curve has zero of order $i$ at the marked point $p_-$. Lemma 2.2 implies that $\psi_i(p)$ has pole of order $i$ at the marked point $p_+$. Then the standard arguments (for details see [1]) show that $\psi_i$ is a meromorphic function on $\Gamma$ having away of the marked point $g$ poles $\gamma_1, \ldots, \gamma_g$ that do not depend on $i$. These analytic properties are the defining properties of the so-called, discrete Baker-Akhiezer (BA) function, introduced in [2]. That establishes a connection of the spectral theory of the lower-triangular operators with the theory of commuting difference operators (see details in [4]), and the theory of algebraic-geometrical solutions of the 2D Toda hierarchy.

The correspondence

$$L \mapsto \{ \Gamma, D = \gamma_1 + \cdots + \gamma_g \}$$

where $\Gamma$ is the spectral curve of the operator $L$ and $D$ is the divisor of poles of the Bloch solution $\psi$ is usually referred as the direct spectral transform. It is one-to-one correspondence of the open everywhere
dense subsets of the spaces of operators and algebraic-geometrical spectral data. The construction of the inverse spectral transform is a particular case of the general construction of algebraic-geometrical solutions of the 2D Toda hierarchy which we now present.

2.2. Algebraic-geometrical solutions of the 2D Toda lattice hierarchy. Let \( \Gamma \) be a smooth genus \( g \) algebraic curve with fixed local coordinates \( z_\pm \) in the neighborhoods of two marked points \( p_\pm \in \Gamma, \ z_\pm(p_\pm) = 0 \), and let \( t = \{ t^\pm_j, j = 1, 2, \ldots \} \) be a set of complex variables (it is assumed that only finite number of them are non-zero). Then, as shown in [3]:

Lemma 2.4. For a generic set of \( g \) points \( \gamma_1, \ldots, \gamma_g \) there is a unique meromorphic function \( \Psi_i(t, p) \), \( p \in \Gamma \) such that: (i) away of the marked points \( p_\pm \) it has at most simple poles at \( \gamma_s \) (if \( \gamma_s \) are distinct); (ii) in the neighborhoods of the marked points it has the form

\[
\Psi_i(t, z_\pm) = z_\pm^{i \pm} e^{\left( \sum_{m} t^\pm_m z_\pm^{-m} \right)} \left( \sum_{s=1}^{\infty} \xi_s^\pm(i, t) z_\pm^s \right), \quad \xi_0^- = 1.
\]

The function \( \Psi_i \) is the particular case of, so-called, two-point multivariable BA function (see the definition of the multi-point multivariable BA function in [11]). Uniqueness of it implies

Theorem 2.5. Let \( \Psi_i(t, p) \) by the BA function corresponding to any set of data above, i.e. \( \{ \Gamma, p_\pm, z_\pm; \gamma_1, \ldots, \gamma_g \} \). Then there exist unique operators \( L^\pm_m \) of the form (1.9, 1.10) with \( \varphi_i(t) := \ln \xi_0^+ (t) \) such that equations (1.8) hold.

Remark. By definition the BA function depends on a choice of local coordinates \( z_\pm \) in the neighborhood of the marked points \( p_\pm \). A change of local coordinate corresponds to triangular transformation of the times \( t^\pm_m \) (compare with the remark in the Introduction).

The algebraic-geometrical solutions of the 2D Toda hierarchy can be explicitly written in terms of the Riemann theta-function. Let us choose a basis of cycles \( a_i, b_i, i = 1, \ldots, g \) on \( \Gamma \) with the canonical matrix of intersections: \( a_i \circ a_j = b_i \circ b_j = 0, a_i \circ b_j = \delta_{ij} \). Then one can consequently define:

(a) a basis of the normalized holomorphic differentials \( \omega_i, \oint_{a_j} \omega_i = \delta_{ij} \);

(b) the matrix of their \( b \)-periods, \( B_{ij} = \oint_{b_j} \omega_i \), and the corresponding Riemann theta function

\[
\theta(z) = \theta(z|B) = \sum_{m \in \mathbb{Z}^g} e^{2\pi i (m, z) + \pi i (Bm, m)}, \quad z = z_1, \ldots, z_g.
\]
(c) the Abel transform $A(p)$ that is a vector with coordinates $A_k(p) = \int^p \omega_k$;
(d) the normalized Abelian differential of the third kind $d\Omega_0, \oint \omega_0 \, d\Omega_0 = 0$, having simple poles with residues $\pm 1$ at $p$, and the normalized abelian differentials of the second kind $d\Omega_{m,\pm}, \oint \omega_{m,\pm} \, d\Omega_{m,\pm} = 0$, having poles at $p_\pm$ of the form $d\Omega_{m,\pm} = d(z_\pm - m + O(z_\pm))$.

**Lemma 2.6.** The Baker-Akhiezer function is given by the formula

$$\Psi_i(t,p) = \frac{\theta(A(p) + iU_0 + \sum U_{m,\pm} t_m^\pm + Z) \theta(A(p) + Z)}{\theta(A(p) - iU_0 + \sum U_{m,\pm} t_m^\pm + Z) \theta(A(p) + Z)} e^{i\Omega_0(p) + \sum t_m \Omega_{m,\pm}(p)}$$

Here the sum is taken over all pairs of indices $(m, \pm)$ and:

$a$) $\Omega_0(p)$ and $\Omega_{m,\pm}(p)$ are the abelian integrals, $\Omega_0(p) = \int^p d\Omega_0$, $\Omega_{m,\pm}(p) = \int^p d\Omega_{m,\pm}$, corresponding to the differentials introduced above and normalized by the condition that in the neighborhood of $p_-$ they have the form

$$\Omega_0(z_-) = \ln z_- + O(z_-), \quad \Omega_{m,-}(z_-) = z_-^m + O(z_-), \quad \Omega_{m,+}(z_-) = O(z_-);$$

$b$) $2\pi iU_0$, $2\pi iU_{\alpha,\beta}$ are the vectors of their $b$-periods, i.e. vectors with the coordinates

$$U_0^k = \frac{1}{2\pi i} \oint_{b_k} d\Omega_0, \quad U_{m,\pm}^k = \frac{1}{2\pi i} \oint_{b_k} d\Omega_{m,\pm};$$

c) $Z$ is an arbitrary vector (it corresponds to the divisor of poles of Baker-Akhiezer function).

Notice, that from the bilinear Riemann relations it follows that $U_0 = A(p_-) - A(p_+)$. Then from the comparison of the evaluation of (2.13) one gets:

**Theorem 2.7.** The algebraic-geometrical solutions of the 2D Toda lattice are given by the formula

$$\varphi_i(t) = \ln \frac{\theta((i + 1)U_0 + \sum U_{m,\pm} t_m^\pm + \bar{Z})}{\theta(iU_0 + \sum U_{m,\pm} t_m^\pm + \bar{Z})} + i\alpha + \sum c_{m,\pm} t_m^\pm$$

where $\bar{Z} = Z + A(p_+)$ is an arbitrary vector, the vectors $U_0$ and $U_{m,\pm}$ are defined in (2.10), and the constants $c_0$ and $c_{m,\pm}$ are the leading coefficients of the expansions of the abelian integrals in the neighborhood of $p_+$:

$$\Omega_0(z_+) = - \ln z_+ + c_0 + O(z_+),$$

$$\Omega_{m,-}(z_+) = c_{m,-} + O(z_+), \quad \Omega_{m,+}(z_+) = z_+^m + c_{m,+} + O(z_+).$$
From (2.20) it is easy to see than in general the algebraic-geometric solution is quasi-periodic function of all the variables including $i$. It is $n$-periodic in the discrete variable $i$ if the vector $nU_0 = n(A(p+) - A(p-))$ is a vector in the lattice defining the Jacobian of the corresponding curve $\Gamma$. The latter constraint is equivalent to the following:

**Lemma 2.8.** Let $\Gamma$ be a smooth algebraic curve on which there is a meromorphic function $w$ with the only pole at some point $p_-$, and zero at another point $p_+$ of order $n$ equal to the order of its pole. Then the BA function corresponding to $\Gamma, p_\pm$ and any divisor $\gamma_s$ satisfies the equation (2.2), and therefore the corresponding solutions of 2D Toda hierarchy are $n$-periodic.

For the proof of the statement it is enough to check that the functions $\Psi_{i-n}$ and $w\Psi_n$ have the same analytical properties and hence coincide.

### 2.3. The dual Baker-Akhiezer function.

For further use, recall also a notion of the dual Baker-Akhiezer function (see details in [11]). First for a non-special degree $g$ divisor $D = \gamma_1 + \cdots + \gamma_g$ on a smooth genus $g$ algebraic curve $\Gamma$ with two marked points one can define a dual degree $g$ effective divisor $D^+ = \gamma_1^+ + \cdots + \gamma_g^+$ as follows: for a given $D$ there exists a unique meromorphic differential $d\Omega$ with simple poles with resides $\pm 1$ at the marked points that is holomorphic everywhere else, and which has zeros at $\gamma_s, d\Omega(\gamma_s) = 0$. The zero divisor of $d\Omega$ is of degree 2$g$. Hence, besides of $\gamma_s$ the differential $d\Omega$ has zeros at $g$ other points $\gamma_s^+$, i.e. $d\Omega(\gamma_s^+) = 0$. In other words, the divisor $D^+$ is defined by the equation $D + D^+ = \mathcal{K} + p_+ + p_- \in J(\Gamma)$ where $\mathcal{K}$ is the canonical class, i.e. the equivalence class of the zero divisor of a holomorphic differential on $\Gamma$.

If the BA function $\Psi_i(t,p)$ is defined by the divisor $D$ then, its dual function $\Psi_i^+(t,p)$ is uniquely defined by the following analytical properties: (i) away of the marked points $p_\pm$ it is meromorphic and has at most simple poles at $\gamma_s^+$ (if $\gamma_s^+$ are distinct); (ii) in the neighborhoods of the marked points it has the form

$$
(2.22) \quad \Psi_i^+(t, z_\pm) = z_\pm^i e^{-(\sum_m t_m z_\pm^{-m})} \left( \sum_{s=1}^{\infty} \chi^+_s(i, t) z_\pm^s \right), \quad \chi_0^- = 1.
$$

It is easy to see that the differential $\Psi_i^+ \Psi_j d\Omega$ is meromorphic with the only possible poles at $p_\pm$. Moreover, for $i > j$ ($i < j$) it is holomorphic at $p_+$ ($p_-$). Since the sum of residues of a meromorphic differential equals zero, we get the equations

$$
(2.23) \quad \text{res}_{p_\pm} \Psi_i^+ \Psi_j d\Omega = \pm \delta_{i,j}
$$
which imply that $\Psi^+$ satisfies the adjoint equations
\begin{equation}
  (\Psi^+ L)_i \equiv \Psi^+_{i+k+1} + a^{(k)}_{i+k} \Psi^+_k + \ldots a^{(1)}_{i+1} \Psi^+_{i+1} = E\psi_i
\end{equation}
and
\begin{equation}
  -\partial_{t_m} \Psi^+ = \Psi^+ L^\pm_m
\end{equation}
The theta-functional formula (2.20) for the dual BA function takes the form:
\begin{equation}
  \psi^+_i(t, p) = \frac{\theta(A(p) - iU_0 - \sum U_{m,\pm} t_m^{\pm} + Z^+) \theta(A(p_-) + Z^+)}{\theta(A(p_-) - iU_0 - \sum U_{m,\pm} t_m^{\pm} + Z^+) \theta(A(p) + Z^+)} \times \\
  \times e^{-i\Omega_0(p) \sum t_m^{\pm} \Omega_{m,\pm}(p)}
\end{equation}
where $Z + Z^+ = K + A(p_+) + A(p_-)$.

From the analytical properties of $\Psi^+$ it easy follows that:

**Lemma 2.9.** Under the assumptions of Lemma 2.8 the dual BA function satisfies the equation
\begin{equation}
  (2.27) \quad \Psi^+_i = w \Psi^+_{i-n}
\end{equation}

**Important remark.** As it was already mentioned above, in the particular case the construction of the algebraic-geometrical solutions of the 2D Toda hierarchy can be seen as the construction of the inverse spectral transform. Indeed, let $\Gamma$ be a curve defined by an equation of the form (2.4), then a simple comparison of the analytic properties shows that the Bloch function of the operator $L$ coincides with the evaluation of the multivariable BA function at the zero value of all continuous times: $\psi_i = \Psi_i(t_k^\pm = 0)$

3. **The Hamiltonian theory of the reduced systems**

The systems in question, namely equations (1.14) and (1.15) were defined as a special reduction of the 2D Toda hierarchy. Therefore, the formulae (2.18), where the Riemann theta-function corresponds to any curve defined by equation (2.4), provides solutions to our reduction of the 2D Toda hierarchy. In this section we develop Hamiltonian theory of this reduced system following the general scheme proposed in [8, 9].

According to that scheme on the space of operators $L$ which is identified with a phase space of the system, one can define a family of two-forms by the formula
\begin{equation}
  (3.1) \quad \omega^{(i)} = -\frac{1}{2} \sum_{\alpha} \text{res}_{p_{\alpha}} E^{-i(\psi^+(w) \delta L \wedge \delta \psi(w))} d\Omega
\end{equation}
where $\delta F(L)$ stands for the differential of a function $F$ on the space of the operators (the BA function with fixed eigenvalue $w$ and fixed...
normalization is such a function), and the sum is taken over the set of points \( p_\alpha \) on the corresponding spectral curve where the expression in the right hand side a’priory has poles: namely at the marked point \( p_\pm \), where the BA function and its dual have poles, and for \( i > 0 \) at the zeros \( p_\ell, \ell = 1, \ldots, k, \) of the function \( E = E(p) \) where \( w = w(p) \) does not vanish, i.e. \( E(p_\ell) = 0, \ w(p_\ell) \neq 0. \)

3.1. The differential \( d\Omega \). Our first goal is to derive a closed expression for \( d\Omega \) introduced above by its analytic properties in terms of Bloch eigenfunction of the operator \( L \) and its adjoint one.

Let us assume first that the coefficients of the operator are \( n \)-periodic. Following a line of arguments in [5] consider the differential \( d\psi \) with respect to the spectral variable. It satisfies the non homogeneous linear equation

\[
(L - E) d\psi = dE \psi
\]

which is just the differential of equation (2.1). Taking the differential of equation (2.2) we get that \( d\psi \) satisfies the following monodromy relation

\[
w_\ell d\psi_i + dw_\ell \psi_i = d\psi_{i-n}
\]

For brevity let us denote the average of a function \( f_i \) over the interval \( l+1 \leq i \leq l+n \) by \( \langle f \rangle_l = \frac{1}{n} \sum_{i=l+1}^{l+n} f_i \) and write \( \langle f \rangle \) when that average does not depend on \( l \), as in the case of \( n \)-periodic functions.

From (3.2) it follows that

\[
E \langle \psi^+ d\psi \rangle_l + dE \langle \psi^+ \psi \rangle = \langle \psi^+(Ld\psi) \rangle_l = \sum_{j=1}^{k+1} \sum_{i=l+1}^{l+n} a_j^{(i)} \psi^+_i d\psi_{i-j}
\]

Equation (2.24) implies

\[
E \langle \psi^+ d\psi \rangle_l = \sum_{j=1}^{k+1} \sum_{i=l+1}^{l+n} a_j^{(i)} \psi^+_i d\psi_i = \sum_{j=1}^{k+1} \sum_{i=l+1+j}^{l+n+j} a_j^{(i)} \psi^+_i d\psi_{i-j}
\]

Subtracting (3.5) from (3.4) and using (3.3) we get

\[
dE \langle \psi^+ \psi \rangle = \frac{dw}{nw} \sum_{j=1}^{k+1} \sum_{i=l+1}^{l+j} a_j^{(i)} \psi^+_i \psi_{i+j}
\]

Notice that the left hand side of (3.6) does not depend on \( l \). Hence, the right hand side of (3.6) is also \( l \)-independent. Taking the average of the right hand side of (3.6) in \( l \) we obtain the equation

\[
dE \langle \psi^+ \psi \rangle = \frac{dw}{nw} \langle \psi^+ (L^{(1)} \psi) \rangle
\]
where

\[ L^{(1)} := \sum_{j=1}^{k+1} j a_i^{(j)} T^{-j} \]

is the difference analog of the first descendent of a differential operator introduced in [5].

From (3.7) it follows that the zeroes of \( dw \) coincide with the zeroes of the meromorphic function \( \langle \psi^+ \psi \rangle \) and the zeros of \( dE \) coincide with the zeros of \( \langle \psi^+(L^{(1)} \psi) \rangle \). Hence:

**Lemma 3.1.** The differential

\[ d\Omega := \frac{dw}{nw \langle \psi^+ \psi \rangle} = \frac{dE}{\langle \psi^+(L^{(1)} \psi) \rangle}. \]

is holomorphic away of the marked points \( p_{\pm} \), and has zeros at the poles of \( \psi \) and \( \psi^+ \); at \( p_{\pm} \) it has simple poles with residues \( \pm 1 \), i.e \( d\Omega \) is the differential introduced above in the definition of the dual BA function.

**Example:** \( k = 1 \)

\[ d\Omega = \frac{dE}{\langle a_i^{(1)} \psi_i^+ \psi_{i-1}^+ + 2 \psi_i^+ \psi_{i-2} \rangle} = \frac{dw}{nw \langle \psi^+ \psi \rangle} \]

**Example:** \( k = 2 \)

\[ d\Omega = \frac{dE}{\langle a_i^{(1)} \psi_i^+ \psi_{i-1}^+ + 2 a_i^{(2)} \psi_i^+ \psi_{i-2} + 3 \psi_i^+ \psi_{i-3} \rangle} = \frac{dw}{nw \langle \psi^+ \psi \rangle} \]

3.2. The symplectic leaves and the Darboux coordinates. It is necessary to emphasize that the form \( \omega^{(i)} \) is not closed and is degenerate on the space of all the operators \( L \). It becomes closed after restriction onto certain subvarieties. As we shall see below, only the forms \( \omega^{(0)} \) and \( \omega^{(1)} \) are non-degenerate on the corresponding subvariety. That allows to regard the total space of operators \( L \) as a Poisson manifold foliated by symplectic leaves of two types. The existence of these two types of foliation reflects, the so-called, bi-Hamiltonian origin of integrable systems.

The constrains defining the symplectic leaves are equivalent to the condition that the form \( \omega^{(i)} \) does not depend on a choice of the normalization of the Bloch eigenvector \( \psi \). The change of normalization is equivalent to the transformation \( \psi_i \rightarrow \psi_i h, \ \psi_i^+ \rightarrow \psi_i^+ h^{-1} \), where \( h = h(w) \) is a scalar function. Under this transformation the differential in the right hand side of (3.11) gets transformed to the differential

\[ E^{-i} \langle \psi^+(w) \delta L \wedge \delta \psi(w) \rangle d\Omega + E^{-i} \langle \psi^+(w) \delta L \psi(w) \rangle \wedge \delta \ln h d\Omega \]
Hence, the form $\omega^{(i)}$ is normalization independent only when the last term in (3.12) is holomorphic near the points $p_\alpha$. From the equation
\[(L - E)\delta\psi(w) = -(\delta L - \delta E(w))\psi\]
and the definition of the adjoint operator it follows that
\[(3.14) \quad \langle \psi^+((\delta L - \delta E)\psi) \rangle = \langle (\psi^+(E - L))\delta\psi \rangle = 0\]
Then using (3.9) we obtain the following statement:

**Lemma 3.2.** The restriction of the form $\omega^{(i)}$ given by (3.1) onto a subvariety of the space of all operators such that on this subvariety the differential $E^{-1}\delta E(w) d\ln w$ is holomorphic in the neighborhoods of points $p_\alpha$ is normalization independent.

**Example** $i = 0$. For $i = 0$ the sum in (3.1) is taken over the marked points $p_\pm$ only. At the point $p_+$ (where $w = 0$) the function $E$ has zero. Hence, the form $Ed\ln w$ has the only pole at $p_-$. Therefore for any operator $L$, the corresponding coefficient of (2.5) vanishes. Namely, $e_{k+1} \equiv 0$.

In the neighborhood of $p_-$ where the function $E$ has pole of order $(k + 1)$ the form $\delta E(w)d\ln w$ has pole of order $(k + 2)$ with no residue. Hence, if for any set $c = (c_1, \ldots, c_k)$ of constants we define $\Lambda_0$ as the subvariety of operators $L$ satisfying the constraints
\[(3.15) \quad \Lambda_0 := \{ L \in \Lambda_0^c \mid e_s(L) = c_s, s = 1, \ldots, k \}\]
where $e_s = e_s(L)$ are the coefficients of expansion (2.5), then:

**Corollary 3.3.** The form $\omega^{(0)}$ restricted to the subvariety $\Lambda_0^c$ is normalization independent.

**Example** $i = 1$. The $E^{-1}\delta E(w) d\ln w$ is holomorphic at the marked point $p_-$. Since, the sum of its residues equals zero, it is holomorphic at the point $p_+$ if it is holomorphic at the points $p_\ell, \ell = 1, \ldots, k$. Using the chain rule we get that the variation of $E(w)$ with fixed $w$ is related with the variation of $w(E)$ with fixed $E$ by the formula
\[
\delta E(w) dw + \delta w(E) dE = 0.
\]
Hence, $\delta \ln E(w) d\ln w$ is holomorphic at $p_\ell$ (the preimages of $E = 0$ where $w \neq 0$) if the equations $\delta w(p_\ell) = 0$ are satisfied. The latter hold along the subvariety
\[(3.16) \quad \Lambda_1 := \{ L \in \Lambda_1^c \mid r_{i,0}(L) = c_i, 1 = 1, \ldots, k \}\]
where $c = (c_1, \ldots, c_k)$ are constants and $r_{i,0}(L) = r_{i,0}$ are the coefficients of the polynomial $\det L(w) = w^{k+1} + \sum_{i=1}^k r_{i,0} w^i$.

**Corollary 3.4.** The form $\omega^{(1)}$ restricted to the subvariety $\Lambda_1^c$ is normalization independent.
Remark 3.5. For \( i > 1 \) the subvarieties \( \Lambda_i^c \), on which the restriction of \( \omega^{(i)} \) is normalization independent, are described in a similar way by a system of \( i(k+1) - 1 \) equations:

\[
\Lambda_i^c := \{ L \in \Lambda_i^c \mid w_{\ell,s} = c_{\ell,s}, s = 1, \ldots, i; w_s = c_s, s = 2, \ldots, i \}
\]

where \( w_{\ell,s} \) are the coefficients of the expansions \( w = \sum_{s=0}^{\infty} w_{\ell,s} E^s \) of \( w \) at the preimages \( p_\ell \) on \( \Gamma \) of \( E = 0 \) at which \( w(p_\ell) \neq 0 \); \( w_s \) are the coefficients of the expansion (2.15) of \( w \) at \( p_+ \) and \( c_{i,s}, c_s \) are constants. Hence \( \Lambda_i^c \) is of dimension \( (n-1)k - i + 1 \). Recall that the family of curves \( \Gamma \) defined by equations of the form (2.4) is of dimension \( k(n+1) \) (the number of the coefficients \( r_{ij} \)). For generic values of coefficients \( r_{ij} \) the curve \( \Gamma \) is smooth and has genus \( g = \frac{k(n+1)}{2} \). Therefore, the correspondence (2.16) restricted to \( \Lambda_i^c \) identifies the latter with the total space of the Jacobian bundle over the space of the corresponding spectral curves. For \( i > 1 \) the dimension of the fiber is bigger then the dimension of the base. Hence the form \( \omega^{(i)} \) restricted to \( \Lambda_i^c \) is degenerate for \( i > 1 \).

3.3. The Darboux coordinates. For completeness, let us present a construction of the Darboux coordinates for the restriction \( \hat{\omega}^{(i)} \) of \( \omega^{(i)} \) onto the subvariety \( \Lambda_i^c \), i.e.

\[
\hat{\omega}^{(i)} := \omega^{(i)} \big|_{\Lambda_i^c}
\]

Theorem 3.6. Let \( \gamma_s \) be the poles of the BA function. Then the equation

\[
\hat{\omega}^{(i)} = \frac{1}{n} \sum_{s=1}^{g} E^{-i}(\gamma_s) \delta E(\gamma_s) \wedge \delta \ln w(\gamma_s).
\]

holds.

The meaning of the right hand side of this formula is as follows. The spectral curve is equipped by definition with the meromorphic functions \( E \) and \( w \). The evaluations \( E(\gamma_s), w(\gamma_s) \) at the points \( \gamma_s \) define functions on the space of \( L \) operators. The wedge product of their external differentials is a two-form on our phase space.

Proof. The proof of the formula (3.20) is very general and does not rely on any specific form of \( L \). Let us present it briefly following the proof of Lemma 5.1 in [7] (more details can be found in [10]). The differential whose residues define \( \omega^{(i)} \) by (3.1) is a meromorphic differential on the
spectral curve $\Gamma$. Therefore, the sum of its residues at the punctures $p_\alpha$ is equal to the negative of the sum of the other residues on $\Gamma$. There are poles of two types. First of all, the differential has poles at the poles $\gamma_s$ of $\psi$. Note that $\delta \psi$ has pole of the second order at $\gamma_s$. Taking into account that $d\Omega$ has zero at $\gamma_s$ we obtain

$$\text{res}_{\gamma_s} E^{-i} \langle \psi^+ \delta L \wedge \delta \psi \rangle d\Omega = \frac{E^{-i} \langle \psi^+ \delta L \psi \rangle}{n \langle \psi^+ \psi \rangle} (\gamma_s) \wedge \delta \ln w(\gamma_s) = \frac{1}{n} E^{-i} (\gamma_s) \delta E(\gamma_s) \wedge \delta \ln w(\gamma_s).$$

The last equality follows from (3.14) which is just the standard formula for the variation of an eigenvalue of an operator.

The second set of poles of the differential in the righthand side of (3.1) is the set of zeros $q_j$ of the differential $dw$. Indeed, in the neighborhood of $q_j$ the local coordinate on the spectral curve is $\sqrt{w - w(q_j)}$ (in general position when the zero is simple). Taking a variation of the Taylor expansion of $\psi$ in that coordinate we get that

$$\delta \psi = -\frac{d\psi}{dw} \delta w(q_j) + O(1).$$

Therefore, $\delta \psi$ has simple pole at $q_j$. In the similar way we have

$$\delta E = -\frac{dE}{dw} \delta w(q_j).$$

Equalities (3.22) and (3.23) imply that

$$\text{res}_{q_j} E^{-i} \langle \psi^+ \delta L \wedge \delta \psi \rangle d\Omega = \text{res}_{q_j} \frac{E^{-i} \langle \psi^+ \delta L \psi \rangle}{n \langle \psi^+ \psi \rangle} \wedge \delta E \ln w = \text{res}_{q_j} \frac{E^{-i} \langle \delta \psi^+ \psi \rangle}{n \langle \psi^+ \psi \rangle} \wedge \delta E \ln w.$$

Due to skew-symmetry of the wedge product we may replace $\delta L$ in (3.24) by $(\delta L - \delta E)$. Then, using the identities $\psi^*(\delta L - \delta E) = \delta \psi^*(E - L)$ and $(E - L)d\psi = -dE \psi$, one gets

$$\text{res}_{q_j} E^{-i} \langle \psi^+ \delta L \wedge \delta \psi \rangle d\Omega = -\text{res}_{q_j} \frac{E^{-i} \langle \delta \psi^+ \psi \rangle}{n \langle \psi^+ \psi \rangle} \wedge \delta E \ln w = \text{res}_{q_j} \frac{E^{-i} \langle \psi^+ \delta \psi \rangle}{n \langle \psi^+ \psi \rangle} \wedge \delta E \ln w,$$

where in the last equality we use the identity $\langle \psi^+ \psi \rangle(q_j) = 0$ (which follows, as we already stressed, from (3.7)). By definition of the locus on which $\omega^{(i)}$ is normalization independent (see Lemma 3.2) the form in the right hand side of (3.25) has no poles at the points $p_\alpha$. Besides of poles at $q_i$ it has poles at $\gamma_s$, only. Hence, after the restriction on the leave we get the equation
Equations (3.21, 3.25, 3.26) directly imply (3.20). The theorem is proved.

3.4. The Hamiltonians. The next step in the construction of the Hamiltonian theory for the systems admitting the Lax representation is to show that the contraction of the form $\omega^{(i)}$, restricted to the subvariety, where it is normalization independent, with the vector field $\partial_t$ defined by the Lax equation is an exact one form, i.e. $\tilde{\omega}^{(i)}(\partial_t, X) = \delta H^{(i)}(X)$. Then, on any subvariety on which the form $\tilde{\omega}^{(i)}$ is non-degenerate the vector-field $\partial_t$ is Hamiltonian with the Hamiltonian $H$.

Below we apply the general scheme to equations (1.14) and (1.15) and explicitly compute the corresponding Hamiltonians. Let $\partial_t$ be the vector field defined by the Lax equations, then

(3.27) \[ \partial_t L = [M, L], \quad \partial_t \psi = M\psi - \psi f \]
where $f$ is a meromorphic function on the spectral curve.

Remark 3.7. The appearance of the term with $f$ in the expression for $\partial_t \psi$ is due to the fact that in the definition of the $\omega^{(i)}$ it is assumed that the normalization of the Bloch function $\psi$ is time-independent: $\psi_0 \equiv 1$. With that normalization if the operator $L$ depends on $t$ according to a Lax equation, then the spectral curve $\Gamma$ is time independent and the time dependence of the pole divisor $D(t)$ of $\psi(t)$ becomes linear after the Abel transform. The latter follows form the relation

(3.28) \[ \psi_i(t, p) = \Psi_i(t, p)\Psi_0^{-1}(t, p) \]
where $\Psi$ is the multi variable BA function given by (2.18). Then the equation (1.8) implies equation (3.27) with $f(t, p) = \partial_t \ln \Psi_0(t, p)$. The function $f$ has poles at the marked points $p_\pm$ of the form

(3.29) \[ f = \sum_{s=1}^{m_\pm} c_s^\pm z^{-s} + O(1) \]
where \( c^\pm \) are constants which in fact parameterize commuting flows of the hierarchy and \( m^\pm \) are positive and negative orders of the operator \( M \).

**Theorem 3.8.** The vector-field \( \partial_{t^h_m} \) defined by Lax equation (1.13) restricted to the subvariety \( \Lambda^c_i \) is Hamiltonian for \( i = 0, 1 \) with respect to the form \( \tilde{\omega}^{(i)} \), and with the Hamiltonian

\[
H^{(0)}_{t^m} = \text{res}_{p_-} z^{-m} E(z) d \ln z = c_{m+k+1}
\]

\[
H^{(1)}_{t^m} = \text{res}_{p_-} z^{-m} \ln E(z) d \ln z
\]

where \( E(z) \) is the series (2.7) with the coefficients defined in Lemma 2.1, and

\[
H^{(i)}_{t^m} = \frac{1}{n} \text{res}_{p_+} E^{-m-i} \ln w(E) dE, \quad i = 0, 1
\]

where \( w(E) \) is defined in (2.13)

**Proof.** The substitution of (3.27) and (3.9 into (3.1) gives

\[
\omega^{(i)}(\partial_t, \cdot) = \frac{1}{2} \sum_{p_{\alpha}} \text{res}_{p_{\alpha}} \left( \langle \psi^+ [M, L] \delta \psi \rangle - \langle \psi^+ \delta L(M \psi - \psi f) \rangle \right) \frac{d \ln w}{n E^i \langle \psi^+ \psi \rangle}
\]

Using once again the equation \((L - E) \delta \psi = -(\delta L - \delta E) \psi\) we get that the differential in the right hand side of (3.33) is equal to

\[
- \frac{1}{2} \left( \langle \psi^+ (M \delta E + \delta L f) \psi \rangle - \langle \psi^+ (\delta L M + M \delta L) \psi \rangle \right) \frac{d \ln w}{n E^i \langle \psi^+ \psi \rangle}
\]

The second term has poles only at the points \( p_{\alpha} \). Hence, the sum of its residues at these points is equal to zero. The first term is equal to

\[
- \frac{1}{2} \langle \psi^+ (2 f + (M - f)) \psi \rangle \delta E \frac{d \ln w}{n E^i \langle \psi^+ \psi \rangle}
\]

From the definition of \( f \) in (3.27) it follows that \( \langle \psi^+ (M - f) \psi \rangle \) is holomorphic at \( p_{\alpha} \). Since the restriction of \( E^{-i} \delta E d \ln w \) onto \( \Lambda^c_i \) is holomorphic at the marked points \( p_{\alpha} \), the second term in (3.35) restricted to \( \Lambda^c_i \) has no residues at \( p_{\alpha} \). Recall that the function \( f \) has poles only at the points \( p_{\pm} \). Using the identity \( \delta E(w) d \ln w = -\delta \ln w(E) dE \) for the residue at \( p_+ \) we finally obtain the equation

\[
\tilde{\omega}^{(i)}(\partial_t, \cdot) = \frac{1}{n} \text{res}_{p_+} f(E) \delta \ln w(E) E^{-i} dE - \frac{1}{n} \text{res}_{p_-} f(w) E^{-i}(w) \delta E(w) d \ln w
\]

Recall, that the choice of the basis vector fields \( \partial_{t^h_m} \) of the hierarchy is fixed by the choice of local coordinates near the marked point \( p_{\pm} \). As it
follows form the constructions of Lemma 2.1 and Lemma 2.2 the most natural choice is \( z = w^{-1/n} \) at \( p_- \) and \( z = E \) at \( p_+ \). With this choice of local the function \( f_\pm^m \) corresponding to \( t = t_\pm^m \) has pole at \( p_\pm \) of the form \( f_\pm^m = E^{-m} + O(E) \) and \( f_\mp^m = z^{-m} + O(z) \), \( z = w^{-1/n} \), respectively. Then (3.36) implies \( \hat{\omega}^{(i)}(\partial_{t_\pm^m}, \cdot) = \delta H_{t_\pm^m}^{(i)} \). The theorem is proved.

4. Special coordinate systems. Examples

We begin this section by presenting certain systems of coordinates on the space of lower triangular operators in which the forms \( \omega^{(\ell)} \), \( \ell = 1, 2 \), have local densities. By latter we mean coordinates \( x^{(j)}_i \) in which the form can be written as \( \omega = \sum f^{(j,j_1)}_{i,i_1} \delta x^{(j)}_i \wedge \delta x^{(j_1)}_i \) where the sum is taken over the set of all the indices with \( |i - i_1| < d_1 \) for some integer \( d_1 \) independent on the period \( n \) of the operator. Moreover it is assumed also that the coefficients \( f^{(j,j_1)}_{i,i_1} \) are functions of \( x^{(j_2)}_{i_2} \) such that \( |i - i_2| < \ell_2 \) for some \( n \)-independent integer \( \ell_2 \).

Remark 4.1. Note, that in the original coordinates on the space of lower triangular operators that are the coefficients \( a^{(j)}_i \) of the operators are not of the type we are looking for. Indeed, by definition the densities of the forms involve the coefficients of the expansions of \( \psi_i \) at the points \( p_\alpha \) which are not local.

4.1. The form \( \omega^{(0)} \). The first coordinate system, in which the form \( \omega^{(0)} \) has local density, we identify with the set of of the first \( k \) coefficients of the expansion (2.7) of the Bloch solution at the marked point \( p_- \). The formulae (2.8, 2.11) for \( s = 1, \ldots k \) can be regarded as the definition of the map

\[
\{\xi^{(s)}_-(i), e_s\} \mapsto \{a^{(j)}_i\}
\]

where the variables functions \( \xi^{(s)}_-(i) \) are defined up to a common shift \( \xi^{(s)}_-(i) \rightarrow \xi^{(s)}_-(i) + c_i \) or equivalently normalized by the constraint \( \xi^{(s)}_- (0) = 0 \).

The form \( \omega^{(0)} \) by definition in (3.1) is an average over \( i \) of some expressions involving \( \xi^{(s)}_- (i-j), j = 0, \ldots k \), and the first \((k-1)\) coefficients of the expansion at \( p_- \) of the function

\[
\psi^{\ast}_i := \frac{\psi^{+}_i}{\langle \psi^+ \psi \rangle}
\]

where \( \psi^+ \) is the dual BA function (2.22). The coefficients of \( \psi^{\ast}_i \) can be found recurrently from the relations

\[
\text{res}_{p_-} \psi^{\ast}_i \psi_{i-j} d \ln z = \delta_{0,j}
\]
which follow from (2.23) and (3.9). Hence, the expression of any of these coefficients in terms of $\xi_s^-$ is local. Then the statement that $\omega^{(0)}$ has local density in the new coordinates is an obvious corollary of the definition.

**Example** $k = 1$ The initial coordinates on the space of $n$ periodic lower triangular operators of order are their coefficients $a_i$: $L = a_i T^{-1} + T^{-2}$. The new coordinates are $x_i := \xi_1^- (i)$ defined up to a common shift and a constant $e_1$. The expression for old coordinates in terms of new ones is given by formula (2.8):

\[(4.4)\]
\[a_i = x_i - x_{i-2} + e_1\]

The substitution of the expansion of $\psi$ and $\psi^+$ into (3.1) gives for $k = 1$ the following expression for the restriction of $\omega^{(0)}$ onto the symplectic leaf $e_1 = const$:

\[(4.5)\]
\[\hat{\omega}^{(0)} = \frac{1}{2} \langle da_i \wedge dx_{i-1} \rangle = \langle dx_i \wedge dx_{i-1} \rangle\]

where as before $\langle \cdot \rangle$ denotes the average of a periodic expression in brackets over the period.

**Remark 4.2.** In the formulae above the differential on the phase space (the space of parameters) was denoted by $\delta$ in order do distinguish it form the differential $d$ with respect to the spectral parameter. After taking the residues of the differential in the spectral parameter, here and below we change $\delta$ in our notations to more conventional one, i.e. $dx_i := \delta x_i$.

According to Theorem 3.8, equations (1.16) restricted to the symplectic leaf $\langle a_i \rangle = \langle e^{\phi_1 - \phi_2} \rangle = e_1 = const$ are Hamiltonin with respect to the symplectic leaf $\hat{\omega}^{(0)}$ with the Hamiltonian $H_{t_1}^{(0)} := e_3$. In order to find its explicit expression in term of the new coordinates we use the equations (2.11).

For $s = 2, k = 1$ we have

\[(4.6)\]
\[\xi_2^- (i) - \xi_2^- (i - 2) + e_1 \xi_2 (i) + e_2 = a_i \xi_2^- (i - 1)\]

Then from (4.4) it follows

\[(4.7)\]
\[\xi_2^- (i) - \xi_2^- (i - 2) + e_2 = x_i x_{i-2} + x_{i-1} x_{i-2} + e_1 (x_{i-1} - x_i)\]

Taking the average of equation (4.7) we get $e_2 = 0$ (recall that in the proof of Lemma 3.2 it was shown that $e_{k+1} = 0$ for any $k$). For $s = 3, k = 1$ equation (2.11) has the form

\[(4.8)\]
\[\xi_3^- (i) - \xi_3^- (i - 2) + e_1 \xi_3^- (i) + e_3 = a_i \xi_2^- (i - 1) = (x_i x_{i-2} + e_1) \xi_2^- (i - 1)\]
Taking the average of (4.8) we get the explicit expression for the Hamiltonian of equation (1.16) in terms of the new coordinates:

\begin{equation}
H_{a_{ij}^{-1}}^{(0)} = e_3 = \langle (x_i - x_{i-2}) \xi_2^-(i - 1) \rangle = \langle x_i (\xi_2^-(i - 1) - \xi_2^-(i + 1)) \rangle = \langle x_i^2 (x_{i-1} - x_{i+1}) \rangle
\end{equation}

where for the last equation we use (4.6).

**Example:** k=2. The expressions of the coefficients of a lower triangular operator of order 3 in terms of the coordinates \( x_i := \xi_1^-(i) \) and \( y_i := \xi_2^-(i) \) are as follow:

\begin{equation}
a_i^{(2)} = x_i - x_{i-3} + e_1
\end{equation}

\begin{equation}
a_i^{(1)} = y_i - y_{i-3} + e_1x_i + e_2 - a_i^{(2)}x_{i-2} = y_i - y_{i-3} - (x_i - x_{i-3})x_{i-2} + e_1(x_i - x_{i-2}) + e_2
\end{equation}

The substitution of expansions for \( \psi \) and \( \psi^+ \) into (3.1) gives the following

\begin{equation}
\omega^{(0)} = \frac{1}{2} \langle da_i^{(1)} \wedge dx_{i-1} + da_i^{(2)} \wedge (\chi_1^-(i) dx_{i-2} + d\xi_2^-(i - 2)) \rangle
\end{equation}

where \( \chi_1^- \) is the first coefficient of the expansion of \( \psi^+ \) at the marked point \( p_- \). Equation (4.3) with \( j = 1 \) implies \( \chi_1^-(i) = -x_{i-1} \). Then after relatively long but straightforward computations we obtain the following expression for \( \omega^{(0)} \) restricted to a leaf along which \( e_1 \) and \( e_2 \) are constants:

\begin{equation}
\hat{\omega}^{(0)} = \langle dy_i \wedge (dx_{i-1} - dx_{i+2}) + d(x_{i-1}x_{i-2}) \wedge dx_i \rangle + e_1 \langle dx_i \wedge dx_{i-1} \rangle
\end{equation}

Equation (1.16) for \( k = 2 \) restricted to a leaf with \( e_1 \) \( e_2 \) being some constants is Hamiltonian with respect to the form (4.13) and with the Hamiltonian \( H_{a_{ij}^{-1}}^{(0)} = e_4 \). Similar to computations of \( e_3 \) above we get the following expression for the Hamiltonian \( H := e_4 \):

\begin{equation}
H = \langle y_{i-1}(y_i - y_{i-3}) \rangle + \langle x_ix_{i-1}x_{i-2}(x_{i-1} - x_i) \rangle + e_1 \langle (x_i^2(x_{i-1} - x_{i+1})) + e_2 \langle x_{i-1}(x_i - x_{i-1}) \rangle + \langle y_i(x_{i+2} - x_{i-1} - x_{i+2}x_{i+1} + x_{i-2}x_{i-1}) \rangle
\end{equation}
4.2. **The form** $\omega^{(1)}$. A system of coordinates in which the form $\omega^{(1)}$ becomes local is suggested by its definition in (3.1) which involves the evaluation of $\psi_i$ at the marked points $p_\ell \in \Gamma$ that are the preimages of $E = 0$ with $w(p_\ell) \neq 0$.

Let $\Phi = \{\phi_\ell^i\}$ be a $(k \times n)$ matrix of rank $k$, i.e. $i = 1, \ldots, n; \ell = 1, \ldots, k$. We call two matrices equivalent $\Phi \sim \Phi'$ if $\Phi' = \Phi \lambda$, where $\lambda = \text{diag}(\lambda_1, \ldots, \lambda_k)$. The space of equivalence classes $[\Phi] := (\Phi/\sim)$ can be seen as the space of *ordered sets* of $k$ distinct points in $(n-1)$-dimensional projective space, $[\phi_\ell^i] \in \mathbb{P}^{n-1}$.

Consider the space of pairs $\{[\Phi], W\}$ where $W = \{w_1, \ldots, w_k\}$ is a set of non-zero numbers, $w_\ell \neq 0$. The symmetric group $S_k$ acts on the space of such pairs by simultaneous permutation of rows of matrix $\Phi$ and coordinates of the vector $W$. Now we are going to define a map from the corresponding factor-space to the space of $n$-periodic operators $L$ of the form (1.3):

\[
\{[\Phi], W\}/S_k \mapsto L
\]  

First note that given a set $W = \{w_1, \ldots, w_k\}$ of non-zero numbers any $(k \times n)$ matrix $\Phi$ can be extended to a unique $(k \times \infty)$ matrix $\phi_\ell^i, i \in \mathbb{Z}$, such that the equation $\phi_{\ell-n}^i = w_\ell \phi_\ell^i$ holds. Then it is easy to see that there is a unique operator $L$ of the form (1.3) such that for any $\ell$ the sequence $\phi_\ell = \{\phi_\ell^i\}$ is a solution of the equation

\[
L \phi_\ell = 0 \iff \sum_{j=1}^{k} a_{i}^{(j)} \phi_{\ell-j}^i = -\phi_{\ell-k-1}^i
\]  

Indeed, for fixed $i$ the system (4.16) is a system of $k$ non-homogeneous linear equations for unknown coefficients of $L$. Hence, by Cramer’s rule:

\[
a_{i}^{(j)} = -\frac{\phi_{i-1}, \ldots, \phi_{i-j+1}, \phi_{i-k-1}, \phi_{i-j-1}, \ldots, \phi_{i-k}}{\phi_{i-1}, \ldots, \phi_{i-j+1}, \phi_{i-j}, \phi_{i-j-1}, \ldots, \phi_{i-k}}
\]  

Here and below we use the following notations: $\phi_i$ is the $k$-dimensional vector with coordinates $\phi_i := \{\phi_\ell^i\}$, and for any set $V_1, \ldots, V_k$ of $k$-dimensional vectors $|V_1, \ldots, V_k|$ stands for the determinant of the corresponding matrix, i.e. $|V_1, \ldots, V_k| := \det (V_\ell^i)$.

Recall that throughout the paper we use parametrization of the leading coefficient $a_i^{(1)}$ by variables $\varphi_i$ such that $a_i^{(1)} = e^{\varphi_i - \varphi_{i-1}}$. Equation (4.17) for $j = 1$ allows to identify these variables with

\[
e^{-\varphi_i} := (-1)^k |\phi_{i-1}, \ldots, \phi_{i-k}|
\]
Then for any $j$ equation (4.17) takes the form

$$a_{i}^{(j)} = (-1)^{ik+1} e^{\varphi_{i}} |\phi_{i-1}, \ldots, \phi_{i-j+1}, \phi_{i-k-1}, \phi_{i-j-1}, \ldots, \phi_{i-k}|$$

**Theorem 4.3.** The map (4.15) defined by formulae (4.18, 4.19) is one-to-one correspondence between open domains. Under this correspondence equation (1.14) and (1.15), restricted to leaves with $w_{\ell}$ being fixed constants, are Hamiltonian with respect to the form

$$\hat{\omega}^{(1)} = \frac{1}{2} \langle d\varphi_{i-1} \wedge d\varphi_{i} - (-1)^{(i-1)k} e^{\varphi_{i}-1} \sum_{j=1}^{k} d\alpha_{i}^{(j)} \wedge |\phi_{i-2}, \ldots, \phi_{i-k}, d\phi_{i-j}| \rangle$$

and with the Hamiltonians

$$H^{-} = \langle a_{i}^{(k)} \rangle ; \quad H^{+} = -\langle a_{i}^{(2)} e^{\varphi_{i}-2-\varphi_{i}} \rangle$$

respectively.

**Proof.** The right hand side of (4.17) is symmetric with respect to the simultaneous permutation of rows of the matrices in the numerator and denominator. Hence, the map (4.15) is well-defined on an open domain when all denominators are non zero. The inverse map is defined by identification of $w_{\ell}$ with non zero roots of the polynomial $R(w, 0) = \det L(w)$ defined in (2.3). In other word $w_{\ell}$ is an evaluation of the function $w(p)$ on the spectral curve $\Gamma$ of $L$ at one of the preimages on $\Gamma$ of $E = 0$, i.e. $p_{\ell} : (w_{\ell}, 0) \in \Gamma$. Under this identification it easy to see that $\phi_{i}$ is just the evaluation of the BA function at $p_{\ell}$, i.e. $\phi_{i}^{\ell} = \psi_{i}(p_{\ell})$. Hence the first statement of the theorem is proved.

Recall that by definition, $\omega^{(1)}$ is equal to the average over $i$ of the sum of residues at $p_{\pm}$ and $p_{\ell}$ of the form

$$-\frac{1}{2n} \sum_{j=1}^{k} \delta a_{i}^{(j)} \wedge (\psi_{i}^{*}\delta \psi_{i-j}) E^{-1} d\ln w$$

The BA function $\psi_{i}$ and its dual $\psi_{i}^{\pm}$ has zero and pole of order $i$ at $p_{-}$, respectively. Since, $E$ at $p_{-}$ has pole of order $k + 1$, the form (4.22) is holomorphic at $p_{-}$. Hence, it has no residue at $p_{-}$. At $p_{+}$ the function $E$ has simple zero. Therefore, the form $E^{-1} d\ln w$ at $p_{+}$ has pole of order 2. At the same time at $p_{+}$ the functions $\psi_{i}^{+}$ and $\psi_{i}$ have zero and pole of order $i$, respectively. Hence, the terms in sum (4.22) with $j > 1$ are holomorphic at $p_{+}$. From (2.12, 2.22) it follows that

$$-\frac{1}{2n} \text{res}_{p_{\pm}} \delta a_{i}^{(1)} \wedge (\psi_{i}^{*}\delta \psi_{i-1}) E^{-1} d\ln w = -\frac{1}{2} \delta(e^{\varphi_{i}-\varphi_{i-1}}) \wedge e^{-\varphi_{i}} \delta(e^{\varphi_{i-1}})$$
\[
\frac{1}{2} \delta \varphi_{i-1} \wedge \delta \varphi_i
\]

Our next goal is to express \( \psi^+ (p_\ell) \) in terms of \( \phi^\ell = \psi (p_\ell) \), which then will allow us to obtain a closed expression of \( \omega (1) \) in terms of \( \phi^\ell \).

**Lemma 4.4.** Let \( r_\ell \) be constants equal \( r_\ell := \text{res}_{p_\ell} E^{-1} d\Omega \). Then

\[
(4.24) \\
r_\ell \psi^+_i (p_\ell) = \frac{(-1)^{\ell+k-1} \det \tilde{\Phi}^{\ell,k}_i}{|\phi_{i-2}, \ldots, \phi_{i-k-1}|}
\]

where \( \tilde{\Phi}_i \) is \((k \times k)\) matrix with columns \((\phi_{i-1}, \ldots, \phi_{i-k})\), and \( \tilde{\Phi}^{\ell,k}_i \) is obtained from \( \tilde{\Phi}_i \) by removing \( \ell\)-th row and the last column.

**Proof.** By definition of \( d\Omega \) the differential \( \psi^+_i \psi_{i-j} E^{-1} d\Omega \) is holomorphic away of the marked points \( p_\pm \) and the points \( p_\ell \) where \( E \) vanishes. For \( 2 \leq j \leq k \) it is holomorphic at \( p_- \) and has simple pole at \( p_+ \) with residue \(-1\). Hence,

\[
(4.25) \\
\sum_{\ell=1}^k \text{res}_{p_\ell} \psi^+_i \psi_{i-j} E^{-1} d\Omega = \sum_\ell r_\ell \psi^+_i (p_\ell) \phi^\ell_{i-j} = 0, \ j = 2, \ldots, k
\]

The differential \( \psi^+_i \psi_{i-j} E^{-1} d\Omega \) is holomorphic at \( p_- \) and has simple pole at \( p_+ \) with residue \(-1\). Hence,

\[
(4.26) \\
\sum_\ell \text{res}_{p_\ell} \psi^+_i \psi_{i-k} E^{-1} d\Omega = \sum_\ell r_\ell \psi^+_i (p_\ell) \phi^\ell_{i-k-q} = 1.
\]

Equations (4.25, 4.26) is a system of linear equations for unknown \( r_\ell \psi_i (p_\ell) \). Then Cramer’s rule implies (4.24). \( \square \)

Note, that multiplying the right hand side of (4.24) by \( d\phi^\ell_{i-j} \) and then taking the sum over \( \ell \) we can identify the latter with an expansion of the determinant below with respect to the last column, i.e.

\[
(4.27) \\
-\frac{1}{2} \sum_{\ell=1}^k r_\ell \psi^+_i (p_\ell) d\phi^\ell_{i-j} = -\frac{1}{2} \left| \phi_{i-2}, \ldots, \phi_{i-k}, d\phi_{i-j} \right| = \left( -1 \right)^{k(i-1)+1} \left| \phi_{i-2}, \ldots, \phi_{i-k}, d\phi_{i-j} \right| e^{\varphi_{i-1}}
\]

The right hand side of (4.20) is equal to the sum of (4.23) and wedge product of (4.27) with \( da^{(j)}_i \). In order to complete the proof of the theorem it remains only to note, that according to Theorem 3.8 the Hamiltonians of equations (1.14) and (1.15) are equal

\[
(4.28) \\
H^- := H_{\partial^-_{1-}} = \text{res}_{z=0} \ln E(z) z^{-2} dz = e_1 = \langle a^{(k)}_i \rangle
\]
and

\begin{equation}
(4.29)
H^+ := H_{\partial_t} = \frac{1}{n} \res_{E=0} \ln w(E)E^{-2}dE = w_1
\end{equation}

where \( w_1 \) are the first coefficient of expansion \((2.15)\). Note, that according to Corollary \(2.3\)

\begin{equation}
(4.30)
n^{-1} \ln w = n^{-1}(\ln \psi_{-n} - \ln \psi_0) = \langle \psi_{i-1} - \psi_i \rangle
\end{equation}

Then, from \((2.12)\) and \((2.13)\) we obtain

\begin{equation}
(4.31)
w_1 = \langle \xi_1^+(i) - \xi_1^+(i) \rangle = -\langle a_i^{(2)} e^{\varphi_{i-1} - \varphi_i} \rangle
\end{equation}

and the Theorem is proved.

**Example** For \( k = 1 \) equation \((4.18)\) takes the form \( e^{-\varphi_i} = (-1)^i \phi_{i-1} \). Then

\begin{equation}
(4.32)
\omega^{(1)} = \frac{1}{2} \langle d\varphi_{i-1} \wedge d\varphi_i - (-1)^{i-1} e^{\varphi_i-1}d(e^{\varphi_i-1} \wedge d\varphi_{i-1}) \rangle = \langle d\varphi_{i-1} \wedge d\varphi_i \rangle
\end{equation}

Note, also that for \( k = 1 \) the coefficient \( a_i^{(2)} = 1 \) and formulae \((4.21)\)
takes the form \((1.18)\).

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