Dual-mode robust MPC for the tracking control of non-holonomic mobile robots

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Abstract

In this paper, a novel dual-mode robust model predictive control (MPC) approach is proposed for solving the tracking control problem of non-holonomic mobile robots with additive bounded disturbance. To reduce the negative effect of disturbance and drive the state of real system closer to the one of nominal system, a robust reference signal is introduced into the cost function of MPC. In order to reduce the computation burden caused by online optimization of MPC and further improve the tracking accuracy, a dual-mode control structure consisting of the robust MPC and the local nonlinear robust control is developed, in which the local nonlinear robust control law is applied within a specified terminal region. Finally, simulation results on the non-holonomic mobile robot are presented to show the validity of the proposed control approach.

Keywords: Non-holonomic mobile robots, robust MPC, dual-mode control, robust reference input signal, tracking control.

1. Introduction

Trajectory tracking control of non-holonomic mobile robots has attracted great attention in recent years, and the applications can be found in broad fields \cite{1-6}. Many control strategies have been proposed to tackle the trajectory tracking problem, such as sliding mode control \cite{7-9}, adaptive control \cite{10-12}, and backstepping control \cite{13}, etc. It is well-known that the physical constraints are often unavoidable for nonholonomic mobile robots, and the constraints are not well-addressed by applying the aforementioned control strategies. Model predictive control (MPC) has achieved extraordinary attention since it is capable of dealing with the state and input constraints while providing an optimal control performance \cite{14-17}.

MPC approach has been studied for the tracking problem of non-holonomic robots. In \cite{18-20}, the MPC is employed for nonholonomic robots by considering the system to be an ideal model. However, when the system is perturbed by additive disturbance, the recursive feasibility of MPC and the stability of closed-loop may be destroyed. To deal with the system with disturbance, several MPC strategies characterised by robustness are proposed. In \cite{21-24}, a min-max MPC is formulated by considering the worst-case of the disturbance to guarantee the recursive feasibility and robust stability. Because of the presence of disturbance in the cost function, the computational complexity of the optimization problem is intractable. To cope with this problem, the tube-based MPC approaches are reported in \cite{25-28}, which take advantages of open-loop desired controller and auxiliary feedback controller to restrain the actual state within a “tube” centered around the optimal nominal state. However, the tube-based MPC is relatively conservative and is sensitive to additive disturbance. Meanwhile, in the optimization problem, the input constraint has to be tightened due to the additive auxiliary controller, which may degrade the optimization performance. Another approach is proposed in \cite{29-32} called robust MPC, which guarantees that the optimization problem based on nominal system is sufficiently robust to additive disturbance. It has to be pointed out that although the conservativeness is reduced compared with tube-based MPC, it is still susceptible because the control law is derived from nominal system.

Considering the limitation of the robustness of MPC, it is interesting to combine MPC with existing robust control approaches. In \cite{33}, the dual-mode robust MPC control scheme is proposed, in which the system state is first driven into a specified terminal region under robust MPC law and then a linear feedback control law obtained by linearized the model is applied. This approach satisfies the constraints while improving the robustness of closed-loop system. What’s more, for the perturbed systems, it is important to alleviate the negative effect of disturbance, and to guarantee that the perturbed system behaves the same as nominal system as possible and the actual state is closer to the nominal state. Therefore, in this paper, a novel dual-mode robust MPC approach is proposed for tracking the control problem of non-holonomic mobile robots with additive bounded disturbance, and the objective is to improve the robustness of MPC and the accuracy of the trajectory tracking control. The main contributions can be summarized as follows:

(1) A robust reference signal is introduced in the cost function to improve the robustness of MPC. By minimizing the cost function, the control law is close to the reference signal, which can reduce the negative effect of disturbance and drive the state of real system closer to the one of nominal system.

(2) A dual-mode control structure consisting of the robust MPC and the local nonlinear robust control is developed, in which the local nonlinear robust control law is applied within a specified terminal region. The approach can relieve the computation burden of online optimization caused by MPC and improve the tracking accuracy. Meanwhile, the satisfaction of constraints is guaranteed in the whole process. The rigorous feasibility of MPC and stability of closed-loop system is conducted.
The notations adopted in this paper are standard. $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^n$ is the $n$-dimensional Euclidean space. $\mathbb{N}$ represents the set of all natural numbers. For a vector $x$, $\|x\| = \sqrt{x^t x}$ and $\|x\|_Q = \sqrt{x^t Q x}$ are the Euclidean norm and weighted $Q$ norm. For a matrix $M$, $\lambda(M)$ and $\lambda(M)$ are the maximum and minimum eigenvalues of $M$ respectively. $(\cdot)^t$ indicates the value of a variable at time $\tau$ predicted from time $t$. The feasible variables are marked as $^*$ and the optimal variables attained by solving optimization problem is marked as $^{\ast}$. To simplify notation, time dependence is omitted when not relevant.

2. Problem Formulation and Preliminaries

2.1. Kinematics of the non-holonomic mobile robot

The mechanical structure of the non-holonomic robot is shown in Fig. 1. The chassis of the robot is rectangle, and $\rho$ is half the distance between the two driving wheels. $v_R$ and $v_L$ are the velocities of right and left driving wheels of the robot, respectively. The velocities of the driving wheels are bounded by $|v_L| \leq a$ and $|v_R| \leq a$, where $a \in \mathbb{R}$ is a known positive constant. Then, the linear velocity $v$ and angular velocity $\omega$ of the robot can be presented as

$$
\begin{align*}
    &\quad v = (v_R + v_L)/2, \\
    &\quad \omega = (v_R - v_L)/(2\rho).
\end{align*}
$$

The kinematics of the center of robot is described by the following continuous-time model:

$$
\dot{\xi} = f(\xi, u) \triangleq \begin{bmatrix} \cos \theta & 0 \\ \sin \theta & 0 \\ 0 & 1 \end{bmatrix} u,
$$

where $\xi = [p^T, \theta]^T \in \mathbb{R}^2 \times (-\pi, \pi)$ is the system state, consisting of the position $p = [x, y]^T$ and the orientation $\theta$, and $u = [v, \omega]^T$ denotes the control input. According to [34], the system (1) is subject to the following input constraint

$$
u \in U = \left\{ [v, \omega]^T | |v|/a + |\omega|/b \leq 1 \right\},$$

with $b = a/\rho$.

![Figure 1: The structure of the nonholonomic mobile robot.](image)

2.2. Tracking error system

To drive the non-holonomic mobile robot to track a desired trajectory, we assume that the trajectory is generated by a virtual robot, which has the same structure (1), and the trajectory can be represented as

$$
\dot{\xi}_e = f(\xi_e, u_e),
$$

where $\xi_e = [p_e^T, \theta_e]^T \in \mathbb{R}^2 \times (-\pi, \pi)$ is the desired state, the desired trajectory is given by $p_e = [x_e, y_e]^T$ and $u_e = [v_e, \omega_e]^T \in \mathbb{U}$ is the control input of virtual robot.

The kinematics of the controlled robot is given by (1). Considering the existence of non-holonomic constraint, we set the reference point of robot at point ‘$h$’ instead of its center, where point ‘$h$’ lies a distance $\rho$ along the perpendicular bisector of the wheel axis ahead of the robot (see Fig. 1). Meanwhile, we consider the disturbance acting on linear velocity due to sideslip while neglecting the disturbance acting on angular velocity. Therefore, the kinematics of the controlled robot at point ‘$h$’ is further formulated as follows:

$$
\dot{\xi}_f = f_b(\xi_f, u_f) + \hat{d},
$$

where $\xi_f = [p_f^T, \theta_f]^T \in \mathbb{R}^2 \times (-\pi, \pi)$ is the state with $p_f = [x_f, y_f]^T$ and $u_f = [v_f, \omega_f]^T \in \mathbb{U}$ is the control input. $\hat{d} = [\cos \theta_f d, \sin \theta_f d, 0]^T$ is the additive disturbance satisfying $|d| \leq \mu$.

By constructing the Frenet-Serret frames, the tracking error $p_{tf} = [x_{tf}, y_{tf}]^T$ between the desired trajectory and controlled robot is given by [19]

$$
p_{tf} = R(-\theta_f)p_t - p_f, \quad \theta_f = \theta_t - \theta_f,
$$

where $R(\theta_f) = \begin{bmatrix} \cos \theta_f & \sin \theta_f \\ -\sin \theta_f & \cos \theta_f \end{bmatrix}$ is the rotation matrix. Taking the derivative of (5) yields the tracking error system

$$
\dot{p}_{tf} = \begin{bmatrix} -\omega_f y_{tf} + v_f \cos \theta_f \\ \omega_f x_{tf} + v_f \sin \theta_f \end{bmatrix} - \begin{bmatrix} v_f \\ \rho \omega_f \end{bmatrix} + \begin{bmatrix} d \\ 0 \end{bmatrix}.
$$

To design a dual-mode robust MPC algorithm for trajectory tracking control, the nominal system representation in MPC is first given.

The nominal system of controlled robot (4) is presented as

$$
\dot{\xi}_f = f_b(\xi_f, \tilde{u}_f),
$$

and thus the nominal form of tracking error system (4) is

$$
\dot{p}_{tf} = \begin{bmatrix} -\omega_f y_{tf} + v_f \cos \theta_f \\ \omega_f x_{tf} + v_f \sin \theta_f \end{bmatrix} - \begin{bmatrix} v_f \\ \rho \omega_f \end{bmatrix}.
$$

2.3. Optimization problem for dual-mode robust MPC

Defining the sampling instant as $\{t_k\}, k \in \mathbb{N}$, and the sampling period as $\delta = t_{k+1} - t_k$, the optimization problem is solved at each sampling instant. If the tracking error $p_{tf}$ is outside the terminal region $\Omega_\varepsilon$ given by $\Omega_\varepsilon = \left\{ p_{tf} \in \mathbb{R}^2 | \|p_{tf}\| \leq \varepsilon, \varepsilon > 0 \right\}$, the robust
MPC is applied. Then, the optimization problem at $t_k$ with prediction horizon $T$ is formulated as follows: \begin{equation}
abla_t f(t_k) \), \bar{u}_f(t) \right), \end{equation}
and subject to \begin{align}
\bar{u}_f(t) &= \bar{u}_f(t_k), \\
\bar{f}_r(t) &= f_r(\bar{y}_r(t), \bar{u}_f(t)), \\
\bar{u}_f(t) &\in \mathcal{U}, \\
\bar{p}_f(t_k + T|t_k) &\in \Omega_c, \\
\end{align}
with $\tau \in [t_k, t_k + T]$. The cost function $J(\bar{p}_f(t_k), \bar{u}_f(t))$ in \textbf{Problem 1} is defined as \begin{equation}
J(\bar{p}_f(t_k), \bar{u}_f(t)) = \int_{t_k}^{t_k+T} L(\bar{p}_f(\tau|t_k), \bar{u}_f(\tau|t_k))d\tau + V_f(\bar{p}_f(t_k + T|t_k))d\tau 
\end{equation}
where $L(\bar{p}_f(\tau|t_k), \bar{u}_f(\tau|t_k)) = \|\bar{p}_f(\tau|t_k)\|^2 + \|\bar{u}_f(\tau|t_k)\|^2$ is stage cost with $Q = \text{diag}(q_1, q_2), R = \text{diag}(r_1, r_2), \bar{u}_f$ is designed as \begin{equation}
\bar{u}_f(\tau|t_k) = \arg \min_{\bar{u}_f(\tau|t_k)} J(\bar{p}_f(t_k), \bar{u}_f(\tau|t_k)),
\end{equation}
and $\bar{u}_f(\tau|t_k)$ is applied as the control law $\tau \in [t_k, t_k + 1)$.

If the tracking error $p_f$ enters the set $\Omega_m$, the following control law is applied \begin{equation}
k_f(\bar{e}_f) = \begin{bmatrix} v_r \cos \bar{\theta}_f + \eta \tan(\bar{\theta} \bar{x}_r) + k_1 \bar{x}_r \\
v_r \sin \bar{\theta}_f + k_2 \bar{y}_f 
\end{bmatrix}
\end{equation}
where $k_1$ and $k_2$ are the controller gain.

Before proceeding further, some lemmas are presented as follows.

\textbf{Lemma 1. }[34] The function $f(\bar{e}_f, \bar{u}_f)$ is locally Lipschitz continuous with respect to its first argument $\bar{e}_f$ and Lipschitz constant is $a$, such that \begin{equation}
\|f(\bar{e}_f, \bar{u}_f) - f(\bar{e}_f, \bar{u}_f)\| \leq a \|\bar{e}_f - \bar{e}_f\|.
\end{equation}

\textbf{Lemma 2. }[35] For the real tracking error system (7) and the nominal system (9) with the same initial state and control input. Then, the state deviation $p_f(t) - \bar{p}_f(t)$ is bounded by \begin{equation}
\|p_f(t) - \bar{p}_f(t)\| \leq a e^{a t} - 1.
\end{equation}

\textbf{Lemma 3. (Gronwall inequality)} If \begin{equation}
x(t) \leq h(t) + \int_{t_0}^{t} \beta(\tau)x(\tau)d\tau, \quad t \in [t_0, T],
\end{equation}
with all the functions involved are continuous on $[t_0, T)$, $T \leq +\infty$ and $\beta(t) \geq 0$, then $x(t)$ satisfies the integral inequality \begin{equation}
x(t) \leq h(t) + \int_{t_0}^{t} \beta(\tau)x(\tau)d\tau, \quad t \in [t_0, T].
\end{equation}

\textbf{Lemma 4. } For the nominal tracking error system (9), if the linear velocity $v_r$ of the virtual robot is limited by \begin{equation}
\max |v_r| = \bar{a} \leq \frac{\eta}{\sqrt{2}}.
\end{equation}
then there exists an invariant set $\Omega_a = \{\bar{p}_f \in \mathbb{R}^2 | \|\bar{p}_f\| \leq \alpha\}$, and local nonlinear control law \begin{equation}
k_f(\bar{e}_f) = \begin{bmatrix} v_r \cos \bar{\theta}_f + \eta \tan(\bar{\theta} \bar{x}_f) + k_1 \bar{x}_f \\
v_r \sin \bar{\theta}_f + k_2 \bar{y}_f 
\end{bmatrix}
\end{equation}
where $\eta \leq \alpha \leq \frac{m}{\sqrt{2}}, m = a - \sqrt{2} \bar{a} - \eta, q_i r_i < \frac{1}{2}, k_i \in \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$ $i = 1, 2$, such that for all $\bar{p}_f \in \Omega_a$, \begin{equation}
V_f(\bar{p}_f) + L(\bar{p}_f, k_f(\bar{p}_f)) \leq 0.
\end{equation}

\textbf{Proof. } The proof consists of three parts: The first part is to show the set $\Omega_a$ is invariant under the control law (22). The Lyapunov function is selected as \begin{equation}
V_1 = \frac{1}{2} P_f \bar{p}_f.
\end{equation}
Taking derivative of (24) and combining (22) yields \begin{equation}
\dot{V}_1 = -k_1 \bar{x}_f^2 - k_2 \bar{y}_f^2 - \eta \bar{x}_f \tan(\bar{\theta} \bar{x}_f) \leq 0,
\end{equation}
which implies the set $\Omega_a$ is invariant. The first part is proven. The second part is to show the feedback control law (22) satisfies the constraint (13). It can be derived that \begin{equation}
\frac{|v_f|}{a} + \frac{|\omega_f|}{b} = \frac{|v_r \cos \bar{\theta}_f + \eta \tan(\bar{\theta} \bar{x}_f) + k_1 \bar{x}_f|}{a} + \frac{|v_r \sin \bar{\theta}_f + k_2 \bar{y}_f|}{pb} \leq \frac{1}{a}(k_1 \bar{x}_f^2 + k_2 \bar{y}_f^2 + \sqrt{2} |v_r| + \eta).
\end{equation}
Since $\alpha \leq \frac{m}{\sqrt{2}}$, then $k_1 \bar{x}_f^2 + k_2 \bar{y}_f^2 \leq m$ holds for all $\bar{p}_f \in \Omega_a$. Thus, \begin{equation}
\frac{|v_f|}{a} + \frac{|\omega_f|}{b} \leq 1
\end{equation}
is proven. The third part is to show inequality (23) holds. It can be shown that \begin{equation}
\dot{V}_f(\bar{p}_f) + L(\bar{p}_f, k_f(\bar{p}_f)) \leq (-k_1 + q_1 r_1) \bar{x}_f^2 + (-k_2 + q_2 + r_2 k_2) \bar{y}_f^2,
\end{equation}
which together with $q_i r_i < \frac{1}{2}, k_i \in \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), i = 1, 2$ implies that the inequality $\dot{V}_f(\bar{p}_f) + L(\bar{p}_f, k_f(\bar{p}_f)) \leq 0$ holds for all $\bar{p}_f \in \Omega_a$.

Therefore, the proof is completed. \qed
Algorithm 1 The dual-mode robust MPC approach

1: Set $k = 0$ and initialize the state of nominal system (8) as the actual one $\xi_I(t_0)$.
2: While $p_{rf} \notin \Omega_e$
3: \hspace{1em} Solve Problem 1
4: \hspace{2em} Apply $u_f(t_k) = \tilde{u}_f^1(t_k|t_k)$ to the real system for $t \in [t_k, t_{k+1})$
5: \hspace{1em} Update $k = k + 1$
6: End while
7: Apply the nonlinear control law $u = \kappa_f(p_{rf})$

Based on the discussion above, the procedure of dual-mode robust MPC algorithm described in Algorithm 1.

Remark 1. It should be noted that different from the robust MPC in [31], the robust reference signal $v_c \cos \tilde{\theta}_f + \eta \tan(\tilde{\theta}_f)$ is introduced in the cost function. By minimizing the cost function, the control input $v_c$ will be close to $v_c \cos \tilde{\theta}_f + \eta \tan(\tilde{\theta}_f)$, in which the term $\eta \tan(\tilde{\theta}_f)$ can alleviate the effect of the additive disturbance and therefore the actual state can be driven closer to the nominal state, i.e. $\|p_{rf} - p_f\| \to 0$. Therefore, the robust MPC part of the proposed approach behaves better performance of disturbance rejection.

3. THEORETICAL ANALYSIS

In this section, the recursive feasibility of the dual-mode robust MPC algorithm and the stability of closed-loop system is investigated.

3.1. Recursive feasibility analysis

Theorem 1. For the tracking error system (7), assume that Problem 1 is feasible at the initial instant $t_0$. Then the Problem 1 is recursive feasible for any instant $t_k$, if the following conditions are satisfied: (1) $\left(\frac{\alpha}{2}(e^{\delta \theta} - 1)\right)^T \leq (\alpha - \epsilon)$, (2) $\lambda(Q') \delta \geq \ln \frac{1}{\tau}$ ($Q' = Q + K^T RK$, $K = [k_1, k_2]^T$).

Proof. To show the recursive feasibility of Problem 1, the candidate solution at time $t_{k+1}$ is

$$\tilde{u}_f(\tau|t_{k+1}) = \begin{cases} \tilde{u}_f^1(\tau|t_k), & \tau \in [t_{k+1}, t_k + T) \\ \kappa_f(\tilde{p}_f(\tau|t_{k+1})), & \tau \in [t_k + T, t_{k+1} + T) \end{cases},$$

and the feasible state trajectory is generated by

$$\tilde{\xi}_f(\tau|t_{k+1}) = f_h(\tilde{\xi}_f(\tau|t_{k+1})), \tilde{u}_f(\tau|t_{k+1}),$$

where $\tau \in [t_{k+1}, t_k + T)$, $\tilde{\xi}_f(t_{k+1}|t_{k+1}) = \xi_f(t_{k+1})$. We will first prove the feasible state satisfying $\tilde{p}_{rf}(t_{k+1} + T|t_{k+1}) \in \Omega_e$ under control law (25).

For $\tau \in [t_{k+1}, t_k + T)$, the difference between feasible state $\tilde{\xi}_f(\tau|t_{k+1})$ and optimal state $\tilde{\xi}_f^*(\tau|t_{k+1})$ is bounded by

$$\|\tilde{\xi}_f(\tau|t_{k+1}) - \tilde{\xi}_f^*(\tau|t_{k+1})\| \leq \frac{\mu}{\alpha}(e^{\delta \theta} - 1)e^{\delta \theta (t_k - t_{k+1})}.$$ 

With the fact

$$\|\tilde{p}_{rf}(\tau|t_{k+1}) - \tilde{p}_{rf}^*(\tau|t_{k+1})\| \leq \|\tilde{\xi}_f(\tau|t_{k+1}) - \tilde{\xi}_f^*(\tau|t_{k+1})\|,$$

and by substituting $\tau = t_k + T$ into (27) and applying triangle inequality, one has

$$\|\tilde{p}_{rf}(t_{k+1} + T|t_{k+1})\| \leq \|\tilde{p}_{rf}^*(t_{k+1} + T|t_{k+1})\| + \frac{\mu}{\alpha}(e^{\delta \theta} - 1)e^{\delta \theta (T - \delta)} \leq \epsilon + (\alpha - \epsilon) = \epsilon,$$

which implies the feasible state $\tilde{p}_{rf}(t_k + T|t_{k+1})$ enters the region $\Omega_e$.

Considering $\tau \in [t_k, t_k + T)$, the control law switches to $\kappa_f(\tilde{p}_f(\tau|t_{k+1}))$. According to Lemma 4, we have

$$V_f(\tilde{p}_f(\tau|t_{k+1})) \leq -2\lambda(Q')V_f(\tilde{p}_f(\tau|t_{k+1})).$$

By applying the comparison principle [35], we have

$$V_f(\tilde{p}_f(\tau|t_{k+1})) \leq V_f(\tilde{p}_f(t_k + T|t_{k+1}))e^{-2\lambda(Q')\delta - \epsilon}.$$

At time $\tau = t_{k+1} + T$, the above inequality can be equivalently written as

$$\|\tilde{p}_{rf}(t_{k+1} + T|t_{k+1})\| \leq \|\tilde{p}_{rf}(t_k + T|t_{k+1})\| e^{\lambda(Q')\delta \tau + \epsilon}.$$

Due to $\|\tilde{p}_{rf}(t_k + T|t_{k+1})\| \leq \alpha$ and $\lambda(Q') \delta \equiv \ln \frac{1}{\tau}$, we obtain

$$\|\tilde{p}_{rf}(t_{k+1} + T|t_{k+1})\| \leq \epsilon,$$

which implies the terminal state constraint $\tilde{p}_{rf}(t_{k+1} + T|t_{k+1}) \in \Omega_e$ is satisfied. Next is to show the candidate solution conforms to input constraint (13).

For $\tau \in [t_{k+1}, t_k + T)$, the control law is derived from $\tilde{u}_f^1(\tau|t_k)$, and thus it satisfies input constraint (13). Meanwhile, by referring to Lemma 4, the terminal control law $\kappa_f(\tilde{p}_f(\tau|t_{k+1}))$ is implemented during $\tau \in [t_{k+1}, t_k + T + T)$, which means the input constraint is also satisfied.

The proof is completed. \hfill \qed

3.2. Stability analysis

To guarantee to stability of closed-loop system, the following theorem is given.

Theorem 2. For system (4) with the dual-mode control input, if the conditions in Theorem 1 and the following condition

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 < 0 \quad (28)$$

are satisfied, where $\varepsilon_1 = \lambda(Q')\left(\frac{\alpha}{2}(e^{\delta \theta} - 1)\right)^T(e^{\theta t_0 - \delta} - 1)\left(\frac{\alpha}{2}(e^{\delta \theta} - 1)\right)\left(2(\alpha - \epsilon)\right)$, $\varepsilon_2 = 2 \lambda(R)(\varepsilon_1 + \varepsilon_2)\gamma_2(T + \delta)$, $\varepsilon_3 = \frac{\mu}{\alpha}(e^{\delta \theta} - 1)e^{\theta t_0 - \delta}(\varepsilon + \alpha)$, $\varepsilon_4 = -\lambda(Q) \delta \varepsilon^2$, $\gamma_1$ and $\gamma_2$ are the upper bound of $\tilde{p}_{rf}$ and $\tilde{u}_{rf}$ respectively, then

1. the tracking error state $p_{rf}$ converges asymptotically to the region $\Omega_e$ under the control law generated by Problem 1 if tracking error $p_{rf}$ is outside $\Omega_e$.

2. the closed-loop tracking error system is input-to-state stable (ISS) under $\kappa_f(p_{rf})$ if the tracking error $p_{rf}$ enters $\Omega_e$. 

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**Proof.** If \( p_{rf} \) is outside \( \Omega_c \), i.e. \( p_{rf} \notin \Omega_c \), define \( \Delta J = J(\tilde{p}_{rf}(t|t_{k+1})), \tilde{u}_{rf}(t|t_{k+1})) - J(p_{rf}(\tau|t_{k})), \tilde{u}_{rf}(\tau|t_{k})) \). Then expanding the term \( \Delta J \) yields that
\[
\Delta J = \int_{t_{k+1}}^{t+T} L(\tilde{p}_{rf}(\tau|t_{k+1})), \tilde{u}_{rf}(\tau|t_{k+1}))d\tau + V_f(\tilde{p}_{rf}(t_{k+1} + T|t_{k+1})) - \int_{t_{k}}^{t+T} L(\tilde{p}_{rf}(\tau|t_{k})), \tilde{u}_{rf}(\tau|t_{k}))d\tau - V_f(\tilde{p}_{rf}(t_k + T|t_{k}))
\]
\[= \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4,
\]
where
\[\Delta_1 = \int_{t_{k+1}}^{t+T} \left( \| \tilde{p}_{rf}(\tau|t_{k+1}) - \tilde{p}_{rf}(\tau|t_{k}) \|_Q^2 - \| \tilde{p}_{rf}(\tau|t_{k}) \|_Q^2 \right)d\tau,
\]
\[\Delta_2 = \int_{t_{k+1}}^{t+T} \left( \| \tilde{u}_{rf}(\tau|t_{k+1}) - \tilde{u}_{rf}(\tau|t_{k}) \|_R^2 - \| \tilde{u}_{rf}(\tau|t_{k}) \|_R^2 \right)d\tau,
\]
\[\Delta_3 = \int_{t_{k+1}}^{t+T} L(\tilde{p}_{rf}(\tau|t_{k+1})), \tilde{u}_{rf}(\tau|t_{k+1}))d\tau + V_f(\tilde{p}_{rf}(t_{k+1} + T|t_{k+1})) - V_f(\tilde{p}_{rf}(t_{k} + T|t_{k}))
\]
\[\Delta_4 = - \int_{t_{k}}^{t+T} L(\tilde{p}_{rf}(\tau|t_{k})), \tilde{u}_{rf}(\tau|t_{k}))d\tau.
\]
For \( \Delta_1 \), it holds that
\[\Delta_1 \leq \int_{t_{k+1}}^{t+T} \left( \| \tilde{p}_{rf}(\tau|t_{k+1}) - \tilde{p}_{rf}(\tau|t_{k}) \|_Q^2 \right)d\tau + \delta \left( \frac{\mu}{a} (e^{at} - 1) e^{2(\tau-t_{k+1})} \right)d\tau
\]
\[\leq \lambda(\delta) \left( \frac{\mu}{a} (e^{at} - 1) e^{2(\tau-t_{k+1})} \right)d\tau
\]
\[\leq \lambda(\delta) \left( \frac{\mu}{a} (e^{at} - 1) \left( e^{2(\tau-t_{k+1})} - 1 \right) \right)
\]
\[\leq \lambda(\delta) \left( \frac{\mu}{a} (e^{at} - 1) \left( e^{2(\tau-t_{k+1})} - 1 \right) \right)
\]
Considering \( \Delta_2 \), we have
\[\Delta_2 \leq \int_{t_{k+1}}^{t+T} \left( \| \tilde{u}_{rf}(\tau|t_{k+1}) - \tilde{u}_{rf}(\tau|t_{k}) \|_R^2 \right)d\tau
\]
\[\leq \lambda(\delta) \left( \frac{\mu}{a} (e^{at} - 1) \left( e^{2(\tau-t_{k+1})} - 1 \right) \right)
\]
\[\leq \lambda(\delta) \left( \frac{\mu}{a} (e^{at} - 1) \left( e^{2(\tau-t_{k+1})} - 1 \right) \right)
\]
The upper bound of \( \Delta_3 \) can be obtained as
\[\Delta_3 = \int_{t_{k+1}}^{t+T} L(\tilde{p}_{rf}(\tau|t_{k+1})), \tilde{u}_{rf}(\tau|t_{k+1}))d\tau + V_f(\tilde{p}_{rf}(t_{k+1} + T|t_{k+1})) - V_f(\tilde{p}_{rf}(t_{k} + T|t_{k}))
\]
\[\leq \int_{t_{k+1}}^{t+T} L(\tilde{p}_{rf}(\tau|t_{k+1})), \tilde{u}_{rf}(\tau|t_{k+1}))d\tau + V_f(\tilde{p}_{rf}(t_{k} + T|t_{k})) - V_f(\tilde{p}_{rf}(t_{k}) + T|t_{k}))
\]
\[\leq \lambda(\delta) \left( \frac{\mu}{a} (e^{at} - 1) \left( e^{2(\tau-t_{k+1})} - 1 \right) \right)
\]
\[\leq \lambda(\delta) \left( \frac{\mu}{a} (e^{at} - 1) \left( e^{2(\tau-t_{k+1})} - 1 \right) \right)
\]
\[\leq \lambda(\delta) \left( \frac{\mu}{a} (e^{at} - 1) \left( e^{2(\tau-t_{k+1})} - 1 \right) \right)
\]
For \( \Delta_4 \), according to [31], it is bounded as
\[\Delta_4 < -\lambda(\delta)e^{2\tau}.
\]
Therefore, it can be deduced that \( \Delta J = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 < 0 \). By following Theorem 2 of [36], \( p_{rf} \) will converge asymptotically to the region \( \Omega_c \).

If the tracking error \( p_{rf} \) enters \( \Omega_c \), the controller law \( \kappa_f(p_{rf}) \) is applied. To prove stability, the Lyapunov candidate for (7) is chosen as
\[V_2 = \frac{1}{2} p_{rf}^T p_{rf} \quad (29)
\]
Differentiating (29) yields that
\[V_2 = p_{rf}^T p_{rf}
\]
\[\leq -k_1 x_{rf}^2 - k_2 y_{rf}^2 - \eta x_{rf} \tanh(\theta x_{rf}) - \mu x_{rf}
\]
\[\leq -k_1 x_{rf}^2 - k_2 y_{rf}^2 - \eta \tanh(\theta x_{rf}) - \mu |x_{rf}|
\]
According to [37], it infers that the closed-loop system of (7) is ISS under \( \kappa_f(p_{rf}) \) and the steady tracking error is bounded as \( \| p_{rf} \| \leq \frac{1}{20} \).

The proof is completed. \( \square \)

**Remark 2.** Noted that in [31], the real tracking error \( p_{rf} \) finally converges into the region \( \Omega_c \). In this paper, the real tracking error \( p_{rf} \) can finally converge to the region \( \| p_{rf} \| \leq \frac{20}{20} \) under the robust control law \( \kappa_f(p_{rf}) \) in the dual-mode structure. By tuning the parameters \( \eta \) and \( \theta \), the region can be largely reduced. Meanwhile, the computation burden can be relieved because the off-line control law \( \kappa_f(p_{rf}) \) is implemented within \( \Omega_c \) instead of robust MPC, and the constraint (2) can be also satisfied.

**Remark 3.** In [33], the local control law \( \kappa_f(\cdot) \) in the dual-mode structure is obtained by linearizing the system model at origin. However, this method can lead to model error and further affect the control accuracy. In this paper, the local control law is generated from the nonlinear tracking error system (7), which can show better tracking accuracy.

### 4. Simulation

In the simulation, the linear speed of the nonholonomic robot is limited by \( \rho = 0.4m/s \) and the half of wheel base by \( \rho = 0.28m \). The control input constraint is given by \( U = \{ s_f/0.4 + \omega_f/1.4286 \leq 1 \} \) with \( b = a/\rho = 1.4286 \). The additive disturbance satisfies \( |d| \leq 0.05 \). The initial configuration is set to be \( \xi_f = [0, 0, x_f/2]^T \). The desired trajectory is chosen as
\[x_s(t) = 0.5 + \sin(t/10), \quad y_s(t) = 1 + 2\sin(t/20)\]
For the optimization problem, the prediction horizon, and the sampling interval are set to be $T = 1.3s$ and $\delta = 0.1s$ respectively. The weighting matrices in the cost function are designed as $Q = \text{diag}(2,2), R = \text{diag}(0.1,0.1)$. The terminal region is set as $\Omega_e = \{p_{tr} \in \mathbb{R}^2 \mid \|p_{tr}\|_2 \leq 0.034\}$. The controller parameters are set as follows $k_1 = k_2 = 2.8, \eta = 0.05, \vartheta = 60$. The simulation example is conducted by following Algorithm 1, and the nonlinear optimization solver ‘IPOPT’ is used. In order to show the control performance, the proposed dual-mode robust MPC approach is compared with the robust MPC in [31]. The simulation results of trajectory tracking are plotted in Fig. 2. The control input of the proposed approach. The control law is switched at ‘63.7s’ and it is clear that the control input satisfies the constraint $\frac{v}{a} + \frac{|u|}{b} \leq 1$ in the whole process. To contrast the tracking accuracy under the two control approaches, we define the following tracking errors given by $x_e = x_f - x_r, y_e = y_f - y_r$. Fig. 4 shows the tracking error results, in which the proposed dual-mode robust control presents better tracking accuracy compared with MPC in [31]. To contrast the performance of disturbance rejection, state differences between the real and nominal system (i.e. $x_{fe} = \bar{x}_f - x_r, y_{fe} = \bar{y}_f - y_r$) are depicted in Fig. 5. It shows that the real state under the dual-mode robust MPC is closer to the nominal state, which implies the proposed approach shows better disturbance rejection.

5. Conclusions

This paper has developed a novel dual-mode robust model predictive control (MPC) approach for the tracking control of nonholonomic mobile robots with additive bounded disturbance. A robust reference term is introduced into the cost function of MPC to reduce the negative effect of disturbance and drives the state of real system closer to the one of nominal system. By adopting the dual-mode control structure, the tracking error is proved to be reduced, while the constraint is satisfied. Finally, the simulations have verified the effectiveness and advantages of the proposed approach.

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