Order one equations with the Painlevé property

Georg Muntingh and Marius van der Put
Department of Mathematics, University of Groningen, P.O.Box 800, 9700 AV Groningen, The Netherlands, georg.muntingh@gmail.com and mvdput@math.rug.nl

Abstract
Differential equations with the Painlevé property have been studied extensively due to their appearance in many branches of mathematics and their applicability in physics. Although a modern, differential algebraic treatment of the order one equations appeared before, the connection with the classical theory did not. Using techniques from algebraic geometry we provide the link between the classical and the modern treatment, and with the help of differential Galois theory a new classification is derived, both for characteristic 0 and \( p \).

1 Introduction
The solutions \( y = y(z) \) of a complex differential equation of order one of the form \( f(y', y, z) = 0 \), where \( y' := \frac{dy}{dz}(z) \) and \( f \) is a polynomial in the first two variables and a rational or algebraic or meromorphic function in the last variable, can have poles, branch points or essential singularities. The equation \( f \) is said to have movable singularities if the set of branch points or the set of essential singularities is not discrete (note that the literature contains various other definitions). In the opposite case \( f \) is said to have the Painlevé property (PP). The purpose of this paper is to give a precise classification of the order one equations with the PP, including complete proofs.

The classical literature starts halfway the nineteenth century with a series of papers by Briot and Bouquet \[BB56a, BB56b\] in which they treat the autonomous case, i.e., \( f(y', y) = 0 \). The general case was solved by L. Fuchs \[Fuc84\], H. Poincaré \[Poi85\] and P. Painlevé \[Pai88\], and essentially correct statements can be found in Painlevé’s Stockholm Lectures \[Pai95\] and in an article by J. Malmquist \[Mal41\]. The classical proofs of Malmquist are, however, incomprehensible for the modern reader. Furthermore, Ince
Inc56, Bieberbach [Bie53] and Hille [Hil76] discuss first order equations with the PP. Unfortunately, their statements and proofs are far from complete and precise.

In 1980 M. Matsuda published a book [Mat80] on first order algebraic differential equations that deserves to be better known. For any characteristic, Matsuda gives a purely algebraic definition for a differential field to have ‘no movable singularities,’ and then gives an essentially correct description of differential fields with this property. However, the link with the above definition of ‘no movable singularities’ is not hinted at.

Now we describe the method of this paper. In the autonomous case, the equation \( f(y, y') = 0 \) defines a curve \( X \) over \( \mathbb{C} \) and a \( \mathbb{C} \)-linear derivation \( D \) on its field of functions \( \mathcal{O}(X) \). In other words, \( D \) is a (meromorphic) vector field on \( X \). The condition that \( f \) has only finitely many branch points turns out to be equivalent to ‘\( D \) has no poles’. The pairs \( (X, D) \) with this property are easily classified and have the PP.

Consider a differential field \((K, \frac{d}{dz})\) which is a finite extension of \((\mathbb{C}(z), \frac{d}{dz})\) and thus \( K \) is the function field of a curve \( Z \) over \( \mathbb{C} \). (In fact, the proofs are valid for more general differential fields). An equation \( f(y, y') = 0 \) over \( K \) defines a curve \( X \) over \( K \) and a derivation \( D \) on its field of functions \( K(X) \) extending \( \frac{d}{dz} \) on \( K \). In general, \( D \) has finitely many poles on \( X \), i.e., closed points \( P \) of \( X \) such that \( D(\mathcal{O}_{X,P}) \not\subset \mathcal{O}_{X,P} \). In particular \( D \) can be interpreted as a meromorphic connection on the line bundle \( \mathcal{O}_X \). Prop. 4.2 states that there are infinitely many branch points for \( f \) on \( Z \) if \( D \) has a pole (this is missing in [Mat80]).

We illustrate this by the example \((y')^2 = y - z^2 \) over the differential field \( K = (\mathbb{C}(z), \frac{d}{dz}) \). Write \( t = y, s = y' \) and consider the differential field \( F = \mathbb{C}(z)(s,t) \) with equation \( s^2 = t - z^2 \) and with the \( \mathbb{C} \)-linear derivation \( D \) given by \( D(z) = 1, D(t) = s \). Thus \( F = \mathbb{C}(z)(s) \) and \( D(s) = \frac{1}{2} - \frac{z}{s} \). We view \( F \) as the function field of the curve \( X = \mathbb{P}^1_K \), i.e., the projective line over \( K \). The derivation \( D \) has a pole at the point \( s = 0 \), which means that the local ring \( \mathcal{O}_{X,0} = K[s]_{(s)} \) at the point \( s = 0 \) is mapped outside itself (indeed, \( D(s) \not\in \mathcal{O}_{X,0} \)). This is the only pole of \( D \) because for \( u = 1/s \) one has \( D(u) = -u^2/2 + zu^3 \). Prop. 4.2 claims the existence of infinitely many values \( z_0 \) such that there exists a branched solution passing through \( z_0 \), i.e., a solution \( y \) using roots of \( (z - z_0) \).

We verify this explicitly by looking at the solutions of the equation as graphs in the \((y, z)\) plane. For any point \((y_0, z_0)\) such that \( y_0 - z_0^2 \neq 0 \) there are locally two holomorphic solutions corresponding to the choices for \( y'(z_0) \) satisfying \( y'(z_0)^2 = y_0 - z_0^2 \). Also for the point \((0, 0)\) one finds two
holomorphic solutions, namely \( y = az^2 \) with \( 4a^2 - a + 1 = 0 \). At any point \((y_0, z_0) \neq (0, 0)\) with \( y_0 - z_0^2 = 0 \) there are locally two branched analytic solutions of the form \( y = y_0 + c(z - z_0)^{3/2} + \cdots \) with \( c^2 = -\frac{8}{7}z_0 \). We conclude that at any point \( z_0 \neq 0 \) there passes a branched solution.

For the classification of the pairs \((X, D)\) where \( D \) has no poles, the field \( K \) can be any differential field and may even have positive characteristic (this is the theme of [Mat80]). In the Appendix we treat the case of positive characteristic. Our main result, Thm. 4.5, states that, after a finite extension of \( K \), the curve \( X \) is defined over \( \mathbb{C} \). However, the derivation \( D \) may use elements of \( K \). In all cases the equation has the PP.

For curves \( X \) of genus 0 or 1 an explicit elementary proof is given. For higher genus, differential Galois theory is applied to the connection \( \nabla \), derived from \( D \) on \( O_X \), on (powers of) the sheaf of relative differential forms \( \Omega_{X/K} \) on \( X \), to prove that, after a finite extension of \( K \), \( X \) is defined over \( \mathbb{C} \). Finally we note that in most cases these finite (separable) extensions of the differential field \( K \) are necessary for the main result.

2 Autonomous Case

Given is a polynomial \( f(S, T) \in \mathbb{C}[S, T] \) involving both \( S \) and \( T \). For the question of whether the equation \( f(y', y) = 0 \) has movable singularities we may and will suppose that \( f \) is irreducible. We use the following notation: \( \mathbb{C}\{z\} \) is the ring of convergent power series in \( z \) (i.e., the germs at \( z = 0 \) of holomorphic functions) and \( \mathbb{C}(\{z\}) \) denotes its field of fractions (the convergent Laurent series at \( z = 0 \)). For any integer \( m \geq 1 \) one considers the ring \( \mathbb{C}\{z^{1/m}\} \) and its field of fraction \( \mathbb{C}(\{z^{1/m}\}) \).

By a branched solution for \( f \) (at \( z = 0 \)) we will mean a \( y \in \mathbb{C}(\{z^{1/m}\}) \) with \( m > 1 \) and \( y \notin \mathbb{C}(\{z\}) \), satisfying \( f(y', y) = 0 \). Since the equation is autonomous, the existence of such a \( y \) will make any point of the complex plane into a branch point for \( f \) and \( f \) does not have the PP. On the other hand we will classify the equations \( f \) having no branched solution (at \( z = 0 \)) and show that they have the PP.

One considers the field \( F := \mathbb{C}(s, t) \), defined by the equation \( f(S, T) = 0 \). It is the function field of a certain curve \( X \) over \( \mathbb{C} \) (or compact Riemann surface). One considers on \( F \) the \( \mathbb{C} \)-linear derivation \( D \) given by \( D(t) = s \). This makes \( F \) into a differential field. Note that \( D \) has an obvious interpretation as (meromorphic) vector field on \( X \). The fields \( \mathbb{C}(\{z^{1/m}\}) \) are considered as differential fields w.r.t. the derivation \( \frac{d}{dz} \). A branched solution is equivalent to a \( \mathbb{C} \)-linear homomorphism of differential fields \( \phi : \mathbb{C}(s, t) \rightarrow \mathbb{C}(s, t) \).
(F, D) → (C(\{z^{1/m}\}), \frac{d}{dz}) with m > 1 and such that the image of φ is not contained in C(\{z\}).

**Theorem 2.1.**

1. Suppose that there exists a branched solution, represented by a homomorphism of differential fields \( \phi : F \rightarrow C(\{z^{1/m}\}) \) with \( m > 1 \) and \( m \) minimal. Then \( \phi \) induces a discrete valuation on \( F \) and thus a (closed) point \( x \in X \).

Let \( O_{X,x} \) be the local ring of the point \( x \). By definition \( \phi \) maps \( O_{X,x} \) to \( C(\{z^{1/m}\}) \). Let \( p \) be a local parameter for \( O_{X,x} \). Then \( O_{X,x}^n \), the analytic local ring at the point \( x \) of \( X \) (seen as Riemann surface), is equal to \( C\{p\} \). Now \( \phi \) uniquely extends to a homomorphism \( \phi^an : O_{X,x}^n \rightarrow C(\{z^{1/m}\}) \). Indeed, \( \phi^an \) is given by the formula \( \phi^an(\sum_{n \geq 0} a_n p^n) = \sum_{n \geq 0} a_n \phi(p)^n \).

The minimality of \( m \) implies that \( \phi^an \) is an isomorphism. Hence \( \phi^an \) induces an isomorphism \( C\{\{p\}\} \rightarrow C(\{z^{1/m}\}) \). The derivation \( D \) on \( F \) induces a derivation \( D^an : C\{p\} \rightarrow C\{\{p\}\} \) by the formula \( D^an(\sum_{n \geq 0} a_n p^n) = \sum_{n \geq 0} na_n p^{n-1} \cdot D(p) \).

From the formula \( \phi \circ D = \frac{d}{dz} \circ \phi \) and the definitions of \( \phi^an \) and \( D^an \) one deduces \( \phi^an \circ D^an = \frac{d}{dz} \circ \phi^an \). Finally, from \( \frac{d}{dz} z^{1/m} = 1/m \cdot z^{-1+1/m} \notin C\{z^{1/m}\} \), one deduces that \( D(p) \notin O_{X,x} \). Thus \( D \) has a pole at \( x \).

On the other hand, suppose that the derivation \( D \) has a pole at the point \( x \in X \). Then \( D(O_{X,x}) \) is not contained in \( O_{X,x} \). As above, we extend \( D \) to a derivation \( D^an : O_{X,x}^n \rightarrow C\{p\} \) which has a pole of order \( 1 - m \) (with \( m > 1 \)).

We want to solve the equation \( a(p) \frac{d}{dp} v = \frac{1}{m} v^{1-m} \) where \( v \) is a local parameter for \( C\{p\} \) (i.e., \( v = c_1 p + c_2 p^2 + \cdots \) with \( c_1 \neq 0 \)). One rewrites the equation as \( \frac{d}{dp} (v^m) = \frac{1}{a(p)} \). There is a \( b(p) \in C\{p\} \) of order \( m \) with \( \frac{d}{dp} b(p) = \frac{1}{a(p)} \). Then \( z := b(p) \) has an \( m \)th root \( v \in C\{p\} \). The order of \( v \)
is 1 and thus \( v \) is a local parameter. One concludes that \( (C(\{p\}), D^{an}) = (C(\{1/m\}), \frac{d}{dz}) \). This defines a branched solution.

(2) Using Riemann-Roch one finds that the sheaf of derivations on \( X \) has a non zero global section only if the genus of \( X \) is 0 or 1. In the first case the field \( F = C(x) \) and \( D = (a_0 + a_1 x + a_2 x^2) \frac{d}{dx} \) is the space of the derivations without poles. For genus 1, the field is as in the statement and the only (non zero) derivations without poles are \( cy \frac{d}{dx} \) with \( c \in C^* \). 

Example 2.2. Let \( f := S^p - T^q \) with \( (p, q) = 1 \). The field \( C(s, t) \) equals \( C(x) \) where \( s = x^q, t = x^p \). The formula \( D(t) = s \) and the last part of the theorem imply that PP is equivalent to \( q - p \in \{-1, 0, 1\} \).

Corollary 2.3. The Riccati differential equation and the Weierstrass differential equation, i.e., the cases (a) and (b) of Theorem 2.1, have the PP.

Proof. We quickly give the well known arguments. A solution to (a) or (b) corresponds to a differential homomorphism \( \phi \) from \((F, D)\) to some differential ring of functions. In case (a) this means that \( g = \phi(x) \) satisfies the Riccati equation \( g' = a_0 + a_1 g + a_2 g^2 \). In case (b) this means that \( g = \phi(y) \) satisfies a Weierstrass equation \((\lambda g')^2 = g^3 + ag + b\). Case (a) with \( a_2 = 0 \) is an inhomogeneous linear equation and the solutions have no other singularities than infinity. Case (a) with \( a_2 \neq 0 \) is the Riccati equation associated to a linear equation of order two. The solutions of the Riccati equation have the form \( g = \frac{v'}{v} \), where \( v \) is a non zero solution of this linear equation. Thus \( g \) can at most have a pole. Finally, the solutions to case (b) are shifts of the Weierstrass function and the singularities in the finite plane are poles.

Remark. In the paper of Briot and Bouquet, the PP is translated into an explicit, necessary and sufficient condition for the irreducible polynomial \( f \in C[S, T] \) [BB56a]. It can be verified that this condition is equivalent to the cases (a) or (b) of Theorem 2.1.

3 Assumptions and Notation

In the general case one considers a differential equation of the form \( f(y', y) = 0 \) where \( f(S, T) \) is a polynomial in \( S \) and \( T \) and has coefficients in some field \( K \) (possibly of germs) of meromorphic functions, defined on some part of the complex plane and closed under \( \frac{d}{dz} \). We restrict ourselves to the most interesting case where \( K \) is a finite extension of \( C(z) \). One can copy the proof below to the case of other fields of meromorphic functions.
Thus $K$ is the field of meromorphic functions on some compact Riemann surface $Z$. A closed point $P$ of $Z$ (or place of $K$) with local parameter $p$ is called ramified for $f$ if there exists a $y \in \mathbb{C}(\{p^{1/e}\})$, not an element of $\mathbb{C}(\{p\})$, satisfying $f(y', y) = 0$. Further we will call $y$ a branched solution at $P$. If there are infinitely many ramified points $P$ for $f$, then certainly $f$ does not have the PP. In what follows we will classify the equations $f$ that have only finitely many ramified points on $Z$ and show that these equations do have the PP. We note that replacing $K$ by a finite extension does not change the property: “there are only finitely many ramified points for $f$.” In particular, we will assume that $f \in K[S, T]$ is absolutely irreducible, i.e., $f$ is irreducible as element of $\overline{K}[S, T]$. Further we will assume that both $S$ and $T$ are present in $f(S, T)$.

Write $K[s, t] = K[S, T]/(f)$. To a branched solution $y$ at $P$ (with local parameter $p$) one associates a homomorphism $\psi : K[s, t] \to \mathbb{C}(\{p^{1/e}\})$, extending the given embedding $K \subset \mathbb{C}(\{p\})$, by sending $y$ to $t$ and $s$ to $y' := \frac{d}{dz}y$. One would like $K[s, t]$ to be a differential ring by introducing the $\mathbb{C}$-linear derivation $D$ by $D(z) = 1$ and $D(t) = s$. Let $\Delta$ denote $\frac{df}{ds}$ (evaluated at $s, t$). In general $K[s, t]$ is not invariant under $D$, but the localization $K[s, t, \frac{1}{\Delta}]$ is seen to be invariant under $D$ and is therefore a differential ring.

There are only finitely many $y$ (obviously algebraic over $K$) such that the corresponding $\psi$ satisfies $\psi(\Delta) = 0$. Omitting these $y$, we may extend the given $\psi : K[s, t] \to \mathbb{C}(\{p^{1/e}\})$ to a $\psi$ defined on the localization $K[s, t, \frac{1}{\Delta}]$. This new $\psi$ has a non trivial kernel, if and only if $y$ is algebraic over $K$. In this case $K(y)$ is a finite extension of $K$ and the kernel of $\psi$ is a maximal ideal of $K[s, t, \frac{1}{\Delta}]$, closed under the differentiation $D$.

Consider a branched solution $y$ such that the corresponding $\psi$ has a trivial kernel (thus $y$ is transcendental over $K$). Let $F$ denote the field of fractions of $K[s, t]$, made into a differential field by the unique extension (also called $D$) of the derivation $D$. Then $\psi$ extends to a homomorphism of differential fields $\phi : (F, D) \to (\mathbb{C}(\{p^{1/e}\}), \frac{d}{dz})$, where $e > 1$ is the smallest natural number for which the image of $\phi$ is contained in $\mathbb{C}(\{p^{1/e}\})$. This is called a transcendental branched solution. One associates to $F$ the non singular, absolutely irreducible, projective curve $X$ over $K$ with function field $F$.

4 General Case

We will need the following result.
Lemma 4.1. Let \( m > 1 \) and \( f \in \mathbb{C}\{z, w\} \) with \( f(0, 0) \neq 0 \) be given. Choose \( c \in \mathbb{C}^* \) with \( c^m = f(0, 0) \). Then the equation \((y^m)' = f(z, y)\) has a solution in \( \mathbb{C}\{z^{1/m}\} \) of the form \( y = cz^{1/m} + \ldots \).

Proof. Consider a formal series \( y = \sum_{i \geq 1} c_i z^{i/m} \). Then
\[
(y^m)' = \sum_{a \geq m} \frac{a}{m} \left\{ \sum_{i_1 + \ldots + i_m = a} c_{i_1} \ldots c_{i_m} \right\} z^{-1+a/m}.
\]
Comparing this expression with \( f(z, \sum c_i z^{i/m}) \) one finds \( c^m = f(0, 0) \). After choosing \( c_1 = c \), one compares the coefficients of \( z^{-1+a/m} \) and one finds that \( c^{m-1}c_{a-m+1} \) equals a polynomial formula in \( c_1, \ldots, c_{a-m} \). Thus we found a unique formal series satisfying the differential equation. One can verify (by brute force) that this series is in fact convergent. More explicitly, we may suppose \( f(0, 0) = 1 \) and \( c = 1 \). Write (for the moment) \( z = t^m \) and \( y = t(1 + h) \) with \( h \in t\mathbb{C}[[t]] \). One obtains a differential equation for \( h \) of the form
\[
\frac{d}{dt} h + mh = \sum_{a \geq 1, b \geq 0} c_{a,b} t^a h^b + \sum_{b \geq 2} c_{b} h^b,
\]
where the coefficients satisfy \( |c_{a,b}| \leq R^{a+b} \) and \( |c_{b}| \leq R^b \) for some \( R > 0 \). After multiplying \( h \) and \( t \) by suitable constants we may suppose that \( R \) is sufficiently small. Write \( h = \sum_{n \geq 1} h_n t^n \), then the recurrence relation for the \( h_n \) takes the form \((n + m)h_n = \) a polynomial in \( h_1, \ldots, h_{n-1} \) of which the absolute value can be estimated. One can verify that \( \lim h_n = 0 \).

Proposition 4.2. Suppose that there exists a closed point \( Q \) of \( X \) such that the local ring \( O_{X,Q} \) is not invariant under \( D \). Then there are infinitely many ramified points (on \( Z \)) for \( f \).

Proof. The assumption and the statement that we want to prove are stable under finite extensions of \( K \). Thus we may suppose that the closed point \( Q \) is rational over \( K \). The field \( F \) can also be seen as the function field of some normal projective surface \( X \) over \( \mathbb{C} \). The inclusion \( K \subset F \) induces a ‘rational map’ \( X \dashrightarrow Z \), where \( Z \) is the non singular, irreducible, projective curve over \( \mathbb{C} \) with function field \( K \). The assumption \( Q \in X(K) \) induces a ‘rational section’ of the above rational map. In the analytic category, one can avoid singularities and describe locally this rational map and section by the following:
Figure 1: A schematic representation of the situation in the proof of Proposition 4.2.

$U \subset Z$ is a disk, identified with $\{z \in \mathbb{C} \mid |z - z_0| < \epsilon\}$. $V \subset X$ is a multidisk $\{z \in \mathbb{C} \mid |z - z_0| < \epsilon\} \times \{u \in \mathbb{C} \mid |u| < \epsilon\}$. The rational map is just $(z, u) \mapsto z$ and the rational section is $z \mapsto (z, 0)$. Further, $u = 0$ is the restriction to $V$ of the divisor on $X$ given by the point $Q$ on $X$. We note that $u$ is a local parameter at $Q$ and the completion of $O_{X,Q}$ equals $K[[u]]$. It is given that $D(u) \notin K[[u]]$. Therefore $D(u) = a_{-m+1}u^{-m+1} + \cdots$ with $m > 1$ and $a_{-m+1} \in K$, different from 0. Then $D(u^m) \in K[[u]]$ and does not lie in $uK[[u]]$.

In analytic terms, $D(u^m)$ is a non zero holomorphic function on $V$. After shrinking $U$ and $V$ we may suppose that $D(u^m)$ has no zeros on $V$. Take now any point $z_1 \in U$. Then we have $D(u^m) = g(z - z_1, u) \in \mathbb{C}\{z - z_1, u\}$ with $g(0, 0) \neq 0$.

According to Lemma 4.1 the differential equation $(h^m)' = g(z - z_1, h)$ has a solution $h = \ast(z - z_1)^{1/m} + \cdots$ lying in $\mathbb{C}\{(z - z_1)^{1/m}\}$. The homomorphism $\mathbb{C}\{z - z_1, u\} \to \mathbb{C}\{(z - z_1)^{1/m}\}$ given by $z - z_1 \mapsto z - z_1, \ u \mapsto h$ induces a homomorphism $\psi : K[u, Du] \to \mathbb{C}\{(z - z_1)^{1/m}\}$ such that $\frac{d}{dz}\psi(u) = \psi(Du)$. If $h$ is not algebraic over $K$, then $\psi$ extends to a differential homomorphism $(F, D) \to \mathbb{C}\{(z - z_1)^{1/e}\}$ for some multiple $e$ of $m$. 

8
Suppose that \( h \) is algebraic over \( K \). Then one can extend \( \psi : K[u, Du] \to \mathbb{C}((z - z_1)^{1/m}) \) to a maximal differential subring \( R \) of \( F \) and with values in a finite extension of the above field. It is easily seen that \( R \) is some local ring \( O_{X, x} \), invariant under \( D \). If \( t \in O_{X, x} \), then \( K[s, t] \subset O_{X, x} \) and the restriction of \( \psi \) to this subring produces a branched algebraic solution.

There is one case that we still have to consider, namely \( t \notin O_{X, x} \). Then \( t^{-1} \) lies in the maximal ideal of \( O_{X, x} \) and \( \psi(t^{-1}) = 0 \). Let \( m_{X, x} \) denote the maximal ideal of \( O_{X, x} \) and \( K(x) \) its residue field. Then \( \psi(m_{X, x}) = 0 \) and \( \psi \) factors as \( O_{X, x} \to K(x) \to \mathbb{C}((z - z_1)^{1/e}) \) with \( e \) a suitable multiple of \( m \).

The place of \( K \) corresponding to \( z_1 \) is ramified in \( K(x) \). As there are only finitely many of these places and finitely many \( x \) with \( t \notin O_{X, x} \) the number of these places \( z_1 \) is finite. Therefore we found infinitely many points \( z_1 \) on \( Z \) which are ramified for \( f \).

\[ \square \]

**Proposition 4.3.** Suppose that the derivation \( D \) on \( F \) is regular, i.e., every local ring \( O_{X, x} \) is invariant under \( D \). Then there exists a finite extension \( L \) of \( K \) and a non singular curve \( X_0 \) over the field of constants \( \mathbb{C} \) of \( K \) such that \( X \times_K L \cong X_0 \times_{\mathbb{C}} L \).

**Proof.** Let \( g \) be the genus of \( X \).

(1) **Suppose that \( g = 0 \).** Then, after possibly a quadratic extension of \( K \), one has \( F = K(x) \). Moreover, “\( D \) regular” is equivalent to “\( K[x] \) and \( K[x^{-1}] \) are invariant under \( D \)”.

Hence \( D \) is regular if and only if \( \psi \) has the form

\[
D(x) = a_0 + a_1 x + a_2 x^2
\]

with \( a_i \in K \).

(2) **Suppose that \( g = 1 \).** After a finite extension of \( K \) we may suppose that \( X \) is given by the affine equation \( y^2 = x(x - 1)(x - a) \) with \( a \in K \), \( a \neq 0, 1 \).

The assumption on \( D \) implies that \( D(x) = A(x) + B(x)y \) with \( A(x), B(x) \in K[x] \). The completion of \( O_{X, \infty} \) has the form \( K[[u]] \) where \( x^{-1} = u^2 \) and \( y = u^{-3}(1 - u^2)(1 - au^2) \). The condition \( D(u) \in K[[u]] \) implies \( 2D(u) = u^{-1}x^2(A(1/u^2) + B(1/u^2)y) = u^3(A(1/u^2) + B(1/u^2)\sqrt{(1 - u^2)(1 - au^2)} \)

has no pole. This implies that \( A(x) = a_0 + a_1 x \) and \( B(x) = b_0 \).

The condition \( D(y) \in K[x, y] \) implies that \( 2yD(y) = D(x(x - 1)(x - a)) \in yK[x, y] \). This expression equals

\[
x(x - 1)(x - a)\left\{ \frac{1}{x} + \frac{1}{x - 1} + \frac{1}{x - a} \right\}(a_0 + a_1 x + b_0 y) - \frac{a'}{x - a} \}
\]

Hence

\[
x(x - 1)(x - a)\left\{ \frac{1}{x} + \frac{1}{x - 1} + \frac{1}{x - a} \right\}(a_0 + a_1 x) - \frac{a'}{x - a} \}
\]

9
lies in \( yK[x, y] \cap K[x] = x(x - 1)(x - a)K[x] \). Therefore \( a_0 = a_1 = a' = 0 \).

This proves the statement and we find moreover that \( D \) has the special form \( D(x) = by \) for some \( b \in K \).

(3) Suppose that \( X \) is a hyperelliptic curve. Here we give an elementary proof and note that (4) contains a more abstract proof. After a finite extension of \( K \) we may suppose that the affine equation of \( X \) has the form \( y^2 = Q \) with \( Q = x(x - 1)(x - a_1) \cdots (x - a_m) \) where \( m > 1 \) is odd and \( 0, 1, a_1, \ldots, a_m \) are distinct elements of \( K \). Then \( D(x) = A + By \) with \( A, B \in K[x] \).

The completion of the local ring at \( \infty \) can be written as \( K[[u]] \) with \( x^{-1} = u^2 \) and \( yu^{m+2} = \sqrt{(1-u^2)(1-u^2a_1) \cdots (1-u^2a_m)} \). The condition \( D(u) \in K[[u]] \) implies that \( B = 0 \) and \( A \) has degree at most one. The condition \( D(y) \in K[x, y] \) implies \( 2yD(y) = D(Q) \in yK[x, y] \cap K[x] = QK[x] \). Now \( \frac{D(Q)}{Q} \in K[x] \) implies that \( A = 0 \) and all \( a_i' = 0 \).

This proves the statement and shows moreover that \( D \) has the special form \( D(x) = D(y) = 0 \).

(4) Suppose that \( g \geq 2 \). The derivation \( D \) on \( F \) extends to a \( \mathbb{C} \)-linear morphism of sheaves \( \nabla : \Omega_{X/K} \to \Omega_{X/K} \) by the formula \( \nabla(\sum f_idg_i) = \sum_i(D(f_i)dg_i + f_id(Dg_i)) \). A verification that this formula is well defined is needed.

Consider an open (affine) subset \( U \) of \( X \). One represents \( O(U) \) as \( K[X_1, \ldots, X_n]/(f_1, \ldots, f_s) = K[x_1, \ldots, x_n] \). Now \( D \) lifts to a derivation \( D^+ \) of \( K[X_1, \ldots, X_n] \) having the property \( D^+(f_1, \ldots, f_s) \subset (f_1, \ldots, f_s) \). Then \( \Omega_{X/K}(U) \) is the free \( O(U) \)-module \( V \) with basis \( dx_1, \ldots, dx_n \) divided out by the \( O(U) \)-submodule \( W \) spanned by the elements \( \{df_i| i = 1, \ldots, s\} \) (or equivalently by all \( df \) with \( f \in (f_1, \ldots, f_s) \) ). One defines \( \nabla_V \) on \( V \) by \( \nabla_V(\sum h_idx_i) = \sum(D(h_i)dx_i + h_id(Dx_i)) \) and has to verify that \( \nabla_V(W) \subset W \). This easily follows from \( D^+(f_1, \ldots, f_s) \subset (f_1, \ldots, f_s) \). There results a \( \nabla \) on \( V/W \) with the required formula.

Now the \( K \)-vector space \( H^0(X, \Omega_{X/K}) \) is invariant under \( \nabla \). This makes \( H^0(X, \Omega_{X/K}) \) into a differential module over \( K \). The same holds for the symmetric powers \( H^0(X, \Omega_{X/K}^\otimes) \). Thus we found line bundles \( \mathcal{L} \) which are invariant under \( D \), i.e., \( D \) maps \( H^0(U, \mathcal{L}) \) to itself for any open \( U \subset X \).

Consider such an \( \mathcal{L} \) which is at the same time very ample. Then the curve \( X \) is equal to \( \text{Proj}(\oplus_{s \geq 0} H^0(X, \mathcal{L}^\otimes s)) \). Each \( K \)-vector space \( H^0(X, \mathcal{L}^\otimes s) \) has an action \( \nabla \) of \( D \) on it, which makes it into a differential module. We take a Picard-Vessiot extension \( U \supset K \) that trivializes these differential modules. In fact, it suffices to trivialize the differential module \( H^0(X, \mathcal{L}) \) since the \( H^0(X, \mathcal{L}^\otimes s) \) are images of the \( s \)th symmetric powers of \( H^0(X, \mathcal{L}) \).
Then $X \times_K U$ is the $\text{Proj}$ of $U \otimes_{\mathbb{C}} H$ where $H = \bigoplus_{s \geq 0} H_s$ is the homogeneous $\mathbb{C}$-algebra, generated by $H_1$, with $H_s = \ker(\nabla, H^0(X \times_K U, \mathcal{L}^{\otimes s}))$ for all $s \geq 0$. Put $X_0 = \text{Proj}(H)$, then $X \times_K U \cong X_0 \times_{\mathbb{C}} U$.

It is well known that in this case there exists also a finite extension $L$ of $K$ such that $X \times_K L \cong X_0 \times \mathbb{C} L$. (Indeed, for this isomorphism the field $U \supset K$ can be replaced by a finitely generated $K$-algebra $R \subset L$. Further $R$ can be replaced by $R/m$ for some maximal ideal $m$).

We add here that for $g \geq 2$, the derivation $D$ has a special form. Write $F$ (after a finite extension of $K$) as $K(x, y)$ where $\mathbb{C}(x, y)$ is the function field of $X_0$. Then $D$ is zero on $\mathbb{C}(x, y)$. \hfill $\Box$

**Corollary 4.4.** Let, as before, the differential field $(F, D)$ correspond to the differential equation $f(y', y) = 0$ with coefficients in the finite extension $K$ of $\mathbb{C}(z)$. If $D$ is regular, then $f$ has the PP.

**Proof.** (1) $F/K$ with genus 0. Then $f$ is equivalent to the Riccati equation $y' + a_0 + a_1 y + a_2 y^2 = 0$ with $a_0, a_1, a_2 \in K$. As in the proof of Corollary 1.3, one shows that the only movable singularities are poles.

(2) $F/K$ with genus 1. Then $f$ is equivalent to an equation of the form $(y')^2 = a y(y - 1)(y - \lambda)$ with $\lambda \in \mathbb{C} \setminus \{0, 1\}$ and $a \in K^*$. We rewrite the latter as $(by')^2 = y(y - 1)(y - \lambda)$ for a suitable $b$ in a suitable $K$. Define a new derivation $d$ on $K$ by $d(f) = bf'$. The equation $d(T) = 1$ has a solution in some Picard-Vessiot extension $L = K(T)$ of $K$. In general, $K(T)$ is a transcendental extension of $K$. If the extension happens to be algebraic, then we regard the differential equation as an equation over $\mathbb{C}(T)$ in the form $(\frac{d}{dT} y)^2 = y(y - 1)(y - \lambda)$. Then, as one knows, the only movable singularities are poles.

If $T$ is transcendental over $K$, then the equation is again seen as an equation over the field $\mathbb{C}(T)$. The solutions are of the form $P(T + c)$ (with $c \in \mathbb{C}$) and have only poles and no branched points.

Take a point $z_0 \in \mathbb{C}$, unramified in $K$. This yields an embedding $K \subset \mathbb{C}(\{z - z_0\})$. If the image of $b$ in this field has no pole then the equation $bf' = f$ has a solution $u \in \mathbb{C}(\{z - z_0\})$. From this one finds the solutions $P(u + c)$ in the field $\mathbb{C}(\{z - z_0\})$. This suffices to show that the only movable singularities of the equation are poles.

(3) $F/K$ with genus $\geq 2$. After a finite extension of $K$, the equation $f$ is equivalent to $y' = 0$. Clearly this equation has the PP. \hfill $\Box$

We summarize these results as follows.
Theorem 4.5. Consider an order one differential equation $f(y', y) = 0$ with coefficients in a finite extension $K$ of $\mathbb{C}(z)$. Let $(K(X), D)$ be the differential field constructed from $f$. Then $f$ has the PP if and only if, after a finite extension of $K$, the differential field $(K(X), D)$ is isomorphic to one of the following:

(a) $(K(u), (a_0 + a_1 u + a_2 u^2) \frac{d}{du})$, with $a_0, a_1, a_2 \in K$ not all zero.

(b) $(K(x, y), f \cdot y \frac{d}{dx})$, with $y^2 = x^3 + ax + b$, $a, b \in \mathbb{C}$ a non-singular elliptic curve and $f \in K^*$.

(c) There is a curve $X_0$ over $\mathbb{C}$ such that $X$ is isomorphic to $X_0 \times_\mathbb{C} K$ and $D = 0$ on $\mathbb{C}(X_0)$.

The three above cases correspond to the following differential equations:

(a') $y' = a_0 + a_1 y + a_2 y^2$, with $a_0, a_1, a_2 \in K$ not all zero.

(b') $(y')^2 = f \cdot (y^3 + ay + b)$, with $a, b \in \mathbb{C}$ and $f \in K^*$.

(c') $y' = 0$.

Remarks 4.6. (1) The results and proofs extend to the case where $K$ is the field of meromorphic functions on some domain of the complex sphere. Also direct limits of such fields of functions, e.g., $\mathbb{C}(\{z\})$, can be taken as base field.

(2) In Proposition 4.3, $\mathbb{C}$ can be replaced by any algebraically closed field of characteristic 0. In fact, Matsuda has proved similar properties for more general base fields [Mat80].

(3) For the case genus 0 and $K$ a finite extension of $\mathbb{C}(z)$ one does not need (in Proposition 4.3) a finite extension of $K$. Indeed, $K$ is a $C_1$-field and a genus 0 curve over $K$ has a $K$-rational point.

However, one can construct a differential field $K$, a curve $X$ over $K$ of genus 0 and derivation $D$ on $K(X)$ without poles, such that a quadratic extension of $K$ is needed in Proposition 4.3.

(4) For the genus 1 case and $K$ a finite extension of $\mathbb{C}(z)$ it can be seen that, in general, an extension of $K$ is needed.

(5) Also for genus $\geq 2$, one needs in Proposition 4.3 (in general) a finite extension of $K$. We construct some examples.

Consider a function field $\mathbb{C}(X_0)$ of a curve $X_0$ of genus $\geq 2$ on which a finite group $G \neq \{1\}$ acts faithfully and such that the curve $X_0/G$ has a
strictly smaller genus. There exists a Galois extension $L \supset \mathbb{C}(z)$ with group $G$. On $R := L \otimes \mathbb{C}(X_0)$ we let the group $G$ act by $g(a \otimes b) = g(a) \otimes g(b)$. We provide $R$ and its field of fractions $F$ with the derivation $D$ given by $D = 0$ on $\mathbb{C}(X_0)$ and $D(z) = 1$. Clearly $D$ commutes with the action of $G$. Moreover $\mathbb{C}(X_0)$ is the field of constants of $(F, D)$.

The ring of invariants $R^G$ and its field of fractions $F_0$ are invariant under $D$. Further $F_0$ is the function field of some curve $X$ over $\mathbb{C}(z)$ and the restriction of $D$ to $F_0$ has the properties of Proposition 4.3. Suppose that there exists a curve $X_1$ (as in the proposition) such that $X \cong X_1 \times \mathbb{C} \mathbb{C}(z)$. Then the field of constants of $F_0$ is $\mathbb{C}(X_1)$. The latter is, however, equal to $\mathbb{C}(X_0)^G$ and this is the function field of the curve $X_0/G$. Now $F = L \otimes F_0$ and the two fields $F$ and $F_0$ have the same genus. Thus we found a contradiction.

(6) The equation $(y')^2 = y^3 + z$, which has constant $j$-invariant, does not have the PP! Indeed, consider the affine ring of the curve $\mathbb{C}(z)[s, t]$ with equation $s^2 = t^3 + z$ and derivation $D$ given by $D(z) = 1$, $D(t) = s$. Then $D(s) = \frac{3st^2 + 1}{2}$ and the latter does not belong to $\mathbb{C}(z)[s, t]$. As in the example of the introduction, one can make this more explicit. A small computation shows that through every point $(y_0, z_0)$ with $z_0 \neq 0$ and $y_0^3 + z_0 = 0$ there are two branched solutions, namely $y = y_0 \pm \frac{2}{3}(z - z_0)^{3/2} + \cdots$.

A Appendix

As in Matsuda’s work [Mat80], we consider the case of differential fields $(K, \partial)$ with characteristic $p > 0$. Two cases are studied here:

(1) $\partial = 0$ and

(2) $[K : K^p] = p$. We choose an element $z \in K \setminus K^p$. Then $K = K^p(z)$ and we may restrict ourselves to $\partial = \frac{d}{dz}$ (i.e., the derivation $\partial$ is defined by $\partial(z) = 1$).

The most important examples for case (2) are:

(a) $K$ is a finite separable extension of $\mathbb{C}(z)$, where $C$ is an algebraically closed field of characteristic $p$. The derivation $\partial$ extends $\frac{d}{dz}$ on $\mathbb{C}(z)$.

(b) $K$ is a finite separable extension of $\mathbb{C}((z))$, where $C$ is an algebraically closed field of characteristic $p$. Further $\partial$ is the extension of $\frac{d}{dz}$ on $\mathbb{C}((z))$. 13
Let $X$ be a smooth, absolutely irreducible, projective curve over $K$. Its function field is denoted by $F$. A derivation $D$ on $F$ extending $\partial$ is given which has the property that for any closed point $Q$ of $X$, the local ring $O_{X,Q}$ is invariant under $D$. We will show that the classification of the possible pairs $(F,D)$ is, up to finite separable extensions of $K$, the same as in the characteristic 0 situation.

Case (1). $D$ is a non zero element of $H^0(X, \text{Der}_{X/K})$. It follows that there are two possibilities:

(1a) Suppose that $X$ has genus 0. After possibly replacing $K$ by a separable extension of degree 2, we may suppose that $F = K(t)$. As before one computes that $D$ must have the form $(a_0 + a_1 t + a_2 t^2) \frac{d}{dt}$ with $a_0, a_1, a_2 \in K$.

(1b) Suppose that $X$ has genus 1. The assumption that $X$ is smooth over $K$ implies that after a finite separable extension of $K$, the curve $X$ becomes an elliptic curve. Then $H^0(X, \text{Der}_{X/K})$ has dimension 1 over $K$ and $D$ is unique up to a $K^*$-multiple.

Case (2).

(2a) Suppose that $X$ has genus 0. As in case (1a), one obtains that, after possibly a separable quadratic extension of $K$, the function field equals $K(t)$ and $D$ has the form $(a_0 + a_1 t + a_2 t^2) \frac{d}{dt}$ with $a_0, a_1, a_2 \in K$.

(2b) Suppose that $X$ has genus 1. After a suitable separable extension of $K$, the function field $F$ has the form $K(s,t)$ with (compare [Si46, p. 324-5]):

$$s^2 = t(t - 1)(t - a)$$

with $a \in K$, $a \neq 0, 1$ for $p > 2$ and for $p = 2$:

(a) $s^2 + st = t^3 + a^2 t^2 + a_6$ with $a_6 \neq 0$

or

(b) $s^2 + a_3 s = t^3 + a_4 t + a_6$ with $a_3 \neq 0$.

For $p > 2$, the proof of Proposition 4.3 remains valid. Therefore $a \in K^p$ and $D$ has the form $D(t) = bs$ for some $b \in K^*$.

For $p = 2$ and case (a) one can transform the equation (using a separable extension of $K$) into $s^2 + st = t^3 + a$ with $a \in K^*$.

Write $D(t) = A + sB$ with $A, B \in K[t]$. Then $tD(s) = s(A + Bt + t^2 B) + (a + t^3)B + t^2 A + a'$ and thus $A$ and $aB + a'$ are in $tK[t]$. A local parameter at $\infty$ is $u = \frac{t}{s}$. Then $D(u) = \frac{t^2 A + a're + t^2 B s}{s^2}$ is supposed to have order $\geq 0$ at $\infty$. Thus both $t^2 A + a'$ and $t^2 B s$ have order $\geq -6$ at $\infty$. This implies $B = 0$ and thus $a' = 0$ since $aB + a' \in tK[t]$. Further $A = bt$ for some $b \in K^*$. We conclude that $a \in K^2$ and $D(t) = bt$ for some $b \in K^*$ or $D = bt A \frac{d}{dt}$.

For $p = 2$ and case (b) one can transform the equation (using a separable extension of $K$) into $s^2 + s = t^3$. The derivation $D$ has the form $D(t) = A + Bs$ with $A, B \in K[t]$ and $D(s) = t^2 D(t)$. The element $u = \frac{t}{s}$ is a
uniformizing parameter at $\infty$. The condition $D(u)$ has order $\geq 0$ at $\infty$ yields (after some calculation) $D(t) = a \in K$. In other words $D = a\frac{d}{dt}$ on $K(s,t)$.

(2c) Suppose that the genus of $X$ is $\geq 2$. Then the sheaf $\Omega_{X/K}^{\otimes 2}$ is very ample. The assumption on $D$ and the reasoning in the proof of Proposition 4.3 implies that $D$ induces a connection $\nabla$ on $M := H^0(X, \Omega_{X/K}^{\otimes 2})$. As before we want to trivialize this connection, using some differential extension of the field $K$. According to [vdP95, Theorem 5.3], there exists a finite separable extension $K_1 \supset K$ such that the connection $K_1 \otimes M$ admits a minimal Picard-Vessiot ring $A$. For notational convenience we write again $K$ for $K_1$. Put $V = \ker(\nabla, A \otimes_K M)$. Thus $V$ is a vector space over the constants $K^p$ of $K$. Moreover, the natural map $A \otimes_{K^p} V \to A \otimes_K M$ is an isomorphism. As in the proof of Proposition 4.3 we find a curve $X_0$ over $K^p$ and an isomorphism $X_0 \times_{K^p} A \to X \times_K A$. The ring $A$ is a local Artin ring with residue field $K$. Dividing by its maximal ideal one obtains an isomorphism $X_0 \times_{K^p} K \to X$. By construction, the derivation $D$ is 0 on the function field $K^p(X_0)$. We conclude the following:

There exists a finite separable extension $K_1 \supset K$ such that $F = K(X) \subset K_1(X) = K_1(X_0) \supset K_1^p(X_0)$, where $X_0$ is a curve over $K_1^p$ and $D$ is zero on $K_1^p(X_0)$.

References

[BB56a] C.A. Briot and J.C. Bouquet. Mémoire sur l’intégration des équations différentielles au moyen des fonctions elliptiques. Journal de l’École (Imperiale) Polytechnique, 36:199–254, 1856.

[BB56b] C.A. Briot and J.C. Bouquet. Recherches sur les propriétés des fonctions définies par des équations différentielles. Journal de l’École (Imperiale) Polytechnique, 36:133–198, 1856.

[Bie53] L. Bieberbach. Theorie der Gewöhnlichen Differentialgleichungen: auf Funktionentheoretische Grundlage Dargestellt. Springer, Berlin, 1953.

[Fuc84] L. Fuchs. über differentialgleichungen, deren integrale feste verzweigungspunkte besitzen. Königlichen Preussischen Akademie der Wissenschaften, 32:699–719, 1884.

[Hil76] E. Hille. Ordinary Differential Equations in the Complex Domain. Wiley-Interscience, New York, 1976.
[Inc56] E.L. Ince. *Ordinary Differential Equations*. Dover Publications, Inc, New York, 1956.

[Mal41] J. Malmquist. Sur les fonctions á un nombre fini de branches satisfaisant á une équation différentielle du premier ordre. *Acta Math.*, 74:175–196, 1941.

[Mat80] M. Matsuda. *First Order Algebraic Differential Equations: A Differential Algebraic Approach*. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1980.

[Pai88] P. Painlevé. Sur les equations différentielles du premier ordre. *Comptes-Rendus Acad. Sc. Paris*, 107:221–224, 320–323, 724–726, 1888.

[Pai95] P. Painlevé. *Leçons sur la Théorie Analytique des Équations Différentielles, Professées a Stockholm*. Hermann, Paris, 1895. Reprinted, Oeuvres de Paul Painlevé, vol. I (Editions du CNRS, Paris, 1973).

[Poi85] H. Poincaré. Sur un théorème de m. fuchs. *Acta Mathematica*, 7:1–32, 1885.

[Sil86] J.H. Silverman. *The Arithmetic of Elliptic Curves*. Springer-Verlag, New York, 1986.

[vdP95] M. van der Put. Differential equations in characteristic $p$. *Compositio Mathematica*, 97:227–251, 1995.