Non Relativistic Limit of a Model of Fermions interacting through a Chern-Simons Field

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Abstract

We study the non relativistic limit of a Model of Fermions interacting through a Chern-Simons Field, from a perspective that resembles the Wilson’s Renormalization Group approach, instead of the more usual approach found in most texts of Field Theory. The solution of some difficulties, and a new understanding of non relativistic models is given.

Models of a Chern-Simons field interacting with non relativistic bosons or fermions have being studied in the literature both for its interest in general understanding of field theory by itself as for its application to Condensed Matter Physics. The use of these models face in general, the difficulties of their non renormalizability. This fact is perhaps, the main reason for the interest, on the results of Bergman and Lozano in one loop, later extended to three loops. Their model consists of a non-relativistic boson field $\phi$ with a

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quartic self interaction and minimally interacting with a Chern-Simons field $A^\mu$:

$$\mathcal{L} = \phi^*(i\frac{d}{dt} + eA^0)\phi - \frac{1}{2m}|(\vec{\nabla} - ie\vec{A})\phi|^2 - \frac{\lambda_0}{4}(\phi^*\phi)^2 + \frac{\theta}{2} \epsilon^{\mu
u\rho} A^\mu \partial^\nu A^\rho. \quad (1)$$

The only primitively divergent Green Function is the boson four point function. Up to one loop, the model can be made finite by the choice of a renormalized coupling constant $\lambda$ through the equation:

$$\lambda_0 = \lambda + \frac{m}{4\pi}\left(\lambda^2 - \frac{4e^2}{m^2\theta^2}\ln\left(\frac{\Lambda}{M}\right)\right) \quad (2)$$

where $\Lambda$ is an ultraviolet (UV) cut-off and $M$ an arbitrary constant (the renormalization constant) with dimension of mass. Their main observation is that at the critical value, $\lambda^* = \frac{2e}{m\theta}$, the one loop contribution vanishes and no renormalization of $\lambda$ is needed. At this choice of $\lambda$ the model regains the scale invariance that it has at classical level, and the relative wave function of the two bosons reproduces the Aharonov-Bohm scattering amplitude up to the second Bohr order. The model of non relativistic fermions interacting with the Chern-Simons field was also discussed in [2] and studied in more details in [3]. In this last paper it is shown that the one loop scattering of two fermions with spins of the same sign (in 2+1 dimension the spin is a pseudo-scalar) is finite in one loop, due to the contact interaction represented by the Pauli interaction, that is already present in the minimal interaction of the fermions with the gauge field. As for the scattering of two fermions of opposite spins the Pauli interaction does not have any role and the amplitude is divergent unless a quartic fermionic interaction of the form $c(\Lambda) \psi^*\psi\phi^*\phi$ where $\psi$ and $\phi$ represent respectively fermions with spin plus and minus 1/2, and $c$ is a constant that depends logarithmically on the UV cut off $\Lambda$. This last fact poses a new problem. If the non relativistic model is thought to be the low energy limit of a more fundamental model of relativistic Dirac fermions interacting with a Chern-Simons field, in the way that this limit is generally taken in most texts [4], it would come from a similar quartic interaction in the relativistic fermions. As is well known one such interaction is non renormalizable! We will show that this is, in fact, a false
problem. No quartic non renormalizable self interaction is needed in the “parent” relativistic model if a new perspective on the non relativistic limit in field theory is taken. Before going to the description of this new limit, lets us briefly resume, in an example, the “Classical Non Relativistic Limit”, and discuss why it is not always correct. Let us consider, in 2+1 dimension, a 2 component Dirac fermion field \( \Psi \), that represents a spin plus fermion and its anti-fermion, interacting with an external electromagnetic field \( A^\mu \), as described by the Lagrangian density (the gamma matrices are \( \gamma^\mu = \sigma^\mu \) where \( \sigma^\mu \) are the Pauli matrices)

\[
L_{rel} = \bar{\Psi} \left\{ \gamma^\mu \left( i \frac{\partial}{\partial x^\mu} + eA^\mu \right) - m \right\} \Psi.
\] (3)

The corresponding equation of motion is:

\[
\left(i \frac{d}{dt} + eA^0\right) \Psi = \left\{ \gamma^0 \gamma \cdot (-i\nabla - e\vec{A}) + \gamma^0 m \right\} \Psi.
\] (4)

Let us now consider a positive energy solution \( \Psi \) of this equation. To make contact with the non relativistic description, in which, the rest energy \( m \) of the particles is not included in the solution, let us make in the above equation of motion, the substitution

\[
\Psi = e^{-imt} \sqrt{2m} \begin{pmatrix} \psi \\ \chi \end{pmatrix}.
\] (5)

The result is:

\[
\left(i \frac{d}{dt} + eA^0\right) \psi = \frac{1}{2m} \left( -i\nabla - e\vec{A} \right)^2 + \frac{e^2}{2m} B \psi,
\] (8)
where \( B = \nabla \wedge \vec{A} \) is the magnetic field. The one component spinor \( \psi \) represent a fermion with spin plus. The last term is the Pauli magnetic moment–magnetic field interaction term of . The Lagrangian density corresponding to this equation of motion is the so called Pauli Schrödinger (PS) propagator of non relativistic fermions in an electromagnetic field:

\[
\mathcal{L}_{\text{class}}^{\text{nonrel}} = \psi^* \left( i \frac{d}{dt} + eA^0 \right) \psi - \frac{1}{2m} \left| \left( -i \nabla - e\vec{A} \right) \psi \right|^2 + \frac{e}{2m} B \psi^* \psi
\]  

The essential facts behind the above non relativistic limit are the assumptions on the strength of \( A^\mu \) and the momentum space support of the field \( \Psi \) (the second assumption is not meaningful without the first, since a low energy initial state of \( \Psi \) can be driven to a relativistic state by the action of a strong \( A^\mu \) field). Suppose now that \( A^\mu \) is not an external, controllable field, but is a dynamical field with dynamics given by a Maxwell or Chern-Simons term (that must be thought as added to the Lagrangian (9)). Let us consider in this theory, the scattering of two low energy fermions, their energy and momenta given by \((p^0 = m + \frac{\vec{p}^2}{2m}, \pm \vec{p})\) with \( |\vec{p}| << m \). On the right side of Figure 1 we draw a possible one loop contribution (among others) to this process. The corresponding amplitude is given by a Feynman integral in the loop momentum \( k^\mu \):

\[
\mathcal{A}_{\text{lowenergy}}^{\text{relativ}} = \cdots \int d^3k \left\{ \cdots \frac{i}{(k^0 + p^0)^2 - (k + \vec{p})^2 - m^2} \right\} \cdots
\]  

The main observation on this equation is that, even if the process we are treating is a low energy one, the amplitude receives contributions of high energy intermediate states, represented by propagators whose dynamics is essentially relativistic, and so, not coming from the Feynman rules of the non relativistic Lagrangian (9). The bigger or lower suppression of the contribution of these high energy states depend on the dynamics of the exchanged \( A^\mu \) field. As we will explicitly see in one example below, for the Chern-Simons field, they effectively give a contribution that can not be understood as coming from the non relativistic Lagrangian (9). What about the description of this same process starting from the non relativistic theory given by (9)? The amplitude for the same process of figure 1 is of the form
where now \( p^0 = |\vec{p}|^2/2m \). Here also, the integral extends up to infinity momenta. High energy intermediate states also contribute to the amplitude, even with a bigger weight than to \( A_{\text{rel}} \), as seen from the worse UV behavior of the Pauli-Schrödinger (PS) propagator. It must yet be observed, that from the viewpoint of the Special Relativity, the PS propagator misses to represent the propagation of high energy intermediate states. Due to these facts, some authors in Field Theory (\cite{10}) take the view of Non Relativistic Field Theory as a cutoff theory. This means that instead of considering, in a wrong way, the contribution of the high energy intermediate states, they prefer to decouple them from the theory by limiting the integration in the Feynman integrals up to a maximal energy-momentum, compatible with the newtonian description provided by the PS propagator. This is also a view taken by some authors in optics (\cite{11}). There, the typical energy involved in the scattering processes are of the order of the ionization energy of the atoms, \( \alpha^2 m \), where \( \alpha \) is the fine structure constant and \( m \) is the electron mass. The cutoff assumed is \( \Lambda = \alpha m \), the inverse of the Bohr radius of the atom, much bigger than the typical energies involved in the scattering processes, and much smaller than the rest energy, \( m \) of the electron.

We will take a slight variation of these ideas, suited for understanding the results on non relativistic models with a CS field in the Coulomb Gauge, as treated in the literature \cite{2,3}. We will consider a non relativistic cutoff, only in the spatial momentum \( \vec{k} \), of the Feynman integrals; that is, we will calculate the Feynman integrals, firstly freely integrating the \( k^0 \) momentum component up to infinity, and then restricting the integration in \( |\vec{k}| \) to the region \((0, \Lambda)\) with \( \Lambda \) chosen so that \( |\vec{p}| << \Lambda << m \), where \( \vec{p} \) is a characteristic momentum of the low energy process in which of interest, and \( m \) is the mass of the fermion field. This choice has the additional technical advantage of not breaking the locality in the time direction. The way, we are proposing, that these cut off non relativistic models are related to originally relativistic ones, is akin to the ideas of the Renormalization Group of Wilson \cite{12}. We will first outline the main ideas in a toy model.
Let us consider a relativistic field theory in 1 space-time dimension, with dynamics given by a Lagrangian $L^{rel}(\Phi)$. Its functional generator is given by

$$Z(j) = \int \mathcal{D}\Phi(p) \exp \left( i \int dp \left( L^{rel}(\Phi) + j(\Phi) \right) \right)$$  \hspace{1cm} (12)

where $j$ is an external source, and $\mathcal{D}\Phi(p) \doteq \prod_0^{\infty} d\Phi(p)$. Suppose that we are only interested in describing “non relativistic” processes involving external particles with momenta $p$ smaller than a certain value $\Lambda << m$, where $m$ is the mass of the field $\Phi$. This limitation can be implemented in the functional generator by choosing the external source to be non null only for the momentum region $(0, \Lambda)$. The $\Phi$ field can be separated in $\Phi(p) = \phi(p) + \bar{\phi}(p)$ where $\phi$ has support in $(0, \Lambda)$ and $\bar{\phi}$ in $(\Lambda, \infty)$. The integration measure goes in $\mathcal{D}\Phi = \mathcal{D}\phi \mathcal{D}\bar{\phi}$, the Lagrangian separates in $L^{rel}(\Phi) = L^{rel}(\phi + \bar{\phi}) = L^{rel}(\phi) + \nabla L(\phi, \bar{\phi})$ and $j \Phi$ gets reduced to $j \phi$. The functional generator becomes

$$Z(j) = \int \mathcal{D}\phi \exp i \int \left( L^{rel}(\phi) + j \phi \right) \int \mathcal{D}\bar{\phi} \exp i \int \left( \nabla L(\phi, \bar{\phi}) \right)$$  \hspace{1cm} (13)

and can be written in the form

$$Z(j) = \int \mathcal{D}\phi \exp i \int \left( L^{eff}(\phi, \Lambda) + j \phi \right)$$  \hspace{1cm} (14)

where $L^{eff}(\phi, \Lambda) = L^{rel}(\phi) + \delta L(\phi, \Lambda)$ and

$$\int \delta L(\phi, \Lambda) = -i \ln \left( \int \mathcal{D}\bar{\phi} \exp i \int \nabla L(\phi, \bar{\phi}) \right)$$  \hspace{1cm} (15)

The effects of the high momenta modes $\bar{\phi}$ are incorporated in the effective dynamic of the low energy ones, through the additional term $\delta L(\phi, \Lambda)$. It is the effective Lagrangian, $L^{eff}$, in which the only remaining free momenta modes are the non relativistic ones, and not the original $L^{rel}$, that will give, through the approximations called Classical Non Relativistic Limit (exemplified above), the same results to low energy processes, as if calculated from the original relativistic model.

The integration in $\bar{\phi}$ in (15) can in general be done by expanding the exponential in a series of powers of $\int \nabla L(\phi, \bar{\phi})$. The result will be a series of Feynman graphs with the
propagator of $\bar{\phi}$ in the internal lines and the field $\phi$ in the external legs. This means that the integration in the loop momenta is restricted to the interval $(\Lambda, \infty)$. The result is in general $\Lambda$ dependent (as we will see the result of Bergman and Lozano is an exception) resulting in an Effective Lagrangian $L^{\text{effect}}$ that is dependent on $\Lambda$.

Let us now return to the models that we want to treat in 2+1 dimensions. We will start with the relativistic Lagrangian

$$L^{\text{relat}} = \bar{\Psi} \left( \gamma^\mu \left( i \frac{\partial}{\partial x^\mu} + e A_\mu \right) - m \right) \Psi$$

$$+ \bar{\Phi} \left( \gamma^\mu \left( i \frac{\partial}{\partial x^\mu} + e A_\mu \right) + m \right) \Phi + \frac{\theta}{2} \varepsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho$$

(16)

where $\Psi$ (or $\Phi$) is a 2 component Dirac field representing a fermion and anti fermion of spin plus (minus). In the Coulomb Gauge, the CS propagator is (indices $\mu, \nu, \ldots$ runs from 0 to 2 and indices $i, j, \ldots$ runs over 1 and 2)

$$\Delta_{\mu\nu} = \langle TA_\mu(p)A_\nu(-p) \rangle = \frac{1}{\theta} \epsilon_{\mu\nu\rho} \frac{k^i}{k^2}$$

(17)

and will be represented by a wavy line. The Dirac propagators of the relativistic fermions will be represented by double straight lines. Through the same steps that led (2) to (7) we get the Classical Non Relativistic limit of this model

$$L^{\text{class}}_{\text{nonrel}} = \bar{\psi} \left( i \frac{d}{dt} + eA^0 \right) \psi - \frac{1}{2m} \left| (i\vec{\nabla} - e\vec{A}) \psi \right|^2 + \frac{e}{2m} B \psi^* \psi$$

$$+ \bar{\phi} \left( i \frac{d}{dt} + eA^0 \right) \phi - \frac{1}{2m} \left| (i\vec{\nabla} - e\vec{A}) \phi \right|^2 - \frac{e}{2m} B \phi^* \phi$$

$$+ \frac{\theta}{2} \epsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho,$$

(19)

where $\psi$ (or $\phi$) are anti commuting one-component fields representing a spin plus (minus) fermion. The fermionic PS propagator will be represented by a single straight line. This model has several different vertices : $F^*F A^0$, $F^*F \vec{A}$, $F^*FA^\mu A_\mu$ and $F^*FB$, where F stands for $\phi$ or $\psi$. The last vertex (Pauli) gives a local interaction between two fermions, mediated by the the propagator.
\[
\Delta_B \doteq <TB(k)A_0(-k)> = \frac{i}{\theta}
\] (20)

that we will represent by an dashed straight line. The fermionic PS Propagator will be represented by a single straight line.

We will leave the result above for future use, and return to the construction of the Effective Non Relativistic Model. This will be done by calculating different low energy processes in the Relativistic Theory and identifying the contributions that come from the low energy intermediate states (and are the same that come from the Classical Non Relativistic Model with a cut off \(\Lambda\)) and the contributions that come from high energy intermediate states, and that must be incorporated in the Effective Non Relativistic Model, through new terms in the Lagrangian. We will restrict the calculations to the one loop order. The sum of Feynman graphs, written as a loop integral can be separated as

\[
A_{\text{low energy}}^{\text{relativ}} = \int d^3 k I\left(k^0, \vec{k}, w(p), \vec{p}\right) = \int_0^\Lambda d^2 k \int_{-\infty}^{\infty} dk^0 I + \int_\Lambda^\infty d^2 k \int_{-\infty}^{\infty} dk^0 I
\] (21)

In the low momenta part, both \(|\vec{p}|\) and \(|\vec{k}|\) are smaller than \(\Lambda \ll m\), and we can safely make the approximation \(w(\vec{q}) = m + \frac{\vec{q}^2}{2m}\), for both \(p\) and \(k\). The propagators and vertices collapse in the correspondent ones, got from the Lagrangian (16). In the high intermediate energy part this approximation can be taken for \(w(p)\) but not for \(w(k)\). As \(|\vec{p}| \ll \Lambda\) and the integral is for \(|\vec{k}| > \Lambda\), the result, \(H(p, \Lambda)\), is analytic in \(p\) and can be expanded in a series in \(p\). Every term of this expansion is a contribution to the process, that can be represented by a (new) local term in the Lagrangian of the Effective Non Relativistic Model. The three processes that require renormalization are the Vacuum Polarization Tensor (Figure 2) the Fermion Self energy (Figure 3) and the Vertex Correction (Figure 4). The calculation, of these quantities in covariant gauges are presented in many papers in the literature (13). In the Coulomb Gauge it was obtained in [6,7]. The results, separating the contributions of the low (first bracket) and of the high (second bracket) intermediate momenta contributions are respectively

8
\[ \Pi_{\mu \nu}^{\text{lowenergy}} = \left[ \text{Zero} + O(1/m^2) \right] + \left[ -i \frac{e^2}{6\pi m} (p^2 g_{\mu \nu} - p_\mu p_\nu) + O(1/m^2) \right] \] (22)

\[ \Sigma_{\psi \phi}^{\text{lowenergy}} = \left[ \text{Zero} + O(1/m^2) \right] + \left[ i \frac{e^2}{4\pi \theta} (\pm \vec{\gamma} \cdot \vec{p} - \frac{p^2}{m}) + O(1/m^2) \right] \] (23)

\[ e A_{\text{external}}^\mu \bar{u}(p') \psi \Gamma_\mu^{\text{lowenergy}} (p' - p) u(p) \psi \phi \]
\[ = \left[ \text{Zero} + O(1/m^2) \right] + \frac{e}{2m} \left[ \frac{e^2}{2\pi \theta} \epsilon^{ij}(p' - p)^j A_{\text{external}}^i + O(1/m^2) \right] \] (24)

As indicated in these formulas, all the contributions to these functions come from the high momenta intermediate states. In fact it is well known that these same functions are zero when calculated in the classical non relativistic model ([2]). As consequence the whole contribution to these functions, come only from the high momenta intermediate states and are independent of the cut off \( \Lambda \). The effects of these terms in correcting the low energy dynamics of the fermions and the CS field are simulated by adding to the Lagrangian (19) the terms

\[ \delta \mathcal{L} = -\frac{1}{4} \frac{e^2}{6\pi m} F_{\mu \nu} F^{\mu \nu} + \frac{e}{2m} \frac{e^2}{2\pi \theta} B \psi^* \psi + \frac{e}{2m} \frac{e^2}{2\pi \theta} B \phi^* \phi. \] (25)

From (25) and (19) we see that the CS field becomes a dynamical propagating field, the so called Maxwell-Chern-Simons field ([1]). We can also see that the magnetic momenta of the spin plus and minus fermions are corrected to (19)

\[ \mu_{\psi \phi} = \frac{e}{2m} \left( \pm 1 + \frac{e^2}{2\pi \theta} \right) \] (26)

(these results where obtained previously in the literature in covariant gauges([18])).

Let us now look at the elastic scattering of two low energy fermions. For simplicity we will work in the Center of Momentum Reference Frame in which the incoming fermions have energy and momenta: \((m + \frac{p^2}{2m}, \vec{p})\) and \((m + \frac{p^2}{2m}, -\vec{p})\) and the outgoing fermions have
\[(m + \frac{p^2}{2m}, \vec{p}') \text{ and } (m + \frac{p^2}{2m}, -\vec{p}') \text{ with } |\vec{p}| = |\vec{p}'| << \Lambda. \text{ The amplitude is a function of } |\vec{p}| \text{ and the angle between } \vec{p} \text{ and } \vec{p}'. \text{ We prefer to write it in terms of } \vec{p} \text{ and the two vectors } \vec{s} = \vec{p} + \vec{p}' \text{ and } \vec{q} = \vec{p}' - \vec{p}. \text{ In Figure 5 are shown the non null graphs contributing to this process.}

For the scattering of one fermion of spin plus and other of spin minus, the contributions of these graphs are listed below, separated in two rows. In the first are the contributions of the low intermediate momenta states, \( A(0, \Lambda) \), and in the second row, the local (independent of p) contributions of the high momenta intermediate states, \( A(\Lambda, \infty) \).

\[
A_{\text{low energy}}^{++ \text{ rel}} = A_{\text{lowene}}^{++ \text{ rel}}(0, \Lambda) + A_{\text{lowene}}^{++ \text{ rel}}(\Lambda, \infty)
\]

Graph 5a = \[
\left[ e^{\frac{\vec{s} \cdot \vec{q}}{m^2}} \right] + \left[ 0 \right]
\]

Graph 5b = \[
\left[ 0 \right] + \left[ \frac{e^4}{6\pi m^2} \right]
\]

Graphs 5c = \[
\left[ 0 \right] + \left[ \frac{e^4}{2\pi m^2} \right] \quad (27)
\]

Graph 5d = \[
\left[ \frac{e^4}{4\pi m^2} \ln \frac{-\vec{q}^2}{\vec{p}^2} \right] + \left[ 0 \right]
\]

Graph 5e = \[
\left[ \frac{e^4}{4\pi m^2} \ln \frac{\Lambda^2}{\vec{q}^2} \right] + \left[ \frac{e^4}{4\pi m^2} \ln \frac{4m^2}{\Lambda^2} \right]
\]

Some observations are in order: 1. The \( A(0, \Lambda) \) parts of each graph (of the Relativistic Model) are the same as calculated from the Classical Non Relativistic Model (19) with a cut off \( \Lambda \), through the graphs drawn on Figure 5, at the right of the corresponding relativistic ones. 2. The \( A(0, \Lambda) \) part of each graph can depend on the non relativistic cut off \( \Lambda \) (see 5e) but the whole graph is independent of \( \Lambda \), as can be seen by adding, for each graph,
the terms of the first and the second row. 3. Had we interpreted \( \Lambda \) as an UV cut off in the usual way, i.e. \( \Lambda \to \infty \), and \( \mathcal{A}(0,\Lambda) \) would be a divergent amplitude. 4. The \( \mathcal{A}(\Lambda,\infty) \) non null contributions of the graphs 5b and 5c could also be get by calculating 5a, starting from the already corrected Effective Lagrangian given by (19) plus (29). 5. The non null \( \mathcal{A}(\Lambda,\infty) \) part of diagram 5e instead, is a new term that must be incorporated in the Effective Lagrangian as a local quartic interaction of the form \( \psi^*\psi \phi^*\phi \). It must be stressed that this term comes from the integration over the high momenta intermediate states of the Renormalizable Relativistic Model; no quartic term of the form \( \Psi^*\Psi \Phi^*\Phi \) is needed in the “parent”Relativistic Model to generate this quartic term in the Effective Non Relativistic Lagrangian. The Effective Non Relativistic Model incorporating all these terms can be written

\[
\mathcal{L}_{\text{nonrel}}^{\text{effect}} = \psi^* \left( i \frac{d}{dt} + eA^0 \right) \psi - \frac{1}{2m} \left( -i \vec{\nabla} - e\vec{A} \right) \psi \left| \psi \right|^2 + \frac{e}{2m} \left( 1 + \frac{e^2}{2\pi\theta} \right) B \psi^* \psi \\
+ \phi^* \left( i \frac{d}{dt} + eA^0 \right) \phi - \frac{1}{2m} \left( -i \vec{\nabla} - e\vec{A} \right) \phi \left| \phi \right|^2 + \frac{e}{2m} \left( -1 + \frac{e^2}{2\pi\theta} \right) B \phi^* \phi \\
+ \frac{\theta}{2} \varepsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho - \frac{1}{46\pi m} F_{\mu\nu} F^{\mu\nu} \\
+ \left( \frac{e^4}{4\pi m\theta} \ln \left( \frac{4m^2}{\Lambda^2} \right) \right) \psi^* \psi \phi^* \phi.
\]

The calculation of the magnetic moment of the fermions, the propagator of the (Maxwell) Chern-Simons, and of the low energy scattering of two fermions, in this theory, using a cut off \( \Lambda \) (up to one loop), give the same results as the calculation of the same quantities starting from the Relativistic Model (10). For example, the amplitude of scattering of one spin plus and one spin minus fermion (the sum of the two rows in equation (27)) gives (14)

\[
\mathcal{A}_{++ \text{nonrel}}^{\text{eff}} \equiv \mathcal{A}_{++ \text{lowene}}^{\text{rel}} = i e^2 \frac{\vec{s} \wedge \vec{q}}{m\theta} + \frac{2e^4}{3\pi m\theta^2} + \frac{e^4}{4\pi m\theta^2} \ln \left( \frac{-4m^2}{\vec{p}^2} \right) 
\]

The calculation starting from the classical Non Relativistic Model, (the sum of terms in the first row in equation (27)) would instead, give the “divergent” result (3)

\[
\mathcal{A}_{++ \text{nonrelat}}^{\text{class}} = i e^2 \frac{\vec{s} \wedge \vec{q}}{m\theta} + \frac{e^4}{4\pi m\theta^2} \ln \left( \frac{-\Lambda^2}{\vec{p}^2} \right)
\]
These results exemplify our main point: taking the non relativistic limit in the Lagrangian and equations of motion (Classical Non Relativistic Limit) and then calculating a process gives in general, a result different than, first calculating the same process in the relativistic theory and later taking the non relativistic limit of the result.

To finish this talk I will turn to the problem that motivated this study: the finite result for $A_{\text{nonrel}}^{\text{class}}$ got in (2) for the scattering of two bosons and its extension (3) to the scattering of two spin plus fermions (we will think that the two fermions are not identical and we don’t need to anti symmetrize the amplitude with respect to the outgoing particles). The non null graphs contributing to this process are the same of figure 5. The result is

$$A_{\text{lowenergy}}^{-+ \text{ rel}} = A_{\text{lowene}}^{-+ \text{ rel}}(0, \Lambda) + A_{\text{lowene}}^{-+ \text{ rel}}(\Lambda, \infty)$$

Graph 5a = \[
\left[ \frac{e^2}{m \theta} \left( 1 + i \frac{s \cdot q}{q^2} \right) \right] + \left[ 0 \right]
\]

Graph 5b = \[
\left[ \frac{e^4}{6 \pi m \theta^2} \right]
\]

Graphs 5c = \[
\left[ \frac{e^4}{2 \pi m \theta^2} \right]
\]

Graph 5d = \[
\frac{e^4}{4 \pi m \theta^2} \ln \left( \frac{s^2}{\Lambda^2} \right) + \left[ \frac{e^4}{4 \pi m \theta} \ln \left( \frac{4m^2}{\Lambda^2} \right) \right]
\]

Graph 5e = \[
\frac{-e^4}{4 \pi m \theta^2} \ln \left( \frac{4m^2}{\Lambda^2} \right) + \left[ \frac{e^4}{4 \pi m \theta^2} \left( \ln \left( \frac{4m^2}{\Lambda^2} \right) - 2 \right) \right]
\]

The differences of these results to the ones in (27) come from the Pauli interaction of the magnetic field of each fermion with the magnetic moment of the other fermion. The effects of these interactions cancel in the scattering of a spin plus and a spin minus fermion and
add in the case of two spin plus fermions. The results for $A^{\text{effect}}$ and $A^{\text{class}}_{\text{nonrel}}$ are now

$$A^{-+\text{nonrel}} = A^{-+\text{rel lowene}} = \frac{e^2}{m\theta} \left( 1 + i \frac{\vec{s} \wedge \vec{q}}{q^2} \right) + \frac{e^4}{6\pi m\theta^2}$$

$$A^{-+\text{class nonrelat}} = \frac{e^2}{m\theta} \left( 1 + i \frac{\vec{s} \wedge \vec{q}}{q^2} \right)$$

The unexpected fact that this last result is finite, independent of $\Lambda$, is in the literature (2) related to the preservation at quantum level, of the scale invariance that the classical non relativistic model presents. In the model of bosons interacting with the CS field this only happens for the special value of the quartic self interaction discussed in the introduction. For fermions the same fact is provided by the Pauli interaction which already appear in the minimal interaction with the CS field; no fine tuning of coupling constants is needed. We here showed another aspect of this independence of $\Lambda$. Unusually, not only $A^{\text{rel lowen}}$ is independent of $\Lambda$: their high and low momenta intermediate energy contributions are separately independents of $\Lambda$. So the difference of the amplitudes got from the Classical or the Effective Non Relativistic Models is only a constant independent of $\Lambda$. If the fermions are identical we must anti symmetrize the amplitudes (32) and (29) in the outgoing particles. In this case no difference at all appears in the final result. The amplitude got from both (32) and (33) is : $i \frac{2e^2 \vec{s} \wedge \vec{q}}{m\theta q^2}$, and gives the Aharonov Bohm scattering amplitude for two identical fermions.

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Figure captions

- Figure 1. Vacuum polarization. The double line represent Dirac fermion propagators, and the wavy line the CS propagator.

- Figure 2. fermion self energy.

- Figure 3. Contributions (in the Coulomb Gauge) to the scattering of a fermion by an external field $A^\mu_{\text{ext}}$. The action of the external field is represented by a cross.

- Figure 4. Example of a one loop graph contributing to the scattering of two fermions.

- Figure 5. Non null graphs contributing to the scattering of two Dirac fermions. On the right of the diagrams are represented the correspondent graphs in the classical non relativistic model.