A PHASELESS INVERSE SCATTERING PROBLEM FOR THE 3-D HELMHOLTZ EQUATION

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Abstract. An inverse scattering problem for the 3-D Helmholtz equation is considered. Only the modulus of the complex valued scattered wave field is assumed to be measured and the phase is not measured. This problem naturally arises in the lensless quality control of fabricated nanostructures. Uniqueness theorem is proved.

1. Introduction. Phaseless Inverse Scattering Problem (PISP) is the problem of recovering a coefficient of a PDE in the case when only the modulus of the complex valued solution of this PDE is measured at a certain surface, while the phase is not measured. We note that traditional statements of inverse scattering problems assume that the whole complex valued wave field is measured outside of scatterers, see, e.g. [4, 6, 9, 10, 25, 26, 27, 28].

Uniqueness for some PISPs for the Schrödinger equation

\[ \Delta v + k^2 v - q(x) v = -\delta(x-y), x \in \mathbb{R}^3 \] (1)

was proven by the author in [15, 16]. As to the PISP for the 1-D version of equation (1), the first uniqueness theorem was proved in [13], also see [14]. In (1) \( q(x) \) is the unknown potential with a compact support and \( k > 0 \) is the wave number.

As to the PISPs for the Helmholtz equation in the 3-D case, a certain uniqueness result was proven in [17]. However, the result of [17] is for the case when the \( \delta \)-function in the right hand side of the Helmholtz equation (which is our case) is replaced with a function \( p(x) \) such that \( p(x) \neq 0 \) in a bounded domain of interest \( \Omega \subset \mathbb{R}^3 \). Uniqueness for the case of the \( \delta \)-function in the right hand side of the 3-D Helmholtz equation is briefly considered in theorem 3 of [18]. However, the term \( O(1/k) \) at \( k \to \infty \) in a certain asymptotic expansion is dropped in [18]: the uniqueness issue is not the main focus of [18].

Thus, the key novelty of this paper is that we prove here uniqueness theorem for a PISP for the 3-D Helmholtz equation in the case when its right hand side is the \( \delta \)-function and no terms are dropped. Unlike [1], the unknown coefficient of the Helmholtz equation is multiplied by \( k^2 \). This causes the main difficulty of the proof of the uniqueness theorem, as compared with [15, 16]. To handle this difficulty, we develop here some new ideas. The key new point is that we use here the apparatus

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of the Riemannian geometry. This apparatus is not used in \cite{15, 16}. We refer to section 5 for a more detailed discussion of some new elements of this paper.

Our PISP is overdetermined. Indeed, the unknown coefficient in our case depends on three variables. At the same time, the data depend on six variables. We note, however, that currently there are no uniqueness theorems even in the traditional statements of inverse scattering problems in 3-D, which would be proven for a non-overdetermined formulation in the case when the $\delta$-function stands in the right hand side of the corresponding PDE. On the other hand, the above mentioned PISP in \cite{17} is non-overdetermined.

Reconstruction procedures for PISPs, for both the 3-D Helmholtz equation and the 3-D Schrödinger equation, were developed in \cite{18, 19, 20, 21}. An essentially modified procedure of \cite{18} was implemented numerically in \cite{22}. In \cite{29, 30} different statements of PISPs were proposed, which led to different uniqueness theorems and different reconstruction procedures.

We now briefly comment the work \cite{30}. In \cite{30} both the incident plane wave and the incident point source were considered. The far field approximation for the wave field was considered in \cite{30}, which is unlike our case, where the far field approximation is not studied. In addition, item 2 of theorem 2.3 of \cite{30} assures uniqueness in the case when the modulus of the total complex valued wave field is measured for a fixed wave number on an open set outside of the support of the unknown potential for all positions of the point source, which runs along another open set outside of that support. For comparison, our Theorem 1 formulated below assures uniqueness in the case when the modulus of the total complex valued wave field is measured for all points belonging to a small ball surrounding a point source, this point source runs along a surface located outside of the support of the unknown function, and these measurements are conducted on an interval of wave numbers.

While we are interested in the reconstruction of the unknown coefficient of a PDE, we also refer to \cite{1, 2, 3, 11, 12} for numerical solutions of PISPs in the case when the surface of a scatterer was reconstructed.

We now discuss applications of PISPs. The increased availability of highly coherent sources of electromagnetic waves allows coherent scattering from all kinds of nanostructures and the resolution at the scale of the used wavelength becomes possible. However, only the square modulus of the resulting complex valued wave field, i.e. the intensity, is measured. The latter makes the phase retrieval necessary to image those nanostructures. The advantage of the inverse scattering approach is that one can obtain an area image directly by computation rather than inspect the nano structures in a slow serial fashion using scanning electron or scanning atomic force microscopes. Thus, the inverse scattering approach opens the possibility of “lensless” imaging at arbitrarily high resolutions, which are ultimately determined by the shortest wavelength sources that can be used. By lensless imaging we mean that not only is there no material with the refracting power that could be used as a lens at these very high frequencies but also that images are not degraded and low-pass filtered as they would be through the use of an imperfect refracting lens. The imaging of nanostructures via solving PISPs is vitally important for the purpose of quality control of fabricated nanostructures. The idea here is to simply create the object’s scattering pattern at these very high frequencies first, and then use computational methods to recover the 3-D image of the spatially distributed dielectric constant of that object with the spatial resolution comparable with the employed wavelength.
Sizes of many nanostructures are usually hundreds of nanometers (nm), \(100\text{nm} = 10^{-7}\text{m} = 0.1\mu\text{m}\), where “m” and “\(\mu\text{m}\)” stand for meter and micron respectively. Hence, in the above application, sizes of objects of interest are typically of the scale of 0.1\(\mu\text{m}\). Therefore, to image these objects for the quality control purpose, the wavelengths of probing electromagnetic waves should also be in the same range. An electromagnetic wave is sent from the vacuum upon the medium containing those nanostructures and the intensity of the resulting wave field is measured outside of that medium at a number of detectors. Either X-rays or optical sources of electromagnetic waves are used. The wavelength of 0.1\(\mu\text{m}\) corresponds to the frequency of 2,997,924.58 Gigahertz, see, e.g. [http://www.photonics.byu.edu/fwnomograph.phtml](http://www.photonics.byu.edu/fwnomograph.phtml). It is well known that the phase of an electromagnetic wave cannot be measured for such very high frequencies [7, 8, 31, 34]. Therefore, we arrive at the problem of the reconstruction of the spatially distributed dielectric constant of a scatterer using only the intensity of the scattered wave field.

When developing numerical methods for PISPs, such as, e.g. [18, 19, 20, 21, 22, 29, 30], it is important to prove uniqueness theorems for these problems. The latter justifies the topic of the current paper from the applied standpoint.

In section 2 we formulate our PISP and also formulate the main result of this paper. In section 3 prove some lemmata. In section 4 we prove the main result. In section 5 we discuss the novelty of this paper.

2. Statement of the PISP and the main result. Let \(\Omega\) and \(G\) be two bounded domains in \(\mathbb{R}^3\) and let \(\Omega \subset G\), \(\partial G \in C^1\) and \(\partial G \cap \partial \Omega = \emptyset\). Let \(\rho > 0\) be a number. For every \(y \in \mathbb{R}^3\) denote \(P'_\rho(y) = \{x \in \mathbb{R}^3: |x - y| < \rho\}\). We assume that our medium, which occupies the entire space \(\mathbb{R}^3\), is isotropic, non-magnetic and is characterized by the spatially distributed dielectric constant \(c(x), x \in \mathbb{R}^3\), which is also called the relative permittivity. It is well known, see, e.g. Chapter 13 in [5], that if \(c(x)\) varies slowly enough on the scales of the wavelength, then the scattering problem for Maxwell’s equations can be approximated by the scattering problem for the Helmholtz equation for a certain component of the electric field.

We assume throughout the paper that the function \(c(x)\) satisfies the following conditions:

\[
(2) \quad c \in C^{15}(\mathbb{R}^3), \quad c(x) \geq c_0 = \text{const.} > 0 \quad \text{in} \quad \mathbb{R}^3,
\]

\[
(3) \quad c(x) = 1 \quad \text{for} \quad x \in \mathbb{R}^3 \setminus \Omega.
\]

The smoothness requirement imposed on the function \(c(x)\) in (2) is due to Lemma 3.1 (below) as well as due to Theorem 2 in [18]. In the course of this proof we use the fundamental solution of a hyperbolic equation with the coefficient \(c(x)\) in the principal part of its operator. The construction of this solution works only if \(c \in C^{15}(\mathbb{R}^3),\) see theorem 4.1 in Chapter 4 of [32]. We also note that usually the minimal smoothness of unknown coefficients is not of a significant concern in uniqueness theorems for multidimensional coefficient inverse problems, see, e.g. [27, 28] and theorem 4.1 in Chapter 4 of [32].

The function \(c(x)\) generates the conformal Riemannian metric,

\[
(4) \quad d\tau = \sqrt{c(x)} |dx|, |dx| = \sqrt{(dx_1)^2 + (dx_2)^2 + (dx_3)^2}.
\]
We assume everywhere below that the following condition holds:

**Condition.** Geodesic lines generated by the metric (4) are regular. In other words, each pair of points \( x, y \in \mathbb{R}^3 \) can be connected by a single geodesic line \( \Gamma(x, y) \).

A sufficient condition for the regularity of geodesic lines was derived in [33], and it can also be found in [18]. Fix an arbitrary point \( y \in \mathbb{R}^3 \) and consider the problem of finding the function \( \tau(x, y) \) from the following conditions:

\[
(5) \quad |\nabla_x \tau(x, y)|^2 = c(x),
\]

\[
(6) \quad \tau(x, y) = O(|x - y|) \quad \text{as} \quad x \to y.
\]

The solution of the problem (5), (6) is \([18, 32]\)

\[
(7) \quad \tau(x, y) = \int_{\Gamma(x,y)} C(\xi) d\sigma,
\]

where \( d\sigma \) is the euclidean arc length. In fact, \( \tau(x, y) \) is the travel time between points \( x \) and \( y \) due to the Riemannian metric (4). The Condition implies that \( \tau(x, y) \) is a single-valued function of both points \( x \) and \( y \) in \( \mathbb{R}^3 \times \mathbb{R}^3 \).

Let \( y \in \mathbb{R}^3 \) be the position of the source. The Helmholtz equation with the radiation condition is:

\[
(8) \quad \Delta u + k^2 c(x)u = -\delta(x - y), \quad x \in \mathbb{R}^3,
\]

\[
(9) \quad \lim_{r \to \infty} r (\partial_r u - iku) = 0,
\]

where \( r = |x - y| \). It follows from theorem 8.7 of [6] that for each \( k > 0 \) problem (8), (9) has unique solution \( u \in C^2(|x - y| \geq \varepsilon), \forall \varepsilon > 0 \). Denote \( \text{dist}(\partial G, \partial \Omega) > 0 \) the Hausdorff distance between the surfaces \( \partial G \) and \( \partial \Omega \).

**Phaseless Inverse Scattering Problem 1 (PISP1).** Let the number \( \rho \in (0, \text{dist}(\partial G, \partial \Omega)/2) \). Let \( u(x, y, k) \) be the solution of the problem (8), (9). Assume that the following function \( F(x, y, k) \) is known

\[
(10) \quad F(x, y, k) = |u(x, y, k)|, \forall y \in \partial G, \forall x \in P_\rho(y), x \neq y, \forall k \in (a, b),
\]

where \( (a, b) \subset \{z > 0\} \) is a certain interval. Determine the unknown coefficient \( c(x) \).

Along with the PISP1 consider

**Phaseless Inverse Scattering Problem 2 (PISP2).** Let the number \( \rho \in (0, \text{dist}(\partial G, \partial \Omega)/2) \) and let the function \( c(x) \) be unknown. Fix an arbitrary pair of points \( y \in \partial G, x \in P_\rho(y), x \neq y \). Assume that the following function \( g_{x,y}(k) \) is known as the function of \( k \):

\[
(11) \quad g_{x,y}(k) = |u(x, y, k)|, \forall k \in (a, b).
\]

Determine the function \( u(x, y, k) \) for all \( k > 0 \) and for this given pair \( x, y \).

**Theorem 1.** The PISP2 has at most one solution. Also, the PISP1 has at most one solution.
3. **Lemmata.** For any complex number \( z \in \mathbb{C} \) we denote by \( \overline{z} \) its complex conjugate. Let \( \omega > 0 \) be a number. Denote
\[
\mathbb{C}_\omega = \{ z \in \mathbb{C} : \text{Im} \, z > -\omega \}, \mathbb{C}_+ = \{ z \in \mathbb{C} : \text{Im} \, z > 0 \}.
\]
Hence, \( \mathbb{C}_+ \) is the upper half plane of the complex plane \( \mathbb{C} \).

3.1. **Some properties of the function** \( u(x, y, k) \). Let \( T > 0 \) be an arbitrary number. Consider the domain \( K(y, T) \) defined as
\[
K(y, T) = \{(x, t) : \tau(x, y) < t < T - \tau(x, y)\}.
\]

**Lemma 3.1.** Let \( y \in \mathbb{R}^3 \) be an arbitrary point. Then there exists a number \( \beta = \beta(y, G) > 0 \) such that for every \( x \in G \) the function \( u(x, y, k) \) can be analytically continued with respect to \( k \) from the half real line \( \mathbb{R}_+ = \{ k : k > 0 \} \) in the half complex plane \( \mathbb{C}_\beta \).

**Proof.** Let \( H(t) \) be the Heaviside function,
\[
H(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases}
\]
Consider the initial value problem for the hyperbolic equation,
\[
\begin{align}
&c(x) w_{tt} = \Delta w + \delta(x - y, t), \quad x \in \mathbb{R}^3, \quad t > 0, \\
&w|_{t<0} = 0.
\end{align}
\]
It was proven in [18] that the solution of the problem (11), (12) has the form
\[
w(x, y, t) = A(x, y) \delta(t - \tau(x, y)) + \tilde{w}(x, y, t) H(t - \tau(x, y)),
\]
where the function \( \tilde{w}(x, y, t) \in C^2(K(y, T)) \) and
\[
A(x, y) > 0, \quad \forall x, y \in \mathbb{R}^3, x \neq y.
\]

Using lemma 6 of Chapter 10 of the book [37] as well as remark 3 after that lemma, we obtain that for any fixed point \( y \in \mathbb{R}^3 \) the function \( w(x, y, t) \) decays exponentially with respect to \( t \to \infty \), together with its \( x, t \)-derivatives up to the second order. This decay is uniform for all \( x \in G \). Hence,
\[
|D^\alpha_{x,t}w(x, y, t)| \leq Z e^{-m \cdot t}, \quad \forall \ t \geq t_0, \forall x \in \overline{G},
\]
where \( Z = Z(G, c) > 0, m = m(G, c) > 0, t_0 = t_0(G, c) > 0 \) are some constants depending only on the domain \( G \) and the function \( c \). In (15) \( \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) is the multi-index with non-negative integer coordinates and \(|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq 2 \).

It follows from (15) that we can apply the operator \( F \) of the Fourier transform to the function \( w \),
\[
\mathcal{F}(w)(x, y, k) = \int_{-\infty}^{\infty} w(x, y, t) \exp(ikt) \, dt := W(x, y, k).
\]
Using theorem 3.3 of [36] and theorem 6 of Chapter 9 of [37], we obtain
\[
W(x, y, k) = u(x, y, k).
\]
In (17) \( u(x, y, k) \) is the solution of the problem \([8], [9]\), which was presented in section 2.
Thus, it follows from (15) and (16) that the function \( u(x, y, k) \) can be analytically continued from \( \mathbb{R}_+ \) in the half plane \( \mathbb{C}_m \). \( \square \)

An analog of Lemma 3.2 was proven in [18] for the case when \( k \) is a real number. We now need \( k \in \mathbb{C}_m \).

**Lemma 3.2.** Let \( A(x, y) \) be the function in (13). The asymptotic behavior of the function \( u(x, y, k) \) is

\[
(18) \quad u(x, y, k) = A(x, y)e^{ik\tau(x, y)} \left(1 + O\left(\frac{1}{k}\right)\right), \quad |k| \to \infty, \quad k \in \mathbb{C}_m, \quad x \in \mathcal{G}.
\]

**Proof.** By (13) and (16)

\[
(19) \quad u(x, y, k) = A(x, y)e^{ik\tau(x, y)} + \int_{\tau(x, y)}^{\infty} \bar{w}(x, y, t) \exp(ikt) \, dt.
\]

Applying the integration by parts in (19) and taking into account (15), we obtain (18). \( \square \)

### 3.2. Four more lemmata.

**Lemma 3.3.** Fix an arbitrary pair of points \( x \in \mathcal{G}, y \in \mathbb{R}^3, x \neq y \) and consider the function \( \varphi(k) = u(x, y, k) \). Then, \( \varphi(k) \) has at most a finite number of zeros in the half complex plane \( \mathbb{C}_m \).

**Proof.** The assertion of this lemma follows immediately from (14) and (18). \( \square \)

**Lemma 3.4.** Let \( f(k) \) be an analytic function in the half plane \( \mathbb{C}_m \). Assume that \( f(k) \) has no zeros in \( \mathbb{C}_+ \cup \mathbb{R} \). Also, let the asymptotic behavior of the function \( f(k) \) be:

\[
f(k) = \frac{C}{k^n} [1 + o(1)] \exp(ikL), \quad |k| \to \infty, \quad k \in \mathbb{C}_+,
\]

where \( C \in \mathbb{C}, n, L \in \mathbb{R} \) are some numbers and \( n \geq 0 \). Then the values of \( |f(k)| \) for \( k \in \mathbb{R} \) uniquely determine the function \( f(k) \) for \( k \in \mathbb{C}_+ \cup \mathbb{R} \).

**Lemma 3.5.** Let the function \( f(k) \) be analytic for all \( k \in \mathbb{R} \). Then the function \( |f(k)| \) can be uniquely determined for all \( k \in \mathbb{R} \) by the values of \( |f(k)| \) for \( k \in (a, b) \).

**Proof.** We have \( |f(k)|^2 = f(k)\overline{f}(k) \). Both \( f(k) \) and \( \overline{f}(k) \) are analytic functions of the variable \( k \in \mathbb{R} \). \( \square \)

**Lemma 3.6.** Consider two finite sets of non-negative integers \( \{p_{j_1}\}_{j_1=1}^{N_1} \) and \( \{q_{j_2}\}_{j_2=1}^{N_2} \). Also, consider two sets of complex numbers \( \{d_{j_1}\}_{j_1=1}^{N_1} \subset (\mathbb{C}_+ \setminus \mathbb{R}) \) and \( \{s_{j_2}\}_{j_2=1}^{N_2} \subset (\overline{\mathbb{C}_+ \setminus \mathbb{R}}) \). Assume that there exist two sets of non-zero complex numbers \( \{a_{j_1}\}_{j_1=1}^{N_1} \) and \( \{b_{j_2}\}_{j_2=1}^{N_2} \) such that

\[
(20) \quad \sum_{j_1=1}^{N_1} a_{j_1} t^{p_{j_1}} \exp(-it\bar{d}_{j_1} t) = \sum_{j_1=1}^{N_1} b_{j_2} t^{q_{j_2}} \exp(-it\bar{s}_{j_2} t), \quad \forall t \geq 0.
\]

Then \( N_1 = N_2 = N \) and numbers involved in (20) can be re-numbered in such a way that

\[ a_j = b_j, p_j = q_j, d_j = s_j, \forall j = 1, ..., N. \]
Proof. Let \( n \geq 0 \) be an integer and let the number \( \alpha \in \mathbb{C}_+ \setminus \mathbb{R} \). Consider the function \( f_{n,\alpha}(t) \),

\[
(21) \quad f_{n,\alpha}(t) = H(t) t^n \exp(-i\alpha t).
\]

Let \( \tilde{f}_{n,\alpha}(k) \) be the Fourier transform of \( f_{n,\alpha}(t) \),

\[
(22) \quad \tilde{f}_{n,\alpha}(k) = \int_{-\infty}^{\infty} f_{n,\alpha}(t) \exp(ikt) \, dt = \mathcal{F}(f_{n,\alpha})(k).
\]

Then the direct calculation leads to

\[
(23) \quad \tilde{f}_{n,\alpha}(k) = (-1)^n n! \cdot \frac{1}{(k - \pi)^{n+1}}.
\]

Hence, multiplying both sides of (20) by the Heaviside function \( H(t) \) and applying then the operator \( \mathcal{F} \) to both sides of the resulting equality, we obtain for all \( k \in \mathbb{R} \)

\[
(24) \quad \sum_{j_1=1}^{N_1} a_{j_1} \frac{(-1)^{p_{j_1}} (p_{j_1})!}{i^{p_{j_1}+1}} \frac{1}{(k - d_{j_1})^{p_{j_1}+1}} = \sum_{j_2=1}^{N_2} b_{j_2} \frac{(-1)^{q_{j_2}} (q_{j_2})!}{i^{q_{j_2}+1}} \frac{1}{(k - \sigma_{j_2})^{q_{j_2}+1}}.
\]

Since (24) is valid for all \( k \in \mathbb{R} \), we can analytically continue both sides of (24) in \( \mathbb{C} \) and obtain then meromorphic functions in both sides of (24). In other words, (24) is valid for all \( k \in \mathbb{C} \), except of poles.

Consider the number \( d_1 \). Without any loss of the generality we assume that the maximal power of \( (k - d_1)^{p_{j_1}+1} \) in (24) is \( p_1 + 1 \). We can have two cases:

Case 1. \( \sigma_{j_2} \neq d_1, \forall j_2 \in [1, N_2] \). Multiply both sides of (24) by \( (k - d_1)^{p_{j_1}+1} \) and set \( k \to d_1 \). Then we obtain that \( a_1 = 0 \). This contradicts to the assumption that all numbers \( a_{j_1}, b_{j_2} \) are non-zeros. Hence, Case 2 takes place:

Case 2. There exists a number \( \sigma_{j_2} \) such that \( \sigma_{j_2} = d_1 \). Without any loss of the generality we assume that \( \sigma_{j_2} = \sigma_1 \) and also that the maximal power of \( (k - \sigma_1)^{q_{j_2}+1} = (k - d_1)^{q_{j_2}+1} \) is \( q_1 + 1 \). Again, without any loss of the generality we can assume that \( q_1 \leq p_1 \). If \( q_1 < p_1 \), then, similarly with Case 1, we obtain that \( a_1 = 0 \), which is again a contradiction. If, however, \( q_1 = p_1 \), then multiplying again both sides of (24) by \( (k - d_1)^{p_{j_1}+1} \) and setting \( k \to d_1 \), we obtain \( a_1 = b_1 \).

Continuing this way, we arrive at the assertion of this lemma. \( \square \)

4. Proof of Theorem 1. First, we consider the PISP2. Consider two arbitrary points \( y \in \partial G, x \in P_{p}(y), x \neq y \) mentioned in the formulation of the PISP2. Denote \( \psi(k) = u(x, y, k), k > 0 \). Suppose that there exist two functions

\[
(25) \quad \psi_1(k) = u_1(x, y, k), \psi_2(k) = u_2(x, y, k), k > 0
\]

such that \( |\psi_1(k)| = |\psi_2(k)| = g_{x,y}(k), k \in (a, b) \). Then by Lemmata 3.1 and 3.5 imply that

\[
(26) \quad |\psi_1(k)| = |\psi_2(k)| = g_{x,y}(k), \forall k \in \mathbb{R}.
\]

In addition by Lemma 3.1 functions \( \psi_1(k) \) and \( \psi_2(k) \) can be analytically continued from \( \mathbb{R}_+ \) in the half plane \( \mathbb{C}_m \subset \mathbb{C} \). We keep the same notations for these analytic continuations.

Below we count each zero of any of two functions \( \psi_1(k) \) and \( \psi_2(k) \) as many times as its multiplicity is. First, we show that real zeros of functions \( \psi_1(k) \) and \( \psi_2(k) \)
Hence, by (26) and (27)\(\|\|\|\|\). Furthermore, since \(\psi_\gamma\) coincide. Let \(\{\gamma^{(1)}_j\}_{j=1}^r\) and \(\{\gamma^{(2)}_s\}_{s=1}^t\) be all real zeros of the functions \(\psi_1(k)\) and \(\psi_2(k)\) respectively. Let \(\gamma^{(1)}_1\) be the real zero of \(\psi_1(k)\) of the multiplicity \(n \geq 1\). Also let \(\gamma^{(1)}_j\) be the zero of the function \(\psi_2(k)\) of the multiplicity \(j \geq 0\). Then

\[
(27) \quad \psi_1(k) = (k - \gamma^{(1)}_1)^n \hat{\psi}_1(k), \psi_2(k) = (k - \gamma^{(1)}_1)^j \hat{\psi}_2(k),
\]

(28) \[
\hat{\psi}_1\left(\gamma^{(1)}_1\right) \neq 0, \hat{\psi}_2\left(\gamma^{(1)}_1\right) \neq 0.
\]

Hence, by (26) and (27)

\[
(29) \quad \left| k - \gamma^{(1)}_1 \right|^n \hat{\psi}_1(k) = \left| k - \gamma^{(1)}_1 \right|^j \hat{\psi}_2(k), \forall k \in \mathbb{R}.
\]

Let, for example \(n > j\). Dividing both sides of (29) by \(\left| k - \gamma^{(1)}_1 \right|^j\), we obtain

\[
\left| k - \gamma^{(1)}_1 \right|^{n-j} \hat{\psi}_1(k) = \hat{\psi}_2(k), \forall k \in \mathbb{R}.
\]

Hence, \(\hat{\psi}_2\left(\gamma^{(1)}_1\right) = 0\), which, however, contradicts to (28). Hence, \(n = j\). Thus, we have proven that sets of real zeros of functions \(\psi_1(k)\) and \(\psi_2(k)\) coincide. Let the set of real zeros of each of these two functions be \(\{\gamma^{(1)}_j\}_{j=1}^r\).

Consider now complex zeros of functions \(\psi_1(k)\) and \(\psi_2(k)\) in the upper half plane \(\mathbb{C}_+\). By Lemma 3.3 each of these two functions has at most a finite number of zeros in \(\mathbb{C}_+\). Let \(\{\theta_j\}_{j=1}^r \subset \mathbb{C}_+\) and \(\{\lambda_j\}_{j=1}^t \subset \mathbb{C}_+\) be those zeros of functions \(\psi_1(k)\) and \(\psi_2(k)\) respectively. Consider functions \(\mu_1(k)\) and \(\mu_2(k)\) defined as

\[
(30) \quad \mu_1(k) = \psi_1(k) \prod_{j=1}^r (k - \gamma_j)^{-1} \cdot \prod_{j=1}^t \left(\frac{k - \theta_j}{k - \theta_j^{-1}}\right), k \in \mathbb{C}_+,
\]

\[
(31) \quad \mu_2(k) = \psi_2(k) \prod_{j=1}^r (k - \gamma_j)^{-1} \cdot \prod_{j=1}^t \left(\frac{k - \lambda_j}{k - \lambda_j^{-1}}\right), k \in \mathbb{C}_+.
\]

Then

\[
(32) \quad \mu_1(k) \neq 0, \mu_2(k) \neq 0 \text{ for } k \in \mathbb{C}_+ \cup \mathbb{R}.
\]

Furthermore, since

\[
\frac{\left| k - \alpha \right|}{k - \alpha} = 1, \forall k \in \mathbb{R}, \forall \alpha \in \mathbb{C},
\]

then (26), (30) and (31) imply that

\[
(33) \quad |\mu_1(k)| = |\mu_2(k)|, \forall k \in \mathbb{R}.
\]

To apply Lemma 3.4, we now should show that functions \(\psi_1(k)\) and \(\psi_2(k)\) have the same first term of their asymptotic expansions for \(|k| \to \infty, k \in \mathbb{C}_+\). It is natural to use Lemma 3.2. However, since functions \(\psi_1(k)\) and \(\psi_2(k)\) supposedly correspond to two different coefficients \(c_1(x)\) and \(c_2(x)\), then they generate different pairs of functions \(A_1(x, y), \tau_1(x, y)\) and \(A_2(x, y), \tau_2(x, y)\). This, in turn generates two different asymptotic behaviors in (18).
Nevertheless, we still can use the asymptotic expansion \((18)\) of Lemma 3.2. Indeed, since \(y \in \partial G, x \in P_\rho (y)\) and also since \(P_\rho (y) \cap \Omega = \emptyset\), then \(c (x) = 1\) for \(x \in P_\rho (y)\). Hence,

\[
(34) \quad \tau (x, y) = |x - y|, \quad x \in P_\rho (y).
\]

Furthermore, it follows from formulae (3.3), (3.4) and (3.9) of [18] that

\[
(35) \quad A (x, y) = \frac{1}{4\pi |x - y|}, \quad y \in \partial G, x \in P_\rho (y), x \neq y.
\]

Hence, using (18), (34) and (35), we obtain

\[
(36) \quad \psi_j (k) = \frac{\exp (ik |x - y|)}{4\pi |x - y|} \left( 1 + O \left( \frac{1}{k} \right) \right), |k| \to \infty, k \in \mathbb{C}_m, j = 1, 2.
\]

Hence, using (30), (31) and (36), we obtain

\[
(37) \quad \mu_j (k) = \frac{1}{k^2} \cdot \frac{\exp (ik |x - y|)}{4\pi |x - y|} \left( 1 + O \left( \frac{1}{k} \right) \right), |k| \to \infty, k \in \mathbb{C}_m, j = 1, 2.
\]

Hence, (32), (33) and (37) imply that we can apply Lemma 3.4 to functions \(\mu_1 (k)\) and \(\mu_2 (k)\). We obtain

\[
(38) \quad \mu_1 (k) = \mu_2 (k), k \in \mathbb{C}_+ \cup \mathbb{R}.
\]

Using (30), (31) and (38), we obtain

\[
(39) \quad \psi_1 (k) \prod_{j=1}^{l_1} \frac{k - \overline{\theta_j}}{k - \theta_j} = \psi_2 (k) \prod_{j=1}^{l_2} \frac{k - \lambda_j}{k - \overline{\lambda_j}}, k \in \mathbb{R}.
\]

One can rewrite (39) in the following equivalent form:

\[
(40) \quad \psi_1 (k) \prod_{j=1}^{l_2} \frac{k - \lambda_j}{k - \overline{\lambda_j}} = \psi_2 (k) \prod_{j=1}^{l_1} \frac{k - \theta_j}{k - \overline{\theta_j}}, k \in \mathbb{R}.
\]

We now want to apply the operator \(F^{-1}\) of the inverse Fourier transform (16) to both sides of (40). To do this, we rewrite (40) as

\[
(41) \quad \psi_1 (k) + \left( \prod_{j=1}^{l_2} \frac{k - \lambda_j}{k - \overline{\lambda_j}} - 1 \right) \psi_1 (k) = \psi_2 (k) + \left( \prod_{j=1}^{l_1} \frac{k - \theta_j}{k - \overline{\theta_j}} - 1 \right) \psi_2 (k), k \in \mathbb{R}.
\]

Consider now functions \(s_1 (k), s_2 (k)\) defined as

\[
s_1 (k) = \prod_{j=1}^{l_1} \frac{k - \theta_j}{k - \overline{\theta_j}} - 1, \quad s_2 (k) = \prod_{j=1}^{l_2} \frac{k - \lambda_j}{k - \overline{\lambda_j}} - 1.
\]

We can rewrite these functions as

\[
s_1 (k) = R_1 (k) \prod_{j=1}^{l_1} \frac{1}{k - \overline{\theta_j}}, \quad s_2 (k) = R_2 (k) \prod_{j=1}^{l_2} \frac{1}{k - \overline{\lambda_j}},
\]
where \( R_2(k) \) is the polynomial of the degree less than \( l_1 \) and \( R_1(k) \) is the polynomial of the degree less than \( l_2 \). Using the partial fraction expansion, we obtain

\[
\begin{align*}
    s_1(k) &= \sum_{j_1=1}^{l_1'} \frac{Y_{j_1}}{(k - \overline{j}_1)}^{p_{j_1}}, \quad s_2(k) = \sum_{j_2=1}^{l_2'} \frac{Z_{j_2}}{(k - \overline{j}_2)}^{q_{j_2}},
\end{align*}
\]

where \( l_1' \leq l_1, l_2' \leq l_2 \), the sets \( \{\overline{j}_1\}_{j_1=1}^{l_1'} \subseteq \{\overline{j}_1\}_{j_1=1}^{l_1} \), \( \{\overline{j}_2\}_{j_2=1}^{l_2'} \subseteq \{\overline{j}_2\}_{j_2=1}^{l_2} \) and \( Y_{j_1}, Z_{j_2} \) are some complex numbers.

Let us calculate inverse Fourier transforms of functions \( s_1 \) and \( s_2 \). Using (21)-(23), we obtain

\[
F^{-1}(s_1) = S_1(t), \quad F^{-1}(s_2) = S_2(t),
\]

\[
egin{align*}
S_1(t) &= H(t) \sum_{j_1=1}^{l_1'} (-1)^{p_{j_1}-1} \frac{i^{p_{j_1}} Y_{j_1}}{(p_{j_1} - 1)!} \cdot t^{p_{j_1}-1} \exp \left(-i\overline{j}_1 t\right),
S_2(t) &= H(t) \sum_{j_2=1}^{l_2'} (-1)^{q_{j_2}-1} \frac{i^{q_{j_2}} Z_{j_2}}{(q_{j_2} - 1)!} \cdot t^{q_{j_2}-1} \exp \left(-i\overline{j}_2 t\right).
\end{align*}
\]

Consider functions \( F^{-1} (\psi_1) \) and \( F^{-1} (\psi_2) \). By (16), (17) and (25)

\[
\begin{align*}
F^{-1} (\psi_1) &= w_1(x,y,t), \quad F^{-1} (\psi_2) = w_2(x,y,t),
\end{align*}
\]

where \( w_1(x,y,t) \) and \( w_2(x,y,t) \) are solutions of the problem (11), (12) with two different coefficients \( c_1(x) \) and \( c_2(x) \).

Since \( x \in P_\rho(y) \), then \( |x - y| < \rho < dist(\partial G, \partial \Omega)/2 \). Let \( t \in (0,\rho) \). Since \( c(x') = 1, \forall x' \in P_\rho(y) \), then

\[
\partial_t^2 w_j = \Delta w_j + \delta(x-y, t), x \in P_\rho(y), t \in (0, \rho), j = 1,2,
\]

\( w_j \big|_{t=0} = 0 \).

Hence, it follows from the method of energy estimates (see, e.g. Chapter 4 of [23] for this method) that

\[
w_1(x,y,t) = w_2(x,y,t) = \frac{\delta(t - |x-y|)}{4\pi |x-y|}, x \in P_\rho(y), t \in (0,\rho).
\]

Hence, using (44) and (45), we obtain

\[
F^{-1}(\psi_1) = F^{-1}(\psi_2) = \frac{\delta(t - |x-y|)}{4\pi |x-y|}, t \in (0,\rho).
\]

We are ready now to apply the operator \( F^{-1} \) to both parts of (41). Using the convolution theorem for the Fourier transform, (44) and (16), we obtain

\[
w_1(x,y,t) + \int_0^t w_1(x,y,t-\tau) S_1(\tau) d\tau = w_2(x,y,t) + \int_0^t w_2(x,y,t-\tau) S_2(\tau) d\tau.
\]

It follows from (45) and (47) that

\[
\int_0^t \delta(t - \tau - |x-y|) S_1(\tau) d\tau = \int_0^t \delta(t - \tau - |x-y|) S_2(\tau) d\tau, \forall t \in (0,\rho).
\]
This is equivalent with

\[(48) \quad S_1(t - |x - y|) = S_2(t - |x - y|), \forall t \in (|x - y|, \rho).
\]

By (42) and (43), both \(S_1(t)\) and \(S_2(t)\) are analytic functions of the real variable \(t > 0\). Hence, (48) is valid for all \(t > |x - y|\). Hence, denoting \(t = t - |x - y|\), we obtain

\[(49) \quad S_1\left(\frac{t}{\rho}\right) = S_2\left(\frac{t}{\rho}\right), \forall \rho > 0.
\]

Finally applying Lemma 3.6 to (49), we obtain that \(l_1 = l_2 := l\) and sets \(\{\theta_1, \theta_2, ..., \theta_l\} = \{\lambda_1, \lambda_2, ..., \lambda_l\}\). Thus, by \(10\), \(\psi_1(k) = \psi_2(k)\) for \(k \in \mathbb{R}\). This and \(25\) lead to

\[(50) \quad u_1(x, y, k) = u_2(x, y, k), k \in \mathbb{R}.
\]

Thus, (50) implies that the PISP2 has at most one solution, which finalizes the proof of the first part of Theorem 1.

We now prove the second part of this theorem, i.e. that the coefficient \(c(x)\) is determined uniquely from the function \(F(x, y, k)\) in (10). Since the function \(F(x, y, k)\) is given, and since \(x\) is an arbitrary point of \(P_\rho(y)\), then (50) holds for all \(x \in P_\rho(y)\). We now return to the original equation (3). We have proven that for each fixed source position \(y \in \partial G\) the function \(u(x, y, k)\) is determined uniquely for all points \(x \in P_\rho(y)\) and for all \(k \in \mathbb{R}\). Fix a point \(y \in \partial G\). Since \((3)\) \(c(x) = 1\) outside of the domain \(\Omega\) and since \(P_\rho(y) \cap \Omega = \emptyset\), then the well-known theorem about the uniqueness of the continuation of the solution of an arbitrary elliptic equation of the second order (see, e.g. Chapter 4 of [24]) implies that the function \(u(x, y, k)\) is determined uniquely for all \(x \in \Omega\). Next, since \(y\) is an arbitrary point of \(\partial G\), then the function \(u(x, y, k)\) is uniquely determined for all pairs \((x, y) \in \{x \in \mathbb{R}^3 \setminus \Omega, y \in \partial G\}\). By (16) and (17) this means, in turn that the function \(w(x, y, t)\) is uniquely determined for all \(y \in \partial G, x \notin \Omega, t > 0\). This and (13) imply that the following function \(\kappa(x, y)\) is known

\[(51) \quad \kappa(x, y) = \tau(x, y), \forall x, y \in \partial G.
\]

The problem of the determination of the function \(c(x)\) from the function \(\tau(x, y)\) known for all \(x, y \in \partial G\) is called the Inverse Kinematic Problem [24], [32], or Travel Time Tomography Problem [35]. In the 3-D case uniqueness of this problem was proven in theorem 3.4 of Chapter 3 of the book [32].

We now show how the latter theorem can be used for the case of (51). Let \(n(x) = \sqrt{c(x)}\). Consider two arbitrary numbers \(n_0\) and \(n_{00}\) such that

\[(52) \quad 0 < n_0 \leq n_{00} < \infty.
\]

Consider the set of functions \(\Lambda(n_0, n_{00})\) which satisfy the following two conditions:

1. Each function \(n(x) \in \Lambda(n_0, n_{00})\) should be such that

\[n(x) \in C^2(\mathbb{G}), n(x) \geq n_0, \forall x \in \mathbb{G} \text{ and } \|n\|_{C^2(\mathbb{G})} \leq n_{00}.
\]

2. For each function \(n(x) \in \Lambda(n_0, n_{00})\) the function \(c(x) = n^2(x)\) must satisfy Condition of section 2 regarding the regularity of geodesic lines.

It follows immediately from theorem 3.4 of Chapter 3 of [32] as well as from the comments of the author of the book [32] just above this theorem that if for any two functions \(n_1, n_2 \in \Lambda(n_0, n_{00})\) their corresponding functions \(\tau_1(x, y), \tau_2(x, y)\) coincide for all \(x, y \in \partial G\), then \(n_1(x) = n_2(x)\) in \(G\).
Suppose now that two functions \( c_1(x) \) and \( c_2(x) \) satisfy conditions (2) and (3) as well as Condition of section 2. Then it follows from (2) and (3) that one can find two numbers \( n_0 = n_0(c_1,c_2) \) and \( n_{00} = n_{00}(c_1,c_2) \) satisfying (52) and such that corresponding functions \( n_1(x) = \sqrt{c_1(x)} \) and \( n_2(x) = \sqrt{c_2(x)} \) belong to the same set \( \Lambda(n_0,n_{00}) \). Hence, assuming that these two functions generate the same function \( \kappa(x,y) \) in (51) and applying theorem 3.4 of Chapter 3 of [32], we obtain that \( c_1(x) = c_2(x) \) in \( G \).

Therefore the uniqueness of the determination of the unknown coefficient \( c(x) \) is established.

5. Summary of main new elements of the mathematical apparatus of this paper. We summarize in this section main new elements of the above mathematical apparatus while leaving aside some secondary new elements.

Lemmas 3.1, 3.2, 3.6 and Theorem 1 are new results. As it is pointed out in the fourth paragraph of Introduction, the main new element of the mathematical apparatus of this paper, compared with [15, 16], is that we use here the Riemannian geometry. Even though the apparatus of the Riemannian geometry is used in [18], it is directed there towards a reconstruction procedure for a corresponding PISP rather than towards a theorem about the uniqueness. The latter causes some important differences here.

As to the proof of Theorem 1, the idea of the part of this proof from (27) to (33) and then from (37) to (43) is similar to the idea of the proof of theorem 1 of the paper [15]. So, this part is given here only for the convenience of the reader.

The rest of the proof of Theorem 1 is new. Indeed, in its part between (33) and (37) we use the asymptotic expansion (18) of Lemma 3.2, which is directly related to the Riemannian geometry. Also, the part of the proof starting from (41) and up to the end of the proof is not present in [15]. It is worthy to notice here that one of important new ideas of this proof linked with (51) was not necessary to use in [15]. Thus, it is not presented in [15]. On the other hand, this idea is directly linked with the Riemannian geometry, since the proof of the above cited theorem 3.4 of Chapter 3 of [32] heavily uses the apparatus of the Riemannian geometry.

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