Time-like pion electromagnetic form factors in $k_T$ factorization with the Next-to-leading-order twist-3 contribution

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We calculate the time-like pion electromagnetic form factor in the $k_T$ factorization formalism with the inclusion of the next-to-leading-order (NLO) corrections to the leading-twist and sub-leading-twist contributions. It’s found that the total NLO correction can enhance (reduce) the magnitude (strong phase) of the leading order form factor by $20\%-30\%$ ($<15^\circ$) in the considered invariant mass squared $q^2 > 5$ GeV$^2$, and the NLO twist-3 correction play the key role to narrow the gap between the pQCD predictions and the measured values for the time-like pion electromagnetic form factor.

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I. INTRODUCTION

As a very important physical observable which may help us to understand the hadrons’ structure and the transition from the perturbative to the non-perturbative region, the electromagnetic (EM) form factor of the pion meson has been the hot subject of numerous experimental and theoretical investigations.

During past four decades, the pion EM form factors$^1$ have been measured frequently by many groups$^{2\text{--}8}$. The space-like pion EM form factor was firstly measured by the Harvard & Cornell collaboration in the range $0.15 \leq Q^2 \leq 10$ GeV$^2$ of the electro-production processes in 1970s$^2$, and then measured by the DESY collaboration at the fixed point ($Q^2 = 0.35$, $0.70$ GeV$^2$) in the similar processes almost at the same time$^3$. In the new century, this space-like form factor was determined separately by the Jefferson Lab $F_{\pi}$ Collaboration in the region $0.60 \leq Q^2 \leq 1.60$ and at the fixed point $Q^2 = 2.45$ GeV$^2$$^4$. For the time-like pion EM form factor, Cyclotron Laboratory reported their result at the point $q^2 = 0.176$ GeV$^2$ in the electro-production process$^5$, then NOVOSIBIRSK collaborations and ORSAY collaboration measured this form factor independently in the region $0.64 \leq q^2 \leq 1.40$ GeV$^2$$^6$ and $1.35 \leq q^2 \leq 2.38$ GeV$^2$$^7$ through the $e^+e^-$ annihilation process respectively. Recently, CLEO Collaboration also reported their precision measurements of this form factor at the relatively large $q^2$ ($q^2 = 9.6$, $13.48$ GeV$^2$)$^8$. A comprehensive summary of experimental measurements for the time-like pion EM form factor can be found in Ref.$^9$.

On the theory side, pion EM form factors also attracted much attentions. The space-like one was studied at different energy regions in QCD by using the different approaches. For example, it
was investigated in high and intermediate energy region in Ref. [10] and Refs. [11, 12] respectively, while its asymptotic behavior at the extremely large $q^2$ was studied in Ref. [13]. In Refs. [14–16], the space-like pion form factor was studied carefully in the perturbative QCD theory, and it was also studied in the sum rules formalism[17]. For the time-like pion EM form factor[18], its high $q^2$ behavior was determined at $q^2 = M_J^2/\Psi$ and it was found that it is larger than the space-like one by a factor of 2 [19]. The Sudakov effect for the time-like form factor was discussed in Refs. [20, 21] and it’s found that the asymptotic behavior of the integrable singularity for the time-like form factor is the same as that for the space-like one. The conformal symmetry was also used to analyze the time-like form factor[22] and it is shown explicitly that the time-like form factor, which was obtained by the analytic continuation of the space-like one, satisfied correctly the dispersion relation. The light-cone QCD investigation recently[23, 24] showed that the effects of the power suppressed sub-leading twist’ and the genuine soft QCD correction’ contributions turn out to be dominant at low- and moderate-energies.

With removing the end-point singularities by the Sudakov factors [25, 26], the $k_T$ factorization theorem[27] is successful to deal with the exclusive processes with a large momentum transfer[28]. In the $k_T$ factorization theorem, the space-like pion EM form factor was re-examined with the inclusion of the Sudakov suppression [29]. Three-parton contribution to pion EM form factor in $k_T$ factorization was also investigated in Refs. [30] and it’s found that such contribution is rather small in size and therefore can be dropped safely.

After completing the NLO calculations for the space-like pion EM form factor at leading twist (twist-2) [31], the authors also studied the NLO twist-2 time-like pion EM form factor [32] and found that the NLO correction to the LO magnitude (strong phase) is lower than 25% ($10^\circ$) at the large invariant mass squared $q^2 > 30 \text{ GeV}^2$ at leading twist. In Refs. [33, 34], the authors calculated the sub-leading twist’s (twist-3) contribution from pion meson distribution amplitudes(DAs) to the exclusive $B \rightarrow \pi$ transition form factors and the space-like pion EM form factor, and they found that this power-suppressed contribution is large in the low and intermediate $q^2$ regions. In this paper, therefore, we will evaluate the NLO twist-3 contribution to the time-like pion EM form factor after the calculation for the NLO correction to the space-like pion EM form factor at twist-3 level [34].

This paper is organized as follows. In Sec. II, we give the Leading order(LO) analysis for the time-like pion EM form factor. In Sec. III, the NLO twist-3 corrections to the time-like form factor will be calculated from the space-like one by analytical continuation. Sec. IV contains the numerical analysis of the NLO effects, and the conclusion will also be given in this section.

II. LO ANALYSIS

In this section we present the LO factorization formula for the time-like pion EM form factor and evaluate the contributions from the two-parton twist-2 and twist-3 pion meson DAs. The LO quark diagram for the relative time-like and space-like pion EM form factor corresponding to the process $\gamma^* \rightarrow \pi\pi(\pi\gamma^* \rightarrow \pi)$ are illustrated in Fig. 1(a) and 1(b), respectively.

One should note that the kinetics for the time-like pion EM form factor are different from that for the space-like form factor, because the two mesons are both outgoing in Fig. 1(a), but one is incoming and another is outgoing in Fig. 1(b). In the light-cone coordinates, the momenta $p_1$ and
$p_2$ in Fig. 1(a) are parameterized as

$$p_1 = (p_1^+, 0, 0), \quad p_2 = (0, p_2^-, 0); \quad p_1^+ = p_2^- = \frac{Q}{\sqrt{2}},$$

$$k_1 = (x_1 p_1^+, 0, 0), \quad k_2 = (0, x_2 p_2^, k_{1T}), \quad q^2 = Q^2 = (p_1 + p_2)^2,$$

with $q^2$ being the invariant mass squared of the intermediate virtual photon, $k_1$ ($k_2$) is the momentum carried by the valence quark (anti-quark) of meson $M_1$ ($M_2$) with the momentum fraction $x_1$ ($x_2$) denoting the strength of the quark (anti-quark) to form the corresponding meson. Then the time-like (space-like) pion EM form factor $G_\pi^T (F_\pi)$ can be specified through the following matrix elements[24]:

$$e(p_1 - p_2)_\mu G_\pi (q^2) = < \pi^+(p_2) \pi^+(p_1) | J^{EM}_\mu (p_1 + p_2) | 0 >,$$

$$e(p_1 + p_2)_\mu F_\pi (Q^2) = < \pi^+(p_2) | J^{EM}_\mu (p_1 - p_2) | \pi^+(p_1) >,$$

where $J^{EM}_\mu$ is the EM current. The space-like momentum transfers in Eq. (4) is $Q^2 = -q^2 = -(p_1 - p_2)^2$, which is different from the time-like one as described in Eq. (2).

From Fig. 1(a) and Fig. 1(b), one can write down the LO time-like and space-like hard kernels

$$H_a^{(0)} (x_1, k_{1T}, Q^2) = \frac{-ie_q 32\pi\alpha_s C_F N_C Q^2}{(p_2 + k_1)^2(k_2 + k_1)^2} \{ x_1 p_1 \phi^A(x_1) \phi^A(x_2) - 2r_0^2 [(p_2 + x_1 p_1) \phi^P(x_1) \phi^P(x_2) - (p_2 - x_1 p_1) \phi^T(x_1) \phi^P(x_2)] \},$$

$$H_b^{(0)} (x_1, k_{1T}, Q^2) = \frac{ie_q 32\pi\alpha_s C_F N_C Q^2}{(p_2 - k_1)^2(k_2 - k_1)^2} \{ x_1 p_1 \phi^A(x_1) \phi^A(x_2) + 2r_0^2 [(p_2 - x_1 p_1) \phi^P(x_1) \phi^P(x_2) - (p_2 + x_1 p_1) \phi^T(x_1) \phi^P(x_2)] \},$$

where $\phi^A(x_1)$ and $\phi^{P,T}(x_1)$ ($\phi^A(x_2)$ and $\phi^{P,T}(x_2)$) represent the twist-2 and twist-3 pion meson DAs for the corresponding meson with the momentum $p_1$ ($p_2$), the chiral parameter is defined as $r_0^2 = m_0^2/Q^2$ with the chiral mass $m_0 = 1.74$ GeV. By comparing Eq. (5) with Eq.(6), we can find that the LO time-like hard kernel has the similar structure with that for the space-like one, the only difference is the direction of the valence quark momentum $k_1$, which will flow into the internal propagators. Then we can obtain the LO time-like hard kernel from the space-like one by direct
replacement $-k_1 \rightarrow k_1$ for the internal propagators, which implied that the time-like form factor can also be obtained from the space-like one by analytical continuation from $-Q^2$ to $Q^2$ in the invariant mass squared $q^2$ space. This is the basic idea being used to calculate the NLO time-like pion EM form factor in this paper.

The LO time-like and space-like pion EM form factor can be obtained by combining Eqs. (3,5) and Eqs. (4,6) respectively, and can then be written in the following forms:

$$Q^2G^{(0)}(x_i, Q^2, k_{iT}) = \frac{128\pi Q^4 \cdot \alpha_s(\mu)}{(p_2 + k_1)^2(k_1 + k_2)^2} \int_0^1 dx_1 dx_2 \int_0^\infty \frac{d^2k_{1T}}{2\pi} \frac{d^2k_{2T}}{2\pi} \cdot \left\{ -x_1\phi^A(x_1)\phi^A(x_2) + 2\beta \left[(1 - x_1)\phi^P(x_1)\phi^P(x_2) + (1 + x_1)\phi^T(x_1)\phi^P(x_2) \right] \right\},$$

(7)

$$Q^2F^{(0)}(x_i, Q^2, k_{iT}) = \frac{128\pi Q^4 \cdot \alpha_s(\mu)}{(p_2 - k_1)^2(k_1 - k_2)^2} \int_0^1 dx_1 dx_2 \int_0^\infty \frac{d^2k_{1T}}{2\pi} \frac{d^2k_{2T}}{2\pi} \cdot \left\{ x_1\phi^A(x_1)\phi^A(x_2) + 2\beta \left[(1 - x_1)\phi^P(x_1)\phi^P(x_2) - (1 + x_1)\phi^T(x_1)\phi^T(x_2) \right] \right\}. $$

(8)

The relation $\phi^T(x) = \partial\phi^A(x)/\partial x$ has been considered in the process to derive out Eq. (7).

For the time-like case, the denominator in Eq. (7) is expanded as

$$(p_2 + k_1)^2(k_1 + k_2)^2 = (x_1Q^2 - k_{1T}^2 + i\epsilon)(x_1x_2Q^2 - |k_{1T} + k_{2T}|^2 + i\epsilon),$$

(9)

and then the internal gluon/quark may go on mass shell, which will generate an image part in the hard kernel according to the principle-value prescription:

$$\frac{1}{k_{1T}^2 - \beta - i\epsilon} = Pr\frac{1}{k_{1T}^2 - \beta} + i\pi \delta(k_{1T}^2 - \beta).$$

(10)

But in the space-like case, no image part would appeared because the internal gluon/quark can’t go on mass shell because the denominator in Eq. (6) should be expanded as

$$(p_2 - k_1)^2(k_1 - k_2)^2 = (x_1Q^2 + k_{1T}^2)(x_1x_2Q^2 + |k_{1T} + k_{2T}|^2).$$

(11)

Now we consider the end-point behaviors of the LO form factors. For the elaboration, we here show the end-point behaviors in Eqs. (7,8) by using the asymptotic pion meson DAs only [34]:

$$\phi^A_\pi(x) = 6 f_\pi x (1 - x), \quad \phi^P_\pi(x) = f_\pi, \quad \phi^T_\pi(x) = f_\pi (1 - 2x).$$

(12)

Then the end-point behaviours of the integrands in Eqs. (7,8) can be expressed roughly as

$$Q^2G^{(0)}(x_i, Q^2, k_{iT}) \propto \frac{-9x_1x_2(1 - x_1)(1 - x_2) + \beta^2}{(p_2 + k_1)^2(k_1 + k_2)^2} x_1x_2Q^4,$$

(13)

$$Q^2F^{(0)}(x_i, Q^2, k_{iT}) \propto \frac{9x_1x_2(1 - x_1)(1 - x_2) + \beta^2}{x_1x_2Q^4},$$

(14)

where the first(second) terms in Eqs. (13,14) are the contributions arose from twist-2(twist-3) DAs. In the expansions of Eq. (14), the transverse momentum contributions in the internal propagators was absorbed into the effective momentum fraction $x_i$. From the expressions in Eqs. (7-14), one can see the following points:
(i) The contribution to the LO pion EM form factor from the twist-2 DAs, no matter for the
time-like case or the space-like one, has no end-point singularity because of the cancelation
of them between the denominator and numerator. The contribution arose from the twist-3
DAs, however, will generate the end-point singularities, although they are power-suppressed
by \( r_\pi^2 \) in the large momentum transfers region. The LO space-like form factor from the
twist-3 DAs is behaved as \( 1/x_1 \), for example, the twist-3 DAs would give the dominate
contribution in the small and intermediate momentum transfers region.

(ii) Since the Sudakov factor from threshold resummation\cite{26} can suppress effectively the end-
point singularity from the twist-3 contribution, a rough estimate shows that the major con-
tribution to the LO space-like form factor in Eq. (14) comes from the region of \( x_1 \sim 0.1 \) and
\( x_2 \sim 0.5 \). Then the contribution to the LO space-like form factor from the twist-2 DAs will
become as large as that from the twist-3 DAs at the point \( Q^2 \sim 7.4 \text{ GeV}^2 \), which has been
confirmed by the numerical result in Ref. [34].

(iii) The second terms in Eq. (13) is proportional to \( 1 - x_1 - x_2^2 \), which is much larger than the
second term in Eq. (14) since this second term is proportioned to \( x_2^2 \sim 10^{-2} \). The end-
point singularity for the time-like form factor induced by the twist-3 DAs, consequently,
is much higher than that for the space-like one. The twist-3 contribution to the time-like form
factor is then much larger than the twist-2 contribution in the low and intermediate \( q^2 \)
region. Simple estimation shows that these two kinds of contributions may become similar
in size in the high \( Q^2 \sim 300 \text{ GeV}^2 \) region.

By making the Fourier transformation for function \( Q^2 G^{(0)}(x_i, Q^2, k_T) \) in Eq. (7) from the
transversal momentum space (\( k_{iT} \)) to the conjugate-parameter space (\( b_i \)), we obtain the standard
do double-b convolution LO time-like pion EM form factor\cite{24, 32, 35}:

\[
Q^2 G^{(0)}_{1T} = \int_0^1 dx_1 dx_2 \int_0^\infty b_1 db_1 b_2 db_2 128\pi Q^4 \cdot \alpha_s(\mu) \cdot \exp[-S(x_i, b_i; Q, \mu)]
\cdot \{-x_1 \phi^A(x_1) \phi^A(x_2) + 2r_0^2 [\{(1 - x_1) \phi^P(x_1) \phi^P(x_2) + (1 + x_2) \phi^P(x_1) \phi^P(x_2)\} \cdot S_t(x_i)\}
\cdot K_0(i\sqrt{x_1 x_2} Q b_2) \cdot [K_0(\sqrt{x_1 Q b_1}) I_0(\sqrt{x_1 Q b_2}) \theta(b_1 - b_2) + (b_1 \leftrightarrow b_2)], \tag{15}
\]

where the Sudakov exponent \( S = S(x_1, b_2; M_B; \mu) + S(x_2, b_2; M_B; \mu) \) is the \( k_T \) resummation
factor, the Sudakov factor \( S_t(x_i) = S_t(x_1) \cdot S_t(x_2) \) refers to the threshold resummation factor, \( K_0 \)
and \( I_0 \) are the Bessel functions:

\[
K_0(iz) = \frac{i\pi}{2} H_0^{(1)}(iz); \quad H_0^{(1)}(iz) = H_0^{(1)}(z) = J_0(z) + iN_0(Z); \quad I_0(z) = J_0(z). \tag{16}
\]

Since the \( k_T \) factorization theorem applies to processes dominated by small \( x \) contribution, so
the NLO correction to the space-like pion EM form factor [31, 34] has been calculated with the
hierarchy \( x_1 Q^2, x_2 Q^2 \gg x_1 x_2 Q^2, k_T^2 \) for convenience. Since there is no end-point singularity
for the LO pion form factor from the twist-2 DAs, we can ignore the transverse momenta for the
internal quark propagator safely for the twist-2 contribution as elaborated in Ref. [32], then the
denominator for the first term in Eq. (7) is reduced to

\[
(p_2 + k_1)^2(k_1 + k_2)^2 = x_1 Q^2(x_1 x_2 Q^2 - |k_{1T} + k_{2T}|^2 + i\epsilon). \tag{17}
\]
The LO time-like pion EM form factor from the twist-2 DAs can be written in a single-b convolution formula as follows:

\[
Q^2 G^{(0)}_{T2,I} = \int_0^1 dx_1dx_2 \int_0^\infty b_1db_1b_2db_2 128\pi Q^4 \cdot \alpha_s(\mu) \cdot \exp[-S(x_1; b_1; Q; \mu)] \\
\cdot \left\{ -x_1\phi^A(x_1)\phi^A(x_2) \right\} \cdot K_0(i\sqrt{x_1x_2}Qb_2).
\] (18)

In Ref. [32], the authors confirmed that the numerical results of form factor in the standard double-b convolution of Eq. (15) is approximately equal to the value of single-b convolution of Eq. (18), which furthermore showed that the major source of the strong phase is produced by the internal gluon propagator for the twist-2 contribution.

The LO time-like form factor from the twist-3 DAs, however, has a high power end-point singularity, the single-b approximation is therefore not valid for the twist-3 DAs’s contribution. So in the next section we have to calculate the NLO twist-3 hard kernel in time-like form factor by using the double-b convolution method.

### III. NLO CORRECTION TO THE TWIST-3 TIME-LIKE PION EM FORM FACTORS

The LO analysis in the last section show that the time-like hard kernel can be obtained from the space-like one by the simple space transfer: \(-Q^2 \rightarrow Q^2\). Because of the Lorentz invariant QCD theory, it’s believed that this analytical continuation should be hold at NLO.

In \(k_T\) factorization theorem, the NLO hard kernel for pion EM form factor is derived by taking the difference of the NLO \(\mathcal{O}(\alpha_s^2)\) quark diagrams and the convolutions of the LO \(\mathcal{O}(\alpha_s)\) hard kernel with the NLO \(\mathcal{O}(\alpha_s)\) effective diagrams for meson wave functions. For the space-like pion EM form factors as described explicitly in Refs. [31, 34], the ultraviolet divergences are just absorbed into the renormalized coupling constant \(\alpha_s(\mu)\) with the massless pion meson, the infrared divergences arose from the soft region are canceled by themselves in the quark diagrams, and the infrared divergences in the collinear region for the quark diagrams can be absorbed into the high order non-perturbative meson wave functions.

For the time-like form factor, the NLO twist-2 hard kernel has been calculated in Ref. ([32]) and then the only unknown NLO correction at present is the one from the twist-3 DAs. With the NLO twist-3 space-like hard kernels calculated in Ref. ([34]), we can obtain the NLO twist-3 time-like hard kernel by the analytical continuation \(-Q^2 \rightarrow Q^2\). For this purpose, we firstly define two types of LO twist-3 time-like hard kernels \(H^{(0)}_{T3,1}(H^{(0)}_{T3,2})\) proportioned to the lorentz structure \(p_{1\mu}(p_{2\mu})\) from Eq. (5):

\[
H^{(0)}_{T3,1}(x_i; k_{IT}, Q^2) = \frac{i\epsilon Q^2}{(p_2 + k_1)^2(k_2 + k_1)^2} \cdot 2i_\mu x_1p_{1\mu} \left[ \phi^P(x_1) + \phi^T(x_1) \right] \phi^P(x_2),
\] (19)

\[
H^{(0)}_{T3,2}(x_i; k_{IT}, Q^2) = \frac{i\epsilon Q^2}{(p_2 + k_1)^2(k_2 + k_1)^2} \cdot 2i_\mu x_2p_{2\mu} \left[ \phi^P(x_1) - \phi^T(x_1) \right] \phi^P(x_2).
\] (20)

By substituting \(Q^2 + i\epsilon\) for the momentum transfers of the virtual photon, and \(x_1x_2Q^2 - (k_{IT} + k_{IT})^2 + i\epsilon(x_1Q^2 - k_{IT}^2 + i\epsilon)\) for the internal gluon(quark), we can obtain the NLO twist-3 hard kernels for the time-like \(\pi^+\pi^-\) production process from the NLO twist-3 space-like one[34]. The NLO twist-3 time-like hard kernels can then be written as the form of

\[
H^{(1)}_{T3,1}(x_i; k_{IT}, Q^2; \mu, \mu_f) = h_{T3,1}(x_i; k_{IT}, Q, \mu, \mu_f) \cdot H^{(0)}_{T3,1}(x_i; k_{IT}, Q^2)
\] (21)

\[
H^{(1)}_{T3,2}(x_i; k_{IT}, Q^2; \mu, \mu_f) = h_{T3,2}(x_i; k_{IT}, Q, \mu, \mu_f) \cdot H^{(0)}_{T3,2}(x_i; k_{IT}, Q^2).
\] (22)
By setting the renormalized and factorized scales both at the internal hard scale \( \mu = \mu_f = t \), and using the follow relations,

\[
\begin{align*}
\ln(-Q^2 - i\epsilon) &= \ln(Q^2) - i\pi, \\
\ln(k_{1T}^2 - x_1Q^2 + i\epsilon) &= \ln(k_{1T}^2 - x_1Q^2) + i\pi \Theta(k_{1T}^2 - x_1Q^2) \\
\ln(k_{2T}^2 - x_1x_2Q^2 + i\epsilon) &= \ln(k_{2T}^2 - x_1x_2Q^2) + i\pi \Theta(k_{2T}^2 - x_1x_2Q^2).
\end{align*}
\]

the relevant correction functions \( h_{T3,1}, h_{T3,2} \) in Eqs. (21,22) can be written as,

\[
egin{align*}
h_{T3,1}(x_i, k_{iT}, Q, t) &= \frac{\alpha_s C_F}{4\pi} \left[ \frac{9}{4} \ln \left( \frac{t^2}{Q^2} \right) - \frac{53}{16} \ln \delta'_{12} - \frac{23}{16} \ln x'_1 - \frac{1}{8} \ln^2 x_2 \\
& \quad - \frac{9}{8} \ln x_2 - \frac{137\pi^2}{96} + \frac{337}{64} + i\pi \frac{5}{2} \right],
\end{align*}
\]

\[
egin{align*}
h_{T3,2}(x_i, k_{iT}, Q, t) &= \frac{\alpha_s C_F}{4\pi} \left[ \frac{9}{4} \ln \left( \frac{t^2}{Q^2} \right) - 4 \ln \delta'_{12} - \frac{1}{2} \ln^2 x'_1 + 2 \ln x_2 \\
& \quad - \frac{15\pi^2}{24} + \ln \frac{2}{4} + \frac{11}{2} + i\pi \left( \frac{7}{4} + \ln x'_1 \right) \right],
\end{align*}
\]

where \( \ln \delta'_{12} \equiv \ln((k_{1T} + k_{2T})^2 - x_1x_2Q^2 + i\epsilon) - \ln Q^2 \) and \( \ln x'_1 \equiv \ln(k_{1T}^2 - x_1Q^2 + i\epsilon) \).

We can then obtain the NLO twist-3 time-like correction functions \( h_{T3,1}, h_{T3,2} \) in the parameter space \( b_i \) by the Fourier transformation from the transverse momentum space \( k_{iT} \) to \( b_i \) space. The correction functions in \( b \) space takes the form of

\[
egin{align*}
h_{T3,1}(x_i, b_i, Q, t) &= \frac{\alpha_s C_F}{4\pi} \left[ \frac{9}{4} \ln \left( \frac{t^2}{Q^2} \right) - \frac{53}{32} \ln \left( \frac{4x_1x_2}{Q^2b_i^2} \right) - \frac{23}{32} \ln \left( \frac{4x_1}{Q^2b_i^2} \right) - \frac{1}{8} \ln^2 x_2 \\
& \quad - \frac{9}{8} \ln x_2 - \frac{137\pi^2}{96} + \frac{19}{4} \gamma_E + \frac{337}{64} + i\pi \frac{39}{8} \right],
\end{align*}
\]

\[
egin{align*}
h_{T3,2}(x_i, b_i, Q, t) &= \frac{\alpha_s C_F}{4\pi} \left[ \frac{9}{4} \ln \left( \frac{t^2}{Q^2} \right) - 2 \ln \left( \frac{4x_1x_2}{Q^2b_i^2} \right) - \frac{1}{8} \ln^2 \left( \frac{4x_1}{Q^2b_i^2} \right) \\
& \quad + \left( \frac{\gamma_E}{2} + \frac{3}{4} i\pi \right) \ln \left( \frac{4x_1}{Q^2b_i^2} \right) + 2 \ln x_2 - \frac{\pi^2}{4} - \frac{\gamma_E^2}{2} + 4\gamma_E \\
& \quad + \frac{\ln 2}{4} + \frac{11}{2} + i\pi \left( \frac{15}{4} - \frac{3}{2} \gamma_E \right) \right],
\end{align*}
\]

where \( \gamma_E \) is the Euler constant.

With the NLO twist-3 correction function in Eqs. (26,27) and the NLO twist-2 correction function in Ref. (32), we can derive the NLO time-like pion EM form factor in \( k_T \) factorization formula as,

\[
Q^2G^{(1)}_{11} = 128\pi Q^4 \cdot \alpha_s(\mu) \cdot \int_0^1 dx_1dx_2 \int_0^\infty b_1db_1b_2db_2 \cdot \exp[-S(x_i; b_i; Q; \mu)] \\
\cdot \left\{ -x_1\phi^A(x_1)\phi^A(x_2) \cdot h_{T2} + 2T_0^2 \left[ (\phi^P(x_1) + \phi^T(x_1)) \phi^P(x_2) \cdot h_{T3,2} \\
+ x_1(\phi^T(x_1) - \phi^P(x_1)) \phi^P(x_2) \cdot h_{T3,3} \right] \cdot S_i(x_i) \right\} \cdot K_0(i\sqrt{x_1x_2}Qb_2) \\
\cdot [K_0(\sqrt{x_1Qb_1})I_0(\sqrt{x_1Qb_2})\theta(b_1 - b_2) + (b_1 \leftrightarrow b_2)],
\]

(28)
where the NLO twist-2 correction function $h_{T2}$ derived from sing-b formula is expressed as the following form [32].

$$h_{T2}(x_i,b,Q,t) = \frac{\alpha_s C_F}{4\pi} \left\{ -\frac{3}{4} \ln \left( \frac{t^2}{Q^2} \right) - \frac{1}{4} \ln^2 \left( \frac{4x_1x_2}{Q^2b_2^2} \right) - \frac{17}{4} \ln^2 x_1 + \frac{27}{8} \ln x_1 \ln x_2 \\
+ \left( \frac{17}{8} \ln x_1 + \frac{23}{16} + \gamma_E + \frac{i\pi}{2} \right) \ln \left( \frac{4x_1x_2}{Q^2b_2^2} \right) - \left( \frac{13}{8} + \frac{17\gamma_E}{4} - \frac{17\pi}{8} \right) \ln x_1 \\
+ \frac{31}{16} \ln x_2 - \frac{\pi^2}{2} + (1 - 2\gamma_E)\pi + \frac{\ln 2}{2} + \frac{53}{4} - \frac{23\gamma_E}{8} - \gamma_E + i\pi \left( \frac{171}{16} + \gamma_E \right) \right\}. \quad (29)$$

### IV. NUMERICAL RESULTS AND DISCUSSIONS

In this section we present the numerical results for the time-like pion EM form factor induced by the distribution amplitudes with different twists at LO and NLO level. Non-asymptotic pion meson DAs as given in Eq. (30) with the inclusion of the high order effects are adopted in our numerical calculation.

$$\phi_A^\pi(x) = \frac{3f_\pi}{\sqrt{6}} x(1 - x) \left[ 1 + a_2^\pi C_2^\pi(u) + a_4^\pi C_4^\pi(u) \right],$$

$$\phi_P^\pi(x) = \frac{f_\pi}{2\sqrt{6}} \left[ 1 + \left( 30\eta_3 - \frac{5}{2}\rho_\pi^2 \right) C_2^\pi(u) - 3 \left( \eta_3\omega_3 + \frac{9}{20}\rho_\pi^2 \right) \left( 1 + 6a_2^\pi \right) C_4^\pi(u) \right],$$

$$\phi_T^\pi(x) = \frac{f_\pi}{2\sqrt{6}} (1 - 2x) \left[ 1 + 6 \left( 5\eta_3 - \frac{1}{2}\eta_3\omega_3 - \frac{7}{20}\rho_\pi^2 - \frac{3}{5}\rho_\pi^2 a_2^\pi \right) \left( 1 - 10x + 10x^2 \right) \right], \quad (30)$$

where the Gegenbauer moments $a_n^\pi$, the parameters $\eta_3, \omega_3$ and $\rho_\pi$ are adopted from Refs. [36–39]:

$$a_2^\pi = 0.25, \quad a_4^\pi = -0.015, \quad \rho_\pi = m_\pi/m_0, \quad \eta_3 = 0.015, \quad \omega_3 = -3.0, \quad (31)$$

with $f_\pi = 0.13$ GeV, $m_\pi = 0.13$ GeV, $m_0 = 1.74$ GeV.

The LO and NLO pQCD predictions for the magnitude and strong phase of time-like pion EM form factor from the twist-2 and twist-3 DAs are illustrated in Fig. 2 and Fig. 3 respectively. By summing up the different twists’ contributions, the total pQCD prediction for these physical quantities are shown in Fig. 4. From Fig. 2, Fig. 3 and Fig. 4, one can see the following points:
FIG. 3. The pQCD predictions for the magnitude and strong phase of time-like pion EM form factors induced by the twist-3 DAs $\phi_{P,T}$. The Rome symbol “II” refers to the form factors calculated in double-b convolution formula as described in Eq. (15).

FIG. 4. The pQCD predictions for the magnitude and strong phase of time-like pion EM form factor as described in Eqs. (15,28) at LO and NLO level. As a comparison, those currently available measured values [6–9] for fixed $q^2$ are also shown in Fig. 4(a).

(1) For the LO form factor induced by the twist-2 DAs, the single-b convolution formula is a good approximation for the region of $q^2 > 30$ GeV$^2$ because the single-b convolution result is close to the standard double-b convolution result in this $q^2$ region. Of course, this approximation can be understood by the fact that the internal gluon propagator carry almost all the strong phase with no end-point singularity for this twist-2 case.

(2) The NLO twist-2 correction to the magnitude (strong phase) of the LO twist-2’s contribution is smaller than 25% (10°) in the region of $q^2 > 30$ GeV$^2$. The NLO twist-3 correction to the magnitude (strong phase) of the LO twist-3’s contribution is smaller than 35% (20°) in the region of $q^2 > 5$ GeV$^2$.

(3) At the LO level, because of the high power singularity, the twist-3 contribution is much larger than the twist-2 part in our considered region of $1 < q^2 < 49$ GeV$^2$. So the obvious NLO twist-3 correction can enhance the LO pQCD prediction and therefore can improve the agreement between the pQCD prediction and the data, especially in the region of $q^2 > 5$ GeV$^2$. The NLO correction with the inclusion of both twist-2 and twist-3 contributions can
enhance (reduce) the magnitude (strong phase) of the LO one by $20\% - 30\% (< 15^\circ)$ in the region of $q^2 > 5 \text{ GeV}^2$. The NLO pQCD prediction for time-like form factor therefore become well consistent with the CLEO data in the of region of $5 < q^2 < 15 \text{ GeV}^2$, as shown explicitly by the solid curve in Fig. 4(a).

Our numerical result at LO is a little smaller than the one in Ref. [32], since we here used the different input DAs and the different choice of the QCD scale $\lambda_{QCD}$. In Ref. ([32]), $\lambda_{QCD}$ is chosen at the fixed value 0.2 GeV. In this paper, however, the QCD scale is varying in the transition process according to the internal hard scale, and $\lambda_{QCD}$ is around 0.25 GeV here.

In this paper, we firstly gave a brief review for the LO time-like and space-like pion EM form factor evaluated in the $k_T$ factorization theorem, and then we calculated the NLO twist-3 correction to the LO time-like pion EM form factor by making the analytic continuation of the NLO twist-3 space-like correction for the corresponding space-like form factor, and finally we made the numerical calculations for the time-like pion EM form factor with the inclusion of the NLO twist-2 and twist-3 corrections.

From the numerical results about the LO and NLO pQCD predictions for the time-like pion EM form factor, we found that:

(i) The LO twist-3 contribution is much larger than the twist-2 one since the high power end-point singularity;

(ii) The NLO twist-2 correction to the LO twist-2 contribution for the magnitude (phase) is lower than $25\% (10^\circ)$ of the LO form factor in the region of $q^2 > 30 \text{ GeV}^2$. The NLO twist-3 correction to the LO twist-3 contribution for the magnitude (phase) of the LO form factor is lower than $35\% (10^\circ)$ in the region of $q^2 > 5 \text{ GeV}^2$.

(iii) The total NLO correction with the inclusion of both the twist-2 and twist-3 contributions can enhance (reduce) the magnitude (phase) of the LO form factor by $20\% - 30\% (< 15^\circ)$ in the region of $q^2 > 5 \text{ GeV}^2$, and consequently the NLO pQCD prediction for the studied pion EM form factor become well consistent with the CLEO data.

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