Fixed-Point Definability and Polynomial Time on Chordal Graphs and Line Graphs

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Abstract

The question of whether there is a logic that captures polynomial time was formulated by Yuri Gurevich in 1988. It is still wide open and regarded as one of the main open problems in finite model theory and database theory. Partial results have been obtained for specific classes of structures. In particular, it is known that fixed-point logic with counting captures polynomial time on all classes of graphs with excluded minors. The introductory part of this paper is a short survey of the state-of-the-art in the quest for a logic capturing polynomial time.

The main part of the paper is concerned with classes of graphs defined by excluding induced subgraphs. Two of the most fundamental such classes are the class of chordal graphs and the class of line graphs. We prove that capturing polynomial time on either of these classes is as hard as capturing it on the class of all graphs. In particular, this implies that fixed-point logic with counting does not capture polynomial time on these classes. Then we prove that fixed-point logic with counting does capture polynomial time on the class of all graphs that are both chordal and line graphs.

1 The quest for a logic capturing PTIME

Descriptive complexity theory started with Fagin’s Theorem [25] from 1974, stating that existential second-order logic captures the complexity class \( \text{NP} \). This means that a property of finite structures is decidable in nondeterministic polynomial time if and only if it is definable in existential second order logic. Similar logical characterisations where later found for most other complexity classes. For example, in 1982 Immerman [44] and independently Vardi [60] characterised the class \( \text{PTIME} \) (polynomial time) in terms of least fixed-point logic, and in 1983 Immerman [46] characterised the classes \( \text{NLOGSPACE} \) (nondeterministic logarithmic space) and \( \text{LOGSPACE} \) (logarithmic space) in terms of transitive closure logic and its deterministic variant. However, these logical characterisations of the classes \( \text{PTIME} \), \( \text{NLOGSPACE} \), and \( \text{LOGSPACE} \), and all other known logical characterisations of complexity classes contained in \( \text{PTIME} \), have a serious drawback: They only apply to properties of ordered structures, that is, relational structures with one distinguished relation that is a linear order of the elements of the structure. It is still an open question whether there are logics that characterise these complexity classes on arbitrary, not necessarily ordered structures. We focus on the class \( \text{PTIME} \) from now on. In this section, which is an updated version of [32], we give a short survey of the quest for a logic capturing \( \text{PTIME} \).

1.1 Logics capturing \( \text{PTIME} \)

The question of whether there is a logic that characterises, or captures, \( \text{PTIME} \) is subtle. If phrased naively, it has a trivial, but completely uninteresting positive answer. Yuri Gurevich [37] was the first to give a precise formulation of the question. Instead of arbitrary finite structures, we restrict our attention to graphs in this paper. This is no serious restriction, because the question of whether there is a logic that captures \( \text{PTIME} \) on arbitrary structures is equivalent to the restriction of the question to graphs. We first need to define what constitutes a logic. Following Gurevich, we take a very liberal, semantically oriented approach. We identify properties of graphs with classes of graphs closed under isomorphism. A logic \( \mathcal{L} \) (on graphs) consists of a computable set of sentences together with a semantics that associates a property \( \mathcal{P}_\varphi \) of graphs with each sentence \( \varphi \). We say that a graph \( G \) satisfies a sentence \( \varphi \), and write \( G \models \varphi \), if \( G \in \mathcal{P}_\varphi \). We say that a property \( \mathcal{P} \) of graphs is definable in \( \mathcal{L} \) if there is a sentence \( \varphi \) such that \( \mathcal{P}_\varphi = \mathcal{P} \).
A logic \( L \) captures \( \text{PTIME} \) if the following two conditions are satisfied:

1. **(G.1)** Every property of graphs that is decidable in \( \text{PTIME} \) is definable in \( L \).
2. **(G.2)** There is a computable function that associates with every \( L \)-sentence \( \phi \) a polynomial \( p(X) \) and an algorithm \( A \) such that \( A \) decides the property \( \mathcal{P}_\phi \) in time \( p(n) \), where \( n \) is the number of vertices of the input graph.

While condition **(G.1)** is obviously necessary, condition **(G.2)** may seem unnecessarily complicated. The natural condition we expect to see instead is the following condition **(G.2')**: Every property of graphs that is definable in \( L \) is decidable in \( \text{PTIME} \). Note that **(G.2)** implies **(G.2')**, but that the converse does not hold. However, **(G.2')** is too weak, as the following example illustrates:

**Example 1.1.** Let \( \mathcal{P}_1, \mathcal{P}_2, \ldots \) be an arbitrary enumeration of all polynomial time decidable properties of graphs. Such an enumeration exists because there are only countably many Turing machines and hence only countably many decidable properties of graphs. Let \( L' \) be the “logic” whose sentences are the natural numbers and whose semantics is defined by letting sentence \( i \) define property \( \mathcal{P}_i \). Then \( L' \) is a logic according to our definition, and it does satisfy **(G.1)** and **(G.2')**. But clearly, \( L' \) is not a “logic capturing \( \text{PTIME} \)” in any interesting sense.

Let me remark that most natural logics that are candidates for capturing \( \text{PTIME} \) trivially satisfy **(G.2)**. The difficulty is to prove that they also satisfy **(G.1)**, that is, define all \( \text{PTIME} \)-properties.

There is a different route that leads to the same question of whether there is a logic capturing \( \text{PTIME} \) from a database-theory perspective: After Aho and Ullman [2] had realised that SQL, the standard query language for relational databases, cannot express all database queries computable in polynomial time, Chandra and Harel [10] asked for a recursive enumeration of the class of all relational database queries computable in polynomial time. It turned out that Chandra and Harel’s question is equivalent to Gurevich’s question for a logic capturing \( \text{PTIME} \), up to a minor technical detail.

The question of whether there is a logic that captures \( \text{PTIME} \) is still wide open, and it is considered one of the main open problems in finite model theory and database theory. Gurevich conjectured that there is no logic capturing \( \text{PTIME} \). This would not only imply that \( \text{PTIME} \neq \text{NP} \) — remember that by Fagin’s Theorem there is a logic capturing \( \text{NP} \) — but it would actually have interesting consequences for the structure of the complexity class \( \text{PTIME} \). Dawar [15] proved a dichotomy theorem stating that, depending on the answer to the question, there are two fundamentally different possibilities: If there is a logic for \( \text{PTIME} \), then the structure of \( \text{PTIME} \) is very simple; all \( \text{PTIME} \)-properties are variants or special cases of just one problem. If there is no logic for \( \text{PTIME} \), then the structure of \( \text{PTIME} \) is so complicated that it eludes all attempts for a classification. The formal statement of the first possibility is that there is a complete problem for \( \text{PTIME} \) under first-order reductions. The formal statement of the second possibility is that the class of \( \text{PTIME} \)-properties is not recursively enumerable.

### 1.2 Fixed-point logics

Fixed-point logics play an important role in finite-model theory, and in particular in the quest for a logic capturing \( \text{PTIME} \). Very briefly, the fixed-point logics considered in this context are extensions of first-order logic by operators that formalise inductive definitions. We have already mentioned that least fixed-point logic \( \text{LFP} \) captures polynomial time on ordered structures; this result is known as the *Immerman-Vardi Theorem*. For us, it will be more convenient to work with *inflationary fixed-point logic* \( \text{IFP} \), which was shown to have the same expressive power as \( \text{LFP} \) on finite structures by Gurevich and Shelah [39] and on infinite structures by Kreutzer [50].

\( \text{IFP} \) does not capture polynomial time on all finite structures. The most immediate reason is the inability of the logic to count. For example, there is no \( \text{IFP} \)-sentence stating that the vertex set of a graph has even...
there is a graph that have no complete graph on five vertices, classes of structures of bounded tree width \cite{34}. In \cite{31}, I proved the same result for the class of all graphs obtained from a subgraph of an important example of such a class is the class of graphs with excluded minors to graph classes defined by excluding induced subgraphs. One of the most basic and proof of the strong perfect graph theorem, the focus of many graph theorists has shifted from graph classes on all classes of graphs that exclude a minor \cite{33}.

planar graphs \cite{30} and around the same time, Julian Mariño and I proved that is an induced subgraph. A graph is claw-free if it does not have a vertex with three pairwise nonadjacent neighbours, that is, if it is a graph whose vertices are the edges of \( G \), with two edges being adjacent in \( G \) if and only if they have a common endvertex. A minor of a graph \( G \) is a graph \( H \) that can be obtained from a subgraph of \( G \) by contracting edges. We say that a class \( C \) of graphs excludes a minor if there is a graph \( H \) that is not a minor of any graph in \( C \). Very recently, I proved that \( \text{IFP+C} \) captures \text{PTIME} on all classes of graphs that exclude a minor \cite{33}.

In the last few years, maybe as a consequence of Chudnowsky, Robertson, Seymour, and Thomas’s \cite{11} proof of the strong perfect graph theorem, the focus of many graph theorists has shifted from graph classes with excluded minors to graph classes defined by excluding induced subgraphs. One of the most basic and important example of such a class is the class of chordal graphs. A cycle \( C \) of a graph \( G \) is chordless if it is an induced subgraph. A graph is chordal (or triangulated) if it has no chordless cycle of length at least four. Figure \ref{fig:chordal} shows an example of a chordal graph. All chordal graphs are perfect, which means that the graphs themselves and all their induced subgraphs have the chromatic number equal to the clique number. Chordal graphs have a nice and simple structure; they can be decomposed into a tree of cliques. A second important example is the class of line graphs. The line graph of a graph \( G \) is the graph \( L(G) \) whose vertices are the edges of \( G \), with two edges being adjacent in \( L(G) \) if they have a common endvertex in \( G \). Figure \ref{fig:line} shows an example of a line graph. The class of all line graphs is closed under taking induced subgraphs. Beineke \cite{5} gave a characterisation of the class of line graphs (more precisely, the class of all graphs isomorphic to a line graph) by a family of nine excluded subgraphs. An extension of the class of line graphs, which has also received a lot of attention in the literature, is the class of claw-free graphs. A graph is claw-free if it does not have a vertex with three pairwise nonadjacent neighbours, that is, if it does not have a claw (displayed in Figure \ref{fig:claw}) as an induced subgraph. It is easy to see that all line graphs are claw-free. Recently, Chudnowsky and Seymour (see \cite{12}) developed a structure theory for claw-free graphs.

It would be tempting to use this structure theory for claw free graphs, or at least the simple treelike structure of chordal graphs, to prove that \( \text{IFP+C} \) captures \text{PTIME} on these classes in a similar way as the structure theory for classes of graphs with excluded minors is used to prove that \( \text{IFP+C} \) captures \text{PTIME} on classes with excluded minors. Unfortunately, this is only possible on the very restricted class of graphs that are both chordal and line graphs (an example of such a graph is shown in Figure \ref{fig:combined} on p.12). We prove
Figure 1.1. (a) a chordal graph, which is not a line graph, and (b) the line graph of $K_4$, which is not chordal

Figure 1.2. A claw

the following theorem:

**Theorem 1.2.**

1. IFP+C does not capture PTIME on the class of chordal graphs or on the class of line graphs.
2. IFP+C captures PTIME on the class of chordal line graphs.

Our construction to prove (1) is so simple that it will apply to any reasonable logic, which means that if a “reasonable” logic captures PTIME on the class of chordal graphs or on the class of line graphs, then it captures PTIME on the class of all graphs.

Further interesting graph classes closed under taking induced subgraphs are various classes of intersection graphs. Very recently, Laubner [51] proved that IFP+C captures PTIME on the class of all interval graphs. To conclude our discussion of classes of graphs on which IFP+C captures PTIME, let me mention a result due to Hella, Kolaitis, and Luosto [41] stating that IFP+C captures PTIME on almost all graphs (in a precise technical sense). Thus it seems that the results for specific classes of graphs are not very surprising, but it should be mentioned that almost no graphs fall in one of the natural graphs classes discussed before.

Instead of capturing all PTIME on a specific class of structures, Otto [55] studied the question of capturing all PTIME properties satisfying certain invariance conditions. Most notably, he proved that bisimulation-invariant properties are decidable in polynomial time if and only if they are definable in the higher-dimensional $\mu$-calculus.

### 1.4 Isomorphism testing and canonisation

As an abstract question, the question of whether there is a logic capturing polynomial time is linked to the graph isomorphism and canonisation problems. Otto [55] was the first to systematically study the connection between canonisation and descriptive complexity theory. Specifically, if there is a polynomial time canonisation algorithm for a class $\mathcal{C}$ of graphs, then there is a logic that captures polynomial time on this class $\mathcal{C}$. This follows from the Immerman-Vardi Theorem. To explain it, let us assume that we represent graphs by their adjacency matrices. A *canonisation mapping* gets as argument some adjacency matrix representing a graph and returns a *canonical* adjacency matrix for this graph, that is, it maps isomorphic adjacency matrices to equal adjacency matrices. As an adjacency matrix for a graph is completely fixed once we specify the ordering of the rows and columns of the matrix, we may view a canonisation as a mapping associating with each graph a canonical ordered copy of the graph. Now we can apply the Immerman-Vardi Theorem to this ordered copy.
Clearly, if there is a polynomial time canonisation mapping for a class of graphs (or other structures) then there is a polynomial time isomorphism test for this class. It is open whether the converse also holds. It is also open whether the existence of a logic for polynomial time implies the existence of a polynomial time isomorphism test or canonisation mapping.

Polynomial time canonisation mappings are known for many natural classes of graphs, for example planar graphs [42, 43], graphs of bounded genus [26, 54], graphs of bounded eigenvalue multiplicity [8], graphs of bounded degree [4, 53], and graphs of bounded tree width [8]. Hence for all these classes there are logics capturing \( \text{PTIME} \). However, the logics obtained through canonisation hardly qualify as natural logics. If a logic is to contribute to our understanding of the complexity class \( \text{PTIME} \)— and from my perspective this is the main reason for being interested in such a logic — we have to look for natural logics that derive their expressiveness from clearly visible basic principles like inductive definability, counting or other combinatorial operations, and maybe fundamental algebraic operations like computing the rank or the determinant of a matrix. If such a logic captures polynomial time on a class of structures, then this shows that all polynomial time properties of structures in this class are based on the principles underlying the logic. Thus even for classes for which we know that there is a logic capturing \( \text{PTIME} \) through a polynomial-time canonisation algorithm, I think it is important to find “natural” logics capturing \( \text{PTIME} \) on these classes. In particular, I view it as an important open problem to find a natural logic that captures \( \text{PTIME} \) on classes of graphs of bounded degree. It is known that IFP+C does not capture \( \text{PTIME} \) on the class of all graphs of maximum degree at most three.

Most known capturing results are proved by showing that there is a canonisation mapping that is definable in some logic. In particular, all capturing results for IFP+C mentioned above are proved this way. It was observed by Cai, Fürer, and Immerman [9] that for classes \( \mathcal{C} \) of structures which admit a canonisation mapping definable in IFP+C, a simple combinatorial algorithm known as the Weisfeiler-Lehman (WL) algorithm [23, 24] can be used as a polynomial time isomorphism test on \( \mathcal{C} \). Thus the the WL-algorithm correctly decides isomorphism on the class of chordal line graphs and on all classes of graphs with excluded minors. A refined version of the same approach was used by Verbitsky and others [35, 49, 61] to obtain parallel isomorphism tests running in polylogarithmic time for planar graphs and graphs of bounded tree width.

1.5 Stronger logics

Early on, a number of results regarding the possibility of capturing polynomial time by adding Lindström quantifiers to first-order logic or fixed-point logic were obtained. Hella [40] proved that adding finitely many Lindström quantifiers (or infinitely many of bounded arity) to fixed-point logic does not suffice to capture polynomial time (also see [17]). Dawar [14] proved that if there is a logic capturing polynomial time, then there is such a logic obtained from fixed-point logic by adding one vectorised family of Lindström quantifiers. Another family of logics that have been studied in this context consists of extensions of fixed-point logic with nondeterministic choice operators [1][18][27].

Currently, the two main candidates for logics capturing \( \text{PTIME} \) are choiceless polynomial time with counting CP+C and inflationary fixed-point logic with a rank operator IFP+R. The logic CP+C was introduced by Blass, Gurevich and Shelah [6] (also see [7] [19]). The formal definition of the logic is carried out in the framework of abstract state machines (see, for example, [38]). Intuitively CP+C may be viewed as a version of IFP+C where quantification and fixed-point operators not only range over elements of a structure, but instead over all objects that can be described by \( O(\log n) \) bits, where \( n \) is the size of the structure. This intuition can be formalised in an expansion of a structure by all hereditarily finite sets which use the elements of the structure as atoms. The logic IFP+R is an extension of IFP by an operator that determines the rank of definable matrices in a structure. This may be viewed as a higher dimensional version of a counting operator. (Counting appears as a special case of diagonal \( \{0,1\} \)-matrices.)

Both CP+C and IFP+R are known to be strictly more expressive than IFP+C. Indeed, both logics can express the property used by Cai, Fürer, and Immerman to separate IFP+C from \( \text{PTIME} \). For both logics it is open whether they capture polynomial time, and it is also open whether one of them semantically contains the other.
2 Preliminaries

\( \mathbb{N}_0 \), and \( \mathbb{N} \) denote the sets of nonnegative integers and natural numbers (that is, positive integers), respectively. For \( m, n \in \mathbb{N}_0 \), we let \( [m, n] := \{ \ell \in \mathbb{N}_0 \mid m \leq \ell \leq n \} \) and \( [n] := [1, n] \). We denote the power set of a set \( S \) by \( 2^S \) and the set of all \( k \)-element subsets of \( S \) by \( \binom{S}{k} \).

We often denote tuples \( (v_1, \ldots, v_k) \) by \( v \). If \( v \) denotes the tuple \( (v_1, \ldots, v_k) \), then by \( v \) we denote the set \( \{v_1, \ldots, v_k\} \). If \( v = (v_1, \ldots, v_k) \) and \( w = (w_1, \ldots, w_t) \), then by \( vw \) we denote the tuple \( (v_1, \ldots, v_k, w_1, \ldots, w_t) \).

2.1 Graphs

Graphs in this paper are always finite, nonempty, and simple, where simple means that there are no loops or parallel edges. Unless explicitly called “directed”, graphs are undirected. The vertex set of a graph \( G \) is denoted by \( V(G) \) and the edge set by \( E(G) \). We view graphs as relational structures with \( E(G) \) being a binary relation on \( V(G) \). However, we often find it convenient to view edges (of undirected graphs) as 2-element subsets of \( V(G) \) and use notations like \( e = \{u, v\} \) and \( v \in e \). Subgraphs, induced subgraphs, union, and intersection of graphs are defined in the usual way. We write \( G[W] \) to denote the induced subgraph of \( G \) with vertex set \( W \subseteq V(G) \), and we write \( G \setminus W \) to denote \( G[V(G) \setminus W] \). The set \( \{w \in V(G) \mid \{v, w\} \in E(G)\} \) of neighbours of a node \( v \) is denoted by \( N^G(v) \), or just \( N(v) \) if \( G \) is clear from the context, and the degree of \( v \) is the cardinality of \( N(v) \). The order of a graph, denoted by \( |G| \), is the number of vertices of \( G \).

The class of all graphs is denoted by \( \mathcal{G} \). A homomorphism from a graph \( G \) to a graph \( H \) is a mapping \( h : V(G) \to V(H) \) that preserves adjacency, and an isomorphism is a bijective homomorphism whose inverse is also a homomorphism.

For every finite nonempty set \( V \), we let \( K[V] \) be the complete graph with vertex set \( V \), and we let \( K_n := K[\{n\}] \). A clique in a graph \( G \) is a set \( W \subseteq V(G) \) such that \( G[W] \) is a complete graph. Paths and cycles in graphs are defined in the usual way. The length of a path or cycle is the number of its edges. Connectedness and connected components are defined in the usual way. A set \( W \subseteq V(G) \) is connected in a graph \( G \) if \( W \neq \emptyset \) and \( G[W] \) is connected. For sets \( W_1, W_2 \subseteq V(G) \), a set \( S \subseteq V(G) \) separates \( W_1 \) from \( W_2 \) if there is no path from a vertex in \( W_1 \setminus S \) to a vertex in \( W_2 \setminus S \) in the graph \( G \setminus S \).

A forest is an undirected acyclic graph, and a tree is a connected forest. It will be a useful convention to call the vertices of trees and forests nodes. A rooted tree is a triple \( T = (V(T), E(T), r(T)) \), where \( (V(T), E(T)) \) is a tree and \( r(T) \in V(T) \) is a distinguished node called the root.

We occasionally have to deal with directed graphs. We allow directed graphs to have loops. We use standard graph theoretic terminology for directed graphs, without going through it in detail. Homomorphisms and isomorphisms of directed graphs preserve the direction of the edges. Paths and cycles in a directed graph are always meant to be directed; otherwise we will call them “paths or cycles of the underlying undirected graph”. Note that cycles in directed graphs may have length 1 or 2. For a directed graph \( D \) and a vertex \( v \in V(D) \), we let \( N^D(v) := \{w \in V(D) \mid \{v, w\} \in E(D)\} \). Directed acyclic graphs will be of particular importance in this paper, and we introduce some additional terminology for them: Let \( D \) be a directed acyclic graph. A node \( w \) is a child of a node \( v \), and \( v \) is a parent of \( w \), if \( (v, w) \in E(D) \). We let \( \preceq^D \) be the reflexive transitive closure of the edge relation \( E(D) \) and \( \prec^D \) its irreflexive version. Then \( \subseteq^D \) is a partial order on \( V(D) \).

A directed tree is a directed acyclic graph \( T \) in which every node has at most one parent, and for which there is a vertex \( r \) called the root such that for all \( t \in V(t) \) there is a path from \( r \) to \( t \). There is an obvious one-to-one correspondence between rooted trees and directed trees: For a rooted tree \( T \) with root \( r := r(T) \) we define the corresponding directed tree \( T' \) by \( V(T') := V(T) \) and \( E(T') := \{\{t, u\} \mid \{t, u\} \in E(T) \} \) and \( t \) occurs on the path \( rTu \). We freely jump back and forth between rooted trees and directed trees, depending on which will be more convenient. In particular, we use the terminology introduced for directed acyclic graphs (parents, children, the partial order \( \preceq, \text{ et cetera} \) for rooted trees.

2.2 Relational structures

A relational structure \( A \) consists of a finite set \( V(A) \) called the universe or vertex set of \( A \) and finitely many relations on \( A \). The only types of structures we will use in this paper are graphs, viewed as structures \( G = (V(G), E(G)) \) with one binary relation \( E(G) \), and ordered graphs, viewed as structures
$G = (V(G), E(G), \leq (G))$ with two binary relations $E(G)$ and $\leq (G)$, where $(V(G), E(G))$ is a graph and $\leq (G)$ is a linear order of the vertex set $V(G)$.

2.3 Logics

We assume that the reader has a basic knowledge in logic. In this section, we will informally introduce the two main logics IFP and IFP+C used in this paper. For background and a precise definition, I refer the reader to one of the textbooks [21, 28, 47, 52]. It will be convenient to start by briefly reviewing first-order logic $\mathbf{FO}$. Formulae of first-order logic in the language of graphs are built from atomic formulae $E(x,y)$ and $x = y$, expressing adjacency and equality of vertices, by the usual Boolean connectives and existential and universal quantifiers ranging over the vertices of a graph. First-order formulae in the language of ordered graphs may also contain atomic formulae of the form $x \leq y$ with the obvious meaning, and formulae in other languages may contain atomic formulae defined for these languages. We write $G \models \varphi(v_1, \ldots, v_k)$ to denote that the free variables of a formula $\varphi$ are among $x_1, \ldots, x_k$. For a graph $G$ and vertices $v_1, \ldots, v_k$, we write $G \models \varphi[v_1, \ldots, v_k]$ to denote that $G$ satisfies $\varphi$ if $x_i$ is interpreted by $v_i$, for all $i \in [k]$

Inflationary fixed-point logic IFP is the extension of FO by a fixed-point operator with an inflationary semantics. To introduce this operator, let $\varphi(X, \vec{x})$ be a formula that, besides a $k$-tuple $\vec{x} = (x_1, \ldots, x_k)$ of free individual variables ranging over the vertices of a graph, has a free $k$-ary relation variable ranging over $k$-ary relations on the vertex set. For every graph $G$ we define a sequence $R_i = R_i(G, \varphi, X, \vec{x})$, for $i \in \mathbb{N}_0$, of $k$-ary relations on $V(G)$ as follows:

$R_0 := \emptyset$

$R_{i+1} := R_i \cup \{ \vec{v} \mid G \models \varphi[R_i, \vec{v}] \}$ for all $i \in \mathbb{N}_0$.

Since we have $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots \subseteq V(G)^k$ and $V(G)$ is finite, the sequence reaches a fixed-point $R_\omega = R_{i+1} = R_i$ for all $i \geq n$, which we denote by $R_\omega = R_\omega(G, \varphi, X, \vec{x})$. The ifp-operator applied to $\varphi, X, \vec{x}$ defines this fixed-point. We use the following syntax:

$$\text{ifp} (X \leftarrow \vec{x} \mid \varphi) \vec{x}.$$  

Here $\vec{x}$ is another $k$-tuple of individual variables, which may coincide with $\vec{x}$. The variables in the tuple $\vec{x}$ are the free variables of the formula $\varphi(\vec{x})$, and for every graph $G$ and every tuple $\vec{v} \in V(G)^k$ of vertices we let $G \models \varphi[\vec{v}] \iff \vec{v} \in R_\omega$. These definitions can easily be extended to a situation where the formula $\varphi$ contains other free variables than $X$ and the variables in $\vec{x}$. These variables remain free variables of $\varphi$. Now formulae of inflationary fixed-point logic IFP in the language of graphs are built from atomic formulae $E(x,y)$, $x = y$, and $X\vec{x}$ for relation variables $X$ and tuples of individual variables $\vec{x}$ whose length matches the arity of $X$, by the usual Boolean connectives and existential and universal quantifiers ranging over the vertices of a graph, and the ifp-operator.

Example 2.1. The IFP-sentence

$$\text{conn} := \forall x_1 \forall x_2 \ \text{ifp} \left( X \leftarrow (x_1, x_2) \mid x_1 = x_2 \lor E(x_1, x_2) \lor \exists x_3 (X(x_1, x_3) \land X(x_3, x_2)) \right) (x_1, x_2)$$

states that a graph is connected.

Inflationary fixed-point logic with counting, IFP+C, is the extension of IFP by counting operators that allow it to speak about cardinalities of definable sets and relations. To define IFP+C, we interpret the logic IFP over two sorted extensions of graphs (or other relational structures) by a numerical sort. For a graph $G$, we let $N(G)$ be the initial segment $[0, |G|]$ of the nonnegative integers. We let $G^+$ be the two-sorted structure $G \cup (N(G), \leq)$, where $\leq$ is the natural linear order on $N(G)$. To avoid confusion, we always assume that $V(G)$ and $N(G)$ are disjoint. We call the elements of the first sort $V(G)$ vertices and the elements of the second sort $N(G)$ numbers. Individual variables of our logic range either over the set $V(G)$ of vertices of $G$ or over the set $N(G)$ of numbers of $G$. Relation variables may range over mixed relations, having certain places for vertices and certain places for numbers. Let us call the resulting logic, inflationary
fixed-point logic over the two-sorted extensions of graphs, IFP+. We may still view IFP+ as a logic over plain graphs, because the extension $G^+$ is uniquely determined by $G$. More precisely, we say that a sentence $\phi$ of IFP+ is satisfied by a graph $G$ if $G^+ \models \phi$. Inflationary fixed-point logic with counting IFP+C is the extension of IFP+ by counting terms formed as follows: For every formula $\phi$ and every vertex variable $x$ we add a term $\#x \phi$; the value of this term is the number of assignments to $x$ such that $\phi$ is satisfied.

With each IFP+C-sentence $\phi$ in the language of graphs we associate the graph property $\mathcal{P}_\phi := \{ G \mid G \models \phi \}$. As the set of all IFP+C-sentences is computable, we may thus view IFP+C as an abstract logic according to the definition given in Section 1.1. It is easy to see that IFP+C satisfies condition $[G.2]$ and therefore condition (G.2)$_\mathcal{C}$ for every class $\mathcal{C}$ of graphs. Thus to prove that IFP+C captures PTIME on a class $\mathcal{C}$ it suffices to verify (G.1)$_\mathcal{C}$.

In the following examples, we use the notational convention that $x$ and variants such as $x_1, x'$ denote vertex variables and that $y$ and variants denote number variables.

**Example 2.2.** The IFP+C-term $0 := \#x \lnot x = x$ defines the number $0 \in N(G)$. The formula

$$\text{succ}(y_1, y_2) := y_1 \leq y_2 \land \lnot y_1 = y_2 \land \forall y(y \leq y_1 \lor y_2 \leq y)$$

defines the successor relation associated with the linear order $\leq$. The following IFP+C-formula defines the set of even numbers in $N(G)$:

$$\text{even}(y) := \text{ifp}\left( Y \leftarrow y \mid y = 0 \lor \exists y' \exists y'' \left( (y') \land \text{succ}(y', y'') \land \text{succ}(y'', y) \right) \right).$$

**Example 2.3.** An Eulerian cycle in a graph is a closed walk on which every edge occurs exactly once. A graph is Eulerian if it has a Eulerian cycle. It is a well-known fact that a graph is Eulerian if and only if it is connected and every vertex has even degree. Then the following IFP+C-sentence defines the class of Eulerian graphs:

$$\text{eulerian} := \text{conn} \land \forall x_1 \text{even}(\#x_2 E(x_1, x_2)),$$

where conn is the sentence from Example 2.1 and even($y$) is the formula from Example 2.2. By standard techniques from finite model theory, it can be proved that the class of Eulerian graphs is neither definable in IFP nor in the counting extension FO+C of first-order logic.

### 2.4 Syntactical interpretations

In the following, $L$ is one of the logics IFP+C, IFP, or FO, and $\lambda, \mu$ are relational languages such as the language $\{ E \}$ of graphs or the language $\{ E, \leq \}$ of ordered graphs. An $L[\lambda]$-formula is an $L$-formula in the language $\lambda$, and similarly for $\mu$. We need some additional notation:

- Let $\approx \approx$ be an equivalence relation on a set $U$. For every $u \in U$, by $u / \approx$ we denote the $\approx$-equivalence class of $u$, and we let $U / \approx := \{ u / \approx \mid u \in U \}$ be the set of all equivalence classes. For a tuple $\bar{u} = (u_1, \ldots, u_k) \in U^k$ we let $\bar{u} / \approx := (u_1 / \approx, \ldots, u_k / \approx)$, and for a relation $R \subseteq U^k$ we let $R / \approx := \{ \bar{u} / \approx \mid \bar{u} \in R \}$.

- Two tuples $\bar{x} = (x_1, \ldots, x_k), \bar{y} = (y_1, \ldots, y_l)$ of individual variables have the same type if $k = \ell$ and for all $i \in [k]$ either both $x_i$ and $y_i$ range over vertices or both $x_i$ and $y_i$ range over numbers. For every structure $G$, we let $G^\bar{x}$ be the set of all tuples $\bar{a} \in (V(G) \cup N(G))^k$ such that for all $i \in [k]$ we have $a_i \in V(G)$ if $x_i$ is a vertex variable and $a_i \in N(G)$ if $x_i$ is a number variable.

**Definition 2.4.**

1. An $L$-interpretation of $\mu$ in $\lambda$ is a tuple

$$\Gamma(\bar{x}) = \left( \gamma_{\text{app}}(\bar{x}), \gamma_{\text{f}}(\bar{x}, \bar{y}), \gamma_{\text{z}}(\bar{x}, y_1, y_2), (\gamma_{R}(\bar{x}, \bar{y}_R))_{R \in \mu} \right),$$

of $L[\lambda]$-formulae, where $\bar{x}, \bar{y}, \bar{y}_1, \bar{y}_2$, and $\bar{y}_R$ for $R \in \mu$ are tuples of individual variables such that $\bar{y}, \bar{y}_1, \bar{y}_2$ all have the same type, and for every $k$-ary $R \in \mu$ the tuple $\bar{y}_R$ can be written as $\bar{y}_{R1} \ldots \bar{y}_{R,k}$, where the $\bar{y}_{R,i}$ have the same type as $\bar{y}$.

In the following, let $\Gamma(\bar{x})$ be an $L$-interpretation of $\mu$ in $\lambda$. Let $G$ be a $\lambda$-structure and $\bar{a} \in G^\bar{x}$:
(3) \( \Gamma(\bar{x}) \) is applicable to \((G, \bar{a})\) if \( G \models \gamma_{app}[\bar{a}] \).

(4) If \( \Gamma(\bar{x}) \) is applicable to \((G, \bar{a})\), we let \( \Gamma[G; \bar{a}] \) be the \( \mu \)-structure with vertex set

\[
V(\Gamma[G; \bar{a}]) := \{ \bar{b} \in G^\delta \mid G \models \psi[\bar{a}, \bar{b}] \} / \approx,
\]

where \( \approx \) is the reflexive, symmetric, transitive closure of \( \{(\bar{b}_1, \bar{b}_2) \in (G^\delta)^2 \mid G \models \gamma[\bar{a}, \bar{b}_1, \bar{b}_2]\} \).

Furthermore, for \( k \)-ary \( R \in \mu \), we let

\[
R(\Gamma[G; \bar{a}]) := \{ (\bar{b}_1, \ldots, \bar{b}_k) \in V(\Gamma[G; \bar{a}]) \mid G \models \gamma[R][\bar{a}, \bar{b}_1, \ldots, \bar{b}_k] \} / \approx.
\]

Syntactical interpretations map \( \lambda \)-structures to \( \mu \)-structures. The crucial observation is that they also induce a reverse translation from \( L[\mu] \)-formulae to \( L[\lambda] \)-formulae.

**Fact 2.5 (Lemma on Syntactical Interpretations).** Let \( \Gamma(\bar{x}) \) be an \( L \)-interpretation of \( \mu \) in \( \lambda \). Then for every \( L[\mu] \)-sentence \( \psi \) there is an \( L[\lambda] \)-formula \( \psi^{-1}(\bar{x}) \) such that the following holds for all \( \lambda \)-structures \( G \) and all tuples \( \bar{a} \in G^\delta \): If \( \Gamma(\bar{x}) \) is applicable to \((G, \bar{a})\), then

\[
G \models \psi^{-1}[\bar{a}] \iff \Gamma[G; \bar{a}] \models \psi.
\]

A proof of this fact for first-order logic can be found in [22]. The proof for the other logics considered here is an easy adaptation of the one for first-order logic.

### 2.5 Definable canonisation

A canonisation mapping for a class of graphs associates with every graph \( G \in \mathcal{C} \) an ordered copy of \( G \), that is, an ordered graph \((H, \leq)\) such that \( H \cong G \). We are interested in canonisation mappings definable in the logic \( \text{IFP+C} \) by syntactical interpretations of \( \{E, \leq\} \) in \( \{E\} \). The easiest way to define a canonisation mapping is by defining a linear order \( \leq \) on the universe of a structure \( G \) and then take \((G, \leq)\) as the canonical copy. However, defining an ordered copy of a structure is not the same as defining a linear order on the universe, as the following example illustrates:

**Example 2.6.** Let \( \mathcal{X} \) be the class of all complete graphs. It is easy to see that there is no \( \text{IFP+C} \)-formula \( \psi(x_1, x_2) \) such that for all \( K \in \mathcal{X} \) the binary relation \( \psi[K; x_1, x_2] \) is a linear order of \( V(K) \).

However, there is an \( \text{FO+C} \)-definable canonisation mapping for the class \( \mathcal{X} \): Let

\[
\Gamma = (\gamma_{app}, \gamma(\bar{y}), \gamma_\approx(\bar{y}_1, \bar{y}_2), \gamma_e(\bar{y}_1, \bar{y}_2), \gamma_c(\bar{y}_1, \bar{y}_2))
\]

be the numerical \( \text{FO+C} \)-interpretation of \( \{E, \leq\} \) in \( \{E\} \) defined by:

- \( \gamma_{app} := \forall x \, x = x \);
- \( \gamma(y) := 1 \leq y \land y \leq \text{ord}, \) where \( \text{ord} := \#x \, x = x \);
- \( \gamma_\approx(\bar{y}_1, \bar{y}_2) := y_1 = y_2 \);
- \( \gamma_e(\bar{y}_1, \bar{y}_2) := \neg y_1 = y_2 \);
- \( \gamma_c(\bar{y}_1, \bar{y}_2) := y_1 \leq y_2 \).

It is easy to see that the mapping \( K \mapsto \Gamma[K] \) is a canonisation mapping for the class \( \mathcal{X} \).

Our notion of definable canonisation slightly relaxes the requirement of defining a canonisation mapping: instead of just one ordered copy, we associate with each structure a parametrised family of polynomially many ordered copies.

**Definition 2.7.**

(1) Let \( \Gamma(\bar{x}) \) be an \( L \)-interpretation of \( \{E, \leq\} \) in \( \{E\} \). Then \( \Gamma(\bar{x}) \) canonises a graph \( G \) if there is at least one tuple \( \bar{a} \in G^\delta \) such that \( \Gamma(\bar{x}) \) is applicable to \((G, \bar{a})\), and for all tuples \( \bar{a} \in G^\delta \) such that \( \Gamma(\bar{x}) \) is applicable to \((G, \bar{a})\) it holds that \( \Gamma[G; \bar{a}] \) is an ordered copy of \( G \).
(2) A class $\mathcal{C}$ of graphs admits $L$-definable canonisation if there is an $L$-interpretation $\Gamma(\bar{x})$ of $\{E, \leq\}$ in $\{E\}$ that canonises all $G \in \mathcal{C}$.

The following well-known fact is a consequence of the Immerman-Vardi Theorem. It is used, at least implicitly, in [30, 31, 34, 48, 55]:

**Fact 2.8.** Let $\mathcal{C}$ be a class of graphs that admits IFP+C-definable canonisation. Then IFP+C captures $\text{PTIME}$ on $\mathcal{C}$.

### 3 Negative results

In this section, we prove that IFP+C does not capture $\text{PTIME}$ on the classes of chordal graphs and line graphs. Actually, our proof yields a more general result: Any logic that captures $\text{PTIME}$ on any of these two classes and that is “closed under first-order reductions” captures $\text{PTIME}$ on the class of all graphs. It will be obvious what we mean by “closed under first-order reductions” from the proofs, and it is also clear that most “natural” logics will satisfy this closure condition. It follows from our constructions that if there is a logic capturing $\text{PTIME}$ on one of the two classes, then there is a logic capturing $\text{PTIME}$ on all graphs.

Our negative results for IFP+C are based on the following theorem:

**Fact 3.1 (Cai, Fürer, and Immerman [9]).** There is a $\text{PTIME}$-decidable property $\mathcal{P}_{\text{CFI}}$ of graphs that is not definable in IFP+C.

Without loss of generality we assume that all $G \in \mathcal{P}_{\text{CFI}}$ are connected and of order at least 4.

#### 3.1 Chordal graphs

Let us denote the class of chordal graphs by $\mathcal{CD}$.

For every graph $G$, we define a graph $\hat{G}$ as follows:

- $V(\hat{G}):= V(G) \cup \{v_e \mid e \in E(G)\}$, where for each $e \in E(G)$ we let $v_e$ be a new vertex;
- $E(\hat{G}):= \left(\frac{|V(G)|}{2}\right) \cup \{v, v_e \mid v \in V(G), e \in E(G), v \in e\}$.

The following lemmas collect the properties of the transformation $G \mapsto \hat{G}$ that we need here. We leave the straightforward proofs to the reader.

**Lemma 3.2.** For every graph $G$ the graph $\hat{G}$ is chordal.

Note that for the graphs $K_2$ and $l_3 := ([3], \emptyset)$ it holds that $\hat{K}_2 \cong \hat{l}_3 \cong K_3$. It turns out that $K_2$ and $l_3$ are the only two nonisomorphic graphs that have isomorphic images under the mapping $G \mapsto \hat{G}$. It is easy to verify this by observing that for $G$ with $|G| \geq 4$ and $v \in V(\hat{G})$, it holds that $v \in V(G)$ if and only if $\deg(v) \geq 3$. Let $\hat{\mathcal{G}}$ be the class of all graphs $H$ such that $H \cong \hat{G}$ for some graph $G$.

**Lemma 3.3.** The class $\hat{\mathcal{G}}$ is polynomial time decidable. Furthermore, there is a polynomial time algorithm that, given a graph $H \in \hat{\mathcal{G}}$, computes the unique (up to isomorphism) graph $G \in \mathcal{G} \setminus \{K \mid K \cong K_2\}$ with $\hat{G} \cong H$.

**Lemma 3.4.** There is an FO-interpretation $\hat{\Gamma}$ of $\{E\}$ in $\{E\}$ such that for all graphs $G$ it holds that $\hat{\Gamma}[G] \cong \hat{G}$.

**Theorem 3.5.** IFP+C does not capture $\text{PTIME}$ on the class $\mathcal{CD}$ of chordal graphs.

**Proof.** Let $\mathcal{P}_{\text{CFI}}$ be the graph property of Fact 3.1 that separates $\text{PTIME}$ from IFP+C. Note that $K_2 \notin \mathcal{P}_{\text{CFI}}$ by our assumption that all graphs in $\mathcal{P}_{\text{CFI}}$ have order at least 4. By Lemma 3.3 the class $\mathcal{G} := \{H \mid H \cong \hat{G} \text{ for some } G \in \mathcal{P}_{\text{CFI}}\}$ is a polynomial time decidable subclass of $\mathcal{CD}$.

Suppose for contradiction that IFP+C captures polynomial time on $\mathcal{CD}$. Then by (G.1) there is an IFP+C-sentence $\phi$ such that for all chordal graphs $G$ it holds that $G \models \phi \iff G \in \mathcal{G}$. We apply the Lemma...
on Syntactical Interpretations to \( \varphi \) and the interpretation \( \hat{\Gamma} \) of Lemma 3.4 and obtain an \( \text{IFP+C} \)-sentence \( \varphi^{-\hat{\Gamma}} \) such that for all graphs \( G \) it holds that

\[
G \models \varphi^{-\hat{\Gamma}} \iff \hat{\Gamma} \cong \varphi.
\]

Thus \( \varphi^{-\hat{\Gamma}} \) defines \( \mathcal{P}_{\text{CFI}} \), which is a contradiction.

3.2 Line graphs

Let \( \mathcal{L} \) denote the class of all line graphs, or more precisely, the class of all graphs \( L \) such that there is a graph \( G \) with \( L \cong L(G) \). Observe that a triangle and a claw have the same line graph, a triangle. Whitney [62] proved that for all nonisomorphic connected graphs \( G, H \) except the claw and triangle, the line graphs of \( G \) and \( H \) are nonisomorphic. The following fact, corresponding to Lemma 3.3, is essentially an algorithmic version of Whitney’s result:

**Fact 3.6 (Roussopoulos [59]).** The class \( \mathcal{L} \) is polynomial time decidable. Furthermore, there is a polynomial time algorithm that, given a connected graph \( H \in \mathcal{L} \), computes the unique (up to isomorphism) graph \( G \in \mathcal{G} \setminus \{K_1, K_2\} \) with \( L(G) \cong H \).

**Lemma 3.7.** There is an \( \text{FO} \)-interpretation \( \Lambda \) of \{\( E \)\} in \{\( E \)\} such that for all graphs \( G \) it holds that \( \Lambda[G] \cong L(G) \).

**Proof.** We define \( \Lambda := (\lambda_{\text{app}}, \lambda_{\nu}(y_1, y_2), \lambda_{\approx}(y_1, y_2, y_1', y_2'), \lambda_{E}(y_1, y_2, y_1', y_2')) \) by:

- \( \lambda_{\text{app}} := \forall x x = x; \)
- \( \lambda_{\nu}(y_1, y_2) := E(y_1, y_2); \)
- \( \lambda_{\approx}(y_1, y_2, y_1', y_2') := (y_1 = y_1' \land y_2 = y_2') \lor (y_1 = y_2' \land y_2 = y_1'); \)
- \( \lambda_{E}(y_1, y_2, y_1', y_2') := (y_1 = y_1' \land \neg y_2 = y_2') \lor (y_2 = y_2' \land \neg y_1 = y_1') \lor (y_1 = y_2' \land \neg y_2 = y_1') \lor (y_2 = y_1' \land \neg y_2 = y_1'). \)

**Theorem 3.8.** \( \text{IFP+C} \) does not capture \( \text{PTIME} \) on the class \( \mathcal{L} \) of line graphs.

**Proof.** The proof is completely analogous to the proof of Theorem 3.5 using Fact 3.6 and Lemma 3.7 instead of Lemmas 3.3 and 3.4

4 Capturing polynomial time on chordal line graphs

In this section, we shall prove that \( \text{IFP+C} \) captures \( \text{PTIME} \) on the class \( \mathcal{C} \cap \mathcal{L} \) of graphs that are both chordal and line graphs. As we will see, such graphs have a simple treelike structure. We can exploit this structure and canonise the graphs in \( \mathcal{C} \cap \mathcal{L} \) in a similar way as trees or graphs of bounded tree width.

**Example 4.1.** Figure 4.1 shows an example of a chordal line graph.

4.1 On the structure of chordal line graphs

It is a well-known fact that chordal graphs can be decomposed into cliques arranged in a tree-like manner. To state this formally, we review tree decompositions of graphs. A tree decomposition of a graph \( G \) is a pair \((T, \beta)\), where \( T \) is a tree and \( \beta : V(T) \to 2^{V(G)} \) is a mapping such that the following two conditions are satisfied:

(T.1) For every \( v \in V(G) \) the set \( \{t \in V(T) \mid v \in \beta(t)\} \) is connected in \( T \).

(T.2) For every \( e \in E(G) \) there is a \( t \in V(T) \) such that \( e \subseteq \beta(t) \).
Lemma 4.3. \( \text{Following lemma:} \) we combine Fact 4.2 with the observations about tree decomposition stated before the fact, we obtain the following:

The set \( \beta(t) \), for \( t \in V(T) \), are called the bags of the decomposition. It will be convenient for us to always assume the tree \( T \) in a tree decomposition to be rooted. This gives us the partial tree order \( \preceq_T \). We introduce some additional notation. Let \( (T, \beta) \) be a tree decomposition of a graph \( G \). For every \( t \in V(T) \) we let:

\[ \gamma(t) := \bigcup_{u \in V(T) \text{ with } t \preceq_T u} \beta(u), \]

The set \( \gamma(t) \) is called the cone of \( (T, \beta) \) at \( t \). It easy to see that for every \( t \in V(T) \setminus \{r(T)\} \) with parent \( s \) the set \( \beta(t) \cap \beta(s) \) separates \( \gamma(t) \) from \( V(G) \setminus \gamma(t) \). Furthermore, for every clique \( X \) of \( G \) there is a \( t \in V(T) \) such that \( X \subseteq \beta(t) \). (See Diestel’s textbook [20] for proofs of these facts and background on tree decompositions.) Another useful fact is that every tree decomposition \( (T, \beta) \) of a graph \( G \) can be transformed into a tree decomposition \( (T', \beta') \) such that for all \( t' \in V(T') \) there exists a \( t \in V(T) \) such that \( \beta'(t') = \beta(t) \), and for all \( t, u \in V(T') \) with \( t \neq u \) it holds that \( \beta'(t) \not\subseteq \beta'(u) \).

**Fact 4.2.** A nonempty graph \( G \) is chordal if and only if \( G \) has a tree decomposition into cliques, that is, a tree decomposition \( (T, \beta) \) such that for all \( t \in V(T) \) the bag \( \beta(t) \) is a clique of \( G \).

For a graph \( G \), we let \( MCL(G) \) be the set of all maximal cliques in \( G \) with respect to set inclusion. If we combine Fact 4.2 with the observations about tree decomposition stated before the fact, we obtain the following lemma:

**Lemma 4.3.** Let \( G \) be a nonempty chordal graph. Then \( G \) has a tree decomposition \( (T, \beta) \) with the following properties:

(i) For every \( t \in V(T) \) it holds that \( \beta(t) \in MCL(G) \).

(ii) For every \( X \in MCL(G) \) there is exactly one \( t \in V(T) \) such that \( \beta(t) = X \).

We call a tree decomposition satisfying conditions (i) and (ii) a good tree decomposition of \( G \).

Let us now turn to line graphs. Let \( L := L(G) \) be the line graph of a graph \( G \). For every \( v \in V(G) \), let \( L(v) := \{e \in E(G) : v \in e\} \subseteq V(L) \). Unless \( v \) is an isolated vertex, \( L(v) \) is a clique in \( L \). Furthermore, we have

\[ L = \bigcup_{v \in V(G)} L[v]. \]

Observe that for all \( v, w \in V(G) \), if \( e := \{v, w\} \in E(G) \) then \( X(v) \cap X(w) = \{e\} \), and if \( \{v, w\} \not\in E(G) \) then \( X(v) \cap X(w) = \emptyset \). The following proposition, which is probably well-known, characterises the line graphs that are chordal:

**Proposition 4.4.** Let \( L = L(G) \in \mathcal{L} \). Then

\[ L \in \mathcal{CD} \iff \text{all cycles in } G \text{ are triangles}. \]
Note that on the right hand side, we do not only consider chordless cycles.

**Proof.** For the forward direction, suppose that $L \in \mathcal{G}$, and let $C \subseteq G$ be a cycle. Then $L[E(C)]$ is a chordless cycle in $L$. Hence $|C| \leq 3$, that is, $C$ is a triangle.

For the backward direction, suppose that all cycles in $G$ are triangles, and let $C \subseteq L$ be a chordless cycle of length $k$. Let $e_1, \ldots, e_k$ be the vertices of $C$ in cyclic order. To simplify the notation, let $e_0 := e_k$. Then for all $i \in [k]$ it holds that $\{e_{i-1}, e_i\} \in E(L)$ and thus $e_{i-1} \cap e_i \neq \emptyset$. Let $v_0, v_1 \in V(G)$ such that $e_1 = \{v_0, v_1\}$, and for $i \in [2, k]$, let $v_i \in e_i \setminus e_{i-1}$. Then $v_i \neq v_j$ for all $j \in [i-2]$, and if $i < k$ even for $j \in [0, i-2]$, because the cycle $C$ is chordless and thus $e_i \cap e_j = \emptyset$. Furthermore, $v_k = v_0$. Thus $\{v_1, \ldots, v_k\}$ is the vertex set of a cycle in $G$, and we have $k = 3$.

**Lemma 4.6.** Let $L = L(G) \in \mathcal{G} \cap \mathcal{L}$, and let $X \in \text{MCL}(L)$ and $e = \{v, w\} \in X$. Then $X = X(v)$ or $X = X(w)$ or there is an $x \in V(G)$ such that $\{x, v\}, \{x, w\} \in E(G)$ and $X = \{e, \{x, v\}, \{x, w\}\}$.

**Proof.** For all $f \in X$, either $v \in f$ or $w \in f$, because $f$ is adjacent to $e$. Hence $X \subseteq X(v) \cup X(w)$. If $X \subseteq X(v)$, then $X = X(v)$ by the maximality of $X$. Similarly, if $X \subseteq X(w)$ then $X = X(w)$. Suppose that $X \setminus X(v) \neq \emptyset$ and $X \setminus X(w) \neq \emptyset$. Let $f \in X \setminus X(v)$ and $g \in X \setminus X(w)$. As $X$ is a clique, we have $\{f, g\} \in E(L)$ and thus $f \cap g \neq \emptyset$. Hence there is an $x \in V(G)$ such that $f = \{x, w\}$ and $g = \{x, v\}$. Furthermore, $X = \{e, f, g\}$. To see this, let $h \in X$. Then $\{h, e\} \in E(L)$ and thus $v \in h$ or $w \in h$. Say, $v \in h$. If $w \in h$, then $h = e$. Otherwise, we have $x \in h$, because $h$ is adjacent to $g$. Thus $h = g$.

**Lemma 4.5.** Let $L = L(G) \in \mathcal{G} \cap \mathcal{L}$, and let $X_1, X_2 \in \text{MCL}(L)$ be distinct. Then $|X_1 \cap X_2| \leq 2$.

**Proof.** Let $L = L(G)$ for some graph $G$. Suppose for contradiction that $|X_1 \cap X_2| \geq 3$. Then $|X_1|, |X_2| \geq 4$, because $X_1$ and $X_2$ are distinct maximal cliques. By Lemma 4.5, it follows that there are vertices $v_1, v_2 \in V(G)$ such that $X_1 = X(v_1)$ and $X_2 = X(v_2)$, which implies $|X_1 \cap X_2| \leq 1$. This is a contradiction.

**Lemma 4.7.** Let $L \in \mathcal{G} \cap \mathcal{L}$, and let $X_1, X_2, X_3 \in \text{MCL}(L)$ be pairwise distinct such that $X_1 \cap X_2 \cap X_3 = \emptyset$. Then there are $i, j, k$ such that $\{i, j, k\} = [3]$ and $X_i \subseteq X_j \cup X_k$ and $|X_i| = 3$.

**Proof.** Let $L = L(G)$ for some graph $G$. Let $e \in X_1 \cap X_2 \cap X_3$. Suppose that $e = \{v, w\} \in E(G)$. As the cliques $X_1, X_2, X_3$ are distinct, it follows from Lemma 4.5 that there is an $i \in [3]$ and an $x \in V(G)$ such that $X_i = \{e, \{x, v\}, \{x, w\}\}$. Choose such $i$ and $x$.

**Claim 1.** For all $j \in [3] \setminus \{i\}$, either $X_j = X(v)$ or $X_j = X(w)$.

**Proof.** Suppose for contradiction that $X_j \neq X(v)$ and $X_j \neq X(w)$. Then by Lemma 4.5, there exists a $y \in V(G)$ such that $\{y, v\}, \{y, w\} \in E(G)$ and $X_j = \{e, \{y, v\}, \{y, w\}\}$. But then $L[\{y, v\}, \{v, x\}, \{x, w\}, \{w, y\}]$ is a chordless cycle in $L$, which contradicts $L$ being chordal. Thus there are $j, k$ such that $\{i, j, k\} = [3]$ and $X_j = X(v)$ and $X_k = X(w)$. Then $X_i \subseteq X_j \cup X_k$.

**Lemma 4.8.** Let $L \in \mathcal{G} \cap \mathcal{L}$. Then every good tree decomposition $(T, B)$ of $L$ satisfies the following conditions (in addition to conditions (i) and (ii) of Lemma 4.3):

(iii) For all $t \in V(T)$,

- either $|B(t)| = 3$ and $t$ has at most three neighbours in $T$ (the neighbours of a node are its children and the parent),
- or for all distinct neighbours $u, u'$ of $t$ in $T$ it holds that $B(u) \cap B(u') = \emptyset$.

(iv) For all $t, u \in V(T)$ with $t \neq u$ it holds that $|B(t) \cap B(u)| \leq 2$.

**Proof.** Let $(T, B)$ be a good tree decomposition of $L$. Such a decomposition exists because $L$ is chordal. As all bags of the decomposition are maximal cliques of $L$, condition (iii) follows from Lemma 4.7 and condition (iv) follows from Lemma 4.6.
4.2 Canonisation

Theorem 4.9. The class \( \mathcal{CD} \cap \mathcal{L} \) of all chordal line graphs admits \( \text{IFP+C} \)-definable canonisation.

Corollary 4.10. \( \text{IFP+C} \) captures \( \text{PTIME} \) on the class of all chordal line graphs.

Proof of Theorem 4.9. The proof resembles the proof that classes of graphs of bounded tree width admit \( \text{IFP+C} \)-definable canonisation [31] and also the proof of Theorem 7.2 (the “Second Lifting Theorem”) in [31]. Both of these proofs are generalisations of the simple proof that the class of trees admits \( \text{IFP+C}\)-definable canonisation (see, for example, [36]). We shall describe an inductive construction that associates with each chordal line graph \( G \) a canonical copy \( G' \) whose universe is an initial segment of the natural numbers. For readers with some experience in finite model theory, it will be straightforward to formalise the construction in \( \text{IFP+C} \). We only describe the canonisation of \( \text{connected} \) chordal line graphs that are not complete graphs. It is easy to extend it to arbitrary chordal line graphs. For complete graphs, which are chordal line graphs, cf. Example 2.6.

To describe the construction, we fix a connected graph \( G \in \mathcal{CD} \cap \mathcal{L} \) that is not a complete graph. Note that this implies \(|G| \geq 3\). Let \( (T, \beta^T) \) be a good tree decomposition of \( G \). As \( G \) is not a complete graph, we have \(|T| \geq 2\). Without loss of generality we may assume that the root \( r(T) \) has exactly one child in \( T \), because every tree has at least one node of degree at most 1 and properties (i), (ii) of a good decomposition do not depend on the choice of the root. It will be convenient to view the rooted tree \( T \) as a directed graph, where the edges are directed from parents to children.

Let \( U \) be the set of all triples \( (u_1, u_2, u_3) \in V(G)^3 \) such that \( u_3 \neq u_1, u_2 \) (possibly, \( u_1 = u_2 \)), and there is a unique \( X \in \text{MCL}(G) \) such that \( u_1, u_2, u_3 \in X \). For all \( \tilde{u} = (u_1, u_2, u_3) \in U \), let \( A(\tilde{u}) \) be the connected component of \( G \setminus \{u_1, u_2\} \) that contains \( u_3 \) (possibly, \( A(\tilde{u}) = G \setminus \{u_1, u_2\} \)). We define mappings \( \sigma^U, \alpha^U, \gamma^U, \beta^U : U \to 2^V(G) \) as follows: For all \( \tilde{u} = (u_1, u_2, u_3) \in U \), we let \( \sigma^U(\tilde{u}) := \{u_1, u_2\} \) and \( \alpha^U(\tilde{u}) := V(A(\tilde{u})) \). We let \( \gamma^U(\tilde{u}) := \sigma^U(\tilde{u}) \cup \alpha^U(\tilde{u}) \), and we let \( \beta^U(\tilde{u}) \) the unique \( X \in \text{MCL}(G) \) with \( u_1, u_2, u_3 \in X \). We define a partial order \( \preceq \) on \( U \) by letting \( \tilde{u} \preceq \tilde{v} \) if and only if \( \tilde{u} = \tilde{v} \) or \( \alpha(\tilde{u}) \supseteq \alpha(\tilde{v}) \). We let \( F \) be the successor relation of \( \preceq \), that is, \( (\tilde{u}, \tilde{v}) \in F \) if \( \tilde{u} \prec \tilde{v} \) and there is no \( \tilde{w} \in U \setminus \{\tilde{u}, \tilde{v}\} \) such that \( \tilde{u} \prec \tilde{w} \prec \tilde{v} \).

Finally, we let \( D := (U, F) \). Then \( D \) is a directed acyclic graph. It is easy to verify that for all \( \tilde{u} \in U \) we have

\[
\beta^U(\tilde{u}) = \gamma^U(\tilde{u}) \setminus \bigcup_{\tilde{v} \in N^D(\tilde{u})} \alpha^U(\tilde{v}),
\]

where \( N^D(\tilde{u}) = \{ \tilde{v} \in U \mid (\tilde{u}, \tilde{v}) \in F \} \).

Recall that we also have mappings \( \beta^T, \gamma^T : V(T) \to 2^V(G) \) derived from the tree decomposition. We define a mapping \( \sigma^T : V(T) \to 2^V(G) \) as follows:

- For a node \( t \in V(T) \setminus \{r(T)\} \) with parent \( s \), we let \( \sigma^T(t) := \beta^T(t) \cup \beta^T(s) \).
- For the root \( r := r(T) \), we first define a set \( S \subseteq V(G) \) by letting \( S := \beta^T(r) \setminus \beta^T(t) \), where \( t \) is the unique child of \( r \). (Remember our assumption that \( r \) has exactly one child.) Then if \(|S| \geq 2\), we choose distinct \( v, v' \in S \) and let \( \sigma^T(r) := \{v, v'\} \), and if \(|S| = 1\) we let \( \sigma^T(r) := S \).

Note that \( \beta^T(t) \setminus \sigma^T(t) \neq \emptyset \) and 1 \leq |\sigma^T(t)| \leq 2 for all \( t \in V(T) \). For the root, this follows immediately from the definition of \( \sigma^T(t) \), and for nodes \( t \in V(T) \setminus \{r(T)\} \) it follows from Lemma 4.8. We define a mapping \( \alpha^T : V(T) \to 2^V(G) \) by letting \( \alpha^T(t) := \gamma^T(t) \setminus \sigma^T(t) \) for all \( t \in V(T) \). We define a mapping \( g : V(T) \to U \) by choosing, for every node \( t \in V(T) \), vertices \( u_1, u_2 \) such that \( \sigma^T(t) = \{u_1, u_2\} \) (possibly \( u_1 = u_2 \)) and a vertex \( u_3 \in \beta(t) \setminus \sigma(t) \) and letting \( g(t) := \{u_1, u_2, u_3\} \). Note that \( \{u_1, u_2, u_3\} \in U \), because \( \beta^T(t) \) is the unique maximal clique in \( \text{MCL}(G) \) that contains \( u_1, u_2, u_3 \).

Claim 1. The mapping \( g \) is a directed graph embedding of \( T \) into \( D \). Furthermore, for all \( t \in V(T) \) it holds that \( \alpha^T(t) = \alpha^U(g(t)) \), \( \beta^T(t) = \beta^U(g(t)) \), \( \gamma^T(t) = \gamma^U(g(t)) \), and \( \sigma^T(t) = \sigma^U(g(t)) \).

Proof. We leave the straightforward inductive proof to the reader.

Let \( \tilde{u}_0 := g(r(T)) \), and let \( U_0 \) be the subset of \( U \) consisting of all \( \tilde{u} \in U \) such that \( \tilde{u}_0 \preceq \tilde{u} \). Let \( D_0 \) be the restriction of \( F \) to \( U_0 \) and \( D_0 := (U_0, D_0) \). Note that \( U_0 \) is upward closed with respect to \( \preceq \) and that \( g(T) \subseteq D_0 \).
Claim 2. There is a mapping \( h : U_0 \to V(T) \) such that \( h \) is a directed graph homomorphism from \( D_0 \) to \( T \) and \( h \circ g \) is the identity mapping on \( V(T) \). Furthermore, for all \( \bar{u} \in U_0 \) it holds that \( \alpha^U(\bar{u}) = \alpha^T(h(\bar{u})) \), \( \beta^U(\bar{u}) = \beta^T(h(\bar{u})) \), \( \gamma^U(\bar{u}) = \gamma^T(h(\bar{u})) \), and \( \sigma^U(\bar{u}) = \sigma^T(h(\bar{u})) \).

Proof. We define \( h \) by induction on the partial order \( \leq \). The unique \( \leq \)-minimal element of \( U_0 \) is \( \bar{u}_0 \). We let \( h(\bar{u}_0) := r(T) \). Now let \( \bar{v} = (v_1, v_2, v_3) \in U_0 \), and suppose that \( h(\bar{u}) \) is defined for all \( \bar{u} \in U_0 \) with \( \bar{u} \leq \bar{v} \). Let \( \bar{u} \in U_0 \) such that \( (\bar{u}, \bar{v}) \in F_0 \), and let \( s := h(\bar{u}) \). By the induction hypothesis, we have \( \alpha^U(\bar{u}) = \alpha^T(s) \), \( \beta^U(\bar{u}) = \beta^T(s) \), \( \gamma^U(\bar{u}) = \gamma^T(s) \), and \( \sigma^U(\bar{v}) = \sigma^T(s) \). The set \( \alpha^U(\bar{v}) \) is the vertex set of a connected component of \( G \setminus \sigma^U(\bar{v}) \) which is contained in \( \alpha^U(\bar{u}) \subseteq \gamma^U(\bar{u}) \subseteq \gamma^T(s) \), and by (4.1) it holds that \( \alpha^U(\bar{v}) \cap \beta^U(\bar{u}) = \emptyset \). Hence there is a child \( t \) of \( s \) such that \( \alpha^U(\bar{v}) \subseteq \alpha^U(t) \). Let \( \bar{v} := g(t) \). If \( \alpha^U(\bar{v}) \subseteq \alpha^T(t) = \alpha^U(\bar{v}) \), then \( \bar{u} \leq \bar{u} \leq \bar{v} \), which contradicts \( (\bar{u}, \bar{v}) \in F_0 \). Hence \( \alpha^U(\bar{v}) = \alpha^T(t) \) and thus \( \sigma^U(\bar{v}) = \sigma^T(t) \). This also implies \( \gamma^U(\bar{v}) = \gamma^T(t) \) and \( \beta^U(\bar{v}) = \beta^T(t) \). We let \( h(\bar{v}) := t \).

To prove that \( h \) is really a homomorphism, it remains to prove that for all \( \bar{u} \in U_0 \) with \( (\bar{u}, \bar{v}) \in F_0 \) we also have \( h(\bar{u}) = s \). So let \( \bar{u} \in U_0 \) with \( (\bar{u}, \bar{v}) \in F_0 \), and let \( s := h(\bar{u}) \). Suppose for contradiction that \( s \neq s' \). If \( s' \not\leq s \) then \( \alpha^U(\bar{v}) \cap \alpha^U(\bar{v}) = \emptyset \) and thus \( \bar{u} \leq \bar{u} \), which contradicts \( (\bar{u}, \bar{v}) \in F_0 \). Thus \( s' \not\leq s \) and similarly \( s \not\leq s' \). But then both \( \sigma^U(s) \) and \( \sigma^U(s) \) separate \( \gamma^U(s) \) from \( \gamma^U(s') \) in \( G \). This contradicts \( \alpha^U(\bar{v}) \subseteq \gamma^T(s) \cap \gamma^T(s') \subseteq (\gamma^U(s) \cap \gamma^U(s')) \setminus (\sigma^U(s) \cap \sigma^U(s')) \).

Thus essentially, the “tree-like” decomposition \( (D_0, \beta^U) \) is the same as the tree decomposition \( (T, \beta^T) \). However, the decomposition \( (D_0, \beta^U) \) is IFP-definable with three parameters fixing the tuple \( t_0 = g(r(T)) \).

Let us now turn to the canonicalisation. For every \( \bar{u} \in U_0 \), we let \( G(\bar{u}) := G(\gamma(\bar{u})) \). Then \( G = G(\bar{u}_0) \). We inductively define for every \( \bar{u} = (u_1, u_2, u_3) \in U_0 \) a graph \( H(\bar{u}) \) with the following properties:

(i) \( V(H(\bar{u})) = [n] \), where \( n := |\gamma(\bar{u})| = |V(G(\bar{u}))| \).

(ii) There is an isomorphism \( f_2 \) from \( G(\bar{u}) \) to \( H(\bar{u}) \) such that if \( u_1 \neq u_2 \) then \( f_2(u_1) = 1 \) and \( f_2(u_2) = 2 \), and if \( u_1 = u_2 \) it holds that \( f_2(u_1) = 1 \).

For the induction basis, let \( \bar{u} \in U_0 \) with \( N^{D_0}(\bar{u}) = \emptyset \). Then \( \gamma^U(\bar{u}) = \beta^U(\bar{u}) \), and \( G(\bar{u}) = K[\beta^U(\bar{u})] \). Let \( n := n := |\gamma(\bar{u})| = |\beta^U(\bar{u})| \) and \( H(\bar{u}) := K_n \). Then (i) and (ii) are obviously satisfied.

For the induction step, let \( \bar{u} \in U_0 \) and \( N^{D_0}(\bar{u}) = \{v_1, \ldots, v_\ell \} \neq \emptyset \). It follows from Claim 2 that for all \( i, j \in [n] \), either \( \gamma(\bar{v}) = \gamma(\bar{v}) \) or \( \gamma(\bar{v}) \cap \gamma(\bar{v}) = \sigma(\bar{v}) \cap \sigma(\bar{v}) \). We may assume without loss of generality that there are \( i_1, \ldots, i_m \in [n] \) such that \( i_1 < i_2 < \ldots < i_m \) and for all \( j, j' \in [m] \) with \( j \neq j' \) we have \( \gamma(\bar{v}) \neq \gamma(\bar{v}) \). For all \( j \in [m] \), \( i \in [j, i_{j+1} - 1] \) we have \( \gamma(\bar{v}) = \gamma(\bar{v}) \). Here and in the following we let \( i_{m+1} := n + 1 \).

The class of all graphs whose vertex set is a subset of \( \mathbb{N} \) may be ordered lexicographically; we let \( H \leq_{\text{lex}} H' \) if either \( V(H) \) is lexicographically smaller than \( V(H') \), that is, the first element of the symmetric difference \( V(H) \triangle V(H') \) belongs to \( V(H') \), or \( V(H) = V(H') \) and \( E(H) \) is lexicographically smaller than \( E(H') \) with respect to the lexicographical ordering of unordered pairs of natural numbers, or \( H = H' \).

Without loss of generality we may assume that for each \( j \in [m] \) it holds that

\[
H(\bar{v}) \leq_{\text{lex}} H(\bar{v}) \leq_{\text{lex}} H(\bar{v}) \leq_{\text{lex}} \ldots \leq_{\text{lex}} H(\bar{v})
\]

and, furthermore,

\[
H(\bar{v}) \leq_{\text{lex}} H(\bar{v}) \leq_{\text{lex}} \ldots \leq_{\text{lex}} H(\bar{v})
\]

Note that, even though the graphs \( G(\bar{v}) \), \( G(\bar{v}) \), \( G(\bar{v}) \), \( G(\bar{v}) \) are vertex disjoint subgraphs of \( G(\bar{u}) \), they may be isomorphic, and hence not all of the inequalities in (4.2) need to be strict. For all \( j \in [m] \), let \( j := \bar{v} \) and \( G_1 := G(\bar{v}) \) an \( H_i := H(\bar{v}) \). Then \( H_1 \leq_{\text{lex}} H_2 \leq_{\text{lex}} \ldots \leq_{\text{lex}} H_m \). Let \( j_1, \ldots, j_\ell \in [m] \) such that \( j_1 < j_2 < \ldots < j_\ell \) and \( H_j := H_{j}\) for all \( i \in [\ell] \), \( j \in [j_1, j_{i+1} - 1] \), where \( j_{i+1} = m + 1 \), and \( H_j \neq H_{j_i} \) for all \( i \in [\ell - 1] \). For all \( i \in [\ell] \), let \( H_j := H_{j_i} \). Furthermore, let \( n_i := |H_j| \) and \( k_i := j_{i+1} - j_i \) and \( q_i := |\sigma^U(\bar{v})| \) and

\[ q := \left| \beta^U(\bar{u}) \cup \bigcup_{j=1}^m \beta^U(\bar{v}_j) \right|. \]
Case 1: For all neighbours \(t, t'\) of \(h(\bar{u})\) in the undirected tree underlying \(T\) it holds that \(\beta^T(t) \cap \beta^T(t') = \emptyset\).

We define \(H(\bar{u})\) by first taking a complete graph \(K_q\), then \(k_1\) copies of \(J_1\), then \(k_2\) copies of \(J_2\), etcetera, and finally \(k_i\) copies of \(J_i\). The universes of all these copies are disjoint, consecutive intervals of natural numbers. Let \(K\) be the union of \([q]\) with the first \(q_i\) vertices of each of the \(k_i\) copies of \(J_i\) for all \(i \in [t]\). Then \(K\) is the set of vertices of \(H(\bar{u})\) that corresponds to the clique \(\beta(\bar{u})\). We add edges among the vertices in \(K\) to turn it into a clique. It is not hard to verify that the resulting structure satisfies (i) and (ii).

Case 2: There are neighbours \(t, t'\) of \(h(\bar{u})\) in the undirected tree underlying \(T\) such that \(\beta^T(t) \cap \beta^T(t') \neq \emptyset\).

Then by Lemma 4.8(iii) we have \(|\beta^U(\bar{u})| = 3\), and \(h(\bar{u})\) has at most two children. Hence \(m \leq 2\), and essentially this means we only have two possibilities of how to combine the parts \(H_1, H_2\) to the graph \(H(\bar{u})\); either \(H_1\) comes first or \(H_2\). We choose the lexicographically smaller possibility. We omit the details.

This completes our description of the construction of the graphs \(H(\bar{u})\).

It remains to prove that \(H(\bar{u})\) is \(\text{IFP+C}\)-definable. We first define \(\text{IFP}\)-formulae \(\theta_U(x, y), \theta_F(x, y), \theta_a(x, y), \theta_\beta(x, y), \theta_\gamma(x, y), \theta_\sigma(x, y)\) such that

\[
U = \{ \bar{u} \in V(G)^3 \mid G \models \theta_U[\bar{u}] \},
F = \{ (\bar{u}, \bar{v}) \in U^2 \mid G \models \theta_F[\bar{u}, \bar{v}] \},
\]

and similarly for \(\beta, \gamma, \sigma\). Then we define formulae \(\theta_U^0(x_0, \bar{x}), \theta_F^0(x_0, \bar{x})\) that define \(D_0\). We have no canonical way of checking that a tuple \(\bar{u}_0\) really is the image \(g(T)\) of the root of a good tree decomposition, but all we need is that the graph \(D^0(\bar{u}_0)\) with vertex set \(\{ \bar{u} \in V(G)^3 \mid G \models \theta_U^0[\bar{u}_0, \bar{u}] \}\) and edge set \(\{ (\bar{u}, \bar{v}) \in U^2 \mid G \models \theta_F[\bar{u}_0, \bar{u}, \bar{v}] \}\) has the properties we derive from \(T\) being a good tree decomposition. In particular, if a node \(\bar{u}\) has a child \(\bar{v}\) with \(\sigma^U(\bar{v}) \cap \sigma^U(\bar{v}) \neq \emptyset\) or children \(\bar{v}_1 \neq \bar{v}_2\) with \(\sigma^U(\bar{v}_1) \cap \sigma^U(\bar{v}_2) \neq \emptyset\), then \(|\beta^U(\bar{u})| \leq 3\). Once we have defined \(D^0\), it is straightforward to formalise the definition of the graphs \(H(\bar{u})\) in \(\text{IFP+C}\) and define an \(\text{IFP+C}\)-interpretation \(\Gamma(\bar{x}_0)\) that canonises \(G\). We leave the (tedious) details to the reader.

Remark 4.11. Implicitly, the previous proof heavily depends on the concepts introduced in [31]. In particular, the definable directed graph \(D\) together with the definable mappings \(\sigma, \alpha\) constitute a definable tree decomposition. However, our theorem does not follow directly from Theorem 7.2 of [31].

The class \(\mathcal{G} \otimes \mathcal{Z}\) of chordal line graphs is fairly restricted, and there may be an easier way to prove the canonicalisation theorem by using Proposition 4.4. The proof given here has the advantage that it generalises to the class of all chordal graphs that have a good tree decomposition where the bags of the neighbours of a node intersect in a “bounded way”. We omit the details.

5 Further research

I mentioned several important open problems related to the quest for a logic capturing \(\text{PTIME}\) in the survey in Section 1. Further open problems can be found in [52]. Here, I will briefly discuss a few open problems related to classes closed under taking induced subgraphs, or equivalently, classes defined by excluding (finitely or infinitely many) induced subgraphs.

A fairly obvious, but not particularly interesting generalisation of our positive capturing result is pointed out in Remark 4.11. I conjecture that our theorem for chordal line graphs can be generalised to the class of chordal claw-free graphs, that is, I conjecture that the class of chordal claw-free graphs admits \(\text{IFP+C}\)-definable canonisation. Further natural classes of graphs closed under taking induced subgraphs are the classes of disk intersection graphs and unit disk intersection graphs. It is open whether \(\text{IFP+C}\) or any other logic captures \(\text{PTIME}\) on these classes. A very interesting and rich family of classes of graphs closed under taking induced subgraphs is the family of classes of graphs of bounded rank width [53], or equivalently, bounded clique width [13]. It is conceivable that \(\text{IFP+C}\) captures polynomial time on all classes of bounded rank width. To the best of my knowledge, currently it is not even known whether isomorphism testing for graphs of bounded rank width is in polynomial time.
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