TESTS FOR FORECAST INSTABILITY AND FORECAST FAILURE UNDER A CONTINUOUS RECORD ASYMPTOTIC FRAMEWORK

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JEL Classification: C10, C22

Keywords: Asymptotic distribution, break date, continuous-time, forecast failure, forecast instability, infill asymptotics, parameter instability, predictive ability, semimartingale

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Abstract
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JEL Classification: C12, C22, C52, C53
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1 Introduction

Since the seminal contribution of Klein (1971; 1969), economic forecasts had been built upon the presumption that the relationships between economic variables remain stable over time. However, the last decades have been subject to many social-economic episodes and technological advancements that have led economists to reconsider the assumption of model stability. The resonant empirical evidences documented in, among others, Perron (1989) and Stock and Watson (1996) [see also the recent survey by Ng and Wright (2013)] have motivated the development of econometric methods that detect such instabilities—most work directed toward structural changes—and estimate the actual dates at which economic relationships change. Yet, the issue of parameter insatiability is not limited to model estimation. In the forecasting literature, there has been a widespread concordance that the major issue that prevents good forecasts for economic variables is parameter instability—and structural changes as a special case—[cf. Banerjee, Marcellino, and Masten (2008), Clements and Hendry (1998, 2006), Elliott and Timmermann (2016), Giacomini (2015), Giacomini and Rossi (2015), Inoue and Rossi (2011), Clark and McCracken (2005), Pesaran, Pettenuzzo, and Timmermann (2006) and Rossi (2013a)].

This paper develops a statistical setting under infill asymptotics to address the issue of testing whether the predictive ability of a given forecast model remains stable over time. Ng and Wright (2013) and Stock and Watson (2003) explain that there has been abundant evidence for which a predictor that has performed well over a certain time period may not perform as well during other subsequent periods. For example, Gilchrist and Zakrajšek (2012) proposed a new credit spread index and showed that a residual component labeled as the excess bond premium—the credit spread adjusted for expected default risk—has considerable predictive content for future economic activity. They documented that this forecasting ability is stronger over the subsample 1985-2010 rather than over the full sample starting from 1973. The latter finding can be attributed to a more developed bond market in the 1985-2010 subsample. Relatedly, Giacomini and Rossi (2010) and Ng and Wright (2013) further examined this finding and found that indeed the predictive ability of commonly used term and credit spreads is unstable and somehow episodic. The latter authors suggested that credit spreads may be more useful predictors of economic activity in a more highly leveraged economy and that recent developments in financial markets translate into credit spreads containing more information than they had previously. We refer to such temporal instability for a given forecasting method as forecast instability or more specifically, as forecast failure.

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1 They reported that structural change tests provide some statistical evidence for a break in a coefficient associated with financial indicators—more specifically the coefficient on the federal funds rate. Given the latter evidence and the well-documented change in the conduct of monetary policy in the late 1970s and the early 1980s, it seems plausible to split the sample in 1985 (see p. 1709 and footnote 11 in their paper).
These terminologies are not new to professional forecasters as they were informally introduced by Clements and Hendry (1998) and generalized in econometric terms by Giacomini and Rossi (2009) who interpreted forecast breakdown (or forecast failure) as a situation in which the out-of-sample performance of a forecast model significantly deteriorates relative to its in-sample performance. Our approach is to formally define forecast instability from the economic forecaster’s perspective.\footnote{We use the terminology “instability” because not only the deterioration but also the improvement of the performance of a given forecast model over time can provide useful information to the forecaster.}

We emphasize that a forecast failure may well result from a short period of instability within the out-of-sample and not necessarily require that the instability be systematic in the sense of persisting throughout the whole out-of-sample period. That is, consistency of a forecast model’s performance with expected performance given the past should hold not only throughout the out-of-sample but also in any sub-sample of the latter. Indeed, many documented episodes of forecast failure seemed to arise from parameters nonconstancy data-generating processes over relatively short time periods compared to the total sample size. Hence, the desire of focusing on statistical tests being able to detect short-lasting instabilities is intuitive: if a test for forecast failure needs the deterioration of the forecasting ability to last for, say, at least half of the total sample in order to have sufficiently high power to reject the null hypotheses, then this test would not perform very well in practice because instability can be short-lasting. Furthermore, the occurrence of recurrent structural instabilities or multiple breaks that compensate each other in the out-of-sample might lead a forecast model to perform, on average, in a similar fashion as in the in-sample period. However, should a forecaster know about those recurrent changes she would conceivably revise its forecast model to adapt to the unstable environment. Hence, we introduce the following definition.

\textbf{Definition 1.1. (Forecast Instability)}

Forecast Instability refers to a situation of either sustained deterioration or improvement of the predictive ability of a given forecast model relative to the historical performance that would had led a forecaster to revise or reconsider its forecast model if she had known the occurrence of such instability. The time lengths of these two distinct periods need not bear any relationship.\footnote{Forecast Failure constitutes a special case of the definition—namely, a sustained deterioration of predictive ability.}

Th definition poses at the center the economic forecaster and consequently it is not merely a statistical definition; rather, it is based on an equilibrium concept. Since forecasting constitutes a decision theoretic problem, it should be from the forecaster perspective that a given forecast model is deemed to have failed. It is implicit from the definition to distinguish between forecasting method and model. Two forecasters may share the same forecast model—the relationship between the variable of interest and the predictor—but use different methods (e.g., recursive scheme versus
rolling scheme). Thus, instability refers to a given method-model pair. The object of the definition is predictive ability. Since the latter can be measured differently by different loss functions, then the definition applies to a given choice of the loss function. A notable aspect of the definition is the reference to the time span of the historical performance and of the putative period of instability. They need not be related. Consider a given forecasting strategy which has performed well during, say, the Great Moderation (i.e., from mid-1980s up to prior the beginning of the Great Recession in 2007). Assume that during the years 2007-2012 this method endures a time of poor performance and returns to perform well thereafter. According to our definition, this episode constitutes an example of forecast instability. However, if one designs the forecasting exercise in such a way that half of the sample is used for estimation and the remaining half for prediction, then this relatively short period of instability gets “averaged-out” from tests which simply compare the in-sample and out-of-sample averages. Conceivably, such tests would not reject the null hypotheses of no forecast failure while it seems that a forecaster would had revised its strategy during the crisis if she had known about such occurring under-performance in the present and immediate future period. Finally, detection of forecast instability does not necessarily mean that a forecast model should be abandoned. In fact, its performance may have improved over time. Yet, even if forecast instability is induced by performance deterioration, a forecaster might not end up switching to a new predictor. For example, entering a state of high variability might lead to poor performance even if the forecast model is still correct. Hence, our definition uses the term reconsider. Continuing with the above example, a forecaster may reconsider the choice of the forecasting window since a longer window may now produce better forecasts while keeping the same forecast model. In other words, knowledge of forecast instability is important because indicates that care must be exercised to assess the source of the changes.\footnote{Economists have documented episodes of forecast failure in many areas of macroeconomics. In the empirical literature on exchange rates a prominent forecast failure is associated with the Meese and Rogoff’s puzzle [cf. Meese and Rogoff (1983), Cheung, Chinn, and Garcia Pascual (2005), and Rossi (2013b) for an up-to-date account]. In the context of inflation forecasting, forecast failures have been reported by Atkeson and Ohanian (2001) and Stock and Watson (2009). For forecast instability concerning other macroeconomic variables see the surveys of Stock and Watson (2003) and Ng and Wright (2013).}

The theoretical implication is that in this paper our tests for forecast instability shall be based on the local behavior of the sequence of realized forecast losses. This is opposite to existing tests for forecast instability—and classical structural change tests more generally—which instead rely on a global and retrospective methodology merely comparing the average of in-sample losses with the average of out-of-sample losses. While maintaining approximately correct nominal size, our class of test statistics achieves substantial gains in statistical power relative to previous methods. Furthermore, as the initial timing of the instability moves away from middle sample toward the tail of the out-of-sample, the gains in power become considerable.
In this paper, we set out a continuous record asymptotic framework for a forecasting environment where \( T \) observations at equidistant time intervals \( h \) are made over a fixed time span \([0, N]\), with \( N = Th \). These observations are realizations from a continuous-time model for the variable to be forecast and for the predictor. From these discretely observed realizations we compute a sequence of forecasts using either a fixed, recursive or rolling scheme. To this sequence of forecasts there corresponds a continuous-time process which satisfies mild regularity conditions and that under the null hypotheses possesses a continuous sample-path. We exploit this pathwise property to base an hypothesis testing problem on the relative performance of a given forecast model over time. Under the hypotheses we expect the sequence of losses to display a smooth and stable path. Any discontinuous or jump behavior followed by a (possibly short) period of substantial discrepancy from the same path over the in-sample period provides evidence against the hypotheses. Our asymptotic theory involves a continuous record of observations where we let the sample size \( T \) grow to infinity by shrinking the sampling interval \( h \) to zero with the time span kept fixed at \( N \), thereby approaching the continuous-time limit.

Our underlying probabilistic model is specified in terms of continuous Itô semimartingales which are standard building blocks for analysis of macro and financial high-frequency data [cf. Andersen, Bollerslev, Diebold, and Labys (2001), Andersen, Fusari, and Todorov (2016), Bandi and Renò (2016) and Barndorff-Nielsen and Shephard (2004)]; the theoretical methodology is thus related to that of Casini and Perron (2017a), Li, Todorov, and Tauchen (2017), Li and Xiu (2016) and Mykland and Zhang (2009).\(^5\) The framework is not only useful for high-frequency data; in particular, recent work of Casini and Perron (2017a, 2017b) has adopted this continuous-time approach for modeling time series regression models with structural changes fitted to low-frequency data (e.g., macroeconomic data that are sampled at weekly, monthly, quarterly, annual frequency, etc.). They have showed that this continuous-time approach delivers a better approximation to the finite-sample distributions of estimators in structural change models and inference is more reliable than previous methods based on classical long-span asymptotics.

The classical approach to economic forecasting for macroeconomic variables is to formulate models in discrete-time and then base inference on long-span asymptotics where the sample size increases without bound and the sampling interval remains fixed [cf. Diebold and Mariano (1995), Giacomini and White (2006) and West (1996)]. There are crucial distinctions between this classical approach and the setting introduced in this paper. Under long-span asymptotics, identification of parameters hinges on assumptions on the distributions or moments of the studied processes [cf. the specification of the null hypotheses in Giacomini and Rossi (2009)], whereas within a

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\(^5\)Recent work by Li and Patton (2017) extends standard methods for testing predictive accuracy of forecasts to a high-frequency financial setting.
continuous-time framework, unknown structural parameters are identified from the sample paths of the studied processes. Hence, we only need to assume rather mild pathwise regularity conditions for the underlying continuous-time model and avoid any ergodic or weak-dependence assumption. As in Casini and Perron (2017a), our framework encompasses any time series regression model allowing for general forms of non-stationarity such as heteroskedasticity and serial correlation.

Given a null hypotheses stated in terms of the path properties of the sequence of losses, we propose a test statistic which compares the local behavior of the sequence of surprise losses defined as the difference between the out-of-sample and in-sample losses. More specifically, our maximum-type statistic examines the smoothness of the sequence of surprise losses as the continuous-time limit is approached. Under the hypotheses, the continuous-time analogue of the sequence of losses follows a continuous motion and any deviation from such smooth path is interpreted as evidence against the hypotheses. The null distribution of the test statistic is non-standard and follows an extreme value distribution. Therefore, our limit theory exploits results from extreme value theory as elaborated by Bickel and Rosenblatt (1973) and Galambos (1987).

We propose two versions of the test statistic: one that is self-normalized and one that uses an appropriate estimator of the asymptotic variance. The test statistic is defined as the maximal deviation between the average surprise losses over asymptotically vanishing time blocks. Further, we consider extensions of each of these statistics which use overlapping rather than non-overlapping blocks. Although they should be asymptotically equivalent, the statistics based on overlapping blocks are more powerful in finite-samples. In a framework where one allows for model misspecification, the problem of non-stationarity such as heteroskedasticity and serial correlation in the forecast losses should be taken seriously. Given the block-based form our test statistics we derive an alternative estimator of the long-run variance of the forecast losses. This estimator differs from the popular estimators of Andrews (1991) and Newey and West (1987) [see Müller (2007) for a review] and it is of independent interest. Finally, we extend results to settings that allow for stochastic volatility, and we conduct a local power analysis and highlight a few differences of our testing framework from the structural change test of Andrews (1993). Related aspects, such as estimating the timing of the instability and covering high-frequency setting with jumps, are being considered in a companion paper.

The rest of the paper is organized as follows. Section 2 introduces the statistical setting, the hypotheses of interest and the test statistics. Section 3 derives the asymptotic null distribution under a continuous record. We discuss the estimation of the asymptotic variance in Section 4. Some extensions and a local power analysis are presented in Section 5. Additional elements that

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6In nonparametric change-point testing, related works are Wu and Zhao (2007) and Bibinger, Jirak, and Vetter (2017).
are covered in our companion paper are briefly described in Section 6. A simulation study is contained in Section 7. Section 8 concludes the paper. The supplemental material to this paper contains all mathematical proofs and additional simulation experiments.

2 The Statistical Environment

Section 2.1 introduces the statistical setting with a description of the forecasting problem and the sampling scheme considered throughout. The underlying continuous-time model and its assumptions are introduced in Section 2.2. In Section 2.3 we set out the testing problem and state the relevant null and alternative hypotheses. The test statistics are presented in Section 2.4. Throughout we adopt the following notational conventions. All limits are taken as \( T \to \infty \), or equivalently as \( h \downarrow 0 \), where \( T \) is the sample size and \( h \) is the sampling interval. All vectors are column vectors and for two vectors \( a \) and \( b \), we write \( a \leq b \) if the inequality holds component-wise. For a sequence of matrices \( \{A_T\} \), we write \( A_T = o_p(1) \) if each of its elements is \( o_p(1) \) and likewise for \( O_p(1) \).

If \( x \) is a non-stochastic vector, \( \|x\| \) denotes the its Euclidean norm, whereas if \( x \) is a stochastic vector, the same notation is used for the \( L^2 \) norm. We use \( \lfloor \cdot \rfloor \) to denote the largest smaller integer function and for a set \( A \), the indicator function of \( A \) is denoted by \( 1_A \). A sequence \( \{u_{kh}\}_{k=1}^T \) is i.i.d. if the \( u_{kh} \) are independent and identically distributed. We use \( \overset{p}{\to}, \overset{\Rightarrow}{\to} \) to denote convergence in probability and weak convergence, respectively. \( \mathcal{M}_{p}^{c\text{i.d.}; p} \) is used for the space of \( p \times p \) positive define real-valued matrices whose elements are c.i.d. The symbol “\( \triangleq \)” is definitional equivalence.

2.1 The Forecasting Problem

The continuous-time stochastic process \( Z \triangleq (Y, X') \) is defined on a filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) and takes value in \( Z \subseteq \mathbb{R}^{q+1} \) where \( \{Y_t\}_{t \geq 0} \) is the variable to be forecast and \( \{X_t\}_{t \geq 0} \) are the predictor variables. The index \( t \) is defined as the continuous-time index and we have \( t \in [0, N] \), where \( N \) is referred to as the time span. In this paper, \( N \) will remain fixed. That is, the unobserved process \( Z_t \) evolves within the fixed time horizon \([0, N] \) and the econometrician records \( T \) of its realizations, with a sampling interval \( h \), at discrete-time points \( h, 2h, \ldots, Th \), where accordingly \( Th = N \). A continuous record asymptotic framework involves letting the sample size \( T \) grow to infinity by shrinking the time interval \( h \) to zero at the same rate so that \( N \) remains fixed.

The index \( k \) is used for the observation (or tick) times \( k = 1, \ldots, T \).

The objective is to generate a series \( \{Y_{(k+\tau)h}\} \) of \( \tau \)-step ahead forecasts. We shall adopt an out-of-sample procedure whereby splitting the time span \([0, N] \) into an in-sample and out-of-sample
window, \([0, N_{\text{in}}]\) and \([N_{\text{in}} + h, N]\), respectively.\(^7\) The latter two time horizons are supposed to be fixed and therefore within the in-sample (or prediction) window a sample of size \(T_m\) is observed whereas within the out-of-sample (or estimation) window the sample is of size \(T_n = T - T_m - \tau + 1\). We consider a general framework that allows for the three traditional forecasting schemes: (1) a fixed forecasting scheme with discrete-time observations \(h, 2h, \ldots, (T_m - 1)h, T_m h = N_{\text{in}}\); (2) a recursive forecasting scheme where at time \(kh\) the prediction sample includes observations \(h, \ldots, (k - 1)h, kh\); (3) a rolling forecasting scheme where the time span of the rolling window is fixed and of the same length as \(N_{\text{in}}\) (i.e., at time \(kh\) the in-sample window includes observations \(kh - T_m h + h, \ldots, (k - 1)h, kh\).\(^8\)

The forecasts may be based on a parametric model whose time-
\(kh\)-parameter estimates are then collected into the \(q \times 1\) random vector \(\hat{\beta}_k\). If no parametric assumption is made, then \(\hat{\beta}_k\) represents whatever semiparametric or nonparametric estimator used for generating the forecasts. The time-
\(kh\) forecast is denoted by \(\hat{f}_k (\hat{\beta}_k) \triangleq f \left( \hat{Z}_{kh}, \hat{Z}_{(k-1)h}, \ldots, \hat{Z}_{(k-m_f+1)h}; \hat{\beta}_k \right)\), where \(f\) is some measurable function. The notation indicates that the \(kh\)-time forecast is generated from information contained in a sample of size \(m_f\).\(^9\)

Next, we introduce a loss function \(L(\cdot)\) which serves for evaluating the performance of a given forecast model. More specifically, each out-of-sample loss \(L_{(k+\tau)h} (\hat{\beta}_k) \triangleq L \left( Y_{(k+\tau)h}, \hat{f}_k (\hat{\beta}_k) \right)\) constitutes a statistical measure of accuracy of the \(\tau\)-step forecast made at time \(kh\). However, given the objective of detecting potential instability of a certain forecasting method over time, we need additionally to introduce the in-sample losses \(L_{jh} (\hat{\beta}_k) \triangleq L \left( Y_{jh}, \hat{y}_j (\hat{\beta}_k) \right)\), where \(\hat{y}_j (\hat{\beta}_k)\) is an in-sample fitted value with \(j\) varying over the specific in-sample window. That is, for each time-
\(kh\) forecast there corresponds a sequence (indexed by \(j\)) of in-sample fitted values \(\hat{y}_j (\hat{\beta}_k)\).\(^10\) Then, the testing problem turns into the detection of any “systematic difference” between the sequence of out-of-sample and in-sample losses; the formal measure of such difference under our context is provided below.

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\(^7\)Indeed, \([0, N_{\text{in}}]\) corresponds to the in-sample window only for the fixed forecasting scheme to be introduced later—e.g., the rolling scheme only uses the most recent span of data of length \(N_{\text{in}}\). A minor and straightforward modification to this notation should be applied when the recursive and rolling schemes are considered. However, for all methods \(N_{\text{in}}\) indicates the artificial separation such that \(N_{\text{in}} + h\) is the beginning of the out-of-sample period.

\(^8\)Equivalently, the observation times within the rolling widow at the \(k\)'s observation are \(k - T_m + 1, \ldots, k\).

\(^9\)\(m_f\) varies with the forecastis scheme; e.g., for the rolling scheme we have \(m_f = T_m\) while for the recursive scheme we have \(m_f = k\).

\(^10\)We have \(j = \tau + 1, \ldots, T_m\) for the fixed scheme, \(j = \tau + 1, \ldots, k\) for the recursive scheme and \(j = k - T_m + \tau + 1, \ldots, k\) for the rolling scheme.
2.2 The Underlying Continuous-Time Model

The process $Z$ is a $\mathbb{R}^{q+1}$-valued semimartingale on $\left( \Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P} \right)$ and we further assume that all processes considered in this paper are càdlàg adapted and possess a $\mathbb{P}$-a.s. continuous path on $[0, N]$. The continuity property represents a key assumption in our setting and implies that $Z$ is a continuous Itô semimartingale. The integral form for $X_t$ is given by,

$$X_t = x_0 + \int_0^t \mu_{X,s} \, ds + \int_0^t \sigma_{X,s} \, dW_{X,s}, \quad (2.1)$$

where $\{W_{X,t}\}_{t \geq 0}$ is a $q \times 1$ Wiener process, $\mu_{X,s} \in \mathbb{R}^q$ and $\sigma_{X,s} \in \mathcal{M}_{q}^{c\bar{a}dl\bar{a}g}$ are the drift and spot covariance process, respectively, and $x_0$ is $\mathcal{F}_0$-measurable. We incorporate model misspecification into our framework by allowing for a large non-zero drift which adds to the residual process:

$$Y_t \triangleq y_0 + (\beta^\ast)' X_{t-} + \int_0^t \mu_{e,s} \, h^{-\vartheta} \, ds + e_t, \quad e_t \triangleq \int_0^t \sigma_{e,s} \, dW_{e,s} \quad (2.2)$$

where $\beta^\ast \in \mathbb{R}^q$, $\{W_{e,t}\}_{t \geq 0}$ is a standard Wiener process, $\sigma_{e,s} \in \mathbb{R}_+$ is its associated volatility, $\mu_{e,s} \in \mathbb{R}$ and $y_0$ is $\mathcal{F}_0$-measurable. In (2.2), the last two terms on the right-hand side account for the residual part of $Y_t$ which is not explained by $X_{t-}$, where $X_{t-} = \lim_{s \uparrow t} X_s$. We assume $\vartheta \in [0, 1/8)$ so that the factor $h^{-\vartheta}$ inflates the infinitesimal mean of the residual component thereby approximating a setting with arbitrary misspecification.

**Remark 2.1.** In (2.2), misspecification manifests itself in the form of (time-varying) non-zero conditional mean of the residual process, and in giving rise to serial dependence in the disturbances which in turn leads to dependence in the sequence of forecast losses. Hence, this specification is similar in spirit to the near-diffusion assumption of Foster and Nelson (1996) who studied the impact of misspecification in ARCH models. On the other hand, Casini and Perron (2017a) introduced a “large-drift” asymptotics with $h^{-1/2}$ to deal with non-identification of the drift in their context. Technically, the latter specification implies that as $h$ becomes small the drift features larger oscillations that add to the local Gaussianity of the stochastic part. Casini and Perron (2017a) referred to this specification as small-dispersion assumption. Finally, note that the presence of $h^{-\vartheta}$ can also be related to the signal plus small Gaussian noise of Ibragimov and Has’minskii (1981) if one sets $\varepsilon_h = h^{\vartheta}$ in their model in Section VII.2.

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11 For accessible treatments of the probabilistic elements used in this section we refer to Aît-Sahalia and Jacod (2014), Jacod and Shiryaev (2003), Jacod and Protter (2012), Karatzas and Shreve (1996) and Protter (2005).

12 Asymptotically, these features can be dealt with basic arguments used in the high-frequency financial statistics literature; however, when $h$ is not too small one needs methods that are robust in finite-samples to such misspecification-induced properties. More precisely, we will propose an appropriate estimator of the long-run variance of the sequence of forecast losses in Section 4.
Assumption 2.1. We have the following assumptions: (i) The processes \( \{X_t\}_{t \geq 0} \) and \( \Sigma^0 \equiv \{\sigma_{X,t}, \sigma_{e,t}\}_{t \geq 0} \) have \( \mathbb{P} \)-a.s. continuous sample paths; (ii) The processes \( \{\mu_{X,t}\}_{t \geq 0} \), \( \{\mu_{e,t}\}_{t \geq 0} \) and \( \{\sigma_{e,t}\}_{t \geq 0} \) are locally bounded; (iii) There exists \( 0 < \sigma_- < \sigma_+ < \infty \) such that \( \mathbb{P} \)-a.s. \( \inf_{t \in [0,N]} \sigma_{V,t}^2 \geq \sigma_-^2 \) and \( \sigma_{V,t}^2 \geq \sup_{t \in [0,N]} \sigma_{V,t}^2 \) with \( V = X, e \); (iv) \( \sigma_{X,t} \in \mathcal{M}^{\text{cadlag}} \) and \( \sigma_{e,t} \in \mathcal{M}_1^{\text{cadlag}} \) and the conditional variance (or spot covariance) is defined as \( \Sigma_{X,t} = \sigma_{X,t}'\sigma_{X,t} \), which for all \( t < \infty \) satisfies \( \int_0^t \Sigma^{(j,j)}_{X,s} \, ds < \infty \), \( j = 1, \ldots, q \) where \( \Sigma^{(p,q)}_{X,t} \) denotes the \( (j,r) \)-th element of \( \Sigma_{X,t} \). Furthermore, for every \( j = 1, \ldots, q \), and \( k = 1, 2, \ldots, T \), the quantity \( h^{-1} \int_{(k-1)h}^{kh} \Sigma_{X,s}^{(j,j)} \, ds \) is bounded away from zero and infinity, uniformly in \( k \) and \( h \); (v) The disturbance process \( e_t \) is orthogonal (in martingale sense) to \( X_t \): \( \langle e, X \rangle_t = 0 \) identically for all \( t \geq 0 \).

Part (i) rules out jump processes from our setting. We relax this restriction in our companion paper; see Section 6. Part (ii) restricts those processes to be locally bounded. These should be viewed as regularity conditions rather than assumptions and are standard in the financial econometrics literature [see Barndorff-Nielsen and Shephard (2004), Li and Xiu (2016) and Li, Todorov, and Tauchen (2017)]; recently, they have been used by Casini and Perron (2017a) in the context of structural change models.

The continuous-time model in (2.1)-(2.2) is not observable. The econometrician only has access to \( T \) realizations of \( Y_t \) and \( X_t \) with a sampling interval \( h > 0 \) over the horizon \([0, N]\). For each \( h > 0 \), \( Z_{kh} \in \mathbb{R}^{q+1} \) is a random vector step function that jumps only at time 0, \( h, 2h, \ldots \), and so on. The discretized processes \( Y_{kh} \) and \( X_{kh} \) are assumed to be adapted to the increasing and right-continuous filtration \( \{\mathcal{F}_t\}_{t \geq 0} \). The increments of a process \( U_t \) are denoted by \( \Delta_h U_k = U_{kh} - U_{(k-1)h} \).

A seminal result known as Doob-Meyer Decomposition [cf. the original sources are Doob (1953) and Meyer (1967); see also Section III.3 in Protter (2005)] allows us to decompose the semimartingale process \( X_t \) into a predictable part and a local martingale part. Hence, it follows that we can write for \( k = 1, \ldots, T \), \( \Delta_h X_k \triangleq \mu_{X,kh} \cdot h + \Delta_h M_{X,k} \) where the drift \( \mu_{X,t} \in \mathbb{R}^q \) is \( \mathcal{F}_{t-h} \)-measurable, and \( M_{X,kh} \in \mathbb{R}^q \) is a continuous local martingale with finite conditional covariance matrix \( \mathbb{P} \)-a.s. \( \mathbb{E} \left( \Delta_h M_{X,kh} \Delta_h M_{X,k} \big| \mathcal{F}_{(k-1)h} \right) = \Sigma_{X,(k-1)h} \cdot h \). Turning to equation (2.2), the error process \( \{\Delta_h e_{k}^*, \mathcal{F}_t\} \), with \( \Delta_h e_{k}^* \triangleq \sigma_{e,(k-1)h} \Delta_h W_{e,k} \), is then a continuous local martingale difference sequence taking its values in \( \mathbb{R} \) with finite conditional variance \( \mathbb{E} \left( (\Delta_h e_{k}^* )^2 \big| \mathcal{F}_{(k-1)h} \right) = \sigma_{e,(k-1)h}^2 \cdot h, \mathbb{P} \)-a.s.

Therefore, we express the discretized analogue of (2.2) as

\[
\Delta_h Y_k = (\beta^*)' \Delta_h X_{k-\tau} + \mu_{e,kh} \cdot h^{1-\theta} + \Delta_h e_k, \quad k = \tau + 1, \ldots, T. \tag{2.3}
\]

Remark 2.2. As explained above, we accommodate possible model misspecification by adding the component \( \mu_{e,k} \cdot h^{1-\theta} \). In the forecasting literature, often one directly imposes restrictions on the
sequence of losses, say, $L(e_k)$ where $e_k = Y_k - \hat{f}_k(\hat{\beta}_k)$ is a forecast error. There are two main differences from our approach. First, in order to facilitate illustrating our novel framework to the reader, we have chosen, without loss of generality, to express directly the relationship between $\Delta_h Y_{k+\tau}$ and $\Delta_h X_k$ while at the same time, allowing for misspecification by including $\mu_{e,kh} \cdot h^{1-\vartheta}$. A second distinction from the classical approach is that the latter imposes restrictions on the sequences of losses such as mixing and ergodicity conditions, covariance stationary and so on. In contrast, our infill asymptotics does not require us to impose any ergodic or mixing condition [cf. Casini and Perron (2017a)].

Finally, we have an additional assumption on the path of the volatility process $\{\sigma^2_{e,t}\}_{t \geq 0}$. This turns out be important because it partly affects the local behavior of the forecast losses.

**Assumption 2.2.** For small $\eta > 0$, define the modulus of continuity of $\{\sigma_{e,t}\}_{t \geq 0}$ on the time horizon $[0, N]$ by $\phi_{\sigma,\eta,N} = \sup_{s,t \in [0, N]} \{|\sigma_t - \sigma_s| : |t - s| < \eta\}$. We assume that $\phi_{\sigma,\eta,\tau_h,N} \leq K_h \eta$ for some sequence of stopping times $\tau_h \to \infty$ and some $\mathbb{P}$-a.s. finite random variable $K_h$.

The assumption essentially states that $\phi_{\sigma,\eta,N}$ is locally bounded and $\{\sigma_{e,t}\}_{t \geq 0}$ is Lipschitz continuous. Lipschitz volatility is a more than reasonable specification for the macroeconomic and financial data to which our analysis is primarily directed. Indeed, the basic case of constant variance $\sigma^2$ is easily accommodated by the assumption. Time-varying volatility is also covered provided $\sigma^2_{e,t}$ is sufficiently smooth. However, the assumption rules out some standard stochastic volatility models often used in finance. We relax that assumption in Section 5, so that we can extend our results to, for example, stochastic volatility models driven by a Wiener process.

### 2.3 The Hypotheses of Interest

As time evolves, a forecast model can suffer instability for multiple reasons. However, incorporating model misspecification into our framework necessarily implies that the exact form of the instability is unknown and thus one has to leave it unspecified. This differs from the classical setting for estimation of structural change models [cf. Bai and Perron (1998) and Casini and Perron (2017a)] where (i) the break date is well-defined as it is part of the definition of the econometric problem, and (ii) the form of the instability is explicitly specified through a discrete shift in a regression parameter. In contrast, under our context we remain agnostic regarding both (i) and (ii). There may be multiple dates at which the forecast model suffers instability and they might be interrelated in a complicated way. Forecast instability may manifest itself in several forms, including gradual, smooth or recurrent changes in the predictive relationship between $Y_{(k+\tau)h}$ and $X_{kh}$; certainly, there could also be discrete shifts in $\beta^*$—arguably the most common case in practice—but this is only a possibility in our setting and not an assumption as in structural change models. A forecast failure
then reflects the forecaster’s failure to recognize the shift in the predictive power of $X_{kh}$ on $Y_{(k+\tau)h}$. On the other hand, even if one can rule out shifts in $\beta^\ast$, a forecast instability may be induced by an increase/decrease in the uncertainty in the data which might result, for example, from changes in the unconditional variance of the target variable. In this case, the predictive ability of $X_{kh}$ on $Y_{(k+\tau)h}$, as described for instance by a parameter $\beta$, remains stable while due to an increase in the unconditional variance of $Y_{(k+\tau)h}$ it might become weak and in turn the forecasting power might breakdown. Tests for forecast failure such as those proposed in this paper and the ones proposed in Giacomini and Rossi (2009) are designed to have power against both of the above hypotheses.  

2.3.1 The Null and Alternative Hypotheses on Forecast Instability

Define at time $(k+\tau)h$ a surprise loss given by the deviation between the time-$(k+\tau)h$ out-of-sample loss and the average in-sample loss: $SL_{(k+\tau)h}(\hat{\beta}_k) \triangleq L_{(k+\tau)h}(\hat{\beta}_k) - L_{kh}(\hat{\beta}_k)$, for $k = T_m, \ldots, T - \tau$, where $L_{kh}(\hat{\beta}_k)$ is the average in-sample loss computed according to the specific forecasting scheme. One can then define the average of the out-of-sample surprise losses

$$\overline{SL}_{N_0}(\hat{\beta}_k) \triangleq N_0^{-1} \sum_{k=T_m}^{T-\tau} SL_{(k+\tau)h}(\hat{\beta}_k),$$

where $N_0 \triangleq N - N_{in} - h$ denotes the time span of the out-of-sample window. In the classical discrete-time setting, under the hypotheses of no forecast instability one would naturally test whether $\overline{SL}_{N_0}(\beta^\ast)$ has zero mean, where $\beta^\ast$ is the pseudo-true value of $\beta$. If the forecasting performance remains stable throughout the whole sample then there should be no systematic surprise losses in the out-of-sample window and thus $E\left[\overline{SL}_{N_0}(\beta^\ast)\right] = 0$. This reasoning motivated the forecast breakdown test of Giacomini and Rossi (2009). Therefore, under the classical asymptotic setting one exploits time series properties of the process $SL_{(k+\tau)h}(\beta^\ast)$ such as ergodicity and mixing together with the representation of the hypotheses by a global moment restriction. By letting the span $N \to \infty$, this method underlies the classical approach to statistical inference but does not directly extend to an infill asymptotic setting. Under continuous-time asymptotics, identification of parameters is achieved by properties of the paths of the involved processes and not by moment conditions. This constitutes the key difference and requires one to recast the above hypotheses into an infill setting thereby making use of assumptions on an underlying continuous-time

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14Recently, Perron and Yamamoto (2018) proposed to apply modified versions of classical structural break tests to the forecast failure setting. However, their testing framework and hence their null hypotheses are different from ours because they do not fix a model-method pair but only fix the forecast model under the null.

15By definition $N_0$ is fixed and should not be confused with $T_n$, which indicates the number of observations in the out-of-sample window. Indeed, $N_0 = T_n h$.

16Global refers to the property that the zero-mean restriction involves the entire sequence of forecast losses.
data-generating mechanism which is assumed to govern the observed data.

We begin with observing that the sequence of losses \( \{L_{kh}(\cdot)\} \) can be viewed as realizations from an underlying continuous-time process \( \{\tilde{L}_t\}_{t \geq 0} \) with \( \tilde{L}_t \equiv \int_0^t L_s(Y_t, X_{t-}; \beta^*) \, ds \). That is, \( \tilde{L}_t \) consists of temporally integrated forecast losses where \( L_t \) is the loss at time \( t \) and is defined by some transformation of the target variable \( Y_t \) and of the predictor \( X_{t-} \).\(^{17}\) In order to provide a general theory, we focus on families of loss functions that depend only on the forecast error.\(^{18}\) We denote this class by \( L_e \) and we say that the loss function \( L(\cdot, \cdot; \cdot) \in L_e \) if \( L_t(Y_t, X_{t-}; \beta) = L_t(e_t; \beta) \) for all \( t \in [0, N] \), where \( e_t = Y_t - \hat{f}_t(\beta) \). The class \( L_e \) comprises the vast majority of loss functions employed in empirical work, including among others the popular Quadratic loss, Absolute error loss and Linex loss. The following examples illustrate how these loss functions are constructed under our setting. For the rest of this section, assume for simplicity \( y_0 = 0, \mu_{e,-} = 0 \) and that \( X_t \) is one-dimensional in (2.2).

**Example.** (QL: Quadratic Loss)
The Mean Squared Error or Quadratic loss function is symmetric and is by far the most commonly used by practitioners. Given (2.2), we have \( e_t = Y_t - \beta^* X_{t-} \). Then \( L(e) = ae^2 \) or \( L_t(Y_t, X_{t-}; \beta^*) = ae_t^2 \) with \( a > 0 \).

**Example.** (LL: Linex Loss)
The Linear-exponential or Linex loss was introduced by Varian (1975) and it is an example of asymmetric loss function. By the same reasoning as in the Quadratic loss case, we have \( L(e) = a_1 (\exp(a_2 e) - a_2 e - 1) \) or \( L_t(Y_t, X_{t-}; \beta^*) = a_1 (\exp(a_2 e_t) - a_2 e_t - 1) \) with \( a_1 > 0, a_2 \neq 0 \).

Below we make very mild pathwise assumptions on the process \( Z \) which imply restrictions on \( \{\tilde{L}_t\}_{t \geq 0} \). We derive asymptotic results under Lipschitz continuity (in \( t \)) of the coefficients of the system of stochastic differential equations driving the data \( \{Z_t\}_{t \geq 0} \). We apply the techniques of stochastic calculus to formulate our testing problem. To avoid clutter, we introduce the notation \( g(Y_t, X_{t-}; \beta^*) = L_t(Y_t, X_{t-}; \beta^*) \) and its shorthand \( g(e_t; \beta^*) = L_t(e_t; \beta^*) \).\(^{19}\) By Itô Lemma, [cf. Section II.7 in Protter (2005)], under smoothness of \( g(e_t; \beta^*) \),

\[
dL_t(e_t; \beta^*) = \frac{\sigma_{e,t}^2}{2} \frac{\partial^2 g(e_t; \beta^*)}{\partial e^2} \, dt + \sigma_{e,t} \frac{\partial g(e_t; \beta^*)}{\partial e} \, dW_{e,t}.
\]

\(^{17}\)The definition of \( \tilde{L}_t \) uses that so long as the forecast step \( \tau \) is small and finite one can approximate \( X_{s-} \) for sufficiently small \( h > 0 \).

\(^{18}\)The most popular loss functions used in economic forecasting are within this category [see Elliott and Timmermann (2016) for a recent incisive account of the literature]. Extension to ad hoc loss functions requires specific treatment that might vary from case to case.

\(^{19}\)The notation implicitly assumes that the same loss function is used for estimation and prediction which in turn implies that the subscript \( t \) in \( L_t(e_t; \beta^*) \) can be omitted since it can be understood from that of the argument \( e_t \).
Let $\mathbb{E}_\sigma$ denote the expectation conditional on the path $\{\sigma_{e,t}\}_{t \geq 0}$. The instantaneous mean of $dL(e_t; \beta^*)$ is $\mathbb{E}_\sigma [dL(e_t; \beta^*) / dt] = 2^{-1} \sigma_{e,t}^2 \mathbb{E}_\sigma (\partial^2 g(e_t; \beta^*) / \partial e^2)$. Note that the latter is a symbolic abbreviation for
\[
\mathbb{E}_\sigma [L_t(e_t; \beta^*) - L_s(e_s; \beta^*)] = \frac{\sigma_{e,t}^2}{2} \mathbb{E}_\sigma \left[ \frac{\partial^2 g(e_t; \beta^*)}{\partial e^2} \right] (t - s) + o(t - s), \quad \text{as } s \uparrow t.
\]

Since the coefficients of the original system of stochastic equations are Lipschitz continuous in $t$, one can verify that $\mathbb{E}_\sigma [dL(e_t; \beta^*) / dt]$ is also Lipschitz upon regularity conditions on $g(\cdot, \beta^*)$ and time-$t$ information.

We denote by $\text{Lip}([0, N])$ the class of Lipschitz continuous functions on $[0, N]$. Let $\{c_t\}_{t \geq 0}$ denote a continuous-time stochastic process that is $\mathbb{P}$-a.s. locally bounded and adapted.

**Definition 2.1.** The process $\{c_t\}_{t \geq 0}$ belongs to $\text{Lip}([0, N])$ if $\sup_{s,t \in [0, \tau_h \wedge N], t \neq s} |c_t - c_s| < K_h |t - s|$ for some sequence of stopping times $\tau_h \to \infty$ and some $\mathbb{P}$-a.s. finite random variable $K_h$.

We are in a position to formulate the testing problem in terms of the pathwise property of $L_t(e_t; \beta^*)$. This implies that the hypotheses are specified in terms of random events which differs from classical hypotheses testing but it is typical under continuous-time asymptotics; see Aït-Sahalia and Jacod (2012) (for many references), Li, Todorov, and Tauchen (2016) and Reiß, Todorov, and Tauchen (2015). We consider the following hypotheses: for any $L(\cdot; \cdot) \in L_e$,\(^{20}\)

\[
H_0 : \left \{ \lim_{s \uparrow t} \mathbb{E}_\sigma [L_t(e_t; \beta^*) - L_s(e_s; \beta^*)] \right \} \in \text{Lip}([N_{in} + h, N]), \tag{2.5}
\]

which means that we wish to discriminate between the following two events that divide $\Omega$:

\[
\Omega_0 \triangleq \left \{ \omega \in \Omega : \left \{ \lim_{s \uparrow t} \mathbb{E}_\sigma [L_t(e_t(\omega); \beta^*) - L_s(e_s(\omega); \beta^*)] \right \} \in \text{Lip}([N_{in} + h, N]) \right \}, \quad \Omega_1 \triangleq \Omega \setminus \Omega_0.
\]

The dependence of the hypotheses on $\omega$ is appropriate because each event $\omega$ generates a certain path of $L(e_t(\cdot); \beta^*)$ on $[0, N]$, where $de_t(\omega) = \sigma_{e,t}(\omega) dW_{e,t}(\omega)$. The hypotheses requires a Lipschitz condition to hold on $[N_{in} + h, N]$, where $N_{in}$ is the usual artificial separation date after which the first forecast is made. $N_{in}$ is taken as given here because the testing problem applies to a specific method-model pair and $N_{in}$ is part of the chosen forecasting method. From a practical standpoint, it would be helpful if this separation date is such that the forecast model is stable on $[0, N_{in}]$ [see Casini and Perron (2017c) for more details]. The latter property is, however, unknown a priori by the practitioner. We cover this case in Section 6.

\(^{20}\)Precise assumptions will be stated below.
Example. (QL; cont’d)
For the Quadratic loss $L(e) = ae^2$, Itô Lemma yields $\mathbb{E}_\sigma [dL_t (e_t; \beta^*) / dt] = a \sigma_{e,t}^2$. If $\sigma_{e,t}$ is Lipschitz continuous, then the hypothesis $H_0$ holds.

Example. (LL; cont’d)
From Itô Lemma, $dL_t (e_t; \beta^*) = a_1 \left\{ a_2 \left[ 2^{-1} a_2 \sigma_{e,t}^2 \exp (a_2 e_t) dt + (\exp (a_2 e_t) - 1) \sigma_{e,t} dW_{e,t} \right] - 1 \right\}$. Consequently, by Itô Isometry [cf. Section 3.3.2 in Karatzas and Shreve (1996) or Lemma 3.1.5 in Øksendal (2000)] $\mathbb{E}_\sigma [dL (e_t; \beta^*) / dt] = a_1 \left\{ a_2 \left( \sigma_{e,t}^2 / 2 \right) \right\} \exp \left( a_2 \left( \int_0^t \sigma_{e,s}^2 ds \right) / 2 \right)$ and hypotheses $H_0$ is seen to hold under Lipschitz continuity of $\sigma_{e,t}$.

We have reduced the forecast instability problem to examination of the local properties of the path of $L_t$. However, we still have to face the question on how to use the data to test $H_0$ in practice. Even if we could observe $\bar{L}_t$, it would not be clear how to formulate a testing problem on the stability of $L_t$ by using path properties of $\bar{L}_t$. The reason is that $\bar{L}_t$ is always absolutely continuous by definition, and thus it would provide little information on the large deviations of the forecast error $e_t$. In order to study the local behavior of $L_t$ one needs to consider the small increments of $L_t$ close to time $t$. Leaving the definition of $\bar{L}_t$ aside for a moment, observe that $\mathbb{P}$-a.s. continuity of $Z_t$ is equivalent to having the relationship between $Y_t$ and $X_t$ holding for any infinitesimal interval of time. For the basic parametric linear model: $dY_t = \beta^* dX_t + de_t$.

Then, the forecast loss is $L(\Delta e_t)$, which is difficult to interpret in rigorous probabilistic terms. However, we can consider its discrete-time analogue. We normalize the forecast error by the factor $\psi_h = h^{1/2}$ and redefine $L_{\psi,kh} (\Delta_h e_k; \beta^*) \triangleq L_{kh} \left( \psi_h^{-1} \Delta_h e_k; \beta^* \right)$. Then, for all $k$, the mean of $L_{\psi,kh} (\Delta_h e_k; \beta^*)$—conditional on $\sigma_{e,kh}$—depends on the parameters of the model and its local behavior can be used as a proxy for the local behavior of the infinitesimal mean of $dL_t (e_t; \beta^*)$. If the corresponding structural parameters of the continuous-time data-generating process satisfy a Lipschitz continuity in $t$, then—knowing $\sigma_{e,kh}$—also $\mathbb{E}_\sigma [L_{\psi,kh} (\Delta_h e_k; \beta^*)] = a \sigma_{e,(k-1)h}^2$.

Example. (QL; cont’d)
Conditional on $\{ \sigma_t \}_{t \geq 0}$, $\Delta_h e_k \sim \mathcal{N} \left( 0, \sigma_{e,(k-1)h}^2 \cdot h \right)$. Thus, $\mathbb{E}_\sigma [L_{\psi,kh} (\Delta_h e_k; \beta^*)] = a \sigma_{e,(k-1)h}^2$. If $\sigma_{e,t}$ is Lipschitz continuous, then the hypothesis $H_0$ holds.

---

21 Recall that composition of Lipschitz functions is Lipschitz and that under our context $\exp \left( a_2 \left( \int_0^t \sigma_{e,s}^2 ds \right) / 2 \right)$ is Lipschitz because (i) $\sigma_{e,s}^2$ is locally bounded and Lipschitz, and (ii) $t \leq N$ and $N$ remains fixed.

22 Alternatively, $L_{\psi,kh} (\Delta_h Y_k, \Delta_h X_k; \beta^*) = L_{kh} (\psi_h^{-1} (\Delta_h Y_k - \beta^* \Delta_h X_k))$.  

15
Example. (LL; cont’d)
Similar to the Quadratic loss case, we have
\[ E_\sigma \left[ L_{\psi,kh} (\Delta h e_k; \beta^*) \right] = a_1 \left( \exp \left( a_2^2 \sigma^2 e, (k-1)/2 \right) - 1 \right). \]
Again, the hypotheses \( H_0 \) is satisfied if \( \sigma_{e,t} \) is Lipschitz.

Both examples demonstrate that pathwise assumptions on the data-generating process implies restrictions on the properties of the sequence of loss functions. For the QL example, if there is a structural break at the observation \( k = T_b \), then this would result in the mean of \( L_{\psi,kh} (\Delta h e_k; \beta^*) \) shifting to a new level after time \( T_b h \). Given that the same reasoning extends to the sequence of surprise losses, one may consider to construct a test statistic on the basis of the local behavior of the surprise losses over time. If there is no instability in the predictive ability of a certain model, then the sequence of out-of-sample surprise losses should display a certain stability. Under the framework of Giacomini and Rossi (2009), this stability is interpreted in a retrospective and global sense as a zero-mean restriction on the sequence over the entire out-of-sample. In contrast, under our continuous-time setting, this stability manifests itself as a continuity property of the path of the continuous-time counterpart of the sequence.

2.4 The Test Statistics

By inspection of the null hypotheses in (2.5), it is evident that a considerable number of forms of instabilities are allowed. These may result from discrete shifts in a model’s structural parameter and/or in structural properties of the processes considered such as conditional and unconditional moments and so on. This first set of non-stationarities relates to the popular case of structural changes which are designed to be detected with high probability by the structural break tests of, among others, Andrews (1993) and Andrews and Ploberger (1994), Bai and Perron (1998) and Elliott and Müller (2006) in univariate settings and of Qu and Perron (2007) in multivariate settings. However, a forecast instability may be generated by many other forms of non-stationarities against which such classical tests for structural breaks are not designed for and consequently they might have little power against. For example, consider the case of smooth changes in model parameters, or in the unconditional variance of \( Y_{kh} \). Even more serious would be the presence of recurrent smooth changes in the marginal distribution of the predictor since in this case the above-mentioned tests are likely to falsely reject \( H_0 \) too often [cf. Hansen (2000)]. Thus, the null hypotheses of no forecast instability calls for a new statistical hypotheses testing framework. Ideally, in this context one needs a test statistic that retains power against any discontinuity, jump and recurrent switch at any point in the out-of-sample and for any magnitude of the shift. We propose a test statistic which aims asymptotically at distinguishing any discontinuity from a regular Lipschitz continuous motion. We introduce a sequence of two-sample t-tests over asymptotically
vanishing adjacent time blocks. This should lead to significant gains in power whenever on fixed
time intervals the out-of-sample losses exhibit instabilities of any form such as breaks, jumps and
relatively large deviations. Such gains are likely to occur especially when instabilities take place
within a small portion of the sample relative to the whole time span—a common case in practice
that has characterized many episodes of forecast failure in economics.

Interestingly, for the Quadratic loss function we can exploit properties of the local quadratic
variation and propose a self-normalized test statistic. Thus, we separate the discussion on the
Quadratic loss from that on general loss functions. Let \( SL_{\psi,(k+\tau)h} (\hat{\beta}_k) \triangleq L_{\psi,(k+\tau)h} (\hat{\beta}_k) - T_{\psi,kh} (\hat{\beta}_k) \),
k = \( T_m, \ldots, T - \tau \). Next, we partition the out-of-sample into \( m_T \triangleq \lceil T_n/n_T \rceil \) blocks each
containing \( n_T \) observations. Let \( B_{h,b} \triangleq n_T^{-1} \sum_{j=1}^{n_T} SL_{\psi,T_m+\tau+bn_T+j-1} (\hat{\beta}_{T_m+bn_T+j-1}) \) and \( \overline{B}_{h,b} \triangleq n_T^{-1} \sum_{j=1}^{n_T} L_{\psi,(T_m+\tau+bn_T+j-1)h} (\hat{\beta}_{T_m+bn_T+j-1}) \) for \( b = 0, \ldots, \lfloor T_n/n_T \rfloor - 1 \).

### 2.4.1 Test Statistics under Quadratic Loss

We propose the following statistic

\[
B_{\text{max},h} (T_n, \tau) \triangleq \max_{b=0,\ldots,\lfloor T_n/n_T \rfloor - 2} \left| \frac{B_{h,b+1} - B_{h,b}}{\overline{B}_{h,b+1}} \right|,
\]

The quantity \( B_{h,b} \) is a local average of the surprise losses within the block \( b \). We have partitioned
the out-of-sample window into \( m_T \) blocks of asymptotically vanishing length \([bn_Th, (b+1)n_Th]\). We consider an asymptotic experiment in which the number of blocks \( m_T \) increases at a controlled
rate to infinity while the per-block sample size \( n_T \) grows without bound at a slower rate than the out-of-sample size \( T_n \). The appeal of the \( B_{\text{max},h} (T_n, \tau) \) statistic is that a large deviation
\( B_{h,b+1} - B_{h,b} \) suggests the existence of either a discontinuity or non-smooth shift in the surprise
losses close to time \( bn_Th \) and thus it provides evidence against \( H_0 \). We comment on the nature
of the normalization \( \overline{B}_{h,b+1} \) in the denominator of \( B_{\text{max},h} \) below, after we introduce a version of
\( B_{\text{max},h} \) statistic which uses all admissible overlapping blocks of length \( n_T h \):

\[
\text{MB}_{\text{max},h} (T_n, \tau) \triangleq \max_{i=n_T,\ldots,T_n-n_T} \left( n_T^{-1} \sum_{j=i-n_T+1}^{i} SL_{\psi,T_m+\tau+j-1} (\hat{\beta}_{T_m+j-1}) - n_T^{-1} \sum_{j=i+1}^{i+n_T} SL_{\psi,T_m+\tau+j-1} (\hat{\beta}_{T_m+j-1}) \right) / \overline{B}_{h,i},
\]

where \( \overline{B}_{h,i} = n_T^{-1} \sum_{j=i+1}^{i+n_T} L_{\psi,T_m+\tau+j-1} (\hat{\beta}_{T_m+j-1}) \). Since under the alternative hypotheses the exact
location of the change-point—or possibly the locations of the multiple change-points—within the
block might actually affect the power of the \( B_{\text{max},h} \)-based test in small samples, we indeed find in
our simulation study that the test statistic $MB_{\text{max},h}$ which uses overlapping blocks is more powerful especially when the instability arises in forms other than the simple one-time structural change. Thus, the power of the $B_{\text{max},h}$ test is slightly sensible to the actual location of the change-point within the block, with higher power achieved when the change-point is close to either the beginning or the end of the block. In contrast, the statistical power of $MB_{\text{max},h}$ is uniform over the location of the change-point in the sample. The latter property is not shared by the exiting test of Giacomini and Rossi (2009) given that its power tends to be substantially lower if the instability is not located at about middle sample.

An important characteristic of both $B_{\text{max},h}$ and $MB_{\text{max},h}$ is that they are self-normalized; no asymptotic variance appears in their definition. The reason for why $B_{h,b+1}$ appears in the denominator of, for example, $B_{\text{max},h}$ is that even though $B_{h,b+1}$ constitutes a more logical self-normalizing term, it might be close to zero in some cases. This would occur under Quadratic loss if, for example, $\sigma_{e,t} = \sigma_e$ for all $t \geq 0$. This is not true for the factor $B_{h,b+1}$.

In addition, observe that allowing for misspecification naturally leads one to deal carefully with artificial serial dependence in the forecast losses in small samples. Thus, we consider a version of the statistics $B_{\text{max},h}$ and $MB_{\text{max},h}$ that are normalized by their asymptotic variance:

$$Q_{\text{max},h}(T_n, \tau) \triangleq \nu^{-1} L_{\text{max}} \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} |B_{h,b+1} - B_{h,b}|,$$

and similarly,

$$MQ_{\text{max},h}(T_n, \tau) \triangleq \nu^{-1} L_{\text{max}} \max_{i=n_T, \ldots, T_n-n_T} \left| \sum_{j=i+1}^{i+n_T} SL_{T_m+\tau+j-1} \left( \hat{\beta}_{T_m+j-1} \right) - n_T^{-1} \sum_{j=1-n_T+1}^{i} SL_{T_m+\tau+j-1} \left( \hat{\beta}_{T_m+j-1} \right) \right|.$$

The quantity $\nu_L$ standardizes the test statistic so that under the null hypotheses we obtain a distribution-free limit. This can be useful because given the fully non-stationary setting together with the possible consequences of misspecification in finite-samples, standardization by the square-root of the asymptotic variance $\nu^2_L$ might lead to a more precise empirical size in small samples. We relegate theoretical details on $\nu_L$ as well as on its estimation to Section 4 where we also present a discussion about its relation with the choice of the number of blocks.

### 2.4.2 Test Statistics under General Loss Function

For general loss $L \in L_e$, we propose the following statistic,

$$G_{\text{max},h}(T_n, \tau) \triangleq \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} \left| \frac{B_{h,b+1} - B_{h,b}}{D_{h,b+1}} \right|,$$
where \( B_{h,b} \), \( B_{h,b+1} \) are defined as in the quadratic case and

\[
D_{h,b+1} \triangleq n_T^{-1} \sum_{j=1}^{n_T} \left( L_{\psi_i(T_m+\tau+(b+1)n_T+j-1)} \left( \hat{\beta}_{T_m+(b+1)n_T+j-1} \right) - L_{\psi_i} \left( \hat{\beta} \right) \right)^2,
\]

with \( L_{\psi_i} \left( \hat{\beta} \right) \triangleq n_T^{-1} \sum_{j=1}^{n_T} L_{\psi_i(T_m+\tau+bn_T+j-1)} \left( \hat{\beta}_{T_m+bn_T+j-1} \right) \). The interpretation of \( G_{\text{max},h} \) is essentially the same as of \( B_{\text{max},h} \), the only difference arising from the denominator \( \sum D_{h,b+1} \) that estimates the within-block variance. A version that uses all overlapping blocks is

\[
MG_{\text{max},h} \left( T_n, \tau \right) \triangleq \max_{i=n_T, \ldots, T_n-n_T} \left| \frac{n_T^{-1} \sum_{j=i+1}^{i+n_T} SL_{\psi_i(T_m+\tau+j-1)} \left( \hat{\beta}_{T_m+j-1} \right) - n_T^{-1} \sum_{j=i-n_T+1}^{i} SL_{\psi_i(T_m+\tau+j-1)} \left( \hat{\beta}_{T_m+j-1} \right) }{\sqrt{D_{h,b+1}}} \right|,
\]

where \( D_{h,b+1} \triangleq n_T^{-1} \sum_{j=i+1}^{i+n_T} \left( L_{\psi_i(T_m+\tau+j-1)} \left( \hat{\beta}_{T_m+j-1} \right) - L_{\psi_i} \left( \hat{\beta} \right) \right)^2 \), with

\[
L_{\psi_i} \left( \hat{\beta} \right) \triangleq n_T^{-1} \sum_{j=i+1}^{i+n_T} L_{\psi_i(T_m+\tau+j-1)} \left( \hat{\beta}_{T_m+j-1} \right).
\]

As argued above, it is useful to consider versions of the statistic \( B_{\text{max},h} \) and \( MB_{\text{max},h} \) that are normalized by their asymptotic variance:

\[
Q_{\text{max},h} \left( T_n, \tau \right) \triangleq \nu_L^{-1} \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor-2} \left| B_{b+1} - B_{h,b} \right|,
\]

and similarly,

\[
MQ_{\text{max},h} \left( T_n, \tau \right) \triangleq \nu_L^{-1} \max_{i=n_T, \ldots, T_n-n_T} \left| \frac{n_T^{-1} \sum_{j=i+1}^{i+n_T} SL_{\psi_i(T_m+\tau+j-1)} \left( \hat{\beta}_{T_m+j-1} \right) - n_T^{-1} \sum_{j=i-n_T+1}^{i} SL_{\psi_i(T_m+\tau+j-1)} \left( \hat{\beta}_{T_m+j-1} \right) }{\sqrt{D_{h,b+1}}} \right|.
\]

3 Continuous Record Distribution Theory for the Test Statistics

3.1 Asymptotic Distribution under the Null Hypotheses

We begin with a set of assumptions. Assumption 3.3 below is a finite-moment condition on the sequence of rescaled forecast losses and and on its first-order derivative. It has a similar scope to A4
in Giacomini and Rossi (2009). Assumption 3.4 is similar to A5 in Giacomini and Rossi (2009) and it imposes the first-order derivative of the forecast losses to be constant over time. It trivially holds when one employs the same loss function for estimation and evaluation. Assumption 3.6 demands the existence of a consistent estimator for $\beta^*$ at the parametric rate $\sqrt{T}$ and it encompasses many estimation procedures. In model (2.3), the popular least-squares method will satisfy the condition [cf. Barndorff-Nielsen and Shephard (2004) and Li, Todorov, and Tauchen (2017)].

**Assumption 3.1.** The process $\{Y_t - (\beta^*)'X_t\}_{t \in [0, N]}$ takes value in an open set $\mathcal{E} \subseteq \mathbb{R}$, and $\beta^*$ takes value in a compact parameter space $\Theta \subset \mathbb{R}^{\dim(\beta)}$.

**Assumption 3.2.** For any $L \in L_e$ we assume $L : \mathcal{E} \times \Theta \mapsto \mathbb{R}$ is a measurable function and $L \in C^{2,2}$ (i.e., twice continuously differentiable in both arguments). For every open set $B$ that contains $\beta^*$ there exists $C < \infty$ such that for all $k \geq 1$, $\sup_{\beta \in B} \|\partial^2 L_{\psi,kh}(\Delta_h e_k; \beta)/\partial \beta \partial \beta'\| < C$.

**Assumption 3.3.** We have $\sup_{k=1,\ldots,T} \mathbb{E}_\sigma \| (L_{\psi,kh}(\Delta_h e_k; \beta^*), \partial L_{\psi,kh}(\Delta_h e_k; \beta^*)/\partial \beta)' \|^4 + \omega \leq \infty$, for $\omega > 0$.

**Assumption 3.4.** For all $k \geq 1$, $\mathbb{E}_\sigma [\partial L_{\psi,kh}(\Delta_h e_k; \beta^*)/\partial e] = \mathcal{K}$, for some $\mathcal{K} < \infty$.

**Assumption 3.5.** For all $k \geq 1$, $|\partial L_{\psi,kh}(e; \cdot)|/\partial e$ is bounded on bounded sets.

**Assumption 3.6.** There exists a sequence $\{\hat{\beta}_k\}_{k=T_m}^{T-\tau}$ such that $\|\hat{\beta}_k - \beta^*\| = O_p \left(1/\sqrt{T}\right)$ uniformly over $k = T_m, \ldots, T - \tau$.

Our asymptotic results are valid under the following conditions on the auxiliary sequence $n_T$.

**Condition 1.** The sequence $\{n_T\}$ satisfies for some $\epsilon > 0$,

$$n_T \to \infty \quad \text{as} \quad T \to \infty \quad \text{and} \quad T^* n_T^{-1} + n_T^{-3/2} h \sqrt{\log (T)} \to 0. \quad (3.1)$$

Condition 1 imposes a lower bound and an upper bound on the growth condition of the sequence $\{n_T\}$. The first part of (3.1) requires $n_T$ to grow to infinity at any faster rate than $T^\epsilon$ with $\epsilon > 0$, which we interpret as saying that the number of observations $n_T$ in each block cannot be too small. The second part of (3.1) provides an upper bound on the growth of $n_T$ and relates to Assumption 2.2 concerning the smoothness of $\{\sigma_{e,t}\}_{t \geq 0}$ thereby ensuring that, for example, the random oscillations of $B_{\max,h}(T_n, \tau)$ can be controlled. As we shall explain in the simulation study of Section 7, we recommend to set $n_T \propto T_n^{2/3-\epsilon}$ for small $\epsilon > 0$. 

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3.1.1 Asymptotic Distribution Under Quadratic Loss Function

**Theorem 3.1.** Let $\gamma_{mT} = [4 \log (m_T) - 2 \log (\log (m_T))]^{1/2}$ and recall $m_T = \lfloor T_n/n_T \rfloor$. Assume Assumption 2.1-2.2, 3.1-3.6, and Condition 1 hold. Let $\mathcal{V}$ denote a random variable defined by $P(\mathcal{V} \leq v) = \exp \left(-\pi^{-1/2} \exp (-v)\right)$. Under $H_0$, we have

(i) $\sqrt{\log (m_T)} \left(2^{-1/2} n_T^{1/2} B_{\max, h} (T_n, \tau) - \gamma_{mT}\right) \Rightarrow \mathcal{V}$;

(ii) $2^{-1/2} \sqrt{\log (m_T)} n_T^{1/2} M B_{\max, h} (T_n, \tau) - 2 \log (m_T) - \frac{1}{2} \log \log (m_T) - \log 3 \Rightarrow \mathcal{V}$.

**Corollary 3.1.** Under the same assumptions of the previous theorem, we have under $H_0$, $\sqrt{\log (m_T)} \left(n_T^{1/2} \nu_L^{-1} Q_{\max, h} (T_n, \tau) - \gamma_{mT}\right) \Rightarrow \mathcal{V}$ and

$$\sqrt{\log (m_T)} \left(n_T^{1/2} \nu_L^{-1}\right) M Q_{\max, h} (T_n, \tau) - 2 \log (m_T) - \frac{1}{2} \log \log (m_T) - \log 3 \Rightarrow \mathcal{V},$$

where $\mathcal{V}$, $m_T$ and $\gamma_{mT}$ are defined as in the previous theorem.

Theorem 3.1 shows that the asymptotic null distribution of our test statistics follows an extreme vale distribution whose critical values can be computed directly. In nonparametric change-point analysis, Wu and Zhao (2007) and Bibinger, Jirak, and Vetter (2017) have derived an extreme value null distribution for tests statistics which share a similar form to ours. As it is stated, the tests statistics are not yet feasible because the asymptotic variances $\nu_L^2$ is unknown. However, we can find statistical consistent estimators which can be used in place of $\nu_L^2$ to make the test feasible. We relegate the treatment of its consistent estimation to Section 4.

3.1.2 Asymptotic Distribution Under General Loss Function

**Theorem 3.2.** Under the same assumptions of the previous theorem and with $\mathcal{V}$, $m_T$ and $\gamma_{mT}$ defined analogously, we have under $H_0$,

(i) $2^{-1/2} \sqrt{\log (m_T)} \left(n_T^{1/2} G_{\max, h} (T_n, \tau) - \gamma_{mT}\right) \Rightarrow \mathcal{V}$;

(ii) $2^{-1/2} \sqrt{\log (m_T)} n_T^{1/2} M G_{\max, h} (T_n, \tau) - 2 \log (m_T) - \frac{1}{2} \log \log (m_T) - \log 3 \Rightarrow \mathcal{V}$.

**Corollary 3.2.** Under the same assumptions of the previous theorem, we have under $H_0$, $\sqrt{\log (m_T)} \left(n_T^{1/2} \nu_L^{-1} Q_{\max, h} (T_n, \tau) - \gamma_{mT}\right) \Rightarrow \mathcal{V}$ and $\sqrt{\log (m_T)} n_T^{1/2} \nu_L^{-1} M Q_{\max, h} (T_n, \tau) - 2 \log (m_T) - \frac{1}{2} \log \log (m_T) - \log 3 \Rightarrow \mathcal{V}$.

4 Estimation of Asymptotic Variance

The purpose of this section is to show how to construct an asymptotically valid estimator of the variance $\nu_L^2$ that enters the definition of our test statistics. This is an important aspect that together
with the selection of the block length might affect statistical inferences based on the proposed tests in finite-samples. Allowing for misspecification is customary in the forecasting literature, and as a consequence this may result in forecast losses that artificially exhibit heteroskedasticity and serial dependence in small samples.

4.1 Estimation of the Asymptotic Variance

We begin with the case of stationary forecast losses, including constant \( \nu_L \) as a special case.

4.1.1 Stationary Forecast Losses

Recall that our test statistics are related to a maximum over blocks of data. Thus, for i.i.d. forecast losses one can use the following estimator for \( \nu_L \) in \( Q_{\max,h} (T_n, \tau) \):

\[
\hat{\nu}_{Q1,b}^2 \triangleq \frac{2}{nT} \sum_{j=1}^{nT} \left[ SL_{\psi,T_m+\tau+bnT+j-1} \left( \hat{\beta}_{T_m+bnT+j-1} \right) - SL_{\psi} \right]^2,
\]

where \( SL_{\psi} \triangleq nT^{-1} \sum_{j=1}^{nT} SL_{\psi,T_m+bnT+j-1} \left( \hat{\beta}_{T_m+bnT+j-1} \right) \). The estimator \( \hat{\nu}_{Q1,b}^2 \) normalizes the difference in the out-of-sample forecast losses between the \( b+1 \) and \( b \) blocks. The statistic \( Q_{\max,h} (T_n, \tau) \) then results in \( Q_{\max,h} (T_n, \tau) = \max_{k=0, \ldots, \lceil T_n/nT \rceil - 2} \left( (B_{h,b+1} - B_{h,b}) / \hat{\nu}_{Q1,b+1} \right)^2 \). For the overlapping blocks case, the estimator is \( \hat{\nu}_{MQ1,i}^2 \triangleq 2nT^{-1} \sum_{j=i+1}^{i+nT} \left( SL_{\psi,T_m+\tau+j-1} \left( \hat{\beta}_{T_m+j-1} \right) - SL_{\psi,i} \right)^2 \), where \( SL_{\psi,i} \triangleq nT^{-1} \sum_{j=i+1}^{i+nT} SL_{\psi,T_m+\tau+j-1} \left( \hat{\beta}_{T_m+j-1} \right) \) so that we can write

\[
MQ_{\max,h} (T_n, \tau) \triangleq \max_{i=nT, \ldots, T_n-nT} \left| \hat{\nu}_{MQ1,i}^{-1} nT^{-1} \sum_{j=i+1}^{i+nT} SL_{\psi,T_m+\tau+j-1} \left( \hat{\beta}_{T_m+j-1} \right) - \sum_{j=i-nT+1}^{i} SL_{\psi,T_m+\tau+j-1} \left( \hat{\beta}_{T_m+j-1} \right) \right|.
\]

Both \( \hat{\nu}_{Q1,b+1}^2 \) and \( \hat{\nu}_{MQ1,i}^2 \) apply a natural block-wise normalization in order to guarantee a distribution-free limit under \( H_0 \). However, it is useful to consider estimators that use all of the observations in the out-of-sample period. Thus, one exploits covariance stationarity of the sequence of forecast losses. Let \( \Phi_{0.75} = 0.647\ldots \) denote the third quartile of the standard normal distribution and define

\[
\hat{\nu}_{2,h} \triangleq \frac{\sqrt{nT}}{2 (m_T - 1)} \sum_{b=1}^{m_T-1} |B_{h,b} - B_{h,b-1}|, \quad \hat{\nu}_{3,h} \triangleq \frac{\sqrt{nT}}{\sqrt{2} \Phi_{0.75}} \left( \sum_{b=1}^{m_T-1} |B_{h,b} - B_{h,b-1}|^2 \right)^{1/2}, \quad \hat{\nu}_{4,h} \triangleq \frac{\sqrt{nT}}{\sqrt{2} \Phi_{0.75}} \text{median} \left( |B_{h,b} - B_{h,b-1}| \right), \quad 1 \leq b \leq m_T - 1.
\]
Note that $\hat{\nu}_{2,h}$, $\hat{\nu}_{3,h}$ and $\hat{\nu}_{4,h}$ can be used to implement both $Q_{\text{max},h}(T_n, \tau)$ and $MQ_{\text{max},h}(T_n, \tau)$. $\hat{\nu}_{3,h}$ is related to Carlstein’s (1986) subseries variance estimate in the context of strong mixing processes and it was also used by Wu and Zhao (2007). Each of the estimators $\hat{\nu}_{2,h}$, $\hat{\nu}_{3,h}$ and $\hat{\nu}_{4,h}$ allows for dependence but requires stationarity. The simulation study in Wu and Zhao (2007) suggests that $\hat{\nu}_{4,h}$ is more robust whereas $\hat{\nu}_{2,h}$ and $\hat{\nu}_{3,h}$ are less precise when there are large instabilities or jumps. For two sequences $\{a_k\}$ and $\{b_k\}$, we write $a_k \approx b_k$ if for some $c \geq 1$, $b_k/c \leq a_k \leq cb_k$ for all $T$.

**Condition 2.** The sequence $\{n_T\}$ satisfies

$$n_T \to \infty \quad \text{as} \quad T \to \infty \quad \text{and} \quad \sqrt{T_n n_T^{-1} \log (T_n)}^3 + n_T T_n^{-2/3} (\log (T))^{1/3} \to 0. \quad (4.1)$$

**Theorem 4.1.** In addition to the assumptions of Theorem 3.1, assume that $\text{Cov} \left( L_{\psi,kh} (\beta^*) , L_{\psi,(k-j)h} (\beta^*) \right)$ depends on $j$ but not on $kh$. Then, under $H_0$, (i) Let $n_T \asymp T^{5/8}$. Then, $\hat{\nu}_{2,h}, \hat{\nu}_{4,h} = \nu_L + O_P \left( T_n^{-1/16} \log (T_n) \right)$; (ii) Let $n_T \asymp T^{1/3}$. Then $\mathbb{E} \left( \left[ \hat{\nu}_{3,h} - \nu_L^2 \right]^2 \right) = O \left( T_n^{-2/3} \right)$.

Under covariance-stationarity, given Theorem 4.1, the results of Corollary 3.1-3.2 are applicable after replacing $\nu_L$ by an appropriate consistent estimator.

### 4.1.2 Non-Stationary Forecast Losses

We now consider estimation of the asymptotic variance in the case the forecast losses are heterogeneous. The estimator $\hat{\nu}$ that we introduce below depends on the specific loss function and thus it can be used for replacing $\nu_L$ in Corollary 3.1-3.2. Non-stationarity implies that $\sigma^2_{e,t}$ is time-varying and thus the results of Theorem 4.1 are not applicable due to the presence of many extra parameters that account for the time-varying structure. To deal with this issue we propose a novel block-wise self-normalization technique which simultaneously addresses two issues. First, the block-wise self-normalization ensures that the difference in forecast losses between two adjacent blocks are asymptotically independent across non-adjacent blocks and that within each block the losses are standardized so that time-varying variances cancel out. Second, by computing an average—over all blocks—of the self-normalized difference in forecast losses we account for possible serial dependence. We derive asymptotic results within a general framework based on the strong invariance principle for stationary processes developed in Wu (2007) and extended to modulated stationary processes by Zhao and Li (2013).

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23 They can be also applied to the test statistics $Q_{\text{max},h}^G(T_n, \tau)$ and $MQ_{\text{max},h}^G(T_n, \tau)$ with $G_{h,b}$ in place of $B_{h,b}$. 

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For each block \( b = 0, \ldots, m_T - 2 \), let

\[
A_{h,b} (\hat{\beta}) \triangleq n_T^{-1} \sum_{j=1}^{n_T} \left( L_{\psi,T_m+\tau+(b+1)n_T+j-1} (\hat{\beta}_{T_m+(b+1)n_T+j-1}) \right),
\]

\[
V_{h,b} (\hat{\beta}) \triangleq n_T^{-1} \sum_{j=1}^{n_T} \left( L_{\psi,T_m+\tau+(b+1)n_T+j-1} (\hat{\beta}) - L_{\psi,b} (\hat{\beta}) \right)^2,
\]

where \( L_{\psi,b} (\hat{\beta}) = n_T^{-1} \sum_{j=1}^{n_T} L_{\psi,T_m+\tau+(b+1)n_T+j-1} (\hat{\beta}) \) and define the statistic

\[
\zeta_{h,b} (\hat{\beta}) \triangleq \sqrt{n_T} \left( A_{h,b} (\hat{\beta}) - A_{h,b-1} (\hat{\beta}) \right) / \sqrt{V_{h,b}}.
\]

Finally, an average—over all blocks \( m_T \)—of the per-block self-normalized statistics \( \zeta_{h,b} \)'s is used to define an estimator of the asymptotic variance:

\[
\nu^2_L \triangleq 2^{-1} (m_T - 1)^{-1} \sum_{b=0}^{m_T-1} \zeta_{h,b}^2.
\]

Let \( \sigma^2_{L,h} \triangleq \text{Var} (L_{\psi,kh} (\beta^*)) \). We also need to introduce the following quantities,

\[
F_{h,b}^* \triangleq \frac{1}{\sigma_{L,(T_m+\tau+(b+1)n_T)h}}, \quad J_{h,b}^* \triangleq \frac{\sigma^2_{L,(T_m+\tau+(b+1)n_T)h}}{\sigma^2_{L,(T_m+\tau+(b+1)n_T)h} \sigma^4_{L,(T_m+\tau+(b+1)n_T)h}} \left( \sum_{j=1}^{n_T} \sigma^2_{L,(T_m+\tau+(b+1)n_T+j)h} \right)^{1/2},
\]

**Theorem 4.2.** Under Condition 2 we have \( \nu^2_L - \nu^2_L = O_P (r^{-1}_h) \), where \( r_h = O_P \left( T^\epsilon_n / (\log (T_n))^2 \right) \) with \( \epsilon \in (0, 1/4) \) such that \( r_h \to \infty \).

The theorem simply states that \( \hat{\nu}_L \) is consistent for \( \nu_L \) and therefore the asymptotic results of Section 3 continue to hold when we replace \( \nu_L \) by \( \hat{\nu}_L \).

## 5 Continuous Semimartingale Volatility and Asymptotic Local Power

### 5.1 Asymptotic Results under Continuous Semimartingale Volatility

In this section we relax the Lipschitz condition on \( \sigma_{e,t} \) and extend the results for the quadratic loss case from Theorem 3.1 to stochastic volatility models driven by a Wiener process. Consequently, this relaxation enables one to utilize the tests proposed in this paper in setting involving high-frequency financial variables. More specifically, we assume that \( \sigma_{e,t} \) is an Itô continuous semimartingale that is almost surely bounded and strictly positive adapted process. We replace Assumption 2.2 by the following.
Assumption 5.1. Under $H_0$ the process $\{\sigma_{\varepsilon,t}\}_{t \geq 0}$ satisfies $\phi_{\sigma,\eta;\tau_h \land N} \leq K_h \eta^\kappa$ for some $\kappa > 0$, some sequence of stopping times $\tau_h \to \infty$ and some $\mathbb{P}$-a.s. finite random variable $K_h$.

The assumption implies that $\sigma_{\varepsilon,t}$ belongs to a rather large class of volatility processes usually considered in financial econometrics. The parameter $\kappa$ plays a key role in the testing framework of this section and we refer to it as the regularity exponent. When $\kappa = 1$ we recover the case of Lipschitz volatility considered in the previous sections while the standard stochastic volatility model without jumps correspond to $\kappa = 1/2 - \epsilon$ for a sufficiently small $\epsilon > 0$. Next, we have a slightly different version of Condition 1.

Condition 3. The sequence $\{n_T\}$ satisfies for some $\epsilon > 0$,

$$n_T \to \infty \quad \text{as} \quad T \to \infty \quad \text{and} \quad T^{n_T^{-1}} + \sqrt{n_T}(n_T \epsilon)^{\kappa} \sqrt{\log(T)} \to 0. \tag{5.1}$$

For Itô continuous semimartingale volatility $\sigma_{\varepsilon,t}$ the condition suggests $n_T \propto T^{1/2-\epsilon}$ for small $\epsilon > 0$. Let $\Gamma_t = \mathbb{E}_\sigma[dL_t(\varepsilon_t; \beta^*)]/dt$.

The more general framework considered here requires us to consider the following null hypotheses: under quadratic loss, $H_0$:

$$\{\Gamma_t\}_{t \in [N_{in} + h, N]} \in \mathcal{C}(\kappa, K_h), \tag{5.2}$$

where $\mathcal{C}(\kappa, K_h)$ is a class of continuous functions on $[N_{in} + h, N]$,

$$\mathcal{C}(\kappa, K_h) \triangleq \left\{ \{\Gamma_t\}_{t \in [N_{in} + h, N]} : \sup_{s,t \in [N_{in} + h, N], |t-s| < \eta} |\Gamma_t - \Gamma_s| \leq K_h \eta^\kappa \right\},$$

where $\kappa > 0$ and $K_h$ is given in Assumption 5.1. Thus, we wish to discriminate between $H_0$ and

$$H_1 : \exists \lambda \in [N_{in} + h, N] \quad \text{with} \quad \{\Gamma_t(\omega)\}_{t \in [N_{in} + h, N]} \in \mathcal{J}_\lambda(\kappa, K_h, d_h), \tag{5.3}$$

where $\mathcal{J}_\lambda(\kappa, K_h, d_h) \triangleq \left\{ \{\Gamma_t\}_{t \in [N_{in} + h, N]} : \{\Gamma_t - \Delta \Gamma_t\}_{t \in [N_{in} + h, N]} \in \mathcal{C}(\kappa, K_h) ; |\Delta \Gamma_{\lambda}| \geq d_h \right\}, \Delta \Gamma_{\lambda} = \Gamma_{\lambda} - \lim_{s \uparrow \lambda} \Gamma_s$ and $\{d_h\}$ is a decreasing sequence. The following theorem extends Theorem 3.1 to the current setting.

Theorem 5.1. Let $m_T$, $\gamma_{m_T}$ and $\psi$ as defined in Theorem 3.1. Assume the assumptions of Theorem 3.1 hold with Assumption 2.2 replaced by Assumption 5.1. Under Condition 3 and quadratic loss, the same results of Theorem 3.1 hold.

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24 For example, for the quadratic loss with $\mu_{\varepsilon,t} = 0$ the notation reduces to $\Gamma_t = \sigma^2_{\varepsilon,t}$. 

5.2 Asymptotic Local Power

In this section we consider the behavior of $\text{MQ}_{\text{max}, h}$ under a sequence of local alternatives.

**Assumption 5.2.** We have the same assumptions as in Theorem 5.1 and assume (i) in model (2.2) we replace $\beta^*$ by $\beta_t = \beta^* + \mu_{\beta,t} / (\log (T_n) n_T)^{1/4}$ where $\mu_{\beta,t} \in \mathbb{R}^q$ is $\mathbb{P}$-a.s. locally bounded and adapted process; (ii) we set $\mu_{e,t} = 0$ for all $t \geq 0$; (iii) we replace Assumption 3.6 by $\|\tilde{\beta}_k - \beta^*\| = \mu_{\beta,kh} / (\log (T_n) n_T)^{1/4} + O_p(T^{-1/2})$ uniformly in $k$.

Part (iii) is a consequence of part (i) as it can be easily verified. Let

$$
\tilde{\text{MQ}}_{\text{max}, h}(T_n, \tau) \triangleq \nu_L^{-1} \max_{i=n_T, \ldots, T_n-n_T} \left| n_T^{-1} \sum_{j=i+1}^{i+n_T} (SL_{\psi,T_m+\tau+j-1}(\tilde{\beta}_{T_m+j-1}) - 2\zeta_{\mu,j,+}) 
- n_T^{-1} \sum_{j=i-n_T+1}^{i} (SL_{\psi,T_m+\tau+j-1}(\tilde{\beta}_{T_m+j-1}) - 2\zeta_{\mu,j,-}) \right|,
$$

where

$$
\zeta_{\mu,j,+} \triangleq \mu'_{\beta,(T_m+\tau+j-1)h}X_{(T_m+\tau+i-1)h} \mu_{\beta,(T_m+\tau+j-1)h} / (\log (T_n) n_T)^{1/2}
$$

and

$$
\zeta_{\mu,j,-} \triangleq \mu'_{\beta,(T_m+\tau+j-1)h}X_{(T_m+\tau+i-n_T-1)h} \mu_{\beta,(T_m+\tau+j-1)h} / (\log (T_n) n_T)^{1/2}.
$$

**Theorem 5.2.** Under Assumption 5.2,

$$
\sqrt{\log (m_T) (n_T^{1/2} \nu_L^{-1})} \tilde{\text{MQ}}_{\text{max}, h}(T_n, \tau) - 2 \log (m_T) - \frac{1}{2} \log \log (m_T) - \log 3 \Rightarrow \mathcal{V},
$$

where $\mathcal{V}$, and $m_T$ are defined as in Theorem 3.1.

**Remark 5.1.** (i) The theorem suggests that under the local alternatives $\beta_t = \beta^* + \mu_{\beta,t} / (\log (T_n) n_T)^{1/4}$ there is a bias term arising from the presence of $\zeta_{\mu}$. This bias term does not vanish asymptotically and results in shifting the center of the distribution. Moreover, it depends on the second moments of the regressors and on the function $\mu_{\beta}$; (ii) The theorem illustrates the sensitivity of the asymptotic power to the form of the alternative. We can attempt to compare Theorem 5.2 with the local power result regarding the sup-Wald test of Andrews (1993). Unlike Theorem 4 in Andrews (1993), our result suggests that the location of the instability should not play any special role and the power should not be sensitive to whether the break in predictive ability occurs at middle sample or toward the tail of the sample. This follows because of the local nature of our test statistic and contrasts with classical tests for parameter instability and structural change since their performance hinges on the location of the break [see Deng and Perron (2008), Kim and Perron (2009)
and Perron and Yamamoto (2016) for additional results on the power of classical structural break tests. However, the magnitude of the break—here shrinking at rate \( (\log (T_n))^{1/4} \)—under our specification of the local alternatives is larger than the one considered by Andrews (1993)—which shrinks at rate \( 1/\sqrt{T} \). This implies a trade-off between location and magnitude of the break, and it is consistent with the evidence provided in our simulation study; (iii) Although not shown here, the local power of the tests is the same when a subset of the vector \( \beta \) is not subject to shift.

Theorem 5.2 can be used to show that our test possesses nontrivial power against alternatives for which the parameter \( \beta_t \) is time-varying and non-smooth.

**Corollary 5.1.** Suppose the assumptions of the previous theorem hold with \( \beta_t = \beta^* + c \mu_{\beta,t}/(\log (T_n))^{1/4} \), where \( c \in \mathbb{R} \). If \( \mu_{\beta} \) and/or \( \{\mu_{\beta,t} \cdot \sigma_{X,t} \}_{t \geq 0} \) is non-smooth, we have

\[
\lim_{c \to \infty} \lim_{h \downarrow 0} \mathbb{P} \left( \sqrt{\log (m_T) n_T} \nu_L^{-1} \text{MQ}_{\text{max},h}(T_n, \tau) - 2 \log (m_T) - \frac{1}{2} \log \log (m_T) - \log 3 > cv_{1-\alpha} \right) = 1,
\]

where \( cv_{1-\alpha} \) is the level \( (1 - \alpha) \) critical value of the distribution of \( \mathcal{V} \) and \( \alpha \in (0, 1) \).

### 6 Extensions

A number of extensions is treated in our companion paper Casini (2018). As explained above, it would be useful to ensure that there are no instabilities in the in-sample period \([0, N_{\text{in}}]\). We propose a procedure that involves a pre-test about instability on \([0, N_{\text{in}}]\) for a given \( N_{\text{in}} \) chosen by the forecaster. Instabilities in the in-sample \([0, N_{\text{in}}]\) are much easier to be detected relative to instabilities in the out-of-sample because they do not face the so-called “contamination effect”. The latter arises, for example under the recursive and rolling scheme, when the instability originally occurring in the out-of-sample eventually enters the moving in-sample window [cf. Casini and Perron (2017c) and Perron and Yamamoto (2018)]. The consequence is that existing tests face substantial power losses. This property is not shared by our test statistics because of their local nature. Our procedure works very well and we show through simulations that instabilities occurring in the in-sample only or occurring both in the in-sample and in the out-of-sample simultaneously, lead easily to rejection of the null hypotheses relative to instabilities occurring in the out-of-sample only—as we consider here.

A second issue is that, in this paper, we have considered processes that have a continuous sample path under the null hypotheses. Thus, it is of interest to extend the results to a setting that involves jump processes which are important in high-frequency financial data. This can be achieved by using techniques that are able to separate the continuous part from the discontinuous part of a Itô semimartingale [see e.g. Li, Todorov, and Tauchen (2017) and Li and Xiu (2016)].
Another important issue is the estimation of the time at which forecast instability occurs. Once the null hypotheses has been rejected, a forecaster may take into consideration the possibility of revising the forecasting method and/or model. Hence, it becomes crucial to learn some information about the timing of the instability. For example, consider the case of a one-time structural change in a parameter of the data-generating process at time $T^0_b = \lfloor T^0 \lambda^0_b \rfloor$, where $T^0_b$ is the break point and $\lambda^0_b \in (0, 1)$ is the fractional break date. Once $H_0$ is rejected, a forecaster would benefit from knowing that the forecast method originally employed is found to statistically either under or over-perform over part of the sample after $T^0_b$ relative to the part prior $T^0_b$. Then, a forecaster would entertain the possibility of modifying the forecast model in order to generate future forecasts for $Y_t$. Not only the forecast model might be revised but most importantly, knowledge of beginning of the instability at $T^0_b$ can be further exploited to design the forecasting method for the future forecasts. It would be inappropriate, for instance, to use a rolling scheme where the rolling window used to construct the forecast include observations prior to $T^0_b$ since those observations provide little informational content for the purpose of predicting $Y_t$ after the change-point $T^0_b$. On the other hand, this line of reasoning is justified by this particular example and indeed in practice many issues arise when dealing with the timing of the insatiability in our context. For example, the exact form of the insatiability may be unknown. Under the latter scenarios, there is no clear-cut break date $T^0_b$ that can be defined. Thus, it is less obvious how a forecaster should proceed in those cases. Nonetheless, one can meaningfully think about the timing of the instability by not just attempting to estimate $T^0_b$—which is not clear how it is defined—but rather attempting to detect the initial date in the sample after which the forecasts become unstable as well as to detect the last date after which the forecasts remain stable relative to the in-sample period. Since our test statistics are local in nature, one can introduce a procedure which sequentially tests the hypotheses $H_0$ in regions of the sample where $H_0$ has not yet been rejected. One then records the number of times for which $H_0$ is rejected and estimates the corresponding change-point dates. After ordering these change-point dates, one has finally access to useful information such has the initial timing of the forecast instability and the last part of the out-of-sample period which remains stable. Such information can arguably be advantageous to the forecaster.

7 Small-Sample Evaluation

We now examine the empirical size and power of our proposed tests and compare them to those of Giacomini and Rossi (2009), abbreviated GR (2009). In particular, we consider both the uncorrected and corrected version of the $t_{\text{stat}}^{T_m,T_n,\tau}$ statistic of GR (2009). Size and power properties

\[25\text{We use a superscript c to indicate the corrected version: } t_{\text{stat,c}}.\]
for the Quadratic loss with fixed scheme are reported in Section 7.1 and Section 7.2, respectively. The Supplemental Material includes corresponding simulation studies for the recursive and rolling schemes and for the Linex loss; these results are not reported here because they are qualitatively equivalent. Overall, one can draw the following conclusions from our simulation study. In terms of size control, the statistics $B_{\text{max},h}$ and $Q_{\text{max},h}$ are comparable with the corrected version $t_{\text{stat},c}$ proposed by GR (2009).\footnote{As shown by GR (2009), the uncorrected version of $t_{\text{stat}}$ can be oversized for models that induce serial dependence in the forecast losses. The authors then proposed a finite-sample correction and did not consider $t_{\text{stat}}$ further in their power analysis. Similarly, GR (2009) showed that just using classical structural break tests in this context is not very helpful as they might have statistical power equals to the size in some cases. Moreover, simulations in Perron and Yamamoto (2018) confirmed that, under rolling and recursive scheme, structural break tests suffer power losses which can be attributed to a so-called “contamination effect” arising when the instability enters the in-sample window [see also Casini and Perron (2017c)].} Moreover, the test $MQ_{\text{max},h}$ that uses overlapping blocks is also comparable in terms of size. The same is not true for $MB_{\text{max},h}$ because often it seems to be somewhat liberal. Turning to the power comparison, each of our test statistics $B_{\text{max},h}$, $Q_{\text{max},h}$ and $MQ_{\text{max},h}$ displays significant power gains over the $t_{\text{stat}}^{m_T,T_n,T_n}$ statistics especially as the period of instability (i) is comparatively short relative to the total sample size and/or (ii) is not located at middle sample. In the latter circumstances, the gains in power are, uniformly over different data-generating processes and over parameter break magnitudes, on the order of 30-40%.

Throughout, we restrict attention to one-step ahead forecast horizon (i.e., $\tau = 1$), and we use the same loss function for estimation and evaluation. We use our asymptotic results as an approximation for the case where $h = 1$ in our theoretical model in (2.3) and consider discrete-time DGPs. In models with serially correlated losses (i.e., S2 and S6 below) for the statistics $Q_{\text{max},h}$ and $MQ_{\text{max},h}$ we employ the long-run variance estimator from Theorem 4.2. With regards to the tests of GR (2009) we use the appropriate version of $t_{\text{stat}}$ and of $t_{\text{stat},c}$.\footnote{As recommended by GR (2009) we set the truncation lag of their HAC estimator equal to $\left\lfloor \frac{T_n^{1/3}}{3} \right\rfloor$; we also the use the truncation lag $\left\lfloor 0.75T_n^{1/3} \right\rfloor$.}

Remark 7.1. Implementation of our tests statistics requires to choose the number of blocks $m_T$—satisfying Condition 1. The finite-sample properties can be sensitive to the choice of $m_T$. This is confirmed in our numerical study, where assigning larger values to $m_T$ than the smallest one allowed by the condition may result in oversized tests. Therefore, we recommend practitioners to set $m_T$ equal to the smallest integer as allowed by Condition 1. This is the strategy we have adopted in the Monte Carlo study of this section, and as we will show, it results in approximately correct size and good power across different data-generating mechanisms.
7.1 Empirical Size

We consider discrete-time DGPs of the form

\[ Y_t = \mu + \beta X_{t-1} + e_t, \quad t = 1, \ldots, T, \]  

(7.1)

for various in-sample and out-of-sample sizes and with a total sample size ranging from \( T = 100 \) to \( T = 500 \). Note that (7.1) is a special case of the theoretical model with a sampling interval \( h = 1 \). We consider six versions of (7.1), where the first and second specification (S1 and S2 below) are calibrated to the Phillips curve model of U.S. inflation from Staiger, Stock, and Watson (1997): S1 involves \( \mu = 2.73 \), \( \beta = -0.44 \), and where \( \{X_t\} \) and \( \{e_t\} \) are independent sequences of zero-mean i.i.d. Gaussian disturbances with unit variance; S2 is the same as S1 but with ARCH errors \( e_t = \sigma_{e,t} u_t, \sigma_{e,t} = 1 + 0.5 e_{t-1}^2 \) with \( u_t \sim \mathcal{N}(0,1) \); S3 specifies \( \{X_t\} \) to follow a zero-mean Gaussian AR(1) with autoregressive coefficient 0.4, \( \beta = 1 \) and \( e_t \sim \mathcal{N}(0,0.49) \) independent of \( X_t \); S5 is a model with a lagged dependent variable \( X_{t-1} = Y_{t-1}, \mu = 0, \beta = 0.3 \) and \( e_t \sim \mathcal{N}(0,0.49) \); S6 involves serially correlated disturbances \( e_t = 0.3 e_{t-1} + u_t, u_t \sim \mathcal{N}(0,1) \).

Table 1-2 report the rejection rates for significance levels \( \alpha = 0.05 \) and 0.10 for model S1-S2. Results for the other DGPs can be found in Table S-1-S-3 in the Supplement. We first focus on i.i.d. forecast losses (i.e., models S1 and S3-S5). Both \( B_{\max,h} \) and \( Q_{\max,h} \) are well-sized. As the sample size increases their performance improves and we note that their rejection frequencies are closer to the nominal level when the in-sample size is one half of the total sample. In model S1, when the in-sample size is 0.25\( T \), \( B_{\max,h} \) and \( Q_{\max,h} \) tend to be slightly conservative while the opposite occurs when in-sample size is 0.75\( T \). The version of \( B_{\max,h} \) that uses overlapping blocks (\( MB_{\max,h} \)) can be quite liberal (cf. models S1 and S3). In contrast, \( MQ_{\max,h} \) seems to control the size well, though it tends to be slightly liberal but that depends on the relative size of the in-sample and out-of-sample windows. We observe that there is no clear pattern in size performance for our test statistics as we raise the sample size \( T \). The reason is straightforward: as we raise \( T \) we also need to adjust the choice of \( m_T \) (the number of blocks) in accordance with Condition 1. This explains why, for example, in Table 1, top panel, the empirical size of \( Q_{\max,h} \) for \( (T_m = 100, T_n = 100) \) is better than for \( (T_m = 150, T_n = 150) \). Turning to the \( t^{stat} \) statistics of Giacomini and Rossi (2009), the uncorrected version performs better than the corrected version since the latter systematically displays an empirical size 2-3\% below the nominal level. We can conclude that in models with i.i.d. errors the statistics \( B_{\max,h}, Q_{\max,h}, MQ_{\max,h} \) and \( t^{stat} \) are comparable in terms of empirical size, whereas \( MB_{\max,h} \) and \( t^{stat,c} \) tend to over-reject and under-reject, respectively.

Let us now turn to models with serially correlated losses. When the disturbances follow an ARCH process, (cf. model S2, Table 2), we observe that both statistics that do not use overlapping
blocks, $B_{\text{max,} h}$ and $Q_{\text{max,} h}$, show reasonable size control. The same feature applies to $MQ_{\text{max,} h}$ while $MB_{\text{max,} h}$ displays rejection rates that are systematically above the significance level. It also appears that the corrected version of the statistic of GR (2009) is now regularly below the nominal level. In contrast, the uncorrected version $t^\text{stat}$ seems to control size well. When the errors follow an autoregressive process (cf. model S6), Table S-3 shows that $t^\text{stat}$ and $MB_{\text{max,} h}$ are arbitrarily oversized for all sample sizes. $MQ_{\text{max,} h}$ and $t^\text{stat,c}$ possess rejection rates frequently below the desired nominal level. The statistic that shows the best empirical sizes across different $T$ is $Q_{\text{max,} h}$.

Overall, our analysis on the size properties of the tests suggests that when the DGP involves i.i.d. errors it is fair to compare $B_{\text{max,} h}$, $Q_{\text{max,} h}$, $MQ_{\text{max,} h}$ and $t^\text{stat}$ whereas the rejection rates of $MB_{\text{max,} h}$ and $t^\text{stat,c}$ tend to deviate systematically from the nominal level. When there are autocorrelated errors, it is difficult to compare $t^\text{stat}$ and $MB_{\text{max,} h}$ with the other statistics because the former can be highly oversized. The statistics that appear to perform better in terms of approximate size control uniformly over different data-generating mechanisms are $Q_{\text{max,} h}$ and $MQ_{\text{max,} h}$.

### 7.2 Empirical Power

We report the small sample power of the tests under various sources of forecast instability. We consider several sample sizes $T$ as well as several designs varying for the distribution of the total sample between in-sample and out-of-sample window. The break date—or the date of the first change-point when more complicated designs are used—is denoted by $T_B^0 = T\lambda_0$, where $\lambda_0 \in (0, 1)$ is the fractional break date. We shall bring special attention to the location of $T_B^0$ in the sample as well as to the duration of the instability (i.e., $T - T_B^0$). We shall see that both factors are actually important for the performance of the methods proposed by Giacomini and Rossi (2009) while our test statistics being local in nature possess essentially uniform power over distinct locations $T_B^0$. Furthermore, our definition of forecast instability does not demand any relationship between the stable and unstable period and thus it is useful to examine the differences in power properties when a one-time change-point is present relative to when short-lasting instabilities arise.

We consider both discrete shifts—a structural break—and recurrent changes in a parameter: model P1a (break in a regression coefficient): $Y_t = 2.73 - 0.44X_{t-1} + \delta X_{t-1}1 \{t > T_B^0\} + e_t$, where $X_{t-1} \sim \text{i.i.d.} \mathcal{N}(0, 1)$ and $e_t \sim \text{i.i.d.} \mathcal{N}(0, 1)$; model P1b: it is the same as model P1a but with $X_{t-1} \sim \text{i.i.d.} \mathcal{N}(1, 1)$; model P2: $Y_t = X_{t-1} + \delta X_{t-1}1 \{t > T_B^0\} + e_t$, where $X_{t-1}$ is a Gaussian AR(1) with autoregressive coefficient 0.4 and unit variance, and $e_t \sim \text{i.i.d.} \mathcal{N}(0, 0.49)$; model P3 (recurrent break in mean): $Y_t = \beta_t + e_t$, where $\beta_t$ switches between $\delta$ and 0 every $p$ periods and $e_t \sim \text{i.i.d.} \mathcal{N}(0, 0.64)$; model P4 (single break in variance): $Y_t = 0.5X_{t-1} + (1 + \delta 1 \{t > T_B^0\}) e_t$ where $X_{t-1} \sim \text{i.i.d.} \mathcal{N}(1, 1)$ and $e_t \sim \text{i.i.d.} \mathcal{N}(0, 1)$; model P5 (recurrent break in variance): $Y_t = \mu + (1 + \beta_t) e_t$, where $\beta_t$ switches between $\delta$ and 0 every $p$ periods and $e_t \sim \text{i.i.d.} \mathcal{N}(0, 0.49)$; model
P6 (lagged dependent variable): \( Y_t = \delta 1 \{ t > T_b^0 \} + 0.3Y_{t-1} + \epsilon_t, \epsilon_t \sim \text{i.i.d.} \mathcal{N} (0, 0.49) \); model P7 (ARCH disturbances): \( Y_t = 2.73 - 0.44X_{t-1} + \delta X_{t-1} 1 \{ t > T_b^0 \} + \epsilon_t \), where \( X_{t-1} \sim \text{i.i.d.} \mathcal{N} (0, 1.5) \) and \( \epsilon_t = \sigma_t u_t, \sigma_t^2 = 0.5 + 0.5e_{t-1}^2, u_t \sim \text{i.i.d.} \mathcal{N} (0, 1) \); model P8 (autocorrelated errors): \( Y_t = 1 + X_{t-1} + \delta X_{t-1} 1 \{ t > T_b^0 \} + \epsilon_t \), where \( X_{t-1} \sim \text{i.i.d.} \mathcal{N} (0, 1.4) \) and \( \epsilon_t = 0.4\epsilon_{t-1} + u_t, u_t \sim \text{i.i.d.} \mathcal{N} (0, 1) \). For models that do not involve recurrent changes we also consider power comparisons when the instability lasts only for some period of time as opposed to the post-\( T_b^0 \) period. This requires replacing \( 1 \{ t > T_b^0 \} \) in models P1-P2, P4 and P6-P8 with \( 1 \{ T_b^0 < t \leq T_b^0 + p \} \) where \( p \) is the number of consecutive observations in which the forecast model is unstable. The value of \( p \) depends on the sample size \( T \). For example, when \( T = 100 \) we set \( p = 10 \); when \( T = 200 \) we set \( p = 20 \) and so on.\(^{28}\) The case of short-term instability is the most prevalent in empirical work because it is very unlikely that a professional forecaster would use a poor-performing predictor or forecast model for many consecutive years (e.g., the whole out-of-sample).

Figure 1-9 in the Appendix plot the power functions for models P1a, P4 and P7. Figure S-1-S-13 in the Supplement plot the power functions for the remaining DGPs. They include several sample sizes ranging from \( T = 100 \) to \( T = 500 \), several in-sample and out-of-sample sizes as well as different locations \( \lambda_0 \) of the breaks. We begin with considering general instabilities first and then move to short-term instabilities. Figure 1-2 reports the results for model P1a. When \( T = 100, 150 \) Figure 1 shows that our tests have good power against model P1a while the tests of GR (2009) seem to be less powerful. For example, when the break date is at \( T_0 = 0.8T \) our tests display reasonable power. However, both \( t_{\text{stat}} \) statistics of GR (2009) perform significantly worse and the associated power curve is bounded away from one even for a very large break size \( \delta = 3 \). This feature disappears when we raise the sample size to \( T \geq 200 \) and maintain the break date at \( T_0 = 0.8T \); see Figure 2. The latter figure also shows that for large sample sizes and instabilities that last for more than 50\% of the out-of-sample (top panels) all tests have good power even though the \( t_{\text{stat}} \) statistics of GR (2009) have slightly higher power. The power turns to be essentially the same when \( \lambda_0 = 0.8 \) (i.e., the instability only lasts for 40\% of the out-of-sample). For model P2, Figure S-3 plots the power functions for \( T = 100, 200 \) and \( \lambda_0 = 0.7, 0.8 \). Except for the pair \( (T = 200, \lambda_0 = 0.7) \) (cf. top-right panel) for which our tests and the \( t_{\text{stat}} \)-type tests display roughly the same power, it is clear that our tests are more powerful than the \( t_{\text{stat}} \) tests (both corrected and uncorrected version). The power gains are substantial and range from 20\% to 40\%. Moreover, as for model P1a and P1b when the instability lasts for less than 50\% of the out-of-sample (cf. \( \lambda_0 = 0.8 \); bottom-left panel) the statistics \( B_{\text{max}, h} \) and \( Q_{\text{max}, h} \) achieve trivial power already for a

\(^{28}\)Note that for \( (T_m = 50, T_n = 50) \) the value \( p = 10 \) corresponds to a period of instability lasting for one-fifth of the out-of-sample; thus, the duration of the instability is nontrivial and consistent with forecasting applications. See the notes to each figure for the other values of \( p \). The title of a figure corresponding to a short-lasting instability is labeled “short-term instability”.

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break magnitude $\delta = 1.5$ whereas the $t^{\text{stat}}$ tests of GR (2009) display rejection rates below 60% even when $\delta = 2$ and yet below 70% when $\delta = 2.5$; that is, their power function does not attain unit power even for very large break magnitudes. These properties characterize all models with i.i.d. errors and extend to models with lagged dependent variables as predictors (cf. model P7; Figure S-10 in the Supplement).

Let us now turn to recurrent breaks in the mean. For recurrent breaks we implement the statistics $\text{MB}_{\text{max},h}$ and $\text{MQ}_{\text{max},h}$ that use overlapping blocks. Figure S-4 plots the power curves for model P3. All tests have power and their performance is essentially the same. Figure 3 corresponds to model P4 (single break in the variance) and shows that when the instability begins in the second half of the out-of-sample (cf. $\lambda_0 = 0.8$; bottom panels) our tests $\text{MB}_{\text{max},h}$ and $\text{MQ}_{\text{max},h}$ achieve good power while the $t^{\text{stat}}$-type tests have little power that does not attain unity even for a large break magnitude $\delta = 1.5$. When there are recurrent breaks in the variance as in model P5, Figure S-6 shows that the our tests $\text{MB}_{\text{max},h}$ and $\text{MQ}_{\text{max},h}$ and the $t^{\text{stat}}$-type tests have all good power and their performance is analogous.

Let us now consider models with either ARCH errors or autocorrelated errors. Observe that the latter models both imply that the forecast losses are serially correlated. Figure 5 shows that when the errors follow an ARCH(1) process the statistic $Q_{\text{max},h}$ based on the asymptotic variance estimator $\hat{\nu}_L^2$ performs well in terms of empirical power. In contrast, the $t^{\text{stat}}$-type tests fail as their power is non-monotonic, never reaches 20% and it decreases to zero as the magnitude of the break rises. We note that the version of $\hat{\nu}_L$ that uses more blocks is less precise. The same results hold true when the disturbances are autocorrelated; see Figure S-11.

Finally, we consider short-term instabilities in Figure 6-9. It is straightforward to recognize a general pattern: the tests of GR (2009) have little power whereas our tests possess good empirical power against all data-generating processes, break locations and sample sizes. Furthermore, the small sample power properties are uniform over the location of the instability and over the relative size of the in-sample and out-of-sample windows. The latter property is important in practice because forecast instabilities are frequently short-lived.

To sum up, our test statistics perform well in controlling the size, even though the versions that use overlapping blocks are somewhat liberal. For our tests, empirical size being close to the significance level is a feature that holds over different DGPs and sample sizes. Turning to power comparison, there is clear evidence that our tests are reliable in that they have good power

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29 We actually implemented the $t^{\text{stat}}$ by using either Andrews’s (1991) or Newey and West’s (1987) estimator of the long-run variance. We also experimented different choices for the truncation lag. The results, however, were unchanged. We suspect that this property depends on the estimation of the long-run variance in our forecasting context which can be challenging due to small sample sizes and to the presence of breaks. The same issues were found in Martins and Perron (2016) and Fossati (2017).
against different form of instabilities. There appears to be substantial power gains relative to existing methods especially when the instability (i) is short-lasting and/or (ii) is located toward the tail of the out-of-sample. These properties characterize both statistics using non-overlapping and overlapping blocks.

8 Conclusions

We have formalized the concepts of forecast instability and forecast failure. Our definition poses at the center the economic forecaster and emphasizes the importance of the time duration of the instability. We assume the data arise as an outcome of an underlying system of stochastic differential equations which then implies that we can approximate the sequence of forecast losses by a continuous-time stochastic process. We have built a testing framework based on the local pathwise properties of that process and have adopted an infill asymptotics to derive the null distribution of the test statistics. The null distribution follows an extreme value distribution. Our results can be used to test whether the predictive ability of a given forecast model changes over time and can be applied in forecasting exercises involving either low-frequency as well as high-frequency macroeconomic and financial variables. The simulation study confirms that there are substantial power gains especially when the instability (i) is short-lasting and/or (ii) is located toward the tail of the out-of-sample. Our framework allows for misspecification, different types of parameter instability and arbitrary forms of non-stationarity such as heteroskedasticity and serial correlation. Our continuous-time specification and associated continuous record asymptotic scheme can provide a promising complementary framework to the classical approach for forecasting in economics.
Tests for Forecast Instability

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## A Appendix

### A.1 Tables

| $\alpha = 0.05$ | $T_m$ | $T_n$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $M_{\text{B},\text{max},h}$ | $M_{\text{Q},\text{max},h}$ |
|-----------------|-------|-------|-------------------------------|-------------------------------|------------------|------------------|------------------|------------------|
| $T = 100$       | 25    | 75    | 0.052                         | 0.038                         | 0.044            | 0.037            | 0.112            | 0.140            |
|                 | 50    | 50    | 0.063                         | 0.030                         | 0.019            | 0.011            | 0.078            | 0.064            |
|                 | 75    | 25    | 0.051                         | 0.046                         | 0.056            | 0.050            | 0.096            | 0.095            |
| $T = 200$       | 50    | 150   | 0.047                         | 0.036                         | 0.029            | 0.032            | 0.136            | 0.083            |
|                 | 100   | 100   | 0.052                         | 0.035                         | 0.032            | 0.030            | 0.110            | 0.070            |
|                 | 150   | 50    | 0.078                         | 0.028                         | 0.060            | 0.058            | 0.059            | 0.058            |
| $T = 300$       | 75    | 225   | 0.045                         | 0.041                         | 0.040            | 0.049            | 0.106            | 0.046            |
|                 | 150   | 150   | 0.054                         | 0.034                         | 0.061            | 0.057            | 0.145            | 0.086            |
|                 | 225   | 75    | 0.072                         | 0.028                         | 0.104            | 0.095            | 0.092            | 0.076            |
| $T = 400$       | 100   | 300   | 0.050                         | 0.048                         | 0.054            | 0.068            | 0.129            | 0.055            |
|                 | 200   | 200   | 0.056                         | 0.042                         | 0.063            | 0.063            | 0.122            | 0.059            |
|                 | 300   | 100   | 0.059                         | 0.030                         | 0.069            | 0.067            | 0.108            | 0.068            |

| $\alpha = 0.10$ | $T_m$ | $T_n$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $M_{\text{B},\text{max},h}$ | $M_{\text{Q},\text{max},h}$ |
|-----------------|-------|-------|-------------------------------|-------------------------------|------------------|------------------|------------------|------------------|
| $T = 100$       | 25    | 75    | 0.154                         | 0.112                         | 0.099            | 0.142            | 0.186            | 0.163            |
|                 | 50    | 50    | 0.102                         | 0.087                         | 0.115            | 0.096            | 0.111            | 0.093            |
|                 | 75    | 25    | 0.137                         | 0.071                         | 0.053            | 0.067            | 0.128            | 0.130            |
| $T = 200$       | 50    | 150   | 0.103                         | 0.106                         | 0.095            | 0.117            | 0.130            | 0.068            |
|                 | 100   | 100   | 0.116                         | 0.096                         | 0.100            | 0.096            | 0.157            | 0.105            |
|                 | 150   | 50    | 0.114                         | 0.076                         | 0.128            | 0.150            | 0.103            | 0.093            |
| $T = 300$       | 75    | 225   | 0.108                         | 0.110                         | 0.077            | 0.109            | 0.167            | 0.085            |
|                 | 150   | 150   | 0.103                         | 0.094                         | 0.116            | 0.106            | 0.204            | 0.130            |
|                 | 225   | 75    | 0.135                         | 0.132                         | 0.142            | 0.118            | 0.194            | 0.163            |
| $T = 400$       | 100   | 300   | 0.098                         | 0.108                         | 0.108            | 0.108            | 0.192            | 0.096            |
|                 | 200   | 200   | 0.105                         | 0.091                         | 0.087            | 0.139            | 0.167            | 0.088            |
|                 | 300   | 100   | 0.112                         | 0.079                         | 0.114            | 0.109            | 0.166            | 0.112            |

The table reports the rejection probabilities of 100$\alpha$%-level tests proposed in the paper and those proposed by Giacomini and Rossi (2009) [(abbreviated GR (2009)] for model S1. For all methods we use the fixed forecasting scheme. $T = T_m + T_n$, where $T$ is the total sample size, $T_m$ is the size of the in-sample window and $T_n$ is the size of the out-of-sample window. $m_T$ is set equal to the smallest integer allowed by Condition 1. Based on 5,000 replications.
Table 2: Empirical small sample size of forecast instability tests based on model S2

|            | GR (2009) | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\max,h}$ | $Q_{\max,h}$ | $MB_{\max,h}$ | $MQ_{\max,h}$ |
|------------|-----------|----------------------------------|----------------------------------|--------------|--------------|--------------|--------------|
| $\alpha = 0.05$ |           |                                  |                                  |              |              |              |              |
| $T = 100$  |           |                                  |                                  |              |              |              |              |
| $T_m$      | $T_n$     |                                  |                                  |              |              |              |              |
| 25         | 75        | 0.049                            | 0.019                            | 0.086        | 0.098        | 0.090        | 0.089        |
| 50         | 50        | 0.069                            | 0.024                            | 0.058        | 0.072        | 0.083        | 0.067        |
| 75         | 25        | 0.039                            | 0.016                            | 0.081        | 0.111        | 0.092        | 0.091        |
| $T = 200$  |           |                                  |                                  |              |              |              |              |
| 50         | 150       | 0.049                            | 0.025                            | 0.076        | 0.072        | 0.138        | 0.089        |
| 100        | 100       | 0.057                            | 0.026                            | 0.070        | 0.073        | 0.106        | 0.068        |
| 150        | 50        | 0.075                            | 0.020                            | 0.055        | 0.070        | 0.082        | 0.070        |
| $T = 300$  |           |                                  |                                  |              |              |              |              |
| 75         | 225       | 0.050                            | 0.029                            | 0.058        | 0.036        | 0.102        | 0.044        |
| 150        | 150       | 0.059                            | 0.032                            | 0.077        | 0.072        | 0.144        | 0.086        |
| 225        | 75        | 0.065                            | 0.025                            | 0.096        | 0.103        | 0.152        | 0.123        |
| $T = 400$  |           |                                  |                                  |              |              |              |              |
| 100        | 300       | 0.054                            | 0.032                            | 0.061        | 0.041        | 0.123        | 0.046        |
| 200        | 200       | 0.051                            | 0.035                            | 0.065        | 0.048        | 0.111        | 0.052        |
| 300        | 100       | 0.068                            | 0.031                            | 0.067        | 0.063        | 0.115        | 0.074        |

| $\alpha = 0.10$ |           |                                  |                                  |              |              |              |              |
| $T = 100$  |           |                                  |                                  |              |              |              |              |
| $T_m$      | $T_n$     |                                  |                                  |              |              |              |              |
| 25         | 75        | 0.109                            | 0.069                            | 0.136        | 0.152        | 0.191        | 0.165        |
| 50         | 50        | 0.107                            | 0.069                            | 0.095        | 0.118        | 0.118        | 0.095        |
| 75         | 25        | 0.134                            | 0.060                            | 0.112        | 0.152        | 0.125        | 0.128        |
| $T = 200$  |           |                                  |                                  |              |              |              |              |
| 50         | 150       | 0.100                            | 0.078                            | 0.125        | 0.113        | 0.199        | 0.133        |
| 100        | 100       | 0.106                            | 0.073                            | 0.108        | 0.111        | 0.160        | 0.108        |
| 150        | 50        | 0.101                            | 0.078                            | 0.101        | 0.105        | 0.112        | 0.091        |
| $T = 300$  |           |                                  |                                  |              |              |              |              |
| 75         | 225       | 0.102                            | 0.081                            | 0.103        | 0.071        | 0.159        | 0.077        |
| 150        | 150       | 0.111                            | 0.079                            | 0.119        | 0.112        | 0.189        | 0.129        |
| 225        | 75        | 0.114                            | 0.068                            | 0.144        | 0.159        | 0.197        | 0.170        |
| $T = 400$  |           |                                  |                                  |              |              |              |              |
| 100        | 300       | 0.097                            | 0.082                            | 0.109        | 0.079        | 0.193        | 0.096        |
| 200        | 200       | 0.089                            | 0.106                            | 0.075        | 0.079        | 0.171        | 0.088        |
| 300        | 100       | 0.112                            | 0.079                            | 0.104        | 0.110        | 0.164        | 0.104        |

Model S2. We use the estimator $\nu_L$ from Theorem 4.2 for the asymptotic variance of $Q_{\max,h}$ and $MQ_{\max,h}$. For the statistics $t_{\text{stat}}$ and $t_{\text{stat},c}$ we use the Newey-West estimator with truncation lags $\left\lfloor T_n^{1/3} \right\rfloor$ as recommended by Giacomini and Rossi (2009). The notes of Table 1 applies.
A.2 Figures

A.2.1 General Instability

Figure 1: Power functions for model P1a: $Y_t = 2.73 - 0.44X_{t-1} + \delta X_{t-1} \mathbb{1}\{t > T^0\} + \epsilon_t$, where $X_{t-1} \sim \text{i.i.d.} \mathcal{N}(0, 1)$, $\epsilon_t \sim \text{i.i.d.} \mathcal{N}(0, 1)$, and $T^0 = T\lambda_0$. $T = 100$ (left panels) and $T = 150$ (right panels). $\lambda_0 = 0.7$ (top panels) and $\lambda_0 = 0.8$ (bottom panels). In-sample size is $T_m = 0.4T$ while out-of-sample size is $T_n = 0.6T$. The green and blue broken lines correspond to $B_{\max,h}$ and $Q_{\max,h}$, respectively. The red and orange broken lines correspond to the $t_{\text{stat}}$ of Giacomini and Rossi (2009), respectively, the uncorrected and corrected version.
Figure 2: Power functions for model P1a. $T = 200$ (left panels) and $T = 300$ (right panels). The notes of Figure 1 apply.

Figure 3: Power functions for model P4 (single break in variance): $Y_t = 0.5X_{t-1} + (1 + \delta 1 \{t > T_0\}) \epsilon_t$ where $X_{t-1} \sim \text{i.i.d.} \mathcal{N}(1, 1)$ and $\epsilon_t \sim \text{i.i.d.} \mathcal{N}(0, 1)$. $T = 200$ (left panels) and $T = 300$ (right panels). $\lambda_0 = 0.6$ (top panels) and $\lambda_0 = 0.8$ (bottom panels). In-sample size is $T_m = 0.3T$ while out-of-sample size is $T_n = 0.7T$. The green and blue broken lines correspond to $B_{\text{max},h}$ and $Q_{\text{max},h}$, respectively. The red and orange broken lines correspond to the $t_{\text{stat}}$ of Giacomini and Rossi (2009), respectively, the uncorrected and corrected version.
Figure 4: Power functions for model P4. $T = 400$ (left panels) and $T = 500$ (right panels). $\lambda_0 = 0.8$ (top panels) and $\lambda_0 = 0.9$ (bottom panels). The notes of Figure 3 apply.

Figure 5: Power functions for model P7 (ARCH errors): $Y_t = 2.73 - 0.44X_{t-1} + \delta X_{t-1}1\{t > T_0\} + e_t$, where $X_{t-1} \sim \text{i.i.d.} \mathcal{N}(0, 1.5)$ and $e_t = \sigma_t u_t$, $\sigma_t^2 = 0.5 + 0.5e_{t-1}^2$, $u_t \sim \text{i.i.d.} \mathcal{N}(0, 1)$. $T = 200$ (left panels) and $T = 300$ (right panels). $\lambda_0 = 0.7$ (top panels) and $\lambda_0 = 0.8$ (bottom panels). $T_m = 0.5T$ and $T_n = 0.5T$. The light-blue and blue broken lines correspond to a version of $Q_{\text{max},h}$ that uses $\tilde{\nu}_L$ but with different choices of $m_T$ (for the light-blue broken line we increase the number of blocks by one relative to the recommended value of $m_T$).
A.2.2 Short-Term Instability

Figure 6: Power functions for model P1a with short-term instability: $Y_t = 2.73 - 0.44X_{t-1} + \delta X_{t-1} 1 \{ T_0^B < t \leq T_0^B + p \} + e_t$, where $X_{t-1} \sim \text{i.i.d.} \mathcal{N}(0, 1)$, $e_t \sim \text{i.i.d.} \mathcal{N}(0, 1)$, and $T_0^B = T\lambda_0$. We set $(T, p) = \{(100, 20), (150, 25)\}$. $\lambda_0 = 0.7$ (top panels) and $\lambda_0 = 0.8$ (bottom panels). $T_m = 0.4T$ and $T_n = 0.6T$. The green and blue broken lines correspond to $B_{\text{max},h}$ and $Q_{\text{max},h}$, respectively. The red and orange broken lines correspond to the $t^{\text{stat}}$ of Giacomini and Rossi (2009), respectively, the uncorrected and corrected version.

Figure 7: Power functions for model P1a. We set $(T, p) = \{(200, 20), (300, 30)\}$. The notes of Figure 6 apply.
Figure 8: Power functions for model P4 (single break in variance) with short-term instability: \( Y_t = 0.5X_{t-1} + (1 + \delta)T_0 \leq t \leq T_0 + p) e_t \) where \( X_{t-1} \sim \text{i.i.d.} \mathcal{N}(1, 1) \) and \( e_t \sim \text{i.i.d.} \mathcal{N}(0, 1) \). We set \((T, p) = \{(200, 30), (300, 30)\} . \) \( \lambda_0 = 0.6 \) (top panels) and \( \lambda_0 = 0.8 \) (bottom panels). \( T_m = 0.3T \) and \( T_n = 0.7T \). The green and blue broken lines correspond to \( B_{\max, h} \) and \( Q_{\max, h} \), respectively. The red and orange broken lines correspond to the \( t_{\max, h} \) of Giacomini and Rossi (2009), respectively, the uncorrected and corrected version.

Figure 9: Power functions for model P4 (single break in variance) with short-term instability We set \((T, p) = \{(400, 30), (500, 30)\} . \) The notes of Figure 8 apply.
Supplemental Material to

Tests for Forecast Instability and Forecast Failure under a Continuous Record Asymptotic Framework

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Abstract

This supplemental material is structured as follows. Section S.A contains the Mathematical Appendix which includes all proofs of the results in the paper. Section S.B reports additional figures and tables from the simulation study of Section 7. In Section S.C we collect additional simulation results pertaining to recursive and rolling schemes and to the Linex loss function.
S.A Mathematical Proofs

The Mathematical Appendix is structured as follows. The proofs of the results in Section 3 and 4 are collected in Section S.A.4 and S.A.5, respectively. The results of Section 5 are covered in Section S.A.6.

S.A.1 Additional Notation

Throughout the proofs, $C$ is a generic constant that may vary from line to line; we may sometime write $C_r$ to emphasize the dependence of $C$ on a scalar $r$. For brevity, we indicate that a sequence $\{U_k\}$ is formed by independent and non-identically distributed random variables by labeling it as i.n.d. For the variables $\Delta_h e_k$ and $\Delta_h X_k$ we use a tilde notation to denote their normalized version: $\Delta_h \tilde{e}_k = h^{-1/2} \Delta_h e_k$ and $\Delta_h \tilde{X}_k = h^{-1/2} \Delta_h X_k$. We use a star superscript ($\ast$) on $\Delta_h \tilde{e}_k$ to indicate the residuals obtained when $\beta = \beta^\ast$: $\Delta_h \tilde{e}^\ast_k = h^{-1/2} (\Delta_h Y_k - (\beta^\ast)' \Delta_h X_{k-\tau})$. We sometime omit the index from $\beta_k$ and simply use $\beta$ when it is clear from the context.

S.A.2 Localization

As it is typical in the high-frequency statistics literature, we use a localization argument [cf. Section I.1.d in Jacod and Shiryaev (2003)]. Thus, we replace Assumption 2.1 and Assumption 2.2 by the following stronger assumption which basically turns the local restrictions into global.

**Assumption S.A.1.** Let Assumption 2.1-2.2, Assumption 3.1-3.6 and Condition 1 hold. When $\mu_{e,t} = 0$ for all $t \geq 0$ the process $\{Z_t\}_{t \geq 0}$ takes value in some compact set; the processes $\{\sigma_X,t, \sigma_e,t\}_{t \geq 0}$ are bounded c\`adl\`ag and $\{\mu_X,t, \mu_e,t\}_{t \geq 0}$ are bounded c\`adl\`ag. Furthermore, $\phi_{\sigma,e,N} \leq C \eta$ for some $C < \infty$.

S.A.3 Preliminary Lemmas

**Lemma S.A.1.** For any $1 \leq r, l \leq q$, and $1 \leq i \leq n_T$, we have

(i) $\sup_{0 \leq s \leq \lfloor T_n/n_T \rfloor} \sum_{j=1}^{T_m+b_n T_l+i-1} \Delta_h X_k^{(r)} \Delta_h e_k^\ast \xrightarrow{P} 0$;

(ii) $\sup_{0 \leq s \leq \lfloor T_n/n_T \rfloor} \left\| \sum_{j=1}^{T_m+b_n T_l+i-1} \Delta_h X_k^{(r)} \Delta_h X_k^{(l)} - \int_0^{N_{nT}+bn T_l} \Sigma_X^{(r,l)} ds \right\| \xrightarrow{P} 0$;

(iii) the central limit theorem in Lemma S.A.5 in Casini and Perron (2017a) holds for $X_1$.

**Proof.** Part (i)-(ii) are a consequence of the law of large numbers for quadratic variation; see Section S.A.3 in Casini and Perron (2017a). For part (iii) see the above referenced theorem. □

S.A.4 Proofs of Section 3

Throughout this section we maintain Assumption S.A.1.

**S.A.4.1 Proof of Theorem 3.1**

The idea behind the proof of both Theorem 3.1-3.2 is the same. Thus, the quadratic loss case serves as a guide and we then use some of these derivations for the general loss case. All the results in this section are proved under $H_0$. 

S-1
**S.A.4.1.1 Proof of part (i) of Theorem 3.1** The theorem is proved through several lemmas. The first step involves showing that the error in replacing $\hat{\beta}$ by $\beta^*$ is asymptotically negligible. We provide the proof of this first step by assuming that $\mu_{e,t} = 0$ in (2.2). That is, in Lemma S.A.2-S.A.3 we have $\mu_{e,t} = 0$ and we show how these results continue to hold without this restriction in Section S.A.4.1.3. We focus for simplicity on the recursive scheme only; the proofs for the other cases are similar and omitted. Let $U_{h,b} \triangleq n_T^{-1} \sum_{j=1}^{n_T} SL_{\psi,T_m+\tau+bn_T+j-1}(\beta^*), \quad U_{h,b} \triangleq n_T^{-1} \sum_{j=1}^{n_T} L_{\psi,T_m+\tau+bn_T+j-1}(\beta^*)$ and $U_{\max,h}(T_n, \tau) \triangleq \max_{b=0,\ldots,\lfloor T_n/n_T \rfloor} (U_{h,b+1} - U_{h,b})/U_{h,b+1}$. In some steps of the proof, we will use the following simple result. For any integer $m \geq 1$, let $c_{1,b}$ and $c_{2,b}$ ($b = 1, \ldots, m$) be arbitrary real numbers, then

$$ |c_{1,b}| \leq |c_{1,b} - c_{2,b}| + |c_{2,b}| \leq \max_{b=1,\ldots,m} |c_{1,b} - c_{2,b}| + \max_{b=1,\ldots,m} |c_{2,b}|. \tag{S.1} $$

**Lemma S.A.2.** As $h \downarrow 0$, $(\log(T_n) n_T)^{1/2} (U_{\max,h}(T_n, \tau) - B_{\max,h}(T_n, \tau)) \overset{P}{\to} 0$.

**Proof.** By the reverse triangle inequality, inequality (S.1) and Lemma S.A.3 below, for some $C_1, C_2 < \infty$,

$$ |U_{\max,h}(T_n, \tau) - B_{\max,h}(T_n, \tau)| 
\leq b=0,\ldots,\lfloor T_n/n_T \rfloor-2 \max n_T^{-1} \sum_{j=1}^{n_T} \left| SL_{\psi,T_m+\tau+bn_T+j-1}(\beta^*)/B_{h,b+1} - SL_{\psi,T_m+\tau+bn_T+j-1}(\hat{\beta})/U_{h,b+1} \right| 
+ \max b=0,\ldots,\lfloor T_n/n_T \rfloor-2 n_T^{-1} \left| \sum_{j=1}^{n_T} SL_{\psi,T_m+\tau+bn_T+j-1}(\beta^*)/B_{h,b+1} - SL_{\psi,T_m+\tau+bn_T+j-1}(\beta)/U_{h,b+1} \right| 
\leq C_1 \max_{b=0,\ldots,\lfloor T_n/n_T \rfloor-2} n_T^{-1} \left( \sum_{j=1}^{n_T} SL_{\psi,T_m+\tau+bn_T+j-1}(\beta^*) - SL_{\psi,T_m+\tau+bn_T+j-1}(\beta) \right)/U_{h,b+1} 
+ C_2 \max_{b=0,\ldots,\lfloor T_n/n_T \rfloor-2} n_T^{-1} \left( \sum_{j=1}^{n_T} SL_{\psi,T_m+\tau+bn_T+j-1}(\beta^*) - SL_{\psi,T_m+\tau+bn_T+j-1}(\beta) \right)/U_{h,b+1} \tag{S.2} $$

Note that for any $j = 1, \ldots, n_T$,

$$ SL_{\psi,T_m+\tau+bn_T+j-1}(\beta^*) - SL_{\psi,T_m+\tau+bn_T+j-1}(\beta) = L_{\psi,T_m+\tau+bn_T+j-1}(\beta^*) - L_{\psi,T_m+\tau+bn_T+j-1}(\beta) + o_P\left(T^{-1/2}\right), $$

where the $o_P\left(T^{-1/2}\right)$ term arises from the proof of Lemma S.A.3. Recall that for $1 \leq j \leq n_T$,

$$ \Delta_{h^j} e_{T_m+\tau+bn_T+j} = \sigma_{e,T_m+\tau+bn_T+j} h \left(h^{-1/2} \Delta_h W_{e,T_m+\tau+bn_T+j}\right), $$

so that

$$ L_{\psi,T_m+\tau+bn_T+j}(\beta^*) - L_{\psi,T_m+\tau+bn_T+j}(\beta) 
= - \left(\beta - \beta^*\right)' \Delta_h \tilde{X}_{T_m+bn_T+j} \Delta_h \tilde{X}_{T_m+bn_T+j} \left(\beta - \beta^*\right) 
+ 2\sigma_{e,T_m+\tau+bn_T+j} h \left(h^{-1/2} \Delta_h W_{e,T_m+\tau+bn_T+j}\right) \left(\beta - \beta^*\right)' \Delta_h \tilde{X}_{T_m+bn_T+j}. \tag{S.3} $$
Recall that $\Delta_h \bar{X}_k = h^{-1/2} \Delta_h X_k$ and thus

$$n_T^{-1} \sum_{j=1}^{n_T} \Delta_h \bar{X}_{T_m + bn_T + j} \Delta_h \bar{X}'_{T_m + bn_T + j} - \Sigma_{X_i(T_m + bn_T)h} = o_p\left(1\right),$$

which follows from Theorem 9.3.2 part (i) in Jacod and Protter (2012). This implies that

$$n_T^{-1} \sum_{j=1}^{n_T} \Delta_h \bar{X}_{T_m + bn_T + j-1} \Delta_h \bar{X}'_{T_m + bn_T + j-1} = O_p\left(1\right),$$

by Assumption 2.1-(iv). By Assumption 2.1-(v) and the aforementioned theorem,

$$n_T^{-1} \sum_{j=1}^{n_T} \left(h^{-1/2} \Delta_h W_{\epsilon,T_m + \tau + bn_T + j-1}\right) \Delta_h \bar{X}_{T_m + bn_T + j-1} \xrightarrow{p} 0.$$

Note that by Assumption 3.6, $\hat{\beta}_k - \beta^*$ = $O_p\left(1/\sqrt{T}\right)$ uniformly in $k \geq T_m$. Therefore, from these arguments we deduce that

$$n_T^{-1} \sum_{j=1}^{n_T} \left(SL_{\psi,T_m + \tau + bn_T + j-1}(\beta^*) - SL_{\psi,T_m + \tau + bn_T + j-1}(\hat{\beta})\right) = o_p\left(1/\sqrt{T}\right). \tag{S.4}$$

Then, for any $\varepsilon > 0$ and any constant $K > 0$, the first term on the right-hand side of (S.2) is

$$P\left(\max_{b=0,\ldots,\left\lfloor T_o/n_T\right\rfloor-2} \left| \frac{\left(\log \left(T_n n_T\right)\right)^{1/2} \left(U_{h,b+1} - B_{h,b+1}\right)}{U_{h,b+1}^2} \right| > \varepsilon \right) \leq P\left(\max_{b=0,\ldots,\left\lfloor T_o/n_T\right\rfloor-2} \left| \frac{\left(\log \left(T_n n_T\right)\right)^{1/2} \left(U_{h,b+1} - B_{h,b+1}\right)}{U_{h,b+1}^2} \right| > \varepsilon/K \right) + P\left(\max_{b=0,\ldots,\left\lfloor T_o/n_T\right\rfloor-2} 1/\left|U_{h,b+1}^2\right| > K \right). \tag{S.5}\right)$$

Given the result on the negligibility of the drift term from Section S.A.4.1.3, we can apply Lemma S.A.4 to $U_{h,b}$. Then, the second probability term above is equal to $P\left(\min_{b=0,\ldots,\left\lfloor T_o/n_T\right\rfloor-2} \left|U_{h,b+1}^2\right| < 1/K \right)$ which converges to zero by letting $K = 4/\sigma^2$. As for the first probability term, we use (S.4) and choose $r > 0$ sufficiently large to deduce that,

$$P\left(\max_{b=0,\ldots,\left\lfloor T_o/n_T\right\rfloor-2} \left| \frac{\left(\log \left(T_n n_T\right)\right)^{1/2} \left(U_{h,b+1} - B_{h,b+1}\right)}{U_{h,b+1}^2} \right| > \varepsilon/K \right) \leq \left(\frac{K}{\varepsilon}\right)^r \sum_{b=0}^{\left\lfloor T_o/n_T\right\rfloor-2} E \left[ \left| \left(\log \left(T_n n_T\right)\right)^{1/2} \left(U_{h,b+1} - B_{h,b+1}\right) \right|^r \right]$$

$$= \left(\frac{K}{\varepsilon}\right)^r \left(\log \left(T_n\right)\right)^{r/2} n_T^{r/2-1} O_p\left(1/T^{r/2-1}\right) \rightarrow 0,$$

in view of Condition 1 and $T_n = O(T)$. We can repeat the same argument for the second term of (S.2). Altogether, this establishes the claim of the lemma. □
Lemma S.A.3. As $h \downarrow 0$,

$$\max_{b=0, \ldots, \lfloor T_n / n_T \rfloor - 2} \frac{1}{n_T} \sum_{j=1}^{n_T} \left( T_{\psi,(T_m+(b+1)n_T+j-1)h}(\tilde{\beta}) - T_{\psi,(T_m+b n_T+j-1)h}(\beta) \right) \xrightarrow{p} 0,$$

and the same result holds with $\beta^*$ in place of $\tilde{\beta}$. Furthermore, as $h \downarrow 0$,

$$\max_{b=0, \ldots, \lfloor T_n / n_T \rfloor - 2} \frac{1}{n_T} \sum_{j=1}^{n_T} \left( T_{\psi,(T_m+(b+1)n_T+j-1)h}(\beta^*) - T_{\psi,(T_m+b n_T+j-1)h}(\beta^*) \right) \xrightarrow{p} 0.$$

Proof. By definition,

$$\left| \frac{1}{n_T} \sum_{j=1}^{n_T} \left( T_{\psi,(T_m+(b+1)n_T+j-1)h}(\beta^*) - T_{\psi,(T_m+b n_T+j-1)h}(\beta^*) \right) \right|$$

$$= \left| \frac{1}{n_T} \sum_{j=1}^{n_T} \left( \frac{T_{m+(b+1)n_T+j-1}}{T_m+(b+1)n_T+j-1} - \frac{T_{m+b n_T+j-1}}{T_m+b n_T+j-1} \right) \right|$$

$$= \sum_{j=1}^{n_T} \left( \frac{T_{m+(b+1)n_T+j-1}}{T_m+(b+1)n_T+j-1} - \frac{T_{m+b n_T+j-1}}{T_m+b n_T+j-1} \right)$$

$$\leq C \left( \frac{n_T}{T_m} \right) + O_p \left( \frac{n_T}{T_m + n_T n_T} \right).$$

Thus, for any $\varepsilon > 0$,

$$\mathbb{P} \left( \max_{b=0, \ldots, \lfloor T_n / n_T \rfloor - 2} \frac{1}{n_T} \sum_{j=1}^{n_T} \left( T_{\psi,(T_m+(b+1)n_T+j-1)h}(\beta^*) - T_{\psi,(T_m+b n_T+j-1)h}(\beta^*) \right) \right) > \varepsilon$$

$$\leq \sum_{b=0}^{\lfloor T_n / n_T \rfloor - 2} \mathbb{P} \left( \frac{1}{n_T} \sum_{j=1}^{n_T} \left( T_{\psi,(T_m+(b+1)n_T+j-1)h}(\beta^*) - T_{\psi,(T_m+b n_T+j-1)h}(\beta^*) \right) \right) > \varepsilon$$

$$\leq \varepsilon^{-r} \sum_{b=0}^{\lfloor T_n / n_T \rfloor - 2} \mathbb{E} \left[ \left( \frac{1}{n_T} \sum_{j=1}^{n_T} \left( T_{\psi,(T_m+(b+1)n_T+j-1)h}(\beta^*) - T_{\psi,(T_m+b n_T+j-1)h}(\beta^*) \right) \right)^r \right]$$

$$\leq \varepsilon^{-r} C \left( \frac{1}{n_T} \sum_{j=1}^{n_T} \left( T_{\psi,(T_m+(b+1)n_T+j-1)h}(\beta^*) - T_{\psi,(T_m+b n_T+j-1)h}(\beta^*) \right) \right)^r \rightarrow 0,$$

for $r > 0$ sufficiently large and in view of Condition 1 since $T_m$ is of the same order as $T_n$. For the last
claim of the lemma, note that

\[ T_{\psi,(T_m+bn_T+j-1)h}(\hat{\beta}) - T_{\psi,(T_m+bn_T+j-1)h}(\beta^*) \]

\[ = \sum_{l=1}^{T_m+bn_T+j-1} \frac{(\Delta_l^e)^2}{T_m + bn_T + j - 1} - \sum_{l=1}^{T_m+bn_T+j-1} \frac{(\Delta_l^e)^2}{T_m + bn_T + j - 1} \]

\[ = \frac{1}{T_m + bn_T + j - 1} \sum_{l=1}^{T_m+bn_T+j-1} (\beta - \beta^*) \Delta_l \tilde{X}_l \Delta_l \tilde{X}_l' (\beta - \beta^*) \]  

(S.7)

\[ - \frac{2}{T_m + bn_T + j - 1} \sum_{l=1}^{T_m+bn_T+j-1-\tau} \Delta_l e_l^x (\beta - \beta^*) \Delta_l \tilde{X}_l. \]  

(S.8)

By Lemma S.A.1, \((T_m + bn_T + j - 1)^{-1} \sum_{l=1}^{T_m+bn_T+j-1-\tau} \Delta_l \tilde{X}_l \Delta_l \tilde{X}_l' = O_P(1)\). Since \(\hat{\beta}_k - \beta^* = O_P\left(\frac{1}{\sqrt{T}}\right)\) uniformly in \(k \geq T_m\) by Assumption 3.6, the term in (S.7) is \(O_P(T^{-1})\) whereas the term (S.8) is \(o_P(T^{-1/2})\) by Lemma S.A.1. Therefore, upon using Condition 1 and the same argument that led to (S.6) we show the last claim of the lemma. The proof of the second claim then follows from combining the result of the first and last claim. \(\square\)

**Lemma S.A.4.** Let \(B_{h,b}^0 = (n_T h)^{-1} \sum_{j=1}^{n_T} \sigma_{e,(T_m+\tau+bn_T-1)h}^2 \Delta_h \tilde{X}_l \Delta_h \tilde{X}_l' (\Delta_h W_{e,T_m+\tau+bn_T+j-1})^2\). For any \(\varepsilon > 0\) and some constant \(K > 0\), \(\mathbb{P}\left(\max_{b=0, \ldots, [T/n_T]-2} \left|\frac{1}{B_{h,b}^0}\right| > K\right) \to 0\).

**Proof.** Note that

\[
\mathbb{P}\left(\max_{b=0, \ldots, [T/n_T]-2} \left|\frac{1}{B_{h,b}^0}\right| > K\right) = \mathbb{P}\left(\min_{b=0, \ldots, [T/n_T]-2} \left|B_{h,b}^0\right| < K^{-1}\right).
\]

\[
= \mathbb{P}\left(\min_{b=0, \ldots, [T/n_T]-2} \frac{1}{n_T h} \sum_{j=1}^{n_T} \sigma_{e,(T_m+\tau+bn_T-1)h}^2 \left(\Delta_h \tilde{X}_l \left(\Delta_h W_{e,T_m+\tau+bn_T+j-1}\right)^2\right) < K^{-1}\right).
\]

\[
\leq \sum_{b=0}^{[T/n_T]-2} \mathbb{P}\left(\frac{1}{n_T h} \sum_{j=1}^{n_T} \sigma_{e,(T_m+\tau+bn_T-1)h}^2 \left(h^{-1/2} \Delta_h W_{e,T_m+\tau+bn_T+j-1}\right)^2 < K^{-1}\right).
\]

With \(K = 2/\sigma_-^2\) [with \(\sigma_-\) defined in Assumption 2.1-(iii)], we can use Markov’s inequality to deduce, for any \(r > 0\),

\[
\mathbb{P}\left(\frac{1}{n_T h} \sum_{j=1}^{n_T} \sigma_{e,(T_m+\tau+bn_T-1)h}^2 \left(h^{-1/2} \Delta_h W_{e,T_m+\tau+bn_T+j-1}\right)^2 < \frac{\sigma_-^2}{2}\right) \leq \mathbb{P}\left(\frac{1}{n_T h} \sum_{j=1}^{n_T} \sigma_{e,(T_m+\tau+bn_T-1)h}^2 \left(\left(h^{-1/2} \Delta_h W_{e,T_m+\tau+bn_T+j-1}\right)^2 - 1\right) < \frac{\sigma_-^2}{2}\right)
\]

\[
\leq \left(\frac{2}{\sigma_-^2}\right)^r n_T^{-r/2} \mathbb{E}\left[\left(n_T^{-1/2} \sum_{j=1}^{n_T} \sigma_{e,(T_m+\tau+bn_T-1)h}^2 \left(\left(h^{-1/2} \Delta_h W_{e,T_m+\tau+bn_T+j-1}\right)^2 - 1\right)\right)^r\right].
\]
From a standard central limit theorem for i.i.d. observations we have
\[ \mathbb{E} \left[ \left| n_T^{-1/2} \sum_{j=1}^{n_T} \left( (h^{-1/2} \Delta_h W_{h,T_m+\tau+bn_T+j-1})^2 - 1 \right) \right|^r \right] < C_{2,r}, \]
where \( C_{2,r} < \infty \). Thus, since we can choose \( r \) sufficiently large we can deduce,
\[
\sum_{b=0}^{[T/n_T]-1} \mathbb{P} \left( \frac{1}{n_T} \sum_{j=1}^{n_T} \left( \sigma_e (T_m+\tau+bn_T-1) h^{-1/2} \Delta_h W_{h,T_m+\tau+bn_T+j-1} \right)^2 < K \right) \\
\leq C_r \left( \frac{2}{\sigma^2} \right)^r (T/n_T)^{-r/2} \to 0,
\]
where we have also used Condition 1. This concludes the proof. \( \square \)

Next, let
\[
B_{\max,h}^0 (T_n, \tau) \triangleq \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} \left( B^0_{h,b+1} - B^0_{h,b} \right) / B^0_{h,b+1},
\]
\[
B_{\max,h}^* (T_n, \tau) \triangleq \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} \left( B^*_{h,b+1} - \bar{U}_{h,b} \right) / \bar{U}_{h,b+1},
\]
where \( B^0_{h,b} = (n_T h)^{-1} \sum_{j=1}^{n_T} \sigma^2 e(T_m+\tau+bn_T-1) h (\Delta_h W_{e,T_m+\tau+bn_T+j-1})^2 \) and \( B^*_{h,b} = n_T^{-1} \sum_{j=1}^{n_T} (\Delta_h W_{T_m+\tau+bn_T+j-1})^2 \).

The following lemma shows that, under \( H_0 \), the difference in the in-sample losses \( T_{\psi,kh} (\beta_k) \) across adjacent blocks is negligible asymptotically.

**Lemma S.A.5.** As \( h \downarrow 0, (\log(T_n) n_T)^{1/2} \left( B_{\max,h}^* (T_n, \tau) - U_{\max,h} (T_n, \tau) \right) \to 0. \)

**Proof.** We begin with the inequality,
\[
\left| B_{\max,h}^* (T_n, \tau) - U_{\max,h} (T_n, \tau) \right| \\
= \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} \left| \frac{B^*_{h,b+1} - B^*_{h,b}}{\bar{U}_{h,b+1} - \bar{U}_{h,b}} \right| \\
\leq \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} \left| \sum_{j=1}^{n_T} \left( T_{\psi,(T_m+(b+1)n_T+j-1) h} (\beta^*_{h,b}) - T_{\psi,(T_m+bn_T+j-1) h} (\beta^*) \right) \right| / \bar{U}_{h,b+1}. 
\]
For any \( \varepsilon > 0 \) and any \( K > 0 \),
\[
\mathbb{P} \left( \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} \left| \frac{\left( \log(T_n) n_T \right)^{1/2} \sum_{j=1}^{n_T} \left( T_{\psi,(T_m+(b+1)n_T+j-1) h} (\beta^*_{h,b}) - T_{\psi,(T_m+bn_T+j-1) h} (\beta^*) \right) \right|}{\bar{U}_{h,b}} > \varepsilon \right) \\
\leq \mathbb{P} \left( \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} \left| \frac{\left( \log(T_n) n_T \right)^{1/2} \sum_{j=1}^{n_T} \left( T_{\psi,(T_m+(b+1)n_T+j-1) h} (\beta^*_{h,b}) - T_{\psi,(T_m+bn_T+j-1) h} (\beta^*) \right) \right|}{\bar{U}_{h,b+1}} > \varepsilon / \sqrt{K} \right) \\
+ \mathbb{P} \left( \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} 1 / \bar{U}_{h,b+1} > \sqrt{K} \right).
\]
By the second result in Lemma S.A.3 the first term converges to zero. As for the second term, it was already treated in (S.5) with \( \tilde{U}_{h,b+1}^2 \) in place of \( \bar{U}_{h,b+1} \), and a similar argument can be applied to yield the same result. □

Lemma S.A.5 implies that the asymptotic behavior of the test statistics under \( H_0 \) is determined by the sequence of out-of-sample losses only. Next, let us define the following quantity which has the volatility shifted back by one block of time-length \( n_T h \):

\[
\tilde{B}_{h,b}^0 = (n_T h)^{-1} \sum_{j=1}^{n_T} \sigma_e(T_m + \tau + (b-1)n_T + j) (\Delta_n W_{e,T_m + \tau + bn_T + j-1})^2,
\]

and use it to define the statistic

\[
\tilde{B}_{max,h}^0(T_n, \tau) \triangleq \max_{b=0, \ldots, \{T_n/n_T\} - 2} \left| \tilde{B}_{h,b+1}^0 - B_{h,b}^0 \right|.
\]

Our final goal is to show that \((\log(T_n) n_T)^{1/2} \left( V_{max,h}(T_n, \tau) - \tilde{B}_{max,h}^0(T_n, \tau) \right)\) converges to zero in probability, where

\[
V_{max,h}(T_n, \tau) \triangleq \max_{b=0, \ldots, \{T_n/n_T\} - 2} \left| \frac{\tilde{B}_{h,b+1}^0 - B_{h,b}^0}{\sigma_e(T_m + \tau + bn_T - 1)h} \right|.
\]

(S.10)

We deduce this result from several small lemmas. We begin by replacing \( B_{max,h}^*(T_n, \tau) \) by \( B_{max,h}^0(T_n, \tau) \).

**Lemma S.A.6.** As \( h \downarrow 0 \), \((\log(T_n) n_T)^{1/2} \left( B_{max,h}^*(T_n, \tau) - B_{max,h}^0(T_n, \tau) \right) \overset{P}{\rightarrow} 0.\)

**Proof.** We begin by using inequality (S.1),

\[
\left| B_{max,h}^*(T_n, \tau) - B_{max,h}^0(T_n, \tau) \right| = \max_{b=0, \ldots, \{T_n/n_T\} - 2} \left| \frac{B_{h,b+1}^* - B_{h,b}^*}{\tilde{U}_{h,b+1}^2} - \frac{B_{h,b+1}^0 - B_{h,b}^0}{\tilde{U}_{h,b+1}^2} \right|.
\]

(S.11)

Consider the second term of (S.11). Let \( K > 0 \). For any \( \varepsilon > 0 \),

\[
\mathbb{P} \left( \max_{b=0, \ldots, \{T_n/n_T\} - 2} \left| (\log(T_n) n_T)^{1/2} \left( B_{h,b}^* - B_{h,b}^0 \right) \right| > \varepsilon \right) \leq \mathbb{P} \left( \max_{b=0, \ldots, \{T_n/n_T\} - 2} \left| (\log(T_n) n_T)^{1/2} \left( B_{h,b}^* - B_{h,b}^0 \right) \right| > \varepsilon / K \right) + \mathbb{P} \left( \max_{b=0, \ldots, \{T_n/n_T\} - 2} \left| 1/B_{h,b+1}^0 \right| > K \right).
\]

(S.12)

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By Lemma S.A.4, \( \mathbb{P} \left( \max_{b=0,\ldots,\lfloor T_n/n_T \rfloor-2} \left| 1/B_{h,b+1}^0 \right| > K \right) = o(1) \) if we set for instance \( K = 2/\sigma^2 \). Let us consider the first term of (S.12). By Itô’s formula,

\[
B_{h,b}^* - B_{h,b}^0 = (n_T h)^{-1} \sum_{j=1}^{n_T} \left( \Delta_t e_{T_m+\tau+bn_T+j-1}^* \right)^2 - (n_T h)^{-1} \sum_{j=1}^{n_T} \sigma_{e_{T_m+\tau+bn_T+j-1}}^2 (\Delta_t W_{e,T_m+\tau+bn_T+j-1})^2
\]

\[
= (n_T h)^{-1} \sum_{j=1}^{n_T} 2 \int_{T_m+\tau+bn_T+j-1}^{(T_m+\tau+bn_T+j-1)h} \left( \sigma_{e_{T_m+\tau+bn_T+j-1}}^2 - \sigma_{e_{T_m+\tau+bn_T+j-2}}^2 \right) ds
\]

Consider the first term of (S.13),

\[
\max_{b=0,\ldots,\lfloor T_n/n_T \rfloor-2} \left| \log (T_n) n_T^{1/2} \left( n_T h \right)^{-1} \sum_{j=1}^{n_T} 2 \int_{T_m+\tau+bn_T+j-1}^{(T_m+\tau+bn_T+j-1)h} \left( \sigma_{e_{T_m+\tau+bn_T+j-1}}^2 - \sigma_{e_{T_m+\tau+bn_T+j-2}}^2 \right) ds \right|
\]

\[
\leq \max_{b=0,\ldots,\lfloor T_n/n_T \rfloor-2} \left( \log (T_n) n_T^{1/2} \left( n_T h \right)^{-1} \sum_{j=1}^{n_T} 2 \int_{T_m+\tau+bn_T+j-1}^{(T_m+\tau+bn_T+j-1)h} \left| \sigma_{e_{T_m+\tau+bn_T+j-1}}^2 - \sigma_{e_{T_m+\tau+bn_T+j-2}}^2 \right| ds \right)
\]

\[
\leq \left( \log (T_n) n_T^{1/2} \left( n_T h \right)^{-1} \sum_{j=1}^{n_T} 2 \int_{T_m+\tau+bn_T+j-1}^{(T_m+\tau+bn_T+j-1)h} \left| e_s - e_{T_m+\tau+bn_T+j-2} \right| ds \right) \mu_{e,s} h^{-\theta} ds.
\]

(S.13)

by Condition 1. Let us now turn to the last term. We have for any integer \( r > 0 \),

\[
\mathbb{P} \left( \max_{b=0,\ldots,\lfloor T_n/n_T \rfloor-2} \left| \log (T_n) n_T^{1/2} \left( n_T h \right)^{-1} \sum_{j=1}^{n_T} 2 \int_{T_m+\tau+bn_T+j-1}^{(T_m+\tau+bn_T+j-1)h} \left( e_s - e_{T_m+\tau+bn_T+j-2} \right) \mu_{e,s} h^{-\theta} ds \right| > \varepsilon / (4K) \right)
\]

\[
\leq \left( \frac{4K}{\varepsilon} \right)^r \sum_{b=0}^{\lfloor T_n/n_T \rfloor-2}
\]

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\[ \times \mathbb{E} \left[ \left( \log(T_n) n_T \right)^{1/2} (n_T h)^{-1} \sum_{j=1}^{n_T} \int_{(T_m + \tau + bn_T + j - 1)h}^{(T_m + \tau + bn_T + j - 2)h} \left( e_s - e(\tau + bn_T + j - 2)h \right) \mu_{e,s} h^{-\vartheta} ds \right] \]

\[ \leq C_r \left( \frac{4K}{\varepsilon} \right) \sum_{b=0}^{n_T} \left( \log(T_n) n_T \right)^{1/2} \left( n_T h \right)^{-1} \left( n_T h^{1+3/8} \right)^r \]

where the last inequality follows from using Jensen’s and Minkowski’s inequalities. By the Burkholder-Davis-Gundy inequality, for any \( s \in [(T_m + \tau + bn_T + j - 2)h, (T_m + \tau + bn_T + j - 1)h] \), we have

\[ \mathbb{E} \left[ \left| e_s - e(\tau + bn_T + j - 2)h \right| \mu_{e,s} \right| h^{-\vartheta} \right] \leq C_r h^{r/2 - \vartheta}, \]

and therefore since \( \vartheta \in [0, 1/8) \),

\[ \mathbb{P} \left( \max_{b=0, \ldots, n_T} \left( \log(T_n) n_T \right)^{1/2} \left( n_T h \right)^{-1} \right) \]

\[ \times \left( \sum_{b=0}^{n_T} \left( \log(T_n) n_T \right)^{1/2} \left( n_T h \right)^{-1} \left( n_T h^{1+3/8} \right)^r \right) \]

\[ \leq C_r \left( \frac{8K}{\varepsilon} \right)^r \sqrt{\log(T_n) T_n^{1/3 + \varepsilon} \left( h^{1/24 + \varepsilon} \right)} \to 0, \]

for \( r > 0 \) sufficiently large and where \( \varepsilon > 0 \) is a small real number. Next, consider the second term of (S.13),

\[ \left( e_s - e(\tau + bn_T + j - 2)h \right) \sigma_{e,s} - \left( W_{e,s} - W_e(\tau + bn_T + j - 2)h \right) \sigma_{e,(\tau + bn_T - 1)h} \]

\( \sigma_{e,s} = \left( W_{e,s} - W_e(\tau + bn_T + j - 2)h \right) \sigma_{e,(\tau + bn_T - 1)h} \)

\( = \sigma_{e,s} \int_{(\tau + bn_T - 1)h}^{(\tau + bn_T + j - 2)h} \sigma_{e,v} dW_{e,v} \)

\[ = \sigma_{e,s} \int_{(\tau + bn_T - 1)h}^{(\tau + bn_T + j - 2)h} \sigma_{e,v} dW_{e,v} \]

\[ + \sigma_{e,s} \int_{(\tau + bn_T - 1)h}^{(\tau + bn_T + j - 2)h} \mu_{e,v} h^{-\vartheta} dv \]

\[ = \left( \sigma_{e,s} - \sigma_{e,(\tau + bn_T - 1)h} \right) \int_{(\tau + bn_T + j - 2)h}^{(\tau + bn_T + j - 1)h} \sigma_{e,v} dW_{e,v} \]

\[ + \sigma_{e,(\tau + bn_T - 1)h} \int_{(\tau + bn_T + j - 2)h}^{(\tau + bn_T + j - 1)h} \left( \sigma_{e,v} - \sigma_{e,(\tau + bn_T - 1)h} \right) dW_{e,v} \]

\[ + \sigma_{e,s} \int_{(\tau + bn_T + j - 2)h}^{(\tau + bn_T + j - 1)h} \mu_{e,v} h^{-\vartheta} dv. \]
For any integer $r > 2$,

$$
\mathbb{P}\left( \max_{b=0, \ldots, \lceil T_n/n_T \rceil - 2} \left( \log \left( \frac{T_n}{n_T} \right) \sqrt{n_T} \right)^{-1}
\leq \frac{\varepsilon}{12K} \sum_{b=1}^{n_T} \left( \log \left( \frac{T_n}{n_T} \right) \sqrt{n_T} \right)^{-1} \int_{\frac{T_n}{n_T} \tau + b \delta_T + j - 2}^{\frac{T_n}{n_T} \tau + (b+1) \delta_T + j - 2} \left( \sigma_{e,s} - \sigma_{e,(\frac{T_n}{n_T} \tau + b \delta_T + j - 1)h} \right) \int_{\frac{T_n}{n_T} \tau + b \delta_T + j - 2}^{s} \left( \sigma_{e,v} dW_{e,v} \right) > \varepsilon \right) \right)
\leq \left( \frac{\varepsilon}{12K} \right)^{-r} \sum_{b=1}^{n_T} \left( \log \left( \frac{T_n}{n_T} \right) \sqrt{n_T} \right)^{-1} \int_{\frac{T_n}{n_T} \tau + b \delta_T + j - 2}^{\frac{T_n}{n_T} \tau + (b+1) \delta_T + j - 2} \left( \sigma_{e,s} - \sigma_{e,(\frac{T_n}{n_T} \tau + b \delta_T + j - 1)h} \right) \int_{\frac{T_n}{n_T} \tau + b \delta_T + j - 2}^{s} \left( \sigma_{e,v} dW_{e,v} \right) \right)^r.
$$

Then, by Hölder’s inequality,

$$
\mathbb{E}\left[ \left( \log \left( \frac{T_n}{n_T} \right) \sqrt{n_T} \right)^{-1} \int_{\frac{T_n}{n_T} \tau + b \delta_T + j - 2}^{\frac{T_n}{n_T} \tau + (b+1) \delta_T + j - 2} \left( \sigma_{e,s} - \sigma_{e,(\frac{T_n}{n_T} \tau + b \delta_T + j - 1)h} \right) \int_{\frac{T_n}{n_T} \tau + b \delta_T + j - 2}^{s} \left( \sigma_{e,v} dW_{e,v} \right) \right]^r
\leq C_r \left( \frac{\sqrt{\log \left( \frac{T_n}{n_T} \right)}}{\sqrt{n_T}} \right)^r
\times \left( \int_{\frac{T_n}{n_T} \tau + b \delta_T + j - 2}^{\frac{T_n}{n_T} \tau + (b+1) \delta_T + j - 2} \left( \sigma_{e,s} - \sigma_{e,(\frac{T_n}{n_T} \tau + b \delta_T + j - 1)h} \right) \int_{\frac{T_n}{n_T} \tau + b \delta_T + j - 2}^{s} \left( \sigma_{e,v} dW_{e,v} \right) \right)^r
\times 1\left\{ \left( \frac{T_n}{n_T} \tau + b \delta_T + j - 1, \frac{T_n}{n_T} \tau + b \delta_T + j - 1 \right) \right\}^2 \right) \right)^{r/2}
\leq C_r \left( \frac{\sqrt{\log \left( \frac{T_n}{n_T} \right)}}{\sqrt{n_T}} \right)^r
\times \left( \int_{\frac{T_n}{n_T} \tau + b \delta_T + j - 2}^{\frac{T_n}{n_T} \tau + (b+1) \delta_T + j - 2} \left( \sigma_{e,s} - \sigma_{e,(\frac{T_n}{n_T} \tau + b \delta_T + j - 1)h} \right) \int_{\frac{T_n}{n_T} \tau + b \delta_T + j - 2}^{s} \left( \sigma_{e,v} dW_{e,v} \right) \right)^r
\times 1\left\{ \left( \frac{T_n}{n_T} \tau + b \delta_T + j - 2, \frac{T_n}{n_T} \tau + b \delta_T + j - 2 \right) \right\}^2 \right) \right)^{r/2}
\leq C_r \left( \frac{\sqrt{\log \left( \frac{T_n}{n_T} \right)}}{\sqrt{n_T}} \right)^r \left( \int_{\frac{T_n}{n_T} \tau + b \delta_T + j - 2}^{\frac{T_n}{n_T} \tau + (b+1) \delta_T + j - 2} \left( \frac{T_n}{n_T} \tau + b \delta_T + j - 1 \right)^2 \right)^{r/2} \right)^{r/2}
\leq C_r \left( \frac{\sqrt{\log \left( \frac{T_n}{n_T} \right)}}{\sqrt{n_T}} \right)^r \left( \int_{\frac{T_n}{n_T} \tau + b \delta_T + j - 2}^{\frac{T_n}{n_T} \tau + (b+1) \delta_T + j - 2} \left( \frac{T_n}{n_T} \tau + b \delta_T + j - 1 \right)^{r/2} \right)^{r/2} \right)^{r/2}.
$$

The same bound holds for the second term in (S.15). Finally, the last term of (S.15) is such that

$$
\mathbb{P}\left( \max_{b=0, \ldots, \lceil T_n/n_T \rceil - 2} \left( \log \left( \frac{T_n}{n_T} \right) \sqrt{n_T} \right)^{-1}
\leq \frac{\varepsilon}{12K} \sum_{b=1}^{n_T} \left( \log \left( \frac{T_n}{n_T} \right) \sqrt{n_T} \right)^{-1} \int_{\frac{T_n}{n_T} \tau + b \delta_T + j - 2}^{\frac{T_n}{n_T} \tau + (b+1) \delta_T + j - 2} \left( \sigma_{e,s} - \sigma_{e,(\frac{T_n}{n_T} \tau + b \delta_T + j - 1)h} \right) \int_{\frac{T_n}{n_T} \tau + b \delta_T + j - 2}^{s} \left( \sigma_{e,v} dW_{e,v} \right) \right)^r.
$$

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and observe that for any integer 

\[
\text{Let } K \leq \frac{\varepsilon}{b} \leq \varepsilon / (12K).
\]

\[
\left[ \frac{T_n}{n_T} \right] - 2 \sum_{b=0}^{n_T} \mathbb{P} \left( (\log (T_n))^{1/2} (n_T h)^{-1} \right) \leq \sum_{b=0}^{n_T} \mathbb{P} \left( (\log (T_n))^{1/2} (n_T h)^{-1} \right)
\]

\[
< \left( \frac{12K}{\varepsilon} \right)^r \sum_{b=0}^{n_T} \mathbb{E} \left( (\log (T_n))^{1/2} (n_T h)^{-1} \right) \left[ (T_m + \tau + b n_T + j - 1) h \right]^{1/2} \left( \sigma_{e,s} \int_{(T_m + \tau + b n_T + j - 2) h}^{s} \mu_{e,v} h^{-\theta} dW_{e,v} \right) > \varepsilon / (12K)
\]

\[
\leq \left( \frac{12K}{\varepsilon} \right)^r \sum_{b=0}^{n_T} \mathbb{E} \left( (\log (T_n))^{1/2} (n_T h)^{-1} \right) \left[ (T_m + \tau + b n_T + j - 1) h \right]^{1/2} \left( \sigma_{e,s} \int_{(T_m + \tau + b n_T + j - 2) h}^{s} \mu_{e,v} h^{-\theta} dW_{e,v} \right).
\]

Let

\[
f_s \triangleq 2^r \sum_{j=1}^{n_T} \sigma_{e,s} \int_{(T_m + \tau + b n_T + j - 2) h}^{s} \mu_{e,v} h^{-\theta} dW_{e,v} \times 1_{[(T_m + \tau + b n_T + j - 2) h, (T_m + \tau + b n_T + j - 1) h]} (s),
\]

and observe that for any integer \( r > 1 \),

\[
\mathbb{E} (f_s^r) = 2^r \sum_{j=1}^{n_T} \mathbb{E} \left( \sigma_{e,s} \left( \int_{(T_m + \tau + b n_T + j - 2) h}^{s} \mu_{e,v} h^{-\theta} dW_{e,v} \right) \right) \left( 1_{[(T_m + \tau + b n_T + j - 2) h, (T_m + \tau + b n_T + j - 1) h]} (s) \right)
\]

\[
\leq 2^r C_r h^r (1 - \theta).
\]

Therefore, the right-hand side of (S.17) is less than

\[
\left( \frac{12K}{\varepsilon} \right)^r \sum_{b=0}^{n_T} \left( \mathbb{E} (f_s^r) \right) \left( (T_m + \tau + b n_T + j - 1) h \right) \left( (T_m + \tau + (b+1) n_T - 1) h \right)^{1/2} \left( \sigma_{e,s} \int_{(T_m + \tau + b n_T + j - 2) h}^{s} \mu_{e,v} h^{-\theta} dW_{e,v} \right) \left( 1_{[(T_m + \tau + b n_T + j - 2) h, (T_m + \tau + b n_T + j - 1) h]} (s) \right)
\]

for \( r \) sufficiently large. This leads to

\[
\mathbb{P} \left( \max_{0 \leq b \leq n_T} \left( \left( \log (T_n) \right)^{1/2} (B_{h,b}^n - B_{h,b}^0) \right) > \varepsilon / K \right) \to 0.
\]

(S.18)
We now turn to the first term on the right hand side of (S.11). Choose any \( \varepsilon > 0 \) and positive \( K < \infty \), and note that

\[
P \left( b = 0, \ldots, \left[ T_n/n_T \right] - 2 \left( \log (T_n) n_T \right)^{1/2} \left| B_{h,b}^* \left( \frac{1}{U_{h,b+1}} - \frac{1}{B_{h,b+1}^0} \right) \right| > \varepsilon \right) \tag{S.19}
\]

\[
\leq P \left( b = 0, \ldots, \left[ T_n/n_T \right] - 2 \left( \log (T_n) n_T \right)^{1/2} \left| B_{h,b}^* \left( U_{h,b+1} - B_{h,b+1}^0 \right) \right| > \varepsilon / K \right)
\]

\[
+ P \left( b = 0, \ldots, \left[ T_n/n_T \right] - 2 \left/ \left| U_{h,b+1} B_{h,b+1}^0 \right| > K \right. \right)
\]

We can manipulate the second term as follows:

\[
P \left( b = 0, \ldots, \left[ T_n/n_T \right] - 2 \left/ \left| U_{h,b+1} B_{h,b+1}^0 \right| > K \right. \right)
\]

\[
= P \left( b = 0, \ldots, \left[ T_n/n_T \right] - 2 \left/ \left| U_{h,b+1} B_{h,b+1}^0 \right| < 1 / K \right. \right)
\]

\[
\leq P \left( b = 0, \ldots, \left[ T_n/n_T \right] - 2 \left/ \left| U_{h,b+1} \right| < 1 / \sqrt{K} \right. \right) + P \left( b = 0, \ldots, \left[ T_n/n_T \right] - 2 \left/ \left| B_{h,b+1}^0 \right| < 1 / \sqrt{K} \right. \right)
\]

\[
\leq P \left( b = 0, \ldots, \left[ T_n/n_T \right] - 2 \left/ \left| U_{h,b+1} - B_{h,b+1}^0 \right| > 1 / \sqrt{K} \right. \right) + P \left( b = 0, \ldots, \left[ T_n/n_T \right] - 2 \left/ \left| B_{h,b+1}^0 \right| < 2 / \sqrt{K} \right. \right)
\]

The second and third term on the right-hand side of the the last inequality converge to zero in view of Lemma S.A.4. Noting that \( U_{h,b} \) coincides with \( B_{h,b}^* \), we can use the same arguments that led to (S.18) which shows a tighter bound since it involves \( \max_{b = 0, \ldots, \left[ T_n/n_T \right] - 2} \left| B_{h,b+1}^* - B_{h,b+1}^0 \right| \) multiplied by \( \left( \log (T_n) n_T \right)^{1/2} \). Turning to the first term in (S.19), note that for any \( K_2 > 0 \),

\[
P \left( b = 0, \ldots, \left[ T_n/n_T \right] - 2 \left/ \left( \log (T_n) n_T \right)^{1/2} \right. \right) \left| B_{h,b}^* \left( U_{h,b+1} - B_{h,b+1}^0 \right) \right| > \varepsilon / K \right)
\]

\[
\leq P \left( b = 0, \ldots, \left[ T_n/n_T \right] - 2 \left/ \left( \log (T_n) n_T \right)^{1/2} \right. \right) \left| U_{h,b+1} - B_{h,b+1}^0 \right| > \varepsilon / (K \cdot K_2) \right)
\]

Noting that \( U_{h,b+1} \) coincides with \( B_{h,b+1}^* \), we can use the same arguments as in (S.18). The second term converges to zero by the same argument as above. Then,

\[
P \left( b = 0, \ldots, \left[ T_n/n_T \right] - 2 \left/ \right. \left| B_{h,b}^* \right| > K_2 \right)
\]

\[
\leq P \left( b = 0, \ldots, \left[ T_n/n_T \right] - 2 \left/ \left| B_{h,b}^* - B_{h,b}^0 \right| > K_2 / 2 \right. \right) + P \left( b = 0, \ldots, \left[ T_n/n_T \right] - 2 \left/ \right. \left| B_{h,b}^0 \right| > K_2 / 2 \right)
\]

The first term was already discussed above whereas the second term converges to zero by invoking again Lemma S.A.4 together with the localization assumption [cf. Assumption 2.1-(iii)] which implies the \( \sigma_{e,t} \)
is bounded from above for all \( t \geq 0 \). \( \square \)

**Lemma S.A.7.** As \( h \downarrow 0 \), \( (\log (T_n) n_T)^{1/2} \left( B^0_{\max,h} (T_n, \tau) - \overline{B}^0_{\max,h} (T_n, \tau) \right) \xrightarrow{\mathbb{P}} 0. \)

**Proof.** By simple rearrangements,

\[
\left| B^0_{\max,h} (T_n, \tau) - \overline{B}^0_{\max,h} (T_n, \tau) \right| \leq \max_{b=0, \ldots, [T_n/n_T] - 2} \left| \frac{B^0_{h,b} (\overline{B}^0_{h,b+1} - B^0_{h,b+1})}{B^0_{h,b} B^0_{h,b+1}} \right|.
\]

We shall show that

\[
\max_{b=0, \ldots, [T_n/n_T] - 2} \left( \log (T_n) n_T \right)^{1/2} \left| B^0_{h,b} (\overline{B}^0_{h,b+1} - B^0_{h,b+1}) \right| = o_{\mathbb{P}} (1). \tag{S.20}
\]

By Lemma S.A.4, \( \mathbb{P} \left( \min_{b=0, \ldots, [T_n/n_T] - 2} |\overline{B}^0_{h,b+1} B^0_{h,b+1}| < 1/K \right) \rightarrow 0 \) for some \( K > 0 \); for example, set \( \sqrt{K} = 2/\sigma^2 \). Turning to the numerator of (S.20), for any \( \varepsilon > 0 \) and any \( K > 0 \),

\[
\mathbb{P} \left( \max_{b=0, \ldots, [T_n/n_T] - 2} \left( \log (T_n) n_T \right)^{1/2} \left| B^0_{h,b} \left( \overline{B}^0_{h,b+1} - B^0_{h,b+1} \right) \right| > \varepsilon \right) \\
\leq \mathbb{P} \left( \max_{b=0, \ldots, [T_n/n_T] - 2} |B^0_{h,b}| > K \right) + \mathbb{P} \left( \max_{b=0, \ldots, [T_n/n_T] - 2} \left( \log (T_n) n_T \right)^{1/2} |B^0_{h,b+1} - B^0_{h,b+1}| > \varepsilon/K \right), \tag{S.21}
\]

where the first term converges to zero by the same argument as in the last part of the proof of Lemma S.A.6. Therefore, it remains to deal with the second term for which

\[
\mathbb{P} \left( \max_{b=0, \ldots, [T_n/n_T] - 2} \left( \log (T_n) n_T \right)^{1/2} |B^0_{h,b+1} - B^0_{h,b+1}| > \varepsilon/K \right) \\
= \mathbb{P} \left( \max_{b=0, \ldots, [T_n/n_T] - 2} \left( \log (T_n) n_T \right)^{1/2} \frac{n_T}{n} \sigma^2_{e.(T_n+\tau+b n_T-1)h} - \sigma^2_{e.(T_n+\tau+(b+1)n_T-1)h} \right) \left| \sum_{j=1}^{n_T} \left( \Delta_h W_{e,T_m+\tau+(b+1)n_T+j-1} \right)^2 > \varepsilon/K \right) \\
\leq \mathbb{P} \left( \max_{b=0, \ldots, [T_n/n_T] - 2} \left( \log (T_n) n_T \right)^{1/2} \sigma^2_{e.(T_n+\tau+b n_T-1)h} - \sigma^2_{e.(T_n+\tau+(b+1)n_T-1)h} > \varepsilon/2K \right) \\
+ \mathbb{P} \left( \max_{b=0, \ldots, [T_n/n_T] - 2} \left( n_T^{-1} \sum_{j=1}^{n_T} \left( \Delta_h W_{e,T_n+\tau+(b+1)n_T+j-1} \right)^2 \right) > 2 \right).
\]

By Assumption 2.2, Markov’s inequality and sufficiently large \( r > 0 \),

\[
\mathbb{P} \left( \max_{b=0, \ldots, [T_n/n_T] - 2} \left( \log (T_n) n_T \right)^{1/2} \sigma^2_{e.(T_n+\tau+b n_T-1)h} - \sigma^2_{e.(T_n+\tau+(b+1)n_T-1)h} > \varepsilon/2K \right) \\
\leq C_r \left( \frac{2K}{\varepsilon} \right)^r \left( \log (T_n) n_T \right)^{r/2} (T_n/n_T) \phi^{r}_{\sigma,n_T,h,N} \rightarrow 0. \tag{S.22}
\]
Finally, for all integers \( r > 0 \),
\[
\mathbb{P} \left( \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} \left[ (n_T h)^{-1} \sum_{j=1}^{n_T} \left( \Delta h W_{e,T_m+\tau+(b+1)n_T+j-1} \right)^2 \right] > 2 \right) \quad (S.23)
\]
\[
\leq \sum_{b=0}^{\lfloor T_n/n_T \rfloor - 2} \mathbb{P} \left( \sum_{j=1}^{n_T} \left( h^{-1/2} \Delta h W_{e,T_m+\tau+(b+1)n_T+j-1} \right)^2 - 1 \right) \geq 1 \}
\]
\[
\leq \sum_{b=0}^{\lfloor T_n/n_T \rfloor - 2} \mathbb{E} \left( \sum_{j=1}^{n_T} \left( h^{-1/2} \Delta h W_{e,T_m+\tau+(b+1)n_T+j-1} \right)^2 - 1 \right) \}}
\]
\[
\leq C_r (T_n/n_T)^{r/2} = C_r T_n n_T^{-1-r/2} \to 0,
\]
in view of Condition 1 by choosing \( r \) sufficiently large. Using this together with (S.22) into (S.21) we deduce (S.20). □

**Lemma S.A.8.** As \( h \downarrow 0 \), \( (\log (T_n) n_T)^{1/2} \left( V_{\max,h} (T_n, \tau) - B_{\max,h}^0 (T_n, \tau) \right) \) \( \overset{\mathcal{P}}{\to} 0 \).

**Proof.** Note that
\[
V_{\max,h} (T_n, \tau) - B_{\max,h}^0 (T_n, \tau) = \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} \left| \frac{B_{h,b+1}^0 - B_{h,h}^0}{\sigma^2_e(T_m+\tau+bn_T-1)h} \right| - \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} B_{h,h}^0 - B_{h,b+1}^0 = \mathcal{O}_p(1).
\]

By the boundedness of \( \sigma_e, t \geq 0 \), and upon using the same arguments as in the previous lemmas for \( B_{h,h}^0 \), the denominator is \( \mathcal{O}_p(1) \). Since for any \( \varepsilon > 0 \),
\[
\mathbb{P} \left( (\log (T_n) n_T)^{1/4} \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} \left| B_{h,b+1}^0 - B_{h,h}^0 \right| > \sqrt{\varepsilon} \right) \]
\[
\leq \mathbb{P} \left( \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} \frac{(\log (T_n) n_T)^{1/4}}{n_T h} \times \sigma^2_e(T_m+\tau+bn_T-1)h \sum_{j=1}^{n_T} \left( (\Delta h W_{e,T_m+\tau+(b+1)n_T+j-1}^2 - (\Delta h W_{e,T_m+\tau+(b+1)n_T+j-1})^2 \right) > \sqrt{\varepsilon} \right) \]
\[
\leq \mathbb{P} \left( \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} \frac{(\log (T_n) n_T)^{1/4}}{n_T h} \sigma^2_e(T_m+\tau+bn_T-1)h \sum_{j=1}^{n_T} (\Delta h W_{e,T_m+\tau+(b+1)n_T+j-1})^2 - 1 \right) > \sqrt{\varepsilon}/2 \right) \]
\[
+ \mathbb{P} \left( \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} \frac{(\log (T_n) n_T)^{1/4}}{n_T h} \sigma^2_e(T_m+\tau+bn_T-1)h \sum_{j=1}^{n_T} (\Delta h W_{e,T_m+\tau+(b+1)n_T+j-1})^2 - 1 \right) > \sqrt{\varepsilon}/2 \right) .
\]

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We consider the first probability term; the argument for the second is analogous. By using a similar argument as in (S.23), the first term is less than

\[
\sum_{b=0}^{T_{n,n_T}-2} \mathbb{P} \left( \left( \log (T_n) n_T \right)^{1/4} \sigma_{e,(T_{m,T}+\tau+b_{n_T+1}) h}^{n_T} \left| \sum_{j=1}^{n_T} \left( h^{-1/2} \Delta_h W_{e,T_{m,T}+\tau+(b+1) n_T+j-1} \right)^2 - 1 \right| > \sqrt{\varepsilon} \right) / 2 \right) \leq C_n \left( \frac{2}{\sqrt{\varepsilon}} \right)^r \left( \log (T_n) n_T \right)^{r/4} \sum_{b=0}^{T_{n,n_T}-2} \mathbb{E} \left( \left| \sum_{j=1}^{n_T} \left( h^{-1/2} \Delta_h W_{e,T_{m,T}+\tau+(b+1) n_T+j-1} \right)^2 - 1 \right| \right) \leq \left( \frac{2}{\sqrt{\varepsilon}} \right)^r \left( \log (T_n) \right)^{r/4} (T_n/n_T)^{-r/4},
\]

which goes to zero by choosing \( r > 0 \) sufficiently large. It remains to show that

\[
\mathbb{P} \left( \left( \log (T_n) n_T \right)^{1/4} \max_{b=0,\ldots,\left[ T_{n,n_T} \right]-2} \left| \tilde{B}_{h,b+1}^{0} - \sigma_{e,(T_{m,T}+\tau+b_{n_T}) h} \right| > \varepsilon^{1/2} \right) \to 0. \tag{S.25}
\]

Simple manipulations yield for some \( C < \infty \), with \( \sigma_+ \leq \sqrt{C} \),

\[
\mathbb{P} \left( \left( \log (T_n) n_T \right)^{1/4} \max_{b=0,\ldots,\left[ T_{n,n_T} \right]-2} \left| \tilde{B}_{h,b+1}^{0} - \sigma_{e,(T_{m,T}+\tau+b_{n_T}) h} \right| > \varepsilon^{1/2} \right) \leq \mathbb{P} \left( \left( \log (T_n) n_T \right)^{1/4} \sum_{b=0}^{T_{n,n_T}-2} \sigma_{e,(T_{m,T}+\tau+b_{n_T}) h} \left| \sum_{j=1}^{n_T} \left( h^{-1/2} \Delta_h W_{e,T_{m,T}+\tau+(b+1) n_T+j-1} \right)^2 - 1 \right| > \sqrt{\varepsilon} \right) \leq C_n \left( \frac{1}{\sqrt{\varepsilon}} \right)^r \left( \log (T_n) n_T \right)^{r/4} \sum_{b=0}^{T_{n,n_T}-2} \mathbb{E} \left( \left| \sum_{j=1}^{n_T} \left( h^{-1/2} \Delta_h W_{e,T_{m,T}+\tau+(b+1) n_T+j-1} \right)^2 - 1 \right| \right) \leq C_n \left( \frac{2}{\sqrt{\varepsilon}} \right)^r \left( \log (T_n) \right)^{r/4} (T_n/n_T)^{-r/4} \to 0.
\]

We have (S.25) and thus (S.24), which concludes the proof. \( \square \)

From Lemma S.A.2-S.A.8 we deduce \( \sqrt{\log (T_n) n_T} \left( B_{\max,h} \left( T_{n,n_T} \right) - V_{\max,h} \left( T_{n,n_T} \right) \right) = o_p(1) \), where \( V_{\max,h} \left( T_{n,n_T} \right) \) was defined in (S.10). By the properties of the Wiener process, for each block \( b \) the variables \( \Delta_h W_{e,T_{m,T}+\tau+b_{n_T}+1} \) are a sequence of \( \chi^2 \) random variables which are independent over \( j \).

After centering these variables, (i.e., \( \left\{ \left( \Delta_h W_{e,T_{m,T}+\tau+b_{n_T}+1} \right)^2 - 1 \right\} \)), we can apply the results in Lemma 1-2 in Wu and Zhao (2007). This leads us to a limit theorem for the statistic \( V_{\max,h} \left( T_{n,n_T} \right) \) which takes a similar form to the statistic in equation (13) of Wu and Zhao (2007). Therefore, in Lemma S.A.10 we provide a limit theorem which adapts Theorem 1 of Wu and Zhao (2007) to our context. The difference hinges on (i) the dependence structure of the variables \( \left\{ \left( \Delta_h W_{e,T_{m,T}+\tau+b_{n_T}+1} \right)^2 - 1 \right\} \) relative to the sequence \( \{ X_k \}_{k \geq 1} \) appearing in Wu and Zhao (2007), and on (ii) the form of our test statistics which allow both for additive and multiplicative structure. For the quadratic loss case, our problem is then similar to that of Bibinger, Jirak, and Vetter (2017) who also uses Lemma 1-2 in Wu and Zhao (2007); yet even in the quadratic loss case our context differs from that of Bibinger, Jirak, and Vetter (2017) because we allow for model misspecification via the additional term \( \mu_{e,d} \) in (2.3) and estimation of \( \hat{\beta}_k \).

**Assumption S.A.2.** The sequence of rescaled forecasts errors \( \{ \Delta_h \hat{c}_k \}_{k \geq 1} \) satisfies, for some \( p \geq 4 \), \( \mathbb{E} \left[ \left| \Delta_h \hat{c}_k \right|^p \right] < \infty \) for all \( k \geq 1 \). Furthermore, the sequence of forecast losses \( \{ L_{\psi,k} \}_{k \geq 1} \) satisfies the same
assumption.

We now explain how to verify Assumption S.A.2.

**Lemma S.A.9.** Given the model in (2.3), Assumption S.A.2 holds.

**Proof.** We know that \( \Delta_h e_k^* = \int_{(k-1)h}^{kh} \mu_{e,s} h^{-\vartheta} ds + \int_{(k-1)h}^{kh} \sigma_{e,s} dW_{e,s} \). Note that conditional on \( \{\mu_{e,t}\}_{t \geq 0} \) and \( \{\sigma_{e,t}\}_{t \geq 0} \),

\[
(\Delta_h e_k^*)^2 = \left( \int_{(k-1)h}^{kh} \mu_{e,s} h^{-\vartheta} ds \right)^2 + \left( \int_{(k-1)h}^{kh} \sigma_{e,s} dW_{e,s} \right)^2 + 2 \int_{(k-1)h}^{kh} \int_{(k-1)h}^{kh} \mu_{e,n} h^{-\vartheta} \sigma_{e,s} dW_{e,s} dW_{e,n} \\
= O\left(h^{2(1-\vartheta)}\right) + \left( \int_{(k-1)h}^{kh} \sigma_{e,s} dW_{e,s} \right)^2 + O_p\left(h^{3/2-\vartheta}\right) \\
= o(h) + \left( \int_{(k-1)h}^{kh} \sigma_{e,s} dW_{e,s} \right)^2 + o_p\left(h\right).
\]

Hence, \( \mathbb{E}\left[|\Delta_h e_k^*|^p \mid \mathcal{F}_{(k-1)h}\right] = \mathbb{E}\left[\left|\int_{(k-1)h}^{kh} \sigma_{e,s} dW_{e,s} \right|^p \mid \mathcal{F}_{(k-1)h}\right] + C_p o_p\left(h^{p/2}\right) \) and Assumption S.A.2 is verified given the properties of the Wiener process and \( \psi_h = h^{1/2} \).

**Lemma S.A.10.** For \( n = 1, \ldots, T_n \), let \( \mu_n = \mu_n\left(nT/T_n\right) \) with \( \mu \in \text{Lip}([0, 1]) \). Let \( \{U_n\}_{n \geq 1} \) denote a sequence of i.i.d. random variables with \( U_n = \mu_n + \tilde{U}_n \), \( \mathbb{E}\left(\tilde{U}_n\right) = 0 \), \( \text{Var}(\tilde{U}_n) = \sigma_{\tilde{U}}^2 \) and \( \mathbb{E}\left[|\tilde{U}_n|^p\right] < \infty \) for some \( p \geq 4 \). Set \( m_T = \lfloor T_n/n_T\rfloor \) and define

\[
B_{\text{max},T_n} \triangleq \frac{1}{n_T} \max_{0 \leq b \leq \lfloor T_n/n_T\rfloor - 2} \left| \sum_{j=1}^{n_T} (U_{(b+1)n_T+j} - U_{bn_T+j}) \right|,
\]

and

\[
MB_{\text{max},T_n} \triangleq \frac{1}{n_T} \max_{n_T \leq i \leq T_n - n_T} \left| \sum_{j=i+1}^{n_T+i} U_j - \sum_{j=i-n_T+1}^{i} U_j \right|.
\]

If the following condition holds,

\[
n_T^{-p/2} T_n = o\left((\log(T_n))^{-p/2}\right), \tag{S.27}
\]

then

\[
\sqrt{\log(m_T)} \left( \frac{\sqrt{n_T}}{\sigma_{\tilde{U}}} B_{\text{max},T_n} - \gamma_{m_T} \right) \Rightarrow \mathcal{Y}, \tag{S.28}
\]

and

\[
\sqrt{\log(m_T)} \left( \frac{\sqrt{n_T}}{\sigma_{\tilde{U}}} MB_{\text{max},T_n} - 2 \log(m_T) - \frac{1}{2} \log \log(m_T) - \log 3 \right) \Rightarrow \mathcal{Y}, \tag{S.29}
\]

where \( \gamma_{m_T} = [4 \log(m_T) - 2 \log(\log(m_T))]/2 \) and \( \mathcal{Y} \) satisfies \( \mathbb{P}(\mathcal{Y} \leq v) = \exp(-\pi^{-1/2} \exp(-v)) \).

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Proof. Without loss of generality, we set $\sigma^2_U = 1$. By the Donsker-Prokhorov invariance principle $T_n^{-1/2} \sum_{j=1}^{[sT_n]} \tilde{U}_j \Rightarrow \mathbb{B}(s)$, where $\{\mathbb{B}(s)\}_{s \in [0,1]}$ is a standard Wiener process on $[0,1]$. Then, we have by definition that $Z_{b+1} \triangleq n^{-1} \{ (b + 1) n_T - \mathbb{B}(b n_T))$, $b = 0, \ldots, m_T - 1$, are i.i.d. standard normal random variables. We have the decomposition

$$n_T^{-1} \sum_{j=1}^{n_T} U_{(b+1)n_T+j} = \frac{Z_{b+1}}{\sqrt{n_T}} + \frac{1}{n_T} \sum_{j=1}^{n_T} \mu_{(b+1)n_T+j} + R_{b+1,n_T},$$

where $R_{b_T,n_T} \triangleq \sum_{j=1}^{bn_T} \tilde{U}_j - \mathbb{B}(bn_T) - \left(\sum_{j=1}^{(b-1)n_T} \tilde{U}_j - \mathbb{B}((b - 1) n_T)\right)$ and recall $\tilde{U}_j = U_j - \mu_j$. By the strong invariance principle of Komlós, Major, and Tusnády (1975), $\max_{b \leq m_T - 1} |R_{b+1,n_T}| = o_{a.s.} \left(T_n^{1/p}\right)$, where we have used the independence structure of $\{\tilde{U}_j\}$. Since $\mu \in \text{Lip}([0,1])$, we have uniformly over $b$ and $j$, $n_T^{-1} \sum_{j=1}^{n_T} \left(\mu_{(b+1)n_T+j} - \mu_{bn_T+j}\right) = O(n_T/m_T)$. Altogether,

$$n_T^{-1} \sum_{j=1}^{n_T} \left(U_{(b+1)n_T+j} - U_{bn_T+j}\right) = Z_{b+1} - Z_b + O_{a.s.} \left(n_T^{-3/2} T_n + n_T^{-1/2} T_n^{1/p}\right)$$

$$= Z_{b+1} - Z_b + o_{a.s.} \left(\log(m_T)\right)^{-1/2}.$$

The result in equation (S.28) then follows from Lemma 1 in Wu and Zhao (2007).

We now turn to the corresponding result for the overlapping case. We redefine $\{Z_j\}_{j \geq 1}$ as being a sequence of standard normal random variables. Then,

$$\max_{n_T \leq i \leq n_T - n_T} \left|\sum_{j=1}^{n_T} (U_j - Z_j) - \sum_{j=n_T-i+1}^{n_T+i} (U_j - Z_j)\right|$$

$$= \max_{n_T \leq i \leq n_T - n_T} \left|\sum_{j=i+1}^{n_T+i} (U_j - \mu_j - Z_j) - \sum_{j=n_T-i+1}^{n_T+i} (U_j - \mu_j - Z_j) + \sum_{j=i+1}^{n_T+i} \mu_j - \sum_{j=n_T-i+1}^{n_T+i} \mu_j\right|$$

$$\leq 4 \max_{n_T \leq i \leq n_T - n_T} \left|\sum_{j=1}^{i} (\tilde{U}_j - Z_j)\right| + \max_{n_T \leq i \leq n_T - n_T} \left|\sum_{j=i+1}^{n_T} \mu_j - \sum_{j=n_T-i+1}^{n_T} \mu_j\right|$$

$$= 4 \max_{n_T \leq i \leq n_T - n_T} \left|\sum_{j=1}^{i} (\tilde{U}_j - Z_j)\right| + O \left(n_T^2 / T_n\right),$$

where the last equality follows from $\mu \in \text{Lip}([0,1])$ and $O(n_T^2 / T_n)$ being uniform. Next, we use Theorem 4 of Komlós, Major, and Tusnády (1976) to derive a bound on the approximation error for the first term above. Let $\{a_{T_n}\}_{T_n \in \mathbb{N}}$ be a positive sequence. By Markov’s inequality,

$$\mathbb{P} \left(\max_{n_T \leq i \leq T_n} \left|\sum_{j=1}^{i} (\tilde{U}_j - Z_j)\right| \geq a_{T_n}\right) \leq C_{1,p} \frac{1}{a_{T_n}^p} \sum_{j=1}^{T_n} \mathbb{E} \left(|\tilde{U}_j|^p\right) \leq C_{2,p} a_{T_n}^{-p} T_n,$$

where $C_{1,p}, C_{2,p} < \infty$. The conditions of Theorem 4 in Komlós, Major, and Tusnády (1976) are satisfied.
if we set \( a_T = \frac{\sqrt{n_T}}{\log(T_n)} \). This leads to

\[
\max_{n_T \leq i \leq T_n - n_T} \left| \frac{n_T}{n_T} \sum_{j=i+1}^{n_T+i} \left( \tilde{U}_j - Z_j \right) - \sum_{j=i-n_T+1}^{i} \left( \tilde{U}_j - Z_j \right) \right| = \text{op}(\sqrt{n_T/\log(T_n)^{-1/2}}), \tag{S.30}
\]

where we have used (S.27). Let \( \mathbb{B}(i) = \sum_{j=i}^{n_T} Z_j \) and define \( H(u) \triangleq (1(0 \leq u < 1) - 1(-1 < u < 0)) / \sqrt{2} \). Use (S.30) to deduce that,

\[
\frac{\sqrt{n_T} \max_{T_n - n_T \leq i \leq T_n}}{\sqrt{2}} = \frac{1}{\sqrt{2n_T}} \max_{n_T \leq i \leq T_n - n_T} \left| \frac{n_T}{n_T} \sum_{j=i+1}^{n_T+i} \tilde{U}_j - \sum_{j=i-n_T+1}^{i} \tilde{U}_j \right| + O\left(\frac{n_T^{3/2}}{T_n}\right)
\]

Therefore, letting \( g_n = \sup \{ |\mathbb{B}(u) - \mathbb{B}(u')| : u, u' \in [0, T_n], |u - u'| \leq 1 \} \), we have

\[
\frac{\sqrt{n_T} \max_{T_n - n_T}}{\sqrt{2}} = \frac{1}{\sqrt{2n_T}} \max_{n_T \leq i \leq T_n - n_T} \left| \int_{\mathbb{R}} H\left(\frac{s-u}{n_T}\right) d\mathbb{B}(u) \right| + O\left(\frac{g_n}{\sqrt{n_T}}\right) + \text{op}(1) / \sqrt{\log(T_n)}.
\]

By the global modulus of continuity of the standard Wiener process [cf. Theorem 2.9.25 in Karatzas and Shreve (1996)], we know that \( g_n = \text{op}\left(\sqrt{\log(T_n)}\right) \). The result for the overlapping case then follows from Lemma 2 in Wu and Zhao (2007) with \( \alpha = 1, D_{H,1} = 3, \) bandwidth \( b_n = n_T^{-1} \) and \( n = T_n \); see their Definition 1 as well and note that their lemma can be applied because \((\log(T_n))^6 = o(n_T)\) holds by condition (S.27). \( \square \)

**Proof of Theorem 3.1-(i).** From Lemma S.A.2-S.A.8, \( \log(T_n) n_T \left( B_{T_n,\tau}(T_n, \tau) - V_{T_n,\tau}(T_n, \tau) \right) = \text{op}(1) \). Lemma S.A.9 shows that Assumption S.A.2 holds. Then, under Condition 1, we can apply Lemma S.A.10 to \( V_{T_n,\tau}(T_n, \tau) \) which in turn leads to the result for \( B_{T_n,\tau}(T_n, \tau) \) in part (i) of Theorem 3.1. \( \square \)

**S.A.4.1.1 Proof of part (ii) of Theorem 3.1** The proof can be simplified considerably by using arguments similar to those of part (i) of Theorem 3.1. Let

\[
MB_{T_n,\tau}(T_n, \tau) = \max_{i=n_T^{-1}, \ldots, n_T} \left| \frac{n_T^{-1}}{n_T^{-1}} \sum_{j=i+1}^{n_T+i} \left( \Delta_h e_{T_n + \tau + j - 1} \right)^2 - 1 \right|, \tag{S.31}
\]

and

\[
MB_{T_n,\tau}^0(T_n, \tau) = \max_{i=n_T^{-1}, \ldots, n_T} \left| \frac{n_T^{-1}}{n_T^{-1}} \sum_{j=i+1}^{n_T+i} \left( \Delta_h e_{T_n + \tau + j - 1} \right)^2 - 1 \right|. \tag{S.32}
\]

**Lemma S.A.11.** \( \sqrt{\log(T_n)} n_T \left( MB_{T_n,\tau}(T_n, \tau) - MB_{T_n,\tau}^0(T_n, \tau) \right) \xrightarrow{p} 0. \)

**Proof.** Note that the choice of overlapping blocks does not alter the results of Lemma S.A.2-S.A.5, which in turn give \((\log(T_n))^2 n_T \left( MB_{T_n,\tau}(T_n, \tau) - MB_{T_n,\tau}^0(T_n, \tau) \right) \xrightarrow{p} 0. \) Thus, we can begin by proving a
result analogous to Lemma S.A.6: \((\log (T_n) n_T)^{1/2} \left( MB_{\text{max}, h}^*(T_n, \tau) - MB_{\text{max}, h}^0(T_n, \tau) \right) \xrightarrow{p} 0\). Note that proceeding as in (S.11), we have

\[ | MB_{\text{max}, h}^*(T_n, \tau) - MB_{\text{max}, h}^0(T_n, \tau) | \leq \max_{i=n_T, \ldots, T-n_T} \left| n_T^{-1} \sum_{j=i-n_T+1}^{i} \left( \Delta h e_{T_m+\tau+j-1}^* \right)^2 - \frac{1}{n_T^{-1} \sum_{j=i+1}^{i+n_T} \sigma_{\epsilon, (T_m+\tau+i-1)h}^2 (h^{-1/2} \Delta h W_e, T_m+\tau+j-1)^2} \right| \]

Then, we can use the same decomposition as in (S.12),

\[ \mathbb{P} \left( \max_{i=n_T, \ldots, T-n_T} (\log (T_n) n_T)^{1/2} \right. \]

\[ \times \left| n_T^{-1} \sum_{j=i-n_T+1}^{i} \left( \Delta h e_{T_m+\tau+j-1}^* \right)^2 - \sigma_{\epsilon, (T_m+\tau+i-n_T-1)h}^2 (h^{-1/2} \Delta h W_e, T_m+\tau+j-1)^2 \right| > \varepsilon \]

\[ \leq \mathbb{P} \left( \max_{i=n_T, \ldots, T-n_T} (\log (T_n) n_T)^{1/2} \right. \]

\[ \times \left| n_T^{-1} \sum_{j=i-n_T+1}^{i} \left( \Delta h e_{T_m+\tau+j-1}^* \right)^2 - \sigma_{\epsilon, (T_m+\tau+i-n_T-1)h}^2 (h^{-1/2} \Delta h W_e, T_m+\tau+j-1)^2 \right| > \varepsilon / K \]

\[ + \mathbb{P} \left( \min_{i=n_T, \ldots, T-n_T} \left| n_T^{-1} \sum_{j=i+1}^{i+n_T} \sigma_{\epsilon, (T_m+\tau+i-1)h}^2 (h^{-1/2} \Delta h W_e, T_m+\tau+j-1)^2 \right| < 1 / K \right) , \]

which holds for any \( \varepsilon > 0 \) and any constant \( K > 0 \). Using the same reasoning as in the proof involving the second term of (S.11) and choosing \( K \) appropriately, we have for the second term,

\[ \mathbb{P} \left( \max_{i=n_T, \ldots, T-n_T} (\log (T_n) n_T)^{1/2} \right. \]

\[ \times \left| n_T^{-1} \sum_{j=i+n_T-1}^{i+n_T} \sigma_{\epsilon, (T_m+\tau+i-1)h}^2 (h^{-1/2} \Delta h W_e, T_m+\tau+j-1)^2 \right| > K \right) \xrightarrow{} 0 . \]

Thus, it remains to consider the first term on the right-hand side above. For the non-overlapping case it was treated in (S.13) and its final bound can be obtained from (S.14)-(S.18). However, for the overlapping block case, the maximum is over a larger number of arguments. Indeed, the final bound is an order \( O(n_T) \) larger than the one for the non-overlapping case. Nonetheless, the same conclusion holds upon choosing \( r \) large enough there:

\[ \mathbb{P} \left( \max_{i=n_T, \ldots, T-n_T} (\log (T_n) n_T)^{1/2} \right. \]

\[ \left. (S.33) \right) \]
\[ \left| n_T^{-1} \sum_{j=i-n_T+1}^{i} \left( \left( \Delta_h e_{T_m+\tau+j-1} \right)^2 - \sigma_{e,(T_m+\tau+i-n_T-1)h}^2 \left( h^{-1/2} \Delta_h W_{e,T_m+\tau+j-1} \right)^2 \right) \right| \geq \varepsilon / K \to 0. \]

Generalizing the arguments that led to (S.33) and noting that the bounds involving the Lipschitz continuity of \( \{\sigma_{e,l}\}_{l \geq 0} \) remain the same as in the non-overlapping case, the corresponding results in Lemma S.A.6-S.A.8 can be verified. This together with Lemma S.A.2-S.A.5—which are valid for both cases with minor changes in notation—yield the conclusion of the lemma. □

**Proof of Theorem 3.1-(ii).** From Lemma S.A.11, \( \sqrt{\log(T_n)} n_T^{-1} \left( \text{MB}_{\max,h} (T_n, \tau) - \text{MB}_0^{\max,h} (T_n, \tau) \right) = o_p(1) \). For the non-overlapping case, Lemma S.A.9 shows that Assumption S.A.2 is satisfied. Given Condition 1, Lemma S.A.10 [cf. the result pertaining to MB_{max,T_n} there] applied to MB_0^{0,h} (T_n, \tau) gives part (ii) of the theorem. □

**S.A.4.1.3 Negligibility of the \( \mu_{e,t} \) term** The negligibility of the drift term can be proven by using similar arguments to those in Section A.3.3 in Casini and Perron (2017a). From the decomposition in (S.26) we have for any \( b = 0, \ldots, [T_n/n_T] - 2 \),

\[
\sum_{j=1}^{n_T} \left( \Delta_h e_{T_m+\tau+bn_T+j-1} \right)^2 = \sum_{j=1}^{n_T} \left( \int_{(T_m+\tau+bn_T+j-1)h}^{(T_m+\tau+bn_T+j)h} \mu_{e,s, h^{-\frac{1}{2}}} \sigma_{e,s,dW_{e,s}} \right)^2 + 2 \sum_{j=1}^{n_T} \int_{(T_m+\tau+bn_T+j-2)h}^{(T_m+\tau+bn_T+j)h} \int_{(T_m+\tau+bn_T+j-2)h}^{(T_m+\tau+bn_T+j-1)h} \mu_{e,v, h^{-\frac{1}{2}}} \sigma_{e,s,dvdW_{e,s}} \right)^2 + o_p (n_T^{3/2}) \tag{S.34}
\]

for small \( \varepsilon > 0 \). The limit theorems involve normalizing the above sums by the factor \( \sqrt{\log(T_n)} n_T^{-1} \), \( n_T^{-1} \), \( h^{-3/2-\varepsilon/2} \). Then the first term is \( o \left( h^{15/12+\varepsilon} \right) \). The bound can be extended to hold for the maximum over blocks \( b = 0, \ldots, [T_n/n_T] - 2 \) by using the same argument as in (S.17). The latter bound also applies to the third term of (S.34) which is even of higher order. Therefore, the results of Lemma S.A.2-S.A.5 still holds when \( \mu_{e,t} \) is not restricted to be null for all \( t \geq 0 \).

**S.A.4.2 Proof of Corollary 3.1**

**Proof.** The proof follows easily from Lemma S.A.10 with \( \sigma_{\tilde{U}} = \nu_L \). That is, we have now \( R_{b,T_n} \triangleq \sum_{j=1}^{b_{n_T}} \tilde{U}_j - \nu_L \mathbb{B} ((bn_T)T) - \left( \sum_{j=1}^{(b-1)n_T} \tilde{U}_j - \nu_L \mathbb{B} ((b-1) n_T) \right) \) which satisfies the same bound as above. Then, proceeding as above,

\[
\nu_L^{-1} n_T^{-1/2} \sum_{j=1}^{n_T} \left( U_{(b+1)n_T+j} - U_{bn_T+j} \right) = Z_{b+1} - Z_b + o_{\text{a.s.}} \left( (\log (n_T))^{-1/2} \right),
\]

and the final result for \( Q_{\max,h} \) can be deduced again from Lemma 1 in Wu and Zhao (2007). □
S.A.4.3 Proof of Theorem 3.2

S.A.4.3.1 Proof of part (i) of Theorem 3.2 Recall the notation for the normalized forecast error \( \Delta_h \tilde{e}_k \equiv \Delta_h e_k / \psi_k \) and for the normalized forecast loss \( L_{\psi,T_{m+\tau+b\tau+j}} (\beta^*) = g (\Delta_h \tilde{e}_{T_{m+\tau+b\tau+j-1}}; \beta^*) \). We use the quantity \( U_{\max,h} (T_n, \tau) \) as defined in the proof for the quadratic case. However, \( \bar{U}_{h,b} \) is now defined as \( D_{h,b} \) but with \( \beta^* \) in place of \( \tilde{\beta} \). Let \( B_{h,b}^* = n_T^{-1} \sum_{j=1}^{n_T} g (\Delta_h \tilde{e}_{T_{m+\tau+b\tau+j-1}}; \beta^*) \). We only provide the proof for the recursive forecasting scheme. As in the proof of Theorem 3.1, we first assume \( \mu_{e,t} = 0 \) in (2.2) and relax such restriction in Section S.A.4.3.3. We again omit the index from \( \tilde{\beta} \) when it is clear from the context.

Lemma S.A.12. For any \( L \in L_e \), the results of Lemma S.A.3 hold.

Proof. By definition and upon using basic manipulations,

\[
|n_T^{-1} \sum_{j=1}^{n_T} (T_{\psi,(T_{m+(b+1)n_T+j-1})h} (\beta^*) - T_{\psi,(T_{m+b\tau+j-1})h} (\beta^*))|
\]

\[
= |n_T^{-1} \sum_{j=1}^{n_T} \sum_{l=1}^{T_{m+(b+1)n_T+j-1}} \frac{g (\Delta_h \tilde{e}_l^*; \beta^*)}{T_{m + (b + 1)n_T + j - 1}} - \sum_{l=1}^{T_{m + bn_T + j - 1}} \frac{g (\Delta_h \tilde{e}_l^*; \beta^*)}{T_{m + bn_T + j - 1}} |
\]

\[
= O_p \left( \frac{n_T}{T_m} \right) + O_p \left( \frac{n_T}{T_m} \right),
\]

where the latter bounds are implied by basic law of large numbers given Assumption 3.3. Then use the same arguments as in the proof of Lemma S.A.3 to yield a bound similar to (S.6). Finally, consider a mean-value expansion of \( g (\Delta_h \tilde{e}_l^*; \tilde{\beta}) \) around \( \beta^* \),

\[
g (\Delta_h \tilde{e}_l^*; \tilde{\beta}) = g (\Delta_h \tilde{e}_l^*; \beta^*) + \frac{\partial g (\Delta_h \tilde{e}_l^*; \beta^*)}{\partial \beta} (\tilde{\beta} - \beta^*) + \frac{1}{2} (\tilde{\beta} - \beta^*) \frac{\partial^2 g (\Delta_h \tilde{e}_l^*; \beta^*)}{\partial \beta \partial \tilde{\beta}} (\tilde{\beta} - \beta^*),
\]

where \( \tilde{\beta} \) is an intermediate point between \( \beta^* \) and \( \tilde{\beta} \). It follows that

\[
T_{\psi,(T_{m+b\tau+j-1})h} (\tilde{\beta}) - T_{\psi,(T_{m+b\tau+j-1})h} (\beta^*)
\]

\[
= \sum_{l=1}^{T_{m+b\tau+j-1}} \frac{g (\Delta_h \tilde{e}_l^*; \tilde{\beta})}{T_{m + bn_T + j - 1}} - \sum_{l=1}^{T_{m+b\tau+j-1}} \frac{g (\Delta_h \tilde{e}_l^*; \beta^*)}{T_{m + bn_T + j - 1}}
\]

\[
= \frac{1}{T_{m + bn_T + j - 1}} \sum_{l=1}^{T_{m+b\tau+j-1}} \left( \frac{\partial g (\Delta_h \tilde{e}_l^*; \beta^*)}{\partial \beta} (\tilde{\beta} - \beta^*) + \frac{1}{2} (\tilde{\beta} - \beta^*) \frac{\partial^2 g (\Delta_h \tilde{e}_l^*; \beta^*)}{\partial \beta \partial \tilde{\beta}} (\tilde{\beta} - \beta^*) \right).
\]

By Assumption 3.2, \( \left| \frac{\partial^2 g (\Delta_h \tilde{e}_l^*; \beta^*)}{\partial \beta \partial \tilde{\beta}} \right| < C \) and thus the second term is \( O_p (1/T) \) uniformly in \( l \).
Assumption 3.3, \(\mathbb{E}((\|\partial g (\Delta_h \bar{e}_l; \beta^*) / \partial \beta\|)^{2+\omega}) < \infty\) uniformly in \(l\). Since
\[
\{\partial g (\Delta_h \bar{e}_l; \beta^*) / \partial \beta - \mathbb{E}(\partial g (\Delta_h \bar{e}_l; \beta^*) / \partial \beta)\}_{l \geq T_m},
\]
forms a martingale difference sequence we can use classical bounds on averages of m.d.s. By Assumption 3.6, \(\hat{\beta} - \beta^* = O_p \left(1/\sqrt{T}\right)\) because \(\hat{\beta}_l - \beta^* = O_p \left(1/\sqrt{T}\right)\) uniformly in \(l \geq T_m\). Thus, \(T_{\psi,(T_m+bn_T+j-1)h} \beta \) - \(T_{\psi,(T_m+bn_T+j-1)h} \beta^*\) = \(O_p \left(1/\sqrt{T}\right)\). Proceeding as in (S.6)-(S.8) one verifies,
\[
\max_{b=0,\ldots, [T_n/n_T]-2} (\log(T_n) n_T)^{1/2} \left| n_T^{-1} \sum_{j=1}^{n_T} \left( T_{\psi,(T_m+bn_T+j-1)h} \beta \right) - T_{\psi,(T_m+bn_T+j-1)h} \beta^* \right| \xrightarrow{p} 0. \square
\]

We now have a corresponding result to Lemma S.A.2.

Lemma S.A.13. As \(h \downarrow 0\), \((\log(T_n) n_T)^{1/2} (U_{\max,h} (T_n, \tau) - G_{\max,h} (T_n, \tau)) \xrightarrow{p} 0.\)

Proof. The same manipulations as in Lemma S.A.2 yield,
\[
|U_{\max,h} (T_n, \tau) - G_{\max,h} (T_n, \tau)| \leq C_1 \max_{b=0,\ldots, [T_n/n_T]-2} \left| \frac{\sum_{j=1}^{n_T} \left( SL_{T_m+\tau+bn_T+j-1}(\beta^*) - SL_{T_m+\tau+bn_T+j-1}(\hat{\beta}) \right)}{U_{b+1,h}} \right| + C_2 \max_{b=0,\ldots, [T_n/n_T]-2} \left| \frac{\sum_{j=1}^{n_T} \left( SL_{T_m+\tau+bn_T+j-1}(\beta^*) - SL_{T_m+\tau+bn_T+j-1}(\hat{\beta}) \right)}{U_{b+1,h}} \right|.
\]

From Lemma S.A.12, for any \(j = 1, \ldots, n_T\),
\[
SL_{T_m+\tau+bn_T+j-1}(\beta^*) - SL_{T_m+\tau+bn_T+j-1}(\hat{\beta}) = L_{\psi,T_m+\tau+bn_T+j-1}(\beta^*) - L_{\psi,T_m+\tau+bn_T+j-1}(\hat{\beta}) + O_P \left(T^{-1/2}\right).
\]

Note that,
\[
L_{\psi,T_m+\tau+bn_T+j-1}(\hat{\beta}) - L_{\psi,T_m+\tau+bn_T+j-1}(\beta^*) = g \left( \Delta_h \bar{e}_{T_m+\tau+bn_T+j-1}; \hat{\beta} \right) - g \left( \sigma_e(T_m+\tau+bn_T-1)h \left(h^{-1/2} \Delta_h W_{e,T_m+\tau+bn_T+j-1}; \beta^*\right) \right),
\]
and taking a mean-value expansion of \(g \left( \Delta_h \bar{e}_{T_m+\tau+bn_T+j-1}; \hat{\beta} \right)\) around \(\beta^*\) we have
\[
g \left( \Delta_h \bar{e}_{T_m+\tau+bn_T+j-1}; \hat{\beta} \right) = g \left( \sigma_e(T_m+\tau+bn_T-1)h \left(h^{-1/2} \Delta_h W_{e,T_m+\tau+bn_T+j-1}; \beta^*\right) \right) \left(\hat{\beta} - \beta^*\right)
\]
Therefore, using the last three relationships above, Assumption 3.2-3.3 and Assumption 3.6 we have for the numerator of (S.35),

\[
\frac{1}{nT} \sum_{j=1}^{nT} \left( SL_{\psi,T_m+\tau+bn_T+j-1} (\beta^*) - SL_{\psi,T_m+\tau+bn_T+j-1} (\hat{\beta}) \right) \]

\[
\leq \left| \frac{1}{nT} \sum_{j=1}^{nT} \left( \frac{\partial g (\sigma_{e,T_m+\tau+bn_T-1}) h (h^{-1/2} \Delta_h W_{e,T_m+\tau+bn_T+j-1})}{\partial \beta} (\hat{\beta} - \beta^*) \right) \right|
\]

\[
+ \frac{1}{2} (\hat{\beta} - \beta^*) \left| \frac{\partial^2 g (\Delta_h \tilde{e}_{T_m+\tau+bn_T+j-1})}{\partial \beta \partial \beta'} (\hat{\beta} - \beta^*) \right|
\]

\[
\leq \| \hat{\beta} - \beta^* \| \left| \frac{1}{nT} \sum_{j=1}^{nT} \left( \frac{\partial g (\sigma_{e,T_m+\tau+bn_T-1}) h (h^{-1/2} \Delta_h W_{e,T_m+\tau+bn_T+j-1})}{\partial \beta} \right) \right|
\]

\[
+ \| \hat{\beta} - \beta^* \|^2 \frac{1}{2nT} \sum_{j=1}^{nT} \left| \frac{\partial^2 g (\Delta_h \tilde{e}_{T_m+\tau+bn_T+j-1})}{\partial \beta \partial \beta'} \right|
\]

Since \( \hat{\beta}_k - \beta^* = O_p \left( \frac{1}{\sqrt{T}} \right) \) uniformly, the first term on the right-hand side above is \( CO_p (T^{-1/2}) \) by Assumption 3.3 while the second term is \( O_p (T^{-1}) \) by Assumption 3.2. Both bounds are uniform in \( b \). Combining the latter two results we have

\[
\frac{1}{nT} \sum_{j=1}^{nT} \left( SL_{\psi,T_m+\tau+bn_T+j-1} (\beta^*) - SL_{\psi,T_m+\tau+bn_T+j-1} (\hat{\beta}) \right) = K \sqrt{\frac{T}{nT}}.
\]

(S.36)

Thus, for any \( \varepsilon > 0 \) and any constant \( K > 0 \), the first term on the right-hand side of (S.35) is such that

\[
P \left( \max_{b=0,\ldots, \lfloor T/n_T \rfloor} \left( \log (T/n_T) \right)^{1/2} n_T^{-1} \sum_{j=1}^{n_T} \left( SL_{\psi,T_m+\tau+(b+1)n_T+j-1} (\beta^*) - SL_{\psi,T_m+\tau+(b+1)n_T+j-1} (\hat{\beta}) \right) \right) > \varepsilon
\]

\[
\leq P \left( \max_{b=0,\ldots, \lfloor T/n_T \rfloor} \left( \log (T/n_T) \right)^{1/2} n_T^{-1} \sum_{j=1}^{n_T} \left( SL_{\psi,T_m+\tau+(b+1)n_T+j-1} (\beta^*) - SL_{\psi,T_m+\tau+(b+1)n_T+j-1} (\hat{\beta}) \right) \right) > \varepsilon/K
\]

\[
+ P \left( \max_{b=0,\ldots, \lfloor T/n_T \rfloor} 1/ \left( U_{b+1,h} \right) < K \right).
\]

(S.37)

By Lemma S.A.16 below, \( P \left( \min_{b=0,\ldots, \lfloor T/n_T \rfloor} \left( U_{b+1,h} \right) < 1/K \right) \) \rightarrow 0 by letting, for example, \( K = 2/\sigma_{L_n}^2 \), where \( \sigma_{L_n} \) was introduced in that proof. As for the first probability term, by using Markov’s
The next quantity that we define is similar to the lemma. Using Condition 1. The argument for the second term of (S.35) is equivalent. This concludes the proof of the lemma. □

Let \( u_{T_m+\tau+b_nT+j-1} \) \( \triangleq \) \( g(e,(T_m+\tau+b_nT+j-1)h)h^{-1/2}(\Delta hW_{e,T_m+\tau+b_nT+j-1}; \beta^*) \) and define \( B_{\text{max},h}^0(T_n, \tau) \) \( \triangleq \) \( \max_{b=0, \ldots, \lfloor T_n/nT \rfloor-2} \left( \frac{\bar{B}_{b+1}^0 - B_{b,h}^0}{\sqrt{D_{b+1}^0}} \right) \), where \( B_{b,h}^0 = n_T^{-1}\sum_{j=1}^{n_T} u_{T_m+\tau+b_nT+j-1} \) and

\[
D_{b,h}^0 \triangleq n_T^{-1}\sum_{j=1}^{n_T} (u_{T_m+\tau+b_nT+j-1} - \bar{g}_b)^2,
\]

with \( \bar{g}_b \triangleq n_T^{-1}\sum_{j=1}^{n_T} u_{T_m+\tau+b_nT+j-1} \).

Similarly, define \( B_{\text{max},h}^* = n_T^{-1}\sum_{j=1}^{n_T} u_{T_m+\tau+b_nT+j-1}^* \), where \( u_{T_m+\tau+b_nT+j-1}^* \) \( \triangleq \) \( g(\Delta h\bar{e}_{T_m+\tau+b_nT+j-1}; \beta^*) \).

The next quantity that we define is similar to \( B_{h,b}^0 \) but has all the parameters shifted back by one block of time length \( n_T \): \( \bar{B}_{h,b}^0 = n_T^{-1}\sum_{j=1}^{n_T} \bar{u}_{T_m+\tau+b_nT+j-1} \)

\[
\bar{u}_{T_m+\tau+b_nT+j-1} \triangleq g(e,(T_m+\tau+(b-1)nT)h)h^{-1/2}\Delta hW_{e,T_m+\tau+b_nT+j-1}; \beta^*),
\]

With this notation we can define the statistic \( \bar{B}_{\text{max},h}^0(T_n, \tau) \) \( \triangleq \) \( \max_{b=0, \ldots, \lfloor T_n/nT \rfloor-2} \left( \frac{\bar{B}_{b+1}^0 - B_{b,h}^0}{\sqrt{D_{b+1}^0}} \right) \), where \( D_{b,h}^0 \triangleq n_T^{-1}\sum_{j=1}^{n_T} (\bar{u}_{T_m+\tau+b_nT+j-1} - \bar{g}_b)^2 \) with \( \bar{g}_b \triangleq n_T^{-1}\sum_{j=1}^{n_T} \bar{u}_{T_m+\tau+b_nT+j-1} \). We want to show that

\[
P \left( (\log (T_n) n_T)^{1/2} \left( V_{\text{max},h}(T_n, \tau) - \bar{B}_{\text{max},h}^0(T_n, \tau) \right) > \varepsilon \right) \to 0,
\]

for any \( \varepsilon > 0 \), where

\[
V_{\text{max},h}(T_n, \tau) \triangleq \max_{b=0, \ldots, \lfloor T_n/nT \rfloor-2} \left| \frac{\bar{B}_{b+1}^0 - B_{b,h}^0}{\sigma_u(T_m+\tau+b_nT+1)h} \right|,
\]

with \( \sigma_{u,(T_m+\tau+b_nT-1)}^2 \) \( \triangleq \) \( \text{Var}(u_{T_m+\tau+b_nT}) \). The normalization by \( \sigma_u(T_m+\tau+b_nT-1)h \) ensures that we obtain a distribution-free limit theory. Note that the localization assumption implies that there exist 0 < \( \sigma_{u,-} < \sigma_{u,+} < \infty \) defined by \( \sigma_{u,-} \triangleq \inf_{b \geq 1} \{ \sigma_{u,bh} \} \) and \( \sigma_{u,+} \triangleq \sup_{b \geq 1} \{ \sigma_{u,bh} \} \). Furthermore, under \( H_0, \sigma_{u,(T_m+\tau+b_nT-1)}h \) is a smooth function of Lipschitz parameters and therefore Condition 3.1 applies to
\[ \sigma_u(T_{m+\tau+bn_T-1}h) \] as well: \( \phi_{\sigma_u,\eta,N} \leq \eta \). Finally, let
\[
B^{*}_{\text{max},h}(T_n, \tau) \triangleq \max_{b=0,\ldots,[T_n/n_T]-2} \left( B^{*}_{h,b+1} - B^{*}_{h,b} \right)/\sqrt{U_{h,b+1}}.
\] (S.38)

We proceed via small lemmas which parallel Lemma S.A.6-S.A.8. The following lemma shows that, under \( H_0 \), the difference in the in-sample losses \( T_{\psi,h} \left( \hat{\beta} \right) \) between adjacent blocks is negligible asymptotically.

**Lemma S.A.14.** As \( h \downarrow 0 \), \( (\log (T_n) n_T)^{1/2} \left( B^{*}_{\text{max},h}(T_n, \tau) - U_{\text{max},h}(T_n, \tau) \right) \xrightarrow{P} 0 \).

**Proof.** Apply (S.1) to yield
\[
\left| B^{*}_{\text{max},h}(T_n, \tau) - U_{\text{max},h}(T_n, \tau) \right| = \left| \max_{b=0,\ldots,[T_n/n_T]-2} \left( B^{*}_{h,b+1} - B^{*}_{h,b} \right)/\sqrt{U_{h,b+1}} \right| - \max_{b=0,\ldots,[T_n/n_T]-2} \left| (U_{h,b+1} - U_{h,b})/\sqrt{U_{h,b+1}} \right| \\
\leq \max_{b=0,\ldots,[T_n/n_T]-2} \frac{\sum_{j=1}^{n_T} \left( T_{\psi,(T_m+(b+1)n_T+j-1)h} (\beta^*) - T_{\psi,(T_m+bn_T+j-1)h} (\beta^*) \right)}{\sqrt{U_{h,b+1}}}.
\]

For any \( \varepsilon > 0 \) and any \( K > 0 \),
\[
\mathbb{P} \left( \max_{b=0,\ldots,[T_n/n_T]-2} \left| (\log (T_n) n_T)^{1/2} \frac{\sum_{j=1}^{n_T} \left( T_{\psi,(T_m+(b+1)n_T+j-1)h} (\beta^*) - T_{\psi,(T_m+bn_T+j-1)h} (\beta^*) \right)}{\sqrt{U_{h,b+1}}} \right| > \varepsilon \right) \\
\leq \mathbb{P} \left( \max_{b=0,\ldots,[T_n/n_T]-2} \frac{\left| (\log (T_n) n_T)^{1/2} \sum_{j=1}^{n_T} \left( T_{\psi,(T_m+(b+1)n_T+j-1)h} (\beta^*) - T_{\psi,(T_m+bn_T+j-1)h} (\beta^*) \right) \right|}{\sqrt{U_{h,b+1}}} > \varepsilon/K \right) \\
+ \mathbb{P} \left( \max_{b=0,\ldots,[T_n/n_T]-2} \left| 1/\sqrt{U_{h,b+1}} \right| > K \right).
\]

By Lemma S.A.12 the first term on the right-hand size converges to zero. As for the second term, use the same argument as in (S.37). The result then follows. \( \square \)

**Lemma S.A.15.** As \( h \downarrow 0 \), \( (\log (T_n) n_T)^{1/2} \left( B^{*}_{\text{max},h}(T_n, \tau) - B^{0}_{\text{max},h}(T_n, \tau) \right) \xrightarrow{P} 0 \).

**Proof.** Note that
\[
\left| B^{*}_{\text{max},h}(T_n, \tau) - B^{0}_{\text{max},h}(T_n, \tau) \right| \\
\leq \max_{b=0,\ldots,[T_n/n_T]-2} \left| B^{*}_{h,b+1} - B^{*}_{h,b} \right| \left( 1/\sqrt{U_{h,b+1}} - 1/\sqrt{D^{0}_{h,b+1}} \right) + \max_{b=0,\ldots,[T_n/n_T]-2} \left| B^{0}_{h,b+1} - B^{0}_{h,b} \right| \left( 1/\sqrt{D^{0}_{h,b+1}} \right) \\
+ \max_{b=0,\ldots,[T_n/n_T]-2} \left| B^{0}_{h,b} \right| \left( 1/\sqrt{D^{0}_{h,b+1}} \right). \quad \text{(S.39)}
\]

Consider the first term of (S.39). We can write for any \( \varepsilon > 0 \), any \( 0 < K, C < \infty \), and some small positive...
number $\varpi < 1/2$,

$$
P \left( \max_{b=0,\ldots,\lfloor T_n/n_T \rfloor-2} \left( \log (T_n) n_T \right)^{1/2} \left( B_{h,b+1}^* - B_{h,b}^* \right) \left( \frac{1}{\sqrt{U_{h,b+1}}} - \frac{1}{\sqrt{D_{h,b+1}^0}} \right) \right) > \varepsilon 
\right)
\leq P \left( \max_{b=0,\ldots,\lfloor T_n/n_T \rfloor-2} \left| n_T \left( B_{h,b+1}^* - B_{h,b}^* \right) \right| > \varepsilon/K \right)
\leq \sum_{b=0}^{\lfloor T_n/n_T \rfloor-2} P \left( \left| n_T \left( B_{h,b+1}^* - B_{h,b}^* \right) \right| > \varepsilon/K \right)
\leq \sum_{b=0}^{\lfloor T_n/n_T \rfloor-2} P \left( \left| n_T \left( B_{h,b+1}^* - \mu_{T_m+\nu+bT-1} \right) \right| > \varepsilon/(3K) \right)
\leq \sum_{b=0}^{\lfloor T_n/n_T \rfloor-2} P \left( \left| n_T \left( \mu_{T_m+\nu+bT} - \mu_{T_m+\nu+bT-1} \right) \right| > \varepsilon/(3K) \right).

(S.41)

Since $B_{h,b}^* \xrightarrow[p]{} \mu_{T_m+\nu+bT-1} \triangleq \mathbb{E}(\nu_{T_m+\nu+bT-1})$ and $\mathbb{E} \left[ \sqrt{n_T} \left( B_{h,b}^* - \mu_{T_m+\nu+bT-1} \right) \right] < \infty$ by a standard CLT, we have for $r > 0$ sufficiently large and by choosing $\varpi$ sufficiently small,

$$
\sum_{b=0}^{\lfloor T_n/n_T \rfloor-2} P \left( \left| n_T \left( B_{h,b}^* - \mu_{T_m+\nu+bT-1} \right) \right| > \varepsilon/(3K) \right) \leq K_r \left( \frac{3K}{\varepsilon} \right)^r \sum_{b=0}^{\lfloor T_n/n_T \rfloor-2} \mathbb{E} \left( \left| n_T \left( B_{h,b}^* - \mu_{T_m+\nu+bT-1} \right) \right|^r \right)
\leq K_r \left( \frac{3K}{\varepsilon} \right)^r T_n n_T^{r(\varpi-1/2)-1} \to 0.
\right)

The term involving $B_{h,b+1}^*$ admits a similar bound. For the last term of (S.41), we use the Lipschitz continuity of $\mu_r$ to yield

$$
\sum_{b=0}^{\lfloor T_n/n_T \rfloor-2} P \left( \left| n_T \left( \mu_{T_m+\nu+(b+1)T-1} - \mu_{T_m+\nu+bT} \right) \right| > \varepsilon/(3K) \right)
\leq \left( \frac{3K}{\varepsilon} \right)^r \sum_{b=0}^{\lfloor T_n/n_T \rfloor-2} \mathbb{E} \left( \left| n_T \left( \mu_{T_m+\nu+(b+1)T-1} - \mu_{T_m+\nu+bT-1} \right) \right|^r \right).
\right)
\right)
\right)
\right)
for $r > 0$ sufficiently large since $\varpi$ is chosen to be small. Thus, $A_{1,h} \to 0$ while Lemma S.A.16 implies that $A_{3,h} \to 0$ by setting $\sqrt{C} = 1/\left(2\sigma_u, ..\right)$. Further, note that $\sigma^2_u(T^*_{m(\tau+(b+1)n_T-1)})^h$ is the limit of both $\overline{U}_{h, b+1}$ and $D_{h, b+1}^0$. Thus, given the i.i.d. structure, we can use a standard CLT to yield

$$
P \left( \sup_{b = 0, \ldots, [T_n/n_T] - 2} \left| \sqrt{\log (T_n) n_T^{1/2}} \sqrt{\overline{U}_{h, b+1} - \sqrt{D_{h, b+1}^0}} \right| > K/C \right) \leq \sum_{b = 0}^{[T_n/n_T] - 2} P \left( \left| \sqrt{\log (T_n) n_T^{1/2}} \left( \sqrt{\overline{U}_{h, b+1} - \sigma_u(T^*_{m+(\tau+(b+1)n_T-1)})^h} \right) \right|^r > (K/2C)^r \right)
$$

$$
+ \sum_{b = 0}^{[T_n/n_T] - 2} P \left( \left| \sqrt{\log (T_n) n_T^{1/2}} \sqrt{D_{h, b+1}^0 - \sigma_u(T^*_{m+(\tau+(b+1)n_T-1)})^h} \right|^r > (K/2C)^r \right)
$$

$$
\leq 2 (K/2C)^{-r} (T_n/n_T)^{O_p} \left( \left( \sqrt{\log (T_n) n_T^{-\varpi}} \right)^r \right) \to 0,
$$

(S.42)

for $r > 1/\varpi$ sufficiently large. This shows that $A_{2,h} \to 0$. It remains to discuss the second term of (S.39); the argument for the third term is equivalent and omitted. Recall the definition of $u^*_{T^*_{m(\tau+(b+1)n_T-1)}}$ and $u^*_{T^*_{m(\tau+(b+1)n_T-1)}}$. By a mean-value expansion,

$$
B^*_{h, b+1} - B^0_{h, b+1} = n^{-1}_T \sum_{j=1}^{n_T} \left( u^*_{T^*_{m+(\tau+(b+1)n_T-1)} - j} - u^*_{T^*_{m+(\tau+(b+1)n_T-1)+j}} \right)
$$

$$
= n^{-1}_T \sum_{j=1}^{n_T} \left( g \left( \Delta h e^*_{T^*_{m+(\tau+(b+1)n_T-1)} - j}, \beta^* \right) - g \left( \sigma e, (T^*_{m+(\tau+(b+1)n_T-1)})^h, h^{-1/2} \left( \Delta h W e, T^*_{m+(\tau+(b+1)n_T-1)} \right), \beta^* \right) \right)
$$

$$
= n^{-1}_T \sum_{j=1}^{n_T} \left[ \partial e \left( \sigma e, (T^*_{m+(\tau+(b+1)n_T-1)})^h, h^{-1/2} \left( \Delta h W e, T^*_{m+(\tau+(b+1)n_T-1)} \right), \beta^* \right) \right]
$$

$$
\times \left( \Delta h e^*_{T^*_{m+(\tau+(b+1)n_T-1)} - j} - \sigma e, (T^*_{m+(\tau+(b+1)n_T-1)})^h, h^{-1/2} \Delta h W e, T^*_{m+(\tau+(b+1)n_T-1)} \right)
$$

$$
+ \partial^2 e \left( \sigma e, (T^*_{m+(\tau+(b+1)n_T-1)})^h, h^{-1/2} \left( \Delta h W e, T^*_{m+(\tau+(b+1)n_T-1)} \right), \beta^* \right)
$$

$$
\times \left( \Delta h e^*_{T^*_{m+(\tau+(b+1)n_T-1)} - j} - \sigma e, (T^*_{m+(\tau+(b+1)n_T-1)})^h, h^{-1/2} \Delta h W e, T^*_{m+(\tau+(b+1)n_T-1)} \right)^2 .
$$

(S.43)

Since for $r = 1, 2$, $|\partial^r e (e; \beta)| < C_r$ for some $C_r < \infty$ by Assumption 3.5, the right-hand side above is less than

$$
(\log (T_n) n_T)^{1/2} C_1 \left( n_T \sqrt{h} \right)^{-1}
$$

$$
\times \sum_{j=1}^{n_T} \left( \Delta h e^*_{T^*_{m+(\tau+(b+1)n_T-1)} - j} - \sigma e, (T^*_{m+(\tau+(b+1)n_T-1)})^h, h^{-1/2} \Delta h W e, T^*_{m+(\tau+(b+1)n_T-1)} \right)
$$

$$
+ (\log (T_n) n_T)^{1/2} C_2 \left( n_T h \right)^{-1}
$$

$$
\times \sum_{j=1}^{n_T} \left( \Delta h e^*_{T^*_{m+(\tau+(b+1)n_T-1)} - j} - \sigma e, (T^*_{m+(\tau+(b+1)n_T-1)})^h, h^{-1/2} \Delta h W e, T^*_{m+(\tau+(b+1)n_T-1)} \right)^2 .
$$

(S.44)
Let us consider the first term of (S.44). By Itô’s formula,

\[
\Delta h e^{T_{m+\tau+(b+1)n_T+j-1} - \sigma_e(T_{m+\tau+(b+1)n_T-1})h} \Delta T_{m+\tau+(b+1)n_T+j-1} W_e
\]

Then, for an integer \( r > 2 \), by Jensen’s inequality,

\[
\mathbb{E}\left[ \left( \log (T_n) n_T \right)^{1/2} \left( n_T \sqrt{h} \right)^{-1} \sum_{j=1}^{n_T} \int_{T_{m+\tau+(b+1)n_T+j-1}}^{T_{m+\tau+(b+1)n_T+j}} \left( \sigma_{e,s} - \sigma_e(T_{m+\tau+(b+1)n_T-1}h) \right) dW_{e,s} \right]^{r/2}
\]

by choosing \( r \) large enough. Next, we consider the second term of (S.45),

\[
\mathbb{E}\left[ \left( \log (T_n) n_T \right)^{1/2} \left( n_T \sqrt{h} \right)^{-1} \sum_{j=1}^{n_T} \int_{T_{m+\tau+(b+1)n_T+j-1}}^{T_{m+\tau+(b+1)n_T+j}} \mu_{e,s} h^{-\vartheta} ds \right]^{r/2}
\]

For the term in the second line of (S.44) apply the same arguments as in (S.46)-(S.47) with \( m = r/2 \) in place of \( r \) above. Choosing \( m \) large enough yields the same result. Thus, using the latter results into the second term of (S.39) via (S.43) we have

\[
\mathbb{P}\left( \max_{b=0,\ldots,[T_n/n_T]-2} \left| \log (T_n) n_T \right|^{1/2} \left| B_{h,b+1}^* - B_{h,b+1}^0 \right| > \varepsilon/K \right) \rightarrow 0.
\]

Note that the same result holds for \( B_{h,b}^* - B_{h,b}^0 \). The first term of (S.39) has been treated above and so the claim of the lemma follows. \( \square \)

**Lemma S.A.16.** Assume \( \mu_{e,t} = 0 \) for all \( t \geq 0 \). Then, \( \mathbb{P}\left( \max_{b=0,\ldots,[T_n/n_T]-2} \left| \mathcal{U}_{h,b} \right| > K \right) \rightarrow 0 \) for some constant \( K > 0 \).
Proof. Note that
\[
\mathbb{P}\left( \max_{b=0,\ldots,\left[T_n/n_T\right]-2} \left| \frac{1}{U_{h,b}} \right| > K \right) = \mathbb{P}\left( \min_{b=0,\ldots,\left[T_n/n_T\right]-2} \left| U_{h,b} \right| < K^{-1} \right)
\]
\[
= \mathbb{P}\left( \min_{b=0,\ldots,\left[T_n/n_T\right]-2} n_T^{-1} \sum_{j=1}^{n_T} \left( L_{\psi,(T_n+\tau+bn_T+j-1)h} (\beta^*) - \mathcal{L}_{\psi,b} (\beta^*) \right)^2 < K^{-1} \right)
\]
\[
\leq \sum_{b=0}^{\left[T/n_T\right]-2} \mathbb{P}\left( n_T^{-1} \sum_{j=1}^{n_T} \left( L_{\psi,(T_n+\tau+bn_T+j-1)h} (\beta^*) - \mathcal{L}_{\psi,b} (\beta^*) \right)^2 < K^{-1} \right).
\]

The rest of the proof continues by setting \( K^{-1} = \sigma_{L,-}^2/2 \) where \( \sigma_{L,-}^2 \triangleq \inf_{k \geq 1} \sigma_{L, kh}^2 \) with \( \sigma_{L, kh}^2 \triangleq \text{Var} (L_{\psi, kh} (\beta^*)) \). We can use Markov’s inequality to deduce for any \( r > 0 \),
\[
\mathbb{P}\left( n_T^{-1} \sum_{j=1}^{n_T} \left( L_{\psi,(T_n+\tau+bn_T+j-1)h} (\beta^*) - \mathcal{L}_{\psi,b} (\beta^*) \right)^2 < \sigma_{L,-}^2/2 \right) \leq \mathbb{P}\left( \left| n_T^{-1} \sum_{j=1}^{n_T} \left( L_{\psi,(T_n+\tau+bn_T+j-1)h} (\beta^*) - \mathcal{L}_{\psi,b} (\beta^*) \right)^2 - \sigma_{L,(T_n+\tau+bn_T-1)h}^2 \right| > \sigma_{L,-}^2/2 \right)
\]
\[
\leq C_r \left( \frac{2}{\sigma_{L,-}^2} \right)^r \mathbb{E}\left[ \left| n_T^{-1} \sum_{j=1}^{n_T} \left( L_{\psi,(T_n+\tau+bn_T+j-1)h} (\beta^*) - \mathcal{L}_{\psi,b} (\beta^*) \right)^2 - \sigma_{L,(T_n+\tau+bn_T-1)h}^2 \right|^r \right].
\]

Observe that, conditional on \( \{\sigma_{e,t}\}_{\geq 0} \), \( \text{Var}_x \left[ L_{\psi,(T_n+\tau+bn_T+j-1)h} (\beta^*) \right] \) is constant across \( j = 1, \ldots, n_T \) for a given \( b \). Then, Assumption S.A.2 implies that we can rely on a basic CLT for i.i.d. observations to yield,
\[
\mathbb{E}\left[ \left| n_T^{-1/2} \sum_{j=1}^{n_T} \left( L_{\psi,(T_n+\tau+bn_T+j-1)h} (\beta^*) - \mathcal{L}_{\psi,b} (\beta^*) \right)^2 - \sigma_{L,(T_n+\tau+bn_T-1)h}^2 \right|^r \right] < C_r \text{, where } C_r < \infty.
\]
Thus, choose \( r \) sufficiently large so that
\[
\sum_{b=0}^{\left[T/n_T\right]-2} \mathbb{P}\left( n_T^{-1} \sum_{j=1}^{n_T} \left( L_{\psi,(T_n+\tau+bn_T+j-1)h} (\beta^*) - \mathcal{L}_{\psi,b} (\beta^*) \right)^2 < K^{-1} \right) \leq C_r \left( \frac{2}{\sigma_{L,-}^2} \right)^r O_{\mathbb{P}} (T_n/n_T)^{-r/2} \to 0,
\]
and the proof is concluded. \( \square \)

Lemma S.A.17. As \( h \downarrow 0 \), \( (\log (T_n))^1/2 \left( B^0_{\text{max}, h} (T_n, \tau) - \bar{B}^0_{\text{max}, h} (T_n, \tau) \right) \overset{\mathbb{P}}{\to} 0. \)

Proof. By basic manipulations,
\[
| B^0_{\text{max}, h} (T_n, \tau) - \bar{B}^0_{\text{max}, h} (T_n, \tau) |
\]
\[
\leq \max_{b=0,\ldots,\left[T_n/n_T\right]-2} \left| \sqrt{D^0_{h,b+1}} (D^0_{h,b+1} - D^0_{h,b}) - \sqrt{D^0_{h,b+1}} (\bar{B}^0_{h,b+1} - D^0_{h,b}) \right| / \sqrt{D^0_{h,b+1} \bar{D}^0_{h,b+1}}
\]
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\[
\leq \max_{b=0, \ldots, \lceil T_n/n_T \rceil - 2} \left| \frac{\sqrt{D_{h,b+1}^0 \left( B_{h,b+1}^0 - \tilde{B}_{h,b+1}^0 \right)} + \left( \tilde{B}_{h,b+1}^0 - B_{h,b}^0 \right) \left( \sqrt{D_{h,b+1}^0 - D_{h,b+1}^0} \right)}{\sqrt{D_{h,b+1}^0 D_{h,b+1}^0}} \right|
\]

\[
\leq \max_{b=0, \ldots, \lceil T_n/n_T \rceil - 2} \left| \frac{\sqrt{D_{h,b+1}^0 \left( B_{h,b+1}^0 - \tilde{B}_{h,b+1}^0 \right)}}{\sqrt{D_{h,b+1}^0 D_{h,b+1}^0}} + \max_{b=0, \ldots, \lceil T_n/n_T \rceil - 2} \left| \frac{\left( \tilde{B}_{h,b+1}^0 - B_{h,b}^0 \right) \left( \sqrt{D_{h,b+1}^0 - D_{h,b+1}^0} \right)}{\sqrt{D_{h,b+1}^0 D_{h,b+1}^0}} \right| \right|
\]

\[\triangleq R_{1,h} + R_{2,h}. \quad (S.49)\]

We begin with showing that \( (\log (T_n) n_T)^{1/2} R_{1,h} \overset{p}{\to} 0, \) or

\[
\max_{b=0, \ldots, \lceil T_n/n_T \rceil - 2} (\log (T_n) n_T)^{1/2} \left| \frac{\sqrt{D_{h,b+1}^0 \left( B_{h,b+1}^0 - \tilde{B}_{h,b+1}^0 \right)}}{\sqrt{D_{h,b+1}^0 D_{h,b+1}^0}} \right| = o_p (1). \quad (S.50)
\]

By Lemma S.A.16, \( \mathbb{P} \left( \min_{b=0, \ldots, \lceil T_n/n_T \rceil - 2} \sqrt{D_{h,b+1}^0} < K^{-1/2} \right) \to 0, \) where, for example, \( \sqrt{K} = 2/\sigma_{u,\cdot}. \)

A similar argument can be used for \( \tilde{D}_{h,b+1}^0 \) and therefore it remains to consider the first term of the following decomposition which is valid for any \( \varepsilon > 0 \) and any \( K > 0, \)

\[
\mathbb{P} \left( \max_{b=0, \ldots, \lceil T_n/n_T \rceil - 2} (\log (T_n) n_T)^{1/2} \left| \frac{\sqrt{D_{h,b+1}^0 \left( B_{h,b+1}^0 - \tilde{B}_{h,b+1}^0 \right)}}{\sqrt{D_{h,b+1}^0 D_{h,b+1}^0}} \right| > \varepsilon / K \right)
\]

\[
\leq \mathbb{P} \left( \max_{b=0, \ldots, \lceil T_n/n_T \rceil - 2} (\log (T_n) n_T)^{1/2} \left| \sqrt{D_{h,b+1}^0 \left( B_{h,b+1}^0 - \tilde{B}_{h,b+1}^0 \right)} \right| > \varepsilon / K \right)
\]

\[
+ \mathbb{P} \left( \max_{b=0, \ldots, \lceil T_n/n_T \rceil - 2} \sqrt{D_{h,b+1}^0} > K \right). \quad (S.51)
\]

We have for any positive \( K_2 < \infty, \)

\[
\mathbb{P} \left( \max_{b=0, \ldots, \lceil T_n/n_T \rceil - 2} (\log (T_n) n_T)^{1/2} \left| \sqrt{D_{h,b+1}^0 \left( B_{h,b+1}^0 - \tilde{B}_{h,b+1}^0 \right)} \right| > \varepsilon / K \right)
\]

\[
\leq \mathbb{P} \left( \max_{b=0, \ldots, \lceil T_n/n_T \rceil - 2} (\log (T_n) n_T)^{1/2} \left| B_{h,b+1}^0 - \tilde{B}_{h,b+1}^0 \right| > \varepsilon / (K \cdot K_2) \right)
\]

\[
+ \mathbb{P} \left( \max_{b=0, \ldots, \lceil T_n/n_T \rceil - 2} \sqrt{D_{h,b+1}^0} > K_2 \right).
\]

It is straightforward to see that Lemma S.A.16 can be applied also to the second term on the right-hand side above. Hence, it is sufficient to focus on the first term only. Recall the definition of \( \tilde{u}_{T_m + \tau + bn_T - 1} \) and \( u_{T_m + \tau + bn_T - 1} \) introduced before Lemma S.A.14. We write

\[
\mathbb{P} \left( \max_{b=0, \ldots, \lceil T_n/n_T \rceil - 2} (\log (T_n) n_T)^{1/2} \left| B_{h,b+1}^0 - \tilde{B}_{h,b+1}^0 \right| > \varepsilon / K \right)
\]
By a mean-value expansion (omitting the second argument of \( g(\cdot, \cdot) \) which is for both terms here equal to \( \beta^* \)),

\[
g\left( \sigma_e(T_m+\tau+(b+1)n_T-1)h^{1/2} \left( \Delta_h W_{e,T_m+\tau+(b+1)n_T+j-1} \right) \right) - g\left( \sigma_e(T_m+\tau+bn_T-1)h^{1/2} \left( \Delta_h W_{e,T_m+\tau+(b+1)n_T+j-1} \right) \right) = g_e \left( \sigma_e(T_m+\tau+bn_T-1)h^{1/2} \Delta_h W_{e,T_m+\tau+(b+1)n_T+j-1} \right) \\
\times \left[ \left( \sigma_e(T_m+\tau+bn_T-1)h - \sigma_e(T_m+\tau+(b+1)n_T-1)h \right) \left( h^{-1/2} \Delta_h W_{e,T_m+\tau+(b+1)n_T+j-1} \right) \right]^2.
\]

In view of Assumption 2.2, for \( r = 1, 2 \),

\[
\left| \sigma_e(T_m+\tau+bn_T-1)h - \sigma_e(T_m+\tau+(b+1)n_T-1)h \right|^r \leq C_r (n_T h)^r,
\]

uniformly in \( b \) where \( C_r < \infty \). Let

\[
\mathcal{C}_1 \triangleq 2 \sup_{k \geq 1} \sup_{t \geq 0} \left| g_e \left( \sigma_{e,t} h^{-1/2} \Delta_h W_{e,k} \right) \right|, \quad \mathcal{C}_2 \triangleq 2 \sup_{k \geq 1} \sup_{t \geq 0} \left| g_{e,e} \left( \sigma_{e,t} h^{-1/2} \Delta_h W_{e,k} \right) \right|.
\]

Then, the right-hand side of (S.52) can be decomposed as follows with \( K_1 = \sqrt{2\mathcal{C}_1} \) and \( K_2 = \sqrt{2\mathcal{C}_2} \),

\[
P\left( \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} (\log(T_n) n_T)^{1/2} n_T^{-1} \sum_{j=1}^{n_T} \left( \bar{u}_{T_m+\tau+(b+1)n_T-1} - u_{T_m+\tau+(b+1)n_T-1} \right) > \varepsilon / K \right)
\]
\[
\leq P\left( \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} (\log(T_n) n_T)^{1/2} \left( \sigma_e(T_m+\tau+bn_T-1)h - \sigma_e(T_m+\tau+(b+1)n_T-1)h \right) > \varepsilon / (K_1 \cdot K) \right)
\]
\[
+ P\left( \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} n_T^{-1} \sum_{j=1}^{n_T} g_e \left( \sigma_e(T_m+\tau+bn_T-1)h h^{-1/2} \Delta_h W_{e,T_m+\tau+(b+1)n_T+j-1} \right) \right)
\]
\[
\times \left( h^{-1/2} \Delta_h W_{e,T_m+\tau+(b+1)n_T+j-1} \right) > K_1
\]
\[
+ P\left( \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} (\log(T_n) n_T)^{1/2} \left( \sigma_e(T_m+\tau+bn_T-1)h - \sigma_e(T_m+\tau+(b+1)n_T-1)h \right)^2 > \varepsilon / (2K_2 \cdot K) \right)
\]
\[
+ P\left( \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} 2^{-1} n_T^{-1} \sum_{j=1}^{n_T} g_{e,e,b,j}(\sigma)(h^{-1/2} \Delta_h W_{e,T_m+\tau+(b+1)n_T+j-1})^2 > K_2 \right)
\]
\[
\triangleq A_{1,h} + A_{2,h} + A_{3,h} + A_{4,h}.
\]

The relationship in (S.53) implies that \( A_{1,h}, A_{3,h} \to 0 \) using Condition 1 because

\[
\max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} (\log(T_n) n_T)^{1/2} \left( \sigma_e(T_m+\tau+bn_T-1)h - \sigma_e(T_m+\tau+(b+1)n_T-1)h \right) \]

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The boundedness of $g_\epsilon(\cdot, \cdot)$ [cf. Assumption 3.5], implies that for $r > 0$ large enough,

$$
\mathbb{P} \left( \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} n_T^{-1} \sum_{j=1}^{n_T} |g_\epsilon \left( \sigma_{e, (T_m + \tau + b n_T - 1) h \Delta h, T_m + \tau + (b + 1) n_T + j - 1} \right) \right.

\times h^{-1/2} \Delta h W_{e, T_m + \tau + (b + 1) n_T + j - 1} > K_1 \left. \right) \leq \mathbb{P} \left( \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} n_T^{-2} C_1^2 \sum_{j=1}^{n_T} \left( h^{-1/2} \Delta h W_{e, T_m + \tau + (b + 1) n_T + j - 1} \right)^2 - 1 \right) < \left( K_1^2 / 2 \right)^{r/2}

\leq (2 / K_1^2)^{r/2} C_r \sum_{b=0}^{\lfloor T_n/n_T \rfloor - 2} \mathbb{E} \left[ n_T^{-2} \sum_{j=1}^{n_T} \left( h^{-1/2} \Delta h W_{e, T_m + \tau + (b + 1) n_T + j - 1} \right)^2 - 1 \right]^{r/2}

\leq (2 / K_1^2)^r C_r n_T^{-1 - 3r/2} \rightarrow 0.

We can apply the same argument with $K_2 = \sqrt{2} C_2$ to $A_{4,h}$ to show that

$$
\mathbb{P} \left( \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} n_T^{-1} \sum_{i=1}^{n_T} g_{\epsilon e, b, j}(\tau) \left( h^{-1/2} \Delta \tau_{T_m + \tau + b n_T + i - 1} W_{e} \right)^2 > K_2 \right)

\leq K_2^2 C_r n_T^{-1 - r} \rightarrow 0,

and so $A_{4,h} \rightarrow 0$. This gives (S.51) and thus (S.50). Next, we consider $R_{2,h}$ and want to show $(\log (T_n))^{1/2} R_{2,h} \mathbb{P} \rightarrow 0$, or

$$
\max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} (\log (T_n))^{1/2} \left( \frac{\tilde{B}_{h,b+1}^0 - B_{h,b}^0}{\sqrt{D_{h,b+1}^0 D_{h,b+1}^0}} \right) = O_P(1).

(S.54)

Proceeding as in (S.51), it is sufficient show

$$
\mathbb{P} \left( \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} (\log (T_n))^{1/2} \left( \frac{\tilde{B}_{h,b+1}^0 - B_{h,b}^0}{\sqrt{D_{h,b+1}^0 D_{h,b+1}^0}} \right) > \varepsilon / K \right) \rightarrow 0.

The argument for $(\log (T_n))^{1/2} \left( \tilde{B}_{h,b+1}^0 - B_{h,b}^0 \right)$ is similar to the one used above, but now one needs an additional step using a Taylor series expansion of $g$; we omit the details. Thus, we have to show

$$
\mathbb{P} \left( \max_{b=0, \ldots, \lfloor T_n/n_T \rfloor - 2} \sqrt{D_{h,b+1}^0} - \sqrt{D_{h,b+1}^0} \right) > C \varepsilon \rightarrow 0,

for some finite $C > 0$. Note that

$$
\sqrt{D_{h,b}^0} - \sqrt{D_{h,b}^0} = \sigma_{u, (T_m + \tau + (b - 1) n_T - 1) h - \sigma_{u, (T_m + \tau + b n_T - 1) h} + O_P \left( n_T^{-1/2} \right)

= \phi_{\sigma_{u, n_T h, N} + O_P \left( n_T^{-1/2} \right)}.

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Since $\phi_{\sigma_u, nT, h, N} \leq C nT h$ uniformly over $h, \ldots, Th = N$, we can show using the same arguments employed above that $\mathbb{P}\left( \mathbb{P}_{b=0, \ldots, \lfloor T/n \rfloor - 2} \left| \left( \bar{D}_{h,b+1} - D_{h,b} \right) \right| > C \varepsilon \right) \to 0$. Therefore, we have $(\log(T_n) nT)^{1/2} R_{2,h} \xrightarrow{P} 0$. The claim of the lemma follows. \hfill $\Box$

**Lemma S.A.18.** As $h \downarrow 0$, $(\log(T_n) nT)^{1/2} \left( V_{\max,h}(T_n, \tau) - \tilde{B}_{\max,h}(T_n, \tau) \right) \xrightarrow{P} 0$.

**Proof.** We have the inequality,

$$V_{\max,h}(T_n, \tau) - \tilde{B}_{\max,h}(T_n, \tau) = \max_{b=0, \ldots, \lfloor T/n \rfloor - 2} \frac{\bar{B}_{h,b+1} - B_{h,b}}{\sigma_{u,(T_m+\tau+bnT-1)h}} - \max_{b=0, \ldots, \lfloor T/n \rfloor - 2} \frac{\tilde{B}_{h,b+1} - B_{h,b}}{\sqrt{D_{h,b+1}}}.$$  

Thus, we want to show that

$$(\log(T_n) nT)^{1/2} \max_{b=0, \ldots, \lfloor T/n \rfloor - 2} \left( \frac{\bar{B}_{h,b+1} - B_{h,b}}{\sqrt{D_{h,b+1}}} - \frac{\tilde{B}_{h,b+1} - B_{h,b}}{\sigma_{u,(T_m+\tau+bnT-1)h}} \right) \rightarrow \mathcal{O}_P(1). \quad (S.55)$$

By Assumption 3.3, $0 < \sigma_{u,(T_m+\tau+bnT-1)h} \leq \infty$ for all $b \geq 0$ while $\bar{D}_{h,b+1}$ was already shown to bounded from below and above. Thus, basic manipulations as in the previous lemmas show that the denominator is also $\mathcal{O}_P(1)$. Turning to the numerator, we have

$$\mathbb{P}\left( \mathbb{E}_{b=0, \ldots, \lfloor T/n \rfloor - 2} \left| \left( \bar{B}_{h,b+1} - B_{h,b} \right) \left( \sqrt{D_{h,b+1}} - \sigma_{u,(T_m+\tau+bnT-1)h} \right) \right| > \varepsilon \right)$$

$$+ \mathbb{P}\left( \mathbb{E}_{b=0, \ldots, \lfloor T/n \rfloor - 2} \left| \tilde{B}_{h,b+1} - B_{h,b} \right| > \sqrt{\varepsilon} \right).$$

In view of the proof of the last part of Lemma S.A.17, we have

$$\mathbb{P}\left( \mathbb{E}_{b=0, \ldots, \lfloor T/n \rfloor - 2} \left| \bar{B}_{h,b+1} - B_{h,b} \right| > \sqrt{\varepsilon} \right) \rightarrow 0.$$  

To conclude the proof of the lemma it remains to show that

$$\mathbb{P}\left( \mathbb{E}_{b=0, \ldots, \lfloor T/n \rfloor - 2} \left| \sqrt{D_{h,b+1}} - \sigma_{u,(T_m+\tau+bnT-1)h} \right| > \sqrt{\varepsilon} \right) \rightarrow 0. \quad (S.56)$$

By the definition of $\bar{D}_{h,b+1}$, the summands $\bar{u}_{T_m+\tau+bnT-1, j}$, $(j = 1, \ldots, nT)$ are independent and each satisfies $\text{Var} \left[ \bar{u}_{T_m+\tau+bnT-1, j} \right] = \sigma_{u,(T_m+\tau+bnT-1)h}^2$. Then,

$$\mathbb{P}\left( \mathbb{E}_{b=0, \ldots, \lfloor T/n \rfloor - 2} \left| \sqrt{D_{h,b+1}} - \sigma_{u,(T_m+\tau+bnT-1)h} \right| > \sqrt{\varepsilon} \right)$$

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\[
\begin{align*}
&\leq \mathbb{P} \left( \max_{b=0,\ldots,\lfloor T_n/n_T \rfloor-2} \left( \frac{1}{n_T} \sum_{j=1}^{n_T} \left( g \left( \sigma_e, (T_m + \tau + b n_T - 1) h \right) \left( \Delta_h W_e, T_m + \tau + (b+1)n_T - j - 1 \right) - \bar{g}_{b+1} \right)^2 - \sigma_u, (T_m + \tau + b n_T - 1) h \right) > \sqrt{\varepsilon} \right) \\
&\times \mathbb{E} \left( \left[ \frac{1}{n_T} \sum_{j=1}^{n_T} \left( g \left( \sigma_e, (T_m + \tau + b n_T - 1) h \right) \left( \Delta_h W_e, T_m + \tau + (b+1)n_T - j - 1 \right) - \bar{g}_{b+1} \right)^2 - \sigma_u, (T_m + \tau + b n_T - 1) h \right] \right)^r \\
&\leq C_r \varepsilon^{-r/2} \sum_{b=0}^{\lfloor T_n/n_T \rfloor-2} \left[ \mathbb{E} \left( \left[ \frac{1}{n_T} \sum_{j=1}^{n_T} \left( g \left( \sigma_e, (T_m + \tau + b n_T - 1) h \right) \left( \Delta_h W_e, T_m + \tau + (b+1)n_T - j - 1 \right) - \bar{g}_{b+1} \right)^2 - \sigma_u, (T_m + \tau + b n_T - 1) h \right] \right)^r \right].
\end{align*}
\]

Note that the variables \( g \left( \sigma_e, (T_m + \tau + b n_T - 1) h \right) \) are independent over \( j \) and their variances are constant and equal to \( \sigma^2_{u, (T_m + \tau + b n_T - 1) h} \). Due to the i.i.d. structure we can rely on a basic CLT for the sample variance which, given Assumption S.A.2, yields

\[
\mathbb{E} \left[ \left( \frac{1}{n_T} \sum_{j=1}^{n_T} \left( g \left( \sigma_e, (T_m + \tau + b n_T - 1) h \right) \left( \Delta_h W_e, T_m + \tau + (b+1)n_T - j - 1 \right) - \bar{g}_{b+1} \right)^2 - \sigma_u, (T_m + \tau + b n_T - 1) h \right] \right] < \infty,
\]

and thus for \( r > 0 \) sufficiently large, we have by Condition 1,

\[
\mathbb{P} \left( \max_{b=0,\ldots,\lfloor T_n/n_T \rfloor-2} \left( \frac{1}{n_T} \sum_{j=1}^{n_T} \left( g \left( \sigma_e, (T_m + \tau + b n_T - 1) h \right) \left( \Delta_h W_e, T_m + \tau + (b+1)n_T - j - 1 \right) - \bar{g}_{b+1} \right)^2 - \sigma_u, (T_m + \tau + b n_T - 1) h \right) > \sqrt{\varepsilon} \right) \\
\leq C_r \varepsilon^{-r/2} O_P \left( \frac{T_n}{n_T} \right) n_T^{-r/2} \rightarrow 0.
\]

Altogether, we have (S.56) and thus (S.55), which concludes the proof. \( \square \)

Proof of Theorem 3.2-(i). By Lemma S.A.13-S.A.18,

\[
(\log (T_n) n_T)^{1/2} \left( G_{\max, h} (T_n, \tau) - V_{\max, h} (T_n, \tau) \right) \xrightarrow{P} 0.
\]

We now apply Lemma S.A.10 to \( V_{\max, h} (T_n, \tau) \). Let

\[
\bar{U}_{T_m + \tau + b n_T - j - 1} \quad \triangleq \quad \frac{\bar{u}_{T_m + \tau + b n_T + j - 1} - \mu_{u, T_m + \tau + (b+1)n_T - 1}}{\sigma_{u, T_m + \tau + (b-1)n_T - 1} h},
\]

\[
U_{T_m + \tau + b n_T - j - 1} \quad \triangleq \quad \frac{u_{T_m + \tau + b n_T + j - 1} - \mu_{u, T_m + \tau + b n_T - 1}}{\sigma_{u, T_m + \tau + b n_T - 1} h},
\]

with \( \mu_{u, T_m + \tau + b n_T - 1} \triangleq \mathbb{E} \left( u_{T_m + \tau + b n_T - 1} \right) \). Then, write

\[
V_{\max, h} (T_n, \tau) = \max_{b=0,\ldots,\lfloor T_n/n_T \rfloor-2} \left( \frac{1}{n_T} \sum_{j=1}^{n_T} \left( \bar{U}_{T_m + \tau + (b+1)n_T - j - 1} - U_{T_m + \tau + b n_T - j - 1} \right) \right).
\]

Observe that the variables \( \bar{U}_{T_m + \tau + (b+1)n_T - j - 1} \) and \( U_{T_m + \tau + b n_T - j - 1} \) have both zero mean, unit variance and are independent over \( b \) and \( j \). Thus, \( V_{\max, h} (T_n, \tau) \) corresponds to \( B_{\max, T_n} \) from Lemma S.A.10. In addition, under Assumption S.A.2 and Condition 1 the final result can be deduced from the same lemma. \( \square \)
S.A.4.3.2 Proof of part (ii) of Theorem 3.2 The proof follows similar steps as those used for \( MQ_{\max,h} \). More specifically, we can repeat the same proof as in Lemma S.A.12 so that corresponding results for a general loss function are still valid. Let \( U_{h,i} \triangleq n_T^{-1} \sum_{j=i+1}^{i+n_T} \left( L_{\psi,(T_m+\tau+j-1)h} (\beta^*) - L_{\psi,i} (\beta^*) \right)^2 \) and define

\[
U_{\max,h} (T_n, \tau) \triangleq \max_{i=n_T, \ldots, T_n-n_T} \left| n_T^{-1} \sum_{j=i+1}^{i+n_T} SL_{\psi,T_m+\tau+j-1} (\beta^*) - n_T^{-1} \sum_{j=i-n_T+1}^{i} SL_{\psi,T_m+\tau+j-1} (\beta^*) \right| \sqrt{U_{h,i}}.
\]

Lemma S.A.19. For any \( L \in L_\epsilon \), we have the results of Lemma S.A.12 and

\[
(\log (T_n) n_T)^{1/2} (U_{\max,h} (T_n, \tau) - MG_{\max,h} (T_n, \tau)) \xrightarrow{P} 0.
\]

Proof. The first claim can be proven in the same fashion as in Lemma S.A.12 with minor changes in notations. Proceeding as in Lemma S.A.13,

\[
|U_{\max,h} (T_n, \tau) - MG_{\max,h} (T_n, \tau)| \leq \max_{i=n_T, \ldots, T_n-n_T} \left| n_T^{-1} \sum_{j=i+1}^{i+n_T} \left( SL_{\psi,T_m+\tau+j-1} (\beta^*) - SL_{\psi,T_m+\tau+j-1} (\beta) \right) \right| \sqrt{U_{h,i} D_{h,i}} + \max_{i=n_T, \ldots, T_n-n_T} \left| n_T^{-1} \sum_{j=i-n_T+1}^{i} \left( SL_{\psi,T_m+\tau+j-1} (\beta^*) - SL_{\psi,T_m+\tau+j-1} (\beta) \right) \right| \sqrt{U_{h,i} D_{h,i}} \leq C_1 \max_{i=n_T, \ldots, T_n-n_T} \left| n_T^{-1} \sum_{j=i+1}^{i+n_T} \left( SL_{\psi,T_m+\tau+j-1} (\beta^*) - SL_{\psi,T_m+\tau+j-1} (\beta) \right) \right| U_{h,i} + C_2 \max_{i=n_T, \ldots, T_n-n_T} \left| n_T^{-1} \sum_{j=i-n_T+1}^{i} \left( SL_{\psi,T_m+\tau+j-1} (\beta^*) - SL_{\psi,T_m+\tau+j-1} (\beta) \right) \right| U_{h,i}.
\]

Following the same derivations as in the non-overlapping case we have a result corresponding to equation (S.36),

\[
\left| n_T^{-1} \sum_{j=i+1}^{i+n_T} \left( SL_{\psi,T_m+\tau+j-1} (\beta^*) - SL_{\psi,T_m+\tau+j-1} (\beta) \right) \right| = KO \left( T^{-1/2} \right).
\]

For any \( \epsilon > 0 \) and any constant \( K > 0 \), we then have the decomposition,

\[
P \left( \max_{i=n_T, \ldots, T_n-n_T} \left| (\log (T_n) n_T)^{1/2} n_T^{-1} \sum_{j=i+1}^{i+n_T} \left( SL_{\psi,T_m+\tau+j-1} (\beta^*) - SL_{\psi,T_m+\tau+j-1} (\beta) \right) \right| > \epsilon \right) \leq P \left( \max_{i=n_T, \ldots, T_n-n_T} \left| (\log (T_n) n_T)^{1/2} n_T^{-1} \sum_{j=i+1}^{i+n_T} \left( SL_{\psi,T_m+\tau+j-1} (\beta^*) - SL_{\psi,T_m+\tau+j-1} (\beta) \right) \right| > \epsilon/K \right) + \P \left( \max_{i=n_T, \ldots, T_n-n_T} 1/ |U_{h,i}| > K \right).
\]
Observe that Lemma S.A.16 remains valid when blocks overlap and so \( \mathbb{P} \left( \max_{i=n_T, \ldots, T_n-n_T} 1/|\mathcal{U}_{h,i}| > K \right) \to 0 \) by setting, for example, \( K = 1/\sigma^2_{L_n} \). Upon using Markov’s inequality, (S.58) (which holds uniformly in \( i \)) and Condition 1 we can conclude the proof with

\[
\mathbb{P} \left( \max_{i=n_T, \ldots, T_n-n_T} \left| \frac{1}{n_T} \sum_{j=i+1}^{i+n_T} \left( SL_{\psi,T_m+\tau+j-1}(\beta^*) - SL_{\psi,T_m+\tau+j-1}(\tilde{\beta}) \right) \right| > \varepsilon/K \right) \\
\leq \frac{K}{\varepsilon} \mathbb{E} \left[ \left( \frac{1}{n_T} \sum_{j=i+1}^{i+n_T} \left( SL_{\psi,T_m+\tau+j-1}(\beta^*) - SL_{\psi,T_m+\tau+j-1}(\tilde{\beta}) \right) \right)^2 \right] \\
= \frac{K}{\varepsilon} \left( \frac{1}{n_T} \sum_{j=i+1}^{i+n_T} \left( SL_{\psi,T_m+\tau+j-1}(\beta^*) - SL_{\psi,T_m+\tau+j-1}(\tilde{\beta}) \right) \right)^2 \to 0. \]

Define

\[
MB_{\text{max},h}^0(T_n, \tau) \triangleq \max_{i=n_T, \ldots, T_n-n_T} \left| \left( \frac{1}{n_T} \sum_{j=i+1}^{i+n_T} g \left( \sigma_{e,(T_m+\tau+i-1)}h^{-1/2} \left( \Delta_h W_{e,T_m+\tau+j-1} \right) \right) \right) \right|.
\]

where

\[
D_{h,i}^0 \triangleq \frac{n_T^{-1}}{\sum_{j=i+1}^{i+n_T} g \left( \sigma_{e,(T_m+\tau+i-1)}h^{-1/2} \left( \Delta_h W_{e,T_m+\tau+j-1} \right) \right)} \right|^2,
\]

with \( \bar{g}_i \triangleq n_T^{-1} \sum_{j=i+1}^{i+n_T} g \left( \sigma_{e,(T_m+\tau+i-1)}h^{-1/2} \left( \Delta_h W_{e,T_m+\tau+j-1} \right) \right) \). Next, let

\[
MB_{\text{max},h}^*(T_n, \tau) \triangleq \max_{i=n_T, \ldots, T_n-n_T} \left| n_T^{-1} \sum_{j=i+1}^{i+n_T} u_{T_m+\tau+j-1} - n_T^{-1} \sum_{j=i-n_T+1}^{i+n_T} u_{T_m+\tau+j-1} \right|,
\]

where \( u_{T_m+\tau+j-1}^* \triangleq g \left( \Delta_h e_{T_m+\tau+j-1}; \beta^* \right) \). Then, define

\[
\widetilde{MB}_{\text{max},h}^0(T_n, \tau) \triangleq \max_{i=n_T, \ldots, T_n-n_T} \left| n_T^{-1} \sum_{j=i+1}^{i+n_T} \left( \bar{u}_{T_m+\tau+j-1} - \bar{g}_i \right) \right|^2 \]

where \( \bar{D}_{h,i}^0 \triangleq n_T^{-1} \sum_{j=i+1}^{i+n_T} \left( \bar{u}_{T_m+\tau+j-1} - \bar{g}_i \right) \), with \( \bar{g}_i \triangleq n_T^{-1} \sum_{j=i+1}^{i+n_T} \bar{u}_{T_m+\tau+j-1} \) and

\[
\bar{u}_{T_m+\tau+j-1} \triangleq g \left( \sigma_{e,(T_m+\tau+i-n_T-1)}h^{-1/2} \Delta_h W_{e,T_m+\tau+j-1}; \beta^* \right).
\]

In the final step we shall show that \( \mathbb{P} \left( \left( \log (T_n) n_T / M_{\text{max},h}(T_n, \tau) - \widetilde{MB}_{\text{max},h}^0(T_n, \tau) \right) > \varepsilon \right) \to 0 \) for any \( \varepsilon > 0 \), where

\[
M_{\text{max},h}(T_n, \tau) \triangleq \max_{i=n_T, \ldots, T_n-n_T} \left| n_T^{-1} \sum_{j=i+1}^{i+n_T} \left( \bar{u}_{T_m+\tau+j-1} - \bar{g}_i \right) \right|^2 \]

\( S-36 \)
with \( \sigma_{u,(T_m+\tau-i-n_T-1)h} \triangleq \left( \text{Var} \left( u_{T_m+\tau+i-n_T} \mid \mathcal{F}(T_m+\tau+i-n_T-1)h \right) \right)^{1/2} \). By Assumption S.A.1 there exist \( 0 < \sigma_{u,-} < \sigma_{u,+} < \infty \) defined by \( \sigma_{u,-} \triangleq \inf_{k \geq 1} \{ \sigma_{u,kh} \} \) and \( \sigma_{u,+} \triangleq \sup_{k \geq 1} \{ \sigma_{u,kh} \} \). In parts of the derivations below we shall use some of the results from the non-overlapping case. In particular, the only difference arises from the fact that now the maximum is over a larger set and therefore the bounds should be adjusted accordingly.

**Lemma S.A.20.** As \( h \downarrow 0 \), \( (\log (T_n) n_T)^{1/2} \left( U_{\max,h} (T_n, \tau) - \text{MB}^0_{\max,h} (T_n, \tau) \right) \xrightarrow{p} 0. \)

**Proof.** First, given Lemma S.A.19 it follows that \( (\log (T_n) n_T)^{1/2} \left( \text{MB}^*_{\max,h} (T_n, \tau) - U_{\max,h} (T_n, \tau) \right) \xrightarrow{p} 0. \) Thus, we have to show

\[
(\log (T_n) n_T)^{1/2} \left( \text{MB}^*_{\max,h} (T_n, \tau) - \text{MB}^0_{\max,h} (T_n, \tau) \right) \xrightarrow{p} 0.
\]

Note that the result of Lemma S.A.16 still holds. Thus, we have decompositions similar to (S.39) and (S.40) and then one can follow the same steps as above. However, the bounds in (S.41) and (S.42) are now different because the maximum is over \( i = n_T, \ldots, T_n - n_T \). The bound in (S.41) is now \( (K/\varepsilon)^r n_T^{\alpha r + r} (2h^{-1}) \) which converges to zero by choosing \( r \) sufficiently large and \( \varepsilon \) small. The bound corresponding to (S.42) also goes to zero for large enough \( r > 0 \). All the steps leading to (S.46) can be repeated with minor changes. Indeed, the bound (S.46) also remains the same because it involves using the condition on Lipschitz continuity, which gives for \( r > 0 \) large enough,

\[
E \left[ \left( (\log (T_n) n_T)^{1/2} \left( n_T^{1/2} \left( n_T^{1/2} \left( \sigma_{e,s} - \sigma_e, (T_m+\tau+i-2)h \right \right) dW_{e,s} \right) \right)^r \right] 
\leq K_r \left( (\log (T_n) n_T)^{1/2} \left( n_T^{1/2} \left( \left( \sigma_{e,s} - \sigma_e, (T_m+\tau+i-2)h \right \right) dW_{e,s} \right) \right)^r 
\leq K_r \left( (\log (T_n))^1/2 \right)^r \tilde{h}^{-1/3+3/3+\varepsilon} \to 0.
\]

(S.61)

Altogether, these arguments can be used to verify the result of the lemma. \( \square \)

**Lemma S.A.21.** As \( h \downarrow 0 \), \( (\log (T_n) n_T)^{1/2} \left( \text{MB}^0_{\max,h} (T_n, \tau) - \text{MV}^0_{\max,h} (T_n, \tau) \right) \xrightarrow{p} 0. \)

**Proof.** The proof follows exactly the same steps as in the proof of Lemma S.A.17-S.A.18. Since some of the bounds need to be adjusted to account for the maximum being over \( i = n_T, \ldots, T_n - n_T \), we can use the same argument as in the previous lemma. Then, all the quantities generalizing the expressions in the proofs of Lemma S.A.17-S.A.18 are controlled thereby yielding \( (\log (T_n) n_T)^{1/2} \left( \text{MB}^0_{\max,h} (T_n, \tau) - \tilde{\text{MB}}^0_{\max,h} (T_n, \tau) \right) \xrightarrow{p} 0 \) and \( (\log (T_n) n_T)^{1/2} \left( \text{MV}^0_{\max,h} (T_n, \tau) - \tilde{\text{MB}}^0_{\max,h} (T_n, \tau) \right) \xrightarrow{p} 0. \) \( \square \)

**Proof of Theorem 3.2-(ii).** From Lemma S.A.19-S.A.21, \( \sqrt{\log (T_n) n_T} (\text{MG}_{\max,h} (T_n, \tau) - \text{MV}_{\max,h} (T_n, \tau)) = o_p (1) \). As for the non-overlapping case, we deduce the limit distribution of \( \text{MV}_{\max,h} \) from that of \( \text{MB}_{\max,T_n} \) derived in Lemma S.A.10. Let

\[
\hat{U}_{T_m+\tau+j-1} \triangleq \begin{cases} \frac{u_{T_m+\tau+i-n_T-1} - \mu_{u,(T_m+\tau+i-n_T-1)h}}{\sigma_{u,(T_m+\tau+i-n_T-1)h}}, & \text{for } j = i + 1, \ldots, i + n_T \\ \frac{u_{T_m+\tau+i-n_T-1} - \mu_{u,(T_m+\tau+i-n_T-1)h}}{\hat{\sigma}_{u,(T_m+\tau+i-n_T-1)h}}, & \text{for } j = i - n_T + 1, \ldots, i. \end{cases}
\]
Then, we have $\mathbb{E} \left( \tilde{U}_{T_m+j} \right) = 0$, $\text{Var} \left( \tilde{U}_{T_m+j} \right) = 1$ and the $\tilde{U}_{T_m+j}$’s are independent across $j$. MV$_{\text{max},h}(T_n, \tau)$ now corresponds to MB$_{\text{max},T_n}$ from Lemma S.A.10. Thus, we can deduce the final result from Lemma S.A.10 since Assumption S.A.2 and Condition 3.1 holds. □

### S.A.4.3.3 Negligibility of the drift term under general loss functions

The reasoning is similar to the quadratic loss case. We only show that the drift component $\mu_{e,t}$ is of higher order. Without estimation uncertainty our tests statistics are simply functions of local averages of $g(\Delta_h e^*_k, \beta^*)$, where $g(\cdot, \cdot)$ is smooth. Note that conditional on $\{\mu_{e,t}\}_{t \geq 0}$ and $\{\sigma_{e,t}\}_{t \geq 0}$,

$$h^{-1/2} \Delta_h e^*_k = h^{-1/2} \int_0^T \mu_{e,s} h^{-\vartheta} ds + h^{-1/2} \int_0^T \sigma_{e,s} dW_{e,s}$$

$$= O \left( h^{-1/2} + h^{-1/2} \right)$$

Since $\vartheta \in [0, 1/8)$ and $\int_0^T \sigma_{e,s} dW_{e,s} \approx \mathcal{N} \left( 0, \int_0^T \sigma_{e,s}^2 ds \right)$, it follows that the first term above is of higher order and should not play any role for the asymptotic results of Lemma S.A.12-S.A.13.

#### S.A.4.4 Proof of Corollary 3.2

**Proof.** It follows the same arguments as for Corollary 3.1. □

### S.A.5 Proofs of Section 4

#### S.A.5.1 Proof of Theorem 4.1

**Proof.** See Theorem 3 in Wu and Zhao (2007). □

#### S.A.5.2 Proof of Theorem 4.2

The initial step in the proof uses a uniform strong approximation result which essentially extends the strong invariance principle of Wu (2007) to our setting. The idea behind the proof is similar to that of Theorem 2.1 in Zhao and Li (2013). Before giving the result, we need to recall the more general framework of Wu (2007).

Let $\{\xi_k\}_{k=1}^{T_n}$ be a sequence of zero-mean independent random variables with $\text{Var} \left( \xi_k \right) = \sigma_k^2$ satisfying $c_\leq \leq \min_{k \geq 1} \{\sigma_k\}$ and $c_\geq \geq \max_{k \geq 1} \{\sigma_k\}$ with $0 < c_\leq < c_\geq < \infty$. Let $\xi_j \triangleq j^{-1} \sum_{k=1}^j \xi_k$, $G_{\xi,j} \triangleq j \xi_j$, and $V_{\xi,j} \triangleq \sum_{k=1}^j \left( \xi_k - \xi_j \right)^2$. Let $\{B_t\}_{t \geq 0}$ and $\{\tilde{B}_t\}_{t \geq 0}$ denote two independent one-dimensional standard Wiener processes which need not be defined on the same probability space. Finally, let $a_{T_n} \triangleq \left| \sigma_{T_n} \right| + \sum_{k=2}^{T_n} \left| \sigma_k - \sigma_{k-1} \right|$, $c_{T_n} \triangleq \left| \sigma_{T_n}^2 \right| + \sum_{k=2}^{T_n} \left| \sigma_k^2 - \sigma_{k-1}^2 \right|$, $\Xi_j \triangleq \sum_{k=1}^j \sigma_k^2$ and $\Xi_j^2 \triangleq \sum_{k=1}^j \sigma_k^4$. We begin with the following lemma involving a strong invariance principle for the process $\{\xi_k\}$ and an uniform approximation for $\{V_{\xi,k}\}$. Without loss of generality assume that $\xi_k = \sigma_k \epsilon_k$, with $\{\epsilon_k\}$ being a zero-mean stationary process with $\mathbb{E} (\epsilon_k^2) = 1$. Further, denote by $\tilde{\sigma}^2$ the long-run variance of $\epsilon_k : \tilde{\sigma}^2 \triangleq \gamma_0 + 2T_n^{-1} \sum_{i=1}^{T_n} \gamma_i$, where $\gamma_i \triangleq \text{Cov} (\epsilon_{k+i}, \epsilon_k)$. Define similarly the long-run variance of $\left( \epsilon_k^2 - 1 \right)$ and denote it by $\tilde{\sigma}^2$. Next, let

$$S_k = \sum_{j=1}^k \epsilon_j, \quad \tilde{S}_k = \sum_{j=1}^k \left( \epsilon_j^2 - 1 \right), \quad k = 1, \ldots, T_n,$$
with the convention $S_0 = S^*_0 = 0$. Then we have the following strong invariance principles [cf. Wu (2007)]:

\[
\max_{1 \leq k \leq T_n} |S_k - \bar{\theta} \bar{B}_k| = o_{a.s.}(\Delta T_n) \quad \text{and} \quad \max_{1 \leq k \leq T_n} |\tilde{S}_k - \bar{\theta} \bar{B}_k| = o_{a.s.}(\Delta T_n),
\]

(S.62)

where $\Delta T_n$ is an approximation error that satisfies $\Delta T_n \to \infty$. Under our context, the order of $\Delta T_n$ is given by the following assumption

**Assumption S.A.3.** Assume $0 < \theta, \tilde{\theta} < \infty$. The relationships in (S.62) holds with $\Delta T_n = T_n^{1/4} \log(T_n)$.  

**Lemma S.A.22.** Given Assumption S.A.2, for any $\eta \in (0, 1]$,

(i) $\max_{T_n/2 \leq j \leq T_n} |G_{\xi,j} - \theta \sum_{k=1}^j \sigma_k (B_k - B_{k-1})| = O_{a.s.}(\Delta T_n)$;

(ii) $\max_{T_n/2 \leq j \leq T_n} |V_{\xi,j} - \Xi_j| = O_{a.s.}\left(\Delta T_n + \tilde{\Xi}_j + \left(\Delta T_n^2 + \Xi_j\right)/T_n\right)$.

**Proof.** To prove part (ii) one needs part (i). However, the same steps in the initial part in the proof of (ii) can be used to prove part (i) as we explain below. Thus, we only prove part (ii). After some simple algebraic manipulations one can verify the decomposition $V_{\xi,j} - \Xi_j = U_j - G^2_{\xi,j}/j$ where $U_j = \sum_{k=1}^j \sigma_k^2 (\epsilon_k^2 - 1)$. By Abel’s formula and $\epsilon_k^2 - 1 = S_k^* - S_{k-1}^*$, we have

\[
U_j = \sum_{k=1}^j \sigma_k^2 (\tilde{S}_k - \tilde{S}_{k-1}) = \left(\sigma_k^2 \tilde{S}_j - \sigma_0^2 \tilde{S}_0\right) - \sum_{k=1}^{j-1} \left(\sigma_{k+1}^2 - \sigma_k^2\right) \tilde{S}_k
\]

and by the rightmost approximation in (S.62) it follows that

\[
U_j = \sigma_j^2 \tilde{\theta} \bar{B}_j - \sum_{k=1}^{j-1} \left(\sigma_{k+1}^2 - \sigma_k^2\right) \tilde{\theta} \bar{B}_k + O_{a.s.}(\Delta T_n)
\]

\[
= \bar{\theta} \sum_{k=1}^j \sigma_k^2 \left(B_k - B_{k-1}\right) + O_{a.s.}(\Delta T_n).
\]

(S.64)

Next, by Kolmogorov’s maximal inequality for independent random variables [cf. Theorem 22.4 in Billingsley (1995)], we have for $C > 0$,

\[
P\left[\max_{1 \leq j \leq T_n} \left|\bar{\theta} \sum_{k=1}^j \sigma_k^2 \left(B_k - B_{k-1}\right)\right| \geq C 2 T_n / \bar{\theta} \sum_{k=1}^j \sigma_k^2 \left(B_k - B_{k-1}\right)\right] \leq \left(\frac{C 2 T_n / \bar{\theta} \sum_{k=1}^j \sigma_k^2 \left(B_k - B_{k-1}\right)}{\sum_{k=1}^j \sigma_k^2 \left(B_k - B_{k-1}\right)}\right)^2 \leq \left(\frac{C}{C^*}\right)^2.
\]

Thus, choosing $C$ large enough shows that $\max_{1 \leq j \leq T_n} \bar{\theta} \sum_{k=1}^j \sigma_k^2 \left(B_k - B_{k-1}\right) \leq C 2 T_n / \bar{\theta} \sum_{k=1}^j \sigma_k^2 \left(B_k - B_{k-1}\right) < \infty$. Use this result into (S.64) to verify that $\max_{1 \leq j \leq T_n} |U_j| = O_{\mathbb{P}}(\tilde{\Xi}_T + \Delta T_n)$. Using the same steps as in (S.63)-(S.64), one verifies $\max_{1 \leq j \leq T_n} |G_{\xi,j}| = O_{\mathbb{P}}(\sqrt{\tilde{\Xi}_T} + \Delta T_n)$. Hence,

\[
V_{\xi,j} - \Xi_j = U_j - G^2_{\xi,j}/j = O_{a.s.}\left(\Delta T_n + \tilde{\Xi}_j + \left(\Delta T_n^2 + \Xi_j\right)/T_n\right),
\]

uniformly in $1 \leq j \leq T_n$, which proves part (ii). $\square$

The first part of the proof uses Lemma S.A.22 applied to the sequence of normalized forecast losses $\{L_{\psi,kh}(\beta^*)\}_{k=T_n}^{T_n+T_n}$. We provide the proof directly for a general loss function; the case of the
quadratic loss function follows as a special case. Since we have already dealt with the discretization error above and have shown that $\mu_{e,h} h^{-\vartheta}$ is negligible for $\vartheta \in [0, 1/8)$, in this section we assume for simplicity that $L_{\psi,kh}(\beta^*) = g(\Delta_{h,k}^2; \beta^*)$, where $\Delta_{h,k}^2 = \sigma_{e,(k-1)h} h^{-1/2} \Delta_h W_{e,k}$. Let $\mu_{T_m + \tau + (b+1)n_T - 1} = \max \left( L_{\psi,(T_m + \tau + (b+1)n_T - 1)h}(\beta^*) \right)$ for $j = 1, \ldots, n_T$, which is justified by the fact that these variables are in the same window.

**Proof of Theorem 4.2.** We shall use Lemma S.A.22-(ii). Let

$$\xi_j \triangleq g \left( \sigma_{e,(T_m + \tau + (b+1)n_T - 1)h} h^{-1/2} \Delta_{h} W_{e,T_m + \tau + (b+1)n_T - 1} \right) - \mu_{T_m + \tau + (b+1)n_T - 1}$$

for $j = 1, \ldots, n_T$. Using basic arguments, we also have $V_{h,b}(\beta) - V_{h,b}^* = O_P(1/\sqrt{n_T})$, where

$$V_{h,b}^* \triangleq n_T^{-1} \sum_{j=1}^{n_T} \left( g \left( \sigma_{e,(T_m + \tau + (b+1)n_T - 1)h} h^{-1/2} \Delta_{T_m + \tau + (b+1)n_T - 1} W_{e} \right) - \mu_{T_m + \tau + (b+1)n_T - 1} \right)^2$$

e.g., use the initial lemmas from the proof of Theorem 3.2 and note that $L_{\psi,b}(\beta^*) - \mu(T_m + \tau + (b+1)n_T - 1)h = O_P(1/\sqrt{n_T})$ by a basic central limit theorem for i.i.d. variables. Then, by Lemma S.A.22-(ii) we have $\max_{0 \leq b \leq |T_n/n_T| - 2} |n_T V_{h,b}(\beta^*) - \Sigma_{h,b}^*| = O_P \left( \Delta_{T_n} + \Sigma_{h,b}^* \right)$, where $\Delta_{T_n} = \frac{T_n^{1/4}}{\log(T_n)}$. Let

$$d_{h,b} \triangleq n_T V_{h,b}(\beta) / \Sigma_{h,b}^* - 1,$$

and note that $d_{h,b} = O_P \left( \left( \Delta_{T_n} + \Sigma_{h,b}^* \right) / \Sigma_{h,b}^* \right)$ by proceeding as in the proof of Lemma S.A.12 and thus using $\hat{\beta}_k - \beta^* = O_P \left( T_n^{-1/2} \right)$ uniformly by Assumption 3.6. By definition $\Sigma_{h,b}^*/\Sigma_{h,b}^* = O_P \left( n_T^{-1/2} \right)$ while by Condition 2 $\Delta_{T_n} / \Sigma_{h,b}^* \rightarrow 0$ so that we deduce $d_{h,b} = o_P(1)$ uniformly over $b$. Let $\{B_t\}_{t \geq 0}$ be a standard Wiener process and define

$$z_{h,b} \triangleq \left( \Sigma_{h,b}^* \right)^{-1/2} \sum_{j=1}^{n_T} \sigma_{L,(T_m + \tau + (b+1)n_T - 1)h} \left( B_{(b+1)n_T - j + 1} - B_{(b+1)n_T - j} \right).$$

By the law of iterated logarithms [cf. Billingsley (1995), Theorem 9.5 in Ch. 1], $\max_{0 \leq b \leq |T_n/n_T| - 2} |z_{h,b}| = O_P \left( \sqrt{\log(T_n)} \right)$. Under $H_0$, by the Lipschitz continuity of $\mu$ we have $\mu_{T_m + \tau + (b+1)n_T - 1} - \mu_{T_m + \tau + bn_T - 1} = O_P \left( n_T h \right)$. This together with applying multiple times the bounds used at the beginning of the proof concerning terms involving $\hat{\beta}$ and $\beta^*$ allows us to that $\zeta_{h,b}(\hat{\beta})$ can be approximated by

$$\zeta_{h,b}^* \triangleq \sqrt{n_T} \left( A_{h,b}(\beta^*) - A_{h,b}(\beta^*) - \mu_{T_m + \tau + (b+1)n_T - 1} \right) / \sqrt{V_{h,b}}$$

because for small $\epsilon > 0$, $\sqrt{n_T} \left( \mu_{T_m + \tau + (b+1)n_T - 1} - \mu_{T_m + \tau + bn_T - 1} \right) = h^\epsilon \rightarrow 0$. Let

$$\tilde{A}_{h,b}(\beta^*) \triangleq n_T \left( A_{h,b}(\beta^*) - \mu_{T_m + \tau + (b+1)n_T - 1} \right).$$

Then,

$$\zeta_{h,b} = \frac{\tilde{A}_{h,b}(\beta^*)}{\sqrt{\Sigma_{h,b}^* \left( \sqrt{n_T V_{h,b} / \sqrt{\Sigma_{h,b}^*}} \right)}} - \frac{\tilde{A}_{h,b}(\beta^*)}{\sqrt{\Sigma_{h,b}^* \left( \sqrt{n_T V_{h,b} / \sqrt{\Sigma_{h,b}^*}} \right)}}$$

S-40
\[ \frac{\tilde{A}_{h,b} (\beta^*)}{\sqrt{\Sigma^*_{h,b} (1 + d_{h,b})}} - \frac{\tilde{A}_{h-1,b} (\beta^*)}{\sqrt{\Sigma^*_{h,b-1} (1 + d_{h,b}) / \Sigma^*_{h,b-1}}}, \]

and given \( d_{h,b} \to 0 \) we know that \( \sqrt{1 + d_{h,b}} = 1 + O_P (d_{h,b}) \). In view of Lemma S.A.22-(i) we have

\[ |\tilde{A}_{h,b}| \leq O_{a.s.} (\Delta T_n). \]

Therefore, the last inequality leads to

\[ \frac{\tilde{A}_{h,b} (\beta^*)}{\sqrt{\Sigma^*_{h,b} (1 + d_{h,b})}} = (1 + d_{h,b}) \]

\[ \times \left[ \frac{\nu L \sum_{j=1}^{n_T} \sigma_L (T_m + \tau + (b+1)n_T + j - 1) h \left( \mathbb{B}_{(b+1)n_T + j} \right. - \left. \mathbb{B}_{(b+1)n_T + j - 1} \right) \sqrt{\Sigma^*_{h,b}}}{\sum_{j=1}^{n_T} \sigma_L (T_m + \tau + (b+1)n_T + j - 1) h \left( \mathbb{B}_{(b+1)n_T + j} \right. - \left. \mathbb{B}_{(b+1)n_T + j - 1} \right) \sqrt{\Sigma^*_{h,b}}} + O_{a.s.} \left( \frac{\Delta T_n}{\sqrt{\Sigma^*_{h,b}}} \right) \right]. \]

A similar argument can be used for the second term while in addition for the denominator we use the fact that \( \Sigma^*_{h,b-1} \) is Lipschitz continuous and therefore \( \Sigma^*_{h,b} - \Sigma^*_{h,b-1} = O_P (n_T h) \), which then gives \( \sqrt{1 + d_{h,b}} \sqrt{1 + O_P (n_T h)} = 1 + O_P (d_{h,b}) \). Let \( \{\mathbb{B}_t\}_{t \geq 0} \) be a standard Wiener process. We can then deduce that

\[ \zeta^*_{b,h} = \frac{\nu L \sum_{j=1}^{n_T} \sigma_L (T_m + \tau + (b+1)n_T + j - 1) h \left( \mathbb{B}_{(b+1)n_T + j} \right. - \left. \mathbb{B}_{(b+1)n_T + j - 1} \right)}{\sum_{j=1}^{n_T} \sigma_L (T_m + \tau + (b+1)n_T + j - 1) h \left( \mathbb{B}_{(b+1)n_T + j} \right. - \left. \mathbb{B}_{(b+1)n_T + j - 1} \right) \sqrt{\Sigma^*_{h,b}}} + (1 + d_{h,b}) O_{a.s.} \left( \frac{\Delta T_n}{\sqrt{\Sigma^*_{h,b-1}}} \right). \]

The stochastic order term in the last equation is, for some small \( \epsilon > 0 \), \( O_P \left( \log (T_n) / \left( T_n^{3/2} \right) \right) \to 0 \) where we have used \( \Delta T_n = T_n^{1/4} \log (T_n) \), Condition 2 and \( \Sigma^*_{h,b} = O_P (n_T) \). Using the properties of the Wiener process, we have

\[ \left( \zeta^*_{b,h} \right)^2 = \left( \nu^2 L \sum_{j=1}^{n_T} \sigma^2_L (T_m + \tau + (b+1)n_T + j - 1) h \right) \frac{\sqrt{\Sigma^*_{h,b}}}{\sum_{j=1}^{n_T} \sigma^2_L (T_m + \tau + (b+1)n_T + j - 1) h} + O_P \left( \log (T_n) / T_n^{3/2} \right) \]

\[ + 2 \nu^2 L = 2 \nu^2 L + O_P \left( \log (T_n) \right)^2 / T_n^2, \]

and therefore, \( \nu^2 L = \nu^2 L + O_P \left( \log (T_n) \right)^2 / T_n^2 \), \( \square \)
S.A.6 Proofs of Section 5

S.A.6.1 Proof of Theorem 5.1

See Casini (2018).

S.A.6.2 Proof of Theorem 5.2

Let \( \Delta_h e_k^o \triangleq \Delta_h Y_k - \beta_k^* \Delta_h X_{k - r} \), where \( \beta_k = \beta^* + \mu_{\beta, kh} / \left( \log \left( T_n \right) n_T \right)^{1/4} \). Let

\[
\overline{MQ}_{\max, h}^* (T_n, \tau) \triangleq \nu_L^{-1} \max_{i = i_T, \ldots, T_n - n_T} \left| n_T^{-1} \sum_{j = i + 1}^{i + n_T} (SL_{\psi, T_m + \tau + j - 1} (\beta^*) - \zeta_{\mu, j, +}) \right|
- \left| n_T^{-1} \sum_{j = i - n_T + 1}^{i} (SL_{\psi, T_m + \tau + j - 1} (\beta^*) - \zeta_{\mu, j, -}) \right|
\]

Our final goal is to show that \( \left( \log \left( T_n \right) n_T \right)^{1/2} \left( V_{\max, h} (T_n, \tau) - MQ_{\max, h} (T_n, \tau) \right) \overset{P}{\to} 0 \), where

\[
V_{\max, h} (T_n, \tau) = \nu_L^{-1} \max_{i = i_T, \ldots, T_n - n_T} \left| n_T^{-1} \sum_{j = i + 1}^{i + n_T} \left( \Delta_h e_{T_m + \tau + j - 1}^o \right)^2 \right|
- \left| n_T^{-1} \sum_{j = i - n_T + 1}^{i} \left( \Delta_h e_{T_m + \tau + j - 1}^o \right)^2 \right|
\]

The result of the theorem then follows from Corollary 3.2.

Lemma S.A.23. \( \left( \log \left( T_n \right) n_T \right)^{1/2} \left( MQ_{\max, h}^* (T_n, \tau) - \overline{MQ}_{\max, h} (T_n, \tau) \right) \overset{P}{\to} 0 \).

Proof. We have

\[
\overline{MQ}_{\max, h}^* (T_n, \tau) - \overline{MQ}_{\max, h} (T_n, \tau)
\leq \nu_L^{-1} \max_{i = i_T, \ldots, T_n - n_T} \left| n_T^{-1} \sum_{j = i + 1}^{i + n_T} \left( SL_{\psi, T_m + \tau + j - 1} (\beta^*) - SL_{\psi, T_m + \tau + j - 1} \left( \beta_{T_m + j - 1} \right) - \zeta_{\mu, j, +} \right) \right|
+ \nu_L^{-1} \max_{i = i_T, \ldots, T_n - n_T} \left| n_T^{-1} \sum_{j = i - n_T + 1}^{i} \left( SL_{\psi, T_m + \tau + j - 1} (\beta^*) - SL_{\psi, T_m + \tau + j - 1} \left( \beta_{T_m + j - 1} \right) - \zeta_{\mu, j, -} \right) \right|
\]

(S.66)

Note that for any \( j = i + 1, \ldots, i + n_T, \)

\[
SL_{\psi, T_m + \tau + j - 1} (\beta^*) - SL_{\psi, T_m + \tau + j - 1} \left( \beta_{T_m + j - 1} \right)
= L_{\psi, T_m + \tau + j - 1} (\beta^*) - L_{\psi, T_m + \tau + j - 1} \left( \beta_{T_m + j - 1} \right) + o_P \left( 1 / \left( \log \left( T_n \right) n_T \right)^{1/2} \right),
\]

where the \( o_P \left( 1 / \left( \log \left( T_n \right) n_T \right)^{1/2} \right) \) term arises from Lemma S.A.24 below. Focusing on the first two terms we have

\[
L_{\psi, T_m + \tau + j - 1} (\beta^*) - L_{\psi, T_m + \tau + j - 1} \left( \beta_{T_m + j - 1} \right) - \zeta_{\mu, j, +}
= \left( \beta^* - \beta_{T_m + j - 1} \right)' \Delta_h \bar{X}_{T_m + j - 1} \Delta_h \bar{X}'_{T_m + j - 1} (\beta^* - \beta_{T_m + j - 1}) - \zeta_{\mu, j, +}
\]
We deal explicitly with the first term on the right-hand side above in (S.76) below and show that it is
\[ \text{op} \left( \left( \log (T_n) n_T \right)^{-1/2} h^{1/4} \right). \]
Moving to the second term, by Theorem 13.3.7 in Jacod and Protter (2012) we have
\[ n_T^{-1/2} \sum_{j=i+1}^{i+n_T} \Delta_h W_{\psi,T_n+\tau+j-1} \Delta_h \bar{X}_{T_n+j-1} = \text{op}(1). \]
Using Assumption 5.2, \( \hat{\beta}_{T_m+j-1} - \beta^* = \text{op} \left( 1/ (\log (T_n) n_T)^{-1/4} \right) \) uniformly in \( j \) and thus, for any \( \varepsilon > 0 \),
\[
P \left( \max_{i=n_T, \ldots, T_n-n_T} \nu_L^{-1} \left( \log (T_n) n_T \right)^{1/2} \left| n_T^{-1} \sum_{j=i+1}^{i+n_T} \left( SL_{T_m+\tau+j-1} (\beta^*) - \sum_{j=i+n_T+1}^{i} \sum_{j=i+1}^{i+n_T} SL_{T_m+\tau+j-1} (\hat{\beta}_{T_m+j-1}) \right) \right| > \varepsilon \right) \leq \left( \frac{C}{\nu_L} \right)^r \left( \log (T_n) n_T \right)^{r/4} \text{op} \left( n_T^{-r/2} \right) \rightarrow 0,
\]
for \( r > 0 \) sufficiently large. The same bound applies to the term in (S.66) and this proves the claim of the lemma. \( \square \)

**Lemma S.A.24.** As \( h \downarrow 0 \),
\[
\nu_L^{-1} \max_{i=n_T, \ldots, T_n-n_T} \left( \log (T_n) n_T \right)^{1/2} \left| n_T^{-1} \sum_{j=i+1}^{i+n_T} T_{\psi,T_n+\tau+j-1} (\beta^*) - \sum_{j=i+n_T+1}^{i} \sum_{j=i+1}^{i+n_T} T_{\psi,T_n+\tau+j-1} (\hat{\beta}_{T_m+j-1}) \right| \xrightarrow{P} 0,
\]
and
\[
\nu_L^{-1} \max_{i=n_T, \ldots, T_n-n_T} \left( \log (T_n) n_T \right)^{1/2} \left| n_T^{-1} \sum_{j=i+1}^{i+n_T} \left( T_{\psi,T_n+\tau+j-1} (\beta^*) - T_{\psi,T_n+\tau+j-1} (\hat{\beta}_{T_m+j-1}) \right) \right| \xrightarrow{P} 0.
\]

**Proof.** By definition,
\[
\left| n_T^{-1} \sum_{j=i+1}^{i+n_T} T_{\psi,T_n+\tau+j-1} (\beta^*) - \sum_{j=i+n_T+1}^{i} \sum_{j=i+1}^{i+n_T} T_{\psi,T_n+\tau+j-1} (\beta^*) \right| \leq n_T^{-1} \sum_{j=i+1}^{i+n_T} \sum_{l=1}^{T_m+j-1-\tau} \frac{\left( \Delta_h e^\circ_{l+\tau} \right)^2}{T_m+j-1} - n_T^{-1} \sum_{j=i+n_T+1}^{i} \sum_{l=1}^{T_m+j-1-\tau} \frac{\left( \Delta_h e^\circ_{l+\tau} \right)^2}{T_m+j-1}
\]
\[
+ n_T^{-1} \sum_{j=i+1}^{i+n_T} \sum_{l=1}^{T_m+j-1-\tau} \frac{\mu_{\beta,(l+\tau)h} \Delta_h \bar{X}_l \Delta_h \bar{X}_l \mu_{\beta,(l+\tau)h}}{(T_m+j-1) (\log T_n n_T)^{1/2}} - n_T^{-1} \sum_{j=i+n_T+1}^{i} \sum_{l=1}^{T_m+j-1-\tau} \frac{\mu_{\beta,(l+\tau)h} \Delta_h \bar{X}_l \Delta_h \bar{X}_l \mu_{\beta,(l+\tau)h}}{(T_m+j-1) (\log T_n n_T)^{1/2}}
\]
\[
+ 2 n_T^{-1} \sum_{j=i+1}^{i+n_T} \sum_{l=1}^{T_m+j-1-\tau} \frac{\Delta_h e^\circ_{l+\tau} \mu_{\beta,(l+\tau)h} \Delta_h \bar{X}_l \Delta_h \bar{X}_l \mu_{\beta,(l+\tau)h}}{(T_m+j-1) (\log T_n n_T)^{1/4}}
\]
\[+ 2 n_T^{-1} \sum_{j=i+1}^{i+n_T} \sum_{l=1}^{T_m+j-1-\tau} \frac{\Delta_h e^\circ_{l+\tau} \mu_{\beta,(l+\tau)h} \Delta_h \bar{X}_l \Delta_h \bar{X}_l \mu_{\beta,(l+\tau)h}}{(T_m+j-1) (\log T_n n_T)^{1/4}}
\]
\[\Delta h_{l+\tau}^\circ = \left(\Delta h_{l+\tau}\right)^2 \left(\frac{1}{T_m + j + n_T - 1} - \frac{1}{T_m + j - 1}\right)\]

\[= 2n_T \sum_{j=i}^{i+n_T} T_{m+j-1} (\beta^*) - n_T^{-1} \sum_{j=i-n_T+1}^{i} T_{\psi,T_{m+j-1}} (\beta^*) \leq C O_P \left(\frac{n_T}{T_m}\right).\]

Thus, for any \(\varepsilon > 0\),

\[\mathbb{P} \left( \nu L \max_{i=n_T,\ldots, T_n-n_T} (\log (T_n) n_T)^{1/2} \left| n_T^{-1} \sum_{j=i}^{i+n_T} T_{\psi,T_{m+j-1}} (\beta^*) - n_T^{-1} \sum_{j=i-n_T+1}^{i} T_{\psi,T_{m+j-1}} (\beta^*) \right| > \varepsilon \right) \leq \varepsilon^{-r} \sum_{i=n_T}^{T_n-n_T} \mathbb{E} \left[ (\log (T_n) n_T)^{r/2} \left| n_T^{-1} \sum_{j=i}^{i+n_T} T_{\psi,T_{m+j-1}} (\beta^*) - n_T^{-1} \sum_{j=i-n_T+1}^{i} T_{\psi,T_{m+j-1}} (\beta^*) \right|^r \right] \leq \varepsilon^{-r} C (\log (T_n) n_T)^{r/2} O_P \left(\frac{n_T^{1-r}}{T_n}\right) \to 0, \tag{S.68}\]

for \(r > 0\) sufficiently large in view of Condition 1 and that \(T_n = O(T_m)\). For the second claim of the lemma, note that

\[T_{\psi,T_{m+j-1}} (\beta^*) = \frac{T_{m+j-1-\tau}}{T_m + j - 1 - \tau} \left(\Delta h_{l+\tau}\right)^2 - \sum_{l=1}^{T_{m+j-1-\tau}} \left(\Delta h_{l+\tau}\right)^2 \left(\frac{1}{T_m + j - 1 - \tau}\right) - \frac{2}{T_m + j - 1 - \tau} \sum_{l=1}^{T_{m+j-1-\tau}} \left(\Delta h_{l+\tau}\right)^2 \left(\frac{1}{T_m + j - 1 - \tau}\right) \tag{S.69}\]

By Lemma S.A.1, \((T_m + j - 1)^{-1} \sum_{l=1}^{T_{m+j-1-\tau}} \Delta h_{l+\tau}\) is \(O_P (1)\) while by Assumption 5.2, \(\hat{\beta}_k - \beta^* = \)
$O_P\left(1/\log (T_n) n_T \right)^{1/4}$ uniformly in $k$. It follows that the term in equation (S.69) is $O_P \left( \log (T_n) n_T \right)^{1/2}$ whereas the term in (S.70) is such that

$$
\frac{2}{T_m + j - 1} \sum_{i=1}^{T_m+j-1} \Delta h e_{i+\tau}^o \left( \beta_{T_m+j-1} - \beta^* \right)' \Delta h \bar{X}_i
$$

$$
\leq 2C \sup_j \left\| \beta_{T_m+j-1} - \beta^* \right\| \frac{1}{T_m + j - 1} \sum_{i=1}^{T_m+j-1} \Delta h e_{i+\tau}^o \left( \beta_{T_m+j-1} - \beta^* \right) \Delta h \bar{X}_i
$$

$$
= o_P \left( T_m^{-1/2} \log (T_n) n_T \right)^{1/4},
$$

where $\epsilon$ is a $q \times 1$ unit vector and we have used the central limit theorem in Lemma S.A.1-(iii). Therefore, upon using the same argument that led to (S.68) and Condition 1 we have the last claim of the lemma.

Lemma S.A.25. $(\log (T_n) n_T)^{1/2} \left( V_{\max,h} (T_n, \tau) - \tilde{\text{MQ}}_{\max,h} - (T_n, \tau) \right) \xrightarrow{p} 0.$

Proof. Note that $SL_{T_m+\tau+j-1} (\beta^*)$ can be expanded as follows:

$$
SL_{T_m+\tau+j-1} (\beta) = L_{\psi,T_m+\tau+j-1} (\beta) - T_{\psi,T_m+\tau+j-1} (\beta) \tag{S.71}
$$

$$
= \left( \Delta h e_{T_m+\tau+j-1}^o \right)^2 + \left( \beta^* - \tilde{\beta}_{T_m+j-1} \right)' \Delta h \bar{X}_{T_m+j-1} \Delta h \bar{X}_{T_m+j-1}^T \left( \beta^* - \tilde{\beta}_{T_m+j-1} \right)
$$

$$
- 2\Delta h e_{T_m+\tau+j-1}^o \left( \beta^* - \tilde{\beta}_{T_m+j-1} \right)' \Delta h \bar{X}_{T_m+j-1} - T_{T_m+\tau+j-1} (\beta). \tag{S.72}
$$

Then we can write (omitting the index from $\tilde{\beta}$),

$$
(\log (T_n) n_T)^{1/2} \left( V_{\max,h} (T_n, \tau) - \tilde{\text{MQ}}_{\max,h} - (T_n, \tau) \right) \leq \max_{i=n_T, \ldots, T_n-n_T} (\log (T_n) n_T)^{1/2} \nu_L^{-1}
$$

$$
\times \left| n_T^{-1} \sum_{j=i+1}^{i+n_T} \left( \beta^* - \tilde{\beta} \right)' \Delta h \bar{X}_{T_m+j-1} \Delta h \bar{X}_{T_m+j-1}^T \left( \beta^* - \tilde{\beta} \right) - \zeta_{\mu,j,+} \right| \tag{S.73}
$$

$$
- n_T^{-1} \sum_{j=i-n_T+1}^{i} \left( \beta^* - \tilde{\beta} \right)' \Delta h \bar{X}_{T_m+j-1} \Delta h \bar{X}_{T_m+j-1}^T \left( \beta^* - \tilde{\beta} \right) - \zeta_{\mu,j,-} \right|
$$

$$
+ \max_{i=n_T, \ldots, T_n-n_T} 2 (\log (T_n) n_T)^{1/2} \nu_L^{-1} \left| n_T^{-1} \sum_{j=i+1}^{i+n_T} \Delta h e_{T_m+j-1}^o \left( \beta^* - \tilde{\beta} \right)' \Delta h \bar{X}_{T_m+j-1}
$$

$$
- n_T^{-1} \sum_{j=i-n_T+1}^{i} \Delta h e_{T_m+j-1}^o \left( \beta^* - \tilde{\beta} \right)' \Delta h \bar{X}_{T_m+j-1} \right|
$$

$$
+ \max_{i=n_T, \ldots, T_n-n_T} (\log (T_n) n_T)^{1/2} \nu_L^{-1}
$$

$$
\times \left| n_T^{-1} \sum_{j=i+1}^{i+n_T} T_{\psi,T_m+j-1} (\beta^*) - n_T^{-1} \sum_{j=i-n_T+1}^{i} T_{\psi,T_m+j-1} (\beta^*) \right| \tag{S.74}
$$

$$
\triangleq A_{1,h} + A_{2,h} + A_{3,h}.
$$
Our goal is to show that \( A_{l,h} \xrightarrow{p} 0 \) for \( l = 1, 2, 3 \). By Lemma S.A.24 we know that \( A_{3,h} \xrightarrow{p} 0 \). Let us focus on \( A_{1,h} \). Note that

\[
A_{1,h} \leq \max_{i=n_T, \ldots, T_n - n_T} \left( \log (T_n) n_T \right)^{1/2} \nu_L^{-1} \times \left| n_T^{-1} \sum_{j=i+1}^{i+n_T} \left( \left( \beta^* - \hat{\beta}_{T_{m+j-1}} \right) \Delta_h \bar{X}_{T_{m+j-1}} \right) \right|
\]

\[
+ \max_{i=n_T, \ldots, T_n - n_T} \left( \log (T_n) n_T \right)^{1/2} \nu_L^{-1} \times \left| -n_T^{-1} \sum_{j=-n_T+1}^{i} \left( \left( \beta^* - \hat{\beta}_{T_{m+j-1}} \right) \right)' \Delta_h \bar{X}_{T_{m+j-1}} \Delta_h \bar{X}'_{T_{m+j-1}} \right|
\]

We have \( \beta^* - \hat{\beta}_{T_{m+j-1}} = \mu_{\beta,(T_{m+j-1})} / (\log (T_n) n_T)^{1/4} \) and

\[
h^{-1/4} \left( \sum_{j=i+1}^{i+n_T} \Delta_h \bar{X}_{T_{m+j-1}} \Delta_h \bar{X}'_{T_{m+j-1}} - \Sigma_{X,(T_m+i)h} \right) = O_p \left( 1 \right),
\]

by Theorem 13.3.7 in Jacod and Protter (2012). Upon using the property of the trace operator we have

\[
\left( \log (T_n) n_T \right)^{-1/2} n_T^{-1} \sum_{j=i+1}^{i+n_T} \mu_{\beta,(T_{m+j-1})} \left( \Delta_h \bar{X}_{T_{m+j-1}} \Delta_h \bar{X}'_{T_{m+j-1}} - \Sigma_{X,(T_m+i)h} \right) \mu_{\beta,(T_{m+j-1})} = O_p \left( \left( \log (T_n) n_T \right)^{-1/2} h^{1/4} \right).
\]

Thus, the first term on the right-hand side of (S.75) is less than

\[
C_r \left( \frac{1}{\nu_L \varepsilon} \right) \sum_{i=n_T}^{T_n-n_T} \mathbb{E} \left[ \left| n_T^{-1} \sum_{j=i+1}^{i+n_T} \left( \Delta_h \bar{X}_{T_{m+j-1}} \right) \right|^r \right] \leq C_r \left( \frac{1}{\nu_L \varepsilon} \right)^r T_n h^{r/4} \rightarrow 0,
\]

for \( r > 0 \) sufficiently large given that \( h = \mathcal{O}(T^{-1}) = \mathcal{O}(T_n^{-1}) \) and \( \varepsilon > 0 \). The same argument can be applied to the second term of (S.75) which then yields \( A_{1,h} = o_p \left( 1 \right) \). It remains to consider \( A_{2,h} \). It is sufficient to show that

\[
\max_{i=n_T, \ldots, T_n - n_T} \left( \log (T_n) n_T \right)^{1/2} \nu_L^{-1} \left| 2n_T^{-1} \sum_{j=i+1}^{i+n_T} \Delta_h \bar{X}_{T_m+j-1} \right| \overset{p}{\rightarrow} 0.
\]

By Theorem 13.3.7 in Jacod and Protter (2012) we now have \( n_T^{-1/2} \sum_{j=i+1}^{i+n_T} \Delta_h \bar{X}_{T_m+j-1} \left( \beta^* - \hat{\beta}_{T_{m+j-1}} \right)' \Delta_h \bar{X}_{T_m+j-1} < \infty \). By Marlov’s inequality, for any \( \varepsilon > 0 \) we have

\[
\mathbb{P} \left( \max_{i=n_T, \ldots, T_n - n_T} \left( \log (T_n) n_T \right)^{1/2} \nu_L^{-1} \left| 2n_T^{-1} \sum_{j=i+1}^{i+n_T} \Delta_h \bar{X}_{T_m+j-1} \right| > \varepsilon \right) \leq C_r \left( \frac{2}{\nu_L \varepsilon} \right)^r \left( \log (T_n) n_T \right)^{r/2 - r/4} \sum_{i=n_T}^{T_n-n_T} \mathbb{E} \left[ \left| n_T^{-1} \sum_{j=i+1}^{i+n_T} \Delta_h \bar{X}_{T_m+j-1} \right|^r \right]
\]

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$$\leq C_r \left(\frac{2}{\nu L E}\right)^r (\log (T_n) n_T)^{r/2-r/4} O_P \left(T_n n_T^{-r/2}\right) \to 0$$

$r > 0$ sufficiently large. This gives (S.77) and thus $A_{2,h} \xrightarrow{P} 0$ which in turn concludes the proof. □

**Proof of Theorem 5.2.** From Lemma S.A.23-S.A.25 $(\log (T_n) n_T)^{1/2} \left(V_{\max,h} (T_n, \tau) - \hat{MQ}_{\max,h} - (T_n, \tau)\right) \xrightarrow{P} 0$. The result then follows from Corollary 3.1. □

**S.A.6.3 Proof of Corollary 5.1**

*Proof.* Since the statistic $\hat{MQ}_{\max,h} (T_n, \tau)$ admits a limit theorem by Theorem 5.2, it is sufficient to show that, conditional on $\{\sigma_{X,i}\}_{i=0}^\infty$, for all $i = n_T, \ldots, T_n - n_T$,

$$(\log (T_n) n_T)^{1/2} c \left| n_T^{-1} \sum_{j=i+1}^{i+n_T} \zeta_{\mu,j,+} - n_T^{-1} \sum_{j=i-n_T+1}^i \zeta_{\mu,j,-} \right| \to \infty,$$

or that

$$n_T^{-1} \sum_{j=i+1}^{i+n_T} \mu'_{\beta,(T_m+\tau+j-1)h} \sigma_{X,(T_m+\tau+i-1)h} h \mu_{\beta,(T_m+\tau+j-1)h} \tag{S.78}\label{eq:S78}$$

$$= n_T^{-1} \sum_{j=i-n_T+1}^i \mu'_{\beta,(T_m+\tau+j-1)h} \sigma_{X,(T_m+\tau+i-n_T-1)h} h \mu_{\beta,(T_m+\tau+j-1)h}$$

for all $i = n_T, \ldots, T_n - n_T$ does not hold. Suppose by contradiction that (S.78) holds. Due to the block-wise structure of the statistic, we know that $\sigma_{X,(T_m+\tau+j-1)h} = \sigma_{X,(T_m+\tau+i-1)h}$ for all $j = i+1, \ldots, i+n_T$ and $\sigma_{X,(T_m+\tau+j-1)h} \neq \sigma_{X,(T_m+\tau+i-n_T-1)h}$ for all $j = i-n_T, \ldots, i$. Then, (S.78) implies

$$\mu'_{\beta,(T_m+\tau+i-1)h} \sigma_{X,(T_m+\tau+i-1)h} h \mu_{\beta,(T_m+\tau+i-1)h} = \mu'_{\beta,(T_m+\tau+i-n_T-1)h} \sigma_{X,(T_m+\tau+i-n_T-1)h} h \mu_{\beta,(T_m+\tau+i-n_T-1)h}$$

for all $i$. This holds if and only if the process $\{z_i\}_{i=n_T}^{T_n-n_T}$ defined by

$$z_i \triangleq \mu'_{\beta,(T_m+\tau+i-1)h} \sigma_{X,(T_m+\tau+i-1)h} h \mu_{\beta,(T_m+\tau+i-1)h}$$

is constant. But this is a contradiction because it is non-smooth by assumption (if only $\mu_{\beta,(T_m+\tau+i-1)h}$ is non-smooth then $z_i$ is still non-smooth because $\sigma_{X,(T_m+\tau+i-1)h} > 0 \, \mathbb{P}\text{-a.s.}$ by assumption.) □
Figure S-1: Small sample power functions for model P1b: \( Y_t = 2.73 - 0.44X_{t-1} + \delta X_{t-1} \mathbf{1} \{ t > T_0^b \} + \epsilon_t \), where \( X_{t-1} \sim \text{i.i.d.} \mathcal{N}(1, 1) \), \( \epsilon_t \sim \text{i.i.d.} \mathcal{N}(0, 1) \), and \( T_0^b = T\lambda_0 \). The sample size is \( T = 100 \) (left panels) and \( T = 150 \) (right panels). The fractional break date is \( \lambda_0 = 0.7 \) (top panels) and \( \lambda_0 = 0.8 \) (bottom panels). In-sample size is \( T_m = 0.4T \) while out-of-sample size is \( T_n = 0.6T \). The green and blue broken lines correspond to \( B_{\text{max},h} \) and \( Q_{\text{max},h} \), respectively. The red and orange broken lines correspond to the \( t_{\text{stat}} \) of Giacomini and Rossi (2009), respectively, the uncorrected and corrected version.
Figure S-2: small sample power functions for model P1b. The sample size is $T = 200$ (left panels) and $T = 300$ (right panels). The notes of Figure S-1 apply.
Figure S-3: Small sample power functions for model P2: $Y_t = X_{t-1} + \delta X_{t-1} \mathbb{1}\{t > T^0_b\} + e_t$ where $X_t$ is a Gaussian AR(1) with autoregressive coefficient 0.4 and unit variance, and $e_t \sim \text{i.i.d. } \mathcal{N}(0, 0.49)$. The sample size is $T = 100$ (left panel) and $T = 200$ (right panel). The fractional break date is $\lambda_0 = 0.7$ (top panel) and $\lambda_0 = 0.8$ (bottom panel). In-sample size is $T_m = 0.4T$ while out-of-sample size is $T_n = 0.6T$. The green and blue broken lines correspond to $B_{\text{max},h}$ and $Q_{\text{max},h}$, respectively. The red and orange broken lines correspond to the $t^{\text{stat}}$ of Giacomini and Rossi (2009), respectively, the uncorrected and corrected version.
Figure S-4: Small sample power functions for model P4 (recurrent break in mean): $Y_t = \beta_t + e_t$, where $\beta_t$ switches between $\delta$ and 0 every $p$ periods and $e_t \sim \text{i.i.d.} \mathcal{N}(0, 0.64)$. We set $(T, p) = \{(200, 30), (300, 40)\}$. The fractional break date is $\lambda_0 = 0.5$ (top panels) and $\lambda_0 = 0.6$ (bottom panels). In-sample size is $T_m = T\lambda_0$ while out-of-sample size is $T_n = T (1 - \lambda_0)$. The green and blue broken lines correspond to $B_{\text{max,}h}$ and $Q_{\text{max,}h}$, respectively. The red and orange broken lines correspond to the $t_{\text{stat}}$ of Giacomini and Rossi (2009), respectively, the uncorrected and corrected version.
Figure S-5: Small sample power functions for model P3. We set \((T, p) = \{(400, 30), (500, 40)\}\). The fractional break date is \(\lambda_0 = 0.7\) (top panels) and \(\lambda_0 = 0.8\) (bottom panels). In-sample size is \(T_m = T \lambda_0\) while out-of-sample size is \(T_n = T (1 - \lambda_0)\). The notes of Figure S-4 apply.
Figure S-6: Small sample power functions for model P6 (recursive break in variance): $Y_t = \mu + (1 + \beta_t) e_t$, where $\beta_t$ switches between $\delta$ and 0 every $p$ periods and $e_t \sim \text{i.i.d.} \mathcal{N}(0, 0.49)$. We set $(T, p) = \{(200, 30), (300, 40)\}$. The fractional break date is $\lambda_0 = 0.5$ (top panels) and $\lambda_0 = 0.6$ (bottom panels). In-sample size is $T_m = T\lambda_0$ while out-of-sample size is $T_n = T\lambda_0$. The green and blue broken lines correspond to $B_{\text{max},h}$ and $Q_{\text{max},h}$, respectively. The red and orange broken lines correspond to the $t_{\text{stat}}$ of Giacomini and Rossi (2009), respectively, the uncorrected and corrected version.
Figure S-7: Small sample power functions for model P4 (single break in variance). The sample size is \( T = 400 \) (left panels) and \( T = 500 \) (right panels). The notes of Figure 3 apply.
Figure S-8: Small sample power functions for model P5 (recursive break in variance). We set \((T, p) = \{(200, 30), (300, 40)\}\). The fractional break date is \(\lambda_0 = 0.7\) (top panels) and \(\lambda_0 = 0.8\) (bottom panels). The notes of Figure S-6 apply.
Figure S-9: Small sample power functions for model P5 (recurrent break in variance). The sample size is $T = 300$ (left panels) and $T = 400$ (right panels). The fractional break date is $\lambda_0 = 0.6$ (top panels) and $\lambda_0 = 0.7$ (bottom panels). The notes of Figure S-6 apply.
Figure S-10: Small sample power functions for model P6 (lagged dependent variables): $Y_t = \delta \mathbf{1}\{ t > T_0 \} + 0.3Y_{t-1} + \epsilon_t$, $\epsilon_t \sim \text{i.i.d.}\mathcal{N}(0, 0.49)$. The sample size is $T = 200$ (left panels) and $T = 300$ (right panels). The fractional break date is $\lambda_0 = 0.7$ (top panels) and $\lambda_0 = 0.8$ (bottom panels). In-sample size is $T_m = 0.4T$ while out-of-sample size is $T_n = 0.6T$. The green and blue broken lines correspond to $B_{\text{max},h}$ and $Q_{\text{max},h}$, respectively. The red and orange broken lines correspond to the $t_{\text{stat}}$ of Giacomini and Rossi (2009), respectively, the uncorrected and corrected version.
Figure S-11: Small sample power functions for model P8 (autocorrelated errors): $Y_t = 1 + X_{t-1} + \delta X_{t-1} \mathbb{1}_{\{t > T_0^h\}} + e_t$, where $X_{t-1} \sim \text{i.i.d.} \mathcal{N}(0, 1.4)$ and $e_t = 0.4u_{t-1} + u_t$, $u_t \sim \text{i.i.d.} \mathcal{N}(0, 1)$. The sample size is $T = 200$ (left panels) and $T = 300$ (right panels). The fractional break date is $\lambda_0 = 0.7$ (top panels) and $\lambda_0 = 0.8$ (bottom panels). In-sample size is $T_m = 0.5T$ while out-of-sample size is $T_n = 0.5T$. The green and blue broken lines correspond to $B_{\text{max},h}$ and $Q_{\text{max},h}$, respectively. The red and orange broken lines correspond to the $t^{\text{stat}}$ of Giacomini and Rossi (2009), respectively, the uncorrected and corrected version.
Figure S-12: Small sample power functions for model P1b with short-term instability: $Y_t = 2.73 - 0.44X_{t-1} + \delta X_{t-1}1\{T_0^b < t \leq T_0^b + p\} + \epsilon_t$, where $X_{t-1} \sim \text{i.i.d.}\mathcal{N}(1, 1)$, $\epsilon_t \sim \text{i.i.d.}\mathcal{N}(0, 1)$, and $T_0^b = T\lambda_0$. We set $(T, p) = \{(100, 20), (150, 25)\}$. The fractional break date is $\lambda_0 = 0.7$ (top panels) and $\lambda_0 = 0.8$ (bottom panels). In-sample size is $T_m = 0.4T$ while out-of-sample size is $T_n = 0.6T$. The green and blue broken lines correspond to $B_{\text{max},h}$ and $Q_{\text{max},h}$, respectively. The red and orange broken lines correspond to the $t_{\text{stat}}$ of Giacomini and Rossi (2009), respectively, the uncorrected and corrected version.
Figure S-13: Small sample power functions for model P2 with short-term instability: \( Y_t = X_{t-1} + \delta X_{t-1} \mathbf{1} \{ T_0^B < t \leq T_0^B + p \} + \epsilon_t \), where \( X_{t-1} \) is a Gaussian AR(1) with autoregressive coefficient 0.4 and unit variance, and \( \epsilon_t \sim \text{i.i.d. } \mathcal{N}(0, 0.49) \), and \( T_0^B = T \lambda_0 \). We set \( (T, p) = \{(100, 20), (200, 30)\} \). The fractional break date is \( \lambda_0 = 0.7 \) (top panels) and \( \lambda_0 = 0.8 \) (bottom panels). In-sample size is \( T_m = 0.4T \) while out-of-sample size is \( T_n = 0.6T \). The green and blue broken lines correspond to \( B_{\text{max}, h} \) and \( Q_{\text{max}, h} \), respectively. The red and orange broken lines correspond to the \( t_{\text{stat}} \) of Giacomini and Rossi (2009), respectively, the uncorrected and corrected version.
S.C  Additional Monte Carlo Studies

This section extends the small-sample analysis of Section 7 to alternative forecasting schemes (recursive and rolling scheme) and to the Linex loss function. Table S-4-S-5 report the empirical sizes at significance level $\alpha = 0.05$ for model S1 for all the tests considered. Let us first consider the case regarding the recursive scheme. The statistic $Q_{\text{max},h}$ appears to be well-sized, even though slightly liberal when the in-sample window is either too small or too large. The statistics $B_{\text{max},h}$ and $\text{MQ}_{\text{max},h}$ seem to display reasonable size control, though they tend to overreject quite a bit whereas $\text{MB}_{\text{max},h}$ is excessively oversized and therefore not comparable with the other tests. Turning to the tests of Giacomini and Rossi (2009), we note that the uncorrected version $t_{\text{stat}}$ performs well while the corrected version is undersized, though not excessively, for all sample sizes. Overall, we can conclude that all the statistics, with exception of $\text{MB}_{\text{max},h}$, are comparable in term of size and consequently it is fair to compare their power properties.

We present the empirical power of the tests in Table S-6-S-9. We consider model P1 with either long-lasting and short-lasting instabilities. Under recursive scheme, Table S-6 shows that all the statistics possess good power against break in a regression coefficient. Since the instability begins at about middle sample (i.e., $\lambda_0 = 0.6$) and lasts for about 40% of the total sample the tests of Giacomini and Rossi (2009) tend to have higher power when the sample size is large. However, when the instability only lasts for few consecutive periods (cf. Table S-7) the gains in statistical power of our tests are substantial. This is equivalent to what observed in the main paper regarding the fixed scheme and it extends to the recursive and rolling scheme (cf. Table S-9). Table S-8 reports the power comparison for the rolling scheme. All tests display good power when the magnitude of the break is large. When $\delta = 0.5$, 1 the statistic $B_{\text{max},h}$ dominates the statistics of Giacomini and Rossi (2009), especially the magnitude of the break is small (i.e., $\delta = 0.5$)—which constitutes a highly relevant case in practice.

Overall, we confirm the same observations relevant for the fixed scheme considered in the main text. Our test statistics show reasonable size control, the only exception is $\text{MB}_{\text{max},h}$ which turns out to be oversized. In terms of power properties, all tests display good power while there are substantial power gains relative to existing methods especially when the instability (i) is short-lasting and/or (ii) is located toward the tail of the out-of-sample.
### Table S-1: Empirical small sample size of forecast instability tests based on model S3

| $\alpha = 0.05$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat,c}}$ (corrected) | GR (2009) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $MB_{\text{max},h}$ | $MQ_{\text{max},h}$ |
|-----------------|----------------------------------|----------------------------------|----------------|-----------------|------------------|-----------------|------------------|
| $T = 100$       | $T_m = 25$  | $T_n = 75$  | 0.049          | 0.019           | 0.086            | 0.098           | 0.090           | 0.089           |
|                 | 50          | 50            | 0.069          | 0.024           | 0.058            | 0.072           | 0.083           | 0.067           |
|                 | 75          | 25            | 0.039          | 0.016           | 0.081            | 0.111           | 0.092           | 0.091           |
| $T = 200$       | 50          | 150           | 0.049          | 0.025           | 0.076            | 0.072           | 0.138           | 0.089           |
|                 | 100         | 100           | 0.057          | 0.026           | 0.070            | 0.073           | 0.106           | 0.068           |
|                 | 150         | 50            | 0.075          | 0.020           | 0.055            | 0.070           | 0.082           | 0.070           |
| $T = 300$       | 75          | 225           | 0.050          | 0.029           | 0.058            | 0.036           | 0.102           | 0.044           |
|                 | 150         | 150           | 0.059          | 0.032           | 0.077            | 0.072           | 0.144           | 0.086           |
|                 | 225         | 75            | 0.065          | 0.025           | 0.096            | 0.103           | 0.152           | 0.123           |
| $T = 400$       | 100         | 300           | 0.054          | 0.032           | 0.061            | 0.041           | 0.123           | 0.046           |
|                 | 200         | 200           | 0.051          | 0.035           | 0.065            | 0.048           | 0.111           | 0.052           |
|                 | 300         | 100           | 0.068          | 0.031           | 0.067            | 0.063           | 0.115           | 0.074           |

| $\alpha = 0.10$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat,c}}$ (corrected) | GR (2009) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $MB_{\text{max},h}$ | $MQ_{\text{max},h}$ |
|-----------------|----------------------------------|----------------------------------|----------------|-----------------|------------------|-----------------|------------------|
| $T = 100$       | $T_m = 25$  | $T_n = 75$  | 0.109          | 0.069           | 0.136            | 0.152           | 0.191           | 0.165           |
|                 | 50          | 50            | 0.107          | 0.069           | 0.095            | 0.118           | 0.118           | 0.095           |
|                 | 75          | 25            | 0.134          | 0.060           | 0.112            | 0.152           | 0.125           | 0.128           |
| $T = 200$       | 50          | 150           | 0.100          | 0.078           | 0.125            | 0.113           | 0.199           | 0.133           |
|                 | 100         | 100           | 0.106          | 0.073           | 0.108            | 0.111           | 0.160           | 0.108           |
|                 | 150         | 50            | 0.101          | 0.078           | 0.101            | 0.105           | 0.112           | 0.091           |
| $T = 300$       | 75          | 225           | 0.102          | 0.081           | 0.103            | 0.071           | 0.159           | 0.077           |
|                 | 150         | 150           | 0.111          | 0.079           | 0.119            | 0.112           | 0.189           | 0.129           |
|                 | 225         | 75            | 0.114          | 0.068           | 0.144            | 0.159           | 0.197           | 0.170           |
| $T = 400$       | 100         | 300           | 0.097          | 0.082           | 0.109            | 0.079           | 0.193           | 0.096           |
|                 | 200         | 200           | 0.089          | 0.106           | 0.075            | 0.079           | 0.171           | 0.088           |
|                 | 300         | 100           | 0.112          | 0.079           | 0.104            | 0.110           | 0.164           | 0.104           |

Model S3. The notes of Table 1 apply.
### Table S-2: Empirical small sample size of forecast instability tests based on model S4

| $\alpha = 0.05$ | $T_m$ | $T_n$ | $t_{stat}$ (uncorrected) | $t_{stat,c}$ (corrected) | $B_{max,h}$ | $Q_{max,h}$ | $M_{B_{max,h}}$ | $M_{Q_{max,h}}$ |
|-----------------|-------|-------|--------------------------|--------------------------|------------|------------|----------------|----------------|
| $T = 100$       | 25    | 75    | 0.055                    | 0.016                    | 0.089      | 0.099      | 0.149          | 0.122          |
|                 | 50    | 50    | 0.077                    | 0.023                    | 0.064      | 0.073      | 0.074          | 0.063          |
|                 | 75    | 25    | 0.066                    | 0.027                    | 0.042      | 0.044      | 0.086          | 0.092          |
| $T = 200$       | 50    | 150   | 0.055                    | 0.028                    | 0.112      | 0.121      | 0.141          | 0.089          |
|                 | 100   | 100   | 0.060                    | 0.029                    | 0.069      | 0.063      | 0.110          | 0.069          |
|                 | 150   | 50    | 0.084                    | 0.020                    | 0.063      | 0.068      | 0.075          | 0.063          |
| $T = 300$       | 75    | 225   | 0.056                    | 0.032                    | 0.067      | 0.040      | 0.108          | 0.048          |
|                 | 150   | 150   | 0.047                    | 0.029                    | 0.062      | 0.039      | 0.144          | 0.085          |
|                 | 225   | 75    | 0.059                    | 0.034                    | 0.045      | 0.020      | 0.147          | 0.120          |
| $T = 400$       | 100   | 300   | 0.051                    | 0.036                    | 0.076      | 0.038      | 0.132          | 0.055          |
|                 | 200   | 200   | 0.052                    | 0.034                    | 0.052      | 0.027      | 0.121          | 0.054          |
|                 | 300   | 100   | 0.051                    | 0.037                    | 0.032      | 0.013      | 0.109          | 0.071          |

| $\alpha = 0.10$ | $T_m$ | $T_n$ | $t_{stat}$ (uncorrected) | $t_{stat,c}$ (corrected) | $B_{max,h}$ | $Q_{max,h}$ | $M_{B_{max,h}}$ | $M_{Q_{max,h}}$ |
|-----------------|-------|-------|--------------------------|--------------------------|------------|------------|----------------|----------------|
| $T = 100$       | 25    | 75    | 0.054                    | 0.072                    | 0.143      | 0.154      | 0.198          | 0.168          |
|                 | 50    | 50    | 0.070                    | 0.065                    | 0.100      | 0.120      | 0.115          | 0.096          |
|                 | 75    | 25    | 0.062                    | 0.069                    | 0.069      | 0.073      | 0.124          | 0.125          |
| $T = 200$       | 50    | 150   | 0.053                    | 0.073                    | 0.120      | 0.118      | 0.202          | 0.137          |
|                 | 100   | 100   | 0.064                    | 0.078                    | 0.107      | 0.104      | 0.152          | 0.106          |
|                 | 150   | 50    | 0.082                    | 0.069                    | 0.095      | 0.119      | 0.118          | 0.097          |
| $T = 300$       | 75    | 225   | 0.046                    | 0.084                    | 0.111      | 0.081      | 0.179          | 0.085          |
|                 | 150   | 150   | 0.060                    | 0.073                    | 0.131      | 0.118      | 0.207          | 0.134          |
|                 | 225   | 75    | 0.068                    | 0.078                    | 0.129      | 0.142      | 0.184          | 0.160          |
| $T = 400$       | 100   | 300   | 0.045                    | 0.085                    | 0.122      | 0.084      | 0.192          | 0.091          |
|                 | 200   | 200   | 0.056                    | 0.088                    | 0.118      | 0.084      | 0.178          | 0.089          |
|                 | 300   | 100   | 0.060                    | 0.086                    | 0.072      | 0.050      | 0.159          | 0.111          |

Model S4. The notes of Table 1 apply.
### Table S-3: Empirical small sample size of forecast instability tests based on model S6

| $\alpha = 0.05$ | $T_m$ | $T_n$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $\text{MB}_{\text{max},h}$ | $\text{MQ}_{\text{max},h}$ |
|-----------------|-------|-------|---------------------------------|---------------------------------|----------------|----------------|----------------|----------------|
| $T = 100$       | 25    | 75    | 0.171                           | 0.005                           | 0.146          | 0.081          | 0.237          | 0.040          |
|                 | 50    | 50    | 0.163                           | 0.005                           | 0.142          | 0.076          | 0.134          | 0.020          |
|                 | 75    | 25    | 0.177                           | 0.035                           | 0.113          | 0.052          | 0.108          | 0.033          |
| $T = 200$       | 50    | 150   | 0.170                           | 0.021                           | 0.155          | 0.059          | 0.087          | 0.022          |
|                 | 100   | 100   | 0.165                           | 0.032                           | 0.121          | 0.045          | 0.193          | 0.024          |
|                 | 150   | 50    | 0.182                           | 0.042                           | 0.092          | 0.044          | 0.127          | 0.018          |
| $T = 300$       | 75    | 225   | 0.172                           | 0.028                           | 0.129          | 0.044          | 0.216          | 0.025          |
|                 | 150   | 150   | 0.164                           | 0.046                           | 0.148          | 0.040          | 0.251          | 0.021          |
|                 | 225   | 75    | 0.168                           | 0.049                           | 0.138          | 0.045          | 0.228          | 0.021          |
| $T = 400$       | 100   | 300   | 0.161                           | 0.034                           | 0.135          | 0.041          | 0.280          | 0.023          |
|                 | 200   | 200   | 0.155                           | 0.046                           | 0.128          | 0.038          | 0.228          | 0.021          |
|                 | 300   | 100   | 0.185                           | 0.061                           | 0.126          | 0.036          | 0.184          | 0.018          |

| $\alpha = 0.10$ | $T_m$ | $T_n$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $\text{MB}_{\text{max},h}$ | $\text{MQ}_{\text{max},h}$ |
|-----------------|-------|-------|---------------------------------|---------------------------------|----------------|----------------|----------------|----------------|
| $T = 100$       | 25    | 75    | 0.251                           | 0.017                           | 0.205          | 0.152          | 0.294          | 0.073          |
|                 | 50    | 50    | 0.235                           | 0.060                           | 0.179          | 0.105          | 0.178          | 0.052          |
|                 | 75    | 25    | 0.252                           | 0.063                           | 0.139          | 0.105          | 0.189          | 0.047          |
| $T = 200$       | 50    | 150   | 0.249                           | 0.041                           | 0.219          | 0.127          | 0.332          | 0.063          |
|                 | 100   | 100   | 0.226                           | 0.060                           | 0.179          | 0.106          | 0.254          | 0.049          |
|                 | 150   | 50    | 0.186                           | 0.057                           | 0.130          | 0.095          | 0.294          | 0.038          |
| $T = 300$       | 75    | 225   | 0.241                           | 0.053                           | 0.186          | 0.113          | 0.301          | 0.060          |
|                 | 150   | 150   | 0.233                           | 0.067                           | 0.198          | 0.101          | 0.329          | 0.041          |
|                 | 225   | 75    | 0.240                           | 0.092                           | 0.188          | 0.093          | 0.288          | 0.037          |
| $T = 400$       | 100   | 300   | 0.227                           | 0.061                           | 0.222          | 0.098          | 0.359          | 0.056          |
|                 | 200   | 200   | 0.229                           | 0.091                           | 0.194          | 0.090          | 0.302          | 0.046          |
|                 | 300   | 100   | 0.236                           | 0.093                           | 0.168          | 0.091          | 0.245          | 0.040          |

Model S6. The notes of Table 1 apply.
### Table S-4: Empirical small sample size of forecast instability tests based on model S1; Recursive scheme

| T  | Tm   | Tn   | t\text{stat} (uncorrected) | t\text{stat,c} (corrected) | B_{max,h} | Q_{max,h} | M_{B, max,h} | M_{Q, max,h} |
|----|------|------|-----------------------------|----------------------------|-----------|-----------|--------------|--------------|
| 100| 25   | 75   | 0.060                       | 0.028                      | 0.108     | 0.071     | 0.194        | 0.111        |
|    | 50   | 50   | 0.067                       | 0.026                      | 0.060     | 0.075     | 0.120        | 0.067        |
|    | 75   | 25   | 0.079                       | 0.021                      | 0.109     | 0.060     | 0.159        | 0.069        |
| 200| 50   | 150  | 0.053                       | 0.033                      | 0.108     | 0.071     | 0.182        | 0.096        |
|    | 100  | 100  | 0.052                       | 0.028                      | 0.089     | 0.064     | 0.146        | 0.076        |
|    | 150  | 50   | 0.055                       | 0.022                      | 0.079     | 0.054     | 0.118        | 0.055        |
| 300| 75   | 225  | 0.063                       | 0.027                      | 0.084     | 0.042     | 0.197        | 0.091        |
|    | 150  | 150  | 0.049                       | 0.033                      | 0.028     | 0.096     | 0.178        | 0.089        |
|    | 225  | 75   | 0.063                       | 0.026                      | 0.108     | 0.072     | 0.197        | 0.091        |
| 400| 100  | 300  | 0.059                       | 0.029                      | 0.078     | 0.050     | 0.156        | 0.080        |
|    | 200  | 200  | 0.055                       | 0.048                      | 0.064     | 0.048     | 0.143        | 0.065        |
|    | 300  | 100  | 0.059                       | 0.025                      | 0.135     | 0.037     | 0.24         | 0.112        |

The table reports the rejection probabilities of 5% -level tests proposed in the paper and those proposed by Giacomini and Rossi (2009) [(abbreviated GR (2009)] for model S1. For all methods we use the recursive forecasting scheme. T = T_m + T_n, where T is the total sample size, T_m is the size of the in-sample window and T_n is the size of the out-of-sample window. m_T is set equal to the smallest integer allowed by Condition 1. Based on 5,000 replications.

### Table S-5: Empirical small sample size of forecast instability tests based on model S1; Rolling scheme

| T  | Tm   | Tn   | t\text{stat} (uncorrected) | t\text{stat,c} (corrected) | B_{max,h} | Q_{max,h} | M_{B, max,h} | M_{Q, max,h} |
|----|------|------|-----------------------------|----------------------------|-----------|-----------|--------------|--------------|
| 100| 25   | 75   | 0.514                       | 0.018                      | 0.126     | 0.128     | 0.185        | 0.128        |
|    | 50   | 50   | 0.064                       | 0.008                      | 0.090     | 0.089     | 0.133        | 0.075        |
|    | 75   | 25   | 0.072                       | 0.019                      | 0.131     | 0.104     | 0.165        | 0.064        |
| 200| 50   | 150  | 0.364                       | 0.038                      | 0.094     | 0.078     | 0.177        | 0.063        |
|    | 100  | 100  | 0.058                       | 0.017                      | 0.082     | 0.073     | 0.132        | 0.064        |
|    | 150  | 50   | 0.078                       | 0.023                      | 0.087     | 0.071     | 0.112        | 0.038        |
| 300| 75   | 225  | 0.230                       | 0.010                      | 0.074     | 0.055     | 0.131        | 0.095        |
|    | 150  | 150  | 0.054                       | 0.036                      | 0.082     | 0.061     | 0.180        | 0.084        |
|    | 225  | 75   | 0.065                       | 0.026                      | 0.118     | 0.072     | 0.193        | 0.069        |
| 400| 100  | 300  | 0.168                       | 0.020                      | 0.084     | 0.060     | 0.099        | 0.163        |
|    | 200  | 200  | 0.057                       | 0.036                      | 0.088     | 0.053     | 0.148        | 0.064        |
|    | 300  | 100  | 0.060                       | 0.023                      | 0.088     | 0.083     | 0.059        | 0.057        |

Model S1; rolling scheme. The notes of Table S-4 apply.
Table S-6: Empirical small sample power of forecast instability tests based on model P1; Recursive scheme

| $\delta = 0.5$ | $T_m$ | $T_n$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $MB_{\text{max},h}$ | $MQ_{\text{max},h}$ |
|----------------|-------|-------|-------------------------------|-------------------------------|-----------------|-----------------|-----------------|-----------------|
| $T = 100$      | 50    | 50    | 0.071                         | 0.056                         | 0.083           | 0.063           | 0.103           | 0.064           |
| $T = 200$      | 100   | 100   | 0.080                         | 0.094                         | 0.101           | 0.069           | 0.161           | 0.099           |
| $T = 300$      | 150   | 150   | 0.098                         | 0.127                         | 0.107           | 0.086           | 0.208           | 0.136           |
| $T = 400$      | 200   | 200   | 0.095                         | 0.125                         | 0.093           | 0.062           | 0.188           | 0.106           |

| $\delta = 1$  | $T_m$ | $T_n$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $MB_{\text{max},h}$ | $MQ_{\text{max},h}$ |
|----------------|-------|-------|-------------------------------|-------------------------------|-----------------|-----------------|-----------------|-----------------|
| $T = 100$      | 50    | 50    | 0.244                         | 0.253                         | 0.206           | 0.151           | 0.167           | 0.117           |
| $T = 200$      | 100   | 100   | 0.445                         | 0.401                         | 0.260           | 0.223           | 0.403           | 0.298           |
| $T = 300$      | 150   | 150   | 0.615                         | 0.679                         | 0.218           | 0.249           | 0.508           | 0.400           |
| $T = 400$      | 200   | 200   | 0.745                         | 0.814                         | 0.240           | 0.268           | 0.588           | 0.432           |

| $\delta = 1.5$| $T_m$ | $T_n$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $MB_{\text{max},h}$ | $MQ_{\text{max},h}$ |
|----------------|-------|-------|-------------------------------|-------------------------------|-----------------|-----------------|-----------------|-----------------|
| $T = 100$      | 50    | 50    | 0.706                         | 0.720                         | 0.483           | 0.350           | 0.308           | 0.208           |
| $T = 200$      | 100   | 100   | 0.946                         | 0.958                         | 0.660           | 0.603           | 0.821           | 0.701           |
| $T = 300$      | 150   | 150   | 0.995                         | 0.997                         | 0.495           | 0.666           | 0.909           | 0.833           |
| $T = 400$      | 200   | 200   | 1                             | 1                             | 0.606           | 0.782           | 0.974           | 0.921           |

| $\delta = 2$  | $T_m$ | $T_n$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $MB_{\text{max},h}$ | $MQ_{\text{max},h}$ |
|----------------|-------|-------|-------------------------------|-------------------------------|-----------------|-----------------|-----------------|-----------------|
| $T = 100$      | 50    | 50    | 0.968                         | 0.970                         | 0.780           | 0.566           | 0.504           | 0.301           |
| $T = 200$      | 100   | 100   | 1                             | 1                             | 0.943           | 0.906           | 0.987           | 0.940           |
| $T = 300$      | 150   | 150   | 1                             | 1                             | 0.846           | 0.948           | 0.997           | 0.989           |
| $T = 400$      | 200   | 200   | 1                             | 1                             | 0.924           | 0.990           | 1               | 1               |

The table reports the rejection probabilities of 5%-level tests proposed in the paper and those proposed by Giacomini and Rossi (2009) [(abbreviated GR (2009)] for model P1 with short-term instability. For all methods we use the rolling forecasting scheme. $T = T_m + T_n$, where $T$ is the total sample size, $T_m$ is the size of the in-sample window and $T_n$ is the size of the out-of-sample window. $\lambda_0 = 0.6$ and $m_T$ is set equal to the smallest integer allowed by Condition 1. Based on 5,000 replications.
Table S-7: Empirical small sample power of forecast instability tests based on model P1; Recursive scheme; Short-term instability

| δ  | T_m | T_n | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $M_{\text{Bmax},h}$ | $M_{\text{Qmax},h}$ |
|----|-----|-----|-------------------------------|---------------------------------|-----------------|-----------------|-----------------|-----------------|
| 0.5| 100 | 50  | 50  | 0.060 | 0.031 | 0.064 | 0.053 | 0.071 | 0.039 |
|    | 200 | 100 | 100 | 0.053 | 0.044 | 0.091 | 0.052 | 0.133 | 0.071 |
|    | 300 | 150 | 150 | 0.058 | 0.053 | 0.093 | 0.051 | 0.184 | 0.106 |
|    | 400 | 200 | 200 | 0.049 | 0.050 | 0.075 | 0.049 | 0.154 | 0.097 |
| 1  | 100 | 50  | 50  | 0.058 | 0.049 | 0.097 | 0.059 | 0.057 | 0.0294 |
|    | 200 | 100 | 100 | 0.073 | 0.088 | 0.244 | 0.155 | 0.269 | 0.179 |
|    | 300 | 150 | 150 | 0.088 | 0.136 | 0.157 | 0.132 | 0.455 | 0.348 |
|    | 400 | 200 | 200 | 0.100 | 0.128 | 0.155 | 0.117 | 0.486 | 0.353 |
| 1.5| 100 | 50  | 50  | 0.086 | 0.090 | 0.193 | 0.112 | 0.072 | 0.040 |
|    | 200 | 100 | 100 | 0.156 | 0.203 | 0.629 | 0.487 | 0.641 | 0.493 |
|    | 300 | 150 | 150 | 0.254 | 0.351 | 0.398 | 0.371 | 0.865 | 0.758 |
|    | 400 | 200 | 200 | 0.329 | 0.415 | 0.432 | 0.434 | 0.942 | 0.870 |
| 2  | 100 | 50  | 50  | 0.130 | 0.151 | 0.347 | 0.181 | 0.107 | 0.052 |
|    | 200 | 100 | 100 | 0.325 | 0.415 | 0.930 | 0.842 | 0.928 | 0.839 |
|    | 300 | 150 | 150 | 0.533 | 0.652 | 0.736 | 0.692 | 0.993 | 0.97  |
|    | 400 | 200 | 200 | 0.724 | 0.819 | 0.809 | 0.805 | 1    | 1    |

Model P1; recursive scheme. The notes of Table S-6 apply.
Table S-8: Empirical small sample power of forecast instability tests based on model P1; Rolling scheme

| $\delta$ = 0.5 | $T_m$ | $T_n$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $MB_{\text{max},h}$ | $MQ_{\text{max},h}$ |
|----------------|------|------|-------------------------------|-------------------------------|-------------------|-------------------|-------------------|-------------------|
| $T$ = 100      | 50   | 50   | 0.158                         | 0.059                         | 0.429             | 0.063             | 0.168             | 0.097             |
| $T$ = 200      | 100  | 100  | 0.144                         | 0.089                         | 0.643             | 0.079             | 0.201             | 0.112             |
| $T$ = 300      | 150  | 150  | 0.098                         | 0.104                         | 0.777             | 0.093             | 0.246             | 0.136             |
| $T$ = 400      | 200  | 200  | 0.052                         | 0.033                         | 0.809             | 0.073             | 0.146             | 0.068             |

| $\delta$ = 1  | $T_m$ | $T_n$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $MB_{\text{max},h}$ | $MQ_{\text{max},h}$ |
|----------------|------|------|-------------------------------|-------------------------------|-------------------|-------------------|-------------------|-------------------|
| $T$ = 100      | 50   | 50   | 0.342                         | 0.165                         | 0.485             | 0.107             | 0.362             | 0.259             |
| $T$ = 200      | 100  | 100  | 0.439                         | 0.329                         | 0.717             | 0.218             | 0.456             | 0.324             |
| $T$ = 300      | 150  | 150  | 0.629                         | 0.571                         | 0.812             | 0.0027            | 0.662             | 0.541             |
| $T$ = 400      | 200  | 200  | 0.878                         | 0.876                         | 0.855             | 0.299             | 0.827             | 0.700             |

| $\delta$ = 1.5| $T_m$ | $T_n$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $MB_{\text{max},h}$ | $MQ_{\text{max},h}$ |
|----------------|------|------|-------------------------------|-------------------------------|-------------------|-------------------|-------------------|-------------------|
| $T$ = 100      | 50   | 50   | 0.686                         | 0.448                         | 0.623             | 0.236             | 0.662             | 0.541             |
| $T$ = 200      | 100  | 100  | 0.879                         | 0.801                         | 0.882             | 0.620             | 0.827             | 0.700             |
| $T$ = 300      | 150  | 150  | 0.968                         | 0.974                         | 0.902             | 0.685             | 0.895             | 0.778             |
| $T$ = 400      | 200  | 200  | 0.991                         | 0.997                         | 0.932             | 0.801             | 0.930             | 0.843             |

| $\delta$ = 2  | $T_m$ | $T_n$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $MB_{\text{max},h}$ | $MQ_{\text{max},h}$ |
|----------------|------|------|-------------------------------|-------------------------------|-------------------|-------------------|-------------------|-------------------|
| $T$ = 100      | 50   | 50   | 0.929                         | 0.800                         | 0.768             | 0.398             | 0.761             | 0.398             |
| $T$ = 200      | 100  | 100  | 0.997                         | 0.993                         | 0.977             | 0.924             | 0.977             | 0.925             |
| $T$ = 300      | 150  | 150  | 1                            | 0.999                         | 0.996             | 0.958             | 0.969             | 0.9580            |
| $T$ = 400      | 200  | 200  | 1                            | 1                            | 0.980             | 0.987             | 0.980             | 0.988             |

The table reports the rejection probabilities of 95%-level tests proposed in the paper and those proposed by Giacomini and Rossi (2009) [(abbreviated GR (2009)] for model P1. For all methods we use the rolling forecasting scheme. $T = T_m + T_n$, where $T$ is the total sample size, $T_m$ is the size of the in-sample window and $T_n$ is the size of the out-of-sample window. $\lambda_0 = 0.6$ and $m_T$ is set equal to the smallest integer allowed by Condition 1. Based on 5,000 replications.
Table S-9: Empirical small sample power of forecast instability tests based on model P1; Rolling scheme; Short-term instability

| $\delta = 0.5$ | $T_m$ | $T_n$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $M_{B_{\text{max},h}}$ | $M_{Q_{\text{max},h}}$ |
|---------------|-------|-------|-------------------------------|-------------------------------|----------------|----------------|----------------|----------------|
| $T = 100$     | 50    | 50    | 0.159                         | 0.059                         | 0.429          | 0.063          | 0.208          | 0.102          |
| $T = 200$     | 100   | 100   | 0.114                         | 0.062                         | 0.631          | 0.067          | 0.132          | 0.072          |
| $T = 300$     | 150   | 150   | 0.106                         | 0.070                         | 0.778          | 0.097          | 0.182          | 0.095          |
| $T = 400$     | 200   | 200   | 0.055                         | 0.048                         |                |                |                |                |

| $\delta = 1$ | $T_m$ | $T_n$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $M_{B_{\text{max},h}}$ | $M_{Q_{\text{max},h}}$ |
|---------------|-------|-------|-------------------------------|-------------------------------|----------------|----------------|----------------|----------------|
| $T = 100$     | 50    | 50    | 0.169                         | 0.071                         | 0.431          | 0.064          | 0.267          | 0.151          |
| $T = 200$     | 100   | 100   | 0.170                         | 0.112                         | 0.702          | 0.193          | 0.261          | 0.149          |
| $T = 300$     | 150   | 150   | 0.186                         | 0.153                         | 0.789          | 0.186          | 0.456          | 0.304          |
| $T = 400$     | 200   | 200   | 0.198                         | 0.173                         | 0.816          | 0.184          | 0.459          | 0.294          |

| $\delta = 1.5$ | $T_m$ | $T_n$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $M_{B_{\text{max},h}}$ | $M_{Q_{\text{max},h}}$ |
|---------------|-------|-------|-------------------------------|-------------------------------|----------------|----------------|----------------|----------------|
| $T = 100$     | 50    | 50    | 0.229                         | 0.117                         | 0.469          | 0.093          | 0.462          | 0.320          |
| $T = 200$     | 100   | 100   | 0.297                         | 0.231                         | 0.875          | 0.601          | 0.621          | 0.408          |
| $T = 300$     | 150   | 150   | 0.402                         | 0.355                         | 0.849          | 0.521          | 0.858          | 0.727          |
| $T = 400$     | 200   | 200   | 0.454                         | 0.439                         | 0.885          | 0.625          | 0.919          | 0.781          |

| $\delta = 2$ | $T_m$ | $T_n$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $M_{B_{\text{max},h}}$ | $M_{Q_{\text{max},h}}$ |
|---------------|-------|-------|-------------------------------|-------------------------------|----------------|----------------|----------------|----------------|
| $T = 100$     | 50    | 50    | 0.328                         | 0.1842                        | 0.518          | 0.149          | 0.854          | 0.748          |
| $T = 200$     | 100   | 100   | 0.508                         | 0.428                         | 0.977          | 0.925          | 0.965          | 0.904          |
| $T = 300$     | 150   | 150   | 0.687                         | 0.645                         | 0.940          | 0.880          | 0.996          | 0.976          |
| $T = 400$     | 200   | 200   | 0.991                         | 0.994                         | 0.966          | 0.952          | 0.999          | 0.990          |

Model P1; rolling scheme. The notes of Table S-8 apply.
Table S-10: Empirical small sample size of forecast instability tests based on model S1; Linex Loss

| $T_m$ | $T_n$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $M_{B_{\text{max},h}}$ | $M_{Q_{\text{max},h}}$ |
|-------|-------|-------------------------------|-------------------------------|----------------|----------------|----------------|----------------|
| 100   | 25    | 0.051                         | 0.010                         | 0.263          | 0.166          | 0.266          | 0.170          |
|       | 50    | 0.083                         | 0.015                         | 0.119          | 0.069          | 0.152          | 0.103          |
|       | 75    | 0.116                         | 0.011                         | 0.143          | 0.098          | 0.164          | 0.128          |
| 200   | 50    | 0.058                         | 0.013                         | 0.185          | 0.091          | 0.309          | 0.158          |
|       | 100   | 0.074                         | 0.016                         | 0.165          | 0.088          | 0.236          | 0.132          |
|       | 150   | 0.102                         | 0.011                         | 0.122          | 0.074          | 0.157          | 0.097          |
| 300   | 75    | 0.057                         | 0.023                         | 0.168          | 0.070          | 0.293          | 0.123          |
|       | 150   | 0.069                         | 0.019                         | 0.192          | 0.094          | 0.322          | 0.164          |
|       | 225   | 0.102                         | 0.015                         | 0.195          | 0.112          | 0.261          | 0.172          |
|       | 100   | 0.058                         | 0.026                         | 0.210          | 0.080          | 0.354          | 0.129          |
|       | 200   | 0.068                         | 0.024                         | 0.183          | 0.083          | 0.297          | 0.137          |
|       | 300   | 0.089                         | 0.016                         | 0.152          | 0.080          | 0.237          | 0.133          |

\(\alpha = 0.10\)

| $T_m$ | $T_n$ | $t_{\text{stat}}$ (uncorrected) | $t_{\text{stat},c}$ (corrected) | $B_{\text{max},h}$ | $Q_{\text{max},h}$ | $M_{B_{\text{max},h}}$ | $M_{Q_{\text{max},h}}$ |
|-------|-------|-------------------------------|-------------------------------|----------------|----------------|----------------|----------------|
| 100   | 25    | 0.096                         | 0.054                         | 0.218          | 0.117          | 0.317          | 0.213          |
|       | 50    | 0.114                         | 0.051                         | 0.150          | 0.084          | 0.190          | 0.128          |
|       | 75    | 0.166                         | 0.044                         | 0.178          | 0.126          | 0.213          | 0.164          |
| 200   | 50    | 0.152                         | 0.051                         | 0.171          | 0.110          | 0.397          | 0.227          |
|       | 100   | 0.128                         | 0.058                         | 0.213          | 0.122          | 0.315          | 0.181          |
|       | 150   | 0.154                         | 0.047                         | 0.179          | 0.117          | 0.206          | 0.134          |
| 300   | 75    | 0.100                         | 0.060                         | 0.240          | 0.110          | 0.373          | 0.166          |
|       | 150   | 0.121                         | 0.063                         | 0.259          | 0.146          | 0.394          | 0.219          |
|       | 225   | 0.145                         | 0.054                         | 0.238          | 0.143          | 0.333          | 0.218          |
|       | 100   | 0.110                         | 0.078                         | 0.270          | 0.120          | 0.463          | 0.212          |
|       | 200   | 0.110                         | 0.071                         | 0.257          | 0.121          | 0.380          | 0.135          |
|       | 300   | 0.136                         | 0.057                         | 0.212          | 0.127          | 0.300          | 0.176          |

The table reports the rejection probabilities of 5%-level tests proposed in the paper and those proposed by Giacomini and Rossi (2009) [(abbreviated GR (2009)] for model S1. For all methods we use the recursive forecasting scheme. \(T = T_m + T_n\), where \(T\) is the total sample size, \(T_m\) is the size of the in-sample window and \(T_n\) is the size of the out-of-sample window. \(m_T\) is set equal to the smallest integer allowed by Condition 1. Based on 5,000 replications.