Locally uniformly rotund renormings of the spaces of continuous functions on Fedorchuk compacts

S.P.Gul’ko$^a$, A.V.Ivanov$^b$, M.S.Shulikina$^c$, S.Troyanski$^d$

$^a$ Faculty of Mechanics and Mathematics, Tomsk State University, Prospect Lenina 36 634010 Tomsk, Russian Federation E-mail address: gulko@math.tsu.ru
$^b$ Institute of Applied Mathematical Research of Karelian Research Centre, Russian Academy of Sciences, Russian Federation, E-mail address: alvlivanov@krc.karelia.ru
$^c$ Department of Mathematics and Computer Science, Tomsk Polytechnic University, Russian Federation, E-mail address: shulikinams@tpu.ru
$^d$ Institute of Mathematics and Informatics, Bulgarian Academy of Science, 1113 Sofia, Bulgaria and Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain, E-mail address: stroya@um.es

Abstract

We show that $C(X)$ admits an equivalent pointwise lower semicontinuous locally uniformly rotund norm provided $X$ is Fedorchuk compact of spectral height 3. In other words $X$ admits a fully closed map $f$ onto a metric compact $Y$ such that $f^{-1}(y)$ is metrizable for all $y \in Y$. A continuous map of compacts $f : X \to Y$ is said to be fully closed if for any disjoint closed subsets $A, B \subset X$ the intersection $f(A) \cap f(B)$ is finite. For instance the projection of the lexicographic square onto the first factor is fully closed and all its fibers are homeomorphic to the closed interval.

1. INTRODUCTION

Let us recall that a Banach space $E$ (or the norm in $E$) is said to be locally uniformly rotund (LUR for short) if $\lim_n ||x_n - x|| = 0$ whenever

$$\lim_n ||(x_n + x)/2|| = \lim_n ||x_n|| = ||x||.$$

The spaces with this property are at the core of renorming theory in Banach spaces and consequently have been extensively studied (see, for example, [3] and [14] and its reference). It is well known that the spaces with a LUR norm have the Kadec property, that is on the unit sphere of $E$ the norm topology and the weak topology coincide. The LUR renorming techniques for a Banach space developed until now, are based in two different approaches. In the first one, for enough convex functions on the Banach space are constructed to apply Deville’s master lemma (see the decomposition method [3, Chapter 7, Lemma 1.1, p.279]), sometimes adding an iteration processes and Banach’s contraction mapping theorem, to finally get an

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equivalent LUR norm. Originally this method use the powerful method of projectional resolutions of the identity (PRI for short) The second one is based to a characterization of those Banach spaces that admit a LUR renorming by means of a linear topological condition of covering type [13]. The existence of such norm is deduced from the existence of some maps acting from normed space $E$ into metric space $Y$. This maps admit some covering properties (see [14]). We present a particular result in this direction which we use latter. This is nonlinear (convex) version for LUR renorming of Banach spaces with strong Markushevich basis.

**Theorem 1.1.** Assume that there is a locally bounded map $\Phi$ from a normed space $E$ into $c_0(\Gamma)$ for some set $\Gamma$ such that:

(i) for every finite set $A \subset \Gamma$, there exists a separable subspace $Z_A$ of $E$ such that:

(a) $Z_A \subset \overline{Z_B}$ whenever $A \subset B \subset \Gamma$;

(b) $x \in \overline{\text{span}\bigcup_{n \in N} Z_{K_n}^{||\cdot||}}$ whenever $x \in E$ and $\{K_n : n \in N\}$ is an increasing sequence (i.e. $K_n \subset K_m$ for $n < m$) of finite subsets of $\Gamma$ with $\overline{\bigcup_{n \in N} K_n} \supset \supp(\Phi x) = \{\gamma \in \Gamma : \Phi x(\gamma) \neq 0\}$;

(ii) there exists norming subspace $F$ of $E^*$ such that for every $\gamma \in \Gamma$ the real function $\delta_\gamma \circ \Phi$ on $E$ is non-negative, convex and $\sigma(E, F)$-lower semi-continuous, where $\delta_\gamma$ is the Dirac measure on $\Gamma$ at $\gamma$.

Then $E$ admits an equivalent $\sigma(E, F)$-lower semi-continuous LUR norm.

**Remark.** Condition (b) is equivalent to

(b’) for every $x \in E$ there exists an increasing sequence $\{K_n : n \in N\}$ of finite subsets of the support $\supp(\Phi x)$ with $x \in \overline{\text{span}\bigcup_{n \in N} Z_{K_n}^{||\cdot||}}$.

**Proof.** Let us consider the condition

(iii) for every $x \in E$ there exists separable subspace $Z(x)$ of $E$ with

$$x \in \overline{\text{span}\bigcup \{Z(x_n) : n \in N\}^{||\cdot||}}$$

whenever $\{x_n : n \in N\}$ is a sequence in $E$ with $\lim_n ||\Phi x_n - \Phi x||_\infty = 0$.

In [14, Theorem 1.15, Theorem 2.14, Corollary 1.21, Theorem 3.28, Corollary 4.34] is shown that (ii) and (iii) imply existence in $E$ an equivalent $\sigma(E, F)$-lower semi-continuous LUR norm. So in order to prove the former Theorem we have to show (i) $\Rightarrow$ (iii).

Let us mention that from (a) it follows that

$$\overline{\text{span}\bigcup_{n \in N} Z_{A_n}} = \overline{\text{span}\bigcup_{n \in N} Z_{B_n}}$$

for any increasing sequences $\{A_n : n \in N\}$, $\{B_n : n \in N\}$ of finite subsets of $\Gamma$ with $\bigcup_{n \in N} A_n = \bigcup_{n \in N} B_n$. So we can define $Z_A$ for any countable subset $A$ of $\Gamma$ by setting

$$Z_A = \overline{\text{span}\bigcup_{n \in N} Z_{A_n}^{||\cdot||}}$$

if $A = \bigcup_{n \in N} A_n$ where $\{A_n : n \in N\}$ an increasing sequence of finite subsets of $\Gamma$. Clearly (a) holds for countable sets as well.

Set $Z(x) = Z_{\text{supp}(\Phi x)}$. Let $\{x_n : n \in N\}$ be a sequence in $E$ with

$$\lim_n ||\Phi x_n - \Phi x||_\infty = 0$$

(1.2)

for some $x \in E$. Using (b’) we can find an increasing sequence $\{K_m : m \in N\}$ of finite subsets of $\text{supp}(\Phi x)$ such that

$$x \in \overline{\text{span}\bigcup_{m \in N} Z_{K_m}^{||\cdot||}}$$

(1.3)
Set \( B_n = \text{supp}(\Phi x_n) \). Fix \( m \in N \). From (1.2) it follows that \( K_m \subset B_n \) for all large enough \( n \). Hence \( Z_{K_m} \subset Z_{B_n} = Z(x_n) \) for all large enough \( n \). So
\[
\cup_{m \in N} Z_{K_m} \subset \cup_{n \in N} Z(x_n).
\]
This and (1.3) imply(1.1). □

**Example 1.2.** Assume that \( E \) has a strong Markushevich basis \( \{e_\gamma : \gamma \in \Gamma \} \) with conjugate system \( \{e_\beta^* : \gamma \in \Gamma \} \), that is \( e_\beta^*(e_\gamma) = \delta_\beta \gamma \), and for every \( x \in E \) we have
\[
x \in \overline{\text{span}\{e_\gamma : e_\beta^*(x) \neq 0, \gamma \in \Gamma \}} \quad \|
\]
For finite set \( A \subset \Gamma \) set \( Z_A = \text{span}\{e_\gamma : \gamma \in A\} \) and define \( \Phi : E \to c_0(\Gamma) \) by formula \( \Phi x(\gamma) = |e_\gamma^*(x)| \). From the definition of strong Markushevich basis for every \( x \in E \) we have
\[
x \in \overline{\text{span}\{e_\gamma : \Phi x(\gamma) = |e_\gamma^*(x)| \neq 0, \gamma \in \Gamma \}} \quad \|
\]
So \( \Phi \) satisfies the hypothesis of the former Theorem.

When Banach space \( E \) admits PRI in the similar way we can construct map \( \Phi \).

Lot of papers are devoted to find different classes of Hausdorff compact spaces (compacts) \( X \) for which \( C(X) \) admits an equivalent pointwise lower semi-continuous LUR norm. Moving on now to topological properties, we say that a compact space \( X \) is *Eberlein* if it is homeomorphic to a weakly compact subset of a Banach space. Equivalently, \( X \) is Eberlein if and only if \( C(X) \) is a weakly compactly generated space. The space \( X \) is called *Talagrand or Gul’ko* compact if \( C(X) \) is weakly \( K\)-analytic or weakly countably determined, respectively. For a full treatment of these concepts, we refer the reader to e.g., [3]. It turns out that if \( X \) hails from one of these three classes of compact spaces then it can be treated as a subset of a pointwise compact cube \([0,1]^\Gamma\), in such a way that, given \( t \in X \), its coordinates \( t(\gamma) \), \( \gamma \in \Gamma \) behave according to certain rules. For instance, \( X \) is Eberlein if and only if we can find such an embedding with the property that for any \( t \in X \) and \( \varepsilon > 0 \), there are only finitely many \( \gamma \in \Gamma \) satisfying \( |t(\gamma)| \geq \varepsilon \). Similarly, spaces from the three corresponding classes of Banach spaces may be endowed with ‘coordinate systems’, which permit the spaces to be carefully analyzed in ways which are not feasible in a fully general, non-separable setting. Every compact space \( X \) from the classes above shares the property that \( X \) may be embedded in \([0,1]^\Gamma\), such that given any \( t \in X \), the support of \( t \) is countable. In general, a space satisfying this property is called *Corson* compact. The space \( X \) is called *Valdivia* compact if it is as above, but, in this case, only a pointwise dense subset of points of \( X \subset [0,1]^\Gamma \) are required to have countable support. These classes have long been relevant to renorming theory. For example, it can be shown that if \( X \) is Valdivia compact then \( C(X) \) admits a pointwise lower semi-continuous LUR norm (cf. [3, Corollary VII.1.10]). In all this cases \( C(X) \) admits PRI. Of course all this result can be obtained directly from Theorem 1.1. We mention that in \( C(X) \) spaces we have a canonical map to \( c_0(\Gamma) \) (see e.g. [14]). Indeed if \( X \subset [0,1]^\Gamma \) the uniform continuity of every \( h \in C(X) \) allows us to define the oscillation map \( \Omega : C(X) \to c_0(\Gamma) \) by formula
\[
\Omega(h)(\gamma) = \sup\{h(t) - h(s) : t, s \in X, (t-s)|_{\Gamma \setminus \{\gamma\}} = 0\},
\]
where \( h \in C(X) \). Map \( \Omega \) was introduced(see [4]) looking for countable sets of coordinates which control a continuous function to obtain extensions of the theorem of Mibu. It is easy to see that \( \Omega \) is a bounded map satisfying condition (ii) of Theorem 1.1. In [14, 2.7] is shown that \( \Omega \) satisfies condition (iii) of Theorem 1.1
when \( X \) is Helly compact of monotone functions on \([0,1]\). In this way is proved that \( C(X) \) is LUR renormable when \( X \) is Helly compact. A generalization of this result when \( X \) is a particular case of Rosenthal compacts can be found in [10], see also [12].

The aim of this note is to find a new class of compact spaces for which the corresponding space of continuous functions is LUR renormable.

The following definition goes back to V.V. Fedorchuk (see e.g. [5, II.1.6])

**Definition 1.3.** Continuous map of compacts \( f : X \to Y \) is said to be fully closed if for every disjoint closed subsets \( A \) and \( B \) of \( X \) the set \( f(A) \cap f(B) \) is finite.

Our main result is next

**Theorem 1.4.** Let \( X \) be a compact space admitting a fully closed map \( f \) onto a metrizable compact \( Y \) such that the fibers \( f^{-1}(y) \) are metrizable for every \( y \in Y \).

Then \( C(X) \) admits an equivalent pointwise lower semi-continuous LUR norm.

The above class of compact spaces is particular case of Fedorchuk compacts. The section 2 is devoted to this class. Now we give an example. Denote with \( L \) the lexicographic square (the projection of this square onto the first factor is fully closed and all its fibers are homeomorphic to the closed interval). In [1] is shown that \( C(L) \) is LUR renormable (for general case of totally ordered compacts see [9] and [14]).

Let \( f : X \to Y \) be a continuous map of compacts. Given \( y \in Y \) define

\[
\text{osc}_{f^{-1}(y)}(h) = \sup_{s,t \in f^{-1}(y)} \{h(t) - h(s)\} = \text{diam}(h(f^{-1}(y)))
\]

for \( h \in C(X) \). Clearly \( \text{osc}_{f^{-1}(y)}(\cdot) \) is a pointwise lower semi-continuous semi-norm in \( C(X) \) and \( \text{osc}_{f^{-1}(y)}(\cdot) \leq 2\|\cdot\|_{\infty} \).

We introduce fiberwise oscillation map \( \Omega_f : C(X) \to l_{\infty}(Y) \) by formula

\[
\Omega_f(h)(y) = \text{osc}_{f^{-1}(y)}(h)
\]

where \( h \in C(X) \).

Since \( \|\Omega_f(h)\|_{\infty} \leq 2\|h\|_{\infty} \) for all \( h \in C(X) \) the map \( \Omega_f \) is bounded. Since \( \text{osc}_{f^{-1}(y)}(\cdot) \) is a pointwise lower semi-continuous semi-norm in \( C(X) \) and \( \delta_y \circ \Omega_f(h) = \text{osc}_{f^{-1}(y)}(h) \) we get that \( \Omega_f \) satisfies condition (ii) of Theorem 1.1. In sections 2 and 3 we show that if \( f \) a fully closed map satisfying the conditions of Theorem 1.4 then \( \Omega_f \) maps \( C(X) \) into \( c_0(Y) \) and satisfies condition (i) of Theorem 1.1.

### 2. FEDORCHUK COMPACTS

The class of Fedorchuk compact spaces was defined in 1984 [11] with the purpose of clarifying the limits of the application of the method of resolutions, which was developed by Fedorchuk (see [5]) and showed exceptional efficiency in constructing counterexamples in general topology. The definition of this class is inextricably linked with the concept of a fully closed mapping introduced by Fedorchuk in the process of developing the method mentioned above. The original definition of fully closed mapping was cumbersome. Later, his author obtained a number of unobvious equivalent formulations (see [5, II. 1.6]). The shortest of them for mappings of Hausdorff compact spaces is the Definition 1.3 above.

**Definition 2.1.** Hausdorff compact space \( X \) is called a Fedorchuk compact (or an \( F \)-compact) if there exists a well-ordered continuous inverse system (an \( F \)-system)
The spectral height $\text{sh}(X)$ of an $F$-compact $X$ is the smallest possible length $\gamma$ of such a system.

We consider only the Fedorchuk compacts of spectral height 3. If $X$ is an $F$-compact and $\text{sh}(X) = 3$, then $X$ is the limit of an $F$-system consisting of three spaces: $X_0$, $X_1$ and $X_2$. The limit of this system coincides with $X_2$, from which it follows that the compact space $X = X_2$ is non-metrizable, since otherwise it would be obtained as the limit of the $F$-system of two compacts: $X_0$ and $X_2$ (the map to a point is always fully closed). Consequently, for any Fedorchuk compact of spectral height 3 there exists a fully closed map $\pi_1^2 : X = X_2 \to X_1$ onto a metric compact space for which the sets $(\pi_1^2)^{-1}(x)$, $x \in X_1$ are metrizable.

Thus, $F$-compacts of spectral height 3 can be characterized as non-metrizable compacts that admit a fully closed map onto a metric compact with metrizable fibers. Note that if such a fully closed map for a compact $X$ exists, then it is almost unique in the following sense: almost all (that is, all but perhaps a countable set) nontrivial fibers of any two such maps coincide (see [6]). Remark, that only a point is an $F$-compact of spectral height 1 and $F$-compacts of spectral height 2 are all non-trivial metric compact spaces.

As noted in the introduction, the lexicographic square $L$ is an $F$-compact of spectral height 3. The classical space "two arrows" (the lexicographic product of unit segment and two-point set $[0, 1] \times \{0, 1\}$) also is $F$-compact of spectral height 3 (the standard projection of this space onto the segment is a fully closed map) as well as the space "Alexandroff double circle" (the projection of "double circle" onto the circle is fully closed).

To the class of $F$-compacts of spectral height 3 belongs a series of various counterexamples constructed by Fedorchuk. Here, first of all, we should mention a group of compacts with non-coinciding dimensions. Among them is the famous example of a two-dimensional compact that has no partitions of lower dimension (see [2], page 314), and also a homogeneous separable compact $X$ with the first axiom of countability with $\dim(X) = 1 < \text{ind}(X) = 2$ and a perfectly normal compact (constructed under the assumption of CH) with same values of the dimensions $\dim$ and $\text{ind}$ (see [5, III. 3.6 and 3.10]).

We will need the following characteristic property of fully closed map obtained by Fedorchuk [5]. Let $f : X \to Y$ be a continuous map, and $A$ be an arbitrary subset of $Y$. Consider a partition of a compact $X$ whose nontrivial elements are sets $f^{-1}(y)$ for $y \in Y \setminus A$. Let $Y_A$ be the quotient space corresponding to this partition (with respect to $f$), that is

$$Y_A = \{f^{-1}(y) : y \in Y \setminus A\} \cup \{\{x\} : f(x) \in A\}.$$ 

Let $f_A : X \to Y_A$ be a quotient map, and let $\pi_A : Y_A \to Y$ be the unique map for which $f = \pi_A \circ f_A$.

**Proposition 2.2.** [5, II. 1.6]. The map $f$ is fully closed if and only if for each $A \subset Y$ the space $Y_A$ is Hausdorff.

In [5, II. 1.7, 1.10] it is also shown that for a fully closed map $f$ the maps $f_A$ and $\pi_A$ are also fully closed for any $A \subset Y$.

**Proposition 2.3.** [5, II. 3.10]. Let $f : X \to Y$ be a fully closed map of the compact $X$ onto a metric compact $Y$ with metrizable fibers $f^{-1}(y)$, $y \in Y$. The
compact $X$ is metrizable if and only if the set of nontrivial fibers (i.e. the fibers $f^{-1}(y)$ that is not a singleton) of $f$ is countable.

3. PROOF OF THEOREM 1.4.

The following assertion gives a characterization of fully closed maps, which plays an important role in what follows.

**Proposition 3.1.** The map $f : X \to Y$ of compacts is fully closed if and only if for any continuous map $g : X \to K$ into a metric compact space $K$ and any $\varepsilon > 0$ the set $H_{g,\varepsilon} = \{ y \in Y : \text{diam}(g(f^{-1}(y))) \geq \varepsilon \}$ is finite.

**Proof.** Necessity. Let $f$ be fully closed. This means that for any disjoint closed subsets $A, B \subset X$ the intersection $f(A) \cap f(B)$ is finite. Suppose that there exists a continuous map $g : X \to K$ to the metric compact $K$ and $\varepsilon > 0$ such that $H_{g,\varepsilon}$ is infinite. Consider the infinite family of closed subsets of $K$ indexed by $y$:

$$G = \{ g(f^{-1}(y)) : y \in H_{g,\varepsilon} \}.$$

Let $F$ be an accumulation point of the family $G$ in the space $\exp(K)$ of nonempty closed subsets of $K$ endowed with the Hausdorff metric. It is obvious that $\text{diam}(F) \geq \varepsilon$. We take in $F$ two distinct points $x_1, x_2$ and their neighborhoods $O_{x_1}, O_{x_2}$ with disjoint closures. By the choice of $F$, there exists an infinite subset $D \subset H_{g,\varepsilon}$ such that $g(f^{-1}(y)) \cap O_{x_i} \neq \emptyset$ for any $y \in D$ for $i = 1, 2$. Consider disjoint closed subsets $A_i = g^{-1}O_{x_i}, i = 1, 2$ in the space $X$. We have $D \subset f(A_1) \cap f(A_2)$, which contradicts that $f$ is fully closed.

Sufficiency. Let $f$ be not fully closed, that is, there exist disjoint closed subsets $A, B$ in $X$ such that the intersection $E = f(A) \cap f(B)$ is infinite. Let $g : X \to [0, 1]$ be a continuous function on $X$ that separates $A$ and $B$. Then the set $H_{g,1}$ contains $E$ and, consequently, is infinite. \(\Box\)

**Corollary 3.2.** For any fully closed map of compacts $f : X \to Y$ the fiberwise oscillation map $\Omega_f$ maps $C(X)$ into $c_0(Y)$.

Let $f : X \to Y$ be a fully closed map of compacts. Given $A \subset Y$ define

$$Z_A = \{ h \in C(X) : \text{supp}(\Omega_f(h)) \subset A \}. \tag{3.1}$$

From the definition of $\Omega_f$ it follows that $Z_A$ is a closed subspace of $C(X)$.

**Lemma 3.3.** The spaces $Z_A$ and $C(Y_A)$ are isomorphically isometric. The linear operator $T_A : Z_A \to C(Y_A)$ defined by formula

$$T_A h = h \circ f_A^{-1} \tag{3.2}$$

give the isometry, i.e.

$$||T_A h||_\infty = ||h||_\infty. \tag{3.3}$$

Moreover for every $y \in A$ and $h \in Z_A$ we have

$$T_A h(\pi_A^{-1}(y)) = h(f^{-1}(y)). \tag{3.4}$$

**Proof.** We have

$$Y_A = \{ f^{-1}(y) : y \in Y \setminus A \} \cup \{ \{x\} : f(x) \in A \}.$$

Pick $h \in Z_A$. Then $\text{osc} f^{-1}(y)(h) = 0$ if $y \in Y \setminus A$. So $(T_A h)(f^{-1}(y)) = h \circ f_A^{-1}(f^{-1}(y)) = h(x)$ (here $f^{-1}(y)$ is a point of $Y_A$) for every $x \in f^{-1}(y)$ and $y \in Y \setminus A$. If $f(x) \in A$ then $(T_A h)(\{x\}) = h(f_A^{-1}(\{x\})) = h(x)$. 


From (3.4) we get
\[ \text{diam}(f^{-1}(y)) = \text{diam}(f^{-1}(\pi_Y^{-1}(y))) = T_A h(\pi_Y^{-1}(y)). \]

\[ \square \]

The main result of this section is

**Proposition 3.4.** Let \( f : X \to Y \) be a fully closed map of compact \( X \) onto a metric compact \( Y \) with metrizable fibers \( f^{-1}(y), y \in Y \). Let \( K \) be a finite subset of a countable set \( A \subset Y \). Then for every \( h \in Z_A \) we have

\[ \text{dist}(h, Z_K) = \inf\{ ||h - g||_\infty : g \in Z_K \} \leq ||\Omega_f h||_{A \setminus K}||_\infty. \tag{3.5} \]

The proof is based on the properties of Fedorchuk compacts. For this reason at first, before the proof, we introduce some concepts and notations. In the subsequent arguments we need the concept of a small image of a set. Recall that a small image \( f^2(B) \) of the set \( B \subset X \) under a map \( f : X \to Y \) is the set \( f^2(B) = Y \setminus f(X \setminus B) \). Clearly \( f^2(B) \subset f(B) \). Let us mention that for any continuous map \( f : X \to Y \) of compact and any open subset \( U \subset X \), a small image \( f^2(U) \) is open in \( Y \), since the set \( f(X \setminus U) \) is closed in \( Y \) as a continuous image of a compact.

We denote with \( \text{Fr}(B) \) the topological boundary of the set \( B \), that is \( \text{Fr}(B) = \mathcal{B} \setminus \text{int}(B) \).

Using the notations of section 2 we consider the fully closed map \( \pi_A : Y_A \to Y \). Consider a partition of compact \( Y_A \) whose nontrivial elements are the sets \( \pi_Y^{-1}(y) \) for \( y \in Y \setminus K \). Let \( Y_{A,K} \) be the quotient space corresponding to this partition, let \( \pi_{A,K} : Y_A \to Y_{A,K} \) be the quotient map, and let \( \pi_{A,K} : Y_{A,K} \to Y \) be the unique map for which \( \pi_A = \pi_{A,K} \circ \pi_{A,K} \). By Proposition 2.3 we have that \( Y_{A,K} \) is metrizable since the set of nontrivial fibers \( \{ \pi_{A,K}^{-1}(y) : y \in K \} \) of the fully closed map \( \pi_{A,K} \) is finite and all this fibers and \( Y \) are metrizable. We fix a metric \( d \) on \( Y_{A,K} \) that is compatible with the topology. We denote with \( B(u, \rho) = \{ y \in Y_{A,K} : d(u, y) < \rho \} \) the open ball centered at \( u \) with radius \( \rho > 0 \).

**Proof of Proposition 3.4.** Pick \( h \in Z_A \) and set

\[ s = ||\Omega_f h||_{A \setminus K}||_\infty, \quad g = T_A h. \]

So \( g = h \circ f_A^{-1} \in C(Y_A) \). Set \( D = A \setminus K \). Enumerate the points \( D \) by the natural numbers: \( D = \{ y_n : n \in N \} \) and denote by \( u_n \) the unique point in \( Y_{A,K} \) with \( \pi_{A,K}(u_n) = y_n \). Set \( E = \{ u_n : n \in N \} \).

Fix \( c > 1 \). For every \( n \in N \) choose an open neighborhood \( O_{y_n} \) of \( \pi_Y^{-1}(y_n) \) in \( Y_A \) in such a way that

\[ \text{diam}(g(O_{y_n})) < c \cdot \text{diam}(g(\pi_Y^{-1}(y_n))). \tag{3.6} \]

From (3.4) we get \( \text{diam}(g(\pi_Y^{-1}(y_n))) = \text{diam}(h(f^{-1}(y_n))) \). Since \( y_n \in A \setminus K \) we get \( \text{diam}(h(f^{-1}(y_n))) \leq s \). So

\[ \text{diam}(g(O_{y_n})) < cs. \tag{3.7} \]

**Claim 1.** There exists a sequence of open sets \( \{ U_n : n \in N \} \) in \( Y_{A,K} \) such that \( E \subset \bigcup_{n \in N} U_n \), the closures of \( U_n \) are pairwise disjoint, \( \overline{U_n} \subset \pi_A^2(O_{y_n}), \text{Fr}(U_n) \cap E = \emptyset \) and \( \text{diam}(\overline{U_n}) < \min\{cs, 1/n\} \).
Proof of Claim 1. We construct the open sets \( U_n, \ n \in \mathbb{N} \) by recursion as follows.

Step 1. Clearly \( u_1 \in \pi_{A,K}^1(O_{y_1}) \). Taking into account that \( E \) is countable, we can choose an open neighborhood \( U_1 \) of the point \( u_1 \) such that \( \overline{U_1} \subset \pi_{A,K}^1(O_{y_1}) \), \( \text{Fr}(U_1) \cap E = \emptyset \) and \( \text{diam}(\overline{U_1}) < \min\{cs, 1\} \). As \( U_1 \) we can take an open ball \( B(u_1, \rho) \) centered at \( u_1 \) with a sufficiently small radius \( \rho \neq d(u_1, u_k), \ k \in \mathbb{N} \). If \( y \in \text{Fr}(U_1) \) then \( d(u_1, y) = \rho \). Therefore \( \text{Fr}(U_1) \cap E = \emptyset \).

Assume that the sets \( U_k \) are already constructed for all \( k < n \) such that their closures are disjoint, \( \overline{U_k} \subset \pi_{A,K}^i(O_{y_k}) \), \( \text{Fr}(U_k) \cap E = \emptyset \), \( \text{diam}(\overline{U_k}) < \min\{cs, 1/k\} \) and \( u_m \in \cup_{k<n} U_k \) for \( m < n \).

Step \( n \). If \( u_n \in \cup_{k<n} U_k \), then put \( U_n = \emptyset \). Otherwise \( u_n \not\in \cup_{k<n} \overline{U_k} \) and we construct a neighborhood \( U_n \) of the point \( u_n \) as in step 1 such that \( \overline{U_n} \subset \pi_{A,K}^i(O_{y_n}) \) \( \cup_{k<n} \overline{U_k} \), \( \text{Fr}(U_n) \cap E = \emptyset \), \( \text{diam}(\overline{U_n}) < \min\{cs, 1/n\} \). \( \square \)

Set \( G = Y_{A,K} \setminus \cup_{n \in \mathbb{N}} U_n \) and \( W = \pi_{A,K}^{-1}(\cup_{n \in \mathbb{N}} U_n) \). The map \( \pi_{A,K} \) is one-to-one on \( Y_A \setminus W \) and \( \pi_{A,K}(Y_A \setminus W) = G \). Define a real function on \( G \) by formula

\[
  r = g \circ (\pi_{A,K}^{-1}|G) .
\]

Set

\[
  a_n = \inf(g(O_{y_n})), \ b_n = \sup(g(O_{y_n})) .
\]

We have

\[
  b_n - a_n = \text{diam}(g(O_{y_n})) < cs .
\]

Thus \( r(\text{Fr}(U_n)) \subset [a_n, b_n] \) since \( \overline{U_n} \subset \pi_{A,K}^i(O_{y_n}) \).

Let us mention that from Proposition 3.1 and (3.6), (3.9) we get

\[
  \lim_{n \to \infty} (b_n - a_n) = 0 .
\]

By the Tietze theorem on the extension of continuous functions defined on a closed subset of a metric space, there is a continuous function \( r_n : \overline{U_n} \to [a_n, b_n] \), which is an extension of the function \( r|_{\text{Fr}(U_n)} \).

Claim 2. The real function

\[
  q(x) = \begin{cases} 
    r(x) & \text{if } x \in G \\
    r_n(x) & \text{if } x \in U_n, \ n \in \mathbb{N} 
  \end{cases}
\]

is continuous on \( Y_{A,K} \).

Proof of Claim 2. Pick \( x \in Y_{A,K} \). If \( x \in \cup U_n \), then the continuity of \( q \) at the point \( x \) is obvious.

Let \( x \in G \setminus \cup \overline{U_n} \). Let \( z \) be the unique point of \( Y_A \) for which \( \pi_{A,K}(z) = x \). Fix \( \varepsilon > 0 \). By the continuity of \( g \), there exists a neighborhood \( V_z \) of \( z \) such that for any \( v \in V_z \) we have \( |g(z) - g(v)| < \varepsilon \). It is clear that \( x \in \pi_{A,K}^i(V_z) \). Choose \( k \in \mathbb{N} \) such that \( B(x, 1/k) \subset \pi_{A,K}^i(V_z) \). Using (3.10) we may assume in addition

\[
  b_n - a_n < \varepsilon \ \text{for } n > k .
\]

We show that \( |q(x) - q(w)| < 2\varepsilon \) for any \( w \in B(x, 1/2k) \setminus \cup_{n \leq 2k} \overline{U_n} \) (the last set is an open neighborhood of the point \( x \)).

If \( w \in G \), then there is a unique point \( v \in V_z \) such that \( \pi_{A,K}(v) = w \) and \( q(w) = g(v) \). Consequently,

\[
  |q(x) - q(w)| = |g(z) - g(v)| < \varepsilon .
\]
If \( w \not\in G \), then \( w \in U_n \) for some \( n > 2k \). Since \( \text{diam}(U_n) < 1/n \leq 1/2k \) and \( w \in B(x, 1/2k) \), we get \( U_n \subset B(x, 1/k) \subset \pi^d_{A,K}(V_z) \). By virtue of (3.11) we have

\[
\text{diam}(g(O_{y_n})) = b_n - a_n < \varepsilon.
\]

Let \( t \in \pi^{-1}_{A,K}(w) \). Then \( t \in O_{y_n} \), since by construction \( U_n \subset \pi^d_{A,K}(O_{y_n}) \). Therefore, \( g(t) \in [a_n, b_n] \). By definition \( q(w) = r_n(w) \in [a_n, b_n] \). Thus, \( |g(z) - g(t)| < \varepsilon \). Since \( t \in V_z \) we have

\[
|g(x) - q(w)| = |g(z) - q(w)| \leq |g(z) - g(t)| + |g(t) - q(w)| < 2\varepsilon.
\]

The continuity of \( q \) at the point \( x \) is proved.

It remains to consider the case when \( x \in Fr(U_n) \) for some \( n \). In this case, we represent \( Y_{A,K} \) as a union of two closed sets:

\[
Y_{A,K} = \overline{U_n} \cup (Y_{A,K} \setminus U_n).
\]

The restriction of \( q \) to \( \overline{U_n} \) is continuous at the point \( x \) by the definition of \( q \). The continuity of the restriction of \( q \) to \( (Y_{A,K} \setminus U_n) \) at the point \( x \) is proved by a verbatim repetition of the above arguments with \( Y_{A,K} \) replaced by \( (Y_{A,K} \setminus U_n) \), \( Y_A \) by \( Y_{A \setminus K}(U_n) \), and \( U_n \) by the empty set. The continuity of \( q \) at the point \( x \) on \( \overline{U_n} \) and \( (Y_{A,K} \setminus U_n) \) implies continuity at the point \( x \) on \( Y_{A,K} \). □

Set \( p = q \circ \pi_{A,K} \circ f_A \). We have \( p \in C(X) \).

Claim 3. We have

(a) \( \|p - h\|_{\infty} \leq c|\Omega f(h)|_{A \setminus K}\|_{\infty} \);

(b) \( \text{supp} \ \Omega f(p) \subset K \subset A \).

Proof of Claim 3. (a) Pick \( x \in X \). Assume that and \( \pi_{A,K}(f_A(x)) \in G \). From the definition of \( q \) we get \( p(x) = h(x) \).

Now let \( \pi_{A,K}(f_A(x)) \not\in G \). In this case \( y = \pi_{A,K}(f_A(x)) \in U_n \) for some \( n \). Since \( U_n \subset \pi^d_{A,K}(O_{y_n}) \) we get \( f_A(x) \in O_{y_n} \) and \( h(x) = g(f_A(x)) \in [a_n, b_n] \) by virtue of (3.9).

On the other hand, when \( y \in U_n \)

\[
p(x) = q(y) = r_n(y) \in [a_n, b_n],
\]

therefore \( |p(x) - h(x)| \leq b_n - a_n \leq cs = c|\Omega f(h)|_{A \setminus K}\|_{\infty} \). Thus \( \|p - h\|_{\infty} \leq c|\Omega f(h)|_{A \setminus K}\|_{\infty} \).

(b) If \( y \not\in K \subset Y \), then \( \pi^{-1}_{A,K}(y) \) is a singleton. Hence for any \( x \in f^{-1}(y) \), \( p(x) = q(t) \). Therefore \( \text{osc}_{f^{-1}(y)}(p) = 0 \) for \( y \not\in K \). Thus \( \text{supp} \ \Omega f(p) \subset K \). □

Since \( c \) is arbitrary number bigger than 1, (a) and (b) complete the proof of proposition. □

In order to finish the proof of Theorem 1.4 it is enough to prove the following

Proposition 3.5. The map of the fiberwise oscillation \( \Omega f : C(X) \to c_0(Y) \) satisfies the conditions (a) and (b) of (i) Theorem 1.1 for any fully closed map \( f : X \to Y \) of compact \( X \) onto a metric compact \( Y \) with metrizable fibers \( f^{-1}(y) \), \( y \in Y \).

Proof. Given a subset \( A \) of \( Y \) we define \( Z_A \) as in (3.1). According Lemma 3.3 \( Z_A \) can be identify with \( C(Y_A) \). If \( A \) is finite or countable we get that \( Y_A \) is metrizable by Proposition 2.3. Hence \( C(Y_A) \) is separable. Evidently condition (a) holds. Pick \( h \in C(X) \) and set

\[
K_m = \{ y \in Y : \Omega f(h)(y) \geq 1/m \}.
\]
Applying Proposition 3.1 we get that $K_m$, $m \in \mathbb{N}$ are finite. Clearly that $\{K_m : m \in \mathbb{N}\}$ is an increasing sequence. From (3.5) we get $\text{dist}(h, Z_{K_m}) \leq 1/m$. So \[ \lim_{m} \text{dist}(h, Z_{K_m}) = 0. \] From this it follows that \[ h \in \text{span} \bigcup_{m \in \mathbb{N}} Z_{K_m}. \] So (b') holds. □

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