A new subtraction scheme for computing QCD jet cross sections at next-to-leading order accuracy

Dedicated to Professor István Lovas on his 75th birthday

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Abstract

We present a new subtraction scheme for computing jet cross sections in electron-positron annihilation at next-to-leading order accuracy in perturbative QCD. The new scheme is motivated by problems emerging in extending the subtraction scheme to the next-to-next-to-leading order. The new scheme is tested by comparing predictions for three-jet event-shape distributions to those obtained by the standard program EVENT.

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1 Introduction

High-energy physics will enter a new era of discovery with the start of LHC operations in 2007. The LHC is a proton-proton collider that will function at the highest energy ever
attained in the laboratory, and will probe a new realm of high-energy physics. The use of
a high-energy hadron collider as a research tool makes substantial demands upon the the-
etical understanding and predictive power of QCD, the theory of the strong interactions
within the Standard Model.

At high $Q^2$ any production rate can be expressed as a series expansion in $\alpha_s$. Because
QCD is asymptotically free, the simplest approximation is to evaluate any series expan-
sion to leading order in $\alpha_s$. However, for most processes a leading-order evaluation yields
unreliable predictions. The next simplest approximation is the inclusion of radiative cor-
rections at the next-to-leading order (NLO) accuracy, which usually warrants a satisfying
assessment of the production rates. In the previous decade, a lot of effort was devoted to
devise process-independent methods and compute rates to NLO accuracy and the problem
is known to be solved [1, 2, 3, 4, 5, 6]. In particular, the dipole subtraction scheme [6]
provides a simple and fully universal way of computing the radiative corrections and has
been implemented in two widely used programs the MCFM [7, 8, 9] and the NLOJET++
[10, 11, 12] codes.

In some cases, typically and most importantly when the NLO corrections are large,
the corrections at next-to-next-to-leading order (NNLO) accuracy are necessary in order to
give a reliable prediction of the production rates. Recently, a lot of effort has been devoted
to the extension of the subtraction schemes used at NLO to NNLO. It was found however,
that the universal NLO schemes cannot be extended to NNLO [13, 14], which motivates
the new method presented in this paper.

2 Subtraction scheme at NLO

The jet cross sections in perturbative QCD are represented by an expansion in the strong
coupling $\alpha_s$. At NLO accuracy we keep the two lowest-order terms,

$$\sigma_{NLO} = \sigma^{LO} + \sigma^{NLO}. \quad (2.1)$$

Assuming an $m$-jet quantity, the leading-order contribution is the integral of the fully
differential Born cross section $d\sigma_m^B$ of $m$ final-state partons over the available $m$-parton
phase space defined by the jet function $J_m$,

$$\sigma^{LO} = \int_m d\sigma_m^B J_m. \quad (2.2)$$

The NLO contribution is a sum of two terms, the real and virtual corrections,

$$\sigma^{NLO} = \int_{m+1} d\sigma_{m+1}^R J_{m+1} + \int_m d\sigma_m^V J_m. \quad (2.3)$$
Here the notation for the integrals indicates that the real correction involves \( m + 1 \) final-state partons, one of those being unresolved, while the virtual correction has \( m \)-parton kinematics, and the phase spaces are restricted by the corresponding jet functions \( J_m \) that define the physical quantity.

In \( d = 4 \) dimensions the two contributions in Eq. (2.3) are separately divergent, but their sum is finite for infrared-safe observables order by order in the expansion in \( \alpha_s \). The requirement of infrared-safety puts constraints on the analytic behaviour of the jet functions that were spelled out explicitly in Ref. [6].

The traditional approach to finding the finite corrections at NLO accuracy is to first continue all integrations to \( d = 4 - 2\epsilon \) \((\epsilon \neq 0)\) dimensions, then regularise the real radiation contribution by subtracting a suitably defined approximate cross section \( d\sigma_{RA}^{m+1} \) such that

1. \( d\sigma_{RA}^{m+1} \) matches the point-wise singular behaviour of \( d\sigma^R \) in the one-parton IR regions of the phase space in any dimensions
2. and it can be integrated over the one-parton phase space of the unresolved parton independently of the jet function, resulting in a Laurent expansion in \( \epsilon \). After performing this integration, the approximate cross section can be combined with the virtual correction \( d\sigma^V \) before integration. We then write

\[
\sigma_{NLO} = \int_{m+1} \left[ d\sigma_{m+1}^R J_{m+1} - d\sigma_{RA}^{m+1} J_m \right]_{\epsilon=0} + \int_{m} \left[ d\sigma_{m}^V + \int_{1} d\sigma_{m+1}^{RA} \right]_{\epsilon=0} J_m .
\]

(Note that \( d\sigma_{RA}^{m+1} \) is multiplied by \( J_m \), therefore, after integration over the phase space of the unresolved parton, it can be combined with \( d\sigma_{RA}^{m+1} \).) Since the first integral on the right hand side of Eq. (2.4) is finite in \( d = 4 \) dimensions by construction, it follows from the Kinoshita-Lee-Nauenberg theorem that the combination of terms in the \( m \)-parton integral is finite as well, provided the jet function defines an infrared-safe observable.

The final result is that one is able to rewrite the two NLO contributions in Eq. (2.3) as a sum of two finite integrals,

\[
\sigma_{NLO} = \int_{m+1} d\sigma_{m+1}^{NLO} + \int_{m} d\sigma_{m}^{NLO},
\]

that are integrable in four dimensions using standard numerical techniques.

### 3 Notation

#### 3.1 Matrix elements

We consider processes with coloured particles (partons) in the final states, while the initial-state particles are colourless (typically electron-positron annihilation into hadrons). Any
number of additional non-coloured final-state particles are allowed, too, but they will be
suppressed in the notation. Resolved partons in the final state are labelled by \(i, k, l, \ldots\),
the unresolved one is denoted by \(r\).

We adopt the colour- and spin-state notation of Ref. [6]. In this notation the amplitude
for a scattering process involving the final-state momenta \(\{p\}\), \(|\mathcal{M}_m(\{p\})|\), is an abstract
vector in colour and spin space, and its normalization is fixed such that the squared am-
plitude summed over colours and spins is

\[
|\mathcal{M}_m|^2 = \langle \mathcal{M}_m | \mathcal{M}_m \rangle .
\]

This matrix element has the following formal loop-expansion:

\[
|\mathcal{M}| = |\mathcal{M}^{(0)}| + |\mathcal{M}^{(1)}| + \ldots ,
\]

where \(|\mathcal{M}^{(0)}|\) denotes the tree-level contribution, \(|\mathcal{M}^{(1)}|\) is the one-loop contribution and
the dots stand for higher-loop contributions, which are not used in this paper.

Colour interactions at the QCD vertices are represented by associating colour charges
\(T_i\) with the emission of a gluon from each parton \(i\). In the colour-state notation, each
vector \(|\mathcal{M}|\) is a colour-singlet state, so colour conservation is simply

\[
\left( \sum_j T_j \right) |\mathcal{M}| = 0 ,
\]

where the sum over \(j\) extends over all the external partons of the state vector \(|\mathcal{M}|\), and
the equation is valid order by order in the loop expansion of Eq. (3.2).

Using the colour-state notation, we can write the two-parton colour-correlated squared
tree amplitudes as

\[
|\mathcal{M}_{i,k}^{(0)}(\{p\})|^2 \equiv \langle \mathcal{M}^{(0)}(\{p\}) | T_i \cdot T_k | \mathcal{M}^{(0)}(\{p\}) \rangle .
\]

The colour-charge algebra for the product \((T_i)^a(T_k)^n\) is:

\[
T_i \cdot T_k = T_k \cdot T_i \quad \text{if} \quad i \neq k ; \quad T_i^2 = C_i .
\]

Here \(C_i\) is the quadratic Casimir operator in the representation of particle \(i\) and we have
\(C_F = T_R(N_c^2 - 1)/N_c = (N_c^2 - 1)/(2N_c)\) in the fundamental and \(C_A = 2T_R N_c = N_c\) in the
adjoint representation, i.e. we are using the customary normalization \(T_R = 1/2\).

3.2 Dimensional regularization, one-loop amplitudes and renor-
malization

We employ conventional dimensional regularization (CDR) in \(d = 4 - 2\varepsilon\) space-time dimen-
sions to regulate both the IR and UV divergences, when quarks (spin-\(\frac{1}{2}\) Dirac fermions)
possess 2 spin polarizations, gluons have $d - 2$ helicity states and all particle momenta are taken as $d$-dimensional.

Turning to the renormalization of the amplitudes, let the perturbative expansion of the unrenormalised scattering amplitude $|A_m\rangle$ in terms of the bare coupling $g_s \equiv \sqrt{4\pi\alpha_s}$ be

$$|A_m\rangle = \left(\frac{\alpha_s^2 \mu^{2\varepsilon}}{4\pi}\right)^{q/2} \left[|A^{(0)}_m\rangle + \left(\frac{\alpha_s^2 \mu^{2\varepsilon}}{4\pi}\right)|A^{(1)}_m\rangle + O((\alpha_s^2)^2)\right],$$

where $q$ is a non-negative integer and $\mu$ is the dimensional-regularization scale. The renormalized amplitudes $|M_m\rangle$ are obtained from the unrenormalized ones by expressing the bare coupling in terms of the running coupling $\alpha_s(\mu^2_R)$ evaluated at the arbitrary renormalization scale $\mu^2_R$ as

$$\alpha_s^2 \mu^{2\varepsilon} = \alpha_s(\mu^2_R) \mu_R^{2\varepsilon} S^{-1}_\varepsilon \left[1 - \left(\frac{\alpha_s(\mu^2_R)}{4\pi}\right) \frac{\beta_0}{\varepsilon} + O(\alpha_s^2(\mu^2_R))\right],$$

where $\beta_0$ is the first coefficient of the QCD $\beta$ function for $n_f$ number of light quark flavours,

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_R n_f.$$

In Eq. (3.7), $S_\varepsilon$ is the phase space factor due to the integral over the $(d-3)$-dimensional solid angle, which is included in the definition of the running coupling in the MS renormalization scheme,*

$$S_\varepsilon = \int \frac{d^{d-3}\Omega}{(2\pi)^{d-3}} = \frac{(4\pi)^\varepsilon}{\Gamma(1 - \varepsilon)}.$$

We always consider the running coupling in the MS scheme defined with the inclusion of this phase space factor.

The relations between the renormalized amplitudes of Eq. (3.10) and the unrenormalized ones are given as follows:

$$|M^{(0)}_m\rangle = \left(\frac{\alpha_s(\mu^2_R) \mu_R^{2\varepsilon} S^{-1}_\varepsilon}{4\pi}\right)^{q/2} |A^{(0)}_m\rangle,$$

$$|M^{(1)}_m\rangle = \left(\frac{\alpha_s(\mu^2_R) \mu_R^{2\varepsilon} S^{-1}_\varepsilon}{4\pi}\right)^{q/2} \alpha_s(\mu^2_R) \frac{S^{-1}_\varepsilon}{\mu_R^2} \left(\mu_R^2 |A^{(1)}_m\rangle - \frac{q}{2} \frac{\beta_0}{\varepsilon} S_\varepsilon |A^{(0)}_m\rangle\right).$$

After UV renormalization, the dependence on $\mu$ turns into a dependence on $\mu_R$, so the physical cross sections depend only on the renormalization scale $\mu_R$. To avoid a cumbersome

*The MS renormalization scheme as often employed in the literature uses $S_\varepsilon = (4\pi)^\varepsilon e^{-\gamma_E}$. It is not difficult to check that the two definitions lead to the same expressions in a computation at the NLO accuracy.
notation, we therefore set $\mu_R = \mu$ in the rest of the paper. Furthermore, after the IR poles are canceled in an NLO computation we may set $\varepsilon = 0$, therefore, the $\mu_R^2\varepsilon$ and $S_{\varepsilon}^{-1}$ factors that accompany the running coupling in the renormalized amplitude do not give any contribution, so we may perform the

$$
\left( \frac{\alpha_s(\mu_R^2)}{4\pi} S_{\varepsilon}^{-1} \right)^{q/2} \left( \frac{\alpha_s(\mu_R^2)}{4\pi} S_{\varepsilon}^{-1} \right)^i \rightarrow \left( \frac{\alpha_s(\mu_R^2)}{4\pi} \right)^{q/2+i}
$$

substitution in Eqs. (3.10)–(3.11).

### 3.3 Cross sections

In our notation the real cross section $d\sigma^R_{m+1}$ is given by

$$
d\sigma^R_{m+1} = N \sum_{\{m+1\}} d\phi_{m+1}(\{p\}) \frac{1}{S_{\{m+1\}}} \langle M_{m+1}^{(0)}(\{p\})|M_{m+1}^{(0)}(\{p\}) \rangle,
$$

where $N$ includes all QCD-independent factors, $\sum_{\{m+1\}}$ denotes a summation over all subprocesses, $S_{\{m+1\}}$ is the Bose symmetry factor for identical particles in the final state and $d\phi_{m+1}(\{p\})$ is the $d$-dimensional phase space for $m+1$ outgoing particles with momenta $\{p\} \equiv \{p_1, \ldots, p_{m+1}\}$ and total momentum $Q$,

$$
d\phi_{m+1}(p_1, \ldots, p_{m+1}; Q) = \left[ \prod_{i=1}^{m+1} \frac{d^d p_i}{(2\pi)^{d-1}} \delta_+(p_i^2)^{\epsilon} \right] (2\pi)^d \delta(d)(p_1 + \cdots + p_{m+1} - Q).
$$

The virtual contribution $d\sigma^V_m$ is

$$
d\sigma^V_m = N \sum_{\{m\}} d\phi_m \frac{1}{S_{\{m\}}} 2 \text{Re} \langle M_m^{(0)}|M_m^{(1)} \rangle.
$$

In the rest of the paper, we define explicitly the approximate cross section $d\sigma^R_{m+1}$ and compute its integral $\int d\sigma^R_{m+1}$.

### 4 The approximate cross section

The construction of the suitable approximate cross section $d\sigma^R_{m+1}$ is made possible by the universal soft and collinear factorization properties of QCD matrix elements [15, 16]. In Ref. [17] we introduced symbolic operators $C_{ir}$, $S_r$ that perform the action of taking the
collinear limit \((p_\mu^i || p_\mu^r)\), or soft limit\(^\dagger\) \((p_\mu^r \to 0)\) of the squared matrix elements, respectively, keeping the leading singular term. Using this notation, we defined the formal expression \(\mathcal{A}|\mathcal{M}_{m+1}^{(0)}|^2\), that matches the singular behaviour of the squared matrix element in all the singly-unresolved regions of the phase space,

\[
\mathcal{A}|\mathcal{M}_{m+1}^{(0)}|^2 = \sum_r \left[ \sum_{i \neq r} \frac{1}{2} C_{ir} + \left( S_r - \sum_{i \neq r} C_{ir} S_r \right) \right] |\mathcal{M}_{m+1}^{(0)}(p_i, p_r, \ldots)|^2. \tag{4.1}
\]

This expression cannot yet serve as a subtraction term because it is defined precisely only in the strict collinear and/or soft limits. It has to be extended over the whole phase space, which requires an exact factorization of the \(m+1\) parton phase space into an \(m\) parton phase space times the phase space measure of the unresolved parton,

\[
d\phi_{m+1} \{ p \} = d\phi_m \{ \tilde{p} \} [dp_1]. \tag{4.2}
\]

With this phase-space factorization we define the approximate cross section as

\[
d\sigma^{R,A}_{m+1} = d\phi_m \{ dp_1 \} \mathcal{A}|\mathcal{M}_{m+1}^{(0)}|^2, \tag{4.3}
\]

where \(\mathcal{A}|\mathcal{M}_{m+1}^{(0)}|^2\) has the same structure as Eq. (4.1),

\[
\mathcal{A}|\mathcal{M}_{m+1}^{(0)}(\{ p \})|^2 = \sum_r \left[ \sum_{i \neq r} \frac{1}{2} C_{ir}(\{ p \}) + \left( S_r(\{ p \}) - \sum_{i \neq r} C_{ir} S_r(\{ p \}) \right) \right]. \tag{4.4}
\]

We now define all terms on the right hand side of Eq. (4.4) precisely.

The collinear counterterm \(C_{ir}(\{ p \})\) reads

\[
C_{ir}(\{ p \}) = 8\pi\alpha_s \mu^{2\varepsilon} \frac{1}{s_{ir}} \langle \mathcal{M}_m (\{ \tilde{p} \}^{(ir)} ) | \hat{P}_{f_f}(z_{i,r}, z_{r,i}, k_{\perp,i,r}; \varepsilon) | \mathcal{M}_m (\{ \tilde{p} \}^{(ir)} ) \rangle, \tag{4.5}
\]

where the \(\hat{P}_{f_f}(z_{i,r}, z_{r,i}, k_{\perp,i,r}; \varepsilon)\) kernels are the \(d\)-dimensional Altarelli-Parisi splitting functions as given in Ref. [17].\(^\dagger\) The momentum fractions \(z_{i,r}\) and \(z_{r,i}\) are

\[
z_{i,r} = \frac{y_{iQ}}{y_{(ir)Q}} \quad \text{and} \quad z_{r,i} = \frac{y_{rQ}}{y_{(ir)Q}}, \tag{4.6}
\]

while the transverse momentum \(k_{\perp,i,r}\) is

\[
k_{\perp,i,r} = \zeta_{i,r} p_{\mu}^r - \zeta_{r,i} p_{\mu}^i + \zeta_{ir} p_{\mu}^{\tilde{r}}, \quad \zeta_{i,r} = z_{i,r} - \frac{y_{ir}}{\alpha_{ir} y_{(ir)Q}}, \quad \zeta_{r,i} = z_{r,i} - \frac{y_{ir}}{\alpha_{ir} y_{(ir)Q}}. \tag{4.7}
\]

\(^\dagger\) For the precise definition of the collinear and soft limits, refer to Refs. [6, 17].

\(^\dagger\) Note in particular that the ordering of the flavour indices and arguments of the Altarelli-Parisi kernels has no meaning in the notation of Ref. [17].
We used the abbreviations $y_{ir} = s_{ir}/Q^2 \equiv 2p_i \cdot p_r/Q^2$, $y_{i(Q)} = y_i + y_r$ with $y_{i(Q)} = 2p_i \cdot Q/Q^2$, $y_{r(Q)} = 2p_r \cdot Q/Q^2$ and $Q^\mu$ is the total four-momentum of the incoming electron and positron, while $\vec{p}_{ir}^\mu$ and $\alpha_{ir}$ are defined below in Eqs. (4.10) and (4.11) respectively. This choice for the transverse momentum is exactly perpendicular to the parent momentum $\vec{p}_{ir}$ and ensures that in the collinear limit $p_{ir}^\mu|p_r^\mu$, the square of $k_{\perp,i,r}^\mu$ behaves as

$$k_{\perp,i,r}^2 \simeq -s_{ir}z_{r,i}z_{i,r}, \quad (4.8)$$

as required (independently of $\zeta_{ir}$). Choosing

$$\zeta_{ir} = \frac{y_{ir}}{\alpha_{ir}}(z_{r,i} - z_{i,r}), \quad (4.9)$$

$k_{\perp,i,r}^\mu \to k_{\perp,i}^\mu$ in the collinear limit as can be shown by substituting the Sudakov parametrization of the momenta into Eq. (4.7) (with properly chosen gauge vector). Note however, that in a NLO computation, fulfilling Eq. (4.8) is sufficient to ensure the correct collinear behaviour of the subtraction term and the longitudinal component that is proportional to $\zeta_{ir}$ does not contribute due to gauge invariance of the matrix elements, so we may choose $\zeta_{ir} = 0$. The $m$ momenta $\{\vec{p}\}^{(ir)} \equiv \{\vec{p}_1, \ldots, \vec{p}_{ir}, \ldots, \vec{p}_{m+1}\}$ entering the matrix elements on the right hand side of Eq. (4.6) are

$$\vec{p}_{ir}^\mu = \frac{1}{1 - \alpha_{ir}}(p_i^\mu + p_r^\mu - \alpha_{ir}Q^\mu), \quad \vec{p}_n^\mu = \frac{1}{1 - \alpha_{ir}}p_n^\mu, \quad n \neq i, r, \quad (4.10)$$

where

$$\alpha_{ir} = \frac{1}{2}[y_{(ir)} - \sqrt{y_{(ir)}^2 - 4y_{ir}}]. \quad (4.11)$$

The soft and soft-collinear counterterms $S_r(\{p\})$ and $C_{ir}S_r(\{p\})$ are

$$S_{r_\nu}(\{p\}) = -8\pi\alpha_s\mu^2\sum_i\sum_{k \neq i} \frac{1}{2}S_{ik}(r)|M_{m,(i,k)}^{(0)}(\{\vec{p}\}^{(r)})|^2, \quad (4.12)$$

$$C_{ir_\nu}S_{r_\nu}(\{p\}) = 8\pi\alpha_s\mu^2\frac{2z_{i,r}}{s_{ir}}T_2^2|M_{m}^{(0)}(\{\vec{p}\}^{(r)})|^2. \quad (4.13)$$

If $r$ is a quark or antiquark, $S_r(\{p\})$ and $C_{ir}S_r(\{p\})$ are both zero. The eikonal factor in Eq. (4.12) is

$$S_{ik}(r) = \frac{2s_{ik}}{s_{ir}s_{rk}}, \quad (4.14)$$

and the momentum fractions entering Eq. (4.13) are given in Eq. (4.6). The $m$ momenta $\{\vec{p}\}^{(r)} \equiv \{\vec{p}_1, \ldots, \vec{p}_{m+1}\}$ ($p_r$ is absent) entering the matrix elements on the right hand sides of Eqs. (4.12) and (4.13) read

$$\vec{p}_n^\mu = \lambda_{r}^\mu[Q, (Q - p_r)/\lambda_r](p_n^\mu/\lambda_r), \quad n \neq r, \quad (4.15)$$
where
\[ \lambda_r = \sqrt{1 - y_r Q}, \] (4.16)
and
\[ \Lambda_\mu^K = g_\mu^K - \frac{2(K + \tilde{K})^\mu(K + \tilde{K})_\nu}{(K + \tilde{K})^2} + \frac{2K_\mu \tilde{K}_\nu}{K^2}. \] (4.17)

The matrix \( \Lambda_\mu^K \) generates a (proper) Lorentz transformation, provided \( K^2 = \tilde{K}^2 \neq 0. \)

The momentum mappings introduced in Eqs. (4.10) and (4.15) both lead to exact phase space factorization in the form of Eq. (4.2) where \([dp_1]\) is a one-parton phase space times a Jacobian factor,

\[ [dp_{1;m}] = J_m(p_r, \{\tilde{p}\}; Q) \frac{d^d p_r}{(2\pi)^{d-1}} \delta_+(p_r^2). \] (4.18)

With our definitions for the momentum mappings the Jacobian factor depends on the number of hard final-state momenta. In the case of the collinear mapping in Eq. (4.10), the Jacobian is

\[ J_{m}^{(ir)}(p_r, \tilde{p}_{ir}; Q) = \frac{(1 - \alpha_{ir})^{(m-1)(d-2)-1} y_{ir} Q}{\sqrt{(y_{ir} \tilde{p}_{ir} + y_{ir} Q - y_r Q)^2 + 4y_{ir} \tilde{p}_{ir} (1 - y_{ir} Q)}} \Theta(1 - \alpha_{ir}), \] (4.19)

and that for the soft mapping of Eq. (4.15) is

\[ J_{m}^{(r)}(p_r; Q) = \lambda_r^{(m-1)(d-2)-2} \Theta(\lambda_r). \] (4.20)

In Eq. (4.19) \( \alpha_{ir} \) is expressed in terms of the variable \( \tilde{p}_{ir} \),

\[ \alpha_{ir} = \frac{\sqrt{(y_{ir} \tilde{p}_{ir} + y_{ir} Q - y_r Q)^2 + 4y_{ir} \tilde{p}_{ir} (1 - y_{ir} Q)} - (y_{ir} \tilde{p}_{ir} + y_{ir} Q - y_r Q)}{2(1 - y_{ir} Q)}. \] (4.21)

This concludes the definition of the approximate cross section in Eq. (4.3). Note that our \( d\sigma^{R,A} \) in general contains fewer subtraction terms than the dipole scheme. Furthermore, we can decrease the number of terms in Eq. (4.4), because the symmetric treatment of the collinear subtractions is convenient for bookkeeping, but not essential in an actual computer code.

## 5 Integral of the approximate cross section

Next we evaluate the integral of the singly-unresolved approximate cross section over the one-parton unresolved phase space. Let us begin with integrating the collinear counterterm
$C_{ir}$. The transverse momentum $k_{\perp, i, r}$ as defined by Eq. (4.1) is orthogonal to $\vec{p}_{ir}$, therefore, the spin correlations generally present in Eq. (4.5) vanish after azimuthal integration \[6\]. Thus when evaluating the integral of the subtraction term $C_{ir}(\{p\})$ over the factorised one-parton phase space $[dp_{1;m}(p_r, \vec{p}_{ir}; Q)]$, we may replace the Altarelli–Parisi splitting functions $\hat{P}_{f,i,r}$ by their azimuthally averaged counterparts $P_{f,i,r}$. Then

\[
\int [dp_{1;m}(p_r, \vec{p}_{ir}; Q)]C_{ir}(\{p\}) = \frac{\alpha_s}{2\pi} \int S_{\epsilon} \left( \frac{\mu^2}{Q^2} \right)^\epsilon C_{ir}(y_{ir}Q; m-1, \epsilon) T_{ir}^2 |M_m^0(\{\vec{p}\}^{(ir)})|^2, \tag{5.1}
\]

where $y_{ir}Q = 2\vec{p}_{ir} \cdot Q$ and\[8\]

\[
\frac{\alpha_s}{2\pi} \int S_{\epsilon} \left( \frac{\mu^2}{Q^2} \right)^\epsilon C_{ir}(y_{ir}Q; m-1, \epsilon) = 8\pi\alpha_s\mu^2 \int [dp_{1;m}(p_r, \vec{p}_{ir}; Q)] \frac{1}{s_{ir}} P_{f,i,r}(\epsilon_{i,r}, \epsilon_{z,i,r}; \epsilon) \frac{1}{T_{ir}^2}. \tag{5.2}
\]

The evaluation of these integrals is discussed in Appendix A and here we give only the final results.

The function $C_{ir}(y_{ir}Q; n, \epsilon)$ depends on the momentum of the parent parton and the flavours of the daughter partons. The three independent flavour combinations are (we have $C_{ir}(y_{ir}Q; n, \epsilon) = C_{ri}(y_{ir}Q; n, \epsilon)$ and $C_{gg}(y_{ir}Q; n, \epsilon) = C_{gg}(y_{ir}Q; n, \epsilon)$)

\[
C_{qq}(x; n, \epsilon) = x^{-2\epsilon} \left[ 2 \left( I_n^{(-1)}(x; \epsilon) - I_n^{(0)}(x; \epsilon) \right) + (1 - \epsilon) I_n^{(1)}(x; \epsilon) \right], \tag{5.3}
\]

\[
C_{qg}(x; n, \epsilon) = \frac{T_R}{C_A} x^{-2\epsilon} \left[ I_n^{(0)}(x; \epsilon) - \frac{2}{1 - \epsilon} \left( I_n^{(1)}(x; \epsilon) - I_n^{(2)}(x; \epsilon) \right) \right], \tag{5.4}
\]

and

\[
C_{gg}(x; n, \epsilon) = 2x^{-2\epsilon} \left[ 2 \left( I_n^{(-1)}(x; \epsilon) - I_n^{(0)}(x; \epsilon) \right) + I_n^{(1)}(x; \epsilon) - I_n^{(2)}(x; \epsilon) \right]. \tag{5.5}
\]

The $I_n^{(k)}(x; \epsilon)$ functions are given explicitly in Ref. \[18\].

The expansion of Eqs. (5.3)–(5.5) in powers of $\epsilon$ is performed using the techniques of \[19\] to obtain

\[
C_{qq}(x; n, \epsilon) = \left[ \frac{1}{\epsilon^2} + \frac{3}{2\epsilon} - \frac{2}{\epsilon} \ln(x) + O(\epsilon^0) \right], \tag{5.6}
\]

\[
C_{gq}(x; n, \epsilon) = \frac{T_R}{C_A} \left[ \frac{2}{3\epsilon} + O(\epsilon^0) \right], \tag{5.7}
\]

\[
C_{gg}(x; n, \epsilon) = \left[ \frac{2}{\epsilon^2} + \frac{11}{3\epsilon} - \frac{4}{\epsilon} \ln(x) + O(\epsilon^0) \right]. \tag{5.8}
\]

\[\text{We chose the dependence on } m \text{ in the argument of this function as } m-1 \text{ because it is due to dependence on } m \text{ in the Jacobian factor in Eq. (4.19), where it appears as } m-1 \text{ in the exponent.}\]
The finite parts, not shown here, depend on $n$ and can be easily found for any given $n$ using the program of Ref. [20]. We quote those for $n = 2, 3$ and 4 in Appendix A.

Next consider the soft counterterm. Defining
\[
\frac{\alpha_s}{2\pi} \epsilon \left( \frac{\mu^2}{Q^2} \right)^\epsilon S_{ik}(y_{ik}, y_{iQ}, y_{kQ}; m - 1, \epsilon) = -8\pi\alpha_s\mu^{2\epsilon} \int [dp_{1;1m}(p_r; Q)] \frac{1}{2} S_{ik}(r),
\]
(5.9)
we obtain
\[
\int [dp_{1;1m}(p_r; Q)] S_r(\{p\}) = \frac{\alpha_s}{2\pi} S_{\xi} \left( \frac{\mu^2}{Q^2} \right)^\epsilon \sum_{i<k} S_{ik}(y_{ik}, y_{iQ}, y_{kQ}; m - 1, \epsilon) |\mathcal{M}_{m;\{i,k\}}^{(0)}(\{\tilde{p}\})|^2.
\]
(5.10)
Finally, integrating the soft-collinear subtraction, Eq. (4.13) we get
\[
\int [dp_{1;1m}(p_r; Q)] C_m S_r(\{p\}) = \frac{\alpha_s}{2\pi} S_{\xi} \left( \frac{\mu^2}{Q^2} \right)^\epsilon \mathcal{C}(m - 1, \epsilon) T^{2}_{\lambda} |\mathcal{M}_{m;\{i,k\}}^{(0)}(\{\tilde{p}\})|^2,
\]
(5.11)
with
\[
\frac{\alpha_s}{2\pi} S_{\xi} \left( \frac{\mu^2}{Q^2} \right)^\epsilon \mathcal{C}(m - 1, \epsilon) = 8\pi\alpha_s\mu^{2\epsilon} \int [dp_{1;1m}(p_r; Q)] \frac{2}{s_{ir}} \frac{z_{ir}}{e^{z_{ir}}}.
\]
(5.12)
The evaluation of the integrals in Eqs. (5.9) and (5.12) is again discussed in Appendix A and here we give only the final results.

The soft functions $S_{ik}(y_{ik}, y_{iQ}, y_{kQ}; n, \epsilon)$ are expressed with the standard beta and hypergeometric functions [21],
\[
S_{ik}(y_{ik}, y_{iQ}, y_{kQ}; n, \epsilon) = -\frac{n(1 - \epsilon)(1 - 2\epsilon)}{\epsilon^2} B(1 - \epsilon, 1 - \epsilon) B(1 - 2\epsilon, n(1 - \epsilon))
\]
\[
\times \frac{y_{ik}}{y_{iQ} y_{kQ}} 2F_1(1, 1, 1 - \epsilon, 1 - \frac{y_{ik}}{y_{iQ} y_{kQ}}),
\]
(5.13)
while
\[
\mathcal{C}(n, \epsilon) = \left[ \frac{n(1 - \epsilon)(1 - 2\epsilon)}{\epsilon^2} + 2 \right] B(1 - \epsilon, 1 - \epsilon) B(1 - 2\epsilon, n(1 - \epsilon)).
\]
(5.14)
Using the expansion
\[
z 2F_1(1, 1, 1 - \epsilon, 1 - z) = z^{-\epsilon} \left[ 1 + \epsilon^2 \text{Li}_2(1 - z) + O(\epsilon^3) \right]
\]
(5.15)
for the hypergeometric function, we find
\[
S_{ik}(y_{ik}, y_{iQ}, y_{kQ}; n, \epsilon) = -\frac{1}{\epsilon^2} - \frac{2}{\epsilon} \sum_{k=1}^{n} \frac{1}{k} + \frac{1}{\epsilon} \ln \frac{y_{ik}}{y_{iQ} y_{kQ}} + O(\epsilon^0),
\]
(5.16)
\[
\mathcal{C}(n, \epsilon) = \frac{1}{\epsilon^2} + \frac{2}{\epsilon} \sum_{k=1}^{n} \frac{1}{k} + O(\epsilon^0).
\]
(5.17)
Notice that
\[
S_{ik}(y_{ik}, y_{kQ}, y_{kQ}; m, \varepsilon) + C S(m, \varepsilon) = \frac{1}{\varepsilon} \ln \frac{y_{ik}}{y_{iQ} y_{kQ}} + O(\varepsilon^0). \tag{5.18}
\]
We quote the finite part of this expansion in Appendix A.

We are now in a position to calculate \( \int_1^\infty d\sigma_{Rm+1}^{RA} \). Let us begin by recalling the form of the fully differential real cross section \( d\sigma_{Rm+1}^{R} \) given in Eq. (3.13). Accordingly, the approximate cross section times the jet function is
\[
d\sigma_{m+1}RA J_m = N \sum_{\{m+1\}} d\phi_{m+1}(\{p\}) \frac{1}{S_{\{m+1\}}} \times \sum_r \left[ \sum_{i \neq r} \frac{1}{2} C_{ir}(\{p\}) J_m(\{\tilde{p}\})^{(ir)} + \left( S_r(\{p\}) - \sum_{i \neq r} C_{ir} S_r(\{p\}) \right) J_m(\{\tilde{p}\})^{(r)} \right]. \tag{5.19}
\]
In order to evaluate \( \int_1^\infty d\sigma_{m+1}^{RA} \) we first use the phase space factorization property of Eq. (4.2), then perform the integration to obtain
\[
\int_1^\infty d\sigma_{m+1}^{RA} J_m = N \sum_{\{m+1\}} d\phi_{m}(\{\tilde{p}\}) \frac{1}{S_{\{m+1\}}} \frac{\alpha_s}{2\pi} S_{\varepsilon} \left( \frac{\mu^2}{Q^2} \right)^\varepsilon \times \sum_r \sum_{i \neq r} \left[ \frac{1}{2} C_{ir}(m - 1, \varepsilon) T^2_i |M^{(0)}_m(\{\tilde{p}\})|^2 J_m(\{\tilde{p}\}) + \sum_{k \neq i, r} S_{ik}(m - 1, \varepsilon) |M^{(0)}_{m;i,k}(\{\tilde{p}\})|^2 J_m(\{\tilde{p}\}) \right. \nonumber \\
- \left. \sum_{k \neq i, r} S_{ik}(m - 1, \varepsilon) |M^{(0)}_{m;i,k}(\{\tilde{p}\})|^2 J_m(\{\tilde{p}\}) \right]. \tag{5.20}
\]
This result is not yet in the form of an \( m \)-parton contribution times a factor. In order to rewrite Eq. (5.20) in such a form we still need to perform the counting of symmetry factors for going from \( m \) partons to \( m + 1 \) partons, which is very similar to the counting in Ref. [6]. We give the details of the calculation in Appendix B. Inserting equation Eq. (B.6) into Eq. (5.20), we obtain
\[
\int_1^\infty d\sigma_{m+1}^{RR,RA} J_m = N \sum_{\{m\}} d\phi_{m}(\{\tilde{p}\}) \frac{1}{S_{\{m\}}} \frac{\alpha_s}{2\pi} S_{\varepsilon} \left( \frac{\mu^2}{Q^2} \right)^\varepsilon \times \sum_i \left[ C_i(m - 1, \varepsilon) T^2_i |M^{(0)}_m(\{\tilde{p}\})|^2 + \sum_{k \neq i} S_{ik}(m - 1, \varepsilon) |M^{(0)}_{m;i,k}(\{\tilde{p}\})|^2 \right], \tag{5.21}
\]
where we have introduced the flavour-dependent functions
\begin{equation}
C_q = C_{qq} - CS, \quad C_g = \frac{1}{2} C_{gg} + n_f C_{qg} - CS.
\end{equation}

We write the final result, dropping the $m$-parton jet function that appears on both sides of Eq. (5.21), as follows
\begin{equation}
\int_1^\infty d\sigma^{RA}_{m+1} = d\sigma^B_m \otimes I(m-1, \varepsilon),
\end{equation}
where $d\sigma^B_m$ is the Born-level cross section for the emission of $m$ partons and
\begin{equation}
I(\{p\}; m-1, \varepsilon) = \frac{\alpha_s}{2\pi} S_\varepsilon \left( \frac{\mu^2}{Q^2} \right)^\varepsilon \times \sum_i \left[ C_i(y_iQ; m-1, \varepsilon) T_i^2 + \sum_{k \neq i} S_{ik}(y_{ik}, y_iQ, y_kQ; m-1, \varepsilon) T_i T_k \right]
\end{equation}
is an operator acting on the colour space of $m$ partons that depends on the colour charges and momenta of the $m$ partons in $|\mathcal{M}_m|^2$ (different from the insertion operator of Ref. [6]). The notation on the right hand side of Eq. (5.23) means that one has to write down the expression for $d\sigma^B_m$ and then replace the Born level squared matrix element $|\mathcal{M}_m|^2 = \langle \mathcal{M}^{(0)}_m | \mathcal{M}^{(0)}_m \rangle$, with
\begin{equation}
\langle \mathcal{M}^{(0)}_m | I(m-1, \varepsilon) | \mathcal{M}^{(0)}_m \rangle.
\end{equation}
The parameter $m-1$ in the argument of the insertion operator matches the arguments of the collinear and soft functions, not the number of coloured legs in the matrix element (in this case it is one less). Using colour conservation ($T_i^2 = -\sum_{k \neq i} T_i T_k$) to combine the $C_i$ and $S_{ik}$ contributions, Eqs. (5.6)--(5.8) and Eq. (5.18), it is straightforward to check that our insertion operator differs from that defined in (7.26) of Ref. [6] only in finite terms,
\begin{equation}
I(\{p\}; m-1, \varepsilon) = I(\{p\}; \varepsilon) + O(\varepsilon^0),
\end{equation}
where
\begin{equation}
I(\{p\}; \varepsilon) = \frac{\alpha_s}{2\pi} S_\varepsilon \left( \frac{\mu^2}{Q^2} \right)^\varepsilon \sum_i \left( T_i^2 \frac{1}{\varepsilon^2} + \gamma_i \frac{1}{\varepsilon} + \sum_{k \neq i} T_i T_k \frac{1}{\varepsilon} \ln y_{ik} \right)
\end{equation}
with the usual flavour constants
\begin{equation}
\gamma_q = \frac{3}{2} C_F, \quad \gamma_g = \frac{\beta_0}{2}.
\end{equation}
It follows that \( \int_1^\infty d\sigma^{RA}_{m+1} \), as defined here, correctly cancels all $\varepsilon$-poles of the virtual cross section $d\sigma^V_m$. As a result, the integrand of the $m$-parton contribution,
\begin{equation}
d\sigma_{m}^{NLO} = \left[ d\sigma_m^V + d\sigma_m^B \otimes I(m-1, \varepsilon) \right]_{\varepsilon=0} J_m,
\end{equation}
is finite and integrable in four dimensions (the potential kinematical singularities are screened by the jet function $J_m$). This finite integrand is given in Appendix A.
6 Checks

The cancellation of the singularities is a strong check on the correctness of the proposed scheme. We have performed such checks by approaching soft or collinear regions of the phase space from a randomly chosen point and computing the ratio of the \((m+1)\) parton squared matrix element and the subtraction terms. This ratio always approaches one. Further checks can be performed by comparing predictions for distributions to the predictions of other well-established computer codes for computing QCD jet cross sections at the NLO accuracy. Currently, our scheme is worked out only for colourless particles in the initial state, therefore, we decided to compare the predictions for the three-jet event-shape distributions thrust \((T)\) and \(C\)-parameter in electron-positron annihilation, when the jet function is a functional

\[
J_n(p_1, \ldots, p_n; O) = \delta(O - O_3(p_1, \ldots, p_n)),
\]

with \(O_3(p_1, \ldots, p_n)\) being the value of either \(\tau \equiv 1 - T\) or \(C\) for a given event \((p_1, \ldots, p_n)\).

Once the phase space integrations in Eq. (2.5) are carried out, the NLO differential cross section for the three-jet observable \(O\) at a fixed scale \(Q\) takes the general form

\[
\Sigma(O) \equiv \frac{1}{\sigma_0} \frac{d\sigma}{dO} \left(O\right) = \frac{\alpha_s(Q)}{2\pi} B_O(O) + \left(\frac{\alpha_s(Q)}{2\pi}\right)^2 C_O(O).
\]

We computed the \(B_O(O)\) Born-level predictions as well as the \(C_O(O)\) correction functions and found complete agreement with the corresponding tables of the benchmark calculation of Kunszt and Nason [22].

7 Conclusions

We have defined a new subtraction scheme for computing NLO corrections to QCD jet cross sections. For NLO computations the new scheme does not provide any particular advantage over the already existing methods and gives identical predictions. The need for the new scheme was motivated by studies in trying to extend the existing NLO subtraction schemes to computations at the NNLO accuracy.

The new scheme is completely general in the sense that any number of massless final state coloured or colourless particles are allowed. It is worked out for processes without coloured partons in the initial state. The extension to deep-inelastic scattering and hadron collisions does not pose conceptual difficulties, but left for later work.
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A Integrals over the factorised single-particle phase
space

In this appendix we compute the collinear and soft functions $C_{ir}$, $S_{r}$ and $CS$ and present
their expansions relevant to three-, four- and five-jet production.

A.1 Collinear integrals

We recall the definition of the collinear functions, Eq. (5.2), from which one trivially gets

$$C_{ir}(y_{ir}Q; m - 1, \varepsilon) = \frac{(4\pi)^2}{S_{\varepsilon}} (Q^2)^{-1+\varepsilon} \int [dp^{(ir)}_{1:m}(p_r, \tilde{p}_{ir}; Q)] \frac{1}{y_{ir}} P_{fi,f_r}(z_{i,r}, z_{r,i}; \varepsilon) \frac{1}{T_{ir}^2}, \quad (A.1)$$

where the azimuthally averaged splitting kernels are

$$P_{g,g_{r}}(z_r) = 2C_A \left[ \frac{1 - z_r}{z_r} + \frac{z_r}{1 - z_r} + (1 - z_r)z_r \right], \quad (A.2)$$

$$P_{q,q_{r}}(z_r; \varepsilon) = T_R \left[ 1 - \frac{2}{1 - \varepsilon}(1 - z_r)z_r \right], \quad (A.3)$$

$$P_{q,g_{r}}(z_r; \varepsilon) = C_F \left[ \frac{1 + (1 - z_r)^2}{z_r} - \varepsilon z_r \right]. \quad (A.4)$$

We parametrise the factorised one-particle phase space in terms of the variables $\alpha_{ir}$, $y_{ir}$
and momentum fraction $z_r \equiv z_{r,i}$, the latter being defined in Eq. (4.6). We find

$$[dp^{(ir)}_{1:m}(p_r, \tilde{p}_{ir}; Q)] = (1 - \alpha_{ir})^{[(m-1)(d-2)-1]} y_{ir}Q \left( Q^2 \right)^{1-\varepsilon} S_{\varepsilon} \frac{(4\pi)^2}{(m-1)(d-2)-1}$$

$$\times \delta \left( y_{ir} - \alpha_{ir} (\alpha_{ir} + y_{ir}Q - \alpha_{ir}y_{ir}Q) \right) d\alpha_{ir} dy_{ir} dz_r$$

$$\times (z_+ - z_-)^{-1+2\varepsilon} \left[ y_{ir} (z_+ - z_r) (z_r - z_-) \right]^{-\varepsilon}$$

$$\times \Theta(1 - \alpha_{ir}) \Theta(\alpha_{ir}) \Theta(z_+ - z_r) \Theta(z_r - z_-), \quad (A.5)$$
The limits of the $z_\nu$-integral are

$$z^{(+)}(\alpha_{ir}, y_{ir}, y_{ir} Q) = \frac{y_{ir}}{\alpha_{ir} (2\alpha_{ir} + y_{ir} Q - \alpha_{ir} y_{ir} Q)},$$

$$z^{(-)}(\alpha_{ir}, y_{ir}, y_{ir} Q) = \frac{y_{ir}}{(\alpha_{ir} + y_{ir} Q - \alpha_{ir} y_{ir} Q)(2\alpha_{ir} + y_{ir} Q - \alpha_{ir} y_{ir} Q)},$$

with $z^{(+)} + z^{(-)} = 1$. We now insert Eqs. (A.2)–(A.5) into Eq. (A.1) and obtain the results presented in Eqs. (5.3)–(5.5), with

$$x^{-2\varepsilon} I_n^{(k)}(x; \varepsilon) = \int_0^1 d\alpha \int_0^1 dy \int_{z^{(\alpha,y,x)}}^{z^{(\alpha,y,x)}} dz \, \delta(y - \alpha (x + x - \alpha x))$$

$$\times (1 - \alpha)^{2n(1-\varepsilon)-1} [z^{+}(\alpha, y, x) - z^{-}(\alpha, y, x)]^{-1+2\varepsilon}$$

$$\times [y (z^{+}(\alpha, y, x) - z) (z - z^{-}(\alpha, y, x))]^{-\varepsilon} \frac{x}{y} z^k. \quad (A.7)$$

These integrals are invariant under the $z \leftrightarrow 1 - z$ transformation, therefore, not all are independent; those with positive and odd $n$ can be expressed with the others. For instance, $I_n^{(1)}(x; \varepsilon) = \frac{1}{2} I_n^{(0)}(x; \varepsilon)$. The integral over $y$ is trivial by making use of the $\delta$ function. After a complicated sequence of integral transformations, the other two integrals can be transformed into known integrals. The results can be found in Ref. [18].

In an actual computation we need the expansion of the collinear functions in $\varepsilon$ to $O(\varepsilon^0)$. The pole terms are independent of $m$ and are given in Eqs. (5.6)–(5.8). Here we give the $O(\varepsilon^0)$ terms, denoted by $\mathcal{FinC}_{i\nu}$. For $n = m - 1 = 1$ we need the functions only at $x = 1$, where

$$\mathcal{FinC}_{qg}(1; 1) = 7 - \frac{\pi^2}{2}, \quad \mathcal{FinC}_{qg}(1; 1) = \frac{2T_R}{3C_A} \frac{3 \ln 2 - 11}{3}. \quad (A.8)$$

For $n = m - 1 \geq 2$ the results for arbitrary $n$ are somewhat cumbersome combinations of elementary, $2F_1$ and $3F_2$ functions. Equivalent simpler expressions can be given in the following way:

$$\mathcal{FinC}_{qg}(x; n) = c_{n\nu}^{\frac{3}{2}} + 2 \left( \ln^2 x + \Phi(1 - x, 2, 2n - 1) + \text{Li}_2(1 - x) - \frac{\pi^2}{4} \right)$$

$$+ \left( d_{n,0} - 3 \right) \ln x - \sum_{i=1}^{n-1} \frac{d_{n,i}}{i} {2F}_1(i, 1, 1 + i, 1 - x)$$

$$+ \sum_{i=0}^{n-1} \frac{d_{n,i}}{2n - 1 - i} {2F}_1(2n - 1 - i, 1, 2n - i, 1 - x), \quad (A.9)$$

$$\mathcal{FinC}_{qg}(x; n) = \frac{2T_R}{3C_A} \left( c_n \frac{\ln x - \frac{1}{2n - 1}}{2n - 1} {2F}_1(2n - 1, 1, 2n, 1 - x) \right.$$ 

$$+ \frac{2n - 1}{4n} \frac{x}{2} {3F}_1 \left( 1, 2n, 2n + 1, \frac{2 - x}{2} \right). \quad (A.10)$$
Table 1: Constants used in the finite part of the expansion of the collinear functions $C_{ir}$

| $m$ | $n$ | $c^g_n$ | $c^n_q$ | $d_{n,0}$ | $d_{n,1}$ | $d_{n,2}$ | $d_{n,3}$ |
|-----|-----|---------|---------|-----------|-----------|-----------|-----------|
| 3   | 2   | $\frac{19}{4}$ | $-\frac{11}{3}$ | $-\frac{3}{2}$ | 1         |           |           |
| 4   | 3   | $\frac{89}{24}$ | $-\frac{17}{4}$ | $-\frac{8}{3}$ | $\frac{3}{2}$ | $\frac{1}{3}$ |           |
| 5   | 4   | $\frac{959}{360}$ | $-\frac{277}{60}$ | $-\frac{17}{5}$ | $\frac{5}{3}$ | $\frac{3}{5}$ | $\frac{1}{6}$ |

The finite parts of the collinear functions for the gluon are not independent from the other two:

$$
\mathcal{F}\text{in}C_{gg}(x; n) = 2\mathcal{F}\text{in}C_{qs}(x; n) - \frac{C_A}{T_R}\mathcal{F}\text{in}C_{q\bar{q}}(x; n) - \frac{2}{3}.
$$

(A.11)

The constants $c^g_n$ and $d_{n,i}$ can be found in Table 1. The function $\Phi(z, s, a)$ is the Lerch transcendent \[\text{\cite{21}},\] defined by the series

$$
\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(a + k)^s},
$$

(A.12)

for which numerical codes for the evaluation exist \[\text{\cite{23}}.\] While numerical codes for the hypergeometric functions also exist, it is actually faster to expand the $_2F_1$ functions using their series expansion because for integer arguments the representation is a sum of polynomials and logarithms.

### A.2 Soft integrals

We recall the definition of the soft functions, Eqs. (5.9) and (5.12), from which one trivially gets

$$
S_{ik}(y_{ik}, y_{iQ}, y_{kQ}; m - 1, \varepsilon) = -\frac{(4\pi)^2}{S_\varepsilon} (Q^2)^{-1+\varepsilon} \int [dp_{1,m}(p_r; Q)] \frac{y_{ik}}{y_{ir}y_{kr}}
$$

(A.13)

and

$$
CS(m - 1, \varepsilon) = \frac{(4\pi)^2}{S_\varepsilon} (Q^2)^{-1+\varepsilon} \int [dp_{1,m}(p_r; Q)] \frac{z_{i,r}}{y_{ir}z_{r,i}}.
$$

(A.14)

Parametrizing the phase space with energy and angles, these integrals can be computed as done in Appendix B of Ref. \cite{6} (see also \cite{24}), leading to Eqs. (5.13) and (5.14). Although, in Eq. (5.22) we have combined the CS functions with the collinear ones because these are
multiplied with the same colour factor, nevertheless one can always use colour conservation
\( T_i^2 = - \sum_{k \neq i} T_i T_k \) to combine the CS and \( S_{ik} \) contributions. Therefore, in presenting the \( O(\varepsilon) \) terms in the \( \varepsilon \)-expansion, we write only the finite term of their sum, which is simpler than the individual contributions,

\[
\mathcal{F} \text{in}[S_{ik}(y_{ik}, y_{iQ}, y_{kQ}; n)] + \text{CS}(n) = \frac{2}{n} + \text{Li}_2 \left( 1 - \frac{y_{iQ} y_{kQ}}{y_{ik}} \right) + 2 \ln \left( \frac{y_{ik}}{y_{iQ} y_{kQ}} \right) \sum_{k=1}^{n} \frac{1}{k}. \quad (A.15)
\]

In order to spell out the finite part of the \( m \)-parton contribution, \( d\sigma_{m}^{\text{NLO}} \), we define the finite part of the one-loop amplitude as

\[
|\mathcal{M}_m^{(1)}(\{p\})| = -\frac{1}{2} I(\{p\}; \varepsilon)|\mathcal{M}_m^{(0)}(\{p\})| + \mathcal{F}\text{in}[\mathcal{M}_m^{(1)}(\{p\})]. \quad (A.16)
\]

Then

\[
[d\sigma_{m}^{V} + d\sigma_{m}^{B} \otimes I(m-1, \varepsilon)]_{\varepsilon=0} = \mathcal{N} \sum_{\{m\}} d\phi_m \frac{1}{S_m} \left\{ 2 \Re\langle\mathcal{M}_m^{(0)}(\{p\})|\mathcal{F}\text{in}|\mathcal{M}_m^{(1)}(\{p\})\rangle^2 \\
+ \frac{\alpha_s}{2\pi} \sum_i \left[ \sum_{k \neq i} \mathcal{F} \text{in} [S_{ik}(y_{ik}, y_{iQ}, y_{kQ}; m-1) + \text{CS}(m-1)] |\mathcal{M}_m^{(0)}(\{p\})|^2 \\
+ \mathcal{F} \text{in} [C_i(y_{iQ}; m-1) + \text{CS}(m-1) T_i^2 |\mathcal{M}_m^{(0)}(\{p\})|^2] \right] \right\}. \quad (A.17)
\]

## B Calculation of symmetry factors

Consider an \( m \)-parton configuration with \( m_f \) quarks of flavour \( f \), \( \bar{m}_f \) antiquarks of flavour \( \bar{f} \) and \( m_g \) gluons. From this configuration we can obtain an \( m+1 \) parton configuration by changing

(i) \( m_g \to m_g + 1 \) or (ii) \( m_f \to m_f + 1, \bar{m}_f \to \bar{m}_f + 1, m_g \to m_g - 1 \). \quad (B.1)

The ratios of symmetry factors corresponding to the two cases are

\[
\frac{S_{\{m\}}^{(i)}}{S_{\{m+1\}}} = \frac{\ldots m_g!}{\ldots (m_g+1)!} = \frac{1}{m_g + 1}, \quad (B.2)
\]

\[
\frac{S_{\{m\}}^{(ii)}}{S_{\{m+1\}}} = \frac{\ldots m_f!\bar{m}_f!m_g!}{\ldots (m_f + 1)! (\bar{m}_f + 1)! (m_g - 1)!} = \frac{m_g}{(m_f + 1)(\bar{m}_f + 1)}. \]
We then have

\[
\sum_{m+1} \frac{1}{S_{m+1}} \sum_i \sum_{r \neq i} \ldots = \sum_{m}^{(i)} \frac{1}{S_{m}} \frac{1}{m_g + 1} \left( \sum_{i=q_f} \sum_{r=q_f} \ldots + \sum_{i=g} \sum_{r=q_f} \ldots \right) \\
+ \sum_{i=q_f} \sum_{r=g} \ldots + \sum_{i=g} \sum_{r=q_f} \ldots + \sum_{i=g} \sum_{r=g} \ldots \\
+ \sum_{m}^{(ii)} \frac{1}{S_{m}} \frac{m_g}{(m_f + 1)(\bar{m}_f + 1)} \left( \sum_{i=q_f} \sum_{r=q_f} \ldots + \sum_{i=g} \sum_{r=q_f} \ldots \right). \tag{B.3}
\]

Also

\[
\sum_{i=q_f} \sum_{r=g} \ldots = (m_g + 1) \sum_{i=q_f} \sum_{r=g} \ldots, \quad \sum_{i=g} \sum_{r=q_f} \ldots = (m_g + 1) \sum_{i=g} \sum_{r=q_f} \ldots, \\
\sum_{i=q_f} \sum_{r=q_f} \ldots = (m_f + 1) (\bar{m}_f + 1) \sum_{i=q_f} \sum_{r=q_f} \ldots, \\
\sum_{i=g} \sum_{r=g} \ldots = \sum_{i=q_f} \sum_{r=q_f} \ldots, \\
\sum_{i=g} \sum_{r=g} \ldots = \sum_{i=q_f} \sum_{r=q_f} \ldots. \tag{B.4}
\]

thus we find

\[
\sum_{m+1} \frac{1}{S_{m+1}} \sum_i \sum_{r \neq i} \ldots = \sum_{m}^{(i)} \frac{1}{S_{m}} \left( \sum_{i=q_f, r=g} \ldots + \sum_{i=q_f, r=\bar{q}_f} \ldots \right) \\
+ \sum_{i=q_f} \sum_{r=g} \ldots + \sum_{i=g} \sum_{r=q_f} \ldots + \sum_{i=g} \sum_{r=g} \ldots \\
+ \sum_{m}^{(ii)} \sum_{i=g} \sum_{r=\bar{q}_f} \ldots + \sum_{i=q_f} \sum_{r=q_f} \ldots \right). \tag{B.6}
\]

The soft contribution to each sum in Eq. (B.3) is nonvanishing only if \( r \) is a gluon, so we have indicated the flavour of \( r \) in each summation in Eq. (B.6).
C Volume of the phase space in $d$ dimensions

In this appendix, we present a simple derivation of the formula, obtained in Ref. [25], for the volume of the phase space of $m$ massless particles in $d$ dimensions, which is a side product of the phase-space factorization presented in Eqs. (4.18) and (4.20).

According to Eqs. (4.2) and (4.18)

$$\int d\phi_{m+1}(Q) = \int d\phi_m(Q) I(Q; m, \varepsilon), \quad (C.1)$$

where, using Eqs. (4.16) and (4.20),

$$I(Q; m, \varepsilon) = \int (1 - y_{rQ})^{1/2} \left[ (m - 1)(d - 2) - 2 \right] \frac{d^d p_r}{(2\pi)^{d-1}} \delta_+(p_r^2). \quad (C.2)$$

Working in the c.m. frame, we parametrise the one-particle phase-space measure with the energy and angles. The integrand, $(1 - y_{rQ}) = (1 - 2E/Q)$, depends only on the energy, therefore,

$$I(Q; m, \varepsilon) = \Omega_{d-1} \frac{1}{(2\pi)^{d-1}} \int_0^{Q/2} \frac{1}{2} E^{d-3} dE \left( 1 - 2E/Q \right)^{1/2} \left[ (m - 1)(d - 2) - 2 \right], \quad (C.3)$$

where

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \quad (C.4)$$

is the volume of the $d$-dimensional hypersurface. Introducing the new variable $x = 2E/Q$, the integral in Eq. (C.3) is readily obtained,

$$I(Q; m, \varepsilon) = (Q^2)^{d/2} \frac{2^{3-2d} \pi^{(d-1)/2}}{\Gamma((d-1)/2)} B\left(d - 2, [(m - 1)(d - 2)]/2 \right). \quad (C.5)$$

We can get rid of the $\sqrt{\pi}$ factors by using the identity,

$$\sqrt{\pi} = 2^{d-3} \frac{\Gamma\left(\frac{d-2}{2}\right) \Gamma\left(\frac{d-1}{2}\right)}{\Gamma(d-2)} \quad (C.6)$$

Thus, we derived the following recursion relation:

$$\int d\phi_{m+1}(Q) = 2^{-d} \pi^{d/2} (Q^2)^{d/2} \frac{\Gamma\left(\frac{(m-1)(d-2)}{2}\right) \Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{(m+1)(d-2)}{2}\right)} \int d\phi_m(Q). \quad (C.7)$$

Starting from the known expression for the two-particle phase space

$$\int d\phi_2(Q) = 2^{1-d} \pi^{1-d/2} (Q^2)^{d/2} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma(d-2)}, \quad (C.8)$$

and using the recursion relation in Eq. (C.7), it is easy to obtain the result quoted in Ref. [25] for $d = 4 - 2\varepsilon$. 

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