Spinodal decomposition during the hadronization stage at RHIC?

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The expansion of strongly interacting matter formed in high-energy nuclear collisions drives the system through the region of phase coexistence. The present study examines the associated spinodal instability and finds that the degree of amplification may be sufficient to raise the prospect of using the spinodal pattern formation as a diagnostic tool for probing the hadronization phase transition.

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I. INTRODUCTION

The central goal of high-energy nuclear collisions is to explore the expected phase transition from the familiar hadronic world to a plasma of quarks and gluons. This phase change of strongly interacting matter is expected to be of first order with a critical temperature $T_c$ in the range of $150 - 200$ MeV, although this remains to be determined experimentally.

When two heavy nuclei collide at sufficiently high energies, such as those provided by the Relativistic Heavy Ion Collider (RHIC), the matter in the central rapidity region is formed well above the critical temperature $T_c$ in a state of rapid longitudinal expansion. The expansion then causes the energy density $\varepsilon$ to continually drop, thus forcing the system through the critical phase region towards hadronization and subsequent chemical and kinetic freezeout. During this evolution the system maintains local equilibrium and one may employ fluid dynamical studies of nuclear collisions to explore the expected phase transition from the familiar hadronic world to a plasma of quarks and gluons.

II. EQUATION OF STATE

To estimate the behavior of $p(\varepsilon)$ through the region of phase coexistence, we employ a cubic spline function to interpolate between the hadron and plasma phases. As is commonly done \[2\ \[3\ \[4\], we approximate the former phase by an ideal gas of hadrons, while we take a bag of non-interacting quarks and gluons for the latter.

In general, the contributions to $p$ and $\varepsilon$ from a non-degenerate particle of mass $m$ are

\begin{align}
  p_\kappa(m) &= \kappa \frac{T^4}{2\pi^2} \sum_{n=1}^{\infty} \frac{\kappa^n}{n^4} \left( \frac{nm}{T} \right)^2 K_2 \left( \frac{nm}{T} \right) \times \frac{\pi^2 T^4}{90}, \\
  \varepsilon_\kappa(m) &= \kappa \frac{T^4}{2\pi^2} \sum_{n=1}^{\infty} \frac{\kappa^n}{n^4} \left[ 3 \left( \frac{nm}{T} \right)^2 K_1 \left( \frac{nm}{T} \right) \right] \times \frac{\pi^2 T^4}{30},
\end{align}

where $\kappa = +$ for bosons and $\kappa = -$ for fermions.

The hadronic phase is approximated as an ideal gas of 14 hadronic species $i = \pi, K, \eta, \rho, \ldots, N, \Lambda, \Sigma, \Delta$ with masses $m_i$ and degeneracies $g_i$ (which include antiparticles where appropriate). Since we are focussing on the central rapidity region of collisions at RHIC, we shall assume that all the chemical potentials vanish, $\mu_B = \mu_Q = \mu_S = 0$. We then have

\begin{align}
  p_{\text{had}}(T) &= \sum_{i=\pi}^{\Delta} g_i \varepsilon_{\pi i}(m_i), \\
  \varepsilon_{\text{had}}(T) &= \sum_{i=\pi}^{\Delta} g_i \varepsilon_{\pi i}(m_i).
\end{align}

The hadronic equation of state is shown in Fig. The critical temperature, for which we use $T_c = 170$ MeV, the pressure is $p_c = 54$ MeV fm$^{-3}$, while the energy density is $\varepsilon_1 = 220$ MeV fm$^{-3}$. The slope of $p(\varepsilon)$ gives the square of the sound speed at $T_c$, $(\partial p_{\text{had}}/\partial \varepsilon)_{T_c} = v_s^2 = 0.27$.

For the plasma phase, we assume that the system can be approximated as an ideal gas of quarks and gluons confined within a bag of suitable (negative) pressure $-B$. The gluons as well as the u and d quarks are taken to be massless, while the s quark is given a mass of $m_s = 150$ MeV (though the results would change little if it were also zero). The corresponding degeneracies are
\( g_g = 16, g_s = 24, \) and \( g_s = 12. \)

\[
p_{\text{qgp}}(T) = [g_g + \frac{7}{8} g_s] \frac{3}{10} T^4 + g_s p -(m_s) - B , \quad (4)
\]

\[
\varepsilon_{\text{qgp}}(T) = [g_g + \frac{7}{8} g_s] \frac{3}{10} T^4 + g_s \varepsilon -(m_s) + B . \quad (5)
\]

Thus the pressure starts at \(-B\), while the energy density starts at \(B\). These quantities increase more rapidly than their hadronic counterparts and the plasma pressure overtakes the hadronic pressure at the critical temperature. This condition determines the bag constant, \(B^{1/4} = 250\) MeV. The resulting plasma equation of state \(p_{\text{qgp}}(\varepsilon)\) is shown in Fig. 1. The critical energy density of the plasma is \(\varepsilon_2 \equiv \varepsilon_{\text{qgp}}(T_c) = 2220\) MeV-fm\(^{-3}\), so that the latent heat of the phase transition is \(\Delta \varepsilon \equiv \varepsilon_2 - \varepsilon_1 = 2000\) MeV-fm\(^{-3}\). The function \(p_{\text{qgp}}(\varepsilon)\) is practically linear (its slight non-linearity arises from \(m_s\)) and it has the slope \((dp_{\text{qgp}}/d\varepsilon)_2 = v_s^2 = 0.33\) at \(T_c\).

Through the region of phase coexistence we employ a cubic spline function that matches the pressure and its derivative (i.e., the speed of sound) at both ends,

\[
p(\varepsilon) = p_+ + \xi(\varepsilon) [(\varepsilon_2 - \varepsilon)v_s^2 - (\varepsilon - \varepsilon_1)v_s^2] \bar{\xi}(\varepsilon) , \quad (6)
\]

where \(\xi \equiv (\varepsilon - \varepsilon_1)/\Delta \varepsilon\) and \(\bar{\xi} \equiv (\varepsilon_2 - \varepsilon)/\Delta \varepsilon\). The resulting curve is shown in Fig. 1. This curve rises initially at \(\varepsilon_1\) with the slope \(v_2\) but it gradually bends over and turns downwards. After crossing the mixed-phase line \(p = p_c\) it starts bending upwards and finally joins the plasma curve at \(\varepsilon_2\) with the slope \(v_s^2\). The crossing occurs at the energy density \(\varepsilon_\ast\) given by \((v_s^2 + v_s^2)\varepsilon_\ast = v_s^2 \varepsilon_2 + v_s^2 \varepsilon_1\). In the cubic spline approximation, the slope \(dp/d\varepsilon\) is parabolic in the phase coexistence region,

\[
\frac{dp}{d\varepsilon} = v_s^2 \left[ 1 - 4 \left( \frac{\varepsilon - \varepsilon_\ast}{\varepsilon_B - \varepsilon_A} \right)^2 \right] \varepsilon_\ast^2 . \quad (7)
\]

The inflection point in \(p(\varepsilon)\) is at \(\varepsilon = (\varepsilon_1 + \varepsilon_\ast + \varepsilon_2)/3\) and its extrema (located at \(\varepsilon_A\) and \(\varepsilon_B\)) are separated by

\[
\varepsilon_B - \varepsilon_A = \frac{\sqrt{3}}{3} [(\varepsilon_2 - \varepsilon_1)^2 + (\varepsilon_\ast - \varepsilon_1)^2 + (\varepsilon_2 - \varepsilon_\ast)^2]^{\frac{1}{2}} . \quad (8)
\]

The steepest slope of \(p(\varepsilon)\) occurs at \(\bar{\varepsilon}\) and is given by

\[
\left( \frac{dp}{d\varepsilon} \right)_{\bar{\varepsilon}} = \bar{v}_s^2 = -\frac{4}{9} [v_s^2 + v_{s'}^2] \left( \frac{\varepsilon_B - \varepsilon_A}{\Delta \varepsilon} \right)^2 \approx 0.150 . \quad (9)
\]

This value is rather robust against changes in \(T_c\) and \(m_s\). The \(\varepsilon\) dependence of the sound speed is shown in Fig. 2.

\[
\text{FIG. 2:} \quad \text{The magnitude of the sound speed, } |v_s| = |\partial p/\partial \varepsilon|^{1/2} , \text{ as a function of } \varepsilon \text{ from zero to above the phase-coexistence region. While } v_s \text{ is approximately constant in each of the single-phase regions, it drops rapidly as the metastable phase-coexistence region is entered, and becomes zero at the spinodal boundaries. It is imaginary inside the spinodal region where the thermodynamical potential is convex and spontaneous phase separation occurs. In the mixed-phase scenario } v_s \text{ is zero throughout the entire phase coexistence region.}
\]

III. ONSET OF SPINODAL DECOMPOSITION

The system is regarded as an ideal fluid with no conserved charges. The energy-momentum tensor \(T^{\mu\nu}\) is then of ideal gas form \(\mathbb{I}\),

\[
T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - pg^{\mu\nu} , \quad (10)
\]

where \(u^\mu = \gamma(1, \mathbf{v})\) is the four-velocity of the energy flow, and local conservation of four-momentum yields the basic equations of motion, \(\partial_\mu T^{\mu\nu} = 0\), i.e.

\[
\partial_t[(\varepsilon + p)\gamma^2 - p] = \nabla \cdot [(\varepsilon + p)\gamma^2 \mathbf{v}] , \quad (11)
\]

\[
\partial_t[(\varepsilon + p)\gamma^2 \mathbf{v}] = \nabla [(\varepsilon + p)\gamma^2 \mathbf{v}^2 + p] . \quad (12)
\]

These coupled equations are closed by the specification of the equation of state \(p(\varepsilon)\).

We now consider a small perturbations on a uniform static system,

\[
\varepsilon(t, \mathbf{r}) = \varepsilon_0 + \delta \varepsilon(t, \mathbf{r}) , \quad \delta \varepsilon(t, \mathbf{r}) \ll \varepsilon_0 . \quad (13)
\]
Since the flow speed $v$ is of the order of $\delta \varepsilon/\varepsilon_0$, it may be regarded as small, $v \ll 1$, and we may ignore terms of order $v^2$ and thus use $\gamma = 1$. Insertion of $\varepsilon(t,r)$ from \[15\] into eqs. \[14\] then yields

\[ 0 \approx \partial_t \varepsilon - \nabla \cdot [(\varepsilon + p)v] \approx \partial_t \delta \varepsilon - (\varepsilon_0 + p_0)\nabla \cdot v, \quad (14) \]

\[ 0 \approx \partial_t [(\varepsilon + p)v] - \nabla p \approx (\varepsilon_0 + p_0)\partial_t v - v^2 \nabla \delta \varepsilon, \quad (15) \]

where we have used $\nabla p(\varepsilon(r)) = (\partial p_0/\partial \varepsilon_0)\nabla \varepsilon = v^2 \nabla \varepsilon$, with $p_0 \equiv p(\varepsilon_0)$ being the pressure in the unperturbed uniform system. The speed of sound was displayed in Fig. 2. Taking the gradient of \[14\] and using the derived relation \[13\] between $\delta \varepsilon$ and $v$ then yields the familiar equation for the propagation of sound waves,

\[ \partial_t^2 \delta \varepsilon(t,r) = \frac{\partial p_0}{\partial \varepsilon_0} \nabla^2 \delta \varepsilon(t,r) \quad . \quad (16) \]

For harmonic perturbations, $\delta \varepsilon \sim \exp(ik \cdot r - i\omega t)$, it yields a linear relation between the wave number and the frequency, $\omega \varepsilon = v \omega k$. Thus, if a physical system is brought into the spinodal region where $\partial p_0/\partial \varepsilon_0 < 0$ (such a quenching can be accomplished by rapid expansion and/or cooling), the frequency is imaginary and small deviations from uniformity evolve exponentially.

The linear dispersion relation becomes unrealistic once the wave length of the perturbation, $\lambda_k = 2\pi/k$, has shrunk to the size of the range of the interaction between the constituents of the fluid. At such small scales, the physical justification for employing fluid dynamics is questionable. However, a physically meaningful treatment can still be made within the framework of fluid dynamics by employing a suitable smoothing, as we shall now explain.

The equation of state $p(\varepsilon)$ gives the pressure in a spatially uniform system as a function of its energy density. When the energy density varies sufficiently gently with position, one may obtain the corresponding local pressure as $p(r) \approx p(\varepsilon(r))$, i.e. one may calculate $p$ as if the energy density had everywhere the same value as at $r$. This local density approximation becomes increasingly inaccurate as the scale of the spatial variations in $\varepsilon(r)$ is decreased, because the finite range of the forces reduces the response to fine ripples. A simple way to account approximately for the effect of the interaction range is to average the pressure over the interaction volume. While this procedure is in fact exact in certain cases (e.g. semiclassical one-body models of nuclear systems [6]), it is adopted here merely as a simple and transparent approximation which needs ultimately to be justified/improved on the basis on microscopic models of the system.

Thus we assume that the local pressure $p(r)$ can be obtained by a convolution with a (normalized) kernel $g(r)$,

\[ p(r) \approx \int d^3 r_2 \ g(|r_1 - r_2|) \ p(\varepsilon(r_2)) \quad . \quad (17) \]

When the perturbation of the energy density is harmonic, $\delta \varepsilon(r) \sim \exp(i k \cdot r)$, the gradient of the local pressure is then proportional to the gradient of the energy density,

\[ \nabla p(r) \approx \frac{\partial p_0}{\partial \varepsilon_0} g_k \nabla \varepsilon(r) = i k g_k \frac{\partial p_0}{\partial \varepsilon_0} \delta \varepsilon(\varepsilon_0) \quad , \quad (18) \]

where $g_k$ is the Fourier transform of $g(r)$. Hence the refinement amounts to simply modulating the local-density term $\partial p_0/\partial \varepsilon_0$ in the dispersion relation by $g_k$, yielding

\[ \omega_k^2 = \frac{\partial p_0}{\partial \varepsilon_0} g_k k^2; \quad \gamma_k = |v_\varepsilon(\varepsilon_0)| k \sqrt{g_k} \quad . \quad (19) \]

As a result of this convenient factorized form, the largest collective frequency occurs at the same wave number $k_0$ for any energy density, thus enhancing the formation of a distinctive spinodal pattern of a characteristic scale.

The spatial scale of the pattern is determined by the range of $g(r)$, which one would expect to be in the range of $0.5 - 1.0$ fm, with the precise value depending on the specific microscopic treatment. For simplicity, we employ here a gaussian form factor, $g(r) \sim \exp(-r^2/2a^2)$, so we have $g_k = \exp(-a^2 k^2/2)$. The growth rates $\gamma_k$ for the spinodal modes can then be written on a simple form,

\[ \gamma_k(\varepsilon) = \left(-\frac{\partial p_0}{\partial \varepsilon}\right) \frac{\varepsilon}{\varepsilon} \left[1 - 4 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon_B - \varepsilon_A}\right)^2\right]^{\frac{1}{2}} k e^{-\frac{1}{2} a^2 k^2} . \quad (20) \]

This dispersion relation is illustrated in Fig. 3.

As the system expands and the energy density traverses the spinodal region, the growth rate $\gamma_k(\varepsilon(t))$ of a given mode increases from zero at $\varepsilon_B$, exhibits a maximum at $\tilde{\varepsilon}$, and then reverts to zero at $\varepsilon_A$. The total growth factor is then approximately equal to $G_k = \exp(\Gamma_k)$, with the amplification coefficient \[\Gamma_k\] being

\[ \Gamma_k \equiv \int_{t_B}^{t_A} \gamma_k(\varepsilon(t)) \ dt \approx \gamma_k(\tilde{\varepsilon}) \Delta t_{\text{eff}} \quad . \quad (21) \]

The effective time during which spinodal instability is encountered is $\Delta t_{\text{eff}} \approx (\pi/4)(t_B - t_A)$ in the cubic spline
approximation. For a quantitative estimate we use the value $t_B - t_A \approx 4 \text{ fm}/c$ gleaned from Ref. [4]. The resulting growth factors $G_k$ are displayed in Fig. 4.

After rapidly reaching a maximum, $G(\lambda)$ subsides relatively slowly, thus allowing some degree of amplification of longer wave lengths. If $a = 0.5 \text{ fm}$, the maximum growth factor of $G_{\text{max}} \approx 8$ is obtained for $\lambda \approx 2 \text{ fm}$, while $a = 0.7 \text{ fm}$ yields $G_{\text{max}} \approx 4$ for $\lambda \approx 3 \text{ fm}$. The corresponding component of the density-density correlation function is proportional to the square of that, $\langle \delta \varepsilon(r_1) \exp(i r_2 \cdot \mathbf{k}) \delta \varepsilon(r_2) \rangle \sim G_k^2$, and can thus reach quite appreciable magnitudes. (Without the finite-range approximation but no characteristic scale would emerge.)

IV. DISCUSSION

Spinodal decomposition is intimately linked to the occurrence of a first-order phase transition and is important in many areas of physics as a mechanism for pattern formation. In nuclear physics it plays a key role for nuclear multifragmentation where it gives rise to highly non-statistical fragment size distributions with a preference for equal masses [7, 8]. The present study is focussed on high-energy nuclear collisions and we have investigated the importance of spinodal decomposition while the expanding matter passes through the phase coexistence region. Working within the framework of relativistic fluid dynamics and basing our analysis on commonly employed assumptions about the equation of state (supplemented with a simple spline procedure), we have found that a significant degree of amplification may occur while the matter passes through the mechanically unstable region of phase coexistence.

Such an onset of spinodal decomposition may have interesting dynamical consequences, most notably the development of characteristic patterns in the energy density, as the most rapidly amplified wavelengths grow predominant. The spatial scale of any such emerging spinodal pattern could in principle be probed by HBT interferometry. It is also conceivable that the azimuthal multipolarity of the favored modes would be reflected in the flow coefficients $v_m$ (of which $v_2$ quantifying elliptic flow is the best explored [2]). Investigation of these prospects would clearly be interesting.

If the appearance of spinodal patterns could in fact be examined experimentally, one could obtain fairly direct information on the equation of state in the critical region, since the relative prominence of various scales would reflect the degree of spinodal instability encountered by the corresponding wave lengths. Conversely, if the absence of such patterns can be established it may be concluded that no spinodal region has been encountered and, consequently, the phase transition cannot be of first order.

In view of the potential utility of spinodal decomposition as a diagnostic tool, it would seem worthwhile to pursue this issue with more realistic dynamical treatments that take better account of the spatio-temporal features of the system (including both the rapid longitudinal expansion and the finite transverse extent which may affect the present simple estimates). In particular, the quantitative importance of spinodal decomposition could be elucidated with existing fluid dynamical models that have been modified to accomodate an equation of state containing the convexity anomaly which is a characteristic feature of first-order phase transitions.

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