Abstract—Highly dynamic tasks that require large accelerations and precise tracking usually rely on accurate models and/or high gain feedback. While kinematic optimization allows for efficient representation and online generation of hitting trajectories, learning to track such dynamic movements with inaccurate models remains an open problem. In particular, stability issues surrounding the learning performance, in the iteration domain, can prevent the successful implementation of model-based learning approaches. To achieve accurate tracking for such tasks in a stable and efficient way, we propose a new adaptive Iterative Learning Control (ILC) algorithm that is implemented efficiently using a recursive approach. Moreover, covariance estimates of model matrices are used to exercise caution during learning. We evaluate the performance of the proposed approach in extensive simulations and in our robotic table tennis platform, where we show how the striking performance of two seven degree of freedom anthropomorphic robot arms can be optimized. Our implementation on the table tennis platform compares favorably with high-gain PD-control, model-free ILC (simple PD feedback type) and model-based ILC without cautious adaptation.

I. INTRODUCTION

Most reaching tasks in control and robotics can be phrased as tracking problems, where the dynamical system needs to follow a certain predefined trajectory in order to reach a goal state. Robotic table tennis in particular [1] consists of planning, generating and executing a series of dynamic single stroke trajectories. In order to reach the hitting state precisely, these high-speed trajectories need to be followed with appropriate motor commands. Computing the right motor commands is a nontrivial task when using cable-driven arms such as the Barrett WAM shown in Figure 1, due to mechanical compliance and low bandwidth.

Iterative Learning Control (ILC) is a learning approach in control theory developed to track (time-varying) reference trajectories. It has been used successfully to follow trajectories under unknown repeating disturbances and model mismatch [3]. The feedforward control inputs are adjusted after each trial based on the resulting deviations from the reference trajectory. The goal is to drive such deviations to zero. ILC can easily incorporate available dynamics models (see e.g., [4], [5]) in a simple and efficient manner.

While there have been many impressive applications of reinforcement learning (RL) [6] to learn robotic tasks [7], RL remains to be computationally and information-theoretically hard in general. Much of control, on the other hand, can be reduced to supervised learning, with the appropriate reference trajectories. Learning in robotics can hence be performed more efficiently with ILC by taking advantage of existing imperfect models and smooth reference trajectories. However, it is rather difficult to ensure a stable learning performance in practice, see Figure 2 for an illustration.

In this paper, we introduce a new learning approach for tracking a variety of fast, dynamic movements efficiently and stably. Efficiency in ensured by using a relatively inexpensive model-based recursive ILC approach. Stability of the updates, or the probability of update monotonicity, is increased by making use of dynamics model covariance estimates. We refer to this as caution throughout the text, and the resulting algorithm is cautious precisely in this sense. This property proves to be critical, as we show the learning performance for fast robot table tennis striking movements. The proposed Bayesian approach, using the posterior over the dynamics model parameters, maintains both adaptation and caution in model-based ILC, while being efficient in terms of learning performance and computational complexity.

Our contributions can be summarized as follows: we propose a new adaptive and cautious model-based ILC algorithm, that is implemented efficiently using a recursive formulation [5]. The proposed approach minimizes expected cost, resulting in a cautious yet efficient learning performance. The expected cost minimization distinguishes the framework remains to be computationally and information-theoretically hard in general. Much of control, on the other hand, can be reduced to supervised learning, with the appropriate reference trajectories. Learning in robotics can hence be performed more efficiently with ILC by taking advantage of existing imperfect models and smooth reference trajectories. However, it is rather difficult to ensure a stable learning performance in practice, see Figure 2 for an illustration.

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Fig. 2. Learning performance of ILC, using inaccurate models without incorporating a notion of uncertainty, may not be monotonic in practice. One can observe ripples that move through the trajectory which can cause instability or damage the robot. In simulations we can create this effect easily by increasing the mismatch scale \( \alpha \) controlling the spectral norm of the difference between the nominal and the actual (lifted) dynamics matrices. The desired trajectory for the first state \( x_1 \) is shown in dashed red on the left hand side for a two dimensional linear time invariant system. The second plot shows the ILC feedforward commands for this particular trajectory and state. The third plot shows the nonmonotonicity of the learning performance, as the mismatch scale \( \alpha \) controlling the norm of the difference is increased. Increasing \( \alpha \) further can prevent even asymptotic stability. The solid lines were generated by direct inversion of the (lifted) model. From the third plot one can observe ripples that move through the trajectory which can cause instability or damage the robot. In simulations we can create this effect easily by increasing the mismatch scale \( \alpha \).

from more conservative min-max approaches, such as the robustly convergent ILC proposed in the literature (using \( H_\infty \) and \( \mu \)-synthesis techniques [8]). Secondly, we discuss model adaptation with linear time varying models and show that Brodgen’s method [9] can be derived from Linear Bayesian Regression (LBR) as the forgetting factor goes to zero. Following the robust learning performance empirically observed in [10], we moreover discuss extensions of ILC to achieve robustness to varying initial conditions and to noise. These considerations can be quite important in practice. Rejection of nonrepeating disturbances are shown to increase with effective weighting of the errors throughout the iterations. The final proposed approach is validated on our robot table tennis setup where we conduct experiments with dynamic hitting movements on two Barrett WAMs.

Related work in the theory and practice of ILC, as well as some more general applications of learning in robotics tasks, are briefly mentioned in the next subsection. The learning control problem is stated afterwards in Section II. Model adaptation using LBR is described with various alternatives in Section III. We formulate a new cautious learning control update law, that is based on expected-cost minimization in Section IV. The resulting adaptive and cautious ILC approach, called bayesILC is described in algorithmic form in Section V, and extensions are discussed for additional robustness to nonrepetitive disturbances. In Section VI we evaluate bayesILC first in extensive simulations, showing that the proposed method is stable, efficient and can outperform other state-of-the-art learning approaches. We then present online learning results on our robot table tennis platform. We discuss the real robot learning results in Section VII and conclude with brief mentions of promising future research directions.

Derivations for the recursive and cautious learning control update introduced in Section IV are given in Appendix A. Appendix B briefly introduces the parameterization of the hitting movements for table tennis.

A. Related Work

Arimoto et al. [11] was one of the first to define Iterative Learning Control with the D-type update law. See [3] and [4] for reviews. Theoretically, most ILC algorithms can be studied as a linear repetitive process using 2D-systems analysis [12]. Monotonic convergence and stability guarantees are of central importance for the practical usefulness of ILC algorithms. They are shown for example in [3], [13], [14] and recently in [8]. In practice, some of the assumptions made in the ILC literature may often be violated. Robustness to varying initial conditions are considered e.g., in [15], [16], [17]. ILC should be used along with a robust feedback controller to reject nonrepeating disturbances, see e.g., [18], [14].

Methods that learn to track (periodic or episodic) trajectories need to compensate for modeling uncertainties and other repetitive disturbances acting on the system to be controlled. However, methods that can efficiently learn the dynamics are model-based (e.g. most of optimization-based ILC [5], [3]) and at least require knowing the correct signs for the linearized dynamics of the system [19], [20].

When executing model-based learning algorithms on dynamical systems, it is essential for stability and safety to incorporate a notion of model uncertainty. Otherwise the learning algorithms can be overconfident and quickly go unstable [14]. One way to achieve a more stable performance in ILC is to filter the high-frequency updates. These robust methods are mostly known as Q-filtering [3] and typically incur a trade-off between stability and performance: the system will often fail to converge to the minimal steady-state error. In this paper, we use a different approach to increase the stability margins of model-based ILC that does not incur such a trade-off.

One of the first papers introducing an optimization based ILC approach incorporating a model of the dynamics is [5]. The recursive implementation first introduced in this paper closely relates to stable plant-inversion approaches [21]. As a more recent example of model-based ILC, Schoellig et al. [22] applied a Kalman-filter based convex optimization rule that avoids direct inversion and showed its performance in quadrocopter flight. An EM-based update law was given in [23] where an impressive application of ILC to a robotic surgical task was presented. ILC has also been combined with robust observers for controlling a heavy-duty hydraulic arm during robotic excavation tasks [24].
Model adaptation in ILC can be studied in the context of solving nonlinear equations. Tracking a fixed reference perfectly corresponds to solving for the control inputs that drive the deviations to zero. Hence, Broyden’s method [9] and generalized secant method [25] were proposed as adaptation methods in the ILC literature to update the plant dynamics. Broyden’s method, without having access to the gradients of a black-box function $f(x) = 0$, maintains a Jacobian matrix approximation $F$. The matrix $F$ is updated at each iteration $k$ in order to satisfy the secant equation

$$f_k - f_{k-1} = F_k(x_k - x_{k-1}),$$

which can then be inverted to yield

$$x_{k+1} = x_k - F_k^b f_k.$$ 

Convergence under restrictive assumptions have been shown for the Broyden’s method. For solving systems nonlinear equations, arguably efficiency rather than stability or monotonic convergence is of importance, and a simple trust-region approach (based on a merit function) suffices to improve stability. We will show how the Broyden’s method can be seen as a limiting case of Linear Bayesian Regression in Section III.

Besides ILC, another learning framework that learns inaccurate models for control is model-based Reinforcement Learning. Including variance fully in the decision-making process may result in efficient and stable learning [26]. However such methods in the ILC literature to update the plant dynamics.

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### II. Problem Statement and Background

Most tasks in robotics can be learned more efficiently whenever feasible trajectories are available. Learning based control approaches focus on tracking these trajectories without relying on accurate models. The goal in trajectory tracking is to track a given reference $r(t)$, $0 \leq t \leq T$, by applying the control inputs $u(t)$. In dynamic robotic tasks, the references are often in the combined state space of joint positions and velocities $\mathbb{R}^n \times \mathbb{R}^m$, and the control inputs $u \in \mathcal{U} \subset \mathbb{R}^m$ are applied for each joint of the robot, i.e., $m = n$. The reference trajectories in table tennis, for instance, enable the execution of hitting and striking motions, i.e., forehand and backhand strokes. Such trajectories can be generated online with nonlinear constrained optimization [2]. Finding the right control inputs to track them accurately is the focus of Iterative Learning Control (ILC).

1) **Linearizing an Inaccurate Model:** Consider the nonlinear robot dynamics of the form

$$\ddot{q} = f(q, \dot{q}, u),$$

$$\ddot{q} = M^{-1}(q)(u + C(q, \dot{q})\dot{q} - G(q)) + e(q, \dot{q}),$$

where on the right hand side are the terms due to the rigid body dynamics model and $e(q, \dot{q})$ are the (unmodeled) disturbances that act on the robot, due to parameter mismatch, viscous friction, stiction, etc. This system can be linearized around a given joint space trajectory $r(t)$, $0 \leq t \leq T$ with nominal inputs $u_{\text{IDM}}(t)$ calculated using the inverse dynamics model [27]. We then obtain the following linear time varying (LTV) representation

$$\dot{e}(t) = A(t)e(t) + B(t)\delta u(t) + d(t, u),$$

where the state vector is the joint angles and velocities $x = [q^T, \dot{q}^T]^T$, the state error is denoted as $e(t) = x(t) - r(t)$, the deviations from the nominal inputs are $\delta u(t) = u(t) - u_{\text{IDM}}(t)$ and the continuous time varying matrices are

$$A(t) = \frac{\partial f}{\partial x}\bigg|_{(r(t), u_{\text{IDM}}(t))}, B(t) = \frac{\partial f}{\partial u}\bigg|_{(r(t), u_{\text{IDM}}(t))}.$$ 

In the error dynamics (2) the additional term $d(t, u)$ accounts for the disturbances and the effects of the linearization (i.e., higher order terms). We can discretize (2-3) with step size $\delta$, $N = T/\delta$ and step index $j = 1, \ldots, N$ to get the following discrete linear system

$$e_{j+1} = A_j e_j + B_j \delta u_j + d_{j+1},$$

where the matrices $A_j, B_j$ are the discretizations of (3). Conventional (discrete) ILC algorithms learn to compensate for the errors incurred along the trajectory by updating the control inputs $\delta u_j$ iteratively. Whenever we refer to the outcome for a particular $k$'th iteration, we will use the first subindex for iterations and the second subindex will be used to denote the (discrete) time step, i.e., the vectors $e_{k,j} \in \mathbb{R}^n$, $\delta u_{k,j} \in \mathbb{R}^m$ denote the deviations and control input compensations at the $j$'th time step during iteration $k$, respectively. The control commands applied at iteration $k + 1$ as

$$u_{k+1,j} = u_{k,j} + \delta u_{k,j},$$

are computed using the deviations $e_{k,j}$ at iteration $k$.

2) **Recursive Norm-Optimal ILC:** Norm-optimal ILC uses the models in (4) to minimize the next iteration errors, where the computed control inputs are optimal with respect to some vector norm. These optimal approaches can learn efficiently by taking advantage of the inaccurate models. Batch methods that compute the compensations using all the deviations, stack the model matrices together to compute (a possibly weighted and dampened) pseudoinverse of this block lower-diagonal matrix. As an alternative, some methods use convex programming to compute these optimal compensations under additional constraints. The condition of this lifted model matrix typically grows exponentially with the horizon size $N$ and computing the pseudoinverse stably becomes very difficult. Typically down-sampling trajectories restores the condition number and a stable inversion becomes much more manageable. As a better alternative, optimization based approaches, depending on the particular optimizer, may avoid computing the pseudoinverse. However such approaches can still be computationally intensive, and may not be suitable for online learning.

As an alternative, the authors in [5] have shown that the direct, batch inversion of the lifted model matrix can be avoided by recursively computing the ILC compensations (in
one pass) using the Linear Quadratic Regulator (LQR) for disturbance rejection [28]. After estimating the disturbances $d_{k+1}$ at the $k'$th trial, the optimal control problem for tracking a desired trajectory can be written as

$$\min_{\delta u} \sum_{j=1}^{N} e_{k+1,j}^T Q_j e_{k+1,j} + \delta u_{k,j}^T R_j \delta u_{k,j},$$

s.t. $e_{k+1, j+1} = A_j e_{k+1,j} + B_j u_{k+1,j} + d_{j+1}$.

Reduction of the ILC problem to the known LQR solution has not attracted much attention however from the control and learning communities, since it was not clear how to study stability and convergence in this formulation.

III. MODEL ADAPTATION

Whenever there is model-mismatch, the linearized model cannot be assumed to hold accurately around the reference trajectory. There is hence a risk that the learning process described in the previous subsections will not be stable. As a remedy, in this section we propose a natural Bayesian adaptation of model matrices and discuss different alternatives in the context of robotics. We show first how discrete model parameter means and variances can be updated with Linear Bayesian Regression (LBR).

A. Recursive Estimation of Model Matrices

The observed deviations from the trajectory $e_{k,j}$ at iteration $k$ can be used to update the linear models $A_{k,j}, B_{k,j}$, that describe the nonlinear dynamics accurately around the trajectory, to first order. Instead of estimating all the parameters together in a costly estimation procedure, the model matrices $A_{k,j}, B_{k,j}$ can rather be updated separately for each $j = 1, \ldots, N$, given the smoothed errors $\hat{e}_{k,j}$

$$\hat{e}_{k,j+1} = A_{k,j} \hat{e}_{k,j} + B_{k,j} u_{k,j} + d_{j+1},$$

which can be rewritten using the Kronecker product and the vectorization operator as follows

$$\hat{e}_{k,j+1} - \hat{e}_{k-1,j+1} \approx X_{k,j} \text{vec} [A_{k,j}, B_{k,j}],$$

$$X_{k,j} = \text{vec} [\hat{e}_{k,j} - \hat{e}_{k-1,j} - \delta u_{k,j}] \otimes I.$$

If we incorporate the belief (including the uncertainty) about the linear dynamics models as Gaussian priors in LBR

$$\theta_{k,j} = \text{vec} [A_{k,j}, B_{k,j}],$$

$$y_{k,j} = \hat{e}_{k,j+1} - \hat{e}_{k-1,j+1},$$

$$\rho(y_{k,j}, \theta_{k,j}) \propto \rho(y_{k,j}, \theta_{k,j}) \rho(\theta_{k,j}),$$

$$\rho(\theta_{k,j}) = \mathcal{N}(\theta_{k,j}, \mu_{k,j}, \Sigma_{k,j}),$$

with a Gaussian likelihood function

$$\rho(y_{k,j}, \theta_{k,j}) = \mathcal{N}(y_{k,j}|X_{k,j} \theta_{k,j}, \sigma^2 I),$$

the models parameter means $\mu_{k,j}$ and variances $\Sigma_{k,j}$ can be updated as

$$\Sigma_{k,j} = (\frac{1}{\sigma^2} X_{k,j}^T X_{k,j} + \Sigma_{k-1,j}^{-1})^{-1},$$

$$\mu_{k,j} = \Sigma_{k,j} (\Sigma_{k-1,j}^{-1} \mu_{k-1,j} + \frac{1}{\sigma^2} X_{k,j}^T y_{k,j}).$$

Smoothened position and velocity error estimates can be obtained, for example, using a zero-phase Butterworth filter.

1) Relation to Broyden’s method: Broyden’s method [9] can be seen as a limiting case of LBR. The mean estimates in (8) are also the solutions of the following linear ridge regression problem

$$\min_{\theta} \frac{1}{\sigma^2} ||y_{k,j} - X_{k,j} \theta||^2_2 + (\theta - \theta_{k,j}) \Sigma_{k,j}^{-1}(\theta - \theta_{k,j}),$$

and as $\sigma^2 \to 0$ we get the (weighted) Broyden’s update for one iteration, which, written in vectorized form, is solving independently for every time step

$$\min_{\theta} (\theta - \theta_{k,j}) \Sigma_{k,j}^{-1}(\theta - \theta_{k,j}),$$

s.t. $y_{k,j} = X_{k,j} \theta$.

Broyden’s method is too sensitive to the sensor noise in robotics tasks as it satisfies the secant rule (9) exactly. On the other hand, LBR in (8) for fixed noise parameter $\sigma^2$, is using all of the past iteration data equally. The norm of the covariance decreases monotonically in each update. For unknown dynamical systems that are highly nonlinear but smooth, to prevent premature shrinking of the covariance matrix, a better alternative is to set an exponential weighting in the adaptation. For a fixed forgetting factor $\lambda \in [0, 1]$, the update in (8) becomes

$$\Sigma_{k,j} = (\frac{1}{\sigma^2} X_{k,j}^T X_{k,j} + \lambda \Sigma_{k-1,j}^{-1})^{-1},$$

$$\mu_{k,j} = \lambda \Sigma_{k,j} (\Sigma_{k-1,j}^{-1} \mu_{k-1,j} + \frac{1}{\sigma^2} X_{k,j}^T y_{k,j}).$$

The forgetting factor $\lambda$ is used to perform exponential weighting of the previous iteration data. As $\lambda \to 0$, we get the (unweighted) Broyden’s method\(^2\), and as $\lambda \to 1$, (10) reduces to (8). See Figure 3 for an illustration.

\(^2\)Unlike the case where $\sigma^2 \to 0$, this equivalence is valid for all the subsequent iterations as well. It can be seen more easily from the filter form of (10).
B. Imposing structure

The structure in the forward dynamics model is not considered in the update rule (10): change in the control inputs directly affects the instantaneous joint accelerations, and only indirectly the joint velocities in the future time steps. By differentiating the smoothed joint velocities, one can instead impose the following model

\[ \tilde{q}_{k,j} - \tilde{q}_{k-1,j} = A_k(\delta_j) e_{k,j} + B_k(\delta_j) \delta u_{k,j}, \]

where we dropped the hat for notational simplicity. The continuous model matrices \( A_k(\delta_j), B_k(\delta_j) \) are members of a reduced parameter space, i.e., \( A_k(\delta_j) \in \mathbb{R}^{n \times 2n}, B_k(\delta_j) \in \mathbb{R}^{n \times m} \), \( j = 1, \ldots, N \). After estimating the continuous model parameter means and variances as in (8), they can be discretized (as discussed before) to form the discrete model parameter means \( A_{k,j} \in \mathbb{R}^{2n \times 2n}, B_{k,j} \in \mathbb{R}^{2n \times m} \) and covariances \( \Sigma_{k,j} \).

As an alternative, note that the rigid body dynamics (1) is parameterized by the link masses, three link center of mass values and six inertia parameters. A total of ten parameters are used for each link to fully parameterize the inverse dynamics model

\[ u = M(q; \theta) \ddot{q} + C(q, \dot{q}; \theta) \dot{q} + G(q; \theta), \]

which can be stacked for each \( j = 1, \ldots, N \) to form the regression model

\[ U_k \approx Y(Q_k, Q_k, \tilde{Q}_k) \theta_k, \]

\[ U_k = \begin{pmatrix} u_{k,1}^T, u_{k,2}^T, \ldots, u_{k,N}^T \end{pmatrix}^T, \]

\[ Q_k^{(l)} = \begin{pmatrix} q_{k,1}^{(l)}, q_{k,2}^{(l)}, \ldots, q_{k,N}^{(l)} \end{pmatrix}^T, \quad l = 0, 1, 2, \]

where \( \theta_k \in \mathbb{R}^{10n} \) appears linearly. Based on the iteration performance (joint position, velocity and acceleration estimates) only the link parameters are updated with LBR as in (8). The forward dynamics model\(^5\) can then be used to sample the means and variances of the continuous LTV matrices, e.g., using Monte Carlo sampling. Discretization as discussed above converts the continuous model parameter means and variances into their discrete form. An advantage of this approach is to compress learning to a lower dimensional space, reducing the variance of the updates at the cost of an introduced bias. Moreover, since the link parameters are invariant throughout the iterations, such an update avoids the flexible yet independent adaptation of the model matrices for each \( j \), and the necessity of introducing a forgetting factor.

IV. CAU TIOUS LEARNING CONTROL

The posterior model covariances \( \Sigma_{k,j} \) can be used to make more cautious decisions within a stochastic control framework. The uncertainty of the model parameters can be seen as a multiplicative noise model and the ILC optimality criterion (5) can be extended to include expectations over them. The multiplicative noise model, unlike the additive noise case, does not lead to certainty-equivalence: the covariance estimates are incorporated in the decision rule. To see how the expected cost minimization leads to caution, note that

\[ \mathbb{E}(e_{k+1,j}^T Q_j e_{k+1,j} \geq \mathbb{E}(e_{k,j}^T Q_j e_{k,j}) \leq \frac{\mathbb{E}(\tilde{e}_{k,j}^T Q_j \tilde{e}_{k,j})}{\mathbb{E}(\tilde{e}_{k,j}^T Q_j \tilde{e}_{k,j})}, \]

which follows from Markov’s inequality. Minimizing the upper bound forces the probability of nonmonotonicity to be low as well.

1) CAUTIOUS ILC: For the expected cost case, where the expectation is taken over the random variables \( A_{k,j} \) and \( B_{k,j} \), for each \( j \), the optimality criterion

\[ \min_{\delta u} \sum_{j=1}^N \mathbb{E} A_{k,j} B_{k,j} [e_{k+1,j}^T Q_j e_{k+1,j} + \delta u_{k,j}^T R_j \delta u_{k,j}], \]

s.t. \( e_{k+1,j+1} = A_{k,j} e_{k+1,j} + B_{k,j} u_{k+1,j} + d_{j+1}, \)

can be solved recursively using dynamic programming [29]

\[ \delta u_{k,j} = K_{k,j} e_{k+1,j} - \Phi_{k,j}^{-1} \ell_{k,j}, \]

\[ K_{k,j} = - \Phi_{k,j}^{-1} \Psi_{k,j}, \]

\[ \Phi_{k,j} = \mathbb{E} A_{k,j} B_{k,j} \left[ B_{k+1,j}^T P_{k+1,j+1} B_{k,j} + R_j \right], \]

\[ \Psi_{k,j} = \mathbb{E} A_{k,j} B_{k,j} \left[ B_{k+1,j}^T P_{k+1,j+1} A_{k,j} \right], \]

\[ \ell_{k,j} = \mathbb{E} A_{k,j} B_{k,j} \left[ B_{k+1,j}^T P_{k+1,j+1} (B_{k,j} u_{k+1,j} + d_{j+1}) + B_{k,j}^T b_{k+1,j} \right], \]

(12)

where \( b_{k,j} \) and the Ricatti matrices \( P_{k,j} \) evolve backwards according to

\[ P_{k,j} = Q_j + M_{k,j} - \Psi_{k,j}^{-1} \Phi_{k,j} \Psi_{k,j}, \]

\[ M_{k,j} = \mathbb{E} A_{k,j} \left[ A_{k,j}^T P_{k+1,j+1} A_{k,j} \right], \]

\[ b_{k,j} = \mathbb{E} A_{k,j} B_{k,j} \left[ A_{k,j} (b_{k,j+1} + P_{k,j+1} B_{k,j} u_{k+1,j} + d_{j+1}) \right], \]

starting from \( P_{k,N} = Q_N \) and \( b_{k,N} = 0 \). The random closed loop system dynamics is given by the matrices

\[ A_{k,j} = A_{k,j} + B_{k,j} K_{k,j}. \]

By a direct comparison to the LQR solution to (5), it can be seen that the control input compensations \( \delta u_{k,j} \) in (12) are computed similarly, with the appropriate expectations added. The ILC update is decomposed into two: a current-iteration feedback term \( u_{k,j} = K_{k,j} e_{k+1,j} \) calculated using the iteration dependent Riccati equations and a feedforward, purely predictive term \( u_{k,j} = - \Phi_{k,j}^{-1} \ell_{k,j} \), solved backwards for each \( j = 1, \ldots, N \). The feedforward terms are responsible for compensating for the estimated random disturbances \( d_j \), calculated using (6).

Cautious update (12) can be implemented without explicitly calculating the disturbances. If the disturbances are taken as random variables defined via the filtered errors \( \tilde{e}_{k,j} \) of the last iteration

\[ d_{j+1} = \tilde{e}_{k,j+1} - A_{k,j} \tilde{e}_{k,j} - B_{k,j} u_{k,j}, \]

the recursion can be simplified by introducing

\[ \nu_{k,j} = b_{k,j} + P_{k,j} \tilde{e}_{k,j}. \]
The feedforward and feedback compensations $\delta u_{k,j}$ can then directly be computed as

$$\delta u_{k,j} = K_{k,j}(e_{k+1,j} - e_{k,j}) - \Phi_{k,j} \Sigma_{k,j}^{-1} E_{k,j} B_{k,j}^T u_{k,j+1} + Q_{j} e_{k,j}. \tag{13}$$

See Appendix A for a detailed derivation. Equation (13) is easier to implement, since the disturbances do not need to be estimated explicitly. The compensations $\delta u_{k,j}$ are added to the total control inputs applied at iteration $k$. In an adaptive implementation, the feedback components of the update, $K_{k,j}(e_{k+1,j} - e_{k,j})$, does not completely subtract the previous feedback controls $K_{k-1,j} e_{k,j}$ from the total control inputs, as the feedback matrices are also adapted over the iterations.

Typically ILC is used to feed the past errors along the trajectory (filtered and multiplied with a learning matrix) back to the system for the next trial as feedforward compensations. A well designed feedback controller, whenever available, is easier to implement, since the disturbances do not need to be estimated explicitly. The compensations $\delta u_{k,j}$ of the model matrices are updated (line 14) before computing the feedforward input compensations $\delta u_{k,j}$ and the feedback matrices $K_{k,j}$. Based on the forgetting factor $\lambda$, the model adaptation strikes a balance between the prior model parameter distribution and the data $y_{k,j} = e_{k,j+1} - e_{k-1,j+1}$ observed in iteration $k$. We discuss the effects of the forgetting factor and the different model adaptation strategies in more detail in Section VI.

The practitioner, wary of the model inaccuracies, can increase robustness and ensure stability by setting large diagonal terms for the initial covariance of model uncertainty, $\Sigma_{0,j} = \gamma I$, $\gamma \gg 1$, $j = 1, \ldots, N$. Moreover, setting large covariances initially helps to observe the inaccuracies of the model and the noise statistics. The covariance will be suitably decreased over the iterations, as adaptation (10) updates the linear models. Observing the noise statistics over the iterations can further help in the design of a good zero-phase filter to reject noise.

The proposed update law takes advantage of the learning efficiency and computational advantages of model-based recursive ILC while being cautious with respect to model mismatch. The computational complexity of the recursive update is $O(Nn^3)$ as opposed to batch norm-optimal ILC, where the batch pseudoinverse operation typically incurs $O(N^3n^3)$ complexity. The batch model-based implementation using the lifted-vector form \cite{6} inverts the input-to-output matrix $F$.

\begin{align*}
U_{k+1} &= U_k - F^T E_k, \\
E_k &= \left(e_{k+1}^T, e_{k+2}^T, \ldots, e_{k,N}^T\right)^T.
\end{align*}

\section*{Algorithm 1 Recursive, adaptive and cautious bayesILC.}

\begin{algorithmic}[1]
\Require{\text{I}_{\text{nom}}, r_j, \lambda, \epsilon > 0, \Sigma_{0,j} \geq 0, R_\gamma > 0, \Sigma_{0,j} > 0}
\Ensure{\text{I}_{\text{nom}}, r_j, \Sigma_{0,j}}
\State Move to initial posture $q_0 = r_0, \dot{q}_0 = 0$.
\State Initialize $k = 1$, $u_{k,j} = 0$, $j = 1, \ldots, N$.
\State Compute mean dyn. parameters $\mu_{0,j}$ by linearizing $f_{\text{nom}}$.
\State Compute feedback $K_{0,j} = LQR(Q_j, R_j, \mu_{0,j}, \Sigma_{0,j})$.
\State Execute with inv. dyn. $u_{\text{nom}}$ and feedback $K_{0,j}$.
\State Filter errors with a zero-phase filter (output: $e_{0,j}$).
\Repeat{ILC operation}
\State 8: Compute cost $J_k = \sum_{j=1}^{N} e_{k,j}^T Q_{j} e_{k,j}$.
\State 9: Compute $\delta u_{k,j}, K_{k,j}$ recursively using (12) - (13).
\State 10: Update feedforward controls $u_{k+1,j} = u_{k,j} + \delta u_{k,j}$.
\State 11: Execute with $u_{\text{nom},j} + u_{k+1,j}$ and feedback $K_{k,j}$.
\State 12: Observe errors $e_{k,j} = x_{k,j} - r_j$.
\State 13: Filter errors with a zero-phase filter (output: $e_{k,j}$).
\State 14: Update model $\mu_{k,j}, \Sigma_{k,j}$ using (10).
\State 15: $k \leftarrow k + 1$.
\Until{$J_k < \epsilon$}
\end{algorithmic}

In this section we algorithmically describe the recursive, adaptive and cautious bayesILC proposed in the last two sections in detail, with the extensions for an online robot learning application. We will consider tracking table tennis trajectories as our application of choice. The online learning algorithm is readily applicable to similar dynamic tasks with real-time constraints, such as throwing, catching skills in sports or fast, demanding manufacturing tasks.
The striking trajectory in table tennis is only an intermediary setup, shown in Figure 1, allows us to use ILC to track the returning trajectory or waiting long enough may suffice to initialize the system close to desired initial conditions, but in some occasions, none of these options may be desirable or available. For example, a robot practicing table tennis with a fixed ballgun running at a fixed rate, may not have time to initialize its desired posture accurately.

Starting from varying initial conditions \( x_{k,0} = [q_{k,0}^T \delta q_{k,0}^T]^T \) one can consider updating the hitting movement \( \mathbf{r}_j \) to take the robot to the same hitting state. For such online updating of trajectories, the invariant trajectory parameters \( \mathbf{p} \) can be used to generate the trajectory from the current joint values. The reference control inputs \( \mathbf{u}_{\text{IDM}} \) can then be recomputed based on the nominal inverse dynamics model. With this correction the total feedforward control commands \( \mathbf{u}_{\text{ILC}} \) at iteration \( k+1 \) are re-computed as

\[
\mathbf{u}_{\text{ILC},j} = \mathbf{u}_{k+1,j} + \mathbf{u}_{\text{IDM},j}(\mathbf{r}_j) - \mathbf{u}_{\text{IDM},j}(\hat{\mathbf{r}}_j),
\]

where \( \hat{\mathbf{r}}_j \) is the updated trajectory starting from the perturbed initial state \( x_{0} + \delta x_{0} \). Using this simple adjustment (16), the stability of the learning performance can be greatly improved.

VI. EVALUATIONS AND EXPERIMENTS

In this section, we demonstrate the effectiveness of the ILC algorithm \( \text{bayesILC} \) presented in Algorithm 1 and described in detail in Section V in the context of tracking table tennis trajectories. We validate the proposed learning control law first in extensive simulations with linear and nonlinear models. In the last section we show real robot experiments with two seven degree of freedom Barrett WAM arms for tracking table tennis striking movements.

A. Verification on Toy Problems

Stability is an important issue in the implementation of different learning controllers in real robot tasks. As a result, we setup extensive simulation experiments to validate the stability and robustness of our learning approach. We also discuss in detail the advantages of the recursive formulation over the batch pseudo-inverse ILC (14).

1) Random Linear Models: We generate here random linear models and random trajectories drawn from Gaussian Processes (GP) with squared exponential kernels. More specifically, the elements of the linear time varying (LTV) model matrices \( \mathbf{A}_j, \mathbf{B}_j \) are drawn from \( (n+m)r \) uncorrelated GPs. The hyperparameters (scale, noise and smoothness parameters) of these GPs are drawn independently from normal distributions.
with fixed means and variances. Moreover, random perturbations of these models (drawn the same way from \((n + m)n\) uncorrelated GP’s) are generated to construct nominal models. Using the proposed random disturbance generation scheme, we can average the results and construct error bars for different ILC algorithms.

The performance of the recursive implementation (i.e., Equation (13) with zero covariances and no adaptation) is shown in Figure 4 on the left hand side, where the results are averaged over ten different trajectories and models. The dimensions of the models are \(n = 2, m = 2\), and the horizon size is set to \(N = 120\). For the LQR and ILC calculations, \(R = 10^{-6}I\) and the weighting matrix \(Q\) was set to the identity. In this case, the batch model-based implementation using the pseudo-inverse \((14)\) is not stable at all without feedback. Applying LQR feedback and adding current iteration ILC in Figure 4 improves the performance (red line in Figure 4), but numerical issues (i.e., large condition number) in inverting the large model matrix \(F\) in lifted form \((15)\) prevents it from stabilizing at steady state error. Tracking performance throughout the experiments is measured with respect to the 2-norm of the deviations \(E_k\).

For the simulation results in Figure 4, the spectral norm of the difference between the nominal and the actual models are each set to \(\alpha \sigma_{\text{max}}(F)\) where \(\alpha = 100\). Increasing \(\alpha\) further increases the probability that the model-based ILC is not monotonically convergent for some trial. For example, one can observe asymptotically but not monotonically convergent ILC behaviour when setting \(\alpha = 900\) for a particular model and trajectory shown in Figure 2. Increasing \(\alpha\) further can prevent even asymptotic stability.

Especially in these cases of high model mismatch, the proposed adaptive and cautious bayesILC offers a stable and convergent ILC behaviour. In Figure 4 on the right hand side, we consider the case where \(\alpha = 1000\). Recursive ILC that is also cautious does not show a stable convergent behaviour, whereas recursive ILC that is not cautious (i.e., covariance of the LTV matrices are zero) is not stable at all. Cautious and adaptive bayesILC, on the other hand, using LBR \((\lambda = 0)\) to update the discrete LTV matrices \(A_{k,j}, B_{k,j}\), shows a monotonic learning performance. The results are again averaged over ten different models and ten trajectories.

2) Gaussian Process Dynamics: The performance of the proposed algorithm bayesILC is evaluated next over random nonlinear models. In these set of experiments, we sample the states from \(n\) uncorrelated GPs with squared exponential kernels and random linear mean functions. The hyperparameters of these GPs are randomized as before. By sampling from such random nonlinear models, we can test the proposed algorithm under nonlinear uncertainties and noisy outputs. The actual model is simulated as follows:

1) Random reference inputs \(v_j \in \mathbb{R}^m, j = 1, \ldots, N\) are drawn \(K\) times from \(m\) GPs.
2) \(n\) oracle GPs are used to sample \(f(x_j, v_j)\) and the generated dynamics is integrated (starting from zero initial conditions) using forward Euler, \(dt = 1/N\), to form \(K\) trajectories. The GPs are conditioned during this process on the generated states \(x_j\) and inputs \(v_j\). These \(n\) oracle GPs constitute the actual but unknown nonlinear dynamics model. Nominal models can be easily generated by using the predictions of the oracle GPs at a subset of the state space. The construction of a nominal model is described in detail below:

1) Another set of control inputs \(u_j, j = 1, \ldots, N\) are drawn from GPs.
2) The mean predictions \(f(x_j, u_j)\) of the oracle GPs at \(u_j\) are used to evolve these control inputs (as in step 2 of the actual model).
3) The \(n\) separate model GPs (with same hyperparameters as the oracle) are conditioned on the resulting trajectory, i.e., the input pairs \((x_j, u_j)\) and the outputs \(f_j = (x_{j+1} - x_j)/N\) for each time step \(j = 1, \ldots, N\). The mean derivative of the model GPs are calculated analytically (using the kernel derivatives). Discretized time varying matrices \(A_j, B_j\) and their variances \(\Sigma_{0,j}\) are constructed for each \(j = 1, \ldots, N\), based on the mean and variance of the GP derivatives.

By sampling \(K = 20\) trajectories for the conditioning of oracle GPs, we can cover a significant part of the state space in \(n = 2\) dimensions. For each ILC iteration thereafter, the mean predictions are used as in step (2) to evolve the trajectory, but without further conditioning of the model GPs. Instead, adaptation is performed as before with LBR, replacing the steps \((3 - 4)\). We can thus avoid the expensive online GP training. Figure 5 shows the learning performance for a horizon size of \(N = 20\). The dimensions of the system is same as before, \(n = 2, m = 2\) and \(R = 10^{-6}I, Q = I\). The results are averaged again over ten experiments. In this nonlinear setting, the recursive ILC that is not cautious shows an unstable behaviour (not shown in Figure 5). Adding adaptation without caution (i.e.,
using the covariances of the LTV matrices) is not stable for some trajectories and can diverge (brown line). Cautious and adaptive \textit{bayesILC}, on the other hand, (blue line), shows a stable convergent learning performance, whereas purely cautious ILC (brown line) is divergent for some of the trajectories.

3) Barrett WAM Model: We next test ILC on striking movements (29) for a seven degree of freedom Barrett WAM simulation model. In the simulations, the robot is started from a fixed initial state $q_0$. The initial posture is chosen from one of the center, right hand side or left hand side resting postures of the robot. The striking parameters (30) are then optimized, based on an incoming table tennis ball with a randomly chosen incoming position and velocity. The link parameters of the Barrett WAM forward dynamics model used to simulate actual trajectories are perturbed randomly to construct nominal models for ILC. The linearization procedure described in Section II produces LTV nominal models that can be used by ILC to reduce the deviations from the desired (fixed) striking movement over the iterations.

The randomization during the optimization guarantees that a variety of hitting movements are tracked throughout the experiments. The performance of the proposed ILC approach \textit{bayesILC} with three different adaptation laws is then evaluated over the striking segment of the optimized (striking and returning) trajectories. The convergence results are averaged over ten such striking movements, as shown in Figure 6. The adaptation of discrete and continuous LTV models are shown in blue and red, respectively, while the adaptation of link parameters is shown in black. Forgetting factor was set to $\lambda = 0.8$ for all of the adaptation laws. One of the desired trajectories, shown in dashed red on the right hand side, is tracked very closely in the final iteration. The blue markers correspond to the time profile of the motion, which are drawn uniformly spaced, one for each 80 milliseconds.

The recursive ILC (without adaptation or caution) is convergent for some of the hitting movements in Figure 6. However, similar to the previous simulation examples, the recursive form of the ILC update, depending on the accuracy of the model along the trajectories, can fail to converge for some trajectories (not shown in the Figure). The proposed recursive, adaptive and cautious algorithm \textit{bayesILC}, with the three adaptation laws shown in Figure 6, shows a better and faster convergence, for a variety of trajectories.

The ILC experiments shown in Figures 6–7 reset the initial posture always to the same desired posture $q_0$. Next, we consider non-repetitive disturbances around the desired initial posture. This would mean, physically, that the robot is not initialized accurately around the resting posture.

Comparisons to the baseline (black line) in Figure 8 illus-
Finally, we perform experiments on our robotic table tennis platform, see Figure 1, where two seven degrees of freedom (DoF) cable-driven, torque-controlled Barrett WAM arms (Ping and Pong) are hanging from the ceiling. The custom made Barrett WAM arms are capable of high speeds and accelerations (approx. up to $10\text{m/s}^2$ in task space). Standard size rackets (16 cm diameter) are mounted on the end-effector of the arms as can be seen in Figures 10 and 11. A vision system consisting of four cameras hanging from the ceiling around each corner of the table is used for tracking the ball [30]. A ball launcher (see Figure 1) is available to throw balls accurately to a fixed position inside the workspace of the robots. The incoming ball arrives with low-variability in desired positions and higher-variability in ball velocities. The whole area to be covered amounts to about 1 m$^2$ circular region surrounding an initial centered posture of the robots.

The realistic simulation environment SL [31] acts both as a simulator and as a real-time interface to the Barrett WAMs in our experiments. The initial positioning is given by a PD controller with high gains on the shoulder joints, which is then toggled off during the experiments with the striking movements, as summarized in Algorithm 2. The high gain PD controller used to initialize the robots was also tested for tracking the striking movements, see Figure 9. When ILC is applied on top of the PD controller, the learning quickly stagnates, leading to oscillations in some of the joints. Instead, a low-gain LQR feedback law is computed for the striking part of the movement with a linearized nominal dynamics model (4). The weighting matrices for this purpose are set to identity, $Q = I$, and the constant penalty matrix is chosen as $R = 0.05I$. Decreasing the scaling of the penalty matrix to 0.03 causes oscillations in the elbow joint, indicating that the nominal model is not very accurate. At the cost of larger initial error, we suggest increasing the input penalties $R$ to improve the stability of ILC in other high degrees-of-freedom
Fig. 10. The Barrett WAM (a.k.a. Ping) is initialized in our experiments in three different starting postures. We make controlled experiments with a simulated ballgun, and generate many different hitting movements, two of them are shown in the above images. The proposed algorithm bayesILC leads to an efficient and stable learning approach for tracking these hitting movements. The right hand side starting posture can be seen on the upper left image. Initially, before learning with ILC starts, the robot performs poorly, and the hitting posture of the robot is shown in the upper central image. After five iterations, the hitting posture is corrected significantly as shown in the upper right image. Similarly, the bottom three images show the operation of the ILC for another trajectory, where the starting posture is fixed on the left hand side of the robot.

Fig. 11. The Barrett WAM (a.k.a. Pong) is initialized in our experiments in three different starting postures. We make controlled experiments with a simulated ballgun, and generate many different hitting movements, one of them is shown in the above images. The proposed algorithm bayesILC leads to an efficient and stable learning approach for tracking these hitting movements. The left hand side starting posture can be seen on the left image. Initially, before learning with ILC starts, the robot performs poorly, and the hitting posture of the robot is shown in the central image. After five iterations, the hitting posture is corrected significantly as shown in the upper right image.

After the visual system outputs a ball estimate, a ball model can be used along with an Extended Kalman Filter to predict a ball trajectory. The ball model accounts for some of the bouncing behavior of the ball and air drag effects. If the predicted ball trajectory coincides with the workspace of the robot, the motion planning system has to then come up with a trajectory that specifies how, where and when to intercept the incoming ball. Desired Cartesian position, velocity and orientations of the racket at the hitting time $T$ impose constraints on the seven joint angles and seven joint velocities of the robot arm at $T$. Along with the desired hitting time $T$ (or the time until impact), these fifteen parameters are used to generate third-order joint space polynomials. These movements can be optimized online in 20 – 30 milliseconds [2], or loaded from a lookup table. In the ILC experiments, the parameters in the lookup table are used without interpolation, to make sure that the same trajectory can be used for balls deviating slightly from their stored values. We make sure that the lookup table is dense enough and that the ballgun is fixed.

Some examples of the generated trajectories are shown in Figures 10 and 11. After a strike, a linear joint trajectory is computed that will take the robots from the current state to the resting posture in $T_{\text{rest}} = 1.0$ seconds. PD feedback control is turned on again for this returning part of the trajectory. When the returning trajectory is executed, SL main thread running the inverse dynamics computations will continue to keep the arms stable around the resting posture, while another thread is detached to run the ILC update⁴. The ILC loop terminates successfully whenever the computed feedforward updates are within the respective torque limits. After a successful termination, if the actual posture is within 0.1 radians distance of the resting posture, the LQR feedback will be turned on again and the robots will start moving to track the same striking motion.

We use a simulated ball to make more controlled experiments, focusing on the control aspect in more detail. If the striking robot movements are executed accurately, then the ball in simulation will be returned close to a desired position on the opponent’s court. At different points in time we have identified three different sets of link parameters for rigid body dynamics. We can use these parameterizations of rigid body dynamics as potential nominal models to kick-start the learning process. We tested these nominal models first in slowed down hitting

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⁴Code is available in the public repository https://gitlab.tuebingen.mpg.de/ okoc/learning-control along with the test scripts used to generate the plots in the previous subsections.
movements, where a slow down rate of two means that
the number of trajectory points double while the hitting time is
held fixed. Cutting down the trajectories to an initial subset
of the movement to restrict potential instabilities, or initial
masking of some of the joints during ILC updates, are other
techniques that we have employed to evaluate these nominal
models in a careful manner. Of the three models, only one
of them was suitable for the local learning that ILC provides.
This model is further adapted with the proposed bayesILC
algorithm in order to improve the tracking of the striking
movements. Adaptation of the trajectories \( r_j \) and the nominal
inputs \( u_{\text{IDM},j} \) was additionally performed on top of ILC, to
stabilize the learning process, since an accurate initialization
of the joints (especially on the wrist and the elbow) was not
possible with the Barrett WAMs.

We have compared bayesILC to two other ILC methods:
batch ILC (14) and ILC with proportional and derivative
(PD) feedback (with constant \( p, d \) values). PD type ILC with
constant \( p \) or \( d \) values is often too simplistic, and did not
yield any improvement in our setup, even after tuning the \( p, d \)
values. Batch ILC was tested with ten times downsampled
trajectories, with adjustable learning rates. We have found
batch ILC to be inferior to the recursive ILC when tested over
multiple trajectories (slowed down and cut versions included)\(^5\). Recursive ILC without any adaptation is monotonically con-
vergent on average for about five iterations, bringing the root
mean squared (RMS) tracking error from about 0.80 to 0.40
on average. Repeating the trajectories for five more iterations,
we note that the tracking error starts increasing slightly due
to introduced oscillations in some of the joints. Introducing
adaptation with recursive and cautious ILC (i.e., the proposed
approach bayesILC) we can decrease the tracking error further,
to about 0.20 monotonically in five more iterations. This
enables us return \( \%40 \) of the simulated balls to the opponent’s
court.

The proposed update law bayesILC evaluated above adapts
the discrete LTV models with a forgetting factor of \( \lambda = 0.8 \).
This value was chosen experimentally, and could be optimized,
e.g., using a dataset of previous ILC performances. The same
parameter values are chosen for the initial covariances as in the
simulation experiments with the Barrett WAM. Adapting the
continuous LTV models, when the trajectories are smoothened
suitably with a zero-phase filter, leads to faster updates with
similar improvements in tracking performance. Using the
online adaptation of the link parameters on the other hand,
leads to poorer convergence in tracking for some of the joints
(most notably, the elbow). This fact leads us to suspect that
the rigid body dynamics model underfits, i.e., the mismatch
for our Barrett WAMs is not purely parametric in nature. We
see that the final cost (as 2-norm of deviations from desired
joint hitting positions and velocities) drops down from 1.70 to
0.20 for bayesILC when the LTV model matrices are adapted
directly. After performing ten more iterations, the percentage
of balls successfully returned to the opponent’s court increases
from \( \%40 \) to about \( \%60 \) on average.

VII. CONCLUSION AND FUTURE WORK

In this paper we presented a novel Iterative Learning Control
(ILC) algorithm that is recursive, cautious and adaptive at
the same time. The closed-form update law (12) that was
presented derives from the adaptive dual control literature
and is sometimes referred to as passive learning [29]. The
algorithm was then recast in a more efficient form (derived
in Appendix A) which does not require the estimation of
disturbances and can be implemented as a recursive ILC
update. The update law makes it easy to introduce caution
with respect to modelling uncertainties and online adaptation
of the linearized model matrices. Unlike typical ILC updates,
feedback matrices for the tracking of striking trajectories are
adapted as well, which are useful for rejecting noise and
varying initial conditions. We believe that the introduced ILC
update yields a principled approach to adapt the models, as
well as their regularizer, based on data.

The proposed algorithm bayesILC was evaluated in dif-
ferent simulations of increasing complexity. Finally in the

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\(^5\)For batch pseudoinverse-based ILC, inversion of the model matrices (4)
around the unstable hitting trajectory causes instability, which is alleviated
by providing an additional current iteration ILC (CI) [3]. CI adds the current
iteration \( k \)'s feedback errors to the feedforward compensations for the next
iteration \( k+1 \). As in our preliminary experiments with the Barrett WAM [10],
we have applied CI in addition to stabilize a downsampled version of batch
model-based ILC.
last subsection we have presented real robot experiments on our robotic table tennis setup with two Barrett WAMs, see Figures 10 and 11. It was shown that the proposed approach leads to an efficient way to learn to track hitting movements online. Hitting movements throughout the experiments are generated in the joint space of the robots and enable them to execute optimal striking motions. Control inputs, as well as a time-varying feedback law, are updated after each trial by using the model based update rule that considers the deviations from the striking trajectory. After the trajectories are executed, the deviations can be used to adapt the model parameter means and variances using Linear Bayesian Regression (LBR). A forgetting factor was considered in addition to make adaptation more flexible. An adaptation of the reference trajectories as well as the nominal inputs was considered on top of bayesILC to render the method more effective and stable for initial posture stabilization errors.

Although we have shown a stable and efficient way to learn to track references with ILC, we have not analyzed its generalization to arbitrary trajectories. In our table tennis setup, we are slowly making progress to having the two robots play against each other. Generalization capacity would play an important role in extending the average game duration between the robots, as the trajectories during the table tennis matches would be generated online [2] according to the state of the game. We believe that in the case where the trajectories are changing, generalizing the learned control commands can be achieved by compressing them to a lower-dimensional input space (i.e., parameters). Learned feedforward commands could be projected to a parameterized feedback matrix, the parameters of which could represent the invariants between the trajectories. An efficient and stable implementation of such parameterizations will be the focus of our future work.

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Solving (20) for the optimal control input compensations, and arranging using the notation in (12)
\[ \delta u_j = K_j e_j - \Phi_j^{-1} \ell_j, \]
\[ K_j = -\Phi_j^{-1} \Psi_j, \]
\[ \Phi_j = R_j + \mathbb{E}_B [B_j^T P_{j+1} B_j], \]
\[ \Psi_j = \mathbb{E}_{A_j, B_j} [B_j^T P_{j+1} A_j], \]
\[ \ell_j = \mathbb{E}_B [B_j^T (P_{j+1} d_{j+1} + b_{j+1})]. \]

In order to derive Riccati-like equation, we can plug (21) into (19), and using (18) get
\[ e^T P_0 e + 2 e^T b_j + c_j = e_j^T Q_j e_j + e_j^T (\Phi_j^{-1} R_j \Psi_j^{-1} \Phi_j) e_j + 2 e_j^T \Phi_j^{-1} R_j \Psi_j^{-1} \Phi_j \ell_j + 2 \mathbb{E}_{A_j, B_j} [(A_j e_j + b_j) P_{j+1} (A_j e_j + b_j)] + 2 \mathbb{E}_{A_j, B_j} [(A_j e_j + b_j)^T b_{j+1}] + c_{j+1}. \]

We can apply dynamic programming to compute the optimal solution recursively
\[ V(e_j) = \min_{u_j} \mathbb{E}_N \left[ e^T P_j e + 2 e^T b_j + c_j \right], \]
\[ \text{s.t. } e_{j+1} = A_j e_j + B_j u_j + d_{j+1}, \]
where the linear time varying system matrices \( A_j, B_j \) are random variables with known means and variances. Rewriting (17) as
\[ e_{j+1} = A_j e_j + B_j u_j + d_{j+1}, \]

and noticing that the Value Function is a quadratic function of the errors along the trajectory,
\[ V(e_j) = e^T P_j e + 2 e^T b_j + c_j, \]

We provide in this section self contained derivations of the cautious ILC update rule, given in Equations (12) and simplified in (13). Consider the following optimal control problem
\[ \min_{\delta u_j} \sum_{j=1}^{N} \mathbb{E}_{A_j, B_j} [e_j^T Q_j e_j + \delta u_j^T R_j \delta u_j], \]
\[ \text{s.t. } e_{j+1} = A_j e_j + B_j u_j + d_{j+1}, \]
where the linear time varying system matrices \( A_j, B_j \) are random variables with known means and variances. Rewriting (17) as
\[ e_{j+1} = A_j e_j + B_j u_j + d_{j+1}, \]
\[ d_{k,j+1} = B_j u_j + d_{j+1}, \]
and noticing that the Value Function is a quadratic function of the errors along the trajectory,
\[ V(e_j) = e^T P_j e + 2 e^T b_j + c_j, \]

We can apply dynamic programming to compute the optimal solution recursively
\[ V(e_j) = \min_{u_j} \mathbb{E}_N \left[ e^T Q_j e_j + \delta u_j^T R_j \delta u_j + \mathbb{E}_{A_j, u_j} V(e_{j+1}, j + 1) \right], \]
\[ V(e_{j+1}, j + 1) = 2 b_j^T P_j e_j + (A_j e_j + B_j d_{j+1} + c_{j+1})^T e_{j+1} \]
\[ + (A_j e_j + B_j d_{j+1} + c_{j+1})^T P_j e_j + (A_j e_j + B_j d_{j+1} + c_{j+1})^T P_j e_j + d_{j+1}. \]

The recursion starts from \( P_N = Q_N \). Taking derivative w.r.t. \( \delta u_j \) of the right hand side, we get
\[ R_j \delta u_j + (\mathbb{E}_{A_j, B_j} [B_j^T P_{j+1} A_j] e_j + \mathbb{E}_{B_j} [B_j^T P_{j+1} B_j] \delta u_j + \mathbb{E}_{B_j} [B_j^T (P_{j+1} d_{j+1} + b_{j+1})]) = 0. \]
since $P_j = Q_j + M_j - \Psi_j^T \Phi_j^{-1} \Psi_j$, the last term becomes
\[
\left( P_j - M_j - K_j^T \Psi_j \right) e_{k,j} = Q_j e_{k,j},
\]
hence, the feedforward recursion defining (24) can be computed independently of disturbance estimates
\[
\nu_j = \mathbb{E}[A_j^T \nu_{j+1}] + Q_j e_{k,j}, \ j = 1, \ldots, N - 1,
\]
starting from $\nu_N = 0$.

**Appendix B**

**Movement Generation for Table Tennis**

In a highly dynamic and complex task such as robot table tennis, one often needs to consider an extension of the standard trajectory tracking task. Based on the varying initial positions and velocities of the robot arm and the trajectory of the incoming ball, each table tennis stroke the robot arm needs to track different trajectories that start from different initial conditions and end with different desired goal states of the arm. Moreover, these trajectories need to be optimized in time to intercept the ball. The striking trajectories $r(t) = [q_{\text{des}}(t), q_{\text{des}}(t)]^T$ are generated online and tracked using the proposed ILC approach.

Striking movement primitives suited to table tennis have been proposed in [32] and [1] as an extension of discrete Dynamic Movement Primitives (DMP). Unlike the original formulation [33], these extensions allow for an arbitrary velocity profile to be attached to the primitives around hitting time. However, these approaches are heavily structured for the problem at hand, introducing and tuning additional domain parameters. In [10] we instead proposed to use rhythmic movement primitives that allow for a limit cycle attractor, which is desirable if we want to maintain the striking motion through goal state. After the striking is completed the DMP can be used to return back to initial state or it can be terminated by setting the forcing terms to zero. An example is shown in Figure 9.

One of the problems with such (kinesthetic) teach-in based approaches is that it is difficult to train heavy robots well for successful performance. For example, the shoulder of the Barrett WAM arm shown in Figure 1 weighs 10 kg alone. It is rather difficult for humans to move the links with heavy inertia. The slower movements of the heavier links are typically compensated with faster movements of the lighter links (such as the wrist). However, tracking these trajectories can also be harder for more demanding wrist movements. An additional difficulty with cable-driven robots such as the Barrett WAM is that the wrists are harder to control.

Based on these considerations, we have worked on a free-final time optimal control based approach to generate minimum acceleration hitting movements for table tennis [2]. In the experiments section, we focus on learning to track these hitting movements. These trajectories are third order polynomials for each degree of freedom of the robot.

We will briefly introduce here the trajectory generation framework introduced in [2]. Consider the following free-time optimal control problem [34]
\[
\begin{align*}
& \min_{\bar{q},\bar{\tau}} \int_0^T \bar{\dot{q}}(t)^T \bar{R} \bar{\dot{q}}(t) \, dt \\
& \text{s.t.} \quad \Psi_{\text{hit}}(\bar{q}(T), T) \in \mathcal{H}, \quad (26) \\
& \psi_{\text{net}}(\bar{q}(T), \bar{q}(T), T) \in \mathcal{N}, \quad (27) \\
& \psi_{\text{hand}}(\bar{q}(T), \bar{q}(T)) \in \mathcal{L}, \quad (28)
\end{align*}
\]
where the final hitting time $T$ is an additional variable to be optimized along with the joint accelerations $\bar{\dot{q}}(t) : [0, T] \rightarrow \mathbb{R}^n$. The weighting matrix $\bar{R}$ for the accelerations is positive definite. Initial conditions for the robot are the joint positions $q_0$ and joint velocities $\dot{q}_0$. The inequality constraints (26) – (27) ensure that the task requirements for table tennis are satisfied, namely, hitting the ball, passing the net, and landing on the opponent’s court.

Solutions of (25) – (28) can be found using Pontryagin’s minimum principle [35]. The optimal $\bar{q}(t)$ in both cases is a third degree polynomial for each degree of freedom. The striking time $T$, the joint position and velocity values at striking time $q_f$ and $\dot{q}_f$ fully parametrize this problem. The time it takes to return to the starting posture, $T_{\text{rest}}$ can be chosen suitably, e.g., based on the speed of the ballgun. The polynomial coefficients for the striking trajectory
\[
\begin{align*}
\bar{q}_{\text{strike}}(t) &= a_3 t^3 + a_2 t^2 + a_1 t + a_0, \quad (29)
\end{align*}
\]
can then be fully determined in joint-space
\[
\begin{align*}
a_3 &= \frac{2}{T^3} (q_0 - q_f) + \frac{1}{T^2} (q_0 + q_f), \\
a_2 &= \frac{3}{T^2} (q_f - q_0) - \frac{1}{T} (q_f + 2q_0),
\end{align*}
\]
for each degree of freedom of the robot.