Prices of anarchy of selfish 2D bin packing games∗

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Abstract

We consider a game-theoretical problem called selfish 2-dimensional bin packing game, a generalization of the 1-dimensional case already treated in the literature. In this game, the items to be packed are rectangles, and the bins are unit squares. The game starts with a set of items arbitrarily packed in bins. The cost of an item is defined as the ratio between its area and the total occupied area of the respective bin. Each item is a selfish player that wants to minimize its cost. A migration of an item to another bin is allowed only when its cost is decreased. We show that this game always converges to a Nash equilibrium (a stable packing where no single item can decrease its cost by migrating to another bin). We show that the pure price of anarchy of this game is unbounded, so we address the particular case where all items are squares. We show that the pure price of anarchy of the selfish square packing game is at least $2 \cdot 3634$ and at most $2 \cdot 3675$. We also present analogous results for the strong Nash equilibrium (a stable packing where no nonempty set of items can simultaneously migrate to another common bin and decrease the cost of each item in the set). We show that the strong price of anarchy when all items are squares is at least $2 \cdot 0747$ and at most $2 \cdot 3605$.

Keywords: Selfish bin packing; square packing; rectangle packing; Nash equilibrium; strong Nash equilibrium; price of anarchy.

1 Introduction

The advent of the Internet and its increasing use have brought new computational tasks and different ways of performing various activities. In such a decentralized environment, many activities involve competition and collaboration over the resources, so users may act selfishly to maximize their benefits. This behavior suggests game-theoretical models for a number of applications.

Many new ideas, approaches, and models of analysis were first proposed to bin packing problems, which motivate the study of game theoretical versions of these problems.

We investigate here a class of packing games which we call selfish 2D bin packing games, in the special case where the bins (or recipients) are unit squares and the items to be packed are rectangles, or more particularly, squares. We call the corresponding games selfish rectangle packing or selfish square packing. (The more general 2D class may include cases in which the items are 2-dimensional objects of other specific forms, such as disks or triangles.) All games

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start with a set of items packed in bins. In our model, the cost of an item is defined as the ratio between its area and the total occupied area of the respective bin. Each item is a selfish player: it wants to minimize its cost and, for that, it may selfishly migrate to a bin with a better occupied area. A stable packing in which no item can decrease its cost by migrating to another bin is called a Nash equilibrium.

Koutsoupias and Papadimitriou [11, 12] were the first to study a measure in a game-theoretic framework that nowadays is known as the price of anarchy, which is the ratio between the worst social cost (number of used bins) of a Nash equilibrium and the optimal social cost (minimum number of bins needed to pack all items). When the game admits coalitions, these concepts are known as the strong Nash equilibrium and strong price of anarchy. In non-cooperative games, the price of anarchy measures the loss of the overall performance due to the decentralized environment and the selfish behavior of the players.

The 1-dimensional (1D) version of the game described above is known as the selfish (1D) bin packing game. It was first investigated by Bilò [1], who proved that this version of the game admits a pure Nash equilibrium. He showed upper bounds on the number of steps to reach a Nash equilibrium from an arbitrary initial configuration and also showed that the price of anarchy is at least 1.6 and at most 1.666. The lower bound and the upper bound have been improved by Yu and Zhang [13] and by Epstein and Kleiman [3] to 1.6416 and 1.6428, respectively. They also showed that the strong price of anarchy is at least 1.6067 and at most 1.6210. There are also results in the literature, with different cost functions. Ma et al. [14] studied the model in which all items in the same bin share the cost equally, that is, if a bin contains $k$ items, then the cost for each item in this bin is $1/k$. They showed that the price of anarchy of any Nash equilibrium under this cost function has an upper bound of 1.7 and also that it is possible to obtain a Nash equilibrium from a feasible packing in $O(n^2)$ steps without increasing its social cost. This result leads to an algorithm that obtains a Nash equilibrium in time $O(n^2)$ with price of anarchy that is at most $1 + \epsilon$, for any given $\epsilon > 0$. For a survey on selfish packing games, we refer the reader to Epstein [3].

In 1D packing, it is easy to decide if an item fits in a unit bin with other items, but this is not the case for higher dimensional packing. So we also consider a parameterized version of the original game in which the players use a specific (polynomial-time) packing algorithm to decide on moving an item. One can think of the original game as one parameterized by an exact packing algorithm. As we will see, some results obtained with the use of specific packing algorithms yield results for the original generic game.

For the rectangle packing game, we consider oriented packing (that is, when rotations are not allowed). We show that, in this case, the game converges to a Nash equilibrium. Then, we prove that its price of anarchy is unbounded. In view of this, we consider two particular cases: (a) the parametric version in which the dimensions of the rectangles are bounded by $1/m$, for an integer $m \geq 2$; and (b) the case in which all items are squares. For the first case, we show that the game parameterized by the well-known NFDH (Next Fit Decreasing Height) algorithm [2] (and the original game as well) has price of anarchy at most $(m/(m - 1))^2$. For the latter case, the selfish square packing game, we prove that the pure price of anarchy is at least 2.3634 and at most 2.6875.

We also present results on the strong Nash equilibrium. We prove that the strong price of anarchy of the selfish square packing game is at least 2.0747 and at most 2.3605.

We have presented some preliminary results on the selfish 2D packing game in an extended abstract for a conference [8]; this paper contains improved results and new ones.

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1 In [11], the authors used the term coordination ratio for the concept, which was later called price of anarchy by Papadimitriou [17].
2 Problem definition and preliminaries

The games considered here are on packing of rectangles (or squares) into unit squares. They are natural generalizations of the selfish 1D packing game, that was introduced by Bilò [1]. We observe that, as we consider oriented packing, there is no loss of generality in the assumption that the bins are unit squares (if not, a scaling of the items and bins will reduce the instance to this case).

We denote by \( \{1, 2, \ldots, n\} \) the set of items (rectangles) in these games, and assume that each item \( i \) has area \( a_i > 0 \) and fits in a bin. Each item is a selfish player that wants to minimize its cost (defined in what follows).

A configuration of a game in a certain moment is a sequence \( p = (p_1, \ldots, p_n) \), where \( p_i \) indicates the bin selected by \( i \). For a bin \( k \), we denote by \( R(k) \) the set of rectangles \( i \) for which \( p_i = k \). If the rectangles in \( R(k) \) fit all together in a bin, then \( a(R(k)) \) denotes the sum of the areas of the rectangles in \( R(k) \), otherwise, we define that \( a(R(k)) = 0 \), and refer to \( k \) as an infeasible bin. We say that \( p \) uses a bin \( k \) if \( a(R(k)) \neq 0 \), and we say that \( p \) is a feasible configuration if \( a(R(p_i)) \neq 0 \) for \( i = 1, \ldots, n \). The cost of an item \( i \) in a configuration \( p \) is given by \( c_i = a_i / a(R(p_i)) \) if \( a(R(p_i)) \neq 0 \), and is infinite otherwise.

As in the Elementary Stepwise System dynamics [2], one at a time players selfishly change their choice of bin, in order to minimize their cost. That is, a player \( i \) in a bin \( k \) has incentive to move to a bin \( \ell \) if \( i \) fits into bin \( \ell \) and \( a(R(\ell) \cup \{i\}) > a(R(k)) \), that is, after the move, its cost decreases. We note that, if a configuration is feasible, then an improving step only happens if the resulting configuration is also feasible. If a configuration is not feasible (because \( a(R(k)) = 0 \) for some bin \( k \)), then an item in an infeasible bin may migrate to another bin (possibly an empty one) and decrease its cost.

The selfish rectangle packing game (SRPG) is defined by a set \( \{1, 2, \ldots, n\} \) of items (rectangles or players), and a set of unit bins into which the items are to be packed, and a cost function that assigns to each item \( i \) a cost \( c_i \), as defined above. The game starts with an arbitrary configuration. Each item wants to minimize its cost, so it can selfishly migrate to decrease its cost. The social cost is the number of used bins (which is precisely \( c_1 + c_2 + \cdots + c_n \) if the configuration is feasible). When all items to be packed are squares, we refer to the selfish square packing game (SSPG).

We say that a configuration of a game is a Nash equilibrium if no player can decrease its cost by moving to (that is, selecting) another bin. Since a game starts at an arbitrary configuration, it is reasonable to ask whether it will always reach a Nash equilibrium. We say that a game converges to a Nash equilibrium if the answer to this question is yes. The next result also holds for the (more general) selfish 2D bin packing game.

**Lemma 2.1.** The selfish rectangle packing game converges to a Nash equilibrium.

**Proof.** We follow the proof presented by Bilò [1], using area instead of length. At each moment of a game, consider the list whose elements are the numbers \( a(R(k)) \) for each used bin \( k \), sorted in non-increasing order. Note that there is a finite number of different such lists (even considering that some bins may be infeasible). After each migration, the new list is always lexicographically greater than the previous. Indeed, when a rectangle migrates from a bin \( k \) to a bin \( \ell \), it is clear that \( a(R(\ell)) \) after the migration is greater than both \( a(R(k)) \) and \( a(R(\ell)) \) before the migration. So, after a finite number of steps, no item can migrate. \( \Box \)

The minimum number of bins needed to pack all the rectangles in a game \( G \) is the optimal social cost, denoted simply by OPT\((G)\), or OPT when \( G \) is clear from the context. Let \( sc(p) \) be the number of bins used in a feasible configuration \( p \). (The acronym \( sc \) stands for social cost.) Let \( \mathcal{N}(G) \) be the set of all Nash equilibria of \( G \).
The (asymptotic) price of anarchy of a class \( G \) of games is defined as

\[
\text{PoA} := \lim_{m \to \infty} \sup_{G \in G, \text{OPT}(G) = m} \max_{p \in \mathcal{N}(G)} \frac{\text{sc}(p)}{m}.
\] (1)
the other with the items with dimensions \((1/2^i, 1−1/2^i)\). So, we have \(sc(p)/2 \geq k/2\). As the number \(k\) of bins can be made as large as we wish, this shows that the price of anarchy of SRPG is unbounded.

Figure 1: (a) A Nash equilibrium using \(k\) bins. (b) Optimal configuration using two bins.

Now we show upper bounds for the price of anarchy of the selfish rectangle packing game using as packing algorithm the well-known NFDH [2]. This algorithm first sorts the set \(L\) of rectangles (to be packed) in decreasing order of height, then packs the rectangles side by side generating levels. The height of a level is the height of the first rectangle in the level. When a rectangle cannot be packed in the current level, if it fits in the same bin on top of the current level, it is packed in a new level above the previous one. If it does not fit, we say that the NFDH algorithm fails to pack the rectangles; otherwise, if the whole set \(L\) can be packed in this way in the bin, then we say that NFDH succeeds.

Note that the configuration in Figure 1(a) is also a Nash equilibrium of the selfish rectangle packing game using NFDH. So PoA(NFDH) is unbounded as well. On the other hand, it is not difficult to show that PoA(NFDH) \(\leq 4\) if we allow only rectangles with dimensions at most \(1/2\). In fact, in what follows we show an upper bound for PoA(NFDH) restricted to rectangles with dimensions at most \(1/m\), for \(m \geq 2\). For that, we use the following result, from which Theorem 3.3 below can be easily obtained.

**Theorem 3.2** (Epstein and Levy [5]). Let \(L \cup \{r\}\) be a set of rectangles with dimensions at most \(1/m\), for \(m \geq 2\). If NFDH packs \(L\) in one bin, but cannot pack \(L \cup \{r\}\) in one bin, then \(a(L) \geq (m−1/m)^2\).

**Theorem 3.3.** For selfish rectangle packing games restricted to rectangles with dimensions at most \(1/m\), for \(m \geq 2\), we have \(PoA(NFDH) \leq \left(\frac{m}{m-1}\right)^2\).

### 3.2 Selfish square packing

We show in this section a lower bound for the price of anarchy of the selfish square packing game. We show a weaker result, and once it is understood, we mention how a slight improvement can be obtained.

First, we describe an optimal packing \(O\) that consists of \(N\) bins with the same configuration, all completely filled with squares of side \((1/i) + \varepsilon\), for some different integer values \(i\), and a very small positive real number \(\varepsilon\). (The value of \(N\) will be defined later.) Then, we describe a packing \(P\) of the same set of squares that is a Nash equilibrium.

The bins of the optimal packing \(O\) have the configuration shown in Figure 2. Each bin contains \(n_i\) square(s) of side \(s_i\), for \(i = 1, \ldots, 7\), as defined ahead and shown in Figure 2 and has the remaining apparently free space completely filled with small squares, which are like sand. These small squares have total area \(\gamma\) (which is approximately 0.0488, considering \(\varepsilon\) very small). These \(N\) bins packed in a completely filled way clearly define an optimal packing.
Now, let us describe the packing $P$. It consists of homogeneous bins. For $0 < \ell \leq 1$, we say that an occupied bin $B$ is $\ell$-homogeneous if all squares packed in $B$ have side $\ell$, and it has the maximal number of such squares.

Note that, for an integer $i > 1$, the wasted space is large for a $(1/i + \varepsilon)$-homogeneous bin for small $i$ and $\varepsilon$. Moreover, no packed in a $(1/j + \varepsilon)$-homogeneous bin can migrate to another $(1/i + \varepsilon)$-homogeneous bin. Indeed, if $j \leq i$, the square does not fit in the other bin. If $j > i$, then, in a $(1/i + \varepsilon)$-homogeneous bin, there are $(i-1)^2$ squares and the occupied area is $(i-1)^2(1/i + \varepsilon)^2$. Moreover, as $j \geq i + 1$, one can check that $(1/j + \varepsilon)^2 + (i-1)^2(1/i + \varepsilon)^2 < (j-1)^2(1/j + \varepsilon)^2$, for $\varepsilon$ small enough. So, a configuration consisting of homogeneous bins is a Nash equilibrium.

Now, we define how many homogeneous bin of each type there are, so that the set of squares packed in the $N$ bins is precisely the set of squares packed in these homogeneous bins. These numbers are:

- $m_1 := N$ bins, each one with precisely one square of side $1/2 + \varepsilon$;
- $m_2 := 3N/4$ bins, each one with precisely 4 squares of side $1/3 + \varepsilon$;
- $m_3 := 2N/9$ bins, each one with precisely 9 squares of side $1/4 + \varepsilon$;
- $m_4 := 2N/16$ bins, each one with precisely 16 squares of side $1/5 + \varepsilon$;
- $m_5 := 5N/36$ bins, each one with precisely 36 squares of side $1/7 + \varepsilon$;
- $m_6 := 2N/49$ bins, each one with precisely 49 squares of side $1/8 + \varepsilon$;
- $m_7 := 5N/144$ bins, each one with precisely 144 squares of side $1/13 + \varepsilon$; and
- $m_8 := \gamma N$ bins, each one completely filled with the very small squares.

Take $N$ such that each $m_i$ is an integer. This configuration is a Nash equilibrium (because of what was argued before), and uses $\sum_{i=1}^{8} m_i > 2.3604N$ bins. As $\text{OPT} = N$, we conclude that $\text{PoA} > 2.3604$.

We obtained the configuration in Figure 2 in the following way. We packed first a square of side $1/2 + \varepsilon$, then we packed the largest possible number of squares of side $1/3 + \varepsilon$, then continued packing the largest possible number of squares of side $1/4 + \varepsilon$, and so on. We stopped the process with squares of side $1/13 + \varepsilon$, as the gain from continuing became very small. At this point, we filled the remaining area with equal size tiny squares (like sand) as that helps to increase a little more the obtained ratio, and also because it makes it obvious that the configuration described is an optimal packing.

This result was obtained before we learned that similar ingredients were used by Epstein and van Stee \cite{stein} to provide a lower bound for the online bounded space hypercube packing problem. In fact, for square packing, the configuration obtained by these authors gives a better bound for our game problem. While we stopped with squares of side $1/13 + \varepsilon$, they proceed up to squares of side $1/43 + \varepsilon$, in the following way (continuing from $m_7$): $m_8 := 2$ squares of side $s_8 := 1/14 + \varepsilon$; $m_9 := 1$ square of side $s_9 := 1/18 + \varepsilon$; $m_{10} := 2$ squares of side $s_{10} := 1/21 + \varepsilon$; $m_{11} := 2$ square of

\[
\begin{align*}
N_\varepsilon &:= 1 \text{ square of side } 1/2 + \varepsilon; \\
n_2 &:= 3 \text{ side } s_2 := 1/3 + \varepsilon; \\
n_3 &:= 2 \text{ side } s_3 := 1/4 + \varepsilon; \\
n_4 &:= 2 \text{ side } s_4 := 1/5 + \varepsilon; \\
n_5 &:= 5 \text{ side } s_5 := 1/7 + \varepsilon; \\
n_6 &:= 2 \text{ side } s_6 := 1/8 + \varepsilon; \\
n_7 &:= 5 \text{ side } s_7 := 1/13 + \varepsilon; \\
\end{align*}
\]
Hence, by Theorem 3.5, a first level of height contains at most two medium squares, which are in \( Z \) be a largest square in \( Z \) the second level. Let \( X \) diction that \( X \) contains at least one medium square. If every square in \( X \) then we can apply Corollary 3.6 to \( X \). Let us analyze why this happens.

For simplicity, for a square \( X \) cannot pack \( X \) for bins containing one item of side greater than 1 2, and also for the other bins containing only squares with side at most 1/2 (by Theorem 3.2). In what follows, we will prove a better upper bound for PoA(NFDH).

The next lemma is the main part of the proof of a result stated without a proof in the preliminary version of this paper [8, Lemma 3.3]. Its proof, as well as that of the following theorem, uses the following terminology.

A square is big if it has side larger than 1/2; medium if it has side larger than 1/3 and at most 1/2; and small if it has side at most 1/3.

Lemma 3.7. Let \( y \) be a small square and \( X \) be a set of small or medium squares. If NFDH cannot pack \( X \cup \{y\} \) in one bin, then \( a(X) \geq 4/9 \).

Proof. For simplicity, for a square \( x \), we use \( x \) to refer to its side. Suppose by contradiction that \( a(X) < 4/9 \). This in particular means that \( X \) contains at most three medium squares.

Let us first argue that \( X \) contains at least one medium square. If every square in \( X \) is small, then we can apply Corollary 3.6 to \( X \cup \{y\} \), with \( m = 3 \). Indeed, as \( a(X \cup \{y\}) < 4/9 + 1/9 = 5/9 = 1/m^2 + (1 - 1/m)^2 \), NFDH packs all squares from \( X \cup \{y\} \) in a bin, a contradiction. So \( X \) contains at least one medium square.

Now let us prove that \( X \) should contain exactly three medium squares. Suppose by contradiction that \( X \) contains one or two medium squares. Let \( X^+ = X \cup \{y\} \). By assumption, NFDH fails to pack \( X^+ \). Let us analyze why this happens.

Observe that all squares have side at most 1/2, thus NFDH fails to pack a square only after the second level. Let \( Z \) be the set of squares in \( X^+ \) that NFDH packs in the first level. Let \( x \) be a largest square in \( Z \), and \( r \) be a largest square in \( X^+ \setminus Z \). Note that \( r \leq 1/3 \), because \( X^+ \) contains at most two medium squares, which are in \( Z \).

Let \( R \) be the rectangle of dimension \((1, 1 - x)\) (corresponding to the region of a bin above a first level of height \( x \)). As NFDH fails to pack \( X^+ \) in a bin, it also fails to pack \( X^+ \setminus Z \) in \( R \). Hence, by Theorem 3.5

\[
a(X) = a(X^+) - y^2 > r^2 + (1-r)(1-x-r) + a(Z) - y^2 \geq r^2 + (1-r)(1-x-r) + x^2.
\]

\[\]

**Theorem 3.4.** The price of anarchy of the selfish square packing game is at least 2.3634.

We now turn to upper bounds for the price of anarchy of the selfish square packing game. For that, let us consider a result that is similar to Theorem 3.2 proved by Meir and Moser [15, Thm. 1] for the special case in which all items are squares.

**Theorem 3.5 (Meir and Moser [15]).** Every set \( S \) of squares whose largest square has side \( \ell \) can be packed by NFDH in a rectangle with dimension \((b, h)\) if \( \ell \leq \min\{b, h\} \) and \( a(S) \leq \ell^2 + (b-\ell)(h-\ell) \).

**Corollary 3.6.** Let \( S \) be a set of squares, each with side at most 1/m for \( m \geq 2 \). If \( a(S) \leq \frac{1}{m^2} + (1 - \frac{1}{m})^2 \), then NFDH can pack \( S \) in a unit square.

Using this result, we can prove that \( \text{PoA}(\text{NFDH}) \leq 4 \). For that, it suffices to note that all bins, except possibly one, have an occupied area of at least 1/4. This area occupation is valid for bins containing one item of side greater than 1/2, and also for the other bins containing only squares with side at most 1/2 (by Theorem 3.2). In what follows, we will prove a better upper bound for \( \text{PoA}(\text{NFDH}) \).

The next lemma is the main part of the proof of a result stated without a proof in the preliminary version of this paper [8, Lemma 3.3]. Its proof, as well as that of the following theorem, uses the following terminology.

A square is big if it has side larger than 1/2; medium if it has side larger than 1/3 and at most 1/2; and small if it has side at most 1/3.

**Lemma 3.7.** Let \( y \) be a small square and \( X \) be a set of small or medium squares. If NFDH cannot pack \( X \cup \{y\} \) in one bin, then \( a(X) \geq 4/9 \).

Proof. For simplicity, for a square \( x \), we use \( x \) to refer to its side. Suppose by contradiction that \( a(X) < 4/9 \). This in particular means that \( X \) contains at most three medium squares.

Let us first argue that \( X \) contains at least one medium square. If every square in \( X \) is small, then we can apply Corollary 3.6 to \( X \cup \{y\} \), with \( m = 3 \). Indeed, as \( a(X \cup \{y\}) < 4/9 + 1/9 = 5/9 = 1/m^2 + (1 - 1/m)^2 \), NFDH packs all squares from \( X \cup \{y\} \) in a bin, a contradiction. So \( X \) contains at least one medium square.

Now let us prove that \( X \) should contain exactly three medium squares. Suppose by contradiction that \( X \) contains one or two medium squares. Let \( X^+ = X \cup \{y\} \). By assumption, NFDH fails to pack \( X^+ \). Let us analyze why this happens.

Observe that all squares have side at most 1/2, thus NFDH fails to pack a square only after the second level. Let \( Z \) be the set of squares in \( X^+ \) that NFDH packs in the first level. Let \( x \) be a largest square in \( Z \), and \( r \) be a largest square in \( X^+ \setminus Z \). Note that \( r \leq 1/3 \), because \( X^+ \) contains at most two medium squares, which are in \( Z \).

Let \( R \) be the rectangle of dimension \((1, 1 - x)\) (corresponding to the region of a bin above a first level of height \( x \)). As NFDH fails to pack \( X^+ \) in a bin, it also fails to pack \( X^+ \setminus Z \) in \( R \). Hence, by Theorem 3.5

\[
a(X) = a(X^+) - y^2 > r^2 + (1-r)(1-x-r) + a(Z) - y^2 \geq r^2 + (1-r)(1-x-r) + x^2.
\]

\[\]
The last inequality follows from the fact that \(a(Z) - y^2 \geq x^2\). This holds because there are at least two squares in \(Z\) and the second largest square in \(Z\) is at least as large as \(y\). Hence,

\[
a(X) > x^2 + 2r^2 - x - 2r + xr + 1
= (x - 1/3)^2 + 2(1/3 - r)^2 - x(1/3 - r) - (2/3)r + 2/3
\geq 2/3 - 1/2(1/3 - r) - (2/3)r \quad \text{(because } x \leq 1/2) \\
= 1/2 - r/6 \geq 4/9 \quad \text{because } r \leq 1/3.
\]

But this is a contradiction, because \(a(X) < 4/9\).

So \(X\) contains three medium squares. As before, let \(X^+ = X \cup \{y\}\). By assumption, NFDH fails to pack \(X^+\). Let us analyze why this happens. Again, NFDH fails to pack a square in \(X^+\) only after the second level. Also, as \(X^+\) contains three squares in \(M\), NFDH packs exactly two squares in the first level (two of the squares in \(X^+ \cap M\)). Let us argue that NFDH packs at least three squares in the second level.

Let \(x_1, x_2, \ldots \) be the squares in \(X\) sorted so that \(x_1 \geq x_2 \geq \cdots \) and let \(x_i = 0\) if \(i > |X|\). If \(X\) had at most three squares, NFDH would succeed in packing \(X^+\). So \(X\) has at least four squares, and \(|X^+| \geq 5\). To prove that there are at least three squares in the second level, it is enough to show that \(x_3 + x_4 \leq 2/3\). Indeed, if \(x_3 + x_4 \leq 2/3\), as \(y \leq 1/3\) and \(x_5 \leq 1/3\), clearly \(x_3 + x_4 + \max\{y, x_5\} \leq 1\). This means that NFDH packs at least three squares in the second level. (Hence \(|X| \geq 5\), otherwise NFDH would not fail to pack \(X^+\).)

In fact, we will prove a stronger statement that will help us also at another point of the proof. We will prove (by contradiction) that \(x_1 + x_3 + x_4 \leq 1\). This implies that \(x_3 + x_4 \leq 1 - x_1 < 2/3\). Suppose that \(x_1 + x_3 + x_4 > 1\), that is, \(x_4 > 1 - x_1 - x_3\). Then

\[
a(X) > x_1^2 + 2x_3^2 + (1 - x_1 - x_3)^2 = 2(x_1 - 1/3)^2 + 3(x_3 - 1/3)^2 + 4/9 + 2x_1x_3 - (2/3)x_1.
\]

But \(2x_1x_3 - (2/3)x_1 \geq 0\) because \(x_3 > 1/3\). So, we can conclude that \(a(X) > 4/9\), a contradiction. Therefore \(x_1 + x_3 + x_4 \leq 1\), as we wished to prove.

Let \(Z\) be the set of squares in \(X^+\) that NFDH packs in the first two levels. As NFDH fails to pack \(X^+ \setminus Z\) is non-empty. Let \(r\) be a largest square in \(X^+ \setminus Z\). Note that \(r \leq x_4\), because \(x_4\) is certainly packed in the second level. Thus \(r \leq 1 - x_1 - x_3\) because \(x_1 + x_3 + x_4 \leq 1\).

The heights of the first and second levels are \(x_1\) and \(x_3\) respectively. So let \(R\) be the rectangle of dimension \((1, 1 - x_1 - x_3)\), corresponding to the region of the bin above the first two levels. As NFDH fails to pack \(X^+\), we know that it also fails to pack \(X^+ \setminus Z\) in \(R\). Hence, using Theorem 3.5, we derive that \(a(X^+ \setminus Z) > r^2 + (1 - r)(1 - x_1 - x_3 - r)\), and

\[
a(X) = a(X^+) - y^2
\geq r^2 + (1 - r)(1 - x_1 - x_3 - r) + a(Z) - y^2
\geq x_1^2 + 2x_3^2 + 2r^2 + (1 - r)(1 - x_1 - x_3 - r).
\] (2)

Inequality (2) holds because \(a(Z) - y^2\) is at least the sum of the areas of the four largest squares in \(X\), which is at least \(x_1^2 + 2x_3^2 + r^2\). Now, as \(r \leq x_1\), from the inequalities above, we get that

\[
a(X) > x_1^2 + 2x_3^2 + 2r^2 + (1 - x_1)(1 - x_1 - x_3 - r)
= x_1^2 + (1 - x_1)^2 + 2x_3^2 + 2r^2 - (1 - x_1)(x_3 + r)
\geq 1/2 + 2x_3^2 + 2r^2 - (1 - x_3)(x_3 + r)
= 3x_3^2 + 2r^2 - x_3 - r + x_3r + 1/2
= 3(x_3 - 1/6)^2 + 2(r - 1/6)^2 + x_3r - r/3 + 13/36
\geq 3(1/6)^2 + r/3 - r/3 + 13/36 = 4/9.
\] (4)
Note that inequality (3) holds because \( x_2^2 + (1 - x_1)^2 \geq 1/2 \) for every \( x_1 \), and (4) holds for every \( x_3 \geq 1/3 \). The equalities are plain mathematical manipulation. But \( a(X) > 4/9 \) contradicts the initial hypothesis, and we conclude the proof that, if NFDH cannot pack \( X \cup \{y\} \) in one bin, then \( a(X) \geq 4/9 \).

Note that the bound 4/9 in Lemma 3.7 is best possible. To see this, consider \( X \) consisting of four squares of side \( 1/3 + \varepsilon \), and \( y \) having side 1/3.

**Theorem 3.8.** For the selfish square packing game with all squares having side at most 1/2, in any Nash equilibrium, the area used in each bin is at least 4/9, except for at most two bins.

**Proof.** Consider a Nash equilibrium of such a game. Let us call bad a used bin whose used area is less than 4/9. We will prove that the number of bad bins is at most two.

There is at most one bad bin that contains only medium squares. This is true because any bad bin can contain at most three medium squares. As NFDH succeeds in packing any four medium squares in a bin, there cannot exist two bad bins containing only medium squares (because if they existed, then any square from the less used bin could migrate to the most used one).

Let us prove that there is at most one bad bin with at least one small square. If we prove this, the number of bad bins is at most two (one with only medium squares and the other containing at least one small square).

Suppose there are two bad bins, \( B_X \) and \( B_Y \), with at least one small square each. Let \( X = R(B_X) \) and \( Y = R(B_Y) \). Suppose \( a(X) \geq a(Y) \), and let \( y \) be a small square in \( Y \). By Lemma 3.7 if \( y \) cannot migrate to \( B_X \), then \( a(X) \geq 4/9 \), which is a contradiction.

Observe that the theorem cannot be improved to state that all but one bin have occupied area of at least 4/9. Indeed, consider one bin with one square of side 1/2 and another bin with three squares of side \( 1/3 + \varepsilon \) and two of side 1/6. Both such bins are bad, and together define a configuration which is a Nash equilibrium.

The following lemma will be used in the proof of the next two theorems.

**Lemma 3.9 (Lemma 4.4 [10]).** For every real numbers \( a, b, \gamma, \delta \) with \( a > 0 \) and \( 0 < \gamma < \delta < 1 \),

\[
\frac{a + b}{\max\{a, \gamma a + \delta b\}} \leq 1 + \frac{1 - \gamma}{\delta}.
\]

Now we are ready to prove the promised improved upper bound for PoA(NFDH).

**Theorem 3.10.** For the selfish square packing game, PoA(NFDH) \( \leq 2.6875 \).

**Proof.** Consider a configuration \( p \) that is a Nash equilibrium for the selfish square packing game using NFDH. Let \( \mathcal{B} \) be the set of bins in this configuration containing a big square, and let \( \mathcal{C} \) be the set of the remaining used bins. Bins in \( \mathcal{B} \) have an occupied area of at least 1/4. By Theorem 3.8 at least \( |\mathcal{C}| - 2 \) bins in \( \mathcal{C} \) have an occupied area of at least 4/9. Set \( N_B = |\mathcal{B}| \) and \( N_C = |\mathcal{C}| - 2 \). Thus, \( \text{OPT} \geq a(\mathcal{B}) + a(\mathcal{C}) \geq \frac{1}{4} N_B + \frac{4}{9} N_C \). On the other hand, \( \text{OPT} \geq N_B \), as each big square has to be packed in a different bin. Thus, \( \text{OPT} \geq \max\{N_B; \frac{1}{4} N_B + \frac{4}{9} N_C\} \).

Putting together these results, we have that

\[
\frac{sc(p)}{\text{OPT}} = \frac{N_B + N_C + 2}{\text{OPT}} \leq \frac{N_B + N_C}{\max\{N_B; \frac{1}{4} N_B + \frac{4}{9} N_C\}} + \frac{2}{\text{OPT}} \leq \frac{43}{16} + \frac{2}{\text{OPT}} = 2.6875 + \frac{2}{\text{OPT}},
\]

where the last inequality follows from Lemma 3.9.
Observe that Theorem 3.10 also holds for any packing algorithm for which the result stated in Lemma 3.7 holds and that succeeds in packing any four medium squares into a bin. In particular, Theorem 3.10 holds for an exact packing algorithm.

We note that all results presented in this section for games using NFDH also hold for the SRPG. This is because all upper bounds were proved using area occupation arguments, which hold also for the SRPG. It should be noted, however, that results on the upper bound for the price of anarchy of a game parameterized by an algorithm may not hold for the non-parameterized game. In the particular case of NFDH, the obtained results imply the following result.

**Corollary 3.11.** For the selfish square packing game, $2.3634 \leq \text{PoA} \leq 2.6875$.

## 4 Bounds for the strong price of anarchy

In this section, we consider the same cost function mentioned in the previous section, but we study strong Nash equilibria. We have the same setting, and the game is analogous, but now coalition is allowed, that is, a group of items (not necessarily from a common bin) may migrate at once if it leads to a cost that is better for all items in the group. It is also allowed that a group of items migrates to a new bin (that is, the number of used bins may increase).

We call strong Nash equilibrium a configuration of the selfish rectangle packing game in which no group of items has incentive to move to decrease the individual cost of its items. Let $\mathcal{SN}(G)$ be the set of all strong Nash equilibria of $G$. For strong Nash equilibria, analogous to the case of Nash equilibria, the following measures are considered. The asymptotic strong price of anarchy of a class $\mathcal{G}$ of games is

$$
\text{SPoA} := \lim_{m \to \infty} \sup_{G \in \mathcal{G}} \max_{p \in \mathcal{N}(G)} \frac{\text{sc}(p)}{m},
$$

(5)

When the game is parameterized by a packing algorithm $A$, we denote the strong price of anarchy of the corresponding game by $\text{SPoA}(A)$.

### 4.1 The strong price of anarchy is bounded

We first note that, as opposed to the pure price of anarchy, the strong price of anarchy is bounded. This is easy to see because one can guarantee that, in any strong Nash equilibrium, each bin has an occupied area of at least $1/4$, except perhaps for a constant number of bins. In fact, this is valid also when the algorithm NFDH is used.

We use the following result, proved by Harren and van Stee [9], that is a generalization of a theorem of Meir and Moser.

**Lemma 4.1.** Let $T$ be a set of rectangles, and $w_{\text{max}}$ (resp. $h_{\text{max}}$) be the maximum width (resp. height) of a rectangle in $T$. If $T$ is packed into a rectangle $R = (a,b)$ by NFDH, then either a total area of at least $(a - w_{\text{max}})(b - h_{\text{max}})$ is packed or the algorithm runs out of items, i.e., all items are packed.

### 4.2 Lower and upper bounds for the strong price of anarchy

We first present a lower bound for the strong price of anarchy of the selfish square packing game. Before that, we note that the configuration described in Section 3.2 consisting of homogeneous bins and used to prove a lower bound on the price of anarchy for this game, is not a strong Nash equilibrium. Indeed, for instance, as long as $\varepsilon$ is small enough, any group of seven squares of side $1/5 + \varepsilon$ has incentive to migrate to a $(1/4 + \varepsilon)$-homogeneous bin.
Theorem 4.2. The strong price of anarchy of the selfish square packing game is at least 2.076.

Proof. For each integer $k \geq 3$, we obtain a lower bound $L_k$ for the strong price of anarchy of the selfish square packing game in which $k$ specific sizes of squares are considered. The idea is similar to the one used in Section 3.2, but requires a more judicious choice of possible square sides. For $i = 1, \ldots, k - 1$, we consider squares of sides $\sigma_i = (1 + \varepsilon)/2^i$, where $\varepsilon$ is a very small positive real number. We also consider squares of side $\sigma_k$, a positive real number that divides 1 and $\sigma_{i-1}$ (and therefore divides any positive value of the form $1 - \sum_{i=1}^{k-1} a_i \sigma_i$ for integer values of $a_i$).

First, we describe an optimal packing $O$ that consists of $N$ bins with the same configuration, all completely filled with squares of sides $\sigma_1, \ldots, \sigma_k$. (The value of $N$ will be defined later.) Then, we describe a packing $P$ of the same set of squares, consisting of only homogeneous bins, that is a strong Nash equilibrium.

The bins of the optimal packing $O$ have the configuration shown in Figure 3. Each bin contains $n_i$ square(s) of side $\sigma_i$, for $i = 1, \ldots, k - 1$, as defined below, and has the remaining apparently free space completely filled with small squares of side $\sigma_k$, which are like sand, as long as $\varepsilon$ is small enough. These $N$ bins packed in a completely filled way clearly define an optimal packing.

![Figure 3: Configuration of a bin in the optimal packing $O$. Each fraction $1/2^i$ represents a square of side $\sigma_i = (1 + \varepsilon)/2^i$.](image)

The maximum number of squares of side $(1 + \varepsilon)/2^i$ that pack in a bin is $(2^i - 1)^2$. So, by packing always the maximum number of squares of side $\sigma_i$ before starting to pack squares of side $\sigma_{i+1}$, we will be able to pack $n_i = (2^i - 1)^2 - 4(2^{i-1} - 1)^2 = 2^{i+1} - 3$ squares of side $\sigma_i$, for $i = 1, \ldots, k - 1$. For instance, as shown in Figure 3, $n_1 = 1$, $n_2 = 2^3 - 3 = 5$, $n_3 = 2^4 - 3 = 13$, and $n_4 = 2^5 - 3 = 29$. As for $n_k$, it is set so that each bin in $O$ is completely filled, that is, $n_k = (1 - (2^{k-1} - 1)^2/\sigma_{k-1}^2)/\sigma_k^2$.

Now, let us describe the packing $P$. It consists of a number of $k$ different types of homogeneous bins, one for each side $\sigma_i$. Since we can pack $(2^i - 1)^2$ squares of side $\sigma_i$ in a bin, for $i = 1, \ldots, k - 1$, the number of $\sigma_i$-homogeneous bins in $P$ is $m_i = n_i N/(2^i - 1)^2$, for $i = 1, \ldots, k - 1$. The maximum number of squares of side $\sigma_k$ that pack in a bin is $1/\sigma_k^2$, as $\sigma_k$ divides 1, so $m_k = n_k N/\sigma_k^2 = (1 - (2^{k-1} - 1)^2/\sigma_{k-1}^2)N$. The value of $N$ is such that every $m_i$ is an integer (for that, it suffices that $N$ be a multiple of $(2^i - 1)^2$ for $i = 1, \ldots, k - 1$).

We claim that $P$ is a strong Nash equilibrium. First, note that items of type $\sigma_k$ do not have incentive to migrate, as, in $P$, all such items are in bins completely filled. Now let $S$ be a group of squares whose sides are in $\{\sigma_1, \ldots, \sigma_{k-1}\}$ and suppose that all squares in $S$ have incentive to migrate to a bin $B$. Let us derive a contradiction by proving that there is a square in $S$ that has no incentive to migrate to $B$. Let $i$ be such that the side of the smallest square in $S$ is $\sigma_i$. 


Clearly, $B$ is a $\sigma_j$-homogeneous bin with $j < i$, otherwise no square in $S$ would fit in a bin with all the squares in $B$. But then $\sigma_i$ divides $\sigma_j$ and also the side of each square in $S$. Thus $B \cup S$ corresponds to a set of squares all of them of side $\sigma_i$, and, if $B \cup S$ can be packed in a bin, such bin would have occupied area at most equal to a $\sigma_i$-homogeneous bin in $\mathcal{P}$. Therefore any square in $S$ of side $\sigma_i$ has no incentive to migrate to $B$. We conclude that $\mathcal{P}$ is indeed a strong Nash equilibrium.

For each $k$, a lower bound $L_k$ is obtained by considering the previously described optimal packing $\mathcal{O}$, consisting of $N$ bins, and the strong Nash equilibrium $\mathcal{P}$, consisting of $m_1 + \cdots + m_k$ bins. More precisely, for $\varepsilon = 1/(2^{k-1} - 1)$, we have that $n_k = m_k = 0$ and

$$L_k = \frac{|\mathcal{P}|}{|\mathcal{O}|} = \frac{m_1 + \cdots + m_{k-1}}{N} = \frac{k-1}{N} \sum_{i=1}^{k-1} \frac{n_i}{(2^i - 1)^2} = \frac{k-1}{N} \sum_{i=1}^{k-1} \frac{2^{i+1} - 3}{(2^i - 1)^2}.$$ 

Thus, for instance, as $L_{20} > 2.076$, the theorem follows. Moreover, $\lim_{k \to \infty} L_k < 2.0761$. 

Now, we show an upper bound for the strong price of anarchy of the selfish square packing game, parameterized by the NFDH algorithm. As we have previously defined, we recall that squares with sides in the intervals $(0, 1/3]$, $(1/3, 1/2]$, and $(1/2, 1]$ are called small, medium, and big, respectively.

First, we show a minimum area occupation of bins in a configuration that is a strong Nash equilibrium, when all squares have side at most $1/k$, for $k \geq 2$.

**Lemma 4.3.** Let $\mathcal{S}$ be a strong Nash equilibrium in which all squares have side at most $1/k$, for an integer $k \geq 2$. Then, each bin in $\mathcal{S}$, except for at most $k^2$ of them, has an occupied area of at least $(k/(k+1))^2$.

**Proof.** Let $\mathcal{S}'$ be the set of bins in $\mathcal{S}$ with at least one square of side in $(1/(k+1), 1/k]$ and $\mathcal{S}''$ be the remaining bins in $\mathcal{S}$. First, let us argue that there is at most one bin in $\mathcal{S}''$ with area occupation less than $(k/(k+1))^2$. Theorem 3.5 guarantees that we can pack in one bin any set of squares whose largest side is at most $1/(k+1)$ and the total area is at most $1/(k+1)^2 + (1 - 1/(k+1))^2$. Thus, except for at most one, every bin in $\mathcal{S}''$ has an area occupation of at least $(1 - 1/(k+1))^2$, that is, at least $(k/(k+1))^2$. Second, note that we can pack in one bin $k^2$ squares that have side in $(1/(k+1), 1/k]$, leading to a total area of at least $(k/(k+1))^2$. Therefore, we can have at most $k^2 - 1$ bins in $\mathcal{S}'$ with occupied area less than $(k/(k+1))^2$, otherwise there would be $k^2$ such squares that could migrate to a new bin, with better area occupation.

**Theorem 4.4.** For the selfish square packing game, $\text{SPoA}(\text{NFDH}) \leq 2.3605$.

**Proof.** Let $p$ be a configuration that is a strong Nash equilibrium (SNE), and let $L$ denote the set of squares packed in the bins of this configuration. Denote by $\mathcal{B}$ the set of bins in the configuration $p$ that contain a big square (and possibly other squares), $\mathcal{M}$ the set of bins containing at least one medium square and no big square, and $\mathcal{S}$ the set of bins containing only small squares.

Since every strong Nash equilibrium is also a Nash equilibrium, all results that we have previously proved for Nash equilibria are also valid here. In particular, by Theorem 3.8, we know that all bins in $\mathcal{M}$, except possibly two of them, have an occupied area of at least $4/9$. By Lemma 4.3, all bins in $\mathcal{S}$, except possibly nine of them, have an occupied area of at least $9/16$. Since all bins in $\mathcal{B}$ have an occupied area larger than $1/4$, the following inequalities hold:

$$\text{OPT}(L) \geq a(L) = a(\mathcal{B}) + a(\mathcal{M}) + a(\mathcal{S}) \geq \frac{1}{4} |\mathcal{B}| + \frac{4}{9} (|\mathcal{M}| - 2) + \frac{9}{16} (|\mathcal{S}| - 9).$$
Next, we show that some of the above fractions (on least occupied area) can be improved, giving a better lower bound for $\text{OPT}(L)$. For that, we analyze two cases. In what follows, we refer to a value $x$ which will be specified later. For the moment, assume $0.62 < x < 0.63$.

**Case 1.** There is at least one bin $B$ in $\mathcal{B}$ containing solely one big square of side at most $x$ and no other squares.

We shall prove that, in this case, all bins in $M$, except for at most three of them, have an occupied area of at least $4(1 - x)^2$. To prove this claim, denote by $M'$ the set of bins in $M$ containing at least one medium square of side at most $1 - x$, and let $M'' = M \setminus M'$.

First we show that all bins in $M'$, except for at most two of them, have an occupied area of at least $\frac{7}{12}$. Indeed, suppose that there are three bins in $M'$ whose occupied area is less than $\frac{7}{12}$. Then, three squares of side at most $1 - x$, each coming from one of these three bins, can migrate to the bin $B$ in $\mathcal{B}$ and, after this migration, the occupied area of $B$ becomes larger than $\frac{1}{4} + 3(1/9) = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$, contradicting the fact that $p$ is an SNE.

Now we show that all bins in $M''$, except for at most three of them, have an occupied area larger than $4(1 - x)^2$. Indeed, suppose there are four bins in $M''$ whose occupied area is less than or equal to $4(1 - x)^2$. In this case, four medium squares with side larger than $1 - x$, each one coming from one of these four bins, can migrate to a new (empty) bin, say $B'$, because, after the migration, $a(B') > 4(1 - x)^2$, a contradiction, as $p$ is an SNE.

We will choose $x$ so that $7/12 > 9/16 \geq 4(1 - x)^2$. Thus we conclude that all bins in $M$, except for at most five of them, have an occupied area of at least $4(1 - x)^2$.

Hence,

$$\text{OPT}(L) \geq a(L) = a(B) + a(M) + a(S) \geq \frac{1}{4}|B| + 4(1 - x)^2(|M| - 5) + \frac{9}{16}(|S| - 9) \geq \frac{1}{4}|B| + 4(1 - x)^2(|M| + |S| - 14).$$

As $\text{OPT}(L) \geq |B|$, we have that $\text{OPT}(L) \geq \max\{|B|, \frac{1}{4}|B| + 4(1 - x)^2(|M| + |S| - 14)\}$.

So,

$$\frac{sc(p)}{\text{OPT}} = \frac{|B| + |M| + |S|}{\text{OPT}} \leq \frac{|B| + |M| + |S| - 14}{\max\{|B|, \frac{1}{4}|B| + 4(1 - x)^2(|M| + |S| - 14)\}} + \frac{14}{\text{OPT}} \leq 1 + \frac{3}{16(1 - x)^2} + \frac{14}{\text{OPT}},$$

by Lemma 3.9.

**Case 2.** There is no bin in $\mathcal{B}$ containing solely one big square of side at most $x$.

In this case, we shall prove that the bins in $\mathcal{B}$ have an occupied area of at least $x^2$, improving on the trivial occupation bound of $1/4$.

Claim A: *All bins in $\mathcal{B}$ containing at least one medium square have an occupied area of at least $x^2$, except for at most two of them.*
Proof of the claim. Suppose there are three bins $B_1$, $B_2$, and $B_3$ in $B$, where $B_i$ contains a big square $b_i$ and a medium square $m_i$ for $i = 1, \ldots, 3$, with occupied area smaller than $7/12$. Assume that $b_1 \leq b_2 \leq b_3$. In this case, the set of squares \{b_1, m_1, m_2, m_3\} can migrate to a new bin, that will have an occupied area of at least $1/4 + 3(1/9) = 7/12 > x^2$, a contradiction, as $p$ is an SNE.

Claim B: All bins in $B$ containing no medium square have an occupied area of at least $x^2$, except for at most 18 of them.

Proof of the claim. For each bin $B \in B$, denote by $b_B$ the big square in $B$. Every bin $b$ in $B$ with $b_B \geq x$ has occupied area at least $x^2$. So let $B'$ be the set of bins $B$ in $B$ with no medium square and $b_B < x$. If $B'$ has at most 17 bins, the claim follows. Thus suppose there are at least 18 bins in $B'$. By the hypothesis of Case 2, there is at least one small square in each bin in $B'$. Let $B'$ be a bin in $B'$ with $a(B')$ minimum and let $B$ be a bin in $B'$ distinct from $B'$. Let $R_B = (1, 1-b_B)$ be the rectangular region above the square $b_B$ in $B$. We must have that $a(B \setminus \{b_B\}) \geq (1-1/3)(1-b_B-1/3)$, otherwise each small item from $B'$ would have incentive to migrate to $B$, as it fits next to $b_B$ and NFDH would be able to pack $B \setminus \{b_B\}$ in the rectangular region $R_B$ by Lemma 4.1. As $(1-1/3)(1-b_B-1/3) \geq (1-1/3)(1-x-1/3)$, we claim that the occupied area of each bin in $B'$, except for possibly $B'$ and at most 17 other bins in $B'$, is at least $x^2$. Otherwise, the small squares in these 17 bins either pack together in a bin, occupying an area of at least $17(1-1/3)(1-x-1/3) \geq 17 \cdot 0.02444 = 0.41548 > x^2$, or a proper subset of them packs in a bin, occupying an area of at least $4/9 > x^2$, by Lemma 3.7. So either all of these small squares or a proper subset of them would have incentive to migrate to a new bin, contradicting the fact that $p$ is a SNE.

From Claims A and B, we have that all bins in $B$ have area least $x^2$, except for at most 19 bins. Proceeding now with the analysis of Case 2, we have that

\[
OPT(L) \geq a(L) = a(B) + a(M) + a(S) \\
\geq x^2(|B|-19) + \frac{4}{9}(|M|-2) + \frac{9}{16}(|S|-9) \\
\geq x^2(|B|-19) + \frac{4}{9}(|M|+|S|-11).
\]

As $OPT(L) \geq |B|$, we have $OPT(L) \geq \max\{|B|-19, x^2(|B|-19) + \frac{4}{9}(|M|+|S|-11)|}$. Thus, by Lemma 3.9

\[
\frac{sc(p)}{OPT} = \frac{|B|+|M|+|S|}{OPT} \leq \frac{|B|+|M|+|S|-30}{\max\{|B|-19, x^2(|B|-19) + \frac{4}{9}(|M|+|S|-11)|} + \frac{30}{OPT} \\
\leq 1 + \frac{9(1-x^2)}{4} + \frac{30}{OPT}.
\]

Making the constant part of the bound of the two cases equal, we obtain the equation $(1-x^2)(1-x)^2 = 1/12$, that has only one solution in the interval $[0, 1]$, namely, $x \approx 0.62876$. We note that the restriction $9/16 \geq 4(1-x^2)$ mentioned in Case 1 is satisfied for this value of $x$.

Thus, using this value of $x$, we conclude that $SPoA(NFDH) \leq 2.3605$.}

As in the previous section, on the pure price of anarchy, the results we have presented for games using NFDH also hold for the SRPG, because all upper bounds were proved using area occupation arguments. Hence, the following result on the strong price of anarchy also holds.

**Corollary 4.5.** For the selfish square packing game, $2.076 \leq SPoA \leq 2.3605$. 

14
5 Concluding remarks

The parameterization of the selfish packing game by a specific packing algorithm has shown to be interesting in various ways. First, depending on the algorithm and the analysis of the (pure or strong) price of anarchy of the corresponding game, the result that one obtains carries over to the generic game. As we have shown, in the case of square packing, the upper bound for PoA(NFDH) and for SPoA(NFDH) have led us to results for PoA and SPoA. This indicates that it would be interesting to consider other packing algorithms and study the behavior of the corresponding games. Possibly, better upper bounds for PoA and SPoA can be obtained this way. Another way of viewing the results for such parameterized games is to consider that they also may be seen as a coordination mechanism to be used (to control selfish decisions of the players), specially if one can show that the price of anarchy of the corresponding game has an acceptable upper bound. In this case, it is desirable to have easy to be implemented polynomial-time algorithms, and this is the case of the algorithms we have considered.

The results we have shown for the 2-dimensional case, with the very natural cost function (proportional to the occupied area), are the first ones in the literature. It would be interesting to further improve the bounds we have shown here, to tighten the gap. Another challenge would be to study the natural generalization of this game to the $d$-dimensional case, for $d \geq 3$.

Also, there are several variants of the problems considered here, such as no repacking versions, items of other forms, like disks, rectangles with rotations allowed, etc. Each such variant requires different strategies and are challenging problems.

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