THE ADDITION OF TEMPORAL NEIGHBORHOOD MAKES THE LOGIC OF PREFIXES AND SUB-INTERVALS EXPSPACE-COMPLETE

LAURA BOZZELLI, ANGELO MONTANARI, ADRIANO PERON, AND PIETRO SALA

Abstract. A classic result by Stockmeyer [Sto74] gives a non-elementary lower bound to the emptiness problem for generalized ∗-free regular expressions. This result is intimately connected to the satisfiability problem for interval temporal logic, notably for formulas that make use of the so-called chop operator. Such an operator may indeed be interpreted as the inverse of the concatenation operation on regular languages, and this correspondence enables reductions between non-emptiness of generalized ∗-free regular expressions and satisfiability of formulas of the interval temporal logic of chop under the homogeneity assumption [HMM83].

In this paper, we study the complexity of the satisfiability problem for suitable weakenings of the chop interval temporal logic, that can be equivalently viewed as fragments of Halpern and Shoham interval logic. We first introduce the logic $\text{BD}_\text{hom}$ featuring modalities $B$, for begins, corresponding to the prefix relation on pairs of intervals, and $D$, for during, corresponding to the infix relation, whose satisfiability problem has been recently shown to be PSPACE-complete [BMPS21b]. The homogeneous models of $\text{BD}_\text{hom}$ naturally correspond to languages defined by restricted forms of generalized ∗-free regular expressions, that use union, complementation, and the inverses of the prefix and infix relations. Then, we focus our attention on the extension of $\text{BD}_\text{hom}$ with the temporal neighborhood modality $A$, corresponding to the Allen relation $\text{Meets}$, and prove that such an addition increases both the expressiveness and the complexity of the logic. In particular, we show that the resulting logic $\text{BDA}_\text{hom}$ is EXPSPACE-complete.

Key words and phrases: Interval Temporal Logic, Star-Free Regular Languages, Satisfiability, Complexity.

* This paper is a revised and extended version of [BMPS20] and [BMPS21a].
1. Introduction

Interval temporal logics (ITLs for short) are versatile and expressive formalisms to specify properties of sequences of states and their duration. When it comes to fundamental problems like satisfiability, their high expressive power is often obtained at the price of undecidability. As an example, the satisfiability problem of the most widely known ITLs, namely, Halpern and Shoham’s HS [HS91], and Venema’s CDT [Ven91a], turn out to be highly undecidable. Despite these negative results, a number of decidable logics have been identified by suitably weakening ITLs (see [BMM+14] for a complete classification of HS fragments). Here the term “weakening” is intended as a set of syntactic and/or semantics restrictions imposed on the formulas of the logic and/or the temporal structures on which such formulas are interpreted, respectively. Among the plethora of possible weakenings, in this paper we focus on (the combination of) the following two natural and well-studied restrictions:

- **Restrict the set of interval relations.** Many decidable fragments of ITLs are obtained by considering a restricted set of Allen’s relations over pairs of intervals. This approach naturally induces fragments of HS with modalities corresponding to the selected subset of interval relations. As an example, the logic of temporal neighborhood (PNL for short) features only 2 modalities, corresponding to 2 interval relations among the possible 13 ones, namely, $A$ (adjacent to the right) and its inverse $\bar{A}$ [CH97]. PNL has been shown to be decidable over all meaningful classes of linear orders [BMSS11, MS12].

- **Restrict the class of models.** As an alternative, it is possible to tame the complexity of ITLs by restricting to classes of models that satisfy certain specific assumptions. An example of such an approach can be found in a recent series of papers that study the model checking problem for ITLs (see, e.g., the seminal paper [MMM+16]), as well as their expressiveness compared to that of classical point-based temporal logics, like LTL, CTL, and CTL* [BMM+19a]. In this setting, models are represented as Kripke structures, and are inherently point-based rather than interval-based. The very same models can be obtained from interval temporal structures by making the so-called homogeneity assumption, that is, by assuming that every proposition letter holds over an interval if and only if it holds at all its points [Roe80]. Under such an assumption, full HS has a decidable satisfiability problem (as a matter of fact, the model checking procedures proposed in the aforementioned series of papers can be easily turned into satisfiability procedures, often retaining the same complexity) [MMM+16]. Because of this, the focus in studying HS fragments under the homogeneity assumption was shifted from decidability to complexity.

Under the homogeneity assumption, a natural connection to generalized $\star$-free regular languages emerges from the analysis of the complexity of ITLs over finite linear orders. A classic result by Stockmeyer states that the emptiness problem for generalized $\star$-free regular expressions is non-elementarily decidable (tower-complete) for unbounded nesting of negation [Sch16, Sto74] (it is $(k-1)$-EXPSPACE-complete for expressions where the nesting of negation is at most $k \in \mathbb{N}^+$). Such a problem can be easily turned into the satisfiability problem for the logic $\mathcal{C}$ of the chop modality, over finite linear orders, under the homogeneity assumption [HMS08, Mos83, Ven91b], and vice versa. $\mathcal{C}$ has one binary modality only, the so-called chop operator, that allows one to split the current interval in two parts and to state what is true over the first part and what over the second one. It can be easily shown that there is a LOGSPACE-reduction of the emptiness problem for
generalized ∗-free regular expressions to the satisfiability problem for \( C \) with unbounded nesting of the chop operator, and vice versa.

The close relationships between formal languages and ITLs have been already pointed out in [MS13a, MS13b], where the ITL counterparts of regular languages, \( \omega \)-regular languages, and extensions of them (\( \omega B \)- and \( \omega S \)-regular languages) have been provided. Here, we focus on some meaningful fragments of \( C \) (under the homogeneity assumption).\(^1\) Modalities for the prefix, suffix, and infix relations over (finite) intervals can be easily defined in \( C \). We have that a formula holds over a prefix of the current interval if and only if it is possible to split the interval in such a way that the formula holds over the first part and the second part contains at least two points. The case of suffixes is completely symmetric. Infixes can be defined in terms of prefixes and suffixes: a proper sub-interval of the current interval is a suffix of one of its prefixes or, equivalently, a prefix of one of its suffixes. The satisfiability problem for the logic \( D_{\text{hom}} \) of the infix relation has been recently shown to be \( \text{PSPACE} \)-complete by a suitable contraction method [BMM+17]. The same problem has been proved to be \( \text{EXPSPACE} \)-hard for the logic \( BE_{\text{hom}} \) of prefixes and suffixes by a polynomial-time reduction from a domino-tiling problem for grids with rows of single exponential length [BMM+19b]. As for the upper bound, the only available one is given by the non-elementary decision procedure for \( HS_{\text{hom}} \) [MMM+16] (\( BE_{\text{hom}} \) is a small fragment of it). Despite several attempts, no progress has been done in the reduction/closure of such a very large gap.\(^2\)

A couple of additional elements that help in understanding why \( BE_{\text{hom}} \) is such a peculiar beast are the following: (i) as shown in [BMM+19b], the only known fragments of \( HS_{\text{hom}} \) whose satisfiability problem has been given an \( \text{EXPSPACE} \) lower bound contain both \( B \) and \( E \) modalities; (ii) the satisfiability problem for the logic \( DE_{\text{hom}} \) (and the symmetric logic \( BD_{\text{hom}} \)), which is a maximal proper fragment of \( BE_{\text{hom}} \), has been recently proved to be \( \text{PSPACE} \)-complete [BMM+17, BMPS20, BMPS21b].

**Goals and structure of the paper.** In this paper, we identify the first \( \text{EXPSPACE} \)-complete fragment of \( HS_{\text{hom}} \) which does not include both \( B \) and \( E \) modalities. Such a fragment is the logic \( BDA_{\text{hom}} \), which extends \( BD_{\text{hom}} \) with the meet (adjacent to the right) modality \( A \). As a preparatory step, we apply the proposed model-theoretic proof technique to the simpler fragment \( BD_{\text{hom}} \); then, we show that it can be tailored to the logic \( BDA_{\text{hom}} \) without any increase in complexity.

The paper is organized as follows. In Section 2, we provide a gentle introduction to ITLs. We first introduce in an informal way the two main propositional ITLs, namely CDT and \( HS \), interpreted over finite linear orders. Then, by making use of a simple example, we compare their expressive power with that of Linear Temporal Logic (LTL). Next, in Section 3, we specify syntax and semantics of \( BD_{\text{hom}} \), and we point out some interesting connections between \( BD_{\text{hom}} \) formulas and restricted forms of generalized ∗-free regular expressions. Then, we prove a small model theorem for the satisfiability of \( BD_{\text{hom}} \) formulas over finite linear orders, which provides a doubly exponential bound (in the size of the formula) on their models. By exploiting such a small model theorem, we show that there exists a decision procedure to check satisfiability of \( BD_{\text{hom}} \) formulas that works in exponential space with respect to the size of the input formula. The proof consists of the following sequence of steps:

---

\(^1\)Hereafter, for any ITL \( X \), we will write \( X_{\text{hom}} \) to indicate that we are considering \( X \) under the homogeneity assumption.

\(^2\)In fact, the only achieved result was a negative one showing that there is no hope in trying to tailor the proof techniques exploited for \( HS_{\text{hom}} \), which are based on the notion of \( BE \)-descriptor, to \( BE_{\text{hom}} \), as it is not possible to give an elementary upper bound on the size of \( BE \)-descriptors (in the case of \( BE_{\text{hom}} \)) [BMP19].
steps. In Section 4, we introduce and discuss a spatial representation of the models of $\text{BD}_{\text{hom}}$ formulas, called compass structure. Then, in Section 5, we prove a series of spatial properties of compass structures for formulas involving modalities $B$ and $D$. Next, in Section 6, by making use of the properties stated in Section 5, we prove the small model theorem for $\text{BD}_{\text{hom}}$, which allows us to devise a procedure to check the satisfiability of $\text{BD}_{\text{hom}}$ formulas over finite linear orders of EXPSPACE complexity. It is worth pointing out that such a decision procedure is sub-optimal, given the results proved in [BMPS21b], where a PSPACE decision procedure for the very same problem is provided; however, it plays an instrumental role in the proof of the main result of the paper about $\text{BDA}_{\text{hom}}$. In Section 7, we introduce modality $A$ and formally define the logic $\text{BDA}_{\text{hom}}$; in addition, we define and discuss its counterpart in terms of generalized $\ast$-free regular expressions. In Section 8, we first prove that the properties stated in Section 5 still holds for $\text{BDA}_{\text{hom}}$, and then we show that an EXPSPACE decision procedure for $\text{BDA}_{\text{hom}}$, over finite linear orders, can be obtained from the one developed in Section 6 with a few small adjustments. In Section 9, we prove EXPSPACE-hardness of the satisfiability problem for $\text{BDA}_{\text{hom}}$, over finite linear orders, by providing a reduction from the exponential corridor tiling problem, thus allowing us to conclude that the EXPSPACE complexity bound for $\text{BDA}_{\text{hom}}$ finite satisfiability is tight. In Section 10, we provide an assessment of the work and outline future research directions.

2. A gentle introduction to Interval Temporal Logics (ITLs)

In this section, we provide a gentle introduction to ITLs, focusing on the features that distinguish them from point-based temporal logics. As a term of comparison, we choose LTL. For the sake of simplicity, we restrict our attention to totally ordered finite models, that is, finite prefixes $0 < 1 < \ldots < N$ of $\mathbb{N}$. With a little abuse of notation, we denote such an order by $N$. In such a setting, the focus is on LTL formulas interpreted on finite traces (we will refer to the set of finite traces simply as models). In the literature, LTL over finite traces is commonly referred to LTL$_f$ [GV13, GMM14].

Let $\text{Prop}$ be a set of proposition letters. The first, crucial difference between ITLs and LTL$_f$ is the way in which $\text{Prop}$ is interpreted over models. Let $\llbracket N \rrbracket = \{[x,y] \mid 0 \leq x \leq y \leq N\}$ be the set of all and only intervals on $N$. In the case of LTL$_f$, we have a function $\pi : N \rightarrow 2^{\text{Prop}}$, while, in the case of ITLs, we have $\mathcal{V} : \llbracket N \rrbracket \rightarrow 2^{\text{Prop}}$. It is easy to see that $\mathcal{V}$ is, in fact, a generalization of $\pi$, as the point-based semantics $\pi$ can be embedded in the interval-based one $\mathcal{V}$ by assuming $\pi(x) = \mathcal{V}([x,x])$, for all $x \in N$. From now on, we will refer to intervals of the form $[x,x]$ as point-intervals and to intervals of the form $[x,y]$, with $x < y$, as strict-intervals. Whenever we will not need to distinguish between point- and strict-intervals, we will simply refer to them as intervals.

In its full generality, ITL interval-based semantics does not impose any constraint on the relationships between the proposition letters that hold over a strict-interval and those that hold over the point-intervals that it includes, that is, the set of proposition letters $\mathcal{V}([x,y])$ that hold over the strict-interval $[x,y]$ may differ from the sets of proposition letters $\mathcal{V}([x,x]), \ldots, \mathcal{V}([y,y])$ that hold on the point-intervals contained in $[x,y]$ (which, obviously, may differ from each other). Similarly, the set of proposition letters $\mathcal{V}([x',y'])$ that hold on a strict-subinterval $[x',y']$ of $[x,y]$, that is, $x \leq x' < y' \leq y$ and $[x',y'] \neq [x,y]$, may differ from those that hold on $[x,y]$. Consider the example of Figure 1, where $\pi$ and $\mathcal{V}$ agree on the labelling of points $0, \ldots, 4$ (they are interpreted as the intervals $[0,0], \ldots, [4,4]$
in the ITL semantics). The evaluation of proposition letters p and q on strict-intervals does not depend on that on their sub-intervals. See, for instance, the interval [1, 4] of Figure 1. Its labelling is \( \mathcal{V}([1, 4]) = \{ p, q \} \) and it features all the possible subsets of \( \{ p, q \} \) as the labels of its point intervals \([1, 1], ..., [4, 4]\). As for its strict-subintervals, it holds that \( \mathcal{V}([1, 2]) = \mathcal{V}([2, 3]) = \mathcal{V}([3, 4]) = \emptyset \), \( \mathcal{V}([1, 3]) = \{ p \} \), and \( \mathcal{V}([2, 4]) = \{ p, q \} \).

One of the first ITLs proposed in literature was CDT [Ven90], whose name comes from its three binary modalities C (Chopping), D (Dawning), and T (Terminating). Their semantics is graphically depicted in Figure 2. Intuitively speaking, if we take a point \( z \) inside an interval \([x, y]\) and we consider the ternary relation \([x, y]\) may be split into \([x, z]\) and \([z, y]\), the three CDT modalities allow one to talk about the properties of such a relation starting from any of the three intervals. More precisely, a formula \( \psi_1 \ C \ \psi_2 \) (chopping between \( \psi_1 \) and \( \psi_2 \)) holds over an interval \([x, y]\) if \([x, y]\) can be split into \([x, z]\) and \([z, y]\), \( \psi_1 \) holds over \([x, z]\), and \( \psi_2 \) holds over \([z, y]\) (topmost part of Figure 2). A formula \( \psi_1 \ D \ \psi_2 \) (dawning \( \psi_2 \)) holds over an interval \([x, y]\) if there exists an interval \([z, x]\) such that \( \psi_1 \) holds over \([z, x]\) and \( \psi_2 \) holds over the interval \([z, y]\) covering both \([z, x]\) and \([x, y]\) (middle part of Figure 2). A formula \( \psi_1 \ T \ \psi_2 \) (terminating \( \psi_2 \)) holds over an interval \([x, y]\) if there exists an interval \([y, z]\) such that \( \psi_1 \) holds over \([y, z]\) and \( \psi_2 \) holds over the interval \([x, z]\) covering both \([x, y]\) and \([y, z]\) (bottom part of Figure 2).

CDT turns out to be very expressive. It can be easily checked that it allows one to specify a number of advanced properties in a straightforward way. As an example, it is easy to write a CDT formula that forces one or more proposition letter to behave like an equivalence relation over the points of the underlying linear order. However, such an expressivity is paid with an undecidable satisfiability problem on every interesting linear order, that is, any linear order but bounded ones, where the problem is trivially decidable. Such a statement holds even if we consider any of the fragments of CDT that contains just one modality among C, D, and T [GMSS06].
A meaningful fragment of CDT is HS [HS91], which features a unary modality for each ordering relation between a pair of intervals (the so-called Allen’s relations [All81]), as shown in Figure 3. For the sake of simplicity, in Figure 3, we deliberately omitted the modality for the inverse of each considered relation, namely $\langle A \rangle$, $\langle B \rangle$, $\langle D \rangle$, $\langle E \rangle$, $\langle L \rangle$, and $\langle O \rangle$. The semantics of each HS modality can be captured by a suitable combination of CDT modalities as shown in Figure 4. The converse is not true. In Figure 4, we make an extensive use of the modal constant $\pi$, which holds over an interval $[x, y]$ if and only if $x = y$, that is, $[x, y]$ is a point-interval. It immediately follows that $\neg\pi$ holds on all and only strict intervals. It is worth pointing out that some HS modalities can be defined as suitable combinations of other ones (a complete account of definability equations for the most significant classes of linear orders is given in [BMM+14, BMM+19c]). For what concerns the HS fragments considered in this paper, namely those featuring unary modalities $\langle A \rangle$, $\langle B \rangle$, and $\langle D \rangle$ (which should not be mistaken with the binary modality $D$ of CDT), we have that modality $\langle L \rangle$ can be defined in terms of modality $\langle A \rangle$ and modality $\langle D \rangle$ can be expressed by means of a suitable combination of modalities $\langle B \rangle$ and $\langle E \rangle$. Notice that the opposite is not true, e.g., $\langle A \rangle$ cannot be expressed by means of modality $\langle L \rangle$. Moreover, in BDA it is not possible to define $\langle A \rangle$ in terms of $\langle L \rangle$, $\langle D \rangle$, and $\langle B \rangle$ and it is not possible to express $\langle B \rangle$ (resp., $\langle D \rangle$) in terms of $\langle A \rangle$ and $\langle D \rangle$ (resp., $\langle B \rangle$).
We conclude the section by showing how both LTL\(_f\) modalities Until \((\psi_1 U \psi_2)\) and Next \((\bigcirc\psi)\) can be easily encoded by means of a combination of modalities \((A)\) and \((B)\) (no need to bring up modality \((D)\)). In Figure 5, we give the formulas that define \(\psi_1 U \psi_2\) (above) and \(\bigcirc\psi\) (below) in AB together with a graphical account of how they “operate” on an interval model. Then, in Figure 6 we applied these encodings to translate the formula \(p U (\neg p \land \neg q)\) (resp., \(\bigcirc(\neg p \land \neg q)\)) into an equivalent formula of AB and, by means of the example of Figure 1, we show how the interval model is constrained when the resulting formula holds over an interval.

As shown in Figure 5 (top), the LTL\(_f\) formula \(\psi_1 U \psi_2\) is translated into the conjunction of \([B](A)(\pi \land \psi_1)\) and \((A)(\pi \land \psi_2)\). Let us recall that \(\psi_1 U \psi_2\) holds at a point \(x\) if there exists a point \(y\), with \(x \leq y\), where \(\psi_2\) holds, and, for each point \(x \leq x_i \leq y\), \(\psi_1\) holds at \(x_i\). The idea behind the translation (a graphical account of it is given in Figure 5) exploits the generality of interval semantics to force the translation of \(\psi_1 U \psi_2\) to hold over the whole interval \([x,y]\). Then, it constrains the formula \(\psi_2\) to hold on \([y,y]\), that is, the right endpoint of the interval, by means of the conjunct \((A)(\pi \land \psi_2)\), which literally says that there exists an interval \([y,y']\), which begins exactly where the current one ends (modality \((A)\)) and is a point-interval (constant \(\pi\)), where \(\psi_2\) holds. Such an interval \([y,y']\) can thus be only the interval \([y,y]\). The first conjunct \([B](A)(\pi \land \psi_1)\) forces the formula \((A)(\pi \land \psi_1)\) to hold on each proper prefix (modality \([B] = \neg\{B\}\)) of the interval \([x,y]\), that is, on each interval \([x,x_i]\), with \(x \leq x_i < y\). Then, by the very same argument we used for \((A)(\pi \land \psi_2)\), we have that \(\psi_1\) holds on each point-interval \([x_i,x_i]\), with \(x \leq x_i < y\).

In Figure 6 (top), we give an example of the application of the proposed translation that makes use of the model of Figure 1. In particular, we analyze the translation of the LTL\(_f\) formula \(p U (\neg p \land \neg q)\), which is true at time point 0 according to the point-based semantics \(\pi\), into the AB formula \(\psi = ([B](A)(\pi \land p) \land (A)(\pi \land \neg p \land \neg q))\), which holds over the interval \([0,3]\) according to the interval-based semantics \(\mathcal{V}\). Let us assume that the formula \(\psi\) holds over the interval \([0,3]\). Its second conjunct \((A)(\pi \land \neg p \land \neg q)\) forces the existence of an interval \([3,y]\), with \(y \geq 3\), where \(\pi\), \(\neg p\), and \(\neg q\) hold. The truth of \(\pi\) on the
Figure 4. A graphical account of the encoding of HS modalities in CDT.
The first two conjuncts \( \neg y \) constrain \( y \). Imposing the existence of an interval \( [0, 3] \) over each proper prefix of \( [0, 3] \) restricts the number of possible candidates to \( [3, 3] \) only; since \( V([3, 3]) = \emptyset \), it immediately follows that both \( \neg p \) and \( \neg q \) hold over \([3, 3]\) as well. The first conjunct \([B](A)(\pi \land p)\) makes use of modality \([B]\), which forces the formula \((A)(\pi \land p)\) to be true over each proper prefix of \([0, 3]\), namely, intervals \([0, 2],[0, 1],\) and \([0, 0]\). This amounts to say that, for each interval \([0, x']\), with \( x' \in \{0, 1, 2\}, (A)(\pi \land p)\) holds on \([0, x']\) if and only if there exists an interval \([x', y]\), with \( y \geq x' \), which makes both \( \pi \) and \( p \) true. As already pointed out, \( \pi \) is true over \([x, y]\) if and only if \( y = x' \). It immediately follows that \( p \) belongs to \( V([x', x']) \) for all point-intervals \([x', x']\), with \( x' \in \{0, 1, 2\} \), that is, \( p \) belongs to \( V([0, 0]), V([1, 1]), \) and \( V([2, 2]) \), as shown in Figure 6 (top).

Let us consider now LTL\(_f\) modality \( \bigcirc \). In Figure 5 (bottom), we provide the translation of \( \bigcirc \psi \) into \( \psi' = (A)(\neg \pi \land \pi \land \pi \land \pi) \). According to the semantics of \( \bigcirc \), \( \pi \) holds at a point \( x \) if and only if \( \pi \) holds at the point \( x + 1 \). As a matter of fact, for the sake of generality and simplicity, the proposed translation \( \psi' \) holds on an interval \([x, y]\) if and only if \( \psi \) holds at the point-interval \([y + 1, y + 1]\) regardless of the fact that \([x, y]\) is a strict-interval or a point-interval. It is possible to force \([x, y]\) to be a point-intervals by adding \( \pi \) as a conjunct of the translation, that is, by defining \( \psi' \) as \( \pi \land (A)(\neg \pi \land [B] \land \pi \land (A)(\pi \land \pi)) \).

A graphical account of the translation is given in Figure 5 (bottom). Whenever \((A)(\neg \pi \land [B] \land \pi \land (A)(\pi \land \pi))\) holds over an interval \([x, y]\), the outermost modality \( A \) imposes the existence of an interval \([y, y']\), with \( y \leq y' \), where \( \neg \pi, [B] \pi, \) and \( (A)(\pi \land \pi) \) hold. The first two conjuncts \( \neg \pi \) and \( [B] \pi \) respectively force \( y' > y (\neg \pi) \) and all proper prefixes \([y, y']\) of \([y, y']\) to be point-intervals \([B] \pi \). The only way to satisfy both conditions is to constrain \( y' \) to be equal to \( y + 1 \). From the truth of \((A)(\pi \land \pi)\) on \([y, y + 1]\), it follows that
there exists an interval \([y + 1, y']\) where both \(\pi\) and \(\psi\) hold. The truth of \(\pi\) over \([y + 1, y']\) allows us to conclude that \(y' = y + 1\), and thus \(\psi\) holds over the point-interval \([y + 1, y + 1]\).

In Figure 6 (bottom), we give an example of the application of the above translation that makes use of the model of Figure 1. We focus our attention on the translation of the LTL\(_f\) formula \(\Diamond(\neg p \land \neg q)\), which is true at time point 2 according to the point-based semantics \(\pi\), into the AB\(_f\) formula \(\psi = \langle A \rangle(\neg \pi \land [B] \pi \land \langle A \rangle(\pi \land \neg p \land \neg q))\), which holds over the interval \([0, 2]\). The outermost modality \(\langle A \rangle\) constrains the three conjuncts \(\neg \pi\), \([B] \pi\), and \(\langle A \rangle(\pi \land \neg p \land \neg q)\) to simultaneously hold over an interval \([2, y]\). From the truth of \(\neg \pi\), it follows that \(y > 2\), and from the truth of \([B] \pi\), we can conclude that \(y = 3\). Now, from the truth of \(\langle A \rangle(\pi \land \neg p \land \neg q)\) over \([2, 3]\), it follows that there exists \(3 \leq y\) such that the conjuncts \(\pi\), \(\neg p\), and \(\neg q\) simultaneously hold over \([3, y]\). Once more, \(\pi\) is true on \([3, y]\) if and only if \(y = 3\) \([3, 3]\), and thus both \(\neg p\) and \(\neg q\) hold over \([3, 3]\).
Last but not least, it is worth pointing out that the truth values of proposition letters on strict-intervals do not come into play in the proposed translations. It immediately follows that such translations still properly work under the homogeneity assumption that we will make in all the following sections.

3. The logic BD of prefixes and infixes

In this section, we introduce the logic BD of prefixes and infixes, we formally state the homogeneity assumption, and we define the relation of finite satisfiability under such an assumption. We conclude the section with a short analysis of the relationships between such a logic and a suitable restriction of generalized *-free regular expressions.

BD formulas are built up from a countable set Prop of proposition letters according to the following grammar: $\varphi ::= p \mid \neg \varphi \mid \varphi \lor \varphi \mid \{B\} \varphi \mid \{D\} \varphi$, where $p \in$ Prop and $\{B\}$ and $\{D\}$ are the modalities for Allen’s relations Begins and During, respectively. In the following, given a formula $\varphi$, we denote by $|\varphi|$ the size of the parse tree for $\varphi$ generated by the above grammar. It is straightforward to show that $|\varphi|$ is less than or equal to the number of symbols used to encode $\varphi$.

Let $N \in \mathbb{N}$ be a natural number and let $\mathbb{I}_N = \{[x, y] : 0 \leq x \leq y \leq N\}$ be the set of all intervals over the prefix $0 \ldots N$ of $\mathbb{N}$. A (finite) model for BD formulas is a pair $M = (\mathbb{I}_N, \mathcal{V})$, where $\mathcal{V} : \mathbb{I}_N \to 2^{\text{Prop}}$ is a valuation that maps intervals in $\mathbb{I}_N$ to sets of proposition letters.

Given a model $M$ and an interval $[x, y]$, the semantics of a BD formula is defined as follows:

- $M, [x, y] \models p$ iff $p \in \mathcal{V}([x, y])$;
- $M, [x, y] \models \neg \varphi$ iff $M, [x, y] \not\models \varphi$;
- $M, [x, y] \models \varphi_1 \land \varphi_2$ iff $M, [x, y] \models \varphi_1$ and $M, [x, y] \models \varphi_2$;
- $M, [x, y] \models \{B\} \varphi$ iff there is $y'$ with $x \leq y' < y$, such that $M, [x, y'] \models \varphi$;
- $M, [x, y] \models \{D\} \varphi$ iff there are $x'$ and $y'$, with $x < x' \leq y' < y$, such that $M, [x', y'] \models \varphi$.

The logical constants $\top$ (true) and $\bot$ (false), the Boolean operators $\lor$, $\land$, and $\iff$, and the (universal) dual modalities $[B]$ and $[D]$ can be derived in the standard way. We say that a BD formula $\varphi$ is finitely satisfiable if and only if there exist a (finite) model $M$ and an interval $[x, y]$ such that $M, [x, y] \models \varphi$ (w.l.o.g., $[x, y]$ can be assumed to be the maximal interval $[0, N]$). Hereafter, whenever we use the term satisfiable, we always mean finite satisfiability, that is, satisfiability over the class of finite linear orders.

**Definition 1 (Homogeneity).** We say that a model $M = (\mathbb{I}_N, \mathcal{V})$ satisfies the homogeneity property ($M$ is homogeneous for short) if and only if

$$\forall p \in \text{Prop} \forall [x, y] \in \mathbb{I}_N \left(p \in \mathcal{V}([x, y]) \iff \forall z \in [x, y] \ p \in \mathcal{V}([z, z])\right).$$

In Figure 7, we given an example of a homogeneous model (a) and of an arbitrary non-homogeneous one (b). For the sake of readability, we will refer to them as $M_a = (\mathbb{I}_7, \mathcal{V}_a)$ and $M_b = (\mathbb{I}_7, \mathcal{V}_b)$, respectively. The complete definitions of $\mathcal{V}_a$ and $\mathcal{V}_b$ are given in Figure 7 below the respective models. It is easy to check that the definition of $\mathcal{V}_a$ satisfies the homogeneity property as stated by Definition 1.

To begin with, we observe that, in homogeneous models, the labelling $\mathcal{V}$ of the intersection of two intervals contains the labellings of the two intervals. This is the case, for instance, with intervals $[1, 4]$ and $[2, 6]$ in Figure 7 (a), whose intersection is the interval $[2, 4]$. This
is not the case with arbitrary models. Consider, for instance, the very same intervals in Fig. 7 (b). Interval $[1, 4]$ violates the homogeneity property because $r \in \mathcal{V}_h(1, 4)$ but $r \notin \mathcal{V}_h(1, 1)$, thus violating the $\Rightarrow$ direction of Def. 1. Interval $[2, 4]$ violates the homogeneity property as well, because $q \in \mathcal{V}_h(2, 2) \cap \mathcal{V}_h(3, 3) \cap \mathcal{V}_h(4, 4)$, but $q \notin \mathcal{V}_h(2, 4)$ (the same for $r$), thus violating the $\Leftarrow$ direction of Definition 1. All the other intervals, including interval $[2, 6]$, in Figure 7 (b) satisfy the homogeneity property, but this is obviously not sufficient to consider the model homogeneous, since each interval of the model must satisfy such a property.

It is worth pointing out that the homogeneity property does not entail, in general, a similar containment property for formulas $\psi \notin \text{Prop}$. As an example, it is easy to check that in the homogeneous model of Figure 7 (a) $\langle B \rangle(p \land \neg q)$ holds over the interval $[1, 4]$, that is, $M_a, [1, 4] \models (B)(p \land \neg q)$, but it does not hold over the interval $[2, 4]$, that is, $M_a, [2, 4] \not\models (B)(p \land \neg q)$, and $(D)(q \land \neg r)$ holds over the interval $[2, 6]$, that is, $M_a, [2, 6] \models (D)(q \land \neg r)$, but it does not hold over the interval $[2, 4]$, that is, $M_a, [2, 4] \not\models (D)(q \land \neg r)$.

Finally, we would like to observe that in homogeneous models, for any proposition letter, the labelling of point-intervals determines that of arbitrary intervals. This is not the case with arbitrary models. Counterexamples are intervals $[1, 4]$ and $[2, 4]$ in Figure 7 (b).

Satisfiability can be relativized to homogeneous models. We say that a BD formula $\varphi$ is satisfiable under homogeneity if there is a homogeneous model $M$ such that $M, [0, N] \models \varphi$. Satisfiability under homogeneity is clearly more restricted than plain satisfiability. We know from [MM14, MMK10] that dropping the homogeneity assumption makes $D$ undecidable. This is not the case with the fragment $B$, whose expressive power is quite limited, which remains decidable [GMS04]. Hereafter, we will always refer to $\text{BD}$ under the homogeneity assumption, denoted by $\text{BD}_\text{hom}$.
We conclude the section with a short account of the relationships between $\text{BD}_{\text{hom}}$ and generalized $\ast$-free regular expressions. Let $\Sigma$ be a finite alphabet. A \textit{generalized $\ast$-free regular expression} (hereafter, simply \textit{expression}) $e$ over $\Sigma$ is a term of the form:

$$e ::= \emptyset | a | \neg e | e + e | e \cdot e,$$

for any $a \in \Sigma$.

We exclude the empty word $\epsilon$ from the syntax, as it makes the correspondence between finite words and finite models of $\text{BD}_{\text{hom}}$ formulas easier (such a simplification is quite common in the literature). An expression $e$ defines a language $\text{Lang}(e) \subseteq \Sigma^+$, which is inductively defined as follows:

- \text{Lang}(\emptyset) = \emptyset;
- \text{Lang}(a) = \{a\}, for every $a \in \Sigma$;
- \text{Lang}(\neg e) = \Sigma^+ \setminus \text{Lang}(e);
- \text{Lang}(e_1 + e_2) = \text{Lang}(e_1) \cup \text{Lang}(e_2);
- \text{Lang}(e_1 \cdot e_2) = \{w_1 w_2 : w_1 \in \text{Lang}(e_1), w_2 \in \text{Lang}(e_2)\}$.

In [Sto74], Stockmeyer proves that the problem of deciding the emptiness of $\text{Lang}(e)$, for a given expression $e$, is non-elementary hard. Let us now consider the logic $C$ of the chop operator (under the homogeneity assumption). As informally described in Section 2, $C$ features one binary modality, the \textit{chop} operator $⟨C⟩$, plus the modus constant $\pi$. For any model $M$ and any interval $[x, y]$, $M, [x, y] \vDash \psi_1(C) \psi_2$ if and only if there exists $z \in [x, y]$ such that $M, [x, z] \vDash \psi_1$ and $M, [z, y] \vDash \psi_2$, and $M, [x, y] \vDash \pi$ if and only if $x = y$.

As already pointed out (see Figure 4), modalities $⟨B⟩$ and $⟨D⟩$ of $\text{BD}_{\text{hom}}$ can be easily encoded in $C$ as follows: $⟨B⟩ \psi = \psi(C) \neg \pi$ and $⟨D⟩ \psi = \neg \pi(⟨C⟩(\psi(C) \neg \pi))$. It can be shown that, for any expression $e$ over $\Sigma$, there exists a formula $\varphi_e$ of $C$ whose set of models is the language $\text{Lang}(e)$, that is, $\text{Lang}(e) = \{V(0, 0) \ldots V(N, N) : (I_N, V) \vDash \varphi_e\}$. Such a formula is the conjunction of two sub-formulas $\psi_{\Sigma}$ and $\psi_e$, where $\psi_{\Sigma}$ guarantees that each unitary interval of the model is labelled by exactly one proposition letter from $\Sigma$, and $\psi_e$ constrains the valuation on the basis of the inductive structure of (the translation of) $e$. As an example, if $e = e_1 \cdot e_2$, then $\psi_e = \psi_{e_1}(C)((\neg \pi \land \neg(\neg \pi(C) \neg \pi))(⟨C⟩\psi_{e_2})$.

Such a mapping of expressions in $C$ formulas allows one to conclude that the satisfiability problem for $C$ is non-elementary hard (its non-elementary decidability follows from the opposite mapping). A careful look at the expression-to-formula mapping reveals that modality $C$ only comes into play in the translation of expressions featuring the operator of concatenation. In view of that, it is worth looking for subclasses of generalized $\ast$-free regular expressions where the concatenation operation is used in a very restricted manner, so as to avoid the need of modality $C$ in the translation.

Let us focus our attention on the following class of \textit{restricted expressions}:

$$e ::= \emptyset | a | \neg e | e + e | \text{Pre}(e) | \text{Inf}(e),$$

where $\text{Pre}(e)$ and $\text{Inf}(e)$ are respectively a shorthand for $e \cdot (\neg \emptyset)$ (thus defining the language $\text{Lang}(\text{Pre}(e)) = \{wv : w \in \text{Lang}(e), v \in \Sigma^+\}$), and $(\neg \emptyset) \cdot e \cdot (\neg \emptyset)$ (thus defining the language $\text{Lang}(\text{Inf}(e)) = \{uwv : u, v \in \Sigma^+, w \in \text{Lang}(e)\}$). Every restricted expression $e$ of the above form can be mapped into an equivalent formula $\varphi_e$ of $\text{BD}_{\text{hom}}$ by applying the usual constructions for empty language, letters, negation, and union, plus the following two rules:

(i) $\varphi_{\text{Pre}(e)} = ⟨B⟩\psi_e$, and (ii) $\varphi_{\text{Inf}(e)} = ⟨D⟩\psi_e$.

In the next sections, we will show that the satisfiability problem for $\text{BD}_{\text{hom}}$ belongs to $\text{EXPSPACE}$. From the above mapping, it immediately follows that the emptiness problem
for the considered subclass of expressions, that only uses prefixes and infixes, can be decided in exponential space (rather than in non-elementary time).

4. Homogeneous Compass Structures

In this section, we introduce a spatial representation of homogeneous models, called homogeneous compass structures, that will considerably simplify the proofs of the next sections.

Let \( \varphi \) be a \( \text{BD}_{\text{hom}} \) formula. We define the closure of \( \varphi \), denoted by \( \text{Cl}(\varphi) \), as the set of all its subformulas and of their negations, plus formulas \( \langle B \rangle \top \) and \( \langle B \rangle \bot \). For every \( \text{BD}_{\text{hom}} \) formula \( \varphi \), it holds that \( \text{Cl}(\varphi) \leq 2|\varphi| + 2 \). A \( \varphi \)-atom (atom for short) is a maximal subset \( F \) of \( \text{Cl}(\varphi) \) that, for all \( \psi \in \text{Cl}(\varphi) \), satisfies the following two conditions (as usual we identify every formula of the form \( \neg \neg \psi \) as \( \psi \)): (i) \( \psi \in F \) if and only if \( \neg \psi \notin F \), and (ii) if \( \psi = \psi_1 \lor \psi_2 \), then \( \psi \in F \) if and only if \( \{\psi_1, \psi_2\} \cap F \neq \emptyset \). Let \( \text{At}(\varphi) \) be the set of all \( \varphi \)-atoms. We have that \( |\text{At}(\varphi)| \leq 2^{|\varphi|+1} \), where \( |\varphi| = |\text{Cl}(\varphi)|/2 \).

It is easy to see that, given a model \( M = (\mathbb{I}_N, \mathcal{V}) \), we can always univocally associate an atom \( F^{[x,y]} \) in \( \text{At}(\varphi) \) with each interval \([x, y] \in \mathbb{I}_N \) by simply put \( F^{[x,y]} = \{ \psi \in \text{Cl}(\varphi) : M, [x, y] \vDash \psi \} \). An example of such an extension of the labelling \( \mathcal{V} \) to atoms is provided in Figure 8 in both a graphical (top) and a tabular (bottom) form. For the sake of readability, in the graphical representation of Figure 8 we only provide the value for positive formulas, since the presence of negative ones follows from the absence of their negation in the atom. As an example, for the interval \([1, 2] \) in Figure 8, \( F^{[1,2]} = \{p, \neg q, \neg r, \langle D \rangle \neg q, \psi_1, \neg \psi_2, \langle B \rangle \psi_1, \langle D \rangle \neg \psi_2, \neg \varphi \} \), where \( \psi_1 = p \land \neg r \) and \( \psi_2 = \neg q \land \langle D \rangle q \).

For \( R \in \{B, D\} \), we introduce the functions \( \text{Req}_R, \text{Obs}_R, \) and \( \text{Box}_R \), that map each atom \( F \in \text{At}(\varphi) \) to the following subsets of \( \text{Cl}(\varphi) \):

- \( \text{Req}_R(F) = \{ \psi \in \text{Cl}(\varphi) : (\langle R \rangle \psi) \in F \} \);
- \( \text{Obs}_R(F) = \{ \psi \in \text{Cl}(\varphi) : \langle R \rangle \psi \in \text{Cl}(\varphi), \psi \in F \} \);
- \( \text{Box}_R(F) = \{ \psi \in \text{Cl}(\varphi) : [\langle R \rangle] \psi \in F \} \).

Notice that, for each \( F \in \text{At}(\varphi) \) and each formula \( \psi \), with \( \psi \in \{ \psi' : \langle B \rangle \psi' \in \text{Cl}(\varphi) \} \), either \( \psi \in \text{Req}_B(F) \) or \( \neg \psi \in \text{Box}_B(F) \); the same for \( D \) (this means that, per se, \( \text{Box}_B(\cdot) \) and \( \text{Box}_D(\cdot) \) are not strictly necessary; we introduce them to simplify some proofs).

Sets \( \text{Req}_R(F), \text{Obs}_R(F), \) and \( \text{Box}_R(F) \) will be extensively used to prove most of the results of the paper. For that reason, we would like to illustrate their behaviour by means of the example in Figure 8.

First, let us observe that all these sets are univocally determined by the atoms in their argument; however, while \( \text{Obs}_R(F) \subseteq F \), this is not the case with \( \text{Req}_R(F) \) and \( \text{Box}_R(F) \). As an example, it holds that \( q \in \text{Req}_D(F^{[1,4]}) \), but \( q \notin F^{[1,4]} \).

Let us consider the case of \( \text{Req}_B(F) \), which extracts the arguments of the \( \langle B \rangle \)-formulas in \( F \). We have that \( \psi_1 \in \text{Req}_B(F^{[0,2]}) \), as \( \langle B \rangle \psi_1 \in F^{[0,2]} \). This means that there must exist a prefix \([0, y] \) of \([0, 2] \), with \( 0 \leq y < 2 \), such that \( \psi_1 \in F^{[0,y]} \) or, equivalently, \( \psi_1 \in \text{Obs}_B(F^{[0,y]}) \). In the considered case, we have that \( y = 1 \). In general, it holds that, for each \( \psi \in \text{Req}_B(F^{[x,y]}) \) (resp., \( \psi \in \text{Req}_D(F^{[x,y]}) \)), there exists a prefix \([x, y'] \) (resp., \( x, y' \in \text{Obs}_B(F^{[x,y]}) \) (resp., \( \text{Obs}_D(F^{[x,y']}) \)).

On the other hand, if \( \langle R \rangle \psi \in \text{Cl}(\varphi) \) and \( \psi \notin \text{Req}_R(F) \), then, necessarily, \( \langle R \rangle \neg \psi \in F \) and thus \( \neg \psi \in \text{Box}_R(F) \). It is easy to prove that \( \text{Box}_R(F) \cap \text{Req}_R(F) = \emptyset \) and \( \text{Box}_R(F) \cup \text{Req}_R(F) = \{ \psi : \langle R \rangle \psi \in \text{Cl}(\varphi) \} \), that is, \( (\text{Req}_R(F), \text{Box}_R(F)) \) is always a
\[
\varphi = \langle B \rangle (p \land \neg r) \land \langle D \rangle (\neg q \land \langle D \rangle q) \\
\]

\[\begin{array}{cccccccccc}
\psi_1 & \psi_2 & p & q & r & \langle D \rangle q & \langle B \rangle q & \langle \langle D \rangle q \rangle q & \langle D \rangle q_1 & \langle D \rangle q_2 & \varphi \\\n0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}\]

Figure 8. A graphical (top) and tabular (bottom) account of the behaviour of \(\text{Req}_R(F), \text{Obs}_R(F),\) and \(\text{Box}_R(F)\), for \(F \in \text{At}(\varphi)\) and \(R \in \{B, D\}\), with \(\varphi = \langle B \rangle (p \land \neg r) \land \langle D \rangle (\neg q \land \langle D \rangle q)\).

Partition of the whole set of temporal requests \(R\) in \(\text{Cl}(\varphi)\). Consider, for instance, for interval \([1, 4]\) in Figure 8. We have that \(\text{Req}_D(F^{[1, 4]}) = \{q\}\) and, since \(\{\psi : \langle D \rangle \psi \in \text{Cl}(\varphi)\} = \{q, \psi_2\}\), it holds that \(\text{Box}_D(F^{[1, 4]}) = \{\neg \psi_2\}\).

As opposed to what we stated above for \(\text{Req}_R\), for every \(\neg \psi \in \text{Box}_D(F^{[x,y]})\) (resp., \(\neg \psi \in \text{Box}_D(F^{[x,y]})\)) and every prefix \([x, y]\) (resp., infix \([x', y']\)) of \([x, y]\), we have that \(\psi \notin \text{Obs}_D(F^{[x,y]})\) (resp., \(\text{Obs}_D(F^{[x,y]})\)). In the considered case, for instance, since \(\neg \psi_2 \in \text{Box}_D(F^{[1,4]})\), we can conclude that \(\psi_2 \notin \text{Obs}_D(F^{[2,2]}) \cup \text{Obs}_D(F^{[2,3]}) \cup \text{Obs}_D(F^{[3,3]})\).

We would like to further explain the relation between \(\text{Req}_R\) and \(\text{Obs}_R\) by considering the example in Figure 8 from another angle. Suppose that, for a given \(N \in \mathbb{N}\) (in our example \(N = 4\)), we want to find, for each \([x, y]\) in \(\mathbb{I}_N\), a “labelling” \(F^{[x,y]} \in \text{At}(\varphi)\) such that:

\[(*)_1 \quad \mathcal{M}, [x, y] \models \psi \text{ if and only if } \psi \in F^{[x,y]},\]
where \( M = (\mathbb{I}, \mathcal{V}) \) and \( \mathcal{V}(\llbracket x, y \rrbracket) = F^{[x,y]} \cap \text{Prop} \). With the additional property:

\[
(\star_2) \quad \varphi \in F^{[0,N]},
\]

such a problem turns out to be the \textit{bounded satisfiability problem}, which is simpler than the problem we are addressing in this paper, namely, the \textit{finite satisfiability problem}. In the latter, indeed, \( N \) is not given as a parameter.

It can be easily shown that the labelings for which the following property holds:

\[
(\star_3) \quad \text{Req}_B(F^{[x,y]}) = \bigcup_{x \leq x' \leq y} \text{Obs}_B(F^{[x',y']}) \quad \text{and} \quad \text{Req}_D(F^{[x,y]}) = \bigcup_{x \leq x' \leq y} \text{Obs}_D(F^{[x',y']}),
\]

all and only the labellings that satisfy property \((\star_1)\).

This means that all the requests that we associate with an interval \([x, y]\) by means of its labelling \( F^{[x,y]} \) must be satisfied (fulfilled). Consider, for instance, the \( B \) relation. It holds that \( \text{Req}_B(F^{[x,y]}) \subseteq \bigcup_{x \leq x' \leq y} \text{Obs}_B(F^{[x',y']}) \). On the other hand, it cannot exist a formula \( \psi \) such that \( \psi \in \bigcup_{x \leq x' \leq y} \text{Obs}_B(F^{[x',y']}) \) and \( \psi \notin \text{Req}_B(F^{[x,y]}) \), as this would mean \( \neg \psi \in \text{Box}_B(F^{[x,y]}) \) which implies \( \psi \notin \bigcup_{x \leq x' \leq y} \text{Obs}_B(F^{[x',y']}) \) (contradiction). Thus, we can conclude \( \text{Req}_B(F^{[x,y]}) \supseteq \bigcup_{x \leq x' \leq y} \text{Obs}_B(F^{[x',y']}) \) as well (consistency).

The very same observations hold for modality \( D \) and all the other \( \text{HS}_{\text{hom}} \) modalities. In fact, this is a general property which holds even without the homogeneity assumption. Thus, we can conclude that \((\star_3)\) is a \textit{necessary and sufficient} condition for \( M \) to satisfy \( \varphi \).

By making use of \( \text{Req}_B, \text{Req}_D, \text{Obs}_B, \) and \( \text{Obs}_D \), we define two binary relations \( \rightarrow_B \) and \( \rightarrow_D \) over \( \text{At}(\varphi) \) as follows.

\textbf{Definition 2.} For all \( F, G \in \text{At}(\varphi) \) we let:

- \( F \rightarrow_B G \) iff \( \text{Req}_B(F) = \text{Req}_B(G) \cup \text{Obs}_B(G) \);
- \( F \rightarrow_D G \) iff \( \text{Req}_D(F) \supseteq \text{Req}_D(G) \cup \text{Obs}_D(G) \).

Relations \( \rightarrow_B \) and \( \rightarrow_D \) are often referred to as \textit{view-to-type} dependencies since they constrain the labelling of a state (an interval) according to the labellings of the states that it can access via certain relations (interval relations). As already pointed out, for every \( \psi \in \{ \psi^j : \langle B \rangle \psi^j \in \text{Cl}(\varphi) \} \) we have either \( \psi \in \text{Req}_B(F) \) or \( \neg \psi \in \text{Box}_B(F) \) (and vice versa). Given two atoms \( F \) and \( G \), with \( F \rightarrow_B G \), and a formula \( \neg \psi \in \text{Box}_B(F) \) it immediately follows that \( \psi \notin \text{Req}_B(F) \) and thus from \( \text{Req}_B(F) = \text{Req}_B(G) \cup \text{Obs}_B(G) \), it follows \( \psi \notin \text{Obs}_B(G) \). Now, from \( \neg \psi \in \text{Box}_B(F) \), it follows that \( \langle B \rangle \psi \in \text{Cl}(\varphi) \), and from \( \langle B \rangle \psi \in \text{Cl}(\varphi) \) and \( \psi \notin \text{Obs}_B(G) \), it follows that \( \psi \notin G \). For the maximality of atoms, it follows that \( \neg \psi \in G \). This allows us to conclude that for every pair of atoms \( F \) and \( G \), with \( F \rightarrow_B G \), we have that \( \text{Box}_B(F) \subseteq G \). The same argument can be applied to the relation \( \rightarrow_D \), and thus for every pair of atoms \( F \) and \( G \), with \( F \rightarrow_D G \), it holds that \( \text{Box}_D(F) \subseteq G \).

In addition, relation \( \rightarrow_D \) is transitive (by definition of atom, from \( \text{Req}_D(F) \supseteq \text{Req}_D(G) \), it immediately follows that \( \text{Box}_R(F) \subseteq \text{Box}_R(G) \)), while \( \rightarrow_B \) is not. A graphical account of relations \( \rightarrow_B \) and \( \rightarrow_D \) is given in Figure 9 and Figure 10, respectively.

As for relation \( \rightarrow_B \) (Figure 9), we may observe how it is used to constrain the \( \text{Req}_B(F^{[x,y]}) \) part of the labelling, for each interval \([x,y]\) and its maximal proper prefix (if any) \([x,y-1]\). This means that, in a “consistent model”, we expect that for each
strict-interval \([x, y]\), \(F^{[x,y]} \rightarrow_B F^{[x,y-1]}\). Notice that \(\rightarrow_B\) is intended to constrain only the maximal prefix of an interval, not all its prefixes.

Now, let us unravel the definition of \(\rightarrow_B\). We have that the following three conditions are satisfied:

1. Box\(_B\)(\(F^{[x,y]}\)) \subseteq Box\(_B\)(\(F^{[x,y-1]}\)), since \(\text{Req}_B(F^{[x,y]}) \supseteq \text{Req}_B(F^{[x,y-1]})\), and thus \(F^{[x,y-1]}\) features at least the same universal requests as \(F^{[x,y]}\);
2. from \(\text{Obs}_B(F^{[x,y-1]}) \subseteq \text{Req}_B(F^{[x,y]})\), it follows that \(F^{[x,y-1]}\) possibly satisfies some of the requests in \(\text{Req}_B(F^{[x,y]})\) and it does not violate any of the universal requests in Box\(_B\)(\(F^{[x,y]}\)); otherwise, we would have \(\text{Obs}_B(F^{[x,y-1]}) \setminus \text{Req}_B(F^{[x,y]}) \neq \emptyset\);
3. from \(\text{Req}_B(F^{[x,y]}) = \text{Req}_B(F^{[x,y-1]}) \cup \text{Obs}_B(F^{[x,y-1]})\), we have that, for each request \(\psi \in \text{Req}_B(F^{[x,y]})\), either \(F^{[x,y]}\) satisfies it without requesting it again, that is, \(\psi \in \text{Obs}_B(F^{[x,y-1]} \setminus \text{Req}_B(F^{[x,y-1]}))\), or \(\psi\) is featured again as a request, that is, \(\psi \in \text{Req}_B(F^{[x,y-1]})\). It is worth noticing that this is the behaviour that one may expect from the labelling of a maximal prefix \([x, y-1]\) of an interval \([x, y]\): a request
As for the observables, compared to atom $F^{[0,1]}$, that is, the two labellings can be swapped without any consequence on the consistency of $B$ requests. In situations like this one, we will say that the involved atoms are $B$-reflexive. Reflexive atoms will play a crucial role in the proof of the results of the next sections. They are denoted by a self-loop in the compass structure of Figure 9. Atom $F^{[0,2]}$ is a different story: it features $\psi_1$ and thus $\psi_1 \in \text{Obs}_B(F^{[0,2]})$. However, $\psi_1 \notin \text{Req}_B(F^{[0,2]})$ and $F^{[0,2]}$ is not reflexive. Clearly, $F^{[0,2]} \rightarrow_B F^{[0,1]}$, since both of them have no $\{B\}$ requests. Atom $F^{[0,3]}$ is the first atom with at least one $\{B\}$ request, namely, $\text{Req}_B(F^{[0,3]}) = \{\psi_1\}$. We have that $F^{[0,3]} \rightarrow_B F^{[0,2]}$, since $\text{Req}_B(F^{[0,3]}) = \text{Req}_B(F^{[0,2]}) \cup \text{Obs}_B(F^{[0,2]})$, that is, $\{\psi_1\} = \emptyset \cup \{\psi_1\}$. Notice that $F^{[0,3]} \rightarrow_B F^{[0,1]}$ does not hold, since $\{\psi_1\} \neq \emptyset \cup \emptyset$, that is, $F^{[0,1]}$ neither satisfies $\psi_1$ nor features $\{B\}\psi_1$. As for the observables, compared to atom $F^{[0,2]}$, atom $F^{[0,3]}$ "loses" $\psi_1$, which is transferred to its $\{B\}\psi_1$ request, but it satisfies two more requests, namely, $\psi_2$ and $\{D\}\psi_1$ in $\text{Obs}_B(F^{[0,3]})$, for the labelling of the intervals that feature $[0,2]$ as proper prefix. As for $F^{[0,4]}$, $\text{Obs}_B(F^{[0,4]}) = \emptyset$ and $\text{Req}_B(F^{[0,4]})$ has two formulas more than $\text{Req}_B(F^{[0,3]})$. Finally, we have that $F^{[0,4]} \rightarrow_B F^{[0,3]}$, that is, $\text{Req}_B(F^{[0,4]}) = \text{Req}_B(F^{[0,2]}) \cup \text{Obs}_B(F^{[0,2]})$, as $\{\psi_1, \psi_2, \{D\}\psi_1\} = \{\psi_1\} \cup \{\psi_2, \{D\}\psi_1\}$; $\psi_1$ is transferred by $F^{[0,2]}$ to the proper prefixes of $[0,2]$ by means of $\psi_1 \in \text{Req}_B(F^{[0,2]})$, while both $\psi_2$ and $\{D\}\psi_1$ are satisfied locally by the observables of $F^{[0,2]}$.

Before proceeding with the analysis of relation $\rightarrow_D$, we state an important lemma that, given an atom $F^{[x,y]}$, determines how many atoms $F^{[x,y+k]}$, with $k \geq 1$, with a distinct pair $(\text{Req}_B(F^{[x,y+k]}), \text{Obs}_B(F^{[x,y+k]}))$ can be placed "above" $F^{[x,y]}$ in a compass structure, that is, may have $F^{[x,y]}$ as a prefix.

---

It is worth pointing out that there may be atoms $F$ and $G$ such that $\text{Req}_B(F) = \text{Req}_B(G) \cup \text{Obs}_B(G)$ (that is, $F \rightarrow_B G$), and $\text{Req}_B(G) \cap \text{Obs}_B(G) \neq \emptyset$, that is, a $(B)$ request may be at the same time locally satisfied by $G$ and featured as request for its proper prefixes.
Lemma 1. Let $\varphi$ be a $\text{BD}_{\text{hom}}$ formula. For any atom $F \in \text{At}(\varphi)$ and any sequence of atoms $F_0 \rightarrow_B \ldots \rightarrow_B F_1 \rightarrow_B F_2 = F$, where, for each $0 \leq i \neq j \leq h$, $\text{Obs}_B(F_i) \setminus \text{Req}_B(F_i) \neq \text{Obs}_B(F_j) \setminus \text{Req}_B(F_j)$ or $\text{Req}_B(F_i) \neq \text{Req}_B(F_j)$, it holds that $h \leq 2|\{\psi : \langle B \rangle \psi \in \text{Cl}(\varphi)\}| - 2|\text{Req}_B(F)| - |\text{Obs}_B(F) \setminus \text{Req}_B(F)|$.

Proof. Let us consider the sequence of pairs $(\text{Req}_B(F_h), \text{Obs}_B(F_h) \setminus \text{Req}_B(F_h)) \ldots (\text{Req}_B(F_0), \text{Obs}_B(F_0) \setminus \text{Req}_B(F_0))$ induced by $F_h \rightarrow_B \ldots \rightarrow_B F_1 \rightarrow_B F_0 = F$. By Definition 2, it holds that $\text{Req}_B(F_i) = \text{Req}_B(F_{i-1}) \cup \text{Obs}_B(F_{i-1})$, for every $0 \leq i \leq h$. Moreover, by recursively unravelling the right part of the equation $\text{Req}_B(F_i) = \text{Req}_B(F_{i-1}) \cup \text{Obs}_B(F_{i-1})$ by replacing $\text{Req}_B(F_{i-j})$ by $\text{Req}_B(F_{i-j-1}) \cup \text{Obs}_B(F_{i-j-1})$, for $1 \leq j < i$, we obtain an alternative formulation of $\text{Req}_B(F_i)$ as $\text{Req}_B(F_0) \cup \bigcup_{0 \leq j < i} \text{Obs}_B(F_j)$.

Now, for each $\psi \in \text{Req}_B(F_h)$, let us define the index $\text{ireq}(\psi) \in \{0, \ldots, h\}$ as follows:

$$\text{ireq}(\psi) = \begin{cases} i & \text{if } \psi \in \text{Req}_B(F_\text{ireq}(\psi)) \setminus \text{Req}_B(F_{\text{ireq}(\psi)-1}); \\ 0 & \text{otherwise.} \end{cases}$$

The fact that $\text{ireq}$ is well defined immediately follows from $\text{Req}_B(F_i) \supseteq \text{Req}_B(F_{i-1})$, for all $0 \leq i \leq h$. Similarly, for each $\psi \in \text{Req}_B(F_h) \cup \text{Obs}_B(F_h)$, let us define the index $\text{iobs}(\psi) \in \{0, \ldots, h\}$ as follows:

$$\text{iobs}(\psi) = \begin{cases} i & \text{if } \psi \in \text{Obs}_B(F_{\text{iobs}(\psi)}) \setminus \text{Req}_B(F_{\text{iobs}(\psi)-1}); \\ 0 & \text{otherwise.} \end{cases}$$

The fact that $\text{iobs}$ is well defined follows from $\text{Req}_B(F_i) \supseteq \text{Req}_B(F_{i-1})$, for all $0 \leq i \leq h$, and $\text{Req}_B(F_i) = \text{Req}_B(F_0) \cup \bigcup_{0 \leq j < i} \text{Obs}_B(F_j)$, for all $0 \leq i \leq h$.

We now prove that there exists an index $i > 0$ such that $i \notin \text{Img}(\text{ireq}) \cup \text{Img}(\text{ioobs})$. By contradiction, let us assume that such an index exists (let us assume $i > 0$; the case $i = 0$ is symmetric). It follows that:

- from $i \notin \text{Img}(\text{ireq})$, it follows that, for each $\psi \in \text{Req}_B(F_h)$, either $\text{ireq}(\psi) > i$, and thus $\psi \notin \text{Req}_B(F_i) \cup \text{Req}_B(F_{i-1})$, or $i > \text{ireq}(\psi)$, and thus $\psi \in \text{Req}_B(F_i) \cap \text{Req}_B(F_{i-1})$, and then $\text{Req}_B(F_i) = \text{Req}_B(F_{i-1})$;
- from $i \notin \text{Img}(\text{ioobs})$, it follows that, for each $\psi \in \text{Req}_B(F_h) \cup \text{Obs}_B(F_h)$, either $\text{ioobs}(\psi) > i$, and thus $\psi \notin \text{Obs}_B(F_i) \cup \text{Obs}_B(F_{i-1}) \cup \text{Req}_B(F_i) \cup \text{Req}_B(F_{i-1})$, or $i > \text{ioobs}(\psi)$, and then $i - 1 > \text{ioobs}(\psi)$, because if $\psi \in \text{Obs}_B(F_{i-1}) \setminus \text{Req}_B(F_{i-1})$, then $\text{ioobs}(\psi) = i$ (contradiction). Hence, $\text{Obs}_B(F_i) \setminus \text{Req}_B(F_i) = \text{Obs}_B(F_{i-1}) \setminus \text{Req}_B(F_{i-1}) = \emptyset$.

From the above two cases, we can conclude that $(\text{Req}_B(F_i), \text{Obs}_B(F_i) \setminus \text{Req}_B(F_i)) = (\text{Req}_B(F_{i-1}), \text{Obs}_B(F_{i-1}) \setminus \text{Req}_B(F_{i-1}))$, and thus we obtain a contradiction. Finally, we have that $h \leq |\text{Img}(\text{ireq})| + |\text{Img}(\text{ioobs})|$, with $|\text{Img}(\text{ireq})| \leq |\{\psi : \langle B \rangle \psi \in \text{Cl}(\varphi)\}| - |\text{Req}_B(F_0)|$ and $|\text{Img}(\text{ioobs})| \leq |\{\psi : \langle B \rangle \psi \in \text{Cl}(\varphi)\}| - |\text{Req}_B(F_0)| - |\text{Obs}_B(F_0) \setminus \text{Req}_B(F_0)|$, and thus $h \leq 2|\{\psi : \langle B \rangle \psi \in \text{Cl}(\varphi)\}| - 2|\text{Req}_B(F_0)| - |\text{Obs}_B(F_0) \setminus \text{Req}_B(F_0)|$.

Let us consider now relation $\rightarrow_D$. By Definition 2, given two atoms $F$ and $G$, the condition imposed by $F \rightarrow_D G$ is weaker than the one imposed by $\rightarrow_B$, that is, containment (superset) instead of full equality of the two sets. This is because with $F \rightarrow_D G$ we want to express the fact that $G$ may label any sub-interval $[x, y]$ of an interval $[x', y']$ ($x' < x \leq y < y'$) not just its maximal proper sub-interval $[x' - 1, y' - 1]$; on the contrary, relation $\rightarrow_B$ only refers to the maximal proper prefix $[x', y' - 1]$ of $[x', y']$. 

\[
\varphi = \langle D \rangle( p \land q ) \land \langle D \rangle( \neg p \land q ) \land \langle D \rangle( p \land \neg q )
\]

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
p & q & v_1 & v_2 & v_3 & \langle D \rangle v_1 & \langle D \rangle v_2 & \langle D \rangle v_3 & \varphi & \text{Req}_D(F^{[x,y]}) & \text{Obs}_D(F^{[x,y]}) \\
\hline
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \{p, \psi_1, \psi_2\} & \emptyset \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \{\psi_1\} & \emptyset \\
2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \{\psi_1\} & \emptyset \\
3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \{\psi_1\} & \emptyset \\
4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \{\psi_1\} & \emptyset \\
\hline
\end{array}
\]

\[
\begin{align*}
\mathcal{L}(0,4) & \rightarrow_D \mathcal{L}(0,4) & \mathcal{L}(0,4) & \rightarrow_D \mathcal{L}(1,3) & \mathcal{L}(0,4) & \rightarrow_D \mathcal{L}(1,2) & \mathcal{L}(0,4) & \rightarrow_D \mathcal{L}(2,3) & \mathcal{L}(0,4) & \rightarrow_D \mathcal{L}(1,1) \\
\mathcal{L}(0,4) & \rightarrow_D \mathcal{L}(2,2) & \mathcal{L}(0,4) & \rightarrow_D \mathcal{L}(3,3) & \mathcal{L}(1,3) & \rightarrow_D \mathcal{L}(1,3) & \mathcal{L}(1,2) & \rightarrow_D \mathcal{L}(1,2) & \mathcal{L}(2,3) & \rightarrow_D \mathcal{L}(2,3) \\
\mathcal{L}(1,1) & \rightarrow_D \mathcal{L}(1,1) & \mathcal{L}(2,2) & \rightarrow_D \mathcal{L}(2,2) & \mathcal{L}(3,3) & \rightarrow_D \mathcal{L}(3,3)
\end{align*}
\]

**Figure 10.** A graphical account of relation \( \rightarrow_D \) from both the interval point of view (left) and the spatial one (right). For the sake of readability, we only highlight the sub-intervals of \([0,4]\).

In Figure 10, both in the interval model and in its compass structure counterpart, we show the labelling of intervals which are required to satisfy relation \( \rightarrow_D \). We cope with three \( \langle D \rangle \) requests: \( \psi_1 = p \land q \), \( \psi_2 = \neg p \land q \), and \( \psi_3 = p \land \neg q \). Let us consider all the proper sub-intervals of the largest interval in the model, namely, sub-intervals \([1,1],[1,2],[1,3],[2,2],[2,3] \), and \([3,3] \) of interval \([0,4]\). At the very bottom of Figure 10, we show when \( F^{[x,y]} \rightarrow_D F^{[x',y']} \), for \( 0 < x \leq x' < 4 \) and \([x, x'] = [0,4]\), is true and it is not. The same pieces of information are graphically depicted in the top right part of Figure 10.

Let us now describe how \( \text{Req}_D(F^{[x,y]}) \) and \( \text{Obs}_D(F^{[x,y]}) \) behave moving from interval \([x,y]\) to its maximal sub-interval \([x+1,y-1]\) starting from the largest interval \([0,4]\). First, we observe that, since \( \text{Obs}_D(F^{[0,4]}) = \emptyset \), it trivially holds that \( \text{Req}_D(F^{[0,4]}) \supseteq \)
proof of the following theorem is straightforward and thus omitted. Given a formula \( \phi \), we say that an atom \( x, y \) are not proper sub-intervals of it, it may be the case that \( \phi \) must be completely "covered" by those holding over its maximal proper sub-interval \( [x + 1, y - 1] \) and the union of all the observables of \( [x + 1, y - 1] \) and \( [x + 2, y - 1] \) (maximal proper prefix of \( [x + 1, y - 1] \)) and \( [x + 2, y - 1] \) (maximal proper suffix of \( [x + 1, y - 1] \)).

As an example, in Figure 10, we have that \( \text{Req}_D(F^{[0,4]}) = \text{Req}_D(F^{[1,3]}) \cup \text{Obs}_D(F^{[0,4]}) \cup \text{Obs}_D(F^{[2,3]}) \) and/or \( F^{[x,y]} \to_B F^{[x+1,y-1]} \) do not hold in a consistent model. For instance, in Figure 10, neither \( F^{[1,3]} \to_D F^{[1,2]} \) nor \( F^{[1,3]} \to_D F^{[2,3]} \) hold.

The next proposition reduces the equality condition for any pair of atoms to the equality of their propositional components and their respective sets of \( B \) and \( D \) requests.

**Proposition 1.** For each pair of atoms \( F, G \in \text{At}(\varphi) \), \( F = G \) if and only if \( \text{Req}_B(F) = \text{Req}_B(G) \), \( \text{Req}_D(F) = \text{Req}_D(G) \), and \( F \cap \text{Prop} = G \cap \text{Prop} \).

The proof of Proposition 1 trivially follows from the fact that, for each atom \( F \) and each \( \psi \in F \), either \( \psi \in \text{Prop} \cup \{ (B) \psi^1 : \psi^1 \in \text{Req}_B(F) \} \cup \{ (D) \psi^1 : \psi^1 \in \text{Req}_D(F) \} \) or \( \psi \) is a Boolean combination of \( \text{Prop} \cup \{ (B) \psi^1 : \psi^1 \in \text{Req}_B(F) \} \cup \{ (D) \psi^1 : \psi^1 \in \text{Req}_D(F) \} \).

Given a formula \( \varphi \), a \( \varphi \)-compass structure (simply compass structure, when \( \varphi \) is clear from the context) is a pair \( \mathcal{G} = (\mathcal{G}_N, \mathcal{L}) \), where \( N \in \mathbb{N} \), \( \mathcal{G}_N = \{(x, y) : 0 \leq x \leq y \leq N \} \), and \( \mathcal{L} : \mathcal{G}_N \to \text{At}(\varphi) \) is a labelling function that satisfies the following conditions:

- (initial formula) \( \varphi \in \mathcal{L}(0, N) \);
- (B-consistency) for all \( 0 \leq x \leq y < N \), \( \mathcal{L}(x, y+1) \to_B \mathcal{L}(x, y) \), and for all \( 0 \leq x \leq N \), \( \text{Req}_B(\mathcal{L}(x, x)) = \emptyset \);
- (D-consistency) for all \( 0 \leq x < x' \leq y' < y \leq N \), \( \mathcal{L}(x, y) \to_D \mathcal{L}(x', y') \);
- (D-fulfilment) for all \( 0 \leq x \leq y \leq N \) and all \( \psi \in \text{Req}_D(\mathcal{L}(x, y)) \), there exist \( x < x' \leq y' < y \) such that \( \psi \in \mathcal{L}(x', y') \).

Observe that the definition of \( \to_B \) and \( B \)-consistency guarantee that all the existential requests via relation \( B \) (hereafter \( B \)-requests) are fulfilled in a compass structure.

We say that an atom \( F \in \text{At}(\varphi) \) is \( B \)-reflexive (resp., \( D \)-reflexive) if \( F \to_B F \) (resp., \( F \to_D F \)). If \( F \) is not \( B \)-reflexive (resp., \( D \)-reflexive), it is \( B \)-irreflexive (resp., \( D \)-irreflexive).

Let \( \mathcal{G} = (\mathcal{G}_N, \mathcal{L}) \) be a compass structure. We define the function \( \mathcal{P} : \mathcal{G}_N \to 2^{\text{Prop}} \) such that \( \mathcal{P}(x, y) = \{ p \in \text{Prop} : p \in \mathcal{L}(x', x') \text{ for all } x \leq x' \leq y \} \). We say that a \( \varphi \)-compass structure \( \mathcal{G} = (\mathcal{G}_N, \mathcal{L}) \) is homogeneous if for all \( (x, y) \in \mathcal{G}_N \), \( \mathcal{L}(x, y) \cap \text{Prop} = \mathcal{P}(x, y) \). The proof of the following theorem is straightforward and thus omitted.
Theorem 1. A $\text{BD}_{\text{hom}}$ formula $\varphi$ is satisfiable if and only if there is a homogeneous $\varphi$-compass structure.

Hereafter, we will often write compass structure for homogeneous $\varphi$-compass structure.

In Figure 11, we depict the homogeneous model $\mathcal{M} = (\mathbb{I}, \mathcal{V})$ (left) with the corresponding compass structure $\mathcal{G} = (\mathcal{G}_7, \mathcal{L})$ (right), for a given formula $\varphi$. We have that $\text{Cl}(\varphi) \cap \text{Prop} = \{p, q\}$, $\{(\mathcal{B})\psi \in \text{Cl}(\varphi)\} = \{\{(\mathcal{B})\top, (\mathcal{B})\neg p\}\}$, and $\{(\mathcal{D})\psi \in \text{Cl}(\varphi)\} = \{\{(\mathcal{D})\neg q\}\}$. We know that, by the homogeneity assumption, the valuation of proposition letters at point-intervals determines that at non-point ones. As an example, if an interval $[x, y]$ contains time point 3, as, e.g., interval $[1, 6]$, then $\{p, q\} \cap \mathcal{V}([x, y]) = \emptyset$. Similarly, if an interval $[x, y]$ contains time point 7 (resp., 0), then it must satisfy $\{p\} \cap \mathcal{V}([x, y]) = \emptyset$ (resp., $\{q\} \cap \mathcal{V}([x, y]) = \emptyset$). As for the compass structure $\mathcal{G}$, we first observe that each interval $[x, y]$ in $\mathcal{M}$ is mapped to a point in the second octant of the $\mathbb{N} \times \mathbb{N}$ grid (in Figure 11, we depict the first quadrant of such a grid, where the first octant is shaded). Thanks to such a mapping, interval relations are mapped into special relations between points (by a slight abuse of terminology, we borrow the names of the interval relations). As an example, in Figure 11 point $(0, 2)$ begins $(0, 3)$. Similarly, as enlightened by the hatched triangle, point $(1, 6)$ has points $(2, 2), (2, 3), (3, 3), (2, 4), (3, 4), (4, 4), (2, 5), (3, 5), (4, 5),$ and $(5, 5)$ as sub-intervals.

In general, all points $(x, x)$ are labelled with irreflexive atoms containing $[B] \bot$, while all points $(x, y)$, with $x < y$, are labelled with atoms containing $(B) \top$. The variety of atoms is exemplified by the following cases. Atom $\mathcal{L}(0, 3)$ is both $B$-irreflexive (Box$_B(\mathcal{L}(0, 3)) = \{p\}$ and $\neg p \in \mathcal{L}(0, 3)$) and $D$-irreflexive (Box$_D(\mathcal{L}(0, 3)) = \{q\}$ and $\neg q \in \mathcal{L}(0, 3)$), atom $\mathcal{L}(4, 6)$ is both $B$-reflexive (Box$_B(\mathcal{L}(4, 6)) = \{p\}$ and $p \in \mathcal{L}(4, 6)$) and $D$-reflexive (Box$_D(\mathcal{L}(4, 6)) = \{q\}$ and $q \in \mathcal{L}(4, 6)$), atom $\mathcal{L}(4, 7)$ is $B$-irreflexive (Box$_B(\mathcal{L}(4, 7)) = \{p\}$ and $\neg p \in \mathcal{L}(4, 7)$) and $D$-reflexive (Box$_D(\mathcal{L}(4, 7)) = \{q\}$ and $q \in \mathcal{L}(4, 7)$), and atom $\mathcal{L}(0, 2)$ is $B$-reflexive (Box$_B(\mathcal{L}(0, 2)) = \{p\}$ and $p \in \mathcal{L}(0, 2)$) and $D$-irreflexive (Box$_D(\mathcal{L}(0, 2)) = \{q\}$ and $\neg q \in \mathcal{L}(0, 2)$). Finally, we would like to highlight that $\mathcal{L}(4, 7) \rightarrow_B \mathcal{L}(4, 6)$ (Req$_B(\mathcal{L}(4, 7)) = \emptyset$).
and $\text{Obs}_B(\mathcal{L}(4,6)) \cup \text{Req}_B(\mathcal{L}(4,6)) = \emptyset$ and $\mathcal{L}(3,0) \rightarrow_D \mathcal{L}(1,2)$ ($\text{Req}_D(\mathcal{L}(3,0)) = \emptyset$ and $\text{Obs}_D(\mathcal{L}(1,2)) \cup \text{Req}_D(\mathcal{L}(1,2)) = \emptyset$).

In the next sections, we will prove a small model theorem about compass structures for an input $\mathbf{BD}_{\text{hom}}$ formula $\varphi$. In particular, we will prove that a model can be built by contracting a larger one in such a way that the resulting model is still a compass structure for $\varphi$. To achieve such a goal, we need to state some spatial properties of compass structures that involves the distinction between $B$-reflexive (resp., $D$-reflexive) and $B$-irreflexive (resp., $D$-irreflexive) atoms. Intuitively, if a point is labelled with an atom which is both $B$-reflexive and $D$-reflexive, its only purpose is to “fill the gaps” in the model, as each $B/D$-request that it possibly solves for other points are transferred to its prefixes/sub-intervals. On the other hand, a point that is $B$-irreflexive, $D$-irreflexive, or both $B$-irreflexive and $D$-irreflexive must be treated carefully since it feature at least one $B/D$-request in its observables that is solved once and for all, and it is not transferred to its prefixes/sub-intervals.

5. Spatial properties of compass structures for $\mathbf{BD}_{\text{hom}}$ formulas

In this section, we prove a series of spatial properties of compass structures that turn out to be very useful in the proofs of the results of Sections 6 and 8. Each property is proved by making use of the previous one as follows:

Section 5.1 - We first show that for any compass structure and any of its $X$-axis coordinate $x$, the sequence $\mathcal{L}(x,0) \ldots \mathcal{L}(x,N)$ is monotonic, that is, for any triplet $0 \leq y_1 < y_2 < y_3 \leq N$, it cannot be the case that $\mathcal{L}(x,y_1) = \mathcal{L}(x,y_3)$ and $\mathcal{L}(x,y_1) \neq \mathcal{L}(x,y_2)$. Such a property allows us to represent relevant information associated with any column $x$ in space (polynomially) bounded in $|\varphi|$.

Section 5.2 - Next, we define an equivalence relation over columns such that two columns are equivalent if they feature the same set of atoms. It is easy to verify that such an equivalence relation is of finite index and its index is exponentially bounded in $|\varphi|$. By exploiting the representation of Section 5.1, we first define a partial order over equivalent columns, and then we prove that, in a compass structure, such a relation totally orders equivalent columns.

Section 5.3 - By exploiting the total order of the elements of each equivalence class, we show a crucial property of the rows of a compass structure, which is the cornerstone of the proof. First, we associate with each point $(x, y)$ on row $y$, with $0 \leq x \leq y$, a tuple consisting of: (i) $\mathcal{L}(x, y)$, (ii) the equivalence class $\sim_x$ of column $x$, and (iii) the set of pairs $(\mathcal{L}(x', y), \sim_{x'})$, for all $x < x' \leq y$, and then we prove that, for every pair of points $(x, y), (x', y)$ that feature the same tuple, $\mathcal{L}(x, y) = \mathcal{L}(x', y)$ for all $y' > y$, that is, columns $x$ and $x'$ behave the same way (i.e., exhibit the same labelling) from $y$ to the upper end.

5.1. A finite characterisation of columns and of their relationships. In this section, we first show that, in every compass structure, the atoms that appear in a column $x$ must respect a certain order, that is, they cannot be interleaved. Let $F, G$, and $H$ be three pairwise distinct atoms. In Figure 12.(a), we give a graphical account of the property that we are going to prove, while, in Figure 12.(b), we show a violation of it (atom $H$ appears before and after atom $G$ moving upward along the column).

We preliminarily prove a fundamental property of $B$-irreflexive atoms.
Lemma 2. Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure. For all $x \leq y < N$, if $\text{Req}_B(\mathcal{L}(x, y)) \subseteq \text{Req}_B(\mathcal{L}(x, y + 1))$, then $\mathcal{L}(x, y)$ is $B$-irreflexive.

Proof. Let us assume by contradiction that $\mathcal{L}(x, y)$ is $B$-reflexive. This means that $\text{Box}_B(\mathcal{L}(x, y)) \subseteq \mathcal{L}(x, y)$. Since $\text{Req}_B(\mathcal{L}(x, y)) \subseteq \text{Req}_B(\mathcal{L}(x, y + 1))$, there exists a formula $\psi \in \text{Req}_B(\mathcal{L}(x, y + 1)) \setminus \text{Req}_B(\mathcal{L}(x, y))$ and thus we have $\lnot \psi \in \text{Box}_B(\mathcal{L}(x, y))$ and, by $B$-reflexivity of $\mathcal{L}(x, y)$, $\lnot \psi \in \mathcal{L}(x, y)$. Since $\mathcal{G}$ is a compass structure, it holds that $\mathcal{L}(x, y) \rightarrow_B \mathcal{L}(x, y - 1) \rightarrow_B \mathcal{L}(x, x)$, and thus $\lnot \psi \in \text{Box}_B(\mathcal{L}(x, y'))$ and $\lnot \psi \in \mathcal{L}(x, y')$, for all $x \leq y' \leq y$. Since, by definition of $\rightarrow_B$, all $B$-requests are fulfilled in a compass structure, we can conclude that $\psi \notin \text{Req}_B(\mathcal{L}(x, y + 1))$ (contradiction).

Let us now provide a bound on the number of distinct atoms that can be placed above a given atom $F$ in a column, that takes into account $B$-requests, $D$-requests, and negative literals in $F$. Formally, we define a function $\Delta_T : \text{At}(\varphi) \rightarrow \mathbb{N}$ as follows:

$$\Delta_T(F) = (\{\text{(B)} \psi \in \text{Cl}(\varphi)\} - 2|\text{Req}_B(F)| - |\text{Obs}_B(F) \setminus \text{Req}_B(F)|) +$$

$$\{|(D) \psi \in \text{Cl}(\varphi)| - |\text{Prop}_D(F)|\} +$$

$$\{|\lnot p : p \in \text{Cl}(\varphi) \cap \text{Prop}\} - \{|\lnot p : p \in \text{Cl}(\varphi) \cap \text{Prop} \land \lnot p \in F\}|$$

The result and the proof of Lemma 1 in Section 4 helps us to understand why the factor 2 comes into play in the case of $B$-requests. Informally, from the proof of Lemma 1 it immediately follows that in order to move down from an atom including $\langle B \rangle \psi$ to an atom including $\lnot \psi, [B] \lnot \psi$ one must pass through an atom including $\psi, [B] \lnot \psi$.

It can be easily checked that, for each $F \in \text{At}(\varphi)$, $0 \leq \Delta_T(F) \leq 4|\varphi| + 1$. To explain how $\Delta_T$ works, we give a simple example. Let $\{\psi : \langle B \rangle \psi \in \text{Cl}(\varphi)\} = \{\psi_1\}$ and let $F_3 \rightarrow_B F_2 \rightarrow_B F_1$, with $\text{Req}_B(F_3) = \{\psi_1\}$ and $\text{Req}_B(F_2) = \text{Req}_B(F_1) = \emptyset$. For simplicity, let $\{\psi : \langle D \rangle \psi \in \text{Cl}(\varphi)\} = \emptyset$, and thus $\text{Prop}_D(F_3) = \text{Prop}_D(F_2) = \text{Prop}_D(F_1) = \emptyset$, and $(F_3 \cap F_2 \cap F_1) \cap \text{Prop} = \text{Prop} = \{p\}$. It holds that $\Delta_T(F_1) = (2 \cdot 1 \cdot 2 \cdot 0 \cdot 0 + 0 \cdot 0 + 1 \cdot 0) = 3$, $\Delta_T(F_2) = (2 \cdot 1 \cdot 2 \cdot 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0) = 2$, and $\Delta_T(F_3) = (2 \cdot 1 \cdot 2 \cdot 1 \cdot 0 + 0 \cdot 0 + 1 \cdot 0) = 1$.

We say that an atom $F$ is initial if and only if $\text{Req}_B(F) = \emptyset$. A $B$-sequence is a sequence of atoms $\text{Sh}_B(F_0 \ldots F_n)$ such that $F_0$ is initial and for all $0 \leq i \leq n$ we have $F_i \rightarrow_B F_{i-1}$, $\text{Req}_D(F_i) \supseteq \text{Req}_D(F_{i-1})$, and $F_i \cap \text{Prop} \subseteq F_{i-1} \cap \text{Prop}$. It is worth pointing out that atoms in a $B$-sequence are monotonically non-increasing in $\Delta_T$, that is, $\Delta_T(F_0) \geq \ldots \geq \Delta_T(F_n)$.

Definition 3. We say that a $B$-sequence $F_0 \ldots F_n$ is flat if and only if it can be written as a sequence $\overline{F}_0^k \ldots \overline{F}_m^k$, where $k_i > 0$, for all $1 \leq i \leq m$, and $\overline{F}_i \neq \overline{F}_j$, for all $1 \leq i < j \leq m$.

For the sake of clarity, it is worth to mention that in this paper $\overline{F}$ it is not used to denote the complement of $F$ but as a simple alias for atoms. Moreover, we say that a flat $B$-sequence $\overline{F}_0 \ldots \overline{F}_m$ is decreasing if and only if $\Delta_T(\overline{F}_0) > \ldots > \Delta_T(\overline{F}_m)$. Flat (decreasing) $B$-sequence are the cornerstone of the following results for compass-structures, they constitute a suitable abstraction for the labelling of intervals $[x, y_1], \ldots, [x, y_n]$ which share the same beginning point. In particular, we will prove that, if we ignore the $k_i$ exponents the representation of a flat (decreasing) $B$-sequence is bounded by size of the input formula $\varphi$.

The following definition is a key piece for allowing us to abstract intervals/points of the type $[x, y_1], \ldots, [x, y_n]$ in a compass structure into flat (decreasing) $B$-sequences.

Definition 4. Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure for $\varphi$ and $0 \leq x \leq N$. We define the shading of $x$ in $\mathcal{G}$, written $\text{Sh}_\mathcal{G}^\mathcal{L}(x)$, as the sequence of atoms $\mathcal{L}(x, x) \ldots \mathcal{L}(x, N)$. 
The next lemma easily follows from Definition 3 and Definition 4. It allows us to abstract the shadings in a compass structure into flat $B$-sequences (proof omitted).

**Lemma 3.** Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure and $0 \leq x \leq N$. It holds that $\text{Sh}^B(x)$ is a $B$-sequence.

The next lemma is the missing piece that allows us to restrict our attention to decreasing flat $B$-sequences when abstracting shadings in a compass-structure.

**Lemma 4.** Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure (for a formula $\varphi$). For every $x \leq y < N$, we have that $\mathcal{L}(x, y) = \mathcal{L}(x, y + 1)$ iff $\mathcal{L}(x, y)$ is $B$-reflexive, $\mathcal{P}(x, y) = \mathcal{P}(x, y + 1)$, and $\text{Req}_D(x, y) = \text{Req}_D(x, y + 1)$.

**Proof.** The left-to-right direction is proved via a case analysis. If $\mathcal{P}(x, y) \neq \mathcal{P}(x, y + 1)$ or $\text{Req}_D(x, y) \neq \text{Req}_D(x, y + 1)$, then $\mathcal{L}(x, y) \neq \mathcal{L}(x, y + 1)$ immediately follows. If $\mathcal{L}(x, y)$ is $B$-irreflexive, then one gets a contradiction by observing that having two occurrences of the same $B$-irreflexive atom stacked one above the other violates the consistency of the compass structure (with respect to the $\rightarrow_B$ relation).

Let us prove now the right-to-left direction. Suppose, by way of contradiction, that $\mathcal{L}(x, y) \neq \mathcal{L}(x, y + 1)$. Then, there exists a formula $\psi \in \text{Cl}(\varphi)$ such that $\psi \in \mathcal{L}(x, y + 1)$ and $\neg \psi \in \mathcal{L}(x, y)$. By Lemma 1, for all $0 \leq x \leq y \leq N$, the truth of $\psi \in \mathcal{L}(x, y)$ is uniquely determined by the truth values of $\mathcal{P}(x, y)$, $\text{Req}_B(x, y)$, and $\text{Req}_D(x, y)$. By the assumption, we get $\text{Req}_B(x, y + 1) \supset \text{Req}_B(x, y)$. To reach the contradiction, we then proceed as in the proof of Lemma 2. 

The next corollary immediately follows from Lemma 1 and Lemma 4. It allows us to give a bound on the distinct atoms that may appear on a shading. More precisely, it states that the shading of each column $x$ in $\mathcal{G}$ is a decreasing flat $B$-sequence, and it gives a polynomial bound on the number of distinct atoms occurring in it.

**Corollary 1.** Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure (for a formula $\varphi$). Then, for all $0 \leq x \leq N$, $\text{Sh}^B(x)$ is a decreasing flat $B$-sequence $F_0^b \ldots F_m^b$, with $0 \leq m \leq 4|\varphi| + 1$. 

\[ \begin{array}{ccc}
H \circ & H \circ & H \circ \\
H \circ & H \circ & G \circ \\
H \circ & H \circ & H \circ \\
G \circ & G \circ & H \circ \\
F \circ & F \circ & F \circ \\
\vdots & \vdots & \vdots \\
\end{array} \]

(a) (b)

**Figure 12.** (a) Monotonicity of atoms along a column in a compass structure, together with a graphical account of the corresponding intervals and of how proposition letters and $B/D$ requests must behave. (b) An example of a violation of monotonicity.

...
5.2. A suitable equivalence relation over columns of a compass structure. By exploiting the above (finite) characterisation of columns, we can define a natural equivalence relation of finite index over columns: we say that two columns $x, x'$ are equivalent if and only if they feature the same set of atoms. Thanks to Corollary 1, if multiple copies of the same atom are present in a column, their occurrences are consecutive, and thus can be represented as blocks. Moreover, these blocks appear in the same order in equivalent columns because of the monotonicity of Req$_B$, Req$_D$, and Prop, the latter being forced by the homogeneity assumption (see Fig. 12.(a)).

In the following, we prove that equivalent columns can be totally ordered according to a given partial order relation over their shadings. Formally, for any two equivalent columns $x$ and $x'$, $\text{Sh}^G(x) < \text{Sh}^G(x')$ if and only if for every row $y$ the atom $\mathcal{L}(x', y)$ is equal to atom $\mathcal{L}(x, y)$ for some row $y'$, with $0 \leq y' \leq y$. Intuitively, this means that moving upward column $x'$ an atom cannot appear until it has appeared on column $x$. In Fig. 13.(a), we depict two equivalent columns that satisfy such a condition. In general, when moving upward, atoms on $x'$ are often "delayed" with respect to atoms in $x$, the limit case being when atoms on the same row are equal. In Fig. 13.(b), a violation of the condition (boxed atoms) is shown.

We are going to prove that this latter situation never occurs in a compass structure.

Let us now define an equivalence relation $\sim$ over decreasing flat $B$-sequences. Two decreasing flat $B$-sequences $\text{Sh}_B = F_0 \cdots F_m$ and $\text{Sh}'_B = G_0 \cdots G_{m'}$ are equivalent, written $\text{Sh}_B \sim \text{Sh}'_B$, if and only if $m = m'$ and, for all $0 \leq i \leq m$, $F_i = G_i$. This amounts to say that two decreasing flat $B$-sequences are equivalent if and only if they feature exactly the same sequence of atoms regardless of their exponents. Then, we can represent equivalence classes as decreasing flat $B$-sequences where each exponent is equal to one, e.g., the $B$-sequence $\overline{F}_0 \cdots \overline{F}_m$ belongs to the equivalence class $[\overline{F}_0 \cdots \overline{F}_m]$. Given an equivalence class $[\overline{F}_0 \cdots \overline{F}_m]$, and $0 \leq i \leq m$, we denote by $[\overline{F}_0 \cdots \overline{F}_m]^i$ the $i$th atom in its sequence, i.e., $[\overline{F}_0 \cdots \overline{F}_m]^i = \overline{F}_i$ for all $0 \leq i \leq m$. We also define a function $\text{next}$ that, given an equivalence class $[\overline{F}_0 \cdots \overline{F}_m]$, and one of its atom $\overline{F}_i$, returns the successor of $\overline{F}_i$ in the sequence $[\overline{F}_0 \cdots \overline{F}_m]$. (for $i = n$, it is undefined). It can be easily checked that $\sim$ is of finite index. From Corollary 1, it follows that its index is (roughly) bounded by $|\text{At}(\varphi)|^{|\varphi|+2} = 2^{|\varphi|+1}(4|\varphi|+2) = 2^{|\varphi|+6}|\varphi|+2$ (function $\Delta_1$ is deterministic, so $\Delta_1(F)$ can assume at most $4|\varphi| + 2$ distinct values).

\begin{figure}
\centering
\begin{tabular}{cc}
\begin{tabular}{cccc}
    & $\bullet$ & $\bullet$ & $\bullet$ \\
$F_1$ & $\rightarrow$ & $F_1$ & $\rightarrow$ \\
$F_2$ & $\rightarrow$ & $F_3$ & $\rightarrow$ \\
$F_3$ & $\rightarrow$ & $F_3$ & $\rightarrow$
\end{tabular} & \hspace{2cm} & 
\begin{tabular}{cccc}
    & $\bullet$ & $\bullet$ & $\bullet$ \\
$F_1$ & $\rightarrow$ & $\bullet$ & $\rightarrow$ \\
$F_2$ & $\rightarrow$ & $F_1$ & $\rightarrow$ \\
$F_3$ & $\rightarrow$ & $F_3$ & $\rightarrow$
\end{tabular}
\end{tabular}
\caption{Two equivalent columns that respect the order (a) and two equivalent columns that violate it (b).}
\end{figure}
Let $\text{Sh}_B = \overline{F}_0^k \ldots \overline{F}_m^k$ be a decreasing flat $B$-sequence. We define the length of $\text{Sh}_B$, written $|\text{Sh}_B|$, as $\sum_{0 \leq i \leq m} k_m$. A partial order $<$ over the elements of each equivalence class $[\text{Sh}_B]_h$ can be defined as follows.

**Definition 5.** Let $\text{Sh}_B = \overline{F}_0^k \ldots \overline{F}_m^k$ and $\text{Sh}_B' = \overline{F}_0^k \ldots \overline{F}_m^k$ be two equivalent decreasing flat $B$-sequences. We say that $\text{Sh}_B$ is dominated by $\text{Sh}_B'$, written $\text{Sh}_B < \text{Sh}_B'$, if and only if (i) $|\text{Sh}_B| > |\text{Sh}_B'|$ and, (ii) for all $0 \leq i \leq m$, $\sum_{0 \leq j \leq i} k_j \leq (|\text{Sh}_B| - |\text{Sh}_B'|) + \sum_{0 \leq j \leq h_j} k_j$.

Let us consider, for instance, the four equivalent decreasing flat $B$-sequences shown in Figure 13, from left to right they are $\text{Sh}_B^0 = F_1 F_2 F_3^2 F_4$, $\text{Sh}_B^1 = F_1 F_2 F_3^2 F_4$, $\text{Sh}_B^2 = F_1 F_2 F_3^2 F_4$, and $\text{Sh}_B^3 = F_1 F_2 F_3 F_4^2$ (for the sake of clarity the exponent 1 is omitted). Let us consider first $\text{Sh}_B^0$ and $\text{Sh}_B^1$, by condition (i) of Definition 5 we have that the only possible domination relation may be $\text{Sh}_B^0 < \text{Sh}_B^1$. Then, let us check if condition condition (ii) of Definition 5 holds for $\text{Sh}_B^0$ and $\text{Sh}_B^1$. In general, one possible possible intuition may be given by the following representation of flat shading. Let us assume that $\text{Sh}_B^0$ is equivalent to $\text{Sh}_B^1$ then in order to prove $\text{Sh}_B^0 < \text{Sh}_B^1$ it suffice to consider the alignment of $\text{Sh}_B^0$ (i.e., the shorter sequence) as a suffix of $\text{Sh}_B^0$. Such an alignment is obtained by prefixing $\text{Sh}_B^1$ with a word of suitable blank symbol ‘_’ with length $|\text{Sh}_B| - |\text{Sh}_B^1|$. Then, in our example, we have that the required alignment of $\text{Sh}_B^0$ is $\text{Sh}_B^1 = F_1 F_2 F_4 F_4$. Now we have that $\text{Sh}_B^0 < \text{Sh}_B^1$ if and only if the first occurrence of each atom $F_i$ in $\text{Sh}_B^0$ does not occur in a strictly smaller position in $\text{Sh}_B^1$. In our example, we have that:

- $F_1$ occurs for the first time at position 0 in $\text{Sh}_B^0$ and at position 2 in $\text{Sh}_B^1$;
- $F_2$ occurs for the first time at position 1 in $\text{Sh}_B^0$ and at position 3 in $\text{Sh}_B^1$;
- $F_3$ occurs for the first time at position 2 in $\text{Sh}_B^0$ and at position 4 in $\text{Sh}_B^1$;
- $F_4$ occurs for the first time at position 5 in $\text{Sh}_B^0$ and at position 6 in $\text{Sh}_B^1$.

Then, since all the first occurrences of atoms in $\text{Sh}_B^1$ are greater or equal than the corresponding ones in $\text{Sh}_B^0$ we may conclude that $\text{Sh}_B^0 < \text{Sh}_B^1$. On the other hand, we have that $\text{Sh}_B^2$ is equivalent to $\text{Sh}_B^3$. However, $\text{Sh}_B^2 \not< \text{Sh}_B^3$ since, if we consider the alignment $\text{Sh}_B^2 = F_1 F_2 F_3 F_4$, we have that atom $F_4$ occurs for the first time at position 5 in $\text{Sh}_B^2$ but at position 6 in $\text{Sh}_B^3$. In the following we will prove that the latter scenario cannot occur in the case of compass structures.

Finally, we introduce a notation for atom retrieval. Let $\text{Sh}_B = \overline{F}_0^k \ldots \overline{F}_m^k$ be a decreasing flat $B$-sequence and $0 \leq i \leq |\text{Sh}_B|$. We put $\text{Sh}_B[i] = \overline{F}_j$, where $j$ is such that $\sum_{0 \leq j < i} k_j < i \leq \sum_{0 \leq j \leq i} k_j$. The next lemma constrains the relationships between pairs of equivalent shadings (decreasing flat $B$-sequences) appearing in a compass structure.

**Lemma 5.** Let $G = (G_N, L)$ be a compass structure. For every pair of columns $0 \leq x < x' \leq N$ such that $\text{Sh}^G(x) \sim \text{Sh}^G(x')$, it holds that $\text{Sh}^G(x) < \text{Sh}^G(x')$.

**Proof.** Let $\Delta = x' - x$, $\text{Sh}^G(x) = \overline{F}_0^k \ldots \overline{F}_m^k$, and $\text{Sh}^G(x') = \overline{F}_0^k \ldots \overline{F}_m^k$. Let us suppose by contradiction that $\text{Sh}^G(x) \not< \text{Sh}^G(x')$. From $\text{Sh}^G(x) \sim \text{Sh}^G(x')$ we have that both $B$-sequences features the same atoms in the same order, they can differ just in their numerosity (i.e., exponents). From $\text{Sh}^G(x) \sim \text{Sh}^G(x')$, $\text{Sh}^G(x) \not< \text{Sh}^G(x')$ and since $|\text{Sh}^G(x)| < |\text{Sh}^G(x')|$ ($x'$ is closer to $N$ than $x$ and thus is shorter) there exists an index $0 \leq i \leq N - x'$ such that one of the following conditions holds:
(1) $\text{Sh}^G(x)[\Delta + i] \cap \text{Prop} \neq \text{Sh}^G(x')[i] \cap \text{Prop}$;
(2) $\text{Req}_D(\text{Sh}^G(x)[\Delta + i]) \neq \text{Req}_D(\text{Sh}^G(x')[i])$;
(3) $\text{Req}_D(\text{Sh}^G(x)[\Delta + i]) = \text{Req}_D(\text{Sh}^G(x')[i])$ and $\text{Req}_D(\text{Sh}^G(x)[\Delta + i])$ is $B$-irreflexive;
(4) $\text{Req}_D(\text{Sh}^G(x)[\Delta + i]) \subseteq \text{Req}_D(\text{Sh}^G(x')[i])$.

The above cases stem from the fact that we are claiming that for a certain index $i$ there exists $j$ such that $\text{Sh}^G(x')[i] = F_j$ and $\text{Sh}^G(x)[\Delta + i] = F_{j-1}$ and thus $\text{Sh}^G(x')[i] \neq \text{Sh}^G(x)[\Delta + i]$. This is the case, for instance, of $x$ and $x'$ in Figure 13 for which we have $\Delta = 2 \text{Sh}^G(x')[3] = F_4$ and $\text{Sh}^G(x)[5] = F_3$.

Let us assume that $i$ is the minimum index which satisfies one of the above conditions.

In the following, we will assume that $\text{Sh}^G(x')[i] = F_j$ and $\text{Sh}^G(x)[\Delta + i] = F_{j-1}$ for some $0 < j \leq m$.

Before proving that in each case we reach a contradiction, let us spend a few more words on how such cases are derived. Since $\text{Sh}^G(x) \sim \text{Sh}^G(x')$ but $\text{Sh}^G(x) \neq \text{Sh}^G(x')$ and since $|\text{Sh}^G(x)| < |\text{Sh}^G(x')|$ we have a situation analogous to the one depicted in Figure 13.(b). In particular, we have that $\text{Sh}^G(x)$ starts “before” $\text{Sh}^G(x')$ by unraveling the common sequence of atoms $F_0 \ldots F_m$ (which is $F_1 \ldots F_4$ in Figure 13.(b)). Due to the fact that $\text{Sh}^G(x) \sim \text{Sh}^G(x')$, When $\text{Sh}^G(x')$ starts it must unravel the same sequence an then it is easy to see that for either for $i$ there exists $k_i \leq k$ such that $\text{Sh}^G(x')[i] = F_{k_i}$ and $\text{Sh}^G(x)[i + \Delta] = F_{k_i}$, i.e., $\text{Sh}^G(x')$ “waits” for $\text{Sh}^G(x)$ before showing any new atom in the sequence $F_0 \ldots F_m$, or there exists $i$ and $k_i > k$ such that $\text{Sh}^G(x')[i] = F_{k_i}$ and $\text{Sh}^G(x)[i + \Delta] = F_{k_i}$, it is the case with $i = 3$, $k = 3$, and $k_i = 4$ in Figure 13.(b)). The first case it is a sufficient condition for concluding that $\text{Sh}^G(x) \equiv \text{Sh}^G(x')$ holds while in the latter case $\text{Sh}^G(x) \neq \text{Sh}^G(x')$ holds. In the latter case, if we take the minimal $i$ that satisfy the condition it is easy to see that $k_i = k + 1$. Then, we have $F_{k_i} \rightarrow_B F_k$ but $F_{k_i} \neq F_k$. The cases (1)-(4) above are all the possible way in which we may have $F_{k_i} \rightarrow_B F_k$ but $F_{k_i} \neq F_k$ with the additional constraint that $F_{k_i} = \mathcal{L}(x, x' + i)$ and $F_k = \mathcal{L}(x, x + \Delta + i)$ which implies, since $[x, x' + i]$ finishes $[x, x + \Delta + i]$ and we are in a compass structures, that $\text{Prop} \cap \mathcal{L}(x, x + \Delta + i) \supseteq \text{Prop} \cap \mathcal{L}(x, x + \Delta + i)$ must satisfy the homogeneity condition and $\text{Req}_D(\mathcal{L}(x, x + \Delta + i)) \subseteq \text{Req}_D(\mathcal{L}(x, x + \Delta + i))$.

For case (1) we have that since $\text{Sh}^G(x)[i] = \mathcal{L}(x, x + \Delta + i)$, $\text{Sh}^G(x')[i] = \mathcal{L}(x', x' + i)$, and $x + \Delta + i = x' + i$ we have, by definition of compass structure, $\mathcal{P}(x, x + \Delta + i) \subseteq \mathcal{P}(x', x' + i)$ but since $k_i > k$ we will have that there exists $i' > i$ for which $\mathcal{P}(x, x + \Delta + i) = \mathcal{P}(x', x' + i)$ and since $[x, x + \Delta + i]$ begins $[x, x + \Delta + i]$ we have $\mathcal{P}(x, x + \Delta + i) \supseteq \mathcal{P}(x, x + \Delta + i) = \mathcal{P}(x', x' + i)$ and thus $\mathcal{P}(x, x + \Delta + i) = \mathcal{P}(x', x' + i)$ (contradiction).

For case (2) we have that since $[x, x' + i]$ is a proper suffix of $[x, x + \Delta + i]$ we have that all the proper sub-intervals of $[x', x' + i]$ are also proper sub-intervals of $[x, x + \Delta + i]$. Then, since $\mathcal{G}$ is a compass structure we have $\text{Req}_D(\text{Sh}^G(x)[\Delta + i]) \supseteq \text{Req}_D(\text{Sh}^G(x')[i])$. From $\text{Req}_D(\text{Sh}^G(x)[\Delta + i]) \neq \text{Req}_D(\text{Sh}^G(x')[i])$ we have $\text{Req}_D(\text{Sh}^G(x)[\Delta + i]) \supset \text{Req}_D(\text{Sh}^G(x')[i])$ which means that $\text{Req}_D(\mathcal{F}_{j-1}) \supset \text{Req}_D(\mathcal{F}_j)$, thus there exists $\psi \in \text{Req}_D(\mathcal{F}_{j-1}) \setminus \text{Req}_D(\mathcal{F}_j)$. This translates into the intervals of a compass structure in $(D)\psi \in \mathcal{L}(x', x' + i)$ and $[D]_{\sim} \psi \in \mathcal{L}(x, x + \Delta + i)$. Since $\mathcal{G}$ is a compass structure we have that there exists a proper sub-interval $x' < x < x' + i$ we have that
$[x^n, x^m]$ is also a proper sub-interval of $[x, x + \Delta + i]$ then since $[D] \neg \psi \in \mathcal{L}(x, x + \Delta + i)$ we have $\neg \psi \in \mathcal{L}(x^n, x^m)$ (contradiction).

Let us assume that $\text{Req}_B(\text{Sh}^G(x)\{\Delta + i\}) = \text{Req}_B(\text{Sh}^G(x')\{i\})$ and $\text{Req}_B(\text{Sh}^G(x)\{\Delta + i\})$ is $B$-irreflexive (case 3). Then, from $\text{Req}_B(\mathcal{F}_j) = \text{Req}_B(\mathcal{F}_{j-1})$, $\mathcal{F}_j \nrightarrow \mathcal{F}_{j-1}$ and $\mathcal{F}_j$ $B$-irreflexive we have that $\mathcal{F}_{j-1}$ is $B$-reflexive. From $\text{Sh}^G(x') \sim \text{Sh}^G(x)$ we have $\text{Sh}^G(x)\{\Delta + i - 1\} = \mathcal{F}_{j-1}$ thus we have $\text{Sh}^G(x)\{\Delta + i\} \cap \text{Prop} = \text{Sh}^G(x')\{i\} \cap \text{Prop} = \text{Sh}^G(x)\{\Delta + i - 1\} \cap \text{Prop}$ and $\text{Req}_D(\text{Sh}^G(x)\{\Delta + i\}) = \text{Req}_D(\text{Sh}^G(x')\{i\}) = \text{Req}_D(\text{Sh}^G(x)\{\Delta + i - 1\})$ thus we can apply Lemma 4 we have $\text{Sh}^G(x)\{\Delta + i\} = \mathcal{F}_{j-1}$ (contradiction). Let us assume $\text{Req}_B(\text{Sh}^G(x)\{\Delta + i\}) \in \text{Req}_B(\text{Sh}^G(x')\{i\})$ (case 4) then from Lemma 2 we have that one among $\mathcal{F}_j$ and $\mathcal{F}_{j-1}$ is $B$-irreflexive. If $\mathcal{F}_j$ is $B$-irreflexive and $\mathcal{F}_{j-1}$ is $B$-reflexive we can use exactly the previous argument to prove a contradiction. If $\mathcal{F}_j$ is $B$-irreflexive and $\mathcal{F}_{j-1}$ is $B$-irreflexive one of the above conditions holds for $i - 1$ violating the minimality of $i$. It remains the case in which $\mathcal{F}_{j-1}$ is $B$-irreflexive and $\mathcal{F}_j$ is $B$-reflexive but in such case we have that one of the above conditions holds for $i - 1$ violating the minimality of $i$ too (contradiction). \qed

5.3. A spatial property of columns in homogeneous compass structures. In this section, we provide a very strong characterization of the rows of a compass structure by making use of a covering property, depicted in Fig. 14, which states that the sequences of atoms on two equivalent columns $x < x'$ must respect a certain order. To start with, we define the intersection of row $y$ and column $x$, with $0 \leq x \leq y$, as the pair consisting of the equivalence class of $x$ and the labelling of $(x, y)$. We associate with each point $(x, y)$ its intersection as well as the set $\mathbb{S}_\sim(x, y)$ of intersections of row $y$ with columns $x'$, for all $x < x' \leq y$. Let us denote by $fp(x, y)$ ($fp$ stands for fingerprint) the triplet associated with point $(x, y)$.

We prove that if a point $(x, y)$ has $n + 1$ columns ($x < x_0 < \ldots < x_n \leq y$ on its right (with $n$ large enough, but polynomially bounded by $|\varphi|$) such that, for all $0 \leq i \leq n$, $fp(x, y)$ is equal to $fp(x, y)$, then the sequence of atoms that goes from $(x, y)$ to $(x, N)$ is exactly the same as the sequence of atoms that goes from $(x_0, y)$ to $(x_0, N)$.

Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure and let $0 \leq x \leq y$. We define $\mathbb{S}_\sim(x, y)$ as the set $\{(\text{Sh}^G(x')\{\sim\}, \mathcal{L}(x', y)) : x' > x\}$. $\mathbb{S}_\sim(x, y)$ collects the equivalence classes of $\sim$ which are witnessed to the right of $x$ on row $y$ plus a “pointer” to the “current atom”, that is, the atom they are exposing on $y$. If $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ is homogeneous (as in our setting), for all $0 \leq x \leq y \leq N$, the number of possible sets $\mathbb{S}_\sim(x, y)$ is bounded by $2^{2^{4|\varphi^2|+6|\varphi^1|+2|\varphi^0|+3}} = 2^{2^{4|\varphi^2|+7|\varphi^1|+3}}$, that is, it is doubly exponential in the size of $|\varphi|$.

The next lemma constrains the way in which two columns $x, x'$, with $x < x'$ and $\text{Sh}^G(x) \sim \text{Sh}^G(x')$, evolve from a given row $y$ on when $\mathbb{S}_\sim(x, y) = \mathbb{S}_\sim(x', y)$.

**Lemma 6.** Let $\mathcal{G} = (\mathbb{G}_N, \mathcal{L})$ be a compass structure and let $0 \leq x < x' \leq y \leq N$. If $fp(x, y) = fp(x', y)$ and $y'$ is the smallest point greater than $y$ such that $\mathcal{L}(x, y') \neq \mathcal{L}(x, y)$, if any, and $N$ otherwise, then, for all $y \leq y'' \leq y'$, $\mathcal{L}(x, y'') = \mathcal{L}(x', y'')$.

**Proof.** Let $\overline{y}$ be the minimum point $\overline{y} > y$ such that $\mathcal{L}(x', y) \neq \mathcal{L}(x', \overline{y})$. Let us assume by contradiction that $\overline{y} \neq y'$. By Lemma 5 we have that $\overline{y} > y'$. Let $\text{Sh}^G(x) = F_0^{k_0} \ldots F_m^{k_m}$.
and let $0 \leq i < m$ be the index such that $\text{Sh}_{G}^{i}(x)[y - x] = \text{Sh}_{G}^{i}(x)[y - x'] = \overline{F}_{i}$. Then we have $\mathcal{L}(x, y) = \overline{F}_{i} = \mathcal{L}(x', y), \mathcal{L}(x, y') = \overline{F}_{i+1}$, and $\mathcal{L}(x', y') = \overline{F}_{i}$. Moreover for every $y \leq y'' < y'$ we have $\mathcal{L}(x, y'') = \mathcal{L}(x', y'') = \overline{F}_{i}$, then $\overline{F}_{i}$ is $B$-reflexive. Let us notice that if $\mathcal{P}(x, y'-1) = \mathcal{P}(x', y') = \overline{F}_{i} \cap \text{Prop}$ then we have that $\mathcal{P}(x, y') = \mathcal{P}(x', y')$. Since $\mathcal{L}(x, y'-1)$ is $B$-reflexive we have that $\text{Req}_{D}(\mathcal{L}(x, y')) \supset \text{Req}_{D}(\mathcal{L}(x', y' - 1)) = \text{Req}_{D}(\mathcal{L}(x', y'))$ otherwise conditions for Lemma 4 apply and $\mathcal{L}(x, y') = \mathcal{L}(x, y'-1)$ (contradiction). This means that there exists $x < \overline{x} < x'$ such that $\psi \in (\mathcal{L}(\overline{x}, y'-1) \cap \text{Req}_{D}(\mathcal{L}(x, y')) \setminus \text{Req}_{D}(\mathcal{L}(x', y')))$. And for every $x' \leq x'' \leq y' - 1$ we have $\psi \notin \mathcal{L}(x', y')$. The simpler case is when $y' = y + 1$. In such a case from $S_{\sim}(x, y) = S_{\sim}(x', y)$ we have that there exists $\overline{x}' > x'$ such that $\mathcal{L}(\overline{x}', y) = \mathcal{L}(\overline{x}, y)$ (contradiction). Let us consider now the case in which $y > y + 1$. Since $\sim \psi \in \text{Box}_{D}(\mathcal{L}(x, y'-1))$ we have that $\psi \notin \mathcal{L}(x', y')$ for every $x < x'' \leq y'' < y' - 1$. Two cases arise:

- there exists $y \leq y'' < y' - 1$ such that $\mathcal{L}(\overline{x}, y'')$ is $B$-reflexive. If it is the case since $\overline{F}_{i} \cap \text{Prop} = \mathcal{P}(x, y') \subseteq \mathcal{P}(\overline{x}, y') \subseteq \mathcal{P}(x', y') = \overline{F}_{i} \cap \text{Prop}$ and $\text{Req}_{D}(\overline{F}_{i}) = \text{Req}_{D}(\mathcal{L}(x, y')) \supset \text{Req}_{D}(\mathcal{L}(\overline{x}, y')) \supset \text{Req}_{D}(\mathcal{L}(x', y')) = \text{Req}_{D}(\mathcal{L}(x', y'))$ for every $y \leq y'' \leq y' - 1$ we have $\mathcal{P}(x, y') = \mathcal{P}(\overline{x}, y') = \mathcal{P}(x', y')$ and $\text{Req}_{D}(\mathcal{L}(\overline{x}, y')) = \text{Req}_{D}(\mathcal{L}(x', y')) = \text{Req}_{D}(\mathcal{L}(x', y'))$ for every $y'' \leq y'' \leq y' - 1$. Then for Lemma 4 we have that $\mathcal{L}(\overline{x}, y'-1) = \mathcal{L}(\overline{x}, y'-2) = \ldots = \mathcal{L}(\overline{x}, y''')$ this means that $\mathcal{L}(\overline{x}, y'-1)$ is not the first atom featuring $\psi$ on the column $\overline{x}$ (contradiction);

- for every $y \leq y'' < y' - 1$ we have that $\mathcal{L}(\overline{x}, y'')$ is $B$-irreflexive. Then, from $S_{\sim}(x, y) = S_{\sim}(x', y)$ there exists $\overline{x}' > x'$ such that $\text{Sh}_{G}^{\overline{x}'}(x') \sim \text{Sh}_{G}^{\overline{x}'}(\overline{x})$ and $\mathcal{L}(\overline{x}, y) = \mathcal{L}(\overline{x}, y)$.

Let us observe that, by definition of $B$-sequence, for every $B$-sequence $F_{0} \ldots F_{n}$ and for every $1 \leq i \leq n$ if $F_{i}$ is $B$-irreflexive then $h_{i} = 1$ (i.e., $B$-irreflexive atoms are unique in every $B$-sequence). Then, we have that for every $y \leq y'' \leq y' - 1$ we have $\mathcal{L}(\overline{x}, y'') = \mathcal{L}(\overline{x}', y'')$ and thus $\psi \in \mathcal{L}(\overline{x}', y'')$ 1 this implies $\psi \in \text{Req}_{D}(\mathcal{L}(x', y'))$ (contradiction).

From Lemma 6, the next corollary follows.

---

**Figure 14.** A graphical account of the behaviour of covered points. We have that $x$ is covered by $x_{0} \ldots x_{n}$ on row $y$ and thus the labelling of points on column $x$ above $(x, y)$ is exactly the same of the correspondent points on column $x_{0}$ above $(x_{0}, y)$, that is, $\mathcal{L}(x, y) = \mathcal{L}(x_{0}, y)$. For $x_{i} y_{i} \mathcal{P}$.
Corollary 2. Let $G = (G_N, L)$ be a compass structure and let $0 \leq x < x' \leq y \leq N$. If $fp(x, y) = fp(x', y)$ and $y'$ is the smallest point greater than $y$ such that $L(x, y') = L(x, y)$, if any, and $N$ otherwise, then, for every pair of points $\bar{x}, \bar{x}'$, with $x < \bar{x} < x' < \bar{x}'$, with $L(\bar{x}, y) = L(\bar{x}', y)$ and $Sh_{\bar{x}}(\bar{x}) \sim Sh_{\bar{x}'}(\bar{x}')$ (with $\bar{x} = \bar{x}'$), it holds that $L(\bar{x}, y') = L(\bar{x}', y')$, for all $y \leq y'' \leq y'$.

The above results lead us to the identification of those points $(x, y)$ whose behaviour perfectly reproduces that of a number of points $(x', y')$ on their right with $fp(x, y) = fp(x', y)$. These points $(x, y)$, like all points “above” them, are useless with respect to fulfilment in a compass structure. We call them covered points.

Definition 6. Let $G = (G_N, L)$ be a compass structure and $0 \leq x \leq y \leq N$. We say that $(x, y)$ is covered if there exist $n + 1 = \Delta_{\uparrow} (L(x, y))$ distinct points $x_0 \ldots x_n \leq y$, with $x \leq x_0$, such that for all $0 \leq i \leq n$, $fp(x, y) = fp(x_i, y)$. In such a case, we say that $x$ is covered by $x_0 < \ldots < x_n$ on $y$.

Lemma 7. Let $G = (G_N, L)$ be a compass structure and let $x, y$, with $0 \leq x \leq y \leq N$, be two points such that $x$ is covered by points $x_0 < \ldots < x_n$ on $y$. Then, for all $y \leq y' \leq N$, it holds that $Sh_{\uparrow}(x)[y] = Sh_{\uparrow}(x_0)[y']$.

Proof. Let $Sh_{\uparrow}(x, y) = F_0 \cdots F_m$, the proof is by induction on $n = \Delta_{\uparrow} (L(x, y))$. If $n = 0$ we have that $L(x, y) = F_m$, since $L(x, y) = L(x_0, y)$ we have $F_m = L(x_0, y)$. Since we are on the last atom of the sequence $Sh_{\uparrow}(x, y)$ and $Sh_{\uparrow}(x, y) \sim Sh_{\uparrow}(x_0, y)$ we have $L(x, y') = L(x_0, y')$ for every $y < y' \leq N$. If $n > 0$, let $L(x, y) = F_i$ with $0 \leq i < m$ (if $i = m$ we can apply the same way of the inductive basis), by Lemma 6 we have that there exists a single minimum point $y'' > y$ for which $L(x, y') = L(x_0, y') = \cdots = L(x_n, y') = F_{i+1}$ and thus for every $y \leq y'' \leq y'$ we have $L(x, y'') = L(x_0, y'')$. Moreover, for Corollary 2 we have that for every $x' > x_n$ such that $Sh_{\uparrow}(x', y') \sim Sh_{\uparrow}(x')$ and there exists $x < x'' < x_n$ such that $Sh_{\uparrow}(x', y') \sim Sh_{\uparrow}(x', y)$ and $L(x', y) = L(x', y')$, then we have that $L(x', y') = L(x'', y')$. Then we have $S_{\downarrow}(x, y') = S_{\downarrow}(x', y')$ for every $0 \leq i < n$ (every one but $x_n$). Since $\Delta_{\uparrow} (F_i) = \Delta_{\uparrow} (F_{i+1})$ we can apply the inductive hypothesis since $x$ is covered by $x_0 < \ldots < x_{n-1}$ on $y'$ and we have that for every $y' \leq y'' \leq N$ we have $L(x, y') = L(x_0, y'')$. 

In Figure 15, we give an intuitive account of the notion of covered point and of the statement of Lemma 7. First of all, we observe that, since $S_{\downarrow}(x, y) = S_{\downarrow}(y, y)$ and, for all $0 \leq j, j' \leq n$, it holds that $(Sh_{\downarrow}(x_j), L(x_j, y)) = (Sh_{\downarrow}(x_j), L(x_j, y))$, there exists $x_n < \hat{x} \leq y$ such that $(Sh_{\downarrow}(x_n), L(x_n, y)) = (Sh_{\downarrow}(\hat{x}), L(\hat{x}, y))$, and $\hat{x}$ is the smallest point greater than $x_n$ that satisfies such a condition. Now, it may happen that $S_{\downarrow}(x, y) \supset S_{\downarrow}(\hat{x}, y)$, and all points $\bar{x}' > x_n$ with $(Sh_{\downarrow}(\bar{x}'), L(\bar{x}', y)) = (Sh_{\downarrow}(\bar{x}'), L(\bar{x}', y))$, for some $x < \bar{x} < x_n$, are such that $x_n < \bar{x} < \hat{x}$. Then, it can be the case that, for all $0 \leq i < n$, $(y_i, y') = F_{i+1}$, as all points $(\bar{x}_i, y')$ satisfy some $D$-request $\psi$ that only belongs to $L(\bar{x}, y' - 1)$. In such a case, as shown in Figure 15, $L(\hat{x}, y') = F_i$, because for all points $(\hat{x}_i, y')$, with $\hat{x} < \hat{x}_i \leq y' < \hat{y} 
\psi \in L(\hat{x}, y')$. Hence, $(\uparrow_{\downarrow}(x_n), F_i) \in S_{\downarrow}(x_n, y')$ for all $0 \leq j < n$, but $(\uparrow_{\downarrow}(x_n), F_{i+1}) \notin S_{\downarrow}(x_n, y')$. Then, by applying Corollary 2, we have that $S_{\downarrow}(x_0, y') = S_{\downarrow}(x_{n-1}, y')$. Since $\Delta_{\downarrow}(F_{i+1}) < \Delta_{\downarrow}(F_i) (\neq n)$, it holds that $\Delta_{\downarrow}(F_{i+1}) \leq n - 1$.

The same argument can then be applied to $x, x_0, \ldots, x_{n-1}$ on $y'$, and so on.
In this section, by exploiting the properties proved in Section 5, we show that the problem of checking whether a $\text{BD}_{\text{hom}}$ formula $\varphi$ is satisfied by some homogeneous model can be decided in exponential space. First, by means of a suitable small model theorem, we prove that either $\varphi$ is unsatisfiable or it is satisfied by a model (a compass structure) of at most doubly-exponential size in $|\varphi|$; then, we show that this model of doubly-exponential size can be guessed in single exponential space.

**Theorem 2.** Let $\varphi$ be a $\text{BD}_{\text{hom}}$ formula. The problem of deciding whether or not it is satisfiable belongs to EXPSPACE.

The proof of Theorem 2 follows from Corollary 3, Lemma 8, and Lemma 9 below.

First of all, thanks to the property proved in Section 5.3, we know that, for every row $y$, there is a finite set of columns $C_y = \{x_1, \ldots, x_n\}$ that behave pairwise differently for the portion of the compass structure above $y$. This means that each column $0 \leq x \leq y$, with $x \notin C_y$, behaves exactly as some $x_i \in C_y$ above $y$; that is, for all $y' > y$, $L(x, y') = L(x_i, y')$. We prove that $n$ is bounded by $|\varphi|$, from which it immediately follows that, in any large enough model, there are two rows $y$ and $y'$, with $\text{Sh}^G(x)[y'] = \text{Sh}^G(x_0)[y'_0]$. Then, we can suitably contract the model into one whose $Y$-size is $y' - y$ shorter. By (possibly) repeatedly applying such a contraction, we obtain a model whose $Y$-size satisfies a doubly exponential bound. To complete the proof, it suffices to show that there exists a procedure that checks whether or not such a model exists in exponential space.

By exploiting Lemma 7, we can show that, for each row $y$, the cardinality of the set of columns $x_1, \ldots, x_m$ which are not covered on $y$ is exponential in $|\varphi|$. Then, the sequence of triplets for non-covered points that appear on $y$ is bounded by an exponential value on $|\varphi|$. It follows that, in a compass structure of size more than doubly exponential in $|\varphi|$, there exist two rows $y, y'$, with $y < y'$, such that the sequences of the triplets for non-covered points that appear on $y$ and $y'$ are exactly the same. This allows us to apply a “contraction” between $y$ and $y'$ on the compass structure.

An example of how contraction works is given in Figure 16. First of all, notice that rows 7 and 11 feature the same sequences for triplets of non-covered points, and that, on any row,
each covered point is connected by an edge to the non-covered point that “behaves” in the same way. More precisely, we have that column 2 behaves as column 4 between \( y = 7 \) and \( y' = 15 \), columns 3, 5, and 7 behave as column 8 between \( y = 11 \) and \( y' = 15 \), and column 4 behaves as column 6 between \( y = 11 \) and \( y' = 15 \). The compass structure in Figure 16.(a) can thus be shrunk into the compass structure in Figure 16.(b), where each column of non-covered points \( x \) on \( y' \) is copied above the corresponding non-covered point \( x' \) on \( y \). Moreover, the column of a non-covered point \( x \) on \( y' \) is copied over all the points which are covered by the non-covered point \( x' \) corresponding to \( x \) on \( y \). This is the case with point 2 in Figure 16.(b) which takes the new column of its “covering” point 4. The resulting compass structure is \( y' - y \) shorter than the original one, and we can repeatedly apply such a contraction until we achieve the desired bound.

The next corollary, which easily follows from Lemma 7, turns out to be crucial for the proof of Theorem 2. Roughly speaking, it states that the property of “being covered” propagates upward.

**Corollary 3.** Let \( \mathcal{G} = (G_N, \mathcal{L}) \) be a compass structure. Then, for every covered point \((x, y)\), it holds that, for all \( y \leq y' \leq N \), point \((x, y')\) is covered as well.

From Corollary 3, it immediately follows that, for every covered point \((x, y)\) and every \( y \leq y' \leq N \), there exists \( y'' > x \) such that \( \mathcal{L}(x', y') = \mathcal{L}(x, y') \). Hence, for all \( \overline{x}, \overline{y} \), with \( \overline{x} < x \leq y'' < \overline{y} \), and any D-request \( \psi \in \text{Req}_D(\mathcal{L}([x, y])) \cap \text{Obs}_D(\mathcal{L}(x, y)) \), we have that \( \psi \in \mathcal{L}(x', y) \), with \( x' > x \). This allows us to conclude that if \((x, y)\) is covered, then all points \((x, y')\), with \( y' \geq y \), are “useless” from the point of view of D-requests.

Let \( \mathcal{G} = (G_N, \mathcal{L}) \) be a compass structure and \( 0 \leq y \leq N \). We define the set of witnesses of \( y \) as the set \( \text{Wit}_G(y) = \{x : (x, y) \text{ is not covered}\} \). Corollary 3 guarantees that, for any row \( y \), the shading \( \text{Sh}^y(x) \) and the labelling \( \mathcal{L}(x, y) \) of witnesses \( x \in \text{Wit}_G(y) \) are sufficient, bounded, and unambiguous pieces of information that one needs to maintain about \( y \).

Given a compass structure \( \mathcal{G} = (G_N, \mathcal{L}) \) and \( 0 \leq y \leq N \), we define the row blueprint of \( y \) in \( \mathcal{G} \), written \( \text{Row}_G(y) \), as the sequence \( \text{Row}_G(y) = ([\text{Sh}^0_B]_-, F_0) \ldots ([\text{Sh}^m_B]_-, F_m) \) such that \( m + 1 = |\text{Wit}_G(y)| \) and there exists a bijection \( b : \text{Wit}_G(y) \to \{0, \ldots, m\} \) such that, for every \( x \in \text{Wit}_G(y) \), it holds that \( \text{Sh}_B^x \in [\text{Sh}^b_x]_- \) and \( F_b(x) = \mathcal{L}(x, y) \).\) Given a compass structure \( \mathcal{G} = (G_N, \mathcal{L}) \), the next lemma allows us to prove the existence of a smaller compass structure \( \mathcal{G}' = (G_{N'}, \mathcal{L}') \) with \( N' < N \) if \( \mathcal{G} \) features two distinct rows \( y < y' \) which share the same blueprint.

**Lemma 8.** Let \( \mathcal{G} = (G_N, \mathcal{L}) \) be a compass structure. If there exist two points \( y, y' \), with \( 0 \leq y < y' \leq N \), such that \( \text{Row}_G(y) = \text{Row}_G(y') \), then there exists a compass structure \( \mathcal{G}' = (G_{N'}, \mathcal{L}') \) with \( N' = N - (y' - y) \).

**Proof.** From \( \text{Row}_G(y) = \text{Row}_G(y') \), by composing bijections, we have that there exists a bijection \( \overline{b} : \text{Wit}_G(y) \to \text{Wit}_G(y') \) such that, for every \( x \in \text{Wit}_G(y) \), we have that \( \mathcal{L}(x, y) = \mathcal{L}(\overline{b}(x), y') \), \( \text{Sh}_B^x \sim \text{Sh}_B^{\overline{b}(x)} \), and \( S_{\leftrightarrow}(x, y) = S_{\leftrightarrow}(\overline{b}(x), y') \). Moreover, for every \( x, x' \in \text{Wit}_G(y) \), we have that \( x \leq x' \iff \overline{b}(x) \leq \overline{b}(x') \). For every point \( 0 \leq x \leq y \), we define the function \( \text{Closest}_\text{wit}(x) : \{0, \ldots, y\} \to \{0, \ldots, y\} \) as follows:

\[ \text{Closest}_\text{wit}(x) = \begin{cases} x & \text{if } x \in \text{Wit}_G(y) \\ \min \left\{ x' : x' > x, x' \in \text{Wit}_G(y), \mathcal{L}(x', y) = \mathcal{L}(x, y), \text{Sh}_B^{x'} \sim \text{Sh}_B^x, S_{\leftrightarrow}(x') = S_{\leftrightarrow}(x) \right\} & \text{otherwise.} \end{cases} \]
Let \( \delta = \overline{y} - y \). We define \( \mathcal{L}' \) as follows:

1. \( \mathcal{L}'(\overline{x}, \overline{y}) = \mathcal{L}(x, y) \) for all \( 0 \leq \overline{x} \leq \overline{y} \leq y \);
2. \( \mathcal{L}'(\overline{x}, \overline{y}) = \mathcal{L}(\overline{x} + \delta, \overline{y} + \delta) \) for all \( y < \overline{x} \leq \overline{y} \leq N \);
3. \( \mathcal{L}'(\overline{x}, \overline{y}) = \mathcal{L}(\text{Closest}_{\text{wit}}(\overline{x}), \overline{y} + \delta) \) for all points \((x, y)\) with \( 0 \leq \overline{x} \leq y \) and \( y < \overline{y} \leq N \).

Now we have to prove that the resulting structure \( \mathcal{G}' = (\mathbb{G}_N, \mathcal{L}') \) is a homogeneous compass structure. This part is omitted, since it is pretty simple but extremely long. Let us just say that it can be proved by exploiting Corollary 3 and the definition of witnesses for a row \( y \).

To conclude the proof of Theorem 2, it suffices to show that if a \( \bf{BD}_{\text{hom}} \) formula is satisfiable, then it is satisfied by a doubly exponential compass structure, whose existence can be checked in exponential space. The following result provides both the small model theorem and the complexity class of checking whether or not a \( \bf{BD}_{\text{hom}} \) formula \( \varphi \) admits it.

**Lemma 9.** Let \( \varphi \) be a \( \bf{BD}_{\text{hom}} \) formula. It holds that \( \varphi \) is satisfiable if and only if there is a compass structure \( \mathcal{G} = (\mathbb{G}_N, \mathcal{L}) \) for it such that \( N \leq 2^{2(4|\varphi|+1)(4|\varphi|^2+7|\varphi|+3)2^{8|\varphi|^3+14|\varphi|+6}} \), whose existence can be checked in \( \text{EXPSPACE} \).

**Proof.** To start with, let us consider the problem of determining how many possible different \( \text{Row}_C(y) \) we can have in a compass structure \( \mathcal{G} = (\mathbb{G}_N, \mathcal{L}) \). Let us first observe that for the monotonicity of the function \( \mathcal{S}_\rightarrow \) we have, for every \( 0 \leq y \leq N \), \( \mathcal{S}_\rightarrow(0, y) \supseteq \ldots \supseteq \mathcal{S}_\rightarrow(y, y) \).

Then, since we cannot have two incomparable, w.r.t. \( \subseteq \) relation, \( \mathcal{S}_\rightarrow(x, y) \) and \( \mathcal{S}_\rightarrow(x', y) \) we have at most \( 2^{4|\varphi|^2+6|\varphi|+2} \cdot 2^{|\varphi|+1} = 2^{4|\varphi|^3+7|\varphi|+3} \) possible distinct \( \mathcal{S}_\rightarrow(x, y) \); that is an
upper bound of the length of the longest possible \( \varepsilon \)-ascending sequence in the set of pairs \( ([\text{Sh}^\varepsilon], F) \) (i.e., equivalence class and atom).

Moreover, each one of the possible witnesses is a pair \( ([\text{Sh}^\varepsilon], F) \) and, since \( \text{Wit}_G(y) \) does not contain covered points, each fingerprint \( f_P(x,y) = ([\text{Sh}^\varepsilon(x)], \mathcal{L}(x,y), S_{\text{hom}}(x,y)) \) can be associated to at most \( 4|\varphi| + 2 \) (i.e., the maximum value for \( \Delta_t \) plus one) distinct points in \( \text{Wit}_G(y) \). Summing up, we have that the maximum length for \( \text{Row}_G(y) \) is bounded by \( 2^{4|\varphi|^2 + 7|\varphi| + 3} \cdot 2^{4|\varphi|^2 + 7|\varphi| + 3} \cdot (4|\varphi| + 2) = (4|\varphi| + 2)2^{8|\varphi|^2 + 14|\varphi| + 6}. \) In each of such positions we can put a pair \( ([\text{Sh}^\varepsilon], F) \) and thus the cardinality of the set of all possible \( \text{Row}_G(y) \) is bounded by \( 2^{4|\varphi|^2 + 7|\varphi| + 3} (4|\varphi| + 2)8^{|\varphi|^2 + 14|\varphi| + 6} \) which is doubly exponential in \( |\varphi| \). Finally, given a \( \varphi \)-compass structure \( G = (N, \mathcal{L}) \), by repeatedly applying Theorem 8, we can obtain a \( \varphi \)-compass structure \( G = (G', \mathcal{L}') \) such that for every \( 0 \leq y < y' \leq N \) we have \( \text{Row}_G(y) = \text{Row}_G(y') \), then, by means of the above considerations on the maximum cardinality for the set of all possible \( \text{Row}_G(y) \), we may conclude that that \( \varphi \) is satisfiable if and only if there is a compass structure \( G = (G, \mathcal{L}) \) for it such that \( N \leq 2^{2(|\varphi|+1)(4|\varphi|^2 + 7|\varphi| + 3)2^{8(|\varphi|^2 + 14|\varphi| + 6}} \).

To complete the proof, it suffices to show that checking the existence of such a doubly exponential compass structure can be done in exponential space.

Let \( M = 2^{2(|\varphi|+1)(4|\varphi|^2 + 7|\varphi| + 3)2^{8(|\varphi|^2 + 14|\varphi| + 6}}} + 1 \) be the bound (plus 1) on the size of a candidate compass structure for the input \( \text{BD}_{\text{hom}} \) formula \( \varphi \), according to the small model theorem just proved. In the following, we briefly describe a decision procedure that decides, for some \( N \leq M \), whether or not there exists a compass structure \( G = (G, \mathcal{L}) \) for the input \( \text{BD}_{\text{hom}} \) formula \( \varphi \). If such a procedure works in exponential space with respect to \( |\varphi| \), we can immediately conclude that the satisfiability problem for \( \text{BD}_{\text{hom}} \) belongs to the \( \text{EXPSPACE} \) complexity class. The decision procedure begins at step \( y = 0 \) by guessing \( \text{Row}_G(y) = ([\text{Sh}^0], F^0) \) where \( F^0 = [\text{Sh}^0] \) and updates \( y \) to \( y + 1 \). For every \( y > 0 \), the procedure proceeds inductively as follows (let \( \text{Row}_G(y) = ([\text{Sh}^0], F^0) \ldots ([\text{Sh}^k], F^k) \)):

1. if there exists \( i \) for which \( \varphi \in F^i \), then return \( \text{true} \);
2. if \( y = M \), then return \( \text{false} \);
3. non-deterministically guess a pair \( ([\text{Sh}^{k+1}], F^{k+1}) \) such that \( F^{k+1} = [\text{Sh}^{k+1}] \);
4. for every \( 0 \leq i \leq k \), let

\[
F^i = \begin{cases}
F^i & \text{if } \text{Req}_{\mathcal{D}}(F^i) = \bigcup_{i < j \leq k} \text{Obs}_{\mathcal{D}}(F^j) \cup \text{Req}_{\mathcal{D}}(F^j) \\
\text{Req}_{\mathcal{B}}(F^i) = \text{Obs}_{\mathcal{B}}(F^i) \cup \text{Req}_{\mathcal{B}}(F^i) \land F^i \land \text{Prop} = \hat{F}^{k+1} \land F^i \land \text{Prop} & \text{otherwise.}
\end{cases}
\]
By Lemma 4, $\overline{F}_i$ is well defined.

(5) if there exists $i$ for which $\overline{F}_i = \bot$, then return $\text{false}$;

(6) let $i_0 < \ldots < i_h$ be the maximal sub-sequence of indexes in $0 \ldots k + 1$ such that, for every $0 \leq j \leq h$, $([Sh_B^i] \ldots, \overline{F}_h)$ is not covered in $([Sh_B^0] \ldots, \overline{F}_0) \ldots ([Sh_B^{i_h}] \ldots, \overline{F}_{i_h})$;

(7) update $y$ to $y + 1$ and restart from step 1.

Soundness and completeness of the above procedure can be proved using the result given in this section. In particular, Corollary 3 comes into play in the completeness proof (item 6 keeps track of all and only the not covered points on row $y + 1$). Moreover, notice that, for each step $0 \leq y \leq M$, we have to keep track of:

1. the current value of $y$, which cannot exceed $2^{2(|\varphi| + 1)(4|\varphi|^2 + 7|\varphi| + 3)28(4|\varphi|^2 + 14|\varphi| + 6)} + 1$ and can be logarithmically encoded using an exponential number of bits;
2. two rows, namely $Row_G(y)$ and $Row_G(y + 1)$, whose maximum length is bounded by $2^{4(|\varphi|^2 + 7|\varphi| + 3) \cdot 2^{4(|\varphi|^2 + 7|\varphi| + 3) \cdot (4|\varphi| + 2)} (4|\varphi| + 2) 2^{4(|\varphi|^2 + 14|\varphi| + 6)}$ (exponential in $|\varphi|$). Moreover, each position in such sequences holds a pair $([\overline{F}_0 \ldots \overline{F}_m], \overline{F}_i)$.

Since $m \leq 4|\varphi| + 1$, we have that each position holds at most $4|\varphi| + 3$ atoms. Each atom can be represented using exactly $|\varphi| + 1$ bits. Summing up, we have that the total space needed for keeping the two rows $y$ and $y + 1$ (step $0 \leq y \leq M$) consists of $2 \cdot (4|\varphi| + 1)(4|\varphi| + 3) \cdot 2^{4(|\varphi|^2 + 7|\varphi| + 3) \cdot 2^{4(|\varphi|^2 + 7|\varphi| + 3) \cdot (4|\varphi| + 2)}$ bits, which, simplified, turns out to be $4(8|\varphi|^3 + 18|\varphi|^2 + 13|\varphi| + 3)2^{8(|\varphi|^2 + 14|\varphi| + 6}$ bits that is still exponential in $|\varphi|$.

This shows that we can decide the satisfiability of $\varphi$ in exponential space. 

\[ 7. \text{Adding Modality } A \text{ to } BD_{hom}: \text{the Logic } BD_{A_{hom}} \]

In this section, we introduce the logic $BD_{A_{hom}}$, that extends $BD_{hom}$ with modality $\langle A \rangle$. The semantics of modality $\langle A \rangle$ has been already given in Section 2 in terms of modality $T$ of CDT as $\langle A \rangle \psi = \psi \ T \ T$. Formally, syntax and semantics of $BD_{A_{hom}}$ are obtained from those of $BD_{hom}$ by simply adding the syntactic rule and the semantic clause for modality $\langle A \rangle$, respectively.

$BD_{A_{hom}}$ formulas are built up from a countable set $\text{Prop}$ of proposition letters according to the grammar: $\varphi ::= p \mid \neg \psi \mid \psi \lor \psi \mid \langle B \rangle \psi \mid \langle D \rangle \psi \mid \langle A \rangle \psi$, where $p \in \text{Prop}$ and $\langle B \rangle, \langle D \rangle, \langle A \rangle$ are the modalities for Allen’s relations $\text{Begins}, \text{During}$, and $\text{Meets}$, respectively.

The semantics of a $BD_{A_{hom}}$ formula is specified by the semantic clauses for $BD_{hom}$ plus the following one:

- $M, [x, y] \models \langle A \rangle \psi$ iff there is $y'$, with $y' \geq y$, such that $M, [y, y'] \models \psi$.

In the rest of section, in analogy to what we did for modalities $\langle B \rangle$ and $\langle D \rangle$ in Section 3, we investigate the counterpart of modality $\langle A \rangle$ in terms of a suitable extension of generalized $\ast$-free regular expressions. Basically, we enrich the semantics of generalized $\ast$-free regular expressions with what we call a “right context”. We will prove that the resulting semantics subsumes the original one, that is, the notion of generalized $\ast$-free regular expression given in Section 3 is just a specialization of it. In particular, the encoding of both $\text{Pre}(e)$ and $\text{Inf}(e)$
Their semantics is defined as follows:

As a preliminary remark, we would like to observe that one may be tempted to interpret modality \( \langle A \rangle \) as a logical counterpart of the concatenation operator. This is wrong. Informally speaking, modality \( \langle A \rangle \) characterizes words with a specific "right context". Such an idea can be formalized as follows.

In order to identify the right generalized \(*\)-free regular expression for modality \( \langle A \rangle \), we provide an alternative, yet equivalent, semantics for these expressions. In such a semantics, the language \( \text{Lang}(e) \) of a generalized \(*\)-free regular expression \( e \) is interpreted over pairs of finite words, that is, \( \text{Lang}(e) \subseteq \Sigma^+ \times \Sigma^* \). A pair \( (w, w') \in \text{Lang}(e) \) represents the word \( w \) belonging to the language \( \text{Lang}(e) \), according to the semantics given in Section 3, together with its "right context" word \( w' \), which is the word that must appear immediately after \( w \).

Formally, generalized \(*\)-free regular expressions of Section 3 are extended as follows:

\[ e ::= \emptyset \mid a \mid \neg e \mid e + e \mid \text{Pre}(e) \mid \text{Inf}(e) \mid \text{Con}(e), \text{ for any } a \in \Sigma \]

Their semantics is defined as follows:

(i) \( \text{Lang}(\emptyset) = \emptyset; \)
(ii) \( \text{Lang}(a) = \{(a, w) : w \in \Sigma^*\}; \)
(iii) \( \text{Lang}(\neg e) = \Sigma^+ \times \Sigma^* \setminus \text{Lang}(e); \)
(iv) \( \text{Lang}(e + e') = \text{Lang}(e) \cup \text{Lang}(e'); \)
(v) \( \text{Lang}((\text{Pre}(e)) = \{(wv, u) : v \in \Sigma^+, (w, vu) \in \text{Lang}(e)\}; \)
(vi) \( \text{Lang}(\text{Inf}(e)) = \{(uwv, z) : u, v \in \Sigma^*, (w, vz) \in \text{Lang}(e)\}; \)
(vii) \( \text{Lang}(\text{Con}(e)) = \{(w, u) : u \in \text{Lang}(e)\}. \)

Let us denote the empty word by \( \varepsilon \). With a little abuse of notation, we say that, for every \( w \in \Sigma^* \), \( w \in \text{Lang}(e) \) if and only if \( (w, \varepsilon) \in \text{Lang}(e) \). Then, it is easy to prove that, for any expression \( e ::= \emptyset \mid a \mid \neg e \mid e + e \mid \text{Pre}(e) \mid \text{Inf}(e) \), \( w \in \text{Lang}(e) \) if and only if \( (w, \varepsilon) \in \text{Lang}(e) \). In such a way, the original (restricted) semantics turns out to be a specialization of the extended one.

It can be easily shown that the extended semantics preserves the mapping from a restricted expression \( e \) to an equivalent \( \text{BD}_{\text{hom}} \) formula \( \varphi_e \) outlined in Section 3. In order to capture the language \( \text{Lang}(\text{Con}(e)) \) in \( \text{BDA}_{\text{hom}} \), we extend the mapping with the rule:

\[ \varphi_{\text{Con}(e)} = \langle A \rangle (\langle B \rangle \top \land [B][B] \bot \land \langle A \rangle \psi_e). \]

Let us assume that \( \varphi_{\text{Con}(e)} \) holds over an interval \([x, y]\). Then, it predicates over "the right context" of \([x, y]\) by stating that there exists an interval \([y, y+1]\) (the constraint on the length of such an interval is imposed by the first two conjuncts \( \langle B \rangle \top \land [B][B] \bot \) which has an adjacent-to-the-right interval \([y+1, y']\) where \( \psi_e \) holds (third conjunct \( \langle A \rangle \psi_e \)).

In order to show the significance of the proposed extension of generalized \(*\)-free regular expression, we explore an interesting correspondence between the operator \( \text{Con} \) (and thus, indirectly, modality \( \langle A \rangle \)) and an operator of the regular expressions typically used in popular programming languages like, for instance, Python [VRDJ95]. It is easy to see that the \( \text{Con} \) operator corresponds to the lookahed operation. Such an operation is usually implemented
as **positive lookahead**, whose syntax is \((? = e)\), and **negative lookahead**, whose syntax is \((? ! e)\), where \(e\) is a regular expression. In many real-world applications, regular expressions are used to execute **pattern matching** inside a long text as an effective alternative to the task of checking whether such a long text belongs to a certain language. This is the case especially in the domain of natural language processing from which the following toy example is borrowed.

Let us suppose that we want to capture a pattern that consists of an English word followed by a list of words in English separated by commas and whose last word is prefixed by the word “and”. An example of a sentence containing such a pattern is the following: “*This paper deals with HS operators meets, begins, and during under homogeneity assumption.*”

In such a toy example, a motivation for matching the word **operators** may be related to the fact that the noun preceding a natural language description of items may represent their type. In the above sentence, “meets”, “begins”, and “during” are indeed of type “operators”. In such an interpretation, we are assuming that the word denoting the type is put immediately before the list of words and thus conjunctions like, e.g., “such as” or “like” are not contemplated. However, they may be captured by regular expressions longer, but not much more complex, than the one we are going to show. For the sake of simplicity, we assume that the number of words in the list is greater than or equal to 3 and each word is a single one. As an example, the pattern “**Concepts such as atoms, compass structures, and requests will be introduced in this section**” is not captured. A regular expression \(re\), which works in any modern programming language, that captures such a pattern is:

\[
re = (\{w\+\})(? = e(? : \{w+, e\}\{2\})and\{w\+\})
\]

where \(\cdot\) is used to highlight the single white space “ “. Since it is outside the scope of this paper, we will not delve too much into the syntax of this kind of regular expressions. For that matter, wonderful websites such as [Reg], exist (they provide a quick reference for syntax and semantics together with examples and, more importantly, a full on-line environment for testing and debugging regular expressions).

Let us briefly explain how \(re\) captures the desired pattern. First of all, we have that \((e)\) is used to capture any pattern in \(e\). The \((? = e)\) operator checks whether the current position is followed by a pattern belonging to the language of \(e\). The \(\{w\}+\) variable represents any word-character, both lower and upper case. The operator + is analogous to the operator \(e* = ee^*\) in standard regular expressions. Thus, \(\{w\}+\) means any single word. The operator \((e)\{n, \}\), with \(n \geq 0\), captures a sequence of \(n\) or more occurrences of pattern \(e\). Finally, the operator \((? : e)\) represents just standard parentheses. A graphical account of the various parts of regular expression \(re\) is shown in Figure 17.

Let \(\Sigma = W \cup S\), where \(W = \{a, \ldots, z, A, \ldots, Z\}\) (word symbols) and \(S = \{\cdot, .., ;, \}'\) (separator symbols). For the sake of brevity, we omit the intermediate phase of translating \(re\) into our \(\ast\)-free restricted fragment and we jump directly to the translation into \(BDA_{hom}\).

For the sake of simplicity, we do not apply the literal translation here; instead, we make use of a shorter, more understandable encoding which is tailored to the structure of the specific regular expression \(re\). As a preliminary step, we provide some shorthands and assumptions that make the encoding formulas more compact. First, we introduce a global modality \([G]\psi\), whose semantics is as follows: given a model \(M = (I_N, V), M, [0, y] \models [G]\psi\) if and only if \(\psi \in V([x, y])\) for every \(0 \leq x, y \leq N\), that is, if \([G]\psi\) holds on an initial interval of the model (an interval whose left endpoint is 0), then \(\psi\) holds on every interval of the model. In \(BDA_{hom}\), we may capture the semantics of \([G]\psi\) by means of the formula \(\psi \land [A]\psi \land [A][A]\psi \land [B]\psi \land [B][A]\psi\). Moreover, in the encoding, we will make use of
A single word is matched if and only if it is immediately followed by a concatenation of the following three elements:

(i) a whitespace;
(ii) a sequence of two or more concatenations of a single word, a comma, and a whitespace;
(iii) the concatenation of the word “and”, a whitespace, and a single word.

Figure 17. A graphical account of $re$ and its sub-expressions.

the shorthands $len_{\geq n}$ and $len_n$ for any $n \in \mathbb{N}$, that constrain the length of the interval on which they hold to be greater than or equal to $n$ and exactly equal to $n$, respectively. More precisely, given a model $M = (I_N, V)$, we have that $[x, y] \models len_{\geq n}$ if and only if $y - x \geq n$, and $M, [x, y] \models len_n$ if and only if $y - x = n$. In $BDA_{hom}$, we may capture the semantics of $len_{\geq n}$ and $len_n$ by means of the formulas $\langle B \rangle^n \pi$ and $len_{\geq n} \land [B]^{n+1} \perp$, respectively.\footnote{Notice that we provide a unary encoding of the length constraints. It is possible to make a binary encoding analogous to the one proposed in [BMM'22].} Since in the proposed encoding we will make use of proposition letters in $\Sigma$ to represent words as points of an interval model (Figure 18), we need to force each point to hold exactly one symbol $\sigma \in \Sigma$. Such a constraint is imposed by putting the formula $\langle G \rangle (\pi \rightarrow \bigvee_{\sigma \in \Sigma} (\sigma \land \bigwedge_{\sigma' \in \Sigma \setminus \{\sigma\}} \lnot \sigma'))$ in conjunction with the encoding of $re$. For the sake of brevity, we will tacitly assume that this is the case. Finally, with a little abuse of notation, in the encoding of $re$ we will make use of $W$ as a shorthand for $\bigvee_{\sigma \in W} \sigma$, which basically allow us to state that a certain (point-)interval holds a word symbol.

Now, we are ready to encode $re$ by a formula $\psi_{re}$. More precisely, we will make use of $\langle D \rangle \psi_{re}$ as the main formula, where $\psi_{re}$ just encodes the matching part. Thus, by “reading” a model $M = (I_N, V)$ for $\langle D \rangle \psi_{re}$, we can easily retrieve every matching by taking all and only those intervals $[x, y]$ such that $M, [x, y] \models \psi_{re}$. As an example, in Figure 18 we have that $M, [0, 90] \models \langle D \rangle \psi_{re}$, while $[24, 34] \not\models \psi_{re}$. In fact, $[24, 34]$ is the only interval that satisfies $\psi_{re}$ in the model of Figure 18 and, as we will see when we will discuss $\psi_{re}$ in more detail, this is determined both by the points belonging to $[24, 34]$ and by the formulas that hold in its “right context”, that is, the intervals $[x, y]$, with $34 \leq x \leq 90$.

Let $\psi_{re} = \psi_{w^m}^{gm} \land \langle A \rangle (\psi_{\odot (w^m)^2}^{gm} \land \langle A \rangle \psi_{\text{and}@w^m}^{gm})$. Intuitively, $\psi_{re}$ requires the presence of three adjacent intervals $[x, y], [y, z], [z, w]$ such that $M, [x, y] \models \psi_{w^m}^{gm}$, $M, [y, z] \models \psi_{\odot (w^m)^2}^{gm}$, and $M, [z, w] \models \psi_{\text{and}@w^m}^{gm}$. Notice that we provide a unary encoding of the length constraints. It is possible to make a binary encoding analogous to the one proposed in [BMM'22].
ψ\textsubscript{gm}^\text{gm} = \langle B \rangle W \land \langle B \rangle (\text{len}_{\geq 1} \rightarrow \langle A \rangle W) \land \langle B \rangle \langle A \rangle W \land \langle A \rangle \neg W.

This formula holds over

\begin{itemize}
\item If \([x, y]\) if and only if point-intervals \([x, x']\) and \([y, y']\) do not hold a word symbol (conjects \(\langle B \rangle \neg W\) and \(\langle A \rangle \neg W\), respectively), but a word symbol holds at all the internal point-intervals \([x', x']\), with \(x < x' < y\) (conjunct \([B](\text{len}_{\geq 1} \rightarrow \langle A \rangle W)\)). Finally, it constrains the interval \([x, y]\) to contain at least one word symbol (conjunct \(\langle B \rangle \langle A \rangle W\)). Intuitively, \(\psi_{\text{gm}}\) encodes an article by means of modality \(\langle A \rangle\). This formula holds over an interval \([x, y]\) if and only if all the following conditions hold:
\begin{enumerate}
\item the symbol \(\ast\) holds on point-interval \([x, x]\) and a word symbol holds on point-interval \([x + 1, x + 1]\) (conjunct \(\langle B \rangle(\text{len}_{\geq 1} \land \langle B \rangle \ast \land \langle A \rangle W)\));
\item the symbol \(\ast\) holds on point-interval \([y, y]\) (conjunct \(\langle A \rangle \ast\));
\item for every strict sub-interval of \([x, y]\) of the form \([x', x']\), we have that either \([x', x']\) is labelled with \(\ast\) and \([x' + 1, x' + 1]\) by a word symbol (disjunct \(\langle B \rangle \ast \land \langle A \rangle W\)), or \([x', x']\) is labelled with \(\ast\) and \([x' + 1, x' + 1]\) with \(\ast\) (disjunct \(\langle B \rangle, \land \langle A \rangle \ast\)), or both \([x', x']\) and \([x' + 1, x' + 1]\) are labelled with a word symbol (disjunct \(\langle B \rangle W \land \langle A \rangle W\)), or \([x', x']\) is labelled with a word symbol and \([x' + 1, x' + 1]\) with \(\ast\) (disjunct \(\langle B \rangle W \land \langle A \rangle \ast\));
\item the symbol \(\ast\) appears as a label of at least two distinct point-intervals \([x', x']\) and \([x'', x'']\) in \([x, y]\), i.e., with \(x < x' < x'' < y\) (conjunct \(\langle B \rangle (\langle A \rangle, \land \langle B \rangle \langle A \rangle, \ast)\)).
\end{enumerate}

In Figure 17, such a condition is satisfied by point-intervals \([40, 40]\) and \([48, 48]\), which are included in the interval \([34, 49]\).

Intuitively, the conjunct \([D](\text{len}_{\geq 1} \rightarrow (\langle B \rangle \ast \land \langle A \rangle W) \lor (\langle B \rangle, \land \langle A \rangle \ast) \lor (\langle B \rangle W \land \langle A \rangle W) \lor (\langle B \rangle W \land \langle A \rangle))\) constrains the word underlying \([x + 1, y + 1]\) to belong to the language of \((\langle w \rangle^*, \ast)^\ast\), while the conjunct \(\langle B \rangle(\langle A \rangle, \langle B \rangle(\langle A \rangle, \ast))\) forces at least two iterations of the \(\ast\) operation in such a language. Thus, together they force such a word to belong to \((\langle w \rangle^+, \ast)^{2+}\).

\begin{itemize}
\item \(\psi_{\text{gm}}^\text{gm} = \langle B \rangle a \land \langle B \rangle(\text{len}_{\geq 1} \land \langle A \rangle n) \land \langle B \rangle(\text{len}_{\geq 2} \land \langle A \rangle d) \land \langle B \rangle(\text{len}_{\geq 3} \land \langle A \rangle \ast) \land \langle B \rangle(\text{len}_{\geq 4} \rightarrow \langle A \rangle W) \land \langle A \rangle \neg W.\) This formula holds over an interval \([x, y]\) if and only if the word underlying the interval \([x, x + 3]\) is exactly \(\ast\) (conjunct \(\langle B \rangle a\), \(\langle B \rangle(\text{len}_{\geq 1} \land \langle A \rangle n)\), \(\langle B \rangle(\text{len}_{\geq 2} \land \langle A \rangle d)\), and \(\langle B \rangle(\text{len}_{\geq 3} \land \langle A \rangle \ast)\) followed by an uninterrupted sequence of word symbols underlying the interval \([x + 4, y - 1]\) (conjunct \(\langle B \rangle(\text{len}_{\geq 4} \rightarrow \langle A \rangle W)\)). In addition, it imposes the word underlying the interval \([x + 4, y - 1]\) to be a greedy match, that is, an entire word is captured, since we force a separator symbol on \([y + 1, y + 1]\) by means of the conjunct \(\langle A \rangle \neg W\).
\end{itemize}

We conclude the section with some remarks about the practical use of regular expressions. To the best of our knowledge, in their implementation the majority of existing programming languages do not support the free use of negation in regular expressions, but they allow for positive/negative lookahead/lookbehind. In this section, we showed how to deal with positive/negative lookahead by means of modality \(\langle A \rangle\). Moreover, we argued that positive/negative lookbehind may be captured by adding modality \(\langle A \rangle\), which is the converse
of modality \(\langle A \rangle\), to \(\text{BDA}_{\text{hom}}\), thus obtaining the logic \(\text{BDA}_{\text{hom}}^{\varphi}\). For the sake of simplicity, we did not take modality \(\langle A \rangle\) into consideration in this work, as its introduction involves a number of technicalities. However, in view of the results established in the next section, we may conjecture with a certain confidence that, under the homogeneity assumption, the satisfiability problem for \(\text{BDA}_{\text{hom}}^{\varphi}\) belongs to the same complexity class as its proper fragment \(\text{BDA}_{\text{hom}}\).

8. The satisfiability problem for \(\text{BDA}_{\text{hom}}\) is decidable in \(\text{EXPSPACE}\)

In this section, we go through the definitions and proofs of Sections 5.1, 5.2, and 5.3 in order to identify the changes that must be made in order to extend them to the fragment \(\text{BDA}_{\text{hom}}\).

To begin with, we state a lemma that establishes a fundamental property of modality \(A\), and will be extensively used in the following definitions and proofs.

**Lemma 10.** For every interval structure \(M = \langle I_N, V \rangle\), every triplet of points \(x \leq y \leq z\) in \(\{0, \ldots, N\}\), and every HS formula \(\psi M, [x, z] \models \langle A \rangle \psi\) if and only if \(M, [y, z] \models \langle A \rangle \psi\).

Figure 18. a graphical account of how a \(\langle A \rangle \psi_{re}\) holds over an interval model representing a text.
we introduce the following specializations for the relations \( \text{MEETS} \) in the bottom of Figure 20. In the example of Figure 20 we have the newly introduced second component \( \alpha \) but \( \text{Req} \) relation is a direct consequence of the fact that Allen relations \( R \) labelling intervals which are in the same functions \( \alpha \) Obs \( \text{sets} \) \( \text{Req} \) in Section 4. For the example in Figure 20, we focus on describing how the behaviour of \( \text{BDA} \).

While, by considering just the first component of a newly defined atom, we keep functions \( \text{F} \) for the sake of simplicity, from now on when we refer to \( \text{F} \alpha \) as a set, we refer to its first component \( \text{F} \). For instance, when we write \( \psi \in \text{F} \alpha \), we mean \( \psi \in \text{F} \). Let \( \text{At}(\varphi) \) be the set of all \( \varphi \)-atoms. We have that \( |\text{At}(\varphi)| \leq 2^{|\varphi|+1} \cdot 2^{|\varphi|-1} = 2^{|\varphi|} \), where \( |\varphi| = |\text{Cl}(\varphi)|/2 \).

While, by considering just the first component of a newly defined atom, we keep functions \( \text{Req}_R \), \( \text{Obs}_R \), and \( \text{Box}_R \) for all \( R \in \{A,B,D\} \) the same as the ones introduced in Section 4 we introduce the following specializations for the relations \( \rightarrow_B \) and \( \rightarrow_D \):

- \( \text{F} \alpha \rightarrow_B \text{G}_\beta \) if \( \text{Req}_B(\text{F}_\alpha) = \text{Req}_B(\text{G}_\beta) \cup \text{Obs}_B(\text{G}_\beta) \) and for every \( \psi \in \text{TF}_A^\varphi \) we have \( \alpha(\psi) = \beta(\psi) \) if \( \beta(\psi) \in \{\blacklozenge,\Box\} \) or \( \psi \notin \text{F} \);

- \( \text{F} \alpha \rightarrow_D \text{G}_\beta \) if \( \text{Req}_D(\text{F}_\alpha) \supseteq \text{Req}_D(\text{G}_\beta) \cup \text{Obs}_D(\text{G}_\beta) \).

In Figure 20 we provide an example of consistent atom labelling of a model of an \( \text{BDA} \text{hom} \) formula \( \varphi \). For what concerns \( \text{Req}_R(\cdot),\text{Box}_R(\cdot),\text{and}\text{Obs}_R(\cdot) \) with \( R \in \{B,D\} \) we can make the same considerations made in the description of the example of Figure 8 in Section 4. For the example in Figure 20, we focus on describing how the behaviour of sets \( \text{Req}_A(\cdot),\text{Box}_A(\cdot),\text{and}\text{Obs}_A(\cdot) \) differ w.r.t. their counterparts \( \text{Req}_R(\cdot),\text{Box}_R(\cdot),\text{and}\text{Obs}_R(\cdot) \) with \( R \in \{B,D\} \) as well as an initial account of the behaviour of the marking functions \( \alpha_{[x,y]} \). Let us observe first that while \( \text{Req}_R \) is “monotone” for \( R \in \{B,D\} \) for atoms labelling intervals which are in the same \( R \)-relation. This claim is not true when \( R = A \) as a direct consequence of the fact that Allen relations \( \text{STARTED-BY} \) and \( \text{CONTAINS} \) are transitive while relation \( \text{MEETS} \) is not. For instance, in Figure 20, we have that \( [0,1] \text{ MEETS } [1,2] \text{ MEETS } [2,3] \) but \( \text{Req}_A(F_{[0,1]}) = \text{Req}_A(F_{[2,3]}) = \emptyset \) and \( \text{Req}_A(F_{[0,1]}) = \{\neg \psi_1\} \). Let us now focus on the newly introduced second component \( \alpha_{[x,y]} \) of each atom which is reported on the very bottom of Figure 20. In the example of Figure 20 we have \( \text{TF}_A^\varphi = \{\neg \psi_1\} \) and thus \( \alpha_{[x,y]} \).
assigns to the interval \([x, y]\) the “status” of \(\neg \psi_1\) on it. More precisely, the fact that 
\(\neg \psi_1 \in \text{Req}_A(F[x,x])\), i.e., the \([G]\)requests pending on point \([x, x]\) which are the same, 
according to Lemma 10, for all the interval of the type \([x', x]\). More precisely, we have 
\(\alpha_{[x,y]}(\neg \psi_1) = \square\) if and only if \(\neg \psi_1 \notin \text{Req}_A(F[x,x])\) which means that 
\(\neg \psi_1\) is not requested by 
\(F[x,x]\) and thus \(\psi_1\) must be satisfied on all the intervals \([x, y]\). This is the case, for instance, 
of intervals \([0, 0], [1, 1], [3, 3], \) and \([4, 4]\) in Figure 20 which impose that 
\(\alpha_{[x,y]}(\neg \psi_1) = \square\), and, consequently, 
\(\psi_1 \in F[x,x]\) for all \([x, y] \in [(x, y) : 0x \leq y \leq 4, x \neq 2]\). If \(\alpha_{[x,y]}(\neg \psi_1) \neq \square\) 
we have that \(\alpha_{[x,y]}(\neg \psi_1) \in \{\Diamond, *\}\) for every \(y \geq x\) which means that the request 
\(\langle A \rangle \neg \psi_1\) is pending on \([x, x]\) (i.e., \(\neg \psi_1 \in \text{Req}_A(F[x,x])\)) and must be satisfied by some interval of
the form \([x, y]\) for some \(y \geq x\). If we take the minimum \(y\) for which \(\neg \psi_1 \in \text{Obs}_A(F^{[x,y]})\) we have that: (1) \(\alpha_{x,y}\)[(-\(\psi_1\)] = \(\emptyset\) for every \(x \leq y' < y\) which means that the pending \(\langle A\rangle\) request \(\neg \psi_1\) is not fulfilled for the intervals ending in \(x\) if we consider the model up to \(y'\); \(\alpha_{x,y}\)[(-\(\psi_1\)] = \(\blacklozenge\) for every \(x \leq y \leq y'\) which means that the pending \(\langle A\rangle\) request \(\neg \psi_1\) is fulfilled for the intervals ending in \(x\) if we consider the model up to \(y'\) and, obviously, will stay fulfilled for such intervals ever after. In Figure 20, this is the case for interval \([2, 2]\) for which \(\neg \psi_1 \in \text{Req}_A(F^{[2,2]})\) holds. However, since we have \(\neg \psi_1 \notin \text{Obs}_A(F^{[2,2]})\) and \(\neg \psi_1 \notin \text{Obs}_A(F^{[2,3]})\) it turns out that \(\alpha_{[2,2]}(\neg \psi_1) = \alpha_{[2,3]}(\neg \psi_1) = \emptyset\). On the other hand \(\neg \psi_1\) appears “for the first time” in \(\text{Obs}_A(F^{[2,y]})\) when \(y = 4\) and thus \(\alpha_{[2,4]}(\neg \psi_1) = \blacklozenge\).

We can extend the claim on labelings made in Section 4 to \(\text{BDA}_{\text{hom}}\) formula and say that all and only the labelings which respect property \((*1)\) are the ones for which the following property holds:

\((*3-b)\) \(\text{Req}_B(F^{[x,y]}) = \bigcup\{x \leq y < y' \text{Obs}_B(F^{[x,y']})\}, \text{Req}_D(F^{[x,y]}) = \bigcup\{x \leq y' < y \text{Obs}_D(F^{[x,y']})\},\) and \(\text{Req}_A(F^{[x,y]}) = \bigcup\{y \leq y' \text{Obs}_A(F^{[y,y']})\}\) for each \([x, y]\) ∈ \(\mathbb{N}\).

An account of how the second component of an atom behaves w.r.t. the relations \(\rightarrow_B\) and \(\rightarrow_D\) is given in Figure 21. Informally speaking, we have that the second component of an atom associated to an interval \([x, y]\) keeps track of the \(A\)-requests featured by \([x, x]\) which have been satisfied by intervals \([x, y']\) with \(y' \leq y\) (i.e., the ones marked with \(\blacklozenge\)) against the ones still pending (i.e., the ones marked with \(\emptyset\)). Moreover, the second component of an atom keeps track of the formulas \(\psi\) that are forced to appear negated in every interval starting in \(x\) due to the presence of \([A]\)\(\psi\) in the labelling of \([x, x]\) (i.e., the formulas marked with \(\blacklozenge\)). Since we cannot consider a model fulfilled until all the \(A\)-requests are satisfied for all points \(x\) in the model we introduce the notion of final atom. An atom \(F_α\) is final iff for every \(\psi \in TF_α^F\) we have \(\alpha(\psi) \in \{\blacklozenge, \sqcap\}\). Now we can provide the notion of compass structures for \(\text{BDA}_{\text{hom}}\) formula of by extending the \(\text{BD}_{\text{hom}}\) with the following requirements:

- \((\text{initial formula})\) \(\varphi \in L(0, N)\);
- \((A\)-consistency\) for all \(0 \leq x \leq y \leq N\), \(\text{Req}_A(L(x, y)) = \text{Req}_A(L(y, y))\);
- \((A\)-fulfilment\) for every \(0 \leq x \leq N\) atom \(L(x, x)\) is final.

Then we have the very same result for compass structures on \(\text{BDA}_{\text{hom}}\) formulas.

**Theorem 3.** A \(\text{BDA}_{\text{hom}}\) formula \(\varphi\) is satisfiable iff there is a homogeneous \(\varphi\)-compass structure.

Now we are ready to point out the minor differences in the steps for generalizing small model theorem of Section 5 to the \(\text{BDA}_{\text{hom}}\) case. First of all it is easy to prove using Lemma 10 that Lemma 2 holds also for \(\text{BDA}_{\text{hom}}\) homogeneous compass structures. In order to take into account the second component of atoms, we redefine the function \(\Delta_{\text{f}} : A(\varphi) \rightarrow N\) as follows:

\[
\Delta_{\text{f}}(F_α) = (2\{|(B)\psi ∈ \text{Cl}(\varphi)| \} - 2|\text{Req}_B(F_α)| + |\text{Obs}_B(F_α) \setminus \text{Req}_B(F_α)| + \{|\{D\}\psi ∈ \text{Cl}(\varphi)| \} - |\text{Req}_D(F_α)| + \{|\neg p : p ∈ \text{Cl}(\varphi) \cap \text{Prop}\} - \{|\neg p : p ∈ \text{Cl}(\varphi) \cap \text{Prop} \wedge \neg p ∈ F_α\}| + \{|\psi ∈ TF_α^F : \alpha(\psi) = \emptyset|\}
\]

The main complication that arises from the introduction of the \(\langle A\rangle\) operator consists of the fact that a \(B\)-sequence sequence that can be instantiated in a compass structure may
The next lemma represents the defining a natural equivalence relation of finite index over columns: we say that two columns of a given compass structure, as it happened for flat decreasing $B$-sequences. A minimal $B$-sequence will not represent the whole sequence of atoms on a “column” $x$ of a given compass structure, as it happened for flat decreasing $B$-sequences in Section 5.

In this case, a minimal $B$-sequence represents the labellings of the sequence of points sharing the same “column” $x$ where the function $\Delta_1(F^i_{\alpha_n})$ decreases as long as we move up on $y$. For capturing such a behaviour we provide the following notion of shading.

Let $G = (N, \mathcal{L})$ be a compass structure for $\varphi$ and $0 \leq x \leq N$. We define the shading of $x$ in $G$, written $Sh^G(x)$, as the sequence of pairs atoms $\langle \mathcal{L}(x, y_0), y_0 \rangle \ldots \langle \mathcal{L}(x, y_m), y_m \rangle$ such that:

1. $y_i < y_{i+1}$ for every $0 \leq i < m$;
2. $\{\Delta_1(\mathcal{L}(x, y)) : 0 \leq y \leq N \} = \{\Delta_1(\mathcal{L}(x, y)) : 0 \leq i \leq m \}$;
3. For every $0 \leq i \leq m$ we have $y_i = \min \{0 \leq y \leq N : \Delta_1(\mathcal{L}(x, y)) = \Delta_1(\mathcal{L}(x, y))\}$, i.e., $y_i$ is the minimum height on the column $x$ that exhibits its value for $\Delta_1$.

For every $0 \leq x \leq N$ let $Sh^G(x) = \mathcal{L}(x, y_0) \ldots \mathcal{L}(x, y_m)$ we denote with $Sh^G_B(x)$ the sequence of atoms $\mathcal{L}(x, y_0) \ldots \mathcal{L}(x, y_m)$, and with $Sh^G_{Bn}(x)$ the sequence of natural numbers $y_0 \ldots y_m$, that is, the projections of $Sh^G(x)$ of on the first and the second components of its elements, respectively.

The next lemma represents the BDA$_{hom}$ counterpart of Lemma 3.

**Lemma 11.** Let $G = (N, \mathcal{L})$ be a compass structure and $0 \leq x \leq N$, then $Sh^G_B(x)$ is a minimal $B$-sequence.

The above (finite) characterisation work just as good as the one defined in Section 5 for defining a natural equivalence relation of finite index over columns: we say that two columns $x, x'$ are equivalent, written $x \sim x'$, if and only if $Sh^G_B(x) = Sh^G_B(x')$. Then taking advantage of Lemma 10 we can prove that Lemma 5 also holds for BDA$_{hom}$ compass structures. The definitions of $S_\sim(x, y)$ and, consequently, of fingerprint $fp(x, y)$ for all $0 \leq x \leq x \leq N$ is
the same as the one given in Section 5. Let us observe that the number of possible sets \( S_\alpha(x, y) \) due to the specialization of atoms is bounded by \( 2^{6|\varphi|^2+2|\varphi|+\frac{5|\varphi|^2+4|\varphi|+\frac{2}{3}}{3}} \) in this case. For two atoms \( F_\alpha \) and \( G_\beta \), we say that they are equivalent modulo \( A \), written \( F_\alpha \equiv_{A \alpha} G_\beta \) if and only if \( F \setminus \text{Req}(A_\alpha) = G \setminus \text{Req}(A_\beta) \) and \( \alpha = \beta \) (i.e., \( F_\alpha \) and \( G_\beta \) have at most different \( \langle A \rangle \) requests). Then we may prove the analogous of Lemma 6 and related Corollary 2 in the case of \( \text{BDA} \). Let \( \alpha(x, y) \) be a compass structure and let 0 \( \leq x < x' \leq y \leq N \). If \( fp(x, y) = fp(x', y) \) and \( y' \) is the smallest point greater than \( y \) such that \( \mathcal{L}(x, y') \not\equiv_{\sim A} \mathcal{L}(x, y) \), if any, and \( N \) otherwise, then, for all \( y \leq y'' \leq y' \), \( \mathcal{L}(x, y'') = \mathcal{L}(x', y') \).

**Corollary 4.** Let \( \mathcal{G} = (N, \mathcal{L}) \) be a compass structure and let 0 \( \leq x < x' \leq y \leq N \). If \( fp(x, y) = fp(x', y) \) and \( y' \) is the smallest point greater than \( y \) such that \( \mathcal{L}(x, y') \not\equiv_{\sim A} \mathcal{L}(x, y) \), if any, and \( N \) otherwise, then, for every pair of points \( \bar{x}, \bar{x}' \), with \( x < \bar{x} < x' < \bar{x}' \), with \( \mathcal{L}(\bar{x}, y) = \mathcal{L}(\bar{x}', y) \) and \( \bar{x} \sim \bar{x}' \vdash x \), it holds that \( \mathcal{L}(\bar{x}, y') = \mathcal{L}(\bar{x}', y') \), for all \( y \leq y'' \leq y' \).

For \( \text{BDA} \) compass structures the definition of covered point, as well as witnesses \( \text{Wit}_\alpha(y) \), and row blueprint \( \text{Row}_\alpha(y) \) is the same of the ones given in Definition 6 and in Section 6, then Lemma 7, Corollary 2, Lemma 7, and Theorem 8 can be proved also in the case of \( \text{BDA} \). On the basis of such results we can provide an algorithm very similar to the one proposed in the proof of Theorem 9 and thus the following analogous result.

**Theorem 4.** Let \( \varphi \) be a \( \text{BDA} \) formula. It holds that \( \varphi \) is satisfiable iff there is a compass structure \( \mathcal{G} = (N, \mathcal{L}) \) for it such that \( N \leq 2^{5|\varphi|\cdot\left(6^{10}|\varphi|^2+4|\varphi|+\frac{2}{3}\right)} \cdot 3^{10|\varphi|^2+4} \cdot 3 \), whose existence can be checked in \( \text{EXPSPACE} \).

**9. EXPSPACE-hardness of \( \text{BDA} \) over finite linear orders**

In this section we prove that the satisfiability problem for \( \text{BDA} \) interpreted over finite linear orders is \( \text{EXPSPACE} \)-hard. The result is obtained by a reduction from the exponential-corridor tiling problem, which is known to be \( \text{EXPSPACE} \)-complete [vEB97]. Such a problem can be stated as follows.

**Problem 1.** Given a tuple \( T = (T, \Rightarrow, \uparrow, C) \) where \( T, C \in \mathbb{N} \) (\( C \) is expressed in binary), and \( \Rightarrow, \uparrow \in \{0, \ldots, T\} \times \{0, \ldots, T\} \), the exponential-corridor tiling problem consists of determining whether or not there exists a function \( \text{tile} : \mathbb{N} \times \{0, \ldots, C\} \rightarrow \{0, \ldots, T\} \) such that:

1. for every \( x \in \mathbb{N} \) we have \( \text{tile}(x, 0) = 0 \) and \( \text{tile}(x, C) = T \);
2. for every \( x \in \mathbb{N} \) and every \( 0 \leq y \leq C \) we have \( (\text{tile}(x, y), \text{tile}(x + 1, y)) \in \Rightarrow \);
3. for every \( x \in \mathbb{N} \) and every \( 0 \leq y < C \) we have \( (\text{tile}(x, y), \text{tile}(x, y + 1)) \in \uparrow \).

The following classical result will be exploited to prove the main goal of this section.

**Theorem 5.** [vEB97] The exponential-corridor tiling problem is \( \text{EXPSPACE} \)-hard.

To define a reduction from Problem 1 to the finite satisfiability of \( \text{BDA} \) we have to face the problem that formulas of \( \text{BDA} \) are interpreted over finite domains, whereas the \( \text{tile} \) functions ranges over an infinite domain. Roughly speaking, we will solve Problem 1 by
means of an infinite “unfolding” of a finite portion of the tiling space that can be encoded by a (finite) model for a suitable $\text{BDA}_{\text{hom}}$ formula. The following result is crucial to that purpose.

**Lemma 13.** Given an instance $\mathcal{T} = (T, \Rightarrow, \emptyset, C)$ of Problem 13 we have that $\mathcal{T}$ is a positive instance if and only if there exists a function $\text{tile} : \{0, \ldots, \text{prefix} + \text{period}\} \rightarrow \{0, \ldots, T\}$ that fulfills conditions 1, 2, and 3 of Problem 1 together with the following one:

1. there exist $\text{prefix} \in \mathbb{N}$ and $\text{period} \in \mathbb{N}^+$ s.t. for every $x \geq \text{prefix}$ and every $0 \leq y \leq C$ we have $\text{tile}(x, y) = \text{tile}(x + \text{period}, y)$.

The proof of Lemma 13 is straightforward and omitted. Lemma 13 allows us to bound the search space for the existence of the function $\text{tile}$ to a finitely representable function $\text{tile} : \{0, \ldots, \text{prefix}, \ldots, \text{prefix} + \text{period}\} \rightarrow \{0, \ldots, T\}$ for some $\text{prefix} \geq 0$ and $\text{period} > 0$. Function $\text{tile}$ witnesses that $\mathcal{T}$ is a positive instance of Problem 1 if it satisfies conditions 1, 2, and 3 restricted to $(x, y) \in \mathbb{N} \times \{0, \ldots, C\}$ with $x < \text{prefix} + \text{period}$ plus the condition that $\text{tile}(\text{prefix}, y) = \text{tile}(\text{prefix} + \text{period}, y)$ for every $y \in \{0, \ldots, C\}$.

Given an instance $\mathcal{T} = (T, \Rightarrow, \emptyset, C)$ of Problem 1 we provide a $\text{BDA}_{\text{hom}}$ formula $\varphi_\mathcal{T}$ that is satisfiable over finite models if and only if there exists a function $\text{tile}$ that satisfies the aforementioned properties an thus, by Lemma 13, if and only if $\mathcal{T}$ is a positive instance of Problem 1. In the proposed encoding we force each point of the model to represent exactly one tile. This is done by exploiting $T + 1$ propositional variables $t_0, \ldots, t_T$, called $\text{tile variables}$, constrained by the following formulas:

\[
\psi_3 = [G] \left( \pi \rightarrow \bigvee_{i=0}^T t_i \right), \text{ given a point in the model at least one tile variable holds over it;}
\]

\[
\psi_1 = [G] \left( \bigwedge_{i=0}^T \left( t_i \land \pi \rightarrow \left( \bigwedge_{j=0,j\neq i}^T \neg t_j \right) \right) \right), \text{ given a point in the model at most one tile variable holds over it (i.e., mutual exclusion).}
\]

Let us assume w.l.o.g. that $C = 2^c - 1$ for some $c \in \mathbb{N}$. Then, we associate to each model point a number in $\{0, \ldots, C\}$ by a binary encoding via $c$-propositional variables $b_1, \ldots, b_c$, where $b_1$ is the most significant bit. Formally, given a model $M = (N, \mathcal{V})$ and a point we define a function with

\[
\text{bit}_\mathcal{V} : \{0, \ldots, N\} \times \{b_1, \ldots, b_c\} \rightarrow \{0, 1\} \text{ where }\text{bit}_\mathcal{V}(n, b_i) = \begin{cases} 1 & \text{if } b_i \in \mathcal{V}([n, n]) \\ 0 & \text{otherwise} \end{cases}.
\]

For the sake of brevity, we denote with $\overline{n}_n$ the natural number whose $c$-bit length binary encoding is $\text{bit}_\mathcal{V}(n, b_1) \ldots \text{bit}_\mathcal{V}(n, b_c)$. We encode the domain of a general function $\text{tile} : \{0, \ldots, \text{prefix}, \ldots, \text{prefix} + \text{period}\} \rightarrow \{0, \ldots, T\}$ into a finite model $M = (N, \mathcal{V})$ by enumerating all the points of the grid $\{0, \ldots, \text{prefix} + \text{suffix}\} \times \{0, \ldots, C\}$ along the timepoints $\{0, \ldots, N\}$ of the model in a lexicographical order. The formula $\psi_\text{tile} = \psi_3 \land \psi_1 \land \psi_{\text{boundaries}} \land \psi_1$ is used to force such constraint where $\psi_{\text{boundaries}}$ and $\psi_1$ are formulas defined as follows:

\[\text{5In the encoding we will make extensive use of the “global” operator whose semantics was introduced in Section 7.}\]
\[
\psi_{\text{boundaries}} = \langle B \rangle \left( \pi \wedge \bigwedge_{i=1}^{c} -b_{i} \right) \wedge \langle A \rangle \left( \bigwedge_{i=1}^{c} b_{i} \right), \quad \text{for every model } M = (N, \mathcal{V}) \text{ for } \psi_{\text{boundaries}} \text{ satisfies } \vec{y}_{0} = 0 \text{ and } \vec{y}_{N} = C; \]

\[
\psi_{1} = [G] \left[ [B] \pi \Rightarrow \left( \bigwedge_{i=1}^{c} (B) b_{i} \wedge \left( \bigwedge_{i=1}^{c} (A) (\pi \rightarrow -b_{i}) \right) \right) \vee \psi_{1}^{*} \right], \quad \text{for every } n \in \{0, \ldots, N\} \text{ if } \vec{y}_{n} = C \text{ then either } n = N \text{ or } \vec{y}_{n+1} = 0, \text{ if } \vec{y}_{n} < N \text{ then } \vec{y}_{n+1} = \vec{y}_{n} + 1; \]

\[
\psi_{+}^{i} = \langle (B) b_{i} \rightarrow (A) (\pi \wedge -b_{i}) \wedge \psi_{+}^{i+1} \rangle \wedge \langle (B) -b_{i} \rightarrow (A) b_{i} \wedge \psi_{+}^{i+1} \rangle, \quad \text{formula } \psi_{+}^{i} \text{ encodes the bit-wise increment for every bit } b_{i}; \text{ it is triggered by } \psi_{1} \text{ on every interval } [n, n+1] \text{ for which } \psi_{+}^{C} \text{ is used for guaranteeing the correct bitwise increment in formulas } \psi_{+}^{i}; \text{ moreover it will be used in the following for correctly identifying tiles which are in the } \Rightarrow \text{ relation.} \]

\[
\psi_{-}^{i} = \neg \pi \wedge \bigwedge_{j=c}^{i} \langle (B) (\pi \wedge b_{i}) \leftrightarrow (A) (\pi \wedge b_{i}) \rangle, \quad \text{formula } \psi_{-}^{i} \text{ holds over an interval } [n, n'] \text{ if and only if } n < n' \text{ and } \text{bit}_{V}(n, b_{j}) = \text{bit}_{V}(n', b_{j}) \text{ for every } i \leq j \leq c; \]

Note that if \( \psi_{+}^{i} \) holds over \([n, n']\) then \( \vec{y}_{n} = \vec{y}_{n'} \). Formula \( \psi_{+}^{i} \) is used for guaranteeing the correct bitwise increment in formulas \( \psi_{+}^{i} \); moreover it will be used in the following for correctly identifying tiles which are in the \( \Rightarrow \) relation.

It is worth noticing that any model \( M = (N, \mathcal{V}) \) that satisfies \( \psi_{\text{tile}} = \psi_{3} \wedge \psi_{1} \wedge \psi_{\text{boundaries}} \wedge \psi_{1} \) fulfills some properties. First of all, the interplay between \( \psi_{\text{boundaries}} \) and \( \psi_{1} \) guarantees that \( N \) is a multiple of \( (C + 1) \) and thus, for suitably chosen \( \text{prefix} \) and \( \text{suffix} \), we can associate each point \( (x, y) \in \{0, \ldots, \text{prefix} + \text{suffix}\} \times \{0, \ldots, C\} \) to a point \( n \in \{0, \ldots, N\} \) by means of a bijection \( \text{map} : \{0, \ldots, \text{prefix} + \text{suffix}\} \times \{0, \ldots, C\} \to \{0, \ldots, N\} \) defined as \( \text{map}(x, y) = x \cdot (C + 1) + y \) (i.e., \( \text{map}^{-1}(n) = \left\lfloor \frac{n}{C+1} \right\rfloor \), \( n \) \% \( C \) \) where \( \% \) is the integer remainder operation). Moreover, let us observe that for every element \( (x, y) \) in the grid, we have that \( x \) is just implicitly encoded in the model by \( \text{map}(x, y) \) (i.e., \( x = \left\lfloor \frac{\text{map}(x, y)}{C+1} \right\rfloor \)), while \( y \) is both implicitly encoded (i.e., \( x = \lfloor \text{map}(x, y) \% C \rfloor \) and explicitly encoded by the the values of variables \( b_{1} \ldots b_{c} \) since it is easy to prove that \( \psi_{\text{boundaries}} \wedge \psi_{1} \) forces \( y = \text{map}(x, y) \). Finally, the conjuncts \( \psi_{3} \wedge \psi_{1} \) ensure that each point in \( n \in \{0, \ldots, N\} \), and thus, by means of \( \text{map} \), any point in the grid, is associated with exactly one tile, that is the unique tile variable that belongs to \( \mathcal{V}(\{n, n\}) \).

For the aforementioned properties, if we consider the function \( f \) that maps a function \( \text{tile} : \{0, \ldots, M\} \times \{0, \ldots, C\} \to \{0, \ldots, T\} \) in the model \( M = (M \cdot (C + 1), \mathcal{V}) \) where for every \( (x, y) \in \{0, \ldots, M\} \times \{0, \ldots, C\} \) we have \( t_{i} \in \mathcal{V}([\text{map}(x, y), \text{map}(x, y)]) \) if and only if \( \text{tile}(x, y) = i \) and \( \vec{y}_{\text{map}(x, y)} = y \), it is easy to prove that \( f \) is a bijection between the set of all such \( \text{tile} \) function, for every \( M \in \mathbb{N}^{+} \), and the set of all finite models for \( \psi_{\text{tile}} \). In summary, the detailed description above shows that any model for \( \psi_{\text{tile}} \) is basically a way to represent
a generic function $\text{tile} : \{0, \ldots, M\} \times \{0, \ldots, C\} \to \{0, \ldots, T\}$ and that, vice versa, each of such functions is represented by exactly one model of $\psi_{\text{tile}}$. The next step is the encoding of the constraints of Lemma 13 in $\mathbf{BA}_{\text{hom}}$ which allow to check whether there exists a function $\text{tile}$ that witnesses that $T$ is a positive instance. Such conditions, restricted to the finite case, are imposed by the following formulas:

\[
\psi_{0, C} = [G]\left(\left(\pi \land \bigwedge_{i=1}^{C} \neg b_i \rightarrow t_0\right) \land \left(\pi \land \bigwedge_{i=1}^{C} b_i \rightarrow t_T\right)\right),
\]

formula $\psi_{0, C}$ forces condition 1 of Problem 1, that is, the bottom tile of each column is 0 and the top tile of each column is $T$;

\[
\psi_{\Rightarrow} = [G]\left(\pi \land \{A\} \Rightarrow \pi \rightarrow \{A\} \left(\psi_{\min}^1 \land \bigvee_{(i,j) \in \Rightarrow} \left(\{B\} t_i \land \{A\} t_j\right)\right)\right),
\]

formula $\psi_{\Rightarrow}$ forces condition 2 of Problem 1, that is, each pair of grid points of type $(x, y), (x+1, y)$ must be labelled with two tiles that are in the $\Rightarrow$ relation. This is done by taking for each point $n < N$ the minimal interval $[n, n']$ with $n < n'$ and $\overline{y}_n = \overline{y}_{n'}$; then, the $\Rightarrow$ relation is forced between the pair of tile variables that hold over $[n, n]$ and $[n', n']$, respectively;

\[
\psi_{\min} = \psi_{\Rightarrow}^1 \land [B] \neg \psi_{\Rightarrow}^1 \overline{y}_n = \overline{y}_{n'}, \text{ and does not exist } n < n' < n' \text{ such that } \overline{y}_n = \overline{y}_{n'}, \text{ Let us notice that, for the constraints imposed by } \psi_{\min}, \text{ we have that } n'-n = C+1 \text{ and thus, according to the definition of } \text{map, we have } \text{map}^{-1}(n') = \left(\left\lfloor \frac{n}{C+1} \right\rfloor + 1, n \mod C\right); \text{ then, } \psi_{\min} \text{ holds on all and only those intervals whose endpoints represent horizontally adjacent points of the original grid;}
\]

\[
\psi_{\|} = [G]\left([B] \pi \land \bigvee_{i=1}^{C} \neg b_i \rightarrow \bigvee_{(i,j) \in \|} \left(\{B\} t_i \land \{A\} t_j\right)\right),
\]

formula $\psi_{\|}$ forces condition 3 of Problem 1, that is, each pair of grid points of type $(x, y), (x, y+1)$ must be labelled with two tiles that are in the $\|$ relation. The constraint can be easily imposed since the encoding ensures that vertical consecutive points in the grid corresponds to consecutive points in the model. The constraint is triggered on all the intervals of the type $[n, n+1]$, with the exception of the of the ones with $\overline{y}_n = C$. The constraint imposes that unique (thanks to $\psi_{\|} \land \psi_{0, C}$) pair of tile variables $(t_i, t_j)$ with $(t_i) \in \mathcal{V}([n, n])$ and $(t_j) \in \mathcal{V}([n', n'])$ must satisfy $(i, j) \in \|$. 
The satisfiability problem for the logic $\text{AB}$. Theorem 7.

Let $\psi$ be a finite linear order does not change if we replace it by full $\text{AB}$, that is, if we remove the one of such columns is the last one. This is done by means of a propositional letter $p$. The first conjunct of formula $\psi_{\text{prefix}}$ imposes that there exists an interval $[n, n']$ in the model for which $p \in \mathcal{V}([n, n'])$, $\overline{y}_n = 0$, and $\overline{y}_{n'} = C$ (i.e., $p$ “covers” at least one column). Moreover, for the homogeneity assumption, we have that $p \in \mathcal{V}([n'', n'])$ for every $n \leq n'' \leq n'$. The second conjunct imposes that for each $p$ labelled points $n$ there must exist a point $n' > n$ with $\overline{y}_n = \overline{y}_{n'}$ (this implicitly implies that $n$ is associated to a grid point which does not belong to the last column). Moreover, formula $[A] \psi^1_n$ imposes that $n'$ must belong to the last column. Finally, it is required that there exists $0 \leq i \leq T$ s.t. $t_i \in \mathcal{V}([n, n]) \cap \mathcal{V}([n', n'])$.

Notice that in the above definitions the use of the $\langle A \rangle$ operator enables us to deal with two key aspects:

1. we can predicate on all the intervals $[n, n']$ for any $n, n' \in \{0, \ldots, N\}$, whereas, by using the $\langle B \rangle$ operator alone, we could predicate only on intervals of the form $[0, n]$;
2. we can predicate on the ending point of any current interval $[n, n']$, i.e., the interval $[n', n']$. Such a feature is missing in the logic $\text{BD}_{\text{hom}}$ where we can predicate only on the beginning point of any current interval. For instance, the logic $\text{BD}_{\text{hom}}$ cannot express properties like $\psi^1_n$ which checks whether the same set of propositional letters holds over the two ending points of an interval.

Let us define now the formula $\varphi_T$ as $\varphi_T = \psi_{\text{tile}} \land \psi_{0,C} \land \psi_{\omega} \land \psi_{\overline{B}} \land \psi_{\text{prefix}}$. Since the models of $\psi_{\text{tile}}$ represent all and only the possible finite tiling functions for $T$ and $\psi_{0,C}, \psi_{\omega}, \psi_{\overline{B}}, \psi_{\text{prefix}}$ select the subset of such functions/models where conditions 1, 2, and 3, of Problem 1 together with condition 13 of Lemma 13 are fulfilled, we can prove the next result.

**Theorem 6.** Let $T = (T, \models, \mathbf{1}, C)$ be an instance of Problem 1. Then, $T$ is a positive instance if and only if the $\text{AB}_{\text{hom}}$ formula $\varphi_T$ is satisfiable over finite linear orders.

It is easy to see that $\varphi_T$ may be generated in LOGSPACE. To this end, it suffices to observe that we may define a multitape Turing Machine that performs the reduction using just a constant amount of working tapes, each one holding either $\lceil \log_2 T \rceil$ bits or $c$ bits. From such an observation and Theorem 5, we obtain the main result of the section.

**Theorem 7.** The satisfiability problem for the logic $\text{AB}_{\text{hom}}$ over finite linear orders is EXPSPACE-hard.

We conclude the section with some remarks that allow us to better understand how the homogeneity assumption affects the satisfiability problem of the considered HS fragments. First of all, we observe that the complexity of the satisfiability problem for $\text{AB}_{\text{hom}}$ over finite linear orders does not change if we replace it by full $\text{AB}$, that is, if we remove the
homogeneity assumption [BMM+14]). Moreover, we would like to point out that the proof of the EXPSPACE-hardness of the satisfiability problem for $\text{AB}_{\text{hom}}$, that is, the proof of Theorem 7 to which this entire section is devoted, does not make use of the homogeneity assumption. On the contrary, the homogeneity assumption marks a deep difference in $\text{BDA}$: we proved that the satisfiability problem for $\text{BDA}_{\text{hom}}$ is decidable in exponential space, whereas the problem is known to be undecidable for full $\text{BDA}$ [MM14, MMK10]. As for model checking, the model checking problem for $\text{AB}_{\text{hom}}$ over finite Kripke structures has been proved to be PSPACE-complete [BMM+19b], while here we proved that the satisfiability checking problem, over finite linear orders, belongs to a higher complexity class, namely, EXPSPACE. The tight complexity bound for the model checking problem over finite Kripke structures for $\text{BDA}_{\text{hom}}$ is still open: we only know that for its three maximal proper fragments $\text{AB}_{\text{hom}}$, $\text{DA}_{\text{hom}}$, and $\text{BD}_{\text{hom}}$ it is PSPACE-complete [BMM+19b, BMPS21b].

10. Conclusions

In this paper, we proved that, under the homogeneity assumption, the satisfiability checking problem for $\text{BDA}_{\text{hom}}$, over finite linear orders, is EXPSPACE-complete. This result stems a number of observations about the complexity landscape of the satisfiability and model checking problems for $\text{HS}$ fragments under homogeneity ($\text{HS}_{\text{hom}}$): (1) it improves the previously-known non-elementary upper bound [MMM+16]; (2) it identifies the first EXPSPACE-complete fragment of $\text{HS}_{\text{hom}}$ with respect to the satisfiability problem [BMM+19b].

For what concerns the satisfiability problem for $\text{BDA}_{\text{hom}}$, we already observed that the homogeneity assumption plays a crucial role only in the proof of the EXPSPACE membership of the problem (upper bound), while it does not play any role in the proof of the EXPSPACE-hardness of the problem (lower bound).

The results for $\text{BDA}_{\text{hom}}$ also shed some light on the problem of determining the exact complexity of the satisfiability checking problem for $\text{BE}_{\text{hom}}$, which is still open. As a matter of fact, $\text{BDA}_{\text{hom}}$ and $\text{BE}_{\text{hom}}$ are not comparable from the point of view of their expressiveness [BMM+14]. However, $\text{BDA}_{\text{hom}}$ captures a fragment of $\text{BE}_{\text{hom}}$, that is, $\text{BD}_{\text{hom}}$ extended with a restricted version of modality $\langle E \rangle$, namely, $\langle E \rangle \pi \psi = \langle A \rangle (\pi \land \psi)$, that allows one to predicate on the right endpoint of an interval. As shown in Section 9, this is the key property that causes the increase in complexity of the satisfiability checking problem from $\text{BD}_{\text{hom}}$ (PSPACE-complete) to $\text{BDA}_{\text{hom}}$ (EXPSPACE-complete). It is easy to see that the result given here can be easily extended to the case of homogeneous structures isomorphic to $\mathbb{N}$.

From a more practical standpoint, we showed how $\text{BDA}_{\text{hom}}$ may encode a very expressive fragment of generalized $\ast$-free regular expression, namely, the fragment that features prefix, infix, and lookahead. Thanks to the result obtained in this work, we have that the emptiness problem for the languages expressed by means of such a fragment is elementary (EXPSPACE-complete) in contrast to the non-elementary-hard result which was known for the the emptiness problem for full generalized $\ast$-free regular expression [Sto74].

As for future work, we plan to investigate the satisfiability/model checking problems for (fragments of) $\text{HS}_{\text{hom}}$, interpreted over the linear orders $\mathbb{Q}$ and $\mathbb{R}$. However, the precise characterization of the complexity of the satisfiability problem for $\text{BE}_{\text{hom}}$ remains the main open problem on the path to determining the exact complexity of the satisfiability problem for full $\text{HS}_{\text{hom}}$, over finite linear orders.
References

[All81] James F Allen. An interval-based representation of temporal knowledge. In *IJCAI*, volume 81, pages 221–226. Citeseer, 1981.

[BMM+14] Davide Bresolin, Dario Della Monica, Angelo Montanari, Pietro Sala, and Guido Sciavicco. Interval temporal logics over strongly discrete linear orders: Expressiveness and complexity. *Theor. Comput. Sci.*, 560:269–291, 2014. doi:10.1016/j.tcs.2014.03.033.

[BMM+17] Laura Bozzelli, Alberto Molinari, Angelo Montanari, Adriano Peron, and Pietro Sala. Satisfiability and model checking for the logic of sub-intervals under the homogeneity assumption. In Ioannis Chatzigiannakis, Piotr Indyk, Fabian Kuhn, and Anca Muscholl, editors, *44th International Colloquium on Automata, Languages, and Programming, ICALP 2017, July 10-14, 2017, Warsaw, Poland*, volume 80 of LIPIcs, pages 120:1–120:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017. doi:10.4230/LIPIcs.ICALP.2017.120.

[BMM+19a] Laura Bozzelli, Alberto Molinari, Angelo Montanari, Adriano Peron, and Pietro Sala. Interval vs. point temporal logic model checking: An expressiveness comparison. *ACM Trans. Comput. Log.*, 20(1):4:1–4:31, 2019. doi:10.1305/ndjfl/1093635589.

[BMM+19b] Laura Bozzelli, Alberto Molinari, Angelo Montanari, Adriano Peron, and Pietro Sala. Which fragments of the interval temporal logic HS are tractable in model checking? *Theor. Comput. Sci.*, 764:125–144, 2019. doi:10.1016/j.tcs.2018.04.011.

[BMM+19c] Davide Bresolin, Dario Della Monica, Angelo Montanari, Pietro Sala, and Guido Sciavicco. Decidability and complexity of the fragments of the modal logic of allen’s relations over the rationals. *Inf. Comput.*, 266:97–125, 2019. doi:10.1016/j.ic.2019.02.002.

[BMP19] Laura Bozzelli, Angelo Montanari, and Adrian Peron. Complexity analysis of a unifying algorithm for model checking interval temporal logic. In Johann Gamper, Sophie Pinchinat, and Guido Sciavicco, editors, *26th International Symposium on Temporal Representation and Reasoning, TIME 2019, October 16-19, 2019, Málaga, Spain*, volume 147 of LIPIcs, pages 18:1–18:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019. doi:10.4230/LIPIcs.TIME.2019.18.

[BMPS20] Laura Bozzelli, Angelo Montanari, Adriano Peron, and Pietro Sala. On a temporal logic of prefixes and infixes. In Javier Esparza and Daniel Král’, editors, *45th International Symposium on Mathematical Foundations of Computer Science, MFCS 2020, August 24-28, 2020, Prague, Czech Republic*, volume 170 of LIPIcs, pages 21:1–21:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPIcs.MFCS.2020.21.

[BMPS21a] Laura Bozzelli, Angelo Montanari, Adriano Peron, and Pietro Sala. Adding the relation meets to the temporal logic of prefixes and infixes makes it expspace-complete. In Pierre Ganty and Davide Bresolin, editors, *Proceedings 12th International Symposium on Games, Automata, Logics, and Formal Verification, GandALF 2021*, Padua, Italy, 20-22 September 2021, volume 346 of EPTCS, pages 179–194, 2021. doi:10.4204/EPTCS.346.12.

[BMPS21b] Laura Bozzelli, Angelo Montanari, Adriano Peron, and Pietro Sala. Pspace-completeness of the temporal logic of sub-intervals and suffixes. To appear in: Carlo Combi, Johan Eder, and Mark Reynolds eds. proceedings of 28th International Symposium on Temporal Representation and Reasoning, TIME 2021, September 27-29, 2021, Klagenfurt, Austria, LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.

[BMS07] Davide Bresolin, Angelo Montanari, and Pietro Sala. An optimal tableau-based decision algorithm for propositional neighborhood logic. In Wolfgang Thomas and Pascal Weil, editors, *STACS 2007, 24th Annual Symposium on Theoretical Aspects of Computer Science, Aachen, Germany, February 22-24, 2007, Proceedings*, volume 4393 of Lecture Notes in Computer Science, pages 549–560. Springer, 2007. doi:10.1007/978-3-540-70918-3_47.

[BMSS11] Davide Bresolin, Angelo Montanari, Pietro Sala, and Guido Sciavicco. Optimal tableau systems for propositional neighborhood logic over all, dense, and discrete linear orders. In Kai Brünnler and George Metcalfe, editors, *Automated Reasoning with Analytic Tableaux and Related Methods - 20th International Conference, TABLEAUX 2011, Bern, Switzerland, July 4-8, 2011. Proceedings,*
volume 6793 of *Lecture Notes in Computer Science*, pages 73–87. Springer, 2011. doi:10.1007/978-3-642-22119-4_8.

[CH97] Zhou Chaochen and Michael R. Hansen. An adequate first order interval logic. In Willem P. de Roever, Hans Langmaack, and Amir Pnueli, editors, *Compositionality: The Significant Difference, International Symposium, COMPOS’97*, Bad Malente, Germany, September 8-12, 1997. Revised Lectures, volume 1536 of *Lecture Notes in Computer Science*, pages 584–608. Springer, 1997. doi:10.1007/978-3-642-49213-5_23.

[GMM14] Giuseppe De Giacomo, Riccardo De Masellis, and Marco Montali. Reasoning on LTL on finite traces: Insensitivity to infiniteness. In Carla E. Brodley and Peter Stone, editors, *Proceedings of the Twenty-Eighth AAAI Conference on Artificial Intelligence*, July 27-31, 2014, Québec City, Québec, Canada, pages 1027–1033. AAAI Press, 2014. URL: http://www.aaai.org/ocs/index.php/AAAI/AAAI14/paper/view/8575.

[GMS04] Valentin Goranko, Angelo Montanari, and Guido Sciavicco. A road map of interval temporal logics and duration calculi. *Journal of Applied Non-Classical Logics*, 14(1-2):9–54, 2004. doi:10.3166/jancl.14.9-54.

[GMS06] Valentin Goranko, Angelo Montanari, Pietro Sala, and Guido Sciavicco. A general tableau method for propositional interval temporal logics: Theory and implementation. *J. Appl. Log.*, 4(3):305–330, 2006. doi:10.1016/j.jal.2005.06.012.

[GV13] Giuseppe De Giacomo and Moshe Y. Vardi. Linear temporal logic and linear dynamic logic on finite traces. In Francesca Rossi, editor, *IJCAI 2013, Proceedings of the 23rd International Joint Conference on Artificial Intelligence*, Beijing, China, August 3-9, 2013, pages 854–860. IJCAI/AAAI, 2013. URL: http://www.aaai.org/ocs/index.php/IJCAI/IJCAI13/paper/view/6997.

[HMM83] Joseph Halpern, Zohar Manna, and Ben Moszkowski. A hardware semantics based on temporal intervals. In Josep Diaz, editor, *Automata, Languages and Programming*, pages 278–291, Berlin, Heidelberg, 1983. Springer Berlin Heidelberg.

[HMS08] Ian Hodkinson, Angelo Montanari, and Guido Sciavicco. Non-finite axiomatizability and undecidability of interval temporal logics with C, D, and T. In CSL, volume 5213 of LNCS, pages 308–322. Springer, 2008. doi:10.1007/978-3-540-87531-4_23.

[HS91] Joseph Y. Halpern and Yoav Shoham. A propositional modal logic of time intervals. *Journal of ACM*, 38(4):935–962, 1991. doi:10.1145/115234.115351.

[MM14] Jerzy Marcinkowski and Jakub Michalisyzn. The undecidability of the logic of subintervals. *Fundam. Inform.*, 131(2):217–240, 2014. doi:10.3233/FI-2014-1011.

[MMK10] Jerzy Marcinkowski, Jakub Michalisyzn, and Emanuel Kieronski. B and D are enough to make the Halpern-Shoham logic undecidable. In Samson Abramsky, Cyril Gavoille, Claude Kirchner, Friedhelm Meyer auf der Heide, and Paul G. Spirakis, editors, *Automata, Languages and Programming, 37th International Colloquium, ICALP*, Bordeaux, France, July 6-10, Proceedings, Part II, volume 6199 of LNCS, pages 357–368. Springer, 2010. doi:10.1007/978-3-642-14162-1_30.

[MMM+16] Alberto Molinari, Angelo Montanari, Aniello Murano, Giuseppe Perelli, and Adriano Peron. Checking interval properties of computations. *Acta Inf.*, 53(6-8):587–619, 2016. doi:10.1007/s00236-015-0250-1.

[Mos83] Ben Moszkowski. *Reasoning About Digital Circuits*. PhD thesis, Stanford University, CA, 1983.

[MS12] Angelo Montanari and Pietro Sala. An optimal tableau system for the logic of temporal neighborhood over the reals. In Ben C. Moszkowski, Mark Reynolds, and Paolo Terenziani, editors, *19th International Symposium on Temporal Representation and Reasoning, TIME 2012, Leicester, United Kingdom, September 12-14, 2012*, pages 39–46. IEEE Computer Society, 2012. doi:10.1109/TIME.2012.18.

[MS13a] Angelo Montanari and Pietro Sala. Adding an equivalence relation to the interval logic AB\(\overline{\text{B}}\): Complexity and expressiveness. In *28th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2013, New Orleans, LA, USA, June 25-28, 2013*, pages 193–202. IEEE Computer Society, 2013. doi:10.1109/LICS.2013.25.

[MS13b] Angelo Montanari and Pietro Sala. Interval logics and \(\omega\)B-regular languages. In Adrian-Horia Dedu, Carlos Martín-Vide, and Bianca Truthe, editors, *Language and Automata Theory and Applications - 7th International Conference, LATA 2013, Bilbao, Spain, April 2-5, 2013.*
Proceedings, volume 7810 of Lecture Notes in Computer Science, pages 431–443. Springer, 2013. doi:10.1007/978-3-642-37064-9_38.

[Reg] regex101: build, test, and debug regex. https://regex101.com. Accessed: 2022-10-31.

[Roe80] Peter Roeper. Intervals and tenses. Journal of Philosophical Logic, 9:451–469, 1980. doi:10.1007/BF00262866.

[Sch16] Sylvain Schmitz. Complexity hierarchies beyond elementary. ACM Transactions on Computation Theory, 8(1):3:1–3:36, 2016. doi:10.1145/2858784.

[Sto74] Larry Joseph Stockmeyer. The complexity of decision problems in automata theory and logic. PhD thesis, Massachusetts Institute of Technology, 1974.

[vEB97] Peter van Emde Boas. The convenience of tilings. CRC Press, 1997.

[Ven90] Yde Venema. Expressiveness and completeness of an interval tense logic. Notre Dame Journal of Formal Logic, 31(4):529–547, 1990. doi:10.1305/ndjfl/1093635589.

[Ven91a] Yde Venema. A Modal Logic for Chopping Intervals. Journal of Logic and Computation, 1(4):453–476, 09 1991. arXiv:http://oup.prod.sis.lan/logcom/article-pdf/1/4/453/3817096/1-4-453.pdf, doi:10.1093/logcom/1.4.453.

[Ven91b] Yde Venema. A modal logic for chopping intervals. Journal of Logic and Computation, 1(4):453–476, 1991. doi:10.1093/logcom/1.4.453.

[VRDJ95] Guido Van Rossum and Fred L. Drake Jr. Python reference manual. Centrum voor Wiskunde en Informatica Amsterdam, 1995.