WEIGHTED NORM INEQUALITIES FOR WEYL MULTIPLIERS AND FOURIER MULTIPLIERS ON THE HEISENBERG GROUP

SAYAN BAGCHI AND SUNDARAM THANGAVELU

Abstract. In this paper we prove weighted norm inequalities for Weyl multipliers satisfying Mauceri’s condition. As an application, we prove certain multiplier theorems on the Heisenberg group and also show in the context of a theorem of Weis on operator valued Fourier multipliers that the R-boundedness of the derivative of the multiplier is not necessary for the boundedness of the multiplier transform.

1. Introduction and the main results

In this paper we are concerned with weighted norm inequalities for Weyl multipliers and the relevance of such inequalities in the study of Fourier multipliers on the Heisenberg group. Building upon a result of Mauceri [12] we prove certain weighted norm inequalities and then investigate the possibility of using them to prove boundedness of Fourier multipliers on the Heisenberg group. Weyl multipliers naturally occur in the context of Fourier multipliers on the Heisenberg group if we view the latter as operator-valued multipliers for the Euclidean Fourier transform. In this context, there is an interesting result of L. Weis [18] which gives a sufficient condition on the (operator-valued) multipliers, so that such Fourier multipliers are bounded on $L^p$ spaces of Banach space valued functions. However, our investigations have led to the conclusion that one of the conditions in the above mentioned theorem of Weis is not necessary for the boundedness of the multiplier transform. Nevertheless, we prove some versions of multiplier theorems on the Heisenberg group.

In order to set-up notation and state our main results, we begin with recalling some basic definitions. Consider the Euclidean Fourier transform defined on $L^1(\mathbb{R}^n)$ by

$$\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot \xi} f(x) dx.$$
It is well known that the map \( f \rightarrow \hat{f} \) extends to the whole of \( L^2(\mathbb{R}^n) \) as a unitary operator. Given a bounded measurable function \( m(\xi) \) on \( \mathbb{R}^n \) we can define a transformation \( T_m \) by setting

\[
\hat{(T_m f)}(\xi) = m(\xi) \hat{f}(\xi), \quad f \in L^2(\mathbb{R}^n).
\]

It is clear that \( T_m \) is a bounded operator on \( L^2(\mathbb{R}^n) \) but without further assumptions it need not extend to \( L^p(\mathbb{R}^n) \) as a bounded operator for \( p \neq 2 \). When it extends we say that \( m \) (or equivalently \( T_m \)) is a Fourier multiplier for \( L^p(\mathbb{R}^n) \). Some sufficient conditions are provided by Hörmander-Mihlin and Marcinkiewicz multiplier theorems, see [3]. For instance, when \( n = 1 \), the boundedness of \( m(\xi) \) together with that of \( \xi m'(\xi) \) is sufficient for the boundedness of \( T_m \) on \( L^p(\mathbb{R}) \) for all \( 1 < p < \infty \).

In the non-commutative set-up we have an analogue of the Fourier transform, namely the Weyl transform, which shares many important properties with the Fourier transform. As is well known, this transform is closely related to the Fourier transform on the Heisenberg group \( \mathbb{H}^n \). For \( f \in L^1 \cap L^2(\mathbb{C}^n) \), its Weyl transform \( W(f) \) is defined as an operator on \( L^2(\mathbb{R}^n) \) by the equation

\[
W(f) = \int_{\mathbb{C}^n} f(z)W(z)dz
\]

where \( W(z) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n) \) is the unitary transformation given by

\[
W(z)\varphi(\xi) = e^{i(x.\xi + \frac{1}{2}x.y)}\varphi(\xi + y), \quad \varphi \in L^2(\mathbb{R}^n).
\]

It is known that \( W \) takes \( L^2(\mathbb{C}^n) \) onto the space of the Hilbert-Schmidt operators on \( L^2(\mathbb{R}^n) \). Analogous to Fourier multipliers one can define Weyl multipliers as follows: given a bounded linear operator \( m \) on \( L^2(\mathbb{R}^n) \) we can define an operator \( T_m \) on \( L^2(\mathbb{C}^n) \) by

\[
W(T_m f) = mW(f)
\]

which is certainly bounded on \( L^2(\mathbb{C}^n) \). If this operator extends to a bounded linear operator on \( L^p(\mathbb{C}^n) \) then we say that \( m \) is a (left) Weyl multiplier for \( L^p(\mathbb{C}^n) \). We can also define right Weyl multipliers.

In [12] Mauceri has obtained sufficient conditions on a bounded linear operator \( m \) on \( L^2(\mathbb{R}^n) \) so that the Weyl multiplier \( T_m \) is bounded on \( L^p(\mathbb{C}^n) \). In order to state this result we need to introduce some notation. The spectral decomposition of the Hermite operator is given by

\[
H = -\Delta + |x|^2 = \sum_{j=0}^{\infty} (2j + n)P_j
\]
where $P_j$ are the projections onto the eigenspaces corresponding to the eigenvalues $(2j+n)$ of the Hermite operator. We can decompose $H$ as

$$H = \frac{1}{2} \sum_{j=1}^{n} (A_j A_j^* + A_j^* A_j)$$

where $A_j = \frac{\partial}{\partial x_j} + x_j$ and $A_j^* = -\frac{\partial}{\partial x_j} + x_j$ are the annihilation and creation operators. In terms of $A_j$ and $A_j^*$ we define noncommutative derivations

$$\delta_j m = [m, A_j], \quad \bar{\delta}_j m = [A_j^*, m].$$

For multi-indices $\alpha, \beta \in \mathbb{N}^n$ we define

$$\delta^\alpha = \delta_1^{\alpha_1} \delta_2^{\alpha_2} ... \delta_n^{\alpha_n}, \quad \bar{\delta}^\beta = \bar{\delta}_1^{\beta_1} \bar{\delta}_2^{\beta_2} ... \bar{\delta}_n^{\beta_n}.$$

We say that an operator $S \in \mathcal{B}(L^2(\mathbb{R}^n))$ is of class $C^k$ if

$$\delta^\alpha \bar{\delta}^\beta S \in \mathcal{B}(L^2(\mathbb{R}^n))$$

for all $\alpha, \beta \in \mathbb{N}^n$ such that $|\alpha| + |\beta| \leq k$. We also define

$$\chi_k = \sum_{2^{k-1} \leq 2j+n < 2^k} P_j$$

The following theorem has been proved in Mauceri [12].

**Theorem 1.1. (Mauceri)** Let $m \in \mathcal{B}(L^2(\mathbb{R}^n))$ be an operator of class $C^{n+1}$ which satisfies the following conditions: For all $\alpha, \beta \in \mathbb{N}^n, |\alpha| + |\beta| \leq n + 1$

$$\sup_{k \in \mathbb{N}^n} 2^{k(|\alpha|+|\beta|-n)|\langle \delta^\alpha \bar{\delta}^\beta m \rangle|^{2}_{HS}} \leq C.$$ 

Then the Weyl multiplier $T_m$ is bounded on $L^p(\mathbb{C}^n), 1 < p \leq 2$. If the above assumption is replaced by

$$\sup_{k \in \mathbb{N}^n} 2^{k(|\alpha|+|\beta|-n)|\langle \delta^\alpha \bar{\delta}^\beta m \rangle|^{2}_{HS}} \leq C.$$ 

then $T_m$ is bounded on $L^p(\mathbb{C}^n), 2 \leq p < \infty$.

In [12] Mauceri has obtained good estimates on the kernels associated to Weyl multipliers. It turns out that with a bit more effort we can do better than this.

**Theorem 1.2.** Let $m \in \mathcal{B}(L^2(\mathbb{R}^n))$ satisfy the condition (1.1) for all $\alpha, \beta \in \mathbb{N}^n, |\alpha| + |\beta| \leq n + 2$. Then the operator $T_m$ is bounded on $L^p(\mathbb{C}^n), 1 < p < \infty$. Moreover, for all $w \in A_{p/2}(\mathbb{C}^n), 2 < p < \infty$, it also satisfies the weighted norm inequality

$$\int_{\mathbb{C}^n} |T_m f(z)|^p w(z)dz \leq C \int_{\mathbb{C}^n} |f(z)|^p w(z)dz$$

for all $f \in L^p(\mathbb{C}^n, w)$. 
Note that the weight function $w$ is taken from $A_{p/2}$, not from $A_p$ as one would expect. If we increase the number of non-commutative derivatives from $n + 2$ to $2n + 2$ then we can allow $A_p$ weights in the weighted norm inequality.

**Theorem 1.3.** Let $m \in B(L^2(\mathbb{R}^n))$ satisfy the condition (1.1) for all $\alpha, \beta \in N^n, |\alpha| + |\beta| \leq 2(n + 1)$. Then, for all $w \in A_p(\mathbb{C}^n), 1 < p < \infty$,

$$
\int_{\mathbb{C}^n} |T_m f(z)|^p w(z) dz \leq C \int_{\mathbb{C}^n} |f(z)|^p w(z) dz
$$

for all $f \in L^p(\mathbb{C}^n, w)$.

In the definition of the Weyl transform we have made use of the unitary operators $W(z)$ acting on $L^2(\mathbb{R}^n)$ and mentioned that these are related to certain representations of the Heisenberg group. As a manifold $H^n = \mathbb{C}^n \times \mathbb{R}$ and the group law on $H^n$ is given by $(z, t)(w, s) = (z + w, t + s + \frac{1}{2} \Im(z \cdot w))$. For each $\lambda \in \mathbb{R} \setminus \{0\}$ we have an irreducible representation $\pi_\lambda$ of $H^n$ realised on $L^2(\mathbb{R}^n)$. The explicit expression for $\pi_\lambda$ is

$$
\pi_\lambda(z, t) \varphi(\xi) = e^{ixt} e^{i\lambda(x \cdot \xi + \frac{1}{2} x \cdot y)} \varphi(\xi + y)
$$

where $\varphi \in L^2(\mathbb{R}^n)$ and $z = x + iy$. It is clear that $\pi_1(z, 0) = W(z)$. Analogous to the Weyl transform we can also define the operators $W_\lambda(f)$ by

$$
W_\lambda(f) = \int_{\mathbb{C}^n} f(z) \pi_\lambda(z, 0) dz.
$$

These are also called the Weyl transforms in the literature.

The Fourier transform of $f \in L^1 \cap L^2(H^n)$ is defined to be the operator valued function

$$
\hat{f}(\lambda) = \int_{H^n} f(z, t) \pi_\lambda(z, t) dz dt.
$$

It is known that for each $\lambda \in \mathbb{R} \setminus \{0\}$, $\hat{f}(\lambda)$ is a Hilbert-Schmidt operator and we have inversion and Plancherel theorems, see e.g [17]. In analogy with Fourier multipliers on $\mathbb{R}^n$ and Weyl multipliers on $\mathbb{C}^n$ we can define multipliers for the (group) Fourier transform on $H^n$. Given a family of bounded linear operators $\{m(\lambda) : \lambda \in \mathbb{R}^*\}$ on $L^2(\mathbb{R}^n)$ we can define $T_m$ on $L^1 \cap L^2(H^n)$ by

$$
(T_m f)(\lambda) = m(\lambda) \hat{f}(\lambda).
$$

If the family $\{m(\lambda) : \lambda \in \mathbb{R}^*\}$ are uniformly bounded on $L^2(\mathbb{R}^n)$, it is clear (from Plancherel theorem) that $T_m$ is bounded on $L^2(H^n)$. We are interested in finding sufficient conditions on $m(\lambda)$ so that $T_m$ will extend to $L^p(H^n)$ as a bounded operator.

In this generality not much is known except for the results proved in the papers by Mauceri-de Michele [14] (for $n = 1$) and Lin [9], for general
Lin has also looked at the boundedness of $T_m$ on Hardy spaces. In [13] Mauceri studied zonal multipliers on the Heisenberg group. When $m(\lambda) = \varphi(H(\lambda))$, $H(\lambda)$ being the scaled Hermite operator $(-\Delta + \lambda^2|x|^2)$ on $\mathbb{R}^n$, the operator $T_m$ becomes $\varphi(L)$, where $L$ is the sublaplacian on $H^n$(which plays the role of $\Delta$ for $H^n$). There are several works on the $L^p$ boundedness of $\varphi(L)$ and the best possible result has been obtained by Muller-Stein [15] and Hebisch [6].

We now bring out the connection between Fourier multipliers on the Heisenberg group and operator valued multipliers for the Euclidean Fourier transform. Recalling the definition of $\hat{f}(\lambda)$ and noting that $\pi_{\lambda}(z, t) = e^{i\lambda t}$ we see that

$$\hat{f}(\lambda) = \int_{\mathbb{C}^n} \left( \int_{-\infty}^{\infty} f(z, t) e^{i\lambda t} dt \right) \pi_{\lambda}(z, 0) dz.$$  

Denoting the inner integral, which is the inverse Fourier transform of $f$ in the central variable, by $f_{\lambda}(z)$ we have $\hat{f}(\lambda) = W_\lambda(f_{\lambda})$. With this notation, the Fourier multiplier $T_m$ takes the form

$$T_m f(z, t) = \int_{-\infty}^{\infty} e^{-i\lambda t} M(\lambda)f_{\lambda}(z) d\lambda$$

where the operator $M(\lambda)$ is related to $m(\lambda)$ by the equation

$$W_\lambda(M(\lambda)) f_{\lambda} = m(\lambda) \hat{f}(\lambda) = m(\lambda) W_\lambda(f_{\lambda}).$$

This means that $M(\lambda)$ is a Weyl multiplier for each $\lambda \in \mathbb{R}^*$. 

We can identify $L^p(\mathbb{H}^n)$ with the space $L^p(\mathbb{R}, L^p(\mathbb{C}^n))$ consisting of all functions $F$ on $\mathbb{R}$ taking values in the Banach space $L^p(\mathbb{C}^n)$ for which the function $t \rightarrow \|F(t)\|_{L^p(\mathbb{C}^n)}$ belongs to $L^p(\mathbb{R})$. The identification is given by the correspondence $\hat{F}(t)(z) = f(z, t)$ for $f \in L^p(\mathbb{H}^n)$. With this identification, note that the function $f_{\lambda} \in L^p(\mathbb{C}^n)$ is nothing but the inverse Fourier transform of $F$:

$$f_{\lambda} = \int_{-\infty}^{\infty} e^{i\lambda t} F(t) dt = \hat{F}(-\lambda)$$

where the integral is taken in the sense of Bochner. Thus the action of the Heisenberg group Fourier multiplier $T_m$ on $f$ can be viewed as

$$T_m f(\cdot, t) = \int_{-\infty}^{\infty} e^{i\lambda t} M(-\lambda) \hat{F}(\lambda) d\lambda.$$ 

This means that Fourier multipliers on the Heisenberg group can be viewed as operator valued Euclidean Fourier multipliers acting on Banach space valued functions.

More generally, suppose $X$ and $Y$ are Banach spaces and $\lambda \rightarrow M(\lambda)$ is a function taking values in $B(X, Y)$, the Banach space of bounded linear operators from $X$ into $Y$. Then we can define operator valued Fourier multipliers
for functions taking values in $X$ by

$$T_M f(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\lambda t} M(\lambda) \hat{f}(\lambda) d\lambda.$$ 

Then for the operator $T_M$ to extend as a bounded linear operator from $L^p(\mathbb{R}, X)$ into $L^p(\mathbb{R}, Y)$ the spaces $X$ and $Y$ have to be UMD spaces. Moreover, unlike the scalar valued case, just the boundedness of the families $\{M(\lambda) : \lambda \in \mathbb{R}\}$ and $\{\lambda M'(\lambda) : \lambda \in \mathbb{R}\}$ is not enough. In this context L. Weis [18] has proved the following multiplier theorem for the Fourier transform.

**Theorem 1.4. (Weis)** Assume that $X$ is UMD and $M(\lambda) \in B(X, X)$ for each $\lambda \in \mathbb{R}$. Suppose $\{M(\lambda) : \lambda \in \mathbb{R}\}$ and $\{\lambda M'(\lambda) : \lambda \in \mathbb{R}\}$ are both R-bounded. Then the operator valued Fourier multiplier $T_M$ defined by

$$T_M f(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\lambda t} M(\lambda) \hat{f}(\lambda)$$

extends to $L^p(\mathbb{R}, X)$ as a bounded operator for all $1 < p < \infty$.

The R-boundedness of a family of operators $\tau \subset B(X, X)$ is defined using Rademacher functions $r_j, j \in \mathbb{N}$. For any sequence $M_j \in \tau$ and $x_j \in X$ it is required that there is a constant $C$ such that

$$\int_0^1 \| \sum_{j=1}^{\infty} r_j(u) M_j x_j \| du \leq C \int_0^1 \| \sum_{j=1}^{\infty} r_j(u) x_j \| du.$$ 

When $X = L^p(\mathbb{R}^n)$ the R-boundedness is equivalent to the vector valued inequality

$$\| \left( \sum_{j=1}^{\infty} |M_j f_j|^2 \right)^{1/2} \|_p \leq C \| \left( \sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \|_p$$

for all choices of $M_j \in \tau$ and $f_j \in L^p(\mathbb{R}^n)$. Given a family $\{M(\lambda) : \lambda \in \mathbb{R}\}$ of bounded linear operators acting on $L^p(\mathbb{R}^n)$ it would be interesting to find some conditions which will imply the R-boundedness. There are some special cases where we do have such conditions guaranteeing the R-boundedness.

Let $H(\lambda) = -\Delta + \lambda^2 |x|^2$ be the scaled Hermite operator on $\mathbb{R}^n$ whose spectrum is $\{2(k + n)|\lambda| : k \in \mathbb{N}\}$. Given a bounded function $\varphi$ defined on the half line $[0, \infty)$ we can define the operator $\varphi(H(\lambda))$ by spectral theorem. Taking $M(\lambda) = \varphi(H(\lambda))$ we can consider the Fourier multiplier

$$T_M f(x, t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\lambda t} \varphi(H(\lambda)) f^\lambda(x) d\lambda$$

where $f \in L^p(\mathbb{R}, \mathbb{R}^n) = L^p(\mathbb{R}^{n+1})$ and $f^\lambda$ stands for the inverse Fourier transform of $f$ in the $t$ variable. In this case the operator $T_M$ can be
interpreted as a spectral multiplier for the Grushin operator $-\Delta - |x|^2 \partial_t^2$ on $\mathbb{R}^{n+1}$ and such multipliers have been studied in [8] and [11]. It has been shown in [8] that standard Hormander conditions on $\varphi$ lead to R-Boundedness of the families $\{M(\lambda) : \lambda \in \mathbb{R}\}$ and $\{\lambda M'(\lambda) : \lambda \in \mathbb{R}\}$.

Another case where the R-boundedness of the multipliers can be proved is given by Weyl multipliers. Using Theorem 1.3 we can easily prove the following result.

**Theorem 1.5.** For each $\lambda \in \mathbb{R}$ let $m(\lambda) \in B(L^2(\mathbb{R}^n))$ and let $\tilde{M}(\lambda)$ be the corresponding Weyl multiplier defined by

$$W(\tilde{M}(\lambda) f) = m(\lambda) W(f).$$

In our earlier notation, $\tilde{M}(\lambda) = T_{m(\lambda)}$. Assume that for each $\lambda$ both $m(\lambda)$ and $\lambda m'(\lambda)$ satisfy the condition (1.1) for all $\alpha, \beta \in \mathbb{N}^n, |\alpha| + |\beta| \leq 2n + 2$.

Then the operator valued Fourier multiplier defined by

$$T_{\tilde{M}} f(z, t) = \int_{-\infty}^{\infty} e^{-i\lambda t} \tilde{M}(\lambda) f(\lambda z) d\lambda$$

extends to be a bounded operator on $L^p(\mathbb{R}, L^p(\mathbb{C}^n))$ for any $1 < p < \infty$.

Since we are considering $X = L^p(\mathbb{C}^n)$ the R-boundedness of the family $\{M(\lambda) : \lambda \in \mathbb{R}\}$ is equivalent to the vector valued inequality for the sequence $\tilde{M}(\lambda_j)$ for any choice of $\lambda_j \in \mathbb{R}$. According to a theorem of Rubio de Francia [16] the vector valued inequality will be a consequence of the weighted norm inequality for $\tilde{M}(\lambda)$:

$$\int_{\mathbb{C}^n} |\tilde{M}(\lambda) f(z)|^2 w(z) dz \leq C \int_{\mathbb{C}^n} |f(z)|^2 w(z) dz$$

for all $w \in A_2(\mathbb{C}^n)$ uniformly in $\lambda$ which will follow from Theorem 1.3.

In [18] Weis has looked at the necessity of the conditions in his theorem. He has proved the following converse to his theorem. Suppose the families $\{M(\lambda) : \lambda \in \mathbb{R}\}$ and $\{\lambda M'(\lambda) : \lambda \in \mathbb{R}\}$ are uniformly bounded on $X$. If the operator $T_M$ is bounded on $L^p(\mathbb{R}, X)$ then for any $a \neq 0$ the family $\{a 2^n M(a 2^n) : n \in \mathbb{Z}\}$ is R-bounded. However, it is not known if the R-boundedness of $\{\lambda M'(\lambda) : \lambda \in \mathbb{R}\}$ is necessary or not. Our investigations on Fourier multipliers on the Heisenberg group have led us to the following result.

**Theorem 1.6.** Let $T_M$ be as in the theorem of Weis. The R-boundedness of $\{\lambda M'(\lambda) : \lambda \in \mathbb{R}\}$ is not necessary for the boundedness of $T_M$ on $L^p(\mathbb{R}, X)$.

As we have noted, when $X = L^p(\mathbb{C}^n)$ the space $L^p(\mathbb{R}, L^p(\mathbb{C}^n))$ can be identified with $L^p(\mathbb{H}^{2n})$ where $\mathbb{H}^{2n}$ is the Heisenberg group. By considering the Riesz transforms on the Heisenberg group which can be realised as operator valued Fourier multipliers, we can prove the theorem stated above.
Coming back to Fourier multipliers on $\mathbb{H}^n$ recall that the transforms $T_m$ can be realised as
\[
T_m f(z,t) = \int_{-\infty}^{\infty} e^{-i\lambda} M(\lambda) f^{\lambda}(z) d\lambda
\]
where $W_{\lambda}(M(\lambda)f^{\lambda}) = m(\lambda) W_{\lambda}(f^{\lambda})$. It is therefore natural to ask whether the R-boundedness of the families \{\$M(\lambda) : \lambda \in \mathbb{R}^s\}$ and \{\$\lambda M'(\lambda) : \lambda \in \mathbb{R}^s\}$ can be guaranteed by some conditions on the multiplier $m(\lambda)$ and its derivative $m'(\lambda)$. When each $m(\lambda)$ is a Euclidean Fourier multiplier on $L^p(\mathbb{R}^n)$ we have a simple result.

**Theorem 1.7.** Let $\{m(\lambda) : \lambda \in \mathbb{R} \setminus \{0\}\}$ be a family of Euclidean Fourier multipliers on $L^p(\mathbb{R}^n), 1 < p < \infty$ such that both the families $\{m(\lambda) : \lambda \in \mathbb{R} \setminus \{0\}\}$ and $\{\lambda m'(\lambda) : \lambda \in \mathbb{R} \setminus \{0\}\}$ are R-bounded on $L^p(\mathbb{R}^n)$. Then the transform $T_m$ defined by $(T_m f)(\lambda) = m(\lambda) f^{\lambda}(\lambda)$ on $L^p \cap L^2(\mathbb{H}^n)$ extends to $L^p(\mathbb{H}^n)$ as a bounded linear operator for $1 < p < \infty$.

But the story of general multipliers is quite different. We need to find sufficient conditions on $m(\lambda)$ and $\lambda m'(\lambda)$ so that the operator families \{\$M(\lambda) : \lambda \in \mathbb{R}^s\}$ and \{\$\lambda M'(\lambda) : \lambda \in \mathbb{R}^s\}$ are R-bounded. As we have to deal with multipliers for $W_{\lambda}$ as well as for $W = W_1$ it is convenient to use the following notation. Given a bounded linear operator $S$ on $L^2(\mathbb{R}^n)$ we use the notation $T^\lambda_S$ to stand for the operator defined by $W_{\lambda}(T^\lambda_S g) = SW_{\lambda}(g)$. In this notation, $M(\lambda) = T^\lambda_{m(\lambda)}$ and we will use both notations depending on the context. It can be shown that $T^\lambda_{S}$ is conjugate to $T^\lambda_{S'}$ for some $S'$ which is related to $S$. We also need a family of non-commutative derivations depending on the parameter $\lambda$.

We define $\delta_j(\lambda) m(\lambda) = |\lambda|^{-1/2} [m(\lambda), A_j(\lambda)]$ where $A_j(\lambda) = \frac{\partial}{\partial \xi_j} + |\lambda| \xi_j$ and $\delta_j(\lambda) m(\lambda) = |\lambda|^{-1/2} [A_j^*(\lambda), m(\lambda)]$, where $A_j^*(\lambda) = -\frac{\partial}{\partial \xi_j} + |\lambda| \xi_j$. Considering the scaled Hermite operator $H(\lambda)$ which can be written as
\[
H(\lambda) = \frac{1}{2} \sum_{j=1}^{n} (A_j(\lambda) A_j^*(\lambda) + A_j^*(\lambda) A_j(\lambda))
\]
we define the dyadic spectral projections $\chi_k(\lambda) = \sum_{2^k-1 \leq 2j+n < 2^k} P_j(\lambda)$. For the sake of brevity, let us say that an operator $m$ satisfies the condition $(M_l(\lambda))$, $M$ for Mauceri, if $m$ is of class $C^l$ and satisfies the following estimates: For all $\alpha, \beta \in \mathbb{N}^n, |\alpha| + |\beta| \leq l$
\[
\sup_{k \in \mathbb{N}^n} 2^{k(|\alpha| + |\beta| - n)} \|(\delta(\lambda)^\alpha \delta(\lambda)^\beta m) \chi_k(\lambda)\|_{HS}^2 \leq C
\]
with a constant $C$ independent of $\lambda$. We simply write $(M_l)$ in place of $(M_l(1))$. 

If \( \delta_r f(z) = f(rz) \) stands for the dilation, then it can be shown that
\[
\delta^{-1} \sqrt{|\lambda|} T^\lambda_{\tilde{m}(\lambda)} = T^1_{\tilde{m}(\lambda)}
\]
where \( \tilde{m}(\lambda) = \delta^{-1} \sqrt{|\lambda|} m(\lambda) \). Equivalently,
\[
(1.3) \quad M(\lambda) = T^\lambda_{\tilde{m}(\lambda)} = \delta \sqrt{|\lambda|} T^1_{\tilde{m}(\lambda)} \delta^{-1} \sqrt{|\lambda|}
\]
Note that \( T^1_{\tilde{m}(\lambda)} \) is a Weyl multiplier:
\[
W(T^1_{\tilde{m}(\lambda)} f) = \tilde{m}(\lambda) W(f), \quad f \in L^2(\mathbb{C}^n).
\]
According to a theorem of Rubio de Francia [16] the R-boundedness of the family \( \{ M(\lambda) = T^\lambda_{\tilde{m}(\lambda)} : \lambda \in \mathbb{R}^* \} \) follows if we can prove the weighted norm inequality
\[
\int_{\mathbb{C}^n} |T^\lambda_{\tilde{m}(\lambda)} f(z)|^2 w(z) dz \leq C_w \int_{\mathbb{C}^n} |f(z)|^2 w(z) dz
\]
for all \( w \in A_2(\mathbb{C}^n) \) uniformly in \( \lambda \). In view of (1.3) it is enough to prove this inequality for \( T^1_{\tilde{m}(\lambda)}, \lambda \in \mathbb{R}^* \). This leads to the following result.

Theorem 1.8. For every \( \lambda \in \mathbb{R}^* \) let \( m(\lambda) \in B(L^2(\mathbb{R}^n)) \) satisfy the condition \((M_{2n+2}(\lambda))\). Then for every \( 1 < p < \infty \), the Weyl multiplier \( M(\lambda) = T^\lambda_{m(\lambda)} \) is R-bounded on \( L^p(\mathbb{C}^n) \).

This takes care of the R-boundedness of the family \( \{ M(\lambda) : \lambda \in \mathbb{R}^* \} \).
The R-boundedness of the family \( \{ \lambda M'_{\lambda}(\lambda) : \lambda \in \mathbb{R}^* \} \) turns out to be even more complicated. Since we have
\[
M(\lambda) = \delta \sqrt{|\lambda|} T^1_{\tilde{m}(\lambda)} \delta^{-1} \sqrt{|\lambda|}
\]
the derivative of \( M(\lambda) \) involves several terms. We will show that
\[
2\lambda \frac{d}{d\lambda} M(\lambda) = [B, M(\lambda)] + T^\lambda_{[x, \nabla, m(\lambda)]} + T^\lambda_{2\lambda \frac{d}{d\lambda} m(\lambda)}
\]
where \( \nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) \) and \( B = \sum_{j=1}^n (z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j}) \). The second and third terms are easy to handle whereas the first term is not. It turns out that \([B, M(\lambda)]\) is not even a Weyl multiplier and hence not accessible to our methods.

The Riesz transforms \( R_j \) on the Heisenberg group \( H^n \) are defined via the multipliers \( A_j(\lambda) H(\lambda)^{-1/2} \) and it is well known that they are bounded on \( L^p(H^n) \). In this case it turns out \( \{ [B, M(\lambda)] : \lambda \in \mathbb{R}^* \} \) is not R-bounded. As a consequence of this we obtain Theorem 1.6. As we discussed earlier, because of the behavior of \([B, M(\lambda)]\) we cannot get any sufficient condition for \( L^p \)-boundedness of the operator \( T_m \) in terms of the condition \((M)\). However we have the following result.
Theorem 1.9. Let the two families of operators \{m(\lambda) : \lambda \in \mathbb{R}^*\}, \{\lambda m'(\lambda) : \lambda \in \mathbb{R}^*\} satisfy the conditions \((M_{2n+3}(\lambda))\) and \((M_{2n+2}(\lambda))\) respectively. Let \(L\) stand for the sublaplacian on \(H^n\). Then the multiplier transform \(T_m\) on \(H^n\) satisfies
\[
\|T_m f\|_p \leq C \|L^{1/2} f\|_p, \quad 1 < p < \infty.
\]

We also have the following result. Let \(T(n) \subset U(n)\) be the torus which acts on \(\mathbb{C}^n\) by \(\rho(\sigma)f(z) = f(e^{i\theta_1}z_1, ..., e^{i\theta_n}z_n)\) if \(\sigma\) is the diagonal matrix with entries \(e^{i\theta_1}, ..., e^{i\theta_n}\). Then
\[
Rf(z) = \int_{T(n)} \rho(\sigma)f(z)d\sigma
\]
is polyradial and it can be easily checked that \(\|Rf\|_p \leq \|f\|_p\).

Theorem 1.10. Let the two families of operators \{m(\lambda) : \lambda \in \mathbb{R}^*\}, and \{\lambda m'(\lambda) : \lambda \in \mathbb{R}^*\} satisfies the condition \((M_{2n+3}(\lambda))\) and \((M_{2n+2}(\lambda))\) respectively. Then for the multiplier transform \(T_m\) on \(H^n\) we have
\[
\|R \circ T_m \circ Rf\|_p \leq C \|f\|_p, \quad 1 < p < \infty.
\]

We conclude the introduction with the following remarks. It is possible to improve slightly the results of Theorems 1.2 and 1.3. In a recent article \cite{1} we have studied the \(L^p\) boundedness of Hermite pseudo-multipliers. In that connection we have introduced modified Mauceri conditions. Following the ideas and techniques used in that paper we can prove theorem 1.2 for multipliers of class \(C^{n+1}\). Also in Theorem 1.3 we can reduce the number of derivatives from \(2n+2\) to \(2n+1\). The plan of the paper is as follows. In the next section we set up notation, recall results from the theory of Weyl transforms and prove some preliminary lemmas needed later. In Section 3 we take up the problem of estimating certain kernels associated to Weyl multipliers. In Section 4 we prove our main results.

2. Preliminaries

We consider the Heisenberg group \(H^n = \mathbb{C}^n \times \mathbb{R}\) equipped with the group law
\[
(z, t)(w, s) = (z + w, t + s + \frac{1}{2} \Im(z \bar{w})).
\]
This is a step two nilpotent Lie group and the Haar measure on \(H^n\) is simply the Lebesgue measure \(dzdt\) on \(\mathbb{C}^n \times \mathbb{R}\). In order to define the Fourier transform on \(H^n\) we need to recall certain families of irreducible unitary representations of \(H^n\).

For each \(\lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}\) and \((z, t) \in H^n\) consider the operator \(\pi_\lambda(z, t)\) defined on \(L^2(\mathbb{R}^n)\) by
\[
\pi_\lambda(z, t) \varphi(\xi) = e^{i\lambda t} e^{i\lambda(x \cdot \xi + \frac{1}{2} x \cdot y)} \varphi(\xi + y)
\]
where \( \varphi \in L^2(\mathbb{R}^n) \). It can be shown that each \( \pi_{\lambda} \) is an irreducible unitary representation of \( H^n \). Moreover, by a theorem of Stone-von Neumann any irreducible unitary representation of \( H^n \) which is non-trivial at the center of \( H^n \) is unitarily equivalent to \( \pi_{\lambda} \) for a unique \( \lambda \in \mathbb{R}^* \). Apart from \( \pi_{\lambda} \), there is another family of one dimensional irreducible unitary representations. As they do not play any role in the Plancherel theorem we do not consider them. See Folland \[5\] and Thangavelu \[17\] for more on these representations.

Given \( f \in L^1(H^n) \) we can define the operator \( \hat{f}(\lambda) = \pi_{\lambda}(f) \) by

\[
\hat{f}(\lambda) = \int_{\mathbb{C}^n} f(z,t)\pi_{\lambda}(z,t)dzdt.
\]

The operator valued function \( \lambda \to \hat{f}(\lambda) \) is called the (group) Fourier transform of \( f \) on \( H^n \). If we let \( f^\lambda \) stand for the inverse Fourier transform of \( f \) in the \( t \)-variable, i.e.

\[
f^\lambda(z) = \int_{-\infty}^{\infty} e^{i\lambda t} f(z,t)dzdt
\]

then we have

\[
\hat{f}(\lambda) = \int_{\mathbb{C}^n} f^\lambda(z)\pi_{\lambda}(z,0)dz.
\]

When \( f \in L^1 \cap L^2(H^n) \), it can be shown that \( \hat{f}(\lambda) \) is a Hilbert-Schmidt operator and

\[
||\hat{f}(\lambda)||^2_{HS} = (2\pi)^n|\lambda|^{-n} \int_{\mathbb{C}^n} |f^\lambda(z)|^2dz.
\]

In view of this Plancherel theorem for the Fourier transform takes the form

\[
||f||^2_2 = \int_{-\infty}^{\infty} ||\hat{f}(\lambda)||^2_{HS}d\mu(z)
\]

where \( d\mu(\lambda) = (2\pi)^{-n-1}|\lambda|^nd\lambda \) is the Plancherel measure. We also have the inversion formula

\[
f(z,t) = \int_{-\infty}^{\infty} tr(\pi_{\lambda}(z,t)^*\hat{f}(\lambda))d\mu(\lambda)
\]

for suitable functions.

Given a family of bounded linear operators \( m(\lambda), \lambda \in \mathbb{R}^* \) we can define \( T_m \) by \( \overline{T_m f}(\lambda) = m(\lambda)\hat{f}(\lambda) \) which can also be written in the form

\[
T_m f(z,t) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-i\lambda t} T^\lambda m f(z)d\lambda.
\]

The study of \( T^\lambda m \) can be reduced to the study of Weyl multipliers using the following lemma.
Lemma 2.1. For each \( \lambda > 0 \) we have \( \delta_{\sqrt{\lambda}}^{-1} T_{m(\lambda)}^\lambda \delta_{\sqrt{\lambda}} = T_{\tilde{m}(\lambda)}^1 \) where \( \tilde{m}(\lambda) = \delta_{\sqrt{\lambda}}^{-1} m(\lambda) \delta_{\sqrt{\lambda}}. \)

Proof. For \( \lambda > 0 \) an easy calculation shows that

\[
\pi_\lambda(z, t) = e^{i\lambda t} \delta_\lambda \pi_1(\sqrt{\lambda}z, 0) \delta_{\sqrt{\lambda}}^{-1}
\]

Since \( W_\lambda(f) \) is defined in terms of \( \pi_\lambda(z, 0) \) it follows that

\[
W_\lambda(\delta_\lambda f) = \lambda^{-n} \delta_\lambda W(f) \delta_{\sqrt{\lambda}}^{-1}
\]

for any \( f \in L^2(\mathbb{C}^n) \). Applying this identity to \( \delta_{\sqrt{\lambda}}^{-1} T_{m(\lambda)}^\lambda \delta_{\sqrt{\lambda}} f \) we see that

\[
m(\lambda) W_\lambda(\delta_\lambda f) = \lambda^{-n} \delta_\lambda W(\delta_{\sqrt{\lambda}}^{-1} T_{m(\lambda)}^\lambda \delta_{\sqrt{\lambda}} f) \delta_{\sqrt{\lambda}}^{-1}
\]

This immediately gives

\[
W(\delta_{\sqrt{\lambda}}^{-1} T_{m(\lambda)}^\lambda \delta_{\sqrt{\lambda}} f) = \delta_{\sqrt{\lambda}}^{-1} m(\lambda) \delta_{\sqrt{\lambda}} W(f)
\]

as desired. \( \square \)

From the above lemma it is clear that, though \( T_{m(\lambda)}^\lambda \) is not a Weyl multiplier, \( \delta_{\sqrt{\lambda}}^{-1} T_{m(\lambda)}^\lambda \delta_{\sqrt{\lambda}} \) is. The noncommutative derivations \( \delta_j \) and \( \tilde{\delta}_j \) acting on \( \tilde{m}(\lambda) \) can be converted into certain derivations acting on \( m(\lambda) \) itself.

Recall that for an operator \( m \) on \( L^2(\mathbb{R}^n) \) we have defined \( \delta_j m = [m, A_j] \) and \( \delta_j m = [A_j^*, m] \) where \( A_j = \frac{\partial}{\partial \xi_j} + \xi_j \) and \( A_j^* = -\frac{\partial}{\partial \xi_j} + \xi_j \). Under the dilation \( \delta_{\sqrt{\lambda}} \) we have

\[
\delta_{\sqrt{\lambda}} A_j \delta_{\sqrt{\lambda}}^{-1} = \lambda^{-1/2} \left( \frac{\partial}{\partial \xi_j} + \lambda \xi_j \right) = \lambda^{-1/2} A_j(\lambda)
\]

and

\[
\delta_{\sqrt{\lambda}} A_j^* \delta_{\sqrt{\lambda}}^{-1} = \lambda^{-1/2} \left( -\frac{\partial}{\partial \xi_j} + \lambda \xi_j \right) = \lambda^{-1/2} A_j^*(\lambda).
\]

In view of these relations, \( \delta_j \tilde{m}(\lambda) = \lambda^{-1/2} \delta_{\sqrt{\lambda}}^{-1}[m(\lambda), A_j(\lambda)] \delta_{\sqrt{\lambda}} \) and \( \tilde{\delta}_j \tilde{m}(\lambda) = \lambda^{-1/2} \delta_{\sqrt{\lambda}}^{-1}[A_j^*(\lambda), m(\lambda)] \delta_{\sqrt{\lambda}}. \)

Lemma 2.2. For \( \lambda > 0 \) we have \( \delta_j \tilde{m}(\lambda) = \lambda^{-1/2} \delta_{\sqrt{\lambda}}^{-1}[\delta_j(\lambda) m(\lambda)] \delta_{\sqrt{\lambda}} \) and \( \tilde{\delta}_j \tilde{m}(\lambda) = \lambda^{-1/2} \delta_{\sqrt{\lambda}}^{-1}[\tilde{\delta}_j(\lambda) m(\lambda)] \delta_{\sqrt{\lambda}} \) where \( \delta_j(\lambda) m(\lambda) = \lambda^{-1/2}[m(\lambda), A_j(\lambda)] \) and \( \tilde{\delta}_j(\lambda) m(\lambda) = \lambda^{-1/2}[A_j^*(\lambda), m(\lambda)]. \)

On the Heisenberg group we have the left invariant vector fields

\[
T = \frac{\partial}{\partial t} \quad X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t} \quad Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t}
\]

which give rise to a family of operators \( Z_j(\lambda) \) and \( Z_j(\lambda) \) as follows:

\[
\frac{1}{2}(X_j - iY_j)(e^{i\lambda t} f(z)) = e^{i\lambda t} Z_j(\lambda) f(z) \quad \text{and} \quad \frac{1}{2}(X_j + iY_j)(e^{i\lambda t} f(z)) = \]

\[
\frac{1}{2}(X_j - iY_j)(e^{i\lambda t} f(z)) = \quad \text{and} \quad \frac{1}{2}(X_j + iY_j)(e^{i\lambda t} f(z)) =
\]
the special Hermite operator which can be written as 

\[ e^{i \lambda t} \bar{Z}_j(\lambda) f(z). \]

Explicitly, we have 

\[ Z_j(\lambda) = \frac{\partial}{\partial z_j} - \frac{i}{4} \bar{z}_j \] and 

\[ \bar{Z}_j(\lambda) = \frac{\partial}{\partial \bar{z}_j} + \frac{i}{4} z_j. \]

We also have their right invariant counterparts 

\[ X_j^R = \frac{\partial}{\partial x_j} + \frac{1}{2} x_j \frac{\partial}{\partial y} \] and 

\[ Y_j^R = \frac{\partial}{\partial y_j} + \frac{1}{2} y_j \frac{\partial}{\partial x}. \]

These gives us the vector fields 

\[ Z_j^R(\lambda) = \frac{\partial}{\partial z_j} + \frac{i}{4} \bar{z}_j \] and 

\[ \bar{Z}_j^R(\lambda) = \frac{\partial}{\partial \bar{z}_j} - \frac{i}{4} z_j. \]

We record the following properties for later use.

**Lemma 2.3.** For any \( \lambda > 0, f \in L^2(\mathbb{C}^n) \) we have:

1. \( W_\lambda(Z_j(\lambda)f) = iW_\lambda(f)A_j^*(\lambda), \quad W_\lambda(\bar{Z}_j(\lambda)f) = iW_\lambda(f)A_j(\lambda), \)
2. \( W_\lambda(Z_j^R(\lambda)f) = iA_j^*(\lambda)W_\lambda(f), \quad W_\lambda(Z_j^R(\lambda)f) = iA_j(\lambda)W_\lambda(f), \)
3. \( \lambda W_\lambda(z_jf) = 2i[W_\lambda(f), A_j^*(\lambda)], \quad \lambda W_\lambda(\bar{z}_jf) = 2i[A_j(\lambda), W_\lambda(f)]. \)

The role of Laplacian \( \Delta \) for \( H^n \) is played by the sublaplacian \( \mathcal{L} \) defined by

\[ \mathcal{L} = -\sum_{j=1}^{n} (X_j^2 + Y_j^2) = -\frac{1}{2} \sum_{j=1}^{n} (X_j - Y_j)(X_j + Y_j) + (X_j + Y_j)(X_j - Y_j). \]

The operator \( L_\lambda \) defined by the condition \( \mathcal{L}(e^{i \lambda t} f(z)) = e^{i \lambda t} L_\lambda(f(z)) \) is called the special Hermite operator which can be written as

\[ L_\lambda = -2 \sum_{j=1}^{n} Z_j(\lambda) \bar{Z}_j(\lambda) + \bar{Z}_j(\lambda)Z_j(\lambda). \]

In view of Lemma 2.3 we have

\[ W_\lambda(L_\lambda f) = W_\lambda(f) \sum_{j=1}^{n} (A_j^*(\lambda)A_j(\lambda) + A_j(\lambda)A_j^*(\lambda)) = W_\lambda(f)H(\lambda) \]

for functions on \( \mathbb{C}^n \). This leads to the equation \( \mathcal{L}(\hat{f}(\lambda)) = \hat{f}(\lambda)H(\lambda) \) for functions on \( H^n \). If \( \mathcal{L}^R \) stand for the right invariant sublaplacian then we have \( \mathcal{L}(\hat{f}(\lambda)) = H(\lambda)\hat{f}(\lambda) \).

In order to study the R-boundedness of the family \( \lambda M_\lambda(\lambda) = \lambda \frac{d}{dx} T_{m(\lambda)} \) we need to get a usable expression for the derivative. We introduce the following operators. Let \( \nabla \) stand for the gradient on \( \mathbb{R}^n, \nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}) \) and let \( \xi \cdot \nabla = \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial x_j} \). On \( \mathbb{C}^n \) we let

\[ B = \sum_{j=1}^{n} (z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j}) = \sum_{j=1}^{n} (x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}). \]

The dilatation operator \( \delta_{\sqrt{\lambda}}, \lambda > 0 \) on \( \mathbb{R}^n \) can be expressed as

\[ \delta_{\sqrt{\lambda}} \varphi(\xi) = e^{\frac{i \xi}{2} \nabla(\log \lambda)} \varphi(\xi) \]

and the same on \( \mathbb{C}^n \) can be written as

\[ \delta_{\sqrt{\lambda}} f(z) = e^{\frac{1}{4} B(\log \lambda)} f(z). \]
Using these expressions we can easily prove the following.

**Lemma 2.4.** For $\lambda > 0$ we have

$$2\lambda \frac{d}{d\lambda} T_\lambda m(\lambda) = [B, T_{\hat{m}(\lambda)}] + T_{[m(\lambda), \xi \cdot \nabla]} + T_{\lambda} \frac{d}{d\lambda} m(\lambda).$$

**Proof.** Differentiating the equation $T_\lambda m(\lambda) = \delta \sqrt{\lambda} T_1 \tilde{m}(\lambda) \delta - 1 \sqrt{\lambda}$ we get

$$2\lambda \frac{d}{d\lambda} T_\lambda m(\lambda) = \delta \sqrt{\lambda} [B, T_{\tilde{m}(\lambda)}] \delta - 1 \sqrt{\lambda} + \delta \sqrt{\lambda} 2\lambda \frac{d}{d\lambda} T_{\lambda} m(\lambda) \delta - 1 \sqrt{\lambda}.$$ 

Since $\tilde{m}(\lambda) = \delta - 1 \sqrt{\lambda} m(\lambda) \delta \sqrt{\lambda}$, by differentiating the equation

$$W(T_{\tilde{m}(\lambda)} f) = \delta^{-1} \sqrt{\lambda} m(\lambda) \delta \sqrt{\lambda} W(f)$$

and using the expression for $\delta \sqrt{\lambda}$ we get

$$2\lambda \frac{d}{d\lambda} \tilde{m}(\lambda) = \delta^{-1} \sqrt{\lambda} 2\lambda \frac{d}{d\lambda} m(\lambda) \delta \sqrt{\lambda} + \delta^{-1} \sqrt{\lambda} [m(\lambda), \xi \cdot \nabla] \delta \sqrt{\lambda}.$$ 

As $B$ and $\xi \cdot \nabla$ commute with $\delta \sqrt{\lambda}$ we get

$$\delta \sqrt{\lambda} [B, T_{\tilde{m}(\lambda)}] \delta^{-1} \sqrt{\lambda} = [B, T_m(\lambda)]$$

and

$$\delta^{-1} \sqrt{\lambda} [m(\lambda), \xi \cdot \nabla] \delta \sqrt{\lambda} = [\tilde{m}(\lambda), \xi \cdot \nabla].$$

This completes the proof. \[\square\]

### 3. Weighted norm estimates for Weyl multipliers

In this section our aim is to show that when the multiplier $m$ satisfies condition $(M_{2(n+1)})$ the Weyl multiplier $T_m$ is bounded on $L^p(\mathbb{C}^n, w)$ for all $w \in A_p(\mathbb{C}^n), 1 < p < \infty$. Any such Weyl multiplier is a twisted convolution operator: $T_m f = k \times f$ for a distribution $k$ on $\mathbb{C}^n$. We will show that conditions on $m$ can be translated into estimates on the kernel $k$ which will then be used to prove the weighted norm inequality. We begin with the following result.

**Theorem 3.1.** Consider the operator $T f = k \times f$ where $k \in L^2(\mathbb{C}^n)$ satisfies the estimate

$$(3.1) \quad |k(z - u)e^{-\frac{i}{2} \Im(z \bar{u})} - k(z)| \leq C \frac{|u|^\delta}{|z|^{2n+\delta}}$$

for some $\delta > 0$ and all $|z| > 2|u|$. Assume that $T$ is bounded on $L^p(\mathbb{C}^n)$ and $T f \in L^p(\mathbb{C}^n, w)$ for a dense class of functions in $L^p(\mathbb{C}^n, w)$ for all $w \in A_p(\mathbb{C}^n), 1 < p < \infty$. Then

$$\int_{\mathbb{C}^n} |T f|^p w(z)dz \leq C \int_{\mathbb{C}^n} |f(z)|^p w(z)dz.$$
The proof of this theorem uses standard arguments. It is well-known that Calderon-Zygmund operators are bounded on $L^p(\mathbb{R}^n, w), w \in A_p(\mathbb{R}^n), 1 < p < \infty$, see e.g. Theorem 7.11 in [3]. Our operator $T$ is sort of an oscillatory singular integral operator and hence the arguments used in proving Theorem 7.11 in [3] can be suitably modified to prove Theorem 3.1.

The sharp maximal function $M^\sharp$ used in the literature needs to be modified. We define the twisted sharp maximal function $\tilde{M}^\sharp$ by

$$\tilde{M}^\sharp f(v) = \sup_{v \in Q} |Q| \int_Q |f(z)e^{-\frac{1}{2}\Im(z \cdot \bar{u})} - \tilde{f}_Q|dz$$

where $f$ is a locally integrable function, $Q$ is a cube, $u$ its center and $\tilde{f}_Q = \frac{1}{|Q|} \int_Q f(z)e^{-\frac{1}{2}\Im(z \cdot \bar{u})}dz$.

For any $s > 1$ let $M_s f = (M|f|^s)^\frac{1}{s}, M$ being the Hardy-Littlewood maximal function.

**Lemma 3.2.** Let $k \in L^2(\mathbb{C}^n)$ satisfies the condition (3.1) in the above theorem. If $Tf = k \times f$ is bounded on $L^p(\mathbb{C}^n), 1 < p < \infty$ then

$$\tilde{M}^\sharp(Tf)(v) \leq C M_s f(v)$$

for any $1 < s < \infty$.

The proof of this lemma is similar to that of Lemma 7.9 in [3]. All we have to do is use $\tilde{f}_Q$ in place of $f_Q$.

Let $M_d$ stand for the dyadic maximal function (see Section 5, Chapter 2 in [3]).

**Lemma 3.3.** Let $w \in A_p(\mathbb{C}^n), 1 \leq p_0 \leq p < \infty$. Then

$$\int_{\mathbb{C}^n} |M_d f(z)|^p w(z)dz \leq C \int_{\mathbb{C}^n} |\tilde{M}^\sharp f(z)|^p w(z)dz$$

whenever $M_d f \in L^{p_0}(\mathbb{C}^n, w)$.

The proof of this lemma depends on good-\(\lambda\) inequality: for some $\delta > 0$

$$w(\{z \in \mathbb{C}^n : M_d f(z) > 2\lambda, \tilde{M}^\sharp f(z) \leq \gamma \lambda\})$$

$$\leq C \gamma^\delta w(\{z \in \mathbb{C}^n : M_d f(z) > \lambda\})$$

Once we have this inequality, the lemma can be proved by expressing the $L^p$ norm of $M_d f$ in terms of its distribution function. See the proof of Lemma 6.9 in [3].

The good-\(\lambda\) inequality with $M^\sharp$ in place of $\tilde{M}^\sharp$ has been proved in [3] (see Lemma 6.10 and Lemma 7.10). The same proof goes through with slight
modifications on account of the 'twist'. We leave the details to the reader.

It is now easy to prove Theorem 3.1. By the hypothesis, there is a dense class \( D \subset L^p(\mathbb{C}^n, w) \) such that \( Tf \in L^p(\mathbb{C}^n, w) \) for \( f \in D \). As \( w \in A_p(\mathbb{C}^n) \), there exists \( s, 1 < s < p \) such that \( w \in A_{p/s}(\mathbb{C}^n) \). For \( f \in D \), \( |Tf(z)| \leq M_d(Tf)(z) \) a.e and hence

\[
\int_{\mathbb{C}^n} |Tf(z)|^p w(z) dz \leq \int_{\mathbb{C}^n} (M_d(Tf)(z))^p w(z) dz.
\]

By Lemma 3.2, 3.3 and the boundedness of \( M_d \) we get

\[
\int_{\mathbb{C}^n} |Tf(z)|^p w(z) dz \leq C \int_{\mathbb{C}^n} |f(z)|^p w(z) dz.
\]

This completes the proof of Theorem 3.1.

The existence of a dense class of functions \( D \subset L^p(\mathbb{C}^n, w) \) such that \( Tf \in L^p(\mathbb{C}^n, w) \), \( f \in D \) is guaranteed once we assume another estimate on the kernel \( k \). Indeed, under the assumption

\[
(3.2) \quad |k(z)| \leq C |z|^{-2n-\theta} \quad \text{for some } \theta > 0
\]

we can show that \( Tf \in L^p(\mathbb{C}^n, w) \) whenever \( f \in C^\infty_0(\mathbb{C}^n) \) (which is dense in \( L^p(\mathbb{C}^n, w) \)). This has been proved for Calderon-Zygmund operators in Theorem 3.11 of [3]. As it only uses the size estimate our assertion is proved.

We can now restate Theorem 3.1 in the following form.

**Theorem 3.4.** Consider the operator \( Tf = k \times f \) where \( k \in L^2(\mathbb{C}^n) \) satisfies (3.1) and (3.2). If \( T \) is bounded on \( L^p(\mathbb{C}^n) \), \( 1 < p < \infty \) then it is also bounded on \( L^p(\mathbb{C}^n, w) \), for all \( w \in A_p(\mathbb{C}^n) \), \( 1 < p < \infty \).

Thus in order to prove weighted norm inequalities for \( T_m \) we only need to prove estimates (3.1) and (3.2) for the kernel \( k \) of \( T_m \).

**Theorem 3.5.** Let \( m \in B(L^2(\mathbb{R}^n)) \) be of class \( C^{2n+2} \) and satisfy Mauceri’s condition \( (M_{2n+2}) \). Then

1. \( |k(z)| \leq C |z|^{-2n-\theta} \) for some \( \theta \geq 0 \),
2. \( |k(z-u)e^{-\frac{1}{2}z \cdot (z-u)} - k(z)| \leq \frac{|u|^{\delta}}{|z|^{2n+\delta}} \)

for some \( \delta > 0 \) and for all \( |z| > 2|u| \) where \( k(z) \) is the kernel of the operator \( M = T_m \).

In order to prove the theorem, let \( t_j = 2^{-j}, j = 1, 2, \ldots \) and consider

\[
S_j = \sum_{k=0}^{\infty} (e^{-2kt_j} - e^{-2kt_{j+1}}) P_k = e^{nt_j}e^{-t_jH} - e^{nt_j+1}e^{-t_{j+1}H}.
\]

Then it follows that \( \sum_{j=1}^{N} S_j = e^{nt_1}e^{-t_1H} - e^{nt_{N+1}}e^{-t_{N+1}H} \) and taking limit as \( N \to \infty \) we get \( I = e^{nt_1}e^{-t_1H} - \sum_{j=1}^{\infty} S_j \). Using this we decompose our
operator \( m \) as
\[
m = m_0 - \sum_{j=1}^{\infty} m_j, \quad m_j = mS_j, \quad m_0 = me^{nt}e^{-tH}.
\]

Let \( k_j \) stand for the kernel of \( m_j, j = 0, 1, 2, \ldots \)

**Proposition 3.6.** For each \( j = 0, 1, 2, \ldots \) we have
1. \(|k_j(z)| \leq C \frac{|z|^{1/4}}{2|z|^{n+1/2}}
2. \(|k_j(z-u)e^{-\frac{i\alpha}{2}(z-u)} - k_j(z)| \leq C \frac{|u|^{1/2}}{|z|^{n+1/2}} \sum_{j=0}^{\infty} \min(\frac{t_{j+1}^{1/4}}{|u|^{1/2}}, \frac{|u|^{1/2}}{t_{j+1}^{1/4}})

for all \(|z| > 2|u|\).

Theorem 3.5 follows immediately once we prove this proposition. Indeed,
\[
|k(z)e^{-\frac{i\alpha}{2}(z-u)} - k(z)| \leq C \frac{|u|^{1/2}}{|z|^{n+1/2}} \sum_{j=0}^{\infty} \min(\frac{t_{j+1}^{1/4}}{|u|^{1/2}}, \frac{|u|^{1/2}}{t_{j+1}^{1/4}})
\]
and splitting the sum into two parts we see that
\[
\frac{|u|^{1/2}}{|z|^{2n+1/2}} \sum_{t_{j+1} \leq |u|^2} t_{j+1}^{1/4}|u|^{-1/2} \leq C \frac{|u|^{1/2}}{|z|^{2n+1/2}}
\]
and also
\[
\frac{|u|^{1/2}}{|z|^{2n+1/2}} \sum_{t_{j+1} > |u|^2} t_{j+1}^{-1/4}|u|^{1/2} \leq C \frac{|u|^{1/2}}{|z|^{2n+1/2}}.
\]
Thus we only need to prove Proposition 3.6.

Coming to the proof of Proposition 3.6 we claim that for all \( z \in \mathbb{C}^n \)
\[
|z|^l|k_j(z)| \leq C \frac{t_{j+1}^{1/2-n}}{t_{j+1}^{1/2}}
\]
whenever \( l \leq 2n + 1 \). In order to estimate \(|z|^l k_j(z)\) it is enough to estimate \( z^\alpha \bar{z}^\beta k_j(z) \) where \(|\alpha| + |\beta| = l\). Under the Weyl transform \( z^\alpha \bar{z}^\beta k_j(z) \) goes into \( \bar{\delta}^\alpha \delta^\beta(mS_j) \) which by Leibniz formula for the derivations \( \bar{\delta}^\alpha \) and \( \delta^\beta \) is a sum of terms of the form
\[
(\bar{\delta}^\mu \delta^\nu (m))\bar{\delta}^\gamma \delta^\rho S_j, \quad |\mu| + |\nu| + |\gamma| + |\rho| = l.
\]

We decompose each of these operators as
\[
\sum_{N=0}^{\infty} (\bar{\delta}^\mu \delta^\nu m)\chi N \bar{\chi} N (\bar{\delta}^\gamma \delta^\rho S_j).
\]

Since \(|f \times g(z)| \leq ||f||_2 ||g||_2\), the \( L^\infty \) norm of the kernel of \((\bar{\delta}^\mu \delta^\nu m)(\bar{\delta}^\gamma \delta^\rho S_j)\) is bounded by
\[
\sum_{N=0}^{\infty} ||(\bar{\delta}^\mu \delta^\nu m)\chi N ||_{HS} ||\bar{\chi} N (\bar{\delta}^\gamma \delta^\rho S_j)||_{HS}.
\]
We now make use of the following lemma which is essentially Lemma 4.4 proved in Mauceri [12].

**Lemma 3.7.** For every $\gamma$ and $\rho$ we have the estimate

$$
\|\chi_N(\delta^\gamma \delta^\rho S_j)\|_{HS}^2 \leq C \ t_j^{2N(n+2-|\gamma|-|\rho|)} f_{\gamma,\rho}^2(2^N t_j+1)
$$

where $f_{\gamma,\rho}$ is a rapidly decreasing function.

In view of this lemma, the kernel of $(\delta^\mu \delta^\nu \delta^\lambda \delta^\rho S_j)$ is bounded by constant times

$$
t_j+1 \sum_{N=0}^{\infty} 2^{N(n-|\mu|-|\nu|)} 2^{N(n+2-|\gamma|-|\rho|)} f_{\gamma,\rho}(2^N t_j+1)
$$

$$
= t_j+1 \sum_{N=0}^{\infty} 2^{N(2n+2-l)} f_{\gamma,\rho}(2^N t_j+1)
$$

$$
\leq C \ t_j^{-(n+\frac{1}{2})} \sum_{N=0}^{\infty} (2^N t_j+1)^{(n+1-\frac{l}{2})} f_{\gamma,\rho}(2^N t_j+1).
$$

Since $f_{\gamma,\rho}(2^N t_j+1)$ has exponential decay $\sum_{N=0}^{\infty} (2^N)^{\alpha} f_{\gamma,\rho}(2^N t_j+1)$ converges leading to the estimate $C t_j^{-(n+\frac{1}{2})}$ for each $\alpha > 0$. Hence the above series can be estimated by $C_{\gamma,\rho} t_j^{-(n+\frac{1}{2})}$.

As this is true for every $\mu, \nu, \gamma$ and $\rho$ satisfying $|\mu| + |\nu| + |\gamma| + |\rho| = |\alpha| + |\beta| = l$ we get the estimate

$$
|z^\alpha z^\beta k_j(z)| \leq C_{\alpha,\beta} t_j^{-(n+\frac{1}{2})}, \ |\alpha| + |\beta| = l
$$

which leads to $|z|^l |k_j(z)| \leq C t_j^{-(n+\frac{1}{2})}$.

When $l = 2n$ we get $|k_j(z)| \leq C |z|^{-2n}$ and when $l = 2n + 1$ we get $|k_j(z)| \leq C t_j^{1/2} |z|^{-2n-1}$ combining these two estimates we obtain

$$
|k_j(z)| \leq C t_j^{1/4} |z|^{-2n-1/2}.
$$

Again if we take $l = 2n + 2$ the above series is estimated by

$$
t_j+1 \sum_{N=0}^{\infty} f_{\gamma,\rho}(2^N t_j+1) \leq \sum_{N=0}^{\infty} (2^N t_j+1) f_{\gamma,\rho}(2^N t_j+1) \leq C
$$

Hence we also obtain the following inequality

$$
|k_j(z)| \leq C |z|^{-2n-2}.
$$

Thus we have proved (1) of Proposition 3.6. In order to prove (2) we need to estimate the gradient of $k_j$ for which we proceed as follows.
Since $\frac{\partial}{\partial z^r} k_j = Z_r k_j - \frac{1}{4} z_j k_j$ and $W(Z_r k_j) = i m S_j A^*_r$, in order to estimate $|z|^l \frac{\partial}{\partial z^r} k_j$ we have to estimate

$$\sum_{N=0}^{\infty} ||(\delta^\mu \delta^\nu m)\chi_N||_{HS} ||\chi_N \delta^\gamma \delta^\rho (S_j A^*_r)||_{HS}$$

where $|\mu| + |\nu| + |\gamma| + |\rho| = l$. Since $\delta_r A^*_r = 0$ and $\delta_r A^*_r = 2I$ it is enough to estimate

$$\sum_{N=0}^{\infty} ||(\delta^\mu \delta^\nu m)\chi_N||_{HS} ||\chi_N \delta^\gamma \delta^\rho (S_j A^*_r)||_{HS}$$

We use the Hermite basis $\Phi_{\alpha, \alpha} \in \mathbb{N}^n$ to calculate the Hilbert-Schmidt norm. Since $A_r^* \Phi_{\alpha} = (2|\alpha| + 2 + n)^{1/2} \Phi_{\alpha + e_r}$ it follows that

$$||\chi_N \delta^\gamma \delta^\rho (S_j A^*_r)||_{HS} \leq C t_j^{2} 2^{N(n+3-|\gamma|+|\rho|)} f_{\gamma, \rho}(2^N t_{j+1})$$

where we have used Lemma 3.5. Therefore,

$$\sum_{N=0}^{\infty} ||(\delta^\mu \delta^\nu m)\chi_N||_{HS} ||\chi_N \delta^\gamma \delta^\rho (S_j A^*_r)||_{HS} \leq C t_j^{1} 2^{N(2n+3-l)} f_{\gamma, \rho}(2^N t_{j+1}) \leq C t_j^{-n+\frac{4}{2}-1/2}.$$

Consequently, we have proved, by taking $l = 2n$ and $l = 2n + 1$, the estimates $|Z_r k_j(z)| \leq C t_j^{-1/2} |z|^{-2n}$ and $|Z_r k_j(z)| \leq C |z|^{-2n-1}$ and combining them we obtain the estimate

$$|Z_r k_j(z)| \leq C t_j^{-1/4} |z|^{-2n-1/2}$$

We also have the estimates $|z_j k_j(z)| \leq C |z||z|^{-2n-2} = C |z|^{-2n-1}$ and $|z_r k_j(z)| \leq C |z||t_{j+1}||z|^{-2n-1}$. Putting all these estimates together we get

$$|\frac{\partial}{\partial z^r} k_j(z)| \leq C t_j^{1/4} |z|^{-2n-1/2} \leq C t_j^{-1/4} |z|^{-2n-1/2}.$$
On the other hand $|k_j(z - u)e^{-\frac{2}{4}3(z, \tilde{u})} - k_j(z)|$ is bounded by

$$|k_j(z - u) - k_j(z)| + |k_j(z)(e^{-\frac{2}{4}3(z, \tilde{u})} - 1)| \leq |u||\nabla k_j(\tilde{z})| + |u||z||k_j(z)|$$

where $\tilde{z}$ is a point on the line segment joining $(z - u)$ and $z$. The gradient term gives the estimate

$$|u||\nabla k_j(z)| \leq C |u|t_{j+1}^{-1/4}|\tilde{z}|^{-2n-1/2} \leq C |u|t_{j+1}^{-1/4}|z|^{-2n-1/2}$$

and the other term is estimated by

$$|u||z||k_j(z)| \leq C |u|t_{j+1}^{-1/4}|z|^{-2n-1/2} \leq C |u|t_{j+1}^{-1/4}|z|^{-2n-1/2}$$

which follows from $|k_j(z)| \leq C |z|^{-2n-2}$ and $|k_j(z)| \leq C t_{j+1}^{1/2}|z|^{-2n-1}$. Thus

$$|k_j(z - u)e^{-\frac{2}{4}3(z, \tilde{u})} - k_j(z)| \leq C |u|t_{j+1}^{-1/4}|z|^{-2n-1/2}$$

which we write as $C \frac{|u|^{1/2}}{|z|^{2n+1/2}} \frac{t_{j+1}^{1/4}}{t_{j+1}^{1/4}}$. Combining the two estimates

$$C \frac{|u|^{1/2}}{|z|^{2n+1/2}} \frac{t_{j+1}^{1/4}}{t_{j+1}^{1/4}} \text{ and } C \frac{|u|^{1/2}}{|z|^{2n+1/2}} \frac{t_{j+1}^{1/4}}{t_{j+1}^{1/4}}$$

we obtain

$$|k_j(z - u)e^{-\frac{2}{4}3(z, \tilde{u})} - k_j(z)| \leq C \frac{|u|^{1/2}}{|z|^{2n+1/2}} \min\left(\frac{t_{j+1}^{1/4}}{|u|^{1/2}}, \frac{t_{j+1}^{1/4}}{|u|^{1/2}}\right).$$

This completes the proof of Proposition 3.6 for all $j \geq 1$. The case $j = 0$ is even simpler since $m_0 = e^{n t_t} e^{-t_t H} = \sum_{k=0}^{\infty} e^{2k t_t}(m P_k)$. It is estimated in a similar fashion and we leave the details to the reader.

We will now prove the following result concerning the commutator of $T^1_m$ with multiplication by a BMO function.

**Theorem 3.8.** Let $m \in B(L^2(\mathbb{R}^n))$ be of class $C^{n+1}$ and satisfy Mauceri’s condition $(M_{2n+2})$. If $b \in BMO(\mathbb{C}^n)$, then there exists a constant $C = C(p, m, n)$ such that

$$|||b, T^1_m|| f||_p \leq C|||b||_* ||f||_p$$

for $1 < p < \infty$.

**Proof.** The main step in the proof of the above theorem is the following estimate:

$$\tilde{M}^r([b, T^1_m]f)(z) \leq C||b||_* (M_r T^1_m f(z) + M_{rs} f(z))$$

where $r, s > 1$ be such that $1 < rs < p$. If we can show that (3.6) is true, then the proof of the theorem is immediate. As the proof of the theorem is similar to the Lemma 11 in [7], we leave the details to the reader. \qed

We now turn our attention towards a proof of Theorem 1.2. In order to prove this theorem we need the following $L^2$ version of Theorem 3.1.
Theorem 3.9. Consider the operator $T = k \times f$ where $k \in L^2(\mathbb{C}^n)$ satisfies the estimate
\begin{equation}
(3.7) \quad \left( \int_{|z| > 2|u|} |z|^{2n+2\delta} |k(z-u)e^{-\frac{i}{2}G(z,\bar{u})} - k(z)|^2 dz \right)^{\frac{1}{2}} \leq C |u|^{\delta}
\end{equation}
for some $\delta > 0$. Then $Tf$ is bounded on $L^p(\mathbb{C}^n)$, $1 < p < \infty$. Moreover, if $Tf \in L^p(\mathbb{C}^n, w)$ for a dense class of functions in $L^p(\mathbb{C}^n, w)$, $w \in A_{p/2}(\mathbb{C}^n)$, $2 < p < \infty$, then
\[ \int_{\mathbb{C}^n} |Tf|^p w(z) dz \leq \int_{\mathbb{C}^n} |f(z)|^p w(z) dz \]
for all $w \in A_{p/2}$, $f \in L^p(\mathbb{C}^n, w)$.

In order to prove the theorem we need the following lemma.

Lemma 3.10. Let $k \in L^2(\mathbb{C}^n)$ satisfies the condition of the above theorem. Then
\[ \tilde{M}^2(Tf)(v) \leq C M_2 f(v). \]

Proof. Let $v \in \mathbb{C}^n$ and $Q$ be a cube containing it. Let $u$ be the center of $Q$. Also, let $f_1 = f\chi_{2Q}$ and $f_2 = f - f_1$. To prove the lemma it is enough to show that
\[ \frac{1}{|Q|} \int_Q |Tf(z)e^{-\frac{i}{2}G(z,\bar{u})} - Tf_2(u)| dz \leq C M_2 f(v). \]

The left hand side can be dominated by
\[ \frac{1}{|Q|} \int_Q |Tf_1(z)| dz + \frac{1}{|Q|} \int_Q |Tf_2(z)e^{-\frac{i}{2}G(z,\bar{u})} - Tf_2(u)| dz. \]

The first term is easy to handle. Indeed, it can be estimated by
\[ \left( \frac{1}{|Q|} \int_Q |Tf_1(z)|^2 dz \right)^{\frac{1}{2}}. \]

Using the $L^2$ boundedness of $T$ we can dominate the above term by
\[ C \left( \frac{1}{|Q|} \int_{2Q} |f(z)|^2 dz \right)^{\frac{1}{2}} \leq 2^n C M_2 f(v). \]

In order to estimate the second term we use the kernel estimate given in the hypothesis. Using the definitions of $T$ and $f_1$, we get
\[ \frac{1}{|Q|} \int_Q \int_{\mathbb{C}^n \setminus 2Q} |k(z-w)e^{\frac{i}{2}G(z,\bar{w}-z\bar{u})} - k(u-w)e^{\frac{i}{2}G(u,\bar{u})}| |f(w)| dw dz. \]

By Hölder’s inequality the inner integral is dominated by the product of
\[ \left( \int_{\mathbb{C}^n} |u-w|^{2n+\delta} |k(u-w-u+z)e^{-\frac{i}{2}G((u-w)(\bar{u}-\bar{z}))} - k(u-w)|^2 dw \right)^{\frac{1}{2}}. \]
Using the hypothesis of the lemma, we can observe that the first integral is bounded by \(|u - z| \delta\) which further can be dominated by \(l(Q) \delta\). The second integral is dominated by
\[
\sum_{k=1}^{\infty} \int_{2^k l(Q) \leq |u - w| < 2^{k+1} l(Q)} \frac{|f(w)|^2}{|u - w|^{2n+2\theta}}.
\]
One can easily see that the above sum is bounded by \(l(Q)^{-\delta} M_2 f(v)\).

Taking average over \(Q\) the lemma is proved.

Now we are ready to prove Theorem 3.9. From (3.7) we can easily deduce that the kernel of \(T\) satisfies the following estimate
\[
\int_{|z| > 2|u|} |k(z - u) e^{-\frac{i}{2} \Im(z, \bar{u})} - k(z)| dz < C.
\]
Hence from Theorem 3.2 of [12] we can conclude that \(T\) is bounded on \(L^p(\mathbb{C}^n)\) for \(1 < p \leq 2\). For \(p > 2\), We will use the above lemma. The point-wise estimate \(|Tf(z)| \leq M_d Tf(z)\) gives us
\[
\int_{\mathbb{C}^n} |Tf(z)|^p dz \leq C \int_{\mathbb{C}^n} (M_d(Tf)(z))^p dz.
\]
As \(T\) is bounded on \(L^2(\mathbb{C}^n)\), we can use Lemma 3.3 to conclude that
\[
\int_{\mathbb{C}^n} |Tf(z)|^p dz \leq C \int_{\mathbb{C}^n} |M^*f(Tf)(z)|^p dz
\]
for any \(p > 2\). Now using Lemma 3.10 and the boundedness of \(M_2\) on \(L^p(\mathbb{C}^n)\), \(p > 2\) one can easily see
\[
\int_{\mathbb{C}^n} |Tf(z)|^p dz \leq C \int_{\mathbb{C}^n} |f(z)|^p dz.
\]
As Lemma 3.3 is true for any \(w \in A_p\), \(1 < p < \infty\), the remaining part of the lemma can be proved by same arguments once we have a dense class of functions appearing in the hypothesis of the theorem.

The existence of a dense class of functions \(D \subset L^p(\mathbb{C}^n, w)\) such that \(Tf \in L^p(\mathbb{C}^n, w)\), \(2 < p < \infty\), is guaranteed once assume the estimate
\[
(3.8) \quad \int_{\mathbb{C}^n} |z|^{2n+2\theta} |k(z)|^{2n+2\theta} dz < C
\]
for some \(\theta > 0\) on the kernel. To see this let us consider the space \(D\) of all smooth functions with compact support. Suppose \(f\) is a function in \(D\)
whose support is contained in $B(0, R)$, the ball of radius $R$ centered at the origin for some $R > 0$. Now, for $\epsilon > 0$, using Hölder’s inequality we see that

$$\int_{|z| < 2R} |Tf(z)|^p w(z) \, dz$$

is bounded by

$$\left( \int_{|z| < 2R} w(z)^{1+\epsilon} \, dz \right)^{\frac{1}{1+\epsilon}} \left( \int_{|z| < 2R} |Tf(z)|^{p(1+\epsilon)/\epsilon} \, dz \right)^{\frac{\epsilon}{1+\epsilon}}.$$  

By the reverse Hölder inequality, we can choose $\epsilon > 0$ such that the first integral is finite. The second integral is finite since $Tf \in L^q, 2 < q < \infty$.

For $|z| > 2R$, applying Hölder’s inequality in the definition of $Tf(z)$ we get

$$|Tf(z)| \leq \left( \int_{\mathbb{C}^n} |z - w|^{2n+2\theta} |k(z - w)|^2 \, dw \right)^{\frac{1}{2}} \left( \int_{|w| < R} |f(w)|^2 \frac{1}{|z - w|^{2n+2\theta}} \, dw \right)^{\frac{1}{2}}.$$  

By (3.8), the right hand side can be dominated by $C_R \|f\|_{\infty} |z|^{-n-\theta}$. The above discussion leads us to the following estimate:

$$\int_{|z| > 2R} |Tf(z)|^p w(z) \, dz \leq C \sum_{j=1}^{\infty} \int_{2^j R < |z| \leq 2^{j+1} R} \frac{w(z)}{|z|^{(n+\theta)p}} \, dz$$

$$\leq C \sum_{j=1}^{\infty} (2^j R)^{-p(n+\theta)} w(B(0, 2^{j+1} R)).$$

As $w \in A_{p/2}, w(B(0, 2^{j+1} R))$ is bounded by $C(2^j R)^{np}$, which implies the above sum is finite. Hence, $Tf \in L^p(\mathbb{C}^n, w)$ for all $f \in \mathcal{D}$.

In view of the above observations, we can restate Theorem 3.9 as follows.

**Theorem 3.11.** Let us consider the operator $T = k \times f$ where the kernel $k \in L^2(\mathbb{C}^n)$ satisfies the hypothesis (3.7) and

$$\left( \int_{\mathbb{C}^n} |z|^{2n+2\theta} |k(z)|^2 \, dz \right)^{\frac{1}{2}} \leq C$$

for some $\delta > 0, \theta > 0$. Then $T$ is bounded on $L^p(\mathbb{C}^n), 1 < p < \infty$. Also, $T$ satisfies the following weighted norm inequality

$$\int_{\mathbb{C}^n} |Tf(z)|^p w(z) \, dz \leq C \int_{\mathbb{C}^n} |f(z)|^p w(z) \, dz$$

where $f \in L^p(\mathbb{C}^n, w), w \in A_{p/2}, 2 < p < \infty$.

Now we are ready to prove Theorem 1.2. From the above discussions we only need to prove the following theorem.
Theorem 3.12. Let $m \in B(L^2(\mathbb{R}^n))$ be of class $C^{n+2}$ and satisfies Mauceri’s condition $(M_{n+2})$. Then

1. $\int_{\mathbb{C}^n} |z|^{2n+1} |k(z)|^2 dz < C$
2. $\left( \int_{|z|>2|u|} |z|^{2n+1} |k(z-u)e^{-\frac{t}{2}\Delta(z,u)} - k(z)|^2 dz \right)^{\frac{1}{2}} \leq C \ |u|^{\frac{1}{2}}.$

In order to prove the above theorem we need the following $L^2$ analogue of Proposition of 3.6. Once we have the following estimates, we immediately get the theorem since the series

$$\sum_{j=1}^{\infty} \min \left( \frac{t_j^{\frac{1}{2}}}{|u|^{\frac{1}{2}}}, \frac{|u|^{\frac{1}{2}}}{t_j^{\frac{1}{2}}}, \frac{|u|^{\frac{1}{2}}}{|u|^{\frac{1}{2}}} \right)$$

converges. Thus we are left with proving the following proposition.

Proposition 3.13. For each $j = 0, 1, 2, \ldots$ we have the estimates:

1. $\left( \int_{\mathbb{C}^n} |z|^{2n+1} |k_j(z)|^2 \right)^{\frac{1}{2}} dz \leq C \ t_j^{\frac{1}{2}}$
2. $\left( \int_{|z|>2|u|} |z|^{2n+1} |k_j(z) - k_j(z)|^2 dz \right)^{\frac{1}{2}}$
   $\leq C \ |u|^{\frac{1}{2}} \min \left( \frac{t_j^{\frac{1}{2}}}{|u|^{\frac{1}{2}}}, \frac{|u|^{\frac{1}{2}}}{t_j^{\frac{1}{2}}}, \frac{|u|^{\frac{1}{2}}}{|u|^{\frac{1}{2}}} \right).$

Proof. To prove (1) we claim that

$$\left( \int_{\mathbb{C}^n} |z|^{2n} |k_j(z)|^2 \right)^{\frac{1}{2}} dz \leq C \ t_j^{\frac{l-n}{2}}$$

whenever $l \leq n + 1$. In order to estimate the $L^2$ norm of $|z|^l |k_j(z)|$, it is enough to estimate the $L^2$ norm of $z^\alpha \bar{z}^\beta k_j(z)$, for $|\alpha| + |\beta| = l$. That is, we have to estimate the Hilbert-Schmidt norm of $\delta^{\alpha} \bar{\delta}^{\beta}(M S_j)$. As we have done in Proposition 3.6, it is enough to estimate

$$\sum_{N=0}^{\infty} (\delta^\mu \bar{\delta}^\nu m) \chi_N \cdot \chi_N (\delta^\gamma \bar{\delta}^{\rho} S_j)$$

where $|\mu| + |\nu| + |\gamma| + |\rho| = l$. The Hilbert-Schmidt norm of the above sum is dominated by

$$\sum_{N=0}^{\infty} ||(\delta^\mu \bar{\delta}^\nu m) \chi_N||_{HS} \cdot ||\chi_N (\delta^\gamma \bar{\delta}^{\rho} S_j)||_{op}.$$ 

Now from Lemma 4.4 in [12] we have

$$||\chi_N (\delta^\gamma \bar{\delta}^{\rho} S_j)||_{op} \leq C \ t_{j+1} 2^{-N(|\gamma|+|\rho|-2)} f_{\gamma,\rho}(2N t_{j+1}).$$

Hence using the above estimate and the hypothesis on $m$ one can get

$$||\sum_{N=0}^{\infty} (\delta^\mu \bar{\delta}^\nu m) \chi_N \cdot \chi_N (\delta^\gamma \bar{\delta}^{\rho} S_j)||_{HS}$$
\[ \leq C \sum_{N=0}^{\infty} 2^{\frac{N(\gamma+|\rho|)}{2}} f_{\gamma, \rho}(2^N t_{j+1}) \]

\[ \leq C t_{j+1} \sum_{N=0}^{\infty} 2^{N(\gamma+|\rho|)} f_{\gamma, \rho}(2^N t_{j+1}) \leq C t_{j+1}^{(l-n)/2}. \]

This proves our claim. Now, when \( l = n, ||z^n k_j(z)||_2 \leq C \) and when \( l = n+1, ||z^{n+1} k_j(z)||_2 \leq C t_{j+1}^{\frac{1}{2}}. \) Combining these two estimates we get

\[ ||z^{n+\frac{1}{2}} k_j(z)||_2 \leq C t_{j+1}^{\frac{1}{2}} \]

which proves (1). Again from the above estimation we see that

\[ ||z^{n+2} K_j(z)||_2 \leq C t_{j+1} \sum_{N=0}^{\infty} f_{\gamma, \rho}(2^N t_{j+1}) \leq C t_{j+1}^{(l-n)/2}. \]

In order to prove (2) we need to estimate the gradient of \( k_j. \) Earlier we have already noted that

\[ \frac{\partial}{\partial z^r} k_j = Z^r k_j - \frac{1}{4} \bar{z}^r k_j. \]

In order to estimate the \( L^2 \) norm of \( |z|^l Z^r k_j \) it is enough to estimate

\[ \sum_{N=0}^{\infty} ||(\delta^\mu \delta^\nu m)\chi_N||_{HS} ||\chi_N(\delta^\gamma \delta^\rho (S_j A^*_r)))||_{op}. \]

From (4.4) in [12] it is not difficult to see that

\[ ||\chi_N(\delta^\gamma \delta^\rho (S_j A^*_r)))||_{op} \leq C t_{j+1} 2^{-N(\gamma+|\rho|)/2} f_{\gamma, \rho}(2^N t_{j+1}). \]

The above estimate and similar arguments used in the proof of (1) lead us to the estimate

\[ ||z^l Z^r k_j||_2 \leq C t_{j+1}^{(l-n-1)/2}. \]

Putting \( l = n \) and \( l = n + 1 \) we get the estimates

\[ ||z^n Z^r k_j||_2 \leq C t_{j+1}^{\frac{1}{2}}, \ ||z^{n+1} Z^r k_j||_2 \leq C \]

respectively. Hence

\[ ||z^{n+\frac{1}{2}} Z^r k_j||_2 \leq C t_{j+1}^{\frac{1}{2}}. \]

For \( z^r k_j(z) \) one can see that

\[ ||z^n z^r k_j||_2 \leq C ||z^{n+1} k_j||_2 \leq C \]

and

\[ ||z^{n+1} z^r k_j||_2 \leq C ||z^{n+2} k_j||_2 \leq C. \]

Thus we have \( ||z^{n+\frac{1}{2}} z^r k_j||_2 \leq C. \) This proves the estimate

\[ ||z^{n+\frac{1}{2}} \nabla k_j(z)||_2 \leq C t_{j+1}^{\frac{1}{2}}. \]
Finally, if \(|z| > 2|u|\), then it follows that \(|z - u| > \frac{1}{2}|z|\). By triangle inequality, \(|k_j(z - u)e^{-\frac{t}{2}z} - k_j(z)| \leq |k_j(z - u)| + |k_j(z)|\) and hence we have

\[
(3.9) \quad \left( \int_{|z| > 2|u|} |z|^{2n+1} |k_j(z - u)e^{-\frac{t}{2}z} - k_j(z)|^2 \right)^{\frac{1}{2}} \leq C |u|^{\frac{t+1}{t}}.
\]

On the other hand, by mean value theorem \(|k_j(z - u)e^{-\frac{t}{2}z} - k_j(z)|\) is bounded by

\[|k_j(z - u) - k_j(z)| + |k_j(z)(e^{\frac{t}{2}z} - 1)| \leq |u||\nabla k_j(z)| + |u||z||k_j(z)|\]

where \(\bar{z}\) is a point on the line segment joining \((z - u)\) and \(z\). Hence we get

\[
(3.10) \quad \left( \int_{|z| > 2|u|} |z|^{2n+1} |k_j(z - u)e^{-\frac{t}{2}z} - k_j(z)|^2 \right)^{\frac{1}{2}} \leq C |u|^{\frac{t+1}{t}} = C |u|^{\frac{1}{t}} |u|^{\frac{1}{t}}.
\]

Comparing (3.9) and (3.10) we get the required result. \(\square\)

4. Fourier multipliers on the Heisenberg group

In this section we prove the theorems stated in the introduction. We begin with Theorem 1.7 which is very easy to prove. The proof is based on the following lemma. Consider convolution operators

\[S(\lambda)f(\xi) = \int_{\mathbb{R}^n} k_\lambda(\xi - \eta)f(\eta)d\eta\]

and denote by \(S_2(\lambda)\) the following operator defined on functions of 2n variables by

\[S_2(\lambda)f(x, y) = \int_{\mathbb{R}^n} k_\lambda(\eta)f(x, y + \eta)d\eta\]

We also let \(e_\lambda\) stand for the operator \((e_\lambda f)(x, y) = e^{(i/2)\lambda x}f(x, y)\).

Lemma 4.1. For every \(\lambda \in \mathbb{R}^*\) we have \(T_{S(\lambda)}^\lambda = e_\lambda S_2(\lambda) e_{-\lambda}\).

Proof. The lemma is proved by simple calculation. We note that

\[W_\lambda(e_\lambda S_2(\lambda)e_{-\lambda} f)\phi(\xi)\]

\[= \int_{\mathbb{C}^n} e^{i(\lambda x \cdot y)} S_2(\lambda) e_{-\lambda} f(x, y) e^{i\lambda(x \cdot \xi + \frac{1}{2}x \cdot y)} \phi(\xi + y) dxdy\]

\[= \int_{\mathbb{C}^n} \int_{\mathbb{R}^n} k_\lambda(\eta)e^{i(\lambda x \cdot (y + \eta))} f(x, y + \eta) e^{i\lambda(x \cdot \xi + x \cdot y)} \phi(\xi + y) dxdy d\eta\]
\[
\int_{\mathbb{C}^n} \int_{\mathbb{R}^n} k_\lambda(\eta)e^{i(\lambda/2)x \cdot y} f(x, y)e^{i\lambda x \cdot (\xi - \eta) + x \cdot y} \phi(\xi - \eta + y) dxdy d\eta
\]

The last integral simplifies to give
\[
\int_{\mathbb{R}^n} k_\lambda(\eta) \int_{\mathbb{C}^n} f(x, y)e^{i\lambda x \cdot (\xi - \eta) + x \cdot y} \phi(\xi - \eta + y) dxdy d\eta =
\int_{\mathbb{R}^n} k_\lambda(\eta) W_\lambda(f) \phi(\xi - \eta) d\eta = S(\lambda) W_\lambda(f) \phi(\xi).
\]

Hence the lemma is proved.

From the lemma we observe that \(T^\lambda_{S(\lambda)}\) is bounded on \(L^p(\mathbb{C}^n)\) whenever \(S(\lambda)\) is bounded on \(L^p(\mathbb{R}^n)\). We also note that when \(f\) is a function on the Heisenberg group, \(e^{-\lambda f^\lambda}(z) = (\tau(x \cdot y)f)^\lambda(z)\) where \(\tau(a)f(z, t) = f(z, t + a/2)\). Consider the multiplier transform
\[
T_S f(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda T^\lambda_{S(\lambda)}f}(z) d\lambda.
\]

In view of the lemma and the above observation we see that
\[
T_S f(z, t + \frac{1}{2}x \cdot y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda S_2(\lambda)(\tau(x \cdot y)f)^\lambda(z)} d\lambda.
\]

This means that
\[
(\tau(x \cdot y)T_S\tau(-x \cdot y)f)(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda S_2(\lambda)f^\lambda(z)} d\lambda.
\]

Under the hypothesis of Theorem 1.7 the families \(S(\lambda)\) and \(\lambda S_2(\lambda)\) are both \(R\)-bounded. Hence the same is true of \(S_2(\lambda)\) and consequently the right hand side of the above equation defines a bounded operator on \(L^p(\mathbb{H}^n)\). As translation in the last variable is a bounded operator on \(L^p(\mathbb{H}^n)\) Theorem 1.7 follows immediately.

Returning to general multipliers on the Heisenberg group recall that
\[
T_m f(z, t) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\lambda T_m^\lambda} f^\lambda(z) d\lambda
\]

and the \(R\)-boundedness of \(M(\lambda) = T_m^\lambda(\lambda)\) can be proved now. By Lemma 2.1 \(\delta_{\sqrt{\lambda}}^{-1/2} T_m^\lambda(\lambda)\delta_{\sqrt{\lambda}} = T_{\tilde{m}(\lambda)}^1\) which means that \(\delta_{\sqrt{\lambda}}^{-1/2} T_m^\lambda(\lambda)\delta_{\sqrt{\lambda}}\) is a Weyl multiplier with multiplier \(\tilde{m}(\lambda) = \delta_{\sqrt{\lambda}}^{-1/2} m(\lambda)\delta_{\sqrt{\lambda}}\). As \(\lambda_{\sqrt{\lambda}}\) is unitary and Hilbert-Schmidt operator norm is unitary invariant, Lemma 2.2 along with the hypothesis on \(m(\lambda)\) stated in Theorem 1.8 allows us to conclude that \(\tilde{m}(\lambda)\) satisfies Maucri's condition \((M_{n+1})\). Consequently, by Theorem 1.3 \(T_{\tilde{m}(\lambda)}^1\) satisfies the weighted norm inequality
\[
\int_{\mathbb{C}^n} |T_{\tilde{m}(\lambda)}^1 f(z)|^p w(z) dz \leq C_w \int_{\mathbb{C}^n} |f(z)|^p w(z) dz,
\]
where $C_w$ depends on $w$ but independent of $\lambda$. The above gives the inequality
\[
\int_{\mathbb{C}^n} |T_m(\lambda)f(z)|^p w(\sqrt{\lambda}z)dz \leq C_w \int_{\mathbb{C}^n} |f(z)|^p w(\sqrt{\lambda}z)dz.
\]
Since $w(\sqrt{\lambda}z)$ satisfies $A_p$ condition with the same norm as $w$, it follows that
\[
\int_{\mathbb{C}^n} |T_m(\lambda)f(z)|^p w(z)dz \leq C_w \int_{\mathbb{C}^n} |f(z)|^p w(z)dz.
\]
By the theorem of Rubio de Francia (see [16]) we get the $R$-boundedness of $M(\lambda) = T_m^\lambda$.

We now turn our attention to the $R$-boundedness of $\lambda^{1/2} M(\lambda)$. According to the Lemma 2.4 we need to treat three families of operators.

**Proposition 4.2.** Under the hypothesis of Theorem 1.9 the families $T_{m(\lambda), \xi, \nabla}^\lambda$ and $T_{\lambda^{1/2}, m(\lambda)}^\lambda$ are $R$-bounded.

**Proof.** The proof of $R$-boundedness of $T_{\lambda^{1/2}, m(\lambda)}^\lambda$ is similar to that of $T_{m(\lambda)}^\lambda$ as $\lambda^{1/2} m(\lambda)$ satisfies the same conditions as $m(\lambda)$. To treat the other one we write
\[
4\lambda \xi_j \frac{\partial}{\partial \xi_j} = (A_j(\lambda) + A_j^*(\lambda))(A_j(\lambda) - A_j^*(\lambda))
\]
\[
= A_j(\lambda)^2 - A_j^*(\lambda)^2 + [A_j^*(\lambda), A_j(\lambda)].
\]
Since $[A_j^*(\lambda), A_j(\lambda)] = -2\lambda I$ we see that
\[
4\lambda [m(\lambda), \xi_j \frac{\partial}{\partial \xi_j}] = [m(\lambda), A_j(\lambda)^2] - [m(\lambda), A_j^*(\lambda)^2]
\]
which can be written as
\[
4\lambda [m(\lambda), \xi_j \frac{\partial}{\partial \xi_j}] = \sqrt{\lambda} (\delta_j(\lambda)m(\lambda)) A_j(\lambda) + \sqrt{\lambda} A_j(\lambda) \delta_j(\lambda)m(\lambda)
\]
\[+ \sqrt{\lambda} (\tilde{\delta}_j(\lambda)m(\lambda)) A_j^*(\lambda) + \sqrt{\lambda} A_j^*(\lambda) \tilde{\delta}_j(\lambda)m(\lambda)
\]
We just consider one family corresponding to the multiplier
\[
\lambda^{-1/2} (\delta_j(\lambda)m(\lambda)) A_j(\lambda) = m_j(\lambda).
\]
Since $\delta_j(\lambda)$ and $\tilde{\delta}_j(\lambda)$ are derivations with $\delta_j(\lambda) A_j(\lambda) = 0$ and $\tilde{\delta}_j(\lambda) A_j(\lambda) = 2\lambda^{1/2} I$ it follows that the above family satisfies condition $(M_{n+1})$. Consequently the operator family $T_{m_j(\lambda)}^\lambda$ is $R$-bounded. The other families are treated in the same way.

Finally, we are left with the family $[B, T_{m(\lambda)}^\lambda]$. Recalling that $B = \sum_{j=1}^n (z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j})$ we consider
\[
[z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j}, T_{m(\lambda)}^\lambda] = [z_j Z_j(\lambda) + \bar{z}_j \bar{Z}_j(\lambda), T_{m(\lambda)}^\lambda].
\]
As \((z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j})\) commutes with dilations we get
\[
\delta^{-1}\sqrt{\lambda} [z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j}, T_{m(\lambda)}^{\lambda}] \delta \sqrt{\lambda} = [z_j Z_j(1) + \bar{z}_j \bar{Z}_j(1), T_{m(\lambda)}^{\lambda}]
\]
where we have used the relations
\[
\delta \sqrt{\lambda} Z_j(1) \delta^{-1} \sqrt{\lambda} = \lambda^{-1/2} Z_j(\lambda), \quad \delta \sqrt{\lambda} \bar{Z}_j(1) \delta^{-1} \sqrt{\lambda} = \lambda^{-1/2} \bar{Z}_j(\lambda).
\]
We observe the following relations:
\[
W(z_j Z_j(1) T_{m(\lambda)}^{\lambda}) f) = -2\delta_j (\tilde{m}(\lambda) W(f) A_j^*),
\]
\[
W(\bar{z}_j \bar{Z}_j(1) T_{m(\lambda)}^{\lambda}) f) = -2\delta_j (\tilde{m}(\lambda) W(f) A_j)
\]
and consequently
\[
W(T_{m(\lambda)}^{\lambda} z_j Z_j(1) f) = -2i\tilde{m}(\lambda) \delta_j (W(f) A_j^*),
\]
\[
W(T_{m(\lambda)}^{\lambda} \bar{z}_j \bar{Z}_j(1) f) = -2i\tilde{m}(\lambda) \delta_j (W(f) A_j).
\]
Therefore, we have
\[
W([z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j}, T_{m(\lambda)}^{\lambda}]) f) = -2\delta_j \tilde{m}(\lambda) W(f) A_j^* - 2\delta_j \tilde{m}(\lambda) W(f) A_j.
\]

The above relation clearly shows that \([B, T_{m(\lambda)}^{\lambda}]\) is not a Weyl multiplier, since \(W(f)\) need not commute with \(A_j\) and \(A_j^*\) in general. Consequently, we cannot hope to show that \([B, T_{m(\lambda)}^{\lambda}]\) is R-bounded. Indeed, when \(m(\lambda) = A_j(\lambda) H(\lambda)^{-1/2}\) which corresponds to the Riesz transforms \(R_j\) on \(H^1\), the operator \([B, T_{m(\lambda)}^{\lambda}]\) is not even bounded on \(L^2(\mathbb{C}^n)\). In spite of this we can easily prove

**Proposition 4.3.** The family \([B, T_{m(\lambda)}^{\lambda}] L^{1/2}_\lambda, \lambda \in \mathbb{R}^*\) is R-bounded on \(L^p(\mathbb{C}^n), 1 < p < \infty\).

**Proof.** Since \(W_\lambda(L^{1/2}_\lambda f) = W_\lambda(f) H(\lambda)^{-1/2}\), we only need to show that the operators \(\tilde{\delta}_j \tilde{m}(\lambda) Z_j(1) L^{-1/2}_1\) and \(\delta_j \tilde{m}(\lambda) \bar{Z}_j(1) L^{-1/2}_1\) satisfies weighted norm inequalities on \(L^p(\mathbb{C}^n)\) for any weight \(w \in A_p(\mathbb{C}^n)\). But \(Z_j(1) L^{-1/2}_1\) and \(\bar{Z}_j(1) L^{-1/2}_1\) are oscillatory singular integral operators and hence satisfy weighted norm inequalities according to the theorem of Lu-Zhang [10]. Since \(\delta_j \tilde{m}(\lambda)\) and \(\tilde{\delta}_j \tilde{m}(\lambda)\) satisfy the condition \((M_{n+1})\) they define the Weyl multipliers which satisfy weighted norm inequalities. This proves the proposition \(\Box\)

Combining Propositions 4.2 and 4.3 and using the fact that \((L^{1/2}_\lambda f)(\lambda) = \hat{f}(\lambda) H(\lambda)^{-1/2}\) we obtain Theorem 1.9. In order to prove Theorem 1.10 we
make use of the following observation. When \( f \) is a polyradial function \( W(f) \) commutes with \( H_j = -\partial^2_{r_i^2} + \xi_j^2, j = 1, 2, \ldots, n \). In view of this we have

\[
2\delta_j \tilde{m}(\lambda) W(f) A^*_j = 2\delta_j \tilde{m}(\lambda) H_j^{1/2} W(f) H^{-1/2} A^*_j
\]

which can be written as (since, \( H_j = \frac{1}{2}(A_j A^*_j + A^*_j A_j) \))

\[
\delta_j \tilde{m}(\lambda) A_j A^*_j H^{-1/2} W(f) H^{-1/2} A^*_j + \delta_j \tilde{m}(\lambda) A^*_j A_j H^{-1/2} W(f) H^{-1/2} A^*_j.
\]

The operator families \( \delta_j \tilde{m}(\lambda) A_j \) and \( \delta_j \tilde{m}(\lambda) A^*_j \) satisfy the condition \((M_{n+1})\) and \( A_j H^{-1/2}, A^*_j H^{-1/2} \) and \( H^{-1/2} A^*_j \) define oscillatory singular integrals (being variations of Riesz transforms).

Let \( T(n) \subset U(n) \) be the torus which acts on \( \mathbb{C}^n \) by \( \rho(\sigma)f(z) = f(e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n) \) if \( \sigma \) is the diagonal matrix with entries \( e^{i\theta_1}, \ldots, e^{i\theta_n} \) then

\[
Rf(z) = \int_{T(n)} \rho(\sigma)f(z)d\sigma
\]

is polyradial and \( ||Rf||_p \leq ||f||_p \). Thus for every \( w \in A_p(\mathbb{C}^n) \) which is polyradial, the operators \( R \circ [B, T^1_m(\lambda)] \circ R \) satisfy weighted norm inequalities. By a theorem of Duandikoetxea et al [4], a polyradial function \( w \) belongs to \( A_p(\mathbb{C}^n) \) if and only if \( w(r_1, \ldots, r_n) \) belongs to \( A_p(\mathbb{R}^n_+, d\mu) \) where \( d\mu(r) = \prod_{j=1}^n r_j dr_j \). Consequently the families \( R \circ T^\lambda_m(\lambda) \circ R \) and \( \lambda \frac{d}{dx} R \circ T^\lambda_m(\lambda) \circ R \) are R-bounded on \( L^p(\mathbb{R}^n_+, d\mu) \). This proves that \( R \circ T^\lambda_m \circ R \) is bounded on \( L^p(H^n) \).

Finally, coming to the proof of the Theorem 1.6 recall that the Riesz transforms which correspond to the multipliers \( m(\lambda) = A_j(\lambda) H(\lambda)^{-1/2} \) are bounded on \( L^p(H^n), 1 < p < \infty \), see [2]. The theorem will be proved if we show that \([B, T^1_m(\lambda)]\) is not bounded on \( L^2(\mathbb{C}^n) \). Note that \( \tilde{m}(\lambda) = A_j H^{-1/2} \) and we have to show that the Hilbert-Schmidt norm of

\[
S = \delta_j(A_j H^{-1/2}) W(f) A^*_j + \delta_j(A_j H^{-1/2}) W(f) A_j
\]

is not bounded by a constant multiple of \( ||f||_2 \). It can be easily seen that \( \delta_j H^{-1/2} = ((H-2)^{-1/2} - H^{-1/2}) A^*_j + \delta_j H^{-1/2} = ((H-2)^{-1/2} - H^{-1/2}) A_j. \) When we take \( f = \Phi_{\alpha,\beta} \) then \( W(f)\varphi = (\varphi, \Phi_{\alpha})\Phi_{\beta} \) and therefore \( S\varphi_{\mu} \) survives only when \( \mu = \alpha + e_j \) or \( \mu = -\alpha - e_j \) where \( e_j \) is the \( j \)-th coordinate vector. Moreover, \( S\Phi_{\alpha+e_j} = (2\alpha_j + 2)^{1/2}\delta_j(A_j H^{-1/2})\Phi_{\beta} \) and \( S\Phi_{\alpha-e_j} = (2\alpha_j)^{1/2}\delta_j(A_j H^{-1/2})\Phi_{\beta} \). This shows that

\[
||S||_{HS}^2 = (2\alpha_j + 2)||\delta_j(A_j H^{-1/2})\Phi_{\beta}||_2^2 + (2\alpha_j)||\delta_j(A_j H^{-1/2})\Phi_{\beta}||_2^2.
\]

Since \( ||f||_2 = 1 \) it is clear that \( ||S||_{HS} \leq C ||f||_2 \) cannot be satisfied. This proves Theorem 1.6.
Acknowledgments

The first author is thankful to CSIR, India, for the financial support. The work of the second author is supported by J. C. Bose Fellowship from the Department of Science and Technology (DST) and also by a grant from UGC via DSA-SAP. Both authors thank the referee for his careful reading and useful comments which were used in revising the manuscript.

References

[1] S. Bagchi and S. Thangavelu, On Hermite pseudo-multipliers, arXiv:1311.5399
[2] M. Christ and D. Geller, Singular integral characterizations of Hardy spaces on homogeneous groups, Duke Math. J. 51 (1984), no. 3, 547-598.
[3] J. Duoandikoetxea, Fourier analysis, Graduate Studies in Mathematics, 29. American Mathematical Society, Providence, RI, 2001
[4] J. Duoandikoetxea, A. Moyua, O. Oruetxebarria and E. Seijo, Radial Ap weights with applications to the disc multiplier and the Bochner-Riesz operators, Indiana Univ. Math. J. 57 (2008), no. 3, 1261-1281.
[5] G. Folland, Harmonic analysis in phase space, Annals of Mathematics Studies,122, Princeton University Press, Princeton, NJ, 1989.
[6] W. Hebisch, Multiplier theorem on generalized Heisenberg groups, Colloq. Math. 65 (1993), no. 2, 231-239
[7] S. Janson, Mean oscillation and commutators of singular integrals operators, Ark. Mat, 16(1978),263-270
[8] K. Jotsaroop, P. K. Sanjay and S. Thangavelu, Riesz transforms and multipliers for the Grushin operator, J. Anal. Math., 119(2013), 255-273
[9] C. Lin, $L^p$ multipliers and their $H^1-L^1$ estimates on the Heisenberg group, Rev Mat Iberoamericana, Vol 11, N. 2, (1995).
[10] S. Lu and Y. Zhang, The weighted norm inequality for a class of oscillatory integral operators, Chinese Science Bulletin, 1992, 37(1): 9-13.
[11] A. Martini and A. Sikora, Weighted Plancherel estimates and sharp spectral multipliers for the Grushin operators, Math. Res. Lett. 19 (2012), no. 5, 1075-1088.
[12] G. Mauceri, The Weyl transform and bounded operators on $L^p(\mathbb{R}^n)$, J. Funct. Anal., 39 (1980), no. 3, 408-429.
[13] G. Mauceri, Zonal multipliers on the Heisenberg group, Pacific J. Math, 95 (1981), no. 1, 143-159.
[14] G. Mauceri and L. de Michele, multipliers on the Heisenberg group, Michigan Math. J., 26 (1979), no. 3, 361-371.
[15] D. Muller and E. M. Stein, On spectral multipliers for Heisenberg and related groups, J. Math. Pures Appl. (9) 73 (1994), no. 4, 413-440.
[16] J. L. Rubio de Francia, Vector-valued inequalities for operators in $L^p$ spaces, Bull. London Math. Soc. 12 (1980), no. 3, 211-215.
[17] S. Thangavelu, Harmonic analysis on the Heisenberg group, Progress in Mathematics, 159, Birkhauser Boston, Inc., Boston, MA, 1998.
[18] L. Weis, Operator valued Fourier multiplier theorems and maximal $L^p$ regularity, Math. Ann. 319(2001), 735-758.

Department of Mathematics, Indian Institute of Science, Bangalore - 560 012, India
E-mail address: sayan@math.iisc.ernet.in
E-mail address: veluma@math.iisc.ernet.in