A Framework for Inferring Causality from Multi-Relational Observational Data using Conditional Independence

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ABSTRACT
The study of causality or causal inference – how much a given treatment causally affects a given outcome in a population – goes way beyond correlation or association analysis of variables, and is critical in making sound data driven decisions and policies in a multitude of applications. The gold standard in causality inference is performing controlled experiments, which often is not possible due to logistical or ethical reasons. As an alternative, inferring causality on observational data based on the Neyman-Rubin potential outcome model has been extensively used in statistics, economics, and social sciences over several decades. In this paper, we present a formal framework for sound causal analysis on observational datasets that are given as multiple relations and where the population under study is obtained by joining these base relations. We study a crucial condition for inferring causality from observational data, called the strong ignorability assumption (the treatment and outcome variables should be independent in the joined relation given the observed covariates), using known conditional independences that hold in the base relations. We also discuss how the structure of the conditional independences in base relations given as graphical models help infer new conditional independences in the joined relation. The proposed framework combines concepts from databases, statistics, and graphical models, and aims to initiate new research directions spanning these fields to facilitate powerful data-driven decisions in today’s big data world.

1. INTRODUCTION
The problem of causal inference goes far beyond simple correlation, association, or model-based prediction analysis, and is practically indispensable in health, medicine, social sciences, and other domains. For example, a medical researcher may want to find out whether a new drug is effective in curing cancer of a certain type. An economist may want to understand whether a job-training program helps improve employment prospects, or whether an economic depression has an effect on the spending habit of people. A sociologist may be interested in measuring the effect of domestic violence on children’s education or the effect of a special curricular activity on their class performances. A public health researcher may want to find out whether giving incentives for not smoking in terms of reduction in insurance premium helps people quit smoking. Causal inference lays the foundation of sound and robust policy making by providing a means to estimate the impact of a certain intervention to the world. For instance, if non-smokers pay reduced insurance premium anyway, and introducing the plan of reduced premium does not help smokers quit smoking, then a simple correlation analysis between people who pay less premium and who do not smoke may not be sufficient to convince policy makers in the government or in insurance companies that the new policy should be introduced – as cited widely in statistical studies, correlation does not imply causation.

The formal study of causality was initiated in statistical science, back in 1920s and 30s by Neyman [32] and Fisher [10], and later investigated by Rubin [49, 51] and Holland [20] among others. The gold standard in causal analysis is performing controlled experiments or randomized trials with the following basic idea: given a population consisting of individual units or subjects (patients, students, people, plots of land, etc.), randomly divide them into treatment (or, the active treatment) and control (or, the control treatment) groups. The units in treatment group receives the treatment whose effect we desire to measure (the actual drug, special training program, discount on the premium, a fertilizer), whereas the units in the control group do not receive it. At the end of the experiment, the difference in the outcome (on the status of the disease, grade in class, smoking status, production of crops) is measured as the causal effect of the treatment. Of course, additional assumptions and considerations are needed in experiment designs to make the results reflect the true causal effect of the treatment [8].

On the other hand, often the causality questions under consideration, including some of the questions mentioned earlier, are difficult or even infeasible to answer by controlled experiments due to ethical or logistical reasons (time, monetary cost, unavailability of instruments to enforce the treatment, etc.). Some extreme examples studied in the past in sociology, psychology, and health sciences include [41] studying effects of criminal violence of laws limiting access to handguns, effects on children from occupational exposures of parents to lead, or long term psychological effects of the death of a close relative, which are not possible to analyze by controlled experiments. Nevertheless, in many such cases we have an observational dataset recording the units, their treatment assignment, and their outcomes, pos-
sibly recorded by a research agency or by the government through surveys. Using such observational data, it is still possible to infer causal relationships between the treatment and the outcome variable under certain assumptions, which is known as the observational study for causal analysis.

In observational studies, however, when units are not assigned treatment or control at random, or when their ‘environment’ selects their treatment (e.g., people in the rich neighborhoods received the special training program as treatment, whereas people in poorer neighborhoods formed the control group), differences in their outcomes may exhibit effects due to these initial selection biases, and not due to the treatment. Some of the sources of such biases may have been measured (called observed covariates, or overt biases [41], e.g., age, gender, neighborhood, etc), whereas some of these biases may remain unmeasured in the observed data (called unobserved covariates, or hidden biases, e.g., some unrecorded health conditions). Observed covariates are taken care of by adjustment in observational studies, e.g., by matching treated and control units with the same or similar values of such covariates (further discussed in Section 2).

1.1 A Framework for Causal Analysis on Multi-Relational Observational Data using Conditional Independence

In the database literature, the study of causality has so far focused on problems like finding tuples or summaries of tuples that affect the query answers [30, 46, 52], abductive reasoning and view updates [4], data provenance [15], why-not analysis [7], and mining causal graphs [54]. Until very recently [45, 53], the topic of causal analysis as done in statistical studies in practice has not been studied in database research. On the other hand, observational causal studies, even as studied nowadays, can significantly benefit from database techniques. For instance, the popular potential outcome model by Neyman-Rubin [51] (discussed in Section 2) can be captured using the relational database model, and one of the most common methods for observational studies – exact matching of treated and control units using their observed covariates – can be efficiently implemented using the group-by operators in SQL queries in any standard database management system, thereby improving the scalability of matching methods [45, 53].

This paper proposes a framework that goes beyond employing database queries to efficiently implement existing techniques for observational studies. The standard observational studies are performed on a single table, storing information about treatment, outcome, and covariates of all units in the population. On the other hand, many available datasets are naturally stored in multi-relational format, where the data is divided into multiple related tables (e.g., author-authorship-publications, student-enrollment-course-background, customer-order-products, restaurant-customer-reviewed-reviews etc.).

These datasets are large (in contrast to relatively smaller datasets mostly used in observational studies recorded by research agencies through surveys), readily available (e.g., DBLP [21], Yelp [23], government [22], or other online data repository used for research [27]), and pose interesting causal questions that can help design policies in schools, businesses, or health.

In addition, causal analysis on observational data inherently depends on the strong ignorability condition (Section 2, i.e., the treatment assignment and the potential outcomes are conditionally independent given a set of observed covariates). The common practice is to try to include as many covariates as possible to match treated and control units in order to ensure this conditional independence. However, one dataset stored as a single relation may have a limited number of possible covariates, whereas integrating with other datasets may extend the set of available confounding covariates for matching (we discuss examples in Section 3.2).

In some other scenarios, additional interesting causal questions may arise by integrating multiple datasets or combining multiple relations. For instance, in a restaurant-customer-reviewed-review dataset like Yelp, one may be interested in finding whether awarding special status to a customer makes her write more favorable reviews with higher ratings. Here the preferred status belongs to the customer table, whereas the ratings of the reviews belong to the review table. Clearly, the set of possible causal questions will be far more limited if only one relation is considered for observational studies.

Extending the scope of observational studies to multi-relational data, however, requires additional challenges to be addressed, and links observational studies to understanding conditional independences in relational databases. Causal analysis on observational data crucially depends on a number of assumptions like SUTVA and strong ignorability involving independences and conditional independences (Section 2). In fact, causal inference can only be done under some causal assumptions that are not testable from data [35], and the results are only valid under these assumptions. Therefore, to perform sound causal analysis on datasets in multiple relational tables, one important task is to understand the notion of conditional independences in the given tables (base relations) and also in the joined table.

The natural approach is to define conditional independence using the probabilistic interpretation: if variables A, B are independent, then \( P(AB) = P(A) \times P(B) \), and if A, B are conditionally independent given C, then \( P(AB|C) = P(A|C) \times P(B|C) \), where in the context of relational databases A, B, C denote subsets of attributes from one or multiple base relations. However, measuring probability values from an exponential space on variable combinations to deduce conditional independences in the joined relation is not only impractical for high-dimensional data, but may also be incorrect when the available data represents only a sample from the actual dataset (e.g., data on 50 patients vs. the data on all people in the world). On the other hand, as explained in the book by Pearl [53], humans are more likely to be able to make probabilistic judgment on a small set of
variables based on domain knowledge, e.g., whether a patient suffering from disease A is conditionally independent on the patient suffering from disease B given her age, which can be obtained much more efficiently and correctly without looking at the data.

Hence one fundamental problem in the framework of multi-relational observational data for observational causal studies is understanding which conditional independences in the joined relation can be inferred from a set of conditional independences given on the base relations, which is the focus of this paper. Unfortunately, testing whether a conditional independence holds given a set of other conditional independences in general is undecidable \[1\] (the conditional independence has to hold in an infinite number of possible distributions), although in certain scenarios, conditional independences are decidable, e.g., when the underlying distribution is graph-isomorph (has an equivalent undirected or directed graphical representations) \[33, 37\] (discussed further in Section 2). Nevertheless, if we still derive certain conditional independences in the joined relation using sound inference rules, we know that a correct conclusion has been made, and can use such conditional independences for covariate selection in observational studies on the joined relation.

1.2 Our Contributions

We propose a framework for performing sound causal analysis on multi-relational observational data with multiple base relations. Toward this goal, we review the concepts of potential outcome model and observational studies for causal analysis as studied in statistics over many decades, and make connections with the relational database model.

Performing sound causal analysis on observational data requires conditional independence assumptions. Using the standard probabilistic interpretation, we study conditional independences that hold in the joined relation for natural joins between two relations in general and special cases (foreign key joins and one-one joins). We show that, for any conditional independence in a base relation of the form “\(X\) is independent of \(Y\) given \(Z\)”, if the set of join attributes is a subset of \(X, Y,\) or \(Z\), then this conditional independence also holds in the joined relation. As applications, we show that (i) conditioning on join attributes (or a superset) satisfies strong ignorability but is not useful since the estimated causal effect will be zero, and (ii) in some cases the join can be avoided to achieve the same estimated causal effect.

We show that, in general, a conditional independence that holds in the base relation may not hold in the joined relation. However, if the conditional independences that hold in the base relation are graph-isomorph \[33, 37\], i.e., if an undirected graph exists (called a perfect map or P-map, see Section 2) such that (i) \(X\) and \(Y\) are separated by a vertex set \(Z\) (\(Z\) forms a cutset between \(X, Y\)), if and only if (ii) \(X\) and \(Y\) are conditionally independent given \(Z\), and if the join is on a single attribute, then all the conditional independences from the base relation propagate to the joined relation.

We also show that, for join between two relations on a single join attribute, the union graph of the P-maps of the base relations, if the given P-maps are connected, gives an independency map or I-map of the joined relation, where separation in the I-map implies a valid conditional independence in the joined relation (although some conditional independences may not be captured in an I-map unlike a P-map). We also review the notions of graphoid axioms and undirected graphical model or Markov networks from the work by Pearl and Paz \[37\] \[38\] that give a toolkit to infer sound conditional independences in the joined relation.

In addition to understanding conditional independences and use of undirected graphical models to their full generality, we discuss four fundamental research directions using our framework: First, we discuss using directed graphical models, or causal Bayesian networks for inferring conditional independences. In contrast to vertex separation in undirected graphs, conditional independences in directed Bayesian networks is given by \(d\)-separation (Pearl and Verma, \[34, 39\]), which may capture additional conditional independences than the undirected model, but introduces new challenges in inferring conditional independence for joined relation. Second, we discuss extension of inferring conditional independences in joined relation to achieving strong ignorability, which involved potential outcomes and missing data (instead of observed outcome). Third, we discuss challenges that may arise in many-to-many joins, which may violate basic causal assumptions, and need to be taken care of in observational studies. Fourth, we discuss whether other weaker concepts of conditional independences like embedded multi-valued dependency (Faigin, \[15\]) is more suitable for causal analysis for multi-relational data, since using the natural probabilistic interpretation, some conditional independences may not propagate to the joined relation, which is contrary to the intuitive idea that the independences are inherent property of different relations and should be unchanged whether or not the relation is integrated with other relations.

1.3 Our Vision for the Framework

We envision several potential impact of our framework in both databases and causality research. (1) It extends the well-studied Neyman-Rubin potential outcome model to multiple relations, enriching the possible set of covariates that can be included or possible set of causal questions that can be asked. Further, using database techniques like group-by queries makes the popular techniques for observational studies like matching more scalable. (2) It presents the rich causality research in statistics to databases, which has significant practical implications in avoiding ‘incorrect causal claims’ in the study of big data and in policy making. (3) It brings together techniques from causality in statistics \[51\] and artificial intelligence \[35\], database theory (e.g., embedded multi-valued dependency \[15\]), database queries (for matching), and undirected and directed graphical models (for conditional independences) \[33, 26\]. Thereby, it creates new research problems
spanning multiple domains, and creates scope of collaboration among database theoreticians and practitioners, statisticians, researchers in artificial intelligence, and domain scientists who are interested in solving causal questions for specific applications.

**Roadmap.** In Section 2 we review some concepts from the causality and graphical models literature. In Section 3 we present our framework and define the notion of conditional independence. Section 4 explores conditional independences in the joined relation, while Section 5 discusses inference using undirected graphs. We conclude by discussing further research directions in Section 6. All proofs and further discussions on related work appear in the appendix.

## 2. BACKGROUND

In this section, we review some material on causality, observational studies, graphoid axioms, and graphical models that we will use in our framework. We use $X \perp Y$ to denote that (sets of) variables $X, Y$ are marginally independent, and $X \perp Y | Z$ or $I(X, Z, Y)$ to denote that $X$ and $Y$ are conditionally independent given $Z$.

### 2.1 Potential Outcome Model

The commonly used model for causal analysis in statistics is known as the potential outcome model (or Neyman-Rubin potential outcome model), which was first proposed in Neyman’s work [52], and was later popularized by Rubin [49, 51]. In causal analysis, the goal is to estimate the causal effect of a treatment $T$ (e.g., a drug to treat fever) on an outcome $Y$ (e.g., the body temperature). The treatment variable $T$ assumes a binary value, where $T = 1$ means that the treatment (or, active treatment) has been applied, and $T = 0$ means that the control treatment (or control) has been applied to the unit. For unit $i$, we denote its outcome and treatment by $Y_i$ and $T_i$. In contrast to predictive analysis, where one computes the distribution (probability or expectation) of $Y$ given $T = t$, in causal analysis, selected units (the treatment group) are administered the treatment, i.e., the value of $Y_i$ is observed by (intuitively) forcing unit $i$ to assume $T_i = 1$: this is called the intervention mechanism. Denoting the value of the outcome as $Y_i(0)$ when $T_i = 0$, and $Y_i(1)$ when $T_i = 1$, the goal is to estimate the causal effect by computing the average treatment effect (ATE):

\[
ATE = E[Y(1) - Y(0)]
\]  

The variables $Y(1), Y(0) = \{ Y_i(1), Y_i(0) \}$ are called the potential outcomes. To estimate the ATE, ideally, we want to estimate the difference in effects of administering both $T_i = 1$ and $T_i = 0$ to the same unit $i$, i.e., we want to compute both $Y_i(1), Y_i(0)$. But the fundamental problem of causal inference is that for each unit we only know either $Y_i(1)$ or $Y_i(0)$ but not both, reducing the causal inference to a missing data problem [20, 42].

The potential outcome model on $n$ units can be represented in a tabular form as shown in Table 1. We will explain the set of covariates $X$ in Section 2.2.

| Unit | $X$ | $T$ | $Y(1)$ | $Y(0)$ | $Y(1) - Y(0)$ |
|------|-----|-----|-------|-------|---------------|
| 1    | $X_1$ | $T_1$ | $Y_1(1)$ | $Y_1(0)$ | $Y_1(1) - Y_1(0)$ |
| 2    | $X_2$ | $T_2$ | $Y_2(1)$ | $Y_2(0)$ | $Y_2(1) - Y_2(0)$ |
| ...  | ... | ... | ... | ... | ... |
| $n$  | $X_n$ | $T_n$ | $Y_n(1)$ | $Y_n(0)$ | $Y_n(1) - Y_n(0)$ |

Table 1: Neyman-Rubin’s potential outcome framework [51]

In randomized controlled experiments, however, randomly assigning treatments to units (each patient is randomly given either the drug for fever or a placebo) gives an unbiased estimate of ATE. In this case, the treatment assignment $T$ is independent of the potential outcomes $Y(1), Y(0)$, i.e.,

\[
T \perp Y(1), Y(0).
\]

Therefore,

\[
E[Y(1)] = E[Y(1)| T = 1] \quad \text{and} \quad E[Y(0)] = E[Y(0)| T = 0],
\]

and from (1) we get,

\[
ATE = E[Y(1)] - E[Y(0)] = E[Y(1)| T = 1] - E[Y(0)| T = 0].
\]

Now the ATE can be estimated by taking the difference of average outcomes of the treated and control units under an additional assumption called SUTVA [51, 11]:

**Definition 2.1. Stable Unit Treatment Value Assumption or SUTVA (Rubin [52], Cuz [77]):**

1. There is no interference among units, i.e., both $Y_i(1), Y_i(0)$ of a unit $i$ are unaffected by what action $T_j$ any other unit $j$ receives.
2. There are no hidden versions of treatments, i.e., no matter how unit $i$ received treatment $T_i = 1$ (resp. 0), the outcome will be observed $Y_i(1)$ (resp. $Y_i(0)$).

### 2.2 Causality for Observational Data

The potential outcome model gives a formal method to reason about ‘potential outcomes’ and estimate ATE in controlled experiments. However, once we attempt to do causal analysis on observational data – a dataset containing $Y_i(1), Y_i(0), T_i$ for each unit $i$ – the independence assumption in (2) typically fails. As mentioned in the introduction, this happens due to selection biases [11], when the treatment assignment depends on the environment of the unit (e.g., it may happen that male patients between age 20-30 received the drug and the other patients received placebo, students already performing well in a class enrolled in a special training program, etc.). In observational studies, some of these potential factors are also recorded in the dataset as variables $X$ (called confounding covariates), while the others may remain unobserved. Table 1 for the potential outcome model also shows $X_i$ for each unit $i$, which is a vector containing possible confounding covariates (e.g., in the study of the drug for fever, $X$ may include age, gender, medical history and conditions, and ethnicity of the patient). A controlled experiment takes care of biases due to both observed and unobserved covariates by randomization. For observational studies, one can
still adjust for selection biases due to observed covariates with additional unconfoundedness assumption as follows:

**Definition 2.2. Strong ignorability assumption**
(Rosenbaum and Rubin, [42]):

1. each individual has a positive probability of being assigned to treatment, and
2. potential outcomes \(Y(0), Y(1))\) and \(T\) are conditionally independent given relevant covariates \(X\):

\[ T \perp Y(0), Y(1) | X \]  

(4)

Using [4], we can write \(ATE\) as

\[ ATE = E[X(Y(1)|T = 1, X) - E[X(Y(0)|T = 0, X)] \]  

(5)

where \(E[X(Y(t)|T = t, X)] = \sum_{X=x} P(X = x)E[Y|T = t, X = x], \ t \in \{0, 1\}\), can be estimated from the observational data, of course if SUTVA (Definition 2.1) holds too. The RHS of Equation 5 is known as the adjusted estimand and is denoted by \(A(T, Y, X)\).

**Definition 2.3. c-equivalence**
(Pearl and Paz, [38]): Two set of variables \(X\) and \(X'\) are called c-equivalent if \(A(T, Y, X) = A(T, Y, X')\), i.e., adjusting based on both \(X\) and \(X'\) would yield a same result.

Thus, if adjusting for one of them, say \(X\), is sufficient for computing the causal effect of \(T\) on \(Y\), adjusting for the \(X'\) is also sufficient. The following is shown in [38].

**Theorem 2.4.** (Pearl and Paz, [38]): A sufficient condition for \(X\) and \(X'\) to be c-equivalent is that they satisfy one of the following two conditions:

(i) \(T \perp X'|X\) and \(Y \perp X'|X', T\)
(ii) \(T \perp X|X'\) and \(Y \perp X'|X, T\)

Matching methods for observational studies. If strong ignorability holds for a set of covariates \(X\), one can estimate ATE using Equation 5 by (a) dividing units with the same value of all covariates in \(X\) into groups (called exact matching), (b) taking the difference of average \(Y\) values of treated and control units for each group. In practice, however, a direct application of this method is impossible, because the data is typically very sparse: for any value \(X = x\) we either have no data values at all, or very few such values, or only treated or only control units for some groups, thereby estimating ATE becomes infeasible. In general, matching methods used in statistics group treated and control units based on the same or similar values of covariates (exact or approximate matchings) to create a balance between treated and control units in each matched group. Popular approximate matching methods are propensity score matching [12] and coarsen exact matching (CEM) [25]. In the former, units with the same value of \(B(X_i) = b\) are matched together, \(B(X) = P[T = 1|X = x]\) being the propensity score. [47]. In the latter, the vector of covariates is coarsened according to a set of user-defined cutpoints and then exact matching used to match units with similar value of the coarsened covariates [25].

Exact matching and CEM on observational data bears high resemblance with the group-by operator in SQL queries, leading to recent applications of database techniques for matching in causal inference [45, 53]. Further discussion on matching techniques can be found in Appendix D.

**Covariate selection for observational studies.** The process of covariate selection chooses a good subset of variables from available variables \(X\) to be used in a matching method for observational studies. Covariate selection is a challenging problem: to maintain strong ignorability, we require that (i) each valid matched group has to have at least one treated and one control unit, favoring selection of as few variables as possible, as well as (ii) the treatment and potential outcomes should be independent given the selected covariates, which may require choosing many covariates. The efficiency of matching in the presence of a large number of covariates is another practical concern. The popular matching techniques aim to select as many variables as possible with the intuition that collecting more information will make the treatment and potential outcomes conditionally independent, whereas a more methodical approach using the underlying graphical causal structure of the covariates has been studied by Pearl and others [35].

### 2.3 Graphoid Axioms and Graphoids

Given a probabilistic model \(P\) on a finite set of variables \(U\) with discrete values, and three subsets \(X, Y, Z \subseteq U\), the variables \(X, Y\) are said to be conditionally independent given \(Z\) (denoted by \(X \perp Y|PZ\) or \(I(X, Z, Y)\)) if for all values \(X = x, Y = y, Z = z, P(x|y, z) = P(x, z)\) whenever \(P(y, z) > 0\). The seminal work on graphoid axioms by Pearl and Paz [33, 37] gives a set of logical conditions for constraining the set of triplets \((X, Y, Z)\) such that \(X \perp Y|PZ\) holds in \(P\).

**Theorem 2.5. Graphoid axioms [33, 37].** For three disjoint subsets of variables \((X, Y, Z)\), if \(I(X, Z, Y)\) holds in some probabilistic model \(P\), then \(I\) must satisfy the following four independent conditions:

- **Symmetry:**
  \[ I(X, Z, Y) \Leftrightarrow I(Y, Z, X) \]  

(6)

- **Decomposition:**
  \[ I(X, Z, Y \cup W) \Rightarrow I(X, Z, Y) \& I(X, Z, W) \]  

(7)

- **Weak Union:**
  \[ I(X, Z, Y \cup W) \Rightarrow I(X, Z \cup W, Y) \]  

(8)

- **Contraction:**
  \[ I(X, Z, Y) \& I(X, Z \cup W, Y) \Rightarrow I(X, Y \cup W, Z) \]  

(9)

If \(P\) is strictly positive, i.e., \(P(x) > 0\) for all combination of variables \(x\), then a fifth condition holds:

- **Intersection:**
  \[ I(X, Z \cup W, Y) \& I(X, Z \cup Y, W) \Rightarrow I(X, Z, Y \cup W) \]  

(10)
Here $Y \cup W$ denotes the union of variable sets $Y, W$. The axiomatic characterization of probabilistic independences give a powerful tool to discuss conditional independences that hold in the model but are not obvious from the numeric values of the probabilities. It also allows us to derive new conditional independences starting with a small set of initial independences, possibly handcrafted by an expert. Although the membership problem – testing whether a triplet $(X, Z, Y)$ satisfies $I(X, Z, Y)$ given a set of conditional independences – is undecidable [1], still if a conditional independence can be derived using the graphoid axioms, we know that it holds in the model.

The graphoid axioms have a correspondence with vertex separation in undirected graphs [37]. If $I(X, Z, Y)$ denotes $Z$ separates $X$ from $Y$ in an undirected graph $G$ (i.e., removing $Z$ destroys all paths between $X, Y$), then it can be easily checked that $I$ satisfies the graphoid axioms. In general, other models in addition to conditional independence relations in probability models can also satisfy the graphoid axioms, and in that case, they are called graphoids:

**Definition 2.6. (Graphoids [37]):** Let $I$ denote an independence relation consisting of a set of triplets $(X, Z, Y)$ where $I(X, Z, Y)$ denotes that $X$ and $Y$ are independent given $Z$. If $I$ satisfies inference rules (7) to (10), it is called a semi-graphoid. If $I$ also satisfies rule (11), it is called a graphoid.

Other than conditional independences and vertex separation in undirected graphs, vertex separation in directed graphs, embedded multi-valued dependencies in relational databases [15] (see Section 6.4), and qualitative constraints [54] are semi-graphoids.

### 2.4 Dependencies by Undirected Graphs

Let $I$ be an arbitrary dependency model $M$ encoding an independence relation $I = I(M)$ with a subset of triplets $(X, Z, Y)$ where $I(X, Z, Y)_M$ denotes $X$ and $Y$ are independent given $Z$. As discussed above, graphoid axioms have a correspondence with vertex separation in graphs. However, it does not say whether a set of given independences $I$ can be captured using vertex separation. Nevertheless, for some dependency models $M$, it is possible to find a graph representation on the variable set such that the conditional independence corresponds to vertex separation.

We denote by $X \perp Y | G \cup Z$ if two subsets of vertices $X, Y$ are separated by $Z$, or $Z$ forms a cutset between $X, Y$. Note that the independence in graph by vertex separation has no relation with independences in a probability space or in a dependency model in general.

**Definition 2.7. (D-map, I-map, P-map) [33]** An undirected graph $G$ on the variable set is a dependency map or D-map of $M$ if for all disjoint subsets $X, Y, Z$, we have

$$I(X, Z, Y)_M \Rightarrow X \perp Y | G \cup Z \quad (11)$$

$G$ is an independence map or I-map if

$$I(X, Z, Y)_M \Rightarrow X \perp Y | G \cup Z \quad (12)$$

$G$ is said to be a perfect map or P-map of $M$ if it is both a D-map and an I-map.

A D-map guarantees that vertices that are connected in $G$ are indeed dependent in $M$, but a pair of separated vertex sets in $G$ may be dependent in $M$. An I-map guarantees that separated vertices in $G$ are independent in $M$, although if they are not separated in $G$, they may still be independent in $M$. Empty graphs are trivial D-maps and complete graphs are trivial I-maps, although they are not useful in practice. In particular, obtaining an I-map encoding as many independences as possible is useful for causal analysis on observational data since it helps in covariate selection satisfying strong ignorability (Definition 2.2). In Section 5 we will obtain I-maps for the joined relation using P-maps of base relations.

Ideally, we want to obtain an I-map that is also a P-map, but there are simple dependency models that do not have any P-maps (e.g., the ones where $I(X, Z, Y)$ but $\neg I(X, Z \cup W, Y)$, since a superset of a cutset is also a cutset). Some such dependencies can be captured using directed graphical models that we discuss in Section 6.1 as a research direction using this framework.

A dependency model $M$ is graph-isomorph if there exists an undirected graph that is a P-map of $M$. The following theorem gives a necessary and sufficient conditions for dependency models to be graph-isomorph.

**Theorem 2.8. (Pearl and Paz [37, 33]):** A necessary and sufficient conditions for a dependency model $M$ to be graph-isomorph is that $I(X, Z, Y) = I(X, Y, Z)_M$ satisfies the following five independent axioms on disjoint set of variables:

- **Symmetry:**
  $$I(X, Z, Y) \Leftrightarrow I(Y, Z, X) \quad (13)$$

- **Decomposition:**
  $$I(X, Z, Y \cup W) \Rightarrow I(X, Z, Y) \land I(X, Z, W) \quad (14)$$

- **Intersection:**
  $$I(X, Z \cup W, Y) \land I(X, Z \cup Y, W) \Rightarrow I(X, Z, Y \cup W) \quad (15)$$

- **Strong Union:**
  $$I(X, Z, Y) \Rightarrow I(X, Z \cup W, Y) \quad (16)$$

- **Transitivity:** For all other variables $\gamma$
  $$I(X, Z, Y) \Rightarrow I(X, Z, \gamma) \lor I(\gamma, Z, Y) \quad (17)$$

It is possible to construct a minimal I-map of a probability distribution (removing any edge would destroy the I-map) and to check if a given graph is an I-map of a given probability distribution by constructing the Markov Network of the probability distribution: when we quantify the links of the undirected graph, the theory of Markov fields allows constructing a complete and consistent quantitative model preserving the dependency structure of an arbitrary graph $G$ [33]. Conditional independences can also be captured using directed acyclic graphs (Bayesian networks and Causal graphs) that we discuss in Section 6.
3. A FORMAL FRAMEWORK FOR
CAUSAL ANALYSIS ON MULTI-
RELATIONAL DATA

Using the concepts in the previous section, we now
describe a framework for causal analysis on observational
data given in multiple relations. First we describe some
notations used in the rest of the paper.

Let D be a database schema with k relations
R1,⋯,Rk (called base relations). We use Ri, where
i ∈ {1,k} both as the name of a relation and the subset
of attributes contained in the relation. A = ∪i Ri denote
the set of all attributes.

Any join without a join condition in the sub-
script denotes the natural join with equality condition
on the common set of attributes, i.e., R ⊙ S = R ⊙ R1,sR1,R2,sR2,⋯,sRk,Rk, where A1,⋯,Ap = R ∩ S
denotes the common set of attributes in R and S. We use
A,B,C,⋯ ∈ A to denote individual attributes, and
unless mentioned otherwise, XY denotes XY denotes X ∪ Y. We will use A ∈ A both as an attribute and as a singleton subset \{ A \}. For a tu-
ples t ∈ R, and attribute(s) A ∈ R, we use t[A] to denote
the value of attribute(s) A of R.

For an attribute A ∈ A and its value a, if A ∈ R, then
N_{R,(A,a)} denotes the number of tuples in R with
A = a, i.e., N_{R,(A,a)} = | \{ t ∈ R : t[A] = a \} |. Unless
mentioned otherwise, we assume the bag semantics,
i.e., the relations can have duplicates, and
projection are duplicate-preserving. For multiple at-
tributes A1,⋯,Ap and their respective values a1,⋯,ap,
N_{R,(A1,a1),⋯,(Ap,ap)} = | \{ t ∈ R : t[A1] = a1,⋯,t[Ap] = ap \}| denotes the number of all tuples matching all the
values of all the attributes. When the attributes are
clear from the context, we will use N_{R,a} instead of
N_{R,(A,a)}, and N_{R,a1,⋯,ap} instead of N_{R,(A1,a1),⋯,(Ap,ap)} for simplicity. N_R denotes the number of tuples in R.

3.1 Conditional Probability and Independence in a Relation

Given an instance of a relation R, the probability dis-
tribution of an attribute A is given by

$$P_R[A = a] = \frac{N_{R,(A,a)}}{N_R} \quad (18)$$

This notion of probability has also been used in [12]
to define information dependencies. The joint probability
distribution of two attributes A, B is given by (similarly,
for two subsets of attributes)

$$P_R[A = a, B = b] = \frac{N_{R,(A,a),(B,b)}}{N_R} \quad (19)$$

Note that for an attribute A ∈ R ∩ S belonging to two
relations R, S, the distribution of A in R and S may be
different, i.e., in general, Pr_R[A = a] ≠ Pr_S[A = a].

Given two attributes A and B, the conditional prob-
ability of A given B is given by

$$Pr_R[A = a | B = b] = \frac{Pr_R[A = a, B = b]}{Pr_R[B = b]} = \frac{N_{R,(A,a),(B,b)}}{N_{R(B=b)}} \quad (20)$$

For every relation R in D, we also have a set of condi-
tional independence statements CI_R defined as follows:

**Definition 3.1. (Conditional independence)** Let X, Y, Z be three mutually disjoint subset of attributes in R. We say that X and Y are conditionally independent given Z, denoted by X ⊥ ⊥ Y|RZ, if for all values x,y,z of X,Y,Z (all
X,Y,Z,x,y,z are subsets of attributes or values),

$$Pr[X = x | Y = y | Z = z] = Pr[X = x, Y = y | Z = z] \quad (21)$$

If X ⊥ ⊥ Y|RZ does not hold in R, we write ¬(X ⊥ ⊥ Y|RZ).

Similarly, X and Y are (marginally) independent if

$$Pr_R[X = x] \times Pr_R[Y = y] = Pr_R[X = x, Y = y]$$

Note that for multi-relational databases, the sub-
set R is important, since even if all of X,Y,Z belong to two
relations R,S, it may hold that X ⊥ ⊥ Y|RZ whereas
¬(X ⊥ ⊥ Y|SZ). We will use CI as an abbreviation of
Conditional Independence. For a CI X ⊥ ⊥ Y|RZ, we say that X,Y belong to the LHS (left hand side) of the
CI, and Z belongs to the RHS (right hand side).

**Entropy.** Under the defined distribution each sub-
set X ⊆ A defines a marginal distribution P(X) with
entropy H(X) = −Σx P(X = x) log P(X = x), where x
denotes a combination of values of attributes in X.

Given X,Y,Z ⊆ A, the other information measures that
we will use in the paper are (e.g., see [10]): (i) con-
ditional entropy: H(X|Y) = H(X) − H(Y); (ii) mutual
information I(X,Y) = H(X) + H(Y) − H(X,Y),
and (iii) conditional mutual information: I(X,Y|Z) =
H(X|Z) − H(X|Y|Z) = H(X) − H(Y|Z).

Note that we are using 1 for independence and I for mutual information.

**Independence in schema vs. instance.** Independence
is typically considered as a property of a schema,
i.e., a conditional or unconditional independence should
hold on all possible instances of a relation (which may
not be true in practice, since we mostly get a sample
of the real world). For instance, if we are looking at
students database, the number of courses taken by a
student and whether the student has done an inter-
ship may not be independent (senior students are likely
to take more courses and also do an internship).
However, given a student or given the seniority of the stu-
ents, these two attributes are conditionally independent.
This conditional independence follows from the
domain knowledge and not necessarily from a specific
instance of the database. In this paper, we focus on con-
ting independence statements that can be inferred from
the schema-level information.

3.2 Why Multi-Relational Data

Note that the potential outcome framework as shown
in Table I resembles a single relation or table in
relational database model. If all required information,
i.e., treatment, outcome, and confounding covariates, is available as a single table, the standard potential outcome model with a single table suffices. However, to do a sound and robust causal analysis on observational data, it is desirable to collect as much information about the units (i.e., possible confounding covariates) as possible. The first example below motivates why it is useful to include additional attributes as covariates by combining multiple relations.

**Example 3.2. (Extension of covariate set by joining multiple relations)** Suppose we have a Students dataset of the form

Students(sid, major, parents_income, gpa),

which stores the id of the student, major, annual income of the parents, and the gpa. The goal is to estimate the causal effect of income of the parents on the performance or gpa of the students, in particular, how much having an annual income of $>100k$ ($T = 1$ if and only if $\text{parents\_income} > 100k$) affects the gpa of the students. If we do causal analysis using this dataset only, the only available covariate is the major of the student. Conditioning on id does not help, since only one tuple will satisfy a given id values, thereby violating strong ignorability condition (Definition 2.2) and making matching unusable, since a group will not have one treated and one control units to estimate the causal effect within a group. Now if we only use major of the student as a covariate, (i) it may not satisfy the strong ignorability condition that $\text{parents\_income} \text{\_gpa\_major}$ (where $\text{parents\_income}'$ $\text{gpa\_major}$ denotes the potential outcomes for income for low and high gpa's), so considering this covariate will give wrong estimate of ATE (1) and (ii) ignoring this covariate will lead to considering all students in a single giant group, thereby again giving an incorrect estimate, since it may not hold that $\text{parents\_income}'$ $\text{gpa}$.

On the other hand, there may be several other datasets available that include additional information about this causal analysis. There may exist a course relation and an enrollment relation storing the courses the students took: Enroll(sid, cid), and Course(cid, year, title, dept, instructor). There also may exist relations storing the names of the parents, and their jobs, educational background, whether they own a house, their ethnicity, and age: Parents(sid, pid, name, job, edu, owns_house, ethnicity, age). For simplicity, we assume that information about only one parent is stored, and revisit this assumption in Section 6.3. Now in the causal analysis, we can include (some of the) attributes $\text{year, title, dept, instructor}$ from the Course relation, and $\text{job, edu, owns\_house, ethnicity, age}$ as additional covariates. The ethnicity, age, education may affect the income of the parent, as well as can also affect the gpa of the student, whereas the information about the courses can affect the gpa. Including these additional as covariates has the potential to make the causal analysis better.

Not only using multiple relations extend the set of available covariates, it also extends the scope of causal analysis where the treatment $T$ belongs to one relation and the outcome $Y$ belongs to another relation, allowing additional causal questions that can be asked.

**Example 3.3. (Extension of causal questions by joining multiple relations)** In Example 3.2, one may be interested in estimating the causal effect of the profession (job) or education level (edu) of the parents (which comes from the ParentInfo relation) in the gpa or major of the student (which comes from the Student relation). Clearly, this question cannot be answered if we look at only single relation, but can be answered if we consider the join of Students, Parents, and ParentInfo relations.

In summary, sometimes the data itself is naturally stored in multiple relations to reduce redundancy, whereas in other cases integrating datasets to gather new information may extend the set of covariates that can be used, or the set of causal questions that can be asked. Therefore, allowing multiple relations in observational data significantly extends the potential outcome framework by Neyman-Rubin (Sections 2.1 and 2.2) to be further useful for practical purposes.

### 3.3 Framework: Causal Analysis with Multiple Relations

Suppose the data is stored in $k$ relations $R_1, \ldots, R_k$, and we want to perform the causal analysis on the joined relation $J = R_1 \times \cdots \times R_k$. If the intended treatment variable $T$ is not already in binary form, we consider a derived attribute $T$, which is 1 if and only if a given predicate $\phi$ on one of the chosen columns $A_T \in \mathcal{A}$ evaluates to true, i.e., $T = 1 \iff \phi(A_T) = \text{true}$. In Example 3.2 $\phi$ is whether $\text{parents\_income} > 100k$. Another chosen column $Y \in \mathcal{A}$ serves as the outcome variable, and may assume real values.

- The final goal is to estimate the causal effect (ATE, (7)) of $T$ on $Y$.

To meet the above goal using techniques from observational studies, we need to solve several sub-goals. The first and foremost sub-goal is the following:

- **Select a set of covariates** $X \subseteq \mathcal{A} \setminus \{TY\}$ such that the strong ignorability (Definition 2.2) holds, i.e.,
  \begin{equation}
  T \perp Y(0), Y(1) | X \tag{22}
  \end{equation}

Unfortunately, due to the fundamental problem of causal analysis [20][22], for any unit only one of $Y(0)$ and $Y(1)$ is observed. If a directed causal graph on the variables is available, Pearl [33] gives sufficient conditions called backdoor criteria (defined in Section 6) to check for admissible covariates $X$ satisfying the above condition. There is a long-standing debate among causality researchers whether the causal graphical model is a practical assumption.
3.4 Valid Units for Causal Analysis from Joins

For doing causal analysis on joined relation, one of the first tasks is to define the units. We illustrate the challenge in this task in the following example.

**Example 3.4.** Consider three relations: \( P(\text{jid}, \text{training}, \text{seniority}) \), \( S(\text{sid}, \text{gpa}, \text{major}) \), \( R(\text{jid}, \text{sid}, \text{class}, \text{sem}, \text{year}, \text{grade}) \), respectively denoting relations for Professor, Student, and Teaches. Here \( \text{tid} \) and \( \text{sid} \) denote the id-s of instructors and students (also foreign keys from \( R \) to \( P, S \)), and other variables are \( \text{training} \) (Boolean variable denoting whether the professor had a special training or went to a top-10 school), \( \text{seniority} \) (of the professor: senior, mid-level, or junior), \( \text{class}, \text{sem}, \text{year}, \text{grade} \) (the class taught by the professor that the student took, semester, year, and grade obtained by the student), \( \text{gpa} \) (average grade of the student), \( \text{major} \) (major of the student). The attributes \( \text{tid}, \text{sid} \) are foreign keys in \( R \) referring to \( P \) and \( S \) respectively.

Suppose one asks the question

- Estimate the causal effect of the training received by instructors on the grades of the students.

There are several tasks to be solved to answer this question: (1) what should be the units, treatment \( T \), and outcome \( Y \), (2) will they satisfy the basic assumption SUTVA, (3) what would be a good choice of confounding covariates \( X \) to satisfy the strong ignorability condition.

From the question, intuitively \( \text{training} \) should be the treatment \( T \) (we revisit this below), but for \( Y \), we have two choices: \( \text{grade} \) from \( P \) and \( \text{gpa} \) from \( S \).

We make the following observation:

**Observation 3.5.** Let \( R \) be the table containing a specified outcome \( Y \) and let \( U \) be the population table containing the units. For SUTVA to hold, each \( Y \)-value from the rows in \( R \) can appear at most once in \( P \), i.e., each tuple in \( R \) can contribute to at most one tuple in \( U \).

Consider Example 3.4. If \( Y = \text{grade} \) and \( T = \text{training} \), \( U = P \bowtie R \), due to the foreign keys from \( R \) to \( P \), the \( Y \) values are not repeated in \( U \). The pairs (student, professor) constitute the units with unique outcome (not value-wise, two students may receive the same grade). Now the standard techniques (e.g., matching on some covariates \( X \) in \( U \) that satisfy ignorability) can be applied to do the causal analysis in \( U \). The same holds if \( U = R \bowtie S \), then also the outcomes are unique.

Now suppose we choose \( Y = \text{gpa} \), \( T = \text{training} \), and population table \( U = P \bowtie R \bowtie S \). Suppose a student \( s \) has been taught by two professors \( p_1, p_2 \), where \( p_1 \) has \( T = 1 \) and \( p_2 \) has \( T = 0 \). Then in \( U \), there will be two tuples \( u_1, u_2 \) for \( s \), one with \( p_1 \) the other with \( p_2 \), both with the same gpa value say \( g_s \). Now the units still are (student, professor) pairs, but the treatment of \( u_1 \) affects the (same) outcome of \( u_2 \), thereby violating SUTVA. Hence in this case the units are not valid.

Note that Observation 3.5 gives a necessary condition for defining valid units satisfying SUTVA making the joined relation amenable to observational causal analysis. For instance, if the treatment \( T \) was in \( R \), and if SUTVA was originally violated in \( R \) (a tuple \( r_1 \in R \) affects the outcome \( Y \) of another tuple \( r_2 \in R \)), then even if each tuple in \( R \) contributes at most once to \( U \), SUTVA will be violated in \( U \) (so this is not a sufficient condition).

On the other hand, there can be another unit table constructed after join satisfying SUTVA: If we choose \( Y = \text{gpa} \), there might be a plausible option to collapse \( P \bowtie R \bowtie S \) to define valid units with unique \( \text{gpa} \) (i.e., the (student) becomes the unit), e.g., by aggregating over different training values to define \( T \) (at least one instructor had training or the majority of the instructors had the training) as well as aggregating different values of covariates from \( P \) or \( R \). We leave this as a direction of future research (Section 6.3) and assume Observation 3.5 holds in this paper.

3.5 Inferring CIs in a Joined Relation

As a stepping stone toward understanding strong ignorability for potential outcomes in the presence of multiple relations, we need to understand how CIs propagate from base relations to the joined relation in the standard relational database model, which is the main focus of this paper. The problem of inferring \( X \perp Y \mid J, Z \) in a related relation \( J = R_1 \bowtie \cdots \bowtie R_k \), even if we ignore the missing data problem arising in the application of causal inference, is non-trivial. As discussed earlier, given the dataset, it may be either inefficient or incorrect to validate this CI in \( J \) by computing the numeric probabilities: (i) \( Z \) may be large, and the independence has to be checked for exponentially many combinations of values of variables in \( Z \), and (ii) the available data itself may not be complete, i.e., the dataset may be a sample from the actual world. On the other hand, prior knowledge or expert knowledge may result in some conditional independences in individual base relations involving a small set of variables or attributes. With this intuition, we define the problem of inferring CIs in a joined relation, which has other potential applications as discussed in Section 3.6.

- Given conditional independences \( CI_i \) in base relations \( R_i \)-s, and three disjoint subset of attributes \( X, Y, Z \), infer whether \( X \perp Y \mid J, Z \) in the joined relation \( J = R_1 \bowtie \cdots \bowtie R_k \).

In other words, instead of directly inferring the CIs in the joined relation, if we have knowledge about CIs that hold in the base relations and the nature of join, how we can infer CIs in the joined relation. Unfortunately, testing CI is undecidable in general. Nevertheless, using properties of join, we can still infer some CIs in
the joined relation and use them for observational causal analysis (Section 3.5). Further, when the base relations are graph-isomorphs, or at least have any non-trivial I-maps (Section 2.4), we can infer a larger classes of CIs in the joined relation, as we illustrate in Section 5 for a special case when the join is on single attribute. We will discuss other sub-goals in Section 6 as further directions of research. The preliminary results on this framework presented in the following two sections primarily use a binary join between two relations R \bowtie S, and we discuss extensions to multiple relations as problems to study in the future.

3.6 Application of Inferring CIs in Query Optimization

Other than helping understand CIs for causal inference with multiple relations, inferring CIs in a joined relation is useful in fine-grained selectivity estimation for query optimization as used in modern query optimizers [17, 37, 14] instead of simply considering textbook assumptions (like uniform distribution or independence among attributes). Knowing additional CIs in base relations reduce the number of different combinations of attribute values that has to be maintained. For instance, if in R(A, B, C, D), we know that A \perp B(C), then by chain rule, \Pr_R(A, B, C) = \Pr_R(A|B)\Pr_R(B|C)\Pr_R(C) = \Pr_R(A|C)\Pr_R(B|C)\Pr_R(C), and instead of keeping frequencies for all possible combinations of A, B, C, we can have a table for C, and two other tables for A|C and B|C. These tables are likely to be much smaller, thereby making the computation of joint distribution more efficient.

Selectivity estimation using graphical models has been studied in different contexts in the literature (probabilistic graphical models for select and foreign key join queries using probabilistic relational model [17], undirected graphical model-based compact synopsis for a single table [14], and model-based approach using directed and undirected graphical models for multiple relations [37]); efficiently learning undirected and directed graphical models has also been extensively studied [26, 35] which unfortunately is computationally expensive [31, 30]. On the other hand, in this paper we study inferring CIs in the joined relation structurally given CIs and graphical models on the base relations (that are either constructed by existing algorithms or obtained using domain knowledge), as well as the properties of the join, without looking again at the data, which can reduce the complexity significantly as well as help in selectivity estimation of subsequent steps in a query optimizer. For instance, we show in Section 5 for a special case of join, an undirected I-map of the joined relation can be obtained by taking the union of P-maps of two base relations.

4. CONDITIONAL INDEPENDENCE IN THE JOINED RELATION

In this section we study the problem stated in the previous section for the special case of binary joins: Given two relations R, S, with conditional independences CI_R, CI_S respectively, (1) Which of the conditional independences in CI_R, CI_S hold in R \bowtie S for arbitrary R and S? (2) What other conditional independences hold in R \bowtie S?

4.1 CI in Joined Relation for Binary Joins

The main result in this section is that, if the join attributes (the common attributes of R, S) belong to one of the three subsets participating in a CI that holds in a base relation, then the CI holds in the joined relation. Further, any pairs of subsets of attributes from the two relations participating in the join are independent in the joined relation given the join attributes:

**Theorem 4.1.** Given two relations R, S, with CIs CI_R, CI_S respectively, the joined relation R \bowtie S satisfies the following CIs:

- For all R’ \subseteq R \setminus S and S’ \subseteq S \setminus R, it holds that R’ \perp S’|_{R\bowtie S}(R \cap S).
- For all CI X \perp Y|RZ \in CI_R, if (R \cap S) \subseteq Z, then X \perp Y|R_{W\bowtie S}Z.
- For all CI X \perp Y|RZ \in CI_R, if (R \cap S) \subseteq X (or Y), then X \perp Y|R_{W\bowtie S}Z.

The proof of the theorem follows from Corollary 4.3, Lemma 4.2 and Lemma 4.6 below, all proofs are given in the appendix. Note that the above theorem does not say that no other CIs hold in the joined relation. In fact, all CIs that can be obtained by applying graphoid axioms (Theorem 2.3) on the CIs stated in the theorem will hold in the joined relation.

**I) CIs conditioning on the join attributes:** The join introduces new CIs in the joined relation R \bowtie S:

**Lemma 4.2.** In the joined relation, (R \setminus S) \perp (S \setminus R)|_{R\bowtie S}(R \cap S).

Proof is in the appendix. The following corollary follows from the graphoid axioms using the decomposition rule 7 multiple times:

**Corollary 4.3.** For any probability space \(P\), if \(X \perp Y|pZ\), then for all subsets \(X’ \subseteq X\) and \(Y’ \subseteq Y\), \(X’ \perp Y’|pZ\).

Using Corollary 4.3 and Lemma 4.2 the following corollary holds:

**Corollary 4.4.** For all subsets of attributes \(R’ \subseteq R \setminus S\) and \(S’ \subseteq S \setminus R\) (including singleton attributes), \(R’ \perp S’|_{R\bowtie S}(R \cap S)\).

**II) CIs with join attributes on the RHS:** Here we show that if the joined attributes belong to the RHS of a CI in a base relation, then the CI propagates to the joined relation.

**Lemma 4.5.** For any \(X \perp Y|RZ\) in CI_R, in the joined relation \(X \perp Y|R_{W\bowtie S}Z\) if \(R \cap S \subseteq Z\), i.e., all join attributes \(R \cap S\) belongs to \(Z\).
Proof is in the appendix.

(III) CIs with join attributes on the LHS: Here we show that if the join attributes belong to one of the subsets of attributes on the LHS, then also the CI propagates to the joined relation.

**Lemma 4.6.** For any \( X \perp\!\!\!\!\!\!\!\!\!\!\!\!\perp Y_{|R} Z \) in \( CI_R \), in the joined relation \( X \perp\!\!\!\!\!\!\!\!\!\!\!\!\perp Y_{|RS} Z \) if \((R \cap S) \subseteq X\), i.e., all join attributes \( R \cap S \) belongs to \( X \) (similarly \( Y \)).

The proof is in the appendix. The following example shows that the conditions in Lemmas 4.5 and 4.6 are necessary in the sense that if the join attribute do not participate in the CI, it may not extend to the joined relation.

**Proposition 4.7.** There exist relation instances for \( R, S \) and a CI such that violating the conditions in both Lemmas 4.5 and 4.6 prohibits the propagation of the CI to the joined relation \( R \times S \).

**Proof.** Consider the relations \( R, S \), and \( R \times S \) below:

| \( R \) | \( S \) | \( R \times S \) |
|-----|-----|-----|
| \( A \) | \( B \) | \( C \) | \( D \) | \( E \) | \( A \) | \( B \) | \( C \) | \( D \) | \( E \) |
| \( a_1 \) | \( b_1 \) | c | \( d_1 \) | e_1 | \( a_1 \) | \( b_1 \) | c | \( d_1 \) | e_1 |
| \( a_1 \) | \( b_2 \) | c | \( d_2 \) | e_2 | \( a_1 \) | \( b_2 \) | c | \( d_2 \) | e_2 |
| \( a_2 \) | \( b_1 \) | c | \( d_3 \) | e_3 | \( a_2 \) | \( b_1 \) | c | \( d_3 \) | e_3 |
| \( a_2 \) | \( b_2 \) | c | \( d_4 \) | e_4 | \( a_2 \) | \( b_2 \) | c | \( d_4 \) | e_4 |

In \( R \), \( A \perp\!\!\!\!\!\!\!\!\!\!\!\!\perp B|R C \), but in \( R \times S \), \( \Pr[A = a_1, B = b_1 | C = c] = \frac{3}{7} \), whereas \( \Pr[A = a_1 | C = c] = \frac{2}{7} \) \( \Pr[B = b_1 | C = c] = \frac{3}{7} \). Hence \( (A \perp\!\!\!\!\!\!\!\!\!\!\!\!\perp B|R C) \) (note that the join attribute \( D \) is not a subset of the RHS of the CI \( A \perp\!\!\!\!\!\!\!\!\!\!\!\!\perp B|R C \)).

On the other hand, \( A \perp\!\!\!\!\!\!\!\!\!\!\!\!\perp B|R C \), which propagates to \( R \times S \) as \( A \perp\!\!\!\!\!\!\!\!\!\!\!\!\perp B|R S C D \) since \( D \) belongs to the RHS of this CI. \( \square \)

However, as we will see in the next section (Theorem 5.2), if the CIs are generated by an undirected graphical model, then all CIs in the base relation propagate to the joined relation.

### 4.2 CI propagation for Foreign-Key Joins and One-one Joins

**Foreign key joins:** Proposition 4.7 shows that not all CIs from a base relation propagates to the joined relation. However, if the joined attributes form a foreign key in \( R \) referring to the primary key in \( S \), then all CIs in \( R \) propagate to \( R \times S \).

**Proposition 4.8.** If the join attributes \((R \cap S)\) in a foreign key in \( R \) referring to the primary key in \( S \), then for all CI \( X \perp\!\!\!\!\!\!\!\!\!\!\!\!\perp Y_{|R} Z \) in \( CI_R \), it holds that \( X \perp\!\!\!\!\!\!\!\!\!\!\!\!\perp Y_{|RS} Z \).

The proof is in the appendix.

**One-one joins:** Note that the propagation rule in Proposition 4.8 is not symmetric, since some CI in \( S \) not satisfying the conditions in Theorem 4.1 may not propagate to \( R \times S \). However, if \((R \cap S)\) is a key (superkey) of both \( R \) and \( S \), and if \( R, S \) have the same set of keys, then the CIs from both relations propagate to \( R \times S \).

**Proposition 4.9.** If \( \pi_{RS} R = \pi_{RS} S \) and if \((R \cap S)\) is a key in both \( R \) and \( S \), then all the CIs from both \( CI_R \) and \( CI_S \) propagate to \( R \times S \).

The proposition directly follows from Proposition 4.8—since \( \pi_{RS} R = \pi_{RS} S \), \( R, S \) must have the same set of primary keys, and therefore \(|R| = |S| = |R \times S|\). Also note that the condition \( \pi_{RS} R = \pi_{RS} S \) is necessary, otherwise some tuples may be lost in the join destroying the CI. Theorem 5.2 in Section 5 states that for relations that are graph-isomorph, the CIs propagate. The above proposition states that even if the relation is not graph-isomorph, but if the join is one-one, then also the CIs propagate.

### 4.3 Application to Observational Studies

Whether a subset of variables \( X \subseteq A \) satisfies strong ignorability, in general, is untestable even for a single relation since the test involves missing data in the form of potential outcomes \( Y(0), Y(1) \) (in contrast to observed outcome \( Y \)), although there are sufficient conditions assuming a causal graphical model ([35], Section 6.2). In this section, first we show a negative result – if the join attributes belong to \( X \), then \( X \) satisfies strong ignorability, but is not useful since the estimated average causal effect of \( T \) on \( Y \) will be zero (Section 4.3.1).

Then we give a positive result (Section 4.3.2) that if we are given a set of variables \( X \) satisfying ignorability spanning multiple relations participating in a join with primary key-foreign keys, it suffices to condition on subset of \( X \) restricted to the relation(s) containing \( T \) and \( Y \), which increases efficiency and reduces variance since the matched groups based on the same values of the covariates will be bigger.

#### 4.3.1 Conditioning on join variables is not useful

Here we show the following proposition, which states that conditioning on any subset of variables that includes the join variables is not useful for causal analysis despite satisfying strong ignorability.

**Proposition 4.10.** If the treatment and the outcome (i.e., also the potential outcome) variables come from two different relations, i.e., \( T \in R \) and \( Y \in S \) (i.e., \( Y(0), Y(1) \in S \)), then given any subset of attributes \( X \subseteq (RS \setminus TY) \) such that \( X \not\subseteq (R \cap S) \), (i) \( T \) and \( Y(0), Y(1) \) are independent (thereby satisfying strong ignorability). However, (ii) \( T \) and \( Y \) are also independent, and therefore (iii) the average treatment effect of \( T \) on \( Y \) using \( X \) is zero.

The proof uses graphoid axioms, and is given in the appendix. Note that the columns \( Y(0), Y(1) \) are hypothetical, since only \( Y = Y(0)(1-T) + Y(1)T \) is observed due to the fundamental problem of causal analysis from missing data. As a special case, the proposition shows that conditioning on \( X \) = all other attributes in \( R \times S \) except \( T, Y \) is not useful (which is often done in statistical causal analysis for applied problems involving a
single relation), since the estimated average treatment effect will be zero. This further motivates the study of CI and understanding ignorability for joined relations (further discussed in Section 6.2).

4.3.2 Avoiding joins

Assume we are given \( k \) relations \( R_1, \ldots, R_k \), where the outcome \( Y \in R_k \). Suppose \( U \) is the universal table obtained by joining these relations. Further, assume that we are given a set of covariates \( X \subseteq U \) that satisfies strong ignorability, i.e., \( Y(1), Y(0) \perp \!\!\!\perp T \mid U, X \). Suppose \( X_j = X \cap R_j, j \in [1, k] \). The following proposition says that, if \( X_i \) contains foreign keys to all other relations, then it suffices to replace \( X \) with \( X_i \) for covariate adjustment, since \( X \) and \( X_i \) are c-equivalent (Definition 2.3).

**Proposition 4.11.** If \( U = R_1 \bowtie \cdots \bowtie R_k, X \subseteq U, X_j = X \cap R_j \) for \( j \in [1, k] \), \( R_i \) contains \( Y \), and \( X_i \) contains foreign keys to all relations \( R_j, j \in [1, k], j \neq i \), then \( X \) and \( X_i \) are c-equivalent.

Intuitively, if \( X_i \) contains the foreign keys to all other relations, then adjusting for \( X_i \) is sufficient, since any two tuple in \( U \) that have the same value of \( X \) will have the same value of \( X_i \). The proof uses entropy and mutual information (Section 3), and uses the fact that \( I(X, Y \mid Z) = 0 \) if and only if \( X \perp Y \mid Z \) [10]: the proof is given in the appendix. However, from Proposition 4.10 if \( T \) and \( Y \) belong to different relations, then conditioning on the join attributes (foreign keys) will result in a zero causal effect. Nevertheless, the above proposition implies that if \( T \) and \( Y \) belong to the same relation \( R_i \), and if \( \neg(T \perp Y \mid R_i, X_i) \), then it is enough to condition on the foreign keys and the covariates in \( R_i \), and the join can be avoided.

The implications of the results established in this section to causal inference are two-fold. *First*, they provide a principled way to reduce the number of covariates needed for estimating causal effect. This can be done by starting with a set of CIs on base relations that propagate to the joined relation (using Theorem 4.1 more CIs propagate if the relations are graph-isomorphic as discussed in the next section). Then Proposition 2.4 can be employed to infer \( X' \), a smaller set of covariates that is c-equivalent with \( X \). A special case is given in Proposition 4.11. It is known that the quality of the matching estimators for causal effect decreases with the number of covariates [13]. Hence, reducing the set of covariates is important for inferring robust causal conclusions. *Second*, Proposition 4.11 shows that under some circumstances it is not required to materialize the joined table involving all relations for collecting more covariates, and it suffices to focus on the subset of the given covariates \( X \) in the relation containing \( Y \). This is not only useful from the efficiency point of view (the matching groups have to be performed on smaller covariates, and more matched groups are likely to be ‘valid’ with at least one treatment and one control unit), but is also interesting because it reveals that it is still possible to make causal inferences, when the values of the covariates in some of the base relations are not recorded or noisy (e.g., when the chema and a valid \( X \) are given by a domain expert but the values in some of the other relations are unavailable).

**Further Questions:** In this section we investigated join of two relations. Understanding the CIs for multiple relations may require investigating the query hypergraph. If the join is on multiple relations where different relations share a subset of attributes from other relations, such complex interaction may prohibit certain CIs to hold on the joined relation. On the other hand, special structure of the query hypergraph (a hypergraph on all attributes where the relations form the hyperedges), like the acyclicity property [2], may allow some CIs in the joined relation.

5. USING UNDIRECTED GRAPHICAL MODELS ON THE BASE RELATIONS

In the previous section, we discussed sufficient conditions to infer CIs in the joined relation when the dependency models satisfied by the two base relations are arbitrary. However, suppose the CIs in the base relations are graph-isomorph, i.e., the base relations \( R, S \) have P-maps \( G_1, G_2 \). The question we study in this relation is whether \( G_1, G_2 \) help generate an I-map \( G \) of \( R \bowtie S \). Since our key motivation is causal analysis, then we will be able to infer correct CIs on \( R \bowtie S \) using \( G \). The other question is whether the information that \( R \) (or \( S \)) is graph-isomorph, helps propagate additional CIs from \( R \) to the joined relation. In this section, we answer these two questions affirmatively when there is only one join attribute: all CIs from the base relation \( R \) propagate to the joined relation \( R \bowtie S \) if \( R \) is graph-isomorph (Theorem 5.2), and in addition, when both \( G_1, G_2 \) are connected, then union of \( G_1, G_2 \) produces an I-map of \( R \bowtie S \) (Theorem 5.12). The extension to multiple base relations and other research questions are discussed as future directions at the end of this section.

Recall that

\[
X \perp Y \mid G, Z
\]

denotes vertex separation in an undirected graph \( G(V, E) \), i.e., if \( X, Y, Z \subseteq V \) are disjoint subsets of vertices, then removing \( Z \) and all the incident edges on \( Z \) from \( G \) (denoted by \( G \mid Z \)) disconnects all paths between all vertices in \( X \) and all vertices in \( Y \) in \( G \) (or \( Z \) is a cutset in \( G \) between \( X, Y \)). Below we state a property used in our proofs:

**Observation 5.1.** In an undirected graph \( G(V, E) \), if \( X \perp Y \mid G, Z \), then for all supersets \( Z' \) of \( Z \), \( X \perp Y \mid G, Z' \).

Although we overload the notation for independence \( \perp \) to also denote vertex separation in graphs, note that, \( X \perp Y \mid G, Z \) by itself does not say anything about CIs of \( X, Y \) given \( Z \). In fact, the results in this section aim to prove that if \( X \perp Y \mid G, Z \), then \( X \perp Y \mid R \bowtie S, Z \).

5.1 CIs from a Relation with a P-Map Propagates to the Joined Relation

Proposition 4.7 shows that not all CIs in \( R \bowtie S \) propagate to the joined relation \( R \bowtie S \). Here we show that if
The proof is given in the appendix, which (along with all other proofs in this section) uses all four symmetry, decomposition, weak union, and contraction properties ($\mathbb{G}-\mathbb{G}$) of graphoid axioms, as well as properties of vertex separation of graphs. To see an example of the application of Lemma 5.6 consider two simple P-maps $A-B-D$ and $D-F$. Note that the independence $A \not\perp F_{[R \bowtie S]}$ does not directly follow from Theorems 5.2 and 4.1 but follows from Lemma 5.6.

Now we move to the general case of separation in $G$. Suppose $X \perp Y|G[Z]$ where $X = X_1, X_2, Y = Y_1, Y_2, Z = Z_1, Z_2$, and $X_1, Y_1, Z_1 \in R, X_2, Y_2, Z_2 \in S$. In the following lemmas we consider different cases when some of $X_1, Y_1, Z_1$-s are empty. All proofs are in the appendix.

**Lemma 5.7.** If $Y_2 = Z_2 = \emptyset$, then $X_1, X_2 \perp Y_1|_{R \bowtie S[Z]}$.

**Lemma 5.8.** If $Y_1 = Z_2 = \emptyset$, then $X_1, X_2 \perp Y_2|_{R \bowtie S[Z]}$.

**Lemma 5.9.** If $Z_2 = \emptyset$, then $X_1, X_2 \perp Y_1|_{R \bowtie S[Z]}$.

**Lemma 5.10.** If $X_2 = Y_2 = Z_1 = \emptyset$, then $X_1 \perp Y_1|_{R \bowtie S[Z]}$.

**Lemma 5.11.** (A) If $X_1 \perp Y_1|G[Z_1, Z_2]$, then $X_1 \perp Y_1|_{R \bowtie S[Z]}$. (B) If $X_1 \perp Y_2|G[Z_1, Z_2]$, then $X_1 \perp Y_2|_{R \bowtie S[Z]}$. (C) If $X_1 X_2 \not\perp Y_1|G[Z_1, Z_2]$, then $X_1 \perp Y_1|_{R \bowtie S[Z]}$. (D) If $X_1 X_2 \not\perp Y_1|G[Z_1, Z_2]$, then $X_1 X_2 \perp Y_1|_{R \bowtie S[Z]}$.

The proof stating the main result of this section follows from the above lemmas (proof in the appendix):

**Theorem 5.12.** Suppose $R$ and $S$ have P-maps $G_1$ and $G_2$ respectively that are connected graphs, $G$ is the union graph of $G_1$, $G_2$, and the join is on a unique common attribute $R \bowtie S = \{D\}$. Then for disjoint subsets of vertices $X, Y, Z \in G$ if $X \not\perp Y|G[Z]$, then $X \not\perp Y|_{R \bowtie S[Z]}$, i.e., $G$ is an I-map for $R \bowtie S$.

**Further Questions.** In this section we showed that CIs from graph-isomorph base relations propagate to joined relation, and the union graph is an I-map if the join is on a single attribute and the given P-maps are connected. Several other questions remain to be explored: (1) If $G_1, G_2$ are P-maps for $R, S$, does a P-map always exist for $R \bowtie S$ (or under what conditions a P-map exists)? Example C.1 shows that new CIs may be generated for some instances, but a schema-based argument will be needed to find a solution to this problem. (2) What if $G_1, G_2$ are only I-maps? In this section, only the proof of Theorem 5.2 uses the properties of P-map in arguing that if a CI does not hold in the base relation $R$, the vertex separation does not hold in $G_1$; all other proofs mentioning P-maps use properties of vertex separation in undirected graphs. This assumption seemed to be necessary in our proof of Theorem 5.2, so the question is whether an alternative proof exists that only uses properties from I-map to infer Theorem 5.12.

$R$ has a P-map $G_1$ (Definition 2.7), i.e., if $X \perp Y|_R Z \Leftrightarrow X \perp Y|_{G_1[Z]}$ for all disjoint subsets $X, Y, Z \subseteq R$, then all CIs in $R$ propagate to $R \bowtie S$ in arbitrary joins.

**Theorem 5.2.** If $R$ has a P-map $G_1$, $D = R \bowtie S$ is the single join attribute, and $X \perp Y|_R Z$ for disjoint $X, Y, Z \subseteq R$, then $X \perp Y|_{R \bowtie S[Z]}$.

Theorem 5.2 is proved using the following lemma; both proofs are given in the appendix.

**Lemma 5.3.** If $R$ has a P-map $G_1$, $D = R \bowtie S$ is the single join attribute, and $X \perp Y|_R Z$ for disjoint $X, Y, Z \subseteq R$, then either $X \not\perp Y|_R D$ or $X \not\perp Y|_R D$.

Note that in $R \bowtie S$, which may not have a P-map in general (Section 6.1 gives an example that can be extended to a join), we can only apply graphoid axioms, whereas since $R$ has a P-map $G_1$, we can apply both graphoid axioms as well as the necessary and sufficient conditions from Theorem 2.8.

We revisit why in Proposition 4.4 the CI $A \perp B|_R Z$ did not propagate to $R \bowtie S$. In relation $R$ in the example, $A \perp B|_R C$. If $R$ had a P-map, the transitivity property of Theorem 2.8 will hold, and we will have either $A \perp D|_R C$ or $D \perp B|_R C$. However, in the example, both do not hold.

On the other hand, an Example 1 in the appendix shows that if we replace the $R$ instance with one that is generated by the P-map $A \rightarrow \neg \neg \neg \rightarrow \neg B \rightarrow \neg C \rightarrow D$, then the CI will propagate to $R \bowtie S$ with the same $S$ instance.

### 5.2 An I-map for Joined Relation

Theorems 5.2 and 4.1 give us sufficient conditions for some of the CIs that hold in the joined relation. The question we study in this section is, whether we can infer additional CIs when both $R$ and $S$ have P-maps $G_1$ and $G_2$. A natural choice is to consider the union graph $G$ of $G_1$, $G_2$, where the set of vertices in $G$ is $R \cup S$ and the set of edges in the union of edges from $G_1$, $G_2$. We use the following two observations in our proofs.

**Observation 5.4.** The join attributes $R \bowtie S$ form a cutset between $R \setminus S$ and $S \setminus R$ in $G$, i.e., all paths between the non-join attributes in two relations must go through $R \setminus S$.

**Observation 5.5.** If $X \not\perp Y|_G Z$ where $X, Y, Z \subseteq R$ and are disjoint, then $X \not\perp Y|_G Z$.

The above observation follows from the fact that $G_1$ has a subset of edges of $G$, and if two subset of nodes are not connected in $G$, they cannot be connected in $G_1$. In this section, we assume that (i) the join is on a single attribute $D$, i.e., $R \setminus S = \{D\}$, and (ii) the P-maps $G_1$, $G_2$ are connected, i.e., there is a path from all vertices in $R$ and $S$ to $D$ in $G_1$, $G_2$ respectively.

First we show that additional CIs inferred from the union graph $G$, not necessarily captured by Theorems 5.2 and 4.1, hold in the joined relation $R \bowtie S$.

**Lemma 5.6.** Suppose $X, Y, Z$ are disjoint set of vertices in the union graph $G$ such that (i) $X$ and $Y$ are disconnected in the graph $G-Z$, and (ii) $X, Z$ belong to one of $R, S$ and $Y$ belongs to the other relation, then $X \not\perp Y|_{R \bowtie S[Z]}$. 

(which uses Theorem 6.2). (3) We also use the assumption that \( G_1, G_2 \) are connected in our proofs - what happens if they are not connected (when some variables are unconditionally independent)? (4) Similar to Section 4 another question is what can be inferred about multiple relations, and whether properties like acyclicity of schemas help in inferring CI sets. Note that if conditions in Theorem 6.2 is weakened, requiring only that \( G_1, G_2 \) are I-maps of \( R, S \), then if a join order on \( k \) exists where every join happens only on one attribute (a special case of join tree for acyclic schema [2]), then the result can be extended to multiple relations, since by combining two I-maps we get another I-map. (5) Extending the results to arbitrary number of join attributes with arbitrary connections is another question to answer.

6. MORE RESEARCH DIRECTIONS FOR THE FRAMEWORK

In addition to the questions stated in the previous sections, here we present three other fundamental directions that need to be explored to make causal analysis on multi-relational data robust and practical.

6.1 Using Directed Graphical Models for CIs

The results in the previous sections suggest that additional knowledge on CIs on base relations help infer additional CIs in the joined relation, and that graphical models on base relation is a convenient technique to infer CIs on the joined relation. Although questions remain to be answered even for undirected graphical models (Section 4, e.g., for join on multiple attributes or with multiple relations), the undirected graphical model has inherent limitations itself in capturing some dependency relations, e.g., when \( I(X, Z, Y) \) but \( \neg I(X, ZW, Y) \) (in undirected graphs, supersets of cutsets are also cutsets). As a special case, consider the scenario when two variables \( X, Y \) are independent (i.e., cannot have any path connecting them in an undirected graph), whereas they are dependent given a third variable \( Z \) (which requires paths between \( X, Z \) and \( Y, Z \)), for instance, when \( X, Y \) denote the random toss of two coins, and \( Z \) denotes the ring of a bell that rings only when the both coins output the same value \( [53] \). These two constraints cannot be satisfied simultaneously in any undirected graph, but they can be captured in a directed graphical model or Bayesian networks by adding two directional edges from \( X \) to \( Z \) and \( Y \) to \( Z \). Bayesian networks are causal when the arrows reflect the direction of causality between variable: two edges from \( X \) to \( Z \) and \( Y \) to \( Z \) imply that both \( X, Y \) causally affect \( Z \) but not each other (mutually independent).

The inference of CIs in Bayesian networks is performed using d-separation proposed by Pearl and Verma [54, 59]. The basic idea is the following (which is extended to general paths in d-separation): for three variables \( X, Y, Z \), if the directions are \( X \to Z \to Y \), \( X \leftarrow Z \leftarrow Y \), or \( X \leftarrow Z \to Y \), then observing \( Z \) makes \( X, Y \) conditionally independent (or the path is blocked), but if the direction is \( X \to Z \leftarrow Y \) (\( Z \) is a collider), and if \( Z \) or any of its descendants are observed, \( X \) and \( Y \) become dependent. In general, \( Z \) is said to d-separate \( X \) from \( Y \) in a directed graph, if all paths from \( X \) to \( Y \) are blocked by \( Z \). Not all CIs can be captured using a directed graph (e.g., CIs captured by a diamond-shaped undirected Markov network). However, the set of chordal graphs, where every cycle of length \( \geq 4 \) has a chord, can be represented by both undirected and directed graphs (chordal graphs have also been studied for acyclic schemas in [6]).

The following example shows that the union of P-maps \( G_1, G_2 \) of base relations \( R, S \) is not a D-map for the joined relation \( R \bowtie S \):

**Example 6.1.** Consider the example when \( G_1 = (A, B) \), i.e., a directed edge from \( A \) to \( B \), and \( G_2 = (C, B) \), i.e., a directed edge from \( C \) to \( B \). Taking the union of these two edges, \( C \) becomes a collider between \( A \) and \( B \), suggesting that \( A \) and \( C \) are not independent given \( B \). This contradicts with Corollary 6.1 that \( A \perp \!\!\!\perp C | R \bowtie S \). Since a CI in the model does not hold in the graph, it is not a D-map.

However, this does not say whether we can have an I-map by union (in the above case, we have a trivial I-map). Getoor et al. proposed the concept of Probabilistic Relational Model (PRM) [17], their construction, and application to selectivity estimation for joins with primary key-foreign keys. Maier et al. [28] proposes the notion of relational d-separation, to infer instance-based CIs in entity-relationship models involving multiple relations. But to the best of our knowledge, inferring schema-based d-separation for joins in Bayesian network, and combining the ones for base relations to get an I-map or a P-map for the joined relation, has not been explored in the literature.

6.2 CIs involving Potential Outcomes and Causal Networks

The CIs discussed in the paper consider columns from base or joined relations. However, for the strong ignorability condition, we need \( Y(0), Y(1) \perp T | X \), where \( Y(0), Y(1) \) are potential outcomes with missing data in the observed relation. In Proposition 4.10 in a two-way join, we showed that conditioning on all attributes from both relations, or attributes containing the join variable satisfies ignorability but is not useful for estimating causal effects. The challenge is that while \( Y(0), Y(1) \) should be independent of \( T \) given \( X \), \( Y \) and \( T \) cannot be independent given \( X \) (otherwise ATE = 0).

In general, strong ignorability is not readily assertable from common knowledge [55]. However, if the underlying variables are represented by a causal directed graph, Pearl [55] gives a sufficient condition called the backdoor criteria for checking ignorability (also called unconfoundedness or admissibility or identifiability): in the DAG, no variable in \( X \) is a descendant of \( T \), and \( X \) ‘blocks’ all paths between \( T \) and \( Y \) that contains an arrow into \( T \) (as done in d-separation, Section 6.1). In other words, using Pearl’s notations, \( Y \) and \( T \) are independent (using d-separation) given \( X \) in the graph \( G' \), which is obtained by deleting all edges emerging
from T (thereby taking care of the fact that T should have a direct effect on Y). Understanding and extending backdoor criteria (and other observations from the causal graphical model from the vast literature by Pearl and co-authors) is an important direction to explore to understand ignorability for joined relations.

6.3 Satisfying Basic Causal Assumptions for Arbitrary Joins

Strong ignorability (Definition 2.2) is one of the necessary conditions for observational causal studies. Another necessary condition is SUTVA (Definition 2.1 required also for controlled experiments), which says that the treatment assignment to one unit does not affect the outcome of another unit. Another hidden assumption is that every unit constitutes one data record (one row in Table 1). We discussed foreign key and one-one joins in Section 4. For foreign key joins, if treatment is assigned to one unit, we get one row for a unit with one T value. The one-one join allows arbitrary selection of Y and T. However, for many-many joins, T and/or Y may repeat in the joined table. In Observation 3.3 we discussed necessary conditions for units obtained by natural joins (tuples containing outcome Y cannot repeat in joined relation). As discussed in Section 3.4, we need to investigate how joined relation can be post-processed to obtain valid units satisfying SUTVA: the same holds for relations originated from more complex queries. For instance, in Example 3.3 if the information about both parents is stored in the ParentsInfo table, and the causal question is how much the jobs of the parents affect the gpas of the students, for a student in the joined table Students \bowtie Parent \bowtie ParentInfo, there may be two rows, both having the same outcome Y = gpa, but potentially different T = parents_income, which should be aggregated to obtain valid units.

6.4 Weaker Notions of CIs

Defining CIs in base and joined relations using probabilistic interpretation of conditional probabilities is a natural choice, although it raises some conceptual questions. As discussed earlier, CIs should be properties of relations when the relations capture well-defined entities, and should not change when we integrate this relation with another relation (e.g., if we are using weather data to reason about flight delays, the CIs that hold among temperature, pressure, humidity should continue to hold in the integrated dataset after joining with flight dataset). However as Proposition 4.7 shows, not all CIs propagate using join, and further, Example C.1 in the appendix shows that new CIs (confined to a base relation) may be generated in the joined relation. However, since all the schema-based CIs do not propagate to the joined relation anyway, the question arises whether other notions of CIs in relations are meaningful that will remain the same whether or not a join has been performed. Indeed, we can consider the combined joint distribution of all attributes from all base relations 17, but the question of inferring CIs in the joined relation starting from the CIs in the base relations still is an interesting and useful question to answer beyond causal analysis (e.g., efficiently learning graphical models on joined relation or selection estimative selection in joined relation, Section 3.3).

One natural choice is that of embedded multi-valued dependency (EMVD) proposed by Fagin 15. A multi-valued dependency (MVD) \( X \rightarrow Y \) holds in a relation R, if for each pair of tuples \( t_1, t_2 \in R \) such that \( t_1[X] = t_2[X] \), there is a tuple t in R such that \( t[X] = t_1[X], t[Y] = t_1[Y], \) and \( t[W] = t_2[W] \), where \( W = R \setminus XY \). An EMVD \( X \rightarrow Y \mid Z \) holds, if in the relation \( \pi_Z R \), the MVD \( X \rightarrow Y \) holds. Basically EMVD gives a weaker form of CIs when in the relation \( \pi_Z R \), the MVD \( X \rightarrow Y \) holds. EMVD gives a weaker form of CIs when in the relation \( \pi_Z R \), the MVD \( X \rightarrow Y \) holds. EMVD gives a weaker form of CIs when in the relation \( \pi_Z R \), the MVD \( X \rightarrow Y \) holds. EMVD gives a weaker form of CIs when in the relation \( \pi_Z R \), the MVD \( X \rightarrow Y \) holds. EMVD gives a weaker form of CIs when in the relation \( \pi_Z R \), the MVD \( X \rightarrow Y \) holds.
7. REFERENCES

[1] C. Beeri. On the membership problem for functional and multivalued dependencies in relational databases. *ACM Trans. Database Syst.*, 5(3):241–259, 1980.

[2] C. Beeri, R. Fagin, D. Maier, and M. Yannakakis. On the desirability of acyclic database schemes. *J. ACM*, 30(3):479–513, July 1983.

[3] C. Beeri, R. Fagin, D. Maier, and M. Yannakakis. On the desirability of acyclic database schemes. *J. ACM*, 30(3):479–513, 1983.

[4] L. Bertossi and B. Salimi. Causes for query answers from databases: Datalog abduction, view-updates, and integrity constraints. To appear in *International Journal of Approximate Reasoning. Corr Arxiv Paper cs.DB/1611.01711*, 2017.

[5] L. Bertossi and B. Salimi. From causes for database queries to repairs and model-based diagnosis and back. *Theory of Computing Systems*, 61(1):191–232, 2017.

[6] F. Chapin. *Experimental Designs in Sociological Research*. Harper; New York, 1947.

[7] A. Chapman and H. V. Jagadish. Why not? In *Proceedings of the 2009 ACM SIGMOD International Conference on Management of Data*, SIGMOD ’09, pages 523–534, 2009.

[8] W. G. Cochran and G. M. Cox. *Experimental designs*. Wiley Classics Library. Wiley, 1992.

[9] W. G. Cochran and D. B. Rubin. Controlling bias in observational studies: A review. *Sankhyā: The Indian Journal of Statistics, Series A*, pages 417–446, 1973.

[10] T. M. Cover and J. A. Thomas. *Elements of Information Theory, 2nd Edition*. Wiley, 2006.

[11] D. Cox. *Planning of experiments*. Wiley series in probability and mathematical statistics: Applied probability and statistics. Wiley, 1958.

[12] M. M. Dalkilic and E. L. Robertson. Information dependencies. In *Proceedings of the Nineteenth ACM SIGMOD-SIGACT-SIGART Symposium on Principles of Database Systems*, May 15-17, 2000, Dallas, Texas, USA, pages 245–253, 2000.

[13] X. De Luna, I. Waernbaum, and T. S. Richardson. Covariate selection for the nonparametric estimation of an average treatment effect. *Biometrika*, 98(4):861–875, 2011.

[14] A. Deshpande, M. Garofalakis, and R. Rastogi. Independence is good: dependency-based histogram synopses for high-dimensional data. *ACM SIGMOD Record*, 30(2):199–210, 2001.

[15] R. Fagin. Multivalued dependencies and a new normal form for relational databases. *ACM Trans. Database Syst.*, 2(3):262–278, 1977.

[16] R. A. Fisher. *The design of experiments*. Oliver and Boyd, Oxford, England, 1935.

[17] L. Getoor, B. Taskar, and D. Koller. Selectivity estimation using probabilistic models. In *ACM SIGMOD Record*, volume 30, pages 461–472.

[18] T. J. Green, G. Karvounarakis, and V. Tannen. Provenance semirings. In *Proceedings of the Twenty-sixth ACM SIGMOD-SIGACT-SIGART Symposium on Principles of Database Systems*, PODS ’07, pages 31–40, 2007.

[19] E. Greenwood. *Experimental sociology: A study in method*. King’s crown Press, 1945.

[20] P. W. Holland. Statistics and causal inference. *Journal of the American Statistical Association*, 81(396):pp. 945–960, 1986.

[21] http://dblp.uni trier.de/xml/. Dblp dataset.

[22] https://www.data.gov. U.s. government’s open data.

[23] https://www.yelp.com/dataset.challenge. Yelp dataset.

[24] S. M. Iacus, G. King, and G. Porro. Causal inference without balance checking: Coarsened exact matching. *Political analysis*, page mpr013, 2011.

[25] S. M. Iacus, G. King, G. Porro, et al. Cem: software for coarsened exact matching. *Journal of Statistical Software*, 30(9):1–27, 2009.

[26] D. Koller and N. Friedman. *Probabilistic Graphical Models - Principles and Techniques*. MIT Press, 2009.

[27] M. Lichman. UCI machine learning repository, [http://archive.ics.uci.edu/ml](http://archive.ics.uci.edu/ml) 2013.

[28] M. E. Maier, K. Marazopoulou, D. T. Arbour, and D. D. Jensen. A sound and complete algorithm for learning causal models from relational data. In *Proceedings of the Twenty-Ninth Conference on Uncertainty in Artificial Intelligence*, UAI 2013, Bellevue, WA, USA, August 11-15, 2013, 2013.

[29] M. E. Maier, B. J. Taylor, H. Oktay, and D. Jensen. Learning causal models of relational domains. In *Proceedings of the Twenty-Fourth AAAI Conference on Artificial Intelligence*, (AAAI), 2010.

[30] A. Meliou, W. Gatterbauer, K. F. Moore, and D. Suciu. The complexity of causality and responsibility for query answers and non-answers. *Proc. VLDB Endow. (PVLDB)*, 4(1):34–45, 2010.

[31] R. E. Neapolitan et al. *Learning bayesian networks*, volume 38. Pearson Prentice Hall Upper Saddle River, NJ, 2004.

[32] J. Neyman. *On the Application of Probability Theory to Agricultural Experiments. Essay on Principles, Section 9*. PhD thesis, Roczniki Nauk Rolniczych Tom X [in Polish], 1923. translated in Statistical Science, 5, page 465-480.

[33] J. Pearl. *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 1988.

[34] J. Pearl. *Probabilistic reasoning in intelligent systems: networks of plausible inference*. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 1988.
A. PROOFS FROM SECTION 4

A.1 Proof of Lemma 4.2

Proof of Lemma 4.2 Fix any value $z$ (as a set) of the attributes $Z = R \cap S$. Given $Z = z$, all tuples $r \in R$ with $r[Z] = z$ join with all tuples $s \in S$ with $s[Z] = z$, and with no other tuples. Let $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$, $T = T_1 \cup T_2$. For any values $X = x, Y = y$, $N_{T,xz} = N_{R,xz} \times N_{S,z}$, $N_{T,yz} = N_{S,yz} \times N_{R,z}$, $N_{T,xyz} = N_{R,xz} \times N_{S,yz}$, and $N_{T,z} = N_{R,z} \times N_{S,z}$. Hence it follows that

$$\frac{N_{T,xyz}}{N_{T,z}} = \frac{N_{T,xz}}{N_{T,z}} \times \frac{N_{T,yz}}{N_{T,z}}$$

i.e., $X \perp Y | T, Z$. □

A.2 Proof of Lemma 4.5

Proof of Lemma 4.5 By the definition of conditional independence, and since $X \perp Y | R \setminus S, Z$, for all values $x, y, z$ of $X, Y, Z$ we have

$$\frac{N_{R,xyz}}{N_{R,z}} = \frac{N_{R,xz}}{N_{R,z}} \times \frac{N_{R,yz}}{N_{R,z}} \tag{23}$$

Fix arbitrary $x, y, z$. Let $Z = Z_1 \cup Z_2$, where $Z_1 = R \cap S$ and $Z_2 = Z \setminus Z_1$. Let $Z_1 = z_1$ and $Z_2 = z_2$ in $z$. Note that $N_{S,(Z_1,z_1)}$ is the number of tuples in $S$ with the value of the join attributes as $z_1$. Each tuple $r$ in $R$ with $Z = z$, i.e., with $Z_1 = z_1$, joins with all tuples in $S$ with $Z_1 = z_1$, and joins with no other tuples in $S$.

[35] J. Pearl. *Causality: models, reasoning, and inference*. Cambridge University Press, 2000.
[36] J. Pearl. *Probabilistic reasoning in intelligent systems: networks of plausible inference*. Morgan Kaufmann, 2014.
[37] J. Pearl and A. Paz. *Graphoids: Graph-based logic for reasoning about relevance relations or when would x tell you more about y if you already know z?* In *ECAI*, pages 357–363, 1986.
[38] J. Pearl and A. Paz. *Confounding equivalence in causal inference*. *Journal of Causal Inference*, 2(1):75–93, 2014.
[39] J. Pearl and T. Verma. The role of representing dependencies by directed graphs. In *Proceedings of the 6th National Conference on Artificial Intelligence*. Seattle, WA, July 1987., pages 374–379, 1987.
[40] M. J. Rattigan, M. E. Maier, and D. Jensen. Relational blocking for causal discovery. In *Proceedings of the Twenty-Fifth AAAI Conference on Artificial Intelligence*, (AAAI), 2011.
[41] P. R. Rosenbaum. *Observational study*, pages 1451–1462. Wiley, Hoboken, N.J., 2005.
[42] P. R. Rosenbaum and D. B. Rubin. The central role of the propensity score in observational studies for causal effects. *Biometrika*, 70(1):pp. 41–55, 1983.
[43] P. R. Rosenbaum and D. B. Rubin. Reducing bias in observational studies using subclassification on the propensity score. *Journal of the American Statistical Association*, 79(387):516–524, 1984.
[44] P. R. Rosenbaum and D. B. Rubin. Constructing a control group using multivariate matched sampling methods that incorporate the propensity score. *The American Statistician*, 39(1):33–38, 1985.
[45] S. Roy, C. Rudin, A. Volfovsky, and T. Wang. FLAME: A Fast Large-scale Almost Matching Exactly Approach to Causal Inference. *ArXiv e-prints*, 1707.06315, July 2017.
[46] S. Roy and D. Suciu. A formal approach to finding explanations for database queries. In *Proceedings of the 2014 ACM SIGMOD International Conference on Management of Data*, SIGMOD ’14, pages 1579–1590, 2014.
[47] D. Rubin. *Matched Sampling for Causal Effects*. Cambridge University Press, 2006.
[48] D. B. Rubin. Matching to remove bias in observational studies. *Biometrics*, pages 159–183, 1973.
[49] D. B. Rubin. Estimating causal effects of treatments in randomized and nonrandomized studies. *Journal of Educational Psychology*, 66(5):688, 1974.
[50] D. B. Rubin. Multivariate matching methods that are equal percent bias reducing. i: Some examples. *Biometrics*, pages 109–120, 1976.
[51] D. B. Rubin. Causal inference using potential outcomes. *Journal of the American Statistical Association*, 100(469):322–331, 2005.
[52] B. Salimi, L. Bertossi, D. Suciu, and G. Van den Broeck. Quantifying causal effects on query answering in databases. In *TaPP*, 2016.
[53] B. Salimi, C. Cole, D. R. Ports, and D. Suciu. Zaliql: Causal inference from observational data at scale. *Proceedings of the VLDB Endowment*, 10(12), 2017.
[54] P. P. Shenoy and G. Shafer. Axioms for probability and belief-function proagation. In *UAI ‘88: Proceedings of the Fourth Annual Conference on Uncertainty in Artificial Intelligence*, Minneapolis, MN, USA, July 10-12, 1988, pages 169–198, 1988.
[55] J. Pearl and T. Verma. The logic of representing dependencies by directed graphs. In *Proceedings of the 6th National Conference on Artificial Intelligence*. Seattle, WA, July 1987., pages 374–379, 1987.
Hence multiplying the numerator and denominator all terms in equation \(23\) above, we get
\[
\frac{N_{RwS_xyz}}{N_{RwS_z}} = \frac{N_{RwS_{x,z}}}{N_{RwS_z}} \times \frac{N_{RwS_{yz}}}{N_{RwS_z}}
\]
In other words, \(X \perp Y|_{RwS}Z\). 

**A.3 Proof of Lemma 4.6**

**Proof of Lemma 4.6** Since \(X \supseteq (R \cap S)\), suppose \(X = W \cup V\) where \(V = (R \cap S)\) and \(W = Z \setminus W\), i.e., \(V\) denotes the set of joined attributes. By the definition of conditional independence, and since \(W \not\perp Y|_{RwS}Z\), for all values \(y, w, v, z\) of \(Y, W, V, Z\), from \(23\) we have:
\[
N_{Rwvz} \times N_{R,vz} = N_{Rwv'y} \times N_{R,z}
\]
(24)

In the joined relation \(T = R \times S\), every tuple with a value \(V = v\) in \(R\) will join with all \(N_{S,v}\) tuples in \(S\) and with no other tuples. Hence,
\[
N_{T,wv'y} \times N_{Tvz} = N_{T,wv'y} \times (\sum_{w',v'} N_{T,wv'y'}z') = (N_{Rwv'y} \times N_{S,v}) \times (\sum_{w',v'} (N_{Rwv'y'}z' \times N_{S,v'})
\]
(25)
\[
= (N_{Rwv'y} \times N_{S,v}) \times (N_{Rv'y} \times N_{S,v'}) \times (\sum_{w',v'} (N_{Rwv'y'}z' \times N_{S,v'})
\]
(26)

And,
\[
N_{T,wv'y} \times N_{Tvz} = (\sum_{y'} N_{T,wy'y}) \times (\sum_{w',v'} N_{T,wv'y'}z')
\]
(27)

**A.4 Proof of Proposition 4.8**

**Proof of Proposition 4.8** With a primary key join, every tuple in \(R\) joins with exactly one tuple in \(S\). Hence for \(T = R \times S\), and for all \(x, y, z\) values of \(X, Y, Z\), if \(N_{RwS_xyz}/N_{RwS_z} = \frac{N_{RwS_{x,z}}}{N_{RwS_z}} \times \frac{N_{RwS_{yz}}}{N_{RwS_z}}\), then \(N_{T,wv'y}/N_{Tvz} = \frac{N_{Tvz}}{N_{Tvz}} \times \frac{N_{Tvz}}{N_{Tvz}}\)

since all frequencies in all numerators and denominators in \(R\) is multiplied by 1 to obtain the frequencies in \(T\). 

**A.5 Proof of Proposition 4.10**

**Proof of Proposition 4.10** Let \(U, V\) denote the subset of attributes in \(R\) and \(S\) respectively in \(X\) that do not belong to \(T, Y, \) or \(R \cap S\). Below, we use \(Y'\) as a placeholder for either \((0, 0)\), \((1, 1)\) (to prove (i)), or \((0, 0)\) (to prove (ii)).
\[
UT \not\perp Y'|_{RwS}(R \cap S)\]
\[
\Rightarrow \quad UT \not\perp Y'|_{RwS}(R \cap S)V\]
\[
\quad \Rightarrow \quad T \not\perp Y'|_{RwS}(R \cap S)VU\]
\[
\quad \Rightarrow \quad T \not\perp Y'|_{RwS}X
\]

since \(X = \{U\} \cup \{V\} \cup (R \cap S)\). This shows that (i) \(T \not\perp Y(0), Y(1)|_{RwS}X\), and (ii) \(T \not\perp Y|_{RwS}X\).

To show (iii), we show that if \(T \not\perp Y|_{RwS}X\), then ATE = 0. If \(T \not\perp Y|X\) then for all \(X = x, Y = y, T = t, P(y|x, t) = P(y|x)\). Thus, \(E[Y|X] = E[Y|X, T = 1] = E[Y|X, T = 0]\). Therefore (see \(25\)), ATE = \(E_x[E[Y|T = 1, X] - E_x[E[Y|T = 0, X]] = E_x[E[Y|T = 1, X] - E_x[E[Y|T = 0, X]] = 0\). This shows (iii). 

**A.6 Proof of Proposition 4.11**

We will use the following properties of entropy \(12\):

**Proposition** A.1. For \(X, Y, Z \subseteq A\)

(a) \(H(X) \geq 0\), \(H(X|Y) \geq 0\), \(I(X, Y) \geq 0\) and \(I(X, Y|Z) \geq 0\)

(b) \(X \not\perp Y\) if and only if \(I(X, Y) = 0\), and \(X \not\perp Y|Z\) if and only if \(I(X, Y|Z) = 0\).

(c) If \(X\) functionally determines \(Y\), i.e., if \(X \rightarrow Y\), then \(H(Y|X) = 0\).

(d) If \(H(Y|X) = 0\) then for any \(Z\), \(H(Y|XZ) = 0\)

(e) If \(X \rightarrow Y\) or, if \(H(Y|X) = H(Y|XZ) = 0\) then for any \(Z\), \(I(Y, Z|X) = 0\).

**Proof**. Proofs of (a, b) can be found in \(12\) (some observations are obvious). To see (c) (also shown in \(12\), note that if \(X\) functionally determines \(Y\) then for any \(x \in E\) and \(y \in Y\) \(P(Y = y, X = x) = P(X = x)\) (follows from the definition of a functional dependency). Thus \(H(X, Y) = H(X)\) and therefore, \(H(Y|X) = H(X, Y) - H(X) = 0\). (d) is obtained from the non-negativity of mutual information. Since \(I(Y, Z|X) \geq 0\) then \(H(Y|X) - H(Y|XZ) \geq 0\), i.e., \(H(Y|X) \geq H(Y|XZ)\). Now if \(H(Y|X) = 0\) then \(H(Y|XZ) = 0\). Since \(H(Y|XZ) \geq 0\), \(H(Y|XZ) = 0\). (e) follows from (c) and (d), since \(I(Y, Z|X) = H(Y|X) - H(Y|XZ)\).
Now we prove Proposition B.11.

**Proof of Proposition B.11.** To prove the claim, we show that one of the conditions in Theorem 2.4 is satisfied by $X$ and $X_i$. In particular, we show that both

\[ T \perp X_i \cup X \]  

(27)

and

\[ Y \perp X \mid \cup X_i T \]  

(28)

hold.

First we prove (27). Since $X_i \subseteq X$, $X$ functionally determines $X_i$ (this is a trivial functional dependency in $U$). Now, from Proposition A.4(c) it follows that, $I(X_i, T \mid X)$. Therefore, $T \perp X_i \mid \cup X$ (Proposition A.1(c)), i.e., (27) holds.

Next we show (28). Since $H(A|B) = H(AB) - H(B)$, $H(X \mid X_i|X_i) = H(X) - H(X_i) = H(X|X_i)$

(29)

Let $FK$ denote the set of foreign keys from $R_i$ to all $R_j$, $j \in [1, k]$, $j \neq i$. Hence $FK$ functionally determines $X_i - X_i$ in $U$, i.e.,

\[ H(X \mid X_i|FK) = 0 \quad \text{(Proposition A.1(c))} \]

\[ \Rightarrow H(X \mid X_i|FKX_i) = 0 \quad \text{(Proposition A.1(d))} \]

\[ \Rightarrow H(X \mid X_i|X_i = 0 \quad \text{(Since FK \subseteq X_i)} \]

\[ \Rightarrow H(X|X_i) = 0 \quad \text{(From (29))} \]

\[ \Rightarrow H(X|X_i, T) = 0 \quad \text{(Proposition A.1(d))} \]

\[ \Rightarrow I(X, Y|X_i, T) = 0 \quad \text{(Proposition A.1(c))} \]

\[ \Rightarrow X \perp Y|X_i, T \quad \text{(Proposition A.1(b))} \]

\[ \square \]

**B. PROOFS FROM SECTION 5**

**B.1 Proof of Lemma 5.3.**

**Proof of Lemma 5.3.** Suppose not, i.e., assume the contradiction that $\neg (X \perp YD|RZ)$ and $\neg (XD \perp Y|RZ)$. Note that $D$ is a single vertex in $G_1$. Since $G_1$ is a P-map for $R$, it follows that $\neg (X \perp YD|G_1 Z)$ and $\neg (XD \perp Y|G_1 Z)$. Since $\neg (X \perp YD|G_1 Z)$, in $G_1$, there is a path from $X$ to $DY$ that does not use any vertex in $Z$. Since $Y \perp Y|RZ \equiv X \perp YG_1 Z$, removing $Z$ disconnects $X$ from $Y$, hence there must be a path $p_1$ from $X$ to $D$ that does not use vertices from $Z$. Similarly, using $\neg (XD \perp YG_1 Z)$, there is a path $p_2$ from $D$ to $Y$ that does not use vertices from $Z$. Combining $p_1$ and $p_2$ (and making it a simple path by removing vertices if needed), there is a path from $X$ to $Y$ in $G_1$ that does not use vertices from $Z$, contradicting the given assumption that $X \perp Y|G_1 Z$, and equivalently $X \perp Y|RZ$. Hence either $X \perp YD|RZ$ or $XD \perp Y|RZ$.

An alternative proof can be obtained using the transitivity property [17]. Since $X \perp Y|G_1 Z$, by strong union [16]

\[ X \perp Y|G_1 Z \]  

(30)

Also,

\[ X \perp Y|G_1 Z \]

\[ \Rightarrow X \perp D[G_1 Z] \text{ or } Y \perp D[G_1 Z] \quad \text{(Transitivity [17])} \]

\[ \Rightarrow X \perp D[G_1, ZY] \text{ or } Y \perp D[G_1, ZX] \quad \text{(31)} \]

(Strong union [16])

\[ \Rightarrow X \perp YD[G_1, Z] \text{ or } XD \perp Y[G_1, Z] \]

(30, 31 and Intersection [15])

\[ \square \]

**B.2 Proof of Theorem 5.2.**

Now we prove Theorem 5.2 using Lemma 5.3.

**Proof of Theorem 5.2.** If joined attributes $D \subseteq X, Y$, or $Z$, then by Theorem 4.1 $X \perp Y|_{R\cap S} Z$.

Otherwise, assume $D \notin X, Y$, and $Z$. By Lemma 5.3 $X \perp Y|_{D}[RZ] \text{ or } XD \perp Y|_{[RZ]}$. Without loss of generality, suppose $X \perp YD|RZ$. Then by Theorem 4.1 $X \perp YD|R_{R\cap S} Z$. Then by the decomposition property of graphoid axioms [16], $X \perp Y|_{R_{R\cap S} S}$.

**B.3 Example B.1.** CIs from Graph-Isomorph Relations Propagate to Joined Relation

**Example B.1.** Suppose $R = (ABCD)$ is given by graph $G_1 = A - B - C - D$, and $S = (DE)$ is given by $D - E$. Here we give an instance of $R$ that conforms to $G_1$: i.e., $A \perp C|_{[R]}[B, B \perp D|C, A \perp D|_BC$ – all can be verified from $R$; $S$ remains the same:

\[ R \times S \]

\[ \begin{array}{|c|c|c|c|c|c|c|c|}
| A | B | C | D | E |
\hline
| a_1 | b_1 | c | d_1 | e_1 |
| a_1 | b_1 | c | d_1 | e_2 |
| a_1 | b_1 | c | d_2 | e_1 |
| a_1 | b_1 | c | d_2 | e_2 |
| a_2 | b_1 | c | d_1 | e_1 |
| a_2 | b_1 | c | d_1 | e_2 |
| a_2 | b_1 | c | d_2 | e_1 |
| a_2 | b_1 | c | d_2 | e_2 |
\end{array} \]

Also, note that, $A \perp C|_{R\cap S} B$ in $R\cap S$, $Pr[A = a_1, B = b_1 | C = c] = 1/4$, whereas $Pr[A = a_1 | C = c] = 1/4$ and $Pr[B = b_1 | C = c] = 1/4$, i.e., the CI now propagates to $R \times S$.

**B.4 Proof of Lemma 5.6.**

**Proof of Lemma 5.6.** Without loss of generality, assume $X, Z \subseteq R$ and $Y \subseteq S$. Let $D = R \cap S$ denote the singleton join attribute. There are different cases:

(i) $D \in Z_i, D \notin Y$: If $D = Z$, the lemma follows from Theorem 4.1. Hence assume $Z = DZ_1$, where $Z_1 \subseteq R$. By Theorem 4.1 $XZ_1 \perp Y|_{R\cap S} D$. By weak union property of graphoid axioms, $X \perp Y|_{R\cap S} DZ_1$, or $X \perp Y|_{R\cap S}$.\[ \square \]

(ii) $D \notin X \text{ (similarly } Y), D \notin Y, Z \text{ If } D \in X$, by Theorem 4.1 the lemma follows. Hence assume $X =
$DX_1$, where $X_1 \subseteq R$ and $DX_1 \perp Y|GZ$. We claim that this case cannot arise. Suppose not. Then in $G$, no path exists between $D$ and $Y$ in $G-Z$. However, $Z \in R$ whereas $D,Y \in S, G_2$ is connected by assumption, and the connectivity of $D$ and $Y$ is not affected by removing $Z$.

(iii) $D \not\subseteq X,Y,Z$: Since $X \not\subseteq Y|GZ$, all paths between $X$ and $Y$ in $G$ go through $Z$. By Observation 5.3, all paths between $X$ and $Y$ also go through $D$. Therefore,

$$X \not\subseteq D|GZ \quad (32)$$

since otherwise, there is a path from $X$ to $D$ that does not go through $Z$, and in conjunction with a path between $D$ to $Y$ in $G_2$ (we assumed that both $G_1, G_2$ are connected), we get a path between $X$ and $Y$ in $G$ that does not go through $Z$ violating the assumption that $X \not\subseteq Y|GZ$.

From (32), since $G_1$ has a subset of edges of $G$ (if a path exists in $G_1$ it must exist in $G$), we have

$$X \not\subseteq D|GZ \quad \Rightarrow \quad X \not\subseteq D|RZ \quad \text{(since $G_1$ is a P-map of $R$)}$$

$$X \not\subseteq D|RWZ \quad \text{(from Theorem 5.2)} \quad (33)$$

The last step follows from the fact that all of $X,Y,Z$ belong to $R$. Now, from Corollary 4.4 we have

$$XZ \not\subseteq Y|RWZ \quad \Rightarrow \quad X \not\subseteq Y|RWZ$$

where the last step follows from weak union of graphoid axioms $[S]$. From (35) and (41), applying the contraction property of the graphoid axioms $[S]$ (assume $X = X, Y = D, Z = Z, \emptyset = Y$),

$$X \not\subseteq YD|RWZ \quad \Rightarrow \quad X \not\subseteq Y|RWZ$$

by the decomposition property of the graphoid axioms $[S]$. This proves the decomposition lemma. $\square$

### B.5 Proof of Lemma 5.7

**Proof of Lemma 5.7** We consider all possible cases w.r.t. the join attribute $D$.

(i) $D \in Z_1, D \not\subseteq X_1, X_2, Y_1$: (i-a) Suppose $Z_1 = D$. Since $X_1,X_2 \not\subseteq Y_1|GZ_1$, we have $X_1 \not\subseteq Y_1|GZ_1$, by Observation 5.3, $X_1 \not\subseteq Y_1|GZ_1$, since $G_1$ is a P-map of $R$, $X_1 \not\subseteq Y_1|GZ_1$, by Theorem 5.1.

$$X_1 \not\subseteq Y_1|RWZ \quad \Rightarrow \quad X_1 \not\subseteq Y_1|RWZ \quad (34)$$

Further, by Corollary 4.4

$$Y_1 \not\subseteq X_2|RWZ \quad (35)$$

By Corollary 4.4, $X_1,Y_1 \not\subseteq X_2|RWZ$, by weak union $[S], X_1 \not\subseteq X_2|RWZ|DY_1$. Using (44) and contraction $[S], X_1 \not\subseteq Y_1|GZ_1$. By weak union $[S], X_1 \not\subseteq Y_1|GZ_2$. By contraction $[S], X_1X_2 \not\subseteq Y_1|GZ_2$. By Observation 5.3, $X_1 \not\subseteq Y_1|GZ_1$, by Theorem 5.1. $\square$

Further, by Corollary 4.4

$$X_1 \not\subseteq Y_1|RWZ \quad (36)$$

By Corollary 4.4, $X_1,Y_1 \not\subseteq X_2|RWZ$, by weak union $[S], X_1 \not\subseteq X_2|RWZ|DY_1$. Using (44) and contraction $[S], X_1 \not\subseteq Y_1|GZ_1$. By weak union $[S], X_1 \not\subseteq Y_1|GZ_2$. By contraction $[S], X_1 \not\subseteq Y_1|GZ_2$. By Observation 5.3, $X_1 \not\subseteq Y_1|GZ_1$, by Theorem 5.1. $\square$

Further, by Corollary 4.4, $X_1,Y_1 \not\subseteq X_2|RWZ$, by weak union $[S], X_1 \not\subseteq X_2|RWZ|DY_1$. Using (44) and contraction $[S], X_1 \not\subseteq Y_1|GZ_1$. By weak union $[S], X_1 \not\subseteq Y_1|GZ_2$. By contraction $[S], X_1 \not\subseteq Y_1|GZ_2$. By Observation 5.3, $X_1 \not\subseteq Y_1|GZ_1$, by Theorem 5.1. $\square$

Further, by Corollary 4.4, $X_1,Y_1 \not\subseteq X_2|RWZ$, by weak union $[S], X_1 \not\subseteq X_2|RWZ|DY_1$. Using (44) and contraction $[S], X_1 \not\subseteq Y_1|GZ_1$. By weak union $[S], X_1 \not\subseteq Y_1|GZ_2$. By contraction $[S], X_1 \not\subseteq Y_1|GZ_2$. By Observation 5.3, $X_1 \not\subseteq Y_1|GZ_1$, by Theorem 5.1. $\square$

Further, by Corollary 4.4, $X_1,Y_1 \not\subseteq X_2|RWZ$, by weak union $[S], X_1 \not\subseteq X_2|RWZ|DY_1$. Using (44) and contraction $[S], X_1 \not\subseteq Y_1|GZ_1$. By weak union $[S], X_1 \not\subseteq Y_1|GZ_2$. By contraction $[S], X_1 \not\subseteq Y_1|GZ_2$. By Observation 5.3, $X_1 \not\subseteq Y_1|GZ_1$, by Theorem 5.1. $\square$

Further, by Corollary 4.4, $X_1,Y_1 \not\subseteq X_2|RWZ$, by weak union $[S], X_1 \not\subseteq X_2|RWZ|DY_1$. Using (44) and contraction $[S], X_1 \not\subseteq Y_1|GZ_1$. By weak union $[S], X_1 \not\subseteq Y_1|GZ_2$. By contraction $[S], X_1 \not\subseteq Y_1|GZ_2$. By Observation 5.3, $X_1 \not\subseteq Y_1|GZ_1$, by Theorem 5.1. $\square$

Further, by Corollary 4.4, $X_1,Y_1 \not\subseteq X_2|RWZ$, by weak union $[S], X_1 \not\subseteq X_2|RWZ|DY_1$. Using (44) and contraction $[S], X_1 \not\subseteq Y_1|GZ_1$. By weak union $[S], X_1 \not\subseteq Y_1|GZ_2$. By contraction $[S], X_1 \not\subseteq Y_1|GZ_2$. By Observation 5.3, $X_1 \not\subseteq Y_1|GZ_1$, by Theorem 5.1. $\square$

Further, by Corollary 4.4, $X_1,Y_1 \not\subseteq X_2|RWZ$, by weak union $[S], X_1 \not\subseteq X_2|RWZ|DY_1$. Using (44) and contraction $[S], X_1 \not\subseteq Y_1|GZ_1$. By weak union $[S], X_1 \not\subseteq Y_1|GZ_2$. By contraction $[S], X_1 \not\subseteq Y_1|GZ_2$. By Observation 5.3, $X_1 \not\subseteq Y_1|GZ_1$, by Theorem 5.1. $\square$

Further, by Corollary 4.4, $X_1,Y_1 \not\subseteq X_2|RWZ$, by weak union $[S], X_1 \not\subseteq X_2|RWZ|DY_1$. Using (44) and contraction $[S], X_1 \not\subseteq Y_1|GZ_1$. By weak union $[S], X_1 \not\subseteq Y_1|GZ_2$. By contraction $[S], X_1 \not\subseteq Y_1|GZ_2$. By Observation 5.3, $X_1 \not\subseteq Y_1|GZ_1$, by Theorem 5.1. $\square$
Also $X_2 \not\subseteq DW_1|GZ_1$. By Lemma 5.6

$$X_2 \not\subseteq DW_1|_{RwS}Z_1$$  \hfill (45)

$X_1W_1Z_1 \not\subseteq X_2|_{RwS}D \Rightarrow X_1 \not\subseteq X_2|_{RwS}DW_1Z_1$ (weak union). By contradiction and (11), $X_1 \not\subseteq DW_1X_2|_{RwS}DW_1Z_1$. By weak union, $X_1 \not\subseteq DW_1|_{RwS}X_2Z_1$. By contradiction and (15), $X_1X_2 \not\subseteq DW_1|_{RwS}Z_1$.

(v) $D \not\subseteq X_1, X_2, Y_1$. Since $X_1X_2 \not\subseteq Y_1|GZ_1$, then by definition of independence in a graph, $X_1 \not\subseteq Y_1|GZ_1$, and since $G_1$ is a subgraph of $G$, $X_1 \not\subseteq X_1|GZ_1$. Since $G_1$ is a P-map of $R$, $X_1 \not\subseteq Y_1|GZ_1$, by the strong union property of Theorem 2.8 $X_1 \not\subseteq Y_1|RDZ_1$, and by Theorem 5.2

$$X_1 \not\subseteq Y_1|_{RwS}Z_1$$  \hfill (46)

Also since $X_1X_2 \not\subseteq Y_1|GZ_1$, by decomposition property of graphoid axioms 7, $X_2 \not\subseteq Y_1|GZ_1$. We claim that $Y_1 \not\subseteq DX_2|GZ_1$. Suppose not. Since $X_2 \not\subseteq Y_1|GZ_1$, there is a path $p_1$ between $D$ and $Y_1$ that does not go through $Z_1$. Since $D$ and $X_2$ are connected by at least one path $p_2$ in $G_2$ (which does not go through $Z_1$ since $Z_1 \not\subseteq R$), by combining $p_1$ and $p_2$ (and simplifying to get a simple path), we get a path from $Y$ to $X_2$ that does not go through $Z_1$, contradicting the assumption that $X_1X_2 \not\subseteq Y_1|GZ_1$. Hence $Y_1 \not\subseteq DX_2|GZ_1$, and since $Y_1 \subseteq Z_1$ is in $R$ and $DX_2$ is in $S$, by Lemma 5.6

$$Y_1 \not\subseteq DX_2|_{RwS}Z_1$$  \hfill (47)

Next note that $X_1Y_1 \not\subseteq X_2|GZ_1$, since $D$ itself is a cutset between $X_1Y_1$ and $X_2$ (Observation 5.4). Since $X_1Y_1 \subseteq R$, $X_2 \subseteq S$, and $Z_1 \subseteq R$, by Lemma 5.6

$$X_1Y_1 \not\subseteq X_2|_{RwS}DZ_1$$  \hfill (48)

Next note that $X_1Y_1 \not\subseteq X_2|_{RwS}DZ_1$, since $D$ is a cutset between $X_1Y_1$ and $X_2$ (Observation 5.4). Since $X_1Y_1 \subseteq R$, $X_2 \subseteq S$, and $Z_1 \subseteq R$, by Lemma 5.6

$$X_1 \not\subseteq X_2|_{RwS}DZ_1$$  \hfill (49)

Note that $W_1$ belongs to $R$ or $G_1$, and does not have any common vertex in $G_2$. In other words, removing $W_1$ cannot affect the connectivity between $X_2, Y_2$. Therefore, it holds that

$$Y_2 \not\subseteq X_2|_{G}D$$  \hfill (50)

Since $X_1X_2 \not\subseteq Y_1|GZ_1$, by definition, $X_1X_2 \not\subseteq Y_1|GZ_1$, and by Lemma 5.7

$$X_1 \not\subseteq Y_1|_{RwS}Z_1$$  \hfill (51)

By Corollary 4.1

$$X_1W_1 \not\subseteq Y_2|_{RwS}D$$  \hfill (52)

Next note that $X_1 \not\subseteq Y_2|_{RwS}D$, since $X_1 \not\subseteq Y_2|_{RwS}D$, by assumption.

By Lemma 5.9 First we argue that the join attribute $D \subseteq Z_1$. Suppose not. Then in $G_{-Z_1}$, there is no path exists between $X_2$ and $Y_2$ in $G$. Since $D \not\subseteq Z_1$, removing $Z_1$ does not remove any edge in $G_2$, implying that no path exists between $X_2$ and $Y_2$ in $G_2$, which contradicts the assumption that $G_2$ is connected.

Hence $D \subseteq Z_1$. Assume $Z_1 = DW_1$ where $W_1 \subseteq R$. Since $X_1 \not\subseteq Y_2|_{GZ_1} \subseteq X_1 \not\subseteq Y_2|_{G}DW_1$, and $W_1$ belongs to $R$ or $G_1$, it holds that $Y_2 \not\subseteq X_2|_{G}D$. Since $G_2$ is a subset of $G$, $Y_2 \not\subseteq X_2|_{G}D$, and since $G_2$ is a P-map of $S$, $Y_2 \not\subseteq X_2|_{S}D$, and by Theorem 5.2

$$Y_2 \not\subseteq X_2|_{RwS}D$$  \hfill (53)

Also, since $X_1X_2 \not\subseteq Y_1|_{G}Z_1$, by decomposition property of graphoid axioms 7, $X_2 \not\subseteq Y_1|_{G}Z_1$. We claim that $Y_1 \not\subseteq DX_2|_{G}Z_1$. Suppose not. Since $X_2 \not\subseteq Y_1|_{G}Z_1$, there is a path $p_1$ between $D$ and $Y_1$ that does not go through $Z_1$. Since $D$ and $X_2$ are connected by at least one path $p_2$ in $G_2$ (which does not go through $Z_1$ since $Z_1 \not\subseteq R$), by combining $p_1$ and $p_2$ (and simplifying to get a simple path), we get a path from $Y$ to $X_2$ that does not go through $Z_1$, contradicting the assumption that $X_1X_2 \not\subseteq Y_1|_{G}Z_1$. Hence $Y_1 \not\subseteq DX_2|_{G}Z_1$, and since $Y_1 \subseteq Z_1$ is in $R$ and $DX_2$ is in $S$, by Lemma 5.6

$$Y_1 \not\subseteq DX_2|_{RwS}Z_1$$  \hfill (54)

Note that $W_1$ belongs to $R$ or $G_1$, and does not have any common vertex in $G_2$. In other words, removing $W_1$ cannot affect the connectivity between $X_2, Y_2$. Therefore, it holds that

$$Y_2 \not\subseteq X_2|_{G}D$$  \hfill (55)

Since $X_1X_2 \not\subseteq Y_1|_{G}Z_1$, by definition, $X_1X_2 \not\subseteq Y_1|_{G}Z_1$, and by Lemma 5.7

$$X_1 \not\subseteq Y_1|_{RwS}Z_1$$  \hfill (56)

By Corollary 4.1

$$X_1W_1 \not\subseteq Y_2|_{RwS}D$$  \hfill (57)

Next note that $X_1 \not\subseteq Y_2|_{RwS}D$, since $X_1 \not\subseteq Y_2|_{RwS}D$, by assumption.

By Corollary 4.1

$$X_1W_1 \not\subseteq Y_2|_{RwS}D$$  \hfill (58)

Next note that $X_1 \not\subseteq Y_2|_{RwS}D$, since $X_1 \not\subseteq Y_2|_{RwS}D$, by assumption.
B.9 Proof of Lemma 5.11

To prove this lemma, we will need additional lemmas:

**Lemma B.2.** (A) If $D \perp X_1|GZ_1Z_2$, then $D \perp X_1|_{RwsZ_1Z_2}$.

(B) If $D \perp X_1X_2|GZ_1Z_2$, then $D \perp X_1|_{RwsZ_1Z_2}$.

**Proof.** (A) By Corollary 4.4, $G$ is a P-map of $X_1|GZ_1Z_1$, and since $G_1$ is a P-map of $R$ and by Theorem 5.2,

$$D \perp X_1|_{RwsZ_1Z_2} \tag{54}$$

By Corollary 4.4, $X_1Z_1 \perp Z_2|_{RwsZ_1D}$. By weak union [8], $X_1 \perp Z_2|_{RwsZ_1D}$. Combining with [54] by contraction [59], $X_1 \perp DZ_2|_{RwsZ_1Z_2}$. By weak union again, $X_1 \perp D|_{RwsZ_1Z_2}$.

(B) $D \perp X_1X_2|GZ_1Z_2 \Rightarrow D \perp X_1|GZ_1$. By Observation 5.1, $D \perp X_1|GZ_1X_2$. By (A) above,

$$D \perp X_1|_{RwsZ_1Z_2} \tag{55}$$

Similarly, $D \perp X_2|GZ_1Z_2$ and

$$D \perp X_2|_{RwsZ_1Z_2} \tag{56}$$

By Corollary 4.4, $X_1Z_1 \perp X_2Z_2|_{RwsZ_1D}$. By weak union [8], $X_1 \perp Z_2|_{RwsZ_1D}$. Combining with [55] by contraction [59], $X_1 \perp DZ_2|_{RwsZ_1Z_2}$. By contraction and [50], $X_1X_2 \perp D|_{RwsZ_1Z_2}$. □

**Lemma B.3.** Suppose the join attribute $D \notin X_1,Y_1,Y_2,Z_1,Z_2$.

(A) If $DX_1 \perp Y_1|GZ_1Z_2$, then $DX_1 \perp Y_1|_{RwsZ_1Z_2}$.

(B) If $DX_1 \perp Y_2|GZ_1Z_2$, then $DX_1 \perp Y_2|_{RwsZ_1Z_2}$.

**Proof.** (A) Since $DX_1 \perp Y_1|GZ_1Z_2$, $D \perp Y_1|GZ_1Z_2$, and by Lemma 5.7,

$$D \perp Y_1|GZ_1Z_2 \tag{57}$$

Also, $X_1 \perp Y_1|GZ_1Z_2$. Hence $X_1 \perp Y_1|GZ_1Z_2D$ (Observation 5.1), and by Lemma 5.7,

$$X_1 \perp Y_1|_{RwsZ_1Z_2} \tag{58}$$

Combining [57] and [58] by contraction, $DX_1 \perp Y_1|_{RwsZ_1Z_2}$.

(B) Since $DX_1 \perp Y_2|GZ_1Z_2$, $D \perp Y_2|GZ_1Z_2$, and by Lemma 5.7,

$$D \perp Y_2|GZ_1Z_2 \tag{59}$$

Also, $X_1 \perp Y_2|GZ_1Z_2$. Hence $X_1 \perp Y_2|GZ_1Z_2D$ (Observation 5.1), and by Lemma 5.8,

$$X_1 \perp Y_2|_{RwsZ_1Z_2} \tag{60}$$

Combining [59] and [60] by contraction, $DX_1 \perp Y_2|_{RwsZ_1Z_2}$. □

**Lemma B.4.** Suppose the join attribute $D \notin X_1,Y_1,Y_2,Z_1,Z_2$.

(A) If $X_1 \perp Y_1|GZ_1Z_2$, then $X_1 \perp Y_1|_{RwsZ_1Z_2}$.

(B) If $X_1 \perp Y_2|GZ_1Z_2$, then $X_1 \perp Y_2|_{RwsZ_1Z_2}$.

**Proof.** (A) If $X_1 \perp Y_1|GZ_1Z_2$, then $X_1 \perp Y_1|_{RwsZ_1Z_2}$. Also $X_1 \perp Z_1|GZ_1Z_2$. Hence $X_1 \perp Y_1|GZ_1Z_2$. By Lemma 5.7, $X_1 \perp Y_1|_{RwsZ_1Z_2}$.

(B) If $X_1 \perp Y_2|GZ_1Z_2$, then $X_1 \perp Y_2|_{RwsZ_1Z_2}$. (D) itself disconnects $X_1$ from $Z_2$. By Lemma 5.9, $X_1 \perp Y_2|_{RwsZ_1Z_2}$. By weak union, $X_1 \perp Y_2|_{RwsZ_1Z_2}$. □

Now we prove Lemma 5.11. There are four non-equivalent cases as stated in the lemma.

**Proof of Lemma 5.11 (A)** If $D \notin Z_1Z_2$, it follows from Lemma 5.2. Since $D \notin X_1$ or $Y_1$, it follows from Lemma 5.2 and 5.3. Hence we assume $D \in Z_1Z_2$. $X_1 \perp Y_1|GZ_1Z_1$. This is because of the fact that $D \notin Z_1Z_2$, and no path between $X_1,Y_1$ in $G$ can go through $Z_2$. In turn, $X_1 \perp Y_1|G_1Z_1$ (Observation 5.1). Hence $X_1 \perp Y_1|GZ_1Z_1$, since $G_1$ is a P-map of $R$, by Theorem 5.2.

$$X_1 \perp Y_1|_{RwsZ_1Z_2} \tag{61}$$

By Corollary 4.4, $X_1Y_1Z_1 \perp Z_2|_{RwsZ_1D}$. By weak union [8], $X_1 \perp Z_2|_{RwsZ_1D}$. Combining with [61] by contraction [59], $X_1 \perp Y_1Z_2|_{RwsZ_1Z_2}$. By decomposition, $X_1 \perp Y_1|_{RwsZ_1Z_2}$.

We claim that either $X_1 \perp D|GZ_1Z_2$ or $Y_1 \perp D|GZ_1Z_2$. Indeed, if both fail, then there is a path from $X_1$ to $Y_1$ in $G\setminus Z_2$, violating the assumption that $X_1 \perp Y_1|_{GZ_1Z_2}$. If $X_1 \perp D|GZ_1Z_2$, by Lemma 5.2, $X_1 \perp D|_{RwsZ_1Z_2}$. Combining with [62] by contraction, $X_1 \perp D |_{RwsZ_1Z_2}$, and by decomposition, $X_1 \perp Y_1|_{RwsZ_1Z_2}$.

If $Y_1 \perp D|GZ_1Z_2$, by similar argument, $DX_1 \perp D|_{RwsZ_1Z_2}$, and by decomposition, $X_1 \perp Y_1|_{RwsZ_1Z_2}$.

(B) If $D \notin Z_1Z_2$, it follows from Lemma 5.2. Since $D \notin X_1$ or $Y_1$, it follows from Lemma 5.2 and 5.3. Hence assume $D \notin Z_1Z_2$. $X_1 \perp Y_1|GZ_1Z_2$. $X_1 \perp Y_2|GZ_1Z_2$, either $X_1 \perp D|GZ_1Z_2$, or $Y_2 \perp D|GZ_1Z_2$, otherwise, a path exists between $X_1$ and $Y_2$ through $D$ that is not in $Z_1Z_2$.

Without loss of generality, assume $X_1 \perp D|GZ_1Z_2$. Then $X_1 \perp D|GZ_1Z_2$ (Observation 5.1), and by Lemma 5.2,

$$X_1 \perp D|_{RwsZ_1Z_2} \tag{63}$$

By Corollary 4.4, $X_1Z_1 \perp Y_2Z_2|_{RwsD}$, $X_1 \perp Y_2|_{RwsZ_1Z_2}$ (weak union). Combining with [63] by contraction, $\Rightarrow X_1 \perp Y_2|_{RwsZ_1Z_2}$, and by decomposition, $X_1 \perp Y_2|_{RwsZ_1Z_2}$.

(C) Since $X_1X_2 \perp Y_1|GZ_1Z_2$, $X_1 \perp Y_1|GZ_1Z_2$, and by case (A) above,

$$X_1 \perp Y_1|_{RwsZ_1Z_2} \tag{64}$$

Also $X_1 \perp Y_1|GZ_1Z_2$, therefore, $X_2 \perp Y_1|GZ_1Z_2X_1$ (Observation 5.1), and by case (B) above,

$$X_2 \perp Y_1|_{RwsZ_1Z_2} \tag{65}$$

Applying contraction [59] on [64] and [65], $X_1X_2 \perp Y_1|GZ_1Z_2$.

(D) Since $X_1X_2 \perp Y_1|GZ_1Z_2$, $X_1X_2 \perp Y_1|GZ_1Z_2$, and by case (C) above,

$$X_1X_2 \perp Y_1|_{RwsZ_1Z_2} \tag{66}$$
Also \(X_1X_2 \perp Y_2|GZ_1Z_2\), therefore, \(X_1X_2 \perp Y_2|GZ_1Z_2Y_1\) (Observation 5.1), and by case (B) above,

\[
X_1X_2 \perp Y_2|_{\text{pos}}(Z_1Z_2)X_1
\]  (67)

Applying contraction (\(\bullet\)) on (66) and (67), \(X_1X_2 \perp Y_1Y_2|_{\text{pos}}Z_1Z_2\).

(D) Consider the remaining case that \(D \notin Z_1Z_2\), \(X_1, X_2, Y_1, Y_2\). If \(X_1X_2 \perp Y_1Y_2|GZ_1Z_2\), either (i) \(X_1 \perp D|GZ_1\), or (ii) \(Y_1 \perp D|GZ_2\) and \(Y_2 \perp D|GZ_2\), otherwise, a path exists between \(X_1\) and either \(Y_1\) or \(Y_2\) through \(D\) that is not in \(Z_1Z_2\).

\[\Box\]

**B.10 Proof of Theorem 5.12**

**Proof of Theorem 5.12.** Suppose \(X = X_1X_2, Y = Y_1Y_2, Z = Z_1Z_2\). Lemma 5.11 covers the (non-equivalent) cases when both \(Z_1, Z_2\) are non-empty. When \(Z_2 = \emptyset\) (equivalently \(Z_1\)): (i) when both \(X, Y\) contain both subsets, the result follows from Lemma 5.9; (ii) when one of \(X, Y\) contain both subsets and the other contain one, the result follows from Lemmas 5.8 and 5.7; (iii) when both \(X, Y\) contain one subset each, the result follows from Lemma 5.9 or Theorem 5.11 (in this case, the CI holds in the base relation since \(G_1\) is a P-map, and therefore propagates to the joined relation).

\[\Box\]

**C. PROOFS FROM SECTION 6**

**C.1 Example C.1: New CIs in the Joined Relation**

**Example C.1.** Consider relations \(R(A, B, C, D, E)\) and \(S(D, F)\). The relation instance contains the tuples \((a_1, b_1, c, d_1, -), (a_2, b_2, c, d_2, -), (a_3, b_3, c, d_3, -), (a_4, b_4, c, d_4, -)\) (with unique values of \(E\) as \(\sim\)) respectively 2, 3, 1, 1 times. For \(C = c\), \(\Pr_R[A = a_1, B = b_1| C = c] = \frac{2}{4}\), whereas \(\Pr_R[A = a_1| C = c] = \frac{5}{14}\) and \(\Pr_R[A = b_1| C = c] = \frac{2}{7}\), so \((A \perp B|_R C)\). Now suppose \(S\) contains 3, 2, 1, 1 tuples respectively of the form \((d_1, -), (d_2, -), (d_3, -), (d_4, -)\), and therefore using the new frequencies in \(J = R \times S\), \(\Pr_J[A = a_1, B = b_1| C = c] = \frac{2}{14}\), whereas \(\Pr_J[A = a_1| C = c] = \frac{5}{14}\) and \(\Pr_J[A = b_1| C = c] = \frac{2}{14}\), thus satisfying the CI \(A \perp B|_{\text{pos}} SC\).

**D. ADDITIONAL RELATED WORK**

**Recent work in the intersection of causality and databases.** While causality has been used as a motivation to explain interesting observations in the area of databases [30, 30], actual causal inference as done in statistical studies using techniques from databases has drawn attention only very recently [13]. In [13], Roy et al. studied efficient matching methods that include a large number of covariates, while ensuring that each group contains at least one treated and one control units, and such that the selected covariates predict the outcome well. Since the goals conflict with each other, the proposed technique aim to match as many units as possible using as many covariates as possible, then drops the ‘less useful’ covariates to match more units in the next round. In [33], Salimi and Suciu proposed a framework for supporting various existing causal inference techniques efficiently inside a database-based engine, for both online and offline settings. The problem of matching units with the same values of covariates has a strong connection with the group-by operator used in SQL queries, therefore, both [43, 44] use database queries to efficiently implement the matching algorithms using standard relational database management systems. However, both [13, 33] consider the problem of efficient causal inference on a single relation, whereas the main focus of the framework proposed in the current paper is extending causal inference to multiple relations.

**Matching for observational studies.** Matching has been studied since 1940s for observational studies [6, 13]. In the 1970s and 1980s, a large literature on different dimension reduction approaches to matching was developed (e.g., [15, 53, 53]). One of the most famous approaches that is still prevalent in current research is matching using propensity score, the distribution of treatment assignment conditional on background covariates. Rosenbaum and Rubin [32] demonstrated that under certain assumptions the propensity score is a balancing score (within each matched group, treated and control units are independent) that allowed for the unbiased estimation of average causal effects, and also is the most coarsened balancing score, thereby allowing valid matching of many units. Beyond naive matching, the propensity score has been integrated into subclassification [47, 43] and multivariate matching schemes [44] for observational data. Relatively recent coarsened exact matching avoids fitting complicated propensity score models by coarsening or discretizing covariates in such a way that the newly constructed covariates allow for exact matching [24]. A general overview of matching techniques can be found in [50].

**Causality and causal graphs in artificial intelligence.** The study of causality in the artificial intelligence community is based on the notion of counterfactuals, where the basic idea is that if the first event (cause) had not occurred, then the second event (effect) would not have occurred (in contrast to the quantitative measure of ATE in the causal analysis in statistics). The work by Pearl and others [35] refined the generally accepted aspects of causality into a rigorous definition using structural equations, which can also be viewed as causal networks. In causal networks, the causal effects and counterfactuals are modeled using a mathematical operator \(\text{do}(X = x)\) to simulate the effect of an intervention (called do-calculus). Pearl proposed a sufficient condition called back-door criterion to identify the distribution \(P(Y = y|\text{do}(T = 1))\) in order to estimate the distribution on the causal effect of the treatment [35].

**Other work on causality in databases.** Motivated by the notion of causality and intervention by Pearl [35], Meilou et al. [30], studied the problem of finding and ranking input tuples as ‘causes’ of
query answers. Roy and Suciu [46] studied finding explanations for aggregate query answers, a summary of tuples that have a large impact on the query answers, with a similar motivation. In [55], Silverstein et al. studied the problem of efficiently determining causal relationship for mining market basket data. Maier et al. [40, 29] identified ways to distinguish statistical association from actual causal dependencies in the relational domain.

To the best of our knowledge, none of the work till date has considered the problem of extending causal analysis and potential outcome model to multiple relations with a rigorous study of the underlying assumptions like SUTVA and strong ignorability, which is the main contribution of the framework proposed in this paper.