Critical capacitance and charge-vortex duality near the superfluid to insulator transition

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Using a generalized reciprocity relation between charge and vortex conductivities at complex frequencies in two space dimensions, we identify the capacitance in the insulating phase as a measure of vortex condensate stiffness. We compute the ratio of boson superfluid stiffness to vortex condensate stiffness at mirror points to be 0.21(1) for the relativistic O(2) model. The product of dynamical conductivities at mirror points is used as a test of charge-vortex duality. We propose the finite wave vector compressibility as an experimental measure of the vortex condensate stiffness for neutral bosons.

Two dimensional superfluid to insulator transitions (SIT) have been observed in diverse systems: e.g., Josephson junction arrays [1], cold atoms trapped in optical lattices [2, 3], and disordered superconducting films [4]. Recent experiments have uncovered important dynamical properties near the quantum critical point: A softening amplitude (Higgs) mode observed in the optical lattice [4], and a critically suppressed threshold frequency seen by terahertz conductivity in superconducting films [6]. These have motivated numerical studies of real-time correlations near criticality [7–9], and novel ideas from holography [10, 11].

Three decades ago, Fisher and Lee [12] showed that the SIT can be described as a Bose condensation of quantum vortices. Despite the appeal of this description, the vortex condensate stiffness, has remained an elusive observable, for which no experimental probe has yet been proposed. Also, to our knowledge, the vortex condensate stiffness, has not been calculated near the critical point, for any microscopic model. In this Letter, we address this problem, by using an exact reciprocity relation between complex dynamical conductivities of bosons ($\sigma$) and vortices ($\sigma_v$):

$$\sigma(\omega) \times \sigma_v(\omega) = q^2/h^2,$$ (1)

where $q$ is the boson charge ( = 2e in superconductors) [13]. At low frequencies, this equation is dominated by the reactive (imaginary) conductivities. The superfluid stiffness $\rho_s$ in the superfluid phase can be measured by the low frequency inductance $L_{sf}$: $\rho_s = h/(2\pi q L_{sf})$, where $q = q^2/h$ is the quantum of conductance. Equation (1) allows us to identify the elusive vortex condensate stiffness with the capacitance per square in the insulating phase, $C_{ins} = h\sigma_q/(2\pi \rho_v)$.

We shall examine quantitatively the charge-vortex duality (CVD) hypothesis [13, 14], which relates the superfluid to the insulating phase. The strict CVD hypothesis predicts a universal critical conductivity at the SIT given by $\sigma_q$. Experiments, however, have measured non-universal values of the critical conductivity [5].

Several factors can spoil CVD in experiments (i) Bosons (in the superfluid) and vortices (in the insulator) have different masses and interaction ranges [15, 16]. (ii) Potential energy (both confining and disordered) couples differently to charges and vortices. (iii) In superconductors, fermionic (Bogoliubov) quasiparticles produce dissipation, which can alter the phase diagram from the purely bosonic theory.

For strict CVD to hold, one expects $\rho_s(\pm \delta g) = \rho_v(\pm \delta g)$, where $\pm \delta g$ are mirror points on either side of the SIT. Figure 1 depicts all the critical energy scales of the relativistic O(2) field theory, obtained by large-scale Monte Carlo simulation. In addition to the Higgs mass $m_H$ and the charge gap $\Delta$, which vanish at the critical point, we compare the energy scales $\rho_s$ and $\rho_v$, which are also critical, but have different relative amplitudes. The ratio $\rho_s(\pm \delta g)/\rho_v(\pm \delta g) = 0.21(1)$. The deviation from unity quantifies the violation of CVD.

It is interesting to ask whether CVD is better satisfied at finite frequencies. To address this, we propose the product function

$$R(z) = \sigma(z, -\delta g) \times \sigma(z, \delta g)/\sigma_q^2,$$ (2)

FIG. 1. Critical energy scales near the SIT computed by QMC. The superfluid is characterized by the mass of the amplitude mode, $m_H$, and the superfluid stiffness, $\rho_s$; the insulator by the single particle gap, $\Delta$, and the vortex condensate stiffness, $\rho_v$. The amplitude ratios $m_H(\pm \delta g)/\Delta(\pm \delta g) = 2.1(3)$, $\rho_s(\pm \delta g)/\Delta(\pm \delta g) = 0.44(1)$, and $\rho_v(\pm \delta g)/\Delta(\pm \delta g) = 2.1(1)$ are universal.
as a measure of CVD between mirror points. Here, $z$ denotes either a real or a Matsubara frequency.

The high frequency conductivity [18] (after removal of cut-off dependent corrections) reaches a universal value $\sigma^* = 0.355(5)\sigma_0$ [19, 11]. We compute the function $R(\omega_m)$ and address its implications to CVD. We conclude by proposing an experimental measure of the vortex condensate stiffness $\rho_v$ for neutral bosons in an optical lattice.

**Vortex transport theory**—Boson charge current $j$ is driven by an electrochemical field $E$. Vortices are point particles in two dimensions. The vorticity current $j_v(t)$ is driven by the Magnus field $E_v$. Hydrodynamics dictate simple relations between electrochemical field and vortex number current, and between boson charge current and Magnus field [19]:

$$E^a = \frac{h}{q} \epsilon^{a\beta\gamma} j^\beta, \quad E^a = \frac{h}{q} \epsilon^{a\beta\gamma} j^\beta, \quad (3)$$

where $\epsilon = i\sigma^g$ is the two dimensional antisymmetric tensor. We note that Eqs. (3) are instantaneous. Conductivity relates currents to their driving fields,

$$j_{(\nu)}(t) = \int_{-\infty}^{t} dt' \sigma_{(\nu)}^{\alpha\beta}(t-t') E^\beta_{(\nu)}(t'), \quad (4)$$

By Fourier transformation, the complex dynamical conductivities obey a reciprocity relation $\epsilon^T \sigma_{\nu} \epsilon = (q^2/h^2)\sigma^{-1}$. For the case of an isotropic longitudinal conductivity $\sigma^{xx} = \sigma^{yy} = \sigma$, one obtains the reciprocity Eq. [1], which can be analytically continued to Matsubara space $\omega \rightarrow i\omega_n$.

**Model and observables**—For numerical simulations we study the discretized partition function $Z = \int D\varphi D\varphi^* e^{-S[\varphi,\varphi^*]}$, where the real action $S$ on Euclidean space-time is

$$S = \sum_{\langle i, j \rangle} \varphi_i \varphi_j^* + c.c + 2\mu \sum_i |\varphi_i|^2 + 4g \sum_i |\varphi_i|^4. \quad (5)$$

Here $\varphi_i$ are complex variables defined on a cubic lattice of size $L \times L \times \beta$. We take $\beta = L$ throughout. For $\mu < 0$, this model undergoes a continuous zero temperature quantum phase transition (QCP) between a superfluid with $\langle \varphi \rangle \neq 0$ for $g < g_c$, and an insulator with $\langle \varphi \rangle = 0$ for $g > g_c$. We define the quantum detuning parameter $\delta g = (g - g_c)/g_c$.

The critical energy scales near the SIT, as shown in Fig[1] in the superfluid phase are the amplitude mode mass $m_H$ and the superfluid stiffness $\rho_s$ [17, 20]. In the insulating phase excitations are gapped, with single-particle gap $\Delta$. The lattice current field is $J_{i,\eta} = -\frac{\delta S}{\delta A_{i,\eta}}$, where we have introduced a U(1) lattice gauge field by Peierls substitution $\varphi_i \varphi_{i+\eta}^* \rightarrow \varphi_i \varphi_{i+\eta}^* e^{iA_{i,\eta}}$. The dynamical conductivity is given by the current-current correlation function,

$$\tilde{\sigma}(\omega_m) = \frac{\Pi_{xx}(\omega_m)}{\omega_m}, \quad \Pi_{xx}(\omega_m) = \frac{1}{L^2 \beta} \sum_{i,j} e^{i\omega_m \tau_{ij}} \frac{\delta (J_{i,x})}{\delta A_{j,x}}, \quad (6)$$

where $\omega_m = \frac{2\pi}{L} m$ is a Matsubara frequency and $\tau_{ij}$ is the discrete imaginary time interval between points $i, j$. Remarkably, in 2+1 dimensions the conductivity has zero scaling dimension [13], such that it is given by a universal amplitude with scaling form

$$\tilde{\sigma}(\omega_m) = \sigma_q \Sigma_\pm (\omega_m/\Delta), \quad (7)$$

where $\Sigma_+$ ($\Sigma_-$) belongs to the insulating (superfluid) phase. Real frequency dynamics can be obtained by analytic continuation $\sigma(\omega) = \tilde{\sigma}(\omega_m \rightarrow -i\omega + 0^+)$.

In the superfluid phase, the reactive conductivity diverges as $\text{Im} \sigma_{sf}(\omega) = 2\pi\sigma_q \rho_h (-\delta g)/(\hbar \omega)$. Previous calculations [21] show that the dissipative component has a small sub-gap contribution below the Higgs mass, $0 < \omega < m_H (-\delta g)$ which goes as $\text{Re}(\sigma_{sf}(\omega)) \sim \omega^5$. This is negligible as $\omega \rightarrow 0$ and the analytic continuation to Matsubara frequency yields

$$\tilde{\sigma}_{sf}(\omega_m) \sim \frac{2\pi\sigma_q \rho_h}{\hbar \omega_m}, \quad (for \ \omega_m \ll m_H). \quad (8)$$

In the insulator, the dissipative conductivity vanishes identically below the chiral gap $\Delta(\delta g)$ [7, 22]. The reactive conductivity vanishes linearly with frequency $\text{Im} \sigma_{ins}(\omega) = -C_{ins}(\omega)$, where $C_{ins}$ is the capacitance per square. Therefore, $\sigma_{ins}$ can be analytically continued to Matsubara space as

$$\tilde{\sigma}_{ins}(\omega_m) \sim C_{ins} \omega_m, \quad (for \ \omega_m \ll \Delta). \quad (9)$$

Eq. (7) implies that the capacitance $C_{ins}$ diverges near the QCP as $C_{ins} \sim 1/\Delta$. The capacitance measures the dielectric response of the insulator. Its divergence reflects the large particle-hole fluctuations near the transition. In the vortex description the insulator is a Bose condensate of vortices, with a low frequency vortex conductivity $\tilde{\sigma}_v(\omega_m) = \rho_v/(\hbar^2 \omega_m)$. As a consequence, $\rho_v$ can be defined in terms of the capacitance by applying Eq. (1),

$$\rho_v = \frac{\hbar \sigma_q}{2\pi C_{ins}}. \quad (10)$$

We shall use this important relation to test for CVD in the 2+1 dimensional O(2) field theory.

**Methods**—A large scale QMC simulation of Eq. [5] is used to evaluate Eq. (9). To suppress the effect of critical slowing down near the phase transition we use the classical Worm Algorithm [23]. This method samples closed loop configurations of a double integer current representation of the partition function. This enables us to consider large systems, of linear size up to $L = 512$, which
is crucial for obtaining universal properties. To validate the universality of our results we performed our analysis on two distinct crossing points of the SIT, by choosing $\mu = 0.5$ and $\mu = -5.89391$ and tuning $g$ across the transition. We found excellent agreement within the error bars. Henceforth we will only present results for $\mu = -5.89391$, a value which has been argued to reduce finite size corrections to scaling \[24\].

First we locate the critical coupling $g_c(\mu)$ with high accuracy. This can achieved by a finite size scaling analysis of the superfluid stiffness $\rho_s = \frac{1}{L} \frac{\partial \mathcal{Z}(\varphi)}{\partial \varphi} |_{\varphi=0}$ \[25\], where $\mathcal{Z}(\varphi)$ is the partition function in the presence of a uniform phase twist $\varphi$. In this work we find $g_c = 7.0284(3)$. We extract the gap $\Delta$ in the insulator by analyzing the asymptotic large imaginary time decay of the two point Green’s function. In a gaped phase this has an exponential decay of the form $G(\tau) \sim e^{-\Delta \tau} + e^{-\Delta (\beta - \tau)}$, where $\beta = 1/T$. We compute $\Delta$ by a fit to this form. Near criticality, the gap is expected to scale as a power law $\Delta = \Delta_0 |\delta g|^{\nu}$, where $\nu$ is the correlation length exponent. We use $\nu = 0.6717(3)$, as obtained by previous high accuracy Monte Carlo studies of the 3D XY model \[26\]. Our results for $\Delta(\delta g)$ are in excellent agreement with the expected scaling, with the non universal pre-factor $\Delta_0 = 2.09(5)$.

Results – In Fig. 2 we present the dynamical conductivity $\sigma(\omega_m)$ as a function of Matsubara frequency, both in the insulator and in the superfluid, for a range of detuning parameters $\delta g$ near the critical point. To suppress finite size effects in the insulator we used an improved estimator, in which we consider only loop configurations with zero winding number \[10, 11\]. We find that the dynamical conductivity as a function of $\omega_m$ in Fig. 2 follows the form of the low frequency reactive conductivity both in the superfluid, Eq. 10, and in the insulator, Eq. 9.

Next we calculate $\rho_s$ and $\rho_c$ in their respective phases. The superfluid stiffness $\rho_s$ was calculated using the standard method of winding number fluctuations \[27\]. In order to extract $\rho_c$ we use the relation in Eq. 10. As a concrete Monte Carlo observable for the capacitance we use the conductivity evaluated at the first non-zero Matsubara frequency:

$$C(\delta g) = \lim_{L \to \infty} \frac{\sigma(\omega_m) = \frac{2\pi}{L} \delta g}{\sigma(\omega_m)}.$$

Both the vortex condensate stiffness $\rho_s$ and the superfluid stiffness $\rho_s$ near the critical point follow a power law behavior $\rho_{(s,v)} \sim \rho^0_{(s,v)} |\delta g|^{\nu_c}$. The non-universal prefactors $\rho^0_{(s,v)}$ and $\rho^0_{(s,v)}$ are extracted by a numerical fit. We find $\rho_s/\rho_c = 0.21(1)$. Surprisingly, this value is close to the value $\rho_s/\rho_c = 0.23$ obtained by a simple one loop weak coupling calculation \[7\].

The universal scaling function of the dynamical conductivity is obtained by rescaling the Matsubara frequency axis by the single particle gap $\Delta$. Curves for different detuning parameters $\delta g$ collapse into a single universal at low frequencies. On the other hand, at high frequencies, $\omega_m$ need not be a negligible fraction of the ultra-violet (UV) cutoff scale $\Lambda$. This leads to non-universal corrections in the conductivity that depend on powers of $\omega_m/\Lambda$. We take these into account by fitting the numerical QMC data to the following scaling form:

$$\sigma_{\pm}(\omega_m, \delta g, \Lambda) = \sigma_q \Sigma_{\pm} \left( \frac{\omega_m}{\Lambda} \right) + A \frac{\omega_m}{\Lambda} + B \left( \frac{\omega_m}{\Lambda} \right)^2.$$

Here, $A$ and $B$ are expected to depend smoothly on the detuning parameter $\delta g$. Since we consider a narrow range of values of $\delta g$, we approximate $A$ and $B$ as constants. This enables us to extract the universal functions $\Sigma_{\pm}$ on both phases by using only two fitting parameters. For further details, see the supplemental material \[28\].

The result of this analysis is shown in Fig. 3(a), where we subtract the non universal part of the conductivity using Eq. 12. The conductivity curves, on each side of the phase transition, collapse, with high accuracy, to the universal conductivity functions $\Sigma_{\pm}(\omega_m/\Delta)$. At high frequencies the universal conductivity curves saturates to a plateau, with $\sigma(\omega \gg \Delta) = 0.354(5) \sigma_q$, in the insulating phase and $\sigma(\omega \gg \Delta) = 0.355(5) \sigma_q$ in the superfluid phase. As a result, we conclude that the high frequency universal conductivity, $\sigma^*$, is a robust quantity across the phase transition. Our scaling correction analysis differs significantly from that of Refs. \[10, 11\], yet the value of the high-frequency conductivity is in agreement with their results.

Finally, we study the CVD as a function of Matsubara frequency. In Fig. 3(b) we depict the product of the Matsubara frequency conductivity evaluated at mirror points across the critical point, $\mathcal{R}(\omega_m) = \sigma(\omega_m, \delta g) \sigma(\omega_m, -\delta g)/\sigma_q^2$. In order to study the critical

![FIG. 2. The conductivity as a function of Matsubara frequency. The curves differ by the detuning parameter $\delta g$. In the insulator, the low frequency conductivity is linear, $\sigma_{\text{ins}} \sim \omega_m$ indicating capacitive behavior. In the superfluid, the conductivity diverges as $\sigma_{\text{sf}} \sim 1/\omega_m$ indicating inductive response.](image-url)
properties we subtract the non-universal cut-off corrections. Note that for \( \omega_m \gg \Delta \), \( R \rightarrow (\sigma^*)^2 \), whereas for \( \omega_m \ll \Delta \), \( R \rightarrow (\sigma_q)^2 \). In both limits, the Matsubara and real frequency results coincide, \( R(\omega) = R(\omega_m) \). By contrast, at intermediate frequencies, determination of \( R(\omega) \) requires analytical continuation. If the CVD were exact then Eq. [1] would imply that this product is frequency independent and equal to 1. Our results display a non trivial frequency dependence and deviate from the predicted CVD value. We attribute this deviation to the different interaction range of charges and vortices.

Discussion and summary - The universal ratio of the reactive conductivity \( C_{ins}/L_{4d} \) motivates future experiments as it provides a direct probe of the charge-vortex duality.

Recent THz spectroscopy measured the complex AC conductivity near the SIT in superconducting InO and NbN thin films [9]. In these systems, the superfluid stiffness in the superconducting phase can be measured from the low frequency reactive response [29, 30]. Detecting the diverging capacitance in the insulator side may require careful subtraction of substrate signal background [31].

Another experimental realization of the SIT is the Mott insulator to superfluid transition of cold atom trapped in an optical lattice. We propose a direct approach to extract the capacitance of the Mott insulator using static measurements. In the insulator, the current and charge response functions are related by the continuity equation, \( \Pi_{xx}(k,\omega) = -\frac{i}{2\pi}\chi_\rho(k,\omega) \), where \( \chi_\rho(k,\omega) \) is the charge susceptibility. Hence, 
\[
C_{ins} = \lim_{\omega \to 0} \lim_{k \to 0} \frac{\Pi_{xx}(k,\omega)}{-\omega^2} = \lim_{k \to 0} \frac{\chi_\rho(k,\omega = 0)}{k^2},
\]
where the \( \omega \to 0 \) and \( k \to 0 \) limits commute since the insulator is gapped [32]. Thus, the capacitance is simply related to the finite \( k \) compressibility of the Mott insulator. This can be measured, e.g. by applying an optical potential at small wave vector \( k \) and probing the rearrangement of boson density using in-situ imaging [33].

Temperature effects are discussed in the supplemental material [28]. Alternatively \( \sigma'(\omega) \), for which experimental protocols were proposed, [10, 34] can be used to compute \( \sigma''(\omega) \) by means of the Kramers-Kronig integral.

In summary, we computed the vortex condensate stiffness \( \rho_v \), the high frequency universal conductivity and provided a quantitative measure for deviation from CVD as a function of Matsubara frequency. In addition, we suggested concrete experiments that test our predictions in Thz spectroscopy of thin superconducting films and in cold atomic systems.

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Supplementary Material for “Critical capacitance and charge-vortex duality near the superfluid to insulator transition”

In the supplemental materials below, we explain the high frequency correction to scaling in Eq. (12) of the main text. In addition, we describe finite temperature effects on the capacitance in the insulating phase, which may prove useful for analysis of experimental data.

High frequency non-universal corrections to the dynamical conductivity

In this section we analyze, in detail, the high frequency non-universal corrections to the dynamical conductivity. The conductivity is a universal amplitude, as a result, the universal scaling function is obtained by rescaling the frequency axis \( \omega_m \) by the critical energy scale \( \Delta \). At low frequency, curves for different detuning parameters \(\delta g \) collapse into a
single universal curve. At high frequency, closer to the ultraviolet cut-off scale $\Lambda$, one expects a substantial deviation from scaling.

The correction to scaling appears only at high frequency, where $\omega_m/\Lambda$ is no longer negligible. In order to eliminate this effect we consider the following ansatz for the scaling function,

$$\sigma(\omega_m, \Delta, \delta g) = \sigma(\omega_m/\Delta, \omega_m/\Lambda)$$

(14)

Expanding in powers of $\omega_m/\Lambda$ gives,

$$\sigma(\omega_m, \Delta, \delta g) \approx \sigma_Q(\omega_m/\Delta) + A\omega_m + B\omega_m^2$$

(15)

Where we have absorbed the $\Lambda$ dependence into the non universal coefficients $A$ and $B$. Since we consider a small range of detuning parameters $\delta g$ we can take $A$, $B$ as constants. To verify our ansatz we performed a joint fit, using all frequency data points. The resulting values of $A$ and $B$ were used in the main text.

![FIG. 4. Numerical fit of the parameters $A$ and $B$ in Eq. 15. Both quantities show little variation with respect to $\Delta$ or $\delta g$. Importantly, they vary smoothly across the phase transition, as the values of $A$ and $B$ do not change significantly from the insulating (INS) phase to the superfluid (SF).](image)

**Charge susceptibility at finite temperature**

In this section we study the charge susceptibility $\chi_\rho(k, T)$ at finite temperature. The charge operator in the O(2) model is defined as $\rho = i(\phi^*\partial_t\phi - \phi\partial_t\phi^*)$.

As an illustrative example, in Fig. 5 we depict $\chi_\rho(k, T)$ for a fixed detuning parameter $\delta g = 1.12 \times 10^{-2}$ and a range of temperatures $T$. At finite temperatures the compressibility $\chi_\rho(k = 0, T)$ is non-zero. In addition, the curvature at zero momentum, $\left.\frac{\partial^2\chi_\rho(k, T)}{\partial k^2}\right|_{k=0}$, decreases at finite temperatures. Both effects should be taken into account in measurements of the capacitance at finite temperature.

The insulator is a gapped phase, therefore we expect that at low temperatures the compressibility will follow an activated behavior [35]. This is demonstrated in Fig. 6, where we plot the compressibility as a function of the inverse temperature $\beta$. The compressibility scales near the critical point as $\chi_\rho(k = 0, T) = \Delta f(\beta \Delta)$, and both axes in Fig. 6 are rescaled to obtain the collapsed universal scaling function $f$. Indeed, we find that at low temperatures, $\beta \Delta \gg 1$, the compressibility decays exponentially $\chi_\rho(k = 0, T) \sim e^{-\Delta/T}$.

We define a generalization of the capacitance to finite temperatures as the curvature of charge susceptibility at zero wave number,

$$C(T) = \left.\frac{\partial^2\chi_\rho(k, T)}{\partial k^2}\right|_{k=0}.$$  

(16)
As $T \to 0$, this coincides with the zero temperature definition. In Fig. 7 we depict the generalized capacitance as a function of the inverse temperature $\beta$. As before, both axes are rescaled to obtain the scaling function. The curves rapidly converge to the zero temperature limit, allowing for an accurate determination of the capacitance for $\beta \Delta > 6$.

As a point of reference, we compare our compressibility to that of a simple analytic calculation based on a free complex Klein-Gordon boson field with mass $\Delta$, with Euclidean action

$$ S = \int_0^\beta d\tau \int d^2 x \left[ |\partial_\tau \varphi|^2 + |\nabla \varphi|^2 + \Delta^2 |\varphi|^2 \right]. $$

(17)

This gives the leading result in a $1/N$ expansion, using the renormalized mass $\Delta$ as an input. The static charge susceptibility is then obtained from a one-loop Feynman diagram calculation, [22]

$$ \chi_\rho(k = 0, \nu_m = 0, T) = \frac{1}{\beta} \sum_{\omega_m} \int \frac{d^2 p}{(2\pi)^2} \left[ \frac{4\omega_m^2}{(\omega_m^2 + p^2 + \Delta^2)(\omega_m^2 + (k+p)^2 + \Delta^2)} - \frac{2}{\omega_m^2 + p^2 + \Delta^2} \right], $$

(18)

where the second term is the diamagnetic contribution. At $k = 0$ this gives the temperature dependent compressibility,

$$ \chi_\rho(k = 0, \nu_m = 0, T) = \int_0^\infty \frac{p \, dp}{2\pi} \frac{\beta}{\cosh(\beta \omega_p) - 1} \sim \frac{\Delta}{\pi} \left( 1 + 1/(\beta \Delta) \right) e^{-\beta \Delta}. $$

(19)

where $\omega_p = \sqrt{p^2 + \Delta^2}$, and where the last expression is asymptotic in the limit $\beta \Delta \gg 1$. This expression is shown in Fig. 7 to match closely the numerical data, despite the simplicity of the model. In addition, from Eq. (18) we compute
FIG. 7. Curvature of the charge susceptibility, $C(T) = \frac{\partial^2 \chi_{\rho}(k,T)}{\partial k^2} |_{k=0}$. For $T \to 0$, this becomes the capacitance. Axes are rescaled to obtain scaling behavior. The noise is dominated by numerical derivatives of the QMC data. The solid line shows an analytic calculation of $C$ for a free Klein-Gordon field.

The finite temperature curvature of the charge susceptibility, Eq. (16). We obtain the integral expression

$$\left. \frac{\partial^2 \chi_{\rho}}{\partial k^2} \right|_{k=0} = \int_0^\infty \frac{p dp}{96 \pi \omega_p^6} \left\{ \beta \omega_p \csch^2 \left( \frac{\beta \omega_p}{2} \right) \left[ 6 \Delta^2 + \beta^2 p^2 \omega_p^2 \right) + 2 \right) - 6 \beta \omega_p^3 \coth \left( \frac{\beta \omega_p}{2} \right) \right) \right\} + 12 \Delta^2 \coth \left( \frac{\beta \omega_p}{2} \right)$$

(20)

where $\csch x \equiv 1/\sinh x$. This is evaluated numerically in Fig. 7. In this case, the analytic result captures the qualitative temperature dependence, although it does not yield the correct overall scale.

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