Classical limit of quantum mechanics for damped driven oscillatory systems: Quantum-classical correspondence

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Abstract

The investigation of quantum-classical correspondence may lead to gain a deeper understanding of the classical limit of quantum theory. We develop a quantum formalism on the basis of a linear-invariant theorem, which gives an exact quantum-classical correspondence for damped oscillatory systems that are perturbed by an arbitrary force. Within our formalism, the quantum trajectory and expectation values of quantum observables are precisely coincide with their classical counterparts in the case where we remove the global quantum constant \( \hbar \) from their quantum results. In particular, we illustrate the correspondence of the quantum energy with the classical one in detail.

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1. Introduction

As is well known, classical mechanics (or Newtonian mechanics) is a special case of a more general theory of physics, the so-called relativistic quantum mechanics in which quantum and relativistic mechanics are merged. The intrinsic outcome of classical mechanics as a low velocity limit of relativistic mechanics has been rigorously tested and there is a common agreement for this consequence in the community of theoretical physics. On the other hand, the classical limit of quantum mechanics is a somewhat subtle problem. Planck’s $\hbar \to 0$ limit [1] and Bohr’s $n \to \infty$ limit [2] are the oldest proposals for the formulation of the classical limit of quantum theory. However, there has been controversy from the early epoch of quantum mechanics concerning this limit through different ideas and thoughts [3–9]. Accordingly, the mechanism on how to interlace the exact correspondence between the quantum and the classical theories has not yet been fully understood. Man’ko and Man’ko argued that the picture of extracting classical mechanics with the simple limitation $\hbar \to 0$ does not have universal applicability [4]. Some physicists believe that quantum mechanics is not concerned with a single particle problem but an ensemble of particles, and its $\hbar \to 0$ limit is not classical mechanics but classical statistical mechanics instead (see Ref. [5] and references therein). For more different opinions concerning the classical limit of quantum mechanics, refer in particular to Refs. [7, 8].

The purpose of this research is to establish a theoretical formalism concerning the classical limit of quantum mechanics for damped driven oscillatory systems, which reveals quantum and classical correspondence, without any approximation or assumption except for the fundamental limitation $\hbar \to 0$. Our theory is based on an invariant operator method [10–13] which is generally used for mathematically treating quantum mechanical systems. This method enables us to derive exact quantum mechanical solutions for time-varying Hamiltonian systems. We will interpret and discuss the physical meanings of our consequences in order to derive an insight for the correspondence principle.
2. Invariant-based Dynamics and Quantum Solutions

To investigate quantum-classical correspondence, we consider a damped driven harmonic oscil-
lator of mass \( m \) and frequency \( \omega_0 \), whose Hamiltonian is given by

\[
\hat{H} = e^{-\gamma t} \frac{\hat{p}^2}{2m} + \frac{1}{2} e^{\gamma t} m [\omega_0^2 \hat{q}^2 - 2f(t)\hat{q}],
\]

where \( \gamma \) is a damping constant and \( f(t) \) is a time-dependent driving force divided by \( m \). In
the case of \( f(t) = 0 \), this becomes the conventional Caldirola-Kanai (CK) Hamiltonian \([14, 15]\)
which has been widely used in a phenomenological approach for the dissipation of the damped
harmonic oscillator.

If we denote the classical solution of the system in configuration space as \( Q(t) \), it can be
written in the form \( Q(t) = Q_h(t) + Q_p(t) \) where \( Q_h(t) \) is a homogeneous solution and \( Q_p(t) \) a
particular solution. From the basic algebra in classical dynamics, we have \([16]\)

\[
Q_h(t) = Q_0 e^{-\gamma t/2} \cos(\omega t + \varphi),
\]

\[
Q_p(t) = \int_0^t \left[ f(t')/\omega \right] e^{-\gamma(t-t')/2} \sin[\omega(t-t')] dt',
\]

where \( Q_0 \) is the amplitude of the mechanical oscillation at \( t = 0 \), \( \omega \) is a modified frequency
which is \( \omega = (\omega_0^2 - \gamma^2/4)^{1/2} \), and \( \varphi \) is an arbitrary phase. The canonical classical solution
in momentum space can also be represented in a similar form: \( P(t) = P_h(t) + P_p(t) \), where
\( P_h(t) = m\dot{Q}_h(t)e^{\gamma t} \) and \( P_p(t) = m\dot{Q}_p(t)e^{\gamma t} \).

In order to describe quantum solutions of the system, it is useful to introduce an invariant
operator which is a powerful tool in elucidating mechanical properties of dynamical systems
that are expressed by a time-dependent Hamiltonian like Eq. \([1]\). A linear invariant operator
of the system can be derived by means of the Liouville-von Neumann equation and it is given
by (see Appendix A)

\[
\hat{I} = c [e^{-\gamma t/2} \hat{p}_p + m(\frac{\gamma}{2} - i\omega)e^{\gamma t/2} \hat{q}_p] e^{i\omega t},
\]

where \( \hat{p}_p = \hat{p} - P_p(t) \), \( \hat{q}_p = \hat{q} - Q_p(t) \), and \( c = (2\hbar m \omega)^{-1/2} e^{i\chi} \) with a real constant phase \( \chi \).
The eigenvalue equation of this operator can be expressed in the form

$$\hat{I}\ket{\phi} = \lambda\ket{\phi},$$

(5)

where $\lambda$ is the eigenvalue and $\ket{\phi}$ is the eigenstate. We have represented the formulae of $\lambda$ and the eigenstate $\langle q|\phi \rangle$ in the configuration space in Appendix A including detailed derivation of them.

According to the Lewis-Riesenfeld theory [10, 17], the wave function that satisfies the Schrödinger equation is closely related to the eigenstate of the invariant operator. In fact, the wave function of the system in the coherent state is represented in terms of $\langle q|\phi \rangle$ as [10]

$$\langle q|\psi \rangle = \langle q|\phi \rangle e^{i\theta(t)},$$

(6)

where $\theta(t)$ is a time-dependent phase. If we insert this equation together with Eq. (1) into the Schrödinger equation, we have $\theta(t) = -\omega t/2$. The wave function described here is necessary for investigating quantum-classical correspondence through the evolution of the system.

3. Correspondence between Quantum and Classical Trajectories

Let us now see whether the expectation values of the position and the momentum operators under this formalism agree with the corresponding classical trajectories or not. Considering that the position operator is represented in terms of $\hat{I}$ as (see Appendix A)

$$\hat{q} = i\sqrt{\hbar/(2m\omega e^{\gamma t})}[\hat{I} e^{-i(\omega t+\chi)} - \hat{I}^* e^{i(\omega t+\chi)}] + Q_p(t),$$

(7)

and using Eq. (6), we can easily verify that

$$\langle \psi|\hat{q}|\psi \rangle = Q(t).$$

(8)

Hence, quantum expectation value of the position operator is exactly the same as the classical trajectory $Q(t)$. In a similar way, the expectation value of the canonical momentum is also derived such that $\langle \psi|\hat{p}|\psi \rangle = m\dot{Q}(t)e^{\gamma t}$. However, in general, the physical momentum in a
damped system is not equivalent to the canonical one. If we define the physical momentum operator in the form $\hat{p}_k = \hat{p}e^{-\gamma t}$ [18], we readily have

$$\langle \psi | \hat{p}_k | \psi \rangle = m \dot{Q}(t) (\equiv P_k(t)),$$

where $P_k(t)$ is the classical physical momentum. We thus confirm that the linear invariant operator theory admits quantum expectation values of $\hat{q}$ and $\hat{p}_k$ in a simple manner, of which results precisely coincide with the corresponding classical values. We can regard this outcome as an initial step for verifying that the invariant formalism of quantum mechanics reconciles with the principle of quantum and classical correspondence.

The above consequence, however, does not mean that the quantum particle (oscillator) follows the exact classical trajectory that is uniquely defined. Quantum mechanics is basically non-local and there are numerous possible paths allowed, within the width of a wave packet, for a quantum particle that has a definite initial condition. It is impossible to indicate exactly which path the quantum particle actually follows, but some paths may be more likely than others, especially those close to the classically predicted path. As a consequence of the Ehrenfest’s theorem [19], the trajectory of the quantum particle can be approximated to that of the classical one only when the width of the quantum wave packet is sufficiently narrow. Details of the Ehrenfest’s theorem for a particular case of the system where the oscillator is driven by a sinusoidal force are shown in Ref. [20].

4. Quantum Energy and Its Classical Limit

As pointed out by Hen and Kalev [19] and some other authors [21], obtaining a quantum-classical correspondence from the test performed at the level concerned expectation values is the key for achieving the genuine correspondence. Hence, it is necessary to compare the expectation values of quantum observables with their counterpart classical quantities. We now analyze the expectation value of the quantum energy which is one of the most common observables in the system. Notice that quantum energy $E(t)$ for a nonconservative system is different from the
FIG. 1: Exact quantum energy (red line), quantum energy with $\hbar \to 0$ (blue line), and classical mechanical energy (circle) of the oscillating cantilever in TMAFM as a function of $t$ where $k = 0.5$, $a_0 = 0.3$, $D_0 = 0.5$, $h = 1$, $m_{\text{eff}} = 1$, $q_0 = 3$, $\gamma = 0.1$, $F_{\text{ext}} = 0.3$, and $\varphi = 0$. The values of $(\omega_0, \omega_d)$ are $(1, 0.3)$ for (A) and $(1.5, 0.6)$ for (B). All values are taken to be dimensionless for convenience; this convention will also be used in subsequent figures.

The expectation value of the Hamiltonian and the expression of the energy operator, in our case, is

$$\hat{E} = e^{-2\gamma t} \hat{p}^2 / (2m) + (1/2)m\omega_0^2 \hat{q}^2.$$  \hspace{1cm} (10)

After representing this operator in terms of $\hat{I}$ and $\hat{I}^\dagger$, we are able to evaluate the expectation value of $\hat{E}$ with the help of Eq. (6). Through this procedure, we finally have (see Appendix B)

$$E(t) = \frac{1}{2} \hbar \Omega + e^{-2\gamma t} \frac{P^2(t)}{2m} + \frac{1}{2} m\omega_0^2 Q^2(t),$$ \hspace{1cm} (11)
FIG. 2: Sawtooth driving force $f(t)$ with $f_0 = 1$, $m = 1$, and $\tau = 1$, where the mathematical formula of $f(t)$ with a period $\tau$ is defined in Appendix D. All values are taken to be dimensionless for convenience. $n$ is the natural number (see Appendix D). We have considered $n$ up to 3 for the blue dashed line and up to 1000 for the red solid line. As $n$ increases, we can have a more exact sawtooth driving force.

where $\Omega = (\omega_0^2/\omega)e^{-\gamma t}$. This is the main consequence of our present research. The first term that contains $\hbar$ is the zero-point energy that cannot be vanished even when the displacement of the oscillator is zero. The (quantum) energy is in general not conserved over time in dissipative systems like this, while it is possible to predict its amount at any given instant in time.

For better understanding of the time behavior of Eq. (11), let us consider a specific system which is the cantilever in the tapping mode atomic force microscopy (TMAFM) \cite{24}. This system is widely used as a dynamic imaging technique. For mechanical description of TMAFM, see Appendix C. The time evolutions of quantum energy for TMAFM are illustrated in Fig. 1 using Eq. (11) with its comparison to the counterpart classical one. This figure exhibits complete consistency between the quantum energy (with $\hbar \rightarrow 0$) and the corresponding classical one. We have also applied our theory to another system which is the familiar damped harmonic
FIG. 3: Exact quantum energy (violet line), quantum energy with $\hbar \to 0$ (green line), and classical mechanical energy (triangle) of the oscillator driven by the sawtooth force as a function of $t$ where $m = 1$, $\hbar = 1$, $\gamma = 0.1$, $\omega_0 = 1$, $\varphi = 0$, and $n = 1000$. The values of $(q_0, \omega_d, f_0)$ are $(3, 0.3, 1)$ for (A) and $(1, 1.2, 2)$ for (B).

oscillator driven by a periodic sawtooth force (see Appendix D and Fig. 2 for its mechanical description). Sawtooth forces or signals are typically observed from atomic force microscopy with biomolecules like proteins [25] and from a modulation of current density in a nuclear-fusion tokamak [26]. Figure 3 shows that the quantum description of this system using our theory also coincides with the classical one. We thus confirm that our formalism of quantum mechanics based on the linear invariant yields exact quantum-classical correspondence.

For further analysis, let us consider the case where the driving force disappears ($f(t) \to 0$).
We can then confirm using Eq. (2) that Eq. (11) reduces to that of Ref. [27], which is of the form

\[ E(t) = \frac{1}{2} h\Omega + E_0 e^{-\gamma t} \left( 1 + \frac{\gamma}{2\omega_0} \cos[2(\omega t + \varphi) - \delta] \right), \]

where \( E_0 = m\omega_0^2 Q_0^2 / 2 \) and \( \delta = \tan^{-1}(2\omega / \gamma) \). Except for the first term which is a purely quantum one, this is the well known formula of the classical mechanical energy for the damped harmonic oscillator. Of course, for the high displacement limit \( Q_0 \gg \hbar / (m\omega) \), it is possible to neglect the quantum effect via the use of the assumption \( \hbar \sim 0 \) and, consequently, the quantum energy can be successfully approximated to the classical one. Though the quantum energy is considered now as a model example in order to explain the correspondence principle, one can easily check, using the formalism developed here, that the analytical expectation values of other observables are also in precise congruence with their classical counterparts under the limit \( \hbar \to 0 \). For other formulae of quantum energies and their interpretation for this reduced system \( (f(t) \to 0) \), that were derived using other methods such as the SU(1,1) Lie algebraic approach, refer to Ref. [28].

5. Uncertainty and Correspondence Principle

An important feature of quantum mechanics, which distinguishes it from classical mechanics, is the appearance of minimum uncertainty product between the arbitrary two noncommutative operators. One cannot simultaneously know the values of position and momentum with arbitrary precision from a quantum measurement, while the classical theory of measurement has nothing to do with such a limitation.

The quantum variance of an observable \( \hat{O} \) in the state \( |\psi\rangle \) is given by \( \Delta \hat{O} = [\langle \psi | \hat{O}^2 |\psi\rangle - \langle \psi | \hat{O} |\psi\rangle^2]^{1/2} \). From this identity and the use of Eq. (6), we can straightforwardly derive the quantum uncertainty product for position and momentum of the system and it results in

\[ \Delta \hat{q} \Delta \hat{p} = \hbar \omega_0 / (2\omega). \]  

Because this consequence is independent of the particular solutions, \( Q_p(t) \) and \( P_p(t) \), the
driving-force does not affect on the uncertainty product. In other words, the uncertainty product of the system is the same as that of the un-driven damped harmonic oscillator $[12]$. Due to the obvious inequality $\omega_0 \geq \omega$, the uncertainty principle holds in this case. For the case $\gamma \to 0$, this uncertainty product reduces to $\bar{\hbar}/2$ which is its minimal value allowed in quantum mechanics for the harmonic oscillator. On the other hand, for $\hbar \to 0$, this becomes zero, showing the classical prediction.

6. Conclusion

The recent trend $[29, 30]$ of the re-implementation of classical mechanics in particle optics using quantum particles is a clear testimony of the close relationship between quantum and classical mechanics. Some essential knowledge of quantum information theory is developed on the basis of classical-like wave properties, while the quantum nature of a physical system is unquestionable especially when nonlocal entanglement is concerned $[31]$. It may be the very common opinion that every new physical theory should not only precisely describe facts that cannot be covered by existing theories, but must also reproduce the predictions of classical mechanics in an appropriate classical limit.

Quantum systems exhibit various nonclassical properties such as entanglement, superposition, nonlocality, and negative Wigner distribution function. While such nonclassicalities are important in the next generation quantum information science, the description of nonclassical properties is valid and reliable only when the underlying quantum formalism used in such descriptions is precise and complete. A formalism of quantum theory may be acceptable only when it gives classical results in the classical limit ($\hbar \to 0$ limit). This is the reason why complete quantum formalism that obeys quantum-classical correspondence is important. Such a formalism may admit to explaining the various characteristics of dynamical systems in a reasonable and consistent way from every possible angle. The result for a correspondence principle that we have developed in this research beyond simple static systems may provide a deep
Appendix A: Derivation of the Linear Invariant Operator

From a straightforward evaluation of the Liouville-von Neumann equation,

$$d\hat{I}/dt = \partial\hat{I}/\partial t + [\hat{I}, \hat{H}]/(i\hbar) = 0,$$  \hspace{1cm} (A1)

using the Hamiltonian given in Eq. (1) in the text, we can easily derive the linear invariant operator $\hat{I}$ that is given in Eq. (4) in the text (see Ref. [13]). Notice that the Hermitian adjoint of this operator, $\hat{I}^\dagger$, is also an invariant operator. From a combined evaluation of the two equations for $\hat{I}$ and $\hat{I}^\dagger$, it is possible to eliminate $\hat{p}$ and, as a consequence, the expression for $\hat{q}$ which appeared in Eq. (7) in the text can be obtained. From a similar method, we can also obtain the expression for $\hat{p}$. By solving the eigenvalue equation of the invariant operator, Eq. (5), in the configuration space on the basis of the technique adopted in Ref. [17], we obtain the eigenvalue as

$$\lambda = \beta e^{i\omega t},$$  \hspace{1cm} (A2)

where $\beta = -i\sqrt{m\omega/(2\hbar)}Q_0e^{-i(\omega t + \varphi - \chi)}$, and the eigenstate of the form

$$\langle q|\phi \rangle = \sqrt{m\omega/\hbar\pi} \exp \left[ e^{\gamma t/2} Aq_p - Bq_p^2 \hbar + C \right],$$  \hspace{1cm} (A3)

where $q_p = q - Q_p(t)$ and

$$A = \sqrt{2\hbar m\omega\beta},$$  \hspace{1cm} (A4)

$$B = \frac{1}{2} me^{\gamma t/2} (\omega + i\gamma/2),$$  \hspace{1cm} (A5)

$$C = \frac{i P_p(t)q}{\hbar} + \frac{\gamma t}{4} - \frac{\beta^2}{2} - \frac{|\beta|^2}{2}.$$  \hspace{1cm} (A6)
Appendix B: Expectation Value of the Energy Operator

We present how to evaluate the expectation value of the energy operator. From a minor evaluation with the energy operator using the expression of $\hat{I}$ (and its Hermitian conjugate $\hat{I}^\dagger$), it is possible to represent the energy operator in terms of $\hat{I}$ and $\hat{I}^\dagger$ such that

$$\hat{E} = \left[ \frac{\hbar}{4} \left( \frac{2\omega_0^2}{\omega} (2\hat{I}^\dagger \hat{I} + 1) - \varepsilon \hat{I}^2 - \varepsilon^* \hat{I}^\dagger \right) \right] e^{-\gamma t} + E_p,$$

where $\varepsilon = \gamma [\gamma/(2\omega) + i] e^{-2i(\omega t + \chi)}$ and

$$\Theta = \left[ \sqrt{\frac{\omega}{m}} e^{-\gamma t/2} \eta P_p(t) \right] + i e^{\gamma t/2} \left[ \sqrt{\frac{m}{\omega}} \omega_0^2 Q_p(t) \right] e^{-i(\omega t + \chi)},$$

and

$$E_p = e^{-2\gamma t} \frac{P_p^2(t)}{2m} + \frac{1}{2} m \omega_0^2 Q_p^2(t),$$

with $\eta = 1 - i\gamma/(2\omega)$. Now by considering the fact that the eigenvalues of $\hat{I}$ and $\hat{I}^\dagger$ are $\lambda$ and $\lambda^*$ respectively, we can easily identify the expectation value of the energy operator, $\langle \psi | \hat{E} | \psi \rangle$, that is given in Eq. (11) in the text. Notice that the $\hbar$ must not be taken simplistically to zero at the initial stage of the evaluation under the pretext of obtaining the classical limit. We should keep it until we arrive at the final representation, Eq. (11).

Appendix C: Cantilever System

Description of the cantilever system appears in Ref. [24]. If we denote the effective mass of the cantilever as $m_{\text{eff}}$, the force acted on the lever is represented in the form

$$f(t) = [F_{\text{ext}} + k(D_0 - a_0 \sin \omega_d t)]/m_{\text{eff}},$$

where $F_{\text{ext}}$ is the tip-sample force, $k(= m_{\text{eff}} \omega_0^2)$ is the cantilever spring constant, $D_0$ is the resting position of the cantilever base, $a_0$ is the driving amplitude, and $\omega_d$ is the drive frequency [24].
Appendix D: Damped Harmonic Oscillator with a Sawtooth Force

We regard the damped harmonic oscillator to which applied an external sawtooth force with
the period \( \tau = 2\pi/\omega_d \). The sawtooth force can be represented as \( f(t) = f_0 t/(m\tau) \) for a period
\(-\tau/2 < t < \tau/2\) (see Fig. 2), where \( f_0 \) is a constant that represents the strength of the force. In this case, \( f(t) \) can be rewritten in terms of an infinite series such that

\[
 f(t) = \left[ f_0/(\pi m) \right] \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\omega_d t). \tag{D1}
\]

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