PROCESI’S CONJECTURE ON THE FORMANEK-WEINGARTEN FUNCTION IS FALSE

MACIEJ DOŁĘGA AND JONATHAN NOVAK

Abstract. In this paper, we disprove a recent monotonicity conjecture of C. Procesi on the generating function for monotone walks on the symmetric group, an object which is equivalent to the Weingarten function of the unitary group.

1. Introduction

Let $\Gamma_d$ be the Cayley graph of the symmetric group $S(d)$ as generated by the conjugacy class of transpositions. Thus $\Gamma_d$ is a $\binom{d}{2}$-regular graded graph with levels $L_0, \ldots, L_{d-1}$, where $L_k$ is the set of permutations which factor into a product of $d-k$ disjoint cycles. Let us mark each edge of $\Gamma_d$ corresponding to the transposition $(i j)$ with $j \in \{2, \ldots, d\}$, the larger of the two symbols interchanged. This edge labeling was first considered by Stanley [8] and Biane [1] in connection with noncrossing partitions and parking functions.

A walk on $\Gamma_d$ is said to be monotone if the labels of the edges it traverses form a weakly increasing sequence. The combinatorics of such walks has been intensively studied in recent years, beginning with the discovery [4] that these trajectories play the role of Feynman diagrams for integration with respect to Haar measure on unitary groups. This is part of a broader subject nowadays known as Weingarten calculus, see [2].

Although non-obvious, it is a fact that the number of monotone walks of given length $r$ between two given permutations $\rho, \sigma \in S(d)$ depends only on the cycle type $\alpha \vdash d$ of the permutation $\rho^{-1} \sigma$. It is therefore sufficient to consider the number $m^r(\alpha)$ of $r$-step monotone walks on $\Gamma_d$ beginning at the identity permutation and ending at a fixed permutation of cycle type $\alpha$. To each partition $\alpha \vdash d$ we associate the generating function

$$M_\alpha(x) = \sum_{r=0}^{\infty} m^r(\alpha)x^r$$

(1.1)

enumerating monotone walks on $\Gamma_d$ of arbitrary length and type $\alpha$. It is known [3] that

$$M_\alpha(x) = \sum_{\lambda \vdash d} \prod_{\square \in \lambda} \frac{\lambda^\lambda_{\square}}{h(\square)(1 - c(\square)x)}$$

(1.2)

MD is supported from Narodowe Centrum Nauki, grant 2021/42/E/ST1/00162/.
JN is supported by NSF grant DMS 1812288 and a Lattimer Fellowship.
where $\chi^\lambda_\alpha$ are the irreducible characters of the symmetric group $S(d)$, with $h(\square)$ and $c(\square)$ being, respectively, the hook length and content of a given cell $\square$ in the Young diagram of $\lambda$ (see [7] for definitions). In particular, $M_\alpha(x)$ is a rational function of $x$ which may be considered as a continuous function of $x$ on the interval $(0, \frac{1}{d-1})$ whose outputs are positive rational numbers. Up to a simple rescaling, the values $M_\alpha\left(\frac{1}{d}\right)$ coincide with the values of the Weingarten function of the unitary group $U(N)$; see [3, 4].

In a recent paper [5], Procesi has pointed out that the function $M_\alpha(x)$ was also studied from the perspective of classical invariant theory by Formanek, and that in this context the values $M_\alpha\left(\frac{1}{d}\right)$ have special significance. Procesi tabulated these numbers for all diagrams $\alpha \vdash d \leq 8$, and on the basis of these computations made the following conjecture.

**Conjecture 1.1.** If $\alpha > \beta$ in lexicographic order, then $M_\alpha\left(\frac{1}{d}\right) > M_\beta\left(\frac{1}{d}\right)$.

In this brief note we give explicit numerical examples which show that Conjecture 1.1 is false.

2. Small $x$

We first clarify that Procesi’s Conjecture 1.1 refers to lexicographic order on partitions viewed as nondecreasing sequences of positive integers, with 1 the first letter in the alphabet, 2 the second letter, and so on. For example, the partitions of six listed in lexicographic order are

111111
11112
1113
1122
114
123
15
222
24
33
6,

and Conjecture 1.1 says that the numbers $M_\alpha\left(\frac{1}{d}\right)$ strictly decrease as $\alpha$ moves down this list, and this is so. However, the pattern fails for sufficiently large degree $d$.

The first sign that Conjecture 1.1 might be false in general is that it is incompatible with the known $x \to 0$ asymptotics of $M_\alpha(x)$. The minimal length of a walk on $\Gamma_d$ from the identity to a permutation of type $\alpha$ is $d - \ell(\alpha)$, and thus by parity the number $m^r(\alpha)$ can only be positive when $r = d - \ell(\alpha) + 2k$ with $k$ a nonnegative integer. We may therefore reparameterize the counts $m^r(\alpha)$ as $m_k(\alpha) := m^{d - \ell(\alpha) + 2k}(\alpha)$ for $k \in \mathbb{N}_0$. The generating function $M_\alpha(x)$ then becomes

\begin{equation}
M_\alpha(x) = x^{d - \ell(\alpha)} \sum_{k=0}^\infty m_k(\alpha)x^{2k}.
\end{equation}

It is then clear that
\[
\lim_{x \to 0} \frac{M_\beta(x)}{M_\alpha(x)} = 0
\]
whenever \(\ell(\alpha) > \ell(\beta)\), which is incompatible with lexicographic order.

One might nevertheless hope that when we compare the small \(x\) behavior of \(M_\alpha(x)\) and \(M_\beta(x)\) with \(\alpha\) and \(\beta\) being partitions of the same length, we find compatibility with lexicographic order. This too is false, as can be seen from the fact \(\ref{counterexample}\) that
\[
m_0(\alpha) = \prod_{i=1}^{\ell(\alpha)} \text{Cat}_{\alpha_i-1},
\]
where \(\text{Cat}_n = \frac{1}{n+1} {2n \choose n}\) is the Catalan number. Then for \(\alpha, \beta \vdash d\) partitions of the same length \(\ell\), we have
\[
\lim_{x \to 0} \frac{M_\beta(x)}{M_\alpha(x)} = \prod_{i=1}^{\ell} \frac{\text{Cat}_{\beta_i-1}}{\text{Cat}_{\alpha_i-1}}.
\]
For small values of \(d\), it does indeed appear to be the case that this product is smaller than 1 when \(\alpha > \beta\), but this is a law of small numbers. Consider the case where
\[
\alpha = (1, 3, \ldots, 3) \quad \text{and} \quad \beta = (2, \ldots, 2, n+1)
\]
Then \(\alpha\) and \(\beta\) are partitions of the same degree \(d = 3n + 1\), they have the same length \(\ell(\alpha) = \ell(\beta) = n + 1\), and \(\alpha\) precedes \(\beta\) in the lexicographic order. However, the ratio of the corresponding Catalan products tends to infinity as \(n \to \infty\),
\[
\frac{\text{Cat}_n}{2^n} \sim \frac{1}{\sqrt{\pi n^{3/2}}} \cdot 2^n.
\]

3. COUNTEREXAMPLES

To give a counterexample to Conjecture \(\ref{conjecture}\) itself, we return to the character formula \(\ref{character_formula}\), which in fact yields counterexamples if one goes a bit farther than the data tabulated in \(\ref{tabulated_data}\). Let \(\alpha^+\) denote the successor of \(\alpha\) in the lexicographic order. The first value of \(d\) for which Conjecture \(\ref{conjecture}\) fails is the famously unlucky number \(d = 13\), for which there exists precisely one violating pair \(\alpha, \alpha^+\). This pair is
\[
M_{(1, 5, 2)} \left( \frac{1}{13} \right) = \frac{13^{13}}{13!^2} \frac{30132115571}{1149266300} < \frac{13^{13}}{13!^2} \frac{426729597219}{16089728200} = M_{(1, 5, 24)} \left( \frac{1}{13} \right).
\]
We have tested Conjecture \(\ref{conjecture}\) for \(d \leq 20\) and it fails for all \(13 \leq d \leq 20\). Moreover the size of the set
\[
G_d := \left\{ \alpha \vdash d: M_\alpha \left( \frac{1}{d} \right) < M_{\alpha^+} \left( \frac{1}{d} \right) \right\}
\]
of consecutive failures at rank $d$ increases with $d$. For instance

$$G_{14} = \{(1^7, 7), (1^5, 2, 7), (1^5, 9)\},$$

$$G_{15} = \{(1^8, 7), (1^6, 2, 7), (1^6, 9), (1^4, 11), (1^3, 2, 10), (1^3, 3, 9)\},$$

$$G_{16} = \{(1^{11}, 5), (1^9, 7), (1^7, 2, 7), (1^7, 9), (1^6, 10), (1^5, 2^2, 7), (1^5, 11), (1^4, 2, 10),$$

$$(1^4, 3, 9), (1^3, 13), (1, 4, 11)\}.$$

Even though Conjecture 1.1 seems to fail for all $d \geq 13$ the structure of the failure set $G_d$ appears to be very interesting: it seems that when $d$ is large, the points in $G_d$ form many short lexicographic intervals and one large lexicographic interval. For instance $|G_{20}| = 45$, so the proportion of the length of a typical interval on which $M_{\alpha\beta}(\frac{1}{10})$ is monotone is equal to $\frac{1}{45}$. Nevertheless, for the interval $((1, 2^2, 4, 11), (2, 5, 13))$, whose cardinality is equal to 151, one has $((1, 2^2, 4, 11), (2, 5, 13)) \cap G_{20} = \{(2, 5, 13)\}$. The number of partitions of size 20 is 627, therefore there exists an interval on which $M_{\alpha\beta}(\frac{1}{20})$ is monotone and which is more than ten times longer than its expected length. This suggests that a weaker version of Conjecture 1.1 might be true. Let $P_d$ denote the set of partitions of size $d$.

**Question 3.1.** Is it true that there exists constant $C > 0$ such that for every positive integer $d$ there exists partitions $\alpha^d > \beta^d \in P_d$ such that for every lexicographic sequence $\alpha^d \geq \alpha > \beta \geq \beta^d$ we have $M_{\alpha\beta}(\frac{1}{d}) > M_{\beta\beta}(\frac{1}{d})$ and $\frac{|\alpha^d \beta^d|}{|P_d|} \geq C$?

We do not know the answer to this question and we leave it wide open. It would also be very interesting to find an explicit description of the set $G_d$, which appears to consists of very specific partitions which might be classifiable. Even though Conjecture 1.1 turned out to be false we believe that the research initiated by Procesi [5] on the behaviour of the function $M_{\alpha\beta}(\frac{1}{d})$ merits further investigation. Indeed, Procesi’s work has added a new and largely unexplored dimension to Weingarten calculus.

**Acknowledgments**

The SageMath computer algebra system [6] has been used for experimentation leading up to the results presented in the paper.

**References**

1. P. Biane, *Parking functions of types A and B*, Electron. J. Combin. 9 (2002), #N7.
2. B. Collins, S. Matsumoto, J. Novak, *The Weingarten calculus*, Not. Amer. Math. Soc., in press.
3. S. Matsumoto, J. Novak, *Jucys-Murphy elements and unitary matrix integrals*, Int. Math. Res. Not. IMRN 2 (2013), 362-397.
4. J. Novak, *Jucys-Murphy elements and the Weingarten function*, Banach Cent. Publ. 89 (2010), 231-235.
5. C. Procesi, *A note on the Formanek Weingarten function*, Note Mat. 41 (2021), 69-109.
6. The Sage Developers, *Sagemath, the Sage Mathematics Software System*, https://www.sagemath.org.
7. R. P. Stanley, *Enumerative Combinatorics*. Vol. 2. Cambridge University Press, New York, 1999.
8. R. P. Stanley, *Parking functions and noncrossing partitions*, Electron. J. Combin. 4 (1997), #2.
Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warszawa, Poland.
Email address: mdołega@impan.pl

Department of Mathematics, University of California, San Diego, USA
Email address: jinovak@ucsd.edu