A brief guide to p-branes

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ABSTRACT

We describe the qualitative properties of p-brane solutions of supergravity theories and present two examples of p-brane solutions, first the dyonic membrane solutions of N=2 D=8 supergravity and second the intersecting M-brane solutions of D=11 supergravity.

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1. Introduction

The last two years have seen remarkable progress towards understanding the non-perturbative properties of superstring theory. This was achieved by making a number of conjectures about the non-perturbative symmetries, called duality symmetries, of the superstring theories (see for example [1-8]). These conjectures are a generalisation of the electromagnetic duality conjecture of Montonen and Olive for Yang-Mills theories. An essential role in understanding the non-perturbative properties of superstring theories is played by their associated effective supergravity theories. This is because first the duality symmetries of superstring theory appear naturally at the level of supergravity theories, in fact they are discrete subgroups of the supergravity duality groups [2]. Second certain classical solutions of supergravity theory, called ‘p-branes’, are the analogue of monopoles and dyons solutions of Yang-Mills theories. Therefore some p-brane solutions of supergravity theories are associated with non-perturbative states in superstring theory. We shall refer to p-branes with $-1 \leq p \leq 2$ as instantons $(p = -1)$, particles $(p = 0)$, strings $(p = 1)$ and membranes $(p = 2)$.

In the first part of this paper, we will explain the qualitative properties of p-brane solutions of supergravity theory. These include the following: the general ansatz that describes a p-brane solution of a D-dimensional supergravity theory, a brief description of an energy bound in the context of p-branes, the definition of the dual $\tilde{p}$-brane of a p-brane, the direct reduction and wrapping of p-branes, and the intersection of p-branes. In the second part of the paper, we will give two examples of p-brane solutions, one will be the dyonic membranes of D=8 N=2 supergravity theory and the other will be a class of solutions of D=11 supergravity with the interpretation of intersecting membranes and 5-branes (M-branes).
2. Qualitative properties of $p$-branes

2.1. Supergravity

The effective theory for the massless modes of a D-dimensional superstring theory is a D-dimensional supergravity theory. The bosonic part of the Lagrangian of such supergravity theory is

$$\mathcal{L} = \mathcal{L}(g, F, \phi)$$

(2.1)

where $g$ is the spacetime metric, $F = \{F^\alpha; \alpha = 1, \ldots, m\}$ are $(p_\alpha + 2)$-form field strengths and $\phi$ are sigma model matter fields with target space a $G/H$ coset. The group $G$ acts on the scalars by its standard left action on the coset $G/H$. It also acts on the field strengths $F$ and their Poincaré duals with some representation in such way that the field equations of the supergravity theory are invariant. (In the full theory the fermions also transform under the action of the group $G$). In what follows, apart from the bosonic sector of the D-dimensional supergravity theory above, we shall use the supersymmetry transformations of the gravitini, $\psi$, and the other spin $1/2$ fermions, $\lambda$, of the theory. Let $D$ be the covariant derivative of the spin connection of the metric $g$. The supersymmetry transformations of $\psi$ and $\lambda$ evaluated at a background for which all fermions vanish takes the form

$$\delta\psi \equiv \hat{D}\eta = D(g, \omega)\eta + T(g, F, \phi)\eta$$
$$\delta\lambda = L(g, \phi, F)\eta,$$

(2.2)

where $\eta$ are the supersymmetry parameters, and $T, L$ are matrices that depend upon the fields of the theory. The action of the group $G$ on the various fields of the supergravity theory induces an action on the supersymmetry transformations.
(2.2). Under this action, the equations (2.2) transform in some representation of $G$ provided that the supersymmetry parameters $\eta$ are transformed in a suitable way. This fact will be used in section 2.3 to argue for the invariance of an energy bound under the action of the group $G$.

The vacua of the above supergravity theory are parameterised by the $G/H$ coset. Using the Dirac quantisation for the charges of the $p$-brane solutions of (2.1), one can argue that the symmetry $G$ of the field equations of the theory is quantum mechanically broken to the discrete subgroup $G(\mathbb{Z})$. The U-duality conjecture then involves the assertion that $G(\mathbb{Z})$ is a symmetry of the full associated superstring theory [2]. The vacua of the theory that lie in the orbits of $G(\mathbb{Z})$ acting on $G/H$ are identified, i.e. quantum mechanically the space of vacua is $G(\mathbb{Z})\backslash G/H$.

2.2. $p$-BRANES

We are seeking solutions of supergravity theories that have the interpretation of parallel infinite planar $p$-dimensional extended objects located in a D-dimensional spacetime, i.e. the spacetime is foliated with parallel leaves that are isometric to $(p+1)$-dimensional Minkowski spacetime. We shall refer to such solutions as ‘$p$-branes’. We shall also require that such solutions should have a well defined notion of mass $M$ and charge $p$ per unit volume, and that they should be extreme, i.e. $M = |p|$. Since the $p$-brane carries charge $p$, it couples naturally to a $(p + 1)$-form gauge potential $A$. Note that apart from the non-trivial spacetime metric $g$ and $(p + 2)$-form field strength $F = dA$ which are necessary to describe a $p$-brane solution, the $p$-brane solution may include some non-constant scalar fields $\phi$.

The above description of the $p$-brane solutions of a supergravity theory requires that they must have a $(p + 1)$-dimensional Poincaré invariance; the Poincaré group acts on the co-ordinates $\{x^\mu; \mu = 0, \ldots, p\}$ which are identified with the world-volume co-ordinates of the $p$-brane. Let $\{y^i; i = 1, \ldots, D_\perp \equiv D - p - 1\}$ be the transverse co-ordinates to the $p$-brane in the D-dimensional spacetime. Then we
have
\[ D = D_⊥ + p + 1 . \] (2.3)

An ansatz for a \( p \)-brane solution of a supergravity theory is the following:
\[
\begin{align*}
  ds^2 &= A^2(y)ds^2(\mathbb{M}^{p+1}) + B^2(y)ds^2(\mathbb{E}^{D⊥}), \\
  F &= \epsilon_{(p+1)} \wedge dC(y) + \xi(y), \\
  \phi &= \varphi(y),
\end{align*}
\] (2.4)

where \( ds^2(\mathbb{M}^{p+1}) \) is the Minkowski metric, \( ds^2(\mathbb{E}^{D⊥}) \) is the Euclidean metric, \( \epsilon_{(p+1)} \) is the volume form on \( \mathbb{M}^{p+1} \), \( A, B \) and \( C \) are functions of the transverse co-ordinates \( y \) and \( \xi \) is a closed \((p+2)\)-form on the transverse space.

The ansatz (2.4) involves several unspecified functions that are determined by solving the Killing spinors equations,
\[ \hat{D}\eta = 0 \]
\[ L(e, \phi, F)\eta = 0 , \] (2.5)

and some of the field equations. The Killing spinor equations are the vanishing conditions of the supersymmetry transformations (2.2) and as we shall explain in the next section are necessary conditions for the \( p \)-brane solutions to be extreme. The \( p \)-brane solutions are then determined in terms of harmonic functions on \( \mathbb{E}^{D⊥} \); the unknown functions of the metric \( ds^2 \) and scalar fields \( \phi \) in (2.4) are usually expressed in terms of powers of harmonic functions, and the field strength \( F \) is expressed in terms of the same harmonic functions and their derivatives. The harmonic functions on Euclidean spaces have point singularities that are interpreted as the positions of the \( p \)-branes.

\* The part of the metric that involves the function \( B \), as stated, does not follow from the requirements that we have imposed on the \( p \)-brane solutions but it turns out that the known \( p \)-brane solutions are always of this form.
The asymptotic behaviour of the metric of a $p$-brane solution as $|y| \to \infty$ depends upon the dimension $D_\perp$ of the transverse space to the $p$-brane. If $D_\perp > 2$, it can be arranged that the spacetime is asymptotically Minkowski. This is due to the fact that the harmonic functions on $\mathbb{R}^{D_\perp}$ for $D_\perp > 2$ can be chosen such that they approach to a constant as $|y| \to \infty$. For $D_\perp \leq 2$ the spacetime exhibits different asymptotic behaviour. In the $D_\perp = 2$ case, either the spacetime is asymptotically $M^{p+1} \times Y$, where $Y$ is a two-dimensional conical space, or the metric exhibits logarithmic behaviour as $|y| \to \infty$ [9,10,11]. For $D_\perp = 1$, the $(D-2)$-brane separates the $D$-dimensional spacetime into two asymptotic regions like a domain wall [12,13,11]. Finally $(D-1)$-branes solutions are identified with the Minkowski vacuum of the associated supergravity theory. The behaviour of the metric at the positions of the $p$-branes is also of interest. The metric is singular at those positions but in some cases these singularities are merely co-ordinate singularities. In addition, if the relevant solution has non-constant scalar fields, then the nature of the singularity at the positions of the $p$-branes depends upon the choice of the frame for the metric. Nevertheless one can deduce important information by studying the singularity structure of the $p$-branes at their positions. For example $p$-brane solutions with time-like naked singularities (in their natural frame) are usually interpreted as fundamental objects while non-singular solutions are interpreted as $p$-brane monopoles and dyons.

2.3. Energy per unit volume bound

As we have already mentioned in the previous section, the $p$-brane solutions of supergravity theories saturate an energy per unit volume bound. Here we shall describe how such a bound can be derived (see refs [14-17]). For this we define the modified Nester tensor

$$\hat{E}^{MN} = \frac{1}{2} \kappa \Gamma^{MNR} \hat{D}_R \kappa + c.c$$

(2.6)

where $\kappa$ is a commuting spinor and $\hat{D}$ is the operator that appears in the gravitini
supersymmetry transformation law (2.2). Then one can show that

\[ D_N \tilde{E}^{MN} = \tilde{D}_N \kappa \Gamma^{MNR} \tilde{D}_R \kappa - \frac{1}{2} \chi \Gamma^M \chi \]  

(2.7)

where \( \chi \) are proportional to the expressions that appear in the supersymmetry transformations (2.2) of the fermions, \( \lambda \) (with \( \eta \), the supersymmetry parameter, replaced by \( \kappa \)).

Next we shall consider a spacetime that asymptotically describes an extended object as we have discussed in the previous section, i.e. one can distinguish the directions along an extended object and the directions transverse to it. Therefore the spacetime is asymptotically \( \mathbb{M}^{p+1} \times \mathbb{E}^{D_\perp} \) along directions transverse to the extended object. (We take here \( D_\perp \geq 3 \), the case for which \( D_\perp \leq 2 \) will be discussed at the end of the section). Next we assume that the \( p \)-brane is wrapped around a large torus, \( \mathbb{T}^p \), so the topology of the spacetime at spatial infinity along the transverse directions (‘transverse spatial infinity’) is \( \mathbb{T}^p \times S^{D_\perp - 1} \). Using this and appropriate decay condition for the other fields, we can define the transverse \( D_\perp \) momentum \( P \) and charge \( q \) per unit volume of such spacetime as follows:

\[ \bar{\kappa}_\infty \Gamma \cdot P \kappa_\infty \equiv \frac{1}{2 V_p \Omega_{D_\perp - 1}} \int_\infty dS^{MN} E_{MN} = \frac{1}{2 \Omega_{D_\perp - 1}} \int_{S^{D_\perp - 1}} dS^{ij} E_{ij} \]  

(2.8)

and

\[ q \equiv \frac{1}{V_p \Omega_{D_\perp - 1}} \int_\infty \tilde{F} = \frac{1}{\Omega_{D_\perp - 1}} \int_{S^{D_\perp - 1}} \tilde{F} , \]  

(2.9)

where \( E \) is the Nester tensor associated with the spin connection of the metric, \( \kappa_\infty \) is the asymptotic value of the spinor \( \kappa \), \( V_p \) is the volume of the unit \( p \)-torus, \( \Omega_{D_\perp - 1} \) is the volume of \( (D_\perp - 1) \)-sphere with unit radius and \( \tilde{F} \) is the Hodge dual.*

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* The definition of the ‘electric’ charge involves the electro-magnetic dual field strength \( G \) of \( F \) rather than its Hodge dual \( \tilde{F} \) that we have used here for simplicity; see for example section 3.
of $F$. Then, one can show that

$$\bar{\kappa}_\infty K \kappa_\infty = \frac{1}{V_p \Omega_{D-1}} \int dS^{MN} \hat{E}_{MN} \geq 0$$

(2.10)

where $K$ is

$$K = \Gamma \cdot P + Z(\langle \phi \rangle, q),$$

(2.11)

and $\langle \phi \rangle$ are the asymptotic values of the scalars $\phi$. The equality in (2.10) can be established by following the definition of the various quantities and $Z$ can be easily determined from the modified Nester tensor. To show the inequality in (2.10) is more involved; for this one makes use of the (2.7) and the ‘Witten condition’ [14]. Finally from (2.11) and (2.10) one can establish a bound for the mass per unit volume of the spacetime in terms of its charges per unit volume. The invariance of the bound under the $G$ group action is an immediate consequence of the invariance of the Nester tensor $\hat{E}$. The latter follows from the transformation properties of the supersymmetry transformations (2.2) under the action of the group $G$. It is clear from (2.7) that a configuration saturates the bound provided that it satisfies the Killing spinor equations (2.5) (with $\eta$, the supersymmetry parameter, replaced by $\kappa$).

Now we turn to examine the case where $D_\perp \leq 2$. For $D_\perp = 2$, many steps in the proof of the bound remain the same. However if one assumes that the asymptotic behaviour of the spacetime along the transverse directions is $\mathbb{R} \times \mathbb{T}^{D-3} \times Y$ where $Y$ is a two-dimensional conical space, then for all configurations that satisfy the Witten condition the bound is always saturated. This follows from a slight modification of the proof of an energy theorem in $2 + 1$ dimensions given in [18]. The $D_\perp = 1$ case involves a spacetime which is topologically $\mathbb{R} \times \mathbb{T}^{D-2} \times \mathbb{R}$ as one approaches the transverse spatial infinity. The transverse spatial infinity in this case is just the disjoint union of two $\mathbb{T}^{D-2}$ and the calculation of the mass and charge per volume simply involves the evaluation of the corresponding expressions at two points at infinity.
2.4. Dual and dyonic $p$-branes

The dual of a $p$-brane coupled to a $(p+1)$-form potential, $A$, in a D-dimensional spacetime, is a $\tilde{p}$-brane coupled to a $(\tilde{p} + 1)$-form potential, $\tilde{A}$, such that the form field strength $G = d\tilde{A}$ has rank $D - p - 2$ and it is related to the Poincaré dual of the $(p + 2)$-form field strength $F = dA^\ast$. It follows from this definition that

$$\tilde{p} = D - p - 4 .$$

(2.12)

It is clear that if the $p$-brane carries ‘electric’ charge $q$, its dual $\tilde{p}$-brane carries ‘magnetic’ charge $p$, where

$$p = \frac{1}{\Omega_{\tilde{D}_\perp -1}} \int_{S^{\tilde{D}_\perp -1}} F ;$$

(2.13)

$\tilde{D}_\perp$ is the number of transverse directions to the $\tilde{p}$-brane.

The possibility arises for dyonic $p$-branes, i.e. those that carry both electric and magnetic charges. For such $p$-branes, $p = \tilde{p}$, and from (2.12), we get

$$p = \frac{D}{2} - 2 .$$

(2.14)

So one gets dyons in $D=4$ [20], dyonic strings in $D=6$ [21,22], dyonic membranes in $D=8$ [17, 23], and finally dyonic 3-branes in ten dimensions [24].

* The precise definition of $G$ can be somewhat complicated and it depends on the way that Bianchi identities and field equations are interchanged under electric-magnetic duality.
2.5. Direct reduction and wrapping of $p$-branes

Given $p$-brane solutions of a supergravity theory in $D$ dimensions, one can construct new $q$-brane solutions in $D - k$ dimensions using ‘direct reduction’ and ‘wrapping’ of $p$-branes. The first procedure involves Kaluza-Klein (KK) reduction in $k$ directions on the co-ordinates of the transverse space. Consistency with KK ansatz requires that the harmonic functions should be independent from these directions. With this procedure one gets a $p$-brane in $D - k$ dimensions. On the other hand wrapping involves KK reduction on $k$ space-like directions of the worldvolume of the $p$-brane. The resulting solution is a $(p - k)$-brane in $D - k$ dimensions. It is also possible to employ simultaneous direct reduction and wrapping. In which case if one wraps a $p$-brane in $k_1$ directions and directly reduces it in $k_2$ directions, the result is a $(p - k_1)$-brane in $d - k$ dimensions, $k = k_1 + k_2$. The above reduction methods for $p$-branes are particularly simple to perform in the context torus compactifications of supergravity theories but they can also be extended to other compactifications of supergravity theories, like $K_3$ and Calabi-Yau ones. The wrapping of $p$-branes, in the latter case, is around the various co-homology co-cycles of the compactifying space. The direction of direct reduction and wrapping of $p$-branes can be reversed. Consequently one can lift $p$-brane solutions of supergravity theories from lower dimensions to higher ones.

2.6. Intersecting $p$-branes

So far we have studied solutions of supergravity theories that have the interpretation as parallel $p$-branes in a $D$-dimensional spacetime. There are, however, solutions of supergravity theories with somewhat different interpretation as intersections of two or more orthogonal $p$-branes. The novelty is that such configurations may be extreme and preserve some of the supersymmetry\(^\dagger\). There are many ways to approach this topic [25-29]; here we will follow [30]. Consider the orthogonal intersection of $p_{\alpha}$-branes, $\alpha = 1, \ldots, L$, on a $r$-brane in $D$ dimensions. The Poincaré

\(^\dagger\) They may though exhibit different asymptotic behaviour from that of parallel $p$-branes.
invariance of the worldvolume of all $L$ orthogonal $p_\alpha$-branes will be broken to that of the $r$-brane that lies in the intersection. Therefore the corresponding solutions will have $(r + 1)$-dimensional Poincaré invariance. The normal bundle of the $r$-brane imbedded in $(D - 1)$-dimensional space can be decomposed into $\ell$ directions along the tangent bundles of the $p_\alpha$-branes, and into the remaining $D_\perp$ directions.†

We shall refer to the former as the ‘relative’ transverse directions and to the latter as the ‘overall’ transverse directions. It is clear that

$$D = r + \ell + D_\perp + 1 .$$  \hspace{1cm} (2.15)

Let $\{x^\mu; \mu = 0, \ldots, r\}$ be the worldvolume coordinates of the $r$-brane, $\{u^a; a = 1, \ldots, \ell\}$ be the co-ordinates along the relative transverse directions and $\{y^i; i = 1, \ldots, D_\perp\}$ be the co-ordinates along the overall transverse directions. Then the ansatz of a spacetime metric that describes $L$ intersecting $p_\alpha$-branes, $\alpha = 1, \ldots, L$, is as follows:

$$ds^2 = A^2(u, y)ds^2(M^{r+1}) + B_{ab}(u, y)du^a du^b + C_{ij}(u, y)dy^i dy^j .$$  \hspace{1cm} (2.16)

The functions $A, B, C$ are determined by solving killing spinor and field equations. It is expected§ that if such solution preserves some supersymmetry the proportion of supersymmetry preserved is $1/2^L$.

To find the (magnetic) dual configuration in $D$ dimensions of $L$ orthogonal $p_\alpha$-branes intersecting on a $r$-brane with number of relative transverse directions $\ell$, we wrap the configuration to $d \equiv (D - \ell)$-dimensions along the relative transverse directions. In $d$ dimensions the solution becomes an $r$-brane solution and from

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† We have used the same symbol, $D_\perp$, to denote both the number of transverse directions of a $p$-brane and the number of overall transverse directions of $L$ intersecting $p_\alpha$-branes. The distinction between the two uses of it will be clear from the context.

§ In the limiting case of the intersection of a membrane and a 5-brane with one to include the other (see section 4), the proportion of the supersymmetry preserved is $1/2$. 

its magnetic dual is a $\tilde{r}$-brane with

$$\tilde{r} = d - r - 4 . \quad (2.17)$$

Now the magnetic dual configuration of the $L$ orthogonal $p_\lambda$-branes intersecting on a $r$-brane (in $D$ dimensions) should reduce to the $\tilde{r}$-brane after wrapping along $\ell$ relative transverse directions to $d$ dimensions. This implies that

$$D = \tilde{r} + \ell + \tilde{D}_\perp + 1, \quad (2.18)$$

where $\tilde{D}_\perp$ is the number of overall transverse directions of the magnetic dual configuration. The equations (2.17) and (2.18) can be solved for $\tilde{r}$ and $\tilde{D}_\perp$ as follows:

$$\tilde{r} = D - \ell - r - 4$$

$$\tilde{D}_\perp = r + 3 . \quad (2.19)$$

Note that the equations which can be obtained from (2.19) by exchanging $(r, D_\perp)$ with $(\tilde{r}, \tilde{D}_\perp)$ are also valid.

### 3. Dyonic membranes

As we have already mentioned membranes in eight dimensions ($D = 8$) can carry both electric and magnetic charges. The possibility then arises for the existence of dyonic membrane solutions in a D=8 supergravity. Such solutions have already been found in [17] in the context of N=2 D=8 supergravity [31]. The relevant Lagrangian is a consistent truncation of the D=8 supergravity Lagrangian with fields the metric $g$, two scalars $\sigma$ and $\rho$ and a 4-form field strength $F$. The truncated Lagrangian is as follows:

$$\mathcal{L} = N \left\{ \sqrt{-g} \left[ R - 2 \partial_\mu \sigma \partial^\mu \sigma - 2 e^{4\sigma} \partial_\mu \rho \partial^\mu \rho - \frac{1}{12} e^{-2\sigma} F_{\alpha\beta\gamma\delta} F^{\alpha\beta\gamma\delta} \right] ight\}$$

$$- \frac{1}{144} \varepsilon^{\mu\nu\rho\sigma\alpha\beta\gamma\delta} \rho F_{\mu\nu\rho\sigma} F^{\alpha\beta\gamma\delta} \right\}, \quad (3.1)$$

for some normalisation factor $N$. The scalar fields $\sigma$ and $\rho$ take values in the coset
\(SL(2, \mathbb{R})/U(1)\) and it is convenient to define the fields

\[
\begin{align*}
\lambda &= 2\rho + ie^{-2\sigma} \\
G &= e^{-\sigma} \tilde{F} - 2\rho F ,
\end{align*}
\]

where \(\tilde{F}\) is the Poincaré dual of \(F\). The new field \(\lambda\) transforms with fractional linear transformations under \(SL(2, \mathbb{R})\) while the pair \((F, G)\) transforms as a doublet under the same group. The field equations of this theory are invariant under this action of \(SL(2, \mathbb{R})\).

We shall be interested in membrane solutions of the equations of motion of (3.1) that are asymptotically flat as one approaches spatial infinity in transverse directions. The transverse spatial infinity in this case is topologically \(S^4 \times \mathbb{R}^2\). The dyonic membrane solutions can be characterised by their magnetic, \(p\), and electric, \(q\), charges. These charges, after an appropriate choice for the normalisation constant \(N\) of the Lagrangian and for the unit of electric charge, are

\[
p = \frac{1}{\Omega_4} \int_{S^4} F , \quad q = \frac{1}{\Omega_4} \int_{S^4} G ,
\]

where the integral is over a 4-sphere cross-section of transverse spatial infinity and \(\Omega_4\) is the volume of the 4-sphere with unit radius. The dyonic membrane solutions of the field equations of (3.1) are [17]

\[
ds^2 = H^{-\frac{1}{2}} ds^2(M^3) + H^\frac{1}{2} ds^2(\mathbb{E}^5) \\
F = \frac{1}{2} e^{\langle \sigma \rangle} \left( \cos \psi \ast dH + \sin \psi \ dH^{-1} \wedge \epsilon_3 \right) \\
\lambda = 2\langle \rho \rangle + e^{-2\langle \sigma \rangle} \cdot \frac{(1 - H) \sin 2\psi + 2i H^\frac{3}{2}}{2(H \cos^2 \psi + \sin^2 \psi)} ,
\]

where star is the Hodge star in \(\mathbb{E}^5\), \(H\) is a harmonic function on \(\mathbb{E}^5\), \(\psi\), \(\langle \rho \rangle\) and \(\langle \sigma \rangle\) are parameters of the solution, and \(\epsilon_3\) is the volume form in \(M^3\). As \(|y| \to \infty\), \(y \in\)
\[ \mathbb{E}^5 \], the metric approaches that of Minkowski spacetime and the field \( \lambda \) approaches the vacuum

\[ \langle \lambda \rangle = 2\langle \rho \rangle + ie^{-2\langle \sigma \rangle}. \]  

(3.5)

The solutions (3.4) preserve half the supersymmetry. The purely ‘electric’ solution in the vacuum \( \langle \lambda \rangle = i \) can be identified as that for which \( \cos \psi = 0 \) in which case the purely ‘magnetic’ one is that with \( \sin \psi = 0 \).

Following the steps described in the previous section, we can derive the bound

\[ M^2 \geq \frac{1}{4} \left[ e^{2\langle \sigma \rangle} (q + 2\langle \rho \rangle p)^2 + e^{-2\langle \sigma \rangle} p^2 \right], \]  

(3.6)

on the mass per unit volume, \( M \); for this we have used

\[ \hat{D}_\kappa = D_\kappa - \frac{1}{2} \Gamma_{\kappa \sigma} e^{2\sigma} d\rho + \frac{1}{96} \Gamma_{\alpha \beta \gamma \delta} \Gamma^{\kappa \sigma} F_{\alpha \beta \gamma \delta}. \]  

(3.7)

This bound is \( SL(2, \mathbb{R}) \) invariant and the dyonic membrane solutions (3.4) saturate this bound as it is easy to show by direct computation.

Dyonic membranes have similar quantum properties to those of dyons. Indeed combining the Nepomechie-Teitelboim (N-T) quantization condition for extended objects with Schwinger-Zwanziger quantization condition for dyons, one can find an analogue of the latter that applies in the context of dyonic \( p \)-branes. This generalized N-T quantization condition for two dyonic membranes with charges \( (p, q) \) and \( (p', q') \) takes the simple (manifestly \( SL(2; \mathbb{R}) \) invariant) form

\[ qp' - q'p \in \mathbb{Z}. \]  

(3.8)

As for dyons in D=4 [32], this formula allows fractional \( q \) for dyonic membranes.

The duality group of the D=8 N=2 supergravity is \( G = SL(3; \mathbb{R}) \times SL(2; \mathbb{R}) \). The 4-form field strength \( F \), and its dual, transform under the \((1,2)\) representation of the supergravity duality group. The U-duality group in this case is
\( G(\mathbb{Z}) = SL(3; \mathbb{Z}) \times SL(2; \mathbb{Z}) \) and includes the T-duality group \( SO(2, 2; \mathbb{Z}) \) of D=8 type II superstring as a proper subgroup. Despite the fact that T-duality is a perturbative symmetry in the context of superstring theory, it acts on \( F \) via a generalised electromagnetic duality transformation. For a more detail discussion of the applications of dyonic membranes in superstring theory see [17].

4. Intersecting M-branes

As we have mentioned, certain solutions of supergravity theories have the interpretation of intersecting \( p \)-branes. Examples of such solutions can be found in D=11 supergravity. The D=11 supergravity has bosonic field content a metric \( g_{(11)} \) and a 4-form field strength \( F_{(11)} \). There are two well known \( p \)-brane solutions of D=11 supergravity the membrane [33] and 5-brane [34]. We will refer to them collectively as M-branes. However apart from these two solutions, D=11 supergravity has additional solutions that solve the Killing spinor equations. A class of such solutions given in [34] is the following:

\[
\begin{align*}
    ds_{(11)}^2 &= -H^{-\frac{2n}{3}} dt^2 + H^{\frac{2n}{3}} d\Sigma^2(\Sigma^{2n}) + H^{\frac{4}{3}} d\Sigma^2(\Sigma^{10-2n}) \\
    F_{(11)} &= -3dt \wedge dH^{-1} \wedge J,
\end{align*}
\]

(4.1)

where \( J \) is a ‘constant’ complex structure in \( \Sigma^{2n} \) and \( n = 1, 2, 3 \). For \( n = 1 \), we get the membrane solution of D=11 supergravity [33]. The remaining two solutions \( (n = 2, 3) \) can be thought as the geometric intersections of two and three membranes at a 0-brane correspondingly [30]; the relative transverse space is \( \Sigma^{2n} \) and the overall transverse space is \( \Sigma^{10-2n} \). The proportion of the supersymmetry preserved by the solutions (4.1) is \( 1/2^n \).

Using this interpretation for the solutions (4.1) and the general arguments about intersecting \( p \)-branes in section 2.6, one can immediately deduce that their
magnetic duals are again intersecting $p$-branes with

$$\tilde{r} = 7 - 2n$$
$$\tilde{D}_\perp = 3 .$$

(4.2)

So for $n = 1$ one gets precisely the 5-brane which is known to be the magnetic dual of membrane. The remaining cases, $n = 2, 3$, can be thought of as two 5-branes intersecting at a 3-brane and three 5-branes intersecting at a string, respectively. Indeed the D=11 supergravity solutions that are the magnetic duals of (4.1) are the following [30]:

$$ds^2_{(11)} = -H^{-\frac{n}{3}} ds^2(M^{8-2n}) + H^{-\frac{n+3}{3}} ds^2(E^{2n}) + H^{\frac{2n}{3}} ds^2(E^3)$$
$$F_{(11)} = \pm 3 \ast dH \wedge J ,$$

(4.3)

where $H$ is a harmonic function of $E^3$, star is the Hodge star in $E^3$ and $n = 1, 2, 3$. For $n = 1$, one gets a special case of the 5-brane solution of D=11 supergravity. (In the 5-brane solution of D=11 supergravity the harmonic function $H$ is a function of $E^5$.) The remaining two solutions are the magnetic duals of the intersecting membrane solutions (4.1) as they have been described above. The proportion of the supersymmetry preserved by the solutions (4.3) is $1/2^n$. Another solution of D=11 supergravity can be found by lifting the dyonic membrane solutions of the previous section to D=11. The D=11 interpretation of the lifted solution is of a membrane lying within a 5-brane [17] and preserve $1/2$ of the supersymmetry. Finally, there are three more known $p$-brane-like solutions of D=11 supergravity the following [35]: (i) a membrane and 5-brane intersecting at a string (preserving 1/4 of the supersymmetry), (ii) two membranes and a 5-brane intersecting at particle (preserving 1/8 of the supersymmetry), and (iii) a membrane and two 5-branes intersecting at a particle (preserving 1/8 of the supersymmetry).

To study the solutions that one gets in D=10 from reducing the $p$-brane-like solutions of D=11 supergravity, we denote with $(r|p_1, \ldots, p_L)$ the solutions of a supergravity theory that have the interpretation of $L$ intersecting $p_\alpha$-branes,
$\alpha = 1, \ldots, L$, with common intersection a $r$-brane. For example $(2|0)$ will denote a membrane solution while $(0|2, 2)$ will denote two intersecting membranes at a 0-brane. There are three different ways to reduce a intersecting $p$-brane solution the following: (i) wrapping the solution along the common intersecting $r$-brane, (ii) wrapping the solution along one of the relative transverse directions and (iii) directly reducing the solution along one of the overall transverse directions. Using the above notation, we denote the $p$-brane-like solutions of D=11 supergravity as follows: $(2|0), (5|0), (2|2, 5), (0|2, 2), (3|5, 5), (1|2, 5), (0|2, 2, 2), (1|5, 5, 5), (0|2, 2, 5)$ and $(0|2, 5, 5)$. Direct reduction on $S^1$ along the overall transverse directions will produce solutions in D=10 with exactly the same interpretation as the $p$-brane-like solutions in D=11. On the other hand the wrapping of solutions along one relative transverse direction does not apply to the membrane and 5-brane solutions but the remaining solutions reduce to D=10 as follows: $(2|2, 4), (0|1, 2), (3|4, 5), (1|1, 5), (1|2, 4) (0|1, 2, 2), (1|4, 5, 5), (0|1, 2, 5), (0|2, 2, 4), (0|1, 5, 5)$ and $(0|2, 4, 5)$. Finally the wrapping of solutions along the $r$-brane lying in the common intersection does not apply to the D=11 M-brane solutions intersecting at a particle but the remaining solutions reduce to D=10 as follows: $(1|0), (4|0), (1|1, 4), (2|4, 4), (0|1, 4)$ and $(0|4, 4, 4)$. It is worth mentioning that the $p$-brane-like solutions of D=11 supergravity reduce to $a = \sqrt{3}, 1, 1/\sqrt{3}$ (electric and magnetic) black hole solutions of D=4 N=8 supergravity [30]. The $a = 0$ D=4 black-hole also has a D=11 interpretation but the relevant D=11 solution involves KK vectors [30, 35].

5. Concluding Remarks

We have presented the qualitative properties of solutions of supergravity theories that saturate a certain bound and consequently satisfy a Killing spinor equation. Such solutions have a $p$-brane interpretation and are the analogues of monopoles and dyons of Yang-Mills theories in the context of supergravity theories. We have also briefly explained the use of the $p$-brane solutions of supergravity theories in the study of non-perturbative superstring theory. The qualitative properties
of $p$-brane solutions that have been described are the following: the general ansatz for constructing such solutions, an energy bound, the ‘magnetic’ dual $\tilde{p}$-brane of a $p$-brane, the direct reduction and wrapping of $p$-branes and finally the intersection of $p$-branes.

Some $p$-brane solutions of $D < 11$ supergravity theories can be obtained from the $p$-brane-like solutions of $D=11$ supergravity using direct reduction and wrapping (there are no supergravity theories in $D > 11$). It is therefore of interest to understand the $p$-brane-like solutions of the $D=11$ supergravity theory. In addition, $D=11$ supergravity is the effective theory of a conjectured M-theory; M-theory upon $S^1$ reduction to $D=10$ gives the type IIA superstring [1,7], and upon $S^1/\mathbb{Z}_2$ reduction to $D=10$ gives the $E_8 \times E_8$ heterotic string [36]. The solutions of $D=11$ supergravity that have been found, so far, with a $p$-brane-like interpretation are the membrane and 5-brane solutions [33, 34] and their intersections [34,30,35]. The latter include the intersection of two and three membranes at a particle, the intersection of a membrane and a 5-brane at a string, the intersection of two membranes and a 5-brane at a particle, the intersection of a membrane and two 5-branes at particle, and the intersection of two and three 5-branes at a 3-brane and a string, respectively. There is also another solution that has the interpretation of a membrane lying within a 5-brane [17]. It may be that this is not the whole story. For example, there has been a classification of all $p$-brane solutions in maximal supergravities [37]. These solutions can be lifted and may produce new $p$-brane-like solutions in $D=11$. Although the emphasis in this paper was on $p$-brane solutions of supergravity theories, a certain class of $p$-branes those that couple to Ramond-Ramond fields may have an explanation within the context of superstring theory as D-branes. For this, one should allow strings with Dirichlet boundary conditions in $D − p − 1$ space-like directions in spacetime; the $p + 1$ directions span the worldvolume of a D-brane (for a review see [38]). It is likely that intersecting D-brane configurations are related to some $p$-brane-like configurations of $D=11$ supergravity.
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