Density of Polynomials in Sub-Bergman Hilbert Spaces

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Abstract. The sub-Bergman Hilbert spaces are analogues of de Branges-Rovnyak spaces in the Bergman space setting. We prove that the polynomials are dense in sub-Bergman Hilbert spaces. This answers the question posted by Zhu in the affirmative.

1. Introduction

Let $A$ be a bounded operator on a Hilbert space $H$. We define the range space $\mathcal{M}(A) = AH$, and endow it with the inner product

$$\langle Af, Ag \rangle_{\mathcal{M}(A)} = \langle f, g \rangle_H, \quad f, g \in H \ominus \text{Ker}A.$$ 

Let $\mathbb{D}$ denote the unit disk. Let $L^2$ denote the Lebesgue space of square integrable functions on the unit circle $\partial \mathbb{D}$. The Hardy space $H^2$ is the subspace of analytic functions on $\mathbb{D}$ whose Taylor coefficients are square summable. Then it can also be identified with the subspace of $L^2$ of functions whose negative Fourier coefficients vanish. The Toeplitz operator on the Hardy space $H^2$ with symbol $f$ in $L^\infty(\mathbb{D})$ is defined by

$$T_f(h) = Pf(h),$$

for $h \in H^2(\mathbb{D})$. Here $P$ be the orthogonal projections from $L^2$ to $H^2$.

Let $b$ be a function in the closed unit ball of $H^\infty(\mathbb{D})$, the space of bounded analytic functions on the unit disk. The de Branges-Rovnyak space $\mathcal{H}(b)$ is defined to be the space $\left(I - T_bT_b^*\right)^{1/2}H^2$. We also define the space $\mathcal{H}(\bar{b})$ in the same way as $\mathcal{H}(b)$, but with the roles of $b$ and $\bar{b}$ interchanged, i.e.

$$\mathcal{H}(\bar{b}) = \left(I - T_{\bar{b}}T_{\bar{b}}^*\right)^{1/2}H^2.$$ 

The spaces $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are also called sub-Hardy Hilbert spaces (the terminology comes from the title of Sarason’s book [4]).

The Bergman space $A^2$ is the space of analytic functions on $\mathbb{D}$ that are square-integrable with respect to the normalized Lebesgue area measure $dA$. For $u \in
$L^\infty(\mathbb{D})$, the Bergman Toeplitz operator $\tilde{T}_u$ with symbol $u$ is the operator on $L^2_\alpha$ defined by

$$\tilde{T}_u h = \tilde{P}(uh).$$

Here $\tilde{P}$ is the orthogonal projection from $L^2(\mathbb{D}, d\alpha)$ onto $A^2$.

In [7], Zhu introduced the sub-Bergman Hilbert spaces. They are defined by

$$\mathcal{A}(b) = (I - \tilde{T}_b \tilde{T}_b^*)^{1/2} A^2$$

and

$$\mathcal{A}(\bar{b}) = (I - \tilde{T}_{\bar{b}} \tilde{T}_{\bar{b}}^*)^{1/2} A^2.$$

Here $b$ is a function in the closed unit ball of $H^\infty(\mathbb{D})$. It is easy to see from the definition that the spaces $\mathcal{A}(b)$ and $\mathcal{A}(\bar{b})$ are contractively contained in $A^2$. But in most cases they are not closed subspaces of $A^2$ (see [7, Corollary 3.13]). Some examples of $\mathcal{A}(b)$ are known: if $||b||_\infty < 1$, then $\mathcal{A}(b)$ is just a renormed version $A^2$; if $b$ is a finite Blaschke product, then $\mathcal{A}(b) = H^2 ([8], [5])$.

Sub-Bergman Hilbert spaces share some common properties with sub-Hardy Hilbert spaces as the way those spaces are defined follows from a general theory on Hilbert space contractions [4]. For instance, both $\mathcal{H}(b)$ and $\mathcal{A}(b)$ are invariant under the corresponding Toeplitz operators with a co-analytic symbol ([4, II-7]). One significant difference between the spaces $\mathcal{A}(b)$ and $\mathcal{H}(b)$ is the multipliers. The theory of $\mathcal{H}(b)$ spaces is pervaded by a fundamental dichotomy, whether $b$ is an extreme point of the unit ball of $H^\infty(\mathbb{D})$. The multiplier structure of de Branges-Rovnyak spaces has been studied extensively by Lotto and Sarason in both the extreme and the nonextreme cases ([1], [2], [3]). However, Zhu [7] showed that every function in $H^\infty$ is a multiplier of $\mathcal{A}(b)$ and $\mathcal{A}(\bar{b})$. As a consequence, the two sub-Bergman Hilbert spaces $\mathcal{A}(b)$ and $\mathcal{A}(\bar{b})$ are norm equivalent.

**Theorem 1.1.** For $b \in H^\infty$ with $||b||_\infty \leq 1$,

$$\mathcal{A}(b) = \mathcal{A}(\bar{b}).$$

**Proof.** The relation

$$||f||_{\mathcal{A}(b)} \leq ||f||_{\mathcal{A}(\bar{b})}$$

follows from Douglas’s criterion [4, I-5] and the operator inequality (see e.g. [6, p. 106])

$$\tilde{T}_b \tilde{T}_{\bar{b}} \leq \tilde{T}_{\bar{b}} \tilde{T}_b.$$

On the other hand, let $f \in \mathcal{A}(\bar{b})$. By [7, Theorem 3.12], $b$ is a multiplier of $\mathcal{A}(b)$ with multiplier norm less than or equal to $||b||_\infty$. Then

$$||f||_{\mathcal{A}(b)}^2 = ||f||_{\mathcal{A}(\bar{b})}^2 + ||bf||_{\mathcal{A}(\bar{b})}^2 \leq ||f||_{\mathcal{A}(\bar{b})}^2 + ||b||_\infty^2 ||f||_{\mathcal{A}(\bar{b})}^2 \leq 2 ||f||_{\mathcal{A}(\bar{b})}^2.$$

Here we used the identity [4, I-8].

In the Hardy space setting, the polynomials belong to $\mathcal{H}(b)$ if and only if $b$ is non-extreme, and in this case polynomials are dense in $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ (see [4, Chapter IV, V]). For the sub-Bergman Hilbert spaces, Zhu in [7] showed that $\mathcal{A}(b)$ always contain $H^\infty$, which includes the polynomials. In the same paper, Zhu
asked whether the polynomials are dense in $A(b)$ or $A(\bar{b})$. In this note, we answer this question in the affirmative.

2. Proof the Main Result

In the theory of $H(b)$ spaces, $H(\bar{b})$ is often more amenable than $H(b)$ because of a representation theorem for $H(\bar{b})$ [4, III-2]. We shall prove a similar version for $A(\bar{b})$.

**Theorem 2.1.** Let $b \in H^\infty$ with $\|b\|_\infty \leq 1$. Let $L_b^2$ denote the space $L^2(\mathbb{D}, (1-|b|^2)dA)$. Let $A_b^2$ be the closure of polynomials in $L_b^2$. Define the operator $S_b$ by

$$S_b g = \tilde{P}((1-|b|^2)g).$$

Then $S_b$ is an isometry from $A_b^2$ onto $A(\bar{b})$.

**Proof.** For any $q \in L_b^2$, $g \in A^2$, we have

$$\langle S_b q, g \rangle_{A^2} = \int_D \tilde{P}((1-|b|^2)q, g) dA = \langle q, g \rangle_{L_b^2},$$

which implies $S_b$ is the natural inclusion from $A_b^2$ into $L_b^2$. Then for any $f, g \in A^2$,

$$\langle S_b S_b^* f, g \rangle_{A^2} = \langle S_b^* f, S_b^* g \rangle_{L_b^2} = \langle f, g \rangle_{L_b^2} = \int_D f\tilde{g}(1-|b|^2)dA = \langle \tilde{T}_1-|b|^2 f, g \rangle_{A^2}. $$

Thus

$$S_b S_b^* = I - \tilde{T}_b \tilde{T}_b.$$

By Douglas’s Criterion,

$$A(\bar{b}) = S_b L_b^2.$$

Notice that every $q \in L_b^2$,

$$(S_b q)(w) = \langle q, k_w \rangle_{L_b^2},$$

where $k_w(z) = \frac{1}{(1-wz)^2}$ is the reproducing kernel of $A^2$. We see that

$$\text{Ker}(S_b) = (\text{Span}_{w \in \mathbb{D}} \{k_w\})^\perp = (A_b^2)^\perp.$$

Hence $S_b$ is an isometry from $A_b^2$ onto $A(\bar{b})$. □

We need the following lemma to prove the main result.

**Lemma 2.1.** Let $b \in H^\infty$ with $\|b\|_\infty \leq 1$. Let $M_n$ denote the closure of the linear span of $\{z^k\}_{k=n}^\infty$ in $A_b^2$. Let $P = \cup_{n \geq 0} (M_n)^\perp$. Then $P$ is dense in $A_b^2$.

**Proof.** If $b$ is a constant, the conclusion is obvious. Next, we assume $b$ is not a constant. Let $f \in A_b^2$ such that $f \perp P$. Then $f \in M_n$ for every $n \geq 0$. It is sufficient to show $f = 0$.

Suppose $f \in A_b^2 \setminus \{0\}$. Then for some $k \geq 1$, $f$ has the power series expansion

$$f(z) = \sum_{j=k}^{\infty} a_j z^j.$$

with \( a_k \neq 0 \). Since \( f \in M_k \), there is a sequence of polynomials \( \{ p_s \}_{s=1}^{\infty} \subset M_k \) for which

\[
p_s \to f \quad \text{in} \quad L^2_{\overline{b}}.
\]

By the definition of the integral norm,

\[
p_s - a_k z^k \to f - a_k z^k \quad \text{in} \quad L^2_{\overline{b}},
\]

and thus \( f - a_k z^k \in M_{k+1} \). But since \( f \in M_{k+1} \) and \( a_k \neq 0 \), it must be the case that \( z^k \in M_{k+1} \). Fix \( r \in (0, 1) \). There exists \( \delta > 0 \) such that

\[
1 - |b(z)|^2 \geq \delta,
\]

for every \( z \) in the disk \( r\mathbb{D} \).

Let

\[
g(z) = \sum_{j=k+1}^{N} b_j z^j, \quad N > k
\]

be a polynomial with minimal degree at least \( k + 1 \). Using the easily established identity

\[
\int_{r\mathbb{D}} z^m \overline{z}^n dA(z) = \begin{cases} z^{2m+2}/(m+1), & m = n, \\ 0, & m \neq n, \end{cases}
\]

we have

\[
||z^k - g(z)||_{A_k^2}^2 = \int_{\mathbb{D}} |z^k - g(z)|^2 (1 - |b(z)|^2) dA(z)
\geq \int_{r\mathbb{D}} |z^k - g(z)|^2 (1 - |b(z)|^2) dA(z)
\geq \delta \int_{r\mathbb{D}} |z^k - g(z)|^2 dA(z)
= \delta (\int_{r\mathbb{D}} |z^k|^2 dA(z) + \int_{r\mathbb{D}} |g(z)|^2 dA(z))
\geq \frac{\delta}{k+1} r^{2k+2} > 0.
\]

By the definition of \( M_{k+1} \), we see that \( z^k \notin M_{k+1} \), which is a contradiction. Thus \( f = 0 \). \( \square \)

**Theorem 2.2.** Polynomials are dense in \( A(b) \).

**Proof.** By Theorem 1.1, it suffices to prove polynomials are dense in \( A(\overline{b}) \). Let \( f \in A(\overline{b}) \) and let \( \varepsilon > 0 \). By Theorem 2.1, there exists \( g \in A_k^2 \) such that \( f = S_h g \). Using Lemma 2.1, we can find \( h \in \mathcal{P} \) with

\[
||g - h||_{A_k^2} < \varepsilon.
\]

By Theorem 2.1, \( S_h : A_k^2 \to A(\overline{b}) \) is an isometry. Thus

\[
||f - S_h h||_{A(\overline{b})} = ||S_h (g - h)||_{A(\overline{b})} = ||g - h||_{A_k^2} < \varepsilon.
\]
It remains to show that $S_b h$ is a polynomial. From the definition of $P$, we see that $h \in (M_n)^\perp$, for some $n \geq 0$. Then
\[
\langle h, z^k \rangle_{A_2^b} = 0,
\]
for every $k \geq n$. If $n = 0$, then $h = 0$. If $n \geq 1$, using (2.1) we have
\[
\langle S_b h, z^k \rangle_{A_2^b} = \langle \tilde{P}((1 - |b|^2)h), z^k \rangle_{A_2^b} = \int_{B} h(z)z^k(1 - |b(z)|^2)dA(z)
= \langle h, z^k \rangle_{A_2^b} = 0,
\]
for every $k \geq n$. Therefore $S_b h$ is a polynomial of degree at most $n - 1$. □

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