Semigroups over Local fields

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Abstract

Let $G$ be a 1-connected, almost-simple Lie group over a local field and $S$ a subsemigroup of $G$ with non-empty interior. The action of the regular hyperbolic elements in the interior of $S$ on the flag manifold $G/P$ and on the associated Euclidean building allows us to prove that the invariant control set exists and is unique. We also provide a characterization of the set of transitivity of the control sets: its elements are the fixed points of type $w$ for a regular hyperbolic isometry, where $w$ is an element of the Weyl group of $G$. Thus, for each $w$ in $W$ there is a control set $D_w$ and $W(S)$ the subgroup of the Weyl group such that the control set $D_w$ coincides with the invariant control set $D_1$ is a Weyl subgroup of $W$. We conclude by showing that the control sets are parameterized by the lateral classes $W(S)\backslash W$.

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5 Semigroups and Control Theory

5.1 Subgroup $W(S)$
1 Semigroups and Control Sets

Our purpose in this text is to study a particular class of semigroups: namely, the semigroup of a Lie group over a local field. This task will be tackled in an essentially geometric way, by means of a detailed study of the action of the semigroup in a suitable space. This approach leads to a rich theory that relates the semigroup to the control sets. We begin with some basic definitions:

Let $X$ be a topological space and $C(X)$ the group of homeomorphisms of $X$, a semigroup (of homeomorphism) is a set $S \subset C(X)$ closed under composition.

Let $S$ be a semigroup of homeomorphism of $X$ and $x$ a point in $X$. The set $Sx = \{gx \in X : g \in S\}$ is called the orbit of $x$ by the action of $S$. We refer to the closure $\text{cl}(Sx)$ of $Sx$ in $X$ as the approximate orbit. A subset of $X$ is $S$-invariant if $SD \subset D$.

We say that the action of $S$ on a set $A \subseteq X$ is approximately transitive if $A \subseteq \text{cl}(Sx)$ for all $x \in A$.

**Definition 1** A control set for $S$ in $X$ is a subset $D \subset X$ which fulfills the following conditions:

1. $D \subset \text{cl}(Sx)$ for all $x \in D$;
2. $D$ is maximal with respect to the first property;
3. $\text{int} D \neq \emptyset$

The first two items of the previous definition of control set say that these sets are maximal among all sets where the action of $S$ is approximately transitive.

We introduce the following partial ordering between control sets: $D_1$ is smaller than $D_2$ if there exist $x \in D_1$ and $s \in S$ such that $sx \in D_2$. A maximal control set with respect to this ordering is called an invariant control set, which is clearly $S$-invariant.

We define the set of transitivity as

$$D_0 = \{x \in D : x \in \text{int}(S^{-1}x)\}$$

**Proposition 2** Let $S$ be a semigroup of homeomorphisms with non-empty interior, $D$ a control set for $S$ in $X$ and $D_0$ the set of transitivity of $D$. Then the following hold:
1. $D_0 = \text{int}(Sx)$ for all $x \in D_0$

2. $\text{cl}(D_0) = D$

3. For any $x, y \in D_0$ there is $s \in S$ with $sx = y$

We point out that item (3) above is the reason why $D_0$ is called the set of transitivity of $D$.

When the topological space $X$ is compact, an application of Zorn’s Lemma yields the existence of an invariant control set $D$.

In the very special case in which $S \subset G$, with $G$ a semi-simple Lie group, San Martin has developed a powerful theory relating the semigroups and the control sets in the flag manifold $G/P$. Among others results San Martin proved that if $\text{int}(S) \neq \emptyset$ then there is a unique control set in $G/P$, and that for each element of the Weyl group $w \in W$ there exist a control set $D_w$ on maximal flag $G/P$, whose elements of the set of transitivity are fixed points of type $w$, for some regular $h$. For more information and another results see [San1], [San2], [San3] and [ST].

We extend these results for algebraic groups over a local field.

In sections 2, 3 and 4 we introduce the basic concepts and definitions used in this work. Since those are quite extensive, the presentations is somehow schematic form. In each of these sections we point our basic bibliographic source but warn the reader that, due to notations used in different contexts, the notations used in this work may differ slightly from the usually adopted in the indicated bibliographical resources.

2 Buildings

The basic bibliographical resource we used for this section are [Bro] and [Ga].

Let $I$ be a finite set of indexes. A Coxeter matrix $M = (m_{ij})$ is a square matrix with values in $\mathbb{N} \cup \{\infty\}$ such that $m_{ij} = 1$ if and only if $i = j$ and $m_{ij} = m_{ji}$.

A Coxeter matrix defines a Coxeter group of type $M$, the group $W(M)$ defined by the presentation

$$W(M) = \langle r_i : r_i^2 = (r_i r_j)^{m_{ij}} = 1, \forall i, j \in I \rangle$$

To simplify the notation we denote $W(M)$ by $W$ wherever it is clear which Coxeter Matrix is been referred to.
We can see identify the set of indexes $I$ with the the set $\{r_i : i \in I\}$ of order two generators of $W$ and call the pair $(W, I)$ a Coxeter system.

A special subgroup of a Coxeter group $W$ with generators $J$ is a subgroup $W(J)$ generated by a subset $J \subseteq I$. A special coset is a coset $\langle wW(J) \rangle$, determined by a special subgroup $W(J)$.

We describe now the chamber complex system, the Coxeter complex associated to a Coxeter system, which plays a crucial rôle in this work. Let $\mathcal{V}$ be a poset with the usual inclusion order. Let $\mathcal{V}$ be a family of finite subsets of $\mathcal{V}$ containing every singleton $\{v\} \subseteq \mathcal{V}$ and satisfying the condition that, if $A \in \mathcal{V}$ and $B \subseteq A$ then $B \in \mathcal{V}$. Such a pair $(\mathcal{V}, \Delta)$ is called a simplicial complex and $\Delta$ a family of simplices. Given simplices $B \subseteq A \in \Delta$ we say that $B$ is a face of $A$.

The cardinality $r$ of a simplex $A$ is said to be its rank and $r - 1$ is called the dimension of $A$. Two simplices $A, B$ in a simplicial complex $\Delta$ are adjacent if they have a codimension 1 face. A gallery is a sequence of maximal simplices in which any two consecutive ones are adjacent.

We say that $\Delta$ is a chamber complex if all maximal simplices have the same dimension and any two can be connected by a gallery. The maximal simplices are called chambers. We say that a chamber complex $\Delta$ is labelled by a set $I$, if there is a surjective function $: V \to I$ that restricts to a bijection in each chamber.

Proposition 3 \[\text{[Hu]}\]

1. The group $W$ acts on $\Sigma(W, I)$ by type-preserving automorphisms.

2. The action of the group $W$ is transitive on the collection of simplices of a given type.

Definition 4 A building is a simplicial complex $\Delta$, together with a family $A$ of subcomplexes (called the set of apartments), satisfying the following axioms:

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Definition 4 A building is a simplicial complex $\Delta$, together with a family $A$ of subcomplexes (called the set of apartments), satisfying the following axioms:
1. Each apartment $\Sigma$ is a Coxeter complex.

2. For any two simplices $A, B \in \Delta$, there is an apartment $\Sigma$ containing both of them.

3. If $\Sigma$ and $\Sigma'$ are two apartments containing $A$ and $B$, there is an isomorphism $\Sigma \to \Sigma'$ fixing $A$ and $B$ pointwise.

4. Every codimension 1 simplex is a face of at least three chambers (thickness axiom).

It is clear, by Axiom B2, that every two apartments are isomorphic. It also implies that two maximal simplices $A$ and $B$ have the same dimension and can be connected by a gallery in the apartment $\Sigma$ containing both of them, and so $\Delta$ is a chamber complex. Furthermore $\Delta$ is a labelled chamber complex labeled by $I$, the set of generators of the Coxeter group of any given apartment $\Sigma$. The isomorphism of apartments $\Sigma \to \Sigma'$ postulated in Axiom B2 can be regarded as a label-preserving isomorphism.

Any collection $\mathcal{A}$ of apartments $\Sigma$ satisfying the former axioms is called a system of apartments for $\Delta$. It’s well known that the apartments are convex, i.e., given two chambers of $\Sigma$, then every minimal gallery of $\Delta$ connecting these chambers is contained in $\Sigma$.

Let $\Delta$ be a building, $A$ a system of apartments for $\Delta$ and $G$ a group acting on $\Delta$ by simplicial label-preserving automorphisms that leave $A$ invariant, i.e., if $\Sigma \in A$ then $g\Sigma \in A$.

We say that the action of $G$ is strongly transitive if $G$ acts transitively in the pairs $(\Sigma, C)$, with $C \in \Sigma$, in other words, if $G$ is transitive in the set of apartments and the stabilizer of an apartment is transitive on the chambers of $\Sigma$.

Henceforward we assume that $G$ acts strongly transitively in $\Delta$, and chose $\Sigma_0 \in A$ and $C_0 \in \Sigma_0$, called the fundamental apartment and the fundamental chamber, respectively.

The following subgroups of $G$ is of particular interest:

\[ B = \{ g \in G : gC_0 = C_0 \} \]  \hspace{1cm} (1)

\[ N = \{ g \in G : g\Sigma_0 = \Sigma_0 \} \]  \hspace{1cm} (2)

\[ T = \{ g \in G : g \text{ fixes } \Sigma_0 \text{ pointwise} \} \]  \hspace{1cm} (3)

\[ W = N/T \]  \hspace{1cm} (4)
Given $J \subseteq I$ we define $P_J \subseteq G$ as the subgroup of $G$ generated by $< B, W(J) >$, where $W(J)$ is the special subgroup with generators indexed by $J$. A face of type $J$ is a simplex of $\Delta$ with vertices labelled by $J$.

The following proposition, although not directly cited in the continuation is crucial in many of the subsequent results in this section.

**Proposition 5** Given $J' \subset J$ and a face $A$ of type $J - J'$, then the stabilizer $P_{J'}$ of $A$ satisfies:

$$P_{J'} = \bigcup_{w \in W'} BwB.$$  

In particular, we have the Bruhat Decomposition of $G$:

$$G = \coprod_{w \in W} BwB$$  

As a consequence of the Bruhat Decomposition and the Axiom of Thickness, we have that $BwB \circ BjB \subseteq BwB \cup BwjB$ and $jBj \subset B$ for every $j \in J$. All these properties are stated as postulates in the definition of $BN$-pairs.

**Definition 6** A *Tits system* or *BN-pair* is a group $G$ with two subgroups $B, N$ satisfying:

BN0. $B$ and $N$ together generate $G$

BN1. $W = N/T$ is a Coxeter group with generators $J = \{j_1, \ldots, j_n\}$, where $T := B \cap N \triangleleft N$

BN2. $BjB \circ BwB \subset BwB \cup BwjB$, for all $w \in W$ and $j \in J$

BN3. $jBj \neq B$ for all $j \in J$

The relation between the structure of buildings and $BN$-pairs is the Tits’ Theorem:

**Theorem 7** Let $\Delta$ be a building where a group $G$ acts strongly transitively, and let $B, C$ and $N$ be defined as in 4. The pair $(B, N)$ is then a Tits system. Conversely, every Tits system $(B, N)$ in a group $G$ defines a building in which the chambers are the cosets of $B$ and the equivalence relation is given by $gB \sim g'B \iff gP_j = g'P_j$. Finally, the action of $G$ is strongly transitive and $N$ stabilizes an apartment.
3 Geometric Realization of Buildings

When the Coxeter group is an isometry group of a space we can give a geometric interpretation to the Coxeter Complex and the associated building.

3.1 Spherical Buildings

Let $V$ be a finite-dimensional real vector space with an inner product, $H \subset V$ an hyperplane and $\alpha$ an unit normal vector to $H$. The reflection with respect to $H$ is the linear transformation defined as $s_H(v) = v - \langle v, \alpha \rangle \alpha$.

A finite group $W$ of linear transformations of $V$ is called a finite reflection group if it is generated by reflections $s_H$, where $H$ ranges over a set $H$ of hyperplanes. In other words, $W$ is a discrete subgroup of the orthogonal group $O(V)$ which is generated by reflections.

We say that a finite reflection group $W$ is essential relative to $V$ if $W$ acts on $V$ with no nonzero fixed points. Given a finite reflection group $W$ and $V_0$ its space of fixed points, It is clear that any subgroup $W$ stabilizes $V_0$ and its orthogonal complement $V_0^\perp$ and $W$ is essential relative to $V_0^\perp$.

A finite reflection group $(W, V)$ is called reducible if $V$ decomposes as $V' \oplus V''$ with $V'$ and $V''$ proper, $W$-invariant subspaces, i.e., $W(V') \subset V'$ and $W(V'') \subset V''$.

Let $F = \{a_H = \langle , \alpha \rangle : H \in \mathcal{H}\}$ the set of all linear functionals associated with $H \in \mathcal{H}$. To each function $\phi : F \to \{+1, -1\}$, we associate the set:

$$C_\phi = \{x \in V : \phi(x) \cdot f(x) > 0, \forall f \in F\}$$

The chambers are the non-empty $C_\phi$. The chambers form a partition of $V \setminus \{H \in \mathcal{H}\}$ into disjoint convex cones.

A set $B$ is called a face of $A$ if its the intersection of $C_\phi$ with a subspace of the form $ST = \{x \in X : f(x) = 0, \forall f \in T \subset \mathcal{H}\}$. In this case we write $B \leq A$. A wall is a maximal proper face.

A Coxeter complex $\Sigma$ is called spherical if it is isomorphic to the complex associated to a finite reflection group. Considering the intersection of apartments, chambers, walls and faces with the unit sphere of $V$ we get the geometric realization of the spherical Coxeter Complex. A building $\Delta_\Sigma$ is called spherical if its apartments are spherical.

The diameter of a spherical building $\Delta_\Sigma$ is finite, and equal to the diameter of any apartment. Two chambers $C, C'$ in a spherical building $\Delta_\Sigma$ are said to be opposite if $d(C, C') = \text{diam}(A)$. 

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3.2 Euclidean Buildings

Let $V$ be a finite-dimensional real vector space with an inner product. The isometry group of $V$ is the semidirect product $O(V) \ltimes V$. An affine reflection group $W$ is a discrete subgroup of $O(V) \ltimes V$ that is generated by reflections in affine hyperplanes.

We define $\mathcal{H} = \{H : s_H \in W\}$ and $\mathcal{F} = \{a_H : H \in \mathcal{H}\}$. We can then define the chambers for the affine reflection group in a similar way as we did for the finite reflection groups. In the affine case the chambers are bounded and they are the open Dirichlet domains for the action of $W$ in $\mathbb{R}^n$.

Henceforth, we let $C_0$ be a fixed chamber (referred to as a fundamental chamber) and let $J$ be the reflections in the walls of $C_0$.

**Proposition 8**

1. The action of $W$ is simply transitive in the chambers.

2. $W$ is generated by $J$.

3. $(W, J)$ is a Coxeter Complex.

4. There exists $x \in V$ such that the stabilizer $W_x$ is isomorphic to a finite Coxeter group $\overline{W}$.

5. $W \simeq \mathbb{Z}^n \rtimes \overline{W}$, with $n$ being the dimension of $V$.

A point $x \in V$ such that $W_x$ is isomorphic to $\overline{W}$ is called a special point.

An abstract Coxeter complex $\Sigma$ is called Euclidean if it is isomorphic to the complex $|\Sigma|$ associated to an affine reflection group. The complex $|\Sigma|$ is called the geometric realization of the complex $\Sigma$. We can choose a norm in the vector space $V$ such that every chamber has diameter 1. This norm is called the canonical norm and with it every simplicial isomorphism $\phi : \Sigma \to \Sigma'$ induces an isometry $|\phi|$ between the corresponding geometric realizations, $|\phi| : |\Sigma| \to |\Sigma'|$.

Let $|\Sigma|$ be the geometric realization of a Euclidean Coxeter complex, and let $\mathcal{H}$ be the associated set of hyperplanes in $V$. Fix $x \in V$ and let $\overline{\mathcal{H}}$ be the set of hyperplanes through $x$ and parallel to some element of $\mathcal{H}$. The set $\overline{\mathcal{H}}$ is finite. Let $\mathcal{F} = \{a_H = \langle \cdot, \alpha \rangle : H \in \overline{\mathcal{H}}\}$ the set of all linear functional associated with $H \in \overline{\mathcal{H}}$. Given a function $\phi : \mathcal{F} \to \{+1, -1\}$, we associate the set:

$$A_\phi = \{y \in V : \phi(x)f(y - x) > 0, \forall f \in \mathcal{F}\}$$

The sectors are the non-empty $A_\phi$. The sectors form a partition of $V$ into disjoint convex cones. Given sectors $A$ and $B$, $A$ is called a subsector of $B$ if $A \subset B$. 

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A set $B$ is called a face of $A$ if its the intersection of $C_\phi$ with a subspace

$$S_T = \{y \in X : f(y - x) = 0, \forall f \in T \subset \overline{H}\}.$$  

In this case we write $B \leq A$. A wall is a maximal proper face. These cells will simply be referred to as conical cells based at $x$. Clearly if $A$ is a sector based at $x$, the set $A - x + y$ is a sector based at $y$.

A building $\Delta_E$ is called Euclidean if its apartments are Euclidean. A geometric realization $|\Delta_E|$ of a Euclidean Building is a building such that each apartment $\Sigma$ is a affine Coxeter complex.

### 3.3 The Geometry of Euclidean Buildings

Euclidean buildings have interesting geometric properties and can be thought of as either an $n$–dimensional generalization of trees and some kind of simplicial countenparts of symmetric space of non-positive curvature.

We start by defining a very special metric in the geometric realization $|\Delta_E|$ of $\Delta_E$. Given two points $x, y \in |\Delta_E|$, the axiom (B1) of buildings says that there exists a apartment $\Sigma$ containing $x$ and $y$. This apartment can be endowed with a metric under which any chamber has diameter. Let $d_\Sigma(x, y)$ denote the distance of $x$ and $y$ in this metric. The canonical metric is then set to be:

$$d(x, y) = d_\Sigma(x, y)$$

A curve $\gamma : I \to |\Delta_E|$ from the unit interval $I$ to the metric space $|\Delta_E|$ is a geodesic if there is a constant $\lambda \geq 0$ such that for any $t \in I$ there is a neighborhood $J$ of $t$ such that for any $t_1, t_2 \in J$ we have:

$$d(\gamma(t_1), \gamma(t_2)) = \lambda|t_1 - t_2|$$

The length of a curve $\gamma$ is defined as

$$L(\gamma) = \sup \left\{\sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1})) : n \in \mathbb{N} \text{ and } a = t_0 < t_1 < \cdots < t_n = b\right\}.$$

A rectifiable curve is a curve with finite length.

In a metric space $(X, d)$, the induced intrinsic metric, $d_i(x, y)$ is the infimum of the lengths of all paths from $x$ to $y$. The length of such a path is defined as explained for rectifiable curves. We set $d_i(x, y) = \infty$ if there is no path of finite length from $x$ to $y$. If $d(x, y) = d_i(x, y)$ for all points $x$ and $y$, we say $(X, d)$ is a length space or a path metric space and the metric $d$ is intrinsic. The geometric realization $|\Delta_E|$ with the canonical metric is a length space.
Proposition 9 ([Ga] pg. 194) Let a morphism \( f : \Delta_E \to \Delta'_E \) of Euclidean buildings be given. Then \(|f| : |\Delta_E| \to |\Delta'_E|\) is an isometry between the corresponding geometric realizations.

The geometric realization \(|\Delta_E|\) with the canonical metric is a CAT(0) space. This means that the curvature of such spaces is smaller or equal to zero, that is, the triangles in such spaces are thinner than in \(\mathbb{R}^2\) in the following sense:

Given three points \(x, y, z\) in a metric space \(X\), a comparison triangle \(\Delta\) in \(\mathbb{R}^2\) is a triangle with vertices \(\overline{x}, \overline{y}, \overline{z}\) such that

\[
d(x, y) = d(\overline{x}, \overline{y}), \quad d(y, z) = d(\overline{y}, \overline{z}) \quad \text{and} \quad d(x, z) = d(\overline{x}, \overline{z})
\]

A metric space \(X\) is called a CAT(0) space if for any geodesical triangle \(\Delta\) in \(X\) and \(\overline{\Delta}\) a comparison triangle in \(\mathbb{R}^2\). We have that \(d(x, y) \leq d(\overline{x}, \overline{y})\) for any \(x, y \in \Delta\).

A complete CAT(0) space is called a Hadamard space. The geometric realization \(|\Delta_E|\) of \(\Delta_E\) is a Hadamard space (see [Ga] p. 197). From now on, a CAT(0) space means a complete one.

The CAT(0) inequality implies the following properties of the building \(|\Delta_E|\):

1. Any two points \(x, y\) are joined by a unique geodesic segment, which varies continuously in its endpoints \(x, y\);

2. Metric balls are convex and contractible; in particular it is a simply connected space and all of its higher homotopy groups are trivial;

3. A flat \(|\Lambda|\) is a set that is isometric to some \(\mathbb{R}^n\). Every maximal flat is an apartment and vice-versa. So we have a geometric description of the apartments in the geometric realization \(|\Delta_E|\).

4. The apartments are convex and every geodesic is contained in an apartment. A geodesic is called regular if it is contained in only one apartment.

3.3.1 The Building at the Infinite

Given \(X\) a locally compact CAT(0) space, two geodesic rays \(\sigma(t)\) and \(\gamma(t)\) are called asymptotic if there exists a constant \(c\) such that \(d(\sigma(t), \gamma(t)) \leq c\) for all \(t \geq 0\). This define an equivalence relation on the set of asymptotic rays we define the ideal boundary \(\partial_\infty X\) or simply \(\partial_\infty X\) as the set of
equivalence classes of asymptotic rays. The union $X \cup \partial_\infty X$ is denoted by $\overline{X}$. Given $x \in X$ and $\xi \in \partial_\infty X$, there exists a geodesic ray $\gamma$ starting at $x$ and such that $\gamma(\infty) = \xi$. This ray is denoted by $\gamma_{x,\xi}$.

We introduce a topology at $\overline{X} = X \cup \partial_\infty X$ using as base to this topology the open sets of $X$ and the following opens sets:

$$U(x, \xi, R, \varepsilon) := \{z \in \overline{X} : z \notin B(x, R) \text{ and } d(\sigma_{x,z}(R), \sigma_{x,\xi}(R)) < \varepsilon\}$$

with $x \in X$, $\xi \in \partial_\infty X$ and $R, \varepsilon$ real positive numbers. This topology is known as the Busemann topology and it does not depend on the choice of the base point, and it turns $\overline{X}$ into a compact space with $X$ open and dense in $\overline{X}$, known as Busemann topology compactification.

A fundamental fact of the topology we have just defined is that given a isometry $\gamma$ of a complete $\text{CAT}(0)$ space, the natural extension of $\gamma$ to $\overline{X}$ is a homeomorphism.

The compactification of the geometric realization of $\Delta_E$, $\partial_\infty(|\Delta_E|)$, can be endowed with a spherical building structure:

Given a conical cell $c \in |\Delta_E|$, it defines a simplex at infinity (or ideal simplex) $c_\infty$ as the set consisting of all equivalence classes $\gamma(\infty)$, where $\gamma$ is a geodesic ray contained in the conical cell $c$. Given an ideal simplex $\sigma$, we denote by $c_{\sigma,x}$ the unique conical cell based at $x$ such that $c_{\sigma,x}(\infty) = \sigma$.

Given two ideal simplices $\sigma, \sigma' \subset \partial_\infty X$, we say that $\sigma'$ is a face of $\sigma$ if the conical cell $c_{\sigma',x}$ is a face of $c_{\sigma,x}$ for some (and hence all) $x \in X$. In that case we write $\sigma' \leq \sigma$. This defines an order relation that turns $\partial_\infty(|\Delta_E|)$ a geometric realization of a spherical building. This is stated in the next theorem and is the key to the study of control sets in this work.

**Theorem 10 ([Ga] p. 279)** Consider the ideal simplex structure $\partial_\infty(|\Delta_E|)$. The face relation is well defined and the poset of ideal simplices of $\partial_\infty(|\Delta_E|)$ is the geometric realization of the spherical building associated with $|\Delta_E|$, i.e., $\partial_\infty(|\Delta_E|) = |\Delta_S|$. Moreover, the apartments of $\partial_\infty(|\Delta_E|)$ are in bijection with the maximal system of apartments of $|\Delta_E|$.

4 **Algebraic Groups over Local Fields**

Basic bibliographic reference for this section is [Hu]. In the first subsection we introduce only notation, taking for granted all definitions.

Let $k$ be a field with $\text{char}(k) = 0$. An ultrametric norm in a field $k$ is a function $| | : k \to \mathbb{Z}_+$, that satisfy

1. $|x| = 0 \iff x = 0$
2. $|xy| = |x||y|$ and $|1| = 1$
3. $|x + y| \leq \max\{|x|, |y|\}$

The last inequality is called **ultrametric inequality** or **strong triangular inequality**.

A field with a norm that satisfies the strong triangular inequality is said to be an **ultrametric field** or **Archimedean field**. A field is said to be complete if it is complete in the topology induced by the norm.

A **local field** is a complete, locally compact ultrametric field.

The $p$-adic field are the paradigmatic examples of local fields. In what follows, we assume that $k$ denotes a local field.

A norm in a local field is induced by a valuation $v : k^* \rightarrow \mathbb{Z}$, in the sense that $|x| = e^{-v(x)}$. Let $A \subset k$ be the subring of elements with $v(x) \geq 0$ and let $\pi \in k$ an element such that $v(\pi) = 1$. The quotient field $A/\pi A$ is called the **residue field** of $k$ with respect to the valuation $v$ (norm $|\cdot|$).

### 4.1 The spherical building of a algebraic group

Let $G$ be a linear algebraic group of dimension $n$ over a field $k$ and $G^0$ the connected component of the identity element $Id \in G$. An algebraic group $T$ over $k$ is a **torus** of rank $n$, if $T$ becomes isomorphic to $(GL_1)^n$ after the extension of the base field to the algebraic closure $k$. A torus is **split** if it is isomorphic over $k$ to $(GL_1)^n$.

A subgroup $H \subset G$ is called parabolic when $G/H$ is a complete variety. A Borel subgroup $P$ is a minimal parabolic subgroup. We choose a Borel subgroup $P$ which contains a maximal split torus $T$. Let $N_{sph}$ denote the normalizer of $T$ in $G$.

**Theorem 11** ([Hu],[Bo]) The triple $(G, P, N_{sph})$ is a BN-pair to which we can associate a spherical building $\Delta_S$.

The following remarkable results can be seen as consequences of the previous theorem

**Proposition 12** [Bro, p. 110](Bruhat Decomposition) Each double class $PgP$, $g \in G$ can be written as $PwP$, $w \in W$. Also, we have that mapping $w \rightarrow PwP$ is a bijection of the Weyl group onto the double class $PgP$, with $g \in G$.

**Proposition 13** [Ga, p. 111] If $gP^{I_1}g^{-1} = P^{I_2}$ then $I_1 = I_2 = I$ and $g \in P^I$. 

A decomposition of an algebraic group as a semidirect product $Z \rtimes N$, with $Z$ reductive and $N$ unipotent is called a **Levi decomposition** and we call $Z$ the **Levi factor** of $G$.

**Proposition 14** (Levi decomposition) Let $N$ be the unipotent radical of $P$ and $Z$ be the centralizer of the torus $T$. Then

$$P = Z \rtimes N$$

### 4.2 The Euclidean building of a algebraic group over a local field

The spherical building described in the previous section depends only on the algebraic structure of $G$. When the base field is a local one, it is possible to assign to $G$ other BN-pair, that yields an Euclidean building for $G$.

**Theorem 15** [TB, Bruhat-Tits Theorem] Let $G$ be a simple and simply connected group over a local field. Then we can associate to $G$ an Euclidean building $\Delta_E$ through a BN-pair, and the group $G$ acts as the group of automorphism of $\Delta_E$.

We give a brief description of the building $\Delta_E$. Let us consider the Weyl group $W_0$ of the spherical building $\Delta_S$. The Weyl group $W$ of $\Delta_E$ is an affine group that has $W_0$ as its linear part: $W := W_0 \rtimes \mathbb{Z}^n$. It acts as a reflection group in $\mathbb{R}^n$ and we denote by $|\Sigma|$ the (geometric realization of the) simplicial structure determined by this action of $W$ on $\mathbb{R}^n$. Then $\Sigma$ is a basic apartment and the structure of $\Delta_E$ is fully determined when we state that the cardinality of

$$\text{adjacent}(C, i) := \{ C' | C \sim_i C' \},$$

independens on the chamber $C$ and the adjacency type $i$ and is constant equal to the cardinality of the residue field $A/\pi A$ of $k$. In other words, there is a prime $p$ such that any given cell has exactly $p$ $i$-adjacent cells for any adjacency type $i$. Reciprocally, considering the description of such a building $\Delta_E$, it is possible to prove that the group of automorphism of $\Delta_E$ is isomorphic to $G$. Full details can be found in [Bro, chapter 5].

Using the Bruhat-Tits theorem we can give the following description of the Euclidean building:

**Theorem 16** ([Bro], pág 163) *The building $\Delta_E$ associated to a group $G$ with a Euclidean BN-pair is isomorphic to the flag complex of the incidence*
geometry consisting of the maximal bounded subgroups of $G$, where two distinct such subgroups $P, Q$ are incident if and only if $P \cap Q$ is a maximal subgroup of $P$.

The subgroups $P_W = \bigcup_{w \in W'} BwB$ of $G$ are called parahoric subgroups. They are open and compact (in the non-Archimedian topology of $G$).

The spherical building at the infinity associated with this Euclidean building coincides with the previous one, in the sense that $\partial_\infty (|\Delta_E|) = |\Delta_S|$.

4.3 The action of $G$ in $\Delta_E$

Assuming the short description of the euclidean building $\Delta_E$ given in the previous section, we present now a more detailed description of the action of $G$ on $\Delta_E$ and the structure it determines in the group $G$ itself.

We define the following subgroups of $G$:

- $N_{aff} = \text{stabilizer of the apartment } \Sigma$
- $B = \text{pointwise fixer of the chamber } C$
- $T = N_{aff} \cap B$
- $P = \text{stabilizer of the chamber } C_\infty$
- $N_{sph} = \text{stabilizer of the apartment } \Sigma_\infty$
- $Z = N_{sph} \cap P$

The subgroup $P$ is denominated minimal parabolic or Borel subgroup, and the subgroup $B$ is the parahoric minimal, or Iwahori subgroup. We remark that $(B, N_{aff})$ is the Tits system that originates the euclidean building and $(P, N_{sph})$ is the Tits system associated with the spherical building.

The stabilizer of the apartment $\Sigma \subset \Delta_E$ and the stabilizer of the apartment $\Sigma_\infty \subset \Delta_S$ coincide, i.e., $N_{aff} = N_{sph}$ [Ga, p. 259]. So the BN decomposition of $G$ associated to both euclidean and spherical buildings, has the same $N$-factor and hence we write simply $N = N_{aff} = N_{sph}$.

We remark also that $W = N/T$ is the affine Weyl group and that $W_0 = N/Z$ is the spherical one.

If we denote by $N_{trans}$ the subgroup of $N$ that acts on the apartment $\Sigma$ by translations, then the Levi factor $Z = N \cap P$ of the minimal parabolic $P$ with respect to the apartment $A_\infty$ is equal to the subgroup $N_{trans}$ [Ga, p. 200].

A sector $A$ with base point $s$ contained in the apartment $\Sigma$ determines a spherical chamber $A^\infty \subset |\Delta_S| = \partial_\infty (|\Delta_E|)$. We define the semigroup $Z_A$ as the subset of $Z = N \cap P$ that sends $s$ to a point in $A$ and $N$ as the unipotent subgroup of $G$ that stabilizes the chamber $A^\infty$ of $\partial_\infty (|\Delta_E|)$.
The group $G$ admits the following decompositions, that are analogous to the Lie case.

**Theorem 17** [Brh], [Ron2, p. 100]

1. (Iwasawa Decomposition) $G = KZN$, and the double classes $K\backslash G/N$ are in bijective correspondence with $Z$.

2. (Cartan Decomposition) $G = KZNK$ and the double classes $K\backslash G/K$ are in bijective correspondence with $Z_A$.

Being $G$ be a simple and simply connected group over a local field, the quotient group of $G$ by any closed subgroup can be provided with a structure of manifold over the local field, so in particular the flag manifolds are manifolds over the local field ([Sc, LG.4.10 and LG.4.11]).

Let $J$ be the set of generators of the spherical Weyl group $W_0$. Given $I \subset J$ we define two special groups $W_I = \langle j_i : i \in I \rangle$ and $W^I = \langle j_k : k \notin I \rangle$, and associate to them the parabolic subgroups $P_I = PW_IP$ and $P^I = PW^IP$. It is well known that $G = \langle P^I, P_I \rangle$ and $P^I \cap P_I = P$. Given the flag manifold $G/P^I$ and the projection $G \to G/P^I$, we denote by $\tilde{g}$ the lateral class $gP^I \subset G/P^I$.

Let $N_I$ be the maximal unipotent subgroup of $P^I$ and let $\overline{N}_I$ be the maximal unipotent subgroup of $P_I$. Let $\sigma \in W_0$ the element with maximal length, i.e., the unique element of the Weyl group $W_0$ that $\sigma$ maps a chamber to its opposite and define $\overline{N} = \sigma N \sigma^{-1}$.

The manifold $\overline{N}_I \cdot \tilde{e}$ can be identified with $\overline{N}_I$ because $n_2n_1^{-1} \in P^I$ iff $n_1 = n_2$. We will show the (well known in the real case) result which assures that $\overline{N}_I \cdot \tilde{e}$ is open and dense in the non-Archimedean topology of $G/P^I$. Given $w \in W$ we define $\overline{N}^w := \overline{N} \cap wNw^{-1}$ and call each $\overline{N}^w$ a Bruhat cell.

The function $gP \to gP^I$ will allow us to restrict to the proof to the case $G/P$ with $P$ a minimal parabolic subgroup.

**Theorem 18** (Bruhat cellular decomposition) Then the mapping $\eta \to \eta \cdot \tilde{w}$ of $\overline{N}^w$ in $G/P$ is injective and $G/P$ is the disjoint union of the cells $\overline{N}^w \tilde{w}$, $w \in W$.

**Proof.** The Bruhat and Levi decompositions yield

$$G = \sigma G = \bigsqcup_{w \in W} \sigma PwP = \bigsqcup_{w \in W} \sigma NZ\sigma^{-1}wP = \bigsqcup_{w \in W} \sigma N\sigma^{-1}wP = \bigsqcup_{w \in W} \overline{N}wP.$$
We will use the fact that the former union is disjoint to construct a decomposition of $G/P$ into disjoint Bruhat cells.

Since $wPw^{-1}$ is the stabilizer of $wP$ in $G$, the stabilizer of $\dot{w} = wP \in G/P$ in $N$ is $N \cap wPw^{-1} = N \cap wNw^{-1}$ and we can therefore identify $NwP$ with $N/(N \cap wNw^{-1})$.

Since $N = (N \cap wNw^{-1})(N \cap wNw^{-1})$, it follows that $NwP = Nw^{\dot{w}}\cdot \dot{w}$. Thus the flag manifold $G/P$ is the disjunct union of the Bruhat cells $Nw\cdot \dot{w}$.

The Bruhat cell $N \cdot \dot{e}$ is open, since it has the same dimension of $G/P$.

Moreover, the Bruhat cells $Nw\cdot \dot{w}$ have codimension at least one and there are finitely many such Bruhat cells, so the cell $N \cdot \dot{e}$ is also dense in $G/P$.

**Corollary 19** The open cell $N \cdot \dot{e}$ is open and dense in $G/P$.

Because of this last corollary, $N \cdot \dot{e}$ is called the open Bruhat cell of $G/P_I$.

Let $\Delta_E$ be the euclidean building associated with $G$ and $|\Delta_E|$ be its geometric realization. As we mentioned, the ideal boundary of $|\Delta_E|$ is the geometric realization $|\Delta_S|$ of the spherical building. Moreover, $|\Delta_E|$ is (open) and dense in the Busemman compactification

$$|\Delta_E^\infty| = |\Delta_E| \cup |\Delta_S| = |\Delta_E| \cup \partial^\infty (|\Delta_E|).$$

We define $\partial^\infty (\Delta_E)$ as the set of barycenters of cells in $|\Delta_S|$. This is a formal definition and clearly the set $\partial^\infty (\Delta_E)$ has a structure of spherical building. If we consider the barycenter of each geometric cell in $|\Delta_E|$, we find that the barycenter of cells in $|\Delta_S|$ are the accumulation points of barycenter of cells in $|\Delta_E|$. Hence we can define the (non-trivial) **Busemann topology** on the spherical building $\Delta_S$ as the topology defined in $\partial^\infty (\Delta_E)$ as a subspace of $|\Delta_E|$.

We can associate to every simplex $S$ an parabolic group $P^I$ and a parabolic type $I$, so that we can define a mapping:

$$\pi : \partial^\infty (\Delta_E) \to \bigcup_{I \subset J} G/P^I$$

$$: S \mapsto P^I$$

The mapping $\pi$ can be decomposed in a family of mappings $\pi^\Theta$, such that

$$\pi^\Theta : \partial^I_{\infty} (\Delta_E) \to G/P^I$$
with \( \partial^I_\infty(\Delta_E) \) being the inverse image \( \pi^{-1}(G/P^I) \). The simplices in \( \partial^I_\infty(\Delta_E) \) are called the simplices of \( I \)-type in \( \partial_\infty(\Delta_E) \).

We note that \( G \) acts transitively on \( \partial^\emptyset_\infty(\Delta_E) \) (or equivalently \( \Delta^\emptyset_S \)) and hence may be identified with \( G/P \) and so we give \( \partial^\emptyset_\infty(\Delta_E) \) the quotient topology, called the ultrametric topology.

**Proposition 20** The topological spaces \( \partial^\emptyset_\infty(\Delta_E) \) with the Busemann topology and \( G/P \) with the ultrametric topology are homeomorphic.

**Proof.** The group \( G \) acts continuously and transitively in \( \partial^\emptyset_\infty(\Delta_E) \) with \( P \) stabilizing a chamber, so we have a continuous bijection \( \pi : G/P \to \partial^\emptyset_\infty(\Delta_E) \)

However, since \( \partial^\emptyset_\infty(\Delta_E) \) is compact, \( \pi \) must be a homeomorphism. ■

The compactness of the flag manifold \( G/P \) is a direct consequence of the previous lemma (compare with [Ma, p. 55]). At last, we can conclude that:

**Proposition 21** Given a chamber \( C \) in \( \partial^\emptyset(\Delta_E) \), \( N \) be the maximal unipotent group fixing \( C \) and \( \overline{N} \) the maximal unipotent group fixing the opposite chamber \( -C \), the orbit \( \overline{N} \cdot C \) is open and dense in \( \partial^\emptyset_\infty(\Delta_E) \).

### 4.4 The action of \( G \) in \( |\Delta_E| \)

From here on, whenever needed, we will always assume (and it will be clear from the context) as a basic assumptions that \( G \) is a simple and simply connected group over a local field and \( \Delta_E \) its associated Euclidean building.

As we know (Section 3.3), the geometric realization \( |\Delta_E| \) of the building \( \Delta_E \) is a \( CAT(0) \) space, in which \( G \) acts as an isometry group. We start stating some fundamentals properties of isometries of \( CAT(0) \) spaces.

The **displacement function** \( d_g : X \to \mathbb{R}^+ \) of an isometry \( g \) is defined by \( d_g(x) = d_g(g(x), x) \). The **translation length** of \( g \) is the number \( |g| := \inf \{d_g(x) : x \in X\} \). The set of points \( x \) such that \( d_g(x) = |g| \) is denoted by \( \text{Min} \ (g) \).

An isometry \( g \) is called **semi-simple** if \( \text{Min} \ (g) \) is non-empty.

**Proposition 22** ([BGS] chapter 2) With the notation above established:

1. \( \text{Min} \ (g) \) is \( g \) invariant;
2. If \( h \) is a isometry of \( X \), then \( |g| = |hgh^{-1}| \) and \( \text{Min} \ (hgh^{-1}) = h \text{Min} \ (g) \);
3. \( \text{Min} \ (g) \) is a closed convex set.
The isometries can be classified as

1. **elliptic** if \( g \) has a fixed point;

2. **hyperbolic** if \( d_g \) attains a strictly positive minimum;

3. **parabolic** if \( d_g \) does not attain its minimum (in other words, if \( \text{Min}(g) = \emptyset \)).

**Proposition 23** \([BGS, \text{chapter 2}]\)

1. An isometry \( g \) of \( |\Delta_E| \) is hyperbolic if, and only if, there is a geodesic \( \gamma : \mathbb{R} \to |\Delta_E| \) which is translated non-trivially by \( g \), namely, there is \( a > 0 \) such that \( g \cdot \gamma(t) = \gamma(t + a) \), for every \( t \in \mathbb{R} \). Such a geodesic is called an **axis** for \( g \).

2. The axes of a hyperbolic isometry \( g \) are parallel to each other and its union is \( \text{Min}(g) \).

A hyperbolic isometry \( h \) of an Euclidean building is said to be **regular** if every axis of \( h \) is contained in just one apartment.

**Lemma 24** Let \( \{\Sigma_i\} \) be a family of apartments that contains an axis \( \gamma \) of \( h \). Then \( U := \bigcup \Sigma_i \) and \( I := \bigcap \Sigma_i \) are both invariants under \( h \) in the sense that \( hU = U \) and \( hI = I \). In particular, if \( h \) is a regular hyperbolic isometry, i.e., every axis of \( h \) is contained in just one apartment, this apartment is invariant by \( h \).

**Proof.** Let \( \Sigma_i \) be an apartment containing the axis \( \gamma \). Then \( h \Sigma_i \) is an apartment containing \( \gamma \) since \( h \) leaves \( \gamma \) invariant.

The family \( \{\Sigma_i\} \) is finite (in the case of a local field) and since \( h \) is a bijection, we see that \( h \) acts as a permutations of the family \( \{\Sigma_i\} \). Consequently, we have

\[
hU = \bigcup h\Sigma_i = \bigcup \Sigma_i = U
\]

and

\[
hI = \bigcap h\Sigma_i = \bigcap \Sigma_i = I
\]

whence they are both invariant under \( h \). \( \blacksquare \)

**Proposition 25** Let \( \{\Sigma_i\}_{i=1}^k \) be a collection of apartments in \( |\Delta_E| \), \( |\Lambda| := \cap \Sigma_i \) be a flat in the euclidean building \( |\Delta_E| \), and \( h \) a hyperbolic isometry of \( |\Lambda| \) that preserves the simplicial structure of \( |\Lambda| \). Then:
1. There exist a hyperbolic isometry \( \tilde{h} \) of \( |\Delta_E| \) such that \( \tilde{h}|_{|\Lambda|} = h \).

2. If \( |\Lambda| \) is not a maximal flat (i.e., an apartment) then any two such extensions \( \tilde{h}_1, \tilde{h}_2 \) differ by an element of \( W_{|\Lambda|} \), i.e., \( \tilde{h}_1(\tilde{h}_2)^{-1} \in W_{|\Lambda|} \), where \( \Sigma \) is an apartment containing \( |\Lambda| \) and \( W_{|\Lambda|} \) stand for the subgroup of \( G \) that fixes \( |\Lambda| \) pointwise.

3. If \( |\Lambda| \) is a maximal flat this extension is unique.

**Proof.** Let \( \Sigma \) be a maximal flat containing \( |\Lambda| \). The flat \( \Sigma \) is isometric to \( \mathbb{R}^n \), with \( n \) the rank of the building, and therefore \( |\Lambda| \) is isometric to a subspace of \( \mathbb{R}^n \). We write \( \Sigma = |\Lambda| \oplus |\Lambda|^\perp \), where \( |\Lambda|^\perp \) is the orthogonal complement of \( |\Lambda| \) in \( \Sigma \). Given a decomposition \( x = x_1 + x_2 \) with \( x_1 \in |\Lambda| \) and \( x_2 \in |\Lambda|^\perp \), the map \( \tilde{h} = hx_1 + x_2 \) is an isometry of \( \Sigma \) that extends \( h \) and preserves the simplicial structure of \( \Sigma \). Now to extend \( \tilde{h} \) to \( |\Delta_E| \), consider \( C \) a chamber in \( \Sigma \) and \( \tilde{h}(C) \) its image. The second and third axioms of buildings (Definition 4) assure the existence of an isometry \( \tilde{h} \) of \( |\Delta_E| \) such that \( \tilde{h}(C) = \tilde{h}(C) \) and \( h(\Sigma) = \Sigma \). This isometry satisfies \( \tilde{h}|_{|\Lambda|} = h \).

By definition, we have that \( \tilde{h}_1(\tilde{h}_2)^{-1}|_{|\Lambda|} \) is the identity, and so, \( \tilde{h}_1(\tilde{h}_2)^{-1} \in W_{|\Lambda|} \). In particular, if \( |\Lambda| = \Sigma \) is a maximal flat we have that the composition \( \tilde{h}_1(\tilde{h}_2)^{-1} \) fixes the apartment \( \Sigma \) pointwise and the uniqueness is established.

In a way analogous to the real Lie group case we have that

\[
H_\Sigma := \{ h \in G : h\Sigma = \Sigma, h \text{ hyperbolic} \}
\]

is a maximal torus (page 15) and an isometry is regular if and only if it is contained in a unique maximal torus. If we consider the action of the Weyl group \( W \) on an apartment we can give another characterization of the regularity of \( h \). Let \( \gamma \) be an axis of \( h \) and \( \Sigma \) an apartment containing \( \gamma \).

Given an hyperplane \( P \subset \Sigma \), we denote by \( s_P \) the reflection in \( P \). Consider the set of hyperplanes that determine reflections in \( W \): \( \mathcal{H} = \{ P : s_P \in W \} \).

Considering the action of \( W \) on \( \Sigma \), a hyperbolic isometry \( h \) is of type \( \Theta \) if \( h \) has an axis invariant by a special subgroup \( W_\Theta \), with \( W_\Theta \) a maximal special subgroup with this property. The set \( H_\Sigma^\Theta := \{ h \in H_\Sigma : h \text{ is of type } \Theta \} \) determines a decomposition \( H_\Sigma = \bigcup_{\Theta' \subset \Theta} H_\Sigma^{\Theta'} \) (disjoint union) with \( H_\Sigma^{\Theta'} \) having the dimension of \( H_\Sigma \) and \( \dim H_\Sigma^{\Theta'} < \dim H_\Sigma^{\Theta''} \) whenever \( \Theta' \nsubseteq \Theta \) (details can be found in [13]). In a similar way, given a flat \( |\Lambda| \) in \( |\Delta_E| \) we define \( H_{|\Lambda|} := \{ h \in G : h|_{|\Lambda|} = |\Lambda|, h \text{ hyperbolic} \} \).

Considering any isometric identification of a flat \( |\Lambda| \) with the vector space \( \mathbb{R}^n \), we fix \( x_0 \in |\Lambda| \) and associate to each hyperbolic isometry \( h \) a
vector $v_h = h(x_0) - x_0$, and denote $V_{H[|\Lambda|]} := \{v_h : h \in H[|\Lambda|]\}$. The action of $\mathbb{Z}$ in $V_{H[|\Lambda|]}$ defined by $nv_h = v_{nh}$ turns $V_{H[|\Lambda|]}$ into a $\mathbb{Z}$–module. To each set of hyperbolic isometries $K \subset H[|\Lambda|]$, we associate a vector space $V_K$ spanned by $\{v_h : h \in K\}$. Since the action of $W$ is irreducible, it follows that $|\Lambda| = \langle v_h : h \in H[|\Lambda|]\rangle$.

Up to the moment we studied the action of hyperbolic isometries on single apartments or flats. Now we want to characterize the action of regular hyperbolic isometries in the Euclidean building and in the spherical building.

We stress that, henceforth we consider only buildings associated to a simple and simply connected group $G$ over a local field and hence, all isometries of the associated buildings will be considered to be automorphisms of the building (contained in $G$).

In all that follows, we use equal symbols to designate a simplex of the Euclidean building $|\Delta_E|$ or of its associated spherical building $\partial_\infty(|\Delta_E|)$, the sole distinction being that the latter are in boldface.

Let $h$ be a regular hyperbolic isometry, $|\Lambda|$ the apartment invariant by $h$ and $|\Lambda| = |\Lambda|_\infty$ the apartment in the spherical building $\partial_\infty(|\Delta_E|)$ that is the boundary of the apartment $|\Lambda|$. The main attractor of a hyperbolic isometry $h$ is the point $\xi \in \partial_\infty(|\Lambda|)$ such that $h^n(x) \to \xi$ for all $x \in X$. If $h$ is regular, there is a unique open chamber at the infinite containing $\xi$, denoted by $C(\xi)$. The main attractor of $h^{-1}$ is called the principal repulsor of $h$, denoted by $-\xi$ and $C(-\xi)$ is the chamber opposite to $C(\xi)$ in $|\Lambda|$, where $|\Lambda|$ is the invariant apartment of $h$.

The main purpose of this section is to understand the behavior of the orbit of an element $\eta \in \partial_\infty(|\Lambda|)$ under the action of a regular hyperbolic isometry $h$.

Our first observation is that when $\eta$ is in the apartment invariants by $h$, i.e., $\eta \in |\Lambda| = \partial_\infty(|\Lambda|)$, the action of $h$ in $\eta$ is trivial: $h\eta = \eta$. So we can restrict ourselves to the case $\eta \notin |\Lambda|$. Next we consider an apartment $|\Lambda'|$ (not necessarily unique) containing $\eta$ and $-\xi$. In such a case, we have

$$h^k(\eta) \in \partial_\infty(h^k(|\Lambda'|)) \setminus \partial_\infty(h^k(|\Lambda|)) = \partial_\infty(h^k(|\Lambda'|)) \setminus \partial_\infty(|\Lambda|)$$

since $|\Lambda|$ is $h$-invariant.

To give a more accurate description of the orbit we will study the behavior of some geodesic segments that converge to the points $\eta$ and $\xi$. In our approach, we will focus on the Euclidian building, the corresponding results for the spherical building being obtained by "projecting at infinity".
Lemma 26 ([Bro], pág. 176) If two apartments $|\Lambda|$ and $|\Lambda'|$ have a common chamber $C$ at infinity ($C \in |\Lambda| \cap |\Lambda'|$), then $|\Lambda| \cap |\Lambda'|$ contains a sector $S$ such that $\partial_\infty(S) = C$.

Lemma 27 Let $h$ be a regular hyperbolic isometry, and $\xi$ and $-\xi$ its main attractor and repulsor, respectively. Let $|\Lambda|$ be the apartment invariant by $h$ and $S \subset |\Lambda|$ a sector with $-\xi \in \partial_\infty(S)$. Then $h^kS$ is a family of increasing nested sectors, in the sense that $h^k(S) \subsetneq h^{k+1}(S)$. Moreover, given $x$ in the interior of $S$ and $\gamma$ a geodesic ray starting at $x$ and not entirely contained in $S$, the length $L(\gamma|_{h^k(S)})$ of the segment of $\gamma$ contained in $h^k(S)$ grows linearly.

Proof. Since $|\Lambda|$ is invariant by $h$ and $|\Lambda|$ is isometric to $\mathbb{R}^n$, we translate the problem to an euclidean setting in the which $S$ is a (simplicial) cone $h$ acts as a translation $hx = x + v_h$ where $-\lambda v_h \in S$ for every $\lambda \geq \lambda_0$ for some $\lambda_0 > 0$ and $\gamma$ a straight line passing through points $y_0$ and $y_1$ with $y_0 \in \partial S$ and $y_1 \notin S$. In this setting the problem becomes an elementary euclidean question and it is easy to see that $h^k(S) \subsetneq h^{k+1}(S)$ and $L(\gamma|_{h^k(S)}) = kd(x_0, hx_0)$ for all $x_0 \in |\Lambda|$.

Given a point $x_0$ in the interior of $S$, an hyperbolic isometry $h \in G$ and a point $\eta \in \partial_\infty(|\Delta_E|)$ and an integer $k$ we define $\gamma^k(t)$ as the geodesic ray that starts at $x_0$ and such that $\gamma^k(\infty) = h^k(\eta)$.

Let $\sigma$ be the geodesic ray entirely contained in $|\Lambda|$ with initial point $x_0$ and that is parallel to the segment $\gamma^0|_{|\Lambda|}$. An initial segment of $\sigma$ ($\{\sigma(t) : 0 \leq t \leq \lambda\}$ for some $\lambda \geq 0$) coincides with an initial segment of $\gamma^0$ and in some point they fork as in a tree, with $\sigma$ being prolonged in $|\Lambda|$ and $\gamma^0$ being prolonged in $|\Lambda'|$. The same happens with $\sigma$ and $\gamma^k$, with the difference that $\gamma^k$ is prolonged in $h^k(|\Lambda'|)$. The lemma below describes the growth of this intersection.

Lemma 28 Let $\sigma$ and $\gamma^k$ be defined as above. Then, the intersection of $\sigma$ and $\gamma^k$ is a geodesic segment whose length grows linearly.

Proof. By construction we know that $\sigma \cap \gamma^0 = \sigma \cap S$. Now we prove that $\sigma \cap \gamma^1 = \sigma \cap h(S)$.

Consider the geodesic segment $h\gamma^0$. This segment passes through $h(x_0)$ and $h(\eta)$, and is contained in $h(|\Lambda'|)$. Also, $h\gamma^0 \cap |\Lambda|$ is parallel to $\sigma$.

Now let $l$ be the geodesic contained in $h(|\Lambda'|)$ that is parallel to $h\gamma^0$ and passes through $x_0$. The segment $l \cap |\Lambda|$ is parallel to $\sigma$, i.e., $\sigma \cap \gamma^1 = \sigma \cap h(S)$. By the same argument we conclude that $\sigma \cap \gamma^k = \sigma \cap h^k(S)$ and, by the
previous lemma, we find that the length of this intersection grows linearly, i.e.,

\[ L(\sigma \cap \gamma^k) = L(\sigma \cap \gamma^0) + kd(x_0, hx_0), \forall x_0 \in |\Lambda|. \]

Now we can state our main result describing the asymptotic action of hyperbolic isometries.

**Proposition 29** Let \( h \) be a regular hyperbolic isometry, \(|\Lambda|\) its invariant flat and \( \xi \) its main attractor. Let \( G = KZN \) be the Iwasawa decomposition defined by \( \xi \) and \( |\Lambda| \) (\( N\xi = \xi \) and \( h \in Z \)). Let \( N \) be the unipotent group that fixes \( -\xi \). Let \( C(\eta) = n\omega_0C(\xi) \in \partial_\infty(\Delta_E) \), with \( w_0 \in W \) and \( n \in N \) its expression relative to the given Iwasawa decomposition. Then:

\[
\lim_{k \to \infty} h^k n\omega_0C(\xi) = w_0C(\xi)
\]

**Proof.** Given \( C(\eta) = n\omega_0C(\xi) \) and \(-C(\xi)\), there is a flat \(|\Lambda|' \subset |\Delta_E|\) such that \( C(\eta), -C(\xi) \in |\Lambda|' := \partial_\infty(|\Lambda|') \). Let \( |\Lambda| \) be the invariant flat of \( h \) and \( |\Lambda| := \partial_\infty(|\Lambda|') \). Since \( -\xi \in |\Lambda| \cap |\Lambda|' \), Lemma 26 assures that the intersection of \(|\Lambda|\) and \(|\Lambda|' \) contains a sector \( S \) such that \( \partial_\infty(S) = C(-\xi) \).

Given \( x_0 \in \text{int} S \) let \( \sigma \) be the geodesic ray with initial point at \( x_0 \) such that \( \sigma(\infty) = n\omega_0\xi \) and for each integer \( k \) define the \( \gamma_k(t) \) as the geodesic ray with \( \gamma^k(0) = x_0 \) and \( \gamma^k(\infty) = h^k(\eta) \).

The rays \( \gamma^k \) and \( \sigma \) coincide in the intersection of the apartments \(|\Lambda| \cap h^k(|\Lambda|')\) that increases linearly (by Lemma 28). Then, [Br, Proposition 8.19, pg. 268] assures this characterizes the convergence in the topology of Busemann, and we have that

\[
\lim_{k \to \infty} h^k n\omega_0\xi = w_0\xi
\]

5 Semigroups and Control Theory

In this section, \( G \) denotes a simple, simply connected algebraic group over a local field. We remark that the group \( G \) is also a Lie group over a local field with the ultrametric topology \((G, \tau_U)\). Our main object of study are the semigroups \( S \subset G \) with non-empty interior in the ultrametric topology.

Let \( \xi \in \partial_\infty(\Delta_E) \) be a regular point in the spherical building and \( C(\xi) \) the unique chamber that contains \( \xi \). By Proposition 20 we can identify the
chambers in $\partial_\infty(\Delta_E)$ with the points in the maximal flag $\mathcal{B}$ and so, by abuse of notation, we write that $\mathcal{C}(\xi) \in \mathcal{B}$, or that $\xi \in \mathcal{B}$.

Given $\xi \in \mathcal{B}$, we denote by $B(\xi)$ the open Bruhat cell containing $\xi$.

**Proposition 30** If a semigroup with non-empty interior $S \subset G$ contains a hyperbolic isometry of $\Delta_E$, then $S$ contains a regular hyperbolic isometry in its interior.

**Proof.** Since the simplicial structure of $\Delta_E$ determines a discrete metric, the infimum of a translation function is attained, and so there are no parabolic isometries in $G$. So we can assume $g \in \text{int}(S)$ as being a elliptic isometry. Let $x$ be a point fixed by $g$ and $K_x$ be the maximal compact subgroup fixing $x$. (This subgroup is a compact open subgroup of $G$! [16]) then $g \in S_x = \text{int}(S) \cap K_x$, and so $S_x$ is a non-empty sub-semigroup of $K_x$. Since $K_x$ is compact it follows that $S_x$ is a open subgroup of $K_x$. Now, given a hyperbolic isometry $h \in S$ since $g \in S_x$ and $S_x$ is a subgroup the isometry $ghg^{-1}$ is hyperbolic in the interior of $S$. The existence of a regular hyperbolic isometry follows then from the usual dimension arguments.

Unlike the real case, there are semigroups with non-empty interior that do not contain any hyperbolic isometry.

**Proposition 31** If a semigroup with non-empty interior $S \subset G$ contains nothing but elliptic isometries, then $S$ is an open subgroup of $G$.

**Proof.** By the previous argument, $S$ contains the inverse of any elliptic isometry $g \in \text{int}(S)$. Therefore $\text{int}(S)$ is an open subgroup of $G$. If $s \in S$, and $g \in \text{int}(S)$, then $gs \in \text{int}(S)$. But $g^{-1} \in \text{int}(S)$ so that $s \in \text{int}(S)$.

We assume henceforth that $S \subset G$ is a semigroup of non-empty interior that contains a hyperbolic isometry.

Let $h \in \text{int}(S)$ be a regular hyperbolic isometry and $\eta$ its main attractor.
Lemma 32 Let $\xi \in B$ and $B(\eta)$ the Bruhat cell based at $\eta$, then exists $g \in S$ such that $g\xi \in B(\eta)$.

Proof. The proof is an immediate consequence from the fact that $B(\eta)$ is open and dense on $B$. Let $\xi \in B$ and $A \subset \text{int}(S)$ be a open set. Then, $A\xi$ is open and intercepts the Bruhat cell $B(\eta)$ that is dense in $B$, i.e., $\exists g \in \text{int}(S)$ such that $g\xi \in B(\eta)$.

The flag manifold $B$ is a compact space ([Ma, pág. 55]) and so the theorem yields the existence of an invariant control set in $B$ for the action of $S$. The following result assures the uniqueness of such a control set in $B$.

Theorem 33 Let $S \subset G$ be a semigroup with $\text{int}(S) \neq \emptyset$ and that contains a hyperbolic isometry. Then the invariant control set in $B$ for the action of $S$ is unique.

Proof. We will prove that there exists $\eta$ such that $\eta \in \text{cl}(S\xi)$ for all $\xi \in B$, i.e., $\eta \in \bigcap_{\xi \in B} \text{cl}(S\xi)$. Let $\eta$ be a main attractor for a regular hyperbolic isometry $h \in \text{int}(S)$. By the previous lemma we can assume that $\xi \in B(\eta)$. So we have that $\lim_{k \to \infty} h^k(\xi) = \eta$, for all $\xi \in B(\eta)$ and $\eta \in \text{cl}(S\xi)$ for all $\xi$. Clearly this implies the uniqueness of the invariant control set.

We denote by $D$ the unique invariant control set in $B$ and by $D_0$ its set of transitivity.

We will now characterize the others control sets in $B$.

We define

$$\Xi = \{ h \in \text{int}(S) : h \text{ is a regular hyperbolic isometry}\}$$

Given a regular hyperbolic isometry $h$, there is a single apartment that contains the axis of $h$ and a corresponding Iwasawa decomposition $G = KAN$. The set of chambers in the spherical building fixed by $h$ is in bijection with $W$, so we can associate to each of these chambers a $w$-type(which depends on the choice of $h$) We will denote by $b(h,w)$ the point fixed by $h$ of $w$-type.

Proposition 34 Let $(D_1)_0 := \{ b(h,1) : h \in \Xi \}$. Then $(D_1)_0 = D_0$.

Proof. Since $(D_1)_0$ is the set of main attractors we have that $(D_1)_0$ is contained in $D_0$. To prove the other inclusion, we consider $\eta = b(h,1) \in D_0$ with $h \in \text{int}(S)$ a regular hyperbolic isometry. Given $\xi \in D_0$ exists $s_1, s_2 \in$
int(S) such that $s_1 \eta = \xi$ and $s_2 \xi = \eta$, because $D_0$ is the set of transitivity of $D$. Now we consider $h_2 = s_1 h^n s_2$. The following lemma proves that $s_1 h^n s_2$ is a regular hyperbolic isometry for sufficiently large $n$. □

**Proposition 35** If $\xi$ is the main attractor of a regular hyperbolic isometry $h$ and $s_1, s_2$ are such that $s_2 s_1$ belongs to the parabolic subgroup that stabilizes $\xi$, then $s_1 h^n s_2$ is a regular hyperbolic isometry for sufficiently large $n$.

**Proof.** Let $g = s_2 s_1$, then $s_1 h^n s_2 = s_1 h^n g s_1^{-1}$. So it is enough to prove that $h^n g$ is hyperbolic when $g$ is an isometry fixing the main attractor $\xi$ of $h$.

Let $|\Lambda|$ be the flat left invariant by $h$ and $|\Lambda|' = g |\Lambda|$, $S$ be a sector in $|\Lambda| \cap |\Lambda|'$ and $S_2 = g^{-1} S \cap S$, so we have $g S_2 \subset S$.

Since $S_2$ and $g S_2$ are sectors in $|\Lambda|$ containing $\xi$, we have that, for $n$ large enough, $h^n g S_2 \subset S_2$, i.e., $h^n g$ is a regular hyperbolic isometry. □

**Theorem 36** For every $w \in W$ there exists a control set $D_w$ in $B$, whose set of transitivity is

$$(D_w)_0 = \{ b(h, w) : h \in \Xi \}$$

and these are all the control sets in $B$.

**Proof.** The theorem will be proved in three steps.

In the first step we will prove that for any control set $D'$ there exists a fixed point of $w$-type $b(h, w)$ in $D'$, for some regular hyperbolic isometry $h$.

In the second step we will prove that, given two points with the same $w$-type $\xi = b(h_1, w)$ and $\eta = b(h_2, w)$, with $w \in W$ and $h_1, h_2 \in \Xi$, then $\xi \in \text{cl}(S \eta)$ and $\eta \in \text{cl}(S \xi)$. Therefore, by Proposition 35, $(D_w)_0$ is contained in the transitivity set of a control set $D_w$.

At last, we will prove that $(D_w)_0 := \{ b(h, w) : h \in \Xi \}$ is the set of transitivity of $D_w$.

(1) Given a control set $D'$ and $\eta \in D'$, let $P$ be the isotropy group of $\eta$. If $\eta \in D'_0$ then $P \cap \text{int}(S) \neq \emptyset$. The subgroup $P$ admits an Iwasawa decompositions as $P = MAN^+$. The subset

$$\sigma = \{ m \in M : \exists h n \in AN^+ \text{ with } m h n \in \text{int}(S) \}$$

has non-empty interior in $M$, since $M$ normalizes $AN$. The fact that $M$ is compact implies that $\sigma$ is a subgroup of $M$. So $\text{int}(S) \cap AN^+ \neq \emptyset$, and hence there exists a regular hyperbolic isometry $g \in \text{int}(S)$ such that $g \eta = \eta$. In particular, $\eta$ is a fixed point of $w$-type for $g$. 26
(2) Given $\xi = b(h_1, w)$ and $\eta = b(h_2, w)$, if $b(h_1, 1) = b(h_2, 1)$ then $b(h_1, w)$ is in the same Bruhat cell that as $b(h_2, w)$, so $\lim_{n \to \infty} (h_2)^n b(h_1, w) = b(h_2, w)$. If $b(h_1, 1) \neq b(h_2, 1)$, the Proposition [34] assures that exists $s_1 \in S$, such that $s_1 b(h_1, 1) = b(h_2, 1)$ since $(D_1)_0$ is a transitivity set, and thereby we reduce to the previous case.

(3) Now we will prove that for any control set $D'$ its set of transitivity is $(D_w)_0$ for some $w$. Let $\eta = b(g, w) \in D'$ be the attractor of $w$-type in $D'$ whose existence was proved in (1).

Given $\xi \in (D_1)_0$, we have $s_1, s_2 \in \text{int}(S)$ such that $s_1 \eta = \xi$ and $s_2 \xi = \eta$. Then $h_2 = s_1 g^n s_2$ is a regular hyperbolic isometry and $\xi = b(s_1 h^n s_2, w_1)$. Thus $D_w$ is the set of transitivity of $D'$.

5.1 Subgroup $W(S)$

In the previous section, we proved that any control set for $S$ in $\mathbb{B}$ is of the form $D_w$ for some $w \in W$.

We now turn to the problem of determining when two such control sets $D_w, D_{w'}$ coincide. To start with, define a subset $W(S)$ of $W$ by the rule

$$W(S) = \{w \in W : D_w = D_1\}$$

By its definition, $W(S)$ depends on the choice of a chamber $C^+$ in the apartment $|A| \subset \partial_{\infty \infty}(X)$ which we fix once and for all. (Notice that if we had chosen the chamber $C^+_1 = gC^+$ instead of $C^+$, then

$$w \in W(S, C^+) \iff gwg^{-1} \in W(S, gC^+).$$

so that conjugation by $g$ defines an isomorphism between $W(S, C^+)$ and $W(S, gC^+)$, which does not depend on the choice of $g$ taking $C^+$ to $gC^+$ since any two such elements differ by an element that fixes pointwise the chamber $C^+$.

From now on, let $b_0$ be the image in $\mathbb{B}$ of the chamber $C^+$ (used to define $W(S) = W(S, C^+)$.)

Lemma 37 Given $b_0 \in (D_1)_0$ Then are equivalent:

1. $w \in W(S)$;

2. $\tilde{w}b_0 \in (D_1)_0$, with $\tilde{w}$ being a representative of $w$ in $M^*$, and $W = M^*/M$. 

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Proof. Let $b_0 \in (D_1)_0$, $w \in W(S)$ and $\tilde{w}$ be a representative of $w$ in $M^*$. By the characterization of $(D_w)$, $\tilde{w} b_0 \in (D_w)_0$, i.e., $\tilde{w} b_0 \in (D_w)_0 = (D_1)_0$. As for the converse, by definition any $w \in W(S)$ is such that $D_w = D_1$, hence $(D_w)_0 = (D_1)_0$. However $(D_w)_0 = \{b(h, w) : h \in \Xi\}$ and therefore $wb_0 \in (D_w)_0$.

Proposition 38 $W(S)$ is a subgroup of $W$.

Proof. Let $b \in (D_1)_0$, $w_1$ and $w_2 \in W(S)$ and $\tilde{w}_1$ and $\tilde{w}_2$ they representatives in $M^*$. By the Lemma 37 $\tilde{w}_1 b \in (D_1)_0$. Another application of 37 for $\tilde{w}_1 b$ allows us to conclude that $(\tilde{w}_1 \tilde{w}_2 (\tilde{w}_1)^{-1}) \tilde{w}_1 b \in D_0$ and therefore $\tilde{w}_1 \tilde{w}_2 b \in D_0$. Hence $W(S)$ is a semigroup of $W$, but since $W$ is finite $W(S)$ is a subgroup.

$W(S)$ is not only a subgroup of $W$, but a Weyl subgroup of $W$, i.e., $W(S) = W_\Theta$ for some subset $\Theta \subset \Pi$.

Theorem 39 $W(S) = W_\Theta$, for some $\Theta \subset \Pi$

Proof. Let $H = \{h \in \text{int}(S) : W(S)h = h\}$.

Suppose that $H \neq \emptyset$ and let $\hat{h} \in H$ be an isometry of maximum regularity in $H$, and $\Theta$ be its type. We will show that $W(S) = W_\Theta$ proving that $W(S)$ acts transitively in $\mathcal{C}_h = \{\text{the chambers that have } \hat{h} \text{ as a wall}\}$.

Since $\hat{h} \in \text{int}(S)$ there is a regular hyperbolic isometry $h_N \in \text{int}(S)$ whose main attractor is the chamber $N \in \mathcal{C}_h$.

So every chamber $N \in \mathcal{C}_h$ belongs to the set of transitivity of the invariant control set in $\mathcal{B}$ and so $S$ acts transitively in $\mathcal{C}_h$.

Suppose now that $H = \emptyset$; we will prove that in this case $W = W(S)$. Let $C$ be a chamber in the set of transitivity $D_0$ of the invariant control set in $\mathcal{B}$ and let $h$ be a regular hyperbolic isometry that has $C$ as its the main attractor.

We denote $h_w = wh^n$ and remark that all $h_w$ are regular hyperbolic isometries for $n$ sufficiently large. Let $K$ be the cone and $K_N$ be the lattice defined as follow:

$$K = \left\{ \sum_{w \in W} a_w v_{h_w} : a_w \in \mathbb{R}^+ \right\}$$

$$K_N = \left\{ \sum_{w \in W} a_w v_{h_w} : a_w \in \mathbb{N} \right\}$$

Notice that $K$ is a vector space, because the absence of fixed points implies that $v_{h_1} + \ldots + v_{h_n} = 0$. And as the action of $W$ is irreducible
and $W(S) \subset W$ acts without fixed points we have that $Kt$ is the whole apartment $\Lambda$.

Since $K_N \subset \text{int}(S)$ is a lattice in $K$ we know that for any chamber $C \in \partial_\infty(|\Lambda|)$ there exists a hyperbolic isometry $h' \in K_N \subset \text{int}(S)$, with main attractor $C$. Therefore $W(S)$ acts transitively in the chambers of $\partial_\infty(|\Lambda|)$. But, as $W$ acts simply transitive in the chambers of $\partial_\infty(|\Lambda|)$, we conclude that $W(S) = W$. ■

The map $w \rightarrow D_w$ defined in the theorem of characterization of $(D_w)_0$ is not necessarily bijective. The following theorem says that we can parametrize the control sets by the lateral classes $W(S) \setminus W$.

**Theorem 40** $D_{w_1} = D_{w_2}$ if and only if $w_1w_2^{-1} \in W(S)$. Hence the control sets of $S$ in $\mathbb{B}$ are in bijection with $W(S) \setminus W$.

**Proof.** It suffices to show that $(D_{w_1})_0 = (D_{w_2})_0$ if and only if $w_1w_2^{-1} \in W(S)$.

$(\Rightarrow)$ Let $b_0 \in D_0 = (D_1)_0$. By the characterization of the control sets, $\tilde{w}_1b_0 \in (D_1)_0$, and by hypothesis we have $(D_{w_1})_0 = (D_{w_2})_0$. Since $\tilde{w}_1b \in (D_{w_2})$, we have by the definition of $(D_{w_2})_0$ and by conjugation, that $(\tilde{w}_1\tilde{w}_2(\tilde{w}_1)^{-1})^{-1}\tilde{w}_1b \in D_0$, and so $(\tilde{w}_2)^{-1}\tilde{w}_1 \in W(S)$.

$(\Leftarrow)$ Let $b' = \tilde{w}_1b_0 \in (D_1)_0$. Then $(\tilde{w}_2)^{-1}b' \in D_0$, since $w_2^{-1}w_1 \in W(S)$. And $(\tilde{w}_1\tilde{w}_2(\tilde{w}_1)^{-1})^{-1}\tilde{w}_1b_0 \in D_0$ and consequently $b' \in (D_{w_2})_0$, by the definition of $(D_{w_2})_0$. ■

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