On Opial-type inequalities via fractional calculus

Portilla, Ana
Saint Louis University, Madrid Campus
Avenida del Valle 34, 28003 Madrid, Spain
aportil2@slu.edu

Rodríguez, José M.
Universidad Carlos III de Madrid, Departamento de Matemáticas
Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain
jomaro@math.uc3m.es

Sigarreta Almira, José M.
Universidad Autónoma de Guerrero, Centro Acapulco
CP 39610, Acapulco de Juárez, Guerrero, Mexico
josemariasigarretaalmira@hotmail.com

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Abstract

Inequalities play an important role in pure and applied mathematics. In particular, Opial inequality plays a main role in the study of the existence and uniqueness of initial and boundary value problems for differential equations. It has several interesting generalizations. In this work we prove some new Opial-type inequalities, and we apply them to generalized Riemann-Liouville-type integral operators.

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1 Introduction

Integral inequalities are used in countless mathematical problems such as approximation theory and spectral analysis, statistical analysis and the theory of
distributions. Studies involving integral inequalities play an important role in several areas of science and engineering.

In recent years there has been a growing interest in the study of many classical inequalities applied to integral operators associated with different types of fractional derivatives, since integral inequalities and their applications play a vital role in the theory of differential equations and applied mathematics. Some of the inequalities studied are Gronwall, Chebyshev, Jensen-type, Hermite-Hadamard-type, Ostrowski-type, Grüss-type, Hardy-type, Gagliardo-Nirenberg-type, reverse Minkowski and reverse Hölder inequalities (see, e.g., [6, 8, 15, 17, 19, 20, 21, 22, 23, 24]).

In this work we obtain new Opial-type inequalities, and we apply them to the generalized Riemann-Liouville-type integral operators defined in [5], which include most of known Riemann-Liouville-type integral operators.

2 Preliminaries

One of the first operators that can be called fractional is the Riemann-Liouville fractional derivative of order \( \alpha \in \mathbb{C} \), with \( \Re(\alpha) > 0 \), defined as follows (see [7]).

**Definition 1** Let \( a < b \) and \( f \in L^1((a, b); \mathbb{R}) \). The right and left side Riemann-Liouville fractional integrals of order \( \alpha \), with \( \Re(\alpha) > 0 \), are defined, respectively, by

\[
\text{RL}J^\alpha_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^a (t-s)^{\alpha-1} f(s) \, ds, \\
\text{RL}J^\alpha_{b-} f(t) = \frac{1}{\Gamma(\alpha)} \int_b^t (s-t)^{\alpha-1} f(s) \, ds,
\]

with \( t \in (a, b) \).

When \( \alpha \in (0, 1) \), their corresponding Riemann-Liouville fractional derivatives are given by

\[
\text{RL}D^\alpha_{a+} f(t) = \frac{d}{dt} \left( \text{RL}J^{1-\alpha}_{a+} f(t) \right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)}{(t-s)^{\alpha}} \, ds, \\
\text{RL}D^\alpha_{b-} f(t) = -\frac{d}{dt} \left( \text{RL}J^{1-\alpha}_{b-} f(t) \right) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{f(s)}{(s-t)^{\alpha}} \, ds.
\]

Other definitions of fractional operators are the following ones.

**Definition 2** Let \( a < b \) and \( f \in L^1((a, b); \mathbb{R}) \). The right and left side Hadamard fractional integrals of order \( \alpha \), with \( \Re(\alpha) > 0 \), are defined, respectively, by

\[
\text{H}^\alpha_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left( \log \frac{t}{s} \right)^{\alpha-1} \frac{f(s)}{s} \, ds,
\]

where \( \alpha \in (0, 1) \).
and
\[ H_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left( \log \frac{s}{t} \right)^{-1} \frac{f(s)}{s} \, ds, \tag{4} \]
with \( t \in (a, b) \).

When \( \alpha \in (0, 1) \), Hadamard fractional derivatives are given by the following expressions:
\[
(H^\alpha D^a_{t+} f)(t) = t \frac{d}{dt} (H^{1-\alpha}_{a+} f(t)) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \left( \log \frac{t}{s} \right)^{-\alpha} \frac{f(s)}{s} \, ds,
\]
\[
(H^\alpha D^a_{t-} f)(t) = -t \frac{d}{dt} (H^{1-\alpha}_{b-} f(t)) = -\frac{1}{\Gamma(1-\alpha)} t \frac{d}{dt} \int_t^b \left( \log \frac{s}{t} \right)^{-\alpha} \frac{f(s)}{s} \, ds,
\]
with \( t \in (a, b) \).

**Definition 3** Let \( 0 < a < b \), \( g : [a, b] \to \mathbb{R} \) an increasing positive function on \( (a, b) \) with continuous derivative on \( (a, b) \), \( f : [a, b] \to \mathbb{R} \) an integrable function, and \( \alpha \in (0, 1) \) a fixed real number. The right and left side fractional integrals in \([10]\) of order \( \alpha \) of \( f \) with respect to \( g \) are defined, respectively, by
\[
I^\alpha_{g, a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g'(s)f(s)}{(g(t) - g(s))^{1-\alpha}} \, ds, \tag{5}
\]
and
\[
I^\alpha_{g, b-} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{g'(s)f(s)}{(g(s) - g(t))^{1-\alpha}} \, ds, \tag{6}
\]
with \( t \in (a, b) \).

There are other definitions of integral operators in the global case, but they are slight modifications of the previous ones.

### 3 General fractional integral of Riemann-Liouville type

Now, we give the definition of a general fractional integral in \([5]\).

**Definition 4** Let \( a < b \) and \( \alpha \in \mathbb{R}^+ \). Let \( g : [a, b] \to \mathbb{R} \) be a positive function on \( (a, b) \) with continuous positive derivative on \( (a, b) \), and \( G : [0, g(b) - g(a)] \times (0, \infty) \to \mathbb{R} \) a continuous function which is positive on \( (0, g(b) - g(a)] \times (0, \infty) \). Let us define the function \( T : [a, b] \times [a, b] \times (0, \infty) \to \mathbb{R} \) by
\[
T(t, s, \alpha) = \frac{G(|g(t) - g(s)|, \alpha)}{g'(s)}.
\]
The right and left integral operators, denoted respectively by $J_{T,a}^\alpha$ and $J_{T,b}^\alpha$, are defined for each measurable function $f$ on $[a, b]$ as

$$J_{T,a}^\alpha f(t) = \int_a^t \frac{f(s)}{T(t,s,\alpha)} \, ds,$$

$$J_{T,b}^\alpha f(t) = \int_t^b \frac{f(s)}{T(t,s,\alpha)} \, ds,$$

with $t \in [a, b]$.

We say that $f \in L^1_T[a, b]$ if $J_{T,a}^\alpha |f|(t), J_{T,b}^\alpha |f|(t) < \infty$ for every $t \in [a, b]$.

Note that these operators generalize the integral operators in Definitions 1, 2 and 3:

(A) If we choose $g(t) = t, \ G(x, \alpha) = \Gamma(\alpha) x^{1-\alpha}, \ T(t, s, \alpha) = \Gamma(\alpha) |t - s|^{1-\alpha}$,

then $J_{T,a}^\alpha$ and $J_{T,b}^\alpha$ are the right and left Riemann-Liouville fractional integrals $RLJ_{a}^\alpha$ and $RLJ_{b}^\alpha$ in 1 and 2, respectively. Its corresponding right and left Riemann-Liouville fractional derivatives are

$$\left( RLD_{a}^\alpha f \right)(t) = \frac{d}{dt} \left( RLJ_{a}^{1-\alpha} f(t) \right), \ \ (RLD_{b}^\alpha f)(t) = -\frac{d}{dt} \left( RLJ_{b}^{1-\alpha} f(t) \right).$$

(B) If we choose

$$g(t) = \log t, \ G(x, \alpha) = \Gamma(\alpha) x^{1-\alpha}, \ T(t, s, \alpha) = \Gamma(\alpha) t \left| \log \frac{t}{s} \right|^{1-\alpha},$$

then $J_{T,a}^\alpha$ and $J_{T,b}^\alpha$ are the right and left Hadamard fractional integrals $H_{a}^\alpha$ and $H_{b}^\alpha$ in 3 and 4, respectively. Its corresponding right and left Hadamard fractional derivatives are

$$\left( HD_{a}^\alpha f \right)(t) = t \frac{d}{dt} \left( H_{a}^{1-\alpha} f(t) \right), \ (HD_{b}^\alpha f)(t) = -t \frac{d}{dt} \left( H_{b}^{1-\alpha} f(t) \right).$$

(C) If we choose a function $g$ with the properties in Definition 4 and

$$G(x, \alpha) = \Gamma(\alpha) x^{1-\alpha}, \ T(t, s, \alpha) = \Gamma(\alpha) \frac{|g(t) - g(s)|^{1-\alpha}}{g'(s)},$$

then $J_{T,a}^\alpha$ and $J_{T,b}^\alpha$ are the right and left fractional integrals $I_{g,a}^\alpha$ and $I_{g,b}^\alpha$ in 5 and 6, respectively.
Definition 5 Let $a < b$ and $\alpha \in \mathbb{R}^+$. Let $g : [a, b] \to \mathbb{R}$ be a positive function on $(a, b]$ with continuous positive derivative on $(a, b)$, and $G : [0, g(b) - g(a)] \times (0, \infty) \to \mathbb{R}$ a continuous function which is positive on $(0, g(b) - g(a)] \times (0, \infty)$. For each function $f \in L_1^1[a, b]$, its right and left generalized derivative of order $\alpha$ are defined, respectively, by

$$D^\alpha_{T,a^+} f(t) = \frac{1}{g'(t)} \frac{d}{dt} \left( J_{1-\alpha}^{1-\alpha} f(t) \right),$$

$$D^\alpha_{T,b^-} f(t) = -\frac{1}{g'(t)} \frac{d}{dt} \left( J_{1-\alpha}^{1-\alpha} f(t) \right).$$

for each $t \in (a, b)$.

Note that if we choose $g(t) = t, G(x, \alpha) = \Gamma(\alpha) x^{1-\alpha}, T(t, s, \alpha) = \Gamma(\alpha) |t - s|^{1-\alpha}$, then $D^\alpha_{T,a^+} f(t) = RL D^\alpha_a f(t)$ and $D^\alpha_{T,b^-} f(t) = RL D^\alpha_b f(t)$. Also, we can obtain Hadamard and others fractional derivatives as particular cases of this generalized derivative.

4 Opial-type inequality

In 1960, Opial [18] proved the following inequality:

If $f \in C^1[0, h]$ satisfies $f(0) = f(h) = 0$ and $f(x) > 0$ for all $x \in (0, h)$, then

$$\int_0^h |f(x)f'(x)| \, dx \leq \frac{h}{T} \int_0^h |f'(x)|^2 \, dx.$$ 

Opial’s inequality and its generalizations play a main role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations [1, 2, 4, 11, 14]. For an extensive survey on these Opial-type inequalities, see [1, 14].

We need the following result in [12, p.44] (see the original proof in [16]). Although the result in [12, p.44] deals with measures on $(0, \infty)$, it can be reformulated for measures on a compact interval (see, e.g., [3, Theorem 3.1]).

Muckenhoupt inequality. Let us consider $1 \leq p \leq q < \infty$ and measures $\mu_0, \mu_1$ on $[a, b]$ with $\mu_0\{\{b\}\} = 0$. Then there exists a positive constant $C$ such that

$$\left\| \int_a^x u(t) \, dt \right\|_{L^p([a, b], \mu_0)} \leq C \left\| u \right\|_{L^q([a, b], \mu_1)}$$

for any measurable function $u$ on $[a, b]$, if and only if

$$B := \sup_{a < x < b} \mu_0([x, b])^{1/q} \left\| (d\mu_1/dx)^{-1} \right\|_{L^{1/(p-1)}((a, x))}^{1/p} < \infty,$$  \hspace{1cm} (10)
where we use the convention $0 \cdot \infty = 0$. Moreover, we can choose

$$C = \begin{cases} B \left( \frac{q}{q-1} \right)^{1/p} q^{1/q}, & \text{if } p > 1, \\ B, & \text{if } p = 1. \end{cases} \quad (11)$$

Muckenhoupt inequality allows to improve Opial inequality in several ways:

1. we allow to integrate with respect to very general measures,
2. we do not need the hypothesis $f(b) = 0$,
3. we do not need the hypothesis $f > 0$ on $(a, b)$,
4. we substitute the hypothesis $f \in C^1[a, b]$ by the weaker one: $f$ is an absolutely continuous function on $[a, b]$.

**Theorem 6** Let us consider $1 \leq p \leq q < \infty$ and measures $\mu_0, \mu_1$ on $[a, b]$ with $\mu_0([b]) = 0$ and

$$B := \sup_{a < x < b} \mu_0([x, b])^{1/q} \left( d\mu_1/dx \right)^{-1} \left\| u \right\|_{L^{1/(p-1)}([a, x])}^{1/p} < \infty.$$

Then

$$\left\| f' \right\|_{L^1([a, b], \mu_0)} \leq B \left( \frac{q}{q-1} \right)^{(p-1)/p} q^{1/q} \left\| f' \right\|_{L^p([a, b], \mu_1)} \left\| f' \right\|_{L^{q/(q-1)}([a, b], \mu_0)}$$

if $p > 1$, and

$$\left\| f' \right\|_{L^1([a, b], \mu_0)} \leq B \left\| f' \right\|_{L^1([a, b], \mu_1)} \left\| f' \right\|_{L^{q/(q-1)}([a, b], \mu_0)}$$

for every absolutely continuous function $f$ on $[a, b]$ with $f(a) = 0$.

**Proof.** By Muckenhoupt inequality, the constant

$$C = \begin{cases} B \left( \frac{q}{q-1} \right)^{(p-1)/p} q^{1/q}, & \text{if } p > 1, \\ B, & \text{if } p = 1. \end{cases}$$

satisfies

$$\left\| \int_a^x u(t) \, dt \right\|_{L^q([a, b], \mu_0)} \leq C \left\| u \right\|_{L^p([a, b], \mu_1)}$$

for any measurable function $u$ on $[a, b]$. For each absolutely continuous function $f$ on $[a, b]$ with $f(a) = 0$, we have that there exists $f'$ a.e. on $[a, b]$, $f' \in L^1[a, b]$ and

$$f(x) = \int_a^x f'(t) \, dt$$

for every $x \in [a, b]$. Consequently,

$$\left\| f \right\|_{L^q([a, b], \mu_0)} \leq C \left\| f' \right\|_{L^p([a, b], \mu_1)}.$$
Hence, Hölder inequality gives

$$\|ff'\|_{L^1([a,b],\mu)} \leq \|f\|_{L^p([a,b],\mu)} \|f'\|_{L^{q/(q-1)}([a,b],\mu)}$$

$$\leq C \|f'\|_{L^p([a,b],\mu)} \|f''\|_{L^{q/(q-1)}([a,b],\mu)}.$$

**Remark 7** For each absolutely continuous function $f$ on $[a,b]$ the set

$$S = \{ x \in [a,b] : \exists f'(x) \}$$

has zero Lebesgue measure, but it is possible to have $\mu_0(S) > 0$ and/or $\mu_1(S) > 0$. The argument in the proof of Theorem 6 gives that the inequality holds for any fixed choice of values of $f'$ on $S$.

Theorem 8 has the following direct consequence.

**Corollary 8** Let us consider $1 \leq p \leq q < \infty$ and a measure $\mu$ on $[a,b]$ with $\mu(\{b\}) = 0$ and

$$B := \sup_{a < x < b} \mu([x,b])^{1/q}(d\mu/dx)^{-1} \|f\|_{L^p([a,b],\mu)}^{1/p} < \infty.$$

Then

$$\|ff'\|_{L^1([a,b],\mu)} \leq B \left( \frac{q}{q-1} \right)^{(p-1)/p} q^{1/q} \|f'\|_{L^p([a,b],\mu)} \|f''\|_{L^{q/(q-1)}([a,b],\mu)}$$

if $p > 1$, and

$$\|ff'\|_{L^1([a,b],\mu)} \leq B \|f'\|_{L^1([a,b],\mu)} \|f''\|_{L^{q/(q-1)}([a,b],\mu)},$$

for every absolutely continuous function $f$ on $[a,b]$ with $f(a) = 0$.

Corollary 8 has the following consequence.

**Corollary 9** Let us consider $1 \leq p \leq 2$ and a measure $\mu$ on $[a,b]$ with $\mu(\{b\}) = 0$ and

$$B := \sup_{a < x < b} \mu([x,b])^{(p-1)/p}(d\mu/dx)^{-1} \|f\|_{L^p([a,b],\mu)}^{1/p} < \infty.$$

If $1 < p \leq 2$, then

$$\|ff'\|_{L^1([a,b],\mu)} \leq B \left( \frac{p^2}{p-1} \right)^{(p-1)/p} \|f'\|_{L^p([a,b],\mu)}^2$$

for every absolutely continuous function $f$ on $[a,b]$ with $f(a) = 0$.

Furthermore, if $\mu$ is a finite measure, then

$$\|ff'\|_{L^1([a,b],\mu)} \leq B \|f'\|_{L^1([a,b],\mu)}^2$$

for every absolutely continuous function $f$ on $[a,b]$ such that $f(a) = 0$. 

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Corollary 8 gives the result, since we have proved that

\[ f \text{ function for every } 1 < p \leq q \leq q \text{ and } \mu \text{ such that } 1/p + 1/q = 1, \]

and so, \( p = q/(q - 1) \) and \( q = p/(p - 1) \). Thus, \( 1 < p \leq 2 \leq q < \infty \) and Corollary gives the result, since

\[ B \left( \frac{q}{q - 1} \right)^{(p - 1)/p} q^{1/q} = B \left( \frac{p}{p - 1} \right)^{(p - 1)/p} = B \left( \frac{p^2}{p - 1} \right)^{(p - 1)/p}. \]

Assume now that \( \mu \) is a finite measure, and fix an absolutely continuous function \( f \) on \([a, b]\) such that \( f(a) = 0 \) and \( f' \in L^{p_0}([a, b], \mu) \) for some \( p_0 > 1 \).

We have proved that

\[ \|f\|_{L^1([a, b], \mu)} \leq B \left( \frac{p^2}{p - 1} \right)^{(p - 1)/p} \|f'\|_{L^p([a, b], \mu)} \]

for every \( 1 < p \leq \min\{p_0, 2\} \).

Let us consider \( B = B(p) \) as a function of \( p \). Thus,

\[ B(p) \leq \mu([a, b])^{(p - 1)/p} \|d\mu/dx\|^{-1}_{L^{1/(p - 1)}([a, b])}. \]

Since \( \mu \) is a finite measure, we have

\[ \lim_{p \to 1^+} B(p) \leq \lim_{p \to 1^+} \mu([a, b])^{(p - 1)/p} \|d\mu/dx\|^{-1}_{L^{1/(p - 1)}([a, b])} = \|d\mu/dx\|^{-1}_{L^\infty([a, b])} = B(1). \]

Since

\[ |f'|^p \leq |f'|^{p_0} \mathds{1}_{\{|f'| \geq 1\}} + \mathds{1}_{\{|f'| < 1\}} \leq |f'|^{p_0} + 1 \in L^1([a, b], \mu) \]

for every \( 1 < p \leq p_0 \), dominated convergence theorem gives

\[ \lim_{p \to 1^+} \|f'\|_{L^p([a, b], \mu)}^2 = \|f'\|_{L^1([a, b], \mu)}^2. \]

Finally, we have

\[ \lim_{p \to 1^+} \left( \frac{p^2}{p - 1} \right)^{(p - 1)/p} = 1, \]

and the desired inequality holds if \( f' \in L^{p_0}([a, b], \mu) \) for some \( p_0 > 1 \).

Let us consider now any absolutely continuous function \( f \) on \([a, b]\) such that \( f(a) = 0 \). Define the measure \( \mu^* \) on \([a, b]\) by \( d\mu^* = d\mu + dx \). Since \( f \) is an absolutely continuous function on \([a, b]\), \( f' \in L^1([a, b], \mu) \). If \( f' \notin L^1([a, b], \mu) \), then the inequality is direct. So, we can assume that \( f' \in L^1([a, b], \mu) \). Thus, there exists a sequence \( \{s_n\} \) of simple functions with

\[ \lim_{n \to \infty} \|f' - s_n\|_{L^1([a, b], \mu^*)} = 0. \]

Hence, there exists \( N \) such that

\[ \|s_n\|_{L^1([a, b], \mu^*)} - \|f'\|_{L^1([a, b], \mu^*)} \leq \|f' - s_n\|_{L^1([a, b], \mu^*)} < 1. \]
for every \( n \geq N \). Therefore,
\[
\|s_n\|_{L^1([a,b],\mu)} \leq \|s_n\|_{L^1([a,b],\mu^*)} \leq \|f'\|_{L^1([a,b],\mu^*)} + 1
\]
for every \( n \geq N \).

Since \( \mu \) is a finite measure, if we define \( f_n(x) = \int_a^x s_n(t) \, dt \), then \( f_n \in C[a,b] \subset L^p([a,b],\mu) \) for every \( p \geq 1 \), and we have proved that
\[
\|f_n f'_n\|_{L^1([a,b],\mu)} \leq B \|f'_n\|^2_{L^1([a,b],\mu)}.
\]
We have for any \( x \in [a, b] \)
\[
|f(x) - f_n(x)| = \left| \int_a^x (f'(t) - s_n(t)) \, dt \right| \leq \int_a^x |s_n(t)| \, dt \leq \|f' - s_n\|_{L^1([a,b],\mu^*)}.
\]
We have
\[
\|ff' - f_n f'_n\|_{L^1([a,b],\mu)} = \int_a^b |ff' - f_n f'_n| \, d\mu
\]
\[
\leq \int_a^b |ff' - f_n| \, d\mu + \int_a^b |f_n f' - f_n f'_n| \, d\mu
\]
\[
\leq \|f\|_\infty \int_a^b |f' - f'_n| \, d\mu + \|f' - s_n\|_{L^1([a,b],\mu^*)} \int_a^b |s_n| \, d\mu
\]
\[
\leq \|f\|_\infty \|f' - s_n\|_{L^1([a,b],\mu^*)} + \|f' - s_n\|_{L^1([a,b],\mu^*)} (\|f'\|_{L^1([a,b],\mu^*)} + 1)
\]
for every \( n \geq N \). Hence,
\[
\lim_{n \to \infty} \|ff' - f_n f'_n\|_{L^1([a,b],\mu)} = 0
\]
and so,
\[
\|ff'\|_{L^1([a,b],\mu)} \leq B \|f'\|^2_{L^1([a,b],\mu)}.
\]

If we choose \( \mu \) as the Lebesgue measure on \([a, b]\), then we obtain the following results.

**Corollary 10** Let us consider \( 1 \leq p \leq q < \infty \). Then
\[
\|ff'\|_{L^1([a,b])} \leq \left( \frac{b - a}{1/q + (p-1)/p} \right)^{1/q + (p-1)/p} \left( \frac{q(p-1)}{p(q-1)} \right)^{(p-1)/p} \|f'\|_{L^p([a,b])} \|f''\|_{L^{p/(q-1)}([a,b])}
\]
if \( p > 1 \), and
\[
\|ff'\|_{L^1([a,b])} \leq (b - a)^{1/q} \|f'\|_{L^1([a,b])} \|f''\|_{L^{p/(q-1)}([a,b])},
\]
for every absolutely continuous function \( f \) on \([a, b]\) with \( f(a) = 0 \).
Proof. Let us compute

$$B = \sup_{a < x < b} (b - x)^{1/q}(x - a)^{(p-1)/p}. $$

For each $\alpha > 0$ and $\beta \geq 0$, consider the function $u$ defined on $[a, b]$ as

$$u(x) = (b - x)^\alpha(x - a)^\beta.$$ 

If $\beta = 0$, then

$$\sup_{a < x < b} u(x) = u(a) = (b - a)^\alpha.$$ 

Assume now that $\beta > 0$. We have for $a < x < b$

$$u'(x) = -\alpha(b - x)^{\alpha-1}(x - a)^\beta + \beta(b - x)^\alpha(x - a)^{\beta-1} = 0\iff\beta(b - x)^\alpha(x - a)^{\beta-1} = \alpha(b - x)^{\alpha-1}(x - a)^\beta$$

$$\iff\beta(b - x) = \alpha(x - a)$$

$$\iff x = \frac{a\alpha + b\beta}{\alpha + \beta}.$$ 

Since $u(a) = u(b) = 0$, we have

$$\sup_{a < x < b} u(x) = \max_{a \leq x \leq b} u(x) = u\left(\frac{a\alpha + b\beta}{\alpha + \beta}\right) = \left(\frac{\alpha(b - a)}{\alpha + \beta}\right)^\alpha \left(\frac{\beta(b - a)}{\alpha + \beta}\right)^\beta = \frac{\alpha^\alpha \beta^\beta}{(\alpha + \beta)^{\alpha + \beta}}(b - a)^{\alpha + \beta}.$$ 

Thus, $B = (b - a)^{1/q}$ if $p = 1$, and

$$B = \frac{(1/q)^{1/q}((p-1)/p)^{(p-1)/p}}{(1/q + (p-1)/p)^{(p-1)/p}}(b - a)^{1/q+(p-1)/p},$$

$$B\left(\frac{q}{q-1}\right)^{(p-1)/p}q^{1/q} = \left(\frac{b - a}{1/q + (p-1)/p}\right)^{1/q+(p-1)/p} \left(\frac{q(p - 1)}{p(q - 1)}\right)^{(p-1)/p}$$

if $p > 1$. Hence, Corollary 8 gives the result.  

Corollary 11 Let us consider $1 \leq p \leq 2$. Then

$$\|ff'\|_{L^1([a,b])} \leq \left(\frac{p(b - a)}{2(p-1)^{1/2}}\right)^{2(p-1)/p}\|f'\|_{L^p([a,b])}^2$$

if $1 < p \leq 2$, and

$$\|ff'\|_{L^1([a,b])} \leq \|f'\|_{L^1([a,b])}^2$$

for every absolutely continuous function $f$ on $[a, b]$ such that $f(a) = 0$. 

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Proof. Assume that \(1 < p \leq 2\). It suffices to consider \(q \geq 2\) such that \(1/p + 1/q = 1\) (recall that \(p = q/(q - 1)\) and \(q = p/(p - 1)\)), and to apply Corollary 10:

\[
\left( \frac{b - a}{1/q + (p - 1)/p} \right)^{1/q + (p - 1)/p} \left( \frac{q(p - 1)}{p(q - 1)} \right)^{(p - 1)/p}
\]

\[
= \left( \frac{b - a}{2(p - 1)/p} \right)^{2(p - 1)/p} \left( \frac{p(p - 1)}{p} \right)^{(p - 1)/p}
\]

\[
= \left( \frac{p(b - a)}{2(p - 1)/p} \right)^{2(p - 1)/p} (p - 1)^{(p - 1)/p} = \left( \frac{p(b - a)}{2(p - 1)/2} \right)^{2(p - 1)/p}.
\]

Let us consider now the case \(p = 1\). Since the Lebesgue measure on \([a, b]\) is finite, Corollary 9 gives

\[
\|ff'\|_{L^1([a, b])} \leq B \|f'\|^2_{L^1([a, b])}
\]

with

\[
B = \sup_{a < x < b} (b - x)^{(p - 1)/p} \|1\|^1_{L^{1/(p - 1)}([a, x])} = \sup_{a < x < b} (b - x)^0 \|1\|_{L^\infty([a, x])} = 1.
\]

\[\blacksquare\]

Remark 12 Note that in the second inequality in Corollary 11:

\[
\|ff'\|_{L^1([a, b])} \leq \|f'\|^2_{L^1([a, b])},
\]

the constant 1 multiplying \(\|f'\|^2_{L^1([a, b])}\) does not depend on the length of the interval \([a, b]\).

Corollaries 8 and 9 have, respectively, the following direct consequences for general fractional integrals of Riemann-Liouville type.

Proposition 13 Let us consider \(1 \leq p \leq q < \infty\). If

\[
B := \sup_{a < x < b} \left( \int_x^b \frac{1}{T(b, s, \alpha)} \, ds \right)^{1/q} \left( \int_a^x T(b, s, \alpha)^{1/(p - 1)} \, ds \right)^{(p - 1)/p} < \infty,
\]

then

\[
\int_a^b \frac{|f(s)f'(s)|}{T(b, s, \alpha)} \, ds \leq B \left( \frac{q}{q - 1} \right)^{(p - 1)/p} q^{1/q} \left( \int_a^b \frac{|f'(s)|^p}{T(b, s, \alpha)} \, ds \right)^{1/p} \left( \int_a^b \frac{|f'(s)|^{q/(q - 1)}}{T(b, s, \alpha)} \, ds \right)^{(q - 1)/q}
\]

if \(p > 1\), and

\[
\int_a^b \frac{|f(s)f'(s)|}{T(b, s, \alpha)} \, ds \leq B \int_a^b \frac{|f'(s)|}{T(b, s, \alpha)} \, ds \left( \int_a^b \frac{|f'(s)|^{q/(q - 1)}}{T(b, s, \alpha)} \, ds \right)^{(q - 1)/q}
\]

for every absolutely continuous function \(f\) on \([a, b]\) with \(f(a) = 0\).
Proposition 14 Let us consider $1 \leq p \leq 2$ and assume that

$$B := \sup_{a < x < b} \left( \int_{x}^{b} \frac{1}{T(b, s, \alpha)} \, ds \right)^{(p-1)/p} \left( \int_{a}^{x} T(b, s, \alpha)^{1/(p-1)} \, ds \right)^{(p-1)/p} < \infty.$$  

If $1 < p \leq 2$, then

$$\int_{a}^{b} \frac{|f(s)f'(s)|}{T(b, s, \alpha)} \, ds \leq B \left( \frac{2}{p-1} \right)^{(p-1)/p} \left( \int_{a}^{b} \frac{|f'(s)|^p}{T(b, s, \alpha)} \, ds \right)^{2/p}$$

for every absolutely continuous function $f$ on $[a, b]$ with $f(a) = 0$.

Furthermore, if

$$\int_{a}^{b} \frac{ds}{T(b, s, \alpha)} < \infty,$$

then

$$\int_{a}^{b} \frac{|f(s)f'(s)|}{T(b, s, \alpha)} \, ds \leq B \left( \int_{a}^{b} \frac{|f'(s)|}{T(b, s, \alpha)} \, ds \right)^2$$

for every absolutely continuous function $f$ on $[a, b]$ with $f(a) = 0$.

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