Parity and Projection from Virtual Knots to Classical Knots

Vassily Olegovich Manturov

September 4, 2012

Abstract

We construct various functorial maps (projections) from virtual knots to classical knots. These maps are defined on diagrams of virtual knots; in terms of Gauss diagram each of them can be represented as a deletion of some chords. The construction relies upon the notion of parity. As corollaries, we prove that the minimal classical crossing number for classical knots.

Such projections can be useful for lifting invariants from classical knots to virtual knots. Different maps satisfy different properties.

MSC: 57M25, 57M27

Keywords: Knot, virtual knot, surface, group, projection, crossing, crossing number, bridge number

1 Introduction. Basic Notions

Classical knot theory studies the embeddings of a circle (several circles) to the plane up to isotopy in three-space. Virtual knot theory studies the embeddings of curves in thickened oriented surfaces of arbitrary genus, up to the addition and removal of empty handles from the surface. Virtual knots have a special diagrammatic theory, described below, that makes handling them very similar to the handling of classical knot diagrams. Many structures in classical knot theory generalize to the virtual domain directly, however, many other required more techniques [19]; nevertheless, many other structures (like Heegaard-Floer homology) have not been generalized to virtual knots so far; the existence of a well-defined projection from virtual knot theory to classical knot theory may help solving such problems.

In the diagrammatic theory of virtual knots one adds a virtual crossing (see Figure 1) that is neither an overcrossing nor an undercrossing. A virtual crossing is represented by two crossing segments with a small circle placed around the crossing point. Figures 1 and 6 are borrowed from [11].

Note that a classical knot vertex is a 4-valent graphical node embedded in the plane with extra structure. The extra structure includes the diagrammatic choice of crossing (indicated by a broken segment) and a specific choice of cyclic order (counterclockwise when embedded in the plane) at the
vertex. By a framing of a four-valent graph we mean a splitting of the four emanating (half)edges into two pairs of opposite (half)edges. The counterclockwise cyclic order includes more information than just a framing. A virtual knot is completely specified by its 4-valent nodes with their cyclic structure if the edges incident to the nodes are labeled so that they can be connected by arcs to form the corresponding graph.

Throughout the paper, all knots are assumed oriented. The results of this paper are about virtual knots, as stated; nevertheless, after a small effort they can be upgraded for the case of virtual links.

A virtual diagram is an immersion of a collection of circles into the plane such that some crossings are structured as classical crossings and some are simply labeled as virtual crossings and indicated by a small circle drawn around the crossing. We regard the resulting diagram as a possible non-planar graph whose only nodes are the classical crossings, with their cyclic structure. Any immersion of such a graph, preserving the cyclic structure at the nodes, will represent the same virtual knot or link. From this, we use the detour move (see below) for arcs with consecutive virtual crossings, so that this equivalence is satisfied. For the projection of the unknot (unlink) without classical crossings we shall also admit a circle instead of graph; thus, we category of graphs includes the circle.

Immersion of each particular circle from the collection gives rise to a component of a virtual link diagram; virtual link diagrams with one component are virtual knot diagrams; we shall deal mostly with virtual knots and their diagrams, unless specified otherwise; (virtual) knots are one-component (virtual) links.

Moves on virtual diagrams generalize the Reidemeister moves (together with obvious planar isotopy) for classical pieces of knot and link diagrams (Figure 1). One can summarize the moves on virtual diagrams by saying that the classical crossings interact with one another according to the usual Reidemeister moves while virtual crossings are artifacts of the attempt to draw the virtual structure in the plane. A segment of diagram consisting of a sequence of consecutive virtual crossings can be excised and a new connection made between the resulting free ends. If the new connecting segment intersects the remaining diagram (transversally) then each new intersection is taken to be virtual. Such an excision and reconnection is called a detour move. Adding the global detour move to the Reidemeister moves completes the description of moves on virtual diagrams. In Figure 1 we illustrate a set of local moves involving virtual crossings. The global detour move is a consequence of moves (B) and (C) in Figure 1. The detour move is illustrated in Figure 2. Virtual knot and link diagrams that can be connected by a finite sequence of these moves are said to be equivalent or virtually isotopic. A virtual knot is an equivalence class of virtual diagrams under these moves.

Another way to understand virtual diagrams is to regard them as representatives for oriented Gauss diagrams [5]. The Gauss diagram encodes the information about classical crossings of a knot diagram and the way they are connected. However, not every Gauss diagram has a planar realization. An attempt to draw the corresponding diagram on the plane leads to the production of the virtual crossings. Gauss diagrams are most convenient for knots, where there is one cycle in the code and one circle in the Gauss diagram. One can work with Gauss diagrams for links with a little bit more care, but we will not touch on this subject.

The detour move makes the particular choice of virtual crossings irrelevant.

Virtual isotopy is the same as the equivalence relation generated on the collection of oriented Gauss
The paper is organized as follows. In the end of the introduction, we present all necessary constructions of Gauss diagrams, band presentation, and parity.

In Section 2, we formulate the main theorem (about projection) and prove it modulo some important auxiliary theorems, one of them due to I.M.Nikonov. We also prove two corollaries from the main theorem.

Section 3 is devoted to the proof of basic lemmas.

In Section 4, we introduce parity groups and discuss other possibilities of constructing projection maps from virtual knots to classical knots.

The paper is concluded by Section 5, where we discuss some obstacles which do not allow us to define the projection uniquely on the diagrammatic level.

1.1 Acknowledgements

I am grateful to L.H.Kauffman, I.M.Nikonov, V.V.Chernov, D.P.Ilyutko for various fruitful discussions.
1.2 Gauss diagrams

**Definition 1.** A *Gauss diagram* is a finite trivalent graph which consists of just an oriented cycle passing through all vertices (this cycle is called the core of the Gauss diagrams) and a collection of oriented edges (chords) connecting crossings to each other. Besides the orientation, every chord is endowed with a sign.

Besides that we consider the empty Gauss diagram which is not a graph, but an oriented circle; this empty Gauss diagram corresponds to the unknot diagram without crossings.

Given a one-component virtual diagram $D$. Let us associate with it the following Gauss diagram $\mathcal{G}(D)$. Let us represent the framed four-valent graph $\Gamma$ of the diagram $D$ as the result of pasting of a closed curve at some points (corresponding to classical crossings) in such a way that the two parts of the neighbourhood of a pasted point are mapped to opposite edges at the crossing.

Thus, we have a map $f : S^1 \to \Gamma$. For the core circle of the chord diagram we take $S^1$, vertices of the chord diagrams are preimages of vertices of $\Gamma$, and chords connect those pairs of vertices having the same image. The orientation of the circle corresponds to the orientation of the knot. Besides, the chord is directed from the preimage of the overcrossing arc to the preimage of an undercrossing arc; the sign of the chord is positive for crossings of type $\overleftarrow{\times}$ and negative for crossings of type $\overrightarrow{\times}$.

We say that a Gauss diagram is *classical* if it can be represented by a classical diagram (embedding of a four-valent graph without virtual crossings). In Fig. 3, Reidemeister moves for Gauss diagrams are drawn without indication of signs and arrows. For Reidemeister-1 (the upper picture) move an addition/removal of a solitary chord of any sign and with any arrow direction is possible. For Reidemeister-2 move (two middle pictures), the chords $a$ and $b$ should have the same orientation, but different signs.

The articulation for the third Reidemeister move (lowest picture) is left for the reader as an exercise.

Note that two the Gauss diagram does not feel the detour move: if two diagrams $K, K'$ are virtually isotopic, then $\mathcal{G}(K) = \mathcal{G}(K')$.

We say that a virtual knot diagram $K_1$ is *smaller* than the diagram $K_2$, if the Gauss diagram of $K_1$ is obtained from that of $K_2$ by a deletion of some chords.

For this, we take the notation $K_1 < K_2$.

As usual, we make no distinction between virtually isotopic diagrams.

This introduces a partial ordering on the set of virtual knot diagrams. The unknot diagram without classical crossings is smaller than any diagram with classical crossings.

Having a Gauss diagram, one gets a collection of classical crossings with an indication how they are connected to each other. So, a Gauss diagram leads to a *virtual equivalence classes* of virtual knot diagrams (note that Gauss diagram carries no information about virtual crossings, so, virtually equivalent diagrams diagrams lead to the same Gauss diagram).

By a *bridge* of a Gauss diagram we mean an arc of the core circle between two adjacent arrowtails (for the edge orientation for the chords of the chord diagram) containing arrowheads only (possibly, none of them). In the corresponding planar diagram, a bridge is a branch of the knot diagram from an undercrossing to the next undercrossing containing overcrossings and virtual crossings only. Thus, every virtual knot diagrams naturally splits into bridges, see Fig. 4.
Figure 3: Reidemeister Moves on Chord Diagrams

Figure 4: The Trefoil Knot and its Bridges
The bridge number of a virtual knot diagram is the minimal number of its bridges. Since the bridge number is defined in terms of Gauss diagram, it does not change under detour moves.

With this, one can define the minimal crossing number and the bridge number for virtual knots to be the minimum of crossing numbers (resp., bridge numbers) over all virtual knot diagrams representing the given knot. When we restrict to classical knots, there we also have the definition when the minima are taken only over classical diagrams.

So, for crossing number and bridge number for classical knots, we have two definitions, the classical one and the virtual one. As we shall see in the present paper (Corollaries 1, 2), these two definitions coincide, moreover, any virtual diagram of a classical knot where the minimal classical crossing number (resp., minimal bridge number) is obtained, is in fact, virtually equivalent to a classical one.

1.3 Band Presentation of Virtual Knots

Note that knots in a thickened surface $S_g \times I$ are encoded by regular projections on $S_g$ with over and undercrossings and no virtual crossings. These diagrams are subject to classical Reidemeister moves which look locally precisely as in the classical case. No detour moves are needed since we have no virtual crossings for such diagrams.

Let $K$ be a (class of a) virtual knot, given by some virtual diagram $K$. Let us describe the band presentation of this knot as a knot in a thickened surface (following N.Kamada and N.Kamada [5]).

We shall construct a surface $S(K)$ corresponding to the diagram $K$, as follows. First, we construct a surface with boundary corresponding to $K$.

With every classical crossing, we associate a “cross” (upper picture in Fig. 5), and with every virtual crossing, we associate a pair of “skew” bands (lower part of Fig. 5).

Connecting these crosses and bands by non-intersecting and non-twisted bands going along the edges of the diagram, we get an oriented 2-manifold with boundary, to be denoted by $S'(K)$ (the orientation is taken from the plane), see Fig. 6.

The diagram $K$ can be drawn on the surface $S'(K)$ in a natural way so that the arcs of the diagram (which may pass through virtual crossings) are located in such a way that the arcs are go along the
middle lines of the band, and classical (flat) crossings correspond to intersection of middle lines inside crossings. Thus we get curve $\delta \subset S'(K)$ (for a link we would get a set of curves). Pasting the boundary components of the manifold $S'(K)$ by discs, we get an oriented manifold $S = S(K)$ without boundary with a curve $\delta$ in it; we call the surface $S(K)$ the *underlying surface for the diagram* $K$. We call the genus of this surface the *underlying diagram genus* of the diagram $K$.

We call the connected components of the boundary of $S'(K)$ the *pasted cycles* or the *rotating cycles*. Originally rotating cycles are defined by using source-sink orientation of $K$, but in this paper we regard them as the boundary of the oriented surface $S'(D)$ since we handle diagrams which do or do not admit a source-sink orientation. These pasted cycles treated as collections of vertices, will be used in the sequel for constructing parity groups.

By the *underlying genus* of a virtual knot we mean the minimum of all underlying genera over all diagrams of this knot.

We say that a diagram $K$ is a *minimal genus diagram* if the genus of the diagram coincides with the genus of the corresponding knot.

As we shall see, some minimal characteristics of virtual knots can be realized only on minimal genus diagrams.

The detour move does not change the band presentation of the knot at all. As for Reidemeister move, the first and the third moves do not change the genus of the knot, whence the second increasing/decreasing move may increase/decrease the genus of the underlying surface (cause stabilization/destabilization).

To define handle stabilization, regard the knot or link as represented by a diagram $D$ on a surface $S$. If $C$ is an embedded curve in $S$ that does not intersect the diagram $D$ and cutting along $D$ does not disconnect the surface, then we cut along $C$ and add two disks to fill in the boundary of the cut surface. This is a handle destabilization move that reduces the genus of the surface to a surface $S'$ containing a new diagram $D'$. The pairs $(S, D)$ and $(S', D')$ represent the same virtual knot or link. The reverse operation that takes $(S', D')$ to $(S, D)$ consists in choosing two disks in $S'$ that are disjoint from $D'$, cutting them out and joining their boundaries by a tube (hence the term handle addition for this direction of stabilization).
We say that two such surface embeddings are *stably equivalent* if one can be obtained from another by isotopy in the thickened surfaces, homeomorphisms of the surfaces and handle stabilization.

**Theorem 1** ([8, 9]). The above description of a band representation leads to a bijection between virtual knots and stably equivalent classes of embeddings of circles in thickened surfaces.

So, we shall deal with the following two equivalences: the usual one (with (de)stabilisation) and the equivalence without (de)stabilisation which preserves the genus of the underlying surface.

The Kuperberg Theorem says that virtual knots can be studied by using their minimal representatives. More precisely, we have

**Theorem 2** (Kuperberg’s Theorem, [12]). A minimal genus diagram of a virtual knot \( K \) is unique up to isotopy; in other words, if two diagrams \( K_1, K_2 \) are of the minimal genus then there is a sequence of Reidemeister moves from \( K_1 \) to \( K_2 \) such that all intermediate diagrams between \( K_1 \) and \( K_2 \) are of the same genus.

### 1.4 Parity

**Definition 2.** Let \( \mathcal{L} \) be a knot theory, i.e., a theory whose objects are encoded by diagrams (four-valent framed graphs, possibly, with further decorations) modulo the three Reidemeister moves (and the detour move) applied to crossings. For every Reidemeister move transforming a diagram \( K \) to a diagram \( K_1 \) there are corresponding crossings: those crossings outside the domain of the Reidemeister move for \( K \) are in one-to-one correspondence with those crossings outside the domain of the Reidemeister move for \( K_1 \). Besides, for every third Reidemeister move \( K \rightarrow K_1 \) there is a natural correspondence between crossings of \( K \) taking part in this move and the resulting crossings of \( K_1 \). By a *parity* for the knot theory \( \mathcal{L} \) we mean a rule for associating 0 or 1 with every (classical) crossing of any diagram \( K \) from the theory \( \mathcal{L} \) in a way such that:

1. For every Reidemeister moves \( K \rightarrow K_1 \) the corresponding crossings have the same parity;
2. For each of the three Reidemeister moves the sum of parities of crossings taking part in this move is zero modulo two.

**Definition 3.** Now, a *parity in a weak sense* is defined in the same way as parity but with the second condition relaxed for the case of the third Reidemeister move. We allow three crossings taking part in the third Reidemeister move to be all odd (so for the third Reidemeister move the only forbidden case is when exactly one of three crossings is odd).

We shall deal with parities for *virtual knots* or for *knots in a given thickened surface*. In the latter case diagrams are drawn on a 2-surface and Reidemeister moves are applied to these diagrams; no “stabilizing” Reidemeister moves changing the genus of the surface are allowed.

We say that two chords of a Gauss diagram \( a, b \) are *linked* if two ends of one chord \( a \) belong to different connected components of the complement to the endpoints of \( b \) in the core circle of the Gauss diagram (it is assumed that no chord is linked with itself). We say that a chord of a Gauss diagram is *even* (with respect to the *Gaussian parity*) if it is linked with evenly many chords; otherwise we say
Figure 7: The parity projection is not idempotent

that this chord is odd (with respect to the Gaussian parity). We shall say that a classical crossing of a virtual knot diagram is even whenever the corresponding chord is even. One can easily check the parity axioms for the Gaussian parity.

For every parity $p$ for virtual knots (or knots in a specific thickened surface), consider a mapping $pr_p : \mathcal{G} \to \mathcal{G}$ from the set of Gauss diagrams $\mathcal{G}$ to itself, defined as follows. For every virtual knot diagram $K$ represented by a Gauss diagram $\mathcal{G}(K)$ we take $pr_p(K)$ to be the virtual knot diagram represented by the Gauss diagram obtained from $\mathcal{G}(K)$ by deleting odd chords with respect to $p$. At the level of planar diagrams this means that we replace odd crossings by virtual crossings.

The following theorem follows from definitions, see, e.g.,[13].

**Theorem 3.** The mapping $pr_p$ is well defined, i.e., if $K$ and $K'$ are equivalent, then so are $pr_p(K)$ and $pr_p(K')$.

The same is true for every parity in a weak sense as discussed above.

Thus, for the Gaussian parity $g$ one has a well-defined projection $pr_g$. Note that if $K$ is a virtual knot diagram, then $pr_g(K)$ might have odd chords: indeed, some crossings which were even in $K$ may become odd in $pr_g(K)$.

However, this map $pr_g$ may take diagrams from one theory to another; for example, if we consider equivalent knots lying in a given thickened surface, their images should not necessarily be realised in the same surface; they will just be equivalent virtual knots. For virtual knots, this is just a map from virtual knots to virtual knots.

Note that $pr_g$ is not an idempotent map. For example, if we take the Gauss diagram with four chords $a, b, c, d$ where $a$ is linked with $b, c$, the chord $b$ is linked with $a, d$, the chord $c$ is linked with $a$, and the chord $d$ is linked with $b$, then after applying $pr_g$, we shall get a diagram with two chords $a, b$, and they will both become odd, see Fig. 7.

Now, let $S_g$ be a surface of genus $g$. Fix a cohomology class $\alpha \in H^1(S_g, \mathbb{Z}_2)$. Let us consider those knots $K$ in $S_g$ for which the total homology class of the knot $K$ in $H_1(S_g, \mathbb{Z}_2)$ is trivial.

With every crossing $v$ of $K$ we associate the two halves $h_{v,1}, h_{v,2}$ (elements of the fundamental group $\pi_1(S_g, v)$) as follows. Let us smooth the diagram $K$ at $v$ according to the orientation of $K$. Thus, we get a two-component oriented link. If $\alpha(h_{v,1}) = \alpha(h_{v,2}) = 0$ we say that the crossing $v$ is even; otherwise we say that it is odd.

In [7] it is proved that this leads to a well-defined parity for knots in $S_g \times I$. Thus, every $\mathbb{Z}_2$-cohomology class of the surface which evaluates trivially on the knot itself, gives rise to a well-defined
parity. We shall call it the homological parity.

2 Statements of Main Results

For every Gauss diagram one can decree some chords (crossings) to be true classical (in an ambiguous way, see discussion in the last section) and remove the other ones, so that the resulting Gauss diagram classical, and this map will give rise to a well-defined projection from virtual knots to classical knots. In Fig. [10] a virtual knot $A$ is drawn in the left part; its band presentation belongs to the thickened torus (see upper part of the right picture); there are four “homologically non-trivial” crossings disappear which leads to the diagram $D$ (virtually isotopic to the one depicted in the lower picture of the right half). This is the classical trefoil knot diagram.

The aim of the present article is the proof of the following

**Theorem 4.** For every virtual diagram $K$ there exists a classical diagram $ar{K}$, such that:

1. $ar{K} < K$;
2. $\bar{K} = K$ if and only if $K$ is classical.
3. If $K_1$ and $K_2$ are equivalent virtual knots, then so do $\bar{K}_1$ and $\bar{K}_2$.
4. The map restricted to non-classical knots is a surjection onto the set of all classical knots.

The discrimination between “true classical” crossings and those crossings which will become virtual is of the topological nature, as we shall see in the proof of Theorem 4.

As usual, we make no distinction between virtually isotopic diagrams: a virtual diagram is said to be classical if the corresponding Gauss diagram represents a classical knot.

Thus, it makes sense to speak about a map from the set of virtual knots to the set of classical knots. This map will be useful for lifting invariants from virtual knots to classical knots.

We shall denote this map by $K \rightarrow f(K)$ where $K$ means the knot type represented by $K$, and $f(K)$ means the resulting knot type of the corresponding classical knots.

The only statement of the theorem which deals with diagrams of knots which are not classical, is 4). Otherwise we could just project all diagrams which do not represent classical knots to the unknot diagram (without classical crossings), and the functorial map would be rather trivial.

Nevertheless, as we shall see, one can construct various maps of this sort. Different proofs of Theorem 4 can be used for constructing various functorial maps and establishing properties of knot invariants.

A desired projection would be one for which there is a well defined mapping at the level of Gauss diagrams, and the projection is such that if any two diagrams which are connected by a Reidemeister moves, their images are connected by the same Reidemeister move or by a detour move. Unfortunately, such projections seem not to exist (see the discussion in the end of the paper); see also Nikonov’s Lemma (Theorem 5).
For example, based on the notion of weak parity and parity groups, we shall construct another projection satisfying the conditions of Theorem 4; the construction will not be in two turns as in the case when Nikonov’s lemma is applied; however, this map will “save” more classical crossings.

From Theorem 4 we have the following two corollaries

**Corollary 1.** Let $K$ be an isotopy class of a classical knot. Then the minimal number of classical crossings for virtual diagrams of $K$ is realized on classical diagrams (and those obtained from them by the detour move). For every non-classical diagram realizing a knot from $K$, the number of classical crossings is strictly greater than the minimal number of classical crossings.

Moreover, minimal classical crossing number of a non-classical virtual knot is realized only on minimal genus diagrams.

Indeed, the projection map from the main theorem decreases the number of classical crossings, and preserves the knot type.

The observation that the following corollary is a consequence from Theorem 4 is due to V.V.Chernov (Tchernov).

**Corollary 2.** Let $K$ be a classical knot class. Then the bridge number for the class $K$ can be realized on classical diagrams of $K$ only.

Moreover, minimal bridge number of a non-classical virtual knot is realized on minimal genus diagrams (here we do not claim that it can not be realized on non-classical diagrams).

*Proof.* Indeed, it suffices to see that if $K' < K$ then $br(K') \leq br(K)$: when replacing a classical crossing with a virtual crossing, the number of bridges cannot be increased; it can only decrease because two bridges can join to form one bridge.

**Remark 1.** We do not claim that the diagram $K'$ representing the class $f(K)$ is unique. In fact, we shall construct many maps satisfying the conditions of Theorem 4. In the last section of the present work we discuss the question, to which extent the diagram $K'$ can be defined uniquely by the diagram $K$, see the discussion in the last section of the paper.

Theorem 4 allows one to lift invariants of classical knots to virtual knots. The straightforward way to do it is to compose the projection with the invariant in question. However, there is another way of doing it where crossings which are not classical, are not completely forgotten (made virtual) but are treated in another way than just usual “true classical” crossings. In similar cases when projection is well defined at the level of diagrams, this was done in [13,1] etc.: in these papers a distinction between even and odd crossings was taken into account to refine many known invariants (note that, according to the parity projection map, one can completely disregard odd crossings; on the other hand, they can be treated as classical crossings as they were from the very beginning).

The proof of Theorem 4 is proved in two steps.

**Theorem 5.** Let $K$ be a virtual diagram, whose underlying diagram genus is not minimal in the class of the knot $K$. Then there exists a diagram $K' < K$ in the same knot class.

**Theorem 6** (I.M.Nikonov). There is a map $pr$ from minimal genus virtual knot diagrams to classical knot diagrams such that for every knot $K$ we have $pr(K) < K$ and if two diagrams $K_1$ and $K_2$ are
related by a Reidemeister move (performed within the given minimal genus diagram) then their images $\text{pr}(K_1)$ and $\text{pr}(K_2)$ are related by a Reidemeister move.

Proof of the Main Theorem (Theorem 4). We shall construct the projection map in two steps.

Let $K$ be a virtual knot diagram. If $K$ is of a minimal genus, then we take $\bar{K}$ to be just $\text{pr}(K)$ as in Theorem 6. Otherwise take a diagram $K'$ instead of $K$ as in Theorem 5. It is of the same knot type as $K$. If the genus of the resulting diagram is still not minimal, we proceed by iterating the operation $K'$, until we get to a diagram $K''$ of minimal genus which represents the class of $K$ and $K'' < K$. Now, set $\bar{K} = \text{pr}(K'')$.

One can easily see that if we insert a small classical knot $L$ inside an edge of a diagram of $K$, then $f(K \# L) = f(K) \# f(L)$. So, the last statement of the theorem holds as well. 

3 Proofs of Key Theorems

3.1 The Proof of Theorem 5

Let $K$ be a virtual knot diagram on a surface $S_g$ of genus $g$. Assume this genus is not minimal for the knot class of $K$. Then by Kuperberg’s theorem it follows that there is a diagram $\tilde{K}$ on $S_g$ representing the same knot as $K$ and a curve $\gamma$ on $S_g$ such that $\tilde{K}$ does not intersect $\gamma$. Indeed, if there were no such diagram $\tilde{K}$, the knot in $S_g \times I$ corresponding to the diagram $K$ would admit no destabilization, and the genus $g$ would be minimal.

The curve $\gamma$ gives rise to a (co)homological parity for knots in $S_g$ homotopic to $K$: a crossing is even if the number if intersections of any of the corresponding halves with $\gamma$ is even, and odd, otherwise.

Since $K$ has underlying diagram genus $g$, there exists at least one odd crossing of the diagram $K$. Let $K$ be the result of $\gamma$-parity projection applied to $K$. We have $K' < K$.

By construction, all crossings of $\tilde{K}$ are even.

Let us construct a chain of Reidemeister moves from $K$ to $\tilde{K}$ and apply the $\gamma$-parity projection to it.

We shall get a chain of Reidemeister moves connecting $K'$ to $\tilde{K}$. So, $K'$ is of the same type as $\tilde{K}$ and $K$. The claim follows.

3.2 The Proof of Theorem 6

Let us construct the projection announced in Theorem 6. Fix a 2-surface $S_g$. Let us consider knots in the thickening of $S_g$ for which genus $g$ is minimal (that is, there is no representative of lower genus for knots in question). Let $K$ be a diagram of such a knot. We shall denote crossings of knot diagrams in $S_g$ and the corresponding points on $S_g$ itself by the same letter (abusing notation).

As above, with every crossing $v$ of $K$ we associate the two halves $h_{v, 1}, h_{v, 2}$, now considered as elements of the fundamental group $\pi_1(S_g, v)$, as follows. Let us smooth the diagram $K$ at $v$ according to the orientation of $K$. Thus, we get a two-component oriented link with components $h_{v, 1}, h_{v, 2}$. Consider every component of this link represented as a loop in $\pi_1(S_g, v)$ and denote them again by $h_{v, 1}, h_{v, 2}$.
Let $\gamma_v, \bar{\gamma}_v$ be the two homotopy classes of the knot $K$ considered as an element of $\pi_1(S_g, v)$: we have two classes because we can start traversing the knot along each of the two edges emanating from $v$. Note that $h_{v,1} \cdot h_{v,2} = \gamma_v$ and $h_{v,2} \cdot h_{v,1} = \bar{\gamma}_v$.

Let us now construct a knot diagram $pr(K)$ from $K$ as follows. If for a crossing $v$ we have $h_{v,1} = \gamma_v^k$ for some $k$ (or, equivalently, $h_{v,2} = \gamma_v^{1-k}$) then this crossing remains classical for $K'$; otherwise, a crossing becomes virtual. Note that it is immaterial whether we take $\gamma_v$ or $\bar{\gamma}_v$ because if $h_{v,1}$ and $h_{v,2}$ are powers of the same element of the fundamental groups, then they obviously commute, which means that $\gamma_v = \bar{\gamma}_v$.

**Statement 1.**

1. For every $K$ as above, $pr(K)$ is a classical diagram;

2. $K = pr(K)$ whenever $K$ is classical

3. If $K_1$ and $K_2$ differ by a Reidemeister move then $pr(K_1)$ and $pr(K_2)$ differ by either a detour move or by a Reidemeister move.

**Proof.** Take $K$ as above and consider $pr(K)$. By construction, all “halves” of all crossings for $pr(K)$ are powers of the same homotopy class. We claim that the underlying surface for $pr(K)$ is a 2-sphere. Indeed, when constructing a band presentation for $pr(K)$, we see that the surface with boundary has cyclic homology group. This happens only for a disc or for the cylinder; in both cases, the corresponding compact surface will be $S^2$.

The situation with the first Reidemeister move is obvious: the new added crossing has one trivial half and the other half equal to the homotopy class of the knot itself.

Now, to prove the last statement, we have to look carefully at the second and the third Reidemeister moves. Namely, if some two crossings $A$ and $B$ participate in a second Reidemeister move, then we have an obvious one-to-one correspondence between their halves such that whenever one half corresponding to $A$ is an power of $\gamma$, so is the corresponding half of $B$.

So, they either both survive in $pr(A), pr(B)$ (do not become virtual) or they both turn into virtual crossings. So, for $pr(A), pr(B)$ we get either the second Reidemeister move, or the detour move. Note that here we deal with the second Reidemeister move which does not change the underlying surface.

Now, let us turn to the third Reidemeister move from $K$ to $K'$, and let $(A, B, C)$ and $(A', B', C')$ be the corresponding triples of crossings. We see that the homotopy classes of halves of $A$ are exactly
those of $A'$, the same about $B, B'$ and $C, C'$. So, the only fact we have to check that the number of surviving crossings among $A, B, C$ is not equal to two (the crossings from the list $A', B', C'$ survive accordingly). This follows from Fig. 8.

Indeed, without loss of generality assume $A$ and $B$ survive. This means that the class $h_{A,1}$ is a power of the class of the whole knot in the fundamental group with the reference point in $A$, and $h_{B,1}$ is a power of class of the knot with the reference point at $B$.

Let us not investigate $h_{C,1}$ (for convenience we have chosen $h_{C,1}$ to be the upper right part of the figure).

We see that $h_{C,1}$ consists of the following paths: $(ca)h_{A,1}(ab)h_{B,1}(cb)^{-1}$, where $(ca), (ab), (cb)$ are non-closed paths connecting the points $A$, $B$, and $C$. Now, we can homotop the above loop to $(ca)h_{A,1}(ca)^{-1}(ca)(ab)h_{B,1}(cb)^{-1}$ and then homotop it to the product of $(ca)h_{A,1}(ca)^{-1}$ and $(cb)h_{B,1}(cb)^{-1}$.

We claim that both these loops are homotopic to $\gamma_{C}^{l}$ and $\gamma_{C}^{m}$ for some exponents $m, l$. Indeed, $h_{A,1}$ is $\gamma_{A}^{k}$ by assumption. Now, it remains to observe that in order to get from $\gamma_{A}$ to $\gamma_{C}$, it suffices to “conjugate” by a path along the knot; one can choose $(ac)$ as such a path. The same holds about $h_{C,1}$.

So, if all crossings $A, B, C$ survive in the projection of $pr(K)$ and $A', B', C'$ survive in $pr(K')$ then we see that $pr(K')$ differs from $pr(K)$ by a third Reidemeister move. If no more than one of $A, B, C$ survives then we have a detour move from $pr(K)$ to $pr(K')$.

\[\square\]

4 The Parity Group, One More Projection, and Connected Sums

In the above text, we have defined parity as a way of decorating crossings by elements of $\mathbb{Z}_2$. It turns out that there is a way to construct an analogue of parity valued in more complicated objects, namely, in groups, depending on the knot diagram. Such “group-valued” parities can be also used for projections, see, e.g., [7].

This group-valued parity can be thought of as a parity in a weak sense: a crossing is even if the corresponding element of the parity group is trivial, and odd otherwise.

However, this can be done for diagrams of some specific genus only.

Let $D$ be a virtual diagram of genus $g$. Now, let us construct the universal parity group $G(D)$. Note that this group will be “universal” only for a specific genus.

Recall that pasted cycles appear in a band–pass presentation of a virtual knot diagram as cycles on the boundary of a surface to be pasted by discs. Every cycle can be treated as a 1-cycle in the 1-frame of the knot diagram graph; the graph itself consists of classical crossings (vertices) and edges between them. Thus, every pasted cycle $C$ gives rise to a collection of classical crossings, it touches.

We shall use the additive notation for this group. For generators of $G(D)$ we take crossings of the diagram $D$. We define two sorts of relations:

1. $2a_i = 0$ for every crossing and there will also be relations correspond to pasted cycles. Namely, a pasted cycle is just a rotating cycle on the 4-valent graph (shadow of the knot)
2. The sum of crossings corresponding to any pasted cycle is zero.

It is obvious that for a classical knot diagram $D$ the group $G(D)$ is trivial (otherwise the reader is referred to Theorem 7 ahead).

Denote the element of the group $G$ corresponding to a crossing $x$ of the knot diagram, by $g(x)$.

In [7] it is proved that the parity group gives rise to a parity in a weak sense: all crossings for which the corresponding element of the group is trivial, are thought of as even crossings, and the other one are thought of as odd crossings. Thus, we get the following

**Theorem 7.** For a virtual diagram $D$ with the surface $S_g$ genus $g$ the group $G(D)$ is the quotient group of $H_1(S_g,\mathbb{Z}_2)$ by the element generated by the knot. In particular, if $D$ is a checkerboard colourable diagram then $G(D) = H_1(S_g,\mathbb{Z}_2)$.

In particular, if $D_1$ and $D_2$ are nonstably equivalent diagrams then $G(D_1) = G(D_2)$.

To prove the theorem, it suffices to associate with every crossing $x$ any of the two halves $h_{x,1}$ or $h_{x,2}$ and consider them as elements of the above mentioned quotient group.

A careful look to the formulation of Theorem 7 shows that:

1. If a crossing $x$ corresponds to the first Reidemeister move, then the corresponding element of the quotient group is equal to zero.

2. If two crossings $x, y$ participate in the second Reidemeister move, then the corresponding elements of the group $G(D)$ are equal to each other.

3. If three crossings $a, b, c$ participate in a third Reidemeister move then $h_a + h_b + h_c = 0$ in $G$.

Thus, the map to the group $G$ gives rise to the parity in a weak sense, which means, in particular, that there is a well-defined projection from knots in $S_g \times I$ to virtual knots.

Let $K$ be a knot diagram in $S_g$. Consider $K$ as a virtual knot diagram (up to virtual equivalence). Now, let $l(K)$ be the diagram obtained from $K$ by making those crossings $x$ of $K$ virtual, for which $h(x) \neq 0 \in G(K)$.

**Theorem 8.** If $K$ and $K_1$ are two diagrams of knots in $S_g \times I$ which differ by one Reidemeister move, then $l(K)$ and $l(K_1)$ either differ by the same Reidemeister move, or coincide (are virtually equivalent).

Moreover $l(K)$ is (virtually equivalent to) $K$ if and only if $K$ is (virtually equivalent to) a classical knot.

The proof follows from general argument concerning parity in a weak sense.

### 4.1 One more projection

Let us now give one more proof of Theorem 4. In fact, the map $f$ from our original proof of Theorem 4 kills too many classical crossings.

For example, if we consider the classical trefoil diagram with three “boxes” shown in Fig. 9. Assume every black box represents a virtual knot diagram lying inside its minimal representative which is
homologically trivial in the corresponding 2-surface, and we put these diagrams into boxes after splitting them at some points. Then, if these diagrams are complicated enough then we see that all three middle classical crossings will become virtual after applying Nikonov’s projection.

On the other hand, since these three virtual knots are homologically trivial, their persistence does not affect the homological triviality of the three crossings depicted in Fig. 9. So, there is a motivation how to find another projection satisfying the condition of Theorem 4 which does not kill the three crossings depicted in this Figure.

The reason is that the Nikonov projection is very restrictive and makes many classical crossings virtual.

Let us now construct another map $g$ from virtual knots to classical knots satisfying all conditions of Theorem 4.

Take a virtual knot diagram $K$. If it is not a minimal genus diagram, apply Theorem 5. We get a diagram $K'$. If $K'$ is not yet of the minimal genus, apply Theorem 5 until we get to a minimal genus diagram. Take this minimal genus diagram $K_m$ and apply the projection with respect to the parity group. Then (if necessary) we again reiterate Theorem 5 to get to the minimal genus diagram, and then apply the parity projection once.

Every time we shall have a mapping which is well defined on the classes of knots: Theorem 5 does not change the class of the knot at all, and the group parity projection is well defined once we know that we are on the minimal genus.

The resulting diagram will be classical. Denote it by $g(K)$.

The reader can easily find virtual knots (1-1 tangles) to be inserted in Fig. 9 so that for the resulting knot $K$, the projection $g(K)$ gives the trefoil knot, whence the projection $f(K)$ is the unknot.

For exact definitions of connected sums, see [18, 10]

**Conjecture 1.** The map $g$ takes connected sum of virtual knots to connected sums of classical knots.

Of course, there are ways to mix the approaches described in the present paper to construct further
projections satisfying the conditions of Theorem 4.

An interesting question is to find “the most careful” projection satisfying all conditions of Theorem 4 which preserves more classical data.

5 Problems with the existence of a well defined map on diagrams

Consider the virtual knot diagram $A$ drawn in the left picture of Fig. 10. If we seek a projection satisfying conditions of the Main Theorem, we may $A$ project to $D$ in the same picture (lower right). Note that $A$ is not classical. However, the two intermediate knots ($B$ and $C$) are both classical: they are drawn on the torus, however, they both fit into a cylinder, and hence, to the plane; so, they will project to themselves.

There is no obvious reason why the projection of $A$ should be exactly $D$ because both $B$ and $C$ are classical; on the other hand there is no obvious way to make a preferred choice between $B$ and $C$ if one decides to take them to be the result of projection of $A$.

So, a bigger diagram projects to a smaller one (we see that $A > B, A > C$ but $B > D, C > D$). This is the lack of naturality which does not allow one to make projection compatible with Reidemeister moves. Of course $A$ differs from $B$ by one Reidemeister move, as well as their images $D$ and $B$, but in the first case the move is decreasing, and in the second case it is increasing.

This is also the reason of ambiguity: in fact, one can also project $A$ to $B$ or $C$ since both these diagrams are classical.

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Figure 10: Virtual Knot and Its Classical Projection

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