Conformal embeddings of $S^2 \to \mathbb{R}^3$ with prescribed mean curvature: A variational approach

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Abstract

Motivated by recent progress on a spinorial analogue of the Yamabe problem in the geometric literature, we study a conformally invariant spinor field equation on the $m$-sphere, $m \geq 2$. Via variational methods and the spinorial Weierstraß representation, we study the problem of prescribing mean curvature for the immersion $S^2 \to \mathbb{R}^3$.

Keywords. Dirac equations, Conformal geometry, Blow-up

1 Introduction

One of the fundamental topics in classical differential geometry concerns the question of whether a smooth Riemannian manifold can be isometrically immersed/embedded into Euclidean spaces. It has been conjectured by Schläfli in 1873 that every $m$-dimensional smooth Riemannian manifold $(M^m, g)$ admits a local isometric embedding into $\mathbb{R}^{N_m}$, with $N_m = \frac{m(m+1)}{2}$. Several important achievements have been made towards this problem in the last century. In one of his outstanding papers, J. Nash [30] showed the existence of a global isometric embedding of any $m$-dimensional smooth Riemannian manifold in $\mathbb{R}^N$ with $N = 3N_m + 4m$ in the case of compact $M$ and $N = (m + 1)(3N_m + 4m)$ in the case of non-compact $M$. Following Nash, we mention the book ”Partial Differential Relations” published in 1986 by Gromov, which contains various problems related to the isometric embedding of Riemannian manifolds. Furthermore, in this book, Gromov showed that $N = N_m + 2m + 3$ is enough for the compact case.

In low dimensions, if one asks the more specific question of whether a given 2-dimensional Riemann surface can be isometrically immersed/embedded into Euclidean 3-space, not too much is known. Besides some general local existence results (see for example [28][33] for real analytic metrics and for smooth metrics under suitable curvature assumptions), non-existence results seem to be easier to come by: for instance, Hilbert’s classical result that the hyperbolic plane does not admit an isometric immersion into $\mathbb{R}^3$, and the fact that a compact non-positively curved Riemann surface cannot be isometrically immersed into $\mathbb{R}^3$.

In the classical differential geometry of surfaces, the Gauß-Codazzi-Mainardi equations consist of a pair of related equations, and it incorporates the extrinsic curvature (mean curvature) of the surface. The equations are precisely the integrability conditions to obtain an immersed
surface in $\mathbb{R}^3$ (locally or on a simply connected domain). With the development of Spin Geometry and the theory of Dirac operators, an equivalent albeit simpler expression becomes well-known, namely the existence of a special spinor field called generalized Killing spinor field (see for instance [25, 32, 36–38] and references therein). This expression is the so-called spinorial Weierstraß representation, and can be understood as a geometrically invariant way for the representation of surfaces in Euclidean 3-space.

In late 1990s, T. Friedrich has written a very nice article in which the spinorial Weierstraß representation is beautifully explained, see [16, Theorem 13]. Eventually, it turns out that there exists a one-to-one correspondence between the existence of an isometric immersion $(\tilde{M}^2, g) \hookrightarrow (\mathbb{R}^3, g_{\mathbb{R}^3})$ with mean curvature $H$ and the existence of a spinor field $\varphi$ with constant length satisfying the following Dirac equation

$$D_g\varphi = H\varphi, \quad |\varphi|_g \equiv 1$$

(1.1)

where $\tilde{M}^2$ is the universal covering of $M^2$, $D_g$ and $|\cdot|_g$ are respectively the Dirac operator and the hermitian metric on the spinor bundle $S(M^2)$ over $(M^2, g)$.

Following this idea, the situation is much clearer for simply connected surfaces. And one may naturally ask the following question:

**Question 1.** Given an arbitrary smooth function $H : S^2 \to \mathbb{R}$, what is the condition on $H$ so that there is an isometric immersion $S^2 \hookrightarrow \mathbb{R}^3$ realizing $H$ as its mean curvature?

Or it is equivalent to consider the following question via the spinorial Weierstraß representation:

**Question 2.** What are necessary and sufficient conditions on $H$ so that there is a solution to Eq. (1.1)?

It seems interesting to see that Question 2 can be investigated via a conformal change of Eq. (1.1). That is, we may begin with the standard Riemann sphere $(S^2, g_{S^2})$ and consider the following nonlinear partial differential equation involving the Dirac operator $D_{g_{S^2}}$:

$$D_{g_{S^2}}\psi = H|\psi|^2_{g_{S^2}} \psi \quad \text{on} \quad (S^2, g_{S^2}).$$

(1.2)

Suppose we have a spinor field $\psi$ that satisfies Eq. (1.2) and that never vanishes. Then one may introduce a conformal metric $g = |\psi|^4_{g_{S^2}} g_{S^2}$ on $S^2$. Through the well-known formula for the Dirac operator under a conformal change of the metrics (see for instance [18, Proposition 1.3.10] and Proposition 2.1 below), we obtain a new spinor field $\varphi := \iota(|\psi|^{-1}_{g_{S^2}} \psi)$ which satisfies

$$D_g\varphi = H\varphi, \quad |\varphi|_g \equiv 1,$$

where $\iota : (S(S^2), g_{S^2}) \to (S(S^2), g)$ is a fiberwise isometry. And thus we find an isometric immersion $(S^2, g) \hookrightarrow (\mathbb{R}^3, g_{\mathbb{R}^3})$ that realizes $H$ as prescribed mean curvature.

In this paper, we will make use of the above spinorial tool in the study of Question 2. From this point of view, it seems natural to consider a two-step procedure:
• find a non-trivial spinor field \( \psi \) satisfying Eq. (1.2),

• show that the zero set under \( |\psi|_{g,S^2} : S^2 \to [0, \infty) \) is empty, i.e. \( \psi^{-1}(0) = \emptyset \).

However this is not an easy task. In general, even if one already knows there exists a non-trivial solution \( \psi \) to Eq. (1.2), it is still difficult to know whether \( \psi \) possesses zeros. This is due to the lack of an adequate replacement for the maximum principle developed for second order elliptic partial differential operators.

Let us return to the problem of finding isometric immersions/embbedings of surfaces. The spinorial Weierstraß representation has transformed the problem into the equivalent problem which amount to solve a PDE on the spinor bundle over the surface. However, still, the picture is not completely clear for immersed surfaces. A further question besides Question 1 and 2 is the following:

**Question 3.** Assume that a function \( H : S^2 \to \mathbb{R} \) is given such that it is the mean curvature of an immersion. Under which condition is this immersion an embedding? And when does it have self-intersections?

In this paper, we are interested in answering Questions 1-3. In order to explain our results in more detail, we begin with the following theorem, which provides a description of the zero set (or nodal set) of a solution to (1.2).

**Theorem 1.1.** On a closed spin surface \((M^2, g)\) of genus \( \gamma \), suppose that \( H : M^2 \to (0, \infty) \) is a smooth function. Let \( \psi \) be a solution of the equation

\[
D_g \psi = H \psi \psi^2 \text{ on } M^2.
\]

(1.3)

Then the number of zeros of \( \psi \) is at most

\[
\gamma - 1 + \int_{M^2} H^2 |\psi|^4 \, d\text{vol}_g.
\]

**Remark 1.2.** We note that, by the regularity arguments in [22] (see also [39]), a weak solution to Eq. (1.3) is smooth provided that \( H \) is smooth. Then it follows from Bär’s theorem (cf. [9]) that the zero set of \( \psi \) should be discrete. Theorem 1.1 may be used to derive upper estimates for the number of nodal points. In order to explain this we write \( H_{max} = \max_{M^2} H \) and we suppose that \( \psi \) satisfies the energy estimate

\[
\int_{M^2} H |\psi|^4 \, d\text{vol}_g < \frac{8\pi}{H_{max}}
\]

(1.4)

and \( M^2 \) has genus 0 (i.e. \( M^2 \) is topologically a sphere), then we have

\[
\# \psi^{-1}(0) \leq -1 + \int_{M^2} H^2 |\psi|^4 \, d\text{vol}_g < -1 + \frac{H_{max} \int_{M^2} H |\psi|^4 \, d\text{vol}_g}{4\pi} < 1.
\]

And thus \( \psi \) will have no zero at all.
After having studied the zero set, we discuss the existence of a (weak) solution to \((1.3)\) on the 2-sphere in the next theorem. In order to state the result, we introduce the following notation:

For a function \(H \in C^2(S^2)\), we set \(H_{\max} := \max_{\xi \in S^2} H(\xi)\) and \(H_{\min} := \min_{\xi \in S^2} H(\xi)\). We then denote the set of maximum points as \(\mathcal{H} := \{\xi \in S^2 : H(\xi) = H_{\max}\}\), and \(\mathcal{H}_\delta := \{\xi \in S^2 : \text{dist}_{g_{S^2}}(\xi, \mathcal{H}) < \delta\}\) its \(\delta\)-neighborhood for \(\delta > 0\).

Our criteria of the function \(H\) will be formulated as

\[ (H) \quad \mathcal{H} \text{ is not contractible in its } \delta\text{-neighborhood } \mathcal{H}_\delta, \text{ for some small } \delta > 0, \text{ but there exits } d \in (\max\{\frac{1}{2}H_{\max}, H_{\min}\}, H_{\max}) \text{ such that } \mathcal{H} \text{ is contractible in } \{\xi \in S^2 : H(\xi) \geq d\}. \]

There is no critical value of \(H\) in the interval \((d, H_{\max})\), and if \(\xi \in S^2\) is a critical point of \(H\) with \(H(\xi) = d\) then the Hessian of \(H\) at \(\xi\) is positive definite.

And then, combining Theorem \([1.1]\), the result is as follows

**Theorem 1.3.** If \(H : S^2 \to (0, \infty)\) is a \(C^2\) function satisfying condition \((H)\), then there is a solution \(\psi\) to Eq. \((1.2)\) with the energy estimate

\[
\frac{4\pi}{H_{\max}} < \int_{S^2} H|\psi|^4_{g_{S^2}} d\text{vol}_{g_{S^2}} < \frac{8\pi}{H_{\max}}.
\]

Particularly, if \(H\) is smooth, then the solution \(\psi\) has no zero at all, i.e. the nodal set \(\psi^{-1}(0)\) is empty.

**Remark 1.4.** By virtue of the spinorial Weierstraß representation, Theorem \([1.3]\) implies the existence of an isometric immersion \(\Pi : (S^2, |\psi|^4_{g_{S^2}} g_{S^2}) \to (\mathbb{R}^3, g_{\mathbb{R}^3})\). In particular, the pull-back of the Euclidean volume form under this immersion will be \(\Pi^*(d\mu) = |\psi|^4_{g_{S^2}} d\text{vol}_{g_{S^2}}\). Notice that the Willmore functional for this immersion, denoted by \(\mathcal{W}(\Pi)\), appears as the integral squared norm of the mean curvature \(H\). Hence, we have the estimate

\[
\mathcal{W}(\Pi) = \int_{S^2} H^2|\psi|^4_{g_{S^2}} d\text{vol}_{g_{S^2}} \leq H_{\max} \int_{S^2} H|\psi|^4_{g_{S^2}} d\text{vol}_{g_{S^2}} < 8\pi.
\]

Due to Li-Yau’s inequality \([27, \text{Theorem 6}]\), the immersion \(\Pi\) covers points in \(\mathbb{R}^3\) at most once. And thus \(\Pi\) is in fact an isometric embedding.

We note that the class of functions \(H\) that satisfy the hypothesis \((H)\) is dense, in \(C^1\)-topology, in the space of positive smooth functions. Then Theorem \([1.3]\) and Remark \([1.4]\) together immediately imply the next corollary, which provides an answer to Question \([1.3]\). For ease of notation, we will denote \([g_{S^2}] = \{f^2 g_{S^2} : f \in C^\infty(S^2), f > 0\}\) the standard conformal class of \(g_{S^2}\).

**Corollary 1.5.** Let \(C^\infty_+\) denote the class of positive smooth functions on \(S^2\) and let

\[ \mathcal{H} = \{H \in C^\infty_+ : H \equiv \text{constant or } H \text{ satisfies condition } (H)\}. \]

Then \(\mathcal{H}\) has the following properties:
Figure 1: A possible shape of function \( H \)'s, in local coordinates, which satisfies the criteria \((H)\).

1. \( \mathcal{H} \) is dense in \( \mathcal{C}^\infty_+ \) with respect to the \( \mathcal{C}^1 \)-topology;

2. for any \( H \in \mathcal{H} \), there exists a conformal metric \( g_H \in [g_{S^2}] \) and an isometric immersion \( \Pi_H : (S^2, g_H) \to (\mathbb{R}^3, g_{\mathbb{R}^3}) \) realizing \( H \) as its mean curvature;

3. the Willmore energy of \( \Pi_H \) satisfies

\[
\mathcal{W}(\Pi_H) = \int_{S^2} H^2 d\text{vol}_{g_H} < 8\pi,
\]

and thus \( \Pi_H \) is an isometric embedding.

We point out that, in [23], T. Isobe also considered the Question 1 and 2. He proved that when the function \( H \) is very close to a constant, that is \( H(\xi) = 1 + \varepsilon Q(\xi) \) on \( S^2 \) with \( Q \) being a specific Morse function and \( \varepsilon > 0 \) is small, then there exists a branched immersion \( \Pi_{\varepsilon,Q} : S^2 \to \mathbb{R}^3 \) that realizes its mean curvature as \( 1 + \varepsilon Q \). Here, by branched immersion we mean that \( \Pi_{\varepsilon,Q} \) is an immersion except at a discrete set of points. The method in [23] is also based on the spinorial Weierstraß representation and the nonlinear Dirac equation (1.2).

Another result concerning this problem is due to M. Anderson [7]. He proved that for any positive function \( H : S^2 \to (0, \infty) \) in the class \( \mathcal{C}^{k,\alpha} \), \( k \geq 0 \) and \( \alpha \in (0, 1) \), there exists a branched immersion \( \Pi : S^2 \to \mathbb{R}^3 \) and an positive affine function \( \ell \) on \( S^2 \) such that the mean curvature of \( \Pi \) is prescribed by \( H + \ell \). In this setting the function \( H > 0 \) lies arbitrarily in the class \( \mathcal{C}^{k,\alpha} \), and thus it can be far away from a constant. However, the affine function \( \ell \) cannot be prescribed.

Note that, in Corollary 1.5, the mean curvature is accurately prescribed by the functions in the class \( \mathcal{H} \) and, particularly, the presence of branch points is ruled out. Hence, comparing with [7, 23], our approach is in a different perspective. The class \( \mathcal{H} \) has its generality in the sense that only a “volcano”-shaped assumption is required and, furthermore, such a class of functions can give an explicit estimate of the Willmore functional from above. However, Corollary 1.5 can not cover any of the results in [7, 23] but has intersection with them. There are many functions lying
outside $\mathcal{H}$ that can be prescribed as mean curvature of certain conformal isometric embeddings. However, there are still several problems that have to be solved.

Last but not least, we want to remind the readers that Eq. (1.2) for an arbitrarily given $H$ does not necessarily admit a non-trivial solution. As was shown in [6], there is an obstruction to the existence of a non-trivial solution to Eq. (1.2). In fact, if there exists a non-trivial solution $\psi$ to Eq. (1.2), then $H$ must satisfy

$$\int_{S^2} (\nabla_X H) |\psi|_{g_{S^2}}^4 d\text{vol}_{g_{S^2}} = 0$$

(1.5)

for any conformal vector field $X$ on $S^2$. In particular, $\nabla_X H$ must change sign on $S^2$. The obstruction (1.5) is quite similar to the Kazdan-Warner obstruction for Gaussian and scalar curvature equations (see [24]).

Let us give an overview of our paper as follows. We will consider the following semilinear Dirac equation with a critical exponent in all dimensions:

$$D_{g_{S^m}} \psi = H |\psi|_{g_{S^m}}^{\frac{2}{m}} \psi \text{ on } (S^m, g_{S^m})$$

(1.6)

under the generalized assumption that $H \in C^2(S^m)$ satisfies

$(H \ast)$ $\mathcal{H}$ is not contractible in its $\delta$-neighborhood $\mathcal{H}_\delta$, for some small $\delta > 0$, but there exists $d \in \left( \max \{ 2^{-\frac{1}{m-1}} H_{max}, H_{min} \}, H_{max} \right)$ such that $\mathcal{H}$ is contractible in $\{ \xi \in S^m : H(\xi) \geq d \}$. There is no critical value of $H$ in the interval $(d, H_{max})$, and if $\xi \in S^m$ is a critical point of $H$ with $H(\xi) = d$ then the Hessian of $H$ at $\xi$ is positive definite.

Here, analogous to the 2-sphere case, $H_{max} := \max_{\xi \in S^m} H(\xi)$ and $H_{min} := \min_{\xi \in S^m} H(\xi)$, $\mathcal{H} := \{ \xi \in S^m : H(\xi) = H_{max} \}$ is the collection of maximum points and $\mathcal{H}_\delta := \{ \xi \in S^m : \text{dist}_{g_{S^m}}(\xi, \mathcal{H}) < \delta \}$ is its $\delta$-neighborhood for $\delta > 0$. Inspired by the idea of Yamabe [43] and Aubin [8] when they deal with the Yamabe problem, here we mainly focus on a specific regularization of Eq. (1.6), namely, a family of related equations with subcritical exponent

$$D_{g_{S^m}} \psi = H |\psi|_{g_{S^m}}^{p-2} \psi \text{ on } (S^m, g_{S^m})$$

(1.7)

where $p \in (2, \frac{2m}{m-1})$. If one considers the variational formulation where solutions of Eq. (1.7) are obtained as critical points of the energy functional

$$\mathcal{L}_p(\psi) := \frac{1}{2} \int_{S^m} (D\psi, \psi) d\text{vol}_{g_{S^m}} - \frac{1}{p} \int_{S^m} H |\psi|^p d\text{vol}_{g_{S^m}}$$

then the main ingredient of this paper lies in the compactness of a family of critical points $\psi_p$ as $p \to \frac{2m}{m-1}$ of $\mathcal{L}_p$. One way to show the compactness is to apply the regularity arguments to get uniform $C^{1,\alpha}$-boundedness for $\psi_p$, and hence by Arzelà-Ascoli theorem, up to a subsequence, the spinor field $\psi_p$ converges to a solution of Eq. (1.6). This has been shown when the energy
\( \mathcal{L}_p(\psi_p) \) is below a so-called blow-up threshold (see [3, 31]). However, in our case, one must allow for blow-up phenomenon, and the situation becomes much more complicated. Whether or not one can obtain a uniform \( C^{1,\alpha} \)-boundedness for these \( \psi_p \) is not clear. Here our strategy is to use build a concentration-compactness alternative for the sequence \( \{\psi_p\} \). To rule out the blow-up phenomenon, one only needs to clarify that concentration will not occur, and this will be done in Section 4.

From the viewpoint of variational calculus, there are at least three different kinds of analytical difficulties when carrying out the above program. The first one is due to the strong indefiniteness of the functional \( \mathcal{L}_p \). It is well-known that the spectrum of the Dirac operator \( D_{g_{SM}} \) is neither bounded from above nor from below. Hence, a critical point of \( \mathcal{L}_p \) has infinite Morse index and infinite co-index. This will cause a problem since the classical variational methods such as the minimizing technique, Morse theory and its variants do not directly apply. The second difficulty is due to the existence of multiple solutions to the subcritical equation (1.7). Note that Eq. (1.7) is invariant under the canonical action of \( S^1 = \{ e^{i\theta} \in \mathbb{C} : \theta \in [0, 2\pi] \} \) on spinors (i.e. if \( \psi \) is a solution of Eq. (1.7) then \( e^{i\theta}\psi \) is also a solution, for every fixed \( \theta \)). Moreover, for the case \( m \equiv 2, 3, 4 \text{ (mod 8)} \), the spinor bundle has a quaternionic structure which commutes with Clifford multiplication, see for instance the construction in [17, Section 1.7] or [26, Page 33, Table III]. Thus in these cases, Eq. (1.7) is invariant under the canonical action of the unit quaternions \( S^3 = \{ q \in \mathbb{H} : |q| = 1 \} \) on spinors. And then one may be interested in the existence of \( G \)-inequivalent solutions of Eq. (1.7), where \( G = S^1 \) or \( S^3 \). Standard variational arguments (see e.g. [34, Chapter 2 Section 5]) show that the symmetric property of Eq. (1.7) under \( G \)-actions ensures multiple \( G \)-orbits of solutions. It can be seen from Remark 2.9 that, among all these \( G \)-orbits of solutions, the ground state energy solutions should blow up when \( p \to \frac{2m}{m-1} \). Whether or not there exists a sequence of solutions \( \{\psi_p\} \) which shows compactness when \( p \) approaches \( \frac{2m}{m-1} \) becomes a problem. Another difficulty lies in the analysis of the zero sets for a solution to Eq. (1.6). Since the equation is defined on the spinor bundle, which is a complex vector bundle, in contrast to the Laplacian on functions, we do not have a maximum principle in order to exclude zeros for eigenspinors.

In what follows, the proof will be carried out as follows: in Section 2, we introduce some notation and recall some known facts about the Dirac operator and blow-up asymptotic behavior of Eq. (1.7); the analysis of zero sets for a solution to (1.6) on surfaces will be given in Section 3; the detailed proof of Theorem 1.3 and Corollary 1.5 are given in Section 4.

## 2 Notation, definitions and known results

### 2.1 The Dirac operator and \( H^1_\sigma \)-spinors

In general, let \( (M, g, \sigma) \) be an \( m \)-dimensional compact spin manifold, where \( g \) is a Riemannian metric on \( M \), \( \sigma : P_{Spin}(M) \to P_{SO}(M) \) is a spin structure on \( M \). We denote \( S(M) = \)
$P_{\text{Spin}}(M) \times \rho S_m$ the spinor bundle equipped with a natural hermitian inner product $(\cdot, \cdot)$. This is a complex vector bundle associated to the principle bundle $P_{\text{Spin}}(M)$ via the fundamental spin representation $\rho: \text{Spin}(m) \to \text{End}(S_m)$. On the spinor bundle $S(M)$, the Dirac operator is then defined as the composition

$$D_g: \Gamma(S(M)) \xrightarrow{\nabla^S} \Gamma(TM \otimes S(M)) \longrightarrow \Gamma(TM \otimes S(M)) \xrightarrow{m} \Gamma(S(M))$$

where $\nabla^S$ is the canonical lift of the Levi-Civita connection on $P_{SO}(M)$ via the double covering $P_{\text{Spin}}(M) \to P_{SO}(M)$ and $m$ denotes the Clifford multiplication $m: X \otimes \psi \mapsto X \cdot \psi$.

The Dirac operator behaves very nicely under conformal changes in the following sense, which was already known to Hitchin [20] and which was worked out in detail in [19]:

**Proposition 2.1.** Let $g_0$ and $g = f^2 g_0$ be two conformal metrics on a Riemannian spin manifold $M$. Then, there exists an isomorphism of vector bundles $\iota: S(M, g_0) \to S(M, g)$ which is a fiberwise isometry such that

$$D_g(\iota(\psi)) = \iota(f^{-\frac{m+1}{2}}D_{g_0}(f^{\frac{m-1}{2}}\psi)),$$

where $S(M, g_0)$ and $S(M, g)$ are spinor bundles on $M$ with respect to the metrics $g_0$ and $g$, respectively, and $D_{g_0}$ and $D_g$ are the associated Dirac operators.

Let us consider the case $M = S^m$ with the standard metric $g_{S^m}$. For simplicity, we denote $D = D_{g_{S^m}}$ the associated Dirac operator on the spinor bundle $S(S^m)$. Let $\text{Spec}(D)$ be its spectrum. It is well-known that $D$ is essentially self-adjoint in $L^2(S^m, S(S^m))$ and has compact resolvents (see [17, 18, 26]). Furthermore, we have

$$\text{Spec}(D) = \{ \pm \left( \frac{m}{2} + j \right) : j = 0, 1, 2, \ldots \}$$

and, for each $j$, the eigenvalues $\pm \left( \frac{m}{2} + j \right)$ have the multiplicity

$$2^{\left\lfloor \frac{m}{2} \right\rfloor} \binom{m + j - 1}{j}$$

(see [35]). This concludes that the spectrum of the Dirac operator is discrete. For notation convenience, we can write $\text{Spec}(D) = \{ \lambda_k \}_{k \in \mathbb{Z} \setminus \{0\}}$ with

$$\lambda_k = \frac{k}{|k|} \left( \frac{m}{2} + |k| - 1 \right).$$

An application of the classical spectral theory implies that the eigenspaces of $D$ form a complete orthonormal decomposition of $L^2(S^m, S(S^m))$, that is,

$$L^2(S^m, S(S^m)) = \bigoplus_{\lambda \in \text{Spec}(D)} \ker(D - \lambda I).$$
Let us denote $m(\lambda_k)$ the multiplicity of $\lambda_k \in \text{Spec}(D)$. It is clear that $L^2(S^m, \mathbb{S}(S^m))$ possesses a normalized Hilbert basis consisting of eigenspinors of $D$, that is, $\{\eta_{k,j}\}$ for $k \in \mathbb{Z} \setminus \{0\}$ and $1 \leq j \leq m(\lambda_k)$ with corresponding eigenvalues $\lambda_k$. Hence $D\eta_{k,j} = \lambda_k\eta_{k,j}$. We can then define the operator $|D|^{1/2} : L^2(S^m, \mathbb{S}(S^m)) \rightarrow L^2(S^m, \mathbb{S}(S^m))$ by

$$|D|^{1/2}\psi = \sum_{k \in \mathbb{Z} \setminus \{0\}} \sum_{j=1}^{m(\lambda_k)} |\lambda_k|^{1/2} a_{k,j} \eta_{k,j},$$

for $\psi = \sum a_{k,j} \eta_{k,j} \in L^2(S^m, \mathbb{S}(S^m))$. Let us set

$$H^{1/2}(S^m, \mathbb{S}(S^m)) := \left\{ \psi = \sum_{k \in \mathbb{Z} \setminus \{0\}} \sum_{j=1}^{m(\lambda_k)} |\lambda_k||a_{k,j}|^2 < \infty \right\}.$$

The space $H^{1/2}(S^m, \mathbb{S}(S^m))$ coincides with the Sobolev space $W^{1/2,2}(S^m, \mathbb{S}(S^m))$ (see [1, 3]), and hence we have the Sobolev embeddings: $H^{1/2}(S^m, \mathbb{S}(S^m)) \hookrightarrow L^q(S^m, \mathbb{S}(S^m))$ for $2 \leq q \leq \frac{2m}{m-1}$. In the sequel, we endow $E := H^{1/2}(S^m, \mathbb{S}(S^m))$ with the inner product

$$\langle \psi, \varphi \rangle = \text{Re}(|D|^{1/2}\psi, |D|^{1/2}\varphi),$$

and the induced norm $\|\cdot\| = (\cdot, \cdot)^{1/2}$, where $(\psi, \varphi)_2 = \int_{S^m} (\psi, \varphi) d\text{vol}_{g_{S^m}}$ is the $L^2$-inner product on spinors. Let us mention here that $\|\cdot\|$ is equivalent to the graph norm for the operator $|D|^{1/2}$, and hence $(E, \|\cdot\|)$ is complete and $C^\infty(S^m, \mathbb{S}(S^m))$ is dense in $E$. Furthermore, $E$ induces a splitting $E = E^+ \oplus E^-$ with

$$E^+ := \overline{\text{span}\{\eta_{k,j} \mid k > 0\}} \quad \text{and} \quad E^- := \overline{\text{span}\{\eta_{k,j} \mid k < 0\},} \quad (2.1)$$

where the closure is taken in the $\|\cdot\|$-topology. It is then clear that these are orthogonal subspaces of $E$ on which the action $\int_{S^m} \langle D\psi, \psi \rangle d\text{vol}_{g_{S^m}}$ is positive or negative. In the sequel, with respect to this decomposition, we will write $\psi = \psi^+ + \psi^-$ for any $\psi \in E$. The dual space of $E$ will be denoted by $E^* := H^{-\frac{1}{2}}(S^m, \mathbb{S}(S^m))$.

### 2.2 Variational settings

As was mentioned before, we consider nonlinear Dirac equations of the following form:

$$D\psi = H|\psi|^{p-2}\psi \quad \text{on} \ S^m, \quad (2.2)$$

where $H : S^m \rightarrow (0, +\infty)$ is a $C^2$ function, $p \in (2, 2^*]$ and $2^* = \frac{2m}{m-1}$.

Eq. (2.2) has a variational structure: $\psi$ is a solution to (2.2) if and only if $\psi$ is a critical point of $L_p$ defined by

$$L_p(\psi) = \frac{1}{2} \int_{S^m} \langle D\psi, \psi \rangle d\text{vol}_{g_{S^m}} - \frac{1}{p} \int_{S^m} H|\psi|^p d\text{vol}_{g_{S^m}}$$

$$= \frac{1}{2} (\|\psi^+\|^2 - \|\psi^-\|^2) - \frac{1}{p} \int_{S^m} H|\psi|^p d\text{vol}_{g_{S^m}}$$
for $\psi = \psi^+ + \psi^- \in E = E^+ \oplus E^-$. 

In order to simplify the notation, we will occasionally use the notation $\mathcal{L}$ instead of $\mathcal{L}_{2^*}$ in the sequel. And for first and second derivatives, we write $\mathcal{L}'_p(\psi)[\varphi]$ for the derivative of $\mathcal{L}_p$ at $\psi$ applied to $\varphi$ and, similarly, we write $\mathcal{L}''_p(\psi)[\varphi, \phi]$ for the second derivative of $\mathcal{L}_p$ at $\psi$ applied to $\varphi$ and $\phi$. By $L^p$ we denote the Banach space $L^p(S^m, \mathcal{S}(S^m))$ for $p \geq 1$ and by $| \cdot |_p$ we denote the usual $L^p$-norm.

Since both the negative and positive parts of $\text{Spec}(D)$ are unbounded, the quadratic form in $L^p$ is of strongly indefinite type. The next result can be viewed as a Lyapunov-Schmidt reduction of the strongly indefinite functional $\mathcal{L}_p$. Specifically, it consists of a 2-step procedure: first to reduce the functional $\mathcal{L}_p$ to the subspace $E^+$ and then to reduce it on a Nehari-Pankov manifold in $E^+$. This procedure will make the subsequent estimates more transparent. The idea of this reduction was already employed to the study of Choquard-Pekar equation in [13, Lemma 2.1 & 2.2] and was also worked out in detail in a recent paper [10] for nonlinear Dirac equations on spin manifolds.

**Proposition 2.2.** (1) There exists a $C^1$ map $h_p : E^+ \to E^-$ such that for $\psi \in E$:

$$\mathcal{L}'_p(\psi)[v] = 0 \quad \forall v \in E^- \quad \implies \quad \psi^- = h_p(\psi^+).$$

Moreover, $h_p(u)$ maximizes $\mathcal{L}_p(u + v)$ over all $v \in E^-$ for $u \in E^+$.

(2) The functional $I_p : E^+ \to \mathbb{R}$, $I_p(u) = \mathcal{L}_p(u + h_p(u))$, satisfies

$$I'_p(u) = 0 \quad \implies \quad \mathcal{L}'_p(u + h_p(u)) = 0.$$

(3) For every $u \in E^+ \setminus \{0\}$, the map $I_{p,u} : [0, \infty) \to \mathbb{R}$, $I_{p,u}(t) = I_p(tu)$, is of class $C^2$ and satisfies

$$I'_{p,u}(t) = 0, \quad t > 0 \quad \implies \quad I''_{p,u}(t) < 0.$$ 

Moreover $I_{p,u}(0) = I'_p(0) = 0$ and $I''_{p,u}(0) > 0$.

**Remark 2.3.** We omit the proof of Proposition 2.2 since it can be done by following the arguments in [10]. Here some remarks are in order. The paper [10] has set up a general variational framework to study the equation $D\psi = \lambda \psi + |\psi|^{2^* - 2}\psi$ and its variations on an arbitrary spin manifold $M$, where $\lambda \in \mathbb{R}$ is a real parameter. Proposition 2.2 can be viewed as a counterpart of [10, Proposition 7.2] by simply taking $\lambda = 0$ and $M = S^m$. The appearance of the positive function $H$ and the subcritical exponent $p$ in the functional $\mathcal{L}_p$ are rather harmless since they do not affect the strict convexity of the super-quadratic part. Hence there is no obstacle to apply the arguments in [10] and to obtain Proposition 2.2.

In the sequel, we shall call $(h_p, I_p)$ the reduction couple for $\mathcal{L}_p$ on $E^+$. From Proposition 2.2(1) and (2), it follows that critical points of $I_p$ and $\mathcal{L}_p$ are in one-to-one correspondence via
the injective map \( u \mapsto u + h_p(u) \) from \( E^+ \) to \( E \). The Nehari-Pankov manifold for \( \mathcal{L}_p \) is then defined as
\[
\mathcal{N}_p = \{ u \in E^+ \setminus \{ 0 \} : I'_p(u)[u] = 0 \} \tag{2.5}
\]
so that the reduced functional \( I_p \) is bounded from below on \( \mathcal{N}_p \). In fact, since \( I_p(u) = \mathcal{L}_p(u + h_p(u)) \), we have
\[
I'_p(u)[w] = \frac{d}{dt} I_p(u + tw) \bigg|_{t=0} = \frac{d}{dt} \mathcal{L}_p(u + tw + h_p(u + tw)) \bigg|_{t=0}
= \langle u, w \rangle - \langle h_p(u), h'_p(u)[w] \rangle - \Re \int_{S^m} H|u + h_p(u)|^{p-2}(u + h_p(u), w + h'_p(u)[w])d\text{vol}_{g_{Sm}}
\]
for any \( w \in E^+ \). Due to Proposition \( 2.2 \) (1), we find \( \mathcal{L}_p(u + h_p(u))[v] = 0 \) for all \( v \in E^- \). This implies
\[
0 = - \langle h_p(u), v \rangle - \Re \int_{S^m} H|u + h_p(u)|^{p-2}(u + h_p(u), v)d\text{vol}_{g_{Sm}}
\]
for all \( v \in E^- \). By noting that the Fréchet derivative of \( h_p \) at a point \( u \in E^+ \), which is denoted by \( h'_p(u) : E^+ \to E^- \), is nothing but a linear operator. By taking \( v = h'_p(u)[w] \) and \( h_p(u) \) respectively in the above expression, we have
\[
- \langle h_p(u), h'_p(u)[w] \rangle - \Re \int_{S^m} H|u + h_p(u)|^{p-2}(u + h_p(u), h'_p(u)[w])d\text{vol}_{g_{Sm}} = 0
\]
and
\[
- \|h_p(u)\|^2 - \Re \int_{S^m} H|u + h_p(u)|^{p-2}(u + h_p(u), h_p(u))d\text{vol}_{g_{Sm}} = 0. \tag{2.6}
\]
Therefore, the expression of \( I'_p(u)[w] \) can be simplified as
\[
I'_p(u)[w] = \langle u, w \rangle - \Re \int_{S^m} H|u + h_p(u)|^{p-2}(u + h_p(u), w)d\text{vol}_{g_{Sm}}, \quad \text{for} \ w \in E^+
\]
and particularly, by taking \( 2.6 \) into account, \( I'_p(u)[u] \) can be further rewritten as
\[
I'_p(u)[u] = \|u\|^2 - \Re \int_{S^m} H|u + h_p(u)|^{p-2}(u + h_p(u), u)d\text{vol}_{g_{Sm}}
= \|u\|^2 - \|h_p(u)\|^2 - \int_{S^m} H|u + h_p(u)|^{p-2}(u + h_p(u), u + h_p(u))d\text{vol}_{g_{Sm}}
= \|u\|^2 - \|h_p(u)\|^2 - \int_{S^m} H|u + h_p(u)|^pd\text{vol}_{g_{Sm}}.
\]
Now, for \( u \in \mathcal{N}_p \), we have \( I'_p(u)[u] = 0 \) and
\[
I_p(u) = I_p(u) - \frac{1}{2} I'_p(u)[u] = \frac{p - 2}{2p} \int_{S^m} H|u + h_p(u)|^pd\text{vol}_{g_{Sm}} \geq 0, \tag{2.7}
\]
where the last inequality comes from the positiveness of \( H \). Furthermore, we notice that \( u \in \mathcal{N}_p \) implies \( u \in E^+ \setminus \{ 0 \} \) and \( h_p(u) \in E^- \) (where \( E^+ \) and \( E^- \) are orthogonal in \( E \)). Then
\( \{ x \in S^m : |u + h_p(u)|(x) > 0 \} \) must have positive measure (otherwise, \( u = -h_p(u) \) a.e. on \( S^m \), which suggests that \( u \in E^+ \cap E^- = \{0\} \)). Hence, (2.7) can be slightly improved as

\[
I_p(u) = I_p(u) - \frac{1}{2} I_p'(u)[u] = \frac{p-2}{2p} \int_{S^m} H|u + h_p(u)|^p d\text{vol}_{g_{S^m}} > 0.
\]

By Proposition 2.2 (3), we have that \( \mathcal{N}_p \) is a \( C^1 \) manifold of codimension 1 in \( E^+ \) and that \( \mathcal{N}_p \) is a natural constraint for the problem of finding non-trivial critical points of \( I_p \) on \( E^+ \). Indeed, one may consider the auxiliary functional \( K_p(u) = I_p'(u)[u] \) and can compute that \( K_p'(u)[u] = I_p'(u)[u] + I_p''(u)[u,u] \). Then, for \( u \in \mathcal{N}_p \), one derives \( K_p(u) = I_p'(u)(1) = 0 \) and \( K_p'(u)[u] = I_p''(u)(1) < 0 \), which suggests \( \mathcal{N}_p \) to be a \( C^1 \) manifold. Moreover, according to the Lagrange multiplier rule, if \( u \in \mathcal{N}_p \) is a constrained critical point of \( I_p \), then there exists \( \lambda \in \mathbb{R} \) such that

\[
I_p'(u)[v] = \lambda K_p'(u)[v] = \lambda (I_p'(u)[v] + I_p''(u)[u,v]) \quad \forall v \in E^+.
\]

Letting \( v = u \), so that \( I_p'(u)(1) = I_p''(u)[u,u] = 0 \), we can conclude from the above relation and the fact \( I_p''(u)(1) = I_p''(u)[u,u] < 0 \) that we must have \( \lambda = 0 \), and hence \( u \) is indeed a critical point of \( I_p \) in \( E^+ \). We also mention that, for \( u \in E^+ \setminus \{0\} \), the function \( I_p(t, u) \) attains its unique positive critical point at \( t = t(p,u) > 0 \) which realizes its maximum (correspondingly, \( t(p,u)u \in \mathcal{N}_p \) is called the projection of \( u \) on \( \mathcal{N}_p \)), and hence we have \( \mathcal{N}_p \) is homeomorphic to the unit sphere of \( E^+ \) and

\[
I_p(u) = \max_{t > 0} I_p(tu) = \max_{\psi \in \text{span}(u) \oplus E^-} \mathcal{L}_p(\psi) \quad \forall u \in \mathcal{N}_p.
\]

In the sequel, it is convenient to consider the reduced functional \( I_p \) restricted to the constraint \( \mathcal{N}_p \). However, an additional difficulty may arise in the variational arguments: it is nearly impossible to calculate the value of \( I_p(u) \) at any point \( u \in \mathcal{N}_p \). Even if we choose an explicit spinor \( \psi \in E \setminus \{0\} \), we cannot directly calculate its projection \( \psi^+ \) in \( E^+ \) and the corresponding projection \( t(p, \psi^+)\psi^+ \in \mathcal{N}_p \), moreover, we know almost nothing about the images of \( h_p(t\psi^+) \) for \( t > 0 \). We state, in the next lemma, a result about an upper bound estimate for the reduced functional \( I_p \) on \( \mathcal{N}_p \) in the directions of “almost critical points” of \( I_p \) that will play a significant role our argument.

**Lemma 2.4.** Let \( p_n \not\rightarrow 2^* \) be a converging sequence and suppose that there exists a sequence of spinor fields \( \{\psi_n\} \subset E \) such that

\[
c_1 \leq \mathcal{L}_{p_n}(\psi_n) \leq c_2 \quad \text{and} \quad \mathcal{L}_{p_n}'(\psi_n) \rightarrow 0
\]

(2.8) for some constants \( c_1, c_2 > 0 \), as \( n \rightarrow \infty \). Then \( \{\psi_n\} \) is bounded in \( E \) such that

\[
\|\psi_n^- - h_{p_n}(\psi_n^+)\| = O(\|\mathcal{L}_p(\psi_n)\|_{E^*})
\]

and

\[
\max_{t > 0} I_p(t\psi_n^+) \leq \mathcal{L}_{p_n}(\psi_n) + O(\|\mathcal{L}_{p_n}(\psi_n)\|^2_{E^*}).
\]
Lemma 2.4 is obtained by collecting the results in [10, Lemma 4.1, Lemma 7.3 - Corollary 7.6] with some minor modifications. The idea in the proof is built upon the strict concavity of $I_{p,n}(t)$ around its maximum point (i.e. (2.4)) and an application of the Inverse Function Theorem to the auxiliary functional $K_{p,n}(u) = I'_{p,n}(u)\|u\|$ near the point $\psi^+_n$. As an immediate consequence of Lemma 2.4 we have

**Corollary 2.5.** Let $p_n \not\rightarrow 2^*$ as $n \to \infty$ and $c_1, c_2 > 0$. For any $\theta > 0$, there exists $\alpha > 0$ such that for all large $n$ and $\psi \in E$ satisfying

\[
c_1 \leq \mathcal{L}_{p_n}(\psi) \leq c_2 \quad \text{and} \quad \|\mathcal{L}'_{p_n}(\psi)\|_{E^*} \leq \alpha
\]

there holds

\[
\max_{t > 0} I_{p_n}(t\psi^+) \leq \mathcal{L}_{p_n}(\psi) + \theta.
\]

For $p \in (2, 2^*]$, let us set

\[
\tau_p := \inf \left\{ I_p(u) : u \in \mathcal{N}_p \right\}.
\]

As a consequence of the following estimate

\[
I_p(u) \geq \mathcal{L}_p(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p} \int_{S^m} H|u|^pdvol_{g_{sm}} \geq \frac{1}{2}\|u\|^2 - C\|u\|^p, \quad \forall u \in E^+
\]

where $C > 0$ is a positive constant depending on $H_{max}$ and the Sobolev embedding $E \hookrightarrow L^p$, it turns out immediately that $\tau_p$ is lower bounded by a positive constant.

**Lemma 2.6.** Let $p_n \not\rightarrow 2^*$ as $n \to \infty$, then $\tau_{p_n} \to \tau_{2^*}$ as $n \to \infty$.

**Proof.** For arbitrary $\varepsilon > 0$ and $k \in \mathbb{N}$, by the Ekeland variational principle [15, 29, 40], we can first choose $u \in \mathcal{N}_{2^*}$ such that $\tau_{2^*} \leq I_{2^*}(u) < \tau_{2^*} + \frac{1}{k}$ and $\|I'_{2^*}(u)\| \leq \frac{1}{k}$.

Let $s_n > 0$ be such that $s_n u \in \mathcal{N}_{p_n}$. It follows from the arguments in [13, see page 182] that \{s_n\} must be bounded (otherwise, $I_{p_n}(s_n u) \to -\infty$ as $s_n \to +\infty$). And hence $\tau_{p_n} \leq I_{p_n}(s_n u)$ is bounded from above.

Note that $\|I'_{2^*}(u)\| = \|\mathcal{L}'_{2^*}(u + h_{2^*}(u))\|$ (see for instance [13, Lemma 2.2]) and that

\[
\mathcal{L}'_{2^*}(u + h_{2^*}(u))\varphi - \mathcal{L}'_{p_n}(u + h_{2^*}(u))\varphi
\]

\[
= \text{Re} \int_{S^m} \left(1 - \|u + h_{2^*}(u)\|^{2^*-p_n}\right)|u + h_{2^*}(u)|^{p_n-2}(u + h_{2^*}(u), \varphi)dvol_{g_{sm}}
\]

\[
= o_n(1)\|\varphi\|
\]

for all $\varphi \in E$, we derive that $\|\mathcal{L}'_{p_n}(u + h_{2^*}(u))\| \leq \frac{2}{k}$ for all large $n$. And thus, by applying Corollary 2.5 we have

\[
\tau_{p_n} \leq I_{p_n}(s_n u) = \max_{s > 0} I_{p_n}(s u) \leq \mathcal{L}_{p_n}(u + h_{2^*}(u)) + \varepsilon
\]

(2.10)
provided that \( k \) is fixed large enough. Since the function \( p \mapsto \int_{S^m} H|u + h_2(u)|^p\text{dvol}_{g_{S^m}} \) is continuous, we have that \( \mathcal{L}_{p_n}(u + h_2(u)) = \mathcal{L}_{2^*}(u + h_2(u)) + o_n(1) = I_{2^*}(u) + o_n(1) \). Therefore, by (2.10) and the arbitrariness of \( \varepsilon \), we have \( \limsup_{n \to \infty} \tau_{p_n} \leq \tau_{2^*} \).

On the other hand, we can choose \( u_n \in \mathcal{M}_{p_n} \) such that \( I_{p_n}(u_n) \leq \tau_{p_n} + \frac{1}{n} \). Let \( t_n > 0 \) be such that \( t_n u_n \in \mathcal{M}_{2^*} \). Then it follows that \( \{t_n\} \) is also bounded. And we can derive that

\[
\tau_{2^*} \leq I_{2^*}(t_n u_n) = \mathcal{L}_{2^*}(t_n u_n + h_2(t_n u_n)) \leq \mathcal{L}_{p_n}(t_n u_n + h_2(t_n u_n)) + o_n(1),
\]

(2.11)

where in the last inequality we have used the fact

\[
\int_{S^m} H|\psi|^{p_n}\text{dvol}_{g_{S^m}} \leq \left( \int_{S^m} H|\psi|^{2^*}\text{dvol}_{g_{S^m}} \right)^{\frac{p_n}{2}} \left( \int_{S^m} H\text{dvol}_{g_{S^m}} \right)^{\frac{2^* - p_n}{2}}
\]

\[
= \int_{S^m} H|\psi|^{2^*}\text{dvol}_{g_{S^m}} + o_n(1)
\]

when \( ||\psi|| \) is bounded independent of \( n \). Hence, we see from (2.11) that \( \tau_{2^*} \leq I_{p_n}(u_n) + o_n(1) \leq \tau_{p_n} + \frac{1}{n} + o_n(1) \) as \( n \to \infty \), which completes the proof.

We conclude this subsection by formulating the exact value of \( \tau_{2^*} \), that we will need in the sequel, i.e.

\[
\tau_{2^*} = \frac{1}{2m(H_{\text{max}})^{m-1}} \left( \frac{m}{2} \right)^m \omega_m
\]

(2.12)

where \( \omega_m \) stands for the volume of \( (S^m, g_{S^m}) \). To see this, we first need to show that \( \tau_{2^*} \geq \frac{1}{2m(H_{\text{max}})^{m-1}} \left( \frac{m}{2} \right)^m \omega_m \). It can be seen from the fact \( H(\xi) \leq H_{\text{max}} \) for \( \xi \in S^m \) that

\[
\mathcal{L}_{2^*}(\psi) \geq \mathcal{L}_{\text{max}}(\psi) := \frac{1}{2} (||\psi^+||^2 - ||\psi^-||^2) - \frac{H_{\text{max}}}{2^*} \int_{S^m} |\psi|^{2^*}\text{dvol}_{g_{S^m}}
\]

(2.13)

for all \( \psi \in E \). And critical points of \( \mathcal{L}_{\text{max}} \) correspond to solutions of

\[
D\psi = H_{\text{max}}|\psi|^{2^* - 2}\psi \quad \text{on } S^m.
\]

(2.14)

Since \( H_{\text{max}} > 0 \), by normalizing Eq. (2.14), we turn to consider

\[
D\psi = |\psi|^{2^* - 2}\psi \quad \text{on } S^m.
\]

(2.15)

Then solutions of Eq. (2.14) and (2.15) are in one-to-one correspondence via the map

\[
\psi \mapsto (H_{\text{max}})^{-\frac{m-1}{2}} \psi.
\]

(2.16)

Recall that \( \mathbb{R}^m \) and \( S^m \setminus \{N\} \) (where \( N \in S^m \) is the North pole) are conformally equivalent, it follows from [22, Proposition 4.1] and (2.16) that if \( \psi \) is a nontrivial solution to Eq. (2.14) then

\[
\mathcal{L}_{\text{max}}(\psi) \geq \frac{1}{2m(H_{\text{max}})^{m-1}} \left( \frac{m}{2} \right)^m \omega_m.
\]

(2.17)
Since it is well-known that Eq. (2.15) has a specific solution, which is a Killing spinor to the constant \(-1/2\) (see e.g. \([5\), Section 5.3]), we can see from (2.16) that the equality in (2.17) is achieved. Furthermore, notice that Proposition 2.2 can be applied to the functional \(\mathcal{L}_{\text{max}}\). Let us denote \(I_{\text{max}}\) the reduced functional for \(\mathcal{L}_{\text{max}}\) on \(E^+\) and \(\mathcal{N}_{\text{max}}\) the associated Nehari-Pankov manifold. Then, by (2.13), we have

\[
\max_{t > 0} I_{2^*}(tu) \geq I_{\text{max}}(u) \quad \text{for all } u \in E^+ \quad \text{and, particularly,}
\]

\[
\max_{t > 0} I_{2^*}(tu) \geq I_{\text{max}}(u) \quad \text{for all } u \in \mathcal{N}_{\text{max}}.
\]

Jointly with the definition of \(\tau_{2^*}\) and (2.17), we obtain

\[
\tau_{2^*} \geq \frac{1}{2m(H_{\text{max}})^{m-1}}(\frac{m}{2})^m \omega_m.
\]

The other inequality requires a delicate upper bound estimate for \(\tau_{2^*}\). In fact, by a careful choice of the parameters, Lemma 4.1 in Section 4 provides a proof, and one may also refer to a similar result in [5].

### 2.3 A concentration-compactness alternative

Again, we choose \(\{p_n\}\) to be a strictly increasing sequence such that \(\lim_{n \to \infty} p_n = 2^*\). In what follows, we shall collect some asymptotic properties of solutions to the equation

\[
D\psi = H|\psi|^{p_n-2}\psi \quad \text{on } S^m
\]

with a specific energy constraints, the proofs will be postponed in the Appendix. First of all, an alternative result comes as follows:

**Proposition 2.7.** Suppose \(\{\psi_n\} \subset E\) is a sequence such that \(\psi_n \to 0\) in \(E\) or \(\psi_n\) converges in \(E\) to a non-trivial solution \(\psi_0\) of Eq. (1.6)

In order to characterize the non-compact case (which corresponds to the weakly convergence \(\psi_n \to 0\) in \(E\)) in the above alternatives, let us introduce the following notations. Let \(\text{Crit}[H] = \{\xi \in S^m : \nabla H(\xi) = 0\}\) denote the critical set of the function \(H\). Let \(\xi \in V \subset S^m\) be an arbitrary point, we denote \(\exp_\xi : U \subset T_\xi S^m \cong \mathbb{R}^m \to V, x = (x_1, \ldots, x_n) \mapsto y = \exp_\xi x\) the exponential map and \((x_1, \ldots, x_m)\) means the normal coordinates. For a sequence of points \(\{a_n\} \subset S^m\) and an arbitrary decreasing sequence \(R_n \searrow 0\), as \(n \to \infty\), we define the rescaled geodesic normal coordinates near each \(a_n\) via \(\mu_n(x) = \exp_{a_n}(R_n x)\). Denoting \(B^0_R = \{x \in \mathbb{R}^m : |x| < R\}\), where \(|\cdot|\) is the Euclidean norm in \(\mathbb{R}^m\), we have a conformal equivalence \((B^0_{R_n}, R_n^{-2} \mu_n^*g_{S^m}) \cong (B_{R_nR(a_n)}, g_{S^m}) \subset S^m\) for all large \(n\). For ease of notation, we set \(g_n = R_n^{-2} \mu_n^*g_{S^m}\). By using the trivialization of the spinor bundle constructed by
Bourguignon and Gauduchon [12], we see that the coordinate map $\mu_n$ induces a spinor bundle identification $(\mu_n) : S_x(B^0_R, g_n) \to S_{\mu_n(x)}(B_{R_n}(a_n), g_{S^m})$ for each $n$. Moreover, due to the compactness of $S^m$, we may assume without loss of generality that $a_n \to a \in S^m$ as $n \to \infty$. Then, for all $n$ large, the aforementioned identifications depend smoothly in $a_n$. Particularly, we have the following blow-up result (the proof is postponed in the Appendix).

**Proposition 2.8.** Let $\{\psi_n\} \subset E$ satisfy (2.19). If $\{\psi_n\}$ does not contain any compact subsequence. Then, up to a subsequence if necessary, there exist a sequence $\{a_n\} \subset S^m$, $a_n \to a \in S^m$, as $n \to \infty$, a sequence of radius $R_n \searrow 0$, a real number $\lambda \in (2^{-\frac{1}{m-1}}, 1]$ and a non-trivial solution $\phi_0$ of the equation

$$D_{g_{S^m}} \phi = \lambda H(a) |\phi|^{2^* - 2} \phi \quad \text{on } \mathbb{R}^m$$

such that

$$\lim_{n \to \infty} R_n^{\frac{m-1}{2^* - p_n}} = \lambda$$

and

$$\psi_n = R_n^{\frac{m-1}{2^* - p_n}} \eta(\cdot)(\mu_n) \circ \phi_0 \circ \mu_n^{-1} + o_n(1) \quad \text{in } E$$

as $n \to \infty$, where $\eta \in C^\infty(S^m)$ is a cut-off function such that $\eta \equiv 1$ on $B_r(a)$ and $\text{supp } \eta \subset B_{2r}(a)$, some $r > 0$. Moreover, we have

$$\mathcal{L}_{p_n}(\psi_n) \geq \frac{1}{2m} \eta(\cdot)(\mu_n) \cdot \eta(\cdot) \cdot \mu_n^{-1} \cdot o_n(1), \quad \text{as } n \to \infty$$

where $\omega_m$ stands for the volume of $(S^m, g_{S^m})$. If $\mathcal{L}_{p_n}(\psi_n) \equiv 0$ for all $n$ in (2.19), then $a \in \text{Crit}[H]$ and $\lambda = 1$.

**Remark 2.9.** (1) Propositions 2.7 and 2.8 are closely linked. More precisely, together with (2.12), we can conclude that, if $\{\psi_n\}$ is a sequence of critical points satisfying the energy estimates in (2.19) and does not contain any compact subsequence, then (up to a subsequence) the associated blow-up point $a \in \text{Crit}[H]$ satisfies $H(a) \geq 2^{-\frac{1}{m-1}} H_{\max} + \theta'$, for some $\theta' > 0$.

(2) It is worth pointing out that, for $H \neq \text{constant}$, the ground state energy solutions of (2.18) (that is, the solutions with energy $\tau_{p_n}$) should blow up when $n \to \infty$. To see this, let us consider generically $\psi_n \in E$ such that $\mathcal{L}_{p_n}(\psi_n) \to \tau_2$, and $\mathcal{L}_{p_n}^*(\psi_n) \to 0$ as $n \to \infty$. By Proposition 2.7, it follows that either (a) $\{\psi_n\}$ converges strongly to $\psi_0 \neq 0$ which satisfies Eq. (1.6) or (b) the sequence $\{\psi_n\}$ blows-up.

If (a) holds, then $\mathcal{L}_{2^*}(\psi_0) = \tau_{2^*}$. To rule out this case, let us first recall the estimate (2.13), where the functional $\mathcal{L}_{\max}$ is introduced. As before, we denote $(h_{\max}, I_{\max})$ the reduction couple for $\mathcal{L}_{\max}$ on $E^+$ and $\mathcal{N}_{\max}$ the associated Nehari-Pankov manifold. Then (2.13) implies that $I_{2^*}(u) \geq I_{\max}(u)$ for all $u \in E^+$. Since $\mathcal{L}_{2^*}(\psi_0) = \tau_{2^*}$ and $\mathcal{L}_{2^*}^*(\psi_0) = 0$, we
have $\psi_0^+ \in \mathcal{N}_{2^*}$. Let $I_{2^*}(\psi_0^+) = \tau_{2^*}$ and $I_{2^*}^\prime(\psi_0^+) = 0$. Let $t_0 > 0$ be such that $t_0 \psi_0^+ \in \mathcal{N}_{2^*}$. We claim that $t_0 \neq 1$. If, by contradiction, we have $t_0 = 1$. Then $\psi_0^+ \in \mathcal{N}_{2^*} \cap \mathcal{N}_{max}$. By (2.17), we find $I_{max}(\psi_0^+) = \tau_{2^*}$ and $I_{max}^\prime(\psi_0^+) = 0$ (this is because $\tau_{2^*}$ equals to the lowest energy level for $I_{max}$ on the constraint $\mathcal{N}_{max}$). Hence, we have

$$
\tau_{2^*} = \frac{1}{2} (\|\psi_0^+\|^2 - \|h_{max}(\psi_0^+)\|^2) - \frac{H_{max}}{2^*} \int_{S^m} |\psi_0^+ + h_{max}(\psi_0^+)|^{2^*} \, dvol_{g_{sm}} \quad (2.20)
$$

and $\tilde{\psi}_0 := \psi_0^+ + h_{max}(\psi_0^+)$ is a nontrivial solution of Eq. (2.14). Then, by the regularity result for weak solutions to Eq. (2.14) (see e.g. [14, 22, 39]) and the Weak Unique Continuation Property for Dirac type operators (see [11, Theorem 2.1]), we find that the zero set of $\tilde{\psi}_0$ does not contain any nonempty open set. And therefore it follows directly from (2.20) that

$$
\tau_{2^*} = \frac{1}{2} (\|\psi_0^+\|^2 - \|h_{max}(\psi_0^+)\|^2) - \frac{H_{max}}{2^*} \int_{S^m} |\psi_0^+ + h_{max}(\psi_0^+)|^{2^*} \, dvol_{g_{sm}}
\leq \frac{1}{2} (\|\psi_0^+\|^2 - \|h_{max}(\psi_0^+)\|^2) - \frac{1}{2^*} \int_{S^m} H |\psi_0^+ + h_{max}(\psi_0^+)|^{2^*} \, dvol_{g_{sm}}
= \mathcal{L}_{2^*}(\tilde{\psi}_0) \leq I_{2^*}(\psi_0^+) = \tau_{2^*}
$$

which is a contradiction, proving the claim. Let’s say $t_0 \neq 1$. Then we can infer

$$
\tau_{2^*} = I_{2^*}(\psi_0^+) > I_{2^*}(t_0 \psi_0^+) \geq I_{max}(t_0 \psi_0^+)
\geq \inf_{u \in \mathcal{N}_{max}} I_{max}(u) = \frac{1}{2m(H_{max})^{m-1}(\frac{m}{2})^m \omega_m} = \tau_{2^*},
$$

which is again a contradiction. Thus we see that only (b) can occur.

## 3 Analysis on the nodal set

In what follows, we shall study the nodal set of solutions to the following nonlinear Dirac equation with critical exponent

$$
D\psi = H|\psi|^{2^* - 2}\psi \quad \text{on } S^m
$$

(3.1)

And let us begin with the following lemma which can be viewed as a generalization of [4, Lemma 4.1] for the case $H \neq \text{constant}$.

**Lemma 3.1.** Let $(M, g)$ be an arbitrary Riemannian spin $m$-manifold (not necessary complete or compact). Assume that there is a spinor $\psi$ of constant length 1 and with $D_g \psi = H \psi$ for a real-valued smooth function $H : M \to \mathbb{R}$, and let $\{e_1, \ldots, e_m\}$ be a local orthonormal frame field. Then

$$
\text{Scal}_g = \frac{4(m-1)}{m} H^2 - 4 \sum_{k=1}^m |\nabla^H_{e_k} \psi|^2
$$
where \( \text{Scal}_g \) is the scalar curvature of \( M \) with respect to \( g \). \( \nabla^H_X \psi := \nabla_X \psi + \frac{H}{m} X \cdot_g \psi \) for \( X \in \Gamma(TM) \) is an induced covariant derivative and \( \cdot_g \) denotes the Clifford multiplication.

**Proof.** The algebraic properties of Clifford multiplication imply that \( \nabla^H \) is metric in the sense that

\[
X(\varphi_1, \varphi_2) = (\nabla^H_X \varphi_1, \varphi_2) + (\varphi_1, \nabla^H_X \varphi_2)
\]

for all \( \varphi_1, \varphi_2 \in \Gamma(S(M)) \) and \( X \in \Gamma(TM) \).

Locally, let \( -\Delta^H = -\sum_{k=1}^m \nabla_{e_k}^H \nabla_{e_k}^H \) denote the Laplace operator corresponds to \( \nabla^H \). Then we have

\[
-\text{Re}(\Delta^H \psi, \psi) = -\text{Re} \sum_{k=1}^m (\nabla_{e_k}^H \nabla_{e_k}^H \psi, \psi)
\]

\[
= -\text{Re} \sum_{k=1}^m [e_k(\nabla_{e_k}^H \psi, \psi) - (\nabla_{e_k}^H \psi, \nabla_{e_k}^H \psi)]
\]

\[
= -\text{Re} \sum_{k=1}^m e_k(\nabla_{e_k}^H \psi, \psi) + \sum_{k=1}^m |\nabla_{e_k}^H \psi|^2. \tag{3.2}
\]

Since \( (\nabla_{e_k}^H \psi, \psi) = (\nabla_{e_k} \psi, \psi) + \frac{H}{m} \cdot (e_k \cdot_g \psi, \psi) \), we have

\[
\text{Re}(\nabla_{e_k}^H \psi, \psi) = \text{Re}(\nabla_{e_k} \psi, \psi) = \frac{1}{2} \text{Re} \langle \text{grad} |\psi|^2, e_k \rangle = 0.
\]

Thus (3.2) gives us

\[
-\text{Re}(\Delta^H \psi, \psi) = \sum_{k=1}^m |\nabla_{e_k}^H \psi|^2. \tag{3.3}
\]

On the other hand, by the twisted version of the Schrödinger-Lichnerowicz formula (see for instance [17, Chapter 5]), we have

\[
(D - \frac{H}{m})^2 = -\Delta^H + \frac{\text{Scal}_g}{4} + \frac{1 - m}{m^2} H^2.
\]

And therefore, by \( D \psi = H \psi, |\psi| \equiv 1 \) and (3.3), we obtain

\[
\frac{(m-1)^2 H^2}{m^2} = -\text{Re}(\Delta^H \psi, \psi) + \frac{\text{Scal}_g}{4} + \frac{1 - m}{m^2} H^2
\]

\[
= \sum_{k=1}^m |\nabla_{e_k}^H \psi|^2 + \frac{\text{Scal}_g}{4} + \frac{1 - m}{m^2} H^2,
\]

which completes the proof. \( \square \)

The following theorem is due to Bär [9]:
Theorem 3.2. Let \((M, g)\) be compact and connected spin \(m\)-manifold and let \(\varphi\) be a solution of
\[D \varphi = P \varphi\]
where \(P\) is a smooth endomorphism. Then the zero set of \(\varphi\) has at most Hausdorff dimension \(m - 2\). And if \(m = 2\), then the zero set is discrete.

Unfortunately, in general, Eq (3.1) does not satisfy Bär’s theorem because \(2^* - 2 = \frac{2}{m-1} \notin \mathbb{Z}\) for \(m \geq 4\) and \(H|\psi|^2 - 2\) could not be smooth even for \(m = 3\). However, in dimension 2, we have \(2^* = 4\) and in this case solutions of Eq (3.1) and the corresponding \(P = H|\psi|^2\) are smooth enough provided that \(H \in C^\infty\). Thus, applying Theorem 3.2, we have the nodal set of a solution \(\psi\) is a discrete subset.

Proposition 3.3. On a compact boundaryless spin surface \((M, g)\) of genus \(\gamma\), suppose that \(H : M \to (0, \infty)\) is a smooth function and \(H_{\text{max}} := \max_{\xi \in M} H(\xi)\). Let \(\psi\) be a solution of the equation
\[D g \psi = H|\psi|^2 \psi\]
on \(M\).
Then the number of zeros of \(\psi\) is at most
\[
\gamma - 1 + \frac{\int_M H^2 |\psi|^4 \text{d} \text{vol}_g}{4\pi}.
\]
In particular, if \(M\) has \(\gamma = 0\) (i.e. \(M\) is homeomorphic to the 2-sphere) and \(\psi\) satisfies
\[
\int_M H|\psi|^4 \text{d} \text{vol}_g < \frac{8\pi}{H_{\text{max}}},
\]
then \(\psi\) has no zero at all.

Proof. The proof is similar to that carried out in [4, Proposition 8.1]. Here, for the sake of completeness, we will sketch the proof as follows.

On \(M \setminus \psi^{-1}(0)\), let us introduce a new metric \(g_1 = |\psi|^4 g\). Then the transformation formula for the Dirac operator under conformal changes (Proposition 2.1) gives us that there is a spinor field \(\varphi_1\) on \(M \setminus \psi^{-1}(0)\) such that
\[D_{g_1} \varphi_1 = H \varphi_1\]
and \(|\varphi_1|_{g_1} = 1\).

By Lemma 3.1 we have the scalar curvature can be estimated by \(S\text{cal}_{g_1} \leq 2H^2\). And hence, we have the estimate for the Gauß curvature, that is \(K_{g_1} \leq H^2\). Moreover, we have
\[
\text{vol}(M \setminus \psi^{-1}(0), g_1) = \int_{M \setminus \psi^{-1}(0)} \text{d} \text{vol}_{g_1} = \int_M |\psi|^4 \text{d} \text{vol}_g.
\]

Let \(\psi^{-1}(0) = \{\xi_1, \ldots, \xi_l\}\) for some \(l \geq 1\) if \(\psi^{-1}(0) \neq \emptyset\). Let \(n_j\) be the order of the first non-vanishing term in the Taylor expansion of \(\psi\) near \(\xi_j, j = 1, \ldots, l\). Then it can be calculated.
that the integral of the geodesic curvature $\kappa_{g_1}$ over a small circle around each $\xi_j$ is close to $-2\pi(2n_j + 1)$. Now, we may remove small open disks around $\xi_j$ from $M$, and we obtain a surface $\tilde{M}$ with boundary. By using the Gauss-Bonnet formula, we can derive
\[
2\pi \chi(\tilde{M}) = \int_{\tilde{M}} K_{g_1} d\text{vol}_{g_1} + \int_{\partial \tilde{M}} \kappa_{g_1} ds \leq \int_{M} H^2 |\psi|^4 d\text{vol}_{g} - \sum_{j=1}^{l} 2\pi(2n_j + 1).
\]
And hence, return to $M$ itself, we have
\[
2\pi(2 - 2\gamma) = 2\pi \chi(M) \leq \int_{M} H^2 |\psi|^4 d\text{vol}_{g} - 4\pi \sum_{j=1}^{l} n_j,
\]
which implies $l$ is at most $\gamma - 1 + \frac{\int_{M} H^2 |\psi|^4 d\text{vol}_{g}}{4\pi}$.

For the case $\gamma = 0$ and $\psi$ satisfies (3.4), we assume $\psi^{-1}(0) = \{\xi_1, \ldots, \xi_l\} \neq \emptyset$. Then we have
\[
1 \leq l \leq -1 + \frac{\int_{M} H^2 |\psi|^4 d\text{vol}_{g}}{4\pi} \leq -1 + \frac{H_{\text{max}}}{4\pi} \int_{M} H |\psi|^4 d\text{vol}_{g} < 1
\]
which is impossible. Therefore $\psi$ has no zero at all.

\section{Proof of Theorem \ref{thm:1.3} and Corollary \ref{cor:1.5}}

\subsection{The existence results}

For notation convenience, we will write $L$ instead of $L^2$. The super-level sets for $H$ will be simply denoted by
\[
\{ H \geq c \} = \{ \xi \in S^m : H(\xi) \geq c \}
\]
for any $c \in \mathbb{R}$. Recalling the hypothesis $(H\ast)$, it follows that $H$ has only finitely many critical points at level $d$. And we may denote the corresponding critical set as
\[
K_d = \{ \xi \in S^m : H(\xi) = d, \nabla H(\xi) = 0 \} = \{ \xi_1, \ldots, \xi_l \}
\]
for some $l \in \mathbb{N}$. We also introduce a $\nu$-neighborhood of $K_d$ as $O_{\nu} := \bigcup_{k=1}^{l} B_{\nu}(\xi_k)$ for $\nu > 0$, where $B_r(\xi)$ is the open ball centered at $\xi \in S^m$ with radius $r > 0$ with respect to the metric $g_{S^m}$. Standard deformation arguments show that hypothesis $(H\ast)$ ensures that $d$ is indeed a critical value for $H$, i.e. $K_d \neq \emptyset$, and $H$ is not contractible in $\{ H \geq d + \sigma \}$ for any $\sigma > 0$. Moreover, for small $\nu > 0$, there exists $\sigma > 0$ such that $H$ is contractible in $\{ H \geq d + \sigma \} \cup O_{\nu}$.

Now we construct the test spinors that will be helpful to characterize the level sets of the energy functional $I_{p_n}$ on the constraint manifold $\mathcal{N}_{p_n}$ (see its definition in Subsection 2.2). Let’s start with an arbitrary $\phi_0 \in \mathbb{S}_m$ (a constant spinor) such that $|\phi_0| = \frac{1}{\sqrt{2}} (\frac{m}{2})^{m+1}$. Then, we define
\[
\phi(x) = f(x) \left( \frac{m}{2} \right) (1 - x) \cdot g_{S^m} \phi_0
\]
where \( f(x) = \frac{2}{1 + |x|^2} \) for \( x \in \mathbb{R}^m \). It is easy to verify that

\[
D_{g_{\mathbb{R}^m}} \phi = \frac{m}{2} f \phi
\]

and

\[
|\phi| = \left( \frac{m}{2} \right)^{\frac{m-1}{2}} f^{\frac{m-1}{2}}. \tag{4.1}
\]

For \( \varepsilon > 0 \), we set

\[
\phi_{\varepsilon}(x) = \varepsilon^{-\frac{m-1}{2}} \phi(x/\varepsilon) \quad \text{and} \quad \phi_{y,\varepsilon}(x) = H(y) \cdot \frac{m-1}{2} \phi_{\varepsilon}(x)
\]

where \( y \in S^m \) is a fixed point. Then we have \( \phi_{y,\varepsilon} \) satisfies the equation

\[
D_{g_{\mathbb{R}^m}} \phi_{y,\varepsilon} = H(y) |\phi_{y,\varepsilon}|^{2^* - 2} \phi_{y,\varepsilon} \quad \text{on} \quad \mathbb{R}^m. \tag{4.2}
\]

To transplant \( \phi_{y,\varepsilon} \) on the sphere \( S^m \), we need the following notation of stereographic projections. Given \( y \in S^m \) we can embed \( S^m \) into \( \mathbb{R}^{m+1} \) in the way that its antipodal point \(-y\) is the North pole. Denoting \( S_y : S^m \setminus \{-y\} \to \mathbb{R}^m \) the stereographic projection, we have \( S_y(y) = 0 \). Moreover, \( S^m \setminus \{-y\} \) and \( \mathbb{R}^m \) are conformally equivalent due to the fact \( (S^{-1}_y)^* g_{S^m} = f^2 g_{\mathbb{R}^m} \) with \( f(x) = \frac{2}{1 + |x|^2} \).

Recall the conformal transformation property mentioned in Proposition 2.1, there is an isomorphism of spinor bundles \( \iota : \mathbb{S}(\mathbb{R}^m, (S^{-1}_y)^* g_{S^m}) \to \mathbb{S}(\mathbb{R}^m, g_{\mathbb{R}^m}) \) such that

\[
D_{g_{\mathbb{R}^m}} (\iota(\varphi)) = \iota \left( f^{\frac{m+1}{2}} D_{(S^{-1}_y)^* g_{S^m}} (f^{-\frac{m-1}{2}} \varphi) \right),
\]

where \( D_{(S^{-1}_y)^* g_{S^m}} \) is the Dirac operator on \( \mathbb{R}^m \) with respect to the metric \( (S^{-1}_y)^* g_{S^m} \). Thus \( \phi_{y,\varepsilon} \) corresponds to a spinor field \( \psi_{y,\varepsilon} \) on \( S^m \) via the formula

\[
\phi_{y,\varepsilon} = \iota \left( f^{-\frac{m-1}{2}} \psi_{y,\varepsilon} \circ S^{-1}_y \right) \tag{4.3}
\]

such that \( \psi_{y,\varepsilon} \) satisfies the equation

\[
D_{g_{S^m}} \psi_{y,\varepsilon} = H(y) |\psi_{y,\varepsilon}|^{2^* - 2} \psi_{y,\varepsilon} \quad \text{on} \quad (S^m, g_{S^m}) \]

Lemma 4.1. For any \( \sigma > 0 \) and \( \theta > 0 \) there exists \( \varepsilon > 0 \) such that

\[
\max_{t > 0} I_{\rho_n}(t \psi_{y,\varepsilon}^+) \leq \frac{1}{2m (d + \sigma)^{m-1}} \left( \frac{m}{2} \right) m \omega_m + 2\theta
\]

uniformly for all large \( n \) and \( y \in \{ H \geq d + \sigma \} \).

Proof. The strategy is to use Corollary 2.5. And to begin with, we first fix \( \alpha > 0 \) associated to \( \theta \) as in Corollary 2.5.

In \( E^s \), we have

\[
\mathcal{L}'(\psi_{y,\varepsilon}) = D\psi_{y,\varepsilon} - H|\psi_{y,\varepsilon}|^{2^* - 2} \psi_{y,\varepsilon} = (H(y) - H)|\psi_{y,\varepsilon}|^{2^* - 2} \psi_{y,\varepsilon}.
\]
Let $\psi \in E$ be such that $||\psi|| \leq 1$, we have

$$
\mathcal{L}'(\psi_{y,\varepsilon})[\psi] = \text{Re} \int_{\mathbb{R}^m} (H(y) - H)|\psi_{y,\varepsilon}|^{2^{*}-2}(\psi_{y,\varepsilon}, \psi)d\nu_{g_{S^m}}
$$

$$
= \text{Re} \int_{\mathbb{R}^m} (H(y) - H \circ S^{-1}_{y})|\phi_{y,\varepsilon}|^{2^{*}-2}(\phi_{y,\varepsilon}, \mathcal{L}_{\psi_{y,\varepsilon}}(\phi_{y,\varepsilon}))d\nu_{g_{S^m}}
$$

$$
\leq C \left( \int_{\mathbb{R}^m} |H(y) - H \circ S^{-1}_{y}|^{2^{*}-1}|\phi_{y,\varepsilon}|^{2^{*}}d\nu_{g_{S^m}} \right)^{\frac{2^{*}-1}{2^{*}}}.
$$

(4.4)

Fix $\delta > 0$ arbitrarily small, we may find that $H \circ S^{-1}_{y}(x) = H(y) + O(\delta)$ uniformly for $|x| \leq \delta$. And hence, by the fact $\int_{\mathbb{R}^m} |\phi_{\varepsilon}|^{2^{*}}d\nu_{g_{S^m}} \equiv \left( \frac{m}{2} \right) m \omega_{m}$, we can get

$$
\int_{B_{\delta}^{0}} |H(y) - H \circ S^{-1}_{y}|^{2^{*}-1}|\phi_{y,\varepsilon}|^{2^{*}}d\nu_{g_{S^m}} \leq O(\delta^{2^{*}}).
$$

(4.5)

On the other hand, we find that

$$
\int_{\mathbb{R}^m \setminus B_{\delta}^{0}} |H(y) - H \circ S^{-1}_{y}|^{2^{*}-1}|\phi_{y,\varepsilon}|^{2^{*}}d\nu_{g_{S^m}} \leq C \int_{\frac{\delta}{2}}^{\infty} \frac{r^{m-1}}{(1 + r^2)^{m}}dr \leq C \left( \frac{\varepsilon}{\delta} \right)^{m}.
$$

(4.6)

Therefore, combining (4.4), (4.5) and (4.6), we have

$$
\|\mathcal{L}'(\psi_{y,\varepsilon})\|_{E^{*}} \leq \frac{\alpha}{2} \quad \text{uniformly for all } y \in S^{m}
$$

(4.7)

provided that $\varepsilon > 0$ is fixed small enough.

Next, we estimate the upper bound of $\mathcal{L}(\psi_{y,\varepsilon})$ in a similar way, that is

$$
\mathcal{L}(\psi_{y,\varepsilon}) = \frac{H(y)}{2m} \int_{S^{m}} |\psi_{y,\varepsilon}|^{2^{*}}d\nu_{g_{S^m}} + \frac{1}{2^{*}} \int_{S^{m}} (H(y) - H)|\psi_{y,\varepsilon}|^{2^{*}}d\nu_{g_{S^m}}
$$

$$
= \frac{H(y)}{2m} \int_{\mathbb{R}^m} |\phi_{y,\varepsilon}|^{2^{*}}d\nu_{g_{S^m}} + \frac{1}{2^{*}} \int_{\mathbb{R}^m} (H(y) - H \circ S^{-1}_{y})|\phi_{y,\varepsilon}|^{2^{*}}d\nu_{g_{S^m}}
$$

$$
\leq \frac{H(y)}{2m} \int_{\mathbb{R}^m} |\phi_{y,\varepsilon}|^{2^{*}}d\nu_{g_{S^m}} + \frac{\theta}{2}
$$

(4.8)

if $\varepsilon$ is small enough. Moreover, by noticing that

$$
\int_{\mathbb{R}^m} |\phi_{y,\varepsilon}|^{2^{*}}d\nu_{g_{S^m}} = H(y)^{-m} \int_{\mathbb{R}^m} |\phi_{\varepsilon}|^{2^{*}}d\nu_{g_{S^2m}} = H(y)^{-m} \left( \frac{m}{2} \right) m \omega_{m},
$$

we soon obtain from (4.8) and $y \in \{ H \geq d + \sigma \}$ that

$$
\mathcal{L}(\psi_{y,\varepsilon}) \leq \frac{1}{2mH(y)^{m-1}} \left( \frac{m}{2} \right) m \omega_{m} + \frac{\theta}{2} \leq \frac{1}{2m(d + \sigma)^{m-1}} \left( \frac{m}{2} \right) m \omega_{m} + \frac{\theta}{2}
$$

(4.10)

uniformly in $y$.

Remark that the function $p \mapsto |\psi|_{p}$ is continuous provided that $\psi \in E$ is fixed. Hence if we fix $\varepsilon$ sufficiently small, then it can be derived from (4.10) and the fact $\{ H \geq d + \sigma \}$ is compact that

$$
0 < c_1 \leq \mathcal{L}_{p_{m}}(\psi_{y,\varepsilon}) \leq \frac{1}{2m(d + \sigma)^{m-1}} \left( \frac{m}{2} \right) m \omega_{m} + \theta
$$

(4.11)
uniformly for all large $n$, where $c_1 > 0$ is some constant.

It remains to evaluate the derivatives of $L_{p_n}$ for large $n$. Take $\psi \in E$ arbitrarily as a test spinor, then it follows that

$$L'_{p_n}(\psi_{y,\varepsilon})[\psi] - L'(\psi_{y,\varepsilon})[\psi]$$

$$= \text{Re} \int_{S^m} H(|\psi_{y,\varepsilon}|^{2^* - 2} |\psi_{y,\varepsilon}|^{p_n - 2}) (|\psi_{y,\varepsilon}|^{p_n - 2}) (\psi_{y,\varepsilon}, \psi) d\nu_{G_{S^m}}$$

$$= \text{Re} \int_{\mathbb{R}^m} (H \circ S_{y}^{-1})(|\phi_{y,\varepsilon}|^{2^* - p_n} - f^{\frac{m-1}{2}}(2^* - p_n)) |\phi_{y,\varepsilon}|^{p_n - 2} (\phi_{y,\varepsilon}, t(f^{\frac{m-1}{2}}(2^* \cdot \psi \circ S_{y}^{-1}) d\nu_{G_{S^m}}.$$ 

Remark that, by (4.1), we have

$$\lim_{n \to \infty} \int_{S^m} |H(|\psi_{y,\varepsilon}|^{2^* - 2} |\psi_{y,\varepsilon}|^{p_n - 2}) (|\psi_{y,\varepsilon}|^{p_n - 2}) (\psi_{y,\varepsilon}, \psi) d\nu_{G_{S^m}}$$

$$\leq o(n(1))$$

Thus

$$|\phi_{y,\varepsilon}(x)| = H(y)^{-\frac{m-1}{2}} \cdot \varepsilon^{-\frac{m-1}{2}} \cdot (\frac{m}{2})^{\frac{m-1}{2}} \cdot (\frac{2\varepsilon^2}{\varepsilon^2 + |x|^2})^{\frac{m-1}{2}}.$$ 

Therefore, due to $p_n \to 2^*$, we obtain $C_{\varepsilon}^{2^* - p_n} = 1 + o(n(1))$ and

$$|\phi_{y,\varepsilon}|^{2^* - p_n} - f^{\frac{m-1}{2}}(2^* - p_n) = o(n(1)) \left(\frac{2}{1 + |x|^2}\right)^{\frac{m-1}{2}}(2^* - p_n) = o(n(1)) f^{\frac{m-1}{2}}(2^* - p_n)$$

uniformly in $y$ as $n \to \infty$. This implies

$$L'_{p_n}(\psi_{y,\varepsilon})[\psi] - L'(\psi_{y,\varepsilon})[\psi]$$

$$= o(n(1)) \text{Re} \int_{\mathbb{R}^m} (H \circ S_{y}^{-1})(f^{\frac{m-1}{2}}(2^* - p_n)) |\phi_{y,\varepsilon}|^{p_n - 2} (\phi_{y,\varepsilon}, t(f^{\frac{m-1}{2}}(2^* \cdot \psi \circ S_{y}^{-1}) d\nu_{G_{S^m}}$$

$$= o(n(1)) \text{Re} \int_{S^m} H|\psi_{y,\varepsilon}|^{p_n - 2} (\psi_{y,\varepsilon}, \psi) d\nu_{G_{S^m}}$$

$$\leq o(n(1)) \|\psi\|$$

uniformly for all $y \in \{H \geq d + \sigma\}$ as $n \to \infty$. Since $\psi \in E$ is arbitrary, we can derive from (4.7) that

$$\|L'_{p_n}(\psi_{y,\varepsilon})\|_{E^*} \leq \|L'(\psi_{y,\varepsilon})\|_{E^*} + \|L'_{p_n}(\psi_{y,\varepsilon}) - L'(\psi_{y,\varepsilon})\|_{E^*} \leq \alpha$$

uniformly for all $y \in \{H \geq d + \sigma\}$ and large $n$.

Combining (4.11) and (4.12), we may then apply Corollary 2.5 to get the desired assertion.

\qed
The next result concerns with the upper bound estimate for \( \max_{t>0} I_{p_n}(t\psi_{y,\varepsilon}^+) \) where \( y \) locates near the critical set \( \mathcal{K}_d \). Recall that, for \( \nu > 0 \), we denote \( \mathcal{O}_\nu \) the \( \nu \)-neighborhood of \( \mathcal{K}_d \).

**Lemma 4.2.** There exist \( C_0, \nu_0 > 0 \) such that
\[
\max_{t>0} I_{p_n}(t\psi_{y,\varepsilon}^+) \leq \frac{1}{2m \cdot d^{m-1}} \left( \frac{m}{2} \right)^m \omega_m - \begin{cases} C_0 \varepsilon^2 |\ln \varepsilon| + O(\varepsilon^2) & m = 2, \\ C_0 \varepsilon^2 + O(\varepsilon^{m+3}) & m \geq 3, \end{cases}
\]
uniformly for all small \( \varepsilon \), large \( n \) and \( y \in \mathcal{O}_{\nu_0} \). Particularly, for all \( y \in \mathcal{O}_{\nu_0} \), the estimate
\[
\max_{t>0} I_{p_n}(t\psi_{y,\varepsilon}^+) \leq \frac{1}{2m \cdot d^{m-1}} \left( \frac{m}{2} \right)^m \omega_m - \theta_0 \quad \text{for some } \theta_0 > 0
\]
holds as long as \( \varepsilon_0 \) is small.

**Proof.** Since \( \mathcal{K}_d = \{\xi_1, \ldots, \xi_t\} \) contains only finitely many points, without loss of generality, from now on we will first restrict ourselves to a neighborhood around \( \xi_i \). Since the hypothesis \( (H_*) \) ensures the Hessian of \( H \) at each \( \xi \in \mathcal{K}_d \) is positive definite. Hence, there exits \( R_1, \delta_0 > 0 \) such that
\[
H(y) \geq d + \delta_0 \operatorname{dist}_{g^{S_m}}(y, \xi_1)^2
\]
for all \( y \in B_{2R_1}(\xi_1) \subset S^m \).

Analogous to (4.8), let us estimate the following quantity
\[
\mathcal{L}(\psi_{y,\varepsilon}) = \frac{H(y)}{2m} \int_{\mathbb{R}^m} |\phi_{y,\varepsilon}|^2 d\operatorname{vol}_{g_{S^m}} + \frac{1}{2^s} \int_{\mathbb{R}^m} \left( H(y) - H \circ S_{y}^{-1} \right) |\phi_{y,\varepsilon}|^2 d\operatorname{vol}_{g_{S^m}}.
\]

And, by (4.9) and (4.13), we find that
\[
\mathcal{L}(\psi_{y,\varepsilon}) \leq \frac{1}{2m \cdot d^{m-1}} \left( \frac{m}{2} \right)^m \omega_m + \frac{1}{2^s} \int_{\mathbb{R}^m} \left( H(y) - H \circ S_{y}^{-1} \right) |\phi_{y,\varepsilon}|^2 d\operatorname{vol}_{g_{S^m}}.
\]

To estimate the second integral, let us recall that \( H(y) = H \circ S_{y}^{-1}(0) \) and \( y \) locates close to \( \xi_1 \), hence we have the Hessian of \( H \circ S_{y}^{-1}(\cdot) \) is positive definite in an open ball \( B_{r_0}^0 \), for some \( r_0 < R_1 \) and \( y \in B_{R_1}(\xi_1) \). In particular, we deduce that
\[
2H(y) - H \circ S_{y}^{-1}(x) - H \circ S_{y}^{-1}(-x) \leq -2\delta_0 |x|^2 \quad \text{for all } x \in B_{r_0}^0.
\]

At this point, let us split the last integral in (4.14) into two parts
\[
\left( \int_{B_{r_0}^0} + \int_{\mathbb{R}^m \setminus B_{r_0}^0} \right) (H(y) - H \circ S_{y}^{-1}) |\phi_{y,\varepsilon}|^2 d\operatorname{vol}_{g_{S^m}}.
\]

And it can be seen from a similar estimate in (4.6) that
\[
\int_{\mathbb{R}^m \setminus B_{r_0}^0} (H(y) - H \circ S_{y}^{-1}) |\phi_{y,\varepsilon}|^2 d\operatorname{vol}_{g_{S^m}} = O(\varepsilon^m).
\]
Notice that $|\phi_{y,\varepsilon}(x)| = |\phi_{y,\varepsilon}(-x)|$ for all $x \in \mathbb{R}^m$, it follows from (4.15) that

$$
\int_{B_{r_0}} (H(y) - H \circ S_{y}^{-1})|\phi_{y,\varepsilon}|^2 d\text{vol}_{g_{\text{Rm}}}
\leq -\delta \int_{B_{r_0}^0} |x|^2|\phi_{y,\varepsilon}|^2 d\text{vol}_{g_{\text{Rm}}} = -C_1 \varepsilon^2 \int_{0}^{r_{0}} \frac{r^{m+1}}{(1 + r^2)^m} dr \leq \begin{cases} -C_1 \varepsilon^2 |\ln \varepsilon| & m = 2, \\ -C_1 \varepsilon^2 & m \geq 3, \end{cases}
$$

for all $\varepsilon$ small, where $C_1 > 0$ is a constant depending only on the dimension $m$ and the value of $H(y)$. Therefore, by (4.14), we obtain immediately

$$
L(\psi_{y,\varepsilon}) \leq \frac{1}{2m} \frac{m}{d^m} \omega_m + O(\varepsilon^m) - \begin{cases} C_1 \varepsilon^2 |\ln \varepsilon| & m = 2, \\ C_1 \varepsilon^2 & m \geq 3, \end{cases}
$$

uniformly for $y \in B_{R_1}(\xi_1)$ and for $\varepsilon$ small.

On the other hand, for the derivative of $L$, we shall use (4.4) to get

$$
\|L'(\psi_{y,\varepsilon})\|_{E^*} \leq C \left( \int_{\mathbb{R}^m} |H(y) - H \circ S_{y}^{-1}|^2 d\text{vol}_{g_{\text{Rm}}} + O(\varepsilon^m) \right)^{\frac{2}{m+1}}
$$

for all $\varepsilon > 0$. Then, it follows that

$$
\|L'(\psi_{y,\varepsilon})\|_{E^*}^{\frac{2}{m+1}} \leq C \|\nabla H\|_{L^\infty(B_{r_0}(y))} \int_{B_{r_0}^0} |x|^2 d\text{vol}_{g_{\text{Rm}}} + O(\varepsilon^m)
$$

as $\varepsilon \to 0$. Remark that

$$
\int_{\mathbb{R}^m} |x|^2 d\text{vol}_{g_{\text{Rm}}} = O(\varepsilon^{2m-1})
$$

as $\varepsilon \to 0$. Combining (4.16) and (4.17), we conclude that

$$
\|L'(\psi_{y,\varepsilon})\|_{E^*} \leq C_2 \|\nabla H\|_{L^\infty(B_{r_0}(y))} \cdot \varepsilon + O(\varepsilon^{\frac{m+1}{2}})
$$

as $\varepsilon \to 0$ with some constant $C_2 > 0$.

Now, given $\varepsilon > 0$ small enough, we see that there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$

$$
L_{p_n}(\psi_{y,\varepsilon}) \leq \frac{1}{2m} \frac{m}{d^m} \omega_m + O(\varepsilon^m) - \begin{cases} \frac{C_1}{2} \varepsilon^2 |\ln \varepsilon| & m = 2, \\ \frac{C_1}{2} \varepsilon^2 & m \geq 3, \end{cases}
$$

and

$$
\|L'_{p_n}(\psi_{y,\varepsilon})\|_{E^*} \leq 2C_2 \|\nabla H\|_{L^\infty(B_{r_0}(y))} \cdot \varepsilon + O(\varepsilon^{\frac{m+1}{2}})
$$

uniformly for $y \in B_{R_1}(\xi_1)$. Since $\nabla H(\xi_1) = 0$, we can derive that

$$
|\nabla H|_{L^\infty(B_{r_0}(y))} \to 0 \quad \text{as} \quad y \to \xi_1 \text{ and } r_0 \to 0.
$$
And therefore, by combining (4.18), (4.19) and Lemma 2.4, we soon have

\[
\max_{t > 0} I_p(t \psi_{y, \varepsilon}^+) \leq \frac{1}{2m} \left( \frac{m}{2} \right)^m \omega_m - \begin{cases} 
\frac{C_1}{4} \varepsilon^2 \ln \varepsilon + O(\varepsilon^2) & m = 2, \\
\frac{C_1}{4} \varepsilon^2 + O(\varepsilon^{m+3}) & m \geq 3,
\end{cases}
\]

uniformly for \( y \in B_{R_1}(\xi_1) \) provided that \( R_1 \) and \( r_0 \) are fixed small enough.

If \( K = \{ \xi_1, \ldots, \xi_l \} \) contains multiple points. By repeating the above argument, one shall get correspondingly \( \hat{\varepsilon} \) large. We also remark that the lemmas hold) gives us that there exists a continuous embedding \( \Phi \) from \( H \) set in \( \Upsilon \):

Now we can introduce a barycenter-type function \( \bar{\psi} \) defined for \( I_p \) (see (2.5)), let us introduce the sub-level sets for \( I_p \) as

\[
I_p^c = \{ u \in \mathcal{N}_p : I_p(u) \leq c \}
\]

for \( c \in \mathbb{R} \). Denoted by

\[
\hat{a} = \frac{1}{2m} \left( \frac{m}{2} \right)^m \omega_m - \theta,
\]

then a combination of Lemma [4.1] and Lemma [4.2] (we fix \( \varepsilon > 0 \) small enough such that both the lemmas hold) gives us that there exists a continuous embedding \( \Phi_n : C_0 \to I_p^{\hat{a}} \) for all \( n \) large. We also remark that \( \hat{a} < 2\tau_2 - \theta \).

Step 2. Due to the hypothesis \((H^*)\), we have \( H_{\min} < d \). Hence if we fix \( \xi_0 \in S^m \) such that \( H(\xi_0) = H_{\min} \), then \( \xi_0 \notin \{ H \geq d \} \). Let \( S_0 : S^m \setminus \{ \xi_0 \} \to \mathbb{R}^m \) be the stereographic projection from \( \xi_0 \) (that is, we treat \( \xi_0 \) as the North pole). We see that the image of \( C_0 \) under \( S_0 \) is a bounded set in \( \mathbb{R}^m \), and thus there exists \( R_0 > 0 \) such that

\[
S_0(C_0) \subset B_{R_0}^0 = \{ x \in \mathbb{R}^m : |x| < R_0 \}.
\]

Now we can introduce a barycenter-type function \( \Upsilon : E \setminus \{0\} \to \mathbb{R}^m \) as

\[
\Upsilon(\psi) = \frac{\int_{S^m} \zeta \circ S_0(\xi) |\psi|^{2^*} \, d\text{vol}_{S^m}}{\int_{S^m} |\psi|^{2^*} \, d\text{vol}_{S^m}},
\]

Proof of Theorem [1.3] The proof will be accomplished by the following four steps:

Step 1. Recall the fact that, by the hypothesis \((H^*)\), for any \( \nu > 0 \) small there exists \( \sigma > 0 \) such that \( H \) is contractible in \( \{ H \geq d + \sigma \} \). Now let us fix \( \nu_0 > 0 \) as in Lemma [4.2], then we get some \( \sigma_0 > 0 \) such that \( H \) is contractible in \( C_0 := \{ H \geq d + \sigma_0 \} \). At this point, we may choose \( \varepsilon_0, \theta_0, \theta > 0 \) small such that

\[
\max \left\{ \frac{1}{2m} \left( \frac{m}{2} \right)^m \omega_m - \frac{1}{2m} \left( \frac{m}{2} \right)^m \omega_m - \theta_0 \right\} < \frac{1}{2m} \left( \frac{m}{2} \right)^m \omega_m - 3\theta
\]

where \( \theta_0 \) depends on \( \varepsilon_0 \) (by Lemma [4.2]).

Recall the the Nehari-Pankov manifold \( \mathcal{N}_p \) defined for \( I_p \) (see (2.5)), let us introduce the sub-level sets for \( I_p \) as

\[
I_p^c = \{ u \in \mathcal{N}_p : I_p(u) \leq c \}
\]

for \( c \in \mathbb{R} \). Denoted by

\[
\hat{a} = \frac{1}{2m} \left( \frac{m}{2} \right)^m \omega_m - \theta,
\]

then a combination of Lemma [4.1] and Lemma [4.2] (we fix \( \varepsilon > 0 \) small enough such that both the lemmas hold) gives us that there exists a continuous embedding \( \Phi_n : C_0 \to I_p^{\hat{a}} \) for all \( n \) large. We also remark that \( \hat{a} < 2\tau_2 - \theta \).

Step 2. Due to the hypothesis \((H^*)\), we have \( H_{\min} < d \). Hence if we fix \( \xi_0 \in S^m \) such that \( H(\xi_0) = H_{\min} \), then \( \xi_0 \notin \{ H \geq d \} \). Let \( S_0 : S^m \setminus \{ \xi_0 \} \to \mathbb{R}^m \) be the stereographic projection from \( \xi_0 \) (that is, we treat \( \xi_0 \) as the North pole). We see that the image of \( C_0 \) under \( S_0 \) is a bounded set in \( \mathbb{R}^m \), and thus there exists \( R_0 > 0 \) such that

\[
S_0(C_0) \subset B_{R_0}^0 = \{ x \in \mathbb{R}^m : |x| < R_0 \}.
\]

Now we can introduce a barycenter-type function \( \Upsilon : E \setminus \{0\} \to \mathbb{R}^m \) as

\[
\Upsilon(\psi) = \frac{\int_{S^m} \zeta \circ S_0(\xi) |\psi|^{2^*} \, d\text{vol}_{S^m}}{\int_{S^m} |\psi|^{2^*} \, d\text{vol}_{S^m}},
\]
where \( \zeta : \mathbb{R}^m \to \mathbb{R}^m \) is defined by

\[
\zeta(x) = \begin{cases} 
  x & |x| < R_0, \\
  \frac{R_0 x}{|x|} & |x| \geq R_0.
\end{cases}
\]

Next, we claim that \( \theta > 0 \) can be chosen sufficiently small such that for all large \( n \) and \( u \in \mathcal{N}_{p_n} \) satisfying

\[
I_{p_n}(u) \leq \tau_2^* + \theta = \frac{1}{2m(H_{\max})^{m-1}} \left( \frac{m}{2} \right)^m \omega_m + \theta,
\]

there holds

\[
\mathcal{S}_0^{-1} \circ \Upsilon(u + h_{p_n}(u)) \in \mathcal{H}_\delta. \tag{4.21}
\]

Suppose to the contrary that there exist \( \theta_n \to 0 \) and \( u_n \in \mathcal{N}_{p_n} \) such that

\[
I_{p_n}(u_n) < \tau_2^* + \theta_n \quad \text{but} \quad \mathcal{S}_0^{-1} \circ \Upsilon(u_n) \not\in \mathcal{H}_\delta, \tag{4.22}
\]

where we set \( \psi_n := u_n + h_{p_n}(u_n) \). Recall the definition of \( \tau_p \) for \( p \in (2, 2^*] \) (see (2.9)), by Lemma 2.6 we can get

\[
I_{p_n}(u_n) \geq \tau_{p_n} = \tau_2^* + o_n(1).
\]

Thus, as \( n \to \infty \), each \( u_n \) almost minimize the functional \( I_{p_n} \) on \( \mathcal{N}_{p_n} \). And it follows from the well-known Ekeland Variational Principle ([15, 29, 40]) that for each \( n \) there exists \( \tilde{u}_n \in \mathcal{N}_{p_n} \) such that

\[
I_{p_n}(\tilde{u}_n) < \tau_{2^*} + 2\theta_n, \quad \|I'_{p_n}|_{\mathcal{N}_{p_n}}(\tilde{u}_n)\| \leq \frac{8\theta_n}{\epsilon} \quad \text{and} \quad \|u_n - \tilde{u}_n\| \leq 2\epsilon. \tag{4.23}
\]

where \( \epsilon > 0 \) is a given small constant whose value will be fixed later. Remark that we may simply rewrite the definition of \( \mathcal{N}_{p_n} \) in (2.5) as \( \mathcal{N}_{p_n} = \{ u \in E^+ : \mathcal{K}_{p_n}(u) = 0 \} \) with \( \mathcal{K}_{p_n}(u) = I'_{p_n}(u)[u] \). Then the Lagrange multiplier rule implies the existence of \( t_n \in \mathbb{R} \) such that

\[
I'_{p_n}|_{\mathcal{N}_{p_n}}(\tilde{u}_n) = I'_{p_n}(\tilde{u}_n) + t_n \mathcal{K}'_{p_n}(\tilde{u}_n).
\]

By using the estimate \( \mathcal{K}'_{p_n}(\tilde{u}_n)[u_n] \leq -\frac{\nu_n-2}{p_n-1} \int_{S^m} H|\tilde{u}_n + h_{p_n}(\tilde{u}_n)|^{p_n} \text{dvol}_{g_{\text{sm}}} \) (a similar inequality has been shown in [10] Lemma 7.4), we can easily conclude from (4.23) that \( t_n \to 0 \) and, hence, \( I'_{p_n}(\tilde{u}_n) \to 0 \) as \( n \to \infty \). Now, set \( \tilde{\psi}_n = \tilde{u}_n + h_{p_n}(\tilde{u}_n) \), it is evident that

\[
\tau_{2^*} + o_n(1) \leq \mathcal{L}_{p_n}(\tilde{\psi}_n) < \tau_{2^*} + 2\theta_n \quad \text{and} \quad \mathcal{L}'_{p_n}(\tilde{\psi}_n) \to 0
\]

as \( n \to \infty \). Then, according to (2) of Remark 2.9 and Proposition 2.7, we have the sequence \( \{\tilde{\psi}_n\} \) must blow-up. Moreover, by Proposition 2.8, there exists a convergent sequence \( \{a_n\} \subset S^m \), \( a_n \to a \in \mathcal{H} \) as \( n \to \infty \) and a sequence of positive numbers \( R_n \searrow 0 \) and a non-trivial solution \( \phi_0 \) of

\[
D_{g_{\text{sm}}} \phi_0 = H_{\max} |\phi_0|^{2^*-2} \phi_0 \quad \text{on } \mathbb{R}^m.
\]
such that
\[ \tilde{\psi}_n = R_n^{m-1} \eta(\cdot) (\mu_n)_* \circ \phi_0 \circ \mu_n^{-1} + o_n(1) \quad \text{in } E \]
as \( n \to \infty \), where \( \mu_n \) and \( \eta \) are as in Proposition 2.8. Therefore, \( \int_{S^m} |\tilde{\psi}_n|^2 d\text{vol}_{g_{S^m}} \) is bounded away from 0 and particularly
\[ \Upsilon(\tilde{\psi}_n) \to \mathcal{S}_0(a) \in \mathcal{S}_0(\mathcal{H}) \quad \text{as } n \to \infty. \]

Recalling the definition of \( \Upsilon \), we see that the derivative \( \Upsilon' : E \setminus \{0\} \to E^* \) exists and the operator norm is bounded around each \( \tilde{\psi}_n \). Hence, by fixing the constant \( \epsilon \) small enough in (4.23), we have
\[ |\Upsilon(\psi_n) - \Upsilon(\tilde{\psi}_n)| < \frac{\epsilon}{2} \]
and therefore we get \( \Upsilon(\psi_n) \in \mathcal{S}_0(\mathcal{H}_\delta) \) for all \( n \) large which contradicts to (4.22).

**Step 3.** We are going to show that \( I_{p_n} \) has a critical point in \( I^{\hat{a}}_{p_n} \setminus I^{\hat{b}}_{p_n} \) where \( \hat{a} \) is defined in (4.20) and
\[ \hat{b} = \frac{1}{2m(H_{\text{max}})^{m-1}} \left( \frac{m}{2} \right)^m \omega_m + \theta. \]
We emphasize here that the embedding \( \Phi_n : C_0 \to \mathcal{N}_{p_n} \) gives us a description of low-energy levels in the sense that \( \Phi_n(\mathcal{H}) \in I^{\hat{b}}_{p_n} \) and \( \mathcal{S}_0^{-1} \circ \Upsilon \circ \Phi_n|_\mathcal{H} \) is homotopic to the inclusion \( \mathcal{H} \hookrightarrow \mathcal{H}_\delta \). Fixing all these notations, we define the family
\[ \Lambda_n = \left\{ \varpi \in C(\mathcal{H} \times [0,1], \mathcal{N}_{p_n}) : \varpi(\cdot,0) = \Phi_n \text{ and } \varpi(\cdot,1) \text{ collapses to a single spinor} \right\}. \]

Clearly, for all \( n \) large, \( \Lambda_n \neq \emptyset \) since the contractibility of \( \mathcal{H} \) in \( C_0 \) gives us a continuous homotopy \( \gamma : \mathcal{H} \times [0,1] \to C_0 \) such that \( \gamma(\cdot,0) = id, \gamma(\cdot,1) \equiv \xi_* \in C_0 \) and \( \Phi_n \circ \gamma \in \Lambda_n \). Since \( E \) is embedded compactly into \( L^p \) for all \( p \in [1,2^*) \), we have \( I_{p_n} \) satisfies the Palais-Smale condition for each \( n \). And thus, by standard variational arguments, one can easily verify
\[ c_n := \inf_{\varpi \in \Lambda_n} \sup_{(\xi,t) \in \mathcal{H} \times [0,1]} I_{p_n}(\varpi(\xi,t)) \]
is a critical value for \( I_{p_n} \). Particularly, \( c_n \in (\hat{b}, \hat{a}] \).

**Step 4.** Let \( \psi_n^+ \) be a critical point of \( I_{p_n} \) at level \( c_n \). It suffices to show that the sequence \( \{\psi_n = \psi_n^+ + h_{p_n}(\psi_n^+)\} \) contains a compact subsequence.

Let us assume to the contrary that any of such sequence \( \{\psi_n\} \) will blow up and let \( a \in S^m \) be the associated blow-up point. According to Proposition 2.8 and the fact \( \mathcal{L}_{p_n}(\psi_n) = I_{p_n}(\psi_n^+) = c_n \leq \hat{a} \), the blow-up point \( a \) should locate inside \( \{H \geq d + \sigma_0\} \). By hypothesis \( (H*) \), there is no critical value of \( H \) in the interval \( (d, H_{\text{max}}) \) and hence we have \( a \in \mathcal{H} \).

Moreover, from Proposition 2.8 again, there is a non-trivial solution \( \phi_0 \) of
\[ D_{g_{S^m}} \phi = H_{\text{max}} |\phi|^{2^* - 2} \phi \quad \text{on } \mathbb{R}^m \]
such that
\[ \psi_n = R_n^{m-1} \eta(\cdot) (\mu_n)_* \circ \phi_0 \circ \mu_n^{-1} + o_n(1) \quad \text{in } E \]
as $n \to \infty$, where $R_n$, $\eta$, $\mu_n$ are as in Proposition 2.8. Moreover
\[
\int_{S^m} H |\psi_n|^p d\text{vol}_{g_{S^m}} = R_n^{m-1(2^{*}-p_n)} \int_{B_{2r}^0 \setminus B_{r}^0} H \circ \mu_n |\phi_0|^p d\text{vol}_{g_n} + o_n(1)
\]
as $n \to \infty$. Since for arbitrary $R > 0$
\[
R_n^{m-1(2^{*}-p_n)} \int_{B_{2r}^0 \setminus B_{r}^0} H \circ \mu_n |\phi_0|^p d\text{vol}_{g_n} \leq C\left( \int_{B_{2r}^0 \setminus B_{r}^0} |\phi_0|^{2^{*}} d\text{vol}_{g_{S^m}} \right)^{\frac{p}{2^{*}}} \left( (2r)^m - (R_n R)^m \right)^{\frac{2^{*}-p_n}{2}}
\]
and
\[
R_n^{m-1(2^{*}-p_n)} \int_{B_{2r}^0 \setminus B_{r}^0} H \circ \mu_n |\phi_0|^p d\text{vol}_{g_n} = H_{\max} \int_{B_{2r}^0} |\phi_0|^{2^{*}} d\text{vol}_{g_n} + o_n(1),
\]
we derive that
\[
\int_{S^m} H |\psi_n|^p d\text{vol}_{g_{S^m}} = H_{\max} \int_{\mathbb{R}^m} |\phi_0|^{2^{*}} d\text{vol}_{g_n} + o_n(1) \tag{4.24}
\]
as $n \to \infty$. Notice that $\phi_0$ also extends to a non-trivial solution $\bar{\phi}_0$ to the equation
\[
D\bar{\phi}_0 = H_{\max} |\bar{\phi}_0|^{2^{*}-2} \bar{\phi}_0 \quad \text{on} \quad S^m, \tag{4.25}
\]
by using the fact $\mathcal{L}_{p_n}(\psi_n) = c_n \leq \hat{a} < 2\tau_{2^{*}}$, we have
\[
H_{\max} \int_{S^m} |\bar{\phi}_0|^{2^{*}} d\text{vol}_{g_{S^m}} = H_{\max} \int_{\mathbb{R}^m} |\phi_0|^{2^{*}} d\text{vol}_{g_n} < \frac{2}{(H_{\max})^{m-1} \left( \frac{m}{2} \right)^m \omega_m}.
\]
In the 2-dimensional case, we have $2^{*} = 4$ and
\[
H_{\max} \int_{S^2} |\bar{\phi}_0|^4 d\text{vol}_{g_{S^2}} = H_{\max} \int_{\mathbb{R}^2} |\phi_0|^4 d\text{vol}_{g_{\mathbb{R}^2}} < \frac{8\pi}{H_{\max}}.
\]
It follows from Proposition 3.3 that $\bar{\phi}_0$ has no zero at all, and from the spinorial Weierstraß representation and Li-Yau’s inequality that $(S^2, |\bar{\phi}_0|^4 g_{S^2})$ is embedded into $\mathbb{R}^3$ with constant mean curvature $H_{\max}$. Now, by the Alexandrov’s theorem [2], we know that $(S^2, |\bar{\phi}_0|^4 g_{S^2})$ must be a round sphere. And hence the Willmore energy satisfies
\[
H_{\max}^2 \int_{S^2} |\bar{\phi}_0|^4 d\text{vol}_{g_{S^2}} = 4\pi.
\]
This and (4.24) imply
\[
c_n = \mathcal{L}_{p_n}(\psi_n) = \frac{p_n - 2}{2p_n} \int_{S^2} H |\psi_n|^p d\text{vol}_{g_{S^2}} = \frac{H_{\max}}{4} \int_{\mathbb{R}^2} |\phi_0|^4 d\text{vol}_{g_{\mathbb{R}^2}} + o_n(1) < \hat{b}
\]
as $n \to \infty$, which is impossible. And therefore, the compactness of the solution sequence $\{\psi_n\}$ follows. $\square$
Remark 4.3. The hypersurfaces with constant mean curvature are much more complicated in higher dimensions, surprising examples have been shown in [21]: there exist infinitely many distinct differentiable immersions of the 3-sphere into Euclidean 4-space having a given positive constant mean curvature, moreover, the total “area” as well as the total integral of the norm of the second fundamental form of such examples can be as large as one wants. This makes the picture of $\tilde{\Phi}_0$ in Step 4 unclear to us when the dimension $m \geq 3$, particularly we do not know whether or not $\tilde{\Phi}_0$ has the ground state energy. And it is also unclear if $\tau_2$ is an isolated critical level of the energy functional for Eq. (4.25). Our approach in this regard up to now have failed.

4.2 Application to 2-sphere in Euclidean 3-space

Thanks to the previous section, we have the existence result for the equation

$$D\psi = H|\psi|^2\psi \quad \text{on } S^2$$

provided $H$ is smooth and satisfy $(H)$. Now let us give a interesting application to the problem of prescribing mean curvature on the 2-sphere.

As a direct application of Proposition 3.3, we soon have

**Corollary 4.4.** On $(S^2, g_{S^2})$, if $H$ is a positive smooth function satisfying criteria $(H)$, then there exists a solution of the equation

$$D\psi = H|\psi|^2\psi, \quad |\psi| > 0.$$

Following the spinorial Weierstraß representation [16, Theorem 13], we apply our existence results now to the problem of prescribing mean curvature on $S^2$:  

**Corollary 4.5.** On $(S^2, g_{S^2})$, if $H$ is a positive smooth function satisfying criteria $(H)$, then there is a conformal metric $g_1$ such that $(S^2, g_1)$ is isometrically immersed into the Euclidean space $\mathbb{R}^3$ with mean curvature $H$.

**Proof.** By Corollary 4.4 we can introduce a conformal metric $g_1 = |\psi|^4 g_{S^2}$ on $S^2$. Then the conformal transformation formula implies that there exists a spinor $\varphi_1$ on $(S^2, g_1)$ such that

$$D_{g_1}\varphi_1 = H\varphi_1 \quad \text{and} \quad |\varphi_1|_{g_1} \equiv 1.$$

Hence by the spinorial Weierstraß representation, we have there is an isometric immersion $(\tilde{S}^2, g) \rightarrow \mathbb{R}^3$ of the universal covering $\tilde{S}^2$ into the Euclidean space $\mathbb{R}^3$ with mean curvature $H$. Since $S^2$ is simply connected, we conclude the assertion immediately.  

5 Appendix

5.1 Proof of Proposition 2.7

The boundedness of \( \{ \psi_n \} \) follows from Lemma 2.4 (or one may follow \cite[Lemma 4.1]{10}). We then assume without loss of generality that \( \psi_n \to \psi_0 \) in \( E \) as \( n \to \infty \). By (2.19), it is not difficult to check that

\[
0 = \lim_{n \to \infty} \mathcal{L}_{p_n}'(\psi_n)[\varphi] = \lim_{n \to \infty} \left( \langle \psi_n^+, \varphi^+ \rangle - \langle \psi_n^-, \varphi^- \rangle - \text{Re} \int_{S^m} H|\psi_n|^{p_n-2}(\psi_n, \varphi)\text{dvol}_{g_{Sm}} \right)
\]

\[
= \langle \psi_0^+, \varphi^+ \rangle - \langle \psi_0^-, \varphi^- \rangle - \text{Re} \int_{S^m} H|\psi_0|^{2^*-2}(\psi_0, \varphi)\text{dvol}_{g_{Sm}} = \mathcal{L}_{2^*}'(\psi_0)[\varphi]
\]

for all \( \varphi \in E \). Hence \( \psi_0 \) turns out to be a solution of the critical equation (1.6).

Set \( \bar{\psi}_n = \psi_n - \psi_0 \). We derive that

\[
D\bar{\psi}_n = H|\psi_n|^{p_n-2}\psi_n - H|\bar{\psi}_n|^{p_n-2}\bar{\psi}_n - H|\psi_0|^{2^*-2}\psi_0
\]

\[
+ H|\psi_0|^{p_n-2}\bar{\psi}_n - H|\psi_0|^{2^*-2}\bar{\psi}_n
\]

\[
+ H|\bar{\psi}_n|^{p_n-2}\bar{\psi}_n + o_n(1)
\]

where the above \( o_n(1) \) term equals to \( \mathcal{L}_{p_n}'(\psi_n) \) as \( n \to \infty \) in \( E^* \). Denoted by

\[
\Phi_n = H|\psi_n|^{p_n-2}\psi_n - H|\bar{\psi}_n|^{p_n-2}\bar{\psi}_n - H|\psi_0|^{p_n-2}\psi_0,
\]

since \( p_n \searrow 2^* \), it turns out that there exists \( C > 0 \) (independent of \( n \)) such that

\[
|\Phi_n| \leq C|\bar{\psi}_n|^{p_n-2}\psi_0| + C|\psi_0|^{p_n-2}|\bar{\psi}_n|.
\]

(5.1)

Thanks to the Egorov theorem, for any \( \epsilon > 0 \), there exists \( \Omega_\epsilon \subset S^m \) such that \( \text{meas}\{S^m\setminus\Omega_\epsilon\} < \epsilon \) and \( \bar{\psi}_n \to 0 \) uniformly on \( \Omega_\epsilon \) as \( n \to \infty \). Therefore, by the boundedness of \( \psi_n \), (5.1) and the Hölder inequality, we have

\[
\text{Re} \int_{S^m}(\Phi_n, \varphi)\text{dvol}_{g_{Sm}} = \text{Re} \int_{S^m \setminus \Omega_\epsilon}(\Phi_n, \varphi)\text{dvol}_{g_{Sm}} + \text{Re} \int_{\Omega_\epsilon}(\Phi_n, \varphi)\text{dvol}_{g_{Sm}}
\]

\[
\leq C \left[ \left( \int_{S^m \setminus \Omega_\epsilon} |\psi_0|^{2^*} \text{dvol}_{g_{Sm}} \right)^{\frac{1}{2^*}} + \left( \int_{S^m \setminus \Omega_\epsilon} |\bar{\psi}_n|^{2^*} \text{dvol}_{g_{Sm}} \right)^{\frac{p_n-2}{2^*-2}} \right]
\]

\[
+ \int_{\Omega_\epsilon} |\Phi_n| \cdot |\varphi| \text{dvol}_{g_{Sm}}.
\]

(5.2)

for arbitrary \( \varphi \in E \) with \( ||\varphi|| \leq 1 \). It is evident that the last integral in (5.2) converges to 0 as \( n \to \infty \) and the remaining integrals tends to 0 uniformly in \( n \) as \( \epsilon \to 0 \). Thus, we get \( \Phi_n = o_n(1) \) in \( E^* \). Noting that \( q \mapsto H(\cdot)|\psi_0|^{q-2}\psi_0 \) is continuous in \( E^* \), we obtain

\[
\mathcal{L}_{p_n}'(\bar{\psi}_n) = D\bar{\psi}_n - H|\bar{\psi}_n|^{p_n-2}\bar{\psi}_n = o_n(1) \quad \text{in} \ E^*.
\]

(5.3)
Now assume $\psi_0 \neq 0$ in $E$ (otherwise we are done). Up to a subsequence, if $L_{p_n}(\bar{\psi}_n) \to 0$ then, by repeating the proof in [10, Lemma 4.1], we have that $\bar{\psi}_n \to 0$ in $E$. This shows the compactness of $\{\psi_n\}$. In what follows, we assume that $L_{p_n}(\bar{\psi}_n)$ is bounded away from 0.

Since $\psi_0$ is a non-trivial solution of Eq. (1.6), we can see that

$$
\tau_{2^*} \leq L_{2^*}(\psi_0) = L_{2^*}(\psi_0) - \frac{1}{2^*} L'_{2^*}(\psi_0)[\psi_0] = \frac{1}{2m} (\|\psi_0^+\|^2 - \|\psi_0^-\|^2).
$$

(5.4)

On the other hand, by (5.3) and $L_{p_n}(\bar{\psi}_n)$ is bounded away from 0, we derive that $\{\bar{\psi}_n\}$ satisfy the condition (2.8) (where the upper bound of $L_{p_n}(\bar{\psi}_n)$ follows directly from the boundedness of $\{\bar{\psi}_n\}$). And hence, by applying Lemma 2.4, we have $\tau_{p_n} \leq \max_{t > 0} I_{p_n}(t\bar{\psi}_n) \leq L_{p_n}(\bar{\psi}_n) + o_n(1)$. This and (5.3) imply

$$
\tau_{p_n} \leq L_{p_n}(\bar{\psi}_n) - \frac{1}{p_n} L'_{p_n}(\bar{\psi}_n)[\bar{\psi}_n] + o_n(1) = \frac{p_n - 2}{2p_n} (\|\bar{\psi}_n^+\|^2 - \|\bar{\psi}_n^-\|^2) + o_n(1).
$$

(5.5)

Now, it follows from (5.4), (5.5) and Lemma 2.6 that

$$
L_{p_n}(\psi_n) = L_{p_n}(\psi_n) - \frac{1}{p_n} L'_{p_n}(\psi_n)[\psi_n] + o_n(1) = \frac{p_n - 2}{2p_n} (\|\psi_n^+\|^2 - \|\psi_n^-\|^2) + o_n(1)
$$

$$
= \frac{p_n - 2}{2p_n} (\|\bar{\psi}_n^+\|^2 - \|\bar{\psi}_n^-\|^2) + \frac{p_n - 2}{2p_n} (\|\psi_0^+\|^2 - \|\psi_0^-\|^2) + o_n(1)
$$

$$
\geq 2\tau_{2^*} + o_n(1)
$$

which contradicts (2.19). Therefore, we must have $L_{p_n}(\bar{\psi}_n) \to 0$ as $n \to \infty$, and the proof is hereby completed.

### 5.2 Proof of Proposition 2.8

We follow the steps in [22, Section 5], with necessary modifications.

To begin with, for some $\delta_0 > 0$, we introduce the singular set of $\{\psi_n\}$:

$$
\Gamma := \left\{ a \in S^m : \lim_{r \to 0} \lim_{n \to \infty} \int_{B_r(a)} |\psi_n|^{p_n} d\text{vol}_{g_{S^m}} \geq \delta_0 \right\}
$$

where $B_r(a) \subset S^m$ is the distance ball of radius $r$ with respect to the metric $g_{S^m}$. As was proved in [22, Lemma 5.3] (the proof can be repeated just adapting the notations in the present paper) that there exists $\delta_0 > 0$ depending only on the geometry of $S^m$ such that $\Gamma \neq \emptyset$.

In order to have a clearer picture of $\psi_n$ near points in $\Gamma$, for $r \geq 0$, we define

$$
\Theta_n(r) = \sup_{a \in S^m} \int_{B_r(a)} |\psi_n|^{p_n} d\text{vol}_{g_{S^m}},
$$

which can be understood as a concentration function for $\psi_n$. Then, by choosing $\bar{\delta} > 0$ small (say $\bar{\delta} < \delta_0$), there exist a decreasing sequence $R_n \searrow 0$ as $n \to \infty$ and $\{a_n\} \subset S^m$ such that

$$
\Theta_n(R_n) = \int_{B_{R_n}(a_n)} |\psi_n|^{p_n} d\text{vol}_{g_{S^m}} = \bar{\delta}.
$$

(5.6)
Up to a subsequence if necessary, we assume that $a_n \to a \in S^m$ as $n \to \infty$.

Recall that we have defined the rescaled geodesic normal coordinates near each $a_n$ by $\mu_n(x) = \exp_{a_n}(R_nx)$ for $x \in \mathbb{R}^m$. Then we have a conformal equivalence $\mathbb{R}^m \supset (B_R^0, g_n) \cong (B_{R_n R(a_n)}, g_S^m)$ for all large $n$, where $g_n = R_n^{-2} \mu_n^* g_S^m$. And, for any $R > 0$, $g_n$ converges to the Euclidean metric in $C^\infty(B_R^0)$ as $n \to \infty$.

Following the idea of local trivialization introduced in [12], the coordinate map $\mu_n$ induces a spinor identification $(\mu_n)_* : S_x(B_R^0, g_n) \to S_{\mu_n(x)}(B_{R_n R(a_n)}, g_S^m)$. In the sequel, we define spinors $\phi_n$ on $B_R^0$ by $\phi_n = R_n^{-1}(\mu_n)_* \circ \psi_0 \circ \mu_n$. By the conformal changes of the Dirac operator, a straightforward calculation shows that $D_{\mu_n} \phi_n = R_n^{-2} (\mu_n)_* \circ (D_{\psi_n} \circ \mu_n)$,

$$\int_{B_R^0} (D_{\mu_n} \phi_n, \phi_n) d\nu_{g_n} = \int_{B_{R_n R(a_n)}} (D_{\psi_n} \psi_n) d\nu_{g_S^m}, \quad (5.7)$$

$$\int_{B_R^0} |\phi_n| d\nu_{g_n} = \int_{B_{R_n R(a_n)}} |\psi_n| d\nu_{g_S^m}, \quad (5.8)$$

and

$$\int_{B_R^0} |\phi_n| p_n d\nu_{g_n} = R_n^{-\frac{m+1}{2} (2^* - p_n)} \int_{B_{R_n R(a_n)}} |\psi_n| p_n d\nu_{g_S^m}. \quad (5.9)$$

Moreover, since $\{\psi_n\}$ is bounded in $E$, we have

$$\sup_{n \geq 1} \int_{B_R^0} |\phi_n| d\nu_{g_n} \leq \sup_{n \geq 1} \int_{S^m} |\psi_n| d\nu_{g_S^m} < +\infty \quad (5.10)$$

for any $R > 0$.

**Lemma 5.1.** There is $\bar{\lambda} > 0$ such that $\bar{\lambda} \leq \liminf_{n \to \infty} R_n^{-\frac{m+1}{2} (2^* - p_n)} \leq \limsup_{n \to \infty} R_n^{-\frac{m+1}{2} (2^* - p_n)} \leq 1$.

**Proof.** It follows from (5.6), (5.9) and Hölder inequality that

$$\bar{\delta} = \int_{B_{R_n R(a_n)}} |\psi_n| p_n d\nu_{g_S^m} \leq \left( \int_{B_R^0} |\phi_n| d\nu_{g_n} \right)^{\frac{2^* - p_n}{2}} \left( \int_{B_R^0} d\nu_{g_n} \right)^{\frac{2^* - p_n}{2}} R_n^{-\frac{m+1}{2} (2^* - p_n)}.$$

Noting that $g_n$ converges to the Euclidean metric in the $C^\infty$-topology on $B_R^0$, we can conclude immediately from $p_n \not\to 2^*$ and (5.10) that $\bar{\delta} \leq C \cdot R_n^{-\frac{m+1}{2} (2^* - p_n)}$ for some constant $C > 0$.

On the other hand, suppose there exists some $\delta > 0$ such that $R_n^{-\frac{m+1}{2} (2^* - p_n)} \geq 1 + \delta$ for all large $n$. Then, we must have $\ln R_n \geq \frac{2 \ln (1 + \delta)}{(m-1)(2^* - p_n)} \to +\infty$ as $n \to \infty$. This implies $R_n \to +\infty$ which is absurd. \qed

Let $\phi_n$ be as above and define $\bar{L}_n = D_{\mu_n} \phi_n - R_n^{-\frac{m+1}{2} (2^* - p_n)} H \circ \mu_n(\cdot)|\phi_n|^{p_n/2} \phi_n$. By using $\mathcal{L}_{p_n}'(\psi_n) \to 0$ in $E^*$ and the conformal changes of the Dirac operator, it is not difficult to check that $\bar{L}_n \to 0$ in $H^2_{\text{loc}}(\mathbb{R}^m, S^m)$.

In what follows, by Lemma 5.1 we may assume that (up to a subsequence if necessary) $R_n^{-\frac{m+1}{2} (2^* - p_n)} \to \lambda \in [\bar{\lambda}, 1]$ as $n \to \infty$. Since $\{\phi_n\}$ is bounded in $H^1_{\text{loc}}(\mathbb{R}^m, S^m)$, we can assume
that \( \phi_n \to \phi_0 \) in \( H^1_{\text{loc}}(\mathbb{R}^m, \mathbb{S}_m) \). And, by (5.10), it is easy to see that \( \phi_0 \in L^2(\mathbb{R}^m, \mathbb{S}_m) \) and satisfies the equation

\[
D_{g_R^n} \phi_0 = \frac{\lambda}{2} |\phi_0|^2 \quad \text{on } \mathbb{R}^m.
\]

(5.11)

At this point, we may apply the local analysis in [22, Lemma 5.5] to see that \( \phi_n \to \phi_0 \) in \( H^1_{\text{loc}}(\mathbb{R}^m, \mathbb{S}_m) \) as \( n \to \infty \), where proof is based on the fact that \( g_n \) converges to \( g_R^n \) in \( C^\infty \)-topology on bounded domains in \( \mathbb{R}^m \). Then, by Lemma 5.1, (5.6) and (5.9), we have

\[
\lambda \int_{B_0^R} |\phi_0|^2 \, dvol_{g_R^n} = \delta,
\]

which implies \( \phi_0 \) is a non-trivial solution of Eq. (5.11). And hence, by the energy gap estimate in [22, Section 4], we can derive via standard rescaling argument that

\[
\int_{\mathbb{R}^m} |\phi_0|^2 \, dx \geq \frac{1}{(\lambda H(a))^m} \left( \frac{m}{2} \right)^m \omega_m.
\]

(5.12)

With these preparations in hand, we may now choose \( \eta \in C^\infty(\mathbb{S}_m) \) be such that \( \eta \equiv 1 \) on \( B_r(a) \) and \( \text{supp} \eta \subset B_{2r}(a) \) for some \( r > 0 \) (for sure \( r \) should not be large in the sense that we assume \( 3r < \text{inj} \mathbb{S}_m \) where \( \text{inj} \mathbb{S}_m \) denotes the injective radius) and define a spinor field \( z_n \in C^\infty(\mathbb{S}_m, \mathbb{S}(\mathbb{S}_m)) \) by \( z_n = R_n^{m-1} \eta(\cdot) \mu_n^{-1} \phi_0 \). Setting \( \psi_n = \psi_n - z_n \), we have

\textbf{Lemma 5.2.}  

(1) \( \varphi_n \to 0 \) in \( E \) as \( n \to \infty \).

(2) \( L_p'(z_n) \to 0 \) and \( L_p'(\varphi_n) \to 0 \) as \( n \to \infty \).

\textbf{Proof.} The proof of (1) is accomplished by showing \( |z_n|_2 \to 0 \) as \( n \to \infty \) since we already have assumed \( \psi_n \to 0 \) in \( E \). To see this, we split the \( L^2 \)-integral of \( z_n \) into two parts

\[
\left( \int_{B_{R_n^m}(a_n)} + \int_{\mathbb{S}_m \setminus B_{R_n^m}(a_n)} \right) |z_n|^2 \, dvol_{g_{R^n}}
\]

where

\[
\int_{B_{R_n^m}(a_n)} |z_n|^2 \, dvol_{g_{R^n}} = R_n^{-m+1} \int_{B_{R_n^m}^0} |\phi_0|^2 \, dvol_{\mu_n^{-1} g_{R^n}} = R_n \int_{B_R^0} |\phi_0|^2 \, dvol_{g_n} \to 0
\]

and

\[
\int_{\mathbb{S}_m \setminus B_{R_n^m}(a_n)} |z_n|^2 \, dvol_{g_{R^n}} \leq C \left( \int_{B_{3r/R_n}^0 \setminus B_R^0} |\phi_0|^2 \, dvol_{g_{R^n}} \right)^{3r/(3r - (R_n R)^m)} \to 0
\]

for some constant \( C > 0 \). We omit the proof of (2) here, since it is standard and is similar to that of [22, Lemma 5.7].
Proof of Proposition\ref{concentrationbehavior}: The concentration behavior. By (2) of Lemma\ref{lemmamain}, we have

\[ \mathcal{L}_{p_n}(z_n) + o_n(1) = \mathcal{L}_{p_n}(z_n) - \frac{1}{p_n} \mathcal{L}'_{p_n}(z_n)[z_n] = \frac{p_n - 2}{2p_n} \left( \|z^+_n\|^2 - \|z^-_n\|^2 \right) \]

and

\[ \mathcal{L}_{p_n}(\varphi_n) + o_n(1) = \mathcal{L}_{p_n}(\varphi_n) - \frac{1}{p_n} \mathcal{L}'_{p_n}(\varphi_n)[\varphi_n] = \frac{p_n - 2}{2p_n} \left( \|\varphi^+_n\|^2 - \|\varphi^-_n\|^2 \right). \]

We claim that

Claim 5.1. \( \mathcal{L}_{p_n}(\psi_n) = \mathcal{L}_{p_n}(z_n) + \mathcal{L}_{p_n}(\varphi_n) + o_n(1) \) as \( n \to \infty \).

Assuming Claim 5.1 for the moment, then we are going to show that \( \mathcal{L}_{p_n}(\varphi_n) \to 0 \) as \( n \to \infty \). Indeed, suppose to the contrary that (up to a subsequence) \( \mathcal{L}_{p_n}(\varphi_n) \geq c > 0 \), it follows from the boundedness of \( \{\varphi_n\} \) in \( E \), Lemma\ref{lemmamain} and \ref{lemmamain2} that

\[ \tau_{2^*} = \tau_{p_n} + o_n(1) \leq \max_{t > 0} I_{p_n}(t\varphi_n^+) + o_n(1) \leq \mathcal{L}_{p_n}(\varphi_n) + o_n(1). \] (5.13)

On the other hand, we have

\[ \mathcal{L}_{p_n}(z_n) = \mathcal{L}_{p_n}(z_n) - \frac{1}{2} \mathcal{L}'_{p_n}(z_n)[z_n] = \frac{p_n - 2}{2p_n} \int_{S^m} H|z_n|^p d\text{vol}_{g_{S^m}} + o_n(1) \]

\[ = \frac{p_n - 2}{2p_n} R_{n}^{m-1}(\varphi^+_n - \varphi^-_n) \int_{B_R^0} \left( H \circ \mu_n \right) |\phi_0|^p d\text{vol}_{g_n} + o_n(1) + o_R(1) \]

\[ = \frac{1}{2m} \lambda H(a) \int_{B_R^0} |\phi_0|^{2^*} d\text{vol}_{g_{S^m}} + o_n(1) + o_R(1) \]

for \( R > 0 \) large. Thus, by (2.12), (5.12), \( \lambda \leq 1 \) and \( H(a) \leq H_{\text{max}} \), we obtain

\[ \mathcal{L}_{p_n}(z_n) \geq \frac{1}{2m(\lambda H(a))^{m-1}} \left( \frac{m}{2} \right)^m \omega_m + o_n(1) \geq \tau_{2^*} + o_n(1). \] (5.14)

Combining Claim 5.1, (5.13) and (5.14), we obtain \( \mathcal{L}_{p_n}(\psi_n) \geq 2\tau_{2^*} + o_n(1) \) which contradicts to (2.19). Therefore, we have \( \mathcal{L}_{p_n}(\varphi_n) \to 0 \) as \( n \to \infty \) and this, together with \( \mathcal{L}'_{p_n}(\varphi_n) \to 0 \), implies \( \varphi_n \to 0 \) in \( E \) as \( n \to \infty \) (follow the proof of [10, Lemma 4.1]). Moreover, we can get a lower bound for \( \lambda \), that is \( \lambda \geq 2^{-\frac{1}{m-1}} \), since \( H(a) \leq H_{\text{max}} \) and \( \mathcal{L}_{p_n}(\psi_n) < 2\tau_{2^*} \).

Now it remains to prove Claim 5.1. We point out here that this is equivalent to show

\[ \int_{S^m} (D\psi, \psi) d\text{vol}_{g_{S^m}} = \int_{S^m} (Dz, z) d\text{vol}_{g_{S^m}} + \int_{S^m} (D\varphi, \varphi) d\text{vol}_{g_{S^m}} + o_n(1). \] (5.15)

And since \( \varphi_n = \psi_n - z_n \), it suffices to prove \( \int_{S^m}(Dz, \varphi) d\text{vol}_{g_{S^m}} = o_n(1) \) as \( n \to \infty \). In fact, for arbitrary \( R > 0 \), we have

\[ \int_{S^m}(Dz, \varphi) d\text{vol}_{g_{S^m}} = \left( \int_{B_{Rn}(a_n)} + \int_{B_{Rn}(a_n) \setminus B_{Rn}(a_n)} \right) (Dz, \varphi) d\text{vol}_{g_{S^m}} \]

\[ = \left( \int_{B_R^0} + \int_{B_{Rn}(a_n) \setminus B_R^0} \right) (D\phi_0, \varphi_0 - \phi_0) d\text{vol}_{g_n}, \]
where the first integral goes to 0 as \( n \to \infty \) since \( \phi_n \to \phi_0 \) in \( H^{\frac{1}{2}}_{\text{loc}}(\mathbb{R}^m, \mathbb{S}_m) \). Meanwhile to estimate the second integral, we first observe that (through the conformal transformation)

\[
\sup_n \int_{B^0_{3r/\alpha_n}} |\phi_n - \phi_0|^2 \, d\rho_{g_{\mathbb{S}_m}} \leq \sup_n \int_{B^0_{3r/(\alpha_n + \delta)}} |\psi_n - z_n|^2 \, d\rho_{g_{\mathbb{S}_m}} < +\infty
\]

Thus, by \( d\rho_{g_n} \leq C d\rho_{g_{\mathbb{S}_m}} \), we have

\[
\left| \int_{B^0_{3r/\alpha_n}} (D_{g_n} \phi_0, \phi_n - \phi_0) \, d\rho_n \right| \leq C \left( \int_{B^0_{3r/\alpha_n}} |\nabla \phi_0|^2 \, d\rho_{g_{\mathbb{S}_m}} \right)^{\frac{m+1}{2m}} \to 0
\]

as \( R \to \infty \). Therefore we obtain (5.15) is valid. \( \square \)

In the above, we proved the concentration behavior \( \psi_n = R^{-\frac{m+1}{2}} \eta(\cdot)(\mu_n) \circ \phi_0 \circ \mu_n^{-1} + o_n(1) \) in \( E \) and the lower estimate for \( L_{pm}(\psi_n) \). It remains to locate the blow-up point \( a \in S^m \) and to evaluate the parameter \( \lambda \in [\lambda, 1] \) that are involved in the analysis.

For arbitrary \( \xi \in S^m \), we embed \( S^m \) into \( \mathbb{R}^{m+1} \) in the way that its antipodal point \( -\xi \) is the North pole. Denoting \( S_\xi \) the stereographic projection from \( S^m \setminus \{ -\xi \} \) to \( \mathbb{R}^m \), we have \( S_\xi(\xi) = 0 \). Particularly, \( S^m \setminus \{ -\xi \} \) and \( \mathbb{R}^m \) are conformally equivalent, where \( (S_\xi^{-1})^* g_{S^m} = f^2 g_{\mathbb{R}^m} \) with \( f(x) = \frac{2}{1 + |x|^2} \).

Let \( \iota : \mathbb{S}(\mathbb{R}^m, (S_\xi^{-1})^* g_{S^m}) \to \mathbb{S}(\mathbb{R}^m, g_{\mathbb{R}^m}) \) denote the isomorphism of spinor bundles induced by Proposition 2.1. If \( \psi \in E \) is a solution to Eq. (2.2) for some \( p \in (2, 2'] \), then \( \phi := \iota(f^{m-1} \psi \circ S_\xi^{-1}) \) will satisfy the transformed equation \( D_{\mathbb{R}^m} \phi = f^{m-1}(2^* - p)(H \circ S_\xi^{-1}) |\phi|^{p-2} \phi \) on \( (\mathbb{R}^m, g_{\mathbb{R}^m}) \).

Let \( \psi_n \) be as before, and now we require further that \( L'_{p_n}(\psi_n) \equiv 0 \). By the regularity results proved in [3], we have \( \psi_n \) is of \( C^{1, \alpha} \) for some \( \alpha \in (0, 1) \) and are classical solutions to (2.18). The proof of Proposition 2.8 will be accomplished by collecting following two results.

**Lemma 5.3.** Let \( a \in S^m \) be the associate blow-up point. Then \( \nabla H(a) = 0 \).

**Lemma 5.4.** Let \( \lambda \in [\lambda, 1] \) be the parameter involved in (5.11). Then \( \lambda = 1 \).

**Proof of Lemma 5.3.** Let us consider the stereographic projection \( S_\alpha : S^m \setminus \{ -a \} \to \mathbb{R}^m \) and the associated bundle isomorphism \( \iota : \mathbb{S}(\mathbb{R}^m, (S_\alpha^{-1})^* g_{S^m}) \to \mathbb{S}(\mathbb{R}^m, g_{\mathbb{R}^m}) \). Denoted by \( \tilde{\phi}_n = \iota(f^{-1} \psi_n \circ S_\alpha^{-1}) \), we have that \( \tilde{\phi}_n \) satisfies

\[
D_{\mathbb{R}^m} \tilde{\phi}_n = H_n(x) |\tilde{\phi}_n|^{p_n - 2} \tilde{\phi}_n \quad \text{on} \quad (\mathbb{R}^m, g_{\mathbb{R}^m}).
\]

(5.16)

where, for ease of notation, we denote \( H_n(x) := f(x) \frac{1}{2^* - p_n}(H \circ S_\alpha^{-1})(x) \).

Take \( \beta \in C^\infty_c(S^m) \) be a cut-off function on \( S^m \) such that \( \beta \equiv 1 \) on \( B_2(a) \) and \( \text{supp} \beta \subset B_3(a) \) where \( r > 0 \) is the fixed radius in the definition of \( z_n \). Then we are allowed to multiply (5.16) by \( \phi_{n,k} = \partial_k((\beta \circ S_\alpha^{-1}) \tilde{\phi}_n) \) as a test spinor for each \( k = 1, 2, \ldots, m \), and consequently

\[
\text{Re} \int_{\mathbb{R}^m} (D_{g_n} \tilde{\phi}_n, \phi_{n,k}) \, d\rho_n = \text{Re} \int_{\mathbb{R}^m} H_n |\phi_{n,k}|^{p_n - 2} \tilde{\phi}_n \, d\rho_{g_n}.
\]

(5.17)
Note that \((\beta \circ S_a^{-1})\tilde{\phi}_n\) has a compact support, we may integrate by parts to get

\[
0 = \Re \int_{\mathbb{R}^m} \partial_k (D_{g_{\mathbb{R}^m}} \tilde{\phi}_n, (\beta \circ S_a^{-1})\tilde{\phi}_n) \, d\text{vol}_{g_{\mathbb{R}^m}} \\
= 2 \Re \int_{\mathbb{R}^m} (D_{g_{\mathbb{R}^m}} \tilde{\phi}_n, \phi_{n,k}) \, d\text{vol}_{g_{\mathbb{R}^m}} + \Re \int_{\mathbb{R}^m} (\partial_k \tilde{\phi}_n, \nabla (\beta \circ S_a^{-1}) \cdot g_{\mathbb{R}^m} \tilde{\phi}_n) \, d\text{vol}_{g_{\mathbb{R}^m}} \\
- \Re \int_{\mathbb{R}^m} (D_{g_{\mathbb{R}^m}} \tilde{\phi}_n, \partial_k (\beta \circ S_a^{-1})\tilde{\phi}_n) \, d\text{vol}_{g_{\mathbb{R}^m}},
\]

(5.18)

where \(\cdot g_{\mathbb{R}^m}\) denotes the Clifford multiplication with respect to \(g_{\mathbb{R}^m}\). Now let us evaluate the last two integrals of the previous equality. First of all we see from the conformal transformation and the regularity results (see [3]) that \(\{\nabla \tilde{\phi}_n\}\) is uniformly bounded in \(L_{m+1}^2(\mathbb{R}^m, S_m)\). Hence, by the concentration behavior of \(\psi_n\), we derive

\[
\left| \int_{\mathbb{R}^m} (\partial_k \tilde{\phi}_n, \nabla (\beta \circ S_a^{-1}) \cdot g_{\mathbb{R}^m} \tilde{\phi}_n) \, d\text{vol}_{g_{\mathbb{R}^m}} \right| \leq C \left( \int_{B_{3r}(a) \setminus B_2(a)} |\psi_n|^2 \, d\text{vol}_{g_{\mathbb{R}^m}} \right)^{\frac{1}{2}} \to 0
\]

as \(n \to \infty\). And analogously, we have \(\left| \int_{\mathbb{R}^m} (D_{g_{\mathbb{R}^m}} \tilde{\phi}_n, \partial_k (\beta \circ S_a^{-1})\tilde{\phi}_n) \, d\text{vol}_{g_{\mathbb{R}^m}} \right| \to 0\) as \(n \to \infty\). Thus, we conclude from (5.18) that

\[
\Re \int_{\mathbb{R}^m} (D_{g_{\mathbb{R}^m}} \tilde{\phi}_n, \phi_{n,k}) \, d\text{vol}_{g_{\mathbb{R}^m}} = o_n(1) \quad \text{as} \quad n \to \infty.
\]

(5.19)

On the other hand, to evaluate the integral in the right side of (5.17), we first observe that

\[
0 = \int_{\mathbb{R}^m} \partial_k \left[ H_n \cdot (\beta \circ S_a^{-1}) |\tilde{\phi}_n|^p \right] \, d\text{vol}_{g_{\mathbb{R}^m}} \\
= \int_{\mathbb{R}^m} \partial_k H_n \cdot (\beta \circ S_a^{-1}) |\tilde{\phi}_n|^p \, d\text{vol}_{g_{\mathbb{R}^m}} + p_n \Re \int_{\mathbb{R}^m} H_n |\tilde{\phi}_n|^{p_n-2} (\tilde{\phi}_n, \phi_{n,k}) \, d\text{vol}_{g_{\mathbb{R}^m}} \\
- (p_n - 1) \int_{\mathbb{R}^m} H_n \cdot \partial_k (\beta \circ S_a^{-1}) |\tilde{\phi}_n|^p \, d\text{vol}_{g_{\mathbb{R}^m}}.
\]

(5.20)

It is evident that the last integral converges to 0 as \(n \to \infty\), since \(\partial_k (\beta \circ S_a^{-1})\) vanishes on \(B^0_{2r}\). We only need to estimate the first integral. Notice that \(f(x) = \frac{2}{1+|x|^2}\) and \(\beta \circ S_a^{-1}\) has a compact support on \(\mathbb{R}^m\), we see that \(H_n, H_n^{-1}\) and \(\nabla H_n\) are bounded uniformly on \(\text{supp}(\beta \circ S_a^{-1})\). Hence, due to the concentration behavior of \(\tilde{\phi}_n\), we obtain

\[
\int_{\mathbb{R}^m} \partial_k H_n \cdot (\beta \circ S_a^{-1}) |\tilde{\phi}_n|^p \, d\text{vol}_{g_{\mathbb{R}^m}} = \partial_k H_n(0) \int_{\mathbb{R}^m} |\tilde{\phi}_n|^p \, d\text{vol}_{g_{\mathbb{R}^m}} + o_n(1)
\]

as \(n \to \infty\). This and (5.20) imply

\[
\Re \int_{\mathbb{R}^m} H_n |\tilde{\phi}_n|^{p_n-2} (\tilde{\phi}_n, \phi_{n,k}) \, d\text{vol}_{g_{\mathbb{R}^m}} = -\frac{1}{p_n} \partial_k H_n(0) \int_{\mathbb{R}^m} |\tilde{\phi}_n|^p \, d\text{vol}_{g_{\mathbb{R}^m}} + o_n(1).
\]

(5.21)

Combining (5.17), (5.19) and (5.21), we conclude that

\[
\partial_k H_n(0) \int_{\mathbb{R}^m} |\tilde{\phi}_n|^p \, d\text{vol}_{g_{\mathbb{R}^m}} = o_n(1)
\]

(5.22)
as \( n \to \infty \). Since we already know from the blow-up analysis that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^m} f^{m-1/(2^*-p_n)} |\tilde{\phi}_n|^{p_n} \, d\text{vol}_{g_m} = \lim_{n \to \infty} \int_{S^m} |\psi_n|^{p_n} \, d\text{vol}_{g_{S^m}} > 0,
\]
(5.22) tells us nothing but \( \partial_k H_n(0) \equiv 0 \). Notice that \( \nabla f(0) = 0 \) and \( k \) can be varying from 1 to \( m \), we have \( \nabla (H \circ S_0^{-1})(0) = 0 \), i.e. \( \nabla H(a) = 0 \) which completes the proof. \( \square \)

**Proof of Lemma 5.4** The proof is very similar to that of Lemma 5.3. Let us recall the equation under stereographic projection (5.16) and consider the conformal transformation of \( \tilde{\phi}_n \) by \( \tilde{\phi}_{n,R}(x) = R^{m-1} \tilde{\phi}_n(Rx) \) for \( R > 0 \). Then we have
\[
D_{g_m} \tilde{\phi}_{n,R} = R^{m-1/(2^*-p_n)} \tilde{H}_{n,R} |\tilde{\phi}_{n,R}|^{p_n-2} \tilde{\phi}_{n,R} \quad \text{on } \mathbb{R}^m
\]
(5.23)
where, for ease of notations, we have denoted \( \tilde{H}_{n,R}(x) = f^{m-1/(2^*-p_n)}(Rx) \cdot H \circ S_0^{-1}(Rx) \).

Let \( \beta \in C^\infty_c(S^m) \) be the same cut-off function as in (5.17), and set \( \hat{\phi}_{n,R}(x) = \beta \circ S_0^{-1}(Rx) \cdot \tilde{\phi}_n(Rx) \). Then we can multiply (5.23) by \( \hat{\phi}_n^* = \frac{\partial}{\partial R} |_{R=R_n} \hat{\phi}_{n,R} \) (where \( R_n > 0 \) was determined in (5.6)) to get
\[
\text{Re} \int_{\mathbb{R}^m} (D_{g_m} \hat{\phi}_{n,R}, \hat{\phi}_n^*) \, d\text{vol}_{g_m} = R_n^{m-1/(2^*-p_n)} \text{Re} \int_{\mathbb{R}^m} \tilde{H}_{n,R} |\tilde{\phi}_{n,R}|^{p_n-2} (\tilde{\phi}_{n,R}, \hat{\phi}_n^*) \, d\text{vol}_{g_m}.
\]
In what follows, we are going to estimate the above two integrals.

Via the conformal transformation property of the Dirac operator, we see that
\[
\int_{\mathbb{R}^m} (D_{g_m} \hat{\phi}_{n,R}, \hat{\phi}_n^*) \, d\text{vol}_{g_m} \equiv \int_{S^m} (D\psi_n, \beta \psi_n) \, d\text{vol}_{g_{S^m}}
\]
(5.24)
and
\[
\int_{\mathbb{R}^m} (\beta \circ S_0^{-1})(Rx) \cdot \tilde{H}_{n,R} |\tilde{\phi}_{n,R}|^{p_n} \, d\text{vol}_{g_m} \equiv R_n^{m-1/(2^*-p_n)} \int_{S^m} \beta H |\psi_n|^{p_n} \, d\text{vol}_{g_{S^m}}
\]
(5.25)
for all \( R > 0 \). Hence, by taking derivative with respect to \( R \) in (5.24) and (5.25), and estimate similar as in the proof of Lemma 5.3, we can obtain
\[
(2^* - p_n)R_n^{-1} \int_{S^m} \beta H |\psi_n|^{p_n} \, d\text{vol}_{g_{S^m}} = O_n(1) \quad \text{as } n \to \infty.
\]
Since the concentration behavior suggests that \( \lim_{n \to \infty} \int_{S^m} \beta H |\psi_n|^{p_n} \, d\text{vol}_{g_{S^m}} > 0 \), we find \( 2^* - p_n = O(R_n) \) as \( n \to \infty \). Therefore we derive
\[
\lambda = \lim_{n \to \infty} R_n^{m-1/(2^*-p_n)} = \lim_{n \to \infty} e^{O(1)R_n \ln R_n} = 1.
\]
This completes the proof. \( \square \)
References

[1] R. Adams, Sobolev Spaces, Academic Press, New York, (1975).

[2] A. D. Alexandrov, Uniqueness theorems for surfaces in the large, Amer. Math. Soc. Transl. Ser. 2, 21, 441-519, (1962).

[3] B. Ammann, A variational problem in conformal spin geometry, Habilitationsschift, Universität Hamburg, (2003).

[4] B. Ammann, The smallest Dirac eigenvalue in a spin-conformal class and cmc immersions, Comm. Anal. Geom. 17 (2009), no. 3, 429-479.

[5] B. Ammann, J.-F. Grossjean, E. Humbert, B. Morel, A spinorial analogue of Aubin’s inequality, Math. Z. 260 (2008), 127-151.

[6] B. Ammann, E. Humbert, M. Ould. Ahmedou, An obstruction for the mean curvature of a conformal immersion $S^n \to \mathbb{R}^{n+1}$, Proc. Amer. Math. Soc. 135 (2007), no. 2, 489-493.

[7] M. Anderson, Conformal immersions of prescribed mean curvature in $\mathbb{R}^3$, Nonlinear Anal. 114 (2015), 142-157.

[8] T. Aubin, Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl. 55 (1976), 269-296.

[9] C. Bär, On nodal sets for Dirac and Laplace operators, Comm. Math. Phys. 188 (1997), 709-721.

[10] T. Bartsch, T. Xu, A spinorial analogue of the Brezis-Nirenberg theorem involving the critical Sobolev exponent, J. Funct. Anal. 280 (2021), 108991.

[11] B. Booss-Bavnbek, M. Marcolli, B.L. Wang, Weak UCP and perturbed monopole equations, Int. J. Math. 13, 987 (2002).

[12] J.-P. Bourguignon, P. Gauduchon, Spineurs, opérateurs de Dirac et variations de métriques, Comm. Math. Phys. 144 (1992), no. 3, 581-599.

[13] B. Buffoni, L. Jeanjean, C.A. Stuart, Existence of a non-trivial solution to a strongly indefinite semilinear equation, Proc. Amer. Math. Soc., 119 (1993), 179-186.

[14] Q. Chen, J. Jost, G. Wang, Nonlinear Dirac equations on Riemann surfaces, Ann. Global Anal. Geom. 33 (2008), 253-270.

[15] I. Ekeland, On the variational principle, J. Math. Anal. Appl. 47 (1974), 324-353.

[16] T. Friedrich, On the spinor representation of surfaces in Euclidean 3-space, J. Geom. Phys. 28 (1998), no. 1-2, 143-157.

[17] T. Friedrich, Dirac Operators in Riemannian Geometry, Grad. Stud. Math., vol 25, Amer. Math. Soc., Providence (2000).
[18] N. Ginoux, The Dirac Spectrum, Lecture Notes in Mathematics, vol. 1976. Springer, Berlin (2009).

[19] O. Hijazi, A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors, Comm. Math. Phys. 104 (1986), 151-162.

[20] N. Hitchin, Harmonic spinors, Adv. Math. 14 (1974), 1-55.

[21] W-Y. Hsiang, Z-H. Teng, W-C. Yu, New examples of constant mean curvature immersions of \((2k-1)\)-spheres into Euclidean \(2k\)-Space, Ann. of Math., 117 (1983), 609-625.

[22] T. Isobe, Nonlinear Dirac equations with critical nonlinearities on compact Spin manifolds, J. Funct. Anal. 260 (2011), no. 1, 253-307.

[23] T. Isobe, A perturbation method for spinorial Yamabe type equations on \(S^m\) and its application, Math. Ann. 355 (2013), no. 4, 1255-1299.

[24] J. Kazdan, F. Warner, Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvature, Ann. Math. 101 (1975), 317-331.

[25] R. Kusner, N. Schmitt, Representation of surfaces in space, [arXiv:dg-ga/9610005] (1996).

[26] H.B. Lawson, M.L. Michelson, Spin Geometry, Princeton University Press (1989).

[27] P. Li, S.T. Yau, A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue on compact surfaces, Invent. Math. 69 (1982), 269-291.

[28] C. S. Lin, The local isometric embedding in \(\mathbb{R}^3\) of two-dimensional Riemannian manifolds with gaussian curvature changing sign cleanly, Comm. Pure Applied Math., 39.6 (1986), 867-887.

[29] J. Mawhin, M. Willem, Critical Point Theory and Hamiltonian Systems, Springer-Verlag, New York, 1989.

[30] J. Nash, The Imbedding Problem for Riemannian Manifolds, Ann. of Math., 63, no. 1 (1956), 20-63.

[31] S. Raulot, A Sobolev-like inequality for the Dirac operator, J. Funct. Anal. 256 (2009), 1588-1617.

[32] J. Roth, Spinorial characterizations of surfaces into 3-homogeneous manifolds, J. Geom. Phys., 60 (2010), 1045-1061.

[33] M.D. Spivak, A comprehensive introduction to differential geometry, Publish or perish, 1970.

[34] M. Struwe, Variational methods, Springer-Verlag, Berlin, 2008.

[35] S. Sulanke, Die Berechnung des Spektrums des Quadrates des Dirac-Operators auf der Sphäre, Doktorarbeit. Humboldt-Universität zu Berlin, Berlin, 1979.
[36] I.A. Taimanov, Surfaces of revolution in terms of solitons, Ann. Global Anal. Geom. 15 (1997), no. 5, 419-435.

[37] I.A. Taimanov, Modified Novikov-Veselov equation and differential geometry of surfaces, Amer. Math. Soc. Transl. Ser. 2, 179 (1997) 133-151.

[38] I.A. Taimanov, The Weierstrass representation of closed surfaces in $\mathbb{R}^3$, Funct. Anal. Appl. 32 (1998), no. 4, 258-267.

[39] C.Y. Wang, A remark on nonlinear Dirac equations, Proc. Amer. Math. Soc. 138 (2010), no. 10, 3753-3758.

[40] M. Willem, Minimax Theorems, Birkhäuser, 1996.

[41] T.J. Willmore, Note on embedded surfaces, An. Sti. Univ. ”Al. I. Cuza” Iasi Sect. I a Mat. (N.S.) 11B (1965), 493-496.

[42] T.J. Willmore, Riemannian Geometry, Oxford Science Publications. Oxford University Press, 1993.

[43] H. Yamabe, On the deformation of Riemannian structures on compact manifolds, Osaka Math. J. 12 (1960) 21-37.

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