Mapping repulsive to attractive interaction in driven–dissipative quantum systems

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Abstract

Repulsive and attractive interactions usually lead to very different physics. Striking exceptions exist in the dynamics of driven–dissipative quantum systems. For the example of a photonic Bose–Hubbard dimer, we establish a one-to-one mapping relating cases of onsite repulsion and attraction. We prove that the mapping is valid for an entire class of Markovian open quantum systems with a time-reversal-invariant Hamiltonian and physically meaningful inverse-sign Hamiltonian. To underline the broad applicability of the mapping, we illustrate the one-to-one correspondence between the nonequilibrium dynamics in a geometrically frustrated spin lattice and those in a non-frustrated partner lattice.

1. Introduction

Photonic quantum systems provide a versatile platform for studying nonequilibrium many-body phenomena of light [1–5], dissipative phase transitions [6–12], and dissipation engineering [13–18]. The nonequilibrium dynamics and steady-state properties of driven–dissipative systems also play a crucial role in the development of quantum information technology for quantum optimal control and open-system state stabilization [15, 18–21]. Despite considerable theoretical and experimental progress in this field, understanding the dynamics at an intuitive level remains challenging. Considerations based on energetically favorable states are not generally appropriate in nonequilibrium, and can in fact be misleading.

We show that this is particularly the case for a driven–dissipative Bose–Hubbard dimer. Specifically, suppose that bosonic excitations are fed coherently into a dimer site, where they are subject to hopping, onsite interaction, and dissipation. How will the physics change when the sign of the onsite interaction is swapped, so that onsite repulsion turns into onsite attraction? In this paper, we demonstrate that there is an exact mapping relating observable expectation values for the repulsive system to those of the attractive system. In other words, while the equilibrium physics of a Bose–Hubbard dimer with a conserved particle number are extremely different for attraction versus repulsion [24], we find that the nonequilibrium dynamics of a driven–dissipative Bose–Hubbard dimer essentially do not distinguish between the repulsive and attractive cases.

This exact mapping does not rely on specific properties of the Bose–Hubbard dimer and similar features have been pointed out for the nonequilibrium dynamics of several other models [8, 22, 25–28]. In fact, the mapping can be generalized and holds for a large class of Markovian open quantum systems with a time-reversal–invariant system Hamiltonian. It relates the nonequilibrium dynamics of an open system $Q_1$, associated with a Hamiltonian $H$, to those of another system $Q_2$, associated with the negative-partner Hamiltonian $-H$. As long as $-H$ has physical meaning (e.g., as an effective Hamiltonian in a rotating frame), the mapping guarantees a one-to-one correspondence between observable expectation values in two different quantum systems $Q_1$ and $Q_2$.

In the remainder of this paper, we first discuss the Bose–Hubbard dimer model with drive and dissipation, approaching the mapping from the point of view of the equations of motion for system observables. We then

Interesting results of a comparison of particles with attractive and repulsive onsite interaction have previously been shown, e.g. for the Fermi–Hubbard model [22, 23].
prove that the result is an instance of a more general mapping, namely Hamiltonian sign inversion (HSI) mapping, which is applicable to a broad range of driven–dissipative systems. We illustrate this point by discussion of another example, namely the mapping of the spin dynamics in a geometrically frustrated lattice onto the corresponding dynamics in a non-frustrated spin lattice.

2. Driven–dissipative Bose–Hubbard dimer: mapping positive to negative \( U \)

We consider a driven–dissipative Bose–Hubbard dimer [29–32] with either repulsive or attractive onsite interaction, \( U > 0 \) or \( U < 0 \), respectively. By inspection of the equations of motion, we will reveal an exact mapping between the cases of positive and negative \( U \), i.e., between dimers with repulsive and attractive onsite interaction. The notion of such a mapping may, at first, seem to contradict the common intuition that attraction and repulsion must lead to entirely different physics. In our following derivation of the HSI mapping for the driven–dissipative Bose–Hubbard dimer, we will carefully discuss how this contradiction is resolved, and what exactly the mapping does and does not imply.

In concrete terms, the Bose–Hubbard dimer is described by the Hamiltonian

\[
H_\pm = \sum_{n=1}^{2} (\omega a_n^\dagger a_n + U_\pm a_n^\dagger a_n^\dagger a_n a_n) + \frac{1}{2} \left( a_1^\dagger a_2 + a_2^\dagger a_1 \right),
\]

and consists of two sites, \( n = 1, 2 \), with onsite energy \( \omega \) (we set \( \hbar = 1 \) throughout) and Hubbard interaction of strength \( U_\pm \geq 0 \). Equation (1) captures both the repulsive case (positive \( U \)) via \( H_+ \), and the attractive case (negative \( U \)) via \( H_- \). Bosonic excitations are created by \( a_n^\dagger \) and can hop between the two sites \( n = 1, 2 \) with rate \( J \).

Simple energetic considerations suggest that repulsion and attraction lead to rather different results: In the positive-\( U \) dimer, bosons repel: the onsite interaction \( U_+ a_n^\dagger a_n^\dagger a_n a_n \) increases the energy quadratically with the number of bosons on each site. For a fixed boson number \( N \gg 1 \), the onsite interaction is minimized by dividing the boson number equally between the two sites. By contrast, in the negative-\( U \) dimer, bosons attract: the Hubbard term \( U_- a_n^\dagger a_n^\dagger a_n a_n \) lowers the energy quadratically with the number of bosons on each site. For a fixed boson number \( N \gg 1 \), the onsite energy can be minimized by having all bosons occupy the same site. (We note that the spectrum of \( H_- \) is not bounded from below if the boson number is not fixed: ultimately, adding more and more bosons will lower the energy indefinitely. In practice, the attractive Bose–Hubbard dimer may serve as an effective model, in which additional nonlinear interactions need to be included when the boson number exceeds a certain threshold. Such additional terms restoring boundeness of the spectrum will naturally be system dependent.)

The above considerations yield the correct picture describing the ground-state physics for a closed-system Bose–Hubbard dimer. However, different physics become important for an open-system dimer, in which bosons are not understood as massive particles but rather as excitations that can be created by a coherent drive, as well as disappear from the system by energy dissipation. Concrete examples of such a system are coupled nonlinear resonators in which photons are the bosonic excitations in question [3, 11, 12]. The HSI mapping we wish to derive becomes meaningful in this open-system setting, where it relates the nonequilibrium dynamics of the positive-\( U \) dimer to those of the negative-\( U \) dimer.

For our derivation, we assume a weak system–bath coupling and validity of the Markov approximation, so that we can describe the time evolution and steady state of the open Bose–Hubbard dimer within the Lindblad master equation formalism [34, 35]. The reduced density matrices \( \rho_\pm \) for positive or negative \( U \) then evolve according to

\[
\frac{d\rho_\pm}{dt} = -i[H_\pm, \rho_\pm] + \gamma \sum_{n=1}^{2} \mathcal{D}[a_n] \rho_\pm,
\]

where \( \mathcal{D}[a_n] \equiv a_n a_n^\dagger - \frac{1}{\gamma} a_n^\dagger a_n^\dagger a_n a_n + \frac{1}{\gamma} a_n a_n^\dagger a_n a_n^\dagger - \frac{1}{\gamma} a_n^\dagger a_n^\dagger a_n a_n^\dagger \) is the dissipator describing the non-unitary evolution induced by the system–bath coupling. The jump operators \( a_n \) produce bosonic excitation loss from each site, as is appropriate, e.g. for describing intrinsic photon loss in transmission-line resonators or optical cavities. We remark that (2) is widely used to describe the open Bose–Hubbard dimer and related models even though, strictly speaking, the employed jump operators do not obey the requirement that jump operators be operators projecting from one eigenstate of the Hamiltonian to another [34]. It is worth noting that the use of such "phenomenological" dissipators has yielded quantitative agreement with experimental data for driven–dissipative photonic systems in specific parameter regimes [3, 11, 12, 36]. A more detailed discussion of this point is beyond the scope of this paper.

\[2\] For example when using the attractive Bose–Hubbard model as an approximation of a transmon qubit, excitations with energies above the maximum of the cosine potential will break the Bose–Hubbard approximation [33]. In that case, a perturbative treatment of the potential is not appropriate.
While both $H_+$ and $H_-$ conserve the total number of bosonic excitations, dissipation induces relaxation of the dimer towards its vacuum state. An external drive can establish a balance between excitation loss and gain. For concreteness, we consider a coherent tone driving the first dimer site as described by the drive Hamiltonian $H_d(t) = \epsilon \{a_d^\dagger e^{-i\omega_d t} + h.c.\}$. Here, $\epsilon$ parametrizes the strength of the drive, and $\omega_d$ its frequency. In the frame co-rotating with the drive, the effective system Hamiltonian is time independent,

$$H_{\text{eff}} = \sum_{n=1}^2 \left[ \delta \omega \ a_n^\dagger a_n + U_n a_n^\dagger a_n a_n a_n + \epsilon (a_n^\dagger + a_n) + J (a_n^\dagger a_{n+1} + a_n^\dagger a_{n+1}) \right],$$

(3)

where $\delta \omega \equiv \omega - \omega_d$ denotes the frequency detuning between the resonator and drive.

We demonstrate the HSI mapping at the level of the expectation values. Consider for instance $(a_1)$, whose real and imaginary parts yield the two field quadratures $I$ and $Q$ in quantum-optics language. For positive-$U$ and negative-$U$ interaction, respectively, the time evolution of $(a_1)$ is governed by

$$\frac{d}{dt}(a_1^+)_r = \left( \delta \omega - \frac{\gamma}{2} \right)(a_1^+)_r + 2U_+ (a_1^+ a_1^+)_r + J (a_2^+)_r + \epsilon,$$

(4)

$$\frac{d}{dt}(a_1^-)_r = \left( \delta \omega - \frac{\gamma}{2} \right)(a_1^-)_r + 2U_- (a_1^- a_1^-)_r + J (a_2^-)_r + \epsilon.$$  

(5)

Our claim, to be substantiated in the following, is that the dynamics for negative-$U$ interaction can be obtained exactly from the dynamics for positive-$U$ interaction. To make this argument, we now consider the positive-$U$ system. For convenience, we introduce the notation $(a_1(p))$, where $p = (U_+, \delta \omega, \epsilon, J)$ collects all external parameters entering the Hamiltonian $H_{\text{eff}}$ (3). (Note that we purposely do not include the dissipation rate $\gamma$ in $p$.) Next, we take the complex conjugate of (4) and write it in the form

$$\frac{d}{dt}(a_1(p)) = \left( -\delta \omega - \frac{\gamma}{2} \right)(a_1(p)) + 2U_- (a_1 a_1^\dagger)_r - J (a_2(p)) - \epsilon,$$

(6)

where we use $-U_+ = U_-$. Comparison with (5) suggests the relation

$$\langle a_1(-p) \rangle_r = \langle a_1(p) \rangle_r^\dagger,$$

(7)

i.e. expectation values for the cases of attractive and repulsive onsite interaction are the same up to the complex conjugation and sign adjustments of remaining Hamiltonian parameters. However, a firm proof of this relation requires that analogous relations also hold for $\langle a_2 \rangle_r$ and $\langle a_1 a_1^\dagger \rangle_r$ and thus, due to the ensuing hierarchy of equations of motion, for all expectation values $\langle A_i \rangle_r = \langle A_i^\dagger \rangle^\dagger_r$. We show in appendix that relation (7) indeed carries over to the general case:

$$\langle A_i^\dagger (-p) \rangle_r = \langle A_i^\dagger (p) \rangle_r^\dagger.$$

(8)

In simple words: every expectation value describing the dynamics for negative-$U$ interaction can be obtained from a corresponding expectation value for positive-$U$ interaction by the following two steps. First, invert the sign of each Hamiltonian parameter, while leaving the signs of decoherence rates unchanged. Second, replace the expectation values by their complex conjugates. Relation (8) therefore establishes a one-to-one map between positive-$U$ and negative-$U$ interaction through HSI. This is summarized by the diagram

$$\begin{array}{c}
Q_1 \text{ system parameters } U_+ > 0; \delta \omega; J; \epsilon \\
\text{damping rate } \gamma \\
\text{expectation values } \langle A \rangle_1 \\
\xrightarrow{\text{HSI}} \\
\text{system parameters } U_- = -U_+; -\delta \omega; -J; -\epsilon \\
\text{damping rate } \gamma \\
\text{expectation values } \langle A \rangle_2 = \langle A \rangle_1^* \\
\end{array}$$

(9)

where the entries in each row specify the corresponding Hamiltonian parameters, damping parameters, and expectation values.

In order to make systems $Q_1$ and $Q_2$ with positive and negative $U$ match even more closely, we may eliminate the sign changes in hopping $J$ and drive strength $\epsilon$ with a gauge transformation, $a_1 \rightarrow -a_1$. At this point, we find that the dynamics of the attractive versus the repulsive driven-damped Bose–Hubbard dimer are exactly the same when switching from a red-detuned to a blue-detuned drive frequency, $\delta \omega \rightarrow -\delta \omega$. (Similar observations for two different driven-damped nonlinear-oscillator systems were made by Dykman [25] and Jin et al [8].) While the ground-state physics of the closed system crucially depend on the sign of the interaction, the nonequilibrium dynamics are identical in the discussed sense. We confirm this statement with multiple numerical simulations. An example of the simulation results is depicted in figure 1. Here, both positive- and negative-$U$ dimers are initialized in a Fock state with one excitation on each site. The dynamics observed for the excitation numbers on the two sites are found to be identical for positive and negative $U$. We confirm independently that the dynamics with different initial states converge to the same steady state for positive and negative $U$. 


It is interesting to note that HSI mapping enables one to extend previous results for the driven–dissipative Bose–Hubbard dimer to the regime with the opposite sign of interaction. For instance, for repulsive interaction, it has been predicted that the steady state of the dimer can undergo spatial symmetry breaking\[30\]. HSI mapping, then, implies that the same symmetry breaking must also be present in the attractive dimer model. Surprisingly, the nature of the interaction appears to play only a secondary role in producing spatial symmetry breaking.

3. HSI mapping

In order to prove the HSI mapping that links positive-\(U\) and negative-\(U\) Bose–Hubbard dimers, we invoke the entire hierarchy of coupled equations of motion for system observables. This approach is cumbersome, and leaves one with the question of whether the HSI mapping relies upon specific properties of the Bose–Hubbard dimer, which would limit its scope to this one particular model. We will demonstrate that this is not the case and show that, rather, the HSI mapping generalizes to arbitrary Markovian open quantum systems with time-reversal-invariant system Hamiltonians \(\mathcal{H}\). (Note that while \(\mathcal{H}\) may be time-reversal-invariant, the coupling of the system to its environment will naturally break the overall time-reversal symmetry.) The HSI mapping establishes a one-to-one correspondence between the dynamics of an open quantum system \(Q_1\) with a system Hamiltonian \(\mathcal{H}\) and the dynamics of a partner system \(Q_2\) with a system Hamiltonian \(-\mathcal{H}\). We will base our discussion on the Lindblad master equation, and show that the HSI mapping can be formulated in a straightforward way that entirely bypasses cumbersome considerations of the hierarchy of equations of motion.

The dynamics of the open system \(Q_1\) are governed by the Lindblad master equation\[34, 35\],

\[
\frac{d}{dt}\rho(t) = -i[H, \rho(t)] + \sum_j \gamma_j \mathcal{D}[c_j] \rho(t),
\]

which describes the time evolution \(t \mapsto \rho(t)\) of the reduced density matrix of \(Q_1\). The dissipation and dephasing processes from coupling to the environment are encoded by rates \(\gamma_j\) and corresponding jump operators \(c_j\). In the absence of coupling to the environment, the system is assumed to be time-reversal-symmetric. As usual, we formalize this symmetry by utilizing the antiunitary time-reversal operator \(T\)\[37, 38\], which must be constructed for each concrete system of interest so that relevant observables obey the appropriate transformation laws, such as \(Tx T^\dagger = x\) and \(Tp T^\dagger = -p\) for generalized position and conjugate momentum operators \(x\) and \(p\). The time-reversal symmetry of the isolated system then amounts to the identity \(TH T^\dagger = \mathcal{H}\).

We construct the general HSI mapping by considering the \(T\)-transform of the density matrix,

\[
\rho_T = T\rho T^\dagger.
\]

We stress that the evolution \(t \mapsto \rho_T(t)\) does not correspond to the backward-in-time evolution of \(t \mapsto \rho(t)\). We obtain the equation of motion for \(\rho_T(t)\) by sandwiching\[10\] with \(T\) and \(T^\dagger\), exploiting the fact that the time-reversal operator obeys \(T T^\dagger = 1\), and invoking the time-reversal symmetry of the system Hamiltonian. This
yields the equation
\[
\frac{d}{dt} \rho_T(t) = -i[H, \rho_T(t)] + \sum_j \gamma_j [Tc_j^\dagger Tj] \rho_T(t),
\tag{12}
\]
which we recognize as having the proper form of a Lindblad master equation. Hence, we may interpret \( \rho_T \) as the density matrix of an open quantum system \( Q_2 \). Comparing this equation (12) with the original master equation (10) for \( \rho \), we see that \( Q_2 \) has a Hamiltonian with an inverted sign, as well as jump operators \( Tc_jT^j \).

We can now relate expectation values \( \langle A \rangle_2 = \text{Tr}(A\rho_T) \) for system \( Q_2 \) back to expectation values for \( Q_1 \). To do so, write \( \langle A \rangle_2 = \text{Tr}(T\bar{A}\rho T^\dagger) \), but note that the cyclic property of the trace does not hold for anti-linear operators such as \( T \). Instead, we simplify the expression further by considering an orthonormal Hilbert space basis of time-reversal-invariant states \( \{|n\} \). In this basis, the action of the time-reversal operator reduces to complex conjugation, such that \[ Tn (\sum_n \alpha_n |n\rangle) = \bar{K} (\sum_n \alpha_n |n\rangle) = \bar{K} (\sum_n \alpha^*_n |n\rangle). \]

With this, we find
\[
\langle A \rangle_2 = \text{Tr}(T\bar{A}\rho T^\dagger) = \sum_n \langle n | \bar{K}A|n\rangle^* = \sum_n \langle m | \bar{K}A|n\rangle |m\rangle = \text{Tr}(T\bar{A}\rho T^\dagger)^*,
\tag{13}
\]
where we temporarily use the shorthand \( \bar{A} = T\bar{A}T^\dagger \). As a result, the correspondence between expectation values in system \( Q_1 \) and \( Q_2 \) takes the form
\[
\langle A \rangle_2 = \text{Tr}(A\rho_T) = \text{Tr}(T\bar{A}\rho T^\dagger)^* = \langle T\bar{A}\rangle_T^\dagger.
\tag{14}
\]

We can summarize the general HSI mapping with the diagram
\[
Q_1 \quad \begin{array}{c|c|c}
\text{system Hamiltonian} & H & -H \\
\text{damping rates, ops.} & \gamma_j, c_j & \gamma_j, Tc_jT^j \\
\text{density matrix} & \rho_1(t) & \rho_2(t) = T\rho_1(t)T^\dagger \\
\text{expectation values} & \langle A \rangle_1 & \langle A \rangle_2 = \langle T\bar{A}\rangle_T^\dagger \\
\end{array} \quad \leftarrow \quad \text{HSI} \quad \rightarrow \quad Q_2
\tag{15}
\]
It is easy to verify that the mapping (9) for the driven–dissipative Bose–Hubbard dimer is a special case of (15). To see this, note that the inversion of the Hamiltonian sign produces the signs change of system parameters recorded in (9); the jump operators considered in the dimer model are time-reversal-invariant, \( Ta_T = a_0 \), and so are the expectation values of the observables \( A_T \). We note that the general HSI mapping immediately extends the correspondence between positive- and negative- \( U \) dimers to Bose–Hubbard lattices of arbitrary size and lattice geometry.

Our results also apply to closed-system Bose–Hubbard lattices (\( \gamma_j = 0 \)), and are in agreement with the recent demonstration of ‘repulsively bound’ pairs [39]. In that work, Winkler et al studied the evolution of a pair of bosons occupying the same site in a three-dimensional optical lattice with repulsive interaction. They showed that such an initial pair Fock state (not an eigenstate of the Hamiltonian) undergoes a time evolution that maintains pairing despite particle repulsion. This phenomenon can be directly inferred by applying the HSI mapping when noting that the initial state is invariant under time reversal and HSI maps repulsive onto attractive interaction. We note further that HSI mapping is not restricted to the bosonic case. In fact, a specific instance of this mapping was reported for the Fermi–Hubbard model in [22].

HSI mapping is mathematically rigorous, but one must check that the partner system \( Q_2 \) is indeed a physically meaningful quantum system. Two aspects are crucial here. First, in infinite-dimensional Hilbert spaces, sign inversion of the Hamiltonian leads to energy spectra not bounded from below. As shown in the dimer example, this is unproblematic if the Hamiltonian is an effective Hamiltonian in a rotating frame, whose eigenvalues only carry the meaning of quasi-energies. Second, HSI mapping may modify jump operators entering the master equation.

A wide range of driven open quantum systems are amenable to HSI mapping, including circuit-QED and ultracold-atoms, which are of interest in studies of phase transitions [6, 8, 10–12] and quantum state preparation [15, 17, 18, 20]. For open quantum systems with finite-dimensional Hilbert space, \( -H \) is always physical. This class of systems covers a number of quantum systems currently being researched, e.g. open spin lattices [6, 40–43], of which we will present one example in the following section. Here, again, the HSI mapping will link two physically different systems and establish a useful one-to-one correspondence between their nonequilibrium dynamics.

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3. The annihilation operator is defined by \( a = \frac{1}{\sqrt{2}} (\xi + i\xi^\dagger) \), where \( \xi \) is a generalized position operator, \( \rho \) is a generalized momentum operator, and \( x_0, p_0 \) are constants. From \( TxT^\dagger = x, TpT^\dagger = -p \), and the anti-linearity of \( T \) follows the time-reversal invariance of \( a \).

4. Justifications must occur on a case-by-case basis depending on the specific model and the parameter regime. Nonetheless, HSI mapping applies to many driven–dissipative systems; see the discussion of jump operators in section 2.
4. Driven–dissipative spin lattice

HSI mapping can easily be applied to driven–dissipative (pseudo)spin lattices which can be realized, for example, by ultracold atoms [43] and circuit-QED devices [44, 45]. We illustrate such an application next, considering an Ising system with one spin per lattice site, each with (Zeeman) energy splitting \( \Omega \). Each spin is driven by a coherent tone with drive strength \( \epsilon_j \) and frequency \( \omega_{d,j} \), and is \( \sigma^z \)-coupled to its nearest neighbors with coupling strength \( J \). The effects of the environment are modeled by spin relaxation with rate \( \Gamma \). In the frame co-rotating with the drive, the system dynamics are governed by the master equation

\[
\frac{d\rho}{dt} = -i[H, \rho] + \Gamma \sum_{j=1}^{N} D[\sigma_j] \rho,
\]

with the system Hamiltonian

\[
H = \sum_{j=1}^{N} \left( \delta \Omega \sigma^z_j \sigma^z_j + \epsilon_j (\sigma^-_j + \sigma^+_j) \right) \pm J \sum_{\langle j,k \rangle} \sigma^+_j \sigma^-_k,
\]

where \( \delta \Omega = \Omega - \omega_{d,j} \) is the detuning.

The Ising coupling strength \( J \) can be designed to be positive (antiferromagnetic coupling) or negative (ferromagnetic coupling), depending on the particular physical realization [45]. We consider the special case where the underlying lattice is not bipartite, such as a triangular or Kagome lattice. In this case, ferromagnetic and antiferromagnetic coupling are well known to lead to very different equilibrium physics: while for \( \Omega = 0 \) the negative-\( J \) ground state is a simple ferromagnet, the positive-\( J \) case faces geometrical frustration of the antiferromagnetic coupling—a phenomenon of great interest in many-body physics, e.g. in the study of spin glasses [46]. Despite the dramatically different ground-state physics of the geometrically frustrated and non-frustrated lattices, one finds that HSI mapping (15) establishes a one-to-one correspondence for the out-of-equilibrium dynamics under driving and damping. (Correspondence between positive \( J \) and negative \( J \) was previously discussed for other nonequilibrium spin systems in [27, 28].) Similar to \( a_j \) in the harmonic oscillator case, the spin-lowering operator \( \sigma^-_j \) for pseudospins is invariant under time reversal. Hence, it is straightforward to apply HSI mapping and obtain

\[
Q_1 \xrightarrow{\text{HSI}} Q_2
\]

where system \( Q_1 \) is the frustrated spin lattice, and \( Q_2 \) the non-frustrated lattice. We can make the Hamiltonian of \( Q_2 \) match that of \( Q_1 \), even more closely by eliminating the sign change in drive strength \( \epsilon_j \) with a gauge transformation, \( \sigma^-_j \rightarrow -\sigma^+_j \) and \( \sigma^+_j \rightarrow -\sigma^-_j \) for all sites \( j \). As a result of this, we find a one-to-one correspondence between the frustrated and non-frustrated spin dynamics and steady state in a driven–dissipative Ising lattice. (The only parameter to be adjusted is the drive-frequency detuning \( \delta \Omega \).)

For numerical confirmation of this result we simulate the dynamics for a triangular plaquette of three spins (see figure 2). (In this simulation, only one of the three sites is driven and the spin excitation number on that particular site is monitored. We initialize the two models in the spin-down state, which is time-reversal-invariant and hence not affected by the HSI mapping.) We find the expected, but non-intuitive, result that the nonequilibrium dynamics are indeed identical in the frustrated and the non-frustrated case.

5. Conclusion

We have proven and illustrated the use of a mapping that establishes a one-to-one correspondence between the nonequilibrium dynamics of one Markovian open quantum system and a second such system whose Hamiltonian carries the opposite sign. This mapping relies on the time-reversal invariance of the system Hamiltonian, and makes the remarkable prediction that the nonequilibrium dynamics of different systems can be essentially identical despite the fact that their equilibrium physics are extremely different. We have demonstrated this HSI mapping for two concrete examples: a driven–dissipative Bose–Hubbard dimer for attractive versus repulsive onsite interaction, and a driven–dissipative Ising spin lattice model with and without geometrical frustration.

HSI mapping is widely applicable to many interesting driven–dissipative quantum models realizable by ultracold atoms and circuit-QED architecture. It allows one to make important predictions not necessarily obvious from simple intuition. For instance, from HSI mapping we can immediately infer that the symmetry-
breaking state predicted for a driven–dissipative repulsive Bose–Hubbard dimer [30] must also occur for the case of an attractive dimer. Moreover, employing HSI mapping onto a driven–dissipative Ising spin lattice shows that the lattice nonequilibrium dynamics are unaffected by geometrical frustration. Establishing such results will not only facilitate a better understanding of open-system dynamics [27, 28], but also motivate new experiments probing nonequilibrium many–body phenomena [8]. We believe that HSI mapping will serve as a valuable tool in the study of nonequilibrium dynamics, steady-state properties, and dissipative phase transitions in open quantum systems.

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Appendix. Equations of motion of the driven–dissipative Bose–Hubbard dimer model

In this appendix, we demonstrate that the expectation values of the general operators $A_{rs}^{p,q} = (a_i^d)^p (a_i^a)_i^q a_i^a_i^d_i^a$ indeed obey the relation

$$\langle A_{rs}^{p,q}(-p) \rangle_- = \langle A_{rs}^{p,q}(p) \rangle_+$$

(see (8) in the main text). We abbreviate the positive- $U$ expectation value by $A_{rs}^{p,q} = \langle A_{rs}^{p,q}(p) \rangle_+$ and deduce the corresponding equation of motion from the master equation (2). We find

$$i \frac{dA_{rs}^{p,q}}{dt} = \left[ (\delta \omega - U)(r + s - p - q) + U(r^2 + s^2 - p^2 - q^2) - \frac{i \gamma}{2} (p + q + r + s) \right] A_{rs}^{p,q}$$

$$+ 2U \left[ (r - p)A_{rs}^{p+1,q} + (s - q)A_{r+1,s}^{p,q+1} \right] + \epsilon \left[ rA_{rs}^{p,q} - pA_{rs}^{p-1,q} \right]$$

$$+ [rA_{r-1,s+1}^{p,q} + sA_{r+1,s-1}^{p,q} - pA_{r-1,s}^{p-1,q+1} - qA_{r,s}^{p+1,q-1}].$$

By complex conjugation of this equation, we obtain

$$i \frac{dA_{rs}^{p,q,*}}{dt} = \left[ (-\delta \omega + U)(r + s - p - q) - U(r^2 + s^2 - p^2 - q^2) - \frac{i \gamma}{2} (p + q + r + s) \right] A_{rs}^{p,q,*}$$

$$- 2U \left[ (r - p)A_{rs}^{p+1,q,*} + (s - q)A_{r+1,s}^{p,q+1,*} \right] - \epsilon \left[ rA_{rs}^{p,q,*} - pA_{rs}^{p-1,q,*} \right]$$

$$- [rA_{r-1,s+1}^{p,q,*} + sA_{r+1,s-1}^{p,q,*} - pA_{r-1,s}^{p-1,q+1,*} - qA_{r,s}^{p+1,q-1,*}].$$

Comparison of the latter equation with the equation of motion for the corresponding negative- $U$ expectation value confirms the proposed relation (A.1).

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