INDEX OF FIBRATIONS AND BRAUER CLASSES THAT NEVER OBSTRUCT THE HASSE PRINCIPLE

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Abstract. Let \( X \) be a smooth projective variety with a fibration into varieties that either satisfy a condition on representability of zero-cycles or that are torsors under an abelian variety. We study the classes in the Brauer group that never obstruct the Hasse principle for \( X \). We prove that if the generic fiber has a zero-cycle of degree \( d \) over the generic point, then the Brauer classes whose orders are prime to \( d \) do not play a role in the Brauer–Manin obstruction. As a result we show that the odd torsion Brauer classes never obstruct the Hasse principle for del Pezzo surfaces of degree 2, certain K3 surfaces, and Kummer varieties.

1. Introduction

Let \( X \) be a smooth projective geometrically integral variety over a number field \( k \). Let \( \mathbb{A} \) denote the adeles of \( k \) and \( X(\mathbb{A}) \) be the set of adelic points on \( X \). Following Manin [Man71], one can use any subset \( H \) of the Brauer group \( \text{Br}(X) := H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{tors}} \) to form a subset \( X(\mathbb{A})^H \subseteq X(\mathbb{A}) \), which contains \( X(k) \) by class field theory. This gives an inclusion reversing map

\[
\{ \text{subsets of } \text{Br}(X) \} \longrightarrow \{ \text{subsets of } X(\mathbb{A}) \},
\]

\[
H \longmapsto X(\mathbb{A})^H.
\]

See [2] for details. Hence, any subset \( H \subseteq \text{Br}(X) \) gives the containments

\[
X(\mathbb{A})^{\text{Br}} := X(\mathbb{A})^{\text{Br}(X)} \subseteq X(\mathbb{A})^H \subseteq X(\mathbb{A}).
\]

For any positive integer \( n \), define the set of prime-to-\( n \) torsion elements by

\[
\text{Br}(X)[n^{-1}] := \{ A \in \text{Br}(X) \mid mA = 0 \text{ for some } (n, m) = 1 \}.
\]

We consider the following general question.

**Question 1.1.** Given a class \( C \) of \( k \)-varieties and an integer \( n > 1 \), does \( X(\mathbb{A}) \neq \emptyset \) imply \( X(\mathbb{A})^{\text{Br}(X)[n^{-1}]} \neq \emptyset \) for all \( X \in C \)? In other words, is it true that \( \text{Br}(X)[n^{-1}] \) never obstructs the Hasse principle?

In this paper, we study some nontrivial cases where the answer to Question 1.1 is positive. Denote by \( \text{Br}_0(X) \) the subgroup in \( \text{Br}(X) \) of constant classes im[\( \text{Br}(k) \to \text{Br}(X) \)], arising from the structure morphism \( X \to \text{Spec}(k) \). If \( \text{Br}(X)[n^{-1}] = \text{Br}_0(X)[n^{-1}] \) for every \( X \in C \), then the answer to Question 1.1 will be trivially positive. Hence a particular case of interest is when \( n \) is chosen so that \( \text{Br}(X)[n^{-1}] \supseteq \text{Br}_0(X)[n^{-1}] \) for some \( X \in C \). For example, we consider the class of degree \( d \) del Pezzo surfaces, where the possible Brauer groups are known.

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Example 1.2. Let $\mathcal{C}$ be the class of del Pezzo surfaces of degree 3 over $k$. Then $\text{Br}(X)/\text{Br}_0(X)$ has exponent 2 or 3 for any $X \in \mathcal{C}$ [SD93]. Moreover if $\text{Br}(X)/\text{Br}_0(X)$ has exponent 2, then $X$ satisfies the Hasse principle [SD93 Corollary 1]. Hence Question 1.1 has a positive answer for $n = 3$.

In general, the possible Brauer groups for a class $\mathcal{C}$ of varieties are difficult to compute; in particular it is already a difficult problem to find an $n$ such that $\text{Br}(X)[n^+] = \text{Br}_0(X)[n^+]$ for all $X \in \mathcal{C}$.

Some recent results on Brauer–Manin obstructions are related to the framework of Question 1.1. For example, in [IS15], Ieronymou and Skorobogatov show that for the class of diagonal quartic surfaces in $\mathbb{P}^3_{\mathbb{Q}}$, the odd torsion part $\text{Br}(X)[2^+\mathbb{Z}]$ of the Brauer group does not obstruct the Hasse principle, i.e., if $X(\mathbb{A}_\mathbb{Q}) \neq \emptyset$ then $X(\mathbb{A}_\mathbb{Q})^{\text{Br}[2^+\mathbb{Z}]} \neq \emptyset$. This gives an answer to Question 1.1 for diagonal quartics over $\mathbb{Q}$ with $n = 2$. They also give conditions on the equation of $X \in \mathcal{C}$ for the group $\text{Br}(X)/\text{Br}_0(X)$ to contain odd torsion elements, so in particular $\text{Br}(X)[2^+\mathbb{Z}] \supseteq \text{Br}_0(X)[2^+\mathbb{Z}]$. In [CV17], Creutz and Viray consider a related but generally logically independent question. For the class $\mathcal{C}$ of degree $d$ $k$-varieties in projective space, they look at the leftmost containment of (1) and consider whether $X(\mathbb{A}_k)^{\text{Br}[d^\mathbb{Z}]} \neq \emptyset \implies X(\mathbb{A}_k)^{\text{Br}} \neq \emptyset$ holds. They write $\text{BM}_d$ for this implication and prove it holds for some classes of varieties such as torsors under abelian varieties and Kummer varieties. This implication is false in general [CV17 Theorem 6.5]. They also write $\text{BM}_d^+$ for the statement there is no prime-to-$d$ Brauer–Manin obstruction. A positive answer to Question 1.1 for $n = d$ is equivalent to $\text{BM}_d^+$. Inspired by Creutz–Viray, one can also ask for a stronger version of Question 1.1 of whether both $\text{BM}_n$ and $\text{BM}_n^+$ hold simultaneously,

Question 1.3. Given a class $\mathcal{C}$ of $k$-varieties and an integer $n > 1$, does $X(\mathbb{A}_k)^B \neq \emptyset$ imply $X(\mathbb{A}_k)^{B + \text{Br}(X)[n^+]} \neq \emptyset$ for all $X \in \mathcal{C}$ and all subgroups $B \in \text{Br}(X)$? In other words, is it true that $\text{Br}(X)[n^+]$ plays no role in obstructing the Hasse principle?

Question 1.1 is a special case of Question 1.3 where $B$ is trivial.

1.1. Main results. For a smooth projective variety $Y$ over a number field $k$, let $A_0(Y)$ be the group of zero-cycles of degree zero, modulo rational equivalence. We say that $Y$ satisfies property (ZC) if for any field extension $K/k$ and $Q \subset Y(K)$, the natural map $Y(K) \to A_0(Y_K)$ given by $P \mapsto (P) - (Q)$ is surjective. For example, smooth projective curves of genus 1 and $k$-rational varieties satisfy (ZC) (see 1.1 1.2 for more examples). Define the index of a variety $Y$ to be the smallest positive integer $d$ such that $Y$ has a zero-cycle of degree $d$. We say that $Y$ satisfies weak approximation if the image of $Y(k)$ is dense in $Y(\mathbb{A})$ with the product topology.

We prove the following theorem, which we use to study Question 1.3 (and hence also Question 1.1) over any number field for all degree 2 del Pezzo surfaces and some diagonal quartic surfaces.

Theorem 1.4. Let $\pi : X \to Z$ be a morphism between smooth projective geometrically integral varieties over a number field $k$. Suppose that $Z$ satisfies weak approximation and there exists a Zariski open set $Z_0 \subseteq Z$ such that the fiber $X_P$ satisfies (ZC) for any $P \in Z_0$. Suppose that the generic fiber over the function field $k(Z)$ has index $d$. If $B \subset \text{Br}(X)$ is a subgroup such that $X(\mathbb{A}_k)^B \neq \emptyset$, then $X(\mathbb{A}_k)^{B + \text{Br}(X)[d^+]} \neq \emptyset$. 

We also prove a similar result when the fibers are torsors under abelian varieties. We use this to study Question 1.1 for Kummer varieties.

**Theorem 1.5.** Let \( \pi: X \to Z \) be a morphism between smooth projective geometrically integral varieties over a number field \( k \). Suppose that \( Z \) satisfies weak approximation. Suppose that the generic fiber \( Y \) is a \( k(Z) \)-torsor under an abelian variety \( A/k(Z) \), and that the order of \( [Y] \in H^1(k(Z), A) \) is \( d \). If \( B \subset \text{Br}(X) \) is a subgroup such that \( X(\mathbb{A})^B \neq \emptyset \), then \( X(\mathbb{A})^{B + \text{Br}(X)[d]} \neq \emptyset \).

**Remark 1.6.** The case when \( Z = \text{Spec} \, k \) is a point was proven in [CV17].

### 1.2. Applications

Colliot-Thélène and Sansuc [CTS80, §V Question k1] have conjectured that for geometrically rational surfaces, the Brauer–Manin obstruction is the only obstruction to the Hasse principle. In particular, it is known that del Pezzo surfaces of degree \( d = 1 \) and \( d \geq 5 \) satisfy the Hasse principle (see, e.g., [VA13]). Some partial results for \( d = 3 \) (see Example 1.2) and \( d = 4 \) (see [Wit07] conditional on Schinzel’s hypothesis and finiteness of Tate-Shafarevich groups) are known. In [Cor07, Question 4.5], Corn asked whether there are any Brauer–Manin obstruction to rational points when \( d = 2 \) and \( \text{Br}(X)/\text{Br}(k) \) is 3 torsion. Moreover he asked if \( X \) satisfies the Hasse principle in this case. If the conjecture is true, a negative answer to the first question would imply a positive answer to the second question.

Let \( C \) be the class of del Pezzo surfaces of degree 2 over a number field \( k \). A consequence of Theorem 1.4 is that Question 1.3 has a positive answer with \( n = 2 \). In other words, the 3-torsion part of the Brauer group will not obstruct the Hasse principle, giving a positive answer to one of Corn’s questions. More concretely, in §4 we prove the following result.

**Corollary 1.7.** Let \( X \) be a degree 2 del Pezzo surface over a number field \( k \). Suppose that \( \text{Br}(X)/\text{Br}_0(X) \) has exponent 3. Then \( X(\mathbb{A}) \neq \emptyset \implies X(\mathbb{A})^{\text{Br}} \neq \emptyset \).

**Remark 1.8.** There exist minimal del Pezzo surfaces over \( k \) with \( \text{Br}(X)/\text{Br}_0(X) \simeq \mathbb{Z}/3\mathbb{Z} \). Hence not every surface occurring in Corollary 1.7 arises as a blow-up of a cubic surface at a rational point. See Example 1.6 which was communicated to us by Elsenhans.

Let \( \mathcal{C} \) be the class of all smooth diagonal quartics in \( \mathbb{P}^3 \) defined by

\[
ax^4 + by^4 + cz^4 + dw^4 = 0,
\]

where \( a, b, c, d \in k^\times \) and \( abcd \in k^{\times 2} \) for some number field \( k \). Then Question 1.3 has a positive answer with \( n = 2 \). This extends a result of Ieronymou and Skorobogatov in [IS13] to any number field, but under the condition that \( abcd \in k^{\times 2} \).

**Corollary 1.9.** Let \( X \) be a smooth diagonal quartic \( [2] \) in \( \mathbb{P}^3_k \) with \( abcd \in k^{\times 2} \). If \( B \subset \text{Br}(X) \) is a subgroup such that \( X(\mathbb{A})^B \neq \emptyset \), then \( X(\mathbb{A})^{B + \text{Br}(X)[2]} \neq \emptyset \).

Given an abelian variety \( A \) of dimension \( \geq 2 \) and a 2-covering of \( A \), one can construct a Kummer variety attached to this 2-covering. Let \( \mathcal{C} \) be the class of all such Kummer varieties over a number field \( k \). It was proven by Skorobogatov and Zarhin that Question 1.4 has positive answer for \( \mathcal{C} \) with \( n = 2 \) [SZ16], and later Skorobogatov extended the result to answer Question 1.3 in [CV17, Theorem A.1]. Their approach relied on proving results about \( \text{Br}(X) \) and its odd torsion part. We use the proof of Theorem 1.5 to give a more direct proof of this, but at the cost of not giving any information about \( \text{Br}(X) \) itself.
Corollary 1.10. Let $A$ be an abelian variety defined over a number field $k$. Let $X$ be the Kummer variety attached to a $2$-covering of $A$ (see [4.2.1]). If $B \subset \Br(X)$ is a subgroup such that $X(A)^B \neq \emptyset$, then $X(A)^{B + \Br(X)[2^+] \neq \emptyset}$.

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2. Proof of Theorem [1.4]

Denote by $\Omega$ the set of places of $k$, and by $v$ the completion of $k$ at $v$ for any $v \in \Omega$. Given any field extension $K/k$, there is a pairing

$$X(K) \times \Br(X) \to \Br(K),$$

$$(P, A) \mapsto \mathcal{A}(P) := \iota^* A.$$

where $i: P \to X$ is the inclusion. This leads to the definition of $X(A)^H$,

$$X(A)^H = \left\{ \{P_v\} \in X(A) \mid \sum_{v \in \Omega} \inv_v A(P_v) = 0 \ \forall A \in H \right\},$$

where $\inv_v: \Br(k_v) \to \mathbb{Q}/\mathbb{Z}$ is the invariant map from local class field theory. In [CT95], Colliot-Thélène extended the above pairing to the group of zero-cycles on $X$. Moreover the pairing respects rational equivalence, thus inducing a pairing on $\text{CH}_0(X_K)$ defined as follows

$$\text{CH}_0(X_K) \times \Br(X) \to \Br(K),$$

$$\left( \sum_i n_i(P_i), A \right) \mapsto \sum_i n_i \text{cores}_{K(P_i)/K} A(P_i).$$

Similarly, by class field theory, if $M \in \text{CH}_0(X)$, its image $\{M_v\} \in \prod_{v \in \Omega} \text{CH}_0(X_{k_v})$ then satisfies $\sum_{v} \inv_v A(M_v) = 0$.

Proof of Theorem [1.4]. By, e.g., [CV17 Lemma 4.8], it suffices to show that $X(A)^B \neq \emptyset \implies X(A)^H \neq \emptyset$ for any finite subgroup $H \subset B + \Br(X)[d^+]$. Let $H \subset B + \Br(X)[d^+]$ be a finite subgroup. Then there is a finite set $S \subset \Omega$ of places in $k$ such that $X$ can be spread out to a scheme $\mathcal{X}$ over $\text{Spec} \mathcal{O}_{k,S}$, where $\mathcal{O}_{k,S}$ denotes the ring of $S$-integers, and any $A \in H$ extends to an element of $\Br(\mathcal{X})$ (see [Sk01, §5]). As a result, the evaluation of any point $P_v \in X(k_v)$ will lie in $\Br(\mathcal{O}_v)$, which is trivial; here $\mathcal{O}_v$ denotes the ring of integers of $k_v$. Since the existence of a model over $\text{Spec} \mathcal{O}_{k,S}$ is stable under a finite field extension, the map $\mathcal{A}(-): X(L_w) \to \Br(L_w)$ is also zero for any finite extension $L_w/k_v$. Hence it follows from the definition that the map $\mathcal{A}(-): \text{CH}_0(X_{k_v}) \to \Br(k_v)$ is also zero.

Let $\sum_i n_i(P_i)$ be a zero-cycle of degree $d$ on the generic fiber. Then each point $P_i$, say of degree $d_i$, gives rise to an irreducible subvariety $E_i \subset X$ such that $E_i \to Z$ is finite surjective of degree $d_i$. By shrinking $Z_0$ if necessary, we can assume $(E_i)_{Z_0} := E_i \times_Z Z_0 \to Z_0$ is finite
étale of degree $d_i$ for each $i$. Since each point in $X(k_v)$ has an analytic neighborhood over $k_v$ for any place $v$, we have $X_Z^0(k_v) \neq \emptyset$, so $X^0_Z(A) \neq \emptyset$ by properness of the fibers.

Let $\{P_v\} \in X(\mathbb{A})^B$. By left-continuity of the Brauer–Manin pairing, we may deform $P_v$ so that $\{P_v\} \in X_Z^0(A) \cap X(\mathbb{A})^B$. By the implicit function theorem, we can find a $v$-adic open set $U_v \subset Z_0$, containing $\pi(P_v)$, where there exists a local section $\rho_v : U_v \rightarrow X$, i.e., $\pi \circ \rho_v = \text{id}$ and $\rho_v(\pi(P_v)) = P_v$. By weak approximation, we can find a rational point $Q \in Z_0(k)$ such that for each $v \in S$, $Q$ is $v$-adically close enough to $\pi(P_v)$ so that $Q \in U_v$ and $B(\rho_v(Q)) = B(P_v)$ for any $B \in H$.

Let $M_i \in \text{CH}_0(X)$ be the zero-cycle of degree $d_i$ corresponding to the subscheme $E_i \times_Z Q \subset X$. By construction, we can also consider $M_i \in \text{CH}_0(F)$ where $F := \pi^{-1}(Q)$ is the fiber above $Q$. Let $M = \sum_i n_i M_i$, which is a zero-cycle of degree $d$, and denote its image in $\text{CH}_0(F_{k_v})$ by $M_v$ for any $v \in \Omega$.

Define the point $\{R_v\} \in X(\mathbb{A})$ as follows. For each $v \notin S$, choose $R_v$ to be any point in $X(k_v)$. For each $v \in S$, first set $O_v := \rho_v(Q) \in F(k_v)$. Let $n \in \mathbb{Z}$ be such that $nd \equiv 1 \mod |H|$ and define the zero-cycle

$$D := n M_v - (nd - 1)(O_v).$$

By the assumption that $F$ satisfies (ZC), there is a point in $R_v \in F(k_v)$ such that $(R_v) = D$ in $\text{CH}_0(F_{k_v})$.

We now show that $\{R_v\} \in X(\mathbb{A})^H$. Fix some $A \in H \cap B$ or $A \in H[d^{-}]$. We have

$$\sum_{v \in \Omega} \text{inv}_v A(M_v) = 0$$

since $M_v$ is the image of the $k$-rational zero-cycle $M$. Recall that by assumption on $S$, the map $A(-) : \text{CH}_0(X_{k_v}) \rightarrow \text{Br}(k_v)$ is zero for $v \notin S$. Hence

$$\sum_{v \in \Omega} \text{inv}_v A(R_v) = \sum_{v \in S} \text{inv}_v A(R_v)$$

$$= \sum_{v \in S} n \text{inv}_v A(M_v) + \sum_{v \in S} (1 - nd) \text{inv}_v A(O_v)$$

$$= \sum_{v \in \Omega} n \text{inv}_v A(M_v) + \sum_{v \in \Omega} (1 - nd) \text{inv}_v A(P_v)$$

$$= 0$$

where the last equality follows from $1 - nd \equiv 0 \mod |H[d^{-}]|$ if $A \in H[d^{-}]$ and from $\{P_v\} \in X(\mathbb{A})^B$ if $A \in B \cap H$. Hence $\{R_v\} \in X(\mathbb{A})^H$. \hfill \Box

3. Proof of Theorem 1.5

The proof follows similar strategy as [2]: the key ingredient is [CV17, Lemma 4.6] whose use was suggested to us by Skorobogatov. We use this to prove a slight variant of [CV17, Corollary 4.3].

**Lemma 3.1.** Let $A$ be an abelian variety over a number field $k$. Let $\psi : Y \rightarrow A$ be a covering. Let $H \subset \text{Br}(Y)$ be a finite subgroup. Suppose $S \subset \Omega$ is a finite set of places such that the group $\text{CH}_0(Y_{k_v})$ pairs trivially with $H$ for all $v \notin S$. 
If there exists \( \{ R_v \} \in \prod_{v \in S} Y(k_v) \) such that
\[
\sum_{v \in S} \text{inv}_v A(R_v) = 0
\]
for all \( A \in H[d^\infty] \), then there exists \( \{ P_v \} \in \prod_{v \in S} Y(k_v) \) such that
\[
\sum_{v \in S} \text{inv}_v A(P_v) = 0.
\]
for all \( A \in H \).

Proof. By [CV17, Lemma 4.6, A.2], there is a map \( \rho: Y \to Y \) such that
\[
\rho^* : \text{Br}(Y) \to \text{Br}(Y)
\]
is the identity on \( H[d^\infty] \) and \( \rho^* H[d^1] \subset \text{Br}_0(Y) = \text{Br}(k) \). Let \( P_v = \rho(R_v) \). For any \( A \in H[d^\infty] \),
\[
\sum_{v \in S} A(P_v) = \sum_{v \in S} \rho^* A(R_v) = \sum_{v \in S} A(R_v) = 0.
\]
Now let \( A \in H[d^1] \) and \( \alpha = \rho^* A \in \text{Br}(k) \). For any \( v \notin S \) and \( M \in \text{CH}_0(Y_{k_v}) \), we have \( \alpha(M) = A(\rho_v M) = 0 \) by assumption. On the other hand, \( \alpha(M) = \text{deg}(M) \alpha_v \) where \( \alpha_v \) is the image of \( \alpha \) under \( \text{Br}(k) \to \text{Br}(k_v) \). Since \( Y_{k_v} \) has a zero-cycle of degree \( d^{2g} \), namely \( \psi^{-1}(0) \), it follows \( \alpha_v \in \text{Br}(k_v)[d^{2g}] \) which shows \( \alpha_v = 0 \). Hence
\[
\sum_{v \in S} \alpha_v = \sum_{v \in \Omega} \sum_{v \in S} \alpha_v = 0.
\]
Thus \( \{ P_v \} \) is orthogonal to \( H[d^\infty] + H[d^1] = H \).

Proof of Theorem 1.5. Let \( H \subset B + \text{Br}(X)[d^1] \) be any finite subgroup. We must show that \( X(A)^H 
eq \emptyset \). Let \( S \subset \Omega \) be a finite set of places such that \( H \) pairs trivially with \( \text{CH}_0(X_{k_v}) \) for all \( v \notin S \) (see [2]). There is a Zariski open set \( U \subset Z \) such that \( A \) has a model \( \mathcal{A} \) over \( U \) and for each point \( P \in U \), the fiber \( X_P \) is a \( k(P) \)-torsor under the abelian variety \( \mathcal{A}_P \). Let \( \{ T_v \} \in X(A)^B \). As in [2], we can find a point \( Q \in U(k) \) that is \( v \)-adically close to \( \pi(T_v) \) so that \( F := \pi^{-1}(Q) \) has a \( k_v \)-point \( R_v \) with \( A(R_v) = \mathcal{A}(T_v) \) for each \( v \in S \) and \( A \in H \). In particular,
\[
\sum_{v \in S} \text{inv}_v A(R_v) = \sum_{v \in S} \text{inv}_v A(T_v) = \sum_{v \in \Omega} \sum_{v \in S} \text{inv}_v A(T_v) = 0
\]
for any \( A \in H[d^\infty] \subset B \). An application of Lemma 3.1 then gives a point \( \{ P_v \} \in \prod_{v \in S} F(k_v) \) such that
\[
\sum_{v \in S} \text{inv}_v A(P_v) = 0
\]
for all \( A \in H \). Setting \( P_v \) to be any point on \( X(k_v) \) for \( v \notin S \), we conclude \( \{ P_v \} \in X(A)^H \).
4. Applications

Before we prove the corollaries listed in §1, we first give examples of varieties satisfying property (ZC).

Example 4.1. Any $k$-rational variety satisfies property (ZC). In fact, any class of varieties that become $K$-rational once they have a $K$-point satisfies (ZC) such as

1. **Quadrics.** In this case we always have a zero-cycle of degree 2. Hence the theorem implies $X(\mathbb{A}) \neq \emptyset \implies X(\mathbb{A})^{Br[*]} \neq \emptyset$ when the fibers are quadrics.
2. **Severi–Brauer varieties.** In this case one can also replace $d$ in Theorem 1.4 by the period of the generic fiber, since index and period have the same prime divisors.
3. **Del Pezzo surfaces of degree $d \geq 5$.**

Example 4.2. Let $X$ be a del Pezzo surface of degree 4. If $X(k) \neq \emptyset$, then there exists a point $P \in X(k)$ not lying on any of the lines on $X$. One can blow up at $P$ to get a cubic surface $\tilde{X}$ containing a line $L$ defined over $k$. The projection from $L$ defines a rational map $\tilde{X} \dashrightarrow \mathbb{P}^1$ which can be extended to a morphism where the fibers are conics. Since there are exactly 10 lines on $\tilde{X}$ intersecting $L$, there are 5 degenerate fibers. By [CTC79] $\tilde{X}$ satisfies (ZC), which implies $X$ satisfies (ZC) by birational invariance of $CH_0(X)$ among smooth projective varieties. Since $X$ can be expressed as intersection of two quadrics in $\mathbb{P}^4$, its index divides 4.

Example 4.3. Let $X$ be a smooth curve of genus 1. Then $X$ satisfies (ZC) by the Riemann–Roch theorem for curves. This is not true for higher dimensional abelian varieties, but one can apply Theorem 1.3 instead in this case.

Remark 4.4. By taking $Z = \text{Spec } k$ in Theorem 1.4 we get that any of the varieties $X$ listed above satisfy $X(\mathbb{A}) \neq \emptyset \implies X(\mathbb{A})^{Br[*]} \neq \emptyset$ where $d$ is the index of $X$. If $X$ is a smooth projective curve of genus 1, then one can take $d$ to be the order of $[X] \in H^1(k, J)$ where $J$ is the Jacobian of $X$.

4.1. Del Pezzo Surfaces. Let $X$ be a del Pezzo surface of degree 2 over a number field $k$. See [Cor07] Theorem 4.1 for a complete list of isomorphism classes of $Br(X)/Br_0(X)$. In particular, $Br(X)/Br_0(X)$ has exponent either 2, 3, or 4, and if it has exponent 3, then it is isomorphic to either $\mathbb{Z}/3\mathbb{Z}$ or $(\mathbb{Z}/3\mathbb{Z})^2$.

**Proof of Corollary 1.7.** We can express $X$ as a smooth hypersurface of degree 4 inside the weighted projective space $\text{Proj } k[w, x, y, z] = \mathbb{P}(2, 1, 1, 1)$. Since char $k \neq 2$, $X$ has an equation of the form

$$w^2 = f(x, y, z),$$

where $f(x, y, z)$ is a homogeneous polynomial of degree 4. Suppose $Br(X)/Br(k) \simeq \mathbb{Z}/3\mathbb{Z}$ or $(\mathbb{Z}/3\mathbb{Z})^2$. Let $\pi: X \dashrightarrow \mathbb{P}^1$ be the rational map given by $[w : x : y : z] \mapsto [y : z]$. The locus of indeterminancy is given by $Z := \{[w : x : y : z] \in X \mid y = z = 0\}$. Let $\tilde{X}$ be the blow up of $X$ along $Z$ giving the following diagram

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\beta} & \mathbb{P}^1 \\
\downarrow & & \\
X & \xrightarrow{\pi} & \mathbb{P}^1
\end{array}$$
Then $\tilde{\pi}$ is a genus 1 fibration, i.e., almost all fibers are smooth projective curves of genus 1. The generic fiber clearly has points of degree 2 over $k(\mathbb{P}^1)$. Hence by Example 4.3 and Theorem 1.4, the subgroup $\text{Br}(\tilde{X})[2^+]$ does not obstruct the Hasse principle. Since $\beta^*\text{Br}(X)[2^+] \subseteq \text{Br}(\tilde{X})[2^+]$, if $\{P_v\} \in \tilde{X}(A)^{\text{Br}(\tilde{X})[2^+]}$, then $\beta(\{P_v\}) \in X(A)^{\text{Br}(X)[2^+]}$. The statement of the corollary follows from the equality $(\text{Br}(X)/\text{Br}_0(X))[2^+] = \text{Br}(X)/\text{Br}_0(X)$. \hfill $\square$

**Remark 4.5.** As genus one curves are torsors under their Jacobian, one can also use Theorem 1.5 to prove the above result.

**Example 4.6.** For any cubic surface $S \subset \mathbb{P}^3$ with 3 torsion in $\text{Br}(S)/\text{Br}_0(S)$, one can blow up a point on $S$ to obtain a del Pezzo surface $X$ of degree 2 with 3 torsion in $\text{Br}(X)/\text{Br}_0(X)$. However, such an $X$ will always have a rational point. An example of a del Pezzo surface over $\mathbb{Q}$ with 3 torsion in Brauer group which does not arise as a blow up of a cubic surface is the following (provided to us by Elsenhans)

$$w^2 = 8x^4 + 16x^3y - 16x^3z + 21x^2y^2 - 30x^2yz + 21x^2z^2 + 14xy^3 - 24xy^2z +$$

$$+ 18xyz - 16xz^3 + 7y^4 - 22y^3z + 9y^2z^2 + 4yz^3 + 2z^4.$$ 

Indeed, the Galois action on $\text{Pic}(\overline{X})$ is given by an order 12 subgroup of the Weil group $W(E_7)$, and for any order 12 subgroup $G \subset W(E_6)$, the Galois cohomology group $H^1(S,G)$ has no elements of order 3. One can check that the above surface has local points everywhere and hence no Brauer–Manin obstruction. Indeed $[w : x : y : z] = [0 : 0 : 1 : 1]$ is a rational point.

### 4.2. K3 Surfaces.

**Proof of Corollary 1.10.** Suppose $X(\mathbb{A}) \neq \emptyset$. Then the quadric $Q$ defined by

$$ar^2 + bs^2 + cu^2 + dv^2 = 0$$ 

is also everywhere locally soluble, and thus has a $k$-point by the Hasse–Minkowski theorem. The condition that $abcd$ is a square implies that there is a ruling $\pi: Q \to \mathbb{P}^1$ over $k$. Under the natural map $\rho: X \to Q$, the pullback of a line in the fiber of $\pi$ is a curve of genus 1, making $\pi \circ \rho$ a genus 1 fibration (See [SD00]). The generic fiber $X_{k(\mathbb{P}^1)}$ is a degree 8 cover of $Q_{k(\mathbb{P}^1)}$. A choice of a section $s: \mathbb{P}^1 \to Q$ gives a point $\text{Spec} k(\mathbb{P}^1) \to Q_{k(\mathbb{P}^1)}$. The projection of $\text{Spec} k(\mathbb{P}^1) \times_{Q_{k(\mathbb{P}^1)}} X_{k(\mathbb{P}^1)}$ to $X_{k(\mathbb{P}^1)}$ gives a degree 8 zero-cycle on $X_{k(\mathbb{P}^1)}$. An application of Theorem 1.4 to $\pi \circ \rho: X \to \mathbb{P}^1$ finishes the proof. \hfill $\square$

#### 4.2.1. Construction of the Kummer varieties in Corollary 1.10.

Let $A$ be an abelian variety of dimension $\geq 2$ defined over $k$. Let $\psi: Y \to A$ be a 2-coverings of $A$. The antipodal involution on $A$ induces an involution $\sigma$ on $Y$. Let $Y' \to Y$ be the blow up along $f^{-1}(0)$. Then $\sigma$ extends to $Y'$; the quotient $X = Y'/\sigma$ is smooth and is called the Kummer variety $\text{Kum}(Y)$ attached to $Y$ (see [SZ16] for more details).

**Proof of Corollary 1.10.** Let $\{P_v\} \in X(\mathbb{A})^H$. It suffices to show that $X(\mathbb{A})^H \neq \emptyset$ for any finite subgroup $H \subset B = \text{Br}(X)[2^+]$. Let $S \subset \Omega$ be a finite set of places such that $H$ pairs trivially with $\text{CH}_0(X_{k_v})$ for any $v \notin S$ (see first paragraph of proof of Theorem 1.4). For each $v \in S$, there is a class $\alpha_v \in H^1(k_v,\mu_2) = k_v^*/k_v^{*2}$ such that $P_v$ lifts to a $k_v$-point on the twist $Y_{\alpha_v}$. By weak approximation, there is a class $\alpha \in H^1(k,\mu_2)$ which restricts to $\alpha_v$ for
each \( v \in S \). Hence the twist \( f : Y'_\alpha \to X \) contains a point in \( \{ Q_v \}_{v \in S} \) mapping to \( \{ P_v \}_{v \in S} \). Since \( Y'_\alpha \) and \( Y_\alpha \) are birational, by abuse of notation, we can identify \( f^* H \) as a subgroup of \( \text{Br}(Y'_\alpha) \) and \( \{ Q_v \}_{v \in S} \) as a point on \( Y'_\alpha \). Then

\[
\sum_{v \in S} \text{inv}_v A(Q_v) = 0
\]

for any \( A \in f^*(H \cap B) = f^* H[2^\infty] \). An application of Lemma 3.1 shows there is a point \( \{ Q'_v \}_{v \in S} \) such that

\[
\sum_{v \in S} \text{inv}_v A(Q'_v) = 0
\]

for any \( A \in f^* H \). Take any point in the preimage in \( Y'_\alpha \) and map it down to \( X \) to obtain a point \( \{ P'_v \}_{v \in S} \). Set \( P'_v \) to be any point on \( X(k_v) \) for \( v /\in S \) to get \( \{ P'_v \} \in X(A)^H \).

\[\square\]

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