On the algorithmic complexity of decomposing graphs into regular/irregular structures

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Abstract

A locally irregular graph is a graph whose adjacent vertices have distinct degrees, a regular graph is a graph where each vertex has the same degree and a locally regular graph is a graph where for every two adjacent vertices u, v, their degrees are equal. In this paper, we investigate the set of all problems which are related to decomposition of graphs into regular, locally regular and/or locally irregular subgraphs and we present some polynomial time algorithms, NP-completeness results, lower bounds and upper bounds for them. Among our results, one of our lower bounds makes use of mutually orthogonal Latin squares which is relatively novel.

Key words: Locally irregular graph; locally regular graph; 1-2-3 Conjecture; graph decomposition; mutually orthogonal Latin squares; semi-coloring; computational complexity.

1 Introduction

For a family F of graphs, an F-decomposition of a graph G is a decomposition of the edge set of the graph G into subgraphs isomorphic to members of the family F. Note that the family F of graphs can be anything, for instance, all regular graphs or all complete graphs. During the last decade, the computational complexity of this problem has received...
a considerable attention. For example, Holyer proved that it is \textbf{NP}-hard to decompose
the edges of a graph into the minimum number of complete subgraphs [22]. For more
examples see [1, 27, 28, 32] and the references therein.

A \textit{locally irregular} graph is a graph whose adjacent vertices have distinct degrees. Also,
a \textit{regular} graph is a graph where each vertex has the same degree and a \textit{locally regular}
graph is a graph where for every two adjacent vertices \( u \) and \( v \), their degrees are equal.
In this work, we consider the set of all problems which are related to decomposition of
graphs into regular and/or locally irregular subgraphs and we present some polynomial
time algorithms, \textbf{NP}-completeness results, upper bounds and lower bounds for them. A
summary of our results and open problems are shown in Table 1 and Table 2. These
graph families have received attention recently because of their relationship to the 1-2-3
Conjecture [24]. (For more information about the 1-2-3 Conjecture and its variations see
[3, 7, 13, 14, 25] and the references therein.)

Before we start we would like to draw the readers attention to the different kinds of
subgraphs that are suitable parts of a decomposition in this paper. When a decomposition
has a \textit{locally irregular} component, this subgraph, \( G_i \) has the property that if \( u \) and \( v \) are
adjacent in \( G_i \) then their degrees in \( G_i \) must be different, \( d_i(u) \neq d_i(v) \). A component,
\( G_i \) is a \textit{regular} subgraph if \( d_i(u) \) is constant for all vertices in \( V(G_i) \). These two are the
quite standard and we have restated them for completeness and clarity. Two other kinds
of component subgraphs are investigated at various points in this paper which are less
traditional and we define them here so that when the reader encounters them herein, she
will have already seen the distinction. A component, \( G_i \) of an edge decomposition is \textit{locally
regular} if for all adjacent \( u, v \in V(G_i) \), their degrees in \( G_i \) must be equal, \( d_i(u) = d_i(v) \).
Clearly all regular graphs are locally regular but the disjoint union of a cycle and an edge
is locally regular without being regular. The final type of component allowed is one in
which each component is permitted to be either regular or locally irregular. Once again a
regular, locally regular or locally irregular graph fits this criterion but the disjoint union of
a cycle, an edge and a non-trivial star satisfies the criterion and is neither regular, locally
regular nor locally irregular.

2 Our results and motivations

2.1 Locally irregular graphs

Motivated by the fact that every connected graph with at least two vertices contains a pair
of vertices of the same degree and the 1-2-3 Conjecture, we consider the locally irregular
graphs. In 2015, Baudon et al. introduced the notion of decomposition into locally irregular subgraphs, where by a decomposition they mean a partitioning of the edges [5]. In such a case, we want to decompose the graph $G$ into locally irregular subgraphs, where by a decomposition of the graph $G$ into $k$ locally irregular subgraphs we refer to a decomposition $E_1, \ldots, E_k$ of $E(G)$ such that the subgraph $G[E_i]$ is locally irregular for every $i = 1, \ldots, k$. The irregular chromatic index of the graph $G$, denoted by $\chi'_\text{irr}$, is the minimum number $k$ such that the graph $G$ can be decomposed into $k$ locally irregular subgraphs.

Baudon et al. identified all connected graphs which cannot be decomposed into locally irregular subgraphs and call them exceptions [5]. They conjectured that apart from these exceptions all other connected graphs can be decomposed into three locally irregular subgraphs [5].

**Conjecture 1** [5] For each non-exception graph $G$, we have $\chi'_\text{irr}(G) \leq 3$.

Afterwards, Bensmail et al. proved that every bipartite graph $G$ which is not an odd length path satisfies $\chi'_\text{irr}(G) \leq 10$ [9]. Recently, Lužar et al. improved the upper bound for bipartite graphs and general graphs, into 7 and 220, respectively [29].

From another point of view, Bensmail and Stevens considered the problem of decomposing the edges of graph into some subgraphs, such that in each subgraph every component is either regular or locally irregular [11]. The regular-irregular chromatic index of graph $G$, denoted by $\chi'_\text{reg-irr}$, is the minimum number $k$ such that $G$ can be decomposed into $k$ subgraphs, such that each component of every subgraph is locally irregular or regular [11]. They conjectured that the edges of every graph can be decomposed into at most two subgraphs, such that each component of every subgraph is regular or locally irregular [11].

**Conjecture 2** [11] For each graph $G$, we have $\chi'_\text{reg-irr}(G) \leq 2$.

How much easier is Conjecture 1 if we relax the problem and only require that each subgraph (instead of each component) should be locally irregular or regular? With this motivation in mind, we consider the problem of partitioning the edges of graph into subgraphs, such that each subgraph is regular or locally irregular. The regular-irregular number of graph $G$, denoted by $\text{reg-irr}(G)$, is the minimum number $k$ such that the graph $G$ can be decomposed into $k$ subgraphs, such that each subgraph is locally irregular or regular.
\[ \chi'_{\text{reg-irr}}(G) \leq \text{reg} - \text{irr}(G) \leq \chi'_{\text{irr}}(G). \] (1)

Motivated by Conjecture 1 and Conjecture 2, we present the following conjecture. With Conjecture 3 we weaken Conjecture 1 and strengthen Conjecture 2.

**Conjecture 3** Each graph can be decomposed into 3 subgraphs, such that each subgraph is locally irregular or regular.

There are infinitely many graphs such that their regular-irregular numbers are three. For example, consider the following tree. First, join two vertices \( v \) and \( u \) by an edge. Then consider four paths of lengths 6,6,2,2 called \( P_1, P_2, P_3, P_4 \), respectively. Identify one of the ends for each of \( P_1, P_2 \) with \( v \), and identify one of the ends for each of \( P_3, P_4 \) with \( u \). Call the resultant tree \( T \). It is easy to check that the tree \( T \) cannot be decomposed into two subgraphs, such that each subgraph is locally irregular or regular. We show that deciding whether a given planar bipartite graph \( G \) with maximum degree three can be decomposed into at most two subgraphs, such that each subgraph is regular or locally irregular is \( \text{NP} \)-complete.

**Theorem 1** Determining whether the regular-irregular number of a given planar bipartite graph \( G \) with maximum degree three is at most two, is \( \text{NP} \)-complete.

From the proof of Theorem 1, one can obtain the following corollary.

**Corollary 1** For a given planar bipartite graph \( G \) with maximum degree three, deciding whether the edge set of \( G \) can be decomposed into two subgraphs \( R \) and \( I \) such that \( R \) is regular and \( I \) is locally irregular, is \( \text{NP} \)-complete.

If \( T \) is a tree which is not an odd length path, then its irregular chromatic index is at most three and there exist infinitely many trees with irregular chromatic index three \([5]\). Baudon et al. proved that the problem of determining the irregular chromatic index of a graph can be handled in linear time when restricted to trees and if \( T \) is a tree with \( \Delta(T) > 4 \), then its irregular chromatic index is at most two \([6]\). Afterwards, Bensmail and Stevens proved that if \( T \) is a tree, then its regular-irregular chromatic index is at most two \([11]\).

Here, for every \( k > 2 \), we construct a tree \( T \) with \( \Delta(T) = k \) such that \( T \) cannot be decomposed into a matching and a locally irregular subgraph and also, we show that every tree can be decomposed into two matchings and a locally irregular subgraph.
Theorem 2
(i) For every $k > 2$, there is a tree with $\Delta(T) = k$ such that $T$ cannot be decomposed into a matching and a locally irregular subgraph.
(ii) Every tree can be decomposed into two subgraphs $P$ and $R$ such that $R$ is a matching and each component of $P$ is an edge or a locally irregular component.
(iii) Every tree can be decomposed into two matchings and a locally irregular subgraph.

In [6], Baudon et al. proved that determining whether a given planar graph $G$, can be decomposed into two locally irregular subgraphs is $\text{NP}$-complete. But their reduction does not preserve the planarity. In this paper, by another reduction, we show that determining whether a given planar graph $G$, can be decomposed into two locally irregular subgraphs is $\text{NP}$-complete.

Theorem 3 Determining whether the irregular chromatic index of a given planar graph $G$ is at most two, is $\text{NP}$-complete.

There is an interesting connection between an edge-labeling which is an additive vertex-coloring and the irregular chromatic index of regular graphs.

Remark 1 Karoński, Łuczak and Thomason initiated the study of edge-labelings which give additive vertex-colorings. That means for every edge $uv$, the sum of labels of the edges incident to $u$ is different from the sum of labels of the edges incident to $v$ [24]. Dudek and Wajc showed that determining whether a given graph has an edge-labeling which is an additive vertex-coloring from $\{1, 2\}$ is $\text{NP}$-complete [18]. Afterwards, Ahadi et al. proved that determining whether a given 3-regular graph $G$ has an edge-labeling which is an additive vertex-coloring from $\{1, 2\}$ is $\text{NP}$-complete [2]. For a given 3-regular graph $G$, it is easy to see that the graph $G$ has an edge-labeling which is an additive vertex-coloring from $\{1, 2\}$ if and only if the edge set of the graph $G$ can be decomposed into at most two locally irregular subgraphs. Thus, for a given 3-regular graph $G$, deciding whether $\chi'_{irr}(G) = 2$ is $\text{NP}$-complete [2].

2.2 Regular graphs

The edge set of every graph can be decomposed such that the subgraph induced by each subset is regular (to obtain a trivial upper bound consider the case that each subgraph is a matching). In 2001, Kulli et al. introduced the regular number of graphs [26]. The regular number of a graph $G$, denoted by $\text{reg}(G)$, is the minimum number of subsets into which the
edge set of the graph $G$ can be decomposed so that the subgraph induced by each subset is regular. Nonempty subsets $E_1, \ldots, E_r$ of $E(G)$ are said to form a regular decomposition of the graph $G$ if the subgraph induced by each subset is regular. The edge chromatic number of a graph, denoted by $\chi'(G)$, is the minimum size of a decomposition of the edge set into 1-regular subgraphs. By Vizing’s theorem the edge chromatic number of a graph $G$ is equal to either $\Delta(G)$ or $\Delta(G) + 1$ (see [33], page 197). Hence, the regular number problem is a generalization of the edge chromatic number and we have the following:

$$
\text{reg}(G) \leq \chi'(G) \leq \Delta(G) + 1.
$$

Determining whether $\text{reg}(G) \leq \Delta(G)$ holds for all connected graphs was posed an open problem in [20]. It was shown that not only there exists a counterexample for the above bound but also for a given connected graph $G$ deciding whether $\text{reg}(G) \leq \Delta(G)$ is NP-complete [15]. Designing an algorithm to decompose a given bipartite graph into the minimum number of regular subgraphs was posed as another problem in [20]. But, it was proved that computation of the regular number is NP-hard for connected bipartite graphs. Also, it was proved that deciding whether $\text{reg}(G) = 2$ for a given connected 3-colorable graph $G$ is NP-complete [15]. Here, we improve the previous results and show that for a given bipartite graph $G$ with maximum degree six, determining whether $\text{reg}(G) = 2$ is NP-complete. Furthermore, we show that there is polynomial time algorithm to decide whether $\text{reg}(G) = 2$ for a given graph $G$ with maximum degree five.

**Theorem 4**

(i) For every number $\alpha \geq 3$, determining whether $\text{reg}(G) = 2$ for a given bipartite graph $G$ with degree set $\{\alpha, 2\alpha\}$, is NP-complete.

(ii) There is polynomial time algorithm to decide whether $\text{reg}(G) = 2$ for a given graph $G$ with maximum degree five.

Also, we consider the problem of determining the regular number for planar graphs. Note that every planar graph $G$ with degree set $\{2, 4\}$ can be decomposed into two regular subgraphs (see the proof of part (ii) of Theorem 4).

**Theorem 5** Determining whether $\text{reg}(G) = 2$ for a given planar graph $G$ with degree set $\{3, 6\}$, is NP-complete.
2.3 Locally regular graphs

We say that a graph $G$ is \textit{locally regular} if each component of $G$ is regular (Note that a regular graph is locally regular but the converse does not hold). The \textit{regular chromatic index} of a graph $G$ denoted by $\chi'_{\text{reg}}$ is the minimum number of subsets into which the edge set of $G$ can be decomposed so that the subgraph induced by each subset is locally regular. From the definitions of locally regular and regular graphs we have the following bound.

$$\chi'_{\text{reg}}(G) \leq \text{reg}(G) \leq \Delta(G) + 1.$$ \hfill (3)

It was shown that determining whether $\text{reg}(G) \leq \Delta(G)$ for a given connected graph $G$ is \textit{NP}-complete [15]. Here, we show that every graph $G$ can be decomposed into $\Delta(G)$ subgraphs such that each subgraph is locally regular. We use the concept of semi-coloring to prove that fact. Daniely and Linial defined a semi-coloring of graphs for the investigation of the tight product of graphs [12]. Afterwards, Furuya \textit{et al.} proved that every graph has a semi-coloring [19].

**Theorem 6** For every graph $G$, $\chi'_{\text{reg}}(G) \leq \Delta(G)$ and this bound is sharp for trees.

**Remark 2** The difference between the regular number and the regular chromatic index of a graph can be arbitrary large. For a fixed $t$, consider a copy of the complete graphs $K_1, \ldots, K_t$. For each $i, i = 1, \ldots, t - 1$, join one of the vertices of the complete graph $K_i$ to one of the vertices of the complete graph $K_{i+1}$. Call the resulting graph $G$. If we put the set of edges of complete graphs $K_1, \ldots, K_t$ in one subgraph and the other edges of $G$ in another subgraph, we obtain an edge decomposition of $G$ into two locally regular subgraphs, so $\chi'_{\text{reg}}(G) \leq 2$. On the other hand, since $G$ has $t + O(1)$ different numbers in its degree set, $\text{reg}(G) \geq \lg t$.

Suppose that $G$ is a connected graph with degree set $\{\alpha, 2\alpha\}$ and the induced graph on the set of vertices of degree $2\alpha$ forms an independent set. It is easy to check that $\chi'_{\text{reg}}(G) = 2$ if and only if $\text{reg}(G) = 2$ (since the induced graph on the set of vertices of degree $2\alpha$ forms an independent set, so in each decomposition every component is $\alpha$-regular). Thus, by the proof of Theorem 4 and Theorem 5, we have the following corollary.

**Corollary 2**

(i) Determining whether $\chi'_{\text{reg}}(G) = 2$ for a given bipartite graph $G$ with maximum degree six, is \textit{NP}-complete.
(ii) Determining whether $\chi'_{\text{reg}}(G) = 2$ for a given planar graph $G$ with degree set $\{3, 6\}$, is $\mathbf{NP}$-complete.

In Theorem 4, we prove that there is polynomial time algorithm to decide whether $\text{reg}(G) = 2$ for a given graph $G$ with maximum degree five. Here, we show that deciding whether a given subcubic graph can be decomposed into two subgraphs such that each subgraph is locally regular, is $\mathbf{NP}$-complete.

**Theorem 7** For a given subcubic graph $G$, determining whether it can be decomposed into two subgraphs such that each subgraph is locally regular, is $\mathbf{NP}$-complete.

**2.4 Locally $k$-irregular graphs**

We say that a graph is *locally $k$-irregular* if and only if for every two adjacent vertices $v$ and $u$, $|d(v) - d(u)| \geq k$. In the following, we would like to decompose $G$ into locally $k$-irregular subgraphs, where by a decomposition of $G$ into $t$ locally $k$-irregular subgraphs we refer to a partition $E_1, \ldots, E_t$ of $E(G)$ such that $G[E_i]$ is locally $k$-irregular for every $i = 1, \cdots, t$. The $k$-irregular chromatic index of $G$, denoted by $\chi'_{k-\text{irr}}$, is the minimum number $t$ such that $G$ can be decomposed into $t$ locally $k$-irregular subgraphs. For every $k$, let $G_k$ be the set of graphs which can be decomposed into locally $k$-irregular subgraphs, define:

$$h(k) = \max_{G \in G_k} \chi'_{k-\text{irr}}(G).$$

Baudon et al. characterized all connected graphs which cannot be decomposed into locally $1$-irregular subgraphs and called them exceptions [5]. They conjectured $h(1) \leq 3$ [5]. Also, Baudon et al. asked the computational complexity of determining $\chi'_{1-\text{irr}} = 2$ for bipartite graphs [6]. We show that for each $k > 1$, deciding whether $\chi'_{k-\text{irr}}(G) = 2$ for a given planar bipartite graph $G$ is $\mathbf{NP}$-complete. For all $k$ we prove the lower bound $h(k) \geq 2k + 1$ and we will use mutually orthogonal Latin squares to prove that $h(k) = \Omega(k^2)$. Finding a better lower bound can be interesting for future work.

**Theorem 8**

(i) For every $k > 1$, determining whether $\chi'_{k-\text{irr}}(G) = 2$ for a given planar bipartite graph $G$ is $\mathbf{NP}$-complete.

(ii) For each $k$, $h(k) \geq 2k + 1$ and if $k > 3$, $h(k) \geq 4k$.

(iii) $h(k) = \Omega(k^2)$.

(iv) For each fixed $k$, there is a polynomial time algorithm to decide whether a given graph $G$ with maximum degree $k + 1$ can be decomposed into two locally $k$-irregular subgraphs.
2.5 Summary of results

A summary of results and open problems are shown in Table 1 and Table 2. In the first table we summarize the recent results on the computational complexity of deciding whether a family of graphs can be decomposed into two subgraphs with some conditions and in the second table we summarize the recent upper bounds and conjectures on the different types of partitioning. For more information about the decomposing the graphs into regular/irregular subgraphs see [4, 8, 10, 29, 30].

Table 1: Recent results on edge decomposing of graphs into two subgraphs. Blue text shows our results.

|                        | Tree                      | Bipartite                  | Planar                     | Subcubic                  |
|------------------------|---------------------------|----------------------------|----------------------------|---------------------------|
| Irregular chromatic index | P [6]            | Open [6]                   | NP-c (Th. 3)               | NP-c (Remark 1)           |
| Regular-irregular number | Open                   | NP-c (Th. 1)               | NP-c (Th. 1)               | NP-c (Th. 1)              |
| 1 regular plus 1 irregular | Open                   | NP-c (Cor. 1)              | NP-c (Cor. 1)              | NP-c (Cor. 1)             |
| Regular number         | P [26]                   | NP-c (Th. 4)               | NP-c (Th. 5)               | P (Th. 4)                 |
| Regular chromatic index | P (Th. 6)                | NP-c (Cor. 2)              | NP-c (Cor. 2)              | NP-c (Th. 7)              |
| k-irregular chromatic index \(k > 1\) | Open                   | NP-c (Th. 8)               | NP-c (Th. 8)               | P (Th. 8)                 |
| Regular-irregular chromatic index | P [11]       | P (Conj. 2 [11])          | P (Conj. 2 [11])          | P (Conj. 2 [11])         |

Table 2: Recent upper bounds and conjectures

|                        | Tree                      | Bipartite graphs | General graphs |
|------------------------|---------------------------|------------------|----------------|
| Irregular chromatic index | 3 [5]                    | 3 (Conj. 1 [5])  | 3 (Conj. 1 [5])|
| Regular-irregular chromatic index | 2 [11]      | 6 [11]           | 2 (Conj. 2 [11])|
| Regular-irregular number | 3 [5]               | 3 (Conj. 3)      | 3 (Conj. 3)    |
| Regular number         | \(\Delta(G)\) [26]      | \(\Delta(G)\) [26]| \(\Delta(G)+1\) [26]|
| Regular chromatic index | \(\Delta(G)\) (Th. 6)   | \(\Delta(G)\) (Th. 6) | \(\Delta(G)\) (Th. 6) |
| k-irregular chromatic index | \(f(k)\) (Prob. 4) | \(f(k)\) (Prob. 4) | \(f(k)\) (Prob. 4) |

3 Proofs

Here, we show that determining whether a given planar bipartite graph \(G\) with maximum degree three can be decomposed into at most two subgraphs, such that each subgraph is regular or locally irregular is \(\text{NP}-\text{complete}\).

Proof of Theorem 1. Clearly, the problem is in \(\text{NP}\). We reduce \textit{Cubic Planar 1-In-3 3-Sat} to our problem. Moore and Robson [31] proved that the following problem is
NP-complete.

Cubic Planar 1-In-3 3-Sat.

Instance: A 3-Sat formula \( \Phi = (X, C) \) such that every variable appears in exactly three clauses, there is no negation in the formula, and the bipartite graph obtained by linking a variable and a clause if and only if the variable appears in the clause, is planar.

Question: Is there a truth assignment for \( X \) such that each clause in \( C \) has exactly one true literal?

Let \( \Phi = (X, C) \) be an instance of Cubic Planar 1-In-3 3-Sat. Without loss of generality suppose that the number of clauses in \( \Phi \) is even and \( C = \{ c_i : i \in \mathbb{Z}_m \} \). Note that we have \( |X| = |C| = m \). We convert the formula \( \Phi \) into a graph \( G_\Phi \) such that the formula \( \Phi \) has a 1-in-3 satisfying assignment if and only if the edge set of the graph \( G_\Phi \) can be decomposed into two subgraphs such that each subgraph is regular or locally irregular. For every pair \((x, i)\), where \( x \in X \) and \( i \in \mathbb{Z}_m \), consider two cycles \( x_i x'_i x''_i x'''_i \) and \( z_i z'_i z''_i z'''_i \), also, put a vertex \( s_i \) and join the vertex \( s_i \) to the vertices \( x''_i, z''_i \). Next for every number \( i, i \in \mathbb{Z}_m \) put a vertex \( r_i \) and join the vertex \( r_i \) to the vertices \( z'''_i, x'''_{i+1 \mod m} \). Also, for each clause \( c_i \), \( c_i \in C \) put a clause vertex \( c_i \) and for every variable \( x \in X \) if \( x \) appears in \( c_i \), then put the edge \( c_i x_i \). Having done these for all variables \( x \) and all clauses \( c_i \), in the resultant graph the degree of every vertex is two or three and the graph is planar. Next, for every vertex \( v \) of degree two put a new vertex \( v_u \) (we will call this new vertex a dummy vertex in our proof) and join the vertex \( v \) to the vertex \( v_u \). Call the resultant graph \( F \). It is easy to check that \( F \) is planar, bipartite and its degree set is \( \{1, 3\} \). See Fig. 1.

![Figure 1: A part of the graph \( F \), when the variable \( x \) appears in \( c_1, c_4 \) and \( c_m \).](image)

Now, consider the following tree. First, join two vertices \( v \) and \( u \) by an edge. Then consider four paths of lengths 4, 4, 2, 2 called \( P_1, P_2, P_3, P_4 \), respectively. Identify one of the ends of each of \( P_1, P_2 \) with \( v \), and identify one of the ends of each of \( P_3, P_4 \) with \( u \). Call the resultant tree \( T \). The graph \( T \) cannot be decomposed into two locally irregular subgraphs. Also, \( T \) cannot be decomposed into two regular subgraphs. Thus, we can only decompose \( T \) into two subgraphs such that one subgraph is regular and the other one is
the vertex \( y \) such that \( \Psi(x) \) is not in the cycle \( x_i \). Since the subgraph \( R \) is 1-regular. For every pair \((x,i)\), where \( x \in X \) and \( i \in \mathbb{Z}_m \), let \( y_i \) be the unique neighbor of the vertex \( x_i \) which is not in the cycle \( x_i x'_i x''_i x'''_i \) (similarly, for every pair \((x',i)\), where \( x \in X \) and \( i \in \mathbb{Z}_m \), let \( y'_{i} \) be the unique neighbor of the vertex \( x'_i \) which is not in the cycle \( x'_i x''_i x'''_i \)). Note that according to the structure of the graph \( G_\phi \), the vertex \( y_i \) is a clause vertex or a dummy vertex. Also, the vertex \( y'_{i} \) is a dummy vertex. Since the subgraph \( R \) is 1-regular, the degree sequence \( d\mathcal{I}(z_i) \) is 2323 or 3232 (otherwise the subgraph \( R \) is not locally irregular). Similarly, the degree sequence \( d\mathcal{I}(z_i')d\mathcal{I}(z_i'')d\mathcal{I}(z_i''') \) for every \( i \), \( E(R) \cap \{x_i y_i\} \neq \emptyset \) or \( E(R) \cap \{x'_i y'_i\} \neq \emptyset \). On the other hand, for every number \( i, i \in \mathbb{Z}_m \), \( s_i x_i''', s_i z_i''', z_i''' ) \in E(G_\phi) \), therefore \( E(R) \cap \{x_i y_i, x_i y'_i : i \in \mathbb{Z}_m \} \) is exactly \( \{x_i y_i : i \in \mathbb{Z}_m \} \) or \( \{x'_i y'_i : i \in \mathbb{Z}_m \} \) (Property A). Also, since \( R \) is 1-regular, for every clause vertex \( c_i, d\mathcal{I}(c_i) \) is 2 or 3. But \( d\mathcal{I}(c_i) \) is not 3. To the contrary, assume that \( d\mathcal{I}(c_i) \) is three and let \( u \) be a neighbor of \( c_i \) in \( R \). We have \( d\mathcal{I}(u) = 3 \), so \( d\mathcal{I}(c_i) = d\mathcal{I}(u) \), but this is a contradiction. Consequently, \( d\mathcal{I}(c_i) = 2 \) (Property B). Define \( \Psi : X \rightarrow \{true, false\} \) such that \( \Psi(x) = true \) if and only if \( x_i c_i \in E(R) \) for some \( i \). By Property A, the function \( \Psi \) is well-defined and by Property B, the function \( \Psi \) is a 1-in-3 satisfying assignment for the formula \( \Phi \).

On the other hand, assume that the function \( \Psi : X \rightarrow \{true, false\} \) is a 1-in-3 satisfying assignment for \( \Phi \). We show that there is a partition for the edge set of \( G_\phi \) into two subgraphs \( R \) and \( I \) such that \( R \) is regular and \( I \) is locally irregular. For every pair \((x,i)\), where \( x \in X \) and \( i \in \mathbb{Z}_m \), let \( w_i (w'_i, w''_i, w'''_i) \) be the unique neighbor of the vertex \( z_i \) (\( z'_i, z''_i, z'''_i \), respectively) which is not in the cycle \( z_i z'_i z''_i z'''_i \). Similarly, let \( y''_i (y'''_i, \text{respect.}) \) be the unique neighbor of the vertex \( x''_i (x'''_i, \text{respect.}) \) which is not in the cycle \( x_i x'_i x''_i x'''_i \). Now, for each \( x \in X \), put \( \{x_i y_i, x''_i y'_i, z''_i w_i, z'''_i w'_i : i \in \mathbb{Z}_m \} \) in \( E(R) \) if \( \Psi(x) = true \) and put \( \{x'_i y'_i, x''_i y''_i, z''_i w_i, z'''_i w'_i : i \in \mathbb{Z}_m \} \) in \( E(R) \) if \( \Psi(x) = false \). One can see that this is a decomposition of the graph \( F \) into two subgraphs \( R \) and \( I \) such that \( R \) is 1-regular and \( I \) is locally irregular. Note that the tree \( T \) can be decomposed into two subgraphs \( R \) and \( I \) such that \( R \) is 1-regular and \( I \) is locally irregular. This completes the proof. □
If $T$ is a tree which is not an odd length path, then its irregular chromatic index is at most three and also, there exist infinitely many trees with irregular chromatic index 3 [5]. Bensmail and Stevens proved that if $T$ is a tree, then its regular-irregular chromatic index is at most two. Here, for every $k > 2$, we construct a tree $T$ with $\Delta(T) = k$ such that $T$ cannot be decomposed into a matching and a locally irregular subgraph and also, we show that every tree can be decomposed into two matchings and a locally irregular subgraph.

**Proof of Theorem 2.** (i) Let $k > 2$ be a fixed number, we construct a tree $T$ with $\Delta(T) = k$ such that the tree $T$ cannot be decomposed into a matching and a locally irregular subgraph. First, consider the following auxiliary tree. Join two vertices $v$ and $u$ by an edge. Then consider four paths of lengths 3,3,3,1 and call them $P_1, P_2, P_3, P_4$, respectively. Identify one of the ends for each of $P_1$, $P_2$ with $v$, and finally identify one of the ends for each of $P_3$, $P_4$ with $u$. Call the resultant tree $T'$. The tree $T'$ has exactly one vertex of degree one such that its neighbor has degree three, call this vertex the bad vertex of the tree $T'$. Now, consider $k$ copies of the tree $T'$ and a new vertex $z$. Join the vertex $z$ to the bad vertex of each copy of $T'$ and call the resultant tree $T$. If the tree $T$ can be decomposed into a matching and a locally irregular subgraph, by the structure of $T'$, in each copy of $T'$, the edge between the bad vertex of $T'$ and $z$ should be in matching, but since the degree of $z$ is $k > 2$, this is a contradiction.

(ii) The proof is by induction on the number of edges in the tree. Assume that, for some integer $m \geq 2$, every tree with $m - 1$ edges can be decomposed into two subgraphs $R$ and $P$ such that $R$ is 1-regular and each component of $P$ is an edge or a locally irregular component. Let $T$ be a tree with $m$ edges. Choose an arbitrary vertex $z$ of $T$, and perform a breadth-first search algorithm from the vertex $z$. This defines a partition $V_0, V_1, \ldots, V_d$ of the vertices of $T$ where each part $V_i$ contains the vertices of $T$ which are at distance exactly $i$ from $z$. Assume that $v \in V_d, u \in V_{d-1}$ and $vu \in E(T)$. Let $T' = T \setminus \{vu\}$, by the inductive hypothesis, $T'$ can be decomposed into two subgraphs $R'$ and $P'$ such that $R'$ is 1-regular and each component of $P'$ is an edge or a locally irregular component. Without loss of generality suppose that $w \in V_{d-2}$ and $uw \in E(T)$. Three cases can be considered:

**Case 1.** If $d_{R'}(u) = 0$. Put $R = R' \cup \{uv\}$. In this case $(P = P', R = R' \cup \{uv\})$ is a suitable partition for $T$.

**Case 2.** If $d_{R'}(u) = 1$ and $uw \in R'$. Put $P = P' \cup \{uv\}$. It is easy to see that $(P = P' \cup \{uv\}, R = R')$ is a suitable partition for $T$.

**Case 3.** If $d_{R'}(u) = 1$ and $uw \notin R'$. Without loss of generality, assume that $e$ is incident with the vertex $u$ and $e \in R'$. Let $P'' = P' \cup \{uv\}$. According to the degrees of $u$ and $w$ in $P'$, one of the two partitions $(P'', R')$ or $(P'' \cup \{e\}, R' \setminus \{e\})$ is a suitable partition for $T$. This completes the proof.
(iii) Let $T$ be a tree, by (ii), $T$ can be decomposed into two subgraphs $\mathcal{P}$ and $\mathcal{R}$ such that $\mathcal{R}$ is a matching and each component of $\mathcal{P}$ is an edge or a locally irregular component. We can decompose $\mathcal{P}$ into a matching and a locally irregular subgraph, thus $T$ can be decomposed into two matchings and a locally irregular subgraph. This completes the proof.

In [6], Baudon et al. proved that determining whether a given planar graph $G$, can be decomposed into two locally irregular subgraphs is \textbf{NP}-complete. But their reduction does not preserve the planarity. We show that determining whether a given planar graph $G$, can be decomposed into two locally irregular subgraphs is \textbf{NP}-complete by a different reduction.

\textbf{Proof of Theorem 3.} We reduce \textbf{Monotone Planar 2-In-4 4-Sat} to our problem. Kara et al. [23] proved that the following problem is \textbf{NP}-complete.

\textit{Monotone Planar 2-In-4 4-Sat.}

\textbf{Instance:} A 4-Sat formula $\Phi = (X, C)$ such that there is no negation in the formula, and the bipartite graph obtained by linking a variable and a clause if and only if the variable appears in the clause, is planar.

\textbf{Question:} Is there a truth assignment for $X$ such that each clause in $C$ has exactly two true literals?

Let $\Psi = (X, C)$ be an instance of \textbf{Monotone Planar 2-In-4 4-Sat}. We denote the number of clauses containing the variable $x$ by $\gamma(x)$. We convert $\Psi$ into a planar graph $G_{\Psi}$ such that $\Psi$ has a 2-in-4 satisfying assignment if and only if the edge set of $G_{\Psi}$ can be decomposed into two locally irregular subgraphs. First, consider the gadget $K$ which is shown in Fig 2. We will use the gadget $K$ several times and it has some important properties.

\textbf{Lemma 1} Suppose that $G$ is a graph and has a copy of $K$ as an induced subgraph. Also, assume that the graph $G$ can be decomposed into two locally irregular subgraphs, then (up to symmetry) the set of black edges (see Fig 2) is in one part and the set of blue edges is in another part.

\textbf{Proof} By considering all possible cases, the proof is straightforward.

We will use the gadget $K$ in order to construct our main gadgets in our reduction.
Consider a cycle of length $\gamma(x)$ with the vertices $x_0, x_1, \ldots, x_{\gamma(x)} - 1$, in that order. For each $i, i = 0, 1, \ldots, \gamma(x) - 1$, replace the edge $x_ix_{(i+1 \mod \gamma(x))}$ of the cycle by a copy of the gadget $L(x_i, x_{(i+1 \mod \gamma(x))})$ ($L(v, u)$ is shown in Fig 2). Finally, for each $i, i = 0, 1, \ldots, \gamma(x) - 1$, put two new vertices $z_i, w_i$ and join the vertex $x_i$ to the vertices $z_i, w_i$. Also, join the vertex $z_i$ to the vertex $w_i$. Call the resultant auxiliary graph $A_{\gamma(x)}(x)$. There are exactly $\gamma(x)$ vertices $x'_0, x'_1, \ldots, x'_{\gamma(x)-1}$ of degree one in $A_{\gamma(x)}(x)$. We will call these vertices the important vertices of $A_{\gamma(x)}(x)$ (see Fig 2). Suppose that $G$ is a graph and has a copy of $A_{\gamma(x)}(x)$ as an induced subgraph. If the graph $G$ can be decomposed into two locally irregular subgraphs, then the set of edges incident with the important vertices is in a same part (Fact 1).

Construction of the clause gadget $B_c$.
Consider a copy of the gadget $A_{18}(v)$; also, add five vertices $c, \alpha_c, \beta_c, \gamma_c, \zeta_c$ and the set of edges $\{c\alpha_c, c\beta_c, c\gamma_c, c\zeta_c\}$. Next, consider 18 paths of lengths 2,2,4,2,2,2,2,4,2,4,4,4,4,4,4,4,4 and call them $P_0, P_1, \ldots, P_{17}$, respectively. Identify one of the ends of $P_0, P_1, P_2$ with $\alpha_c$. Identify one of the ends of $P_3, P_4, \ldots, P_8$ with $\beta_c$. Identify one of the ends of $P_9, P_{10}, P_{11}$ with $\gamma_c$ and identify one of the ends of $P_{12}, \ldots, P_{17}$ with $\zeta_c$. For each $i, i = 0, 1, \ldots, 17$, identify the other end of path $P_i$ with the important vertex $v'_i$ of $A_{18}(v)$. Call the resultant gadget $B_c$. See Fig. 3.

By the structure of the gadget $B_c$, if the graph $G$ can be decomposed into two locally irregular subgraphs $I_1, I_2$, then exactly two of the edges $c\alpha_c, c\beta_c, c\gamma_c, c\zeta_c$ are in $I_1$ (Fact 2). Note that the vertices $\alpha_c, \beta_c, \gamma_c, \zeta_c$ are incident with 4,7,4,7 edges respectively and exactly 3,6,1,1 of them are in $I_1$ or vice-versa respectively (Fact 3).
Now, we are ready to define the graph $G_\Psi$. For every variable $x \in X$, put a copy of the gadget $A_\gamma(x)$ and for each clause $c \in C$, put a copy of the gadget $B_c$. For every pair $(x, c)$, if $x$ appears in $c$, then put an edge between the vertex $c$ of the gadget $B_c$ and one of the important vertices $x'_0, x'_1, \ldots, x'_{\gamma(x)}$ of the gadget $A_\gamma(x)$, such that having done this procedure for all pairs, the degree of each important vertex is two. Call the resultant planar graph $G_\Psi$. Let $c = (x \lor y \lor z \lor w)$ be an arbitrary clause and without loss of generality suppose that $cx'_0, cy'_0, cz'_0, cw'_0 \in E(G_\Psi)$. By Fact 2 and Fact 3, if the graph $G_\Psi$ can be decomposed into two locally irregular subgraphs $I_1, I_2$, then exactly two of the edges $cx'_0, cy'_0, cz'_0, cw'_0$ are in $I_1$. Thus, we can find a 2-in-4 satisfying assignment. On the other hand, assume that the formula $\Psi$ has a 2-in-4 satisfying assignment $\Gamma$. For a given clause $c = (x \lor y \lor z \lor w)$, without loss of generality suppose that $cx'_0, cy'_0, cz'_0, cw'_0 \in E(G_\Psi)$. Then for each literal $v$ ($v \in \{x, y, z, w\}$) put $cv'_0$ in $I_1$ if and only if $\Gamma(v) = true$. One can extend this to a proper decomposition. This completes the proof.

\[\square\]

It was proved that computation of the regular number is $\textbf{NP}$-hard for connected bipartite graphs \[15\]. Also, it was shown that deciding whether $\text{reg}(G) = 2$ for a given connected 3-colorable graph $G$ is $\textbf{NP}$-complete \[15\]. We improve these two results, and show that for a given bipartite graph $G$ with maximum degree six, deciding whether $\text{reg}(G) = 2$ is $\textbf{NP}$-complete. Furthermore, we present a polynomial time algorithm to decide whether $\text{reg}(G) = 2$ for a given graph $G$ with maximum degree five.

\textbf{Proof of Theorem 4.} (i) It has been shown that the following version of \textit{Not-All-Equal}
(NAE) satisfying assignment problem is NP-complete [15].

Cubic Monotone NAE (2,3)-Sat.

Instance: Set $X$ of variables, collection $C$ of clauses over $X$ such that each clause $c \in C$ has $|c| \in \{2,3\}$, every variable appears in exactly three clauses and there is no negation in the formula.

Question: Is there a truth assignment for $X$ such that each clause in $C$ has at least one true literal and at least one false literal?

Let $\alpha \geq 3$ be a fixed number. We reduce Cubic Monotone NAE (2,3)-Sat to our problem in polynomial time. Consider an instance $\Phi$, we transform this into a bipartite graph $G_\Phi$ in polynomial time such that $reg(G_\Phi) = 2$ if and only if $\Phi$ has an NAE truth assignment. We use two auxiliary gadgets $H^\alpha_c$ and $I^\alpha_c$. See Fig. 4.

Figure 4: The two auxiliary gadgets $H^3_c$ and $I^3_c$. The gadget $H^3_c$ is on the left side.

Construction of the gadget $I^\alpha_c$.

Consider two copies of the complete bipartite graph $K_{\alpha,\alpha}$ and call them $K[X,Y], K'[X',Y']$, where $X = \{x_i : 1 \leq i \leq \alpha\}, X' = \{x'_i : 1 \leq i \leq \alpha\}, Y = \{y_i : 1 \leq i \leq \alpha\}, Y' = \{y'_i : 1 \leq i \leq \alpha\}$. Add two vertices $b_c$ and $b'_c$. Join the vertex $b_c$ to the vertices $x_1, \ldots, x_{\alpha-1}, x'_1, \ldots, x'_{\alpha-1}$ and join the vertex $b'_c$ to the vertices $y_1, \ldots, y_{\alpha-1}, y'_1, \ldots, y'_{\alpha-1}$. Finally, remove a perfect matching between the two sets of vertices $x_1, \ldots, x_{\alpha-1}, x'_1, \ldots, x'_{\alpha-1}$ and $y_1, \ldots, y_{\alpha-1}, y'_1, \ldots, y'_{\alpha-1}$. Call the resulting gadget $I^\alpha_c$. See Fig. 4.

Note that in the gadget $I^\alpha_c$ the degrees of all vertices except $b_c$ and $b'_c$ are $\alpha$. Also, the degrees of the vertices $b_c$ and $b'_c$ are $2\alpha - 2$.

Construction of the gadget $H^\alpha_c$.

Consider two copies of the complete bipartite graph $K_{\alpha,\alpha}$ and call them $K[X,Y], K'[X',Y']$,
where \( X = \{ x_i : 1 \leq i \leq \alpha \} \), \( X' = \{ x'_i : 1 \leq i \leq \alpha \} \), \( Y = \{ y_i : 1 \leq i \leq \alpha \} \), \( Y' = \{ y'_i : 1 \leq i \leq \alpha \} \). Add two vertices \( a_c \) and \( a'_c \). Identify the vertex \( x_\alpha \) with the vertex \( x'_\alpha \). Join the vertex \( a_c \) to the vertices \( x_1, \ldots, x_{\alpha-1}, x'_1, \ldots, x'_{\alpha-1} \) and join the vertex \( a'_c \) to the vertices \( y_1, \ldots, y_{\alpha-1}, y'_1, \ldots, y'_{\alpha-1} \). Finally, remove a perfect matching between the set of vertices \( x_1, \ldots, x_{\alpha-1}, x'_1, \ldots, x'_{\alpha-2} \) and the set of vertices \( y_1, \ldots, y_{\alpha-1}, y'_1, \ldots, y'_{\alpha-2} \). Call the resulting gadget \( \mathcal{H}_c^\alpha \). See Fig. 4.

Note that in the gadget \( \mathcal{H}_c^\alpha \) the degrees of all vertices except \( a_c \) and \( a'_c \) are \( \alpha \). Also, the degrees of the vertices \( a_c \) and \( a'_c \) are \( 2\alpha - 3 \). The graph \( G_\Phi \) has a copy of the gadget \( \mathcal{H}_c^\alpha \) for each clause \( c \in C \) with \( | c | = 3 \), and a copy of the gadget \( \mathcal{T}_c^\alpha \) for each clause \( c \in C \) with \( | c | = 2 \). Also, for each variable \( x \in X \), put two vertices \( x \) and \( x' \) and consider a copy of the complete bipartite graph \( K_{\alpha, \alpha} \) and call it \( K[X, Y] \), where \( X = \{ x_i : 1 \leq i \leq \alpha \} \), \( Y = \{ y_i : 1 \leq i \leq \alpha \} \). Join the vertex \( x \) to the vertices \( x_1, \ldots, x_{\alpha-3} \) and join the vertex \( x' \) to the vertices \( y_1, \ldots, y_{\alpha-3} \). Next, remove a perfect matching between the set of vertices \( x_1, \ldots, x_{\alpha-3} \) and the set of vertices \( y_1, \ldots, y_{\alpha-3} \). Finally, for each clause \( c = (y \lor z \lor w) \), where \( y, w, z \in X \) add the edges \( a_c y, a'_c y', a_c z, a'_c z', a_c w \) and \( a'_c w' \). Also, for each clause \( c = (y \lor z) \), where \( y, w \in X \) add the edges \( b_c y, b'_c y', b_c z \) and \( b'_c z' \). Call the resultant graph \( G_\Phi \). The degree of every vertex in \( G_\Phi \) is \( \alpha \) or \( 2\alpha \) and the graph is bipartite. Note that there are no two adjacent vertices of degree \( 2\alpha \).

First, assume that reg\((G_\Phi) = 2 \) and let \( G_1 \) and \( G_2 \) be a regular decomposition of the graph \( G_\Phi \) such that \( G_i \) is \((r_i)\)-regular, for each \( i, i = 1, 2 \). The graph \( G_\Phi \) has vertices with degrees \( \alpha \) and \( 2\alpha \), so \( r_1 = r_2 = \alpha \). For every \( x, x \in X \), the vertex \( x \) has degree \( \alpha \), so all edges incident with the vertex \( x \) are in the same part. For every \( x \in X \), if all edges incident with the vertex \( x \) are in \( G_1 \), put \( \Gamma(x) = true \) and if all edges incident with the vertex \( x \) are in \( G_2 \), put \( \Gamma(x) = false \). According to the construction of \( \mathcal{H}_c^\alpha \), \( (\mathcal{T}_c^\alpha \), respectively), the set of edges \( \{a_c x_1, \ldots, a_c x_{\alpha-1}\} \) \( \{b_c x_1, \ldots, b_c x_{\alpha-1}\} \), respectively) is in one part and the set of edges \( \{a_c x'_1, \ldots, a_c x'_{\alpha-2}\} \) \( \{b_c x'_1, \ldots, b_c x'_{\alpha-2}\} \), respectively) is in another part. Therefore, for every clause \( c = (y \lor z \lor w) \), at most two of the three edges \( a_c y, a_c z \) and \( a_c w \) are in \( G_1 \). Also, at most two of the three edges \( a_c y, a_c z \) and \( a_c w \) are in \( G_2 \). Similarly, for every clause \( c = (y \lor z) \), exactly one of the two edges \( b_c y \) and \( b_c z \) is in \( G_1 \) (Note that for every clause \( c = (y \lor z \lor w) \), at most two of the three edges \( a'_c y', a'_c z' \) and \( a'_c w' \) are in \( G_1 \) and similarly exactly one of the two edges \( b'_c y' \) and \( b'_c z' \) is in \( G_1 \)). Hence, \( \Gamma \) is an NAE satisfying assignment. On the other hand, suppose that the formula \( \Phi \) has an NAE satisfying assignment \( \Gamma : X \rightarrow \{true, false\} \). For every variable \( x \in X \), put all edges incident with the the vertices \( x \) and \( x' \) in \( G_1 \) if and only if \( \Gamma(x) = true \). By this method, it is easy to show that \( G_\Phi \) can be decomposed into two regular subgraphs. This completes the proof.

(ii) For a given connected graph \( G \), assume that \( \Delta(G) = 5 \) and let \( G_1 \) and \( G_2 \) be a
regular decomposition of the graph $G$ such that $G_i$ is $(r_i)$-regular, for each $i$, $i = 1, 2$. If $G$ is not a 5-regular graph, then two cases can be considered:

Case 1.1: $r_1 = 2$ and $r_2 = 3$. In this case the degree set of the graph must be a subset of $\{2, 3, 5\}$. Let $G'$ be the induced graph on the set of vertices of degrees two and five. A subgraph $F$ of a graph $G'$ is called a factor of $G'$ if $F$ is a spanning subgraph of $G'$. If a factor $F$ has all of its degrees equal to $k$, it is called a $k$-factor. Thus a 2-factor is a disjoint union of finitely many cycles that cover all vertices of $G'$. It is well-known that the problem of finding a 2-factor can be solved in polynomial time by matching techniques. In other words, a 2-factor can be constructed in polynomial time if the answer is YES (see [33], page 141). The graph $G$ can be decomposed into a 2-regular and a 3-regular graphs if and only if the graph $G'$ has a 2-factor.

Case 1.2: $r_1 = 1$ and $r_2 = 4$. In this case the degree set of graph must be a subset of $\{1, 4, 5\}$. Let $G'$ be the induced graph on the set of vertices of degrees one and five. It is easy to see that the graph $G$ can be decomposed into a 1-regular and a 4-regular graphs if and only if the graph $G'$ has a perfect matching. Finding the maximum matching is in $P$ (see [33], page 145), hence, the proof is completed.

Now, assume that $\Delta(G) = 4$. If the graph $G$ is not a 4-regular graph, then two cases can be considered:

Case 2.1: $r_1 = 2$ and $r_2 = 2$. In this case the degree set of graph must be a subset of $\{2, 4\}$. Consider the graph $H$ with the vertex set $\{v|d(v) = 4, v \in V(G)\}$ and join two vertices $v$ and $u$ in $H$ if and only if there is a path $vx_1 \ldots x_tu$ in $G$ such that $d(x_1) = \cdots = d(x_t) = 2$ (note that if $vu \in E(G)$, then we join the vertex $v$ to the vertex $u$). A $k$-factorization of $H$ is a partition of the edges of $H$ into disjoint $k$-factors. For $k \geq 1$, every $2k$-regular graph admits a 2-factorization (see [33], page 140), therefore $H$ has a 2-factor, thus the graph $G$ has a 2-factor. Consequently, in this case the graph $G$ always can be decomposed into two regular subgraphs.

Case 2.2: $r_1 = 1$ and $r_2 = 3$. In this case the degree set of graph must be a subset of $\{1, 3, 4\}$. Let $G'$ be the induced graph on the set of vertices of degree one and four. It is easy to see that the graph $G$ can be decomposed into a 1-regular and a 3-regular subgraphs if and only if the graph $G'$ has a perfect matching. Finding the maximum matching is in $P$ (see [33], page 145), consequently the proof is completed.

Now, assume that $\Delta(G) = 3$. If $G$ is not a 3-regular graph, then $r_1 = 1$ and $r_2 = 2$. Let $G'$ be the induced graph on the set of vertices of degrees one and three. It is easy to see that the graph $G$ can be decomposed into a 1-regular and a 2-regular graphs if and only if the graph $G'$ has a perfect matching. The other cases for $\Delta(G) = 1$ and $\Delta(G) = 2$ are trivial. This completes the proof.
Note that in the proof of previous theorem from Case 2.1, we have the following corollary.

**Corollary 3** Let $G$ be graph with degree set $\{2, 4\}$. The edge set of the graph $G$ can be decomposed into two subgraphs such that each subgraph is 2-regular.

Next, we consider the problem of determining the regular number for planar graphs.

**Proof of Theorem 5.** Clearly, the problem is in NP. We reduce *Monotone Planar 2-In-4 4-Sat* to our problem. Kara et al. [23] proved that the following problem is NP-complete.

*Monotone Planar 2-In-4 4-Sat.*

**Instance:** A 4-Sat formula $\Psi = (X, C)$ such that there is no negation in the formula, and the bipartite graph obtained by linking a variable and a clause if and only if the variable appears in the clause, is planar.

**Question:** Is there a truth assignment for $X$ such that each clause in $C$ has exactly two true literals?

Let $\Psi = (X, C)$ be an instance of *Monotone Planar 2-In-4 4-Sat.* We denote the number of the clauses containing the variable $x$ by $\gamma(x)$. We convert $\Psi$ into a planar graph $G_\Psi$ with degree set $\{3, 6\}$ such that $\Psi$ has a 2-in-4 satisfying assignment if and only if the edge set of $G_\Psi$ can be decomposed into two regular subgraphs. For each variable $x$, consider a cycle of length $\gamma(x)$ with the vertices $x_1, \ldots, x_{\gamma(x)}$, in that order and for each clause $c$ consider a copy the gadget $Z_c$ which is shown in Fig 5. Finally, for every pair $(x, c)$, where $x \in X$ and $c \in C$, if $x$ appears in $c$, join the vertex $a_c$ to one of the vertices $x_1, x_2, \ldots, x_{\gamma(x)}$, such that in the resulting graph the degree of every vertex in $\{x_1, x_2, \ldots, x_{\gamma(x)}\}$ is three. Call the resultant planar graph $G_\Psi$.

![Figure 5: The auxiliary gadget $Z_c$.](image-url)
First, assume that \( \text{reg}(G_\Psi) = 2 \) and let \( G_1 \) and \( G_2 \) be a regular decomposition of the graph \( G_\Psi \) such that \( G_i \) is \( (r_i) \)-regular, for each \( i, i = 1, 2 \). The graph \( G_\Psi \) has vertices with degrees 3 and 6, so \( r_1 = r_2 = 3 \). For every \( x \in X \), the degrees of vertices \( x_1, x_2, \ldots, x_{\gamma(x)} \) are 3, thus, all edges incident with one of the vertices \( x_1, x_2, \ldots, x_{\gamma(x)} \) are in the same part. For every \( x \in X \), if all edges incident with the vertex \( x_1 \) are in \( G_1 \), put \( \Gamma(x) = \text{true} \) and if all edges incident with the vertex \( x_1 \) are in \( G_2 \), put \( \Gamma(x) = \text{false} \). According to the construction of \( Z_c \) the set of black edges are in one part and the set of blue edges are in another part. Therefore, for every clause \( c = (x \lor y \lor z \lor w) \), exactly two edges from the four edges \( a_c x, a_c y, a_c z \) and \( a_c w \) are in the subgraph \( G_1 \). Hence, the function \( \Gamma \) is a 2-in-4 satisfying assignment. On the other hand, suppose that the formula \( \Phi \) is 2-in-4 satisfiable with the satisfying assignment \( \Gamma : X \to \{\text{true}, \text{false}\} \). For every variable \( x \in X \), put all edges incident with the vertices \( x_1, x_2, \ldots, x_{\gamma(x)} \) in \( G_1 \) if and only if \( \Gamma(x) = \text{true} \), also for every gadget \( Z_c \), put the black edges in \( G_1 \) and the blue edges in \( G_2 \). By this method, it is easy to show that the graph \( G_\Psi \) can be decomposed into two regular subgraphs. This completes the proof.

\[\square\]

It was shown that determining whether \( \text{reg}(G) \leq \Delta(G) \) for a given connected graph \( G \) is \( \text{NP} \)-complete [15]. Here, we show that every graph \( G \) can be decomposed into \( \Delta(G) \) subgraphs such that each subgraph is locally regular and this bound is sharp for trees.

**Proof of Theorem 6.** We use the concept of semi-coloring to prove our theorem. Daniely and Linial defined a semi-coloring of graphs for the investigation of the tight product of graphs [12]. Afterwards, Furuya et al. proved that every graph has a semi-coloring [19]. Let \( G \) be a graph. For \( i \in \{1, 2\} \), let \( \binom{\Delta(G)}{i} \) denotes the family of subsets of \( \{1, 2, \ldots, \Delta(G)\} \) with cardinality exactly \( i \). A semi-coloring of the graph \( G \) is a coloring

\[\ell : E(G) \to \binom{\Delta(G)}{1} \cup \binom{\Delta(G)}{2}\]

such that for every \( v \in V(G) \),

1. For each \( i, i = 1, 2, \ldots, \Delta(G) \), \( \sum_{e \ni v} w_i(e) \in 0, 1 \), where

\[w_i(e) = \begin{cases} \frac{1}{|x(e)|} & i \in \ell(e) \\ 0 & i \notin \ell(e) \end{cases}\]

and

2. For any \( 1 \leq i < j \leq \Delta(G) \), \( |\{e : e \ni v, \ell(e) = \{i, j\}\}| \in \{0, 2\} \).
Let $G$ be a graph with maximum degree $\Delta(G)$ and semi-coloring $\ell$. Define the following decomposition for the edges of the graph $G$.

For each $i$, $i = 1, 2, \ldots, \Delta(G)$, $\mathcal{P}_i = \{e : \ell(e) = \{i\}\} \cup \{e : \ell(e) = \{i, j\}, i < j\}$.

For any semi-coloring $\ell$ of the graph $G$ and any $i, j$, where $1 \leq i < j \leq \Delta(G)$, each component of the subgraph of $G$ induced by edges with the color $\{i, j\}$ is a singleton or a cycle. So, each component of each part $\mathcal{P}_i$ is an edge or a cycle. Thus, the regular chromatic index of the graph $G$ is at most $\Delta(G)$. Since every graph has a semi-coloring [19], so our proof is completed.

Now, let $T$ be a tree. The tree $T$ does not have any cycle, so in every edge decomposition, each part is a matching. On the other hand, the edge chromatic number of the tree $T$ is equal to $\Delta(T)$ [33], therefore, $\chi_{\text{reg}}(T) = \Delta(T)$. This completes the proof.

In Theorem 4, we prove that there is polynomial time algorithm to decide whether $\text{reg}(G) = 2$ for a given graph $G$ with maximum degree five. Here, we show that deciding whether a given subcubic graph can be decomposed into two subgraphs such that each subgraph is locally regular, is $\text{NP}$-complete.

**Proof of Theorem 7.** We reduce Cubic Monotone NAE $(2,3)$-Sat to our problem. It is shown that the following version of NAE satisfying assignment problem is $\text{NP}$-complete [15].

**Cubic Monotone NAE $(2,3)$-Sat.**

**Instance:** Set $X$ of variables, collection $C$ of clauses over $X$ such that each clause $c \in C$ has $|c| \in \{2, 3\}$, every variable appears in exactly three clauses and there is no negation in the formula.

**Question:** Is there a truth assignment for $X$ such that each clause in $C$ has at least one true literal and at least one false literal?

Let $\Phi = (X, C)$ be an instance of Cubic Monotone NAE $(2,3)$-Sat. We convert the formula $\Phi$ into a graph $G$ such that $\Phi$ has an NAE satisfying assignment if and only if the edge set of $G$ can be decomposed into two subgraphs such that each subgraph is locally regular. We use two gadgets $U(c)$ and $W(x)$. The gadget $U(c)$ is shown in Fig. 6.

**Construction of $W(x)$**

Consider a cycle $C_6$, with vertices $v_1, u_1, v_2, u_2, v_3, u_3$, in that order. For each $i$, $1 \leq i \leq 3$, put a triangle and join one the vertices of that triangle to the vertex $u_i$. The resultant graph has nine vertices with degree two and six vertices with degree three. The three
edges between triangles and $C_6$ are called blue edges and other edges of that graph are called black edges. Call the resultant graph $S(x)$. Now consider two copies of $S(x)$ and rename the vertices $v_1, v_2, v_3$ ($v_1, v_2, v_3$) in the first copy (second copy) of $S(x)$, by $x_1, x_2, x_3$ ($x'_1, x'_2, x'_3$), respectively. Call the resulting gadget $W(x)$.

Construction of the graph $G$

Let $\Phi = (X, C)$ be an instance of Cubic Monotone NAE $(2,3)$-Sat. For every clause $c \in C$, if $|c| = 3$, then put a copy of the gadget $U(c)$ and if $|c| = 2$ put two vertices $c$ and $c'$. Also, for every variable $x \in X$, put a copy of the gadget $W(x)$. Now, for any clause $c$ containing $x$, choose an index $i$, $1 \leq i \leq 3$, if $|c| = 2$, then put the two edges $cx_i, c'x'_i$ and if $|c| = 3$ choose an index $j$, $1 \leq j \leq 3$ and add the two edges $c_jx_i, c'_jx'_i$. Do these so that in the resultant graph $G$, the degree of every vertex is at most 3.

Suppose that the graph $G$ can be decomposed into two subgraphs such that each subgraph is locally regular. Since $G$ is subcubic, each component of every subgraph is an edge or a cycle. Thus, in each copy of the gadget $W(x)$, the set of blue edges is in one subgraph and the set of black edges is in another subgraph (Property 1). For each clause $c = (x, y)$, without loss of generality suppose that $cx_1, cy_1 \in E(G)$. By the structure of $G$, the two edges $cx_1$ and $cy_1$ are in different subgraphs, similarly, for every clause $c = (x, y, z)$, without loss of generality suppose that $c_1x_1, c_2y_1, c_3z_1 \in E(G)$. By the structure of $U(c)$, the edges $c_1x_1, c_2y_1, c_3z_1$ are not in the same subgraph (Property 2). Now, assume that $G$ can be decomposed into two subgraphs $G_1$ and $G_2$ such that each subgraph is locally regular. For every $x \in X$, if the two edges incident with $x_1$ in the gadget $W(x)$ are in $G_1$, put $\Gamma(x) = true$ and otherwise put $\Gamma(x) = false$. By Property 1 and Property 2, it is easy to check that $\Gamma(x)$ is an NAE assignment.

□
Here, we prove that for each $k > 1$, deciding whether $\chi_{k-irr}^t(G) = 2$ for a given planar bipartite graph $G$ is NP-complete. For all $k$ we prove the lower bound $h(k) \geq 2k + 1$ and we will use mutually orthogonal Latin squares and prove that $h(k) = \Omega(k^2)$.

**Proof of Theorem 8.** (i) We reduce *Monotone Planar 2-In-4 4-Sat* to our problem. Kara *et al.* [23] proved that the following problem is NP-complete.

**Monotone Planar 2-In-4 4-Sat.**

**Instance:** A 4-Sat formula $\Phi = (X, C)$ such that there is no negation in the formula, and the bipartite graph obtained by linking a variable and a clause if and only if the variable appears in the clause, is planar.

**Question:** Is there a truth assignment for $X$ such that each clause in $C$ has exactly two true literals?

Assume that $k \geq 2$ is a fixed integer. Let $\Phi = (X, C)$ be an instance of *Monotone Planar 2-In-4 4-Sat*. We convert the formula $\Phi$ into a graph $G_k$ such that the formula $\Phi$ has a 2-in-4 satisfying assignment if and only if the edge set of the graph $G_k$ can be decomposed into two locally $k$-irregular subgraphs. First, we introduce two useful gadgets $A_\alpha$ and $B(c)$. We denote the number of the clauses containing the variable $x$ by $\gamma(x)$.

**Construction of $A_\alpha$**

Let $C_{4\alpha}$ be a cycle with $4\alpha$ vertices $v_1, v'_1, u_1, u'_1, \ldots, v_\alpha, v'_\alpha, u_\alpha, u'_\alpha$ in that order. For each $i, i = 1, 2, \ldots, \alpha$, put $2k - 2$ new vertices $w_i^1, z_i^1, \ldots, w_{k-1}^i, z_{k-1}^i$ and join the vertex $v_i$ to the vertices $w_1^i, \ldots, w_{k-1}^i$; also, join the vertex $u_i$ to the vertices $z_1^i, \ldots, z_{k-1}^i$. Call the resulting bipartite graph $A_\alpha$. In the gadget $A_\alpha$, call the set of vertices $w_1^i, w_1^2, w_1^3, \ldots, w_1^\alpha$, *main vertices* and call the set of edges incident with the main vertices, *main edges*.

![Figure 7: The auxiliary gadget $A_4$, where $k = 4.$](image-url)
Construction of $\mathcal{B}(c)$

Let $P_5$ be a path with vertices $p_1, p_2, \ldots, p_5$, in that order. Put $2k - 2$ new vertices $q_1, q'_1, q_2, q'_2, \ldots, q_{k-1}, q'_{k-1}$, join the vertex $p_2$ to the vertices $q_1, q_2, \ldots, q_{k-1}$ and join the vertex $p_4$ to the vertices $q'_1, q'_2, \ldots, q'_{k-1}$. Call the resulting graph $\mathcal{D}$. Note that in $\mathcal{D}$, we have $d(p_1) = d(p_5) = 1$. Now put a new vertex $c$ and $k - 1$ copies of $\mathcal{D}$. In each copy of $\mathcal{D}$ join the vertices $p_1$ and $p_5$ to the vertex $c$. Call the resulting graph $\mathcal{B}(c)$. See Fig. 8.

![Figure 8: The auxiliary gadget $\mathcal{B}(c)$, where $k = 4$.](image)

In the next, we introduce the construction of the graph $G_k$. For every variable $x \in X$, put a copy of the gadget $\mathcal{A}_\gamma(x)$ (we call this copy of the gadget $\mathcal{A}_\gamma(x)$, the gadget corresponds to the variable $x$) and for each clause $c \in C$, put a copy of the gadget $\mathcal{B}(c)$. Now, for any clause $c$ containing $x$, connect one of the main vertices of a copy of $\mathcal{A}_\gamma(x)$ corresponding to the variable $x$ to the vertex $c$ of $\mathcal{B}(c)$. Do these procedures for all variables $x$ and all clauses $c$, in such a way that in the resultant graph $G_k$ the degree of every main vertex is two and for every $c \in C$ the degree of vertex $c$ is $2k + 2$.

First, suppose that the graph $G_k$ can be decomposed into two locally $k$-irregular subgraphs $\mathcal{I}$ and $\mathcal{I}'$. Since the degree set of $G_k$ is $\{1, 2, k + 1, 2k + 2\}$, by the structure of the graph $G_k$, for every vertex $v$ of degree two, if $e, e' \ni v$, then $e \in E(\mathcal{I})$ and $e \in E(\mathcal{I}')$ or vice versa. By the structure of $\mathcal{A}_\alpha$, for each copy of $\mathcal{A}_\alpha$, the set of its main edges is in $\mathcal{I}$ or $\mathcal{I}'$ (Fact 1). Furthermore, by the structure of $\mathcal{D}$, if we consider the induced graph on the set of vertices $V(\mathcal{B}(c))$, we have $d_{\mathcal{I}}(c) = d_{\mathcal{I}'}(c) = k - 1$ (Fact 2).

Now, assume that $G_k$ can be decomposed into two locally $k$-irregular subgraphs $\mathcal{I}$, $\mathcal{I}'$. Let $\Gamma : X \to \{true, false\}$ be a function such that $\Gamma(x) = true$ if and only if the set of main edges of the gadget $\mathcal{A}_\gamma(x)$ corresponding to the variable $x$ is in $\mathcal{I}$. By Fact 1 and 2, it is easy to see that $\Gamma$ is a 2-in-4 satisfying assignment. Next, suppose that $\Gamma : X \to \{true, false\}$ is a 2-in-4 satisfying assignment. For every variable $x$ put the
set of main edges of the gadget $A_{v(x)}$ corresponds to the variable $x$ in $I$ if and only if $\Gamma(x) = true$. It is easy to see that $I$ can be extended to a $k$-irregular graph such that $G \setminus E(I)$ is also a $k$-irregular graph. This completes the proof.

(ii) In order to show that $h(k) \geq 2k + 1$, it is enough to present a graph $G$ such that $\chi'_{k-irr}(G) \geq 2k + 1$. Consider a cycle $C = v_1v_2v_3$. Put $k - 1$ new vertices and join them to the vertex $v_1$, also put $2k$ new vertices and join them to the vertex $v_2$. Next, put $2k - 1$ vertices $u_1, u_2, \ldots, u_{2k-1}$ and join them to the vertex $v_3$. Finally, for every $i, i = 1, 2, \ldots, 2k - 1$ put $k$ new vertices and join them to the vertex $u_i$. Call the resultant graph $G$. It is easy to check that $G$ can be decomposed into locally $k$-irregular graphs. Suppose that $\chi'_{k-irr}(G) < 2k + 1$ and let $E_1, E_2, \ldots, E_t$ be a decomposition of $E(G)$ such that $G[E_i]$ is locally $k$-irregular for every $i = 1, 2, \ldots, t$.

There are vertices of degree one in the neighbors of vertex $u_1$, choose one of these vertices call it $z$. Let $I$ be the induced graph on the set of edges $E_1$ and without loss of generality assume that $d_{I}(z) = 1$. Since $d_{G}(u_1) = k + 1$, we have $d_{I}(u_1) = k + 1$. Since $\chi'_{k-irr}(G) < 2k + 1$, we have $d_{I}(v_3) = 2k + 1$. Therefore $v_1v_3, v_2v_3 \in E_1$. It is easy to see that $d_{I}(v_1) = k + 1$ (otherwise we obtain a contradiction). Hence $v_1v_2 \in E_1$. Therefore $d_{I}(v_2) \geq 2$, but this is a contradiction. Thus $\chi'_{k-irr}(G) \geq 2k + 1$.

Now, assume that $k \geq 4$. Consider a cycle $C = v_1v_2v_3$. Put $k - 2$ new vertices and join them to the vertex $v_1$, also put $2k - 2$ new vertices $u_1, u_2, \ldots, u_{2k-2}$ and join them to the vertex $v_2$. Next, put $3k - 2$ vertices and join them to the vertex $v_3$. Finally, for every $i, i = 1, 2, \ldots, 2k - 3$ (note that $i \neq 2k - 2$), put $k$ new vertices and join them to the vertex $u_i$. Call the resultant gadget $S$. Now, consider $4k$ copies of $S$ and three isolated vertices $z_1, z_2, z_3$. Next, for every $i, i = 1, 2, 3$, join the vertex $z_i$ to the vertex $v_i$ in each copy of $S$. Call the resulting graph $G$. It is easy to check that $G$ can be decomposed into locally $k$-irregular subgraphs and in every decomposition of $G$, all edges of a copy of the gadget $S$ are in the same subgraph. Thus, by the structure of the graph $G$, $\chi'_{k-irr}(G) = 4k$.

(iii) Let $k$ be a sufficiently large number and $p$ be a prime number such that $0.1k \leq p \leq 0.2k$. (Bertrand’s postulate states that for any integer $d > 3$, there always exists at least one prime number $p$ with $d < p < 2d - 2$ [17]).

A Latin square of order $n$ is an $n \times n$ matrix such that every element of $\{1, 2, \ldots, n\}$ occurs exactly once in each row and each column. A set of Latin squares is called mutually orthogonal Latin squares (MOLS) if every pair of its element Latin squares is orthogonal to each other [21]. (Two Latin squares $L_1$ and $L_2$ are orthogonal if for any $(i, j)$, there exists unique $(k, l)$ such that $L_1(k, l) = i$ and $L_2(k, l) = j$.)
Let $\mathcal{L}^1, \ldots, \mathcal{L}^{p-1}$ be a set of MOLS of order $p$, with elements from $\{1, \ldots, p\}$ (If $p$ is prime, then there exist $p-1$ MOLSs of order $p$ [21]). Consider $p$ copies of the complete graph $K_{[k/2]+1}$ and let the vertex set of the $\theta$th copy of $K_{[k/2]+1}$ be $\{v_i^\theta : 1 \leq i \leq [k/2]+1\}$. For each pair $(\alpha, i)$, where $1 \leq \alpha \leq p$ and $1 \leq i \leq [k/2]+1$, add $[k/2]+i$ new vertices $\{w_{ij}^\alpha : 1 \leq j \leq [k/2]+i\}$ to the graph and join them to the vertex $v_i^\theta$ (so the degree of the vertex $v_i^\theta$ is $k+i$). Finally, for every $i, j, r$ and $a$ identify the vertices $w_{ij}^1$ and $w_{ar}^j$ if and only if the $(i, j)$ element of $(r-1)$th Latin square is $a$ (for a Latin square $L$ of order $n$, the element on the $i$th row and the $j$th column is denoted by $(i, j)$ element of $L$). Call the resultant graph $G_k$.

Now, we show $\chi'_{k-irr}(G_k) \geq \Omega(k^2)$. By the structure of $K_{[k/2]+1}$ and their incident leaves, all of the edges $v_{ij}^\alpha w_{ij}^\alpha$, $1 \leq j \leq [k/2]+i$ appear in one part; and the degree of the vertex $w_{ij}^\alpha$ in that part should be one (Property A).

Also, by the structure of the graph, for every $\alpha$, the two edges $v_{ij}^\alpha w_{ij}^\alpha$ and $v_{ij}^\alpha w_{ij}^\beta$ should appear in different parts (Property B).

By the structure of MOLS, for every $\alpha, \alpha', \beta, \beta'$, $2 \leq \alpha < \alpha' \leq p, 1 \leq \beta \leq \beta' \leq p$ there are $i, j$ such that $(i, j)$ element of $(\alpha-1)$th Latin square is $\beta$ and $(i, j)$ element of $(\alpha'-1)$th Latin square is $\beta'$. So, $w_{ij}^1$ and $w_{ij}^\alpha$ were merged; also $w_{ij}^1$ and $w_{ij}^\beta$ were merged. Thus, two vertices $w_{ij}^\alpha$ and $w_{ij}^\beta$ were merged. Therefore by Property A, the two edges $v_{ij}^\beta w_{ij}^\beta$ and $v_{ij}^\beta w_{ij}^\beta$ appear in two different parts. By this fact and Property B, we have $\chi'_{k-irr}(G_k) \geq \Omega(k^2)$.

(iv) Let $k$ be a fixed number and $G$ be a graph with maximum degree $k+1$. If $G$ can be decomposed into two locally $k$-irregular subgraphs, then $G$ meets the following three necessary conditions.

**Condition A.** There are no two adjacent vertices of degrees less than $k+1$.

**Condition B.** There are no two adjacent vertices of degrees $k+1$.

**Condition C.** If $u$ and $v$ are two adjacent vertices and $d(u) = k+1$, then $d(v) \leq 2$.

Suppose that $G$ has the above three conditions. Note that these conditions can be checked in polynomial time. Let $S = \{v : d_G(v) = k+1\}$, and construct the graph $G^*$ with the vertex set $S$. For every two distinct vertices $u, v \in S$, join the vertex $u$ to the vertex $v$ in $G^*$, if and only if there is a vertex $z$ in $G$ such that $uz, uz \in E(G)$ and $d_G(z) = 2$. It is easy to check that $G$ can be decomposed into two locally $k$-irregular subgraphs if and only if $G^*$ is bipartite. Since, there is a polynomial-time algorithm for determining whether a given graph has a chromatic number at most 2, therefore, our proof is completed.

\[\square\]
4 Concluding remarks and further research

In this work, we considered the set of problems which is related to decomposition of graphs into regular, locally regular and locally irregular subgraphs and we presented some polynomial time algorithms, \( \text{NP} \)-completeness results, lower bounds and upper bounds for them. A summary of results and open problems were shown in Table 1 and Table 2. Here, we present some remarks.

There exist infinitely many trees with irregular chromatic index three \([5]\). Baudon et al. proved that the problem of determining the irregular chromatic index of a graph can be handled in linear time when restricted to trees \([6]\). It is then natural to ask if the same holds for bipartite graphs. The following problem which was also asked by Baudon et al. in \([6]\) remains unsolved.

**Problem 1.** (Baudon et al. \([6]\)) Determine the computational complexity of deciding whether \( \chi'_{\text{irr}} = 2 \) for bipartite graphs.

Baudon et al. characterized all connected graphs which cannot be decomposed into locally 1-irregular subgraphs and call them exceptions \([5]\). For each \( k > 1 \), it would be interesting to characterize all connected graphs which cannot be decomposed into locally \( k \)-irregular subgraphs.

**Problem 2.** For each \( k \), characterize all connected graphs which cannot be decomposed into locally \( k \)-irregular subgraphs.

**Problem 3.** For each \( k \), can one decide in polynomial time whether a given graph \( G \), can be decomposed into locally \( k \)-irregular subgraphs?

Baudon et al. conjectured \( h(1) \leq 3 \) \([5]\). Here, we proved that \( h(k) \geq 2k + 1 \) and \( h(k) = \Omega(k^2) \). The next question is to determine whether there is a function \( f \) in terms of \( k \) such that \( h(k) \leq f(k) \).

**Problem 4.** Does there is a function \( f \) in terms of \( k \) such that \( h(k) \leq f(k) \)?

In this work, by using mutually orthogonal Latin squares we proved that \( h(k) = \Omega(k^2) \). Finding a better lower bound can be interesting.

It was shown that determining whether \( \text{reg}(G) \leq \Delta(G) \) for a given connected graph \( G \) with minimum degree one is \( \text{NP} \)-complete \([15]\). Does there exist a graph \( G \) with minimum degree two, such that \( \text{reg}(G) = \Delta(G) + 1 \)?
Problem 5. For each \( t \), does there exist a graph \( G \) with minimum degree \( t \), such that \( \text{reg}(G) = \Delta(G) + 1 \)?

We proved that every graph \( G \) can be decomposed into \( \Delta(G) \) subgraphs such that each subgraph locally regular and this bound is sharp for trees. Characterizing all connected graphs which cannot be decomposed into \( \Delta(G) - 1 \) subgraphs such that each subgraph is locally regular is interesting.

Problem 6. Characterize all connected graphs which cannot be decomposed into \( \Delta(G) - 1 \) subgraphs such that each subgraph is locally regular.

Every planar graph \( G \) with degree set \( \{2, 4\} \) can be decomposed into two regular subgraphs. We proved that determining whether \( \text{reg}(G) = 2 \) for a given planar graph \( G \) with degree set \( \{3, 6\} \) is \( \text{NP} \)-complete. With a similar technique we can show that deciding whether \( \text{reg}(G) = 2 \) for a given planar graph \( G \) with degree set \( \{5, 10\} \) is \( \text{NP} \)-complete. Since there is no 6-regular planar graph, the only remaining case is planar graphs with degree set \( \{4, 8\} \).

Problem 7. Determine the computational complexity of deciding whether \( \text{reg}(G) = 2 \) for planar graphs with degree set \( \{3, 6\} \).

In this work, for every \( k > 2 \), we constructed a tree \( T \) with \( \Delta(T) = k \) such that \( T \) cannot be decomposed into a matching and a locally irregular subgraph and also, we proved that every tree can be decomposed into two matchings and a locally irregular subgraph. Computational complexity of determining whether a given tree can be decomposed into a matchings and a locally irregular subgraph remains open.

Problem 8. Determine the computational complexity of deciding whether a given tree can be decomposed into a matching and a locally irregular subgraph.

Regarding the error that we found in [6], we should mention that in order to preserve the planarity it would be sufficient to show that if all variables are inside a clause cycle (a cycle which connects all clauses in an arbitrary order) and the graph is planar, then 1-in-3 Satisfiability is \( \text{NP} \)-complete. It is interesting to mention that although Planar 3-SAT is \( \text{NP} \)-complete if all clauses are inside the variable cycle and the graph is planar, then the problem is polynomial-time solvable [16]. A special case is when all clauses are also connected in a path. Then the problem is still in \( \text{P} \) because we can show that this implies that clauses are all on one side of the variable cycle [16]. We conjecture that the 1-in-3 Satisfiability in that case is also polynomial-time solvable.
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