Single-particle Green’s functions of the Calogero-Sutherland model at couplings $\lambda = 1/2$, 1 and 2

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At coupling strengths $\lambda = 1/2$, 1, or 2, the Calogero-Sutherland model (CSM) is related to Brownian motion in a Wigner-Dyson random matrix ensemble with orthogonal, unitary, or symplectic symmetry. Using this relation in conjunction with superanalytic techniques developed in mesoscopic conductor physics, we derive an exact integral representation for the CSM two-particle Green’s function in the thermodynamic limit. Simple closed expressions for the single-particle Green’s functions are extracted by separation of points. For the advanced part, where a particle is added to the ground state and later removed, a sum of two contributions is found: the expected one with just one particle excitation present, plus an extra term arising from fractionalization of the single particle into a number of elementary particle and hole excitations.

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I. INTRODUCTION

While the study of integrable models of interacting particle systems in one dimension has a long history, going back to Bethe [1], it has proved difficult to extract explicit expressions for their correlation functions from the solutions. Recently, however, the correlation functions of one class of such models, those with inverse-square interactions, have started to become available. These models have a special simplicity as compared to the integrable models of the Bethe type, and combine some of the special properties of an ideal gas with non-trivial statistics.

The original model of this type is the Calogero-Sutherland model (CSM) of a gas of spinless non-relativistic particles on a ring, with a scale-invariant interaction proportional to the inverse-square chord distance between particles. Different members of the CSM family are characterized by a dimensionless parameter $\lambda$, where the wavefunction vanishes as $(x_i - x_j)^{\lambda}$ as two particles approach each other. At the coupling $\lambda = 1$ the interaction vanishes, and the model reduces to a gas of free spinless fermions. Early progress in obtaining correlation functions of the model came from Dyson’s work [2] on correlations of the eigenvalues of random matrices: at the special couplings $\lambda = 1/2$, 1, and 2, where the CSM is related to the orthogonal, unitary, and symplectic random matrix ensembles. Dyson’s work led to explicit expressions [3] for the static density-density correlations at $\lambda = 1/2$ and 2, and for the density matrix at $\lambda = 2$. For many years, these were the only explicitly-obtained correlations of integrable models.

The recent breakthrough derives from the work of Simon, Lee and Al’tshuler [4] (SLA), and again has its origin in the study of random matrix problems. Using the functional integral formalism introduced by Efetov [5], SLA obtained expressions that they initially conjectured were the dynamical density-density correlations at $\lambda = 1/2$ and 2, since the analogous quantity at $\lambda = 1$ was indeed that quantity for the free Fermi gas, and the equal-time limit of their result coincided with the results known from Dyson’s work. Their remarkable conjecture was later verified. A variant of the SLA methodology was used by us [6] (HZ) to obtain the retarded part of the bosonic single-particle Green’s function at $\lambda = 2$, which at equal times reduces to the previously-known expression for the density matrix. When rewritten in terms of form factors, the SLA results become especially simple, and led to a conjecture by one of us [8] of their generalization to all rational coupling parameters $\lambda$. This conjecture was recently proved using Jack polynomial techniques [9] (and was independently made at integral values of $\lambda$, using group-theoretical arguments [10]). Jack polynomial methods were introduced in this context by Forrester [11], who recently [12] used them to prove a similar conjecture extending the HZ result to all integer $\lambda$. Ha [13] has also extended this to arbitrary rational $\lambda$.

While Jack polynomial methods are not restricted to particular values of $\lambda$, they require further development. In the meantime, the random matrix techniques provide powerful computational methods at couplings $\lambda = 1/2$ and 2. In this paper, we extend our earlier result [6] by also giving the advanced part of the bosonic single-particle Green’s function at $\lambda = 2$, and extending the calculation to the other coupling $\lambda = 1/2$.

The paper is organized as follows. Sect. [2] reviews the physics of the CSM, with particular emphasis given to the identification of its elementary particle and hole excitations. Anyonic particle creation and annihilation operators are defined in Sect. [3]. Our results are stated in Sect. [4]. The two-particle Green’s function defined in Sect. [3] is rewritten as a harmonic oscillator correlation...
II. THE CALOGERO-SUTHERLAND MODEL

The Calogero-Sutherland model \[3\] is a model of a gas of non-relativistic impenetrable particles with inverse-square interactions. It is convenient to impose periodic boundary conditions on a ring of length \(L\), although the simple results for the correlation functions will only be given in the thermodynamic limit. The Hamiltonian is

\[
H_{CS} = \sum_i \frac{p_i^2}{2m} + \sum_{i<j} \frac{g}{d(x_i - x_j)^2} \tag{1}
\]

where \(d(x) = (L/\pi) \sin(\pi x/L)\) is the chord distance between particles on the ring. The model is fully defined by the boundary condition on the wavefunction when two particles approach each other:

\[
|\Psi| \propto |x_i - x_j|^\lambda, \quad |x_i - x_j| \to 0,
\]

where \(\lambda \geq 0\), and

\[
g = \frac{h^2 \lambda(\lambda - 1)}{m}.
\]

Thus \(\lambda \to 0\) corresponds to the free Bose gas, and \(\lambda \to 1\) to the free (spinless) Fermi gas. (For \(-1/2 < g \leq 0\), there are two different choices of \(\lambda\) for each \(g\).)

The energy spectra \(3\) are given by the Bethe-Ansatz like equations, and can be viewed as adiabatic deformations of the free-electron spectrum for \(\lambda = 1\): for \(i = 1, \ldots, N\)

\[
k_i L = 2\pi I_i + \pi (\lambda - 1) \sum_j \text{sign}(k_i - k_j), \tag{2}
\]

where the \(\{I_i\}\) are distinct integers (or for more general boundary conditions, all \(I_i - I_j\) are integral). Energy and momentum of the eigenstate are then given by

\[
E = \sum_i \frac{(h k_i)^2}{2m}, \quad P = h \sum_i k_i. \tag{3}
\]

In the ground state, the \(I_i\)'s are consecutive integers, and the \(k_i\)'s are equally spaced, with spacing \(2\pi \lambda /L\). If an excited state is made by creating a “hole” in this “pseudo-Fermi sea”, by removing one of the \(I_i\)'s, the gap between the two consecutive \(k_i\) on either side of the hole is \(2\pi(\lambda + 1)/L = (2\pi \lambda /L)(1 + \lambda^{-1})\). There is a very simple interpretation of this: the dressed charge of the hole is \(-\lambda^{-1}\), as the extra width of the pseudo-Fermi sea associated with the presence of the hole is \(\lambda^{-1}\) of the width per particle. (Here the bare particles are taken to have unit charge). The other kind of excitation is to add a particle in a momentum state outside the pseudo-Fermi sea. In this case, the charge is +1, the bare charge. In the free fermion limit, \(\lambda = 1\), these identifications coincide with the usual ones for the Fermi gas.

Besides carrying fractional charge, the excitations have a natural interpretation as particles carrying fractional statistics (see e.g., \[23\]). This is perhaps easiest to see from the wavefunctions for the periodic model, which have the form \[3\]

\[
\Psi = \phi(z_1, \ldots, z_N) \prod_{i<j} (z_i - z_j)^\lambda \prod_i z_i^{(J+\alpha)} \tag{4}
\]

where \(z_i = e^{2\pi ix_i/L}\), \(\phi(\{z_i\})\) is a symmetric polynomial that is not divisible by \(z_i\) (called a Jack polynomial \(11\)), \(J\) is any integer, and \(\alpha\) is fixed (modulo an integer) by the choice of generalized periodic boundary condition (the phase change of \(\Psi\) when \(x_i \to x_i + L\) in the older literature \[3\]). In the free fermion limit, it was usual to replace \((z_i - z_j)^\lambda\) by \(|z_i - z_j|^\lambda\) or \(|z_i - z_j|^{-1}\), and call the bare particles bosons or fermions, but in the light of the striking resemblance of the CSM wavefunction to the Laughlin wavefunction \[14\], and the wavefunction for “anyons”, the form \(4\) seems more appropriate. This strongly suggests that the bare particles of the CSM be identified as anyons, with “statistical parameter" \(\Theta = \pi \lambda\). Polychronakos \[15\] has also given arguments for this based on scattering phase shifts.

The model for anyons is that of particles carrying both charge and magnetic flux \[14\]. We thus can identify the bare CSM particle as effectively carrying “electric” charge \(Q\) and “magnetic” flux \(\Phi\), so the statistical angle \(\Theta\) is \(Q\Phi/2\hbar\), where

\[
\text{particle: } Q = e, \quad \Phi = \lambda h/e, \quad \Theta = \pi \lambda.
\]

What about the hole excitation? A “coherent” state with a single hole has the wavefunction

\[
\Psi = \prod_i (z_i - Z) \Psi_0
\]

where \(\Psi_0\) is the ground state wavefunction. Here \(Z\) (if unimodular) parameterizes the position of the hole; expansion of this wavefunction in powers of \(Z\) gives its expansion in states of definite momentum which are the one-hole eigenstates of the Hamiltonian. All this is essentially the same as in the Laughlin states and the fractional quantum Hall effect (FQHE) \[14\]. Taking over the charge-statistics results from the FQHE \[17\], leads to the identification of the charge \(Q_h\), flux \(\Phi_h\), and statistical angle \(\Theta_h\) of the hole:

\[
\text{hole: } Q_h = -\lambda^{-1} e, \quad \Phi_h = -h/e, \quad \Theta_h = \pi \lambda^{-1}.
\]

Again, these identifications reduce to the trivially expected ones when \(\lambda = 1\), but are otherwise highly non-trivial.
The CSM has Galilean invariance, and it is straightforward to compute the effective mass of the holes from the equations (2,3). (The inertial mass of the particle excitation is unchanged from the bare value.) The hole has inertial mass $m_0 = -m\lambda^{-1}$, so that, as expected in a Galilean-invariant system, $Q_h/m_h = Q/m$. This is further corroboration of the identification of the fractional charge of the hole. Since the particle and hole excitations have different masses, the appropriate dynamical variable with which to describe them is their velocity. Holes have velocities in the range $-v_s < v < v_s$, where $v_s = \pi\lambda\hbar\rho_0/m$ is the speed of sound (long wavelength density fluctuations) in the model, and $\rho_0 = N/L$ is the mean density. Left-moving low-energy holes $v \approx -v_s$ and right-moving low-energy holes $v \approx v_s$ are continuously connected to each other. On the other hand, the particle excitations have $|v| > v_s$, and the left- and right-moving states are disjoint.

If an eigenstate is characterized by a set of particle excitations with velocities $\{v_i\}$ and a set of hole excitations with velocities $\{\bar{v}_i\}$, momentum and energy (relative to the ground state) are given by

$$P = \sum_i mv_i + \sum_i m\bar{v}_i,$$

$$E = \sum_i \frac{1}{2}m(v_i^2 - \bar{v}_i^2) + \sum_i \frac{1}{2}m(v_i^2 - \bar{v}_i^2). \quad (5)$$

Let us now consider the implication of these identifications for the form factors: what types of excitation are produced when the density operator

$$\rho(x) = \sum_i \delta(x - x_i)$$

acts on the ground state? The density operator carries no charge, and is a bosonic operator. For $\lambda = 1$, we know the answer: the action of $\rho(x)$ on the ground state produces exactly one hole and one particle, with a total charge $Q + Q_h = 0$, and a total flux $\Phi + \Phi_h = 0$.

The recent calculations of Simons, Lee, and Altshuler show, when suitably interpreted, give the non-trivial answer for $\lambda = 2$ and $\lambda = 1/2$. For $\lambda = 2$, one particle and two holes are produced, while for $\lambda = 1/2$, two particles and one hole are produced. This count corresponds to the number of free velocity parameters in the form factors implied by the SLA results. As expected, these are consistent with a total charge of zero and a total flux of zero. The generalization of this is that at rational coupling $\lambda = p/q$, the action of the local density operator on the CSM ground state will produce excited states with exactly $q$ particles and $p$ holes. A further restriction is found: in the $\lambda = 1/2$ CSM, the results show that both the particles have velocities in the same direction (momenta on the same side of momentum space with respect to the pseudo-Fermi sea). The generalization of this is the result that all the $q$ particle excitations have velocities in the same direction (all left-movers or all right-movers). Thus the simplest possible excitations consistent with charge neutrality account for the complete spectral weight of the density-density correlation function. This is a generalization of the free Fermi gas property of the $\lambda = 1$ model, where just one particle and one hole are produced by the action of the local density operator. In a more general interacting model, there would be contributions from two particle-hole pairs, three particle-hole pairs, and so on. The remarkable structure of the non-trivial solutions found at $\lambda = 1/2$ and 2 is that the only intermediate states that contribute are the minimally-excited ones, where the smallest number of excitations consistent with charge neutrality are made. In this sense, the CSM behaves like a generalized ideal gas model.

In the case of the single-particle Green’s function, there are two terms to consider: the advanced term, where a particle is added to the system, and removed at a later time, and the retarded term where it is removed and later restored. In these cases, the intermediate states must again have an excitation content with the same total charge and flux as the operator that initially acts. Thus in the advanced case, the total charge is $e$, and the flux is $\hbar/e$: the negatives of these apply in the retarded case. At coupling $\lambda = p/q$, we find the expected contribution with just one particle excitation, plus a second term with one particle in the forward direction, $q$ particles in the backwards direction, and $p$ holes. In the retarded case, the general result found by Ha is $q - 1$ particles (in the same direction) and $p$ holes.

### III. ANYON CREATION AND ANNIHILATION OPERATORS

We have reviewed the arguments in favor of the interpretation of the CSM particles as anyons when $\lambda$ is not an integer. This is the interpretation we adopt.

To calculate the anyonic CSM Green’s functions, we shall need to define the action of the particle creation and annihilation operators on $N$-anyon states. Recall first how the particle annihilation operator $\psi_x$ acts on a state of $N$ bosons ($s = +1$) or fermions ($s = -1$):

$$\langle\{x_i\}|\psi_x|\Psi_N\rangle = \sum_{i=1}^N s^i\Psi_N(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_N).$$

Now, the Hilbert space of $N$-anyon CSM wavefunctions is spanned by states of the form

$$\langle\{x_i\}|\Psi_N\rangle = S_N(\{x_i\}) \prod_{i<j} d(x_i - x_j)^\lambda, \quad (6)$$

where $S_N(\{x_i\})$ is a symmetric function. When $\lambda$ is an even (odd) integer, these wavefunctions are bosonic (fermionic), and in these cases insertion into the previous equation gives (up to normalization):
Note that the function $S_{N,x}^{(\lambda)}$ is a symmetric function of its $N$ arguments, not only for integral $\lambda$ but for any $\lambda$. Therefore, it makes sense to define the annihilation operator $\psi^\dagger_x$ by (7) at any $\lambda \geq 0$, and this is what we do. The operator so defined properly maps the $(N+1)$-anyon Hilbert space onto the $N$-anyon Hilbert space. Similarly, by starting from the formula

$$
\langle \{x_i\} | \psi^\dagger_x | \Psi_N \rangle = \sum_{i=1}^{N+1} s^i \delta(x-x_i) \Psi_N (...) \psi^\dagger_{x_{i+1}} (...) \psi^\dagger_{x_{i+1}} ...
$$

for bosons or fermions, we obtain

$$
S_{N,x}^{(\lambda)}(\{x_i\}) = S_{N+1}(x, \{x_i\}) \prod_i d(x-x_i)^\lambda.
$$

(7)

Again, the last equations make sense, and give a valid definition of the anyon creation operator $\psi_x^\dagger$, for any $\lambda$.

$$
-\infty < v_1 \ldots < v_q < -v_s < \bar{v}_1 \ldots < -\bar{v}_p < +v_s < v_{q+1} < +\infty.
$$

Wave number $k = P/h$ and frequency $\omega = E/h$ have been given in (3). The “form factor” $|F|^2 = |\langle V, \bar{V} \psi^\dagger_x | 0 \rangle|^2$ consists of two parts: $F = F^{(0)} \times F^{(1)}$. The first one,

$$
|F^{(0)}(n_p, n_h, \lambda | V, \bar{V})|^2 = \prod_{i>j} (v_i - v_j)^{2\lambda} \prod_{i>j} (\bar{v}_i - \bar{v}_j)^{2/\lambda} \prod_{i=1}^{n_p} (v_i^2 - v_s^2)^{1-\lambda} \prod_{i=1}^{n_h} (v_i^2 - v_i^3)^{1-\lambda} \prod_{i=1}^{n_q} (v_i - v_{q+1})^2
$$

arises in our calculation as a Jacobian or measure and in this sense is purely statistical, doing no more than measuring the phase space available to the elementary excitations. $F^{(0)}$ is completely determined by the root system of an underlying Lie superalgebra and is therefore universal. (More precisely speaking, if $n_p$ and $n_h$ are the numbers of particles and holes an excitation is composed of, the form factor of that excitation will contain a factor $F^{(0)}(n_p, n_h, ...)$. Note also the duality symmetry

$$
F^{(0)}(n_p, n_h, \lambda | V, \bar{V}) = F^{(0)}(n_h, n_p, \lambda^{-1} | V, \bar{V}).
$$

(10)

The other factor is specific to the advanced part of the single-particle Green’s function and is given by

$$
F^{(1)}_2 = C_2 \left. \frac{d}{dv} \left( \frac{(v - \bar{v}_1)(v - v_3)}{(v - v_1)^2} \right) \right|_{v = v_2},
$$

$$
F^{(1)}_1 = 0,
$$

$$
F^{(1)}_{1/2} = C_{1/2} \int_0^\infty dv \sqrt{v - v_3} \left( \frac{v - \bar{v}_1}{\prod_{i=1}^2 \sqrt{v - v_i}} - 1 \right).
$$

The normalization constants are $C_2 = \sqrt{m/8\pi h}$ and $C_{1/2} = \sqrt{m/4\pi^3 h}$.

Our calculation of $F^{(1)}_\lambda$ for $\lambda = 1/2, 1, 2$ is suggestive of generalization and leads us to propose the following conjecture for integral $\lambda = p > 1$:

$$
F^{(1)}_\lambda = C_\lambda \left( \frac{d}{dv} \right)^{\lambda-1} \prod_{i=1}^p \frac{(v - \bar{v}_i)}{(v - v_1)^\lambda} \bigg|_{v = v_2}.
$$

(11)

Verification of this conjecture is posed as a challenge to the theory of Jack symmetric functions.

V. TWO-PARTICLE GREEN’S FUNCTION

The objective of this paper is the calculation of the advanced part of the single-particle Green’s function. For reasons of generality and convenience of presentation, we
will set up our computational machinery for the two-particle (or particle-hole) Green’s function \((t \geq t')\)

\[
G_{xywz}(t - t') = \langle 0 | \hat{\psi}_x^\dagger(t) \hat{\psi}_y(t) \hat{\psi}_w^\dagger(t') \hat{\psi}_z(t') | 0 \rangle.
\]

For \(x = y\) and \(w = z\), this Green’s function reduces to the dynamical density correlation function calculated in \([5, 9]\). When the pair \(x, w\) is taken to be remote from the pair \(y, z\), we expect \(G\) to separate:

\[
G_{xywz}(t) \rightarrow G_p(x - w; t)G_h(y - z; t)
\]

where \(G_p\) and \(G_h\) are the retarded and advanced parts of single-particle Green’s function.

Our first task is to describe the action of the particle-hole creation operator \(\psi_k^\dagger \psi_k\) on the \(N\)-particle ground state \(|0\rangle\). By combining the definitions \([5, 9]\) we see that this action is simply multiplication with a function of the coordinates \(X := \{x_1, \ldots, x_N\}\):

\[
\langle X | \psi_{x_1}^\dagger \psi_{y_1} | 0 \rangle = O_{ab}(X)\langle X | 0 \rangle,
\]

\[
O_{ab}(X) = \sum_{j=1}^N \delta(a - x_j) \prod_{k \neq j}^N \frac{d^\lambda(b - x_k)}{d^\lambda(a - x_k)} \tag{12}
\]

As it stands, the “particle-hole function” \(O_{ab}(X)\) is not in a form suitable for calculation by the technique used in the present paper, which is why we make the following modification. Consider for \(\varepsilon > 0\) the function \(\delta_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}\)

\[
\delta_\varepsilon(x) = \Re[(2i\varepsilon)^\lambda(2\pi i)^{-\lambda}(x + i\varepsilon)^{-\lambda}(x - i\varepsilon)^{-1}].
\]

Since \(f(0) = \lim_{\varepsilon \to 0} \int_\mathbb{R} f(x)\delta_\varepsilon(x)dx\) for any smooth function \(f\), \(\delta_\varepsilon\) is a valid regularization of Dirac’s delta distribution centered at zero. We make \(\delta_\varepsilon\) periodic by replacing distance by chord distance:

\[
\delta_{\varepsilon}(x) = \Re \frac{(2i\varepsilon)^\lambda/2\pi i}{d^\lambda(x + i\varepsilon)d^\lambda(x - i\varepsilon)}.
\]

Inserting this into the expression for \(O_{ab}(X)\), we obtain

\[
O_{ab}(X) = \Re \frac{(2i\varepsilon)^\lambda/2\pi i}{d^\lambda(b - a + i\varepsilon)d^\lambda(b - a - i\varepsilon)} \times \sum_{j=1}^N \frac{1}{d(a - x_j - i\varepsilon)} \prod_{k=1}^N \frac{d^\lambda(b - x_k + i\varepsilon)}{d^\lambda(a - x_k + i\varepsilon)} \tag{13}
\]

which differs from \([2]\) by terms that become negligible in the limit \(\varepsilon \to 0\). Introducing

\[
K(X, X'; t) = \langle 0 | X | e^{-itHCS}/h | X' \rangle \langle X' | 0 \rangle
\]

we can now write the two-particle Green’s function in the form

\[
G_{xywz}(t) = \int dX \int dX' O_{xy}(X)K(X, X'; t)O_{wz}(X').
\]

Here, \(dX = \prod dx_i\) and the integration range is that sector of \([0, L]^N\) where \(x_i < x_{i+1}\) for \(i = 1, \ldots, N - 1\).

VI. MAPPING ON A SYSTEM OF HARMONIC OSCILLATORS

It has long been known \([13]\) that the CSM at coupling strengths \(\lambda = 1/2, 1, \) and \(2\) is equivalent to free-particle motion on a symmetric space \(U(N)/O(N), U(N)\) and \(U(2N)/Sp(2N)\), respectively. More precisely speaking, for \(\lambda = 1/2, 1, \) or \(2\) there exists a similarity transformation taking \(H_{CS}\) into the radial part of the kinetic energy operator on the corresponding symmetric space. We will now review this transformation for the case \(\lambda = 2\).

Let \(C = 1_n \otimes i\sigma^y\) be the symplectic unit acting on the \(2N\)-dimensional linear space \(\mathbb{C}^N \otimes \mathbb{C}^2\). With \(g\) running through the group \(U(2N)\) we consider the set, \(M_N\), of matrices \(S\) of the form

\[
S(g) = gCg^TC^{-1}.
\]

Such matrices are unitary and obey the self-duality constraint \(S = CS^TC^{-1}\). They do not form a group. Rather, since \(S(gk) = S(g)\) for any element \(k\) of the symplectic subgroup \(Sp(2N) \subset U(2N)\) defined by the condition \(kCk^T = C\), \(M_N\) is isomorphic to the coset space \(U(2N)/Sp(2N)\). \(U(2N)\) acts as a transformation group on \(M_N\) by \(S(g) \mapsto S(gh)\) \((h \in U(2N))\). When endowed with its natural \(U(2N)\)-invariant metric \(-\frac{1}{2}\text{Tr} dSdS^\dagger\) and probability measure \(d\mu(S), M_N\) is known in random matrix theory as Dyson’s Circular Symplectic Ensemble (CSE) \([1, 13]\).

Dyson, in his classic paper \([1]\), proved the following two results. (i) Any element \(S \in M_N\) has a “polar” decomposition \(S = ke^{i\theta}k^{-1}\) where \(k \in Sp(2N)\), and \(\theta = \text{diag}(\theta_1, \ldots, \theta_N) \otimes 1_2\) is diagonal. (We refer to \(e^{i\theta}\) and \(k\) as the “radial” and “angular” parts of \(S\). Note that every eigenvalue of \(S\) occurs with multiplicity 2.) (ii) Polar decomposition takes the invariant integral on \(M_N\) into

\[
\int_{M_N} f(S)d\mu(S) = \int_{[0, 2\pi)^N} \left( \int_{Sp(2N)} f(ke^{i\theta}k^{-1})dk \right) J(\theta) d\theta
\]

where \(d\theta = \prod_{i=1}^N d\theta_i\), and \(dk\) is a suitably normalized Haar measure for \(Sp(2N)\). The Jacobian \(J\) for the transformation to polar coordinates is

\[
J(\theta) = \prod_{i<j} \sin^4[(\theta_i - \theta_j)/2].
\]

Now, let \(L\) be the Laplacian (or minus the operator for kinetic energy) of the Riemannian manifold \(M_N\). If \(L\)
is the radial part of $L$, we have $Lf = L_r f$ for any radial function $f$. Using

$$\text{Tr} \, dSd\Sigma = -2 \sum_{i=1}^{N} \partial_i \theta_i^2 + \text{Tr}[k^{-1} dk, e^{i\theta}]^2$$

in conjunction with the standard coordinate expression for the Laplacian on a Riemannian space, we easily get

$$L_r = J(\theta)^{-1} \sum_{i=1}^{N} \frac{\partial}{\partial \theta_i} J(\theta) \frac{\partial}{\partial \theta_i}.$$

The differential operator $L_r$ is self-adjoint w.r.t. the radial measure $J(\theta) d\theta$. It is made self-adjoint w.r.t. the flat measure $d\theta$ by carrying out the similarity transformation $L_r \to J^{1/2} L_r J^{-1/2}$. With the help of the similarity transformation $w_{ij} w_{jk} + w_{jk} w_{ki} + w_{ki} w_{ij} = 1$ for $w_{ij} = \cot((\theta_i - \theta_j)/2)$, and by making the identification $\theta_i = 2\pi x_i/L$, we obtain

$$H_{\text{CS}}^{(\lambda=2)} = \frac{(2\pi \hbar)^2}{2mL^2} \left(-J^{1/2} L_r J^{-1/2} + N(N^2 - 1)/3\right).$$

This is the desired relation between the CSM at $\lambda = 2$ and free-particle radial motion on the symmetric space $\mathcal{M}_N \simeq U(2N)/\text{Sp}(2N)$.

For $\lambda = 2$, the function $O_{ab}(X)$ of Eq. (13) can be viewed as a radial function

$$O_{ab}(S) = \Re \frac{2i\varepsilon^2/\pi}{d^2 (b - a + i\varepsilon)} \times$$

$$\frac{1}{2} \text{Tr} \left[ \frac{1}{d(a - L/2\pi \ln S - i\varepsilon)} \frac{\partial}{\partial a} \frac{1}{d(a - L/2\pi \ln S + i\varepsilon)} \right].$$

on $\mathcal{M}_N$. Hence, the particle-hole Green’s function of the CSM can be calculated as a dynamical correlation function for free-particle motion on $\mathcal{M}_N$. Note that this description of the latter is not entirely trivial, since $\mathcal{M}_N$ is a curved space.

A further simplification arises in the thermodynamic limit $N \to \infty$. In this limit, Dyson’s circular ensembles are known to become locally equivalent to the corresponding Gaussian ensembles [19]. For the dynamical correlation functions, the equivalence can be understood by the following argument. We go back to our starting point and think of the eigenvalues $\theta_i \in [0, 2\pi)$ of $S \in \mathcal{M}_N$ as a strongly correlated gas of $N$ particles with repulsive long-range forces. (These are the particles of the $\lambda = 2$ Calogero-Sutherland model, except that now they are confined to a ring of circumference $2\pi$ instead of $L$.) An important consequence of the strong correlations is that the total system size becomes an irrelevant length scale of the gas in the large-$N$, or high-density, limit. This comes about because the (impenetrable) particles of the correlated gas never stray far from their average positions and therefore do not sense the periodic boundary conditions. There is, then, just one length scale in this problem for large $N$, which is the mean distance, $\Delta$, between nearest neighbors. Similarly, there is only one time scale: the time it takes for the wave packet of a free quantum particle with kinetic energy $-\partial^2/\partial \theta^2$ to spread over a distance $\Delta$. The interaction, owing to its scale-invariant $1/r^2$ short-distance form, does not introduce an extra time scale. For these reasons, and because correlations always decay, the dynamical correlation functions of the particle gas will go to zero on space and time scales large compared to $\Delta$ and $\Delta^2$, respectively. Since the spacing of $N$ particles uniformly distributed on the unit circle is $\Delta = 2\pi/N$, we conclude that the dynamical correlation functions are fully determined by the physics at short distances (of order $1/N$) and short times (of order $1/N^2$). Note that this is an indication of universality.

Returning to the equivalent problem of free-particle motion on the symmetric space $\mathcal{M}_N$, we infer that the curvature of $\mathcal{M}_N$ becomes an irrelevant feature in the large-$N$ limit. In concrete terms, for the purpose of computing the dynamical correlations near some point $p = e^{i\theta} \in \mathcal{M}_N$, we may as well replace $\mathcal{M}_N$ by its tangent space $T_p \mathcal{M}_N$ at $p$. In this way the problem of free-particle motion on a curved space gets simplified to free-particle motion on a flat space. There is one complication however. Its nature, and remedy, are again most clearly seen in the particle gas picture. If the unit circle $S^1$ is replaced by its tangent space, which is the real line $\mathbb{R}$, there is nothing that holds the gas together to produce a finite ground state density. Fortunately, we can easily remedy the situation by simply enclosing the gas in a box or any kind of confining well. More precisely speaking, what we are going to do is the following. If particles were added to the confining well with the size of the well kept fixed, the particle density would increase. Therefore, to have a well-defined limit when sending $N \to \infty$, we rescale the confining potential so as to maintain a constant particle density near the center of the well. Then, in the limit $N \to \infty$ a translationally invariant region will form around this center, the system will become ignorant of the confinement scale, and the dynamical correlations will approach the universal form we are trying to compute.

It is now clear how we should proceed. All points of a symmetric space are equivalent, so without loss we can take $p$ to be the origin, $p = e^{i\theta} = 0$. We linearize $S = \exp iQ$ around $S = o$, with $iQ$ parameterizing the tangent space $T_o \mathcal{M}_N$. From the defining equations of $\mathcal{M}_N$ (i.e. $S^\dagger = S^{-1}$ and $S = CS^T C^{-1}$) we infer the linear conditions

$$Q = Q^\dagger = CQ^T C^{-1}. \quad (14)$$

The correct physical dimension of length is restored by replacing $Q \to (L/2\pi \theta) Q$. The expression for the particle-hole function $O_{xy}$ then takes the form
\[ O_{xy}(Q) = \mathbb{R} \frac{2i\varepsilon^2/\pi}{(y - x + i\varepsilon)^2} C_{xy}^\sigma(Q), \]
\[ C_{xy}^\sigma(Q) = \frac{1}{2} \text{Tr} \frac{1}{x - i\varepsilon - Q} \text{Det} \frac{y + i\varepsilon - Q}{x + i\varepsilon - Q}. \] (15)

To obtain the Green's functions of the CSM at \( \lambda = 2 \) and for \( N \to \infty \), we have to solve a one-particle problem with Hamiltonian

\[ H = \frac{\hbar^2}{2m} \mathcal{L}_Q + V(Q) \]

where \( \mathcal{L}_Q \) is the Laplacian of the Euclidean space \( T_oM_N \) with metric \( \mathcal{g} = \frac{1}{2} \text{Tr} dQ dQ \), and \( V(Q) = V(k Q k^{-1}) \) is a confining radial potential with a global minimum at \( Q = 0 \). If \( \langle Q|0 \rangle \) is the ground state wavefunction of \( H \) and

\[ K(Q, Q'; t) = \langle 0|Q \rangle \langle Q'e^{-iHt/\hbar}|Q' \rangle \langle Q'|0 \rangle, \]

we need to calculate the double integral

\[ \int dQ \int dQ' \, O_{xy}(Q) K(Q, Q'; t) O_{wz}(Q') \] (16)

where \( dQ \) is a Euclidean measure for \( T_oM_N \). We take the thermodynamic limit \( N \to \infty \) while rescaling \( V \) in such a way that the density \( \rho_0 = \langle 0|\frac{1}{2} \text{Tr} \delta(Q)|0 \rangle \) remains fixed. By the argument given above, the expression (14) with \( x, y, w, z \) at a distance of order \( N^0 \) from the minimum of the confining well, will then converge to the CSM particle-hole Green's function \( G_{xywz}(t) \) in this very limit.

For technical convenience we choose the potential to be harmonic: \( V(Q) = m \omega^2 \text{Tr} Q^2/2\hbar \). We then get a harmonic oscillator with frequency \( \omega \), ground state wavefunction \( \langle Q|0 \rangle = \text{const} \times \exp(-m\omega \text{Tr} Q^2/2\hbar) \) and propagator

\[ \langle Q|e^{-iHt/\hbar}|Q' \rangle = \sqrt{\frac{m\omega}{2\pi\hbar \sin \omega t}} \frac{\dim M_N}{\text{dim} M_N} \]

\[ \times \exp \left[ \frac{i m \omega}{4 \hbar \sin \omega t} \text{Tr} \left( (Q^2 + Q'^2) \cos \omega t - 2QQ' \right) \right]. \]

When time is analytically continued to the imaginary axis, this propagator turns into the diffusion kernel for Brownian motion in which is called the Gaussian Symplectic Ensemble (GSE) of random matrix theory. For future reference we note that the basic time-ordered harmonic oscillator correlation function for self-dual sources \( A \) and \( B \) is

\[ \langle 0|\text{Tr} B Q(t) \text{Tr} A Q(t')|0 \rangle = \frac{\hbar}{m \omega} e^{-i\omega|t-t'|} \text{Tr} AB. \] (17)

The average density of the GSE is known to be of semicircular shape [4] and in the center is given by

\[ \rho_0 = \sqrt{N m \omega / \hbar \pi^2}, \]

so we are going to take the thermodynamic limit holding the product \( N \omega \) fixed.

All of the above steps are easily transcribed to the cases \( \lambda = 1 \) and \( 1/2 \). For \( \lambda = 1 \) we end up with a harmonic oscillator model of complex hermitean matrices \( Q \), while for \( \lambda = 1/2 \) the matrices \( Q \) are real symmetric.

**VII. SUPERSYMMETRIC CALCULATION OF GREEN’S FUNCTIONS FOR \( \lambda = 2 \)**

We have seen that the particle-hole Green’s function \( G_{xywz}(t) \) of the CSM at couplings \( \lambda = 1/2, 1, \) and \( 2 \) can be computed as a harmonic oscillator correlation function

\[ G_{xywz}(t - t') = \langle 0|\text{Tr} O_{xy}(Q(t)) O_{wz}(Q(t'))|0 \rangle \]

in the thermodynamic limit. For \( \lambda = 2 \), the harmonic oscillator variable \( Q \) is a self-dual hermitean \( 2N \times 2N \) matrix, and by using (13) we get

\[ G_{xywz}(t) = \frac{(2i\varepsilon^2/\pi)^2}{(x - y)^2(w - z)^2} \sum_{\sigma,\tau=\pm} \sigma \tau \, g_{xywz}^\sigma(\varepsilon \omega), \]

\[ g_{xywz}^\sigma(t - t') = \langle 0|\text{Tr} C_{xy}^\sigma(Q(t)) C_{wz}^\sigma(Q(t'))|0 \rangle \] (18)

Similar formulas hold for \( \lambda = 1/2 \) and 1. Eqs. (18) express the particle-hole Green’s function as a sum of four terms. In Sects. VII A-VII H we calculate \( g^{++} \), from which \( \gamma^+ = \gamma^+ \) is obtained by complex conjugation followed by the time reversal operation. The modifications necessary for \( g^{++} \) and \( g^{--} \) are described in Sect. VII I. The changes that occur for \( \lambda = 1/2 \) and 1 are sketched briefly in Sect. VII J.

**A. Gaussian superintegrals**

Because \( C_{xy}^\sigma(Q) \) depends on \( Q \) in a rather complicated way, the problem of calculating the Green’s function \( g^{++} \) is not immediately tractable. As a preparatory step, we are now going to make \( Q \) appear in the argument of an exponential function. This will reduce the problem to calculation of Gaussian integrals.

Let \( v_{k\alpha} \) and \( \chi_{k\alpha} \) \((k = 1, ..., N; \alpha = \uparrow, \downarrow)\) be a set of complex commuting and anticommuting (or Grassmann) variables. Our starting point are the formulas

\[ \text{Det}(x \pm i\varepsilon - Q)^{-1} = \int \, dv \, \exp \pm i\varepsilon(x \pm i\varepsilon - Q)v, \]

\[ \text{Det}(y - Q) = \int \, d\chi \, \exp \chi(y - Q)\chi, \]

where \( dv \) and \( d\chi \) are suitably normalized (Euclidean) measures, and

\[ \varepsilon(x \pm i\varepsilon - Q) = \varepsilon_{k\alpha}((x \pm i\varepsilon)\delta_{k\alpha} - Q_{k\alpha,\beta})\varepsilon_{\beta}, \]

\[ \chi(y - Q) \chi = \chi_{k\alpha}(y \delta_{k\alpha} - Q_{k\alpha,\beta})\chi_{\beta}. \]

Summation over repeated indices is implied.
In later subsections a plethora of tensor products will appear, so to avoid writing a lot of indices we are now going to switch to basis-free notation. This do by interpreting the integration variables $v_{\alpha}$ and $\chi_{\alpha}$ as the matrix elements of a $(1+1) \times 2N$-dimensional supermatrix $W$. More formally, we proceed as follows. We introduce (canonical) bases \( \{ e^k \}_{k=1, \ldots, N} \), \( \{ e^\alpha \}_{\alpha=\uparrow, \downarrow} \), and \( \{ e^B, e^F \} \), of particle-coordinate space $\mathbb{C}^N$, “spin” space $\mathbb{C}^2$, and super (or Boson-Fermion) space $V_{bf} = \mathbb{C}^{1|1}$. Elements of the dual bases are denoted by $\theta^a$. We then take $W: \mathbb{C}^N \otimes \mathbb{C}^2 \rightarrow V_{bf}$ to be the linear function defined by

\[
W = e^B \otimes v_{\alpha} \theta^k \otimes \theta^\alpha + e^F \otimes \chi_{\beta} \theta^l \otimes \theta^3.
\]

Similarly, we define $\tilde{W}: V_{bf} \rightarrow \mathbb{C}^N \otimes \mathbb{C}^2$ by

\[
\text{Det} \frac{y-Q}{x \pm i\varepsilon - Q} = \int dW d\tilde{W} \exp \pm i \text{Tr} \tilde{W}(gW - W Q^T). \tag{19}
\]

By differentiating both sides with respect to $x$ and then setting $y = x \pm i \varepsilon$, we obtain

\[
\text{Tr} \frac{1}{x \pm i\varepsilon - Q} = \mp i \int dW d\tilde{W} \text{Tr} \tilde{W} e^{B B W} \exp \pm i \text{Tr} \tilde{W}((x \pm i\varepsilon)W - W Q^T). \tag{20}
\]

### B. Implementing self-duality

Formulas (19) and (20) allow us to write the particle-hole function $C_{xy}^{\sigma}(Q(t))$ as a product of Gaussian integrals over two supermatrices $W^{At}$ and $W^{Rt}$. Doing the same for its complex conjugate $C_{xy}^{\sigma}(Q(t))$, we can carry out the integration over $Q's$ in the expression for the Green’s function $g^{+-}$ by setting

\[
A = i \tilde{W}^{At} W^{At} - i \tilde{W}^{Rt} W^{Rt},
\]

\[
B = i \tilde{W}^{At'} W^{At'} - i \tilde{W}^{Rt'} W^{Rt'},
\]

and using the harmonic oscillator identity

\[
\langle 0 \rangle \exp(\text{Tr} A Q(t)) \exp(\text{Tr} B Q(t')) \langle 0 \rangle = \exp \frac{1}{2} \langle 0 \rangle \left( \text{Tr} A Q(t) + \text{Tr} B Q(t') \right)^2 \langle 0 \rangle. \tag{21}
\]

Unfortunately, since $A$ and $B$ fail to be self-dual, we are not permitted to use the simple correlator \( \langle 0 \rangle \) but must work with a more complicated version thereof. As a result, the Hubbard-Stratonovitch transformation (Sect. 7D) is unnecessarily complicated and the orthogonal symmetry of the external surface emerging in the thermodynamic limit (Sect. 7E) is obscured. To improve the situation we make the following modification \( \langle 20 \rangle \).

The idea is to enlarge the supermatrix $W$ in such a way that $P := \tilde{W} W$ does satisfy the self-duality condition $P = C P^T C^{-1}$. To that end, we introduce an additional degree of freedom, called “quasispin”, which takes two values and is thus formally identical to a spin $1/2$. Quasispin space is denoted by $V_{cd}$. (It is the cooperon-diffuson freedom of mesoscopic conductor physics.) $W$ is extended to a supermatrix $W : \mathbb{C}^N \otimes \mathbb{C}^2 \rightarrow V_{bf} \otimes V_{cd}$ by adding an extra row index running through two values (for the two-dimensional space $V_{cd} \simeq \mathbb{C}^2$).

Let $W^T$ denote the supertranspose of the supermatrix $W$. Note that $W^{TT} = \sigma W$ and $W^{TT} = W^T \sigma$, where $\sigma := (E^{BB} - E^{FF}) \otimes 1$ stands for superparity. (These double transposition rules are implied by the relation $(WW)^T = W^T W^T$, which is needed for reasonable supercalculus.) The product $\tilde{W} W$ will be self-dual if we impose the conditions

\[
\tilde{W} = C W^T \tau^{-1}, \quad W = \tau \tilde{W}^T C^{-1}, \tag{22}
\]

with some $c$-number matrix. Consistency requires $W^T = C^{-1} \tilde{W}^T = C^{-1} \tilde{W}^T \tau T$. Since $C^T = -C$, it follows that $\tau$ must satisfy $\tau = -\sigma \tau T$. The choice we make is $\tau = E^{BB} \otimes i \sigma y + E^{FF} \otimes 1$.

The extension to quasispin space in combination with imposition of the constraint $W = C W^T \tau^{-1}$, leaves the number of independent components of $W$ unchanged. It is easy to see by inspection that formulas (19) and (20) remain valid if we modify $W$ in the manner described, extend $q$ to quasispin space in the trivial way $(q \rightarrow q \otimes 1)$ and place an extra factor of $1/2$ in the exponent.

### C. Integral Representation of $g^{+-}$

We now assemble the various factors needed for the Gaussian integral representation of the Green’s function $g^{+-}$ by (13), (15), (19) and (20). For the trace of the resolvent and the ratio of determinants at time $t$, we introduce two supermatrices $W^{At}$ and $W^{Rt}$, respectively.
Similarly, at time \( t' \) we introduce a pair of supermatrices \( W^{A\tau} \) and \( W^{R\tau} \).

Since the multitude of supermatrices involved leads to lengthy expressions, we further compactify our notation. We do this by arranging all supermatrices vertically as a single giant supermatrix, denoted again by \( W \). In tensor-product notation,

\[
W : C^N \otimes C^2 \rightarrow V_{bf} \otimes V_{cd} \otimes V_{ar} \otimes V_{T},
\]

\[
W = e^\theta \otimes e^{\tau'} \otimes e^{\lambda} \otimes e^{W_{\mu\nu,i\alpha}} \otimes \theta \otimes \theta',
\]

where \( V_T \simeq C^2 \simeq V_{ar} \) and the subscripts “\( T \)” and “\( ar \)” refer to the two-Time and advanced-retarded (\( \pm \epsilon \)) structures. The new indices take the values \( \rho = t, t' \) and \( \lambda = A, R \). To assemble all factors, we define

\[
q = [(x - i\epsilon)(E^{BB} + E^{FF}) \otimes E^{AA} \otimes E^{tt} + ((x + i\epsilon)E^{BB} + (y + i\epsilon)E^{EF}) \otimes E^{RR} \otimes E^{tt} + ((w - i\epsilon)E^{BB} + (z - i\epsilon)E^{EF}) \otimes E^{AA} \otimes E^{tt'} + (w + i\epsilon)(E^{BB} + E^{EF}) \otimes E^{RR} \otimes E^{tt'}] \otimes 1_{cd},
\]

and we introduce the matrices

\[
I = E^{BB} \otimes 1_{cd} \otimes E^{AA} \otimes E^{tt}, \quad \pi_t = 1_{bf \times cd \times ar} \otimes E^{tt},
\]

\[
l' = E^{BB} \otimes 1_{cd} \otimes E^{RR} \otimes E^{tt'}, \quad \pi_{t'} = 1_{bf \times cd \times ar} \otimes E^{tt'}.
\]

(Here and in the following we occasionally take liberty to change the order of the tensor products, for notational simplicity. What is meant will always be clear from the index nomenclature used.) With these definitions, \( g^{+-} \) has the expression

\[
g_{xyuv}^{+-}(t - t') = 2^{-4} \int dWd\tilde{W} \times \frac{\text{Tr}(W IW)\text{Tr}(\tilde{W}l'(W)e^{-i\text{Tr}W\eta W}/2)}{(0\langle T \exp(\text{Tr} AQ(t)) \text{exp}(\text{Tr} BQ(t'))\langle 0\rangle \exp(\frac{1}{\hbar} \text{Tr} W\eta W(t + t') WQ(t'))\langle 0\rangle)}.
\]

A diagonal matrix \( \eta \) containing elements \( \pm 1 \) was inserted for convergence of the bosonic integrations, see [17] and [24]. The sign of these matrix elements is fixed on the bosonic subspace but is arbitrary on the fermionic one. The “good” choice for \( \eta \) turns out to be [21]

\[
\eta = (E^{BB} \otimes \sigma_{ar}^{\ast} + E^{FF} \otimes 1_{ar}) \otimes 1_{cd \times T}.
\]

We make this choice because it leads without further ado to the metric of the extremal surface of Sect. [112] being Riemannian (as opposed to being indefinite). In contrast, if we made the more natural choice \( \eta' = \sigma_{cd}^{\ast} \otimes 1_{bf \times cd \times T} \), we would eventually be forced to rotate integration contours to achieve positive definiteness.

We add a brief discussion of symmetries. Recall that \( W \) and \( \tilde{W} \) are related by \( \tilde{W} = CW^{T}\tau^{-1} \) with

\[
\tau = (E^{BB} \otimes i\sigma_{cd}^{y} + E^{FF} \otimes 1_{cd}) \otimes 1_{ar \times T}.
\]

For \( t = t' \) and \( q \) a multiple of the unit matrix, the exponential part of the integrand in [23] depends on \( W \) only through the combination \( \tilde{W}\eta W \), which is invariant under transformations \( W \rightarrow gW \), \( \tilde{W} \rightarrow \tilde{W}g^{\dagger} \) if \( g \) satisfies \( g^{\dagger}g = 1 \). Invariance of the constraint \( \tilde{W} = CW^{T}\tau^{-1} \) requires \( g^{\dagger} = g^{\dagger}T\tau^{-1} \). The two conditions on \( g \) can be combined into the equations

\[
g = (\tau g)^{-1}T(\tau g^{-1}T^{-1}) = g^{\dagger}T\tau^{-1}.
\]

The first of these defines an orthosymplectic complex Lie supergroup Osp(8|8). The second fixes a noncompact (pseudo-unitary) real subgroup Osp(4,4|4), which we shall denote by \( G \) for short. \( G \) is the symmetry group of our problem and will play a central role later.

D. Hubbard-Stratonovitch transformation

Formula [23] is useful because it enables us to integrate out the matrix variables \( Q \) and \( W \) and be left with a small number (independent of \( N \)) of integrals to do. This is achieved as follows.

Combination of the harmonic oscillator identities [17] and [24] gives

\[
\langle 0\langle T \exp(\text{Tr} AQ(t)) \text{exp}(\text{Tr} BQ(t'))\langle 0\rangle \exp(h \text{Tr}(A^2 + B^2 + 2ABe^{-i\omega|t-t'|})/2m\omega).
\]

We apply this relation to [23] by identifying

\[
i\tilde{W}\eta\pi_{t}W/2 = A, \quad i\tilde{W}\eta\pi_{t'}W/2 = B.
\]

Putting \( a := i\pi_{t}W\tilde{W}\eta/2 \) and \( b := i\pi_{t'}W\tilde{W}\eta/2 \) and using the cyclic invariance of the trace, we rewrite the right-hand side of [23] as

\[
\exp(h \text{Str}(a^2 + b^2 + 2abe^{-i\omega|t-t'|})/2m\omega).
\]

The symbol \( \text{Str} \) denotes the supertrace. The next step is to linearize the quadratic form \( a^2 + b^2 + ... \) by the introduction of a Hubbard-Stratonovitch field \( M \). Let \( M \) be a supermatrix which has the same symmetries as the composite object \( W\tilde{W}\eta \), and let a Gaussian \( M \)-average be defined by

\[
\langle \bullet \rangle_{M} = \int dM \bullet \exp(-m\omega V(M))/2\hbar,
\]

\[
V(M) = \text{Str}(M_{tt} + M_{tt'} + 2e^{i\omega|t-t'|}M_{tt}M_{tt'}),
\]

where \( M_{ij} = \pi_{t}M_{ji} \) for \( i, j = t, t' \). We then claim that the expression [24] is equal to

\[
\langle \exp(\text{Str} M(a + b)) \rangle_{M} = \langle \exp\frac{1}{2} \text{Tr} \tilde{W}\eta MW \rangle_{M}.
\]

Formal verification of this claim is the simple matter of doing a Gaussian integral by completing the square. On a rigorous level, however, one needs to demonstrate that the integration contours for the bosonic-bosonic variables in \( M \) can be chosen to be uniformly convergent in \( W \).
This is a tricky matter which will not be discussed here but is well understood; see Ref. [2], Sect. 5.2.

Aside from being Gaussian in \( \hat{W} \), the last expression has the crucial property of being invariant under unitary transformations \( W \to W U, \hat{W} \to U \hat{W} U^\dagger (U \in \text{Sp}(2N) \subset U(2N)) \). This means that the particle-coordinate and spin degrees of freedom have now been completely decoupled. Hence, doing the Gaussian integral over the supermatrix \( W \) subject to the constraint \( \hat{W} = \hat{C} W^+ \tau^{-1} \), we get the inverse of a superdeterminant [22] raised to the \( N^\text{th} \) power:

\[
\int dW d\hat{W} \ e^{i \text{Tr} \hat{W} (M - q) W / 2} = \text{SDet}^{-N} (M - q).
\]

By differentiating this equation with respect to the parameters \((x - i\varepsilon)\) and \((w + i\varepsilon)\) which are contained in \( q \) and couple to \( \text{Tr} \hat{W} \eta IW \) and \( \text{Tr} \hat{W} \eta IW \), we can re-express \( g^{\pm} \) in (23) as an \( M \)-average:

\[
g^{\pm} = \frac{N^2}{4} (Z(M))_M, \quad Z(M) = \text{STr} \frac{I}{M - q} \text{STr} \frac{I'}{M - q} \text{SDet}^{-N} (M - q). \tag{28}
\]

Note the close similarity to the starting expression (18) and \([13]\) for \( g^{\pm} \): the resolvents and determinants in \( Q \)-space have simply been replaced by resolvents and superdeterminants in \( M \)-space.

E. Thermodynamic limit

The Gaussian integral transformation from \( Q \) to \( M \) has reduced the number of integration variables enormously, from order \( N^2 \) to order \( N^0 \). The large number \( N \) now appears explicitly in the integrand, allowing us to make a saddle-point approximation which becomes exact in the thermodynamic limit \( N \to \infty \).

Recall that we take \( N \to \infty \) holding the particle density \( \rho_0 = \sqrt{N m \omega / \hbar^2} \) fixed. We anticipate that the supermatrix \( M \) will be of order \( N^1 \) at and near the saddle point, whereas \( q \) (containing the positions \( x, y, w, z \)) and \( (t - t') \) are of order \( N^0 \). To extract the leading large-\( N \) behavior of the integral (23) and (24), we look for those \( M \) where the superfunction

\[
\Omega(M) = \text{SDet}^{-N} (M) \exp - m \omega \text{STr} M^2 / 2 \hbar. \tag{29}
\]

is maximal. By the relation

\[
\text{SDet}^{-N} (M) = \exp - N \text{STr} \ln M,
\]

such supermatrices are solutions of the saddle-point equation \( NM^{-1} + m \omega M / \hbar = 0 \) or, equivalently,

\[
M^2 = -(N/k_0)^2
\]

with \( k_0 = \pi \rho_0 \). They form supermanifolds that are invariant under “rotations” \( M \to g M g^{-1} \) by elements \( g \) of the symmetry group \( G \).

Let us now make \( M \) dimensionless by substituting for it the rescaled expression \( (i N/k_0) M \). The solution spaces of the rescaled saddle-point equation \( M^2 = 1 \) can be labelled by the eigenvalues of \( M \), which are equal to either \(+1\) or \(-1\). It turns out that such eigenvalues do not lie on the integration contour that is suggested by the requirement of uniform convergence for the bosonic-bosonic variables. However, a subset of the surfaces on which \( \Omega(M) \) is maximal can be reached by deformation of the integration contour using Cauchy’s theorem. As far as the bosonic-bosonic sector is concerned, the signs of the eigenvalues of \( M \) on such a “reachable” surface are uniquely determined by the pole structure of \( \Omega(M) \), which in turn is determined by the small imaginary parts \( \pm i \varepsilon \), omitted in (29); see [21], Sect. 5.3. As for the eigenvalues of \( M \) associated with the fermionic-fermionic sector, no constraints from convergence and analyticity exist. In this case, the determining agent is the limit \( N \to \infty \), which singles out the extremal surface with the largest dimension \( d = d_0 - d_1 \) (bosonic dimension minus fermionic dimension). By inspection one finds that \( d \) is largest for the surface containing the element

\[
\Lambda = \sigma_d \otimes 1_{bf} \chi_{cd} \chi_T.
\]

We restrict the integration in (23) to this surface by setting \( M = g \Lambda g^{-1} \). Expanding \( \text{SDet}^{-N} (i N k_0^{-1} M - q) \) with respect to \( q \) and \( \exp \omega |t - t'| \) with respect to \( |t - t'| \), and keeping only the terms that survive in the thermodynamic limit, we find

\[
g^{\pm} = -(k_0 / 2)^2 \int d\mu(M) \text{STr} (IM) \text{STr} (I'M) \times \exp -i k_0 \text{STr} q M + i \omega_0 |t - t'| \text{STr} M_{it} M_{it'} \tag{30}
\]

with \( \omega_0 = \hbar k_0^2 / m \). As usual, integration over the Gaussian fluctuations around the surface produces the uniform (or \( G \)-invariant) integration measure \( d\mu(M) \).

F. Choice of polar coordinates

What we did up to now was standard technology [3, 21, 24] and obvious to the expert. Now the real work begins! Our task is to calculate the integral (31). We begin with some notational and conceptual preparations.

The supermatrix \( M \) has dimension \( 16 \times 16 \), since the spaces \( V_{bf}, V_{cd}, V_{ar} \) and \( V_T \), on the tensor product of which \( M \) acts, are all two-dimensional. Recall the definition of the symmetry group \( G \) as the set of \( 16 \times 16 \) supermatrices \( g \) that obey the conditions (24). We endow \( G \) with its natural metric structure given by \( \text{STr} d\mu g^{-1} \). Let \( K \) be the subgroup of elements \( k \in G \) that commute with \( \Lambda \). Since \( M = g \Lambda g^{-1} \) is invariant under \( g \to gk \) \((k \in K)\), the integrand in (31) is a function on the coset space \( G/K \). Elements of this coset space are denoted by
$gK$, and the uniform integration measure on $G/K$ is denoted by $dg_K$. Thus, the integral \[ (30) \] is of the general form
\[
\int_{G/K} f(gK)dg_K.
\] (31)

To calculate such an integral we need to choose a suitable system of coordinates, or parameterization, of $G/K$. In making this choice we are guided by the general form the particle-hole Green’s function is expected to have: a good coordinate system should contain the velocities of the elementary excitations produced by the particle-hole operator as a subset, and it should make the integral separate into two form factors corresponding to the final time $t$ and initial time $t'$. In the present subsection and the one that follows, we will describe the parameterization that meets these expectations.

Let $\tau_3$ be defined by
\[
\tau_3 = \pi_t - \pi_t' = 1_{bf \times cd \times xar} \otimes \sigma_T^\tau.
\]

We denote by $G_e$ the subgroup of $G$ whose elements commute with $\tau_3$. In other words, $G_e$ is that part of the symmetry group which operates at fixed time. Furthermore, we denote by $A$ the six-dimensional abelian subgroup of $G$ generated by $(i, 1, 2)$
\[
H_1 = E^{BB} \otimes 1_{cd} \otimes (C^{+R}_A)^{T_{xar}},
\]
\[
H_2 = E^{BB} \otimes 1_{cd} \otimes (C^{+R}_A)^{T_{xar}},
\]
\[
H_{1+2i} = E^{FF} \otimes E^{ij}_p \otimes (C^{+R}_A)^{T_{xar}},
\]
\[
H_{2+2i} = E^{FF} \otimes E^{ij}_p \otimes (C^{+R}_A)^{T_{xar}},
\]
with $C^{+}_{ij} = E^{u'} \otimes E^{ij} \pm E^{u'i} \otimes E^{i}$. These generators are off-diagonal with respect to the chosen bases of $V_T$ and $V_{ar}$, which implies that they anticommute with $\tau_3$ and $\Lambda$. Therefore, the elements $a$ of the group $A$ satisfy
\[
a = \tau_3 a^{-1} \tau_3 = \Lambda a^{-1} \Lambda.
\]

Technically speaking, $A$ is a maximal abelian subgroup for the Cartan decomposition of $G$ w.r.t. $K$.

Given $G_e$ and $A$, we introduce nonstandard polar coordinates on $G/K$ by the map
\[
\phi : G_e/M \times A^+ \to G/K,
\]
\[
(gM, a) \mapsto g_0K,
\]
where $M$ is the subgroup of elements of $K_e = K \cap G_e$ which commute with all elements of $A$, and $A^+$ is some connected open subset of $A$ such that $\phi$ is bijective. The abelian factor $a$ contains the degrees of freedom which are “radial” with respect to the polar coordinate decomposition, while $g \in G_e$ will be parameterized by a set of “angular coordinates”.

By the substitution rule for superintegrals, the polar coordinate map $\phi$ transforms the invariant integral \[ (31) \] into
\[
\int_{A^+} \left( \int_{G_e} f(gaK)dg \right) J(a)da + ... \] (33)

Here $dg$ is a (suitably normalized) Haar-Berezin measure for $G_e$, $da$ is a Euclidean measure on $A$, and $J$ is the superjacobian of the transformation. The dots indicate correction terms (so-called “boundary terms”), which are due to the super nature of the integral and will be discussed later.

The Jacobian $J$ is calculated in Appendix A. If we put $\ln a = \sum_{i=1}^{2} \theta_iH_i + \sum_{i=1}^{3} \varphi_iH_i + 2$ and introduce
\[
v_1 = -v_s \cosh 2\theta_1, \quad v_2 = v_s \cosh 2\theta_2,
\]
\[
\bar{v}_1 = -v_s \cos 2\varphi_1, \quad \bar{v}_3 = v_s \cos 2\varphi_2,
\]
\[
\bar{v}_2 = -v_s \cos 2\varphi_3, \quad \bar{v}_4 = v_s \cos 2\varphi_4,
\]

our result for $J = |F(0)|^2$ (up to normalization) is given by $\int A$ with $\lambda = 2, n_p = 2$ and $n_h = 4$. We choose for $A^+$ the fundamental positive domain defined by
\[-\infty < v_1 < -v_s < \bar{v}_1 < \bar{v}_2 < \bar{v}_3 < \bar{v}_4 < v_s < v_2 < \infty.
\]

We will now extract from the argument of the exponential function in \[ (31) \] those parts which depend solely on the abelian factor $a$ parameterized by $\{v_1, \bar{v}_j\}$. The term multiplying $[t - t']$ is evaluated as follows:
\[-4 \text{STr} M_{1'i'}M_{1't} = - \text{STr}(1 + \tau_3)M(1 - \tau_3)M
\]
\[= \tau_3 M \tau_3 M = \tau_3 (a\Lambda a^{-1} - 1) = \tau_3 a^4 \]
\[= \tau_3 \text{cosh} 4a = 2 \text{STr} a \text{cosh}^2 2a \]
\[= \frac{8}{v_s^2} \sum_i v_i^2 - \frac{4}{v_s^2} \sum_j \bar{v}_j^2 = 16E(a)/mv_s^2, \] (35)

The term $\text{STr} qM$ containing the coordinates $x, y, w, z$ of the points of particle creation and annihilation is rewritten as
\[\text{STr} qM = \text{STr} q(ga^2\Lambda g^{-1})
\]
\[= \text{STr} q(ga^2\Lambda g^{-1} - a^2\Lambda) + \text{STr} A q \text{cosh} \text{2} \ln a.
\]

Defining $m_h = -m/2$ and introducing
\[P_p = m(v_1 + v_2) + m_h(\bar{v}_1 + \bar{v}_2), \quad P_h = m_h(\bar{v}_1 + \bar{v}_4),
\]
we can write the $g$-independent term in the form
\[-mv_s \text{STr} A q \text{cosh} \text{2} \ln a = (x - u)P_p(a) + (y - z)P_h(a).
\]

G. Identification of the form factor

The introduction of polar coordinates on $G/K$ by the map $\phi$ separates the integral \[ (33) \] over $M$ into a (radial) integral over the abelian group $A$ and an (angular) integral over the symmetry group $G_e$ operating at fixed time. The latter, in turn, separates into two commuting
factors, \( G_e = G_t \times G_{t'} \), one for each time. Defining for \( i = t, t' \) the projections
\[
q_i = \pi_i q, \quad K_i(a) = k_0 \pi_i \Lambda \cosh 2 \ln a,
\]
we get
\[
\text{STr } g(a^2 \Lambda g^{-1}) = \sum_{i=t,t'} \text{STr } g_i K_i(a) g_i^{-1}.
\]
Hence, the integral over \( G_e \) splits into two factors, one for \( G_t \) and another for \( G_{t'} \). Since these are formally identical, let us concentrate on the first one. We introduce \( \Lambda_t = \pi_t \Lambda \) and \( \pi_h = E^{FF} \otimes E^{RR} \otimes 1_{cd} \otimes E^{tt} \), \( \pi_p = \pi_t - \pi_h \).

\[
q_t = x \pi_p + y \pi_h - i \varepsilon \Lambda_t
\]
and, since \( \text{STr } g^{-1} \pi_t g = \text{STr } \pi_t = 0 \) for \( g \in G_t \), the integral over \( G_t \) is given by
\[
f_t^+(x - y; a) = \int_{G_t} dg \text{STr } (IgK_t(a)g^{-1}) \exp \left[-i \text{STr} \left(2(x - y)(\pi_p - \pi_h) - i \varepsilon \Lambda_t \right) \right] (36)
\]
To summarize, our complete expression for \( g^{+-} \) is
\[
g^{+-}_{xywz}(t) = \int_{A^+} f_t^+(x - y; a) f_0^+(w - z; a) \exp[-itE(a)/\hbar + i(x - w)P_p(a)/\hbar + i(y - z)P_h(a)/\hbar] J(a) da. (37)
\]
Having gone through a long and indirect derivation, we are finally in a position to interpret the mathematical structures at hand and translate them into physics. We see that a particle-hole excitation of the ground state of the Calogero-Sutherland model at \( \lambda = 2 \) fractionalizes into six elementary excitations. The velocities of these excitations appear as the degrees of freedom of a six-dimensional abelian group \( A \) in our treatment. There are two particle excitations with velocities \( v_1, v_2 \) and four hole excitations with velocities \( \bar{v}_1, \ldots, \bar{v}_4 \). Note that \( v_1 < -v_2 \) and \( v_2 > +v_s \) are on opposite sides of the pseudo-Fermi sea. The radial functions
\[
P(a) = \frac{1}{2} mv_s \text{STr } \pi_t \Lambda \cosh 2 \ln a,
\]
\[
E(a) = \frac{1}{2} mv_s^2 \text{STr } \cosh^2 2 \ln a,
\]
are the total momentum and energy of the particle-hole excitation. The equation \( P = P_p + P_h \) decomposes the momentum into its particle and hole components. Our expressions for \( P_h \) and \( P_p \) show that a single hole fractionalizes into two hole excitations (with velocities \( \bar{v}_3, \bar{v}_4 \)) whereas a single particle fractionalizes into two particle and two hole excitations (with velocities \( v_1, v_2; \bar{v}_1, \bar{v}_2 \)). Finally,
\[
F(x; a) = \lim_{\varepsilon \to 0} \frac{\varepsilon^2}{\pi x^2} f_t^+(x; a) \sqrt{J(a)} (38)
\]
is identified as the form factor:
\[
\langle 0 | \psi_y^+ | \psi_x^+ | a \rangle = \langle a | \psi_x^+ | \psi_y^+ | 0 \rangle = F(x - y; a) e^{ixP_p(a)/\hbar + iyP_h(a)/\hbar}.
\]
The last statement is not affected by the fact that \( g^{+-} \) is only one out of four contributions \( g^{\sigma \tau} (\sigma \tau = \pm 1) \) to the particle-hole Green’s function. The other contributions yield the same form factor; see Sect. \([\text{VII}]\).
It is possible, though by no means easy, to calculate the form factor \( F \) in closed analytic form by doing the integral in \([\text{X}]\). The result is neither simple nor illuminating, and we will not give it here. Substantial simplifications occur for \( |x - y| \gg 1/k_0 \). In this limit, \( F(x - y; a) \simeq F_p(a) \times F_h(a) \) becomes independent of \( x - y \) and separates into two factors for a single particle and a single hole. The technical steps of taking \( |x - y| \to \infty \), \( \varepsilon \to 0 \), and carrying out the final integrations, are done in Appendix B. Our result for the single-particle form factor has been presented in Sect. \([\text{X}]\). The result for the single-hole form factor is the one given previously \([\text{I}]\).

**H. Boundary terms**

Berezin’s proof \([\text{22}]\) of the substitution rule for superintegrals goes through without modification only for functions with compact support, or if the integration domain has no boundary. For polar-coordinate superintegrals such as \([\text{23}]\), whose radial domain \( A^+ \) does have a boundary, one expects extra terms (indicated by dots) to appear in general. The structure of such “boundary terms” was first investigated systematically by the mathematician Rothstein \([\text{24}]\). Unfortunately, Rothstein’s general theory is too implicit to be useful for anything but the simplest applications. In the present problem we are dealing with the polar-coordinate integral on a supermanifold with the rather special structure of a symmetric space. For this case, a powerful method for construction of all boundary terms has recently been developed \([\text{25}]\). The basic idea underlying this method is sketched for a simple example in Appendix C. Postponing the full exposition of the method to a future publication, we are now going to state the result obtained.

It turns out that all the boundary terms that occur for polar-coordinate superintegrals of the type \([\text{3}]\), are associated with nonintegrable singularities of the radial integration measure \( J(a) da \). While such singularities never
exist for ordinary (i.e. nonsuper) polar-coordinate integrals, they do exist in the present case because the superjacobian, being a superdeterminant, puts factors both in the numerator and in the denominator. By the nature of the polar coordinate map, the singularities all lie on the boundary \( \partial \mathbf{A}^+ \) of the radial space \( \mathbf{A}^+ \). For our choice of \( \mathbf{A}^+ \), nonintegrable singularities occur for (1) \( \psi_1 = \bar{\psi}_1 = \psi_2 = -\psi_3 \), and (2) \( \bar{\psi}_3 = \bar{\psi}_1 = \bar{\psi}_2 = \psi_4 \). Each of these conditions defines a three-dimensional subspace \( \mathbf{A}^+_i \). Their intersection defines a zero-dimensional subspace, which is the unit element to unity – i.e. no elementary excitations are present – the last boundary term, where the abelian factor citations, in agreement with the findings of [5]. Finally, other words, this term is equal to the product of ground results from setting \( \psi_1 = \bar{\psi}_1 = \bar{\psi}_2 = \psi_4 \). Correspondingly, there are three boundary terms in the present problem. To formulate them, let \( \mathbf{M}_i \) be the subgroup of elements of \( \mathbf{K}_0 \) which commute with all elements of \( \mathbf{A}^+_i \). Furthermore, let \( \mathbf{J}_i : \mathbf{A}^+_i \to \mathbf{R} \) be the reduced jacobian which is obtained by omitting from \( \mathbf{J} \) all singular factors. With these definitions, the complete formula for transformation to polar coordinates is

\[
\int_{G/K} f(gK)dg_K = \int_{A^+} \left( \int_{G_\mathbf{M}/M} f(gaK)dg_M \right) J(a)da + \sum_{i=1}^2 \int_{A^+_i} \left( \int_{G_{A^+_i}/M_i} f(gaK)dg_{M_i} \right) J_i(a)da + \int_{G_{A^+_i}/K_0} f(gK)dg_{K_0}.
\]

When the function \( f \) is given by the integrand of (30), this formula is interpreted as follows. The first term on the right-hand side (the “regular” term) corresponds to fractionalization of \( \psi_1^i \psi_2^i \psi_y^i \psi_y^i |0 \rangle \) into six elementary excitations, as discussed above. What the appearance of two boundary integrals with \( \dim \mathbf{A}^+_i = 3 \) means is that there exists the additional possibility for \( \psi_1^i \psi_2^i \psi_y^i \psi_y^i \) to fractionalize into three elementary excitations. Two of these are holes, and one is a particle. (It is not hard to show that the regular term vanishes by symmetry for \( x = y \), so that the dynamical density correlation function is completely determined by fractionalization into three elementary excitations, in agreement with the findings of (30)). Finally, the last boundary term, where the abelian factor \( a \) is set to unity – i.e. no elementary excitations are present – accounts for the fact that the ground state is contained in \( \psi_1^i \psi_2^i \psi_y^i \psi_y^i |0 \rangle \) with nonvanishing amplitude in general. In other words, this term is equal to the product of ground state expectation values \( \langle 0 | \psi_1^i \psi_2^i \psi_y^i \psi_y^i |0 \rangle \langle 0 | \psi_1^i \psi_2^i \psi_y^i \psi_y^i |0 \rangle \) which determines the long-time limit of the dynamical correlation function.

All of the boundary terms make nonvanishing contributions to \( g^{++} \), and hence to the full particle-hole Green’s function, in general. However, when the points \( x \) and \( y \) (or \( w \) and \( z \)) are removed from each other, only one of the boundary terms survives. This is the one that results from setting \( \psi_1 = \bar{\psi}_1 = \bar{\psi}_2 = \psi_4 \). To extract its contribution to the advanced part of the single-particle Green’s function, we proceed as in Appendix B. The reduction in dimensionality leads to somewhat simplified calculations in comparison with the regular term. The form factor is now a function of just a single variable \( v = v_2 \). Doing the calculation, and including the contribution from \( g^{-} \), we obtain the result for the elementary term in the single-particle Green’s function given in Sect. [V].

I. The terms \( g^{++} \) and \( g^{-} \)

Formula (18) for the particle-hole Green’s function

\[
G_{xywz}(t-t') = \langle 0 | \psi^i_x(t) \psi^i_w(t') \psi^i_z(t') |0 \rangle
\]

involves four terms: \( g^{++} \), \( g^{-} \), \( g^{--} \), and \( g^{-} \). We have shown in detail how to calculate \( g^{-} \), which is related to \( g^{++} \) by complex conjugation followed by time reversal. The calculation of \( g^{-} \) is identical except for the minor changes we are now going to indicate.

\( g^{-} \) differs from \( g^{++} \) only by \( C_{xy}^{-} (Q(t)) \) replacing \( C_{xy}^{++} (Q(t)) \) in (18). Consequently, to get \( g^{-} \) from \( g^{++} \), all we need to do is to replace the definitions of \( I \) and \( q \) by

\[
I = E^{BB} \otimes 1_{cd} \otimes E^{RR} \otimes E^{tt},
q = [(x-i\epsilon)E^{BB} + (y-i\epsilon)E^{FF} \otimes E^{AA} \otimes E^{tt} + (x+i\epsilon)(E^{BB} + E^{FF}) \otimes E^{RR} \otimes E^{tt} + ...] \otimes 1_{cd}.
\]

The supertrace in the expression for the phase factor \( \exp(-ik_0 q \Lambda \cosh 2 \ln a) \) changes to

\[
-q \Lambda \cosh 2 \ln a = 2(x-w)(v_1 + v_2) - (y-w)(\bar{v}_1 + \bar{v}_2) - (x-z)(\bar{v}_3 + \bar{v}_4).
\]

This equation illuminates the physical content of \( g^{-} \): since the variables \( v_i \) and \( \bar{v}_i \) are the velocities of the elementary excitations present in \( \psi^i_x(t') \psi^i_w(t') \psi^i_z(t') |0 \rangle \), we see that we are now dealing with a situation where the two holes created by \( \psi^i_z(t') \) are annihilated by \( \psi^i_x(t) \) (rather than by \( \psi^i_x(t) \) as before), while the two holes created by \( \psi^i_w(t') \) are annihilated by \( \psi^i_x(t) \). It is clear from this picture, and it is not hard to verify by direct calculation, that \( g^{-} \) vanishes in the limit \( |x-y| \to \infty \), \( x-w \) and \( y-z \) held fixed. The same statement applies to \( g^{++} \). Therefore, the single-particle Green’s function receives no contribution from \( g^{-} \) and \( g^{++} \) but is solely determined by \( g^{--} \) and \( g^{-} \).

J. The cases \( \lambda = 1/2 \) and 1

A major benefit from our basis-free approach is the ease with which the transcription to the case \( \lambda = 1/2 \) can be made. We now briefly describe the essential modifications, paying no attention to changes of normalization factors etc.
The harmonic oscillator variable $Q$ becomes a real symmetric matrix of dimension $N \times N$, while the basic time-ordered harmonic oscillator correlation function remains formally the same (for symmetric $A$ and $B$). The expression for the particle-hole function $O_{xy}$ is replaced by

$$O_{xy}(Q) = \Re \frac{\sqrt{2i\pi/2\pi i}}{y - x + i\varepsilon} C_{xy}^\varepsilon(Q),$$

$$C_{xy}^\varepsilon(Q) = \Tr \frac{1}{x - i\varepsilon - Q} \sqrt{\det (y + i\varepsilon - Q)}.$$

The trace of the resolvent is treated in the same way as before. To implement the symmetry requirement $(WW)^T = WW$, we take $W$ to be a linear function $W : C^N \to V_{bf} \otimes V_{cd}$ subject to the constraint

$$W = \tau \tilde{W}^T, \quad \tilde{W} = W^T \tau^{-1}.$$

Self-consistency now implies $\tau = +\sigma \tau^T$, and we choose

$$\tau = E^{BB} \otimes 1 + E^{FF} \otimes i\sigma^y.$$

Note that this is different from the case $\lambda = 2$ only by the exchange of the bosonic-bosonic and fermionic-fermionic sectors. Consequently, the cases $\lambda = 2$ and $\lambda = 1/2$ are connected by a duality transformation that simply exchanges these sectors. (See (10).)

To deal with the square root of the determinant, we write it as

$$\sqrt{\det(y + i\varepsilon - Q)} \sqrt{\det(x + i\varepsilon - Q)}.$$

For the numerator we introduce a complex fermion, and for the denominator two real bosons. (This is the same count as in the advanced $(-i\varepsilon)$ sector.)

The Green’s function $g^{+-}$ defined in (13) is still given by (23) but with

$$q = [(x - i\varepsilon)1_{bf} \otimes E^{AA} + (y + i\varepsilon)(E^{BB} \otimes E^{22} + E^{FF} \otimes 1_{cd}) \otimes E^{RR} + (x + i\varepsilon)(E^{BB} \otimes E^{11} \otimes E^{RR}) \otimes E^{tt} + ...].$$

The dots indicate analogous terms corresponding to time $t'$, which are unchanged. Hubbard-Stratonovitch transformation and thermodynamic limit remain the same.

The symmetry group $G$ is defined by (24) (with the modified expression for $\tau$), and the abelian group $A$ appearing in the polar decomposition of $G$ is again six-dimensional. From the duality symmetry connecting $\lambda = 1/2$ with $\lambda = 2$, we readily see that four of the six degrees of freedom of $A$ are now particles and two are holes. We choose the fundamental positive domain $A^+$

$$-\infty < v_1 < v_2 < -v_3 < \bar{v}_1 < \bar{v}_2 < v_3 < v_4 < \infty.$$
were developed in the context of random matrix theory and mesoscopic conductor physics. The CSM particle-hole Green’s function for \( \lambda = 1/2 \) and \( \lambda = 2 \) was shown to be a sum of four structurally identical terms, each of which is an integral over a Riemannian symmetric super-space \( G/K \). The spaces \( G/K \) for both couplings are real sections of the complex (ified) supermanifold \( G_c/K_c = \text{Osp}(8|8)/\text{Osp}(4|4) \times \text{Osp}(4|4) \). They transform into each other by a kind of duality (or particle-hole) transformation which exchanges the bosonic-bosonic and fermionic-fermionic sectors. The space \( G/K \) for \( \lambda = 1 \) is a real section of \( \text{Gl}(2\,\mathbb{C})/\text{Gl}(2\,\mathbb{C}) \times \text{Gl}(2\,\mathbb{C}) \).

The key to further progress was the identification of a suitable polar decomposition of \( G/K \) into a maximal totally geodesic flat submanifold (“radial” space) and a transverse manifold (“angular” space). Making this decomposition corresponds to expanding the particle-hole Green’s function in a basis of energy eigenstates:

\[
\langle 0 | \psi_y \psi_x | 0 \rangle = \sum_{\nu} \langle \nu | \psi_x \psi_y | 0 \rangle e^{-i(\xi - \lambda^2)} E_c / K \langle \nu | \psi_x \psi_y | 0 \rangle.
\]

The radial coordinates of \( G/K \) were interpreted as the velocities of the elementary particle and hole excitations which \( \psi_x \psi_y | 0 \rangle \) decomposes into when \( N \to \infty \). Thus, doing the radial integrals amounts to summing over \( \nu \) in that limit. Moreover, the transverse manifold was seen to be a direct product of two supermanifolds, one associated with initial time \( t' \) and the other with final time \( t \). Integration over (either) one of these yields the particle-hole form factor \( \langle \nu | \psi_x \psi_y | 0 \rangle \), see \((38,39)\).

The radial space of \( G/K \) has both compact and noncompact degrees of freedom. On making the identification with elementary excitations of the CSM, the former translate into holes, and the latter into particles. For \( \lambda = 1/2 \) (\( \lambda = 2 \)) we find two (four) holes and four (two) particles. The generalization of this to rational \( \lambda = p/q \) is that there will be \( 2p \) holes and \( 2q \) particles in \( \psi_x \psi_y | 0 \rangle \) in general. However, this is not the whole story yet. When the substitution rule is applied to a superintegral \( \int_{G/K} I(\xi) d\xi \), it is found that such an integral expressed in polar coordinates, must be corrected by the addition of several lower-dimensional integrals (the “boundary” terms). These corrections are a novel feature of superintegration, and are needed to restore invariance of the integral under translations \( \xi \mapsto g : \xi \) \((g \in G)\), which is lost when the naive substitution rule involving only the “regular” term is assumed. In our case there are three distinct boundary terms. For \( \lambda = 1/2 \) (\( \lambda = 2 \)), two of them have a radial integration domain which is three-dimensional, with two (one) of the radial degrees of freedom being noncompact, i.e. particle-like, and one (two) compact or hole-like. The remaining boundary term accounts for the ground state component in \( \psi_x \psi_y | 0 \rangle \). For \( x = y \) the regular term in the polar-coordinate integral vanishes, and we recover the result that the dynamical density correlations at \( \lambda = p/q \) are exhausted by contributions from \( p \) holes and \( q \) particles. The single-particle form factors were extracted from \( \langle \nu | \psi_x \psi_y | 0 \rangle \) by separation of points \((|x - y| \to \infty)\).

The methods described in this paper readily extend to many-particle Green’s functions. By following the steps of Sects. \[\text{[5][7]}\] we can, in principle, derive an exact expression for the form factor of any excited state \( |\nu \rangle \) which is produced by several actions of \( \psi \) and \( \psi^\dagger \) on the ground state. Such a form factor always separates into two parts. The first one, \( F^{(0)} \), is universal and is given by \((3)\). The second one, \( F^{(1)} \), is given in the form of an integral representation analogous to \((38)\). For example, for \( \lambda = 2 \) we get

\[
\langle \nu | \psi_{x_1} \cdots \psi_{x_n} | 0 \rangle = \left( F^{(0)} (\nu) F^{(1)} (x_1 \ldots x_n | \nu) \right).
\]

As a final remark, we are intrigued by the finding that the \( p \) holes and \( q \) particles excited by the CSM density operator, can be interpreted as radial degrees of freedom of a symmetric supermanifold \( G/K \) for \( \lambda = 1/2, 1, 2 \). We wonder whether there could exist a “quantum” deformation (with deformation parameter \( \lambda = p/q \)) of the symmetric supermanifold, such that the maximal “torus” of the deformed manifold had \( p \) compact and \( q \) noncompact directions. If such a deformation existed, the treatment of our paper might carry over to more general values of \( \lambda \), and the problem of calculating correlation functions for the Laughlin wavefunction in one dimension and in the thermodynamic limit, might be solved.

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**APPENDIX A: CALCULATION OF A JACOBIAN**

We are going to calculate the jacobian \( J \) associated with the map \( \phi : G_c/M \times A^+ \to G/K \). According to its definition by the substitution rule for superintegrals \((22)\), \( J \) is the superdeterminant of the supermatrix that expresses the differential of \( \phi \) in an orthonormal basis of the tangent spaces of \( G_c/M \times A^+ \) and \( G/K \). Let \( G, K, A, G_c, \) and \( M \) be the Lie algebras of \( G, K, A, G_c, \) and \( M \), respectively, and let \( P \) be the tangent space of \( G/K \) at the origin \( 1 \cdot K \). Since the invariant measure on \( G/K \) is not changed by left translation, \( J(a) \) equals the superdeterminant of the linear transformation \( T_a : (G_c - M) \times A \to P \) given by
where $\text{Ad}(a)Z = aZa^{-1}$ denotes the adjoint action, and $Z_\mathcal{P}$ means the $\mathcal{P}$-component of $Z \in \mathcal{G}$ with respect to the orthogonal decomposition $\mathcal{G} = \mathcal{K} + \mathcal{P}$.

To calculate $J(a) = \text{SDet} T_a$, we further decompose $\mathcal{K}$ and $\mathcal{P}$ into their even parts, $\mathcal{K}_e$ and $\mathcal{P}_e$, and odd parts, $\mathcal{K}_o$ and $\mathcal{P}_o$, with respect to the involutory automorphism $Z \mapsto \tau_3 Z \tau_3$. Since the elements of $\mathcal{A} \subset \mathcal{P}_o$ anticommute with both $\Lambda$ and $\tau_3$, the following commutation relations are obvious:

$$[\mathcal{A}, \mathcal{P}_e] \subset \mathcal{K}_o, \quad [\mathcal{A}, \mathcal{K}_e - \mathcal{M}] \subset \mathcal{P}_o - \mathcal{A}, \quad [\mathcal{A}, \mathcal{K}_o] \subset \mathcal{P}_e, \quad [\mathcal{A}, \mathcal{P}_o - \mathcal{A}] \subset \mathcal{K}_e - \mathcal{M}.$$ 

From these we see that, if $Z = X + Y$ is the decomposition of $Z$ by $\mathcal{G}_e - \mathcal{M} = \mathcal{P}_e + (\mathcal{K}_e - \mathcal{M})$, then

$$T_a(X + Y, H) = H + \cosh \text{ad}(\ln a) \ X - \sinh \text{ad}(\ln a) \ Y$$

where we have introduced $\text{ad}(\ln a) X := [\ln a, X]$ (commutator) and used the relation $\text{Ad}(a) = \exp(\text{ad}(\ln a))$. It follows that

$$J(a) = \text{SDet} \left( \begin{array}{c} \cosh \text{ad}(\ln a) \mid_{\mathcal{P}_e - \mathcal{P}_o} \times \vspace{1em} \\ \sinh \text{ad}(\ln a) \mid_{\mathcal{K}_e - \mathcal{M} - \mathcal{P}_e - \mathcal{A}} \end{array} \right).$$

The superdeterminant of a supermatrix equals the product of its bosonic eigenvalues divided by the product of its fermionic ones. We thus need the eigenvalues of the linear operators

$$\text{ad}(\ln a) : \mathcal{K}_o + \mathcal{P}_e \rightarrow \mathcal{P}_e + \mathcal{K}_o, \quad \text{ad}(\ln a) : \mathcal{K}_o + \mathcal{P}_o \rightarrow \mathcal{P}_o + \mathcal{K}_o.$$

Because $\text{ad}(\ln a)$ is block off-diagonal, these occur in pairs with opposite sign. Let us denote a set of positive eigenvalues by $\Delta_+^a$ and $\Delta_+^o$, respectively. Furthermore, let $|m_a|$ denote the multiplicity of eigenvalue $a$, and put sign$(m_a) = +1$ ($-1$) for a bosonic (fermionic). Then the expression for $J(a)$ becomes

$$J(a) = \prod_{a \in \Delta_+^o} \left( \sinh a(\ln a) \right)^{m_a} \prod_{\beta \in \Delta_+^o} \left( \cosh a(\ln a) \right)^{m_\beta}.$$ 

Note that the derivation of this formula has used no more than the generic geometric structures underlying any symmetric space $G/\mathbb{K}$ with an involution $\tau_3$. It is therefore valid for all of the cases $\lambda = 1/2$, $\lambda = 1$ and $\lambda = 2$.

Calculation of eigenvalues $\alpha$ (also called roots) and multiplicities $m_a$ is a standard exercise in linear algebra. Adopting the parametrization $\ln a = \sum_{i=1}^2 \theta_i H_i + \sum_{i=1}^2 \phi_i H_{i+2}$ with $H_i$ given in (32) we find the following eigenvalues (resp. multiplicities) for $\lambda = 2$:

$$\Delta_+^b : \quad \theta_1 \pm \theta_2 (\pm 1),$$

$$\Delta_+^o : \quad \theta_1 \pm \theta_2 (\pm 1), \quad \theta_1 \pm \theta_2 (\pm 2) \quad (k + l \text{ odd}),$$

$$\Delta_+^b : \quad 2\theta_1 (\pm 3) \quad (k = 1, 2),$$

$$\Delta_+^o : \quad \theta_1 \pm \theta_2 (\pm 1) \quad (k < l; \ k + l \text{ even}),$$

$$\theta_1 \pm \theta_2 (\pm 2) \quad (k + l \text{ even}).$$

The roots $\alpha$ for $\lambda = 1/2$ are obtained from this by a simple duality transformation $\alpha \mapsto \imath \alpha$.

The Jacobian $J$ is easily converted to the form given in (31), $J \propto |F(0)|$, by using the trigonometric identities

$$2 \sinh(A + B) \sinh(A - B) = \cosh(2A) - \cosh(2B),$$

$$2 \cosh(A + B) \cosh(A - B) = \cosh(2A) + \cosh(2B),$$

and recalling the substitutions (34). The duality symmetry (31) relating $\lambda = 2$ with $\lambda = 1/2$ is a consequence of the duality of the corresponding root systems ($\alpha \leftrightarrow \imath \alpha$).

**APPENDIX B: CALCULATION OF THE SINGLE-PARTICLE FORM FACTOR**

We are going to extract the single-particle form factor $F_p(\rho) = \langle 0|\psi_\rho|a \rangle \exp(-iP_p(a)/\hbar)$ from the particle-hole form factor $F(x - y; a)$ by taking $|x - y| \rightarrow \infty$ in (30) and (33). In this limit, the integrals over some of the degrees of freedom of $G_t \simeq Uosp(2,2|4)$ can be done by stationary phase. We begin by isolating these degrees of freedom.

First of all, we make the transformation $g \rightarrow g^{-1}$ which turns the term coupling to $x - y$ into $\text{STr} K_t(a)(\pi_p - \pi_h)g^{-1}$ and, being an isometry, leaves $dg$ unchanged. We abbreviate $\pi_p - \pi_h := D$. Next, we decompose $G_t$ by $G_t \simeq (G_t/\mathbb{H}) \times \mathbb{H}$ where

$$H = \{ h \in G_t | h D h^{-1} = D \},$$

and we write $g \in G_t$ as $g = sh$, accordingly. The factor $h \in \mathbb{H}$ drops out of the expression $g D g^{-1} = s D s^{-1}$. Moreover, the measures match under factorization, i.e. $dg$ turns into $ds_d h dh$, with $ds_d$ ($dh$) being the invariant measure on $G_t/\mathbb{H}$ (H). Now, because the phase of

$$A(s) = \exp(-\frac{1}{2}(x - y) \text{STr} K_t(a) (s D s^{-1} - D))$$

oscillates rapidly for $|x - y| \gg 1/k_0$, the integral over $G_t/\mathbb{H}$ can be done by using the stationary phase approximation. Setting $s = \exp X$ and doing the Gaussian integral that results upon expansion to lowest nonvanishing order in $X$, we get

$$\langle x - y \rangle^{-2} \int_{G_t/\mathbb{H}} A(s) ds_d h |x - y|^{-2 \infty}$$

$$\prod_{i=1}^4 \left( \frac{\bar{v}_2 - \bar{v}_1}{\bar{v}_1 - \bar{v}_2} \right)^{1/2} \left( \bar{v}_1 - \bar{v}_2 \right)^{1/2}$$
This expression precisely cancels that part of the phase space factor \( \sqrt{J} \) which connects the velocities of the “particle set” \( \{v_1, v_2, \bar{v}_1, \bar{v}_2\} \) with those of the “hole set” \( \{\bar{v}_3, \bar{v}_4\} \), so that \( \sqrt{J} \) gets replaced by the product \( \sqrt{J_p} \times \sqrt{J_h} = F_p(0) \times F_h(0) \) (Eq. (3)). By the particle-hole structure of \( D = \pi_p \times \pi_h \), the group \( H \) decomposes as

\[
    H \simeq H_p \times H_h = \text{Uosp}(2,2|2) \times O(2).
\]

Hence, the form factor separates as expected: \( F \to F_p \times F_h \), and we have \( F_p = F_p(0) \times F_p'(1) \), \( F_h = F_h(0) \times F_h(1) \). The factor pertaining to the single hole is trivial,

\[
    F_h^{(1)}(a) = \int_{H_h} dh = \text{const},
\]

while the single-particle form factor \( F_p^{(1)}(a) \) equals

\[
    \varepsilon^2 \int_{H_p} dh \, \text{STr}(K_i(a) h h^{-1} \varepsilon) e^{-\varepsilon \text{STr} K_i(a)(\varepsilon \Lambda_i h^{-1} - \Lambda_i)},
\]

where a normalization constant has been omitted.

The remaining task is to carry out the integration over the group \( H_p \). Simplifying the notation by writing \( K \) for \( K_i(a) \) and omitting a term \( \varepsilon \text{STr} \Lambda_i K \), which vanishes in the limit \( \varepsilon \to 0 \), we encounter the integral

\[
    \varepsilon^2 \int_{H_p} dh \, \text{STr}(K h h^{-1} \varepsilon) \exp(-\varepsilon \text{STr} K h \Lambda_i h^{-1}).
\]

We will do this integral in several steps. The first one is to factor \( H_p \) by \( H_p \simeq (H_p/\Lambda_i) \times \Lambda_i \) with \( H_\Lambda \simeq \text{Uosp}(2|2) \times \text{Sp}(2) \) being the subgroup of \( H_p \) that fixes \( \Lambda_i \) w.r.t. the action \( \Lambda_i \mapsto h \Lambda_i h^{-1} \). Again, the invariant measures match. \( H_\Lambda \) is a maximal compact subgroup of \( H_p \), so \( H_p/\Lambda_i \) is a (super-)symmetric space of the noncompact type. The integrand is constant on \( \text{Sp}(2) \) and depends only on \( \text{Uosp}(2|2) \) through the combination \( h h^{-1} \varepsilon \) in the pre-exponential factor. Therefore, the integral over \( H_\Lambda \) is trivial and can be done by “orthogonality”. (By orthogonality we mean the relation

\[
    \int_G dU \, D(U)_{\alpha\beta} D(U^{-1})_{\gamma\delta} = c_0(-1)^{[\beta][\delta]} \delta_{\alpha\delta} \delta_{\beta\gamma},
\]

valid for any irreducible representation \( D \) of a compact supergroup \( G \).) Thus, with \( \pi_+ \) being the \( \text{Uosp}(2|2) \) scalar \( \pi_+ = 1_{\Lambda_i} \otimes E^{AA} \otimes E^{tt} \), integration over \( \Lambda_i \) simply causes the substitution \( h h^{-1} \varepsilon \to c_0 h_+ \varepsilon h^{-1} \), followed by restriction of the integration domain from \( H_p \) to \( H_p/\Lambda_i \).

To do the integral over the latter, we exploit the simplifications that arise from the limit \( \varepsilon \to 0 \). What must happen is that the integral over the noncompact manifold \( H_p/\Lambda_i \) becomes singular as \( \varepsilon^2 \) for \( \varepsilon \to 0 \), thereby cancelling the prefactor \( \varepsilon^2 \). This singular behavior implies that the integral is dominated by the contributions from the asymptotic region on \( H_p/\Lambda_i \). We can therefore replace the integral by its contraction to the asymptotic region. Using the relation \( h(\Lambda_i - 2\pi_+) h^{-1} = \Lambda_i - 2\pi_+ \)

we see that \( \pi_+ \) may be replaced by \( \Lambda_i/2 \) in the integrand with an error that becomes negligible in the limit \( \varepsilon \to 0 \). We set \( Q := \varepsilon \Lambda_i h^{-1} \), which satisfies \( Q^2 = \varepsilon^2 \), so that \( Q^2 = 0 \) on the contracted (or “light-cone”) manifold that emerges in the limit \( \varepsilon \to 0 \). The invariant measure \( \varepsilon \cdot dh_{H_p}/\Lambda_i \) can be shown to contract to the invariant measure \( dQ \) on the light-cone \( Q^2 = 0 \). Hence, the integral giving the single-particle form factor \( F_p^{(1)} \) reduces to

\[
    \int_{Q^2=0} dQ \, \text{STr} K Q e^{-\text{STr} K Q} = -\frac{d}{dt} \bigg|_{t=1} \int_{Q^2=0} dQ \, e^{-i \text{STr} K Q}.
\]

To do the integral over \( Q \), we write

\[
    Q = \sum_{i,j=A,R} (Q_{ij})_{bf} \otimes E^{ij}_{ar} \otimes E^{tt},
\]

and solve the constraint \( Q^2 = 0 \) by putting

\[
    Q_{AA} = ZZ^\dagger, \quad Q_{AR} = -Z\sqrt{ZZ^\dagger}, \quad Q_{RA} = \sqrt{ZZ^\dagger} Z, \quad Q_{RR} = -Z^\dagger Z,
\]

with

\[
    Z = E^{BB} \otimes \left( \begin{pmatrix} z_1 & z_2 \\ -z_2 & \bar{z}_1 \end{pmatrix} \right)_{cd} + E^{tt} \otimes \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right)_{cd}.
\]

The invariant measure \( dQ \) in these variables is the Euclidean measure \( dZ dZ^\dagger = d z_1 d z_2 d\bar{z}_1 d\bar{z}_2 \partial z_1 \partial\bar{z}_1 \partial z_2 \partial\bar{z}_2 \) multiplied by \( \text{STr} Z Z^\dagger \). Upon making the decomposition

\[
    K = \sum_{i=A,R} (K_i)_{bf} \otimes E^{tt}_{ar} \otimes E^{tt},
\]

we get \( \text{STr} K Q = \text{STr} Z Z^\dagger (K_A Z - Z K_R) \), and our integral becomes

\[
    -\frac{d}{dt} \bigg|_{t=1} \int dZ dZ^\dagger \, \text{STr} Z Z^\dagger e^{-i \text{STr} Z Z^\dagger (K_A Z - Z K_R)}. \tag{39}
\]

By the property of perfect grading (i.e. equal number of bosonic and fermionic degrees of freedom), the Euclidean measure \( dZ dZ^\dagger \) is invariant under scale transformations \( Z \to Z/\sqrt{t} \). Therefore, carrying out this transformation and then taking the derivative at \( t = 1 \), we obtain

\[
    \int dZ dZ^\dagger \, \text{STr} Z Z^\dagger e^{-i \text{STr} Z Z^\dagger (K_A Z - Z K_R)}.
\]

This is a Gaussian integral which is easy to calculate. The result is the one given in Sect. [17].

We conclude this appendix by sketching the evidence in favor of the conjecture \([30]\). When proper meaning is given to the quantities \( K \) and \( Q \), the single-particle form factor \( F_p^{(1)} \) can be written in the form \([39]\) for all the cases \( \lambda = 1/2, 1, 2 \) accessible to our
method. The invariant measure $dQ$ takes the general form $dZdZ^\dagger(\text{STr } ZZ^\dagger)^{\lambda-1}$. By doing the same manipulations as before, we obtain

$$F_p^{(1)} = (\lambda - 1) \int dZdZ^\dagger(\text{STr } ZZ^\dagger)^{\lambda-1} e^{\text{STr } Z^\dagger \{Z K_R - K_A Z\}}.$$ 

For integral $\lambda$, the pre-exponential factor $(\text{STr } ZZ^\dagger)^{\lambda-1}$ can be generated by shifting $K_R \to K_R + (k_0v/v) \cdot 1$ and taking $\lambda - 1$ derivatives w.r.t. $v$ at $v = 0$. If we switch to the notation of Sect. [VI], the resulting Gaussian integral equals

$$\int dZdZ^\dagger e^{-\text{STr } Z^\dagger \{K_A Z - Z K_R - Z k_0 v/v_s\}} = \prod_{i=1}^{p} (v_{p+1} + v - v_{i}) \prod_{j=1}^{p} (v_{p+1} + v - v_{j})^\lambda$$

in all cases. This is why we believe that $[18]$ holds true in general.

**APPENDIX C: CONSTRUCTION OF BOUNDARY TERMS**

We sketch our general procedure for constructing the boundary terms that are associated with polar-coordinate superintegrals on supersymmetric spaces. Consider the simple example of the Euclidean superplane $E_{2|2}$ with complex supercoordinates $(z, \bar{z}, \zeta, \bar{\zeta})$ and invariant Berezin integral

$$\int_{E_{2|2}} f d\mu = (4\pi i)^{-1} \int_{\mathbb{C}} \left( \partial_z \partial_{\bar{z}} f(z, \bar{z}, \zeta, \bar{\zeta}) \right) dzd\bar{z}.$$ 

The metric of $E_{2|2}$, $d\Omega^2 + d\zeta d\bar{\zeta}$, is preserved by transformations

$$\left( \begin{array}{c} z \\ \bar{z} \\ \zeta \\ \bar{\zeta} \end{array} \right) \mapsto \left( \begin{array}{c} k z \\ k \bar{z} \\ \zeta \\ \bar{\zeta} \end{array} \right)$$

if $k \in U(1|1)$. We introduce polar coordinates $(k, r) = \phi^{-1}(z, \bar{z}, \zeta, \bar{\zeta})$ via the diffeomorphism $\phi : U(1|1)/U(1) \times \mathbb{R}^+ \to E_{2|2}$ defined by

$$\left( \begin{array}{c} z \\ \bar{z} \\ \zeta \\ \bar{\zeta} \end{array} \right) = k \left( \begin{array}{c} r \\ 0 \\ 0 \\ 0 \end{array} \right).$$

The substitution rule for superintegrals yields

$$\int_{E_{2|2}} f d\mu = \int_{\mathbb{R}^+} \left( \int_{U(1|1)} (f \circ \phi)(k, r) dk \right) r^{-1} dr + \mathcal{R}[f]$$

where we have extended the angular integration to the group $U(1|1)$ for convenience. The normalization constant $1/4\pi i$ has been absorbed into the Haar-Berezin measure $dk$. $\mathcal{R}[f]$ is a correction term which is due to the radial space $\mathbb{R}^+$ having a boundary. To construct it, we exploit the translational invariance of the Berezin integral on $E_{2|2}$ as follows. We write $X := (z, \bar{z}, \zeta, \bar{\zeta})$ for short and denote the scalar product of two vectors $X, X'$ by

$$\langle X, X' \rangle = \Re \langle \bar{z}' + \bar{\zeta}' \rangle.$$ 

If $X_0 = (z_0, \bar{z}_0, \zeta_0, \bar{\zeta}_0)$ is some set of parameters, we define a directional derivative $D_0 f$ by

$$D_0 f = \langle X_0, \text{grad} f \rangle.$$ 

From $\int_{E_{2|2}} (D_0 f) d\mu = 0$ we deduce

$$\mathcal{R}[D_0 f] = - \int_{\mathbb{R}^+} \left( \int_{U(1|1)} (D_0 f \circ \phi)(k, r) dk \right) r^{-1} dr.$$ 

By first decomposing the directional derivative into its radial and angular parts, then partially integrating, and finally recombinining terms, we obtain the relation

$$\frac{1}{r} \int_{U(1|1)} (D_0 f \circ \phi)(k, r) dk = \frac{\partial}{\partial r} \left( \frac{1}{r} \int_{U(1|1)} \langle k^{-1} \cdot X_0, e_1 \rangle (f \circ \phi)(r, k) dk \right)$$

where $e_1$ is the unit vector $e_1 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$, and $k^{-1} \cdot X_0$ denotes the element $X_0$ rotated by $k^{-1}$. We now integrate the total radial derivative to get

$$\mathcal{R}[D_0 f] = \lim_{r \to 0} \frac{1}{r} \int_{U(1|1)} \langle k^{-1} \cdot X_0, e_1 \rangle (f \circ \phi)(k, r) dk.$$ 

By Taylor expanding $f$ to linear order in $r$ and doing the resulting integral over $U(1|1)$, we obtain the expression

$$\mathcal{R}[D_0 f] = \langle X_0, \text{grad} f(0) \rangle = (D_0 f)(0),$$

from which conclude

$$\mathcal{R}[f] = f(0).$$

This elementary result is well-known; for a direct derivation see e.g. [29]. The above construction has the great virtue of using no more than the generic structures at hand. It therefore generalizes easily to more complicated situations. In particular, it can be used to construct the boundary terms associated with the polar-coordinate integral $[33]$. The result is as stated in the text.

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