GENERATING THE LEVEL 2 SUBGROUP BY INVOLUTIONS

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Abstract. We obtain a minimal generating set of involutions for the level 2 subgroup of the mapping class group of a closed nonorientable surface.

1. Introduction

Let \( N_g \) be a closed nonorientable surface of genus \( g \geq 2 \). The mapping class group \( \text{Mod}(N_g) \) is defined to be the group of isotopy classes of all diffeomorphisms of \( N_g \). The first homology group \( H_1(N_g; \mathbb{Z}) \) is generated by \( \{x_1, x_2, \ldots, x_g\} \), where \( x_i \) for \( 1 \leq i \leq g \) are the homology classes of one-sided curves as depicted in Figure 1.

![Figure 1: Generators of \( H_1(N_g; \mathbb{Z}) \).](image)

The \( \mathbb{Z}_2 \)-homology classes \( \pi_i \) of these curves form a basis for \( H_1(N_g; \mathbb{Z}/2\mathbb{Z}) \). The \( \mathbb{Z}_2 \)-valued intersection pairing is a symmetric bilinear form \( \langle \cdot, \cdot \rangle \) on \( H_1(N_g; \mathbb{Z}/2\mathbb{Z}) \) satisfying \( \langle \pi_i, \pi_j \rangle = \delta_{ij} \) for \( 1 \leq i, j \leq g \). For more on automorphisms of \( H_1(N_g; \mathbb{Z}/2\mathbb{Z}) \) and \( \mathbb{Z}_2 \)-valued intersection pairings we refer the reader to [2]. Let \( \text{Iso}(H_1(N_g; \mathbb{Z}/2\mathbb{Z})) \) be the group of automorphisms of \( H_1(N_g; \mathbb{Z}/2\mathbb{Z}) \) which preserve \( \langle \cdot, \cdot \rangle \). The level 2 subgroup \( \Gamma_2(N_g) \) of \( \text{Mod}(N_g) \) is the group of isotopy classes of diffeomorphisms which act trivially on \( H_1(N_g; \mathbb{Z}/2\mathbb{Z}) \). It fits into the following short exact sequence:

\[
1 \longrightarrow \Gamma_2(N_g) \longrightarrow \text{Mod}(N_g) \longrightarrow \text{Iso}(H_1(N_g; \mathbb{Z}/2\mathbb{Z})) \longrightarrow 1.
\]

For a two-sided simple closed curve \( \alpha \) and a one-sided simple closed curve \( \mu \) which intersect in one point, let \( K \) denote a regular neighborhood of \( \mu \cup \alpha \) that is homeomorphic to the Klein bottle with a hole. Let \( M \subset K \) be a regular neighborhood of \( \mu \), which is a Möbius strip. We define the crosscap slide (also called \( Y \)-homeomorphism) \( Y_{\mu, \alpha} \) as the self-diffeomorphism of \( N_g \) obtained from sliding \( M \) once along \( \alpha \) and fixing each point of the boundary of \( K \) (Figure 2).

For \( I = \{i_1, i_2, \ldots, i_k\} \) a subset of \( \{1, 2, \ldots, g\} \), let \( \alpha I \) be the simple closed curve shown in Figure 3. Throughout the paper, we introduce the following notations:

- \( Y_{i_1;i_2;\ldots;i_k} = Y_{\alpha_{i_1}^{i_2}\cdots\alpha_{i_k}} \),

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Szepietowski proved that $\Gamma_2(N_g)$ is equal to the subgroup of Mod($N_g$) generated by all crosscap slides [3, Theorem 5.5]. Moreover, he proved that $\Gamma_2(N_g)$ can be generated by (infinitely many) involutions [3, Theorem 3.7]. In [4], Szepietowski also gave a finite set of generators for $\Gamma_2(N_g)$. Later, Hirose and Sato reduced the number of generators of $\Gamma_2(N_g)$, their generating set is as follows [1, Theorem 1.2].

**Theorem 1.1.** For $g \geq 4$, the level 2 subgroup $\Gamma_2(N_g)$ is generated by the following two types of elements:

1. $Y_{i,j}$ for $i \in \{1, 2, \ldots, g-1\}$, $j \in \{1, 2, \ldots, g\}$ and $i \neq j$;
2. $T_{i,j,k,l}^2$ for $1 < j < k < l$.

Note that when $g = 3$, the group $\Gamma_2(N_3)$ is generated only by the elements of type (1). Hirose and Sato [1, Theorem 1.4] also showed that for $g \geq 4$

$$H_1(\Gamma_2(N_g); Z) \cong (\mathbb{Z}/2\mathbb{Z})(1/2) + (1),$$

which in turn implies that the above generating set is minimal.

In this paper, our purpose is to give a minimal generating set of involutions for the level 2 subgroup $\Gamma_2(N_g)$. 
2. A generating set for $\Gamma_2(N_g)$

Let us start this section by introducing bar notation for two-sided simple closed curves. In the remainder of this paper, let $\alpha_{1,i,j,k}$ and $\alpha_{i,j}$ be two sided simple closed curves depicted in Figure 4. Observe that when we put a bar over a two-sided simple closed curve it passes below the in-between crosscaps. For the ease of notation, we also use the following notations:

- $\overline{Y}_{i,j} = Y_{\alpha_{i},\alpha_{i,j}}$,
- $\overline{T}_{1,i,j,k} = T_{\alpha_{1,i,j,k}}$.

Recall that $\Gamma_2(N_g)$ is generated by all crosscap slides [3, Theorem 5.5]. Let $\mathcal{Y}$ and $\overline{\mathcal{Y}}$ be the subgroups of $\Gamma_2(N_g)$ generated by elements of the form $Y_{i,j}$ and $\overline{Y}_{i,j}$, for $i \in \{1, 2, \ldots, g - 1\}$, $j \in \{1, 2, \ldots, g\}$ and $i \neq j$, respectively.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{The curves $\overline{\alpha}_{i,j}$ and $\alpha_{i,j}$ for $1 < j < k < l$.}
\end{figure}

Lemma 2.1. The subgroups $\mathcal{Y}$ and $\overline{\mathcal{Y}}$ are equal to each other.

Proof. Let us first show that $\overline{\mathcal{Y}} \subseteq \mathcal{Y}$. For $\overline{Y}_{i,j} \in \overline{\mathcal{Y}}$, if we assume that $|i - j| = 1$, since

$$
\overline{Y}_{i,i+1} = Y_{\alpha_i,\alpha_{i,i+1}}, \quad \overline{Y}_{i+1,i} = Y_{\alpha_{i+1},\alpha_{i,i+1}}
$$

for all $i = 1, 2, \ldots, g - 1$, we have $\overline{Y}_{i,j} \in \mathcal{Y}$. Assume now that $|i - j| > 1$: For $i < j$, let us first consider the case $j - i = 2$. It is easy to verify that

$$
\overline{Y}_{i+1,i+2}^{-1}(\alpha_i, \alpha_{i,i+2}) = (\alpha_i, \alpha_{i,i+2}),
$$

for all $i = 1, \ldots, g - 2$ (see Figure 5). Using $\overline{Y}_{i+1,i+2} = Y_{i+1,i+2} \in \mathcal{Y}$, we have

$$
Y_{i,i+2}^{-1}Y_{i+1,i+2}^{-1}Y_{i,i+2}Y_{i+1,i+2} \in \mathcal{Y}.
$$

For the case $j - i = 3$, one can see that (see Figure 6)

$$
\overline{Y}_{i+1,i+3}^{-1}\overline{Y}_{i+2,i+3}^{-1}(\alpha_i, \alpha_{i,i+3}) = (\alpha_i, \alpha_{i,i+3}).
$$

Now, since $\overline{Y}_{i+2,i+3}$ and $\overline{Y}_{i+1,i+3}$ are all contained in $\mathcal{Y}$ we have

$$
\overline{Y}_{i,i+3} = (\overline{Y}_{i+1,i+3}^{-1}\overline{Y}_{i+2,i+3}^{-1}Y_{i,i+2}\overline{Y}_{i+1,i+3}^{-1}\overline{Y}_{i+2,i+3}^{-1})^{-1} \in \mathcal{Y}
$$
for \( i = 1, \ldots, g - 3 \). For the remaining \( i < j \) cases, one can see that
\[
\overline{Y}_{i,j} = (\overline{Y}_{i+1,j}^{-1} \overline{Y}_{i+2,j}^{-1} \cdots \overline{Y}_{j-1,j}^{-1})Y_{i,j}(\overline{Y}_{i+1,j}^{-1} \overline{Y}_{i+2,j}^{-1} \cdots \overline{Y}_{j-1,j}^{-1})^{-1} \in \mathcal{Y}
\]
for all \( i = 1, \ldots, g - 1, \ j = 1, \ldots, g \).

Now, we consider the cases where \( i > j \). For \( i - j > 2 \), we have (see Figure 7)
\[
\overline{Y}_{i+1,i}(\alpha_{i+2}, \alpha_{i,i+2}) = (\alpha_{i+2}, \alpha_{i,i+2}).
\]

Using \( \overline{Y}_{i+1,i} \in \mathcal{Y} \) for \( i = 1, \ldots, g - 3 \), we get
\[
\overline{Y}_{i+2,i} = \overline{Y}_{i+1,i} \overline{Y}_{i+2,i} \overline{Y}_{i+1,i}^{-1} \in \mathcal{Y}.
\]
As before, for all \( i = 1, \ldots, g - 1 \) and \( j = 1, \ldots, g - 2 \), we have
\[
\overline{Y}_{i,j} = (\overline{Y}_{i,j-1} \cdots \overline{Y}_{i,i+1})Y_{i,j}(\overline{Y}_{i,j-1} \cdots \overline{Y}_{i,i+1})^{-1} \in \mathcal{Y}.
\]
Thus, $\mathcal{Y}_{i,j} \in \mathcal{Y}$ for $1 \leq i < j \leq g$. Since we cover all the cases, we have $\mathcal{Y} \subseteq \mathcal{Y}$. For the reverse inclusion, note that we have the following equalities

\begin{align*}
(1) \quad Y_{i,j} &= \left\{ \begin{array}{ll}
(Y_{i+1,j} \cdots Y_{j-1,j})^{-1}Y_{i,j}(Y_{i+1,j} \cdots Y_{j-1,j})^{-1} & \text{if } i < j, \\
(Y_{i,j}^{-1} \cdots Y_{i+1})^{-1}Y_{i,j}(Y_{i,j}^{-1} \cdots Y_{i+1})^{-1} & \text{if } i > j,
\end{array} \right.
\end{align*}

which immediately imply that $\mathcal{Y} \subseteq \mathcal{Y}$. □

Next, we present a minimal generating set for the level 2 subgroup $\Gamma_2(N_g)$ (cf. [1, Theorem 1.2]).

**Theorem 2.2.** For $g \geq 4$, the level 2 subgroup $\Gamma_2(N_g)$ can be generated by

1. $\mathcal{Y}_{i,j}$ for $i \in \{1, \ldots, g-1\}$, $j \in \{1, \ldots, g\}$ and $i \neq j$,
2. $T_{1,i,j,k}^2$ for $1 < i < j < k$.

**Proof.** Let $G$ be the subgroup of $\Gamma_2(N_g)$ generated by the elements given in (1) and (2). Since by Lemma 2.1 we have $\mathcal{Y} = \mathcal{Y}$, it is enough to prove that $T_{1,i,j,k}^2$ is contained in the subgroup $G$ for $1 < i < j < k$.

It is easy to check that

$$Y_{i+1,j}^{-1} \cdots Y_{j-2,j}^{-1}Y_{j-1,j}^{-1}(\alpha_{1,i,j,k}) = \alpha_{1,i,j,k}.$$ 

Thus

$$T_{1,i,j,k}^2 = (Y_{i+1,j}^{-1} \cdots Y_{j-2,j}^{-1}Y_{j-1,j}^{-1})T_{1,i,j,k}^2(Y_{i+1,j}^{-1} \cdots Y_{j-2,j}^{-1}Y_{j-1,j}^{-1})^{-1},$$

which implies that

$$T_{1,i,j,k}^2 = (Y_{i+1,j}^{-1} \cdots Y_{j-2,j}^{-1}Y_{j-1,j}^{-1})^{-1}T_{1,i,j,k}^2(Y_{i+1,j}^{-1} \cdots Y_{j-2,j}^{-1}Y_{j-1,j}^{-1}) \in G$$

for $1 < i < j < k$. This completes the proof. □

### 3. Involution Generators for $\Gamma_2(N_g)$

In this section, we give a generating set of involutions for $\Gamma_2(N_g)$. Throughout this section, consider the surface $N_g$ as shown in Figure 8 so that it is invariant under the reflection $R$ about the indicated plane. Note that, $R$ acts trivially on $H_1(N_g; \mathbb{Z}/2\mathbb{Z})$, which implies that it is an element of the subgroup $\Gamma_2(N_g)$.

![Figure 8. The reflection $R$.](image)

**Proposition 3.1.** For $g \geq 4$, the group $\Gamma_2(N_g)$ can be generated by

1. $R$,
2. $R\mathcal{Y}_{i,j}$ for $i \in \{1, \ldots, g-1\}$, $j \in \{1, \ldots, g\}$ and $i \neq j$,
3. $R\mathcal{Y}_{1,i}T_{1,i,j,k}^2$ for $1 < i < j < k$. 

Proof. Let $G$ be the subgroup generated by the elements listed in the statement of the proposition. Since the subgroup $G$ contains $R$ and $R Y_{i,j}$, it also contains

$$Y_{i,j} = R(Y_{i,j})$$

for $i \in \{1, \ldots, g - 1\}$, $j \in \{1, \ldots, g\}$ and $i \neq j$. Recall that $\mathcal{Y}$ is generated by such elements, hence $\mathcal{Y} \subseteq G$. By Theorem 2.2, it remains to prove that $T^2_{1,i,j,k}$ also belongs to $G$. Now, it is easy to see that $G$ contains

$$Y_{1,i}Y_{\alpha_k,\pi_j,k}^{-1}T^2_{1,i,j,k} = R(Y_{1,i}Y_{\alpha_k,\pi_j,k}^{-1}T^2_{1,i,j,k}).$$

The elements $Y_{\alpha_k,\pi_j,k}$ are contained in $\mathcal{Y} = \mathcal{Y}$ by [4, Lemma 3.5] and Lemma 2.1. Since the elements $Y_{1,i}$ are also contained in $G$, one can conclude that $T^2_{1,i,j,k} \in G$ for $1 < i < j < k$, which finishes the proof. \qed

**Lemma 3.2.** The reflection $R$ can be expressed as a product of finitely $Y$-homeomorphisms. In particular

$$R = Y_{g-1,g}Y_{g-2,g} \cdots Y_{1,g}.$$ 

**Proof.** It follows from the proof of [3, Lemma 3.4] that $R$ can be written as

$$R = Y_{g-1,g}^{-1}T_{g-1,g}^{-1}Y_{g-2,g-1}^{-1}T_{g-2,g-1}(T_{\alpha_i+1,i+2}T_{\alpha_{i+2,i+3}} \cdots T_{\alpha_{g-1,g}})^{-1}Y_{i,i+1}(T_{\alpha_i+1,i+2}T_{\alpha_{i+2,i+3}} \cdots T_{\alpha_{g-1,g}})^{-1}Y_{1,2}(T_{\alpha_2,3} \cdots T_{\alpha_{g-1,g}}).$$

It is easy to see that

$$(T_{\alpha_i+1,i+2}T_{\alpha_{i+2,i+3}} \cdots T_{\alpha_{g-1,g}})^{-1}(\alpha_i,\alpha_{i,i+1}) = (\alpha_{i,i},g),$$

from which we obtain

$$Y_{i,g} = (T_{\alpha_i+1,i+2}T_{\alpha_{i+2,i+3}} \cdots T_{\alpha_{g-1,g}})^{-1}Y_{i,i+1}(T_{\alpha_i+1,i+2}T_{\alpha_{i+2,i+3}} \cdots T_{\alpha_{g-1,g}}),$$

for $i \in \{1, \ldots, g-1\}$. This completes the proof. \qed

Next, we show that the elements mentioned in Theorem 3.1 are all involutions. We already know that the reflection $R$ is an involution.

**Lemma 3.3.** If $g \geq 4$, then the elements $R Y_{i,j}^{-1}$ are all involutions for $i \in \{1, \ldots, g-1\}$, $j \in \{1, \ldots, g\}$ and $i \neq j$.

**Proof.** It is enough to see that $R(\alpha_i,\pi_{i,j}) = (\alpha_{i},\pi_{i,j})^{-1}$. \qed

**Lemma 3.4.** If $g \geq 4$, then the elements $R Y_{1,i}^{-1}Y_{\alpha_k,\pi_j,k}^{-1}T^2_{1,i,j,k}$ are all involutions for $1 < i < j < k$.

**Proof.** First of all, it is easy verify that

$$R(\pi_{1,i},\pi_{j,k}) = (\pi_{1,i},\pi_{j,k})^{-1}$$

and

$$R(\alpha_i,\alpha_k) = (\alpha_i,\alpha_k)^{-1}.$$ 

Then we have the following:

$$R Y_{1,i}^{-1}Y_{\alpha_k,\pi_j,k}^{-1}R^{-1} = Y_{1,i}^{-1}Y_{\alpha_k,\pi_j,k}^{-1}Y_{1,i}^{-1} = Y_{\alpha_k,\pi_j,k}^{-1}Y_{1,i}^{-1},$$

where the last identity follows from the commutativity of crosscap slides $Y_{1,i}$ and $Y_{\alpha_k,\pi_j,k}$. Observe that, this implies $R Y_{1,i}^{-1}Y_{\alpha_k,\pi_j,k}$ is an involution. Moreover, since

$$R Y_{1,i}^{-1}Y_{\alpha_k,\pi_j,k}(\pi_{1,i,j,k}) = \pi_{1,i,j,k}^{-1},$$
it follows that $R \overline{Y}_{1,i}Y_{\alpha_k,\pi_{j,k}} \overline{T}_{1,i,j,k}^2$ is also an involution.

Finally, we present our involution generators. Note that in the following, the number of involution generators is equal to \( \binom{2}{2} + \binom{3}{2} \) which is the minimal possible number of generators for $\Gamma_2(N_g)$.

**Theorem 3.5.** For $g \geq 5$ and odd, $\Gamma_2(N_g)$ is generated by the following involutions:

1. $R \overline{Y}_{1,g}, R \overline{Y}_{2,g}, \ldots, R \overline{Y}_{g-2,g}, R \overline{Y}_{g-1,g}$,
2. $R \overline{Y}_{i,j}$ for $i, j \in \{1, 2, \ldots, g-1\}$ and $i \neq j$,
3. $R \overline{Y}_{1,i}Y_{\alpha_k,\pi_{j,k}} \overline{T}_{1,i,j,k}^2$ for $1 < i < j < k$.

For $g \geq 4$ and even, $\Gamma_2(N_g)$ is generated by the following involutions:

1. $R, R \overline{Y}_{1,g}, R \overline{Y}_{2,g}, \ldots, R \overline{Y}_{g-2,g}$,
2. $R \overline{Y}_{i,j}$ for $i, j \in \{1, 2, \ldots, g-1\}$ and $i \neq j$,
3. $R \overline{Y}_{1,i}Y_{\alpha_k,\pi_{j,k}} \overline{T}_{1,i,j,k}^2$ for $1 < i < j < k$.

**Proof.** Let $G$ denote the subgroup of $\Gamma_2(N_g)$ generated by the elements listed in Theorem 3.5. It follows from lemmata 3.3 and 3.4 that the generators of the group $G$ are involutions.

Let us first assume that $g \geq 5$ and odd. By Proposition 3.1, it is enough to prove that $R$ is contained in the subgroup $G$. It follows from Lemma 3.2 the reflection $R$ can be expresses as

$$R = \overline{Y}_{g-1,g} \overline{Y}_{g-2,g} \cdots \overline{Y}_{1,g}$$

Then

$$R = \overline{Y}_{g-1,g} \overline{Y}_{g-2,g} \overline{Y}_{g-3,g} \cdots \overline{Y}_{2,g} \overline{Y}_{1,g}$$

which is contained in the subgroup $G$ using $R \overline{Y}_{i,g}^{-1} R = \overline{Y}_{i,g}$.

Assume now that $g \geq 4$ and even. In this case, by Proposition 3.1, it suffices to show that the subgroup $G$ contains the element $\overline{Y}_{g-1,g}$. The following element is contained in the subgroup $G$:

$$R(R \overline{Y}_{g-2,g} \overline{Y}_{g-3,g} \overline{Y}_{g-4,g} R \cdots \overline{Y}_{2,g} \overline{Y}_{1,g})$$

$$= R(R \overline{Y}_{g-2,g} \overline{Y}_{g-3,g} \overline{Y}_{g-4,g} \cdots \overline{Y}_{2,g} \overline{Y}_{1,g})$$

$$= \overline{Y}_{g-2,g} \overline{Y}_{g-3,g} \overline{Y}_{g-4,g} \cdots \overline{Y}_{2,g} \overline{Y}_{1,g},$$

using again $R \overline{Y}_{i,g}^{-1} R = \overline{Y}_{i,g}$. One can conclude that $\overline{Y}_{g-1,g} \in G$ since $R \in G$ by Lemma 3.2, which finishes the proof. \[ \Box \]

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