CONJUGATE LINEAR PERTURBATIONS OF DIRAC OPERATORS AND MAJORANA FERMIONS

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ABSTRACT. We study a canonical class of perturbations of Dirac operators that are defined in any dimension and on any Hermitian Clifford module bundle. These operators generalize the 2-dimensional Jackiw–Rossi operator, which describes electronic excitations on topological superconductors. We also describe the low energy spectrum of these operators on complete surfaces, under mild hypotheses.

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INTRODUCTION

In [11], Jackiw and Rossi introduced a Dirac-type equation in two spatial dimensions which corresponds to a Lagrangian that couples Dirac fermions to the superconducting order parameter of an $s$-wave superconductor. Ground states of this theory are interpreted as Majorana fermions pinned to vortices [4]. Furthermore, this theory has potential applications in quantum computing; cf. [8, 10, 14].

In this paper, we reformulate the classical Jackiw–Rossi theory in terms of spin geometry and generalize the Jackiw–Rossi (Hamiltonian) operator to more general fields and higher dimensions, and study the spectral properties of this theory. These generalized Jackiw–Rossi...
operators have the form

$$H = D + \mathcal{A},$$

where $D$ is a Dirac-type operator on and $\mathcal{A}$ is a conjugate linear bundle map. Since $H$ is not complex linear, its eigenspaces are not complex (but only real) subspaces of the Hilbert space. Moreover, eigenspinors of $H$ can be viewed as Majorana fermions; cf. [4]. Furthermore, in certain cases the above operators have been studied in the context of pseudo-holomorphic curves; cf. [6,7,9,12,18,20,21].

After the introducing the general theory, we study the Jackiw–Rossi equation

$$H\Psi = m\Psi,$$

on complete surfaces, with $|m|$ small. The key analytic observation in studying the Jackiw–Rossi equation is that the planar Jackiw–Rossi operator comes from a concentrating pair, in the sense of Maridakis; cf. [15, Definition 2.1]. In other words, the operator

$$\mathcal{B} := D^* \circ \mathcal{A} + \mathcal{A}^* \circ D,$$

is a bundle map, as opposed to a first order differential operator. This allows us to prove strong contraction results for eigenspinors; cf. [15, 17].

In the final section of the paper, we give a few applications of our results:

1. We give a short proof of the Riemann–Roch theorem.
2. We construct solutions to the (generalized) Jackiw–Rossi equation in higher dimensions.
3. Construct a canonical projective space bundle over symmetric powers of Riemann surfaces.

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Organization of the paper. In Section 1, we introduce the generalized Jackiw–Rossi Hamiltonians. In Section 3, we construct solutions to the Jackiw–Rossi equations on closed surfaces.
and curved planes. Section 4 is devoted to four applications of Theorem 3.4. For completeness, in Appendix A, we summarize the necessary notation and background from the representation theory of Clifford algebras and modules. In Appendix B, we also construct generalizations of the Bogoliubov–de Gennes equation.

1. Generalized Jackiw–Rossi Theory on Pseudo-Riemannian Manifolds

In this section, we generalize the above construction to bundles of Clifford modules. Let \((X, g)\) be a \(d\)-dimensional, smooth, oriented, pseudo-Riemannian manifold of signature \((t, s)\). As before, let \(r = t - s\). Let \(\text{CL}(TX, g)\) be the bundle of (real) Clifford algebras given by

\[
\forall x \in X : \text{CL}(TX, g)_x := \text{CL}(T_x X, g_x).
\]

Finally, let \(E \to X\) be a Hermitian vector bundle with a Clifford structure. In other words, there is a Clifford multiplication, \(\text{cl} : \text{CL}(TX, g) \to \text{End}(E)\), that is a unital homomorphism of algebras, such that for all \(v \in T^* X\), \(\text{cl}(v) \in \text{su}(E)\). We can think of \(\text{cl}\) as a map from \(\bigwedge^*(X) \otimes E\) to \(E\). The associated Hermitian vector bundle, \(F\), is defined via

\[
E \otimes E \cong \begin{cases} 
\bigwedge^\text{even}(X) \otimes F, & \text{if } d \text{ is odd}, \\
\bigwedge^\text{even}(X) \otimes F, & \text{if } d \text{ is even}.
\end{cases}
\]

We remark that the isomorphism above is canonical; cf. [16, Proposition 11.1.27.]. Thus, in particular, we again get a pairing \(\mathcal{B}_E : E \otimes E \to F\). Furthermore, \(F^*\) decomposes as \(F^* \cong F^+ \oplus F^-\), to symmetric and anti-symmetric parts, as in Appendix A.2. Note that \(F\) need not be a tensor-square of another vector bundle.

We call a connection on \(E\) compatible, if it is unitary and the Clifford multiplication is parallel. Let \((\nabla, \Phi)\) be a pair of a compatible connection \(\nabla\) and a smooth section of \(F\), \(\Phi\). We define two operators on sections of \(E\) as follows:

1. Using \(\nabla\), we define the twisted Dirac operator as

\[
\mathcal{D}_\nabla : L^2(E) \to L^2(E); \Psi \mapsto \mathcal{D}_\nabla \Psi = \text{cl}(\nabla \Psi).
\]

2. Using \(\Phi\), we define the perturbation term \(\mathcal{A}_\Phi\) as

\[
\hbar \mathcal{E}(\mathcal{A}_\Phi \Psi_1, \Psi_2) = \Phi(\mathcal{B}_E(\Psi_1, \Psi_2)).
\]

Example 1.1. Let \(X\) be spin and \(E = \$\) a spinor bundle corresponding to a spin structure. Then \(F^*\) is the trivial line bundle. When equipped with the product connection, \(\mathcal{D} = \mathcal{D}_\nabla\) is just the ordinary Dirac operator. If \(\Phi\) is a covariantly constant and unit length, then \(\mathcal{A} = \mathcal{A}_\Phi\) is just the canonical (up to scalar) isomorphism from \(\$\) to its conjugate. Similarly, if \(X\) is spin\(^c\), then \(F^*\)
is the associated line bundle, and a connection and a section of $F^*$ provides further examples of the above operators. In both cases $F^* = F^+$.  

**Example 1.2.** Let $S$ be a spinor bundle and $E$ be a real or a quaternionic vector bundle and $F = S \otimes E$. Then $F^* = E^* \otimes E^*$, and the real or quaternionic structure defines a section, $\Phi_0$, of $F^*$. In fact $\Phi_0$ takes values in $F^+$ in the case of a real structure, and in $F^-$ in the case of a quaternionic structure. Thus $s_\Phi_0 = \pm s_r$, accordingly.

A particular case of the above examples are known in physics as the Jackiw–Rossi theory; cf. [4].

**Example 1.3.** Let $X = \Sigma$ be an oriented, Riemannian surface, $L$ a Hermitian line bundle over $\Sigma$, and $\Theta$ be a “square root” of the canonical line bundle of $\Sigma$. Then $S^\pm_{\Sigma} = L \Theta^\pm_1$ defines a spin$^c$ spinor bundle, and the corresponding line bundle is $F = F^+ = L^{-2}$.

For each compatible connection, $\nabla$, on $F$, we get a connection on $S^\pm_{\Sigma}$, and thus a twisted Dirac operator 

$$D_\nabla \begin{pmatrix} \Psi^+ \\ \Psi^- \end{pmatrix} = \begin{pmatrix} \overline{\partial}_\nabla \Psi^- \\ \partial_\nabla \Psi^+ \end{pmatrix}.$$  

For each section, $\Phi$, of $F$, we also get an algebraic operator

$$A_\Phi \begin{pmatrix} \Psi^+ \\ \Psi^- \end{pmatrix} = \begin{pmatrix} \Phi(\overline{\partial}_\nabla \Psi^-) \\ \Phi(\partial_\nabla \Psi^+) \end{pmatrix}.$$  

The operator $D_\nabla + A_\Phi$ is the (real, symmetric) Jackiw–Rossi operator in [4] and for each $\mu \in \mathbb{R}$ the functional 

$$\Gamma(\Psi) \to \mathbb{R}; \quad \Psi \mapsto h^8(\Psi, D_\nabla \Psi) + 2 \text{Re}(h^8(A_\Phi \Psi, \Psi)) - \mu |\Psi|^2,$$  

is the energy density in [4, Equation (1)].

Let now $X := \mathbb{R}^1 \times \Sigma$ be equipped with $g_X := -dt^2 + g$, to make it a Lorentzian 3-manifold. The pullback of $(E, h, \text{cl})$ canonically defines a spin$^c$ spinor bundle with 

$$\text{cl}(dt) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$  

which is the (pullback of the) parity operator. Using a slight abuse of notation, let $(\nabla, \Phi)$ denote also the pullback fields to $X$. The corresponding Jackiw–Rossi equation can then be described as follows:

$$\nabla_{\partial_t} \Psi^+ = H_{\nabla, \Phi} \Psi^-,$$

$$\nabla_{\partial_t} \Psi^- = -H_{\nabla, \Phi}^+ \Psi^+.$$  

which is equivalent to [11, Equation (2.8)].
The last example is the central motivation for this paper, in general, and for the next definition, in particular.

**Definition 1.4.** Let \((X, g)\) be an oriented, Riemannian \(d\)-manifold, and let \((\mathcal{E}, h, c\ell)\) be a bundle of Hermitian Clifford modules and \(\mathcal{F}\) be the associated vector bundle. Furthermore, let \((\nabla, \Phi)\) be a pair of as above. Then the (generalized) Jackiw–Rossi operator, associated to the data \((X, g, \mathcal{E}, h, c\ell, \nabla, \Phi)\), is

\[
H_{\nabla, \Phi} = D_\nabla + \mathcal{A}_\Phi : L^2(\mathcal{E}) \to L^2(\mathcal{E}).
\]

(1.3)

Similarly, we call the equation

\[
H_{\nabla, \Phi} \Psi = 0,
\]

(1.4)

the Jackiw–Rossi equation.

In the following lemma we prove that any conjugate linear perturbation of a Dirac-type operator can be viewed as (half of) a generalized Jackiw–Rossi operator.

**Lemma 1.5.** Let \(E_1\) and \(E_2\) be Hermitian vector bundles over a smooth manifold, \(X\). Let \(D : \Gamma(E_1) \to \Gamma(E_2)\), be a Dirac-type operator, and \(A : \Gamma(E_1) \to \Gamma(E_2)\), be a conjugate linear bundle map.

Then \(E := E_1 \oplus E_2\) is a Hermitian Clifford module bundle, where the Clifford multiplication is induced by the symbol of \(D\), c. Furthermore, let

\[
H := \begin{pmatrix}
0 & D^* + A^* \\
D + A & 0
\end{pmatrix},
\]

(1.5)

is a generalized Jackiw–Rossi operator as in Definition 1.4, with the connection, \(\nabla\), defined uniquely by \(D\), and the values of \(\Phi\) are given by Lemma A.3 through the bilinear maps

\[
\forall x \in X : \quad B_x \begin{pmatrix}
\Psi_{E_1,1} \\
\Psi_{E_2,1}
\end{pmatrix}, \begin{pmatrix}
\Psi_{E_1,2} \\
\Psi_{E_2,2}
\end{pmatrix} := h_x^{E_1} (A_x \Psi_{E_1,1}, \Psi_{E_2,2}) + h_x^{E_2} (A_x \Psi_{E_1,2}, \Psi_{E_2,1}).
\]

(1.6)

Finally,

\[
\ker(H) = \ker(D + A) \oplus \ker((D + A)^*) \cong \ker(D + A) \oplus \text{coker}(D + A),
\]

\[
\text{Spec}(H) - \{0\} = \left\{ \lambda \in \mathbb{R} \mid \lambda^2 \in \text{Spec}((D + A)^*(D + A)) - \{0\} \right\},
\]

\[
\forall \lambda \in \text{Spec}(H) - \{0\} : \quad \ker(H - \lambda) = \left\{ \Psi \left| \frac{\lambda}{(H + A)\Psi} \in \ker((D + A)^*(D + A) - \lambda^2) \right. \right\}.
\]
Proof. Clearly, $E$ is a Clifford module bundle. Let $\mathcal{F}$ defined through equation (1.1). Let $\nabla^1$ be a unitary connection induced by $D$, that is a unitary connection on $E_1$, such that $D = c \circ \nabla^1$. Such a connection exists and is unique, as $D$ is Dirac-type. Similarly, let $\nabla^2$ induced by $D^*$, and let $\nabla := \nabla^1 \oplus \nabla^2$. Then the twisted Dirac operator on $E$ is

$$
D = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}.
$$

Similarly, if $\Phi$ is given via equation (1.6), then

$$
\mathcal{A}_\Phi = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix},
$$

thus $H = D \circ \mathcal{A}_\Phi$ is of the form (1.5), and hence $H$ is a generalized Jackiw–Rossi operator. The claims about the spectra and eigenspinors are then straightforward.

For each $r$, let us define $\sigma_r, s_r \in \{-1, 1\}$ as

$$
\mathcal{A} \circ \text{cl} = \sigma_r \text{cl}(\cdot) \circ \mathcal{A}, \quad \& \quad \mathcal{A}^2 = s_r \mathbb{1},
$$

(1.7)

In even dimensions, where $\sigma_r$ is also a choice, our conventions are

$$
\sigma_0 = -1, \quad \sigma_2 = -1, \quad \sigma_4 = 1, \quad \sigma_6 = 1.
$$

Let us now prove a technical result about Jackiw–Rossi operators.

**Lemma 1.6.** Let $H_{\mathcal{F}, \Phi}$ be as in Definition 1.4, and let $\text{cl}(\mathcal{A}_{\mathcal{F}, \Phi}) = \sum_{i=1}^d \text{cl}(dx^i)\mathcal{A}_{\mathcal{F}, i\Phi}$. Then we have that

$$
D_\mathcal{F} \circ \mathcal{A}_\Phi = \text{cl}(\mathcal{A}_{\mathcal{F}, \Phi}) + \sigma_r \mathcal{A}_\Phi \circ D_\mathcal{F}.
$$

Thus, if $\Phi$ takes values only in $\mathcal{F}^\pm$, then

$$
D_\mathcal{F}^* \circ \mathcal{A}_\Phi + \mathcal{A}_\Phi^* \circ D_\mathcal{F} = \text{cl}(\mathcal{A}_{\mathcal{F}, \Phi}) + (\sigma_r \pm s_r) \mathcal{A}_\Phi^* \circ D_\mathcal{F}.
$$

**Proof.** Let us pick a local coordinate chart. Since $D_\mathcal{F}$ is self-adjoint and $\mathcal{A}_\Phi^* = \pm s_r$, we have

$$
D_\mathcal{F}^* \circ \mathcal{A}_\Phi + \mathcal{A}_\Phi^* \circ D_\mathcal{F} = D_\mathcal{F} \circ \mathcal{A}_\Phi \pm s_r \mathcal{A}_\Phi \circ D_\mathcal{F}
$$

$$
= \sum_{i=1}^d \left( \text{cl}(dx^i) \circ \mathcal{A}_\Phi \pm s_r \mathcal{A}_\Phi \circ \text{cl}(dx^i) \circ \nabla_i \right)
$$

$$
= \sum_{i=1}^d \left( \text{cl}(dx^i) \circ (\nabla_i \mathcal{A}_\Phi) + \text{cl}(dx^i) \circ \mathcal{A}_\Phi \circ \nabla_i \pm s_r \mathcal{A}_\Phi \circ \text{cl}(dx^i) \circ \nabla_i \right)
$$

$$
= \sum_{i=1}^d \left( \text{cl}(dx^i) \circ \mathcal{A}_{\mathcal{F}, i\Phi} + (\sigma_r \pm s_r) \mathcal{A}_\Phi \circ \text{cl}(dx^i) \circ \nabla_i \right)
$$

$$
= \text{cl}(\mathcal{A}_{\mathcal{F}, \Phi}) + (\sigma_r \pm s_r) \mathcal{A}_\Phi^* \circ D_\mathcal{F}
$$
which concludes the proof.

Using Lemma 1.6 we prove a concentration property for Jackiw–Rossi operators.

**Theorem 1.7.** Let $X$ now be closed, $\pm := -\sigma r s$, (that is, $\sigma r \pm s = 0$), $\Phi$ be a section $\mathcal{F}^\pm$, $H_{\mathcal{V}, \Phi}$ be as in Definition 1.4, $(t_n)_{n \in \mathbb{N}}$ be a sequence of real numbers that converges to infinity, and $\Phi_n := t_n \Phi$. Let $Z_\Phi$ be the degenerate locus of $\Phi$, that is $Z_\Phi := \{x \in X \mid \{0, x\} \neq \ker(\mathcal{A}_\Phi) \leqslant \mathcal{E}_x\}$.

Assume that for each $n \in \mathbb{N}$ we have an eigenspinors of $H_{\mathcal{V}, \Phi_n}$, call $\Psi_n \in \Gamma(\mathcal{F})$, and

$$B := \limsup_{n \in \mathbb{N}} \frac{\|H_{\mathcal{V}, \Phi_n} \Psi_n\|_{L^2(X, g)}^2}{t_n \|H_{\mathcal{V}, \Phi_n} \Psi_n\|_{L^2(X, g)}^2} < \infty.$$ 

Then, for all positive integers $k, l$ there are constants $C = C(X, g, \mathcal{E}, h, \text{cl}, \nabla, B, k, l) > 0$, such that if $\Omega_n \subseteq X$ is a sequence of open sets with $\text{dist}(\Omega_n, Z_\Phi) > 0$, then

$$\|\Psi_n\|_{L^2_k(\Omega_n)} \leqslant \frac{C}{(\text{dist}(\Omega_n, Z_\Phi)^2 t_n)^l} \|\Psi_n\|_{L^2(X)}.$$ 

**Proof.** The parity of $\Phi$ is chosen, using Lemma 1.6, so that $\text{cl}$ and $\mathcal{A}_\Phi$ form a concentrating pair, in the sense of Maridakis; cf. [15, Definition 2.1]. Then the result is a special case of [15, Corollary 2.6].

The final lemma of this section shows that while the theory is not conformally invariant, its kernel is conformally equivariant.

**Lemma 1.8.** Let $(X, g, \mathcal{E}, h, \text{cl}, \nabla, \Phi)$ be as in Definition 1.4. Let $H_{\mathcal{V}, \Phi}$ be the corresponding Jackiw–Rossi operator. Let $\varphi$ be a smooth, real-valued function on $X$ and $d := \dim_{\mathbb{R}}(X)$.

Conformally changed Riemannian manifold, $(X, g^\varphi := e^{2\varphi} g)$, has a Hermitian Clifford module bundle $(\mathcal{F}^\varphi, h^\varphi, \text{cl}^\varphi)$ that is canonically isomorphic, as a smooth vector bundle, to $\mathcal{E}$, and moreover, under this isomorphism we have

$$h^\varphi = e^{-\varphi} h, \quad \text{cl}^\varphi = e^{-\varphi} \text{cl}.$$ 

Let us define

$$\nabla^\varphi := \nabla + \frac{1}{4} [\text{cl}(d\varphi), \text{cl}(\cdot)].$$

Then the Jackiw–Rossi operator, $H_{\mathcal{V}^\varphi, \Phi}$, corresponding to $(X, g^\varphi, \mathcal{F}^\varphi, h^\varphi, \text{cl}^\varphi, \nabla^\varphi, \Phi)$ satisfies, under the above mentioned canonical isomorphism, that

$$H_{\mathcal{V}^\varphi, \Phi} = e^{-\frac{d+1}{2} \varphi} H_{\mathcal{V}, \Phi} \circ e^{\frac{d-1}{2} \varphi}.$$ 

In particular, there is a map

$$\ker(H_{\mathcal{V}, \Phi}) \rightarrow \ker(H_{\mathcal{V}^\varphi, \Phi}); \Psi \mapsto e^{-\frac{d-1}{2} \varphi} \Psi,$$ 

(1.10)
which is an \(L^2\)-unitary isomorphism.

Proof. The proof adapts the ideas [5, Appendix D] to the case at hand.

The statements about \((E^{\phi}, h^{\phi}, c^{\phi})\), in particular equation (1.8) are standard. The statement about \(\nabla^{\phi}\) follows from [13, page 110]. Thus we have

\[
e^{d-1/2} \phi \circ \left( \sum_{i=1}^{d} c^{\phi}(dx^i) \nabla_{\partial_i}^{\phi} \right) \circ e^{-d-1/2} \phi = e^{d-1/2} \phi \circ \left( \sum_{i=1}^{d} e^{-\phi} c(\nabla \partial_i \phi) \left[ c(d\phi), c(dx^i) \right] \right) \circ e^{-d-1/2} \phi
\]

\[
= e^{d-1/2} \phi \circ \left( \sum_{i=1}^{d} c(dx^i) \nabla_{\partial_i} \phi \circ e^{-d-1/2} \phi + \frac{1}{4} c(dx^i) \left[ c(d\phi), c(dx^i) \right] \right)
\]

\[
= \sum_{i=1}^{d} \left( c(dx^i) \nabla_{\partial_i} - \frac{d-1}{2} \partial_i \phi c(dx^i) + \frac{1}{4} c(dx^i) \left[ c(d\phi), c(dx^i) \right] \right)
\]

\[
= \nabla \phi - \frac{d-1}{2} c(d\phi) + \frac{1}{4} \sum_{i=1}^{d} c(dx^i) \left[ c(d\phi), c(dx^i) \right].
\]

Let us pick a local normal chart for \(g\). Then

\[
\sum_{i=1}^{d} c(dx^i) \left[ c(d\phi), c(dx^i) \right] = \sum_{i,j=1}^{d} \partial_j \phi c(dx^i) \left[ c(dx^i), c(dx^i) \right]
\]

\[
= 2 \sum_{i,j=1, i \neq j}^{d} \partial_j \phi c(dx^i)
\]

\[
= 2(d-1) c(d\phi),
\]

and thus

\[
\sum_{i=1}^{d} c^{\phi}(dx^i) \nabla_{\partial_i}^{\phi} = e^{-d-1/2} \phi \circ \nabla \phi \circ e^{d-1/2} \phi,
\]

which proves the claim for the twisted Dirac operator part of \(H_{\nabla, \phi}\). For the algebraic part note that by equation (1.8) we also have \(\mathcal{A}_\phi\) is invariant under the conformal change. Furthermore, \(\mathcal{A}_\phi\) is clearly (complex) linear in \(\Phi\), thus

\[
\mathcal{A}_{\phi \Psi} = \mathcal{A}_{e^{2\phi} \Psi} = e^{2\phi} \mathcal{A}_\phi = e^{-d-1/2} \phi \circ \mathcal{A}_\phi \circ e^{d-1/2} \phi,
\]

which concludes the proof of equation (1.9). The proof of equation (1.10) is now trivial. \( \square \)

2. The 2-dimensional Jackiw–Rossi theory

2.1. The model case. In this section we analyze a simple case which serves as the model and main ingredient for the results of the next section.

Consider \(\mathbb{C}\) with its flat metric and let \(dA\) be its area form. Let \(\alpha\) be a smooth, complex valued, compactly supported function on \(\mathbb{C}\), and \(\Phi\) be a homogeneous, complex polynomial
on \( \mathbb{C} \) degree \( m \in \mathbb{N}_+ \) with an isolated zero at the origin of index \( k \in \mathbb{Z} \), that is
\[
\Phi(z, \overline{z}) = Tz^{\frac{m+k}{2}} \overline{z}^{\frac{m-k}{2}} + \varphi(z, \overline{z}),
\]
where \( \varphi \) is also a homogeneous, complex polynomial of degree \( m \), and for some \( \epsilon \in (0, |T|) \), it satisfies
\[
|\varphi(z, \overline{z})| \leqslant |T|(1 - \epsilon) |z|^m.
\]
Note that \( \frac{m+k}{2} \) are assumed to be integers.

For \( f \in L^2(\mathbb{C}) \), let us consider (the weak formulation of) the equation:
\[
\frac{\partial f}{\partial z} + \alpha f + \Phi \overline{f} = 0. \tag{2.1}
\]

The main result of this section is the following theorem.

**Theorem 2.1.** There is \( c = c(\epsilon) > 0 \), such that if \( \|\alpha\|_{L^\infty(\mathbb{C})} \leqslant c \sqrt{|T|} \), then the dimension of the space of solutions of equation (2.1) is a \( \max(|k|, 0) \). Moreover, solutions of equation (2.1) decay exponentially.

**Remark 2.2.** The case when \( \alpha = 0 \) and \( \varphi = 0 \) was considered by Rauch in [18]. However for the purposes of this paper we need to assume that neither \( \alpha \), nor \( \varphi \) vanish, and these requirements pose some nontrivial technical difficulties.

To prove Theorem 2.1, we first prove a few technical results below.

First we consider the \( \alpha = 0 \), but \( \varphi \neq 0 \) case. To simplify our notation, for each \( f \in C_\text{cpt}^\infty(\mathbb{C}) \), let
\[
\tilde{D}^+ f = \frac{\partial f}{\partial z} + \Phi \overline{f},
\]
\[
\tilde{D}^- f = -\frac{\partial f}{\partial z} + \Phi \overline{f}.
\]

These define discontinuous operators on \( L^2(\mathbb{C}) \) with dense domain, that also satisfy the following Weitzenböck-type identities for any \( f \in C_\text{cpt}^\infty(\mathbb{C}) \):
\[
\tilde{D}^- \tilde{D}^+ f = \Delta f + |\Phi|^2 f - (\partial \Phi) \overline{f}, \tag{2.2a}
\]
\[
\tilde{D}^+ \tilde{D}^- f = \Delta f + |\Phi|^2 f + (\overline{\partial} \Phi) \overline{f}, \tag{2.2b}
\]
and thus
\[
\|\tilde{D}^+_f\|_{L^2(\mathbb{C})}^2 = \|df\|_{L^2(\mathbb{C})}^2 + \|\Phi f\|_{L^2(\mathbb{C})}^2 - \int_{\mathbb{C}} (\partial \Phi) |f|^2 \, dA, \tag{2.3a}
\]
\[
\|\tilde{D}^-_f\|_{L^2(\mathbb{C})}^2 = \|df\|_{L^2(\mathbb{C})}^2 + \|\Phi f\|_{L^2(\mathbb{C})}^2 + \int_{\mathbb{C}} (\overline{\partial} \Phi) |f|^2 \, dA. \tag{2.3b}
\]
Furthermore, since their duals are also densely defined, both $\tilde{D}^\pm$ are closeable, in fact $D^\pm = (\tilde{D}^\pm)^*$ are the unique closed extensions of $\tilde{D}^\pm$. Let $\mathcal{H}^\pm = \text{dom}(D^\pm)$ equipped with the graph norm (which defines a real Hilbert space structure).

Now we are ready to prove the following theorem, which a fortiori implies Theorem 2.1 in the $\alpha = 0$ case.

**Theorem 2.3.** Both $\tilde{D}^\pm$ have unique closed extensions, $D^\pm$. Let $\mathcal{H}^\pm := \text{dom}(D^\pm)$ equipped with the graph norm (which defines a real Hilbert space structure). The operators $D^\pm : \mathcal{H}^\pm \to L^2(\mathbb{C})$ are Fredholm and

$$\dim_{\mathbb{R}}(\ker(D^\pm)) = \max(\{\pm k, 0\}).$$

In particular, the Fredholm index of $D^\pm$ is $\pm k$.

Furthermore, elements of $\ker(D^\pm)$ decay exponentially.

**Proof.** To show that $\tilde{D}^\pm$ is closeable, we need to prove that for any sequence $(f^\pm_n)_{n \in \mathbb{N}}$, such that $f^\pm_n \in C^\infty_{\text{cpt}}(\mathbb{C})$, $f^\pm_n \to 0$ in $L^2(\mathbb{C})$, and $(D^\pm f^\pm_n)_{n \in \mathbb{N}}$ is Cauchy in $L^2(\mathbb{C})$, we have that $D^\pm f^\pm_n \to 0$. Using equations (2.3a) and (2.3b), we get

$$\|\tilde{D}^+(f^+_m - f^+_n)\|_{L^2(\mathbb{C})}^2 = \|d(f^+_m - f^+_n)\|^2_{L^2(\mathbb{C})} + \|\Phi(f^+_m - f^+_n)\|^2_{L^2(\mathbb{C})} - \int_{\mathbb{C}} (\partial \Phi)(f^+_m - f^+_n)^2 \ dA,$$

$$\|\tilde{D}^-(f^-_m - f^-_n)\|_{L^2(\mathbb{C})}^2 = \|d(f^-_m - f^-_n)\|^2_{L^2(\mathbb{C})} + \|\Phi(f^-_m - f^-_n)\|^2_{L^2(\mathbb{C})} + \int_{\mathbb{C}} (\partial \Phi)(f^-_m - f^-_n)^2 \ dA,$$

Thus

$$\|\tilde{D}^\pm(f^\pm_m - f^\pm_n)\|_{L^2(\mathbb{C})}^2 = \|d(f^\pm_m - f^\pm_n)\|_{L^2(\mathbb{C})}^2 + \int_{\mathbb{C}} (|\Phi|^2 + |d\Phi|) |f^\pm_m - f^\pm_n|^2 \ dA.$$

Let $R > 0$ such that $|\Phi|^2 \geq 2|d\Phi|$ on $\mathbb{C} - B_R(0)$. Such an $R$ always is due to the assumptions on $\Phi$. Then we get that

$$\|d(f_m - f_n)\|_{L^2(\mathbb{C})}^2 + \|\Phi(f^+_m - f^+_n)\|_{L^2(\mathbb{C})}^2 \leq 2\|\tilde{D}^\pm(f_m - f_n)\|_{L^2(\mathbb{C})}^2 + C\|f_m - f_n\|_{L^2(\mathbb{C})}^2,$$

which means that $(f^\pm_n)_{n \in \mathbb{N}}$ is convergent in $L^2_1(\mathbb{C})$ and $(|\Phi|f^\pm_n)_{n \in \mathbb{N}}$ is convergent in $L^2(\mathbb{C})$. Furthermore, both are necessarily converging to the zero function, and thus so is $D^\pm f^\pm_n$. Hence both $\tilde{D}^\pm$ are closeable, and since they are densely defined, the closed extensions, $D^\pm$, are unique. Note that the above proof also shows that $\mathcal{H}^\pm = \text{dom}(D^\pm) \subseteq L^2_1(\mathbb{C})$.

Next we prove that all $f \in \ker(D^\pm)$ are smooth. We present the proof for $f \in \ker(D^+)$, as the two cases are analogous. Since $f \in \mathcal{H}^+ = \text{dom}(D^+) \subseteq L^2(\mathbb{C}) \subseteq L^2_{1,\text{loc}}(\mathbb{C})$ and $\Phi$ is smooth, we have that

$$\frac{\partial f}{\partial \bar{z}} = -\Phi f \in L^2_{1,\text{loc}}(\mathbb{C}),$$

thus by usual elliptic regularity and bootstrap, $f$ is smooth.
Next we show that each \( f \in \ker(D^+) \) decays exponentially. We again only show for the + case. By equation (2.1) and the smoothness of \( f \) we have that
\[
\Delta f = -\frac{\partial f}{\partial z\overline{z}} = \frac{\partial \Phi}{\partial z} f - |\Phi|^2 f,
\]
and thus
\[
\left(\frac{1}{2}\Delta + |\Phi|^2 - \left|\frac{\partial \Phi}{\partial z}\right|\right)|f|^2 \leq 0.
\]
By the assumptions, for some \( R \) large enough, \( \Phi \) satisfies on \( \mathbb{C} - B_R(0) \) that
\[
(\Delta + 1)|f|^2 \leq 0,
\]
and thus \( |f| \) decays exponentially as \( |z| \to \infty \).

Next we show that the dimension of \( \ker(D^+) \) is at most \( k \). Let \( p = m + k^2 \) and \( q = m - k^2 \). Since \( 0 \leq |k| \leq m \), we have that \( p \) and \( q \) are both nonnegative. Let \( T \in \mathbb{C} \) be the coefficient of \( z^p\overline{z}^q \) in \( \Phi \), and write \( \varphi = \Phi - T z^p\overline{z}^q \). By the hypotheses on \( \Phi \) we have that \( \varphi \) is also a homogeneous, complex polynomial of degree \( m \), and for some \( \epsilon \in (0,|T|) \), it satisfies
\[
|\varphi(z,\overline{z})| \leq |T|(1 - \epsilon)|z|^m.
\]
Now we have
\[
-\frac{1}{2} \frac{\partial}{\partial z} \left( f^2 z^{k} \right) = -\frac{f^2}{z^k} |\Phi| z^k = |f|^2 \left( T|z|^{2q} + \frac{\varphi}{z^k} \right) \geq |T|\epsilon|z|^{2q}|f|^2.
\]
Hence, using the exponential decay of \( f \), we get
\[
|T|\epsilon \int_\mathbb{C} |z|^{2q}|f|^2 \, dA \leq -\frac{1}{2} \lim_{r \to 0^+} \lim_{R \to \infty} \int_{B_R(0) - B_r(0)} \frac{\partial}{\partial z} \left( f^2 z^{k} \right) \, dA
\]
\[
= i \frac{1}{2} \left( \lim_{R \to \infty} \int_{S_R(0)} \frac{f^2}{z^k} \, dz - \lim_{r \to 0^+} \int_{S_r(0)} \frac{f^2}{z^k} \, dz \right)
\]
\[
= -\frac{1}{2i} \lim_{r \to 0^+} \int_{S_r(0)} \frac{f^2}{z^k} \, dz.
\]
Note that when \( q = 0 \), this formula resembles the Samols’ Localization Formula; cf. [19]. Since \( f \) is smooth, when \( k \leq 0 \), then the right hand side is zero, hence \( f \equiv 0 \). If \( k > 0 \), then let us have the \( k \)th order Taylor expansion of \( f \)
\[
f(z,\overline{z}) = \sum_{a=0}^k \sum_{b=0} a f_{ab} z^a \overline{z}^b + O\left(|z|^{k+1}\right).
\]
Plugging this into inequality (2.4), we get that
\[
\int_\mathbb{C} |z|^{2q}|f|^2 \, dA \leq \pi \lim_{r \to 0^+} \frac{1}{2\pi i} \int_{S_r(0)} \frac{f^2}{z^k} \, dz = \pi \sum_{a=0}^{k-1} f_{a0} f_{(k-1-a)0}.
\]
Hence the kernel of the linear map

\[ f \mapsto \sum_{a=0}^{k-1} f_{a0} z^a, \]

is trivial, and thus the dimension of \( \ker(D^+) \) is at most \( k \).

Next, we show that there is \( C^\pm = c^\pm(\epsilon) m\sqrt{\pi} > 0 \), such that for all \( f \in (\ker(D^\pm))^\perp \leq \mathcal{H}^\pm \):

\[ \|D^\pm f\|_{L^2(\mathbb{C})} \geq C^\pm \|f\|_{L^2(\mathbb{C})}. \]  

(2.5)

Fix \( R > 0 \) so that \( |\Phi|^2 \geq |d\Phi| + 1 \) on \( \mathbb{C} - B_R(0) \). Assume that \( f \in \mathcal{H}^\pm \) is supported outside of \( B_R(0) \). We carry out the proof only for the + sign, as the − can be done similarly. Let \( \left( f_n^\pm \right)_{n \in \mathbb{N}} \) be a sequence in \( C^\infty_{\text{cpt}}(\mathbb{C} - B_R(0)) \) that converges to \( f \) in the topology of \( \mathcal{H}^+ \). Then we have, using equation (2.2a)

\[
\|D^+ f_n\|_{L^2(\mathbb{C})}^2 = \|df_n\|_{L^2(\mathbb{C})}^2 + \|\Phi f_n\|_{L^2(\mathbb{C})}^2 - \int_{\mathbb{C}} (\partial \Phi) \overline{f_n} \, dA \\
\geq 0 + \int_{\mathbb{C} - B_R(0)} \left( |\Phi|^2 - \left| \frac{\partial \Phi}{\partial z} \right|^2 \right) |f_n|^2 \, dA \\
\geq \|f_n\|_{L^2(\mathbb{C})}^2.
\]

Using that both sides are continuous in the topology of \( \mathcal{H}^+ \) and taking the limit as \( n \to \infty \) gives us the result.

Next we prove inequality (2.5) in the case when \( f \) is smooth, unit \( L^2 \)-norm, and \( \text{supp}(f) \subseteq B_{2R}(0) \). Assume, by contradiction, that inequality (2.5) does not hold for any \( C^\pm > 0 \), and pick a sequence of smooth functions, \( \left( f_n^\pm \right)_{n \in \mathbb{N}} \), such that each \( f_n \) has unit \( L^2 \)-norm, \( \text{supp}(f_n) \subseteq B_{2R}(0) \), and \( D^\pm f_n \) converges to zero in \( L^2(\mathbb{C}) \). Since \( f_n \) is supported in \( B_{2R}(0) \), we have that both \( \overline{\partial} f_n^+ \) and \( \partial f_n^- \) are bounded in \( L^2(B_{2R}(0)) \), and thus \( f_n \) is bounded in \( L^2(B_{2R}(0)) \). Since the embedding \( L^2(B_{2R}(0)) \to L^2(B_{2R}(0)) \) is compact, without any loss of generality, we can assume that \( f_n \) converges to some function \( f \in L^2(B_{2R}(0)) \). But then again \( D^\pm f_n \) converges also, and thus \( f_n \) is convergent in \( L^2_{1}(B_{2R}(0)) \), and thus \( f \in L^2_{1}(B_{2R}(0)) \), and

\[
D^\pm f = \lim_{n \to \infty} D^\pm f_n = \lim_{n \to \infty} g_n = 0,
\]

and hence \( f \in \ker(D^+) \cap (\ker(D^+))^\perp \), which contradicts that \( f \) has unit norm.

Finally, let \( f \in \mathcal{H}^\pm \) be any. Let \( R \) be as above and pick \( \chi_R : \mathbb{C} \to [0, 1] \) such that it is smooth, identically 1 on \( B_R(0) \), vanishes on \( \mathbb{C} - B_{2R}(0) \), and \( \|d\chi_R\|_{L^\infty(\mathbb{C})} \leq \frac{2}{R} \). Then we have

\[
\|D^\pm f\|_{L^2(\mathbb{C})}^2 = \|D^\pm (\chi_R f + (1 - \chi_R) f)\|_{L^2(\mathbb{C})}^2 \\
= \|D^\pm (\chi_R f)\|_{L^2(\mathbb{C})}^2 + \|D^\pm ((1 - \chi_R) f)\|_{L^2(\mathbb{C})}^2 \\
= \|D^\pm (\chi_R f)\|_{L^2(\mathbb{C})}^2 + \|D^\pm ((1 - \chi_R) f)\|_{L^2(\mathbb{C})}^2.
\]
\[ + 2 \text{Re} \left( \int_{\mathbb{C}} D^\pm(\chi_R f)D^\pm(1 - \chi_R f) \, d\lambda \right) \]
\[ \geq C \left( \| \chi_R f \|_{L^2(\mathbb{C})}^2 + \| (1 - \chi_R) f \|_{L^2(\mathbb{C})}^2 \right) \]
\[ - 2 \int_{\mathbb{C}} (\chi_R (1 - \chi_R)|D^\pm f|^2 + 2|d\chi_R||f||D^\pm f| + |d\chi_R|^2|f|^2) \, d\lambda \]
\[ \geq \left( \frac{C}{2} - \frac{4}{R^2} \right) \| f \|_{L^2(\mathbb{C})}^2 - \| D^\pm f \|_{L^2(\mathbb{C})}^2 - \frac{8}{R} \| f \|_{L^2(\mathbb{C})} \| D^\pm f \|_{L^2(\mathbb{C})}. \]

From this we get, for \( f \neq 0 \), that
\[ \frac{\| D^\pm f \|_{L^2(\mathbb{C})}}{\| f \|_{L^2(\mathbb{C})}} \geq \frac{\sqrt{CR^2 + 16} - 4}{2R} > 0, \]
which proves inequality (2.5). The dependence of \( C^\pm \) on \( |T| \) is clear from the proof.

Now we prove that both \( D^\pm \) are Fredholm. We have already proved that they are continuous, and have finite dimensional kernels. It is then enough to prove that their cokernels are finite dimensional. We do that in two steps:

1. Prove that the images of \( D^\pm \) are closed, and thus \( \text{coker}(D^\pm) \equiv (\text{im}(D^\pm))^\perp. \)
2. We show that \( (\text{im}(D^\pm))^\perp = \ker(D^\mp). \)

To prove (1), let \( \Pi^\pm : \mathcal{H}^\pm \to \ker(D^\pm) \) be the orthogonal projection, which is compact, as the kernels are finite dimensional (in fact, one of them is always trivial). Then, using inequality (2.5), we have that
\[ \forall f \in \mathcal{H}^\pm : \quad \| f \|_{\mathcal{H}^\pm}^2 = \| D^\pm f \|_{L^2(\mathbb{C})}^2 + \| f \|_{L^2(\mathbb{C})}^2 \]
\[ = \| D^\pm (f - \Pi^\pm f) \|_{L^2(\mathbb{C})}^2 + \| (f - \Pi^\pm f) \|_{L^2(\mathbb{C})}^2 + \| \Pi^\pm f \|_{L^2(\mathbb{C})}^2 \]
\[ \leq (1 + (C^\pm)^2) \| D^\pm (f - \Pi^\pm f) \|_{L^2(\mathbb{C})}^2 + \| \Pi^\pm f \|_{L^2(\mathbb{C})}^2 \]
\[ = (1 + (C^\pm)^2) \| D^\pm f \|_{L^2(\mathbb{C})}^2 + \| \Pi^\pm f \|_{L^2(\mathbb{C})}^2. \]

Thus by [3, Lemma 4.3.9], \( \text{im}(D^\pm) \) is closed.

Finally, we show that \( (\text{im}(D^\pm))^\perp = \ker(D^\mp) \). The fact that \( (\text{im}(D^\pm))^\perp \supseteq \ker(D^\mp) \) is straightforward. In order to prove the other inclusion, note first that any \( u \in L^2(\mathbb{C}) \) defines a continuous functional on \( \mathcal{H}^\pm \) by \( f \mapsto \langle u | f \rangle_{L^2(\mathbb{C})} \), because the norm on \( \mathcal{H}^\pm \) is strictly, pointwise greater than the one on \( L^2(\mathbb{C}) \). Thus, by the Reisz Representation Theorem, there is a unique \( v^\pm_u \in \mathcal{H}^\pm \), such that
\[ \forall f \in \mathcal{H}^\pm : \quad \langle u | f \rangle_{L^2(\mathbb{C})} = \langle v^\pm_u | f \rangle_{\mathcal{H}^\pm} = \langle D^\pm v^\pm_u | D^\pm f \rangle_{L^2(\mathbb{C})} + \langle v^\pm_u | f \rangle_{L^2(\mathbb{C})}. \]
Note that equation (2.6) is the weak formulation of the Poisson type equation
\[
(D^\pm D^\pm + 1)v^\pm_u = u. \tag{2.7}
\]
Since \(v^\pm_u \in \mathcal{H}^\pm \leq L^2_z(\mathbb{C})\), elliptic regularity guarantees that \(v_u \in L^2_{2,\text{loc}}(\mathbb{C})\).

Let now \(u^\pm \in (\text{im}(D^\pm))^\perp\), or equivalently
\[
\forall f \in \mathcal{H}^\pm:\quad \langle u^\pm | D^\pm f \rangle_{L^2_z(\mathbb{C})} = 0.
\]
Let \(v^\pm_u\) as above, and define \(w^\pm_u = D^\pm v^\pm_u \in L^2_{1,\text{loc}}(\mathbb{C})\). Now \(w^\pm_u\) is a weak solution of
\[
(D^\pm D^\pm + 1)w^\pm_u = 0.
\]
Thus \(w^\pm_u\) is smooth, and hence so is \(v^\pm_u\), again by elliptic regularity. This means, that \(v^\pm_u\) is a strong solution of equation (2.7), and thus \(u\) is smooth, and standard argument shows that it satisfies \(D^\pm u = 0\), and so \((\text{im}(D^\pm))^\perp \leq \ker(D^\pm)\).

Now let \(\Phi = Tz^p \overline{z}^q + \varphi\). Without any loss of generality (using Lemma 1.8 and rescaling), we can assume that \(T = 1\), and define
\[
\Phi_t := z^p \overline{z}^q + (1 - t)\varphi.
\]
This defines a homotopy between \(\Phi_0 = \Phi\) and \(\Phi_1 = z^p \overline{z}^q\), such that for every \(t \in [0, 1]\), \(\Phi_t\) is a degree \(m\) homogeneous polynomial with a unique zero of index \(k\) at the origin. Furthermore, the corresponding operators, call \(\{D_t^\pm\}_{t \in [0, 1]}\) form a norm-continuous family of operators that, by the above arguments, are all Fredholm, and thus
\[
\text{index}_\mathbb{R}(D^\pm) = \text{index}_\mathbb{R}(D^\pm_0) = \text{index}_\mathbb{R}(D^\pm_1).
\]
The index of \(D^\pm_1\) was computed by Rauch in [18, Proposition 3.1] and is equal to \(\pm k\). This, together with the triviality of \(\ker(D^\pm \text{sign}(k))\), concludes the proof. \(\square\)

Finally, we are ready to prove Theorem 2.1.

**Proof of Theorem 2.1:** First, let \(\alpha \in C^\infty_{\text{cpt}}(\mathbb{C})\) arbitrary. The operators defined as
\[
D_\alpha^+ f = \frac{\partial f}{\partial z} + \alpha f + \overline{\Phi f},
\]
\[
D_\alpha^- f = -\frac{\partial f}{\partial \overline{z}} + \overline{\alpha} f + \Phi \overline{f}.
\]
are compact perturbations of \(D^\pm\) thus are themselves Fredholm operators and have the same Fredholm indices, that is \(\pm k\). Again, it is enough to consider the \(k \geq 0\) case. For any \(f_1, f_2 \in C^\infty_{\text{cpt}}(\mathbb{C})\) we have
\[
\langle f_1 | D_\alpha^+ f_2 \rangle_{L^2(\mathbb{C})} = \langle D_\alpha^- f_1 | f_2 \rangle_{L^2(\mathbb{C})},
\]
and thus it is enough to show that \( D_\alpha^- \) has trivial kernel. Assume that \( f \in \mathcal{H}^- \) and \( D_\alpha^- f = 0 \), or equivalently
\[
D^- f = -\overline{\alpha} f.
\]
Using inequality (2.5), we get that
\[
\|f\|_{L^2(\mathbb{C})} \leq \frac{1}{c^-(e)^{m+1/|\Gamma|}} \|D^- f\|_{L^2(\mathbb{C})} = \frac{1}{c^-(e)^{m+1/|\Gamma|}} \|\overline{\alpha} f\|_{L^2(\mathbb{C})} \leq \frac{\|\alpha\|_{L^\infty(\mathbb{C})}}{c^-(e)^{m+1/|\Gamma|}} \|f\|_{L^2(\mathbb{C})}.
\]
Thus if
\[
\|\alpha\|_{L^\infty(\mathbb{C})} < c^-(e)^{m+1/|\Gamma|},
\]
then \( f = 0 \). The claim about the exponential decay of the solutions can be proven as in the proof of Theorem 2.3, as \( \alpha \) is compactly supported.

3. Generalized Jackiw–Rossi Theory on Complete Surfaces

In this section, we use the results of Section 2.1 to classify and construct solutions of Jackiw–Rossi equation (1.4).

Let \( X \) be a oriented, complete Riemannian surface, and let \( \mathcal{S}_{\mathcal{L}} \) be a spinor bundle corresponding to a spin\(^c\) structure of \( X \) with associated line bundle \( \mathcal{L}^2 \). Let \( \nabla \) be a compatible connection on \( \mathcal{S}_{\mathcal{L}} \), and \( \Phi \) be a smooth section of \( \mathcal{F}^+ = \mathcal{L}^{-2} \). Thus we can define the Jackiw–Rossi operator \( H_{\nabla,\Phi} = D_{\nabla} + s \Phi \) as before. Recall that in dimension two, every connection induces a holomorphic structure. We use the usual notations, \( \partial \nabla := \nabla^{1,0} \) and \( \overline{\partial} \nabla := \nabla^{0,1} \). In particular, in the splitting \( \mathcal{S}_{\mathcal{L}} = \mathcal{S}_{\mathcal{L}}^+ \oplus \mathcal{S}_{\mathcal{L}}^- \) where \( \mathcal{S}_{\mathcal{L}}^\pm = \mathcal{L} \Theta^\pm \) with \( \Theta^2 = K_X \), we have
\[
D_{\nabla} = \begin{pmatrix} 0 & \overline{\partial} \nabla \\ \partial \nabla & 0 \end{pmatrix}.
\]

Let \( \mathcal{Z}_{\Phi} := \{ x \in X \mid \Phi(x) = 0 \} \) be the zero locus of \( \Phi \). Assume for the rest of the section the following:

A1: The curvature of \( \nabla \) is bounded.

A2: \( \nabla \Phi \) is bounded.

A3: \( \mathcal{Z}_{\Phi} \) is finite. Let
\[
\delta_{\Phi} := \frac{1}{4} \inf \{ \text{dist}(x, y) \mid x, y \in \mathcal{Z}_{\Phi} \& x \neq y \} > 0.
\]

A4: The infimum of \( |\Phi| \) on the set \( \{ x \in X \mid \text{dist}(x, \mathcal{Z}_{\Phi}) > \delta \} \) is positive. Then \( |\Phi(x)| > c \). \(^1\)

A5: For all \( x \in \mathcal{Z}_{\Phi} \), there is a homogeneous, complex polynomial on \( \mathbb{C} \), \( \phi_x \), of degree \( m(x) \in \mathbb{N}_+ \) with an isolated zero at the origin of index \( k(x) \in \mathbb{Z} \), a local, holomorphic, normal chart \( z \), and a local, normal trivialization of \( \mathcal{L} \), \( \nu_x \), such that for all \( y \in X \) close

\(^1\)When \( X \) is compact, then this condition is implied by assumption A2, but when \( X \) is noncompact, then it excludes, for example, the cases when \( \Phi \) decays at “infinity".
enough points we have
\[ \Phi(y) = \phi_x(z(y), z(y))v_x(y) + O(|z|^{m(x)+1}), \]
\[ \nabla \Phi(y) = O(|z|^{m(x)-1}). \]

In other words, the leading order term of \( \Phi \) at any of its zero is of the form that we studied in Section 2.1.

As in Section 2.1, let us define the (real) Hilbert spaces \( \mathcal{H}_{V, \Phi}^\pm \) as the norm-completions of the vector space of smooth, compactly supported sections of \( \mathcal{S}_{\mathcal{L}}^\pm \), with respect to the \( L^2 \)-graph norm of \( H_{V, \Phi}^\pm \), that is:
\[
\forall \Psi \in C^\infty_{\text{cpt}}(\mathcal{S}_{\mathcal{L}}^\pm) : \| \Psi \|_{\mathcal{H}_{V, \Phi}^\pm} = \left( \| H_{V, \Phi} \Psi \|_{L^2(\mathcal{S}_{\mathcal{L}})}^2 + \| \Psi \|_{L^2(\mathcal{S}_{\mathcal{L}})}^2 \right)^{\frac{1}{2}},
\]
and let \( \mathcal{H}_{V, \Phi} = \mathcal{H}_{V, \Phi}^+ \cap \mathcal{H}_{V, \Phi}^- \).

The next two lemmata are technical results that are necessary to prove our main theorem.

**Lemma 3.1.** Under the assumptions **A1** through **A5**, we have the following:

1. \( \mathcal{H}_{V, \Phi} \subseteq L^2_1(\mathcal{S}_{\mathcal{L}}) \) and the inclusion is a continuous embedding. If \( \Phi \) is bounded, then \( \mathcal{H}_{V, \Phi} = L^2_1(\mathcal{S}_{\mathcal{L}}) \).

2. Considered as a densely defined operator on \( L^2(\mathcal{S}_{\mathcal{L}}) \), \( H_{V, \Phi} \) (with domain being \( \mathcal{H}_{V, \Phi} \)) is self-adjoint.

**Proof.** Simple computation yields the following Weitzenböck identity:
\[
( H_{V, \Phi})^* H_{V, \Phi} \Psi = \nabla^* \nabla \Psi + c(F_\Psi) \Psi + c(\mathcal{A}_V \Phi) \Psi + |\Phi|^2 \Psi. \tag{3.2}
\]

Thus, if \( \Psi \) is a smooth and compactly supported section on \( \mathcal{S}_{\mathcal{L}} \), then we have
\[
\| \nabla \Psi \|_{L^2(T^* X \otimes \mathcal{S}_{\mathcal{L}})}^2 = \| H_{V, \Phi} \Psi \|_{L^2(\mathcal{S}_{\mathcal{L}})}^2 - \langle \Psi | c(F_\Psi) \Psi \rangle_{L^2(\mathcal{S}_{\mathcal{L}})} - \langle \Psi | c(\mathcal{A}_V \Phi) \Psi \rangle_{L^2(\mathcal{S}_{\mathcal{L}})} - \| \mathcal{A}_V \Psi \|_{L^2(\mathcal{S}_{\mathcal{L}})}^2 \leq \| H_{V, \Phi} \Psi \|_{L^2(\mathcal{S}_{\mathcal{L}})}^2 + 2(\| \nabla \Phi \|_{L^\infty} + \| F_\Psi \|_{L^\infty}) \| \Psi \|_{L^2(\mathcal{S}_{\mathcal{L}})}^2.
\]

Using assumptions **A1** and **A2**, we get \( \| \Psi \|_{L^2(\mathcal{S}_{\mathcal{L}})} \leq C(\nabla, \Phi) \sqrt{t} \| \Psi \|_{\mathcal{H}_{V, \Phi}} \). Furthermore, if \( \Phi \) is bounded, then the above equations also gives \( \| \Psi \|_{\mathcal{H}_{V, \Phi}} \leq C'(\nabla, \Phi) \sqrt{t} \| \Psi \|_{L^2_1(\mathcal{S}_{\mathcal{L}})} \). This proves the first claim.

Using the notation of [3], let \( V := \mathcal{H}_{V, \Phi} \), \( H := L^2(\mathcal{S}_{\mathcal{L}}) \), and
\[
B : V \times V \to \mathbb{R}, \quad (\Psi_1, \Psi_2) \mapsto \langle \Psi_1 | H_{V, \Phi} \Psi_2 \rangle_{L^2(\mathcal{S}_{\mathcal{L}})}.
\]

By the proof of the first claim, \((V, H, V \hookrightarrow H)\) is a Gelfand triple (cf. [3, Definition 6.3.7]), and the bilinear form \( B \) satisfies the conditions of [3, Theorem 6.3.8] for \( t \) large enough (with constants \( c = C = C(\nabla, \Phi) \sqrt{t} \) and \( \delta = 0 \) in the reference's notation). This concludes the proof of the second claim. \( \square \)
**Lemma 3.2.** Under the assumptions A1 through A5, for all $\mathcal{E}_0 > 0$, there is a positive number $T$, only dependent on $(X, g, \mathcal{L}, \nabla, \Phi, \mathcal{E}_0)$, such that for all $t > T$, the intersection $[-\mathcal{E}_0, \mathcal{E}_0] \cap \text{Spec}(H_{\nabla, r\Phi})$ contains only eigenvalues.

Furthermore, if $\lambda \in [-\mathcal{E}_0, \mathcal{E}_0] \cap \text{Spec}(H_{\nabla, r\Phi})$, then $\ker(H_{\nabla, r\Phi} - \lambda \mathbb{1}_{L^2(\mathcal{L})})$ is a finite dimensional subspace of $C^\infty(\mathcal{L}) \cap L^2(\mathcal{L})$.

**Proof.** Let $\chi$ be the nonnegative function that satisfies

$$\chi^2 = \max\{1 - |\Phi|^2, 0\}.$$ 

By assumption, $\chi$ is compactly supported, and thus the map

$$K : \mathcal{H}_{\nabla, r\Phi} \to L^2(\mathcal{L}); \Psi \mapsto \chi \Psi,$$

is compact. Then for all $\Psi \in \mathcal{H}_{\nabla, r\Phi}$ we have

$$\|\Psi\|_{L^2(\mathcal{L})}^2 \leq \|\alpha_{\Phi}\|_{L^2(\mathcal{L})}^2 + \|K\Psi\|_{L^2(\mathcal{L})}^2.$$

Using the Weitzenböck identity from the proof of Lemma 3.1, we get for any $\lambda \in \mathbb{R}$ that

$$t^2 \|\alpha_{\Phi}\|_{L^2(\mathcal{L})}^2 \leq \|H_{\nabla, r\Phi} \Psi\|_{L^2(\mathcal{L})}^2 + (2t \|\nabla \Phi\|_{L^\infty} + 2||F\|_{L^\infty}) \|\Psi\|_{L^2(\mathcal{L})}^2 \leq \|H_{\nabla, r\Phi} - \lambda \mathbb{1}_{L^2(\mathcal{L})} \Psi\|_{L^2(\mathcal{L})}^2 + 2\lambda \langle \Psi | (H_{\nabla, r\Phi} - \lambda \mathbb{1}_{L^2(\mathcal{L})}) \Psi \rangle_{L^2(\mathcal{L})} + (2t \|\nabla \Phi\|_{L^\infty} + 2||F\|_{L^\infty} + \lambda^2) \|\Psi\|_{L^2(\mathcal{L})}^2 \leq \left(\|H_{\nabla, r\Phi} - \lambda \mathbb{1}_{L^2(\mathcal{L})} \Psi\|_{L^2(\mathcal{L})}^2 + \sqrt{2t \|\nabla \Phi\|_{L^\infty} + 2||F\|_{L^\infty} + \lambda^2} \|\Psi\|_{L^2(\mathcal{L})}^2\right) \leq \left(\|H_{\nabla, r\Phi} - \lambda \mathbb{1}_{L^2(\mathcal{L})} \Psi\|_{L^2(\mathcal{L})}^2 + \sqrt{2t \|\nabla \Phi\|_{L^\infty} + 2||F\|_{L^\infty} + \lambda^2} \|\Psi\|_{L^2(\mathcal{L})}^2\right)^2.$$ 

Thus, we have

$$\|\Psi\|_{\mathcal{H}_{\nabla, r\Phi}} \leq \frac{1}{t} \|H_{\nabla, r\Phi} - \lambda \mathbb{1}_{L^2(\mathcal{L})} \Psi\|_{L^2(\mathcal{L})}^2 + \|K\Psi\|_{L^2(\mathcal{L})}^2 + \sqrt{\frac{2t \|\nabla \Phi\|_{L^\infty} + 2||F\|_{L^\infty} + \lambda^2}{t}} \|\Psi\|_{L^2(\mathcal{L})}^2.$$ 

Thus if $|\lambda| < \mathcal{E}_0$ and $t > T = O\left(\sqrt{\|\nabla \Phi\|_{L^\infty}^2 + \|F\|_{L^\infty} + \mathcal{E}_0}\right)$, then the last term can be absorbed on the left, and we get (for some $c > 0$)

$$\|\Psi\|_{\mathcal{H}_{\nabla, r\Phi}} \leq c \left(\|H_{\nabla, r\Phi} - \lambda \mathbb{1}_{L^2(\mathcal{L})} \Psi\|_{L^2(\mathcal{L})}^2 + \|K\Psi\|_{L^2(\mathcal{L})}^2\right).$$ 

Using [3, Lemma 4.3.9] with $X = \mathcal{H}_{\nabla, r\Phi}$, $Y = Z = L^2(\mathcal{L})$, and $\Lambda = H_{\nabla, r\Phi}$, this implies that $H_{\nabla, r\Phi} - \lambda \mathbb{1}_{L^2(\mathcal{L})}$ closed image and finite dimensional kernel.

Now if $|\lambda| < \mathcal{E}_0$ and $t > T$, then $H_{\nabla, r\Phi} - \lambda \mathbb{1}_{L^2(\mathcal{L})}$ is self-adjoint and has closed image, thus $\lambda$ is either an eigenvalue of $H_{\nabla, r\Phi}$ or the image $H_{\nabla, r\Phi} - \lambda \mathbb{1}_{L^2(\mathcal{L})}$ is dense, and thus all of $L^2(\mathcal{L})$,
in which case $\lambda$ is not in the spectrum. This proves that $\lambda \in [-\epsilon_0, \epsilon_0] \cap \text{Spec}(H_{V, t\Phi})$ contains only eigenvalues.

Since $(V, \Phi)$ is a smooth pair and $H_{V, \Phi}$ is elliptic, each $\Psi \in \ker (H_{V, t\Phi} - \lambda I_{L^2(S_{\mathcal{L}})})$ is smooth. Furthermore, using equation (3.2) we get that $|\nabla^* \nabla \Psi| \leq C(\lambda, t, \nabla, \Phi) |\Psi|$, thus $\Psi \in L^2(S_{\mathcal{L}})$. This concludes the proof. □

**Corollary 3.3.** The operators $H_{V, \Phi}^\pm: \mathcal{H}_{V, t\Phi}^\pm \rightarrow L^2(S_{\mathcal{L}})$ are Fredholm and

$$\text{index}_{\mathbb{R}}(H_{V, \Phi}^\pm) = -\text{index}_{\mathbb{R}}(H_{V, \Phi}^\mp).$$

**Proof.** By Lemma 3.2, for large $t$, $H_{V, t\Phi}: \mathcal{H}_{V, t\Phi} \rightarrow L^2(S_{\mathcal{L}})$ has finite dimensional kernel and closed image. Since as a densely defined map on $L^2(S_{\mathcal{L}})$, $H_{V, \Phi}$ is self-adjoint, its index is zero. Finally, as $H_{V, \Phi}$ is odd, with respect to the parity grading, we get equation (3.3). □

For the rest of the paper, let

$$\tau_x(t) := \frac{m(x) + \sqrt{t}}{2},$$

$$M := \max (\{ m(x) | x \in Z_{\Phi} \}),$$

$$\tau(t) := \frac{M + \sqrt{t}}{2} = \min (\{ \tau_x(t) | x \in Z_{\Phi} \}).$$

We are now ready to state our main theorem.

**Theorem 3.4.** Under the assumptions $A1$ through $A5$, there are positive numbers, $\epsilon$ and $T$, both dependent only on $(X, g, \mathcal{L}, \nabla, \Phi)$, such that for all $t > T$, there are subspaces $\mathcal{K}_t \subseteq \mathcal{H}_{V, t\Phi}$ with the following properties:

1. $\mathcal{K}_t$ and $\mathcal{K}_t^\perp$ are invariant subspaces of $H_{V, t\Phi}$ and the parity operator $\mathbb{P}$.
2. The dimension of $\mathcal{K}_t$ is given by

$$\dim_{\mathbb{R}}(\mathcal{K}_t) = \sum_{x \in Z_{\Phi}} |k(x)| < \infty.$$

3. For all $\Psi \in \mathcal{K}_t^\perp$, we have that

$$\|H_{V, t\Phi} \Psi\|_{L^2(S_{\mathcal{L}})} \geq \epsilon_0 \tau(t) \|\Psi\|_{L^2(S_{\mathcal{L}})}.$$

In particular, $\ker (H_{V, t\Phi}) \subseteq \mathcal{K}_t$.
4. For all $\Psi \in \mathcal{K}_t$, we have that

$$\|H_{V, t\Phi} \Psi\|_{L^2(S_{\mathcal{L}})} \geq \frac{1}{\epsilon \tau(t)} \|\Psi\|_{L^2(S_{\mathcal{L}})}.$$

In particular, $\ker (H_{V, t\Phi}) \subseteq \mathcal{K}_t$. 18
Furthermore, we have that for all \( t \in \mathbb{R}_+ \)
\[
\text{index}_\mathbb{R}(H_{V_i,t\Phi}^+ \rho) = \pm \sum_{x \in \mathcal{Z}_\Phi} k(x),
\]  
(3.7)

**Proof.** In order to make the rest of the computations simpler, let us conformally flatten out a neighborhood of \( \mathcal{Z}_\Phi \) that is contained in the \( 2\delta_\Phi \)-ball around \( \mathcal{Z}_\Phi \). Since X is 2-dimensional, this can always be done, and due to Lemma 1.8. By assumption A3, \( \mathcal{Z}_\Psi \mathcal{Z}_\Phi \) is finite, so the conformal factor can be chosen to be bounded, and as it is independent of \( t \), it is enough to prove the claims in this new metric. Without any loss of generality we can also assume that X is flat on the ball of radius two around \( \mathcal{Z}_\Phi \), and it is a tubular neighborhood.

We construct a finite dimensional subspace, \( \mathcal{V}_t \subseteq H_{V_i,t\Phi} \) that, for \( t \) large enough, satisfies (for some \( \epsilon' > 0 \), independent of \( t \))
\[
\dim(\mathcal{V}_t) = \sum_{x \in \mathcal{Z}_\Phi} |k(x)|,
\]
\[
\forall \Psi \in \mathcal{V}_t : \|H_{V_i,t\Phi}\Psi\|_{L^2(S^2_{\mathcal{Z}^+})} \leq \frac{1}{t^{\delta}(t)} \|\Psi\|_{L^2(S_{\mathcal{Z}^+})}, \tag{3.8a}
\]
\[
\forall \Psi \in \mathcal{V}_t^\perp : \|H_{V_i,t\Phi}\Psi\|_{L^2(S^2_{\mathcal{Z}^+})} \geq \epsilon' t(t) \|\Psi\|_{L^2(S_{\mathcal{Z}^+})}. \tag{3.8b}
\]

Note that this subspace \( \mathcal{V}_t \) almost satisfies the requirements of \( \mathcal{H}_t \), expect it is not necessarily invariant with respect to \( H_{V_i,t\Phi} \) and \( \mathbb{P} \). In order to construct \( \mathcal{V}_t \), let us first make a few choices. For each \( x \in \mathcal{Z}_\Phi \), choose a smooth function, \( \chi_x \), with values in \([0, 1]\), so that \( \text{supp}(\chi_x) \subseteq B_2(x) \) and \( \chi_x|_{B_1(x)} \equiv 1 \). Fix \( x \in \mathcal{Z}_\Phi \) and assume, without any loss of generality, that \( k(x) \geq 0 \). Let \( \theta \) be a local, flat section of \( \Theta, \nu_x \) and \( z \) as in assumption A5, \( \alpha := (\chi_x h(\nu_x, \xi^+(\nabla v_x))) \circ z^{-1} \) (extended by zero to all of \( \mathbb{C} \)), and \( f_t \in L^2_1(\mathbb{C}) \) be a solution to
\[
\frac{\partial f_t}{\partial z} + \alpha f_t + t\phi_x x^{-1} = 0. \tag{3.9}
\]

Note that this is an instance of equation (2.1). Since \( f_t \) decays exponentially and \( \phi_x \) is homogeneous of degree \( m(x) \), we can see (through a simple rescaling argument) that there is a \( c > 0 \), independent of \( f_t \), such that for all \( w \in \mathbb{C} \)
\[
|f_t(w)| \leq c \tau_x(t) \exp\left( -\frac{\tau_x(t)}{c} |w| \right) \|f_t\|_{L^2(\mathbb{C})}, \tag{3.10}
\]

Let now \( \tilde{\Psi}_0^x_{x,f_t} := \chi_x (f_t \circ oz)\theta_x \nu_x \in \Gamma(S_{\mathcal{Z}^+}) \), extended as zero to the rest of \( X \). Using inequality (3.10) we can see that (after potentially increasing \( c \))
\[
\left\langle \tilde{\Psi}_0^x_{x,f_t}, \Psi_{0,y,f_t}' \right\rangle_{L^2(S_{\mathcal{Z}^+})} = \begin{cases} 
\langle f_t | f_t \rangle_{L^2(\mathbb{C})} \left( 1 + O\left( \frac{1}{\tau_x(t)} \right) \right), & \text{if } x = y, \\
\exp\left( -\frac{\tau(t)}{c} \text{dist}(x, y) \right) \|f_t\|_{L^2(\mathbb{C})} \|f_t\|_{L^2(\mathbb{C})}, & \text{otherwise}.
\end{cases}
\]

Thus, repeating the construction for all \( x \in \mathcal{Z}_\Phi \), the resulting spinor span we get a (real) subspace of \( H_{V_i,t\Phi} \) of dimension \( \sum_{x \in \mathcal{Z}_\Phi} |k(x)| \). Let us call this subspace \( \mathcal{V}_t \).
Next we prove inequalities (3.8a) and (3.8b). On the ball of radius 2 around $\mathcal{Z}_\Phi$ the model operator

$$ H^\text{model}_t : f^+ \theta_x \nu_x + f^- \overline{\theta}_x \nu_x \mapsto \left( - \frac{\partial f_t}{\partial z} + \alpha f_t + t \phi x f_t \right) \theta_x \nu_x + \left( \frac{\partial f_t}{\partial z} - \alpha f_t + t \phi x f_t \right) \overline{\theta}_x \nu_x $$

and $H_{\nu, \phi}$ differ only by an algebraic term of norm, close to $x \in \mathcal{Z}_\Phi$ is $O(t |z|^{m(x)+1})$. Thus for any $\Psi \in \mathcal{V}_t$, of the form $\Psi = \Psi_x f_t$, we have using inequality (3.10) that

$$ |H_{\nu, \phi} \Psi| \leq O \left( \frac{1}{\tau_x(t)} \right) \| \Psi \|_{L^2(\mathcal{Z}_\Phi)} \leq O \left( \frac{1}{\tau(t)} \right) \| \Psi \|_{L^2(\mathcal{Z}_\Phi)} $$

Integrating over the neighborhood yields

$$ \| H_{\nu, \phi} \Psi \|_{L^2(\mathcal{Z}_\Phi)} \leq O \left( \frac{1}{\tau_x(t)} \right) \| \Psi \|_{L^2(\mathcal{Z}_\Phi)} \leq O \left( \frac{1}{\tau(t)} \right) \| \Psi \|_{L^2(\mathcal{Z}_\Phi)} $$

Since $\mathcal{V}_t$ is a finite span of such spinors, we get inequality (3.8a).

Let now $\Psi \in \mathcal{V}_t^\perp$ and let $\chi := 1 - \sum_{x \in \mathcal{Z}_\Phi} \chi_x$. Using equation (3.2) and assumption A5 we get, for $t$ large enough, that

$$ \| H_{\nu, \phi} \chi \Psi \|_{L^2(\mathcal{Z}_\Phi)}^2 = \| \nabla(\chi \Psi) \|_{L^2(\mathcal{Z}_\Phi)}^2 + \langle (\chi \Psi) | c(\nabla \chi) (\chi \Psi) \rangle_{L^2(\mathcal{Z}_\Phi)} + t \langle (\chi \Psi) | c \exp \left( -\frac{\tau_x(t)}{c} \right) (\chi \Psi) \rangle_{L^2(\mathcal{Z}_\Phi)} $$

$$ \geq c^2 t^2 \| \chi \Psi \|_{L^2(\mathcal{Z}_\Phi)}^2. $$

For each $x \in \mathcal{Z}_\Phi$, let $f^\pm_x$ be the functions on $\mathbb{C}$ that satisfy

$$ f^+ o \ impeccable = h(\theta_x \nu_x, \chi_x \Psi), $$

$$ f^- o \ impeccable = h(\overline{\theta}_x \nu_x, \chi_x \Psi). $$

Then $\Psi \in \mathcal{V}_t^\perp$ implies that the $L^2$-orthogonal projections of $f^\pm$ onto the space of solutions to equation (3.9) (or to the dual equation), $f^\pm_\parallel$, satisfies $\| f^\pm_\parallel \|_{L^2(\mathcal{Z}_\Phi)} = O \left( \exp \left( -\frac{\tau_x(t)}{c} \right) \| \Psi \|_{L^2(\mathcal{Z}_\Phi)} \right)$. Thus, if $f^\pm = f^\pm_\parallel + f^\pm_\perp$, then, using inequality (2.5), we get

$$ \| H_{\nu, \phi} (\chi_x \Psi) \|_{L^2(\mathcal{Z}_\Phi)} \geq c \left( \| D_\alpha f^+ \|_{L^2(\mathbb{C})} + \| D_\alpha f^+ \|_{L^2(\mathbb{C})} \right) - O \left( \| \chi_x \Psi \|_{L^2(\mathcal{Z}_\Phi)} \right) $$

$$ \geq c' \tau_x \left( \| f^+ \|_{L^2(\mathbb{C})} + \| f^+ \|_{L^2(\mathbb{C})} \right) - O \left( \exp \left( -\frac{\tau_x(t)}{c} \right) \| \chi_x \Psi \|_{L^2(\mathcal{Z}_\Phi)} \right) $$

$$ \geq c'' \tau_x(t) \| \chi_x \Psi \|_{L^2(\mathcal{Z}_\Phi)}. $$

Since $\chi = 1 - \sum_{x \in \mathcal{Z}_\Phi} \chi_x$ and if $x \neq y$, then $\chi_x \chi_y = 0$, we get that (for some constants $c_i > 0$)

$$ \| H_{\nu, \phi} \Psi \|_{L^2(\mathcal{Z}_\Phi)}^2 = \| H_{\nu, \phi} (\chi \Psi) + H_{\nu, \phi} \left( (1 - \chi) \Psi \right) \|_{L^2(\mathcal{Z}_\Phi)}^2 $$

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\[
= \|H_{\nu, \rho}(\chi \Psi)\|^2_{L^2(S, \nu)} + \sum_{x \in \mathcal{Z}_\phi} \|H_{\nu, \rho}(\chi_x \Psi)\|^2_{L^2(S, \nu)}
+ 2\langle H_{\nu, \rho}(\chi \Psi) | H_{\nu, \rho}((1 - \chi) \Psi) \rangle_{L^2(S, \nu)}
\geq c_1 t^2 \|\chi \Psi\|^2_{L^2(S, \nu)} + c_2 \sum_{x \in \mathcal{Z}_\phi} (\tau_x(t))^2 \|\chi_x \Psi\|^2_{L^2(S, \nu)}
- \|\text{det}(d\chi)\Psi\|^2_{L^2(S, \nu)} + 2\int_{\chi} (1 - \chi) |H_{\nu, \rho}\Psi|^2 \text{vol}_g
+ 2\langle \text{det}(d\chi)\Psi | (1 - 2\chi)H_{\nu, \rho}((1 - \chi) \Psi) \rangle_{L^2(S, \nu)}
\geq c_3 (\tau(t))^2 \int_{\chi} \left( |\chi|^2 + \sum_{x \in \mathcal{Z}_\phi} |\chi_x|^2 \right) \|\Psi\|^2 \text{vol}_g - c_4 \|\Psi\|^2_{L^2(S, \nu)}
- c_5 \|\Psi\|_{L^2(S, \nu)} \|H_{\nu, \rho} \Psi\|_{L^2(S, \nu)}
\geq c_6 (\tau(t))^2 \|\Psi\|^2_{L^2(S, \nu)} - c_5 \|\Psi\|_{L^2(S, \nu)} \|H_{\nu, \rho} \Psi\|_{L^2(S, \nu)}.
\]

Rearranging the above inequality proves inequality (3.8b).

We are now ready to define \( \mathcal{K}_t \). By Lemma 3.2, for any \( \mathcal{E}_0 \), the set \([-\mathcal{E}_0, \mathcal{E}_0] \cap \text{Spec}(H_{\nu, \rho})\) contains only eigenvalues. Let \( T \) be the maximum of the ones given in Lemma 3.2 for \( \mathcal{E}_0 = 1 \) and required for the definition of \( \mathcal{V}_t \) above. Furthermore, if need, increase \( T \) so that it is greater than \( \frac{4}{c}\). Since decreasing \( \mathcal{E}_0 \) does not change the fact that \([-\mathcal{E}_0, \mathcal{E}_0] \cap \text{Spec}(H_{\nu, \rho})\) contains only eigenvalues, let \( \mathcal{E}_0 = \frac{2}{\text{det}(d\tau(t))} \) and let \( \mathcal{K}_t \) be the span of the corresponding eigenvectors. By construction, \( \mathcal{K}_t \) is invariant under \( H_{\nu, \rho} \) and since \( \mathbb{P} \) reverses eigenvectors, \( \mathcal{K}_t \) is also invariant under \( \mathbb{P} \). Furthermore, inequality (3.5) is immediately satisfied with \( \mathcal{E} = \frac{E_t}{c} \). Note that both inequalities (3.5) and (3.6) remain true if \( \mathcal{E} \) is increased.

Next we show that \( \dim_{\mathbb{R}}(\mathcal{K}_t) = \dim_{\mathbb{R}}(\mathcal{V}_t) = \sum_{x \in \mathcal{K}_t} |k(x)| \). By contradiction, if \( \dim_{\mathbb{R}}(\mathcal{K}_t) > \sum_{x \in \mathcal{K}_t} |k(x)| \), then \( \mathcal{K}_t \cap \mathcal{V}_t^\perp \) is nontrivial, but that would contradict inequality (3.8b). Similarly, if \( \dim_{\mathbb{R}}(\mathcal{K}_t) < \sum_{x \in \mathcal{K}_t} |k(x)| \), then \( \mathcal{K}_t^\perp \cap \mathcal{V}_t \) is nontrivial, but that would contradict inequality (3.8a). This proves equation (3.4).

In order to show inequality (3.6), let \( \Pi_t \) be the orthogonal projection from \( L^2(S, \nu) \) onto \( \mathcal{V}_t \). If \( \Psi \in \mathcal{V}_t \), then, using inequalities (3.8a) and (3.8b)
\[
\frac{1}{c_3 (\tau(t))^2} \|\Psi\|^2_{L^2(S, \nu)} \geq \|H_{\nu, \rho} \Psi\|^2_{L^2(S, \nu)}
= \|H_{\nu, \rho}\left(1 - \Pi_t\right) \Psi\|^2_{L^2(S, \nu)}
\geq \|H_{\nu, \rho}\left(1 - \Pi_t\right) \Psi\|^2_{L^2(S, \nu)} - \|H_{\nu, \rho} \Pi_t \Psi\|^2_{L^2(S, \nu)}
\geq \|H_{\nu, \rho}\left(1 - \Pi_t\right) \Psi\|^2_{L^2(S, \nu)} - \frac{1}{c_3 (\tau(t))^2} \|\Pi_t \Psi\|^2_{L^2(S, \nu)}
\geq \|H_{\nu, \rho}\left(1 - \Pi_t\right) \Psi\|^2_{L^2(S, \nu)} - \frac{1}{c_3 (\tau(t))^2} \|\Psi\|^2_{L^2(S, \nu)},
\]
thus
\[ \| (1 - \Pi_t) \Psi \|_{L^2(\mathcal{S}_\omega)} \leq \frac{3}{(\varepsilon^* t(t))^2} \| \Psi \|_{L^2(\mathcal{S}_\omega)}. \] (3.11)

This implies that (using \( t > T \) and potentially increasing \( T \) once more)
\[
\| \Pi_t \Psi \|_{L^2(\mathcal{S}_\omega)} = \sqrt{\| \Psi \|_{L^2(\mathcal{S}_\omega)}^2 - \| (1 - \Pi_t) \Psi \|_{L^2(\mathcal{S}_\omega)}^2} \\
\geq \sqrt{1 - \frac{9}{(\varepsilon^* t(t))^2}} \| \Psi \|_{L^2(\mathcal{S}_\omega)} \\
\geq \left(1 - O\left(\frac{1}{t^2}\right)\right) \| \Psi \|_{L^2(\mathcal{S}_\omega)}.
\]

Thus \( \Pi_t \) induces a linear isomorphism that is almost isometric.

Now let \( \Psi \in \mathcal{K}_f^\perp \) and let pick a (unique) \( \Psi_0 \in \mathcal{K}_f \), such that \( \Pi_t \Psi = \Pi_t \Psi_0 \). Then, using inequality (3.11)
\[
\| \Pi_t \Psi \|_{L^2(\mathcal{S}_\omega)} = \langle \Psi | \Pi_t \Psi \rangle_{L^2(\mathcal{S}_\omega)} \\
= \langle \Psi | \Pi_t \Psi_0 \rangle_{L^2(\mathcal{S}_\omega)} \\
= -\langle \Psi | (1 - \Pi_t) \Psi_0 \rangle_{L^2(\mathcal{S}_\omega)} \\
\leq \| \Psi \|_{L^2(\mathcal{S}_\omega)} \| (1 - \Pi_t) \Psi_0 \|_{L^2(\mathcal{S}_\omega)} \\
\leq \| \Psi \|_{L^2(\mathcal{S}_\omega)} \frac{(\varepsilon^* t(t))^2}{9} \| \Psi_0 \|_{L^2(\mathcal{S}_\omega)} \\
\leq \| \Psi \|_{L^2(\mathcal{S}_\omega)} \frac{O(1)}{(\varepsilon^* t(t))^2} \| \Pi_t \Psi_0 \|_{L^2(\mathcal{S}_\omega)} \\
= \| \Psi \|_{L^2(\mathcal{S}_\omega)} \frac{O(1)}{(\varepsilon^* t(t))^2} \| \Pi_t \Psi \|_{L^2(\mathcal{S}_\omega)},
\]
and thus
\[
\| \Pi_t \Psi \|_{L^2(\mathcal{S}_\omega)} \leq \frac{O(1)}{(\varepsilon^* t(t))^2} \| \Psi \|_{L^2(\mathcal{S}_\omega)}.
\]

Hence
\[
\| H_{V_\omega,\delta^0} \Psi \|_{L^2(\mathcal{S}_\omega)} = \| H_{V_\omega,\delta^0} (1 - \Pi_t) + \Pi_t) \Psi \|_{L^2(\mathcal{S}_\omega)} \\
\geq \| H_{V_\omega,\delta^0} (1 - \Pi_t) \Psi \|_{L^2(\mathcal{S}_\omega)} - \| H_{V_\omega,\delta^0} \Pi_t \Psi \|_{L^2(\mathcal{S}_\omega)} \\
\geq \mathcal{E}' \tau(t) \| \Psi \|_{L^2(\mathcal{S}_\omega)} - \| \Psi \|_{L^2(\mathcal{S}_\omega)} - \| \Pi_t \Psi \|_{L^2(\mathcal{S}_\omega)} \\
\geq \mathcal{E}' \tau(t) \| \Psi \|_{L^2(\mathcal{S}_\omega)} - \| \Psi \|_{L^2(\mathcal{S}_\omega)} \\
\geq \mathcal{E}' \tau(t) \| \Psi \|_{L^2(\mathcal{S}_\omega)} - \| \Psi \|_{L^2(\mathcal{S}_\omega)},
\]
which, for some \( \varepsilon^* > 0 \), yields inequality (3.6).

Now, let us prove equation (3.7). Let \( \mathcal{V}_f^\pm \) be the (anti-)chiral part of \( \mathcal{V}_f \) and \( \Pi_t^\pm \) be the \( L^2 \)-orthogonal projections from \( \mathcal{H}_{V_\omega,\delta^0}^\pm \) onto \( \mathcal{V}_f^\pm \). The operators
\[
\tilde{H}^\pm := (1 - \Pi_t^\pm) \circ H_{V_\omega,\delta^0}^\pm \circ (1 - \Pi_t^\pm) : \mathcal{H}_{V_\omega,\delta^0}^\pm \to L^2(\mathcal{S}_\omega).
\]

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are compact (in fact, finite rank) perturbations of $H^\pm_{\nabla, \rho\Phi}$, thus they are also Fredholm and have the same index. By construction, $\ker(\tilde{H}^\pm) = \mathcal{V}^\pm_t$, and hence
\[
\dim_{\mathbb{R}}(\ker(\tilde{H}^\pm)) = \dim_{\mathbb{R}}(\mathcal{V}^\pm_t) = \sum_{x \in \mathbb{Z}_\Phi} \pm k(x),
\]
and similarly, it is easy to see that, $\coker(\tilde{H}^\pm) = \mathcal{V}^{\mp}_t$, and hence
\[
\dim_{\mathbb{R}}(\coker(\tilde{H}^\pm)) = \dim_{\mathbb{R}}(\mathcal{V}^{\mp}_t) = \sum_{x \in \mathbb{Z}_\Phi} \mp k(x).
\]
Hence the index
\[
\text{index}_{\mathbb{R}}(H^\pm_{\nabla, \rho\Phi}) = \text{index}_{\mathbb{R}}(\tilde{H}) = \sum_{x \in \mathbb{Z}_\Phi} \pm k(x) - \sum_{x \in \mathbb{Z}_\Phi} \mp k(x) = \pm \sum_{x \in \mathbb{Z}_\Phi} k(x),
\]
which concludes the proof of equation (3.7).

\[\square\]

4. Applications

4.1. The Riemann–Roch Theorem. In this section we use Theorem 3.4 to give a short proof of the Riemann–Roch Theorem. Note that this is not entirely novel; cf. [21, Section 7].

**Theorem 4.1** (Riemann–Roch Theorem). Let $\Sigma$ be a compact Riemann surface and $\mathcal{L} \to \Sigma$ be a holomorphic line bundle of degree $d \in \mathbb{Z}$. Then
\[
\dim_{\mathbb{C}}(H^0(\mathcal{L})) - \dim_{\mathbb{C}}(H^1(\mathcal{L})) = c_1(\mathcal{L})[\Sigma] + 1 - \text{genus}(\Sigma).
\]

**Proof.** Let $\mathcal{K}$ be the canonical bundle of $\Sigma$ and $\tilde{\delta} : \Gamma(\mathcal{L}) \to \Gamma(\mathcal{L} \mathcal{K})$ be the Cauchy–Riemann operator of $\mathcal{L}$. Then
\[
H^0(\mathcal{L}) = \ker(\tilde{\delta}), \quad \text{and} \quad H^1(\mathcal{L}) = \coker(\tilde{\delta}),
\]
and thus
\[
\dim_{\mathbb{C}}(H^0(\mathcal{L})) - \dim_{\mathbb{C}}(H^1(\mathcal{L})) = \text{index}_{\mathbb{C}}(\tilde{\delta}).
\]
Next, note that $c_1(\mathcal{L})[\Sigma] + 1 - \text{genus}(\Sigma) = \frac{1}{2} c_1(\mathcal{L}^2 \mathcal{K})[\Sigma]$.

Pick a $\Phi \in \Gamma(\mathcal{L}^2 \mathcal{K})$ so that it has exactly $|c_1(\mathcal{L}^2 \mathcal{K})[\Sigma]|$ zeros, and around each zero it is holomorphic (resp. anti-holomorphic), if $c_1(\mathcal{L}^2 \mathcal{K})[\Sigma] > 0$ (resp. $c_1(\mathcal{L}^2 \mathcal{K})[\Sigma] < 0$). Such $\Phi$ always exists. Equip $\mathcal{L}$ and $\mathcal{K}$ (and thus $\mathcal{L}^2 \mathcal{K}$) with a compatible Hermitian structures and consider the Clifford module bundle
\[
\mathcal{E} := \mathcal{L} \oplus \mathcal{L} \mathcal{K}.
\]
The induced Dirac operator is
\[ D = \begin{pmatrix} 0 & \bar{\delta}^* \\ \delta & 0 \end{pmatrix}. \]
The canonical bundle of \( E \) is \( F = (L^2 K)^* \), thus \( \Phi \in F^* \) induces an odd, conjugate linear bundle map, \( A_\Phi \), and the operator
\[ H := D + A_\Phi = \begin{pmatrix} 0 & \bar{\delta}^* + A_\Phi^- \\ \delta + A_\Phi^+ & 0 \end{pmatrix}, \]
is a generalized Jackiw–Rossi operator. Since \( \Sigma \) is closed, \( D \) if a compact perturbation of \( H \).

Thus, using equation (3.7), we get
\[ \text{index}_{\text{CA}}(\partial) = \frac{1}{2} \text{index}_{\text{CA}}(\partial + A_\Phi^+) - \frac{1}{2} \dim_{\text{CA}}(\ker(\partial + A_\Phi^+)) \]
\[ = \frac{1}{2} \dim_{\text{CA}}(\ker(\partial + A_\Phi^+)) - \frac{1}{2} \dim_{\text{CA}}(\ker(\partial + A_\Phi^+)) \]
\[ = \frac{1}{2} c_1(L^2 K)[\Sigma], \]
which concludes the proof. \( \square \)

4.2. Solutions on spacetimes. In this section we use Theorem 3.4, to construct solutions in higher dimensions.

Let \( \Sigma \) be a oriented, complete Riemannian surface, and let \( S_L \) be a spinor bundle corresponding to a spin\( ^c \) structure of \( X \) with associated line bundle \( L^2 \). Let \( \nabla \) be a compatible connection on \( S_L \) and \( \Phi \) be a smooth section of \( F^+ = L^{-2} \), both satisfying the conditions A1 through A5 in Section 3. Thus we get a generalized Jackiw–Rossi operator, \( H_{\nabla, \Phi} \).

Let \( Y \) be a smooth, oriented, Lorentzian surface, that is a pseudo-Riemannian manifold of signature (1, 1). Assume that \( E_Y \rightarrow Y \) is a Hermitian Clifford module bundle with compatible connection \( \nabla^Y \), and a flat, compatible, real structure, call \( A_Y \). Let \( \ker(D_Y) \) be the vector space of smooth solutions of the Dirac equation on \( E_Y \). We impose no integrability requirements this time. Note that \( A_Y \) induces a real structures on \( \ker(D_Y^+) \) and if \( \Psi^Y, \pm \in \ker(D_Y^\pm) \), then \( \frac{1}{2}(\Psi^Y, \pm + A_Y^\pm \Psi^Y, \pm) \in \ker(D_Y^\pm) \) is a real vector with respect to \( A_Y^\pm \).

Remark 4.2. Examples of \( E_Y \rightarrow Y \) include the spinor bundle of \( Y \) when it exists. Furthermore, the spinor bundle can be twisted by flat, real bundles.

Let \( X = Y \times \Sigma \), which is a Lorentzian 4-manifold, that is a spacetime. Following [1, §6], let
\[ E_X := E^+_X \oplus E^-_X, \quad \text{with} \quad E^\pm_X := (\pi^+_Y(E^+_Y) \otimes \pi^+_\Sigma(E^+_\Sigma)) \oplus (\pi^-_Y(E^-_Y) \otimes \pi^-_\Sigma(E^-_\Sigma)). \]
Now $E_X$ is a Clifford module bundle over $X$ (via [1, Equation (6.1)]), equipped with a compatible connection $\nabla^X$ induced by $\nabla$ and $\nabla^Y$, and a conjugate linear map $A_X := A_Y \otimes A_\Phi$. It is easy to see that $A_X$ has the form $A_{\Phi X}$ from equation (1.2) with $\Phi^X$ being the pullback of $\Phi$ to $X$. In particular, the zero locus of $\Phi^X$ is $Y \times \mathcal{Z}_\Phi$, which is a collection of embedded Lorentzian minimal surfaces, that is worldsheets.

The above data gives us a generalized Jackiw–Rossi operator on $E_X \rightarrow X$:

$$H_{\nabla^X, \Phi} = D_{\nabla^X} + A_{\Phi X}.$$  

We are now ready to state the theorem of this section.

**Theorem 4.3.** Let

$$\Psi^Y := \begin{pmatrix} \Psi_{Y, +}^Y \cr \Psi_{Y, -}^Y \end{pmatrix} \in \ker \left( D_{\nabla^Y}^+ \right) \oplus \ker \left( D_{\nabla^Y}^- \right),$$

and assume that $A_Y \Psi^Y = \Psi^Y$. Then for each $\lambda \in \text{Spec}(H_{\nabla^Y, \Phi})$ and for each

$$\Psi^\Sigma := \begin{pmatrix} \Psi_{\Sigma, +} \cr \Psi_{\Sigma, -} \end{pmatrix} \in \ker \left( H_{\nabla^Y, \Phi} - \lambda \mathbb{1} \right),$$

the spinor

$$\Psi^X := \begin{pmatrix} \Psi_{Y, +}^Y \otimes \Psi_{\Sigma, +} + \Psi_{Y, -}^Y \otimes \Psi_{\Sigma, -} \\ \Psi_{Y, +}^Y \otimes \Psi_{\Sigma, -} + \Psi_{Y, -}^Y \otimes \Psi_{\Sigma, +} \end{pmatrix},$$

is in the kernel of $H_{\nabla^X, \Phi} - \lambda \mathbb{1}$. In particular, when $\Phi$ is replaced by $t\Phi$, then these spinors concentrate around $Y \oplus \mathcal{Z}_\Phi$ as $t \rightarrow \infty$, according to Theorem 3.4.

**Remark 4.4.** Similar constructions can be performed when $Y$ is a pseudo-Riemannian manifold of any type $(t, s)$, equipped with a Hermitian Clifford module bundle with compatible connection, and a flat, compatible, real structure. We focused on the case of a Lorentzian manifold for the obvious mathematical physical importance.

4.3. **The Majorana bundle over the vortex moduli space.** Once a Hermitian Clifford module bundle over a (pseudo-)Riemannian manifold is fixed, the only remaining parameters of generalized Jackiw–Rossi operators are a compatible connection $\nabla$ is a section $\Phi$. In certain situations there are (mathematically and physically) relevant such families that form a smooth moduli.

One such example is Ginzburg–Landau theory, more concretely, $\tau$-vortices. Let us briefly summarize the theory of $\tau$-vortices in dimension two. Let $(\Sigma, g)$ be a closed, oriented Riemannian surface and $V \rightarrow \Sigma$ be a Hermitian line bundle of positive degree $d$. Let $\omega$ be the
Kähler/volume form and $\Lambda : \Lambda^2 \to \Lambda^0$ be the contraction by $\omega$. For each smooth unitary connection $\nabla$ and smooth section $\Phi$ on $\mathcal{V}$ the (critically coupled) Ginzburg–Landau energy is
\[
E_\tau(\nabla, \Phi) = \int_\Sigma \left( |\nabla \tau|^2 + |\nabla \Phi|^2 + \frac{1}{4}(\tau - |\Phi|^2)^2 \right) \text{vol}. \tag{4.1}
\]
One can perform a “Bogomolny trick” and rewrite the energy (4.1) as
\[
E_\tau(\nabla, \Phi) = \int_\Sigma \left( |i\Lambda_\nabla - \frac{1}{2}(\tau - |\Phi|^2)|^2 + 2|\overline{\nabla}_\nabla \Phi|^2 \right) \text{vol} + 2\pi \tau d.
\]
Thus, absolute minimizers of the energy (4.1) are characterized by
\[
i\Lambda_\nabla = \frac{1}{2}(\tau - |\Phi|^2), \tag{4.2a}
\]
\[
\overline{\nabla}_\nabla \Phi = 0. \tag{4.2b}
\]
We call equations (4.2a) and (4.2b) the $\tau$-vortex equations and solutions $(\nabla, \Phi)$ $\tau$-vortex fields.

Let $\tau_0 := \frac{4\pi d}{\text{Area}(\Sigma)}$. In [2, Theorem 4.6] Bradlow showed that when $\tau < \tau_0$, then there are no $\tau$-vortex fields, when $\tau = \tau_0$, then $(\nabla, \Phi)$ is a $\tau$-vortex field exactly when $\nabla$ is a $U(1)$ Yang–Mills field and $\Phi = 0$, and when $\tau > \tau_0$ then all $\tau$-vortex fields are irreducible, that is $\Phi \neq 0$, and the moduli space of gauge equivalence classes of $\tau$-vortex fields is canonically isomorphic to $\text{Sym}^d(\Sigma)$. Note that $\Phi$ is holomorphic, and let $\mathcal{D}_\Phi \in \text{Sym}^d(\Sigma)$ be its (effective) divisor. The canonical isomorphism is then given by $[\nabla, \Phi] \mapsto \mathcal{D}_\Phi$. Here gauge equivalence refers to the action of $\gamma \in \mathcal{G} := L^2_2(\Sigma; U(1))$ via
\[
\gamma(\nabla, \Phi) = (\gamma \circ \nabla \circ \gamma^{-1}, \gamma \Phi).
\]
Using the metric $g$ and a reference point $x_0 \in \Sigma$, we can canonically factor $\mathcal{G}$ as follows: Let us identify $H^1(\Sigma; \mathbb{Z})$ with the group of harmonic functions from $\Sigma$ to $U(1)$ that are one at $x_0$ and $L^2_{2, x_0}(\Sigma; \mathbb{R})$ be the space of $L^2_1$ function that vanish at $x_0$. Then the following map is an isomorphism
\[
U(1) \oplus H^1(\Sigma; \mathbb{Z}) \oplus L^2_{2, x_0}(\Sigma; i\mathbb{R}) \to \mathcal{G}; \quad (\lambda, \gamma_H, if) \mapsto \lambda \gamma_H \exp(if).
\]
Let us fix $\tau > \tau_0$ and let $\mathcal{N}$ be space of all $\tau$-vortex fields determined by these data, that is
\[
\mathcal{N} := \{ (\nabla, \Phi) \mid (\nabla, \Phi) \text{ solves equations (4.2a) and (4.2b)} \}.
\]
Then $\mathcal{N}$ is a Hilbert manifold modeled on the $L^2_1$-completion of $i\Omega^1 \oplus \Gamma(\mathcal{L}^2_2)$, and the factor $\mathcal{N}/L^2_{2, x_0}(\Sigma; i\mathbb{R})$ is a finite dimensional manifold of (real) dimension $2d + 1$. The actions of $U(1)$ and $H^1(\Sigma; \mathbb{Z})$ are also easy to understand: Note that $H^1(\Sigma; \mathbb{Z}) \cong \pi_1(\text{Sym}^d(\Sigma))$ and thus $\mathcal{N}/\left( H^1(\Sigma; \mathbb{Z}) \oplus L^2_{2, x_0}(\Sigma; i\mathbb{R}) \right)$ is a principal $U(1)$ bundle over $\text{Sym}^d(\Sigma)$. 
Let now $d = 2k > 0$ be positive and even, write $\mathcal{V} = \mathcal{L}^{-2}$ (thus the degree of $\mathcal{L}$ is $-k < 0$), and let $\mathcal{S}_\mathcal{V}$ be as in Section 1. We can then associate a Jackiw–Rossi operator to each $\tau$-vortex field, $(\nabla, \Phi)$. Let us prove a technical lemma.

**Lemma 4.5.** For any $\nabla$ and $\Phi$ we have

$$\text{cl}^+(\mathcal{A}_{\nabla}\Phi) = \text{cl}^+(\mathcal{A}_{\nabla}^{0,\Phi}),$$

$$\text{cl}^-(\mathcal{A}_{\nabla}\Phi) = \text{cl}^-(\mathcal{A}_{\nabla}^{0,\Phi}).$$

Thus if $\mathcal{S}_\nabla\Phi = 0$, then for all $\Psi^+ \in \mathcal{H}^+_{\nabla,\Phi}$

$$\|H^+_{\Psi,\Phi}\|^2_{L^2(\mathcal{S}_\mathcal{V})} = \|\mathcal{D}^+\Psi^+\|^2_{L^2(\mathcal{S}_\mathcal{V})} + \|\mathcal{A}_{\nabla}^+\Psi^+\|^2_{L^2(\mathcal{S}_\mathcal{V})},$$

and, in particular, if $\Phi$ is not identically zero, then $\ker(H^+_{\nabla,\Phi})$ is trivial and $\ker(H^-_{\nabla,\Phi})$ has real dimension $d$.

**Proof.** This is a local computation. 

Now the assignment $(\nabla, \Phi) \mapsto \ker(H^-_{\nabla,\Phi})$ defines a smooth, real, rank $k$ vector bundle over $\mathcal{N}$, which we call $\tilde{\mathcal{H}}$. In order to descend this bundle to the vortex moduli space, we have to understand how gauge transformations act on $\tilde{\mathcal{H}}$.

Assume that $\gamma = \mu^2$ for some $\mu \in \mathcal{G}$. Then $\Psi \in \ker(H^-_{\nabla,\Phi})$ exactly if $\mu\Psi \in \ker(H^-_{\nabla,\Phi})$. Thus, to descend the kernel bundle $\tilde{\mathcal{H}}$ to the vortex modulo space, one needs to take square roots of gauge transformations, but that cannot always be done for two independent reasons:

1. Let us define a loop in $U(1) \subseteq \mathcal{G}$ via $t \mapsto \gamma_t := \exp(2it\pi)$ as $t$ runs from zero to $2\pi$. Then $t \mapsto \mu_t := \exp(it\pi)$ is also a loop in $\mathcal{G}$ and satisfies $\mu_t^2 = \gamma_t$, but $\mu_{2\pi} = -1 \neq 1$. Thus nonzero elements of $\tilde{\mathcal{H}}$ would be identified with their opposites. This issue can be remedied if we only attempt to descend the projectivization of $\tilde{\mathcal{H}}$.

2. If $\gamma_{11}$ is a harmonic, $U(1)$-valued function that represents an integer cohomology class that is not the double of another class, then $\gamma_{11}$ is a gauge transformation without a square root.

As $L^2_{2,0}(\Sigma; i\mathbb{R})$ is a vector space, taking square roots of gauge transformations coming from $L^2_{2,0}(\Sigma; i\mathbb{R})$ can always be done.

Note that $2H^1(\Sigma; \mathbb{Z}) = (\Sigma; 2\mathbb{Z})$, and let $\mathcal{G}' := \mathcal{G}^2 \cong U(1) \oplus H^1(\Sigma; 2\mathbb{Z}) \oplus L^2_{2,0}(\Sigma; i\mathbb{R})$. The above arguments prove the following:

**Theorem 4.6.** Let $\tilde{\mathcal{M}} := \mathcal{N}/\left[H^1(\Sigma; 2\mathbb{Z}) \oplus L^2_{2,0}(\Sigma; i\mathbb{R})\right]$. Then $\tilde{\mathcal{M}}$ is a smooth, closed manifold of (real) dimension $4k+1$ and the vector bundle $\tilde{\mathcal{H}}$ descends smoothly to $\tilde{\mathcal{M}}$, call $\mathcal{H}'$. Furthermore, $U(1)$ has a smooth, free action on $\tilde{\mathcal{M}}$, which locally ascends to $\mathcal{H}'$, with monodromy $-1$. 

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Let $\mathcal{M} := \mathcal{N}/\mathcal{G} \cong \tilde{\mathcal{M}}/U(1)$. Then $\mathcal{M}$ is a principal $H^1(\Sigma;\mathbb{Z}_2) \cong \mathbb{Z}_2^{h_1(\Sigma)}$ bundle over $\text{Sym}^{2k}(\Sigma)$, corresponding to the $H^1(\Sigma;2\mathbb{Z}) \subset \pi_1(\text{Sym}^{2k}(\Sigma))$. Furthermore, the projectivization of $\mathcal{K}$, call $\mathbb{P}(\mathcal{K})$, descends to $\mathcal{M}$.

As a concluding thought, we look at the simplest example, that is when $\Sigma$ is the $2$-sphere. Let $\Sigma = S^2$ with the round metric or radius one, let $\mathcal{L} \cong \mathcal{O}(-1)$. Then $H^1(\Sigma;\mathbb{Z})$ is trivial and for any positive integer $d$, we have $\text{Sym}^{d}(S^2) \cong \mathbb{C}\mathbb{P}^n$. The Coulomb bundle over $\text{Sym}^{d}(S^2)$ is the Hopf fibration $S^{2d+1} \to \mathbb{C}\mathbb{P}^d$. Thus, as $b_1(S^2) = 0$, if $d = 2k$, then $\mathcal{M} = S^{4k+1}$ and $\mathcal{M} := \text{Sym}^{2k}(S^2) \cong \mathbb{C}\mathbb{P}^{2k}$. Furthermore, $\mathcal{K}$ is a rank $2k$ real vector bundle over $S^{4k+1}$. Such bundles are necessarily orientable and thus are classified by homotopy types of maps from $S^{4k}$ to $\text{SO}(2k)$, that is, by $\pi_{4k}(\text{SO}(2k))$. When $k = 1$, this group is $\pi_4(\text{SO}(2)) = \pi_4(S^1)$ which is trivial. Thus in this case $\mathcal{K}$ is a trivial bundle. When $k = 2$, then $\pi_8(\text{SO}(4)) \cong \mathbb{Z}_2^2$ is the Klein group, thus we cannot conclude the triviality of $\mathcal{K}$.

## Appendix A. Bilinear Pairings and Conjugate Linear Maps

### A.1. Pairings of spinor modules.

Let $\mathcal{S} = \mathcal{S}_{t,s}$ be the spinor representation of the $d := (t + s)$-dimensional (real) Clifford algebra, $\text{CL}_{t,s} = \text{CL}_{t,s}^0 \oplus \text{CL}_{t,s}^1$. Here $t$ is the number of timelike (negative definite) and $s$ is the number of spacelike (positive definite) directions, and let $r := t - s$. Denote the Hermitian structure and the Clifford multiplication on $\mathcal{S}$, by $h$ and $\mathcal{C}$, respectively. In even dimensions the spinor representation is graded, $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$. The grading is also called the parity, and the grading operator is denoted by $\mathcal{P}$. An operator acting on spinors is even, if it preserves parity (commutes with $\mathcal{P}$), and odd, if it interchanges it (anti-commutes with $\mathcal{P}$).

In every dimension there are (essentially unique) nondegenerate, $\text{CL}_{t,s}^0$-invariant, metric compatible, bilinear forms on $\mathcal{S}$, which we call $\mathcal{B}$. A concise way to think about these pairing is given via the (essentially canonical) isomorphisms of representations:

$$\mathcal{S} \otimes \mathcal{S} \cong \begin{cases} \wedge^*_{\mathbb{C}}(\mathbb{R}^{t,s}), & \text{if } t + s \text{ is even,} \\ \wedge^{t,s}_{\mathbb{C}}(\mathbb{R}^{t,s}), & \text{if } t + s \text{ is odd.} \end{cases}$$

Denote the metric dual of $\mathcal{B}$ by $\mathcal{A}$, which is then an anti-unitary map of $\mathcal{S}$. The $\mathcal{B}$ can be recovered from $\mathcal{A}$ via

$$\mathcal{B}(\Psi_1, \Psi_2) = h(\mathcal{A}\Psi_1, \Psi_2).$$

The choice of $\mathcal{A}$ is always ambiguous up to the action of $U(1)$ via $\mathcal{A} \rightarrow \lambda \mathcal{A}$. Note that $(\lambda \mathcal{A})^2 = \mathcal{A}^2$. In odd dimensions these maps (and thus the forms) are unique, up to the action of $U(1)$, but in even dimensions there is another sign ambiguity, which we explain below. Let $\equiv_n$ be the modulo $n$ equivalence relation of integers and let us denote the restrictions of $\mathcal{A}$ to $\mathcal{S}^\pm$ by $\mathcal{A}^\pm$, in even dimensions. Following [22, Theorem 6.5.7], we have the following:
Lemma A.1. The above bilinear form can be extended to forms as follows:

- If \( t + s \) is odd, then the map
  \[
  \tilde{B} : S \otimes S \to \bigwedge_{\mathbb{C}}^{\text{even}}(\mathbb{R}^{t,s});
  \]
  \[
  \Psi_1 \otimes \Psi_2 \mapsto \sum_{0 \leq ||I|| \leq d} B(\chi(I)) \Psi_1, \Psi_2) \, dx^I,
  \]
  where \( \Psi_1, \Psi_2 \) are elements of \( \bigwedge_{\mathbb{C}}^* (\mathbb{R}^{t,s}) \), is an isomorphism. Analogous isomorphism exists for odd forms.

- If \( t + s \) is even, then the map
  \[
  \tilde{B} : S \otimes S \to \bigwedge_{\mathbb{C}}^*(\mathbb{R}^{t,s});
  \]
  \[
  \Psi_1 \otimes \Psi_2 \mapsto \sum_{0 \leq ||I|| \leq d} B(\chi(I)) \Psi_1, \Psi_2) \, dx^I,
  \]
  is an isomorphism.

Furthermore, if \( \pi_0 : \bigwedge_{\mathbb{C}}^{\text{even}}(\mathbb{R}^{t,s}) \to \mathbb{C} \) is the orthogonal projection onto \( \bigwedge_{\mathbb{C}}^{0}(\mathbb{R}^{t,s}) \cong \mathbb{C} \), then \( B = \pi_0 \circ \tilde{B} \).

Proof. The above maps are linear injections between finite dimensional vector spaces, so one can prove that they are isomorphisms by simple dimension counts. \( \square \)

Remark A.2. Let \( \text{vol} \) be the volume form of \( \mathbb{R}^{t,s} \), that is a unit norm element of \( \bigwedge_{\mathbb{C}}^{t+s}(\mathbb{R}^{t,s}) \). Then \( A' = \chi(\text{vol}) \circ A \) is also a \( \text{CL}_{t,s} \)-invariant conjugate linear map. In odd dimensions \( \chi(\text{vol}) \) is proportional to the identity, but in even dimensions \( \chi(\text{vol}) \) is proportional to the parity operator, and if \( A = \chi(\cdot) \circ A \), then \( A' = -\sigma \chi(\cdot) \circ A' \).
A.2. More general Clifford modules. Let now $E$ be any Hermitian $\text{CL}_{t,s}$ module. There always is a Hermitian vector space $E$, such that $E \cong S \otimes E$; cf. [16, Corollary 11.1.22.], in fact $E \cong \text{Hom}_{\text{CL}_d}(S, E)$. In particular, if $d$ is even, then $E$ is also $\mathbb{Z}_2$-graded, which we still denote as $E \cong E^+ \oplus E^-$ with $E^\pm \cong S^\pm \otimes E$.

Let $F := E \otimes E$. Then we have the following generalization of the pairing from Appendix A:

$$\mathcal{B}_E : E \otimes E \to F; (\psi_1 \otimes e_1) \otimes (\psi_2 \otimes e_2) \mapsto \mathcal{B}(\psi_1, \psi_2)e_1 \otimes e_2,$$

and more generally we define maps $\widehat{\mathcal{B}}_E$, with values in $\bigwedge^*_{\text{CL}_d}(\mathbb{R}^{t,s}) \otimes F$, similarly to Lemma A.1. Moreover, we have the following simple lemma.

**Lemma A.3.** For any complex bilinear form, $B$, on $E$, there exists a $\Phi$ in $\left(\bigwedge^*_{\text{CL}_d}(\mathbb{R}^{t,s}) \otimes F\right)^*$, such that

$$B = \Phi \circ \widehat{\mathcal{B}}_E.$$ 

If $r = t - s$ is even, then $\Phi$ is unique. If $r = t - s$ is odd, then $\Phi$ is unique among even forms (or odd forms for the analogous isomorphism).

Furthermore, if $\Phi$ is in $\left(\bigwedge^k_{\text{CL}_d}(\mathbb{R}^{t,s}) \otimes F\right)^*$, (where $k$ is even, if $t + s$ is odd), then

$$B(\Psi_1, \Psi_2) = (-1)^{\frac{k(k+1)}{2} - s} \sigma_k \Phi B(\Psi_2, \Psi_1).$$

**Proof.** The proof involves straightforward computations and we only present the key steps here.

In all cases, the correspondence

$$\Phi \mapsto \Phi \circ \widehat{\mathcal{B}}_E,$$

is a linear injection between finite dimensional vector spaces, so one can prove that they are isomorphisms by simple dimension counts. This proves equation (A.1) and the uniqueness of $\Phi$.

Finally, proving equation (A.2) is then straightforward, using equation (1.7) and that the values of $\text{cl}$ on 1-forms are skew-adjoint with respect to the Hermitian structure of $E$. 

Now, given an element $\Phi \in \left(\bigwedge^*_{\text{CL}_d}(\mathbb{R}^{t,s}) \otimes F\right)^*$, we can now define a conjugate linear map $\mathcal{A}_\Phi : E \to E$ via

$$h^E(\mathcal{A}_\Phi \Psi_1, \Psi_2) = \Phi(\widehat{\mathcal{B}}_E(\Psi_1, \Psi_2)).$$

Now let us consider the splitting $E \otimes E \cong \bigwedge^2 E \oplus \bigwedge^2 E$. These subspaces are the $\pm 1$ eigenspaces of the involution $\Psi_1 \otimes \Psi_2 \mapsto \Psi_2 \otimes \Psi_1$. Similarly, we get a splitting $F = F^+ \oplus F^-$. Assume now that $\Phi^\pm \in F^\pm$. Then simple computation shows that

$$h^E(\mathcal{A}_\Phi^\pm \Psi_1, \Psi_2) = \pm s \sigma_r h^E(\mathcal{A}_\Phi^\pm \Psi_2, \Psi_1).$$
A metric compatible real or a quaternionic structure on $E$ can be regarded as an element of $\mathcal{F}^+$ or $\mathcal{F}^-$, respectively.

### Appendix B. The Bogoliubov–de Gennes Equation

In this section we show how the above theory also leads to a geometric framework (and generalization) to the Bogoliubov–de Gennes equations; cf [4, Equation (55)].

For simplicity, let $X$ be a closed, oriented, Riemannian manifold. The generalized Jackiw–Rossi operator in equation (1.3) has two uncommon properties: it is not complex linear and it may not be (real) self-adjoint. There is however a canonical way to construct a (in fact, two) complex linear and (complex) self-adjoint operator(s) out of this data.

Let $p \in \{-1, 1\}$. Using the notation of Definition 1.4, let $\hat{E} := E \oplus \overline{E}$. Since $\mathcal{A}_\Phi$ is a complex linear map from $E$ to $\overline{E}$, we have that the operator

$$\hat{H}_{V, \Phi, p} := \begin{pmatrix} D_V & \mathcal{A}_\Phi \\ \mathcal{A}_\Phi^* & pD_V \end{pmatrix} = \begin{pmatrix} D_V & \mathcal{A}_\Phi \\ \mathcal{A}_\Phi^* & pD_V \end{pmatrix},$$

is a complex linear map from $L^2_1(\hat{E})$ to $L^2(\hat{E})$ and self-adjoint as a densely defined operator on $L^2(\hat{E})$. Motivated by [4, Equations (9) and (11)], we call $\hat{H}_{V, \Phi}$ the Bogoliubov–de Gennes operator.

For each $m \in \mathbb{R}$, the equation

$$\hat{H}_{V, \Phi, p} \begin{pmatrix} \Psi_1 \\ \overline{\Psi_2} \end{pmatrix} = m \begin{pmatrix} \overline{\Psi_1} \\ \Psi_2 \end{pmatrix},$$

is the Euler–Lagrange equation of the energy density

$$E_{V, \Phi}(\Psi_1, \Psi_2) = h(\Psi_1, D_V \Psi_1) + ph(\Psi_2, D_V \Psi_2) + 2\text{Re}(\Phi(\mathcal{B}(\Psi_1 \otimes \Psi_2))) - m(|\Psi_1|^2 + |\Psi_2|^2).$$

Let us define a unitary operator on $\hat{E}$ as

$$\mathcal{C} \begin{pmatrix} \Psi_1 \\ \overline{\Psi_2} \end{pmatrix} := \begin{pmatrix} \Psi_2 \\ \overline{\Psi_1} \end{pmatrix}.$$ 

Then we have

$$\hat{H}_{V, \Phi, p} \circ \mathcal{C} = p\mathcal{C} \circ \hat{H}_{V, \Phi, p}, \quad \mathcal{C}^2 = \mathcal{S}_\Phi \mathcal{P}_{\hat{E}}.$$  \hfill (B.1)

By equation (B.1), $\mathcal{C}$ preserves the kernel of $\hat{H}_{V, \Phi, p}$ and by equation (B.2), $\mathcal{C}$ is either a real ($\mathcal{S}_\Phi \mathcal{P} = 1$) or a quaternionic ($\mathcal{S}_\Phi \mathcal{P} = -1$) structure on $\ker(\hat{H}_{V, \Phi, p})$.  

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If $p = 1$, then equation (B.1) can be interpreted as $\hat{H}_{V,\Phi,p}$ having a *time-reversal* symmetry, which is represented by $\hat{C}$, and, by equation (B.2), this symmetry is bosonic if $s_\Phi = 1$ and fermionic if $s_\Phi = -1$.

If $p = -1$, then equation (B.1) can be interpreted as $\hat{H}_{V,\Phi,p}$ having a *particle-hole* symmetry, which is represented by $\hat{C}$, and, by equation (B.2), this symmetry is bosonic if $s_\Phi = -1$ and fermionic if $s_\Phi = 1$.

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