ON THE NON-VANISHING OF DIRICHLET $L$-FUNCTIONS AT THE CENTRAL POINT

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Abstract. We investigate the consequences of natural conjectures of Montgomery type on the non-vanishing of Dirichlet $L$-functions at the central point. We first justify these conjectures using probabilistic arguments. We then show using a result of Bombieri, Friedlander and Iwaniec and a result of the author that they imply that almost all Dirichlet $L$-functions do not vanish at the central point. We also deduce a quantitative upper bound for the proportion of Dirichlet $L$-functions for which $L(1/2, \chi) = 0$.

1. Introduction and statement of results

The central values of $L$-functions and their derivatives are of crucial importance in number theory. Perhaps the most important example are the values $L(k)(E, 1)$ for an elliptic curve $E$, which are strongly linked with important invariants of $E$. For $k = 1$ this is the Gross-Zagier Formula, and for $k \leq r(E)$ (the rank of $E$), this is the Birch and Swinnerton-Dyer Conjecture.

It is widely believed that the vanishing of $L$-functions at the central point should be explained by arithmetical reasons. The Birch and Swinnerton-Dyer Conjecture is such a reason, and another type of reason is the value of the root number. Indeed, self-dual $L$-functions whose root number is $-1$ must vanish to odd order at the central point. As for Dirichlet $L$-functions, it is believed that we always have $L(1/2, \chi) \neq 0$; this was first conjectured by Chowla [Ch] for real primitive characters $\chi$. A good reason to believe this conjecture is that the root number of self-dual Dirichlet $L$-functions, that is $L(s, \chi)$ with $\chi$ real and primitive, can never equal $-1$.

While Chowla’s Conjecture is still open, there has been substantial progress towards this question. A famous result of Soundararajan [So] states that the proportion of Dirichlet $L$-functions $L(s, \chi_{8d})$ with $d$ odd and squarefree which do not vanish at $s = 1/2$ is at least $7/8$; this result was extended by Conrey and Soundararajan [CS] to show that at least 20% of these $L$-functions do not vanish on the whole interval $s \in [0, 1]$.

As for general Dirichlet characters $\chi \mod q$, Balasubramanian and K. Murty [BM], improving on [Ba], have shown that at least 4% of the Dirichlet $L$-functions with $\chi \mod q$ do not vanish at $s = 1/2$. This proportion was subsequently improved to $1/3$ by Iwaniec and Sarnak [IS1], and more recently to 34.11% by Bui [Bu]. Under GRH, Murty [M] (see also

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1 This can be deduced by an exact Gauss sum computation (see Chapters 2 and 9 of [Da])

2 These authors have also shown [IS2] that in certain families of newforms of either varying weight or level, at least $\frac{1}{3}$ of the members satisfy $L(1/2, f \otimes \chi_D) > 0$, for any fixed $D$. Iwaniec and Sarnak further proved that any improvement of the constant $\frac{1}{3}$ would imply a significant bound on Landau-Siegel zeros.
has shown that this proportion is at least $50\%$\(^3\). Sarnak\(^4\) noticed that Montgomery’s Conjecture on primes in arithmetic progressions implies the Katz-Sarnak prediction for the 1-level density for any finite support, and as a consequence almost all Dirichlet $L$-functions do not vanish at the central point. However as we will see below, Montgomery’s Conjecture heavily depends on the assumption that $L(\frac{1}{2}, \chi) \neq 0$. The goal of the current paper is to formulate an analogue of Montgomery’s Conjecture which is independent of real zeros. From this we will deduce that $L(\frac{1}{2}, \chi) \neq 0$ for almost all $\chi$ mod $q$, with $Q < q \leq 2Q$.

We should also mention that corresponding questions for the derivatives $L^{(k)}(\frac{1}{2}, \chi)$ have been studied. Bui and Milinovich have shown that asymptotically for $q$ and $k$ tending to infinity, $L^{(k)}(\frac{1}{2}, \chi) \neq 0$ for almost all $\chi$ mod $q$. A corresponding result for completed Dirichlet $L$-functions $\Lambda(s, \chi)$ had earlier been obtained by Michel and VanderKam \cite{MV}, but with limiting proportion $\frac{2}{3}$.

The unconditional results mentioned earlier rely heavily on mollification methods, who have greatly flourished in the past years. The goal of the current paper is to take a different viewpoint to the vanishing of $L(s, \chi)$ at the central point, by inputting probabilistic arguments.

Bombieri, Friedlander and Iwaniec have shown \cite{BFI} that in the range $Q_0 < Q \leq 2Q_0$, with $Q_0 = x^{\frac{1}{2}}(\log x)^A$ and $a \neq 0$ a fixed integer,

$$\sum_{Q < q \leq 2Q, (q,a) = 1} \left| \psi(x; q, a) - \frac{\psi(x; \chi_0)}{\phi(q)} \right| \ll x \left( \frac{\log \log x}{\log x} \right)^2. \tag{1}$$

As a consequence, taking $a = 1$ and using the orthogonality relations we obtain the bound

$$\sum_{Q < q \leq 2Q} \left| \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \sum_{\rho_\chi} x^{\rho_\chi} \right| \ll x \left( \frac{\log \log x}{\log x} \right)^2, \tag{2}$$

where $\rho_\chi$ runs through the nontrivial zeros of $L(s, \chi)$. If $\rho_\chi \notin \mathbb{R}$, then the term $x^{\rho_\chi}/\rho_\chi$ oscillates; however potential real zeros $\rho_\chi$ would result in non-oscillating terms on the left hand side of (2). It is therefore natural to believe that a better bound holds after removing the real zeros - or at least it is very natural to believe that

$$\sum_{Q < q \leq 2Q} \left| \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0, \rho_\chi \notin \mathbb{R}} \sum_{\rho_\chi} x^{\rho_\chi} \right| \ll x \left( \frac{\log \log x}{\log x} \right)^2. \tag{3}$$

We first remark that this last bound implies the non-vanishing of almost all Dirichlet $L$-functions at the central point.

**Proposition 1.1.** Fix $A > -2$, and assume that (3) holds in the range $x^{\frac{1}{2}}(\log x)^A < Q \leq 2x^{\frac{1}{2}}(\log x)^A$. Then almost all Dirichlet $L$-functions do not vanish at the central point. More precisely, for $Q$ large enough we have

$$\frac{1}{Q^2} \sum_{q \leq Q} \sum_{\chi \mod q} z(\chi) \ll \frac{(\log \log Q)^2}{(\log Q)^{2+A}}, \tag{4}$$

where $z(\chi)$ is the number of real zeros of $L(s, \chi)$ in the critical strip, counted with multiplicity.

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\(^3\)One can interpret this result as an asymptotic for the 1-level density of low-lying zeros of Dirichlet $L$-functions for test function whose Fourier transform has support contained in $(-2, 2)$.

\(^4\)Private conversation.
Remark 1.2. Montgomery’s probabilistic argument (see below) supports (3) (and predicts a stronger bound). As for (2) (which is known unconditionally), one would need to add the assumption that $L(\frac{1}{2}, \chi) \neq 0$ for Montgomery’s argument to support this bound.

We now investigate the implications of a more powerful conjecture than (3) on the nonvanishing of Dirichlet $L$-functions at the central point. Montgomery’s Conjecture, which is motivated by a probabilistic argument, states that in a certain range of $q$ and $x$ with $(a,q) = 1$,

$$\psi(x; q, a) - \psi(x; \chi_0) = \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \chi(a) \psi(x; \chi) \ll x^{\frac{1}{2} + \epsilon} q^{-\frac{1}{4}}. $$

This conjecture is based on the fact that under GRH we have

$$x^{-\frac{1}{2}} \sum_{\chi \neq \chi_0} \chi(a) \psi(x; \chi) = -\frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \chi(a) \sum_{\rho_X} \frac{x^{\gamma_X}}{\rho_X} + O(x^{-\frac{1}{2}}(\log qx)^2),$$

and one can show (see Appendix A) that if the $\gamma_X$ are distinct and nonzero, then the first term on the right hand side of (5) has a limiting logarithmic distribution with zero mean and variance $\sim \phi(q)^{-1} \log q$. Hence we believe that this term should not exceed $q^{-\frac{1}{2} + \epsilon}$. If we remove the assumption that the $\gamma_X$ are nonzero, then we need to reformulate Montgomery’s Conjecture. Indeed if the proportion of $\chi \mod q$ such that $L(\frac{1}{2}, \chi) = 0$ is not exactly zero, then Montgomery’s Conjecture is false. We now reformulate this conjecture, depending on a parameter $0 < \eta < 1$.

Hypothesis 1.3 (Modified Montgomery Conjecture). Fix $\epsilon > 0$. In the range $q \leq x^\eta$, we have for $(a,q) = 1$ that

$$\frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \chi(a) \sum_{\rho_X \in R} \frac{x^{\rho_X}}{\rho_X} \ll x^{\frac{1}{2} + \epsilon} q^{-\frac{1}{2}}.$$

We will show that this hypothesis implies a strong non-vanishing result on Dirichlet $L$-functions at the central point.

Theorem 1.4. Fix $\epsilon > 0$. Assume GRH, and assume that for some $\frac{1}{2} < \eta < 1$, Hypothesis 1.3 holds. Then we have that

$$\frac{1}{Q^2} \sum_{q \leq Q} \sum_{\chi \mod q} z(\chi) \ll \frac{1}{Q^{\frac{1}{2} - \epsilon}},$$

where $z(\chi)$ is the order of vanishing of $L(s, \chi)$ at $s = \frac{1}{2}$.

Remark 1.5. In contrast with Montgomery’s Conjecture, Hypothesis 1.3 does not imply GRH, but rather implies that the nonreal zeros of $L(s, \chi)$ lie on the line $\Re(s) = \frac{1}{2}$. This last statement was used as a hypothesis in the work of Sarnak and Zaharescu [SZ], who showed that it implies an effective bound on the class number of imaginary quadratic fields.

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5The contrapositive of this statement follows from Theorem 2.13 of [PM]. Indeed, taking the test function $\eta_\kappa(y) := (\sin(\kappa \pi y)/\kappa \pi y)^2$, whose Fourier transform is supported in the interval $[-\kappa, \kappa]$, the 1-level density is asymptotically $\tilde{\eta}_\kappa(0) = 1/\kappa$, and taking arbitrarily large values of $\kappa$ gives the desired conclusion.

6It is actually sufficient to assume that (5) holds on average over $q \leq Q$, with $Q \leq x^\eta$ and $a = 1$. 
Remark 1.6. As mentioned earlier in Footnote 5, Montgomery’s Conjecture implies the Katz-Sarnak prediction for the 1-level density in the family of Dirichlet $L$-functions modulo $q$, and thus it follows that almost all members of this family do not vanish at the central point. However, Montgomery’s Conjecture has the assumption that $L\left(\frac{1}{2}, \chi\right) \neq 0$ built in, and moreover the Katz-Sarnak density conjecture does not allow one to obtain an explicit error term as in (4) and (7). Finally, the range $x^{\frac{1}{2}-o(1)} < Q < x^{\frac{1}{2}+o(1)}$ in which we are working in Proposition 1.1 corresponds in the Katz-Sarnak problem to test functions whose Fourier transform is supported in $(-2-\epsilon, -2+\epsilon) \cup (2-\epsilon, 2+\epsilon)$, and thus does not allow one to tackle the rest of the support, which is needed to obtain the non-vanishing of almost-all Dirichlet $L$-functions at the central point.

Proposition 1.1 follows from a fairly straightforward argument. As for Theorem 1.4, the proof is more involved and relies on the properties of the sum

$$S(Q; x) := - \sum_{Q < q \leq 2Q} \left( \psi(x; q, 1) - \frac{\psi(x, \chi_0)}{\phi(q)} \right),$$

which we will study using two different techniques. We now record one of the resulting estimates which we believe is of independent interest.

Proposition 1.7. Fix $\epsilon > 0$, assume GRH and assume that Hypothesis 1.3 holds for some $\frac{1}{2} < \eta < 1$. Then in the range $x^{\frac{1}{2}} < Q \leq x$ we have that

$$S(Q; x) = \frac{Q}{2} \log(x/Q) + C_3 Q + O_\epsilon \left( \frac{x^{1+\epsilon}}{Q^{\frac{3}{2}}} + x^{-\frac{1}{2}+\epsilon} \right),$$

where

$$C_3 := \frac{1}{2} \left( \log 2\pi + \gamma + \sum_p \frac{\log p}{p-1} + 1 \right) - \log 2.$$

(Note that this gives an asymptotic for $S(Q, x)$ in the range $x^{\frac{1}{2}+o(1)} < Q = o(x)$, and that the error term is independent of $\eta$.)

2. An application of the Bombieri-Friedlander-Iwaniec Theorem

In this section we prove Proposition 1.1.

Proof of Proposition 1.1. Applying the triangle inequality twice gives that in the range $x^{\frac{1}{4}} (\log x)^A < Q \leq 2x^{\frac{1}{4}} (\log x)^A$,

$$\left| \sum_{Q < q \leq 2Q} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \sum_{\rho \in \mathbb{R}} \frac{x^{\rho_x}}{\rho_X} \right| \leq \sum_{Q < q \leq 2Q} \left| \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \sum_{\rho \in \mathbb{R}} \frac{x^{\rho_x}}{\rho_X} \right| + \sum_{Q < q \leq 2Q} \left| \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \sum_{\rho \in \mathbb{R}} \frac{x^{\rho_x}}{\rho_X} \right| \ll x \left( \frac{\log \log x}{\log x} \right)^2,$$

by (2) and (3). We therefore have that

$$x \left( \frac{\log \log x}{\log x} \right)^2 \gg \sum_{Q < q \leq 2Q} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \sum_{\rho \in \mathbb{R}} \frac{x^{\rho_x}}{\rho_X} \geq \sum_{Q < q \leq 2Q} \frac{1}{q} \sum_{\chi \neq \chi_0} \sum_{\rho \in \mathbb{R}} \frac{x^{\rho_x}}{\rho_X}. \quad (8)$$
We now note that if \( \rho \in \mathbb{R} \) is a zero of \( L(s, \chi) \), then \( 1 - \rho \) is also a zero of \( L(s, \overline{\chi}) \) with the same multiplicity \( m_\rho \), hence this pair of zeros give a contribution of

\[
m_\rho \frac{x^\rho}{\rho} + m_\rho \frac{x^{1-\rho}}{1-\rho} \geq m_\rho x^{\frac{1}{2}}.
\]

Therefore grouping characters by conjugate pairs in (8), we obtain that the last term on the right is

\[
\geq \sum_{Q < q \leq 2Q} \frac{1}{q} \sum_{\chi \neq \chi_0} \sum_{\rho \in \mathbb{R}} \frac{1}{2} x^{\frac{1}{2}} \geq \frac{x^{\frac{1}{2}}}{4Q} \sum_{Q < q \leq 2Q} \sum_{\chi \neq \chi_0} z(\chi).
\]

We conclude that

\[
\frac{1}{Q^2} \sum_{Q < q \leq 2Q} \sum_{\chi \mod q} z(\chi) \ll \frac{x^{\frac{1}{2}}}{Q} \left( \frac{\log \log x}{\log x} \right)^{2}.
\]

A standard argument using dyadic intervals gives the claimed bound. \( \square \)

3. Applications of Montgomery’s Conjecture

In this section we study the quantity

\[
S(Q; x) = - \sum_{Q < q \leq 2Q} \left( \psi(x; q, 1) - \psi(x; \chi_0) \right),
\]

using two different techniques. The proof of Theorem 1.4 will follow by comparing these two estimates.

We first give a conditional bound on \( S(Q; x) \) using techniques of [Fi1], which ultimately relies on Hooley’s variant of the divisor switching method [H].

**Lemma 3.1.** Fix \( \epsilon > 0 \) and assume GRH. In the range \( x^{\frac{3}{4}} \leq Q \leq x \), we have the estimate

\[
S(Q; x) = \frac{Q}{2} \log(x/Q) + C_3 Q + O_\epsilon \left( \frac{x^{\frac{3}{7}} (\log x)^2}{Q} + Q^{\frac{5}{7}} - x^{-\frac{1}{2} + \epsilon} \right),
\]

where \( S(Q; x) \) is defined in (9) and

\[
C_3 := \frac{1}{2} \left( \log 2\pi + \gamma + \sum_p \frac{\log p}{p(p-1)} + 1 \right) - \log 2.
\]

(Note that this is gives an asymptotic for \( S(Q; x) \) in the range \( x^{\frac{3}{4}} \log x = o(Q), Q \leq x \).)

**Proof.** We evaluate \( S(Q, x) \) by following the argument in the proof of Proposition 6.1 of [Fi1] (see also [FG, FG3, FGHM, H]). We first write

\[
S(Q, x) = \sum_{2Q < q \leq x} \psi(x; q, 1) - \sum_{Q < q \leq x} \psi(x; q, 1) + \psi(x; \chi_0) \sum_{Q < q \leq 2Q} \frac{1}{\phi(q)} = I - II + III.
\]

Lemma 5.2 of [Fi1] combined with the Riemann Hypothesis implies that

\[
III = C_1 x \log 2 + O \left( \frac{x \log Q}{Q} + x^{\frac{1}{4}} (\log x)^2 \right),
\]
where \( C_1 := \frac{\zeta(2)\zeta(3)}{\zeta(6)} \). We treat \( I \) and \( II \) as follows (see Lemma 5.1 of [Fi1]):

\[
II = \sum_{Q < \eta \leq x} \sum_{Q < n \leq \eta} \Lambda(n) = \sum_{1 \leq r < (x-1)/Q} \sum_{rQ+1 < n \leq x} \Lambda(n)
\]

\[
= \sum_{1 \leq r < (x-1)/Q} \left( \psi(x; r, 1) - \psi(rQ + 1; r, 1) \right)
\]

\[
= \sum_{1 \leq r < (x-1)/Q} \frac{x - rQ - 1}{\phi(r)} + O \left( \frac{x^{\frac{3}{2}}(\log x)^2}{Q} \right)
\]

\[
= x \left( C_1 \log(x/Q) + C_2 + \frac{\log(x/Q)}{2x/Q} + C_0 \frac{Q}{x} + O_\epsilon \left( \frac{Q^{\frac{3}{2} - \epsilon}}{x^{\frac{3}{2}}} + \frac{x^{\frac{3}{2}}(\log x)^2}{Q} \right) \right)
\]

by GRH and Lemma 5.9 of [Fi1]. Here,

\[
C_2 := C_1 \left( \gamma - 1 - \sum_p \frac{\log p}{p^2 - p + 1} \right), \quad C_0 := \frac{1}{2} \left( \log 2\pi + \gamma + \sum_p \frac{\log p}{p(p - 1) + 1} \right).
\]

Note that at this point we cannot apply Hypothesis 1.3 in going from (11) to (12), since we have no information on the real zeros of \( L(s, \chi) \). Later we will reiterate this proof and apply our non-vanishing results at this step to get a better error term.

We conclude the proof by collecting our estimates for \( I, II \) and \( III \):

\[
S(Q, x) = \frac{Q}{2} \log(x/Q) + Q(C_0 - \log 2) + O_\epsilon \left( \frac{1}{x^{\frac{3}{2}}} + \frac{x^{\frac{3}{2}}(\log x)^2}{Q} \right).
\]

We now combine Lemma 3.1 with Hypothesis 1.3 to obtain a first non-vanishing result.

**Lemma 3.2.** Fix \( \epsilon > 0 \). Assume GRH, and assume that for some \( \frac{1}{2} \leq \eta < 1 \), Hypothesis 1.3 hold\(^7\). Then we have that

\[
\frac{1}{Q^2} \sum_{Q < q \leq 2Q} \sum_{\chi \mod q} z(\chi) \ll_\epsilon \frac{1}{Q^{\min(\frac{1}{2} - \frac{1}{2\eta}; 1)} - \epsilon}.
\]

where \( z(\chi) \) is the number of real zeros of \( L(s, \chi) \) in the critical strip, counted with multiplicity. (Note that if \( \frac{2}{3} \leq \eta < 1 \), then the right hand side of (13) equals \( Q^{-\frac{1}{2} + \epsilon} \).

**Proof.** We study the quantity

\[
S(Q; x) = - \sum_{Q < q \leq 2Q} \left( \psi(x; q, 1) - \psi(x, \chi_0) \right),
\]

in the range \( x^{\eta}/3 \leq Q \leq x^{\eta}/2 \).

\(^7\)It is actually sufficient to assume that for \( Q \approx x^{\eta} \),

\[
\sum_{Q < q \leq 2Q} \frac{\psi(x; q, 1)}{\phi(q)} \sum_{\chi \neq \chi_0} \sum_{\rho, \rho_x \notin \mathbb{R}} \frac{z^{\alpha}_x}{\rho_x} \ll \epsilon Q^{\frac{3}{2}} x^{\frac{1}{2} + \epsilon}.
\]
On one hand, we apply the explicit formula and GRH:

\[
S(Q; x) = \sum_{Q < q \leq 2Q} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \sum_{\rho_x} \frac{x^{\rho_x}}{\rho_x} + O(Q(\log Qx)^2)
\]

\[
= 2x^{\frac{1}{2}} \sum_{Q < q \leq 2Q} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} z(\chi) + \sum_{Q < q \leq 2Q} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \sum_{\rho_x \notin \mathbb{R}} \frac{x^{\rho_x}}{\rho_x} + O(Q(\log Qx)^2)
\]

\[
\geq \frac{x^{\frac{1}{2}}}{Q} \sum_{Q < q \leq 2Q} \sum_{\chi \neq \chi_0} z(\chi) + O_\epsilon(x^{\frac{3}{2} + \frac{1}{2}Q^{\frac{1}{2}}}),
\]

(14)

by Hypothesis 1.3.

On the other hand, we compare this with the estimate for \( S(Q; x) \) in Lemma 3.1, yielding

\[
\frac{1}{Q^2} \sum_{Q < q \leq 2Q} \sum_{\chi \neq \chi_0} \sum_{\chi \neq \chi_0} z(\chi) \ll_\epsilon x^{\frac{1}{2}} Q^{-\frac{1}{2}} + \frac{\log x}{x^{\gamma}} + \frac{x(\log x)^2}{Q^2} \ll_\epsilon Q^{\frac{1}{2} - \epsilon} + Q^{-2 + \epsilon}.
\]

\[\square\]

We now refine Lemma 3.2 by re-inserting Hypothesis 1.3 in its proof. We will iterate this process several times, until we reach the error term appearing in Theorem 1.4.

**Lemma 3.3.** Fix \( \epsilon > 0 \), assume GRH and assume that Hypothesis 1.3 holds\(^8\) for some \( \frac{1}{2} < \eta < 1 \). Assume further that for \( \kappa(\eta) \) a function of \( \eta \) satisfying \( 0 < \kappa(\eta) < \frac{1}{2} \), we have

\[
\frac{1}{Q^2} \sum_{Q < q \leq 2Q} \sum_{\chi \neq \chi_0} z(\chi) \ll_\epsilon \frac{1}{Q^{\frac{1}{2} - \epsilon}} + \frac{1}{Q^{\kappa(\eta) - \epsilon}}.
\]

(15)

Then it follows that

\[
\frac{1}{Q^2} \sum_{Q < q \leq 2Q} \sum_{\chi \neq \chi_0} z(\chi) \ll_\epsilon \frac{1}{Q^{\frac{1}{2} - \epsilon}} + \frac{1}{Q^{2 - \frac{1}{\eta} - \kappa(\eta)(1 - \frac{1}{\eta}) - \epsilon}}.
\]

(16)

**Proof.** We set \( x^{\eta}/3 \leq Q \leq x^{\eta}/2 \) and follow the proofs of Lemmas 3.1 and 3.2, applying (15) in going from (11) to (12). Note that (15), GRH and Hypothesis 1.3 imply that

\[
\sum_{1 \leq r < (x-1)/Q} \left( \psi(x; r, 1) - \psi(rQ + 1; r, 1) - \frac{\psi(x, \chi_0) - \psi(rQ + 1, \chi_0)}{\phi(r)} \right)
\]

\[
= \sum_{1 \leq r < (x-1)/Q} \frac{1}{\phi(r)} \sum_{\chi \neq \chi_0} (\psi(x, \chi) - \psi(rQ + 1, \chi))
\]

\[
= \sum_{1 \leq r < (x-1)/Q} \frac{1}{\phi(r)} \sum_{\chi \neq \chi_0} \sum_{\gamma \neq 0} x^{\frac{1}{2} + i\gamma} - (rQ + 1)^{\frac{1}{2} + i\gamma}
\]

\[
+ 2 \sum_{1 \leq r < (x-1)/Q} \frac{1}{\phi(r)} \sum_{\chi \neq \chi_0} z(\chi)(x^{\frac{1}{2}} - (rQ + 1)^{\frac{1}{2}})
\]

\[
\ll_\epsilon \frac{x^{1 + \epsilon}}{Q^{\frac{1}{2}}} + x^{\frac{1}{2} - \kappa(\eta) + \epsilon} Q^{-1 + \kappa(\eta)},
\]

\(^8\)Again it is sufficient to assume that (15) holds on average over \( q \leq Q \), with \( Q \leq x^{\eta} \) and \( a = 1 \).
since for $r < (x - 1)/Q$ we always have $r \leq (rQ + 1)^n$, thanks to the fact that $1/2 < \eta < 1$. Also in applying (15) we used a dyadic decomposition of the sum over $r$. Following the subsequent steps of the proofs of Lemmas 3.1 and 3.2 we obtain that since $Q \geq x^{1/2}$,

$$\frac{1}{Q^2} \sum_{Q < q \leq 2Q} \sum_{\chi \equiv \chi_0} z(\chi) \ll \epsilon \frac{x^{3/2}}{Q^{3/2}} + x^{1-\kappa(\eta)+\frac{1}{2}Q^{-2+\kappa(\eta)}} + x^{1/2+\epsilon} Q^{-3/2} \ll \epsilon \frac{1}{Q^{\min(1/2, 2-1/\eta - \kappa(\eta)(1-1/\eta)) - \epsilon}}.$$

\[
\square
\]

We now show that starting from Lemma 3.2 with a fixed $1/2 < \eta < 1$ and applying Lemma 3.3 iteratively, we eventually obtain the error term $Q^{-1/2+\epsilon}$.

**Lemma 3.4.** Fix $1/2 < \eta < 1$, and define $f(t) := 2 - \frac{1}{\eta} - t(1 - \frac{1}{\eta})$. Then for $n$ large enough (depending on $\eta$), we have that

$$f^{(n)} \left(2 - \frac{1}{\eta}\right) > \frac{1}{2},$$

where $f^{(n)}$ is the $n$-th iterate of $f$.

**Proof.** One easily shows the following formula:

$$f^{(n)} \left(2 - \frac{1}{\eta}\right) = \left(2 - \frac{1}{\eta}\right) \sum_{k=0}^{n} \left(\frac{1}{\eta} - 1\right)^k = 1 - \left(\frac{1}{\eta} - 1\right)^{n+1}.$$

It follows that for any fixed $1/2 < \eta < \infty$,

$$\lim_{n \to \infty} f^{(n)} \left(2 - \frac{1}{\eta}\right) = 1.$$

\[
\square
\]

We are now ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** Fix $1/2 < \eta < 1$. By Lemma 3.2, we have that

$$\frac{1}{Q^2} \sum_{Q < q \leq 2Q} \sum_{\chi \equiv \chi_0} z(\chi) \ll \epsilon \frac{1}{Q^{1/2-\epsilon}} + \frac{1}{Q^{2-1/\eta - \epsilon}}.$$  \hspace{1cm} (17)

We apply Lemma 3.3 iteratively to this estimate; Lemma 3.4 implies that after a finite number of steps we will obtain the bound

$$\frac{1}{Q^2} \sum_{Q < q \leq 2Q} \sum_{\chi \equiv \chi_0} z(\chi) \ll \epsilon \frac{1}{Q^{1/2-\epsilon}}.$$  \hspace{1cm} (18)

The desired estimate follows from a decomposition into dyadic intervals.

\[
\square
\]

**Proof of Proposition 1.7.** We follow the proof of Lemma 3.1. We apply Theorem 1.4 in going from (11) to (12); as seen in the proof of Lemma 3.3 this will yield that

$$S(Q; x) = \frac{Q}{2} \log(x/Q) + C_3 Q + O_\epsilon \left(x^{1/2} (\log x)^2 + Q^{3/2-\epsilon} x^{-1/2+\epsilon} + \frac{x^{1+\epsilon}}{Q^{3/2}}\right).$$

The proof follows since $x^{1/2} < Q \leq x$.  \hspace{1cm} □
APPENDIX A. THE DISTRIBUTION OF THE ERROR TERM IN THE PRIME NUMBER THEOREM IN ARITHMETIC PROGRESSIONS

In this appendix we study the limiting logarithmic distribution of the term on the left hand side of (6), and justify Hypothesis [13]. Let us first study the remainder term in the prime number theorem for arithmetic progressions:

\[ T(x; q, a) := -x^{-\frac{1}{2}} \left( \psi(x; q, a) - \frac{\psi(x, \chi_0)}{\phi(q)} \right) = \frac{x^{-\frac{1}{2}}}{\phi(q)} \sum_{\chi \neq \chi_0} \overline{\chi}(a) \sum_{\rho_\chi} x^{\rho_\chi} \rho_\chi + o(1). \]

Assuming GRH, one can show that \( T(x; q, a) \) has a limiting logarithmic distribution \( \mu_{q,a} \), a probability measure whose associated random variable will be denoted by \( X_{q,a} \).

**Proposition A.1.** Assume GRH. Then \( T(e^y; q, a) \) has a limiting probability distribution \( \mu_{q,a} \) as \( y \to \infty \), whose mean is given by

\[ \mathbb{E}[X_{q,a}] = \int_{\mathbb{R}} td\mu_{q,a}(t) = \frac{2}{\phi(q)} \sum_{\chi \neq \chi_0} \overline{\chi}(a) z(\chi), \]

where \( z(\chi) \) is the order of vanishing of \( L(s, \chi) \) at \( s = \frac{1}{2} \). The variance of \( X_{q,a} \) is given by

\[ \text{Var}[X_{q,a}] = \int_{\mathbb{R}} (t - \mathbb{E}[X_{q,a}])^2 d\mu_{q,a}(t) = \frac{1}{\phi(q)^2} \sum_{\chi \neq \chi_0} |\chi(a)|^2 \sum_{\gamma_\chi \neq 0} \frac{m_{\gamma_\chi}^2}{\frac{1}{4} + \gamma_\chi^2}, \]

where \( m_{\gamma_\chi} \) denotes the multiplicity of \( \gamma_\chi \) in the multiset \( S(q) := \{ \gamma_\chi \in \mathbb{R} : L(\frac{1}{2} + i\gamma_\chi, \chi) = 0, \chi \mod q \} \).

**Proof.** The existence of the limiting distribution follows from [ANS]. The computation of the first two moments is almost identical to that in Lemmas 2.4 and 2.5 of [Fi2].

We now study the left hand side of (6) by defining

\[ T^*(x; q, a) := \frac{x^{-\frac{1}{2}}}{\phi(q)} \sum_{\chi \neq \chi_0} \overline{\chi}(a) \sum_{\rho_\chi \notin \mathbb{R}} x^{\rho_\chi}. \]

Similarly as in Proposition [A.1] one shows under GRH that \( T^*(x; q, a) \) has a limiting logarithmic distribution whose mean is exactly zero and whose variance is given by

\[ V^*(q; a) := \frac{1}{\phi(q)^2} \sum_{\chi \neq \chi_0} |\chi(a)|^2 \sum_{\gamma_\chi \neq 0} \frac{m_{\gamma_\chi}^2}{\frac{1}{4} + \gamma_\chi^2}. \]

Assuming that the \( m_{\gamma_\chi} \) are uniformly bounded, we deduce using the Riemann-von Mangoldt formula that \( V^*(q; a) \ll \phi(q)^{-1} \log q \). Hence, if \( \Psi(q) \) is any function tending to infinity, then Chebyshev’s Inequality gives

\[ \text{Prob}[|X_{q,a}| \geq \Psi(q)\phi(q)^{-\frac{1}{2}}(\log q)^{\frac{1}{2}}] \ll \frac{1}{\Psi(q)^2}, \]

that is \( X_{q,a} \) is normally bounded above by \( \phi(q)^{-\frac{1}{2}}(\log q)^{\frac{1}{2}} \). We need however to be careful in making conjectures about the size of \( T^*(x; q, a) \), since even though very rare, ’Littlewood phenomena’ do happen. For this reason we add the \( x^\varepsilon \) factor, which gives (6). We should also be careful with the range \( q \leq x^\eta \) in (6), since by the work of Friedlander and Granville
[FG1] [FG2], the primes up to \(x\) are not equidistributed in arithmetic progressions modulo \(q\) when \(q \asymp x/(\log x)^B\).

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