PERIOD POLYNOMIAL RELATIONS BETWEEN
FORMAL DOUBLE ZETA VALUES OF ODD WEIGHT

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ABSTRACT. For odd $k$, we give a formula for the relations between double zeta value $\zeta(r, k - r)$ with $r$ even. This formula provides a connection with the space of cusp forms on $\text{SL}_2(\mathbb{Z})$. This is the odd weight analogue of a result in [3] by Gangl, Kaneko and Zagier. We also provide an answer of the question asked by Zagier in [9] about the left kernel of $B_K$. Although the restricted sum statement in [3] fails in the odd weight case, we provide an asymptotical statement the replaces it. Our statement works more generally for restricted sums with any congruence condition on the first entry of the double zeta value.

1. INTRODUCTION AND MAIN RESULT

The double zeta values, which are defined for integers $r \geq 2$, $s \geq 1$ by

$$\zeta(r, s) = \sum_{m > n > 0} \frac{1}{m^r n^s} ,$$

satisfy numerous relations. The double shuffle relations on double zeta values are given by the following two sets of well-known relations (cf., e.g., [4], [5], [8]):

$$\zeta(r, s) + \zeta(s, r) = \zeta(r)\zeta(s) - \zeta(k) \quad (r + s = k, r, s \geq 2), \quad (2)$$

$$\sum_{r=2}^{k-1} \left[ \binom{r-1}{j-1} + \binom{r-1}{k-j-1} \right] \zeta(r, k - r) = \zeta(j)\zeta(k - j) \quad (2 \leq j \leq \frac{k}{2}). \quad (3)$$

Often people work on the formal double zeta space $\mathcal{D}_k$ generated by formal symbols $Z_{r,s}$, $P_{r,s}$ and $Z_k$ satisfying the above two sets of relations, with $\zeta(r, s), \zeta(r)\zeta(s)$ and $\zeta(k)$ replaced by $Z_{r,s}$, $P_{r,s}$ and $Z_k$, respectively. The advantage for this space is that we can work in it purely algebraically, since the double zeta values may satisfy other relations than those generated by (2) and (3).

Many authors have studied the relations which can be deduced from the above two sets of relations. One of the most famous results in this area concerns the following sort of relations, which gives the first connection with modular forms:

**Theorem** (Theorem 3 (Rough statement) in [3]). The values $\zeta(\text{odd}, \text{odd})$ of weight $k$ satisfy at least $\dim S_k$ linearly independent relations, where $S_k$ denotes the space of cusp forms of weight $k$ on $\text{SL}_2(\mathbb{Z})$. 

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Example. For \( k = 12 \) and \( k = 16 \), the first two cases for which there are non-zero cusp forms on \( \text{SL}_2(\mathbb{Z}) \), we have the following identities.

\[
\begin{align*}
\frac{5197}{691} \zeta(12) &= 28 \zeta(9, 3) + 150 \zeta(7, 5) + 168 \zeta(5, 7) \\
\frac{78967}{3617} \zeta(16) &= 66 \zeta(13, 3) + 375 \zeta(11, 5) + 686 \zeta(9, 7) + 675 \zeta(7, 9) + 396 \zeta(5, 11).
\end{align*}
\]

In their paper [3], Gangl, Kaneko and Zagier proved the following general result for the formal double zeta space \( D_k \) instead. We denote by \( P_{\text{ev, even}} \) the subspace of \( D_k \) spanned by the \( P_{\text{even, even}} \). Let \( W_k^- \) denote the space of even period polynomials of weight \( k \) (see Section 2).

**Theorem** (Theorem 3 in [3]). The spaces \( P_{\text{ev}}^k \) and \( W_k^- \) are canonically isomorphic to each other. More precisely, to each \( p \in W_k^- \) we associate the coefficients \( q_{r,s} \) (\( r + s = k \)) which are defined by

\[
p(X + Y, Y) = \sum_{r+s=k} q_{r,s} X^{r-1} Y^{s-1} - 2 \sum_{r+s=k} q_{r,s} Z_{r,s} \quad (\text{mod } Z_k),
\]

and conversely, an element \( \sum_{r,s \text{ odd}} c_{r,s} Z_{r,s} \in D_k \) belongs to \( P_{\text{ev}}^k \) if and only if \( c_{r,s} = q_{r,s} \) arising in this way.

By taking the double zeta value realization, the above result for double zeta values follows directly.

Although people know the above result for double zeta values of even weight, the direct connection between double zeta values of odd weight and the space of cusp form is still unknown. In [9], Zagier proved the following result:

**Theorem** (Theorem 3 in [9]). For each odd integer \( k = 2K + 1 \geq 5 \), the numbers \( \{ \zeta(k - 2r - 1, 2r + 1) \mid r = 0, \ldots, K - 1 \} \) satisfy \( \dim S_{k-1} + \dim S_{k+1} \) relations, where \( S_i \) denotes the space of cusp forms of weight \( i \) on \( \text{SL}_2(\mathbb{Z}) \).

The above result suggests a connection between the relations of odd weight double zeta values and the space of cusp forms. In this paper, we will prove the following results, which will establish the “missing” direct connection as in the even weight case. One of them provides a relation from \( S_{k-1} \), and the other one from \( S_{k+1} \).

**Theorem 1.** Let \( k \geq 12 \) be an even integer. To each \( p \in W_k^+ \) we associate the coefficients \( b_{r,s} \) (\( r + s = k + 1 \)) which are defined by

\[
p(X + Y, Y) = \sum_{r+s=k+1} \binom{k-1}{r-1} b_{r,s} X^{r-1} Y^{s-2}.
\]
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Then

\[ \sum_{4 \leq r \leq k-2 \text{ even}, \ r + s = k+1} (b_{r,s} - b_{s,r})Z_{r,s} \equiv 0 \pmod{Z_{k+1}}. \] (5)

\textbf{Theorem 2.} Let \( k \geq 12 \) be an even integer. To each \( p \in W_k^- \) we associate the coefficients 
\( c_{r,s} \) \((r + s = k - 1)\) which are defined by

\[ \frac{\partial}{\partial X} p(X + Y, Y) = \sum_{r+s=k-1} \binom{k-3}{r-1} c_{r,s} X^{r-1} Y^{s-1}. \]

Then

\[ \sum_{4 \leq r \leq k-4 \text{ even}, \ r + s = k-1} (c_{r,s} - c_{s,r})Z_{r,s} \equiv 0 \pmod{Z_{k-1}}. \] (6)

\textbf{Theorem 3} (Rough statement). Let \( k \geq 7 \) be an odd integer. Up to rational multiples of 
\( \zeta(k) \), the values \( \{\zeta(r,s)\} \ r \text{ even}, \ 4 \leq r \leq k-3 \), \( r + s = k \) satisfy \( \dim S_{k-1} + \dim S_{k+1} \)
relations, where \( S_i \) denotes the space of cusp forms of weight \( i \) on \( \text{SL}_2(\mathbb{Z}) \).

Noticed that the first entry of the double zeta value in this result is an even integer between
4 and \( k - 3 \), while in Zagier’s result it is an even integer between 2 and \( k - 1 \). At this moment,
we are not able to show that those relations are linearly independent. But conjecturally, they
are. It is worth to point out that our result is compatible with the decomposition of double
zeta values of odd weight into single zeta values by Euler in [2].

\textbf{Example 4.} For \( k = 11, k = 13 \) and \( k = 15 \), the only cases when \( \dim S_{k-1} + \dim S_{k+1} = 1 \),
we have

\[ -3\zeta(11) = 28\zeta(8, 3) + 20\zeta(6, 5) - 42\zeta(4, 7); \]
\[ -3\zeta(13) = 24\zeta(10, 3) + 28\zeta(8, 5) - 10\zeta(6, 7) - 36\zeta(4, 9); \]
\[ -3\zeta(15) = 22\zeta(12, 3) + 30\zeta(10, 5) + 7\zeta(8, 7) - 20\zeta(6, 9) - 33\zeta(4, 11). \]

For \( k = 17 \), the first case when \( \dim S_{k-1} = \dim S_{k+1} = 1 \), we have

\[ -23\zeta(17) = \]
\[ 156\zeta(14, 3) + 242\zeta(12, 5) + 153\zeta(10, 7) - 56\zeta(8, 9) - 215\zeta(6, 11) - 234\zeta(4, 13); \]
\[ -597\zeta(17) = \]
\[ 4004\zeta(14, 3) + 6358\zeta(12, 5) + 4347\zeta(10, 7) - 1624\zeta(8, 9) - 5885\zeta(6, 11) - 6006\zeta(4, 13), \]

where the first identity comes from \( S_{16} \), and the second one from \( S_{18} \).

Gangl, Kaneko and Zagier also proved the following statement in their paper [3].
Theorem (Theorem 1 in [3]). For even $k > 2$, one has
\[ \sum_{r=2}^{k-1} Z_{r,k-r} = \frac{3}{4} Z_k, \quad \sum_{r=2}^{k-1} Z_{r,k-r} = \frac{1}{4} Z_k. \]

The double zeta value realization of the above statement tells us that for even $k > 2$, we have
\[ \sum_{r=2}^{k-1} \zeta(r, k-r) = \frac{3}{4} \zeta(k), \quad \sum_{r=2}^{k-1} \zeta(r, k-r) = \frac{1}{4} \zeta(k). \]

It is easy to see that the above statement does not hold for double zeta values of odd weight. But asymptotically, it is still correct, i.e., we have the following statement.

Theorem 5. For any integer $d \geq 2$, and any $i$ satisfying $0 \leq i \leq d - 1$, we have
\[ \lim_{k \to \infty} \frac{\sum_{r=2}^{k-1} \zeta(r, k-r)}{\zeta(k)} = C_d^{(i)}, \]
where
\[ C_d^{(i)} = \sum_{j=2}^{\infty} (\zeta(j) - 1). \]

Remark. Notice that we do not require our $k$ to be odd in this case. It is worth pointing out that our result above is compatible with Corollary 1.2 in [6] by Machide.

Example 6. In particular, when $d = 2$ and $i = 0, 1$, we have (cf., e.g., [1])
\[ C_2^{(0)} = \sum_{n=1}^{\infty} (\zeta(2n) - 1) = \frac{3}{4}, \]
\[ C_2^{(1)} = \sum_{n=1}^{\infty} (\zeta(2n+1) - 1) = \frac{1}{4}. \]

Therefore,
\[ \frac{\sum_{r=2}^{k-1} \zeta(r, k-r)}{\zeta(k)} \to \frac{3}{4}, \quad \text{as } k \to \infty; \]
\[ \sum_{r=2}^{k-1} \zeta(r, k-r) \sum_{r \text{ odd}} \frac{\zeta(k)}{r} \rightarrow \frac{1}{4}, \quad \text{as } k \rightarrow \infty. \]

In Section 2 we provide some background on the formal double zeta space and the 
PGL_2(\mathbb{Z})-action on the space of homogeneous polynomials. In Section 3 we provide the
proof of Theorem 1. Theorem 2 can be proved using almost the same method, so we will
only provide the construction and skip the detailed proof. Some double zeta values examples
for Theorem 1 and Theorem 2 will be provided in Section 4. In Section 5, we will explain how
to use our theorems to obtain information about left kernel of Zagier’s matrix \( B_K \). Finally,
in Section 6, we will prove Theorem 5 and provide some more examples of restricted sums
of double zeta values.

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2. Background

We begin by reviewing the definition of the formal double zeta space (cf., [4], [5], [8]). Let
\( k > 2 \) be an integer. We introduce formal variables \( Z_{r,s}, P_{r,s} \) and \( Z_k \) and impose the relations
\[ Z_{r,s} + Z_{s,r} = P_{r,s} - Z_k \quad (r + s = k), \]
\[ \sum_{r+s=k} \left[ \binom{r-1}{1} + \binom{r-1}{j-1} \right] Z_{r,s} = P_{i,j} \quad (i + j = k). \]
(From now on, whenever we write \( r + s = k \) or \( i + j = k \) without comment, it is assumed
that the variables are integers \( \geq 1 \).

The formal double zeta space is defined as the \( \mathbb{Q} \)-vector space
\[ D_k = \{ \text{\( \mathbb{Q} \)-linear combinations of formal symbols } Z_{r,s}, P_{r,s}, Z_k \} \langle \text{relations (7) and (8)} \rangle. \]

The double zeta realization we consider in this paper is the following realization \( D_k \rightarrow \mathbb{R} \)
of the formal double zeta space.

\[
Z_{r,s} \mapsto \begin{cases} 
\zeta(r, s), & \text{if } r > 1, \\
\kappa, & \text{if } r = 1,
\end{cases}
\]
\[
P_{r,s} \mapsto \begin{cases} 
\zeta(r)\zeta(s), & \text{if } r, s > 1, \\
\kappa + \zeta(k-1, 1) + \zeta(k), & \text{if } r = 1 \text{ or } s = 1,
\end{cases}
\]
\[ Z_k \mapsto \zeta(k), \]
where $\kappa \in \mathbb{R}$ can be chosen to be any real number.

One basic way of working with $\mathcal{D}_k$ is by studying the relations among the $Z_{r,s}$. We first introduce some basic notation. For each even $k$, let $V_k = \langle X^{r-1}Y^{s-1} \mid r + s = k \rangle$ be the space of homogeneous polynomials of degree $k - 2$ in two variables. Let $W_k \subset V_k$ be the subspace of polynomials satisfying the relations

\[ P(X,Y) + P(-Y,X) = 0 \]
\[ P(X,Y) + P(X - Y,X) + P(Y,Y - X) = 0. \]

We call $P \in W_k$ a period polynomial. This period polynomial space splits as the direct sum of subspaces $W^+_k$ and $W^-_k$ of polynomials which are symmetric and antisymmetric with respect to $X \leftrightarrow Y$. We call them odd and even period polynomials. The Eichler-Shimura-Manin theory tells us that there are canonical isomorphisms over $\mathbb{C}$ between $S_k$ and $W^+_k$ and between $M_k$ and $W^-_k$.

In [3], Gangl, Kaneko and Zagier proved the following statement, which is important in understanding the connection between relations of $Z_{r,s}$ up to $Z_k$ and the period polynomials.

**Proposition 7** (Proposition 2 in [3]). Let $a_{r,s}$ and $\lambda$ be rational numbers. Then the following two statements are equivalent:

1. The relation
   \begin{equation}
   \sum_{r+s=k} a_{r,s} Z_{r,s} = \lambda Z_k
   \end{equation}

   holds in $\mathcal{D}_k$.

2. The generating function
   \begin{equation}
   A(X,Y) = \sum_{r+s=k} \binom{k-2}{r-1} a_{r,s} X^{r-1} Y^{s-1} \in V_k
   \end{equation}

   can be written as $H(X,X + Y) - H(X,Y)$ for some symmetric homogeneous polynomial $H \in \mathbb{Q}[X,Y]$ of degree $k - 2$, and

   \[ \lambda = \frac{k - 1}{2} \int_0^1 H(t,1-t)dt. \]

The last thing we want to review is the $\text{PGL}_2(\mathbb{Z})$-action on $V_k$ and an alternative definition of $W_k$ using this action. Let $F \in V_k$ be a homogeneous polynomial of degree $k - 2$ in $X$ and $Y$, and $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{PGL}_2(\mathbb{Z})$. The $\text{PGL}_2(\mathbb{Z})$-action on $V_k$ is defined to be

\[ (F|\gamma)(X,Y) = F(aX + bY,cX + dY). \]

There are 5 important elements in $\text{PGL}_2(\mathbb{Z})$ which will be used later.

\[ \varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = US = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T' = U^2S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \]
By using the above matrices, the space $W_k$ can also be defined as

$$W_k = \ker(1 + S) \cap \ker(1 + U + U^2) \subset V_k. \tag{12}$$

### 3. Proof of Theorem 1

Having reviewed formal double zeta space, period polynomials and the $\text{PGL}_2(\mathbb{Z})$-action, we now turn our attention to proving our main theorem. We will only give the detailed proof for Theorem 1 here. Theorem 2 can be treated by the same method, so we only provide the corresponding construction in a remark.

**Proof of Theorem 1.** Let $q = p|T$. Since $p$ is an odd period polynomial, it must be symmetric. We have $p(X + Y, X) = p(X, X + Y)$. Let $f = q \cdot Y - q \varepsilon \cdot X$. First we want to show that $f = f|ST'$. By a direct computation, we have

$$f|ST' - f = (q \cdot Y - q \varepsilon \cdot X)|ST' - (q \cdot Y - q \varepsilon \cdot X)$$

$$= q|ST' \cdot X - q \varepsilon ST' \cdot (-(X + Y)) - (q \cdot Y - q \varepsilon \cdot X)$$

$$= (q \varepsilon + q|ST' + q \varepsilon ST') \cdot X + (q \varepsilon ST' - q) \cdot Y.$$

I claim that two terms in parentheses are both zero.

$$q| \varepsilon + q|ST' + q| \varepsilon ST' = p|T \varepsilon + p|TST' + p|T \varepsilon ST'$$

$$= p(X + Y, X) + p(-Y, X) + p(-Y, -X - Y)$$

$$= p(X, X + Y) - p(X, Y) + p(X + Y, Y)$$

$$= 0;$$

$$q| \varepsilon ST' - q = p|T \varepsilon ST' - p|T$$

$$= p(-Y, -X - Y) - p(X + Y, Y)$$

$$= p(X + Y, Y) - p(X + Y, Y)$$

$$= 0.$$

Hence we have shown that $f|ST' = f$. Now let us consider the function $f|S$. Since

$$(f|S)| \varepsilon - f|S = (q \cdot Y - q \varepsilon \cdot X)|S \varepsilon - (q \cdot Y - q \varepsilon \cdot X)|S$$

$$= (q| \varepsilon S \varepsilon - q|S) \cdot X + (q|S \varepsilon - q| \varepsilon S) \cdot Y$$

$$= 0,$$

we know that $f|S$ is a symmetric homogeneous polynomial of degree $k - 1$. By applying Proposition 4 to the following identity

$$f - f|S = f|ST' - f|S = (f|S)|T' - 1),$$

the coefficients of $f - f|S$ will give us a relation between $Z_{r,s}$ ($r + s = k + 1$) up to a scalar multiple of $Z_{k+1}$. The last thing we need to show is that the only nonzero terms of $X^{r-1}Y^{s-1}$ appearing in $f - f|S$ are $2\binom{k-1}{r-1}(b_{r,s} - b_{s,r})X^{r-1}Y^{s-1}$ for even $r$ satisfying $4 \leq r \leq k - 2$. 

Since $f|S$ is symmetric, $f - f|S = f|SS - f|S$ only contains the terms with odd powers of $X$ between 3 and $k - 3$, and those coefficients will be double of the corresponding ones in $f$. (There are no terms of odd powers of $X$ of degree 1 and $k - 1$ since $q$ itself already does not have such terms.) According to the definition, the coefficient of $X^{r-1}Y^{s-1}$ in $f = q \cdot Y - q|\varepsilon \cdot X$ is

$$\binom{k-1}{r-1} b_{r,s} - \binom{k-1}{s-1} b_{s,r} = \binom{k-1}{r-1} (b_{r,s} - b_{s,r}).$$

Therefore, after dividing by 2, we have shown the exact relation claimed in Theorem 1. □

**Remark.** For the proof of theorem 2, we need to take $q = \frac{\partial}{\partial X} p|T$ and $f = q - q|\varepsilon$. Again we have $f = f|ST' \implies f - f|S = f|ST' - f|S = (f|S)|(T' - 1)$.

4. **Examples**

Now we will provide some examples for Theorem 1 and Theorem 2. By taking the double zeta value realization, we obtain the examples for double zeta values given in Example 4

**Example 8.** The space $W^+_{12}$ is 1-dimensional, spanned by the odd period polynomial $p(x) = 4x^9 - 25x^7 + 42x^5 - 25x^3 + 4x$. We have $p(x + 1) = 4x^9 + 36x^8 + 119x^7 + 161x^6 + 21x^5 - 161x^4 - 144x^3 - 36x^2$, so the $b_{r,s}$ of the theorem are given (after multiplication by 330) by the table

| $s$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|----|
| $r$ | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 |
| 330$b_{r,s}$ | 24 | 72 | 119 | 115 | 15 | -161 | -288 | -216 |

The relations in the theorem, divided by 10, become

$$24Z_{10,3} + 28Z_{8,5} - 10Z_{6,7} - 36Z_{4,9} \equiv 0 \mod Z_{13}.$$ 

In the double zeta value realization, this can be written as

$$24\zeta(10, 3) + 28\zeta(8, 5) - 10\zeta(6, 7) - 36\zeta(4, 9) = -3\zeta(13). \quad (13)$$

**Example 9.** The space $W^-_{12}$ is 2-dimensional, spanned by the even period polynomial $p(x) = x^{10} - 1$ and $x^8 - 3x^6 + 3x^4 - x^2$. For the latter, we have $p'(x + 1) = 8x^7 + 56x^6 + 150x^5 + 190x^4 + 112x^3 + 24x^2$, so the $c_{r,s}$ of the theorem are given (after multiplication by 63) by the table

| $s$ | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|
| $r$ | 8 | 7 | 6 | 5 | 4 | 3 |
| 63$c_{r,s}$ | 14 | 42 | 75 | 95 | 84 | 42 |
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The relations in the theorem, divided by \(-1\), become

\[ 28Z_{8,3} + 20Z_{6,5} - 42Z_{4,7} \equiv 0 \mod Z_{11}. \]

In the double zeta value realization, this can be written as

\[ 28\zeta(8, 3) + 20\zeta(6, 5) - 42\zeta(4, 7) = -3\zeta(11). \]  \( (14) \)

**Example 10.** The space \( W_{16} \) is 2-dimensional, spanned by the even period polynomial \( p(x) = x^{14} - 1 \) and \( 2x^{12} - 7x^{10} + 11x^8 - 11x^6 + 7x^4 - 2x^2 \). For the latter, we have \( p'(x + 1) = 24x^{11} + 264x^{10} + 1250x^9 + 3330x^8 + 5488x^7 + 5824x^6 + 4050x^5 + 1850x^4 + 528x^3 + 72x^2 \), so the \( c_{r,s} \) of the theorem are given (after multiplication by \( \frac{420}{2} \)) by the table

| \( r \) | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|
| \( s \) | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| \( \frac{420}{2}c_{r,s} \) | 66 | 198 | 375 | 555 | 686 | 728 | 675 | 555 | 396 | 198 |

The relations in the theorem, divided by \(-6\), become

\[ 22Z_{12,3} + 30Z_{10,5} + 7Z_{8,7} - 20Z_{6,9} - 33Z_{4,11} \equiv 0 \mod Z_{15}. \]

In the double zeta value realization, this can be written as

\[ 22\zeta(12, 3) + 30\zeta(10, 5) + 7\zeta(8, 7) - 20\zeta(6, 9) - 33\zeta(4, 11) = -3\zeta(15). \]  \( (15) \)

**Remark.** The reason why we do not consider the even period polynomials \( x^{10} - 1 \) and \( x^{14} - 1 \) in Example 8 and Example 10 is that they will always give us the trivial relation.

5. Relation to Zagier’s problem

Let us see how our relations (5) and (6) are related to Zagier’s matrix \( B_K \). Let \( k = 2K + 1 \). In [9], Zagier obtained the following relation

\[ \zeta(k - 2m - 1, 2m + 1) \]

\[ = \sum_{n=1}^{K} \left[ \delta_{n,m} + \delta_{n,K} - \frac{2n}{2m} - \frac{2n}{2K - 2m - 1} \right] \zeta(2n + 1)\zeta(k - 2n - 1), \]  \( (16) \)

where \( \zeta(0) = -\frac{1}{2} \) by convention.

Let \( B_K \) be the matrix coming from the above relations (16). For example, when \( k = 11 \), the above relations can be written as

\[
\begin{pmatrix}
\zeta(10, 1) \\
\zeta(8, 3) \\
\zeta(6, 5) \\
\zeta(4, 7) \\
\zeta(2, 9)
\end{pmatrix}
= 
\begin{pmatrix}
5 & -1 & -1 & -1 & -1 \\
82 & 0 & -6 & -15 & -36 \\
\frac{461}{2} & 0 & 0 & -21 & -126 \\
\frac{329}{2} & 0 & -4 & -20 & -84 \\
27 & -2 & -4 & -6 & -8
\end{pmatrix}
\begin{pmatrix}
\zeta(11) \\
\zeta(8)\zeta(3) \\
\zeta(6)\zeta(5) \\
\zeta(4)\zeta(7) \\
\zeta(2)\zeta(9)
\end{pmatrix}
= 
B_5
\begin{pmatrix}
\zeta(11) \\
\zeta(8)\zeta(3) \\
\zeta(6)\zeta(5) \\
\zeta(4)\zeta(7) \\
\zeta(2)\zeta(9)
\end{pmatrix}.
\]
Let us consider the following submatrix $B_5^{(1)}$ of $B_5$.

$$
\begin{pmatrix}
5 & -1 & -1 & -1 & -1 \\
82 & 0 & -6 & -15 & -36 \\
\frac{461}{2} & 0 & 0 & -21 & -126 \\
\frac{329}{2} & 0 & -4 & -20 & -84 \\
27 & -2 & -4 & -6 & -8
\end{pmatrix}
$$

Since this submatrix $B_5^{(1)}$ corresponds to the linear expressions of $\{\zeta(8, 3), \zeta(6, 5), \zeta(4, 7)\}$ in terms of $\{\zeta(6), \zeta(5), \zeta(4), \zeta(7), \zeta(2), \zeta(9)\}$ up to scalar multiples of $\zeta(11)$, the relation (14) can be translated into the fact that the vector $(28, 20, -42)$ lies in the left kernel of $B_5^{(1)}$. In general, the above argument proves the following statement.

**Proposition 11.** Let $k = 2K + 1$ be an odd integer. Let $B_k^{(1)}$ be the $(K - 2) \times (K - 2)$-minor of $B_K$ obtained by deleting the first two columns, the first and last row of $B_K$. Then the vectors obtained from the coefficients of $Z_{r,s}$ in the linear relations (5) and (6) belong to the left kernel of $B_k^{(1)}$.

Now let us see how to use our relation (14) to get Zagier’s relation

$$-6\zeta(10, 1) + 17\zeta(8, 3) + 13\zeta(6, 5) - 27\zeta(4, 7) + 3\zeta(2, 9) = 0. \quad (17)$$

Or more generally for any weight $k = 2K + 1 \geq 11$, let us see how to use our relations from Theorem 1 and Theorem 2 to get nontrivial elements in the left kernel of $B_K$ (i.e., kernel of $B_k'$).

We define the following canonical relations of double zeta values of odd weight.

**Definition 12.** For any odd integer $k \geq 5$, we call the following relation the canonical relation in weight $k$.

$$2(k - 2)\zeta(k - 1, 1) + \sum_{r+s=k \atop r \text{ even}} (r - s)\zeta(r, s) - (k - 2)\zeta(2, k - 2) = -\frac{3}{4}(k - 3)\zeta(k) \quad (18)$$

Since we have already known from (16) the expression of $\zeta(r, s)$ in terms of linear combination of products of single zeta values when $r$ is even, we can see that the above canonical relations are equalities by a direct computation using (16).

**Example 13.** The first few canonical relations in lower weights are listed below.

$$-\frac{3}{2}\zeta(5) = 6\zeta(4, 1) - 3\zeta(2, 3);$$

$$-3\zeta(7) = 10\zeta(6, 1) - \zeta(4, 3) - 5\zeta(2, 5);$$

$$-\frac{9}{2}\zeta(9) = 14\zeta(8, 1) + 3\zeta(6, 3) - \zeta(4, 5) - 7\zeta(2, 7);$$
In particular, in weight 11, we have both canonical relation in weight 11 and our relation

\[-6\zeta(11) = 18\zeta(10, 1) + 5\zeta(8, 3) + \zeta(6, 5) - 3\zeta(4, 7) - 9\zeta(2, 9);\]

\[-3\zeta(11) = 28\zeta(8, 3) + 20\zeta(6, 5) - 42\zeta(4, 7).\]

Now we can easily see that by cancelling \(\zeta(11)\) from the above two relations, we get exactly Zagier’s relation (17)

\[-6\zeta(10, 1) + 17\zeta(8, 3) + 13\zeta(6, 5) - 27\zeta(4, 7) + 3\zeta(2, 9) = 0.\]

In general, for any odd integer \(k = 2K + 1 \geq 11\), by cancelling \(\zeta(k)\) from both canonical relation of weight \(k\) and some weight \(k\) relation obtained from Theorem 1 or Theorem 2, we will get a nontrivial element in the left kernel of \(B_K\).

6. PROOF OF THEOREM 5

In this section, we will give the proof of Theorem 5. The examples about all restricted sum with \(d = 3, 4\) will be given at the end of the section.

**Proof of Theorem 5.** When \(d = 1\), the result is immediate from the following results due to Euler, Borwein, Bradley, and Crandall (cf., [1])

\[\sum_{r=2}^{k-1} \zeta(r, k-r) = \zeta(k), \quad \sum_{n=2}^{\infty} (\zeta(n) - 1) = 1.\]

Without loss of generality, we may assume that \(d \geq 2\). Since \(\lim_{k \to \infty} \zeta(k) = 1\), we only need to show the equality

\[\lim_{k \to \infty} \sum_{r=2}^{k-1} \zeta(r, k-r) = C_d^{(i)} := \sum_{j=2}^{\infty} (\zeta(j) - 1).\]

We can rewrite this limit as

\[\lim_{k \to \infty} \sum_{r=2}^{k-1} \left( \zeta(r, k-r) - (\zeta(r) - 1) \right) = 0. \tag{19}\]

Let \(s = k - r\). By the definition of double zeta values, we have

\[\zeta(r, s) - (\zeta(r) - 1) = \frac{1}{3^r} \left( \frac{1}{2^{2s}} \right) + \frac{1}{4^r} \left( \frac{1}{2^s} + \frac{1}{3^s} \right) + \cdots + \frac{1}{m^r} \left( \frac{1}{2^s} + \cdots + \frac{1}{(m-1)^s} \right) + \cdots\]
\[
\frac{1}{3^r}(\zeta(s) - 1) + \frac{1}{4^r}(\zeta(s) - 1) + \cdots + \frac{1}{m^r}(\zeta(s) - 1) + \cdots \\
< (\zeta(r) - 1)(\zeta(s) - 1).
\]

Let’s assume that \(i + Nd \leq k < i + (N + 1)d\) for some \(N\). Taking the sum over all qualifying \(r\) lying between 2 and \(k\), we get
\[
\sum_{r=2}^{k-1} \left( \zeta(r, k-r) - (\zeta(r) - 1) \right) < \sum_{2 \leq i + dj \leq k-2} \left( \zeta(i + dj) - 1 \right) \left( \zeta(k - i - dj) - 1 \right)
\]
\[
\leq (\zeta(2) - 1) \cdot (N + 1) \cdot \left( \zeta\left(\frac{k}{2}\right) - 1 \right)
\]
\[
\leq (\zeta(2) - 1) \cdot (N + 1) \cdot (\zeta(N) - 1)
\]
Since the right hand side goes to zero as \(N\) goes to infinity by the following comparison
\[
\lim_{N \to \infty} (N + 1)(\zeta(N) - 1) = \lim_{N \to \infty} N(\zeta(N) - 1)
\]
\[
= \lim_{N \to \infty} N \left( \frac{1}{2^N} + \frac{1}{3^N} + \frac{1}{4^N} + \cdots \right)
\]
\[
\leq \lim_{N \to \infty} N \left( \frac{1}{2^N} + \frac{1}{2^N} + \frac{1}{4^N} + \frac{1}{4^N} + \frac{1}{4^N} + \frac{1}{4^N} + \frac{1}{8^N} + \cdots \right)
\]
\[
= \lim_{N \to \infty} N \left( \frac{1}{2^{N-1}} + \frac{1}{4^{N-1}} + \frac{1}{8^{N-1}} + \cdots \right)
\]
\[
= \lim_{N \to \infty} \frac{N}{2^{N-1} - 1}
\]
\[
= 0,
\]
we have shown (19), i.e., we have proven Theorem 5. \(\square\)

**Example 14.** For \(d = 1, 2, 3,\) and \(4\), we have the following computations.

| \(d\) | \(i = 0\) | \(i = 1\) | \(i = 2\) | \(i = 3\) |
|-------|------------|------------|------------|------------|
| 1     | 1          |            |            |            |
| 2     | 0.75       | 0.25       |            |            |
| 3     | 0.22168939 \ldots | 0.09180726 \ldots | 0.68650334 \ldots |            |
| 4     | 0.08666297 \ldots | 0.03906700 \ldots | 0.66333702 \ldots | 0.21093299 \ldots |
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