SYMMETRIES OF CROSS CAPS

ATSUFUMI HONDA, KOSUKE NAOKAWA, KENTARO SAJI, MASAAKI UMEHARA, AND KOTARO YAMADA

Abstract. It is well-known that cross caps on surfaces in the Euclidean 3-space can be expressed in Bruce-West’s normal form, which is a special local coordinate system centered at the singular point. In this paper, we show a certain kind of uniqueness of such a coordinate system. In particular, the functions associated with this coordinate system produce new invariants on cross cap singular points. Using them, we characterize the possible symmetries on cross caps.

Introduction

Cross caps (which are also called Whitney’s umbrellas, see Figure 1, left) are the only singular points of stable maps of surfaces to 3-manifolds, and investigated by several geometers [1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 14].

Let \( f: \mathbb{R}^2 \to \mathbb{R}^3 \) be a \( C^\infty \)-map from a domain \( U \) in the Euclidean plane \( \mathbb{R}^2 \) into the Euclidean space \( \mathbb{R}^3 \) having a cross cap at \( p \in U \). Then there exists a local coordinate system \((u, v)\) centered at \( p \) satisfying \( f(v(0,0) = 0 \), where \( f_v := \frac{\partial f}{\partial v} \) and \( 0 := (0,0,0) \). Since the rank of the Jacobi matrix of \( f \) is one, we have \( f_u(0,0) \neq 0 \). By the well-known criterion for a cross cap by Whitney, \( \{f_u, f_{uv}, f_{vv}\} \) is linearly independent at \( (0,0) \).

Figure 1. The image of the standard cross cap (left) and the tangent and normal lines, and principal and normal planes of a cross cap with three symmetries with \( a_{20} > 0 \) (right)
Definition 0.1. We fix a cross cap $f : (U; u, v) \rightarrow \mathbb{R}^3$ satisfying $f_u(0, 0) = 0$. We call the line \[ \{ f(0, 0) + tf_u(0, 0) : t \in \mathbb{R} \} \] the tangent line at the cross cap. The plane in $\mathbb{R}^3$ passing through $f(0, 0)$ spanned by $f_u(0, 0)$ and $f_v(0, 0)$ is called the principal plane or the co-normal plane. On the other hand, the plane passing through $f(0, 0)$ perpendicular to the tangent line is called the normal plane (see Figure 1 right). Moreover, the line obtained as the intersection of the principal plane and the normal plane is called the normal line.

We note that the tangent line, the principal plane and the normal plane do not depend on the choice of admissible coordinate systems (cf. [10] and [9]).

A local coordinate system $(u, v) = (u(r_1, r_2), v(r_1, r_2))$ of $(\mathbb{R}^2; r_1, r_2)$ is called positive (resp. negative) if the Jacobian $u_{r_1}v_{r_2} - v_{r_1}u_{r_2}$ is positive (resp. negative). The following fact is known:

Fact 0.2 (cf. [2] and [15], see also [4] and [12]). Let $f : U \rightarrow \mathbb{R}^3$ be a $C^\infty$-map having a cross cap at $p \in U$. Then there exist a positive local coordinate system $(u, v)$ centered at $p$ and $C^\infty$-functions $a(u, v), b(v)$ satisfying

\begin{align*}
& b(0) = b'(0) = b''(0) = 0, \quad a(0, 0) = a_u(0, 0) = a_v(0, 0) = 0, \quad a_{vv}(0, 0) > 0 \\
& \text{such that}
\end{align*}

\begin{align*}
& f(u, v) = (u, uv + b(v), a(u, v))
\end{align*}

after composing an appropriate isometric motion in $\mathbb{R}^3$. In this expression, $e_1 := (1, 0, 0)$ points in the direction of the tangent line, $e_3 := (0, 0, 1)$ and $e_1$ span the principal plane, and $e_2 := (0, 1, 0)$ and $e_3$ span the normal plane.

The coordinate system $(u, v)$ in Fact 0.2 is called a Bruce-West coordinate system, and the equation (0.2) is called the normal form of a cross cap.

The geometric invariance of the coefficients of Maclaurin series of $a(u, v)$ and $b(v)$ has been pointed out and discussed in [4] [10] [12]. However, these functions are determined up to additions of flat functions in general, and there are no references which assert that the functions $a(u, v)$ and $b(v)$ themselves are geometric invariants as far as the authors know. The purpose of this paper is to prove the following local rigidity theorem of Bruce-West’s coordinates, which removes these ambiguities in $a(u, v)$ and $b(v)$ by flat functions:

Theorem A (Rigidity of Bruce-West’s coordinates). Let $(U_i; u_i, v_i) (i = 1, 2)$ be positive coordinate neighborhoods centered at $p_i$ in $\mathbb{R}^2$, and $f_i : U_i \rightarrow \mathbb{R}^3$ $C^\infty$-maps satisfying $f_i(0, 0) = 0$ each of which has a cross cap singular point at $p_i \in U_i$. Suppose that $f_i (i = 1, 2)$ are written in the following normal forms

\begin{align*}
& f_i(u_i, v_i) = (u_i, u_iv_i + b_i(v_i), a_i(u_i, v_i)) \quad (i = 1, 2)
\end{align*}

and $f_1(U_1) \subset f_2(U_2)$. Then there exist a pair of neighborhoods $V_i(\subset U_i)$ of $p_i (i = 1, 2)$ and an orientation preserving diffeomorphism $\varphi : V_1 \rightarrow V_2$ such that

\begin{align*}
& f_1 = f_2 \circ \varphi, \quad u_1 = u_2 \circ \varphi(u_1, v_1), \quad v_1 = v_2 \circ \varphi(u_1, v_1).
\end{align*}

As a consequence, there exists a positive number $\varepsilon$ such that

\begin{align*}
& a_1(u, v) = a_2(u, v), \quad b_1(v) = b_2(v)
\end{align*}

hold for $u, v$ satisfying $|u|, |v| < \varepsilon$.

Regarding the above results, we give the following definition:

Definition 0.3. We call $a(u, v)$ the first characteristic function and $b(v)$ the second characteristic function.

We also give the following definition:
Definition 0.4. Let $U_i$ ($i = 1, 2$) be a neighborhood of $p_i$ in $R^3$. For each $i = 1, 2$, let $f_i : U_i \to R^3$ be a $C^\infty$-map having a cross cap singular point at $p_i \in U_i$. We say that $f_2$ is congruent to $f_1$ as a map germ if there exist an isometry $T$ of $R^3$ and a diffeomorphism germ $\varphi$ satisfying $\varphi(p_1) = p_2$ such that $f_1 = T \circ f_2 \circ \varphi$ holds on a sufficiently small neighborhood of $p_1$. Moreover, if $T$ and $\varphi$ are both orientation preserving, then $f_2$ is said to be positively congruent to $f_1$.

As an application of Theorem A, we obtain the following:

Theorem B. The two characteristic functions $a(u, v)$ and $b(v)$ can be considered as geometric invariants of positive congruence classes on cross cap germs.

Moreover, as an application, we show the following three corollaries:

Corollary C-1. Let $f : U \to R^3$ be a $C^\infty$-map defined on an open subset $U$ of $R^2$ and $p \in U$ a cross cap singular point. Then the following three assertions are equivalent:

(i) The image germ of $f$ at $p$ is invariant under the reflection $T_1$ with respect to the principal plane.

(ii) There exist a connected open neighborhood $W(\subset U)$ of $p$ and an involutive orientation reversing $C^\infty$-diffeomorphism $\varphi : W \to W$ such that $f \circ \varphi = T_1 \circ f$ on $W$.

(iii) The characteristic functions satisfy $a(u, -v) = a(u, v)$ and $b(v) = -b(-v)$.

Corollary C-2. Let $f : U \to R^3$ be a $C^\infty$-map defined on an open subset $U$ of $R^2$ and $p \in U$ a cross cap singular point. Then the following three assertions are equivalent:

(i) The image germ of $f$ at $p$ is invariant under the reflection $T_2$ with respect to the normal plane.

(ii) There exist a connected open neighborhood $W(\subset U)$ of $p$ and an involutive orientation preserving $C^\infty$-diffeomorphism $\varphi : W \to W$ such that $f \circ \varphi = T_2 \circ f$ on $W$.

(iii) The characteristic functions satisfy $a(u, v) = a(-u, -v)$ and $b(v) = b(-v)$.

Corollary C-3. Let $f : U \to R^3$ be a $C^\infty$-map defined on an open subset $U$ of $R^2$ and $p \in U$ a cross cap singular point. Then the following three assertions are equivalent:

(i) The image germ of $f$ at $p$ is invariant under the $180^\circ$-rotation $T_3$ with respect to the normal line.

(ii) There exist a connected open neighborhood $W(\subset U)$ of $p$ and an involutive orientation reversing $C^\infty$-diffeomorphism $\varphi : W \to W$ such that $f \circ \varphi = T_3 \circ f$ on $W$.

(iii) The characteristic functions satisfy $a(-u, v) = a(u, v)$ and $b(v) = 0$.

Finally, we remark that Corollary C-j ($j = 1, 2, 3$) can be generalized for other space forms: We denote by $M^3(c)$ the simply connected complete Riemannian 3-manifold of constant sectional curvature $c$. For a germ of cross cap in $M^3(c)$ ($c \neq 0$), we can define the notions of tangent line, normal line principal plane and normal plane, like as in the case of the Euclidean 3-space. We then denote by $T_1$ (resp. $T_2$) the reflection with respect to the principal plane (resp. normal plane). On the other hand, let $T_3$ be the $180^\circ$-rotation with respect to the normal line. The following assertion holds:

Proposition D. Let $f : U \to M^3(c)$ be a $C^\infty$-map defined on an open subset $U$ of $R^2$ and $p \in U$ a cross cap singular point. Then the following three assertions are equivalent for each $j = 1, 2, 3$:

(i) The image germ of $f$ at $p$ is invariant under the isometry $T_j$ of $M^3(c)$.

(ii) There exist a connected open neighborhood $W(\subset U)$ of $p$ and a $C^\infty$-involution $\varphi : W \to W$ such that $f \circ \varphi = T_j \circ f$ on $W$.

(iii) The two characteristic functions $a(u, v)$ and $b(v)$ (cf. (2)] satisfy (iii) of Corollary C-j,
1. Proofs

We first prepare a definition and a lemma:

**Definition 1.1.** Let $f : U \to \mathbb{R}^3$ be a $C^\infty$-map with a cross cap singular point at $p \in U$ having a unit normal vector field $\nu$ of $f$ defined on $U \setminus \{p\}$. Then an open neighborhood $V(\subset U)$ of $p$ is said to be **admissible** if

1. the closure $\overline{V}$ of $f$ is compact and contained in $U$,
2. $f|_{\overline{V}} = \{p\}$ where $f|_V$ is the restriction of $f$ to the subset $V$,
3. the map $L := (f, \nu) : V \setminus \{p\} \to \mathbb{R}^3 \times S^2$ is an injective immersion, and
4. the set

\[
A := \left\{ q \in V \setminus \{p\} : \exists q' \in V \setminus \{p, q\} \text{ such that } f(q) = f(q') \right\} \cup \{p\}
\]

is the image of a regular curve in $V$ passing through $p$ such that $V \setminus A$ consists of two connected components.

**Lemma 1.2.** Let $f : U \to \mathbb{R}^3$ be a $C^\infty$-map with a cross cap singular point at $p \in U$. Then there exists an admissible neighborhood $V(\subset U)$ of $p$.

**Proof.** The standard cross cap $f_0 : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $f_0(u, v) := (u, uv, v^2)$ (see Figure 1 left). Since $f$ as a map germ at $p$ is right-left equivalent to the standard cross cap, there exist

- an open neighborhood $U_0(\subset \mathbb{R}^2)$ of the origin $(0, 0)$,
- a $C^\infty$-map $\varphi : U_0 \to \mathbb{R}^2$ giving a diffeomorphism between $U_0$ and $\varphi(U_0)(\subset U)$,
- an open neighborhood $\Omega(\subset \mathbb{R}^3)$ of $(0)\cdot (p)$ and
- a $C^\infty$-map $\Phi : (\Omega, f(p)) \to (\mathbb{R}^3, 0)$ giving a diffeomorphism between $\Omega$ and $\Phi(\Omega)$

such that $\Phi \circ f \circ \varphi = f_0$ holds on $U_0$. Then the closure of the open disk $D_r$ of radius $r$ centered at the origin is contained in $U_0$ for sufficiently small $r(>0)$. We fix such an $r$ and set $V_0 := D_r$. Since $f_0^{-1}(f_0(0, 0)) = \{(0, 0)\}$, the map $f$ satisfies (a1) and (a2). On the other hand, we consider the regular curve parametrizing the $v$-axis $\gamma_0(v) := \{(0, v) \in V_0; v \in \mathbb{R}\}$, whose image is the closure of the self-intersection set of $f_0$ in $V_0$, because $f_0(0, v) = f_0(0, -v)$. Moreover,

\[
\nu_0(u, v) := \frac{1}{\sqrt{u^2 + 4v^2 + 4v^4}}\ (2v^2, -2v, u) \quad ((u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\})
\]

is a unit normal vector field of $f_0$ on $\mathbb{R}^2 \setminus \{(0, 0)\}$. By computing $\nu_0$ at each point of $A_0(\subset \mathbb{R}^2)$, it can be easily observed that, at each point $q_0 \in A_0 \setminus \{(0, 0)\}$, two sheets of $f_0$ at $f_0(q_0)$ meet transversally. We set $V := \varphi(V_0)$. Since $\Phi \circ f \circ \varphi = f_0$ the set (as the image of the regular curve $\Phi \circ \gamma_0$) $A := \varphi(A_0)$ coincides with the set given in (1.1). So (A) satisfies (a4). Moreover, at each point $q \in A \setminus \{p\}$, two sheets of $f$ at $f(q)$ meet transversally, as well as $f_0$. So the map $L$ satisfies (a3).

**Corollary 1.3.** Let $f$ be as above, and let $V(\subset U)$ be an admissible neighborhood of $p$. If $D_\varepsilon(p)$ is the open disk of radius $\varepsilon(>0)$ centered at $p$ satisfying $D_\varepsilon(p) \subset V$, then $L = (f, \nu)$ gives a homeomorphism between $B := \overline{V} \setminus D_\varepsilon(p)$ and $L(B)$. Moreover, by setting $O := V \setminus \overline{D_\varepsilon(p)}$, the restriction $L|_O$ of $L$ to the open subset $O$ is an embedding.

**Proof.** Since $B$ is compact, $L|_O : B \to L(B)$ is a homeomorphism, because of the fact that a continuous map from a compact space to a Hausdorff space is a closed map. Since $O$ is a subset of $\overline{V} \setminus D_\varepsilon$, the map $L|_O$ gives a homeomorphism between $O$ and $L(O)$. Since $L$ is an injective immersion on $V$, we can conclude that $L|_O$ is an embedding.

We let $(U_i; u_i, v_i) \ (i = 1, 2)$ be a coordinate neighborhood centered at $p_i$ in $\mathbb{R}^2$ and $f_i : U_i \to \mathbb{R}^3$ a $C^\infty$-map satisfying $f_i(0, 0) = 0$ which has a cross cap singular point at $p_i \in U_i$. Suppose that $f_i \ (i = 1, 2)$ are written in the normal forms and $f_1(U_1)$ is a subset of $f_2(U_2)$. By Lemma 1.2 for each $i = 1, 2$, we can take an admissible neighborhood $V_i(\subset U_i)$ of $p_i$. Since the canonical form $f_0 : \mathbb{R}^2 \to \mathbb{R}^3$ is a proper map, by [11] Corollary 1.15, we may assume that
(f_i|_{V_i})^{-1}(f_i(p_i)) = \{p_i\} and f_i is V_i-proper in the sense of [11, Definition 1.1] for each i = 1, 2. Then, by [11, Theorem 1.16], the condition f_i(U_i) \subset f_2(U_2) implies that there exist relatively compact neighborhoods W_i(\subset V_i) (i = 1, 2) such that W_i \subset V_i and and f_i(W_i) \subset f_2(W_2).

Again, applying Lemma 1.2, we take an admissible neighborhood W_i of p_i such that \overline{W_i} \subset W_i.

Here, we replace W_1 by W_1 and W_2 by V_2, and then obtain admissible neighborhoods W_i (i = 1, 2) of p_i satisfying

\overline{W_1} \subset U_1, \quad \overline{W_2} \subset U_2, \quad f_i(W_i) \subset f_2(W_2).

For each i \in \{1, 2\}, we set

\begin{equation}
A_i := \{q \in W_i \setminus \{p_i\} : \exists q' \in W_i \setminus \{p_i, q\} \text{ such that } f_i(q) = f_i(q')\} \cup \{p_i\}.
\end{equation}

Since W_i (i = 1, 2) is admissible, (a4) in Definition 1.1 implies that W_i \setminus A_i consists of two connected components, denoted by W'_i and W''_i. Since f_1(W_1) \subset f_2(W_2), it is obvious that f_1(A_1) \subset f_2(A_2). So we may assume that f_1(W'_1) \subset f_2(W_2) and f_1(W''_1) \subset f_2(W''_2). We then replace by \nu_2 by \nu_2 if necessary, and may also assume that L_1(W'_1) \subset L_2(W_2) and L_1(W''_1) \subset L_2(W''_2).

(For example, for the standard cross cap f_0, \nu_0(u, 0) = (0, 0, 1) (resp. \nu_0(u, 0) = (0, 0, -1)) given in [12] is the inward (resp. outward) normal vector when u > 0 (resp. u < 0), see Figure 1 left.) So we have

\begin{equation}
L_1(W'_1) \subset L_2(W_2),
\end{equation}

where W'_1 := W_i \setminus \{p_i\}. Since L_2 is injective (cf. (a3)), the map \varphi := L_2^{-1} \circ L_1 from W'_1 to W''_1 is defined. The following assertion holds:

**Lemma 1.4.** The set \varphi(W'_1) is open in W'_2, and \varphi gives a diffeomorphism between W'_1 and \varphi(W'_1).

**Proof.** We let \varepsilon_i (i = 1, 2) be sufficiently small positive numbers. We set

\begin{equation*}
O_i := W_i \setminus \overline{D_{\varepsilon_i}(p_i)}, \quad B_i := \overline{W_i} \setminus \overline{D_{\varepsilon_i}(p_i)} \quad (i = 1, 2).
\end{equation*}

We can choose \varepsilon_2 > 0 so that

\begin{align*}
L_1(O_1) & \subset L_2(O_2), & L_1(B_1) & \subset L_2(B_2).
\end{align*}

In fact, suppose the first (resp. the second) equality fails for arbitrary \varepsilon_2(> 0). By regarding (1.3), there exist q_n \in O_1 (resp. q_n \in B_1) and q'_n \in D_{1/n}(p_2) (resp. q'_n \in \overline{D_{1/n}(p_2)}) such that L_1(q_n) = L_2(q'_n). Since B_1 is compact, we can take a convergent subsequence \{q_{n_i}\}_{i=1}^{\infty} converging to a point q_{\infty} \in B_1. Since \lim_{n \to \infty} q'_n = p_2, we have that

\begin{equation*}
f_1(q_{\infty}) = \lim_{n \to \infty} f_1(q_{n_i}) = \lim_{n \to \infty} f_2(q'_n) = 0,
\end{equation*}

and the fact \begin{align*} f_i^{-1}(f_i(p_i)) = \{p_i\} \implies q_{\infty} = p_i, \text{ which contradicts that } q_{\infty} \in B_1. \text{ Thus, } L_1(O_1) \subset L_2(O_2) \text{ (resp. } L_1(B_1) \subset L_2(B_2)) \text{ holds for some } \varepsilon_2.
\end{align*}

By the first assertion of Corollary 1.3 each L_i|_{B_i} (i = 1, 2) gives a homeomorphism between B_i and L_i(B_i), and so, \varphi|_{B_1} = L_2^{-1} \circ L_1 is an injective continuous map from the compact set B_1 to the Hausdorff space B_2. So \varphi|_{B_1} gives a homeomorphism between B_1 and \varphi(B_1).

Since O_1 is an open subset of \mathbb{R}^2, by the invariance of domain, O'_1 := \varphi(O_1) is also an open subset of O_2. Thus, \varphi gives a homeomorphism between O_1 and O'_1. By the second assertion of Corollary 1.3 L_i|_{O_i} (i = 1, 2) are embeddings, and so, \varphi|_{O_1} gives a diffeomorphism between O_1 and O'_1. Since \varepsilon_1(> 0) can be arbitrarily chosen, we can conclude that \varphi(W'_1) is an open subset of W_2, and \varphi is a diffeomorphism between W'_1 and \varphi(W'_1). \hfill \Box

Under the preparation above, we prove Theorem A:
Proof of Theorem A. Let \( \{q_n\}_{n=1}^{\infty} \) be a sequence in \( W_1 \setminus \{p_1\} \) converging to \( p_1 \in W_1 \). Then \( Q_n := f_1(q_n) \) converges to the origin \( 0 \) by the continuity of \( f_1 \). Since \( f_1(W_1) \subset f_2(W_2) \), we can take a point \( q'_n \) on \( W_2(\subset W_2) \) such that \( f_2(q'_n) = Q_n \). Since \( W_2 \) is compact, \( \{q'_n\} \) has an accumulation point \( q'_{\infty} \in W_2 \), and the continuity of \( f_2 \) yields

\[
0 = \lim_{n \to \infty} Q_n = \lim_{n \to \infty} f_2(q'_n) = f_2(q'_{\infty}).
\]

By (a2), we have \( q'_{\infty} = p_2 \), which implies that \( \varphi \) can be uniquely extended to a continuous map \( \varphi : W_1 \to \varphi(W_1) \) satisfying

\[
(1.5) \quad \varphi(p_1) = p_2.
\]

So \( f_1 = f_2 \circ \varphi \) holds on \( W_1 \). Since \((u_i, v_i) (i = 1, 2)\) are positive coordinate systems, \( \varphi \) is orientation preserving. We set

\[
u := u_1, \quad v := v_1
\]

and

\[
x(u,v) := u_2 \circ \varphi(u,v), \quad y(u,v) := v_2 \circ \varphi(u,v).
\]

Since \((u_i, v_i) (i = 1, 2)\) are both Bruce-West’s coordinates, \( f_1 = f_2 \circ \varphi \) implies that

\[
(1.6) \quad x(u,v) = u,
\]

\[
(1.7) \quad x(u,v)y(u,v) + b_2(y(u,v)) = uv + b_1(v)
\]

hold for \((u,v) \in W_1\). To obtain the conclusion, it is sufficient to show that \( y(u,v) = v \) on a certain open neighborhood \( \Omega(\subset W_1) \) of \((0,0)\). (In fact, suppose that it is true. By setting \( \psi := (u_2, v_2) \), the map \( \psi \circ \varphi \) coincides with the identity map on \( W_1 \). Since \( \psi \) is a local coordinate system of \( W_2 \), it is a local diffeomorphism, and so, \( \varphi(= \psi^{-1}) \) is the desired local diffeomorphism at the origin.

By (1.6) and (1.7), we have

\[
(1.8) \quad uy(u,v) + b_2(y(u,v)) = uv + b_1(v).
\]

We set

\[
b_1(v) := v^3 \beta_1(v), \quad b_2(y) := y^3 \beta_2(y).
\]

It is well-known that, for any \( C^\infty \)-function \( A(x) \) of one variable defined on an open interval containing \( x = 0 \), there exists a \( C^\infty \)-function \( B(x,y) \) defined on a neighborhood of \((0,0)\) satisfying \( A(y) - A(x) = (y-x)B(x,y) \) on the neighborhood. So there exist \( C^\infty \)-functions \( \xi(w_1, w_2) \) and \( \eta(w_1, w_2) \) defined on a neighborhood of \((w_1, w_2) = (0,0)\) such that

\[
(1.9) \quad b_2(w_2) - b_2(w_1) = (w_2 - w_1)(\xi(w_1, w_2), \quad \beta_2(w_2) - \beta_2(w_1) = (w_2 - w_1)\eta(w_1, w_2).
\]

We set

\[
(1.10) \quad \zeta(u,v) := \xi(v, y(u,v)), \quad G(u,v) := u + \zeta(u,v),
\]

which are defined on a rectangular domain

\[
D := I \times J(\subset W_1)
\]

containing \((0,0)\), where \( I \) and \( J \) are open intervals containing the origin \( 0 \in R \). By (1.8), we have

\[
(1.11) \quad b_2(y(u,v)) - b_1(v) = -u(y(u,v) - v).
\]

On the other hand, we have that

\[
(1.12) \quad (v - y(u,v))G(u,v) = uv - uy(u,v) + (v - y(u,v))\xi(v, y(u,v))
\]

\[
= \left( b_2(y(u,v)) - b_1(v) \right) + \left( b_2(v) - b_2(y(u,v)) \right)
\]

\[
= b_2(v) - b_1(v).
\]

By (1.11) and (1.12), we have

\[
(1.13) \quad (y(u,v) - v)\zeta(u,v) = b_2(y(u,v)) - b_2(v),
\]
Substituting $u$ and so
\[
(y(u,v) - v)\zeta(u,v) = b_2(y(u,v)) - b_2(v)
\]
\[
= y(u,v)^3b_2(y(u,v)) - v^3b_2(v)
\]
\[
= y(u,v)^3(\beta_2(y(u,v)) - \beta_2(v)) + \beta_2(v)(y(u,v)^3 - v^3)
\]
on $D$. So, under the assumption that $y(u,v) \neq v$, we have
\[
\zeta(u,v) = y(u,v)^3\eta(v,y(u,v)) + \beta_2(v)(y(u,v)^3 - v^3)
\]
(1.14)

We now suppose that Theorem A fails. Then there exists a sequence \((u_n,v_n)\) on $D$ satisfying $y(u_n,v_n) \neq v_n$ and $\lim_{n \to \infty} (u_n,v_n) = (0,0)$. Since $y(0,0) = 0$ (cf. (1.5)), the equality (1.14) implies that
\[
\lim_{n \to \infty} \zeta(u_n,v_n) = 0.
\]
So we may assume that $G_n(u_n,v_n) \neq 0$ holds for sufficiently large $n$. We fix such an $n$. Since $(u_n,v_n) \neq (0,0)$, by the implicit function theorem, there exist a neighborhood $O(\subset D)$ of $(u_n,v_n)$ and a $C^\infty$-function $u := u(v)$ defined on an open interval $J_n$ containing $v_n$ such that $u(v_n) = u_n$ and
\[
Z := \{(u,v) ; v \in J_n\} = \{(u,v) \in O ; G(u,v) = 0\}.
\]
Substituting $u = u(v)$ to (1.12), we have
\[
0 = (v - y(u(v),v))G(u(v),v) = b_2(v) - b_1(v),
\]
and so $b_1 = b_2$ holds on $J_n$. Since $G(u,v) \neq 0$ on $O \backslash Z$, (1.12) implies that $v = y(u,v) = 0$.

Then, by the continuity of $y$, we have
\[
v - y(u,v) = 0 \quad ((u,v) \in O),
\]
contradicting the fact $y(u_n,v_n) \neq v_n$. \qed

Proof of Theorem B. Let $f_i$ $(i = 1,2)$ be two $C^\infty$-maps defined on a neighborhood $U$ of the origin $(0,0) \in \mathbb{R}^2$ each of which has a cross cap singular point at $(0,0)$. We are thinking that $f_i$ $(i = 1,2)$ are map germs of cross cap singularities, and assume that $f_2$ is congruent to $f_1$ as a map germ (cf. Definition 4.1). Then there exist an orientation preserving isometry $T$ of $\mathbb{R}^3$ and an orientation preserving local diffeomorphism $\varphi$ such that $f_2 = T \circ f_1 \circ \varphi$. Without loss of generality, we may assume that $f_1$ and $f_2$ are written in normal forms centered at $(0,0)$, that is, they are expressed as (1.3) by setting $u_1 = u_2$ and $v_1 = v_2$.

We may regard that each point of $\mathbb{R}^3$ consists of column vectors. Then $T$ can be identified with an orthogonal matrix by multiplying them from the left. Since $T$ maps the tangent line and the principal and normal planes of $f$ to those of $g$, it is written in the form
\[
T = \begin{pmatrix}
\varepsilon_1 & 0 & 0 \\
0 & \varepsilon_2 & 0 \\
0 & 0 & \varepsilon_3
\end{pmatrix},
\]
where $\varepsilon_j \in \{1,-1\}$ for $j = 1,2,3$. By the condition $a_{02} > 0$, we have $\varepsilon_3 = 1$. So the possibility of $T$ is
\[
T_0 := \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix}, \quad T_1 := \begin{pmatrix} 1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1 \end{pmatrix}, \quad T_2 := \begin{pmatrix} -1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix}, \quad T_3 := \begin{pmatrix} -1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1 \end{pmatrix}.
\]
Suppose that $T = T_0$ (resp. $T = T_1$). Then we have
\[
T_0 \circ f(u,v) = (u,uv + b_1(v),a_1(u,v))
\]
(1.15)
\[
\text{resp. } T_1 \circ f(u,-v) = (u,uv - b_1(-v),a_1(u,-v))
\]
(1.16)
Comparing this with
\[ f_2(u, v) = (u, uv + b_2(v), a_2(u, v)), \]
we obtain (cf. Theorem A)
\[ \varphi(u, v) = (u, v), \quad b_2(v) = b_1(v), \quad a_2(u, v) = a_1(u, v) \]
\[ \text{(resp. } \varphi(u, v) = (u, -v), \quad b_2(v) = -b_1(-v), \quad a_2(u, v) = a_1(u, -v)) \].

We next suppose that \( T = T_2 \) (or \( T = T_3 \)). Then we have
\[ T_2 \circ f(-u, -v) = (u, uv + b_1(-v), a_1(-u, -v)) \]
\[ \text{(resp. } T_3 \circ f(-u, -v) = (u, uv - b_1(v), a_1(-u, v))) \].

Again, comparing this with \( f_1 \), we obtain
\[ \varphi(u, v) = (-u, -v), \quad b_2(v) = b_1(-v), \quad a_2(u, v) = a_1(-u, -v) \]
\[ \text{(resp. } \varphi(u, v) = (-u, v), \quad b_2(v) = -b_1(v), \quad a_2(u, v) = a_1(-u, v))) \].

If \( g \) is positively congruent to \( f \), then \( T \) and \( \varphi \) must be orientation preserving maps. Since \( T_1, T_2 \) are orientation reversing, we have \( T = T_0 \) or \( T_2 \). However, when \( T = T_3 \), we have seen that \( \varphi(u, v) = (-u, v) \), that is, \( \varphi \) is orientation reversing. So only the possibility is the case that \( T \) and \( \varphi \) are the identity maps. So we obtain \( (1.18) \), proving the assertion.

**Proof of Corollaries C-1, C-2, C-3.** In the above proof of Theorem B, we may set \( f := f_1 = f_2 \). Then, it can be directly observed that

1. \( T_1 \) is the reflection with respect to the principal plane of \( f \),
2. \( T_2 \) is the reflection with respect to the normal plane,
3. \( T_3 \) is the \( 180^\circ \)-rotation with respect to the normal line.

Regarding this, we first show that (i) implies (ii). So we suppose that (i) of Corollaries C-\( j \) \((j = 1, 2, 3)\) hold. Then \( T_j(f(V)) \subset f(U) \) holds for a sufficiently small neighborhood \( V(\subset U) \) of \( p \). By Fact 1.1 and Theorem A, \( T_j \circ f \circ \varphi_j = f \) \((j = 1, 2)\) holds for a local orientation preserving diffeomorphism \( \varphi_j \) on \( \mathbb{R}^2 \). Since \( T_j \) is an involution, so is \( \varphi_j \) are involutions. Thus, we obtain (ii) of Corollary C-\( j \).

If \( f \) has the symmetry \( T_1 \), then \( (1.19) \) implies
\[ -b(-v) = b(v), \quad a(u, -v) = a(u, v). \]

Similarly, if \( f \) has the symmetry \( T_2 \) (resp. \( T_3 \)), then \( (1.22) \) (resp. \( (1.23) \)) implies
\[ b(v) = b(-v), \quad a(u, v) = a(-u, v) \]
\[ \text{(resp. } b(v) = 0, \quad a(u, v) = a(-u, v)). \]

So we get the relations of \( a \) and \( b \) as in the statements. Conversely, if \( a \) and \( b \) satisfy (iii) of Corollary C-\( j \), then \( f \) satisfies (1) obviously.

**Example 1.5.** Consider a family of cross caps \( f(u, v) := (u, uv + v^3, cu^2 + v^2) \), where \( c \) is a real number. In this example, the characteristic functions are given by
\[ b(v) = v^3, \quad a(u, v) = cu^2 + v^2. \]

Since \( T_1 \circ f(u, v) = f(u, v) \), this example is the case of Corollary C-1. In fact, \( b(v) \) is an odd function and \( v \mapsto a(u, v) \) is an even function.

**Example 1.6.** Consider a family of cross caps \( f(u, v) := (u, uv + v^4, cu^2 + v^2) \), where \( c \) is a real number. In this example, the characteristic functions are given by
\[ b(v) = v^4, \quad a(u, v) = cu^2 + v^2. \]
Since $T_2 \circ f(-u, -v) = f(u, v)$, this example is the case of Corollary C-2. In fact, $b(v)$ and $a(u, v) = a(-u, -v)$ are even functions.

**Example 1.7.** Consider a family of cross caps $f(u, v) := (u, uv, cu^2 + v^2)$, where $c$ is a real number. The characteristic functions are given by

$$b(v) = 0, \quad a(u, v) = cu^2 + v^2.$$  

Since $T_3 \circ f(-u, -v) = f(u, v)$, this example is the case of Corollary C-3. In fact, $b(v)$ vanishes identically and $a(u, v) = a(-u, v)$ holds. Since $a(u, v)$ is an even function with respect to $u$ and $v$, this example has the property that

$$T_1 \circ f(-u, -v) = f(u, v), \quad T_2 \circ f(-u, -v) = f(u, v),$$

that is, the image of $f$ is invariant under the three isometries $T_1$, $T_2$, and $T_3$.

**Proof of Proposition D.** We fix an orientable Riemannian 3-manifold $(M^3, g)$ with a fixed point $o \in M^3$. Let $\{e_1, e_2, e_3\}$ be an oriented orthonormal basis of the tangent space $T_oM^3$ of $M^3$ at $o$, and let $(x_1, x_2, x_3)$ be a geodesic normal coordinate system of $M^3$ at $o$ induced by the exponential map at $o$. We let $f : U \to M^3$ be a $C^\infty$-map which has a cross cap singular point $p \in U$ satisfying $f(p) = o$. Then by a suitable choice of $\{e_1, e_2, e_3\}$ and by a choice of an oriented local coordinate system $(u, v)$, we may assume that $f(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v))$ is written as

$$x_1(u, v) = u, \quad x_2(u, v) = uv + b(v), \quad x_3(u, v) = a(u, v).$$

We then consider the case that $(M^3, g)$ is $M^3(c)$ with constant curvature $c \in \mathbb{R})$. Regarding $T_oM^3(c)$ as a column vector space, each isometry of $M^3(c)$ fixing $o$ corresponds to a left-multiplication of a $3 \times 3$ orthogonal matrix. Moreover, the left-multiplications of the orthogonal matrices $T_i$ $(i = 1, 2, 3)$ on the coordinate system $(x_1, x_2, x_3)$ give isometric motions of $M^3(c)$ fixing $f(p)$. Furthermore, the matrices $T_1$, $T_2$ and $T_3$ satisfy the conditions $(t_1)$, $(t_2)$ and $(t_3)$, respectively. Thus the assertions of Proposition D can be proved by modifying the proofs of Corollaries C-1, C-2, C-3.

**Remark 1.8.** Even when $(M^3, g)$ is an arbitrarily given Riemannian 3-manifold, the functions $a(u, v)$ and $b(v)$ given in (1.24) can be considered as geometric invariants for cross caps. (In fact, the choice of $\{e_1, e_2, e_3\}$ corresponds to the coordinate changes of normal coordinates $(x_1, x_2, x_3)$ via a left-multiplication of the orthogonal matrix with determinant 1. Thus, $a(u, v)$ and $b(v)$ are determined independently of a choice of the local coordinates $(u, v)$ and of a choice of $\{e_1, e_2, e_3\}$ giving the expression (1.24) for $f$ as a consequence of Theorem A.)

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**References**

[1] J. W. Bruce and F. Tari, *On binary differential equations*, Nonlinearity **8** (1995), 255–271.

[2] J. W. Bruce and J. M. West, *Functions on a crosscap*, Math. Proc. Cambridge Philos. Soc. **123** (1998), 19–39.

[3] F. S. Dias and F. Tari, *On the geometry of the cross-cap in the Minkoswki 3-space and binary differential equations*, Tohoku Math. J. (2) **68** (2016), 293–328.

[4] T. Fukui, M. Hasegawa, *Fronts of Whitney umbrella – differential geometric approach via blowing up*, J. Singul. **4** (2012), 35–67.

[5] T. Fukui and J. J. Nuno Ballesteros, *Isolated roundings and flattennings of submanifolds in Euclidian paces*, Tohoku Math. J. **57** (2005), 469–503.

[6] R. Garcia, C. Gutierrez and J. Sotomayor, *Lines of Principal curvature around umbilics and Whitney umbrellas*, Tohoku Math. J. **52** (2000), 163–172.

[7] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Mathematics, **14** Springer-Verlag, 1973.

[8] C. Gutierrez and J. Sotomayor, *Lines of principal curvature for mappings with Whitney umbrella singularities*, Tohoku Math. J. **38** (1986), 551–559.

[9] M. Hasegawa, A. Honda, K. Naokawa, K. Saji, M. Umehara and K. Yamada, *Intrinsic properties of surfaces with singularities*, Int. J. Math. **26** (2015), 1540008.
[10] M. Hasegawa, A. Honda, K. Naokawa, M. Umehara and K. Yamada, *Intrinsic invariants of cross caps*, Selecta Math. New Ser. **20** (2014), 769–785.

[11] A. Honda, K. Naokawa, K. Saji, M. Umehara and K. Yamada, *A generalization of Zakalyukin's lemma, and symmetries of surface singularities*, to appear in J. Singul. (arXiv:2104.08505).

[12] A. Honda, K. Naokawa, M. Umehara, K. Yamada, *Isometric realization of cross caps as formal power series and its applications*, Hokkaido Math. J. **48** (2019), 1–44.

[13] J. M. Oliver, *On pairs of foliations of a parabolic cross-cap*, Qual. Theory Dyn. Syst. **10** (2011), 139–166.

[14] F. Tari, *Pairs of geometric foliations on a cross-cap*, Tohoku Math. J. **59** (2007), 233–258.

[15] J. West, *The differential geometry of the cross-cap*, Ph. D. thesis, University of Liverpool (1995)

(Atsufumi Honda) Department of Applied Mathematics, Faculty of Engineering, Yokohama National University, 79-5 Tokiwadai, Hodogaya, Yokohama 240-8501, Japan

Email address: honda-atsufumi-kp@ynu.ac.jp

(Kosuke Naokawa) Department of Computer Science, Faculty of Applied Information Science, Hiroshima Institute of Technology, 2-1-1 Miyake, Saeki, Hiroshima 731-5193, Japan

Email address: k.naokawa.ec@cc.it-hiroshima.ac.jp

(Kentaro Saji) Department of Mathematics, Faculty of Science, Kobe University, Rokko, Kobe 657-8501

Email address: saji@math.kobe-u.ac.jp

(Masaaki Umehara) Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo 152-8552, Japan

Email address: umehara@is.titech.ac.jp

(Kotaro Yamada) Department of Mathematics, Tokyo Institute of Technology, Tokyo 152-8551, Japan

Email address: kotaro@math.titech.ac.jp