\( \mathcal{PT} \)-symmetric quantum systems with positive \( \mathcal{P} \)

Miloslav Znojil  
Nuclear Physics Institute ASCR,  
250 68 Řež,  
Czech Republic*  
and  
Hendrik B. Geyer  
Institute of Theoretical Physics, University of Stellenbosch and  
Stellenbosch Institute for Advanced Study,  
7600 Stellenbosch,  
South Africa†

December 21, 2013

* e-mail: znojil@ujf.cas.cz  
† e-mail: hbg@sun.ac.za
Abstract

A new version of $\mathcal{PT}$-symmetric quantum theory is proposed and illustrated by an $N$-site-lattice Legendre oscillator. The essence of the innovation lies in the replacement of parity $\mathcal{P}$ (serving as an indefinite metric in an auxiliary Krein space $\mathcal{K}_P$) by its non-involutory alternative $\mathcal{P}^{(\text{positive})} := Q > 0$ playing the role of a positive-definite nontrivial metric $Q \neq I$ in an auxiliary, redundant, unphysical Hilbert space $\mathcal{K}_Q$. It is shown that the $\mathcal{P}^{(\text{positive})} \mathcal{T}$-symmetry of this form remains appealing and technically useful.

\textit{PACS}: 03.65.-w; 03.65.Ca; 03.65.Ta
\textit{Keywords}: PT-symmetry; Hilbert-space metric; positive $\mathcal{P} := Q$; Legendre oscillator; quasi-Hermitian observables.
1 The formalism

The operators of parity $\mathcal{P}$ and charge $\mathcal{C}$ entering the product

$$\Theta^{(PT)} = \mathcal{P}\mathcal{C}$$

play a key role in the $\mathcal{PT}$-symmetric quantum theory (PTQT) of Bender et al [1]. The formalism may briefly be characterized as a specific implementation of the standard quantum theory in which the physical Hilbert space $\mathcal{H}^{(PT)}$ of states is assumed Hamiltonian-dependent in the sense that the usual “Dirac” (i.e., “friendly but false” [2]) inner product $\langle \eta|\Phi \rangle^{(F)}$ of elements $|\eta\rangle$ and $|\Phi\rangle$ is replaced by the Hamiltonian-adapted [3] inner product

$$\langle \eta|\Phi \rangle^{(PT)} := \langle \eta|\Theta^{(PT)}|\Phi \rangle^{(F)}$$

which, under certain assumptions [4], is unique.

In applications the systematically developed theory starts from an arbitrary Hamiltonian $H$ with real spectrum which satisfies the $\mathcal{PT}$-symmetry requirement

$$\mathcal{PT}H = H\mathcal{PT}$$

where the anti-linear operator $\mathcal{T}$ mediates the reversal of time [5]. The model is given its full physical content by a self-consistent construction of a second observable, viz., of the operator of charge $\mathcal{C} = \mathcal{C}^{(PT)}(H)$ [3, 6].

The chosen or “input” Hamiltonian $H$ is allowed to be non-Hermitian in the Dirac sense, $H \neq H^\dagger$. Bender and Boettcher’s [5] $\mathcal{PT}$-symmetric oscillator $H^{(BB)} = p^2 + gx^2$ with $g = g(x) = (ix)^\delta$ and $\delta > 0$ serves as an often quoted example. Still, characterizing the PTQT models primarily as non-Hermitian would be misleading. Indeed, these models are strictly Hermitian in the physical Hilbert space $\mathcal{H}^{(PT)}$ while only appearing to be non-Hermitian in the (irrelevant and manifestly unphysical, auxiliary) representation space $\mathcal{H}^{(F)}$ endowed, by definition, with the trivial, “friendly but false” Dirac metric $\Theta^{(Dirac)} = I \neq \Theta^{(PT)}$.

The overall methodical framework remains unchanged when we replace the involutory operator of parity (with the property $\mathcal{P}^2 = I$) by another indefinite (and invertible) operator $\mathcal{P}$. In the language of mathematics the correspondingly generalized $\mathcal{PT}$-symmetry requirement [3] still remains practically useful and mathematically tractable as equivalent to the so called pseudo-hermiticity of $H$ [7] or to the Hermiticity of $H$ in another auxiliary space, viz., in a Krein space $\mathcal{K}^\mathcal{P}$ endowed with the metric $\mathcal{P}$ (often called pseudo-metric) which is, by definition, indefinite [1, 8].

The parity-resembling pseudo-metric $\mathcal{P}$ may possess just a finite number $k > 0$ of anomalous (i.e., say, negative) eigenvalues. Even then, the validity of Eq. (3)
entitles us to speak about a PTQT model and, in a purely formal sense, about the Hermiticity of $H$ in the auxiliary Krein (or, at finite $k$, Pontryagin) space $K^P$.

In our present note we shall demonstrate specific methodical as well as phenomenological appeal of the extreme (and, as it seems, not yet contemplated) choice of the non-involutionary invertible operator $P$ which proves, in addition, positive definite (mimicking the extreme choice of $k = 0$). Of course, once we replace the symbol $P$ (reserved, conventionally, for the indefinite $k > 0$ pseudometrics) by $Q$ (the symbol characterizing a positive definite operator), our auxiliary space $K^Q$ will cease to be a Krein (or Pontryagin) space, re-acquiring the perceivably less complicated Hilbert-space mathematical status.

In the PTQT spirit, the physical status is expected to be re-acquired by some other, ad hoc Hilbert space $\mathcal{H}^{(PT)}$ (or rather $\mathcal{H}^{(QT)}$, to be specified below). In other words, we shall describe here just a version of the PTQT model-building scheme in which

[p1] the “input” operator $P^{(positive)} = Q$ will play the role of a positive-definite metric determining an auxiliary, unphysical Hilbert space $K^Q$;

[p2] the “input” Hamiltonian $H$ will be assumed $QT$-symmetric, i.e., compatible with Eq. (3) where $P$ is replaced by $Q$;

[p3] we shall follow Ref. [7] and reinterpret Eq. (1) as a mere definition of the operator $C^{(QT)} = Q^{-1}\Theta^{(QT)}$, with a true physical metric $\Theta^{(QT)} \neq Q$ yet to be specified;

[p4] finally, we shall recall Ref. [3] and broaden the class of admissible observables yielding, typically, a weakening of the physical role played by $C^{(QT)}$ itself.

2 An exactly solvable illustrative example

A solvable example with properties [p1] -[p4] is derived here from the recurrences which are satisfied by the Legendre orthogonal polynomials $P_n(x)$. These recurrences may formally be rewritten as an infinite linear system $H^{(\infty)}|\psi_n\rangle = E_n|\psi_n\rangle$ satisfied by the column vector-like array

$$|\psi_n\rangle = \begin{pmatrix} \langle\psi_n\rangle_1 \\ \langle\psi_n\rangle_2 \\ \langle\psi_n\rangle_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} P_0(E_n) \\ P_1(E_n) \\ P_2(E_n) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ E_n \\ \frac{1}{2}(3E_n^2 - 1) \\ \vdots \end{pmatrix}$$
at an arbitrary real parameter $E_n$ (the range of which is usually restricted to the interval $(-1,1)$). The system’s real matrix

$$H^{(\infty)} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & \ldots \\
1/3 & 0 & 2/3 & 0 & 0 & \ldots \\
0 & 2/5 & 0 & 3/5 & 0 & \ldots \\
0 & 0 & 3/7 & 0 & 4/7 & \ldots \\
0 & 0 & 0 & 4/9 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{bmatrix}$$

is asymmetric and, therefore, it cannot be interpreted as Hermitian in the Dirac sense.

### 2.1 Requirements [p1] and [p2]

The infinite system of equations $H^{(\infty)}|\psi_n\rangle = E_n|\psi_n\rangle$ is not suitable for our present purposes because all of its formal “eigenvectors” $|\psi_n\rangle$ would have an infinite Dirac norm. This can be deduced from the completeness of the Legendre-polynomial basis in $L^2(-1,1)$. Thus, we have to turn attention to a truncated, finite-dimensional version of our linear system.

Firstly we find that using the $N$ by $N$ truncated submatrices $H^{(N)}$ of $H^{(\infty)}$ at any $N = 1, 2, \ldots$, the corresponding Schrödinger-like equation

$$H^{(N)}|\psi^{(N)}_n\rangle = E^{(N)}_n|\psi^{(N)}_n\rangle, \quad n = 0, 1, \ldots, N-1$$

may only be satisfied by the truncated vector (2) if we manage to guarantee that the next-neighbor vector element vanishes,

$$(|\psi_n\rangle)_{N+1} = P_N(E_n) = 0.$$  \hspace{1cm} (5)

*Vice versa*, the $N$-plet of roots of the latter polynomial equation is nondegenerate and real so that it strictly coincides with the spectrum of $H^{(N)}$. The eigenvalue part of the problem is thus settled.

Secondly, once we eliminate $T$ from Eq. (3) and replace $P$ by $Q$ yielding

$$[H^{(N)}]^\dagger Q = QH^{(N)}$$

we may conclude that all of our Hamiltonians $H^{(N)}$ are $Q^{-T}$-symmetric (i.e., compatible with Eq. (6)), provided only that the specific choice is made of the diagonal and positive definite matrix $Q = Q^{(N)}$ with the truncation-independent matrix elements

$$[Q]_{11} = \frac{1}{2}, \quad [Q]_{22} = \frac{3}{2}, \quad \ldots, \quad [Q]_{NN} = \frac{N-1/2}{(N-1)!}.$$  \hspace{1cm} (7)
At any fixed $N$ this matrix is positive definite and may be interpreted as a metric in Hilbert space, therefore. Once this metric is used to define the inner products, the resulting Hilbert space will be treated here as a direct parallel of the auxiliary Krein space $\mathcal{K}^P$ of PTQT. This parallelism will be underlined not only by the notation (our Hilbert space with the inner products defined via matrices (7) will be denoted by the symbol $\mathcal{K}^Q$) but also by the interpretation (preserving the analogy, also our auxiliary Hilbert space $\mathcal{K}^Q$ will be assumed and declared *unphysical*).

### 2.2 Requirements [p3] and [p4]

We assume that the time evolution of our quantum system *in spe* (more precisely, of all of its eligible finite-dimensional ket vectors $|\psi^{(N)}\rangle$) is generated by the Hamiltonian, the manifest non-Hermiticity $H^{(N)} \neq [H^{(N)}]^\dagger$ of which forces us to complement Eq. (4), for the sake of mathematical completeness, by the second, isospectral Schrödinger equation

$$\left[H^{(N)}\right]^\dagger |\psi_n^{(N)}\rangle = E_n^{(N)} |\psi_n^{(N)}\rangle, \quad n = 0, 1, \ldots, N - 1. \quad (8)$$

Its “ketket” solutions $|\psi_n^{(N)}\rangle$ are in general different from the kets $|\psi_n^{(N)}\rangle$ of Eq. (4). Nevertheless, the $QT-$symmetry requirement (6) may be recalled to simplify the second Schrödinger Eq. (8),

$$H^{(N)} \left[Q^{-1}|\psi_n^{(N)}\rangle\right] = E_n^{(N)} \left[Q^{-1}|\psi_n^{(N)}\rangle\right], \quad n = 0, 1, \ldots, N - 1. \quad (9)$$

Due to the non-degenerate nature of the spectrum, the bracketed vectors $\left[Q^{-1}|\psi_n^{(N)}\rangle\right]$ must be proportional, at every subscript, to their predecessors given by the first Schrödinger Eq. (4).

We are free to choose the proportionality constants equal to one. In other words, the second Schrödinger equation (8) may be declared solved by ketkets

$$|\psi_n^{(N)}\rangle = Q|\psi_n^{(N)}\rangle, \quad n = 0, 1, \ldots, N - 1. \quad (10)$$

This is a useful convention because now, the *complete* set of the eligible PTQT (or, rather, QTQT) metrics may be defined by Mostafazadeh’s expression [11, 12]

$$\Theta^{QT} (H^{(N)}) = \sum_{j=0}^{N-1} |\psi_j^{(N)}\rangle \kappa_j^{QT} \langle \psi_j^{(N)}| \quad (11)$$

As long as we work with the finite dimensions $N < \infty$, our choice of the $N-$plet $\kappa^{QT}$ of the optional, variable parameters $\kappa_j^{QT} > 0$ is entirely arbitrary.
2.3 QTQT models with nontrivial “charge” $C^{(QT)} \neq I$

In the next step we take the elementary decomposition of the unit operator

$$I^{(N)} = \sum_{j=0}^{N-1} \frac{1}{\langle \psi_j^{(N)} | Q | \psi_j^{(N)} \rangle} \langle \psi_j^{(N)} | Q | \psi_j^{(N)} \rangle |\psi_j^{(N)}\rangle,$$  \hspace{1cm} (12)

and multiply it by $Q$ from the left, yielding another identity

$$Q = Q^{(N)} = \sum_{j=0}^{N-1} \frac{1}{\langle \psi_j^{(N)} | Q | \psi_j^{(N)} \rangle} \langle \psi_j^{(N)} | Q | \psi_j^{(N)} \rangle |\psi_j^{(N)}\rangle.$$  \hspace{1cm} (13)

A comparison of this expression with Eq. (11) specifies the exceptional set of constants

$$\kappa_j^{(exc.)} = \frac{1}{\langle \psi_j^{(N)} | Q | \psi_j^{(N)} \rangle}, \quad j = 0, 1, \ldots, N - 1$$  \hspace{1cm} (14)

at which the charge becomes trivial, $C^{(exc.)} = I$, and at which our two alternative Hilbert spaces, viz., spaces $K^Q$ and $H^{(QT)}$ would coincide.

*Vice versa*, a nontrivial QTQT operator $C^{(QT)} = Q^{-1} \Theta^{(QT)} \neq I$ is obtained whenever we choose, in (11), any other $N$-plet of parameters,

$$\exists j, \quad \kappa_j^{(QT)} \neq \kappa_j^{(exc.)} = \frac{1}{\langle \psi_j^{(N)} | Q | \psi_j^{(N)} \rangle}, \quad 1 \leq j \leq N - 1.$$  \hspace{1cm} (15)

The equivalence between our two alternative Hilbert spaces $K^Q$ and $H^{(QT)}$ becomes broken. In parallel, the nontrivial QTQT operator $C^{(QT)} \neq I$ must be found an appropriate interpretation (obviously, calling it still a “charge” could be misleading).

Summarizing [2], we may now comply with the overall PTQT or QTQT philosophy and declare just the latter metric $\Theta^{(QT)}$ and the related standard Hilbert space $H^{(S)}$ “physical”.

3 Discussion

3.1 The criterion of observability in $H^{(QT)}$

One of the remarkable consequences of the most popular choice of the metric $\Theta^{(Dirac)} = I$ is that it trivializes the test of the Hermiticity of any given set of observables $\Lambda_j$ in the “friendly” space $H^{(F)}$. On the contrary, whenever we select a nontrivial $\Theta^{(QT)} \neq I$ in the physical space $H^{(QT)}$, a similar test requires the verification of validity of the Dieudonné’s [13] equation(s)

$$\Lambda_j^\dagger \Theta^{(QT)} = \Theta^{(QT)} \Lambda_j.$$  \hspace{1cm} (16)
where $\Theta^{(QT)}$ is given by formula (11). Thus, assuming that we know the (curly bra-ket denoted) eigenstates and spectral representation of
\[
\Lambda = \sum_{j=0}^{N-1} |\lambda_j^{(N)}\rangle \langle \lambda_j^{(N)}| \approx \langle \lambda_j^{(N)}| \lambda_j^{(N)}\rangle
\]
we may introduce two overlap matrices $U(\vec{\kappa})$ and $V(\vec{\lambda})$ with elements
\[
U_{jk}(\vec{\kappa}) = \frac{1}{\kappa_j} \langle \psi_j^{(N)}| \lambda_k^{(N)} \rangle \approx \langle \lambda_j^{(N)}| \lambda_k^{(N)}\rangle, \quad V_{jk}(\vec{\lambda}) = \frac{\lambda_j}{\langle \lambda_j^{(N)}| \lambda_k^{(N)}\rangle} \approx \langle \lambda_j^{(N)}| \psi_k^{(N)}\rangle
\]
and rewrite Eq. (16) as the requirement of the Hermiticity of the product
\[
U(\vec{\kappa})V(\vec{\lambda}) = M = M^\dagger.
\]
A simplified version of the observability criterion (19) may be developed for the special Hilbert spaces $\mathcal{H}^{(QT)}$ in which the nontrivial physical metric $\Theta^{(QT)} \neq I$ acquires a special sparse-matrix form. Incidentally, our toy-model Hamiltonians $H^{(N)}$ of section 2 prove suitable for an explicit constructive illustration of such an anomalous scenario.

### 3.2 The emergence of hidden horizons

A straightforward application of symbolic-manipulation software reveals and confirms that at any dimension $N$, the tridiagonal-matrix restriction imposed upon Eq. (11) yields the one-parametric family of the tridiagonal Legendre-oscillator metrics
\[
\Theta^{(QT)}_{(\text{sparse})} = Q^{(\text{diagonal})} + \begin{pmatrix}
0 & \kappa_1 & 0 & \ldots & 0 & 0 \\
\kappa_1 & 0 & \kappa_2 & 0 & \ldots & 0 \\
0 & \kappa_2 & 0 & \kappa_3 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \kappa_{N-2} & 0 & \kappa_{N-1} \\
0 & 0 & \ldots & 0 & \kappa_{N-1} & 0
\end{pmatrix}
\]
where $\kappa_j = \alpha/(j-1)!$ while $Q^{(\text{diagonal})}$ is defined by Eq. (7). In such a particular illustration the condition of positivity of metrics (20) restricts the admissible range of the optional parameter to a finite and $N$-dependent interval, $\alpha \in (-\gamma^{(N)}, \gamma^{(N)})$. The boundary values $\gamma^{(N)}$ must be determined numerically. In our tests they even seemed to converge to a positive limit $\gamma^{(\infty)} > 0$.

From the point of view of physics the latter observation (which follows from our special choice of the form of the metric (20)) might look like a paradox. Indeed, we
know that the spectrum of energies themselves remains real at any \( \alpha \). Fortunately, an explanation of such an apparent paradox is simple. Once we have fixed the form of the metric \( \Theta^{(QT)}_{\text{sparse}} = \Theta^{(QT)}_{\text{sparse}}(\alpha) \) which ceases to be invertible at \( \alpha = \pm \gamma(N) \), we may simply connect this irregularity, via Dieudonné’s Eq. (16), with the loss of reality of the spectrum of one of the other observables \( \Lambda = \Lambda(\alpha) \). Thus, although the energies \( E_n \) themselves remain, by construction, manifestly insensitive to any changes of \( \alpha \), the eigenvalues of \( \Lambda \) will depend on \( \alpha \) in general. In this sense, we may invert the arrow of argument and reinterpret Eq. (16) as an implicit definition of the “admissible” metric for a particular observable \( \Lambda = \Lambda(\alpha) \). Then, naturally, with the change of \( \alpha \) we may reach the physical horizon (called also an exceptional point in parametric space [14]) at which the one-parametric spectrum \( \{\lambda_j(\alpha)\}_{j=0}^{\infty} \) ceases to be real.

As long as the spectrum of our toy-model Hamiltonian itself remains real at any \( \alpha \) (i.e., formally, its own physical horizon lies at \( \alpha = \pm \infty \)), we might call the other, hidden horizons (i.e., those caused by the boundaries of observability of the other, not necessarily explicitly, or completely known, observables \( \Lambda \)) “secondary”.

4 Conclusions

The proposal of consistency of working with quantum observables which fail to be self-adjoint in a manifestly unphysical, auxiliary Hilbert space \( \mathcal{K}^{\mathcal{Q}} \) dates back to the nuclear physics inspired analysis in [3]. This approach has recently been reformulated, in Ref. [2], as a fairly universal recipe working with a triplet of Hilbert spaces. One of these spaces (denoted, in [2], as \( \mathcal{H}^{(P)} \) and being “microscopic” or “fermionic” in the nuclear physics context of Ref. [3]) may be characterized as “physical” but “prohibitively complicated”. In contrast, the other two (both “bosonic” in [3]) formed the pair of the “first auxiliary” space \( \mathcal{H}^{(F)} \) and the “second auxiliary” space \( \mathcal{H}^{(S)} \).

The main distinctive feature of this three-Hilbert-space pattern may be seen in a complete absence of the concept of \( \mathcal{PT} \) symmetry of the given (and, in the Dirac-metric space \( \mathcal{H}^{(F)} \), manifestly non-Hermitian) Hamiltonian \( \mathcal{H} \). Still, in a historical perspective, the addition of the requirement of the \( \mathcal{PT} \) symmetry and of the postulate of the observability of the charge \( \mathcal{C}^{(PT)} \) proved remarkably productive.

In the original PTQT formalism using the genuine, indefinite parity \( \mathcal{P} \) one starts from the auxiliary Krein space \( \mathcal{K}^{\mathcal{Q}} \) and constructs the Hamiltonian-dependent charge \( \mathcal{C} \) and the metric (1). The gains (e.g., the productivity and flexibility of the recipe) are accompanied by certain losses (e.g., the unexpected emergence of the long-ranged causality-violating effects caused by the charge \( \mathcal{C} = \mathcal{C}(\mathcal{H}) \) in the scattering dynamical
regime [15]).

Similar limitations may be expected to occur when one turns attention to the positive definite metrics $Q$. A partial encouragement of such a move may be sought in the recent success of a simultaneous introduction of the short-range non-localities in both the (genuinely $PT$-symmetric) Hamiltonians $H \neq H^\dagger$ and band-matrix metrics $\Theta^{(PT)}$ [16]. In the “standard”, parity-related $PT$-symmetric theory, several families of certain finite-dimensional Hamiltonians $H$ were already assigned short-ranged metrics $\Theta$ constructively [17].

All of these PTQT developments sound encouraging and one might expect that parallel developments could be also realized in the present QTQT context; we think that our present illustrative example may also find multiple descendants. One should emphasize that irrespectively of the indefiniteness or positivity of $P$ resp. $Q$, the underlying paradigm of the Hamiltonian-controlled choice of Hilbert space seems to be fairly efficient.

After all, as we explained, the recipe is not incompatible even with the most traditional quantum-theory textbooks. Indeed, in all of the current implementations of the PTQT/QTQT idea, the apparent non-Hermiticity $H \neq H^\dagger$ of the “input” Hamiltonian seems to be an almost irrelevant, half-hidden byproduct of our habitual and tacit preference of the traditional and friendly Hilbert space $\mathcal{H}^{(F)} := L^2(\mathbb{R})$ of square-integrable functions. In our present proposal, the preference of the Hilbert spaces in their “simplest possible” representations $\mathcal{K}^Q$ has just been given an innovative reinterpretation, with the simplified initial metric $Q$ entering, via Eq. (10), the closed-form definition of the biorthogonal basis. Subsequently, such knowledge of the basis converted formula (11) into an explicit definition of all of the eligible metrics and physical Hilbert spaces $\mathcal{H}^{(QT)}$.

To summarize, what we have proposed is, in essence, a double Hilbert space construction of a quantum system from its given Hamiltonian $H$. Our proposal was inspired by the success of the methodically productive concept of $PT$-symmetry in quantum mechanics. With the applicability illustrated via an exactly solvable Legendre quantum lattice oscillator, our QTQT proposal refers to ideas and assumptions which are quite similar to their PTQT predecessors.

The main innovation may be seen in the replacement of the auxiliary, unphysical Krein space $\mathcal{K}^P$ (in which the indefinite operator $P$ served as its pseudo-metric) by an equally unphysical Hilbert space $\mathcal{K}^Q$. For this purpose we required the positive definiteness of the parity-replacing auxiliary metric $Q$. Still, we tried to keep as many analogies between $P$ and $Q$ as possible. In particular, we insisted on the evolution of the system being rendered unitary via a factorized metric $\Theta^{(QT)} = QC^{(QT)}$. This being said, we partly accepted the philosophy of review [7] and deviated from the
traditional PTQT prescriptions by not trying to make such a metric unique. This
new flexibility enabled us to preserve the nontriviality of the charge-like operator
\( \mathcal{C}^{(QT)} \neq I \), albeit accompanied by a weakening of the constraints imposed upon this
particular operator.

**Acknowledgement**

MZ (partially supported by GAČR grant Nr. P203/11/1433 and by MŠMT “Doppler
Institute” project Nr. LC06002) acknowledges the atmosphere and hospitality of
STIAS, Stellenbosch where this work has been mainly performed.
References

[1] C. M. Bender, Rep. Prog. Phys. 70 (2007) 947.

[2] M. Znojil, SIGMA 5 (2009) 001.

[3] F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, Ann. Phys. (NY) 213 (1992) 74.

[4] P. Siegl, PRAMANA - J. Phys. 73 (2009) 279; S. Hassi and S. Kuzhel, arXiv:1101.0046.

[5] C. M. Bender and K. A. Milton, Phys. Rev. D 55 (1997) R3255; C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80 (1998) 4243; F. M. Fernández, R. Guardiola, J. Ros and M. Znojil, J. Phys. A: Math. Gen. 31 (1998) 10105; G. Lévai and M. Znojil, J. Phys. A: Math. Gen. 33 (2000) 7165.

[6] C. M. Bender, D. C. Brody and H. F. Jones, Phys. Rev. Lett. 89 (2002) 0270401; C. M. Bender and H. F. Jones, Phys. Lett. A 328 (2004) 102; D. Krejcirik: J. Phys. A: Math. Theor. 41 (2008) 244012.

[7] A. Mostafazadeh, Int. J. Geom. Meth. Mod. Phys. 7 (2010) 1191.

[8] H. Langer and Ch. Tretter, Czech. J. Phys. 54 (2004) 1113.

[9] M. Znojil, J. Math. Syst. Sci., to appear (arXiv: 1110.1218).

[10] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (Dover, New York, 1970).

[11] A. Mostafazadeh, J. Math. Phys. 43 (2002) 205 and 2814.

[12] M. Znojil, SIGMA 4 (2008) 001.

[13] J. Dieudonne, Proc. Int. Symp. Lin. Spaces (Pergamon, Oxford, 1961), p. 115.

[14] T. Kato, Perturbation theory for linear operators (Springer, Berlin, 1966), p. 64; M. Znojil, Phys. Lett. B 647 (2007) 225; U. Guenther, H. Langer and Ch. Tretter, SIAM J. Math. Anal. 42 (2010) 1413.
[15] H. F. Jones, Phys. Rev. D 76 (2007) 125003;
    M. Znojil, J. Phys. A: Math. Theor. 41 (2008) 292002;
    M. Znojil, SIGMA 5 (2009) 085.

[16] M. Znojil, Phys. Rev. D. 80 (2009) 045022.

[17] P. E. G. Assis and A. Fring, J. Phys. A: Math. Theor. 41 (2008) 244001;
    M. Znojil, J. Math. Phys. 50 (2009) 122105;
    H. Schomerus, Phys. Rev. Lett. 104 (2010) 233601;
    M. Znojil, Phys. Rev. A 82 (2010) 052113;
    Y. N. Joglekar, D. Scott, M. Babbey and A. Saxena, Phys. Rev. A 82 (2010) 030103(R);
    M. Znojil, Phys. Lett. A 375 (2011) 2503;
    M. Znojil, J. Phys. A: Math. Theor. 44 (2011) 075302.