Continuification Control of Large-Scale Multiagent Systems in a Ring

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Abstract—In this Letter, we propose a method to control large-scale multiagent systems swarming in a ring. Specifically, we use a continuification-based approach that transforms the microscopic, agent-level description of the system dynamics into a macroscopic, continuum-level representation, which we employ to synthesize a control action towards a desired distribution of the agents. The continuum-level control action is then discretized at the agent-level in order to practically implement it. To confirm the effectiveness and the robustness of the proposed approach, we complement theoretical derivations with numerical simulations.

Index Terms—Agents-based systems, distributed parameter systems, large-scale systems.

I. INTRODUCTION AND BACKGROUND

A PRESSING open challenge in control theory is to find methods to steer the collective behavior of large-scale multiagent systems consisting of many dynamical units (or agents) interacting with a given, and possibly time-varying, network topology. Examples of this problem include multirobot systems [1], [2], [3], cell populations, [4], [5], and human networks [6], [7]. Typically, in these applications, the goal is to control some macroscopic observables of the emerging collective behavior. However, control needs to be practically exerted at the microscopic, individual agent-level. Developing methods that translate macroscopic-level control goals into microscopic-level control actions is a critical challenge [8].

In Statistical Physics [9], [10], mean field approaches are often used to describe large-scale systems. Through a mean field approximation, one can obtain a macroscopic description of the emergent behavior of the system in terms of appropriate distributed parameter models, derived from the microscopic ordinary differential equations (ODEs) models describing the agents’ dynamics. Such mean field approaches have been used to control the collective behavior of multiagent systems [11], [12], [13], [14]. Also, in Applied Mathematics, problems related to the control of crowds, herding, and flocking agents were solved by finding a mean field description of the agents of interest [11], [15]. Macroscopic descriptions were also used in some multiagent reinforcement learning scenarios [16], [17]. Other methodologies recently proposed in the literature are based on the use of graphons [18] and data and manifold learning [19], [20].

Inspired from the paradigm proposed in [21] (see Fig. 1), here we adopt a continuification approach in which a macroscopic model, derived from the agents’ dynamics, is used to design a control strategy at the macroscopic level. Such a macroscopic control action is then discretized in order to be deployed on the agents at the microscopic level. As a representative case of study, we address the problem of steering the dynamics of a group of interacting agents on a one-dimensional periodic domain (a ring). Our goal is to control the agents so that they achieve some desired configuration, independently of their interactions (repulsive, attractive, etc.) and their initial configuration. Such a problem has important ramifications in traffic dynamics [22], [23], swarming robots [12], and natural systems, including animals’ collective motion [24], [25], [26], [27], [28], [29], and cell populations [5].

After presenting the microscopic description of the agents’ behavior, we derive a macroscopic, partial differential equations (PDE) model for their emergent behavior and we solve the problem of designing a control strategy to achieve a desired agents’ configuration. We propose a mathematical proof of convergence at the macroscopic level, and then we discretize the control action to obtain the required control inputs acting on the individual agents. In contrast with [21],...
the microscopic control inputs are obtained by spatially sampling the macroscopic control action at the agents’ positions. Theoretical derivations are complemented by numerical simulations validating the effectiveness and robustness of the proposed strategy.

II. MODEL AND PROBLEM STATEMENT

A. The Model

Let \( X \) be a group of \( N \) identical mobile agents moving on the unit circle \( S = [-\pi, \pi] \). By making the kinematic assumption widely used in the literature [30] (that is, neglecting acceleration and considering a drag force proportional to the velocity), the dynamics of the \( i \)-th agent can be expressed as

\[
\dot{x}_i = N \sum_{j=1}^{N} f(\{x_i, x_j\}) + u_i, \quad (1)
\]

where \( x_i \) is the angular position of agent \( i \) on \( S \), \( \{x_i, x_j\} \mod (2\pi) - \pi \) is the angular distance between agents \( i \) and \( j \) wrapped on \( S \), \( u_i \) is the velocity control input affecting its behavior, and \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a velocity interaction kernel modeling pairwise interactions between the agents.

Assuming the number of agents to be sufficiently large, we describe the macroscopic collective behavior emerging from the microscopic agents’ dynamics in terms of the density profile of agents on \( S \) at time \( t \), say \( \rho : S \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \). This function is such that, when integrated over a subset of \( S \), it returns the number of agents in that subset. By definition, we require that \( \int_S \rho(x, t) \, dx = N \) for any \( t \).

The function \( f \) takes the relative angular distance between two agents and returns a velocity. Similar to [28], [29], we assume that \( f \) is a vanishing odd function, discontinuous at the origin, where it takes zero value, with a bounded derivative everywhere. As shown in Fig. 2, the kernel can take different functional forms to model various types of interactions.

In open-loop, where the control input \( u_i \) is set to zero for all the agents, four types of emerging behaviors can occur depending on the initial configuration of the agents and on the different functional form of the interaction kernel (modeling different ranges of attraction and/or repulsion) [28], [29]: spreading (see Fig. 2a), collapsing (see Fig. 2b), clustering, (see Fig. 2c), or stable aggregation (see Fig. 2d).

B. Problem Statement

The problem is to select a set of distributed control inputs \( u_i \) acting at the microscopic, agent-level in order for the agents to organize themselves into a desired macroscopic configuration on \( S \). Specifically, given some desired periodic smooth density profile, say \( \rho^d(x, t) \), associated with the target agents’ configuration, the problem can be reformulated as that of finding a set of distributed control inputs \( u_i, \, i = 1, 2, \ldots, N \) in (1) such that

\[
\lim_{t \to \infty} \|\rho^d(\cdot, t) - \rho(\cdot, t)\|_2 = 0, \quad (2)
\]

for agents starting from an initial configuration \( x_i(0) = x_{i0}, \, i = 1, \ldots, N \) that is proximal to the one prescribed by \( \rho^d(x, 0) \); here, \( \| \cdot \|_2 \) is the \( L^2 \) norm on \( S \).

III. CONTROL DESIGN

We assume the multiagent system of interest to be described by a large set of \( N \) coupled ordinary differential equations (ODEs), and adopt an approach based on continuification\(^1\) (or continuation) [21]. We describe next how each of the four steps depicted in Fig. 1 can be implemented to solve the problem of interest.

A. Continuification

Following [28], [29], [31], we can derive the macroscopic model describing the open-loop dynamics of (1) when \( u_i = 0 \), as the mass conservation law,

\[
\rho(x, t) + [\rho(x, t)V(x, t)]_x = 0, \quad \forall x \in [-\pi, \pi], \, \forall t \geq 0, \quad (3)
\]

\(^1\)Here, we use the term continuification instead of continuation, used in [21], to distinguish this procedure from the parametric continuation of dynamical systems.

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**Fig. 1.** Continuification-based control approach used in this letter, inspired by [21].

**Fig. 2.** Typical emerging behaviors arising from (1) rotated around the origin. The asymptotic agents’ density is shown in each panel together with their distribution on the ring (left inset) and the corresponding interaction kernel (right inset).
where $V$ is the velocity field characterising the advection term of (3), which can be expressed as

$$V(x, t) = \int_{-\pi}^{\pi} f((x, y)_{\pi}) \rho(y, t) \, dy = (f * \rho)(x, t),$$

where “*” is the circular convolution operator. For the problem to be well-posed, we shall impose boundary and initial conditions, which read

$$\rho(-\pi, t) = \rho(\pi, t), \quad \forall t \geq 0$$

$$\rho(x, 0) = \rho^0(x), \quad \forall x \in [-\pi, \pi].$$

for a periodic function $\rho$ over the domain $S$, such as the one we are searching for, the velocity will also be periodic by construction. Hence, the flux is also periodic and the total mass is constant in time. Specifically, integrating (3) and using (5), we establish that $\int_{S} \rho(t, x) \, dx = 0$.

**B. Macroscopic Control Design**

To achieve asymptotic convergence, we consider the addition to (3) of a control input $q$, representing a mass source/sink term. The resulting closed-loop macroscopic model is

$$\dot{\rho}(x, t) + \left[ \rho(x, t) V(x, t) \right]_x = q(x, t).$$

We design $q$ by assuming the desired density profile $\rho^d$, fulfills the following reference dynamics:

$$\dot{\rho}^d(x, t) + \left[ \rho^d(x, t) V^d(x, t) \right]_x = 0,$$

where we define

$$V^d(x, t) = \int_{-\pi}^{\pi} f((x, y)_{\pi}) \rho^d(y, t) \, dy = (f * \rho^d)(x, t).$$

Equation (8) is a mass conservation law, assumed to fulfill periodic boundary conditions similar to (5).

Letting $e(x, t)$ be the mismatch between the agents’ density and the reference density, defined as $e(x, t) := \rho^d(x, t) - \rho(x, t)$, subtracting (7) from (8) yields

$$e(x, t) + \left[ e(x, t)(-V^e(x, t)) \right]_x + \left[ e(x, t) V^d(x, t) \right]_x = -q(x, t),$$

subject to the boundary and initial conditions

$$e(-\pi, t) = e(\pi, t), \quad \forall t \geq 0,$$

$$e(x, 0) = \rho^d(x, 0) - \rho^0(x), \quad \forall x \in [-\pi, \pi].$$

**Theorem 1:** Choosing the control input in (7) as the following linear form in the mismatch $e(\cdot, t)$,

$$q(x, t) = K_p e(x, t) - \left[ e(x, t) V^d(x, t) \right]_x$$

$$- \left[ \rho^d(x, t) V^e(x, t) \right]_x,$$

where $K_p$ is a positive constant gain and

$$V^e(x, t) = \int_{-\pi}^{\pi} f((x, y)_{\pi}) e(y, t) \, dy = (f * e)(x, t),$$

makes the error dynamics (10) locally asymptotically converge to 0.

**Proof:** Under the control action (13) and considering (8), the error dynamics becomes

$$e(x, t) + \left[ e(x, t)(-V^e(x, t)) \right]_x = -K_p e(x, t).$$

Choosing as a candidate Lyapunov function the square of the $L^2$ norm of the error, $\|e\|^2_2 = \int_{-\pi}^{\pi} e^2(x, t) \, dx$, we obtain

$$\frac{d}{dt} \|e\|^2_2 = 2 \int_{-\pi}^{\pi} e(x, t) (e(x, t) V^e(x, t))_x \, dx$$

$$= \int_{-\pi}^{\pi} 2e^2(x, t) V^e(x, t) + \left[ e^2(x, t) \right]_x V^e(x, t) \, dx$$

Integrating by parts the second integrand in $\Psi$ and taking into account the periodicity condition, we can write

$$\Psi = \int_{-\pi}^{\pi} 2e^2(x, t) V^e(x, t) \, dx.$$

Now, the spatial derivative of the velocity can be written as $V^e(x, t) = (f * e)(x, t)$, whose $\infty$-norm can be bounded using Young’s inequality [32] as follows:

$$\|V^e_\infty\| \leq \|f_\infty\|_2 \|e_\infty\|_2.$$

where we note that $\|f_\infty\|_2$ is independent of time. Using (18) in (16) along with the bound in (19), we establish the following inequality:

$$\left( \|e\|^2_2 \right)_t \leq -2K_p \|e\|^2_2 \|\|f_\infty\|_2\|_2 \|e\|^2_2.$$

For any $\gamma > 0$, if $\|e\|_2 < \gamma$, then the error will approach zero for $K_p$ sufficiently large ($K_p > \gamma / \|f_\infty\|_2$).

Note that the feedback control action $q(x, t)$ consists of three terms. The first two terms are local control actions, while the latter is non-local, involving the convolution between the interaction kernel and the error. Such non-locality is practically mitigated by considering a vanishing interaction kernel (see Fig. 2).

Also, we remark that the error dynamics becomes globally asymptotically convergent to 0 if the control input in (13) includes the extra term $-\{e(x, t) V^e(x, t)\}_x$, which is a nonlinear (quadratic) function of the mismatch. In this case, (15) simplifies to $e_t(x, t) = -K_p e(x, t)$.

**C. Discretization and Microscopic Control**

Next, we need to discretize the macroscopic control action in order to obtain the control inputs $u_t$ that can be deployed to steer the agents’ microscopic dynamics (1). Firstly, we recast the macroscopic controlled model in (7) to include $q(x, t)$ as a control action on the velocity as

$$\rho_t(x, t) + \left[ \rho(x, t) (V(x, t) + U(x, t)) \right]_x = 0,$$

Clearly, the control action cannot change the total mass of the system: the choice of writing $q$ as a mass source/sink is only a matter of simplicity of the derivations, and we will ultimately incorporate this term as a control input on the velocity.
where $U$ is an auxiliary function computed from the linear PDE (assuming $\rho \neq 0$)

$$\rho(x, t)U(x, t) = -q(x, t). \quad (22)$$

The case $\rho = 0$ corresponds to a case in which $q$ is effectively behaving as a source/sink, changing the mass of the system (impossible without affecting the total number of agents in the system). Integrating (22), we obtain

$$U(x, t) = -\frac{1}{\rho(x, t)} \left[ \int_{-\pi}^{\pi} q(y, t) \, dy + q(-\pi, t) \right]. \quad (23)$$

To obtain the control action to be applied to the individual agent, we consider the agents as particles of a continuum, although not tagged with any label. We compute the velocity input acting on agent $i$ by sampling $U(x_i, t)$ at $x_i$, that is,

$$u_i(t) = U(x_i, t), \quad i = 1, 2, \ldots, N. \quad (24)$$

We remark the following. (i) In a decentralised scenario, each agent needs to possess, exchange, or estimate enough information about the others to compute $U(x_i, t)$, which is non-local. For instance, $U(x_i, t)$ could be computed discretizing (23), by using information on the control inputs computed by preceding agents. (ii) The assumption that $\rho$ is nonzero is reasonable, as agents will estimate the density from their own positions, and hence we can choose a smoothing kernel such that $U$ is always well-defined. Moreover, since we are interested in discretizing the spatial control action, we know that $\rho$ will be different from zero at least where there are effectively agents to control, that is, $U$ is surely well-defined where we need to discretize it. (iii) The discretized controller will fulfill asymptotic convergence of agents’ density to the desired one only in the limit of infinite number of agents.

IV. VALIDATION

We validate the proposed strategy by selecting as a representative case of study the interaction kernel derived from a Morse potential, often used in the literature [28], [29], [33],

$$f(z) = \text{sgn}(z) \left[ -Ge^{-|z|/L} + e^{-|z|} \right], \quad (25)$$

where, $G > 0$ and $L > 0$ modulate the strength and characteristic distance of an attractive term, while the second term models repulsion normalized to have unitary repulsive strength and length scale as in [28], [29]. We choose $G$ and $L$ so that the repulsive interaction is dominant (as for example depicted in Fig. 2a). In particular we select $G = L = 0.5$. We then address the problem of steering $N = 50$ agents to converge towards different stable aggregation scenarios, against their mutual repulsion. We choose $K_p = 10$ and set the initial positions of the agents as evenly distributed in $S$, $\rho^0(x) = N/2\pi$. We consider both regulation and tracking scenarios where agents need to converge towards a time-invariant or time-varying desired density profile. We also assess convergence and robustness of the control strategy.

In order to quantify the steady-state error associated to a given trial, we use the Kullback-Leibler (KL) divergence [34] to estimate the distance between the desired density profile and the density estimated from the positions of the agents, $D_{KL}(\hat{\rho}(\cdot, t) \| \hat{\rho}^d(\cdot, t))$, where $\hat{\rho}(\cdot, t)$ and $\hat{\rho}^d(\cdot, t)$ are the normalised versions of $\rho(\cdot, t)$ and $\rho^d(\cdot, t)$ such that their integral is 1, rather than $N$. With respect to the implementation, we discretize the spatial domain into 150 grid points. Agents are not constrained to move on such stencils, so that we use a linear interpolation of $U(x, t)$ when computing $u_i(t)$. Convolutions are evaluated as in [35], integrals are approximated with the trapezoidal method, and derivatives as central differences. For estimating densities from the agents’ positions, we use a Gaussian $msn$ (minimum of standard deviation and interquantile range) kernel estimation, adapted to the circular domain, as in [36].

A. Regulation

First, we study a regulation task where agents should achieve a desired time-invariant density profile, $\rho^d(x, t) = \rho^d(x)$. We consider both a monomodal and a bimodal desired density.

As an initial trial, we choose the desired density as the following von Mises function [37] with prescribed mean $\mu$

$$\mu = \frac{1}{3} \pi, \quad \frac{1}{3} \pi, \quad \frac{2}{3} \pi$$

where $\mu$ is a vector of the desired mean parameters.

![Fig. 3. Regulation to a monomodal distribution: (a) initial and (b) final configuration of the agents with their associated density (in blue) and desired density (in orange); (c) time and space evolution of the absolute value of the error function; (d) KL divergence between the desired normalised density and the normalised density.](image)

![Fig. 4. Control inputs computed as in (24) acting on the agents for the monomodal regulation.](image)
and concentration coefficient $k$:
\[
\rho^d(x) = \frac{N e^{k \cos(x-\mu)}}{2\pi I_0(k)},
\]
(26)

where $N$ is used to let the desired density sum to the total number of agents and $I_0$ is the modified Bessel function of the first kind of order 0 [37]. We set $\mu = 0$ and $k = 4$.

Figures 3(a)-(b) show the initial and final configuration of the agents and their associated density (compared with the desired one). The evolution of the error absolute value in space and time is depicted in Fig. 3c, while the time evolution of the KL divergence is shown in Fig. 3d. The results confirm the effectiveness of the proposed strategy with the control error converging quickly (in less than one time unit) to a small value and the agents achieving the desired configuration. We also show the control inputs at the microscopic level in Fig. 4, resulting in signals converging to constant values. Note that the nonzero residual error shown in Fig. 3d is an effect of the discretization of the macroscopic control action and does indeed converge to zero as the number of agents increases.

As a further test, we also consider the problem of achieving a reference density which is the bimodal combination of two von Mises functions with the same concentration parameters and different means,
\[
\rho^d(x) = \frac{N}{4\pi I_0(k)} \left[ e^{\cos(x-\mu_1)} + e^{\cos(x-\mu_2)} \right],
\]
(27)

with $k = 8$ and $\mu_1 = \pi/2$ and $\mu_2 = -\pi/2$. In Fig. 5, we show the results of our simulations confirming the ability of the proposed strategy to achieve the desired control goal. For brevity, we omit the evolution of the control inputs, which are qualitatively similar to those in Fig. 4.

**B. Tracking**

To evaluate the ability of our strategy to track a time-varying desired density profile, we choose as target configuration a von Mises function such as (26) with a constant concentration parameter but a time-varying mean $\mu(t)$ which is null for the first 0.5 time units of the simulation and then increases with rate $\dot{\mu} = 1.47$ rad per time unit, until reaching the value $\pi/3$. From that time instant, $\mu$ decreases with rate $\dot{\mu} = -1.47$ rad per time unit until reaching the value $-\pi/3$, increasing again with rate $\dot{\mu} = 1.47$ rad per time unit, until returning to zero. The evolution of the error absolute value in space and time and the KL divergence between $\rho(x,t)$ and $\rho^d(x,t)$ is shown in Fig. 6, confirming the viability of the proposed strategy and its effectiveness in steering agents’ behavior towards the desired time-varying configuration.

**C. Robustness**

Finally, we test the robustness of the proposed strategy to changes in the number of agents and to measurement noise.

1) **Scalability**: To assess the scalability of the proposed control strategy to different numbers of agents, we run simulations for different values of $N$, and for each trial we record the value of $D_{KL}$ at the end of the simulation, at time $t_f$. We consider as test scenario the one discussed in Section IV-A. We report the results of such test in Fig. 7a, for $N$ spanning from 1 to $10^3$. As the proposed strategy is based on a continuum approximation of the discrete set of agents of interest, we notice that, as expected, the steady state error becomes smaller as $N$ increases, becoming sufficiently smaller than 0.2 after $N$ gets
larger than 5. Notice that the case \( N = +\infty \) is reported as well in Fig. 7a. This result was obtained performing a finite difference approximation of the continuied controlled model in (7). The numerical implementation was inspired by that reported in [35].

2) Measurement Noise: In order to assess the robustness of the proposed strategy, we perform additional simulations adding to (13) some white noise with different power \( P \) (measured in dBW). For each of these simulations, which use the same set-up considered in Section IV-A, we record the KL divergence at the end of the trials. Results are reported in Fig. 7b. We observe that, as the noise power increases past \( P = 60 \text{ dBW} \), the steady-state mismatch between the agents' density and the desired one worsens.

V. CONCLUSION AND FUTURE WORK

We developed a continuification-based control strategy for a swarm of agents moving on a periodic bounded domain. We started by deriving a macroscopic model of agents’ distribution on the ring and designed a control action able to steer it to a desired configuration, proving its convergence. The microscopic control strategy was then obtained by spatially sampling the macroscopic control function at the agents’ positions. Numerical simulations confirmed the effectiveness and robustness of the proposed approach. The extension to higher dimensional scenarios and to a fully distributed and localized framework are the subject of ongoing work.

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