ON THE CAHN-HILLIARD/ALLEN-CAHN EQUATIONS WITH SINGULAR POTENTIALS

ALAIN MIRANVILLE
Université de Poitiers
Laboratoire de Mathématiques et Applications
UMR CNRS 7348, SP2MI
Boulevard Marie et Pierre Curie, Téléport 2
F-86962 Chasseneuil Futuroscope Cedex, France
Xiamen University, School of Mathematical Sciences
Xiamen, Fujian, China

WAFA SAOUD
Université de Poitiers
Laboratoire de Mathématiques et Applications
UMR CNRS 7348, SP2MI
Boulevard Marie et Pierre Curie, Téléport 2
F-86962 Chasseneuil Futuroscope Cedex, France

RAAFAT TALHOUK
Université Libanaise,
Laboratoire de Mathématiques - EDST
Faculté des Sciences
Hadath, Liban

(Communicated by Tomas Caraballo)

ABSTRACT. The purpose of this work is to prove the existence and uniqueness of the solution for a Cahn-Hilliard/Allen-Cahn system with singular potentials (and, in particular, the thermodynamically relevant logarithmic potentials). We also prove the existence of the global attractor. Finally, we show further regularity results and we prove a strict separation property (from the pure states) in one space dimension.

1. Introduction. The following Cahn-Hilliard/Allen-Cahn model was introduced by A. Novick-Cohen and J. Cahn in [4]:

\[
\frac{\partial u}{\partial t} = h^2 \Delta (f(u + v) + f(u - v) - h^2 \Delta u),
\]

\[
\frac{\partial v}{\partial t} = -f(u + v) + f(u - v) - \alpha v + h^2 \Delta v,
\]

to model simultaneous order-disorder and phase separation in binary alloys on a BCC lattice in the neighborhood of the triple point. These authors explored two phenomenological approaches leading to systems of coupled Allen-Cahn/Cahn-Hilliard (AC/CH) equations (see [4] for more details).

2010 Mathematics Subject Classification. 35K51, 35B41, 35B45.
Key words and phrases. Allen-Cahn/Cahn-Hilliard equations, logarithmic potential, global attractor, regularity properties, strict separation property.
Here, $u$ denotes the concentration of one of the components and is a conserved quantity, while $v$ is an order parameter. Furthermore, $h$ is a (positive) parameter which represents the lattice spacing, the parameter $\alpha$ reflects the location of the system within the phase diagram and may be either positive or negative, and the nonlinear term $f$ is the derivative of a double-well potential $F$. We further note that the system is a gradient flow in $(H^1)^\prime \times L^2$ for the free energy

$$J(u,v) = \int_\Omega \left\{ F(u+v) + F(u-v) + \frac{\alpha}{2} v^2 + \frac{1}{2} h^2 (|\nabla u|^2 + |\nabla v|^2) \right\} dx,$$

where $f = F'$. These equations, endowed with Neumann boundary conditions, have been studied in [3] by A. Novick-Cohen, D. Brochet, and D. Hilhorst who proved the well-posedness and the existence of maximal attractors and inertial sets (i.e., exponential attractors) for the usual cubic nonlinear term $f(s) = s^3 - \beta s$ in three space dimensions.

These results were improved in [12], taking initial conditions in $H^2(\Omega)$, which allowed to prove the existence of exponential attractors (and, thus, of the finite-dimensional global attractor) for a large class of nonlinear terms containing polynomials of arbitrary odd degree with a strictly positive leading coefficient in three space dimensions.

A similar system, with a non-constant mobility, was treated in [7] where the existence of weak solutions for the Neumann problem for a degenerate parabolic system consisting of a fourth-order and a second-order equations with singular lower-order terms in one space dimension was proved. In addition, asymptotics for a similar system with a non-constant mobility, proposed as a diffuse interface model for simultaneous order-disorder and phase separation, was studied in [17]. There, A. Novick-Cohen focused on motion in the plane. This framework yields both sharp interface and diffuse interface models of sintering of small grains and thermal grains boundary grooving in polycrystalline films. This work was extended in [18], where the authors studied the partial wetting case, and their analysis accounts for motion in three space dimensions.

We also mention that numerical methods to solve coupled (AC/CH) systems were studied in, e.g., [4, 9, 19, 22, 23, 24]. Furthermore, a NKS method for the implicit solution of a coupled (AC/CH) system was proposed in [25].

In this work, we take $h = 1$ and $\alpha = 0$ and obtain the following system:

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta (f(u+v) + f(u-v)) = 0, \quad (1)$$

$$\frac{\partial v}{\partial t} - \Delta (f(u+v) - f(u-v)) = 0, \quad (2)$$

$$u = \Delta u = v = 0 \text{ on } \Gamma, \quad (3)$$

$$u|_{t=0} = u_0, \quad v|_{t=0} = v_0, \quad (4)$$

where $\Omega$ is a bounded domain of $\mathbb{R}^N$ ($N = 1, 2, \text{ or } 3$) with smooth boundary $\Gamma$.

In the Cahn-Hilliard theory, a thermodynamically relevant potential $F$ is the following logarithmic function which follows from a mean-field model (see [13, 14]):

$$F(s) = \frac{\theta_c}{2} \left( 1 - s^2 \right) + \frac{\theta}{2} \left[ (1-s) \ln \left( \frac{1-s}{2} \right) + (1+s) \ln \left( \frac{1+s}{2} \right) \right], \quad (5)$$

$s \in (-1, 1), \quad 0 < \theta < \theta_c.$
Furthermore,

\[ f(s) = (F'(s)) = -\theta_c s + \frac{\theta}{2} \ln \left( \frac{1 + s}{1 - s} \right). \]  \hfill (6)

The logarithmic potential \( F \) is very often approximated by regular ones, typically,

\[ F(s) = \frac{1}{4} (s^2 - 1)^2, \]  \hfill (7)

leading to the following cubic nonlinear term

\[ f(s) = s^3 - s. \]  \hfill (8)

Note, however, that such an approximation is reasonable when the quench is shallow, i.e. when the absolute temperature \( \theta \) is close to the critical one \( \theta_c \). Also note that the nonlinear term \( f \) leads to essential difficulties, due to the fact that we need to prove that the order parameter remains in the physically relevant interval \((-1, 1)\) (see, e.g. [6, 13, 15]).

In this article, we first prove the existence of weak solutions to the (AC/CH) equations with singular nonlinear terms. To do so, we approximate the singular nonlinear terms by regular ones and prove the convergence of the solutions to the approximated problems to that to the limit singular one. Then, we prove the uniqueness of the solution, which allows us to define the corresponding semigroup and prove the existence of the global attractor. We finally prove some higher-order regularity results which lead to a strict separation property in one space dimension.

2. Setting of the Problem. As far as the nonlinear term \( f \) is concerned, we assume more generally that

\[ f \in C^1(-1, 1), \ f(0) = 0, \]  \hfill (9)

\[ \lim_{s \to \pm 1} f(s) = \pm \infty, \ \text{and} \ \lim_{s \to \pm 1} f'(s) = +\infty. \]  \hfill (10)

In particular, it follows from these assumptions that

\[ f' \geq -c_0, \ c_0 \geq 0, \]  \hfill (11)

and

\[ -c_1 \leq F(s) \leq f(s)s + c_2, \ c_1, c_2 \geq 0, \ s \in (-1, 1), \]  \hfill (12)

where \( F(s) = \int_0^s f(r)dr \) (in particular, in order to obtain the right-hand side of (12), we can study the variations of the function \( s \mapsto f(s)s - F(s) + \frac{\theta}{2} s^2 \), whose derivative has, owing to (11), the sign of \( s \).

Remark 1. In particular, the thermodynamically relevant logarithmic functions (6) satisfy the above assumptions.

Next, we define, for \( N \in \mathbb{N} \), the approximated function \( f_N \in C^1(\mathbb{R}) \) by

\[ f_N(s) = \begin{cases} f(-1 + \frac{1}{N}) + f'(-1 + \frac{1}{N}) (s + 1 - \frac{1}{N}), & s < -1 + \frac{1}{N}, \\ f(s), & |s| \leq 1 - \frac{1}{N}, \\ f(1 - \frac{1}{N}) + f'(1 - \frac{1}{N})(s - 1 + \frac{1}{N}), & s > 1 - \frac{1}{N}. \end{cases} \]

We thus have

\[ f_N' \geq -c_0 \]  \hfill (13)

and, setting \( F_N(s) = \int_0^s f_N(r)dr \),

\[ -c_3 \leq F_N(s) \leq c_4 f_N(s)s + c_5, \ c_4 > 0, \ c_3, c_5 \geq 0, \ s \in \mathbb{R}, \]  \hfill (14)

\[ f_N(s)s \geq c_6 |f_N(s)| - c_7, \ c_6 > 0, \ c_7 \geq 0, \ s \in \mathbb{R}, \]  \hfill (15)
where the constants $c_i$, $i = 3, \ldots, 7$, are independent of $N$ (see [15]).

Then, we introduce the approximated problem:

$$\frac{\partial u_N}{\partial t} + \Delta^2 u_N - \Delta (f_N(u_N + v_N) + f_N(u_N - v_N)) = 0,$$  
(16)

$$\frac{\partial v_N}{\partial t} - \Delta v_N + f_N(u_N + v_N) - f_N(u_N - v_N) = 0,$$  
(17)

$u_N = \Delta u_N = v_N = 0$ on $\Gamma$,

(18)

$u_N|_{t=0} = u_0$, $v_N|_{t=0} = v_0$.

(19)

3. **A priori estimates.** Our aim in this section is to derive a priori estimates for the solutions $u_N$ and $v_N$ to (16)-(19). These a priori estimates are independent of $N$ and are formal, i.e. we assume that $u_N$ and $v_N$ are as smooth as needed. In particular, we will derive a priori estimates which are independent of $N$ on $f_N(u_N + v_N)$ and $f_N(u_N - v_N)$ in $L^2((0,T) \times \Omega), T > 0$. These a priori estimates allow us to obtain the existence of a solution to (16)-(19) by implementation of a Galerkin approximation. This will also allow us to pass to the limit $N \to +\infty$ in the approximated system (16)-(19).

We start by assuming that:

$$\|u_0 + v_0\|_{L^\infty(\Omega)} \leq 1 - \delta,$$  
(20)

and

$$\|u_0 - v_0\|_{L^\infty(\Omega)} \leq 1 - \delta', \delta, \delta' \in (0,1),$$  
(21)

where $\delta$ and $\delta'$ are fixed positive constants.

We now write equation (16) in the equivalent form

$$(-\Delta)^{-1} \frac{\partial u_N}{\partial t} - \Delta u_N + f_N(u_N + v_N) + f_N(u_N - v_N) = 0.$$  
(22)

We multiply (22) by $u_N$, (17) by $v_N$ and sum the resulting equalities to obtain

$$\frac{1}{2} \frac{d}{dt} (\|u_N\|_{L^2}^2 + \|v_N\|_{L^2}^2) + \|\nabla u_N\|^2 + \|\nabla v_N\|^2$$

$$+ ((f_N(u_N + v_N), u_N + v_N)) + ((f_N(u_N - v_N), u_N - v_N)) = 0,$$  
(23)

where $((\ , \ , \ ))$ and $\|\|$ are the usual $L^2$-scalar product and associated norm. Furthermore, $\|\|_{-1}$ is the $H^{-1}$ norm. More generally, $\|\|_X$ denotes the norm in the Banach space $X$.

Using (15), the previous equation gives

$$\frac{1}{2} \frac{d}{dt} (\|u_N\|_{L^2}^2 + \|v_N\|_{L^2}^2) + \|\nabla u_N\|^2 + \|\nabla v_N\|^2$$

$$+ \|f_N(u_N + v_N)\|_{L^1(\Omega)} + \|f_N(u_N - v_N)\|_{L^1(\Omega)} \leq c'.$$  
(24)

Then, we multiply (22) by $\frac{\partial u_N}{\partial t}$ and (17) by $\frac{\partial v_N}{\partial t}$. We sum the resulting equalities and obtain

$$\frac{1}{2} \frac{d}{dt} (\|\nabla u_N\|^2 + \|\nabla v_N\|^2) + \left| \frac{\partial u_N}{\partial t} \right|_{-1}^2 + \left| \frac{\partial v_N}{\partial t} \right|^2$$

$$+ ((f_N(u_N + v_N), \frac{\partial}{\partial t}(u_N + v_N))) + ((f_N(u_N - v_N), \frac{\partial}{\partial t}(u_N - v_N))) = 0,$$  
(25)
which implies
\[
\frac{1}{2} \frac{d}{dt} \left( ||\nabla u_N||^2 + ||\nabla v_N||^2 + 2 \int_\Omega F_N(u_N + v_N)dx + 2 \int_\Omega F_N(u_N - v_N)dx \right) \\
+ \left( \left| \frac{\partial u_N}{\partial t} \right|_{-1}^2 + \left| \frac{\partial v_N}{\partial t} \right|^2 \right) = 0.
\] (25)

Owing to (14), equation (23) yields
\[
\frac{1}{2} \frac{d}{dt} (||u_N||^2_1 + ||v_N||^2) + ||\nabla u_N||^2 + ||\nabla v_N||^2 \\
+ c \int_\Omega \left( F_N(u_N + v_N) + F_N(u_N - v_N) \right) dx \leq c', \ c > 0, c' \geq 0.
\] (26)

The sum of (24), (25), and (26) gives
\[
\frac{d}{dt} E_N + c \left( E_N + ||f_N(u_N + v_N)||_{L^1(\Omega)} + ||f_N(u_N - v_N)||_{L^1(\Omega)} \\
+ \left( \left| \frac{\partial u_N}{\partial t} \right|_{-1}^2 + \left| \frac{\partial v_N}{\partial t} \right|^2 \right) \right) \leq c', \ c > 0, c' \geq 0,
\] (27)

where
\[
E_N = ||u_N||^2_1 + ||v_N||^2 + ||\nabla u_N||^2 + ||\nabla v_N||^2 \\
+ 2 \int_\Omega \left( F_N(u_N + v_N) + F_N(u_N - v_N) \right) dx.
\]

We note that (27) and Gronwall’s lemma imply the dissipative estimate
\[
E_N(t) \leq e^{-ct} E_N(0) + c', \ c > 0, \ t \geq 0.
\] (28)

Integrating now (27) with respect to time, we have
\[
||f_N(u_N + v_N)||_{L^1((0,T) \times \Omega)} + ||f_N(u_N - v_N)||_{L^1((0,T) \times \Omega)} \\
\leq ce^{-ct} \left( ||u_0||^2_1 + ||v_0||^2 + ||\nabla u_0||^2 + ||\nabla v_0||^2 \right) \\
+ c \int_\Omega \left( F_N(u_0 + v_0) + F_N(u_0 - v_0) \right) dx + c''.
\] (29)

Furthermore, for every \( r > 0 \),
\[
\int_t^{t+r} \left( \left| \frac{\partial u_N}{\partial t} \right|_{-1}^2 + \left| \frac{\partial v_N}{\partial t} \right|^2 \right) d\tau \leq ce^{-ct} \left( ||u_0||^2_1 + ||v_0||^2 + ||\nabla u_0||^2 + ||\nabla v_0||^2 \right) \\
+ c \int_\Omega \left( F_N(u_0 + v_0) + F_N(u_0 - v_0) \right) dx + c''(r), \ c' > 0, \ t \geq 0.
\] (30)

Also note that
\[
\int_\Omega \left( F_N(u_0 + v_0) + F_N(u_0 - v_0) \right) dx \leq c,
\] (31)

since ||u_0 + v_0||_{L^\infty} \leq 1 - \delta and ||u_0 - v_0||_{L^\infty} \leq 1 - \delta'. Therefore, (29) yields
\[
||f_N(u_N + v_N)||_{L^1((0,T) \times \Omega)} + ||f_N(u_N - v_N)||_{L^1((0,T) \times \Omega)} \\
\leq ce^{-ct} \left( ||u_0||^2_1 + ||v_0||^2 + ||\nabla u_0||^2 + ||\nabla v_0||^2 + c'' \right) + c.
\] (32)
We have thus found an estimate on the $L^1$-norm of $f_N(u_N + v_N)$ and $f_N(u_N - v_N)$. We now need to derive an estimate of its $L^2$-norm.

Integrating (27) over $[0,t)$, we obtain
\[
\int_0^t \left( \left\| \frac{\partial u_N}{\partial t} \right\|^2 + \left\| \frac{\partial v_N}{\partial t} \right\|^2 \right) dt \leq c' e^{-c'' t} (\|u_0\|^2 + \|v_0\|^2 + \|\nabla u_0\|^2 + \|\nabla v_0\|^2 + c'').
\] (33)

We next multiply (22) by $-\Delta u_N$, (17) by $-\Delta v_N$, and sum the resulting equations to find:
\[
\frac{1}{2} \frac{d}{dt} (\|u_N\|^2 + \|\nabla v_N\|^2) + \|\Delta u_N\|^2 + \|\Delta v_N\|^2 + \left( (f'N + v_N)\nabla(u_N + v_N), \nabla(u_N + v_N) \right) \\
+ \left( (f'(u_N - v_N)\nabla(u_N - v_N), \nabla(u_N - v_N) \right) = 0,
\] (34)

which gives, owing to (13),
\[
\frac{1}{2} \frac{d}{dt} (\|u_N\|^2 + \|\nabla v_N\|^2) + \|\Delta u_N\|^2 + \|\Delta v_N\|^2 \leq c (\|\nabla u_N\|^2 + \|\nabla v_N\|^2).
\] (35)

Furthermore,
\[
\|u_N\|^2_{H^1(\Omega)} \leq c \|u_N\| \|u_N\|_{H^2(\Omega)}, \quad c > 0,
\]
\[
\leq \frac{1}{2} \|\Delta u_N\|^2 + c \|u_N\|^2.
\]

Therefore,
\[
\frac{d}{dt} (\|u_N\|^2 + \|\nabla v_N\|^2) + c (\|\Delta u_N\|^2 + \|\Delta v_N\|^2) \leq c' (\|u_N\|^2 + \|\nabla v_N\|^2).
\] (36)

Using now Gronwall’s lemma, we obtain
\[
\|u_N(t)\|^2 + \|\nabla v_N(t)\|^2 \leq e^{c't} (\|u_0\|^2 + \|\nabla v_0\|^2).
\] (37)

It follows from (36) that
\[
\int_0^t (\|\Delta u_N\|^2 + \|\Delta v_N\|^2) dt \leq c e^{c't} (\|u_0\|^2 + \|\nabla v_0\|^2).
\] (38)

Hence,
\[
\|u_N\|_{L^2(0,T;H^2(\Omega))}^2 + \|v_N\|_{L^2(0,T;H^2(\Omega))}^2 \leq c e^{c't} (\|u_0\|^2 + \|\nabla v_0\|^2).
\] (39)

We now rewrite (22) and (17) as
\[
f_N(u_N + v_N) + f_N(u_N - v_N) = -(-\Delta)^{-1} \frac{\partial u_N}{\partial t} + \Delta u_N,
\] (40)
\[
f_N(u_N + v_N) - f_N(u_N - v_N) = -\frac{\partial v_N}{\partial t} + \Delta v_N.
\] (41)

We sum (40) and (41), then subtract them, to have
\[
2f_N(u_N + v_N) = -(-\Delta)^{-1} \frac{\partial u_N}{\partial t} + \Delta u_N - \frac{\partial v_N}{\partial t} + \Delta v_N,
\]
\[
2f_N(u_N - v_N) = -(-\Delta)^{-1} \frac{\partial u_N}{\partial t} + \Delta u_N + \frac{\partial v_N}{\partial t} - \Delta v_N.
\]
Integrating the previous equations with respect to time and using (30) and (39), we obtain
\[ \|f_N(u_N + v_N)\|_{L^2((0,T) \times \Omega)} + \|f_N(u_N - v_N)\|_{L^2((0,T) \times \Omega)} \]
\[ \leq ce^{c't} \left( \|u_0\|_{L^2(\Omega)}^2 + \|v_0\|_{L^2(\Omega)}^2 + \|\nabla u_0\|_{L^2(\Omega)}^2 + \|\nabla v_0\|_{L^2(\Omega)}^2 + c'' \right). \]
We can then write (28) as
\[ \|u_N(t)\|_{L^2(\Omega)}^2 + \|v_N(t)\|_{L^2(\Omega)}^2 + \|\nabla u_N(t)\|_{L^2(\Omega)}^2 + \|\nabla v_N(t)\|_{L^2(\Omega)}^2 \]
\[ \leq e^{-ct} \left( \|u_0\|_{L^2(\Omega)}^2 + \|v_0\|_{L^2(\Omega)}^2 \right) \]
\[ + \|\nabla u_0\|_{L^2(\Omega)}^2 + \|\nabla v_0\|_{L^2(\Omega)}^2 + 2 \int_\Omega \left( F_N(u_0 + v_0) + F_N(u_0 - v_0) \right) \, dx + c'. \]
Recalling that
\[ \int_\Omega \left( F_N(u_0 + v_0) + F_N(u_0 - v_0) \right) \, dx \leq c, \]
because, noting that if \( N \) is large enough, \( F_N(s_0) = F(s_0) \), and owing to (20) and (21), the previous relation becomes
\[ \|u_N(t)\|_{L^2(\Omega)}^2 + \|v_N(t)\|_{L^2(\Omega)}^2 + \|\nabla u_N(t)\|_{L^2(\Omega)}^2 + \|\nabla v_N(t)\|_{L^2(\Omega)}^2 \]
\[ \leq ce^{-ct} \left( \|\nabla u_0\|_{L^2(\Omega)}^2 + \|\nabla v_0\|_{L^2(\Omega)}^2 + c'' \right). \]

4. Existence and uniqueness of solutions.

**Theorem 4.1.** We assume that \( u_0 \) and \( v_0 \) are given such that \( (u_0, v_0) \in H^1(\Omega)^2 \), \( \|u_0 + v_0\|_{L^\infty(\Omega)} < 1 \), and \( \|u_0 - v_0\|_{L^\infty(\Omega)} < 1 \). Then, (1)-(4) possesses a unique (weak) solution \((u, v)\) such that, \( \forall \, T > 0 \),
\[ (u, v) \in C_w([0,T]; H_0^1(\Omega)^2) \cap L^2(0,T; H^2(\Omega)^2) \cap L^\infty(0,T; H_0^1(\Omega)^2) \]
and
\[ \left( \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \right) \in L^2(0,T; H^{-1}(\Omega) \times L^2(\Omega)), \]
where the subscript \( w \) stands for the weak topology, and
\[ \frac{d}{dt}((u, q))_1 + ((\nabla u, \nabla q)) + ((f(u + v) + f(u - v), q)) = 0, \]
\[ \frac{d}{dt}((v, q)) + ((\nabla v, \nabla q)) + ((f(u + v) - f(u - v), q)) = 0, \]
a.e. \( t \in [0,T], \forall \, q \in C_c^\infty(\Omega). \)

**Proof.** a) Existence

We consider the solution \((u_N, v_N)\) to the approximated problem (16)-(19) (as already mentioned in the previous section, the proof of existence of such a solution having the above regularity can be obtained by a standard Galerkin scheme).

Furthermore, since the estimates derived in the previous section are independent of \( N \), this solution converges, up to a subsequence which we do not relabel, to a limit function \((u, v)\) in the following sense:
\[ u_N \to u \in L^2(0,T; H^2(\Omega)) \text{ weakly owing to (39)}, \]
\[ \frac{\partial u_N}{\partial t} \to \frac{\partial u}{\partial t} \in L^2(0,T; H^{-1}(\Omega)) \text{ weakly owing to (33) and a.e.}, \]
\[ v_N \to v \in L^2(0,T; H^2(\Omega)) \text{ weakly owing to (39) and a.e.}, \]
Here, the constant $c$ and $f$ nonlinear terms containing pactness results.

The only difficulty, when passing to the limit, is to pass to the limit in the nonlinear terms containing $f_N$. First, it follows from (32) that $f_N(u_N + v_N)$ and $f_N(u_N - v_N)$ are bounded, independently of $N$, in $L^1((0,T) \times \Omega)$. Then, it follows from the explicit expression of $f_N$ that

$$\text{meas}(F_{N,M}) \leq \mu(\frac{1}{N}), \quad N \leq M,$$

where

$$F_{N,M} = \{(t,x) \in (0,T) \times \Omega, \; |u_M(t,x) + v_M(t,x)| > 1 - \frac{1}{N}\}$$

and

$$\mu(s) = \frac{c}{\min(|f(1-s)|, |f(s-1)|)}.$$  

Here, the constant $c$ is independent of $N$ and $M$. Note that there holds

$$\int_0^T \int_\Omega |f_M(u_M + v_M)| dx \; dt \geq \int_{F_{N,M}} |f_M(u_M + v_M)| dx \; dt$$

$$\geq c' \text{meas}(F_{N,M}) \frac{1}{\mu(\frac{1}{N})},$$

where the constant $c'$ is independent of $N$ and $M$.

We can pass to the limit $M \to \infty$ (employing Fatou’s lemma on (44)) and then $N \to \infty$ (noting that $\lim_{s \to 0} \mu(s) = 0$) to find

$$\text{meas}\{(t,x) \in (0,T) \times \Omega, |u(t,x) + v(t,x)| \geq 1\} = 0,$$

so that

$$-1 < u(t,x) + v(t,x) < 1 \text{ a.e. } (t,x).$$

In the same way, we can prove that

$$-1 < u(t,x) - v(t,x) < 1 \text{ a.e. } (t,x).$$

Next, it follows from the above almost everywhere convergence of $u_N$ and $v_N$, from (45), (46), and the explicit expression of $f_N$ that

$$f_N(u_N + v_N) \to f(u + v) \text{ a.e. } (t,x) \in (0,T) \times \Omega$$

and

$$f_N(u_N - v_N) \to f(u - v) \text{ a.e. } (t,x) \in (0,T) \times \Omega.$$

Finally, since, owing to (42), $f_N(u_N + v_N)$ and $f_N(u_N - v_N)$ are bounded, independently of $N$, in $L^2((0,T) \times \Omega)$, it follows from (47) and (48) that $f_N(u_N + v_N) \to f(u + v)$ and $f_N(u_N - v_N) \to f(u - v)$ in $L^2((0,T) \times \Omega)$ weakly, which finishes the proof of the passage to the limit (the weak continuity property follows from Strauss’s lemma, see, e.g. [20]).

We also note that it follows from (43) that $(u,v) \in L^\infty(0,T; H^1_0(\Omega)^2)$.

b) Uniqueness

Let $(u_1, v_1)$ and $(u_2, v_2)$ be two solutions to (1)-(4) with initial data $(u_{01}, v_{01})$ and $(u_{02}, v_{02})$, respectively. We set $(u,v) = (u_1 - u_2, v_1 - v_2)$ and $(\tilde{u}, \tilde{v}) = (u_{01} - u_{02}, v_{01} - v_{02})$ and have

$$(-\Delta)^{-1} \frac{\partial u}{\partial t} - \Delta u + f(u_1 + v_1) - f(u_2 + v_2)$$

$$+ f(u_1 - v_1) - f(u_2 - v_2) = 0,$$

(49)
\[ \frac{\partial v}{\partial t} - \Delta v + f(u_1 + v_1) - f(u_2 + v_2) \]
\[ - f(u_1 - v_1) + f(u_2 - v_2) = 0, \]
\[ u = \Delta u = v = 0 \text{ on } \Gamma, \]
\[ u|_{t=0} = u_0, \ v|_{t=0} = v_0. \] (51)

We multiply (49) by \( u \), (50) by \( v \), and sum to obtain
\[ \frac{1}{2} \frac{d}{dt} (||u||^2_{-1} + ||v||^2) + ||\nabla u||^2 + ||\nabla v||^2 \]
\[ + ((f(u_1 + v_1) - f(u_2 + v_2), u + v)) \]
\[ + ((f(u_1 - v_1) - f(u_2 - v_2), u - v)) = 0. \] (53)

Let \( p = u_1 + v_1, \ q = u_2 + v_2, \ h = u_1 - v_1, \text{ and } l = u_2 - v_2 \). We have
\[ ((f(u_1 + v_1) - f(u_2 + v_2), u + v)) = ((f(p) - f(q), p - q)) \]
\[ = ((f'(\xi)(p - q), p - q)) \]
\[ \geq -c_0||p - q||^2 \]
\[ \geq -c_0||u + v||^2, \] (54)
and
\[ ((f(u_1 - v_1) - f(u_2 - v_2), u - v)) = ((f(h) - f(l), h - l)) \]
\[ \geq -c_0||u - v||^2. \] (55)

Hence, using the previous inequalities, equation (53) yields
\[ \frac{1}{2} \frac{d}{dt} (||u||^2_{-1} + ||v||^2) + ||\nabla u||^2 + ||\nabla v||^2 \]
\[ = -(f(u_1 + v_1) - f(u_2 + v_2), u + v)) \]
\[ - ((f(u_1 - v_1) - f(u_2 - v_2), u - v)) \]
\[ \leq c_0||u + v||^2 + c_0||u - v||^2 \]
\[ \leq c(||u||^2 + ||v||^2). \] (56)

Employing the interpolation inequality
\[ ||u||^2 \leq c \ ||u||_{-1} ||\nabla u|| \leq \frac{1}{2} ||\nabla u||^2 + c ||u||^2_{-1}, \]
we deduce that
\[ \frac{d}{dt}(||u||^2_{-1} + ||v||^2) + ||\nabla u||^2 + ||\nabla v||^2 \leq c'(||u||^2_{-1} + ||v||^2). \] (57)

It finally follows from Gronwall’s lemma that
\[ (||u||^2_{-1} + ||v||^2) \leq e^{c't}(||u_0||^2_{-1} + ||v_0||^2), \] (58)

hence the uniqueness (taking \( (u_0, v_0) = (0, 0) \)), as well as the continuous dependence with respect to the initial data in \( H^{-1}(\Omega) \times L^2(\Omega) \). \qed
5. **Existence of the global attractor.** It follows from Theorem 4.1 that we can define the continuous family of operators

\[ S(t) : \Phi_1 \to \Phi \]

\[ (u_0, v_0) \to (u(t), v(t)), \]

where

\[ \Phi = \{(u, v) \in H^1(\Omega)^2; |u + v| < 1 \text{ and } |u - v| < 1 \text{ a.e.}\} \]

and

\[ \Phi_1 = \Phi \cap \{(u, v) \in L^\infty(\Omega)^2; ||u + v||_{L^\infty(\Omega)} < 1 \text{ and } ||u - v||_{L^\infty(\Omega)} < 1\}. \]

**Theorem 5.1.** The family of operators associated with (1)-(2) is dissipative in \( H^1(\Omega)^2 \) in the sense that it possesses a bounded absorbing set \( B_1 \subset H^1(\Omega)^2 \), i.e.

\[ \forall B \subset \Phi_1 \text{ bounded, } \exists t_0 = t_0(B) \text{ such that } t \geq t_0 \text{ implies } S(t)B \subset B_1. \]

**Proof.** The dissipativity of \( S(t) \) and the existence of a bounded absorbing set in \( H^1(\Omega)^2 \) immediately follow from (43).

We now assume that

\[ \lim_{s \to \pm 1} F(s) = c, \quad (59) \]

where \( c \) is a constant (note that this holds for the thermodynamically relevant logarithmic potentials). Then, \( S(t) \) as defined above is now a semigroup on \( \Phi \) (i.e. \( S(0) = I \) (identity operator) and \( S(t + \tau) = S(t) \circ S(\tau), t, \tau \geq 0 \)).

As a consequence of Theorem 5.1 and of (59), it follows from standard results (see, e.g. [2, 10, 14, 21]) that we have the following theorem.

**Theorem 5.2.** The semigroup \( S(t) \) possesses the global attractor \( \mathcal{A} \) on \( \Phi \) (i.e. \( \mathcal{A} \) is compact in \( H^{-1}(\Omega) \times L^2(\Omega) \), bounded in \( \Phi \), invariant and attracts the images of all bounded subsets of \( \Phi \) with respect to the topology of \( H^{-1}(\Omega) \times L^2(\Omega) \)).

**Remark 2.** In order to prove that one has the global attractor and, in particular, the attraction property in the natural topology of the phase space \( \Phi \), one would need additional regularity on the solutions or the strict separation property from the singular values \( \pm 1 \) which we are not able to prove, except in one space dimension (see Section 6 below). We can also note that it follows from (58) that we can extend, in a unique way and by continuity, the semigroup \( S(t) \) to the closure of \( \Phi \) in the \( H^{-1}(\Omega) \times L^2(\Omega) \)-topology, namely to

\[ \overline{\Phi} = \{(u, v) \in L^\infty(\Omega)^2; ||u + v||_{L^\infty(\Omega)} \leq 1 \text{ and } ||u - v||_{L^\infty(\Omega)} \leq 1\}. \]

The corresponding semigroup again possesses the global attractor which is precisely \( \mathcal{A} \).

6. **Further regularity results.** In what follows, we set \( V = H^1_0(\Omega) \). We also denote by \( V' \) its dual space and by \( ||.||_{V'} \) its norm.

We can decompose the singular potential \( F \) as

\[ F(x) = S(x) + \frac{\theta}{2}(1 - x^2), \]

with

\[ \lim_{x \to -1} S'(x) = -\infty, \lim_{x \to 1} S'(x) = +\infty, S''(x) \geq \theta > 0, \forall x \in (-1, 1), \quad (60) \]
and we let
\[ \theta_c - \theta = \alpha > 0. \]

We also require that \(S\) satisfies
\[ |S''(x)| \leq e^{c|S'(x)|+c}, \forall x \in (-1,1), \tag{61} \]
for some positive constant \(c\), and \(S''\) is convex.

We mention below a Trudinger-Moser type inequality (see, e.g. \[16\]) which is needed later on.

**Lemma 6.1.** Let \(\Omega\) be a bounded smooth domain of \(\mathbb{R}^2\). Then, there exists a unique solution \((u,v)\) such that
\[ \int_\Omega e^{c|u|^2} \, dx \leq c e^{c|u|^2}, \forall u \in V. \tag{62} \]

Let us now define the free energy functional
\[ \mathcal{E}(u + v) = \frac{1}{2}||\nabla u||^2 + \frac{1}{2}||\nabla v||^2 + 2 \int_\Omega F(u + v) \, dx + 2 \int_\Omega F(u - v) \, dx. \]

**Theorem 6.2.** Let \(u_0\) and \(v_0\) be in \(V\) such that \(F(u_0 + v_0)\) and \(F(u_0 - v_0) \in L^1(\Omega)\). Then, there exists a unique solution \((u,v)\) in \(C([0,T]; H^{-1}_w(\Omega) \times L^2_w(\Omega))\) which fulfills the estimate
\[ \mathcal{E}(u + v(t)) + \int_t^{t+1} (||\nabla \mu(s)||^2 + ||v_t(s)||^2) \, ds \leq \mathcal{E}(u_0 + v_0), \forall t \geq 0. \tag{63} \]

**Proof.** The existence and uniqueness of the solution can be proved in the same way as in section 4. Therefore, we confine ourselves only to the proof of (63).

We rewrite equation (1) in the form
\[ (-\Delta)^{-1} \frac{\partial u}{\partial t} - \Delta u + f(u + v) + f(u - v) = 0. \tag{64} \]

We start by differentiating equations (64) and (2) with respect to time to find
\[ (-\Delta)^{-1} \frac{\partial u}{\partial t} - \Delta \frac{\partial u}{\partial t} + f'(u + v)(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t}) + f'(u - v)(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}) = 0, \tag{65} \]
\[ \frac{\partial}{\partial t} \frac{\partial v}{\partial t} - \Delta \frac{\partial v}{\partial t} + f'(u + v)(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t}) + f'(u - v)(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}) = 0, \tag{66} \]
\[ \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0 \text{ on } \Gamma. \tag{67} \]

Summing (65) times \(t \frac{\partial u}{\partial t}\) and (66) times \(t \frac{\partial v}{\partial t}\), then using (11), we obtain
\[ \frac{d}{dt} \left( \frac{1}{2} ||\frac{\partial u}{\partial t}||^2_{L^2} + \frac{1}{2} ||\frac{\partial v}{\partial t}||^2_{L^2} \right) + t ||\nabla \frac{\partial u}{\partial t}||^2 + t ||\nabla \frac{\partial v}{\partial t}||^2 \leq \frac{1}{2} ||\frac{\partial u}{\partial t}||^2_{L^2} + \frac{1}{2} ||\frac{\partial v}{\partial t}||^2 + c't \left( ||\frac{\partial u}{\partial t}||^2 + ||\frac{\partial v}{\partial t}||^2 \right). \]

Employing the interpolation inequality
\[ ||\frac{\partial u}{\partial t}||^2 \leq c' ||\frac{\partial u}{\partial t}||_{L^2} ||\nabla \frac{\partial u}{\partial t}|| \]
\[ \leq c' ||\frac{\partial u}{\partial t}||^2_{L^2} + \frac{1}{2} ||\nabla \frac{\partial u}{\partial t}||^2, \]

\[ ||\frac{\partial u}{\partial t}||^2 \leq c' ||\frac{\partial u}{\partial t}||_{L^2} ||\nabla \frac{\partial u}{\partial t}|| \]
\[ \leq c' ||\frac{\partial u}{\partial t}||^2_{L^2} + \frac{1}{2} ||\nabla \frac{\partial u}{\partial t}||^2, \]
we find
\[ \frac{1}{2} \frac{d}{dt} \left( \| \frac{\partial u}{\partial t} \|_{L^2}^2 + \| \frac{\partial v}{\partial t} \|_{L^2}^2 \right) + c t \left( \| \nabla u \|_{L^2}^2 + \| \nabla v \|_{L^2}^2 \right) \leq \frac{1}{2} \| u \|_{H^1}^2 + \frac{1}{2} \| v \|_{H^1}^2 + c \left( \frac{1}{2} \| \frac{\partial u}{\partial t} \|_{L^2}^2 + \| \frac{\partial v}{\partial t} \|_{L^2}^2 \right). \] (68)

We deduce from (30), (31) (which hold when \( N \to +\infty \)), (68), and Gronwall’s lemma that
\[ \frac{1}{2} \| u(t) \|_{H^1}^2 + \frac{1}{2} \| v(t) \|_{H^1}^2 \leq \frac{1}{t} Q(\| u_0 \|_{H^1(\Omega)}, \| v_0 \|_{H^1(\Omega)}), \quad t \in (0, 1). \] (69)

Summing then (65) times \( \frac{\partial u}{\partial t} \) to (66) times \( \frac{\partial v}{\partial t} \) and using (11) and an interpolation inequality, we have
\[ \frac{d}{dt} \left( \frac{1}{2} \| \frac{\partial u}{\partial t} \|_{L^2}^2 + \frac{1}{2} \| \frac{\partial v}{\partial t} \|_{L^2}^2 \right) + 2 \left( \| \nabla u \|_{L^2}^2 + \| \nabla v \|_{L^2}^2 \right) \leq c \left( \frac{1}{2} \| \frac{\partial u}{\partial t} \|_{L^2}^2 + \| \frac{\partial v}{\partial t} \|_{L^2}^2 \right). \] (70)

Finally, we conclude from (69), (70), and Gronwall’s lemma that
\[ \frac{1}{2} \| u(t) \|_{H^1}^2 + \| \nabla u(t) \|_{L^2}^2 \leq e^{c t} Q(\| u_0 \|_{H^1(\Omega)}, \| v_0 \|_{H^1(\Omega)}), \quad t \geq 1. \] (71)

where \( Q \) denotes a monotone increasing function.

Integrating now (70) between \( t \) and \( t + 1 \) and using (71), we find
\[ \int_t^{t+1} \left( \| \nabla \frac{\partial u}{\partial t} \|_{L^2}^2 + \| \nabla \frac{\partial v}{\partial t} \|_{L^2}^2 \right) dt \leq e^{c t} Q(\| u_0 \|_{H^1(\Omega)}, \| v_0 \|_{H^1(\Omega)}). \] (72)

for \( t \geq 1 \). Therefore, \( (\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}) \in L^2(t, t + 1; H^1_0(\Omega)^2) \).

Next, we rewrite equations (1) and (2) as
\[ u_t = \Delta \mu, \] (73)
where
\[ \mu = -\Delta u + F'(u + v) + F'(u - v), \]
and
\[ v_t + F'(u + v) - F'(u - v) = \Delta v. \] (74)

Equations (73) and (74) are equivalent to
\[ ((u_t, q)) + ((\nabla \mu, \nabla q)) = 0, \quad \forall \ q \in H^1_0(\Omega), \] (75)
and
\[ ((v_t, q')) + ((F'(u + v), q')) - ((F'(u - v), q')) \]
\[ + ((\nabla v, q')) = 0, \quad \forall \ q' \in H^1_0(\Omega). \] (76)

Using equations (75)-(76) and the standard chain rule in \( L^2(0, T; V) \cap H^1(0, T; V') \), we get
\[ ((u_t, \mu)) + ((\nabla \mu, \nabla \mu)) + ((v_t, v_t)) + ((F'(u + v), v_t)) - ((F'(u - v), v_t)) \]
\[ + ((\nabla v, \nabla v_t)) = 0, \]
which gives the energy equality
\[ \frac{d}{dt} \mathcal{E}(u + v) + \| \nabla \mu \|^2 + \| v_t \|^2 = 0. \]
It finally follows from the Gronwall lemma that
\[ \mathcal{E}((u + v)(t)) + \int_t^{t+1} (||\nabla \mu(s)||^2 + ||v_t(s)||^2) ds \leq \mathcal{E}(u_0 + v_0), \quad \forall \ t \geq 0. \]

\[ \square \]

In what follows, according to (63), the generic positive constant \( c \) may also depend on the initial energy \( \mathcal{E}(u_0 + v_0) \). In particular, we will use
\[ \mathcal{E}((u + v)(t)) + \int_t^{t+1} (||\nabla \mu(s)||^2 + ||v_t(s)||^2) ds \leq c, \quad \forall \ t \geq 0. \quad (77) \]

**Theorem 6.3.** Let the assumptions of Theorem 6.1 hold. Then there exists a positive constant \( c \) such that
\[ ||\mu||_{L^\infty(1,t;V)} + ||v_t||_{L^\infty(1,t;L^2(\Omega))} \leq c, \quad \forall \ t \geq 1, \quad (80) \]
and
\[ ||u_t + v_t||_{L^\infty(1,t';V')} + ||u_t - v_t||_{L^\infty(1,t';V')} + ||u||_{L^2(t,t+1;V')} + ||v||_{L^2(t,t+1;V')} \leq c, \quad \forall \ t \geq 1. \quad (79) \]

**Proof.** We start by differentiating equation (74) with respect to time, which yields
\[ \frac{\partial}{\partial t} v_t + \frac{\partial}{\partial t} [F'(u + v) - F'(u - v)] = \Delta v_t. \quad (80) \]

Testing (73) by \( \mu_t \) and (80) by \( v_t \), then summing the resulting equations, we obtain
\[
\frac{1}{2} \frac{d}{dt} ( ||\nabla \mu||^2 + ||v_t||^2 ) + ||\nabla v_t||^2 + ( (u_t - \Delta u_t) + (F''(u + v)(u_t + v_t), u_t + v_t) + (F''(u - v)(u_t - v_t), u_t - v_t) = 0. \]

We observe that
\[
( (u_t, - \Delta u_t) + (F''(u + v)(u_t + v_t), u_t + v_t) + (F''(u - v)(u_t - v_t), u_t - v_t) ) \]
\[
\geq ||\nabla u_t||^2 - \alpha ||u_t + v_t||^2 - \alpha ||u_t - v_t||^2 \]
\[
\geq \frac{1}{2} ||\nabla u_t||^2 - c ||\nabla v_t||^2 - c ||u_t + v_t||^2_{V'} - c ||u_t - v_t||^2_{V'}. \quad (81) \]

Accordingly, setting
\[ \psi(t) = \frac{1}{2} ||\nabla \mu(t)||^2 + \frac{1}{2} ||v_t(t)||^2, \]
we end up with the differential inequality
\[ \frac{d}{dt} \psi + \frac{1}{2} ||\nabla u_t||^2 + c ||\nabla v_t||^2 \leq c ||u_t + v_t||^2_{V'} + c ||u_t - v_t||^2_{V'}. \quad (82) \]

Using (77), the definition of \( \mathcal{E} \), and (12) we can deduce that
\[ \int_t^{t+1} \psi(s) ds \leq c, \quad c \geq 0. \quad (83) \]

Furthermore, using (72),
\[ \int_t^{t+1} ||u_t + v_t||^2_{V'} ds \leq c \quad (84) \]
and
\[ \int_t^{t+1} ||u_t - v_t||^2_{V'} ds \leq c. \quad (85) \]
Therefore, the uniform Gronwall lemma leads to
\[ \psi(t) \leq c, \quad \forall \ t \geq 1. \]
In particular, we have the bound
\[ ||\mu||_{L^\infty(1,t;V)} + ||v_t||_{L^\infty(1,t;L^2(\Omega))} \leq c, \quad \forall \ t \geq 1, \]
which in turn gives
\[ ||u_t||_{L^\infty(1,t;V')} + ||v_t||_{L^\infty(1,t;V')} \leq c, \quad \forall \ t \geq 1. \]
The desired conclusion (79) follows from an integration in time of (82) on \((t, t+1), \ t \geq 1,\) combined with the previous inequality.

**Remark 3.** The proof of Theorem 6.1 is formal, but it can be justified within a Galerkin scheme as in the proof of Theorem 4.1. More precisely, all the computations can be rigorously performed within the Galerkin scheme. Given that \(F''\) is controlled from below, the estimates turn out to be independent of the approximation parameter and a final passage to the limit gives the result.

**Lemma 6.4.** Let \(N = 2\) and \(u_0\) and \(v_0 \in V\) be such that \(F(u_0 + v_0)\) and \(F(u_0 - v_0) \in L^1(\Omega).\) Then, for any \(p \geq 1,\) there exist two positive constants \(c\) and \(c'\) depending on \(p\) such that
\[ ||S''(u + v)||_{L^p(t,t+1;L^p(\Omega))} \leq c(p), \quad \forall \ t \geq 1, \tag{86} \]
and
\[ ||S''(u - v)||_{L^p(t,t+1;L^p(\Omega))} \leq c'(p), \quad \forall \ t \geq 1. \tag{87} \]

**Proof.** Equations (1) and (2) can be written in the equivalent form
\[ -\Delta u + S'(u + v) + S'(u - v) = \tilde{\mu}, \tag{88} \]
\[ -\Delta v + S'(u + v) - S'(u - v) = \tilde{\mu}', \tag{89} \]
where
\[ \tilde{\mu} = -(-\Delta)^{-1}\frac{\partial u}{\partial t} + 2\theta_c u \]
and
\[ \tilde{\mu}' = -\frac{\partial v}{\partial t} + 2\theta_c v. \]
Combining (88) and (89), we obtain
\[ -\Delta (u + v) + 2S'(u + v) = \tilde{\mu} + \tilde{\mu}', \tag{90} \]
and
\[ -\Delta (u - v) + 2S'(u - v) = \tilde{\mu} - \tilde{\mu}'. \tag{91} \]
For any \(L > 0,\) we consider
\[ g_1 = S'(u + v)e^{L|S'(u+v)|} \]
and
\[ g_2 = S'(u - v)e^{L|S'(u-v)|}. \]
We observe that
\[ \nabla g_1 = S''(u + v)[1 + L|S'(u + v)|e^{L|S'(u+v)|}] \nabla (u + v). \]
Then we consider equation (90) and test it with \( g_1 \), which yields
\[
\int_{\Omega} \nabla(u + v) \nabla(u + v) S''(u + v)[1 + L] S'(u + v) e^{L |S'(u+v)|} \, dx
\]
\[
+ 2 \int_{\Omega} S'(u + v)^2 e^{L |S(u+v)|} \, dx = \int_{\Omega} \mu g_1 \, dx + \int_{\Omega} \mu' g_1 \, dx.
\]
The first term on the left-hand side is nonnegative (\( S'' \) is convex). Therefore, the previous relation is equivalent to
\[
2 \int_{\Omega} S'(u + v)^2 e^{L |S(u+v)|} \, dx \leq \int_{\Omega} \mu g_1 \, dx + \int_{\Omega} \mu' g_1 \, dx.
\]
Now, the right-hand side can be controlled by means of a generalized Young’s inequality (see [1], Section 8.2) as follows:
\[
\int_{\Omega} \mu S'(u + v) e^{L |S(u+v)|} \, dx \leq \int_{\Omega} \mu |S'(u + v)| e^{L |S(u+v)|} \, dx
\]
\[
\leq \frac{1}{2} \int_{\Omega} S'(u + v)^2 e^{L |S(u+v)|} \, dx + \int_{\Omega} \mu c(L) \, dx + c.
\]
In the same way,
\[
\int_{\Omega} \mu' g_1 \, dx \leq \frac{1}{2} \int_{\Omega} S'(u + v)^2 e^{L |S(u+v)|} \, dx + \int_{\Omega} \mu c(L) \, dx + c.
\]
Using Lemma 6.1, we end up with
\[
2 \int_{\Omega} S'(u + v)^2 e^{L |S(u+v)|} \, dx \leq \int_{\Omega} S'(u + v)^2 e^{L |S(u+v)|} \, dx + 2c,
\]
whence
\[
\int_{\Omega} S'(u + v)^2 e^{L |S(u+v)|} \, dx \leq c,
\]
where \( c \) depends on \( L \). On account of (61), we observe that, for any \( p \geq 1 \),
\[
|S''(x)|^p \leq c e^{pc} + |S'(x)|^2 e^{pc[S'(x)]}, \quad \forall x \in (-1,1).
\]
Combining finally (61) and (93) and taking \( L = pc \), we deduce that
\[
\|S''(u + v)\|_{L^p(t,t+1;L^p(\Omega))} \leq c(p), \quad \forall t \geq 1.
\]
Similarly, if we consider equation (91) and test it with \( g_2 \), we can easily obtain
\[
\|S''(u - v)\|_{L^p(t,t+1;L^p(\Omega))} \leq c(p), \quad \forall t \geq 1.
\]

\[ \square \]

**Lemma 6.5.** Let the assumptions of Lemma 6.2 hold. Then, there exists a constant \( c \) such that
\[
\|u_t\|_{L^\infty(2,t;L^2(\Omega))} + \|u_t\|_{L^2(2,t+1,H^2(\Omega))} + \|v_t\|_{L^\infty(2,t;H^1(\Omega))}
\]
\[
+ \|v_t\|_{L^2(2,t+1,H^2(\Omega))} \leq c, \quad \forall t \geq 2.
\]

**Proof.** Differentiating equations (73) and (74) with respect to time, we obtain
\[
\frac{\partial}{\partial t} u_t = \Delta u_t
\]
and
\[
\frac{\partial}{\partial t} v_t + \frac{\partial}{\partial t} [S'(u + v) - S'(u - v)] - 2\theta_c v_t = \Delta v_t.
\]
Hence, using H"older and Young's inequalities and Lemma 6.2, we have
\[ \frac{1}{2} \frac{d}{dt} (|u_t|^2 + |\nabla v_t|^2) + ||\Delta v_t||^2 = 2\theta_c ||\nabla v_t||^2 + \int_\Omega \partial_t |S'(u + v) - S'(u - v)| \Delta v_t \, dx + ((\Delta u_t, u_t)). \tag{97} \]

Note that
\[(\Delta u_t, u_t) = -((\Delta u_t, \Delta u_t)) - 2((\theta_c \Delta u_t, u_t)) + ((\partial_t [S'(u + v) + S'(u - v)], \Delta u_t)). \tag{98} \]

Moreover, using Hölder and Young's inequalities and Lemma 6.2, we have
\[ \int_\Omega S''(u + v)(u_t + v_t) \Delta v_t \, dx \leq ||S''(u + v)||_{L^3(\Omega)} ||u_t + v_t||_{L^6(\Omega)} ||\Delta v_t||_{L^2(\Omega)} \leq \frac{3c(p)^2}{4} ||u_t + v_t||_{L^2(\Omega)}^2 + \frac{1}{4} ||\Delta v_t||^2 \tag{99} \]
and similarly
\[ \int_\Omega S''(u - v)(u_t - v_t) \Delta v_t \, dx \leq ||S''(u - v)||_{L^3(\Omega)} ||u_t - v_t||_{L^6(\Omega)} ||\Delta v_t||_{L^2(\Omega)} \leq \frac{3c(p)^2}{4} ||u_t - v_t||_{L^2(\Omega)}^2 + \frac{1}{4} ||\Delta u_t||^2, \tag{100} \]
\[ \int_\Omega S''(u + v)(u_t + v_t) \Delta u_t \, dx \leq ||S''(u + v)||_{L^3(\Omega)} ||u_t + v_t||_{L^6(\Omega)} ||\Delta u_t||_{L^2(\Omega)} \leq \frac{3c(p)^2}{4} ||u_t + v_t||_{L^2(\Omega)}^2 + \frac{1}{4} ||\Delta u_t||^2, \tag{101} \]
\[ \int_\Omega S''(u - v)(u_t - v_t) \Delta u_t \, dx \leq ||S''(u - v)||_{L^3(\Omega)} ||u_t - v_t||_{L^6(\Omega)} ||\Delta u_t||_{L^2(\Omega)} \leq \frac{3c(p)^2}{4} ||u_t - v_t||_{L^2(\Omega)}^2 + \frac{1}{4} ||\Delta u_t||^2. \tag{102} \]

We deduce from (97)-(102) that
\[ \frac{d}{dt} (|u_t|^2 + |\nabla v_t|^2) + ||\Delta v_t||^2 + ||\Delta u_t||^2 \leq c||u_t + v_t||_{L^2(\Omega)}^2 + c||u_t - v_t||_{L^2(\Omega)}^2, \tag{103} \]
where $c$ depends on $p$. Then, owing to the continuous embedding $H^1(\Omega) \subset L^6(\Omega)$, we have
\[ \frac{1}{2} \frac{d}{dt} (|u_t|^2 + |\nabla v_t|^2) + ||\Delta v_t||^2 + ||\Delta u_t||^2 \leq c||\nabla v_t||^2 + c||\nabla u_t||^2. \tag{104} \]
Furthermore,
\[ ||u_t||_{H^1(\Omega)}^2 \leq c||u_t||_{H^2(\Omega)} \leq c||u_t||^2 + c||\Delta u_t||^2, \quad c > 0. \]

Hence
\[ \frac{1}{2} \frac{d}{dt} (|u_t|^2 + |\nabla v_t|^2) + ||\Delta v_t||^2 + ||\Delta u_t||^2 \leq c(||u_t||^2 + ||\nabla v_t||^2). \tag{105} \]

Using (78), (79), and the uniform Gronwall lemma on (104), we obtain the desired result. \[\square\]
Remark 4. The proof of Lemma 6.3 is carried out with the solution itself since (86) and (87) cannot be guaranteed within a Galerkin scheme.

Theorem 6.6. Let \( N = 1 \) and let \( f \) satisfy assumption (10). Then, there exist two positive constants \( \delta \) and \( \delta' \) and \( T > 0 \) such that

\[
|| (u + v)(t) ||_{L^\infty(\Omega)} \leq 1 - \delta, \quad \forall \ 2 \leq t \leq T, \tag{105}
\]

and

\[
|| (u - v)(t) ||_{L^\infty(\Omega)} \leq 1 - \delta', \quad \forall \ 2 \leq t \leq T. \tag{106}
\]

Proof. Since we are in one space dimension, we have the Sobolev embedding \( H^1(\Omega) \subset C(\overline{\Omega}) \). Therefore, it follows from (77) and (79) that

\[
|| \tilde{\mu} ||_{L^\infty(2, T; L^\infty(\Omega))} \leq c, \quad \forall \ 2 \leq t \leq T.
\]

Testing (90) by \( |S'(u + v)|^{p-2} S'(u + v) \), we get

\[
(p - 1) \int_{\Omega} |S'(u + v)|^{p-2} S''(u + v)|\nabla (u + v)|^2 dx + 2||S'(u + v)||_{L^p(\Omega)}^p \]

\[= \int_{\Omega} \tilde{\mu} |S'(u + v)|^{p-2} S'(u + v) dx - \int_{\Omega} v_1 |S'(u + v)|^{p-2} S'(u + v) dx\]

\[+ 2\theta_c \int_{\Omega} |S'(u + v)|^{p-2} S'(u + v) dx.
\]

The first term on the left-hand side is nonnegative, so that an application of the Hölder inequality yields

\[
||S'(u + v)||_{L^p(\Omega)} \leq c(||\tilde{\mu}||_{L^p(\Omega)} + ||v_1||_{L^p(\Omega)} + ||v||_{L^p(\Omega)}). \tag{107}
\]

Recalling equation (36) which holds when \( N \to +\infty \), we have

\[
\frac{d}{dt} (||u||^2 + ||\nabla v||^2) + c(||\Delta u||^2 + ||\Delta v||^2) \leq c'(||u||^2 + ||\nabla v||^2). \tag{108}
\]

Integrating (37) (which also holds when \( N \to +\infty \)) yields

\[
\int_{t}^{t+1} (||u||^2 + ||\nabla v||^2) d\tau \leq c, \quad t \geq 0. \tag{109}
\]

Then, applying the uniform Gronwall’s lemma to (108) and using (109), we find

\[
||u(t)||^2 + ||\nabla v(t)||^2 \leq c, \quad t \geq 1. \tag{110}
\]

Hence, using again the continuous embedding \( H^1(\Omega) \subset C(\overline{\Omega}) \), integrating (107) in time from \( t \) to \( t + 1 \) and using (78) and (110), we have

\[
||S'(u + v)||_{L^p(\Omega \times (t, t+1))} \leq c, \quad \forall \ 2 \leq t \leq T,
\]

where \( c \) is independent of \( p \) and \( t \). Applying Theorem 2.14 in [1], we obtain

\[
||S'(u + v)||_{L^\infty(\Omega \times (t, t+1))} \leq c, \quad \forall \ 2 \leq t \leq T.
\]

This implies that there exists \( \delta > 0 \) such that

\[
||u + v||_{L^\infty(\Omega \times (t, t+1))} \leq 1 - \delta, \quad \forall \ 2 \leq t \leq T.
\]

Since \( u + v \in L^\infty(0, t; C(\overline{\Omega})) \) for all \( t \geq 0 \), we also infer that

\[
||u + v||_{L^\infty(\Omega \times (2, t))} \leq 1 - \delta, \quad \forall \ 2 \leq t \leq T.
\]

Finally, we deduce (105) from the continuity in time.

In the same way, we can obtain the second inequality (106) using equation (91) instead of (90). \( \square \)
An immediate consequence is the following

**Corollary 1.** Let the assumptions of Theorem 6.3 hold. Then, there exists a positive constant $C$ such that

$$\|u(t)\|_{H^4(\Omega)} \leq C, \quad \forall \, t \geq 3.$$ 

**Corollary 2.** It also follows from Theorem 6.3 that $u$ and $v$ are strictly separated from the pure states, i.e. there exist two positive constants $\gamma$ and $\gamma'$ such that

$$|u(t,x)| \leq 1 - \gamma, \quad \forall \, (x,t) \in \Omega \times (2,T),$$

and

$$|v(t,x)| \leq 1 - \gamma', \quad \forall \, (x,t) \in \Omega \times (2,T).$$

**Remark 5.** We can note that the strict separation property in two space dimensions can be obtained from the estimates of section 6 provided that we can prove that $\|v\|_{L^\infty} \leq c$ and $\|v_t\|_{L^\infty} \leq c'$, where $c$ and $c'$ are two constants which are independent of $p$ (see [8]; see also [11, 13]). Such estimates seem difficult to derive, due to the coupling, and will be addressed elsewhere.

**REFERENCES**

[1] A. Adams and J. Fournier, *Sobolev Spaces*, 2nd edition, Academic Press, 2003.

[2] A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, 1st edition, Elsevier, Amsterdam, 1992.

[3] D. Brochet, D. Hilhorst and A. Novick-Cohen, *Finite-Dimensional exponential attractor for a model for order-disorder and phase separation*, Appl. Math. Lett., 7 (1994), 83–87.

[4] J. W. Cahn and A. Novick-Cohen, Evolution equations for phase separation and ordering in binary alloys, *Statistical Phys.*, 76 (1994), 877–909.

[5] J. W. Cahn and J. E. Hilliard, *Free energy of a nonuniform system I, Interfacial free energy*, J. Chem. Phys., 28 (1958), 258–267.

[6] L. Cherfils, A. Miranville and S. Zelik, *The Cahn-Hilliard equation with logarithmic nonlinear terms*, Milan J. Math., 79 (2011), 561–596.

[7] R. Dal Passo, L. Giacomelli and A. Novick-Cohen, *Existence for an Allen-Cahn/Cahn-Hilliard system with degenerate mobility*, Interfaces Free Boundaries, 1 (1999), 199–226.

[8] A. Giorgini, M. Grasselli and A. Miranville, *The Cahn-Hilliard-Oono equation with singular potential*, Math. Models Methods Appl. Sci., 27 (2017), 2485–2510.

[9] P. C. Millett, S. Rokkam, A. El-Azab, M. Tonks and D. Wolf, *Void nucleation and growth in irradiated polycrystalline metals: A phase-field model*, Modelling Simul. Mater. Sci. Eng., 17 (2009), 0064–003.

[10] A. Miranville, *Finite dimensional global attractor for a class of doubly nonlinear parabolic equations*, Cent. Eur. J. Math., 4 (2006), 163–182.

[11] A. Miranville, *The Cahn-Hilliard equation and some of its variants*, AIMS Math., 2 (2017), 479–544.

[12] A. Miranville, W. Saoud and R. Talhouk, *Asymptotic behavior of a model for order-disorder and phase separation*, Asymptot. Anal., 103 (2017), 57–76.

[13] A. Miranville and S. Zelik, *Robust exponential attractors for Cahn-Hilliard type equations with singular potentials*, Math. Methods Appl. Sci., 27 (2004), 545–582.

[14] A. Miranville and S. Zelik, *Attractors for dissipative partial differential equations in bounded and unbounded domains*, in *Handbook of Differential Equations, Evolutionary Partial Differential Equations* (eds. C.M. Dafermos and M. Pokorny), Elsevier, Amsterdam, (2008), 103–200.

[15] A. Miranville and S. Zelik, *The Cahn-Hilliard equation with singular potentials and dynamic boundary conditions*, Discrete Cont. Dyn. Sys., 28 (2010), 275–310.

[16] T. Nagai, T. Senba and K. Yoshida, *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac., 40 (1997), 411–433.

[17] A. Novick-Cohen, *Triple-junction motion for an Allen-Cahn/Cahn-Hilliard system*, Phys. D, 137 (2000), 1–24.
[18] A. Novick-Cohen and L. Peres Hari, Geometric motion for a degenerate Allen-Cahn/Cahn-Hilliard system: The partial wetting case, Phys. D, 209 (2005), 205–235.

[19] S. Rokkam, A. El-Azab, P. Millett and D. Wolf, Phase field modeling of void nucleation and growth in irradiated metals, Modelling Simul. Mater. Sci. Eng., 17 (2009), 0064–002.

[20] R. Temam, Navier-Stokes Equations: Theory and Numerical Analysis, AMS Chelsea Publishing, American Mathematical Society, 2001.

[21] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, 2nd edition, Applied Mathematical Sciences, Volume 68, Springer-Verlag, New York, 1997.

[22] M. R. Tonks, D. Gaston, P. C. Millett, D. Andrs and P. Talbot, An object-oriented finite element framework for multiphysics phase field simulations, Comput. Mater. Sci., 51 (2012), 20–29.

[23] L. Wang, J. Lee, M. Anitescu, A. E. Azab, L. C. McInnes, T. Munson and B. Smith, A differential variational inequality approach for the simulation of heterogeneous materials, in Proc. SciDAC, 2011.

[24] Y. Xia, Y. Xu and C. W. Shu, Application of the local discontinuous Galerkin method for the Allen-Cahn/Cahn-Hilliard system, Commun. Comput. Phys., 5 (2009), 821–835.

[25] C. Yang, X. C. Cai, D. E. Keyes and M. Pernice, NKS method for the implicit solution of a coupled allen-cahn/cahn-hilliard system, Domain Decomposition Methods in Science and Engineering, 21 (2014), 819–827.

Received February 2018; revised June 2018.

E-mail address: alain.miranville@math.univ-poitiers.fr
E-mail address: wafa.saoud@hotmail.com
E-mail address: rtaalhouk@ul.edu.lb