Non-perturbative fixed point in a non-equilibrium phase transition

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(Dated: October 24, 2018)

PACS numbers: 05.10.Cc 64.60.Ak 64.60.Ht 82.20.-w

We apply the non-perturbative renormalization group method to a class of out-of-equilibrium phase transitions (usually called “parity conserving” or, more properly, “generalized voter” class) which is out of the reach of perturbative approaches. We show the existence of a genuinely non-perturbative fixed point, i.e. a critical point which does not seem to be Gaussian in any dimension.

Our understanding of equilibrium phase transitions is largely due to the success of perturbative renormalization group (RG) methods performed around some (upper or lower) critical dimension $d_c$ and to the existence of integrability and conformal symmetry properties in $d = 2$. The situation is far less satisfactory out-of-equilibrium, where the relevant ingredients determining universality classes are sometimes not even known. There are a number of technical reasons for this: (i) many systems possess neither a lower $d_c$ nor a low-dimensional exact solution; (ii) even at the critical point, models with a Langevin-like dynamics, that is, those that involve only one time derivative in their kinetic terms cannot be conformal invariant; (iii) contrary to equilibrium, no RG calculation is in general available at and above three loop order and this prevents from using series re-summation techniques to compute accurately universal quantities in low dimensions. If, moreover, one keeps in mind that features not accessible to perturbative RG methods may play a crucial rôle, then the so-called non-perturbative renormalization group (NPRG) approach appears as a method of choice out-of-equilibrium.

Such non-perturbative effects were evidenced recently in a study of the classic reaction-diffusion problem where particles $A$ diffuse, branch ($A \rightarrow 2A$) and annihilate ($2A \rightarrow \emptyset$) with rates $D$, $\sigma$, and $\lambda$. Whereas, in an important work, Cardy and Täuber had shown that perturbative RG calculations led to conclude that no finite-$\sigma$ transition to the empty absorbing state is possible for $d > 2$, an NPRG study at the non-universal level showed that such absorbing phase transitions exist in any finite dimension (with their critical properties in the directed percolation class, as expected from perturbative RG).

In this Letter, we apply the NPRG method to a similar class of absorbing phase transitions, one for which even the universal properties are out of the reach of perturbative approaches. We put forward the existence of a genuinely non-perturbative fixed point, i.e. a critical point which does not seem to be Gaussian in any dimension. Our calculations unveil the structure of the RG flow, reveal clearly why perturbative methods are doomed to failure in this case, and provide estimates of critical exponents heretofore accessible only via numerical simulations or series expansions.

We actually treat two classes of systems known to be equivalent in $d = 1$, the physical dimension where the non-perturbative fixed point alluded to above is relevant. The first group includes the reaction-diffusion system described above but where the branching reaction now creates pairs of particles ($A \rightarrow 3A$), so that, incidently, the parity of the total number of particles is conserved. Improperly named “parity-conserving” (PC) class (as argued in [8], where this conservation law was shown to have no influence on similar reaction-diffusion systems), it is best characterized by the second group of problems, that of phase transitions into one out of two $Z_2$-symmetric absorbing states. In $d = 1$, the particles of the PC model can indeed be seen as interfaces between “+” and “−” domains. In this “spin” language, domains evolve and compete under $Z_2$-symmetric rules with noise only acting at interfaces, the definition of the “generalized voter” (GV) class as given in [12].

The reaction-diffusion problem ($A \rightarrow 3A$, $2A \rightarrow \emptyset$) was also studied in [2]. Using the Doi-Peliti formalism [13], one can obtain the following microscopic action [13]:

$$S[\phi, \bar{\phi}] = \int_{x,t} \bar{\phi}(\partial_t - D \nabla^2)\phi - \lambda(1 - \bar{\phi}^2)\phi^2 + \sigma(1 - \bar{\phi}^2)^2\bar{\phi} \phi$$

in terms of the “physical” density field $\phi$ and the associated response field $\bar{\phi}$. Cardy and Täuber first performed an expansion around the upper critical dimension $d_c = 2$ where the transition occurs at zero branching rate $\sigma$, so that the fixed point then is that of the pure annihilation problem $2A \rightarrow \emptyset$. They showed that this annihilation fixed point remains relevant down to $d = \frac{4}{3}$ (at one-loop order), where it becomes also attractive in the $\sigma$ direction. In another expansion, this time performed directly in $d = 1$, they were able to identify an appropriate combi-
nation of the coupling constants $\lambda, \sigma$ which does admit a fixed point for $d \leq \frac{4}{3}$, although the flow diagram with respect to these original variables is rather peculiar. Even if their results do suggest that a phase transition should exist at $\sigma \neq 0$ — as observed in numerical simulations and mean-field-based expansions of many microscopic models in $d = 1$ — the critical exponents remain poorly determined and, worse, the very possibility of computations beyond one loop-order appears to be problematic.

As for the GV class, the following Langevin equation was recently proposed:

$$\partial_t \phi = (-\sigma \phi + \mu \dot{\phi}^2) (1 - \phi^2) + D \nabla^2 \phi + \sqrt{2 \lambda (1 - \phi^2)} \eta$$

with $\phi \in [-1, 1]$ and $\eta$ is a delta-correlated Gaussian noise of unit variance. The $(1 - \phi^2)$ factors, appearing both in the deterministic force and in the noise amplitude, impose $\phi = \pm 1$ to be symmetric absorbing states. In $d = 1$, only one type of transition is observed by varying $\sigma$ for any value of $\mu$, and its critical properties are indeed those observed for the so-called parity-conserving models. Taking $\mu = 0$ for the sake of simplicity, the generating functional associated with the simplified Langevin equation is nothing but

$$\hat{\Gamma}(\phi) = \int \mathcal{D}\phi \exp(-\int dt \left( -\frac{1}{2} \nabla^2 \phi + \sqrt{2 \lambda} \psi \right)^2)$$

where $\hat{\Gamma}(\phi)$ is the symmetric, off-diagonal, $2 \times 2$ matrix of element $R_k$ and $\bar{\Gamma}_{k}^{(2)}[\psi, \bar{\psi}]$ the $2 \times 2$ matrix of second derivatives of $\Gamma_k$ with respect to $\psi$ and $\bar{\psi}$. Obviously, Eq. (3) cannot be solved exactly and one usually truncates it. A standard truncation is the derivative expansion $(4)$ in which $\Gamma_k$ is expanded as a power series in $\nabla$ and $\partial_t$. The local potential approximation (LPA), which is the simplest such truncation, consists in keeping only a potential term in $\Gamma_k$ while neglecting any field renormalization:

$$\Gamma_k^{\text{LPA}} = \int_{\mathcal{V}} \left\{ U_k(\psi, \bar{\psi}) + \psi (\partial_t - D \nabla^2) \psi \right\}. \quad (6)$$

If the anomalous dimensions are not too large, the LPA already provides a good description of the effective potential as well as a rather accurate estimate of the exponent $\nu$ governing the divergence of the correlation length. Since our main goal is to identify the non-perturbative fixed point governing the PC/GV transition in $d = 1$, we shall restrict ourselves, in what follows, to the LPA.

The NPRG equation for the effective potential, valid for all reaction-diffusion processes involving a single species, has been established in [15]. Studying a particular model amounts to solving this equation in a subspace defined by the symmetries of the problem, starting with the corresponding microscopic action $S$. The flow equation for the dimensionless potential $u \equiv k^{d+2} U_k$, expressed in terms of the dimensionless fields $\psi \rightarrow k^{-d} \psi$ and $\bar{\psi} \rightarrow \bar{\psi}$, reads (to lighten notations, we omit the implicit dependence on $u$ on the running scale):

$$\partial_s u = -(d + 2) u d \psi u^{(1,0)} - V_d \left[ 1 - \frac{u^{(2,0)} u^{(0,2)}}{(1 + u^{(1,1)})^2} \right]^\frac{1}{2}, \quad (7)$$

where $u^{(n,p)} = \frac{\partial^n u d^{p} u}{\partial \psi^n \partial \bar{\psi}^p}$ and $V_d = \frac{2^{d+1} - d^{d-2}}{d! (d/2)!}$. In our problem, the effective potential must remain unchanged under the simultaneous transformations $\psi \rightarrow -\psi$ and $\bar{\psi} \rightarrow -\bar{\psi}$, (“parity-conservation”/Z$_2$ symmetry of the PC/GV models). This leads to the existence of three quadratic invariant quantities, $\bar{\psi}^2$, $\bar{\psi}^2$, and $\psi \bar{\psi}$, from which all other invariant combinations of the fields can be built. Action (11) can be expressed in terms of these invariants, but it also possesses additional features: the potential of the microscopic action $S$ is proportional to $1 - \bar{\psi}^2$ and vanishes for $\psi = 0$ (in the PC language). One can check that this structure is preserved by the renormalization flow, which further constrains the functional subspace in which the running potential evolves.
To summarize, the structure of the running potential defining our problem is

\[ u(\psi, d) = (1 - \psi^2) F(\psi^2, \psi \bar{\psi}) \]  

Postponing the numerical resolution of the partial differential equation \( \psi \) in the functional subspace defined above, we now perform a Taylor series expansion of the type fixed point \( F^* \) whose eigenvalues are both negative for \( d < 1.3784 \ldots \) and complex-conjugated at larger \( d \) (only the real part is plotted).

The Gaussian fixed point \( F_G \), of eigenvalues \((2, 2 - d)\) is relevant above \( d_c = 2 \), where it coincides with \( F_A \). For \( d \in \left[ \frac{4}{3}, 2 \right) \), \( F_A \), whose eigenvalues are \((d - 2, 3d - 4)\), is relevant. At \( d = \frac{4}{3} \), \( F_A \) and the non-trivial fixed point \( F^* \) coincide and exchange stability, so that \( F^* \) is the relevant fixed point for \( d < \frac{4}{3} \) (Fig. 1). Note that then \( \sigma^* > 0 \), and thus \( F^* \) is in the physical region of parameter space, whereas it plays no role for the physics of reaction-diffusion systems when \( d > \frac{4}{3} \). Note also that \( F^* \) is not Gaussian in any dimension (at least at this order), and is thus out of the reach of any perturbative expansion. The PC/GV fixed point in \( d = 1 \) is thus \( F^* \), a genuinely non-perturbative fixed point. Its associated critical exponent, given by the inverse of its negative eigenvalue is \( \nu = \frac{12}{\sqrt{28 - 3d}} \approx 2.30 \). The flow diagram in this dimension is shown in Fig. 2. The once unstable manifold of \( F^* \), connected to \( F_G \), is the critical “surface” separating the absorbing and the active phases. The flow around \( F_A \) is rather peculiar: as \( d \) is decreased from \( \frac{4}{3} \), the eigenvector of \( F_A \) which is not parallel to the \( \lambda \) axis rotates and becomes parallel, in \( d = 1 \), to this axis. Thus, \( F_A \) is degenerate for \( d = 1 \), since its two eigenvectors coincide (Fig. 2). This implies in particular that every point flowing in the absorbing phase reaches \( F_A \) along the \( \lambda \) axis. It is not clear to us, at this point, what might be the physical signature of this for microscopic models.

The simplest truncation, that we now analyze in some detail, consists in keeping in \( u \) only the two coupling constants already present in \( S \). Inserting this Ansatz in Eq. (7), we obtain the following non-trivial flows for the running constants \( 21) \):

\[
\partial_d \lambda = -\lambda (2 - d) + 2V_d \frac{\lambda^2 (1 + 22\sigma)}{(1 - 2\sigma)^3} \quad (9)
\]

\[
\partial_d \sigma = -2\sigma + 6V_d \frac{\lambda\sigma}{(1 - 2\sigma)^2} . \quad (10)
\]

This RG flow possesses three fixed points: the trivial, Gaussian, fixed point \( F_G \) \((\lambda^*_G = \sigma^*_G = 0)\), the annihilation fixed point \( F_A \) \((\lambda^*_A = \frac{2 - d}{12}, \sigma^*_A = 0)\), and the non-trivial fixed point \( F^* \) of coordinates:

\[
\lambda^* = \frac{192}{V_d (28 - 3d)^2} , \quad \sigma^* = \frac{4 - 3d}{56 - 6d} . \quad (11)
\]

The table below shows the values of exponent \( \nu \) with the order \( n \) of the LPA truncation (see text). The column “min” refers to the minimal truncation \((\lambda, \sigma)\). The last column is a conservative estimate taken from various Monte-Carlo simulations.

| TABLE I: Values of exponent \( \nu \) with the order \( n \) of the LPA truncation (see text). The column “min” refers to the minimal truncation \((\lambda, \sigma)\). The last column is a conservative estimate taken from various Monte-Carlo simulations. |
|---|---|---|---|---|---|---|---|---|---|---|
| \( n \) | \( \text{min} \) | 3 | 4 | 5 | 6 | 7 | 8 | 9 | MC |
| \( \nu \) | 2.30 | 2.48 | 2.20 | 2.23 | 2.11 | 2.057 | 2.015 | 2.0017 | 1.85(10) |

We now report on the results obtained for truncations of the potential \( u \) that go far beyond the simplest trun-
cognition described above. Of course, there exist many ways to organize a polynomial expansion of $u$ around $\psi = 0$, $\bar{\psi}^2 = 1$ in terms of the three quadratic invariants $\psi \bar{\psi}$, $1 - \bar{\psi}^2$, $\psi^2$ which abides the $Z_2$-symmetry of the PC/GV class. Equivalently, thanks to Eq. (3), one can use any basis spanned by monomials in $\psi^2, \bar{\psi} \psi$, and we have tried several choices for it. In all cases, we stress that the qualitative picture unveiled at the minimal level is preserved at higher orders. It turns out that the fastest convergence of the exponent $\nu$ is obtained with the basis which is also the more transparent from the physical point of view. At order $n$, we consider all possible branching $(n - 2m)A \rightarrow nA$ and annihilating reactions $nA \rightarrow (n - 2k)A$ involving at most $n$ particles. These elementary reactions respectively correspond to all the terms $(1 - \bar{\psi}^{2m})(\psi \bar{\psi})^{n - 2m}$ and $(1 - \bar{\psi}^{2k})\psi^n \bar{\psi}^{n - 2k}$, and are anyhow ineluctably generated under renormalization. Table I shows the exponent and are anyhow ineluctably generated under renormalization.

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Note that discarding the (somewhat trivial and arbitrary) dimensional part of Eq. (4), the flow equation of the effective potential is invariant under the exchange $\psi \leftrightarrow \bar{\psi}$. The equivalence between the PC and GV classes is thus preserved under renormalization.

Eqs. (10) show a pole at $\sigma = \frac{1}{2}$. However, if we integrate the RG flow from a region of small $\sigma$, it never hits the pole. We also find that the relevant fixed point always lies in the small-$\sigma$ region. In fact, the pole would be problem-atic only if we wanted to follow the flow of a microscopic action with $\sigma \geq \frac{1}{2}$ from $k = \Lambda$. In this case, our equations would have to be modified in the large $k$ regime since then a cut-off function $R_k$ diverging at $k = \Lambda$ would have to be taken, which would automatically eliminate the pole. Note that at any rate, such modifications do not change the fixed points, exponents, etc.

Cardy and Tauber found from their RG analysis of the “massive theory” performed at fixed dimension a flow in $d = 1$ which differs qualitatively from ours, albeit the two eigendirections around $F_\Lambda$ were also parallel there.

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