On The Structure Of The Chan-Paton Factors
For D-Branes In Type II Orientifolds

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Abstract

We determine the structure of the Chan-Paton factors of the open strings ending on space filling D-branes in Type II orientifolds. Through the analysis, we obtain a rule concerning possible distribution of O-plane types. The result is applied to classify the topology of D-branes in terms of Fredholm operators and K-theory, deriving a proposal made earlier and extending it to more general cases. It is also applied to compactifications with $\mathcal{N} = 1$ supersymmetry in four-dimensions. We adapt and develop the language of category in this context, and use it to describe some decay channels.
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1 Introduction

The Chan-Paton factors [1–4] carry the gauge quantum numbers for Yang-Mills type fields on D-branes and are important ingredients in any theory including open strings. \( N \) coincident BPS D-branes in Type II string theory have \( U(N) \) gauge symmetry on their worldvolume. In Type II orientifolds [5–8], \( N \) BPS D-branes on top of an orientifold plane have either \( O(N) \) or \( USp(N) \) gauge symmetry depending on the choice of orientifold action on the Chan-Paton factor. The choice is referred to as the type of \( O \)-plane and is denoted by \( O^- \) for \( O(N) \) and \( O^+ \) for \( USp(N) \), reflecting the sign of the tension of the plane. In practice, one may be interested in D-branes which are not on top of the \( O \)-plane, and also, there can be several \( O \)-planes of different types in a given theory. However, the orientifold projection condition of open string states is known only in simple examples, and usually only in the bosonic sector. More fundamentally, a general condition on allowed distributions of \( O \)-plane types for a given involution are not known. In this paper, we approach these problems by studying the structure of the Chan-Paton factor of the open string ending on space filling D-branes (i.e. D9-branes).

One motivation of this work comes from classification of D-brane charges via K-theory [9]. For Type I, Type IIB resp. Type IIA string theory on a spacetime \( X \), D-brane charges take values in the group \( KO(X) \), \( K(X) \) resp. \( K^{-1}(X) \) [10, 11]. This can be derived very naturally [11, 12] from the study of topology of D9-brane configurations including tachyon condensation [13]. For Type II orientifolds on a spacetime \( X \) with an involution, similar classification exists in terms of KR-theory [14]: If the orientifold planes have \( O^- \)-type and codimension \( k \) (modulo 8) or/and \( O^+ \)-type and codimension \( (k \pm 4) \), D-brane charges take values in \( KR^{-k}(X) \) [15, 16]. However, that is just a proposal based on consistency with T-duality. Direct derivation from D9-brane configurations has been missing. Furthermore, the proposal does not cover more general situations such as coexistence of \( O^- \) and \( O^+ \)-planes of the same dimension or of \( O \)-planes of the same type but with the dimensions differing by 4. One goal of the present paper is to fill the gap by finding the structure of the D9-brane Chan-Paton factor in a general Type II orientifold.

Another motivation comes from four-dimensional \( \mathcal{N} = 1 \) supersymmetric compactifications with D-branes. Orientifold is an indispensable element in the tadpole cancellation [17], which is required for models with non-zero gravitational coupling. We would like to have an approach to systematically construct and analyze such models. One possible approach would be to realize D-branes as supersymmetric configurations on space filling D-branes in Type IIB string theory. At least before orientifolding, it comes with a useful mathematical language, that of \( D \)-brane category, to describe an important part of the low
energy effective theory on D-branes, such as the low lying spectrum, the tree level superpotential and the D-flatness condition [18]. In order to adapt the approach to consistent string compactifications, solid knowledge on the structure of the Chan-Paton factors for space filling D-branes in Type IIB orientifold is required. Applying the understanding obtained in this paper and building on earlier steps taken in [19, 20], we develop the theory and put it into the right context. We summarize the structure in the categorical language, with the expectation that it can be used in non-geometric regimes such as orbifolds and Gepner models, as well as in Type IIA models.

The primary tool of our study is the consistency condition on the parity operator $P$, with which we define the orientifold projection: It must square to a gauge transformation $g$,

$$P^2 = g,$$ 

(1.1)
on open strings stretched between all possible pairs of D-branes. For the orientifold by an involution, which we consider in the present paper, $g$ is the GSO operator $(-1)^F$ on the Neveu-Schwarz sector and the identity on the Ramond sector. This condition was employed by Gimon-Polchinski in [8] to determine the gauge groups of D5 and D1-branes in Type I string theory. Our work applies this method to study the local as well as global properties of orientifold projection conditions in more general Type II orientifolds. As important cases, we manage to find the structure of the Chan-Paton factors for D-branes of all dimensions in Type I string theory, extending earlier results by [8, 11, 21, 22].

In the next few paragraphs, we summarize the structure of the Chan-Paton factors which we find in this paper.

In an orientifold, we must consider an invariant configuration of D-branes. Namely, a configuration $B$ of D-branes and its orientifold image $P(B)$ must be the “same” or, to be more precise, isomorphic. In fact, the isomorphism itself, $P(B) \cong B$, carries an important information as that is used to define the orientifold projection of open string states. We shall refer to it as the orientifold isomorphism, or the o-isomorphism for short, of the D-branes. We would like to find the possible form of parity transform, $B \mapsto P(B)$, and the condition on the isomorphisms $P(B) \cong B$, for configurations $B$ of space-filling D-branes in the Type II orientifold on $X$ with an involution $\tau : X \rightarrow X$.

A D9-anti-D9-brane configuration in Type IIB string theory on $X$ is determined by a choice of superconnection data [23], i.e., a $\mathbb{Z}_2$-graded hermitian vector bundle $E$ on $X$ (the Chan-Paton bundle), an even unitary connection $A$ of $E$ (the gauge field), and an odd hermitian section $T$ of $\text{End}(E)$ (the tachyon). A configuration of non-BPS D9-branes in Type IIA is determined by $(E, A, T)$ as above with a distinguished section $\xi$ of $\text{End}(E)$
which is odd and obeys \( \xi^2 = \text{id}_E \), \([\xi, A] = 0\) and \( \{\xi, T\} = 0 \). The data \((E, A, T, \xi)\) can be obtained from an ungraded data \((\tilde{E}, \tilde{A}, \tilde{T})\), via \( E = \tilde{E} \oplus \tilde{E} \) and

\[
A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{A} \end{pmatrix}, \quad T = \begin{pmatrix} 0 & \tilde{T} \\ \tilde{T} & 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\]

A parity exchanges the right and the left ends of the open string and therefore must involve the transpose of the Chan-Paton factor. Here, it is natural to use a \( \mathbb{Z}_2 \)-graded version of the transpose, \( f \mapsto f^T \), with the property \( (fg)^T = (-1)^{|f||g|} g^T f^T \). We shall find that the transform \( \mathcal{B} \mapsto \mathcal{P}(\mathcal{B}) \) is given by

\[
\begin{align*}
\mathcal{P}(E) &= \tau^* E^* \otimes \mathcal{L}, \\
\mathcal{P}(A) &= -\tau^* A^T + \alpha, \\
\mathcal{P}(T) &= \varepsilon \tau^* T^T.
\end{align*}
\]

Here, \( \varepsilon \) is a phase, \( i \) or \(-i\), that is associated with the parity action on the worldsheet fermion. \((\mathcal{L}, \alpha)\) is a hermitian line bundle with a unitary connection, which we call \textit{twist}. Invariance of the worldsheet action, which includes the B-field term \( \int \Sigma x^* B \), requires the constraint \( d\alpha = \tau^* B + B \).

A natural candidate for the o-isomorphism \( \mathcal{P}(\mathcal{B}) \cong \mathcal{B} \) is a unitary map \( U : \mathcal{P}(E) \to E \) that transforms \( \mathcal{P}(A) \) and \( \mathcal{P}(T) \) back to \( A \) and \( T \),

\[
U(-\tau^* A^T + \alpha) U^{-1} + i^{-1} U d U^{-1} = A,
\]

\[
(-1)^{|U|} U (\varepsilon \tau^* T^T) U^{-1} = T.
\]

We have two maps, \( U = U_{(i)} \) and \( U_{(-i)} \), corresponding to the two phases, \( \varepsilon = i \) and \(-i\). This fact will be particularly important to define the orientifold projection condition in the Ramond sector. The requirement (1.1) on the parity operator \( \mathcal{P} \) defined via the o-isomorphism \( U \) yields a condition of the form

\[
U (\tau^* U^T)^{-1} \iota = c \cdot \sigma,
\]

for a section \( c \) of \( \tau^* \mathcal{L} \otimes \mathcal{L}^* \) which is parallel with respect to the connection \( \tau^* \alpha - \alpha \). Here, \( \iota \) is the natural isomorphism \( E \to E^{**} \) and \( \sigma \) is the grading operator on \( E \) which assigns 1 and \(-1\) on even and odd elements. For Type IIA, we need an additional condition

\[
(-1)^{|U|} U \tau^* \xi^T U^{-1} = \mu \xi,
\]

where \( \mu \) is a phase \( \pm i \) which is independent of \( \varepsilon \).
On the \( \tau \)-fixed point set, \( \tau^* \mathcal{L} \) is canonically isomorphic to \( \mathcal{L} \) and \( \tau^* \alpha - \alpha \) vanishes in the tangent direction. This means that the parallel section \( c \) of \( \tau^* \mathcal{L} \otimes \mathcal{L}^* \) can be regarded as a complex number at each O-plane. This number is related to the dimension and the type of the O-plane. Let \( k \) be the codimension of the O-plane, which is even (resp. odd) for Type IIB (resp. IIA) orientifold, and put

\[
[k] := \begin{cases} 
  k \pmod{8} & \text{for } O^- \\
  k + 4 \pmod{8} & \text{for } O^+.
\end{cases} \tag{1.6}
\]

Then,

\[
c = \begin{cases} 
  \varepsilon \frac{[k]}{2} & \text{for } \text{even (IIB),} \\
  \varepsilon \frac{[k] - \varepsilon}{2} & \text{for } \text{odd (IIA).}
\end{cases} \tag{1.7}
\]

When the twist \((\mathcal{L}, \alpha)\) is trivial, \( c \) is a constant number over the entire spacetime and all the O-planes must have the same \([k]\). For example, \( O^{-} p \) can coexist with \( O^{-} p + 4 \) but not with \( O^{+} p \) nor \( O^{-} p + 4 \). For a non-trivial twist, the value of \( c \) may differ from one O-plane to another and such “forbidden mixture” becomes possible.

The local behaviour of the condition on the tachyon near an O-plane can be written more explicitly. We choose a trivialization of the Chan-Paton vector bundle as well as the twist line bundle \( \mathcal{L} \) in a neighborhood of a point of an O-plane. For Type IIB, the condition reads

\[
T = U \tau^* T^t U^{-1}, \quad U \text{ even, } \begin{cases} 
  U = \tau^* U^t \quad [k] = 0 \quad (O9^-/O5^+/O1^-) \\
  U = -\tau^* U^t \quad [k] = 4 \quad (O9^+/O5^-/O1^+). 
\end{cases} \tag{1.8}
\]

\[
T = -U \tau^* T^t U^{-1}, \quad U \text{ odd, } \begin{cases} 
  U = \tau^* U^t \quad [k] = 2 \quad (O7^-/O3^+) \\
  U = -\tau^* U^t \quad [k] = 6 \quad (O7^+/O3^-). 
\end{cases}
\]

For Type IIA, we write the condition on the ungraded data \( \check{T} \):

\[
\check{T} = \check{U} \tau^* \check{T}^t \check{U}^{-1}, \quad \begin{cases} 
  \check{U} = \tau^* \check{U}^t \quad [k] = 7 \quad (O6^+/O2^-) \\
  \check{U} = -\tau^* \check{U}^t \quad [k] = 3 \quad (O6^-/O2^+). 
\end{cases} \tag{1.9}
\]

\[
\check{T} = -\check{U} \tau^* \check{T}^t \check{U}^{-1}, \quad \begin{cases} 
  \check{U} = \tau^* \check{U}^t \quad [k] = 1 \quad (O8^-/O4^+/O0^-) \\
  \check{U} = -\tau^* \check{U}^t \quad [k] = 5 \quad (O8^+/O4^-/O0^+). 
\end{cases}
\]

\( U \) and \( \check{U} \) are determined from \( U \), and we use here the ordinary matrix transpose \( f^t \) rather than the graded transpose \( f^f \).

The structure (1.8)-(1.9) can be described in terms of Fredholm operators of a Hilbert space with Clifford algebra action [24, 25], which are relevant in a formulation of K-theory
KR$^{-[k]}$. In particular, (1.8)-(1.9) would follow from the proposal in [15, 16] concerning the classification of D-brane charges in terms of the KR-theory. This is in fact the first way we obtained this structure (summer, 1999). In this paper, we directly derive this structure using a worldsheet analysis, from which the proposal in [15, 16] follows. In addition, we will also be able to find K-theory classification of D-brane charges in the case with a non-trivial twist where the local behaviour changes from one O-plane to another. We would also like to point out that the pattern (1.8)-(1.9) appears in the classification of random matrix ensembles or many body systems. These eight cases plus two from Type IIB and Type IIA string theories match the “ten-fold way” classification based on symmetry properties [26, 27]. The tachyon $T$ resp. $\tilde{T}$ corresponds to the random matrix (or the Hamiltonian) in a system with resp. without chirality, and $U$ resp. $\tilde{U}$ corresponds to the unitary matrix that enters into either the time reversal or the charge conjugation symmetry. In addition, we note that the structure for the case $[k] = 2$ versus 6 played an important rôle in [28].

The rest of the paper is organized as follows.

We make preliminary remarks in Section 2. We fix our convention on the spin structure and parity action at the boundary of the upper-half plane, introducing the phase $\epsilon = \mp i$. We also describe the Chan-Paton structure of space filling D-branes in Type II string theory and duality in the category of graded vector spaces.

In Section 3, we explain all the structures summarized above except the formula (1.7). We also discuss the orientifold projection in the Ramond sector, which is defined using both of the two o-isomorphisms, $U(i)$ and $U(-i)$.

In Section 4, we study orientifold action on systems with boundary fermions. This section provides the background for our treatment of non-BPS D-branes. We also study the D9-brane configuration that represents D-branes on top of the O-plane, and find an evidence of the formula (1.7).

Section 5 is the main section in which we determine the structure of space filling D-branes in Type II orientifolds on the flat Minkowski space with a single O-plane. We derive the formula (1.7) using the consistency condition (1.1).

In Section 6, we determine the tachyonic and massless spectrum on D-branes in Type I string theory.

In Section 7, we illustrate how non-trivial twists give rise to “forbidden mixture” of O-plane types in explicit examples of toroidal and Calabi-Yau compactifications. We classify the orientifold data $(\tau, B, \mathcal{L}, \alpha, c)$ and find agreement in well-studied examples as well as
some new results.

In Section 8, we classify the topology of D9-brane configurations in terms of Fredholm operators on a Hilbert space and/or K-theory. We see how it is organized in terms of the Clifford algebras. This leads to the K-theory classification proposed in [15, 16] for the cases with trivial twist. We also introduce new K-theory in order to describe the cases with non-trivial twists.

In Section 9, we consider Type IIB orientifold on Calabi-Yau manifolds with holomorphic involutions, with a focus on $\mathcal{N} = 1$ spacetime supersymmetry. We shall study the condition for the orientifold projection to be compatible with $\mathcal{N} = 1$ supersymmetry and find that we can focus on D-branes with quasi-o-isomorphisms of a certain degree. We also develop and/or adapt the language of category and use it to describe some decay channels.

Throughout this paper, we set $\alpha' = 1$.

2 Preliminaries

We make remarks on three independent subjects, (i) Worldsheet spin structures and parity action on spinors, (ii) Structure of the Chan-Paton factors on space filling D-branes in Type II string theory, and (iii) Transpose of linear maps between $\mathbb{Z}_2$-graded vector spaces. The main purpose is to fix the convention and notation.

2.1 Worldsheet Spin Structures

A Type II string theory is obtained by a chiral GSO projection — gauging the independent sign flips of left-handed and right-handed spinors on the worldsheet, $(-1)^{F_L}$ and $(-1)^{F_R}$. This involves a sum over different spin structures. In a flat cylinder region, the parallel transport along the non-trivial circle is either the sign flip (Neveu-Schwarz (NS) sector) or the identity (Ramond (R) sector) for each of the two chiralities. A closed string thus has four sectors in total, NS-NS, R-R, R-NS and NS-R. If the worldsheet has a boundary, in order to specify a boundary condition, we must choose an identification between left-handed and right-handed spinors at the boundary. The choice is two-fold, related by a sign. This is the boundary analog of the spin structure. In a flat strip region, the identification at one boundary is sent by the parallel transport to the one at the other boundary (R-sector) or to the one opposite to it (NS-sector). Closed string states in the
R-NS and NS-R sectors and open string states in the R-sector correspond to spacetime fermions, while states in the other sectors correspond to spacetime bosons.

To each open string state corresponds a boundary vertex operator. The boundary spin structure is continuous at the insertion point of an NS-vertex operator, while it flips by a sign at a R-vertex operator. This rule appears opposite to the one on the

![Figure 1: State-operator correspondence](image)

strip, but that is because the worldsheets defining the correspondence (Figure 1) have the curvature \( \frac{1}{4\pi} \int R \sqrt{g} d^2\sigma = \mp \frac{1}{2} \). Likewise, to each closed string state corresponds a bulk vertex operator. The holonomy around the insertion point is opposite to the one along the closed string, as the worldsheets defining the correspondence (the doubles of the surfaces in Figure 1) have the curvature \( \frac{1}{4\pi} \int R \sqrt{g} d^2\sigma = \mp 1 \). For example, the holonomy is trivial around the insertion point of an NS-NS vertex operator.

We now introduce conventions concerning boundary spin structures and parity transforms. We consider the strip, with the space coordinate \( \sigma^1 = \sigma \) spanning the interval \(-\pi \leq \sigma \leq 0\) and the time coordinate \( \sigma^0 = t \). Let us first look at the boundary on the right, \( \sigma = 0 \). The superpartner of the boundary value of the coordinate field, \( x(t, 0) \), is either of the following two

\[
(\pm) : \psi(t) = \psi_+(t, 0) \pm \psi_-(t, 0). \tag{2.1}
\]

These two possibilities can be regarded as coming from the two different choices of boundary spin structure. More generally, the choice can be characterized in terms of the boundary condition for the \( \mathcal{N} = (1, 1) \) supercurrents, \((\pm) : G_1^\pm \pm G_1^\mp = 0 \). Here \( G_1^\pm \) is the normal component of the supercurrent, which reads as \( G_1^\pm = \mp \psi_\pm \cdot (\partial_0 \pm \partial_1)x \) for the sigma model. Let us next look at the boundary on the left, \( \sigma = -\pi \). Again there are two possibilities for the superpartner of the boundary value of \( x(t, -\pi) \),

\[
(\pm) : \psi(t) = \psi_+(t, -\pi) \mp \psi_-(t, -\pi), \tag{2.2}
\]
or more generally, two possibilities for the boundary condition on the supercurrent \((\pm)\) :
\[ G^1_+ \mp G^1_- = 0. \] In the NS-sector, the spin structures on the two boundaries are \((++)\) or \((--\)) , i.e., both \((+)\) or both \((-\)). In the R-sector, they are \((--\)) or \((++\)).

Let us go from the Minkowski to the Euclidean strip by the Wick rotation, \(t \to -it_E\), and then to the upper-half plane, \(\text{Im}(z) \geq 0\), by
\[ z = \tau^1 + i\tau^2 = e^{t_E - i\sigma}. \]
The right and the left boundaries are mapped to real positive \(z\) and real negative \(z\).
The fermionic fields may be expressed as \(\psi^{\text{strip}}_-(dz) \frac{1}{2} = \psi^{\text{plane}}_-(d\tau^1, \tau^2)\) and \(\psi^{\text{strip}}_+(dz) \frac{1}{2} = \psi^{\text{plane}}_+(d\tau^1, \tau^2)\), from which we find the relation between the field components
\[ \psi^{\text{strip}}_\pm(t_E, \sigma) = e^{\frac{i}{2}(t_E \pm i\sigma)} \psi^{\text{plane}}_\pm(\tau^1, \tau^2). \quad (2.3) \]
In particular, \(\psi^{\text{strip}}_\pm = e^{tE/2} \psi^{\text{plane}}_\pm\) at the right boundary and \(\psi^{\text{strip}}_\pm = \mp i e^{tE/2} \psi^{\text{plane}}_\pm\) at the left boundary. The superpartner of the coordinate field \(x\) is
\[ (\pm) : \psi^{\text{plane}}_\pm(\tau) = \psi^{\text{plane}}_\pm(\tau, 0) \pm \psi^{\text{plane}}_\pm(\tau, 0) \quad (2.4) \]
at the boundary of the upper-half plane.

An orientifold is obtained by gauging a transformation of fields that involves a parity of the worldsheet. Note that a parity swaps the chirality of spinors. On the Minkowski strip, a parity acts on the fermions as
\[ \Omega : \psi_\pm(t, \sigma) \to \mp \psi_\mp(t, -\pi - \sigma), \quad (2.5) \]
or as \((-1)^F\Omega\), \((-1)^{FR}\Omega\) and \((-1)^{FL}\Omega\) which have sign factor \(\pm, +\) and \(-\) instead of \(\mp\) on the right hand side. They obey
\[ \Omega^2 = ((-1)^F\Omega)^2 = (-1)^F, \quad (2.6) \]
\[ ((-1)^{FR}\Omega)^2 = ((-1)^{FL}\Omega)^2 = \text{id}. \quad (2.7) \]
\(\Omega\) and \((-1)^{FR}\Omega\) map the \((\pm)\) spin structure on one boundary to the \((\pm)\) on the other, and thus lift to transformations of NS sector. On the other hand, \((-1)^{FR}\Omega\) and \((-1)^{FL}\Omega\) map the \((\pm)\) on one to the \((\mp)\) on the other, and lift to transformations of Ramond sector.

The action on the field components on the upper-half plane can be found from the relation \((2.3)\). \(\Omega\) does \(\psi^{\text{plane}}_\pm(\tau^1, \tau^2) \to -i \psi^{\text{plane}}_\mp(-\tau^1, \tau^2)\), and the other three parities have phases \(+i\), \(\pm i\) and \(\mp i\) instead of \(-i\). The spin structure \((\pm)\) is invariant under \(\Omega\) and \((-1)^{FR}\Omega\) while it is flipped under \((-1)^{FR}\Omega\) and \((-1)^{FL}\Omega\). The field \((2.4)\) transforms as
\[ \psi^{\text{plane}}_\pm(\tau) \to \varepsilon \psi^{\text{plane}}_\mp(-\tau), \quad (2.8) \]
where
\[ \varepsilon = \mp i \quad \text{for} \quad \left\{ \begin{array}{l}
\Omega : (\pm) \to (\pm) \\
(-1)^F \Omega : (\mp) \to (\mp) \\
(-1)^{FlR} \Omega : (\pm) \to (\mp) \\
(-1)^{Fl} \Omega : (\mp) \to (\pm) 
\end{array} \right. \quad (2.9) \]

This transformation rule may also be understood as follows. Let \( z \) be the complex coordinate as above. The identification of the spinor bundles of the opposite chirality is given either by \( (+) : \sqrt{dz} = \sqrt{dz} \) or \( (-) : \sqrt{dz} = -\sqrt{dz} \). Under the parity \( z \mapsto -\bar{z} \), \((dz,d\bar{z})\) is mapped to \((-d\bar{z},-dz)\) and hence \((\sqrt{dz},\sqrt{d\bar{z}})\) is mapped to \((-i\sqrt{d\bar{z}},-i\sqrt{dz})\) or to the other three combinations of the phases, corresponding to \( \Omega \) and the other three parities. \( \Omega \) preserves the identification \((\pm) : \sqrt{dz} = \pm \sqrt{dz} \), and maps \( \sqrt{dz} \) to \( \mp i \sqrt{dz} \). This together with the consideration on the other parities reproduces (2.8) with (2.9).

This argument is useful in finding the parity transform of other fields. For example, the spin \((\frac{3}{2},-\frac{1}{2})\) super-ghost system transforms as
\[ \beta_{\text{plane}}(\tau) \to \epsilon^3 \beta_{\text{plane}}(-\tau), \quad \gamma_{\text{plane}}(\tau) \to \epsilon^{-1} \gamma_{\text{plane}}(-\tau). \quad (2.10) \]

For later use, let us record the \( \Omega \) parity action on the Fourier modes of the fermions for the open string stretched between D9-branes. The mode expansions are
\[ \psi_{\pm}(t,\sigma) = \left\{ \begin{array}{l}
\sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi_r(t) e^{\mp ir\sigma} \quad (++) \\
\pm \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi_r(t) e^{\mp ir\sigma} \quad (--) \\
\sum_{n \in \mathbb{Z}} \psi_n(t) e^{\mp in\sigma} \quad (-+) \\
\pm \sum_{n \in \mathbb{Z}} \psi_n(t) e^{\mp in\sigma} \quad (+-) 
\end{array} \right. \quad (2.11) \]

and the parity action is
\[ \Omega : \left\{ \begin{array}{l}
\psi_r \quad \text{in} \quad (++) \quad \to \quad e^{i\pi r} \psi_r \quad \text{in} \quad (++) \\
\psi_r \quad \text{in} \quad (--) \quad \to \quad -e^{i\pi r} \psi_r \quad \text{in} \quad (--) \\
\psi_n \quad \text{in} \quad (++) \quad \to \quad (-1)^n \psi_n \quad \text{in} \quad (+-) \\
\psi_n \quad \text{in} \quad (--) \quad \to \quad -(-1)^n \psi_n \quad \text{in} \quad (+-) 
\end{array} \right. \quad (2.12) \]

We stress that the notations we have introduced, \((\pm), \Omega \) and \( \varepsilon \), are simply to fix the convention in which we discuss parity transform on an open string or on a neighborhood of boundary vertex operators. They are by no means canonical, as the meaning can be easily changed, say, with a redefinition of the frame.

### 2.2 D9-Branes In Type II String Theory

We describe the structure of the Chan-Paton factors on space filling D-branes in Type II string theory and write down the corresponding worldsheet boundary interaction.
A D9-anti-D9-brane system in Type IIB string theory supports a $\mathbb{Z}_2$-graded vector bundle on $X$,

$$E = E^0 \oplus E^1, \quad \text{(2.13)}$$

with a hermitian inner product. $E^0$ and $E^1$ are the Chan-Paton bundles of branes and antibranes respectively. They are distinguished by the $\mathbb{Z}_2$-grading operator

$$\sigma = \begin{pmatrix} \text{id}_{E^0} & 0 \\ 0 & -\text{id}_{E^1} \end{pmatrix}. \quad \text{(2.14)}$$

The tachyon is an odd endomorphism of $E$, that is, a linear map $T : E \to E$ that exchanges $E^0$ and $E^1$. It is assumed to be hermitian, $T^\dagger = T$. The gauge field $A$ is an even unitary connection of $E$, $A = A^\dagger$. They can be written as

$$T = \begin{pmatrix} 0 & T_{01} \\ T_{10} & 0 \end{pmatrix}, \quad A = \begin{pmatrix} A^0 & 0 \\ 0 & A^1 \end{pmatrix}. \quad \text{(2.14)}$$

The boundary interaction corresponding to the configuration $(T, A)$ is given by the path-ordered exponential,

$$\mathbf{P} \exp \left( -i \int_{t_i}^{t_f} A_t \, dt \right) \quad \text{(2.15)}$$

with

$$\mathcal{A}_t = \dot{x}^\mu A_\mu - i \frac{\psi^\mu \psi^\nu}{4} F_{\mu\nu} + i \frac{\psi^\mu}{2} D_\mu T + \frac{1}{2} T^2. \quad \text{(2.16)}$$

It depends on the the boundary value $x^\mu(t)$ of the sigma model field via $T = T(x(t))$ and $A_\mu = A_\mu(x(t))$ as well as the boundary value $\psi^\mu(t)$ of a linear combination of the fermionic field given by (2.1), (2.2) or (2.4). $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu]$ is the field strength and $D_\mu T$ is the covariant derivative $\partial_\mu T + i [A_\mu, T]$. The boundary interaction has $\mathcal{N} = 1$ supersymmetry

$$\delta x^\mu = i \epsilon_1 \psi^\mu, \quad \delta \psi^\mu = -2 \epsilon_1 \dot{x}^\mu, \quad \text{(2.17)}$$

since it varies as

$$\delta \mathcal{A}_t = -i D_t (\epsilon_1 (T - \psi \cdot A)) + i \epsilon_1 T. \quad \text{(2.18)}$$

For a closed boundary component of the worldsheet, $S^1 \subset \partial \Sigma$, with anti-periodic or periodic spin structure, the interaction enters into the path-integral weight as the trace or the supertrace factor,

$$\text{tr} \left[ \mathbf{P} \exp \left( -i \int_{S^1} \mathcal{A}_t \, dt \right) \right] \quad \text{or} \quad \text{tr} \left[ \sigma \mathbf{P} \exp \left( -i \int_{S^1} \mathcal{A}_t \, dt \right) \right]. \quad \text{(2.19)}$$
We refer the reader to Section 4.2 for the background of what is said below.

To describe a system of non-BPS D9-branes in Type IIA string theory on $X$ we need to choose a hermitian vector bundle $\tilde{E}$ on $X$ without $\mathbb{Z}_2$-grading. This $\tilde{E}$ is not exactly the Chan-Paton bundle but "$1/\sqrt{2}$ of it". The corresponding worldsheet boundary has an extra degrees of freedom and it is appropriate to introduce the $\mathbb{Z}_2$-graded double of $\tilde{E}$:

$$E = \tilde{E} \oplus \tilde{E}. \quad (2.20)$$

The boundary interaction is given again by (2.16) with the condition that it commutes with the odd operator

$$\xi := i \begin{pmatrix} 0 & -\mathrm{id} \tilde{E} \\ \mathrm{id} \tilde{E} & 0 \end{pmatrix}. \quad (2.21)$$

This means that the gauge field $A$ commutes with $\xi$ and the tachyon $T$ \textit{anticommutes} with $\xi$ since $\xi$ is odd and hence anticommutes with $\psi^\mu$. Namely, the tachyon and the gauge field can be written as

$$T = \begin{pmatrix} 0 & \tilde{T} \\ \tilde{T} & 0 \end{pmatrix}, \quad A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{A} \end{pmatrix}, \quad (2.22)$$

for a unitary connection $\tilde{A}$ of $\tilde{E}$ and a hermitian section $\tilde{T}$ of $\text{End}(\tilde{E})$. The system has $\mathcal{N} = 1$ worldsheet supersymmetry (2.17). For a boundary circle $S^1 \subset \partial \Sigma$ with anti-periodic or periodic spin structure, we have

$$\frac{1}{\sqrt{2}} \text{tr} \left[ \text{P exp} \left( -i \int_{S^1} A_i \mathrm{d}t \right) \right] \quad \text{or} \quad \# \sqrt{2} \text{tr} \left[ i\sigma \xi \text{P exp} \left( -i \int_{S^1} A_i \mathrm{d}t \right) \right] \quad (2.23)$$

where $\#$ is a certain phase.

Replacing a brane by its antibrane is done by $\sigma \rightarrow -\sigma$ with $\xi$ being fixed. Couplings to all the RR sector states change by a sign since the second expression in (2.23) does so. By conjugation with $\xi$, this operation is equivalent to keeping $\sigma$ and $\xi$ but doing $T \rightarrow -T$. That is,

$$(\tilde{E}, \tilde{A}, \tilde{T}) \mapsto (\tilde{E}, \tilde{A}, -\tilde{T}).$$

\section{2.3 Linear Algebra — Graded Duality}

We comment on duality operations in the category of $\mathbb{Z}_2$-graded vector spaces, in which a minus sign shows up when two odd elements exchange their positions. As usual, $|v| = 0$ or 1 (modulo 2) if $v$ is even or odd with respect to a given grading.
First of all, the dual $V^*$ of a $\mathbb{Z}_2$-graded vector space $V$ has a natural $\mathbb{Z}_2$-grading — even elements of $V^*$ are orthogonal to odd elements of $V$, and vice versa.

For a linear map between graded vector spaces, $f : V \to W$, we define its graded transpose (or simply transpose), $f^T : W^* \to V^*$, by

$$\langle f^T(w^*), v \rangle = (-1)^{|f||w^*|}\langle w^*, f(v) \rangle,$$

(2.24)

for $v \in V$ and $w^* \in W^*$. If $f$ is expressed as

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(2.25)

with respect to basis of $V$ and $W$ such that even elements precede odd elements, then, with respect to the dual basis, the graded transpose is expressed as

$$f^T = \begin{pmatrix} a^t & -c^t \\ b^t & d^t \end{pmatrix},$$

(2.26)

where $a^t, b^t, \ldots$ are transpose matrices for $a, b, \ldots$. The graded transpose has the desired property under the composition of maps, say $g : U \to V$ and $f : V \to W$,

$$(f \circ g)^T = (-1)^{|f||g|} g^T \circ f^T.$$  

(2.27)

In particular, we have $(f^T)^{-1} = (-1)^{|f|} (f^{-1})^T$. For an even map $f$, we may simply write $f^{-T}$ for the inverse transpose.

The definition (2.24) is by no means unique. For example, we may take $\langle f^T(w^*), v \rangle = (-1)^{|f||v|}\langle w^*, f(v) \rangle$, which also satisfies the property (2.27). However, we must make some choice, and (2.24) is the choice we make throughout this paper.

There is a natural isomorphism $\iota$ from a vector space $V$ to its double dual $V^{**}$,

$$\langle \iota(v), v^* \rangle = (-1)^{|v||v^*|}\langle v^*, v \rangle,$$

(2.28)

for $v \in V$ and $v^* \in V^*$. It is even with respect to the natural gradings. The natural isomorphism for the dual, $\iota_{V^*} : V^* \to V^{**}$, is equal to the inverse transpose of $\iota = \iota_V : V \to V^{**}$,

$$\iota_{V^*} = \iota^{-T}_V.$$  

(2.29)

For $f : V \to W$, its double transpose $f^{TT} : V^{**} \to W^{**}$ is conjugate to $f$,

$$\iota_W^{-1} f^{TT} \iota_V = f.$$  

(2.30)
If the $\mathbb{Z}_2$-gradings are changed, the graded transpose of course changes. Let us show the gradings explicitly as the subscript as $f^T_{\sigma_V,\sigma_W}$ and $|f|_{\sigma_V,\sigma_W}$ for $f : V \to W$. By definition, we have $\langle f^T_{\sigma_V,\sigma_W}(w^*), v \rangle = (-1)^{|f|_{\sigma_V,\sigma_W}|w^*|_{\sigma_W}} \langle w^*, f(v) \rangle$. The change in the transpose is

$$
f^T_{-\sigma_V,\sigma_W} = f^T_{\sigma_V,\sigma_W},
$$

$$
f^T_{\sigma_V,-\sigma_W} = -\sigma_V^* f^T_{\sigma_V,\sigma_W},
$$

$$
f^T_{-\sigma_V,-\sigma_W} = \sigma_V^* f^T_{\sigma_V,\sigma_W} \sigma_W^*,
$$

where $\sigma_{V^*}$ and $\sigma_{W^*}$ are the natural gradings of $V^*$ and $W^*$ induced by $\sigma_V$ and $\sigma_W$.

### 3 D-Branes In Type II Orientifolds

In this section, we discuss how we want to define D-branes in Type II orientifolds in terms of D9-brane configurations.

#### 3.1 Parity Actions On Boundary Interactions

We now discuss the parity action on the boundary interaction determined by the configuration $(T, A)$. Let us first look at the parity action on the on-shell vertex operators for tachyons and massless vector bosons. The corresponding states are

$$
k \cdot \psi_{-\frac{1}{2}} |k; i, j\rangle_{(\pm \pm)}, \quad \left(\zeta \cdot \alpha_{-1} - \sqrt{2} \zeta \cdot \psi_{-\frac{1}{2}} k \cdot \psi_{-\frac{1}{2}}\right) |k; i, j\rangle_{(\pm \pm)},
$$

where the subscript $(\pm \pm)$ shows the spin structure at the two boundaries. The parity $\Omega$ transforms these states to

$$
\mp i k \cdot \psi_{-\frac{1}{2}} |k; j, i\rangle_{(\pm \pm)}, \quad -\left(\zeta \cdot \alpha_{-1} - \sqrt{2} \zeta \cdot \psi_{-\frac{1}{2}} k \cdot \psi_{-\frac{1}{2}}\right) |k; j, i\rangle_{(\pm \pm)},
$$

possibly up to a linear transformation of Chan-Paton vectors. The factor $\mp i$ in the transformation of tachyon comes from the parity action on $\psi_{-\frac{1}{2}}$, see (2.12). The Chan-Paton indices $i$ and $j$ are swapped because the orientation of the string is reversed. This would correspond to the transpose of the Chan-Paton factor. However, since we are considering $\mathbb{Z}_2$-graded Chan-Paton vector spaces, it would be more natural to take the graded transpose. Thus, we find that the parity $\Omega$ transforms the on-shell fluctuation of $T$ and $A$ as

$$
\delta T \longrightarrow \mp i \delta T^T, \quad \delta A \longrightarrow -\delta A^T.
$$

(3.1)
Note that this is compatible with the hermiticity of $T$ — if $\delta T$ is odd and hermitian then $\delta T^T$ is anti-hermitian, as can be seen by the matrix representation for the graded transpose, \((2.26)\). Thus $\mp i \delta T^T$ is also hermitian.

Let us now look at the parity action on the boundary interaction for finite $T$ and $A$. For concreteness, we consider the upper-half plane, $\text{Im}(z) \geq 0$, on which the parity acts as $z \mapsto -\bar{z}$. The boundary interaction takes the form

\[
W(\tau_f, \tau_i) = \text{P exp} \left( -i \int_{\tau_i}^{\tau_f} A_\tau \, d\tau \right)
\]

(3.2)

with

\[
iA_\tau = i \frac{dx^\mu}{d\tau} A_\mu - \frac{i}{4} \psi^\mu \psi^\nu F_{\mu\nu} + \frac{i}{2} \psi^\mu D_\mu T + \frac{1}{2} T^2,
\]

(3.3)

which is the Wick rotated version of \((2.16)\). Note that $\psi^\mu$ is the $\mu$-th component of the fermion defined in \((2.4)\). $W(\tau_f, \tau_i)$ is a linear map from $E$ at $x(\tau_i)$ to $E$ at $x(\tau_f)$. In the computation of correlation functions, it is to be multiplied to the Chan-Paton factor at $\tau = \tau_i$ and to be followed by the Chan-Paton factor at $\tau = \tau_f$. The parity reverses the orientation of the boundary and acts on the Chan-Paton factors by graded transpose. Thus, it acts on the boundary interaction as $W(\tau_f, \tau_i) \mapsto W(\tau_f, \tau_i)^T$ in addition to the action on the fields $x(\tau)$ and $\psi(\tau)$. The parity transforms the relevant fields as follows (see \((2.8)\) with \((2.9)\)):

\[
x^\mu(\tau) \to x^\mu(-\tau), \quad \psi^\mu(\tau) \to \varepsilon \psi^\mu(-\tau).
\]

The outcome is

\[
\text{P exp} \left( -i \int_{\tau_i}^{\tau_f} A_\tau \, d\tau \right) \mapsto \text{P exp} \left( -i \int_{-\tau_f}^{-\tau_i} \tilde{A}_\tau \, d\tau \right)
\]

with

\[
i\tilde{A}_\tau = -i \frac{dx^\mu}{d\tau} A^T_\mu - \frac{i}{4} \varepsilon^2 \psi^\mu \psi^\nu F^T_{\mu\nu} + \frac{i}{2} \varepsilon \psi^\mu D_\mu T^T + \frac{1}{2} (T^2)^T.
\]

(3.4)

Note that we have a sign in the relation $(T \circ T)^T = -T^T \circ T^T$, since the tachyon $T$ is odd. Namely, $(T^2)^T = (\varepsilon T^T)^T$. We see that the effect of the parity action is

\[
T \mapsto \varepsilon T^T, \quad A \mapsto -A^T.
\]

(3.5)

This is nothing but the off-shell and finite version of \((3.1)\). Note that the Chan-Paton bundle $E$ has transformed to its dual, $E^*$.

If the parity is combined with an involution $\tau : X \to X$ of the spacetime, then the transformation rule \((3.5)\) is dressed by the pull back, $T \to \varepsilon \tau^* T^T$, $A \to -\tau^* A^T$, and
$E \mapsto \tau^*E^*$. Furthermore, we may also combine it with a shift of the gauge field, which is in fact enforced when there is a nonzero $B$-field. The $B$-field enters into the (Euclidean) action as

$$S_B = -i \int_S x^* B + \frac{i}{4} \int_{\partial S} B_{\mu\nu}(x) \psi^\mu \psi^\nu d\tau.$$  \hfill (3.6)

Under the parity combined with $x \mapsto \tau \circ x$ and $\psi \mapsto \tau^* \psi$, this transforms to

$$S_B \mapsto -i \int_S \Omega x^* \tau^* B + \frac{i}{4} \int_{\partial S} [(\tau^* B)_{\mu\nu}(x) \psi^\mu \psi^\nu](-\tau)d\tau$$

$$= i \int_S x^* (\tau^* B) - \frac{i}{4} \int_{\partial S} [(\tau^* B)_{\mu\nu} \psi^\mu \psi^\nu](\tau)d\tau$$

where we have used the fact that $\Omega$ reverses the orientation of the worldsheet. Thus the change in the action is

$$\Delta S_B = i \int_S x^*(\tau^* B + B) - \frac{i}{4} \int_{\partial S} (\tau^* B + B)_{\mu\nu} \psi^\mu \psi^\nu d\tau.$$  \hfill (3.7)

At this point, we recall the condition coming from the invariance of the weight $\exp \left(i \int_S x^* B\right)$ under the parity $x \mapsto \tau \circ x \circ \Omega$, for a closed worldsheet $\Sigma$ with an orientation reversing involution $\Omega : \Sigma \to \Sigma$. The condition is $\exp \left(-i \int_S x^* (\tau^* B + B)\right) = 1$. That is, $\tau^* B + B$ has value $2\pi$ times an integer on any 2-cycle of $X$, i.e.,

$$[\tau^* B + B] \in H^2(X, 2\pi \mathbb{Z}).$$  \hfill (3.8)

This means that there is a complex line bundle $\mathcal{L}$ with a $U(1)$ connection $\alpha$ such that

$$d\alpha = \tau^* B + B,$$  \hfill (3.9)

so that $-\left[\tau^* B + B\right]/2\pi$ is the first Chern class of $\mathcal{L}$. Given the expression (3.9), the change $\Delta S_B$ can be written as a boundary term which is equal to $iA_\tau$ for $(E, A, T) = (\mathcal{L}, \alpha, 0)$. The net effect is therefore the shift of the gauge field by $\alpha$. We find that the parity transform is

$$T \mapsto \varepsilon \tau^* T^T, \quad A \mapsto -\tau^* A^T + \alpha.$$  \hfill (3.10)

The Chan-Paton bundle $E$ is transformed to $\tau^* E^* \otimes \mathcal{L}$.

Note that the line bundle with connection $(\mathcal{L}, \alpha)$ obeying (3.9) is not unique if $X$ is not simply connected — shift of $\alpha$ by a flat connection preserves the condition (3.9). Thus, we must make a choice of $(\mathcal{L}, \alpha)$, and that is so even when the B-field vanishes. This is an important part of the data of the orientifold, which we call the twist. As we will discuss in the next subsection, there is a severe constraint on the twist $(\mathcal{L}, \alpha)$ in order to be able to impose a $\mathbb{Z}_2$ orientifold projection. Note that the B-field gauge transformation,
$B \rightarrow B + d\Lambda$ and $A \rightarrow A + \Lambda$, shifts the twist connection as $\alpha \rightarrow \alpha + \Lambda + \tau^*\Lambda$. Here, $\Lambda$ is a connection of a $U(1)$ bundle and in particular $d\Lambda$ must represent an element of $H^2(X, 2\pi \mathbb{Z})$.

Let us discuss the parity mapping of open string states. We consider the open string stretched between two D-branes $B_i$ ($i = 1, 2$) determined by the data $(E_i, A_i, T_i)$. The wavefunctional for a string configuration $x : [0, 1] \rightarrow X$ (here we suppress the fermionic configuration $\psi_\pm$ from the notation) is a linear map from $E_1$ at $x(0)$ to $E_2$ at $x(1)$,

$$\Phi[x] \in \text{Hom}(E_{1x(0)}, E_{2x(1)}).$$

Naïvely, the parity image of this state is its transpose combined with $x \mapsto \tau \Omega(x) := \tau \circ x \circ \Omega$ for $\Omega(\sigma) = 1 - \sigma$,

$$\Phi^T[\tau \Omega(x)] \in \text{Hom}(E^*_2 \tau(x(0)), E^*_1 \tau(x(1))) = \text{Hom}(\tau^*E^*_2x(0), \tau^*E^*_1x(1)).$$

However, what we want as the parity image must take value in

$$\text{Hom}((\tau^*E^*_2 \otimes \mathcal{L})_{x(0)}, (\tau^*E^*_1 \otimes \mathcal{L})_{x(1)}) \cong \text{Hom}(\tau^*E^*_2x(0), \tau^*E^*_1x(1)) \otimes \text{Hom}(\mathcal{L}_{x(0)}, \mathcal{L}_{x(1)}).$$

Thus, we have to amend $\Phi^T[\tau \Omega(x)]$ by an element of $\text{Hom}(\mathcal{L}_{x(0)}, \mathcal{L}_{x(1)})$. One and only one natural candidate is the parallel transport along the path $x([0, 1])$ with respect to the connection $\alpha$:

$$h_\alpha[x] = \exp \left( -i \int_{x[0,1]} \alpha \right) : \mathcal{L}_{x(0)} \rightarrow \mathcal{L}_{x(1)}. \quad (3.11)$$

We therefore define the parity mapping as

$$\Phi[x] \mapsto \Phi^T[\tau \Omega(x)] \otimes h_\alpha[x]. \quad (3.12)$$

### 3.2 The Orientifold Isomorphism

We now discuss how to define D-branes in Type II orientifolds in which a parity symmetry is gauged. First of all, a D-brane must be invariant under the parity, that is, the parity image must be physically equivalent to the original brane. Second, it is not enough that the two are just physically equivalent, but an explicit isomorphism must be specified. This is needed in order to impose orientifold projection that selects invariant open string states. To see the necessity, let us take a D-brane $\mathcal{B}$ and denote its parity image by $\mathcal{P}(\mathcal{B})$. The parity maps the space of states of the open string ending on $\mathcal{B}$, $\mathcal{H}_\mathcal{B}$, to the space of states of the open string ending on $\mathcal{P}(\mathcal{B})$, which is another space $\mathcal{H}_{\mathcal{P}(\mathcal{B})}$. For the orientifold projection, however, we need a parity operator acting on
the same space. That would be provided by a map from \( \mathcal{H}_{P(B), P(B)} \) back to \( \mathcal{H}_{B, B} \), which can be defined if an isomorphism
\[
P(B) \xleftrightarrow{\cong} B
\] (3.13)
is specified. We shall call it an orientifold isomorphism (or \( o \)-isomorphism for short). Thirdly, we would like to choose the \( o \)-isomorphism so that the parity operator acting on the open string states is an involution. To be more precise, we would like the isomorphism to respect the algebra of parity actions. For example, if \( P = P(\tau \Omega) \) denote the parity operator corresponding to \( \tau \Omega \), we want it to respect the relation \((\tau \Omega)^2 = (-1)^F\) that follows from (2.6):
\[
P^2 = (-1)^F.
\] (3.14)
This yields an important constraint on the possible form of the \( o \)-isomorphisms.

Let us consider D-branes determined by D9-brane configurations and their parity actions of the form (3.10). The typical case in which a brane \( B = (E,A,T) \) and its parity image \( \mathcal{P}(B) = (\tau^* E^* \otimes \mathcal{L}, -\tau^* A^T + \alpha, \epsilon \tau^* T^T) \) determine physically equivalent D-branes is when there is a gauge transformation between them, that is, a unitary map
\[
U : \tau^* E^* \otimes \mathcal{L} \rightarrow E
\] (3.15)
that transforms the boundary interaction \( \mathcal{P}(A) \) for the image brane \( \mathcal{P}(B) \) to the one \( A \) for the original brane \( B \):
\[
A_\tau = U(x)\mathcal{P}(A)_\tau U(x)^{-1} + i^{-1}U(x)\frac{d}{d\tau}U(x)^{-1}.
\]
Namely,
\[
T = (-1)^{|U|}U(\epsilon \tau^* T^T)U^{-1},
\] (3.16)
\[
A = U(-\tau^* A^T + \alpha)U^{-1} + i^{-1}UdU^{-1}.
\] (3.17)
The sign factor \((-1)^{|U|}\) comes from the reordering of \( U(x) \) and \( \psi^\mu \) in \( U(x)\mathcal{P}(A)_\tau U(x)^{-1} \).

We see from (3.16) that \( U \) should depend on the phase \( \epsilon = \mp i \). If we want to be specific, we write \( U = U_{(\epsilon)} \). The relation of the form \( U_{(i)} \propto U_{(-i)} \circ \sigma^T \), as well as \( U_{(i)} \propto \xi \circ U_{(-i)} \circ \sigma^T \) for Type IIA, is consistent with the conditions (3.16) and (3.17).

We would like to regard such \( U \) as an orientifold isomorphism with which we define the parity operator. Let us discuss the action of \( P = P(\tau \Omega) \) on the NS sector. We should take \( U = U_{(-i)} \) or \( U_{(i)} \) for the spin structure \((++)\) or \((-\cdots)\) respectively, since \( \epsilon = \mp i \) for \( \Omega \) on \((\pm)\), see (2.9). We have already defined a map from \( \mathcal{H}_{B, B} \) to \( \mathcal{H}_{P(B), P(B)} \), as shown in (3.12). We want to compose it with a map \( \mathcal{H}_{P(B), P(B)} \rightarrow \mathcal{H}_{B, B} \), which is obtained by
composition with $U(x(1))$ and $U(x(0))^{-1}$. The parity image of the state $\Phi$ is thus given by

$$P(\Phi)[x] = U(x(1)) \circ (\Phi^T[\tau \Omega(x)] \otimes h_\alpha[x]) \circ U(x(0))^{-1}(-1)^{|\Phi||U|}$$

If we introduce the evaluation map $ev_\sigma$ that associates to a string $x : [0, 1] \to X$ the value at $\sigma$, $ev_\sigma(x) = x(\sigma)$, we have a more concise expression

$$P(\Phi) = ev^*_1 U \circ ((\tau \Omega)^* \Phi^T \otimes h_\alpha) \circ ev^*_0 U^{-1}(-1)^{|\Phi||U|}. \quad (3.18)$$

Let us compute the parity squared,

$$P^2(\Phi) = P(ev^*_1 U \circ ((\tau \Omega)^* \Phi^T \otimes h_\alpha) \circ ev^*_0 U^{-1}(-1)^{|\Phi||U|})$$

$$= ev^*_1 U \circ ((\tau \Omega)^*(ev^*_1 U \circ ((\tau \Omega)^* \Phi^T \otimes h_\alpha) \circ ev^*_0 U^{-1} \otimes h_\alpha) \circ ev^*_0 U^{-1}$$

$$= ev^*_1 (U \circ \tau^*(U^T)^{-1}) \circ ((\tau \Omega)^* \Phi^T \otimes (\tau \Omega)^* h_\alpha^T \otimes h_\alpha) \circ ev^*_0 (\tau^*U^T \otimes U^{-1})$$

Here, we used the relation $(\tau \Omega)^*ev^*_1 U = ev^*_0 \tau^*U$, etc, that results from $(ev_1 \circ \tau \Omega)(x) = \tau(x(0)) = (\tau \circ ev_0)(x)$, etc. We have also used the identities that involve the graded transpose, $(U\Phi^T U^{-1})^T = (-1)^{|U||U^{-1}|} \Phi^T U T = (U^T)^{-1} \Phi^{TT} U T$. We may further use the identity $\Phi^{TT} = I \circ \Phi \circ I^{-1}$ from (2.30). Note that $h_\alpha^T[\tau \Omega(x)]$ can be regarded as the parallel transport of $\tau^*\mathcal{L}^*$ along the path $x[0, 1]$ with respect to the connection $-\tau^*\alpha$,

$$(\tau \Omega)^* h_\alpha^T = h_{-\tau^*\alpha}^T.$$ Collecting all, we obtain the expression for the parity squared

$$P^2(\Phi) = ev^*_1 (U \tau^*(U^T)^{-1} I) \circ ((\tau \Omega)^{2*} \Phi \otimes h_{-\tau^*\alpha - \alpha}) \circ ev^*_0 (U \tau^*(U^T)^{-1} I)^{-1}. \quad (3.19)$$

Let us now impose the basic requirement (3.14): $P^2 = (-1)^F$. Note that $(-1)^F$ can be realized on the wavefunctional $\Phi[x]$ by the action of $(\tau \Omega)^2$ on $x$ combined with the conjugation by the $\mathbb{Z}_2$-grading operator $\sigma$ on the Chan-Paton factor. Therefore we would like (3.19) to be equal to $\sigma \circ (\tau \Omega)^2 \Phi \circ \sigma^{-1}$. This would be the case if

$$U \tau^*(U^T)^{-1} I = \sigma \otimes c \quad (3.20)$$

where $c$ is a “scalar” such that

$$c(x(1)) \cdot h_{-\tau^*\alpha + \alpha}[x] \cdot c(x(0))^{-1} = 1 \quad (3.21)$$

for any open string configuration $x : [0, 1] \to X$. Note that it may depend on the phase $\varepsilon$, $c = c(\varepsilon)$, corresponding to $U = U(\varepsilon)$. Equation (3.20) is the condition for $U$ to be an $\alpha$-isomorphism.
It follows from the definition of $U$ as a map $\tau^*E^* \otimes \mathcal{L} \to E$ that the “scalar” $c$ in (3.20) can be regarded as a section of the line bundle $(\tau^*\mathcal{L}^* \otimes \mathcal{L})^{-1}$. Then, the condition (3.21) means that $c^{-1}$ is a globally defined parallel section of $\tau^*\mathcal{L}^* \otimes \mathcal{L}$ with respect to the connection $-\tau^*\alpha + \alpha$. This in particular means that the connection $-\tau^*\alpha + \alpha$ must be flat and have trivial holonomy along any loop. This provides a severe constraint on the choice of $(\mathcal{L}, \alpha)$.

The parallel section $c$ must be common for all D-branes in the theory. To see this, note that a formula like (3.19) holds also for a wavefunction $\Phi$ of the open string stretched between different D-branes, $B_1$ and $B_2$,

$$P^2(\Phi) = ev_1^*(U_2\tau^*(U_2^T)^{-1}\iota_2) \circ ((\tau\Omega)^2 \otimes h_{-\tau^*\alpha+\alpha}) \circ ev_0^*(U_1\tau^*(U_1^T)^{-1}\iota_1)^{-1}. \quad (3.22)$$

It then follows from the requirement $P^2(\Phi) = (-1)^F \Phi$ that the parallel section $c$ for $B_1$ must be the same as the one for $B_2$.

**Type IIA Case**

As stated in Section 2.2, non-BPS D9-branes in Type IIA string theory supports a Chan-Paton bundle $E$ with a special structure $\xi : E \to E$, see (2.20) and (2.21). The tachyon and the gauge field obey the constraint $\{T, \xi\} = 0$ and $[A, \xi] = 0$. In fact, as will be explained in Section 4.2, all states must obey such a constraint, i.e.,

$$\xi \circ \Phi = (-1)^{|\Phi|} \Phi \circ \xi. \quad (3.23)$$

The parity operation (3.10) with (3.12) preserves this structure — we have $\tau^*\xi^T$ on the parity image $\tau^*E^* \otimes \mathcal{L}$. We require that the $o$-isomorphism $U$ maps it back to $\xi$, namely,

$$(-1)^{|U|} U \tau^*\xi^T U^{-1} = \mu \xi, \quad (3.24)$$

for some proportionality constant $\mu$ which must be $i$ or $-i$ by the hermiticity of $\xi$. The two choices of $\mu$ are related by the exchange $U \leftrightarrow \xi \circ U$ because $(-1)^{|\xi| U|} = -(-1)^{|U|}$. Note that the conditions (3.16) and (3.17) are maintained by this exchange, thanks to $\xi T \xi^{-1} = -T$ and $\xi A \xi^{-1} = A$. More generally, $U$ and $\xi \circ U$ give rise to the same parity operator on open string states that obey the constraint (3.23). Therefore $\mu = i$ and $\mu = -i$ are physically equivalent although we need to make a choice once and for all.

**Isomorphisms**

Let $(B_1, U_1)$ and $(B_2, U_2)$ be D-branes with $o$-isomorphisms, where $B_i = (E_i, A_i, T_i)$ for $i = 1, 2$. We would like to discuss the condition for the two to be physically equivalent as
D-branes in the orientifold? Suppose there is an even and unitary bundle map $f: E_1 \to E_2$ that sends $(A_1, T_1)$ to $(A_2, T_2)$ in the obvious sense (and $\xi$ of $B_1$ to $\xi$ of $B_2$ for Type IIA). What should $f$ do on the o-isomorphisms? We require that, given a third brane, $(B_3, U_3)$, the parity operator $P: \mathcal{H}_{B_1, B_3} \to \mathcal{H}_{B_3, B_1}$ is equal to $P: \mathcal{H}_{B_2, B_3} \to \mathcal{H}_{B_3, B_2}$ under the natural relations between the domains and the targets which are determined by $f$. That is, for any state $\Phi \in \mathcal{H}_{B_1, B_3}$, we require $\text{ev}^* f \circ P(\Phi) = P(\Phi \circ \text{ev}_0^* f^{-1})$ at $\mathcal{H}_{B_3, B_2}$. A direct computation shows that this condition is

$$f \circ U_1 = U_2 \circ \tau^*(f^{-1})^T.$$  \hspace{1cm} (3.25)

We shall call such an $f$ an *isomorphism* from $(B_1, U_1)$ to $(B_2, U_2)$.

### 3.3 The Type Of O-Planes

We learned that we must choose a twist, i.e., a hermitian line bundle $\mathcal{L}$ with a unitary connection $\alpha$ such that (i) the curvature equals $\tau^* B + B$ and (ii) the connection $-\tau^* \alpha + \alpha$ of $\tau^* \mathcal{L} \otimes \mathcal{L}$, which is flat by (i), has trivial holonomy along closed loops. Furthermore, we also found that we must specify an o-isomorphism, i.e., an isomorphism $U: \tau^* \mathcal{E} \otimes \mathcal{L} \to \mathcal{E}$ which obeys $U\tau^* (U^T)^{-1} = \sigma \otimes c$. Here, $c$ is a parallel section, common for all D-branes, of the line bundle $(\tau^* \mathcal{L} \otimes \mathcal{L})^{-1}$ with respect to the flat connection $\tau^* \alpha - \alpha$. Note that the identity $U^{TT} \sim U$ (2.30) yields the constraint on it,

$$\tau^* c \cdot c = (-1)^{|U|}.$$  \hspace{1cm} (3.26)

One very important fact is that $\tau^* \mathcal{L}$ is canonically isomorphic to $\mathcal{L}$ over the fixed point set $X^\tau$ of the involution. To see this, we first recall that the total space of the pull back $\tau^* \mathcal{L}$ is defined as the subspace of $X \times \mathcal{L}$ consisting of points $(x, v)$ such that $v$ is in the fibre of $\tau(x)$, $v \in \mathcal{L}|_{\tau(x)}$. Then the canonical isomorphism is given by

$$(x, v) \in \tau^* \mathcal{L}|_x \longleftrightarrow v \in \mathcal{L}|_x, \quad \forall x \in X^\tau.$$  \hspace{1cm} (3.27)

In other words, $\tau^* \mathcal{L} \otimes \mathcal{L}$ is canonically trivial over $X^\tau$. Also, the connection $-\tau^* \alpha + \alpha$ is canonically flat when restricted to $X^\tau$. In particular, the parallel section $c$ of $(\tau^* \mathcal{L} \otimes \mathcal{L})^{-1}$ can be defined on $X^\tau$ as a locally constant function with values in complex numbers. The possible values are constrained by $c^2 = (-1)^{|U|}$ from (3.26) — $\pm 1$ if $U$ is even and $\pm i$ if $U$ is odd. We claim that it is the sign of this value that determines the type of the O-plane.
To be precise, we claim that the value of $c$ at an O-plane is related to its type and dimension by (1.7), which we repeat here:

\[
c = \begin{cases} 
\pm \varepsilon \frac{k}{2} & \text{at O}(9-k)^{\mp}\text{-plane (Type IIB)}, \\
\pm \varepsilon \frac{k-\mu}{2} & \text{at O}(9-k)^{\pm}\text{-plane (Type IIA)}. 
\end{cases}
\] (3.28)

In particular, we have

\[
(-1)^{|U|} = \begin{cases} 
(-1)^{\frac{k}{2}} & \text{(Type IIB)}, \\
(-1)^{\frac{k-\mu}{2}} & \text{(Type IIA)}. 
\end{cases}
\] (3.29)

Due to its local nature, it is enough to prove the formula (3.28) in the simplest case of orientifold of the Minkowski space with a single flat O$(9-k)$-plane. This will be done in Section 5.

The value of $c$ can be different at different O-plane components if the twist $(\mathcal{L}, \alpha)$ is non-trivial. This leads, via the formula (3.28), to mixture of O-plane types. One of the first examples of mixed type O-planes in the literature is Type IIA orientifold with one O8$^-$ and one O8$^+$ at antipodal points of a circle. This theory is T-dual to Type IIB orientifold on the dual circle with a half-period shift [32, 33]. Back in the Type IIA orientifold, the half-period shift occurs on the Wilson line, and that is nothing but a non-trivial twist $\alpha$. This example is in fact how we discovered that mixed O-plane types can be made possible by non-trivial twists. We shall describe more examples in Section 7, including the details of O8$^-$-O8$^+$.

In what follows, we shall call $c$ the crosscap section.

The Four Cases

According to (3.29), the statistics of the o-isomorphism $U$ is determined by the codimension of the O-plane modulo 4. This in particular means that the components of the fixed point set $X^\tau$ must have the same codimensions modulo 4. This is guaranteed when the involution $\tau : X \to X$ has a lift $\tau_S$ to an action on Majorana spinors on $X$. To see this, suppose there is a codimension $k$ O-plane and let us choose local coordinates so that $\tau$ acts as the sign flip of $x^1, ..., x^k$. At this O-plane, the lift $\tau_S$ is realized by the multiplication by $\pm \Gamma^1 \cdots \Gamma^k$ if $k$ is even, and by $\pm i \Gamma_{11} \Gamma^1 \cdots \Gamma^k$ if $k$ is odd. Here $\Gamma^i$’s are the Gamma matrices, $\{\Gamma^\mu, \Gamma^\nu\} = -2\eta^{\mu\nu}$ (with the $(- + + +)$ convention for $\eta_{\mu\nu}$), and $\Gamma_{11} = \Gamma^0 \Gamma^1 \cdots \Gamma^9$. Then its square is

\[
\tau^2_S = \begin{cases} 
(\pm \Gamma^1 \cdots \Gamma^k)^2 = (-1)^{\frac{k}{2}} & k \text{ even} \\
(\pm i \Gamma_{11} \Gamma^1 \cdots \Gamma^k)^2 = (-1)^{\frac{k+1}{2}} & k \text{ odd}. 
\end{cases}
\] (3.30)
Since $\tau^2_S$ is either 1 or $-1$ globally, the codimension $k$ modulo 4 must be common to all O-planes.

Let us classify the possibilities into four cases

- $(B_{\pm})$ $\tau$ is orientation preserving and $\tau^2_S = \pm \text{id}$,
- $(A_{\pm})$ $\tau$ is orientation reversing and $\tau^2_S = \pm \text{id}$.

$(B_{\pm})$ and $(A_{\pm})$ are for Type IIB and Type IIA orientifolds respectively. O-planes that can appear are determined by (3.30), i.e.

- $(B_{\pm})$ O9/O5/O1
- $(B_{-})$ O7/O3
- $(A_{+})$ O6/O2
- $(A_{-})$ O8/O4/O0

The statistics of the o-isomorphisms is

- $(B_{\pm})$ $(-1)^{|U_{(-i)}|} = (-1)^{|U_{(i)}|} = \pm 1$,
- $(A_{\pm})$ $(-1)^{|U_{(-i)}|} = -(1)^{|U_{(i)}|} = \pm i \cdot \mu$.

This allows us to generalize the structure of the D9-brane Chan-Paton factor to the case when the involution $\tau : X \to X$ is fixed point free — the case without O-plane.

### 3.4 Ramond Sector

Let us discuss the parity action on the Ramond sector. For orientifold projection, we need to use the operator corresponding to the parity, $(-1)^{F_R} \tau \Omega$ or $(-1)^{F_L} \tau \Omega$, that lifts to an action on spinors in this sector, i.e., preserves each of the $(\pm)$ and $(\mp)$ spin structures of the strip. Let us discuss the action of $\tau \tilde{\Omega} = (-1)^{F_R} \tau \Omega$ which has $\varepsilon = \mp i$ for $(\pm) \to (\mp)$, see (2.9). The corresponding operator $\tilde{\mathcal{P}} = \mathcal{P}(\tau \tilde{\Omega})$ in the $(\pm)$ sector is defined by

$$\tilde{\mathcal{P}}(\Phi) = \text{ev}_{1}^{*} U_{(-i)} \circ ((\tau \tilde{\Omega})^{*} \Phi^{T} \otimes h_{\alpha}) \circ \text{ev}_{0}^{*} U_{(i)}^{-1} (-1)^{|U_{(i)}||\tau \tilde{\Omega}^{*} \Phi|}.$$

The definition in the $(\mp)$ sector is the same except that $U_{(i)}$ and $U_{(-i)}$ must be interchanged. As we will see in Section 5.4, the operator $(\tau \tilde{\Omega})^{*}$ can be odd in the Ramond sector — it is odd for Type IIA and even for Type IIB — and that is why the sign factor is written as $(-1)^{|U_{(i)}||\tau \tilde{\Omega}^{*} \Phi|}$ rather than $(-1)^{|U_{(i)}||\Phi|}$. In order for the total parity to be
even, the statistics of \( U(i) \) and \( U(-i) \) must be equal for Type IIB and opposite for Type IIA. A natural relation between \( U(i) \) and \( U(-i) \) is then

\[
\text{IIB} : \quad U(i) = \kappa U(-i) \circ \sigma^T, \quad (3.33)
\]

\[
\text{IIA} : \quad U(i) = \mu \kappa \xi \circ U(-i) \circ \sigma^T, \quad (3.34)
\]

for some constant phase \( \kappa \). Recall that such relations are consistent with the conditions (3.16) and (3.17).

In the Type IIA case, there is an additional reason that the relation must be (3.34) rather than (3.33). The constraint (3.23), which we require also in the Ramond sector, is preserved under the parity (3.32) only if \( U(i) \) and \( U(-i) \) obey (3.24) with the same phase \( \mu \). This is the case if the relation is \( U(i) \propto \xi U(-i) \sigma^T \) but not if \( U(i) \propto U(-i) \sigma^T \). That \( U(i) \) and \( U(-i) \) have opposite statistics in Type IIA orientifolds is also suggested in the formula (3.29) (or (3.31)). We must also make sure that the operator \( \tilde{P} \) is independent of the choice of \( \mu (i \) or \(-i) \). We noted earlier that \( \mu \rightarrow -\mu \) is implemented by \( U \rightarrow \xi U \), but we have not fixed the proportionality constant. In order for the parity \( \tilde{P} \) to be the same, we need \( U(\pm i) \rightarrow \pm \xi U(\pm i) \) up to an overall constant. We have placed \( \mu \) in the relation (3.34) so that we do not need to change \( \kappa \) as \( \mu \rightarrow -\mu \).

We would like the operator \( \tilde{P} = P(\tau \tilde{\Omega}) \) to obey the same algebraic relation as \( \tau \tilde{\Omega} \) (c.f. (2.7)),

\[
\tilde{P}^2 = \text{id.} \quad (3.35)
\]

Let us compute the left hand side:

\[
\tilde{P}^2(\Phi) = ev^*_i(U(-i)\tau^*(U^T(-i)^{-1}) \circ (\tau \tilde{\Omega})^{*2}\Phi^{TT} \otimes h_{-\tau^*}\alpha + \alpha) \circ ev^*_\bar{0}(U(i)\tau^*(U^T(-i)^{-1})^{-1} \times (-1)^{|(\tau \tilde{\Omega})^*|+(\tau \tilde{\Omega})^*|\Phi|}
\]

\[
= \begin{cases} 
\kappa^{-2}(-1)^{|U(-0)|}(\tau \tilde{\Omega})^{*2}\Phi & \text{(IIB)} \\
\kappa^{-2}\mu^{-1}(-1)^{|U(-0)|}\xi^{-1} \circ (\tau \tilde{\Omega})^{*2}\Phi \circ \xi(\tau \tilde{\Omega})^{*2}\Phi & \text{(IIA)}
\end{cases}
\]

\[
= (\tau \tilde{\Omega})^{*2}\Phi \times \begin{cases} 
\kappa^{-2}(-1)^{|U|} & \text{(IIB)} \\
\kappa^{-2}\mu^{-1}(-1)^{|U(-0)|} & \text{(IIA)}
\end{cases}
\]

In the second equality, we used the relation (3.33)-(3.34) and (3.24), along with (3.20) and (3.21). In the third equality, we used (3.23). The consistency condition \( \tilde{P}^2 = \text{id} \) determines the phase \( \kappa \) up to a sign.

Alternatively, we may define \( \tilde{P} \) as the composition \( (-1)^{FR} \circ P \) or equivalently as \( P \circ (-1)^{FL} \). Note that each of \( P \), \( (-1)^{FR} \) and \( (-1)^{FL} \) exchanges \((+-)\) and \((-+)\) but
a product of two of them preserves them. The operator $P : (+-) \to (+-)$ is defined
by the same expression as (3.32) except that $\tau \Omega$ is replaced by $\tau \Omega$, and similarly for
$\tilde{P} : (-+) \to (+-)$. The square $P^2 : (+-) \to (+-)$ is

$$ P^2(\Phi) = \text{ev}_1^*(U_{(i)}^{\tau^* U_{(i)}^T})^{-1} \circ ((\tau \Omega)^{2*}\Phi \otimes h_{\tau^*\alpha+\alpha}) \circ \text{ev}_0^*(U_{(-i)}^{\tau^* U_{(-i)}^T})^{-1} $$

$$ = \sigma \circ (\tau \Omega)^{2\Phi} \circ \sigma^{-1} \times c_{(i)}c_{(-i)}. $$

The ratio $c_{(i)}/c_{(-i)}$ can be found from (3.33) and (3.34) along with (3.24):

$$ c_{(i)}c_{(-i)}^{-1} = \begin{cases} (-1)^{|U_i|} & \text{(IIB)} \\ \mu(-1)^{|U_{(-i)}|} & \text{(IA)}. \end{cases} \quad (3.37) $$

If we use (3.31) it may further be evaluated to be $\pm 1$ for $(B_\pm)$ and $\mp i$ for $(A_\pm)$. As we
will see in Section 5.4, $\sigma \circ (\tau \Omega)^{2\Phi} \circ \sigma^{-1}$ is not always equal to $(-1)^F \Phi$ but only up to
a certain phase which exactly cancels this phase, so that the relation $P^2(\Phi) = (-1)^F \Phi$
holds. Let us next discuss the definition of the operators $(-1)^{F_R}$ and $(-1)^{F_L}$. Note that
they transform the field (2.4) as $(-1)^{F_R} : \psi^{\mu} \to \psi^{\mu}$ and $(-1)^{F_L} : \psi^{\mu} \to -\psi^{\mu}$. In order to be a symmetry of the boundary interaction, which includes $\oint \psi^{\mu} D_{\mu} T$, $(-1)^{F_R}$ resp. $(-1)^{F_L}$
may be defined to act on the Chan-Paton factor as the identity resp. the $\mathbb{Z}_2$-grading. Now
that we have $P$, $(-1)^{F_R}$ and $(-1)^{F_L}$, the operator $\tilde{P}$ can be defined as, say, $(-1)^{F_R} \circ P$.
The consistency condition $\tilde{P}^2 = \text{id}$ follows from $P^2 = (-1)^F$ provided that the relation
$(-1)^{F_R} \circ P = P \circ (-1)^{F_L}$ holds. However, the last relation is not automatic but imposes
a constraint on the square of the relative phase $\kappa$. As it must be the case, that constraint
agrees with the one from $\tilde{P}^2 = \text{id}$ via (3.36).

Repeating the same analysis for the open string between different branes, say $B_1$ and
$B_2$, we find that the product $\kappa_1\kappa_2$ obeys the same condition as the squares, $\kappa_1^2$ and $\kappa_2^2$.
This in particular means $\kappa_1 = \kappa_2$: All D-branes must have the same value of $\kappa$. Since $\kappa^2$
is determined, we only have to choose the sign of (common) $\kappa$. To be precise, the sign of $\kappa$
for a fixed definition of $(\tau \Omega)^*$. We next argue that this is related to the “orientation of
the orientifold”.

The Phase $\kappa$ And The Orientation Of Orientifold

The key is the open-closed channel duality. Let $|C_\star\rangle$ denote the crosscap state for a
parity operator $\star$, and $|B\rangle$ be the boundary state of an invariant brane. Then we have

$$ \text{Tr}_{\text{NS}} P q_H^{H_o} \propto \langle C_\star q_c^{H_c} | B \rangle_{\text{NSNS}}, $$

$$ \text{Tr}_R \tilde{P} q_H^{H_o} \propto \langle C_\star q_c^{H_c} | B \rangle_{\text{RR}}, $$

$$ \quad (3.38) $$

24
where $H_o$ and $q_o$ (resp. $H_c$ and $q_c$) are the Hamiltonian and the modular parameter in the open string (resp. closed string) channel.

Now, the effect of the sign flip $\kappa \rightarrow -\kappa$ is to flip the sign of the parity operator $\tilde{P}$ in the Ramond sector while keeping the operator $P$ in the Neveu-Schwarz sector. By the channel duality (3.38), this is to change the sign of either $|B\rangle_{RR}$ or $|C_p\rangle$ while keeping $|B\rangle_{NSNS}$ and $|C_p\rangle$ untouched. Reversing the sign of the RR-part of the boundary state while keeping the NSNS part is nothing but replacing the brane by its antibrane, i.e., reversing the orientation of the brane. However, the sign flip of $\kappa$ has no such effect. Therefore, we must conclude that the sign flip of $\kappa$ corresponds to the sign flip of the RR-part $|C_p\rangle$ of the crosscap state. This may be regarded as the orientation reversal of the orientifold — it is nothing but the orientation reversal of the O-planes when they exist. In this sense, the phase $\kappa$ can be interpreted as the parameter for the orientation of the orientifold.

As a side remark, we note that our formulation can directly derive the consequence of the channel duality (3.38) that the brane orientation reversal flips the sign of the parity operator $\tilde{P}$ in the Ramond sector while keeping $P$ in the NS sector. In Type IIB string theory, the orientation reversal of a brane is simply to flip the $\mathbb{Z}_2$-grading, $\sigma \rightarrow \sigma = -\sigma$. That changes the graded transpose according to (2.31). In order to maintain the condition (3.16) we need to change the o-isomorphism as $U \rightarrow \overline{U} = U\sigma^T$ up to a multiplicative constant. This constant must be opposite between $\epsilon = i$ and $-i$, say,

$$\overline{U}_{(i)} = U_{(i)}\sigma^T \quad \text{and} \quad \overline{U}_{(-i)} = -U_{(-i)}\sigma^T,$$

in order for the brane and its antibrane to have the same $\kappa$, i.e., for $U_{(i)} = \kappa U_{(-i)}\sigma^T$ and $\overline{U}_{(i)} = \kappa \overline{U}_{(-i)}\sigma^T$ to hold at the same time. Under the grading flip with $U_{(\pm i)} \rightarrow \overline{U}_{(\pm i)} = \pm U_{(\pm i)}\sigma^T$, the parity operator $P$ in the NS sector remains the same but the operator $\tilde{P}$ in the Ramond sector is reversed. In Type IIA string theory, the brane orientation reversal is done by $(\sigma, \xi) \rightarrow (-\sigma, \xi)$ and it again leads to the transformation of the o-isomorphism as $U_{(\pm i)} \rightarrow \pm U_{(\pm i)}\sigma^T$. We find the same effect.

**Worldsheet Supersymmetry**

There is an $\mathcal{N} = 1$ supersymmetry in Ramond sector. The expression for the supercharge is found by the Noether procedure — on a $(-+)$ sector state $\Phi$, it acts as

$$Q_1 \Phi = \int_0^1 d\sigma \left( \psi \cdot (p + B \cdot x') + \tilde{\psi} \cdot x' \right) \Phi$$

$$+ \text{ev}_1^*(\psi \cdot A - T) \circ \Phi - (-1)^{|\Phi|} \Phi \circ \text{ev}_0^*(\psi \cdot A + iT),$$

(3.40)

25
where \( p \) is the conjugate momentum for \( x \), and we used the notation \( \psi = \psi_+ + \psi_- \), \( \tilde{\psi} = g \cdot (\psi_+ - \psi_-) \) and \( x' = \partial_\sigma x \). The appearance of \( B \) and \( A \) is due to the relation between the time derivative of \( x \) and the conjugate momentum \( p \),

\[
g \cdot \dot{x}(\sigma) = p(\sigma) + B \cdot x'(\sigma) + A \tilde{\delta}(\sigma - 1) \circ \circ A \delta(\sigma).
\]

The appearance of \( T \) is due to the \( i \dot{\epsilon}_1 T \) term in \( \delta A_t \), see (2.18). We have \(-i T \) inside \( \text{ev}_0^* \), unlike \( T \) inside \( \text{ev}_1^* \), because the time runs in the opposite direction on the left boundary — the precise phase relation, \(-i \) versus 1, can be found by the relation (2.3) between the field variables on the strip and those on the upper-half plane. It is important for the hermiticity of \( Q_1 \) that \( T \) is multiplied by \( i \) in the last term, since \( \Phi \mapsto (-1)^{\Phi_1} \Phi \circ T \) is an anti-hermitian operator if \( T \) is hermitian and odd. It is straightforward to see the supersymmetry relation

\[
(Q_1)^2 = 2H, \tag{3.41}
\]

where the Hamiltonian \( H \) includes the action on the Chan-Paton factor \( \Phi \mapsto \text{ev}_1^* (A_t) \circ \Phi - \Phi \circ \text{ev}_0^* (A_t) \).

The parity operator \( \tilde{P} = P(\tau \tilde{\Omega}) \) commutes with the supercharge,

\[
Q_1 \tilde{P} = \tilde{P} Q_1. \tag{3.42}
\]

To prove this relation, all of (3.16), (3.17) as well as (3.9) are required. For example, \( Q_1 \) acts on \( h_\alpha \) that enters in the definition of \( \tilde{P} \), and produces the factor

\[
Q_1 \left( -i \int_0^1 d\sigma \frac{dx^\mu}{d\sigma} \alpha_\mu(x) \right) = -\int_0^1 d\sigma \left( \frac{d\psi^\mu}{d\sigma} \alpha_\mu(x) + \frac{dx^\mu}{d\sigma} \psi_\nu \partial_\nu \alpha_\mu(x) \right)
\]

\[
= -\text{ev}_1^* (\psi \cdot \alpha) + \text{ev}_0^* (\psi \cdot \alpha) - \int_0^1 d\sigma \psi \cdot (\tau^* B + B) \cdot x',
\]

where (3.9) is used in the last equality. The terms \( \text{ev}_1^* (\psi \cdot \alpha) \) contribute in cancellation of the terms from the \( Q_1 \) action of \( U \), via the relation (3.17). The term involving \( (\tau^* B + B) \) cancels with the B-field term in the expression (3.40) for \( Q_1 \). The complete proof of (3.42) is left as an exercise to the reader.

### 3.5 The Data — Summary

Through the analysis, we have identified the data to specify the orientifold itself and D-branes in it. It can be summarized as follows.

We consider the Type II orientifold on a ten dimensional spin manifold \( X \) by an involution \( \tau : X \to X \) with a lift \( \tau_S \) to an action on Majorana spinors. It is classified into
four cases, \((B_\pm)\) and \((A_\pm)\), depending on whether \(\tau\) is orientation preserving or reversing and whether \(\tau^2\) is 1 or \(-1\). To specify the theory, we need to choose additional data — the B-field \(B\), a hermitian line bundle \(L\) over \(X\) (the twist bundle), a unitary connecton \(\alpha\) of \(L\) (the twist connection), and a section \(c\) of \(\tau^*L \otimes \tilde{L}^*\) (the crosscap section). These are required to obey the constraints:

(i) \(d\alpha = B + \tau^*B\),  
(ii) the connection \(\tau^*\alpha - \alpha\) of \(\tau^*L \otimes \tilde{L}^*\) has trivial holonomy along any loop,  
(iii) \(c\) is a parallel section with respect to that connection such that \(\tau^*c \cdot c = 1\).

On a \(\tau\)-fixed locus, the section \(c\) can be regarded as a number, \(+1\) or \(-1\), which determines the type of the O-plane:

| O-plane | O9\(\mp\) | O8\(\mp\) | O7\(\mp\) | O6\(\mp\) | O5\(\mp\) | O4\(\mp\) | O3\(\mp\) | O2\(\mp\) | O1\(\mp\) | O0\(\mp\) |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| \(c\)    | \(\pm1\) | \(\pm1\) | \(\mp1\) | \(\mp1\) | \(\mp1\) | \(\mp1\) | \(\pm1\) | \(\pm1\) | \(\pm1\) | \(\pm1\) |

The data for D-branes depend on the cases:

\((B_\pm)\): a \(\mathbb{Z}_2\)-graded hermitian vector bundle \(E\) with an even unitary connection \(A\) and an odd hermitian endomorphism \(T\), and a unitary isomorphism \(U: \tau^*E^* \otimes L \to E\) such that

\[
U = c \cdot \tau^*U^t, \quad (-1)^{|U|} = \pm1, \\
A = U(-\tau^*A^t + \alpha)U^{-1} + i^{-1}UdU^{-1}, \\
T = \pm U\tau^*T^tU^{-1}. 
\]  

\((A_\pm)\): an ungraded hermitian vector bundle \(\tilde{E}\) with a unitary connection \(\tilde{A}\) and a hermitian endomorphism \(\tilde{T}\), and a unitary isomorphism \(\tilde{U}: \tau^*\tilde{E}^* \otimes L \to \tilde{E}\) such that

\[
\tilde{U} = c \cdot \tau^*\tilde{U}^t, \\
\tilde{A} = \tilde{U}(-\tau^*\tilde{A}^t + \alpha)\tilde{U}^{-1} + i^{-1}\tilde{U}d\tilde{U}^{-1}, \\
\tilde{T} = \pm \tilde{U}\tau^*\tilde{T}^t\tilde{U}^{-1}. 
\]  

The section \(c\) is independent of the phase \(\varepsilon\) and is related to the one we introduced earlier by \(c = c, \varepsilon c, \varepsilon^{\frac{\tau - \tilde{\tau}}{2}}c\) and \(\varepsilon^{\frac{1+\tau}{2}}c\), for \((B_+), (B_-), (A_+)\) and \((A_-)\) respectively. The
D-brane o-isomorphism $U$ is obtained from $U$ or $\hat{U}$ via

\[
(B_+) : \quad U_{(-i)} = \begin{pmatrix} U_{00} & 0 \\ 0 & iU_{11} \end{pmatrix}, \quad U_{(i)} = \kappa \left( \begin{array}{cc} U_{00} & 0 \\ 0 & -iU_{11} \end{array} \right),
\]

\[
(B_-) : \quad U_{(-i)} = \begin{pmatrix} 0 & iU_{01} \\ U_{10} & 0 \end{pmatrix}, \quad U_{(i)} = \kappa \left( \begin{array}{cc} 0 & -iU_{01} \\ U_{10} & 0 \end{array} \right),
\]

\[
(A_+) : \quad U_{(-i)} = \begin{pmatrix} \bar{U} & 0 \\ 0 & i\bar{U} \end{pmatrix}, \quad U_{(i)} = \kappa \left( \begin{array}{cc} 0 & i\bar{U} \\ \bar{U} & 0 \end{array} \right),
\]

\[
(A_-) : \quad U_{(-i)} = \begin{pmatrix} 0 & i\bar{U} \\ \bar{U} & 0 \end{pmatrix}, \quad U_{(i)} = -\kappa \left( \begin{array}{cc} \bar{U} & 0 \\ 0 & i\bar{U} \end{array} \right).
\]

$U_{ij}$ in Case $(B_\pm)$ are the blocks of $U$ with respect to the decomposition $E = E^0 \oplus E^1$. The expressions for $U$ in Case $(A_\pm)$ are for the choice $\mu = -i$. The expressions for the other choice $\mu = +i$ can be obtained by the replacement $U_{(\pm i)} \to \pm \xi U_{(\pm i)}$. The conditions (1.8)-(1.9) in the introduction section are written in terms of $U$ and $\hat{U}$ in this summary. Note that there is an ambiguity in $U$: it could be replaced by $\sigma U$ in which case the equation involving the tachyon has an extra sign, i.e. we have $T = T \tau^* T^* U^{-1}$ for $(B_\pm)$, while the section $c$ becomes $\pm c$ for $(B_\pm)$. Of course, this is simply a matter of convention and has no physical effect. In an announcement [34] of the present work, we reported the result partly in this different convention.

**Gauge Transformations**

As discussed earlier, the B-field gauge transformation, $B \to B + d\Lambda$ and $A \to A + \Lambda$, shifts the twist connection $\alpha$ by $\Lambda + \tau^* \Lambda$. Accordingly, the Chan-Paton bundle $E$ and the twist bundle $L$ are mapped to $E \otimes L$ and $L \otimes L \otimes \tau^* L$ respectively, where $L$ is the hermitian line bundle which has $\Lambda$ as a unitary connection. The o-isomorphism is now from $\tau^* (E \otimes L)^* \otimes (L \otimes L \otimes \tau^* L) \cong \tau^* E^* \otimes L \otimes L$ to $E \otimes L$ and can be taken as $U \otimes \text{id}_L$, which we write simply as $U$. In particular the crosscap section $c$ does not change under this transformation. We may also consider ordinary gauge transformation of the twist connection, $i\alpha \to i\alpha + \lambda^{-1} d\lambda$, for a $U(1)$-valued function $\lambda$. The simplest way to maintain the relation (3.17) is to transform the o-isomorphism as $U \to \lambda U$. The section $c$ is then transformed to $\lambda \cdot \tau^* \lambda^{-1} \cdot c$. Note that $\lambda \cdot \tau^* \lambda^{-1} = 1$ at $\tau$-fixed points, in accordance with the fact that the O-plane type cannot change under gauge transformations. To summarize, we have found gauge transformations which map the orientifold data $(B, L, (B, L, \alpha, c) \to$ (B $+ d\Lambda, L \otimes L \otimes \tau^* L, 11, \alpha + \Lambda + \tau^* \Lambda, c)$ and $(B, L, \alpha - i\lambda^{-1} d\lambda, \lambda \cdot \tau^* \lambda^{-1} \cdot c)$, (3.47)
and the D-brane data \((E, A, T, U)\) to
\[
(E \otimes L, A + \Lambda, T, U) \quad \text{and} \quad (E, A, T, \lambda \cdot U).
\] (3.48)

Classification of orientifold data has been discussed in \[35\] and \[36\]. It would be interesting to find relation of the present result to these works.

4 Boundary Fermions And Parity Actions

In this section, we study parity operation on worldsheet theory with boundary fermions, which has boundary action of the form
\[
S_{\text{bdry}} = \int dt \left\{ \frac{i}{4} \sum_{i=1}^{s} \xi_i^{(R)} \frac{d}{dt} \xi_i^{(R)} + \cdots \right\}_{\text{right}} + \int dt \left\{ -\frac{i}{4} \sum_{i'=1}^{r} \xi_{i'}^{(L)} \frac{d}{dt} \xi_{i'}^{(L)} + \cdots \right\}_{\text{left}}.
\] (4.1)
where the ellipses are interaction terms of \(\xi_i\)'s and the boundary values of the bulk fields. This study serves as a preparation for a part of the direct CFT determination of the structure of the D9-brane Chan-Paton factor to be done in next section. It also provides a background for the treatment of non-BPS D-branes in Type II string theory, such as D9-branes in Type IIA, which is employed in the present paper. At the end of this section, we study a particular configuration on unstable D9-branes that represents BPS D-branes on top of an orientifold plane.

Boundary fermion realization of Chan-Paton factors was first studied in \[37\]. Parity action on boundary fermion system was discussed in \[38\].

4.1 Open String States

Let us consider an open string with \(s\) fermions on the right boundary and \(r\) fermions on the left boundary. The boundary action takes the form
\[
S_{\text{bdry}} = \int dt \left\{ \frac{i}{4} \sum_{i=1}^{s} \xi_i^{(R)} \frac{d}{dt} \xi_i^{(R)} + \cdots \right\}_{\text{right}} + \int dt \left\{ -\frac{i}{4} \sum_{i'=1}^{r} \xi_{i'}^{(L)} \frac{d}{dt} \xi_{i'}^{(L)} + \cdots \right\}_{\text{left}}.
\] (4.2)
The sign of the kinetic terms for \(\xi_{i'}^{(L)}\)'s is opposite to the standard one since the natural orientation of the left boundary is pointing toward the “past”. The canonical anticommutation relations are
\[
\{\xi_i^{(R)}, \xi_j^{(R)}\} = 2\delta_{i,j}, \quad \{\xi_{i'}^{(L)}, \xi_{j'}^{(L)}\} = -2\delta_{i',j'}, \quad \{\xi_i^{(R)}, \xi_{j'}^{(L)}\} = 0.
\] (4.3)
and the hermiticity is
\[ \xi_i^{(R)^\dagger} = \xi_i^{(R)}, \quad \xi_i^{(L)^\dagger} = -\xi_i^{(L)}. \] (4.4)

The numbers \( r \) and \( s \) must be related to the boundary conditions on the bulk fields, say \( \beta \) (right) and \( \alpha \) (left), so that the space of open string states has a GSO operator — a \( \mathbb{Z}_2 \)-grading operator \( (-1)^F \) that anticommutes with \( \xi_i^{(R)}, \xi_i^{(L)} \) as well as the bulk fermions. When \( r + s \) is even, the boundary fermion algebra (4.3) has a unique \( \mathbb{Z}_2 \)-graded irreducible representation. Then, the boundary conditions \( (\alpha, \beta) \) must be such that the bulk fields have their own quantization with a \( \mathbb{Z}_2 \)-graded space of states, so that the space of total open string states is the (graded) tensor product of the boundary fermion factor and the bulk factor
\[ \mathcal{H}^{\text{tot}}_{(r,\alpha),(s,\beta)} = \mathcal{H}^{\text{b.f.}}_{r,s} \otimes \mathcal{H}^{\text{bulk}}_{\alpha,\beta}. \] (4.5)

When \( r + s \) is odd, the algebra (4.3) has two distinct irreducible representations and neither of them is \( \mathbb{Z}_2 \)-graded. In this case, the boundary conditions \( (\alpha, \beta) \) must be such that the bulk fields have an unpaired fermionic mode, so that the combined bulk-boundary system has a unique irreducible representation with a GSO operator \( (-1)^F \). The present discussion follows [11] where the open string stretched between a BPS D9-brane and a non-BPS D0-brane in Type IIB or Type I string theory is found to have odd number of bulk fermion zero modes. This was recognized as a problem against natural quantization with GSO projection and, as a solution, it was proposed to place a single fermion on the D0 boundary. We shall say more on this momentarily.

Before giving a more detailed description of the space of states, let us introduce some notations on representations of the Clifford algebra
\[ \{ \xi_i, \xi_j \} = 2\delta_{i,j}, \quad i, j = 1, \ldots, s, \] (4.6)
with even \( s \). We take complex combinations of the generators, \( \eta_i = \frac{1}{2}(\xi_{2i-1} + i\xi_{2i}) \) and \( \overline{\eta}_i = \frac{1}{2}(\xi_{2i-1} - i\xi_{2i}) \) \( (i = 1, \ldots, \frac{s}{2}) \), which obey the relations
\[ \{ \eta_i, \overline{\eta}_j \} = \delta_{i,j}, \quad \{ \eta_i, \eta_j \} = \{ \overline{\eta}_i, \overline{\eta}_j \} = 0. \]

An irreducible representation is build on a vector \( |0\rangle \) annihilated by all \( \eta_i \)'s, and is spanned by \( |0\rangle, \overline{\eta}_i|0\rangle, \eta_i\overline{\eta}_j|0\rangle, \ldots, \overline{\eta}_i \cdots \overline{\eta}_{\frac{s}{2}}|0\rangle \). It has a \( \mathbb{Z}_2 \)-grading. For example, even multiples of \( \overline{\eta}_i \)'s on \( |0\rangle \) are even and odd multiples are odd. The hermitian inner product such that the above \( 2^\frac{s}{2} \) vectors form an orthonormal basis has the property \( \eta_i^\dagger = \overline{\eta}_i \) (equivalently \( \xi_j^\dagger = \xi_j \)). We shall denote this graded irreducible representation with inner product by \( V_s \).
(i) \( r \) and \( s \) even

The boundary fermion factor in (4.5) is naturally of the Chan-Paton form, that is, the space of linear maps

\[ \mathcal{H}_{r,s}^{bf} = \text{Hom}_C(V_r, V_s). \]  

On this space, \( \xi^{(R)}_i \)'s and \( \xi^{(L)}_{i'} \)'s act as

\[ \xi^{(R)}_i \phi = \xi_i \circ \phi, \quad \xi^{(L)}_{i'} \phi = (-1)^\phi \phi \circ \xi_{i'}. \]  

It is straightforward to check the anticommutation relation (4.3), as well as the hermiticity (4.4) with respect to the inner product \( (\phi_1, \phi_2) = \text{tr}_{V_r}(\phi_1^\dagger \phi_2) = \text{tr}_{V_s}(\phi_2 \phi_1^\dagger) \). The space \( \text{Hom}_C(V_r, V_s) \) has a natural \( \mathbb{Z}_2 \)-grading induced by those of \( V_r \) and \( V_s \).

(ii) \( r \) and \( s \) odd

The boundary fermion factor in (4.5), though \( \mathbb{Z}_2 \)-graded by itself, does not have the structure of the space of linear maps between \( \mathbb{Z}_2 \)-graded vector spaces. Such a Chan-Paton form would be advantageous though, for example, in the consideration of product of open string states. For this purpose, we introduce auxiliary boundary fermions, one at each boundary — \( \xi^{(R)} \) on the right and \( \xi^{(L)} \) on the left, with the action

\[ S_{\text{aux}} = \int_{-\infty}^{\infty} dt \frac{i}{4} \xi^{(R)} \frac{d}{dt} \xi^{(R)} - \int_{-\infty}^{\infty} dt \frac{i}{4} \xi^{(L)} \frac{d}{dt} \xi^{(L)}. \]  

The space of states of the extended system is of the Chan-Paton type, \( \text{Hom}_C(V_{r+1}, V_{s+1}) \). It can alternatively be defined as the (graded) tensor product of the original space and the space from the auxiliary fermions. The second factor is a 2-dimensional space consisting of one even and one odd states. We suppose that the even state \( |0\rangle_{\text{aux}} \) satisfies the continuity condition \( \xi^{(R)} |0\rangle_{\text{aux}} = \xi^{(L)} |0\rangle_{\text{aux}} \). To get back the original space, we select only the states of the form \( \phi' \otimes |0\rangle_{\text{aux}} \). This is equivalent to imposing the projection condition

\[ \xi^{(R)} \xi^{(L)} = 1. \]  

The boundary fermion sector can thus be realized as

\[ \mathcal{H}_{r,s}^{bf} = \text{Hom}_C(V_{r+1}, V_{s+1}) \bigg|_{\xi^{(R)} \xi^{(L)} = 1}. \]  

Note that the \( \xi^{(R)} \xi^{(L)} = 1 \) condition amounts for \( \phi \in \text{Hom}_C(V_{r+1}, V_{s+1}) \) to

\[ \xi \circ \phi = (-1)^\phi \phi \circ \xi, \]  

where \( \xi \) acts on \( V_{r+1} \) and \( V_{s+1} \) as the “last” Clifford generator. We see that this is compatible with the product of open string states.
(iii) \( r \) even and \( s \) odd (resp. \( r \) odd and \( s \) even)

In this case, as remarked above, the boundary conditions \((\alpha, \beta)\) must be such that the bulk fields have an unpaired fermionic mode. We introduce a pair of auxiliary fermions, \( \xi \) and \( \xi' \), which shall be included into the boundary and the bulk sectors respectively. We choose the sign of the kinetic term of \( \xi \) to be the same as the one for the odd boundary fermions, i.e. positive if \( s \) is odd (resp. negative if \( r \) is odd), and opposite to the one for \( \xi' \). Then the extended boundary fermion system has a \( \mathbb{Z}_2 \)-grading and is of the Chan-Paton form

\[
H_{r,s}^{b.f.+aux} = \text{Hom}_\mathbb{C}(V_r, V_{s+1}) \quad \text{(resp. } \text{Hom}_\mathbb{C}(V_{r+1}, V_s))
\]

Likewise the extended bulk system also has a \( \mathbb{Z}_2 \)-grading. To remove the extra degrees of freedom coming from the auxiliary fermions, we impose the projection condition \( \xi \xi' = 1 \).

The space of states is therefore

\[
H_{(r,\alpha),(s,\beta)}^{tot} = H_{r,s}^{b.f.+aux} \otimes H_{\alpha,\beta}^{bulk+aux} \bigg|_{\xi \xi' = 1}
\]

### 4.2 Non-BPS D-Branes In Type II Superstrings

It is a good point to provide a background for the description of non-BPS D-branes used in this paper.

Type IIA (IIB) string theory has non-BPS \( D_p \)-branes in addition to BPS \( D_q \)-branes, where \( p \) is odd (even) and \( q \) is even (odd). As mentioned above, it was found in \cite{11} that a natural quantization of open strings with GSO projection is possible by placing an odd number of fermions along the boundary for non-BPS D-branes. More generally, we may also have an additional \( \mathbb{Z}_2 \)-graded vector space \( V' \) along such a boundary. If we allow this, we may assume that the number of boundary fermions is one, since the additional, even number of boundary fermions may be included in \( V' \). The single boundary fermion \( \xi_1 \) and \( V' \) may have interaction among themselves and with the boundary values of the bulk fields.

In order to present the total degrees of freedom at the boundary, i.e., \( \xi_1 \) and \( V' \), in the more standard Chan-Paton form, we introduce a single auxiliary boundary fermion \( \xi \). Then, we have a \( \mathbb{Z}_2 \)-graded Chan-Paton vector space \( V \) which has twice as many dimensions as \( V' \). Of course, we have to make sure that the introduction of \( \xi \) does not change the content of the theory. One necessary condition is that \( \xi \) has only the kinetic term and has no interaction with other fields. On the open string states, we need to impose an appropriate projection condition, as we discuss below.
On a boundary circle $C$ without insertion of open string state, the inclusion of auxiliary fermion has the effect of multiplying “1”, provided that the normalization is done correctly. When the boundary circle $C$ has the anti-periodic spin structure, we use

$$\frac{1}{\sqrt{2}} \int \mathcal{D}\xi \exp \left( i \int_C \frac{i}{4} \xi \frac{d}{d\tau} \xi d\tau \right) = 1.$$  \hspace{1cm} (4.15)

The factor of $\frac{1}{\sqrt{2}}$ is required for the reason explained in [11]. For the periodic spin structure, we use

$$\frac{\#}{\sqrt{2}} \int \mathcal{D}\xi \xi(\tau_0) \exp \left( i \int_C \frac{i}{4} \xi \frac{d}{d\tau} \xi d\tau \right) = 1,$$  \hspace{1cm} (4.16)

for some phase $\#$ where $\tau_0$ is an arbitrary point of the circle. If we perform the path-integral of $\xi$ together with $\xi_1$ and include $V^\prime$ as well, we have the trace (for antiperiodic circle) or the supertrace (for periodic circle) over the $\mathbb{Z}_2$-graded Chan-Paton space $V$ of a Wilson line $P \exp \left( -i \int_C A \right)$. This results in the factors of the form (2.23). Note that $A$ commutes with the auxiliary fermion $\xi$ since the original boundary interaction does not involve $\xi$. In particular, if $A$ is written in the form (2.16), then the tachyon $T$ anticommutes with $\xi$ and the gauge field $A$ commutes with $\xi$. That is, they are of the form (2.22).

Next, let us consider a boundary circle $C$ with insertion of open string states, assuming that all the segments correspond to non-BPS D-branes. Thus, all the open string states are of the type (ii), and the auxiliary fermion $\xi$ goes around the circle $C$. As discussed in the previous subsection, we require that the open string states in the extended system obey the constraint $\xi(\tau_0) \xi(L) = 1$. This means that it is of the factorized form $\Phi \otimes |0\rangle_{\text{aux}}$ where $\Phi$ is the state of the original system and $|0\rangle_{\text{aux}}$ is the state of the auxiliary fermion system obeying $\xi(\tau_0) |0\rangle_{\text{aux}} = \xi(L) |0\rangle_{\text{aux}}$. Then, the auxiliary path-integral factors out and, with the correct normalization, gives us 1 again, leaving us with the system before the extension. Thus, the extension by auxiliary fermion again has no effect as long as the constraint $\xi(\tau_0) \xi(L) = 1$ is satisfied on the open string states.

Finally, let us consider the boundary circle $C$ where some segments correspond to BPS D-branes while others correspond to non-BPS D-branes. In this case, open string states of the type (iii) must be present. Following the previous subsection, along the segment corresponding to non-BPS D-branes, we introduce additional auxiliary fermion $\xi'$ with kinetic term of the opposite sign compared to the one for $\xi$. The resulting amplitude in the extended system is of the form as depicted in Figure 2 for the case of a disc. The dashed lines correspond to the $\xi'$-lines. With the requirement of $\xi' \xi' = 1$ at (iii) and $\xi(\tau_0) \xi'(L) = \xi'(\tau_0) \xi(L) = 1$ at (ii), the $\xi-\xi'$ path-integrals factor out to give “1” and we get back the amplitude of the original system.
The resulting description is similar to the one for Type IIB and Type I D0-branes given by A. Sen \cite{13,39,41}. If we set $V' = C$ and turn off boundary interaction, our construction has the same Chan-Paton structure and the open string interaction rule as the ones given by Sen, which were first proposed in \cite{40} and later rationalized in \cite{13,41} using the $(-1)^{F_L}$ orbifold. Here $(-1)^{F_L}$ is the mod 2 number of spacetime fermions from the left movers, and the corresponding orbifold maps Type IIA to Type IIB and vice versa. However, unlike in the Green-Schwarz formalism, this orbifold is not natural in the NSR formalism we are working with. We have given a natural derivation of the structure within the NSR formalism.

4.3 Parity Action

We now study parity actions. First, we would like to find parity transformations of boundary fermions that leave invariant the boundary action which, on the boundary of the (Euclidean) upper-half plane, takes the form

$$iS_{\text{bdry}} = \int_{\partial \Sigma} d\tau \left\{ \frac{i}{4} \sum_{i=1}^{s} \xi_i \frac{d}{d\tau} \xi_i + \cdots \right\}. \quad (4.17)$$

We assume a linear transformation which then must be of the form

$$\xi_i(\tau) \rightarrow \varepsilon \sum_{i=1}^{s} O_{ij} \xi_j(-\tau), \quad (4.18)$$

where $\varepsilon$ is the phase (2.9) that appears in the transformation $\psi(\tau) \rightarrow \varepsilon \tau^* \psi(-\tau)$. This is so that the orientifold projection condition on GSO even expressions $(\psi)^n(\xi)^m \ (n + m \ \text{even})$ does not depend on the choice of boundary spin structure. Invariance of the kinetic
term in (4.17) requires that $O_{ij}$ is an orthogonal matrix, $O^tO = 1_s$. The algebra of the parity transformations, $(\tau \Omega)^2 = (-1)^F$, $((-1)^F \tau \Omega)^2 = \text{id}$ etc, requires that it is involutive, $O^2 = 1_s$. It follows from unitarity of the parity operator on open strings that $O$ must be real, $O^* = O$ (see below). In particular, with a real orthogonal transformation of $\xi_i$’s, the matrix $O$ can be made into the diagonal form

$$O = \text{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1). \quad (4.19)$$

Additional condition on $O$ may come from the invariance of the interaction terms, “+⋯” in (4.17).

We study the parity action on the open string states, in the set-up of Section 4.1. For concreteness, we consider $P = P(\tau \Omega)$ for $\Omega$ as in (2.5). We assume for simplicity that the boundary conditions for the bulk fields, $\alpha$ and $\beta$, are also invariant under the parity. We would like to find an even operator

$$P : \mathcal{H}_{(r, \alpha), (s, \beta)}^{\text{tot}} \rightarrow \mathcal{H}_{(s, \beta), (r, \alpha)}^{\text{tot}}, \quad (4.20)$$

corresponding to the parity $\tau \Omega$ that transforms the boundary fermions as

$$\xi_i^{(R)} \rightarrow P \xi_i^{(R)} P^{-1} = \varepsilon \sum_{j=1}^{s} O_{ij} \xi_j^{(L)}, \quad (4.21)$$

$$\xi_i^{(L)} \rightarrow P \xi_i^{(L)} P^{-1} = \varepsilon' \sum_{j'=1}^{r} O'_{ij'} \xi_j^{(R)}. \quad (4.22)$$

Here $\varepsilon$ and $\varepsilon'$ are $\mp i$ depending on the spin structures of the right and the left boundaries of the domain strip. In view of the hermiticity (4.4) of $\xi_i^{(R)}$ and $\xi_i^{(L)}$, unitarity of the operator $P$ indeed requires that the matrices $O_{ij}$ and $O'_{ij'}$ must be real. In Case (ii) and (iii), we also need to specify the transformation of the auxiliary fermions. We look for a parity operator that preserves the factorized form of the space of states, (4.5) or (4.14),

$$P(\phi \otimes \psi) = (-1)^{|\phi||P_{\text{bulk}}|} P_{\text{CP}}(\phi) \otimes P_{\text{bulk}}(\psi). \quad (4.23)$$

We shall determine the Chan-Paton part $P_{\text{CP}}$ below.

(i) $r$ and $s$ even

We look for an operator $P_{\text{CP}} : \text{Hom}(V_r, V_s) \rightarrow \text{Hom}(V_s, V_r)$ of the from

$$P_{\text{CP}}(\phi) = (-1)^{|\phi| |U|} U' \circ \phi^T \circ U^{-1}, \quad (4.24)$$

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for linear maps $U : V_s^* \to V_s$ and $U' : V_s^* \to V_s$. As $\xi^{(L)}_i$ and $\xi^{(R)}_i$ are given by (4.8), the transformation (4.21) is realized if

$$U\xi^T_i U^{-1} = (-1)^{|U|}\varepsilon \sum_{j=1}^s O_{ij}\xi_j. \quad (4.25)$$

Since $V_s$ is an irreducible representation of the Clifford algebra, this condition uniquely fixes $U$ up to a scalar multiplication. The same can be said on $U'$. We would like $U$ and $U'$ to play the rôle of the o-isomorphisms of the boundary fermion systems.

Let us explicitly construct such $U$ in the case where $O_{ij} = \delta_{ij}$. The equation (4.25) reads for $\eta_i$ and $\overline{\eta}_i$ ($i = 1, \ldots, \frac{d}{2}$) as

$$U\eta^T_i U^{-1} = (-1)^{|U|}\varepsilon \eta_i, \quad U\overline{\eta}^T_i U^{-1} = (-1)^{|U|}\varepsilon \overline{\eta}_i. \quad (4.26)$$

From the first set of equations, we find $\eta^T_i U^{-1}|0\rangle = \text{const} \times U^{-1}\eta_i|0\rangle = 0$, which means that $U^{-1}|0\rangle$ is proportional to $|0\rangle \eta_1 \cdots \eta_{\frac{d}{2}}$. Using the rest of the conditions, we find

$$U^{-1}\overline{\eta}_{i_1} \cdots \overline{\eta}_{i_a}|0\rangle = \varepsilon^{-a} \langle 0|0 \cdots \eta_{\frac{d}{2}} \overline{\eta}_{i_1} \cdots \overline{\eta}_{i_a}, \quad (4.27)$$

for $1 \leq i_1 < \cdots < i_a \leq \frac{d}{2}$. This is the solution to (4.40). We shall denote it by $U_s$. Note that the right hand side of (4.27) changes by a factor of $(-1)^a$ if we switch the sign of $\varepsilon$. Therefore $U_s$'s for the opposite phase $\varepsilon$ are related by $U_{s(i)} = \sigma \circ U_{s(-i)}$. Note that

$$(-1)^{|U_s|} = (-1)^{\frac{d}{2}}. \quad (4.28)$$

Let us see if $U = U_s$ satisfies the condition (3.20) for an o-isomorphism. Let us compute the pairing $\langle (U^T)^{-1}v, w \rangle$ for $v = \overline{\eta}_{i_1} \cdots \overline{\eta}_{i_a}|0\rangle$ and $w = \overline{\eta}_{j_1} \cdots \overline{\eta}_{j_b}|0\rangle$:

$$\langle (U^T)^{-1}v, w \rangle = (-1)^{|U|}\langle (U^{-1})^T v, w \rangle = (-1)^{|U|}\varepsilon^{-a}\langle w, U^{-1}v \rangle = (-1)^{|U|}\varepsilon^{-a}\langle U^{-1}v, w \rangle = (-1)^{|U|+a}\langle 0|0 \cdots \eta_{\frac{d}{2}} \overline{\eta}_{j_1} \cdots \overline{\eta}_{j_b} \overline{\eta}_{i_1} \cdots \overline{\eta}_{i_a}|0\rangle$$

On the other hand, we have

$$\langle U^{-1}v, w \rangle = (-1)^a\langle U^{-1}v, w \rangle = (-1)^a\varepsilon^{-a}\langle 0|0 \cdots \eta_{\frac{d}{2}} \overline{\eta}_{i_1} \cdots \overline{\eta}_{i_a} \overline{\eta}_{j_1} \cdots \overline{\eta}_{j_b}|0\rangle = (-1)^{a+ab}\varepsilon^{-a}\langle 0|0 \cdots \eta_{\frac{d}{2}} \overline{\eta}_{j_1} \cdots \overline{\eta}_{j_b} \overline{\eta}_{i_1} \cdots \overline{\eta}_{i_a}|0\rangle$$
Note that these are non-zero only when $a + b = \frac{s}{2}$. Using this fact and also the relation $|U| \equiv \frac{s}{2} \pmod{2}$, we find

$$\langle (U^T)^{-1}w, w \rangle = (-1)^{\frac{s}{2} + a} \varepsilon^{-b + a} \langle U^{-1}\sigma v, w \rangle = \varepsilon^{\frac{s}{2}} \langle U^{-1}\sigma v, w \rangle.$$

This proves that (3.20) is indeed satisfied,

$$U_s(U^T_s)^{-1} = \varepsilon^{\frac{s}{2}} \sigma. \quad (4.29)$$

The construction can be done in the same way in the general case (4.19). When the multiplicity $s_+$ of eigenvalue 1 is even (and hence so is $s_-$), we can find complex combinations of $\xi_i$'s, $\eta_i$ and $\overline{\eta}_i (i = 1, \ldots, \frac{s}{2})$, such that $U\eta_i^T U^{-1}$ is proportional to $\eta_i$ for all $i$. Then we find that $U^{-1}|0\rangle$ is proportional to $\langle 0 | \eta_1 \cdots \eta_{\frac{s}{2}}$. When $s_+$ is odd (and hence so is $s_-$), one can find complex combinations such that $U\eta_i^T U^{-1}$ is proportional to $\eta_i$ for all but one $i$, say $i = 1$, where it is proportional to $\overline{\eta}_1$. We then find that $U^{-1}|0\rangle$ is proportional to $\langle 0 | \eta_2 \cdots \eta_{\frac{s}{2}}$. The $U^{-1}$ transform of other vectors can be obtained by using (4.25). We can show

$$(-1)^{|U|} = (-1)^{\frac{s_+ + s_-}{2}},$$

$$U(U^T)^{-1} = \varepsilon^{\frac{s_+ + s_-}{2}} \sigma. \quad (4.30)$$

(ii) $r$ and $s$ odd

Let us first discuss the parity transformation of the auxiliary fermions, $\xi^{(R)}$ and $\xi^{(L)}$. We require that the $\xi^{(R)} \xi^{(L)} = 1$ condition is maintained so that the parity preserves the from $\phi \otimes |0\rangle_{\text{aux}}$ of vectors. We must also require the condition $P^2 = (-1)^F$ and that preserves the hermiticity $\xi^{(R)} = \xi^{(R)}$ and $\xi^{(L)} = -\xi^{(L)}$. This is satisfied if the transformation is

$$\xi^{(R)} \longrightarrow \mu \xi^{(L)}, \quad \xi^{(L)} \longrightarrow \mu \xi^{(R)}, \quad (4.31)$$

where $\mu$ is $i$ or $-i$ and is independent of the phase $\varepsilon$. Note that the reasoning for the correlation of parity transformation of the physical boundary fermions $\xi_i$’s with the phase $\varepsilon$ does not apply to to auxiliary fermions. Also, the $\xi^{(R)} \xi^{(L)} = 1$ condition would be violated in the Ramond-sector if we insisted such a correlation.

If the parity on the Chan-Paton factor is written as (4.24), then $U = U(\varepsilon) \mu$ must satisfy (4.25) as well as

$$U\xi^T U^{-1} = (-1)^{|U|} \mu \xi \quad (4.32)$$
for both $\varepsilon = -i$ and $+i$. If we put $\xi_{s+1} = \xi$ as the last member of the extended Clifford algebra, we have
\[
U(\varepsilon)\xi^TU(\varepsilon)^{-1} = (-1)^{|U(\varepsilon)|}\varepsilon \sum_{j=1}^{s+1} O_{ij} \xi_j
\]
with
\[
O(\mp i) = \begin{pmatrix} O & 0 \\ 0^t & \pm i\mu \end{pmatrix}.
\]
The solutions for $U(i)$ and $U(-i)$ must be related by
\[
U(i) = \xi \circ \sigma \circ U(-i),
\]
up to scalar multiplication. In particular, they have opposite statistics. Using (4.30) we find
\[
U(\mp i)(U(\mp i)^T)^{-1} = (\mp i)^{\frac{s_{+} - s_{-} \pm i\mu}{2}} \sigma,
\]
and
\[
(-1)^{|U(\mp i)|} = (-1)^{\frac{s_{+} - s_{-} \pm i\mu}{2}}.
\]
“Atiyah-Bott-Shapiro (ABS) construction” in string theory literature. It can be described as a system of \( k \) boundary fermions, \( \xi_1, \ldots, \xi_k \), with the action

\[
S_{\text{bdry}} = \int_{\partial \Sigma} dt \left\{ \frac{i}{4} \sum_{i=1}^{k} \xi_i \frac{d}{dt} \xi_i - \frac{i}{2} \sum_{i=1}^{k} \psi^i \xi_i - \frac{1}{2} \sum_{i=1}^{k} (x^i)^2 \right\}.
\]

(4.37)

After quantizing \( \xi_i \)'s (together with the auxiliary fermion \( \xi \) for Type IIA (\( k \) odd)), the boundary interaction takes the form (2.16) where the gauge field is trivial, \( A = 0 \), and the tachyon has the profile

\[
T(x) = \vec{x} \cdot \vec{\xi} = \sum_{i=1}^{k} x^i \xi_i.
\]

(4.38)

It is represented on the trivial vector bundle with the fibre

\[
V = \begin{cases} 
V_k & \text{IIB (} k \text{ even),} \\
V_{k+1} & \text{IIA (} k \text{ odd).}
\end{cases}
\]

Now we put this configuration on top of the \( O(9-k) \)-plane of the involution \( \tau \) that acts by the sign flip of \( x^1, \ldots, x^k \). Since the fermions \( \psi^i(\tau) \) transform under \( \tau \Omega \) into \( -\epsilon \psi^i(-\tau) \) for \( i = 1, \ldots, k \), the invariance of the boundary interaction \( \int_{\partial \Sigma} d\tau \sum_{i=1}^{k} \psi^i(\tau) \xi_i(\tau) \) requires that the parity transform of the boundary fermions is

\[
\xi_i(\tau) \rightarrow \epsilon \xi_i(-\tau).
\]

(4.39)

Then the condition for the o-isomorphism (4.25) reads

\[
U \xi^T_i U^{-1} = (-1)^{|U|} \epsilon \xi_i, \quad i = 1, \ldots, k.
\]

(4.40)

This also follows from the condition (3.16) of the orientifold invariance of the tachyon profile, which reads for (4.38) as

\[
\sum_{i=1}^{k} x^i \xi_i = (-1)^{|U|} U \left( \epsilon \sum_{i=1}^{k} (-x^i) \xi^T_i \right) U^{-1}.
\]

For \( k \) odd (Type IIA), we also need

\[
(-1)^{|U|} U \xi^T U^{-1} = \mu \xi.
\]

The solutions are obtained in the previous subsection and are found to satisfy the o-isomorphism condition (3.20). Let us write them down together with the values of the crosscap section \( c \). For \( k \) even (Type IIB),

\[
U = U_k, \quad c = \epsilon^\frac{k}{2}.
\]

(4.41)
For $k$ odd (Type IIA),

$$
\begin{align*}
U &= U_{k+1}, & c &= \varepsilon \frac{k+1}{2} \quad \text{for } \varepsilon = \mu, \\
U &= \xi \circ U_{k+1}, & c &= \varepsilon \frac{k-1}{2} \quad \text{for } \varepsilon = -\mu.
\end{align*}
$$

(4.42)

Let us next put $N$ BPS D-branes on top of the O-plane. An open string state has a general form $|\psi, ij\rangle$ where $\psi$ is the state of the conformal field theory and $i, j = 1, \ldots, N$ are the Chan-Paton indices. The parity acts on the states as

$$
P : |\psi, ij\rangle \mapsto -\sum_{i', j'=1}^{N} \gamma_{j'j} \gamma_{i'i}^{-1} \tau \Omega(\psi), j'i'\rangle \gamma_{ii'}^{-1}.
$$

(4.43)

In terms of the D9-brane configuration, the $N$ D-branes can be realized on the Chan-Paton space $V = V_{\text{ABS}} \otimes \mathbb{C}^N$ with the grading $\sigma = \sigma_{\text{ABS}} \otimes 1_N$ and the tachyon profile $T(x) = T_{\text{ABS}}(x) \otimes 1_N$. Here we put the subscript “ABS” for all the quantities found above for a single D-brane. The parity action (4.43) corresponds to the choice

$$
U = U_{\text{ABS}} \otimes \gamma
$$

(4.44)

of an o-isomorphism on the D9-branes. For this $U$ we have

$$
U(U^T)^{-1} = U_{\text{ABS}}(U_{\text{ABS}}^T)^{-1} \otimes \gamma(\gamma^t)^{-1} = c_{\text{ABS}} \sigma_{\text{ABS}} \otimes \gamma(\gamma^t)^{-1}.
$$

(4.45)

Compare the number $c_{\text{ABS}}$, given in (4.41) and (4.42), and the number $c$ in the formula (1.7). We see that they agree for the $O^-$-type and is opposite in sign for the $O^+$-type. Therefore, the formula (1.7) leads to the condition

$$
\gamma(\gamma^t)^{-1} = \pm 1_N \quad \text{for } O^\pm\text{-type.}
$$

(4.46)

Namely, $\gamma$ is symmetric for $O^-$ and antisymmetric for $O^+$. By a suitable basis change of $\mathbb{C}^N$, which does $\gamma \rightarrow M \gamma M^t$, we can set $\gamma = 1_N$ for $O^-$ and $\gamma = J_N$ for $O^+$, where

$$
J_N = \begin{pmatrix}
0 & -1_{N/2} \\
1_{N/2} & 0
\end{pmatrix}.
$$

This means that the gauge group on the D-branes is $O(N)$ for $O^-$ and $USp(N)$ for $O^+$. We have seen that the formula (1.7) leads to this standard fact on BPS D-branes on top of the O-plane.

Turning around the logic, let us require that the $N$ BPS D-branes on top of the O-plane have gauge group $O(N)$ for $O^-$ and $USp(N)$ for $O^+$, namely, that $\gamma$ must obey the equation (4.46). Via the relation (4.45), this means that the number $c$ is given by (1.7)
for this particular configuration. As remarked in Section 3.2, the section $c$ is common for all D-branes in the theory. We therefore conclude that the formula (1.7) must hold in any D9-brane configuration.

This may be regarded as the first though indirect derivation of the formula (1.7). In the next section, we will give a direct derivation of the formula by studying D9-branes without excitation.

### 5 The Structure Of D9-Brane Chan-Paton Factor

In this section, we study the consistency condition of the $\alpha$-isomorphism for the D9-branes in the Type II orientifold on $\mathbb{R}^{10}$ associated with the involution

\[ \tau : (x^0, x^1, \ldots, x^k, x^{k+1}, \ldots, x^9) \mapsto (x^0, -x^1, \ldots, -x^k, x^{k+1}, \ldots, x^9) \]

which has a single $O(9-k)$-plane. In particular, we will prove the formula (1.7) that relates the value of the crosscap section $c$ at the O-plane to its type and dimension.

The main ground of study will be the open strings stretched between the $D(9-k)$-branes at the $O(9-k)$-plane and the D9-branes. The idea is to use the basic requirement $P^2 = (-1)^F$ on these sectors together with the knowledge on the Chan-Paton factor for the $D(9-k)$-branes. However, it is extremely subtle to define the parity on such boundary changing sectors, especially with the freedom of choosing a phase factor. Facing this problem, we follow Gimon-Polchinski and employ the operator product rule

\[ P(\Phi_2 \cdot \Phi_1) = (-1)^{\Phi_1 \cdot |\Phi_2|} P(\Phi_1) \cdot P(\Phi_2) \] (5.1)

which holds if at least one of $\Phi_1$ and $\Phi_2$ is in the NS sector. (In this section, we denote open string states and the corresponding vertex operators using the same symbols. We shall also abbreviate $(-1)^{\Phi}$ as $(-1)^{\Phi}$ when there is no danger of confusion.) Applying this in the set-up of Figure 3, we find another form of the basic requirement

\[ P(P(\Phi) \cdot \Phi) = (-1)^\Phi P(\Phi) \cdot P^2(\Phi) = P(\Phi) \cdot \Phi. \] (5.2)

If $\Phi$ is in the $D(9-k)$-D9 sector, then the product $P(\Phi) \cdot \Phi$ is in the $D(9-k)$-D$(9-k)$ sector in which we know very well about the parity operator. The equation (5.2) will give us a strong constraint on the structure of Chan-Paton factor of the D9-branes.

A simple reinterpretation of this analysis will also determine the structure of Chan-Paton factor for D-branes of all dimensions in Type I string theory. In addition, in
Section 5.4 we shall use the same type of argument to show the relation $P^2 = (-1)^F$ in the Ramond sector for the D9-D9 string, as promised in Section 3.4.

For the most part, where we discuss the product $NS \times NS \to NS$, we shall use vertex operators in the 0-picture [42] so that we can ignore the ghost sector — all the states and vertex operators that appear in the discussion will be the ones from the “matter” sector, i.e., the $c = 15$ superconformal sigma model on the ten dimensional Minkowski space. In Section 5.4, we consider the product of Ramond vertex operators and the ghost sector needs to be included in the discussion.

### 5.1 Dp-Dq Strings

We first record the mode expansions of the fermions and the parity action on the modes, for the open string stretched between a flat D$q$-brane in $\mathbb{R}^{10}$ and a flat D$p$-brane inside it ($p \leq q$). The boundary condition of the field $x^\mu$ at the two ends of the string is, depending of the direction $\mu$, NN, ND (or DN), or DD, where “N” and “D” stand for Neumann and Dirichlet respectively. The mode expansion of fermions is, for the NS-sector of the type $(++)$,

$$
\psi_\mu(t, \sigma) = \begin{cases} 
\sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi^\mu_r(t) e^{\pm i r \sigma} & \text{NN}, \\
\pm \sum_{n \in \mathbb{Z}} \psi^\mu_n(t) e^{\pm i n \sigma} & \text{ND}, \\
\sum_{n \in \mathbb{Z}} \psi^\mu_n(t) e^{\mp i n \sigma} & \text{DN}, \\
\pm \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi^\mu_r(t) e^{\mp i r \sigma} & \text{DD}. 
\end{cases}
$$

The mode expansion for the other pairs of spin structures is easy to obtain from this by noting that the replacement $(+)$ → $(-)$ can be implemented by $N \to D$ and $D \to N$. For example, the expansion of an NN direction in the R-sector of the type $(--)$ is the same as the one for a DN direction in (5.3).

The space of states has a degeneracy due to the fermionic zero modes that obey
the Clifford algebra relations, $\{\psi_0^\mu, \psi_0^{\mu'}\} = \eta^{\mu\mu'}$. The space of states is thus in a spinor representation of $SO(|p - q|)$ in the NS-sector, and of $SO(p, 1) \times SO(9 - q)$ in the R-sector. When $|p - q|$ is even, there is a $\mathbb{Z}_2$-grading operator $(-1)^F$ with which we can define the GSO projection. When $|p - q|$ is odd, there is no $\mathbb{Z}_2$-grading as there are odd number of zero modes. This requires us to have an odd number of fermions from boundaries, as suggested in [11], so that the total space of states has a $\mathbb{Z}_2$-grading.

Let us now look at the action of the parity on the mode that appear in (5.3). The $\Omega$ parity, $\psi_\pm(t, \sigma) \to \mp \psi_\mp(t, -\sigma - \pi)$, transforms the modes in the NS-sector of the type $(++)$ as

$$\begin{align*}
\text{NN} \to \text{NN} : \psi_\mu \to e^{\pi ir} \psi_\mu, \\
\text{ND} \to \text{DN} : \psi_n \to (-1)^n \psi_n, \\
\text{DN} \to \text{ND} : \psi_n \to (-1)^n \psi_n, \\
\text{DD} \to \text{DD} : \psi_r \to -e^{\pi ir} \psi_r.
\end{align*}$$

(5.4) (5.5) (5.6)

Let us describe the mode expansions and parity transform of fields $\psi^\mu_\pm$ for $\mu = 1, \ldots, k$ in the set-up of Figure 3 such that the state $\Phi$ is in the NS-sector. Let $\psi_{\pm}^{\text{plane}}$ be the components on the upper-half plane of one of these fields. For the $(+)$ spin structure at the boundary $\text{Im}(z) = 0$, they obey the boundary condition $\psi_{\pm}^{\text{plane}} = \psi_{\mp}^{\text{plane}}$ on the segment $0 < z < 1$ and the opposite boundary condition $\psi_{\pm}^{\text{plane}} = -\psi_{\mp}^{\text{plane}}$ on the other parts $z < 0$ and $z > 1$. We consider three kinds of mode expansion as depicted in Figure 4.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{mode_expansions.png}
\caption{The mode expansions}
\end{figure}

$$\begin{align*}
\psi_-^{\text{plane}} &= -\sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi_r (z - \frac{1}{2})^{-r - \frac{1}{2}} = \sum_{n \in \mathbb{Z}} \psi'_n z^{-n - \frac{1}{2}} = -\sum_{n \in \mathbb{Z}} \psi''_n (z - 1)^{-n - \frac{1}{2}}, \\
\psi_+^{\text{plane}} &= \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi_r (\bar{z} - \frac{1}{2})^{-r - \frac{1}{2}} = \sum_{n \in \mathbb{Z}} \psi'_n \bar{z}^{-n - \frac{1}{2}} = \sum_{n \in \mathbb{Z}} \psi''_n (\bar{z} - 1)^{-n - \frac{1}{2}}.
\end{align*}$$

(5.7)
where the square roots are defined so that $\sqrt{a}$ is real positive for real positive $a$. Let us look at the action of the parity $z \to 1 - \bar{z}$ that exchanges $z = 0$ and $z = 1$. The $\tau \Omega$ parity transforms the fields as $\psi_{\pm}^{\text{plane}}(z, \bar{z}) \to -(-i)\psi_{\mp}^{\text{plane}}(1 - \bar{z}, 1 - z)$. The first minus sign comes from the involution $\tau$ and the phase $(-i)$ is from the definition of $\Omega$, see (2.8). The transformation of the modes are

\begin{align*}
\psi_{r} & \longrightarrow e^{\pi ir} \psi_{r}, \\
\psi'_{n} & \longrightarrow -(-1)^{n} \psi''_{n}, \\
\psi''_{n} & \longrightarrow (-1)^{n} \psi'_{n}.
\end{align*}

For the other spin structure $(-)$ of the boundary, there is an extra minus sign for the expansion of $\psi_{\text{plane}}$ in (5.7) and on the hand sides in all of (5.8) (5.9) and (5.10).

### 5.2 $k$ Even (Type IIB)

We work with the formulation in which an open string state takes the form

$$\Phi = \phi \otimes \psi,$$

where $\phi$ is from the Chan-Paton factor and $\psi$ is a state in the bulk sector. The product is given by

$$\left(\phi_{1} \otimes \psi_{1}\right) \cdot \left(\phi_{2} \otimes \psi_{2}\right) = (-1)\phi_{1}^{\phi_{2}}(\phi_{1} \cdot \phi_{2}) \otimes (\psi_{1} \cdot \psi_{2}),$$

and the parity takes the form

$$P(\phi \otimes \psi) = P_{\text{CP}}(\phi) \otimes P_{\text{bulk}}(\psi) (-1)^{\phi} P_{\text{bulk}}.$$

Let $\Phi$ be in the NS sector of the D($9-k$)-D9 string and let us take the product with its parity image $P(\Phi)$,

$$P(\Phi) \cdot \Phi = (-1)^{\phi\psi} P_{\text{CP}}(\phi) \otimes P_{\text{bulk}}(\psi).$$

It is in the NS sector of the D($9-k$)-D($9-k$) string. The bulk parity $P_{\text{bulk}}$ is even in this sector and hence the parity image of $P(\Phi) \cdot \Phi$ is

$$P(P(\Phi) \cdot \Phi) = (-1)^{\phi\psi} P_{\text{CP}}(\phi) \otimes P_{\text{bulk}}(\psi).$$

In view of the requirement (5.2), we would like to compare the expressions (5.14) and (5.15). To simplify the Chan-Paton factor of (5.15), we note the following property of the parity operators of the form (4.24):

$$P_{\text{CP}}(\phi_{2} \cdot \phi_{1}) = (-1)^{\phi_{1}^{\phi_{2}} \phi_{2}^{\phi_{1}}} P_{\text{CP}}^{b,c} P_{\text{CP}}(\phi_{1}) \cdot P_{\text{CP}}(\phi_{2}).$$
Here $\phi_1$ and $\phi_2$ are the Chan-Paton factors for the $a$-$b$ and $b$-$c$ strings and we put the superscript as $P_{CP}^{a,b}$ in order to show the domain sector of the parity. The property (5.16) can be proved by a straightforward computation:

\[
\text{LHS} = U_a(\phi_2 \cdot \phi_1)^T U_c^{-1}(-1)^{U_c(\phi_1 + \phi_2)}
\]

\[
= (-1)^{\phi_1 \phi_2} U_a \phi_1^T U_b^{-1} \cdot U_b \phi_2^T U_c^{-1}(-1)^{U_c(\phi_1 + \phi_2)}
\]

\[
= (-1)^{\phi_1 \phi_2} P_{CP}^{a,b}(\phi_1)(-1)^{\phi_1} U_b \cdot P_{CP}^{b,c}(\phi_2)(-1)^{\phi_2} U_c
\]

\[
= (-1)^{\phi_1 \phi_2} P_{CP}^{a,b}(\phi_1) \cdot P_{CP}^{b,c}(\phi_2)(-1)^{\phi_1(\phi_b + \phi_c)} = \text{RHS}.
\]

Using (5.16) we find that the Chan-Paton factor of (5.15) is given by

\[
P_{CP}(P_{CP}(\phi) \cdot \phi) = (-1)^{\phi} P_{CP}(\phi) \cdot P_{CP}^2(\phi). \tag{5.17}
\]

The requirement (5.2) on (5.14) and (5.15) yields the condition that

\[
P_{bulk}(P_{bulk}(\psi) \cdot \psi) = \star P_{bulk}(\psi) \cdot \psi, \tag{5.18}
\]

for some scalar $\star \in \mathbb{C}$, and that

\[
P_{CP}^2(\phi) = \star^{-1}(-1)^{\phi} \phi. \tag{5.19}
\]

The latter gives a strong constraint on the structure of the Chan-Paton factor for the D9-brane. For this purpose, it is important to find the scalar $\star$ in the relation (5.18).

We first consider the case $k = 2$. We are interested in the NS sector of the D7-D9 and D9-D7 strings. Due to the fermionic zero modes in the DN/ND directions, $\psi_0^1$ and $\psi_0^2$, the ground states are two-fold degenerate. Let $|\uparrow\rangle$ and $|\downarrow\rangle$ be the ground states which are characterized by

\[
(\psi^1_0 + i \psi^2_0) |\downarrow\rangle = 0 \quad \text{and} \quad |\uparrow\rangle = (\psi^1_0 - i \psi^2_0) |\downarrow\rangle.
\]

With respect to the $U(1)$ symmetry of rotations in the $\mu = 1, 2$ directions, the two states have different charges — they differ by the charge of $(\psi^1_0 - i \psi^2_0)$, which we normalize to be 1. In addition, the spectrum must be symmetric under the charge conjugation. Thus, the ground states $|\uparrow\rangle$ and $|\downarrow\rangle$ have charges $\frac{1}{2}$ and $-\frac{1}{2}$ respectively. Note that the parity action $\tau \Omega$ commutes with the rotational symmetry and hence conserves the charges. We thus find that $P_{bulk}|\alpha\rangle_{79} \propto |\alpha\rangle_{97}$, for both $\alpha = \uparrow$ and $\downarrow$. In particular the product $P_{bulk}|\alpha\rangle_{79} \cdot |\alpha\rangle_{79}$ is proportional to $|\alpha\rangle_{97} \cdot |\alpha\rangle_{79}$ and hence has charge $\frac{\epsilon_\uparrow + \epsilon_\downarrow}{2} = \epsilon_\alpha$, where we introduced $\epsilon_\uparrow = 1$ and $\epsilon_\downarrow = -1$. The leading term in the operator product expansion is the primary state in that charge sector,

\[
P_{bulk}|\alpha\rangle_{79} \cdot |\alpha\rangle_{79} \sim (\psi^1_{\alpha \downarrow} - i \epsilon_\alpha \psi^2_{\alpha \downarrow}) |0\rangle_{77}. \tag{5.20}
\]

45
Since the right hand side is odd, we find that the parity $P_{\text{bulk}}$ is odd in the D7-D9 sector. Let us look at the parity action on the state (5.20). We refer to (5.8) for the $\tau \Omega$ parity transform of the modes $\psi^{\mu}_{\frac{1}{2}}$ with $\mu = 1, 2$. In the NS sector of the type $(++)$, it is $\psi^{\mu}_{\frac{1}{2}} \to e^{\mp \frac{2\pi i}{79}}\psi^{\mu}_{\frac{1}{2}} = -i\psi^{\mu}_{\frac{1}{2}}$. In the NS sector of the type $(-\cdot)$ we have the opposite sign. Therefore, $P_{\text{bulk}}$ acts on $(\psi^{1}_{\frac{1}{2}} - i\epsilon_{\alpha}\psi^{2}_{\frac{1}{2}})|0\rangle_{\tau \gamma}$ as multiplication by $\mp i$ on the $(\pm \pm)$ sector, i.e., by $\epsilon$ as defined in (2.9). We thus found

$$P_{\text{bulk}}(P_{\text{bulk}}|\alpha\rangle_{\tau \gamma} \cdot |\alpha\rangle_{\tau \gamma}) = \epsilon P_{\text{bulk}}|\alpha\rangle_{\tau \gamma} \cdot |\alpha\rangle_{\tau \gamma}.$$ 

This shows that the scalar in (5.18) is $\star = \epsilon$.

Generalization to all even $k$ is straightforward. We look at the D$(9-k)$-D9 ground state of the form $|\vec{\alpha}\rangle_{(9-k)\bar{g}} = |\alpha_{1} \ldots \alpha_{k-1} \frac{1}{2}(9-k)\bar{g}\rangle$ which is annihilated by $(\psi^{2j-1}_{0} - i\epsilon_{\alpha_{j}}\psi^{2j}_{0})$ for all $j = 1, \ldots, \frac{k-1}{2}$. The operator product with its parity image has the leading term

$$P_{\text{bulk}}|\vec{\alpha}\rangle_{(9-k)\bar{g}} \cdot |\vec{\alpha}\rangle_{(9-k)\bar{g}} \sim (\psi^{1}_{\frac{1}{2}} - i\epsilon_{\alpha_{1}}\psi^{2}_{\frac{1}{2}}) \cdots (\psi^{k-1}_{\frac{1}{2}} - i\epsilon_{\alpha_{k-1}}\psi^{k}_{\frac{1}{2}})|0\rangle_{(9-k)(9-k)}.$$ 

The parity $P_{\text{bulk}}$ acts on this state as multiplication by $\epsilon \cdots \epsilon \ (\frac{k}{2} \text{ times})$, which shows that the scalar in (5.18) is $\star = \epsilon^{\frac{k}{2}}$. Also, we find $(-1)^{P_{\text{bulk}}} = (-1)^{\frac{k}{2}}$ in the D$(9-k)$-D9 sector.

Let us now shift our attention to the Chan-Paton factor. We know it for the BPS D$(9-k)$-branes on top of the O-plane (see Section 4.4): The Chan-Paton vector space is a purely even space $C^{N}$ and the o-isomorphism is given by a matrix $\gamma$ such that

$$\gamma^{\dagger}\gamma^{-1} = \pm 1 \quad \text{for } O^{\mp}-\text{type.}$$

As the D9-branes, we take the conformally invariant boundary condition for which the gauge and the tachyon fields have trivial profile. Let $V$ be the Chan-Paton vector space and $U : V^{*} \to V$ be the o-isomorphism. The Chan-Paton part of the parity operator is then defined by

$$P_{\text{CP}}(\phi) = \gamma \circ \phi^{T} \circ U^{-1}(-1)^{\phi U}$$ 

for $\phi \in \text{Hom}_{C}(C^{N}, V)$. As $\gamma$ is even, the statistics of $U$ coincides with that of $P_{\text{CP}}$, which in turn is equal to that of $P_{\text{bulk}}$ since the total parity $P = P_{\text{CP}} \otimes P_{\text{bulk}}$ must be even. Therefore we have

$$(-1)^{U} = (-1)^{P_{\text{CP}}} = (-1)^{P_{\text{bulk}}} = (-1)^{\frac{k}{2}}.$$ 

The square of $P_{\text{CP}}$ is given by

$$P_{\text{CP}}^{2}(\phi) = (-1)^{P_{\text{CP}}} (U(U^{T})^{-1}I) \circ \phi \circ (\gamma^{\dagger}\gamma^{-1}).$$ 

Since we had found $\star = \epsilon^{\frac{k}{2}}$, the constraint (5.19) leads to

$$U(U^{T})^{-1}I = \pm \epsilon^{\frac{k}{2}}\sigma \quad \text{for } O^{\mp}-\text{type.}$$ 

This derives the formula (1.7) for the D9-branes in the Type IIB orientifold.
5.3 $k$ Odd (Type IIA)

We next consider the case where $k$ is odd. There is an odd number of fermionic zero modes in the D$(9-k)$-D9 and D9-D$(9-k)$ strings, and we need to place an odd number of fermions at the boundary for the D9-branes so that we have a $\mathbb{Z}_2$-grading in the total sector.

5.3.1 O8$^-$

We begin with the O8$^-$ case ($k = 1$). Let us see if a single boundary fermion $\xi_1$ can do the job, i.e., if we can find a parity transformation of $\xi_1$ so that we have an even parity operator $P$ on the D8-D9 and D9-D8 sectors satisfying the requirement $P^2 = (-1)^F$. We first work with the formulation that does not introduce auxiliary fermions of Section 4. To simplify the story, we start with a single D9-brane with $\xi_1$. We may also focus on a single D8-brane since $\gamma = 1$ for the D8-branes on top of O8$^-$. Thus, we may assume that the open string has a trivial Chan-Paton factor.

In the NS-sector of each of the D8-D9 and D9-D8 strings, there is a single fermionic zero mode from $\psi^1_{\pm}$. In the set up of Figure 3 and Figure 4 (with $D_{9-k} = D_8$) it is $\psi_0'$ on D8-D9 and $\psi_0''$ on D9-D8 in the expansion (5.7) of $\psi_{\pm}^\text{plane}$. Together with the fermion $\xi_1$ which runs along the boundary of the D9-brane condition, they obey the algebra

$$\{\psi_0', \psi_0''\} = 1, \quad (\xi_1)^2 = 1, \quad \{\psi_0', \xi_1\} = 0 \quad \text{on D8-D9},$$

$$\{\psi_0'', \psi_0''\} = 1, \quad (\xi_1)^2 = -1, \quad \{\psi_0'', \xi_1\} = 0 \quad \text{on D9-D8}.$$

$\psi_0'$ and $\xi_1$ are both hermitian in the D8-D9 string sector while $\psi_0''$ is hermitian and $\xi_1$ is antihermitian in the D9-D8 string sector. Due to these modes, the ground states are two-fold degenerate in each sector, one even and one odd. Let $|\pm\>_{s9}$ and $|\pm\>_{s8}$ be the D8-D9 and D9-D8 ground states characterized by

$$\left(\sqrt{2}\psi_0' \pm i\xi_1\right) |\pm\>_{s9} = 0, \quad (5.25)$$

$$\left(\sqrt{2}\psi_0'' + \xi_1\right) |\pm\>_{s8} = 0. \quad (5.26)$$

Of course, being ground states, they are annihilated by all positive frequency modes including $\psi_0'$ or $\psi_0''$ with $n \geq 1$ (for $|\pm\>_{s9}$ or $|\pm\>_{s8}$). We shall establish the following operator product rule,

$$|+\>_{s8} \cdot |+\>_{s9} \sim |+\>_{s8} \cdot |-\>_{s9} \sim |0\>_{s8}, \quad (5.27)$$

$$|-\>_{s8} \cdot |+\>_{s9} \sim |+\>_{s8} \cdot |-\>_{s9} \sim \psi_{-\frac{1}{2}} |0\>_{s8}. \quad (5.28)$$
This implies, and in fact is equivalent to, the statement concerning the relation between the gradings in the D8-D9 and D9-D8 sectors: \( |+\rangle_{89} \) and \( |+\rangle_{98} \) are both even (or both odd). In other words,

\[
(-1)^F_{89} = \sqrt{2}i\xi_1 \psi'_0 (-1)^F_{98} \quad \text{and} \quad (-1)^F_{98} = -\sqrt{2}\xi_1 \psi''_0 (-1)^F_{89}
\]

(or the simultaneous sign flip), where \((-1)^F_{89}\) and \((-1)^F_{98}\) are the canonical \(\mathbb{Z}_2\)-gradings for the non-zero modes.

For the proof of (5.27) and (5.28), we may focus on the \(c = \frac{1}{2}\) conformal field theory of a single Majorana fermion (i.e. \(\psi_\pm = \psi^{\dagger}_\pm\)) with a single boundary fermion (\(\xi_1\)) on the segment \(0 < z < 1\). We first consider a conformal mapping of the upper-half plane

\[
w = \frac{z}{1-z},
\]

that maps \(z = 0, 1\) to \(w = 0, \infty\). We consider the mode expansion of \(\psi_\pm\) with respect to \(w\) (and \(\overline{w}\))

\[
\psi_- = -\sum_{n \in \mathbb{Z}} \psi_n w^{-n} \sqrt{\frac{dw}{w}}, \quad \psi_+ = \sum_{n \in \mathbb{Z}} \psi_n \overline{w}^{-n} \sqrt{\frac{d\overline{w}}{\overline{w}}},
\]

and compare them with

\[
\psi_- = \psi_-^{\text{plane}} \sqrt{dz}, \quad \psi_+ = \psi_+^{\text{plane}} \sqrt{dz},
\]

where \(\psi^{\text{plane}}_\pm\) are given by (5.7). To make it more precise, we have \(\sqrt{dw} = \sqrt{d\overline{w}}\) and \(\sqrt{dz} = \sqrt{d\overline{z}}\) at the boundary with the (+) spin structure. Also the square roots that appear in the expansions are defined so that \(\sqrt{a}\) is real positive for real positive \(a\). Then we find the following relations among the modes

\[
\psi'_n = \psi_n + a_1 \psi_{n+1} + a_2 \psi_{n+2} + \cdots, \quad (5.31)
\]

\[
\psi''_n = -i(-1)^n \psi_{-n} + b_1 \psi_{-n-1} + b_2 \psi_{-n-2} + \cdots. \quad (5.32)
\]

Note that the state \(|+\rangle_{89}\) is annihilated by \((\sqrt{2}\psi_0 + i\xi_1)\) and \(\psi_n\) for all \(n \geq 1\). Taking the complex conjugation, we have

\[
s_9 \langle + \mid \left(\sqrt{2}\psi_0 - i\xi_1\right) = 0, \quad s_9 \langle + \mid \psi_{-n} = 0 \quad (\forall n \geq 1), \quad (5.33)
\]

where we have used the fact that \(\psi_0\) and \(\xi_1\) are both hermitian and that \(\psi^{\dagger}_n = \psi_{-n}\). These can be regarded as conditions on a state inserted at \(w = \infty\). When mapped back to the \(z\)-plane, using (5.32), these become the following conditions at \(z = 1\):

\[
\left(\sqrt{2}\psi''_0 - \xi_1\right) \mid ? \rangle_{98} = 0, \quad \psi''_n \mid ? \rangle_{98} = 0 \quad (\forall n \geq 1). \quad (5.34)
\]
These are nothing but the conditions that define the state $|+\rangle_{98}$. The fact that $s_9 \langle +|+\rangle_{s9}$ is nonzero means that the two point function of $|+\rangle_{s9}$ and $|+\rangle_{98}$ on the upper-half plane, or equivalently on the disc, is non-zero. This proves that their operator product starts with $|0\rangle_{88}$. The same argument holds for the product of $|\rangle_{s9}$ and $|\rangle_{98}$. Thus we have established (5.27). This also shows that $|+\rangle_{s9}$ and $|+\rangle_{98}$ must be both even or both odd, establishing (5.29) up to a simultaneous sign flip. In particular, the product of $|+\rangle_{s9}$ and $|\rangle_{98}$ must be odd. Their operator product must start with $\psi_{-\frac{1}{2}}|0\rangle_{88}$, which is the unique odd primary state in this sector of the $c = \frac{1}{2}$ boundary conformal field theory. This is also supported by the fact that $s_9 \langle -|0\rangle_{s9}$ is non-zero. This establishes (5.28).

Let us now discuss the parity operator. Recall the transformation of the modes $\psi_n'$ and $\psi_n''$ for the spin structure (+) from (5.9) and (5.10):

$$\psi_n' \rightarrow -(-1)^n \psi_n'', \quad \psi_n'' \rightarrow (-1)^n \psi_n'.$$

As for $\xi_1$, there are two possibilities for the parity transform: $\xi_1 \rightarrow +i\xi_1$ or $\xi_1 \rightarrow -i\xi_1$. The factor of $i$ is required since $\xi_1$ is hermitian at D8-D9 and antihermitian at D9-D8. If we use $\xi_1 \rightarrow -i\xi_1$, the condition for $|\rangle_{s9}$ in (5.25) is mapped to the condition for $|\rangle_{98}$ in (5.26). That is, we find $P|\rangle_{s9} \propto |\rangle_{98}$, which means that $P$ is even. Applying this to the operator product rule (5.27), we find $P(|\rangle_{s9} \cdot |\rangle_{s9}) \sim |0\rangle_{88}$. Since $P|0\rangle_{88} = |0\rangle_{88}$, we see that

$$P(P|\rangle_{s9} \cdot |\rangle_{s9} ) = P|\rangle_{s9} \cdot |\rangle_{s9}.$$

Thus, we successfully find an even parity operator satisfying (5.2) and hence $P^2 = (-1)^F$. If we use $\xi_1 \rightarrow +i\xi_1$ instead, the condition for $|\rangle_{s9}$ is mapped to the condition for $|\rangle_{98}$. That is, $P|\rangle_{s9} \propto |\rangle_{98}$, and hence $P$ is odd. Applying this to (5.28), we find $P|\rangle_{s9} \cdot |\rangle_{s9} \sim \psi_{-\frac{1}{2}}|0\rangle_{88}$. Since $P\psi_{-\frac{1}{2}}|0\rangle_{88} = -i\psi_{-\frac{1}{2}}|0\rangle_{88}$, we see that (5.2) fails but instead we have

$$P(P|\rangle_{s9} \cdot |\rangle_{s9} ) = -iP|\rangle_{s9} \cdot |\rangle_{s9}.$$

For the spin structure (−), $\xi_1 \rightarrow +i\xi_1$ yields an even parity operator $P$ which satisfies (5.2) while $\xi_1 \rightarrow -i\xi_1$ yields an odd parity $P$ such that (5.2) fails but $P(P|\rangle_{s9} \cdot |\rangle_{s9} ) = +iP|\rangle_{s9} \cdot |\rangle_{s9}$ holds.

To summarize: For $O8^-$, a single D9-brane with a single boundary fermion $\xi_1$ is admissible, as long as the parity transform is $\xi_1 \rightarrow \varepsilon \xi_1$.

Let us discuss the Chan-Paton structure in the formalism that includes auxiliary fermions. The D8-D9 or D9-D8 string with a single boundary fermion $\xi_1$ on D9 is of the type (iii) in the terminology of Section 4. Thus we introduce a pair of auxiliary fermions, $\xi$ and $\xi'$, and impose the projection condition $\xi \xi' = 1$. These two fermions
transform oppositely under the parity: if $\xi \rightarrow \mu \xi$ then $\xi' \rightarrow -\mu \xi'$, where $\mu$ is fixed ($i$ or $-i$) independently of the boundary spin structure. $\xi$ is included in the Chan-Paton factor while $\xi'$ is quantized together with the bulk modes. Thus, the Chan-Paton factor consists of $\xi_1$ and $\xi$ that transform under the parity as

$$\xi_1 \rightarrow \varepsilon \xi_1, \quad \xi \rightarrow \mu \xi.$$ 

The extended Chan-Paton vector space is therefore the 2-dimensional space $V_2$ of the irreducible representation of the Clifford algebra generated by $\xi_1$ and $\xi$. When $\varepsilon = \mu$, the two transform in the same way, and therefore the o-isomorphism is $U = U_{2(\mu)}$. Here $U_{s(\pm i)}$ is the isomorphism introduced in Section 4.3. When $\varepsilon = -\mu$, they transform oppositely and hence we take $U = \xi U_{2(-\mu)}$ as the o-isomorphism. Note that they obey

$$U(U^T)^{-1} = \begin{cases} \mu \sigma & \text{for } U = U_{2(\mu)} \quad (\varepsilon = \mu), \\ \sigma & \text{for } U = \xi U_{2(-\mu)} \quad (\varepsilon = -\mu). \end{cases} \quad (5.35)$$

This shows the formula (1.7) for the O8$^-$ case ($[k] = 1$).

### 5.3.2 The General Case

It is straightforward to extend the above analysis for the general (odd) codimension $k$ and the type of the O-plane.

Let us first study the parity operator on the open strings stretched between a single D$(9-k)$-brane and a single D9-brane equipped with a single boundary fermion $\xi_1$. On the D$(9-k)$-D9 and D9-D$(9-k)$ strings, there are $k$ fermionic zero modes, $\psi_0^1, \ldots, \psi_0^k$. As the basis of the ground states, we take

$$|\bar{\alpha}, \pm\rangle_{(9-k)\bar{9}} := |\alpha_1 \cdots \alpha_{k-1}, \pm\rangle_{(9-k)\bar{9}}, \quad |\bar{\alpha}, \pm\rangle_{9(9-k)} := |\alpha_1 \cdots \alpha_{k-1}, \pm\rangle_{9(9-k)}.$$ 

These are annihilated by $$(\psi_0^{k-1} - i \epsilon_1 \psi_0^j)$$ for $j = 1, \ldots, k-1$, where $\epsilon_1 = 1$ and $\epsilon_1 = -1$, and satisfy the conditions of the form (5.25) and (5.26), in which $\psi_0'$ and $\psi_0''$ are the zero modes of $\psi_{\pm}^k$. Their operator product expansions are of the form

$$|\bar{\alpha}, \pm\rangle_{9(9-k)} \cdot |\bar{\alpha}, \pm\rangle_{9(9-k)} \sim (\psi_0^{k-1} - i \epsilon_1 \psi_0^1) \cdots (\psi_0^{k-2} - i \epsilon_1 \psi_0^{k-1}) |0\rangle_{9(9-k)}.$$ 

(5.36)

If we choose $\xi_1 \rightarrow \varepsilon \xi_1$ as the parity transform, then the corresponding parity operator $P_1$ maps $|\bar{\alpha}, \pm\rangle_{9(9-k)}$ to $|\bar{\alpha}, \pm\rangle_{9(9-k)}$. In view of the operator product rule (5.36), we see that

$$(-1)^{P_1} = (-1)^{\frac{k-1}{2}}$$

and that

$$P_1(P_1 |\bar{\alpha}, \pm\rangle_{9(9-k)} \cdot |\bar{\alpha}, \pm\rangle_{9(9-k)}) = \varepsilon^{\frac{k-1}{2}} P_1 |\bar{\alpha}, \pm\rangle_{9(9-k)} \cdot |\bar{\alpha}, \pm\rangle_{9(9-k)}.$$ 

(5.37)
For the other choice, \( \xi_1 \to -\varepsilon \xi_1 \), the corresponding operator \( P_1 \) maps \(|\tilde{\alpha}, \pm\rangle_{(9-k)9}\) to \(|\tilde{\alpha}, \mp\rangle_{(9-k)9}\). This together with (5.36) yields \((-1)^{P_1} = (-1)^{\frac{k-1}{2}}\) and a formula of the form (5.37) in which \( \varepsilon \frac{k-1}{2} \) is replaced by \( \varepsilon \frac{k+1}{2} \).

In order to find an even parity operator \( P \) satisfying (5.2), we include the Chan-Paton factor into the discussion. We consider \( N \) D\((9-k)\)-branes and place a graded vector space \( V' \) on the D9-brane boundary. A D\((9-k)\)-D9 string state is of the form \( \phi' \otimes \psi \) where \( \phi' \) is a map \( C^N \to V' \) and \( \psi \) is a state like \(|\tilde{\alpha}, \pm\rangle_{(9-k)9}\). We consider the parity of the form \( P(\phi' \otimes \psi) = (-1)^{P_1} P_{CP}'(\phi') \otimes P_1(\psi) \), in which \( P_{CP}' \) is given by \( P_{CP}'(\phi') = \gamma \phi'^T U^{-1}(1)\phi' U' \) for the o-isomorphism \( \gamma \) of the \( N \) D\((9-k)\)-brane and \( U' : V' \to V' \). If we choose \( \xi_1 \to \varepsilon \xi_1 \) as the parity transform, the condition for the Chan-Paton part is \((-1)^{P_{CP}} = (-1)^{\frac{k-1}{2}} \) and \( P_{CP}'(\phi') = \varepsilon^{-\frac{k-1}{2}}(-1)\phi' \). This is achieved when \((-1)^{U'} = (-1)^{\frac{k+1}{2}}\) and

\[
U'(U'^T)^{-1} = \pm \varepsilon^{\frac{k+1}{2}} \sigma' \quad \text{for O}^\mp \text{-type.} \tag{5.38}
\]

For the other choice \( \xi_1 \to -\varepsilon \xi_1 \), \( U' \) has the opposite statistics, \((-1)^{U'} = (-1)^{\frac{k+1}{2}}\), and satisfies (5.38) in which the phase \( \varepsilon^{\frac{k+1}{2}} \) is replaced by \( \varepsilon^{-\frac{k+1}{2}} \).

Let us now include the auxiliary fermions, \( \xi \) and \( \xi' \). The extended Chan-Paton vector space \( V \) is the tensor product of \( V' \) and the 2-dimensional space \( V_2 \) coming from \((\xi_1, \xi)\), and the o-isomorphism \( U \) is the tensor product of \( U' \) and the one for \((\xi_1, \xi)\). If we take \( \xi_1 \to \varepsilon \xi_1 \), then the latter factor is identical to the one chosen in the case of \( O8^- \), i.e., \( U_{2(\mu)} \) for \( \varepsilon = \mu \) and \( \xi U_{2(-\mu)} \) for \( \varepsilon = -\mu \). Then, \( U(U'^T)^{-1} \) is the product of (5.38) and (5.35),

\[
U(U'^T)^{-1} = \begin{cases} 
\pm \varepsilon^{\frac{k+1}{2}} \sigma, & \varepsilon = \mu, \\
\pm \varepsilon^{\frac{k+1}{2}} \sigma, & \varepsilon = -\mu,
\end{cases} \quad \text{for O}^\mp \text{-type.} \tag{5.39}
\]

This proves the formula (1.7) for the D9-branes in the Type IIA orientifold.

A Family Of Solutions

Let us record a solution to the condition in the theory with \( O(9-k)^- \)-plane for arbitrary (even or odd) \( k \). It is to have \( k \) boundary fermions, \( \xi_1, \ldots, \xi_k \), that transform under the parity as

\[
\xi_j(\tau) \to \varepsilon \xi_j(-\tau), \quad j = 1, \ldots, k. \tag{5.40}
\]

Indeed, for even \( k \), this leads to the o-isomorphism \( U_k \) that satisfies the condition, i.e., (5.24) with the plus sign. The condition for odd \( k \), (5.38) with the plus sign, is solved by \( U' = U_{k-1} \), which is associated with \( k-1 \) boundary fermions that transforms as
\( \xi_j(\tau) \to \varepsilon \xi_j(-\tau) \). Including the single boundary fermion \( \xi_1(\tau) \), we have (5.40). Note that the solution (5.40) can be obtained from the ABS configuration for the D-brane on top of the O-plane, (4.37) with (4.39), by turning off the tachyon.

### 5.4 Ramond Sector

As another application of the analysis developed in Sections 5.2 and 5.3, we study the parity operator on the Ramond-sector of the D9-D9 string, both in Type IIB \((k\text{ even})\) and Type IIA \((k\text{ odd})\). We shall show, as promised in Section 3.4, that the relation \( P^2 = (-1)^F \) does indeed hold under the formula (1.7) of the Chan-Paton factor.

There are two novelties in this discussion. One is that the operator product rule (5.1) is modified by a sign if both of the two states are in the Ramond sector, i.e., spacetime fermions, \( P(\psi_2 \cdot \psi_1) = -(-1)^{|\psi_1|+|\psi_2|} P(\psi_1) \cdot P(\psi_2) \). In particular, the basic requirement takes the form

\[
P(P(\psi) \cdot \psi) = -P(\psi) \cdot \psi.
\]

Second, vertex operators in the Ramond sector must be in half-integer pictures and thus the ghost sector cannot be ignored. We shall consider the product of two Ramond vertex operators in the \((-\frac{1}{2})\)-picture that results in an NS vertex operator in the \((-1)\)-picture.

We recall that the mode expansions of the fermions \( \psi_{\pm} \) in the D9-D9 string are given in (2.11) and that the \( \tau \Omega \) parity action on the modes is

\[
\tau \Omega : \begin{cases} 
\psi_r \text{ in } (++) & \to e^{i\tau \Omega} \psi_r \text{ in } (++) \\
\psi_r \text{ in } (--) & \to -e^{i\tau \Omega} \psi_r \text{ in } (--) \\
\psi_n \text{ in } (++) & \to (-1)^n \psi_n \text{ in } (++) \\
\psi_n \text{ in } (++) & \to (-1)^n \psi_n \text{ in } (++)
\end{cases}
\]

where

\[
(\tau \Omega \psi)^\mu = \begin{cases} 
-\psi^\mu & \mu = 1, \ldots, k \\
+\psi^\mu & \mu \neq 1, \ldots, k.
\end{cases}
\]

We shall sometimes write \(-i\psi_{r0}^{10} \) for \( \psi_{r0}^{10} \). We find zero modes \( \psi_{\mu}^{\pm} \) in the Ramond-sector, i.e., in the \((+-)\) or \((-+)\) sector. There are massless spacetime fermions, labeled by a quintuplet \( \vec{\alpha} = \alpha_1 \cdots \alpha_5 \) of ups and downs, \( \alpha_j = \uparrow, \downarrow \), which satisfy

\[
(\psi_{0j}^{2j-1} - ie_{\alpha_j} \psi_{0j}^{2j}) |\vec{\alpha}\rangle_{(\pm\mp)} = 0, \quad j = 1, \ldots, 5.
\]

We have chosen them to be in the \((-\frac{1}{2})\)-picture, annihilated by \( \beta_n \) and \( \gamma_{n+1} \) for all \( n \geq 0 \). When \( k \) is even, these defining conditions are invariant under the parity. Thus \( P_{\text{bulk}} \) maps
\[|\bar{\alpha}\rangle_{(+)} \text{ to } |\bar{\alpha}\rangle_{(-)} \text{ up to a constant, and vice versa. When } k \text{ is odd, the condition changes at } j = \frac{k+1}{2}. \] That is, the arrow \( \alpha_j \) flips at \( j = \frac{k+1}{2} \) and remains the same for the other \( j \)'s. For example, for \( k = 3 \), we have

\[ \mathbf{P}_{\text{bulk}} |\uparrow\uparrow\uparrow\uparrow\rangle_{(\pm \pm)} \propto |\uparrow\downarrow\uparrow\uparrow\rangle_{(\pm \pm)}. \] (5.45)

We therefore find that

\[ (-1)\mathbf{P}_{\text{bulk}} = (-1)^k. \] (5.46)

For even \( k \), the operator product of a ground state and its parity image is

\[ \mathbf{P}_{\text{bulk}}|\bar{\alpha}\rangle_{(+-)} \cdot |\bar{\alpha}\rangle_{(+)} \sim \left(\psi_{\frac{1}{2}} - i\epsilon_{\alpha_1} \psi_{\frac{1}{2}}^2\right) \cdots \left(\psi_{\frac{9}{2}} - i\epsilon_{\alpha_5} \psi_{\frac{10}{2}}\right) |\bar{\alpha}\rangle_{(+)} \] (5.47)

where \( |\bar{\alpha}\rangle_{(+)} \) is the vacuum in the \((-1)\)-picture, annihilated by \( \beta_r \) and \( \gamma_r \) for all \( r \geq \frac{1}{2} \). Using \( \psi_{\frac{r}{2}} \rightarrow e^{-\frac{\pi i}{2}} \tau_r \psi_{\frac{r}{2}} \) from (5.42), we see that the parity acts on this state by multiplication by \( i^k \cdot (-i)^{5-k} \cdot (-i) = -(-1)\frac{k}{2} \). The last factor of \((-i)\) is from the transformation of the vertex operator \( \delta(\gamma) \) corresponding to \( |\bar{\alpha}\rangle_{(+)} \), see (2.10). For odd \( k \), a ground state and its parity image have opposite arrows at \( j = \frac{k+1}{2} \), and therefore their product misses the factor \( \left(\psi_{\frac{k}{2}} - i\epsilon_{\alpha_{k+1}} \psi_{\frac{k+1}{2}}\right) \) compared to (5.47). Hence the parity action on the product is multiplication by \( i^k \cdot (-i)^{5-k} \cdot (-i) = i(-1)\frac{k+1}{2} \). Thus we found

\[ \mathbf{P}_{\text{bulk}}(\mathbf{P}_{\text{bulk}}(\psi) \cdot \psi) = \mathbf{P}_{\text{bulk}}(\psi) \cdot \psi \times \left\{ \begin{array}{ll} -(-1)^k & k \text{ even} \\ i(-1)^{\frac{k+1}{2}} & k \text{ odd} \end{array} \right. \] (5.48)

for a bulk state \( \psi \) in the \((+-)\) sector.

On the other hand, we have \( \mathbf{P}_{\text{CP}}^2(\phi) = (-1)^k \mathbf{P}_{\text{CP}}(c_{(i)} c_{(-i)}^{-1})(-1)^{\phi} \phi \) on the Chan-Paton factor in the \((+-)\) sector. Using the formula (3.37) in which we insert (3.29) for the value of \((-1)^{|U_{(i)}|}\), or directly using (1.7), we find \( c_{(i)} c_{(-i)}^{-1} = (-1)^{\frac{k}{2}} \) for \( k \) even and \(-i(-1)^{\frac{k+1}{2}} \) for \( k \) odd. Applying the identity (5.16), we find

\[ \mathbf{P}_{\text{CP}}(\mathbf{P}_{\text{CP}}(\phi) \cdot \phi) = \mathbf{P}_{\text{CP}}(\phi) \cdot \phi \times \left\{ \begin{array}{ll} -(-1)^k & k \text{ even} \\ i(-1)^{\frac{k+1}{2}} & k \text{ odd} \end{array} \right. \] (5.49)

Note that the formula (5.15) holds in the present case since \( \mathbf{P}_{\text{bulk}} \) in the NS sector of the D9-D9 string is even. Inserting (5.48) and (5.49) into that formula, we see that the required relation (5.41) holds. Thus, we have proved the promised relation \( \mathbf{P}^2 = (-1)^F \) in the Ramond-sector. We remark that we needed to use the ten-dimensionality of the spacetime in this discussion.
6 D-Branes In Type I String Theory

In this section, we analyze the massless and tachyonic spectrum on D-branes of various dimensions in Type I string theory. We use two descriptions — conformal field theory with the standard D-brane boundary condition on the one hand, and the D9-brane configurations with nontrivial tachyon profiles on the other. The former is a direct application of the result of the previous section. We will encounter an ambiguity in the parity action in the Ramond sector, which may be fixed by an input from spacetime physics. In the second approach, we find no ambiguity and can honestly derive the spectrum.

6.1 CFT Analysis Of The Spectrum

What is done in the previous section can be interpreted as the study of the Chan-Paton factor of D-branes of all dimensions in Type I string theory. In particular, as the o-isomorphisms for Type I $D_p$-brane, we can use the ones obtained for the D9-branes in the presence of $O_p^{-}$-plane, which may be taken as follows:

\[
\begin{align*}
    p &= 9, 1 \\
    U_{(\mp i)} &= \begin{pmatrix} 1_{N^0} & 0 \\ 0 & \pm i1_{N^1} \end{pmatrix} \\
    p &= 7 \\
    U_{(i)} &= \begin{pmatrix} 0 & \pm i1_N \\ 1_N & 0 \end{pmatrix} \\
    p &= 5 \\
    U_{(i)} &= \begin{pmatrix} J_{N^0} & 0 \\ 0 & \pm iJ_{N^1} \end{pmatrix} \\
    p &= 3 \\
    U_{(i)} &= \begin{pmatrix} 0 & \mp i1_N \\ 1_N & 0 \end{pmatrix}
\end{align*}
\]

where we have chosen $\mu = -i$, see (3.46). The relative phase between $U_{(i)}$ and $U_{(-i)}$ has been chosen arbitrarily.

This enters into the parity operator with which we define the orientifold projection of the degrees of freedom on the $D_p$-brane worldvolume. Let us look at the tachyons and massless particles from the $p$-$p$ strings. The parity, $P = P(\Omega)$ in the NS-sector and $\tilde{P} = (-1)^{F_R}P(\Omega)$ in the R-sector, acts on the relevant states as follows

\[
\begin{align*}
    k_t \cdot \psi |k_t\rangle_{(++)} &\mapsto -ik_t \cdot \psi |k_t\rangle_{(++)}, \\
    (\zeta \cdot \alpha_{-1} + \cdots) |k_b\rangle_{(++)} &\mapsto -(-1)^{|k|}(\zeta \cdot \alpha_{-1} + \cdots) |k_b\rangle_{(++)}, \\
    |k_f, \alpha\rangle_{(-+)} &\mapsto z_{\alpha} |k_f, \alpha^\prime\rangle_{(-+)}
\end{align*}
\]

(6.2)
where \( k_t^2 = 1, k_b^2 = k_f^2 = 0 \) and \( \zeta \cdot k_b = 0 \). In the second line, \((-1)^{|\zeta|} = +1 \) resp. \(-1\) if \( \zeta \) is tangent resp. transverse to the brane. The important information in the Ramond sector analysis is the transformation of the fermionic zero modes which is, in the \((-+)\) sector,

\[
\psi^\mu_0 \longrightarrow \begin{cases} 
    \psi^\mu_0 & \text{if } x^\mu \text{ is tangent to the brane,} \\
    -\psi^\mu_0 & \text{if } x^\mu \text{ is normal to the brane.}
\end{cases} \quad (6.3)
\]

\( \alpha \) and \( \alpha' \) in (6.2) are the labels of the spin which are equal for odd \( p \) and different for even \( p \), as in (5.45). \( z_{\alpha} \) is some phase. For odd \( p \), it is of the form

\[
z_{\alpha} = e^{i\theta} \chi_n(\alpha),
\]

where \( e^{i\theta} \) is an \( \alpha \)-independent phase and \( \chi_n(\alpha) = \pm 1 \) is the chirality of \( \alpha \) in the directions normal to the brane — it comes from the minus sign in (6.3). (In the \((+-)\) sector, the transformation is opposite to (6.3) and we have the chirality in the tangent directions. However, after GSO projection, that is equal to the chirality in the normal directions.)

The action (6.2) is to be combined with the action on the Chan-Paton factor

\[
\phi \longrightarrow \begin{cases} 
    U_{(-i)} \circ \phi^T \circ U^{-1}_{(-i)}(-1)^{|U_{(-i)}|}|\phi| & \text{(NS),} \\
    U_{(i)} \circ \phi^T \circ U^{-1}_{(i)}(-1)^{|U_{(-i)}|}|\phi| & \text{(R).}
\end{cases} \quad (6.4)
\]

The spectrum analysis in the NS-sector (spacetime bosons) is straightforward, and only the result will be presented. The analysis in the R-sector (spacetime fermions) is more interesting. The phase \( z_{\alpha} \) must be determined in order to specify the orientifold projection. The consistency condition \( \tilde{P}^2 = \text{id} \) fixes it only up to a sign. As far as the spectrum analysis is concerned, this sign is irrelevant for even \( p \) cases, since the orientifold projection simply relates the Chan-Paton factors multiplying the two different vectors, \( |\alpha\rangle \) and \( |\alpha'\rangle \). The sign turns out to be irrelevant also in the cases \( p = 7, 3 \). The sign does affect the spectrum for \( p = 9, 5, 1 \) and thus we need to know it for an honest analysis. At this moment, we do not know how to determine it purely within the conformal field theory — the analysis as in the previous section is not sufficient. Facing this problem, for now, we resort for help to the information of spacetime physics, in particular spacetime supersymmetry. In the next subsection, we will see that the sign can be determined by our formulation based on tachyon configurations on D9-branes.

For \( p = 9, 5, 1 \), the o-isomorphisms \( U_{(\pm i)} \) are even and hence the even part (Dp-branes) and odd part (Dp-branes) are individually invariant under the orientifold. The projection conditions for fermions in these two sectors are opposite due to the difference between \( U_{(i)} \) and \( U_{(-i)} \) — if one is symmetric then the other is antisymmetric — as it must be
the case by the open-closed channel duality (see Section 3.4). The sign of $e^{i\theta}$ determines which is which. Note that the distinction between branes and antibranes are up to us, and we take the convention that the D9-brane preserves the same supersymmetry as the O9-plane, and that the D9-D5 and D9-D1 strings have massless fermions of positive chirality in $5+1$ and $1+1$ dimensions respectively. Then, information about spacetime supersymmetry can tell us which fermions are supposed to survive the orientifold projection: (i) D9-D9 spectrum must include superpartners of the gauge bosons, (ii) D5-D5 spectrum must include a $(1,0)$ gauge multiplet, and the gaugino in it must have negative chirality so that D9-D5 string yields a massless $(1,0)$ hypermultiplet [43], and (iii) on D1-brane the spacetime supersymmetry has positive chirality and hence the superpartner of the massless scalar resp. vector must have negative resp. positive chirality [44]. These requirements fix the sign as $e^{i\theta} = -1$ for $p = 9, 1$ and $e^{i\theta} = 1$ for $p = 5$ under the relative phase given in (6.1) in which we have $U_{(i)} = \sigma U_{(-i)}$. The result is summarized in the table below.

| $p$ | gauge group | tachyon | massless scalar | massless fermion |
|-----|-------------|---------|----------------|-----------------|
| 9   | $O(N^0) \times O(N^1)$ | bi      | none           | $(A, 1)_+, (1, S)_+, \text{bi}_-$ |
| 8   | $O(N)$      | A       | S              | A, S           |
| 7   | $U(N)$      | A       | adj            | adj            |
| 6   | $USp(N)$    | A       | A              | A, S           |
| 5   | $USp(N^0) \times USp(N^1)$ | bi     | $(A, 1), (1, A)$ | $(A, 1)_+, (1, S)_+, (S, 1)_-, (1, A)_-, \text{bi}_-, \text{bi}_+$ |
| 4   | $USp(N)$    | S       | A              | A, S           |
| 3   | $U(N)$      | S       | adj            | adj            |
| 2   | $O(N)$      | S       | S              | A, S           |
| 1   | $O(N^0) \times O(N^1)$ | bi     | $(S, 1), (1, S)$ | $(A, 1)_+, (1, S)_+, (S, 1)_-, (1, A)_-, \text{bi}_-, \text{bi}_+$ |
| 0   | $O(N)$      | A       | S              | A, S           |

‘bi’, ‘adj’, ‘1’, ‘S’ and ‘A’ stand for the bifundamental, adjoint, singlet, symmetric tensor and antisymmetric tensor representations respectively. Note that adj = A for $O(n)$ and adj $\cong$ S for $USp(n)$. The massless scalar is tensored with a normal vector to the brane. The massless fermion is a worldvolume spinor tensored with a spinor of the normal bundle. For $p = 9, 5, 1$, there is a restriction on the chirality in the tangent resp. normal directions, shown by the subscript resp. superscript. For $p = 7, 3$, the massless fermion in the adjoint representation of $U(N)$ can have all four chirality pairs.
Parts of the result had been obtained earlier. Gimon-Polchinski [8] and Witten [11] applied the same method to find the projection condition on the bosonic sector in $p$ odd cases. References [21] and [22] proposed/obtained the list of tachyons and massless bosons using different methods. Ref. [45] determined the orientifold projection of massless fermions from the $\mathcal{D}9-\mathcal{D}9$ string using the channel duality argument.

### 6.2 Spectrum Via D9-Brane Configurations

Let us realize the same D-branes as D9-anti-D9-brane systems and study the orientifold projection of massless fermions, in particular for the cases $p = 9, 5, 1$.

#### 6.2.1 The Configurations

A D$p$-brane in Type IIB string theory is provided by the tachyon configuration (4.38),

$$T(x) = \sum_{i=1}^{k} x^i \xi_i,$$

for $k = 9 - p$, represented on a graded vector space $V$. To have it as a configuration in Type I string theory, we need an $\mathcal{O}$-isomorphism $U : V^* \to V$ which is even and has $c = 1$. One way to achieve this is to introduce $k$ additional boundary fermions $\xi_{k+1}, \ldots, \xi_{2k}$ and take

$$V = V_{2k} \quad \text{and} \quad O = \begin{pmatrix} -1_k & 1_k \end{pmatrix},$$

where $O$ is the matrix (4.18) that specifies the parity transformation. The first $k$ eigenvalues are chosen to be $-1$ for the invariance of the boundary interaction $\int d\tau \sum_{i=1}^{k} \psi^i x^i \xi_i$, or equivalently, for the $\mathcal{O}$-isomorphism condition $U(\varepsilon T(x)^T)U^{-1} = T(x)$; we have the opposite sign compared to (4.40) since the Type I involution is $x^i \to x^i$ rather than $x^i \to -x^i$. Then, having eigenvalue $+1$ with multiplicity $k$ guarantees $(-1)^{|U|} = 1$ and $c = 1$, see (4.30).

The boundary theory flows in the infra-red limit to the D$p$-brane boundary condition for $x^\mu$ and $\psi^\mu_{\pm}$, and only $\xi_{k+1}, \ldots, \xi_{2k}$ remain as the boundary degrees of freedom. They transform as $\xi_i(\tau) \to \varepsilon \xi_i(-\tau)$ under the parity and hence are nothing but (the Type I versions of) the consistent boundary degrees of freedom recorded in (5.40). These boundary fermions are represented on $V_{\text{IR}} = V_k$ for even $k$ and, together with an auxiliary fermion $\xi$, on $V_{\text{IR}} = V_{k+1}$ for odd $k$.  

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Of course, (6.5) is not the only solution and is not even the minimal one except for low values of \( k \). In the notation of Table (6.1), it corresponds to \( N = 2^{\frac{k-1}{2}} \) for odd \( k \) (i.e., even \( p \)), \( N = 2^{\frac{k}{2}}-1 \) for \( k = 2, 6 \) \((p = 7, 3)\) and \( N^0 = N^1 = 2^{\frac{k}{2}}-1 \) for \( k = 4, 8 \) \((p = 5, 1)\). We can of course construct other cases including the minimal ones. To be specific, however, we shall discuss the orientifold projection for the solution (6.5).

### 6.2.2 The Massless Fermions

Next, we write down the wavefunctions for the supersymmetric ground states in the Ramond sector. We employ the zero mode approximation which is sufficient for the purpose of finding the parity action. We denote by \( S \) the spinor representation of the algebra \( \{\psi^\mu_0, \psi^\nu_0\} = \eta^{\mu\nu} \) of the fermionic zero modes \( \psi^\mu_0 (\mu = 0, 1, \ldots, 9) \). We first consider the wavefunction for the \( p-p \) string. We represent it as \( \Psi_{pp} = \phi \otimes s \) where \( \phi \) and \( s \) take values in \( \text{Hom}(V, V) \) and \( S \) respectively. The supercharge (3.40) acts on this state as

\[
Q_1 (\phi \otimes s) = -i \sum_{\mu=0}^{9} \psi^\mu_0 \frac{\partial}{\partial x^\mu} (\phi \otimes s) - (T \circ \phi) \otimes s - (-1)^{|\phi|} i (\phi \circ T) \otimes s. \tag{6.6}
\]

A general solution to the supersymmetry condition \( Q_1 \Psi_{pp} = 0 \) is

\[
\Psi_{pp} = e^{-\sum_{i=1}^{k} (x^i)^2} \prod_{j=1}^{k} \left( 1 + (i-1)\xi_j \psi^j_0 \right) \tilde{\phi}(\xi_{k+1}, \ldots, \xi_{2k}) \otimes s(x) \tag{6.7}
\]

where \( \tilde{\phi}(\xi_{k+1}, \ldots, \xi_{2k}) \) is a sum of products of \( \xi_{k+1}, \ldots, \xi_{2k} \) only, and \( s(x) \) solves the 5 + 1 dimensional massless Dirac equation \( \sum_j \psi^j_0 \partial_j s(x) = 0 \) (where \( j \) runs over 0, \( k+1, \ldots, 9 \)). This corresponds to the state \( \tilde{\phi} \otimes |s\rangle \) in the CFT description, where \( \tilde{\phi} \) is regarded as an element of \( \text{Hom}(V_{IR}, V_{IR}) \) with the constraint of the graded commutativity with \( \xi \) in the odd \( k \) case, and \( |s\rangle \) is the state corresponding to the solution \( s(x) \) of the Dirac equation.

For our purpose, we also need to know the wavefunction for the Ramond ground states of the 9-\( p \) string, for \( p = 5, 1 \) (i.e., \( k = 4, 8 \)). Assuming that the number of D9-branes is 1, the wavefunction \( \Psi_{9p} \) takes values in \( V \otimes S \). The supersymmetry condition reads \( Q_1 \Psi_{9p} = -i \sum_{\mu=0}^{9} \psi^\mu_0 \frac{\partial}{\partial x^\mu} \Psi_{9p} - T \Psi_{9p} = 0 \) and a general solution is of the form

\[
\Psi_{9p} = e^{-\frac{1}{\sqrt{2}} \sum_{i=1}^{k} (x^i)^2} \prod_{j=1}^{k} \left( 1 + i \sqrt{2} \xi_j \psi^j_0 \right) v \otimes s(x) \tag{6.8}
\]

where \( s(x) \) solves the 5 + 1 dimensional Dirac equation. Quantization of the \( 2k + 10 \) fermions \( \xi_i \) and \( \psi^\mu_0 \) can be grouped into the \( k \) pairs, \( (\xi_i, \psi^i_0) \) for \( i = 1, \ldots, k \), and the
remaining ten. The factor \((1 + i\sqrt{2} \xi_i \psi^i_0)\) acts as the projection operator into one out of two states in the \((\xi_i, \psi^i_0)\) system. Thus, the dimension of the space of solutions is \(2^5\), which is exactly what we expect in the CFT description.

Let us fix our convention of the GSO projection \((-1)^F = 1\). We define the \(Z_2\)-grading on the vector space \(V\) by

\[
\sigma = i^k \xi_1 \cdots \xi_{2k}
\]

so that the GSO operator is given by \((-1)^F := \sigma \otimes \Gamma_{9+1}\), where the second factor is the ten-dimensional chirality \(\Gamma_{9+1} := 2^5 \psi^0_0 \psi^1_0 \cdots \psi^9_0\). Let us look at the GSO projection of the \(9+p\) string. Using \((i\sqrt{2} \xi_j \psi^j_0)^2 = 1\) we find, for the state given by (6.8), \((-1)^F \Psi_{9p} = (-1)^{\frac{k(k-1)}{2}} \xi_{k+1} \cdots \xi_{2k} \cdot 2^{5-\frac{1}{2}} \psi^0_0 \psi^{k+1} \cdots \psi^9_0 \Psi_{9p}\). In the cases of our interest, \(k = 4, 8\) (i.e., \(p = 5, 1\)), this may be written as

\[
(-1)^F \Psi_{9p} = \xi_{k+1} \cdots \xi_{2k} \Gamma_{p+1} \Psi_{9p},
\]

where \(\Gamma_{p+1} := 2^{\frac{p+1}{2}} \psi^0_0 \psi^{k+1} \cdots \psi^9_0\) is the chirality in \(p+1\) dimensions. Thus, the convention taken in Section 6.1 (that D9-D5 and D9-D1 strings yield massless fermions of positive chirality in \(5+1\) and \(1+1\) dimensions) corresponds to the choice

\[
\sigma_{IR} = \xi_{k+1} \cdots \xi_{2k}
\]

for the \(Z_2\)-grading in the infra-red Chan-Paton vector space \(V_{IR} = V_k\) \((k = 4, 8)\).

### 6.2.3 Orientifold Projection

Now we look at the orientifold projection of massless fermions \(\Psi = \Psi_{pp}\) for \(p = 5\) and 1. For the factorized expression of the state, \(\Psi = \phi \otimes s\), we have \(\check{P}(\Psi) = \check{P}_{CP}(\phi) \otimes \check{P}_{99}(s)\) where, in the \((-+)\) sector

\[
\check{P}_{CP}(\phi) = U_{(i)} \circ \phi^T \circ U_{(-i)}^{-1}
\]

and \(\check{P}_{99}\) is an action on \(S\) associated with the parity \(\check{\Omega}_{99}\) on the worldsheet fermions with Neumann boundary condition at both boundaries. The reason we put the subscript “99” is to distinguish it from the parity action on the fermions with Dp-brane boundary condition at both boundaries. Note that the action on the modes is \(\check{\Omega}_{99} : \psi^\mu_0 \rightarrow \psi^\mu_0\) for \(\mu = 0, 1, \ldots, 9\), in the \((-+)\)-sector. Hence \(\check{P}_{99}\) is equal to the identity up to a phase,
\[ \tilde{P}_{99}(s) = e^{i\delta}s. \] Let us compute \( \bar{P}(\Psi) \). Using \( U_{(i)} = \kappa U_{(-i)} \sigma^T = \kappa \sigma U_{(-i)} \), we find

\[
U_{(i)} \left[ \prod_{j=1}^{k} (1 + (i - 1)\xi_j \psi_0^j) \bar{\phi} \right] U_{(-i)}^{-1}
\]

\[
= \kappa \sigma \prod_{j=1}^{k} \left[ i^{j} e^{\pi i \sqrt{2} \xi_j \psi_0^j} \left( 1 - (i - 1)\xi_j \psi_0^j \right) \right] U_{(-i)} \bar{\phi}^T U_{(-i)}^{-1}
\]

\[
= \kappa \cdot i^3 k \cdot e^{\frac{\pi i}{4}k} \xi_{k+1} \cdots \xi_{2k} \cdot 2^\frac{k}{2} \psi_0^1 \cdots \psi_0^k \prod_{j=1}^{k} \left( 1 - (i - 1)\xi_j \psi_0^j \right) U_{(-i)} \bar{\phi}^T U_{(-i)}^{-1}
\]

\[
= \kappa \cdot e^{\frac{\pi i}{4}k} \prod_{j=1}^{k} \left( 1 + (i - 1)\xi_j \psi_0^j \right) U_{IR(i)} \bar{\phi}^T U_{IR(-i)}^{-1} \cdot 2^\frac{k}{2} \psi_0^1 \cdots \psi_0^k.
\]

Here \( U_{IR(\pm i)} \) are the \( \xi_{k+1}, \ldots, \xi_{2k} \) parts of \( U_{(\pm i)} \), and we decided to take them to obey the relation \( U_{IR(i)} = \xi_{k+1} \cdots \xi_{2k} U_{IR(-i)} \) which means

\[
U_{IR(i)} = \sigma_{IR} U_{IR(-i)} \tag{6.12}
\]

in view of (6.11). Thus, \( \Psi \mapsto \bar{P}(\Psi) \) corresponds in the CFT description to

\[
\bar{\phi} \otimes |s\rangle \mapsto \kappa e^{\frac{\pi i}{4}k} U_{IR(i)} \bar{\phi}^T U_{IR(-i)}^{-1} \otimes \chi_{n}(s) e^{i\delta}|s\rangle, \tag{6.13}
\]

where \( \chi_{n}(s) \) is the chirality of \( s \) in the normal directions, \( \chi_{n}(s)|s\rangle = 2^\frac{k}{2} \psi_0^1 \cdots \psi_0^k |s\rangle \). We read from this that the phase \( e^{i\theta} \) is given by

\[
e^{i\theta} = \kappa e^{\frac{\pi i}{4}k+i\delta}.
\]

Since all branes have the same value of \( \kappa \), we see that \( e^{i\theta} \) for \( k = 4 \) is opposite to the one for \( k = 0, 8 \). Orientifold projection with these values of \( e^{i\theta} \) and with the relation (6.12) is exactly what we have seen to be consistent with spacetime supersymmetry. Thus, we obtained the “correct” spectrum of massless spacetime fermions without any input from spacetime physics. This computation exhibits the power of our formulation.

### 7 Twists — Illustration By Examples

We illustrate our general considerations on the orientifold data, in particular the relation between the twisting and mixed type O-planes, in explicit examples of toroidal and
Calabi-Yau compactifications. We classify the orientifold data \((\tau, B, L, \alpha, c)\) on tori and discuss T-duality relations. It matches with the known results in the well-studied examples of \(S^1\) and \(T^2\) and also leads to new results for higher dimensional tori. For orientifolds of Calabi-Yau manifolds by holomorphic involutions, we find a convenient way to read off the type of O-planes using holomorphy.

### 7.1 Circle

As the first example, let us consider Type II orientifolds on \(S^1 \times \mathbb{R}^9\). We parametrize the circle by a coordinate \(x\) with periodicity \(x \equiv x + 1\). There are three inequivalent involutions: (i) the identity, \(x \mapsto x\), (ii) the half-period shift, \(x \mapsto x + \frac{1}{2}\), and (iii) the inversion, \(x \mapsto -x\). (i) and (ii) are orientation preserving and are for Type IIB orientifolds while (iii) is orientation reversing and is for Type IIA orientifolds. In (i) the whole spacetime is the fixed point set (O9-plane), and (ii) is fixed point free (no O-plane). (iii) has two fixed point sets, one at \(x = 0\) and another at \(x = \frac{1}{2}\) (two O8-planes).

Let us classify possible choices of the data \((B, L, \alpha, c)\). Note that we may assume that the B-field is zero, \(B = 0\), since any flat B-field on \(S^1 \times \mathbb{R}^9\) is exact and can be gauged away. Likewise, we may assume that \(L\) is the trivial line bundle. Since \(B = 0\), the twist connection \(\alpha\) must be flat, and we write \(\alpha = \alpha dx\) for a real parameter \(\alpha\), with the gauge equivalence relation \(\alpha \sim \alpha + 2\pi\). The \(\Lambda\) gauge transformation that preserves \(B = 0\) must also be flat, \(\Lambda = \lambda dx\), and it acts on the twist connection as \(\alpha \rightarrow \alpha + 2\lambda\) for (i) and (ii) and trivially, \(\alpha \rightarrow \alpha\), for (iii).

(i) The identity.
We can turn off the twist connection \(\alpha\) by the gauge transformation \(\Lambda = -\frac{1}{2} \alpha dx\). The crosscap section \(c\) is thus a constant function that squares to 1. Thus there are two cases: \(c = 1\) and \(c = -1\), giving rise to the O9\(^{-}\) and O9\(^{+}\) planes respectively.

(ii) The half-period shift.
Again \(\alpha\) can be turned off and there are two cases: \(c = 1\) and \(c = -1\). But these two cases are equivalent since they are related by a combination of the \(\Lambda\) and \(\lambda\) gauge transformations (3.47) — take \(\Lambda = -\pi dx\) and \(\lambda = e^{2\pi i x}\).

(iii) The inversion.
The holonomy of \(\alpha - \tau^* \alpha\) along the circle is \(e^{2i\alpha}\) and it must be trivial, \(e^{2i\alpha} = 1\). Up to gauge equivalence we find two possibilities: \(\alpha = 0\) or \(\pi\) (mod \(2\pi\)). The crosscap section \(c\)
is given by
\[ c(x) = \exp \left( -i \int_0^x (\tau^* \alpha - \alpha) \right) \cdot c(0) = e^{2i\alpha x} c(0). \tag{7.1} \]

This expression is in reference to the frame of \( \tau^* \mathcal{L} \otimes \mathcal{L}^{-1} \) that comes from the original trivialization of \( \mathcal{L} \), and this frame matches with the canonical trivialization of \( \tau^* \mathcal{L} \otimes \mathcal{L}^{-1} \) at the two fixed points. Therefore the values of \( c(0) \) and \( c(\frac{1}{2}) \) according to (7.1) directly show the type of the O8-planes. For \( \alpha = 0 \), we have \( c(0) = c(\frac{1}{2}) = 1 \) (both \( \text{O8}^- \)) or \( c(0) = -c(\frac{1}{2}) = -1 \) (both \( \text{O8}^+ \)). For \( \alpha = \pi \), we have \( c(0) = -c(\frac{1}{2}) = 1 \) (\( \text{O8}^- \) at \( x = 0 \) and \( \text{O8}^+ \) at \( x = \frac{1}{2} \)) or \( c(0) = -c(\frac{1}{2}) = -1 \) (\( \text{O8}^- \) at \( x = 0 \) and \( \text{O8}^- \) at \( x = \frac{1}{2} \)). The last two cases are obviously equivalent as they are related by a diffeomorphism, \( x \mapsto x + \frac{1}{2} \).

Let us look at T-duality relation among the above orientifolds. We first recall that the T-dual of a circle \( S^1 \) is its dual circle \( \tilde{S}^1 = H^1(S^1, U(1)) \) which parametrizes a flat \( U(1) \) bundle on \( S^1 \). The orientifold action on a \( U(1) \) gauge field, \( A \rightarrow -\tau^* A + \alpha \), reads for the flat field \( A = a dx \) as \( a \rightarrow a + \alpha \) (with \( \alpha = 0 \) or \( \pi \)) for (iii) and \( a \rightarrow -a \) for (i) and (ii). Thus we find that (i) is T-dual to (iii) with \( \alpha = 0 \) while (ii) is T-dual to (iii) with \( \alpha = \pi \). This is consistent with the known facts: Type I string theory on a circle ((i) with \( c = 1 \)) is T-dual to Type IIA orientifold on the dual circle with two \( \text{O8}^- \) planes, while Type IIB orientifold on a circle by a half-period shift is T-dual to Type IIA orientifold with \( \text{O8}^- \) and \( \text{O8}^+ \) \[32, 33\]. Note that the gauge field shifts as \( A \mapsto A - \pi dx \) under the \((\Lambda, \lambda) = (-\pi dx, e^{2\pi i x})\) transformation that relates the two cases in (ii), see (3.48). This corresponds under T-duality to the fact that the two cases in (iii) with \( \alpha = \pi \) are related by the shift of the coordinate \( x \mapsto x + \frac{1}{2} \).

Let us find some simple D9-brane configurations, say, in Type IIA orientifolds of (iii). The condition for \((\tilde{A}, \tilde{T}, \tilde{U})\) reads
\[
\tilde{U}(x) = c(x) \cdot \tilde{U}(-x)^t, \\
\tilde{A}_x(x) = \tilde{U}(x)(\tilde{A}_x(-x)^t + \alpha)\tilde{U}(x)^{-1} + i^{-1} \tilde{U}(x) \frac{d}{dx} \tilde{U}(x)^{-1}, \\
\tilde{T}(x) = -\tilde{U}(x) \tilde{T}(-x)^t \tilde{U}(x)^{-1}.
\]

\( \text{O8}^- \) and \( \text{O8}^- \) (\( \alpha = 0, \ c = 1 \))
There is a rank one solution
\[
\tilde{U} = 1, \ \tilde{A}_x = a, \ \tilde{T} = f(x), \tag{7.2}
\]
where \( f(x) \) is any odd and periodic function. For a generic \( f(x) \), this corresponds to D8 and anti-D8 branes at points of the circle — at the zeroes of \( f(x) \) with positive and negative slopes respectively. (There is a single D8 at an \( \text{O8}^- \) and a single anti-D8 at the
other O8\(^-\). This is inconsistent\(^{46,47}\) for a reason that cannot be detected from the open string tree-level analysis.) Note that the tachyon must vanish if we insist it to be a constant. But non-zero constant tachyon is allowed if take the sum of the two,

\[
\begin{align*}
\hat{U} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{A}_x = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad \hat{T} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\end{align*}
\]

In the T-dual picture, Type I on \(\tilde{S}^1\), the solution (7.2) with \(\hat{T} = 0\) corresponds to a single non-BPS D8-brane at \(a \in \tilde{S}^1\). Absence and presence of constant tachyon for the rank one and two cases correspond to the fact that the non-BPS D8-brane in Type I is stable but its charge is conserved only modulo 2\(^{11,39}\).

\(\text{O8}^+\) and \(\text{O8}^+\) \((\alpha = 0, \ c = -1)\)

We see that \(\hat{U}(0)\) and \(\hat{U}(\frac{1}{2})\) are both antisymmetric and hence there is no rank one solution. There are solutions for even ranks. For example, a rank two solution is,

\[
\begin{align*}
\hat{U} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \hat{A}_x = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad \hat{T} = \begin{pmatrix} t_1 & t_2 \\ t_2^* & -t_1 \end{pmatrix}.
\end{align*}
\]

(7.3)

This corresponds to the non-BPS D8-brane in the T-dual theory, the USp-version of Type I\(^{45}\).

\(\text{O8}^-\) and \(\text{O8}^+\) \((\alpha = \pi, \ c(x) = e^{2\pi ix})\)

Note that \(\hat{U}(0)\) is symmetric but \(\hat{U}(\frac{1}{2})\) is antisymmetric, and again there is no rank one solution. There is a rank two solution

\[
\begin{align*}
\hat{U} &= \begin{pmatrix} 0 & e^{2\pi ix} \\ 1 & 0 \end{pmatrix}, \quad \hat{A}_x = \begin{pmatrix} a & 0 \\ 0 & a + \pi \end{pmatrix}, \quad \hat{T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
\]

(7.4)

In the T-dual Type IIB orientifold on \(\tilde{S}^1\) by the half-period shift, this corresponds to two non-BPS D8-branes at the opposite points of \(\tilde{S}^1\), one at \(a\) and the other at \(a + \pi\). See Figure [5]. Existence of non-zero constant tachyon corresponds to the instability of the non-BPS D8-branes in Type IIB string theory.

### 7.2 Two-Torus

Let us next study Type II orientifolds on \(T^2 \times \mathbb{R}^8\). We use the coordinates \((x, y)\) of \(T^2\) which have periodicity \((x, y) \equiv (x + 1, y) \equiv (x, y + 1)\). Up to diffeomorphisms, there are six distinct involutions: (i) the identity, \((x, y) \mapsto (x, y)\); (ii) the half-period shift in one direction, \((x, y) \mapsto (x + \frac{1}{2}, y)\); (iii) the reflection of the type \((x, y) \mapsto (-x, y)\); (iv) the
Figure 5: T-duality between Type IIA (left) and Type IIB (right) orientifolds

reflection plus shift, \((x, y) \mapsto (-x, y + \frac{1}{2})\), (v) the reflection of the type \((x, y) \mapsto (y, x)\), and (vi) the inversion, \((x, y) \mapsto (-x, -y)\). The properties of these involutions are summarized in the table:

| involution | (i) | (ii) | (iii) | (iv) | (v) | (vi) |
|------------|-----|------|-------|------|-----|------|
| Type       | B   | B    | A     | A    | A   | B    |
| O-plane    | O9  | none | 2×O8  | none | 1×O8| 4×O7 |

Type B (resp. A) means that the involution preserves (resp. reverses) the orientation and thus can be used to define Type IIB (resp. IIA) orientifolds.

Let us classify possible choices of the data \((B, \mathcal{L}, \alpha, c)\). We parametrize the B-field as

\[ B = b \, dx \wedge dy, \]

where we may take \(b\) from the range \(0 \leq b < 2\pi\). The condition \([B + \tau^*B] \in H^2(T^2, 2\pi\mathbb{Z})\) requires \(2b = \in 2\pi\mathbb{Z}\) (i.e. \(b = 0\) or \(\pi\)) for Type B but impose no constraint for Type A. Let us first consider Type B with \(b = 0\) and Type A for which the twist connection must be flat and can be written as

\[ \alpha = \alpha_x dx + \alpha_y dy. \]

For the involutions (i) and (ii), we can turn off \(\alpha\) using a flat \(\Lambda\) gauge transformation. For (vi), we find that \(\alpha_x\) and \(\alpha_y\) must be 0 or \(\pi\) by the constraint that \(\alpha - \tau^*\alpha = 2\alpha\) has trivial holonomy. For (iii) and (iv), a flat \(\Lambda\) gauge transformation shifts \(\alpha\) by \(\Lambda + \tau^*\Lambda = 2\lambda_y dy\) and hence \(\alpha_y\) can be turned off. The constraint that \(\alpha - \tau^*\alpha = 2\alpha_x dx\) is trivial requires \(\alpha_x = 0\) or \(\pi\). However, in (iv), the gauge transformation \(\Lambda = 2\pi y dx\) turns off \(\alpha_x = \pi\). Thus, we may set \(\alpha = 0\) in this case. On the other hand, in (iii), \(\alpha_x\) cannot be turned off by a \(\Lambda\) gauge transformation. For (v), a flat \(\Lambda\) gauge transformation shifts \(\alpha\) by \(\Lambda + \tau^*\Lambda = (\lambda_x + \lambda_y)(dx + dy)\) and hence we may set \(\alpha_y = 0\) again. The constraint that
\( \alpha - \tau^*\alpha = \alpha_x(dx - dy) \) has trivial holonomy requires \( \alpha_x = 0 \). Thus in this case we can turn off the twist connection, \( \alpha = 0 \).

It remains to consider Type B with \( b = \pi \). In this case, \( d\alpha = 2\pi dx \wedge dy \) and hence \( \mathcal{L} \) is a complex line bundle with first Chern class \(-1\). In fact, with any choice of complex structure of \( T^2 \) (there is one natural choice for a given metric), \( (\mathcal{L}, \alpha) \) can be regarded as a holomorphic line bundle of degree \(-1\), namely, \( \mathcal{O}(-p) \) for a point \( p \) of \( T^2 \) — the holomorphic line bundle that has a meromorphic section with a simple pole at \( p \) and without zero. We note that \( (\mathcal{L} \otimes \tau^*\mathcal{L}^*, \alpha - \tau^*\alpha) \cong \mathcal{O}(-p + \tau(p)) \) and that it is trivial if and only if \( \tau(p) = p \). For the involution (ii), there is no \( \alpha \) that obey the condition, since \( \tau(p) \neq p \) for any point \( p \). For (i), the condition \( \tau(p) = p \) is satisfied for any \( p \) and hence any \( \alpha \) will do. In this case, however, \( \mathcal{O}(-p) \) for all \( p \)'s are related by flat \( \Lambda \) gauge transformations. Therefore there is only one choice. For (vi), \( \tau(p) = p \) requires that \( p \) must be one of the four fixed points.

Let us discuss what types of orientifold planes are possible in each case. If \( \alpha = 0 \), then the crosscap section \( c \) is constant and hence all O-planes (if there exist) are of the same type. If \( \alpha \) is a 2-torsion, i.e., if \( \alpha \) is flat and non-trivial but \( 2\alpha \) is trivial, a half of the O-planes are of the type \( \text{O}^- \) and the other half is of the type \( \text{O}^+ \). In the case (vi) with \( b = \pi \), where \( (\mathcal{L}, \alpha) \cong \mathcal{O}(-p) \) for one of the four fixed points \( p \), the O7-plane at \( p \) is of the opposite type compared to the other three O7-planes. We shall see this last point by an explicit construction below, and also in Section [7.4] applying a general argument for holomorphic involutions.

For illustration, let us explicitly construct a twist connection for the case (vi) with \( b = \pi \). Note that \( d\alpha = 2\pi dx \wedge dy \) is solved by

\[
\alpha = 2\pi x dy.
\] (7.5)

This determines a connection of a line bundle \( \mathcal{L} \) over \( T^2 = \mathbb{R}^2/\mathbb{Z}^2 \) which is defined as the quotient of the trivial line bundle over \( \mathbb{R}^2 \) with a global frame \( \sigma(x,y) \), by the relations

\[
\sigma(x,y) \equiv \sigma(x+1,y) e^{-2\pi i y} \equiv \sigma(x,y+1).
\] (7.6)

The pull back connection \( \tau^*\alpha \) on \( \tau^*\mathcal{L} \) has an expression \( 2\pi(-x) d(-y) \) with respect to the pull-back frame \( \tau^*\sigma(x,y) = ([x,y], \sigma(-x,-y)) \). Note that this 1-form is exactly the same as (7.5) and the frame \( \tau^*\sigma(x,y) \) obeys exactly the same relations as (7.6). Therefore the line bundle \( \tau^*\mathcal{L} \otimes \mathcal{L}^{-1} \) has a global frame \( u([x,y]) = \tau^*\sigma(x,y) \otimes \sigma(x,y)^{-1} \) and the connection \( \tau^*\alpha - \alpha \) is represented by 0 with respect to it. That is, \( u \) is a parallel section. Thus, this \( (\mathcal{L}, \alpha) \) satisfies the condition for a twist connection.
Let us evaluate $u$ at the four fixed points, $p_1 = [0, 0]$, $p_0 = [\frac{1}{2}, 0]$, $p_2 = [0, \frac{1}{2}]$ and $p_3 = [\frac{1}{2}, \frac{1}{2}]$:

$$u(p_1) = (p_1, \sigma(0, 0)) \otimes \sigma(0, 0)^{-1} = 1,$$

$$u(p_0) = (p_0, \sigma(-\frac{1}{2}, 0)) \otimes \sigma(\frac{1}{2}, 0)^{-1} = (p_0, \sigma(\frac{1}{2}, 0)) \otimes \sigma(\frac{1}{2}, 0)^{-1} = 1,$$

$$u(p_2) = (p_2, \sigma(0, -\frac{1}{2})) \otimes \sigma(0, \frac{1}{2})^{-1} = (p_2, \sigma(0, \frac{1}{2})) \otimes \sigma(0, \frac{1}{2})^{-1} = 1,$$

$$u(p_3) = (p_3, \sigma(-\frac{1}{2}, \frac{1}{2})) \otimes \sigma(\frac{1}{2}, \frac{1}{2})^{-1} = (p_3, \sigma(\frac{1}{2}, \frac{1}{2})(-1)) \otimes \sigma(\frac{1}{2}, \frac{1}{2})^{-1} = -1.$$

We have used the defining relations (7.6) in the latter three lines, and also the canonical isomorphism (3.27) for the evaluation. We see that the value at $p_3$ is opposite to the value at the other three points. Since the crosscap section $c$ is proportional to $u$, the type of O7-plane at $p_3$ is opposite to the type of the other three O7-planes.

We now show that, for any choice of complex structure of $T^2$, this twist connection $\alpha$ determines a holomorphic structure on $L$ which is isomorphic to $O(-p_3)$. Let us take $z = -y + \pi x$ as a complex coordinate (Im($\tau$) > 0). It is straightforward to see that $\sigma(x, y)^{-1} e^{\pi i x^2} \vartheta_3(-y + \tau x, \tau)$, where $\vartheta_3$ is Jacobi theta function

$$\vartheta_3(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 + 2\pi i n z},$$

is invariant under $(x, y) \rightarrow (x + 1, y)$ and $(x, y + 1)$. Thus, it defines a global section of $L^{-1}$. We can also see that this section is holomorphic with respect to the holomorphic structure determined by the connection $-\alpha$. Since $\vartheta_3(z, \tau)$ has a simple zero at $z = \frac{1}{2} + \frac{1}{2}\tau$ (mod $\mathbb{Z} + \tau\mathbb{Z}$), we find that $(L^{-1}, -\alpha) \cong O(p_3)$, or equivalently, $(L, \alpha) \cong O(-p_3)$.

Other twist connections must differ from (7.5) by a 2-torsion and hence are given by $\alpha_0 = 2\pi x dy + \pi dx$, $\alpha_1 = 2\pi x dy + \pi (dx + dy)$ and $\alpha_2 = 2\pi x dy + \pi dy$. Repeating the above analysis, we find for the twist ($L, \alpha_i$) ($i = 0, 1, 2$) that the O-plane at $p_i$ is of the opposite type compared to the other three O-planes, and also that $(L, \alpha_i) \cong O(-p_i)$.

The classification is summarized in the table below:

| involution | (i) | (ii) |
|------------|-----|------|
| $b$        | 0   | $\pi$ |
| $(L, \alpha)$ | trivial | $c_1 = -1$ | trivial |
| O-plane    | O9− | O9+ | O9− | O9+ | none |

| involution | (iii) | (iv) | (v) |
|------------|-------|------|-----|
| $b$        | arbitrary | arbitrary | arbitrary |
| $(L, \alpha)$ | trivial | 2-torsion | trivial |
| O-plane    | 2·O8− | 2·O8+ | O8− & O8+ | none | O8− | O8+ |
| involution | (vi) |
|------------|-----|
| \(b\)     | 0   | \(\pi\) |
| \((\mathcal{L}, \alpha)\) | trivial | 2-torsion | \(O(-p), \tau(p) = p\) |
| O-plane    | \(4\cdot O^7\) | \(4\cdot O^7^+\) | \(2\cdot O^7^- \& 2\cdot O^7^+\) |
|            | \(O^7^+ \& 3\cdot O^7^-\) | \(O^7^- \& 3\cdot O^7^+\) |

There are T-duality relations among them. \((i), (iii)\) and \((vi)\) with trivial twist are obviously T-dual to one another. When T-duality is applied to \((ii), (x, y) \mapsto (x + \frac{1}{2}, y),\) in the \(x\) direction, as in the case of the circle, we find \((iii)\) with 2-torsion twist. The latter in turn is T-dual to \((vi)\) with 2-torsion twist. When T-duality is applied to \((ii)\) in the \(y\) direction, we find \((iv)\). Finally, \((i)\) with \(b = \pi\) and \((vi)\) with \(b = \pi\) are both T-dual to \((v)\).

To see this, let us take \((7.5)\) as the twist connection for \((i)\) and \((vi)\) with \(b = \pi\). Since that expression is invariant under translations in \(y\), we may perform T-duality in the \(y\) direction. Since the orientifold action on the Wilson lines is

\[
(i) \quad a_x dx + a_y dy \mapsto -(a_x dx + a_y dy) + 2\pi x dy = -a_x dx + (-a_y + 2\pi x) dy,
\]

\[
(vi) \quad a_x dx + a_y dy \mapsto -(a_x dx - a_y dy) + 2\pi x dy = a_x dx + (a_y + 2\pi x) dy,
\]

the action on the T-dual coordinates \((x, \tilde{y}) = (x, a_y/2\pi)\) is

\[
(i) \quad (x, \tilde{y}) \mapsto (x, -\tilde{y} + x),
\]

\[
(vi) \quad (x, \tilde{y}) \mapsto (-x, \tilde{y} + x).
\]

These actions are equivalent to the involution \((v), (x, y) \mapsto (y, x),\) under the coordinate change

\[
(x_v, y_v) = (x_i - \tilde{y}_i, \tilde{y}_i) = (x_{vi} + \tilde{y}_{vi}, \tilde{y}_{vi}).
\]

This shows that \((v)\) is obtained from \((i)\) and \((vi)\) with \(b = \pi\) by T-duality. Note that the \(\tilde{y}_vi\) and \(\tilde{y}_i\) directions are respectively parallel and orthogonal to the fixed line \(x_v = y_v\). Thus, we may also say that \((vi)\) with \(b = \pi\) and \((i)\) with \(b = \pi\) are obtained from \((v)\) by T-duality in these two directions.

Let us comment on the structure group of the Chan-Paton bundle \(E\) in Case \((i)\) where the involution \(\tau\) is the identity. We suppose that \(E\) is purely even and has rank \(N\) (we know that we must set \(N = 32\) for tadpole cancellation). In the \(b = 0\) case, the twist is trivial and the orientifold isomorphism defines a unitary map \(U : E^* \to E\) such that \(U^t = U\) or \(-U\). This reduces the structure group of \(E\) from \(U(N)\) to \(G = O(N)\) or \(USp(N)\) — an orthonormal frame \(\sigma\) of \(E\) is \(G\)-admissible if \(U\) maps the dual frame \(\sigma^*\) to \(\sigma\) times the identity matrix \(1_N\) or the symplectic matrix \(J_N\). The condition \(A = U(-A^t)U^{-1} + i^{-1}UdU^{-1}\) says that the connection \(A\) preserves the reduction, i.e., \(A\) can
be regarded as an $O(N)$ or $USp(N)$ gauge field. In the $b = \pi$ case, the twist is non-trivial, $c_1(L) = -1$, and the orientifold isomorphism defines a unitary map $U : E^* \otimes L \rightarrow E$ such that $U^t = U$ or $-U$. This defines a principal bundle with the structure group

$$G' = O(N)/\{\pm 1_N\} \text{ or } USp(N)/\{\pm 1_N\}$$

—a local section is given by an expression $\sigma \otimes u^{-\frac{1}{2}}$ where $\sigma$ is an orthonormal frame of $E$ and $u$ is a frame of $L$ with unit length such that $U$ maps $\sigma^* \otimes u$ to $\sigma \cdot 1_N$ or $\sigma \cdot J_N$. The condition $A = U(-A^t + \alpha)U^{-1} + i^{-1}UdU^{-1}$ reads

$$A' = U(-A'^t)U^{-1} + i^{-1}UdU^{-1} \quad \text{for} \quad A' = A - \frac{1}{2}\alpha.$$ 

It says that $A'$ defines a connection of the principal $G'$-bundle. This $G'$-bundle does not lift to a $G$-bundle since $c_1(L)$ is odd and there is an obstruction to define a square root $L^{\frac{1}{2}}$. We find that Case (i) with $b = \pi$ is a compactification without a vector structure, which is the assertion made earlier in [33, 48] for O9$^\perp$. T-duality to (vi) with $b = \pi$ was originally argued in [33]. T-duality to (v) was discussed more recently in [49].

### 7.3 Higher Dimensional Torus

Let us classify orientifolds on higher dimensional torus $T^n$, $n \geq 3$. Instead of looking for orientifolds by all possible involutions, we just look for equivalence classes under T-duality and diffeomorphisms. We use the coordinates $x^1, \ldots, x^n$ of periods 1.

By T-duality, we can map any involution to the identity or a half-period shift. Thus, we only have to consider such involutions. Note that $B + \tau^*B = 2B$ for such involutions and hence the B-field components $B_{ij}$ must be 0 or $\pi$. Also, once $(\tau, B)$ is specified, all choices of allowed twist connection are equivalent up to flat $\Lambda$ gauge transformations (as $\Lambda + \tau^*\Lambda = 2\Lambda$). Thus we only need to classify admissible data $(\tau, B)$, i.e., those which admit a twist connection. Since the analysis is straightforward, we just record the result of classification:

If $(\tau', B')$ is admissible on $T'^n$ for $n' < n$, then it determines an admissible data $(\tau, B)$ on $T^n$ — we set $(\tau, B) = (\tau', B')$ on the first $n'$ coordinates and $(\tau, B) = (\text{id}, 0)$ for the remaining coordinates and components. As we increase the dimension by one, exactly one new class of admissible $(\tau, B)$ appears. The new class that appears for $T^n$ at even $n$ has $\tau = \text{the identity}$ and the B-field of maximal rank, say

$$B = \sum_{i=1}^{\frac{n}{2}} \pi \, dx^{2i-1} \wedge dx^{2i} = \pi \, dx^1 \wedge dx^2 + \cdots + \pi \, dx^{n-1} \wedge dx^n.$$
For this \((\tau, B)\), the two choices of crosscap section, \(c\) and \(-c\), are inequivalent. The new class that appears at odd \(n\) has \(\tau = a\) half-period shift, say, in the \(x^n\) direction, and a maximal rank B-field in the transverse directions, \(B = \pi dx^1 \wedge dx^2 + \cdots + \pi dx^{n-2} \wedge dx^{n-1}\) for example. For this \((\tau, B)\), the two choices of crosscap section are equivalent.

We can see this equally easily in the T-dual picture in which the involution \(\tau\) is the inversion \(x \rightarrow -x\). Let us describe the new class of orientifolds that appears at even \(n\) in this picture. It has a maximal rank B-field, say \(B = \pi dx^1 \wedge dx^2 + \cdots + \pi dx^{n-1} \wedge dx^n\). To describe the twist connection, we view the torus \(T^n\) as the product \(T^2 \times T^2 \times \cdots \times T^2\), where \(T^2\) is the two-torus in the \(x^i-x^j\) directions. Then the twist connection is the sum \(\alpha_{12} + \alpha_{34} + \cdots + \alpha_{(n-1)n}\), where \(\alpha_{ij}\) is the connection on \(T^2\) of the type that appears in Case (vi) with \(b = \pi\) in the previous subsection. The \(2^n\) O-planes can be labeled by \(\vec{p} = (p_{12}, p_{34}, \ldots, p_{(n-1)n})\) where \(p_{ij}\) is one of the four fixed points of the inversion of \(T^2\).

Let us denote the four fixed points by \(p_{ij}^{(0)}, p_{ij}^{(1)}, p_{ij}^{(2)}, p_{ij}^{(3)}\) and let us assume that \(\alpha_{ij}\) is such that \(p_{ij}^{(3)}\) is the distinguished point, just as in the explicit construction on \(T^2\). Then, the O-plane type is classified according to whether the number \(n_{\vec{p}}\) of \(\vec{p}\) components is even or odd. Let us count the number of O-plane with odd \(n_{\vec{p}}\): For \(n_{\vec{p}} = 1\), we have \(2\) choices for the \(p_{ij} = p_{ij}^{(3)}\) component and, for each of them, there are \(3^2 - 1\) choices of the fixed points in other components. Generalization to \(n_{\vec{p}} \geq 3\) is obvious, and the total is

\[
\sum_{i \text{ odd}, 1 \leq i \leq n} \left(\frac{\frac{n}{2}}{i} \right) 3^\frac{n}{2} - i = 2^{n-1} - 2^\frac{n}{2} - 1.
\]

This number is 1, 6, 28, 120 for \(n = 2, 4, 6, 8\). Thus, the number of \(O^\pm\)-planes in this orientifold is as below (or the opposite):

| \(n\) | 2  | 4  | 6  | 8  |
|-------|----|----|----|----|
| O-plane | \(O7^+\ & 3 \cdot O7^-\) | \(6 \cdot O5^+ \& 10 \cdot O5^-\) | \(28 \cdot O3^+ \& 36 \cdot O3^-\) | \(120 \cdot O1^+ \& 136 \cdot O1^-\) |

The new class of orientifolds that appears at odd \(n\) has maximal rank B-field, say \(B = \pi dx^1 \wedge dx^2 + \cdots + \pi dx^{n-2} \wedge dx^{n-1}\), and \(\alpha = \alpha_{12} + \cdots + \alpha_{(n-2)(n-1)} + \pi dx^n\). It has equal number of O-planes. The distribution of the O-planes at \(x^n = 0\) is as in the table above and the one at \(x^n = \frac{1}{2}\) is opposite to it.

The classification is summarized in the table below, in the T-duality frame in which
τ is the inversion and there are $2^n O(9 - n)$-planes.

|    | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|----|---|---|---|---|---|---|---|---|---|---|
| (1, 0) | (2, 0) | (4, 0) | (8, 0) | (16, 0) | (32, 0) | (64, 0) | (128, 0) | (256, 0) | (512, 0) |
| (1, 1) | (2, 2) | (4, 4) | (8, 8) | (16, 16) | (32, 32) | (64, 64) | (128, 128) | (256, 256) |
| (3, 1) | (6, 2) | (12, 4) | (24, 8) | (48, 16) | (96, 32) | (192, 64) | (384, 128) |
| (4, 4) | (8, 8) | (16, 16) | (32, 32) | (64, 64) | (128, 128) | (256, 256) |
| (10, 6) | (20, 12) | (40, 24) | (80, 48) | (160, 96) | (320, 192) |
| (16, 16) | (32, 32) | (64, 64) | (128, 128) | (256, 256) |
| (36, 28) | (72, 56) | (144, 112) | (288, 224) | (576, 448) |
| (64, 64) | (128, 128) | (256, 256) |
| (136, 120) | (272, 240) | (544, 480) |
| (256, 256) |

$(a, b)_m$ is a theory in which there are $a$ O$^-$-planes and $b$ O$^+$-planes (or the opposite) and the B-field is of rank $m$. No subscript means $B = 0$.

Toroidal compactifications of Type I with non-zero B-fields was originally studied in [50] and parts of this table were indeed constructed there. See also [51]. Classification of type distributions of O-planes on $T^n/Z_2$ had been given in [47, 52] for the case $n = 3$ and our result reproduces it. The methods used in these references both look computationally more involved for higher $n$ and classification had not been carried out. In contrast, our construction is very simple and quickly led us to the above result.

### 7.4 Holomorphic Involutions

As the last class of examples, we consider Type II orientifold on $M \times \mathbb{R}^{10 - 2n}$, where $M$ is an $n$-dimensional compact complex manifold, by an involution $\tau$ which is holomorphic on $M$ and trivial on $\mathbb{R}^{10 - 2n}$. We assume that the B-field is a $(1, 1)$ form on $M$. Then, a twist connection $\alpha$ has a curvature without $(0, 2)$-form component, $\partial_2^{\alpha} = 0$, and hence defines a holomorphic structure on $\mathcal{L}$. The assumption is automatic if $H^{2,0}(M) = H^{0,2}(M) = 0$, e.g., when $M$ is a simply connected Calabi-Yau three-fold, in which case any $B$ can be made into a $(1, 1)$ form by a $\Lambda$ gauge transformation. Note also that there is a unique twist $(\mathcal{L}, \alpha)$ for any B-field such that $[B + \tau^*B] \in H^2(M, 2\pi\mathbb{Z})$ as long as $M$ is simply connected.

A parallel section of $\tau^*\mathcal{L} \otimes \mathcal{L}^{-1}$ with respect to $\tau^*\alpha - \alpha$ is of course holomorphic. Conversely, a holomorphic section of $\tau^*\mathcal{L} \otimes \mathcal{L}^{-1}$ is necessarily parallel, since its ratio with a parallel section must be a holomorphic function of a compact complex manifold and
must be a constant. Thus, a holomorphic section \( c \) of \( \tau^* L \otimes L^{-1} \) qualifies as a crosscap section if it satisfies \( \tau^* c \cdot c = 1 \). It can be regarded as a linear map \( c : L \to \tau^* L \) and also there is a canonical map \( \tau : \tau^* L \to L \) over the map on the base \( \tau : M \to M \). By composing the two, we find a lift \( \tilde{\tau}_c \) of \( \tau \) to \( L 
abla_7 \)

\[
\begin{array}{ccc}
\tilde{\tau}_c : L & \xrightarrow{c} & \tau^* L \\
\downarrow & & \downarrow \\
M & \overset{\text{id}}{\longrightarrow} & M \\
& & \xrightarrow{\tau} M \\
\end{array}
\] (7.7)

The condition \( \tau^* c \cdot c = 1 \) is equivalent to the statement that \( \tilde{\tau}_c \) is an involution of \( L 
abla_8 \),

\[
\tilde{\tau}_c \circ \tilde{\tau}_c = \text{id}_L.
\] (7.8)

The value of \( c \) at a fixed point of \( \tau \) is equal to the value of \( \tilde{\tau}_c \) at that point. Thus, a crosscap section may be regarded as a holomorphic and involutive lift of \( \tau \) to \( L \), and the type of an O-plane is determined by its value according to (3.43).

This observation is very useful to find the types of the O-planes. For illustration, let us consider a particular Calabi-Yau manifold with a class of holomorphic involutions which were studied in detail in \([53]\). We first introduce a toric variety \( X \) realized as a symplectic quotient of \( \mathbb{C}^6 \) by \( U(1) \times U(1) \) with the action \( x \mapsto \epsilon_i x_i \) for \( i = 1, \ldots, 5 \) and \( x_6 \mapsto x_6 \). Fixed points are found by solving the equation \( \tau(x) = x \cdot (g, h) \).

\[
x = (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto x \cdot (g, h) := (hx_1, hx_2, gx_3, gx_4, gx_5, gh^{-2}x_6).
\] (7.9)

The solutions are listed below, together with \((g, h)\) needed for (7.9):

\[
x_i^8 x_6^4 + x_2^8 x_6^4 + x_3^4 + x_4^4 + x_5^4 = 0.
\]
Any of these involutions acts trivially on the second cohomology group, as one can see by noting that the divisors defining the generating line bundles $O(1,0)$ and $O(0,1)$ are invariant under the sign flips of the coordinates $x_i$. In particular, $B = b_1 c_1 (O(1,0)) + b_2 c_1 (O(0,1))$ satisfies the condition $[B + \tau^* B] \in H^2(M, 2\pi \mathbb{Z})$ if and only if $2b_1, 2b_2 \in 2\pi \mathbb{Z}$. Thus, we may set $b_1 = \pi q_1$ and $b_2 = \pi q_2$ where $q_1, q_2 = 0$ or $1$. For this $B$, the twist is by a holomorphic line bundle isomorphic to $O(-q_1, -q_2)$.

As the lift of the involution $\tau$ to the line bundle $L \cong O(-q_1, -q_2)$, we may take

$$\tilde{\tau} [x, v] = [\tau(x), v].$$

The value at a fixed point can be expressed in terms of the element $(g, h)$ that realizes (7.9): $\tilde{\tau}[x, v] = [x \cdot (g, h), v] = [x, g^{q_1} h^{q_2} v]$. That is

$$\tilde{\tau} = g^{q_1} h^{q_2} \quad \text{at the fixed point obeying (7.9).} \quad (7.11)$$

For this choice of lift, $\tilde{\tau}_c = -\tilde{\tau}$, the O-plane types are shown in the table below for each value of $(b_1, b_2)$. For the opposite choice, $\tilde{\tau}_c = -\tilde{\tau}$, the types are all opposite to the table.

\begin{table}
\begin{tabular}{|c|c|c|c|}
\hline
involution & solution & $(g, h)$ & description \\
\hline
$(+++++)$ & any $x$ & $(1,1)$ & the whole $M$ \\
$(++-++)$ & $x_3 = x_4 = 0$ & $(1,1)$ & a divisor \\
$(++-++)$ & $x_3 = x_4 = x_5 = 0$ & $(1,1)$ & eight points \\
$(++--+)$ & $x_2 = 0$ & $(1,1)$ & a divisor (K3) \\
$(+-+++)$ & $x_1 = 0$ & $(-1,1)$ & a divisor (K3) \\
$(+-++-)$ & $x_2 = x_3 = 0$ & $(1,1)$ & a curve ($C_3$) \\
$(+-++-)$ & $x_1 = x_3 = 0$ & $(1,-1)$ & a curve ($C_3$) \\
$(+-+-+)$ & $x_2 = x_3 = x_4 = 0$ & $(1,1)$ & four points \\
$(+-+-+)$ & $x_1 = x_3 = x_4 = 0$ & $(1,-1)$ & four points \\
$(+-+-+)$ & $x_2 = x_5 = x_6 = 0$ & $(-1,1)$ & four points \\
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\hline
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$C_g$ in the table stands for a curve of genus $g$. 

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Any of these involutions acts trivially on the second cohomology group, as one can see by noting that the divisors defining the generating line bundles $O(1,0)$ and $O(0,1)$ are invariant under the sign flips of the coordinates $x_i$. In particular, $B = b_1 c_1 (O(1,0)) + b_2 c_1 (O(0,1))$ satisfies the condition $[B + \tau^* B] \in H^2(M, 2\pi \mathbb{Z})$ if and only if $2b_1, 2b_2 \in 2\pi \mathbb{Z}$. Thus, we may set $b_1 = \pi q_1$ and $b_2 = \pi q_2$ where $q_1, q_2 = 0$ or $1$. For this $B$, the twist is by a holomorphic line bundle isomorphic to $O(-q_1, -q_2)$.

As the lift of the involution $\tau$ to the line bundle $L \cong O(-q_1, -q_2)$, we may take

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We see that the value of $\tilde{\tau}$ is $-1$ at $p$ and $+1$ at the other three fixed points. This shows that the O7-plane at $p$ is of the opposite type compared to the three other O7-planes.

The result for the values $(b_1, b_2) = (0, \pi)$ and $(\pi, \pi)$ matches with the result obtained in [53] based on continuation of RR-charges to the Gepner point and the tadpole cancellation condition there. The same problem was studied in [54].

As another application, let us revisit the problem of finding the O-plane types in the orientifold of two-torus, for Case (vi) with $b = \pi$. In that case we found that the twist connection determines the holomorphic line bundle $\mathcal{O}(-p)$ where $p$ is one of the four fixed points. Let us choose a flat coordinate $z$ defined on a neighborhood $U_0$ of the point $p$ such that $z(p) = 0$. The inversion $\tau$ acts on it as $z \mapsto -z$. The line bundle $\mathcal{O}(-p)$ has a local frame $\sigma_0$ on $U_0$ and another frame $\sigma_1$ on a complement of $p$, $U_1 = T^2 - \{p\}$, which are related on the overlap by

$$\sigma_0(x) = \sigma_1(x) \cdot z(x), \quad x \in U_0 \cap U_1. \quad (7.12)$$

We may assume that $U_0$ is $\tau$-invariant. As a lift of $\tau$, we can take

$$\tilde{\tau}(\sigma_0(x)) := \sigma_0(\tau(x)) \cdot (-1),$$

$$\tilde{\tau}(\sigma_1(x)) := \sigma_1(\tau(x)).$$

Note that the relation (7.12) is maintained by a minus sign in one of the two equations.

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We see that the value of $\tilde{\tau}$ is $-1$ at $p$ and $+1$ at the other three fixed points. This shows that the O7-plane at $p$ is of the opposite type compared to the three other O7-planes.
8 Topology Of D-Branes

The goal of this section is to classify the configurations of the space filling D-branes up to continuous deformations including tachyon condensation.

8.1 Tachyons And Fredholm Operators

Let $H$ be a separable Hilbert space over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. One can introduce a topology in the set of bounded linear operators on $H$, and subsets therein, using the norm $||f|| := \sup_{v \neq 0} |f(v)|/|v|$. Let $GL(H)$ be the group of bounded operators of $H$ with bounded inverses and let $U(H)$ be the subgroup consisting of unitary operators. The most important fact for us is Kuiper’s theorem:

*The groups $GL(H)$ and $U(H)$ are contractible to a point.*

In particular, any vector bundle over a space $X$ with the fibre $H$ and the structure group $GL(H)$ or $U(H)$ is trivializable.

For the real and quaternionic fields, $U(H)$ may as well be denoted by $O(H)$ and $USp(H)$ respectively. In what follows, all of the three fields appear. To avoid confusion, we shall put the field as the subscript, as $H_\mathbb{R}$, $H_\mathbb{C}$ and $H_\mathbb{H}$.

8.1.1 Type IIB — $\mathcal{F}(H_\mathbb{C})$

In Type IIB string theory, D-brane/anti-D-brane pair is regarded as the vacuum without a D-brane if the tachyon defines a linear isomorphism between the Chan-Paton bundles supported by the branes and antibranes [11, 56]. One way to motivate this from the worldsheet point of view is to look at the long distance behaviour of the boundary interaction (2.16): the tachyon enters as the potential term, $\frac{1}{2} T^2$, and its positive values have the effect to contract the worldsheet boundary. Thus, as long as the topology is concerned, we can freely add or remove such trivial brane-antibrane pairs, finite or infinite. We may also consider finite deformation of the tachyon itself. For example, even if the tachyon is not originally an isomorphism, if we can make it into an isomorphism by a finite deformation, the brane-antibrane system is continuously connected to the vacuum.

Let $(E, A, T)$ be a D9-brane configuration. Recall that $E = E^0 \oplus E^1$ and that the tachyon $T$ determines a linear bundle map $T_{10} : E^0 \to E^1$. Let us add infinitely many trivial brane-antibrane pairs. This is done, for example, by adding a (trivial) Hilbert
bundle \( H'_C = X \times H'_C \) to both of \( E^0 \) and \( E^1 \) and extending \( T_{10} \) by the identity of \( H'_C \):

\[
\begin{align*}
H'_C & \xrightarrow{\text{id}} H'_C \\
\oplus & \oplus \\
E^0 & \xrightarrow{T_{10}} E^1.
\end{align*}
\]

(8.1)

By Kuiper’s theorem the two vector bundles can be trivialized, \( E^0 \oplus H'_C \cong H_C \cong E^1 \oplus H'_C \). Then, (8.1) can be regarded as an endomorphism of the trivial bundle \( H_C = X \times H_C \), i.e., we have a function of \( X \) with values in the operators of \( H_C \), denoted again by \( T_{10}(x) \). The kernel and the cokernel of \( T_{10}(x) \) are finite dimensional for any \( x \), as they are bounded by the ranks of \( E^0 \) and \( E^1 \). That is, \( T_{10}(x) \) is a Fredholm operator. Thus, we obtained a continuous map

\[
T_{10} : X \rightarrow \mathcal{F}(H_C),
\]

(8.2)

where \( \mathcal{F}(H_C) \) is the space of Fredholm operators on \( H_C \). Continuous deformation of the original D9-brane configuration \( (E, A, T) \) corresponds to continuous deformation of the map (8.2), and vice versa. Furthermore addition or subtraction of trivial brane-antibrane pairs to or from \( (E, A, T) \) results in the same map \( T_{10} \) or at least a map that can be connected by continuous deformation. In this sense, the set of homotopy classes of the maps (8.2), which is denoted by

\[
[X, \mathcal{F}(H_C)],
\]

(8.3)

classifies the topology of D-branes. The space \( \mathcal{F}(H_C) \) is closed under composition of operators. This induces the structure of a semi-group in the set (8.3).

The spacetime for string theory is non-compact in almost all cases and we typically impose some conditions on the configurations of fields, branes, etc. For example, we often assume translational invariance in some of the dimensions, say the time plus a part of the space, in order to describe static configurations of particles and branes. In such a case, we simply ignore such ‘irrelevant’ dimensions. Also, if we have spatial infinities, we usually impose certain boundary condition in order to achieve finite energy or finite tension. In such a case, the map (8.2) must obey the respective boundary condition, or alternatively we take one point compactification of the relevant dimensions and impose the condition at the infinity point. In what follows, we shall often assume that the ‘spacetime’ \( X \) is compact or compactified for these reasons.

8.1.2 Type IIA — \( \widehat{\mathcal{F}}_s(H_C) \)

Let us next consider Type IIA string theory. A D9-brane configuration is regarded as the vacuum if the tachyon defines an isomorphism of the Chan-Paton bundle to itself.
Note that any vector bundle admits an isomorphism — the identity. Therefore, if the spacetime \( X \) is compact, any D9-brane supporting a finite rank vector bundle is continuously connected to the vacuum since there is a finite deformation of the tachyon to the identity, say

\[
\hat{T}_t = t \text{id} + (1 - t)\hat{T}.
\]

(8.4)

Thus, unlike in the Type IIB case, in order to have a non-trivial D-brane configuration on a compact space \( X \), the vector bundle \( \hat{E} \) must be of infinite rank to start with. If \( \hat{E} \) is indeed of infinite rank, it can be trivialized by Kuiper’s theorem, and the tachyon \( \hat{T} \) can be regarded as a continuous function with values in self-adjoint operators on a Hilbert space \( H_{C} \). In order to have finite energy, we need \( \text{Ker} \hat{T}(x) \cong \text{Coker} \hat{T}(x) \) to be finite dimensional at any point \( x \). Therefore \( \hat{T}(x) \) is a Fredholm operator. Furthermore, we may assume that the tachyon has infinitely many positive and infinitely many negative eigenvalues. To motivate this, suppose that \( \hat{T} \) has only finitely many negative eigenvalues. Then, (8.4) defines a homotopy between \( \hat{T} \) and the identity operator, and hence the brane is continuously deformable to the vacuum. The similar homotopy works for those with finitely many positive eigenvalues — we just replace \( \text{id} \) by \( -\text{id} \). Following the literature, we denote by \( \hat{\mathcal{F}}_*(H_{C}) \) the space of \textit{skew-adjoint} Fredholm operators on \( H_{C} \) which have infinitely many positive imaginary and infinitely many negative imaginary eigenvalues. Then, we have a map

\[
\hat{T} : X \longrightarrow i^{-1}\hat{\mathcal{F}}_*(H_{C}),
\]

(8.5)

and the set of homotopy classes of such maps

\[
[X, \hat{\mathcal{F}}_*(H_{C})]
\]

(8.6)

classifies the topology of D-branes.

For illustration, let us consider the case \( X = S^1 \times \mathbb{R}^9 \) where we ignore the dependence in the \( \mathbb{R}^9 \) direction. We use the coordinate \( x \) of \( S^1 \) which have periodicity \( x \equiv x + 1 \). We consider the tachyon \( \hat{T}(x) \) whose eigenvalues are

\[
\lambda_n(x) = \lambda(x - x_0 + n).
\]

(8.7)

where \( n \) runs over all integers and \( \lambda \) is a real number. This represents a single BPS D8-brane at \( x = x_0 \). Note that this operator \( \hat{T}(x) \) is not bounded, but the usual trick, \( \hat{T} \to \hat{T}/\sqrt{1 + \hat{T}^*\hat{T}} \), turns it into a bounded operator. In fact, \( |T| = \infty \) is the natural value at the vacuum, in the framework in which the tachyon \( T \) appears in the boundary interaction as (2.16). Thus, we always assume this trick in order to put things in the context of Fredholm operators. Another remark that has to be made here is that a single
BPS D8-brane at a point of a circle violate the tadpole cancellation condition and is inconsistent in the full string theory. However, the tadpole condition can be ignored in the classical limit where the string coupling constant is set equal to zero. Alternatively, the above $\tilde{T}(x)$ enters as a building block into the D9-brane configuration for a BPS Dp-brane at a point of $S^1$, which has no tadpole problem for $p = 0, 2, 4$.

If $X$ is non-compact, the configuration can be non-trivial even when rank $\hat{E}$ is finite. For example, a BPS D8-brane in $\mathbb{R}^{10}$ can be realized by a tachyon configuration on a rank one vector bundle. In Section 4.4, it is provided by the linear profile $\tilde{T}(x^1) = x^1$. If we apply $\tilde{T} \to \tilde{T}/\sqrt{1 + \tilde{T}^2}$ to it, we obtain a kink as shown in Figure 6 (left). We see that the topology of the configuration is stable under finite deformation. For example, the zero point of the tachyon, i.e., the location of the BPS D8-brane, cannot be removed. Pairs of new zeroes may be created, but the number of zeroes with positive slope minus the number of zeroes with negative slope is always 1. In non-compact situations, we often partially compactify the space by attaching one point to the relevant boundary directions. Let us see whether or how the configuration can be extended to the infinity point in the present example. We compactify the real line $\mathbb{R}^1$ of $x^1$ to the circle $S^1$ by attaching one point at infinity. The original kink does not extend to the infinity point since the limiting values at the two boundaries, $x^1 \to +\infty$ and $x^1 \to -\infty$, are different — they even have opposite signs. To cure this problem, we may add trivial D9-brane configurations, i.e. those whose tachyons are everywhere non-zero. We immediately notice that infinitely many trivial D9-branes are required, in order for the spectrum to have the same limit as $x^1 \to +\infty$ and $x^1 \to -\infty$. In the end, we have infinitely many positive and infinitely many negative tachyon values, as shown in Figure 6 (right). (The resulting configuration is essentially the same as the periodic configuration (8.7) on the circle.) We are automatically led to a map to $i^{-1}\hat{\mathcal{F}}_*(H_C)$ by the attempt to extend the configuration to the infinity point.
8.1.3 Type II Orientifolds Without Twist — $T^k$

Let us next discuss the topology of D9-brane configurations in Type II orientifolds with trivial twist. Before starting, let us collect some useful facts on linear algebra of hermitian vector spaces and its duals.

Some Linear Algebras

We shall take the convention that a hermitian inner product on a complex vector space $V$, denoted by $(v, v')_V$, is antilinear in the left entry and linear in the right entry. We define an antilinear map $h_V : V \rightarrow V^*$, by

$$\langle h_V(v), v' \rangle = (v, v')_V.$$  \hspace{1cm} (8.8)

It is easy to prove that $h_{V^*} = h_V^{-1}$, say, by choosing an orthonormal basis and its dual basis. For a linear map $f : V \rightarrow W$, we have

$$h_W \circ f \circ h_V^{-1} = (f^\dagger)^t = (f^\dagger)^\dagger.$$  \hspace{1cm} (8.9)

This can be proved by straightforward computation; $\langle h_W f h_V^{-1}(v^*), w \rangle = \langle f h_V^{-1}(v^*), w \rangle_W = \langle h_V^{-1}(v^*), f^\dagger(w) \rangle_V = \langle v^*, f^\dagger(w) \rangle = \langle (f^\dagger)^t(v^*), w \rangle$. For a linear map $U : V^* \rightarrow V$ we define an antilinear map from $V$ to itself by

$$\varsigma = U \circ h_V : V \xrightarrow{h_V} V^* \xrightarrow{U} V.$$  \hspace{1cm} (8.10)

Using the above results, we find the property $\varsigma^2 = U h_V U h_V = U h_V U h_V^{-1} = U (U^\dagger)^t$. If $U$ is unitary, $U^\dagger = U^{-1}$, then we have

$$U = \pm U^t \implies \varsigma^2 = \pm \text{id}_V.$$  \hspace{1cm} (8.11)

If an antilinear map $\varsigma : V \rightarrow V$ obeys $\varsigma^2 = \text{id}_V$, then $V$ has the structure of the complexification of a real vector space, $V \cong V_R \otimes \mathbb{C}$, and $\varsigma$ is the complex conjugation. Indeed let us define a subspace $V_R \subset V$ as a real vector space by the set of vectors $v$ such that $\varsigma(v) = v$. Then, any vector $v \in V$ can be written as $\text{Re}(v) + i \text{Im}(v)$ for $\text{Re}(v) = (v + \varsigma(v))/2 \in V_R$ and $\text{Im}(v) = (v - \varsigma(v))/2i \in V_R$.

If an antilinear map $\varsigma : V \rightarrow V$ obeys $\varsigma^2 = -\text{id}_V$, then $V$ has the structure of a quaternionic vector space, $V = V_H$, and $\varsigma$ is the multiplication by $j \in \mathbb{H}$. Indeed, we define the $i, j, k$ multiplications by $i \cdot v = iv, j \cdot v = \varsigma(v), k \cdot v = i\varsigma(v)$ respectively. It is straightforward to check that these obey the quaternion algebra, $ij = -ji = k, k^2 = -1$, etc.
We define the hermitian conjugate $f^\dagger$ of an antilinear map $f : V \to W$ by

$$(f^\dagger(w), v)_V = (f(v), w)_W. \quad (8.12)$$

If $U$ is unitary, $\varsigma$ is also unitary

$$\varsigma^\dagger = \varsigma^{-1}, \quad (8.13)$$

i.e., $(\varsigma(v), \varsigma(v'))_V = (Uh_V(v), Uh_V(v'))_V = (h_V(v), h_V(v'))_V = (v, h_V(v')) = (v', v)_V$.

Vector spaces in the above remarks are implicitly assumed to be finite dimensional. Let $\mathcal{H}_\mathbb{C}$ be a complex separable Hilbert space. One can define a map $h_{\mathcal{H}_\mathbb{C}} : \mathcal{H}_\mathbb{C} \to \text{Hom}_\mathbb{C}(\mathcal{H}_\mathbb{C}, \mathbb{C})$ just as in the finite dimensional cases (8.8). We define the subspace $\mathcal{H}_\mathbb{C}^* \subset \text{Hom}_\mathbb{C}(\mathcal{H}_\mathbb{C}, \mathbb{C})$ by the image of $h_{\mathcal{H}_\mathbb{C}}$. Then, $\mathcal{H}_\mathbb{C}^*$ itself is naturally a separable complex Hilbert space. Under this definition of the dual, all of the above remarks apply to Hilbert spaces as well.  

The Space With Involution $\mathcal{P}^k$

Let us take a D9-brane configuration in a Type II orientifold on a space $X$ with an involution $\tau : X \to X$. In addition to the gauge field and the tachyon, we have a unitary bundle map, $U : \tau^*E^* \to E$ for Type IIB and $\check{U} : \tau^*\check{E}^* \to \check{E}$ for Type IIA, that obey certain conditions depending on a mod 8 integer $k$ (denoted by $[k]$ in Introduction) that is determined by the codimension and the type of the O-planes. By adding infinitely many empty branes if necessary, we can trivialize the resulting complex Hilbert bundle(s) by Kuiper’s theorem. In particular, we have families of Fredholm and unitary operators over $X$,

$$k \text{ even : } T_{10}(x) \in \mathcal{F}(\mathcal{H}_\mathbb{C}) \text{ and } U(x) : H_\mathbb{C}^* \oplus \mathcal{H}_\mathbb{C} \to \mathcal{H}_\mathbb{C} \oplus H_\mathbb{C},$$

$$k \text{ odd : } \check{T}(x) \in i^{-1}\check{\mathcal{F}}(\mathcal{H}_\mathbb{C}) \text{ and } \check{U}(x) : H_\mathbb{C}^* \to \mathcal{H}_\mathbb{C}.$$ 

\footnote{In the standard terminology, our $\mathcal{H}_\mathbb{C}$ is nothing but the conjugate vector space $\check{\mathbb{H}}_\mathbb{C}$. The latter is equal to $H_\mathbb{C}$ as a set, but with the scalar multiplication rule modified by complex conjugation. We would also like to warn the reader on possibly confusing use of notation: We write $f^*$ for the complex conjugation of a linear map/operator $f$ on $H_\mathbb{R} \otimes \mathbb{C}$.}
These obey the following conditions from (1.8)-(1.9), where \( T = \begin{pmatrix} 0 & T_{10}^\dagger \\ T_{10} & 0 \end{pmatrix} \).

| \( k \) | Conditions                                                                 |
|-------|---------------------------------------------------------------------------|
| 0     | \( U \) even, \( U = \tau^* U^t \), \( T = U \tau^* T^t U^{-1} \)       |
| 1     | \( \bar{U} = \tau^* \bar{U}^t \), \( \bar{T} = -\bar{U} \tau^* \bar{T}^t \bar{U}^{-1} \) |
| 2     | \( U \) odd, \( U = \tau^* U^t \), \( T = -U \tau^* T^t U^{-1} \)    |
| 3     | \( \bar{U} = -\tau^* \bar{U}^t \), \( \bar{T} = \bar{U} \tau^* \bar{T}^t \bar{U}^{-1} \) |
| 4     | \( U \) even, \( U = -\tau^* U^t \), \( T = U \tau^* T^t U^{-1} \)    |
| 5     | \( \bar{U} = -\tau^* \bar{U}^t \), \( \bar{T} = -\bar{U} \tau^* \bar{T}^t \bar{U}^{-1} \) |
| 6     | \( U \) odd, \( U = -\tau^* U^t \), \( T = -U \tau^* T^t U^{-1} \)    |
| 7     | \( \bar{U} = \tau^* \bar{U}^t \), \( \bar{T} = \bar{U} \tau^* \bar{T}^t \bar{U}^{-1} \) |

Using again Kuiper’s theorem, now for \( C \), \( R \) and \( H \), we can prove that there is a choice of trivialization of the Hilbert bundle(s) such that \( U(x) \) or \( \bar{U}(x) \) is \( x \)-independent. Since the argument is the same for \( k \) even and \( k \) odd cases, we shall only spell out the proof in the latter cases.

The first step is to focus on the fixed point set \( X^\tau \) of the involution. If \( X^\tau \) is empty, this step is absent — go to the next step. On this set, the condition \( \bar{U} = \pm \tau^* \bar{U}^t \) becomes \( \bar{U} = \pm \bar{U}^t \). Defining the family of antilinear operators \( \varsigma(x) \) on \( H_C \) as in (8.10), we find using (8.11) that \( \varsigma(x)^2 = \pm 1 \) for any \( x \). In particular, for each point \( x \in X^\tau \), \( \varsigma(x) \) provides \( H_C \) with the structure of the complexification of a real Hilbert space \( H'_R(x) \) or of a quaternionic Hilbert space \( H'_H(x) \). Thus, we find a Hilbert bundle

\[
\bigcup_{x \in X^\tau} H'_R(x) \quad \text{or} \quad \bigcup_{x \in X^\tau} H'_H(x).
\]

Using Kuiper’s theorem for \( R \) or \( H \), we see that we can trivialize it as the real or quaternionic Hilbert bundle. Namely, we have found a frame in which the antilinear operator \( \varsigma(x) \) is independent of \( x \). This in turn means that \( \bar{U}(x) \) is independent of \( x \).

The next step is to construct the frame over the open subset \( X - X^\tau \). We take a cell decomposition of \( X \) which is \( \tau \)-compatible, i.e., the \( \tau \) image of any cell is another cell and any open cell is either inside or outside \( X^\tau \). The strategy is to construct the frame recursively with respect to the dimension of the cells, so that \( \bar{U} \) is a constant. Let us choose such a constant \( \bar{U}_* \). It must satisfy \( \bar{U}_* = \pm \bar{U}_*^t \) if the condition for \( \bar{U} \) is \( \bar{U} = \pm \tau^* \bar{U}^t \). Recall that we have already made a choice of frame on the cells inside \( X^\tau \), on which we
may assume $\bar{U} \equiv \bar{U}_\ast$. Note that a frame change is given by a unitary operator $\bar{M}(x)$ that acts on $\bar{U}(x)$ by

$$\bar{U}(x) \longmapsto \bar{M}(x)\bar{U}(x)\bar{M}(\tau(x))^t.$$ 

Let us start with the 0-dimensional cells. For a 0-cell $x_i \in X - X^\tau$, we choose unitary operators $\bar{M}(x_i)$ and $\bar{M}(\tau(x_i))$ such that $\bar{M}(x_i)\bar{U}(x_i)\bar{M}(\tau(x_i))^t = \bar{U}_\ast$. Then at the mirror 0-cell, $x_{\tau(i)} = \tau(x_i)$, we also find

$$\bar{M}(x_{\tau(i)})\bar{U}(x_{\tau(i)})\bar{M}(\tau(x_{\tau(i)}))^t = \bar{M}(\tau(x_i))\bar{U}(\tau(x_i))\bar{M}(x_i)^t$$

$$= \pm\bar{M}(\tau(x_i))\bar{U}(x_i)^t\bar{M}(x_i)^t = \pm(\bar{M}(x_i)\bar{U}(x_i)\bar{M}(\tau(x_i))^t)^t = \pm\bar{U}_\ast = \bar{U}_\ast.$$

Thus, we are done with the 0-cells. Next let us move on to 1-cells. Take a 1-cell $\gamma$ which is not inside $X^\tau$. At the two end points of $\gamma$, say $x_i$ and $x_j$, and their mirror points, the frame changing operators $\bar{M}$ are already chosen. We choose a path $\bar{M}(\gamma)$ in $U(H_C)$ that connects $\bar{M}(x_i)$ and $\bar{M}(x_j)$. This is possible because the unitary group $U(H_C)$ is connected. Then we choose $\bar{M}$ along the mirror 1-cell $\tau(\gamma)$ so that $\bar{M}\bar{U}\tau^*\bar{M}^t \equiv \bar{U}_\ast$ holds along $\gamma$. Then it also holds along the mirror $\tau(\gamma)$ as we can show just as for the 0-cells. Now we are done for 1-cells. This recursive procedure never fails since the unitary group $U(H_C)$ is contractible (Kuiper’s theorem). This is what we wanted to show.

We can now assume that the antilinear operator, $\varsigma = U \circ h_{H_C \oplus H_C}$ on $H_C \oplus H_C$ (resp. $\varsigma = \bar{U} \circ h_{H_C}$ on $H_C$), is constant over $X$. We also know that it is even/odd and satisfies $\varsigma^2 = \pm 1$ if the o-isomorphism is even/odd and obeys $U = \pm\tau^*U^t$ (resp. $\bar{U} = \pm\tau^*\bar{U}^t$). Using (8.9) and the hermiticity of the tachyon, we see that the condition on the tachyon reads as

$$\tau^*T = (-1)^{\frac{k}{2}}\varsigma \circ T \circ \varsigma^{-1} \quad \text{(resp. } \tau^*\bar{T} = (-1)^{\frac{k+1}{2}}\varsigma \circ \bar{T} \circ \varsigma^{-1}). \quad (8.14)$$

Let us see what it means for each $k \in \mathbb{Z}/8\mathbb{Z}$.

$k = 0$

The antilinear operator $\varsigma$ on $H_C \oplus H_C$ is even and squares to 1. Therefore it introduces the structure of the complexification of a real Hilbert space $H_R$ in each of the first and the second Hilbert spaces, and $\varsigma$ is simply the complex conjugation operator. The condition of the tachyon is therefore $\tau^*T = T^*$. In particular, $T := T_{10} : H_R \otimes \mathbb{C} \to H_R \otimes \mathbb{C}$ also satisfies

$$\tau^*T = T^*.$$
The antilinear operator \( \varsigma \) on \( H_C \) squares to 1. Thus \( H_C \) has the structure of the complexification of a real Hilbert space \( H_R \) and \( \varsigma \) is simply the complex conjugation operator. The condition for \( T := \check{T} : H_R \otimes C \to H_R \otimes C \) is therefore
\[
\tau^* T = -T^* \quad (resp. \quad \tau^* T = T^*).
\]

The antilinear operator \( \varsigma \) on \( H_C \oplus H_C \) is odd and squares to 1 (resp. \(-1\)). Note that
\[
\varsigma T \varsigma^{-1} = \varsigma T \varsigma^{-1} = \varsigma T^\dagger \varsigma^\dagger = \varsigma (\varsigma T)^\dagger,
\]
where we have used the hermiticity of \( T \) and unitarity (8.13) of \( \varsigma \). Thus, \( T \to -\varsigma T \varsigma^{-1} \) does \( \varsigma T \to -\varsigma^2 (\varsigma T)^\dagger = - (\varsigma T)^\dagger \) (resp. \( (\varsigma T)^\dagger \)). Note that \( \varsigma T \) is even and antilinear. Let us denote by \( T : H_C \to H_C \) its restriction to the first copy of \( H_C \). The condition on the tachyon reads for this antilinear operator as
\[
\tau^* T = -T^\dagger \quad (resp. \quad \tau^* T = T^\dagger).
\]

The antilinear operator \( \varsigma \) on \( H_C \oplus H_C \) is even and squares to \(-1\). Therefore it introduces the structure of a quaternionic Hilbert space, \( H_C = H_H \), and \( \varsigma \) is multiplication by \( j \in H \). The condition for \( T := \check{T} : H_H \to H_H \) is therefore
\[
\tau^* T = j T j^{-1} \quad (resp. \quad \tau^* T = -j T j^{-1}).
\]

The antilinear operator \( \varsigma \) on \( H_C \oplus H_C \) is even and squares to \(-1\). Therefore it introduces the structure of a quaternionic Hilbert space \( H_H \) in each of the first and the second Hilbert spaces, and \( \varsigma \) is simply multiplication by \( j \in H \). The condition of the tachyon is therefore \( \tau^* T = j T j^{-1} \). In particular, \( T := T_{10} : H_H \to H_H \) also satisfies
\[
\tau^* T = j T j^{-1}.
\]

Let us introduce a space with an involution, \( \mathcal{T}^k = \mathcal{T}^k(H) \), consisting of certain type
of Fredholm operators on a Hilbert space $H$ as follows:

| $k$ | Fredholm operators | The involution |
|-----|--------------------|----------------|
| 0   | $T : H_R \otimes C \rightarrow H_R \otimes C$, C-linear | $J : T \mapsto T^*$ |
| 1   | $T : H_R \otimes C \rightarrow H_R \otimes C$, C-linear, self-adjoint, ipin | $J : T \mapsto -T^*$ |
| 2   | $T : H_C \rightarrow H_C$, C-antilinear | $J : T \mapsto -T^\dagger$ |
| 3   | $T : H_H \rightarrow H_H$, C-linear, self-adjoint, ipin | $J : T \mapsto j T j^{-1}$ |
| 4   | $T : H_H \rightarrow H_H$, C-linear | $J : T \mapsto j T j^{-1}$ |
| 5   | $T : H_C \rightarrow H_C$, C-antilinear | $J : T \mapsto -j T j^{-1}$ |
| 6   | $T : H_C \rightarrow H_C$, C-linear, self-adjoint, ipin | $J : T \mapsto T^\dagger$ |
| 7   | $T : H_R \otimes C \rightarrow H_R \otimes C$, C-linear, self-adjoint, ipin | $J : T \mapsto T^*$ |

Here “ipin” means that the operator has infinitely many positive and infinitely many negative eigenvalues. We have seen that the tachyon determines a map $T$ from $X$ to the space $\mathcal{T}^k$ which is equivariant with respect to the involution $\tau$ on $X$ and the involution $J$ on $\mathcal{T}^k$, that is, $T(\tau(x)) = J(T(x))$. The set of such $\mathbb{Z}_2$-equivariant maps modulo $\mathbb{Z}_2$-equivariant homotopies, denoted by

$$[X, \mathcal{T}^k]_{\mathbb{Z}_2}, \quad (8.15)$$

classifies the topology of D-branes in the orientifold.

### 8.2 Clifford Algebras

We shall show that the spaces with involution, $\mathcal{T}^k$, can be characterized in terms of Clifford algebras.

Let $C_n$ be the Clifford algebra over $R$ generated by $J_1, \ldots, J_n$ which obey the relations $\{J_i, J_j\} = -2 \delta_{i,j}$. It is isomorphic to the following algebra:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|---|
| $C_n$ | $R$ | $C$ | $H$ | $H \oplus H$ | $H(2)$ | $C(4)$ | $R(8)$ | $R(8) \oplus R(8)$ |

$k(m)$ stands for the algebra of $m \times m$ matrices over a field $k$. Also, $C_{n+8} \cong C_n(16)$ where the operation $\mathcal{A} \mapsto \mathcal{A}(16)$ replaces each simple factor $k(m)$ of an algebra $\mathcal{A}$ by $k(16m)$. For example, $C_8 \cong R(16)$, $C_{11} \cong H(16) \oplus H(16)$, etc. Irreducible representations of $C_n$ are $k^m$ for each simple factor $k(m)$ and any representation consists of a sum of these. We refer to the paper [24] by Atiyah-Bott-Shapiro for these facts.
For $k \geq 1$, let $H_R$ be a real Hilbert space with a representation of $C_{k-1}$ such that the generators $J_i$ act as skew-adjoint operators and each irreducible representation has infinite multiplicity. Let $\mathcal{P}_*^k(H_R \otimes C)$ be the space of skew-adjoint Fredholm operators $A$ on $H_R \otimes C$ that satisfy the following conditions:

(i) $A$ anticommutes with $J_1, \ldots, J_{k-1}$.

(ii) For odd $k$, (i) implies that the operator $w(A) = J_1 J_2 \cdots J_{k-1} A$ commutes with $J_1, \ldots, J_{k-1}$ and $A$. It is self-adjoint for $k = -1 \bmod 4$ and skew-adjoint for $k = 1 \bmod 4$. The condition is that $w(A)$ (resp. $i^{-1} w(A)$) has infinitely many positive and infinitely many negative eigenvalues.

Conditions (i) and (ii) are vacuous for $k = 1$, and hence we have $\mathcal{P}_*^1(H_R \otimes C) = \hat{\mathcal{P}}_*(H_R \otimes C)$. For $k = 0$, by definition, we put $\mathcal{P}_*^0(H_R \otimes C) := \mathcal{F}(H_R \otimes C)$ for a real Hilbert space $H_R$. An important point for us is that the complex conjugation, $A \mapsto A^*$, defines an involution of the space $\mathcal{P}_*^k(H_R \otimes C)$. Indeed Fredholm property and the conditions (i), (ii) are invariant under the complex conjugation since $\text{Ker} A^* = (\text{Ker} A)^*$, $J_i^* = J_i$ and $w(A^*) = w(A)^*$.

The Hilbert space $H_R$ with a representation of $C_{k-1}$ of the above type is unique up to isomorphisms. Therefore we may simply write $\mathcal{P}_*^k$ for $\mathcal{P}_*^k(H_R \otimes C)$. Also, because of the mod 8 periodicity of the Clifford algebras and their representations, we have $\mathcal{P}_*^{k+8} \cong \mathcal{P}_*^k$.

We claim that $\mathcal{P}^k$ can be identified with $\mathcal{P}_*^k$ as the space with involution. In fact, we shall provide a bijection

$$\mathcal{P}_*^k(H_R \otimes C) \xrightarrow{\cong} \mathcal{P}^k(H)$$

(8.16)

for a certain Hilbert space $H$ related to $H_R$, which is equivariant with respect to the complex conjugation of $\mathcal{P}_*^k(H_R \otimes C)$ and the involution $J$ of $\mathcal{P}^k(H)$. The proof that the map indeed sends $\mathcal{P}_*^k(H_R \otimes C)$ to $\mathcal{P}^k(H)$, is $\mathbb{Z}_2$-equivariant and is bijective is straightforward and is left as an exercise for the reader.

$k = 0$

$\mathcal{P}_*^0(H_R \otimes C)$ and $\mathcal{P}^0(H_R \otimes C)$ are identical as the space with involution.
The map is $\mathcal{F}_k^k(\mathbb{H}_R \otimes \mathbb{C}) \longrightarrow \mathcal{T}_k(\mathbb{H}_R \otimes \mathbb{C})$ given by $A \mapsto T = i^{-1}A$.

For $k = 2$

By $C_1 \cong \mathbb{C}$ (where $J_1 \leftrightarrow i$), the Hilbert space $\mathbb{H}_R$ itself has the structure of a complex Hilbert space, $\mathbb{H}_R = \mathbb{H}_C$. The map is

$$A = A_1 + iA_2 \in \mathcal{F}_k^2(\mathbb{H}_R \otimes \mathbb{C}) \longrightarrow T = A_1 + iA_2 \in \mathcal{T}_2(\mathbb{H}_C),$$

where $A_1$ and $A_2$ are the real and the imaginary part of $A$ (we shall use this notation in what follows as well).

For $k = 3$

By $C_2 \cong \mathbb{H}$ (say, $J_1 \leftrightarrow i$, $J_2 \leftrightarrow j$ and $J_1J_2 \leftrightarrow k$), the Hilbert space $\mathbb{H}_R$ can be regarded as a quaternionic Hilbert space, $\mathbb{H}_R = \mathbb{H}_H$. The map is

$$A = A_1 + iA_2 \in \mathcal{F}_k^3(\mathbb{H}_R \otimes \mathbb{C}) \longrightarrow T = w(A_1) + iw(A_2) \in \mathcal{T}_3(\mathbb{H}_H).$$

For $k = 4$

By $C_3 \cong \mathbb{H} \oplus \mathbb{H}$ (say, $J_1 \leftrightarrow (i, -i)$, $J_2 \leftrightarrow (j, -j)$ and $J_3 \leftrightarrow (k, -k)$) and the assumption on the multiplicity of the irreducible representations of the simple factors, we may write $\mathbb{H}_R = \mathbb{H}_H^+ \oplus \mathbb{H}_H^-$, where $\mathbb{H}_H^+$ (resp. $\mathbb{H}_H^-$) is the subspace on which the first (resp. second) $\mathbb{H}$ acts non-trivially. Note that any operator from $\mathbb{H}_H^+$ to $\mathbb{H}_H^-$ that anticommutes with $J_i$'s is $\mathbb{H}$-linear. The map is

$$A = A_1 + iA_2 \in \mathcal{F}_k^4(\mathbb{H}_R \otimes \mathbb{C}) \longrightarrow T = A_1 + iA_2 \in \mathcal{T}_4(\mathbb{H}_H^+, \mathbb{H}_H^-).$$

For $k = 5$

By $C_4 \cong \mathbb{H}(2)$, we may write $\mathbb{H}_R = \mathbb{H}^2 \otimes \mathbb{H} \mathbb{H}_H$ for a quaternionic Hilbert space $\mathbb{H}_H$. For an operator $f$ of $\mathbb{H}_R$ that commutes with $C_4$, there is an $\mathbb{H}$-linear operator $\hat{f}$ of $\mathbb{H}_H$ such that $f = \text{id}_{\mathbb{H}_2} \otimes \hat{f}$. The map to

$$A = A_1 + iA_2 \in \mathcal{F}_k^5(\mathbb{H}_R \otimes \mathbb{C}) \longrightarrow T = i^{-1}\left(\overline{w(A_1)} + i\overline{w(A_2)}\right) \in \mathcal{T}_5(\mathbb{H}_H).$$
By $C_5 \cong C(4)$, we may write $H_R = C^i \otimes_C H_C$ for a complex Hilbert space $H_C$. Also, the automorphism of $C_5$ that flips the sign of $J_i$’s can be mapped to the automorphism $\varphi \mapsto J \varphi^* J^{-1}$ of $C(4)$ where $J$ is a real antisymmetric $4 \times 4$ matrix. For an operator $f$ of $H_R$ that anticommutes with $J_i$’s, there is a $C$-antilinear operator $\hat{f}$ of $H_C$ such that $f(v \otimes h) = J(v^*) \otimes \hat{f}(h)$. The map is

$$A = A_1 + iA_2 \in \mathcal{F}_\ast^6(H_R \otimes C) \mapsto T = \hat{A}_1 + i\hat{A}_2 \in \mathcal{F}^6(H_C).$$

$k = 7$

By $C_6 \cong R(8)$, we may write $H_R = R^8 \otimes_R H'_R$ for a real Hilbert space $H'_R$. For an operator $f$ of $H_R$ that commutes with $C_6$, there is an $R$-linear operator $\hat{f}$ of $H'_R$ such that $f = \text{id}_{R^8} \otimes \hat{f}$. The map is

$$A = A_1 + iA_2 \in \mathcal{F}_7^7(H_R \otimes C) \mapsto T = \hat{w}(A_1) + i\hat{w}(A_2) \in \mathcal{F}^7(H'_R \otimes C).$$

### 8.3 Fredholm Operators And K-Theory

Let $K(X)$ be the Grothendieck group of the category of finite rank complex vector bundles over a compact space $X$. It may be defined as the set of pairs $(E^0, E^1)$ of isomorphism classes of vector bundles modulo the equivalence relations generated by $(E^0, E^1) \sim (E^0 \oplus F, E^1 \oplus F)$. It is a group under the sum $(E^0, E^1) + (F^0, F^1) := (E^0 \oplus F^0, E^1 \oplus F^1)$ — the zero is $(F, F)$, and the negative is $-(E^0, E^1) = (E^1, E^0)$. For a space with a base point, $X = (X, x_0)$, we denote by $\tilde{K}(X)$ the subgroup of $K(X)$ consisting of elements that restrict to zero at the base point $x_0$. For a closed subspace $Y$ of a space $X$, we denote by $X/Y$ the space with a base point obtained from $X$ by contracting $Y$ to one point which becomes the base point. Then we put $K(X, Y) := \tilde{K}(X/Y)$. For a space with a base point $X = (X, x_0)$, we define its reduced suspension by

$$S^iX = \frac{I \times X}{(\partial I \times X) \cup (I \times x_0)}.$$ 

$S^iX$ denotes the $i$-times operation of $S$ on $X$. For a space $X$, we denote by $X^+$ the disjoint union of $X$ and a point $*$ which is regarded as the base point of $X^+$. It is easy to see that $S^iX^+ = (I^i \times X)/(\partial I^i \times X)$. We put

$$K^{-i}(X) := \tilde{K}(S^iX^+) = K(I^i \times X, \partial I^i \times X),$$

$$K^{-i}(X, Y) := \tilde{K}(S^i(X/Y)).$$

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For a space $X$ with an involution $\tau : X \to X$, a \textit{Real vector bundle} $E$ is a complex vector bundle with an antilinear involution over $\tau$, i.e., the involution is an antilinear map of the fibre over $x$ to the fibre over $\tau(x)$. The Grothendieck group of the category of finite rank Real vector bundles over a compact space with involution, $X = (X, \tau)$, is denoted by $\text{KR}(X)$. We also have $\text{KR}(X, Y)$ for an invariant closed subspace $Y$ of $X$. We define $\text{KR}^{-i}(X), \text{KR}^{-i}(X, Y)$ as before where we assume trivial action of the involution on the $I$ factors. When the involution $\tau$ is the identity map of $X$, a Real vector bundle $E$ is the complexification of a real vector bundle on which the involution acts as the complex conjugation. See the paragraph below (8.11). Hence the category of Real vector bundles is identical to the category of real vector bundles. In such a case, we write $\text{KO}^{-i}$ for $\text{KR}^{-i}$.

These K-theory functors enjoy the property of generalized cohomology theory, such as the long exact sequence for a pair $Y \subset X$ and Mayer-Vietoris property. They also obey Bott periodicity, $\text{K}^{-i} \cong \text{K}^{-(i+2)}$ and $\text{KR}^{-i} \cong \text{KR}^{-i+8}$.

D-brane charges in Type I, Type IIB and Type IIA string theories are classified by K-theories, $\text{KO}, \text{K}$, and $\text{K}^{-1}$ respectively, as proposed and shown in [10, 11] (for I and IIB) and in [11, 12] (for IIA). It was also proposed and argued in [11, 15, 16] that D-brane charges in Type II orientifold with $\text{Op}^-$ (resp. $\text{Op}^+$) planes only is classified by $\text{KR}^{-9-p}$ (resp. $\text{KR}^{-5-p}$).

8.3.1 The Theorem of Atiyah and Jänich

The interpretation of $\text{K}(X)$ as the lattice of D-brane charges in Type IIB string theory on $X$ is very natural — the pair $(E^0, E^1)$ consists of the Chan-Paton vector bundles supported on branes and antibranes, and the relation $(E^0, E^1) \sim (E^0 \oplus F, E^1 \oplus F)$ corresponds to brane-antibrane creation and annihilation. On the other hand, we have seen that the semi-group $[X, \mathscr{F}(H_C)]$ classifies the topology of D-branes in the same theory. This implies that the two classifying (semi-)groups, $\text{K}(X)$ and $[X, \mathscr{F}(H_C)]$, must be equivalent in some way.

In fact, a direct link between the two (semi-)groups had been established a long time ago (around 1964) by Atiyah and Jänich [9, 57]: \textit{There is a natural isomorphism of semi-groups}

\[ \text{index} : [X, \mathscr{F}(H_C)] \longrightarrow \text{K}(X). \]  

(8.17)

For a map $T : X \to \mathscr{F}(H_C)$, the kernel and the cokernel of $T_x = T(x)$ are finite dimensional for any $x \in X$. If their dimensions are constant, they form vector bundles, denoted by $\text{Ker}(T)$ and $\text{Coker}(T)$. Then, we put $\text{index}(T) = (\text{Ker}(T), \text{Coker}(T)) \in \text{K}(X)$. In gen-
eral, the dimensions of Ker $T_x$ and Coker $T_x$ may jump as $x$ varies while their difference, the index of $T_x$, stays the same. In such a case, $\text{index}(T)$ is defined as follows [9]. First, for any point $x \in X$ we denote by $V_x$ the orthogonal complement of Ker $T_x$ in $H_C$. One can show that there is an open neighborhood $U_x$ of $x$ such that Ker $T_y \cap V_x = \{0\}$ for any $y \in U_x$ and that the family of vector spaces $H_C/T_y(V_x)$ parametrized by $y \in U_x$ form a trivial vector bundle over $U_x$. Since $X$ is compact, it can be covered by finite number of such open subsets, say $U_{x_i}$'s. Denote the intersection of all $V_{x_i}$'s by $V$. It is a closed subspace of $H_C$ of finite codimension. Then, we have Ker $T_x \cap V = \{0\}$ for any $x \in X$ and the family of vector spaces $H_C/T_x(V)$ form a vector bundle over $X$, denoted by $H_C/T(V)$. The index map is injective since the kernel can be shown to be equal to $[X, GL(H_C)]$, and that is one point by Kuiper's theorem.

The definition of the index map is very natural from the tachyon condensation picture. Also, we had already constructed the inverse map: Given $(E^0, E^1) \in K(X)$ we can find a family of Fredholm operators over $X$ by (8.1) in which we may set $T_{10} = 0$. It is not difficult to see that the resulting family gives rise to $(E^0, E^1)$ under the index map. Thus, the index map is precisely what we expected as the relation between the two (semi-)groups through D9-brane configurations in Type IIB string theory.

The same holds for D-branes in Type I string theory. The relevant map is

$$\text{index} : [X, \widehat{\mathcal{F}}(H_R)] \longrightarrow KO(X). \quad (8.18)$$

### 8.3.2 The Atiyah-Singer Theorem

The interpretation of $K^{-1}(X)$ as the group of D-brane charges in Type IIA string theory is less straightforward. In fact, it is best to go through $[X, \widehat{\mathcal{F}}_*(H_C)]$ which we had already established in Section 8.1.2 as the set that classifies the topology of D-branes. Again, the relevant mathematical fact for us had been obtained a long time ago (1969) by Atiyah-Singer [25]: There is a homotopy equivalence

$$\alpha : \widehat{\mathcal{F}}_*(H_C) \longrightarrow \Omega \mathcal{F}(H_C), \quad (8.19)$$

where $\Omega \mathcal{F}(H_C)$ is the based loop space at id$_{H_C}$, i.e., the space of maps $f : I \longrightarrow \mathcal{F}(H_C)$ with the boundary condition $f(0) = f(1) = \text{id}_{H_C}$. Note that $[X, \Omega \mathcal{F}(H_C)]$ is identical to the set $[SX^+, \mathcal{F}(H_C)]_0$ of homotopy classes of maps from $SX^+ = (I \times X)/(\partial I \times X)$ to $\mathcal{F}(H_C)$ that send the base point $\partial I \times X$ to the base point id$_{H_C}$. By the index map, the latter set is mapped bijectively onto $\tilde{K}(SX^+) = K^{-1}(X)$. Thus, we find a natural bijection

$$[X, \widehat{\mathcal{F}}_*(H_C)] \xrightarrow{\alpha} [X, \Omega \mathcal{F}(H_C)] = [SX^+, \mathcal{F}(H_C)]_0 \xrightarrow{\text{index}} K^{-1}(X). \quad (8.20)$$
This interpretation of $K^{-1}(X)$ as the classifying set of D-brane charges was used in [15] in the definition of T-duality map. It was also revisited in [58].

The paper [25] also shows something that will be important for us. Let $H_R$ be the real Hilbert space with a $C_{k-1}$ action as in the definition of $\mathcal{F}^k_*(H_R \otimes \mathbb{C})$. We denote by $\mathcal{F}^k_*(H_R)$ the space of skew-adjoint Fredholm operators on $H_R$ satisfying (i) and, for $k = -1 \mod 4$, (ii). It can be regarded as the subspace of $\mathcal{F}^k_*(H_R \otimes \mathbb{C})$ consisting of real operators, i.e., the fixed point set of the complex conjugation on $\mathcal{F}^k_*(H_R \otimes \mathbb{C})$. Let $H_C$ be the complex analog of $H_R$, which can be realized, say, by $H_C = H_R \otimes \mathbb{C}$. We denote by $\mathcal{F}^k_*(H_C)$ the space of skew-adjoint Fredholm operators on $H_C$ satisfying (i) and (ii). When $H_C = H_R \otimes \mathbb{C}$, it is simply the same space as $\mathcal{F}^k_*(H_R \otimes \mathbb{C})$ in which we forget about the real structure. Let $H$ be such $H_R$ resp. $H_C$. By the assumption on the multiplicity of the irreducible representations, the representation of $C_{k-1}$ on $H$ extends to a representation of $C_{k+1} \supset C_{k-1}$. In particular, $J_k$, regarded as an operator of $H$, belongs to $\mathcal{F}^k_*(H)$ (condition (ii) is satisfied since $w(J_k)J_{k+1} = -J_{k+1}w(J_k)$ for odd $k$), and is taken as its base point. We put $J_0 = \text{id}_H$ for $k = 0$. Let $\Omega \mathcal{F}^{k-1}_*(H)$ be the based loop space at $J_{k-1}$. The result of [25] is that there is a homotopy equivalence

$$\alpha : \mathcal{F}^k_*(H) \longrightarrow \Omega \mathcal{F}^{k-1}_*(H),$$

given by

$$\alpha(A)(t) = \begin{cases} J_{k-1} \cos(2\pi t) + A \sin(2\pi t) & 0 \leq t \leq \frac{1}{2} \\ J_{k-1} \cos(2\pi(1-t)) + J_k \sin(2\pi(1-t)) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

The base point $J_k$ of $\mathcal{F}^k_*(H)$ is mapped to a loop that is contractible in $\mathcal{F}^{k-1}_*(H)$ to the constant loop at $J_{k-1}$. In particular, we have

$$[X, \mathcal{F}^k_*(H)] \cong [X^+, \Omega \mathcal{F}^{k-1}_*(H)] \cong [S X^+, \mathcal{F}^{k-1}_*(H)]$$

where we used the standard relation $[(X, x_0), \Omega(Y, y_0)] \cong [S(X, x_0), (Y, y_0)]$ in the last step. Applying this repeatedly, and applying the index map at the end, we find that $\mathcal{F}^k_*(H)$ is a classifying space for $KO^{-k}$ resp. $K^{-k}$:

$$[X, \mathcal{F}^k_*(H)] \cong KO^{-k}(X),$$

$$[X, \mathcal{F}^k_*(H_C)] \cong K^{-k}(X).$$

8.3.3 $\mathcal{T}^k$ and $KR^{-k}$

We shall now show that $\mathcal{T}^k \cong \mathcal{F}^k_*$ is the classifying $\mathbb{Z}_2$-space for $KR^{-k}$. We start with the case $k = 0$ where $\mathcal{F}^0_*(H_R \otimes \mathbb{C}) = \mathcal{F}(H_R \otimes \mathbb{C})$. It is known [59] that there is a natural
Let us first define this map. Note that the trivial bundle \( X \times H_R \otimes C \) has a Real structure \( \hat{\tau} : (x, h) \mapsto (\tau(x), h^*) \).

For a \( \mathbb{Z}_2 \)-map \( T : X \to \mathcal{F}(H_R \otimes C) \), i.e., \( T_{\tau(x)} = T_x^* \), the map \( \hat{\tau} \) induces Real structures in the families of vector spaces, \( \ker T_x \) and \( \text{coker } T_x \). If they define complex vector bundles, then, index(\( T \)) = (\( \ker(T), \text{coker}(T) \)) is an element of \( \text{KR}(X) \). In general, we proceed as in the definition of (8.17). We can find finite number of points \( x_i \) with the neighborhoods \( U_{x_i} \) satisfying the condition as in the complex case such that \( U_{x_i} \)'s and \( U_{\tau(x_i)} \)'s cover \( X \). Define \( V \) to be the intersection of \( V_{x_i} \)'s and \( V_{\tau(x_i)} \)'s. Since \( V_{x_i} = V_{\tau(x_i)} \), it is invariant under the complex conjugation, \( V = V \). The trivial bundle \( H_C/V \) is then a trivial Real bundle by \( \hat{\tau} \). The same map \( \hat{\tau} \) introduces the structure of a Real bundle on \( H_C/T(V) \) as well, since \( T_x(V)^* = T_x^*(V) = T_{\tau(x)}(V) \). Then we put index(\( T \)) = (\( H_C/V, H_C/T(V) \)) \( \in \text{KR}(X) \).

A Real bundle pair \( (E^0, E^1) \) over \( X \) defines a \( k = 0 \) D9-brane configuration with vanishing tachyon. Applying the construction of Section 8.1.3 to it, we find a \( \mathbb{Z}_2 \)-map \( X \to \mathcal{F}(H_R \otimes C) \) whose index is \( (E^0, E^1) \in \text{KR}(X) \). This gives a right inverse to the index map. Therefore, to see that they are bijections, we need to show that the index map (8.25) is injective.

The kernel of the map (8.25) is the set \( [X, GL(H_R \otimes C)]_{\mathbb{Z}_2} \) where the involution of \( GL(H_R \otimes C) \) is the complex conjugation, as one can show by adapting the argument used in [9]. We want to show that it consists of one point, i.e., that any \( \mathbb{Z}_2 \)-map \( f : X \to GL(H_R \otimes C) \) can be deformed by a \( \mathbb{Z}_2 \)-homotopy to the constant map to the identity element \( 1 \). This is true by the \( \mathbb{Z}_2 \)-equivariant contractibility of \( GL(H_R \otimes C) \) [60]. For a space \( X \) with a \( \tau \)-compatible cell decomposition, we may also proceed as in Section 8.1.3 where we proved that \( \bar{U}(x) \) can be made constant. First we focus on the fixed point set \( X^\tau \). On this set the \( \mathbb{Z}_2 \)-map satisfies \( f(x) = f(x)^* \), that is, it is a map into \( GL(H_R) \subset GL(H_R \otimes C) \). Using Kuiper’s theorem for \( R \) we can find a homotopy from \( f|_{X^\tau} \) to the constant map to \( 1 \). Next, we extend it to a \( \mathbb{Z}_2 \)-homotopy on \( X - X^\tau \), using a \( \tau \)-compatible cell decomposition. This is possible thanks to Kuiper’s theorem for \( C \). This shows that the set \( [X, GL(H_R \otimes C)]_{\mathbb{Z}_2} \) consists of one point, thus proving injectivity and hence bijectivity of the index map (8.25).

Let us next proceed to higher \( k \). The map (8.21) for \( H = H_R \otimes C \) is equivariant with respect to the complex conjugation. By the Atiyah-Singer theorem, it is a homotopy
equivalence of ordinary spaces and also induces a homotopy equivalence between the subspaces of $\mathbb{Z}_2$-fixed points. Then, is it a homotopy equivalence of $\mathbb{Z}_2$-spaces? A question of this type had been asked in [61–63]: Let $G$ be a compact group and let $f : Y \to Z$ be a $G$-equivariant map between $G$-spaces. Suppose $f : Y^H \to Z^H$ is a homotopy equivalence for any closed subgroup $H \subset G$. Then, is $f$ a homotopy equivalence of $G$-spaces? Affirmative answers of various levels were obtained under various additional assumptions. For us the following from Theorem (5.4) of [61], Chapter II (see also [62]) suffices: If $X$ is a $G$-space with a $G$-compatible cell decomposition, then $f$ induces a bijection $[X, Y]_G \cong [X, Z]_G$. Applying this to our problem, assuming that $X$ has a $\mathbb{Z}_2$-compatible cell decomposition, we find the bijection in the middle,

$$\begin{align*}
[X, \mathcal{F}_s^k(H \otimes C)]_{\mathbb{Z}_2} \\
\| \\
[X^+, \mathcal{F}_s^k(H \otimes C)]_{\mathbb{Z}_2, 0} \cong \ [X^+, \Omega \mathcal{F}_s^{k-1}(H \otimes C)]_{\mathbb{Z}_2, 0} \\
\| \\
[ S X^+, \mathcal{F}_s^{k-1}(H \otimes C)]_{\mathbb{Z}_2, 0}.
\end{align*}$$

Applying this repeatedly and applying the index map at the end, we find a bijection

$$[X, \mathcal{F}_s^k(H \otimes C)]_{\mathbb{Z}_2} \cong KR^{-k}(X). \quad (8.26)$$

Namely, $\mathcal{F}^k \cong \mathcal{F}_s^k(H \otimes C)$ is a classifying $\mathbb{Z}_2$-space for $KR^{-k}$.

Since we have seen that $[X, \mathcal{F}^k]_{\mathbb{Z}_2}$ classifies the topology of D-branes in the Type II orientifold on $X$, so does $KR^{-k}(X)$. In particular, the proposal of [15] is derived.

### 8.4 Type II Orientifolds With Twists

Let us now consider Type II orientifolds with non-trivial twist, in which the D9-branes have the structure as summarized in Section 3.5. The goal is to describe the classification of the topology of D-branes in terms of a certain kind of K-theory.

#### 8.4.1 Twisted Real Bundles

Let $X$ be a compact space with an involution $\tau : X \to X$. Let $\mathcal{L}$ be a complex line bundle over $X$ such that $\tau^*\overline{\mathcal{L}} \otimes \mathcal{L}$ is topologically trivial. We choose a trivialization $c : \tau^*\overline{\mathcal{L}} \otimes \mathcal{L} \to \mathbb{C}$ such that $\tau^*c$ is equal to $c^*$, the complex conjugate of $c$. We shall call a space $X$ with such data, $\tau, \mathcal{L}$ and $c$, a twisted Real space. A twisted Real vector bundle over a twisted Real space $(X, \tau, \mathcal{L}, c)$ is a complex vector bundle $E$ over $X$ equipped with
an antilinear map $\tilde{\tau} : E \to E \otimes L^{-1}$ over $\tau$ that squares to $c$. That is, we have an antilinear map $\tilde{\tau}_x : E_x \to (E \otimes L^{-1})_{\tau(x)}$ for each $x \in X$ such that the composition

$$E_x \xrightarrow{\tilde{\tau}_x} E_{\tau(x)} \otimes L^{-1}_{\tau(x)} \xrightarrow{\tilde{\tau}(x)} E_x \otimes L^{-1}_x \otimes L^{-1}_{\tau(x)}$$

(8.27)

is equal to multiplication by $c(x) \in (\tau^* \overline{L} \otimes L)^x$. An isomorphism between two twisted Real bundles, say, from $(E_1, \tilde{\tau}_1)$ to $(E_2, \tilde{\tau}_2)$, is provided by a linear map $f : E_1 \to E_2$ such that $\tilde{\tau}_{2x} \circ f_x = f_{\tau(x)} \circ \tilde{\tau}_{1x}$. We denote by $\text{KR}(X, \tau, L, c)$ or more simply by $\text{KR}(X; c)$ the Grothendieck group of the category of finite rank twisted Real bundles over the twisted Real space $(X, \tau, L, c)$. For an ordinary Real space $(X, \tau)$, i.e., when $L$ is the trivial bundle $X \times \mathbb{C}$ and if $c = 1$, then a twisted Real bundle is an ordinary Real bundle over $(X, \tau)$ and the group $\text{KR}(X; c)$ is equal to the ordinary KR group $\text{KR}(X)$. Also, we shall later define $\text{KR}^{-i}(X, \tau, L, c) = \text{KR}^{-i}(X; c)$ which agrees with $\text{KR}^{-i}(X)$ for an ordinary Real space.

Let $(B, L, \alpha, c)$ be the data for a Type II orientifold on $(X, \tau)$. Then, $(X, \tau, L, c)$ satisfies the condition for a twisted Real space. Indeed, the hermitian inner product on $L$ yields an isomorphism $\overline{L} \cong L^*(= L^{-1})$, and we recover the conditions $c : \tau^* \overline{L} \otimes L \cong \mathbb{C}$ and $\tau^* c = c^*$ from the properties (ii) and (iii) of $(L, \alpha, c)$ from Section [3.5]. Conversely, given a twisted Real space $(X, \tau, L, c)$ we can find a hermitian metric $h$ on $L$ and a unitary connection $\alpha$ of $(L, h)$ such that the properties (ii) and (iii) from Section [3.5] hold, where $c$ is regarded as a trivialization of $\tau^* L^* \otimes L$ via the isomorphism $\overline{L} \cong L^*$ provided by $h$. Thus, the part $(L, \alpha, c)$ of the orientifold data can be identified as the twisted Real structure $(L, c)$.

Given a D-brane data, say $(E, A, T, U)$ for Type IIB orientifold $(X, \tau, B, L, \alpha, c)$, we can construct a twisted Real vector bundle over the corresponding twisted Real space. Indeed, the antilinear map is defined by

$$\tilde{\tau}_x := U_{\tau(x)} \circ h_{E_x} : E_x \xrightarrow{h_{E_x}} E_x \xrightarrow{U_{\tau(x)}} (E \otimes L^{-1})_{\tau(x)}.$$  

(8.28)

That it squares to $c$ follows from the identity (8.9), the unitarity of $U$, and the condition $U = c \cdot \tau^* U^t$. Conversely, given a twisted Real bundle $(E, \tilde{\tau})$ over a twisted Real space $(X, \tau, L, c)$, we can find the $(E, U)$-part of a D-brane data for the corresponding orientifold. That is, we can find a hermitian metric $h_E$ on $E$ and a unitary map $U : \tau^* E^* \otimes L \to E$ that gives $\tilde{\tau}$ by (8.28). Indeed, if the anti-linear map $\tilde{\tau}_x$ is represented by a matrix $M(\tau x)$ with respect to a frame of $E$, its image frame of $\tau^* E$ and a unitary frame of $\tau^* L^{-1}$, then, we put $h_E = 2(1 + MM^t)^{-1}$ with respect to the same frame of $E$, and define $U$ via (8.28). Then, one can show that $U$ is indeed unitary (and obeys
$U = c \cdot \tau^* U^t$). Furthermore, there is an isomorphism between twisted Real bundles if and only if there is an isomorphism between the corresponding $(E, U)$’s. Here we say that $(E_1, U_1)$ is isomorphic to $(E_2, U_2)$ if there is a unitary map $f : E_1 \to E_2$ such that (c.f. (3.25))

$$f \circ U_1 \circ \tau^* f^t = U_2.$$  

In what follows, we shall also refer to the $(E, U)$ part of the D-brane data as a (twisted) Real bundle.

As the counterpart of (3.47), there are gauge transformations of the twisted Real structure $(L, c) \to (L \otimes L \otimes \tau^* T^{-1}, c)$ and $(L, \lambda \cdot \tau^* \lambda^\ast \cdot c)$, where $L$ is a complex line bundle and $\lambda$ is a $\mathbb{C}^\times$-valued function with $\lambda^\ast$ being its complex conjugate. The transformations

$$(E, \tilde{\tau}) \to (E \otimes L, \tilde{\tau}) \quad \text{and} \quad (E, \tau^* \lambda \cdot \tilde{\tau})$$

(8.31)

send twisted Real bundles over the original twisted Real space to those over the transformed spaces. These in particular determine isomorphisms of the group $\text{KR}(X, \tau, L, c)$ to the ones corresponding to the transformed data (8.30).

We claim that the D-brane charges in the Type II orientifold with data $(X, \tau, B, L, \alpha, c)$ are classified by the Grothendieck groups, $\text{KR}^{-i}(X; c) = \text{KR}^{-i}(X, \tau, L, c)$. To be precise,

$$(B_+) : \quad \text{KR}(X; c) \cong \text{KR}^{-4}(X; -c),$$

$$(A_-) : \quad \text{KR}^{-1}(X; c) \cong \text{KR}^{-5}(X; -c),$$

$$(B_-) : \quad \text{KR}^{-2}(X; c) \cong \text{KR}^{-6}(X; -c),$$

$$(A_+) : \quad \text{KR}^{-7}(X; c) \cong \text{KR}^{-3}(X; -c).$$

(8.32)

Recall that $(B_\pm)$ and $(A_\pm)$ label the distinction concerning whether $\tau$ is orientation preserving or not (i.e. IIB or IIA) and whether the lift $\tau_S$ to Majorana spinors squares to 1 or $-1$. The isomorphism between K-theory groups

$$\text{KR}^{-i}(X; c) \cong \text{KR}^{-i-4}(X; -c)$$

(8.33)

is the Bott periodicity in this context.

In Case $(B_+)$, it is easy to understand that the group $\text{KR}(X; c)$ appears as the classification of the D-brane charges. In this case, the Chan-Paton bundle for D9-branes is graded, $E = E^0 \oplus E^1$, and the $\circ$-isomorphism is even, $U = \text{diag}(U_0, U_1)$. Then, we have
a pair of twisted Real bundles, \((E^0, U_0), (E^1, U_1)\), which represents an element of the group \(\text{KR}(X; c)\) as long as \(E\) is of finite rank. The classification in the other three cases can also be guessed by taking the decompactification limit and matching with the classification in the untwisted cases — observe that (8.32) reduces to the result of the previous subsection when the twist \((\mathcal{L}, \alpha)\) is trivial so that \(c \equiv 1\) or \(-1\). In what follows, we shall give a derivation of the claim (8.32).

8.4.2 The Hyperbolic Bundle

As before, we add infinitely many empty branes if necessary so that we have a Hilbert bundle as the Chan-Paton bundle on which the tachyon acts as a Fredholm operator. The first step in the classification is to establish that there is no freedom in the choice of underlying Real bundle. In general, a \textit{hyperbolic bundle} is the twisted Real bundle \((H_F, U_F)\) for some complex vector bundle \(F\) where

\[
H_F = F \oplus (\tau^* F^* \otimes \mathcal{L}),
\]

\[
U_F = \begin{pmatrix} 0 & \text{id}_F \otimes c \\ \text{id}_{\tau^* F^* \otimes \mathcal{L}} & 0 \end{pmatrix}.
\] (8.34)

We will show that any twisted Real Hilbert bundle \((E, U)\) is isomorphic to a hyperbolic bundle \((H_F, U_F)\) for some Hilbert bundle \(F\). Note that \(F\) is necessarily trivial, \(F \cong \mathbb{H}_\mathbb{C}\), by Kuiper’s theorem.

When \(E\) is graded and \(U\) is odd (Case (B\(_-\))), \(E = E^0 \oplus E^1\),

\[
U = \begin{pmatrix} 0 & U_{01} \\ U_{10} & 0 \end{pmatrix}, \quad U_{01} : \tau^* E^1^* \otimes \mathcal{L} \xrightarrow{=} E^0, \quad U_{10} : \tau^* E^0^* \otimes \mathcal{L} \xrightarrow{=} E^1,
\]

\((E, U)\) is indeed isomorphic to the hyperbolic bundle \((H_{E^0}, U_{E^0})\). For Case (B\(_+\)), we need to show that both of \((E^0, U_0)\) and \((E^1, U_1)\) are isomorphic to hyperbolic bundles. The vector bundle \(\tilde{E}\) in Type IIA cases (A\(_\pm\)) is ungraded. Thus, the main task is to show the assertion when \(E\) is ungraded.

To this end, we introduce the notion of “Lagrangian subbundles”. A subbundle \(V \subset E\) is a \textit{Lagrangian} subbundle of a twisted Real bundle \((E, U)\) when the following is an exact sequence of vector bundles

\[
0 \rightarrow V \xrightarrow{i} E \xrightarrow{\tau^* \circ U^{-1}} \tau^* V^* \otimes \mathcal{L} \rightarrow 0.
\] (8.35)

Note that \(F \subset H_F\) is a Lagrangian subbundle of \((H_F, U_F)\). We shall show that (i) a twisted Real bundle having a Lagrangian subbundle \(V\) is isomorphic to the hyperbolic
bundle \((H_V, U_V)\), and (ii) a Lagrangian subbundle always exists in a twisted Real Hilbert bundle.

**Proof Of (i)**

Suppose \(V \subset E\) is a Lagrangian subbundle of \((E, U)\). Using the hermitian inner products on the vector bundles involved, we can find a splitting of the exact sequence (8.35). I.e., an exact sequence in the opposite direction

\[
0 \leftarrow V \leftarrow^t E \leftarrow^s \tau^* V^* \otimes \mathcal{L} \leftarrow 0,
\]

such that \(t \circ i = \text{id}_V\), \(j \circ s = \text{id}_{\tau^* V^* \otimes \mathcal{L}}\) and \(i \circ t + s \circ j = \text{id}_E\), for \(j = \tau^* i^t \circ U^{-1}\). Then, we have an isomorphism of vector bundles \((i, s) : V \oplus (\tau^* V^* \otimes \mathcal{L}) \to E\). This defines an isomorphism of twisted Real bundles, \((H_V, U_V) \cong (E, U)\), in the sense of (8.29). That is,

\[
(i, s) \begin{pmatrix}
0 & \text{id}_V \otimes c \\
\text{id}_{\tau^* V^* \otimes \mathcal{L}} & 0
\end{pmatrix} \begin{pmatrix}
\tau^* i^t \\
\tau^* s^t
\end{pmatrix} = U.
\]

This can be shown using the condition \(U = c \cdot \tau^* U^t\), the unitarity of \(U\) and the defining properties of the splitting.

**Proof Of (ii)**

Let \((E, U)\) be a twisted Real Hilbert bundle. The first step is to find a Lagrangian subbundle over the fixed point set \(X^\tau\) on which \(c\) can be canonically identified as a number which is either +1 or −1 in each component. On this set, \(U\) defines a non-degenerate bilinear form \(\beta : E \times E \to \mathcal{L}\) which is symmetric or antisymmetric depending on \(c \equiv +1\) or \(c \equiv -1\). Then, “Lagrangian” is in the usual sense — a subbundle of \(E\) is Lagrangian if it is equal to its own orthocomplement with respect to \(\beta\). We take a cell decomposition of \(X^\tau\) which is fine enough so that, over each cell, \(\mathcal{L}\) is trivialized and \(\beta\) takes values in \(\mathbb{C}\). We try to construct a Lagrangian subbundle recursively with respect to the dimension of the cells. The question is whether a Lagrangian subbundle over the boundary of a cell can be extended to the interior. The answer is yes because the space of all Lagrangian subspaces in \((H_\mathbb{C}, \beta)\) (the Lagrangian Grassmannian \(\Lambda(H_\mathbb{C}, \beta)\)) is contractible, as we show below. Here \(H_\mathbb{C}\) is a complex Hilbert space and \(\beta\) is a symmetric or antisymmetric bilinear form which is compatible with the inner product (i.e. the map \(H_\mathbb{C}^* \to H_\mathbb{C}\) defined by \(\beta\) is unitary).

\(\beta\) symmetric In this case, \(H_\mathbb{C}\) is a complexification of a real Hilbert space \(H_\mathbb{R}\) on which \(\beta\) agrees with the inner product. Let us take an orthogonal complex structure \(J\) of \(H_\mathbb{R}\).
For example, take an orthonormal basis \( \{ e_n \}_{n=1}^{\infty} \subset \mathbb{H}_{\mathbb{R}} \) and put \( J(e_{2m-1}) = e_{2m} \) and \( J(e_{2m}) = -e_{2m-1} \) for all \( m \). Then, we have a Lagrangian subspace \( V_J \subset \mathbb{H}_{\mathbb{C}} = \mathbb{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \) consisting of antiholomorphic vectors, \( v \in V_J \iff J(v) = -iv \). Conversely any Lagrangian \( V \subset \mathbb{H}_{\mathbb{C}} \) determines an orthogonal complex structure \( J_V \) of \( \mathbb{H}_{\mathbb{R}} \). To see this, take an orthonormal basis \( \{ v_m \}_{m=1}^{\infty} \subset V \) with respect to the hermitian inner product, and put \( e_{2m-1} = \sqrt{2} \text{Re}(v_m) \) and \( e_{2m} = \sqrt{2} \text{Im}(v_m) \). These vectors \( e_n \) define an orthonormal basis of \( \mathbb{H}_{\mathbb{R}} \). The complex structure \( J_V \) is obtained by applying the above construction to this basis \( \{ e_n \}_{n=1}^{\infty} \). It is straightforward to see that \( J \mapsto V_J \) and \( V \mapsto J_V \) are inverse to each other. Thus, we find
\[
\Lambda(\mathbb{H}_{\mathbb{C}}, \beta) \cong O(\mathbb{H}_{\mathbb{R}})/U(\mathbb{H}_{\mathbb{R}}, J_0)
\]
where \( U(\mathbb{H}_{\mathbb{R}}, J_0) \) is the subgroup of \( O(\mathbb{H}_{\mathbb{R}}) \) that commutes with a fixed complex structure \( J_0 \) and is isomorphic to the group \( U(V_{J_0}) \) of unitary operators of the Hilbert space \( V_{J_0} \). Both \( O(\mathbb{H}_{\mathbb{R}}) \) and \( U(V_{J_0}) \) are contractible by Kuiper’s theorem and hence so is the Grassmannian.

\( \beta \) antisymmetric In this case, \( \mathbb{H}_{\mathbb{C}} \) has the structure of a quaternionic Hilbert space \( \mathbb{H}_{\mathbb{H}} \) (with \( \mathbb{H} \) acting from the right, say), such that the quaternionic and hermitian inner products are related by \( (v, w)_{\mathbb{H}} = (v, w)_{\mathbb{C}} - j\beta(v, w) \). Let us take an orthonormal basis \( \{ e_n \}_{n=1}^{\infty} \) of \( \mathbb{H}_{\mathbb{H}} \). Then, it spans over \( \mathbb{C} \) a Lagrangian subspace of \( (\mathbb{H}_{\mathbb{C}}, \beta) \). Conversely, given a Lagrangian subspace \( V \) of \( (\mathbb{H}_{\mathbb{C}}, \beta) \) choose an orthonormal basis of \( V \) with respect to the hermitian inner product \( (\ , \)\)\( _{\mathbb{C}} \). Then it is an orthonormal basis of \( \mathbb{H}_{\mathbb{H}} \). Therefore, we have a one to one correspondence between Lagrangian subspaces of \( (\mathbb{H}_{\mathbb{C}}, \beta) \) and orthonormal bases of \( \mathbb{H}_{\mathbb{H}} \) up to complex unitary base changes. This shows
\[
\Lambda(\mathbb{H}_{\mathbb{C}}, \beta) \cong USp(\mathbb{H}_{\mathbb{H}})/U(V_0)
\]
where \( V_0 \) is a fixed Lagrangian subspace of \( (\mathbb{H}_{\mathbb{C}}, \beta) \). Both \( USp(\mathbb{H}_{\mathbb{H}}) \) and \( U(V_0) \) are contractible by Kuiper’s theorem and hence so is the Grassmannian.

Having constructed a Lagrangian subbundle on each component of the fixed point set \( X^\tau \), the next task is to extend it over the entire space \( X \). We do it by recursive construction using a \( \mathbb{Z}_2 \)-compatible cell decomposition, as in Section 8.1.3. By a moment of thought, we see that there is no obstruction in the recursive process. This establishes the existence of a Lagrangian subbundle of \( (E, U) \).

8.4.3 The Classification

Since there is no freedom in the underlying twisted Real bundle, the focus of the classification of D-branes is that of the tachyon configurations. Let us write down the
condition for the tachyons, $T$ (for Type IIB) or $\hat{T}$ (for Type IIA), which is a Fredholm
operator at each point $x \in X$.

(B+) We can take $(E^0, U_0) = (E^1, U_1) = (H_{H_C}, U_{H_C})$, and the condition $T = U_{H_C} \tau^* T^i U_{H_C}^{-1}$
reads for $T_{10} : H_{H_C} \to H_{H_C}$ as

$$T_{10} = U_{H_C} \circ \tau^* T_{10}^i \circ U_{H_C}^{-1}. \quad (8.36)$$

(B-) We can take $(E, U) = (H_{H_C}, U_{H_C})$ and the condition is $T = -U_{H_C} \tau^* T^i U_{H_C}^{-1}$. In
particular it reads as

$$T_{10} = -c^{-1} \cdot \tau^* T_{10}^i. \quad (8.37)$$

(A±) We can take $(\dot{E}, \dot{U}) = (H_{H_C}, U_{H_C})$ and the condition $\dot{T} = \pm \dot{U} \tau^* T^i \dot{U}^{-1}$ reads for
$\dot{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as

$$A = \pm \tau^* D^i, \quad B = \pm c \cdot \tau^* B^i. \quad (8.38)$$

Note that $A = A^\dagger, C = B^\dagger, D = D^\dagger$ by the hermiticity of $\dot{T}$. We recall that, at each point
$x, \dot{T}(x)$ must have infinitely many positive and infinitely many negative eigenvalues.

These conditions can be recast into another form using Clifford algebras. Let $\mathcal{F}(H_{H_C}) = \mathcal{F}_*^0(H_{H_C})$ be
the bundle of Fredholm operators on $H_{H_C}$ — the fibre at $x \in X$ is the space of Fredholm operators on the Hilbert space $H_{H_C}|_x$. For $k \geq 1$, let $\mathcal{F}_*^k(H_{H_C})$ be the bundle of skew-adjoint Fredholm operators of $H_{H_C}$ satisfying the conditions (i) and (ii) as in
Section 8.2. Here we assume that the Clifford algebra $C_{k-1}$ acts on $H_{H_C} = H_C \oplus (H_C \otimes L)$ as

$$J_i = \begin{pmatrix} J_i & 0 \\ 0 & -J_i \otimes \text{id} \end{pmatrix}, \quad i = 1, \ldots, k-1, \quad (8.39)$$

where $J_i$ are skew-adjoint $C$-linear operators on $H_C$ that determine a complex representation of $C_{k-1}$ on $H_C$. We assume that each irreducible representation $C_{k-1}$ occurs in $H_C$ with infinite multiplicity. These $J_i$’s are chosen so that they commute with the twisted
Real structure $\hat{\tau}$ determined by $U_{H_C}$,

$$\hat{\tau} \circ J_i = J_i \circ \hat{\tau}.$$

It follows that the conjugation $A \mapsto \hat{\tau} \circ A \circ \hat{\tau}^{-1}$ defines an involution of $\mathcal{F}_*^k(H_{H_C})$ over
the one $\tau$ on the base $X$.

The assumption on the multiplicity of $C_{k-1}$ representations in $H_C$ is vacuous when $(k-1)$ is even since there is only one irreducible representation. When $(k-1)$ is odd, there
are the two representations of $C_{k-1}$ distinguished by the value of the center $J_1 \cdots J_{k-1}$.

Since 
\[ (-J_1^t) \cdots (-J_{k-1}^t) = (-1)^{\frac{k(k-1)}{2}} (J_1 \cdots J_{k-1})^t, \]
the representations determined by $J_i$’s and $-J_i^t$’s are the same for $(k - 1) = 3 \mod 4$ and are opposite for $(k - 1) = 1 \mod 4$. This means that the assumption on the multiplicity in $H_C$ is unnecessary when $(k - 1) = 1 \mod 4$. To see this, suppose that only one of the two representations occurs in $H_C$. Then, $H_C \otimes L$ consists of the other representation. Let us take a decomposition $H_C = H_1 \oplus H_2$, where $H_1$ and $H_2$ are both infinite dimensional and invariant under $C_{k-1}$. We choose a trivialization $H'_C \otimes L \cong H'_C$ and put $H'_C = H_1 \oplus H_2$. Then, $(H_{H_C}, U_{H_C})$ is isomorphic to $(H'_C, U_{H'_C})$ as a twisted Real bundle and each of the two representations occurs in $H'_C$ with infinite multiplicity. This does not happen when $(k - 1) = 3 \mod 4$ — the assumption on the multiplicity in $H_C$ is necessary.

When the twist $L$ is trivial and $c = 1$, $\mathcal{F}^k(H_{H_C})$ is the trivial bundle with fibre $\mathcal{F}^k_*(H_R \otimes C)$ as defined in Section 8.2. To see this, we choose a complex structure $J$ of $H_R$ that commutes with the $C_{k-1}$ generators. Then, we identify $H_C$ resp. $H'_C \otimes C$ as the complex subspaces $H_{R,0}^1$ resp. $H_{R,0}^{0,1}$ of $H_R \otimes C$ consisting of vectors satisfying $Jv = -\tau v$ resp. $Jv = -iv$, so that $H_{H_C} = H_R \otimes C$. Under this identification, the conjugation by $\hat{\tau}$ is equal to the complex conjugation.

We shall consider sections of $\mathcal{F}^k_*(H_{H_C})$ that are equivariant with respect to the involution $\tau$ on $X$ and the conjugation by $\hat{\tau}$, i.e.,
\[ \tau^* A = \hat{\tau} \circ A \circ \hat{\tau}^{-1}, \] (8.40)
or equivalently, $A = U_{H_C} \circ \tau^* A \circ U_{H_C}^{-1}$. We denote by $\Gamma(X, \mathcal{F}^k_*(H_{H_C}))_{Z_2(c)}$ the space of such sections. Now we can state the main result on the classification: Tachyon configurations are in one to one correspondence with equivariant sections of Fredholm bundles of the following types

\begin{align*}
(B_+) : \quad & \Gamma(X, \mathcal{F}(H_{H_C}))_{Z_2(c)}, \\
(A_-) : \quad & \Gamma(X, \mathcal{F}_*(H_{H_C}))_{Z_2(c)}, \\
(B_-) : \quad & \Gamma(X, \mathcal{F}^2_*(H_{H_C}))_{Z_2(c)}, \\
(A_+) : \quad & \Gamma(X, \mathcal{F}^3_*(H_{H_C}))_{Z_2(-c)}.
\end{align*}

(8.41)

For Case (B_+), the correspondence is obviously $T_{10} = A$ — the condition (8.36) is nothing but (8.40). To see the rest, let us write down the condition (8.40) for the skew adjoint operator $A$ written as $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ according to the decomposition $H_{H_C} = H_C \oplus (H'_C \otimes L)$,
\[ \alpha = -\tau^* \delta^t, \quad \beta = -c \cdot \tau^* \beta^t. \] (8.42)
Note that $\alpha = -\alpha^\dagger$, $\gamma = -\beta^\dagger$ and $\delta = -\delta^\dagger$ by the skew-adjointness of $A$.

(A) The correspondence is given by $\tilde{T} = i^{-1}A$. The condition (8.38) agrees with (8.42) and the conditions concerning the eigenvalues also match.

(B) The two complex representations of $C_1$ are $J_1 = i$ and $J_1 = -i$. By the remark above for the case $(k - 1) = 1 \mod 4$, we may assume that the operator $J_1$ on $H_{\mathbb{C}}$ is given by

$$J_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Anticommutativity $AJ_1 = -J_1A$ requires $\alpha = \delta = 0$ and the condition for $\beta$ agrees with the one for $T_{01} = T_{10}^\dagger$. Thus, we find a correspondence $T = i^{-1}A$.

(A+) We may take

$$J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

for a decomposition $H_{\mathbb{C}} = H_1 \oplus H_1$. Anticommutativity $AJ_i = -J_iA$ ($i = 1, 2$) requires

$$\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & -\alpha_1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & \beta_1 \\ \beta_1 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & \gamma_1 \\ \gamma_1 & 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} \delta_1 & 0 \\ 0 & -\delta_1 \end{pmatrix}.$$

The condition (8.42) takes the same form, $\alpha_1 = -\tau^*\delta_1^t$ and $\beta_1 = -c\cdot\tau^*\beta_1^t$. Let us compute

$$w(A) = J_1J_2A:
\begin{align*}
w(A) &= -i \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_1 & -\beta_1 \\ -\gamma_1 & -\delta_1 \\ \gamma_1 & -\delta_1 \end{pmatrix} \\ &\approx -i \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & -\delta_1 \\ \alpha_1 & \beta_1 \\ \gamma_1 & -\delta_1 \end{pmatrix},
\end{align*}$$

where we made a unitary basis change in the second equality. If we put

$$\tilde{T} = -i \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & -\delta_1 \end{pmatrix},$$

the conditions on $\tilde{T} : H_{H_1} \to H_{H_1}$ agree with those on $A \in \Gamma(X, \mathcal{F}_\ast^k(H_{H_\mathbb{C}}))_{\mathbb{Z}_2(-c)}$, including Fredholmness as well as the condition on the eigenvalues.

*Bott Periodicity*

We next show the periodicity

$$\Gamma(X, \mathcal{F}_\ast^k(H_{H_{\mathbb{C}}}))_{\mathbb{Z}_2(c)} \cong \Gamma(X, \mathcal{F}_\ast^{k+4}(H_{H_{\mathbb{C}}}))_{\mathbb{Z}_2(-c)}, \quad (8.43)$$

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which in particular leads to mod 8 periodicity. The key is the isomorphism $C_{k+3} \cong C_{k-1} \otimes C_4$

$$e_i \leftrightarrow \begin{cases} e_i \otimes e_1e_2e_3e_4 & i = 1, \ldots, k-1, \\ 1 \otimes e_{1-(k-1)} & i = k, \ldots, k+3. \end{cases} \quad (8.44)$$

Suppose $H_C$ is the Hilbert space with an admissible $C_{k-1}$ representation, i.e., each irreducible representation appears with infinite multiplicity. Then, a Hilbert space with an admissible $C_{k+3}$ representation is obtained by

$$H'_C = H_C \otimes V_4 \quad \text{where} \quad V_4 \cong C_4$$

is the (unique) irreducible representation of $C_4$. The isomorphism (8.43) is given by

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \leftrightarrow A' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} \alpha \otimes J_{1234} & \beta \otimes u \\ \gamma \otimes u^\dagger & \delta \otimes J_{1234}^t \end{pmatrix}, \quad (8.45)$$

where $J_{1234} : V_4 \rightarrow V_4$ is defined by $J_1J_2J_3J_4$ and $u : V_4^* \rightarrow V_4$ is such that $J_iu = uJ_i^t$ for $i = 1, 2, 3, 4$. It is straightforward to check that $C_{k-1}$ anticommutativity of A corresponds to $C_{k+3}$ anticommutativity of $A'$. Repeating the computation in Section 4.3, we find

$$u^t = -u,$$

from which it follows that

$$\beta = -c \cdot \tau^* \beta' \iff \beta' = c \cdot \tau^* \beta''.$$ 

Since $\alpha = -\tau^* \delta'$ is obviously equivalent to $\alpha' = -\tau^* \delta''$, we find that (8.45) indeed gives rise to the isomorphism (8.43).

**K-Theory**

Let $[X, \mathcal{F}^k(H_{H_C})]_{\mathbb{Z}_2(c)}$ be the set of connected components of the space $\Gamma(X, \mathcal{F}^k(H_{H_C}))_{\mathbb{Z}_2(c)}$. For $k = 0$ there is a bijection

$$\text{index} : [X, \mathcal{F}(H_{H_C})]_{\mathbb{Z}_2(c)} \rightarrow \text{KR}(X; c). \quad (8.46)$$

It is defined as follows. Let $T$ be an equivariant section of $\mathcal{F}(H_{H_C})$, i.e., $\hat{T}_x \circ T_x = T_{T_x} \circ \hat{T}_x$. We see that $\hat{T}_x$ maps $\text{Ker} T_x$ to $\text{Ker} T_{T(x)} \otimes \mathcal{L}^{-1}_{T(x)}$, and the same holds on $\text{Coker} T$. Thus, if Ker $T$ and Coker $T$ have constant ranks, the pair $(\text{Ker} T, \text{Coker} T)$ determines an element of $\text{KR}(X; c)$. In general, we can find a subspace $V \subset H_C$ of finite codimension such that $H_{H_C}/T(H_V)$ defines a finite rank twisted Real vector bundle, and we put $\text{index}(T) = (H_{H_C}/V, H_{H_C}/T(H_V)) \in \text{KR}(X; c)$. The proof that it is a bijection is similar to the earlier cases and is omitted here.

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For $k \geq 1$, we may put by definition

$$KR^{-k}(X; c) = [X, \mathcal{F}_k^*(H_{\mathbb{H}_{\mathbb{C}}})]_{\mathbb{Z}_2(c)}. \quad (8.47)$$

Then, the classification (8.32) and the periodicity (8.33) follow from (8.41) and (8.43). Also, when the twist $L$ is trivial and $c = 1$, $KR^{-k}(X; c)$ agrees with $KR^{-k}(X)$.

Alternatively, we may seek for a definition within the category of finite rank twisted Real bundles. Suspension cannot be used in general, since the twist $L$ may be non-trivial along a subspace that is to be contracted. Here we quote an alternative definition of the group $K(X, Y)$ as the Grotherndieck group of the category of the pair $(E, \psi)$ of a finite rank vector bundle on $X$ which is trivial over $Y$ and a trivialization $\psi : E|_Y \rightarrow Y \times \mathbb{C}^r$ over $Y$. We would like to define $KR(X, Y; c)$ in the similar way. The question is when do we say that a twisted Real bundle is trivial? Here we propose to say that $(E,U)$ is trivial when it is isomorphic to the hyperbolic bundle $(H_{\mathbb{C}^r},U_{\mathbb{C}^r})$ for the trivial vector bundle $\mathbb{C}^r$ of arbitrary rank. This is so that the index map

$$\text{index} : [(X, Y), (\mathcal{F}(H_{\mathbb{H}_{\mathbb{C}}}), \text{id})]_{\mathbb{Z}_2(c)} \rightarrow KR(X, Y; c)$$

becomes a bijection, where the domain stands for the set of connected components of the space of equivariant sections of $\mathcal{F}(H_{\mathbb{H}_{\mathbb{C}}})$ which is the identity over $Y$. Then, we define $KR^{-k}(X; c)$ by $KR(I^k \times X, \partial I^k \times X; c)$, where we extend $(L, c)$ uniformly in the $I^k$ direction.

Do the two definitions agree? The key to this question is whether there is a homotopy equivalence like (8.21) also in the twisted case. We first note that we can take $J_k$, obtained by extension of Clifford algebra action on $H_{\mathbb{C}}$, as the base section of the bundle $\mathcal{F}_k^*(H_{\mathbb{H}_{\mathbb{C}}})$ (we put $J_0 = \text{id}$ for $k = 0$). Then, the formula (8.22) defines a map

$$[(X, Y), (\mathcal{F}_k^*(H_{\mathbb{H}_{\mathbb{C}}}), J_k)]_{\mathbb{Z}_2(c)} \rightarrow [(I \times X, (I \times Y) \cup (\partial I \times X)), (\mathcal{F}^{k-1}_*(H_{\mathbb{H}_{\mathbb{C}}}), J_{k-1})]_{\mathbb{Z}_2(c)}. \quad (8.48)$$

We claim without proof that it is a bijection. If that is indeed the case, using it iteratively, we find that the two definitions agree,

$$[X, \mathcal{F}_k^*(H_{\mathbb{H}_{\mathbb{C}}})]_{\mathbb{Z}_2(c)} \cong [(I^k \times X, \partial I^k \times X), (\mathcal{F}(H_{\mathbb{H}_{\mathbb{C}}}), \text{id})]_{\mathbb{Z}_2(c)}. \quad (8.49)$$

9 \quad $\mathcal{N} = 1$ Supersymmetry

Type II orientifold on a Calabi-Yau three-fold with D-branes is one way to obtain $\mathcal{N} = 1$ supersymmetric theories in $3 + 1$ dimensions with non-zero Newton’s constant.
In this section, we shall study the structure of Chan-Paton facts for space-filling D-branes in Type IIB orientifolds by holomorphic involutions, with a focus on $\mathcal{N} = 1$ supersymmetry and categorical description.

9.1 $\mathcal{N} = 2$ Worldsheet Supersymmetry

To start with, let us focus on the supersymmetric sigma model on the internal space $M$ which we take for now to be an $n$-dimensional Kähler manifold. It has an extended $\mathcal{N} = (2,2)$ supersymmetry. We are interested in D-branes that preserve a diagonal $\mathcal{N} = 2_B$ subalgebra with $U(1)$ $R$-symmetry which acts on the worldsheet fields as

$$\delta x^i = \epsilon \psi^i, \quad \delta x^\tau = -\bar{\tau} \bar{\psi}^\tau, \quad \delta \psi^i = -2 \bar{\tau} \psi^\tau, \quad \delta \bar{\psi}^\dot{i} = 2 i \bar{\epsilon} \psi^\tau.$$

We shall summarize the condition and properties of boundary interactions with this extended supersymmetry. We refer the reader to [64] for more detail.

The Conditions

The condition for the D9-brane configuration $(E, A, T)$ to preserve the symmetry is as follows: $E$ has a $\mathbb{Z}$-grading that reduces modulo 2 to the original $\mathbb{Z}_2$-grading, the gauge field $A$ has degree 0 and has a $(1,1)$-form curvature, and the tachyon $T$ can be written as a sum $T = iQ - i\bar{Q}^\dagger$ where $Q$ has degree 1, is holomorphic $D_\tau Q = 0$, and squares to zero $Q^2 = 0$. Such a data defines a complex of holomorphic vector bundles.

$$\cdots \rightarrow \mathcal{E}^i \xrightarrow{Q} \mathcal{E}^{i+1} \xrightarrow{Q} \mathcal{E}^{i+2} \rightarrow \cdots$$

For the purpose of our discussion, it is appropriate to generalize the boundary interaction $\mathcal{A}$ so that it has higher powers of the fermions $\psi$. $\mathcal{N} = 1$ supersymmetry requires the form

$$\mathcal{A}_t = -\dot{x}^\mu \frac{\partial}{\partial \psi^\mu} T + \frac{i}{2} \psi^{i} \frac{\partial}{\partial x^\mu} T + \frac{1}{2} T^2$$

where $T$ depends both on $x$ and $\psi$. The condition of $\mathcal{N} = 2_B$ supersymmetry with $U(1)$ $R$-symmetry is that $E$ has a $\mathbb{Z}$-grading and $T = iQ - i\bar{Q}^\dagger$ where

$Q$ has degree 1,

$$\frac{\partial}{\partial \psi^i} Q = 0,$$

$$\psi^\tau \frac{\partial}{\partial x^\tau} Q + Q^2 = 0.$$
By the first two conditions, we may write

\[ Q = Q^{(0)} + Q^{(1)} + Q^{(2)} + \cdots + Q^{(n)}, \quad Q^{(p)} = \frac{1}{p!} \psi^{\tau_1} \cdots \psi^{\tau_p} Q_{\tau_1 \cdots \tau_p}(x), \] (9.7)

where \( Q_{\tau_1 \cdots \tau_p} \) has degree \((1 - p)\), i.e., it maps \( E^i \) to \( E^{i+1-p} \). For the two term case \( Q = Q^{(0)} + Q^{(1)} \), we find, after rewriting \( Q^{(0)} = Q \) and \( Q^{(1)} = i\psi^\tau A_\tau \), that the boundary interaction (9.3) agrees with the one (2.16) for \((E, A, T)\) where \( A = A^t dx^t + A^i dx^i \) and \( T = iQ - iQ^\dagger \). Also, the last equation (9.6) splits to \( Q^2 = 0, D_\tau Q = 0 \) and \( F_{\tau\bar{\tau}} = 0 \), which are indeed the condition quoted above for \((E, A, T)\). In general, (9.6) splits to \((n + 1)\) equations starting from \( Q^{(0)2} = 0 \).

**D-term Deformations And Brane-Antibrane Annihilation**

Our primary interest is the infra-red limit of the boundary interaction \( A_t \). There are two types of operations that do not change the low energy behaviour. One is the boundary D-term deformation — deformation of \( A_t \) by the terms of the form \( Q Q_\lambda \) and \( Q Q_\lambda^\dagger \) for some expressions \( \epsilon, \epsilon' \) of \( x, \psi \) and its derivatives, where

\[ iQ_\lambda = iQ_{x,\psi} \lambda + Q\lambda - (-1)^{|\lambda|} \lambda Q, \] (9.8)

\[ i\overline{Q}_\lambda = i\overline{Q}_{x,\psi} \lambda - Q^\dagger \lambda + (-1)^{|\lambda|} \lambda Q^\dagger. \] (9.9)

\( Q_{x,\psi} \) etc are the supersymmetry variations of \( x^\mu, \psi^\mu \), i.e., \( \delta = i\epsilon \overline{Q}_{x,\psi} - i\tau Q_{x,\psi} \) in (9.1). Note that any deformation of \( Q \) leads to \( \delta A_t = \frac{i}{2} Q \delta Q^\dagger - \frac{i}{2} \overline{Q} \delta Q \). In particular, deformation of \( Q \) of the form \( \delta Q = iQ \beta \) is a boundary D-term deformation. Such a \( \beta \) must be of degree 0 and be independent of \( \psi^i \) (but it can depend on \( \psi^\tau \)), in order for the deformed \( Q \) to satisfy (9.4) and (9.5). The deformation then takes the following form under which (9.6) is manifestly invariant:

\[ \delta Q = \psi^\tau \frac{\partial}{\partial x^\tau} \beta + [Q, \beta]. \] (9.10)

The other operation is brane-antibrane annihilation, which is to discard a part of D-brane that is empty in the infra-red limit. Note that the boundary interaction \( A \) includes the potential term

\[ V = \frac{1}{2} \{ Q^{(0)}, Q^{(0)\dagger} \}, \] (9.11)

where \( Q^{(0)} \) is the leading term of \( Q \). When all of its eigenvalues are positive everywhere on \( M \), then the boundary has no degree of freedom at low enough energies. Thus, such a brane is empty in the infra-red limit. Below, we will propose a more general characterization of empty branes in terms of homological algebra.
The cohomology classes of boundary NS vertex operators with respect to the supercharge $Q$ form a ring, the boundary chiral ring. It is protected from renormalization and is isomorphic to the ring of boundary chiral primary fields in the infra-red superconformal field theory. Let us describe it in the zero mode approximation.

Under the replacement $\psi^i \rightarrow dx^i$, $Q$ can be regarded as a differential form with values in endomorphisms of $E$, with $Q^{(p)} \in \Omega^{0,p}(M, Hom^{1-p}(E, E))$, and the condition (9.6) can be written as

$$\overline{\partial} Q + Q^2 = 0.$$  \hspace{1cm} (9.12)

Let us consider the open string between D-branes $B_1 = (E_1, Q_1)$ and $B_2 = (E_2, Q_2)$ satisfying the conditions above. In the zero mode approximation, NS vertex operators of canonical $R$-charge $i$ are represented by differential forms in

$$C^i(B_1, B_2) := \bigoplus_{p+q=i} \Omega^{0,p}(M, Hom^q(E_1, E_2)),$$  \hspace{1cm} (9.13)

and the supercharge $Q$ is represented by the Dolbeault type operator

$$iQ\phi = \overline{\partial}\phi + Q_2 \phi - (-1)^{\partial i}\phi Q_1.$$  \hspace{1cm} (9.14)

We denote the spaces of $Q$-closed elements and $Q$-cohomology classes by $Z^i(B_1, B_2)$ and $\mathcal{H}^i(B_1, B_2)$ respectively. For three D-branes $B_1$, $B_2$ and $B_3$, we have a product

$$\mathcal{H}^i(B_1, B_2) \times \mathcal{H}^j(B_2, B_3) \longrightarrow \mathcal{H}^{i+j}(B_1, B_3)$$  \hspace{1cm} (9.15)

induced from the obvious product in $C(B_a, B_b)$. This is the boundary chiral ring.

It is important to keep in mind that elements of $Hom^q(E, F)$ with $q$ odd are given odd statistics here — for example, they anticommute with $dx^i$’s. This is assumed in the products discussed above, including the ones in (9.12) and (9.14). However, we may work with a formulation in which no such statistics is given to odd homomorphisms. This is done, for example, by placing all $\psi^i$’s on the left of homomorphisms before identifying them as differential forms. One advantage of this choice is that $\psi^i \partial_\psi \phi$ can be identified with $\overline{\partial}\phi$ without a sign. The product in this formulation, denoted by “$\wedge$”, is related to the graded product used above by

$$\varphi_1 \varphi_2 = (-1)^{q_1p_2} \varphi_1 \wedge \varphi_2,$$  \hspace{1cm} (9.16)

for $\varphi_1 \in \Omega^{0,p_1}(M, Hom^{q_1}(F, G))$ and $\varphi_2 \in \Omega^{0,p_2}(M, Hom^{q_2}(E, F))$. This remark is particularly useful when we discuss shifts of grading, which we do next.

---

1“Canonical R-charge” is the naive, ultra-violet R-charge. It may not be the same as the actual R-charge in the infra-red conformal field theory. The terminology is after “canonical dimension”.
The “shift by one to the left”, \(B = (E, \mathcal{Q}) \mapsto B[1] = (E[1], \mathcal{Q}[1])\), is defined by
\[
E[1]^i = E^{i+1}, \quad \mathcal{Q}[1]^{(p)} = (-1)^{p+1} \mathcal{Q}^{(p)}.
\]
(9.17)

The conditions (9.4)-(9.6) are preserved under this operation. For an integer \(i\), we write \(B[i]\) for the \(i\)-times shift of \(B\). The space \(\mathcal{H}^j(B_1, B_2)\) is isomorphic to \(\mathcal{H}^{j-i}(B_1, B_2[i])\) and also to \(\mathcal{H}^{j-i}(B_1[-i], B_2)\). This follows from the isomorphisms of complexes
\[
\mathcal{C}(B_1, B_2) \cong \mathcal{C}(B_1, B_2[i])[-i] \cong \mathcal{C}(B_1[-i], B_2)[-i]
\]
\[
\phi^{(p)} \longleftrightarrow (-1)^{ip} \phi^{(p)} \longleftrightarrow (-1)^{i(j-i)} \phi^{(p)},
\]
(9.18)
where \(\phi^{(p)}\) is the \(p\)-form component of a degree \(j\) element \(\phi \in \mathcal{C}^j(B_1, B_2)\). The shift of the complexes, \(\mathcal{C} \mapsto \mathcal{C}[-i]\), is the standard one; \(\mathcal{C}[-i]^j = \mathcal{C}^{j-i}\) and \(Q[-i] = (-1)^i Q\).

To understand the significance of the signs, let us check \(\mathcal{C}(B_1, B_2) \cong \mathcal{C}(B_1[-i], B_2)[−i]\). For \(\phi = \sum_{p=0}^n \phi^{(p)} \in \mathcal{C}^j(B_1, B_2)\), we have
\[
i(Q\phi)^{(p)} = \overline{\partial} \phi^{(p-1)} + \sum_{k+l=p} \left( \mathcal{Q}_2^{(k)} \phi^{(l)} - (-1)^j \phi^{(l)} \mathcal{Q}_1^{(k)} \right)
\]
\[
= \overline{\partial} \phi^{(p-1)} + \sum_{k+l=p} \left( (-1)^{j-k} \mathcal{Q}_2^{(k)} \wedge \phi^{(l)} - (-1)^j (-1)^{(j-i)k} \phi^{(l)} \wedge \mathcal{Q}_1^{(k)} \right).
\]
If we regard \(\phi\) as an element of \(\mathcal{C}^{j-i}(B_1[-i], B_2)\), where \(\mathcal{Q}_1[-i]^{(k)} = (-1)^{i(k+1)} \mathcal{Q}_1^{(k)}\) by definition (9.17), we have
\[
i(Q\phi)^{(p)} = \overline{\partial} \phi^{(p-1)} + \sum_{k+l=p} \left( \mathcal{Q}_2^{(k)} \phi^{(l)} - (-1)^{j-i} \phi^{(l)} \mathcal{Q}_1[-i]^{(k)} \right)
\]
\[
= \overline{\partial} \phi^{(p-1)} + \sum_{k+l=p} \left( (-1)^{j-k} \mathcal{Q}_2^{(k)} \wedge \phi^{(l)} - (-1)^j (-1)^{(j-i)k} \phi^{(l)} \wedge \mathcal{Q}_1[-i]^{(k)} \right).
\]
The two indeed agree. This demonstrates the necessity of the sign \((-1)^{p+1}\) in the shift (9.17). The sign \((-1)^{(j-i)}\) in (9.18) is just to guarantee that the sign in \(Q[-i] = (-1)^i Q\) is reproduced correctly. The other relation \(\mathcal{C}(B_1, B_2) \cong \mathcal{C}(B_1, B_2[i])[-i]\) can be shown in the same way, in which the sign \((-1)^p\) in (9.18) plays a more important rôle. The apparent asymmetry in the rôle of the signs originates from the choice (9.16) of convention. Note also that there is an ambiguity in the choice of signs in (9.18) — they can be modified by factors depending only on \(i\).

Using the two isomorphisms in (9.18) repeatedly, we find an isomorphism of complexes
\[
[i] : \mathcal{C}(B_1, B_2) \xrightarrow{\cong} \mathcal{C}(B_1[i], B_2[i])
\]
\[
\phi^{(p)} \longmapsto \phi[i]^{(p)} = (-1)^{i(|\phi|+p)} \phi^{(p)}.
\]
(9.19)
The sign \((-1)^{i(|\phi|+p)}\) is uniquely determined by the condition that the shift preserves the composition rule, \((\varphi_1\varphi_2)[i] = \varphi_1[i]\varphi_2[i]\), and by \([i] \circ [j] = [i + j]\). A useful way to write the shift (9.17) and (9.19) is
\[
Q[1] = -Q|_{\psi \rightarrow -\psi} = \sigma Q\sigma, \quad \phi[1] = (-1)^{|\phi|} \phi|_{\psi \rightarrow -\psi} = \sigma_2 \phi \sigma_1. \quad (9.20)
\]

**Cone**

For an element \(\phi \in \mathcal{Z}^0(B_1, B_2)\), its cone \(\text{Cone}(\phi) = (E_\phi, Q_\phi)\) is defined by
\[
E_{\phi} = E_{\phi}[1] \oplus E_2,
Q_{\phi} = \begin{pmatrix} Q_{\phi}[1] & 0 \\ \phi & Q_2 \end{pmatrix}.
\]
in the expression for \(Q_{\phi}\) is regarded as an element of \(\mathcal{Z}^1(B_1[1], B_2)\) by (9.18). It is straightforward to check that it satisfies the condition \(\overline{\partial} Q_{\phi} + Q_{\phi}^2 = 0\).

**Zero Objects**

A D-brane \(B\) is said to be a zero object if the cohomology space vanish
\[
\mathcal{H}(B, B) = 0.
\]
It follows that \(\mathcal{H}(B, B') = \mathcal{H}(B', B) = 0\) for any brane \(B'\). For example, a complex of vector bundles is a zero object if and only if its cohomology sheaves all vanish, that is, it is an exact complex. Note that for a bounded exact complex \((\mathcal{E}, Q)\) the boundary potential \(V = \frac{1}{2}\{Q, Q^\dagger\}\) is positive definite everywhere, and therefore the corresponding D-brane is empty in the infra-red limit. This motivates us to propose that the D-brane corresponding to a zero object is infra-red empty.

**Quasi-Isomorphisms**

An element \(s \in \mathcal{Z}^0(B_1, B_2)\) is said to be a quasi-isomorphism if it represents an isomorphism in \(\mathcal{H}^0(B_1, B_2)\), that is, if it has an inverse — an element \(s^{-1} \in \mathcal{Z}^0(B_2, B_1)\) such that \(ss^{-1} \simeq \text{id}_{E_2}\) and \(s^{-1}s \simeq \text{id}_{E_1}\) where \(\simeq\) means equality modulo \(Q\)-exact terms. Multiplication by a quasi-isomorphism \(s\) induces the isomorphisms
\[
\mathcal{H}^i(B_3, B_1) \xrightarrow{\cong} \mathcal{H}^i(B_3, B_2), \quad \mathcal{H}^i(B_2, B_3) \xrightarrow{\cong} \mathcal{H}^i(B_1, B_3),
\]
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for any $B_3$. We may also define a \textit{quasi-isomorphism of degree $j$} in an analogous way — an element $s \in \mathcal{Z}^j(B_1, B_2)$ with an “inverse” $s^{-1} \in \mathcal{Z}^{-j}(B_1, B_2)$ in the same sense as above. By (9.18), it can be regarded as a quasi-isomorphism (of degree 0) from $B_1$ to $B_2[j]$, or from $B_1[-j]$ to $B_2$.

For example, let us consider D-branes given by complexes of vector bundles, $\mathcal{E}_1$ and $\mathcal{E}_2$. A cochain map $s' : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ can be regarded an element of $\mathcal{Z}^0(\mathcal{E}_1, \mathcal{E}_2)$. It is a quasi-isomorphism if it induces an isomorphism of the cohomology sheaves at each degree. In fact, this is the traditional meaning of quasi-isomorphism.

An important fact is: \textit{A Q-closed element is a quasi-isomorphism if and only if its cone is a zero object}. The proof of “only if” part is a straightforward computation. “If” part can be seen by noting that the identity of the cone must be $Q$-exact.

If there is a quasi-isomorphism from $B_1$ to $B_2$, there is a chain of D-term deformations and brane-antibrane annihilation that connects the two D-branes. This was shown in [64] for quasi-isomorphisms in the traditional sense, but the derivation there goes through for quasi-isomorphisms in the present sense as well, provided that the cone, a zero object, indeed corresponds to an empty brane which can be annihilated. In particular, the two D-branes flow to the same fixed point in the infra-red limit. In this sense a quasi-isomorphism \textit{yields an isomorphism between the low energy D-branes.}

\textit{Ramond Ground States}

Cohomological description is possible also for Ramond ground states which, in a superstring theory on $M \times \mathbb{R}^D$ ($D + 2n = 10$), give rise to massless fermions in the space-time $\mathbb{R}^D$. Let us consider the Ramond sector of the open string between two D-branes $B_i = (E_i, Q_i), i = 1, 2$. To be specific, we choose the ($-+$) spin structure of the open string. In the zero mode approximation, wavefunctions are represented by spinors with values in $\text{Hom}(E_1, E_2)$, on which $\psi^i_0$ and $\bar{\psi}_0^\dagger$ act as the Gamma matrices. On a Kähler manifold $M$, the spin bundle is isomorphic to $\sqrt{K} \otimes \bigwedge T_M^*$ where $\sqrt{K}$ is a line bundle which squares to the canonical bundle $K = \det(T_M^*)$. The choice of $\sqrt{K}$ corresponds to the choice of the spin structure of $M$. Thus, zero mode wavefunctions are elements of

$$\mathcal{C}^i_R(B_1, B_2) := \bigoplus_{p+q=i} \Omega^{0,p}(M, \sqrt{K} \otimes \text{Hom}^q(E_1, E_2)),$$

(9.22)

where $i$ shows the canonical R-charge. The supercharge $Q$ is represented by

$$iQ\phi_R = \bar{\phi}_R + Q_2\phi_R - (-1)^{\phi_R} |\phi_R|_{\psi \rightarrow i\psi},$$

(9.23)
where $|\phi_R|$ is the canonical R-charge plus $\frac{n}{2}$. We have the factor $-iQ_1|_{\psi \to i\psi}$ rather than $Q_1$ in the last term, because the left boundary has the opposite orientation, as in the case of $\mathcal{N} = 1$ supersymmetry (3.40). The spaces of $Q$-cohomology classes, which correspond to Ramond ground states, are denoted by $\mathcal{H}_R^i(B_1, B_2)$. Note that the chiral ring naturally acts on them,

$$\mathcal{H}_R^i(B_1, B_2) \times \mathcal{H}_R^j(B_2, B_3) \times \mathcal{H}_R^k(B_3, B_4) \longrightarrow \mathcal{H}_R^{i+j+k}(B_1, B_4).$$

(9.24)

The left action is the naïve one and the right action is via $\psi \to i\psi$.

### 9.2 O-Isomorphisms From Quasi-Isomorphisms

Let us consider the Type II orientifold of $M \times \mathbb{R}^D$ with respect to a holomorphic involution $\tau$ of $M$ and a twist $(\mathcal{L}, \alpha)$ which is holomorphic, $\overline{\partial}_\alpha = 0$. The parity transform (3.10) of the $\mathcal{N} = 2$ $D$-brane $B = (E, Q)$ reads

$$Q \mapsto \mathcal{P}(Q) = \varepsilon \tau^* Q^T|_{\psi \to \varepsilon \psi} + i\psi \bar{A}_\tau.$$

(9.25)

The image bundle can be given a $\mathbb{Z}$-grading compatible with the $\mathbb{Z}_2$,

$$\mathcal{P}(E)^i = \tau^* (E^{-i})^* \otimes \mathcal{L},$$

(9.26)

with respect to which $\mathcal{P}(Q)$ has degree 1. $\mathcal{P}(Q)$ is obviously independent of $\psi^i$. It also obeys (9.6): $\psi \bar{A}_\tau Q + Q^2 = 0$ implies $\psi \bar{A}_\tau Q^T - (Q^T)^2 = 0$ which, after $\psi \to \varepsilon \psi$, yields $\psi \bar{A}_\tau (\varepsilon Q^T|_{\psi \to \varepsilon \psi}) + (\varepsilon Q^T|_{\psi \to \varepsilon \psi})^2 = 0$. This relation is preserved by the pull back by the holomorphic map $\tau$ and also by the shift by the holomorphic connection $i\psi \bar{A}_\tau$. Thus, the image brane $\mathcal{P}(B)$ preserves $\mathcal{N} = 2$ supersymmetry with $U(1)$ R-symmetry.

To define the parity operator on open string states, an o-isomorphism must be specified. Let $U: \mathcal{P}(E) \to E$ be a unitary o-isomorphism of a D-brane given by a complex of vector bundles, $Q = Q + i\psi \bar{A}_\tau$. The condition that it maps the boundary interaction for $\mathcal{P}(B)$ to the one for $B$, (3.16) and (3.17), can be written as

$$QU = 0 \quad \text{and} \quad Q^\dagger U = 0$$

(9.27)

for $U$ regarded as an element of $\mathcal{C}(\mathcal{P}(B), B)$. Here $Q^\dagger$ is defined in the analogous way to (9.14). In particular, the parity operator defined via $U$ preserves $\mathcal{N} = 2$ supersymmetry, and also the $U(1)$ R-symmetry if $U$ has a definite degree.

Such an example can be obtained from a “holomorphic o-isomorphism”, i.e., an isomorphism between a complex of holomorphic vector bundles and its parity image,
\( U_{\text{hol}} : (\mathcal{P}(\mathcal{E}), \mathcal{P}(Q)) \to (\mathcal{E}, Q) \), which satisfies (3.20). Indeed, one can construct a hermitian metric of \( \mathcal{E} \) with respect to which \( U_{\text{hol}} \) is unitary: We first construct it on the \( \tau \)-fixed point set \( M^\tau \). This is possible essentially due to the fact that \( G_C/G \) is contractible, for \( G = O(N) \) or \( USp(N) \). We then extend it to the entire \( M \), recursively using a \( \tau \)-compatible cell decomposition (as in analogous constructions in Section 8). Because of the contractibility of \( G_C/G \), for \( G \) as above and \( U(N) \), the metric is unique up to continuous deformation. Being holomorphic and unitary, \( U_{\text{hol}} \) sends the hermitian connection of \( \mathcal{P}(\mathcal{E}) \) to that of \( \mathcal{E} \). In this way we obtain a unitary \( o \)-isomorphism.

What we are really interested in, however, are the infra-red fixed points of the boundary interactions and the orientifold action on them. It is enough that we have an isomorphism between the infra-red limit of the brane and its parity image. In the above example of complex of holomorphic vector bundles, a unitary \( o \)-isomorphism of course does the job, but that is not necessary. For example, we could have made a “wrong” choice of the fibre metric so that \( U_{\text{hol}} \) fails to be unitary, in which case \( \mathbf{Q}U_{\text{hol}} = 0 \) holds but \( \mathbf{Q}^T U_{\text{hol}} = 0 \) fails. Such a brane is connected to the one with a unitary \( o \)-isomorphism by a continuous deformation of the metric, and that induces an isomorphism in the infra-red limit. Therefore, we have an infra-red isomorphism between the brane and its parity image. More generally, as we discussed, a quasi-isomorphism induces an isomorphism in the infra-red superconformal field theory. Therefore, it is enough that we have a “quasi-o-isomorphism”, i.e., a quasi-isomorphism between the brane and its parity image.

It is rare that we explicitly know the infra-red isomorphism induced from a quasi-isomorphism. Therefore, a quasi-o-isomorphism is usually helpless for writing down the parity operator on all open string states. However, it does help us write down the parity operator on the chiral sector, as we now describe.

First, the parity transform of NS vertex operators in the zero mode approximation, \( \mathcal{P} : \mathcal{C}(B_1, B_2) \to \mathcal{C}(\mathcal{P}(B_2), \mathcal{P}(B_1)) \), is given by

\[
\mathcal{P}(\phi) = \tau^* \phi^T \big|_{\psi \to \varepsilon \psi}. \tag{9.28}
\]

Unlike in (3.12) we do not need the parallel transport factor \( h_\alpha \) in the zero mode sector. It has the property

\[
\mathcal{P}(i\mathbf{Q}\phi) = \tau^* (\overline{\partial} \phi + \mathbf{Q}_2 \phi - (-1)^{[\phi]} \phi \mathbf{Q}_1)^T \big|_{\psi \to \varepsilon \psi} = \tau^* ( \overline{\partial} \phi^T + (-1)^{[\phi]} \phi^T \mathbf{Q}_2^T - \mathbf{Q}_1^T \phi^T ) \big|_{\psi \to \varepsilon \psi} = \varepsilon \overline{\partial} \mathcal{P}(\phi) - \varepsilon (-1)^{[\phi]} \mathcal{P}(\phi) \mathcal{P}(\mathbf{Q}_2) + \varepsilon \mathcal{P}(\mathbf{Q}_1) \mathcal{P}(\phi) = \varepsilon i \mathbf{Q} \mathcal{P}(\phi). \tag{9.29}
\]
Note also that
\[ \mathcal{P}(\phi_1 \phi_2) = (-1)^{|\phi_1||\phi_2|} \mathcal{P}(\phi_2) \mathcal{P}(\phi_1), \]  
for \( \phi_1 \in \mathcal{C}(\mathcal{B}_2, \mathcal{B}_3) \) and \( \phi_2 \in \mathcal{C}(\mathcal{B}_1, \mathcal{B}_2) \).

Let \( \mathcal{B} \) be a D-brane and let \( \mathcal{s} : \mathcal{P}(\mathcal{B}) \to \mathcal{B} \) be a quasi-isomorphism, i.e. an element \( \mathcal{s} \in \mathcal{Z}(\mathcal{P}(\mathcal{B}), \mathcal{B}) \) with an “inverse” \( \mathcal{s}^{-1} \in \mathcal{Z}(\mathcal{B}, \mathcal{P}(\mathcal{B})) \) in the sense explained earlier. Let us put
\[ \mathcal{P}(\phi) = \mathcal{s} \mathcal{P}(\phi) \mathcal{s}^{-1} (-1)^{|\mathcal{s}| |\phi|}. \]  
The property (9.29) leads to the commutation relation
\[ \mathcal{P} \mathcal{Q} = \varepsilon \mathcal{Q} \mathcal{P}. \]  
In particular, \( \mathcal{P} \) maps \( \mathcal{Q} \)-closed resp. exact elements to \( \mathcal{Q} \)-closed resp. exact elements. Also, if \( \phi \) is \( \mathcal{Q} \)-closed, shifts of \( \mathcal{s} \) and \( \mathcal{s}^{-1} \) by \( \mathcal{Q} \)-exact terms only affect \( \mathcal{P}(\phi) \) by \( \mathcal{Q} \)-exact terms and hence do not affect its \( \mathcal{Q} \)-cohomology class. Thus, (9.31) defines a parity operator \( \mathcal{P} : \mathcal{H}(\mathcal{B}, \mathcal{B}) \to \mathcal{H}(\mathcal{B}, \mathcal{B}) \). This represents the parity action on chiral primary fields in the infra-red superconformal field theory, as \( \mathcal{Q} \) is expected to flow to the superconformal generator \( G_{-\frac{1}{2}} \). The relation (9.32) is compatible with this expectation since \( G_{-\frac{1}{2}} \) satisfies \( \mathcal{P} G_{-\frac{1}{2}} \mathcal{P}^{-1} = \varepsilon G_{-\frac{1}{2}} \).

The condition for \( \mathcal{P}^2 = (-1)^F \) can be found in the same way as in Section 3.2. By definition, \( \mathcal{P}^2(\phi) = \mathcal{s} \mathcal{P}(\mathcal{s}^{-1}) \mathcal{P}^2(\phi) \mathcal{P}(\mathcal{s}) \mathcal{s}^{-1} (-1)^{|\mathcal{s}|}. \) Using \( \mathcal{P}(\mathcal{s}^{-1})(-1)^{|\mathcal{s}|} \simeq \mathcal{P}(\mathcal{s})^{-1} \) and
\[ \mathcal{P}^2(\phi) = \phi^{TT}|_{\psi \rightarrow -\psi} = \iota \phi^{-1}|_{\psi \rightarrow -\psi} = (-1)^{|\phi|} \iota \sigma^{-1} \phi \sigma^{-1}; \]  
we find \( \mathcal{P}^2(\phi) \simeq (-1)^{|\phi|} t \phi t^{-1} \) for \( t = \mathcal{s} \mathcal{P}(\mathcal{s})^{-1} \iota \sigma^{-1} \). The condition is thus
\[ \mathcal{s} \simeq (\sigma^{-1} \otimes \mathbf{c}) \mathcal{P}(\mathcal{s}), \]  
for some nowhere vanishing holomorphic section \( \mathbf{c} \) of \( \tau^* \mathcal{L} \otimes \mathcal{L}^* \). Note that, for such a \( \mathbf{c} \), the factor \( (\sigma^{-1} \otimes \mathbf{c}) \) is a (quasi-)isomorphism from \( \mathcal{P}^2(\mathcal{B}) \) back to \( \mathcal{B} \). For a pair of D-branes with quasi-o-isomorphisms, say \( (\mathcal{B}_1, \mathbf{s}_1) \) and \( (\mathcal{B}_2, \mathbf{s}_2) \), the parity operator \( \mathcal{P} : \mathcal{H}(\mathcal{B}_1, \mathcal{B}_2) \to \mathcal{H}(\mathcal{B}_2, \mathcal{B}_1) \) is defined by \( \mathcal{P}(\phi) = \mathbf{s}_1 \mathcal{P}(\phi) \mathbf{s}_2^{-1} (-1)^{|\phi| |\mathbf{s}_2|} \). The consistency condition \( \mathcal{P}^2 = (-1)^F \) requires that \( \mathbf{s}_1 \) and \( \mathbf{s}_2 \) both satisfy (9.34) with a common \( \mathbf{c} \).

The condition (9.34) is the quasi-o-isomorphism version of the condition (3.20) for unitary o-isomorphisms. The consistency condition requires that the section \( \mathbf{c} \) in (9.34) must be equal to the one for unitary o-isomorphisms, i.e., the crosscap section, as the notation already implies. In particular, \( \mathbf{c} \) and the mod 2 degree of \( \mathbf{s} \) are correlated with the types and the dimensions of O-planes as (3.28) and (3.29).
Parity On Ramond Ground States

Let us describe the parity action on Ramond ground states. For this purpose, we must manifest the dependence on the phase \( \varepsilon = \mp i \), say, by the subscript, \( P(\varepsilon) \), \( s(\varepsilon) \) etc. Following the discussion in Section 3.4 we propose that the quasi-o-isomorphisms for the two phases are related by

\[
s(i) = \kappa s(-i)\sigma^T.
\]  

Indeed, under this relation \( s(i) \) is \( Q \)-closed as an element of \( \mathcal{C}(P(i)(B), B) \) if and only if \( s(-i) \) is \( Q \)-closed as an element of \( \mathcal{C}(P(-i)(B), B) \). That can be shown using \( P(i)(Q) = \sigma^T P(-i)(Q)\sigma^T \) which holds since \( Q \) is odd, \( \sigma Q|\psi\rightarrow\epsilon\psi \sigma = -Q \). Let \( (B_i, s_i) \) be D-branes with quasi-o-isomorphisms, for \( i = 1, 2 \). The parity \( \tilde{P} \) is represented on the zero mode Ramond sector sector \((9.22)\) by

\[
\tilde{P}(\phi_R) = s_{2(i)}\tau^*\psi^T_R \left( s_{1(-i)}|\psi\rightarrow\epsilon\psi \right) (-1)^{\phi_R||s_i}
\]  

It is straightforward to check that it commutes with the supercharge \( Q \),

\[
\tilde{P}Q = Q\tilde{P}.
\]  

In particular, \( \tilde{P} \) acts on the \( Q \)-cohomology classes. This represents the parity action on the Ramond ground states. The relation \((9.37)\) is compatible with the expectation that \( Q \) flows to the superconformal zero mode \( G_0 \) which commutes with \( \tilde{P} \).

9.2.1 The Degree Of O-Isomorphisms

Let \( B \) be a D-brane with a quasi-o-isomorphism \( s: P(B) \rightarrow B \) of a certain degree. Let us see if the brane shifted by one to the left, \( B[1] \), defined by \((9.17)\), also has a quasi-o-isomorphism. Note that we have a quasi-isomorphism \( s[1]: P(B)[1] \rightarrow B[1] \) of the same degree as \( s \) (see \((9.19)\) for the definition). But we want \( P(B[1]) \) in the place of \( P(B)[1] \). How are they related? The former has

\[
P(E[1])^i = \tau^*(E[1]^{-i}* \otimes \mathcal{L} = \tau^*(E^{-i+1}* \otimes \mathcal{L},
\]

and \( P(Q[1]) = \tau^*(Q[1])^T_{E[1]|\psi\rightarrow\epsilon\psi + i\psi^\alpha\tau} \). Here we have denoted the transpose by \((-)^{T_{E[1]}}\) to emphasize that it is with respect to the shifted degree. The relation to the one before the twist can be found from \((2.31)\): \( f^{T_{E[1]}} = \sigma E^* f^{T_E} \sigma E^* \) where \( \sigma E^* = \sigma E^T = \sigma^T \). Note also that \( Q[1] = \sigma Q \sigma \) by \((9.20)\). Thus, we find that \((Q[1])^{T_{E[1]}}\) is equal to \( Q^{T_E} = Q^T \) and hence that \( P(Q[1]) = P(Q) \) in the end. On the other hand, \( P(B)[1] \) has

\[
P(E)[1]^i = P(E)^{i+1} = \tau^*(E^{-i-1}* \otimes \mathcal{L},
\]
and \( \mathcal{P}(Q)[1] = \sigma^T \mathcal{P}(Q) \sigma^T \). Thus, we have a degree \(-2\) isomorphism

\[
\sigma^T : \mathcal{P}(B[1]) \longrightarrow \mathcal{P}(B)[1].
\]  

(9.38)

Composing this with \( s[1] \), we obtain a quasi-isomorphism

\[
s[1] \circ \sigma^T : \mathcal{P}(B[1]) \longrightarrow B[1]
\]  

(9.39)

of degree \( |s| - 2 \). It is straightforward to see that it satisfies the condition (9.34) for the same \( c \) as \( s \). Thus, it is a quasi-o-isomorphism of \( B[1] \). Note that \( B[1] \) may be interpreted as the antibrane of \( B \). Then \( s \rightarrow s[1] \circ \sigma^T \) is compatible with the general formula (3.39) that relates the unitary o-isomorphisms between a brane and its antibrane. As in that case, in order to maintain the relation (9.35), we need to choose an opposite sign for opposite \( \varepsilon \), say, \( \bar{s}_{(\mp i)} = \pm s[1]_{(\mp i)} \circ \sigma^T \). Applying this repeatedly, we find that \( B[i] \) has a quasi-o-isomorphism of degree \( |s| - 2i \).

The shift by even integers, \( B \mapsto B[2i] \), which results in \( |s| \rightarrow |s| - 4i \), has no effect on the resulting D-brane in the string theory. Thus, we can always bring the degree of the quasi-o-isomorphism into one of the two values in a window of length four. For example, we may always assume \( |s| = 0 \) or \( 2 \) (resp. \( |s| = \pm 1 \)) if \( |s| \) is even (resp. odd). It is also true that shift by odd integers, say \( B \mapsto B[1] \), does not change the infra-red boundary interaction itself. However, \( B[1] \) must be distinguished from \( B \) in string theory — it is the antibrane of \( B \), which yields the opposite GSO projection for the open string with other branes. For example, the orientifold projection condition is opposite in the Ramond sector. To summarize, all branes are classified into two classes, distinguished by the degree of quasi-o-isomorphisms mod 4. We shall discuss the significance of this in Section 9.3 from the viewpoint of spacetime supersymmetry.

The degree of o-isomorphisms may be traded into the degree of the parity transform. Suppose we have a D-brane \( B \) with a quasi-o-isomorphism \( s : \mathcal{P}(B) \rightarrow B \) of degree \( |s| = -r_o \). Using (9.18), we may regard it as a quasi-isomorphism of degree 0 from \( \mathcal{P}(B)[r_o] \) to \( B \). This motivates us to change the definition of the parity transform to

\[
\mathcal{P}_{r_o}(B) = \mathcal{P}(B)[r_o].
\]  

(9.40)

Accordingly, we change the definition of the parity transform of vertex operators to \( \mathcal{P}_{r_o}(\phi) = \mathcal{P}(\phi)[r_o] \). In this formulation, the parity operator is defined by

\[
\mathcal{P}(\phi) = s \mathcal{P}_{r_o}(\phi) s^{-1}.
\]  

(9.41)

There is no sign factor \((-1)^{|s||\phi|}\) since \( s \) is regarded to have degree 0 here. Using (9.18) and (9.19) along with (9.16), one can check that this parity operator is identical to the
one in (9.31). The condition (9.34) for $P^2 = (-1)^F$ translates in this formulation to

$$s \simeq \begin{cases} 
(\sigma_1^{-1} \otimes c)P_{r_o}(s) & r_o \text{ even}, \\
-(\tau^{-1} \otimes c)P_{r_o}(s) & r_o \text{ odd}.
\end{cases}$$

(9.42)

The precise sign on the right hand side is important if we want $c$ here to be identical to the crosscap section. If we are interested only in branes within one of the two classes mentioned above, we may fix $r_o$ and restrict our attention to those with degree zero quasi-o-isomorphisms.

9.2.2 An Example

For illustration, let us present an example in which there is a quasi-o-isomorphism but not a unitary o-isomorphism. It is in Type IIB orientifold on $T^2 \times \mathbb{R}^8$ by $\tau = \text{the inversion of } T^2$, with all four O7-planes of type O$^-$ (for this, we must take $B = 0$, the trivial twist, and $c \equiv \epsilon$). In this case, the degree of quasi-o-isomorphisms is odd. Alternatively, we may take the parity transform to be $P_{r_o}$ for $r_o = \pm 1$, say, and seek for branes with degree 0 quasi-o-isomorphisms. Let $p \in T^2$ be one of the four fixed points. The holomorphic line bundle $O(p)$ has a section $\vartheta_p$ (a theta function) that vanishes at $p$. The brane we consider is given by the following complex

$$B : \quad O \xrightarrow{\vartheta_p} O(p)$$

(9.43)

where the underline shows the position of the degree 0 component. Its image by the transform $P_1$ is isomorphic to the following complex

$$P_1(B)' : \quad O(-p) \xrightarrow{\vartheta_p} O$$

(9.44)

via $\tau^*O(p)^* \cong O(-p)$ and $\tau^*O^* \cong O$ which we assume in what follows. Both (9.43) and (9.44) represent a D7-brane at $p$ and hence the two must be isomorphic. However, one cannot find a holomorphic isomorphism between them — the two vector bundles, $O(p) \oplus O$ and $O \oplus O(-p)$, are simply not isomorphic. But there is a quasi-isomorphism.

---

1A single D7-brane at an O7$^-$ is actually inconsistent [46, 47]. However, at the classical level, we should be able to construct such a configuration. Alternatively, we may consider a single D3-brane at one of the four O3$^-$-planes in orientifold of $T^2 \times \mathbb{R}^8$ by an involution which flips the sign of four coordinates of $\mathbb{R}^8$ in addition to the two of $T^2$. We also note here that we do not (need to) respect other constraints such as those of the type discussed in [65].
An element \( s \in C^0(\mathcal{P}_1(B)', B) \) given by

\[
\begin{array}{c}
\mathcal{O} \xrightarrow{\partial_p} \mathcal{O}(p) \\
\mathcal{O}(-p) \xrightarrow{\epsilon_{p_0}} \mathcal{O}
\end{array}
\]  

(9.45)

is \( Q \)-closed if \( s^{(0)}_{-1} = \varepsilon s^{(0)}_0 \) and \( \overline{\partial} s^{(0)} = \partial_p s^{(1)}_0 \). A solution is determined for any choice of \( s^{(0)}_0 \) which is holomorphic in a neighborhood of the point \( p \) at which \( \partial_p \) vanishes. This is a quasi-isomorphism, i.e., there is an inverse \( s^{-1} \in C^0(B, \mathcal{P}_1(B)' , B) \), \( ss^{-1} \simeq \text{id}_B \) and \( s^{-1}s \simeq \text{id}_{\mathcal{P}_1(B)'}, \) as long as \( s^{(0)}_0 \) is non-vanishing at \( p \) (note that such an \( s^{(0)}_0 \) cannot be globally holomorphic). The condition (9.42) for \( \rho = 1 \) reads

\[
(-\varepsilon \tau s^{(0)}_{t-1}, -\tau s^{(1)}_0, \varepsilon \tau s^{(0)}_0) \simeq (s^{(0)}_0, s^{(1)}_0, s^{(0)}_{-1}),
\]

and is satisfied for a suitable choice of \( s^{(0)}_0 \). Thus, \( B \) has a quasi-o-isomorphism.

Existence of such an example raises a question concerning our classification of topology in Section 8 in which we assumed that all branes have unitary o-isomorphisms. Does it miss some of the branes? In fact, there would be no problem if each brane has a representative, if not itself, which does admit a unitary o-isomorphism. Let us examine such a possibility for the example (9.43).

Unfortunately, no representative of (9.43) admits a unitary o-isomorphism if its Chan-Paton bundle is of finite rank. To see this, suppose \( \mathcal{B}_F = (F, Q_F) \), with \( F \) of finite rank, has a unitary o-isomorphism. Then, we have isomorphisms of vector bundles, \( F^{ev} \simeq \tau^* F^{od*} \) and \( F^{od} \simeq \tau^* F^{ev*} \). It then follows that \( F^{ev} \) and \( F^{od} \) have the same rank and their first Chern classes are related by \( c_1(F^{ev}) = \tau^* c_1(F^{od*}) = -c_1(F^{od}) \), where we have used the fact that \( \tau^* \) is the identity on second cohomology classes. In particular, the Chern character of the brane is \( \text{ch}(\mathcal{B}_F) = c_1(F^{ev}) - c_1(F^{od}) = 2c_1(F^{ev}) \). On the other hand, if \( \mathcal{B}_F \) and \( \mathcal{B} \) are quasi-isomorphic, their Chern characters must also agree. However, \( \text{ch}(\mathcal{B}) \) is an integral generator of \( H^2(T^2, \mathbb{Z}) \) and cannot be equal to \( \text{ch}(\mathcal{B}_F) \in 2H^2(T^2, \mathbb{Z}) \).

However, there is an infinite rank representative which admits a unitary o-isomorphism. It is given by a complex

\[
\mathcal{B}_\infty: \quad E^{-1} \xrightarrow{Q} E^0
\]

where the vector bundle \( E^i \) is the quotient of the trivial bundle over the complex plane \( \mathbb{C} \) with fibre \( \oplus_{n,m \in \mathbb{Z}} \mathbb{C}|n,m\rangle \), by the equivalence relation

\[
(z, |n, m\rangle_i) \sim (z + 1, |n - 1, m\rangle_i) \sim (z + \tau, |n, m - 1\rangle_i),
\]
and $Q$ is given by
\[
Q : (z, |n, m\rangle_{-1}) \mapsto (z, |n, m\rangle_o) \cdot (z + n + \tau m).
\] (9.46)

Note that $Q$ vanishes exactly at one point $[z] = [0] \in T^2$ on a rank one subbundle. That is, it represents a D7-brane at $[0]$. In particular, it represents the same brane as (9.43), if we identify $p$ as the point $[0]$. Let us next describe the orientifold image $\mathcal{P}_1(B_\infty)$. The Chan-Paton bundle consists of degree $-1$ component $\tau^*E^0$ and degree 0 component $\tau^*E^{-1}$. The fibre of $\tau^*E^0$ at $[z] \in T^2$ is spanned by $\tau^*(z, [n, m]) := ([z], (z, [n, m]))$, where $\{(z, [n, m])\}_{n,m}$ form the dual frame to $\{(z, |n, m\rangle_i)\}_{n,m}$. The tachyon configuration is obtained from \[ \mathcal{P}_1(Q) : \tau^*(z, [n, m]) \mapsto \tau^*(z, [-1, [n, m]] \cdot \epsilon(z - n - \tau m). \]

There is a unitary o-isomorphism $U : \mathcal{P}_1(B_\infty) \to B_\infty$ given by
\[
\tau^*(z, [n, m]) \mapsto \epsilon(z, [-n, -m\rangle_{-1}),
\tau^*(z, [-1, [n, m]] \mapsto (z, [-n, -m\rangle_o).
\]

It is straightforward to check that it sends $\mathcal{P}_1(Q)$ to $Q$ and satisfies the condition (9.42) for $r_o = 1$. The brane $(B_\infty, U)$ is isomorphic to $(B, s)$ as D-branes in the orientifold, i.e., there is a quasi-isomorphism $f \in C^0(B_\infty, B)$ such that
\[
f \circ U \circ \mathcal{P}_1(f) \simeq s, \tag{9.47}
\]

which is the quasi-isomorphism version of (3.25). To construct such $f$, let us choose local frames of $\mathcal{O}$ and $\mathcal{O}(p)$ in the neighborhood $|z| < 3\epsilon$ of $p = [0]$ with respect to which we have $\vartheta_p = z$. Let us put $f = 0$ on $(z, |n, m\rangle_i)$ if $|z + n + \tau m| > 2\epsilon$ and
\[
f : (z, |0, 0\rangle_{-1}) \in E^{-1} \mapsto \rho \in \Omega^0(\mathcal{O}),
f : (z, |0, 0\rangle_o) \in E^0 \mapsto \left(\rho, \frac{1}{z} \bar{\vartheta}\rho\right) \in \Omega^0(\mathcal{O}(p)) \oplus \Omega^{0,1}(\mathcal{O}),
\]
where $\rho$ is a smeared step function which is constantly 1 for $|z| < \epsilon$ and zero outside $|z| < 2\epsilon$. Then $f$ is $Q$-closed. And it satisfies (9.47) if $s_0 = \rho \cdot \tau^* \rho$.

This exercise illustrates that even if a brane does not admit a unitary o-isomorphism, there can be another representative that admits one. However, we have not given a proof of general existence. We leave this question open in this paper.
9.3 The Spectral Flow

When the first Chern class of $M$ vanishes, $c_1(M) = 0$, the non-linear sigma model on $M$ admits the topological B-twist. We shall use this to define a one to one correspondence between Ramond ground states and chiral ring elements, known as the spectral flow \[66\], and study the spacetime supersymmetry that results from it.

The topological B-twist turns the spinors $\psi^±_i$ and $\psi^±_i$ into scalars and one-forms respectively \[67\]. The scalars are denoted by $\eta^i$ and $\theta_i$ and the one-forms are written as $\rho^i$. The spinor supercharge $Q$ becomes a scalar, with the transformation rule

$$\begin{align*}
\delta x^i &= 0, \\
\delta x^i &= -i\bar{c}\eta^i \\
\delta \rho^i &= -2i\bar{c}dx^i, \\
\delta \eta^i &= 0, \\
\delta \theta_i &= 0,
\end{align*}$$

(9.48)

for $\delta = -i\bar{c}Q$. In the absence of spinors, there is no need to choose spin structure, and in particular, there is no (+) versus (−) distinction in the boundary condition for a D-brane. For the space-filling brane with data $(E, Q)$, we impose the boundary condition $\theta_i = 0$ and $\rho^i_n = 0$ compatible with (9.48), and make the replacement $\psi^± \rightarrow \eta^±$ and $\psi^± \rightarrow \rho^±_i$ inside the boundary interaction $A_r$.

On a flat region of the worldsheet, the twisted model is indistinguishable from the original $\mathcal{N} = 2$ theory, provided we choose an appropriate boundary condition in the latter. For example, at the boundary of the upper-half plane, vertex operators of the twisted model correspond to NS vertex operators in the untwisted model. As another example, the twisted model on a flat strip corresponds to the Ramond sector of the untwisted model. These two flat geometries can be connected by the ‘quarter-sphere’ diagram as shown in Fig. 7. In the twisted model, the scalar supercharge $Q$ is conserved

![Figure 7: The spectral flow](image)

and has a closed one form as its current, even in the curved region. The diagram therefore determines a one to one correspondence between $Q$-cohomology classes of vertex operators and $Q$-cohomology classes of states, i.e. between chiral ring elements and Ramond ground states. This is the spectral flow.
To be precise, there is a minor difference between the twisted and untwisted models on the flat strip. It is in the boundary interaction $A_t$ on the left. Because the time runs in the opposite direction on the left boundary, we make a replacement $\psi^\tau \rightarrow i\psi^\tau$ and $\psi^i \rightarrow i\psi^i$ inside $A_t$ in the untwisted model (here we take the $(-+)$ spin structure — see Section 3.4). In the twisted model, on the other hand, we do $\eta^\tau \rightarrow \eta^\tau$ and $\rho^i_t \rightarrow -\rho^i_t$. The difference is $\psi^\tau \rightarrow i\psi^\tau$ and $\psi^i \rightarrow -i\psi^i$, which is nothing but the R-symmetry transformation with phase $i$. At this point, we introduce the representation of the R-symmetry on the Chan-Paton bundle, $R_B(\lambda) : E \rightarrow E$, which acts as multiplication by $\lambda^i$ on the degree $i$ component $E^i$. The condition that $Q$ has degree 1 can be expressed as $R_B(\lambda)Q|_{\psi^\tau} = \lambda Q|_{\psi^\tau}$. At this point, we introduce the representation of the R-symmetry on the Chan-Paton bundle, $R_B(\lambda) : E \rightarrow E$, which acts as multiplication by $\lambda^i$ on the degree $i$ component $E^i$. The condition that $Q$ has degree 1 can be expressed as $R_B(\lambda)Q|_{\psi^\tau} = \lambda Q|_{\psi^\tau}$.

Based on this consideration, we consider the following map of NS vertex operators to Ramond sector states in the zero mode approximation, which sends $C^i(B_1, B_2)$ to $C_{R}^{i-\frac{2}{\kappa}}(B_1, B_2)$,

$$\mathcal{U} : \phi \mapsto \sqrt{\Omega} \otimes \phi \circ R_{B_1}(i).$$

(9.49)

To be precise, this is a map from $(++)$ to $(-+)$. Here $\sqrt{\Omega}$ is a holomorphic trivialization of the line bundle $\sqrt{K}$ (we are choosing a spin structure of $M$ that admits it). It can be regarded as a square root of a holomorphic volume form $\Omega$ of $M$ which exists when $c_1(M) = 0$. The presence of $\sqrt{\Omega}$ in (9.49) is consistent with the definition of the topological B-model measure under which the worldsheet of Euler number $\chi$ comes with the factor $\Omega^\chi$; the sphere has $\chi = 2$ and hence the quarter-sphere may be assigned $\chi = \frac{1}{2}$. The map (9.49) indeed commutes with the supercharge,

$$Q \circ \mathcal{U} = \mathcal{U} \circ Q,$$

where $Q$ on the left resp. right hand side is defined by (9.23) resp. (9.14). In particular, it determines a one to one correspondence between chiral ring elements and Ramond ground states.

Using this map let us compare the parity $\tilde{P}$ on Ramond ground states and the parity $P$ on chiral ring elements. Note first that, under the holomorphic involution $\tau$, the holomorphic volume form $\Omega$ is invariant up to a sign, $\tau^* \Omega = \pm \Omega$, and hence its square root $\sqrt{\Omega}$ is invariant up to a phase, $\tau^* \sqrt{\Omega} = g_S \sqrt{\Omega}$ with $g_S^2 = \pm 1$. By a straightforward computation, we find, for a brane with a quasi-o-isomorphism $(B, s)$

$$\tilde{P}(\mathcal{U}(\phi)) = \kappa \epsilon_{|s|} i^{\epsilon_{|s|}} \mathcal{U}(P(\phi)),$$

(9.50)

where $\epsilon_{|s|} := g_S i^{-|s|}$. The choice of the phase $g_S$ is a part of the choice of the lift $\tau_S$ discussed in Section 3.3. Since we consider the trivial involution on $R^D$, we must have
\[ \varrho_S^2 = \pm 1 \] in Case \((B_\pm)\). Recall also that \(|s|\) is even in Case \((B_+)\) and odd in Case \((B_-)\). Therefore, the phase \(\epsilon_{|s|}\) is a sign.

The formula (9.50) exhibits the spacetime supersymmetry. For concreteness let us consider the case \(D = 4\) and \(n = 3\), i.e., compactification on a Calabi-Yau three-fold \(M\) down to four dimensions. We consider a brane \(\mathcal{B}\) such that chiral ring elements flow to chiral primary fields with the actual R-charge very close to the canonical R-charge. The states corresponding to massless fermions are of the form \(\phi_R \otimes |u,k\rangle_R\) where \(\phi_R\) is a Ramond ground state for the sigma model on \(M\) and \(|u,k\rangle_R\) is a 4d massless spinor state. Note that \(\tilde{P}\) does not act trivially on \(|u,k\rangle_R\). For this choice, the phase \(\kappa\) is a sign, and also, the orientifold projection condition takes the form

\[ \tilde{P}(\phi_R) = \phi_R. \]

On the other hand, massless vectors correspond, in the 0-picture, to states of the form \(\phi_0 \otimes (\zeta \cdot \alpha_{-1} + \cdots) |k\rangle_{NS}\) where \(\phi_0\) is the identity times the Chan-Paton factor, and nearly massless scalars correspond to the states of the form \(\phi_1 \otimes k \cdot \psi_{-4} |k\rangle_{NS}\) where \(\phi_1\) is a chiral primary state of the R-charge close to \(|\phi_1| = 1\). Recall that the parity \(P\) acts on the states \(\langle \zeta \cdot \alpha_{-1} + \cdots |k\rangle_{NS}\) and \(k \cdot \psi_{-4} |k\rangle_{NS}\) by multiplication by \((-1)\) and \((-i)\) respectively in the \((++)\) spin structure. Thus, orientifold projection condition on these light bosons takes the form

\[ P(\phi_0) = -\phi_0 \quad \text{and} \quad P(\phi_1) = i\phi_1. \]

Inserting these to (9.50), we find \(\tilde{P}(\mathcal{U}(\phi_0)) = -\kappa \epsilon_{|s|} \mathcal{U}(\phi_0)\) and \(\tilde{P}(\mathcal{U}(\phi_1)) = -\kappa \epsilon_{|s|} \mathcal{U}(\phi_1)\). When \(\kappa \epsilon_{|s|} = -1\), these are nothing but the orientifold projection condition for the corresponding massless fermions. For this choice of \(\kappa\), the unprojected degrees of freedom form the vector and chiral multiplets of a possibly broken \(\mathcal{N} = 1\) supersymmetry. For the other choice, \(\kappa \epsilon_{|s|} = 1\), the orientifold projection conditions for the bosons and fermions do not match — the branes and the orientifold preserve completely different supersymmetries. Of course, in that case, we may replace the brane \(\mathcal{B}\) by its antibrane, say \(\mathcal{B}[1]\). If we do so, the quasi-o-isomorphism has a shifted degree, \(|s| - 2\), as we have discussed in Section 9.2.1. The shift \(|s| \to |s| - 2\) flips the sign of \(\epsilon_{|s|} = \varrho_S i^{-|s|}\), and indeed we recover the condition \(\kappa \epsilon_{|s| - 2} = -1\).

This discussion shows the significance of the division of branes into two classes by the mod 4 degree of quasi-o-isomorphisms (Section 9.2.1). For a given \(\kappa\), branes from only one class is compatible with the spacetime supersymmetry that is preserved by the
orientifold with orientation $\kappa$. I.e., there is a one to one correspondence between bosons and fermions. Note that the supersymmetry may be broken — scalars may be massive or tachyonic while fermions are always massless. On the other hand, the other class of branes have no chance; there is not even a correspondence between bosons and fermions. If we replace $\kappa$ by $-\kappa$, the rôles of the two classes are exchanged.

### 9.4 A Categorical Description

D-branes in a theory of oriented strings form a category $\mathcal{C}$; objects are D-brane data, morphisms for a pair of objects are open string states, and the composition of morphisms represents gluing of two open strings into one. The categorical description may also be extended to D-branes in orientifolds [19]. In the most basic form, it goes as follows.

An orientifold transform of D-branes can be represented as a contravariant functor $P : \mathcal{C} \to \mathcal{C}$. We require that the square $P^2 = P \circ P$ is isomorphic to the identity functor of $\mathcal{C}$, and we choose such an isomorphism

$$c : P^2 \xrightarrow{\cong} \text{Id}_\mathcal{C}. \quad (9.51)$$

That is, to each object $X \in \mathcal{C}$ is assigned an isomorphism $c_X \in \text{Hom}_\mathcal{C}(P^2(X), X)$ in such a way that the following diagram commutes for each morphism $f \in \text{Hom}_\mathcal{C}(X, Y)$,

$$
\begin{array}{ccc}
P^2(X) & \xrightarrow{c_X} & X \\
p^2(f) \downarrow & & \downarrow f \\
P^2(Y) & \xleftarrow{c_Y} & Y \\
\end{array}
$$

We further require that it satisfies

$$c_{P(X)} \cdot P(c_X) = \text{id}_{P(X)}. \quad (9.53)$$

Then, we may define the category of D-branes in the orientifold, $\mathcal{O} = \mathcal{O}(\mathcal{C}; P, c)$, as follows. An object is a brane with an o-isomorphism, i.e., a pair $(X, s)$ of an object $X \in \mathcal{C}$ and an isomorphism $s \in \text{Hom}_\mathcal{C}(P(X), X)$ satisfying the condition

$$s = c_X \cdot P(s). \quad (9.54)$$

The space of morphisms $\text{Hom}_\mathcal{O}((X, s), (Y, t))$ is the space $\text{Hom}_\mathcal{C}(X, Y)$. It is equipped with the operator $P : \text{Hom}_\mathcal{C}(X, Y) \to \text{Hom}_\mathcal{C}(Y, X)$ defined by $P(f) = s \cdot P(f) \cdot t^{-1}$. This operator is an involution, $P^2 = \text{id}$, thanks to the condition (9.54) and the commutativity of the
diagram (9.52). An isomorphism from \((X, s)\) to \((Y, t)\) is an isomorphism \(f \in \text{Hom}_C(X, Y)\) such that

\[ f \cdot s \cdot P(f) = t. \tag{9.55} \]

Some remarks are in order.

(i) If the category is graded, i.e., if the space of morphisms has a \(\mathbb{Z}_2\) grading, it is natural to assume that \(P\) is a contravariant functor in the graded sense, i.e., there is a sign on the product of morphisms, \(P(f \cdot g) = (-1)^{|f||g|}P(g) \cdot P(f)\). Furthermore, we may need a sign in the definition of the parity operators as \(P(f) = s \cdot P(f) \cdot t^{-1}(-1)^{|f||t|} \cdot t\), if the isomorphism \(t\) has a non-zero degree.

(ii) The property (9.53) is required for existence of isomorphisms obeying (9.54) — use (9.54) in itself.

(iii) The structure is partly motivated by the results of the present paper and in fact can be used to summarize them as discussed below. For example, the isomorphism condition (9.55) is motivated by (3.25).

(iv) A pair \((P, c)\) satisfying (9.51) and (9.53) is known as a duality of the category \(C\). See, for example, [68].

As far as the chiral ring sector is concerned, the structure found in this section can be summarized in this language, although the presence of worldsheet spinors requires a minor modification. We take the category \(C = \mathcal{D}(M)\), which has D-brane data \(\mathcal{B} = (E, Q)\) as objects, cohomology classes in \(\mathcal{H}(\mathcal{B}_1, \mathcal{B}_2)\) as morphisms between objects, and the product (9.15) as the composition of morphisms. The parity functor is the transform \(P\) given by (9.25) and (9.26). It depends on the choice of the phase \(\varepsilon\) (i.e., of the boundary spin structure), \(P = P(\varepsilon)\). Also, its square \(P^2\) is not isomorphic to the identity but to the functor \((-1)^F\) which acts as the identity on objects but as the \(\mathbb{Z}_2\)-grading on morphisms. Indeed, given a crosscap section \(c\), the assignment

\[ \mathcal{B} = (E, Q) \mapsto c_{\mathcal{B}} = \sigma_E t_E^{-1} \otimes c \]

provides an isomorphism

\[ c : P^2 \xrightarrow{\cong} (-1)^F. \]

That is, the diagram (9.52) commutes if the vertical arrow \(f\) on the right is replaced by \((-1)^{|f|}f\). This was indeed seen in (9.33). The condition for \(\alpha\)-isomorphisms (9.54) appears in (9.34).

If we restrict our attention to D-branes compatible with the spacetime supersymmetry preserved by the orientifold, we may assume that all \(\alpha\)-isomorphisms have the same degree.
Alternatively, we may take a shifted parity functor

\[ \mathcal{P}_{r_0} = T^r \circ \mathcal{P}, \]

and restrict our attention to branes with \( \sigma \)-isomorphisms of degree zero. \( T^j \) is the shift functor that acts as \( B \mapsto B[j] \) on objects and as \( \phi \mapsto \phi[j] \) on morphisms. The relevant isomorphism \( \mathcal{P}_{r_0}^2 \cong (-1)^F \) is provided by

\[ c_{r_0,(E,Q)} = (-1)^r \sigma_{E}^{r_0+1} t_E^{-1} \otimes c. \]

See (9.42). We shall denote the category of such D-branes by \( \mathcal{O}(\mathcal{D}(M), \mathcal{P}_{r_0}, c_{r_0}) \). Translating the discussion in Section 9.2.1, given an \( \sigma \)-isomorphism \( s : \mathcal{P}_{r_0}(\mathcal{B}) \to \mathcal{B} \) of degree zero, we have an \( \sigma \)-isomorphism \( s[1] \circ \sigma^T : \mathcal{P}_{r_0+2}(\mathcal{B}[1]) \to \mathcal{B}[1] \) again of degree zero. This defines an equivalence

\[ \mathcal{O}(\mathcal{D}(M), \mathcal{P}_{r_0}, c_{r_0}) \cong \mathcal{O}(\mathcal{D}(M), \mathcal{P}_{r_0+2}, c_{r_0+2}) \]

between the categories of D-branes compatible with the opposite orientation of the orientifold.

**Digression: General Type II Orientifolds**

We make a brief digression to discuss whether the categorical description is possible for the entire sector of more general Type II orientifolds, as those considered in Section 3. For this we need to define the product of open string states, as it is used in the condition like (9.54) as well as in the definition of the parity operator where the isomorphism \( c_X \) and \( \sigma \)-isomorphisms are regarded as open string states. One possibility is to use the \( \ast \)-product in open string field theory, but this may not be useful in the current status where even the identity element is represented by a complicated state. As an alternative, we take the product of boundary vertex operators, which has to be taken rather informally as it depends on the insertion points of the operators. A parity functor \( \mathcal{P} \) is given by (3.10) and (3.12). For a brane \( \mathcal{B} = (E, A, T) \), an isomorphism \( \mathcal{P}^2(\mathcal{B}) \to \mathcal{B} \) is provided by the “vertex operator” \( \sigma_{E}^{t_E^{-1}} \otimes c \). It appears in the following form in the path-integral weight:

\[ W_B(\tau_f, \tau) \circ (\sigma_{E}^{t_E^{-1}} \otimes c)|_{\tau} \circ W_{\mathcal{P}^2(\mathcal{B})}(\tau, \tau_i), \]

where \( W_B(\tau_f, \tau_i) \) is the boundary interaction (3.2) for the brane \( \mathcal{B} \). It is independent of the insertion point \( \tau \) by the fact that \( c \) is flat with respect to \( \tau^* \alpha - \alpha \). Likewise, an \( \sigma \)-isomorphism \( U : \mathcal{P}(\mathcal{B}) \to \mathcal{B} \) may be regarded as a “vertex operator” which appears as

\[ W_B(\tau_f, \tau) \circ U|_{\tau} \circ W_{\mathcal{P}(\mathcal{B})}(\tau, \tau_i). \]
Again, it is independent of the insertion point $\tau$ thanks to the equations (3.16)-(3.17). By the presence of Ramond sector states, we need to consider both of the two boundary spin structures. This is unlike in the discussion of the chiral ring, which is a part of the Neveu-Schwarz sector, where we were able to work with a fixed one. We have functors that flip the boundary spin structure, $(-1)^{FR}$ and $(-1)^{FL}$, which we discussed in Section 3.4. They obey the relations of the form $(-1)^{FL} \circ (-1)^{FR} = (-1)^{FL} \circ (-1)^{FR} = (-1)^F$ and $(-1)^{FR} \circ \mathcal{P} \cong \mathcal{P} \circ (-1)^{FL}$. The hom space has two $\mathbb{Z}_2$-gradings. One is the usual worldsheet $\mathbb{Z}_2$-grading, and the other is of spacetime nature — Neveu-Schwarz states are even and Ramond states are odd. Accordingly, we have

$$\mathcal{P}(\Phi \cdot \Psi) = (-1)^{|\Phi|+|\Psi|}|\mathcal{P}(\Psi) \cdot \mathcal{P}(\Phi)|.$$ 

The spacetime sign factor has appeared in (5.41) for example.

**Topological B-Model**

As we discussed earlier, the topological model has no worldsheet spinors and hence there is no need of choosing spin structure nor summing over the choices (i.e. no GSO projection). In particular, there is a unique boundary condition for a D-brane and a unique parity transformation for an orientifold — there is no $(\pm)$ versus $(-)$ nor $\Omega$ versus $(-1)^{FR}\Omega$, etc. Vertex operators are always of “NS” type and the states are always of “Ramond” type, and the parity must square to the identity, not to $(-1)^F$. This indicates that the categorical description in the basic form applies without modification.

Let us first determine the parity transform of the worldsheet fields. The guiding principle is to preserve the scalar supersymmetry (9.48). We notice that $\eta^i$ is the partner of $x^i$ and $dx^i$ is the partner of $\rho^i$. This implies that the parity transform $\Omega$ which preserves the symmetry (9.48) is

$$x \rightarrow x \circ \Omega, \quad \eta^i \rightarrow \eta^\sigma \circ \Omega, \quad \theta_i \rightarrow -\theta_i \circ \Omega, \quad \rho^i \rightarrow \Omega^* \rho^i.$$  

(9.59)

The boundary interaction includes the terms

$$i\mathcal{A}_\tau = \cdots - \frac{1}{2}\eta \partial_i Q + \frac{1}{2}\eta \partial_i Q^\dagger + \frac{1}{2}\{Q, Q^\dagger\},$$

where $Q = Q(x, \eta)$ and $Q^\dagger = Q^\dagger(x, \rho_\tau)$. Under the parity $\Omega$, which reverses the orientation of the boundary, composed with the transpose, the interaction transforms as

$$i\mathcal{A}_\tau \mapsto \left( \cdots - \frac{1}{2}(-\rho^i)\partial_i Q + \frac{1}{2}\eta \partial_i Q^\dagger|_{\rho_\tau \rightarrow -\rho_\tau} + \frac{1}{2}\{Q, Q^\dagger\}|_{\rho_\tau \rightarrow -\rho_\tau} \right)^T \circ \Omega.$$ 

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We see that the effect of the parity transform is

\[ Q \mapsto -Q^T, \quad Q^\dagger \mapsto (Q^\dagger)^T|_{\rho_r \rightarrow -\rho_r}. \]  

(9.60)

This is consistent with the hermiticity relation in the theory before the topological twist between the components of \( Q \) and \( Q^\dagger \), since \((f^\dagger)^T = (-1)^{f_1}(f^T)^\dagger\).

Combined with the involution \( \tau : M \rightarrow M \) and the twist by \((L, \alpha)\), we find a parity functor \( P_0 : D(M) \rightarrow D(M) \),

\[ E \mapsto P_0(E) = \tau^*E^* \otimes \mathcal{L}, \]

\[ Q \mapsto P_0(Q) = -\tau^*Q^T + i\eta^\tau \alpha, \]  

(9.61)

We may also consider the shifted versions, \( P_{r_o} = T_{r_o} \circ P_0 \). The square \( P_{r_o}^2 \) is isomorphic to the identity functor by

\[ c_{r_o}(E, Q) = \sigma_{E}^{r_o} \tau\eta^{-1}_E \otimes c_{top} \]  

(9.62)

where \( c_{top} \) is a holomorphic section of \( \tau^*\mathcal{L} \otimes \mathcal{L}^* \) with the property \( c_{top} \cdot \tau^*c_{top} = 1 \) for (9.53). This gives us an orientifold category consisting of pairs \((B, s)\) where \( s \) is of degree zero and obeys \( s \simeq c_{r_o}B \cdot P_{r_o}(s) \). The resulting parity operator, \( P_{top} \), squares to the identity, \( P_{top}^2 = id \). The shift functor \( T \) yields an equivalence of categories

\[ D(D(M), P_{r_o}, c_{r_o}) \cong D(D(M), P_{r_o+2}, -c_{r_o+2}), \]  

(9.63)

if we take the common \( c_{top} \) for both \( c_{r_o} \) and \( c_{r_o+2} \). Note the appearance of a minus sign for the isomorphism \( c \), in contrast with (9.58). That means that the section \( c_{top} \) does not simply reflect the type of the O-planes in the corresponding Type II model. We now see a more explicit relation.

**Relation To Type II**

Let us see how the information of Type II orientifold can be recovered from the D-brane category for topological orientifold. To this end, let us compare the parity transform of the fields at the boundary:

**Type II** : \[ \psi^T \rightarrow \epsilon\psi^T \circ \Omega, \quad \psi^i \rightarrow \epsilon\psi^i \circ \Omega, \]

**topological** : \[ \eta^T \rightarrow \eta^T \circ \Omega, \quad \rho^i \rightarrow -\rho^i \circ \Omega. \]

We see that the two are related by the R-symmetry transform, \( \psi^T \rightarrow \epsilon^{-1}\psi^T \) and \( \psi^i \rightarrow \epsilon\psi^i \).

This prompts us to look into the representation \( R_{B}(\lambda) \) of the R-symmetry on the Chan-Paton bundle. Note that \( R_{T-r_o(B)}(\lambda) = \lambda^{-r_o}R_{B}(\lambda) \) and \( R_{T(B)}(\lambda) = R_{P_0(B)}(\lambda) = R_{B}(\lambda)^{-T} \).
Comparison of (9.25) and (9.28) on the one hand and (9.61) on the other shows the relation

\[ R_{\mathcal{P}(\mathcal{B})}(\epsilon)\mathcal{P}(\mathcal{Q})R_{\mathcal{P}(\mathcal{B})}(\epsilon)^{-1} = \mathcal{P}_0(\mathcal{Q}), \]
\[ R_{\mathcal{P}(\mathcal{B}_1)}(\epsilon)\mathcal{P}(\phi)R_{\mathcal{P}(\mathcal{B}_2)}(\epsilon)^{-1} = \mathcal{P}_0(\phi). \]

The same holds for \( \mathcal{P}_{r_{\alpha}} \) versus \( \mathcal{P}_{r_{\beta}} \). The o-isomorphisms in the Type II and topological theories can also be related. Let \( s \) and \( s' \) be o-isomorphisms of an object \( B \) in the two categories based on \( (\mathcal{P}_{r_{\alpha}}, \mathcal{C}_{r_{\alpha}}) \) and \( (\mathcal{P}_{r_{\beta}}, \mathcal{C}_{r_{\beta}}) \). If we put

\[ s \propto s R_{\mathcal{P}_{r_{\alpha}}(\mathcal{B})}(\epsilon), \quad (9.64) \]

then \( s \) and \( s' \) obey the condition (9.54) in the respective categories at the same time. (9.64) is consistent with the relation (9.35) between \( s(i) \) and \( s(-i) \) since \( R_{\mathcal{P}_{r_{\alpha}}(\mathcal{B})}(-1) \propto \sigma^T_B \).

The resulting parity operators in the two theories are simply related by

\[ \mathcal{P}(\phi) = \epsilon^{|\phi|} \mathcal{P}_{top}(\phi). \quad (9.65) \]

Combining this with (9.50), or by a direct comparison, we also find the relation to the parity operator on the Ramond ground states,

\[ \tilde{\mathcal{P}} = \kappa \epsilon_{-r_{\alpha}} \mathcal{U} \circ \mathcal{P}_{top} \circ \mathcal{U}^{-1}. \quad (9.66) \]

We also find from (9.64) the precise relation between the crosscap section \( c \) in the Type II theory and the topological counterpart \( c_{top} \),

\[ c = \epsilon^{-r_{\alpha}} c_{top}. \quad (9.67) \]

This relation reproduces the fact (9.63) that \( c_{top} \) changes by a sign under the shift \( r_{\alpha} \rightarrow r_{\alpha} + 2 \) (since \( c \) is invariant).

**Triangles**

We shall make a comment on the category \( \mathcal{C} = \mathfrak{D}(M) \) and the parity functor \( \mathcal{P}_0 \) defined in (9.61) or its shifts \( \mathcal{P}_j = T^j \mathcal{P} \). Here and in what follows, we take only degree zero morphisms unless otherwise stated, i.e., we redefine the morphism space as \( \text{Hom}_C(X, Y) = \mathfrak{H}^0(X, Y) \). For a morphism \( u \in \text{Hom}_C(X, Y) \), we have a sequence of objects and morphisms,

\[ X \xrightarrow{u} Y \xrightarrow{{}^y} \text{Cone}(u) \xrightarrow{{}^{(1,0)}} T(X), \quad (9.68) \]
A triangle is a sequence of the form \( A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A) \) which is isomorphic to the above for some \( u : X \rightarrow Y \). The functor \( T \) and the set of triangles obey a certain set of axioms and make the category \( C \) a triangulated category. We refer the reader to [69] for details. One of the axioms is the rotation axiom: if \( X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X) \) is a triangle, then \( Y \xrightarrow{v} Z \xrightarrow{w} T(X) \xrightarrow{-T(u)} T(Y) \) is also a triangle. It is very important to be careful about the minus sign on the last arrow. For example, under the same assumption, \( T(X) \xrightarrow{T(u)} T(Y) \xrightarrow{T(v)} T(Z) \xrightarrow{T(w)} T^2(X) \) is not always a triangle but is so when a minus sign is placed at each arrow, or alternatively, a minus sign at one of the three arrows — we can flip the sign of two arrows by a change of basis (which is an isomorphism).

Given a triangle \( X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X) \), its image under the parity \( P = P_j \), more precisely,

\[
\begin{align*}
PT(X) & \xrightarrow{P(u)} P(Z) \xrightarrow{P(v)} P(Y) \xrightarrow{\sigma_{P(X)} \circ P(u)} TPT(X) \\
& \xrightarrow{(9.69)}
\end{align*}
\]

is also a triangle, for any \( j \in \mathbb{Z} \). Note that \( \sigma_{P(X)} \) defines an isomorphism \( P(X) \cong TPT(X) \).

We shall call such a category with duality \( (\mathcal{C}, P, \epsilon) \) a triangulated category with duality. Here \( \epsilon = \epsilon_j \) for \( P = P_j \) defined in (9.62).

The category \( \mathcal{D}(M) \) is equivalent as a triangulated category to the full subcategory of the bounded derived category of sheaves of \( \mathcal{O}_M \) modules consisting of complexes with coherent cohomology sheaves [70, 71]. If \( M \) is algebraic, that is equivalent to the derived category of coherent sheaves on \( M \). For the case \( \tau = \text{id}_M \), the parity functor \( P_0 \) is essentially the same as the duality functor \( \mathbb{R} \text{hom}(-, \mathcal{L}) \).

### 9.5 Some Binding/Decay Channels

Let us discuss possible channels of bound state formation or decay. We put \( (\mathcal{C}, P, \epsilon) = (\mathcal{D}(M), P_j, \epsilon_j) \) for some \( j \).

Invariant Cones

The superposition of a brane and its orientifold image gives rise to an invariant brane. Indeed, for any object \( L \in \mathcal{C} \), the direct sum \( H_L = P(L) \oplus L \) has an \( \alpha \)-isomorphism \( s_L : P^2(L) \oplus P(L) \rightarrow P(L) \oplus L \),

\[
s_L = \begin{pmatrix}
0 & \text{id}_{P(L)} \\
\epsilon_L & 0
\end{pmatrix}.
\]

(9.70)
Note that it satisfies the condition (9.54),

\[ c_{HL} p(s_L) = \begin{pmatrix} c_{p(L)} & 0 \\ 0 & c_L \end{pmatrix} \begin{pmatrix} 0 & p(c_L) \\ p(id_{p(L)}) & 0 \end{pmatrix} = \begin{pmatrix} 0 & c_{p(L)} p(c_L) \\ c_L id_{p^2(L)} & 0 \end{pmatrix} = s_L, \]

where we used (9.53). An invariant object of this form is called \textit{hyperbolic}. This construction works if \((C, P, c)\) is a general category with duality.

When the brane and its orientifold image are bound together by an open string state, does it form an invariant brane? More specifically, does the cone of a morphism \(u : T^{-1}P(L) \to L\) admits an \(o\)-isomorphism? As the simplest candidate, let us see if \(s_L\) in (9.70) can serve as an \(o\)-isomorphism of the cone \(C = \text{Cone}(u)\). It is enough to check whether it is \(Q\)-closed. Note that

\[ Q_C = \begin{pmatrix} Q_{p(L)} & 0 \\ u & Q_L \end{pmatrix}, \quad Q_P(C) = \begin{pmatrix} Q_{p^2(L)} & -P(u) \\ 0 & Q_{p(L)} \end{pmatrix} \]

where \(u\) here is regarded as a degree 1 map \(P(L) \to L\). The condition for \(Q s_L = 0\) is

\[ u + c_L \cdot P(u) = 0. \quad (9.71) \]

Under this condition, \((C, s_L)\) is an invariant object. We shall call it an \textit{invariant cone} of \(u \in \mathbb{Z}^1(T^{-1}P(L), L)\) satisfying (9.71), and denote it by \(\text{Cone}(u, L)\). When the actual R-charge of \(u\) is smaller than 1, \(L\) and \(P(L)\) are bound together to form the invariant cone \(\text{Cone}(u, L)\). When the charge is greater than 1, the invariant cone splits to \(L\) and \(P(L)\).

The construction can be extended to a general triangulated category with duality. For this purpose, we first rewrite (9.71) as a condition for \(u\) regarded as a degree zero morphism \(u : T^{-1}P(L) \to L\). It reads

\[ u = c_L \cdot \sigma_{p^2(L)} \cdot T^{-1}P(u), \quad (9.72) \]

where \(\sigma_{p^2(L)}\) is regarded as an isomorphism \(T^{-1}P T^{-1}P(L) \to P^2(L)\). The morphism \(u\) extends to a triangle \(T^{-1}P(L) \xrightarrow{u} L \xrightarrow{v} C \xrightarrow{w} P(L)\). Applying \(P\) to it and rotating once, we have another triangle which appears as the top line in the diagram below. It is important that no sign is needed on the arrows.

\[
\begin{array}{c}
T^{-1}P(L) \\
\downarrow \\
T^{-1}P(L)
\end{array}
\xrightarrow{\sigma_{p^2(L)} T^{-1}P(u)}
\begin{array}{c}
P^2(L) \\
\epsilon_L \\
P(L)
\end{array}
\xrightarrow{p(u)}
\begin{array}{c}
P(C) \\
\exists \phi \\
P(L)
\end{array}
\xrightarrow{p(v)}
\begin{array}{c}
P(L)
\end{array}
\]

\[
\begin{array}{c}
\downarrow id \\
\downarrow \epsilon \\
\downarrow \exists \phi \\
\downarrow id
\end{array}
\]

\[
\begin{array}{c}
T^{-1}P(L) \\
\downarrow \\
T^{-1}P(L)
\end{array}
\xrightarrow{u} 
\begin{array}{c}
L \\
\rightarrow \\
C
\end{array}
\xrightarrow{v} 
\begin{array}{c}
P(L)
\end{array}
\]
The left square commutes because of (9.72). Then, by one of the axioms of triangulated category, there exists a morphism, denoted \( \varphi \) in the diagram, such that the remaining two squares also commute. Applying the functor \( P \) to the two squares, and using the fact that \( c \) is an isomorphism of \( P^2 \) to \( \text{Id}_C \) obeying (9.53), we find that the diagram still commutes even if \( \varphi \) is replaced by \( c \circ \varphi \circ c^{-1} \). This means \( \varphi' = c \circ \varphi \circ c^{-1} \). Therefore, we may assume from the beginning that \( \varphi \) obeys this equation, replacing it by the average \( \varphi' \) if necessary. The fact that the identities and \( c \) are isomorphisms means that \( \varphi \) is also an isomorphism. That is, \( \varphi : P(C) \to C \) is an \( o \)-isomorphism! One can also show that \( (C, \varphi) \) is unique up to an isomorphism in \( O(C, P, c) \). We may also call it the invariant cone of \( u : T^{-1}P(L) \to L \) obeying (9.72) and denote it by \( \text{Cone}(u, L) \). This procedure is taken from P. Balmer’s work [72] in which it is called the “symmetric cone construction”. An earlier source is M. Knebusch’s work [73] on the category \( \mathfrak{Bi}(M) \) of holomorphic vector bundles with symmetric bilinear forms over \( M \). See the review [74] and a survey in [75] by Balmer. “Invariant objects” here are called “bilinear space” by Knebusch and “symmetric space” by Balmer.

An invariant object of this type is called \emph{metabolic}. The object \( L \) is called the \emph{Lagrangian} of the invariant cone \( \text{Cone}(u, L) \) because it fits into a triangle that includes

\[
L \xrightarrow{v} C \xrightarrow{P(v) \circ \varphi^{-1}} P(L).
\]

Compare this with the exact sequence (8.35) that defines Lagrangian subbundle of a twisted Real bundle. In the present language, the assertions (i) and (ii) in Section 8.4.2 are stated as (i) any metabolic object is hyperbolic in the category of topological twisted Real bundles, and (ii) any object is metabolic (and hence hyperbolic by (i)) in the category of topological twisted Real Hilbert bundles. In general, and in particular for \( (C, P, c) = (\mathcal{O}(M), P_j, c_j) \), neither is true: there are metabolic but non-hyperbolic objects and there are non-metabolic objects. See the survey by Balmer in [75] and a reference therein for examples on an elliptic curve.

\section*{Binding Invariant Objects}

For two invariant branes, \((X, s)\) and \((Y, t)\), their direct sum \((X \oplus Y, s \oplus t)\) is obviously an invariant brane. Let us see if they can be bound together by the cone construction. For \( f : T^{-1}X \to Y \), its cone \( Z = \text{Cone}(f) \) and its orientifold image have \( Q \)-profiles

\[
Q_Z = \begin{pmatrix}
Q_X & 0 \\
f & Q_Y
\end{pmatrix}, \quad Q_{P(Z)} = \begin{pmatrix}
Q_{P(X)} & -P(f) \\
0 & Q_{P(Y)}
\end{pmatrix}.
\]
For a trial o-isomorphism \( \varphi = \begin{pmatrix} s + \Delta s & a \\ b & t + \Delta t \end{pmatrix} \), the condition for \( Q \varphi = 0 \) is

\[
Q \Delta s = 0, \quad (s + \Delta s)P(f) + iQa = 0, \\
f(s + \Delta s) + iQb = 0, \quad fu + vP(f) + iQ\Delta t = 0,
\]

and the condition (9.54) is \( \Delta s = c_X P(\Delta s), \Delta t = c_Y P(\Delta t), a = c_X P(b) \) and \( b = c_Y P(a) \). If we assume \( \Delta s = 0 \), or if \( s + \Delta s \) is still a quasi-isomorphism, we must conclude that \( f \) is \( Q \)-exact. It then follows that the resulting object is isomorphic to the direct sum. Indeed, if \( f = Qg \) (and \( \Delta s = 0 \) for simplicity), we may take \( a = -sP(g), b = -gs \) and \( \Delta t = gsP(g) \) as the solution, for which \((Z, \varphi)\) is isomorphic to \((X \oplus Y, s \oplus t)\) by

\[
\begin{pmatrix} s & -sP(g) \\ -gs & t + gsP(g) \end{pmatrix} = \begin{pmatrix} \text{id}_X & 0 \\ -g & \text{id}_Y \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} \text{id}_{P(X)} & -P(g) \\ 0 & \text{id}_{P(Y)} \end{pmatrix}.
\]

The only way out would be to find \( \Delta s \) such that \( s + \Delta s \) is no longer a quasi-isomorphism and that \( \varphi \) is a quasi-isomorphism. However, there is no general way to find such a \( \Delta s \). Thus, the cone construction does not lead to anything new in general.

This of course does not mean that it is impossible to bind two invariant branes together. A rather trivial example is to bind two hyperbolic objects \((H_{L_1}, s_{L_1})\) and \((H_{L_2}, s_{L_2})\) into another hyperbolic object \((H_L, s_L)\) where \( L \) is the cone of a morphism \( f : T^{-1}L_1 \rightarrow L_2 \). Note that \( H_L \) is not of the form of a cone since the binding arrows go in both ways — \( f \) goes from \( H_{L_1} \) to \( H_{L_2} \) while \( -P(f) \) goes oppositely. A slightly less trivial example is obtained by replacing the hyperbolic objects by invariant cones in this construction. A yet another example is to bind an invariant object \((X, s)\) and a hyperbolic object \((H_L, s_L)\). Let us try the following \( Q \)-profile and o-isomorphism

\[
Q = \begin{pmatrix} Q_X & w \\ v & Q_{P(L)} \end{pmatrix}, \quad \varphi = \begin{pmatrix} s & \text{id}_{P(L)} \\ c_L & u \end{pmatrix},
\]

for \( v \in C^1(X, L) \), \( w \in C^1(P(L), X) \) and \( u \in C^1(P(L), L) \). The condition for this to determine an invariant brane is

\[
Qv = 0, \quad Qw = 0, \quad vw + iQu = 0, \\
w + sP(v) = 0, \quad u + c_LP(u) = 0.
\]

This channel may correspond to a “tertiary vertex” in flow trees for orientiholes [76].
9.6 K-Theory, Revisited

In Section 8, we discussed the classification of the topology of D-brane configurations in terms of K-theory, where the machinery of Grothendieck group is applied to the categories of topological vector bundles (with additional structures). We may also apply it to the type of categories discussed in the present Section.

The Grothendieck group \([77]\) of a triangulated category \(\mathcal{C}\), denoted by \(K(\mathcal{C})\), is the free Abelian group of isomorphism classes of objects of \(\mathcal{C}\) divided by the relation \([X] - [Y] + [Z] = 0\) for each triangle \(X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)\). For the category \(\mathcal{C} = \mathfrak{D}(M)\), it is equal to the Grothendieck group \(K_\omega(M)\) of holomorphic vector bundles over \(M\). There is a forgetful map to the topological K-theory \(f : K_\omega(M) \to K(M)\). \((9.74)\)

In general, it is neither injective nor surjective.

The Grothendieck group of a category with duality is referred to as the Hermitian K-theory, or the Grothendieck-Witt group, and comes with another version, called the Witt group. They have origins in surgery theory and the theory of symmetric or antisymmetric bilinear forms. The theory is introduced and developed by Wall, Novikov, Karoubi, and many people. See [75, 78–81], for example. The theory for the category \(\mathfrak{Bi}(M)\) was developed by Knebusch [73]. Relevant for us is the case of triangulated categories with duality [72, 82], which we now describe.

The Grothendieck-Witt group (or the Hermitian K-theory) of a triangulated category with duality \((\mathcal{C}, P, c)\), denoted by \(GW(\mathcal{C}, P, c)\), is the free Abelian group of isomorphism classes of objects of \(\mathfrak{D}(\mathcal{C}, P, c)\) divided by the relations \([\{(X, s)\} + \{(Y, t)\} = \{(X \oplus Y), (s \oplus t)\}\) and \([\text{Cone}(u, L)] = [(H_L, s_L)]\). The class of an invariant cone is identified with the underlying hyperbolic object. The Witt group of \((\mathcal{C}, P, c)\), denoted by \(W(\mathcal{C}, P, c)\), is obtained by dividing further by the relation \([\text{Cone}(u, L)] = 0\). Only “truly invariant” objects are non-zero in this group. There is an exact sequence of groups \(K(\mathcal{C}) \to GW(\mathcal{C}, P, c) \to W(\mathcal{C}, P, c) \to 0\). \((9.75)\)

The first map sends \([X]\) to \([(H_X, s_X)]\).

This can be applied to our category with dualities \((\mathfrak{D}(M), P_j, c_j)\). Let us regard them as a series where we use a fixed section \(c_{\text{top}}\) to define \(c_j\). It is 4-periodic by (9.63). If \((\mathcal{C}, P, c)\) is one of them, we define \(GW^i(\mathcal{C}, P, c)\) and \(W^i(\mathcal{C}, P, c)\) as the Grothendieck-Witt and Witt groups for the duality at the “\(i\)-steps ahead”. For example, \(GW^i(\mathfrak{D}(M), P_0, c_0) = \ldots\)
GW(\mathcal{O}(M), P_i, c_i). Note that (9.63) implies that \(GW^{i+2}(\mathcal{C}, P, c) \cong GW^i(\mathcal{C}, P, -c)\) and similarly for the Witt group. These series of groups are of course 4-periodic. Here we would like to quote from [72] a cohomological interpretation of the Witt groups \(W^i\). First, note that \(c_j\) and \(c_{j-1}\) are related by \(c_{j-1}X = c_jX \cdot \sigma_{P_j^2(X)}\) where \(\sigma_{P_j^2(X)}\) is regarded as an isomorphism \(P_j^2(X) \rightarrow P_j^2(X)\). Then, for the case \((P, c) = (P_j, c_j)\) the condition (9.72) can be written as

\[
u = c_{j-1}L \cdot P_{j-1}(u)
\]

for \(u\) regarded as a degree 0 map \(u : P_{j-1}(L) \rightarrow L\). Thus, the invariant cone construction is to construct from a brane with an “o-morphism” for the duality \((P_{j-1}, c_{j-1})\) a brane with an o-isomorphism of the next duality \((P_j, c_j)\). The image of this map consists of metabolic objects for \((P_j, c_j)\). The kernel of the next map consists of brane with o-morphisms for \((P_j, c_j)\) whose cones are trivial. Note that the cone of a morphism is trivial if and only if the morphism is an isomorphism. Therefore, the kernel consists of branes with o-isomorphisms, i.e., invariant branes. Therefore, cohomology classes are invariant objects modulo metabolic ones, which are nothing but elements of the Witt group \(W(\mathcal{O}(M), P_j, c_j) = W^i(\mathcal{O}(M), P_0, c_0)\).

Let us consider D-branes in Type II orientifold on \(M \times \mathbb{R}^D\) with data \((B, \mathcal{L}, \alpha, c)\). We assume that everything non-trivial occurs in the \(M\)-component. Then, the relevant K-group is \(GW(\mathcal{O}(M), P_r, c_r)\) where \(r_o\) is even for \((B_+)\) and odd for \((B_-)\). Here \(c_r\) or \(c_{\text{top}}\) is related to the crosscap section \(c\) by (9.67), i.e., \(c = \epsilon^{-r_o}c_{\text{top}}\). At this point, we recall the crosscap section \(c\) introduced in Section 3.5, \(c = c\) for \((B_+)\) and \(c = \epsilon c\) for \((B_-)\), which is used in many places including Section 8.4 for K-theory classification of topology. If we identify \(c_{\text{top}}\) with \(c\), i.e., if we put

\[
\mu_{r_o} = \epsilon^{-r_o}c,
\]

then, we must take \(r_o = 0\) for \((B_+)\) and \(r_o = -1\) for \((B_-)\). That is, the relevant Hermitian K-theory is

\[
\begin{align*}
(B_+) : & \quad GW^0(\mathcal{O}(M), P_0, c_0) \cong GW^2(\mathcal{O}(M), P_0, -c_0), \\
(B_-) : & \quad GW^{-1}(\mathcal{O}(M), P_0, c_0) \cong GW^1(\mathcal{O}(M), P_0, -c_0),
\end{align*}
\]

(9.76)

Compare this with the topological classification (8.32). This implies that the group \(GW^{-i}(\mathcal{O}(M), P_0, c_0)\) is related to \(KR^{-2i}(M, c)\) in a way similar to the relation between the algebraic and topological K-theories. We might have a forgetful map between them like (9.74), but that requires us to find a representative with a holomorphic o-isomorphism for each invariant object with a quasi-o-isomorphism. See the discussion in the example 9.2.2.

K-theory discussed in this subsection and the topological K-theory discussed in Section 8 carry different information, as shown by the fact that the forgetful map (9.74) is in
general neither injective nor surjective. We may ask whether the category (with duality) also knows about the topological K-theory. The answer is known to be no in general, as long as we view the category only as the triangulated category. However, in many cases it comes from a differential graded category. For our example of D-branes on \( M \), as the space of morphisms we may take the whole \( \mathbb{Q} \)-complex \( \mathcal{C}(\mathcal{B}_1, \mathcal{B}_2) \), with (9.13) as the \( i \)-th component, instead of the 0-th cohomology space. Alternatively (and equivalently [70, 71]), we may take the differential graded category of perfect complexes on \( M \). Then, it is known that, as long as \( M \) is a (smooth) projective variety, the topological K-theory of the underlying topological space, \( \text{K}(M) \), can be recovered from the differential graded category. (This was shown in [84] based on “Semi-topological K-theory” by Friedlander and Walker [83]). It is an interesting question if that can be extended to the differential graded category with duality.

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