Extended Curie-Weiss law: a nonextensive perspective

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Abstract

In the framework of the Tsallis nonextensive statistical mechanics we study an assembly of $N$ spins, first in a background magnetic field, and then assuming them to interact via a long-range homogeneous mean field. To take into account the spin fluctuations the dynamical field coefficient is considered to be linearly dependent on the temperature. The physical quantities are evaluated using a perturbative expansion in the nonextensivity parameter $(1 - q)$. The extended Curie-Weiss law in the mean field case has been generalized. The critical temperature and the Curie-Weiss constant are found to be dependent on the nonextensivity parameter $(1 - q)$.

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I Introduction

A generalization of the Boltzmann-Gibbs extensive statistical mechanics was proposed by Tsallis [1] via a deformation of the functional form of entropy

\[ S = k \ln_q W, \quad q \in \mathbb{R}_+, \] (1.1)

where \( k \) is the Boltzmann constant, and \( W \) is the weight factor. The stability of Tsallis entropy (1.1) is ensured by maintaining the deformation parameter \( q \) as a positive real number [2]. The deformed \( q \)-logarithm and its inverse the \( q \)-exponential function read

\[ \ln_q x = \frac{x^{1-q} - 1}{1 - q}, \quad \exp_q(x) = \left[ 1 + (1 - q)x \right]^{\frac{1}{1-q}}. \] (1.2)

The extensive classical Boltzmann-Gibbs entropy is recovered from (1.1) in the \( q \to 1 \) limit. Nonextensive statistical mechanics has found applications to a wide variety of fields such as anomalous diffusion [5,6], quantum information theory [7,8], astrophysical problems [9,10], and biological systems [11,12].

The introduction of physically appropriate constraints while the nonextensive entropy (1.1) of the system is maximized has been achieved in two alternate ways. In the so-called second constraint picture [3] one works with the unnormalized \( q \)-expectation values of a physical variable \( O \):

\[ \langle O \rangle_q^{(2)} = \sum_j \left( p_j^{(2)}(\beta) \right)^q O_j, \quad \langle 1 \rangle_q^{(2)} \equiv c^{(2)}(\beta) = \sum_j \left( p_j^{(2)}(\beta) \right)^q, \] (1.3)

where \( p_j^{(2)}(\beta) \) is the ensemble probability of the microstate \( j \) in the second constraint picture:

\[ p_j^{(2)}(\beta) = \frac{\exp_q(-\beta E_i)}{Z_q^{(2)}(\beta)}, \quad Z_q^{(2)}(\beta) = \sum_i \exp_q(-\beta E_i). \] (1.4)

It is well-known that in the second constraint formalism the expectation value of the unit operator is not preserved, and, as given in (1.3), it equals to the sum of the \( q \)-weights \( c^{(2)}(\beta) \).

In contrast the third constraint scenario [4] employs appropriately normalized escort probabilities resulting in the following \( q \)-expectation values:

\[ \langle O \rangle_q^{(3)} = \sum_j \left( p_j^{(3)}(\beta) \right)^q O_j, \quad c^{(3)}(\beta) = \sum_j \left( p_j^{(3)}(\beta) \right)^q, \] (1.5)

where \( p_j^{(3)}(\beta) \) is the corresponding probability of the microstate \( j \):

\[ p_j^{(3)}(\beta) = \frac{1}{Z_q^{(3)}} \exp_q \left( -\beta \frac{E_i - U_q^{(3)}}{c^{(3)}(\beta)} \right), \quad Z_q^{(3)}(\beta) = \sum_i \exp_q \left( -\beta \frac{E_i - U_q^{(3)}}{c^{(3)}(\beta)} \right). \] (1.6)
and $c^{(3)}(\beta)$ is the sum of $q$-weights. The thermodynamic averages of the physical quantities are obtained via the generalized partition function $\bar{Z}_q^{(3)}(\beta)$ that relates to the sum of the $q$-weights as

$$\bar{Z}_q^{(3)}(\beta) = \left( c^{(3)}(\beta) \right)^{1/q}. \tag{1.7}$$

In both the above cases the inverse temperature $\beta \equiv (kT)^{-1}$ is associated with the Lagrange multiplier corresponding to the internal energy. In the light of the fact that in the third constraint formalism the unit operator trivially preserves its norm for an arbitrary $q$, it is considered to be fully satisfactory. Fortunately, the ensemble probabilities in the two pictures may be interrelated by the following equivalence relation:

$$p_i^{(3)}(\beta) = p_i^{(2)}(\beta'), \tag{1.8}$$

where the general recipe for constructing the transformation to the auxiliary temperature $\beta'$ reads

$$\beta = \beta' \frac{c^{(2)}(\beta')}{1 - (1 - q)\beta'U_q^{(2)}(\beta')}, \tag{1.9}$$

where $U_q^{(2)}(\beta)$ is the internal energy in the second constraint picture. Its explicit computation pertinent to our models will be discussed later. Here we note that as a result of the above equivalence property the dynamical quantities obtained in the second constraint framework may be translated to their respective values corresponding to the choice of the third constraint.

Application of nonextensive statistical mechanics to spin systems was initiated in [13], where an assembly of $N$ noninteracting spin-$\frac{1}{2}$ particles in a background field was studied in the second constraint formalism, associating the inverse temperature with the Lagrange multiplier corresponding to the energy. A similar study based on the third constraint formalism was also performed [14]. The magnetic susceptibility in this model showed the interesting feature referred to as dark magnetism, indicating that the apparent number of spins are different from the actual number of spins. Employing a high temperature limit, it was observed in [14] that in the domain $q > 1$ ($q < 1$) the effective number of spins $N_{\text{eff}} > N$ ($N_{\text{eff}} < N$). In the study of manganites nonextensive statistical mechanics has been observed [15] to fit the experimental data on magnetization better than the standard Boltzmann-Gibbs statistics. A numerical analysis of a multilevel spin model has been done [16] following Tsallis statistics. A collection of spin clusters has been examined [18,19] in the third constraint picture. These authors obtained generalized paramagnetic susceptibility in the noninteracting regime, and a nonextensive modification of the Curie-Weiss law in the context of the mean field model.

In our current work we make a slight departure from the Refs. [13,14]. We examine a classical arbitrary $N$-spin system in a weak background magnetic field without adopting the high temperature limit. The thermodynamic quantities in the second and the third constraint pictures are evaluated as a perturbative series in the nonextensivity parameter $(1 - q)$ by disentangling the $q$-exponential (1.2). This process of perturbative expansion, that may be continued to an arbitrary order, is based on the technique developed in [20] and previously used in [21,22]. In passing from the second constraint picture to
that of the third constraint we employ a transformation \[22\] procedure that allows us to express the physical quantities in the later scenario directly in terms of the former. In our perturbative expansion for the spins in the background field we retain terms at all orders of temperature. Subsequently we study interacting spins in an extended form of the mean field model \[23\] where the field strength coefficient (the proportionality factor) is made temperature dependent to accommodate quantum fluctuations among allowed configurations \[24\]. The critical temperature and the Curie-Weiss constant have been evaluated in the third constraint framework by retaining terms in the perturbation scheme up to the order \((1 - q)^2\). In particular, the critical temperature in the nonextensive regime increases (decreases) compared to its value given by the Boltzmann-Gibbs statistics for the domain \(q > 1\) \((q < 1)\). Our observation qualitatively agrees with the results obtained in \[17\] where a different definition of temperature for the nonextensive spin system has been used. The plan of this article is as follows: Spins in the presence of a weak external magnetic field is considered in Sec. II. This is followed by the consideration of an extended mean field model in Sec. III. Our concluding remarks are given in Sec. IV.

II Spins in a weak background field

The classical Hamiltonian of a system of \(N\) spins in the presence of a background magnetic field \(H\) is given by

\[
E = -\mu H \sum_{i=1}^{N} \cos \theta_i, \tag{2.1}
\]

where \(\mu\) is the magnetic moment of the spins oriented at polar angles \((\theta_i, \phi_i|i = 1, \ldots, N)\) with the field. The partition function of the system in the second constraint picture reads

\[
Z_{(2)}^{(q)}(\beta) = \int_{\theta_i=0}^{\pi} \int_{\phi_i=0}^{2\pi} \left[ 1 + (1 - q)\beta \mu H \sum_{i=1}^{N} \cos \theta_i \right] \frac{1}{N} \prod_{i=1}^{N} \sin \theta_i \, d\theta_i \, d\phi_i. \tag{2.2}
\]

In contrast to the extensive case the available phase space of integration in (2.2) depends on the strength of the magnetic field. In the regime of weak magnetic field

\[
\hat{\beta} \equiv \beta \mu H < \frac{1}{|1 - q| N}, \tag{2.3}
\]

the integrand is real and positive in the whole phase space, and the partition function may be obtained as follows:

\[
Z_{(2)}^{(q)}(\beta) = (2\pi)^N \left( \frac{\Theta}{\hat{\beta}^N} \right) \mathcal{S}_1(\hat{\beta}), \quad \Phi = \prod_{\ell=1}^{N} \left[ 1 + (1 - q)\ell \right]^{-1}, \tag{2.4}
\]

where the binomial sum \(\mathcal{S}_1(x)\) involving the \(q\)-exponentials is given by

\[
\mathcal{S}_1(x) = \sum_{n=0}^{N} \binom{N}{n} (-1)^n \left( \exp_q((N - 2n)x) \right)^{\Lambda}, \quad \Lambda = 1 + (1 - q)N. \tag{2.5}
\]
In the extensive $q \to 1$ limit above sum approaches its well-known classical value $\mathcal{G}_1(x) \to 2^N \sinh^N x$. The expression (2.4) for the partition function exhibits singularities at $q = 1 + \frac{1}{n}$ for $n = 1, \ldots, N$. These singularities are similar to the ones observed in [21,22], and reflect the fact that the number of degrees of freedom plays a key role in determining the allowed range of the nonextensivity parameter $q$.

The internal energy in the second constraint picture [3]

$$U_q^{(2)}(\beta) = -\frac{\partial}{\partial \beta} \ln_q Z_q^{(2)}(\beta)$$

is evaluated by employing the partition function (2.4):

$$U_q^{(2)}(\beta) = \mu H \Re (\mathcal{G}_1(\hat{\beta}))^{-q} \hat{\beta}^{(q-1)N} \left( N \hat{\beta}^{-1} \mathcal{G}_1(\hat{\beta}) - \mathcal{G}_1'(\hat{\beta}) \right),$$

where

$$\Re = (2\pi)^{(1-q)N} \Phi^{1-q}.$$  

Here and elsewhere the primed functions indicate derivatives with respect to their arguments. The corresponding magnetization is obtained via the defining relation [13]

$$M_q^{(2)} = \frac{1}{\beta} \frac{\partial}{\partial H} \ln_q Z_q^{(2)}.$$  

On subsequent use of a $q$-deformed Langevin function $L_q(x)$ the magnetization (2.9) assumes the form

$$M_q^{(2)} = \mu \, N \, \Re \, L_q(\hat{\beta}),$$

where

$$L_q(x) = x^{(q-1)N} (\mathcal{G}_1(x))^{1-q} \left( \coth_q(x; N) - x^{-1} \right), \quad \coth_q(x; N) = \frac{\mathcal{G}_1'(x)}{N \, \mathcal{G}_1(x)}.$$  

In the extensive $q \to 1$ limit, the magnetization (2.10) reduces to its well-known classical value. The magnetization in the second constraint as a perturbative series in $(1-q)$ reads

$$M_q^{(2)} = \mu \, \Re \left( L(\hat{\beta}) + (1-q) \, M_1 + (1-q)^2 \, M_2 + \ldots \right), \quad L(x) = \coth(x) - x^{-1}.$$  

The coefficients in the perturbative series up to $(1-q)^2$ are given by

$$M_1 = N^2 \, L(\hat{\beta}) \, \ln Z(\hat{\beta}) - N(1-N) \, \hat{\beta}^2 \coth \hat{\beta} \cosech^2 \hat{\beta} + N^2 \coth^2 \hat{\beta}$$

$$+ N(1 - 2N) \hat{\beta} \coth^2 \hat{\beta} - N(1-N) \hat{\beta},$$

$$M_2 = -N^2(1-2N)\hat{\beta} + N(1-N)\hat{\beta}^3 - 6N(1-N+N^2)\hat{\beta}^3 \coth^2 \hat{\beta} \cosech^2 \hat{\beta}$$

$$- N^3 \coth \hat{\beta} - (N - 5N^2)\hat{\beta}^3 \coth^4 \hat{\beta} + \frac{N}{2}(1-N-12N^3)\hat{\beta} \coth^2 \hat{\beta}$$

$$+ \frac{N^3}{2} \hat{\beta}^2 \coth^3 \hat{\beta} + N(3-5N+2N^2)\hat{\beta}^4 \coth^3 \hat{\beta} \cosech^2 \hat{\beta}$$

$$+ \frac{N}{2} \left( (4-13N+10N^2) - (4-6N+2N^2)\hat{\beta}^2 \right) \hat{\beta}^2 \coth \hat{\beta} \cosech^2 \hat{\beta}$$

$$+ \frac{N}{3} (2-6N+3N^2)\hat{\beta}^3 \coth^2 \hat{\beta} \cosech^2 \hat{\beta} + \frac{N^3}{2} (\ln Z(\hat{\beta}))^2 \hat{\beta}$$

$$- N^2 \hat{\beta} \coth \hat{\beta} \ln Z(\hat{\beta}) \, L(\hat{\beta}) + N^2(1-N)\hat{\beta} \coth^2 \hat{\beta} \ln Z(\hat{\beta})$$

$$+ N^2 \hat{\beta}^2 \coth \hat{\beta} \cosech^2 \hat{\beta} \ln Z(\hat{\beta}),$$  

(2.13)
where $Z(x) = 2 \sinh(x)/x$. The perturbative evaluation of the internal energy in the second constraint framework is found to follow the standard thermodynamic relation

$$U = -M H. \quad (2.14)$$

For the sake of brevity we refrain from quoting it explicitly.

Our task now is to translate the previous results to the third constraint picture. As evident from the context of (1.9) the sum of the $q$-weights plays a seminal role in enacting this transformation. The definitions (1.3) and (1.4) lead to the relation

$$c^{(2)}(\beta) = \Omega(\beta) (Z_q^{(2)})^{-q}. \quad (2.15)$$

In the above equation the sum $\Omega(\beta)$ is given by

$$\Omega(\beta) \equiv \sum_i [1 - (1 - q)\beta E_i]^{1-q} = (2\pi)^N \Phi \Lambda \beta^{-N} \mathcal{S}_2(\beta), \quad (2.16)$$

where the binomial sum $\mathcal{S}_2(x)$ reads

$$\mathcal{S}_2(x) = \sum_{n=0}^{N} \left(\begin{array}{c} N \\ n \end{array}\right) (-1)^n \left(\exp\left((N - 2n)x\right)\right)^{1-q}. \quad (2.17)$$

Employing (2.16) and (2.4) we may express the sum of $q$-weights in the second constraint picture as

$$c^{(2)}(\beta) = \mathfrak{M} \Lambda \beta(q-1)^N \mathcal{S}_2(\beta) \mathcal{S}_1(\beta)^{-q}, \quad (2.18)$$

and its perturbative evaluation up to terms $O(1 - q)^3$ reads:

$$c^{(2)}(\beta) = \mathfrak{M} \left(1 + (1 - q) \mathcal{P}_1 + (1 - q)^2 \mathcal{P}_2 + (1 - q)^3 \mathcal{P}_3 + \ldots\right), \quad (2.19)$$

where the perturbative coefficients may be enlisted as

$$\begin{align*}
\mathcal{P}_1 &= N \ln Z(\hat{\beta}) - N \hat{\beta} \coth \hat{\beta} + N, \\
\mathcal{P}_2 &= \frac{N^2}{2} \hat{\beta}^2 \coth^2 \hat{\beta} - \frac{N}{2} \left(1 - 2N\right) \frac{\hat{\beta}}{\hat{\beta} \coth \hat{\beta}} \left(1 - 2N\hat{\beta} \coth \hat{\beta}\right) \hat{\beta}^2 \coth^2 \hat{\beta} \\
&\quad + \frac{N^2}{2} \left(2 - 2 \hat{\beta} \coth \hat{\beta} + \ln Z(\hat{\beta})\right) \ln Z(\hat{\beta}), \\
\mathcal{P}_3 &= N^3 \hat{\beta} \left(1 + 4 \hat{\beta}^2\right) \coth \hat{\beta} + N \left(3 - 5N - 2N^2\right) \hat{\beta}^5 \coth^3 \hat{\beta} \coth^2 \hat{\beta} \\
&\quad + \frac{N}{12} \left(17 - 33N + 12N^2\right) \hat{\beta}^4 \coth^2 \hat{\beta} \coth \hat{\beta} \\
&\quad + \frac{N}{4} \left(17 - 39N + 22N^2\right) \hat{\beta}^4 \coth^2 \hat{\beta} \coth \hat{\beta} \\
&\quad + \frac{N^3}{6} \ln Z(\hat{\beta}) \left(3 \hat{\beta}^2 \coth^2 \hat{\beta} + 3(1 - \hat{\beta} \coth \hat{\beta}) \ln Z(\hat{\beta}) + (\ln Z(\hat{\beta}))^2\right) \\
&\quad - \frac{N^2(1-N)}{2} \left(1 - 2 \hat{\beta} \coth \hat{\beta}\right) \hat{\beta}^2 \coth^2 \hat{\beta} \ln Z(\hat{\beta}). \quad (2.20)
\end{align*}$$

Following an approach used in [22] we interrelate the dynamical quantities such as the internal energy and the magnetization directly from the second to the third constraint.
picture. The equivalence of the ensemble probabilities (1.8) leads to a ready translation of the expectation values of an observable \( O \) in the two constraint frameworks as

\[
O^{(3)}_q(\beta) = \frac{O^{(2)}_q(\beta')}{c^{(2)}(\beta')}. \tag{2.21}
\]

To fruitfully employ this procedure we need to invert the transformation relation (1.9) of the temperature. As a closed form inversion rule is not at hand, we adopt a perturbative technique \[21\], and express the auxiliary temperature \( \beta' \) in terms of the physical temperature \( \beta \) retaining terms up to second order in \((1 - q)\):

\[
\beta' = \frac{\beta}{N} \left(1 + (1 - q) g(\beta) + (1 - q)^2 h(\beta) + \ldots \right), \tag{2.22}
\]

where the rescaled temperature is given by \( \overline{\beta} = \beta N^{-1} \). The perturbative terms of the transformation relation read

\[
g(\beta) = -2N(1 - \overline{\beta} \coth \overline{\beta}) - N \ln Z(\overline{\beta}),
\]

\[
h(\beta) = -\frac{3N}{2} \beta^2 (1 - \coth^2 \overline{\beta}) - 2N(1 + N) \overline{\beta}^3 \coth \beta \cosh \overline{\beta}^3 - 5N^2 \overline{\beta} \coth \overline{\beta}
+ 5N^2 \beta^3 \coth \overline{\beta}^3 + 3N^2 \beta^2 \coth \overline{\beta}^3 - N^2 (1 - \beta^2) + \frac{N^2}{2} (\ln Z(\overline{\beta}))^2
- 3N^2 \beta \coth \overline{\beta} \ln Z(\overline{\beta}) + 2N^2 \beta^2 \coth^2 \overline{\beta} \ln Z(\overline{\beta}). \tag{2.23}
\]

Aided by the inverse transformation series (2.22) we employ (2.21) for obtaining a perturbative evaluation of the magnetization in the third constraint approach to the order \( O((1 - q)^2) \):

\[
M^{(3)}_q(\beta) = \mu \left(N L(\overline{\beta}) + (1 - q) M_1 + (1 - q)^2 M_2 + \ldots \right), \tag{2.24}
\]

where the coefficients \( M_1 \) and \( M_2 \) read

\[
M_1 = N^2 \left(2\overline{\beta}^2 - 1\right) L(\overline{\beta}) - N^2 \overline{\beta} \ln Z(\overline{\beta}) L'(\overline{\beta}),
\]

\[
M_2 = 2N^3 L(\overline{\beta}) + 4N^3 \overline{\beta} - \frac{N^2}{2} \overline{\beta}^3 - N \overline{\beta}^3 - (2N + 5N^2) \overline{\beta}^3 \coth \overline{\beta} \cosh \overline{\beta}^3
+ \frac{N}{2} \left(4 + 10N - (4 + 9N^2) \overline{\beta}^4\right) \overline{\beta}^2 \coth \overline{\beta} \cosh \overline{\beta}^2
+ 2N(6 + N + N^2) \overline{\beta}^4 \coth \overline{\beta}^4 \cosh \overline{\beta}^4 - \frac{N}{2} (9N^2 + 4N \overline{\beta}^2 - 8) \overline{\beta} \coth \overline{\beta}^2
- \frac{3N}{2} (2 + N) \overline{\beta}^3 \coth \overline{\beta}^3 + \frac{N^2}{6} \left(9 - 36N - (4 - 6N) \overline{\beta}^2\right) \overline{\beta} \cosh \overline{\beta} \cosh \overline{\beta}^2
+ N \overline{\beta} \coth \overline{\beta} \cosh \overline{\beta}. \tag{2.25}
\]

The internal energy \( U^{(3)}_q(\beta) \) in the third constraint picture may be read off directly from the corresponding magnetization \( M^{(3)}_q(\beta) \) via the general thermodynamic relation (2.14). We do not quote it explicitly. Magnetic susceptibility is defined as

\[
\chi^{(3)}_q(\beta) \equiv \frac{\partial M^{(3)}_q}{\partial H}, \tag{2.26}
\]
and we now employ (2.24) to compute it:

\[ \chi_3^{(3)}(\beta) = \frac{\mu^2}{\beta} \left( \left( \frac{N}{\beta} \right)^2 - N \coth^2 \beta + (1 - q) \vartheta_1 + (1 - q)^2 \vartheta_2 + \ldots \right). \]  

(2.27)

The above perturbative coefficients up to the order \(O((1 - q)^3)\) are given below:

\[ \begin{align*}
\vartheta_1 &= \frac{N^2}{\beta^3} \ln \mathcal{Z}(\beta) + N \left( 1 + N + N \ln \mathcal{Z}(\beta) \right) \coth^2 \beta - 2N^2 \coth^2 \beta - 2N^2 \coth^2 \beta \\
&\quad + N \left( 1 + N \right) \coth^2 \beta + 2N \left( 1 + N \right) \coth^2 \beta - 3N^2 \coth^2 \beta \\
&\quad - N \left( 4N^2 - 4 \beta \right) \coth \beta \coth^2 \beta,
\end{align*} \]

(2.28)

The specific heat in the third constraint framework is obtained via the corresponding internal energy:

\[ C_q^{(3)}(\beta) = \frac{\partial U_q^{(3)}}{\partial T} = k \frac{\beta^2}{N} \left( \left( \frac{N}{\beta} \right)^2 - N \coth^2 \beta + (1 - q) \vartheta_1 + (1 - q)^2 \vartheta_2 + \ldots \right). \]

(2.29)

The thermodynamic quantities in the weak field limit \(\hat{\beta} \ll 1\) follows directly:

\[ U_q^{(3)}(\beta) = -\frac{N_{\text{eff}}}{3} \beta \mu^2 H^2, \quad C_q^{(3)} = \frac{N_{\text{eff}}}{3} k \beta^2, \]

(2.30)
Figure 1: Dependence of the ratio $N_{\text{eff}}/N$ on $q$ for various values of $N$.

where the $q$-dependent effective number of spins $N_{\text{eff}}$ reads

$$N_{\text{eff}} = \frac{N}{\mathcal{R}} \left(1 - (1 - q)(1 + N \ln 2) - \frac{(1 - q)^2}{2} \left(N - 2N \ln 2 + N^2 - (N \ln 2)^2\right) + \ldots\right).$$

Substituting the value (2.8) of the $q$-dependent scale factor $\mathcal{R}$ in (2.31), we notice that up to the perturbative order $(1 - q)^2$ it follows $N_{\text{eff}} > N$ ($N_{\text{eff}} < N$) for the region $q > 1$ ($q < 1$). This is evident from the Fig. 1.

Turning to the magnetization (2.24) in a weak field $\beta \ll 1$ regime, we obtain

$$M_q^{(3)} = \frac{N_{\text{eff}} \mu^2 H}{3kT},$$

where the corresponding susceptibility may be viewed as a nonextensive generalization of Curie’s law:

$$\chi_q^{(3)} = \frac{c_{\text{eff}}}{T}, \quad c_{\text{eff}} = \frac{N_{\text{eff}} \mu^2}{3k}.$$

The thermodynamic quantities evaluated above (2.30–2.33) show a nonlinear dependence on the number of spins $N$, and the nonextensivity parameter $(1 - q)$. This effect of nonextensivity manifest in the aforesaid inequality between $N_{\text{eff}}$ and $N$ leads to the phenomenon of dark magnetism discussed in [13].

The specific heat, the internal energy and the magnetization have also been evaluated using the approach discussed in [4], where the sum of the $q$-weights plays a central role. The relation between the ensemble probabilities (1.8) enables us to obtain $c_q^{(3)}(\beta)$ as a perturbative series

$$c_q^{(3)}(\beta) = \mathcal{R} \left(1 + (1 - q) \mathcal{P}_1 + (1 - q)^2 \mathcal{P}_2 + (1 - q)^3 \mathcal{P}_3 + \ldots\right),$$

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where the coefficients may be listed as

\[ \mathcal{P}_1 = N \ln Z(\beta) - N(\beta \coth \beta - 1), \]
\[ \mathcal{P}_2 = \frac{N^2}{2} \left( \ln Z(\beta) + 4N^2 - 2\beta^2 \coth^2 \beta - 2\beta \coth \beta \right) \ln Z(\beta) \]
\[ + \frac{N^2}{2} \left( 4 - 4\beta \coth \beta - \beta^3 \right) + N(1 + N)\beta^3 \coth \beta \cosech^2 \beta \]
\[ - \frac{N}{2}(1 + N)\beta^2 \cosech ^2 \beta, \]
\[ \mathcal{P}_3 = 5N^3 - \frac{N^3}{2} \beta (\beta + 8 \coth \beta) + \frac{N^2}{2} (1 - 2N - 6N^2) \beta^2 \cosech^2 \beta \]
\[ + \frac{N}{12} (17 - 12N + 24N^2) \beta^4 \cosech^4 \beta - 2N(1 + N + N^2)\beta^5 \coth \beta \cosech^2 \beta \]
\[ - N(3N^2 - \beta^3 \coth^3 \beta - N^2(1 + 3N)\beta^5 \coth \beta \cosech^2 \beta \]
\[ - \frac{N^3}{6} \beta^3 \coth^3 \beta - \frac{N}{4} (17 - 23N - 12N^2) \beta^4 \coth^2 \beta \cosech^2 \beta + 4N^3 \ln Z(\beta) \]
\[ - N^2 (1 - 2N) \beta^2 \cosech^2 \beta \ln Z(\beta) - 3N^3 \beta \coth \beta \ln Z(\beta) \]
\[ - N^2 (3 + 2N) \beta^3 \coth \beta \cosech \beta \ln Z(\beta) + \frac{N^3}{2} \beta^2 \coth^2 \beta \ln Z(\beta) \]
\[ - N^2 (3 - N)\beta \coth^2 \beta \cosech^2 \beta \ln Z(\beta). \] (2.35)

The exponent property (1.7) in conjunction with our perturbative evaluation (2.34) of the sum of the \( q \)-weights, now allows us to employ the formulation specified in [4] as a consistency check on our results. Examining the generalized partition function a differential equation involving the internal energy has been established in [4]:

\[ \beta \frac{\partial U_q^{(3)}}{\partial \beta} = \frac{\partial}{\partial \beta} \ln Z_q^{(3)}(\beta) = \frac{\partial}{\partial \beta} c^{(3)}(\beta) - \frac{1}{1 - q}, \] (2.36)

where we have used the exponent relation (1.7) in the last equality. Substituting the internal energy that may be readily obtained via the equations (2.14) (2.24), and the sum of the \( q \)-weights given in (2.34), it may be explicitly verified that the differential equation (2.36) holds order by order in our perturbation theory. This is a nontrivial consistency check on our results for physical quantities. It has been noted earlier [22] that, as a consequence of the exponent relation (1.7), evaluations of the thermodynamic quantities such as specific heat, say, up to the second order in \((1 - q)\) necessitate computing the sum of the \( q \)-weights \( c^{(3)}(\beta) \) till the third order. That counting has been maintained in (2.34).

Furthermore, the differential equation (2.36) may be used for a direct extraction of the susceptibility in our model. As suggested by (2.24) we postulate the general functional dependence of magnetization as

\[ M_q^{(3)} = \mu f(\beta). \] (2.37)

The relations (2.14), (2.36) and (2.37) now readily produce the magnetic susceptibility:

\[ \chi_q^{(3)} = - \frac{\mu}{N H} \frac{\partial}{\partial \beta} \left( \frac{c^{(3)} - 1}{1 - q} \right). \] (2.38)
The derivation following from (2.38) precisely agrees with the magnetic susceptibility obtained earlier in (2.27). This confirms the validity of our perturbative procedure.

III Mean field model: temperature dependent effective field coefficient

Physical reasoning tells us that effects of nonextensivity is likely to be pronounced for systems embodying long-range interactions between the constituents. For an interacting spin system a good first approximation is provided by the mean field model where the long-range component of the interaction between the spins is taken into account via a homogeneous magnetic field \((H_m)\) that is assumed to be directly proportional to the magnetization per spin. The Hamiltonian of the system reads

\[
E = -\mu (H + H_m) \sum_{i=1}^{N} \cos \theta_i, \tag{3.1}
\]

where \(H_m\) is the dynamical field resulting from the long range interactions between the spins as envisaged in the mean field model. The generalized partition function in the third constraint reads

\[
\bar{Z}_q^{(3)} (\beta) = \exp_q (\epsilon) \int_{\theta_i=0}^{\pi} \int_{\phi_i=0}^{2\pi} \left[ 1 + (1 - q) \tilde{\beta} \sum_{i=1}^{N} \cos \theta_i \right]^{1-q} \prod_{i=1}^{N} \sin \theta_i d\theta_i d\phi_i, \tag{3.2}
\]

where \(\epsilon = \frac{\beta U_q^{(3)}}{U_q^{(3)}}\), and the scaled dimensionless variable \(\tilde{\beta}\) is given by

\[
\tilde{\beta} = \frac{\beta \mu (H + H_m)}{\epsilon^{(3)} (\beta) + (1 - q) \beta U_q^{(3)}}. \tag{3.3}
\]

In the regime \(|1 - q| < 1\) we assume that the integrand is real and positive in the entire phase space, and obtain the generalized partition function as follows:

\[
\bar{Z}_q^{(3)} (\beta) = (2\pi)^N \Phi \exp_q (\epsilon) \bar{\beta}^{-N} \mathcal{G}_1 (\bar{\beta}). \tag{3.4}
\]

The primary definition of a thermodynamic observable (1.5) leads to the following integral form of the internal energy in the third constraint picture:

\[
U_q^{(3)} (\beta) = -\mu (H + H_m) \left( \bar{Z}_q^{(3)} (\beta) \right)^{-1} \left( \exp_q (\epsilon) \right)^q \int_{\theta_i=0}^{\pi} \int_{\phi_i=0}^{2\pi} \left[ 1 + (1 - q) \tilde{\beta} \sum_{i=1}^{N} \cos \theta_i \right]^{1-q} \prod_{i=1}^{N} \sin \theta_i d\theta_i d\phi_i. \tag{3.5}
\]

Implementing the above phase space integration we obtain the internal energy as

\[
U_q^{(3)} (\beta) = \left( \frac{2\pi}{\beta} \right)^N \Phi \mu (H + H_m) \left( N \tilde{\beta}^{-1} \mathcal{G}_1 (\tilde{\beta}) - \mathcal{G}_1^{(3)} (\tilde{\beta}) \right) \frac{(\exp_q (\epsilon))^q}{\bar{Z}_q^{(3)}}. \tag{3.6}
\]
On substituting the generalized partition function (3.4) the implicit equation (3.6) may be recast as follows:

\[ U_q^{(3)}(\beta) = \frac{\mu(H + H_m)}{1 + (1 - q) \varepsilon} \left( \frac{N}{\beta} - \frac{\mathcal{S}_1(\tilde{\beta})}{\mathcal{S}_1(\beta)} \right). \]  

(3.7)

We expand the term in the parenthesis up to second order in the nonextensivity parameter \((1 - q)\) and fourth order in the dynamical variable \(\beta \mu(H + H_m)\). The nonzero contributions in the resulting expansion read

\[ U_q^{(3)}(\beta) = -\frac{N \beta \mu^2(H + H_m)^2}{3 \, c^{(3)}(\beta)} \left( \mathcal{U}_1 - \frac{\mathcal{U}_2}{15} \left( \frac{\beta \mu(H + H_m)}{c^{(3)}(\beta)} \right)^2 + \ldots \right), \]  

(3.8)

where the perturbative coefficients are given by

\[
\begin{align*}
\mathcal{U}_1 & = 1 - (1 - q) \left( 1 + 2 \varepsilon \right) + (1 - q)^2 \varepsilon \left( 2 + 3 \varepsilon \right) + \ldots, \\
\mathcal{U}_2 & = 1 - (1 - q) \left( 6 - 10 N + 4 \varepsilon \right) + (1 - q)^2 \left( 11 - 25 N + 24 \varepsilon - 40 \varepsilon + 10 N \varepsilon^2 \right) + \ldots .
\end{align*}
\]  

(3.9)

The phase space integral corresponding to the sum of the \(q\)-weights may be read off via (1.6) and (1.7):

\[ c^{(3)}(\beta) = \left( \frac{\exp_q(\varepsilon)}{Z_q^{(3)}(\beta)} \right)^q \int_{\theta_i=0}^{\pi} \int_{\phi_i=0}^{2\pi} \left[ 1 + (1 - q) \tilde{\beta} \sum_{i=1}^N \cos \theta_i \right]^{\frac{N}{\beta \mu(H + H_m)^2}} \prod_{i=1}^N \sin \theta_i \, d\theta_i \, d\phi_i. \]  

(3.10)

Performing the above integrations and subsequently substituting the generalized partition function (3.4) above sum of the \(q\)-weights assumes the form

\[ c^{(3)}(\beta) = \mathcal{R} \Lambda \, \tilde{\beta}^{(q-1)N} \, \mathcal{S}_2(\tilde{\beta})^{-q} \mathcal{S}_1(\tilde{\beta})^{-q}. \]  

(3.11)

As done before in the instance of the internal energy in (3.8) the rhs of the expression (3.11) is expanded perturbatively up to second order in the nonextensivity parameter \((1 - q)\) and fourth order in the variable \(\beta \mu(H + H_m)\):

\[ c^{(3)}(\beta) = \mathcal{R} \left( 1 + (1 - q) N \ln 2 + (1 - q)^2 \frac{N}{2} \left( 1 + N + N \ln(2)^2 \right) \right) \]  

\[ + \varphi_1(\varepsilon) \left( \frac{\beta \mu(H + H_m)}{c^{(3)}(\beta)} \right)^2 + \varphi_2(\varepsilon) \left( \frac{\beta \mu(H + H_m)}{c^{(3)}(\beta)} \right)^4 + \ldots , \]  

(3.12)

where the perturbative expansion of the coefficients of monomials of the field variable read

\[
\begin{align*}
\varphi_1(\varepsilon) & = -\frac{N}{6} \left( (1 - q) + (1 - q)^2 \left( 1 - N \ln 2 + 2 \varepsilon \right) + \ldots \right), \\
\varphi_2(\varepsilon) & = \frac{N}{60} \left( (1 - q) - (1 - q)^2 \left( 6 - \frac{15 N}{2} - N \ln 2 + 4 \varepsilon \right) + \ldots \right). 
\end{align*}
\]  

(3.13)
Towards obtaining the magnetization we first explicitly obtain the quantities \( U_q^{(3)}(\beta) \) and \( c^{(3)}(\beta) \) by solving the pair of simultaneous implicit equations (3.8) and (3.12). We can systematically obtain their solutions in an order by order perturbation theory where we retain terms up to \((1 - q)^2\) in the nonextensivity parameter and \((\beta \mu (H + H_m))^4\) in the field variable. The relevant expression for the internal energy reads

\[
U_q^{(3)}(\beta) = -\frac{N \beta \mu^2 (H + H_m)^2}{9 \Gamma} \Gamma + \frac{N \beta^3 \mu^4 (H + H_m)^4}{9 \Xi} \Xi + \ldots,
\]

(3.14)

where the \( q \)-dependent numerical factors \( \Gamma \) and \( \Xi \) are given by

\[
\Gamma = 1 - (1 - q) (1 + N \ln 2) - (1 - q)^2 \frac{N}{2} (1 + N - 2 \ln 2 - N (\ln 2)^2) + \ldots,
\]

(3.15)

\[
\Xi = 1 - (1 - q) \left( 6 + \frac{5N}{2} + 3N \ln 2 \right) + (1 - q)^2 \left( 11 - \frac{3N}{2} (N + 1) + \frac{N \ln 2}{2} (36 + 15N + 9N \ln 2) \right) + \ldots.
\]

(3.16)

The perturbative expansion also yields the sum of \( q \)-weights as

\[
c^{(3)}(\beta) = \mathcal{R} \left( 1 + (1 - q) N \ln 2 + (1 - q)^2 \frac{N}{2} (1 + N + N (\ln 2)^2) \right.
\]

\[
+ \Pi_1 \left( \frac{\beta \mu (H + H_m)}{9} \right)^2 + \Pi_2 \left( \frac{\beta \mu (H + H_m)}{9} \right)^4 + \ldots \right),
\]

(3.17)

where the coefficients may be listed as follows:

\[
\Pi_1 = -\frac{N}{6} \left((1 - q) + (1 - q)^2 (1 - N \ln 2) + \ldots\right),
\]

\[
\Pi_2 = \frac{N}{60} \left((1 - q) - (1 - q)^2 \left( 6 + \frac{5N}{2} + 3N \ln 2 \right) + \ldots \right).
\]

(3.18)

As the internal energy and the magnetization are related via the standard thermodynamic expression (2.14) we may now readily obtain a perturbative expansion for the magnetization by employing the corresponding series (3.14) for the internal energy:

\[
M_q^{(3)} = \mu \left( \frac{N}{3 \mathcal{R}} \beta \mu (H + H_m) \right) \Gamma - \frac{N}{45 \mathcal{R}^3} \left( \beta \mu (H + H_m) \right)^3 \Xi \right). \]

(3.19)

The homogeneous magnetic field \( H_m \) is taken to be proportional to the magnetic moment per spin. To account for the quantum effect of spin fluctuations the proportionality factor, known as the effective field coefficient \( \lambda(T) \), is considered to be temperature dependent [24,23]. Following [23] we here study a mean field spin model with the field coefficient \( \lambda(T) \) being linearly dependent on temperature. The above discussion leads to the field variable

\[
H_m = \lambda(T) m_q^{(3)}, \quad m_q^{(3)} = \frac{M_q^{(3)}}{N}, \quad \lambda(T) = \xi + \zeta T,
\]

(3.20)
where the coefficients $\xi$ and $\zeta$ characterize the long range spin interaction. Substituting (3.20) in (3.19) and considering the vanishing limit of the external field $H = 0$, the magnetization reads

$$m_q^{(3)} \Theta \left( \frac{3 \Xi}{5 \Gamma^3} \mu^2 \left( 1 + \frac{\zeta T}{\xi} \right)^3 \Theta ^2 \left( \frac{T_c^{(3)}}{T} \right)^3 \left( m_q^{(3)} \right)^2 + 1 - \frac{T_c^{(3)}}{T} \right) = 0, \quad (3.21)$$

where the critical temperature is

$$T_c^{(3)} = \frac{\xi \mu^2 \Gamma}{3k \mathfrak{R} \Theta}, \quad \Theta = 1 - \frac{\zeta \mu^2 \Gamma}{3k \mathfrak{R}}. \quad (3.22)$$

To visualize the variation of the critical temperature with respect to the nonextensivity parameter $(1 - q)$, we, in Fig.2, plot the ratio $\kappa$ defined as

$$\kappa \equiv \frac{\beta_c^{(3)} + \delta}{\beta_c^{(3)} |_{q=1} + \delta}, \quad \delta = \frac{\zeta}{\xi}. \quad (3.23)$$

From Fig.2 it may be inferred that the critical temperature increases (decreases) compared to its standard Boltzmann-Gibbs value in the regime $q > 1$ ($q < 1$). This result is qualitatively similar to the observation in [17] where these authors have adopted an alternate definition of the temperature for the nonextensive spin system. The mean field model studied by these authors is slightly different from the one considered here in that we assume the dynamical field strength to be temperature dependent in order to accommodate the quantum fluctuations in spin configurations.

Equation (3.21) suggests that above the critical temperature $T > T_c^{(3)}$, the only real solution for the magnetization in null external field condition is given by

$$m_q^{(3)} |_{H=0} = 0 \quad (3.24)$$

that corresponds to the paramagnetic phase. Using the standard definition (2.26) in conjunction with the relations (3.19) and (3.24) we now obtain the magnetic susceptibility in the paramagnetic regime:

$$\chi_q^{(3)} = \frac{C}{T - T_c^{(3)}}, \quad C = \frac{N \mu^2 \Gamma}{3k \mathfrak{R} \Theta}. \quad (3.25)$$

It is evident from (3.21) that below the critical temperature $T < T_c^{(3)}$ the magnetization in the null external field limit is given by two stable real values given by

$$m_q^{(3)} |_{H=0} = \pm \mu \sqrt{\frac{5 \Gamma^3}{3 \Xi}} \Theta^{-1} \left( 1 + \frac{\zeta T}{\xi} \right)^{-3/2} \frac{T}{T_c^{(3)}} \sqrt{1 - \frac{T}{T_c^{(3)}}}. \quad (3.26)$$

This corresponds to the ferromagnetic transition as observed in the generalized mean field approach in the nonextensive framework. In the regime $T \to T_c^{(3)}$ the susceptibility in the null external field limit may be obtained via (3.19) and (3.26):

$$\chi_q^{(3)} = \frac{C/2}{T_c^{(3)} - T}. \quad (3.27)$$
As characteristic of the mean field approach, the divergence of the susceptibility follows critical exponent law
\[ \chi_q^{(3)} \sim \frac{1}{|T - T_c^{(3)}|}, \]
where the critical temperature depends on the nonextensivity parameter \((1 - q)\). Of course in the extensive \(q \to 1\) limit, the standard physical quantities corresponding to the Boltzmann-Gibbs statistics are recovered. We also quote the results obtained in the usual mean field model, where the dynamical proportionality factor \(\lambda\) is regarded as independent of temperature \(i.e.\) the linear coefficient in (3.20) assumes the value \(\zeta = 0\). In this limit the critical temperature and the Curie-Weiss constant in the nonextensive scenario may be arrived at via (3.22) and (3.25):
\[ T_c^{(3)} = \frac{\xi \mu^2 \Gamma}{3k N}, \quad \mathcal{C} = \frac{N \mu^2 \Gamma}{3k N}. \]

IV Remarks

Our main focus in the present work has been to study a system of spins with long range interactions approximated by a mean field model governed by the nonextensive Tsallis statistics. To incorporate the quantum spin fluctuations the mean field model investigated here is assumed to have the dynamical field strength coefficient depending linearly on temperature. The nonextensivity is implemented by using a perturbative technique, where the implicit simultaneous equations involving the internal energy and the sum of
$q$-weights were solved explicitly as series expansions up to the order $(1 - q)^2$ in the nonextensivity parameter. The perturbation method developed here may be continued to an arbitrary order in the parameter $(1 - q)$. The signature of the nonextensivity is evident as the critical temperature is found to depend on the number of spins $N$ and the deformation variable $(1 - q)$. Compared to its standard Boltzmann-Gibbs value the critical temperature increases (decreases) for the domain $q > 1$ ($q < 1$). The extended Curie-Weiss law characterizing the susceptibility in the regions above and below the critical temperature has been generalized to the nonextensive case. Analogous to the critical temperature the Curie-Weiss constant also embodies the effects of nonextensivity. Another interesting feature reflecting nonextensivity emerges for noninteracting spins in the presence of a background field, where we observe the presence of dark magnetism. This supports the results [13,14] obtained earlier. Parallel to the observation in [14] our analysis indicates that the effective number of spins $N_{\text{eff}} > N$ ($N_{\text{eff}} < N$) for the regime $q > 1$ ($q < 1$).

The mean field technique used in the current work may be fruitfully employed in calculating the magnetic properties of systems which form clusters [25]. A cluster is typically a macroscopic region consisting of a large number of spins whose interactions are described via locally homogeneous mean fields in the domain of each cluster. Then each cluster is approximated by a single effective spin forming an ensemble of varying spins whose magnetic properties may be studied by adopting a suitable model. The results obtained here may facilitate studying such models in the context of Tsallis statistics.

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