Three dimensional dilatonic gravity’s rainbow: exact solutions

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Deep relations of dark energy scenario and string theory results into dilaton gravity, on one hand, and the connection between quantum gravity with gravity’s rainbow, on the other hand, motivate us to consider three dimensional dilatonic black hole solutions in gravity’s rainbow. We obtain two classes of the solutions which are polynomial and logarithmic forms. We also calculate conserved and thermodynamic quantities, and examine the first law of thermodynamics for both classes. In addition, we study thermal stability and show that one of the classes is thermally stable while the other one is unstable.

I. INTRODUCTION

In the context of AdS/CFT correspondence, the Hawking-Page phase transition is often applied for investigating a confinement-deconfinement phase transition. In other words, in lower than a critical temperature ($T_c$), quarks are confined to be grouped together in pairs or triples configurations, while for $T >> T_c$ they are in a deconfined phase (quark-gluon plasma). In the black hole language, a transition from unstable to thermally stable solutions is the so-called Hawking-Page phase transition. The Hawking-Page phase transition is one of the primitive attempts for obtaining a quantum theory of gravity. In general, black hole thermodynamics is a bridge between classical viewpoint of general relativity with quantum gravity. Although we could not already find a full description of quantum theory of gravity, the attempts for reducing the gaps between the theory of general relativity and quantum mechanics are still alive. Gravity’s rainbow is one of such attempts that arises from a deep insight to remove the gap between classical theory of gravity and its quantum nature. Recently, it has also been observed that gravity’s rainbow has some interesting consequences. Among them one may recall black hole remnant [1], information paradox [2] and nonsingular universe [3–5].

On the other hand, one of the most common results of quantum gravity is the existence of a minimum length [6, 7]. Also, this minimum length may be naturally arisen in string theory [8–10]. The existence of a minimum measurable length can be translated to an upper limit of energy probe in high energy physics. In addition to the constant velocity of light, one may regard invariant Planck energy to obtain doubly special relativity [11–13]. The doubly special relativity is motivated by the following generalized energy-momentum dispersion relation

$$E^2 f^2(\varepsilon) - p^2 g^2(\varepsilon) = m^2,$$

where $\varepsilon = E/E_P$, $E_P$ is the Planck energy and $E$ is the energy probe of the test particle. The functions $f(\varepsilon)$ and $g(\varepsilon)$ are called rainbow functions, and they should satisfy the following constraint

$$\lim_{\varepsilon \to 0} f(\varepsilon) = 1, \quad \lim_{\varepsilon \to 0} g(\varepsilon) = 1.$$  \hspace{1cm} (2)

This condition guarantees that one can reproduce the standard dispersion relation in the infrared limit. Basically, it has also been possible to extend doubly special relativity to the case of curved spacetime and the resultant theory is known as doubly general relativity or gravity’s rainbow [14, 15]. The arbitrary metric $G(E)$ in gravity’s rainbow can be written as [15]

$$G(E) = \eta^{ab} e_a(E) \otimes e_b(E),$$

where the energy dependency of the frame fields are

$$e_0(E) = \frac{1}{f(\varepsilon)} \hat{e}_0, \quad e_i(E) = \frac{1}{g(\varepsilon)} \hat{e}_i,$$

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in which the tilde quantities refer to the energy independent frame fields as well. As we mentioned, the maximum attainable energy is the Planck energy \( E_P \) and the energy at which spacetime is probed is represented by \( E \). Therefore, considering a test particle which is probing the geometry of spacetime, one can regard \( E \) as the energy of that test particle.

On the other hand, it is important to study the UV completion of general relativity such that it reduces to the general relativity in the IR limit. In this regard, one may generalize the Einstein theory to Horava-Lifshitz gravity [16, 17]. The Horava-Lifshitz gravity is also motivated by a deformation of the usual energy-momentum dispersion relation in the UV limit. So, there is a close relation between gravity’s rainbow and Horava-Lifshitz gravity [18]. This relation is due to the fact that the foundations of both theories are based on breaking of the usual energy-momentum dispersion relation in the UV limit. Considering the UV deformation of geometries that occurs in string theory and the close relation between Horava-Lifshitz gravity with gravity’s rainbow [18], one is motivated to regard modified energy dependent metrics.

One of the main motivations of considering the string theory, at present, is furnishing a combined quantum theory of gravity and gauge field interactions [14–21]. It was shown that, in the context of string theory, the usual gravitational field tensors are mixed with a scalar field partner; the so-called dilaton. Taking into account the dilaton field, one may ask for its coupling with other fields. It is considered that dilaton may couple with gravitational field in Jordan frame (Brans-Dicke theory), while it couples to matter field in Einstein frame (dilaton gravity). Actions and field equations of both Brans-Dicke theory and dilaton gravity are conformally related. Therefore, one may use the solutions of Einstein frame to obtain the corresponding ones in Jordan frame [22–25].

The works of Fierz, Jordan, Brans and Dicke [26–28] were the pioneer attempts for coupling the scalar field with gravity. Dilaton gravity theories were renewed in the 1990ies for explaining the observational results of supernovae at high values of the redshift [29–32], which are related to the evidences for a nonvanishing dark energy [33–37].

Excessive interests of quantum gravity and dark energy scenarios with their relations to dilaton gravity, motivate one to study analytical solutions of Einstein gravity in the presence of a dilaton field. In this paper, we obtain three dimensional black hole solutions in the context of Einstein-dilaton theory with an energy-dependent metric.

Three dimensional (charged) solutions, the so-called BTZ (Banados—Teitelboim—Zanelli) black holes, have attracted a lot of interest [38–44]. Generalization of BTZ solutions to nonlinear electrodynamics, \( F(R) \) theory and massive gravity have been investigated in Refs. [45–52]. A class of higher dimensional modified nonlinear charged solutions, BTZ-like black holes and wormholes, were obtained in literature [53, 54]. Here, we are going to generalize the solutions of [55] to the case of energy dependent spacetime.

II. THREE DIMENSIONAL DILATONIC BLACK HOLES IN GRAVITY’S RAINBOW

Regarding a minimal coupling between Einstein gravity and dilaton field \( (\Phi) \), in addition to the Ricci scalar \( (R) \) (which is related to Einstein gravity), one may regard a kinetic term \( ((\nabla \Phi)^2) \) as well as a potential term \( (V(\Phi)) \). Thus the suitable Lagrangian density of Einstein-dilaton gravity can be written as

\[
\mathcal{L} = R - 2(\nabla \Phi)^2 - V(\Phi).
\]

(4)

Using the variational method, we obtain the following equations of motion

\[
R_{\mu \nu} = 2\partial_\mu \Phi \partial_\nu \Phi + g_{\mu \nu} V(\Phi),
\]

(5)

\[
\nabla^2 \Phi = \frac{1}{4} \frac{\partial V}{\partial \Phi},
\]

(6)

in which we will use the following Liouville-type dilaton potential

\[
V(\Phi) = 2\Lambda e^{2\alpha \Phi},
\]

(7)

where \( \Lambda \) is a constant which is referred to the cosmological constant and \( \alpha \) is the dilaton parameter. Now, we consider the following three dimensional energy-dependent static metric

\[
ds^2 = -\frac{\Psi(r)}{f^2(\varepsilon)} dt^2 + \frac{dr^2}{g^2(\varepsilon)\Psi(r)} + r^2 R^2(\varepsilon) d\phi^2,
\]

(8)

where \( f^2(\varepsilon) \) and \( g^2(\varepsilon) \) are rainbow functions.
Taking into account the field equations with the metric (5), we find that \( t t, \, r r \) and \( \phi \phi \) components (the nonzero components) of Eq. (5) can be, respectively, simplified as
\[
e_{tt} = \Psi''(r) + \frac{\Psi'(r)}{r} + \frac{\Psi'(r) R'(r)}{R(r)} + \frac{2V(\Phi)}{g^2(\varepsilon)} = 0, \tag{9}
\]
\[
e_{rr} = e_{tt} + 2\Psi(r) \left( \frac{2R'(r)}{rR(r)} + \frac{R''(r)}{R(r)} + 24\Phi^2(r) \right) = 0, \tag{10}
\]
\[
e_{\phi\phi} = \frac{\Psi'(r)}{r} + \frac{\Psi'(r) R'(r)}{R(r)} + \frac{2\Psi(r) R'(r)}{rR(r)} + \frac{\Psi(r) R''(r)}{R(r)} + \frac{V(\Phi)}{g^2(\varepsilon)} = 0. \tag{11}
\]

Besides, Eq. (9) may be rewritten as
\[
4\Psi'(r)\Psi'(r) + 4\Psi(r) \left( \Phi''(r) + \frac{\Phi'(r)}{r} + \frac{R'(r)\Psi'(r)}{R(r)} \right) - \frac{1}{g^2(\varepsilon)} \frac{dV(\Phi)}{d\Phi} = 0. \tag{12}
\]

Considering both Eqs. (9) and (10), simultaneously (regarding \( e_{tt} - e_{rr} \)), one can find that the second part of Eq. (10) should vanish, separately
\[
e_{tt} - e_{rr} = \frac{2R'(r)}{rR(r)} + \frac{R''(r)}{R(r)} + 2\Phi^2(r) = 0, \tag{13}
\]
which is independent of the metric function (\( \Psi(r) \)). After some manipulations, we find
\[
e_{tt} - e_{rr} = \frac{2}{r} \frac{d}{dr} \ln R(r) + \frac{d^2}{dr^2} \ln R(r) + \left[ \frac{d}{dr} \ln R(r) \right]^2 + 2\Phi^2(r) = 0. \tag{14}
\]

Taking into account Eq. (14), one finds that \( R(r) \) is an exponential function of \( \Phi(r) \). In what follows, we regard two classes of \( R(r) \) function to obtain exact solutions.

A. Case I: \( R(r) = e^{\beta r} \) with \( \beta \neq 2 \):

Now, we regard the ansatz \( R(r) = e^{\beta r} \) to obtain the unknown functions. Using the field equations with the line element (5), we obtain
\[
\Phi(r) = \sqrt{\gamma(1 - 2\gamma)} \ln \left( \frac{b}{r} \right), \tag{15}
\]
\[
\Psi(r) = \frac{-\Lambda r^2}{\gamma(6\gamma - 1)g^2(\varepsilon)} \left( \frac{b}{r} \right)^{2(1-2\gamma)} - \frac{m}{r^{2\gamma-1}}, \tag{16}
\]
where \( b \) and \( m \) are integration constants, \( \gamma = 1/(2 + \beta^2) \) (with constraint \( \gamma < \frac{1}{2} \)) and \( \Lambda = -l^2 \). Inserting Eqs. (15) and (16) in the field equations (5) and (6) with arbitrary \( \beta \neq 2 \) (\( \gamma \neq 1/6 \)), one can show that all the equations are satisfied only for \( \beta = \alpha \).

Looking for the black hole solutions, one should study the curvature scalars. Calculations show that the Kretschmann scalar diverges at the origin \( (r = 0) \) and is finite for \( r > 0 \). Thus one concludes that there is an essential singularity located at \( r = 0 \). This singularity can be covered with an event horizon, and therefore, we can interpret the solution as black hole. The root of the metric function is located at
\[
r_+ = \left( \frac{b^2(1-2\gamma)}{(\gamma(6\gamma - 1)l^2mg^2(\varepsilon))^{\frac{1}{1-2\gamma}}} \right)^{-1}. \tag{17}
\]

Eq. (17) shows that in order to have a real horizon radius with positive mass and rainbow function, we should regard \( 1/6 < \gamma < 1/2 \).

It is notable that in the absence of dilaton field (\( \alpha = \beta = 0 \) and \( \gamma = 1/2 \)), obtained solution reduces to the BTZ black hole with AdS asymptote. In order to investigate the effects of dilaton field, we plot the metric function. Regarding Figs. 1 and 2 one finds that although the singularity is spacelike in the absence of dilaton field, dilatonic solutions contain a null-like singularity at the origin. In other words, metric function has a real positive root for \( \alpha = 0 \), while there are two real nonnegative roots for dilatonic solution, in which inner horizon is located at \( r = 0 \) (\( \lim_{r \to 0^+} \Psi(r) = 0 \)). Furthermore, one can find that although rainbow functions may affect the location of the event horizon, they do not change the type of singularity (see Fig. 2). In addition, dilaton field affects the asymptotical behavior and obtained solution is not asymptotically AdS for \( \alpha \neq 0 \).
B. Case II: $R(r) = e^{\beta \Phi}$ with $\beta = 2$

In this section, we consider a special case, which was ill-defined in previous solutions (Eq. (16)). Regarding $\beta = 2$, one can find new field equations. It is straightforward to show that Eqs. (5) and (6) have the following solutions for $\beta = 2$ ($R(r) = e^{2\Phi}$)

$$\Phi(r) = \frac{1}{3} \ln \left( \frac{b}{r} \right),$$

(18)

$$\Psi(r) = \left[ -\frac{6\Lambda b^{4/3}}{g^2(\varepsilon)} \ln \left( \frac{r}{L} \right) - m \right] r^{2/3},$$

(19)

where $m$ is an integration constant and $L$ is an arbitrary constant with length dimension. In order to have consistent solutions, we should set $\alpha = \beta = 2$. Calculation of scalar curvatures shows that there is a null singularity at the origin. Such as previous case, this singularity can be covered by an event horizon with the following real positive radius

$$r_+ = L \exp \left( \frac{ml^2 g^2(\varepsilon)}{6b^{4/3}} \right),$$

(20)

where confirms that the event horizon of this black hole is sensitive to variations of $L$ and rainbow functions (see Figs. 1 and 2 for more details). Regarding the asymptotical behavior of the metric function for large $r$, we find that although $\Psi(r) \propto \Lambda r^{2/3} \ln r$ confirms that the dominant term of Eq. (19) is $\Lambda$–term, it differs from the behavior of asymptotically AdS spacetime, in which $\Psi(r) \propto \Lambda r^2$ for large $r$. 

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**FIG. 1:** The behavior of $\Psi(r)$ versus $r$ for $b = 2$, $l = 0.5$, $m = 1.4$ and $g(\varepsilon) = 1.01$.

Eq. (16): $\alpha = 0.5$ (solid line) and $\alpha = 0.7$ (bold line).

Eq. (19): $L = 0.09$ (dotted line) and $L = 0.11$ (dashed line).

**FIG. 2:** The behavior of $\Psi(r)$ versus $r$ for $b = 2$, $l = 0.5$ and $m = 1.4$.

Eq. (16): $\alpha = 0.5$, $g(\varepsilon) = 1.001$ (solid line) and $g(\varepsilon) = 1.2$ (bold line).

Eq. (19): $L = 0.11$, $g(\varepsilon) = 1.001$ (dotted line) and $g(\varepsilon) = 1.2$ (dashed line).
III. THERMODYNAMICS OF THE SOLUTIONS

In this section, we are going to obtain thermodynamic and conserved quantities related to the solutions. At first, we calculate the Hawking temperature. Using the surface gravity interpretation, one finds that

\[ T_H = \frac{g(\varepsilon) \Psi'(r)}{4\pi f(\varepsilon)} \bigg|_{r=r_+}. \]  

(21)

After some simplifications, we obtain

\[ T_H = \frac{1}{2\pi l^2 g(\varepsilon)} \left\{ \frac{b^{2(1-2\gamma)}}{r_+^{4\gamma-1}}, \quad \alpha \neq 2 \right\} \]

\[ \frac{b^{2/3} r_+^{1/3}}{r_+}, \quad \alpha = 2, \]  

(22)

where \( r_+ \) was obtained for each branches (see Eqs. (17) and (20)). Regarding positive rainbow functions, one obtains positive definite temperature.

Now, we should obtain finite mass. AdS/CFT correspondence \([56, 57]\) guarantees that we can apply the counterterm method for asymptotically AdS solutions to calculate finite conserved quantities. Although, obtained solutions are not asymptotically AdS, it was shown that \([58, 59]\) one can find an appropriate counterterm for removing the divergences in horizon-flat spacetimes. The action of three-dimensional Einstein-dilaton gravity with suitable Gibbons-Hawking surface term and counterterm action can be written as

\[ I_{\text{total}} = I_{\text{bulk}} + I_{\text{boundary}} + I_{\text{counterterm}}, \]  

(23)

in which

\[ I_{\text{bulk}} = -\frac{1}{16\pi} \int_{\mathcal{M}} d^3x \sqrt{-g} \left( R - 2(\nabla \Phi)^2 - V(\Phi) \right), \]

\[ I_{\text{boundary}} = -\frac{1}{8\pi} \int_{\partial\mathcal{M}} d^2x \sqrt{-h} \Theta(h), \]

\[ I_{\text{counterterm}} = -\frac{1}{8\pi} \int_{\partial\mathcal{M}} d^2x \sqrt{-h} \left( \frac{1}{L_{\text{eff}}} \right), \]  

(24)

where \( \Theta \) represents the trace of the extrinsic curvature of the boundary \( \partial\mathcal{M} \) with induced metric \( h^{ab} \) and suitable effective length is

\[ L_{\text{eff}} = \sqrt{\frac{2\alpha^2 - 2}{V(\Phi)}}, \]  

(25)

which reduces to \( l \) in the absence of the dilaton field (\( \alpha = 0 \)), as expected. Considering the total finite action, we apply the Brown-York method for constructing a divergence free stress-energy tensor \([60]\)

\[ T^{ab} = \frac{1}{8\pi} \left( \Theta^{ab} - \Theta h^{ab} - \frac{h^{ab}}{L_{\text{eff}}} \right). \]  

(26)

Taking into account temporal Killing vector \( \partial/\partial t \), one can find its associated conserved quantity, which is the quasilocal mass

\[ M = \frac{m}{4f(\varepsilon)} \times \begin{cases} b^{1-2\gamma}, & \alpha \neq 2 \\ b^{2/3}, & \alpha = 2 \end{cases}. \]  

(27)

It is known that the area law is valid in Einstein-dilaton gravity. In addition, the validity of the area law is examined in spacetimes with zero-curvature boundary of \( t = cte \) and \( r = cte \) in the context of non-Einsteinian gravity. Therefore, we regard the area law for our solutions to obtain entropy as

\[ S = \frac{\pi}{2g(\varepsilon)} \times \begin{cases} b^{1-2\gamma} r^{2\gamma}, & \alpha \neq 2 \\ b^{2/3} r_+^{1/3}, & \alpha = 2 \end{cases}. \]  

(28)
Here, we are going to check the validity of the first law of thermodynamics. To do this, we can obtain the mass \( M \) as a function of the only extensive quantity \( S \). Regarding the fact that metric function should vanish at the event horizon \( f(r_+) = 0 \), we obtain the following Smarr-type formula
\[
M(S) = \frac{-\Lambda b^2}{4f(\varepsilon)g^2(\varepsilon)} \times \left\{ \begin{array}{ll}
\frac{1}{(6\gamma-11)} \left( \frac{b}{r_+} \right)^{1-6\gamma}, & \alpha \neq 2 \\
\alpha = 2,
\end{array} \right.
\]
where
\[
r_+ = \begin{cases}
\left( \frac{2g(\varepsilon)}{\pi b^{4/3}} \right)^{1/2\gamma}, & \alpha \neq 2 \\
\left( \frac{2g(\varepsilon)}{\pi b^{4/3}} \right)^{3}, & \alpha = 2.
\end{cases}
\]
Now, we can calculate the Hawking temperature as the intensive quantity conjugate to \( S \)
\[
T_H = \frac{dM}{dS} = \frac{\partial M}{\partial r_+} \left( \frac{\partial S}{\partial r_+} \right)^{-1}
\]
which is in agreement with Eq. (22), and therefore, the first law of thermodynamics is valid as
\[
dM = T_H dS.
\]

IV. THERMODYNAMIC STABILITY

It is well-known that asymptotically flat uncharged black holes (Schwarzschild solutions) are thermally unstable, and therefore, in order to obtain stable black holes, one can insert cosmological constant or electric charge to the solutions. Here, we are going to study the effects of dilaton field on thermal stability of neutral black holes.

Thermal stability of a system can be discussed in various ensembles. In the canonical ensemble, thermal stability of a system is determined by its heat capacity. The positivity of the heat capacity \( C = T(\partial S/\partial T) \) is sufficient to ensure thermal stability of a thermodynamical system. Regarding obtained results for the temperature and entropy, one finds
\[
C = \frac{T_H}{\partial S} = \begin{cases}
\frac{\pi b^{1-2\gamma} r_+^{2\gamma}}{g(\varepsilon)(4\gamma-1)}), & \alpha \neq 2 \\
-\frac{\pi b^{1/3} r_+^{1/3}}{2g(\varepsilon)}, & \alpha = 2.
\end{cases}
\]

According to Eq. (33), we find that the solutions are unstable for \( \alpha = 2 \) and \( |\alpha| > \sqrt{2} \). In other words, although BTZ black hole is stable everywhere, three dimensional uncharged dilatonic black holes are stable for \( \alpha \in (-\sqrt{2}, +\sqrt{2}) \) (or equivalently \( 1/4 < \gamma < 1/2 \)). As a result, one may regard a phase transition for the critical value of the dilaton parameter \( (\alpha_c = \sqrt{2}) \) (see Fig. 3 in which shows an increasing behavior for positive \( T \) and a divergence point for the heat capacity).

V. CONCLUSION

In this paper, we have considered three dimensional gravity’s rainbow in the presence of a dilaton field. In order to obtain consistent solutions, we have regarded a class of Liouville-type dilaton potential. We have obtained two classes of the solutions with polynomial and logarithmic forms. We also showed that the solutions can be considered as a BTZ dilatonic black hole only for \( \alpha^2 \leq 4 \).

Taking into account the area law and the surface gravity interpretation, we have calculated the entropy and the temperature of the black hole solutions. We have also added a suitable counterterm to the gravitational action to obtain finite mass. In addition, using the Smarr-type formula for the mass, we have found that the first law of thermodynamics is valid for both classes of the solutions.

Then, we have investigated thermal stability of the solutions and calculated the heat capacity. We have found that logarithmic black hole solution is unstable while polynomial one may be thermally stable.

Finally, it is worthwhile to investigate stationary and/or charged solutions as well as their extension to massive gravity \([61, 62]\). In addition, it is interesting to study thermodynamic geometry of the solutions based on HPEM (Hendi, Panahiyan, Eslam Panah and Momennia) method \([63, 64]\). We will address these issues in the forthcoming work.
FIG. 3: Case $\alpha \neq 2$: The heat capacity (solid line) and temperature (bold line) versus $\alpha$ for $b = 1$, $l = 0.5$, $f(\varepsilon) = 1$, $g(\varepsilon) = 1.1$, and $r_+ = 1.2$.

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