On the internal soliton propagation over slope-shelf topography

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Abstract. Dynamics and properties of internal soliton propagation over slope-shelf topography is investigated. We derive a nonlinear wave equation based on the two-layer fluid model, which produce a variable-coefficient perturbed Korteweg-deVries (vP-KdV) equation. A special solution in term of single-soliton will be highlighted.

1. Introduction
Internal wave transmits energy to the ocean interior up to the sea floor. Like waves of sea surface, the internal waves will experience a transformation of energy at a time when approaching the beach. The transformation of energy depends on the sea bottom topography. The depth of the sea will change from deep to shallow with varied forms. The most common is in the shape of a slope-shelf.

The propagation of internal wave, especially internal solitary waves (ISW) or internal soliton, from shelf break has remarkable properties. The internal waves would propagate into much shallower water and transform to the elevation waves [1–3]. The ISW propagation that usually was described by Korteweg de Vries (KdV) equation will change sign in its coefficient. The comparison of observation by using satellite image and in-situ experiment showed the sign is negative in deep water and positive in shallow water. The wave transform from waves of depression to waves of elevation, for example the depression propagating from a deep part of a basin onto the shelf, will break when the wave amplitude is larger than a half of the water depth minus the undisturbed depth of the isopycnal of maximum depression [4–7].

In this paper, we will develop an analytic model of the ISW propagation over slope-shelf topography. The two-layered fluid model will be used to derive KdV form. The special solution will be used to examine the nature of dynamics of ISW such as run-up and the breaking waves.

2. Derivation of nonlinear internal waves propagation over slope-shelf topography
The derivation of ISW propagation over uneven bottom has been done [4]. In this paper, we will derive another approach and show the limitation of previous derivation. We consider a two-layer fluid bounded above by a rigid horizontal plane (the rigid-lid approximation), \( z = h_1 \) and below by a rigid horizontally-varying boundary \( z = -h_2(x) \). Each layer consists of incompressible, inviscid and irrotational fluid of a constant density \( \rho_1 \) for the upper layer and \( \rho_2 \) for the lower layer. The free interface between the layer is denoted by \( z = \eta(x, t) \). The geometry of boundary
Figure 1. The boundary problem of two-layer fluid with rigid lid on the upper layer and uneven bottom on the lower layer.

value problems is depicted in figure 1. The derivation of internal soliton wave equation will follow procedures as discussed by [8].

We begin with the equations of motion

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + w_i \frac{\partial u_i}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p_i}{\partial x},$$  \hspace{1cm} (1)

$$\frac{\partial w_i}{\partial t} + u_i \frac{\partial w_i}{\partial x} + w_i \frac{\partial w_i}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p_i}{\partial z},$$  \hspace{1cm} (2)

$$\frac{\partial u_i}{\partial x} + \frac{\partial w_i}{\partial z} = 0,$$  \hspace{1cm} (3)

and their boundary values are given by

$$w_1|_{z=h_1} = 0$$  \hspace{1cm} (4)

$$w_2|_{z=-h_2} = -u_2 \frac{\partial h_2(x)}{\partial x}$$  \hspace{1cm} (5)

$$w_i|_{z=\eta} = \frac{\partial \eta}{\partial t} + u_i \frac{\partial \eta}{\partial x}$$  \hspace{1cm} (6)

$$(p_2 - p_1)|_{z=\eta} = (\rho_2 - \rho_1) g \eta$$  \hspace{1cm} (7)

where $u_i$, $w_i$ are orbital velocities for horizontal and vertical respectively, and $\rho_i$ is the fluid density; here index $i$ takes the value 1 for the upper layer and 2 for the lower layer. Now, we introduce a scaling as follows: $[t] = T$, $[x] = L$, $[z] = H_0$, $[u] = \epsilon c_0$, $[w] = \epsilon c_0 H_0 / L$, $[\eta] = \epsilon H_0$ and $[p] = \rho_0 \epsilon c_0^2$, where $\epsilon$ is a small parameter and $c$ is phase velocity. The phase speed is given by $c_0^2 = \bar{g} H_0$ with $\bar{g} = g(\rho_2 - \rho_1)/\rho_2$ and $H_0 = h_1 + h_2$ for constant $h_2$.

Using these scales, we can cast (1) and (2) in a nondimensional form

$$\frac{\partial u_i}{\partial t} + \epsilon u_i \frac{\partial u_i}{\partial x} + \epsilon w_i \frac{\partial u_i}{\partial z} = -\frac{\partial p_i}{\partial x},$$  \hspace{1cm} (8)

$$\frac{\partial w_i}{\partial t} + \epsilon u_i \frac{\partial w_i}{\partial x} + \epsilon w_i \frac{\partial w_i}{\partial z} = -\frac{1}{\delta} \frac{\partial p_i}{\partial z},$$  \hspace{1cm} (9)

$$\frac{\partial u_i}{\partial x} + \frac{\partial w_i}{\partial z} = 0$$  \hspace{1cm} (10)
where $\delta = H_0^2/L^2$. The boundary conditions can be written in the nondimensional form,

$$ w_1|_{z=0} = 0 \quad (11) $$

$$ w_2|_{z=(\alpha-1)H} = -\frac{u_2}{D} \partial h_2(x) \quad (12) $$

$$ w_i|_{z=\eta} = \frac{\partial \eta}{\partial t} + \epsilon u_i \frac{\partial \eta}{\partial x} \quad (13) $$

$$ (p_2 - p_1)|_{z=\eta} = \eta \quad (14) $$

Comparing with (13) we have

$$ \frac{\partial \eta}{\partial t} + \epsilon u_i \frac{\partial \eta}{\partial x} = \frac{\partial \eta}{\partial t} + \epsilon u_i \frac{\partial \eta}{\partial x} + \mathcal{O}(\epsilon^2) \quad . $$

Comparing with (13) we have

$$ w_i|_{z=0} = \frac{\partial \eta}{\partial t} + \epsilon \frac{\partial (nu_i)}{\partial x} + \mathcal{O}(\epsilon^2) . \quad (17) $$

The boundary condition (14) can be written as

$$ \eta = (p_2 - p_1)|_{z=0} + \epsilon \eta \left( \frac{\partial p_2}{\partial z} - \frac{\partial p_1}{\partial z} \right) \bigg|_{z=0} + \mathcal{O}(\epsilon^2) . \quad (18) $$

We also define a shear cross-interface as $\tilde{u} = (u_2 - u_1)|_{z=\eta}$ and its Taylor expansion as follows

$$ \tilde{u} = (u_2 - u_1)|_{z=0} + \epsilon \eta \left( \frac{\partial u_2}{\partial z} - \frac{\partial u_1}{\partial z} \right) \bigg|_{z=0} + \mathcal{O}(\epsilon^2) . \quad (19) $$

We develop variables $p_i, u_i, w_i$ and $\eta$ in a series, in which $\epsilon$ serves as small parameter:

$$ \begin{pmatrix} u_i \\ w_i \\ \eta_i \\ p_i \end{pmatrix} = \begin{pmatrix} u_i^{(0)} \\ u_i^{(1)} \\ \eta_i^{(0)} \\ \eta_i^{(1)} \\ p_i^{(0)} \\ p_i^{(1)} \end{pmatrix} + \epsilon \begin{pmatrix} u_i^{(0)} \\ u_i^{(1)} \\ \eta_i^{(0)} \\ \eta_i^{(1)} \\ p_i^{(0)} \\ p_i^{(1)} \end{pmatrix} + \mathcal{O}(\epsilon^2) . \quad (20) $$

By using this expansion, the equations of motion (8), (9) and (10) become

$$ \frac{\partial u_i^{(0)}}{\partial t} + \frac{\partial p_i^{(0)}}{\partial x} + \epsilon \frac{\partial u_i^{(1)}}{\partial t} + \epsilon u_i^{(0)} \frac{\partial u_i^{(0)}}{\partial x} + \epsilon w_i^{(0)} \frac{\partial u_i^{(0)}}{\partial x} + \epsilon \frac{\partial p_i^{(1)}}{\partial x} + \mathcal{O}(\epsilon^2) = 0, \quad (21) $$

$$ \frac{\partial p_i^{(0)}}{\partial z} + \epsilon \frac{\partial w_i^{(0)}}{\partial x} + \epsilon \frac{\partial p_i^{(1)}}{\partial x} + \mathcal{O}(\epsilon^2) = 0, \quad (22) $$

$$ \frac{\partial u_i^{(0)}}{\partial x} + \frac{\partial w_i^{(0)}}{\partial z} + \epsilon \left( \frac{\partial u_i^{(1)}}{\partial x} + \frac{\partial w_i^{(1)}}{\partial z} \right) + \mathcal{O}(\epsilon^2) = 0 , \quad (23) $$
and the boundary conditions (11), (12), (17), (27) and (19) give

\begin{align}
    w_1^{(0)} |_{z=\alpha} + \varepsilon w_1^{(1)} |_{z=\alpha} + O(\varepsilon^2) &= 0, \\
    w_2^{(0)} |_{z=(\alpha-1)\bar{H}} + \varepsilon w_2^{(1)} |_{z=(\alpha-1)\bar{H}} + \varepsilon u_2^{(0)} \frac{\partial \bar{h}_2(x)}{\partial x} + O(\varepsilon^2) &= 0, \\
    w_1^{(0)} |_{z=0} + \varepsilon w_1^{(1)} |_{z=0} - \frac{\partial \eta^{(0)}}{\partial t} - \varepsilon \left( \frac{\partial \eta^{(1)}}{\partial t} - \alpha \frac{\partial \eta^{(0)}}{\partial x} \right) + O(\varepsilon^2) &= 0, \\
    \eta^{(0)} + \varepsilon \eta^{(1)} &= \left( p_2^{(0)} - p_1^{(0)} \right) |_{z=0} + \varepsilon \left( p_2^{(1)} - p_1^{(1)} \right) |_{z=0} + \varepsilon \eta^{(0)} \left( \frac{\partial p_2^{(0)}}{\partial z} - \frac{\partial p_1^{(0)}}{\partial z} \right) |_{z=0} + O(\varepsilon^2), \\
    \bar{u}^{(0)} + \varepsilon \bar{u}^{(1)} &= \left( u_2^{(0)} - u_1^{(0)} \right) |_{z=0} + \varepsilon \left( u_2^{(1)} - u_1^{(1)} \right) |_{z=0} + \varepsilon \eta^{(0)} \left( \frac{\partial u_2^{(0)}}{\partial z} - \frac{\partial u_1^{(0)}}{\partial z} \right) |_{z=0} + O(\varepsilon^2),
\end{align}

where we have assumed that the variation of bottom topography acts as a perturbation of solitary wave, which means \( \bar{h}_2(x) = \epsilon \bar{h}_2(x) \).

At the lowest order, \( \varepsilon^{(0)} \), the equations of motion and their boundary conditions read

\begin{align}
    \frac{\partial u_1^{(0)}}{\partial t} + \frac{\partial p_1^{(0)}}{\partial x} &= 0, \quad \frac{\partial p_1^{(0)}}{\partial z} = 0, \quad \frac{\partial u_1^{(0)}}{\partial x} + \frac{\partial v_1^{(0)}}{\partial z} = 0, \quad \frac{\partial w_1^{(0)}}{\partial z} |_{z=\alpha} = 0, \quad \frac{\partial w_2^{(0)}}{\partial z} |_{z=(\alpha-1)\bar{H}} = 0, \\
    \frac{\partial w_1^{(0)}}{\partial t} |_{z=0} &= 0, \quad \frac{\partial \eta^{(0)}}{\partial x} = \left( p_2^{(0)} - p_1^{(0)} \right) |_{z=0}, \quad \frac{\partial \eta^{(0)}}{\partial x} = \left( u_2^{(0)} - u_1^{(0)} \right) |_{z=0}.
\end{align}

We see that, in the lowest order, \( u_1^{(0)} \) and \( p_1^{(0)} \) do not depend on vertical coordinate \( z \). Writing down the first and the second of (29) into their components, we get

\begin{align}
    \frac{\partial \eta_1^{(0)}}{\partial t} + \frac{\partial p_1^{(0)}}{\partial x} &= 0, \quad \frac{\partial u_2^{(0)}}{\partial t} + \frac{\partial p_2^{(0)}}{\partial x} = 0.
\end{align}

Subtracting the two equations yields

\begin{align}
    \frac{\partial \bar{u}^{(0)}}{\partial t} + \frac{\partial \eta^{(0)}}{\partial x} = 0.
\end{align}

Next, we integrate the continuity equation, i.e, the third equation of (29), with respect to vertical coordinate with the upper and lower bounds. For the upper layer,

\begin{align}
    \frac{\partial u_1^{(0)}}{\partial x} (\alpha - 0) + \frac{\partial w_1^{(0)}}{\partial z} |_{z=\alpha} - \frac{\partial w_1^{(0)}}{\partial z} |_{z=0} = 0.
\end{align}

By using the boundary condition, we get

\begin{align}
    \alpha \frac{\partial u_1^{(0)}}{\partial x} - \frac{\partial \eta^{(0)}}{\partial t} = 0.
\end{align}

For the lower layer,

\begin{align}
    \frac{\partial u_2^{(0)}}{\partial x} (0 - (\alpha - 1)\bar{H}) + \frac{\partial w_2^{(0)}}{\partial z} |_{z=0} - \frac{\partial w_2^{(0)}}{\partial z} |_{z=(\alpha-1)\bar{H}} = 0.
\end{align}
By using the boundary condition and multiplying (34) by \((\alpha - 1)\bar{H}\), and adding up the results we get
\[
\alpha(\alpha - 1)\bar{H}\frac{\partial \bar{u}^{(0)}}{\partial x} + ((\alpha - 1)\bar{H} - \alpha)\frac{\partial \eta^{(0)}}{\partial t} = 0.
\]
(36)

Finally, by using (34), we get
\[
\frac{\partial^2 \eta^{(0)}}{\partial t^2} - c_0^2 \frac{\partial^2 \eta^{(0)}}{\partial x^2} = 0,
\]
where \(c_0^2 = (\alpha(\alpha - 1)\bar{H})/(\alpha - 1)\bar{H} - \alpha\). When \(h_2(x) = h_2\) with \(h_2\) being constant and \(\bar{H} = 1\), we have the phase speed of linear long wave for two-fluid system. It is well known that (37) describes linear long waves, with both left and right wards propagating. Its general solution can be written as \(\eta^{(0)} = F(\xi_-) + G(\xi_+)\), where \(\xi_- = x - c_0t\) represents the right propagation, and \(\xi_+ = x + c_0t\), the left propagation.

Now, we will arrive at the crucial blackpoint, namely the first order. At the first order, \(\epsilon^{(1)}\), we have
\[
\frac{\partial u_i^{(1)}}{\partial t} + u_i^{(0)} \frac{\partial u_i^{(0)}}{\partial x} + \frac{\partial p_i^{(1)}}{\partial x} = 0
\]
(38)
\[
\delta_x \frac{\partial u_i^{(0)}}{\partial t} + \frac{\partial p_i^{(1)}}{\partial z} = 0
\]
(39)
\[
\frac{\partial u_i^{(1)}}{\partial x} + \frac{\partial u_i^{(1)}}{\partial z} = 0
\]
(40)

for the equations of motion (note that we have used the fact that \(\partial (u_i^{(0)}, p_i^{(0)})/\partial z = 0\)) and
\[
w_i^{(1)}|_{z=\alpha} = 0, w_i^{(1)}|_{z=(\alpha-1)\bar{H}} = -u_2^{(0)} \frac{\partial h_2(x)}{\partial x}
\]
(41)
\[
w_i^{(1)}|_{z=0} = \frac{\partial \eta^{(1)}}{\partial t} + \frac{\partial (\eta^{(0)} u_i^{(0)})}{\partial x}
\]
(42)
\[
\eta^{(1)} = (p_2^{(1)} - p_1^{(1)})|_{z=0}, \bar{u}^{(1)} = (u_2^{(1)} - u_1^{(1)})|_{z=0}
\]
(43)

for the boundary conditions. Before we work further with the first order, we require the following relations
\[
u_i^{(0)} = \frac{(1 - \alpha)\bar{H}}{(\alpha - 1)\bar{H} - \alpha} \bar{u}^{(0)}, \quad u_2^{(0)} = \frac{\alpha}{(1 - \alpha)\bar{H} + \alpha} \bar{u}^{(0)}.
\]
(44)

These relations can be obtained from the zeroth order. By doing integration from \(\alpha\) to \(z\) and using (34) for the upper layer, and similarly doing integration from \((\alpha - 1)\bar{H}\) to \(z\) and using boundary condition of the zeroth order for the lower layer, we get
\[
w_i^{(0)}(z) = \left(\frac{\alpha - z}{\alpha}\right) \frac{\partial \eta^{(0)}}{\partial t}, \quad w_2^{(0)}(z) = \left(\frac{(\alpha - 1)\bar{H} - z}{(\alpha - 1)\bar{H}}\right) \frac{\partial \eta^{(0)}}{\partial t}.
\]
(45)

We write the horizontal momentum equation, (38), for each layer, and then subtract and use boundary condition of the first order and (44), we get
\[
\frac{\partial \bar{u}^{(1)}}{\partial t} + \Lambda \bar{u}^{(0)} \frac{\partial \bar{u}^{(0)}}{\partial x} + \frac{\partial \eta^{(1)}}{\partial x} = 0,
\]
(46)
where $\Lambda = (\alpha^2 - (\alpha - 1)^2 \ddot{H}^2)/(\alpha - 1) \ddot{H} - \alpha^2$. Taking differential operation with respect to $z$ for the horizontal momentum equation and to $x$ for the vertical momentum equation, followed by subtracting these results and taking integration with respect to $t$ yield

$$
\delta \frac{\partial u_1^{(0)}}{\partial x} = \frac{\partial u_2^{(1)}}{\partial z}.
$$

(47)

By substituting (45) and taking integral operation with respect to $z$, we obtain

$$
u_1^{(1)} = \frac{\delta^*}{\alpha} \left( \alpha z - \frac{1}{2} \right) \frac{\partial^2 \eta^{(0)}}{\partial x \partial t} + u_1^{(1)} |_{z=0}
$$

(48)

$$
u_2^{(1)} = \frac{\delta^*}{(\alpha - 1) \ddot{H}} \left( (\alpha - 1) \ddot{H} z - \frac{1}{2} \right) \frac{\partial^2 \eta^{(0)}}{\partial x \partial t} + u_2^{(1)} |_{z=0}.
$$

(49)

Same as the lowest order, integrating the continuity equation (40) with respect to $z$ and using (48) give

$$
\frac{\partial \eta^{(1)}}{\partial t} + \frac{\partial}{\partial x} \left( \eta^{(0)} u_1^{(0)} \right) - \alpha \frac{\partial u_1^{(1)}}{\partial x} |_{z=0} - \frac{1}{3} \delta_1 \alpha^2 \frac{\partial^2 \eta^{(0)}}{\partial x^2 \partial t} = 0.
$$

(50)

By using the first order of boundary condition and (42), we have

$$
\frac{\partial \eta^{(1)}}{\partial t} + \frac{\partial}{\partial x} \left( \eta^{(0)} u_2^{(0)} \right) + u_2^{(1)} |_{z=0} + u_2^{(1)} |_{z=\alpha} - w_2^{(1)} |_{z=0} = 0.
$$

(51)

In the same manner, we get the lower layer as follows

$$
-\frac{\partial}{\partial x} \left[ \frac{\delta^*}{3\alpha} (\alpha - 1)^2 H^2 \frac{\partial^2 \eta^{(0)}}{\partial x \partial t} + (\alpha - 1) \ddot{H} u_2^{(1)} |_{z=0} \right] + w_2^{(1)} |_{z=0} - w_2^{(1)} |_{z=(\alpha-1)\ddot{H}} = 0.
$$

By using (41) and (42), we arrive at

$$
\frac{\partial \eta^{(1)}}{\partial t} + \frac{\partial}{\partial x} \left( \eta^{(0)} u_2^{(0)} \right) + u_2^{(1)} \frac{\partial h_2}{\partial x} + (1 - \alpha) \frac{\partial \ddot{H}}{\partial x} u_2^{(1)} |_{z=0} + (1 - \alpha) \ddot{H} \frac{\partial u_2^{(1)}}{\partial x} |_{z=0}
$$

$$
- \frac{\delta^*}{3} (\alpha - 1)^2 H^2 \frac{\partial^2 \eta^{(0)}}{\partial x^2 \partial t} - \frac{\delta^*}{3} (\alpha - 1)^2 H^2 \frac{\partial^2 \eta^{(0)}}{\partial x^2 \partial t} = 0.
$$

(52)

By assuming that $H \gg h_2$ and considering that the varying depth is a perturbation of internal waves, then we can ignore the derivative of $H$; thus by substituting (44), we get

$$
\frac{\partial \eta^{(1)}}{\partial t} + \frac{\partial}{\partial x} \left( \eta^{(0)} u_2^{(0)} \right) + \frac{\alpha}{(1 - \alpha) \ddot{H} + \alpha} \frac{\partial h_2}{\partial x} + (1 - \alpha) \ddot{H} \frac{\partial u_2^{(1)}}{\partial x} |_{z=0} - \frac{\delta^*}{3} (\alpha - 1)^2 \ddot{H} \frac{\partial^2 \eta^{(0)}}{\partial x^2 \partial t} = 0.
$$

(53)

Multiplying (51) by $(1 - \alpha) \ddot{H}$, (53) with $\alpha$, followed by adding them up and using (43), we obtain

$$
(\alpha + (1 - \alpha) \ddot{H}) \frac{\partial \eta^{(1)}}{\partial t} + \alpha (1 - \alpha) \ddot{H} \frac{\partial u_2^{(1)}}{\partial x} + \frac{\alpha^2}{(1 - \alpha) \ddot{H} + \alpha} \frac{\partial h_2}{\partial x} + \alpha \frac{\partial (u_2^{(0)} \eta^{(0)})}{\partial x} + (1 - \alpha) \ddot{H} \frac{\partial (\bar{\nu}_1^{(0)} \eta^{(0)})}{\partial x} - \frac{\delta^*}{3} \ddot{H} (\alpha + (1 - 2\alpha) \ddot{H}) \frac{\partial^2 \eta^{(0)}}{\partial x^2 \partial t} = 0.
$$

(54)
Finally, after thorough effort, we get evolution equation for \( \eta \) as

\[
\frac{\partial^2 \eta^{(1)}}{\partial t^2} - c_0^2 \frac{\partial^2 \eta^{(1)}}{\partial x^2} + 2\Lambda \frac{\partial}{\partial x} \left( \eta^{(0)} \frac{\partial \eta^{(0)}}{\partial x} \right) - \Delta_1 \frac{\partial}{\partial x} \left( \frac{\partial h_2}{\partial x} \eta^{(0)} \right) + \Delta_2 \frac{\partial^4 \eta^{(0)}}{\partial x^4} = 0 ,
\]

(55)

where \( \Delta_1 = \alpha^2/((\alpha - 1)\bar{H} - \alpha)^2 \) and \( \Delta_2 = (\delta_4 \alpha \bar{H} (\alpha + (1 - 2\alpha)\bar{H}) c_0^2)/(3(\alpha - 1)\bar{H} - \alpha) \). Because the function of \( \eta^{(0)} \) also meets the wave equation (37), then the derivative with respect to \( x \) can be split as follows

\[
\frac{\partial}{\partial x} \eta^{(0)} = -\frac{1}{2c_0} \left( \frac{\partial}{\partial t} - c_0 \frac{\partial}{\partial x} \right) \eta^{(0)} .
\]

(56)

Using the operator we write

\[
\left( \frac{\partial}{\partial t} - c_0 \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + c_0 \frac{\partial}{\partial x} \right) \eta^{(1)} - \frac{\Lambda}{c_0} \left( \frac{\partial}{\partial t} - c_0 \frac{\partial}{\partial x} \right) \left( \eta^{(0)} \frac{\partial \eta^{(0)}}{\partial x} \right) - \frac{\Delta_1}{2c_0} \left( \frac{\partial}{\partial t} - c_0 \frac{\partial}{\partial x} \right) \left( \frac{\partial h_2}{\partial x} \eta^{(0)} \right) - \frac{\Delta_2}{2c_0} \frac{\partial^3 \eta^{(0)}}{\partial x^3} = 0 ,
\]

(57)

with integration constant being zero; this yields

\[
\left( \frac{\partial}{\partial t} + c_0 \frac{\partial}{\partial x} \right) \eta^{(1)} - \frac{\Lambda}{c_0} \eta^{(0)} \frac{\partial \eta^{(0)}}{\partial x} - \Delta_1 \frac{\partial h_2}{\partial x} \eta^{(0)} - \Delta_2 \frac{\partial^3 \eta^{(0)}}{\partial x^3} = 0 .
\]

(58)

Finally, after thorough effort, we get evolution equation for \( \eta \) as

\[
\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} - \frac{\Lambda}{c_0} \frac{\partial \eta}{\partial x} - \frac{\Delta_1}{2c_0} \frac{\partial h_2}{\partial x} \eta - \frac{\Delta_2}{2c_0} \frac{\partial^3 \eta}{\partial x^3} = 0 .
\]

(59)

This is called the variable-coefficient perturbed Korteweg-deVries (vP-KdV) equation.

### 3. Special solution of ISW propagation over slope-shelf topography

Before we solve (59), let us write the equation in the form of

\[
\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} + \gamma_1 \frac{\partial \eta}{\partial x} + \gamma_2 \frac{\partial^3 \eta}{\partial x^3} + F(x)\eta = 0 ,
\]

(60)

where \( \gamma_1 = -\Lambda/c_0 \), \( \gamma_2 = -\Delta_2/(2c_0) \) and \( F(x) = -\Delta_1/(2c_0)(\partial h_2/\partial x) \). The single-soliton solution of the equation is well-known when \( F(x) = 0 \) and the coefficients are constant. It has been shown that by using transformation

\[
\psi = \exp \left[ \int \frac{2F(x')}{c_0} dx' \right] \eta , \quad \tau = \int \frac{dx'}{c_0} , \quad \varsigma = \tau - t ,
\]

(61)

equation (60) reduces to [9]

\[
\frac{\partial \psi}{\partial \tau} + \delta_1(\tau) \psi \frac{\partial \psi}{\partial \varsigma} + \delta_2(\tau) \frac{\partial^3 \psi}{\partial \varsigma^3} = 0 ,
\]

(62)

where \( \delta_1 = \gamma_1/(c_0 \exp \left[ \int 2F/c_0 dx \right]) \) and \( \delta_2 = \gamma_2/c_0^3 \). Now, the coefficients are only function of \( \tau \) alone. An analytic solution based on slowly varying topography approximation has been done.
by several researcher [9]. The slowly varying topography means that the topography undulation is larger than solitary wave wavelength. The solution can be expressed as follows

$$\psi = \psi_0(\bar{\tau}, \bar{\varsigma}) + \epsilon \psi_1(\bar{\tau}, \bar{\varsigma}) + o(\epsilon^2),$$

(63)

where $\bar{\tau} = \epsilon \tau$ and $\bar{\varsigma} = \epsilon \varsigma$. The zeroth order leads to single-soliton solution given by [4]

$$\psi_0(\bar{\varsigma}, \bar{\tau}) = A_0 \text{sech}^2 \left[ \gamma (\bar{\varsigma} - V_0 \bar{\tau}) \right],$$

(64)

where $A_0 = 3V_0/\delta_1$ and $\gamma = \sqrt{\delta_1 A_0/(12\delta_2)}$. The first order satisfies the third-order ordinary differential equation with $\psi_0$ coefficient. By using the conservation of mass and momentum, the solution is given by

$$\psi_1(\bar{\varsigma}, \bar{\tau}) = A_1 \text{sech}^2 \left[ \sqrt{\frac{A_1}{2}} (\bar{\varsigma} - V_1 \bar{\tau}) \right],$$

(65)

where $A_1 = \sqrt{V_1^2/4 + 2A_0 + 2A_0 + A_0^2 - V_0^2/2}$ and $V_1 = 2A_0 + 2A_1$. Another way to find the analytic solution of variable-coefficient KdV equation has been developed by expansion methods based on the Riccati ordinary differential equation. The solution is given by [10]

$$\psi(x, t) = \psi_0 \text{sech} \left[ \gamma (\varsigma - V(\tau)\tau) \right],$$

(66)

where

$$V(\tau) = \frac{1}{\tau} \left( \int \delta_1(\tau) \int \delta_2(\tau') d\tau' d\tau + \int \frac{\psi_0 \delta_1}{3} d\tau \right),$$

(67)

With the special form of the topography, then solution could be obtained.

4. Conclusion

Propagation of internal solitary wave in two layer fluid have been derived. We obtain the perturbed variable coefficient Kortweg and de Vries equation. By using slowly varying bottom topography the soliton solution can be obtained.

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