Flux Tubes on Higgs Branches in SUSY Gauge Theories

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Abstract

We study flux tubes on Higgs branches with curved geometry in supersymmetric gauge theories. As a first example we consider $\mathcal{N}=1$ QED with one flavor of charges and with Higgs branch curved by adding a Fayet-Iliopoulos (FI) term. We show that in a generic vacuum on the Higgs branch flux tubes exist but become “thick”. Their internal structure in the plane orthogonal to the string is determined by “BPS core” formed by heavy fields and long range “tail” associated with light fields living on the Higgs branch. The string tension is given by the tension of “BPS core” plus contribution coming from the “tail”. Next we consider $\mathcal{N}=2$ QCD with gauge group $SU(2)$ and $N_f = 2$ flavors of fundamental matter (quarks) with the same mass. We perturb this theory by the mass term for the adjoint field which to the leading order in perturbation parameter do not break $\mathcal{N}=2$ supersymmetry and reduces to FI term. The Higgs branch has Eguchi-Hanson geometry. We work out string solution in the generic vacuum on the Higgs branch and calculate its string tension. We also discuss if these strings can turn into semilocal strings, the possibility related to the confinement/deconfinement phase transition.
1 Introduction

The mechanism of confinement as a dual Meissner effect arising upon condensation of monopoles was suggested a long ago [1]. Once monopoles condense the electric flux is confined in the (dual) Abrikosov–Nielsen–Olesen (ANO) flux tube [2, 3] connecting heavy trial charge and anti-charge. The flux tube has constant energy per unit length (string tension ). This ensures that the confining potential between heavy charge and anti-charge increases linearly with their separation. However, because the dynamics of monopoles is hard to control in strong coupling gauge theories no quantitative description of this phenomenon was constructed in QCD.

The revival of interest to this problem occurs after the work of Seiberg and Witten [4, 5]. They showed that in supersymmetric gauge theories the holomorphy and electromagnetic duality are powerful tools to study non-Abelian dynamics at strong coupling. Using the exact solution of $N = 2$ supersymmetric gauge theory they were able to demonstrate that the condensation of monopoles do really occurs near the monopole vacuum once $N = 2$ theory is slightly perturbed by the mass term of the adjoint matter [4].

Since then a lot of papers study confinement and formation of flux tubes in supersymmetric gauge theories [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. One important particular feature of supersymmetric theories is the presence of modular spaces – manifolds on which scalar fields can develop arbitrary VEV’s. If the gauge group is broken down completely by scalar VEV’s such vacuum manifolds are called Higgs branches. In this paper we study the formation of flux tubes in supersymmetric gauge theories with Higgs branches.

In fact, this problem was addressed earlier in ref. [10] for the case of Higgs branches with flat geometry. In particularly, the formation of flux tubes on the Higgs branch of $N=2$ QCD with gauge group $SU(2)$ and $N_f = 2$ flavors of fundamental matter hypermultiplets (quarks) with the same mass was studied. The Higgs branch in this case represent an extreme type I superconductor with vanishing Higgs mass. It was shown in [10] that the flux tubes still exist in this set up although becomes logarithmically “thick” due to the presence of light scalar field. Because of this the confining potential behaves as

$$V(L) \sim \frac{L}{\log L},$$

with the separation $L$ between two heavy well-separated (magnetic) charges. This potential is still confining but is no longer linear in $L$. Note, that in this
case we have condensation of scalar components of quarks (electric charges) so these are monopoles which are confined. As the potential between heavy trial charges is an order parameter to distinguish between different phases of the theory, we see that we have a new confining phase with a non-linear potential which is specific for Higgs branches in supersymmetric gauge theories.

In this paper we continue to study flux tubes on Higgs branches concentrating on the case of Higgs branches with curved geometry. The curvature on a Higgs branch is induced by adding of the Fayet-Iliopoulos (FI) term to the theory.

Our first example is $\mathcal{N}=1$ QED with one flavor of quarks and the FI D-term added. This theory has a two dimensional Higgs branch of a hyperbolic form. In order to regularize this theory in the infrared one would like to lift the Higgs branch making scalars which fluctuate along the vacuum manifold slightly massive. It turns out that this can be done without destroying of $\mathcal{N}=1$ supersymmetry. We start with $\mathcal{N}=2$ QED with one quark flavor which has no Higgs branch. Then we add a mass term for the neutral matter field breaking $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$. Then in the limit of mass parameter of this field $\mu \to \infty$ we recover $\mathcal{N}=1$ QED with its Higgs branch. At large but finite $\mu$ we have Higgs branch lifted but the potential along the would be Higgs branch becoming more and more flat as we increase $\mu$.

In this set up we study the formation of flux tubes in a generic vacuum on the would be Higgs branch. We show that flux tubes exist and have the following structure in the plane orthogonal to the string axis. They are formed by “BPS core” which size is determined by the FI-parameter and long range “tail” formed by light scalar fields. The string tension is given by the tension of “BPS core” plus contribution coming from the “tail”. If the VEV of scalar fields squared is of the same order as the FI-parameter the contribution of the ”tail” becomes a small correction to the tension coming from the ”BPS core”. That is why we can call this string “almost BPS”.

This string gives rise to the confinement of monopoles with the potential which depends on the monopole separation $L$ as

$$V(L) \sim L \left(1 + \frac{\text{const}}{\log L}\right).$$

Here the first term comes from the “BPS core” while the second one is determined by the “tail”.

\[ (1.2) \]
Next we consider $\mathcal{N}=2$ QCD with gauge group $SU(2)$ and two flavors of quarks with the same mass. This theory has four dimensional Higgs branch \cite{5}. We perturb this theory by the mass term for the adjoint field which to the leading order in perturbation parameter do not break $\mathcal{N}=2$ supersymmetry and reduces to FI term \cite{7,11}. The non-zero FI-parameter induces curvature on the Higgs branch. It becomes a hyper-Kahler manifold with Eguchi-Hanson \cite{17} geometry.

We work out the ANO string solution in a generic point on the Higgs branch and calculate the string tension. Similar to the $\mathcal{N}=1$ case it is given by the contribution of “BPS core” plus correction coming from the “tail” of light fields living on the Higgs branch.

We also discuss if this ANO string can turn into semilocal string (see \cite{18} for a review on semilocal strings) by increasing the size of its “BPS core”. If this happen this would ruin confinement making the potential between monopoles to fall-off as a power of $L$ at large $L$. We show that if VEV of the quark fields is much larger then the FI-parameter the semilocal string is not developed. ANO string appears to be stable so monopoles are confined by the potential \cite{12}.

However, for vacua at the base $S_2$ cycle of the Eguchi-Hanson manifold for which quark VEV is equal to the FI-parameter ANO strings turn into semilocal strings \cite{15,16}. For these vacua we have deconfinement phase. We put forward a conjecture that confinement/deconfinement phase transition occurs exactly at the base cycle. Finally we consider effects which break $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$ and show that in this case the deconfinement phase disappears.

The paper is organized as follows. In Sect. 2 we review flux tubes on Higgs branches with flat geometry \cite{10}. In Sect. 3 we introduce $\mathcal{N}=1$ QED with FI term and in Sect. 4 study flux tubes on the Higgs branch in this theory. In Sect. 5 we consider $\mathcal{N}=2$ QCD with gauge group $SU(2)$ and two flavors of quarks with the same mass and study flux tubes on the Higgs branch in this theory. In Sect. 6 we discuss the issue of semilocal strings. Sect. 7 contains our brief conclusions.

\section{Extreme type I strings}

In this section we review classical solution for ANO vortices in theories with flat Higgs potential which arises in supersymmetric settings \cite{10}. In particu-
lar, in ref. [10] flux tubes on the Higgs branch of $\mathcal{N}=2$ QCD with two flavors of fundamental matter (quarks) were studied. Consider the Abelian Higgs model

$$S_{AH} = \int d^4x \left\{ \frac{1}{4g^2} F_{\mu\nu}^2 + |\nabla_\mu q|^2 + \lambda(|q|^2 - v^2)^2 \right\}, \quad (2.3)$$

for the single complex field $q$ with quartic coupling $\lambda = 0$. Here $\nabla_\mu = \partial_\mu - i n_e A_\mu$, where $n_e$ is the electric charge of the field $q$. Following [10] we consider first this model with small $\lambda$, then we take the limit $\lambda \to 0$.

The field $q$ develops VEV, $q = v$, breaking down the U(1) gauge group. Photon acquires the mass

$$m_\gamma^2 = 2n_e^2 g^2 v^2, \quad (2.4)$$

while the Higgs mass is equal to

$$m_q^2 = 4\lambda v^2. \quad (2.5)$$

The model (2.3) is the standard Abelian-Higgs model which admits ANO strings [2, 3]. For an arbitrary $\lambda$ the Higgs mass differs from that of the photon. The ratio of the photon mass to the Higgs mass is an important parameter, in the theory of superconductivity it characterizes the type of superconductor. Namely, for $m_q < m_\gamma$ we have the type I superconductor in which two well separated ANO strings attract each other. Instead for $m_q > m_\gamma$ we have the type II one in which two well separated strings repel each other. This is related to the fact that scalar field produces an attraction for two vortices, while the electromagnetic field produces a repulsion.

Now we consider the extreme type I limit in which

$$m_q \ll m_\gamma. \quad (2.6)$$

We also assume the week coupling regime in the model (2.3) $\lambda \ll g^2 \ll 1$.

The general idea to find the string solution is to separate behavior of different fields at different scales present in the problem due to the condition (2.6). This method goes back to the paper by Abrikosov [2] in which the tension of type II string under condition $m_q \gg m_\gamma$ has been calculated. The similar idea was used in [10] to calculate the tension of the type I string under condition $m_q \ll m_\gamma$.

To the leading order in $\log m_\gamma/m_q$ the vortex solution has the following structure in the plane orthogonal to the string axis [10]. The electromagnetic field is confined in a core with the radius

$$R_g \sim \frac{1}{m_\gamma} \log \frac{m_\gamma}{m_q}. \quad (2.7)$$
The scalar field is close to zero inside the core. Instead, outside the core, the electromagnetic field is vanishingly small while the scalar field behaves as

\[ q = v \left( 1 - \frac{K_0(m_q r)}{\log 1/m_q R_g} \right) e^{i\alpha}, \tag{2.8} \]

where \( r \) and \( \alpha \) are polar coordinates in the plane orthogonal to the string axis. Here \( K_0 \) is the Bessel function with exponential fall-off at infinity and logarithmic behavior at small arguments, \( K_0(x) \sim \log 1/x \) at \( x \to 0 \). The reason for this behavior is that in the absence of the electromagnetic field outside the core the scalar field satisfies a free equation of motion and (2.8) is a solution to this equation. From (2.8) we see that the scalar field slowly (logarithmically) approaches its boundary value \( v \).

The result for the string tension is \[ T = \frac{2\pi v^2}{\log m_\gamma/m_q}. \tag{2.9} \]

The main contribution to the tension in (2.9) comes from the logarithmic “tail” of the scalar \( q \). It is given by the kinetic term for the scalar field in (2.3). This term contains a logarithmic integral over \( r \). Other terms in the action are suppressed by powers of \( \log m_\gamma/m_q \) as compared with the one in (2.9).

The results in (2.7), (2.9) mean that if we naively take the limit \( m_q \to 0 \) the string becomes infinitely thick and its tension goes to zero [10]. This means that there are no strings in the limit \( m_q = 0 \). The absence of ANO strings in theories with flat Higgs potential was first noticed in [19].

One might think that the absence of ANO strings means that there is no confinement of monopoles in theories with Higgs branches. As we will see now this is not the case [10]. So far we have considered infinitely long ANO strings. However the setup for the confinement problem is slightly different. We have to consider monopole–anti-monopole pair at large but finite separation \( L \). Our aim is to take the limit \( m_q \to 0 \). To do so let us consider ANO string of the finite length \( L \) within the region

\[ \frac{1}{m_\gamma} \ll L \ll \frac{1}{m_q}. \tag{2.10} \]

Then it turns out that \( 1/L \) plays the role of the IR-cutoff in Eqs. (2.7) and (2.9) instead of \( m_q \) [10]. The reason for this is that for \( r \ll L \) the problem
is two dimensional and the solution of the two dimensional free equation of motion for a scalar field is given by (2.8). If we naively put \( m = 0 \) in this solution the Bessel function reduces to the logarithmic function which cannot reach a finite boundary value at infinity. Thus as we mentioned above the infinitely long flux tubes do not exist. This was noticed in [19]. However for \( r \gg L \) the problem becomes three dimensional. The solution for the three dimensional free scalar equation of motion behaves as \( q - v \sim 1/|x| \), where \( x_n, n = 1, 2, 3 \) is a coordinate in the three dimensional space. We see that with this behavior the scalar field reaches its boundary value at infinity. Clearly \( 1/L \) plays a role of IR cutoff for the logarithmic behavior of the scalar field.

Now we can safely put \( m = 0 \). The result for the electromagnetic core of the vortex becomes

\[
R_g \sim \frac{1}{m} \log m L, \tag{2.11}
\]

while its string tension is given by [10]

\[
T = \frac{2\pi v^2}{\log m L}. \tag{2.12}
\]

We see that the ANO string becomes "thick" but still its transverse size \( R_g \) is much less than its length \( L \), \( R_g \ll L \). As a result the potential between heavy well separated monopole and anti-monopole is still confining but is no longer linear in \( L \). It behaves as [10]

\[
V(L) = 2\pi v^2 \frac{L}{\log m L}. \tag{2.13}
\]

As we already explained in the Introduction the potential \( V(L) \) is an order parameter which distinguishes different phases of a theory (see, for example, review [20]). We conclude that we have a new confining phase on the Higgs branches. It is clear that this phase can arise only in supersymmetric theories because we do not have Higgs branches without supersymmetry.

### 3 \( \mathcal{N}=1 \) QED with Fayet-Iliopoulos term

#### 3.1 The model

The field content of \( \mathcal{N}=1 \) QED is as follows. The vector multiplet contains \( U(1) \) gauge field \( A_\mu \) and Weyl fermion \( \lambda^\alpha, \alpha = 1, 2 \). The chiral matter
multiplet contains two complex scalar fields $q$ and $\tilde{q}$ as well as two complex Weyl fermions $\psi^\alpha$ and $\tilde{\psi}^\alpha$. The bosonic part of the action reads

$$S_{QED} = \int d^4x \left\{ \frac{1}{4g^2} F_{\mu\nu}^2 + \nabla_\mu q \nabla_\mu q + \nabla_\mu \tilde{q} \nabla_\mu \tilde{q} + V(q, \tilde{q}) \right\}, \quad (3.1)$$

where $\nabla_\mu = \partial_\mu - \frac{i}{2} A_\mu$, $\nabla_\mu = \partial_\mu + \frac{i}{2} A_\mu$, so we assume that that matter fields have electric charge $n_e = 1/2$. The potential of this theory comes from the $D$-term and given by

$$V(q, \tilde{q}) = \frac{g^2}{8} \left( |q|^2 - |\tilde{q}|^2 - \xi_3 \right)^2, \quad (3.2)$$

where parameter $\xi_3$ arises if we include FI $D$-term in our theory. We denote here the FI $D$-term parameter $\xi_3$. This notation will become clear later in $\mathcal{N}=2$ setup where $D$ and $F$ FI parameters form a triplet with respect to global $SU(2)_R$.

The vacuum manifold of the theory (3.1) is a Higgs branch determined by the condition

$$|\langle q \rangle|^2 - |\langle \tilde{q} \rangle|^2 = \xi_3. \quad (3.3)$$

The dimension of this Higgs branch is two. To see this note, that we have two complex scalars (four real variables) subject to one constraint (3.3). Also we have to subtract one gauge phase, thus we have 4-1-1=2.

Now let us consider the mass spectrum of the QED (3.1). As it is clear from (3.3) at any non-zero $\xi_3$ scalar fields develop VEV’s breaking $U(1)$ gauge symmetry. The photon mass is given by

$$m_\gamma^2 = \frac{1}{2} g^2 v^2, \quad (3.4)$$

where we introduce the VEV of scalar field

$$v^2 = |\langle q \rangle|^2 + |\langle \tilde{q} \rangle|^2. \quad (3.5)$$

To find matter masses we diagonalize the $4 \times 4$ mass matrix for scalar fields in (3.2). It has three zero eigenvalues one of which is ”eaten” by the Higgs mechanism while two others correspond to chiral massless multiplet living on the Higgs branch. The remaining fourth eigenvalue is equal to the photon mass (3.4),

$$m_H = m_\gamma. \quad (3.6)$$
This eigenvalue corresponds to the scalar superpartner of the photon in the massive $\mathcal{N}=1$ vector multiplet.

Our aim is to study string solutions in a generic vacuum on the Higgs branch (3.3). It is clear that this solution is formed by both massive electromagnetic and scalar fields as well as by massless scalars living on the Higgs branch. Similar to the approach of ref. [10] (reviewed above in Sect. 2) we would like to regularize the problem at hand in the infrared giving these massless scalars a small mass $m_L$ and then taking the limit $m_L \to 0$. We will do it in the next subsection.

### 3.2 Softly broken $\mathcal{N}=2$ QED

It turns out that we can lift the Higgs branch of $\mathcal{N}=1$ QED considered in the previous subsection giving the massless fields small masses without breaking $\mathcal{N}=1$ supersymmetry. To do so following ref. [11] let us consider $\mathcal{N}=2$ QED. This theory besides fields which enter $\mathcal{N}=1$ QED contains also neutral chiral field $A$ which interacts with charged matter via superpotential

$$W_{N=2} = \frac{1}{\sqrt{2}} \tilde{Q} A Q - \frac{1}{2\sqrt{2}} \eta A,$$  \hspace{1cm} (3.7)

where $Q$ and $\tilde{Q}$ are charge superfields and we also introduce the FI $F$-term proportional to the complex FI parameter

$$\eta = \xi_1 + i \xi_2.$$  \hspace{1cm} (3.8)

Two FI $F$-term parameters $\xi_{1,2}$ together with one $D$-term parameter $\xi_3$ form a triplet with respect to global $SU(2)_R$ symmetry of $\mathcal{N}=2$ QED.

Let us break $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$ adding a mass term for the chiral field $A$ via additional superpotential

$$\delta W = \frac{\mu}{2} A^2.$$  \hspace{1cm} (3.9)

Now if we take the limit $\mu \to \infty$ we can integrate out heavy $A$-field and end up with $\mathcal{N}=1$ QED considered in the previous subsection. Namely, adding (3.9) to (3.7) and integrating out $A$ we get the superpotential

$$W = -\frac{1}{4\mu} \left( \tilde{Q} Q - \frac{\eta}{2} \right)^2.$$  \hspace{1cm} (3.10)
This superpotential leads to the following scalar potential in $\mathcal{N}=1$ QED

$$V(q, \bar{q}) = \frac{g^2}{8}(|q|^2 - |\bar{q}|^2 - \xi_3)^2 + \frac{1}{4\mu^2}(|q|^2 + |\bar{q}|^2)|q\bar{q} - \frac{\eta}{2}|^2$$  \hspace{1cm} (3.11)

The first term here comes from the D-components of the gauge multiplet, while the second one comes from the superpotential above. We expand $\eta = \xi_1 + i\xi_2$, where $\xi_1$ and $\xi_2$ are real and use $SU(2)_R$ rotation to put $\xi_2 = 0$.

We see that in the limit $\mu \to \infty$ this potential reduces to the one in (3.2). However, for any finite $\mu$ the potential (3.11) has no Higgs branch and the vacuum state is uniquely defined. Namely, the potential (3.11) has a minimum at

$$<q> = \sqrt{\xi_3 \cosh \rho_0},$$
$$<\bar{q}> = \sqrt{\xi_3 \sinh \rho_0},$$  \hspace{1cm} (3.12)

where we introduce parameter $\rho_0$ defined via

$$\sinh 2\rho_0 = \frac{\xi_1}{\xi_3}$$  \hspace{1cm} (3.13)

Note, that if $\xi_2 = 0$ we can use gauge freedom to make both $<q>$ and $<\bar{q}>$ positive.

Calculating the $4 \times 4$ scalar mass matrix near this vacuum we obtain one zero eigenvalue (corresponding to the state ”eaten” by the Higgs mechanism) and another one equal to the mass of the photon, see (3.4), (3.6). As we explained above the corresponding scalar together with the photon form a bosonic part of $\mathcal{N}=1$ vector massive multiplet. The mass of this multiplet remains unchanged by our IR regularization. The only modification is that the VEV of the scalar field is now fixed by the parameters of the theory. Namely, it is defined by (3.5) which can be now expressed in terms of FI parameters as

$$v^4 = \xi_1^2 + \xi_3^2.$$  \hspace{1cm} (3.14)

The other two eigenvalues of the scalar mass matrix are given by

$$m_L = \frac{\xi_1}{\mu}.$$  \hspace{1cm} (3.15)

This is the mass of one $\mathcal{N}=1$ chiral multiplet (containing two real scalars). In the limit of large $\mu$ that we consider here

$$m_L \ll m_\gamma.$$  \hspace{1cm} (3.16)
This means that we have one heavy scalar in our problem (with the mass equal to the mass of the photon, \( m_H = m_\gamma \)) and two light scalars. In particular, in the limit of \( \mu \to \infty \) \( m_L \) goes to zero and we recover the Higgs branch of \( \mathcal{N}=1 \) QED.

Note that the form of the second term in the potential (3.11) is not important for our purposes. It serves as a IR regularization which lifts the Higgs branch and gives would be massless moduli fields a small mass (3.15).

4 String solution

In this section we consider solutions for flux tubes which arise in a generic point on the Higgs branch of \( \mathcal{N}=1 \) QED. First we start with a base point on the Higgs branch, which corresponds to \( < \tilde{q} > = 0 \). Next we consider the generic point on the vacuum manifold.

4.1 BPS strings

Consider the particular vacuum with \( < \tilde{q} > = 0 \) in \( \mathcal{N}=1 \) QED (3.1). It is well known that there is a BPS ANO string solution for this particular choice of vacuum [21].

We can easily recover this solution in the softly broken QED with the scalar potential (3.11) taking the limit \( \xi_1 = 0 \), while keeping \( v = \xi_3 \) non-zero. It is clear that light fields do not play any role and we can look for the string solution using the following ansatz

\[
\tilde{q} = 0. \tag{4.1}
\]

With this substitution the bosonic part of softly broken \( \mathcal{N}=2 \) QED reduces to the standard Abelian Higgs model (2.3) for one complex scalar field \( q \) interacting with the electromagnetic field, at the particular value of the quartic coupling \( \lambda = g^2/8 \). This value of \( \lambda \) ensures that the mass of the scalar field in (2.3) is equal to the mass of the photon, see (3.6). As we already explained in the previous section this is a consequence of \( \mathcal{N}=1 \) supersymmetry.

Thus our model is on the boundary separating superconductors of the I and II type. In this case vortices do not interact. It is well known that vanishing of the interaction at \( m_H = m_\gamma \) can be explained by the BPS nature of the ANO strings. The ANO string satisfy the first order equations and
saturate the Bogomolny bound [22]. This bound follows from the following representation for the string tension $T$,

$$
T = 2\pi \xi_3 n + \int d^2 x \left\{ \frac{1}{2g} F_{ik} + \frac{g}{4} \left( |q|^2 - \xi_3 \right) \varepsilon_{ik} \right\}^2 + \frac{1}{2} |\nabla_i q + i \varepsilon_{ik} \nabla_k q|^2 .
$$

(4.2)

Here indices $i, k = 1, 2$ denote coordinates transverse to the axis of the vortex. The minimal value of the tension is reached when both positive terms in the integrand of Eq. (4.2) vanish. The string tension becomes

$$
T_{BPSS} = 2\pi \xi_3 n ,
$$

(4.3)

where the winding number $n$ counts the magnetic flux $2\pi n$ (we assume positive $n$). The linear dependence of string tensions on $n$ implies the absence of their interactions.

For simplicity we consider the case $n = 1$. Putting the integrand in Eq. (4.2) to zero gives two first order differential equations,

$$
r \frac{d}{dr} \phi(r) - f(r) \phi(r) = 0 ,
$$

$$
- \frac{1}{r} \frac{d}{dr} f(r) + \frac{g^2}{4} \left( \phi^2(r) - \xi_3 \right) = 0 ,
$$

(4.4)

where the profile functions $\phi(r)$ and $f(r)$ are introduced in a standard way,

$$
q(x) = \phi(r) e^{i\alpha} ,
$$

$$
A_i(x) = -2\epsilon_{ij} \frac{x_j}{r^2} \left[ 1 - f(r) \right] .
$$

(4.5)

Here $r = \sqrt{x_i^2}$ is the distance and $\alpha$ is the polar angle in the (1,2) plane. The profile functions are real and satisfy the boundary conditions

$$
\phi(0) = 0 , \quad f(0) = 1 ,
$$

$$
\phi(\infty) = \sqrt{\xi_3} , \quad f(\infty) = 0 ,
$$

(4.6)

which ensures that the scalar field reaches its vev $\sqrt{\xi_3}$ at the infinity and the vortex carries one unit of the magnetic flux. The equations (4.4) with boundary conditions (4.6) lead to the unique solution for the profile functions (although an analytic form of this solution is not found).
In QED with the $\mathcal{N}=2$ supersymmetry broken down to $\mathcal{N}=1$ the emergence of the first order equations (4.4) signals that some (half) of the remaining four SUSY charges of $\mathcal{N}=1$ algebra act trivially on the ANO solution (cf. [23, 24, 21, 11]). In this case the Bogomolny (topological) bound for the string tension coincides with the central charge of SUSY algebra. Note that at $\mu = 0$ we have BPS strings in $\mathcal{N}=2$ QED [7, 11]. As we increase $\mu$ but keep $\xi_1 = 0$ they become BPS strings of $\mathcal{N}=1$ QED. As the number of states in string multiplets cannot jump we conclude that we get two BPS string multiplets at large $\mu$ from one BPS string in $\mathcal{N}=2$ theory at $\mu = 0$.

4.2 String in a generic vacuum

Now let us turn to the generic case when both $\xi_1$ and $\xi_3$ are non-zero. It is clear that in this case the ANO string is no longer BPS saturated. To see this, note that now we cannot take the light scalar fields with mass $m_L$ to be zero on the vortex solution. This means that the vortex has a long range “tail” in (1, 2) plane formed by the light scalars. This ensures attraction of different vortices and type I superconductivity.

The string tension for non-BPS string is bounded from below by the central charge of the $\mathcal{N}=1$ algebra

$$T \geq 2\pi \xi_3,$$

which is determined by the FI D-term parameter $\xi_3$ [21].

Now let us work out the approximate solution for the ANO vortex and calculate its tension. The difference of the problem at hand with the one studied in [10] and reviewed in Sect. 2 is that in [10] was considered the case which corresponds to $\xi_3 = 0$ in the low energy QED. In this case heavy scalar does not develop VEV and can be put to zero on the vortex solution. The only scalar which is present in the problem is the light one.

In the present case, when both $\xi_1$ and $\xi_3$ are non-zero both heavy and light scalars are non-zero. To solve the problem we adopt the following model for the structure of the vortex in (1, 2) plane. The electromagnetic field together with the heavy scalar form a core of relatively small radius. Let us call it $R_c$. Outside this core heavy fields are almost zero, while light components produce a “tail” of size $1/m_L$. To be more specific we consider separately three different regions: $r \ll R_c, R_c \ll r \ll 1/m_L$ and $r \gtrsim 1/m_L$.

In the first region (inside the core)

$$r \lesssim R_c$$

(4.8)
only heavy scalar and electromagnetic fields are non-zero. while the light
scalars are almost zero. This suggests that we can look for the vortex solution
with
\[ \tilde{q} = 0. \] (4.9)
With this ansatz our model reduces to the standard Abelian Higgs model
(2.3) at the special value of quartic coupling \( \lambda = g^2 / 8 \), see Sect 4.1. This
means that the solution for the vortex inside the core is given by the BPS
solution we reviewed in the previous subsection. In particular, the field \( q \)
and \( A_i \) are given by (4.5) with the profile functions \( \phi \) and \( f \) subject to the
first order equations (4.4) and boundary conditions (4.6). Now the bound-
dary condition (4.6) at infinity should be understood as a condition at the
boundary of the core, namely, at \( r > R_c \). Say, for scalar fields this means
\[ q(r > R_c) = \sqrt{\xi}, \] (4.10)
while \( \tilde{q} \) is zero. This means that outside the core scalar fields go to the base
point on the vacuum manifold.
As soon as the size of this BPS string is given by \( 1 / \sqrt{g^2 \xi} \) we conclude
that
\[ R_c = \frac{1}{\sqrt{g^2 \xi}}. \] (4.11)
The string tension of the vortex with \( n = 1 \) is given by
\[ T = 2\pi \xi + T_{tail}. \] (4.12)
Here the first term is "BPS" contribution (4.3) coming from the region (4.3)
inside the core, while \( T_{tail} \) stands for the contribution of the light scalar "tail"
coming from the region outside the core. Let us work out this contribution.
Consider the region of intermediate \( r \) outside the core,
\[ R_c \ll r \ll 1/m_L. \] (4.13)
In this region we can neglect the second term in the potential (3.11). The
motion of scalar fields is restricted by the constraint
\[ |q|^2 - |\tilde{q}|^2 = \xi \] (4.14)
enforced by the first term in the potential (3.11), see also (3.2). The con-
straint (4.14) ensures that the only light scalar modes (actually massless,
once we neglect $m_L$) are exited in the region (4.13). To put it another way, the constraint (4.14) reduces the motion of scalar fields to the motion along the Higgs branch given by (4.14). From (4.14) we see that this Higgs branch has non-flat metric (see also eq. (4.18) below). This makes a difference with the case studied in [10] where Higgs branch was flat.

Now let us work out the sigma model on the Higgs branch. The constraint (4.14) can be solved by the substitution

$$q = \sqrt{\xi_3} e^{i\alpha + i\beta(x)} \cosh \rho(r),$$

$$\tilde{q} = \sqrt{\xi_3} e^{i\alpha - i\beta(x)} \sinh \rho(r).$$

Two complex fields $q$ and $\tilde{q}$ have two different phases. We use the gauge freedom to fix their average to be equal to the polar angle $\alpha$ in order to ensure the correct flux of the solution. The phase difference contains arbitrary function $\beta(x)$. The boundary condition for functions $\rho$ and $\beta$ is fixed by VEV’s of scalar fields (3.12).

$$\rho(r \sim R_c) = 0,$$

$$\rho(r \sim 1/m_L) = \rho_0, \beta(r \sim 1/m_L) = 0.$$  

(4.16)

Now let us take the low energy limit formally sending $m_\gamma \to \infty$. In this limit we can integrate out the gauge field

$$A_i = -i \frac{\bar{q} \partial_i q - \partial_i \bar{q} q + \bar{q} \partial_i \tilde{q} - \partial_i \bar{q} \tilde{q}}{\bar{q} \tilde{q} + \bar{q} \bar{q}}$$

$$= 2 \left( \partial_i \alpha + \frac{\partial_i \beta}{\cosh 2\rho} \right),$$

(4.17)

where we use the substitution (4.15). Then our model reduces to the 2d sigma model

$$T_{tail} = \xi_3 \int d^2 x \cosh 2\rho \left\{ (\partial_i \rho)^2 + (\partial_i \beta)^2 \tanh^2 2\rho \right\}$$

(4.18)

with non-flat metric of the target space.

Now the problem is to find the classical solution for two dimensional sigma model (4.18) with boundary conditions (4.16). To do this in more general setting we consider a sigma model with arbitrary metric

$$T_{tail} = \xi_3 \int d^2 x g_{MN} \partial_i \varphi^N \partial_i \varphi^N,$$

(4.19)
where $N, M$ numerates light scalar fields. Now we assume that these fields depend only on the radial coordinate $r$. This leads us to the one dimensional sigma model which determine the tension of the string ”tail”

$$T_{tail} = \frac{2\pi \xi_3}{\log 1/m_L R_c} \int_0^1 dt g_{MN} \partial_t \phi^N \partial_t \phi^N, \quad (4.20)$$

where we introduce normalized logarithmic time

$$t = \frac{\log r/R_c}{\log 1/m_L R_c}. \quad (4.21)$$

The equations of motion for this sigma model define a geodesic line

$$\partial_t^2 \phi^N + \Gamma_{MK}^N \partial_t \phi^M \partial_t \phi^K = 0, \quad (4.22)$$

where $\Gamma_{MK}^N$ denotes the connection on the target manifold. The energy conservation shows that the action on this line is determined by the square of the length of this line

$$T_{tail} = \frac{2\pi \xi_3}{\log 1/m_L R_c} l^2. \quad (4.23)$$

Here the length of the geodesic line reads

$$l = \int_0^1 dt \sqrt{g_{MN} \partial_t \phi^N \partial_t \phi^N}. \quad (4.24)$$

For our model (4.18) at hand the geodesic line is particularly simple. Clearly,

$$\beta = 0 \quad (4.25)$$

at the geodesic line and its length on the Higgs branch becomes

$$l = \int_0^{\rho_0} d\rho \sqrt{\cosh 2\rho}, \quad (4.26)$$

where the upper limit is defined by (3.13). This length determine our final result for the tension of the string

$$T = 2\pi \xi_3 + \frac{2\pi \xi_3}{\log (g \sqrt{\xi_3}/m_L)} l^2. \quad (4.27)$$

It is easy to check that the third region $r \gtrsim 1/m_L$ (where scalar fields approach their VEV’s exponentially) gives corrections to this result suppressed by powers of $\log g \sqrt{\xi_3}/m_L$, see also [10].
Let us note that this string solution "feels" not only the VEV of the scalar field $v^2$ but the whole structure of the Higgs branch. In particular, the size of the core $R_c$ is determined by $g \sqrt{\xi_3}$ (see (4.11)) rather than by the mass of the photon (3.4) (determined by $gv$).

Of course, if we send $\xi_1$ to zero going to the base point on the Higgs branch $<\tilde{q}> = 0$ the second term in (4.27) vanishes and we will get the BPS string. Note however, that the long non-BPS string multiplet contains two bosonic and two fermionic states, while the short BPS multiplet contains one bosonic and one fermionic state. As the number of states cannot jump the long multiplet reduces to two short BPS multiplets at $\xi_1 = 0$. This is in accordance with our conclusion in the end of the previous subsection where we considered breaking of $\mathcal{N}=2$ supersymmetry by turning on $\mu$. There we saw that one $\mathcal{N}=2$ BPS multiplet reduces to two $\mathcal{N}=1$ BPS multiplets as we switch on $\mu$ at zero $\xi_1$.

If $\xi_3$ and $v^2$ are of the same order the second term in (4.27) becomes small as compared with the first one due to the logarithmic suppression. This is the reason why we can call this string "almost BPS". Note however, that this string is not a BPS one. It does not belong to a short BPS multiplet and all four supercharges act on this solution non-trivially.

To make contact with the case discussed in Sect. 2 let us consider the limit $\xi_3 \ll v^2$. In this limit the geometry of the Higgs branch becomes flat and $l^2 \to v^2/\xi_3$. Then our result (4.27) reduces to eq. (2.9). Note that as we send $\xi_3$ to zero the size of the core of the string $R_c$ grows and eventually freezes as it reaches the value $R_g$, see (2.7). At $g \sqrt{\xi_3} \ll 1/R_g$ the core is determined by the electromagnetic field [10] and the heavy scalar plays no role any longer.

Let us now remove our IR regularization sending $\mu \to \infty$. Then we recover the Higgs branch of $\mathcal{N}=1$ QED and light scalars become strictly massless. Repeating the arguments in Sect. 2 we see that infinitely long strings do not exist any longer in the generic point on the Higgs branch. However, strings of finite length still exist. Their string tension is now given by

$$T = 2\pi \xi_3 + \frac{2\pi \xi_3}{\log (g \sqrt{\xi_3} L)} l^2. \quad (4.28)$$

They give rise to the confinement of monopoles with the potential of type (1.2).

To conclude this section let us make a comment on literature. In ref. [15, 16] vortices in $\mathcal{N}=1$ QED were considered and it was concluded that in
generic point on the Higgs branch the string is unstable. The only vacuum which support string solutions is the base point of the Higgs branch $\langle \tilde{q} \rangle = 0$. The so called "vacuum selection rule" was put forward in \cite{15, 16} to ensure this property. We would like to stress that our results here and in ref. \cite{10} do not contradict above mentioned papers. As we already explained infinitely long strings do not exist in a generic point on the modular space, indeed. However this does not mean that we loose confinement. As we explained in Sect. 2 to study confinement we have to consider strings of finite length. Strings of finite length do exist in a generic point on the Higgs branch and produce confining potential \cite{12}.

5 \textbf{$\mathcal{N}=2$ QCD}

In this section we study another example of flux tubes on Higgs branches. We consider $\mathcal{N}=2$ QCD with gauge group $SU(2)$ and two flavors of fundamental matter (quarks). If masses of these quarks are equal then there is a Higgs branch in this theory. First we review the effective low energy theory on the Higgs branch \textcircled{\textit{5}} and then construct the string solution.

5.1 Higgs branch

The $\mathcal{N}=2$ vector multiplet of the theory at hand on the component level consists of the gauge field $A_{\mu}^a$, two Weyl fermions $\lambda_1^{aa}$ and $\lambda_2^{aa}$ ($\alpha = 1, 2$) and the complex scalar $\varphi^a$, where $a = 1, 2, 3$ is the color index. Fermions form a doublet $\lambda_f^{aa}$ with respect to global $SU(2)_R$ group, $f = 1, 2$.

The scalar potential of this theory has a flat direction. The adjoint scalar field develop an arbitrary VEV along this direction breaking $SU(2)$ gauge group down to $U(1)$. We choose $\langle \varphi^a \rangle = \delta^{a3} \langle a \rangle$. The complex parameter $\langle a \rangle$ parameterize the Coulomb branch. The low energy effective theory generically contains only the photon $A_{\mu} = A_3^{a\mu}$ and its superpartners: two Weyl fermions $\lambda_f^{a\alpha}$ and the complex scalar $a$. This is massless short vector $\mathcal{N}=2$ multiplet. It contains 4 boson + 4 fermion states. W-boson and its superpartners are massive with masses of order of $\langle a \rangle$.

Quark hypermultiplets have the following structure. They consist of complex scalars $q^{kA}$, $\bar{q}_{Ak}$ and fermions $\psi^{k\alpha A}$, $\bar{\psi}_{A\alpha k}$, where $k = 1, 2$ is the color index and $A = 1, \ldots, N_F$ is the flavor one. Scalars $q^{kA}$, $\bar{q}^{kA}$ form a doublet $q^{kfA}$, $f = 1, 2$ with respect to $SU(2)_R$ group. All these states are in the BPS short
representations of $\mathcal{N}=2$ algebra on the Coulomb branch with $4 \times N_c \times N_f = 16$ real boson states ($+16$ fermion states).

Coulomb branch has three singular points where monopoles, dyons or charges become massless. Two of them correspond to monopole and dyon singularities of the pure gauge theory. Their positions on the Coulomb branch are given by

$$u_{m,d} = \pm 2m\Lambda_2 - \frac{1}{2}\Lambda_2^2,$$  \hspace{1cm} (5.1)

where $u = \frac{1}{2}\langle \phi^a \phi^a \rangle$ and $\Lambda_2$ is the scale of the theory with $N_f = 2$. In the large $m$ limit $u_{m,d}$ are approximately given by their values in the pure gauge theory $u_{m,d} \simeq \pm 2m\Lambda_2 = \pm 2\Lambda_2^2$, where $\Lambda$ is the scale of $N_f = 0$ theory.

The charge singularity corresponds to the point where half of quark states becomes massless. We denote them $q^{fA}$ and $\psi^{\alpha A}$, $\bar{\psi}_A^\alpha$ dropping the color index. They form $N_f = 2$ short hypermultiplets with $4 \times N_f = 8$ real boson states. The rest of quark states acquire large mass $2m$ and we ignore them in the low energy description. The charge singularity appears at the point

$$a = -\sqrt{2} \ m$$  \hspace{1cm} (5.2)

on the Coulomb branch. In terms of variable $u$ (5.2) reads

$$u_c = m^2 + \frac{1}{2}\Lambda_2^2.$$  \hspace{1cm} (5.3)

Strictly speaking, we have $2 + N_f = 4$ singularities on the Coulomb branch. However, two of them coincides for the case of two flavors of matter with the same mass.

The effective theory on the Coulomb branch near the charge singularity (5.2) is given by $\mathcal{N} = 2$ QED with light matter fields $q^{fA}$, and their superpartners as well as the photon multiplet.

We deform the underlying non-Abelian theory adding a superpotential

$$\delta W = \frac{\mu}{2} \Phi^{a2},$$  \hspace{1cm} (5.4)

which is a mass term for the adjoint chiral field $\Phi^a$. Generally speaking this perturbation breaks $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$. The Coulomb branch shrinks to three above mentioned singular points which we call $\mathcal{N}=1$ vacua. In particularly, here we will be interested in quark vacuum (5.3) which is far away from the origin in weak coupling provided the mass of quarks is large $m \gg \Lambda_2$. 

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In the low energy effective theory near quark vacuum the superpotential (5.4) can be expanded as

\[ \delta W = -\frac{1}{2\sqrt{2}} \eta \delta A + \cdots, \]  

(5.5)

where \( A \) is the neutral chiral superfield with the lowest component \( a \), while

\[ \eta = -2\sqrt{2} < a >= 4\mu m. \]  

(5.6)

Dots in (5.5) denote higher orders in \( \delta A = A + \sqrt{2}m \).

It turns out that if in the limit of small \( \mu \) we truncate the series in (5.5) restricting ourselves only to the leading in \( \delta A \) term then the perturbation (5.5) does not break \( \mathcal{N}=2 \) supersymmetry in the low energy QED \[7, 11\]. The reason for this is that the leading term in (5.5) is linear in \( \delta A \). It is FI F-term which does not break \( \mathcal{N}=2 \) supersymmetry.

The bosonic part of the effective \( \mathcal{N}=2 \) QED near quark vacuum looks like

\[ S^{QED} = \int d^4x \left\{ \frac{1}{4g^2} F_{\mu\nu}^2 + \frac{1}{g^2} |\partial_\mu a|^2 + \nabla_\mu \bar{q}_A \nabla_\mu q^{FA} + \frac{g^2}{8} \left[ \text{Tr} (\bar{q}_m q) - \xi_m \right]^2 \right\} + \frac{1}{2} |q^{FA}|^2 |a + \sqrt{2}m_A|^2, \]  

(5.7)

where trace is calculated over flavor and \( SU(2)_R \) indices while \( \xi_m, m = 1, 2, 3 \) is a \( SU(2)_R \) triplet of FI parameters. In particular, for the choice (5.5) \( \xi_3 = 0 \) while \( \xi_1 \) and \( \xi_2 \) are real and imaginary parts of the complex parameter \( \eta \), see (3.8). The scalar potential in the theory (5.7) comes from the elimination of \( D \) and \( F \) terms. In more transparent notations it reads

\[ V = \frac{g^2}{8} (|q^A|^2 - |\bar{q}_A|^2)^2 + \frac{g^2}{2} |\bar{q}_A q^A - \frac{\xi_1}{2}|^2 + \frac{1}{2} (|q^A|^2 + |\bar{q}_A|^2) |a + \sqrt{2}m_A|^2. \]  

(5.8)

The QED coupling constant \( g^2 \) is small near the quark vacuum in (5.7) if \( m \gg \Lambda_2 \).

The charge singularity (5.2) is the root of the Higgs branch [5]. To find it we look for zeros of the potential in (5.7). We have

\[ \bar{q}_A (\tau_m)^P q^{fA} = \xi_m, \quad m = 1, 2, 3. \]  

(5.9)
(Here $m$ is an adjoint $SU(2)_R$ index, not to be confused with color indices.)
This equation determines the Higgs branch (manifold with $\langle q \rangle \neq 0$) which touches the Coulomb branch at the point \((5.2)\). It has non-trivial solutions for $N_f \geq 2$ \cite{5}. This is the reason why we choose $N_f = 2$ for our discussion.

Once quark fields develop non-zero VEV’s on the Higgs branch the $U(1)$ gauge group in \((5.7)\) is broken and the photon acquires the mass

$$m^2_\gamma = \frac{1}{2}g^2v^2,$$ \hspace{1cm} (5.10)

where we introduce the quark VEV

$$|\langle q^{fA} \rangle|^2 = |\langle q^A \rangle|^2 + |\langle \tilde{q}_A \rangle|^2 = v^2.$$ \hspace{1cm} (5.11)

As soon as the potential is zero on the fields which satisfy constraint \((5.9)\) the moduli fields which develop VEV’s on the Higgs branch are massless.

The number of these massless moduli (dimension of the Higgs branch) is four. To see this, note that we have eight real scalars subject to three conditions \((5.9)\). Also one phase is gauged. Overall we have 8-3-1=4, which gives us the dimension of the Higgs branch. Four massless scalars correspond to the lowest components of one short hypermultiplet. The other quark fields (4 real boson states + fermions) acquire the mass of the photon \((5.10)\). Together with states from the photon multiplet they form one long (non-BPS) $\mathcal{N}=2$ multiplet (cf. \cite{11}). It has 8 boson + 8 fermion states.

If we consider energies much less then the photon mass we can integrate out heavy scalars and electromagnetic field. Then we are left with the effective sigma model for light fields living on the Higgs branch. The four dimensional Higgs branch is a hyper-Kahler manifold. At non-zero $\xi_m$ it has Eguchi-Hanson geometry \cite{17}. Like in Sect. 3 we use $SU(2)_R$ rotations to put $\xi_2 = 0$ so $\eta$ is real, $\eta = \xi_1$. The convenient parametrization of the metric is as follows \cite{25}

$$S_\sigma = \int d^4x \left\{ \left[ 1 - \left( \frac{\xi_1}{w^2} \right)^2 \right]^{-1} \left( \partial_\mu w \right)^2 + w^2 \left[ \left( \partial_\mu \theta \right)^2 + \sin^2 \theta \left( \partial_\mu \delta \right)^2 \right] 
+ w^2 \left[ 1 - \left( \frac{\xi_1}{w^2} \right)^2 \right] \left( \partial_\mu \gamma + \cos \theta \partial_\mu \delta \right)^2 \right\}. \hspace{1cm} (5.12)$$

Here the Higgs branch is parametrized by one modulus field $w$ (which takes values from $\sqrt{\xi_1}$ to $\infty$)

$$w^2 = |q^{fA}|^2 = |q^A|^2 + |\tilde{q}_A|^2.$$ \hspace{1cm} (5.13)
and three phases $\theta$ (with values in the interval $(0, \pi)$) and $\delta$, $\gamma$ with values in the interval $(0, 2\pi)$.

We use three rotations of broken $SU(2)$ subgroup of global $SU(N_f = 2) \times SU(2)_R$ group to put VEV’s of the scalar fields on the Higgs branch in the form

$$< w > = v, < \theta > = 0, < \gamma > = 0.$$  \hfill (5.14)

One important distinction of the Higgs branch at hand with the one in $\mathcal{N}=1$ QED is that the base of the Higgs branch in the present case is a compact manifold rather then a point. The base of the Higgs branch is defined as a submanifold with the minimal $|q|^2$. For the case of Higgs branch (3.3) in $\mathcal{N}=1$ QED the base is defined by the condition $\tilde{q} = 0$ which reduces the Higgs branch to a single point $\rho = 0$. For the case of the Higgs branch (5.9) this condition becomes

$$q^A = \tilde{q}^A, \hfill (5.15)$$

which can be obtained from the condition $\tilde{q} = 0$ by $SU(2)_R$ rotation transforming $\xi_3$ into $\xi_1$, see [11]. The condition (5.15) reduces the number of real scalars from eight to four. They are subject to constraint (5.9) which boils down to a single condition $2|q^A|^2 = \xi_1$. Subtracting $U(1)$ phase we get $4-1-1=2$ which is the dimension of the base of the Higgs branch. Clearly this manifold is a two dimensional sphere $S_2$. In terms of the coordinates in (5.12) it is parametrized by angles $\theta$ and $\delta$, while

$$w = \sqrt{\xi_1}, \hfill (5.16)$$

on the base manifold.

### 5.2 Flux tubes

Now let us consider ANO strings in a generic point on the Higgs branch in $\mathcal{N}=2$ QED (5.7). The field $a$ is frozen at its VEV

$$a = -\sqrt{m} \hfill (5.17)$$

and does not play any role in the string solution. As we discussed in the previous subsection we have massive scalars with the mass equal to the mass of the photon (5.10) and four massless scalars. Therefore to find the string solution we use the same method as in Sect. 4.
Namely, our string consist of a BPS core formed by heavy fields and a tail formed by light fields. To find the BPS core we impose condition

\[ q^A = \bar{q}^A = \frac{1}{\sqrt{2}} \varphi^A, \]  

(5.18)

which reduces the QED (5.7) to the Abelian Higgs model with two complex flavors \( \varphi^A \) with the potential

\[ V = \frac{g^2}{8} (|\varphi^A|^2 - \xi_1)^2. \]  

(5.19)

Clearly this model possess standard BPS strings \(^1\) with the tension

\[ T_{BPS} = 2\pi \xi_1 \]  

(5.20)

which satisfy boundary conditions

\[ w(r \gtrsim R_c) = \sqrt{\xi_1}, \quad \theta(r \gtrsim R_c) = \theta_0, \quad \delta(r \gtrsim R_c) = \delta_0, \]  

(5.21)

outside the core. Here the size of the core \( R_c \) is given by

\[ R_c = \frac{1}{g\sqrt{\xi_1}}. \]  

(5.22)

Outside the core heavy fields are almost zero and the string is determined by the classical solution of the one dimensional sigma model (4.20) (with \( \xi_3 \) replaced by \( \xi_1 \)) with target space geometry (5.12). Namely, the tension of the tail is given by

\[ T_{tail} = \frac{2\pi}{\log L/R_c} l^2, \]  

(5.23)

where \( L \) is the length of the string and \( l \) is the length of the geodesic line on the Higgs branch (5.12) between points (5.21) and (5.14).

The initial point of this geodesic line (5.21) is not fixed yet. In principle, it could be any point on the base submanifold. To fix it we require the string to have the minimal tension. This boils down to the condition of having geodesic line of the minimal length between point (5.14) and the point in question on the base submanifold.

\(^1\)See however next section for the discussion on semilocal strings
This requirement ensures that this point has
\[ w(r \gtrsim R_c) = \sqrt{\xi_1}, \quad (r \gtrsim R_c) = 0. \] (5.24)

Moreover, clearly the phases \( \theta, \delta \) and \( \gamma \) are zero on the whole geodesic trajectory. We checked this explicitly solving the Hamilton-Jacobi equation for this geodesic line following the method of ref. [26]. This means that the length of the geodesic line is given by
\[ l = \int^{v} \frac{dw}{\sqrt{1 - \left( \frac{\xi_1}{w} \right)^2}}. \] (5.25)

With this length the final answer for the tension of the string becomes
\[ T = 2\pi\xi_1 + \frac{2\pi}{\log (g\sqrt{\xi_1}L)} l^2. \] (5.26)

If we take the VEV’s of scalar fields on the base submanifold \((v = \sqrt{\xi_1})\) the second term in (5.26) vanishes and we get tension of BPS string. Note, that like in Sect. 4 we get two short BPS multiplets in this limit from one long non-BPS string multiplet which exist in a generic point on the Higgs branch. In the opposite limit \(\xi_1 \to 0\) the Eguchi-Hanson manifold becomes flat and \(l \to v\). Besides that the BPS core contribution in (5.26) vanishes and we get (2.12) obtained in [10] for the case of Higgs branch without FI term
\(^2\).

To conclude this section we can reexpress the boundary conditions on the base manifold (5.24) which serves as a conditions ”at infinity” for the BPS core of the string in terms of the original quark fields \(\varphi^A\) of the model with potential (5.19). Using the standard relation of unit vector on \(S_2\) in \(O(3)\) sigma model with quark variables
\[ \cos \theta = \frac{1}{\xi_1} \bar{\varphi}_A (\tau^3)^A_B \varphi^B, \]
\[ \sin \theta \cos \delta = \frac{1}{\xi_1} \bar{\varphi}_A (\tau^1)^A_B \varphi^B, \]
\[ \sin \theta \sin \delta = \frac{1}{\xi_1} \bar{\varphi}_A (\tau^2)^A_B \varphi^B \] (5.27)

\(^2\)Note that the size of electromagnetic core in the limit \(\xi_1 \to 0\) is frozen on the value \(R_g\), see [21,11]
we obtain that point \((5.24)\) on the base submanifold corresponds to the following values of quark fields

\[
\varphi^{A=1} = \sqrt{\xi_1}, \quad \varphi^{A=2} = 0.
\]

This means that with our choice of scalar VEV’s \((5.14)\) the BPS core is formed only by the first flavor, while the second one remains unexcited.

6 Semilocal strings

So far our study of flux tubes in \(\mathcal{N}=2\) QED was not complete. The point is that BPS strings in theories with extended global symmetry possess an additional zero mode associated with their transverse size \(r_0\). The string parametrized by the size \(r_0\) is called semilocal string (see [18] for a review). Semilocal string interpolates between ANO string and 2d sigma model instanton lifted in four dimensions (lump). At non-zero \(r_0\) the semilocal string has power fall-off of the profile functions at infinity, instead of the exponential fall-off for ANO string at \(r_0 = 0\).

As we explain below this leads to a dramatic physical effect - we loose confinement if a semilocal string is developed instead of ANO string. In the next subsection we briefly review semilocal strings and then consider the possibility of their formation in a generic point on the Higgs branch in \(\mathcal{N}=2\) QCD with non-zero FI term.

6.1 BPS semilocal strings

Let us recall basic features of semilocal strings [18]. The simplest model where they appear is the Abelian Higgs model with two complex flavors

\[
S_{AH} = \int d^4x \left\{ \frac{1}{4g^2} F_{\mu\nu}^2 + |\nabla_\mu \varphi^A|^2 + \frac{g^2}{8} (|\varphi^A|^2 - \xi_1)^2 \right\}. \tag{6.1}
\]

Note, that we already considered this model in Sect. 5.2 discussing the BPS core of the string on the Higgs branch. It arises from \(\mathcal{N}=2\) QED \((5.7)\) when we restrict ourselves to the ansatz \((5.18)\).

The topological reason for existence of ANO vortices is that for gauge group \(U(1)\) \(\pi_1[U(1)] = Z\). On the other hand we can go to the low energy limit in \((6.1)\) restricting ourselves to the vacuum manifold \(|\varphi^A|^2 = \xi_1\).
The vacuum manifold has dimension 4-1-1=2, where we subtract one real condition mentioned above as well as one gauge phase. It represents two dimensional sphere $S_2$. Thus, the low energy limit of theory (6.1) is $O(3)$ sigma model. Now recall that $\pi_2[S_2] = \pi_1[U(1)] = Z$ and this is a topological reason for existence of instantons in two dimensional $O(3)$ sigma model. Lifted in four dimensions they become a string-like objects (lumps).

So now the question is what is the relation between ANO flux tubes of QED (6.1) and lumps of $O(3)$ sigma model. Clearly the model (6.1) has ANO strings. Say, if we put the second flavor to zero this model reduces to the one flavor model considered in Sect. 4.1. Then ANO string is given by eqs. (4.5), (4.4) for the first flavor while the second one is zero. However, it turns out (see [27]) that this solution has zero mode associated with exiting of the second flavor. This zero mode is parametrized by the parameter $r_0$ which plays the role of the size of the string in (1,2)-plane.

To find this solution let us modify the standard parametrization (4.5) for the ANO string including the second flavor

$$\begin{align*}
\varphi^1(x) &= \phi_1(r) e^{i \alpha}, \\
\varphi^2(x) &= \phi_2(r), \\
A_i(x) &= -2 \epsilon_{ij} \frac{x_j}{r^2} [1 - f(r)].
\end{align*}$$

(6.2)

Note, that the second flavor does not wind at infinity. Therefore, it has boundary condition $\varphi_2(\infty) = 0$ while at $r = 0$ it can be non-zero.

The first order equations for the profile functions here look like

$$\begin{align*}
r \frac{d}{dr} \phi_1(r) - f(r) \phi_1(r) &= 0, \\
r \frac{d}{dr} \phi_2(r) - (f(r) - 1) \phi_2(r) &= 0, \\
- \frac{1}{r} \frac{d}{dr} f(r) + \frac{g^2}{4} \left( \phi_1^2(r) + \phi_2^2(r) - \xi_1 \right) &= 0.
\end{align*}$$

(6.3)

The solution to these equations [27, 28] at non-zero $r_0$ is very different from that of ANO string. It has long range power fall-off at infinity for all profile functions. In particularly, in the limit of large transverse size of the string $r_0 \gg 1/g \sqrt{\xi_1}$ it has the form

$$\phi_1(r) = \sqrt{\xi_1} \frac{r}{\sqrt{r^2 + r_0^2}},$$

25
\[ \phi_2(r) = \sqrt{\xi_1} \frac{r_0}{\sqrt{r^2 + r_0^2}}, \]

\[ f = \frac{r_0^2}{r^2 + r_0^2}. \]  

(6.4)

This solution has the same tension as ANO string

\[ T = 2\pi \xi_1. \]  

(6.5)

We see that scalar fields on the solution (6.4) belong to the vacuum manifold \(|\phi^A|^2 = \xi_1\). This means that we can relate this solution to the \(O(3)\) sigma model lump. To do this we first use the relation between fields \(\phi^A\) and the unit vector on \(S_2\) (5.27) and then represent this vector in terms of complex field \(\omega(x)\) via standard relations

\[
\cos \theta = \frac{1 - |\omega|^2}{1 + |\omega|^2},
\]

\[
\sin \theta \cos \delta = 2 \frac{Re \omega}{1 + |\omega|^2},
\]

\[
\sin \theta \sin \delta = 2 \frac{Im \omega}{1 + |\omega|^2}.
\]  

(6.6)

With this substitution the low energy limit of (6.1) becomes an \(O(3)\) sigma model

\[ S_{eff} = \int d^4x \frac{|\partial_\mu \omega|^2}{(1 + |\omega|^2)^2}. \]  

(6.7)

The standard lump (instanton) solution of this model with center at zero and size \(r_0\) looks like

\[ \omega_{\text{lump}} = \frac{r_0}{x_1 + ix_2}, \]  

(6.8)

where parameter \(r_0\) can be made real by the shift in the polar angle \(\alpha\).

If we now reexpress this solution in terms of quark fields using relations (6.6), (5.27) we arrive at the solution (6.4). Thus we identified the semilocal string in the limit of large \(r_0\) (6.1) with the lump solution of \(O(3)\) sigma model (up to a factorization over \(Z_2\)). The reason for this identification is that \(O(3)\) sigma model is a low energy effective theory for two-flavor Abelian Higgs model (6.1).
Now let us come back to our $\mathcal{N}=2$ QED (5.7) and consider first the vacuum which belongs to the base submanifold $v = \sqrt{\xi_1}$ of the Higgs branch. This was done in [15, 16]. In this case the ANO string considered in the Sect. 5.2 becomes BPS. As we already mentioned for these VEV’s we can look for string solution using the *ansatz* (5.18) which reduces the bosonic part of our theory to the two flavor model (6.1). This model has semilocal strings so our BPS ANO string is just a particular case of a semilocal string at $r_0 = 0$ [15, 16]. Say, in the opposite limit $r_0 \gg 1/g\sqrt{\xi_1}$ the semilocal string is given by the profile functions (6.4).

The most physically important consequence of emergence of semilocal strings is that we loose the monopole confinement. As we already explained to study confinement we have to consider a string of a finite length $L$ stretched between heavy monopole and anti-monopole. Clearly the problem now becomes three dimensional. The string size $r_0$ does not correspond to a zero mode any longer. Instead, it is fixed by the separation $L$ at large value $r_0 \sim L$. Thus the monopole flux is not trapped into a narrow flux tube. Instead, it is spread over a large three dimensional volume of size of order of $L$. Clearly this produces a potential between monopole and anti-monopole with power-like fall-off at large separations, $V(L) \sim L^{-\gamma}$ ($\gamma > 0$). Such potential does not confine.

We see that formation of semilocal strings at a base point on the Higgs branch lead to a dramatic physical effect – deconfinement. Therefore it is quite important to study the possibility of formation of semilocal strings at a generic point on the Higgs branch. We do it in the next subsection.

### 6.2 Semilocal strings at a generic vacuum on the Higgs branch

Now we assume that semilocal string can be formed at a generic vacuum on the Higgs branch of $\mathcal{N}=2$ QCD and study the dependence of its tension on its size $r_0$. As we mentioned in the previous subsection there is no such dependence for BPS string in the vacuum which belongs to a base submanifold, $r_0$ is associated with the zero mode. We will see below that this is not the case for a semilocal string at a generic vacuum on the Higgs branch.

To study the possibility of formation of semilocal strings it is sufficient to consider a semilocal string in the limit of large size $r_0 \gg 1/g\sqrt{\xi_1}$ because in this limit the semilocal string shows its crucial distinctions from the ANO
As we discussed in the previous subsection in this limit the semilocal string is formed by light fields only. It does not have core formed by heavy fields and looks like sigma model lump.

Therefore to study semilocal string in the limit $r_0 \gg g/\sqrt{\xi_1}$ we can go to the low energy limit in $\mathcal{N}=2$ QED (5.7) which is given by sigma model (5.12) with Eguchi-Hanson geometry. Assuming that all fields in this sigma model depend only on coordinates in $(1,2)$-plane we rewrite the tension of our semilocal string as follows

$$T_{\text{semilocal}} = \int d^2x \left\{ \left[ 1 - \left( \frac{\xi_1}{w^2} \right)^2 \right]^{-1} (\partial_\mu w)^2 + w^2 \left[ (\partial_\mu \theta)^2 + \sin^2 \theta (\partial_\mu \delta)^2 \right] 
+ w^2 \left[ 1 - \left( \frac{\xi_1}{w^2} \right)^2 \right] (\partial_\mu \gamma + \cos \theta \partial_\mu \delta)^2 \right\}.$$  \hspace{1cm} \hspace{1cm} (6.9)

Now our aim is to find a classical solution of this model which represents a kind of hybrid of the "tail" solution of Sect. 5.2 with lump (instanton) solution of Sect. 6.1. The boundary conditions of this solution at infinity are given by the VEV’s (5.14)

$$w(\infty) = v, \theta(\infty) = 0, \gamma(\infty) = 0,$$ \hspace{1cm} \hspace{1cm} (6.10)
while the boundary conditions at zero are given by those of the lump (6.4), namely $\varphi^1(0) = 0$ and $\varphi^2(0) = \sqrt{\xi_1}$. Rewriting this in terms of fields entering the model (6.9) using (5.27) we get

$$w(0) = \sqrt{\xi_1}, \theta(0) = \pi, \gamma(0) = 0.$$ \hspace{1cm} \hspace{1cm} (6.11)

We see that our solution winds around base cycle of the Eguchi-Hanson manifold ($S_2$) and then goes away in a non-compact direction (along $w$). Awaiting for the full solution of the problem here in this section we consider the case when the VEV of scalar fields is much larger than the FI parameter,

$$v \gg \sqrt{\xi_1}.$$ \hspace{1cm} \hspace{1cm} (6.12)

In this limit the field $w$ goes far away from the base (at $w = \sqrt{\xi_1}$) along the non-compact manifold. When $w \gg \sqrt{\xi_1}$ the geometry in (6.9) becomes flat and the problem reduces to solving of free equations of motion. For the radial coordinate $w$ the solution of the equation $\partial^2 w = 0$ reads (cf. (2.8))

$$w(r) = v \frac{\log r/r_0}{\log L/r_0},$$ \hspace{1cm} \hspace{1cm} (6.13)
while
\[ \theta(x) = 0, \ \delta(x) = 0, \ \gamma(x) = 0 \quad (6.14) \]
like for the "tail" solution of Sect. 5.2. Here we use the lump size \( r_0 \) as a UV
cutoff for the logarithmic behavior of \( w \). This solution is valid at large \( r \),
\( R \ll r \ll L \), where parameter \( R \) will be determined shortly.

Instead for small \( r \), \( r \lesssim R \) the solution is given by a certain deformation
of the instanton solution of Sect. 6.1 which we denote as
\[
(w(x), \theta(x), \delta(x), \gamma(x)) = (w_{\text{inst}}(x), \theta_{\text{inst}}(x), \delta_{\text{inst}}(x), \gamma_{\text{inst}}(x)), \quad (6.15)
\]
where \( w_{\text{inst}}(x) \sim \sqrt{\xi_1} \). Clearly the solution changes its behavior from (6.15)
to (6.13), (6.14) when the coordinate \( w \) given by the logarithmic expression
(6.13) becomes much larger than \( \sqrt{\xi_1} \). This gives
\[
R = r_0 \exp \left( \frac{\sqrt{\xi_1}}{v} \log \frac{L}{r_0} \right). \quad (6.16)
\]

The tension of our semilocal string is now given by the sum of tensions
of deformed instanton (6.15) and the logarithmic "tail" (6.13), (6.14),
\[
T_{\text{semilocal}} = T_{\text{inst}}(r_0) + \frac{2\pi v^2}{\log L/r_0}, \quad (6.17)
\]
where the calculation of the tail contribution is similar to that in Sect. 2 (see [10]).

Of course we do not know the function \( T_{\text{inst}}(r_0) \) without knowing the
solution (6.15). What we know is that at \( r_0 = 0 \) the semilocal string reduces
to ANO string and deformed instanton (6.15) goes into BPS core of the string
discussed in Sect. 5.2. Thus,
\[
T_{\text{inst}}(r_0 = 0) = 2\pi \xi_1, \quad (6.18)
\]
see (5.20). Moreover, the central charge of \( \mathcal{N}=2 \) algebra gives us a lower
bound for this tension (see, for example, [11] for details) \(^3\)
\[
T_{\text{inst}}(r_0) \geq 2\pi \xi_1. \quad (6.19)
\]

\(^3\)To prove this formally we can construct configuration given by (6.15) at \( r \lesssim R \)
and extrapolated by constant fields at \( r \gg R \). Then we use the SUSY bound for this configu-
ration.
We see that at least the second term in (6.17) depend on $r_0$, so $r_0$ is not associated with conformal zero mode any longer. The reason for this breaking of conformal invariance at the classical level is the non-compactness of the Higgs branch. To see this note, that in the last subsection we were dealing with lump solution which maps two dimensional space onto compact base submanifold $S_2$ of the Eguchi-Hanson Higgs branch. The tension of this lump solution is a constant (6.5) determined by the volume of $S_2$. The collective coordinate $r_0$ in this case is associated with the conformal zero mode.

As we move to a generic vacuum on the Higgs branch our semilocal string solution becomes a map onto a non-compact manifold. In order to get a finite string tension we use a cutoff in the target space introduced by the VEV of the scalar fields $v$. This cutoff in the target space requires an IR cutoff $L$ in the two dimensional $(1,2)$-plane. We already discussed how this works in Sect.2. The logarithmic solution to the free equations of motion (6.13) cannot go to its VEV at infinity. In order to insure that the scalar fields reach their VEV’s at infinity we have to introduce a IR cutoff, say small masses for light fields (like in Sect. 2.4) or the finite length of the string $L$. Clearly this breaks the conformal invariance and produces the dependence of the string tension (6.17) on $r_0$.

Now minimizing the string tension (6.17) with respect to $r_0$ using (6.18) and (6.19) we find that

$$r_0 = 0.$$

(6.20)

In the limit $r_0 \to 0$ the semilocal string reduces to ANO string considered in Sect. 5.2. In particular, the string tension is given by (5.26) in the limit $l \to v$. Note, that quantum fluctuations of $r_0$ are negligibly small, suppressed by the world sheet volume of the string.

This means that semilocal strings are not formed in the generic vacuum on the Higgs branch at least in the limit $v \gg \sqrt{\xi_1}$. This conclusion is in accordance with the general expectation [18] that semilocal strings are not stable in type I superconductor and reduce to ANO strings.

### 6.3 Deconfinement phase transition

The conclusion made in the previous subsection that semilocal strings are not developed at a generic vacuum on the Higgs branch (at least if $v \gg \sqrt{\xi_1}$) and we are dealing with ANO strings has rather important physical consequences. This means that we have a confinement phase for monopoles in these vacua.
with confining potential determined by the ANO string tension (5.20), see (1.2). Instead, for the vacua on the base submanifold \( v = \sqrt{\xi_1} \) semilocal strings are formed and we have deconfinement phase.

Therefore, it is clear that we can expect confinement/deconfinement phase transition at certain intermediate critical value \( v_c \). To find this critical value we have to study semilocal string solution for vacua which are not far away from the base submanifold, \( v \sim \sqrt{\xi_1} \). This is left for a future work. Here we put forward a plausible conjecture that in fact

\[
v_c = \sqrt{\xi_1},
\]  

(6.21)

which means that we have confinement phase for all vacua on the Higgs branch except the base submanifold. The reason for this conjecture is that nothing special happens at any intermediate value \( v_c \). Instead, the value \( v_c = \sqrt{\xi_1} \) is physically distinct because at this value of scalar VEV strings become BPS-saturated (one long non-BPS string multiplet becomes two short BPS multiplets, see Sect. 5.2).

This perfectly matches results of [10] where unbroken \( \mathcal{N}=2 \) QCD with two degenerative flavors at \( \xi_1 = 0 \) was considered. At zero \( \xi_1 \) the geometry of Higgs branch becomes flat, given by \( R^4/Z_2 \) and the \( S_2 \) cycle shrinks to a singular point at the origin. It was shown in [10] that monopoles are in the confinement phase on the Higgs branch at any non-zero \( v \). As \( v \) goes to zero the transverse size of ANO string becomes infinite while its tension goes to zero, see (2.11), (2.12), so monopole confinement becomes unobservable. Introducing the non-zero FI parameter \( \xi_1 \) we resolve the singularity at the origin and see that at the base cycle \( S_2 \) we have deconfinement phase while on the rest of the Higgs branch we have monopole confinement.

To conclude this section we would like to stress that the above discussion of confinement/deconfinement phase transition refers only to \( \mathcal{N}=2 \) QED (5.7) considered on its own right. In contrast, if we start from non-Abelian softly broken \( \mathcal{N}=2 \) SU(2) QCD the story is rather different. The point is that (as we mentioned in Sect. 5.1) the \( \mathcal{N}=2 \) QED (5.7) is the low energy description of the underlying non-Abelian theory only in the special limit

\[
\mu \to 0, \ m \to \infty, \ \xi_1 = \text{const},
\]  

(6.22)

which ensures that \( \mathcal{N}=2 \) supersymmetry is preserved. In this case the superconductivity on the base submanifold is on the border between type I and
type II and we have BPS strings which eventually become semilocal strings in the presence of SU(2) global flavor symmetry.

Generically if we take into account corrections in $\mu/m$ the $\mathcal{N}=2$ supersymmetry is broken down to $\mathcal{N}=1$. As it is shown in [11] superconductivity becomes of type I. For the type I superconductor semilocal strings become unstable [18]. In particular, we expect that the string tension even for the string in the vacuum on the base submanifold becomes a function of $r_0$

\[ T_{\text{lump}} \sim 2\pi \xi_1 \left( 1 + \text{const} \ r_0^2 (\delta m)^2 + \ldots \right), \tag{6.23} \]

where $\delta m$ is the splitting of masses in the former $\mathcal{N}=2$ multiplet. This splitting is given by [11]

\[ \delta m \sim \mu. \tag{6.24} \]

We conclude that in this case the semilocal string is not developed. Minimization of (6.23) with respect to $r_0$ gives

\[ r_0 = 0. \tag{6.25} \]

Note again that quantum fluctuation in $r_0$ are suppressed by the volume of the string world sheet.

This means that we do not have deconfinement phase even on the base submanifold of the Higgs branch in QCD with $\mathcal{N}=2$ supersymmetry softly broken down to $\mathcal{N}=1$. All vacua on the Higgs branch of this theory give rise to the confinement phase for monopoles.

### 7 Conclusions

In this paper we studied flux tubes on Higgs branches in supersymmetric gauge theories. Our main conclusion is that although Higgs branches ensure presence of massless scalars flux tubes still exist and give rise to a confining potential. However due to the presence of massless scalars the string tension is no longer a constant. It becomes a slow (logarithmic) function of the length of the string $L$. Still it produces a confining potential of type (1.2).

First we studied $\mathcal{N}=1$ QED with FI $D$-term. This theory has a two dimensional Higgs branch. The solution for the string in a generic point on this Higgs branch is given by "BPS core" formed by heavy fields and a "tail" formed by light fields living on the Higgs branch. Finding the solution for the "tail" reduces to finding a classical solutions of the sigma model on the
Higgs branch. The tension of the string is given by the sum of tensions of "BPS core" and the "tail", see (4.28).

Next we studied $\mathcal{N}=2$ QCD with gauge group $SU(2)$ and two flavors of quarks with the same mass. This theory has a four dimensional Higgs branch. We deformed this theory with the mass term for the adjoint field. To the leading order in the deformation parameter $\mu$ $\mathcal{N}=2$ supersymmetry is not broken in the effective QED describing the low energy limit of this theory. Higgs branch has an Eguchi-Hanson geometry.

We showed that far away from the base cycle $S_2$ of the Higgs branch stable ANO strings are formed. Their tension is again given by the sum of tensions of "BPS core" and the "tail", see (5.26). These strings produce confinement of monopoles with the confining potential of type (1.2). However, in the vacua on the base cycle $S_2$ of the Higgs branch the story is rather different. Semilocal BPS strings are developed in vacua on the base submanifold leading to a deconfinement regime. We conjecture that the confinement/deconfinement phase transition occurs exactly at the base cycle of the Eguchi-Hanson space.

If we introduce breaking of $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$ in this theory it turns out that the deconfinement phase disappears and we have monopole confinement on the whole Higgs branch.

Acknowledgments

We are grateful to Alexander Gorsky, Mikhail Shifman, David Tong and Arkady Vainshtein for helpful discussions. This work is supported in part by INTAS grant No. 00-00334. The work of A. Y. is also supported by the Russian Foundation for Basic Research grant No. 02-02-17115 and by Theoretical Physics Institute at the University of Minnesota.

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