Unknown Quantum States and Operations, a Bayesian View

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The classical de Finetti theorem provides an operational definition of the concept of an unknown probability in Bayesian probability theory, where probabilities are taken to be degrees of belief instead of objective states of nature. In this paper, we motivate and review two results that generalize de Finetti’s theorem to the quantum mechanical setting: Namely a de Finetti theorem for quantum states and a de Finetti theorem for quantum operations. The quantum-state theorem, in a closely analogous fashion to the original de Finetti theorem, deals with exchangeable density-operator assignments and provides an operational definition of the concept of an “unknown quantum state” in quantum-state tomography. Similarly, the quantum-operation theorem gives an operational definition of an “unknown quantum operation” in quantum-process tomography. These results are especially important for a Bayesian interpretation of quantum mechanics, where quantum states and (at least some) quantum operations are taken to be states of belief rather than states of nature.

I. INTRODUCTION

What is a quantum state?\textsuperscript{1} Since the earliest days of quantum theory, it has been understood that the quantum state can be used (through the Born rule) to derive probability distributions for the outcomes of all measurements that can be performed on a quantum system. But is it more than that? Is a quantum state an actual property of the system it describes? The Bayesian view of quantum states [1–6,13–22] is that it is not: The quantum state is not something the system itself possesses. Rather it is solely a function of the observer (or, better, agent) who contemplates the predictions, gambles, decisions, or actions he might make with regard to those quantum measurements.

\textsuperscript{1}This paper represents predominantly a culling of the material in Refs. [1–3]. Everything, however, has been updated to accommodate the major shift in our thinking represented in Refs. [4–6]. In particular, it reflects a change in our views of quantum probabilities from that of an objective Bayesianism of the type promoted by E. T. Jaynes [7] to a subjective or personalistic Bayesianism of the type promoted by B. de Finetti, L. J. Savage, J. M. Bernardo and A. F. M. Smith, and R. Jeffrey [8–12]. This makes all the difference in the world with regard to the meaning of quantum states, operations, and their usage within statistical theory.
What distinguishes this view from a more traditional “Copenhagen-interpretation style” view—for instance the view expressed so clearly and carefully in Ref. [23]—is that there is no pretense that a quantum state represents a physical fact. Quantum states come logically before that: They represent the temporary and provisional beliefs a physicist holds as he travels down the road of inquiry. It is the outcomes of quantum measurements that represent physical facts within quantum theory, not the quantum states. In particular, there is no fact of nature to prohibit two different agents from using distinct pure states $|\psi\rangle$ and $|\phi\rangle$ for a single quantum system.\(^2\) Difficult though this may be to accept for someone trained in the traditional presentation of quantum mechanics, the only thing it demonstrates is a careful distinction between the terms belief and fact.

Quantum states are not facts.\(^3\) But if so, then what is an “unknown quantum state”? There is hardly a paper in the field of quantum information that does not make use of the phrase. Unknown quantum states are teleported [28,29], protected with quantum error correcting codes [30,31], and used to check for quantum eavesdropping [32,33]. The list of uses grows each day. Are all these papers nonsense? In a Bayesian view of quantum states, the phrase is an oxymoron, a contradiction in terms: If quantum states are states of belief rather than states of nature, then a state is known by someone—at the very least, by the agent who holds it.

Thus for a quantum Bayesian, if a phenomenon ostensibly invokes the concept of an unknown state in its formulation, the unknown state must be a kind of shorthand for a more involved story. In other words, the usage should be viewed a call to arms, an opportunity for further analysis. For any phenomenon using the idea of an unknown quantum state, the quantum Bayesian should demand that either:

1. The owner of the unknown state—some further agent—be explicitly identified. (In this case, the unknown state is merely a stand-in for the unknown state of belief of an essential player who went unrecognized in the original formulation.) Or,

2. If there is clearly no further agent upon the scene, then a way must be found to reexpress the phenomenon with the term “unknown state” banished from the formulation. (In this case, the end-product will be a single quantum state used for describing the phenomenon—namely, the state that actually captures the initial agent’s overall beliefs throughout.)

In this paper, we will analyze the particular use of unknown states that comes from quantum-state tomography [34–36]. Beyond that, we will also argue for the necessity of (and carry out) a similar analysis for quantum-process tomography [37–39].

\(^2\) Contrast this to the treatment of Refs. [24–26]. In any case, the present point does not imply that a single agent can believe willy-nilly anything he wishes. To quote D. M. Appleby, “You know, it is really hard to believe something you don’t actually believe.”

\(^3\) For a selection of papers that we believe help shore up this statement in various ways—though most of them are not explicitly Bayesian in their view of probability—see Ref. [27].
The usual, non-Bayesian description of quantum-state tomography is this. A device of some sort repeatedly prepares many instances of a quantum system in a fixed quantum state $\rho$, pure or mixed. An experimentalist who wishes to characterize the operation of the device or to calibrate it for future use may be able to perform measurements on the systems it prepares even if he cannot get at the device itself. This can be useful if the experimenter has some prior knowledge of the device’s operation that can be translated into a probability distribution over states. Then learning about the state will also be learning about the device. Most importantly, though, this description of tomography assumes the state $\rho$ is unknown. The goal of the experimenter is to perform enough measurements, and enough kinds of measurements (on a large enough sample), to estimate the identity of $\rho$.

![Diagram](image)

**FIG. 1.** What can the term “unknown state” mean if quantum states are taken to be solely compendia of Bayesian expectations rather than states of nature? When we say that a system has an unknown state, must we always imagine a further agent whose state of belief is symbolized by some $|\psi\rangle$, and it is the identity of that belief which we are ignorant of?

This is clearly an example where there is no further player on whom to pin the unknown state as a state of belief or judgment. Any attempt to find such a missing player would be entirely artificial: Where would the player be placed? On the inside of the device the tomographer is trying to characterize? The only available course is the second strategy above—to banish the idea of the unknown state from the formulation of tomography.
FIG. 2. To make sense of quantum tomography, must we resort to imagining a “man in the box” who has a better description of the systems than we do? How contrived would the Bayesian story be if this were so!

To do this, we take a cue from the field of Bayesian probability theory itself [8–11,40,41]. In Bayesian theory, probabilities are not objective states of nature, but measures of personalistic belief. The overarching Bayesian theme is to identify the conditions under which a set of decision-making agents can come to a common belief or probability assignment for a random variable even though their initial beliefs differ. Bernardo and Smith [11] make the point forcefully:

[I]ndividual degrees of belief, expressed as probabilities, are inescapably the starting point for descriptions of uncertainty. There can be no theories without theoreticians; no learning without learners; in general, no science without scientists. It follows that learning processes, whatever their particular concerns and fashions at any given point in time, are necessarily reasoning processes which take place in the minds of individuals. To be sure, the object of attention and interest may well be an assumed external, objective reality: but the actuality of the learning process consists in the evolution of individual, subjective beliefs about that reality. However, it is important to emphasize . . . that the primitive and fundamental notions of individual preference and belief will typically provide the starting point for interpersonal communication and reporting processes. . . [W]e shall therefore often be concerned to identify and examine features of the individual learning process which relate to interpersonal issues, such as the conditions under which an approximate consensus of beliefs might occur in a population of individuals.

Following that theme is the key to understanding quantum tomography from a Bayesian point of view.

The offending classical concept is an “unknown probability,” an oxymoron for the precisely same reason as an unknown quantum state. The procedure analogous to quantum-state tomography is the estimation of an unknown probability from the results of repeated

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4If a quantum state is nothing more than a compendium of probabilities—Bayesian probabilities—then, of course, it is for precisely the same reason.
trials on “identically prepared systems,” all of which are said to be described by the same, but unknown probability.

Let us first consider a trivial example to clinch the idea of subjective probabilities before moving to a full tomographic setting. Suppose a die is thrown 10 times. Let $k \in \{1, \ldots, 6\}$ represent the outcome of a single throw of the die, and suppose the results in the 10 throws were these:

- $k = 1$ appeared 1 times,
- $k = 2$ appeared 4 times,
- $k = 3$ appeared 2 times,
- $k = 4$ appeared 2 times,
- $k = 5$ appeared 1 times,
- $k = 6$ appeared 0 times.

A typical inference problem is to assign a probability, $p$, to the outcome $k = 6$ in the next throw of the die, given these data. But clearly the answer cannot be determined by the data alone. It depends on a prior probability assignment. Possibilities include:

1. The assumption that the die is fair. In this case $p$ is independent of the data and given by $p = 1/6$.

2. A totally uninformative prior. In this case, a generalization of the Laplace rule of succession [7] gives $p = 1/16$.

3. The die came from a box that contains only trick dice of two types: The first type never comes up 1, and the other type never comes up 6. In this case, $p = 0$.

The point of this trivial example cannot be stressed enough. Data alone is never enough to specify a probability distribution. The only way to “derive” a probability distribution from data is to cheat—to make use of an implicit probability assignment made prior to the collection of the data. This holds as much for a problem like the present one, whose setting is a finite number of trials, as for a problem containing a potentially infinite number of trials.

The way to eliminate unknown probabilities from the discussion of infinite numbers of trials was introduced by Bruno de Finetti in the early 1930s [9,41]. The idea is to make explicit the implicit class of priors generally used in such problems. He did this by focusing on the meaning of the \textit{equivalence} of repeated trials. What could equivalent trials mean—de Finetti asked—but that a probability assignment for multiple trials should be symmetric under the permutation of those trials? With his classical representation theorem, de Finetti [9] showed that a multi-trial probability assignment that is permutation-symmetric for an arbitrarily large number of trials—he called such multi-trial probabilities \textit{exchangeable}—is equivalent to a probability for the “unknown probabilities.” Thus the unsatisfactory concept of an unknown probability vanishes from the description in favor of the fundamental idea of assigning an exchangeable probability distribution to multiple trials.

This cue in hand, it is easy to see how to reword the description of quantum-state tomography to meet our goals. What is relevant is simply a judgment on the part of the experimenter—notice the essential subjective character of this “judgment”—that there is no
distinction between the systems the device is preparing. In operational terms, this is the judgment that all the systems are and will be the same as far as observational predictions are concerned. At first glance this statement might seem to be contentless, but the important point is this: To make this statement, one need never use the notion of an unknown state—a completely operational description is good enough. Putting it into technical terms, the statement is that if the experimenter judges a collection of \( N \) of the device’s outputs to have an overall quantum state \( \rho^{(N)} \), he will also judge any permutation of those outputs to have the same quantum state \( \rho^{(N)} \). Moreover, he will do this no matter how large the number \( N \) is. This, complemented only by the consistency condition that for any \( N \) the state \( \rho^{(N)} \) be derivable from \( \rho^{(N+1)} \), makes for the complete story.

The words “quantum state” appear in this formulation, just as in the original formulation of tomography, but there is no longer any mention of unknown quantum states. The state \( \rho^{(N)} \) is known by the experimenter (if no one else), for it represents his state of belief. More importantly, the experimenter is in a position to make an unambiguous statement about the structure of the whole sequence of states \( \rho^{(N)} \): Each of the states \( \rho^{(N)} \) has a kind of permutation invariance over its factors. The content of the quantum de Finetti representation theorem [42,43]—which we will demonstrate in a later section—is that a sequence of states \( \rho^{(N)} \) can have these properties, which are said to make it an exchangeable sequence, if and only if each term in it can also be written in the form

\[
\rho^{(N)} = \int P(\rho) \rho^{\otimes N} d\rho ,
\]

where \( \rho^{\otimes N} = \rho \otimes \rho \otimes \cdots \otimes \rho \) is an \( N \)-fold tensor product and \( P(\rho) \) is a fixed probability distribution over density operators.\(^5\)

The interpretive import of this theorem is paramount. It alone gives a mandate to the term unknown state in the usual description of tomography. It says that the experimenter can act as if his state of belief \( \rho^{(N)} \) comes about because he knows there is a “man in the box,” hidden from view, repeatedly preparing the same state \( \rho \). He does not know which such state, and the best he can say about the unknown state is captured in the probability distribution \( P(\rho) \).

The quantum de Finetti theorem furthermore makes a connection to the overarching theme of Bayesianism stressed above. It guarantees for two independent observers—as long as they have a rather minimal agreement in their initial beliefs—that the outcomes of a sufficiently informative set of measurements will force a convergence in their state assignments for the remaining systems [2]. This “minimal” agreement is characterized by a judgment on the part of both parties that the sequence of systems is exchangeable, as described above, and a promise that the observers are not absolutely inflexible in their opinions. Quantitatively, the latter means that though \( P(\rho) \) might be arbitrarily close to zero, it can never vanish.

This coming to agreement works because an exchangeable density operator sequence can be updated to reflect information gathered from measurements by a quantum version of

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\(^5\)For further technical elaborations on the quantum de Finetti theorem, see Ref. [44].
Bayes’s rule for updating probabilities. Specifically, suppose the starting point of a quantum tomography experiment is a prior state

$$\rho^{(N+M)} = \int P(\rho) \rho^{\otimes(N+M)} d\rho$$

(1.2)

for \(N+M\) copies of the system. Then the first \(N\) systems are measured. Say the measurement outcomes are represented by a vector \(\alpha = (\alpha_1, \ldots, \alpha_N)\). It can be shown that the post-measurement state of the remaining \(M\) copies conditioned on the outcome \(\alpha\) is of the form

$$\rho^{(M)} = \int P(\rho|\alpha) \rho^{\otimes M} d\rho,$$

(1.3)

where \(P(\rho|\alpha)\) is given by a quantum Bayes rule [2]. In the special case that the same measurement, \(\{E_\alpha\}\), is measured on all \(N\) copies, the quantum Bayes rule takes the simple form

$$P(\rho|\alpha) = \frac{P(\rho)P(\alpha|\rho)}{P(\alpha)},$$

(1.4)

where

$$P(\alpha|\rho) = \text{tr}(\rho^{\otimes N} E_{\alpha_1} \otimes \cdots \otimes E_{\alpha_N})$$

(1.5)

and

$$P(\alpha) = \int P(\rho)P(\alpha|\rho) d\rho.$$  

(1.6)

For a sufficiently informative set of measurements, as \(N\) becomes large, the updated probability \(P(\rho|\alpha)\) becomes highly peaked on a particular state \(\rho_\alpha\) dictated by the measurement results, regardless of the prior probability \(P(\rho)\), as long as \(P(\rho)\) is nonzero in a neighborhood of \(\rho_\alpha\). Suppose the two observers have different initial beliefs, encapsulated in different priors \(P_i(\rho), i = 1, 2\). The measurement results force them with high probability to a common state of belief in which any number \(M\) of additional systems are assigned the product state \(\rho_\alpha^{\otimes M}\), i.e.,

$$\int P_i(\rho|\alpha) \rho^{\otimes M} d\rho \longrightarrow \rho_\alpha^{\otimes M}$$

(1.7)

for \(N\) sufficiently large.

This shifts the perspective on the purpose of quantum-state tomography: It is not about uncovering some “unknown state of nature,” but rather about the various observers’ coming to agreement over future probabilistic predictions. In this connection, it is interesting to note that the quantum de Finetti theorem and the conclusions just drawn from it work only within the framework of complex vector-space quantum mechanics. For quantum mechanics

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6For an emphasis of this point in the setting of quantum cryptography, see Ref. [45].
based on real and quaternionic Hilbert spaces, the connection between exchangeable density operators and unknown quantum states does not hold [1].

The plan of the remainder of the paper is as follows. In Sec. II, we discuss the classical de Finetti representation theorem [9,46] in the context of Bayesian probability theory. In Sec. III, we introduce the Bayesian formulation of tomography in terms of exchangeable multi-system density operators, accompanied by a critical discussion of objectivist formulations of tomography. Furthermore, we state the quantum-state de Finetti representation theorem. Section IV presents an elementary proof of the quantum de Finetti theorem. There, also, we introduce a novel measurement technique for tomography based upon generalized quantum measurements. In Sec. V, we come to an intermezzo, mentioning possible extensions of the main theorem. In Sec. VI, we change course to consider the issue of quantum operations. In particular, we argue that (at least some) quantum operations should be considered subjective states of belief, just as quantum states themselves. This brings to the fore the issue of “unknown quantum operations” within a Bayesian formulation of quantum mechanics. In Sec. VII, we pose the need for a version of a quantum de Finetti theorem for quantum operations in order to make sense of quantum-process tomography from a Bayesian point of view. In Sec. VIII, we make the statement of the theorem precise. And in Sec. IX, we run through the proof of this quantum-process de Finetti theorem. Finally in Sec. X, we conclude with a discussion of where this research program is going. In particular, we defend ourselves against the (glib) “shot gun” reaction that all of this amounts to a rejection of realism altogether: It simply does not.

II. THE CLASSICAL DE FINETTI THEOREM

The tension between the objectivist and Bayesian points of view is not new with quantum mechanics. It arises already in classical probability theory in the form of the war between “objective” and “subjective” interpretations [47]. According to the subjective or Bayesian interpretation, probabilities are measures of personal belief, reflecting how an agent would behave or bet in a certain situation. On the other hand, the objective interpretations—in all their varied forms, from frequency interpretations to propensity interpretations—attempt to view probabilities as real states of affairs or “states of nature” that have nothing to do with an agent at all. Following our discussion in Sec. I, it will come as no surprise to the reader that the authors wholeheartedly adopt the Bayesian approach. For us, the reason is simply our experience with this question, part of which is an appreciation that objective interpretations inevitably run into insurmountable difficulties. (See Refs. [8,10,11,40] for a sampling of criticisms of the objectivist approach.)

We will note briefly, however, that the game of roulette provides an illuminating example. In the European version of the game, the possible outcomes are the numbers 0, 1, . . . , 36. For a player without any privileged information, all 37 outcomes have the same probability \( p = 1/37 \). But suppose that shortly after the ball is launched by the croupier, another player obtains information about the ball’s position and velocity relative to the wheel. Using the
information obtained, this other player can make more accurate predictions than the first. His probability is peaked around some group of numbers. The probabilities are thus different for two players with different states of belief.

Whose probability is the true probability? From the Bayesian viewpoint, this question is meaningless: There is no such thing as a true probability. All probability assignments are subjective assignments based specifically upon one’s prior data and beliefs.

For sufficiently precise data—including precise initial data on positions and velocities and probably also including other details such as surface properties of the wheel—Newtonian mechanics assures us that the outcome can be predicted with certainty. This is an important point: The determinism of classical physics provides a strong reason for adopting the subjectivist view of probabilities [49]. If the conditions of a trial are exactly specified, the outcomes are predictable with certainty, and all probabilities are 0 or 1. In a deterministic theory, all probabilities strictly greater than 0 and less than 1 arise as a consequence of incomplete information and depend upon their assigner’s state of belief.

Of course, we should keep in mind that our ultimate goal is to consider the status of quantum states and, by way of them, quantum probabilities. One can ask, “Does this not change the flavor of these considerations?” Quantum mechanics is avowedly not a theory of one’s ignorance of a set of hidden variables [50]: So how can its probabilities be subjective? In Sec. III we argue that despite the intrinsic indeterminism of quantum mechanics, the essence of the above discussion carries over to the quantum setting intact. Furthermore, there are specifically quantum-motivated arguments for a Bayesian interpretation of quantum probabilities.

For the present, though, let us consider in some detail the general problem of a repeated experiment—spinning a roulette wheel \(N\) times is an example. As discussed briefly in Sec. I, this allows us to make a conceptual connection to quantum-state tomography. Here the individual trials are described by discrete random variables \(x_n \in \{1, 2, \ldots, k\}, n = 1, \ldots, N;\) that is to say, there are \(N\) random variables, each of which can assume \(k\) discrete values. In an objectivist theory, such an experiment has a standard formulation in which the probability in the multi-trial hypothesis space is given by an independent, identically distributed distribution

\[
p(x_1, x_2, \ldots, x_N) = p_{x_1} p_{x_2} \cdots p_{x_N} = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k},
\]

where \(n_j\) is the number of times outcome \(j\) is listed in the vector \((x_1, x_2, \ldots, x_N)\), so that \(\sum_j n_j = N\). The number \(p_j\) \((j = 1, \ldots, k)\) describes the objective, “true” probability that the result of a single experiment will be \(j\) \((j = 1, \ldots, k)\). This simple description—for the objectivist—only describes the situation from a kind of “God’s eye” point of view. To the experimentalist, the “true” probabilities \(p_1, \ldots, p_k\) will very often be unknown at the outset.

\[7\] An entertaining account of a serious attempt to make money from this idea can be found in Ref. [48].

\[8\] Perhaps, more carefully, we should have added, “without a stretch of the imagination.” For a stretch of the imagination, see Ref. [51].
Thus, his burden is to estimate the unknown probabilities by a statistical analysis of the experiment’s outcomes.

In the Bayesian approach, it does not make sense to talk about estimating a true probability. Instead, a Bayesian assigns a prior probability distribution \( p(x_1, x_2, \ldots, x_N) \) on the multi-trial hypothesis space and then uses Bayes’s theorem to update the distribution in the light of measurement results. A common criticism from the objectivist camp is that the choice of distribution \( p(x_1, x_2, \ldots, x_N) \) with which to start the process seems overly arbitrary to them. On what can it be grounded, they would ask? From the Bayesian viewpoint, the subjectivity of the prior is a strength rather than a weakness, because assigning a prior amounts to laying bare the necessarily subjective assumptions behind any probabilistic argument, be it Bayesian or objectivist. Choosing a prior among all possible distributions on the multi-trial hypothesis space is, however, a daunting task. As we will now see, this task becomes tractable by the de Finetti representation theorem.

It is very often the case that one or more features of a problem stand out so clearly that there is no question about how to incorporate them into an initial assignment. In the present case, the key feature is contained in the assumption that an arbitrary number of repeated trials are equivalent. This means that one has no reason to believe there will be a difference between one trial and the next. In this case, the prior distribution is judged to have the sort of permutation symmetry discussed briefly in Sec. I, which de Finetti [41] called exchangeability. The rigorous definition of exchangeability proceeds in two stages.

A probability distribution \( p(x_1, x_2, \ldots, x_N) \) is said to be symmetric (or finitely exchangeable) if it is invariant under permutations of its arguments, i.e., if

\[
p(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(N)}) = p(x_1, x_2, \ldots, x_N) \quad (2.2)
\]

for any permutation \( \pi \) of the set \( \{1, \ldots, N\} \). The distribution \( p(x_1, x_2, \ldots, x_N) \) is called exchangeable (or infinitely exchangeable) if it is symmetric and if for any integer \( M > 0 \), there is a symmetric distribution \( p_{N+M}(x_1, x_2, \ldots, x_{N+M}) \) such that

\[
p(x_1, x_2, \ldots, x_N) = \sum_{x_{N+1}, \ldots, x_{N+M}} p_{N+M}(x_1, \ldots, x_N, x_{N+1}, \ldots, x_{N+M}) \quad (2.3)
\]

This last statement means that the distribution \( p \) can be extended to a symmetric distribution of arbitrarily many random variables. Expressed informally, an exchangeable distribution can be thought of as arising from an infinite sequence of random variables whose order is irrelevant.

We now come to the main statement of this section: If a probability distribution \( p(x_1, x_2, \ldots, x_N) \) is exchangeable, then it can be written uniquely in the form

\[
p(x_1, x_2, \ldots, x_N) = \int_{S_k} P(p) p_{x_1 p_{x_2} \cdots p_{x_N}} dp = \int_{S_k} P(p) p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} dp , \quad (2.4)
\]

where \( p = (p_1, p_2, \ldots, p_k) \), and the integral is taken over the probability simplex

\[
S_k = \left\{ p : p_j \geq 0 \text{ for all } j \text{ and } \sum_{j=1}^k p_j = 1 \right\} . \quad (2.5)
\]
Furthermore, the function $P(p) \geq 0$ is required to be a probability density function on the simplex:

$$\int_{S_k} P(p) \, dp = 1.$$  \hfill (2.6)

Equation (2.4) comprises the classical de Finetti representation theorem for discrete random variables. (A simple proof of this theorem in the case of binary random variables can be found in Refs. [46,1].)

Let us reiterate the importance of this result for the present considerations. It says that an agent, making solely the judgment of exchangeability for a sequence of random variables $x_j$, can proceed as if his state of belief had instead come about through ignorance of an unknown, but objectively existent set of probabilities $p$. His precise ignorance of $p$ is captured by the “probability on probabilities” $P(p)$. This is in direct analogy to what we desire of a solution to the problem of the unknown quantum state in quantum-state tomography.

As a final note before finally addressing the quantum problem in Sec. III, we point out that both conditions in the definition of exchangeability are crucial for the proof of the de Finetti theorem. In particular, there are probability distributions $p(x_1, x_2, \ldots, x_N)$ that are symmetric, but not exchangeable. A simple example is the distribution $p(x_1, x_2)$ of two binary random variables $x_1, x_2 \in \{0, 1\}$,

$$p(0, 0) = p(1, 1) = 0,$$

$$p(0, 1) = p(1, 0) = \frac{1}{2}.$$ \hfill (2.7) \hfill (2.8)

One can easily check that $p(x_1, x_2)$ cannot be written as the marginal of a symmetric distribution of three variables, as in Eq. (2.3). Therefore it can have no representation along the lines of Eq. (2.4). (For an extended discussion of this, see Ref. [52].) Indeed, Eqs. (2.7) and (2.8) characterize a perfect “anticorrelation” of the two variables, in contrast to the positive correlation implied by distributions of de Finetti form.

### III. THE QUANTUM DE FINETTI REPRESENTATION

Let us now return to the problem of quantum-state tomography described in Sec. I. In the objectivist formulation of the problem, a device repeatedly prepares copies of a system in the same quantum state $\rho$. This is generally a mixed-state density operator on a Hilbert space $\mathcal{H}_d$ of $d$ dimensions. We call the totality of such density operators $\mathcal{D}_d$. The joint quantum state of the $N$ systems prepared by the device is then given by

$$\rho^{\otimes N} = \rho \otimes \rho \otimes \cdots \otimes \rho,$$ \hfill (3.1)

the $N$-fold tensor product of $\rho$ with itself. This, of course, is a very restricted example of a density operator on the tensor-product Hilbert space $\mathcal{H}_d^{\otimes N} \equiv \mathcal{H}_d \otimes \cdots \otimes \mathcal{H}_d$. The experimenter, who performs quantum-state tomography, tries to determine $\rho$ as precisely as possible. Depending upon the version of the argument, $\rho$ is interpreted as the “true” state of each of the systems or as a description of the “true” preparation procedure.
We have already articulated our dissatisfaction with this way of stating the problem, but we give here a further sense of why both interpretations above are untenable. Let us deal first with the version where $\rho$ is regarded as the true, objective state of each of the systems. In this discussion it is useful to consider separately the cases of mixed and pure states $\rho$. The arguments against regarding mixed states as objective properties of a quantum system are essentially the same as those against regarding probabilities as objective. In analogy to the roulette example given in the previous section, we can say that, whenever an observer assigns a mixed state to a physical system, one can think of another observer who assigns a different state based on privileged information.

The quantum argument becomes yet more compelling if the apparently nonlocal nature of quantum states is taken into consideration. Consider two parties, $A$ and $B$, who are far apart in space, say several light years apart. Each party possesses a spin-$1/2$ particle. Initially the joint state of the two particles is the maximally entangled pure state $\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$. Consequently, $A$ assigns the totally mixed state $\frac{1}{2}(|0\rangle\langle0| + |1\rangle\langle1|)$ to her own particle. Now $B$ makes a measurement on his particle, finds the result 0, and assigns to $A$’s particle the pure state $|0\rangle$. Is this now the “true,” objective state of $A$’s particle? At what precise time does the objective state of $A$’s particle change from totally mixed to pure? If the answer is “simultaneously with $B$’s measurement,” then what frame of reference should be used to determine simultaneity? These questions and potential paradoxes are avoided if states are interpreted as states of belief. In our example, $A$ and $B$ have different states of belief and therefore assign different states. For a detailed analysis of this example, see Ref. [53]; for an experimental investigation see Ref. [54].

If one admits that mixed states cannot be objective properties, because another observer, possessing privileged information, can know which pure state underlies the mixed state, then it becomes very tempting to regard the pure states as giving the “true” state of a system. Probabilities that come from pure states would then be regarded as objective, and the probabilities for pure states within an ensemble decomposition of mixed states would be regarded as subjective, expressing our ignorance of which pure state is the “true” state of the system. An immediate and, in our view, irremediable problem with this idea is that a mixed state has infinitely many ensemble decompositions into pure states [55–57], so the distinction between subjective and objective becomes hopelessly blurred.

This problem can be made concrete by the example of a spin-$\frac{1}{2}$ particle. Any pure state of the particle can be written in terms of the Pauli matrices as

$$|n\rangle\langle n| = \frac{1}{2}(I + n \cdot \sigma) = \frac{1}{2}(I + n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3), \tag{3.2}$$

where the unit vector $n = n_1e_1 + n_2e_2 + n_3e_3$ labels the pure state, and $I$ denotes the unit operator. An arbitrary state $\rho$, mixed or pure, of the particle can be expressed as

$$\rho = \frac{1}{2}(I + S \cdot \sigma), \tag{3.3}$$

where $0 \leq |S| \leq 1$. If $|S| < 1$, there is an infinite number of ways in which $S$ can be written in the form $S = \sum_j p_jn_j$, $|n_j| = 1$, with the numbers $p_j$ comprising a probability distribution, and hence an infinite number of ensemble decompositions of $\rho$. 

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\[ \rho = \sum_j p_j \frac{1}{2} (I + n_j \cdot \sigma) = \sum_j p_j |n_j\rangle \langle n_j| . \]  

(3.4)

Suppose for specificity that the particle’s state is a mixed state with \( S = \frac{1}{2} e_3 \). Writing \( S = \frac{3}{4} e_3 + \frac{1}{4} (-e_3) \) gives the eigendecomposition,

\[ \rho = \frac{3}{4} |e_3\rangle \langle e_3| + \frac{1}{4} |-e_3\rangle \langle -e_3| , \]

(3.5)

where we are to regard the probabilities \( \frac{3}{4} \) and \( \frac{1}{4} \) as subjective expressions of ignorance about which eigenstate is the “true” state of the particle. Writing \( S = \frac{1}{2} n_+ + \frac{1}{2} n_- \), where \( n_{\pm} = \frac{1}{2} e_3 \pm \sqrt{2} e_1 \), gives another ensemble decomposition,

\[ \rho = \frac{1}{2} |n_+\rangle \langle n_+| + \frac{1}{2} |n_-\rangle \langle n_-| , \]

(3.6)

where we are now to regard the two probabilities of \( 1/2 \) as expressing ignorance of whether the “true” state is \( |n_+\rangle \) or \( |n_-\rangle \).

The problem becomes acute when we ask for the probability that a measurement of the \( z \) component of spin yields spin up; this probability is given by \( \langle e_3|\rho|e_3\rangle = \frac{1}{2} (1 + \frac{1}{2} \langle e_3|\sigma_3|e_3\rangle) = \frac{3}{4} \). The eigendecomposition gets this probability by the route

\[ \langle e_3|\rho|e_3\rangle = \frac{3}{4} |\langle e_3|e_3\rangle|^2 + \frac{1}{4} |\langle e_3|-e_3\rangle|^2 . \]

(3.7)

Here the objective quantum probabilities, calculated from the eigenstates, report that the particle definitely has spin up or definitely has spin down; the overall probability of \( \frac{3}{4} \) comes from mixing these objective probabilities with the subjective probabilities for the eigenstates. The decomposition (3.6) gets the same overall probability by a different route,

\[ \langle e_3|\rho|e_3\rangle = \frac{1}{2} |\langle e_3|n_+\rangle|^2 + \frac{1}{2} |\langle e_3|n_-\rangle|^2 . \]

(3.8)

Now the quantum probabilities tell us that the objective probability for the particle to have spin up is \( \frac{3}{4} \). This simple example illustrates the folly of trying to have two kinds of probabilities in quantum mechanics. The lesson is that if a density operator is even partially a reflection of one’s state of belief, the multiplicity of ensemble decomposition means that a pure state must also be a state of belief.

Return now to the second version of the objectivist formulation of tomography, in which the experimenter is said to be using quantum-state tomography to determine an unknown preparation procedure. Imagine that the tomographic reconstruction results in the mixed state \( \rho \), rather than a pure state, as in fact all actual laboratory procedures do. Now there is a serious problem, because a mixed state does not correspond to a well-defined procedure, but is itself a probabilistic mixture of well-defined procedures, i.e., pure states. The experimenter is thus trying to determine an unknown procedure that has no unique decomposition into well defined procedures. Thus he cannot be said to be determining an unknown procedure at
all. This problem does not arise in a Bayesian interpretation, according to which all quantum states, pure or mixed, are states of belief. In analogy to the classical case, the quantum de Finetti representation provides an operational definition for the idea of an unknown quantum state in this case.

Let us therefore turn to the Bayesian formulation of the quantum-state tomography problem. Before the tomographic measurements, the Bayesian experimenter assigns a prior quantum state to the joint system composed of the $N$ systems, reflecting his prior state of belief. Just as in the classical case, this is a daunting task unless the assumption of exchangeability is justified.

The definition of the quantum version of exchangeability is closely analogous to the classical definition. Again, the definition proceeds in two stages. First, a joint state $\rho^{(N)}$ of $N$ systems is said to be symmetric (or finitely exchangeable) if it is invariant under any permutation of the systems. To see what this means formally, first write out $\rho^{(N)}$ with respect to any orthonormal tensor-product basis on $\mathcal{H}_d^\otimes N$, say $|i_1\rangle|i_2\rangle\cdots|i_N\rangle$, where $i_k \in \{1, 2, \ldots, d\}$ for all $k$. The joint state takes the form

$$\rho^{(N)} = \sum_{i_1, \ldots, i_N, j_1, \ldots, j_N} R^{(N)}_{i_1, \ldots, i_N, j_1, \ldots, j_N} |i_1\rangle \cdots |i_N\rangle \langle j_1| \cdots \langle j_N|,$$  \hspace{1cm} (3.9)

where $R^{(N)}_{i_1, \ldots, i_N, j_1, \ldots, j_N}$ is the density matrix in this representation. What we demand is that for any permutation $\pi$ of the set $\{1, \ldots, N\}$,

$$\rho^{(N)} = \sum_{i_1, \ldots, i_N, j_1, \ldots, j_N} R^{(N)}_{i_{\pi(1)}, \ldots, i_{\pi(N)}, j_{\pi(1)}, \ldots, j_{\pi(N)}} |i_1\rangle \cdots |i_N\rangle \langle j_1| \cdots \langle j_N|,$$  \hspace{1cm} (3.10)

which is equivalent to

$$R^{(N)}_{i_{\pi(1)}, \ldots, i_{\pi(N)}, j_{\pi(1)}, \ldots, j_{\pi(N)}} = R^{(N)}_{i_1, \ldots, i_N, j_1, \ldots, j_N}.$$  \hspace{1cm} (3.11)

The state $\rho^{(N)}$ is said to be exchangeable (or infinitely exchangeable) if it is symmetric and if, for any $M > 0$, there is a symmetric state $\rho^{(N+M)}$ of $N + M$ systems such that the marginal density operator for $N$ systems is $\rho^{(N)}$, i.e.,

$$\rho^{(N)} = \text{tr}_M \rho^{(N+M)},$$  \hspace{1cm} (3.12)

where the trace is taken over the additional $M$ systems. In explicit basis-dependent notation, this requirement is

$$\rho^{(N)} = \sum_{i_1, \ldots, i_N, j_1, \ldots, j_N} \left( \sum_{i_{N+1}, \ldots, i_{N+M}} R^{(N+M)}_{i_1, \ldots, i_{N+1}, \ldots, i_{N+M}, j_1, \ldots, j_{N+1}, \ldots, j_{N+M}} \right) |i_1\rangle \cdots |i_N\rangle \langle j_1| \cdots \langle j_N|. \hspace{1cm} (3.13)$$

In analogy to the classical case, an exchangeable density operator can be thought of informally as the description of a subsystem of an infinite sequence of systems whose order is irrelevant.

The precise statement of the quantum de Finetti representation theorem [42,58] is that any exchangeable state of $N$ systems can be written uniquely in the form
\( \rho^{(N)} = \int_{\mathcal{D}_d} P(\rho) \rho^{\otimes N} \, d\rho . \) \hspace{1cm} (3.14)

Here \( P(\rho) \geq 0 \) is normalized by

\[ \int_{\mathcal{D}_d} P(\rho) \, d\rho = 1 , \] \hspace{1cm} (3.15)

with \( d\rho \) being a suitable measure on density operator space \( \mathcal{D}_d \) [e.g., one could choose \( d\rho = dS \, d\Omega \) in the parameterization (3.3) for a spin-1/2 particle]. The upshot of the theorem, as already advertised, is that it makes it possible to think of an exchangeable quantum-state assignment \textit{as if} it were a probabilistic mixture characterized by a probability density \( P(\rho) \) for the product states \( \rho^{\otimes N} \).

Just as in the classical case, both components of the definition of exchangeability are crucial for arriving at the representation theorem of Eq. (3.14). The reason now, however, is much more interesting than it was previously. In the classical case, extendibility was used solely to exclude anticorrelated probability distributions. Here extendibility is necessary to exclude the possibility of Bell inequality violations for measurements on the separate systems. This is because the assumption of symmetry alone for an \( N \)-party quantum system does not exclude the possibility of quantum entanglement, and all states that can be written as a mixture of product states—of which Eq. (3.14) is an example—have no entanglement \cite{59}.

A simple example for a state that is symmetric but not exchangeable is the Greenberger-Horne-Zeilinger state of three spin-\( \frac{1}{2} \) particles \cite{60},

\[ |\text{GHZ} \rangle = \frac{1}{\sqrt{2}} \left( |0\rangle |0\rangle |0\rangle + |1\rangle |1\rangle |1\rangle \right) , \] \hspace{1cm} (3.16)

which is not extendible to a symmetric state on four systems. This follows because the only states of four particles that marginalize to a three-particle pure state, like the GHZ state, are product states of the form \( |\text{GHZ}\rangle \langle \text{GHZ}| \otimes \rho \), where \( \rho \) is the state of the fourth particle; such states clearly cannot be symmetric. These considerations show that in order for the proposed theorem to be valid, it must be the case that as \( M \) increases in Eq. (3.12), the possibilities for entanglement in the separate systems compensatingly decrease \cite{61}.

**IV. PROOF OF THE QUANTUM DE FINETTI THEOREM**

To prove the quantum version of the de Finetti theorem, we rely on the classical theorem as much as possible. We start from an exchangeable density operator \( \rho^{(N)} \) defined on \( N \) copies of a system. We bring the classical theorem to our aid by imagining a sequence of identical quantum measurements on the separate systems and considering the outcome probabilities they would produce. Because \( \rho^{(N)} \) is assumed exchangeable, such identical measurements give rise to an exchangeable probability distribution for the outcomes. The trick is to recover enough information from the statistics of these measurements to characterize the exchangeable density operator.

With this in mind, the proof is expedited by making use of the theory of generalized quantum measurements or positive operator-valued measures (POVMs) \cite{62–64}. POVMs
generalize the textbook notion of measurement by distilling the essential properties that make the Born rule work. The generalized notion of measurement is this: Any set $\mathcal{E} = \{E_\alpha\}$ of positive-semidefinite operators on $\mathcal{H}_d$ that forms a resolution of the identity, i.e., that satisfies

$$\langle \psi | E_\alpha | \psi \rangle \geq 0, \quad \text{for all } |\psi\rangle \in \mathcal{H}_d \quad (4.1)$$

and

$$\sum_\alpha E_\alpha = I, \quad (4.2)$$

corresponds to at least one laboratory procedure counting as a measurement. The outcomes of the measurement are identified with the indices $\alpha$, and the probabilities of those outcomes are computed according to the generalized Born rule,

$$p_\alpha = \text{tr}(\rho E_\alpha). \quad (4.3)$$

The set $\mathcal{E}$ is called a POVM, and the operators $E_\alpha$ are called POVM elements. Unlike standard or von Neumann measurements, there is no limitation on the number of values $\alpha$ can take, the operators $E_\alpha$ need not be rank-1, and there is no requirement that the $E_\alpha$ be idempotent and mutually orthogonal. This definition has important content because the older notion of measurement is simply too restrictive: there are laboratory procedures that clearly should be called “measurements,” but that cannot be expressed in terms of the von Neumann measurement process alone.

One might wonder whether the existence of POVMs contradicts everything taught about standard measurements in the traditional graduate textbooks [65] and the well-known classics [66]. It does not. The reason is that any POVM can be represented formally as a standard measurement on an ancillary system that has interacted in the past with the system of main interest. Thus in a certain sense, von Neumann measurements capture everything that can be said about quantum measurements [63]. A way to think about this is that by learning something about the ancillary system through a standard measurement, one in turn learns something about the system of real interest. Indirect though this might seem, it can be a very powerful technique, sometimes revealing information that could not have been revealed otherwise [67].

For instance, by considering POVMs, one can consider measurements with an outcome cardinality that exceeds the dimensionality of the Hilbert space. What this means is that whereas the statistics of a von Neumann measurement can only reveal information about the $d$ diagonal elements of a density operator $\rho$, through the probabilities $\text{tr}(\rho \Pi_i)$, the statistics of a POVM generally can reveal things about the off-diagonal elements, too. It is precisely this property that we take advantage of in our proof of the quantum de Finetti theorem.

Our problem hinges on finding a special kind of POVM, one for which any set of outcome probabilities specifies a unique operator. This boils down to a problem of pure linear algebra. The space of operators on $\mathcal{H}_d$ is itself a linear vector space of dimension $d^2$. The quantity $\text{tr}(A^\dagger B)$ serves as an inner product on that space. If the POVM elements $E_\alpha$ span the space of operators—there must be at least $d^2$ POVM elements in the set—the measurement probabilities $p_\alpha = \text{tr}(\rho E_\alpha)$—now thought of as projections in the directions $E_\alpha$—are sufficient
to specify a unique operator $\rho$. Two distinct density operators $\rho$ and $\sigma$ must give rise to different measurement statistics. Such measurements, which might be called *informationally complete*, have been studied for some time [68].

For our proof we need a slightly refined notion—that of a *minimal* informationally complete measurement. If an informationally complete POVM has more than $d^2$ operators $E_\alpha$, these operators form an overcomplete set. This means that given a set of outcome probabilities $p_\alpha$, there is generally no operator $A$ that generates them according to $p_\alpha = \text{tr}(AE_\alpha)$. Our proof requires the existence of such an operator, so we need a POVM that has precisely $d^2$ linearly independent POVM elements $E_\alpha$. Such a POVM has the minimal number of POVM elements to be informationally complete. Given a set of outcome probabilities $p_\alpha$, there is a unique operator $A$ such that $p_\alpha = \text{tr}(AE_\alpha)$, even though, as we discuss below, $A$ is not guaranteed to be a density operator.

Do minimal informationally complete POVMs exist? The answer is yes. We give here a simple way to produce one, though there are surely more elegant ways with greater symmetry [69,70]. Start with a complete orthonormal basis $|e_j\rangle$ on $\mathcal{H}_d$, and let $\Gamma_{jk} = |e_j\rangle\langle e_k|$. It is easy to check that the following $d^2$ rank-1 projectors $\Pi_\alpha$ form a linearly independent set.

1. For $\alpha = 1, \ldots, d$, let
   \[ \Pi_\alpha \equiv \Gamma_{jj} , \]  
   where $j$, too, runs over the values $1, \ldots, d$.

2. For $\alpha = d + 1, \ldots, \frac{1}{2}d(d + 1)$, let
   \[ \Pi_\alpha \equiv \Gamma_{(1)jk}^{(1)} = \frac{1}{2}(|e_j\rangle + |e_k\rangle)(\langle e_j| + \langle e_k|) = \frac{1}{2}(\Gamma_{jj} + \Gamma_{kk} + \Gamma_{jk} + \Gamma_{kj}) , \]  
   where $j < k$.

3. Finally, for $\alpha = \frac{1}{2}d(d + 1) + 1, \ldots, d^2$, let
   \[ \Pi_\alpha \equiv \Gamma_{(2)jk}^{(2)} = \frac{1}{2}(|e_j\rangle + i|e_k\rangle)(\langle e_j| - i\langle e_k|) = \frac{1}{2}(\Gamma_{jj} + \Gamma_{kk} - i\Gamma_{jk} + i\Gamma_{kj}) , \]  
   where again $j < k$.

All that remains is to transform these (positive-semidefinite) linearly independent operators $\Pi_\alpha$ into a proper POVM. This can be done by considering the positive semidefinite operator $G$ defined by

\[ G = \sum_{\alpha=1}^{d^2} \Pi_\alpha . \]  

It is straightforward to show that $\langle \psi|G|\psi\rangle > 0$ for all $|\psi\rangle \neq 0$, thus establishing that $G$ is positive definite and hence invertible. Applying the (invertible) linear transformation $X \rightarrow G^{-1/2}XG^{-1/2}$ to Eq. (4.7), we find a valid decomposition of the identity,
\[ I = \sum_{\alpha=1}^{d^2} G^{-1/2} \Pi_\alpha G^{-1/2} . \]  

(4.8)

The operators

\[ E_\alpha = G^{-1/2} \Pi_\alpha G^{-1/2} \]  

(4.9)

satisfy the conditions of a POVM, Eqs. (4.1) and (4.2), and moreover, they retain the rank and linear independence of the original \( \Pi_\alpha \).

It is worthwhile noting a special property of all minimal informationally complete POVMs \( \{ E_\alpha \} \): For no quantum state \( \rho \) is it ever the case that \( \text{tr} \rho E_\alpha = 1 \). Let us show this for the case where all the \( E_\alpha = k_\alpha |\psi_\alpha\rangle \langle \psi_\alpha| \) are rank one. Suppose it were the case that for some \( \rho \), \( \text{tr} \rho E_0 = 1 \). Then it would also have to be the case that \( E_0 = \rho = |\psi\rangle \langle \psi| \) for some vector \( |\psi\rangle \). But then, because \( \sum E_\alpha = I \), it follows that \( \langle \psi_\alpha |\psi\rangle = 0 \) for all \( \alpha \neq 0 \). That is, all the \( |\psi_\alpha\rangle \) must lie in a \((d-1)\)-dimensional subspace. But then it follows that at most \((d-1)^2 + 1 \neq d^2\), we have a contradiction with the assumption that \( \{ E_\alpha \} \) is an informationally complete POVM. (In particular, for the case of the particular, minimal informationally complete POVM in Eq. (4.9), it can be shown that \[ P(h) \leq \left[ d - \frac{1}{2} \left( 1 + \cot \frac{3\pi}{4d} \right) \right]^{-1} < 1 . \]  

(4.10)

For large \( d \), this bound asymptotes to roughly \((0.79d)^{-1}\). What this means generically is that for no informationally complete POVM can the vectors \( p_\alpha \) completely fill the probability simplex.

\[ \text{FIG. 3. The planar surface represents the space of all probability distributions over } d^2 \text{ outcomes. If the probability distributions are imagined to be generated by a minimal informationally complete measurement, the set of valid quantum states in this representation represents a convex region strictly smaller than the whole simplex.} \]
With this generalized measurement (or any other one like it), we can return to the main line of proof. Recall we assumed that we captured our state of belief by an exchangeable density operator \( \rho^{(N)} \). Consequently, repeated application of the (imagined) measurement \( \mathcal{E} \) must give rise to an exchangeable probability distribution over the \( N \) random variables \( \alpha_n \in \{1, 2, \ldots, d^2\}, \ n = 1, \ldots, N \). We now analyze these probabilities.

Quantum mechanically, it is valid to think of the \( N \) repeated measurements of \( \mathcal{E} \) as a single measurement on the Hilbert space \( \mathcal{H}_d^\otimes N \equiv \mathcal{H}_d \otimes \cdots \otimes \mathcal{H}_d \). This measurement, which we denote \( \mathcal{E}^\otimes N \), consists of \( d^2N \) POVM elements of the form \( E_{\alpha_1} \otimes \cdots \otimes E_{\alpha_N} \). The probability of any particular outcome sequence of length \( N \), namely \( \alpha \equiv (\alpha_1, \ldots, \alpha_N) \), is given by the standard quantum rule,

\[
p^{(N)}(\alpha) = \text{tr}[\rho^{(N)}(E_{\alpha_1} \otimes \cdots \otimes E_{\alpha_N})] .
\]  

Because the distribution \( p^{(N)}(\alpha) \) is exchangeable, we have by the classical de Finetti theorem [see Eq. (2.4)] that there exists a unique probability density \( P(p) \) on \( S_{d^2} \) such that

\[
p^{(N)}(\alpha) = \int_{S_{d^2}} P(p) p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_N} \, dp .
\]  

It should now begin to be apparent why we chose to imagine a measurement \( \mathcal{E} \) consisting of precisely \( d^2 \) linearly independent elements. This allows us to assert the existence of a unique operator \( A_p \) on \( \mathcal{H}_d \) corresponding to each point \( p \) in the domain of the integral. The ultimate goal here is to turn Eqs. (4.11) and (4.12) into a single operator equation.

With that in mind, let us define \( A_p \) as the unique operator satisfying the following \( d^2 \) linear equations:

\[
\text{tr}(A_p E_\alpha) = p_\alpha , \quad \alpha = 1, \ldots, d^2 .
\]  

Inserting this definition into Eq. (4.12) and manipulating it according to the algebraic rules of tensor products—namely \( (A \otimes B)(C \otimes D) = AC \otimes BD \) and \( \text{tr}(A \otimes B) = (\text{tr}A)(\text{tr}B) \)—we see that

\[
p^{(N)}(\alpha) = \int_{S_{d^2}} P(p) \text{tr}(A_p E_{\alpha_1}) \cdots \text{tr}(A_p E_{\alpha_N}) \, dp
\]

\[
= \int_{S_{d^2}} P(p) \text{tr}(A_p E_{\alpha_1} \otimes \cdots \otimes A_p E_{\alpha_N}) \, dp
\]

\[
= \int_{S_{d^2}} P(p) \text{tr}[A_p^\otimes N (E_{\alpha_1} \otimes \cdots \otimes E_{\alpha_N})] \, dp .
\]  

If we further use the linearity of the trace, we can write the same expression as

\[
p^{(N)}(\alpha) = \text{tr}\left[ \left( \int_{S_{d^2}} P(p) A_p^\otimes N \, dp \right) E_{\alpha_1} \otimes \cdots \otimes E_{\alpha_N} \right] .
\]  

The identity between Eqs. (4.11) and (4.15) must hold for all sequences \( \alpha \). It follows that

\[
\rho^{(N)} = \int_{S_{d^2}} P(p) A_p^\otimes N \, dp .
\]
This is because the operators $E_{\alpha_1} \otimes \cdots \otimes E_{\alpha_N}$ form a complete basis for the vector space of operators on $\mathcal{H}_d^{\otimes N}$.

Equation (4.16) already looks very much like our sought after goal, but we are not there quite yet. At this stage one has no right to assume that the $A_p$ are density operators. Indeed they generally are not: the integral (4.12) ranges over some points $p$ in $S_d^2$ that cannot be generated by applying the measurement $\mathcal{E}$ to any quantum state. Hence some of the $A_p$ in the integral representation cannot correspond to "unknown quantum states."

The solution to this conundrum is provided by the overall requirement that $\rho^{(N)}$ be a valid density operator. This requirement places a significantly more stringent constraint on the distribution $P(p)$ than was the case in the classical representation theorem. In particular, it must be the case that $P(p)$ vanishes whenever the corresponding $A_p$ is not a proper density operator. Let us move toward showing that.

We first need to delineate two properties of the operators $A_p$. One is that they are Hermitian. The argument is simply

$$\text{tr}(E_\alpha A_p^\dagger) = \text{tr}[(A_p E_\alpha)^\dagger] = \text{tr}(A_p E_\alpha)^* = \text{tr}(A_p E_\alpha),$$

where the last step follows from Eq. (4.13). Because the $E_\alpha$ are a complete set of linearly independent operators, it follows that $A_p^\dagger = A_p$. The second property tells us something about the eigenvalues of $A_p$:

$$1 = \sum_\alpha p_\alpha = \text{tr} \left( A_p \sum_\alpha E_\alpha \right) = \text{tr} A_p.$$

In other words the (real) eigenvalues of $A_p$ must sum to unity.

We now show that these two facts go together to imply that if there are any non-quantum-state $A_p$ with positive weight $P(p)$ in Eq. (4.16), then one can find a measurement for which $\rho^{(N)}$ produces illegal "probabilities" for sufficiently large $N$. For instance, take a particular $A_q$ in Eq. (4.16) that has at least one negative eigenvalue $-\lambda < 0$. Let $|\psi\rangle$ be a normalized eigenvector corresponding to that eigenvalue and consider the binary-valued POVM consisting of the elements $\bar{\Pi} = |\psi\rangle\langle\psi|$ and $\Pi = I - \bar{\Pi}$. Since $\text{tr}(A_q \bar{\Pi}) = -\lambda < 0$, it is true by Eq. (4.18) that $\text{tr}(A_q \Pi) = 1 + \lambda > 1$. Consider repeating this measurement over and over. In particular, let us tabulate the probability of getting outcome $\Pi$ for every single trial to the exclusion of all other outcomes.

The gist of the contradiction is most easily seen by imagining that Eq. (4.16) is really a discrete sum:

$$\rho^{(N)} = P(q) A_q^{\otimes N} + \sum_{p \neq q} P(p) A_p^{\otimes N}.$$  

The probability of $N$ occurrences of the outcome $\Pi$ is thus

$$\text{tr}(\rho^{(N)} \Pi^{\otimes N}) = P(q) \text{tr}(A_q^{\otimes N} \Pi^{\otimes N}) + \sum_{p \neq q} P(p) \text{tr}(A_p^{\otimes N} \Pi^{\otimes N})$$

$$= P(q) [\text{tr}(A_q \Pi)]^N + \sum_{p \neq q} P(p) [\text{tr}(A_p \Pi)]^N$$

$$= P(q)(1 + \lambda)^N + \sum_{p \neq q} P(p) [\text{tr}(A_p \Pi)]^N. \quad (4.20)$$
There are no assurances in general that the right-hand term in Eq. (4.20) is positive, but if 
\( N \) is an even number it must be. It follows that if \( P(q) \geq 0 \), for sufficiently large \( \text{even} \ \ N \),
\[
\text{tr}(\rho^{(N)}\Pi^{\otimes N}) > 1 , \tag{4.21}
\]
contradicting the assumption that it should always be a probability.

All we need to do now is transcribe the argument leading to Eq. (4.21) to the general
integral case of Eq. (4.16). Note that by Eq. (4.13), the quantity \( \text{tr}(A_p\Pi) \) is a (linear)
continuous function of the parameter \( p \). Therefore, for any \( \epsilon > 0 \), there exists a \( \delta > 0 \) such
that \( |\text{tr}(A_p\Pi) - \text{tr}(A_q\Pi)| \leq \epsilon \) whenever \( |p - q| \leq \delta \), i.e., whenever \( p \) is contained
within an open ball \( B_\delta(q) \) centered at \( q \). Choose \( \epsilon < \lambda \), and define \( \overline{B}_\delta \) to be
the intersection of \( B_\delta(q) \) with the probability simplex. For \( p \in \overline{B}_\delta \), it follows that
\[
\text{tr}(A_p\Pi) \geq 1 + \lambda - \epsilon > 1 . \tag{4.22}
\]
If we consider an \( N \) that is even, \( [\text{tr}(A_p\Pi)]^N \) is nonnegative in all of \( S_{d^2} \), and we have that
the probability of the outcome \( \Pi^{\otimes N} \) satisfies
\[
\text{tr}(\rho^{(N)}\Pi^{\otimes N}) = \int_{S_{d^2}} P(p) [\text{tr}(A_p\Pi)]^N \, dp \\
= \int_{S_{d^2} - \overline{B}_\delta} P(p) [\text{tr}(A_p\Pi)]^N \, dp + \int_{\overline{B}_\delta} P(p) [\text{tr}(A_p\Pi)]^N \, dp \\
\geq \int_{\overline{B}_\delta} P(p) [\text{tr}(A_p\Pi)]^N \, dp \\
\geq (1 + \lambda - \epsilon)^N \int_{\overline{B}_\delta} P(p) \, dp . \tag{4.23}
\]
Unless
\[
\int_{\overline{B}_\delta} P(p) \, dp = 0 , \tag{4.24}
\]
the lower bound (4.23) for the probability of the outcome \( \Pi^{\otimes N} \) becomes arbitrarily large as
\( N \to \infty \). Thus we conclude that the requirement that \( \rho^{(N)} \) be a proper density operator
constrains \( P(p) \) to vanish almost everywhere in \( \overline{B}_\delta \) and, consequently, to vanish almost
everywhere that \( A_p \) is not a physical state.

Using Eq. (4.13), we can trivially transform the integral representation (4.16) to one
directly over the convex set of density operators \( D_d \) and be left with the following statement.
Under the sole assumption that the density operator \( \rho^{(N)} \) is exchangeable, there exists a
unique probability density \( P(\rho) \) such that
\[
\rho^{(N)} = \int_{D_d} P(\rho) \rho^{\otimes N} \, d\rho . \tag{4.25}
\]
This concludes the proof of the quantum de Finetti representation theorem.
V. INTERMEZZO

In classical probability theory, exchangeability characterizes those situations where the only data relevant for updating a probability distribution are frequency data, i.e., the numbers $n_j$ in Eq. (2.4) which tell how often the result $j$ occurred. The quantum de Finetti representation shows that the same is true in quantum mechanics: Frequency data (with respect to a sufficiently robust measurement) are sufficient for updating an exchangeable state to the point where nothing more can be learned from sequential measurements; that is, one obtains a convergence of the form (1.7), so that ultimately any further measurements on the individual systems are statistically independent.

Beyond the aesthetic point of showing the consistency of the Bayesian conception of quantum states, we also believe the technical methods exhibited here will be of interest in the practical arena. Recently there has been a large literature on which classes of measurements have various advantages for tomographic purposes [71,72]. To our knowledge, the work in Ref. [1] reiterated here was the first to consider tomographic reconstruction based upon minimal informationally complete POVMs. One can imagine several advantages to this approach via the fact that such POVMs with rank-one elements are automatically extreme points in the convex set of all measurements [73].

Furthermore, the classical de Finetti theorem is only the beginning with respect to general questions in classical statistics to do with exchangeability and generalizations of the concept [74]. One should expect no less of quantum exchangeability studies. In particular, we are thinking of such issues as representation theorems for finitely exchangeable distributions [52,75]. Just as our method for proving the quantum de Finetti theorem was able to rely heavily on the classical theorem, so one might expect similar benefits from the classical results in the case of quantum finite exchangeability—although, there will certainly be new aspects to the quantum case due to the possibility of entanglement in finite exchangeable states [76]. Finally, a practical application of such representation theorems could be their potential to contribute to the solution of some outstanding problems in constructing security proofs for various quantum key distribution schemes [77,78].

VI. SUBJECTIVITY OF QUANTUM OPERATIONS

At this point we turn our attention to quantum operations. So far, we have made much to-do of the subjectivity of quantum states. But what of quantum operations? Should this structural element within quantum theory be considered of the nature of a fact—like the outcome of a quantum measurement—or, like the quantum state, should it be recognized as a statement of an agent’s belief or expectation [4]?

The usual presentation of the concept of a quantum operation is that it is the most general quantum-state evolution allowed by the laws of quantum mechanics [63,79]. Upon the process of measurement of some POVM $\{E_\alpha\}$, depending upon the particular details of a measurement interaction and the outcome $\alpha$, an initial quantum state $\rho$ can change to any new state of the form

$$
\Phi_\alpha(\rho) = \frac{1}{\text{tr}\rho E_\alpha} \sum_i A_{\alpha i} \rho A_{\alpha i}^\dagger,
$$

(6.1)
where $A_{\alpha i}$ can be any operators whatsoever, as long as they satisfy

$$E_{\alpha} = \sum_i A_{\alpha i}^\dagger A_{\alpha i} . \quad (6.2)$$

In the case where the POVM is the completely uninformative one—i.e., the POVM $\{I\}$ comprised solely of the identity operator—one recovers the general expression for all possible quantum mechanical time evolutions in terms of trace-preserving completely positive maps:

$$\rho \rightarrow \Phi(\rho) = \frac{1}{\text{tr}\rho E_{\alpha}} \sum_{\alpha} A_{\alpha} \rho A_{\alpha}^\dagger , \quad (6.3)$$

where the $A_{\alpha}$ are any operators that satisfy

$$I = \sum_{\alpha} A_{\alpha}^\dagger A_{\alpha} . \quad (6.4)$$

Let us first focus on the latter type of evolution. A quantum Bayesian should become suspicious that it contains a subjective component because of another representation theorem of Kraus [63]. For any trace preserving completely positive map $\Phi$ on a system, one can always imagine an ancillary system $A$, a quantum state $\sigma$ for that ancillary system, and unitary interaction $U$ between the system and the ancilla, such that

$$\Phi(\rho) = \text{tr}_A \left( U(\rho \otimes \sigma) U^\dagger \right) , \quad (6.5)$$

where $\text{tr}_A$ represents a partial trace over the ancilla’s Hilbert space. The Bayesian should ask, “Whose state of belief is $\sigma$?”

But worse than that, the representation in Eq. (6.5) is not unique. There can well be distinct density operators $\tau$ for the ancilla and distinct unitary interactions $V$ such that

$$\Phi(\rho) = \text{tr}_A \left( V(\rho \otimes \tau) V^\dagger \right) . \quad (6.6)$$

What is to be made of this? If one accepts that a quantum state is a subjective judgment, then it would seem one should also be compelled to accept that the map $\Phi$ can be thought of as having (at least) a subjective component through the quantum state $\sigma$, and that the subjective judgment even leaks into unitary operation $U$, nominally describing the interaction.

Without belaboring that particular point, let us now change focus to the state-change rule associated with the von Neumann collapse postulate. In that context, the measurement operators $E_{\alpha} = |\alpha\rangle\langle\alpha|$ correspond to projectors onto an orthonormal basis $\{ |\alpha\rangle \}$ and

$$\Phi_\alpha(\rho) = \frac{1}{\text{tr}_\rho E_{\alpha}} E_{\alpha} \rho E_{\alpha} = |\alpha\rangle\langle\alpha| . \quad (6.7)$$

The salient point is that, conditioned on the measurement outcome, the final quantum state for the system is uniquely determined after such a measurement.

Note what this implies. If the quantum operation associated with a measurement device is an objective fact (i.e., of the same nature as the measurement outcome), then so too must be the posterior quantum state $E_{\alpha} = |\alpha\rangle\langle\alpha|$. But the whole foundation of this paper is that
quantum states are not objective facts. It follows that the quantum operation $\Phi_\alpha$ cannot be an objective fact either. Instead, the von Neumann collapse rule associated with a projective measurement must be as much of a subjective judgment as a quantum state is in the first place.

The moral: (At least some) quantum operations are not facts, and general quantum time evolutions have a subjective component. But, if so, then what are the “unknown quantum operations” that experimentalists routinely measure in the laboratory?

VII. QUANTUM PROCESS TOMOGRAPHY

In quantum process tomography [37–39], an experimenter lets an incompletely specified device act on a quantum system prepared in an input state of his choice, and then performs a measurement (also of his choice) on the output system. This procedure is repeated many times over, with possibly different input states and different measurements, in order to accumulate enough statistics to assign a quantum operation to the device. Here and throughout this section, by a quantum operation we mean a trace-preserving completely positive linear map. Quantum process tomography has been demonstrated experimentally in liquid state nuclear magnetic resonance [80,81], and recently a number of optical experiments [82,84,83] have implemented entanglement-assisted quantum process tomography. The latter is a procedure that exploits the fact that quantum process tomography is equivalent to quantum state tomography in a larger state space [85–88].

In the usual description of process tomography, it is assumed that the device performs the same unknown quantum operation $\Phi$ every time it is used, and an experimenter’s prior information about the device is expressed via a probability density $p(\Phi)$ over all possible operations. What, however, is the operational meaning of an unknown quantum operation? When does the action of a device leave off from an initial input so that the next input can be sent through? In particular, what gives the right to suppose that a device does not have memory or, for instance, does not entangle the successive inputs passing through it? These questions boil down to the need to explore a single issue: What essential assumptions must be made so that quantum process tomography is a logically coherent notion?

What is called for is a method of posing quantum process tomography that never requires the invocation of the concept of an unknown quantum operation. This can be done by focusing upon the action of a single known quantum operation $\Phi^{(N)}$, which acts upon $N$ nominal inputs. In particular, we identify conditions under which $\Phi^{(N)}$, $(N = 1, 2, \ldots)$, can be represented as

$$\Phi^{(N)} = \int p(\Phi) \Phi^{\otimes N} d\Phi, \quad \text{(7.1)}$$

---

9 Ref. [4], in fact, tries to argue for more: Namely, that all quantum operations are subjective states of belief, just like all quantum states are subjective states of belief. The present work, however, refrains from attempting that larger task.
for some probability density $p(\Phi)$, and where the integration extends over all single-system quantum operations $\Phi$. With this theorem established, the conditions under which an experimenter can act as if his prior $\Phi(N)$ corresponds to ignorance of a “true” but unknown quantum operation are made precise.

Our starting point is the closely aligned and similarly motivated de Finetti representation theorem for quantum states of the previous sections. Here, we make use of the correspondence between quantum process tomography and quantum-state tomography mentioned above to derive a de Finetti representation theorem for sequences of quantum operations.

VIII. THE PROCESS-TOMOGRAPHY THEOREM

In this section and the next, we restrict our attention to devices for which the input and output have the same Hilbert space dimension, $D$. In the following, $\mathcal{H}_D$ denotes a $D$-dimensional Hilbert space, $\mathcal{H}_D^\otimes N = \mathcal{H}_D \otimes \cdots \otimes \mathcal{H}_D$ denotes its $N$-fold tensor product, and $\mathcal{L}(\mathcal{V})$ denotes the space of linear operators on a linear space $\mathcal{V}$. The set of density operators for a $D$-dimensional quantum system is a convex subset of $\mathcal{L}(\mathcal{H}_D)$.

The action of a device on $N$ nominal inputs systems is then described by a trace-preserving completely positive map $\Phi(N): \mathcal{L}(\mathcal{H}_D^\otimes N) \rightarrow \mathcal{L}(\mathcal{H}_D^\otimes N)$, which maps the state of the $N$ input systems to the state of the $N$ output systems. We will say, in analogy to the definition of exchangeability for quantum states, that a quantum operation $\Phi(N)$ is exchangeable if it is a member of an exchangeable sequence of quantum operations.

To define exchangeability for a sequence of quantum operations in a natural way, we reduce the properties of symmetry and extendibility for sequences of operations to the corresponding properties for sequences of states. In the following, we will use bold letters to denote vectors of indices, e.g. $j = (j_1, \ldots, j_N)$. We will use $\pi$ to denote a permutation of the set $\{1, \ldots, N\}$, where the cardinality $N$ will depend on the context. The action of the permutation $\pi$ on the vector $j$ is defined by $\pi j = (j_{\pi(1)}, \ldots, j_{\pi(N)})$.

Any $N$-system density operator $\rho(N)$ can be expanded in the form

$$\rho(N) = \sum_{j, l} r_{j, l}^{(N)} \otimes |j_{Q_i}^i\rangle \langle l_{Q_i}^i| = \sum_{j, l} r_{j, l}^{(N)} |j_{Q_i}^i\rangle \langle l_{Q_i}^i| \otimes \cdots \otimes |j_{Q_N}^N\rangle \langle l_{Q_N}^N|,$$  

where $\{|1_Q^i\rangle, \ldots, |D_Q^i\rangle\}$ denotes an orthonormal basis for the Hilbert space $\mathcal{H}_D$ of the $i$th system, and $r_{j, l}^{(N)}$ are the matrix elements of $\rho(N)$ in the tensor product basis. We define the action of the permutation $\pi$ on the state $\rho(N)$ by

$$\pi \rho(N) = \sum_{j, l} r_{\pi j, \pi l}^{(N)} \otimes |j_{\pi^{-1}(i)}^i\rangle \langle l_{\pi^{-1}(i)}^i| = \sum_{j, l} r_{j, l}^{(N)} \otimes |j_{\pi^{-1}(i)}^i\rangle \langle l_{\pi^{-1}(i)}^i|.$$  

With this notation, we can make the following definition. A sequence of quantum operations, $\Phi(k): \mathcal{L}(\mathcal{H}_D^\otimes k) \rightarrow \mathcal{L}(\mathcal{H}_D^\otimes k)$, is called exchangeable if, for $k = 1, 2, \ldots$,
1. \( \Phi^{(k)} \) is symmetric, i.e.,

\[
\Phi^{(k)}(\rho^{(k)}) = \pi \left( \Phi^{(k)}(\pi^{-1} \rho^{(k)}) \right)
\]

(8.4)

for any permutation \( \pi \) of the set \( \{1, \ldots, k\} \) and for any density operator \( \rho^{(k)} \in \mathcal{L}(\mathcal{H}_D^\otimes k) \), and

2. \( \Phi^{(k)} \) is extendible, i.e.,

\[
\Phi^{(k)}(\text{tr}_{k+1} \rho^{(k+1)}) = \text{tr}_{k+1} \left( \Phi^{(k+1)}(\rho^{(k+1)}) \right)
\]

(8.5)

for any state \( \rho^{(k+1)} \).

In words, these conditions amount to the following. Condition (1) is equivalent to the requirement that the quantum operation \( \Phi^{(k)} \) commutes with any permutation operator \( \pi \) acting on the states \( \rho^{(k)} \): It does not matter what order we send our systems through the device; as long as we rearrange them at the end into the original order, the resulting evolution will be the same. Condition (2) says that it does not matter if we consider a larger map \( \Phi^{(N+1)} \) acting on a larger collection of systems (possibly entangled), or a smaller \( \Phi^{(N)} \) on some subset of those systems: The upshot of the evolution will be the same for the relevant systems.

We are now in a position to formulate the de Finetti representation theorem for quantum operations. A quantum operation \( \Phi^{(N)} : \mathcal{L}(\mathcal{H}_D^\otimes N) \to \mathcal{L}(\mathcal{H}_D^\otimes N) \) is an element of an exchangeable sequence if and only if it can be written in the form

\[
\Phi^{(N)} = \int p(\Phi) \Phi^{\otimes N} d\Phi \quad \text{for all } N,
\]

(8.6)

where the integral ranges over all single-shot quantum operations \( \Phi : \mathcal{L}(\mathcal{H}_D) \to \mathcal{L}(\mathcal{H}_D) \), \( d\Phi \) is a suitable measure on the space of quantum operations, and the probability density \( p(\Phi) \geq 0 \) is unique. The tensor product \( \Phi^{\otimes N} \) is defined by \( \Phi^{\otimes N}(\rho_1 \otimes \cdots \otimes \rho_N) = \Phi(\rho_1) \otimes \cdots \otimes \Phi(\rho_N) \) for all \( \rho_1, \ldots, \rho_N \) and by linear extension for arbitrary arguments.

Just as with the original quantum de Finetti theorem \([42,1]\), this result allows a certain latitude in how quantum process tomography can be described. One is free to use the language of an unknown quantum operation if the condition of exchangeability is met by one’s prior \( \Phi^{(N)} \) but it is not required: For the quantum Bayesian in particular, the known quantum operation \( \Phi^{(N)} \) is the only meaningful quantum operation in the problem.

**IX. PROOF OF THE PROCESS-TOMOGRAPHY THEOREM**

Let \( \Phi^{(N)}, N = 1, 2, \ldots, \) be an exchangeable sequence of quantum operations. \( \Phi^{(N)} \) can be characterized in terms of its action on the elements of a basis of \( \mathcal{L}(\mathcal{H}_D^\otimes N) \) as follows.

\[
\Phi^{(N)} \left( \bigotimes_{i=1}^{N} |l_i^{Q_i}\rangle \langle k_i^{Q_i}| \right) = \sum_{l, m} s^{(N)}_{l, m, k} \bigotimes_{i=1}^{N} |p_i^{Q_i}\rangle \langle m_i^{Q_i}|.
\]

(9.1)
The coefficients $S_{l,j,m,k}^{(N)}$ specify $\Phi^{(N)}$ uniquely. It follows from a construction due to Choi [89] that the $S_{l,j,m,k}^{(N)}$ can be regarded as the matrix elements of a density operator on $D^{2N}$-dimensional Hilbert space $H_{D^{2N}}$. This can be seen as follows. Let

$$|\Psi\rangle = \frac{1}{\sqrt{D}} \sum_{k=1}^{D} |k^{R_1}\rangle |k^{Q_1}\rangle \in H_D \otimes H_D = H_{D^2}$$

be a maximally entangled state in $H_{D^2}$, where the $|k^{R_i}\rangle$ ($k = 1, \ldots, D$) form orthonormal bases for the ancillary systems labelled $R_i$ ($i = 1, \ldots, N$). The corresponding density operator is

$$|\Psi\rangle \langle \Psi| = \frac{1}{D} \sum_{j,k} |j^{R_i}\rangle \langle j^{R_i}| \otimes |j^{Q_i}\rangle \langle j^{Q_i}| \in \mathcal{L}(H_{D^2}).$$

Similarly, we define a map, $J$, from the set of quantum operations on $H_D^{\otimes N}$ to the set of density operators on $H_D^{\otimes N}$ by

$$J(\Phi^{(N)}) \equiv (I^{(N)} \otimes \Phi^{(N)})(|\Psi\rangle \langle \Psi|^{\otimes N})$$

$$= \frac{1}{D^N} (I^{(N)} \otimes \Phi^{(N)})(\sum_{j,k}^{N} \bigotimes_{i=1}^{N} |j_i^{R_i}\rangle \langle k_i^{R_i}| \otimes |j_i^{Q_i}\rangle \langle k_i^{Q_i}|))$$

$$= \frac{1}{D^N} \sum_{l,j,m,k} S_{l,j,m,k}^{(N)} \bigotimes_{i=1}^{N} |j_i^{R_i}\rangle \langle k_i^{R_i}| \otimes |j_i^{Q_i}\rangle \langle k_i^{Q_i}|).$$

In this definition, $I^{(N)}$ denotes the identity operation acting on the ancillary systems $R_1, \ldots, R_N$. The map $J$ is injective, i.e. $J(\Phi_1^{(N)}) = J(\Phi_2^{(N)})$ if and only if $\Phi_1^{(N)} = \Phi_2^{(N)}$.

The first stage of the proof of the de Finetti theorem for operations is to show that the density operators $J(\Phi^{(N)})$, $N = 1, 2, \ldots$, form an exchangeable sequence when regarded as $N$-system states, with $R_i$ and $Q_i$ jointly forming the $i$th system. To do this, we first show that $J(\Phi^{(N)})$ is symmetric, i.e., invariant under an arbitrary permutation $\pi$ of the $N$ systems.

Note that since the density operators $\rho^{(N)}$ actually span the whole vector space $\mathcal{L}(H_D^{\otimes N})$, enforcing Definition 1 above amounts to identifying the linear maps on the left- and right-hand sides of Eqs. (8.4) and (8.5). I.e.,

$$\Phi^{(k)} = \pi \circ \Phi^{(k)} \circ \pi^{-1}$$

and

$$\Phi^{(k)} \circ \text{tr}_{k+1} = \text{tr}_{k+1} \circ \Phi^{(k+1)}$$

Thus in much that we do it suffices to consider the action of these maps on an arbitrary basis state $E^{(N)} = \bigotimes_{i=1}^{N} |j_i^{Q_i}\rangle \langle k_i^{Q_i}|$ for arbitrary $j$, $k$. In particular,

$$\pi\left(\Phi^{(N)}(\pi^{-1}E^{(N)})\right) = \pi\left(\Phi^{(N)}\left(\bigotimes_{i=1}^{N} |x^{Q_i}_{\pi(i)}\rangle \langle k_i^{Q_i}_{\pi(i)}|\right)\right)$$

27
According to the quantum de Finetti theorem for density operators, we can write

\[ \pi \sum_{l,m} S^{(N)}_{l,\pi,l,j,m,k} \bigotimes_{i=1}^{N} |l_i^{Q_i}\rangle \langle m_i^{Q_i}| \]

\[ = \sum_{l,m} S^{(N)}_{l,\pi,l,j,m,k} \bigotimes_{i=1}^{N} |l_i^{Q_i}\rangle \langle m_i^{Q_i}| . \]  

(9.7)

Assuming Eq. (8.4), i.e., symmetry of \( \Phi^{(N)} \), for all \( j \) and \( k \), it follows that

\[ S^{(N)}_{l,\pi,l,j,m,k} = S^{(N)}_{l,j,m,k} \]  

(9.8)

for all \( l,j,m,k \), which, using Eq. (9.4), implies that

\[ \pi(J(\Phi^{(N)})) = J(\Phi^{(N)}) , \]

i.e., symmetry of \( J(\Phi^{(N)}) \).

To prove extendibility of \( J(\Phi^{(N)}) \), we introduce the following notation for partial traces: we denote by \( \text{tr}_R \) the partial trace over the subsystem \( R_{N+1} \), and by \( \text{tr}_Q \) the partial trace over the subsystem \( Q_{N+1} \). In this notation, we need to show that \( \text{tr}_R^N \text{tr}_Q J(\Phi^{(N+1)}) = J(\Phi^{(N)}) \). Using Eqs. (8.5) and (9.4),

\[ \text{tr}_R^N \text{tr}_Q J(\Phi^{(N+1)}) = \text{tr}_R^N \text{tr}_Q^N \]

\[ = \text{tr}_R^N \text{tr}_Q^N \frac{1}{D^{N+1}} \left( I^{(N+1)} \bigotimes \Phi^{(N+1)} \right) \left( \sum_{j,k,k_{N+1}}^{N+1} \bigotimes_{i=1}^{N} |j_i^{R_i}\rangle \langle k_i^{R_i}| \otimes |j_i^{Q_i}\rangle \langle k_i^{Q_i}| \right) \]

\[ = \frac{1}{D^{N+1}} \sum_{j,k,k_{N+1}}^{N+1} \left( \bigotimes_{i=1}^{N} |j_i^{R_i}\rangle \langle k_i^{R_i}| \right) \otimes \text{tr}_Q^N \Phi^{(N+1)} \left( \bigotimes_{i=1}^{N} |j_i^{Q_i}\rangle \langle k_i^{Q_i}| \right) \]

\[ = \frac{1}{D^{N+1}} \left( I^{(N)} \bigotimes \Phi^{(N)} \right) \left( \sum_{j,k,k_{N+1}}^{N+1} \bigotimes_{i=1}^{N} |j_i^{R_i}\rangle \langle k_i^{R_i}| \otimes |j_i^{Q_i}\rangle \langle k_i^{Q_i}| \right) \]

\[ = \frac{1}{D^{N}} \left( I^{(N)} \bigotimes \Phi^{(N)} \right) \left( \sum_{j,k}^{N} \bigotimes_{i=1}^{N} |j_i^{R_i}\rangle \langle k_i^{R_i}| \otimes |j_i^{Q_i}\rangle \langle k_i^{Q_i}| \right) \]

\[ = J(\Phi^{(N)}) . \]  

(9.10)

We have thus shown that \( J(\Phi^{(N)}) \), \( N = 1, 2, \ldots \), form an exchangeable sequence. According to the quantum de Finetti theorem for density operators, we can write

\[ J(\Phi^{(N)}) = \int p(\rho) \rho^{\otimes N} \, d\rho , \]  

(9.11)

where \( p(\rho) \geq 0 \) is unique, and \( \int d\rho \, p(\rho) = 1 \). With the parameterization

\[ \rho = \frac{1}{D} \sum_{l,j,m,k} S^{(l)}_{l,j,m,k} |j_i^{R_i}\rangle \langle k_i^{R_i}| \otimes |l_i^{Q_i}\rangle \langle m_i^{Q_i}| , \]  

(9.12)
Eq. (9.11) takes the form

\[ J(\Phi^{(N)}) = \frac{1}{D^N} \int_{\mathcal{D}} dS \, p(S) \left( \sum_{l_j,m,k} S^{(1)}_{l_j,m,k} |j^R_k\rangle \langle k^Q_l| \otimes |l^Q_l\rangle \langle m^Q_l| \right)^\otimes N \]

\[ = \frac{1}{D^N} \int_{\mathcal{D}} dS \, p(S) \bigotimes_{i=1}^N \sum_{l_j,m_k} S^{(1)}_{l_j,m_k} |j^R_k\rangle \langle k^Q_l| \otimes |l^Q_l\rangle \langle m^Q_l| \]

\[ = \frac{1}{D^N} \sum_{l_j,m_k} \int_{\mathcal{D}} dS \, p(S) \bigotimes_{i=1}^N S^{(1)}_{l_j,m_k} |j^R_k\rangle \langle k^Q_l| \otimes |l^Q_l\rangle \langle m^Q_l| , \quad (9.13) \]

where the integration variable is a vector with $D^4$ components, $S = (S_{1,1,1,1}^{(1)}, \ldots, S_{D,D,D,D}^{(1)})$, and where the integration domain, $\mathcal{D}$, is the set of all $S$ that represent matrix elements of a density operator. The function $p(S)$ is unique, $p(S) \geq 0$, and $\int_{\mathcal{D}} dS \, p(S) = 1$. Notice the slight abuse of notation in the first line of Eq. (9.13), where the superscripts $R$ and $Q$ label the entire sequences of systems $R_1, \ldots, R_N$ and $Q_1, \ldots, Q_N$, respectively.

Comparing Eq. (9.13) with Eq. (9.4), we can express the coefficients $S^{(N)}_{l_j,m_k}$ specifying the quantum operation $\Phi^{(N)}$ [see Eq. (9.1)] in terms of the integral above:

\[ S^{(N)}_{l_j,m_k} = \int_{\mathcal{D}} dS \, p(S) \prod_{i=1}^N S^{(1)}_{l_i,m_i,k_i} . \quad (9.14) \]

Hence, for any $j$ and $k$,

\[ \Phi^{(N)} \left( \bigotimes_{i=1}^N |j^Q_i\rangle \langle k^Q_i| \right) = \sum_{l,m} \int_{\mathcal{D}} dS \, p(S) \left( \prod_{i=1}^N S^{(1)}_{l_i,m_i,k_i} \right) \bigotimes_{i=1}^N |l^Q_i\rangle \langle m^Q_i| \]

\[ = \int_{\mathcal{D}} dS \, p(S) \bigotimes_{i=1}^N S^{(1)}_{l_i,m_i,k_i} |l^Q_i\rangle \langle m^Q_i| . \quad (9.15) \]

The $D^4$ coefficients, $S^{(1)}_{l_j,m_k}$, of the vector $S$ define a single-system map, $\Phi_S$, via

\[ \Phi_S(|j^Q\rangle \langle k^Q|) \equiv \sum_{l,m} S^{(1)}_{l_j,m_k} |l^Q\rangle \langle m^Q| \quad (j,k = 1, \ldots, D) . \quad (9.16) \]

Hence

\[ \Phi^{(N)} \left( \bigotimes_{i=1}^N |j^Q_i\rangle \langle k^Q_i| \right) = \int_{\mathcal{D}} dS \, p(S) \bigotimes_{i=1}^N \Phi_S(|j^Q_i\rangle \langle k^Q_i|) \]

\[ = \int_{\mathcal{D}} dS \, p(S) \Phi_S^{\otimes N} \left( \bigotimes_{i=1}^N |j^Q_i\rangle \langle k^Q_i| \right) . \quad (9.17) \]

Since this equality holds for arbitrary $j$ and $k$, it implies the representation

\[ \Phi^{(N)} = \int_{\mathcal{D}} dS \, p(S) \Phi_S^{\otimes N} . \quad (9.18) \]

For all $S \in \mathcal{D}$, the map $\Phi_S$ is completely positive. This can be seen by considering

\[ J(\Phi_S) = (I \otimes \Phi_S)(|\Psi\rangle \langle \Psi|) = \frac{1}{D} \sum_{l_j,m_k} S^{(1)}_{l_j,m_k} |j^R_k\rangle \langle k^Q_l| \otimes |l^Q_l\rangle \langle m^Q_l| , \]
which, by definition of $\mathcal{D}$, is a density operator and therefore positive. It follows from a theorem by Choi [89] that $\Phi_S$ is completely positive.

To complete the proof, we will now show that $p(S) = 0$ almost everywhere unless $\Phi_S$ is trace-preserving, i.e., a quantum operation. More precisely, we show that if $U \in \mathcal{D}$ is such that $\Phi_U$ is not trace-preserving, then there exists an open ball $B$ containing $U$ such that $p(S) = 0$ in $B \cap \mathcal{D}$.

The essence of the argument can be most easily explained in the special case that the integral (7.1) takes the form of a sum,

$$
\Phi^{(N)} = \sum_i p_i \Phi_i^{\otimes N},
$$

(9.19)

where $p_i > 0$. It follows that

$$
1 = \sum_i p_i (\text{tr}[\Phi_i(\rho)])^N
$$

(9.20)

for all single-system density operators $\rho$. Now assume that the sum extends over some nontrace-preserving operation, which we take to be $\Phi_1$ without loss of generality. This means that

$$
\text{tr}[\Phi_1(\rho)] \neq 1
$$

(9.21)

for some single-system density operator $\rho$. Now either $\text{tr}[\Phi_1(\rho)] < 1$, in which case normalization of $\Phi^{(1)}(\rho)$ implies that $\text{tr}[\Phi_k(\rho)] > 1$ for some $k \neq 1$, or $\text{tr}[\Phi_1(\rho)] > 1$, in which case we set $k = 1$. In both cases

$$
p_k (\text{tr}[\Phi_k(\rho)])^N \to \infty,
$$

(9.22)

which contradicts Eq. (9.20). We have thus shown that the sum (9.19) extends only over trace-preserving operations.

Now let us return to the general case where $\Phi^{(N)}$ is represented by an integral. For $\delta > 0$ and $U \in \mathcal{D}$, we define $B_\delta(U)$ to be the set of all $S$ such that $|S - U| < \delta$, i.e., $B_\delta(U)$ is the open ball of radius $\delta$ centered at $U$. Furthermore, we define $\bar{B}_\delta(U) = B_\delta(U) \cap \mathcal{D}$.

Let $U \in \mathcal{D}$ be such that $\Phi_U$ is not trace-preserving, i.e., there exists a density operator $\rho$ for which $\text{tr}[\Phi_U(\rho)] \neq 1$. We distinguish two cases.

**Case (i):** $\text{tr}[\Phi_U(\rho)] = 1 + \epsilon$, where $\epsilon > 0$. Since $\text{tr}[\Phi_S(\rho)]$ is a linear and therefore continuous function of the vector $S$, there exists $\delta > 0$ such that

$$
|\text{tr}[\Phi_S(\rho)] - \text{tr}[\Phi_U(\rho)]| < \epsilon/2
$$

(9.23)

whenever $S \in B_\delta(U)$. For $S \in B_\delta(U)$,

$$
\text{tr}[\Phi_S(\rho)] > 1 + \epsilon - \epsilon/2 = 1 + \epsilon/2.
$$

(9.24)

Therefore

$$
\text{tr}[\Phi^{(N)}(\rho^{\otimes N})] = \text{tr} \left[ \int_\mathcal{D} dS \ p(S) \ \Phi_S^{\otimes N}(\rho^{\otimes N}) \right]
$$

30
\[ = \int_D dS \ p(S) \ (\text{tr}[\Phi_S(\rho)])^N \]
\[ = \int_{D \setminus \bar{B}_\delta(U)} dS \ p(S) \ (\text{tr}[\Phi_S(\rho)])^N + \int_{\bar{B}_\delta(U)} dS \ p(S) \ (\text{tr}[\Phi_S(\rho)])^N \]
\[ \geq \int_{\bar{B}_\delta(U)} dS \ p(S) \ (\text{tr}[\Phi_S(\rho)])^N \]
\[ > (1 + \epsilon/2)^N \int_{\bar{B}_\delta(U)} dS \ p(S) . \] (9.25)

Unless \( \int_{\bar{B}_\delta(U)} dS \ p(S) = 0 \), there exists \( N \) such that \( \text{tr}[\Phi^{(N)}(\rho^{\otimes N})] > 1 \), which contradicts the assumption that \( \Phi^{(N)} \) is trace-preserving. Hence \( p(S) = 0 \) almost everywhere in \( \bar{B}_\delta(U) \).

**Case (ii):** \( \text{tr}[\Phi_U(\rho)] = 1 - \epsilon \), where \( 0 < \epsilon \leq 1 \). Because of continuity, there exists \( \delta > 0 \) such that
\[ |\text{tr}[\Phi_S(\rho)] - \text{tr}[\Phi_U(\rho)]| < \epsilon/2 \] (9.26)
whenever \( S \in \bar{B}_\delta(U) \). Hence, for \( S \in \bar{B}_\delta(U) \),
\[ \text{tr}[\Phi_S(\rho)] < 1 - \epsilon + \epsilon/2 = 1 - \epsilon/2 \] (9.27)

Now assume that \( \int_{\bar{B}_\delta(U)} dS \ p(S) = \eta > 0 \). Then, letting \( N = 1 \),
\[ 1 = \text{tr}[\Phi^{(1)}(\rho)] = \text{tr} \left[ \int_D dS \ p(S) \ \Phi_S(\rho) \right] \]
\[ = \int_{D \setminus \bar{B}_\delta(U)} dS \ p(S) \ \text{tr}[\Phi_S(\rho)] + \int_{\bar{B}_\delta(U)} dS \ p(S) \ \text{tr}[\Phi_S(\rho)] \]
\[ < \int_{D \setminus \bar{B}_\delta(U)} dS \ p(S) \ \text{tr}[\Phi_S(\rho)] + \eta(1 - \epsilon/2) , \] (9.28)

which implies that
\[ \int_{D \setminus \bar{B}_\delta(U)} dS \ p(S) \ \text{tr}[\Phi_S(\rho)] > 1 - \eta + \eta \epsilon/2 > 1 - \eta . \] (9.29)

Since
\[ \int_{D \setminus \bar{B}_\delta(U)} dS \ p(S) = 1 - \eta , \] (9.30)
it follows that there exist \( \zeta > 0 \) and a point \( V \in D \setminus \bar{B}_\delta(U) \) such that \( \text{tr}[\Phi_V(\rho)] > 1 \) and
\[ \int_{\bar{B}_\delta(V)} dS \ p(S) > 0 \text{ for all } \xi \leq \zeta . \] (9.31)

We are thus back to case (i) above. Repeating the argument of case (i) one can show that this contradicts the assumption that \( \Phi^{(N)} \) is trace preserving for large \( N \). It follows that \( \eta = 0 \), i.e., \( p(S) = 0 \) almost everywhere in \( \bar{B}_\delta(U) \). This concludes the proof of the de Finetti theorem for quantum operations.
What we have proven here is a representation theorem. It shows us when an experimenter is warranted to think of his (prior) known quantum operation assignment as built out of a lack of knowledge of a “true” but unknown one. In that way, the theorem has the same kind of attraction as the previous de Finetti theorem for quantum states.

In particular for a Bayesian interpretation of quantum mechanics, it may be a necessary ingredient for its very consistency. In Refs. [4,6], it has been argued strenuously that quantum operations should be considered of essentially the same physical meaning and status as quantum states themselves: They are Bayesian expressions of an experimenter’s judgment. This could be captured in the slogan “a quantum operation is really a quantum state in disguise.” In other words, the Choi representation theorem [89] is not just a mathematical nicety, but is instead of deep physical significance.10

Therefore, just as an unknown quantum state is an oxymoron in a Bayesian interpretation of quantum mechanics, so should be an unknown quantum operation. In the case of quantum states, the conundrum is solved by the existence of a de Finetti theorem for quantum tomography. In this section we have shown that the conundrum in quantum process tomography can be solved in the same way.

X. CONCLUDING REMARKS

Is there something in nature even when there are no observers or agents about? At the practical level, it would seem hard to deny this, and neither of the authors wish to be viewed as doing so. The world persists without the observer—there is no doubt in either of our minds about that. But then, does that require that two of the most celebrated elements (namely, quantum states and operations) in quantum theory—our best, most all-encompassing scientific theory to date—must be viewed as objective, agent-independent constructs? There is no reason to do so, we say. In fact, we think there is everything to be gained from carefully delineating which part of the structure of quantum theory is about the world and which part is about the agent’s interface with the world.

From this perspective, much—but not all—of quantum mechanics is about disciplined uncertainty accounting, just as is Bayesian probability theory in general. Bernardo and Smith [11] write this of Bayesian theory,

What is the nature and scope of Bayesian Statistics ... ?

Bayesian Statistics offers a rationalist theory of personalistic beliefs in contexts of uncertainty, with the central aim of characterising how an individual should act in order to avoid certain kinds of undesirable behavioural inconsistencies. The theory establishes that expected utility maximization provides the basis for rational decision making and that Bayes’ theorem provides the key to the ways in which beliefs should fit together in the light of changing evidence. The goal, in effect, is to establish rules and procedures for individuals concerned with

10There have been a few pieces of recent technical work that may be useful for shoring up this idea. On the contingency, see Ref. [90].
disciplined uncertainty accounting. The theory is not descriptive, in the sense of claiming to model actual behaviour. Rather, it is prescriptive, in the sense of saying “if you wish to avoid the possibility of these undesirable consequences you must act in the following way.

In fact, one might go further and say of quantum theory, that in those cases where it is not just Bayesian probability theory full stop, it is a theory of stimulation and response [4,22]. The agent, through the process of quantum measurement stimulates the world external to himself. The world, in return, stimulates a response in the agent that is quantified by a change in his beliefs—i.e., by a change from a prior to a posterior quantum state. Somewhere in the structure of those belief changes lies quantum theory’s most direct statement about what we believe of the world as it is without agents.

The present effort, showing how a Bayesian account of quantum states and operations is fully consistent with the laboratory practices of quantum-state and process tomography, is a necessary exercise along the way to pinpointing that direct statement.

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[1] C. M. Caves, C. A. Fuchs, and R. Schack, “Unknown Quantum States: The Quantum de Finetti Representation,” J. Math. Phys. 43, 4537 (2002).
[2] R. Schack, T. A. Brun, and C. M. Caves, “Quantum Bayes Rule,” Phys. Rev. A 64, 014305 (2001).
[3] C. A. Fuchs, R. Schack, and P. F. Scudo, “A de Finetti Representation Theorem for Quantum Process Tomography,” to appear in Phys. Rev. A. See also quant-ph/0307198.
[4] C. A. Fuchs, “Quantum States: What the Hell Are They?,” posted at http://netlib.bell-labs.com/who/cafuchs.
[5] C. M. Caves, C. A. Fuchs and R. Schack, “Conditions for Compatibility of Quantum-State Assignments,” Phys. Rev. A 66, 062111 (2002).
[6] C. A. Fuchs, “Quantum Mechanics as Quantum Information (and only a little more),” quant-ph/0205039.
[7] E. T. Jaynes, Probability Theory: The Logic of Science, edited by G. L. Bretthorst (Cambridge University Press, Cambridge, 2003).
[8] B. de Finetti, “Probabilism,” Erkenntnis 31, 169–223 (1989).
[9] B. de Finetti, Theory of Probability (Wiley, New York, 1990).
[10] L. J. Savage, The Foundations of Statistics (Dover, New York, 1972).
[11] J. M. Bernardo and A. F. M. Smith, Bayesian Theory (Wiley, Chichester, 1994).
[12] R. Jeffrey, *Subjective Probability (The Real Thing)*, posted at [http://www.princeton.edu/~bayesway/](http://www.princeton.edu/~bayesway/).

[13] C. A. Fuchs and A. Peres, “Quantum Theory Needs No ‘Interpretation’,” Phys. Today 53(3), 70 (2000); “Quantum Theory – Interpretation, Formulation, Inspiration: Fuchs and Peres Reply,” 53(9), 14 (2000).

[14] C. M. Caves, C. A. Fuchs and R. Schack, “Quantum Probabilities as Bayesian Probabilities,” Phys. Rev. A 65, 022305 (2002).

[15] C. M. Caves and C. A. Fuchs, “Quantum Information: How Much Information in a State Vector?,” in *The Dilemma of Einstein, Podolsky and Rosen - 60 Years Later (An International Symposium in Honour of Nathan Rosen - Haifa, March 1995)*, edited by A. Mann and M. Revzen, Ann. Israel Phys. Soc. 12, 226 (1996).

[16] T. A. Brun, C. M. Caves, and R. Schack, “Entanglement Purification of Unknown Quantum States,” Phys. Rev. A 63, 042309 (2001).

[17] C. A. Fuchs, “Quantum Mechanics as Quantum Information, Mostly,” J. Mod. Opt. 50, 987–1023 (2003).

[18] R. Schack, “Quantum Theory from Four of Hardy’s Axioms,” Found. Phys. 33, 1461–1468 (2003).

[19] D. M. Appleby, “Facts, Values and Quanta,” *quant-ph/0402015*.

[20] S. J. van Enk and C. A. Fuchs, “Quantum State of an Ideal Propagating Laser Field,” Phys. Rev. Lett. 88, 027902-1–027902-4 (2002); S. J. van Enk and C. A. Fuchs, “Quantum State of a Propagating Laser Field,” Quant. Info. Comp. 2, 151–165 (2002).

[21] C. A. Fuchs, “The Anti-Växjö Interpretation of Quantum Mechanics,” *quant-ph/0204146*.

[22] C. A. Fuchs, *Notes on a Paulian Idea: Foundational, Historical, Anecdotal & Forward-Looking Thoughts on the Quantum*, with foreword by N. David Mermin, (Växjö University Press, Växjö, Sweden, 2003). See also *quant-ph/0105039*.

[23] A. Peres, “What Is a State Vector?,” Am. J. Phys. 52, 644–650 (1984).

[24] N. D. Mermin, “Whose Knowledge?,” in *Quantum (Un)speakables: From Bell to Quantum Information*, edited by R. A. Bertlmann and A. Zeilinger (Springer-Verlag, Berlin, 2002).

[25] T. A. Brun, J. Finkelstein, and N. D. Mermin, “How Much State Assignments Can Differ,” Phys. Rev. A 65, 032315 (2002).

[26] N. D. Mermin, “Compatibility of State Assignments,” J. Math. Phys. 43, 4560–4566 (2002).

[27] R. W. Spekkens, “In Defense of the Epistemic View of Quantum States: A Toy Theory,” *quant-ph/0401052*; L. Hardy, “Quantum Theory From Five Reasonable Axioms,” *quant-ph/0101012*; O. Cohen, “Classical Teleportation of Classical States,” *quant-ph/0310017*; R. Duvenhage, “The Nature of Information in Quantum Mechanics,” Found. Phys. 32, 1399–1417 (2002); P. G. L. Mana, “Why Can States and Measurement Outcomes Be Represented as Vectors?,” *quant-ph/0305117*; A. Duwell, “Quantum Information Does Not Exist,” Stud. Hist. Phil. Mod. Phys. 34(3), 479–499 (2003); I. Pitowski, “Betting on the Outcomes of Measurements: A Bayesian Theory of Quantum Probability,” Stud. Hist. Phil. Mod. Phys. 34(3), 395–414 (2003); R. Clifton, J. Bub, and H. Halvorson, “Characterizing Quantum Theory in Terms of Information-Theoretic Constraints,” Found. Phys. 33, 1561–1591 (2003); J. Bub, “Why the Quantum?,” to appear in Stud. Hist. Phil. Mod. Phys., see also *quant-ph/0402149*; D. M. Appleby, “The Bell-Kochen-Specker Theorem,” *quant-ph/0308114*; G. Bacciagaluppi, “Classical Extensions, Classical Representations and Bayesian Updating in Quantum Mechanics,” *quant-ph/0403055*. 

34
[28] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, “Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels,” Phys. Rev. Lett. 70, 1895 (1993).

[29] D. Boschi, S. Branca, F. De Martini, L. Hardy, and S. Popescu, “Experimental Realization of Teleporting an Unknown Pure Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels,” Phys. Rev. Lett. 80, 1121 (1998); D. Bouwmeester, J.-W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, “Experimental Quantum Teleportation,” Nature 390, 575 (1997); A. Furusawa, J. L. Sorensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik, “Unconditional Quantum Teleportation,” Science 282, 706 (1998).

[30] P. W. Shor, “Scheme for Reducing Decoherence in Quantum Computer Memory,” Phys. Rev. A 52, R2493 (1995).

[31] A. M. Steane, “Error Correcting Codes in Quantum Theory,” Phys. Rev. Lett. 77, 793 (1996).

[32] C. H. Bennett and G. Brassard, “Quantum Cryptography: Public Key Distribution and Coin Tossing,” in Proc. IEEE International Conference on Computers, Systems and Signal Processing (IEEE Press, New York, 1984), p. 175, IEEE, 1984; C. H. Bennett, “Quantum Cryptography Using Any Two Nonorthogonal States,” Phys. Rev. Lett. 68, 3121 (1992).

[33] A. Muller, H. Zbinden, and N. Gisin, “Underwater Quantum Coding,” Nature 378, 449 (1995); W. T. Buttlar, R. J. Hughes, P. G. Kwiat, S. K. Lamoreaux, G. G. Luther, G. L. Morgan, J. E. Nordholt, C. G. Peterson, and C. M. Simmons, “Practical Free-Space Quantum Key Distribution over 1km,” Phys. Rev. Lett. 81, 3283 (1998); R. J. Hughes, G. L. Morgan, and C. G. Peterson, “Practical Quantum Key Distribution over a 48-km Optical Fibre Network,” J. Mod. Opt. 47, 533 (2000).

[34] K. Vogel and H. Risken, “Determination of Quasiprobability Distributions in Terms of Probability Distributions for the Rotated Quadrature Phase,” Phys. Rev. A 40, 2847 (1989).

[35] D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani, “Measurement of the Wigner Distribution and the Density Matrix of a Light Mode Using Optical Homodyne Tomography: Application to Squeezed States and the Vacuum,” Phys. Rev. Lett. 70, 1244 (1993).

[36] U. Leonhardt, “Quantum-State Tomography and Discrete Wigner Function,” Phys. Rev. Lett. 74, 4101 (1995).

[37] Q. A. Turchette et al., “Measurement of Conditional Phase Shifts for Quantum Logic,” Phys. Rev. Lett. 75, 4710 (1995).

[38] I. L. Chuang and M. A. Nielsen, “Prescription for Experimental Determination of the Dynamics of a Quantum Black Box,” J. Mod. Opt. 44, 2455 (1997).

[39] J. F. Poyatos, J. I. Cirac, and P. Zoller, “Complete Characterization of a Quantum Process: the Two-Bit Quantum Gate,” Phys. Rev. Lett. 78, 390 (1997).

[40] H. E. Kyburg, Jr. and H. E. Smokler, eds., Studies in Subjective Probability, Second Edition (Robert E. Krieger Publishing, Huntington, NY, 1980).

[41] For a collection of de Finetti’s original papers and their translations into English, see P. Monari and D. Cocchi, editors, Probabilità e Induzione—Induction and Probability (Biblioteca di Statistica, CLUEB, Bologna, 1993).

[42] R. L. Hudson and G. R. Moody, “Locally Normal Symmetric States and an Analogue of de Finetti’s Theorem,” Z. Wahrschein. verw. Geb. 33, 343 (1976).

[43] R. L. Hudson, “Analogs of de Finetti’s Theorem and Interpretative Problems of Quantum Mechanics,” Found. Phys. 11, 805 (1981).

[44] D. Petz, “A de Finetti-Type Theorem with $m$-Dependent States,” Prob. Th. Rel. Fields 85,
1 (1990); A. Bach, “De Finetti’s Theorem and Bell-Type Correlation Inequalities,” Europhys. Lett. 16, 513 (1991); L. Accardi and Y. G. Lu, “A Continuous Version of de Finetti’s Theorem,” Ann. Prob. 21, 1478–1493 (1993); A. Bach, Indistinguishable Classical Particles (Springer, Berlin, 1997), Lecture Notes in Mathematics, New Series, Vol. 44; R. L. Hudson, “Some Properties of Reduced Density Operators,” Int. J. Quant. Chem. 74, 595 (1999).

[45] C. A. Fuchs and K. Jacobs, “Information Tradeoff Relations for Finite-Strength Quantum Measurements,” Phys. Rev. A 63, 062305 (2001).

[46] D. Heath and W. Sudderth, “De Finetti’s Theorem on Exchangeable Variables,” Am. Stat. 30(4), 188 (1976).

[47] L. Daston, “How Probabilities Came To Be Objective and Subjective,” Hist. Math. 21, 330 (1994).

[48] T. A. Bass, The Newtonian Casino (Penguin Books, London, 1991), previously published as The Eudaemonic Pie: Or why Would Anyone Play Roulette without a Computer in His Shoe (Houghton Mifflin, New York, 1985).

[49] R. N. Giere, “Objective Single-Case Probabilities and the Foundations of Statistics,” in Logic, Methodology and Philosophy of Science IV, edited by P. Suppes, L. Henkin, A. Jojo, and G. C. Moisil (North-Holland, Amsterdam, 1973), p. 467.

[50] J. S. Bell, Speakable and Unspeakable in Quantum Mechanics: Collected Papers on Quantum Philosophy (Cambridge U. Press, Cambridge, 1987).

[51] J. T. Cushing, A. Fine, and S. Goldstein, editors, Bohmian Mechanics and Quantum Theory: An Appraisal (Kluwer, Dordrecht, 1996).

[52] E. T. Jaynes, “Some Applications and Extensions of the de Finetti Representation Theorem,” in Bayesian Inference and Decision Techniques, edited by P. Goel and A. Zellner (Elsevier, Amsterdam, 1986), p. 31.

[53] A. Peres, “Classical Interventions in Quantum Systems. I. The Measuring Process,” Phys. Rev. A 61, 022116 (2000).

[54] V. Scarani, W. Tittel, H. Zbinden, and N. Gisin, “The Speed of Quantum Information and the Preferred Frame: Analysis of Experimental Data,” Phys. Lett. A 276, 1 (2000).

[55] E. Schrödinger, “Probability relations between separated systems,” Proc. Cam. Philo. Soc. 32, 446 (1936).

[56] E. T. Jaynes, “Information Theory and Statistical Mechanics. II,” Phys. Rev. 108, 171 (1957).

[57] L. P. Hughston, R. Jozsa, and W. K. Wootters, “A complete classification of quantum ensembles having a given density matrix,” Phys. Lett. A 183, 14 (1993).

[58] E. Størmer, “Symmetric States of Infinite Tensor Products of C*-algebras,” J. Func. Anal. 3, 48 (1969).

[59] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, “Mixed-State Entanglement and Quantum Error Correction,” Phys. Rev. A 54, 3824 (1996).

[60] N. D. Mermin, “What’s Wrong with These Elements of Reality?,” Phys. Today 43(6), 9 (1990).

[61] M. Koashi, V. Bužek, and N. Imoto, “Entangled Webs: Tight Bound for Symmetric Sharing of Entanglement,” Phys. Rev. A 62, 050302 (2000).

[62] E. B. Davies and J. T. Lewis, “An Operational Approach to Quantum Probability,” Comm. Math. Phys. 17, 239–260 (1970).

[63] K. Kraus, States, Effects, and Operations. Fundamental Notions of Quantum Theory (Springer, Berlin, 1983), Lecture Notes in Physics Vol. 190.

[64] A. Peres, Quantum Theory: Concepts and Methods (Kluwer Academic Publishers, Dordrecht,
The Netherlands, 1993).

[65] E. Merzbacher, *Quantum Mechanics*, 2nd edition (Wiley, New York, 1970); C. Cohen-Tannoudji, *Quantum Mechanics*, 2nd revised enlarged edition (Wiley, New York, 1977).

[66] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, translated by T. A. Beyer (Princeton U. Press, Princeton, 1955); P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th edition (Oxford U. Press, Oxford, 1958).

[67] A. S. Holevo, “Information-Theoretical Aspects of Quantum Measurement,” Prob. Info. Trans. 9, 110 (1973).

[68] E. Prugovecki, “Information-Theoretical Aspects of Quantum Measurements,” Int. J. Theo. Phys. 16, 321 (1977).

[69] G. M. D’Ariano, P. Perinotti, M. F. Sacchi, “Informationally complete measurements and groups representation,” quant-ph/0310013.

[70] J. M. Renes, R. Blume-Kohout, A. J. Scott, C. M. Caves, “Symmetric Informationally Complete Quantum Measurements,” to appear in J. Math. Phys. (2004); see also quant-ph/0310075.

[71] For a small sampling of more recent considerations, see: G. M. D’Ariano, L. Maccone, and M. G. A. Paris, “Quorum of Observables for Universal Quantum Estimation,” J. Phys. A 34, 93 (2001); S. Weigert, “Quantum Time Evolution in Terms of Nonredundant Probabilities,” Phys. Rev. Lett. 84, 802 (2000); V. Bužek, G. Drobný, R. Derka, G. Adam, and H. Wiedemann, “Quantum State Reconstruction from Incomplete Data,” Chaos Sol. Fract. 10, 981 (1999).

[72] This question appears to have been considered much earlier than the current interest: W. Band and J. L. Park, “The Empirical Determination of Quantum States,” Found. Phys. 1, 133 (1970); J. L. Park and W. Band, “A General Method of Empirical State Determination in Quantum Physics: Part I,” Found. Phys. 1, 211 (1971); W. Band and J. L. Park, “A General Method of Empirical State Determination in Quantum Physics: Part II,” Found. Phys. 1, 339 (1971).

[73] A. Fujiwara and H. Nagaoka, “Operational Capacity and Pseudoclassicality of a Quantum Channel,” IEEE Trans. Inf. Theory 44, 1071 (1998).

[74] One can get a feeling for this from the large review article, D. J. Aldous, “Exchangeability and Related Topics,” in École d’Été de Probabilités de Saint-Flour XIII – 1983, edited by P. L. Hennequin, Lecture Notes in Mathematics Vol. 1117 (Springer-Verlag, Berlin, 1985), pp. 1–198.

[75] P. Diaconis, “Finite Forms of de Finetti’s Theorem on Exchangeability,” Synthese 36, 271 (1977); P. Diaconis and D. Freedman, “Finite Exchangeable Sequences,” Ann. Prob. 8, 745–764 (1980).

[76] G. G. Emch, “Is There a Quantum de Finetti Programme?,” in Proceedings of the Conference: Foundations of Probability and Physics – 2, edited A. Khrennikov (Växjö University Press, Växjö, Sweden, 2002), pp. 159–178.

[77] H.-K. Lo, H. F. Chau, and M. Ardehali, “Efficient Quantum Key Distribution Scheme And Proof of Its Unconditional Security,” quant-ph/0011056.

[78] K. Tamaki, M. Koashi, and N. Imoto, “Unconditionally Secure Key Distribution Based on Two Nonorthogonal States,” Phys. Rev. Lett. 90, 167904 (2003).

[79] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, (Cambridge University Press, Cambridge, 2000).

[80] M. A. Nielsen, E. Knill, and R. Laflamme, “Complete Quantum Teleportation Using Nuclear Magnetic Resonance,” Nature 396, 52 (1998).

[81] A. M. Childs, I. L. Chuang, and D. W. Leung, “Realization of quantum process tomography
in NMR,” Phys. Rev. A 64, 012314 (2001).

[82] Y. Nambu et al., “Experimental Investigation of Pulsed Entangled Photons and Photonic Quantum Channels,” in Quantum Optics in Computing and Communications, Proceedings of SPIE Vol. 4917, edited by S. Liu, G. Guo, H.-K. Lo, and N. Imoto (2002), pp. 13–24, e-print quant-ph/0210147.

[83] J. B. Altepeter et al., “Ancilla-assisted Quantum Process Tomography,” Phys. Rev. Lett. 90, 193601 (2003).

[84] F. De Martini, A. Mazzei, M. Ricci, and G. M. D’Ariano, “Exploiting Quantum Parallelism of Entanglement for a Complete Experimental Quantum Characterization of a Single Qubit Device,” Phys. Rev. A 67, 062307 (2003).

[85] D. W. Leung, “Towards Robust Quantum Computation,” PhD thesis, Stanford University (2000), e-print cs.CC/0012017.

[86] D. W. Leung, “Choi’s Proof and Quantum Process Tomography,” J. Math. Phys. 44, 528 (2003).

[87] W. D¨ur and J. I. Cirac, “Non-local Operations: Purification, Storage, Compression, Tomography, and Probabilistic Implementation,” Phys. Rev. A 64, 012317 (2001).

[88] G. M. D’Ariano and P. Lo Presti, “Tomography of Quantum Operations,” Phys. Rev. Lett. 86, 4195 (2001).

[89] M.-D. Choi, “Completely Positive Linear Maps on Complex Matrices,” Lin. Alg. App. 10, 285 (1975).

[90] G. M. D’Ariano and P. Lo Presti, “Imprinting a Complete Information about a Quantum Channel on Its Output State,” quant-ph/0211133; K. Życzkowski and I. Bengtsson, “On duality between Quantum Maps and Quantum States,” quant-ph/0401119; P. Arrighi and C. Patricot, “On Quantum Operations as Quantum States,” quant-ph/0307024; J. Oppenheim and B. Reznik, “A Probabilistic and Information Theoretic Interpretation of Quantum Evolutions,” quant-ph/0312149; A. Fujiwara and P. Algoet, “One-to-one Parameterization of Quantum Channels,” Phys. Rev. A 59, 3290–3294 (1999).