Curved space resolution of singularity of fractional D3-branes on conifold

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Abstract

We construct a supergravity dual to the cascading $SU(N + M) \times SU(N)$ supersymmetric gauge theory (related to fractional D3-branes on conifold according to Klebanov et al) in the case when the 3-space is compactified on $S^3$ and in the phase with unbroken chiral symmetry. The size of $S^3$ serves as an infrared cutoff on the gauge theory dynamics. For a sufficiently large $S^3$ the dual supergravity background is expected to be nonsingular. We demonstrate that this is indeed the case: we find a smooth type IIB supergravity solution using a perturbation theory that is valid when the radius of $S^3$ is large. We consider also the case with the euclidean world-volume being $S^4$ instead of $R \times S^3$, where the supergravity solution is again found to be regular. This “curved space” resolution of the singularity of the fractional D3-branes on conifold solution is analogous to the one in the non-extremal (finite temperature) case discussed in our previous work.

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1. Introduction

Gauge theory – gravity duality relates a gauge theory on the world volume of a large number of D-branes to supergravity backgrounds where the branes are replaced by the corresponding fluxes. In a particular realization of this duality, Klebanov and Witten (KW) \[2\] considered \( N \) regular D3-branes placed at a conical singularity in type IIB string theory. At small \( \text{t Hooft coupling } g_s N \ll 1 \), the system is best described by open strings and realizes \( SU(N) \times SU(N) \mathcal{N} = 1 \) supersymmetric gauge theory with two pairs of chiral multiplets \( A_i, B_j \) and a quartic superpotential at an infrared superconformal fixed point. In the limit of strong \( \text{t Hooft coupling } \) this gauge theory is best described by type IIB supergravity compactified on \( AdS_5 \times T^{1,1} \), with \( N \) units of the RR 5-form flux through the \( T^{1,1} \). If this is a genuine equivalence, then phenomena observed on the gauge theory side should have dual description in string theory on \( AdS_5 \times T^{1,1} \). In particular, any deformation of the gauge theory visible in the large \( N \) limit should have a counterpart in the dual gravitational description, and vice versa.

Certain deformations, trivial on the gravity side, may have highly nontrivial analogs in the gauge theory dynamics. For example, the presence of the \( AdS_5 \) factor in the KW geometry is a reflection of the conformal symmetry of the dual gauge theory. In the Poincare coordinates in \( AdS_5 \), its boundary, and thus the space-time where the gauge theory is formulated, is \( R^{1,3} \). In the global parameterization of \( AdS_5 \) the boundary is \( R \times S^3 \). Such gravitational background should correspond to the the superconformal KW gauge theory defined on \( R \times S^3 \). From the supergravity perspective, going from the Poincare to the global coordinates is a simple local coordinate transformation. However, on the gauge theory side, this “deformation” drastically modifies the dynamics. Defined on a round 3-sphere the gauge theory will have no zero modes; \( \mathbb{E} \) it will have a mass gap in the spectrum of order of inverse radius of \( S^3 \). The modification of the spectrum of the theory substantially modifies its thermodynamics. As in a similar system studied in \( \mathbb{H} \), we expect a thermal phase transition in the \( S^3 \)-compactified KW model, which, on the gravity side, should map into the Hawking-Page phase transition. We would like to emphasize that such a phase transition should occur only for the gauge theory defined on a curved space like \( S^3 \).

\[ \text{1 For reviews and references see, e.g., [1].} \]

\[ \text{2 For the scalars, this follows from their coupling to the scalar curvature, required by the conformal invariance.} \]
It is not known how to translate a generic gauge theory deformation into the dual supergravity description. For the particular deformations for which the dictionary is known, one typically encounters a naked singularity in the corresponding deformed geometry. Consider, for example, wrapping $M$ D5-branes on the collapsed 2-cycle of the conifold, in addition to $N$ D3-branes put at its apex \[3\]. On the gauge theory side this deformation corresponds to changing the gauge group to $SU(N + M) \times SU(N)$ with the same set of chiral multiplets and the superpotential as in the $M = 0$ case. The dual supergravity background found in \[5\] was shown to have a naked singularity. Another example with a naked singularity in the bulk is provided by a large number on NS5-branes wrapping a 2-cycle of the resolved conifold in type IIB string theory \[6\]. The field-theory dual of this system can be interpreted as a compactification (in our language – a deformation) of the little string theory on $S^2$. Yet another, probably the simplest, example of generation of IR singularity is a mass deformation of the $N = 4$ $SU(N)$ SYM theory dual to $AdS_5 \times S^5$ compactification of type IIB string theory. Turning on a mass deformation on the gauge theory side translates into turning on 3-form fluxes on the gravity side \[7,8\]. At the linearized level, the fluxes diverge in the bulk, leading to a naked singularity.

A common feature of the discussed singularities is that they are produced by a well-defined deformation in the dual gauge-theory system. On the gravity side they occur in the bulk (as opposed to the boundary) of the geometry, which, according to the familiar UV/IR correspondence \[10\] expected in gauge–gravity duals, should reflect the IR physics of the gauge theory. If we can resolve the IR singularity induced by the deformation on the gauge theory side, then the translation of the resolution mechanism to the gravity side should cure the naked singularity there as well.

This philosophy is rooted in the belief that there is a genuine equivalence between the two dual descriptions, and it was recently successfully applied, in particular, in refs. \[8,11,6\] and in \[12,13,14,15,16\]. These two groups of papers differ in the type of mechanism used for the singularity resolution. In the former case, the singularity in the deformed gauge theory is resolved by non-perturbative phenomena, intrinsic to gauge theory, namely, the confinement and the chiral symmetry breaking. The resolution of the singularity proposed in the second group of papers is extrinsic to gauge theory: one puts the system at (sufficiently high) finite temperature.

In this paper we propose a more unified perspective on the issue of singularity resolution in gauge–gravity duals, and present a new specific example of the resolution mechanism. Although we shall concentrate on the case of the fractional D3-branes on conifold
geometry (KT background for short), we believe that our approach is generic and should
be applicable to other cases as well.

An overview of the singularity resolution approaches given above underscores the
similarity in all resolution mechanisms. As we have emphasized, in all cases the singularity
is an IR phenomenon when viewed from the gauge theory perspective. Then a natural
way to resolve the singularity is to disallow the gauge theory to access low-energy states.
This can be achieved as a result of a dynamical gauge-theory effect (generation of a mass
gap in the spectrum due to confinement as in [8,9,10]) or by introducing an IR cutoff
"by hand" (turning on a finite temperature as in [12,13,14,15,16]). It is clear from this
perspective that there should be many other ways to resolve the singularity: all one has
to do is to introduce an IR cutoff on the field theory side and to understand what that
cutoff translates into on the gravity side of the duality. The corresponding supergravity
background should contain an extra scale (the deformed conifold scale in [11], or the non-
extremality parameter in [12,13,14,15,16], or the curvature of the “longitudinal” space in the
examples considered below).

One possibility to introduce an IR cutoff is by “compactifying” the space on which
the gauge theory is defined. As a specific realization of this proposal we shall consider the
resolution of the singularity of the KT background by defining the dual gauge theory on
$R \times S^3$ instead of 4-d Minkowski space. The space compactification should provide an IR
cutoff, and so for sufficiently large radius of the 3-sphere we should expect the restoration of
chiral symmetry in the dual field theory, and thus a smooth dual supergravity background.

It should be emphasized that not all of space compactifications (that provide an IR
cutoff) may resolve the singularity of the supergravity dual. For example, compactifying
the $SU(N+M) \times SU(N)$ gauge theory on a 3-torus $T^3$ will not resolve the singularity.
We expect that a “good” (singularity-resolving) compactification is the one that lifts the zero
modes of all of the gauge-theory fields, i.e. gauge bosons, fermions and scalars. Let the
space on which the gauge theory is defined be compactified on a $d$-dimensional manifold

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3 The proposal to use a finite temperature as an IR cutoff to cloak naked singularities in five
dimensional gravity coupled to scalars was put forward in [17].

4 The singularity of the KT background is related to the chiral symmetry breaking in the
dual field theory. This symmetry (reflected in the $U(1)$ fiber symmetry of $T^{1,1}$) will be present in
the generalized KT background we will construct.

5 The gravity dual will be the original KT solution with the spatial coordinates of the
D3-brane world-volume periodically identified.
\( M_d \). There will not be massless gauge-boson modes, provided the first Betty number of \( M_d \) vanishes. The scalars will not have zero modes provided they are coupled to a non-zero scalar curvature of \( M_d \). Thus the second condition is a nonvanishing Ricci scalar of \( M_d \). One is also to make sure that there is no fermionic zero modes. While the \( S^3 \) compactification satisfies these conditions, the \( T^3 \) one fails to do that.

One may also consider a Euclidean version and define the gauge theory on a curved 4-d space-time, e.g., \( S^4 \) or \( K3 \). Then \( S^4 \) will lead to a resolution of the singularity (as we shall see below), but \( K3 \) will not, since it has \( R_{mn} = 0 \) and thus does not lift the zero modes of the scalars.

Let us comment also on a peculiar relation between the space on which gauge theory is defined and its counterpart in the dual supergravity description. On the gauge theory side we think of space-time being a manifold of fixed size. In the context of gauge theory – gravity duality, the space-time where the gauge theory “lives” should be identified with a boundary submanifold of the dual 10-d supergravity space-time. The size of this submanifold may obviously dependent on other (transverse) directions. One example is \( AdS_5 \times S^5 \) in global parametrization of the \( AdS_5 \), where the size of the spatial part of the boundary \( S^3 \) changes with the radial coordinate of \( AdS_5 \). Another example is provided by the duality discussed in \([6]\), where the gauge theory arises from compactification of the little string theory on \( S^2 \) of fixed size. The size of the corresponding 2-sphere in the dual supergravity background changes logarithmically with the radial coordinate \([6]\) .

The rest of the paper is organized as follows. In section 2 we discuss the generalizations of the KT ansatz for the supergravity background dual to the cascading gauge theory compactified on \( R \times S^3 \) and \( S^4 \). Following the approach of \([3,13,14]\), in section 3 we derive the corresponding 1-d effective action that generates the equations for the radial evolution of the functions parametrizing the background metric and matter fields. We then discuss the simplest supersymmetric solutions of these equations realizing the extremal fractional D3-brane KT background \([3]\) and the \( AdS_5 \times T^{1,1} \) gravity dual to the KW gauge theory \([2]\) compactified on \( R \times S^3 \) or \( S^4 \).

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\( ^6 \) Related observations can be made in the case of other, more familiar, deformations of gauge theory. In \([8]\) the authors studied the duality in the context of the mass deformed \( \mathcal{N} = 4 \) \( SU(N) \) SYM theory. There, a constant mass deformation on the gauge theory side translated into a variable 3-form flux in the gravity dual.
We then consider the deformations of $M_4 = R \times S^3$ and $M_4 = S^4$ compactifications of the KW model caused by switching on $P \neq 0$ units of fractional 3-brane flux. As in the closely related work \cite{14} on the non-extremal generalization of the KT background, being unable to solve the resulting equations exactly, we resort to a perturbation theory valid in the regime when the effective D3-brane charge (or the 5-form flux) $K_\star$ is much larger than the fractional 3-brane charge, $K_\star \gg P^2$. Physically, this approximation amounts to introducing an IR cutoff in the dual gauge theory at an energy scale high enough to mask the low energy chiral symmetry breaking, which is responsible for the generation of the KT singularity \cite{11}.

In section 4 we construct a smooth supergravity solution interpolating between the $S^4$ compactification of the KW model in the IR and the KT model in the UV. In section 5 we address the same problem in a technically more challenging case of the $R \times S^3$ compactification of the KT model. Both examples of regular compactifications of the KT model provide support to the general idea of resolving naked singularities in the bulk of gravitational duals to gauge theories by an IR cutoff produces by a “boundary” space compactification.

We conclude in section 6 with comments on constructing a gravitational dual to mass-deformed conformally compactified $\mathcal{N} = 4$ supersymmetric Yang-Mills theories.

2. $R \times S^3$ and $S^4$ generalizations of the KT background

Our aim will be to explore the generalization of the KT solution \cite{3} for fractional D3-brane on conifold to the case when the constant radial distance slices of the “parallel” part of the metric have geometry $R \times S^3$ or $S^4$ (we shall consider the case of Euclidean signature). We shall argue that the corresponding solutions are regular (for large enough D3-brane charge compared to fractional 3-brane charge).

We shall start with the same ansatz as in \cite{3,13} and simply replace 1+3 “longitudinal” directions by $R \times S^3$ or by $S^4$. The treatment of the two cases will be very similar, and we will discuss them in parallel. There will be direct analogy with the non-extremal (finite temperature) case considered in \cite{12,13,14}.

As in \cite{3} we will impose the requirement that the background has abelian symmetry associated with the $U(1)$ fiber of $T^{1,1}$ as we will consider a phase where chiral symmetry breaking

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\footnote{This is also the region of validity of the non-extremal deformation, i.e. of the finite temperature resolution of the KT singularity due to the chiral symmetry restoration studied in \cite{14}.}
is restored. In the case of $R \times S^3$ our general ansatz for a 10-d (Euclidean-signature) Einstein-frame metric will involve 4 functions $x, y, z$ and $w$ of radial coordinate $u$

$$ds_{10E}^2 = e^{2z}(dM_4)^2 + e^{-2z}[e^{10y}du^2 + e^{2y}(dM_5)^2] ,$$  \hspace{1cm} (2.1)$$

$$(dM_4)^2 = e^{-6x}dX_0^2 + e^{2x}(dS^3)^2 ,$$  \hspace{1cm} (2.2)$$

$$(dS^3)^2 = d\alpha^2 + \sin^2 \alpha (d\beta^2 + \sin^2 \beta d\gamma^2) .$$  \hspace{1cm} (2.3)$$

Here the 3-sphere replaces the 3 “flat” longitudinal directions of the 3-brane and $M_5$ is a deformation of the $T^{1,1}$ metric

$$(dM_5)^2 = e^{-8w}e_\psi^2 + e^{2w} (e_{\theta_1}^2 + e_{\phi_1}^2 + e_{\theta_2}^2 + e_{\phi_2}^2) ,$$  \hspace{1cm} (2.4)$$

$$e_\psi = \frac{1}{3} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2) , \hspace{0.5cm} e_{\theta_i} = \frac{1}{\sqrt{6}} d\theta_i , \hspace{0.5cm} e_{\phi_i} = \frac{1}{\sqrt{6}} \sin \theta_i d\phi_i .$$

We choose the radius of $S^3$ to be 1 as it can be absorbed into a shift of $x$ (and a rescaling of $X_0$).

In the case of $S^3$ replaced by $R^3$ (i.e. in the $x \to x + x_0, \ x_0 \to \infty$ limit) this becomes the ansatz of [13], where the non-extremal D3-brane case was considered. The extremal D3-brane on the standard conifold and the more general fractional D3-brane KT solution have $x = w = 0$. While in [13] a non-constant function $x(u)(= au)$ was reflecting the non-extremality of the background, in the present $R \times S^3$ case it will be non-trivial as a consequence of the curvature of $S^3$.

The ansatz in the $S^4$ case is the same as (2.1) but with $(dM_4)^2$ given by

$$(dM_4)^2 = (dS^4)^2 = d\alpha^2 + \sin^2 \alpha (d\beta^2 + \sin^2 \beta (d\gamma^2 + \sin^2 \gamma d\delta^2)) ,$$  \hspace{1cm} (2.5)$$

where the radius of $S^4$ is again chosen to be 1. Here there is no function $x$, i.e. the number of functions in the metric is the same as in the extremal case (however, in contrast to the standard KT case, here $w$ will, in general, be non-trivial).

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8 This metric can be brought into a more familiar form $ds_{10E}^2 = h^{-1/2}(r)(dM_4)^2 + \ h^{1/2}(r)[g^{-1}(r)dr^2 + r^2 ds_5^2]$ , where $h = e^{-4z-4x}, \ r = e^{y+x+w}, \ g = e^{-8x}, \ e^{10y+2x}du^2 = g^{-1}(r)dr^2$. When $w = 0$ and $e^{4y} = r^4 = \frac{1}{4\alpha}$, the transverse 6-d space is the standard conifold with $M_5 = T^{1,1}$. Small $u$ thus corresponds to large distances in 5-d and vice versa. In the $AdS_5$ region large $u$ is near the origin of $AdS_5$ space, while $u = 0$ is its boundary.
As for the matter fields, we shall assume that the dilaton $\Phi$ may depend on $u$, and our ansatz for the $p$-form fields (the same in the $R \times S^3$ and $S^4$ cases) will be exactly as in the extremal KT case \cite{5} and in \cite{13}:

\[
F_3 = Pe_\psi \wedge (e_{\theta_1} \wedge e_{\phi_1} - e_{\theta_2} \wedge e_{\phi_2}) , \quad B_2 = f(u)(e_{\theta_1} \wedge e_{\phi_1} - e_{\theta_2} \wedge e_{\phi_2}) , \quad (2.6)
\]

\[
F_5 = \mathcal{F} + \ast \mathcal{F} , \quad \mathcal{F} = K(u)e_\psi \wedge e_{\theta_1} \wedge e_{\phi_1} \wedge e_{\theta_2} \wedge e_{\phi_2} , \quad K(u) = Q + 2P f(u) , \quad (2.7)
\]

where, as in \cite{6}, the expression for $K$ follows from the Bianchi identity for the 5-form. The constants $Q$ and $P$ are proportional to the numbers $N$ and $M$ of standard and fractional D3-branes; their precise normalizations (see \cite{18}) will not be important here.

In what follows, we shall first derive the corresponding system of type IIB supergravity equations of motion describing the radial evolution of the six unknown functions of $u$: $x, y, z, w, K, \Phi$ (five functions $y, z, w, K, \Phi$ in the $S^4$ case).

We shall then discuss its solutions aiming to show that there exists a smooth interpolation (in radial coordinate only) between (i) a non-singular short-distance region where the 10-d background is approximately $AdS_5 \times T^{1,1}$ written in the coordinates where the $u =$ const slice is $R \times S^3$ or $S^4$, and (ii) a long-distance region where the 10-d background approaches the KT solution. We shall start with the short-distance ($u = \infty$ or $\rho = 0$) region, i.e. $AdS_5 \times T^{1,1}$ space (with the radius determined by the effective charge $K_*$) and show that by doing perturbation theory in the small parameter $\frac{P^2}{K_*} \ll 1$ one can match it onto the KT asymptotics at large distances ($u \to 0$ or $\rho \to \infty$). The crucial point will be that $O(\frac{P^2}{K_*})$ perturbations will be regular at small distances. This will be exactly parallel to the discussion of the non-extremal case in \cite{14} where the starting point in the IR was a regular non-extremal D3-brane (black hole in $AdS_5$) solution with large (above critical) Hawking temperature.

We shall assume, for notational simplicity, that the value of the radius $L$ of the short-distance limit space $AdS_5 \times T^{1,1}$ is 1. That corresponds to the choice of the normalizations where the effective 3-brane charge is ($g_s = 1$)

\[
K_* = 4 , \quad \text{i.e.} \quad L = 1 . \quad (2.8)
\]

9 The reason why the form of the ansatz is the same is that it is formulated in terms of the transverse space geometry only (the “parallel” or “electric” part of $F_5$ is then fixed by the selfduality condition).
In discussing the $O(P^2)$ deformation it will be useful to compare the three possible regular starting points – the $AdS_5 \times T^{1,1}$ space in the three different parametrizations, where the constant radial slice is $R^4$, $R \times S^3$ and $S^4$ respectively:

\begin{align}
    ds_{10}^2 &= e^{2\rho}(dR^4)^2 + d\rho^2 + (dT^{1,1})^2, \quad -\infty < \rho < \infty, \quad (2.9) \\
    ds_{10}^2 &= \cosh^2 \rho \, dX_0^2 + \sinh^2 \rho \, (dS^3)^2 + d\rho^2 + (dT^{1,1})^2, \quad 0 < \rho < \infty, \quad (2.10) \\
    ds_{10}^2 &= \sinh^2 \rho \, (dS^4)^2 + d\rho^2 + (dT^{1,1})^2, \quad 0 < \rho < \infty, \quad (2.11)
\end{align}

While these three spaces (with the Euclidean $AdS_5$ metric written in Poincare, global and “hyperboloid” parametrizations) are related locally by the coordinate transformations, these involve changing all five of the coordinates, i.e. the radial, but also the angular ones.

It is the assumption that the 10-d deformation $\mathcal{L}_{11}$ of the factorized metrics $(2.9),(2.10)$ and $(2.11)$ under switching on the 3-form flux $\mathcal{L}_{11}$ depends only on the corresponding radial coordinate $\rho$ (which is different in the three cases) that makes the resulting solutions different. Since the 10-d metric is no longer a direct product, different choices of the radial coordinate (or of the metric on the $\rho = \text{const}$ slice) lead to inequivalent 10-d equations and thus inequivalent D3-brane solutions no longer related by a local coordinate transformation beyond the short-distance $AdS_5 \times T^{1,1}$ limit.

In particular, while the Poincare patch metric $(2.9)$ leads to the KT solution which is singular in the IR (for $\rho \to -\infty$), that singularity is resolved in the $R \times S^3$ and $S^4$ where the $\rho \to 0$ limit is described by $(2.10)$ and $(2.11)$, respectively.

3. Action for equations of radial evolution and special cases

As in $\mathcal{L}_{11}$ we shall first derive the 1-d effective action that generates the equations for the radial evolution of unknown functions. Computing the scalar curvature for the metric $(2.4)$ we find that in the $R \times S^3$ case $(2.4),(2.5)$ $\int d^{10}x \sqrt{G}R$ is proportional to $I_{gr} = \int du \, L_{gr}$, where

\begin{equation}
    L_{gr}(R \times S^3) = 5y'^2 - 3x'^2 - 2z'^2 - 5w'^2 + \frac{3}{2}e^{-2x+10y-4z} + e^{8y}(6e^{-2w} - e^{-12w}). \quad (3.1)
\end{equation}

The corresponding expression in the $S^4$ case $(2.7),(2.9)$ is

\begin{equation}
    L_{gr}(S^4) = 5y'^2 - 2z'^2 - 5w'^2 + 3e^{10y-4z} + e^{8y}(6e^{-2w} - e^{-12w}). \quad (3.2)
\end{equation}
Note that it can be formally obtained from (3.1) by setting\textsuperscript{10}
\[ x = \text{const} , \quad e^{-2x} = 2 . \]  

The new term in $L_{gr}$ (3.1) compared to the (non)extremal $R \times R^3$ case in [5,13] is the first potential term that reflects the curvature of $R \to S^3$ (or $S^4$).\textsuperscript{11}

The matter part $L_m$ of the effective type IIB Lagrangian (contributions of the dilaton, 3-form fields and the 5-form following from (2.6),(2.7)) is essentially the same as in the KT case [5] and [13] since $L_m$ does not depend on the function $x$ and the structure of $M_4$. As a result, $L = L_{gr} + L_m = T - V$, where
\begin{align}
  T &= 5y'^2 - 3x'^2 - 2z'^2 - 5w'^2 - \frac{1}{8} \Phi'^2 - \frac{1}{4} e^{-\Phi+4z-4y-4w} \frac{K'^2}{4P^2} , \\
  V &= -\frac{3}{2} e^{-2x+10y-4z} - e^{8y} (6e^{-2w} - e^{-12w}) + \frac{1}{4} e^{\Phi+4z+4y+4w} P^2 + \frac{1}{8} e^{8z} K^2 .
\end{align}

The equations of motion that follow from $L$ should be supplemented by the “zero-energy” constraint $T + V = 0$. As in [14], we will use the 5-form flux function $K(u) = Q + 2P f(u)$ instead of $f(u)$ in (2.4).

The new potential term $e^{-2x+10y-4z}$ in (3.3) associated with the scalar curvature of the 4-space, in general, leads to breaking of supersymmetry and thus to a non-trivial modification of the extremal KT solution. In the non-extremal case discussed in [13] this term was absent and the equation for $x$ was simply giving $x = au$, with $a$ being the non-extremality parameter. In the $R \times S^3$ case with the function $x$ is no longer a “modulus” – it cannot be easily decoupled. In the $S^4$ case the new potential term in (3.2) provides a non-trivial mixing between the $y, z$.

Let us first consider some special solutions of the equations following from this action.

\textsuperscript{10} The coefficient 2 accounts for the ratio of the values of the Ricci scalars of $S^3$ and $S^4$.

\textsuperscript{11} Its scaling under shifts of $x, y, z$ follows directly from the structure of the metric (2.1). Shifting $x$ or $z$ to restore explicitly the inverse radius parameter of $S^3$ or $S^4$ as its coefficient, one may then recover the $R \times R^3$ case in the limit when this parameter goes to zero. As in [13], in the absence of matter terms $w = 0$ is a consistent fixed point of the equations of motion, corresponding to $M_5$ in (2.4) replaced by the standard $T^{1,1}$. Note also that a special solution of the equations $R_{mn} = 0$ that follow from this gravitational action is $R$ times a cone over $S^3 \times T^{1,1}$ or a cone over $S^4 \times T^{1,1}$.
3.1. Flat 4-space case: extremal KT solution

Let us first recall the solution in the \( M_4 = R \times R^3 \) case, (corresponding formally to the “infinite radius” limit \( x = \infty \) of (3.3)). The crucial observation made in [5] is that in the absence of the \( e^{-2x+10y-4z} \) term the Lagrangian (3.4),(3.5) admits a superpotential, i.e. \( L = g_{ij}(\phi'^i + g^{ik} \partial_k W)(\phi'^j + g^{jk} \partial_k W) - 2W' \). As a result, there exists a special BPS solution satisfying \( \phi'^i + g^{ik} \partial_k W = 0 \) and thus also the zero-energy constraint. In the present case [5,19]

\[
W = \frac{1}{4} e^{4y}(3e^{4w} + 2e^{-6w}) - \frac{1}{8} e^{4z}K , \tag{3.6}
\]

and the corresponding system of 1-st order equations is

\[
x' = 0 , \quad \frac{1}{5}e^{4y}(3e^{4w} + 2e^{-6w}) = 0 , \quad w' - \frac{3}{5}e^{4y}(e^{4w} - e^{-6w}) = 0 , \tag{3.7}
\]

\[
\Phi' = 0 , \quad K' + 2P^2 e^{\Phi+4y+4w} = 0 , \quad z' + \frac{1}{4} e^{4z}K = 0 . \tag{3.8}
\]

Choosing the special solution \( w = 0 \) we then find [5,12]

\[
x = w = \Phi = 0 , \quad e^{-4y} = 4u , \quad K = K_0 - \frac{P^2}{2} \ln u , \tag{3.9}
\]

\[
e^{-4z} = h = h_0 + (K_0 + \frac{P^2}{2})u - \frac{P^2}{2} u \ln u , \tag{3.10}
\]

where \( h_0 = 0 \) if we omit the standard asymptotically flat region (as we shall assume below).

3.2. \( K = \text{const} \) (\( P = 0 \)) case: \( \text{AdS}_5 \times T^{1,1 \text{r}} \) with \( M_4 = R \times S^3 \) or \( M_4 = S^4 \)

Setting first fractional 3-brane flux to zero \( P = 0 \) (i.e. \( K = K_* = \text{const} \) and also \( \Phi = f = 0 \)) we get from (3.4),(3.5):

\[
L = 5y'^2 - 3x'^2 - 2z'^2 - 5w'^2 + \frac{3}{2} e^{-2x+10y-4z} + e^{8y}(6e^{-2w} - e^{-12w}) - \frac{1}{8} K_*^2 e^{8z} . \tag{3.11}
\]

Here the first term in the potential is the contribution of the curvature of \( S^3 \), the second is the curvature of the \((w\text{-deformed}) T^{1,1 \text{r}} \) space, and the last one is the negative 5-d cosmological constant originating from the 5-form flux contribution. Shifting \( z \) and \( x \) we may set the D3-brane charge parameter \( K_* \) to some fixed value, e.g., \( K_* = 4 \) as in (2.8).

Since \( w = 0 \) is an obvious special solution, in this case we get

\[
L = 5y'^2 - 3x'^2 - 2z'^2 + \frac{3}{2} e^{-2x+10y-4z} + 5e^{8y} - 2e^{8z} . \tag{3.12}
\]

\[\text{12} \quad u = \frac{1}{4r^4} \text{ where } r \text{ is the standard radial coordinate in D3-brane solution.}\]
In the standard “flat” D3-brane case, i.e. in the absence of the $e^{-2x+10y-4z}$ term, this system is easily integrated giving us extremal ($x = 0$) or non-extremal ($x = au$) D3-brane on conifold solution. The case of

$$y = z \tag{3.13}$$

then corresponds to the $AdS_5 \times T^{1,1}$ limit (2.9) where the $M_5$ part of the metric (2.1) factorizes.

In general, while it is not clear how to solve the system that follows from (3.12) analytically, it is easy to see that the 5+5 factorized case (3.13) is still a special solution. Here we end up with

$$L = 3(y'^2 - x'^2 + \frac{1}{2} e^{-2x+6y} + e^{8y}) \tag{3.14}$$

The corresponding equations have the following solution

$$e^{4x} = \tanh \rho \ , \quad e^{4y} = \sinh \rho \ \cosh \rho \ , \quad d\rho = -e^{4y} du \tag{3.15}$$

where we have set the only integration constant (the origin of $\rho$) to zero.\footnote{Note that while for $q \neq 0$ or $y \neq z$ (3.12) does not admit a superpotential, it exists for (3.14) (cf. (3.6)) $W = \frac{3}{4}(\frac{1}{2} e^{-2x+2y} + e^{4y})$.}

In the $S^4$ case (3.2) setting $K = K_* = \text{const}$ gives (e.g., using (3.3) in (3.11))

$$L = 5y'^2 - 2z'^2 - 5w'^2 + 3e^{10y-4z} + e^{8y}(6e^{-2w} - e^{-12w}) - \frac{1}{8} K^2_* e^{8z} \tag{3.16}$$

or, for $w = 0$, and $K_* = 4$,

$$L = 5y'^2 - 2z'^2 + 3e^{10y-4z} + 5e^{8y} - 2e^{8z} \tag{3.17}$$

The meaning of the three terms in the potential is again the curvature of $S^4$, the curvature of $T^{1,1}$ and negative cosmological term produced by the $F_5$ flux. Equivalently,

$$L = 3n'^2 - 30m'^2 + 3e^{6n} + e^{8n}(5e^{-16m} - 2e^{-40m}) , \quad z = n - 5m , \quad y = n - 2m . \tag{3.18}$$

In general, this system does not admit a superpotential (wrapping the Euclidean 3-brane world-volume over $S^4$ breaks supersymmetry). The special easily solvable case is the fixed

\footnote{Here $u = \ln \tanh \rho + \sinh^{-2} \rho$, so that $u(\rho \to \infty) \to 2e^{-2\rho}$ and $u(\rho \to 0) \to \frac{1}{2\rho^2}$.}
point \( m = 0 \), i.e. \( y = z \) (3.13) or the case of factorization \( M_{10} \rightarrow M_5 \times T^{1,1} \). Here we are left with just with one function \( y \) satisfying the zero-energy constraint (there is thus an obvious superpotential, cf. (3.14))

\[
y'' = e^{6y} + e^{8y}, \tag{3.19}
\]

so that

\[
z = y = \ln \sinh \rho, \quad d\rho = -e^{4y} du, \tag{3.20}
\]

where we again set \( \rho_0 = 0 \). Then the metric becomes equal to (2.11), with the \( AdS_5 \) part written in the parametrization where the topology of the radial slices is \( S^4 \).

It is useful to stress again that the three \( AdS_5 \times T^{1,1} \) metrics (2.9),(2.10), and (2.11), though related locally by the coordinate transformations, are obtained from inequivalent 1-d actions. This reflects inequivalence of the corresponding radial coordinates, and leads also to very different properties of the corresponding fractional brane \((P \neq 0)\) deformations of these backgrounds discussed below.

4. Strategy of finding \( P \neq 0 \) solution and \( S^4 \) case

Being unable to solve the system of equations that follows from (3.4),(3.5) in general, we need to resort to perturbation theory similar to the one used in [14]. Our aim will be to show that starting from the asymptotic KT geometry at large \( \rho \) one may smoothly interpolate to a regular \( AdS_5 \times T^{1,1} \) geometry (with large enough effective charge \( K_* \gg P^2 \)) at small \( \rho \) with the metric having a non-trivial scalar curvature of \( \rho = \text{const} \) slices, i.e. (2.10) or (2.11).

Following the same strategy as used in in [14] in finite temperature case we shall start with the \( AdS_5 \times T^{1,1} \) background (2.11) expected to be a good approximation in the small \( \rho \) region if \( K_* = K(\rho \rightarrow 0) \) is sufficiently large, and solve the supergravity equations perturbatively to leading order in \( \frac{P^2}{K_*} \ll 1 \). We shall see that the leading deformation of the \( AdS_5 \times T^{1,1} \) background will be regular at small \( \rho \).

If one starts instead with the “flat” \( AdS_5 \times T^{1,1} \) metric (2.9), such perturbative expansion reproduces the exact form of the KT solution already at the first order of perturbation theory in \( \frac{P^2}{K_*} \) (note that the correction terms in (3.9),(3.10) are linear in \( P^2 \)). Here, however, the perturbation (and the exact solution) is singular in the short-distance region

\[
u = \frac{\cosh \rho (1-2 \sinh^2 \rho)}{3 \sinh^3 \rho} + \frac{2}{3}, \quad \text{so that } u(\rho \rightarrow 0) \rightarrow \frac{1}{3 \rho^3}, \quad u(\rho \rightarrow \infty) \rightarrow 4 e^{-4\rho}.
\]

\[15\]
(which in the case of (2.9) corresponds to $\rho \to -\infty$). As was explained in [14], introducing non-extremality (i.e. replacing $AdS_5$ by the black hole background with sufficiently high temperature) resolves the singularity, making the perturbative solution regular. We shall see that a similar resolution is provided by the curvature of the “parallel” 3-brane directions.

As was already mentioned above, to simplify the presentation we shall assume that the value of the 5-form flux at $\rho \to 0$ is fixed as in (2.8), so that the radius of $AdS_5$ is 1 as in (2.9)–(2.11). The expansion parameter is then simply $P^2$.

The full system of 2-nd order equations following from (3.4), (3.5) in the $R \times S^3$ case is similar to the one in [14]

$$ x'' - \frac{1}{2} e^{-2x-4z+10y} = 0 , \quad (4.1) $$

$$ 10y'' - 8e^{8y}(6e^{-2w} - e^{-12w}) - 30x'' + \Phi'' = 0 , \quad (4.2) $$

$$ 10w'' - 12e^{8y}(e^{-2w} - e^{-12w}) - \Phi'' = 0 , \quad (4.3) $$

$$ \Phi'' + e^{-\Phi+4z-4y-4w}(\frac{K'^2}{4P^2} - e^{2\Phi+8y+8w}P^2) = 0 , \quad (4.4) $$

$$ 4z'' - K^2 e^{8z} - e^{-\Phi+4z-4y-4w}(\frac{K'^2}{4P^2} + e^{2\Phi+8y+8w}P^2) - 12x'' = 0 , \quad (4.5) $$

$$ (e^{-\Phi+4z-4y-4w}K')' - 2P^2 K e^{8z} = 0 . \quad (4.6) $$

The integration constants are subject to the zero-energy constraint $T + V = 0$. It is easy to see that because of the extra $S^3$-curvature term $e^{-2x-4z+10y}$ in the potential this system does not (in contrast to the non-extremal case [12]) admit a special solution with constant dilaton and self-dual 3-forms. In [13] we needed to relax this 1-st order system to get a non-singular non-extremal solution. Here we do not have a choice – all functions (in particular, $w$) are to be non-trivial in general.

In the $S^4$ case we get instead\(^\text{13}\)

$$ 10y'' - 8e^{8y}(6e^{-2w} - e^{-12w}) - 30e^{10y-4z} + \Phi'' = 0 , \quad (4.7) $$

$$ 10w'' - 12e^{8y}(e^{-2w} - e^{-12w}) - \Phi'' = 0 , \quad (4.8) $$

\(^{16}\) If we set $K'^2 - 4P^3 e^{2\Phi+8y+8w} = 0$, i.e. $K' = -2P^2 e^{\Phi+4y+4w}$, then (4.6) implies that $z$ should be subject to the first-order equation in (3.8), but this is not consistent with (4.3) unless $x'' = 0$.

\(^{17}\) This system is related to the $R \times S^3$ one by setting $e^{-2x} = 2$ in (4.1)–(4.6) after using (1.1) in (4.2), (4.3).
\begin{equation}
\Phi'' + e^{-\Phi+4z-4y-4w} \left( \frac{K'^2}{4P^2} - e^{2\Phi+8y+8w} P^2 \right) = 0 , \tag{4.9}
\end{equation}

\begin{equation}
4z'' - K^2 e^{8z} - e^{-\Phi+4z-4y-4w} \left( \frac{K'^2}{4P^2} + e^{2\Phi+8y+8w} P^2 \right) - 12 e^{10y-4z} = 0 , \tag{4.10}
\end{equation}

\begin{equation}
\left( e^{-\Phi+4z-4y-4w} K' \right)' - 2P^2 K e^{8z} = 0 , \tag{4.11}
\end{equation}

with the zero-energy constraint

\begin{align*}
5y'^2 - 2z'^2 - 5w'^2 - \frac{1}{8} \Phi'^2 - \frac{1}{4} e^{-\Phi+4z-4y-4w} \frac{K'^2}{4P^2} \\
- 3e^{10y-4z} - e^{8y} (6e^{-2w} - e^{-12w}) + \frac{1}{4} e^{4z+4y+4w} P^2 + \frac{1}{8} e^{8z} K^2 = 0 . \tag{4.12}
\end{align*}

This system is simpler than in the \( R \times S^3 \) case, and in the remainder of this section we shall concentrate on its solution for the first \( O(P^2) \) deformation away from the \( AdS_5 \times T^{1,1} \) metric (2.11).

4.1. Asymptotics of regular \( S^4 \) solution

Let us first discuss the expected behavior of the solution in the two asymptotic regions: \( \rho \to 0 \) (\( u \to \infty \)) and \( \rho \to \infty \) (\( u \to 0 \)), i.e. in the short-distance and long-distance limits in 10-d space. We would like the solution to have a regular short-distance limit which has the form (2.11) (up to possible constant rescalings)

\begin{equation}
\rho \to 0 : \ y \to \ln \rho + y_* , \ z \to \ln \rho + z_* , \ w \to w_* , \ \Phi \to \Phi_* , \ K \to K_* . \tag{4.13}
\end{equation}

At large distances (\( \rho \to \infty \)) the solution is expected to approach the extremal KT solution (3.9), (3.10), i.e. (note that according to (3.20) \( u(\rho \to \infty) \to 4e^{-4\rho} \))

\begin{equation}
\rho \to \infty : \ w \to 0 , \ \Phi \to 0 , \ e^y \to 1 \frac{e^\rho}{2} , \ K \to 2P^2 \rho , \ e^{-4z} \to 8P^2 \rho e^{-4\rho} . \tag{4.14}
\end{equation}

To demonstrate the existence of a regular solution which interpolates between these two asymptotics we shall start with (2.11) which is valid for \( P = 0 \), and find its deformation order by order in \( P^2 \). We shall see that (under a proper choice of integration constants) the leading \( O(P^2) \) perturbations are regular at \( \rho \to 0 \), so that we indeed match onto the short-distance asymptotics (4.13). It turns out that the leading \( O(P^2) \) correction is already enough to match onto the expected KT long-distance asymptotics (4.14).

Our ansatz for the leading perturbative solution that differs from (2.11) by the \( O(P^2) \) terms will be

\begin{equation}
K = 4 + 2P^2 k(\rho) , \ \Phi = P^2 \phi(\rho) , \ w = P^2 w(\rho) , \tag{4.15}
\end{equation}
\[ y = y_0(\rho) + P^2 \xi(\rho) , \quad z = y_0(\rho) + P^2 \eta(\rho) , \quad y_0(\rho) \equiv \ln \sinh \rho , \quad (4.16) \]

We shall look for solutions for the perturbations \( k, \phi, w, \xi, \eta \) which are regular at \( \rho \to 0 \)
\[ \rho \to 0 : \quad k, \phi, w, \xi, \eta \to \text{const} , \quad (4.17) \]
in agreement with (4.13). We will find then that at large \( \rho \) the solution matches onto the KT asymptotics (4.14)
\[ \rho \to \infty : \quad w, \phi, \xi \to 0 , \quad k \to \rho , \quad \eta \to -\frac{1}{8} \rho . \quad (4.18) \]

4.2. Solution for \( O(P^2) \) perturbations

Substituting (4.17) into the system (4.7)–(4.12) we get
\[ 10 \xi'' - 320 e^{8y_0} \xi - 60 e^{6y_0} (5 \xi - 2 \eta) + \phi'' + O(P^2) = 0 , \quad (4.19) \]
\[ 10 w'' - 120 e^{8y_0} w - \phi'' + O(P^2) = 0 , \quad (4.20) \]
\[ \phi'' + k'^2 - e^{8y_0} + O(P^2) = 0 , \quad (4.21) \]
\[ 4 \eta'' - 128 e^{8y_0} \eta - 24 e^{6y_0} (5 \xi - 2 \eta) - (16k + 1)e^{8y_0} - k'^2 + O(P^2) = 0 , \quad (4.22) \]
\[ k'' - 4 e^{8y_0} + O(P^2) = 0 , \quad (4.23) \]
\[ 2y_0 (5 \xi' - 2 \eta') - \frac{1}{4} k'^2 + e^{8y_0} \left[ \frac{1}{4} + 2k - 8(5 \xi - 2 \eta) \right] - 6e^{6y_0} (5 \xi - 2 \eta) + O(P^2) = 0 . \quad (4.24) \]

Here prime stands for the derivative over \( u \), with \( du = -e^{-4y_0} d\rho \) (see (3.20)). Changing to the derivatives over \( \rho \) we finish with
\[ k'' + 4y_0 k' - 4 = 0 , \quad k' \equiv \frac{dk}{d\rho} = -e^{-4y_0} \frac{dk}{du} , \quad y_0 = \coth \rho , \quad (4.25) \]
\[ \phi'' + 4y_0' \phi' + k'^2 - 1 = 0 , \quad (4.26) \]
\[ w'' + 4y_0' w' - 12w + \frac{1}{10} (k'^2 - 1) = 0 , \quad (4.27) \]
\[ \xi'' + 4y_0' \xi' - 32 \xi - 6e^{-2y_0} (5 \xi - 2 \eta) - \frac{1}{10} (k'^2 - 1) = 0 , \quad (4.28) \]
\[ \eta'' + 4y_0' \eta' - 32 \eta - 6e^{-2y_0} (5 \xi - 2 \eta) - \frac{1}{4} (k'^2 + 1 + 16k) = 0 , \quad (4.29) \]
\[ y_0' (5 \xi' - 2 \eta') - (3e^{-2y_0} + 4)(5 \xi - 2 \eta) - \frac{1}{8} (k'^2 - 1 - 8k) = 0 . \quad (4.30) \]
The deformation of the background is thus driven by the perturbation \( k(\rho) \) of the effective 3-brane charge \( K \); solving for \( k(\rho) \) first we then determine the source terms in the linear equations for the remaining perturbations. The equation (4.25) is readily solved:

\[
k = -\frac{5}{6} + \rho \coth \rho \left( 1 - \frac{1}{2 \sinh^2 \rho} \right) + \frac{1}{2 \sinh^2 \rho}, \tag{4.31}
\]

where we have fixed the two integration constants so that to satisfy the condition (4.17) of regularity at small \( \rho \): \( k(0) = 0 \). Indeed, \( k(\rho \to 0) \to \frac{2}{5}\rho^2 + O(\rho^4) \). We also get the expected matching onto the KT asymptotics (4.18): \( k(\rho \to \infty) \to \rho \).

The solution for the dilaton perturbation (4.26) is then:

\[
\phi = \frac{13}{72} - \frac{1}{2 \sinh^2 \rho} + \frac{3\rho^2 + 2\rho \coth \rho}{8 \sinh^4 \rho} - \frac{\rho^2}{8 \sinh^6 \rho}, \tag{4.32}
\]

where again we have fixed the integration constants so that to have the regularity at small \( \rho \), \( \phi(\rho \to 0) = \frac{\rho^2}{10} + O(\rho^4) \). At large \( \rho \) the dilaton perturbation exponentially approaches zero, in agreement with (4.18).

The three equations for the gravitational perturbations \( w, \xi, \eta \) have a similar structure (as was also the case in [14]). For \( w \) we get

\[
w'' + 4 \coth \rho \ w' - 12w + q(\rho) = 0, \tag{4.33}
\]

\[
q \equiv \frac{1}{10} (k'^2 - 1) = \frac{1}{10} \left[ \frac{(12\rho - 8 \sinh 2\rho + \sinh 4\rho)^2}{640 \sinh^8 \rho} - 1 \right].
\]

Note that the source term is regular at small \( \rho \): \( q(\rho \to 0) \to -\frac{1}{10} + \frac{8}{125} \rho^2 + ... \), and \( q(\rho \to \infty) \to \frac{12}{5} e^{-2\rho} + O(\rho e^{-4\rho}) \). As a result, this equation has a regular solution near \( \rho = 0 \): \( w = w_* + (\frac{6}{5}w_* + \frac{1}{100})\rho^2 + ... \).\(^{18} \) It is easy to see (following the analysis in [14] or by numerical integration) that this regular short-distance asymptotics is smoothly connected to the long-distance asymptotics \( w \to \frac{3}{20} e^{-2\rho} \to 0 \).

The equations for \( \xi \) (4.28) and \( \eta \) (4.29) are coupled though the \( 5\xi - 2\eta \) term, but the equation for this combination can be easily integrated. In fact, its solution can be found from the constraint (4.30):

\[
\nu' + p_1(\rho) \nu + p_2(\rho) = 0, \quad \nu \equiv 5\xi - 2\eta, \tag{4.34}
\]

\(^{18} \) Note that (4.33) can be put also in the following form (which is of the same type that appeared in [14]): \( \ddot{w}'' - 2(6 + \frac{\cosh 2\rho}{\sinh^2 \rho})\dot{w} + \sinh^2 \rho \ q(\rho) = 0 \), \( w = e^{-2y_0} \tilde{w} = \sinh^{-2} \rho \ \tilde{w} \).
\[ p_1 \equiv -\left( \frac{3}{\sinh^2 \rho} + 4 \right) \tanh \rho , \quad p_2 \equiv -\frac{1}{8} \tanh \rho \left( k^2 - 1 - 8k \right) . \]

This gives:

\[ \nu = -S(\rho) \int d\rho \ S^{-1}(\rho) \ p_2(\rho) , \quad S \equiv e^{-\int d\rho \ p_1(\rho)} = \sinh^3 \rho \ \cosh \rho . \quad (4.35) \]

The resulting expression for \( \nu \) (given in terms of the dilogarithm function) is regular at small \( \rho \): \( \nu(\rho \to 0) = \frac{1}{8} \rho^2 + O(\rho^3) \), while for large \( \rho \) we get \( \nu \to \frac{1}{4} \rho \), in agreement with (4.18).

We are left with only one equation to solve – for \( \xi \) (or for \( \eta \)) (4.33)

\[ \xi'' + 4 \coth \rho \ \xi' - 32 \xi + v(\rho) = 0 , \quad (4.36) \]

\[ v \equiv -\tanh \rho \left[ \frac{6}{\sinh^2 \rho} \nu + \frac{1}{10} (k^2 - 1) \right] . \]

Its analysis is the same as for (4.33). Since the source \( v \) here is again regular at \( \rho \to 0 \): \( v = v_0 + O(\rho^2) \), \( v_0 = -\frac{13}{20} \), the solution for \( \xi \) is also regular, \( \xi = \xi_* + \frac{16}{5} \xi_* - \frac{19}{10} \rho^2 + O(\rho^4) \). As in the case of (4.33), we are also able to connect this to the required large \( \rho \) asymptotics (4.18), i.e. \( \xi \sim e^{-2\rho} \to 0 \).

We conclude that both the matter the gravitational perturbations are regular at small \( \rho \), and match onto the KT solution at large \( \rho \).

It is instructive to see explicitly why replacing \( S^4 \) by \( R^4 \), i.e. going back to the original KT ansatz, gives singular solution, i.e. why repeating the above perturbative analysis in the \( R^4 \) case leads to singular \( O(P^2) \) corrections, even though the starting point \(- AdS_5 \times T^{1,1} \) space in Poincare coordinates (2.9) is non-singular (see also [12]). Omitting the potential term associated with the curvature of \( S^4 \) in (1.7),(1.25),(1.12) and using the ansatz (1.13),(1.16) with \( y_0 = -\frac{1}{4} \ln(4u) = \rho \) (cf. (3.9),(2.9)) we finish with the following system of equations that replaces (1.23)–(1.30) \( (y'_0 = 1) \)

\[ k'' + 4k' - 4 = 0 , \quad \phi'' + 4\phi' + k^2 - 1 = 0 , \quad (4.37) \]

\[ w'' + 4w' - 12w + \frac{1}{10} (k^2 - 1) = 0 , \quad \xi'' + 4\xi' - 32\xi - \frac{1}{10} (k^2 - 1) = 0 , \quad (4.38) \]

\[ \eta'' + 4\eta' - 32\eta - \frac{1}{4} (k^2 + 1 + 16k) = 0 , \quad (4.39) \]

\[ ^{19} \text{The derivative here is over } \rho \text{ that here takes values } -\infty < \rho < \infty, \text{ with } \rho \to -\infty \text{ being the short-distance limit.} \]
\[ 5\xi' - 2\eta' - 4(5\xi - 2\eta) - \frac{1}{8}(k'^2 - 1 - 8k) = 0. \] (4.40)

Fixing the integration constants so that to achieve maximal possible regularity of functions at \( \rho = -\infty \) we get

\[ k = \rho, \quad \phi = 0, \quad w = 0, \quad \eta = -\frac{1}{32} - \frac{1}{8}\rho, \quad \xi = 0. \] (4.41)

This reproduces (3.3), (3.10) (note that \( e^{-4z} = e^{-4y_0}(1 + P^2\eta + ...) \)), and thus leads to a short-distance singularity at \( \rho \to -\infty \). It is the singular behaviour of the “source function” \( k \) that translates into the singularity of the gravitational perturbation \( \eta \). At the same time, in the non-extremal case in [14] and in the present \( S^4 \) case (4.31) (and \( R \times S^3 \) case discussed below) the function \( k \) has regular short-distance limit.

5. \( P \neq 0 \) solution in \( R \times S^3 \) case

The case of compactification on \( S^3 \) though technically more complicated, can be analyzed analogously to the \( S^4 \) case. We will construct a smooth supergravity RG flow interpolating between a conformal compactification of the KW geometry at the origin, and the asymptotically KT geometry to the leading order in \( P^2 \). The full second order system is given by (1.1) - (1.6). The starting point for the deformation by the 3-form fluxes is the \( AdS_5 \times T^{1,1} \) space in the global parametrization (2.10). In what follows we will use the radial coordinate \( t \) related to \( \rho \) in (2.10) as

\[ t = \tanh^2 \rho, \] (5.1)

and to \( u \) in (1.1) - (4.6) as

\[ \frac{du}{dt} = \frac{e^{z-5y}}{2\sqrt{t(1-t)}}. \] (5.2)

Here \( t \to 0_+ \) and \( t \to 1_- \) are the short-distance and the long distance limits of the 10-d space, respectively.\[21\] Let us also introduce the functions

\[ f_1 = e^{12x-4z}, \quad f_2 = t^2e^{-4z-4x}, \quad f_3 = e^{4y-16w-4z}, \quad f_4 = e^{4y+4w-4z}, \] (5.3)

\[ \text{The integration constants are subject to the zero-energy constraint as explained above.} \]

\[ \text{These are correspondingly the IR and the UV regimes of the holographically dual gauge theory.} \]
so that the deformed 10-d metric (2.1) takes the form
\[ ds_{10E}^2 = f_1^{-1/2} dX_0^2 + t f_2^{-1/2} (dS^3)^2 + \frac{dt^2}{4t(1-t)^2} + f_3^{1/2} e^{2\psi} + f_4^{1/2} (e^{2\theta_1} + e^{2\varphi_1} + e^{2\theta_2} + e^{2\varphi_2}) . \] (5.4)

The reason for the redefinitions (5.2), (5.3) is that using \( f_i(t) \) it is easier to construct perturbative in \( P^2 \) solution to (4.1) - (4.6). For \( P = 0 \) we recover the \( AdS_5 \times T^{1,1} \) space in the global parametrization (2.10)
\[ f_1 = f_2 = (1-t)^2, \quad f_3 = f_4 = 1, \] (5.5)
with unit radius corresponding to the choice of \( K = 4 \).

Our ansatz for a perturbative solution that differs from (5.5) by \( O(P^2) \) terms will be similar to (4.15), (4.16)
\[ f_1(t) = (1-t)^2 + P^2 \varphi_1(t), \quad f_2(t) = (1-t)^2 + P^2 \varphi_2(t) , \] (5.6)
\[ f_3(t) = 1 + P^2 \varphi_3(t), \quad f_4(t) = 1 + P^2 \varphi_4(t) , \]
\[ K(t) = 4 + 2P^2 k(t), \quad \Phi(t) = P^2 \phi(t) . \]

From (5.4) it is clear that to avoid a singularity in the metric at \( t \to 0_+ \) we should have
\[ \varphi_2(t) \to 0, \quad \varphi_{1,3,4}(t) \to \text{const} . \] (5.7)

Also, to reproduce the \( P = 0 \) values of the dilaton and of the regular D3-brane charge \( K \) at \( t = 0 \) we shall assume that
\[ \phi(t) \to 0 , \quad k(t) \to 0 . \] (5.8)

At large distances (\( t \to 1_- \)) the solution is expected to approach the extremal KT solution (3.9), (3.10)
\[ \varphi_1(t) \to 2k(t)e^{-4k(t)} , \quad \varphi_2(t) \to 2k(t)e^{-4k(t)} , \] (5.9)
\[ \varphi_3(t) \to \frac{1}{2} k(t), \quad \varphi_4(t) \to \frac{1}{2} k(t) , \quad \phi(t) \to 0 , \quad k(t) \to +\infty . \]

Notice that because \( k(t \to 1_-) \to +\infty \), the perturbative expansion (5.6) necessarily breaks down there, so that, strictly speaking, we should not expect to reproduce the precise form of the KT asymptotics (5.9). This is indeed what we will find. We will recover asymptotically
the warped product of the two factors \( R \times S^3 \) (with a finite \( S^3 \)) and \( T^{1,1} \), with the warp factors differing from the corresponding ones in the KT geometry by subleading logarithmic corrections. The same phenomenon was also observed in \([13]\).

Now, changing the radial coordinate according to (5.2), performing the redefinitions (5.3) in (4.4) - (4.6), and substituting the expansion (5.6) into the resulting system of equations, we obtain a coupled system of second-order equations for \( \varphi_{1,2,3,4}(t), \phi(t), k(t) \). In particular, for \( k(t) \) we find

\[
t(1-t)^2k'' + (1-t)(2-t)k' - 1 = 0. \tag{5.10}
\]

The solution of (5.10) with the correct boundary conditions is

\[
k(t) = -\frac{1}{2} \ln(1-t). \tag{5.11}
\]

For the dilaton perturbation we find

\[
t(t-1)^2\phi'' + (1-t)(2-t)\phi' + \frac{1}{4}(t-1) = 0, \tag{5.12}
\]

and its appropriate solution is

\[
\phi(t) = -\frac{1}{4t} [t \ln(1-t) + \ln(1-t)\ln(1-t+\ln t)]. \tag{5.13}
\]

Next, let us consider the equations for the \( \varphi_3 \) and \( \varphi_4 \). Introducing

\[
\varphi_{34}(t) \equiv \varphi_4 - \varphi_3, \tag{5.14}
\]

we obtain (using the already determined functions)

\[
2t(1-t)^2\varphi_{34}'' + 2(1-t)(2-t)\varphi_{34}' - \frac{2}{3}\varphi_{34} + (t-1) = 0. \tag{5.15}
\]

The solution of (5.15) with the correct asymptotics is

\[
\varphi_{34}(t) = \frac{t+2}{2(1-t)} [\ln(1-t)\ln t] - \frac{5t+1}{4t} \ln(1-t) - \frac{3}{2}. \tag{5.16}
\]

Substituting the already determined functions into the equation for \( \varphi_3 \) we find

\[
(1-t)^4(\frac{t^2}{1-t} \varphi_3')' - 8t(1-t)\varphi_3 + \frac{1}{4}t(t^2 - 28t + 27) - 2t(t+2)\ln(1-t)\ln t + 2(7t^2 - 6t - 1)\ln(1-t) - 4(t^2 + 2)\ln(1-t) + t = 0. \tag{5.17}
\]
Though (5.17) looks complicated, the general solution can still be found

\[ \varphi_3(t) = \frac{1}{12t(1-t)^2} [I_1(t) + I_2(t)] + \frac{t^2 + 6t + 3}{(1-t)^2} (\alpha_1 + 3\alpha_2 \ln t) + \alpha_2 \frac{51t^2 + 48t + 1}{t(1-t)^2}, \quad (5.18) \]

where

\[ I_1(t) = -t(t^2 + 6t + 3) \int_0^t \frac{dx}{x(1-x)^6} [51x^2 + 48x + 1 + 3x(x^2 + 6x + 3) \ln x] \times [x^3 - 3x^2 + 27x + 4(7x^2 - 6x - 1) \ln(1-x) - 8x(x + 2) \ln(x) \ln(1-x) - 24x^2 \text{Li}_2(x)], \quad (5.19) \]

and

\[ I_2(t) = [51t^2 + 48t + 1 + 3t(t^2 + 6t + 3) \ln t] \int_0^t \frac{dx}{(1-x)^6} (x^2 + 6x + 3) \]

\[ \times [x^3 - 28x^2 + 27x + 4(7x^2 + 6x - 1) \ln(1-x) - 8x(x + 2) \ln(x) \ln(1-x) - 24x^2 \text{Li}_2(x)] . \]

Both integration constants \( \alpha_1 \) and \( \alpha_2 \) are uniquely fixed by the boundary conditions. For \( t \to 0 \) we find

\[ \varphi_3(t) = \frac{\alpha_2}{t} + \alpha_2 (50 + 9 \ln t) + 3\alpha_1 + \alpha_2 O(t \ln t) + O(t) . \quad (5.21) \]

From (5.21) we see that the analyticity of \( \varphi_3 \) at the origin requires \( \alpha_2 = 0 \). In the limit \( t \to 1_- \) we get

\[ \varphi_3(t) = \frac{10s}{(1-t)^2} - \frac{8s}{(1-t)^2} s + \frac{1}{8} - \frac{1}{4} \ln(1-t) + O((1-t) \ln(1-t)) , \quad (5.22) \]

where

\[ s \equiv \alpha_1 + \frac{1}{120} [I_1(t \to 1_-) + I_2(t \to 1_-)] . \quad (5.23) \]

It is straightforward to verify that the sum \( I_1(t \to 1_-) + I_2(t \to 1_-) \) is actually finite.\footnote{22 Numerically, we find that \( I_1(t \to 1_-) + I_2(t \to 1_-) \approx -7.753297 \).}

From (5.22) we conclude that to get the KT asymptotic for the \( \varphi_3 \) as given by (5.9) we have to tune \( s = 0 \). Then (5.23) uniquely fixes the coefficient \( \alpha_1 \).

We did not find the exact analytical solutions for \( \varphi_1, \varphi_2 \), but it is possible to show that the regularity at \( t \to 0 \) fixes all the integration constants but one. In general, one finds

\[ \varphi_1(t) = \gamma (1-t)^2 + \sum_{i=1}^{\infty} d_{1i} t^i , \quad \varphi_2(t) = \sum_{i=1}^{\infty} d_{2i} t^i , \quad (5.24) \]
where $d_{1i}$ and $d_{3i}$ are some (uniquely) determined coefficients and $\gamma$ is an arbitrary integration constant. The presence of $\gamma$ reflects the freedom of rescaling of the time coordinate $X_0$ in (5.4). This arbitrary constant has no effect on the UV ($t \to 1_-$) asymptotic, where we find
\[ \phi_1(t) \to \frac{1}{16} (1-t)^2 [\ln(1-t)]^2, \quad \phi_2(t) \to \phi_1(t). \] (5.25)

Unlike the solution for the $\phi_3$-perturbation, which precisely reproduces the corresponding KT asymptotic, the precise form of the KT asymptotics for $\phi_1, \phi_2$ would be (see (5.9), (5.11))
\[ \phi_1(t) \to -(1-t)^2 \ln(1-t), \quad \phi_2(t) \to \phi_1(t). \] (5.26)

The (subleading) difference between (5.25) and (5.26) should not be surprising. Much like what happens in the nonextremal deformation of the KT solution [14], our perturbative expansion breaks down at $t \to 1_-$. 

6. Concluding remarks

In this paper we have argued that naked bulk singularities of gravitational backgrounds dual to gauge theories can be resolved by introducing an analog of an IR cutoff in gauge theories into the supergravity background. As a new explicit realization of this proposal we demonstrated the resolution of the singularity of fractional D3-branes on conifold background by the compactification of the gauge-theory space-time on $R \times S^3$ or $S^4$ with sufficiently large radius.

Unlike the original KT solution [5], the resulting supergravity backgrounds discussed here are nonsupersymmetric. This should not be too surprising, as our solutions are certain deformations of the KT background which had only $\mathcal{N} = 1$ supersymmetry in four dimensions. An interesting question is whether one can preserve supersymmetry in the process compactification of gauge theories with reduced supersymmetry, and what would be their gravity duals.

A promising starting point to address this question is the so called $\mathcal{N} = 2^*$ RG flow describing a mass deformed $\mathcal{N} = 4$ gauge theory. The corresponding supergravity solution was found in [20] (PW), and the realization of the gauge-gravity duality in this case was explained in detail [21,22]. It is straightforward to construct a linearized solution for the

23 Recall that this function determines the asymptotic warp factor of the $T^{1,1}$ space in the $R \times S^3$ compactification of the KT geometry.
gravitational background dual to the mass deformed $R \times S^3$ compactified $\mathcal{N} = 4$ SYM theory. In fact, the solution (and its supersymmetries) are precisely the same as in the original PW construction. The physical explanation for this is that the linearized solution effectively probes the UV dynamics of the gauge theory, where the compactification is actually irrelevant.

A highly nontrivial question is whether one can find the full nonlinear (supersymmetric?) solution in this case. An intuitive reason for why this solution may exist is the following. As explained in [21], the PW flow is dual in the IR to a special vacuum point in the $\mathcal{N} = 2^*$ moduli space. Neither the gauge theory nor gravity is able to explain what picks out this particular vacuum. The problem would be resolved if we assume the existence of an analogous RG flow for the compactified $\mathcal{N} = 4$ SYM theory. Indeed, the adjoint scalar coupling to the curvature of $S^3$ would lift all of the moduli space, apart from an isolated point; the conjecture is that the $\mathcal{N} = 2^*$ vacuum of the PW flow is precisely the one surviving under the $S^3$ compactification. Finally, there is an interesting enhancement phenomenon in the PW geometry. The size of the $S^3$ in the “compactified” flow produces a new mass scale in the geometry. One could imagine a phase transition originating from an interplay between the mass scale in the $\mathcal{N} = 4$ deformation and the scale introduced by the $S^3$.

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