Family of analytic entanglement monotones

A. Delgado\textsuperscript{\textsuperscript{*}} and T. Tessier\textsuperscript{\textsuperscript{†}}

Department of Physics and Astronomy, University of New Mexico, 800 Yale Boulevard, 87131 Albuquerque, USA

(Dated: November 12, 2018)

We derive a family of entanglement monotones, one member of which turns out to be the negativity. Two others are shown to be lower bounds on the I-concurrence, and on the I-tangle, respectively [P. Rungta and C. M. Caves, to appear in Phys. Rev. A]. We compare these bounds with the I-concurrence and I-tangle on the isotropic states, and on rank-two density operators resulting from a Tavis-Cummings interaction. Our results provide a global structure relating several different entanglement measures. Additionally, they possess analytic forms which are easily evaluated in the most general cases.

PACS numbers: 03.67.-a, 03.65.Ud

Entanglement has proven to be a useful resource in the implementation of quantum information processing protocols \cite{1}. This observation, coupled with the important role played by entanglement in the foundations of quantum theory, has led to the search for a quantitative theory of entanglement. In this context, a deep connection between the concepts of separability and of positive maps has been established \cite{2}, \cite{3}, and several different measures of entanglement have been proposed \cite{4}.

The concurrence and the tangle are two related entanglement measures yielding analytic forms in the case of two qubit quantum systems \cite{5}. Due to their intimate connection with the entanglement of formation, and with the phenomenon of entanglement-sharing \cite{6}, they have proven to be particularly useful tools for studying fundamental issues in quantum mechanics. Recently, analytic forms for these entanglement measures have been found for bipartite quantum systems of arbitrary dimensions in an overall pure state \cite{7}. However, the generalizations of these quantities, known respectively as the I-concurrence and the I-tangle \cite{8}, to mixed states of a bipartite system of arbitrary dimensions, involves a difficult minimization over ensemble decompositions. So far, it has been possible to calculate analytically the I-concurrence and the I-tangle only for the isotropic states \cite{9}, and in the special case that the density matrix of an arbitrary bipartite quantum system has a rank no greater than two \cite{10}.

Except for the cases mentioned above, analytic forms for the I-concurrence and the I-tangle are lacking. In this work we present a new family of entanglement monotones, certain members of which constitute easily computable lower bounds for the I-concurrence and for the I-tangle. In order to construct these functions we make use of the fact that the I-concurrence and the I-tangle are convex-roof extensions of the generalized concurrence for pure states of arbitrary dimensions, and of the square of this quantity, respectively. Therefore, the I-concurrence (I-tangle) can be bounded from below by any convex function which agrees with the I-concurrence (I-tangle) on the set of bipartite pure states. This requirement is seen to be satisfied by specific members of the set of entanglement measures introduced herein. Each is a simple function of the negative eigenvalues generated via the partial transposition operation, and may be easily calculated with any standard linear algebra package.

In order to estimate the quality of these functions as lower bounds, we compare their values with the values of the I-concurrence and the I-tangle on the family of isotropic states. The I-tangle, and the corresponding lower bound, are also compared numerically for rank-two density matrices arising in the context of the Tavis-Cummings model \cite{10}.

The construction that follows is based on the theory of majorization. This formalism has become a very useful tool in characterizing the relationships among density matrices, ensemble decompositions, and measurement processes \cite{11}, and has led to new insights in the structure of quantum algorithms \cite{12}, and in the problem of Hamiltonian simulation \cite{13}.

The following is a brief review of the main tenets of majorization theory. The reader is referred to \cite{14} for extensive background on the subject. Given two vectors $x$ and $y$ in $\mathbb{R}^n$, we say that the vector $x$ is majorized by the vector $y$, denoted by the expression $x \prec y$, when the following two conditions hold:

\begin{align}
\sum_{i=1}^{k} x_i ^\downarrow & \leq \sum_{i=1}^{k} y_i ^\downarrow, \forall k = 1, \ldots, n \\
\sum_{i=1}^{n} x_i ^\downarrow & = \sum_{i=1}^{n} y_i ^\downarrow. 
\end{align}

Here, the symbol $\downarrow$ indicates that the vector coefficients are arranged in decreasing order.

Majorization is naturally connected with the idea of comparative disorder \cite{14}. In fact, $x \prec y$ if and only if there exists a doubly stochastic matrix $A$ such that $x = Ay$. A matrix $A$ is doubly stochastic if its coefficients $a_{ij}$ are non-negative and $\sum_{j} a_{ik} = \sum_{j} a_{kj} = 1, \forall k$. If we consider $x$ and $y$ to be probability distributions, then the fact that $x$ is majorized by $y$ expresses the idea that $x$ is more disordered, in a quantifiable sense, than $y$.  

\textsuperscript{*}Corresponding author.

\textsuperscript{†}Electronic address: tessier@unm.edu
In the case that only condition (1) holds, we say that $x$ is weakly submajorized by $y$. This is denoted by the expression $x \prec_w y$. We will make use of the following two results concerning majorization.$^4$

$$x \prec_w y \in \mathbb{R}^n \Rightarrow x^+ \prec_w y^+$$

$$x \prec_w y \in \mathbb{R}^n_+ \Rightarrow x^p \prec_w y^p, \forall p \geq 1,$$

where the operations $x^p$ and $x^+$ act on each component of $x$ individually. The $x^+$ operation simply converts each of the negative entries in $x$ into a zero.

The following relation allows us to construct a family of convex functions of the negative eigenvalues of a hermitian matrix. Given any two hermitian matrices $A$ and $B,$

$$\lambda(A + B) \prec \lambda^+(A) + \lambda^+(B),$$

where $\lambda(T)$ denotes the vector whose coefficients are eigenvalues of $T$. Let us now define the vectors $\tilde{\lambda}(T) = -\lambda(T) = \lambda(-T)$, such that the negative coefficients in $\lambda(T)$ become positive in $\tilde{\lambda}(T)$. Clearly Eq. (3) also holds for the vectors $\tilde{\lambda}(T),$ i.e.,

$$\tilde{\lambda}(A + B) \prec \tilde{\lambda}^+(A) + \tilde{\lambda}^+(B).$$

The coefficients of the vectors in Eq. (7) are, by definition, members of $\mathbb{R}^n_+.$ Thus, using property (4) we obtain

$$\left[\left(\tilde{\lambda}(A + B)\right)^+\right]^p \prec_w \left[\left(\tilde{\lambda}^+(A) + \tilde{\lambda}^+(B)\right)^+\right]^p.$$  \hspace{1cm} (8)

According to condition (1) for $k = n$, we have

$$\sum_{i=1}^{n} \left(\tilde{\lambda}^+(A + B)\right)^p \leq \sum_{i=1}^{n} \left[\left(\tilde{\lambda}^+(A) + \tilde{\lambda}^+(B)\right)^+\right]^p,$$  \hspace{1cm} (9)

where we have removed the ordering of the vector on the left hand side. The term inside square brackets on the right hand side of Eq. (9) can be bounded from above by $\tilde{\lambda}^+(A) + \tilde{\lambda}^+(B)$, yielding

$$\sum_{i=1}^{n} \left(\tilde{\lambda}^+(A + B)\right)^p \leq \sum_{i=1}^{n} \left(\tilde{\lambda}^+(A) + \tilde{\lambda}^+(B)\right)^p.$$  \hspace{1cm} (10)

Taking the $p$-th root of Eq. (10), and using Minkowski’s inequality $^{12}$, we obtain

$$\left[\sum_{i=1}^{n} \left(\tilde{\lambda}^+(A + B)\right)^p\right]^{1/p} \leq \left[\sum_{i=1}^{n} \left(\tilde{\lambda}^+(A)\right)^p\right]^{1/p} + \left[\sum_{i=1}^{n} \left(\tilde{\lambda}^+(B)\right)^p\right]^{1/p}.$$  \hspace{1cm} (11)

The terms in square brackets on the right hand side of Eq. (11) are the sums of the positive coefficients of $\tilde{\lambda}(A)$ ($\tilde{\lambda}(B)$) to the $p$-th power, or equivalently, to the sums of the absolute values of the negative coefficients of $\lambda(A)$ ($\lambda(B)$) to the $p$-th power. Thus, we see that the quantity

$$M_p(A) = \left(\sum_{\lambda(A)<0} |\lambda(A)|^p\right)^{1/p}$$  \hspace{1cm} (12)

obeys the triangle inequality on the set of hermitian matrices for any $p \geq 1$. In particular, $M_p(A)$ is a convex function, i.e.,

$$M_p(\alpha A + \beta B) \leq \alpha M_p(A) + \beta M_p(B)$$  \hspace{1cm} (13)

for $\alpha$ and $\beta$ in the interval $[0,1]$ such that $\alpha + \beta = 1$. Similar results may also be shown to hold for the set of functions

$$N_p(A) \equiv M_p(A)^p, \forall p \geq 1.$$  \hspace{1cm} (14)

The I-concurrence $C(\rho)$ of a density matrix $\rho$ acting on a bipartite $d$-dimensional Hilbert space $\mathcal{H}$ is defined by $^8$

$$C(\rho) = \min_{\{p_i, \Psi_i\}} \left\{ \sum_i p_i C(\Psi_i) \left| \left| \rho = \sum_i p_i |\Psi_i\rangle \langle \Psi_i| \right| \right\},$$  \hspace{1cm} (15)

with the concurrence $C(\Psi)$ of a bipartite pure state $|\Psi\rangle$ given by

$$C(\Psi) = 2 \left(\sum_{i<j} c_i^2 c_j^2\right)^{1/2}.$$  \hspace{1cm} (16)

The $c_i$’s in Eq. (16) are the coefficients of the state $|\Psi\rangle$, written in the Schmidt decomposition $^1$. The I-concurrence is the convex-roof extension of $C(\Psi)$, and represents the average value of the pure state concurrences for an ensemble decomposition of $\rho$, minimized over all possible ensemble decompositions. The I-tangle has a similar construction, and is given by

$$\tau(\rho) = \min_{\{p_i, \Psi_i\}} \left\{ \sum_i p_i C^2(\Psi_i) \left| \left| \rho = \sum_i p_i |\Psi_i\rangle \langle \Psi_i| \right| \right\}.$$  \hspace{1cm} (17)

Due to a result by Uhlmann $^{14}$, the I-concurrence and the I-tangle are known to be the largest convex functions defined on the set of density operators which agree with $C(\Psi)$ and $C^2(\Psi)$, respectively, over the set of bipartite pure states. Therefore, if we are able to find a convex function which agrees with $C(\Psi)$ ($C^2(\Psi)$) over the set of bipartite pure states, then it will automatically constitute a lower bound for the I-concurrence (I-tangle). This can be accomplished by observing that the coefficients in Eq. (16) are the absolute values of the negative eigenvalues.
of the partial transpose of $|\Psi\rangle$. Since the partial transposition operation is linear, the function $M_2^{pt}(\rho)$ defined by

$$M_2^{pt}(\rho) = 2M_2(\rho^{pt}) \tag{18}$$

is a convex function which agrees with the I-concurrence on the set of bipartite pure states. Consequently, $M_2^{pt}(\rho)$ is a lower bound for the I-concurrence, i.e.,

$$M_2^{pt}(\rho) \leq C(\rho). \tag{19}$$

In a similar way it can be shown that the function $N_2^{pt}(\rho) \equiv [M_2^{pt}(\rho)]^2$ is a lower bound for the I-tangle, i.e.,

$$N_2^{pt}(\rho) \leq \tau(\rho). \tag{20}$$

These results hold for arbitrary dimensional bipartite quantum systems.

A similar result was obtained in [16] for the case of two qubits where $d = 4$. Specifically, it was shown that the negativity $N(\rho)$ is a lower bound on the concurrence $C$. The negativity is defined to be the absolute value of the sum of the negative eigenvalues of the partial transpose of $\rho$. Thus, the negativity is seen to be one member of our family of entanglement monotones, i.e., $N(\rho) \equiv M_1^{pt}(\rho)$. Additionally, in the case $d = 4$ the partial transpose of $\rho$ has at most one negative eigenvalue, implying that $M_2^{pt}(\rho)$ also reduces to the negativity in this situation.

The lower bounds given by Eqs. (19) and (20) are functions of the negativity eigenvalues produced via the partial transposition operation. Hence, they are entanglement monotones in their own right. This follows directly from the monotonicity of $N(\rho)$. Specifically, it may be shown that monotonicity is preserved in Eqs. (19) and (20) for choices of $p$ other than one. Further, these quantities have the additional advantage that they may be evaluated in a straightforward manner with the help of a standard linear algebra package.

It has been shown that the positive partial transposition criterion is a necessary and sufficient condition for separability for $d \leq 6$ [8]. In higher dimensions, positivity under partial transposition is a necessary, but not sufficient, condition for separability. However, this is not a serious drawback for the usefulness of these lower bounds. Indeed, it has been shown by numerical experiments and theoretical results that the volume of the set of density operators with positive partial transpose decreases exponentially with the dimension $d$ of the Hilbert space [3].

The exact values of the I-tangle for the isotropic states $\rho_F$ was analytically calculated in [8]. Isotropic states describe a quantum system composed of two subsystems of equal dimension $d$. They are mixtures formed by the convex combination of a maximally mixed state and a maximally entangled pure state, i.e.,

$$\rho_F = (1 - \lambda)\frac{1}{d^2}I \otimes I + \lambda|\Psi^+\rangle\langle\Psi^+|. \tag{21}$$

Here, $I$ is the identity operator acting on a $d$-dimensional Hilbert space, and $|\Psi^+\rangle$ is the state given by

$$|\Psi^+\rangle = \sum_{i=1}^{d} \frac{1}{\sqrt{d}} |i\rangle \otimes |i\rangle. \tag{22}$$

The parameter $\lambda$ in Eq. (21) can be related to the fidelity $F$ of $\rho_F$ with respect to the state $|\Psi^+\rangle$, (where $F \equiv |\langle\Psi^+|\rho_F|\Psi^+\rangle| \in [0, 1]$), via the relation

$$\lambda = \frac{d^2F - 1}{d^2 - 1}. \tag{23}$$

It has been shown that the isotropic states are separable for $F \leq 1/d$ [19].

The value of the lower bounds $M_2^{pt}(\rho)$ and $N_2^{pt}(\rho)$ on the isotropic states can be calculated easily. Since the partial transposition operation is linear, and since the identity operator is invariant under this operation, the eigenvalues of $\rho_F^{pt}$ are readily obtained. They are given by $(1 - \lambda)/d^2 \pm \lambda/d$ with multiplicity $d(d+1)/2$, respectively. The negative eigenvalues $(1 - \lambda)/d^2 - \lambda/d$ become positive when $\lambda \leq 1/(d+1)$, or equivalently, when $F \leq 1/d$. Thus,

$$M_2^{pt}(\rho_F) = \begin{cases} \frac{d}{2} \left(\frac{\lambda}{d} - \lambda + \lambda \sqrt{\frac{d(d-1)}{2}}\right) & \lambda > 1/(d + 1) \\ 0 & \lambda \leq 1/(d + 1) \end{cases} \tag{24}$$

and

$$N_2^{pt}(\rho_F) = [M_2^{pt}(\rho_F)]^2. \tag{25}$$

The behaviors of the I-tangle $\tau(\rho_F)$ and of $N_2^{pt}(\rho_F)$ for the isotropic states are depicted in Fig. 1. For $d = 2$, the two functions assume the same values, while for larger dimensions and constant fidelity, the difference between the lower bound and the I-tangle increases. In
the two-atom TCM as functions of the effective time $g t$. The upper curve shows the evolution of $\tau (\rho_{af})$, and the lower curve demonstrates the similar qualitative behavior of the lower bound $N_2^{pl}(\rho_{af})$.

Consequently, the entanglement between the field and the atoms results in a reduced density operator for the state of the overall system is pure, tracing over one of the atoms. This agreement is a direct result of the monotonicity of the I-tangle $[9]$. The I-tangle and its lower bound show clear differences in their magnitudes. However, the lower bound preserves the basic structure present in the evolution of the tangle. This agreement is a direct result of the monotonicity of $N_2^{pl}(\rho)$.

The entanglement measures analyzed above belong to larger classes of monotones defined by taking $A = \rho^{pt}$ in Eqs. (12) and (14). The negativity defined for a system of two qubits is seen to be one member of this set. Other instances correspond to lower bounds on the quantities I-concurrence and I-tangle, which are useful tools for investigations of quantum information theoretic concepts, and of fundamental quantum mechanics. Apart from offering a larger structure from which to view these different entanglement measures, this new family of functions also possesses analytic forms which are easily computed even in the most general cases. It will be interesting to see if further connections between seemingly unrelated measures of entanglement can be found using this formalism.

The authors are grateful to I. Deutsch, A. Scott and P. Rungta for many helpful discussions and comments. This work was supported under the auspices of the Office of Naval Research, Grant No. N00014-00-1-0575.

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* Electronic address: delgado@info.phys.unm.edu
† Electronic address: tessiert@info.phys.unm.edu

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