SHARP ESTIMATES OF UNIMODULAR FOURIER MULTIPLIERS ON WIENER AMALGAM SPACES

WEICHAO GUO AND GUOPING ZHAO

Abstract. We study the boundedness on the Wiener amalgam spaces $W_p^s$ of Fourier multipliers with symbols of the type $e^{i\mu(\xi)}$, for some real-valued functions $\mu(\xi)$ whose prototype is $|\xi|^{\beta}$ with $\beta \in (0, 2]$. Under some suitable assumptions on $\mu$, we give the characterization of $W_p^s \to W_p^s$ boundedness of $e^{i\mu(D)}$, for arbitrary pairs of $0 < p, q \leq \infty$. Our results are new on both sides of sufficiency and necessity, even for the special case $\mu(\xi) = |\xi|^\beta$ with $1 < \beta < 2$.

1. INTRODUCTION

The main aim of this paper is to study certain unimodular Fourier multipliers on the Wiener amalgam spaces. For two function spaces $X$ and $Y$, we call a tempered distribution $m$ a Fourier multiplier from $X$ to $Y$, if there exists a constant $C > 0$ such that
\[ \|T_m(f)\|_Y \leq C\|f\|_X, \]
for all $f$ in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, where
\[ T_m f = m(D)f = \mathcal{F}^{-1}(m\mathcal{F}f) \]
is the Fourier multiplier operator associated with $m$, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and inverse Fourier transform respectively, and $m$ is called the symbol or multiplier of $T_m$.

In particular, unimodular multipliers arise naturally when one solves the Cauchy problem for some dispersive equations. For example, consider the Cauchy problem of dispersive equation
\[
\begin{cases}
  i\partial_t u + (-\Delta)^{\beta/2} u = 0 \\
  u(0, x) = u_0
\end{cases}
\]
$(t, x) \in \mathbb{R} \times \mathbb{R}^n$, the formal solution is given by $u(t, x) = e^{i|D|^{\beta}} u_0(x)$, where $e^{i|D|^{\beta}}$ is just the unimodular Fourier multiplier defined by $e^{i|D|^{\beta}} f = \mathcal{F}^{-1}(e^{i|\xi|^{\beta}} \mathcal{F}f)$. Note that $e^{i|D|^{\beta}} u_0$ is the free solution of the Schrödinger equation when $\beta = 2$, and the free solution of the wave equation when $\beta = 1$. Hence, in order to study the dispersive equation, it is of great interest to study the boundedness of unimodular Fourier multipliers on several function spaces.

2000 Mathematics Subject Classification. 42B15, 42B35.
Key words and phrases. unimodular Fourier multipliers, wiener amalgam spaces, sharp potential loss.
Taking \( e^{i|D|^\beta} \) as the prototype, the unimodular Fourier multiplier in \( \mathbb{R}^n \) is defined by
\[
e^{i\mu(D)} f(x) := \int_{\mathbb{R}^n} e^{2\pi i x \xi} e^{i\mu(\xi)} \hat{f}(\xi) d\xi,
\]
where \( \mu \) is a real-valued function (with some suitable assumptions).

Since \( e^{i|\xi|^\beta} \in L^\infty(\mathbb{R}^n) \), the unimodular multiplier \( e^{i|D|^\beta} \) preserves the \( L^2(\mathbb{R}^n) \)-norm. But in generally, \( e^{i|D|^\beta} \) is not bounded on \( L^p(\mathbb{R}^n) \) if \( p \neq 2 \). Surprising, it was proved by Bényi-Grochenig-Okoudjou-Rogers \( [1] \) that \( e^{i|D|^\beta}(0 < \beta \leq 2) \) is bounded on \( M^s_{p,q} \) for all \( 1 \leq p,q \leq \infty \), \( s \in \mathbb{R} \). Furthermore, in the case \( \beta > 2 \), Miyachi-Nicola-Rivetti-Tabacco-Tomita \( [13] \) showed that, for \( 1 \leq p,q \leq \infty \) and \( s \in \mathbb{R} \), \( e^{i|D|^\beta} \) is bounded from \( M^s_{p,q} \) to \( M_{p,q} \) if and only if \( s \geq (\beta - 2)n(1/p - 1/2) \).

Roughly speaking, the boundedness behavior of \( e^{i|D|^\beta} \) is better on the modulation spaces than on the Lebesgue spaces. For recent research in this connection one can see e.g. \( [3, 12, 20] \).

Among numerous references, we observe that in a recent work by Nicola-Primo-Tabacco \( [15] \), the authors use a more structured approach to deal with the boundedness of \( e^{i\mu(D)} \). In fact, inspired by Concetti-Toft \( [2] \), Nicola-Primo-Tabacco use the Taylor expansion to send the regularity information from \( \mu \) to \( e^{i\mu} \). Note that this process was in fact hidden in the action of "taking derivation for \( e^{i\mu(\xi)} \), in many previous reference (see \( [13, 20] \).

In contrast to the fully development of research for the boundedness properties of \( e^{i\mu(D)} \) on the modulation spaces \( M^s_{p,q} \), the behavior of unimodular multipliers on the Wiener amalgma spaces \( W^s_{p,q} \) still remains on both sides of sufficiency and necessity. As far as we know, there are only few results concerning this topic. One can find some boundedness results of \( e^{i|D|^\beta} \) in \( [4] \), and see \( [13] \) for the sharp estimate of Schrödinger operator \( e^{i|D|^2} \).

In the present work, we consider the sufficiency and necessity for the boundedness on the Wiener amalgma spaces \( W^s_{p,q} \) of \( e^{i\mu(D)} \) whose prototype is \( e^{i|D|^\beta}(0 < \beta \leq 2) \). First, we state our results for sufficiency. Denote by \( \dot{p} := \min\{1,p\}, \dot{q} := \min\{1,q\} \) and \( \dot{p} \land \dot{q} = \min\{\dot{p}, \dot{q}\} \).

**Theorem 1.1** (Potential persistence). Suppose \( 0 < p,q \leq \infty \). Let \( \mu \in C^1(\mathbb{R}^n) \) be a real-valued function satisfying
\[
\nabla \mu \in W_{n/(\dot{p} \land \dot{q})+\varepsilon}^{\infty, \infty}
\]
for some \( \varepsilon > 0 \). Then \( e^{i\mu(D)} \) is bounded on \( W^p_{\dot{p}, \dot{q}} \).

**Theorem 1.2** (Potential loss). Suppose \( 0 < p,q \leq \infty \). Let \( \mu \) be a real-valued \( C^1(\mathbb{R}^n) \) function satisfying
\[
\langle \xi \rangle^{-s} \nabla \mu(\xi) \in (W_{n/(\dot{p} \land \dot{q})+\varepsilon})^n
\]
for some \( s, \varepsilon > 0 \). Then \( e^{i\mu(D)} : W^p_{\dot{p}, \dot{q}} \to W^p_{\dot{p}, \dot{q}} \) is bounded for \( \delta > s\varepsilon/(\dot{p} \land \dot{q}) \).

**Theorem 1.3** (Interpolation case). Suppose \( 0 < p,q \leq \infty \). Let \( \mu \) be a real-valued \( C^2(\mathbb{R}^n) \) function satisfying
\[
\langle \xi \rangle^{-s} \nabla \mu(\xi) \in (W_{n/(\dot{p} \land \dot{q})+\varepsilon})^n, \quad \partial^2 \mu \in W_{n/\delta+\varepsilon}^{\infty, \infty} \quad (|\gamma| = 2),
\]
for some \( s, \varepsilon > 0 \). Then \( e^{i\mu(D)} : W^p_{\dot{p}, \dot{q}} \to W^p_{\dot{p}, \dot{q}} \) is bounded for \( \delta \geq s\varepsilon/|1/p - 1/q| \) with strict inequality when \( p \neq q \).
Corollary 1.4 (Derivative condition). Suppose \( 0 < p, q \leq \infty \). Let \( \epsilon > n(1/p - 1) \), \( \beta \in (0, 2] \). Let \( \mu \) be a real-valued function of class \( C^{1/n(\hat{p} \wedge \hat{q}) + 3} \) on \( \mathbb{R}^n \setminus \{0\} \) which satisfies

\[
|\partial^\gamma \mu(\xi)| \leq C_\gamma |\xi|^{\epsilon - |\gamma|}, \quad 0 < |\xi| \leq 1, \ |\gamma| \leq [n/(1/p - 1/2)] + 1, \tag{1.2}
\]

and

\[
\begin{cases}
|\partial^\gamma \mu(\xi)| \leq C_\gamma |\xi|^{\beta - |\gamma|}, & |\xi| > 1, \ 1 \leq |\gamma| \leq [n/(\hat{p} \wedge \hat{q})] + 2, \text{ if } \beta \in (0, 1], \\
|\partial^\gamma \mu(\xi)| \leq C_\gamma |\xi|^{\beta - |\gamma|}, & |\xi| > 1, \ 1 \leq |\gamma| \leq [n/(\hat{p} \wedge \hat{q})] + 3, \text{ if } \beta \in (1, 2].
\end{cases}
\tag{1.3}
\]

Then \( e^{i\mu(D)} : W^{p,q}_\delta \to W^{p,q} \) is bounded for \( \delta \geq n|1/p - 1/q| \max\{\beta - 1, 0\} \) with strict inequality when \( \beta > 1, p \neq q \).

Next, we give the following theorem for the necessity part.

Theorem 1.5 (Sharp potential loss). Suppose \( 0 < p, q \leq \infty \). Let \( 1 < \beta \leq 2 \), and let \( \mu \) be a real-valued \( C^2(\mathbb{R}^n \setminus \{0\}) \) function satisfying

\[
(1 - \rho_0) \partial^\gamma \mu \in W^{\infty, \infty}_{n/(\hat{p} \wedge \hat{q}) + \epsilon} (|\gamma| = 2) \quad \text{for some } \epsilon > 0,
\]

where \( \rho_0 \) is a \( C^\infty \) function supported in \( B(0, 1/2) \) and satisfies \( \rho_0(\xi) = 1 \) on \( B(0, 1/4) \). Suppose that the Hessian determinant of \( \mu \) is not zero at some point \( \kappa_0 \) with \( |\kappa_0| = 1 \). Moreover,

\[
\mu(\lambda \xi) = \lambda^{\beta} \mu(\xi), \quad \lambda \geq 1, \ \xi \in B(\kappa_0, r_0) \cap \mathbb{S}^{n-1}
\]

for some \( r_0 > 0 \). Then the boundedness of \( e^{i\mu(D)} : W^{p,q}_\delta \to W^{p,q} \) implies

\[
s \geq n(\beta - 1)|1/p - 1/q|
\]

with strict inequality when \( p \neq q \).

As an application, we give the following characterization for the boundedness of our prototype.

Corollary 1.6 (Return to the prototype). Let \( 1 \leq p \leq \infty \), \( 0 < q < \infty \), \( \beta \in (0, 2] \). We have \( e^{i|D|^\beta} : W^{p,q}_\delta \to W^{p,q} \) is bounded if and only if

\[
\delta \geq n|1/p - 1/q| \max\{\beta - 1, 0\}
\]

with strict inequality when \( \beta > 1, p \neq q \).

Our new contribution are listed as follows.

(1) For the sufficiency part, we first give a useful criterion for the boundedness of \( e^{i\mu(D)} \) on the Wiener amalgam spaces in the full range \( 0 < p, q \leq \infty \).

(2) For the necessity, we first give the sharp potential loss for a large family of unimodular multipliers with non-degenerate Hessian matrixes.

(3) As an application, we obtain the sharp boundedness results for the operators \( e^{i|D|^\beta} \) with \( \beta \in (0, 2] \). Even in this special case, our results are an essential improvement of the previous results for \( \beta \in (1, 2) \).

The rest of this paper is organized as follows. In Section 2, we collected a number of definitions and auxiliary results, including the dilation and convolution properties of certain Wiener amalgam spaces.

Section 3 is devoted to the proof of theorems and corollaries for the sufficiency part. The first key point is to find a suitable working space for \( e^{i\mu} \), such
that the functions in this working space can be localized in the time plane. Although the convolution relations $W_{\delta}^{p,q,\infty} \ast W_{\delta}^{p,q} \subset W_{\delta}^{p,q}$ supply a natural working space $M^{\infty,\beta \wedge q} = \mathcal{F}W_{\delta}^{\beta \wedge q,\infty}$ for $\langle \cdot \rangle^{-\beta} e^{i\mu}$ to live in, we observe that $M^{\infty,\beta \wedge q}$ is not a suitable space for time localization. In consideration of the time localization property, and the optimal embedding relation with $M^{\infty,\beta \wedge q}$, we choose $W_{n/(\beta \wedge q) + \epsilon}^{\infty} = M_{n/(\beta \wedge q) + \epsilon}^{\infty}$ as our desirable working space. Then, by time localization and Taylor expansion (of order 1), we give the proofs of Theorem 1.1 and Theorem 1.2. Combining with the boundedness property on the diagonal line $p = q$, we further give the proof of Theorem 1.3 by an interpolation argument. Finally, Corollary 1.4 is proved by establishing an embedding relations between the working space and the function space with bounded derivative.

Section 4 contains the proof of Theorem 1.5 and Corollary 1.6. Under the assumption of non-degenerate of Hessian matrix, we find that the set $\{\nabla \mu(\xi) : \xi \in \mathbb{R}^n\}$ is scattered in the sense that the distance between $\nabla \mu(\xi_1)$ and $\nabla \mu(\xi_2)$ has a lower bound according to the distance between $\xi_1$ and $\xi_2$, and according to the value of $\beta$. In fact, the scattered degree of $\{\nabla \mu(\xi) : \xi \in \mathbb{R}^n\}$ is increasing as the value of $\beta \in (1, 2]$. In particularly, when $\beta = 2$, the scattered degree of $\{\nabla \mu(\xi) : \xi \in \mathbb{R}^n\}$ is just like that of $\{\xi : \xi \in \mathbb{R}^n\}$. By this observation, we give a detailed scattered property in Lemma 4.1, which can be further used to obtain some useful estimates of special functions in Lemma 4.2. Then, by a rotation trick, the boundedness of $e^{i\mu(\beta)}$ can be reduced to the simple embedding of certain discrete sequences, which leads to our final conclusion of Theorem 1.5. As a return to the prototype, Corollary 1.6 is the direct conclusion of Corollary 1.4 and Theorem 1.5.

Some complements are prepared in the last section. In the high growth case of $\mu$, keep the prototype $\mu(\xi) = |\xi|^\beta (\beta > 2)$ under consideration, we find that the working space should be replaced by some Wiener amalgam space without potential, just like the working space used in the modulation case. Based on this observation, we give some boundedness results for the high growth case.

2. PRELIMINARIES

We start this section by recalling some notations. Let $C$ be a positive constant that may depend on $n, p_i, q_i, s_i, \alpha, \beta$. The notation $X \lesssim Y$ denotes the statement that $X \leq CY$, the notation $X \sim Y$ means the statement $X \lesssim Y \lesssim X$, and the notation $X \asymp Y$ denotes the statement $X = CY$. For a multi-index $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$, we denote

$|k|_\infty := \max_{i=1,2\ldots,n} |k_i|, \quad \langle k \rangle := (1 + |k|^2)^{\frac{\beta}{2}}$.

In this paper, for the sake of simplicity, we use the notation ”$\mathcal{S}$” to denote some large positive number which may be changed corresponding to the exact environment.

Let $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ be the Schwartz space and $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^n)$ be the space of tempered distributions. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx, \quad \mathcal{F}^{-1}f(x) = \check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x \cdot \xi} d\xi$.

We recall some definitions of the function spaces treated in this paper.
Definition 2.1. Let $0 < p \leq \infty$, $s \in \mathbb{R}$. The weighted Lebesgue space $L^p_{s,k}$ consists of all measurable functions $f$ such that

$$\|f\|_{L^p_{s,k}} = \begin{cases} \left( \int_{\mathbb{R}^n} |f(x)|^p \langle x \rangle^{ps} dx \right)^{1/p}, & p < \infty \\ \text{ess sup}_{x \in \mathbb{R}^n} |f(x)\langle x \rangle^s|, & p = \infty \end{cases} \tag{2.2}$$

is finite. If $f$ is defined on $\mathbb{Z}^n$, we denote

$$\|f\|_{l^p_{s,k}} = \begin{cases} \left( \sum_{k \in \mathbb{Z}^n} |f(k)|^p \langle k \rangle^{ps} \right)^{1/p}, & p < \infty \\ \sup_{k \in \mathbb{Z}^n} |f(k)\langle k \rangle^s|, & p = \infty \end{cases} \tag{2.4}$$

and $l^p_{s,k}$ as the (quasi) Banach space of functions $f : \mathbb{Z}^n \to \mathbb{C}$ whose $l^p_{s,k}$ norm is finite. We write $L^p_{s,k}$, $l^p_{s,k}$ for short, respectively, if there is no confusion.

We recall an embedding relation between weighted sequences. This lemma is easy to be verified, so we omit the proof here.

Lemma 2.2. Suppose $0 < q_1, q_2 \leq \infty$, $s_1, s_2 \in \mathbb{R}$. Then

$$l^q_{s_1} \subset l^q_{s_2}$$

holds if and only if

$$\frac{1}{q_2} \leq \frac{1}{q_1}, \quad s_2 \leq s_1, \tag{2.5}$$

$$\frac{1}{q_2} > \frac{1}{q_1}, \quad \frac{1}{q_2} + \frac{s_2}{n} < \frac{1}{q_1} + \frac{s_1}{n}. \tag{2.6}$$

Now, we turn to the definition of modulation and Wiener amalgam space. Fixed a nonzero function $\phi \in \mathcal{S}$, the short-time Fourier transform of $f \in \mathcal{S}'$ with respect to the window $\phi$ is given by

$$V_\phi f(x, \xi) = \langle f, M_\xi T_x \phi \rangle = \int_{\mathbb{R}^n} f(y) \overline{\phi(y-x)} e^{-2\pi i y \cdot \xi} dy, \tag{2.7}$$

where the translation operator is defined as $T_{x_0} f(x) := f(x-x_0)$ and the modulation operator is defined as $M_\xi f(x) := e^{2\pi i x \cdot \xi} f(x)$, for $x, x_0, \xi \in \mathbb{R}^n$.

Definition 2.3. Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$. Given a window function $\phi \in \mathcal{S}\setminus\{0\}$, the modulation space $M^p_{s,q}$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the norm

$$\|f\|_{M^p_{s,q}} = \left( \|V_\phi f(x, \xi)\|_{L^q_{s,k}} \right)^{q/p} \tag{2.8}$$

is finite, with the usual modification when $p = \infty$ or $q = \infty$. In addition, we write $M^p_{s,q} := M^p_{s,0}$.

The above definition of $M^p_{s,q}$ is independent of the choice of window function $\phi$. One can see this fact in [7] for the case $(p, q) \in [1, \infty]^2$, and in [6] for the case $(p, q) \in (0, \infty]^2 \setminus [1, \infty]^2$. More properties of modulation spaces can be founded in [19]. One can also see [5] for a survey of modulation spaces.
Applying the frequency-uniform localization techniques, one can give an alternative definition of modulation spaces (see [17, 18] for details).

We denote by $Q_k$ the unit cube with the center at $k$. Then the family $\{Q_k\}_{k \in \mathbb{Z}^n}$ constitutes a decomposition of $\mathbb{R}^n$. Let $\rho \in \mathcal{S}(\mathbb{R}^n)$, $\rho : \mathbb{R}^n \to [0,1]$ be a smooth function satisfying that $\rho(\xi) = 1$ for $|\xi|_\infty \leq 1/2$ and $\rho(\xi) = 0$ for $|\xi| \geq 3/4$. Let

$$
\rho_k(\xi) = \rho(\xi - k), \quad k \in \mathbb{Z}^n
$$

be a translation of $\rho$. Since $\rho_k(\xi) = 1$ in $Q_k$, we have that $\sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \geq 1$ for all $\xi \in \mathbb{R}^n$. Denote

$$
\sigma_k(\xi) = \rho_k(\xi) \left( \sum_{l \in \mathbb{Z}^n} \rho_l(\xi) \right)^{-1}, \quad k \in \mathbb{Z}^n.
$$

It is easy to know that $\{\sigma_k(\xi)\}_{k \in \mathbb{Z}^n}$ constitutes a smooth decomposition of $\mathbb{R}^n$, and $\sigma_k(\xi) = \sigma(\xi - k)$. The frequency-uniform decomposition operators can be defined by

$$
\square_k := \mathcal{F}^{-1}\sigma_k\mathcal{F}
$$

for $k \in \mathbb{Z}^n$. Now, we give the (discrete) definition of modulation space $M^{p,q}_{n}$.

**Definition 2.4.** Let $s \in \mathbb{R}, 0 < p, q \leq \infty$. The modulation space $M^{p,q}_{n}$ consists of all $f \in \mathcal{S}'$ such that the (quasi-)norm

$$
\|f\|_{M^{p,q}_{n}} := \left( \sum_{k \in \mathbb{Z}^n} (\langle k \rangle^{sq}\|\square_k f\|_{L^p_k})^q \right)^{1/q}
$$

is finite. We write $M^{p,q}_{n} := M^{p,q}_{0}$ for short.

**Remark 2.5.** We remark that the above definition is independent of the choice of $\sigma$. So, we can choose suitable $\sigma$ fitting our case. In the definition above, the function sequence $\{\sigma_k(\xi)\}_{k \in \mathbb{Z}^n}$ satisfies $\sigma_k(\xi) = 1$ and $\sigma_k(\xi)\sigma_l(\xi) = 0$ if $k \neq l$, where $|\xi - k|_\infty \leq 1/4$. We also recall that the definition of $M^{p,q}_{n}$ and $M^{p,q}_{n}$ are equivalent. In this paper, we mainly use $M^{p,q}_{n}$ to denote the modulation space.

**Definition 2.6.** Let $0 < p, q \leq \infty, s \in \mathbb{R}$. Given a window function $\phi \in \mathcal{S}\setminus\{0\}$, the Wiener amalgam space $W^{p,q}_{s,n}$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the norm

$$
\|f\|_{W^{p,q}_{s,n}} = \left( \|V_\phi f(x,\xi)\|_{L^q_{\xi,x}} \right)^{p/q}
$$

is finite, with the usual modifications when $p = \infty$ or $q = \infty$.

**Lemma 2.7.** (Equivalent norm of $W^{p,q}_{s,n}$). Let $0 < p, q \leq \infty, s \in \mathbb{R}$. Then

$$
\|f\|_{W^{p,q}_{s,n}} \sim \left( \sum_{k \in \mathbb{Z}^n} \|\sigma_k f\|_{L^p_{\xi,x}}^p \right)^{1/p} \sim \left( \sum_{k \in \mathbb{Z}^n} \|\sigma_k f\|_{L^p_k}^p \right)^{1/p}
$$

with usual modification when $p = \infty$.

Following we collect some useful properties on Wiener amalgam spaces. The first one is the convolution and product relations. We put the proof in Appendix A.

**Lemma 2.8** (Convolution relations). Let $p, q \in (0,\infty]$, $s > n, \delta \in \mathbb{R}, \epsilon > 0$, we have
Lemma 2.9 (Dilation property of $W^{s,\infty}_a$, [11 Theorem 3.3]). Let $p \in (0, \infty]$, $\epsilon > 0$, $s > n$ we have
\[ \| f \|_{W^{s,\infty}_a} \lesssim \| f \|_{W^{s,\infty}_a}, \]
where $f_t(x) := f(tx)$, $t \in (0, 1]$.

Proof. For the self-containing of this paper, we give a short proof here. Write
\[ \| f \|_{W^{s,\infty}_a} = \| f \|_{M^{s,\infty}_a} = \sup_{k \in \mathbb{Z}^n} \langle k \rangle^s \| \Box_k f_t \|_{L^\infty}. \]
Denote by
\[ A_{k,t} := \{ l \in \mathbb{Z}^n : \sigma_l(\xi) \sigma_k(t\xi) \neq 0 \}. \]
Then
\[ \Box_k f_t \|_{L^\infty} = \| t_n \mathcal{F}^{-1}(\sigma_k(\xi) f(\frac{\xi}{t})) \|_{L^\infty} = \| \mathcal{F}^{-1}(\sigma_k(\xi) f(\xi)) \|_{L^\infty} = \| \mathcal{F}^{-1}(\sum_{l \in A_{k,t}} \sigma_l(\xi) \sigma_k(t\xi) f(\xi)) \|_{L^\infty} \]
For $k$ near the origin, we have
\[ \langle k \rangle^s \| \Box_k f_t \|_{L^\infty} \sim \| \Box_k f_t \|_{L^\infty} \lesssim |A_{k,t}| \sup_{l \in A_{k,t}} \| \mathcal{F}^{-1}(\sigma_l(\xi) f(\xi)) \|_{L^\infty} \lesssim \sum_{l \in \mathbb{Z}^n} \langle l \rangle^{-s} \| l \rangle^s \| \Box_k f_t \|_{L^\infty} \lesssim \sum_{l \in \mathbb{Z}^n} \langle l \rangle^{-s} \| f \|_{W^{s,\infty}_a} \lesssim \| f \|_{W^{s,\infty}_a}. \]
For $k$ far away for the origin, observe that $|A_{k,t}| \sim t^{-n}$, and $|l| \sim |k/t|$ for $l \in A_{k,t}$. We have
\[ \| \Box_k f_t \|_{L^\infty} \lesssim |A_{k,t}| \sup_{l \in A_{k,t}} \| \mathcal{F}^{-1}(\sigma_l(\xi) f(\xi)) \|_{L^\infty} \lesssim t^{-n} \sup_{l \in A_{k,t}} \langle l \rangle^{-s} \| l \rangle^s \| \Box_k f_t \|_{L^\infty} \lesssim t^{-n} \langle k/t \rangle^{-s} \| f \|_{W^{s,\infty}_a} \lesssim \langle k \rangle^{-s} \| f \|_{W^{s,\infty}_a}. \]
The above two estimates yield that
\[ \| f_t \|_{W^{s,\infty}_a} \sim \sup_{k \in \mathbb{Z}^n} \langle k \rangle^s \| \Box_k f_t \|_{L^\infty} \lesssim \| f \|_{W^{s,\infty}_a}. \]
\[ \Box \]

Lemma 2.10 (Rotation free). Let $p,q \in (0, \infty]$, $s \in \mathbb{R}$. Denote by $f_P(x) = f(P^{-1}x)$. We have
\[ \| f_P \|_{W^{p,q}_a} \sim \| f \|_{W^{p,q}_a}. \]
Proof. Without loss of generality, we assume that $\phi$ is a radial function. By a direct calculation we get

$$V_{\phi} f_P(x, \xi) = \int_{\mathbb{R}^n} \hat{f}(P\eta) \hat{\phi}(\eta - \xi)e^{2\pi i x \eta} d\eta$$

$$= \int_{\mathbb{R}^n} \hat{f}(\eta) \hat{\phi}(P^{-1}\eta - \xi)e^{2\pi i x P^{-1}\eta} d\eta$$

$$= \int_{\mathbb{R}^n} \hat{f}(\eta) \hat{\phi}(\eta - P\xi)e^{2\pi i P x \eta} d\eta$$

where we denote $f_P(x) := f(Px)$. It follows that

$$\|f_P\|_{W^{p,q}_s} \sim \|f\|_{W^{p,q}_s}.$$ 

Finally, we recall a disjoint property for the $\alpha$-covering, which will be used in the proof of scattered property of $\nabla \mu$.

Lemma 2.11 (See Lemma A.1 in [9]). Let $\alpha \in [0, 1)$ be arbitrary. Then there is some $\delta > 0$ such that the family

$$(B^\delta_k)_{k \in \mathbb{Z}^n} := (B((k + \alpha \pi k, \delta(k + \alpha \pi)))_{k \in \mathbb{Z}^n}$$

is pairwise disjoint.

3. Boundedness Results

In this section, we establish some boundedness results of the umodular Fourier multiplier on Wiener amalgam spaces, including the proof of potential persistence case Theorem 1.1, and the proof of potential loss case Theorem 1.2 and 1.3. Then, by establishing Lemma 3.3, we give the proof of Corollary 1.4.

3.1. Potential persistence: low growth of $\mu$. This subsection is devoted to the proof of Theorems 1.1.

Proof of Theorems 1.1. By the convolution relation (see Lemma 2.8)

$$W^{p,q} * W^{\dot{p},\dot{q},\infty} \subset W^{p,q},$$

we only need to verify that $\mathcal{F}^{-1} e^{i\mu} \subset W^{\dot{p},\dot{q},\infty}$, or equivalently,

$$e^{i\mu} \in M^{\infty,\dot{p},\dot{q}}.$$

Here, by the assumption of $\nabla \mu \in (W^{\infty,\infty}_n/(p\wedge q) + \epsilon)^n$, we will show that

$$e^{i\mu} \in W^{\infty,\infty}_n/(p\wedge q) + \epsilon.$$ 

Then (3.1) follows by the fact $W^{\infty,\infty}_n/(p\wedge q) + \epsilon \subset M^{\infty,\dot{p},\dot{q}}$. By an equivalent norm of Wiener amalgam space (see Lemma 2.7) we have

$$\|e^{i\mu}\|_{W^{\infty,\infty}_n/(p\wedge q) + \epsilon} \sim \sup_{k \in \mathbb{Z}^n} \|\sigma_k e^{i\mu}\|_{W^{\infty,\infty}_n/(p\wedge q) + \epsilon}.$$
The Taylor’s formula yields that
\[ \mu(k + \xi) = \mu(k) + \int_0^1 \xi \cdot \nabla \mu(k + t\xi) dt =: \mu(k) + R_k. \]

Then,
\[ \| \sigma_k e^{it\mu} \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} = \| \sigma_0 e^{i\sigma_0(k+) \mathbb{R}} \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} = \| \sigma_0 e^{iR_k} \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} = \| \sigma_0 e^{i\sigma_0 R_k} \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}}, \]

where \( \sigma_k^* := \sum_{\gamma \neq 0} \sigma_1 \). By the algebraic property \( W^{\infty,\infty}_{n/(p,q)+\epsilon} \cdot W^{\infty,\infty}_{n/(p,q)+\epsilon} \subset W^{\infty,\infty}_{n/(p,q)+\epsilon} \) (see Lemma 2.8), there exists a constant \( C \geq 1 \) such that
\[ \| f \cdot f \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} \leq C \| f \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}}^2. \]

This implies that
\[ \| \sigma_0 e^{i\sigma_0 R_k} \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} = \| \sigma_0 + \sigma_0(e^{i\sigma_0 R_k} - 1) \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} \approx 1 + \| \sigma_0 + e^{i\sigma_0 R_k} - 1 \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} \lesssim 1 + \| e^{i\sigma_0 R_k} - 1 \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} \]
\[ \lesssim 1 + \sum_{m=1}^{\infty} \frac{(i\sigma_0 R_k)^m}{m!} \| W^{\infty,\infty}_{n/(p,q)+\epsilon} \| \leq \sum_{m=0}^{\infty} \frac{C^m}{m!} \| \sigma_0^* R_k \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}}^m \leq \exp \left( C \| \sigma_0^* R_k \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} \right). \]

Finally,
\[ \| \sigma_0 R_k \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} = \left\| \sigma_0 \int_0^1 \xi \cdot \nabla \mu(k + t\xi) dt \right\|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} \]
\[ \lesssim \sum_{|\gamma| = 1} \| \sigma_0 \xi^\gamma \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} \cdot \left\| \int_0^1 \partial^\gamma \mu(k + t\xi) dt \right\|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} \]
\[ \lesssim \sum_{|\gamma| = 1} \| \int_0^1 \partial^\gamma \mu(k + t\xi) dt \right\|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} \]
\[ \lesssim \sum_{|\gamma| = 1} \int_0^1 \| \partial^\gamma \mu(k + t\xi) \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} dt \]
\[ = \sum_{|\gamma| = 1} \int_0^1 \| \partial^\gamma \mu(t\xi) \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} dt \leq \sum_{|\gamma| = 1} \| \partial^\gamma \mu \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} \]

where in the last inequality we use the dilation property Lemma 2.8.

Combining the above estimates yields that
\[ \| e^{it\mu} \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} \sim \sup_{k \in \mathbb{Z}^n} \| \sigma_k e^{it\mu} \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} \lesssim \sup_{k \in \mathbb{Z}^n} \left( C \| \sigma_0^* R_k \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} \right) \]
\[ \lesssim \exp \left( C \sum_{|\gamma| = 1} \| \partial^\gamma \mu \|_{W^{\infty,\infty}_{n/(p,q)+\epsilon}} \right) \lesssim 1. \]

We have now completed this proof. \( \square \)
3.2. Potential loss: mild growth of $\mu$.

Proof of Theorems 1.2. Without loss of generality, we assume

$$s\epsilon + sn/(\hat{p} \wedge \hat{q}) - \delta < 0$$

for fixed $s > 0$ and $\delta > sn/(\hat{p} \wedge \hat{q})$. Denote by

$$P_t(x) := \langle x \rangle^t, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}.$$

By the convolution relations $W_{\delta}^{p,q} * W_{\hat{p} \wedge \hat{q}}^{\infty} \subset W^{p,q}$, we only need to verify that

$$\mathcal{F}^{-1} e^{i\mu} \in W_{\hat{p} \wedge \hat{q}}^{\infty}.$$

Note that $W_{n/(\hat{p} \wedge \hat{q}) + \epsilon}^{\infty} \subset M_{\hat{p} \wedge \hat{q}}$, if we have

$$P_{-e} e^{i\mu} \in W_{n/(\hat{p} \wedge \hat{q}) + \epsilon},$$

the desired conclusion follows by

$$\mathcal{F}^{-1} e^{i\mu} \in W_{\hat{p} \wedge \hat{q}}^{\infty}.$$

Write

$$\|P_{-e} e^{i\mu}\|_{W_{n/(\hat{p} \wedge \hat{q}) + \epsilon}} \sim \sup_{k \in \mathbb{Z}^n} \|\sigma_k P_{-e} e^{i\mu}\|_{W_{n/(\hat{p} \wedge \hat{q}) + \epsilon}} \sim \sup_{k \in \mathbb{Z}^n} \langle k \rangle^{-\delta} \|\sigma_k e^{i\mu}\|_{W_{n/(\hat{p} \wedge \hat{q}) + \epsilon}}.$$

Denote by

$$A_k = \{ l \in \mathbb{Z}^n : \sigma_l \cdot \sigma_k(\frac{\cdot}{\langle k \rangle^s}) \neq 0 \}.$$

Then

$$\|\sigma_k e^{i\mu}\|_{W_{n/(\hat{p} \wedge \hat{q}) + \epsilon}} = \|\sigma_k \sum_{l \in A_k} \sigma_l(\langle k \rangle^s) e^{i\mu}\|_{W_{n/(\hat{p} \wedge \hat{q}) + \epsilon}}$$

$$\lesssim \|\sigma_k\|_{W_{n/(\hat{p} \wedge \hat{q}) + \epsilon}} \|\sum_{l \in A_k} \sigma_l(\langle k \rangle^s) e^{i\mu}\|_{W_{n/(\hat{p} \wedge \hat{q}) + \epsilon}}$$

$$\lesssim \|\sum_{l \in A_k} \sigma_l(\langle k \rangle^s) e^{i\mu}\|_{W_{n/(\hat{p} \wedge \hat{q}) + \epsilon}}^1,$$

where we use the fact

$$\|\sigma_k\|_{W_{n/(\hat{p} \wedge \hat{q}) + \epsilon}} = \|\sigma_0\|_{W_{n/(\hat{p} \wedge \hat{q}) + \epsilon}} \lesssim 1.$$

Denote

$$\mu_k(x) := \mu(\frac{x}{\langle k \rangle^s}).$$
Let us turn to the estimate of $\| \sum_{l \in \mathbb{A}_k} \sigma_l \langle \langle k \rangle^s \cdot \rangle e^{i\mu} \|_{W_{n/\mu}^{\infty, \infty}}$ as follows:

$$
\| \sum_{l \in \mathbb{A}_k} \sigma_l \langle \langle k \rangle^s \cdot \rangle e^{i\mu} \|_{W_{n/\mu}^{\infty, \infty}} \leq \| \sum_{l \in \mathbb{A}_k} \sigma_l \langle \langle k \rangle^s \cdot \rangle e^{i\mu} \|_{\mathcal{F} L_{n/\mu}^{\infty, \infty}}
$$

$$
= \| (k)^{-sn} \mathcal{F}^{-1} \left( \sum_{l \in \mathbb{A}_k} \sigma_l e^{i\mu} \right) (x) (x)^{n/\mu} \|_{L^{\infty}}
$$

$$
\lesssim \langle k \rangle^{-sn} \langle k \rangle^{s(n/\mu)+\epsilon} \| \mathcal{F}^{-1} \left( \sum_{l \in \mathbb{A}_k} \sigma_l e^{i\mu} \right) (x) (x)^{n/\mu} \|_{L^{\infty}}
$$

Observe that

$$
|\mathbb{A}_k| \lesssim \langle k \rangle^{sn}.
$$

We further have

$$
\| \sum_{l \in \mathbb{A}_k} \sigma_l \langle \langle k \rangle^s \cdot \rangle e^{i\mu} \|_{W_{n/\mu}^{\infty, \infty}} \leq \langle k \rangle^{s(1/\mu) - 1} \langle k \rangle^{se} \sum_{l \in \mathbb{A}_k} \| \mathcal{F}^{-1} (\sigma_l e^{i\mu}) (x) (x)^{n/\mu} \|_{L^{\infty}}
$$

$$
= \langle k \rangle^{s(\epsilon + n/\mu)} \sup_{l \in \mathbb{A}_k} \| \sigma_l e^{i\mu} \|_{W_{n/\mu}^{\infty, \infty}}
$$

Write

$$
\mu_k (\xi) = \mu_k (l) + \int_0^1 (\xi - l) \cdot \nabla \mu_k ((1 - t)l + t\xi) \, dt =: \mu_k (l) + R^k_t.
$$

Then

$$
\| \sigma_l e^{i\mu} \|_{W_{n/\mu}^{\infty, \infty}} \leq \| \sigma_l R^k_t \|_{W_{n/\mu}^{\infty, \infty}}
$$

$$
\lesssim \| \sigma_l \|_{W_{n/\mu}^{\infty, \infty}} \exp \left( C \| \sigma_l^* R^k_t \|_{W_{n/\mu}^{\infty, \infty}} \right) \leq \exp \left( C \| \sigma_l^* R^k_t \|_{W_{n/\mu}^{\infty, \infty}} \right).
$$
Take $h$ to be a $C^\infty_c(\mathbb{R}^n)$ function such that $h(\xi) = 1$ on the support of $\sigma^*_0$. We have
\[
\|\sigma^* R^k_\ell\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}} = \|\sigma^*_0 \int_0^1 \xi \cdot \nabla \mu_k(l + t\xi)h(t\xi)dt\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}}
\lesssim \sum_{|\gamma|=1} \|\sigma^*_0 \xi^\gamma\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}} \int_0^1 h(t\xi) \partial^\gamma \mu_k(l + t\xi)dt\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}}
\lesssim \sum_{|\gamma|=1} \int_0^1 \|h(t\xi) \partial^\gamma \mu_k(l + t\xi)\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}} dt
\lesssim \sum_{|\gamma|=1} \|h(\xi) \partial^\gamma \mu_k(l + \xi)\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}} = \sum_{|\gamma|=1} \|h(\cdot - l) \partial^\gamma \mu_k\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}}.
\]

To estimate $\|h(\cdot - l) \partial^\gamma \mu_k\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}}$, we will use the assumption that $\langle \xi \rangle^{-s} \nabla \mu(\xi) \in (W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon})^n$. Choose a function $C^\infty_c(\mathbb{R}^n)$ function $g$ such that
\[
g\left(\frac{\xi}{|k|}\right) - k)h(\xi - l) = h(\xi - l), \quad l \in A_k.
\]
Then
\[
\langle k \rangle^{-s}\|g(\cdot - k) \partial^\gamma \mu\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}} \sim \|g(\cdot - k) P_{-s} \partial^\gamma \mu\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}}\lesssim \|g(\cdot - k)\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}} \|P_{-s} \partial^\gamma \mu\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}}\lesssim \|P_{-s} \partial^\gamma \mu\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}} \lesssim 1.
\]

By dilation property of $W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}$, we get
\[
\|g\left(\frac{\cdot}{|k|}\right) - k) \partial^\gamma \mu\left(\frac{\cdot}{|k|}\right)\|_{W^{\infty,\infty}_{0,n/(p\wedge q)+\varepsilon}} \lesssim \|g(\cdot - k) \partial^\gamma \mu\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}}.
\]
Then
\[
\|g\left(\frac{\cdot}{|k|}\right) - k) (\partial^\gamma \mu_k(\cdot)\|_{W^{\infty,\infty}_{0,n/(p\wedge q)+\varepsilon}} = \langle k \rangle^{-s}\|g\left(\frac{\cdot}{|k|}\right) - k) \partial^\gamma \mu\left(\frac{\cdot}{|k|}\right)\|_{W^{\infty,\infty}_{0,n/(p\wedge q)+\varepsilon}}\lesssim \langle k \rangle^{-s}\|g(\cdot - k) \partial^\gamma \mu\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}}\lesssim \|P_{-s} \partial^\gamma \mu\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}} \lesssim 1.
\]
For every $l \in A_k$, we get
\[
\|h(\cdot - l) \partial^\gamma \mu_k\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}} = \|h(\cdot - l)g\left(\frac{\cdot}{|k|}\right) - k) \partial^\gamma \mu_k\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}}\lesssim \|h(\cdot - l)\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}} \|g\left(\frac{\cdot}{|k|}\right) - k) \partial^\gamma \mu_k\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}} \lesssim 1.
\]
Combining with the above estimates yields that
\[
\|P_{-\delta} e^{i\mu}\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}} \sim \sup_{k \in \mathbb{Z}^n} \langle k \rangle^{-\delta}\|\sigma_k e^{i\mu}\|_{W^{\infty,\infty}_{n/(p\wedge q)+\varepsilon}}\lesssim \sup_{k \in \mathbb{Z}^n} \langle k \rangle^{\alpha + \alpha n/(p\wedge q) - \delta} \lesssim 1,
\]
where we use the assumption (3.2) in the last inequality. \qed
3.3. Interpolation with modulation case. This subsection is devoted to the proof of Theorem 1.3. Thanks to the additional information of the second order derivative of \( \mu \) such like the prototype \( |\xi|^\beta (\beta \in (1, 2)) \), the conclusion in Theorem 1.2 can be improved to be a more sharpen one. First, we establish the following lemma under the assumption associated with the second derivative of \( \mu \). This lemma is a slight generalization of [15] Theorem 1.1.

Lemma 3.1. Suppose \( 0 < p, q \leq \infty \). Let \( \mu \in C^2(\mathbb{R}^n) \) be a real-valued function satisfying

\[
\partial^\gamma \mu \in W^{\infty, p}_{n/p + \epsilon} (|\gamma| = 2).
\]

Then \( e^{i(\mu)} \) is bounded on \( M^{p,q} \).

Proof. By the convolution relation

\[
M^{p,q} \ast M^{p,\infty} \subset M^{p,q},
\]

we only need to verify that \( \mathcal{F}^{-1} e^{i\mu} \in M^{p,\infty} \), or equivalently, \( e^{i\mu} \in W^{\infty, p} \). Write

\[
\|e^{i\mu}\|_{W^{\infty, p}} \sim \sup_{k \in \mathbb{Z}^n} \||\sigma_k e^{i\mu}\|_{W^{\infty, p}} = \sup_{k \in \mathbb{Z}^n} \|\sigma_0 e^{i\mu(-k)}\|_{W^{\infty, p}} = \sup_{k \in \mathbb{Z}^n} \|\sigma_0 e^{iR_k}\|_{W^{\infty, p}},
\]

where

\[
R_k(\xi) := \mu(k + \xi) - \mu(k) - \nabla \mu(k) \cdot \xi = \sum_{|\gamma| = 2} \frac{2\xi^\gamma}{\gamma!} \int_0^1 (1 - t) \partial^\gamma \mu(k + t\xi) dt.
\]

Next,

\[
\|\sigma_0 e^{iR_k}\|_{W^{\infty, p}} \lesssim \|\sigma_0 e^{i\sigma_0 R_k}\|_{W^{\infty, p}} = \|\sigma_0 e^{i\sigma_0 R_k}\|_{W^{\infty, p}}.
\]

Using Lemma 2.8 we further conclude that

\[
\|\sigma_0 e^{i\sigma_0 R_k}\|_{W^{\infty, p}} \lesssim \|\sigma_0\|_{W^{\infty, p}} \|e^{i\sigma_0 R_k}\|_{W^{\infty, p}} \lesssim \|e^{i\sigma_0 R_k}\|_{W^{\infty, p}} \lesssim \exp \left( C \|\sigma_0 R_k\|_{W^{\infty, p}} \right).
\]

We continue this estimate as follows:

\[
\|\sigma_0^p R_k\|_{W^{\infty, p}} = \left\|\sigma_0^p \sum_{|\gamma| = 2} \frac{2\xi^\gamma}{\gamma!} \int_0^1 (1 - t) \partial^\gamma \mu(k + t\xi) dt \right\|_{W^{\infty, p}} \lesssim \sum_{|\gamma| = 2} \|\sigma_0^p \xi^\gamma\|_{W^{\infty, p}} \left\|\int_0^1 (1 - t) \partial^\gamma \mu(k + t\xi) dt \right\|_{W^{\infty, p}} \lesssim \sum_{|\gamma| = 2} \left\|\int_0^1 (1 - t) \partial^\gamma \mu(k + t\xi) dt \right\|_{W^{\infty, p}} \lesssim \sum_{|\gamma| = 2} \int_0^1 \|\partial^\gamma \mu(k + t\xi)\|_{W^{\infty, p}} dt
\]

where in the last inequality we use the dilation property property in Lemma 2.9.
Combining the above estimates yields that
\[
\| e^{i\mu} \|_{W^{\infty,p}} \sim \sup_{k \in \mathbb{Z}^n} \| |\sigma_k e^{i\mu}| \|_{W^{\infty,p}} \\
\lesssim \sup_{k \in \mathbb{Z}^n} \| |\sigma_0 e^{iR_k}| \|_{W^{\infty,p}} \\
\lesssim \sup_{k \in \mathbb{Z}^n} \exp \left( C \| |\sigma_R R_k| \|_{W^{\infty,p}} \right) \lesssim \sup_{k \in \mathbb{Z}^n} \exp \left( C \sum_{|\gamma|=2} \| \partial^\gamma \mu \|_{W^{\infty,p}} \right).
\]

Proof of Theorem 1.3. Note that $M^{p,p} = W^{p,p}$. By Lemma 3.1 we obtain the boundedness of $e^{i\mu(D)}$ on $W^{p,p}$. The desired conclusion then follows by the interpolation between this and the boundedness of $e^{i\mu(D)} : W^{p,\infty}_{\delta} \rightarrow W^{p,q}(\delta > sn/(\hat{p}\wedge \hat{q}))$ derived by Theorem 1.2.

\[ \square \]

3.4. An application to derivative condition. We give the proof of Corollary 1.4 in this section. By an embedding lemma, the boundedness result in Theorem 1.3 yields the conclusion in Corollary 1.4. First, we recall a useful lemma for dealing with the multiplier near the origin.

Lemma 3.2 (see [14, 16]). Suppose $0 < p \leq \infty$, and $\mu > n(1/\hat{p} - 1)$. Let $\mu$ be a $C^{[n(1/\hat{p}-1/2)+1]}(\mathbb{R}^n\setminus\{0\})$ function with compact support, satisfying
\[
|\partial^\gamma \mu(\xi)| \leq C_\gamma |\xi|^{-|\gamma|}, \quad |\xi| \neq 0, \quad |\gamma| \leq [n(1/\hat{p} - 1/2)] + 1,
\]
then $m \in \mathcal{F}L^p$.

Next, we establish the following embedding relations which will be used in the proof of Corollary 1.4.

Lemma 3.3. Let $N \in \mathbb{N}$, $f \in C^N$, where $C^N$ is defined by
\[
C^N := \{ g \in C^N : |\partial^\gamma g| \lesssim 1 \text{ for all } |\gamma| \leq N \},
\]
with the norm $\| g \|_{C^N} := \sum_{|\gamma| \leq N} \| \partial^\gamma g \|_{L^\infty}$. We have $f \in W^{\infty,n}_N$ and
\[
\| f \|_{W^{n,\infty}_N} \lesssim \| f \|_{C^N}.
\]

Proof. Write
\[
\| f \|_{W^{n,\infty}_N} \sim \sup_{k \in \mathbb{Z}^n} \| |\sigma_k f| \|_{W^{n,\infty}} \sim \sup_{k \in \mathbb{Z}^n} \| \mathcal{F}^{-1} |\sigma_k f| \|_{L^\infty}.
\]

To estimate $\| \mathcal{F}^{-1} |\sigma_k f| \|_{L^\infty}$, we define the invariant derivative
\[
L_x := \frac{x \cdot \nabla}{2\pi i |x|^2}.
\]

Then
\[
L_{e^{2\pi i x}}(e^{2\pi i x} \xi) = e^{2\pi i x} \xi.
\]

We also have
\[
L_x = -L_x.
\]

Then
\[
\mathcal{F}^{-1} \sigma_k f(x) = \int_{\mathbb{R}^n} \sigma_k(\xi) f(\xi) (L_x)^N (e^{2\pi i x} \xi) d\xi \\
= \int_{\mathbb{R}^n} (L_x^*)^N (\sigma_k(\xi) f(\xi)) e^{2\pi i x} \xi d\xi.
\]
From this, we further have
\[ |\mathcal{F}^{-1}\sigma_k f(x)| \lesssim \int_{\mathbb{R}^n} |(L^*)^N(\sigma_k(\xi)f(\xi))|d\xi \]
\[ \lesssim |x|^{-N} \int_{\mathbb{R}^n} \sum_{|\gamma|+|\gamma|_2=N} |\partial^{\gamma_1}\sigma_k(\xi)\partial^{\gamma_2}f(\xi)|d\xi \]
\[ \lesssim |x|^{-N} \|f\|_{C^N} \int_{\mathbb{R}^n} \sum_{|\gamma|+|\gamma|_2=N} |\partial^{\gamma_1}\sigma_k(\xi)|d\xi \lesssim \|f\|_{C^N} |x|^{-N}. \]

This and the fact \[ |\mathcal{F}^{-1}\sigma_k f(x)| \lesssim \|\sigma_k f\|_{L^1} \lesssim \|f\|_{C^N} \] yield that
\[ |\mathcal{F}^{-1}\sigma_k f(x)| \lesssim \|f\|_{C^N} |x|^{-N}. \]

From this, we get the desired conclusion
\[ \|f\|_{W^{\infty,0}} \sim \sup_{k \in \mathbb{Z}^n} \|\mathcal{F}^{-1}\sigma_k f(\cdot)\|_{L^\infty} \lesssim \|f\|_{C^N}. \]

\[ \square \]

**Proof of Corollary 1.4.** Let \( \rho_0 \) be a smooth function supported on \( B(0,1) \), and denote by
\[ \mu_1 := \rho_0 \mu, \quad \mu_2 := (1-\rho_0)\mu. \]
Take \( \rho_0^* \) to be a smooth function supported on \( B(0,2) \), satisfying that \( \rho_0 \cdot \rho_0^* = \rho_0 \).

Note that
\[ |\partial^\gamma [\rho_0^*(e^{i\mu_1} - 1)](\xi)| \lesssim |\xi|^{-|\gamma|}, \quad |\xi| \neq 0, \quad |\gamma| \leq [n/(\hat{p} - 1/2)] + 1. \]

Using Lemma 3.2, we have \( \rho_0^*(e^{i\mu_1} - 1) \in \mathcal{F} L^\hat{p} \). Then,
\[ e^{i\mu_1} = \rho_0^*(e^{i\mu_1} - 1) + \rho_0^* \in \mathcal{F} L^\hat{p}. \]

Moreover, observing that \( e^{i\mu_1} \) have compact support, we further have
\[ \|\mathcal{F}^{-1}e^{i\mu_1}\|_{W^{p,\infty}} \sim \|e^{i\mu_1}\|_{M^{p,\infty}} \sim \|e^{i\mu_1}\|_{\mathcal{F} L^p} \lesssim 1. \]

It follows by \( W^{p,q} * W^{p,\infty} \subset W^{p,q} \) that \( e^{i\mu_1(D)} \) is bounded on \( W^{p,q} \).

Now, we turn to deal with \( e^{i\mu_2} \).

If \( \beta \in (0,1] \), we have,
\[ |\partial^\gamma \mu_2(\xi)| \lesssim 1, \quad 1 \leq |\gamma| \leq [n/(\hat{p} \wedge \hat{q})] + 2. \]

Thus
\[ \nabla \mu_2 \in (C^{[n/(\hat{p} \wedge \hat{q})]+1})^n. \]

There exists a small positive constant \( \epsilon \) such that \( n/(\hat{p} \wedge \hat{q}) + \epsilon \leq [n/(\hat{p} \wedge \hat{q})] + 1 \) and
\[ W^{n/(\hat{p} \wedge \hat{q})+1}_{[n/(\hat{p} \wedge \hat{q})]+1} \subset W^n_{[n/(\hat{p} \wedge \hat{q})]+1}. \]

From this and the embedding relation \( C^{[n/(\hat{p} \wedge \hat{q})]+1} \subset W^n_{[n/(\hat{p} \wedge \hat{q})]+1} \) derived by Lemma 3.3, we further deduce that
\[ \nabla \mu_2 \in (C^{[n/(\hat{p} \wedge \hat{q})]+1})^n \subset (W^n_{n/(\hat{p} \wedge \hat{q})+\epsilon})^n. \]

It then follows by Theorem 1.1 that \( e^{i\mu_2(D)} \) is bounded on \( W^{p,q} \).

If \( \beta \in (1,2] \), we have
\[ |\partial^\gamma \mu_2(\xi)| \lesssim |\xi|^{\beta - |\gamma|}, \quad 1 \leq |\gamma| \leq [n/\hat{p}] + 3. \]
Denote by $s := \beta - 1$. We have
\[
(\xi)^{-s}\nabla \mu_2 \in (C^{[n/(\hat{p}\hat{q})]+1})^n, \quad \partial^\gamma \mu_2 \in C^{[n/(\hat{p}\hat{q})]+1} \ (|\gamma| = 2).
\]
By the similar method as in the above case $\beta \in (0, 1]$, we further deduce that
\[
(\xi)^{-s}\nabla \mu_2 \in (W^{\infty,\infty}_{n/(\hat{p}\hat{q})+\epsilon})^n, \quad \partial^\gamma \mu_2 \in W^{\infty,\infty}_{n/(\hat{p}\hat{q})+\epsilon} \ (|\gamma| = 2).
\]
This and Theorem \ref{1.3} imply the boundedness of $e^{i\mu_2(D)}$ from $W^p,q_{\delta}^r$ to $W^p,q$, where
\[
\delta \geq sn[1/p - 1/q] = n(\beta - 1)[1/p - 1/q] \text{ with strict inequality when } p \neq q.
\]
Combining the above two cases we have that $e^{i\mu_2(D)} : W^p,q_{\delta}^r \rightarrow W^p,q$ is bounded, where
\[
\delta \geq n[1/p - 1/q] \max\{\beta - 1, 0\} \text{ with strict inequality when } \beta > 1, p \neq q.
\]
This and the boundedness of $e^{i\mu_1(D)}$ yield the final conclusion of Corollary \ref{4.4} \hfill \Box

4. SHARP LOSS OF POTENTIAL

This section is devoted to the proof of Theorem \ref{1.3}. To this end, we first develop the scattered property of $\nabla \mu$ in Lemma \ref{1.1} and then use this to establish some useful estimates of certain special functions in Lemma \ref{1.2}. Thanks to these estimates and a rotation trick, the boundedness results on Wiener amalgam spaces can be reduced to the simple embedding relations of weighted sequences. This yields the final conclusion in Theorem \ref{1.3}

Lemma 4.1 (Scattered set). Let $1 < \beta \leq 2$, and let $\mu$ be a real-valued $C^2(\mathbb{R}^n \setminus \{0\})$ function. Suppose that the Hessian determinant of $\mu$ is not zero at some point $\kappa_0$ with $|\kappa_0| = 1$. Moreover,
\[
\mu(\lambda \xi) = \lambda^\beta \mu(\xi), \quad \lambda \geq 1, \quad \xi \in B(\kappa_0, r_0) \cap \mathbb{S}^{n-1}
\]
for some $r_0 > 0$. Then there exists a cone $\Gamma$ with vertex at the origin such that the set \{\nabla \mu(\xi)\}_{\xi \in \Gamma}$ is scattered in the following sense:
\[
|\nabla \mu(\xi_k) - \nabla \mu(\xi_l)| \geq C > 0
\]
for $k, l \in A$, $k \neq l$, where
\[
E := \{l \in \mathbb{Z}^n \setminus \{0\} : \langle l \rangle^{\frac{2-\beta}{2-\beta}} \in \Gamma\}, \quad \xi_l := \langle l \rangle^{\frac{2-\beta}{2-\beta}} l.
\]
Proof. Note that the Hessian matrix of $\mu$ is a real symmetric matrix. This and the assumption $|\text{Hess}_\mu(\kappa_0)| \neq 0$ imply that all the eigenvalues of $\text{Hess}_\mu(\kappa_0)$, denoted by $\lambda_j, j = 1, 2, \cdots, n$, are nonzero real numbers. Write
\[
\text{Hess}_\mu(\kappa_0) = P^{-1} \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) P,
\]
where $P$ is a orthogonal matrix. Take
\[
B = \text{diag}(\text{sgn}\lambda_1, \text{sgn}\lambda_2, \cdots, \text{sgn}\lambda_n), \quad Q = P^{-1} B P.
\]
Denote by $A(\xi) := Q(\text{Hess}_\mu(\xi))$. Then
\[
A(\kappa_0) := Q(\text{Hess}_\mu(\kappa_0)) = P^{-1} B(\text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)) P
\]
\[
= P^{-1} (\text{diag}(|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|)) P.
\]
By the continuity of $\partial_{1,2}\mu$, we can write
\[
\text{Hess}_\mu(\xi) = \text{Hess}_\mu(\kappa_0) + R(\xi - \kappa_0),
\]
where $R$ is a matrix satisfying
\[
\lim_{\xi \rightarrow 0} ||R(\xi)|| = 0. \quad (4.1)
\]
For any two fixed point \( \xi \),

\[
| x^T A(\kappa_0) x | = | x^T P^{-1}(\text{diag}(|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|)) P x |
\]

\[
= | (P x)^T \text{diag}(|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|) P x |
\]

\[
= \sum_{j=1}^n |\lambda_j| \cdot |(P x)_j|^2 \geq \min_{1 \leq j \leq n} |\lambda_j| \cdot |P x|^2 = \min_{1 \leq j \leq n} |\lambda_j| \cdot |x|^2
\]

and

\[
| x^T (A(\xi) - A(\kappa_0)) x | = | x^T Q(\text{Hess} \mu(\xi) - \text{Hess} \mu(\kappa_0)) x |
\]

\[
= | x^T (QR(\xi - \kappa_0)) x |
\]

\[
\leq \| QR(\xi - \kappa_0) \| |x|^2 = \| R(\xi - \kappa_0) \| |x|^2.
\]

From the above two estimates and (1.1), we can choose a small constant \( r \in (0, r_0) \) such that for \( \xi \in S^{n-1} \cap B(\kappa_0, r) \),

\[
| x^T A(\xi) x | \geq | x^T A(\kappa_0) x | - | x^T (A(\xi) - A(\kappa_0)) x |
\]

\[
\geq \min_{1 \leq j \leq n} |\lambda_j| \cdot |x|^2 - \| R(\xi - \kappa_0) \| |x|^2
\]

\[
\geq \frac{1}{2} \min_{1 \leq j \leq n} |\lambda_j| \cdot |x|^2.
\]  \( \text{(4.2)} \)

Denote by \( \Gamma \) the cone containing all vectors \( \xi \) such that \( \frac{\xi}{|\xi|} \in S^{n-1} \cap B(\kappa_0, r) \). By the assumption, we know that for every \( |\gamma| = 2 \),

\[
\partial^\gamma \mu(\lambda \xi) = \lambda^{\beta-2} \partial^\gamma \mu(\xi), \quad \lambda \geq 1, \quad \xi \in S^{n-1} \cap B(\kappa_0, r),
\]

which implies that

\[
A(\lambda \xi) = Q(\text{Hess} \mu(\lambda \xi)) = \lambda^{\beta-2} Q \text{Hess} \mu(\xi) = \lambda^{\beta-2} A(\xi), \quad \lambda \geq 1, \quad \xi \in S^{n-1} \cap B(\kappa_0, r).
\]

From this and (4.2), we know that for \( \xi \in \Gamma \setminus B(0, 1) \)

\[
| x^T A(\xi) x | \geq |\xi|^{\beta-2} \frac{1}{2} \min_{1 \leq j \leq n} |\lambda_j| \cdot |x|^2 \geq |\xi|^{\beta-2} |x|^2.
\]  \( \text{(4.3)} \)

Set

\[
E := \{ l \in \mathbb{Z}^n \setminus \{0\} : \langle l \rangle^{\frac{2-n}{\beta-1}} l \in \Gamma \}, \quad \xi_l := \langle l \rangle^{\frac{2-n}{\beta-1}} l.
\]

Next, we will proof that \( \{ \nabla \mu(\xi_l) \}_{l \in A} \) is a scattered set in the sense that

\[
| \nabla \mu(\xi_k) - \nabla \mu(\xi_l) | \geq C, \quad k, l \in A, \quad k \neq l.
\]

For any two fixed point \( \xi_l, \xi_k \) we set

\[
F(\theta) = \nabla \mu(\xi_l + \theta(\xi_k - \xi_l)) \cdot (Q(\xi_k - \xi_l)).
\]
A direct calculation yields that

\[
F'(\theta) = \left( \sum_{j=1}^{n} \partial_j \mu(\xi + \theta(\xi - \xi)) Q(\xi_k - \xi) \right)'
\]

\[
= \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \partial_{ij} \mu(\xi + \theta(\xi - \xi)) Q(\xi_k - \xi) \right) \cdot (\xi_k - \xi)_{ij}
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{ij} \mu(\xi + \theta(\xi - \xi)) Q(\xi_k - \xi) \cdot (\xi_k - \xi)_{ij}
\]

\[
= (Q(\xi_k - \xi))^{T} \text{Hess} \mu(\xi + \theta(\xi - \xi))(\xi_k - \xi).
\]

Observe that \((Q(\xi_k - \xi))^T = (\xi_k - \xi)^T Q^T = (\xi_k - \xi)^T Q\), we have

\[
F'(\theta) = (\xi_k - \xi)^T Q \text{Hess} \mu(\xi + \theta(\xi - \xi))(\xi_k - \xi)
\]

\[
= (\xi_k - \xi)^T A(\xi + \theta(\xi - \xi))(\xi_k - \xi).
\]

Note that \(\xi + \theta(\xi - \xi) \in \Gamma \setminus B(0, 1)\). We use (4.3) to further obtain

\[
F'(\theta) \gtrsim |\xi + \theta(\xi - \xi)|^2 |\xi_k - \xi|^2.
\]

Next,

\[
F(1) - F(0) = \int_{0}^{1} F'(\theta) d\theta
\]

\[
\geq |\xi_k - \xi|^{2} \int_{0}^{1} |\xi + \theta(\xi - \xi)|^2 d\theta.
\]

By the following identity

\[
4 \left( \frac{|\xi|}{2|\xi_k - \xi|} \right) + \left( 1 - \frac{2|\xi|}{|\xi_k - \xi|} \right) = 1,
\]

we have

\[
\max \left\{ 4 \left( \frac{|\xi|}{2|\xi_k - \xi|} \right), 1 - \frac{2|\xi|}{|\xi_k - \xi|} \right\} \geq 1/2.
\]

If \(4 \left( \frac{|\xi|}{2|\xi_k - \xi|} \right) \geq 1/2\), we have

\[
|\xi| \geq 2\theta|\xi_k - \xi|, \quad (\theta \leq 1/8).
\]

Then

\[
\int_{0}^{1} |\xi + \theta(\xi_k - \xi)|^2 d\theta \leq \int_{0}^{1/8} |\xi + \theta(\xi_k - \xi)|^2 d\theta
\]

\[
\geq \int_{0}^{1/8} |\xi|^2 d\theta \sim |\xi|^2.
\]

If \(1 - \frac{2|\xi|}{|\xi_k - \xi|} \geq 1/2\), we have

\[
\theta|\xi_k - \xi| \geq 2|\xi|, \quad (1/2 \leq \theta \leq 1).
\]
Then
\[
\int_0^1 |\xi_l + \theta(\xi_k - \xi_l)|^{\beta-2} d\theta \geq \int_{1/2}^1 |\xi_l + \theta(\xi_k - \xi_l)|^{\beta-2} d\theta \\
\geq \int_{1/2}^1 |\theta(\xi_k - \xi_l)|^{\beta-2} d\theta \gtrsim |\xi_k - \xi_l|^{\beta-2}.
\] (4.6)

The combination of (4.4), (4.5) and (4.6) yields that
\[
|F(1) - F(0)| \gtrsim |\xi_k - \xi_l|^2 \min\{|\xi_l|^{\beta-2}, |\xi_k - \xi_l|^{\beta-2}\}.
\]

On the other hand
\[
|\xi_k - \xi_l|^2 \min\{|\xi_l|^{\beta-2}, |\xi_k - \xi_l|^{\beta-2}\}.
\]

It follows by the above two estimates that
\[
\min\{|\xi_k - \xi_l|^{\beta-2}, |\xi_k - \xi_l|^{\beta-2}\}.
\]

Note denotes in Lemma 4.1 there exists a constant \(\delta > 0\) such that
\[
B(\xi_l, \delta|\xi_l|^{2-\beta}) \cap B(\xi_k, \delta|\xi_k|^{2-\beta}) = \emptyset,
\]
which implies that
\[
|\xi_k - \xi_l| \gtrsim |\xi_l|^{2-\beta}.
\]

Hence,
\[
|\xi_k - \xi_l| \gtrsim |\xi_l|^{2-\beta}.
\]

This completes the proof of Lemma 4.1.

**Lemma 4.2** (Estimates of special functions). Let \(0 < p, q \leq \infty, s \in \mathbb{R}\). Let \(\mu\) be a real-valued function satisfying the assumptions of Theorem 1.3. Let \(E\) be the set mentioned in Lemma 4.1. For every fixed nonnegative truncated (only finite nonzero items) sequence \(\{a_k\}_{k \in \mathbb{E}}\), there exists a Schwartz function \(F\) corresponding to \(\{a_k\}_{k \in \mathbb{E}}\), such that the following two estimates is valid

1. \(\|e^{i\theta(D)} F\|_{W^{p,q}} \sim \|\{a_k\}_{k \in \mathbb{E}}\|_{W^p};\)
2. \(\|F\|_{W^{p,q}_{s}} \sim \|\{a_k\}_{k \in \mathbb{E}}\|_{W^p_{s/(s-1)}};\)

Moreover, the above estimates is uniformly for all \(\{a_k\}_{k \in \mathbb{E}}\).

**Proof.** It follows by Lemma 4.1 that there exists a constant \(R > 0\) such that the family \(\{B(\nabla \mu(\xi_k), R)\}_{k \in \mathbb{E}}\) is pairwise disjoint. In additional, by Lemma 2.11 there exists a positive constant \(r < 1/2\) such that the family \(\{B(\xi_k, r)\}_{k \in \mathbb{Z}^n}\) is pairwise disjoint. Take \(\hat{h}\) and \(\hat{h}^*\) be two real-valued nonnegative Schwartz functions satisfying
\[
\hat{h}(\xi) = \hat{h}^*(\xi) = 1\] on \(B(0, r/8)\), \(\supp\hat{h} \subset \supp\hat{h}^* \subset B(0, r/4)\), \(\hat{h} \cdot \hat{h}^* = \hat{h} \).

Denote by \(\hat{h}_k(\xi) := \hat{h}(\xi - \xi_k)\) and \(\hat{h}_k^*(\xi) := \hat{h}^*(\xi - \xi_k)\). For any nonnegative truncated (only finite nonzero items) sequence \(\{a_k\}_{k \in \mathbb{E}}\), we set
\[
\hat{F} := \sum_{k \in \mathbb{A}} a_k \hat{h}_k.
\]
Write
\[
e^{i\mu F} = e^{i\mu} \sum_{k \in A} a_k \hat{h}_k = e^{i\mu} \sum_{k \in A} a_k \hat{h}_k \hat{h}_k^* \\
= \left( \sum_{k \in A} \hat{h}_k^*(\xi) e^{i\mu(\xi - i\nabla \mu(\xi_k)\xi)} \right) \cdot \left( \sum_{k \in A} a_k \hat{h}_k(\xi) e^{i\nabla \mu(\xi_k)\xi} \right)
\]
=: m(\xi) \cdot \hat{G}(\xi).
\]

We claim that \( \|m\|_{W^{\infty,\infty}_{n/(p,q)+\varepsilon}} \lesssim 1 \). As in the proof of Theorem \ref{thm1.1} we write
\[
\|m\|_{W^{\infty,\infty}_{n/(p,q)+\varepsilon}} = \sup_{k \in A} \|\hat{h}_k(\xi) e^{i\mu(\xi - i\nabla \mu(\xi_k)\xi)}\|_{W^{\infty,\infty}_{n/(p,q)+\varepsilon}}
\]
\[
= \sup_{k \in A} \|\hat{h}_k^*(\xi) e^{i\mu(\xi - i\nabla \mu(\xi_k)\xi - i\mu(\xi))}\|_{W^{\infty,\infty}_{n/(p,q)+\varepsilon}}
\]
\[
= \sup_{k \in A} \|\hat{h}_k^*(\xi) e^{i\mu(\xi + \xi_k) - i\nabla \mu(\xi_k)\xi - i\mu(\xi)}\|_{W^{\infty,\infty}_{n/(p,q)+\varepsilon}}
\]
\[
= \sup_{k \in A} \|\hat{h}_k^*(\xi) e^{iR_k(\xi)}\|_{W^{\infty,\infty}_{n/(p,q)+\varepsilon}},
\]
where
\[
R_k(\xi) = \mu(\xi_k + \xi) - \mu(\xi_k) - \nabla \mu(\xi_k)\xi = \sum_{|\gamma| = 2} \frac{2\xi_\gamma}{\gamma!} \int_0^1 (1 - t) \partial^\gamma \mu(\xi_k + t\xi) dt.
\]

Take \( \psi \) to be a \( C^\infty_c(\mathbb{R}^n) \) function supported on \( B(0, 1/2) \) such that \( \hat{\psi} \hat{h}^* = \hat{h}^* \). By the similar argument as in the proof of Theorem \ref{thm1.1} we get
\[
\|\hat{h}_k^*(\xi) e^{iR_k(\xi)}\|_{W^{\infty,\infty}_{n/(p,q)+\varepsilon}}
\]
\[
\lesssim \exp \left( C \left\| \psi(\xi) \sum_{|\gamma| = 2} \frac{2\xi_\gamma}{\gamma!} \int_0^1 (1 - t) \psi(t\xi) \partial^\gamma \mu(\xi_k + t\xi) dt \right\|_{W^{\infty,\infty}_{n/(p,q)+\varepsilon}} \right)
\]
\[
\lesssim \exp \left( C \left\| \sum_{|\gamma| = 2} \int_0^1 (1 - t) \psi(t\xi) \partial^\gamma \mu(\xi_k + t\xi) dt \right\|_{W^{\infty,\infty}_{n/(p,q)+\varepsilon}} \right)
\]
\[
\lesssim \exp \left( C \left\| \psi(\cdot - \xi_k) \partial^\gamma \mu(\cdot) \right\|_{W^{\infty,\infty}_{n/(p,q)+\varepsilon}} \right)
\]

Together with this and the following estimate
\[
\sum_{|\gamma|=2} \|\psi(-\xi_k)\partial^\gamma \mu(\cdot)\|_{W^{\infty,\infty}_{0,(p,q)+\epsilon}} = \sum_{|\gamma|=2} \|\psi(-\xi_k)(1-\rho_0)\partial^\gamma \mu(\cdot)\|_{W^{\infty,\infty}_{0,(p,q)+\epsilon}} \\
\lesssim \sum_{|\gamma|=2} \|\psi(-\xi_k)\|_{W^{\infty,\infty}_{0,(p,q)+\epsilon}} \cdot \|(1-\rho_0)\partial^\gamma \mu(\cdot)\|_{W^{\infty,\infty}_{0,(p,q)+\epsilon}} \\
= \sum_{|\gamma|=2} \|\psi\|_{W^{\infty,\infty}_{0,(p,q)+\epsilon}} \cdot \|(1-\rho_0)\partial^\gamma \mu\|_{W^{\infty,\infty}_{0,(p,q)+\epsilon}} \\
\lesssim \sum_{|\gamma|=2} \|(1-\rho_0)\partial^\gamma \mu\|_{W^{\infty,\infty}_{0,(p,q)+\epsilon}},
\]
the desired estimate follows by
\[
\|m\|_{W^{\infty,\infty}_{0,(p,q)+\epsilon}} = \sup_{k \in \mathbb{Z}^n} \|\hat{h}^*(\xi)e^{i\nu \mu(\xi)}\|_{W^{\infty,\infty}_{0,(p,q)+\epsilon}} \\
\lesssim \sup_{k \in \mathbb{Z}^n} \exp \left( C \sum_{|\gamma|=2} \|\psi(-\xi_k)\partial^\gamma \mu(\cdot)\|_{W^{\infty,\infty}_{0,(p,q)+\epsilon}} \right) \\
\lesssim \sup_{k \in \mathbb{Z}^n} \exp \left( C \sum_{|\gamma|=2} \|(1-\rho_0)\partial^\gamma \mu\|_{W^{\infty,\infty}_{0,(p,q)+\epsilon}} \right) \lesssim 1,
\]
where in the last estimate we use the assumption that \((1-\rho_0)\partial^\gamma \mu \in W^{\infty,\infty}_{0,(p,q)+\epsilon}
for |\gamma| = 2\). Hence,
\[
\|e^{i\nu(D)}F\|_{W^{p,q}} = \|m(D)G\|_{W^{p,q}} \\
\lesssim \|\mathcal{F}^{-1}m\|_{W^{\infty,\infty}} \|G\|_{W^{p,q}} \quad (4.7)
\]
\[
= \|m\|_{M^{\infty,\infty}} \|G\|_{W^{p,q}} \lesssim \|m\|_{W^{\infty,\infty}_{0,(p,q)+\epsilon}} \|G\|_{W^{p,q}} \lesssim \|G\|_{W^{p,q}}.
\]

On the other hand, observe
\[
\hat{G} = \sum_{k \in A} a_k \hat{h}(\xi)e^{i\nu \mu(\xi)} \\
= \left( \sum_{k \in A} \hat{h}^*(\xi)e^{-i\nu \mu(\xi)} + i\nu \mu(\xi) \right) \cdot \left( e^{i\nu \mu} \sum_{k \in A} a_k \hat{h}(\xi) \right) = \overline{\mu} e^{i\mu} \hat{F}.
\]
A similar argument yields that
\[
\|\overline{\mu}\|_{W^{\infty,\infty}_{0,(p,q)+\epsilon}} \lesssim 1,
\]
and
\[
\|G\|_{W^{p,q}} \lesssim \|\overline{\mu}\|_{W^{\infty,\infty}_{0,(p,q)+\epsilon}} \|e^{i\mu(D)}F\|_{W^{p,q}} \lesssim \|e^{i\mu(D)}F\|_{W^{p,q}} \quad (4.8)
\]
Then \(\|G\|_{W^{p,q}} \sim \|e^{i\mu(D)}F\|_{W^{p,q}}\) follows by (4.7) and (4.8). Let us turn to the estimate of \(\|G\|_{W^{p,q}}\).

**Lower estimate of \(\|G\|_{W^{p,q}}\).** Take \(\hat{\phi}\) to be a real-valued function supported on \(B(0,3R/4)\) and satisfying
\[
\hat{\phi}(\xi) = 1, \quad \xi \in B(0,r/2).
\]
Thus, for $\eta \in B(\xi, r/4)$,

$$
\hat{G}(\eta)\phi(\eta - \xi) = \sum_{k \in A} a_k \hat{h}_k(\eta)e^{i\nabla \mu(\xi_k)\eta}\hat{\phi}(\eta - \xi)
= a_l \hat{h}_l(\eta)e^{i\nabla \mu(\xi_l)\eta}
$$

Then for $\eta \in B(\xi, r/4)$,

$$
|V_{\phi}G(x, \xi)| = \left| \int_{\mathbb{R}^n} \hat{G}(\eta)\phi(\eta - \xi)e^{2\pi i\eta \cdot x} d\eta \right|
= |a_l| \int_{\mathbb{R}^n} \hat{h}_l(\eta)e^{i\nabla \mu(\xi_l)\eta}e^{2\pi i\eta \cdot x} d\eta
= |a_l| |h(x + \nabla \mu(\xi_l)/2\pi)|.
$$

Next,

$$
\|V_{\phi}G(x, \xi)\|_{L^q_x} \geq \left( \sum_{k \in A} \int_{B(\xi, r/4)} |V_{\phi}G(x, \xi)|^q d\xi \right)^{1/q}
\sim \left( \sum_{k \in A} a_k^q |h(x + \nabla \mu(\xi_k)/2\pi)|^q \right)^{1/q}.
$$

By Lemma 4.1, we know that the family $\{B(-\nabla \mu(\xi_k)/2\pi, R/2\pi)\}_{k \in A}$ is pairwise disjoint. Then the desired lower estimate follows by

$$
\|G\|_{L^{p,q}} = \|V_{\phi}G(x, \xi)\|_{L^q_x} \|L^p_x
\geq \left( \sum_{k \in A} a_k^p |h(x + \nabla \mu(\xi_k)/2\pi)|^q \right)^{1/p}.
$$

**Upper estimate of $\|G\|_{L^{p,q}}$.** By the definition of $h_k$ and $\phi$, we know that for any $k \in A$,

$$
\{ \xi \in \mathbb{R}^n : \hat{h}_k\hat{\phi}(\eta - \xi) \neq 0 \} \subset B(\xi_k, r).
$$
Moreover the family \( \{ B(\xi_k, r) \}_{k \in \mathbb{Z}^n} \) is pairwise disjoint. For \( \xi \in B(\xi_k, r) \),

\[
|V_{\delta} G(x, \xi)| = \left| \int_{\mathbb{R}^n} \hat{G}(\eta) \hat{\varphi}(\eta - \xi) e^{2\pi i \eta \cdot x} d\eta \right|
\]
\[
= a_k \int_{\mathbb{R}^n} \hat{h}_k(\eta) \hat{\varphi}(\eta - \xi) e^{i \mu(\xi_k) \eta} e^{2\pi i \eta \cdot x} d\eta
\]
\[
= \left| a_k \int_{\mathbb{R}^n} \hat{h}_k(\eta) \hat{\varphi}(\eta - \xi) e^{i \mu(\xi_k) \eta} e^{2\pi i \eta \cdot x} d\eta \right|
\]
\[
= a_k |h_k| \cdot |\mathcal{M}_\xi \varphi(x + \nabla \mu(\xi_k))/2\pi| \lesssim a_k |h_k| \cdot |\varphi(x + \nabla \mu(\xi_k))/2\pi| \lesssim (x + \nabla \mu(\xi_k)/2\pi)^{-\mathcal{L}}.
\]

It follows that

\[
\|V_{\delta} G(x, \xi)\|_{L^\infty} \leq \left( \sum_{k \in E} \int_{B(\xi_k, r)} |V_{\delta} G(x, \xi)|^q d\xi \right)^{1/q} 
\]
\[
\lesssim \left( \sum_{k \in E} \int_{B(\xi_k, r)} a_k^q (x + \nabla \mu(\xi_k)/2\pi)^{-\mathcal{L}} d\xi \right)^{1/q} 
\]
\[
\lesssim \left( \sum_{k \in E} a_k^q (x + \nabla \mu(\xi_k)/2\pi)^{-\mathcal{L}} \right)^{1/q} \tag{4.9}
\]
\[
\lesssim \sup_{k \in E} a_k (x + \nabla \mu(\xi_k)/2\pi)^{-\mathcal{L}} \left( \sum_{k \in E} (x + \nabla \mu(\xi_k)/2\pi)^{-\mathcal{L}} \right)^{1/q}.
\]

By Lemma 4.1, we know that

\[
|E_{0,x}| := |k \in \mathbb{Z}^n : |x + \nabla \mu(\xi)/2\pi| \leq 1| \lesssim 1
\]

and

\[
|E_{j,x}| := |k \in \mathbb{Z}^n : 2^{j-1} \leq |x + \nabla \mu(\xi)/2\pi| \leq 2^j| \lesssim 2^{jn},
\]

uniformly for all \( x \in \mathbb{R}^n \) and \( j \geq 1 \). From this, we have

\[
\sum_{k \in E} (x + \nabla \mu(\xi_k)/2\pi)^{-\mathcal{L}} \lesssim \sum_{j=0}^{\infty} \sum_{k \in E_{j,x}} (x + \nabla \mu(\xi_k)/2\pi)^{-\mathcal{L}}
\]
\[
\lesssim \sum_{j=0}^{\infty} |E_{j,x}| 2^{-jn\mathcal{L}} \lesssim \sum_{j=0}^{\infty} 2^{jn} 2^{-jn\mathcal{L}}.
\]

This and (4.9) imply that

\[
\|V_{\delta} G(x, \xi)\|_{L^\infty} \lesssim \sup_{k \in E} a_k (x + \nabla \mu(\xi_k)/2\pi)^{-\mathcal{L}}.
\]
Then, 

\[ \|G\|_{W^{p,q}} = \left\| V_\phi G(x, \xi) \right\|_{L^p_x} \left\| G \right\|_{L^q_x} \]

\[ \lesssim \sup_{k \in E} a_k (\ell + \nabla \mu(\xi_k)/\pi)^{-2} \left\| L^p \right\| 

\[ \lesssim \left( \sum_{k \in E} a_k^p (\ell + \nabla \mu(\xi_k)/\pi)^{-2} \right)^{1/p} \]

\[ = \left( \sum_{k \in E} a_k^p \| \ell + \nabla \mu(\xi_k)/\pi \|_{L^p_x} \right)^{1/p} \lesssim \left( \sum_{k \in E} a_k^p \right)^{1/p} \].

Next, we consider the estimate of \( \|F\|_{W^{p,q}} \).

**Lower estimate of \( \|F\|_{W^{p,q}} \).**

By the definition of \( h_k \) and \( \phi \), we know that for \( \eta \in B(\xi, \sqrt{4}) \),

\[ \hat{F}(\eta) \hat{\phi}(\eta - \xi) = \sum_{k \in E} a_k \hat{h}_k(\eta) \hat{\phi}(\eta - \xi) = a_1 \hat{h}_1(\eta). \]

Then for \( \eta \in B(\xi, \sqrt{4}) \),

\[ |V_\phi F(x, \xi)| = \left| \int_{\mathbb{R}^n} \hat{F}(\eta) \hat{\phi}(\eta - \xi) e^{2\pi i \eta \cdot x} d\eta \right| = a_1 \int_{\mathbb{R}^n} \hat{h}_1(\eta) e^{2\pi i \eta \cdot x} d\eta = a_1 |h_1(x)| = a_1 |h(x)|. \]

Form this we further deduce that

\[ \left\| V_\phi F(x, \xi) \right\|_{L^{q,x}} \geq \left( \sum_{k \in E} \int_{B(\xi_k, r/4)} |V_\phi F(x, \xi)|^q (\xi_k)^{sq} d\xi \right)^{1/q} \]

\[ \sim \left( \sum_{k \in E} a_k^q (\xi_k)^{sq} \right)^{1/q} |h(x)| \sim \left( \sum_{k \in E} a_k^q (\xi_k)^{sq/(\beta - 1)} \right)^{1/q} |h(x)|. \]

Then the desired estimate follows by

\[ \|F\|_{W^{p,q}} = \| |V_\phi F(x, \xi)| \|_{L^{q,x}} \|_{L^p_x} \gtrsim \left( \sum_{k \in E} a_k^q (\xi_k)^{sq/(\beta - 1)} \right)^{1/q} \|h\|_{L^p} \sim \| \{a_k\}_{k \in E} \|_{L^q_{x/(\beta - 1)}}. \]

**Upper estimate of \( \|F\|_{W^{p,q}} \).** Next, we consider the estimate of \( \|F\|_{W^{p,q}} \). By the definition of \( h_k \) and \( \phi \), we know that for any \( k \in A \),

\[ \{ \xi \in \mathbb{R}^n : \hat{h}_k \phi(\eta - \xi) \neq 0 \} \subset B(\xi_k, r), \]
where the family \( \{ B(\xi, r) \}_{k \in \mathbb{Z}^n} \) is pairwise disjoint. For \( \xi \in B(\xi, 1) \),

\[
|V_\phi F(x, \xi)|^q = \left| \int_{\mathbb{R}^n} \hat{F}(\eta) \hat{\phi}(\eta - \xi) e^{2\pi i x \cdot \eta} d\eta \right|^q = \left| a_k \int_{\mathbb{R}^n} \hat{h}_k(\eta) \hat{\phi}(\eta - \xi) e^{2\pi i x \cdot \eta} d\eta \right|^q = \left| a_k \int_{\mathbb{R}^n} \hat{h}_k(\eta) \hat{M}_\xi \phi(\eta) e^{2\pi i x \cdot \eta} d\eta \right|^q = a_k^q |h_k * M_\xi \phi(x)|^q.
\]

For the last term, we further have

\[
a_k^q |h_k * M_\xi \phi(x)|^q \leq a_k^q |h_k| * |M_\xi \phi|(x)^q = a_k^q |h| * |\phi|(x)^q \lesssim a_k^q |x|^{-q \mathcal{L}},
\]

where in the last inequality we use the fact that both \( h \) and \( \phi \) are Schwartz functions, \( \mathcal{L} \) indicates a sufficiently large number.

Now, we have the following estimate

\[
\|V_\phi F(x, \xi)\|_{L^q} = \sum_{k \in E} \int_{B(\xi, r)} |V_\phi F(x, \xi)|^q (\xi)^{sq} d\xi
\]

\[
\lesssim \sum_{k \in E} \int_{B(\xi, r)} a_k^q |x|^{-q \mathcal{L}} (\xi)^{sq} d\xi
\]

\[
\lesssim \sum_{k \in E} a_k^q |\xi_k|^{sq} |x|^{-q \mathcal{L}} \lesssim \sum_{k \in E} a_k^q (k)^{\frac{sq}{\pi^2}} |x|^{-q \mathcal{L}}.
\]

Then,

\[
\|F\|_{W^{p,q}} = \left\| \|V_\phi F(x, \xi)\|_{L^q} \right\|_{L^p}
\]

\[
\leq \left\| \sum_{k \in E} a_k^q (k)^{\frac{pq}{\pi^2}} \right\|_{L^p}^{1/q} \sim \left\| \{a_k\}_{k \in E} \right\|_{\ell^q_{\pi^2/(\beta-1)}}.
\]

**Lemma 4.3** (Rotation trick). Let \( \mu \) be a real-valued functions satisfying the assumptions of Theorem 4.3. For every nonnegative sequence \( \{a_k\}_{k \in \mathbb{Z}^n} \), we have

\[
\left\| \left\{ a_k \right\}_{k \in \mathbb{Z}^n} \right\|_{\ell^p} \lesssim \left\| \left\{ a_k \right\}_{k \in \mathbb{Z}^n} \right\|_{\ell^p_{\pi^2/(\beta-1)}}.
\]

**Proof.** If \( e^{i\mu(D)} : W^{p,q}_x \to W^{p,q} \) is bounded, for any Schwartz function we have

\[
\|e^{i\mu(D)}f\|_{W^{p,q}} \lesssim \|f\|_{W^{p,q}_x}, \quad \|f\|_{W^{p,q}_x} \lesssim \|e^{i\mu(D)}f\|_{W^{p,q}}.
\]

This and Lemma 4.2 yield that

\[
\left\| \left\{ a_k \right\}_{k \in E} \right\|_{\ell^p} \lesssim \left\| \left\{ a_k \right\}_{k \in E} \right\|_{\ell^p_{\pi^2/(\beta-1)}}, \quad \left\| \left\{ a_k \right\}_{k \in E} \right\|_{\ell^p_{\pi^2/(\beta-1)}} \lesssim \left\| \{a_k\}_{k \in E} \right\|_{\ell^p}.
\]

In the above two inequalities, the set \( E \) will be replaced by \( \mathbb{Z}^n \), by using a rotation trick as follows.
Denote by $\mu_P(\xi) := \mu(P^{-1}\xi)$, where $P$ is a orthogonal matrix. By a direct calculation we get

$$e^{i\mu_P(D)}f = P(e^{i\mu(D)}f_{P^{-1}})$$

where $f_{P^{-1}}(x) := f(Px)$. Using Lemma 2.10 we get

$$\|e^{i\mu_P(D)}f\|_{W^{p,q}} = \|P(e^{i\mu(D)}f_{P^{-1}})\|_{W^{p,q}}$$

$$\sim \|e^{i\mu(D)}f_{P^{-1}}\|_{W^{p,q}} \lesssim \|f_{P^{-1}}\|_{W^{p,q}} \sim \|f\|_{W^{p,q}}.$$  

Now, we have proved that the operator $e^{i\mu_P(D)} : W^{p,q} \to W^{p,q}$ is bounded uniformly for all orthogonal matrix $P$. We also have

$$\text{Hess} \mu_P(\xi) = \text{Hess} \mu(P^{-1}\xi).$$

From the above arguments we claim that $\mu_P$ satisfies all the assumption of Theorem 1.5 when $\kappa_0$ is replaced by $P\kappa_0$.

Then we apply the same argument of Lemma 4.1 and 4.2 to the new operator $e^{i\mu_P}$. We get

$$\|\{a_k\}_{k\in E_p}\|_{l^p} \lesssim \|\{a_k\}_{k\in E_p}\|_{r_{s/(\beta-1)}^p}\|\{a_k\}_{k\in E_p}\|_{l^p} \lesssim \|\{a_k\}_{k\in E_p}\|_{l^p}. \quad (4.10)$$

where $E_p := \{l \in \mathbb{Z}^n \setminus \{0\} : \langle l, P \rangle \leq 2^{-\beta} l, z_l := \langle l, \frac{2^{-\beta}}{\beta} \rangle$. The cone chosen in the proof of Lemma 4.1. Note that there exist finite orthogonal matrix, denoted by $P_i$ such that $\bigcup E_{P_i} = \mathbb{Z}^n \setminus \{0\}$. From this and (4.10), we get

$$\|\{a_k\}_{k\in \mathbb{Z}^n}\|_{l^p} \lesssim \sum_i \|\{a_k\}_{k\in E_{P_i}}\|_{l^p} + |a_0|$$

$$\lesssim \sum_i \|\{a_k\}_{k\in E_{P_i}}\|_{r_{s/(\beta-1)}^p} + |a_0| \lesssim \|\{a_k\}_{k\in \mathbb{Z}^n}\|_{r_{s/(\beta-1)}^p}.$$ 

A similar argument yields another desired conclusion:

$$\|\{a_k\}_{k\in \mathbb{Z}^n}\|_{r_{s/(\beta-1)}^p} \lesssim \|\{a_k\}_{k\in \mathbb{Z}^n}\|_{l^p}.$$ 

$\square$

**Proof of Theorem 1.5.** If $p \leq q$, By Lemma 4.3 we have

$$l^q_{s/(\beta-1)} \subset l^p.$$ 

From this and Lemma 2.2 we further obtain

$$1/p \leq 1/q + s/n(\beta - 1) \iff s \geq n(\beta - 1)(1/p - 1/q) = n(\beta - 1)|1/p - 1/q|$$

with strict inequality when $p < q$.

If $p > q$, we use Lemma 4.3 to get

$$l^p \subset l^q_{s/(\beta-1)}.$$ 

Then Lemma 2.2 further imply that

$$1/q - s/n(\beta - 1) < 1/p \iff s > n(\beta - 1)(1/q - 1/p) = n(\beta - 1)|1/p - 1/q|.$$ 

$\square$
Proof of Corollary 1.6 First, the sufficiency follows by Corollary 1.4. Note that \( e^{iD^\beta} \) and \( e^{-iD^\beta} \) are bounded on \( W^{p,q} \) when \( \beta \in (0,1] \). If \( e^{iD^\beta} : W^{\delta,p} \to W^{p,q} \) is bounded, we have

\[
\|f\|_{W^{p,q}} = \|e^{-iD^\beta}(e^{iD^\beta}f)\|_{W^{p,q}} \lesssim \|e^{iD^\beta}\|_{W^{p,q}} \lesssim \|f\|_{W^{\delta,p}},
\]

which implies that \( \delta \geq 0 \). Finally, when \( \beta \in (1,2] \), by Lemma 3.3 we have \( (1 - \rho_0)\partial^\gamma \mu \in W^{\infty,\infty}_{n/(p\wedge q)+\epsilon} (|\gamma| = 2) \) for some \( \epsilon > 0 \). Then the necessity follows by Theorem 1.5. \( \square \)

5. Complements: high growth of \( \mu \)

Keep the prototype \( \mu(\xi) = |\xi|^\beta \) under consideration, if \( \beta > 2 \), we find that the previous working space \( W^{\infty,\infty}_{n/(p\wedge q)+\epsilon} \) should be replaced by a more reasonable one fitting this high growth case. In fact, in the high growth case, the working space is expected to be a function space in which the functions can not only be localized in time, but also be invariant under the modulation operator. By this observation, the Wiener amalgam space without potential, such like \( W^{p,q} \), may be a good choice.

As in [4], to establish the boundedness result on Wiener amalgam spaces, another approach is to use the boundedness result on modulation spaces. Note that the natural working space for modulation case is just a Wiener amalgam space without potential, one can see the natural working space \( W^{\infty,1} \) used in [15]. Here, we first give a generalization of Theorem 1.2 in [15], then by an embedding relations between modulation and Wiener amalgam spaces, we obtain the boundedness results on Wiener amalgam spaces.

Lemma 5.1. Suppose \( 0 < p, q \leq \infty \). Let \( \mu \) be a real-valued \( C^2(\mathbb{R}^n) \) function satisfying

\[
\begin{cases}
  |\xi|^{-s}\partial^\gamma \mu \in W^{\infty,1}_{n/(p\wedge q)+\epsilon}, & \text{if } p \geq 1; \\
  |\xi|^{-s}\partial^\gamma \mu \in W^{n/p-n+\epsilon}_{n/(p\wedge q)+\epsilon} (|\gamma| = 2), & \text{if } p < 1,
\end{cases}
\]

for some \( s, \epsilon > 0 \). Then \( e^{i\mu(D)} : M^{p,q}_\delta \to M^{p,q} \) is bounded for \( \delta \geq sn|1/p - 1/2| \).

Proof. We only give the sketch of this proof, since it is similar as the proof of Theorem 1.2 and Lemma 3.1. By the convolution relation (see [8, Corollary 4.2])

\[
M^{p,q}_\delta * M^{\infty}_{-\delta} \subset M^{p,q},
\]

we only need to verify that \( \mathcal{F}^{-1}e^{i\mu} \in M^{\infty}_{-\delta} \), or equivalently, \( |\xi|^{-\delta}e^{i\mu} \in W^{\infty,p} \).

Write

\[
\|P_\delta e^{i\mu}\|_{W^{\infty,p}} \sim \sup_{k \in \mathbb{Z}^n} \|\sigma_k P_\delta e^{i\mu}\|_{W^{\infty,p}} = \sup_{k \in \mathbb{Z}^n} (k)^{-\delta} \|\sigma_k e^{i\mu}\|_{W^{\infty,p}}.
\]

Set

\[
B_k := \{ l \in \mathbb{Z}^n : \sigma_l \cdot \sigma_k \left( \frac{l}{(k)^{3/2}} \right) \neq 0 \}.
\]
Observe $|B_k| \sim \langle k \rangle^{sn/2}$, and recall that $W^{\infty, \hat{p}}$ is a Banach algebra (see [3, Corollary 4.2]). We have

$$
\| \sigma_k e^{i\mu} \|_{W^{\infty, \hat{p}}} = \| \sigma_k \sum_{l \in B_k} \sigma_l(\langle k \rangle^{s/2}) e^{i\mu} \|_{W^{\infty, \hat{p}}}
\lesssim \| \sigma_k \|_{W^{\infty, \hat{p}}} \sum_{l \in B_k} \| \sigma_l(\langle k \rangle^{s/2}) e^{i\mu} \|_{W^{\infty, \hat{p}}}
\lesssim \| \sum_{l \in B_k} \sigma_l(\langle k \rangle^{s/2}) e^{i\mu} \|_{W^{\infty, \hat{p}}}
\lesssim \left( \sum_{l \in B_k} \| \sigma_l(\langle k \rangle^{s/2}) e^{i\mu} \|_{W^{\infty, \hat{p}}}^p \right)^{1/p} \lesssim \langle k \rangle^{\frac{sn}{\hat{p}}} \sup_{l \in B_k} \| \sigma_l(\langle k \rangle^{s/2}) e^{i\mu} \|_{W^{\infty, \hat{p}}}
$$

Denote by $\mu_k(x) := \mu(\frac{x}{\langle k \rangle^\gamma})$, and

$$R^k_l(\xi) := \mu_k(\xi - l) - \nabla \mu(l)(\xi - l) = \sum_{|\gamma| = 2} \frac{2(\xi - l)^\gamma}{\gamma!} \int_0^1 (1 - t)^{\gamma} \mu((1 - t)l + t\xi) dt.
$$

For $l \in B_k$, we further have

$$
\| \sigma_l(\langle k \rangle^{s/2}) e^{i\mu} \|_{W^{\infty, \hat{p}}} = \| \sigma_l(\langle k \rangle^{s/2}) e^{i\mu} \|_{\mathcal{F}L^p}
= \langle k \rangle^{\frac{sn}{\hat{p}}(1/\hat{p}-1)} \| \sigma_l e^{i\mu} \|_{\mathcal{F}L^p}
= \langle k \rangle^{\frac{sn}{\hat{p}}(1/\hat{p}-1)} \| \sigma_l R^k_l \|_{\mathcal{F}L^p}
\sim \langle k \rangle^{\frac{sn}{\hat{p}}(1/\hat{p}-1)} \| \sigma_l R^k_l \|_{W^{\infty, \hat{p}}}
\lesssim \langle k \rangle^{\frac{sn}{\hat{p}}(1/\hat{p}-1)} \exp(C\| \sigma_l R^k_l \|_{W^{\infty, \hat{p}}}) \lesssim \langle k \rangle^{\frac{sn}{\hat{p}}(1/\hat{p}-1)},
$$

where in the last inequality we use $\| \sigma_l R^k_l \|_{W^{\infty, \hat{p}}} = \| \sigma_l R^k_l \|_{W^{\infty, 1}} \lesssim 1$ for $p \geq 1$, and $\| \sigma_l R^k_l \|_{W^{\infty, \hat{p}}} \lesssim \| \sigma_l R^k_l \|_{W^{\infty, 1}} \lesssim 1$ for $p < 1$, which can be derived by a similar argument as in the proof of Theorem [2].

Combining the above estimates yields that for $p = \infty$ or $p \leq 1$,

$$
P^{-\delta} e^{i\mu} \|_{W^{\infty, \hat{p}}} \sim \sup_{k \in \mathbb{Z}^n} \langle k \rangle^{-\delta} \| \sigma_k e^{i\mu} \|_{W^{\infty, \hat{p}}}
\lesssim \sup_{k \in \mathbb{Z}^n} \langle k \rangle^{-\delta} \langle k \rangle^{\frac{sn}{\hat{p}}(1/\hat{p}-1)} \sup_{l \in B_k} \| \sigma_l(\langle k \rangle^{s/2}) e^{i\mu} \|_{W^{\infty, \hat{p}}}
\lesssim \sup_{k \in \mathbb{Z}^n} \langle k \rangle^{-\delta} \langle k \rangle^{\frac{sn}{\hat{p}}(1/\hat{p}-1)} \sim \sup_{k \in \mathbb{Z}^n} \langle k \rangle^{-\delta + sn(1/\hat{p}-1/2)} \lesssim 1,
$$

where we use the assumption $\delta \geq sn[1/p - 1/2] = sn(1/\hat{p}-1/2)$ as $p = \infty$ or $p \leq 1$.

Note that when $p = 2$, $e^{i\mu(D)}$ is bounded on $M^{p,q}$. The final conclusion then follows by an interpolation among the cases of $p = 2$, $p = \infty$ and $p \leq 1$.

Then, we recall an embedding relations between modulation and Wiener amalgam spaces. The proof is easy, so we omit here.

**Lemma 5.2.** Let $p, q \in (0, \infty)$, then

$$M^{p,q} \subset W^{p,q}, \text{ if } p \geq q,
M^{p,q}_\delta \subset W^{p,q}, \text{ if } p < q, \delta > n[1/p - 1/2],$$

where $W^{p,q}$ is the Wiener amalgam space.
and
\[ W^{p,q} \subset M^{p,q}, \text{ if } p \leq q; \]
\[ W^{p,q}_\delta \subset M^{p,q}, \text{ if } p > q, \delta > n[1/p - 1/q]. \]

Now, we are in a position to give our desired conclusion.

**Theorem 5.3** (high growth of \( \mu \)). Suppose \( 0 < p, q \leq \infty \). Let \( \mu \) be a real-valued \( C^2(\mathbb{R}^n) \) function satisfying
\[
\begin{cases}
(\xi)^{-s} \partial^\gamma \mu \in W^{s,1}, & \text{if } p \geq 1; \\
(\xi)^{-s} \partial^\gamma \mu \in W^{s,n/p-n+\epsilon}, & \text{if } p < 1,
\end{cases}
\]
for some \( s, \epsilon > 0 \) and all \( |\gamma| = 2 \). Then \( e^{i\mu(D)} : W^{p,q}_\delta \rightarrow W^{p,q} \) is bounded for \( \delta \geq sn[1/p - 1/2] + n[1/p - 1/q] \) with strict inequality when \( p \neq q \).

**Proof.** The case \( p = q \) follows by Lemma 5.1 and the fact \( W^{p,p} = M^{p,p} \).

For \( p > q \), we use Lemma 5.2 to deduce
\[
\|e^{i\mu(D)}f\|_{W^{p,q}} \lesssim \|e^{i\mu(D)}f\|_{M^{p,q}} \lesssim \|f\|_{M^{p,q}_{\delta-n[1/p - 1/2]}} \lesssim \|f\|_{W^{p,q}_\delta}.
\]

For \( p < q \), we use Lemma 5.2 to deduce
\[
\|e^{i\mu(D)}f\|_{W^{p,q}} \lesssim \|e^{i\mu(D)}f\|_{M^{p,q}_{\delta-n[1/p - 1/2]}} \lesssim \|f\|_{M^{p,q}_{\delta}} \lesssim \|f\|_{W^{p,q}_\delta}.
\]

\[ \square \]

**Remark 5.4.** As one can see, in the high growth case, the potential loss comes from two aspects. The first one \( n[1/p - 1/2] \) can be viewed as the result of the scattered property of \( \nabla \mu \), and the second one \( sn[1/p - 1/2] \) comes from the second order derivative of \( \mu \) as in the modulation case. This more complex composition may add more difficulties to determine the sharp loss of potential in this high growth case. One can see [4] for a partial result in this direction.

As in the Corollary 1.4, we can also establish the conclusions fitting more detailed derivative condition of \( \mu \).

**Corollary 5.5.** Suppose \( 0 < p, q \leq \infty \). Let \( \epsilon > n(1/\hat{p} - 1), \beta > 2 \). Let \( \mu \) be a real-valued function of class \( C^{[n(1/\hat{p} - 1) + 3]} \) on \( \mathbb{R}^n \setminus \{0\} \) which satisfies
\[
\begin{align*}
|\partial^\gamma \mu(\xi)| &\leq C_\gamma |\xi|^{\beta - |\gamma|}, & 0 < |\xi| \leq 1, \ |\gamma| \leq n(1/\hat{p} - 1/2) + 1, \ (5.1) \\
\end{align*}
\]
and
\[
\begin{align*}
|\partial^\gamma \mu(\xi)| &\leq C_\gamma |\xi|^{\beta - |\gamma|}, & |\xi| > 1, \ 2 \leq |\gamma| \leq n(1/\hat{p} - 1/2) + 3 \ (5.2)
\end{align*}
\]
Then \( e^{i\mu(D)} : W^{p,q}_\delta \rightarrow W^{p,q} \) is bounded for \( \delta \geq sn[1/p - 1/2] + n[1/p - 1/q] \) with strict inequality when \( p \neq q \).

**Proof of Corollary 5.5.** Let \( \rho_0 \) be a smooth function supported on \( B(0,1) \), and denote by
\[ \mu_1 := \rho_0 \mu, \quad \mu_2 := (1 - \rho_0) \mu. \]

The boundedness of \( e^{i\mu_1(D)} \) follows by the same argument as in the proof of Corollary 1.4. Now we turn to the estimate of \( e^{i\mu_2(D)} \).

Denote \( s := \beta - 2 \). We claim that
\[
\begin{cases}
(\xi)^{-s} \partial^\gamma \mu \in W^{s,1}, & \text{if } p \geq 1; \\
(\xi)^{-s} \partial^\gamma \mu \in W^{s,n/p-n+\epsilon}, & \text{if } p < 1,
\end{cases}
\]
for some $\epsilon > 0$ and all $|\gamma| = 2$. Then the final conclusion follows by this claim and Theorem 5.3.

For any fixed $\gamma_0$ with $|\gamma_0| = 2$, we denote $g_{\gamma_0} := \langle \xi \rangle^{-\delta} \partial^{\gamma_0} \mu_2$. It follows by the assumption that

$$|\partial^\gamma g_{\gamma_0}(\xi)| \lesssim 1, \quad |\xi| > 1, \quad |\gamma| \leq |n(1/p - 1/2)| + 1. \quad (5.3)$$

Thus, for $p \geq 1$, we use the H"older inequality to deduce that

$$\|g_{\gamma_0}\|_{W^{\infty,1}} = \sup_{k \in \mathbb{Z}^n} \|\sigma_k g_{\gamma_0}\|_{\mathcal{F}^1}$$

$$= \sup_{k \in \mathbb{Z}^n} \|\mathcal{F}^{-1}(\sigma_k g_{\gamma_0})\|_{L^1}$$

$$\lesssim \sup_{k \in \mathbb{Z}^n} \|\langle x \rangle^{\lceil n/2 \rceil + 1} \mathcal{F}^{-1}(\sigma_k g_{\gamma_0})(x)\|_{L^2}$$

$$\sim \sup_{k \in \mathbb{Z}^n} \|\sigma_k g_{\gamma_0}\|_{H^{\lceil n/2 \rceil + 1}} \sim \sup_{k \in \mathbb{Z}^n} \sum_{|\gamma| \leq |n(1/p - 1/2)| + 1} \|\partial^\gamma (\sigma_k g_{\gamma_0})\|_{L^2} \lesssim 1.$$

For $p < 1$, choose $\epsilon$ such that $n(1/p - 1/2) + \epsilon < |n(1/p - 1/2)| + 1$, then

$$\|g_{\gamma_0}\|_{W^{\infty,1}_{n/p - n/2 + \epsilon}} = \sup_{k \in \mathbb{Z}^n} \|\sigma_k g_{\gamma_0}\|_{\mathcal{F}^1_{n/p - n/2 + \epsilon}}$$

$$= \sup_{k \in \mathbb{Z}^n} \|\langle x \rangle^{n/p - n/2 + \epsilon} \mathcal{F}^{-1}(\sigma_k g_{\gamma_0})(x)\|_{L^1}$$

$$= \sup_{k \in \mathbb{Z}^n} \|\langle x \rangle^{n(1/p - 1/2) + \epsilon - (n(1/p - 1/2)) + 1/2} \mathcal{F}^{-1}(\sigma_k g_{\gamma_0})(x)\|_{L^1}$$

$$\lesssim \sup_{k \in \mathbb{Z}^n} \|\langle x \rangle^{n(1/p - 1/2) + 1} \mathcal{F}^{-1}(\sigma_k g_{\gamma_0})(x)\|_{L^2}$$

$$\sim \sup_{k \in \mathbb{Z}^n} \|\sigma_k g_{\gamma_0}\|_{H^{n(1/p - 1/2) + 1}} \sim \sup_{k \in \mathbb{Z}^n} \sum_{|\gamma| \leq |n(1/p - 1/2)| + 1} \|\partial^\gamma (\sigma_k g_{\gamma_0})\|_{L^2} \lesssim 1.$$

We have now verified the claim and completed this proof. \hfill \Box

Applying Corollary 1.4 and 5.5 to the prototype $\mu(\xi) = |\xi|^{\beta}$, we deduce the following conclusion.

**Corollary 5.6.** Let $0 < p, q \leq \infty$, $\beta > n(1/p - 1)$. We have $e^{i|D|^{\beta}} : W^p_{p,q} \to W^p_{p,q}$ is bounded if

$$\delta \geq n[1/p - 1/q] \max\{\beta - 1, 0\} + n[1/p - 1/2] \max\{\beta - 2, 0\}$$

with strict inequality when $\beta > 1, p \neq q$.

**APPENDIX A.**

In order to prove Lemma 4.5 in \cite{1}, we first recall the sharp version of Young's inequality of discrete form.

**Lemma A.1** (Lemma 4.5 in \cite{1}). Suppose $0 < q, q_1, q_2 \leq \infty$. Set $S := \{j \in \mathbb{Z} : q_j \geq 1, 1 \leq j \leq 2\}$. Then

$$l_{q_1} * l_{q_2} \subset l_q$$

holds if and only if

$$\begin{cases} (|S| - 1) + 1/q \leq 1/q_1 + 1/q_2, \\ 1/q \leq 1/q_1, 1/q \leq 1/q_2. \end{cases}$$

By this lemma, we further have following useful inequality.
Lemma A.2. Let $0 < p, q \leq \infty$. We have
\[ l^{\frac{p}{q}} \ast l^{\frac{p}{q}} \subset l^{\frac{p}{q}} \]

Proof. Denote by $r_1 = r := \frac{q}{p}$, $r_2 := \frac{\tilde{p} \tilde{q}}{\tilde{p}}$. We have
\[ 1/r_1 \leq 1/r, \quad 1/r \leq 1/r_2. \tag{A.1} \]

We divide this proof into two cases.

Case 1: $r < 1$ or $r_2 < 1$. The desired conclusion follows by Lemma A.1 and (A.1).

Case 2: $r, r_2 \geq 1$. We only need to check
\[ 1 + 1/r \leq 1/r_1 + 1/r_2, \]
which is equivalent to
\[ 1 \leq 1/r_2 \iff \frac{\tilde{p} \tilde{q}}{\tilde{p}} \leq 1. \]
\[ \square \]

Then, we give the following product relation on modulation space.

Lemma A.3. Let $0 < p, q \leq \infty$, we have
\[ M_{\tilde{p}, \tilde{q}} \cdot M^\infty_{\tilde{p} \wedge \tilde{q}} \subset M_{\tilde{p}, \tilde{q}}. \]

Proof. Using the almost orthogonality of the frequency projections $\sigma_k$, we have that for all $k \in \mathbb{Z}^n$,
\[ \Box_k(fg) = \sum_{i,j \in \mathbb{Z}^n} \Box_k(\Box_i f \cdot \Box_j g) = \sum_{|l| \leq c(n)} \sum_{i+j=k+l} \Box_k(\Box_i f \cdot \Box_j g), \]
where $c(n)$ is a constant depending only on $n$. By the fact that $\Box_k$ is an $L^p$ multiplier, we use Hölder’s inequality to deduce that
\[ \| \Box_k(fg) \|_{L^p} = \left\| \sum_{|l| \leq c(n)} \sum_{i+j=k+l} \Box_k(\Box_i f \cdot \Box_j g) \right\|_{L^p} \]
\[ \lesssim \left( \sum_{|l| \leq c(n)} \sum_{i+j=k+l} \| \Box_k(\Box_i f \cdot \Box_j g) \|_{L^p} \right)^{1/p} \]
\[ \lesssim \left( \sum_{|l| \leq c(n)} \sum_{i+j=k+l} \| \Box_i f \cdot \Box_j g \|_{L^p} \right)^{1/p} \]
\[ \lesssim \left( \sum_{|l| \leq c(n)} \sum_{i+j=k+l} \| \Box_i f \|_{L^p} \| \Box_j g \|_{L^\infty} \right)^{1/p}. \tag{A.2} \]

By the definition of modulation space, we further have
\[ \| fg \|_{M_{\tilde{p}, \tilde{q}}} = \left\| \{ \Box_k(fg) \} \right\|_{L^p} \]
\[ \lesssim \left\| \{ \Box_i f \} \|_{L^{p_1}} \right\|_{L^{t_1/p}} \left\| \{ \Box_j g \} \|_{L^{p_2}} \right\|_{L^{t_2/p}} \tag{A.3} \]
\[ \lesssim \left\| \{ \Box_i f \} \right\|_{L^{p_1}} \left\| \{ \Box_j g \} \right\|_{L^{p_2}} \]
where in the last inequality we use the convolution relation $l^{\frac{p}{q}} \ast l^{\frac{p}{q}} \subset l^{\frac{p}{q}}$. in Lemma A.2 \[ \square \]
Now, we are in a position to give the proof of Lemma 2.8.

Proof of Lemma 2.8. The second conclusion $W_{s,\infty}^{\infty,\infty} \cdot W_{s,\infty}^{\infty,\infty} \subset W_{s,\infty}^{\infty,\infty}$ can be found in [8, Corollary 4.2].

Following, we focus on the proof of $W_{p,q}^{\delta,\delta} \ast W_{p,\infty}^{\infty,\infty} \subset W_{p,q}^{\infty,\infty}$. For $f \in W_{p,q}^{\delta,\delta}$ and $g \in W_{p,\infty}^{\infty,\infty}$, denote by $F = \langle D \rangle^\delta f$ and $G = \langle D \rangle^{-\delta} g$. Then

$$\|f\|_{W_{p,q}^{\delta,\delta}} \sim \|F\|_{W_{p,q}^{\infty,\infty}}, \quad \|g\|_{W_{p,\infty}^{\infty,\infty}} \sim \|G\|_{W_{p,q}^{\infty,\infty}}, \quad f \ast g = F \ast G.$$ 

From this, we only need to verify the following equivalent relation

$$W_{p,q}^{\delta,\delta} \ast W_{p,\infty}^{\infty,\infty} \subset W_{p,q}^{\infty,\infty},$$

which is equivalent to the following product inequality on modulation space

$$M_{q,p}^{q,p} \cdot M_{\infty,\infty}^{\infty,\infty} \subset M_{q,p}^{q,p}.$$

This is just the conclusion in Lemma A.3. □

References

[1] Á. Bényi, K. Gröchenig, K. A. Okoudjou, and L. G. Rogers. Unimodular Fourier multipliers for modulation spaces. *Journal of Functional Analysis*, 246(2):366–384, may 2007.

[2] F. Concetti and J. Toft. Schatten von Neumann properties for Fourier integral operators with non-smooth symbols. *Arkiv för Matematik*, 47(2):295–312, oct 2009.

[3] E. Cordero and F. Nicola. Boundedness of schrödinger type propagators on modulation spaces. *Journal of Fourier Analysis and Applications*, 16(3):311–339, 2010.

[4] J. Cunanan and M. Sugimoto. Unimodular fourier multipliers on wiener amalgam spaces. *Journal of Mathematical Analysis and Applications*, 419(2):738–747, nov 2014.

[5] H. G. Feichtinger. Modulation Spaces: Looking Back and Ahead. *Sampling Theory in signal and Image Processing*, 5(2):109–140, 2006.

[6] Y. V. Galperin and S. Samarah. Time-frequency analysis on modulation spaces $M^{q,p}_{\infty,\infty}$, $0 < p,q$. *Applied and Computational Harmonic Analysis*, 16(1):1–18, 2004.

[7] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Springer Science & Business Media, 2013.

[8] W. Guo, J. Chen, D. Fan, and G. Zhao. Characterization of Some Properties on Weighted Modulation and Wiener amalgam spaces. *arXiv:1602.02871* To appear in Michigan Mathematical Journal.

[9] W. Guo, D. Fan, and G. Zhao. Full characterization of the embedding relations between $\alpha$-modulation spaces. *Science China Mathematics*, 61(7):1243–1272, jul 2018.

[10] W. Guo, H. Wu, Q. Yang, and G. Zhao. Characterization of inclusion relations between Wiener amalgam and some classical spaces. *Journal of Functional Analysis*, 273(1), 2017.

[11] J. HAN and B. WANG. $\alpha$-modulation spaces (I) scaling, embedding and algebraic properties. *Journal of the Mathematical Society of Japan*, 66(4):1315–1373, oct 2014.

[12] Q. Huang, J. Chen, D. Fan, and X. Zhu. Hörmander-type theorems on unimodular multipliers and applications to modulation spaces. *Banach Journal of Mathematical Analysis*, 12(1):85–103, jan 2018.

[13] T. Kato and N. Tomita. A remark on the Schrödinger operator on Wiener amalgam spaces. *arXiv:1711.06395*, pages 1–8, 2017.

[14] A. Miyachi, F. Nicola, S. Rivetti, A. Tabacco, and N. Tomita. Estimates for unimodular Fourier multipliers on modulation spaces. *Proc. Amer. Math. Soc.*, 137(11):3869–3883, 2009.

[15] F. Nicola, E. Primo, and A. Tabacco. Highly oscillatory unimodular Fourier multipliers on modulation spaces. *arXiv:1801.06422* pages 1–19, 2018.

[16] N. Tomita. Unimodular Fourier multipliers on modulation spaces $M_{p,\infty}$ for $0 < p < 1$. *Harmonic analysis and nonlinear partial differential equations, in RIMS Kokyuroku Bessatsu, B18*, Res. Inst. Math. Sci. (RIMS), Kyoto, pages 125–131, 2010.

[17] H. Triebel. Modulation Spaces on the Euclidean n-Space. *Zeitschrift für Analysis und ihre Anwendungen*, 2(5):445–457, 1983.

[18] B. Wang and H. Hudzik. The global Cauchy problem for the NLS and NLKG with small rough data. *Journal of Differential Equations*, 232(1):36–73, jan 2007.
[19] B. Wang, Z. Huo, C. Hao, and Z. Guo. Harmonic Analysis Method for Nonlinear Evolution Equations, I. WORLD SCIENTIFIC, 2011.

[20] G. Zhao, J. Chen, D. Fan, and W. Guo. Sharp estimates of unimodular multipliers on frequency decomposition spaces. Nonlinear Analysis, 142:26–47, sep 2016.

SCHOOL OF MATHEMATICS AND INFORMATION SCIENCES, GUANGZHOU UNIVERSITY, GUANGZHOU, 510006, P.R.CHINA
E-mail address: weichaoguomath@gmail.com

SCHOOL OF APPLIED MATHEMATICS, XIAMEN UNIVERSITY OF TECHNOLOGY, XIAMEN, 361024, P.R.CHINA
E-mail address: guopingzhaomath@gmail.com