Abstract

We provide a formula for the cardinality constraint, a lower bound in the form of $|F| \geq K$, in such a way that the conjunction of the BFC and the OFC (explained below) can be solved simultaneously. The parameter $K$ can vary from instance to instance. We assume the availability of a successor predicate, and that the domain is ordered. To our knowledge, no polynomially solvable expression has been developed so far, even for a simple problem such as Matching. However, once such a formula is developed for Matching, it can also be applied to harder problems such as Clique.

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1 Introduction

We represent decision versions of optimization problems as a conjunction of a single objective function constraint (OFC) and a set of basic feasibility constraints (BFC).

The OFC comprises the single constraint $|F| \geq K$ for decision problems based on maximization, and $|F| \leq K$ for those based on minimization. Here $|F|$ is a measure of the size of the objective function.

The OFC can also be called the cardinality constraint, since it expresses an upper or lower bound on the size of the set $F$. In this paper, we will consider the upper bound, $|F| \leq K$.

The arity of the predicate $F$ can vary. For instance, for a graph $G = (V, E)$, if $F \subseteq V$, then $F$ will be unary (of arity one). However, if $F \subseteq E$, then $F$ will be binary (of arity two).

Example: Suppose $V$ consists of the vertices $i, 1 \leq i \leq 7$. Considering predicate $F$, assume that $F(1), F(3), F(6)$ and $F(7)$ are true, and let the rest, $F(2), F(4)$ and $F(5)$ be false. In other words, $F = \{1, 3, 6, 7\}$. If $K = 5$, then the cardinality constraint $|F| \leq K$ is true. On the other hand, if $K = 3$, then the constraint has been violated.
1.1 List of Predicates Used

There are two types of predicates, the *knowns* and the *unknowns*.

The *knowns* can be determined from the graph input $G = (V, E)$, or from the particular value that a universal quantifier can assume in a certain clause (we use $s$, $t$, $x$ and $y$ for universal quantifiers).

The *unknowns*, called “decision variables” in optimization terminology, are the ones to be determined by some SAT algorithm.

The Horn condition (that the number of positive literals in a clause must be at most one) applies only to the *unknown* relations, not the *known* ones.

1.1.1 The knowns

Let’s assume that the vertices in $V$ are ordered in a sequence.

(a) Based on the ordering of $V$, we can define a successor predicate $\text{succ}V$ on $V$. Then $\text{succ}V(x, y)$ is true if and only if vertex $y$ is the successor of vertex $x$ in $V$.

The undirected edges in $E$ can also be defined as ordered pairs $(s, t)$ such that $s < t$. Thus if $s \geq t$, we can immediately conclude that $(s, t) \notin E$ (or, we say that $E(s, t)$ is false).

If edge $(s, t) \in E$, then the relation $E(s, t)$ is true, and hence $(s < t)$ must be true.

(b) Define another set $N$, also based on $V$, as follows:

$$N = \{0, 1, 2, \cdots, |V|\}.$$  

The difference is that $V$ is a set of vertices whereas $N$ is a set of non-negative integers.

Similar to $\text{succ}V$, we can define a successor relation $\text{succ}N$ based on the ordering of $N$. Then $\text{succ}N(x, y)$ is true if and only if $y$ is the successor of $x$ in $N$.

(c) $D(x, y, s, t)$ and $D(s, t)$ (Sec. 1.2.5) and $P(x, y, s, t)$ and $P(s, t)$ (Sec. 1.2.6).

(d) The truth values of relations such as $(x < y)$, $(u > v)$, $\text{succ}N(x, y)$, $\text{succ}V(u, v)$, .... can be determined from the particular value assumed by $u$, $v$, $x$ and $y$ in a certain clause.

1.1.2 The unknowns

(e) Unary $F$, defined in the introduction.

(f) Binary $L$, defined in Sec. 1.2.

1.2 Expression for the constraint $|F| \geq K$

Recall that the Horn condition applies only to the *unknown* relations, not the *known* ones.

Define a binary relation $L(x, y)$ such that $x \in N$ and $y \in V$.

$L$ is not a function; for any $x \in N$, it is possible that both $L(x, s)$ and $L(x, t)$ are true when $s \neq t$. But the inverse of $L$ is a function; which means that in $L(x, s)$, given $s \in V$, it reverse maps to a unique $x \in N$.

Given the ordered sequence $V$, let $\text{min}$ and $\text{max}$ be the first and last vertices in $V$ respectively.
1.2.1 The base case

In order to get the recursion going, let us expand $V$ and $N$ by one element each. Add the element “base” to $V$, such that $min$ is the successor of $base$. Expand $N$ to include zero. After these two additions, we still have $|N| = |V|$. Let us set

$$L(0, base) \equiv SuccN(0, 1) \equiv SuccV(base, min) \equiv TRUE.$$ (1)

The other two conditions, that $SuccN(0, 1) \equiv SuccV(base, min) \equiv TRUE$, can be substituted into the recursion formula below. From here on, the recursion will take over and build $L$ correctly.

We could also consider “$L(0, base)$” as one of the clauses in the formula for the cardinality constraint (in which case it will be set to “true” by a SAT algorithm).

1.2.2 Directed bipartite graph

It will be helpful to look at the relation $L$ as a directed bipartite graph $G_b = (N, V, L)$. The elements of $N$ can be placed on the left in a vertical column, with vertex 0 at the bottom and vertex $|V|$ at the top. Similarly on the right side, we have vertices from $V$ in a column with vertex “base” at the bottom and “max” at the top.

The arc for the base case is the bottom-most. From this, we build our way to the top.

The directed edges go from left to right (from $N$ to $V$). For a vertex $x \in N$, its out-degree is given by $\text{Out-degree}(x) \geq 0$. However, for a vertex $s \in V$, its in-degree must be one (exactly one arc arrives at $s$), if the problem instance is satisfiable.

Many of the constraints below can be called either “triangle” constraints or “rectangle” constraints.

**Triangle constraint**: There is one vertex on the left side of the bipartite graph (in $N$) and two on the right (in $V$). The two arcs involved are outgoing from the vertex in $N$. As for the two vertices on the right, one is a successor of the other; we will name them as $(u)$ and $(u + 1)$, for example.

**Rectangle constraint**: Two vertices each on the left and right. On both sides of the bipartite graph, one vertex is a successor of the other. For instance, we will call them $(x)$ and $(x + 1)$ on the left, and $(u)$ and $(u + 1)$ on the right. The two arcs involved are “parallel lines adjacent to each other”.

The rectangle type of constraints are Horn. However, some of the triangle type of constraints are non-Horn.

1.2.3 Arcs-do-not-cross constraint

Two arcs (lines) of $L$ can meet at a vertex, but they cannot cross:

$$\eta(1) \equiv \forall x \forall y \forall s \forall t [L(x, s) \land L(y, t) \land (x \leq y)] \rightarrow (s \leq t)$$
$$\equiv \forall x \forall y \forall s \forall t \neg L(x, s) \lor \neg L(y, t) \lor \neg (x \leq y) \lor (s \leq t).$$ (2)

$$\eta(2) \equiv \forall x \forall y \forall s \forall t [L(x, s) \land L(y, t) \land (s \leq t)] \rightarrow (x \leq y)$$
$$\equiv \forall x \forall y \forall s \forall t \neg L(x, s) \lor \neg L(y, t) \lor \neg (s \leq t) \lor (x \leq y).$$ (3)
(s ≤ t) is shorthand for “(s = t) ∨ (s < t)”.

However, when s, t ∈ V (on the right side) are adjacent to each other, that is, when SuccV(s, t) is true, then either x = y, or y is the successor of x:

\[
\begin{align*}
\eta(3) & \equiv \forall x \forall y \forall t \ [L(x, s) \land L(y, t) \land SuccV(s, t)] \rightarrow [SuccN(x, y) \lor (y = x)] \\
& \equiv \forall x \forall y \forall s \forall t \ [L(x, s) \lor -L(y, t) \lor -SuccV(s, t) \lor SuccN(x, y) \lor (y = x)] \\
& \equiv \forall x \forall y \forall s \forall t \ [L(x, s) \lor -L(y, t) \lor -SuccV(s, t) \lor (y = x + 1) \lor (y = x)] \\
& \equiv \forall x \forall y \forall s \ [L(x, s) \land L(y, s + 1)] \rightarrow [(y = x + 1) \lor (y = x)].
\end{align*}
\]

This is important for building L from the bottom to the top. We have also provided the “t = s + 1 and y = x + 1” version of the constraint in the last two lines in (4).

1.2.4 Miscellaneous (book keeping) constraints

(a) If there are two outgoing arcs L(x, s) and L(x, s + 1) from the same vertex x ∈ N, then F(s + 1) must be false:

\[
\begin{align*}
\eta(5) & \equiv \forall x \forall s \forall t \ [L(x, s) \land L(x, t) \land SuccV(s, t)] \rightarrow -F(t) \\
& \equiv \forall x \forall s \forall t \ [L(x, s) \lor -L(x, t) \lor -SuccV(s, t) \lor -F(t)] \\
& \equiv \forall x \forall s \ [L(x, s) \lor -L(x, s + 1) \lor -F(s + 1)].
\end{align*}
\]

We have substituted t = s + 1 (as well as y = x + 1, below).

(b) The following is a rectangle constraint. If there are two parallel lines adjacent to each other, then the line at the top should arrive at a vertex t for which F(t) is true:

\[
\begin{align*}
\eta(6) & \equiv \forall x \forall y \forall s \forall t \ [L(x, s) \land L(y, t) \land SuccN(x, y) \land SuccV(s, t)] \rightarrow F(t) \\
& \equiv \forall x \forall y \forall s \forall t \ [L(x, s) \lor -L(y, t) \lor -SuccN(x, y) \lor SuccV(s, t) \lor F(t)] \\
& \equiv \forall x \forall s \ [L(x, s) \lor -L(x, s + 1) \lor -L(x + 1, s + 1) \lor F(s + 1)] \\
& \equiv \forall x \forall s \ [L(x, s) \land L(x + 1, s + 1)] \rightarrow F(s + 1).
\end{align*}
\]

(c) Every s ∈ V reverse maps to a unique x ∈ N:

\[
\begin{align*}
\eta(7) & \equiv \forall x \forall y \forall s \ [L(x, s) \land L(y, s)] \rightarrow (x = y) \\
& \equiv \forall x \forall y \forall s \ [L(x, s) \land L(y, s)] \rightarrow (x = y). \\
\end{align*}
\]

1.2.5 Forward recursion

L scans the entire domain V from min to max; as it does so, it keeps a running count of the number of elements in V that are also members of F. (Note that base has been assigned an index of zero in (4).)
The following is a rectangle constraint. If \( t \in F \), then \( s \) and \( t \) have different indices (which are \( x \) and \( y \) respectively):

\[
\eta(8) \equiv \forall x \forall y \forall s \forall t \ [D(x, y, s, t) \land L(x, s) \land F(t)] \rightarrow L(y, t) \\
\equiv \forall x \forall y \forall s \forall t \neg D(x, y, s, t) \lor \neg L(x, s) \lor \neg F(t) \lor L(y, t).
\]

\( t \) share the same index in \( N \), which is \( x \):

\[
\eta(9) \equiv \forall x \forall y \forall s \forall t \ [D(s, t) \land L(x, s) \land \neg F(t)] \rightarrow L(x, t) \\
\equiv \forall x \forall y \forall s \forall t \neg D(s, t) \lor \neg L(x, s) \lor F(t) \lor L(x, t).
\]

This constraint is not Horn since it has two positive literals (the last two).

### 1.2.6 Backward recursion

Start from \( L(x, \text{max}) \) for every \( x \geq k \), and work backwards down to \( L(0, \text{min}) \).

This is a rectangle constraint. If \( t \in F \), then \( s \) and \( t \) have different indices (\( x \) and \( y \) respectively):

\[
\eta(10) \equiv \forall x \forall y \forall s \forall t \ [P(x, y, s, t) \land L(y, t) \land F(t)] \rightarrow L(x, s) \\
\equiv \forall x \forall y \forall s \forall t \neg P(x, y, s, t) \lor \neg L(y, t) \lor \neg F(t) \lor L(x, s).
\]

The following is a triangle constraint. If \( t \notin F \), then \( x \) and \( t \) share the same index, which is \( y \):

\[
\eta(11) \equiv \forall y \forall s \forall t \ [P(s, t) \land L(y, t) \land \neg F(t)] \rightarrow L(y, s) \\
\equiv \forall y \forall s \forall t \neg P(s, t) \lor \neg L(y, t) \lor F(t) \lor L(y, s) \\
\equiv \forall y \forall t \neg P(t - 1, t) \lor \neg L(y, t) \lor F(t) \lor L(y, t - 1).
\]

This constraint is non-Horn.

### 1.2.7 Index of Max is at least \( K \)

One way of stating that the index of \( \text{max} \) is at least \( k \):

\[
\eta(12) \equiv \forall x \ L(x, \text{max}) \rightarrow (x \geq k) \equiv \forall x \ \neg L(x, \text{max}) \lor (x \geq k).
\]

A different way, to include every possible \( L \):

\[
\eta(12) \equiv \forall x \forall s \ [L(x, s) \land (s = \text{max})] \rightarrow (x \geq k) \equiv \forall x \forall s \ \neg L(x, s) \lor (s \neq \text{max}) \lor (x \geq k).
\]
1.3 The cardinality constraint

Just the conjunctions of all the $\eta$’s. All constraints are polynomial in the size of $V$.

1.4 Applications of the cardinality constraint

This has applications for polynomially solvable problems as well as NP-complete ones. Consider the basic feasibility constraints (BFC) mentioned in the Introduction.

The BFC for unweighted non-bipartite Matching (decision version) can be written as a Horn formula:

$$\phi_M = \forall x \forall y [M(x) \land M(y)] \rightarrow \neg W(x, y).$$

$M(x)$ is true if and only if the edge $x$ is matched. $W(x, y)$ is true if and only if edges $x$ and $y$ share a common vertex. $W$ is a known predicate whereas $M$ is unknown.

Thus we can do a conjunct of the cardinality constraint with the BFC formula $\phi_M$, to obtain an expression that represents Matching.

The BFC for Maximum Clique (decision version) can also be written as a Horn formula:

$$\phi_C = \forall x \forall y [F(x) \land F(y)] \rightarrow E(x, y).$$

$F(x)$ is true if and only if the vertex $x$ is marked (or chosen). $E(x, y)$ is true if and only if $(x, y)$ is an edge in the given graph $G = (V, E)$. $E$ is a known predicate whereas $F$ is unknown.

Thus we can do a conjunct of the cardinality constraint with the BFC’s of polynomially solvable problems, as well as NP-complete ones.

References

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