Statistical field theories deformed within different calculi

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Abstract. Within framework of basic-deformed and finite-difference calculi, as well as deformation procedures proposed by Tsallis, Abe, and Kaniadakis to be generalized by Naudts, we develop field-theoretical schemes of statistically distributed fields. We construct a set of generating functionals and find their connection with corresponding correlators for basic-deformed, finite-difference, and Kaniadakis calculi. Moreover, we introduce pair of additive functionals, whose expansions into deformed series yield both Green functions and their irreducible proper vertices. We find as well formal equations, governing by the generating functionals of systems which possess a symmetry with respect to a field variation and are subjected to an arbitrary constrain. Finally, we generalize field-theoretical schemes inherent in concrete calculi in the Naudts spirit. From the physical point of view, we study dependences of both one-site partition function and variance of free fields on deformations. We show that within the basic-deformed statistics dependence of the specific partition function on deformation has in logarithmic axes symmetrical form with respect to maximum related to deformation absence; in case of the finite-difference statistics, the partition function takes non-deformed value; for the Kaniadakis statistics, curves of related dependences have convex symmetrical form at small curvatures of the effective action and concave form at large ones. We demonstrate that the only moment of the second order of free fields takes non-zero values to be proportional to inverse curvature of effective action. In dependence of the deformation parameter, the free field variance has linearly arising form for the basic-deformed distribution and increases non-linearly rapidly in case of the finite-difference statistics; for more complicated case of the Kaniadakis distribution, related dependence has double-well form.

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1 Introduction

In the course of the complex system investigations, vast variety of statistical theories has been developed \textsuperscript{[1,2,3,4,5,6,7,8,9,10]}\textsuperscript{[12]}. The approaches related are based on the principle peculiarity of the statistical behavior of complex systems which is known to be their complicated dynamics being spanned in the fractal phase space governed by long-range interaction or long-time memory effects \textsuperscript{[11,12,13,14]}. These complications have been overcome within framework of the standard statistical approach \textsuperscript{[15]} by means of modification of the Boltzmann-Gibbs distribution due to deformations of the exponential function. Formally, the approaches proposed are based on using the extremum principle for a deformed entropic forms where the ordinary logarithm function is substituted with some versions modified according to Tsallis \textsuperscript{[1]}, Abe \textsuperscript{[4]}, Kaniadakis \textsuperscript{[5]}, Naudts \textsuperscript{[6]}, basic deformation procedure \textsuperscript{[10]}, et cetera. To our knowledge, the only effort along a direct using the field-theoretical method for development of a statistical scheme has been attempted in the work \textsuperscript{[16]} for the Tsallis thermostatistics. The present article is undertaken with the purpose to generalize the standard statistical field theory \textsuperscript{[17,18]} within different versions of calculi elaborated to this moment.

Historically, the first example of such a calculus gives the basic-deformed calculus ($q$-calculus, in other words) which has been originally introduced by Heine and Jackson \textsuperscript{[19,20]} in the study of the basic hypergeometric series \textsuperscript{[21,22]}. It appears the $q$-calculus does not only play a central role in the the quantum groups and algebra belonging to mathematical branches, but have a deep physical meaning \textsuperscript{[23,24]}. In this context, the studying $q$-deformed bosons and fermions shows that thermodynamics can be built on the formalism of $q$-calculus where the ordinary derivative is substituted by the use of an appropriate Jackson derivative and $q$-integral \textsuperscript{[25]}. Moreover, being based on a scale transformation related to the Jackson $q$-derivative and $q$-integral, the basic-deformed calculus is very well suited to describe multifractal sets \textsuperscript{[26,27]}. Displaying critical phenomena of the type of growth processes, rupture, earthquake, financial crashes, these systems reveal a
discrete scale invariance with the existence of log-periodic oscillations deriving from a partial breakdown of the continuous scale invariance symmetry into a discrete one — as occurs, for example, in hierarchical lattices [25,29,30,31].

Concerning the formalism on which is based our method, one needs to point out that it reduces to using a generating functional which presents the Fourier-Laplace transform of the partition function from the dependence on the fluctuating distribution of an order parameter to an auxiliary field [17,18]. Due to the exponential character of this transform determination of correlators of the order parameter is provided by differentiation of the generating functional over auxiliary field. As was mentioned above, fractality of the phase space of complex systems contains calculations related to deformed Gamma functional over auxiliary field. In Section 6, we generalize the field-theoretical schemes elaborated in previous Sections. The mutual complementarity of the basic exponentials (1) is apparent from the multiplication rule [32]

\[ E_q(x)e_q(y) = e_q(x+y) \]  

accoring to which

\[ E_q(x)e_q(-x) = 1. \]  

The principle peculiarity of the exponentials (1) is to keep their forms under action of the Jackson derivative

\[ \mathcal{D}_q^a f(x) \equiv \frac{d_q f(x)}{d_q x} := \frac{f(qx) - f(x)}{(q-1)x}. \]  

Really, one has for arbitrary constants \( a \) and \( b \) [32]

\[ \mathcal{D}_q^a e_q(ax + b) = ae_q(ax + b), \]  

\[ \mathcal{D}_q^a E_q(ax + b) = aE_q(qax + b). \]  

In this way, for arbitrary functions \( f(x) \) and \( g(x) \) the Leibnitz rule reads:

\[ \mathcal{D}_q^a [f(x)g(x)] = g(x)\mathcal{D}_q^a f(x) + f(x)\mathcal{D}_q^a g(x) = g(x)\mathcal{D}_q^a f(x) + f(x)\mathcal{D}_q^a g(x). \]  

Let us write as well the useful equations

\[ e_q(q^n x) = \left(1 + (q-1)x\right)_q^\infty e_q(x), \]  

\[ E_q(q^n x) = \frac{E_q(x)}{(1 - (q-1)x)_q^\infty} \]  

following from the definition (6) and the equalities (7) at \( a = 1 \) and \( b = 0 \). Here, we use the basic-deformed binomial

\[ (x+y)^n_q := (x+y)(x+qy)\ldots(x+q^{n-1}y). \]  

In addition to the basic exponentials (1) being invariant with respect to action of the Jackson derivative (6), its eigen-functions are known to represent the homogeneous functions determined with the property

\[ h(\lambda x) = \lambda^q h(x) \]  

where an exponent \( q \) plays the role of the self-similarity degree, \( \lambda \) is an arbitrary factor playing the role of deformation of self-similar systems for which the homogeneous
functions are the basis of the statistical theory related to the Jackson derivative.
To this end, the eigen-values of the Jackson derivative are determined on the set of the homogeneous functions represent the basic numbers:

$$D^q_\lambda h(x) = [q]_\lambda h(x), \quad [q]_\lambda = \frac{\lambda^q - 1}{\lambda - 1}. \quad (12)$$

Apart from invariant action of the Jackson derivative $D^q_\lambda$, the exponentials are persistent as well under action of the basic integral $I^q_\lambda$, whose property $D^q_\lambda I^q_\lambda = 1$ relates to the explicit definition:

$$I^q_\lambda f(x) \equiv \int f(x) d_q x := (1 - q) x \sum_{n=0}^{\infty} f(q^n a) q^n. \quad (13)$$

Meaning $D^q_\lambda$ and $I^q_\lambda$ as transformation operators of the Lee group, let us introduce the generators related:

$$u^q_\lambda := \ln_q (D^q_\lambda), \quad j^q_\lambda := \ln_q (I^q_\lambda); \quad U^q_\lambda := \ln_q (D^q_\lambda), \quad J^q_\lambda := \ln_q (I^q_\lambda). \quad (14)$$

Here, the pair of the basic logarithmic functions $\ln_q (x)$ and $\ln_q (x)$ is used as inverse functions to the exponentials. Using the properties $u^q_\lambda + j^q_\lambda = 0$ and $U^q_\lambda + J^q_\lambda = 0$, one obtains the functional expressions of the basic integral through the Jackson derivative:

$$I^q_\lambda = e_q (-u^q_\lambda) = e_q [-\ln_q (D^q_\lambda)], \quad I^q_\lambda = E_q (-U^q_\lambda) = E_q [-\ln_q (D^q_\lambda)]. \quad (15)$$

Let us list now main cases of deformations that complete the basic deformation related to the exponentials.

Above, we have demonstrated the dual pair of these exponentials is obeyed the condition $[n]_{1/q} = [n]_q$, so that the dual exponential $E_q(x) = e_{1/q}(x)$ coincides with the original one, $e_q(x)$. The symmetrized $q$-calculus is a basis of the Abe statistics.

The third example gives the $h$-exponential that is specified by the expression:

$$e_h(x) := (1 + h)^x. \quad (17)$$

This function is inverse to the $h$-logarithm:

$$\ln_h(x) = \frac{h \ln(x)}{\ln(1 + h)} \quad (18)$$

with $\ln(x)$ being the ordinary logarithm function. It is easy to convince the expression may be presented in form of the series if the basic numbers are substituted by the $h$-numbers:

$$[n]_h := \frac{hn}{\ln(1 + h)}. \quad (19)$$

The dual $h$-exponential can be defined as $E_h(x) := e_{-h}(x)$ to obey the trivial rule

$$e_h(x + y) = e_h(x)e_h(y) = E_{-h}(x)e_h(y) \quad (20)$$

instead of Eq. $[4]$. Setting the symmetry with respect to change of the $h$ sign, we obtain the self-dual exponential $e_h(x) = E_h(x) \equiv e_{-h}(x)$, with whose consideration we restrict ourselves further. In the limit $h \to 0$, the $h$-calculus reduces naturally to the usual one.

The $h$-exponential is obviously invariant with respect to action of the $h$-derivative:

$$D_{\partial}^h f(x) := \frac{f(x + h) - f(x)}{h}. \quad (21)$$

However, the properties take the form

$$D_{\partial}^h e_h(ax + b) = d_h(a)e_h(ax + b), \quad d_h(a) \equiv e_h(ah) - 1 = \frac{(1 + h)^a - 1}{h} \quad (22)$$

complicated with the factor $d_h(a)$. Although this factor has the ordinary limit $d_{h \to 0}(a) \to a$, action of the $h$-derivative on the exponential does not obey a condition of the type for arbitrary values of a constant $a$. To restore such a condition we shall use the definition of the $h$-derivative:

$$D_{\partial}^h := [1]_h \partial_x, \quad [1]_h = \frac{h}{\ln(1 + h)}, \quad \partial_x \equiv \frac{\partial}{\partial x} \quad (23)$$

instead of Eq. $[21]$. Being applied to the exponential, this derivative ensures obviously the first property with index $q$ substituted by $h$. Respectively, the $h$-integral, being inverse to the derivative, is defined as:

$$I^h_{\partial} f(x) = \int f(x) d_h x, \quad d_h x = [1]_{1/h}^{-1} dx. \quad (24)$$

Quite different example represents the Tsallis exponential:

$$\exp_q(x) := [1 + (1 - q)x]^{1/q} \quad (25)$$

characterized by the deformed number

$$[n]_q = \frac{n}{1 + (1 - q)(n - 1)}. \quad (26)$$

Here, the dual number relates to the exponential:

$$\exp_q(x) = \exp_{1/q}(x) = [\exp_q(-x/q)]^{-q}, \quad (27)$$

whose inverse value is proportional to the escort probability being the basis of the Tsallis thermostatistics $[3, 14]$. It is principally important for our aims the following: i) the Tsallis exponential is not invariant with respect to the Jackson derivative, whereas action of the ordinary derivative gives $\frac{\partial}{\partial x} \exp_q(x) = \exp_q'(x)$; ii) the dual exponential is obeyed the condition $\exp_q'(1/q)(-qx) \exp_q(x) = 1$ that does not coincide with the condition. In the limit $q \to 1$, the Tsallis calculus reduces to the ordinary one.
while it coincides with the $h$-calculus at arbitrary values of the deformation parameter $q = 1 - h/x$. It is worthwhile to stress, however, the Tsallis exponential increases with the $x$-growth according to the power law $[25]$, while the $h$-exponential $[17]$ varies exponentially. The statistical field theory based on the Tsallis calculus has been developed in work $[16]$. As the following example, we consider the case of the Kaniadakis deformation when exponential and logarithm functions are defined as follows $[5]$:

$$\exp_\kappa(x) := \left[\kappa x + \sqrt{1 + (\kappa x)^2}\right]^{1/\kappa},$$

$$\ln_\kappa(x) := \frac{x^\kappa - x^{-\kappa}}{2\kappa}.$$  \hspace{1cm} (28)

Here, the deformation parameter $\kappa$ belongs to the interval $(-1, 1)$ and the limit $\kappa \to 0$ relates to the ordinary functions $\exp(x)$ and $\ln(x)$. The exponential (28) is self-dual in the sense that it is obeyed the condition $\exp_\kappa(x) \exp_\kappa(-x) = 1$ of the type $[5]$. However, the multiplication rule $[4]$ takes the form $[33]$

$$\exp_\kappa(x) \exp_\kappa(y) = \exp_\kappa\left(x \oplus y\right)$$  \hspace{1cm} (30)

where the sum is deformed as

$$x \oplus y := x \sqrt{1 + (\kappa y)^2} + y \sqrt{1 + (\kappa x)^2}.$$  \hspace{1cm} (31)

Remarkably, the rule of the type (30) takes place also for the Tsallis $q$-calculus where the deformed sum $[31]$ is written as $[37]$

$$x \oplus_q y := x + y + (1 - q)xy.$$  \hspace{1cm} (32)

Unfortunately, the exponential (28) may not be presented in the form of any deformed series $[1]$ with a number of the type $[2]$. However, it is easy to convince that this exponential is invariant with respect to action of the derivation operator $[35]$

$$D^\kappa_x \equiv \frac{\partial}{\partial_\kappa x} := \sqrt{1 + (\kappa x)^2} \frac{\partial}{\partial x}.$$  \hspace{1cm} (33)

Moreover, for arbitrary constant $a$ one obtains

$$D^\kappa_x \exp_\kappa(ax) = a \exp_\kappa(ax)$$  \hspace{1cm} (34)

instead of Eqs. ($[7]$ and $[22]$). The integration operator being inverse to the derivative $[33]$ is defined in the relativistic form $[38]$

$$I^\kappa_x f(x) := \int f(x) d_\kappa x, \quad d_\kappa x \equiv \frac{dx}{\sqrt{1 + (\kappa x)^2}}.$$  \hspace{1cm} (35)

The derivative property $[34]$ accompanied with the integral definition $[35]$ will be shown to be formal statement for development of the field-theoretical scheme based on the Kaniadakis calculus.

It is worthwhile to stress the logarithm (29) can be generalized to the form $[34]$

$$\ln_{\kappa\tau\varsigma}(x) := \frac{x^\tau \left[ (\varsigma x)^\kappa - (\varsigma x)^{-\kappa} \right] - (\varsigma^\kappa - \varsigma^{-\kappa})}{(\kappa + \tau)\varsigma^\kappa + (\kappa - \tau)\varsigma^{-\kappa}}$$  \hspace{1cm} (36)

being solution of the functional equation

$$\partial_\kappa [x\Lambda(x)] = \lambda \Lambda (x/\alpha) + \eta$$  \hspace{1cm} (37)

with parameters

$$\alpha = \left(\frac{1 + \tau - \varsigma}{1 + \tau + \varsigma}\right)\frac{1}{\varsigma},$$

$$\lambda = \left(\frac{1 + \tau - \varsigma}{1 + \tau + \varsigma}\right)\frac{\varsigma}{\varsigma + \tau},$$

$$\eta = (\lambda - 1)\varsigma^{\kappa - \varsigma^{-\kappa}} = \frac{\varsigma^{\kappa - \varsigma^{-\kappa}}}{(\kappa + \tau)\varsigma^\kappa + (\kappa - \tau)\varsigma^{-\kappa}}.$$  \hspace{1cm} (38)

Remarkably, the generalized logarithm (36) yields known cases of the following deformations: i) the choice of parameters $\tau = 0$, $\varsigma = 1$, and $\kappa \in (-1, 1)$ relates to the Kaniadakis deformation $[5]$; ii) $\kappa = -\tau = (1 - q)/2$ – the Tsallis deformation with parameter $q$ $[4]$; iii) $\kappa = (q - 1)/2$, $\tau = (q + 1)/2$, and $\varsigma = 1$ – the Abe deformation with parameter $q$ $[4]$; iv) the choice $\varsigma = 1$ relates to the two-parameter logarithm proposed by Mittal, Sharma, and Taneja $[35,36]$; v) the case $\tau = 0$ relates to the scaled two-parameter logarithm proposed by Kaniadakis $[34]$.

Finally, we note one more example of generalized logarithms – the functionally deformed logarithm $[6]$

$$\ln_\phi(x) := \int_1^x \frac{dx'}{\phi(x')}$$  \hspace{1cm} (39)

specified with a function $\phi(x)$. As usually, corresponding exponential function $e_\phi(x)$ is defined by the condition $e_\phi[\ln_\phi(x)] = x$. Formally, we may as well define a derivation operator $D^\phi_x$ which keeps the form of the functionally deformed exponential according to the equation $D^\phi_x e_\phi(x) = \eta_\phi e_\phi(x)$ with an eigen-value $\eta_\phi$ fixed by the $\phi(x)$ function choice.

Above considered examples show that a generalized calculus can be built as a result of the following steps:

1. Choose a deformed exponential $e_\kappa(x)$ and find its dual form $E_\kappa(x)$ to be obeyed the multiplication rule

$$E_\kappa(x)e_\kappa(y) = e_\kappa(x+y).$$  \hspace{1cm} (40)

If above exponentials may be expanded into the Taylor series of the type $[1]$, their choice is fixed by numbers $[\eta_\kappa]$, generalizing the expressions $[2,16]$, and $[19]$ with parameter $q$ being substituted with a deformation $\lambda$. In the case of the type of both Tsallis and Kaniadakis deformations, more convenient to use a self-dual exponential obeying the multiplication rule

$$e_\lambda(x)e_\lambda(y) = e_\lambda\left(x \oplus y\right)$$  \hspace{1cm} (41)

defined by specifying deformed sum $x \oplus y$ [see, for example, Eqs. (31) and (32)].
2. Define a deformed differentiation operator $D^q_{\lambda}$ type of the Jackson derivative \(^{[3]}\) according to the condition that this operator keeps invariant forms of the generalized exponentials $e_x(x)$ and $E_{\lambda}(x)$.

3. Introduce a deformed integration operator according to the definitions
\[
I^q_{\lambda} = e_x [-\ln_{\lambda} (D^q_{\lambda})] = E_{\lambda} [-\ln_{\lambda} (D^q_{\lambda})] \tag{42}
\]
generalizing Eqs. \(^{[15]}\) (here, deformed logarithmic functions $\ln_{\lambda}(x)$ and $\ln_{\lambda}(x)$ are defined to be inverse to the generalized exponentials related).

As a result, we achieve the position to develop a statistical field theory that is based on the use of a generating functional being a generalization of the characteristic function \(^{[17,18]}\). This function is known to be presented by the Fourier-Laplace transform
\[
p(j) := I^q_{\lambda} [p(x)E_{\lambda}(jx)] = \int p(x)E_{\lambda}(jx)d_\lambda x \tag{43}
\]
of the probability distribution $p(x)$. The key point is that deformed exponential standing within integrand of the characteristic function \(^{[43]}\) is eigen-function of the deformed differentiation operator $D^{\lambda(q)}_{\lambda}$ with eigen value $d_\lambda(x)$. As a result, multiple differentiation of this function over auxiliary variable $j$ keeps its exponential form to yield the moments
\[
[d_\lambda(x)]^n := \int [d_\lambda(x)]^n p(x)d_\lambda x = (D^q_{\lambda})^n p(j) \bigg|_{j=0} \tag{44}
\]
of an order parameter $d_\lambda(x)$.

### 3 Basic-deformed statistics

Let us consider a statistical system, whose distribution over states $x = \{r_a, p_a\}$ in the phase space of particles $a = 1, \ldots, N, N \to \infty$ with coordinates $r_a$ and momenta $p_a$ is determined by a Hamiltonian $H = H(x)$. We are interested in study of the coarse space distribution $\phi(x)$ of an order parameter $\phi$. Within the coarse grain approximation, thermostatistics of the deformed system is governed by the partition functional
\[
Z_q\{\phi\} := \int e_{\lambda} [-\beta H(x)] \delta [\phi - \phi(x)] d_q x = e_{\lambda} (-S\{\phi\}) \tag{45}
\]
Here, $d_q x = (q - 1)x$ stands for the basic-deformed differential, $S = S(\phi)$ is an effective action, and one takes into account also that thermostatistical distribution of a basic-deformed system is proportional to the $q$-deformed exponential $e_{\lambda} [-\beta H(x)]$ with the inverse temperature $\beta$ measured in the energy units \(^{[10]}\).

The principle peculiarity of the definition \(^{[45]}\) is that both thermostatistical exponential and integral over the phase space are the basic deformed ones. To this end, we need to use the basic-deformed Laplace transform
\[
Z_q\{J\} := \int Z_q\{\phi\} E_q (J \cdot \phi) \{d_\phi\} = \int e_q (-S\{\phi\} + J \cdot \phi) \{d_q \phi\} \tag{46}
\]
where the last equation is written with accounting the property \(^{[4]}\). For the sake of simplicity, we use the lattice representation to describe the coordinate dependence by means of the index $i = 1, \ldots, N$ in the shorthands $J \cdot \phi \equiv \sum_i J_i \phi_i$ and $\{d_q \phi\} \equiv \prod_i d_i \phi_i$.

According to the rules \(^{[7]}\) the $n$-fold differentiation of the last expression for the generating functional \(^{[46]}\) yields
\[
(D^q_{\lambda_1} \ldots D^q_{\lambda_n}) Z_q\{J\} = \int (\phi_{i_1} \ldots \phi_{i_n}) e_q (-S\{\phi\} + J \cdot \phi) \{d_q \phi\}. \tag{47}
\]
The right hand site of this equality determines the correlator
\[
\langle \phi_{i_1} \ldots \phi_{i_n} \rangle_q = Z_q^{-1} \int (\phi_{i_1} \ldots \phi_{i_n}) Z_q\{\phi\} \{d_q \phi\} \tag{48}
\]
where the coefficient is inversely proportional to the partition function
\[
Z_q := \int Z_q\{\phi\}\{d_q \phi\} = \int e_q (-S\{\phi\}) \{d_q \phi\}. \tag{49}
\]
Combination of the last equalities allows for one to express an arbitrary correlator through the basic-deformed derivatives of the generating functional \(^{[46]}\):
\[
\langle \phi_{i_1} \ldots \phi_{i_n} \rangle_q = Z_q^{-1} (D^q_{\lambda_1} \ldots D^q_{\lambda_n}) Z_q\{J\} \bigg|_{J_1, \ldots, J_n = 0}. \tag{50}
\]

Within framework of the harmonic approach, effective action takes the deformed parabolic form
\[
S^{(0)}\{\phi\} = \sum_i S^{(0)}\{\phi_i\}, \quad S^{(0)}\{\phi\} = (\phi_i)^2 q \Delta^2 \tag{51}
\]
where the deformed binom \(^{[10]}\) is used and $\Delta^2$ stands for the inverse curvature. To apply the rule \(^{[4]}\) for the basic-deformed exponential in the generating functional \(^{[46]}\) let us suppose the symmetry with respect to substituting the deformation parameter $q$ by the inverse value $1/q$. Then, it is convenient to separate whole lattice into odd sites $i'$ and even ones $i''$ and use Eq. \(^{[4]}\) for each of couples $i', i''$.

To this end, the exponential in the partition function \(^{[49]}\) is transformed as follows:
\[
e_q (-S^{(0)}\{\phi\}) = e_q \left( \sum_{i'} \phi_{i'} \right) e_{1/q} \left( \sum_{i''} \phi_{i''} \right) \tag{52}
\]
where square brackets denote the integer of the fraction $N/2$. As a result, the generating functional (46) takes the multiplicative form

$$Z_q^{(0)}\{J\} = \prod_{\nu} z_q^{(0)}(J_{\nu}) \prod_{\mu} z_{q^{1/2}}^{(0)}(J_{\mu}).$$

(53)

As show simple calculations in Appendix A, each of multipliers related to one site is determined by the expression

$$z_q^{(0)}(J) = \frac{2\Delta}{\sqrt{\left[2\gamma_q^2\right]}} \gamma_q \left(\frac{1}{2}\right) E_q \left[\frac{q}{\left[2\gamma_q^2\right]} (\Delta J)^2\right].$$

(54)

where the basic-deformed $\gamma$-function is defined by the first equation (A.1). Respectively, specific partition function $z_q^{(0)} \equiv z_q^{(0)}(J = 0)$ reads:

$$z_q^{(0)} = \frac{2\Delta}{\sqrt{\left[2\gamma_q^2\right]}} \gamma_q \left(\frac{1}{2}\right), \quad \gamma_q = 1 + q.$$

(55)

As shows Figure 1a, the dependence of this function on the deformation parameter has in logarithmic axes symmetrical form with respect to the maximum point $q = 1$.

According to the definition (56), within the harmonic approach the order parameter is determined as

$$\langle \phi \rangle_q^{(0)} = \frac{q}{\left[2\gamma_q^2\right]} \left(1 + q - q^2 (q - 1) \frac{\Delta^2 J^2}{\left[2\gamma_q^2\right]^2} \right) \left[\frac{q}{\left[2\gamma_q^2\right]} (\Delta J)^2\right] \bigg|_{J=0} = 0$$

where the second relation (7) is used. In similar manner, cumbersome but simple calculations give the field variance

$$\langle \phi^2 \rangle_q^{(0)} = \Delta^2 q.$$  

(56)

Thus, the basic-deformed distribution of free fields has zero moment of the first order and the variance, being proportional to the inverse curvature of the related action (51) and dependent on the deformation parameter linearly.

To develop a deformed perturbation theory one ought, as usually, to pick out an unharmonic contribution $V = V\{\phi\}$ in total action $S = S^{(0)} + V \{18\}$. Then, the basic-deformed exponential can be written as follows:

$$e_q(-S) = E_q(-V) e_q(-S^{(0)}).$$

(57)

As a result, the formal expansion of the exponential $E_q(-V)$ in power series with consequent use of the differentiation

"Generally speaking, a statistical ensemble might comprise of odd number of particles $N$ that must arrive at unmatched factor in products $\{52\}$. However, this factor is negligible within the thermodynamic limit $N \to \infty$."

Fig. 1. Dependences of the one-site partition functions on the deformation parameters: a) within the basic-deformed statistics at $\Delta = 0.5, 1, 2$ (curves 1, 2, 3, respectively); b) within the Kaniadakis statistics (curves 1–5 correspond to $\Delta = 0.1, 0.25, 0.5, 1.0, 2.0$).

Further, making use of the perturbation scheme with implementation of related diagram technics is straightforward \[18\]. Moreover, the thermodynamic limit $N \to \infty$ allows for one to use the Wick theorem to express higher correlators through the variance (56).

Similarly to the ordinary field scheme [18], an inconvenience of the above approach is that the generating functional (46) is non-additive value. To escape this drawback one should introduce the Green functional

$$G_q := \ln_q (Z_q)$$

(59)

being deformed logarithm of the functional (46). It worthwhile to note the function of the deformed logarithm has not an explicit form to be defined by the inverse exponential function $Z_q = e_q (G_q)$ given by the first series (1).
The pair of the functionals $G_q(J)$ and $Γ_q(φ)$ plays the role of related kernels, whose basic-deformed variation yields the state equations

$$φ_i = D^q J, G_q ⇔ J_i = D^q φ_i, Γ_q.$$  

(61)

Being analytical functional, these potentials can be presented by the following series:

$$G_q(J) = \sum_{n=1}^{∞} \frac{1}{[n]!} \sum G^{(n)}_{q, i_1...i_n} J_{i_1}...J_{i_n},$$  

(62)

$$Γ_q(φ) = \sum_{n=1}^{∞} \frac{1}{[n]!} \sum Γ^{(n)}_{q, i_1...i_n} φ_{i_1}...φ_{i_n},$$  

(63)

To this end, related kernels $G^{(n)}_{q, i_1...i_n}$ and $Γ^{(n)}_{q, i_1...i_n}$ reduce to $n$-particle Green function and its irreducible part, respectively. Within the diagram representation, these kernels are depicted in Figure 2.

Following the standard field scheme [18], we show further that the generating functional [46] obeys some formal relations. The first displays a system symmetry with respect to the basic-deformed variation in the form

$$δ_q φ_i = e_q f_i(φ)$$  

(64)

given by an analytical functional $f_i(φ)$ in the limit $e_q → 0$.

Due to this variation the integrand of the last expression of the functional [46] is transformed in the following manner:

$$e_q (-S[φ + δ_q φ] + J · (φ + δ_q φ))$$

$$≃ e_q \left[ (-S[φ] + J · φ) + \left( \left( \frac{∂S}{∂φ_i} + J_i \right) δ_q φ_i \right) \right]$$

$$= e_q \left[ (-S[φ] + J · φ) E_q \left[ \left( \left( \frac{∂S}{∂φ_i} + J_i \right) δ_q φ_i \right) \right] \right]$$

$$≃ e_q \left[ (-S[φ] + J · φ) \left( 1 + \left( \left( \frac{∂S}{∂φ_i} + J_i \right) e_q f_i(φ) \right) \right) \right]$$  

(65)

where the sum over repeated indexes is implied and the second expansion [1] is taken into account. On the other hand, the Jacobian determinant appearing due to passage from $φ$ to $φ + δ_q φ$ gives the factor $1 + (∂f_i/∂φ_i) e_q$. As a result, collecting multipliers which include the infinitesimal value $e_q$, one obtains from the invariance property of the generating functional [46]:

$$\left[ f_i \{D^q J\} \left( \frac{∂S}{∂φ_i} \{D^q J\} - J_i \right) - \frac{∂f_i}{∂φ_i} \{D^q J\} \right] Z_q(J) = 0.$$

(66)

Here, we use the first of the state equations [61] for operator representations of the type $f_i(φ) e_q(-S[φ] + J · φ) = f_i(D^q J) e_q(-S[φ] + J · φ)$. At condition $f_i(φ)$ is const., the equation (66) takes the simplified form, following directly from the generating functional [46] after variation over the field φ.

The second of above pointed equations allows for one to take into account an arbitrary condition $F_i(φ) = 0$, $j = 1, 2, ..., N$ of the set of fields $φ_i$, $i = 1, 2, ..., N$ to be found. Taking into account of this condition is achieved by inserting the δ-functional $δ(F)$ into the integrand of the expression (46) that results in introducing the prolonged form

$$Z^{(F)}_q(J) := \int e_q [-S[φ]] E_q \{J · φ + λ · F\} \{d_q φ\} \{d_q λ\}.$$  

(67)

Then, variation over an auxiliary variables $λ_j, j = 1, 2, ..., N$ yields the desired result

$$F_i \{D^q J\} Z^{(F)}_q(J) = 0.$$  

(68)

In comparison with the standard field theory [18], the principal peculiarity of the equalities (66) and (68) is that they contain the Jackson derivative $D^q J$ instead of the ordinary variation $δ/J$.

### 4 Finite-difference statistics

Within framework of the finite-difference statistics, the field scheme is based on the definitions [17 - 20, 23] and [24] inherent in the h-calculus [32] to be developed in analogy with consideration stated in the previous section. With using the $h$-exponential [17], the partition functional takes the form of the $h$-integral

$$Z_h[φ] := \int Z_h \{φ\} e_h \{J · φ\} \{d_h φ\},$$  

(69)

instead of the expression (45). Respectively, the generating functional [46] is written as

$$Z_h(J) := \int Z_h \{φ\} e_h \{J · φ\} \{d_h φ\}.$$  

(70)

F. Fig. 2. Diagram representations of Green functions (a) and their irreducible parts (b).
Respectively, the variance related is written as 
\[ \langle \phi_1 \ldots \phi_n \rangle_h \]
= \[ Z_h^{-1} \left( \mathcal{D}^h_{J_1} \ldots \mathcal{D}^h_{J_n} \right) Z_h \{ J \} |_{J_1, \ldots, J_n = 0} . \] (71)

As in Eq. (51), the harmonic action
\[ S^{(0)} \{ \phi \} = \sum_i^N \phi_i^2 / [2h \Delta^2] , \quad [2h] = \frac{2h}{\ln(1 + h)} \] (72)
is determined with the inverse curvature \( \Delta^2 \). Then, the generating functional (70) is expressed by the product
\[ Z_h^{(0)} \{ J \} = \prod_i^N e_h \{ J_i \} \] (73)
of the type (53) with the one-particle factor (A.8). Taking into account the property (A.7) of the \( h \)-gamma function, one obtains the expression
\[ z_h^{(0)} \{ J \} = \sqrt{2\pi \Delta e_h} \left[ \frac{[1]_h}{2} (\Delta J)^2 \right] \] (74)
reducing to the standard form in the limit \( h \to 0 \). Moreover, the specific partition function
\[ z_h^{(0)} = \int_{-\infty}^{+\infty} e_h \left( -\frac{\phi^2}{[2h] \Delta^2} \right) d_h \phi \] (75)
related to \( J = 0 \) takes the non-deformed value \( z_h^{(0)} = \sqrt{2\pi \Delta} \) for arbitrary parameters \( h \).

According to the definition (71), the mean value of \( h \)-deformed free fields equals
\[ \langle \phi \rangle_h^{(0)} = [1]_h e_h \left( \frac{[1]_h}{2} (\Delta J)^2 \right) \Delta^2 J \bigg|_{J = 0} = 0. \] (76)
Respectively, the variance related is written as
\[ \langle \phi^2 \rangle_h^{(0)} = \Delta^2 [1]_h^2 , \quad [1]_h = \frac{h}{\ln(1 + h)} \] (77)
instead of Eq. (56). Thus, similarly to the basic-deformed distribution the \( h \)-deformed free fields have zero moment of the first order and the variance being proportional to the inverse curvature of the action (72) related. However, dependence of the variance (77) on the deformation parameter appears to be more strong than the linear dependence (56) related to the case of the basic deformation (see Figure 3).

The \( h \)-deformed perturbation theory is built in the complete accordance with the scheme stated in the previous section, with the only difference that the dual \( q \)-exponential \( E_q(x) \) should be substituted with the self-dual \( h \)-exponential \( e_h(x) \). Along this line, making use of the symbolic perturbation expansion of the type (57) yields the generating functional
\[ Z_h \{ J \} = e_h \left( -V \{ D^h_j \} \right) Z_h^{(0)} \{ J \} \] (78)
instead of Eq. (58). As well, application of the diagram technics and the Wick theorem appears to be straightforward. Moreover, following to the standard line [18], one should supplement the non-additive functional (70) by the Green functional \( G_h := l_h (Z_h) \) where the \( h \)-logarithm is defined as (18) to be inverse to the \( h \)-exponential (17).

The functional \( \Gamma_h = \Gamma_h \{ \phi \} \) conjugated to the Green functional \( G_h \{ J \} \) and the state equations related are defined by the Legendre transformation (60) and Eqs. (61), respectively, where subscripts \( q \) are substituted by indexes \( h \). With this substitution, the kernels of the series (62) and (63) are reduced to the \( n \)-particle Green function \( G^{(n)}_{\phi_h} \) and its irreducible part \( I^{(n)}_{\phi_h} \) as is depicted graphically in Figure 2. Finally, the formal relations for the generating functional (70) take the forms of equations (66), (67), and (68) to display a system symmetry with respect.
to the \( h \)-deformed variation \( \delta_h \phi_i(x) = \phi_i(x + h) - \phi_i(x) \) and take into account an arbitrary condition \( F_1(\phi) = 0 \), \( j = 1, 2, \ldots \) for the set of fields \( \{ \phi_i \} , i = 1, 2, \ldots , N \) (in above pointed equations, one should again substitute \( q \) by \( h \)).

5 Kaniadakis statistics

Taking into account close likeness between basic-deformed, Tsallis and Kaniadakis statistics, we present the latter basing on the field-theoretical schemes developed in Section 3 and Ref. [16]. In so doing, one needs to substitute the Tsallis deformed exponentials (25) by the Kaniadakis ones (28) and take into account the multiplication rule (30) defined with the deformed sum (31). As a result, the generating functional (46) takes the form

\[
Z_\kappa \{ J \} := \int Z_\kappa \{ \phi \} \exp_\kappa \{ J \cdot \phi \} \{ d_\kappa \phi \} = \int \exp_\kappa \left( -S(\phi) + J \cdot \phi \right) \{ d_\kappa \phi \} . \tag{79}
\]

According to the property (54), the correlator [cf. with Eq. (50)]

\[
\langle \phi_{i_1} \cdots \phi_{i_n} \rangle_\kappa = Z^{-1}_\kappa \left( D_{\kappa, i_1} \cdots D_{\kappa, i_n} \right) Z_\kappa \{ J \} \big|_{J_1, \ldots , J_n = 0} \tag{80}
\]

is determined by the partition function \( Z_\kappa = Z_\kappa \{ J = 0 \} \) and the Kaniadakis derivative (33). Then, with using the harmonic action

\[
S^{(0)}(\phi) = \sum_i \frac{\phi_i^2}{2\Delta^2} \tag{81}
\]

the generating functional (79) reduces to the product

\[
Z^{(0)}_\kappa \{ J \} = \prod_i z^{(0)}_\kappa (J_i) \quad \tag{82}
\]

of the one-site factors

\[
z^{(0)}_\kappa (J) = \int_{-\infty}^\infty \exp_\kappa \left( -\frac{\phi^2}{2\Delta^2} \right) \exp_\kappa (J \phi) \{ d_\kappa \phi \} . \tag{83}
\]

The specific partition function \( z^{(0)}_\kappa = z^{(0)}_\kappa (J = 0) \) takes the explicit form

\[
z^{(0)}_\kappa = \int_{-\infty}^\infty \left[ \sqrt{1 + \left( \frac{\kappa \phi^2}{2\Delta^2} \right)^2 - \frac{\kappa \phi^2}{2\Delta^2}} \right] \frac{d\phi}{\sqrt{1 + (\kappa \phi^2)^2}} . \tag{84}
\]

According to Eqs. (80) and (83), the first moment is \( \langle \phi \rangle^{(0)}_\kappa = 0 \), while the definition of the free field variance \( \langle \phi^2 \rangle^{(0)}_\kappa \) is achieved by inserting the \( \phi^2 \) factor into the integrand of the integral [84] and dividing by \( z^{(0)}_\kappa \).

As shows Figure 1, dependence of the one-site partition function (84) on the deformation parameter \( \kappa \) has a symmetrical form with respect to the point \( \kappa = 0 \). Characteristically, the \( |\kappa| \) arising results in the \( z^{(0)}_\kappa \) increase at small values \( \Delta^2 \) of the inverse curvature of action [51], while the \( \Delta \) growth transforms the concave curve of the dependence \( z^{(0)}(\kappa) \) into the convex one. What about the dependence \( \langle \phi^2 \rangle^{(0)}(\kappa) \) for the variance of the \( \kappa \)-deformed free fields, Figure 3b visualizes more complicated curve: first, \( |\kappa| \) growing results in the \( \langle \phi^2 \rangle^{(0)}_\kappa \) decrease from the value \( \langle \phi^2 \rangle^{(0)}_{\kappa = 0} = \Delta^2 \), after that the field variance increases before an anomalous value \( \langle \phi^2 \rangle^{(0)}_{|\kappa| = \Delta^2} \).

6 Generally deformed statistics

The examples considered in Sections 3–5 and Ref. [16] show the generalization of the deformed statistics should be performed along the two different lines. The first generalizes the basic-deformed and \( h \)-statistics, the second makes the same for statistics of the type proposed by Tsallis and Kaniadakis. Such a partition is stipulated by the principle difference between the multiplication rules (4) and (30). In the first case, this rule is provided by means of finding an exponential \( \lambda \) being dual to the initial one \( \exp(\phi) \) (this case is implemented for the basic- and \( h \)-calculus, and the latter relates to the self-dual exponential); the second class requires to deform the sum standing in the exponent of r.h.s. of Eq. (41) according to rules of the types (41) and (42). Let us consider above cases separately.

Following to the method developed in the end of Section 2, a generalization of both basic-deformed and \( h \)-statistics is carried out in the straightforward manner. However, as shows comparison of the considerations stated in Sections 3 and 4 to ensure the passage to the expression

\[
Z_\lambda \{ J \} = \int \exp_\lambda \left( -S(\phi) + J \cdot \phi \right) \{ d_\lambda \phi \} \tag{85}
\]

that contains the ordinary sum of exponents one follows to use the deformed Laplace transform

\[
Z_\lambda \{ J \} := \int Z_\lambda \{ \phi \} E_\lambda \{ J \cdot \phi \} \{ d_\lambda \phi \} \tag{86}
\]

with the dual exponential \( E_\lambda(\phi) \Rightarrow E_q(\phi) \), being inherent in the basic-deformed calculus with \( q = \lambda \), or the expression

\[
Z_\lambda \{ J \} := \int Z_\lambda \{ \phi \} e_\lambda \{ J \cdot \phi \} \{ d_\lambda \phi \} \tag{87}
\]

with the initial exponential \( e_\lambda(\phi) \Rightarrow e_q(\phi) \), taking place in the \( h \)-deformed calculus with \( h = \lambda \) [cf. Eqs. (46) and (70)]. In all above cases, the \( n \)-fold derivative yields

\[
\left( D_{\lambda, 1} \cdots D_{\lambda, n} \right) Z_\lambda \{ J \} \tag{88}
\]

\[
= \int \left( \eta_\lambda(\phi_1) \cdots \eta_\lambda(\phi_n) \right) e_\lambda \left( -S(\phi) + J \cdot \phi \right) \{ d_\lambda \phi \} .
\]
Here, we take into account the differentiation rule
\[ D^b_{\lambda} e_\lambda \left( -S + J \cdot \phi \right) = \eta_\lambda (\phi) e_\lambda \left( -S + J \cdot \phi \right) \]  
(89)
where an eigen-value \( \eta_\lambda (\phi) \) is determined by action of a
generalized derivative \( D^b_{\lambda} \) with respect to a auxiliary field \( J_i \) (in the simple cases of both basic- and \( h \)-deformed cal-
culi, one has \( \eta_\lambda (\phi) = \phi_i \), while the Tsallis calculus relates
with \( \eta_\lambda (\phi) = \ln_{2-\lambda} (\phi_i) \) \[16\]. As a result, the correlator
\[ \langle \eta_\lambda (\phi_1) \cdots \eta_\lambda (\phi_n) \rangle_\lambda \]
(90)
with the partition function
\[ Z_\lambda := Z_{\lambda}^{-1} \int (\eta_\lambda (\phi_1) \cdots \eta_\lambda (\phi_n)) \cdot Z_\lambda \{ \phi \} \{ d_\lambda \phi \} \]
is expressed in the form
\[ \langle \eta_\lambda (\phi_1) \cdots \eta_\lambda (\phi_n) \rangle_\lambda = Z_{\lambda}^{-1} \left( D^b_{\lambda} \cdots D^b_{\lambda \phi_1} \right) Z_\lambda \left( J, \ldots, J_n = 0 \right). \]
Within the harmonic approach, the action \[51\] is written
in the square-law form
\[ S^{(0)} = \sum_{i=1}^{N} \frac{(\phi_i^2)}{2 |\lambda|} \]
(93)
where a field set \( \{ \phi_i \} \) is distributed with the unit variance
and \( \lambda \)-deformed square \( (\phi_i^2) \) and number \( 2 |\lambda| \) are used inste-
ded of Eqs. \[2\] and \[10\]. Then, the generating functional \[86\] takes
the form
\[ Z_\lambda^{(0)} \{ J \} = \prod_{i'} z_\lambda^{(0)} (J_{i'}) \prod_{i''} z_\lambda^{(0)} (J_{i''}) \]
(94)
of the type \[53\]. Respectively, for the generating functional \[87\] one
obtains
\[ Z_\lambda^{(0)} \{ J \} = \prod_{i} z_\lambda^{(0)} (J_i). \]
(95)
Here, each of multipliers related to one site is determined by the expression
\[ z_\lambda^{(0)} (J) = \sqrt{2 |\lambda| \pi} e_{\lambda} \left( \frac{J^2}{2 |\lambda| \pi} \right) \]
(96)
where \( \Delta_\lambda \) is a \( \lambda \)-deformed variance taking the value \( \Delta_{\lambda 0} = 1 \)
in the non-deformed limit \( \lambda \rightarrow \lambda_0 \); in turn, \( \lambda \)-deformed
\( \pi \)-number is defined by equation of the type \[1.5\] with \( \lambda \)
standing instead of \( \eta \) (as pointed out above, a dual \( \lambda \)-
exponential \( E_{\lambda}(x) \) should be used in the case of the type \( q \)-
calculus and a self-dual \( \lambda \)-exponential \( e_{\lambda}(x) \) – for the class
of \( h \)-type calculus). By this, the specific partition function
\[ z_\lambda^{(0)} = e_{\lambda}^{(0)} (J = 0) \]
is given by the simple expression
\[ z_\lambda^{(0)} = \sqrt{2 |\lambda| \pi}. \]
(97)
Similarly to the \( q \)- and \( h \)-statistical field theories, the mean
value of free fields equals \[ \langle \eta (\phi) \rangle^{(0)}_\lambda \propto J_{\mid J=0} = 0, \]
while the variance \[ \langle \eta^2 (\phi) \rangle^{(0)}_\lambda \] appears to be monotonically
increasing function of the deformation parameter \( \lambda \). Calculation
of explicit form of dependences \[66\] and \[57\], as well as the
variance \[ \langle \eta^2 (\phi) \rangle^{(0)}_\lambda \] of order parameter \( \eta = \eta (\phi) \) related
to the derivation rule \[89\] needs to specify a concrete form of
the \( \lambda \)-deformed exponentials.

The perturbation theory is based on the equation type
\[67\] and \[58\], as well as the diagram technics and the
Wick theorem are built similarly to above consid-
ered field-theoretical schemes. Moreover, the additive
generating functional related to the non-additive ones, \[86\]
and \[87\], is expressed by the Green functional \( G^{(0)}_\lambda \{ J \} \)
in the form \[59\]. The passage from this functional, de-
pendent on an auxiliary field \( J \), to the conjugate func-
tional \( \Gamma_\lambda = \Gamma_\lambda \{ \phi \} \), being a functional of an order param-
eter \( \phi \), is achieved by the Legendre transformation type of
\[60\], while the state equations have the form \[61\]. Respectively,
series of sorts \[62\] and \[63\] have kernels that reduce to the
\( n \)-particle Green function and its irreducible part.
Within the diagram representation, these kernels look like
depicted in Figure \[2\]. What about the formal equations
for the generating functionals \[86\] and \[87\], they take the
forms \[66\], \[67\], and \[68\] to display a system sym-
metry with respect to the \( \lambda \)-deformed variation \( \delta_\lambda \phi_i (x) \)
and take into account an arbitrary condition \( F^j \{ \phi \} = 0, \)
\( j = 1, 2, \ldots \) for the set of fields \( \{ \phi_i \}, i = 1, 2, \ldots, N \).
As pointed out above, in all above expressions the index
\( q \) should be substituted by the subscript \( \lambda \), and the \( \lambda \)-
exponentials \( e_{\lambda}(x) \) and \( E_{\lambda}(x) \) should be used instead of the
\( q \)-exponentials \( e_q(x) \) and \( E_q(x) \).

Finally, we state main principles of generalization of
the Kaniadakis-type statistics. As was stressed in the be-
ning of this Section, the corner stone of the Kaniadakis
calculus is that the multiplication rule \[50\] is defined by
deformed sum of the type \[31\]. As a result, the generating
functional \[79\] takes the form
\[ Z_\lambda \{ J \} := \int Z_\lambda \{ \phi \} e_{\lambda} \{ J \cdot \phi \} \{ d_\lambda \phi \} \]
(98)
determined by a generally deformed sum standing in the
last exponent. Respectively, the correlator \[80\] is written
as follows:
\[ \langle \eta_\lambda (\phi_1) \cdots \eta_\lambda (\phi_n) \rangle_\lambda = Z_{\lambda}^{-1} \left( D^b_{\lambda \phi_1} \cdots D^b_{\lambda \phi_n} \right) Z_\lambda \left( J \right) \mid_{J_1, \ldots, J_n = 0} \]
(99)
with the partition function \( Z_\lambda = Z_\lambda \{ J = 0 \} \). Instead of
Eq. \[80\], we take into account here the differentiation rule
\[ D^a_{\lambda} e_{\lambda} (ax) = \eta_\lambda (a) e_{\lambda} (ax) \]
(100)
with an eigen-value \( \eta_\lambda (a) \) and a deformation \( \lambda (a) \), being
deefined by action of a generalized derivative \( D^a_{\lambda} \) with
respect to a variable $x$ at arbitrary constant $a$ (for the Kaniadakis calculus, one has $\eta_q(a) = a$ and $\lambda(a) = \lambda a$).

By analogy with above considered field-theoretical schemes, one finds partition function, perturbation theory, diagram techniques, additive and conjugated Green functional, many-particle Green functions and their irreducible parts, as well as formal equations for the generating functional of systems that display a symmetry with respect to field variation and have some constraints.

7 Concluding remarks

Before a discussion of the results obtained we should stress uppermost that quantum algebra and quantum groups, on whose basis our approach is founded, have been the subject of intense research in different fields of the $q$-deformed quantum theory (see [38] and references therein). Moreover, using the $q$-deformed algebra has allowed to develop multifractal theory [39,27] and thermostatistics of deformed bosons and fermions [28]. In our consideration, we restrict ourselves with generalization of the only classical thermostatistics, whose version has been developed first in work [10].

The principle peculiarity of the basic-deformed distribution arising in this model is to exhibit a cut-off in the energy spectrum which is generally expected in complex systems, whose underlying dynamics is governed by long-range interactions.

It is worthwhile to note in this connection the deformed distribution proposed by Tsallis [11] that is characterized by the power-law asymptotic behavior. The Tsallis picture is known to be inherent in self-similar statistical systems, whose field theory has been built by using both Mellin transform of the Tsallis exponential and Jackson derivative [10]. Contrary to the examples considered in Sections 3–6, a fluctuating order parameter of self-similar systems has non-zero mean value that reduces to the Tsallis deformed logarithm of the amplitude of a hydrodynamic mode. Formally, this is caused by using the Mellin transform with the power-law kernel $\phi^\alpha$ instead of the Fourier-Laplace one with the exponential kernel $\exp(J\phi)$.

Above, we develop field-theoretical schemes founded on the basis of basic-deformed and finite-difference calculi [32], as well as within framework of deformation procedures proposed by Tsallis, Abe, Kaniadakis, and Naudts [11,15,6]. We construct generating functionals related and find their connection with corresponding correlators for basic-deformed, $h$-, and Kaniadakis calculi. Moreover, we introduce pair of additive functionals whose expansions into deformed series yield both Green functions and their irreducible proper vertices, as well as find formal equations, governing by the generating functionals of systems which possess a symmetry with respect to a field variation and are subjected to an arbitrary constrain. Finally, we generalize in the Naudts spirit the field-theoretical schemes inherent in concrete calculi.

Concerning the physical results above obtained, one should point out peculiarities of dependences of both one-site partition function and variance of free fields on deformations (see Figures 1 and 3, respectively). In the case of basic deformation, the specific partition function has in logarithmic axes symmetrical form with respect to the maximum point $q = 1$ (by this, the $\Delta^2$ increase shifts up the curves of dependences related logarithmically equidistantly). For the $h$-deformation, the specific partition function takes non-deformed value. In the case of the Kaniadakis deformation, the dependence of the one-site partition function $z_{\kappa}(0)$ on the deformation parameter $\kappa$ has a symmetrical form with respect to the point $\kappa = 0$ (by this, the $\kappa$ growth results in the $z_{\kappa}(0)$ increase at small inverse curvature $\Delta^2$ of the effective action, while the $\Delta$ growth transforms the concave curve of the dependence $z_{\kappa}(0)(\kappa)$ into the convex one). What about the correlators of free-distributed fields, the only moment of the second order takes non-zero values. For all distributions, this moment is proportional to the inverse curvature $\Delta^2$ of the action related to increase with the deformation parameter growth linearly in the case of the basic-deformed statistics and non-linearly rapidly for the $h$-statistics. More complicated behaviour takes place for the Kaniadakis deformation, when the variance related decreases first and increases then up to an anomalous value.

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Appendix A

Within framework of the basic-deformed calculus, the pair of dual $q$-gamma functions is defined by the integrals

$$\gamma_q(\alpha) := \int_0^{+\infty} x^{\alpha-1} e_q(-x) d_q x,$$

(A.1)

$$G_q(\alpha) := \int_0^{+\infty} x^{-\alpha} E_q(-qx) d_q x$$

(A.2)

where the upper limit equals $\frac{1}{1-q}$ for $|q| < 1$. These definitions are principle different from the introduced in Ref. [32] because we substitute the ordinary power function $x^{\alpha-1}$ by the deformed one $x^{-\alpha}$ which generalizes the deformed binom $[10]$ to substitute integer $n$ by arbitrary exponent $\alpha - 1$. The definitions (A.1) and (A.2) ensure the properties

$$G_q(\alpha + 1) = [\alpha]_q G_q(\alpha),$$

(A.3)

$$\gamma_q(\alpha + 1) = [\alpha]_q \gamma_q(\alpha) q^{-\alpha} = -[\alpha]_q \gamma_q(\alpha)$$

(A.4)

with $\gamma_q(0) = \gamma_q(1) = 1$ and $G_q(0) = G_q(1) = 1$. At $\alpha = 1/2$, the definition (A.1) gives the deformed Poisson
The specific generating functional is calculated as:

\[
\gamma_h(\alpha) := \int_0^{+\infty} x^{\alpha-1} e_h(-x)dx = [1]^{\alpha-1}_h \Gamma(\alpha) \quad (A.7)
\]

appears to be proportional to the ordinary one \( \Gamma(\alpha) \). By analogy with the calculations \( A.6 \), the one-site generating functional is expressed as:

\[
z_h^{(0)}(J) = 2 \left[ \frac{1}{2} \right]_h [\gamma_h] \left[ \frac{1}{2} \right]_h (\Delta J)^2 \Delta
\]

Taking into account the property \( A.7 \), one obtains the simple result \( 74 \).

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