ON NEWTON'S METHOD FOR ENTIRE FUNCTIONS

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Abstract. The Newton map $N_f$ of an entire function $f$ turns the roots of $f$ into attracting fixed points. Let $U$ be the immediate attracting basin for such a fixed point of $N_f$.

We study the behavior of $N_f$ in a component $V$ of $\mathbb{C} \setminus U$. If $V$ can be surrounded by an invariant curve within $U$ and satisfies the condition that for all $z \in \hat{\mathbb{C}}$, $N_f^{-1}(\{z\}) \cap V$ is a finite set, we show that $V$ contains another immediate basin of $N_f$ or a virtual immediate basin (Definition 3.8).

1. Introduction

Newton’s method is a classical way to approximate roots of entire functions by an iterative procedure. Trying to understand this method may very well be called the founding problem of holomorphic dynamics [18, p. 51].

Newton’s method for a complex polynomial $p$ is the iteration of a rational function $N_p$ on the Riemann sphere. Such dynamical systems have been extensively studied in recent years. Tan Lei [22] gave a complete classification of Newton maps of cubic polynomials. In 1992, Manning [15] constructed a finite set of starting values for $N_p$ that depends only on the degree of $p$, such that for any appropriately normalized polynomial with degree $d \geq 10$, the set contains at least one point that converges to a root of $p$ under iteration of $N_p$. Hubbard, Schleicher and Sutherland [13] extended this by constructing a small set of starting values that depends only on the degree $d \geq 2$ and trivial normalizations and finds all roots of $p$.

If $f$ is a transcendental entire function, the associated Newton map $N_f$ will generally be transcendental meromorphic, except in the special case $f = pe^q$ with polynomials $p$ and $q$ (see Proposition 2.11) which was studied by Haruta [10]. Bergweiler [2] proved a no-wandering-domains theorem for transcendental Newton maps that satisfy several finiteness assumptions. Mayer and Schleicher [17] have shown that immediate basins for Newton maps of entire functions are simply connected and unbounded, extending a result of Przytycki [19] in the polynomial case. They have also shown that Newton maps of transcendental functions may exhibit a type of Fatou component that does not appear for Newton maps of polynomials, so called virtual immediate basins (Definition 3.8) in which the dynamics converges to $\infty$. The thesis [16] investigates the Newton map of the transcendental function $z \mapsto ze^z$ and shows that it exhibits virtual immediate basins; see Figure 1 for an illustration. While immediate basins of roots are by definition related to zeroes of $f$ (compare Definition 2.1), under mild technical assumptions a virtual immediate basin leads to an asymptotic zero of $f$; in other words, a virtual immediate basin often contains an asymptotic path of an asymptotic value at 0 for $f$ [4].
Newton’s map for $z \mapsto ze^z$. The immediate basin of 0 has infinitely many accesses to the right. Any two of them surround a virtual immediate basin. More precisely, all curves of the form $(2k + 1)i + [2, \infty]$ are contained in a virtual immediate basin; the virtual basins for $k_1 \neq k_2$ are disjoint and separated by an access to $\infty$ of the immediate basin of 0. The visible area is from $-8 - 10i$ to $12 + 10i$.

In this paper, we continue the work of [17] and investigate the behavior of Newton maps in the complement of an immediate basin. Our main result (Theorem 5.1) is that if a complementary component can be surrounded by an invariant curve through $\infty$, then it contains another immediate basin or virtual immediate basin, unless it maps infinite-to-one onto at least one point of $\hat{\mathbb{C}}$. We believe that the last “unless”-condition is unnecessary, but our methods do not allow us to show this.

An immediate corollary for Newton maps of polynomials is that between any two “channels” of any root, there is always another root. This is folklore, but we do not know of a published reference. This result can be viewed as a first step towards a classification of polynomial Newton maps.

Our paper is structured in the following way: In Section 2, we give an introduction to some general properties of Newton maps. In Section 3, we investigate homotopy classes of curves to $\infty$ in immediate basins and prove some auxiliary results. In Section 4, we prove a fixed point theorem which we will need and which might be interesting in its own right. In Section 5, we state and prove our main result.

2. Newton’s Method as a Dynamical System
2.1. Immediate Basins. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function and $N_f$ its associated (meromorphic) Newton map

$$N_f = \text{id} - \frac{f}{f'}.$$
If $f$ is a polynomial, then $N_f$ extends to a rational map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$. If $\xi$ is a root of $f$ with multiplicity $m \geq 1$, then it is an attracting fixed point of $N_f$ with multiplier $\frac{m-1}{m}$. Conversely, every fixed point $\xi \in \mathbb{C}$ of $N_f$ is attracting and a root of $f$.

**Definition 2.1** (Immediate Basin). Let $\xi$ be an attracting fixed point of $N_f$. The basin of $\xi$ is $\{z \in \mathbb{C} : \lim_{n \to \infty} N_f^\circ_n(z) = \xi\}$, the open set of points which converge to $\xi$ under iteration. The connected component $U$ of the basin that contains $\xi$ is called its immediate basin.

Immediate basins are $N_f$-invariant because they are Fatou components and contain a fixed point. The following theorem is the main result (Theorem 2.7) of [17].

**Theorem 2.2** (Immediate Basins Simply Connected). If $\xi$ is an attracting fixed point of the Newton map $N_f$, then its immediate basin $U$ is simply connected and unbounded.

We will use the following notation throughout the paper:
If $\gamma$ is a curve, the symbol $\gamma$ denotes the mapping $\gamma : I \to \mathbb{C}$ from an interval into the plane as well as its image $\gamma(I) \subset \mathbb{C}$. By a tail of an unbounded curve we mean any unbounded connected part of its image.

For $r > 0$ and $z \in \mathbb{C}$, the symbol $B_r(z)$ designates the disk of radius $r$ centered at $z$.

The full preimage of a point $z \in \hat{\mathbb{C}}$ is the set $N_f^{-1}(\{z\})$. Its only accumulation point can be $\infty$ by the identity theorem. Any point $z' \in N_f^{-1}(\{z\})$ is called a preimage of $z$.

Unless stated otherwise, the boundary and the closure of a set are considered in $\hat{\mathbb{C}}$.

### 2.2. Singular Values

Since the concept of singular values is crucial for the study of dynamical systems, we give a brief reminder of the most important types. In particular, we state some properties of asymptotic values; these appear only for transcendental maps.

**Definition 2.3** (Singular Value). Let $h : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function. We call a point $p \in \mathbb{C}$ a regular point of $h$ if $p$ has a neighborhood on which $h$ is injective. Otherwise, we call $p$ a critical point. A point $v \in \hat{\mathbb{C}}$ is called a regular value if there exists a neighborhood $V$ of $v$ such that for every component $W$ of $h^{-1}(V)$, $h^{-1}|_V : V \to W$ is a single-valued meromorphic function. Otherwise, $v$ is called a singular value.

The image of a critical point is a singular value and is called a critical value.

Critical points in $\mathbb{C}$ are exactly the zeroes of the first derivative. For a rational map, all singular values are critical values.

**Definition 2.4** (Asymptotic Value). Let $h : \mathbb{C} \to \hat{\mathbb{C}}$ be a transcendental meromorphic function. A point $a \in \mathbb{C}$ is called an asymptotic value of $h$ if there exists a curve $\Gamma : \mathbb{R}_+ \to \mathbb{C}$ with $\lim_{t \to \infty} \Gamma(t) = \infty$ such that $\lim_{t \to \infty} h(\Gamma(t)) = a$. We call $\Gamma$ an asymptotic path of $a$.

In general, an asymptotic value is defined by having an asymptotic path towards any essential singularity. Note that in our definition, the set of singular values is the closure of the set of critical and asymptotic values.

We follow [5] in the classification of asymptotic values.
Definition 2.5 (Direct and Indirect Singularity). Let \( h : \mathbb{C} \to \hat{\mathbb{C}} \) be a meromorphic function and \( a \in \mathbb{C} \) be a finite asymptotic value with asymptotic path \( \Gamma \). For each \( r > 0 \), let \( U_r \) be the unbounded component of \( h^{-1}(B_r(a)) \) that contains an unbounded end of \( \Gamma \).

We say that \( a \) is a direct singularity (with respect to \( \Gamma \)) if there is an \( r > 0 \) such that \( h(z) \neq a \) for all \( z \in U_r \). We call \( a \) an indirect singularity if for all \( r > 0 \), there is a \( z \in U_r \) such that \( h(z) = a \) (then there are infinitely many such \( z \) in \( U_r \)).

Theorem 2.6 (Direct Singularities). \([12, \text{Theorem 5}]\). The set of direct singularities of a meromorphic function is always countable. \(\square\)

It is possible however that the set of (direct and indirect) singularities is the entire extended plane: Eremenko \([8]\) constructed meromorphic functions of prescribed finite order whose set of asymptotic values is all of \( \hat{\mathbb{C}} \).

Lemma 2.7 (Unbounded Preimage). Let \( h : \mathbb{C} \to \hat{\mathbb{C}} \) be a meromorphic function and \( B \subset \mathbb{C} \) a bounded topological disk whose boundary is a simple closed curve \( \beta \). Suppose that \( \beta \) contains no critical values and that \( \tilde{B} \) is an unbounded preimage component of \( B \). Then \( \partial \tilde{B} \) contains an unbounded curve \( \tilde{\beta} \) with \( h(\tilde{\beta}) \subset \beta \) such that either \( h|_{\tilde{\beta}} : \tilde{\beta} \to \beta \) is a universal covering map or \( h(\tilde{\beta}) \) lands at an asymptotic value on \( \beta \).

Proof. Let \( w \in \partial \tilde{B} \). Clearly, \( h(w) \in \beta \) and by assumption, \( h \) is a local homeomorphism in a neighborhood of \( w \). It follows that the closed and unbounded set \( \partial \tilde{B} \) is locally an arc everywhere; therefore it cannot accumulate in any compact subset of \( \mathbb{C} \) and must contain an arc \( \tilde{\beta} \) that converges to \( \infty \). The curve \( \tilde{\beta} \) contains no critical points. If \( h|_{\tilde{\beta}} : \tilde{\beta} \to \beta \) is not a universal covering map, then it must land at an asymptotic value. \(\square\)

2.3. Newton Maps. We show that there is only one class of entire functions that have rational Newton maps. This class contains all polynomials. We give a classification of the dynamics within immediate basins for Newton maps of polynomials.

First, we investigate under which conditions a meromorphic function is the Newton map of an entire function. The following proposition uses ideas of Matthias Görner and extends a similar result for rational maps (see below) and certain transcendental functions \( [2, \text{page 3}] \). We do not know if the proposition is new; however, we certainly do not know of a published reference.

Proposition 2.8 (Newton Maps). Let \( N : \mathbb{C} \to \hat{\mathbb{C}} \) be a meromorphic function. It is the Newton map of an entire function \( f : \mathbb{C} \to \mathbb{C} \) if and only if for each fixed point \( N(\xi) = \xi \in \mathbb{C} \), there is a natural number \( m \in \mathbb{N} \) such that \( N'(\xi) = \frac{m-1}{m} \). In this case, there exists \( c \in \mathbb{C} \setminus \{0\} \) such that

\[
\frac{d\zeta}{\zeta - N(\zeta)}
\]

Two entire functions \( f, g \) have the same Newton maps if and only if \( f = c \cdot g \) for a constant \( c \in \mathbb{C} \setminus \{0\} \).

Proof. We start with the last claim: \( f \) and \( cf \) have the same Newton map \( \text{id} - f/f' = \text{id} - 1/(\ln f)' \). Conversely, if \( f \) and \( g \) have the same Newton maps, then \( (\ln f)' = (\ln g)' \), and the claim follows.
It is easy to check that every Newton map satisfies the criterion on derivatives at fixed points.

For the other direction, we construct a map \( f \) such that \( N_f = N \). Let \( z_0 \in \mathbb{C} \) be any base point and define \( \tilde{f}(z) = \int_{\gamma} \frac{d\zeta}{\zeta - N(\zeta)} \), where \( \gamma : [0; 1] \to \mathbb{C} \) is any integration path from \( z_0 \) to \( z \) that avoids the fixed points of \( N \). This defines \( \tilde{f} \) up to \( 2\pi i k \): if \( \gamma' \) is another choice of integration path, the residue theorem shows that

\[
\frac{1}{2\pi i} \int_{\gamma' \circ \gamma^{-1}} \frac{d\zeta}{\zeta - N(\zeta)} = \sum_{N(\zeta) = \xi} \text{Res}_\xi \left( \frac{1}{\zeta - N(\zeta)} \right),
\]

where the sum is taken over the finitely many fixed points of \( N \) that are contained in the compact regions bounded by the closed path \( \gamma' \circ \gamma^{-1} \). Near a fixed point \( \xi \), it is easy to show that \( z - N(z) = \frac{1}{m}(z - \xi) + o(z - \xi) \). Hence we get \( \text{Res}_\xi \left( \frac{1}{z - N(z)} \right) = m \in \mathbb{N} \).

It follows that the map \( f = \exp(\tilde{f}) \) is well defined and holomorphic outside the fixed points of \( N \). Near such a fixed point \( \xi \), \( \tilde{f} \) has the form \( m \log(z - \xi) + O(1) \). Clearly, the real part of this converges to \(-\infty\) for \( z \to \xi \), hence setting \( f(\xi) = 0 \) makes \( f \) an entire function as desired. An easy calculation then shows that \( N_f = N \). A different choice of base point \( z_0 \) will change \( f \) by a multiplicative constant and lead to the same Newton map \( N_f \).

The following corollary is essentially due to Janet Head ([11, Proposition 2.1.2], [22, Lemma 2.2]).

**Corollary 2.9 (Rational Newton Maps).** A rational map \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) of degree \( d \geq 2 \) is the Newton map of a polynomial of degree at least two if and only if \( f(\infty) = \infty \) and for all other fixed points \( a_1, \ldots, a_d \in \mathbb{C} \) there exists a number \( m_j \in \mathbb{N} \) such that \( f'(a_j) = \frac{m_j^{-1}}{m_j} < 1 \). Then, \( f \) is the Newton map of the polynomial

\[
p(z) = a \prod_{j=1}^{d} (z - a_j)^{m_j}
\]

for any complex \( a \neq 0 \).

**Sketch of Proof:** Let \( a \in \mathbb{C} \setminus \{0\} \). Since \( N_p \) and \( f \) have the same fixed points with identical multiplicities, the residuals of the maps \( \tilde{f} := (f - \text{id})^{-1} \) and \( \tilde{N} := (N_p - \text{id})^{-1} \) at their common simple poles \( a_1, \ldots, a_d \in \mathbb{C} \) agree, and thus also those at \( \infty \). Hence, \( \tilde{f} - \tilde{N} \) is a polynomial with \( \lim_{z \to \infty}(\tilde{f} - \tilde{N})(z) = 0 \). Hence \( \tilde{f} = \tilde{N} \) and the claim follows.

We want to exclude the trivial case of Newton maps with degree one.

**Lemma 2.10 (One Root).** Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function such that its Newton map \( N_f \) has an attracting fixed point \( \xi \in \mathbb{C} \) with immediate basin \( U = \mathbb{C} \). Then, there exist \( d > 0 \) and \( a \in \mathbb{C} \) such that \( f(z) = a(z - \xi)^d \).

**Proof:** Since \( N_f \) has no periodic points of minimal period at least 2, it cannot be transcendental [11, Theorem 2]. Hence \( N_f \) is rational and its fixed points can only be \( \xi \) and \( \infty \), both of which must be simple. It follows that \( N_f \) has degree at most one and since it has no poles in \( \mathbb{C} \), it is a polynomial. The claim now follows from Proposition 2.8.
In the rest of this paper, we will assume that \( N_f \) is not a Möbius transformation. Theorem 2.2 implies then that for each immediate basin \( U \) of \( N_f \), there exists a Riemann map \( \varphi : \mathbb{D} \to U \) with \( \varphi(0) = \xi \).

The following simple proposition classifies rational Newton maps of entire functions. Its first half is stated without proof in [1].

**Proposition 2.11** (Rational Newton Map). Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function. Its Newton map \( N_f \) is rational if and only if there are polynomials \( p \) and \( q \) such that \( f \) has the form \( f = pe^q \). In this case, \( \infty \) is a repelling or parabolic fixed point.

More precisely, let \( m,n \geq 0 \) be the degrees of \( p \) and \( q \), respectively. If \( n = 0 \) and \( m \geq 2 \), then \( \infty \) is repelling with multiplier \( \frac{m}{m-1} \). If \( n = 0 \) and \( m = 1 \), then \( N_f \) is constant. If \( n > 0 \), then \( \infty \) is parabolic with multiplier \( +1 \) and multiplicity \( n+1 \geq 2 \).

**Proof.** By [18, Corollary 12.7], every rational function of degree at least 2 has a repelling or parabolic fixed point. Since \( N_f \) is a Newton map, this non-attracting fixed point is unique and must be at \( \infty \). In addition to this, there are finitely many attracting fixed points \( a_1, \ldots, a_n \in \mathbb{C} \) with associated natural numbers \( m_1, \ldots, m_n \in \mathbb{N} \) such that the multipliers satisfy \( N_f'(a_i) = \frac{-m_i}{m_i-1} \). Let \( p(z) = \prod_{i=1}^{n}(z-a_i)^{m_i} \).

Since attracting fixed points of \( N_f \) correspond exactly to the roots of \( f \), \( f \) has the form \( f = pe^h \) for an entire function \( h \). If \( h \) was transcendental, so would be

\[
N_f = \text{id} - \frac{p e^h}{p' e^h + h' p e^h} = \text{id} - \frac{p}{p' + h' p},
\]

a contradiction. The other direction follows by direct calculation and the rest of the proof is left to the reader. \( \square \)

**Figure 2.** Newton map for a polynomial of degree 9. The channels are clearly visible.

For Newton maps of \( f = pe^q \), the area of every immediate basin is finite if \( \deg q \geq 3 \) [10] and infinite if \( p(z) = z \) and \( \deg q \in \{0,1\} \) [6].

The dynamics within immediate basins of Newton maps of polynomials has an easy classification, because all singular values are critical values.
Theorem 2.12 (Polynomial Newton Maps). Let $p$ be a polynomial of degree $d > 1$, normalized so that its roots are contained in the unit disk $\mathbb{D}$. Let $\xi$ be a root of $p$ and $U$ its immediate basin for $N_p$. Then, $U$ contains $0 < k < d$ critical points of $N_p$ and $N_p|_U$ is a proper self-map of degree $k + 1$. Outside the disk $B_2(0)$, $N_f$ is conformally conjugate to multiplication by $d^{-1}/d$. Finally, $U \setminus B_2(0)$ has exactly $k$ unbounded components, so called channels, each of which maps over itself under $N_f$. \hfill $\Box$

Figure 2 illustrates this theorem.

3. Accesses in Immediate Basins

3.1. Invariant Accesses. We investigate the immediate basins of attraction for the attracting fixed points of $N_f$. If $f$ is a polynomial, we have seen in Theorem 2.12 that immediate basins have an easy geometric structure. In the general case, $N_f$ has an essential singularity at $\infty$ and immediate basins may well have infinitely many accesses to $\infty$. We use prime end theory to distinguish them.

Under a finiteness assumption, we have some control over the image of a sequence that converges to $\infty$ through an immediate basin.

Lemma 3.1 (Invariant Boundary). Let $U$ be an immediate basin of the Newton map $N_f$ and $U_R$ an unbounded component of $U \setminus B_R(0)$ with the property that no point has infinitely many preimages in $U_R$. Then for any sequence $(z_n) \subset U_R$ with $z_n \to \infty$, all limit points of $N_f(z_n)$ are contained in $\partial U \cup \{\infty\}$.

The condition is necessary, because if there exists a point $p \in U$ with infinitely many preimages $p_1, p_2, \ldots \in U_R$, we have $p_n \to \infty$ and $N_f(p_n) = p \in U$ for all $n \in \mathbb{N}$.

Proof. Assume there exists a sequence $(z_n) \subset U_R$ that converges to $\infty$ with $N_f(z_n) \to p \in U$. Let $B \subset U$ be a closed neighborhood of $p$ inside $U$ such that its boundary $\partial B$ is a simple closed curve $\beta$ that contains no direct singularities (this is possible by Theorem 2.6) nor critical values.

Suppose first that $p \notin N_f(\partial B_R(0))$. Then we may choose $B$ small enough such that $B \cap N_f(\partial B_R(0)) = \emptyset$. The image of the first finitely many $z_n$ need not be in $B$; ignoring those, each $z_n$ is contained in a component $W_n$ of $N_f^{-1}(B) \cap U$. If a $W_n$ is bounded, it maps surjectively onto $B$ under $N_f$. Therefore, by the finiteness assumption, there can be only finitely many bounded $W_n$. Each bounded $W_n$ contains finitely many $z_n$; hence there must be an $n$ such that $W_n$ is unbounded. By Lemma 2.14 and again because of the finiteness assumption, $\partial W_n$ contains an asymptotic path of an asymptotic value on $\beta$. But this asymptotic value must be an indirect singularity, which also contradicts the finiteness assumption.

If $p \in N_f(\partial B_R(0))$, a small homotopy of the curve $\partial B_R(0)$ in a neighborhood of $p$ solves the problem. \hfill $\Box$

Figure 1 suggests that immediate basins can reach out to infinity in several different directions. We make this precise in the following definitions that generalize the concept of a channel in the polynomial case.

Definition 3.2 (Invariant Access). Let $\xi$ be an attracting fixed point of $N_f$ and $U$ its immediate basin. An access to $\infty$ of $U$ is a homotopy class of curves within $U$ that begin at $\xi$, land at $\infty$ and are homotopic with fixed endpoints.
An invariant access to $\infty$ is an access with the additional property that for each representative $\gamma$, its image $N_f(\gamma)$ belongs to the access as well.

**Lemma 3.3 (Access Induces Prime End).** Let $[\gamma]$ be an access to $\infty$ in $U$. Then $[\gamma]$ induces a prime end $P$ in $U$ with impression $\{\infty\}$. If $[\gamma]$ is invariant, then $N_f(P) = P$.

**Proof.** Let $\gamma \subset U$ be a curve representing $[\gamma]$ that starts at the fixed point $\xi$ and lands at $\infty$. For $n \in \mathbb{N}$, let $W_n$ be the component of $U \setminus B_n(0)$ that contains a tail of $\gamma$. The $W_n$ represent a prime end $P$ with impression $\infty$. Now a curve $\gamma' \subset U$ that starts at $\xi$ and lands at $\infty$ is homotopic to $\gamma$ if and only if a tail of it is contained in $W_n$ for $n$ large enough. Hence the prime end $P$ of $[\gamma]$ is well-defined. The last claim follows immediately from the definition. $\square$

It is clear that different accesses induce different prime ends. We state one more well-known topological fact about the boundary behavior of Riemann maps before using prime ends to characterize invariant accesses.

**Lemma 3.4 (Accesses Separate Disk).** Let $U \subseteq \mathbb{C}$ be a simply connected unbounded domain and $\gamma_1, \gamma_2 : \mathbb{R}_0^+ \to U \cup \{\infty\}$ two non-homotopic curves that land at $\infty$ and are disjoint except for their common base point $z_0 = \gamma_1(0) = \gamma_2(0) \in U$. Let $C$ be a component of $\mathbb{C} \setminus (\gamma_1 \cup \gamma_2)$ and $\varphi : \mathbb{D} \to U$ a Riemann map with $\varphi(0) = z_0$.

Then $\varphi^{-1}(\gamma_1)$ and $\varphi^{-1}(\gamma_2)$ land at distinct points $\zeta_1$ and $\zeta_2$ of $\partial \mathbb{D}$. Furthermore, $\partial U \cap C \subset \hat{\mathbb{C}}$ corresponds under $\varphi^{-1}$ to a closed interval on $\partial \mathbb{D}$ that is bounded by $\zeta_1$ and $\zeta_2$.

This follows immediately because $\varphi$ extends to a homeomorphism from $\hat{\mathbb{D}}$ to the Carathéodory compactification of $U$, see [18, Theorem 17.12].

If $f$ is a polynomial, it follows from Theorem [18, Proposition 3.5] that every immediate basin contains a curve that lands at $\infty$, is homotopic to its image and induces an invariant access. In the general case, it is a priori not even clear that a curve that lands at $\infty$ and is homotopic within $U$ to its image induces an invariant access. The following proposition deals with this issue.

**Proposition 3.5 (Curve Induces Invariant Access).** Let $\gamma \subset U \cup \{\infty\}$ be a curve connecting the fixed point $\xi$ to $\infty$ such that $N_f(\gamma)$ is homotopic to $\gamma$ in $U$ with endpoints fixed. Let $W_n$ be a sequence of fundamental neighborhoods representing the prime end $P$ induced by $[\gamma]$. Then $\gamma$ defines an invariant access to $\infty$ if and only if there is no $z \in \hat{\mathbb{C}}$ that has infinitely many preimages in all $W_n$.

**Proof.** Suppose that $\gamma$ defines an invariant access, i.e., if $\gamma'$ is homotopic in $U$ to $\gamma$, then $N_f(\gamma')$ is homotopic to $N_f(\gamma)$. Assume there is a point $z_0 \in \hat{\mathbb{C}}$ with the property that $N_f^{-1}(\{z_0\}) \cap W_n$ is an infinite set for all $W_n$. Without loss of generality, we may assume that for all $n \in \mathbb{N}$, $W_n \setminus W_{n+1}$ contains one preimage of $z_0$. Then we can find a curve $\gamma'$ with a tail contained in each $W_n$ that goes through a preimage of $z_0$ in each $W_n \setminus W_{n+1}$. Clearly, $\gamma'$ is homotopic to $\gamma$, while its image does not land at $\infty$ and can therefore not be homotopic to $N_f(\gamma)$ with endpoints fixed, a contradiction.

Now suppose that no point has infinitely many preimages in all $W_n$. Since the $W_n$ are nested, no point can have infinitely many preimages in any $W_n$ for $n$ sufficiently large. We uniformize $U$ to the unit disk via a Riemann map $\varphi : \mathbb{D} \to U$ such that $\varphi(0) = \xi$ and consider the induced dynamics $g = \varphi^{-1} \circ N_f \circ \varphi : \mathbb{D} \to \mathbb{D}$. 


By [18, Corollary 17.10], $\varphi^{-1}(\gamma)$ and $\varphi^{-1}(N_f(\gamma))$ land on $\partial D$. Since the curves are homotopic, they even land at the same point $\zeta \in \partial D$. Now by assumption, there exists an $\varepsilon > 0$ such that within $B_\varepsilon(\zeta)$, no $g$-preimage of any point in $D$ accumulates. By Lemma 3.1 it follows that the $g$-image of any sequence converging to $\partial D$ inside $B_\varepsilon(\zeta) \cap D$ will also converge to $\partial D$. Hence we can use the Schwarz Reflection Principle [20, Theorem 11.14] to extend $g$ holomorphically to a neighborhood of $\zeta$ in $\mathbb{C}$. It follows that for the extended map, $\zeta$ is a repelling fixed point with positive real multiplier: if the multiplier was not positive real, $g$ would map points in $B_\varepsilon(\zeta) \cap D$ out of $D$. Also, $\zeta$ cannot be attracting or parabolic, because in this case it would attract points in $D$, which all converge to 0 under iteration.

Since $D$ is simply connected, all curves in $D$ from 0 to $\zeta$ will be homotopic to each other and their $g$-images. A curve in $D$ that starts at 0 lands at $\zeta$ if and only if its $\varphi$-image in $U$ is homotopic to $\gamma$ with endpoints fixed, because $\varphi^{-1}(P)$ is a prime end in $D$ with impression $\zeta$. □

Remark 3.6. We have shown that each invariant access defines a boundary fixed point in the conjugated dynamics on the unit disk, and the dynamics can be extended to a neighborhood of this boundary fixed point, necessarily yielding a repelling fixed point. By [18, Corollary 17.10] it follows that different invariant accesses induce distinct boundary fixed points.

If $f$ is a polynomial, there exists a one-to-one correspondence between accesses to $\infty$ of $U$ and boundary fixed points of the induced map $g$ [13, Proposition 6].

Corollary 3.7 (Invariant Curve). Each invariant access has an invariant representative, i.e. a curve $\gamma: \mathbb{R}^+_0 \to U$ that lands at $\infty$ with $\gamma(0) = \xi$ and $N_f(\gamma) = \gamma$.

Proof. For the extension of $g$ to a neighborhood of $\zeta \in \partial D$, the multiplier of $\zeta$ is positive real. A short piece of straight line in linearizing coordinates around $\zeta$ maps over itself under $g$. Its forward orbit lands at the fixed point. □

Since there are uncountably many choices of such invariant curves, we can always find one that contains no critical or direct asymptotic values outside a sufficiently large disk.

3.2. Virtual Basins. If $f$ is a polynomial and $U \subset \mathbb{C}$ an invariant Fatou component of $N_f$, then $U$ is the immediate basin of a root of $f$, because the Julia set of $N_f$ is connected [21], all finite fixed points are attracting and the fixed point at $\infty$ is repelling. If $f$ is transcendental entire, $N_f$ may possess invariant unbounded Fatou domains in which the dynamics converges to $\infty$. Such components are Baker domains or attracting petals of an indifferent fixed point at infinity. In many cases, such components contain an asymptotic path of an asymptotic value at 0 for $f$ [4].

Definition 3.8 (Virtual Basin). An unbounded domain $V \subset \mathbb{C}$ is called virtual immediate basin of $N_f$ if it is maximal (among domains in $\mathbb{C}$) with respect to the following properties:

1. $\lim_{n \to \infty} N_f^m(z) = \infty$ for all $z \in V$;
2. there is a connected and simply connected subdomain $S_0 \subset V$ such that $N_f(S_0) \subset S_0$ and for all $z \in V$ there is an $m \in \mathbb{N}$ such that $N_f^m(z) \in S_0$.

We call the domain $S_0$ an absorbing set for $V$.

Clearly, virtual immediate basins are forward invariant.
Theorem 3.9 (Virtual Basin Simply Connected). [17, Theorem 3.4] Virtual immediate basins are simply connected.

It might be possible to extend Shishikura’s theorem [21] to show that for Newton maps of entire functions, all Fatou components are simply connected. Taixes has announced partial results in this direction, in particular he rules out the existence of cycles of Herman rings (see also Corollary 4.9 below). If it were also known that Baker domains are always simply connected, then a result of Cowen [7, Theorem 3.2] would imply that every invariant Fatou component of a Newton map is an immediate basin or a virtual immediate basin (see [17, Remark 3.5]).

4. A Fixed Point Theorem

Let $X$ be a compact, connected and triangulable real $n$-manifold and let $f : X \to X$ be continuous with finitely many fixed points. Each fixed point of $f$ has a well-defined Lefschetz index, and $f$ has a global Lefschetz number. The classical Lefschetz fixed point formula says that the sum of the Lefschetz indices is equal to the Lefschetz number of $f$, up to a factor of $(-1)^n$ [14, 5].

In [9, Lemma 3.7], Goldberg and Milnor give a version of this theorem for weakly polynomial-like mappings $f : D \to \mathbb{C}$. We prove a similar result for a class of maps $f : \Delta \to \hat{\mathbb{C}}$, where $\Delta \subset \hat{\mathbb{C}}$ is a closed topological disk. By extending the range of $f$ to $\hat{\mathbb{C}}$, we allow poles and have to take more boundary components into account than Goldberg and Milnor.

Definition 4.1 (Lefschetz Map). Let $\Delta \subset \hat{\mathbb{C}}$ be a closed topological disk with boundary curve $\partial \Delta$ and $f : \Delta \to \hat{\mathbb{C}}$ an orientation preserving open mapping with isolated fixed points. We call $f$ a Lefschetz map if it satisfies the following conditions:

- for every $z \in \hat{\mathbb{C}}$, the full preimage $f^{-1}(\{z\}) \subset \Delta$ is a finite set;
- $f(\partial \Delta)$ is a simple closed curve so that $f|_{\partial \Delta} : \partial \Delta \to f(\partial \Delta)$ is a covering map of finite degree;
- $f(\partial \Delta) \cap \hat{\mathbb{C}} = \emptyset$;
- if $\xi \in \partial \Delta$ is a fixed point of $f$, then $\xi$ has a neighborhood $U$ such that $f(\partial \Delta \cap U) \subset \partial \Delta$, and $f$ is expanding on $\partial \Delta \cap U$.

Remark 4.2. The definition of “expanding” is with respect to the local parametrization of $\partial \Delta$ near $\xi$ so that $f|_{\partial \Delta \cap U}$ is topologically conjugate to $x \mapsto 2x$ in a neighborhood of 0.

In this case, the map $f$ can be extended continuously to $U \cup \Delta$ so that $f$ on $U \setminus \hat{\Delta}$ is topologically conjugate to $z \mapsto 2z$ on the half disk $\{z \in \mathbb{C} : |z| < 1$ and $\text{Im}(z) \geq 0\}$ (possibly after shrinking $U$). Such an extension will be called the simple extension outside of $\Delta$ near $\xi$.

Definition 4.3 (Lefschetz Index). Let $W \subset \mathbb{C}$ be a closed topological disk and $f : W \to \mathbb{C}$ be continuous with an isolated fixed point at $\xi \in \bar{W}$. With $g(z) = f(z) - z$, we assign to $\xi$ its Lefschetz index

$$\iota(\xi, f) := \lim_{\epsilon \to 0} \frac{1}{2\pi i} \oint_{\partial B_{\epsilon}(\xi)} \frac{d\zeta}{g(\partial B_{\epsilon}(\xi))}.$$  

This is the number of full turns that the vector $f(z) - z$ makes when $z$ goes once around $\xi$ in a sufficiently small neighborhood.
If \( \xi \in \partial W \cap \mathbb{C} \) is an isolated boundary fixed point which has a simple extension outside of \( W \), then we define its Lefschetz index as above for this simple extension.

For an interior fixed point, it is easy to see that the limit exists and is invariant under homotopies of \( f \) that avoid additional fixed points. Strictly speaking, the curve \( g(\partial B_\epsilon(\xi)) \) need not be an admissible integration path (i.e. rectifiable), but because of homotopy invariance, we may ignore this problem, and we will often do so in what follows.

The Lefschetz index is clearly a local topological invariant; for boundary fixed points, it does not depend on the details of the extension. Therefore, the index is also defined if \( \xi = \infty \), using local topological coordinates. Note that for boundary points, the simple extension as defined above generates the least possible Lefschetz index for all extensions of \( f \) to a neighborhood of \( \xi \).

If \( f \) is holomorphic in a neighborhood of a fixed point \( \xi \), then \( i(\xi, f) \) is the multiplicity of \( \xi \) as a fixed point.

**Definition 4.4 (Lefschetz Number).** Let \( f: \Delta \to \bar{\mathbb{C}} \) be a Lefschetz map, let \( V \) be the component of \( \bar{\mathbb{C}} \setminus f(\partial \Delta) \) containing \( \Delta \) and let \( \gamma_k \) denote the components of \( \partial f^{-1}(V) \). The **Lefschetz number** \( L(f) \) of \( f \) is then defined as

\[
L(f) := \sum_k |\text{deg}(f|_{\gamma_k}: \gamma_k \to \gamma)|.
\]

**Remark 4.5.** In this definition, the orientations of all \( \gamma_k \) are irrelevant. The \( \gamma_k \) are exactly the components of \( f^{-1}(f(\partial \Delta)) \), possibly with the exception of \( \partial \Delta \) itself: this latter curve is counted only if points in \( \Delta \) near the boundary of \( \Delta \) are mapped into \( V \) (in the other case, one can imagine \( \bar{\mathbb{C}} \setminus \mathbb{C} \) as an omitted component of \( f^{-1}(V) \), and consequently we also omit its boundary curve). As a result, \( L(f) \geq 0 \), with equality iff \( f^{-1}(V) \cap \Delta = \emptyset \), i.e., the sum over the \( \gamma_k \) is empty.

The Lefschetz number is invariant under topological conjugacies. We may thus choose coordinates so that \( \overline{V} \subset \mathbb{C} \).

**Lemma 4.6 (Mapping Degree on Curve).** Let \( \gamma \subset \mathbb{C} \) be a Jordan curve and \( f: \gamma \to \mathbb{C} \) continuous so that \( f: \gamma \to f(\gamma) \) is a covering map and \( \gamma \setminus f(\gamma) \) is contained in the bounded component of \( \mathbb{C} \setminus f(\gamma) \). If \( \gamma \) contains no fixed points of \( f \), then the mapping degree of \( f|_\gamma: \gamma \to f(\gamma) \) satisfies

\[
\deg(f|_\gamma: \gamma \to f(\gamma)) = \frac{1}{2\pi i} \oint_{g(\gamma)} \frac{d\zeta}{\zeta},
\]

where \( g(z) = f(z) - z \), and \( \gamma \) and \( f(\gamma) \) inherit their orientations from \( \mathbb{C} \).

**Proof.** Let \( \Delta \) be the bounded component of \( \mathbb{C} \setminus \gamma \) and \( w_0 \in \Delta \) any base point. Since \( \Delta \) is contractible to \( w_0 \) within \( \Delta \), \( g|_{\partial \Delta} = (f - \text{id})|_{\partial \Delta} \) is homotopic to \( (f - w_0)|_{\partial \Delta} \) in \( \mathbb{C} \setminus \{0\} \).

The integral \( \oint \) counts the number of full turns of \( f(z) - z \) as \( z \) runs around \( \gamma = \partial \Delta \). By homotopy invariance, this is equal to the number of full turns \( f(\gamma) \) makes around \( w_0 \), and this equals the desired mapping degree of \( f|_\gamma \). \( \square \)

**Lemma 4.7 (Equality in \( \mathbb{C} \)).** Let \( V \subset \mathbb{C} \) be a simply connected and bounded domain with piecewise \( C^1 \) boundary and let \( f: \overline{V} \to f(\overline{V}) \subset \mathbb{C} \) be a continuous map with

\[
\text{deg}(f|_\gamma: \gamma \to f(\gamma)) = \frac{1}{2\pi i} \oint_{g(\gamma)} \frac{d\zeta}{\zeta},
\]

where \( g(z) = f(z) - z \), and \( \gamma \) and \( f(\gamma) \) inherit their orientations from \( \mathbb{C} \).

**Proof.** Let \( \Delta \) be the bounded component of \( \mathbb{C} \setminus \gamma \) and \( w_0 \in \Delta \) any base point. Since \( \Delta \) is contractible to \( w_0 \) within \( \Delta \), \( g|_{\partial \Delta} = (f - \text{id})|_{\partial \Delta} \) is homotopic to \( (f - w_0)|_{\partial \Delta} \) in \( \mathbb{C} \setminus \{0\} \).

The integral \( \oint \) counts the number of full turns of \( f(z) - z \) as \( z \) runs around \( \gamma = \partial \Delta \). By homotopy invariance, this is equal to the number of full turns \( f(\gamma) \) makes around \( w_0 \), and this equals the desired mapping degree of \( f|_\gamma \). \( \square \)
finitely many fixed points, none of which are on ∂V. Then
\[ \sum_{f(\xi) = \xi} \iota(\xi, f) = \frac{1}{2\pi i} \oint_{f(\partial V)} \frac{d\zeta}{\zeta}, \]
where again \( g = f - \text{id}. \)

**Proof.** Break up \( V \) into finitely many disjoint simply connected open pieces \( V_j \) with piecewise \( C^1 \) boundaries so that each \( V_j \) either contains a single fixed point of \( f \) or \( f(V_j) \cap V_j = \emptyset \), and each fixed point of \( f \) is contained in some \( V_j \). This can be done by first choosing disjoint neighborhoods for all fixed points and then partitioning their compact complement in \( V \) into pieces of diameter less than \( \theta \), where \( \theta \) is chosen in such a way that \( |f - \text{id}| > \theta \) in this complement. Set
\[ c_j := \frac{1}{2\pi i} \oint_{\partial V_j} \frac{d\zeta}{\zeta}. \]
Then
\[ \frac{1}{2\pi i} \oint_{\partial V_j} \frac{d\zeta}{\zeta} = \sum_j \frac{1}{2\pi i} \oint_{\partial V_j} \frac{d\zeta}{\zeta} = \sum_j c_j. \]
On the pieces with \( f(V_j) \cap V_j = \emptyset \), we have \( c_j = 0 \), and on a piece \( V_j \) with fixed point \( \xi_j \), we have \( \iota(\xi_j, f) = c_j \) by definition. The claim follows. \( \square \)

**Theorem 4.8 (Fixed Point Count).** Let \( f : \Delta \to \hat{\Delta} \) be a Lefschetz map with Lefschetz number \( L(f) \in \mathbb{N} \). Then
\[ L(f) = \sum_{f(\xi) = \xi} \iota(\xi, f). \]

**Proof.** Let \( V \) be the component of \( \hat{\Delta} \setminus f(\partial \Delta) \) containing \( \Delta \), and choose coordinates of \( \hat{\Delta} \) such that \( V \) is bounded.

Suppose first that \( f \) has no fixed points on \( \partial \Delta \). Let \( \{U_i\} \) be the collection of components of \( f^{-1}(\hat{\Delta} \setminus V) \) and let \( \{V_j\} \) be the collection of components of \( f^{-1}(V) \). Since \( f \) is open, each \( U_i \) maps onto \( \hat{\Delta} \setminus V \) and each \( V_j \) maps onto \( V \) as a proper map. It follows that there are only finitely many \( U_i \) and \( V_j \), and they satisfy \( f(\partial U_i) = f(\partial V_j) = f(\partial \Delta) \) for each \( U_i \) and each \( V_j \). Note that every fixed point of \( f \) must be in some \( V_j \).

Subdivide the \( V_j \) into finitely many simply connected pieces so that no fixed points of \( f \) are on the boundaries; call these subdivided domains \( V'_j \). The orientation of \( \hat{\Delta} \) induces a boundary orientation on the \( V'_j \).

Set again \( g := f - \text{id}. \) Then, applying Lemma 1.17 to \( V'_j \subset \Delta \subset \hat{\Delta} \subset \mathbb{C} \) yields
\[ \sum_{f(\xi) = \xi} \iota(\xi, f) = \sum_{j} \frac{1}{2\pi i} \oint_{\partial V'_j} \frac{d\zeta}{\zeta} = \sum_j \frac{1}{2\pi i} \oint_{\partial V_j} \frac{d\zeta}{\zeta}. \]
Let \( \gamma := \partial V \) and \( \gamma_k \subset \Delta \) be the components of \( \partial V_k \) (these are exactly the curves from Definition 4.4), with the orientation they inherit from \( \partial V_k \). Then
\[ \sum_j \frac{1}{2\pi i} \oint_{\partial V_j} \frac{d\zeta}{\zeta} = \sum_k \frac{1}{2\pi i} \oint_{\partial(\gamma_k)} \frac{d\zeta}{\zeta}. \]

The curves \( \gamma_k \) come in two different kinds: those which surround the component \( V_j \) of which they form part of the boundary, and those which are surrounded by
their component \( V_j \). The first kind has the same orientation as the orientation it would inherit as a simple closed curve in \( \mathbb{C} \), and the second kind has the opposite orientation. On the other hand, it is easy to check that \( f|_{\gamma_k} : \gamma_k \to \gamma \) has positive mapping degree (with respect to this orientation of \( \gamma_k \), and the standard orientation in \( \mathbb{C} \) for \( \gamma \)) exactly for curves \( \gamma_k \) of the first kind. As a result, Lemma 4.6 implies in both cases

\[
\frac{1}{2\pi i} \oint_{g(\gamma_k)} \frac{d\zeta}{\zeta} = |\deg (f|_{\gamma_k} : \gamma_k \to \gamma)| .
\]

This implies the claim if \( f \) has no fixed points on \( \partial \Delta \).

If \( f \) has boundary fixed points, we employ a simple extension outside of \( \Delta \) in a small neighborhood of each such fixed point. In order for the extended map to be a Lefschetz map, the preimages need to be extended as well. If the extended neighborhoods are sufficiently small, this does not change the Lefschetz number of \( f \). □

As an immediate corollary of this theorem, we observe that Newton maps do not have fixed Herman rings. Note that Taulis has announced a more general result: he uses quasiconformal surgery to rule out any periodic cycles of Herman rings for Newton maps.

**Corollary 4.9 (No Fixed Herman Rings).** Newton maps of entire functions have no fixed Herman rings.

**Proof.** By [21], we may assume that \( N : \mathbb{C} \to \hat{\mathbb{C}} \) is a transcendental meromorphic Newton map. Suppose it has a fixed Herman ring, i.e. an invariant Fatou component \( H \) such that \( N|_H \) is conjugate to an irrational rotation of an annulus of finite modulus. Then, \( H \) contains an invariant and essential simple closed curve \( \gamma \). Clearly, \( \deg(N : \gamma \to \gamma) = +1 \). Let \( \Delta \) be the bounded component of \( \mathbb{C} \setminus \gamma \). Then, \( N|_{\bar{\mathbb{C}}} \) is a Lefschetz map and by Theorem 4.8, \( \Delta \) contains a fixed point. This is a contradiction, because all fixed points of \( N \) have an unbounded immediate basin (Theorem 2.2). □

### 5. Between Accesses of an Immediate Basin

In this section, we state and prove our main result. Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function and \( N_f \) its Newton map. Let \( \xi \in \mathbb{C} \) be a fixed point of \( N_f \) and \( U \) its immediate basin. Suppose that \( U \) has two distinct invariant accesses, represented by \( N_f \)-invariant curves \( \Gamma_1 \) and \( \Gamma_2 \). Consider an unbounded component \( \bar{V} \) of \( \mathbb{C} \setminus (\Gamma_1 \cup \Gamma_2) \). We keep this notation for the entire section.

**Theorem 5.1 (Main Theorem).** If no point in \( \hat{\mathbb{C}} \) has infinitely many preimages within \( \bar{V} \), then the set \( V := \bar{V} \setminus U \) contains an immediate basin or a virtual immediate basin of \( N_f \).

Note that we do not assume that \( V \) is connected.

**Corollary 5.2 (Polynomial Case).** If \( N_f \) is the Newton map of a polynomial \( f \), then each component of \( \mathbb{C} \setminus U \) contains the immediate basin of another root of \( f \).

**Proof of Corollary 5.2.** If \( f \) is a polynomial, \( N_f \) is a rational map. It has finite mapping degree and there exists \( R > 0 \) such that all components of \( U \setminus B_R(0) \) contain exactly one invariant access. Furthermore, all accesses are invariant [18].
Proposition 6]. Since \( \infty \) is a repelling fixed point of \( N_f \), there are no virtual immediate basins.

The rest of this section will be devoted to the proof of Theorem 5.1. This proof will be based on the fixed point formula in Theorem 4.8. In order to be able to use it in our setting, we will need some preliminary statements.

**Proposition 5.3 (Pole on Boundary).** If no point \( z \in U \) has infinitely many preimages within \( \hat{V} \cap U \), then \( \partial V = \partial U \cap \overline{V} \) contains at least one pole of \( N_f \).

In particular, if \( \partial V \) is connected or \( N_f | U \) has finite degree, then \( \partial V \) contains a pole of \( N_f \). Every pole on \( \partial V \) is arcwise accessible from within \( U \).

**Proof.** Let \( \varphi : D \to U \) be a Riemann map for the immediate basin \( U \) with \( \varphi(0) = \xi \). It conjugates the dynamics of \( N_f \) on \( U \) to the induced map \( g = \varphi^{-1} \circ N_f \circ \varphi : D \to D \).

By Lemma 5.4, the Carathéodory extension \( \varphi^{-1} \) maps \( \partial V \subset \partial U \cup \{ \infty \} \) to a closed interval \( I \subset \partial D \) that is bounded by the landing points \( \zeta_1 \) and \( \zeta_2 \) of \( \varphi^{-1}(\Gamma_1) \) and \( \varphi^{-1}(\Gamma_2) \). By assumption, there is an open neighborhood of \( \hat{I} \) in \( D \) which contains only finitely many \( g \)-preimages of every \( z \in D \). By Proposition 5.5, there is a neighborhood \( \hat{W'} \) of \( \hat{I} \) in \( D \) with the same property. Consider a sequence \( (z_n) \subset D \) whose accumulation set is in \( \hat{W'} \cap \partial D \). By Lemma 5.6, all limit points of \( (g(z_n)) \) are in \( \partial D \). Hence there is a neighborhood \( \hat{W} \) of \( \hat{I} \) in \( D \) such that we can extend \( g \) by Schwarz reflection to a holomorphic map \( \tilde{g} : W \to \hat{C} \) that coincides with \( g \) on \( \hat{W} \cap \partial D \). The endpoints \( \zeta_1 \) and \( \zeta_2 \) of \( I \) are fixed under this map, because each is the landing point of an invariant curve. They are repelling, because otherwise they would attract points from within \( D \).

Clearly, \( \tilde{g}(I) \subset \partial D \). If \( \tilde{g}(I) = I \), then \( \tilde{g} \) has to have an additional fixed point on \( \hat{I} \). This is a contradiction, because all points in \( \partial D \) converge to 0 under iteration of \( g \).

If \( I \) contained a critical point \( c \) of \( \tilde{g} \), points in \( D \) arbitrarily close to \( c \) would be mapped out of \( D \) by \( \tilde{g} \), again a contradiction. Hence \( \tilde{g} : I \to \partial D \) is surjective and there are points \( z_1, z_2 \in \hat{I} \) such that \( \tilde{g}(z_1) = \zeta_1, \tilde{g}(z_2) = \zeta_2 \).

For \( i = 1, 2 \), let \( \beta_i : [0, 1) \to D \) be the radial line from 0 to \( z_i \). Then, \( \varphi(\beta_i) \) accumulates at a continuum \( X_i \subset \partial V \) while \( N_f(\varphi(\beta_i)) = \varphi(\tilde{g}(\beta_i)) \) lands at \( \infty \) in the access of \( \Gamma_i \). By continuity, \( N_f(X_i) = \{ \infty \} \); the identity theorem shows that \( X_i = \{ p_i \} \) is a pole and \( \varphi(\beta_i) \) lands at \( p_i \).

We use the following general lemma to show that \( N_f | \hat{V} \) can be continuously extended to \( \infty \).

**Lemma 5.4 (Extension Lemma).** Let \( h : \hat{C} \to \hat{C} \) be a meromorphic function and \( G \subset \hat{C} \) an unbounded domain. Suppose that \( \partial G \) can be parametrized by two asymptotic paths of the asymptotic value \( \infty \) and that no point has infinitely many preimages within \( G \). Then, \( h | G \) can be continuously extended to \( \infty \).

**Proof.** Since \( h(\partial G) \) is unbounded, the only possible continuous extension is to set \( h(\infty) = \infty \).

If \( h \) cannot be continuously extended to \( \infty \), there exists a sequence \( z_n \to \infty \) in \( G \) such that \( h(z_n) \to p \in \hat{C} \). Let \( S > |p| \) and pick \( R > 0 \) such that \( |h(z)| \geq S \) for all \( z \in \partial G \) with \( |z| \geq R \), and \( p \notin h(\partial B_S(0)) \). We may suppose that all \( |z_n| > R \). Then we can choose a closed neighborhood \( B \subset B_S(0) \) of \( p \) whose boundary is a simple closed curve that contains no critical values or direct singularities and so that \( B \) is
disjoint from \( h(\partial B_R(0)) \). Now let \( W_n \) be the component of \( h^{-1}(B) \) that contains \( z_n \). Then \( W_n \subset \mathbb{C} \setminus B_R(0) \). Since \( z_n \in G \), it follows that all \( W_n \subset G \setminus B_R(0) \).

If all \( W_n \) are bounded, each can contain only finitely many \( z_k \) and there must be infinitely many such components. Since bounded \( W_n \) map onto \( B \), this would contradict the finiteness assumption. Hence there is an unbounded preimage component \( W_0 \). By Lemma 2.4 \( G \setminus B_R(0) \) then contains an asymptotic path of an indirect singularity on \( \partial B \), which also contradicts the finiteness assumption. \( \square \)

In the next proposition, we show that \( N_f |_{\mathbb{H}} \) is injective near \( \infty \). For the proof, we use an extremal length argument in the half-strip

\[ Y := [0, \infty) \times [0, 1] \]

in which we measure the modulus of a quadrilateral by curves connecting the left boundary arc to the right. For \( x \in \mathbb{R} \), define

\[ \mathbb{H}_x := \{ z \in \mathbb{C} : \text{Re}(z) \geq x \} \]

First, we prove a technical lemma.

**Lemma 5.5 (Bound on Modulus).** Let \( 0 < t \leq s \), let \( \beta \subset Y \) an injective curve from \((t, 1)\) to \((s, 0)\) and let \( Q \) the bounded component of \( Y \setminus \beta \). Let \( (0, 0), (s, 0), (t, 1) \) and \((0, 1)\) be the vertices of the quadrilateral \( Q \). Then, \( \text{mod}(Q) \leq t + 1 \).

**Proof.** Let \( R \subset Y \) be the rectangle with vertices \((0, 0), (0, 1), (t + 1, 1), (t + 1, 0)\). Its area and modulus are both equal to \( t + 1 \). In particular, \( \text{area}(Q \cap R) \leq \text{area}(R) = t + 1 \). Using the admissible density \( \rho(x) = \frac{1}{\sqrt{\text{area}(Q \cap R)}} \cdot \chi_{Q \cap R}(x) \), we get the estimate

\[ \frac{1}{\text{mod}(Q)} \geq \frac{1}{\text{area}(Q \cap R)} \geq \frac{1}{\text{area}(R)} = \frac{1}{t + 1} \]

because \( \int_{\gamma} \rho \, d\gamma \geq \frac{1}{\sqrt{\text{area}(Q \cap R)}} \) for this density and all rectifiable curves \( \gamma \) that connect the upper to the lower boundaries. \( \square \)

**Proposition 5.6 (Invariance Near \( \infty \)).** Suppose that every \( z \in \widehat{\mathbb{C}} \) has only finitely many \( N_f \)-preimages in \( \widehat{V} \). Then there exists \( R_0 > 0 \) such that for all \( R > R_0 \), the map \( N_f \) is injective on \( \widehat{V} \setminus B_R(0) \). Moreover, there exists \( S > 0 \) with the property that

\[ N_f(\widehat{V} \setminus B_R(0)) \setminus B_S(0) = \widehat{V} \setminus B_S(0) \]

**Proof.** Choose \( R_0 > \max\{|z| : z \in N_f^{-1}(\infty) \cap \overline{V}\} \). It follows from the open mapping principle and invariance of \( \partial V \) that there exists \( S_0 > 0 \) such that \( \partial N_f(\widehat{V} \setminus B_{R_0}(0)) \setminus B_{S_0}(0) \subset \partial \widehat{V} \). Since there are points \( z \in \widehat{V} \) with arbitrarily large \( |z| \) such that \( N_f(z) \in \widehat{V} \), it follows that either \( N_f(\widehat{V} \setminus B_R(0)) \subset \widehat{V} \) or \( N_f(\widehat{V} \setminus B_R(0)) \) contains a punctured neighborhood of \( \infty \) within \( \widehat{\mathbb{C}} \).

In the first case, the claims follow easily. By way of contradiction, we may thus assume that we are in the second case.

We consider the situation in logarithmic coordinates: with an arbitrary but fixed choice of branch, let \( C \subset \mathbb{H}_{\text{log}(R)} \) be the unique unbounded component of \( \text{log}(\widehat{V} \setminus B_R(0)) \). This is a closed set whose boundary consists of two analytic curves \( \gamma_1 \) and \( \gamma_2 \) and a subset of the vertical line at real part \( \text{log}(R) \). Define a holomorphic map \( g : C \to \mathbb{C} \) by \( g(z) = \text{log}(N_f(e^z)) \), choosing the branch such that \( \gamma_1 \subset g(\gamma_1) \).
This is possible because $\Gamma_1 = e^{\gamma_1}$ is $N_f$-invariant. Since $e^{\gamma_2}$ is also $N_f$-invariant, there exists $k \in \mathbb{Z}$ such that with $\gamma_4 = g(\gamma_2), \gamma_4 = \gamma_2 + 2\pi ik$. Define also $\gamma_3 := \gamma_1 + 2\pi ik$. Since $N_f(V \setminus B_R(0))$ contains a neighborhood of $\infty$, we get $k \neq 0$. See Figure 3 for an illustration of the notations and note that $\gamma_1, \gamma_2, \gamma_3$ and $\gamma_4$ are pairwise disjoint. These four curves have a natural vertical order induced by the observation that each curve separates sufficiently far right half planes into two unbounded components. To fix ideas, suppose that $\gamma_2$ is below $\gamma_1$. Then $\gamma_4$ is below $\gamma_3$. The construction implies that $\gamma_4$ is below $\gamma_1$, and no curve is between $\gamma_1$ and $\gamma_2$. Then, the vertical order is $\gamma_1, \gamma_2, \gamma_3, \gamma_4$.

![Figure 3. Illustrating the notations for Proposition 5.6.](image)

By Lemma 5.4, $g$ is continuous at $+\infty$. By the open mapping principle, there exists $T > \log(R)$ such that $\partial g(C) \cap \mathbb{H}_T \subset \gamma_1 \cup \gamma_4$. Let $C_1$ be the unique unbounded component of $C \cap \mathbb{H}_T, C_2 = C_1 + 2\pi ik$ and $C_T := g(C) \cap \mathbb{H}_T$. Note that $C_1 \cup C_2 \subset C_T$. We may choose $T$ in such a way that $\partial \mathbb{H}_T$ does not contain any critical values of $g$. Define $C'' = g^{-1}(C_T) \subset C$. Then, $g : C'' \to C_T$ is a proper map and therefore has well-defined degree. Since $g$ is injective on $\partial C''$ and has no pole, this degree is one and $g$ is univalent.

The idea of the proof is as follows: the curves $\gamma_2$ and $\gamma_3$ subdivide $C_T$ into three parts which are unbounded to the right. With an appropriate bound to the right, we obtain a large bounded quadrilateral consisting of three sub-quadrilaterals. Two of these sub-quadrilaterals, the upper and the lower ones, have moduli comparable to the modulus of the entire quadrilateral. This is a contradiction to the Grötzsch inequality if the right boundaries are sufficiently far out.

Define a homeomorphism $\psi : Y \to C_1$ that is biholomorphic on the interior and normalized so that it preserves the boundary vertex $\infty$ and the other two boundary vertices. We denote by $\mu_x \subset Y$ the vertical line segment at real part $x$. There exists an $x_0$ such that for $x \geq x_0, \psi(\mu_x) \subset C'$. For $x > x_0$, we denote by $Q_x$ the rectangle in $Y$ that is bounded by $\mu_x$, $\mu_{x'}$, $\mu_{x''}$ and $d = (x, 0)$ and $d = (x, 0)$, its modulus is equal to $x - x_0$. We denote the vertices of its image $Q'_x := \psi(Q_x)$ by $a', b', c', d'$, respectively. Let $a'' = g(a')$ and $d'' = g(d')$. Since $g$ and $\psi$ are univalent, $\text{mod}(g(Q'_x)) = x - x_0$. 

The curve $g(\psi(\mu_x))$ is a boundary curve of $g(Q_x')$; it connects $a''$ and $d''$ within $C_T$. Let $e_-$ be the intersection point $g(\psi(\mu_x)) \cap \gamma_2$ closest to $a''$ along $g(\varphi(\mu_x))$, and let $e_+$ be the intersection point furthest to the left along $\gamma_2$. Let $C'_1$ be the bounded subdomain of $C_1$ bounded by $g(\psi(\mu_x))$ between $a''$ and $e_-$, viewed as a quadrilateral with vertices $a''$ and $e_-$ and two more vertices on $\partial E_T$. Similarly, let $C''_1$ be the bounded subdomain of $\mathbb{C}$ bounded by $\partial E_T$, $\gamma_1$, the part of $\gamma_2$ to the left of $e_+$, and the part of $g(\varphi(\mu_x))$ between $a''$ and $e_+$, with right vertices $a''$ and $e_+$. Finally, let $C'''_1 := C'_1 \cup C''_1$ with right vertices $a''$ and $e_-$, and let $C''''_1 := C'_1$ but with right vertices $a''$ and $e_+$ (instead of $a''$ and $e_-$).

If $g(\psi(\mu_x))$ intersects $\gamma_2$ only once, then $e_- = e_+$ and $C'_1 = C''_1 = C'''_1 = C''''_1$. In general, the three domains $C'_1, C''_1, C'''_1$ may be different. However, we have $\text{mod}(C'_1) \geq \text{mod}(C''_1) \geq \text{mod}(C'''_1) \geq \text{mod}(C''''_1)$: the first inequality holds because $C'_1 \subset C''_1$, the second describes identical domains but with one boundary vertex moved, and the third follows again from the inclusion $C'''_1 \subset C''_1$, but this time the domain is extended on the “right” side of the domain, rather than on the “lower” side because the boundary vertex has moved.

Pulling back under $\psi$, we find that $\text{Re}(\psi^{-1}(a'')) \leq \text{Re}(a) = x$, because the map $\psi^{-1} \circ g \circ \psi$ repels points away from $\infty$. By Lemma 5.5 it follows that $\text{mod}(C''''_1) \leq \text{mod}(C'_1) \leq x + 1$.

Similar considerations on the left end of $C_1$, as well as for $C_2$, allow to subdivide $g(Q_x')$ by a single curve segment of $\gamma_2$ and $\gamma_3$ into three sub-quadrilaterals, two of which have modulus at most $x+1$. But the Grötzsch inequality implies that

$$\frac{1}{x - x_0} = \frac{1}{\text{mod}(g(Q_x'))} \geq \frac{1}{x + 1} + \frac{1}{x + 1},$$

hence $x \leq 2x_0 + 1$ which is a contradiction for large $x$. \hfill \Box

**Proof of Theorem 5.1.** In order to use Theorem 1.8 we construct an injective curve that surrounds an unbounded domain in $V$ such that the image of the curve does not intersect this domain. Consider a Riemann map $\varphi : \mathbb{D} \to U$ with $\varphi(0) = \xi$ and the induced dynamics $g = \varphi^{-1} \circ N_f \circ \varphi$ on $\mathbb{D}$. By the Remark after Proposition 5.3 the curves $\varphi^{-1}(\Gamma_1)$ and $\varphi^{-1}(\Gamma_2)$ land at points $\zeta_1, \zeta_2 \in \partial \mathbb{D}$, and $g$ extends to a neighborhood of $\zeta_1$ and $\zeta_2$ so that $\zeta_1$ and $\zeta_2$ become repelling fixed points. These fixed points have linearizing neighborhoods in which the curves $\varphi^{-1}(\Gamma_1)$, respectively $\varphi^{-1}(\Gamma_2)$, are straight lines in linearizing coordinates. If $0 < r < 1$ is large enough, these two curves intersect the circle at radius $r$ only once and we can join them by a circle segment at radius $r$ to an injective curve $\Gamma' \subset \mathbb{D}$ in such a way that $\Gamma := \varphi(\Gamma')$ separates $V$ from $\xi$. Let $W$ be the closure in $\mathbb{C}$ of the connected component of $\mathbb{C} \setminus \Gamma$ that contains $V$ (Figure 1). Note that no component of $N_f^{-1}(\Gamma)$ that intersects $W$ can leave $W$: in $\mathbb{D}$, any such component would have to intersect $\Gamma'$. But by the Schwarz Lemma, $g^{-1}(\Gamma')$ has greater absolute value than $r$ everywhere and $\Gamma'$ has only one $g$-preimage within the linearizing neighborhood of $\zeta_1$; this preimage is contained in $\Gamma'$. The same is true at $\zeta_2$.

By Proposition 5.4 $W$ contains an unbounded preimage component $W'$ of itself such that the boundary $\partial W'$ is contained in $\Gamma_1 \cup \Gamma_2$ outside a sufficiently large disk. Make $W'$ simply connected by filling in all bounded complementary components.

We claim that $\partial W'$ contains at least one finite pole on $\partial U$: if it did not, then $\partial W' \subset U$ and $N_f|_{\partial W'} : \partial W' \to \Gamma$ would be injective for all choices of $r$ above.
In the limit for \( r \to 1 \), this would imply that \( N_f|_{\partial V} \) was injective, contradicting Proposition \( \ref{prop:injectivity} \).

Therefore, \( \partial W' \) maps onto \( \Gamma \) with covering degree at least 2. If \( \infty \) is not an isolated fixed point in \( W' \), we are done. Otherwise it is easy to see that \( N_f|_{W'} \) is a Lefschetz map: there is a single boundary fixed point \( \infty \); the conditions on this boundary fixed point are satisfied because \( \Gamma_1, \Gamma_2 \subset U \), where the dynamics is expanding away from \( \infty \). Now Theorem \( \ref{thm:lefschetz} \) implies that \( W' \) contains fixed points of combined Lefschetz indices at least 2, because \( \partial W' \) contains a pole. If \( W' \) contains a finite fixed point, we are done. If not, it follows that the fixed point at \( \infty \) has Lefschetz index at least 2. Consider a Riemann map \( \psi : W' \to \mathbb{H}^+ \) that uniformizes \( W' \) to the upper half plane and maps \( \infty \) to 0; this map preserves the Lefschetz index. By Proposition \( \ref{prop:lefschetz-index} \), the map \( g = \psi \circ N_f \circ \psi^{-1} \) is defined in a relative neighborhood of 0 in \( \mathbb{H}^+ \). If a sequence converges to \( \mathbb{R} \) in this neighborhood, then so will the image of this sequence. Hence we can extend \( g \) to a neighborhood of 0 in \( \mathbb{C} \) by reflection. This extension does not reduce the Lefschetz index of 0: for a boundary fixed point, the index is defined by extending \( g \) to the lower half-plane in the way which generates the least possible fixed point index (compare Definition \( \ref{def:lefschetz-index} \)). Reflection however may increase the Lefschetz index. Therefore, 0 is a parabolic (since multiple) fixed point of the extended map, and it is easily seen that \( \partial \mathbb{H}^+ \) is in the repelling direction. By the Fatou flower theorem \( \ref{thm:fatou-flower} \), 0 has an attracting petal in \( \mathbb{H}^+ \) that induces a virtual immediate basin inside \( V \). \( \square \)

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ON NEWTON’S METHOD FOR ENTIRE FUNCTIONS

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