VISCOUS FRW COSMOLOGY IN MODIFIED GRAVITY

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Abstract

We discuss a modified form of gravity implying that the action
contains a power α of the scalar curvature. Coupling with the cosmic
fluid is assumed. As equation of state for the fluid, we take the sim-
plest version where the pressure is proportional to the dens ity. Based
upon a natural ansatz for the time variation of the scale factor, we
show that the equations of motion are satisfied for general α. Also
the condition of conservation of energy and momentum is satisfied.
Moreover, we investigate the case where the fluid is allowed to possess
a bulk viscosity, and find the noteworthy fact that consistency of the
formalism requires the bulk viscosity to be proportional to the power
(2α−1) of the scalar expansion. In Einstein’s gravity, where α = 1,
this means that the bulk viscosity is proportional to the scalar ex-
pansion. This mathematical result is of physical interest; as discussed
recently by the present authors, there exists in principle a viscosity-
driven transition of the fluid from the quintessence region into the
phantom region, implying a future Big Rip singularity.

1 Introduction

Recent years have witnessed a considerable interest in modified versions of
Einstein’s gravity. The motivation for this is coming largely from the evi-
dence that has accumulated about the ongoing accelerated expansion of the
universe. This evidence is seen in the high redshifts of type Ia supernovae [11].

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as well as in the anisotropy power spectrum of the cosmic microwave background [2]. These observations lead naturally to the idea of a dark energy, believed to be 73% of the total energy content of the universe. Thermodynamically, the dark energy concept implies rather bizarre consequences; thus the equation of state parameter \( w \) of the cosmic fluid may be less than -1, implying a singularity in the future called a Big Rip [3], and it leads also to negative entropies [4]. In the papers listed in Ref. [3], actually two different types of Big Rip were investigated. In the recent paper of Nojiri et al. [5], a classification of the known types of Big Rip were made and two new types of Big Rip were discovered, so that we know now four types in all.

The intention of introducing modified gravity models is that one may obtain a gravitational alternative to the conventional description of dark energy. The possibility may look attractive as the presence of dark energy may thereby be a consequence of the expansion of the universe [6]. Various aspects of the modified gravity idea have recently been discussed in Refs. [7, 8, 9, 10] (for an application to the de Sitter space, see Ref. [11].) In the present paper we shall study the following modified gravity model:

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} (f_0 R^\alpha + L_m). \tag{1}
\]

Here \( \kappa^2 = 8\pi G \), \( f_0 \) and \( \alpha \) are constants (\( \alpha \) may in principle be negative), and \( L_m \) is the matter Lagrangian. This kind of system was recently studied by Abdalla et al. [6]. Note that the dimension of \( f_0 \) is cm\(^2(\alpha-1)\).

It is interesting to note that the string/M-theory effective action may produce terms with negative powers of \( R \) [12]. Now, modified gravity with an \( 1/R \) is in itself not stable [13], but the inclusion of a \( R^2 \) term in the action [7], or the inclusion of quantum effects [14], makes the theory stable. As we are interested in the late time dynamics of our theory we may drop the \( R^2 \) terms and work only with terms that are dominant at late times.

It turns out that even this apparently simple generalization of Einstein’s theory is not quite trivial. We will focus attention on the following two points:

- How the equations of motion as derived from the action (1) are compatible with a scale factor \( a \) varying with time according to the ansatz of Eq. (6) below. The fluid is in the first part of our work taken to be non-viscous, and we adopt for simplicity the conventional equation of state

\[
p = (\gamma - 1)\rho = w\rho. \tag{2}
\]
It turns out that the equations of motion are actually satisfied, for arbitrary values of $\alpha$. Moreover, the conservation equation for energy and momentum are satisfied also. This is reassuring, as regards the adaptability of the conventional formalism to the modified gravity theory.

- Our second point is to introduce a bulk viscosity $\zeta$ in the fluid. It turns out that in order to obtain a consistent description, one arrives in a natural way at a time dependent bulk viscosity model; see Eq. (29) below. In particular, in the case of Einstein’s gravity ($\alpha = 1$), the bulk viscosity becomes simply proportional to the scalar expansion. An interesting fact is that this is precisely the relationship recently found in Ref. [16] to be compatible with a viscosity-driven transfer of the fluid from the quintessence region ($w > -1$) into the phantom region ($w < -1$).

## 2 Solutions of the Equations of Motion. Non-Viscous Case

We start from the spatially flat FRW metric,

$$ds^2 = -dt^2 + a^2(t) \, dx^2,$$

and set the cosmological constant $\Lambda$ equal to zero. The Hubble parameter is $H = \dot{a}/a$, where an overdot means differentiation with respect to $t$. The scalar expansion is $\theta = U^\mu_{\mu} = 3H$, where $U^\mu$ is the four-velocity of the fluid.

We shall be primarily interested in the development of the late universe, from $t = t_0$ onwards. For simplicity we choose the origin of time such that $t_0 = 0$. The corresponding energy density is $\rho_0$; this quantity is assumed to be known. Similarly we assume the initial scalar expansion $\theta_0$ to be known. In Einstein’s gravity [16, 17]

$$\rho = \rho_0 \left(1 + \frac{1}{2} \gamma \theta_0 t\right)^{-2},$$

$$a = a_0 \left(1 + \frac{1}{2} \gamma \theta_0 t\right)^{2/3\gamma}.$$  

In the recent modified gravity model of Abdalla et al. [6], the exponent $2/3\gamma$ in Eq. (5) is replaced by $2\alpha/3\gamma$. It becomes thus natural to assume the following time variation of the scale factor:

$$a = a_0 \left(1 + \beta t\right)^{2\alpha/3\gamma}$$
in the present case. We shall adopt this form in the following. The parameter $\beta$ has to be determined from the equations of motion. Taking the (00)-component of these equations, it is in principle possible to relate $\beta$ to the initial energy density $\rho_0$ at $t = 0$. Instead of Eq. (6), we may equally well write

$$a = a_0 \left(1 + \frac{\gamma}{2\alpha} \theta_0 t \right)^{2\alpha/3\gamma},$$  \hspace{1cm} (7)

$\theta_0 = 3H_0$ being the scalar expansion at $t = 0$. The explicit time dependence of $\theta$ thus follows at once,

$$\theta = \theta_0 \left(1 + \frac{\gamma}{2\alpha} \theta_0 t \right)^{-1}. \hspace{1cm} (8)$$

We see that $\beta$ and $\theta_0$ are equivalent parameters to be determined from the equations of motion as they are related through a constant factor,

$$\beta = \frac{\gamma}{2\alpha} \theta_0. \hspace{1cm} (9)$$

In Einstein’s gravity,

$$\beta = \frac{1}{2} \gamma \theta_0, \hspace{0.5cm} \alpha = 1. \hspace{1cm} (10)$$

In this case we find from Riemann’s energy equation that $\theta$ is related to $\rho$ in a very simple way,

$$\theta^2 = 3\kappa^2 \rho; \hspace{1cm} (11)$$

cf., for instance, Refs. [16, 17]. Equation (11) holds for all values of $t$.

From the action (1) we obtain in the general case the equations of motion[6]

$$-\frac{1}{2} f_0 g_{\mu\nu} R^\alpha + \alpha f_0 R_{\mu\nu} R^{\alpha-1} - \alpha f_0 \nabla_\mu \nabla_\nu R^{\alpha-1} + \alpha f_0 g_{\mu\nu} \nabla^2 R^{\alpha-1} = \kappa^2 T_{\mu\nu}, \hspace{1cm} (12)$$

where $T_{\mu\nu}$ is the energy-momentum tensor corresponding to $L_m$ (there is a printing error before Eq. (3.2) in Ref. [6]). The notation is chosen such that the values $\alpha = 1, f_0 = 1$ correspond to Einstein’s gravity.

Let us put $\mu = \nu = 0$. In coordinate basis we have

$$R_{00} = -\frac{3\dot{a}}{a}, \hspace{1cm} \dot{R} = 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right), \hspace{1cm} (13)$$

$$\nabla_0^2 R^{\alpha-1} = (\alpha - 1) \partial_0 (R^{\alpha-2} \dot{R}), \hspace{1cm} (14)$$
\[ \nabla^2 R^{\alpha-1} = -\frac{\alpha - 1}{a^3} \partial_0 (a^3 R^{\alpha-2} \dot{R}). \]  
(15)

Then, since \( T_{00} = \rho \), the (00)-component of the equations of motion becomes

\[ \frac{1}{2} f_0 R^\alpha - \alpha f_0 \frac{3\ddot{a}}{a} R^{\alpha-1} - f_0 \alpha (\alpha - 1) \partial_0 (R^{\alpha-2} \dot{R}) \]
\[ - \alpha f_0 \frac{\alpha - 1}{a^3} \partial_0 (a^3 R^{\alpha-2} \dot{R}) = \kappa^2 \rho. \]
(16)

Now taking into account our ansatz (6) for the scale factor we calculate

\[ R = \frac{4\alpha \beta^2 / \gamma}{(1 + \beta t)^2} \left( \frac{4\alpha}{3\gamma} - 1 \right), \]
(17)

\[ \frac{3\ddot{a}}{a} R^{\alpha-1} = \frac{(2\beta)^{2\alpha}(2\alpha/3\gamma - 1)}{2(1 + \beta t)^{2\alpha}} \left[ \frac{\alpha}{\gamma} \left( \frac{4\alpha}{3\gamma} - 1 \right) \right]^{\alpha-1}, \]
(18)

\[ \nabla_0^2 R^{\alpha-1} = \frac{(2\beta)^{2\alpha}(\alpha - 1)(2\alpha - 1)}{2(1 + \beta t)^{2\alpha}} \left[ \frac{\alpha}{\gamma} \left( \frac{4\alpha}{3\gamma} - 1 \right) \right]^{\alpha-1}, \]
(19)

\[ \nabla_0^2 R^{\alpha-1} = -\frac{(2\beta)^{2\alpha}(\alpha - 1)(2\alpha - 2\alpha / \gamma - 1)}{2(1 + \beta t)^{2\alpha}} \left[ \frac{\alpha}{\gamma} \left( \frac{4\alpha}{3\gamma} - 1 \right) \right]^{\alpha-1}, \]
(20)

whereby we arrive at the equation

\[ f_0 \theta_0^{2\alpha} (\gamma/\alpha)^2 (4\alpha/3\gamma - 1)^{\alpha-1} \left\{ (2 - \alpha) \frac{2\alpha}{3\gamma} - (\alpha - 1)(2\alpha - 1) \right\} = \kappa^2 \rho, \]
(21)

reinstalling \( \theta_0 \) by using Eq. (9). As is seen from Eq. (21), the energy density varies with time as

\[ \rho = \frac{\rho_0}{(1 + \beta t)^{2\alpha}}. \]
(22)

If the parameter set \( \{ f_0, \alpha, \rho_0 \} \) is given, the value of the initial scalar expansion \( \theta_0 \) can in principle be determined from Eq. (21), setting \( t = 0 \). Equation (21) agrees with Ref. (6) (the notation, and the initial assumption (6) for the scale factor, are different).

The case of Einstein’s gravity, \( \alpha = 1 \), \( f_0 = 1 \), yields

\[ \theta_0^2 = 3\kappa^2 \rho_0, \]
(23)

which is in accordance with Eq. (11) when \( t = 0 \). It is seen that \( \theta^2 \) and \( \rho \) are both varying with time as \((1 + \beta t)^{-2}\).
Next, we may for general $\alpha$ put $\mu = \nu = r$ in Eq. (12), to calculate the pressure component of the equations of motion. We have, in coordinate basis, $T_{rr} = pg_{rr} = pa^2$. We abstain, however, from giving the details here. The most important point in our context is that the pressure is from this equation found to vary with time in the same manner as the density $\rho$,

$$p = \frac{p_0}{(1 + \beta t)^{2\alpha}}.$$  

Thus, if the equation of state (2) holds at $t = 0$, it holds for all later values of $t$ also, with the same constant values for $\gamma$ and $w$. This is important for the consistency and applicability of the modified gravity theory.

### 2.1 A remark on energy-momentum conservation

So far, we have not taken into account the condition coming from conservation of energy and momentum. A characteristic property shows up if one takes the covariant divergence on both sides of Eq. (12): terms on the left hand side of the equation combine to yield zero. This is shown explicitly in a recent calculation of Koivisto [15]. Thus we obtain the same conservation equation for energy-momentum as in ordinary relativity:

$$\nabla^\nu T_{\mu\nu} = 0,$$  

which in turn implies

$$\frac{d\rho}{dt} = -\rho + p \frac{d}{a^3} a^3.$$  

Inserting the forms (22) and (24) for $\rho$ and $p$ we see that Eq. (26) is satisfied identically. Again, this is a point supporting the consistency of the modified gravity.

We may moreover note that the relation

$$\frac{p}{p_0} = \left( \frac{a}{a_0} \right)^{-3\gamma},$$  

known from ordinary relativity, is satisfied also here, for arbitrary values of $\alpha$. 

6
3 Introduction of Viscosity

Assume now that the fluid is viscous. Because of the assumed spherical symmetry the shear viscosity plays no role, and only the bulk viscosity \( \zeta \) has to be considered. On thermodynamical grounds, in conventional physics \( \zeta \) has to be positive; this being a consequence of the positive entropy change in irreversible processes. We shall assume \( \zeta > 0 \) also here. In general, the presence of \( \zeta \) does not have any influence upon the (00)-component of the equations of motion. The only change in the formalism because of viscosity is that the thermodynamical pressure \( p \) becomes replaced with the effective pressure \( \tilde{p} \), defined as

\[
\tilde{p} = p - \zeta \theta
\]  

(cf., for instance, Refs. [16, 17]). This follows from the expression for the stress tensor for a viscous fluid. It can be easily shown that the \((rr)\)-component of Eq. (12) (the pressure component of the equations of motion) becomes satisfied for all \( t \) if \( \zeta \) is taken to be time dependent, \( \zeta \propto \theta^{2\alpha - 1} \), or

\[
\zeta = \frac{\zeta_0}{(1 + \beta t)^{2\alpha - 1}}.
\]  

Namely, the product \( \zeta \theta \) then varies with \( t \) in the same manner as \( p \),

\[
\zeta \theta = \frac{\zeta_0 \theta_0}{(1 + \beta t)^{2\alpha}};
\]  

cf. Eq. (24). We do not need here to involve the pressure equation explicitly, but can confine ourselves to the energy equation (21), making use of the equation of state (2). In the presence of viscosity the right hand side of Eq. (21) thereby becomes changed to

\[
\kappa^2 \rho = \frac{\kappa^2}{\gamma - 1} p \rightarrow \frac{\kappa^2}{\gamma - 1} (p - \zeta \theta).
\]  

Thus the generalized equation from which \( \theta_0 \) can be determined in the viscous case, becomes

\[
\frac{1}{2} f_0 \left( \frac{\gamma}{\alpha} \right)^\alpha \left( \frac{4\alpha}{3\gamma} - 1 \right)^{\alpha - 1} \left\{ (2 - \alpha) \frac{2\alpha}{3\gamma} - (\alpha - 1)(2\alpha - 1) \right\} \theta_0^{2\alpha} = \kappa^2 \left( \rho_0 - \frac{\zeta_0 \theta_0}{\gamma - 1} \right).
\]  

(32)
Let us here consider two special cases. As usual, the most important case is that of Einstein’s gravity, \( \alpha = 1, f_0 = 1 \), whereby Eq. (32) reduces to a quadratic equation in \( \theta_0 \). We get

\[
\theta_0 = \frac{-3\kappa^2 \zeta_0 + \kappa^2 \sqrt{9 \zeta_0^2 + 12(\gamma - 1)^2 \rho_0 / \kappa^2}}{2(\gamma - 1)}.
\] (33)

The case \( \alpha = 1 \) turns out to have a bearing on the transition of the fluid through the barrier \( w = 1 \) into the phantom region. In the notation of Ref. [16] we can write in this case

\[
\zeta = \tau \theta,
\] (34)

\( \tau \) being a positive constant. As found in Ref. [16], if the magnitude of the bulk viscosity is so large that the condition

\[
3\kappa^2 \tau > \gamma
\] (35)

is satisfied, then the fluid will be driven into the phantom region \( (w < -1) \) and the Big Rip singularity, even if it started out in the quintessence region \( (w > -1) \). This result follows directly from Friedmann’s equations for a viscous fluid, and shows up mathematically as a singularity in the density \( \rho \) occurring at a finite time \( t \).

As our second example we will consider the case \( \alpha = 1/2 \), meaning that the gravitational action (1) involves the square root of the scalar curvature. Mathematically, Eq. (32) then becomes quite simple and yields the solution

\[
\theta_0 = \frac{\kappa^2 \rho_0}{f_0 \sqrt{\frac{3}{10(1 - 3\gamma/2)} + \frac{\kappa^2 \zeta_0}{\gamma - 1}}}.
\] (36)

The initial scalar expansion becomes thus proportional to the initial density. The same proportionality holds true for the time dependent quantities \( \theta \) and \( \rho \); cf. Eqs. (3) and (22).

4 Summary

We started out with a spatially flat FRW space, adopting a modified gravity action in the form of Eq. (11); this form being recently studied in Ref. [6]. The
corresponding equations of motion are given in Eq. (12). We made the basic ansatz (6) for the time variation of the scale factor $a$. This ansatz turned out to function well throughout, thus showing in general the flexibility and usefulness of the modified gravity formalism. We considered also the energy-momentum conservation, with similar satisfactory results.

As far as viscosity is concerned, the formalism admits the presence of a bulk viscosity $\zeta$ without any problems if $\zeta$ is taken to be proportional to the scalar expansion $\theta$ raised to the power $(2\alpha - 1)$; cf. Eq. (29). Then the product $\zeta \theta$ varies with time in the same manner as the pressure $p$, and the equations of motion are satisfied. This behavior could hardly have been seen beforehand, without a detailed consideration of the formalism. In the special case of Einstein’s gravity ($\alpha = 1$) this result fits nicely into the viscous cosmology description recently given in Ref. [16]: there exists in principle a viscosity-driven transition of the fluid from the quintessence region into the phantom region, implying a future Big Rip singularity.

Finally we mention that our inclusion of bulk viscosity may be used in more general modified gravities, for instance, those introduced by Allemandi et al. [18].

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