APPENDIX

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1

We shall follow the notation introduced in the paper of A.E. Parker [P]. It is then well known that for any finite dimensional $G$-module $M$ we have that

$$(M, \nabla(\mu)) = \sum_i (-1)^i \dim \text{Ext}_G^i(M, \nabla(\mu))$$

(1.0.1)

Now the Lusztig conjecture (which is still unproved) can be formulated as a statement about

$$\dim \text{Ext}_G^l(L(\lambda), \nabla(\mu))$$

for $\lambda$ and $\mu$ in a certain region of the dominant weights.

Especially the Lusztig conjecture would imply that for certain $w, y \in W_p$

$$\dim \text{Ext}_G^{l(w) - l(y)}(L(w, \lambda), \nabla(y, \mu)) = 1$$

Our aim is to prove this statement for all $w, y \in W_p$, i.e. even for weights in the region where the Lusztig conjecture is known not to hold.

Our main methods to obtain this result will be the exact sequences from Corollary 3.4 together with a result of Jantzen on the translation functors applied to simple modules. It is a pleasure to thank the referee for many useful suggestions and especially for pointing out this result, which replaces our original argument. I also wish to thank H. H. Andersen for useful discussion related to this work.

2. Translation arguments

Recall the translation functors $T^\lambda_\mu$ and $T^\mu_\lambda$ from section 3 of the paper. Choose $\lambda$ a regular, and $\mu$ a semiregular weight. Choose moreover $w \in W_p$ such that $ws < w$ for a simple reflection $s$. It is then an easy, well known, fact that $T^\lambda_\mu L(w, \mu)$ has simple head and socle, both

\footnote{Supported by the TMR-Network Algebraic Lie Theory ERB FMRX-CT97-0100 and EPSRC Grant M22536}
isomorphic to $L(ws\lambda)$. Following [A] one can thus define modules $U(w\lambda)$ and $Q(w\lambda)$ by the exact sequences

$$0 \rightarrow L(ws\lambda) \rightarrow T^s_\mu L(w\mu) \rightarrow Q(w\lambda) \rightarrow 0 \quad (2.0.2)$$

$$0 \rightarrow U(w\lambda) \rightarrow Q(w\lambda) \rightarrow L(ws\lambda) \rightarrow 0 \quad (2.0.3)$$

It should be noticed that the action of the affine Weyl group on the weights is here the one of the above paper, rather than the one of [A].

The following result is well known, see e.g. [J1], II.6.20.

**Lemma 2.1.** For $w, y \in W_p$ such that $w\lambda \in X(T)^+$ we have for $i > l(w) - l(y)$ that

$$\text{Ext}^i_G(L(w\lambda), \nabla(y\lambda)) = 0$$

2.1. The next two Lemmas relate the Ext groups we are interested in to an Ext group involving $U(w\lambda)$.

**Lemma 2.2.** Assume $ys < y$. Then the following holds

$$\text{Ext}^{l(w)-l(y)}_G(U(w\lambda), \nabla(y\lambda)) \cong \text{Ext}^{l(w)-l(y)}_G(L(ws\lambda), \nabla(ys\lambda)) \text{ if } ys\lambda \in X(T)^+$$

$$\text{Ext}^{l(w)-l(y)}_G(U(w\lambda), \nabla(y\lambda)) \cong \text{Ext}^{l(w)-l(y)-1}_G(L(ws\lambda), \nabla(y\lambda)) \text{ if } ys\lambda \notin X(T)^+$$

**Proof.** By Lemma 2.6 of [A] we have that

$$\text{Ext}^i_G(Q(w\lambda), \nabla(y\lambda)) \cong \text{Ext}^i_G(L(ws\lambda)), \nabla(ys\lambda)) \text{ if } ys\lambda \in X(T)^+$$

$$\text{Ext}^i_G(Q(w\lambda), \nabla(y\lambda)) \cong \text{Ext}^{i-1}_G(L(ws\lambda), \nabla(y\lambda)) \text{ if } ys\lambda \notin X(T)^+$$

We insert this information into the long exact sequence that arises from the application of $\text{Hom}_G(-, \nabla(y\lambda))$ to (2.1.7). If $ys\lambda \in X(T)^+$, part of the resulting sequence is

$$\rightarrow \text{Ext}^{l(w)-l(y)}_G(L(ws\lambda), \nabla(y\lambda)) \rightarrow \text{Ext}^{l(w)-l(y)}_G(L(ws\lambda), \nabla(ys\lambda)) \rightarrow \text{Ext}^{l(w)-l(y)}_G(U(w\lambda), \nabla(y\lambda)) \rightarrow \text{Ext}^{l(w)-l(y)+1}_G(L(ws\lambda), \nabla(y\lambda)) \rightarrow$$

while for $ys\lambda \notin X(T)^+$ part of the resulting sequence is

$$\rightarrow \text{Ext}^{l(w)-l(y)}_G(L(ws\lambda), \nabla(y\lambda)) \rightarrow \text{Ext}^{l(w)-l(w)-1}_G(L(ws\lambda), \nabla(y\lambda)) \rightarrow \text{Ext}^{l(w)-l(y)}_G(U(w\lambda), \nabla(y\lambda)) \rightarrow \text{Ext}^{l(w)-l(y)+1}_G(L(ws\lambda), \nabla(y\lambda)) \rightarrow$$

But by Lemma 2.1 the first and last terms in the two sequences are zero; the Lemma is proved.

**Lemma 2.3.** Assume $sy > y$. Then the following holds

$$\text{Ext}^{l(w)-l(y)}_G(U(w\lambda), \nabla(y\lambda)) \cong \text{Ext}^{l(w)-l(y)-1}_G(L(ws\lambda), \nabla(y\lambda))$$

**Proof.** This follows from the last Lemma and the exact sequence of Corollary 3 of the paper, together with the fact that $T^s_\lambda U(w\lambda) = 0$. 

2.2. After these preparatory Lemmas we can prove our main result.

Theorem 2.4. For all \( y, w \in W_p \) such that \( y \leq w \) and \( y, \lambda, w, \lambda \in X(T)^+ \) we have

\[
\dim \text{Ext}_G^{l(w) - l(y)}(L(w, \lambda), \nabla(y, \lambda)) = 1
\]

Proof. We proceed by induction on \( l(w) \). If \( w = 1 \Rightarrow y = 1 \) and

\[
\dim \text{Hom}_G(L(\lambda), \nabla(\lambda)) = 1
\]

We then assume the result for \( w' \) with \( l(w') < l(w) \) and choose \( s \) with \( ws < w \) and \( ws, \lambda \in X(T)^+ \). Then the Theorem holds for \( ws \) and we get from Lemma 2.2 and Lemma 2.3 that

\[
\dim \text{Ext}_G^{l(w') - l(y)}(U(w, \lambda), \nabla(y, \lambda)) = 1
\]

So the Theorem would be a consequence of the isomorphism

\[
\text{Ext}_G^{l(w) - l(y)}(U(w, \lambda), \nabla(y, \lambda)) \cong \text{Ext}_G^{l(w) - l(y)}(L(w, \lambda), \nabla(y, \lambda)) \tag{2.2.1}
\]

We now claim that

\[
[U(w, \lambda), L(z, \lambda)] \neq 0, \quad z \neq w \Rightarrow l(w) - l(z) \geq 2 \tag{2.2.2}
\]

Believing this we would have from Lemma 2.1 that

\[
\text{Ext}_G^{l(w) - l(y) - k}(L(z, \lambda), \nabla(y, \lambda)) = 0 \text{ for } k = 0, 1 \tag{2.2.3}
\]

Now it is easy to see, (proof of prop. 2.8 (ii) of \([A]\)) that

\[
[U(w, \lambda), L(w, \lambda)] = 1 \tag{2.2.4}
\]

And then (2.4.1) would follow by filtering \( U(w, \lambda) \) and considering the terms of index \( l(w) - l(y) \) and \( l(w) - l(y) - 1 \) in the long exact sequence given by the application of \( \text{Hom}_G(-, \nabla(y, \lambda)) \). So we must prove (2.2.2).

But this follows from a result of Jantzen, part (b) of the proposition on page 299 in \([J2]\), saying that

\[
[U(w, \lambda), L(z, \lambda)] \leq 2 [\nabla(ws, \lambda), L(z, \lambda)], \quad z \neq w \tag{2.2.5}
\]

□
REFERENCES

[A] H.H. Andersen, An inversion formula for the Kazhdan-Lusztig polynomials for affine Weyl groups. Advances in mathematics, vol 60 No. 2, May 1986, 125-153

[J1] J.C. Jantzen, Representation of algebraic groups, Pure and appl. Math 131, Academic Press (1987).

[J2] J.C. Jantzen, Weyl modules for groups of Lie type. M. Collins (ed.), ”Finite simple groups II” (Proceedings Durham 1978 ), London/New York (1980) (Academic Press):2

[KL] D.Kazhdan, G.Lusztig, Coxeter Groups and Hecke algebras, Invent. Math., vol 39, 165-184, 1979

[P] A.E. Parker, On the good filtration dimension of Weyl modules for a linear algebraic group, J. reine angew. Math. 562 (2003), 5-21.

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