Nearly-Kähler 6-Manifolds of Cohomogeneity Two: Local Theory

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October 2018

Abstract

We study nearly-Kähler 6-manifolds equipped with a cohomogeneity-two Lie group action for which the principal orbits are coisotropic. If the metric is complete, then we show that this last condition is automatically satisfied, and both the acting Lie group and the principal orbits are finite quotients of $S^3 \times S^1$.

We partition the class of such nearly-Kähler structures into three types (called I, II, III) and prove a local existence and generality result for each type. Metrics of Types I and II are shown to be incomplete.

We also derive a quasilinear elliptic PDE system on the 2-dimensional orbit space which nearly-Kähler structures of Type I must satisfy. Finally, we remark on a relatively simple one-parameter family of Type III structures that turn out to be incomplete metrics that are cohomogeneity-one under the action of a larger group.

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1 Introduction

Nearly-Kähler 6-manifolds are a class of Riemannian 6-manifolds \((M^6, g)\) whose geometry is in some sense modeled on the round 6-sphere \(S^6 \cong G_2/SU(3)\). Like the round \(S^6\), they carry a triple \((J, \Omega, \Upsilon)\) consisting of a compatible almost-complex structure \(J\), a non-degenerate 2-form \(\Omega\), and a \((3,0)\)-form \(\Upsilon\), and these are asked to satisfy the defining differential equations

\[d\Omega = 3 \text{Im}(\Upsilon)\]
\[d\text{Re}(\Upsilon) = 2 \Omega \wedge \Omega.\]

Here, the almost-complex structure \(J\) is not integrable, and the 2-form \(\Omega\) is not closed.

Yet, in spite of these two shortcomings, nearly-Kähler 6-manifolds enjoy several remarkable characterizations that have led to increased attention as of late, especially in connection with exceptional holonomy metrics and real Killing spinors. Indeed:

**Theorem (Bär [1], Grunewald [20]):** Let \((M^6, g)\) be a simply-connected, spin, complete Riemannian 6-manifold. Let \(\text{Cone}(M, g) = (\mathbb{R}^+ \times M, dt^2 + t^2 g)\) be its Riemannian cone. The following are equivalent:

(i) \((M, g)\) admits a nearly-Kähler structure.
(ii) \((M, g)\) has a real Killing spinor.
(iii) \(\text{Cone}(M, g)\) has a parallel spinor.
(iv) \(\text{Hol}(\text{Cone}(M, g)) = G_2\) or \((M^6, g) \cong (S^6, g_{\text{round}})\).

In fact, nearly-Kähler 6-manifolds are Einstein of positive scalar curvature. Thus, complete examples are compact with finite fundamental group (by Bonnet-Myers).

A central problem in the study of nearly-Kähler 6-manifolds is the present dearth of compact, simply-connected examples. Indeed, as of this writing, only six such examples are known. Four of these are the homogeneous spaces \([27]\)

\[S^6 = \frac{G_2}{SU(3)}, \quad S^3 \times S^3 = \frac{\text{SU}(2)^3}{\Delta \text{SU}(2)}, \quad \text{CP}^3 = \frac{\text{Sp}(2)}{U(1) \times \text{Sp}(1)} \times \frac{U(1)}{T^2}, \quad \text{Flag}(\mathbb{C}^3) = \frac{\text{SU}(3)}{T^2},\]

and it has been shown \([9]\) that these are the only possible homogeneous examples. Here, we caution that the metric on \(S^3 \times S^3\) is not the product metric, and the almost-complex structure on \(\text{CP}^3\) is not the standard one.

Following work of Conti and Salamon \([11]\), Fernández, Ivanov, Muñoz and Ugarte \([14]\), and Podestà and Spiro \([22, 23]\), recently Foscolo and Haskins \([16]\) succeeded in constructing inhomogeneous nearly-Kähler metrics on \(S^6\) and \(S^3 \times S^3\) that are cohomogeneity-one under an \((\text{SU}(2) \times \text{SU}(2))\)-action. Their approach involves cohomogeneity-one techniques, drawing on methods of Eschenburg and Wang \([13]\) and Böhm \([4]\), guided by the idea that such examples might arise as desingularizations of the sine-cone over the Sasaki-Einstein \(S^2 \times S^3\).

From the point of view of symmetries, the next natural question is the existence of compact simply-connected examples of cohomogeneity-two. This remains a difficult open problem, and is the primary motivation for this work.

1.1 Methods and Main Results

In this work, we study the geometry of cohomogeneity-two nearly-Kähler 6-manifolds. That is, we study nearly-Kähler 6-manifolds \(M\) equipped with a faithful \(G\)-action whose generic orbits have
codimension two. We always suppose that $G$ is a closed, connected subgroup of the isometry group of $M$, and that the $G$-action preserves $(J, \Omega, \Upsilon)$.

We will restrict attention to the case where the principal $G$-orbits are coisotropic, meaning that the 4-form $\Omega \wedge \Omega$ vanishes on these (4-dimensional) orbits. Our first result shows that, in fact, this case is the one of most interest:

**Theorem 1.1:** Let $M$ be a nearly-Kähler 6-manifold. Suppose that a connected Lie group $G$, closed in the isometry group of $M$, acts faithfully on $M$ with cohomogeneity two, preserving $(J, \Omega, \Upsilon)$.

(a) If $M$ is complete, then the principal $G$-orbits in $M$ are coisotropic.

(b) If the principal $G$-orbits are coisotropic, then $G$ is 4-dimensional and non-abelian.

(c) If $M$ is complete, then both $G$ and the principal $G$-orbits in $M$ are finite quotients of $S^3 \times S^1$.

Next, we turn to the question of local existence. That is, on sufficiently small open sets of $\mathbb{R}^6$ we ask whether cohomogeneity-two nearly-Kähler metrics can exist at all. If so, what is the initial data required to construct these metrics as solutions to a (sequence of) Cauchy problem(s)?

We approach this problem by an application of Cartan’s Third Theorem [8]. This result generalizes Lie’s Third Theorem on the “integration” of Lie algebras to local Lie groups. Its primary hypothesis is that “mixed partials commute,” meaning the satisfaction of a set of integrability conditions (analogous to the Jacobi identity for Lie algebras).

In the case of cohomogeneity-two nearly-Kähler metrics with coisotropic principal orbits, these integrability conditions form a system of 80 quadratic equations for 55 unknown functions. Careful study of this system leads us to partition the class of metrics under consideration into three types, called Types I, II, and III.

We will show that metrics of each Type exist locally and in abundance: each Type is an infinite-dimensional family. More precisely:

**Theorem 1.2:** On sufficiently small open sets in $\mathbb{R}^6$:

(a) Nearly-Kähler structures of Type I exist, depending on 2 arbitrary functions of 1 variable. If $M$ is of Type I, then $G$ is a discrete quotient of $H_3 \times \mathbb{R}$, where $H_3$ is the real Heisenberg group. In particular, metrics of Type I are incomplete.

(b) Nearly-Kähler structures of Type II exist, depending on 2 arbitrary functions of 1 variable. If $M$ is of Type II, then $G$ is solvable. In particular, metrics of Type II are incomplete.

(c) Nearly-Kähler structures of Type III with $G = (S^3 \times S^1)/(\text{Finite})$ exist, depending on 2 arbitrary functions of 1 variable.

The dependence on 2 arbitrary functions of 1 variable — the same initial data (or “local generality”) required to construct holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ — suggests the possibility that cohomogeneity-two nearly-Kähler 6-manifolds may be recovered from holomorphic data. More precisely, one can ask:

1. Can cohomogeneity-two nearly-Kähler structures be reconstructed from solutions to an elliptic PDE system on a Riemann surface $\Sigma$?
2. Can solutions to this elliptic PDE system, in turn, be reinterpreted as pseudo-holomorphic curves in some almost-complex manifold?
3. Can this elliptic PDE system be recast as a single second-order elliptic PDE on $\Sigma$?
4. Can cohomogeneity-two nearly-Kähler structures be reconstructed from holomorphic data by means of a Weierstrass representation formula?
These questions provide interesting directions for further study. In this work, we offer the following first steps in the Type I setting:

**Proposition 1.3:** Across their principal loci, nearly-Kähler structures of Type I are solutions to a certain quasilinear elliptic PDE system (5.11) on a Riemann surface.

The derivation of the PDE system (5.11) can be mimicked to obtain similar elliptic PDE systems in the Type II and Type III settings as well. However, as we hope to make plain, the PDE systems in those settings are extremely cumbersome to write down explicitly.

We expect an affirmative answer to Question 2, and are optimistic about an affirmative answer to Question 3, as well. Of course, Question 4 is significantly more speculative.

At the end of this report, we study a particular subclass of the Type III nearly-Kähler metrics with \(G = (S^3 \times S^1)/(\text{Finite})\) in some detail. This subclass turns out to be a one-parameter family of metrics \(\{g_c : c \in [\sqrt{3}, \infty]\}\), and we derive an explicit formula for \(g_{\infty}\). Unfortunately, the metric \(g_{\sqrt{3}}\) is homogeneous, while the metrics \(g_c\) for \(c \in (\sqrt{3}, \infty]\) are incomplete and cohomogeneity-one under the action of a larger group.

### 1.2 Organization

This work is organized as follows. In §2, we review the fundamentals of \(H\)-structures and intrinsic torsion, the language in which this work is phrased. In particular, we state Cartan’s Third Theorem (labeled Theorem 2.2), our main tool for proving local existence/generality results.

In §3.1, we compare and contrast various definitions of “nearly-Kähler 6-manifold” encountered in the literature, clarifying our own conventions. The material of §2 and §3.1 is standard, and experts may wish to skip these. In §3.2, we prove Theorem 1.1(a) (labeled Proposition 3.3).

Section 4 sets up the moving frame apparatus that we will use in our study. In §4.2, we adapt frames to the (coisotropic) principal \(G\)-orbits. This frame adaptation defines an \(O(2)\)-structure, and its study is central to this work. In §4.3, we describe the intrinsic torsion of our \(O(2)\)-structure, concrete geometric interpretations of which are offered in §4.5.

In §4.4, we prove Theorem 1.1(b) and 1.1(c) (labeled Proposition 4.5). Section 4.6 sets up the machinery needed to describe cohomogeneity-two nearly-Kähler structures as solutions to an elliptic PDE system on a Riemann surface.

In §5, we describe our partition into Types I, II, and III. In §5.1, we examine Type I structures and prove Theorem 1.2(a) (labeled Theorem 5.4 and Proposition 5.5) and Proposition 1.3 (labeled Proposition 5.6). Similarly, §5.2 pertains to Type II structures and contains a proof of Theorem 1.2(b) (labeled Theorem 5.9 and Proposition 5.10), and §5.3 contains a proof of Theorem 1.2(c) (labeled Theorem 5.15).

Finally, in §6, we study the one-parameter family of metrics \(\{g_c : c \in [\sqrt{3}, \infty]\}\) mentioned above.

**Notation:** The following notation and terminology will be used throughout.

- Let \(\pi : P \to M\) be a submersion. A \(k\)-form \(\theta \in \Omega^k(P)\) is \(\pi\)-semibasic if \(X \cdot \theta = 0\) for all vectors \(X \in TP\) tangent to the \(\pi\)-fibers. We will simply say “semibasic” when \(\pi\) is clear from context.
• When \( \omega = (\omega^1, \ldots, \omega^n) \) denotes the tautological 1-form on an H-structure \( B \to M^n \), we will use the shorthand
\[
\omega^{ij} := \omega^i \wedge \omega^j, \quad \omega^{ijk} = \omega^i \wedge \omega^j \wedge \omega^k, \quad \text{etc.}
\]
to denote wedge products.

• For 1-forms \( \alpha_1, \ldots, \alpha_k \in \Omega^1(M) \), we let \( \langle \alpha_1, \ldots, \alpha_k \rangle \) denote the differential ideal in \( \Omega^*(M) \) generated by these 1-forms. In particular, \( \langle \alpha_1, \alpha_2 \rangle \) denotes an ideal (not an inner product).

Acknowledgements: This work forms part of the author’s 2018 Ph.D. thesis at Stanford University. I would like to thank Robert Bryant for suggesting this problem and generously sharing his many insights. Without his tireless guidance, this work would not have been possible. I would also like to thank Rick Schoen for supporting my research throughout the duration of this project, and for all the inspiring lectures, seminars, and discussions on geometric analysis.

This work has also benefited from helpful conversations with Gavin Ball, Jason DeVito, Christos Mantoulidis, Rafe Mazzeo, Gonçalo Oliveira, McKenzie Wang, and Wolfgang Ziller.

2 H-Structures and Cartan’s Third Theorem

Much of this work will be phrased in the language of H-structures, intrinsic torsion, and augmented coframings. As such, we use this section to recall this terminology, set notation, and describe our primary technical tool for proving local existence. The material in this section is standard; more information can be found in [3], [17], and [25].

2.1 H-Structures and Intrinsic Torsion

Let \( M \) be a smooth \( n \)-manifold. A coframe at \( x \in M \) is a vector space isomorphism \( u: T_x M \to \mathbb{R}^n \). We let \( \pi_1: \mathcal{F}M \to M \) denote the general coframe bundle, which is the principal right \( \text{GL}_n(\mathbb{R}) \)-bundle over \( M \) whose fiber at \( x \in M \) consists of the coframes at \( x \). Here, the right \( \text{GL}_n(\mathbb{R}) \)-action on \( \mathcal{F}M \) is by composition: for \( g \in \text{GL}_n(\mathbb{R}) \) and \( u \in \mathcal{F}M \), we set
\[
u \cdot g := g^{-1} \circ u.
\]

A coframing on an open set \( U \subset M \) is an \( n \)-tuple \( \eta = (\eta^1, \ldots, \eta^n) \) of linearly independent 1-forms on \( U \). We think of coframings as \( \mathbb{R}^n \)-valued 1-forms \( \eta \in \Omega^1(U; \mathbb{R}^n) \) for which each \( \eta_x: T_x U \to \mathbb{R}^n \) is a coframe. Alternatively, we regard coframings as local sections \( \sigma_\eta \in \Gamma(U; \mathcal{F}M) \) or as local trivializations \( \psi_\eta: U \times \text{GL}_n(\mathbb{R}) \to \mathcal{F}M|_U \) via \( \psi_\eta(x, g) = \eta \cdot g \).

To a local diffeomorphism \( f: M_1 \to M_2 \), we associate the bundle map \( f^{(1)}: \mathcal{F}M_1 \to \mathcal{F}M_2 \) defined by
\[
f^{(1)}(u) = u \circ (f|_{\pi_1(u)})^{-1}.
\]
One can check that \( f \mapsto f^{(1)} \) is functorial.

For a subgroup \( H \leq \text{GL}_n(\mathbb{R}) \), an H-structure \( B \) on an \( n \)-manifold \( M^n \) is an H-subbundle of the general coframe bundle \( B \subset \mathcal{F}M \). Note that, despite the terminology, an H-structure depends on the representation of \( H \) on \( \mathbb{R}^n \), not just on the abstract group itself.

We say that H-structures \( \pi_1: B_1 \to M \) and \( \pi_2: B_2 \to M_2 \) are (locally) equivalent if there is a (local) diffeomorphism \( f: M_1 \to M_2 \) for which \( f^{(1)}(B_1) = B_2 \).
The tautological 1-form on an \( H \)-structure \( B \) is the \( \mathbb{R}^n \)-valued 1-form \( \omega = (\omega^1, \ldots, \omega^n) \in \Omega^1(B; \mathbb{R}^n) \) given by
\[
\omega(v) = u(\pi_*(v)), \quad \text{for } v \in T_uB.
\]
The tautological 1-form “reproduces” all of the local coframings of \( M \), in that it satisfies the following property: For any coframing \( \eta \in \Omega^1(U; \mathbb{R}^n) \), we have \( \sigma_\eta^*(\omega^1, \ldots, \omega^n) = (\eta^1, \ldots, \eta^n) \), or equivalently, \( \psi_\eta^*(\omega^1, \ldots, \omega^n)|_{(x, h)} = (\eta^1, \ldots, \eta^n)|_x \cdot h \).

One can show [17] that if \( H \) is connected, a smooth map \( F : B_1 \to B_2 \) between \( H \)-structures is a local equivalence if and only if \( F^*(\omega_2) = \omega_1 \).

A connection on an \( H \)-structure \( B \) is simply a connection on the principal \( H \)-bundle \( B \). That is, it is an \( \mathfrak{h} \)-valued 1-form \( \phi \in \Omega^1(B; \mathfrak{h}) \) that sends \( H \)-action vector fields to their Lie algebra generators and is \( H \)-equivariant:
\[
\phi(X^\#) = X, \quad \text{for all } X \in \mathfrak{h}
\]
\[
R_h^\phi(\phi) = \text{Ad}_h^{-1}(\phi), \quad \text{for all } h \in H.
\]
Note that the first condition implies that \( \phi \) restricts to each \( H \)-fiber to be the Maurer-Cartan form on \( H \).

Given an \( H \)-structure \( \pi : B \to M \) with connection \( \phi \in \Omega^1(B; \mathfrak{h}) \), one can differentiate the equation \( \psi_\eta^*(\omega) = \eta \cdot h \) to derive Cartan’s first structure equation
\[
d\omega = -\phi \wedge \omega + \frac{1}{2}T_\phi(\omega \wedge \omega),
\]
where \( T_\phi : B \to \mathbb{R}^n \otimes \Lambda^2(\mathbb{R}^n)^* \) is a function called the torsion of the connection \( \phi \). To emphasize the distinction between \( \mathbb{R}^n \) and \((\mathbb{R}^n)^*\), let us write \( V = \mathbb{R}^n \).

Let \( \phi_1, \phi_2 \) be two connections on \( B \), with torsion functions \( T_{\phi_1}, T_{\phi_2} \), respectively. The difference \( \phi_1 - \phi_2 \) is \( \pi \)-semibasic and so can be written \( \phi_1 - \phi_2 = p(\omega) \) for some function \( p : B \to \mathfrak{h} \otimes V^* \). A calculation shows [17, 25] that the difference in the torsion functions is
\[
T_{\phi_1} - T_{\phi_2} = \delta(p),
\]
where \( \delta : \mathfrak{h} \otimes V^* \hookrightarrow V \otimes V^* \otimes V^* \to V \otimes \Lambda^2 V^* \) is the \( H \)-equivariant linear map given by skew-symmetrization. Thus, the composite map
\[
T : B \to V \otimes \Lambda^2 V^* \to \frac{V \otimes \Lambda^2 V^*}{\delta(\mathfrak{h} \otimes V^*)} =: H^{0,2}(\mathfrak{h})
\]
is independent of the choice of connection \( \phi \). We refer to \( T \) as the intrinsic torsion of the \( H \)-structure, and the codomain \( H^{0,2}(\mathfrak{h}) = (V \otimes \Lambda^2 V^*)/\text{Im}(\delta) \) as the intrinsic torsion space.

**Remark:** The vector space \( H^{0,2}(\mathfrak{h}) \) can be regarded as a Spencer cohomology group, which explains the reason for the notation. □

### 2.2 The Case of \( H \leq \text{SO}(n) \)

Suppose now that \( H \leq \text{SO}(n) \). We regard \( B \subset F_{\text{SO}(n)} \), where \( F_{\text{SO}(n)} \) is the orthonormal frame bundle corresponding to the underlying \( \text{SO}(n) \)-structure. Let \( \theta \in \Omega^1(F_{\text{SO}(n)}; \text{so}(n)) \) denote the Levi-Civita connection. On \( F_{\text{SO}(n)} \), the Fundamental Lemma of Riemannian Geometry gives
\[
d\omega = -\theta \wedge \omega.
\]
Let us split $\mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{h}^\perp$ with respect to the Killing form of $\mathfrak{so}(n)$. Accordingly, we split

$$\theta|_B = \gamma_H + \tau_H,$$

where $\gamma_H \in \Omega^1(B; \mathfrak{h})$ and $\tau_H \in \Omega^1(B; \mathfrak{h}^\perp)$. One can check that $\gamma_H$ is a connection on the $H$-structure $B$, while $\tau_H = t(\omega)$ for some $t: B \to \mathfrak{h}^\perp \otimes V^\ast$. Thus, on $B$,

$$d\omega = -\gamma_H \wedge \omega + \frac{1}{2} \delta(t)(\omega \wedge \omega),$$

and so the torsion of the connection $\gamma_H$ takes values in $\delta(\mathfrak{h}^\perp \otimes V^\ast)$. In fact, since $\delta: \mathfrak{so}(n) \otimes V^\ast \to V \otimes \Lambda^2 V^\ast$ is injective, and since $\Lambda^2 V^\ast \cong \mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{h}^\perp$, we have

$$V \otimes \Lambda^2 V^\ast \cong V \otimes (\mathfrak{h} \oplus \mathfrak{h}^\perp) = (V \otimes \mathfrak{h}) \oplus (V \otimes \mathfrak{h}^\perp) \cong \delta(\mathfrak{h} \otimes V) \oplus \delta(\mathfrak{h}^\perp \otimes V),$$

whence

$$H^{0,2}(\mathfrak{h}) \cong \delta(\mathfrak{h}^\perp \otimes V) \cong \mathfrak{h}^\perp \otimes V.$$

We will return to this formula in §3.1 in the cases $H = U(3) \leq SO(6)$ and $H = SU(3) \leq SO(6)$.

### 2.3 Group Actions on $H$-Structures

We will be concerned with $H$-structures on manifolds $M$ equipped with a $G$-action that preserves the $H$-structure. In this regard, we make a simple preliminary observation.

A $G$-action on $M$ induces $G$-actions on both $T^*M$ and $FM$. Explicitly, the $G$-action on $FM$ is

$$g \cdot u = (g^{-1})^*u = u \circ (g^{-1})_*.$$

Note that if $g \in G$ stabilizes a coframe $u \in FM|_x$, then $gx = x$ and $(g^{-1})^*u = u$, so that $g$ acts as the identity on $T^*_xM$. From this, we observe:

**Lemma 2.1:** Let $P \to M^n$ be an $H$-structure, where $H \leq SO(n)$. Suppose $M$ is equipped with a $G$-action that preserves the $H$-structure and acts by cohomogeneity-$k$ on $M$. Then $n - k \leq \dim(G) \leq n + \dim(H)$.

**Proof:** Since $G$ acts with cohomogeneity-$k$ on $M^n$, so $G$ acts transitively on the $(n - k)$-dimensional principal orbits in $M$, so $\dim(G) \geq n - k$.

On the other hand, if $g$ stabilizes a coframe $u \in FM|_x$, then $g$ acts as the identity on $T^*_xM$. Since $g$ is an isometry (because $H \leq SO(n)$), so $g = \text{Id}$, so the $G$-action on $P$ is free. Thus, $\dim(G) \leq \dim(P) = n + \dim(H)$. ◇

### 2.4 Cartan’s Third Theorem

In order to prove the local existence of $H$-structures with desired properties, we encode the data of an $H$-structure in terms of an “augmented coframing.”

**Definition:** An augmented coframing on an $n$-manifold $P$ is a tuple $(\eta, a, b)$, where $\eta = (\eta^1, \ldots, \eta^n)$ is a coframing on $P$, and $a = (a^1, \ldots, a^s): P \to \mathbb{R}^s$ and $b = (b^1, \ldots, b^r): P \to \mathbb{R}^r$ are smooth functions.

The functions $a^1, \ldots, a^s: P \to \mathbb{R}$ are called the primary invariants of the augmented coframing, while the functions $b^1, \ldots, b^r: P \to \mathbb{R}$ are called free derivatives.

For the rest of this section, we fix index ranges $1 \leq i, j, k \leq n$ and $1 \leq \alpha, \beta \leq s$ and $1 \leq \rho \leq r$. 

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We will be interested in augmented coframings that satisfy a given set of structure equations, by which we mean a set of equations of the form

\[ \begin{align*} 
    d\eta^i &= -\frac{1}{2}C^i_{jk}(a) \eta^j \wedge \eta^k \\
    da^\alpha &= F^\alpha_i(a, b) \eta^i 
\end{align*} \]  

for some given functions \( C^i_{jk}(u) = -C^i_{jk}(u) \) on \( \mathbb{R}^s \) and \( F^\alpha_i(u, v) \) on \( \mathbb{R}^s \times \mathbb{R}^r \).

Let \( \pi: B \to M^n \) and \( \theta \in \Omega^1(B; \mathfrak{h}) \) be an \( H \)-structure-with-connection. Let \( \omega \in \Omega^1(B; \mathbb{R}^n) \) denote the tautological 1-form on \( B \). Then \( \eta = (\omega, \theta) = (\omega^i, \theta^i) \): \( TB \to \mathbb{R}^n \oplus \mathfrak{h} \) is a coframing of \( B \) whose exterior derivatives satisfy equations of the form

\[ \begin{align*} 
    d\omega^i &= -\theta^i_j \wedge \omega^j + T^i_{jk} \omega^j \wedge \omega^k \\
    d\theta^i_j &= -\theta^i_k \wedge \theta^k_j + R^i_{jk\ell} \omega^k \wedge \omega^\ell \\
    dT^i_{jk} &= A^i_{jk\ell}(T) \theta^\ell + B^i_{jk\ell} \omega^\ell \\
    dR^i_{jk\ell} &= C^i_{jk\ell m}(R) \theta^m + D^i_{jk\ell m} \omega^m. 
\end{align*} \]  

for some functions \( T = (T^i_{jk}): B \to V \otimes \Lambda^2 V^* \) and \( R = (R^i_{jk\ell}): B \to \mathfrak{h} \otimes \Lambda^2 V^* \).

Conversely, suppose \( \tilde{P} \) is a manifold with a coframing \( \eta = (\omega, \theta): TP \to \mathbb{R}^n \oplus \mathfrak{h} \) and functions \( T = (T^i_{jk}): \tilde{P} \to V \otimes \Lambda^2 V^* \) and \( R = (R^i_{jk\ell}): \tilde{P} \to \mathfrak{h} \otimes \Lambda^2 V^* \) satisfying (2.3a)-(2.3d). From (2.3a), there is a submersion \( \pi: \tilde{P} \to M \) whose fibers are integral manifolds of the (Frobenius) ideal \( \langle \omega^1, \ldots, \omega^n \rangle \). Further, one can construct a local diffeomorphism \( \sigma: \tilde{P} \to FM \) whose image is an \( H \)-structure \( B \subset FM \) such that \( \sigma \) sends \( \pi \)-fibers to \( H \)-orbits and has \( \sigma^*(\omega_0) = \omega \), where \( \omega_0 \) is the tautological form on \( B \).

To prove the local existence of augmented coframings satisfying prescribed structure equations (2.2), we will appeal to a very general result. This theorem, due to Cartan, is a vast generalization of the converse to Lie’s Third Theorem on the “integration” of a Lie algebra to a local Lie group. Roughly, the theorem says that the necessary first-order conditions for existence — namely, \( d(\eta^j) = 0 \) and \( d(da^\alpha) = 0 \) — are very close to sufficient.

Let us be more explicit. The equations \( d(\eta^j) = 0 \), meaning \( d(C^i_{jk}(a) \eta^j \wedge \eta^k) = 0 \), expand to

\[ F^\alpha_j \frac{\partial C^i_{kl}}{\partial u^\alpha} + F^\beta_k \frac{\partial C^i_{lj}}{\partial u^\beta} + F^\ell_i \frac{\partial C^i_{jk}}{\partial u^\ell} = C^m_{kij} C^m_{k\ell j} + C^m_{mkj} C^m_{\ell j} + C^m_{mkl} C^m_{jk}. \]

(2.4)

Similarly, the equations \( d(da^\alpha) = 0 \), meaning \( d(F^\alpha_i(a, b) \eta^i) = 0 \), expand to

\[ 0 = \frac{\partial F^\alpha_i}{\partial u^\rho} dB^\rho \wedge \eta^i + \frac{1}{2} \left( F^\gamma_i \frac{\partial F^\alpha_i}{\partial u^\gamma} - F^\gamma_j \frac{\partial F^\alpha_i}{\partial u^\beta} - C^\ell_{ij} F^\alpha_\ell \right) \eta^i \wedge \eta^j. \]

Since we lack formulas for \( dB^\rho \), it is not immediately clear how to satisfy this condition. However, if there exist functions \( G^\rho_j \) on \( \mathbb{R}^s \times \mathbb{R}^r \) for which

\[ F^\gamma_i \frac{\partial F^\alpha_i}{\partial u^\gamma} - F^\gamma_j \frac{\partial F^\alpha_i}{\partial u^\beta} - C^\ell_{ij} F^\alpha_\ell = \frac{\partial F^\alpha_i}{\partial u^\rho} G^\rho_j - \frac{\partial F^\alpha_j}{\partial u^\rho} G^\rho_i, \]

(2.5)

then \( d(da^\alpha) = 0 \) reads simply

\[ 0 = \frac{\partial F^\alpha_i}{\partial u^\rho} \left( dB^\rho - G^\rho_j \eta^j \right) \wedge \eta^i. \]
Thus, if functions $G_\rho^j$ exist which satisfy (2.5), then there will exist an expression of the $db^\rho$ in terms of $\eta^i$ that will fulfill $d(da^\alpha) = 0$. We need one last piece of terminology before stating the theorem.

Definition: The tableau of free derivatives of the equations (2.2) at a point $(u, v) \in \mathbb{R}^s \times \mathbb{R}^r$ is the linear subspace $A(u, v) \subset \text{Hom}(\mathbb{R}^n, \mathbb{R}^s)$ given by

$$A(u, v) = \text{span} \left\{ \frac{\partial F_\alpha^i}{\partial v^\rho}(u, v) e^i \otimes f^\rho : 1 \leq \rho \leq r \right\},$$

where here $\{e_1, \ldots, e_s\}$ is a basis of $\mathbb{R}^s$ and $\{f^1, \ldots, f^n\}$ is a basis of $(\mathbb{R}^n)^*$.

**Theorem 2.2 (Cartan):** Fix real-analytic functions $C_{jk}^i = -C_{kj}^i$ on $\mathbb{R}^s$ and $F_\alpha^i$ on $\mathbb{R}^s \times \mathbb{R}^r$. Suppose that:
- The functions $C_{jk}^i$ and $F_\alpha^i$ satisfy (2.4).
- There exist real-analytic functions $G_\rho^i$ on $\mathbb{R}^s \times \mathbb{R}^r$ satisfying (2.5).
- The tableau of free derivatives $A(u, v)$ is involutive, has dimension $r$, and has Cartan characters $(\tilde{s}_1, \ldots, \tilde{s}_n)$ for all $(u, v) \in \mathbb{R}^s \times \mathbb{R}^r$.

Then for any $(a_0, b_0) \in \mathbb{R}^s \times \mathbb{R}^r$, there exists a real-analytic augmented coframing $(\eta, a, b)$ on an open neighborhood of $0 \in \mathbb{R}^n$ that satisfies (2.2) and has $(a(0), b(0)) = (a_0, b_0)$.

Moreover, augmented coframings satisfying (2.2) depend (modulo diffeomorphism) on $\tilde{s}_p$ functions of $p$ variables (in the sense of exterior differential systems) where $\tilde{s}_p$ is the last non-zero Cartan character of $A(u, v)$.

**Remark:** In outline, the proof of Theorem 2.2 is as follows: One constructs an exterior differential system on the manifold $\text{GL}_n(\mathbb{R}) \times \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^r$ whose integral $n$-manifolds are in bijection with augmented coframings satisfying (2.2). An application of the Cartan-Kähler Theorem then yields the desired integral $n$-manifolds, and these depend on $\tilde{s}_p$ functions of $p$ variables. For details, see [8].

The Cartan-Kähler Theorem requires real-analyticity, which is why Theorem 2.2 does, too. However, since we will be using Theorem 2.2 to construct Einstein metrics — which are real-analytic in harmonic coordinates [12] — the real-analyticity hypothesis is not a significant limitation.

## 3 Nearly-Kähler 6-Manifolds

### 3.1 Nearly-Kähler 6-Manifolds

There are, at present, (at least) three inequivalent definitions of “nearly-Kähler 6-manifold” encountered in the literature. We take this opportunity to compare and contrast the various notions, and also put our work in its proper context.

In Gray’s original formulation [18], a nearly-Kähler structure on a smooth 6-manifold $M^6$ referred to a certain kind of $\text{U}(3)$-structure on $M^6$. A $\text{U}(3)$-structure $B \subset FM$ is equivalent to specifying on $M$ a triple $(g, J, \Omega)$ consisting of a Riemannian metric $g$, an almost-complex structure $J$, and a non-degenerate 2-form $\Omega$ satisfying the compatibility condition $g(u, v) = \Omega(u, Jv)$. A 6-manifold with $\text{U}(3)$-structure is called an almost-Hermitian 6-manifold.

In [19], the intrinsic torsion space of a $\text{U}(3)$-structure was calculated to be of the form

$$H^{0,2}(u(3)) = u(3) \perp \mathbb{R}^6 = W_1 \oplus W_2 \oplus W_3 \oplus W_4,$$
where $W_1, W_2, W_3, W_4$ are certain irreducible $U(3)$-modules of real dimensions 2, 16, 12, 6, respectively.

A $U(3)$-structure was then defined to be nearly-Kähler if its intrinsic torsion function $T: B \to H^{0,2}(u(3))$ takes values in $W_1$, the lowest-dimensional piece in the decomposition. This is equivalent (see [13], [24]) to requiring that $\nabla J$ satisfies $(\nabla X J)(X) = 0$ for all vector fields $X \in \Gamma(TM)$, or equivalently that $\nabla \hat{\Omega} = \frac{1}{3} d\hat{\Omega}$, where $\nabla$ is the Levi-Civita connection of the metric $g$.

**Remark:** Note that an almost-Hermitian 6-manifold is Kähler if its intrinsic torsion is identically zero. Equivalently, $\nabla J = 0$, or equivalently $\nabla \hat{\Omega} = 0$. □

In this work, we will adopt a different definition of “nearly-Kähler” also encountered in the literature (see, e.g., [7] and [16]) which entails an additional bit of structure. For us, a “nearly-Kähler structure” refers to a certain kind of $SU(3)$-structure.

An $SU(3)$-structure $B \subset FM$ is equivalent to specifying on $M$ a triple $(\omega, J, \hat{\Omega})$ as above together with a $(3,0)$-form $\Upsilon$ such that $\Upsilon \wedge \Upsilon = -\frac{4}{3} i \hat{\Omega}^3$. In fact, the data $(\hat{\Omega}, \Upsilon)$, subject to appropriate algebraic conditions, is enough to reconstruct $(\omega, J)$. Thus, an $SU(3)$-structure may be regarded as a pair $\hat{\Omega} \in \Omega^2(M)$ and $\Upsilon \in \Omega^3(M; \mathbb{C})$ such that at each $x \in M$, there is an isomorphism $u: T_x M \to \mathbb{R}^6$ for which

$$\hat{\Omega}|_x = u^*(dx^1 \wedge dx^4 + dx^2 \wedge dx^5 + dx^3 \wedge dx^6)$$

$$\Upsilon|_x = u^*(dz^1 \wedge dx^2 \wedge dz^3)$$

where $(z^1, z^2, z^3) = (x^1 + ix^4, x^2 + ix^5, x^3 + ix^6)$ are the standard coordinates on $\mathbb{C}^3 \cong \mathbb{R}^6$.

One can show [10] that the intrinsic torsion space of an $SU(3)$-structure is of the form

$$H^{0,2}(su(3)) = su(3)^\perp \otimes \mathbb{R}^6 = X_0^+ \oplus X_0^- \oplus X_2^+ \oplus X_2^- \oplus X_3 \oplus X_4 \oplus X_5,$$

where $X_0^+, X_2^+, X_3, X_4, X_5$ are certain irreducible $SU(3)$-modules of real dimensions 1, 8, 12, 6, respectively. Following [2], we can give a more concrete description of $H^{0,2}(su(3))$ via exterior algebra. Indeed, the $SU(3)$-modules $\Lambda^2(\mathbb{R}^6)$ and $\Lambda^3(\mathbb{R}^6)$ decompose into irreducibles as ([2], [15])

$$\Lambda^2(\mathbb{R}^6) = \mathbb{R} \Omega \oplus \Lambda_8^2 \oplus \Lambda_8^2$$

$$\Lambda^3(\mathbb{R}^6) = \mathbb{R} \mathrm{Re}(\Upsilon) \oplus \mathbb{R} \mathrm{Im}(\Upsilon) \oplus \Lambda_6^3 \oplus \Lambda_{12}^3,$$

where

$$\Lambda_8^2 = \{*(\alpha \wedge \mathrm{Re}(\Upsilon)) : \alpha \in \Lambda^1\}$$

$$\Lambda_8^2 = \{\beta \in \Lambda^2 : \beta \wedge \mathrm{Re}(\Upsilon) = 0 \text{ and } *\beta = -\beta \wedge \Omega\}$$

$$\Lambda_6^3 = \{\alpha \wedge \Omega : \alpha \in \Lambda^1\} = \{\gamma \in \Lambda^3 : *\gamma = \gamma\}$$

$$\Lambda_{12}^3 = \{\gamma \in \Lambda^3 : \gamma \wedge \Omega = 0 \text{ and } \gamma \wedge \mathrm{Re}(\Upsilon) = 0 \text{ and } \gamma \wedge \mathrm{Im}(\Upsilon) = 0\}.$$

This gives the description

$$H^{0,2}(su(3)) \cong \mathbb{R} \oplus \mathbb{R} \oplus \Lambda_8^2 \oplus \Lambda_8^2 \oplus \Lambda_6^3 \oplus \Lambda_{12}^3 \oplus \Lambda^1 \oplus \Lambda^1.$$

It can be shown [10] that the intrinsic torsion of the $SU(3)$-structure can be completely encoded in the exterior derivatives of $\Omega$ and $\Upsilon$. Moreover, borrowing the notation of [15], these exterior derivatives decompose as

$$d\Omega = 3\tau_0 \mathrm{Re}(\Upsilon) + 3\tau_1 \mathrm{Im}(\Upsilon) + \tau_2 + \tau_4 \wedge \Omega$$

$$d\mathrm{Re}(\Upsilon) = 2\tau_0 \Omega^2 + \tau_5 \wedge \mathrm{Re}(\Upsilon) + \tau_2 \wedge \Omega$$

$$d\mathrm{Im}(\Upsilon) = -2\tau_0 \Omega^2 - J\tau_5 \wedge \mathrm{Re}(\Upsilon) + \tau_2 \wedge \Omega,$$
where \( \tau_0, \tilde{\tau}_0 \in \Omega^0, \tau_2, \tilde{\tau}_2 \in \Omega^2_0, \tau_3 \in \Omega^3_2, \) and \( \tau_4, \tau_5 \in \Gamma(TM), \) and \( \Omega^k_\ell = \Gamma(\Lambda^k_\ell(T^*M)). \) This leads to:

**Definition:** Let \( M^6 \) be a real 6-manifold.

A nearly-Kähler structure on \( M \) is an SU(3)-structure \( B \subset FM \) whose intrinsic torsion function \( T: B \to H^{0,2}(\mathfrak{su}(3)) \) takes values in \( X_0 \).

That is: A nearly-Kähler structure on \( M \) is an SU(3)-structure \( (\Omega, \Upsilon) \) such that \( \tau_0 = \tau_2 = \tilde{\tau}_2 = \tau_3 = \tau_4 = \tau_5 = 0 \) and \( \tilde{\tau}_0 = c \) is constant. In other words, it is an SU(3)-structure \( (\Omega, \Upsilon) \) that satisfies

\[
\begin{align*}
 d\Omega &= 3c \text{Im}(\Upsilon) \\
 d\text{Re}(\Upsilon) &= 2c \Omega^2 \\
 d\text{Im}(\Upsilon) &= 0.
\end{align*}
\]

Of course, the third equation is a consequence of the first.

**Remark:** Note that other works (e.g. [16]) instead take \( \tau_0 = c \) constant and all other torsion forms equal to zero. □

Note that a nearly-Kähler structure has \( c = 0 \) if and only if it is Calabi-Yau. Those with \( c \neq 0 \) are sometimes called strict nearly-Kähler structures. In this case, by rescaling the metric, we may take the constant \( c = 1 \). For simplicity, and following [15] and [16], we enact the following:

**Convention:** In this work, by a “nearly-Kähler structure” we will always mean a “strict nearly-Kähler structure, scaled so that \( c = 1 \).”

### 3.2 The Coisotropic Orbit Condition

We will need to understand how 4-planes in \( \mathbb{R}^6 \) behave under the usual SU(3)-action. This requires some linear algebraic preliminaries.

Consider \((\mathbb{R}^6, g_0, \Omega_0)\) with the standard metric \( g_0 \), symplectic form \( \Omega_0 \), and orientation. Let * denote the corresponding Hodge star operator. We let \((e_1, \ldots, e_6)\) be the standard basis of \( \mathbb{R}^6 \), and we identify \( \mathbb{C}^3 \cong \mathbb{R}^6 \) via \((z^1, z^2, z^3) = (x^1 + ix^4, x^2 + ix^5, x^3 + ix^6)\).

Explicitly,

\[
\begin{align*}
 g_0 &= (dx^1)^2 + \cdots + (dx^6)^2 \\
 \Omega_0 &= dx^1 \wedge dx^4 + dx^2 \wedge dx^5 + dx^3 \wedge dx^6.
\end{align*}
\]

In particular, we observe that

\[
*\Omega_0 = \frac{1}{2} \Omega_0 \wedge \Omega_0. \tag{3.1}
\]

Let \( V_k(\mathbb{R}^6) \) denote the Stiefel manifold of ordered orthonormal \( k \)-frames in \( \mathbb{R}^6 \), and let \( \text{Gr}_k(\mathbb{R}^6) \) denote the Grassmannian of real \( k \)-planes in \( \mathbb{R}^6 \). Recall that the symplectic complement and orthogonal complement of a \( k \)-plane \( E \in \text{Gr}_k(\mathbb{R}^6) \) are the respective subspaces

\[
E^{\Omega} := \{ v \in \mathbb{R}^6 : \Omega_0(v, w) = 0, \forall w \in E \}
\]

\[
E^{\perp} := \{ v \in \mathbb{R}^6 : g_0(v, w) = 0, \forall w \in E \}.
\]

We say that \( E \) is isotropic if \( E \subset E^{\Omega} \), and that \( E \) is coisotropic if \( E \supset E^{\perp} \). Using (3.1), we see that for a 4-plane \( E \in \text{Gr}_4(\mathbb{R}^6) \):

\[
(\Omega_0 \wedge \Omega_0)|_E = 0 \iff E^{\perp} \text{ is an isotropic 2-plane } \iff E \text{ is a coisotropic 4-plane}.
\]

In particular, coisotropic 4-planes are in bijection with isotropic 2-planes.
We now seek to understand the $\text{SU}(3)$-action on 2-planes (equivalently, 4-planes) in $\mathbb{R}^6$. For $\theta \in [0, \pi]$, let us set

$$V_2(\theta) = \text{SU}(3) \cdot (e_1, \cos(\theta)e_4 + \sin(\theta)e_2) \subset V_2(\mathbb{R}^6)$$

$$\text{Gr}_2(\theta) = \text{SU}(3) \cdot \text{span}(e_1, \cos(\theta)e_4 + \sin(\theta)e_2) \subset \text{Gr}_2(\mathbb{R}^6).$$

Of particular interest to us is the orbit

$$\text{Gr}_2(\frac{\pi}{2}) = \text{SU}(3) \cdot \text{span}(e_1, e_2) = \{ E \in \text{Gr}_2(\mathbb{R}^6) : E \text{ isotropic} \}.$$

**Lemma 3.1:**

(a) Every $(v, w) \in V_2(\mathbb{R}^6)$ belongs to exactly one of the orbits $V_2(\theta)$, where $\theta \in [0, \pi]$. The transitive $\text{SU}(3)$-actions on $V_2(0)$ and $V_2(\pi)$ have stabilizer $\text{SU}(2)$. For $\theta \in (0, \pi)$, the transitive $\text{SU}(3)$-action on $V_2(\theta)$ is free.

(b) Every $E \in \text{Gr}_2(\mathbb{R}^6)$ belongs to exactly one of the orbits $\text{Gr}_2(\theta)$, where $\theta \in [0, \pi]$. The transitive $\text{SU}(3)$-action on $\text{Gr}_2(0) \cong \mathbb{CP}^2$ has stabilizer $\text{U}(2)$. For $\theta \in (0, \pi)$, the transitive $\text{SU}(3)$-action on $\text{Gr}_2(\theta)$ has stabilizer $\text{O}(2)$.

(c) In particular, $\text{SU}(3)$ acts transitively on

$$\text{Gr}_2(\frac{\pi}{2}) \cong \{ E \in \text{Gr}_4(\mathbb{R}^6) : E \text{ coisotropic} \}$$

with stabilizer

$$\text{O}(2) = \begin{pmatrix} \cos \theta & \mp \sin \theta & 0 \\ \sin \theta & \mp \cos \theta & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \leq \text{SU}(3) \leq \text{SO}(6).$$

**Proof:** (a) We first show that every $(v, w) \in V_2(\mathbb{R}^6)$ belongs to some $V_2(\theta)$.

Let $(v, w) \in V_2(\mathbb{R}^6)$. Since $\text{SU}(3)$ acts transitively on $V_1(\mathbb{R}^6) \cong S^5$, there exists $A \in \text{SU}(3)$ with $Av = e_1$, so $A(v, w) = (e_1, Aw)$. Since $Aw \perp e_1$, so $Aw \in \mathbb{R}e_4 \oplus \mathbb{C}^2$, where $\mathbb{C}^2 = \text{span}_{\mathbb{R}}(e_2, e_5, e_3, e_6)$.

Now, the subgroup of $\text{SU}(3)$ that fixes $e_1 \in \mathbb{R}^6$ is a copy of $\text{SU}(2)$. This $\text{SU}(2)$ acts on the orthogonal $\mathbb{R}e_4 \oplus \mathbb{C}^2$ in the usual way: it acts trivially $\mathbb{R}e_4$ and in the standard way on $\mathbb{C}^2$. In particular, every $x \in \mathbb{R}e_4 \oplus \mathbb{C}^2$ is $\text{SU}(2)$-conjugate to an element of the form $c_4e_4 + c_2e_2$, where $c_1 \in \mathbb{R}$ and $c_2 \geq 0$.

Thus, there exists $B \in \text{SU}(2) \leq \text{SU}(3)$ with $B \cdot Aw = c_4e_4 + c_2e_2$ for some $c_4 \in \mathbb{R}$ and $c_2 \geq 0$, so $BA \cdot (v, w) = (e_1, c_4e_4 + c_2e_2)$. Since $1 = \|w\|^2 = \|BAw\|^2 = c_4^2 + c_2^2$, so we may write $(c_4, c_2) = (\cos \theta, \sin \theta)$ for some $\theta \in [0, \pi]$. Thus, $(v, w) \in V_2(\theta)$.

To see that the orbits are disjoint, note that the composition $\Omega_0 : V_2(\mathbb{R}^6) \hookrightarrow \mathbb{R}^6 \times \mathbb{R}^6 \rightarrow \mathbb{R}$ is an $\text{SU}(3)$-invariant function, so is constant on the $\text{SU}(3)$-orbits $V_2(\theta)$. Indeed,

$$\Omega_0(e_1, \cos(\theta)e_4 + \sin(\theta)e_2) = \cos(\theta).$$

In particular, if $(v, w) \in V_2(\theta_1) \cap V_2(\theta_2)$, then $\cos(\theta_1) = \cos(\theta_2)$, so $\theta_1 = \theta_2$.

Note that $A \in \text{SU}(3)$ stabilizes $(e_1, \cos(\theta)e_4 + \sin(\theta)e_2)$ if and only if $Ae_1 = e_1$ (so $Ae_4 = e_4$) and $\sin(\theta)Ae_2 = \sin(\theta)e_2$. For $\theta = 0$ and $\theta = \pi$, this describes $\text{SU}(2)$. For $\theta \in (0, \pi)$, this describes
the identity subgroup.

(b) This follows from part (a) and the fibration $O(2) \rightarrow V_2(\mathbb{R}^6) \rightarrow \text{Gr}_2(\mathbb{R}^6)$.

(c) Note that if $A \in SU(3)$ stabilizes $\text{span}(e_1, e_2)$, then $A$ also stabilizes $\text{span}(e_4, e_5)$, which forces $A$ to lie in the $O(2)$ subgroup described above.

Thus, there are two geometrically natural first-order conditions that one could impose on the real 4-folds in a nearly-Kähler 6-manifold. In one direction, we could ask that the 4-fold be pseudo-holomorphic (normal planes lie in $\text{Gr}_2(0)$). However, such submanifolds do not exist, even locally \[6\]. In the other direction, we could ask that the 4-fold be coisotropic (normal planes lie in $\text{Gr}_2(\pi/2)$).

There is, however, another reason to study coisotropic 4-folds: any complete nearly-Kähler 6-manifold of cohomogeneity-two must have coisotropic principal orbits, as we now show.

**Lemma 3.2:** Let $N^n$ be a compact $G$-homogeneous Riemannian manifold. If $\chi \in \Omega^n(N)$ is a $G$-invariant exact $n$-form on $N$, then $\chi = 0$.

**Proof:** Let $\chi$ be such a $G$-invariant exact $n$-form. Write $\chi = f \text{ vol}_N$ for some function $f \in C^\infty(N)$. Since $\chi$ is $G$-invariant, so $f$ is $G$-invariant. Since the $G$-action is transitive, so $f$ is constant. Since $N$ is compact and $\chi$ is exact, Stokes’ Theorem gives

$$0 = \int_N \chi = \int_N f \text{ vol}_N = f \cdot \text{vol}(N).$$

Thus, $f = 0$, whence $\chi = 0$. ♦

**Proposition 3.3:** Let $M^6$ be a nearly-Kähler 6-manifold equipped with a $G$-action of cohomogeneity-two that preserves the $SU(3)$-structure, where $G \leq \text{Isom}(M,g)$ is closed.

If $M$ is complete, then $M$ is compact, $G$ is compact, the quotient space $M/G$ is compact Hausdorff, and the principal $G$-orbits in $M$ are coisotropic.

**Proof:** Suppose $M$ is complete. Since $M$ is Einstein of positive scalar curvature, by Bonnet-Myers, $M$ is compact. By Myers-Steenrod \[21\], the isometry group $\text{Isom}(M,g)$ is compact, so $G$ is compact.

Let $N^4$ be any principal $G$-orbit in $M$. Note that $N$ is a compact, $G$-homogeneous Riemannian manifold. Moreover, $\Omega^2 = \frac{1}{2}d(\text{Im}(\Upsilon))$ is a $G$-invariant exact 4-form on $N$. Thus, by Lemma 3.2, we have $\Omega^2|_N = 0$, meaning that $N$ is coisotropic. ♦

**Remark:** If, moreover, $M$ is connected and simply-connected, and the Lie group $G$ is connected, then the quotient space $M/G$ is simply-connected. See, e.g., \[5\]. □

Finally, although we will not need it here, we remark that the same argument establishes:

**Proposition 3.4:** Let $M^6$ be a nearly-Kähler 6-manifold equipped with a $G$-action of cohomogeneity-three that preserves the $SU(3)$-structure, where $G \leq \text{Isom}(M,g)$ is closed.

If $M$ is complete, then $M$ is compact, $G$ is compact, the quotient space $M/G$ is compact Hausdorff, and the 3-form $\text{Im}(\Upsilon)$ vanishes on the principal $G$-orbits.
4 Moving Frame Setup

4.1 The First Structure Equations of a Nearly-Kähler 6-Manifold

Let \( \pi: B \to M \) be an SU(3)-structure on a 6-manifold \( M \). Let \( \omega = (\omega^1, \ldots, \omega^6) \in \Omega^1(B; \mathbb{R}^6) \) denote the tautological 1-form. We will identify \( \mathbb{C}^3 \cong \mathbb{R}^6 \) via
\[
(z^1, z^2, z^3) = (x^1 + ix^4, x^2 + ix^5, x^3 + ix^6)
\]
and let \( \zeta = (\zeta^1, \zeta^2, \zeta^3) \in \Omega^1(B; \mathbb{C}^3) \) denote the \( \mathbb{C} \)-valued tautological 1-form:
\[
(\zeta^1, \zeta^2, \zeta^3) = (\omega^1 + i\omega^4, \omega^2 + i\omega^5, \omega^3 + i\omega^6).
\]
Since \( B \) is an SU(3)-structure, the 6-manifold \( M \) is endowed with a metric \( g \), a non-degenerate 2-form \( \Omega \), and a complex volume form \( \Upsilon \). Pulled up to \( B \), these are exactly:
\[
\pi^* g = \sum (\zeta_j \circ \zeta_j)^2 = (\omega^1)^2 + \cdots + (\omega^6)^2
\]
\[
\pi^* \Omega = \frac{i}{2} \sum \zeta_j \wedge \zeta_j = \omega^{14} + \omega^{25} + \omega^{36}
\]
\[
\pi^* \Upsilon = \zeta_1 \wedge \zeta_2 \wedge \zeta_3 = (\omega^1 + i\omega^4) \land (\omega^2 + i\omega^5) \land (\omega^3 + i\omega^6).
\]

In the special case where the SU(3)-structure \( B \) is nearly-Kähler, the exterior derivatives \( d\zeta_i \) satisfy the first structure equations (see \[7\], \[28\]) given by
\[
d\zeta_i = -\kappa_{ij} \wedge \zeta_j + \zeta_i \wedge \zeta_k,
\]
where \( \kappa = (\kappa_{ij}) \in \Omega^1(B; \mathfrak{su}(3)) \) is a connection 1-form, and where \((i, j, k)\) is an even permutation of \((1, 2, 3)\). In terms of the basis \((\omega^1, \ldots, \omega^6)\) for the \( \pi \)-semibasic 1-forms, the structure equations (4.1) read
\[
d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \end{pmatrix} = - \begin{pmatrix} 0 & \alpha_3 & -\alpha_2 & -\beta_{11} & -\beta_{12} & -\beta_{13} \\ -\alpha_3 & 0 & \alpha_1 & -\beta_{21} & -\beta_{22} & -\beta_{23} \\ \alpha_2 & -\alpha_1 & 0 & -\beta_{31} & -\beta_{32} & -\beta_{33} \\ \beta_{11} & \beta_{12} & \beta_{13} & 0 & \alpha_3 & -\alpha_2 \\ \beta_{21} & \beta_{22} & \beta_{23} & -\alpha_3 & 0 & \alpha_1 \\ \beta_{31} & \beta_{32} & \beta_{33} & \alpha_2 & -\alpha_1 & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \end{pmatrix} + \begin{pmatrix} \omega^{23} - \omega^{56} \\ -\omega^{13} + \omega^{46} \\ \omega^{12} - \omega^{45} \\ -\omega^{26} + \omega^{35} \\ \omega^{16} - \omega^{34} \\ -\omega^{15} + \omega^{24} \end{pmatrix},
\]
where \( \alpha_i, \beta_{ij} \in \Omega^1(B; \mathbb{R}) \) are connection 1-forms with \( \beta_{ij} = \beta_{ji} \) and \( \sum \beta_{ii} = 0 \).

In this work, however, it will be convenient to express (4.1) in a different form. Indeed, in light of the \( O(2) \)-representation on \( \mathbb{R}^6 \) described in Lemma 3.1(c), we will often prefer the basis of \( \pi \)-semibasic 1-forms given by \((\eta, \overline{\eta}, \omega^3, \theta, \overline{\theta}, \omega^6)\), where
\[
\eta = \omega^1 + i\omega^2
\]
\[
\theta = \omega^4 + i\omega^5.
\]
In terms of this basis, (4.1) is equivalent to
\[
d \begin{pmatrix} \eta \\ \omega^3 \\ \theta \\ \omega^6 \end{pmatrix} = - \begin{pmatrix} -i\alpha & 0 & 2i\xi_1 & -\xi_0 & -\xi_3 & -2\xi_2 \\ -i\xi_1 & -i\xi_1 & 0 & -\xi_0 & -\xi_3 & -2\xi_2 \\ \xi_0 & \xi_3 & 2\xi_2 & -i\alpha & 0 & 2i\xi_1 \\ \xi_2 & \xi_2 & -2\xi_0 & i\xi_1 & -i\xi_1 & 0 \end{pmatrix} \wedge \begin{pmatrix} \eta \\ \omega^3 \\ \theta \\ \omega^6 \end{pmatrix} + i \begin{pmatrix} -\eta \wedge \omega^3 + \theta \wedge \omega^6 \\ \frac{1}{2}(\eta \wedge \overline{\eta} - \theta \wedge \overline{\theta} + \omega^3 \wedge \theta + \eta \wedge \omega^6) \\ \frac{1}{2}(\overline{\eta} \wedge \theta - \eta \wedge \overline{\theta}) \\ \frac{1}{2}(\theta \wedge \eta - \theta \wedge \overline{\eta}) \end{pmatrix},
\]
where \( \alpha, \xi_0 \in \Omega^1(B; \mathbb{R}) \) and \( \xi_1, \xi_2, \xi_3 \in \Omega^1(B; \mathbb{C}) \) are connection 1-forms. The structure equations (4.3) will be central to our calculations.
4.2 Frame Adaptation: The O(2)-Bundle $P$

Let $M$ be a nearly-Kähler 6-manifold acted upon by a connected Lie group $G$ with cohomogeneity-two. We suppose that this $G$-action preserves the SU(3)-structure and that the principal $G$-orbits are coisotropic. For simplicity, the following two conventions will be in force for the rest of this work.

**Convention 4.1:** Without loss of generality, we suppose that $G$ acts faithfully on $M$, and that $G$ is a closed subgroup of the isometry group of $M$. □

**Convention 4.2:** We restrict our attention entirely to the principal locus of $M$, by which we mean the union of principal $G$-orbits $Gx$ in $M$. Henceforth, when we refer to the manifold $M$, we shall always mean the principal locus of $M$. □

We begin our study by adapting coframes to the foliation of $M$ by coisotropic 4-folds. Define the subbundle $P \subseteq B$ of SU(3)-coframes $u = (u^1, \ldots, u^6) : T_xM \rightarrow \mathbb{R}^6$ for which $T_xGx = \text{Ker}(u^1, u^2)$. In other words, letting $\{e_1, \ldots, e_6\}$ denote the standard basis of $\mathbb{R}^6$, we set

$$P = \{u \in B : u(T_xGx) = \text{span}(e_3, e_4, e_5)\} \subset B.$$ 

Since SU(3) acts transitively on the Grassmannian of coisotropic 4-planes in $\mathbb{R}^6$ (Lemma 3.1), this adaptation is well-defined. Note that $P$ is an O(2)-subbundle, where the inclusion $O(2) \leq SU(3) \hookrightarrow GL_6(\mathbb{R})$ is the one described in Lemma 3.1(c).

**Remark:** The Lie group $G$ is contained in the group $\text{Aut}_{O(2)}$ of automorphisms that preserve the foliation of $M$ by coisotropic 4-folds, which is itself contained in the full automorphism group $\text{Aut}_{SU(3)}$ of the SU(3)-structure:

$$G \leq \text{Aut}_{O(2)}(M) \leq \text{Aut}_{SU(3)}(M).$$

By Lemma 2.1, we see that:

$$4 \leq \dim(G) \leq 5, \quad 4 \leq \dim(\text{Aut}_{O(2)}) \leq 7, \quad 4 \leq \dim(\text{Aut}_{SU(3)}) \leq 14. \quad \Box$$

Henceforth, we work on the O(2)-subbundle $P \subseteq B$ and use the same letter $\pi : P \rightarrow M$ to denote the restricted projection map. Now, the connection 1-form $\kappa \in \Omega^1(B; \text{su}(3))$ does not remain a connection form when restricted to $P$. Indeed, for a choice of splitting $\text{su}(3) = \mathfrak{so}(2) \oplus W$, the 1-form $\kappa|_P \in \Omega^1(P; \text{su}(3))$ decomposes as

$$\kappa|_P = \gamma_{O(2)} + \tau_{O(2)},$$

where $\gamma_{O(2)} \in \Omega^1(P; \mathfrak{so}(2))$ is a connection 1-form and $\tau_{O(2)} \in \Omega^1(P; W)$ is $\pi$-semibasic. In terms of the basis $(\eta, \bar{\eta}, \omega^3, \theta, \bar{\theta}, \omega^6)$, this splitting reads

$$\kappa|_P = \begin{pmatrix} -i\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 2i\xi_1 & -\xi_0 & -\xi_3 & -2\xi_2 \\ i\xi_1 & -i\xi_1 & 0 & -\xi_2 & -\xi_2 & 2\xi_2 \\ \xi_0 & \xi_3 & 2\xi_2 & 0 & 0 & 2i\xi_1 \\ -\xi_2 & -\xi_2 & 2\xi_2 & 0 & 0 & 2i\xi_1 \\ i\xi_1 & -i\xi_1 & 0 & -\xi_2 & -\xi_2 & 2\xi_2 \\ i\xi_1 & -i\xi_1 & 0 & -\xi_2 & -\xi_2 & 2\xi_2 \end{pmatrix}$$

In particular, $\alpha$ is a connection 1-form for the O(2)-bundle $\pi : P \rightarrow M$, so $(\eta, \bar{\eta}, \omega^3, \theta, \bar{\theta}, \omega^6, \alpha)$ is a coframing on $P$. 

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On the other hand, the 1-forms $\xi_0 \in \Omega^1(P; \mathbb{R})$ and $\xi_1, \xi_2, \xi_3 \in \Omega^1(P; \mathbb{C})$ are $\pi$-semibasic, so that we may write

$$
2\xi_1 = a_{11}\eta + a_{12}\bar{\eta} + a_{13}\omega^3 + a_{14}\theta + a_{15}\bar{\theta} + a_{16}\omega^6
$$

$$
2\xi_2 = a_{21}\eta + a_{22}\bar{\eta} + a_{23}\omega^3 + a_{24}\theta + a_{25}\bar{\theta} + a_{26}\omega^6
$$

$$
2\xi_3 = a_{31}\eta + a_{32}\bar{\eta} + a_{33}\omega^3 + a_{34}\theta + a_{35}\bar{\theta} + a_{36}\omega^6
$$

$$
\xi_0 = a_{01}\eta + a_{02}\bar{\eta} + a_{03}\omega^3 + a_{04}\theta + a_{05}\bar{\theta} + a_{06}\omega^6
$$

(4.4)

for some 24 $G$-invariant functions $a_{ij}: P \rightarrow \mathbb{C}$. We will refer to these 24 functions as the torsion functions of the $O(2)$-structure. In the next section, we will see (Lemma 4.4) that they are not independent of one another.

4.3 The Torsion of the $O(2)$-Structure

We continue with the setup from §4.2. The purpose of this section is to derive relations (Lemma 4.4) on the 24 functions $a_{ij}: P \rightarrow \mathbb{C}$ of (4.4). In the next section, we will use this information to show (Proposition 4.5) that the acting Lie group $G$ is 4-dimensional and non-abelian.

We begin with the following observation: Unlike a generic $O(2)$-structure, the $O(2)$-structures in our situation enjoy a special geometric feature. Namely, the (real) 4-plane field $\text{Ker}(\omega^1, \omega^2) = \text{Ker}(\eta, \bar{\eta})$ on $M^6$ is integrable, its leaf space is the orbit space $\Sigma = M/G$ (recall Convention 4.2), and the quadratic form $(\omega^1)^2 + (\omega^2)^2 = \eta \circ \bar{\eta}$ descends to a Riemannian metric on $\Sigma$. Consequently:

**Lemma 4.3:** There exist a 1-form $\phi \in \Omega^1(P; \mathbb{R})$ and a function $K \in \Omega^0(P; \mathbb{R})$ such that

$$
d\eta = i\phi \wedge \eta
$$

(4.5)

$$
d\phi = \frac{i}{2} K \eta \wedge \bar{\eta}.
$$

(4.6)

**Proof:** The quadratic form $\eta \circ \bar{\eta} \in \Gamma(\text{Sym}^2(T^*P))$ is both $O(2)$-invariant and $G$-invariant, so descends to a Riemannian metric $g_{\Sigma}$ on the surface $\Sigma$. Let $\varpi: F \rightarrow \Sigma$ denote the orthonormal frame bundle of $g_{\Sigma}$; and let $\bar{\eta} \in \Omega^1(F; \mathbb{C})$ denote the $\mathbb{C}$-valued tautological 1-form on $F$.

By the Fundamental Lemma of Riemannian Geometry, there exists a unique 1-form $\bar{\phi} \in \Omega^1(F)$, the Levi-Civita connection of $g_{\Sigma}$, for which

$$
d\bar{\eta} = i\bar{\phi} \wedge \bar{\eta}.
$$

Now, the quotient map $\text{pr}: M \rightarrow \Sigma$ induces a map $\bar{\text{pr}}: P \rightarrow F$ via $\bar{\text{pr}}(u)(v) := u(\bar{v})$, where $\bar{v} \in TM$ is the horizontal lift of $v \in T\Sigma$. Unwinding the definitions shows that $\bar{\text{pr}}^*(\bar{\eta}) = \eta$, whence the equation $d\eta = i\bar{\text{pr}}^*(\bar{\phi}) \wedge \eta$ holds on $P$. Setting $\phi := \bar{\text{pr}}^*(\bar{\phi})$ establishes (4.5).

Equation (4.6) now follows by differentiating (4.5). That is, $K$ is the Gauss curvature of $g_{\Sigma}$.

**Lemma 4.4:** There exist seven $\mathbb{C}$-valued functions $p_1, p_2, p_3, p_4, q_1, q_2, r: P \rightarrow \mathbb{C}$ and four $\mathbb{R}$-valued functions $h_1, h_2, h_3, h_4: P \rightarrow \mathbb{R}$ for which

$$
2\xi_1 = h_1\eta - ih_2\bar{\theta} + q_1\bar{\theta} + p_1\omega^3 + p_2\omega^6
$$

$$
2\xi_2 = ih_4\eta + (h_3 - i)\theta + q_2\bar{\theta} - ip_2\omega^3 + p_3\omega^6
$$

$$
\xi_3 = p_4\theta - ir\bar{\theta} - iq_1\omega^3 + q_2\omega^6
$$

$$
\xi_0 = \bar{p}_4\bar{\theta} + p_4\bar{\theta} - h_2\omega^3 + h_3\omega^6.
$$

(4.7)
Their exterior derivatives modulo \(\langle \omega^1, \omega^2 \rangle = \langle \eta, \bar{\eta} \rangle\) satisfy

\[
\begin{align*}
dp_1 \equiv ip_1\phi & \quad dq_1 \equiv 2iq_1\phi & \quad dh_1 \equiv 0 \\
dp_2 \equiv ip_2\phi & \quad dq_2 \equiv 2iq_2\phi & \quad dh_2 \equiv 0 \\
dp_3 \equiv ip_3\phi & \quad dq_3 \equiv 2iq_3\phi & \quad dh_3 \equiv 0 \\
dp_4 \equiv ip_4\phi & \quad dr \equiv 3ir\phi & \quad dh_4 \equiv 0.
\end{align*}
\]  

Moreover, we have the formula

\[
\phi = \alpha + (h_1 + 1)\omega^3 - h_4\omega^6. \tag{4.9}
\]

The upshot is that we have re-expressed the torsion of the O(2)-structure in terms of just seven \(\mathbb{C}\)-valued functions and four \(\mathbb{R}\)-valued functions \(p_i, q_i, r, h_i\) on \(P\), all of which are \(G\)-invariant and O(2)-equivariant (for the O(2)-actions indicated by \(4.8\)). Accordingly, we will refer to \(T = (p_1, p_2, p_3, p_4, q_1, q_2, r, h_1, h_2, h_3, h_4) : P \to \mathbb{C}^7 \oplus \mathbb{R}^4\) as the (intrinsic) torsion of the O(2)-structure. Geometrically, the function \(T\) describes the 1-jet of the O(2)-structure (or the 2-jet of the underlying SU(3)-structure) up to diffeomorphism.

Equation (4.9) shows in particular that \((\omega^1, \ldots, \omega^6, \phi) : TP \to \mathbb{R}^7\) is a coframing on \(P\). Going forward, we prefer to work with the coframings \((\omega^1, \ldots, \omega^6, \phi)\) and \((\eta, \overline{\eta}, \omega^3, \theta, \overline{\theta}, \omega^6, \phi)\) rather than with the original \((\omega^1, \ldots, \omega^6, \alpha)\) and \((\eta, \overline{\eta}, \omega^3, \theta, \overline{\theta}, \omega^6, \alpha)\).

**Proof of Lemma 4.4:** From \((4.3)\) and \((4.4)\), we have

\[
d\eta = i(\alpha + ia_{01}\theta + ia_{31}\overline{\theta} + (a_{11} + 1)\omega^3 + ia_{21}\omega^6) \wedge \eta \tag{4.10}
\]

\[
- ia_{12}\overline{\eta} \wedge \omega^3 + a_{22}\overline{\eta} \wedge \omega^6 + a_{32}\overline{\eta} \wedge \overline{\theta} + a_{02}\overline{\eta} \wedge \theta
\]

\[
+ (a_{05} - a_{34})\overline{\theta} \wedge \theta + (a_{03} + ia_{14})\omega^3 \wedge \theta + (a_{33} + ia_{15})\omega^3 \wedge \overline{\theta}
\]

\[
+ (a_{24} - a_{06} + i)\theta \wedge \omega^6 - (a_{25} - a_{36})\omega^6 \wedge \overline{\theta} + (ia_{16} + a_{23})\omega^3 \wedge \omega^6.
\]

Equating this with \(d\eta = i\phi \wedge \eta\) yields the following relations:

\[
\begin{align*}
a_{12} &= 0 & a_{32} &= 0 & a_{15} &= ia_{33} & a_{05} &= a_{34} & a_{25} &= a_{36} \\
a_{22} &= 0 & a_{02} &= 0 & a_{06} &= a_{24} + i & a_{03} &= -ia_{14} & a_{16} &= ia_{23}
\end{align*}
\]

From these relations, we may define

\[
\begin{align*}
p_1 &= a_{13} & q_1 &= a_{15} = ia_{33} & h_1 &= a_{11} \\
p_2 &= a_{16} = ia_{23} & q_2 &= a_{25} = a_{36} & h_2 &= -a_{03} = ia_{14} \\
p_3 &= a_{26} & q_3 &= a_{05} = a_{34} & h_3 &= a_{06} = a_{24} + i \\
p_4 &= a_{05} = a_{34} & r &= ia_{35} & h_4 &= -ia_{21}.
\end{align*}
\]

Moreover, since \(\xi_0\) is real-valued, we see that \(a_{03}\) and \(a_{06}\) are real-valued, whence \(h_2\) and \(h_3\) are real-valued. The reality of \(\xi_0\) also yields \(a_{01} = \overline{a}_{02} = 0\) and \(a_{04} = \overline{a}_{05} = \overline{a}_{14}\).

In this new notation, \((4.10)\) and its complex conjugate now read as follows:

\[
d\eta = i(\alpha + ia_{31}\overline{\theta} + (h_1 + 1)\omega^3 - h_4\omega^6) \wedge \eta
\]

\[
d\overline{\eta} = -i(\alpha - ia_{31}\theta + (\overline{h}_1 + 1)\omega^3 - \overline{h}_4\omega^6) \wedge \overline{\eta}.
\]
Again equating with $d\eta = i\phi \wedge \eta$ and $d\bar{\eta} = -i\phi \wedge \bar{\eta}$, we see that $a_{31} = 0$, that $h_1$ and $h_4$ are real-valued, and that

$$\phi = \alpha + (h_1 + 1) \omega^3 - h_4 \omega^6.$$ 

This proves (4.7) and (4.9). The proof of (4.8) is a direct calculation. ◊

4.4 The Acting Lie Group

**Proposition 4.5:** The Lie group $G$ is 4-dimensional and non-abelian. In particular, if $M$ is complete, then both $G$ and the principal $G$-orbits in $M$ are finite quotients of $SU(2) \times U(1) \cong S^3 \times S^1$.

**Proof:** For $X \in \mathfrak{g}$, let $X^\# \in \Gamma(TP)$ be the corresponding $G$-action vector field on $P$, by which we mean $X^\#|_p = \frac{d}{dt}|_{t=0} (\exp txX) \cdot p$.

Since $X^\#$ is tangent to the pre-images $\pi^{-1}(Gx) \subset P$, we have $\omega^1(X^\#) = \omega^2(X^\#) = 0$. From the real part (4.5), we have $d\omega^1 = -\phi \wedge \omega^2$, whence

$$0 = \mathcal{L}_{X^\#} \omega^1 = \iota_{X^\#}(d\omega^1) + d(\iota_{X^\#} \omega^1) = \iota_{X^\#}(-\phi \wedge \omega^2) = -\phi(X^\#)\omega^2,$$

whence $\phi(X^\#) = 0$. Thus, at each $p \in P$, we have

$$\mathfrak{g} \cong \{X^\#|_p \in T_pP: X \in \mathfrak{g}\} \subset \ker(\omega^1, \omega^2, \phi)|_p,$$

whence $\dim(G) \leq 4$. Since $\dim(G) \geq 4$, we have equality. In particular, the inclusion in (4.11) is an equality, so the $G$-orbits in $P$ are the integral 4-folds of $\mathcal{I}_G := \langle \omega^1, \omega^2, \phi \rangle$.

Let us now identify $G$ with an integral 4-fold of $\mathcal{I}_G$. Via this identification, $\{\omega^3, \omega^4, \omega^5, \omega^6\}$ is a basis of left-invariant 1-forms on $G$. Let $\{X_3, X_4, X_5, X_6\}$ be a basis of $\mathfrak{g} = \{\text{left-invariant vector fields on } G\}$ whose dual basis is $\{\omega^3, \omega^4, \omega^5, \omega^6\}$. From (4.3) and (4.7), one may calculate that

$$d\omega^3 \equiv -\frac{3}{2} i \theta \wedge \bar{\theta} = -3 \omega^4 \wedge \omega^5 \quad (\text{mod } \eta, \bar{\eta}, \omega^3, \omega^6, \phi).$$

Thus, $[X_4, X_5] = 3X_3$, so $G$ is non-abelian.

If $M$ is complete, then (Proposition 3.3) $G$ is a compact 4-dimensional non-abelian Lie group. Hence, by the classification of compact Lie groups [5], $G$ must be a finite quotient of $SU(2) \times U(1)$. In this case, since the principal $G$-orbits are 4-dimensional $G$-homogeneous spaces, they must also be finite quotients of $SU(2) \times U(1)$. ◊

4.5 Geometric Interpretation of the Torsion

We pause to interpret the torsion functions $p_1, p_2, p_3, p_4, q_1, q_2, r, h_1, h_2, h_3, h_4$ geometrically. This section is parenthetical to the rest of this work: the main results in §5 will not draw on these remarks.

4.5.1 Background: Riemannian Submersions

Let $\text{pr}: (M^n, g) \to (\Sigma^k, g_2)$ denote an arbitrary Riemannian submersion between Riemannian manifolds $M$ and $\Sigma$. Recall that the vertical distribution $\mathcal{V} = \ker(\text{pr}_*)$ is given by the tangent spaces to the pr-fibers, and the horizontal distribution $\mathcal{H} = \mathcal{V}^\perp$ is the orthogonal complement.

The geometry of the submersion $\text{pr}$ is governed by the two $(1, 2)$-tensor fields on $M$, called the $A$-tensor and $T$-tensor, given by

$$A(X, Y) = (\nabla_{X^{\text{Hor}}} Y)^{\text{Ver}} + (\nabla_{X^{\text{Hor}}} Y)^{\text{Hor}}$$

and

$$T(X, Y) = (\nabla_{X^{\text{Ver}}} Y)^{\text{Hor}} + (\nabla_{X^{\text{Ver}}} Y)^{\text{Ver}}.$$
for \( X, Y \in \Gamma(TM) \), where \( \nabla \) is the Levi-Civita connection on \( TM \), and where \( \text{Ver}: TM \to V \) and \( \text{Hor}: TM \to H \) are the projections onto the vertical and horizontal distributions, respectively.

Note that \( A \equiv 0 \) if and only if the horizontal distribution \( H \) is integrable. Indeed, \( A(X, Y) = \frac{1}{2}[X, Y]^{\text{Ver}} \) for \( X, Y \in \Gamma(H) \). Meanwhile, the T-tensor is essentially the second fundamental form \( II \) of the pr-fibers. Indeed, \( T(X, Y) = II(X, Y) \) for \( X, Y \in \Gamma(V) \).

Finally, we point out that \( A(X, \cdot) = 0 \) for all \( X \in \Gamma(V) \), and similarly \( T(X, \cdot) = 0 \) for all \( X \in \Gamma(H) \). Thus, the A- and T-tensors are recovered, respectively, from the knowledge of \( A(X, \cdot) \) for \( X \in \Gamma(H) \) and \( T(X, \cdot) \) for \( X \in \Gamma(V) \). For more information, see [3].

### 4.5.2 Geometric Interpretation of the Torsion Functions

We now return to our usual setting, which is that of a cohomogeneity-two nearly-Kähler 6-manifold \((M^6, g)\) with coisotropic principal orbits.

Let \( \text{pr}: (M^6, g) \to \Sigma \) denote the projection to the orbit space. As in Lemma 4.3, we equip \( \Sigma \) with the Riemannian metric \( g_\Sigma = \eta \circ \bar{\eta} \), so that \( \text{pr} \) is a Riemannian submersion. We claim that:

**Proposition 4.6:**

(a) The torsion functions \( h_1, h_4 \) determine the A-tensor, and conversely.

(b) The torsion functions \( p_1, p_2, p_3, p_4, q_1, q_2, r, h_2, h_3 \) determine the T-tensor, and conversely.

To see this, let \( F_{SO(6)} M \) denote the oriented orthonormal frame bundle of the metric \( g \). By the Fundamental Lemma of Riemannian Geometry, there is a unique 1-form \( \psi \in \Omega^1(F_{SO(6)} M; \mathfrak{o}(6)) \), the Levi-Civita connection, such that

\[
d\omega = -\psi \wedge \omega.
\]

One can check that, using the notation of (4.2), the Levi-Civita connection (restricted to \( P \)) is

\[
\psi = \begin{pmatrix}
0 & \alpha_3 + \frac{1}{2}\omega^3 & -\alpha_2 - \frac{1}{2}\omega^2 & -\beta_{11} & -\beta_{12} - \frac{1}{2}\omega^6 & -\beta_{13} + \frac{1}{2}\omega^5 \\
\alpha_2 - \frac{1}{2}\omega^2 & -\alpha_3 - \frac{1}{2}\omega^3 & \alpha_1 + \frac{1}{2}\omega^1 & \beta_{11} & \beta_{12} - \frac{1}{2}\omega^6 & \beta_{13} - \frac{1}{2}\omega^5 \\
\alpha_1 - \frac{1}{2}\omega^1 & \alpha_2 - \frac{1}{2}\omega^2 & \alpha_3 - \frac{1}{2}\omega^3 & \beta_{13} + \frac{1}{2}\omega^5 & 0 & \beta_{23} - \frac{1}{2}\omega^4 \\
-\beta_{13} + \frac{1}{2}\omega^5 & \beta_{12} - \frac{1}{2}\omega^6 & \beta_{23} - \frac{1}{2}\omega^4 & -\alpha_3 - \frac{1}{2}\omega^3 & -\alpha_2 - \frac{1}{2}\omega^2 & 0 \\
\beta_{13} - \frac{1}{2}\omega^5 & \beta_{12} - \frac{1}{2}\omega^6 & \beta_{23} - \frac{1}{2}\omega^4 & \beta_{23} - \frac{1}{2}\omega^4 & -\alpha_3 + \frac{1}{2}\omega^3 & 0 \\
\beta_{12} + \frac{1}{2}\omega^6 & \beta_{22} & \beta_{33} & \beta_{33} & \alpha_2 - \frac{1}{2}\omega^2 & -\alpha_1 + \frac{1}{2}\omega^1 \\
\beta_{12} + \frac{1}{2}\omega^6 & \beta_{22} & \beta_{33} & \beta_{33} & \alpha_2 - \frac{1}{2}\omega^2 & -\alpha_1 + \frac{1}{2}\omega^1 \\
\beta_{13} - \frac{1}{2}\omega^5 & \beta_{23} - \frac{1}{2}\omega^4 & \beta_{33} & \beta_{33} & \alpha_2 - \frac{1}{2}\omega^2 & -\alpha_1 + \frac{1}{2}\omega^1 \\
\end{pmatrix}.
\]

Thus, letting \( \nabla \) denote the corresponding covariant derivative operator on \( TM \), we have

\[
(\nabla_X e_i)^\text{Hor} = \psi_i^1(X)e_1 + \psi_i^2(X)e_2
\]

\[
(\nabla_X e_i)^\text{Ver} = \psi_i^3(X)e_3 + \psi_i^4(X)e_4 + \psi_i^5(X)e_5 + \psi_i^6(X)e_6
\]

where \((e_1, \ldots, e_6)\) is any local O(2)-frame field. Using these formulas, together with (4.7), one can compute the A- and T-tensors.

For example, a calculation shows that

\[
A(e_1 + ie_2, e_1) = i \left[ (h_1 + \frac{1}{2})e_3 - h_4 e_6 \right] \quad A(e_1 + ie_2, e_4) = 0
\]

\[
A(e_1 + ie_2, e_2) = -i \left[ (h_1 + \frac{1}{2})e_3 - h_4 e_6 \right] \quad A(e_1 + ie_2, e_5) = 0
\]

\[
A(e_1 + ie_2, e_3) = -i(h_1 + \frac{1}{2})(e_1 + ie_2) \quad A(e_1 + ie_2, e_6) = ih_4(e_1 + ie_2)
\]

where we have extended \( A \) to be \( \mathbb{C} \)-bilinear. In particular, we observe that the 2-plane field \( \text{Ker}(\omega^3, \omega^4, \omega^5, \omega^6) \) in \( M \) normal to the principal \( G \)-orbits is integrable if and only if \( A = 0 \), or equivalently

\[
h_1 = -\frac{1}{2} \quad \text{and} \quad h_4 = 0.
\]
Computations by the author suggest that this integrability cannot happen (locally), but the details require closer examination.

We now exhibit the second fundamental form $\mathbb{II}$ of the principal orbits. This is the normal bundle-valued quadratic form $\mathbb{II} \in \Gamma(\text{Sym}^2(T^*M) \otimes NM)$ given by

$$\mathbb{II} = h_{ij}^1 \omega^i \omega^j \otimes e_1 + h_{ij}^2 \omega^i \omega^j \otimes e_2$$

where

$$h_{ij}^1 = \begin{pmatrix} -\text{Im}(p_1) & -\text{Im}(q_1) + h_2 & \text{Re}(q_1) & -\text{Im}(p_2) \\ -\text{Im}(q_1) + h_2 & -3\text{Re}(p_4) - \text{Im}(r) & -\text{Im}(p_4) + \text{Re}(r) & -\text{Re}(q_4) - h_3 \\ \text{Re}(q_1) & -\text{Im}(p_4) + \text{Re}(r) & -\text{Re}(p_4) + \text{Im}(r) & -\text{Im}(q_2) - h_3/2 \\ -\text{Im}(p_2) & -\text{Re}(q_2) - h_3 & -\text{Im}(q_2) - 1/2 & -\text{Im}(p_3) \end{pmatrix}$$

$$h_{ij}^2 = \begin{pmatrix} \text{Re}(p_1) & \text{Re}(q_1) & \text{Im}(q_1) + h_2 & \text{Re}(p_2) \\ \text{Re}(q_1) & -\text{Im}(p_4) + \text{Re}(r) & -\text{Re}(p_4) + \text{Im}(r) & -\text{Im}(q_2) + 1/2 \\ \text{Im}(q_1) + h_2 & -\text{Re}(p_4) + \text{Re}(r) & -3\text{Im}(p_4) - \text{Re}(r) & \text{Re}(q_2) - h_3 \\ \text{Re}(p_2) & -\text{Im}(q_2) + 1/2 & \text{Re}(q_2) - h_3 & -\text{Im}(p_3) \end{pmatrix}.$$

Conversely, one can invert these formulas to recover $p_1, p_2, p_3, p_4, q_1, q_2, r, h_2, h_3$ in terms of $h_{ij}^1, h_{ij}^2$. Indeed,

$$p_1 = h_{11}^1 - ih_{11}^1 \quad 2q_1 = 2h_{13}^1 + i(h_{13}^2 - h_{12}^1) \quad 2h_2 = h_{12}^1 + h_{13}^2$$
$$p_2 = h_{14}^1 - ih_{14}^1 \quad 2q_2 = (h_{34}^2 - h_{24}^1) - i(1 + 2h_{34}^1) \quad 2h_3 = -h_{24}^1 - h_{34}^2$$
$$p_3 = -h_{44}^1 - ih_{44}^2 \quad 4p_4 = -(h_{22}^1 + h_{33}^1) - i(h_{23}^1 - h_{33}^2) \quad 4r = (3h_{23}^1 - h_{33}^2) + i(3h_{33}^1 - h_{22}^1)$$

illustrating that these torsion functions are simply affine-linear combinations of second fundamental form coefficients. We also note that the mean curvature of the principal orbits is

$$H = -\text{Im}(p_1) + \text{Re}(p_4) + 4\text{Re}(p_4) e_1 + (\text{Re}(p_1) - \text{Im}(p_3) - 4\text{Im}(p_4)) e_2,$$

so that a principal orbit is minimal if and only if $p_1 + ip_3 + 4ip_4 = 0$. Finally, we point out that the presence of the $1/2$ terms in $(h_{ij}^1)$ and $(h_{ij}^2)$ implies that they cannot vanish simultaneously, so that none of the principal orbits can be totally-geodesic.

### 4.5.3 Descent to $M$

We caution the reader that the 1-forms $\eta, \bar{\eta}, \omega^3, \theta, \bar{\theta}, \omega^6$ and torsion functions $p_j, q_j, r, h_j$ are defined on the bundle $P$, not on the base manifold $M$. However, $O(2)$-invariant combinations of these will descend to be well-defined (possibly up to sign) on $M$.

For example, the quadratic forms $\eta \circ \bar{\eta}, (\omega^3)^2, \theta \circ \bar{\theta}$, and $(\omega^6)^2$ descend to $M$, and the differential forms $\eta \wedge \bar{\eta}, \omega^3, \theta \wedge \bar{\theta}$, and $\omega^6$ descend to be well-defined $up to sign$. Similarly, the norms of the torsion functions $|p_1|, |p_2|, |p_3|, |p_4|, |q_1|, |q_2|, |r|, |h_1|, |h_2|, |h_3|, |h_4|$ are well-defined on $M$, while the 1-forms, quadratic forms, and cubic forms

$$\bar{p}_1 \eta, \bar{p}_2 \eta, \bar{p}_3 \eta, \bar{p}_4 \eta, \bar{q}_1 \eta \circ \eta, \bar{q}_2 \eta \circ \eta, \bar{q}_3 \eta \circ \eta \circ \eta$$

descend to be well-defined up to sign.
4.6 Calculus on the Orbit Space $\Sigma$

In this section, we describe the geometric structure on the orbit space $\Sigma$ of the principal locus in terms of our moving frame apparatus. This will be used in §5 to derive the elliptic PDE systems on $\Sigma$ satisfied by cohomogeneity-two nearly-Kähler structures.

4.6.1 Operators on Holomorphic Line Bundles over $\Sigma$

Recall that the quadratic form $\eta \circ \eta = (\omega^1)^2 + (\omega^2)^2$ descends to a well-defined Riemannian metric $g_{\Sigma}$ on the quotient surface $\Sigma$. Let $\omega: F \to \Sigma$ denote the corresponding orthonormal coframe bundle over $\Sigma$. Recall also from the proof of Lemma 4.3 that the Levi-Civita connection $\phi \in \Omega^1(F)$ and Gauss curvature $K: F \to \mathbb{R}$ of the metric $g_{\Sigma}$ satisfy

$$
\begin{align*}
  d\eta &= i \phi \wedge \eta \\
  d\phi &= \frac{i}{2} K \eta \wedge \eta,
\end{align*}
$$

Remark: Here, it would perhaps be more proper to write $\overline{\phi}$ and $\overline{\eta}$ in place of $\phi$ and $\eta$, as in the proof of Lemma 4.3. However, we will often follow a convention common in the method of moving frames, denoting forms on $F$ and their pullbacks by $\overline{pr}: P \to F$ by the same notation. □

Define an almost-complex structure on $\Sigma$ as follows: For $\sigma \in \Lambda^1(\Sigma; \mathbb{C}) = T^*\Sigma^\mathbb{C}$, we declare that $\sigma \in \Lambda^{1,0}(\Sigma)$ iff $\varpi^*(\sigma) \in \text{span}_\mathbb{C}(\eta)$. For dimension reasons, this almost-complex structure is integrable, and by construction, it is compatible with the metric. Since the associated 2-form $\frac{i}{2} \eta \wedge \overline{\eta}$ is closed (again by dimension reasons), so $\Sigma$ is Kähler.

$$
T^*F^\mathbb{C} \xrightarrow{\cong} T^*\Sigma^\mathbb{C} \xrightarrow{\varpi^*} K_{\Sigma}^n \otimes T^*\Sigma^\mathbb{C}
$$

By construction, $K_{\Sigma} = \Lambda^{1,0}(\Sigma) \to \Sigma$ is a holomorphic line bundle, as are its tensor powers $K_{\Sigma}^n \cong \text{Sym}^n(\Lambda^{1,0}(\Sigma))$. In particular, each $K_{\Sigma}^n$ admits a $\partial$-operator:

$$
\partial: \Gamma(K_{\Sigma}^n) \to \Gamma(K_{\Sigma}^n \otimes \Lambda^{0,1}(\Sigma)).
$$

The Levi-Civita connection $\phi \in \Gamma(T^*F^\mathbb{C})$ of the metric on $\Sigma$ induces a covariant derivative operator on $T^*\Sigma^\mathbb{C}$. Since $\Sigma$ is Kähler, there is an induced covariant derivative operator on $K_{\Sigma} = \Lambda^{1,0}(\Sigma)$, and hence also on all of its tensor powers:

$$
\nabla: \Gamma(K_{\Sigma}^n) \to \Gamma(K_{\Sigma}^n \otimes T^*\Sigma^\mathbb{C}).
$$

Let us give a more explicit description of $\nabla$. Let $\sigma \in \Gamma(K_{\Sigma}^n)$ be a smooth section, say $\varpi^*(\sigma) = f\eta^n$ for some function $f \in \Omega^0(F; \mathbb{C})$. Write

$$
df = f'\eta + f''\overline{\eta} - f_0i\phi.
$$

Then $\nabla\sigma \in \Gamma(\Sigma; K_{\Sigma}^n \otimes T^*\Sigma^\mathbb{C})$ is the section such that $\varpi^*(\nabla\sigma) \in \Gamma(F; \varpi^*(K_{\Sigma}^n \otimes T^*\Sigma^\mathbb{C}))$ is given by

$$
\varpi^*(\nabla\sigma) = \eta^n \otimes (df + f_0i\phi) = \eta^n \otimes (f'\eta + f''\overline{\eta}).
$$

Since $\Sigma$ is Kähler, the Levi-Civita connection on $T^*\Sigma$ coincides with the Chern connection on $\Lambda^{1,0}\Sigma$ (under the isomorphism $T^*\Sigma \cong \Lambda^{1,0}(\Sigma)$), so that $\nabla$ is compatible with both the holomorphic structure and Hermitian structure on $K_{\Sigma} = \Lambda^{1,0}(\Sigma)$. In particular,

$$
\varpi^*(\partial\sigma) = \varpi^*(\nabla^{0,1}\sigma) = f''\eta^n \otimes \overline{\eta}.
$$
4.6.2 The Laplacian of the Conformal Factor

Let \((z)\) be a local holomorphic coordinate on \(\Sigma\). We can write
\[
\lambda \eta = \varpi^*(dz)
\]
for some non-zero function \(\lambda = e^{u+iv}: F \rightarrow \mathbb{C}\), where here \(u, v: F \rightarrow \mathbb{R}\). Note that \(|\lambda|^2 = e^{2u}\), and hence \(u = \frac{1}{2} \log |\lambda|^2\), both descend to well-defined functions on \(\Sigma\). We also have
\[
e^{2u} \eta \circ \bar{\eta} = \varpi^*(dz \circ d\bar{z})
e^{2u} \eta \wedge \bar{\eta} = \varpi^*(dz \wedge d\bar{z}).
\]

Let \(*\) be the Hodge star of the metric and orientation on \(\Sigma\). The Hodge Laplacian \(\Delta\) on functions is then \(\Delta = -*d*d\). The following result is classical, but we prove it for completeness.

**Lemma 4.7:** We have \(\Delta u = -K\).

**Proof:** We begin by observing that
\[
i \phi \wedge \eta = d\eta = d\left(\frac{1}{\lambda} \varpi^*(dz)\right) = -\frac{d\lambda}{\lambda^2} \wedge \varpi^*(dz) = -\frac{d\lambda}{\lambda} \wedge \eta,
\]
so
\[
\left(n + \frac{d\lambda}{\lambda}\right) \wedge \eta = 0.
\]
Thus, by Cartan’s Lemma, there exists a function \(h: F \rightarrow \mathbb{C}\) with
\[
d\frac{\lambda}{\lambda} = -i\phi + h\eta,
\]
Write the exterior derivative of \(h\) in the form \(dh = h'\eta + h''\bar{\eta} + h_0\phi\) for some functions \(h', h'', h_0: F \rightarrow \mathbb{C}\). Then
\[
-\frac{1}{2} K \eta \wedge \bar{\eta} = d(i\phi) = d\left(-\frac{d\lambda}{\lambda}\right) + d(h\eta) = dh \wedge \eta + h d\eta = -h'' \eta \wedge \bar{\eta} + (h_0 + ih) \phi \wedge \eta
\]
shows that
\[
dh = h'\eta + \frac{1}{2} K \bar{\eta} - ih\phi.
\]
Finally, noting that
\[
d(|\lambda|^2) = \frac{d\lambda}{\lambda} + \frac{d\bar{\lambda}}{\lambda} = h\eta + \bar{\eta}\bar{\eta}
\]
we may calculate
\[
\Delta u = \frac{1}{2} \Delta (|\lambda|^2) = -\frac{1}{2} * d * d(|\lambda|^2) = -\frac{1}{2} * d * (h\eta + \bar{\eta}\bar{\eta}) = -\frac{1}{2} *(iK \eta \wedge \bar{\eta}) = -K.
\]
\[\diamondsuit\]
5 Local Existence and Generality

We continue with the setup of §4, which we reiterate for clarity. We let $M$ be a nearly-Kähler 6-manifold acted upon by a connected Lie group $G$ with cohomogeneity-two. We suppose that this $G$-action preserves the SU(3)-structure $(J, \Omega, \Upsilon)$ and that the principal $G$-orbits are coisotropic. Conventions 4.1 and 4.2 (stated in §4.2) remain in force.

We continue to work on the principal $O(2)$-bundle $\pi: P \to M$, defined in §4.2 as a frame adaptation. On $P$, we work with either of the global coframings $(\omega^1, \ldots, \omega^6, \phi)$ or $(\eta, \overline{\eta}, \omega^3, \theta, \overline{\theta}, \omega^6, \phi)$, and their exterior derivatives are given by (4.3) and (4.7). Finally, the intrinsic torsion of the $O(2)$-structure has been encoded as a function

$$T = (p_1, p_2, p_3, p_4, q_1, q_2, r, h_1, h_2, h_3, h_4): P \to \mathbb{C}^7 \oplus \mathbb{R}^4$$

which satisfies the $O(2)$-equivariance described in (4.8).

Our primary objective is to prove a local existence/generality theorem for nearly-Kähler 6-manifolds of cohomogeneity-two (always assuming the principal orbits are coisotropic) by appealing to Cartan’s Third Theorem (Theorem 2.2). Concretely, this means satisfying the integrability conditions

$$d(d\theta) = 0$$

as well as ensuring the involutivity and correct dimension of the tableau of free derivatives. Fortunately, the equations $d(d\eta) = d(d\overline{\eta}) = 0$ are already satisfied (by Lemma 4.3).

By contrast, the integrability conditions (5.1) are quite complicated, consisting of $4\binom{6}{3} = 80$ quadratic equations on 55 real-valued functions: the 18 real and imaginary parts of the torsion functions, their 36 “directional derivatives” in the two directions normal to the $G$-orbits, and the Gauss curvature $K$ of the orbit space $\Sigma$. Thus, arranging for (5.1) will occupy us for some time.

5.0.1 The Three Types

We begin by solving two of the simpler quadratic equations arising in (5.1). Namely, we calculate

$$0 = d(d\theta) \wedge \eta \wedge \overline{\eta} \wedge \theta = -8 [q_1(h_1 + 3 + ih_3) - q_2(h_2 + ih_4)] \omega^{123456}$$

yielding the quadratics

$$q_1(h_1 + 3 + ih_3) - q_2(h_2 + ih_4) = 0$$

$$q_1\overline{q}_2 + \overline{q}_1 q_2 = 2 (h_1 h_2 + 3 h_2 + h_3 h_4).$$

To solve this system, we introduce the $\mathbb{C}$-valued functions

$$s_1 = (h_1 + 3) + ih_3$$

$$s_2 = h_2 + ih_4.$$
For \( z, w \in \mathbb{C} \), we let \( \langle z, w \rangle = \text{Re}(z\overline{w}) \) denote the euclidean inner product on \( \mathbb{C} \cong \mathbb{R}^2 \), and let \( \|z\| = \sqrt{z\overline{z}} \) denote the euclidean norm. Then equations (5.5a)-(5.5b) are simply

\begin{align}
q_1s_1 - q_2s_2 &= 0 \quad (5.6a) \\
\langle q_1, q_2 \rangle &= \langle s_2, s_1 \rangle. \quad (5.6b)
\end{align}

The solution to (5.6a)-(5.6b) is provided by the following geometric fact.

Lemma 5.1: Let \( a, b, c, d \in \mathbb{C} \) be complex numbers satisfying both

\[
ad - bc = 0 \quad (a, b) = (c, d).
\]

Then exactly one of the following holds:

(i) \( a = b = c = d = 0 \).

(ii) \( \langle a, b \rangle = \langle c, d \rangle \neq 0 \) and \( \|a\| = \|c\| \) and \( \|b\| = \|d\| \).

(iii) \( \langle a, b \rangle = \langle c, d \rangle = 0 \) and \( (a, b, c, d) \neq (0, 0, 0, 0) \).

Accordingly, we partition the class of manifolds under consideration into three types:

Definition: Let \( M \) be a nearly-Kähler 6-manifold of cohomogeneity-two with coisotropic principal orbits. We say that \( M \) is of:

- **Type I** if \( q_1 = q_2 = s_1 = s_2 = 0 \).
- **Type II** if \( \langle q_1, q_2 \rangle = \langle s_2, s_1 \rangle \neq 0 \) and \( \|q_1\| = \|s_2\| \) and \( \|q_2\| = \|s_1\| \).
- **Type III** if \( \langle q_1, q_2 \rangle = \langle s_2, s_1 \rangle = 0 \) and \( (q_1, q_2, s_1, s_2) \neq (0, 0, 0, 0) \).

Remark: The Type conditions are pointwise: to be precise, we should speak of being “Type I at \( p \in P \),” and so on. Since the Type conditions are \( \text{O}(2) \)-invariant, it makes sense to speak of \( M \) as being “Type I at \( m \in M \),” etc. It is conceivable for a nearly-Kähler structure on \( M \) to be of (say) Type I at some points of \( M \) and be of Type II at others. □

In the sequel, we study each Type of cohomogeneity-two nearly-Kähler structure separately. In each case, the primary challenge will be solving the quadratic equations (5.1). Once this is done, we will solve (5.2), (5.3), (5.4) and draw conclusions.

We will see that the algebra involved in solving (5.1) is fairly simple for Type I structures, but is significantly more labor intensive in the Type II case, and even more so in the Type III case.

5.1 **Type I**

In this section, we study nearly-Kähler structures of Type I. In particular, we prove a local existence/generality result (Theorem 5.4) for these structures. We then show that for this Type, the acting Lie group \( G \) is nilpotent (Proposition 5.5), and hence the underlying metrics are incomplete. We conclude the section by deriving an elliptic PDE system (Proposition 5.6) on the quotient surface \( \Sigma \) which Type I structures must satisfy.

5.1.1 **The Integrability Conditions**

Our first task is to make explicit the integrability conditions (5.1), which amount to quadratic equations on both the torsion functions \( p_i, q_i, r, h_i \) and on their first derivatives. In preparation, we
express the exterior derivatives of \( p_4 \) and \( r \) as

\[
\begin{align*}
dp_4 &= p'_4 \eta + p''_4 \overline{\eta} + ip_4 \phi \\
\ dr &= r' \eta + r'' \overline{\eta} + 3ir \phi
\end{align*}
\]

for some functions \( p'_4, p''_4, r', r'' : P \to \mathbb{C} \). In doing so, we have used the \( G \)-invariance and \( O(2) \)-equivariance (4.8) of \( p_4 \) and \( r \).

**Lemma 5.2:** Let \( M \) be a nearly-Kähler manifold of Type I.

(a) On the \( O(2) \)-coframe bundle \( P \), the following algebraic relations hold:

\[
\begin{align*}
(p_1, p_2, p_3, p_4) &= (-4ip_4, 0, 4p_4, p_4) \\
(q_1, q_2) &= (0, 0) \\
(h_1, h_2, h_3, h_4) &= (-3, 0, 0, 0).
\end{align*}
\]

Thus, the torsion can be expressed in terms of the functions \( p_4 \) and \( r \).

(b) On the \( O(2) \)-coframe bundle \( P \), the following differential relations hold:

\[
\begin{align*}
p'_4 &= -6|p_4|^2 - \frac{3}{2} \\
p''_4 &= -5p_4^2 + ip_4 r \\
r' &= -5ip_4^2 - \overline{p}_4 r \\
r'' &= -5p_4 + 5p_4^2 - ip_4 r \\
K &= 2(6 + |r|^2 - |p_4|^2).
\end{align*}
\]

**Proof:** The equations \((q_1, q_2) = (0, 0)\) and \((h_1, h_2, h_3, h_4) = (-3, 0, 0, 0)\) are immediate from the definition of “Type I.” For the others, we calculate (using (4.3) and (4.7))

\[
\begin{align*}
0 &= d(d\theta) \wedge \theta \wedge \overline{\theta} \wedge \omega^6 = 8p_2 \omega^{123456} \\
0 &= d(d\theta) \wedge \eta \wedge \overline{\eta} \wedge \omega^6 = 6(p_1 + ip_3) \omega^{123456} \\
0 &= d(d\theta) \wedge \eta \wedge \theta \wedge \omega^6 = 8(p_1 + 4ip_4) \omega^{123456}
\end{align*}
\]

and

\[
\begin{align*}
0 &= d(d\omega^3) \wedge \overline{\eta} \wedge \omega^3 \wedge \theta = 16\left(p'_4 + 6|p_4|^2 + \frac{3}{2}\right) \omega^{123456} \\
0 &= d(d\omega^3) \wedge \eta \wedge \theta \wedge \omega^3 = 16\left(p''_4 + 5p_4^2 - ip_4 r\right) \omega^{123456} \\
0 &= d(d\theta) \wedge \omega^3 \wedge \overline{\theta} \wedge \omega^6 = 2\left(12 + 2|r|^2 - 2|p_4|^2 - K\right) \omega^{123456} \\
0 &= d(d\theta) \wedge \omega^3 \wedge \theta \wedge \omega^6 = 4(ir' + p''_4) \omega^{123456}
\end{align*}
\]

from which the result follows. ♦

A calculation using MAPLE shows that if the equations (5.7)-(5.8) of Lemma 5.2 hold, then the integrability conditions (5.1), (5.2), and (5.3) are all satisfied, and that

\[
\begin{align*}
d(dr) &= (F \eta \wedge \overline{\eta} - 4ir'' \phi \wedge \overline{\eta}) + dr'' \wedge \eta \\
\end{align*}
\]

where

\[
F = -\left(13|p_4|^2 r + \frac{39}{2} r + 3r|r|^2 + 50ip_4^2 - \overline{p}_4 r''\right).
\]

We summarize our discussion so far.
Summary 5.3: Nearly-Kähler structures of Type I are encoded by augmented coframings \(((\eta, \overline{\eta}, \omega^3, \theta, \overline{\theta}, \omega^6, \phi), (p, r), r'')\) on \(P\) satisfying the following structure equations:

\[\begin{align*}
 d\eta &= i\phi \wedge \eta \\
 d\phi &= i(6 + |r|^2 - |p|^2) \eta \wedge \overline{\eta} \\
 d\theta &= i\phi \wedge \theta + \left(\overline{r} \theta \wedge \eta - p \theta \wedge \eta\right) - 2p \text{Re}(\theta \wedge \overline{\eta}) - 4i\omega^6 \wedge \eta \\
 d\omega^3 &= \frac{5}{2}i\overline{\eta} \wedge \eta + \frac{3}{2}i\overline{\theta} \wedge \theta + 4 \text{Re}(p \omega^3 \wedge \overline{\eta}) + 8 \text{Re}(p \omega^6 \wedge \overline{\theta}) \\
 d\omega^6 &= \frac{3}{2}i\overline{\eta} \wedge \theta - \frac{3}{2}i \eta \wedge \overline{\theta} - 4 \text{Re}(p \omega^6 \wedge \overline{\eta})
\end{align*}\]

\[\begin{align*}
 dp &= -\left(6|p|^2 + \frac{3}{2}\right) \eta - \left(5p^2 - i\overline{p}r\right) \overline{\eta} + ip\phi \\
 dr &= -\left(5ip^2 + \overline{pr}\right) \eta + r''\overline{\eta} + 3ir\phi.
\end{align*}\]

where for ease of notation, we have set \(p = p_4\).

Augmented coframings satisfying the structure equations (5.10a)-(5.10b) will satisfy the integrability conditions (5.1), (5.2), (5.3), as well as (5.9a)-(5.9b). In the language of §2.4, the functions \(p\) and \(r\) are the “primary invariants,” while \(r''\) is the “free derivative.”

Remark: The formulas of §4.5.2 simplify considerably in the Type I setting. In particular, we remark that the principal \(G\)-orbits in \(M\) have mean curvature vector

\[H = -4 \left(\text{Re}(p)e_1 + \text{Im}(p)e_2\right)\]

and have scalar curvature

\[\text{Scal} = -\frac{9}{4} - 16|p|^2 < 0.\]

Thus, \(p\) is essentially the mean curvature (or scalar curvature) of the principal orbits. □

Remark: From our remarks in §4.5.3, we see that for Type I nearly-Kähler structures, the 2-plane distribution normal to the principal \(G\)-orbits in \(M\) is never integrable. □

5.1.2 Local Existence/Generality

We are now ready to state a local existence and generality theorem for Type I structures.

Theorem 5.4: Nearly-Kähler structures of Type I exist locally and depend on 2 functions of 1 variable in the sense of exterior differential systems. In fact:

For any \(x \in \mathbb{R}^6\) and \((a_0, b_0) \in \mathbb{C}^2 \times \mathbb{C}\), there exists a Type I nearly-Kähler structure on an open neighborhood \(U \subset \mathbb{R}^6\) of \(x\) and an \(O(2)\)-coframe \(f_x \in P|_x\) at \(x\) for which

\[(p, r)(f_x) = a_0 \quad \text{and} \quad r''(f_x) = b_0.\]

Remark: In a certain sense [8], the space of diffeomorphism classes of \(k\)-jets of Type I nearly-Kähler structures has dimension \(2k + 4\). □
Proof: The discussion in §5.1.1 shows that hypotheses (2.4) and (2.5) of Cartan’s Third Theorem (Theorem 2.2) are satisfied. It remains to examine the tableau of free derivatives. At a point \((u, v) \in \mathbb{R}^4 \times \mathbb{R}^2\), this is the vector subspace \(A(u, v) \subset \text{Hom}(\mathbb{R}^7; \mathbb{R}^4) \cong \text{Mat}_{4 \times 7}(\mathbb{R})\) given by

\[
A(u, v) = \left\{ \begin{pmatrix} x & y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x & y & 0 & 0 & 0 & 0 & 0 \\ y & -x & 0 & 0 & 0 & 0 & 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}.
\]

Since \(A(u, v)\) is independent of the point \((u, v) \in \mathbb{R}^4 \times \mathbb{R}^2\), we can write \(A = A(u, v)\) without ambiguity. We observe that \(A\) is 2-dimensional and has Cartan characters \(\tilde{s}_1 = 2\) and \(\tilde{s}_k = 0\) for \(k \geq 2\). One can check that \(A\) is an involutive tableau, meaning that its prolongation \(A^{(1)}\) satisfies \(\dim(A^{(1)}) = 2 = \tilde{s}_1 + 2\tilde{s}_2 + \cdots + 7\tilde{s}_7\). Thus, Cartan’s Third Theorem applies, and we conclude the result. \(\Diamond\)

Remark: The complex characteristic variety of the tableau \(A\) is

\[
\Xi^C_A = \{ [\xi] \in \mathbb{P}(\mathbb{C}^7) : w \otimes \xi \in A \text{ for some } w \in \mathbb{R}^4, w \neq 0 \} \\
= \{ [\xi] \in \mathbb{P}(\mathbb{C}^7) : (\xi_1 + i\xi_2)(\xi_1 - i\xi_2) = 0, \xi_3 = \cdots = \xi_7 = 0 \}.
\]

The fact that the local generality of Type I structures is 2 functions of 1 variable, with complex characteristic variety consisting of two complex conjugate points, strongly suggests the possibility of a holomorphic interpretation of these structures. \(\Box\)

5.1.3 Incompleteness

Nearly-Kähler structures of Type I cannot arise from a complete metric, as we now show. Recall that the real Heisenberg group is the (non-compact) Lie group

\[
H_3 = \left\{ \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} : x_i \in \mathbb{R} \right\} \leq \text{GL}_3(\mathbb{R}).
\]

Proposition 5.5: If \(M\) is of Type I, then the universal cover of the acting Lie group \(G\) is \(\tilde{G} = H_3 \times \mathbb{R}\). In particular, the metric on \(M\) is incomplete.

Proof: As in the proof of Proposition 4.5, we identify \(G\) with an integral 4-fold of the ideal \(\mathcal{I}_G = \langle \omega^1, \omega^2, \phi \rangle = \langle \eta, \overline{\eta}, \phi \rangle\). Under this identification, \(\{\omega^3, \omega^4, \omega^5, \omega^6\}\) is a basis of left-invariant 1-forms on \(G\). From (5.11), their exterior derivatives (mod \(\mathcal{I}_G\)) are given by

\[
d\omega^3 = d\omega^4 = -3\omega^4 \equiv 3\omega^4 - 8\text{Re}(p)\omega^5 - 8\text{Im}(p)\omega^6
\]

\[
d\omega^4 = d\omega^5 = d\omega^6 = 0.
\]

Let \(\{X_3, X_4, X_5, X_6\}\) be a basis of \(\mathfrak{g} = \{\text{left-invariant vector fields on } G\}\) whose dual basis is \(\{\omega^3, \omega^4, \omega^5, \omega^6\}\). Let \(Y = \frac{2}{3}\text{Im}(p)X_4 - \frac{5}{3}\text{Re}(p)X_5 + X_6\). Then \(\{X_3, X_4, X_5, Y\}\) is a basis of \(\mathfrak{g}\) with

\[
\begin{align*}
[X_4, X_5] &= 3X_3 \\
[X_4, Y] &= 0 \\
[X_5, Y] &= 0
\end{align*}\]

\[
\begin{align*}
[X_3, X_4] &= 0 \\
[X_3, X_5] &= 0 \\
[X_3, Y] &= 0.
\end{align*}\]

This exhibits \(\mathfrak{g}\) as the Lie algebra of the Lie group \(H_3 \times \mathbb{R}\), and so the universal cover of \(G\) is \(\tilde{G} = H_3 \times \mathbb{R}\). Thus, Proposition 3.3 implies that the underlying metric is incomplete. \(\Diamond\)
5.1.4 A Quasilinear Elliptic PDE System

We now derive a quasilinear elliptic PDE system on the orbit space $\Sigma$ which Type I nearly-Kähler structures must satisfy. For this, we make use of the setup of §4.6, which we recall briefly. Let $(z)$ be a local holomorphic coordinate on $\Sigma$, and write

$$\lambda \eta = \varpi^*(dz)$$

for some non-zero function $\lambda = e^{u+iv} \colon F \to \mathbb{C}$, where $u, v \colon F \to \mathbb{R}$. Note that

$$e^{2u} \eta \circ \eta = \varpi^*(dz \circ d\overline{z}).$$

In the Type I setting, we have that

$$p \eta = \varpi^*(\alpha) = \varpi^*(fdz)$$
$$r \eta^3 = \varpi^*(\beta) = \varpi^*(gdz^3)$$

for some polynomial forms $\alpha \in \Gamma(K_{\Sigma})$, $\beta \in \Gamma(K_{\Sigma} \otimes \mathbb{C})$ and functions $f, g \colon \Sigma \to \mathbb{C}$.

Proposition 5.6: For any nearly-Kähler structure of Type I, the functions $f, g, u \colon \Sigma \to \mathbb{C}$ satisfy the following quasilinear elliptic PDE system:

$$\frac{\partial f}{\partial \overline{z}} = -6|f|^2 - \frac{3}{2}e^{-2u}$$
$$\frac{\partial g}{\partial \overline{z}} = 5i f^2 e^{-2u} - \overline{f} g$$
$$\Delta u = -(6 + |g|^2 e^{6u} - |f|^2 e^{2u})$$

Proof: We calculate

$$\overline{\partial} \alpha = \overline{\partial}(f \, dz) = \nabla^{0,1}(f \, dz) = \frac{\partial f}{\partial \overline{z}} d\overline{z} \otimes dz$$
$$\varpi^*(\overline{\partial} \alpha) = -\left(6ip\overline{p} + \frac{3}{2}\right) \overline{\eta} \otimes \eta = \varpi^* \left(\left(-6|f|^2 - \frac{3}{2}e^{-2u}\right) d\overline{z} \otimes dz\right)$$

and

$$\overline{\partial} \beta = \overline{\partial}(g \, dz^3) = \nabla^{0,1}(g \, dz^3) = \frac{\partial g}{\partial \overline{z}} d\overline{z} \otimes dz^3$$
$$\varpi^*(\overline{\partial} \beta) = (5ip^2 - pr) \overline{\eta} \otimes \eta^3 = \varpi^* \left((5ie^{-2u} f^2 - \overline{f} g) d\overline{z} \otimes dz^3\right).$$

This gives the first two equations. For the last equation, we simply apply Lemma 4.7:

$$\Delta u = -K = -(6 + |r|^2 - |p|^2) = -(6 + e^{6u} \varpi^*|g|^2 - e^{2u} \varpi^*|f|^2).$$

◊

Remark: We have shown that a Type I structure determines a Riemann surface $\Sigma$ and functions $f, g, u \colon \Sigma \to \mathbb{C}$ which satisfy (5.11). Conversely, given a Riemann surface $\Sigma$ and functions $f, g, u \colon \Sigma \to \mathbb{C}$ that solve (5.11), one can clearly recover 1-forms $\eta, \overline{\eta}, \phi$ and functions $p, r$ satisfying

$$d\eta = i\phi \wedge \eta$$
$$d\phi = i(6 + |r|^2 - |p|^2) \eta \wedge \overline{\eta}$$
$$dp = -\left(6|p|^2 + \frac{3}{2}\right) \eta - (5p^2 - ipr) \overline{\eta} + ip\phi$$
$$dr = -(5ip^2 + pr) \eta + r'' \overline{\eta} + 3ir\phi.$$
We expect that, in fact, one can go further recover and 1-forms $\omega^3, \theta, \bar{\theta}, \omega^6$ satisfying the remainder of the structure equations in (5.10a), and in this way recover the entire nearly-Kähler structure. Showing this will require a deeper understanding of the structure equations (5.10a)-(5.10b). □

5.2 Type II

We now examine nearly-Kähler structures of Type II. The integrability conditions for Type II structures are significantly more complicated than those for Type I. To satisfy them, we will make a further frame adaptation and a change-of-variable.

Ultimately, we will draw two conclusions. First, we obtain (Theorem 5.10) a local existence/generality theorem for Type II structures. Second, will show that the Lie group $G$ is solvable (Proposition 5.11), and hence that the underlying metrics are incomplete.

5.2.1 A Frame Adaptation

By definition, Type II structures are those with $\|q_1\| = \|s_2\|$ and $\|q_2\| = \|s_1\|$ and $\langle q_1, q_2 \rangle = \langle s_1, s_2 \rangle \neq 0$. Thus, the $O(2)$-equivariant function $q_1 s_2 = q_2 s_1 : P \to \mathbb{C}$ maps into the unit circle $\mathbb{S}^1 \subset \mathbb{C}$. Accordingly, we may adapt frames as follows: define the $\mathbb{Z}_2$-subbundle $P_1 = \{u \in P: q_1(u) = is_2(u)\} \subset P$.

We refer to elements of $P_1$ as $\mathbb{Z}_2$-coframes. For the remainder of §5.2, we work on $P_1$.

The price we pay for this adaptation is the presence of additional torsion functions. Indeed, on $P_1$ the 1-form $\phi$ is no longer a connection form, but rather

$$\phi = \ell_1 \omega^1 + \ell_2 \omega^2$$

for some new $G$-invariant functions $\ell_1, \ell_2 : P_1 \to \mathbb{R}$.

5.2.2 The Integrability Conditions

We now move to solve the integrability conditions (5.1). For this, we make the following change-of-variables. Rather than work with $p_1, p_2, p_3, p_4, r : P_1 \to \mathbb{C}$, we will work with $t_1, \ldots, t_8, r_1, r_2 : P_1 \to \mathbb{R}$ defined by:

$$

t_1 = \text{Re}(p_1 + 4ip_4) \\
t_2 = \frac{1}{24} \text{Re}(p_3 + 4p_4) \\
t_3 = \text{Re}(p_2) \\
t_4 = \text{Im}(p_4) \\
t_5 = \text{Im}(p_1 + 4ip_4) \\
t_6 = \text{Im}(p_3 + 4p_4) \\
t_7 = \text{Im}(p_2) \\
t_8 = \text{Re}(p_4). \\
r_1 = \text{Re}(r) \\
r_2 = \text{Im}(r).
$$

The factor of $\frac{1}{24}$ appearing in $t_2$ is merely for the sake of clearing future denominators. Since each $t_i, h_i,$ and $\ell_i$ is $G$-invariant, we can write their exterior derivatives as

$$

dt_i = t_{i1} \omega^1 + t_{i2} \omega^2 \\
dh_i = h_{i1} \omega^1 + h_{i2} \omega^2 \\
d\ell_i = \ell_{i1} \omega^1 + \ell_{i2} \omega^2.
$$

We now state the Type II analogue of Lemma 5.2.
Lemma 5.7: Let $M$ be a nearly-Kähler manifold of Type II.

(a) On the $\mathbb{Z}_2$-coframe bundle $P_1$, the following 12 algebraic equations hold:

\[
\begin{align*}
\text{Re}(q_1) &= -h_4 & t_5 &= 0 & h_2 &= -4t_2t_3 \\
\text{Im}(q_1) &= h_2 & t_6 &= t_1 + 8t_4 - 64t_1t_2^2 & h_3 &= -4t_1t_2 \\
\text{Re}(q_2) &= -h_3 & t_7 &= 0 & r_1 &= t_1 + 4t_2 \\
\text{Im}(q_2) &= h_1 + 3 & t_8 &= -t_2(2h_1 + 3) & r_2 &= t_2 + t_8 + 24t_2
\end{align*}
\]

Thus, the torsion is expressible in terms of the 8 real-valued functions

\[
t_1, t_2, t_3, t_4 \quad \text{and} \quad h_1, h_4, \ell_1, \ell_2.
\]

(b) The integrability conditions $d(d\omega^i) = 0$ are equivalent to the 12 algebraic equations (5.12) together with the equations

\[
\begin{align*}
t_{11} &= t_1(\ell_2 - 32h_1t_2) & h_{11} &= H_{11}(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2) \\
t_{12} &= t_1^2(64t_2^2 + 2) + t_1(4t_4 - \ell_1) + 2t_3^2 + 6(h_1 + 3) & h_{12} &= H_{12}(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2) \\
t_{21} &= 4t_2^2(2h_1 - 9) - t_2t_2 - \frac{3}{2} & h_{41} &= H_{41}(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2) \\
t_{22} &= t_2t_1 & h_{42} &= H_{42}(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2) \\
t_{31} &= 16t_2(h_4t_1 - h_1t_3) + t_3t_2 & h_{11} &= u_1 + \frac{1}{2}G_1(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2) \\
t_{32} &= t_3(192t_1t_2^2 - 12t_4 - \ell_1) - 6h_4 & h_{12} &= u_2 + \frac{1}{2}G_2(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2) \\
t_{41} &= T_{41}(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2) & h_{41} &= u_2 - \frac{1}{2}G_2(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2) \\
t_{42} &= T_{42}(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2) & h_{42} &= -u_1 + \frac{1}{2}G_1(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2)
\end{align*}
\]

where we set $(u_1, u_2) = (\frac{1}{2}(\ell_{11} - \ell_{22}), (\frac{1}{2}(\ell_{12} + \ell_{21}))$, and where the functions $T_{41}, T_{42}$ and $H_{11}, H_{12}, H_{41}, H_{42}$ and $G_1, G_2$ appearing on the right-hand sides of (5.13) are polynomial functions (of degree \leq 5) whose explicit formulas are listed in the appendix.

Proof: (a) The left-most equations for $q_1$ and $q_2$ define our frame adaptation $P_1 \subset P$. For the remaining eight equations, a calculation shows:

\[
\begin{align*}
0 &= d(d\omega^4) \wedge \omega^{126} \quad \implies \quad (h_1 + 3)t_7 + h_4t_5 = 0 \\
0 &= d(d\omega^5) \wedge \omega^{126} \quad \implies \quad h_2t_5 - h_3t_7 = 0
\end{align*}
\]

We rewrite (5.14a)-(5.14b) as

\[
\begin{pmatrix}
(h_1 + 3) & h_4 \\
h_2 & -h_3
\end{pmatrix}
\begin{pmatrix}
t_7 \\
t_5
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

Since $M$ is of Type II, we have $(h_1 + 3)h_2 + h_3h_4 = \langle s_1, s_2 \rangle \neq 0$, from which it follows that
Equations (5.15a)-(5.15b) now give the formulas for $h_2$ and $h_3$, while (5.16a)-(5.16b) give the formulas for $t_6$ and $t_8$. Finally, since $M$ is of Type II, we have $(h_2)^2 + (h_4)^2 = \|s_2\|^2 \neq 0$. Thus, equations (5.17a)-(5.17d) give the remaining two equations.

(b) This is a direct check of the equations remaining in $d(d\omega^i) = 0$. ◯

A calculation using MAPLE shows that if the equations (5.12)-(5.13) of Lemma 5.7 hold, then $d(dt_i) = 0$ and $d(dh_i) = 0$ are also satisfied, and that

$$
\begin{align*}
  d(d\ell_1) &= F_1 \omega^{12} + (du_1 \wedge \omega^1 + du_2 \wedge \omega^2) \\
  d(d\ell_2) &= F_2 \omega^{12} + (du_2 \wedge \omega^1 - du_1 \wedge \omega^2),
\end{align*}
$$

where $F_1$, $F_2$ are certain polynomial functions (of degree \leq 8) of $t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2$ and $u_1, u_2$ whose explicit formulas we will not list here.

**Summary 5.8:** Nearly-Kähler structures of Type II are encoded by augmented coframings $((\omega^i, \ldots, \omega^6), (t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2), (u_1, u_2))$ on the $\mathbb{Z}_2$-bundle $P_1 \rightarrow M$ satisfying the structure equations

$$
\begin{align*}
  d\omega^1 &= -\ell_1 \omega^{12} \\
  d\omega^2 &= -\ell_2 \omega^{12}
\end{align*}
$$

and

$$
\begin{align*}
  d\omega^3 &= (2h_1 + 1) \omega^{12} - 4t_8 \omega^{13} + 2h_4 \omega^{15} - (t_1 + 4t_4) \omega^{23} - 2h_2 \omega^{25} - t_3 \omega^{26} \\
  &\quad - t_3 \omega^{35} + 2h_2 \omega^{36} - 3 \omega^{45} - 24t_2 \omega^{46} - t_6 \omega^{56} \\
  d\omega^4 &= (\ell_2 + 4t_8 + 24t_2) \omega^{14} - 2\ell_1 \omega^{15} + 2h_4 \omega^{23} - \ell_1 \omega^{24} - 2(\ell_2 + 12t_2) \omega^{25} + 2(h_1 + 1) \omega^{26} \\
  &\quad + 2(h_1 + 3) \omega^{35} - 8(t_8 - 3t_2) \omega^{36} + 2h_4 \omega^{56} \\
  d\omega^5 &= -(\ell_2 + 24t_2) \omega^{15} + 4\omega^{16} - 2h_2 \omega^{23} - 24t_2 \omega^{24} + (\ell_1 + 4t_4) \omega^{25} + 2h_3 \omega^{26} \\
  &\quad + 2h_3 \omega^{35} + (t_6 - t_1 - 8t_4) \omega^{36} - 2h_2 \omega^{56} \\
  d\omega^6 &= -2h_4 \omega^{12} + (2h_1 + 3) \omega^{15} - 4(t_8 - 6t_2) \omega^{16} - t_3 \omega^{23} + 3 \omega^{24} + 2h_3 \omega^{25} + (t_6 - 4t_4) \omega^{26} \\
  &\quad + t_1 \omega^{35} - 2h_3 \omega^{36} - t_3 \omega^{56}
\end{align*}
$$

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where \( t_6, t_8, h_2, h_3 \) are given by (5.12), and
\[
\begin{align*}
\omega_i &= t^{i1}_1 \omega^1 + t^{i2}_2 \omega^2 \\
\omega_i &= h^{i1}_1 \omega^1 + h^{i2}_2 \omega^2 \\
\omega_i &= \ell^{i1}_1 \omega^1 + \ell^{i2}_2 \omega^2
\end{align*}
\] (5.19c)

where \( t^{i1}_{11}, \ldots, t^{i1}_{42}, h^{i1}_{11}, h^{i2}_{12}, h^{i1}_{41}, h^{i2}_{42} \) and \( \ell^{i1}_{11}, \ell^{i1}_{12}, \ell^{i2}_{21}, \ell^{i2}_{22} \) are given by (5.13).

Augmented coframings satisfying the structure equations (5.19a)-(5.19c) and (5.12)-(5.13) will satisfy \( d\left(d\omega^i\right) = 0 \) and \( d(dt^i) = d(dh^i_1) = d(dh^i_4) = 0 \) and (5.18). In the language of §2.4, the functions \( t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2 \) are the “primary invariants,” while \( u_1, u_2 \) are the “free derivatives.”

### 5.2.3 Local Existence/Generality

We may now state the corresponding local existence/generality result for Type II structures.

**Theorem 5.9:** Nearly-Kähler structures of Type II exist locally and depend on 2 functions of 1 variable in the sense of exterior differential systems. In fact:

For any \( x \in \mathbb{R}^6 \) and any \((a_0, b_0) \in \mathbb{R}^8 \times \mathbb{R}^2\), there exists a Type II nearly-Kähler structure on an open neighborhood \( U \subset \mathbb{R}^6 \) of \( x \) and a \( \mathbb{Z}_2\)-coframe \( f_x \in P_1|x \) at \( x \) for which
\[
(t_1, t_2, t_3, t_4, h_1, h_4, \ell_1, \ell_2)(f_x) = a_0 \quad \text{and} \quad (u_1, u_2)(f_x) = b_0
\]

**Proof:** The discussion in §5.2.2 shows that hypotheses (2.4) and (2.5) of Cartan’s Third Theorem (Theorem 2.2) are satisfied. It remains to examine the tableau of free derivatives. At a point \((u,v) \in \mathbb{R}^8 \times \mathbb{R}^2\), this is the vector subspace \( A(u,v) \subset \text{Hom}(\mathbb{R}^6; \mathbb{R}^8) \cong \text{Mat}_{8\times 6}(\mathbb{R}) \) given by
\[
A(u,v) = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}
\]

Since \( A(u,v) \) is independent of the point \((u,v) \in \mathbb{R}^8 \times \mathbb{R}^2\), we can write \( A = A(u,v) \) without ambiguity. We observe that \( A \) is 2-dimensional and has Cartan characters \( \tilde{s}_1 = 2 \) and \( \tilde{s}_k = 0 \) for \( k \geq 2 \). One can also check that \( A \) is an involutive tableau, meaning that its prolongation \( A^{(1)} \) satisfies \( \dim(A^{(1)}) = 2 = \tilde{s}_1 + 2\tilde{s}_2 + \cdots + 6\tilde{s}_6 \). Thus, from Cartan’s Third Theorem, we conclude the result. \( \diamond \)

### 5.2.4 Incompleteness

As in the Type I setting, the non-compactness of the Lie group \( G \) will prevent metrics of Type II from being complete.

**Proposition 5.10:** If \( M \) is of Type II, then the Lie algebra \( \mathfrak{g} = \text{Lie}(G) \) is solvable. In particular, the metric on \( M \) is incomplete.

**Proof:** As in the proof of Proposition 4.5, we identify \( G \) with an integral 4-fold of the differential ideal \( \mathcal{I}_G = \langle \omega^1, \omega^2, \phi \rangle = \langle \eta, \overline{\eta}, \phi \rangle \). Under this identification, \( \{\omega^3, \omega^4, \omega^5, \omega^6\} \) is a basis of \( \mathfrak{g}^* = \{\text{left-invariant 1-forms on } G\} \).
Let $\zeta = \omega^5 + 8t_2\omega^6$, so that $\{\omega^3, \omega^4, \omega^6, \zeta\}$ is also a basis for $\mathfrak{g}^*$. One can check that their exterior derivatives (mod $\langle \omega^1, \omega^2 \rangle$) are

$$
d\omega^3 \equiv -t_3 \omega^3 \wedge \zeta - 3 \omega^4 \wedge \zeta + t_6 \omega^6 \wedge \zeta \\
d\omega^4 \equiv 2(h_1 + 3) \omega^3 \wedge \zeta - 2h_4 \omega^6 \wedge \zeta \\
d\omega^6 \equiv t_1 \omega^3 \wedge \zeta + t_3 \omega^6 \wedge \zeta \\
d\zeta \equiv 0,
$$

where we recall $t_6 = t_1 + 8t_4 - 64t_1t_2^2$.

Let $\{X_3, X_4, X_5, Z\}$ be a basis of $\mathfrak{g} = \{\text{left-invariant vector fields on } G\}$ whose dual basis is $\{\omega^3, \omega^4, \omega^6, \zeta\}$. Their Lie brackets are then

$$
[X_3, Z] = t_3X_3 - 2(h_1 + 3)X_4 - t_1X_6 \\
[X_4, Z] = 3X_4 \\
[X_6, Z] = -t_6X_3 + 2h_4X_4 - t_3X_6
$$

From this, it is clear that $[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = 0$, so that $\mathfrak{g}$ is solvable. (Note, however, that $\mathfrak{g}$ is not nilpotent in general.) Thus, Proposition 3.3 implies that the underlying metric is incomplete. ♦

5.3 Type III

We now consider nearly-Kähler structures of Type III. This is perhaps the most interesting case, as there is the possibility for complete metrics to exist in this class. Unfortunately, the integrability conditions (5.1)-(5.4) are even more complicated than those of Type II.

Examining these conditions leads us to several changes-of-variable (§5.3.1, §5.3.2). The upshot is that the intrinsic torsion

$$
(p_1, p_2, p_3, p_4, q_1, q_2, r; h_1, h_2, h_3, h_4): P \to \mathbb{C}^7 \oplus \mathbb{R}^4
$$

will be recast as a function

$$
(u, v, z, r; t_0, t_1, t_2): P \to \mathbb{C}^4 \oplus \mathbb{R}^3.
$$

Even with this repackaging, however, we find it difficult to solve (5.1) in general. As such, we will impose the ansatz that $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ (which by Proposition 3.3 is the case of most interest anyway). This will let us normalize the function $t_2$, which simplifies (5.1) further.

Finally, after this normalization, we are able to solve all the integrability conditions by means of a computer algebra system (Lemma 5.14), thus yielding a local existence/generality result (Theorem 5.15) for the nearly-Kähler structures with $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$.

5.3.1 A First Change-of-Variable

By definition, Type III structures are those with $\langle q_1, q_2 \rangle = \langle s_2, s_1 \rangle = 0$ and $q_1, q_2, s_1, s_2$ not all zero. Recalling also that $q_1 s_1 - q_2 s_2 = 0$, we have:

$$
\text{rank} \begin{pmatrix} q_1 & q_2 \\ s_2 & s_1 \end{pmatrix} = 1, \text{ and } q_1 \bar{q}_2 \text{ and } s_2 \bar{s}_1 \text{ both pure imaginary.}
$$

This leads us to factor

$$
\begin{pmatrix} q_1 & q_2 \\ s_2 & s_1 \end{pmatrix} = \begin{pmatrix} z \\ w \end{pmatrix} \begin{pmatrix} it_1 & t_2 \\ it_1 w & t_2 w \end{pmatrix},
$$

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where \( z, w : P \to \mathbb{C} \) and \( t_1, t_2 : P \to \mathbb{R} \). Note that by definition of Type III, we cannot have \( (z, w) = (0, 0) \), nor can we have \( (t_1, t_2) = (0, 0) \).

Remark: We caution that the functions \( z, w, t_1, t_2 \) are not uniquely defined: we may replace \((z, w, t_1, t_2)\) with \((cz, cw, it_1/c, t_2/c)\) for any non-vanishing function \( c : P \to \mathbb{R} \). \( \Box \)

### 5.3.2 A Second Change-of-Variable

We now solve \( 4 \left( \binom{4}{2} + \binom{3}{2} + \binom{3}{3} \right) = 24 \) of the \( 4 \binom{6}{2} = 80 \) equations arising in (5.1). Namely, we will solve \( d(d\nu) \wedge \eta \wedge \bar{\eta} = 0 \) and \( d(d\nu) \wedge \eta \wedge \omega^3 = 0 \) and \( d(d\nu) \wedge \bar{\eta} \wedge \omega^3 = 0 \) for \( \nu \in \{\omega^3, \theta, \bar{\theta}, \omega^6\} \). This is accomplished in the following lemma:

**Lemma 5.11**: Let \( M \) be a nearly-Kähler manifold of Type III.

(a) There exist functions \( u, v, \bar{v} : P \to \mathbb{C} \) such that:

\[
\begin{align*}
p_1 + 4ip_4 &= 6\iota_2u \\
p_2 &= -6\iota_1u \\
p_3 + 4p_4 &= 24v.
\end{align*}
\]

(b) On the \( O(2) \)-coframe bundle \( P \), the following algebraic relations hold:

\[
\begin{align*}
\text{Im}(w) &= -24 \text{Re}(u\bar{v}) \\
12\bar{v} &= z(t_1^2\bar{u} + 4it_2\bar{v}) - iw(t_1^2u - 4it_2v) - 3it_2u. 
\end{align*}
\]

**Proof**: The existence of \( v, \bar{v} \) is immediate. Let us set \( y = p_1 + 4ip_4 \) and expand the identities \( d(d\omega^3) \wedge \eta \wedge \bar{\eta} = 0 \) and \( d(d\omega^6) \wedge \eta \wedge \bar{\eta} = 0 \). This yields, for example,

\[
\begin{align*}
d(d\omega^6) \wedge \eta \wedge \bar{\eta} \wedge \theta &= 0 \quad \implies \quad is_2y - q_1\bar{y} - s_1p_2 - iq_2\bar{p}_2 = 0 \quad (5.22a) \\
\end{align*}
\]

and

\[
\begin{align*}
d(d\omega^3) \wedge \eta \wedge \bar{\eta} \wedge \omega^3 &= 0 \quad \implies \quad 4(\bar{p}_2v + p_2\bar{v}) = -(s_2 + \bar{s}_2) \quad (5.23a) \\
d(d\omega^6) \wedge \eta \wedge \bar{\eta} \wedge \omega^3 &= 0 \quad \implies \quad 4(\bar{v}y + v\bar{y}) = is_1 - \bar{s}_1 \quad (5.23b) \\
d(d\omega^6) \wedge \eta \wedge \bar{\eta} \wedge \omega^6 &= 0 \quad \implies \quad p_2y - p_2\bar{y} = 0. \quad (5.24)
\end{align*}
\]

In light of (5.24), we see that in order for the linear system (5.23a)-(5.23b) to have solutions, we must have

\[
\begin{pmatrix}
is_2 \\
- q_1 \\
s_2 + \bar{s}_2 \\
0
\end{pmatrix} + \begin{pmatrix}
s_2 \\
iq_2 \\
i(s_1 - \bar{s}_1) \\
0
\end{pmatrix} \begin{pmatrix}
y \\
\bar{y} \\
p_2 \\
p_2 \end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

Regard the equations (5.22a), (5.22b), and (5.25) as a homogeneous linear system:

\[
\begin{pmatrix}
is_2 & - q_1 & - s_1 & -iq_2 \\
- \bar{q}_1 & is_2 & i\bar{s}_2 & \bar{s}_1 \\
0 & i(s_1 - \bar{s}_1) & 0 & i(s_1 - \bar{s}_1)
\end{pmatrix} \begin{pmatrix}
y \\
\bar{y} \\
p_2 \\
p_2 \end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

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The solutions to this system are of the form

\[
\begin{pmatrix}
y \\
y \\
p_2 \\
p_2
\end{pmatrix} = 6u \begin{pmatrix} t_2 \\ 0 \\ -t_1 \\ 0 \end{pmatrix} + 6\overline{u} \begin{pmatrix} 0 \\ t_2 \\ 0 \\ -t_1 \end{pmatrix}
\]

for some \( u : P \to \mathbb{C} \). This proves (a). The only equations left in \( d(d\nu)\wedge\eta\wedge\eta = 0, d(d\nu)\wedge\eta\wedge\omega^{36} = 0 \) and \( d(d\nu)\wedge\overline{\eta}\wedge\omega^{36} = 0 \) for \( \nu \in \{\omega^3, \theta, \overline{\theta}, \omega^6\} \) are exactly those in the statement of (b). ♦

**Bookkeeping 5.12:** We pause to unwind the notational changes. By definition of Type III and by Lemma 5.11(a), our torsion functions \( (p_i, q_i, r, s_i) \) are now expressed as follows:

\[
\begin{align*}
p_1 &= 6t_2u - 12i(v + \overline{v}) \\
p_2 &= -6t_1u \\
p_3 &= 12(v - \overline{v}) \\
p_4 &= 3(v + \overline{v}) \\
q_1 &= it_1z \\
q_2 &= t_2z \\
s_1 &= t_2w \\
s_2 &= it_1w.
\end{align*}
\]

That is, the torsion is expressed in terms of \( u, v, \overline{v}, z, w, r, t_1, t_2 : P \to \mathbb{C} \).

By Lemma 5.11(b), the functions \( \overline{v} \) and \( \text{Im}(w) \) can be expressed in terms of the others. Hence, setting \( t_0 = \text{Re}(w) \), we regard the torsion as a function

\((u, v, z, r; t_0, t_1, t_2) : P \to \mathbb{C}^4 \oplus \mathbb{R}^3.\)

One can check that these functions are \( \text{O}(2) \)-equivariant. Indeed, modulo \( \langle \omega^1, \omega^2 \rangle = \langle \eta, \overline{\eta} \rangle \):

\[
\begin{align*}
du &\equiv iu\phi \\
dv &\equiv iv\phi \\
dz &\equiv 2iz\phi \\
dr &\equiv 3ir\phi \\
dt_1 &\equiv 0 \\
dt_2 &\equiv 0.
\end{align*}
\]

### 5.3.3 The Ansatz \( g = su(2) \oplus u(1) \)

Having solved the 24 of the 80 equations arising in (5.1), we aim to solve the remaining 56 of them. To this end, we restrict attention to the case where \( G \) is a finite quotient of \( \text{SU}(2) \times \text{U}(1) \).

This ansatz imposes inequalities on the torsion which allow us to normalize \( t_2 \).

**Lemma 5.13:** Let \( M \) be of Type III. If \( g = su(2) \oplus u(1) \), then the (real) 1-form

\[
\sigma_6 := 3(iu\overline{z} - u\overline{w})\theta - 3(i\overline{u}z + uw)\overline{\theta} + (|w|^2 - |z|^2)\omega^6
\]

is non-vanishing, and the (real) symmetric matrix

\[
Q := \begin{pmatrix}
\frac{1}{3}t_2 & -2t_1\text{Im}(u) & -2t_1\text{Re}(u) \\
-2t_1\text{Im}(u) & 48\text{Im}(u)\text{Re}(v) - \text{Im}(z) + t_0 & 24\text{Re}(uv) - \text{Re}(z) \\
-2t_1\text{Re}(u) & 24\text{Re}(uv) - \text{Re}(z) & -48\text{Re}(u)\text{Im}(v) + \text{Im}(z) + t_0
\end{pmatrix}
\]

is positive-definite or negative-definite. In particular, \( t_2 \) is nowhere-vanishing.

**Proof:** As in the proof of Proposition 4.5, we identify \( G \) with an integral 4-fold of the differential ideal \( \mathcal{I}_G = \langle \omega^1, \omega^2, \phi \rangle = \langle \eta, \overline{\eta}, \phi \rangle \). Under this identification, \( \{\omega^3, \omega^4, \omega^5, \omega^6\} \) is a basis of \( g^* = \)
\{\text{left-invariant} \ 1\text{-forms on} \ G\}. 

Suppose that $g = su(2) \oplus u(1)$. Then $g$ has a non-zero center, so there exists a non-zero element of $g^*$ which is closed. A calculation shows that the only elements of $g^*$ which are closed are multiples of 

$$\sigma_6 := 3(iu\bar{z} - \overline{uw}) \theta - 3(iu\bar{z} + uw) \bar{\theta} + (|w|^2 - |z|^2) \omega^6.$$ 

Thus, $\sigma_6$ is non-vanishing.

We now observe that $\{\sigma_3, \sigma_4, \sigma_5, \sigma_6\}$ is a basis for $g^*$, where we are defining 

$$(\sigma_3, \sigma_4, \sigma_5) := (t_2\omega^3 - t_1\omega^6, \text{Im}(\chi), \text{Re}(\chi)) \quad \chi := -3(2t_1u\omega^3 - \theta - 8iv\omega^6).$$ 

One can calculate that modulo $\mathcal{I}_G$,

$$d\begin{pmatrix} \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{pmatrix} \equiv Q \begin{pmatrix} \sigma_4 \wedge \sigma_5 \\ \sigma_5 \wedge \sigma_3 \\ \sigma_3 \wedge \sigma_4 \end{pmatrix} \quad ds_6 \equiv 0.$$ 

Since $\{\sigma_3, \sigma_4, \sigma_5\}$ is a basis of $su(2)^*$, this coefficient matrix $Q$ must be positive-definite or negative-definite, and hence $t_2$ is nowhere-vanishing. $\diamond$

### 5.3.4 Local Existence/Generality

We now move to solve the integrability conditions (5.1) in the case of $g = su(2) \oplus u(1)$. By Lemma 5.13, the function $t_2$ is nowhere-vanishing. Recalling that $z, w, t_1, t_2$ are only defined up to scaling by a nowhere-vanishing function $c: P \to \mathbb{R}$, we shall choose $c$ so that 

$$t_2 = 1.$$ 

Thus, the torsion of the $O(2)$-structure is now encoded by $(u, v, z, r; t_0, t_1): P \to \mathbb{C}^4 \oplus \mathbb{R}^2$. Since $u, v, z, r, t_0, t_1$ are $G$-invariant and $O(2)$-equivariant (by (5.26)), we may express their exterior derivatives as 

$$du = u'\eta + u''\bar{\eta} + iu\phi \quad dz = z'\eta + z''\bar{\eta} + 2iz\phi \quad dt_1 = t'_1\eta + t''_1\bar{\eta}$$ 

$$dv = v'\eta + v''\bar{\eta} + iv\phi \quad dr = r'\eta + r''\bar{\eta} + 3ir\phi \quad dt_0 = t'_0\eta + t''_0\bar{\eta}.$$ 

The Type III analogue of Lemma 5.2(b) and Lemma 5.7(b) is the following:

**Lemma 5.14:** Let $M$ be a Type III nearly-Kähler structure with $g = su(2) \oplus u(1)$. With the equations of Lemma 5.11 imposed and with the normalization $t_2 = 1$ in place, the integrability conditions (5.1) are equivalent to 

$$u' = f_1(u, \overline{u}, v, \overline{v}, z, \overline{z}, r, \overline{r}, t_0, t_1) \quad u'' = f_2(u, \overline{u}, v, \overline{v}, z, \overline{z}, r, \overline{r}, t_0, t_1)$$ 

$$v' = f_3(u, \overline{u}, v, \overline{v}, z, \overline{z}, r, \overline{r}, t_0, t_1) \quad v'' = f_4(u, \overline{u}, v, \overline{v}, z, \overline{z}, r, \overline{r}, t_0, t_1)$$ 

$$z' = f_5(u, \overline{u}, v, \overline{v}, z, \overline{z}, r, \overline{r}, t_0, t_1) \quad z'' = f_6(u, \overline{u}, v, \overline{v}, z, \overline{z}, r, \overline{r}, t_0, t_1)$$ 

$$r' = f_7(u, \overline{u}, v, \overline{v}, z, \overline{z}, r, \overline{r}, t_0, t_1) \quad t'_0 = f_8(u, \overline{u}, v, \overline{v}, z, \overline{z}, r, \overline{r}, t_0, t_1)$$ 

$$t'_1 = f_{10}(u, \overline{u}, v, \overline{v}, z, \overline{z}, r, \overline{r}, t_0, t_1) \quad t''_0 = f_9(u, \overline{u}, v, \overline{v}, z, \overline{z}, r, \overline{r}, t_0, t_1)$$ 

$$t''_1 = f_{11}(u, \overline{u}, v, \overline{v}, z, \overline{z}, r, \overline{r}, t_0, t_1) \quad K = f_{12}(u, \overline{u}, v, \overline{v}, z, \overline{z}, r, \overline{r}, t_0, t_1).$$

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for certain functions $f_1, \ldots, f_{12}$ of $u, \overline{u}, v, \overline{v}, z, \overline{r}, r, t_0, t_1$.

**Remark:** The explicit expressions for $f_1, \ldots, f_{12}$ are sufficiently cumbersome that we will not list them here. They turn out to be polynomial functions of degree $\leq 10$. □

**Proof:** With the equations of Lemma 5.11 imposed, and with $t_2 = 1$ imposed, there are $56 = 80 - 24$ polynomial equations remaining in (5.1) involving the functions

$$ u, \overline{u}, v, \overline{v}, z, \overline{r}, r, t_0, t_1 \quad \text{and} \quad u', u'', \overline{v}', \overline{v}'', v', v'', \overline{v}', \overline{v}'', z', z'', r', r'', r', t_0', t_1', t_1'', K. $$

A direct application of a computer algebra system (we used MAPLE) will solve these 56 equations, yielding lengthy explicit formulas for $u, u'', \ldots, t_1', t_1'', K$ in terms of $u, \overline{u}, v, \overline{v}, z, \overline{r}, r, t_0, t_1$. ◊

As in the Type I and Type II settings, a calculation using MAPLE now shows that if the equations (5.27) of Lemma 5.14 hold, then $d(d\phi) = 0$ and $d(dv) = d(dz) = d(dt_0) = d(dt_1) = 0$ are all satisfied, and that

$$ d(dr) = (F \eta \wedge \overline{\eta} - 4ir'' \phi \wedge \overline{\eta}) + dr'' \wedge \eta $$

where $F$ is a certain polynomial function (of degree 14) of $u, \overline{u}, v, \overline{v}, z, \overline{r}, r, t_0, t_1$ whose explicit formula we will not list here.

The upshot of this discussion is that the integrability conditions (2.4) and (2.5) of Cartan’s Third Theorem are finally satisfied. In particular, we obtain the following local existence/generality result:

**Theorem 5.15:** Nearly-Kähler structures (of Type III) for which $G$ is a finite quotient of $SU(2) \times U(1)$ exist locally and depend on 2 functions of 1 variable in the sense of exterior differential systems. In fact:

For any $x \in \mathbb{R}^6$ and $(a_1, a_2, a_3, a_4, \overline{a}_5, \overline{a}_6, b_0) \in \mathbb{C}^4 \times \mathbb{R}^2 \times \mathbb{C}$ with $(a_3, \overline{a}_5, \text{Re}(a_1 \overline{a}_2)) \neq (0, 0, 0)$, there exists a (Type III) nearly-Kähler structure with $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ on an open neighborhood $U \subset \mathbb{R}^6$ of $x$ and an $O(2)$-coframe $f_x \in P|_x$ at $x$ for which

$$ (u, v, z, r)(f_x) = (a_1, a_2, a_3, a_4) \quad \text{and} \quad (t_0, t_1)(f_x) = (\overline{a}_5, \overline{a}_6) \quad \text{and} \quad r''(f_x) = b_0. $$

**Remark:** The unusual looking requirement $(a_3, \overline{a}_5, \text{Re}(a_1 \overline{a}_2)) \neq (0, 0, 0)$ is simply the condition $(z, \text{Re}(w), \text{Im}(w)) \neq (0, 0, 0)$ mentioned at the start of §5.3.1. That is, it is exactly the condition $(q_1, q_2, s_1, s_2) \neq (0, 0, 0)$ forming part of the definition of “Type III.” □

**Proof:** It remains only to examine the tableau of free derivatives. This proceeds exactly as in the cases of Types I and II, so we omit the details. ◊

### 6 A One-Parameter Family of Incomplete Cohomogeneity-One Metrics

With the issue of local existence settled, it is natural to seek special solutions. That is, one would like to find explicit examples of nearly-Kähler structures with $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ with one or more of the invariants $(u, v, z, r, t_0, t_1)$ prescribed in advance. For example, it is natural to ask that the intrinsic torsion

$$ (u, v, z, r; t_0, t_1) : P \to \mathbb{C}^4 \oplus \mathbb{R}^2 $$

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take values in a prescribed $O(2)$-invariant subset of $\mathbb{C}^4 \oplus \mathbb{R}^2$.

In this section, we comment on those nearly-Kähler structures with $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ that satisfy $u = 0$. A priori, it is not at all clear how many such structures exist (or that any exist at all).

Let $M$ be a nearly-Kähler 6-manifold with $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ satisfying $u = 0$. The condition $u = 0$ turns out to be quite strong: a calculation using MAPLE shows that the integrability conditions resulting from $u = 0$ imply that $z = 0$, $r = 0$, and $t_0 = 3$. Thus, the torsion is encoded in the function

$$A = (a_1, a_2, a_0) := (24 \operatorname{Re}(v), 24 \operatorname{Im}(v), 3t_1): P \to \mathbb{R}^3.$$

**Remark:** Unwinding the various changes-of-variable, we find that we are exactly in the situation of

$$(p_1, p_2, p_3, p_4) = (0, 0, 24v, 0) \quad r = 0 \quad (h_1, h_2, h_3, h_4) = (0, 0, 0, 3t_1).$$

That is, $(a_1, a_2, a_0) = (\operatorname{Re}(p_3), \operatorname{Im}(p_3), h_4)$. \[\square\]

The structure equations on $P$ now read as follows:

\begin{align}
    d\omega^1 &= -\phi \wedge \omega^2 & d\omega^3 &= \omega^{12} + a_0 \omega^{15} - a_0 \omega^{24} - 3\omega^{45} - a_1 \omega^{46} - a_2 \omega^{56} \\
    d\omega^2 &= \phi \wedge \omega^1 & d\omega^4 &= a_0 \omega^{23} - \omega^{26} + 3\omega^{35} + a_1 \omega^{36} + a_0 \omega^{56} - \phi \wedge \omega^5 \\
    d\phi &= 3(a_0^2 + 1) \omega^{12} & d\omega^5 &= -a_0 \omega^{13} + \omega^{16} - 3\omega^{34} + a_2 \omega^{36} - a_0 \omega^{46} + \phi \wedge \omega^4 \\
    & & d\omega^6 &= -2a_0 \omega^{12} + a_1 \omega^{16} + a_2 \omega^{26}
\end{align}

and

\begin{equation}
    dA = \begin{pmatrix}
        da_1 \\
        da_2 \\
        da_0
    \end{pmatrix} = \begin{pmatrix}
        a_0^2 - a_1^2 - 3 & -a_1a_2 & -a_2 \\
        -a_1a_2 & a_0^2 - a_2^2 - 3 & a_1 \\
        -2a_0a_1 & -2a_0a_2 & 0
    \end{pmatrix}
    \begin{pmatrix}
        \omega^1 \\
        \omega^2 \\
        \phi
    \end{pmatrix}.
\end{equation}

These equations satisfy $d(d\omega^i) = d(d\phi) = 0$ and $d(da_i) = 0$. Thus, Cartan’s Third Theorem yields:

**Theorem 6.1:** Nearly-Kähler structures of Type III for which $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ and $u = 0$ exist locally. In fact:

For any $(x_0, y_0, z_0) \in \mathbb{R}^3$, there exists a Type III nearly-Kähler structure with $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ and $u = 0$ on an open neighborhood $U \subset \mathbb{R}^6$ of $x$ and an $O(2)$-coframe $f_x \in P|_x$ at $x$ for which

$$(a_1, a_2, a_0)(f_x) = (x_0, y_0, z_0).$$

We can completely describe the induced metric on the principal orbits:

**Proposition 6.2:** Let $M$ be a Type III nearly-Kähler 6-manifold with $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ and $u = 0$. Then the principal $G$-orbits in $M$ are locally isometric to $S^3(\frac{2}{\sqrt{3}}) \times S^1$.

**Proof:** Let $N^4$ be a principal $G$-orbit. We will work on the $O(2)$-bundle given by $\pi^{-1}(N) \to N$. Note that on $\pi^{-1}(N)$, the functions $a_0, a_1, a_2$ are all constant, and that $\omega^1 = \omega^2 = 0$. Therefore,
on $\pi^{-1}(N)$, the structure equations (6.1) read:
\[
\begin{pmatrix}
\omega^3 \\
\omega^4 \\
\omega^5 \\
\omega^6
\end{pmatrix} = - \begin{pmatrix}
0 & 3\omega^3 + a_1\omega_6 & -3\omega^4 + a_2\omega_6 & 0 \\
-3\omega^5 - a_1\omega_6 & 0 & 3\omega^4 - a_0\omega_6 - \phi & 0 \\
\frac{2}{3}\omega^4 - a_2\omega_6 & \frac{3}{2}\omega^3 + a_0\omega_6 + \phi & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \wedge
\begin{pmatrix}
\omega^3 \\
\omega^4 \\
\omega^5 \\
\omega^6
\end{pmatrix}
\]
\[d\phi = 0.\]

Since $d\omega^6 = 0$, locally it is the case that $\omega^6 = df$ for some submersion $f: N^4 \to S^1$. The equations (6.3) show that the $f$-horizontal distribution $\langle \omega^3, \omega^4, \omega^5 \rangle$ is integrable. Moreover, a calculation using (6.3) shows that the $f$-fibers are totally-geodesic and have constant curvature $\frac{9}{4}$. Thus, $N$ is locally isometric to a product $S^3(\frac{2}{3}) \times S^1$. \(\diamondsuit\)

Before stating our main result on Type III nearly-Kähler structures with $u = 0$, we make two preliminary observations.

First, note that if $A: P \to \mathbb{R}^3$ has rank $r$, then $\dim(\text{Aut}_{O(2)}) = 7 - r$. Indeed, on the one hand, $\text{Aut}_{O(2)}$ acts freely on $P$. On the other hand, it is a general fact \[17\] that $\text{Aut}_{O(2)}$ acts transitively on the level sets of $A$, which by hypothesis are $(7 - r)$-dimensional.

Second, note that the substitution
\[
\begin{align*}
\beta_3 &:= 3\omega^3 - a_0\omega^6 - \phi \\
\beta_4 &:= 3\omega^4 - a_2\omega^6 - a_0\omega^1 \\
\beta_5 &:= 3\omega^5 + a_1\omega^6 - a_0\omega^2
\end{align*}
\]
greatly simplifies the structure equations (6.1):
\[
\begin{align*}
d\omega^1 &= -\phi \wedge \omega^2 & d\beta_3 &= -\beta_4 \wedge \beta_5 \\
d\omega^2 &= \phi \wedge \omega^1 & d\beta_4 &= -\beta_5 \wedge \beta_3 \\
d\phi &= 3(a_0^2 + 1)\omega^{12} & d\beta_5 &= -\beta_3 \wedge \beta_4 \\
d\omega^6 &= -2a_0\omega^{12} + (a_1\omega^1 + a_2\omega^2) \wedge \omega^6.
\end{align*}
\]

Our result is:

**Theorem 6.3:** Let $M$ be a Type III nearly-Kähler 6-manifold with $g = \mathfrak{su}(2) \oplus u(1)$ and $u = 0$.

(a) The image of $A: P \to \mathbb{R}^3$ is one of the following:

(i) One of the points $(0, 0, \pm \sqrt{3})$.

(ii) An open subset of the 2-sphere $\{a_1^2 + a_2^2 + (a_0 - c)^2 = c^2 - 3\}$, where $\sqrt{3} < |c| < \infty$.

(iii) An open subset of the 2-plane $\{a_0 = 0\}$.

(b) In case (i), the nearly-Kähler structure on $M$ is locally homogeneous under the action of the 7-dimensional group $\text{Aut}_{O(2)}$. In fact, $M$ is locally isomorphic to $\text{Aut}_{O(2)}/O(2)$, and $\text{Aut}_{O(2)}$ is a finite quotient of $\text{SU}(2) \times \text{SU}(2) \times U(1)$. The quotient surface $\Sigma$ is locally isometric to the round sphere $S^2(\frac{1}{2\sqrt{3}})$.

(c) In cases (ii) and (iii), the nearly-Kähler structure on $M$ is cohomogeneity-one under the action of the 5-dimensional group $\text{Aut}_{O(2)}$. The underlying metric on $M$ is incomplete.

In case (ii), the surface $\Sigma$ admits a Killing vector field and its Gauss curvature $K \geq 3$ lies in the interval $[6(c^2 - 1 - c\sqrt{c^2 - 3}), 6(c^2 - 1 + c\sqrt{c^2 - 3})]$. In case (iii), the surface $\Sigma$ is locally isometric to the round sphere $S^2(\frac{1}{\sqrt{3}})$. 

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Proof: (a) We observe that
\[ d \left( \frac{a_0}{a_0^2 + a_1^2 + a_2^2 + 3} \right) = 0, \]
so that on a connected set
\[ \frac{a_0}{a_0^2 + a_1^2 + a_2^2 + 3} = \frac{1}{2c} \]
where \( c \in (\mathbb{R} - 0) \cup \{ \infty \} \) is a constant. Thus, the image of \( A: P \rightarrow \mathbb{R}^3 \) is a subset of:
(i) \(|c| = \sqrt{3}\): One of the points \((0, 0, \pm \sqrt{3})\).
(ii) \(\sqrt{3} < |c| < \infty\): The 2-sphere \(\{a_1^2 + a_2^2 + (a_0 - c)^2 = c^2 - 3\}\).
(iii) \(|c| = \infty\): The 2-plane \(\{a_0 = 0\}\).

It remains to check the openness claims in (ii) and (iii). For this, note first that either \(\operatorname{rank}(A) = 0\) or \(\operatorname{rank}(A) = 2\) if and only if the image of \(A\) is one of the points \((0, 0, \pm \sqrt{3})\). Consequently, \(\operatorname{rank}(A) = 2\) if and only if the image of \(A\) is a subset of the 2-sphere or 2-plane described above.

(b) In case (i), we have \((a_1, a_2, a_0) = (0, 0, \pm \sqrt{3})\). Set \((\beta_1, \beta_2) := (2\sqrt{3}\omega^1, 2\sqrt{3}\omega^2)\) and \(\beta_6 := 6\omega^6 + \sqrt{3}\phi\). Then the structure equations (6.5) are:
\[
\begin{align*}
  d\beta_1 &= \beta_2 \wedge \phi \\
  d\beta_2 &= \phi \wedge \beta_1 \\
  d\phi &= \beta_1 \wedge \beta_2 \\
  d\beta_3 &= -\beta_4 \wedge \beta_5 \\
  d\beta_4 &= -\beta_5 \wedge \beta_3 \\
  d\beta_5 &= -\beta_3 \wedge \beta_4.
\end{align*}
\]
Thus, \(P\) with the coframing \((\omega^1, \ldots, \omega^6, \phi)\) is locally isomorphic to \(SU(2) \times SU(2) \times U(1)\) with its left-invariant coframing. In particular, \(M^6\) is (locally) homogeneous. The surface \(\Sigma\) has Gauss curvature \(K = 3(a_0^2 + 1) = 12\), hence is locally isometric to \(S^2(\frac{1}{2\sqrt{3}})\).

(c) We now consider cases (ii) and (iii). In these cases, \(\operatorname{rank}(A) = 2\), so \(H := \operatorname{Aut}_{O(2)}\) is a 5-dimensional group. Since \(\dim(H) = 5\), the \(H\)-action on \(M^6\) must have cohomogeneity \(\geq 1\). On the other hand, since \(H \supset G\) and \(G\) acts with cohomogeneity two, the \(H\)-action must have cohomogeneity \(\leq 2\). If \(H\) acted on \(M\) with cohomogeneity two, then Proposition 4.5 would imply that \(\dim(H) = 4\), a contradiction. Thus, \(M\) is cohomogeneity one under the \(H\)-action.

From Podestá and Spiro’s study \[22\] of cohomogeneity-one nearly-Kähler metrics, we deduce that the underlying metric is incomplete.

Let \(\varpi: F^3 \rightarrow S^2\) denote the orthonormal coframe bundle of \(\Sigma\), as in §4.6. In case (ii), a direct calculation using (6.1) shows that a vector field \(X \in \Gamma(TF)\) satisfies \(L_X \omega^1 = L_X \omega^2 = L_X \phi = 0\) and \(L_X A = 0\) if and only if
\[
\begin{bmatrix}
\omega^1(X) \\
\omega^2(X) \\
\phi(X)
\end{bmatrix} = C \begin{bmatrix}
-a_2 \\
-a_1 \\
3 - a_0^2
\end{bmatrix}
\]
for some constant \(C\). Such a vector field \(X \in \Gamma(TF)\) is \(\varpi\)-related to a (non-zero) Killing vector field on \(\Sigma\). Moreover, the Gauss curvature of \(\Sigma\) is \(K = 3(a_0^2 + 1)\), where \((a_1, a_2, a_0)\) lies on the 2-sphere \(\{a_1^2 + a_2^2 + (a_0 - c)^2 = c^2 - 3\}\), whence \(a_0 \in [c - \sqrt{c^2 - 3}, c + \sqrt{c^2 - 3}]\).

In case (iii), the surface \(\Sigma\) has Gauss curvature \(K = 3\), so is locally isometric to \(S^2(\frac{1}{2\sqrt{3}})\). ♦
6.1 An Explicit Example

In case (iii), where \( a_0 = 0 \), we can go further and “integrate” the structure equations (6.5) and (6.2). Since \((a_1, a_2) : P \to \mathbb{R}^2\) is an \( O(2) \)-equivariant function, we may adapt frames as follows: consider the subset
\[
P_1 := \{ p \in P : a_2(p) = 0 \}.
\]
Over the open dense subset \( M^* = \{ m \in M : ((a_1)^2 + (a_2)^2)(m) \neq 0 \} \subset M \), the projection \( P_1 \to M^* \) is a \( \mathbb{Z}_2 \)-bundle. For the remainder of this section, we work on \( P_1 \). For ease of notation, set \( a := a_1 \).

On \( P_1 \), the 1-form \( \phi \) is no longer a connection form, but rather
\[
\phi = k_1 \omega^1 + k_2 \omega^2
\]
for some new \( G \)-invariant functions \( k_1, k_2 : P_1 \to \mathbb{R} \). Thus, equations (6.2) now read:
\[
da = -(a^2 + 3) \omega^1 \\
0 = -3 \omega^2 + a \phi = ak_1 \omega^1 + (ak_2 - 3) \omega^2,
\]
whence \((k_1, k_2) = (0, 3/a)\), and thus \( \phi = \frac{3}{a} \omega^2 \). The structure equations (6.5) now read:
\[
d\omega^1 = 0 \\
d\omega^2 = -\frac{3}{a} \omega^1 \land \omega^2 \\
d\omega^6 = a \omega^1 \land \omega^6 \\
d\beta_3 = -\beta_4 \land \beta_5 \\
d\beta_4 = -\beta_5 \land \beta_3 \\
d\beta_5 = -\beta_3 \land \beta_4.
\]
Since
\[
\omega^1 = -\frac{1}{a^2 + 3} da,
\]
we see that
\[
d\omega^2 = d \left( \log \frac{a}{\sqrt{a^2 + 3}} \right) \land \omega^2 \\
d\omega^6 = d \left( \log \frac{1}{\sqrt{a^2 + 3}} \right) \land \omega^6,
\]
and thus
\[
\omega^2 = \frac{a}{\sqrt{a^2 + 3}} dt \\
\omega^6 = \frac{1}{\sqrt{a^2 + 3}} ds
\]
for some locally-defined functions \( t, s : P_1 \to \mathbb{R} \). Unwinding the notational changes, our substitution (6.3) is simply:
\[
\beta_3 = 3 \omega^3 - \frac{3}{\sqrt{a^2 + 3}} dt \\
\beta_4 = 3 \omega^4 \\
\beta_5 = 3 \omega^5 + \frac{a}{\sqrt{a^2 + 3}} ds \\
\omega^3 = \frac{1}{3} \left( \beta_3 + \frac{3}{\sqrt{a^2 + 3}} dt \right) \\
\omega^4 = \frac{1}{3} \beta_4 \\
\omega^5 = \frac{1}{3} \left( \beta_5 - \frac{a}{\sqrt{a^2 + 3}} ds \right)
\]
We draw the following conclusion:

**Theorem 6.4:** Let \((a, t, s) : \mathbb{R}^3 \times S^3 \to \mathbb{R}^3\) denote the projection to the \( \mathbb{R}^3 \)-factor, and let \((\beta_3, \beta_4, \beta_5)\) denote the left-invariant coframing on \( SU(2) \simeq S^3 \). Then the Riemannian metric \( g_\infty \) on \( \mathbb{R}^3 \times S^3 \)
given by

\[ g_{\mathfrak{co}} = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 + (\omega^4)^2 + (\omega^5)^2 \]

\[ = \frac{1}{(a^2 + 3)^2} \, da^2 + \frac{a^2}{a^2 + 3} \, dt^2 + \frac{1}{a^2 + 3} \, ds^2 \]

\[ + \frac{1}{9} \left( \beta_3 + \frac{3}{\sqrt{a^2 + 3}} \, dt \right)^2 + \frac{1}{9} \beta_4^2 + \frac{1}{9} \left( \beta_5 - \frac{a}{\sqrt{a^2 + 3}} \right) \, ds^2 \]

is a Type III nearly-Kähler metric with \( g = \mathfrak{su}(2) \oplus \mathfrak{u}(1), \) \( u = 0, \) and \( a_0 = 0. \) In fact, it is the unique such metric up to diffeomorphism.

7 Appendix

In Lemma 5.7(b), a calculation using MAPLE shows that the polynomial functions \( T_{41}, T_{42}, H_{11}, H_{12}, H_{41}, H_{41}, \) and \( G_1, G_2 \) are:

\[ T_{41} = -t_1(64t_2^2(2h_1 + 9) + 16t_2^2\ell_2 - 4t_2(h_1 - 2)) + t_2(6\ell_1 - 4h_4 t_3) + \ell_2 t_4 \]
\[ T_{42} = 8t_2^2(2t_1(\ell_1 + 12t_4) + 4t_1^2 + 6h_1 + 27) - t_4(4t_1 + \ell_1 + 12t_4) - \frac{1}{2}(t_1^2 + t_3^2 + 3) + 6\ell_2 t_2 \]

and

\[ H_{11} = 8t_2(t_1\ell_1 - 4h_4^2 - 12h_1) + 2\ell_2(h_1 + 3) \]
\[ H_{12} = 2t_1(16t_2^2(2h_1 + 9) + 4\ell_2 t_2 + h_1 + 2) - 2\ell_1(h_1 + 3) - 2t_3 h_4 \]
\[ H_{21} = -8t_2(t_3\ell_1 + 4h_1 h_4 + 6h_4) + 2\ell_2 h_4 \]
\[ H_{22} = -2t_3(16t_2^2(2h_1 + 9) + 4\ell_2 t_2 + h_1 + 2) + 2h_4(64t_2 t_4^2 - 8t_4 - t_1 - \ell_1) \]

and

\[ G_1 = 256t_1 t_2^2(2h_1 + 9) + 8t_2(2h_1 t_1 - 2h_4 t_3 - 15\ell_1 + 4t_1) + 64\ell_2 t_1 t_2^2 \]
\[ G_2 = 96\ell_2^2(2h_1 - 9) + 4t_2^2(2h_1 - 21) - 3(\ell_1^2 + \ell_2^2) - 4(h_1^2 + h_4^2) - 4\ell_1 t_4 - 12h_1 - 12. \]

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