A one-sample location test based on weighted averaging of two test statistics in high-dimensional data

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We discuss a one-sample location test that can be used in the case of high-dimensional data. For high-dimensional data, the power of Hotelling’s test decreases when the dimension is close to the sample size. To address this loss of power, some non-exact approaches were proposed, e.g., Dempster (1958, 1960), Bai and Saranadasa (1996) and Srivastava and Du (2006). In this paper, we focus on Hotelling’s test and Dempster’s test. The comparative merits and demerits of these two tests vary according to the local parameters. In particular, we consider the situation where it is difficult to determine which test should be used, that is, where the two tests are asymptotically equivalent in terms of local power. We propose a new statistic based on the weighted averaging of Hotelling’s $T^2$ statistic and Dempster’s statistic that can be applied in such a situation. Our weight is determined on the basis of the maximum local asymptotic power on a restricted parameter space that induces local asymptotic equivalence between Hotelling’s test and Dempster’s test. In addition, some good asymptotic properties with respect to the local power are shown. Numerical results show that our test is more stable than Hotelling’s $T^2$ statistic and Dempster’s statistic in most parameter settings.

**Key Words** Asymptotic power, Dempster’s test, High-dimensional data, One-sample location test, $T^2$-statistic, .
1 Introduction

Let \( x_1, x_2, \cdots, x_N \) be \( p \) dimensional observation vectors from \( \mathcal{N}_p(\mu, \Sigma) \). We consider the following one-sample hypothesis test

\[
H_0 : \mu = \mu_0 \quad \text{vs.} \quad H_1 : \mu \neq \mu_0.
\]

To test the hypothesis \( H_0 \), traditionally Hotelling’s test statistic \((T^2\text{-statistic})\) is used, which is defined by

\[
T^2 = N(\bar{x} - \mu_0)'S^{-1}(\bar{x} - \mu_0),
\]

where

\[
\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i, \quad S = \frac{1}{n} \sum_{i=1}^{N} (x_i - \bar{x})(x_i - \bar{x})',
\]

and \( n = N - 1 \). It is well known that under the null hypothesis \( H_0 \), \((N - p)/(np)T^2\) has an \( F \)-distribution with degrees of freedom \( p \) and \( N - p \). Let the significance level be chosen as \( \alpha \) and the threshold be denoted by \( F_{p,N-p}(\alpha) \). Then Hotelling’s test rejects \( H_0 \) if

\[
\frac{N - p}{np}T^2 > F_{p,N-p}(\alpha).
\]

However, Hotelling’s test has the serious defect that the \( T^2 \) statistic is undefined when the dimension of the data is greater than the sample size. In subsequent years, a number of improvements on Hotelling’s test in the high-dimensional setting were discussed, see e.g., Dempster (1958, 1960), Bai and Saranadasa (1998), Srivastava (2007), Srivastava and Du (2008). In this paper, we focus on Dempster’s non-exact test. Dempster (1958, 1960) proposed a non-exact test for the hypothesis \( H_0 \), where the dimension \( p \) is possibly greater than the sample size \( N \). Dempster’s test statistic (D-statistic) is defined as

\[
D_n = \frac{(\bar{x} - \mu_0)'(\bar{x} - \mu_0)}{\text{tr}S}.
\]
However, the exact null distribution of $D_n$ was not derived. Therefore, Fujikoshi et al. (2004) proposed an approximate test procedure based on the asymptotic normality

$$
\sqrt{n} \frac{D_n - 1}{\sqrt{2 \hat{a}_2 / (\hat{c} \hat{a}_1^2)}} \rightarrow \mathcal{N}(0, 1), \quad (1.1)
$$

under $H_0$, and the assumptions

\begin{align*}
(A1) & \quad n, p \to \infty \text{ with } \frac{p}{n} \to c \in (0, 1), \\
(A2) & \quad 0 < \lim_{p \to \infty} a_i = \lim_{p \to \infty} \frac{\text{tr} \Sigma^i}{p} < \infty, \quad i = 1, \cdots, 6.
\end{align*}

Here, $\hat{c} = p/n$ and

$$
\hat{a}_1 = \frac{\text{tr} S}{p}, \quad \hat{a}_2 = \frac{n^2}{p(n-1)(n-2)} \left( \text{tr} S^2 - \frac{(\text{tr} S)^2}{n} \right)
$$

are the unbiased and consistent estimators of $a_1$ and $a_2$. Based on the asymptotic normality (1.1), the approximate Dempster’s test rejects $H_0$ if

$$
\sqrt{n} \frac{D_n - 1}{\sqrt{2 \hat{a}_2 / (\hat{c} \hat{a}_1^2)}} \geq z(\alpha),
$$

where the selected significance level is $\alpha$ and the threshold is denoted by $z(\alpha)$.

Hotelling’s test is powerful when the dimension of the data set is sufficiently small as compared with the sample size. However, even when $p \leq n$, Hotelling’s test is known to perform poorly if $p$ is close to $n$. This behavior was demonstrated by Bai and Saranadasa (1996), who studied the performance of Hotelling’s test under $p, n \to \infty$ with $p/n \to c < 1$, and showed that the asymptotic power of the test is decreased for large values of $c$. In a comparison of the two tests it can be seen that the power of Hotelling’s test increases much more slowly than that of Dempster’s test, as the non-central parameter increases when $c$ is close to one. The conclusion drawn from these results is that the comparative merits and demerits of Hotelling’s test and Dempster’s test vary according to the non-central parameter and $c$. The contribution of this paper is that a new statistic that possesses both these properties
asymptotically is proposed; that is, we propose the following statistic which is a weighted average of the $T^2$ statistic and D-statistic:

$$T(\rho) = \rho \sqrt{n} \left( \frac{T^2}{n} - \frac{p}{n - p} \right) + (1 - \rho) \sqrt{n}(D_n - 1),$$

where $\rho \in [0, 1]$. Then, the method used for determining the weight $\rho$ is an important issue. In our study, the weight is determined on the basis of the maximum local asymptotic power. The only difficulty is that the true optimal weight depends on the true mean vector, which is unobservable. One method for erasing the information of the true mean vector is to restrict the parameter space that induces local asymptotic equivalence between Hotelling’s test and Dempster’s test. This parameter space results in a situation where it is not easy to determine which test may be used. Further, the local asymptotic power on this parameter space is evaluated under the condition of a high dimensional framework, that is, the sample size and the dimension simultaneously go to infinity under the condition that $p/n \rightarrow c \in (0, 1)$. Large sample asymptotics assume that the dimension $p$ is finite and fixed, while the sample size $N$ grows indefinitely. This asymptotic yields a bad approximation in many real-world situations where the dimension $p$ is of the same order as the sample size $N$. However, it is well known that the high dimensional approximation performs well in not only a high dimensional situation, but also a large sample situation. This fact explains why high dimensional approximation is used. We maximize the local asymptotic power and find the optimal weight as a function of $\Sigma$; then, we obtain its consistent estimator. We also show that replacing the true optimal weight with a consistent estimator makes no difference asymptotically. In addition, when the parameter constraint is removed, our statistic is comparable to Hotelling’s test and Dempster’s test. Our test outperforms both tests in terms of local asymptotic power; that is, we can guarantee that our test does not have the lowest local asymptotic power among the three tests.

This paper is organized as follows. In Section 2, we introduce the asymptotic property of Hotelling’s test and Dempster’s test, and propose the asymptotically
optimal weight $\rho$ for $T(\rho)$ to address the situation where their local asymptotic powers are equal. In addition, we give the sufficient condition of a parameter space that allows our test to outperform Dempster’s test and Hotelling’s test in terms of local asymptotic power. In Section 3, we investigate the performances of our test through numerical studies. The conclusion of our study is summarized in Section 4. Some preliminary results and proofs are given in the appendix.

2 Description of the weighted averaging test statistic and its asymptotic properties

In this section, we propose a weighted averaging test statistic of $D$-statistic and $T^2$-statistic. We consider the class of weighted averaging test statistics

$$
T = \left\{ T(\rho) \left| T(\rho) = \rho \sqrt{n} \left( \frac{T^2}{n} - \frac{p}{N-p} \right) + (1-\rho) \sqrt{n}(D_n - 1), \rho \in [0, 1] \right. \right\}.
$$

We note that class $T$ includes the $D$-statistic ($\rho = 0$) and $T^2$-statistic ($\rho = 1$).

First, we propose the optimal weight on the parameter space such that determining the appropriate use of Dempster’s test and Hotelling’s test is difficult, that is, where Dempster’s test and Hotelling’s test have same local asymptotic power. In order to derive the local asymptotic power of a test statistic belonging to class $T$, we assume the conditions (A1), (A2), and

\begin{equation}
(A3) \quad 0 < \lim_{n,p \to \infty} n^{1/2} \Delta^2 < \infty, \quad 0 < \lim_{n,p \to \infty} n^{1/2} \Delta_I^2 < \infty, \quad 0 < \lim_{n,p \to \infty} n^{1/2} \Delta_{\Sigma}^2 < \infty,
\end{equation}

where

$$
\Delta^2 = (\mu - \mu_0)'\Sigma^{-1}(\mu - \mu_0), \quad \Delta_I^2 = (\mu - \mu_0)'(\mu - \mu_0), \quad \Delta_{\Sigma}^2 = (\mu - \mu_0)'\Sigma(\mu - \mu_0).
$$

The following lemma provides the asymptotic normality of $T(\rho)$ under local alternatives.

**Lemma 2.1.** Assume conditions (A1), (A2), and (A3). For any $\rho \in [0, 1]$, it holds that

$$
\frac{1}{\sigma(\rho, \hat{c}, \hat{a}_1, \hat{a}_2)} \left[ T(\rho) - \sqrt{n} \left\{ \rho \frac{\Delta^2}{1-c} + (1-\rho) \frac{\Delta_I^2}{ca_1} \right\} \right] \xrightarrow{d} N(0, 1),
$$

where $\sigma(\rho, \hat{c}, \hat{a}_1, \hat{a}_2)$ is the standard deviation of $T(\rho)$.
where
\[ \sigma^2(\rho, c, a_1, a_2) = \rho^2 \frac{2c}{(1-c)^3} + (1-\rho)^2 \frac{2a_2}{ca_1^2} + 2\rho(1-\rho) \frac{2}{1-c}. \]

(Proof) See, Appendix A.2.

Due to Lemma 2.1, the test based on \( T(\rho) \) rejects \( H_0 \) if
\[ \frac{T(\rho)}{\sigma(\rho, \hat{c}, \hat{a}_1, \hat{a}_2)} \geq z(\alpha). \]  
(2.1)

Now, consider the power for testing procedure (2.1). Let
\[ \delta(\rho| \Delta^2, \Delta^2_I, a_1, a_2) = \frac{\rho \Delta^2/(1-c) + \sqrt{n}(1-\rho)\Delta_I^2/(a_1c)}{\sqrt{2\rho^2c(1-c)^{-3} + 2(1-\rho)^2a_2/(a_1^2c) + 4\rho(1-\rho)(1-c)^{-1}}} \]

By using asymptotic normality of \( T(\rho) \) (Lemma 2.1), we have
\[ \Pr \left( \frac{T(\rho)}{\sigma(\rho, \hat{c}, \hat{a}_1, \hat{a}_2)} \geq z(\alpha) \right) \to \Phi \left( \sqrt{n} \delta(\rho| \Delta^2, \Delta^2_I, a_1, a_2) - z(\alpha) \right) \]  
(2.2)
under conditions (A1), (A2), and (A3).

Our objective is to determine the weight \( \rho \) that maximizes the local asymptotic power (2.2). Specifically, we assume a restricted parameter space such that the local asymptotic power of Hotelling’s test and of Dempster’s test are asymptotically equivalent. By using Lemma 2.1, under assumptions (A1)-(A3) and
\[ (\mu, \Sigma) \in \Omega_0 = \left\{ (\mu, \Sigma) \mid \frac{\Delta^2}{\Delta_I^2} = \frac{1}{\sqrt{(1-c)a_2}} \right\}, \]
it holds that
\[ \Pr \left( \frac{N-p}{np} T^2 \geq F_{p,N-p}(\alpha) \right) - \Pr \left( \sqrt{n} \frac{D_n - 1}{\sqrt{2a_2/(\hat{c}^2\hat{a}_1)}} \geq z(\alpha) \right) \to 0; \]  
(2.3)
this is, their powers are asymptotically equivalent when \( (\mu, \Sigma) \in \Omega_0 \). In the following proposition, we obtain the optimal weight on the parameter space \( \Omega_0 \).

**Proposition 2.1.** Assume the conditions \( (\mu, \Sigma) \in \Omega_0 \) and (A1)-(A3). Then, the statistics
\[ T(\rho^*(c, a_1, a_2)) = \rho^*(c, a_1, a_2) \sqrt{n} \left( \frac{T^2}{n} - \frac{p}{N-p} \right) + (1-\rho^*(c, a_1, a_2)) \sqrt{n}(D_n - 1) \]  
(2.4)
has maximum local asymptotic power

\[ \Phi \left( \frac{\sqrt{n(1-c)\Delta^2}}{\sqrt{c(a_1/\sqrt{a_2(1-c)} + 1)}} \right) \]

in class \( \mathcal{T} \). Here,

\[ \rho^*(c, a_1, a_2) = \left( \frac{a_1c\sqrt{a_2(1-c)}}{a_2(1-c)^2} + 1 \right)^{-1}. \]

(Proof) See, Appendix A.3.

In practice, it is necessary to replace the unknown parameters \( a_1 \) and \( a_2 \) in (2.4) with their consistent estimators \( \hat{a}_1 \) and \( \hat{a}_2 \). Nishiyama et al. (2013) provided the following unbiased and consistent estimators of \( a_1, a_2, a_3 \):

\[ \hat{a}_1 = \frac{\text{tr}S}{p}, \]

\[ \hat{a}_2 = \frac{n^2}{p(n+2)(n-1)} \left\{ \text{tr}S^2 - \frac{(\text{tr}S)^2}{n} \right\}, \]

\[ \hat{a}_3 = \frac{n^2}{(n+4)(n+2)(n-1)(n-2)p} \{n^2\text{tr}S^3 - 3n\text{tr}S^2\text{tr}S + 2(\text{tr}S)^3\}. \]

In this study, \( \hat{a}_3 \) is used (2.8). The following lemma shows the asymptotic properties of these estimators.

**Lemma 2.2.** Assume conditions (A1) and (A2). Then, it holds that

\[ \hat{a}_i = a_i + O_p(n^{-1}), \ i = 1, 2, 3. \]

(Proof) See, Hyodo et al. (2014).

Using Lemma 2.2, we propose an adapted version of (2.4):

\[ T(\rho^*(\hat{c}, \hat{a}_1, \hat{a}_2)) = \rho^*(\hat{c}, \hat{a}_1, \hat{a}_2)\sqrt{n} \left( \frac{T^2}{n} - \frac{p}{N-p} \right) + (1 - \rho^*(\hat{c}, \hat{a}_1, \hat{a}_2))\sqrt{n}(D_n - 1). (2.5) \]

Further, we denote \( \rho^*(\hat{c}, \hat{a}_1, \hat{a}_2) \) by \( \hat{\rho}^* \) and \( \rho^*(c, a_1, a_2) \) simply by \( \rho^* \).

According to the asymptotic normality of \( T(\hat{\rho}^*) \) under the null hypothesis \( H_0 \), we propose the test rejects \( H_0 \) if

\[ \frac{T(\hat{\rho}^*)}{\sigma(\hat{\rho}^*, \hat{c}, \hat{a}_1, \hat{a}_2)} \geq z(\alpha). \] (2.6)
Since $\hat{\rho}^* = \rho^* + o_p(1)$, we obtain the asymptotic power of (2.6) as

$$\Pr\left( \frac{T(\hat{\rho}^*)}{\sigma(\hat{\rho}^*, \hat{c}, \hat{a}_1, \hat{a}_2)} \ge z(\alpha) \right) \to \Phi \left( \frac{\sqrt{n(1-c)\Delta^2}}{\sqrt{c(a_1/\sqrt{a_2(1-c) + 1})}} \right).$$

Thus, the power of $T(\hat{\rho}^*)$ is asymptotically equivalent to that of $T(\rho^*)$.

From Proposition 2.1 and the above results, we derive the asymptotic null distribution of the proposed test statistic $T(\hat{\rho}^*)$; the improved estimator of the critical point of our test is derived by using the Cornish-Fisher expansion. The following proposition provides the asymptotic null distribution of $T(\hat{\rho}^*)/\sigma(\hat{\rho}^*, \hat{c}, \hat{a}_1, \hat{a}_2)$.

**Proposition 2.2.** Assume assumptions (A1) and (A2) and $H_0$. Then, it holds that

$$\Pr\left( \frac{T(\hat{\rho}^*)}{\sigma(\hat{\rho}^*, \hat{c}, \hat{a}_1, \hat{a}_2)} \le x \right) = \Phi(x) - \frac{\phi(x)}{\sqrt{n}} \left\{ \frac{b_1(c)(it)}{\sigma(\rho^*, c, a_1, a_2)} + \frac{b_3(\rho^*, c, a_1, a_2, a_3)(it)^3}{\sigma^3(\rho^*, c, a_1, a_2)} (x^2 - 1) \right\} + o\left( n^{-1/2} \right),$$

where

$$b_1(c) = \nu_1(c), \quad b_3(c, a_1, a_2, a_3) = \frac{\nu_3(c, a_1, a_2, a_3)}{6} - \frac{\nu_1(c)}{2}.\tag{2.7}$$

Here,

$$\nu_1(c) = \frac{2\rho^* c}{(1-c)^2},$$

$$\nu_3(c, a_1, a_2, a_3) = \frac{4\rho^* c(5c + 2)}{(1-c)^5} + \frac{24\rho^* (1-\rho^*)(c+1)}{(1-c)^3} + \frac{12\rho^* (1-\rho^*)^2 a_2(2-c)}{a_1^2(1-c)^2c}$$

$$+ \frac{8(1-\rho^*)^3 a_3}{a_1^3 c^2}.$$

**(Proof)** See, Appendix A.4.

Let $x(\alpha)$ be the upper $100\alpha$ percentile of the statistic $T(\hat{\rho}^*)/\sigma(\hat{\rho}^*, \hat{c}, \hat{a}_1, \hat{a}_2)$. In addition, the Cornish-Fisher expansion of the true upper $100\alpha$ percentile is obtained by

$$x(\alpha) \approx z(\alpha) + \frac{1}{\sqrt{n}} \left\{ \frac{a_1(c)}{\sigma(\rho^*, c, a_1, a_2)} + \frac{a_3(\rho^*, c, a_1, a_2, a_3)}{\sigma^3(\rho^*, c, a_1, a_2)} (z(\alpha)^2 - 1) \right\}. \tag{2.7}$$
In practice, it is necessary to replace the unknown parameters $a_1$, $a_2$, and $a_3$ in (2.7) with their consistent estimators $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$. We replace the $a_i$’s in (2.7) with their unbiased and consistent estimator $\hat{a}_i$, and propose an approximate upper 100\% percentile

$$\hat{x}(\alpha) = z(\alpha) + \frac{1}{\sqrt{n}} \left\{ \frac{a_1(\hat{c})}{\sigma(\hat{\rho}^*, \hat{c}, \hat{a}_1, \hat{a}_2)} + \frac{a_3(\hat{\rho}^*, \hat{c}, \hat{a}_1, \hat{a}_2, \hat{a}_3)}{\sigma^3(\hat{\rho}^*, \hat{c}, \hat{a}_1, \hat{a}_2)} (z(\alpha)^2 - 1) \right\}. \quad (2.8)$$

Applying (2.8), the test rejects $H_0$ if

$$\frac{T(\hat{\rho}^*)}{\sigma(\hat{\rho}^*, \hat{c}, \hat{a}_1, \hat{a}_2)} \geq \hat{x}(\alpha). \quad (2.9)$$

Finally, we compare Hotelling’s test and Dempster’s test with our test (2.6) (or (2.9)). In the following proposition, we give the sufficient condition that allows our test to have the highest local asymptotic power among the three tests. Furthermore, even when a sufficient condition does not hold, we can guarantee that our test does not have the lowest local asymptotic power among the three tests.

**Proposition 2.3.** Assume (A1), (A2), and (A3). The proposed test (2.6) (or (2.9)) has the highest local asymptotic power among the three tests under the condition

$$(C1) \frac{\Delta^2_{2, \alpha}}{\Delta^2_{1, \alpha}} \in \left[ \frac{\sqrt{2} \left( 1 + a_1 \sqrt{(1-c)/a_2} \right)^{1/2} - 1}{\sqrt{a_2(1-c)}} , \frac{\sqrt{2} \left( 1 + a_1 \sqrt{(1-c)/a_2} \right)^{1/2} - 1}{\sqrt{a_2(1-c)}} \right].$$

Furthermore, the local asymptotic power of our test (2.6) (or (2.9)) is second highest among the three tests when condition (C1) does not hold.

**Proof** See, Appendix A.5.

### 3 Numerical results

In this section, we investigate the finite sample behavior of the proposed test and compare it with the $T^2$ test and Dempster’s test. To compare the three tests, we need
to define the Attained Significance Level (ASL) and the empirical powers. We draw an independent sample of size \( N = 40i + p \), where \( i = 1, \ldots, 10 \) valid \( p \)-dimensional normal distributions \( \mathcal{N}_p(\mu, \Sigma) \) under the null hypothesis \( H_0 : \mu = 0 \). Further, we set the covariance structures \( \Sigma = (\eta^{i-j}) \), where \( \eta = 0.2, 0.4, 0.6 \), respectively. We replicate this \( r = 10^5 \) times, and, using \( T^2, D_n, \) and \( T(\hat{\rho}^*) \), calculate

\[
\begin{align*}
\text{ASL}_\alpha(T^2) & = \frac{\# \text{ of } ((N - p)/(np)T^2 > F_{p,N-p}(\alpha))}{r}, \\
\text{ASL}_\alpha(D_n) & = \frac{\# \text{ of } (\sqrt{n}(D_n - 1)/\sqrt{2\hat{a}_2/(\hat{c}\hat{a}_1^2)} > y(\alpha))}{r},
\end{align*}
\]

denoting the ASL of \( T^2, D_n, \) and \( T(\hat{\rho}^*) \), respectively. Here, \( y(\alpha) \) is the improved estimator of the critical point for \( D_n \), which was provided by Nishiyama et al. (2013) and defined as

\[
y(\alpha) = z(\alpha) + \frac{1}{\sqrt{p}}q_1(z(\alpha)) + \frac{1}{p}q_2(z(\alpha)) + \frac{1}{n}q_3(z(\alpha)),
\]

and

\[
\begin{align*}
q_1(z(\alpha)) & = \frac{\sqrt{2\hat{a}_3}}{3\sqrt{\hat{a}_2^3}}(z(\alpha)^2 - 1), \\
q_2(z(\alpha)) & = \frac{\hat{a}_4}{2\hat{a}_2^2}z(\alpha)(z(\alpha)^2 - 3) - \frac{2\hat{a}_3^2}{9\hat{a}_2^4}z(\alpha)(2z(\alpha)^2 - 5), \\
q_3(z(\alpha)) & = \frac{z(\alpha)}{2},
\end{align*}
\]

where \( \hat{a}_4 \) is the consistent estimator of \( a_4 \). For details, see Nishiyama et al. (2013).

The attained significance levels specified by the selection of set \( (p, \eta) \) are given in Tables 1-6. Since Hotelling’s test is an exact test under the multivariate normality assumptions, we focus on Dempster’s test and our test. Tables 1-6 show that the attained significance levels of both tests approximate the nominal level \( \alpha \) reasonably well in all cases. In addition, we note that, according to these results, our test has a tendency to become conservative. To compute the empirical powers, we select

\[
\mu = \left( \frac{2}{n^{1/4}\sqrt{p}}, \ldots, \frac{2}{n^{1/4}\sqrt{p}} \right).
\]
Using the same number of replications as above, we draw independent samples of size $N$ from $\mathcal{N}_p(\mu, \Sigma)$, and calculate the empirical power as

$$EP_{\alpha}(T^2) = \frac{\# \text{ of } ((N - p)/(np)T^2 > F_{p,N-p}(\alpha))}{r},$$

$$EP_{\alpha}(D_n) = \frac{\# \text{ of } \left(\sqrt{n}(D_n - 1)/\sqrt{2\hat{a}_2/(\hat{c}\hat{a}_1^2)} > y(\alpha)\right)}{r},$$

and

$$EP_{\alpha}(T(\hat{\rho}^*)) = \frac{\# \text{ of } (T(\hat{\rho}^*)/\sigma(\hat{\rho}^*, \hat{c}, \hat{a}_1, \hat{a}_2) > \hat{x}(\alpha))}{r}.$$

The results for the empirical power are summarized in Tables 7 to 12, where bold face marks the highest power among the three tests. These tables show that, while our test statistic has the highest power among the three tests in many cases, the other two tests have the highest power in some cases. Specifically, among the three tests, the performance of Dempster’s test is comparatively good when $N$ is small, and that of Hotelling’s test is comparatively good when $N$ is large. Although the power of our test is not always the highest, it is close to being so. In other words, the weight behaves such that our statistic is comparable with whichever statistic has the relatively higher power, the D-statistic or the $T^2$-statistic.

4 Conclusion

We proposed a new test statistic for the one-sample location test in high-dimensional data. Our proposed test statistic uses the weighted averaging of Hotelling’s $T^2$ statistic and Dempster’s statistic. Some asymptotic properties of this statistic were also shown. The important issue is that the local asymptotic power of our test does not become lower than that of Hotelling’s test and Dempster’s test. In addition, simulations indicate that the newly derived test statistic is relatively stable as compared with the D-statistic and $T^2$-statistic. When the difference in the power of the D-statistic and the $T^2$-statistic is large, it can be seen that our statistic is comparable with whichever statistic has the relatively higher power, the D-statistic or the $T^2$-statistic.
$T^2$-statistic. In conclusion, we recommend that our test statistic be applied instead of the D-statistic and $T^2$-statistic over a wide range.

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Appendix A.

A.1. Some preliminary result

**Lemma A. 1** (The central limit theorem for quadratic forms). Let $z = (z_1, \ldots, z_p)'$ be distributed $p$-dimensional standardized normal random variable and $\Omega = \text{diag}(\omega_1, \ldots, \omega_p)$ be arbitrary $p \times p$ non random diagonal matrix. Suppose that $T = z'\Omega z - \text{tr} \Omega$ and $\sigma_p^2 = 2\text{tr} \Omega^2$. Then, $T/\sigma_p \xrightarrow{d} \mathcal{N}(0,1)$ as $p \to \infty$ if the following condition is satisfied:

$$\frac{\text{tr} \Omega^4}{(\text{tr} \Omega^2)^2} \to 0 \text{ as } p \to \infty. \quad (A.1)$$

**(Proof)**

It can be expressed that

$$T = (z'\Omega z - \text{tr} \Omega) = \sum_{i=1}^n (\omega_i z_i^2 - \omega_i).$$

Let $Y_i = \omega_i z_i^2 - \omega_i$, $i = 1, 2, \ldots, p$. Then $T = \sum_{i=1}^n Y_i$ and the moment of $Y_i$ is calculated by

$$E[Y_i^2] = 2\omega_i^2, \quad E[Y_i^4] = 60\omega_i^4.$$ 

We wish to give sufficient conditions that ensure $T/\sigma_p \xrightarrow{d} \mathcal{N}(0,1)$. For now, we check only the Lyapunov Condition. The Lyapunov Condition for sequences $\{Y_i\}_{i=1}^p$ states that there exists $\eta \in \mathbb{N}$ such that

$$\frac{\sum_{i=1}^p E[Y_i^{2+\eta}]}{\sigma_p^{2+\eta}} \to 0 \text{ as } p \to \infty.$$ 

Based on the first and second moments of $Y_i$, we can calculate

$$\sum_{i=1}^p E[Y_i^2] = 2\text{tr} \Omega^2 (\equiv \sigma_p^2), \quad \sum_{i=1}^p E[Y_i^4] = 60\text{tr} \Omega^4. \quad (A.2)$$

From (A.2), under the condition (A.1),

$$\frac{\sum_{i=1}^p E[Y_i^4]}{\sigma_p^4} = \frac{60\text{tr} \Omega^4}{(2\text{tr} \Omega^2)^2} \to 0$$
as \( p \to \infty \). This result show that the condition (A.1) implie Lyapunov Condition. Thus, the Lyapunov Condition also implies \( T/\sigma_p \xrightarrow{d} \mathcal{N}(0,1) \). \( \square \)

**Lemma A. 2** (Some moments for quadratic forms). Let \( z \) be distributed \( p \)-dimensi onal standard normal random variable and \( A_i, \ i = 1, 2, 3 \) be arbitrary \( p \times p \) diagonal matrix. Then it holds that

(i) \( E[z^tA_1z] = \text{tr} A_1 \),

(ii) \( E[z^tA_1zz^tA_2z] = 2\text{tr} A_1A_2 + \text{tr} A_1\text{tr} A_2 \),

(iii) \( E[z^tA_1zz^tA_2zz^tA_3z] = \text{tr} A_1\text{tr} A_2\text{tr} A_3 + 2\text{tr} A_3\text{tr} A_1A_2 \\
+ 2\text{tr} A_2\text{tr} A_1A_3 + 2\text{tr} A_1\text{tr} A_2A_3 + 8\text{tr} A_1A_2A_3 \).

(Proof) See e.g. Mathai et al. (1995).

**A.2. Proof of Lemma 2.1.**

At first, we expand \( T^2 \) stochastically. Suppose that \( \Gamma = (\gamma_1, \cdots, \gamma_p) \) is an orthog- onal matrix such that \( \Sigma = \Gamma \Lambda \Gamma^t \), where \( \Lambda = \text{diag} (\lambda_1, \ldots, \lambda_p) \) and, for \( i = 1, \ldots, p \), \( \lambda_i \) is \( i \)-th eigenvalue of \( \Sigma \). Define the random variables \( u \) and \( W \) by

\[
 u = \sqrt{N}\Gamma'(\Sigma^{-1/2}(x - \mu_0) - \tau), \quad W = n\Gamma\Sigma^{-1/2}S\Sigma^{-1/2}\Gamma.
\]

It is seen that \( u \) and \( W \) are mutually independently distributed as \( u \sim \mathcal{N}_p(0, I_p) \), respectively, where \( \tau = \Sigma^{-1/2}(\mu - \mu_0) \). Then the statistic \( T^2/n \) is denoted by

\[
 (\Gamma u + \sqrt{N}\tau)^tW^{-1}(\Gamma u + \sqrt{N}\tau) \overset{d}{=} \frac{(\Gamma u + \sqrt{N}\tau)^t(\Gamma u + \sqrt{N}\tau)}{v'v}.
\]

where \( v \sim \mathcal{N}_{N-p}(0, I_{N-p}) \), and \( u \) and \( v \) are mutually independent. Then the the statistic \( T^2/n \) can be expanded as

\[
 \frac{T^2}{\sqrt{n}} \overset{d}{=} \sqrt{n}(u'u - p) + 2\sqrt{N}\tau'\Gamma u + N\tau'\tau + p \left( 1 + \frac{v'v - (N-p)}{N-p} \right)^{-1} \\
= \sqrt{n} \left( \frac{N\Delta^2}{N-p} + \frac{p}{N-p} \right) + \frac{u'u - p}{\sqrt{n}(1-c)} - \frac{c}{1-c} \frac{v'v - n(1-c)}{\sqrt{n}(1-c)} + o_p(1).
\]
Thus, we have
\[
\sqrt{n} \left\{ \frac{T^2}{n} - \left( \frac{N \Delta^2}{N - p} + \frac{p}{N - p} \right) \right\} \xrightarrow{d} \frac{u^t u - p}{\sqrt{n}(1 - c)} - \frac{c}{(1 - c)^2} \frac{v^t v - n(1 - c)}{\sqrt{n}} + o_p(1). \tag{A.3}
\]

Next, we expand $D_n$ stochastically as following
\[
N(\bar{x} - \mu_0)'(\bar{x} - \mu_0) \overset{d}{=} \frac{u^t \Lambda u + N \tau^t \Sigma \tau}{\text{tr} \Sigma} = \left( 1 + \frac{\Delta_i^2}{ca_1} \right) + \frac{u^t \Lambda u - pa_1}{pa_1} + o_p(n^{-1/2}).
\]
Thus, we can obtain
\[
\sqrt{n} \left\{ D - \left( 1 + \frac{\Delta_i^2}{ca_1} \right) \right\} \overset{d}{=} \frac{u^t \Lambda u - pa_1}{\sqrt{nca_1}} + o_p(1). \tag{A.4}
\]
From (A.3) and (A.4), we expand $T(\hat{\rho}^*)$ stochastically as following
\[
T(\rho) = \rho \left( \frac{u^t u - p}{\sqrt{n}(1 - c)} - \frac{c}{(1 - c)^2} \frac{v^t v - n(1 - c)}{\sqrt{n}} \right) + (1 - \rho) \frac{u^t \Lambda u - pa_1}{\sqrt{nca_1}} + o_p(1) \\
= u^t \left( \frac{\rho}{\sqrt{n}(1 - c)} I_p + \frac{1 - \rho}{\sqrt{nca_1}} \Lambda \right) u - \text{tr} \left( \frac{\rho}{\sqrt{n}(1 - c)} I_p + \frac{1 - \rho}{\sqrt{nca_1}} \Lambda \right) \\
- \left\{ v^t \left( \frac{c \rho}{\sqrt{n}(1 - c)^2} I_{N-p} \right) v - \text{tr} \left( \frac{c \rho}{\sqrt{n}(1 - c)^2} I_{N-p} \right) \right\} + o_p(1).
\]
By using Lemma A.1 and the independency $u$ and $v$, we obtain Lemma 2.1.

\section*{A.3. Proof of Proposition 2.1.}
Assume that $(\mu, \Sigma) \in \Omega_0$ i.e. $\Delta_i^2/\Delta^2 = \sqrt{(1 - c)a_2}$. By using Lemma 2.1, we have
\[
\Pr \left( \frac{T(\rho)}{\sigma(\rho, \hat{c}, \hat{a}_1, \hat{a}_2)} \geq z(\alpha) \right) \rightarrow \Phi \left( \sqrt{n} f(\rho) \Delta^2 - z(\alpha) \right),
\]
where
\[
f(\rho) = \frac{\rho/(1 - c) + (1 - \rho)\sqrt{a_2(1 - c)/(a_1 c)}}{\sqrt{2\rho^2c/(1 - c)^3 + 2(1 - \rho)^2a_2/(a_1^2c) + 4\rho(1 - \rho)/(1 - c)}}.
\]
To obtain the optimal $T(\rho)$ which maximize the local asymptotic power function, we consider the optimization problem: $\max_{\rho \in [0,1]} f(\rho)$, because $\Phi(\cdot)$ is monotonically
increasing on $\mathbb{R}$. We find $f'(\rho)$ and set it equal to zero. Solving $f'(\rho) = 0$ for $\rho$ gives us

$$\rho^*(c, a_1, a_2) = \left(\frac{a_1 c \sqrt{a_2 (1 - c)}}{a_2 (1 - c)^2} + 1\right)^{-1}.$$ 

The second derivative is given by

$$f''(\rho) = \Delta^2 \left\{ (1 - c)^2 \left( \sqrt{a_2 (1 - c)} - a_2 / a_1 \right) + a_1 c (1 - c) - c \sqrt{a_2 (1 - c)} \right\}^4 / \sqrt{2 a_1 (1 - c)^3 c} \left\{ c \ell \left( a_1^2 (1 - c) \ell + \left( \sqrt{a_2 (1 - c)} - a_1 (1 - c) \right)^2 \right) \right\}^{3/2} < 0,$$

so $f(\rho^*(c, a_1, a_2))$ is a local maximum value. Here, $\ell = a_2 / a_1^2 + c - 1$. Since $f'(\lambda)$ is monotone decreasing function on $[0, 1]$, we can get $\rho^*(c, a_1, a_2)$ as the solution to $\max_{\rho \in [0, 1]} f(\rho)$. Thus, the optimal linear combination is given by

$$T(\rho^*(c, a_1, a_2)) = \rho^*(c, a_1, a_2) \sqrt{n} \left( \frac{T^2}{n} - \frac{p}{N - p} \right) + (1 - \rho^*(c, a_1, a_2)) \sqrt{n} (D_n - 1)$$

and its asymptotic power is

$$\Pr \left( \frac{T(\rho^*(c, a_1, a_2))}{\sigma(\rho^*(c, a_1, a_2), c, a_1, a_2)} \geq z(\alpha) \right) \rightarrow \Phi \left( \frac{\sqrt{n (1 - c)} \Delta^2}{c (a_1 / \sqrt{a_2 (1 - c)^{1/2}} + 1) - z(\alpha)} \right). \quad \Box$$

**A.4. Proof of Proposition 2.2.**

Define the variables

$$h_1 = \frac{u' u - p}{\sqrt{2p}}, h_2 = \frac{v' v - (N - p)}{\sqrt{2(N - p)}}, h_3 = \frac{u' \Lambda u - p a_1}{\sqrt{2a_2 p}}, h_4 = \frac{\sqrt{n p} (\hat{a}_1 - a_1)}{\sqrt{2a_2}}.$$ 

Since $\hat{a}_i = a_i + o_p(n^{-1/2})$ for $i = 1, 2$, it holds that $\hat{\rho}^* = \rho^* + o_p(n^{-1/2})$. Thus we obtain

$$T(\hat{\rho}^*) = \rho^* \left( T_1 + \frac{T_2}{\sqrt{n}} \right) + (1 - \rho^*) \left( D_1 + \frac{D_2}{\sqrt{n}} \right) + o_p(n^{-1/2}),$$

where

$$T_1 = \frac{c}{1 - c} \left( \frac{\sqrt{2} h_1}{\sqrt{c} - \sqrt{1 - c}} \right), T_2 = \frac{c}{1 - c} \left( \frac{2h_2^2}{1 - c - \sqrt{1 - c} \sqrt{c}} \right),$$

$$D_1 = \frac{\sqrt{2a_2 h_3}}{a_1 \sqrt{c}}, D_2 = -\frac{\sqrt{2a_2 h_4}}{a_1 \sqrt{c}}.$$
Then the moment of order \(i (= 1, 2, 3)\) of \(T(\rho^*)\) denotes

\[
E[T(\rho^*)] = E \left[ \frac{\sqrt{2} c \rho^* h_1}{1 - c} - \frac{\sqrt{2} c \rho^* h_2}{(1 - c)^{3/2}} + \frac{\sqrt{2} a_2 (1 - \rho^*) h_3}{a_1 \sqrt{c}} \right] + o(n^{-1/2}),
\]

\[
E[T^2(\rho^*)] = E \left[ \frac{2 \rho^* c^2 h_2^2}{(1 - c)^2} + \frac{2 \rho^* c^2 h_2^2}{(1 - c)^3} + \frac{2 (1 - \rho^*)^2 a_2 h_3^2}{a_1^2 c} - \frac{4 \rho^*^2 c^3/2 h_1 h_2}{(1 - c)^{5/2}} \right] + o(n^{-1/2}),
\]

\[
E[T^3(\rho^*)] = E \left[ \frac{2 \sqrt{2} \rho^* c^{3/2} h_1^3}{(1 - c)^3} - \frac{2 \sqrt{2} \rho^* c^{3/2} h_2^3}{(1 - c)^4} + \frac{6 \sqrt{2} a_2 (1 - \rho^*)^3 h_3^2}{a_1^2 (1 - c)^2} \right] + o(n^{-1/2}).
\]

By using Lemma A.2, we have

\[
E[h_1] = 0, E[h_2] = 0, E[h_3] = 0, E[h_4] = 0, E[h_1^2] = 1, E[h_2^2] = 1, E[h_3^2] = 1, E[h_1^2] = 1,
\]

\[
E[h_3^2] = \frac{2 \sqrt{2}}{\sqrt{p}}, E[h_3^2] = \frac{2 \sqrt{2}}{\sqrt{N - p}}, E[h_3^2] = \frac{2 \sqrt{2} \text{tr} \Sigma^3}{(\text{tr} \Sigma^2)^{3/2}}, E[h_3^2] = \frac{2 \sqrt{2} \text{tr} \Sigma}{p \sqrt{\text{tr} \Sigma^2}},
\]

\[
E[h_1 h_3^2] = \frac{2 \sqrt{2}}{\sqrt{p}}.
\]

Hence, the moments can be calculated by

\[
E[T(\rho^*)] = \frac{\nu_1(c)}{\sqrt{n}} + o\left(n^{-1/2}\right),
\]

\[
E[T^2(\rho^*)] = \sigma^2(c, a_1, a_2) + o\left(n^{-1/2}\right),
\]

\[
E[T^3(\rho^*)] = \frac{\nu_3(c, a_1, a_2, a_3)}{\sqrt{n}} + o\left(n^{-1/2}\right).
\]

The relationship between the first three moments and cumulants, obtained by extracting coefficients from the expansion, is as follows:

\[
\kappa_1(T(\rho^*)) = E[T(\rho^*)],
\]

\[
\kappa_2(T^2(\rho^*)) = E[T^2(\rho^*)] - (E[T(\rho^*)])^2,
\]

\[
\kappa_3(T^3(\rho^*)) = E[T^3(\rho^*)] - 3E[T^2(\rho^*)]E[T(\rho^*)] + 2(E[T(\rho^*)])^3.
\]
From the above three relationships (A.8)-(A.10) and (A.5)-(A.7), the first three cumulants of $T(\rho^*)$ are obtained by

$$
\kappa_1(T(\rho^*)) = \frac{1}{\sqrt{n}} b_1(c) + o\left(n^{-1/2}\right),
$$

$$
\kappa_2(T^2(\rho^*)) = \sigma^2(c, a_1, a_2) + o\left(n^{-1/2}\right),
$$

$$
\kappa_3(T^3(\rho^*)) = \frac{6}{\sqrt{n}} b_3(c, a_1, a_2, a_3) + o\left(n^{-1/2}\right).
$$

Hence, the characteristic function of $T(\rho^*)/\sigma(\hat{c}, \hat{a}_1, \hat{a}_2)$ can be expressed as

$$
C(t) = \exp \left( \sum_{j=1}^{3} \frac{t^j}{j!} \frac{\kappa_j(T(\rho^*))}{\sigma^j(c, a_1, a_2)} \right) + o\left(n^{-1/2}\right)
$$

$$
= \exp \left( -\frac{t^2}{2} \right) \left[ 1 + \frac{1}{\sqrt{n}} \left\{ \frac{b_1(c)(it)}{\sigma(c, a_1, a_2)} + \frac{b_3(c)(it)^3}{\sigma^3(c, a_1, a_2)} \right\} \right] + o\left(n^{-1/2}\right).
$$

This result show Proposition 2.2. □

A.5. Proof of Proposition 2.3.

By using Lemma 2.1, we have

$$
\Pr \left( \frac{N - p}{p} T^2 \geq F_{p,N-p}(\alpha) \right) \to \Phi \left( \frac{\sqrt{n(1-c)\Delta^2_{\Sigma^{-1}}}}{\sqrt{2c}} - z(\alpha) \right), \quad (A.11)
$$

$$
\Pr \left( \frac{n^{-1/2} \hat{\Delta}}{\sigma(\hat{c}, \hat{a}_1, \hat{a}_2)} \geq z(\alpha) \right) \to \Phi \left( \frac{\sqrt{n\hat\Delta^2}}{2ca_2} - z(\alpha) \right), \quad (A.12)
$$

$$
\Pr \left( \frac{T(\hat{\rho}^*)}{\sigma(\hat{c}, \hat{a}_1, \hat{a}_2)} \geq z(\alpha) \right) \to \Phi \left( \frac{\sqrt{\frac{a_2(1-c)\Delta^2_{\Sigma^{-1}} + \hat\Delta^2}{2\{(1-c)\sqrt{a_1a_2} + a_2\}}} - z(\alpha)}{\sqrt{n}} \right). \quad (A.13)
$$

From (A.11) and (A.13), we have

$$
\Phi \left( \frac{\sqrt{n(1-c)\Delta^2_{\Sigma^{-1}}}}{\sqrt{2c}} - z(\alpha) \right) \leq \Phi \left( \frac{\sqrt{a_2(1-c)\Delta^2_{\Sigma^{-1}} + \hat\Delta^2}}{2\sqrt{\{(1-c)\sqrt{a_1a_2} + a_2\}}} - z(\alpha) \right)
$$

$$
\leq \frac{\sqrt{\frac{1-c}{2c} \Delta^2_{\Sigma^{-1}}}}{\sqrt{2}} \leq \frac{\sqrt{a_2(1-c)\Delta^2_{\Sigma^{-1}} + \hat\Delta^2}}{2\sqrt{\{(1-c)\sqrt{a_1a_2} + a_2\}}}.
$$

$$
\frac{\Delta^2_{\Sigma^{-1}}}{\Delta^2_{\hat{r}}} \leq \frac{\sqrt{2} \left( 1 + a_1\sqrt{(1-c)/a_2} \right)^{1/2} - 1}{\sqrt{a_2(1-c)}}.
$$

(A.14)
Therefore, condition (A.14) are necessary and sufficient conditions for the condition that the local asymptotic power of our test is superior to the local asymptotic power of $T^2$-test. Similarly, we have that

$$\Phi \left( \frac{\sqrt{n} \Delta_j^2}{\sqrt{2ca_2}} - z(\alpha) \right) \leq \Phi \left( \frac{\sqrt{n} a_2 (1-c) \Delta_{\Sigma^{-1}}^2 + \Delta_j^2}{2 \sqrt{(1-c)^{1/2} a_1 a_2^{1/2} + a_2} c} - z(\alpha) \right)$$

$$\iff \frac{\Delta_j^2}{\sqrt{2ca_2}} \leq \frac{\sqrt{a_2 (1-c) \Delta_{\Sigma^{-1}}^2 + \Delta_j^2}}{2 \sqrt{(1-c)^{1/2} a_1 a_2^{1/2} + a_2} c}$$

$$\iff \frac{\sqrt{2} \left( 1 + a_1 \sqrt{(1-c)/a_2} \right)^{1/2} - 1}{\sqrt{a_2 (1-c)}} \leq \frac{\Delta_{\Sigma^{-1}}^2}{\Delta_j^2} \tag{A.15}$$

by using (A.11) and (A.13). Therefore, condition (A.15) are necessary and sufficient conditions for the condition that the local asymptotic power of our test is superior to the local asymptotic power of D-test. Since $0 \leq \sqrt{2} \left( 1 + a_1 \sqrt{(1-c)/a_2} \right)^{1/2} - 1 \leq 1$, we have

$$\frac{\sqrt{2} \left( 1 + a_1 \sqrt{(1-c)/a_2} \right)^{1/2} - 1}{\sqrt{a_2 (1-c)}} \leq \left\{ \frac{\sqrt{2} \left( 1 + a_1 \sqrt{(1-c)/a_2} \right)^{1/2} - 1}{\sqrt{a_2 (1-c)}} \right\}^{-1} \tag{A.16}$$

Combining (A.14)-(A.16), we obtain

(i) \begin{align*}
\frac{\sqrt{2} \left( 1 + a_1 \sqrt{(1-c)/a_2} \right)^{1/2} - 1}{\sqrt{a_2 (1-c)}} & \leq \frac{\Delta_{\Sigma^{-1}}^2}{\Delta_j^2} \\
\iff \lim_{n,p \to \infty} \Pr \left( \frac{T(\hat{\rho}^*)}{\sigma(c, \hat{a}_1, \hat{a}_2)} \geq z(\alpha) \right) & > \text{max} \left\{ \lim_{n,p \to \infty} \Pr \left( \sqrt{n} \frac{D_n}{\sigma_2(\hat{a}_1, \hat{a}_2)} \geq z(\alpha) \right), \lim_{n,p \to \infty} \Pr \left( \frac{N - p T^2}{n} \geq F_{p,N-p}(\alpha) \right) \right\},
\end{align*}

(ii) \begin{align*}
\frac{\sqrt{2} \left( 1 + a_1 \sqrt{(1-c)/a_2} \right)^{1/2} - 1}{\sqrt{a_2 (1-c)}} & > \frac{\Delta_{\Sigma^{-1}}^2}{\Delta_j^2} \\
\iff \lim_{n,p \to \infty} \Pr \left( \sqrt{n} \frac{D_n}{\sigma_2(\hat{a}_1, \hat{a}_2)} \geq z(\alpha) \right) & > \lim_{n,p \to \infty} \Pr \left( \frac{T(\hat{\rho}^*)}{\sigma(c, \hat{a}_1, \hat{a}_2)} \geq z(\alpha) \right) \\
& > \lim_{n,p \to \infty} \Pr \left( \frac{N - p T^2}{n} \geq F_{p,N-p}(\alpha) \right).
\end{align*}
\[
\left\{ \frac{\sqrt{2} \left( 1 + a_1 \sqrt{(1-c)/a_2} \right)^{1/2} - 1}{\sqrt{a_2(1-c)}} \right\}^{-1} < \frac{\Delta^2_{\nu-1}}{\Delta^2_{I}}
\]

\[
\Leftrightarrow \lim_{n,p \to \infty} \Pr \left( \frac{N - p T^2}{p n} \geq F_{p, N-p} (\alpha) \right) > \lim_{n,p \to \infty} \Pr \left( \frac{T(\hat{\rho}^*)}{\sigma(c, \hat{a}_1, \hat{a}_2)} \geq z(\alpha) \right) > \lim_{n,p \to \infty} \Pr \left( \sqrt{n} \frac{D_n - 1}{\sigma^2(\hat{a}_1, \hat{a}_2)} \geq z(\alpha) \right).
\]

These results prove Proposition 2.3. \qed

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Table 1. ASL in the case of $(\eta, p) = (0.2, 50)$

| $\alpha \setminus N$ |  70  |  110 |  150 |  190 |  230 |
|---------------------|------|------|------|------|------|
| $T^2$               | 0.009| 0.010| 0.010| 0.010| 0.010|
| $D_n$               | 0.011| 0.010| 0.010| 0.010| 0.010|
| $T(\rho)$           | 0.009| 0.009| 0.009| 0.009| 0.009|
| $T^2$               | 0.046| 0.049| 0.050| 0.051| 0.050|
| $D_n$               | 0.051| 0.051| 0.050| 0.050| 0.050|
| $T(\rho)$           | 0.043| 0.046| 0.047| 0.048| 0.048|
| $T^2$               | 0.095| 0.099| 0.100| 0.101| 0.100|
| $D_n$               | 0.101| 0.100| 0.101| 0.100| 0.100|
| $T(\rho)$           | 0.089| 0.094| 0.096| 0.097| 0.097|

Table 2. ASL in the case of $(\eta, p) = (0.4, 50)$

| $\alpha \setminus N$ |  70  |  110 |  150 |  190 |  230 |
|---------------------|------|------|------|------|------|
| $T^2$               | 0.009| 0.010| 0.010| 0.009| 0.011|
| $D_n$               | 0.011| 0.010| 0.010| 0.010| 0.010|
| $T(\rho)$           | 0.009| 0.008| 0.009| 0.009| 0.010|
| $T^2$               | 0.046| 0.050| 0.050| 0.050| 0.051|
| $D_n$               | 0.052| 0.051| 0.050| 0.049| 0.051|
| $T(\rho)$           | 0.044| 0.045| 0.046| 0.046| 0.049|
| $T^2$               | 0.095| 0.100| 0.101| 0.099| 0.100|
| $D_n$               | 0.102| 0.102| 0.099| 0.099| 0.101|
| $T(\rho)$           | 0.091| 0.095| 0.095| 0.094| 0.097|

Table 3. ASL in the case of $(\eta, p) = (0.6, 50)$

| $\alpha \setminus N$ |  70  |  110 |  150 |  190 |  230 |
|---------------------|------|------|------|------|------|
| $T^2$               | 0.009| 0.010| 0.009| 0.011| 0.010|
| $D_n$               | 0.010| 0.010| 0.010| 0.009| 0.010|
| $T(\rho)$           | 0.008| 0.008| 0.009| 0.010| 0.010|
| $T^2$               | 0.046| 0.049| 0.048| 0.049| 0.050|
| $D_n$               | 0.051| 0.050| 0.050| 0.050| 0.049|
| $T(\rho)$           | 0.043| 0.045| 0.046| 0.047| 0.047|
| $T^2$               | 0.095| 0.100| 0.099| 0.100| 0.100|
| $D_n$               | 0.102| 0.100| 0.100| 0.100| 0.099|
| $T(\rho)$           | 0.089| 0.093| 0.096| 0.095| 0.095|
Table 4. ASL in the case of $(\eta, p) = (0.2, 100)$

| $\alpha \setminus N$ | 120 | 160 | 200 | 240 | 280 |
|----------------------|-----|-----|-----|-----|-----|
| $T^2$                | 0.008 | 0.009 | 0.010 | 0.010 | 0.010 |
| $D_n$                | 0.009 | 0.010 | 0.010 | 0.010 | 0.010 |
| $T(\rho)$            | 0.008 | 0.007 | 0.009 | 0.008 | 0.009 |
| $T^2$                | 0.046 | 0.048 | 0.049 | 0.050 | 0.050 |
| $D_n$                | 0.050 | 0.049 | 0.050 | 0.050 | 0.049 |
| $T(\rho)$            | 0.041 | 0.043 | 0.045 | 0.046 | 0.047 |
| $T^2$                | 0.094 | 0.096 | 0.098 | 0.101 | 0.099 |
| $D_n$                | 0.099 | 0.099 | 0.100 | 0.099 | 0.099 |
| $T(\rho)$            | 0.088 | 0.089 | 0.093 | 0.097 | 0.095 |

Table 5. ASL in the case of $(\eta, p) = (0.4, 100)$

| $\alpha \setminus N$ | 120 | 160 | 200 | 240 | 280 |
|----------------------|-----|-----|-----|-----|-----|
| $T^2$                | 0.008 | 0.010 | 0.010 | 0.010 | 0.010 |
| $D_n$                | 0.009 | 0.011 | 0.010 | 0.010 | 0.010 |
| $T(\rho)$            | 0.008 | 0.008 | 0.009 | 0.009 | 0.009 |
| $T^2$                | 0.046 | 0.049 | 0.050 | 0.050 | 0.051 |
| $D_n$                | 0.051 | 0.050 | 0.051 | 0.050 | 0.051 |
| $T(\rho)$            | 0.041 | 0.044 | 0.046 | 0.046 | 0.048 |
| $T^2$                | 0.094 | 0.098 | 0.100 | 0.099 | 0.102 |
| $D_n$                | 0.101 | 0.100 | 0.099 | 0.100 | 0.100 |
| $T(\rho)$            | 0.087 | 0.092 | 0.095 | 0.096 | 0.097 |

Table 6. ASL in the case of $(\eta, p) = (0.6, 100)$

| $\alpha \setminus N$ | 120 | 160 | 200 | 240 | 280 |
|----------------------|-----|-----|-----|-----|-----|
| $T^2$                | 0.008 | 0.010 | 0.010 | 0.010 | 0.010 |
| $D_n$                | 0.010 | 0.010 | 0.010 | 0.010 | 0.010 |
| $T(\rho)$            | 0.008 | 0.008 | 0.008 | 0.008 | 0.009 |
| $T^2$                | 0.046 | 0.049 | 0.050 | 0.049 | 0.050 |
| $D_n$                | 0.051 | 0.050 | 0.050 | 0.050 | 0.050 |
| $T(\rho)$            | 0.041 | 0.044 | 0.045 | 0.046 | 0.048 |
| $T^2$                | 0.095 | 0.099 | 0.099 | 0.099 | 0.100 |
| $D_n$                | 0.100 | 0.100 | 0.100 | 0.100 | 0.100 |
| $T(\rho)$            | 0.087 | 0.091 | 0.094 | 0.094 | 0.097 |
Table 7. Empirical powers with $(\eta, p) = (0.2, 50)$

| $\alpha \backslash N$ | 70   | 110  | 150  | 190  | 230  |
|-----------------------|------|------|------|------|------|
| $T^2$                 | 0.06 | 0.23 | 0.40 | 0.53 | 0.65 |
| $D_n$                 | 0.17 | 0.25 | 0.32 | 0.39 | 0.46 |
| $T(\rho)$             | 0.12 | 0.29 | 0.42 | 0.53 | 0.63 |
|                       | 0.21 | 0.49 | 0.67 | 0.78 | 0.85 |
| $D_n$                 | 0.30 | 0.41 | 0.50 | 0.57 | 0.64 |
| $T(\rho)$             | 0.33 | 0.55 | 0.69 | 0.77 | 0.84 |
|                       | 0.34 | 0.64 | 0.79 | 0.87 | 0.92 |
| $D_n$                 | 0.51 | 0.63 | 0.71 | 0.77 | 0.81 |
| $T(\rho)$             | 0.48 | 0.69 | 0.80 | 0.87 | 0.91 |

Table 8. Empirical powers with $(\eta, p) = (0.4, 50)$

| $\alpha \backslash N$ | 70   | 110  | 150  | 190  | 230  |
|-----------------------|------|------|------|------|------|
| $T^2$                 | 0.04 | 0.12 | 0.21 | 0.30 | 0.39 |
| $D_n$                 | 0.15 | 0.22 | 0.27 | 0.33 | 0.39 |
| $T(\rho)$             | 0.09 | 0.21 | 0.30 | 0.39 | 0.47 |
|                       | 0.15 | 0.33 | 0.46 | 0.56 | 0.65 |
| $D_n$                 | 0.26 | 0.36 | 0.43 | 0.49 | 0.56 |
| $T(\rho)$             | 0.27 | 0.44 | 0.55 | 0.64 | 0.71 |
|                       | 0.26 | 0.47 | 0.60 | 0.70 | 0.77 |
| $D_n$                 | 0.48 | 0.58 | 0.65 | 0.72 | 0.76 |
| $T(\rho)$             | 0.41 | 0.58 | 0.68 | 0.76 | 0.81 |

Table 9. Empirical powers with $(\eta, p) = (0.6, 50)$

| $\alpha \backslash N$ | 70   | 110  | 150  | 190  | 230  |
|-----------------------|------|------|------|------|------|
| $T^2$                 | 0.04 | 0.13 | 0.22 | 0.31 | 0.40 |
| $D_n$                 | 0.10 | 0.14 | 0.18 | 0.21 | 0.25 |
| $T(\rho)$             | 0.08 | 0.17 | 0.25 | 0.33 | 0.40 |
|                       | 0.15 | 0.33 | 0.47 | 0.57 | 0.66 |
| $D_n$                 | 0.18 | 0.24 | 0.29 | 0.33 | 0.37 |
| $T(\rho)$             | 0.23 | 0.39 | 0.50 | 0.58 | 0.65 |
|                       | 0.26 | 0.48 | 0.61 | 0.70 | 0.78 |
| $D_n$                 | 0.38 | 0.45 | 0.51 | 0.56 | 0.61 |
| $T(\rho)$             | 0.37 | 0.53 | 0.63 | 0.70 | 0.77 |
Table 10. Empirical powers with \((\eta, p) = (0.2, 100)\)

| \(\alpha \setminus N\) | \(120\) | \(160\) | \(200\) | \(240\) | \(280\) |
|------------------------|--------|--------|--------|--------|--------|
| \(T^2\)               | 0.04   | 0.13   | 0.24   | 0.35   | **0.45** |
| 0.01 \(D_n\)          | **0.14** | 0.18   | 0.22   | 0.26   | 0.30   |
| \(T(\rho)\)           | 0.09   | **0.19** | **0.28** | **0.37** | 0.44   |
| \(T^2\)               | 0.16   | 0.35   | 0.50   | 0.62   | **0.71** |
| 0.05 \(D_n\)          | **0.29** | 0.35   | 0.40   | 0.45   | 0.50   |
| \(T(\rho)\)           | 0.27   | **0.44** | **0.55** | **0.64** | 0.70   |
| \(T^2\)               | 0.27   | 0.50   | 0.65   | 0.75   | **0.82** |
| 0.10 \(D_n\)          | **0.48** | 0.55   | 0.60   | 0.65   | 0.69   |
| \(T(\rho)\)           | 0.42   | **0.59** | **0.69** | **0.77** | **0.82** |

Table 11. Empirical powers with \((\eta, p) = (0.4, 100)\)

| \(\alpha \setminus N\) | \(120\) | \(160\) | \(200\) | \(240\) | \(280\) |
|------------------------|--------|--------|--------|--------|--------|
| \(T^2\)               | 0.03   | 0.09   | 0.15   | 0.22   | 0.28   |
| 0.01 \(D_n\)          | **0.14** | **0.17** | 0.21   | 0.25   | 0.28   |
| \(T(\rho)\)           | 0.07   | 0.15   | **0.22** | **0.29** | **0.35** |
| \(T^2\)               | 0.12   | 0.26   | 0.37   | 0.47   | 0.54   |
| 0.05 \(D_n\)          | **0.27** | 0.32   | 0.37   | 0.42   | 0.46   |
| \(T(\rho)\)           | 0.24   | **0.37** | **0.47** | **0.55** | **0.61** |
| \(T^2\)               | 0.22   | 0.39   | 0.51   | 0.61   | 0.68   |
| 0.10 \(D_n\)          | **0.46** | **0.52** | 0.57   | 0.62   | 0.66   |
| \(T(\rho)\)           | 0.38   | **0.52** | **0.61** | **0.68** | **0.73** |

Table 12. Empirical powers with \((\eta, p) = (0.6, 100)\)

| \(\alpha \setminus N\) | \(120\) | \(160\) | \(200\) | \(240\) | \(280\) |
|------------------------|--------|--------|--------|--------|--------|
| \(T^2\)               | 0.03   | 0.07   | 0.12   | 0.16   | 0.22   |
| 0.01 \(D_n\)          | **0.11** | **0.14** | **0.16** | **0.19** | 0.21   |
| \(T(\rho)\)           | 0.06   | 0.09   | 0.15   | 0.18   | **0.27** |
| \(T^2\)               | 0.11   | 0.22   | 0.31   | 0.39   | 0.46   |
| 0.05 \(D_n\)          | **0.21** | 0.25   | 0.29   | 0.33   | 0.36   |
| \(T(\rho)\)           | **0.21** | **0.32** | **0.40** | **0.46** | **0.52** |
| \(T^2\)               | 0.20   | 0.34   | 0.45   | 0.53   | 0.61   |
| 0.10 \(D_n\)          | **0.39** | **0.44** | **0.49** | **0.53** | **0.56** |
| \(T(\rho)\)           | 0.33   | **0.46** | **0.54** | **0.60** | **0.66** |