ELEMENTS OF UNIFORMLY BOUNDED WORD-LENGTH IN GROUPS

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Abstract. We study a characteristic subgroup of finitely generated groups, consisting of elements with uniform upper bound for word-lengths. For a group $G$, we denote this subgroup $G_{\text{bound}}$. We give sufficient criteria for triviality and finiteness of $G_{\text{bound}}$. We prove that if $G$ is virtually abelian then $G_{\text{bound}}$ is finite. In contrast with numerous examples where $G_{\text{bound}}$ is trivial, we show that for every finite group $A$, there exists an infinite group $G$ with $G_{\text{bound}} = A$. This group $G$ can be chosen among torsion groups. We also study the group $G_{\text{bound}}(d)$ of elements with uniformly bounded word-length for generating sets of cardinality less than $d$.

1. Introduction

Let $G$ be a finitely generated group and $S = \{s_1^{\pm 1}, \ldots, s_k^{\pm 1}\}$ a finite symmetric generating set of $G$. The word-length of an element $g \in G$ with respect to the generating set $S$ denoted $l_S(g)$, is the minimal integer $n$ such that there exists $s_{i_1}, \ldots, s_{i_n}$ in $S$ satisfying $g = s_{i_1} \cdots s_{i_n}$.

It is clear that the word-length depends on the generating set $S$. If $S_1$ and $S_2$ are two finite generating sets of $G$, then the word-lengths $l_{S_1}$ and $l_{S_2}$ are bi-lipschitz equivalent. This means that there exist two constants $C_1$ and $C_2$, such that $C_1 l_{S_2} \leq l_{S_1} \leq C_2 l_{S_2}$. Given a length $l$, bi-lipschitz equivalent to $l_{S_1}$, one can ask when $l$ is realisable as $l_{S_1}$ for some finite generating set $S_1$ of $G$. There are numerous obstacles for such $l$ to be realisable. One of such obstacles can be formulated in terms of uniform exponential growth, see for example [4, 5, 8, 13, 14]. Another possible obstacle is uniform non-amenability see [2, 18, 19].

In this paper, we show that in some groups the value of $l$ at one single element can be an obstacle for $l$ to be a word-length. Our goal is to study such elements.

Definition 1. An element $g \in G$ has uniformly bounded word-length if there exists a constant $M > 0$ such that $l_S(g) < M$ for every finite generating set $S$ of $G$. The set of elements of uniformly bounded word-length of $G$ is denoted $G_{\text{bound}}$.

We recall that a characteristic subgroup $H$ of a group $G$ is a subgroup such that for every automorphism $\phi$ of $G$, $\phi(H) = H$.

Key words and phrases. Word-length, word metrics, characteristic subgroups, FC-center, virtually abelian groups, torsion groups, group identities, Burnside groups, hyperbolic groups.

This work is partially supported by the ERC grant GroIsRan.
We recall that the FC-center of a finitely generated group $G$ is the set of elements of $G$ with finite number of conjugates. We denote it $FC(G)$. It follows from the definition that the FC-center is a characteristic subgroup of $G$. It is known that every finitely generated group $G$ such that $FC(G) = G$ is virtually abelian, (see for instance [20], Theorem 4.32).

In fact, $G_{\text{bound}}$ is a characteristic subgroup of $G$ contained in $FC(G)$, see Lemma 1 in Section 2. It follows that every finitely generated subgroup of $G_{\text{bound}}$ is virtually abelian see Corollary 1. In Section 3 we give a sufficient criteria for triviality of $G_{\text{bound}}$ in terms of girth of $G$. Combining this condition with a result of Olshanskii and Sapir concerning the girth of hyperbolic groups, see Theorem 2 in [17], we obtain that $G_{\text{bound}}$ is trivial when $G$ is non-elementary hyperbolic.

The main result in Section 4 is the following.

**Theorem.** If $G$ is a finitely generated virtually abelian group, then $G_{\text{bound}}$ is finite.

This implies that for every infinite finitely generated group $G$, the subgroup $G_{\text{bound}}$ has infinite index, see Corollary 4.

In Section 5, we prove the following theorem.

**Theorem.** Let $A$ be a finite group, there exists a finitely generated infinite group $G$ such that $G_{\text{bound}} = A$. This group can be chosen among torsion groups.

In Section 6 we provide examples of elements with prescribed length in finitely generated groups. In Section 7 we study a generalization of $G_{\text{bound}}$ by fixing the cardinality of the considered generating sets in the definition of $G_{\text{bound}}$.

It seems natural to ask whether there exists a finitely generated group $G$ such that $G_{\text{bound}}$ is infinite.

2. Basic properties

**Lemma 1.** $G_{\text{bound}}$ is a characteristic subgroup of $G$.

**Proof.** It follows from the definition of $G_{\text{bound}}$ that it contains the unit element of $G$.

Consider $g \in G_{\text{bound}}$. We have $m = \sup_{S} l_{S}(g) = \sup_{S} l_{S}(g^{-1})$ because $l_{S}(g) = l_{S}(g^{-1})$. Therefore, if $g \in G_{\text{bound}}$, then $g^{-1} \in G_{\text{bound}}$. Consider now $g_{1}, g_{2} \in G_{\text{bound}}$. We denote $m_{1} = \sup_{S} l_{S}(g_{1})$ and $m_{2} = \sup_{S} l_{S}(g_{2})$.

For every finite generating set $S$, we have

$$l_{S}(g_{1}g_{2}) \leq l_{S}(g_{1}) + l_{S}(g_{2}) \leq m_{1} + m_{2}.$$ 

Thus, $g_{1}g_{2} \in G_{\text{bound}}$. Hence, $G_{\text{bound}}$ is a subgroup of $G$.

Let us consider $g \in G_{\text{bound}}$, $h \in G$. Let $S = \{s_{1}, \ldots, s_{k}\}$ be a finite generating set of $G$, such that $g = s_{i_{1}} \cdots s_{i_{n}}$ where $s_{i_{j}} \in S$ for every $1 \leq j \leq n$. Given an automorphism $A$ of $G$, we have $A(g) = A(s_{i_{1}} \cdots s_{i_{n}}) = \ldots$
A(s_{i_1}) \ldots A(s_{i_n}) and the set \( A(S) = \{ A(s_1), \ldots, A(s_k) \} \) generates \( G \). We remark that \( l_{A(S)}(A(g)) = l_S(g) \). For a finite generating set \( E \) of \( G \), \( l_E(A(g)) = l_{A^{-1}(E)}(g) \). We conclude that if \( m = \sup_S l_S(g) \), then \( l_E(A(g)) = l_{A^{-1}(E)}(g) \leq m \). It implies that \( A(g) \in G_{\text{bound}} \) and \( G_{\text{bound}} \) is a characteristic subgroup of \( G \).

**Lemma 2.** If \( G = A \times B \), then \( G_{\text{bound}} < A_{\text{bound}} \times B_{\text{bound}} \).

**Proof.** Consider \( g = (a, b) \in G_{\text{bound}} \). For a finite generating set \( E \) of \( A \) and a finite generating set \( F \) of \( B \), \( M = E \times \{ e_B \} \cup \{ e_A \} \times F \) is a finite generating set of \( G \). We have \( l_M(g) = l_E(a) + l_F(b) \). The inclusion \( G_{\text{bound}} \subseteq A_{\text{bound}} \times B_{\text{bound}} \) follows from \( l_E(a) \leq l_M(g) \) and \( l_F(b) \leq l_M(g) \).

For some groups we do not have equality \( (A \times B)_{\text{bound}} = A_{\text{bound}} \times B_{\text{bound}} \) as we see in the following example.

**Example 1.** Consider \( A = \mathbb{Z} \) and \( B = \mathbb{Z}/q\mathbb{Z} \) where \( q \) is an integer such that \( q > 2 \), \( G = A \times B \). We have \( G_{\text{bound}} \) trivial and \( A_{\text{bound}} \times B_{\text{bound}} = \{ 0 \} \times \mathbb{Z}/q\mathbb{Z} \).

Indeed, since \( B_{\text{bound}} = B \), the element \((0, 1)\) is contained in \( A_{\text{bound}} \times B_{\text{bound}} \). Let \( p \) be a prime number such that \( p > q + 1 \). It follows that \( S = \{ \pm(p, 1), \pm(q + 1, 0) \} \) is a generating set of \( G \). Hence, \((0, 1) = (p + q + 1) - p(q + 1, 0)\). Consequently, we have \( l_S(0, 1) = p + q + 1 \). This implies that \( l_S(0, 1) \) tends to infinity when \( p \) tends to infinity. Thus, \((0, 1)\) is not in \( G_{\text{bound}} \). It follows that \( \{ 0 \} \times B \), which equals \( \{ 0 \} \times B_{\text{bound}} \) is not contained in \( G_{\text{bound}} \). Consequently, \( A_{\text{bound}} \times B_{\text{bound}} \) is not a subgroup of \( G_{\text{bound}} \).

**Lemma 3.** If \( G \) is a finitely generated group, then \( G_{\text{bound}} \) is a subgroup of \( FC(G) \).

**Proof.** Let \( S \) be a finite generating set of \( G \). We want to prove that if \( g \in G \) and \( g \notin FC(G) \), then \( g \notin G_{\text{bound}} \). Consider \( g \in G \) with length \( l_S(g) = n \) and \( s_{i_1}, \ldots, s_{i_n} \in S \) such that \( g = s_{i_1} \ldots s_{i_n} \). Let us consider \( h \in G \) such that \( h \neq e \). Since the map \( A : x \mapsto h^{-1}xh \) is an automorphism of \( G \), the finite set \( S' = A(S) = h^{-1}Sh = \{ h^{-1}s_1h, \ldots, h^{-1}s_nh \} \) generates \( G \). We obtain \( l_{A(S)}(A(g)) = l_{h^{-1}Sh}(h^{-1}gh) = l_{h^{-1}Sh}(h^{-1}s_1h \ldots h^{-1}s_nh) = l_S(g) = n \).

Let us consider now \( g \in G \) and \( g \notin FC(G) \). This means that there exists a sequence \( \{ n_h \} \) satisfying \( lim_{n \to \infty} l_{S}(h_nh^{-1}) = \infty \). It follows that \( lim_{n \to \infty} l_{h^{-1}Sh_n}(g) = \infty \). Since \( \{ h^{-1}Sh_n \} \) is a sequence of finite generating sets of \( G \), the element \( g \) is not in \( G_{\text{bound}} \). From this, we conclude that \( G_{\text{bound}} \subseteq FC(G) \).

**Corollary 1.** Let \( G \) be a finitely generated group. Every finitely generated subgroup of \( G_{\text{bound}} \) is virtually abelian.

**Proof.** Let \( H \) be a finitely generated subgroup of \( G_{\text{bound}} \). Since \( (FC(G) \cap H) \subseteq FC(H) \) and \( H \subseteq G_{\text{bound}} \subseteq FC(G) \), then \( H \subseteq FC(H) \). Therefore, \( H = FC(H) \). The claim that \( H \) is virtually abelian follows from Theorem 4.3.1 in [20]. This theorem asserts that a finitely generated group \( G \) such
that $FC(G) = G$ is virtually abelian.

\section{Groups with infinite girth}

Consider a group $G$ generated by a finite set $S$. We recall that a simple loop in the corresponding Cayley graph $\Gamma(G, S)$, is a non-trivial path starting and ending at $e$ without self intersections. The \textit{girth} of a Cayley graph $\Gamma(G, S)$ is the minimal length of a simple loop in this graph. A group has \textit{infinite girth} if for every fixed constant $k > 0$, there is a finite generating set $S$ of $G$, such that the girth of $\Gamma(G, S)$ is greater than $k$.

\begin{lemma}
Let $G$ be a finitely generated group with finite generating set $S$ and $\Gamma(G, S)$ the corresponding Cayley graph. Suppose $g \in G$ an element of order $n$ and $w$ a word with respect to $S$ representing $g$ with minimal length $k$. Consider $w'$ the shortest word among conjugates of $w$. The path obtained by concatenation of $n$ times $w'$ is a simple loop.
\end{lemma}

\begin{proof}
Suppose that $w' = uu^{-1}$ is the shortest word among conjugates of $w$. Assume that there exist a sub-word of $w''$ of $w'$ satisfying $w' = sw''s^{-1}$. It follows that there exists a word $u$ such that $w = u^{-1}sw''s^{-1}u$ and we obtain that $w''$ is a shorter conjugate of $w$ than $w'$. It is a contradiction. We conclude that $w''$ is a simple loop.
\end{proof}

\begin{lemma}
If a group $G$ has infinite girth, then $G_{\text{bound}}$ does not contain torsion elements.
\end{lemma}

\begin{proof}
Consider $g \in G$, such that there exists a minimal integer $n(g) > 1$ satisfying $g^{n(g)} = e$. Assume that $g \in G_{\text{bound}}$. There exists some constant $M > 0$ such that for every finite generating set $S = \{s_1, \ldots, s_m\}$ of $G$, we have $l_S(g) \leq M$. Therefore, for every generating set $S$ of $G$, we have $l_S(g^{n(g)}) \leq n(g)M$. We know that there exists a word $w$ over $S$, representing $g$, such that $w = s_{i_1} \ldots s_{i_k}$ corresponding to a path starting at $e$ and ending at $g$ in the Cayley graph $\Gamma(G, S)$. Let $w'$ be the shortest word among conjugates of $w$. The path corresponding to the concatenation of $n$ copies of the word $w' \ldots w'$, starts and ends in $e$. It is a non-trivial simple loop as shown in Lemma[4]. Therefore, for every finite generating set, we have a loop of length bounded by $n(g)M$. It contradicts the infinite girth of $G$.
\end{proof}

\begin{corollary}
If $G$ is a finitely generated torsion group with infinite girth, then $G_{\text{bound}}$ is trivial.
\end{corollary}

There are numerous examples of groups satisfying Lemma[5]. It is the case for the first Grigorchuk group. In fact, this group is a torsion group see Theorem 2.1 in [11], (see also Chapter 8, Theorem 17 in [4]) and has infinite girth as shown in corollary 6.12 in [3]. But a stronger statement holds for the first Grigorchuk group. This group has trivial FC-center. In fact, it is a ”just infinite” group, see Theorem 8.1 in [11]. It means that every normal subgroup has finite index. Since the FC-center is a normal subgroup of $G$, it is finitely generated and then virtually abelian. It implies that if FC-center is non-trivial, the first Grigorchuk
group is virtually abelian. It is known that infinite abelian groups contains elements of infinite order and the first Grigorchuk group is a torsion group, so it doesn’t contain infinite abelian subgroups. As a consequence, $FC(G)$ is trivial.

However, there are examples with non-trivial FC-center and the Lemma remains applicable, as it holds for a hyperbolic groups.

Now, we want to prove that the assumption of Lemma is true for every hyperbolic groups.

First, we recall some basic definitions.

A geodesic triangle with vertices $x_1, x_2, x_3$ in a geodesic metric space $X$, is the union of three geodesic segments $[x_1, x_2], [x_1, x_3]$ and $[x_2, x_3]$, where $[p, q]$ denotes a geodesic with endpoints $p$ and $q$. The $\delta$-neighbourhood of a subset $A \subset X$, is $N_\delta(A) = \{ x \in X | \inf_{y \in A} d(x, y) \leq \delta \}$.

A geodesic triangle is said to be $\delta$-thin if every side $[x_j, x_i]$ of the triangle is contained in the $\delta$-neighbourhood of the union of the two other sides, i.e $[x_1, x_2] \subset N_\delta([x_2, x_3] \cup [x_1, x_3])$ and $[x_2, x_3] \subset N_\delta([x_2, x_1] \cup [x_1, x_3])$ and $[x_2, x_3] \subset N_\delta([x_2, x_1] \cup [x_1, x_3])$.

Definition 2. Let $\delta > 0$ be a positive constant. A geodesic metric space $X$ is said $\delta$-hyperbolic if every geodesic triangle in $X$ is $\delta$-thin.

Definition 3. A group $G$ generated by a finite set $S$ is $\delta$-hyperbolic if it is a $\delta$-hyperbolic metric space for the word metric given by $S$.

For properties of hyperbolic groups see [12], see also [10].

Lemma 6. If $G$ is a hyperbolic group, then $FC(G)$ is finite.

Proof. We know that a subgroup of a hyperbolic group is infinite if and only if it contains a hyperbolic element (i.e element of infinite order), see for example Corollary 36, Chapter 8 in [10]. We know also that if $h \in G$ is a hyperbolic element, we can find an integer $n_1 \in \mathbb{N}$ and $x \in G$ such that the subgroup generated by $\{xh^{n_1}x^{-1}, h^{n_1}\}$ is free of rank two, see proposition 5.5 in [13]. Since $FC(G)$ is normal in $G$, it follows that if $FC(G)$ is infinite, it contains a free subgroup of rank two. This is impossible because, by definition, every element of $FC(G)$ has finitely many conjugates.

We conclude that $FC(G)$ is finite. □

Corollary 3. Non-elementary hyperbolic groups have trivial $G_{\text{bound}}$.

Proof. Since $G_{\text{bound}} \subset FC(G)$ as shown in Lemma, we conclude that $G_{\text{bound}}$ is finite. It is known that non-elementary hyperbolic groups have infinite girth see Theorem 2.6 in [1]. Therefore, it follows from Lemma that $G_{\text{bound}}$ is trivial. □

4. Virtually abelian groups

Lemma 7. If $G$ is an infinite finitely generated abelian group, $G_{\text{bound}}$ is trivial.

Proof. Since $G$ is a finitely generated abelian group, there exist integers $\{p_1 \leq p_2 \leq \cdots \leq p_n\}$ such that $G$ is isomorphic to $\mathbb{Z}^d \times \mathbb{Z}/p_1\mathbb{Z} \times \cdots \times \mathbb{Z}/p_n\mathbb{Z}$ with minimal $n$. Therefore, we will prove that groups of this form have
trivial $G_{\text{bound}}$. We consider $G = \mathbb{Z}^d \times \mathbb{Z}/p_1 \mathbb{Z} \times \ldots \times \mathbb{Z}/p_n \mathbb{Z}$ and $A = \mathbb{Z} \times \mathbb{Z}/p_1 \mathbb{Z} \times \ldots \times \mathbb{Z}/p_n \mathbb{Z}$. We have $G = \mathbb{Z}^{d-1} \times A$.

By Lemma 2, we know that $G_{\text{bound}} < (\mathbb{Z}^{d-1})_{\text{bound}} \times A_{\text{bound}}$. We will prove that $(\mathbb{Z}^{d-1})_{\text{bound}}$ and $A_{\text{bound}}$ are trivial.

We start by proving that $\mathbb{Z}_{\text{bound}}^{d-1}$ is trivial for every integer $d > 0$. First, observe that $\mathbb{Z}_{\text{bound}}$ is trivial. Indeed, let $n$ be an integer. Consider two different primes $p, q$ such that $q > p > |n|$. There exist $a, b \in \mathbb{Z} \setminus \{0\}$ such that $ap + bq = n$ and $|a| + |b|$ is minimal. Therefore, $S = \{\pm p, \pm q\}$ is a generating set of $\mathbb{Z}$. We have $ap = n - bq$ so, $|a| = \frac{|n - bq|}{p}$. If $q$ tends to infinity, $l_S(n) = |a| + |b|$ tends to infinity. Therefore, for every $d > 1$, $\mathbb{Z}_{\text{bound}} = \{0\}$.

Since $\mathbb{Z}_{\text{bound}} = 0$, by Lemma 2, we have $\mathbb{Z}_{\text{bound}}^d = 0$.

Consider $s_0 = (1, 0, \ldots, 0), s_1 = (0, 1, 0, \ldots, 0), \ldots, s_n = (0, 0, \ldots, 0, 1)$ and $S = \{s_i\}_{0 \leq i \leq n}$ the standard generating set of $A$.

Let $g = (x_0, x_1, \ldots, x_n) \in A$ be a non-trivial element. If $x_i = 0$ for every $i \in \{1, \ldots, n\}$, then we have a similar case than $G = \mathbb{Z}$ and we consider the generating set

$$E = \{(p, 0, \ldots, 0), (q, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)\}.$$ 

It follows that $l_E(g) = |\alpha| + |\beta|$ where $ap + \beta q = x_0$. As shown above, this length tends to infinity when $q$ tends to infinity.

If there exists $i \in \{1, \ldots, n\}$ such that $x_i \neq 0$, then we consider the generating set $E_q$ obtained by replacing $s_i$ by $e_i = (q, 0, \ldots, 0, 1, 0, \ldots, 0)$ in $S$ i.e $E_q = (S \setminus \{s_i\}) \cup \{s_i + qs_0\}$, this means that

$$e_0 = s_0, e_1 = s_1, \ldots, e_{i-1} = s_{i-1}, e_i = s_i + qs_0, e_{i+1} = s_{i+1}, \ldots, e_n = s_n.$$ 

For a prime number $q$, there exist integers $\alpha, \beta \in \mathbb{Z}$ such that the word $x_0 = \alpha + \beta q$ with $\beta = x_i \mod p_i$ and $|\alpha| + |\beta|$ minimal. In fact, in order to obtain $x_i$ at the $i^{th}$ component of $g$, we should have $\beta = x_i \mod p_i$. In this case, the shortest expression of $g$ with respect to $E_q$ is

$$g = \alpha e_0 + \beta e_i + \sum_{j=1, j \neq i}^{n} x_j e_j.$$ 

As a consequence, we have $|\alpha| = |x_0 - \beta q|$ and

$$l_{E_q}(g) = |\alpha| + |\beta| + \sum_{j=1, j \neq i}^{n} |x_j|.$$ 

Since $|\alpha|$ tends to infinity when $q$ tends to infinity, we have

$$\lim_{q \to +\infty} l_{E_q}(g) = +\infty.$$ 

We conclude that $A_{\text{bound}} = \{e\}$. Since $G_{\text{bound}} < (\mathbb{Z}^{d-1})_{\text{bound}} \times A_{\text{bound}}$ and $(\mathbb{Z}^{d-1})_{\text{bound}} = \{e\}$, we conclude that $G_{\text{bound}} = \{e\}$.

Notice that for every abelian group $G$, the subgroup $G_{\text{bound}}$ is trivial and $FC(G) = G$. 

\[\square\]
Theorem 1. If $G$ is a finitely generated virtually abelian group, then $G_{\text{bound}}$ is finite.

Proof. Assume that $H = \mathbb{Z}^d$ is a normal subgroup of $G$, such that $G/H$ is finite. We can suppose that $H$ is a normal subgroup of $G$. We want to prove that $G_{\text{bound}} \cap H$ is trivial. We consider $S_H = \{z_1^{\pm 1}, ..., z_d^{\pm 1}\}$ a free abelian generating set of $H$. For $x, z \in H$, $< z, x >$ is the inner product in $\mathbb{Z}^d = H$. Since for every non trivial $h \in H$ there exists $i \in \{1, ..., d\}$ such that $< h, z_i > \neq 0$, we will prove that for every constant $K > 1$, there exists a generating set $S$ such that $l_S(h) \geq K$.

Without loss of generality, we suppose that $< h, z_1 > \neq 0$. We denote $\eta = < h, z_1 >$. We will prove that if $h \in G_{\text{bound}}$, then $\eta \neq < h, z_1 >$. Fix a constant integer $K > 1$. Consider $T = \{t_1, ..., t_n\}$ a transverse of $G/H$. Let $A_K$ be the set of products $t_1 \cdots t_k$ such that $t_i \cdots t_k \in H$ and $k \in \{0, ..., K\}$, i.e., words of length less than $K$ over $T \cup T^{-1}$ contained in $H$. We denote $C = \max\{|< a, z_1 > | : a \in AK\}$.

Now, we define the generating set $S$. Consider $\Gamma = \{\theta_i\}_{1 \leq i \leq n}$ such that $\theta_i = t_i z_i^{u_i}$ where for every $i \in \{1, ..., d\}$, $u_i$ is a positive prime number. We choose $(u_i)_{1 \leq i \leq n}$ growing fast enough. It is sufficient to choose $(u_i)_{1 \leq i \leq n}$ such that $u_1 > |\eta| + C$ and $u_{i+1} > 2K u_i$.

Fix a constant $M > 2$. We consider

$$S = \{z_1^{\pm p_1}, z_2^{\pm q_1}, ..., z_d^{\pm q_1}\}_{1 \leq i \leq d} \cup \Gamma \cup \Gamma^{-1}$$

where $(p_i, q_i)_{1 \leq i \leq d}$ are distinct prime numbers such that $p_{j+1} > p_j M$, $q_{j+1} > M q_j$, $q_1 > M p_d$ and $p_1 > u_n M$.

We will prove that if there exists an integer $K > 1$ such that $l_S(h) < K$ for every $(p_i, q_i)_{1 \leq i \leq d}$ and $(u_i)_{1 \leq i \leq n}$, then it contradicts the fact that $< h, z_1 > = \eta$.

Since the set $S_H$ is a free generating set of $H = \mathbb{Z}^d$, for every primes $(p_i, q_i)_{1 \leq i \leq d}$, every word representing $z_1$ with respect to $S$ contains powers of $z_1$ as sub-words and since $< h, z_1 > \neq 0$, every word $w$ over $S$ representing $h$ contains power of $z_1$ as sub-words and $< w, z_1 > = \eta$.

We can write $h$ as a product of elements of $S$:

$$h = \theta_{i_1} z_1^{\eta_1} \theta_{i_2} z_2^{\eta_2} \cdots \theta_{i_m} z_m^{\eta_m}.$$ 

This word can be rewritten as

$$h = \theta_{i_1} z_1^{\eta_1} \theta_{i_2} z_2^{\eta_2} \cdots \theta_{i_m} z_m^{\eta_m} = \theta_{i_1} \theta_{i_2} \cdots \theta_{i_m}.$$

Observe that since $H$ is a normal subgroup of $G$, $\theta_{i_j} z_k \theta_{i_j}^{-1} \in H$ and $z = H_{i_1} .. H_{i_m} \in H$. Since $G/H$ is a group, we know that a product of elements of $T$ is another element of $T$ multiplied by an element of $H$. In fact, for every $t_{i_1}, t_{i_2}$, there exists $t_j \in T$ and $y \in H$ such that $t_{i_1} t_{i_2} = t_j y$. Therefore, for $t \in T$ and $y, x \in H$, $(ty)^{-1} = t y^{-1} t^{-1} = t x t^{-1} = t x t^{-1}$ because $H$ is abelian. This implies that $t x t^{-1}$ depends only of the class of $t$ in $G/H$.

Now, we return to the proof. According to the observation above, there exists $(t_{j_1}, ..., t_{j_m})$ such that

$$h = t_{j_1} z_{i_1} t_{j_1}^{-1} t_{j_2} z_{i_2} t_{j_2}^{-1} \cdots t_{j_m} z_{i_m} t_{j_m}^{-1} \theta_{i_1} \theta_{i_2} \cdots \theta_{i_m}.$$
We have
\[ \theta_{i_1 \ldots i_m} = t_{i_1}^u z_{i_1}^{u_1} \ldots t_{i_m}^{u_m} = (t_{i_1}z_{i_1}^{-1})^{u_1}(t_{i_2}z_{i_2}^{-1})^{u_2} \ldots (t_{i_m}z_{i_m}^{-1})^{u_m}(t_{i_1} \ldots t_{i_m})^{-1}(t_{i_1} \ldots t_{i_m}). \]

We denote \( \alpha_j^\gamma = z_{i_1} \ldots z_{i_m} \). We have the following equations:

\[ \eta = \langle h, z_1 \rangle = \begin{cases} < z_1, (t_{j_1}z_{j_1}^{-1}) \ldots (t_{j_m}z_{j_m}^{-1}) > + < z_1, \theta_{i_1 \ldots i_m} > \\ = \sum_{h=1}^{m} \langle z_1, t_{j_h} z_{j_h}^{-1} > + \sum_{h=1}^{m} \langle z_1, t_{j_h} z_{j_h}^{-1} > + < z_1, t_{i_1} \ldots t_{i_m} > \\ = \sum_{h=1}^{m} k_{i_h} \alpha_{j_h} + \sum_{h=1}^{m} u_{i_h} \alpha_{j_h} + < z_1, z > \\ = \sum_{h=1}^{m} (x_{i_h} q_{i_h} + y_{i_h} p_{i_h}) \alpha_{j_h} + \sum_{h=1}^{m} u_{i_h} \alpha_{j_h} + < z_1, z > . \end{cases} \]

This implies that there exist integers \( \{X_i, Y_i\}_{1 \leq i \leq d} \) and \( \{R_i\}_{1 \leq i \leq n} \) such that

\[ \sum_{h=1}^{m} (x_{i_h} q_{i_h} + y_{i_h} p_{i_h}) \alpha_{j_h} = \sum_{i=1}^{d} X_i q_i + Y_i p_i \]

and

\[ \sum_{h=1}^{m} u_{i_h} \alpha_{j_h} = \sum_{j=1}^{n} R_j u_j. \]

This allows us to write:

\[ \eta = \langle h, z_1 \rangle = \sum_{i=1}^{d} (X_i q_i + Y_i p_i) + \sum_{j=1}^{n} R_j u_j + < z_1, z > . \]

Suppose that \( l_S(h) \leq K \). It implies that \( z = t_{i_1} \ldots t_{i_m} \in A_K \) and

\[ | < z_1, z > | \leq C. \]

Let us prove that \( \sum_{i=1}^{d} X_i q_i + Y_i p_i \neq 0 \) for the shortest expression of \( h \) with respect to \( S \) with length less than \( K \). The numbers \( (u_{i_h})_{1 \leq h \leq n} \) are chosen to guarantee that \( h \) can not be expressed as a word of length less than \( K \) over \( \Gamma \cup \Gamma^{-1} \). In fact, if it was the case, i.e \( l_{\Gamma \cup \Gamma^{-1}}(h) < K \), then \( \sum_{h=1}^{m} |R_j| < K \). It follows that

\[ \eta = < z_1, z > + \sum_{j=1}^{n} R_j u_j \]

and

\[ |\eta| + C \geq \sum_{j=1}^{n} R_j u_j. \]

We denote \( \gamma = \max \{j \mid R_j \neq 0\} \). We have \( \gamma > 1 \). In fact, if \( \gamma = 1 \), then
Since we supposed $\eta = z_1, z > +R_1 u_1$. It follows that $|\eta| + C \geq |R_1 u_1| > |\eta| + C$ and this can not hold. We have

$$|\eta| + C \geq \left| \sum_{j=1}^{\gamma} R_j u_j \right| \geq ||R_{\gamma} u_{\gamma} - \sum_{j=1}^{\gamma-1} R_j u_j|| \geq |u_\gamma - \sum_{j=1}^{\gamma-1} R_j u_j| \geq |u_\gamma - u_{\gamma-1} \sum_{j=1}^{\gamma-1} R_j| \geq |u_\gamma - K u_{\gamma-1}| \geq K u_{\gamma-1} \geq K(|\eta| + C).$$

Since $K > 1$, this is a contradiction. It is clear that in this case we have $\sum_{h=1}^{m} k_n A_j^{\nu_h} \neq 0$ and as a consequence $\sum_{i=1}^{d} X_i q_i + Y_i p_i \neq 0$.

We have

$$|\eta| + C \geq \left| \sum_{i=1}^{d} (X_i q_i + Y_i p_i) + \sum_{j=1}^{n} R_j u_j \right| \geq ||\sum_{i=1}^{d} X_i q_i - \sum_{i=1}^{d} Y_i p_i - \sum_{j=1}^{n} R_j|| \geq ||\sum_{i=1}^{d} X_i M^{i+1} q_i - \sum_{i=1}^{d} Y_i p_i - u_n \sum_{j=1}^{n} R_j|| \geq p_d \left| \sum_{i=1}^{d} X_i M^{i+1} - \sum_{i=1}^{d} Y_i \right| - \sum_{j=1}^{n} |R_j|.$$ 

Since we supposed $l_5(h) \leq K$, we have $\sum_{i=1}^{d} |X_i| + |Y_i| + \sum_{j=1}^{n} |R_j| \leq K$. When $M$ tends to infinity, the following inequality

$$C + |\eta| \geq p_d \sum_{i=1}^{d} (|X_i| M^{i+1} - |Y_i|) - \sum_{j=1}^{n} |R_j|$$

can not hold. We conclude that if there exists an integer $K > 1$ such that $l_5(h) < K$ for every $\{p_i, q_i\}_{1 \leq i \leq d}$ and $\{u_i\}_{1 \leq i \leq n}$, then it contradicts the fact that $< h, z_1 > = \eta$.

Therefore, $G_{\text{bound}} \cap H = \{e\}$. It implies that the restriction of the map $G \to G/H$ to $G_{\text{bound}}$ is injective. Since $H$ is of finite index, we conclude that $G_{\text{bound}}$ is finite.

**Corollary 4.** Let $G$ be a finitely generated group. Either $G_{\text{bound}}$ has infinite index or $G$ is finite and we have equality $G = G_{\text{bound}}$.

**Proof.** Suppose that $G_{\text{bound}}$ has finite index and $G$ infinite. It implies that $G_{\text{bound}}$ is infinite and finitely generated. From Lemma 3 we know that in this case, $G_{\text{bound}}$ is virtually abelian. Therefore, $G$ is virtually abelian. Nevertheless, we know that if $G$ is virtually abelian, $G_{\text{bound}}$ is finite. It is a contradiction. As a consequence, $G_{\text{bound}}$ has infinite index if and only if $G$ is infinite. Finally, if $G$ is finite, then for every $g \in G$ and every generating set $S$ of $G$, $l_5(g) \leq \#G$ and we conclude that $G_{\text{bound}} = G$. ☐

The following example gives a virtually abelian groups with non-trivial $G_{\text{bound}}$. We denote $D_8$ the dihedral group with 8 elements.

**Example 2.** The group $G = Z \times D_8$ has non trivial $G_{\text{bound}}$. This group is non-abelian and nilpotent of class 2. The center of $D_8$ is $C(D_8) = Z/2Z = \{0, z\}$.

Let $S$ be a finite generating set of $G$. Since $G$ is non abelian, there exist $x = (a, u), y = (b, v) \in S$ such that $[x, y] \neq e$. Since $D_8$ is nilpotent of
class 2, \(\forall u, v, w \in D_4\) we have \([u, v], w = e\). Therefore, \(\forall r \in D_8\), we have \([x, y], \{0, r\} = (0, [u, v], r) = (0, e)\) then, \([u, v] \in C(D_4) \setminus \{0\} = z\). It implies that
\[
l_S(0, z) = l_s(0, [u, v]) = l_s([x, y]) \leq 2l_s(x) + 2l_s(y) = 4.
\]
Therefore, \((0, z) \in G_{bound}\). As a consequence of Lemma \([2]\) and Lemma \([7]\), \(\mathbb{Z}/2\mathbb{Z} \subset G_{bound}\).

5. Groups with prescribed finite \(G_{bound}\)

**Theorem 2.** Let \(A\) be a finite group, there exists a finitely generated infinite group \(G\) such that \(G_{bound} = A\). This group can be chosen among torsion groups.

**Proof.** A result of Duguid and Mclain, see \([7]\) and \([15]\), says that a finitely generated group \(G\) has an infinite quotient \(H\) with trivial FC-center if and only if \(G\) is not virtually nilpotent. For an exposition of this result, see also \([9]\). It is known that finitely generated torsion virtually nilpotent groups are finite, see proposition 2.19 in \([14]\). Since Burnside groups are finitely generated torsion groups, every infinite Burnside group \(B\) has an infinite quotient \(H\) such that FC\((H)\) is trivial. We consider for example \(B(2, p)\) the free Burnside group obtained as a quotient of \(F_2\), the free group of rank 2, by \(F =< \{x^p[x \in F_2]\}>\). There exists a constant \(\gamma > 1\) such that \(p > \gamma\), the group \(B(2, p)\) is infinite, see \([16]\) and \([6]\). We consider \(p\) a prime number such that \(p > \max\{\gamma, \#A\}\) where \(#A\) is the cardinality of \(A\).

Since \(B(2, p)\) is infinite and finitely generated, it follows that it has an infinite quotient \(H\) such that FC\((H)\) is trivial. Notice that \(\forall h \in H\), \(h^p = e\). We have \(H_{bound} < FC(H) = \{e\}\).

Consider \(G = H \times A\). Let us prove that \(G_{bound} = \{e\} \times A\).

Firstly we will show that \(A \subset G_{bound}\). Let \(g = (h, a)\) be an element of \(G\). We denote \(q\) the order of \(a\) in \(A\). We have \(g^p = (h^p, a^p \text{mod} q) = (e, a^p \text{mod} q)\). There exist \(\alpha \in \mathbb{N}\) and \(\beta \in \mathbb{Z}\) such that \(\alpha p + \beta q = 1\) and \(\alpha \leq q\). It implies that \(g^{\alpha p} = (e, a)\).

Let \(\pi_A: H \times A \rightarrow A\), \(\pi_H: H \times A \rightarrow H\) be the standard surjections. Consider a finite generating set \(S_G = \{(h_1, a_1), \ldots, (h_n, a_n)\}\) of \(G\), we denote \(s_i = (h_i, a_i)\). The set \(\pi_A(S_G) = \pi_A = \{a_1, \ldots, a_n\}\) is a generating set of \(A\).

Consider a word \(w = s_{i_1} \ldots s_{i_n}\) over \(S_G\) with minimal length such that \(\pi_A(w) = a\), i.e. \(\exists h \in H\) such that \(w = (h, a) = (h, a_{i_1} \ldots a_{i_n})\). It follows that \(l_{S_G}(w) = l_{S_A}(a)\). Since \(w^{\alpha p} = (e, a)\), we have
\[
l_{S_G}(e, a) \leq \alpha p l_{S_G}(w) \leq \alpha p l_{S_A}(a) \leq \alpha p \#(A) \leq q p \#(A) \leq p \#(A)^2.
\]
Since this is true for every generating set \(S_G\) of \(G\), we conclude that \((e, a) \in G_{bound}\) and \((e) \times A < G_{bound}\).

Now, consider \((h, a) \in G_{bound}\). There exists a constant \(M > 0\) such that \(l_{S}(h, a) < M\) for every generating set \(S\) of \(G\). It follows that \(h \in H_{bound}\). In fact, for every generating set \(S\) of \(G\), \(l_{\pi_H(S)}(h) \leq l_S(h, a) \leq M\). Suppose that \(h\) is not in \(H_{bound}\), i.e. there exists a sequence \((S_H)_{h \in \mathcal{N}}\) of finite generating sets of \(H\) such that \(\lim_{i \to +\infty} l_{S_H}(h) = +\infty\). Consider \(S_{G_i} = S_H \times \{e\} \cup \{e\} \times A\).

It is a generating set of \(G\). We have
\[
l_{S_{G_i}}(h, a) = l_{S_H}(h) + 1
\]
and
\[ \lim_{t \to +\infty} l_{S_G}(h, a) = +\infty. \]

This contradicts \((h, a) \in G_{\text{bound}}\). Finally, since \(H_{\text{bound}} = \{e\}, G_{\text{bound}} = A\). □

**Remark 1.** Notice the following:

1. In the group \(G\) considered in Theorem 2, the finite subgroup \(A\) is contained in the ball of radius \(p\|A\|^2\) of every Cayley graph \(\Gamma(G, S)\) where \(S\) is a finite generating set of \(G\).
2. \(H\) and \(G\) are commensurable but \(H_{\text{bound}} = \{e\}\) and \(G_{\text{bound}} = A\) which is non-trivial.

6. **Elements with prescribed length**

Consider a finitely generated group \(G\), an element \(g \in G \setminus G_{\text{bound}}\) and an integer \(k > 0\). We want to answer the following question: can we construct a generating set \(S\) of \(G\), such that \(l_S(g) = k\)?

For example: \(G = \mathbb{Z}^2\). We denote \(g = (1, 0)\). Given \(n \in \mathbb{N}\) and \(S = \{(\pm 1, n), (0, \pm 1)\}\), we have

\[ l_S(g) = l_S((1, n) + n(0, -1)) = n + 1. \]

The answer is positive for non-trivial elements of free groups.

**Example 3.** Let \(g \in F_k\) be a non-trivial element and an integer \(l\). There exists a generating set \(E\) of \(F_k\) such that \(l_E(g) = l + 1\).

In fact, consider \(g \in F_k, l \in \mathbb{N} \setminus \{0\}, p = 2l + 1, S = \{x_1, \ldots, x_k\}\) the standard free generating set of \(F_k\) and \(u, v\) two distinct prime numbers, such that \(v > u > p\). We denote \(A = \{x_i^u, x_i^{-u}\}_{1 \leq i \leq k}\) and \(E = \{g^2, g^p\} \cup A\). This is clearly a generating set of \(F_k\) because \(\{x_i^u, x_i^{-u}\}_{1 \leq i \leq k}\) generates \(S\). Let us prove that \(g = g^{p-2l}\) is the shortest expression of \(g\) with respect to \(E\). Since \(S\) is a free generating set of \(F_k\), if an element \(x_j\) is used for the shortest expression of \(g\) with respect to \(E\), then there exists an expression of \(g\) with respect to \(A\) such that \(l_A(g) < l + 1\). This means that

\[ g = x_{i_1}^{n_1} \ldots x_{i_m}^{n_m} \]

where \(n_i \in \mathbb{Z}\). Notice that this expression is unique. It follows that there exists \(\alpha_j, \beta_j \in \mathbb{Z}\) such that \(\alpha_j u + \beta_j v = n_i\) and minimal \(|\alpha_j| + |\beta_j|\). Hence the shortest expression of \(g\) with respect to \(E\) is

\[ g = x_{i_1}^{\alpha_1 u + \beta_1 v} \ldots x_{i_m}^{\alpha_m u + \beta_m v}. \]

It follows that \(l_E(g) = \sum_{j=1}^m |\alpha_j| + |\beta_j|\). We have \(|\alpha_j| = \frac{|n_i - \beta_j v|}{u}\). When \(v > 3n_i u, l_A(g) > l + 1\). In this case, \(g = g^{p-2l}\) is the shortest expression of \(g\) with respect to \(E\) and \(l_E(g) = l + 1\).

We can use the same argument in order to prove that the length of elements of free abelian groups \(\mathbb{Z}^d\) can be prescribed.
Example 4. Let \( g \in \mathbb{Z}^d \) be a non-trivial element and an integer \( l \). There exists a generating set \( E \) of \( \mathbb{Z}^d \) such that \( l_E(g) = l + 1 \).

In fact, consider \( S = \{z_1, \ldots, z_n\} \) a free abelian generating set of \( \mathbb{Z}^d \). Consider \( A = \{u z_i, v z_i\}_{1 \leq i \leq k} \) and \( E = \{2 g, p g\} \cup A \) where \( p = l + 1 \) and \( u, v \) two prime numbers such that \( v > u > p \). Using the same argument as in the precedent example, we obtain that the shortest word representing \( g \) with respect to \( E \) is \( g = (p - 2 l) g \). It follows that \( l_E(g) = l + 1 \).

7. Generalization: \( G_{\text{bound}}(d) \)

We denote \( m(G) \) the minimal number of elements contained in a symmetric generating set of \( G \). Notice that if \( S \) is a finite symmetric generating set of a group \( G \) i.e \( S = S^{-1} \), then \( S \) has an even cardinality.

Definition 4. Let \( G \) be a finitely generated group.

\[
G_{\text{bound}}(d) = \{ g \in G \mid \exists M > 0 \text{ such that for every finite generating set } S \text{ of } G \text{ with cardinality less or equal to } d, \ l_S(g) \leq M \}.
\]

It is clear that for \( d \geq m(G) \),

\[
G_{\text{bound}} \subset G_{\text{bound}}(d) \subset G_{\text{bound}}(d - 1) \cdots \subset G_{\text{bound}}(m(G))
\]

and \( G_{\text{bound}} = \cap_{d \geq m(G)} G_{\text{bound}}(d) \).

7.1. Properties.

Lemma 8. For every \( d \geq m(G) \), the following properties of \( G_{\text{bound}} \) remain true for \( G_{\text{bound}}(d) \):

1. It is a characteristic subgroup of \( G \).
2. It is contained in \( FC(G) \).
3. Every finitely generated subgroup of \( G_{\text{bound}}(d) \) is virtually abelian.

Proof. Consider an element \( g \in G_{\text{bound}}(d) \). There exists a constant \( m > 0 \) such that for every generating set of cardinality less or equal to \( d \), \( l_S(g) \leq m \). Given an automorphism \( A \) of \( G \), and a generating set \( E \) of \( G \) we have seen in Lemma 1 that \( l_E(A(g)) = l_{A^{-1}(E)}(g) \leq m \). The generating sets \( A^{-1}(E) \) and \( E \) have the same cardinality, in particular this is true if the automorphism \( A \) is a conjugation. Using the same argument as in Lemma 2, it follows that for every \( d \geq m(G) \), we have \( G_{\text{bound}}(d) \) is a subgroup of \( FC(G) \). As we showed in Corollary 1 and Lemma 3 property (3) is a direct consequence of property (2). \( \square \)

Corollary 5. We have the following:

1. Every non-virtually cyclic (resp non-cyclic torsion free) hyperbolic group \( G \) has trivial \( G_{\text{bound}}(d) \) for every \( d \geq m(G) + 1 \) (resp \( d \geq m(G) \)).
2. If \( G \) is the first Grigorchuk group, then for every \( d \geq m(g) \), \( G_{\text{bound}}(d) \) is trivial.

Proof. (1) Consider a non-virtualy cyclic (resp non-cyclic torsion free) hyperbolic group \( G \). A consequence of Theorem 2 in [17] is that for every \( d \geq m(G) + 1 \) (resp \( d \geq m(G) \)), \( G \) has a sequence of generating sets of cardinality \( d \), \( \{S_i(d)\}_{i \in \mathbb{N}} \), such that the length of minimal loop in the Cayley...
Consider $\alpha p$ implies that for fixed integers $\alpha, \beta$ isomorphism. Therefore, there exist $A \in G$ such that $G_{\text{bound}}(d) \subseteq FC(G)$, we obtain that $G_{\text{bound}}(d)$ is trivial.

(2) Let $G$ be the first Grigorchuk group. It is known that $G$ is "just infinite" see Theorem 8.1 in [11]. Therefore, if $FC(G)$ is non-trivial, then it has finite index. It follows that $FC(G)$ is finitely generated and virtually abelian, see Theorem 4.3.1 in [20]. As a consequence, $G$ is a virtually abelian group. Since $G$ is a finitely generated torsion group, it can not be an infinite virtually abelian group. It is a contradiction. We conclude that $FC(G)$ is trivial.

The fact that for every $d \geq m(G)$, we have $G_{\text{bound}}(d) \subseteq FC(G)$, implies that $G_{\text{bound}}(d)$ is trivial.

\[\square\]

7.2. Examples.

**Proposition 1.** Consider $G = \mathbb{Z}^d$.

(1) For $d = 1$, if $m = 2$, $G_{\text{bound}}(m) = G$ else $G_{\text{bound}}(m) = \{0\}$.

(2) For $d \geq 2$ and $m \geq 2d$, $G_{\text{bound}}(m) = \{0\}$.

**Proof.** (1) Consider $d = 1$ and $G = \mathbb{Z}$. The generating set of $\mathbb{Z}$ with cardinality 2 is $S = \{\pm 1\}$. In this case, the length of an integer $n \in \mathbb{Z} \setminus \{0\}$ is $l_S(n) = n$ and $G_{\text{bound}}(2) = \mathbb{Z}$.

Let $p, q$ be two different prime numbers. There exist $a, b \in \mathbb{Z}$ such that $ap + bq = n$. For every integers $K > |n|, p > K + 1$ and $q > p!$, we have:

\[|a| = \frac{|bq - n|}{p} \geq \left| \frac{bp!}{p} - \frac{n}{p} \right| \geq |b(p - 1)! - 1| \geq K! - 1 \geq K.\]

We know that $S = \{\pm p, \pm q\}$ is a generating set of $\mathbb{Z}$ and $l_S(1) = |a| + |b|$. From this, we conclude that for every integers $n \in \mathbb{Z}$ and $K > |n|$, there exists a generating set $S$ such that $l_S(n) > K$. It implies that for $m \geq 4$, $G_{\text{bound}}(m) = \{0\}$.

(2) Let $X = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ non zero i.e there exists $i \in \{1, \ldots, d\}$ such that $x_i \neq 0$. Using an element of $SL(d, \mathbb{Z})$, we can suppose $x_1 \geq 1$.

Consider $p, q$ two distinct primes, and $a, b \in \mathbb{Z}$ such that $bp - aq = 1$. We have $A = \left( \begin{array}{cc} p & q \\ \alpha & b \end{array} \right) \in SL(2, \mathbb{Z})$ and the map $v \in \mathbb{Z}^2 \rightarrow Av \in \mathbb{Z}^2$ is an isomorphism. Therefore, there exist $\alpha, \beta \in \mathbb{Z}$ such that $A \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) = \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right)$. It implies that $ap + \beta q = x_1$.

For fixed integers $K > 1, p > \max\{ |x_1|, K + 1 \}$ and $q > p!$, we have:

\[|\alpha| = \frac{|\beta q - x_1|}{p} \geq \left| \frac{\beta p!}{p} - \frac{x_1}{p} \right| \geq |\beta|(p - 1)! - 1 \geq K! - 1 \geq K.\]

Consider the generating set of $\mathbb{Z}^d$ with $2d$ elements: $S_{(p, q)} = \{\pm v_1, \ldots, \pm v_d\}$ where $v_1 = (p, a, 0, \ldots, 0), v_2 = (q, b, 0, \ldots, 0), v_3 = (0, 0, 1, 0, \ldots, 0), \ldots, v_d = (0, 0, \ldots, 1)$.
With respect to generating set $S_{(p,q)}$, we have

$$X = \alpha v_1 + \beta v_2 + \sum_{j=3}^{d} x_j v_j.$$ 

For every $K > 1$, consider $(p,q)$ as above. We have

$$l_{S_{(p,q)}}(X) = |\alpha| + |\beta| + \sum_{j=3}^{d} |x_j| \geq K.$$ 

This implies that for $d \geq 2$ and $m \geq 2d$, $G_{\text{bound}}(m) = \{0\}$. 

**Proposition 2.** Let $G = \langle a, b, c \mid [a, b] = c, [a, c] = [c, b] = 1 \rangle$ be Heisenberg’s group. We have:

1. $G_{\text{bound}(4)} = \mathbb{Z} = Z(G)$ the center of $G$.
2. For $d \geq 6$, $G_{\text{bound}(d)}$ is trivial.

**Proof.** (1) Let $S = \{x^{\pm 1}, y^{\pm 1}\}$ be a generating set of $G$. Denote $z = [x, y]$. Since $G$ is nilpotent of degree 2, for every $g \in G$, we have $[[x, y], g] = [z, g] = c$. This implies that $z \in Z(G) = \mathbb{Z}$. We know that $G/Z(G) = \mathbb{Z}^2$. It induces a surjective homomorphism:

$$\pi: G \rightarrow G/Z(G) = \mathbb{Z}^2.$$ 

We want to show that $Z(G)$ is generated by $z = [x, y]$. It is known that $Z(G)$ is generated by $c = [a, b]$ with the presentation above. We have $z \in Z(G) = \mathbb{Z}$ and $c$ a generator of $Z(G)$. It follows that there exists an integer $i$ such that $z = c^i$. Let us show that $g = x^\alpha y^\beta$ with $\alpha \beta \neq 0$, is not an element of $Z(G)$.

The set $\pi\{x^{\pm 1}, y^{\pm 1}\}$ is a generating set of $G/Z(G) = \mathbb{Z}^2$. For $(\alpha, \beta) \neq (0,0)$, $\pi(g) = \alpha \pi(x) + \beta \pi(y) \neq 0$. It implies that $g$ is not an element of $Z(G)$. Therefore, for every element $g \in G$, there exist $\alpha, \beta, \gamma \in \mathbb{Z}$ such that $g = x^{\alpha} y^{\beta} z^{\gamma}$.

We can write $a = x^{\alpha_1} y^{\beta_1} z^{\gamma_1}$ and $b = x^{\alpha_2} y^{\beta_2} z^{\gamma_2}$. It follows,

$$[a, b] = x^{\alpha_1} y^{\beta_1} x^{\alpha_2} y^{\beta_2} y^{-\beta_1} x^{-\alpha_1} y^{-\beta_2} x^{-\alpha_2} = (x^{\alpha_1} y^{\beta_1} x^{-\alpha_1} y^{-\beta_1})(y^{\beta_1} x^{\alpha_2} y^{-\beta_1} x^{-\alpha_1} x^{-\alpha_2}) = x^{\alpha_2} x^{\alpha_1} y^{\beta_1} y^{\beta_2} y^{-\beta_1} x^{-\alpha_1} y^{-\beta_2} x^{-\alpha_2}.$$ 

Since $G$ is nilpotent of degree 2, for every $x, y \in G$ and $\alpha, \beta \in \mathbb{Z}$, we have $[x^\alpha, y^\beta] = x^{\alpha} y^{\beta} x^{-\alpha} y^{-\beta} = [x, y]^{\alpha \beta}$.

It follows that:

$$x^{\alpha_1} y^{\beta_1} x^{-\alpha_1} y^{-\beta_1} = z^{u_1},$$

$$y^{\beta_1} x^{\alpha_2} y^{-\beta_1} x^{-\alpha_1} x^{-\alpha_2} = z^{u_2},$$

$$x^{\alpha_1} y^{\beta_2} x^{-\alpha_1} y^{-\beta_2} = z^{u_3}.$$ 

Where $u_1 = \alpha_1 \beta_1$, $u_2 = -\beta_1 (\alpha_1 + \alpha_2)$ and $u_3 = \alpha_1 \beta_2$. We denote $j = u_1 u_2 u_3$. Using these equations in the expression of $[a, b]$, we obtain the following:

$$[a, b] = z^{u_1 u_2} z^{u_2} (x^{\alpha_1} y^{\beta_2} x^{-\alpha_1} y^{-\beta_2}) = z^{u_1 u_2 u_3} z^j.$$
Accordingly, \( c^j = c \). This implies that \(|i| = 1\) and \( z \) generates \( Z(G) \). It follows that for every generating set \( S \) of \( G \) with 4 elements, \( l_S(z) \leq 4 \). As a result \( Z(G) \subset G_{\text{bound}}(4) \). We recall that \( G_{\text{bound}}(4) \subset FC(G) \). We will show that \( FC(G) = Z(G) \).

It is known that every element of \( G \) has a unique expression of the form \( g = a^i b^j c^l \) where \( i, j, l \in \mathbb{Z} \). Therefore, if \( j \neq 0 \), then \( \forall n \in \mathbb{Z} \),

\[
a^n(a^i b^j c^l)a^{-n} = a^i(a^n b^j a^{-n})c^l.
\]

Since \( a^n b^j a^{-n} = c^{nj} b^j \), it follows that

\[
a^n(a^i b^j c^l)a^{-n} = a^i(c^{nj} b^j)c^l = a^i b^j c^{l+nj}.
\]

If \( i \neq 0 \), then \( \forall n \in \mathbb{Z} \) we have

\[b^n(a^i b^j c^l)b^{-n} = a^i b^j c^{l-ni}.
\]

Consider \( h = a^n b^m c^k \) a non-trivial element. We have

\[
hgh^{-1} = a^n b^m a^i b^j c^{l-m} a^n = a^n(b^m(a^i b^j c^{l-m})b^{-m})a^n = a^n(a^i b^j c^{l-mi})a^{-n} = a^i b^j c^{l-mi+nj}.
\]

It implies that elements which are not in \( Z(G) \) have an infinite number of conjugates. So, they are not in \( FC(G) \). We conclude that \( FC(G) = Z(G) \).

(2) We know that \( G_{\text{bound}}(6) < FC(G) \). Consider two prime number \( p, q \). The set \( S = \{a^{\pm p}, a^{\pm q}, b^{\pm 1}\} \) is a generating set of \( G \). Let us prove that for every \( n \in \mathbb{Z} \), \( c^n \notin G_{\text{bound}}(6) \). For every \( u, v \in \mathbb{Z} \) such that \( uv = n \) we have \( c^n = [a^n, b^v] \). There exist \( \alpha_u, \beta_u \in \mathbb{Z} \) such that \( \alpha_u p + \beta_u q = u \). It follows that \( c^n = [a^n, b^v] = a^{\alpha_u p + \beta_u q} b^v a^{-\alpha_u p - \beta_u q} b^{-v} \). Consider \( A \) the set of divisors of \( n \). This set is finite. We have \( l_S(c^n) \geq \min\{|\alpha_u| + |\beta_u| \mid u \in A\} \) which tends to infinity when \( p \) tends to infinity. Since for every \( d \geq 6 \), \( G_{\text{bound}}(d) \subset G_{\text{bound}}(6) \), we conclude that \( G_{\text{bound}}(d) \) is trivial for \( d \geq 6 \).

**Corollary 6.** The Heisenberg group has trivial \( G_{\text{bound}} \).

**Proof.** It follows from \( G_{\text{bound}} = \cap_{d \geq m(G)} G_{\text{bound}}(d) \) and \( G_{\text{bound}}(6) \) is trivial.

**Acknowledgements**

I would like to express my deep gratitude to my supervisor, Anna Erschler, for the advices and help she gives me in my work. I am very grateful to Markus Steenbock for his remarks on the preliminary version and discussions that have improved the exposition. I thank Arman Darbinyan for his comments.
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