Exact results for perturbative partition functions of
theories with $SU(2|4)$ symmetry

Yuhma Asano, Goro Ishiki, Takashi Okada, and Shinji Shimasaki

Department of Physics, Kyoto University
Kyoto, 606-8502, Japan

Abstract

In this paper, we study the theories with $SU(2|4)$ symmetry which consist of
the plane wave matrix model (PWMM), super Yang-Mills theory (SYM) on $R \times S^2$
and SYM on $R \times S^3/Z_k$. The last two theories can be realized as theories around
particular vacua in PWMM, through the commutative limit of fuzzy sphere and
Taylor’s T-duality. We apply the localization method to PWMM to reduce the
partition function and the expectation values of a class of supersymmetric operators
to matrix integrals. By taking the commutative limit and performing the T-duality,
we also obtain the matrix integrals for SYM on $R \times S^2$ and SYM on $R \times S^3/Z_k$.
In this calculation, we ignore possible instanton effects and our matrix integrals
describe the perturbative part exactly. In terms of the matrix integrals, we also
provide a nonperturbative proof of the large-$N$ reduction for circular Wilson loop
operator and free energy in $\mathcal{N} = 4$ SYM on $R \times S^3$.

* e-mail address : yuhma@gauge.scphys.kyoto-u.ac.jp
† e-mail address : ishiki@gauge.scphys.kyoto-u.ac.jp
‡ e-mail address : okada@gauge.scphys.kyoto-u.ac.jp
§ e-mail address : shinji@gauge.scphys.kyoto-u.ac.jp
1 Introduction

Recently, there has been increasing interest in localization in quantum field theory, which enables us to exactly compute a certain class of physical observables. The exact computations of the partition function and the vacuum expectation value (vev) of a Wilson loop have been done, for instance, in $\mathcal{N} = 2$ or $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theories in four dimensions \cite{1,2} and $\mathcal{N} = 2$ quiver Chern-Simons-matter theories in three dimensions \cite{3}. These exact results have not only provided a nontrivial evidence of AdS/CFT duality, but also revealed a surprising relationship between $\mathcal{N} = 2$ SYM on $S^4$ and Liouville/Toda CFT \cite{5,6}. More recently, the localization was also applied to the computation of the partition function of $\mathcal{N} = 1$ SYM in five dimensions to examine its relation to M5-brane \cite{7,10}.

The localization technique should be useful also for a matrix quantum mechanics or a matrix model since they, in general, involve complicated interactions. For instance, the partition functions of matrix models of Yang-Mills type in zero dimension having $[X_m, X_n]^2$ interactions were computed by using the localization in \cite{11,12}. In this paper, we apply the localization to the plane wave matrix model (PWMM) \cite{13}, which was originally proposed as a matrix quantum mechanics describing M-theory on the pp-wave spacetime in the light cone frame. This theory is a mass deformation of the BFSS matrix theory \cite{14} with maximal supersymmetries preserved. In contrast to the BFSS matrix theory, PWMM has no flat directions because of the mass deformation. The vacua of PWMM are discrete and given by fuzzy spheres.

PWMM is also known as one of theories with $SU(2\mid 4)$ symmetry \cite{15}, which consist of $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$, 2+1 SYM on $R \times S^2$ \cite{16} and PWMM. All these theories are obtained from $\mathcal{N} = 4$ SYM on $R \times S^3$ by a consistent truncation and have common features that they have mass gap, discrete spectrum and many discrete vacua. Among the $SU(2\mid 4)$ symmetric theories the following relations hold (See Fig. 1) \cite{16,17}. (a) the theory around each vacuum of 2+1 SYM on $R \times S^2$ is equivalent to the theory around a certain vacuum of PWMM and (b) the theory around each vacuum of $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ is equivalent to the theory around a certain vacuum of 2+1 SYM on $R \times S^2$ with an orbifold condition imposed.

\textsuperscript{1} Some extensions of these relations have been discussed in \cite{18,19}.
The relation (a) shows that the commutative limit of concentric fuzzy spheres with different radii in PWMM corresponds to multiple monopoles in 2+1 SYM on $R \times S^2$. If PWMM and 2+1 SYM on $R \times S^2$ are regarded as theories on D0-branes and D2-branes, respectively, then the relation (a) corresponds to the Myers effect [20]. Namely, D0-branes become polarized into fuzzy spheres by a background flux. The commutative limit of fuzzy spheres realizes a D0-D2 bound state. The monopole charges in 2+1 SYM on $R \times S^2$ are identified with the D0-charges in the D0-D2 bound state.

The relation (b) can be regarded as the Taylor’s T-duality in gauge theories on D-branes [21]. While it was originally proposed for gauge theories on flat spacetime, the relation (b) provides an extension to the case of a nontrivial $U(1)$ bundle, $S^3/Z_k \to S^2$. The orbifolding condition effectively yields the circle along which the T-duality is performed.

These relations were shown directly in the gauge theory side in [16,17]. In [15], Lin and Maldacena investigated the gauge/gravity duality for theories with $SU(2|4)$ symmetry and developed a unified method for providing the gravity dual for each vacuum of these theories. In this gravity dual picture, it was shown that the relations (a) and (b) are also satisfied [17].

By combining (a) and (b), one obtains the following relation [17]: (c) the theory around each vacuum of $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ is equivalent to the theory around a certain vacuum of PWMM with an orbifolding condition imposed.

In this paper, we obtain exact results of PWMM by using the localization. In addition, by making use of the relations (a) and (c), we obtain exact results of 2+1 SYM on $R \times S^2$ and $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ from PWMM.

We perform the localization for PWMM by constructing equivariant cohomology following [2]. Since PWMM has a noncompact time direction unlike theories considered in [2], we have to specify boundary conditions of fields at the future and the past infinities. In this paper, we demand that all fields are finite at both infinities such that the action of PWMM is finite. Once the boundary conditions are specified, the localization can be performed as usual. We construct off-shell supersymmetries in PWMM, which is denoted by $Q$ in the following, and add to the action a $Q$-exact term. The theory does not

\footnote{See also [22,24] for the integrability structure of the theories with $SU(2|4)$ symmetry and [25] for the connection to the little string theory.}
depend on the coefficient of the $Q$-exact term. Sending the coefficient of the $Q$-exact term to infinity reduces the computation of the vev of $Q$-closed operators to a one-loop integral around zeros of the $Q$-exact term. If we ignore the instanton configurations discussed below, each saddle point is labeled by a representation of $SU(2)$ algebra and a constant hermitian matrix $M$. In the end, the vev of $Q$-closed operators amounts to a sum of terms each of which is labeled by an $SU(2)$ representation and given by a matrix integral of $M$. Since each vacuum of PWMM is also labeled by an $SU(2)$ representation, each term in the sum is thought of as the contribution from the theory around the corresponding vacuum of PWMM. We then use the relations (a) and (c) to obtain exact results of 2+1 SYM on $R \times S^2$ and $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$. By extracting the contribution from the $SU(2)$ representation used in the relations (a) and (c) from the vev of a $Q$-closed operator in PWMM, we obtain the vev of the corresponding operators in SYM on $R \times S^2$ and SYM on $R \times S^3/Z_k$.

As mentioned above, there can be contributions to the saddle points from instantons and anti-instantons localizing at the past and the future infinities, respectively. This is reminiscent of the situation in $\mathcal{N} = 2$ or $\mathcal{N} = 4$ SYM on $S^4$ [2], where the instantons and the anti-instantons are localizing at the South and the North poles, respectively. In this paper, we simply ignore the instanton contributions. The evaluation of the instanton part is technically difficult and it is beyond the scope of this paper. Nevertheless, if we
restrict ourselves to the 't Hooft limit, where the instantons are suppressed, our results become exact.

As a consistency check of our computation, we reproduce a one-loop result of PWMM around the trivial background. Furthermore, we show that PWMM around the background corresponding to $\mathcal{N} = 4$ SYM on $R \times S^3$ through the relation (c) becomes a Gaussian matrix model. This is consistent with the results in [2,26,27].

In terms of the matrix integral obtained through the localization in PWMM, we also check the validity of the nonperturbative formulation of the planar $\mathcal{N} = 4$ SYM on $R \times S^3$ proposed in [28], which should be important in the context of the AdS/CFT correspondence [29]. This formulation is based on the combination of (a) and another relation (b’) in Fig. 1. Since the orbifolding condition in (b) needs infinitely large gauge group from the beginning, the relation (c) by itself can not provide a regularization of $\mathcal{N} = 4$ SYM. However, in the 't Hooft limit, one has an alternative relation (b’) which is based on the large-$N$ reduction. The large-$N$ reduction was first proposed by Eguchi and Kawai for theories on flat space [31] and the relation (b’) can be regarded as an extension of the large-$N$ reduction to the case of a nontrivial $U(1)$ bundle [28]. The large-$N$ equivalence (b’) holds only in the planar limit. However, it does not need the orbifolding condition so that its combination with (a), which we call (c’) in Fig. 1, enables us to regularize the planar $\mathcal{N} = 4$ SYM on $R \times S^3$ nonperturbatively in terms of PWMM. It should be remarked that this regularization preserves 16 supersymmetries, half of supersymmetries of the original $\mathcal{N} = 4$ SYM.

The validity of the nonperturbative formulation has been checked by performing perturbative calculations [28,32,33] and by numerical simulations [34,35]. In [32], the vev of the circular Wilson loop in $\mathcal{N} = 4$ SYM is also reproduced from PWMM within the ladder approximation. In this paper, we test this formulation by computing the free energy and the vev of the circular Wilson loop operator nonperturbatively.

This paper is organized as follows. In Section 2, we review the relations among the

---

3 Some preliminary results of numerical simulations of $\mathcal{N} = 4$ SYM in this formulation are reported in [36,37], where correlation functions and Wilson loops are numerically computed and compared with the results predicted from the gravity side.

4 The same kind of the large $N$ equivalence between $\mathcal{N} = 2$ Chern-Simons-matter theories on $S^3$ and their dimensionally reduced models was also investigated in [38,41].
theories with $SU(2|4)$ symmetry. We first perform the consistent truncation of $\mathcal{N} = 4$ SYM on $R \times S^3$ and obtain $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$, 2+1 SYM on $R \times S^2$ and PWMM. We then explain how $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ and 2+1 SYM on $R \times S^2$ can be retrieved from PWMM, namely, how the relations in Fig. 1 hold. We also discuss supersymmetric Wilson loops in these relations. In Section 3, we perform the localization in PWMM. We construct off-shell supersymmetries in PWMM and add a $Q$-exact term to the action. After the one-loop integration around saddle points, we obtain a matrix integral expression of the partition function and the vev of a Wilson loop operator in PWMM. In Section 4, using the relations (a), (c) and (c'), we obtain the partition functions and the vev of a Wilson loop operators in 2+1 SYM on $R \times S^2$ and $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ from those in PWMM. In Section 5, we summarize our results. Some useful formulae and perturbative check of our result are summarized in Appendices.

### Summary of notations

The indices used in this paper are summarized as follows:

\[
M, N, \cdots = 1, 2, \cdots, 9, 0, \quad M', N', \cdots = 1, 2, \cdots, 9, \\
a, b, \cdots = 1, 2, 3, 4, \quad a', b', \cdots = 2, 3, 4, \\
m, n, \cdots = 5, 6, 7, 8, 9, 0, \quad m', n', \cdots = 5, 6, 7, 8, 
\] (1.1)

where $M, N, \cdots$ are the indices of $SO(9,1) = SO(4) \times SO(5,1)$, $a, b, \cdots$ are the indices of the $SO(4)$ and $m, n, \cdots$ are the indices of the $SO(5,1)$.

The gauge group of the theories we consider in this paper is always a unitary group. The ranks of the gauge groups are denoted by $N$, $N_{S2}$ and $N_{PW}$ for $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$, 2+1 SYM on $R \times S^2$ and PWMM, respectively. The coupling constants for these theories are denoted by $g$, $g_{S2}$ and $g_{PW}$, respectively.

## 2 Relations among theories with $SU(2|4)$ symmetries

In this section, we review the relations among $SU(2|4)$ symmetric theories. In Section 2.1, we make consistent truncations of $\mathcal{N} = 4$ SYM on $R \times S^3$ to obtain $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$, 2+1 SYM on $R \times S^2$ and PWMM, which all have $SU(2|4)$ symmetry, many
discrete vacua and mass gap\[15,30\]. In Section 2.2, we explain how higher dimensional theories can be obtained from lower dimensional theories \[17,28\], namely, the relations in Fig. 1. In Section 2.3, we explain Wilson loops in these relations \[32\], which are computable by using the localization.

### 2.1 Theories with $SU(2|4)$ symmetries

We start with $\mathcal{N}=4$ SYM on $R \times S^3$. We follow the same notation as \[2\]. The metric of $R \times S^3$ and gamma matrices are summarized in Appendix A. We set the radius of $S^3$ to be 1. The action of $\mathcal{N}=4$ SYM on $R \times S^3$ is given by

$$S_{R \times S^3} = \frac{1}{g^2} \int d\tau d\Omega_3 \text{Tr} \left( -\frac{1}{4} F_{MN} F^{MN} - \frac{1}{2} X_m X^m - \frac{i}{2} \psi \Gamma^M D_M \psi \right),$$

(2.1)

where

$$F_{ab} = \nabla_a X_b - \nabla_b X_a - i[X_a, X_b], \quad F_{am} = D_a X_m, \quad F_{mn} = -i[X_m, X_n],$$

$$D_a = \nabla_a - i[X_a, ], \quad D_m = -i[X_m, ].$$

(2.2)

$a, b, \cdots = 1, 2, 3, 4$ are the local Lorentz indices of $SO(4)$ and $m, n, \cdots = 5, 6, \cdots, 9, 0$ are the indices of $SO(5, 1)$ R-symmetry. $a = 1$ corresponds to $R$ direction, $\tau$, while $a = 2, 3, 4$ correspond to $S^3$ direction, $(\theta, \varphi, \psi)$. $X_a$ are gauge fields, $X_m$ are scalar fields and $\psi$ is a Majorana spinor with 16 components. Because of the conformal coupling to the curvature, this theory is massive and the vacuum is trivial and unique. At the moment, we work in Lorentzian signature, so that $X_0 = -X^0$. Later, we move to Euclidean signature by regarding $X_0$ as an anti-hermitian matrix\[5\].

For later convenience, we take vielbein as right-invariant 1-form defined in Appendix B and expand the gauge field on $S^3$ in terms of it. In this local Lorentz frame, the action takes the form

$$S_{R \times S^3} = \frac{1}{g^2} \int d\tau d\Omega_3 \text{Tr} \left[ -\frac{1}{2} (\partial_1 X_{b'}) - i L_{b'} X_1 - i [X_1, X_{b'}] \right]^2$$

$$-\frac{1}{4} (2 \varepsilon_{a'b'c'} X_{c'} + i L_{a'} X_{b'} - i L_{b'} X_{a'} - i [X_{a'}, X_{b'}])^2$$

$$-\frac{1}{2} (D_a X_m)^2 - \frac{1}{2} X_m X^m - \frac{i}{2} \psi \Gamma^1 \partial_1 \psi + \frac{1}{2} \psi \Gamma^a \mathcal{L}_a \psi - \frac{3i}{8} \psi \Gamma^{234} \psi$$

---

5 As in \[2\], the integrand of the path integral is defined by $\exp(S)$. After the Wick rotation, the action $S$ becomes negative definite.
\[ + \frac{1}{4} [X_m, X_n] [X^m, X^n] - \frac{1}{2} \Psi \Gamma^m [X_m, \Psi] \], \quad (2.3) \]

where \( L_{\alpha'} \) are the Killing vectors defined in (B.8).

The action is invariant under the following supersymmetry transformations

\[ \delta_s X_M = -i \Psi \Gamma_M \epsilon, \]
\[ \delta_s \Psi = \left( \frac{1}{2} F_{MN} \Gamma^{MN} - \frac{1}{2} X_m \tilde{\Gamma}^m \Gamma^a \nabla_a \right) \epsilon. \quad (2.4) \]

Here \( \epsilon \) is a conformal Killing spinor satisfying

\[ \nabla_a \epsilon = \tilde{\Gamma}_a \epsilon, \quad (2.5) \]

where \( \tilde{\epsilon} \) is another spinor satisfying

\[ \Gamma^a \nabla_a \tilde{\epsilon} = -\frac{1}{2} \epsilon. \quad (2.6) \]

Here \( \epsilon \) is Grassmann even, so that \( \delta_s \) is Grassmann odd. One can easily solve these equations with the ansatz \( \tilde{\epsilon} = \pm \frac{1}{2} \Gamma^{19} \epsilon \), for which (2.5) and (2.6) become

\[ \nabla_a \epsilon = \pm \frac{1}{2} \Gamma^{a} \Gamma^{19} \epsilon. \quad (2.7) \]

Then, the solution is given by

\[ \epsilon_+ = \begin{pmatrix} e^{\frac{\tau}{2}} \eta_1 \\ e^{\frac{\tau}{2} g} \eta_2 \\ e^{-\frac{\tau}{2}} \eta_3 \\ e^{-\frac{\tau}{2} g} \eta_4 \end{pmatrix} \quad \text{and} \quad \epsilon_- = \begin{pmatrix} e^{-\frac{\tau}{2}} g \eta_1 \\ e^\frac{\tau}{2} \eta_2 \\ e^{\frac{\tau}{2} g} \eta_3 \\ e^{\frac{\tau}{2}} \eta_4 \end{pmatrix}, \quad (2.8) \]

for the upper and the lower sign in (2.7), respectively. \( \eta_{1,2,3,4} \) are four-component constant spinors and \( g \) and \( \bar{g} \) are defined by

\[ g = e^{\frac{\tau}{2} J_3} e^{\frac{\tau}{2} J_4} e^{-\frac{\tau}{2} J_1}, \]
\[ \bar{g} = e^{-\frac{\tau}{2} J_4} e^{-\frac{\tau}{2}} J_3 e^{-\frac{\tau}{2} J_1}, \quad (2.9) \]

where \( J_3, \bar{J}_3, J_4 \) and \( \bar{J}_4 \) are defined in Appendix A. For each case, there exist \( 4 \times 4 = 16 \) constant spinors, and thus the theory totally possesses 32 supersymmetries. Note that, for each case, half of Killing spinors do not depend on the coordinates of \( S^3 \). They will, therefore, survive even in theories with \( SU(2|4) \) symmetry, which are obtained by a consistent truncation of \( \mathcal{N} = 4 \) SYM on \( R \times S^3 \). Thus, all the \( SU(2|4) \) symmetric theories possess 16 supersymmetries.
First, we consider $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$. The $Z_k$ acts on the $S^1$ fiber of $S^3$. SYM on $R \times S^3/Z_k$ is therefore obtained by making a consistent truncation for the fields so that only the modes which have the periodicity $(\theta, \varphi, \psi) \sim (\theta, \varphi, \psi + 4\pi/k)$ are surviving. The action takes the same form as (2.1) or (2.3). The vacuum of this theory is determined by the flat connection on $S^3/Z_k$ and so characterized by the holonomy $U$ along the $S^1$ fiber up to gauge transformation. Since $\pi_1(S^3/Z_k) = \mathbb{Z}_k$, $U$ satisfies $U^k = 1$. Hence $U$ can be written as

$$U = \text{diag}(1_{M_1}, e^{2\pi i/k}1_{M_2}, e^{2\pi i \times 2/k}1_{M_3}, \ldots, e^{2\pi i(k-1)/k}1_{M_k}),$$

(2.10)

where the sum of the multiplicities is equal to the rank of the gauge group, $N = \sum_i M_i$. The vacua of $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ are parametrized by a set of the multiplicities $\{M_i | i = 1, 2, \ldots, k, \sum_i M_i = N\}$.

Second, we consider 2+1 SYM on $R \times S^2$. This theory is easily obtained by taking $k \to \infty$ limit for $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ or just dropping the fiber dependence of the fields in (2.3).

$$S_{R \times S^2} = \frac{1}{g_{S^2}^2} \int d\tau d\Omega_2 \text{Tr} \left(-\frac{1}{2} (\partial_1 X_{b'} - i L_{b'}^{(0)} X_1 - i [X_1, X_{b'}])^2ight.\
\left.-\frac{1}{4} (2 \varepsilon_{a'b'c'} X_c + i L_{a'}^{(0)} X_{b'} - i L_{b'}^{(0)} X_{a'} - i [X_{a'}, X_{b'}])^2\right.\
\left.-\frac{1}{2} (D_a X_m)^2 - \frac{1}{2} X_m X^m - \frac{i}{2} \Psi \Gamma^1 \partial_\tau \Psi + \frac{1}{2} \Psi \Gamma^{a'} L_{a'}^{(0)} \Psi - \frac{3i}{8} \Psi \Gamma^{234} \Psi\right.\
\left.+ \frac{1}{4} [X_m, X_n] [X_m, X^n] - \frac{1}{2} \Psi \Gamma^m [X_m, \Psi]\right),
$$

(2.11)

where $L_{a'}^{(0)}$ are ordinary angular momentum operators, which are defined in Appendix C.

It follows from (12.6) that the radius of $S^2$ is $\frac{1}{2}$. One can rewrite $X_{a'}$ in terms of gauge fields and a scalar field on $S^2$ by decomposing $X_{a'}$ into horizontal and vertical components;

$$\vec{X} = \Phi \vec{e}_r + a_2 \vec{e}_\varphi - a_3 \vec{e}_\theta,$$

(2.12)

where $\vec{X} = (X_2, X_3, X_4)$, $\vec{e}_r = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$, $\vec{e}_\varphi = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta)$ and $\vec{e}_\theta = (-\sin \varphi, \cos \varphi, 0)$. $a_2$ and $a_3$ are the gauge fields in the local Lorentz frame and
\( \Phi \) is the scalar field on \( S^2 \). Then, the first two lines in (2.11) are rewritten as

\[
\int d\tau d\Omega_2 \text{Tr} \left( -\sum_{i=2,3} \frac{1}{2}(f_{ii})^2 - \frac{1}{2}(f_{23} - 2\Phi)^2 - \frac{1}{2}(D_i \Phi)^2 - \sum_{i=2,3} \frac{1}{2}(D_i \Phi)^2 \right),
\]

where \( f_{ii} \) \((i = 2, 3)\) and \( f_{23} \) are the field strength on \( R \times S^2 \). The vacuum of this theory is determined by

\[
f_{ii} = 0, \quad f_{23} - 2\Phi = 0, \quad D_i \Phi = 0, \quad X_m = 0 \quad (i = 2, 3).
\]

In the gauge in which \( X_1 = 0 \) and \( \Phi \) is diagonal, the first four equations are solved by

\[
\hat{a}_2 = 0, \quad \hat{a}_3 = -\frac{\cos \theta \mp 1}{\sin \theta} \hat{\Phi}, \quad \hat{\Phi} = 2 \text{ diag}(q_{-\Lambda/2} 1_{N_{-\Lambda/2}}, \cdots, q_{s} 1_{N_s}, \cdots, q_{\Lambda/2} 1_{N_{\Lambda/2}}),
\]

where \( s = -\Lambda/2, -\Lambda/2 + 1, \cdots, \Lambda/2 \) and \( \Lambda \) is an even number. The sum of the multiplicities is equal to the rank of the gauge group, \( N_{S^2} = \sum_s N_s \). The upper and lower signs correspond to the patch I \((0 \leq \theta < \pi)\) and the patch II \((0 < \theta \leq \pi)\), respectively. Each diagonal configuration is nothing but the Dirac monopole with monopole charge \( q_s \). The charge quantization condition imposes \( q_s \) to be an integer or a half-integer. One can easily translate the solution (2.15) into that in terms of \( X_{a'} \),

\[
\hat{X}_2 = \frac{1 \pm \cos \theta}{\sin \theta} \cos \varphi \cdot \hat{\Phi}, \quad \hat{X}_3 = \frac{1 \pm \cos \theta}{\sin \theta} \sin \varphi \cdot \hat{\Phi}, \quad \hat{X}_4 = \mp \hat{\Phi}.
\]

These backgrounds are combined with angular momentum operators into those in a monopole background as

\[
L_{a'}^{(0)} + \hat{X}_{a'} = \text{diag}(L_{a'}^{(q_{-\Lambda/2})} 1_{N_{-\Lambda/2}}, \cdots, L_{a'}^{(q_s)} 1_{N_s}, \cdots, L_{a'}^{(q_{\Lambda/2})} 1_{N_{\Lambda/2}}),
\]

where \( L_{a'}^{(q)} \) is defined in (C.3).

**Plane wave matrix model**

Finally, PWMM is obtained by dropping the coordinate dependence of \( S^2 \) in \( 2 + 1 \) SYM on \( R \times S^2 \),

\[
S_{PW} = \frac{1}{g_{PW}^2} \int d\tau \text{Tr} \left( -\frac{1}{4} F_{MN} F^{MN} - \frac{1}{2} X_m X^m - \frac{i}{2} \bar{\Psi} \Gamma^M D_M \Psi \right),
\]

10
where
\[ F_{1M} = D_1 X_M = \partial_1 X_M - i[X_1, X_M] \quad (M \neq 1), \]
\[ F_{a'b'} = 2 \varepsilon_{a'b'c'} X_{c'} - i[X_{a'}, X_{b'}], \quad F_{a'm} = D_{a'} X_m = -i[X_{a'}, X_m], \quad F_{mn} = -i[X_m, X_n], \]
\[ D_1 \Psi = \partial_1 \Psi - i[X_1, \Psi], \quad D_{a'} \Psi = \frac{1}{4} \varepsilon_{a'b'c'} \Gamma^{b'c'} \Psi - i[X_{a'}, \Psi], \quad D_m \Psi = -i[X_m, \Psi]. \quad (2.19) \]

They are obtained by dropping derivatives in (2.2) in the right-invariant frame.

When both \( X_0 \) and \( X_1 \) are Wick rotated so that the theory has the ordinary Lorenzian signature, PWMM has \( R \times SO(3) \times SO(6) \) symmetry as the bosonic subgroup of \( SU(2|4) \). The first factor, \( R \), corresponds to the translation of the \( \tau \) direction and the second and the third factors correspond to the rotations for \( X_{a'} \) and \( X_m \), respectively. In this paper, we will construct an equivariant cohomology with respect to the action of a \( U(1) \) subgroup of the bosonic subgroup combined with a gauge transformation.

The vacuum of PWMM is given by the solution to the following equations
\[ \partial_1 X_{b'} - i[X_1, X_{b'}] = 0, \quad 2 \varepsilon_{a'b'c'} X_{c'} - i[X_{a'}, X_{b'}] = 0, \quad X_m = 0. \quad (2.20) \]

In \( X_1 = 0 \) gauge, the first two equations are solved by
\[ X_{a'} = -2 L_{a'}, \quad (2.21) \]
where \( L_{a'} \) is a representation of \( SU(2) \) algebra; \([L_{a'}, L_{b'}] = i \varepsilon_{a'b'c'} L_{c'}\). \( L_{a'} \) are in general reducible and can be represented as
\[ L_{a'} = \begin{pmatrix} 1_{N_{-\Lambda/2}} \otimes L_{a'}^{[j-\Lambda/2]} \\ \vdots \\ 1_{N_s} \otimes L_{a'}^{[j_s]} \\ \vdots \\ 1_{N_{\Lambda/2}} \otimes L_{a'}^{[j_{\Lambda/2}]} \end{pmatrix}, \quad (2.22) \]
where \( s = -\Lambda/2, -\Lambda/2 + 1, \ldots, \Lambda/2 \) and \( \Lambda \) is an even number. \( L_{a'}^{[j]} \) is the spin \( j \) representation matrix of \( SU(2) \) algebra and \( N_{PW} = \sum_s (2j_s + 1) N_s \).

### 2.2 \( \mathcal{N} = 4 \) SYM on \( R \times S^3/Z_k \) and 2+1 SYM on \( R \times S^2 \) from PWMM

Here we explain the relations in Fig. 1.
2.2.1 2+1 SYM on $R \times S^2$ from PWMM

First, let us review the relation (a). In order to see (a), one can utilize the harmonic expansion of the two theories in (a).

We first consider the theory expanded around the fuzzy sphere background (2.22) in PWMM. To analyze this, it is convenient to decompose the fluctuation fields around the background into blocks according to the block structure in (2.22). We call the block with size $(N_s \times N_t) \otimes ((2j_s + 1) \times (2j_t + 1))$ as $(s,t)$-block. For each block, there is a suitable matrix basis called fuzzy spherical harmonics $\hat{Y}_{J_m(jj')}$, which behave as an irreducible representation of $SU(2)$ under the adjoint action of (2.22). Several properties of fuzzy spherical harmonics are summarized in Appendix D. For instance, the $(s,t)$-block of scalars $X^{(s,t)}(\tau)$ can be expanded as

$$X^{(s,t)}(\tau) = \sum_{J=|j_s-j_t|}^{j_s+j_t} \sum_{m=-J}^{J} X_{J_m}^{(s,t)}(\tau) \otimes \hat{Y}_{J_m(j_s,j_t)},$$  \hspace{1cm} (2.23)

where $X_{J_m}^{(s,t)}(\tau)$ is a $N_s \times N_t$ matrix.

Next, let us see the theory expanded around the monopole background (2.16) in 2+1 SYM on $R \times S^2$. We decompose the fluctuation fields into blocks according to (2.17), where $(s,t)$-block is now $N_s \times N_t$ matrix. Since all the fields are in the adjoint representation, the $(s,t)$-block couples with gauge fields of the monopole background with monopole charge $q_s - q_t$. In this case, a useful basis is the monopole spherical harmonics defined in Appendix C, which form a basis of sections of a complex line bundle on $S^2$. Under the action of the angular momentum operator in the presence of a monopole, they behave as an irreducible representation of $SU(2)$. The $(s,t)$-block of scalars $X^{(s,t)}(\tau, \Omega)$ can be expanded as

$$X^{(s,t)}(\tau, \Omega) = \sum_{J=|q_s-q_t|}^{\infty} \sum_{m=-J}^{J} X_{J_m}^{(s,t)}(\tau)Y_{J_m(q_s-q_t)}(\Omega).$$  \hspace{1cm} (2.24)

The angular momentum is bounded below because the background magnetic field carries nonzero angular momentum.

Notice the similarity between (2.23) and (2.24). The angular momentum of fields in SYM on $R \times S^2$ (2.24) is bounded below by $|q_s - q_t|$ while that in PWMM (2.23) is
bounded below by \( |j_s - j_t| \) and also bounded above by \( j_s + j_t \). Thus, in (2.23) we take the limit in which
\[
2j_s + 1 = n + 2q_s \left( -\frac{\Lambda}{2} \leq s \leq \frac{\Lambda}{2} \right), \quad n \to \infty \quad \text{with} \quad \frac{g^2_{PW}}{n} = \frac{g^2_s}{4\pi} = \text{fixed} \quad (2.25)
\]
and \( N_s \) and \( \Lambda \) in PWMM are identified with those in 2+1 SYM on \( R \times S^2 \). Under this limit, \( |j_s - j_t| = \frac{1}{2} |q_s - q_t| \) and \( j_s + j_t \to \infty \) are realized. Then one can see that (2.23) coincides with (2.21). This shows that the spectrum of 2+1 SYM on \( R \times S^2 \) is completely reproduced from PWMM.

It also turns out that the interaction terms of both theories are coincident in the limit (2.25). In the mode expansion in PWMM, the coefficients of the interaction terms involve the trace of the product of three fuzzy spherical harmonics
\[
\hat{C}_{J_1m_1(J_s j_t)J_2m_2(j_s j_u)J_3m_3(j_u j_s)} \\
\equiv \text{Tr}(\hat{Y}_{J_1m_1(j_s j_t)}\hat{Y}_{J_2m_2(j_s j_u)}\hat{Y}_{J_3m_3(j_u j_s)}) \\
= (-1)^{2j_t+J_1+J_2-J_3} \sqrt{(2J_1+1)(2J_2+1)(2J_3+1)} \left( \begin{array}{ccc} J_1 & J_2 & J_3 \\ m_1 & m_2 & m_3 \end{array} \right) \left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_u & j_s & j_t \end{array} \right\}. \quad (2.26)
\]
Similarly, interaction terms in 2+1 SYM on \( R \times S^2 \) have the integral over \( S^2 \) of the product of three monopole spherical harmonics
\[
\mathcal{C}_{J_1m_1q_1J_2m_2q_2J_3m_3q_3} \equiv \int d\Omega Y_{J_1m_1q_1}(\Omega)Y_{J_2m_2q_2}(\Omega)Y_{J_3m_3q_3}(\Omega) \\
= \sqrt{(2J_1+1)(2J_2+1)(2J_3+1)} \left( \begin{array}{ccc} J_1 & J_2 & J_3 \\ m_1 & m_2 & m_3 \end{array} \right) \left( \begin{array}{ccc} J_1 & J_2 & J_3 \\ q_1 & q_2 & q_3 \end{array} \right). \quad (2.27)
\]
where \( \left( \begin{array}{ccc} J_1 & J_2 & J_3 \\ m_1 & m_2 & m_3 \end{array} \right) \) and \( \left\{ \begin{array}{ccc} J_1 & J_2 & J_3 \\ j_u & j_s & j_t \end{array} \right\} \) are the Wigner’s 3j- and 6j-symbol, respectively. In the limit (2.25), by putting \( j_u - j_s = q_1, j_t - j_u = q_2 \) and \( j_s - j_t = q_3 \) and using (12)
\[
\left\{ \begin{array}{ccc} a & b & c \\ d + R & e + R & f + R \end{array} \right\} \approx \frac{(-1)^{a+b+c+2(d+e+f+R)}}{\sqrt{2R}} \left( \begin{array}{ccc} a & b & c \\ e - f & f - d & d - f \end{array} \right), \quad (2.28)
\]
one can show that
\[
\sqrt{n}\hat{C}_{J_1m_1(j_s j_t)J_2m_2(j_s j_u)J_3m_3(j_u j_s)} \to \mathcal{C}_{J_1m_1q_1J_2m_2q_2J_3m_3q_3}. \quad (2.29)
\]
By renormalizing the fields in PWMM as, \( X \to \sqrt{n}X \), one can correctly reproduce all the interaction terms of 2+1 SYM from PWMM.

Thus, the theory around (2.16) of 2+1 SYM on \( R \times S^2 \) is equivalent to the theory around (2.22) of PWMM in the limit (2.25).
2.2.2 $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ from 2+1 SYM on $R \times S^2$

Taylor’s T-duality

Next, let us consider the relation (b) in Fig. 1. It states that SYM on $R \times S^3/Z_k$ can be equivalently described by SYM on $R \times S^2$ around appropriate monopole background with the orbifolding condition imposed. This is an extension of the T-duality in gauge theory a la Taylor to that on a $U(1)$ bundle on $S^2$.

Let us first consider SYM on $R \times S^3/Z_k$ around the trivial background. $S^3/Z_k$ can be regarded as an $S^1$-bundle on $S^2$ and one can make the Kaluza-Klein (KK) expansion along the fiber $S^1$ direction. Since $S^3/Z_k$ is a nontrivial fiber bundle, the KK expansion can be made locally. The theory thus obtained is the theory on $R \times S^2$ with infinite number of KK modes. These KK modes are sections of a complex line bundle on $S^2$ and can be regarded as fluctuations around a monopole background in 2+1 SYM on $R \times S^2$, where the monopole charge is identified with the KK momentum. Therefore, $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ can be obtained by expanding 2+1 SYM on $R \times S^2$ around an appropriate monopole background so that all the KK modes of $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ are reproduced. This is achieved in the following manner. First, we take the background (2.15) in 2+1 SYM on $R \times S^2$ with

$$q_s = \frac{k_s}{2}, \quad N_s = N \quad \text{for} \quad -\infty \leq s \leq \infty,$$

(2.30)

where $\Lambda$ in (2.15) is set to infinity from the beginning. Then, we make the identification among blocks of fluctuations around (2.30) as

$$X^{(s,t)}(\tau, \Omega) = X^{(s+1,t+1)}(\tau, \Omega) \quad \text{for} \quad -\infty < s, \forall t < \infty.$$

(2.31)

In the end, we can retrieve (an infinite copies of) $\mathcal{N} = 4$ $U(N)$ SYM on $R \times S^3/Z_k$ around the trivial background. In fact, the classical action of SYM on $R \times S^2$ becomes equal to that of SYM on $R \times S^3/Z_k$ if the infinite multiplicity, $\sum_s$, is absorbed by the renormalization of the coupling constant as

$$\frac{\pi g_{S^2}^2}{2 \sum_s} \to g^2.$$

(2.32)

The same argument holds for $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ around a nontrivial background [18]. To realize the theory around the background specified by the holonomy
we introduce a further internal structure to \( \hat{\Phi} \) in (2.15) by replacing
\[
q_s 1_{N_s} \to \text{diag}(q_s^{(1)} 1_{N_s^{(1)}}, \ldots, q_s^{(k)} 1_{N_s^{(k)}}),
\]
for all \( s \in \mathbb{Z} \). Then the appropriate background for (2.10) is given by
\[
q_s^{(i)} = \frac{ks}{2} + \frac{(i - 1)s}{2}, \quad N_s^{(i)} = M_i
\]
for \( i = 1, 2, \ldots, k \) and \( s \in \mathbb{Z} \). By expanding the theory on \( R \times S^2 \) around the monopole background with (2.34), one obtains the theory on \( R \times S^3 / Z_k \) with the holonomy (2.10).

Note that the matrix size of 2+1 SYM has to be infinity to perform the T-duality due to the orbifolding condition (2.31), for which \( \Lambda \to \infty \) is necessary. Thus, this can not be applied to 2+1 SYM on \( R \times S^2 \) with finite matrix size.

**Large-\( N \) reduction**

If we restrict ourselves to the planar limit, we have an alternative way, the large-\( N \) reduction, to realize the theory on \( R \times S^3 / Z_k \). See the relation (b') in Fig. 1. This method does not need the orbifolding condition (2.31) and hence it can be applied to SYM on \( R \times S^2 \) with finite matrix size. This implies that if one finds a good UV regularization for SYM on \( R \times S^2 \), one can also regularize the planar SYM on \( R \times S^3 / Z_k \) with the same regularization by using the large-\( N \) equivalence (b'). The matrix size on \( R \times S^2 \) corresponds to the UV cutoff for the momentum along the fiber direction. As we have seen above, the theory on \( R \times S^2 \) can be regularized by PWMM through the relation (a). Hence, in terms of the large-\( N \) reduction, the planar SYM on \( R \times S^3 / Z_k \) can be regularized by PWMM as we will see in Section 2.2.3.

Let us review the large-\( N \) reduction. It is shown in [28] that the planar limit of \( \mathcal{N} = 4 \) SYM on \( R \times S^3 / Z_k \) around the trivial background can be retrieved from 2+1 SYM on \( R \times S^2 \) in the following way. We first expand 2+1 SYM around the background (2.15) with
\[
q_s = \frac{ks}{2}, \quad N_s = N \quad \text{for} \quad -\frac{\Lambda}{2} \leq s \leq \frac{\Lambda}{2}.
\]

At the end of our calculations, we take the limit in which
\[
\Lambda \to \infty, \quad N \to \infty, \quad \text{with} \quad \frac{\pi g_{s^2}^2 N}{2} = g^2 N = \text{fixed}.
\]
Then, we retrieve the planar limit of $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ around the trivial background.

The theory around nontrivial background with the holonomy \((2.10)\) would be also obtained by replacing the distribution of the monopole charges as in \((2.34)\).

### 2.2.3 $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ from PWMM

#### Taylor’s T-duality and fuzzy sphere

It is then clear how one can obtain $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ from PWMM. This is achieved by the relation (c) in Fig. 1 which is given by the combination of (a) and (b). Let us first consider the theory on $R \times S^3/Z_k$ around the trivial background. This theory is realized from PWMM though the relation (c) as follows. We first expand PWMM around the particular background \((2.22)\) in which the spin $j_s$ of the $s$-th block satisfies $2j_s + 1 = n + ks$. All the multiplicities $N_s$ are set to $N$. We then impose the orbifolding condition on the fluctuations in PWMM. Through (a), the resultant theory is equivalent to SYM on $R \times S^2$ with monopole charges \((2.30)\) with the orbifolding condition \((2.31)\) imposed on the fluctuations. Then through Taylor’s T-duality, this theory is equivalent to $U(N) \mathcal{N} = 4$ SYM on $R \times S^3/Z_k$. The coupling constant should be renormalized as

$$\frac{2\pi^2 g_{PW}^2}{n \sum_s} \to g^2. \quad (2.37)$$

For the theory around the nontrivial background labeled by the holonomy \((2.10)\), we replace the distribution of the monopole charges to \((2.34)\).

Note that in order to perform the T-duality, we have to start with PWMM with infinitely large matrices. Namely, the formal parameters $n$ and $\Lambda$ should be infinite.

#### Large-$N$ reduction on $S^3/Z_k$

In order to realize $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ from PWMM with finite matrix size, we can make use of the relation (c') in Fig. 1 obtained by combining the relations (a) and (b'); We expand PWMM around the background \((2.22)\) with

$$2j_s + 1 = n + ks, \quad N_s = N \quad \text{for} \quad -\frac{\Lambda}{2} \leq s \leq \frac{\Lambda}{2} \quad (2.38)$$

16
and take the limit in which

\[ n \to \infty, \quad \Lambda \to \infty, \quad n - \Lambda \to \infty, \quad N \to \infty \]

with

\[ \frac{2\pi^2 g_F^2 N}{n} = g^2 N = \text{fixed}. \] (2.39)

Then, the planar limit of \( N = 4 \) SYM on \( R \times S^3 / \mathbb{Z}_k \) around the trivial background is retrieved. The theory around nontrivial background would be also obtained by the modification shown in (2.34).

Note that before one takes the continuum limit, the theory is described by a matrix quantum mechanics with finite matrix size. Hence, this relation provides a non-perturbative formulation of the planar \( N = 4 \) SYM on \( R \times S^3 / \mathbb{Z}_k \) in terms of PWMM, which is alternative to the lattice formulation. The parameters \( n \) and \( \Lambda \) correspond to the UV momentum cutoffs for the \( S^2 \) and the \( S^1 \) directions, respectively.

### 2.3 Wilson loop

Let us consider supersymmetric Wilson loops in \( N = 4 \) SYM on \( R \times S^3 \). The supersymmetric Wilson loop in \( N = 4 \) SYM on \( R \times S^3 \) takes the form

\[
W(C) = \frac{1}{N} \text{Tr} P \exp \left( i \int_0^1 ds \left\{ \dot{x}^\mu(s) e_\mu^a(x(s)) X_a(x(s)) + i |\dot{x}(s)| \Theta^m(s) X_m(x(s)) \right\} \right)
\] (2.40)

where the contour \( C \) is parametrized by \( x^\mu : [0, 1] \to C \) and \( \Theta^m(s) \) is a vector satisfying \( \eta_{mn} \Theta^m \Theta^n = 1 \). In order for the Wilson loop to be invariant under (2.4), a Killing spinor in (2.8) has to satisfy

\[
\{ \dot{x}^\mu(s) e_\mu^a \Gamma_a + i |\dot{x}(s)| \Theta^m(s) \Gamma_m \} \epsilon(x) = 0.
\] (2.41)

We parametrize the great circle of \( S^3 \) by

\[
x^\mu(s) : (\tau(s), \theta(s), \varphi(s), \psi(s)) = (0, 0, 0, 4\pi s).
\] (2.42)

The Wilson loop on this great circle with \( \Theta^m = i\delta^{m0} \) is written as

\[
W(\text{circle}) = \frac{1}{N} \text{Tr} P \exp \left( 2\pi i \int_0^1 ds \left\{ X_4(x(s)) - X_0(x(s)) \right\} \right).
\] (2.43)
Then (2.44) becomes

\[(\Gamma_4 - \Gamma_0)\epsilon = 0.\] (2.44)

In either case of \(\epsilon_+\) or \(\epsilon_-\) in (2.8), \(\eta_3 = -J_4\eta_1\) and \(\eta_4 = -\bar{J}_4\eta_2\) solves (2.41). Hence, the Wilson loop along the great circle of \(S^3\) is half-BPS. It is shown in [2] that the vev of the circular Wilson loop can be computed by using the localization as

\[
\langle W(\text{circle}) \rangle = \frac{1}{N} \langle \text{Tr} e^{2\pi M} \rangle \equiv \frac{1}{Z} \int dM \frac{1}{N} \text{Tr} e^{2\pi M} e^{-\frac{4\pi^2}{g^2} \text{Tr} M^2},
\] (2.45)

where \(\lambda\) is the \('t\) Hooft coupling.

The Wilson loop in \(\mathcal{N} = 4\) SYM on \(R \times S^3/Z_k\) takes the same form as (2.40) except that the contour is on \(R \times S^3/Z_k\) and only the modes which respect the periodicity \((\theta, \varphi, \psi) \sim (\theta, \varphi, \psi + \frac{4\pi}{k})\) are left. The contour (2.42) is considered as the one on \(S^3/Z_k\) which winds \(k\) times around the nontrivial cycle on \(S^3/Z_k\).

One can construct the operators in 2+1 SYM on \(R \times S^2\) and PWMM which are equivalent, through the relations in Fig. 1, to the Wilson loop (2.40) in SYM on \(R \times S^3/Z_k\) such that its contour is closed in \(S^3\) [13]. They are obtained by applying the consistent truncation to (2.40). For the circular Wilson loop operator (2.43), they can be constructed as follows. The operator in 2+1 SYM on \(R \times S^2\) can be obtained by dropping the coordinate dependence of the \(S^1\)-fiber of \(S^3\),

\[
W_{R \times S^2} = \frac{1}{N_{S^2}} \text{Tr} \exp (2\pi i (X_4 - X_0)) \bigg|_{(\tau, \theta, \varphi) = (0, 0, 0)}. \tag{2.46}
\]

Since we have dimensionally reduced the \(S^1\)-fiber direction where the Wilson loop was winding, \(X_4\) in (2.46) contains only the vertical component \(\Phi\) in (2.12). Therefore, this operator is just a local operator of scalar fields on \(R \times S^2\). The operator in PWMM is obtained from (2.46) by the dimensional reduction and takes the same form,

\[
W_{PWMM} = \frac{1}{N_{PW}} \text{Tr} \exp (2\pi i (X_4 - X_0)) \big|_{\tau = 0}. \tag{2.47}
\]

The operators (2.46) and (2.47) preserve half of the supersymmetries in \(SU(2|4)\) symmetric theories.

Note that the matrix sizes \(N_{S^2}\) and \(N_{PW}\) have to be infinite for (b) and (c), so that (2.46) and (2.47) are not well-defined in these cases. However, one can treat the matrix
sizes as formal products, $N_{S^2} = N\Lambda$ and $N_{PW} = Nn\Lambda$, to see the following equivalence \[21\] \[43\].

Based on the relations in Fig. 1, one can show the following equivalence between the operators (2.43), (2.46) and (2.47). It follows from the relation (a) that the vev of (2.46) in SYM on $R \times S^2$ around (2.16) is equivalent to the vev of (2.47) in PWMM around (2.22) in the limit (2.25). In addition, it also follows from the relation (c) that the vev of (2.43) in $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ around the vacuum labeled by (2.10) is equivalent to the vev of (2.47) in PWMM around the corresponding vacuum with the orbifolding condition imposed (See Section 2.2.3). The similar equivalence holds for the relation (c’) if one takes the fuzzy sphere vacuum in PWMM with (2.38) and takes the continuum limit (2.39). The statement for (c’) was checked explicitly in [32] to all orders of the perturbation theory within the ladder approximation.

3 Localization in PWMM

In this section, we calculate the partition function of PWMM up to the instanton part by applying the localization method. We first construct off-shell supersymmetries in PWMM and then add a SUSY exact term to the action. Then the path integral is dominated by the saddle point configuration of the exact term as usual. The saddle point is given by the fuzzy sphere configuration labeled by the representation of $SU(2)$ Lie algebra. After the one-loop integral is performed, the total partition function is given, up to the instanton part, by a sum over all the representations of $SU(2)$ whose dimensions are equal to the matrix size $N_{PW}$ of PWMM,

$$Z = \sum_\mathcal{R} Z_{\mathcal{R}},$$

where $\mathcal{R}$ is the $N_{PW}$ dimensional representation of $SU(2)$ and $Z_{\mathcal{R}}$ is contribution from the corresponding saddle point. We will see that, for every $\mathcal{R}$, $Z_{\mathcal{R}}$ is written as an eigenvalue integral.

3.1 Supersymmetry

Since PWMM is obtained by the dimensional reduction of $\mathcal{N} = 4$ SYM on $R \times S^3$ to one dimension, the Killing spinors in PWMM are also obtained from those in $\mathcal{N} = 4$ SYM.
on $R \times S^3$ by dropping the components depending on the coordinates of $S^3$. Namely, $\epsilon_+$ with $\eta_2 = \eta_4 = 0$ and $\epsilon_-$ with $\eta_1 = \eta_3 = 0$ in (2.8) are the Killing spinors in PWMM. We consider only $\epsilon_+$ (with $\eta_2 = \eta_4 = 0$) and omit the subscript $+$ in the following. We impose a further condition on $\epsilon$ that the supersymmetry transformation by $\epsilon$ leaves the Wilson loop in PWMM (2.47) invariant. Consequently, $\epsilon$ takes the following form

$$\epsilon = e^{\frac{x}{2} \Gamma^{09}} e^{-\frac{x}{2} \Gamma^{49}} \begin{pmatrix} \eta_1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $\eta_1$ is a four-component constant spinor normalized as $\eta_1 \eta_1 = 1$. In fact, the Killing vector $v^M = \epsilon \Gamma^M \epsilon$ constructed from (3.2) has components,

$$v^0 = 2 \cosh \tau, \quad v^4 = -2, \quad v^9 = 2 \sinh \tau,$$

with all the other elements zero. The field defined by

$$\phi := v^M X_M$$

is invariant under this supersymmetry because of (A.9) and hence the Wilson loop (2.47) defined as the exponential of $\phi$ is also invariant.

We extend this supersymmetry to off-shell following [44]. We introduce seven auxiliary fields $K_i (i = 1, 2, \cdots, 7)$ and modify the supersymmetry transformations in PWMM to

$$\delta_s X_M = -i \Psi \Gamma_M \epsilon,$$

$$\delta_s \Psi = \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon - X_m \Gamma^m \Gamma^{19} \epsilon + K^i \nu_i,$$

$$\delta_s K_i = i \nu_i \Gamma^M D_M \Psi.$$  

(3.5)

Here, $\nu_i$ are bosonic spinors determined by the closure of the transformations. In fact, the closure requires $\nu_i$ to satisfy

$$\epsilon \Gamma^M \nu_i = 0,$$

$$\frac{1}{2} (\epsilon \Gamma^N \epsilon) \Gamma^N_{\alpha \beta} = \nu^i \nu^j_{\alpha \beta} + \epsilon_\alpha \epsilon_\beta,$$

$$\nu_i \Gamma^M \nu_j = \delta_{ij} \epsilon \Gamma^M \epsilon.$$  

(3.6)
Conversely, if these equations are satisfied, the supersymmetry (3.5) together with the bosonic symmetries in PWMM form a closed algebra. For a given supersymmetry parameter $\epsilon$, the spinors $\{\nu_i | i = 1, \ldots, 7\}$ can be determined by solving (3.6). When $\epsilon$ is given by (3.2), the equations (3.6) are solved by

$$\nu_i = \sqrt{2} e^{\frac{\pi}{4} \Gamma_{109}} e^{-\frac{\pi}{4} \Gamma_{49}} \Gamma_{i8} \left( \eta_1 \right), \quad (i = 1, 2, \ldots, 7)$$

(3.7)

This is easily checked by noting that (3.2) and (3.7) are equal to constant spinors up to a local Lorentz transformation represented by $e^{\frac{\pi}{4} \Gamma_{109}} e^{-\frac{\pi}{4} \Gamma_{49}}$. Since these constant spinors satisfy (3.6) and the equations (3.6) are Lorentz covariant, (3.2) and (3.7) also satisfy (3.6).

In order to make the action of PWMM invariant under (3.5), quadratic terms,

$$\frac{1}{g_{FW}^2} \int d\tau \frac{1}{2} \text{Tr} K_i K_i,$$

(3.8)

should be added to the action (2.18). Since these terms have the wrong sign, $K_i$’s should be integrated over the imaginary axis.

In the following computation, we put

$$\eta_1 = (1, 0, 0, 0)^T$$

(3.9)

for simplicity.

For later convenience, we make a change of variables of the path integral for the fermion field. Since $\{\Gamma^{M'} \epsilon, \nu^i | M' = 1, \ldots, 9, i = 1, \ldots, 7\}$ forms the orthogonal basis of 16 component spinors, $\Psi$ can be decomposed as

$$\Psi = \Psi_{M'} \Gamma^{M'} \epsilon + \Upsilon_i \nu^i.$$  

(3.10)

We treat $\{\Psi_{M'}, \Upsilon_i\}$ as the new variables in the path integral. The supersymmetry transformations are rewritten using the new variables as

$$\delta_s X_{M'} = -i (\epsilon \epsilon) \Psi_{M'}, \quad (\epsilon \epsilon) \delta_s \Psi_{M'} = (\delta_\phi + \delta_{U(1)}) X_{M'},$$

\footnote{Rigorously speaking, we treat $\Psi_{M'} \sqrt{\epsilon \epsilon}$ and $\Upsilon_i \sqrt{\epsilon \epsilon}$ as the new variables to have a trivial Jacobian in the path integral measure.}
\[ (e\epsilon)\delta_s Y_i := H_i, \quad \delta_s H_i = -i(e\epsilon)(\delta_\phi + \delta_{U(1)})Y_i, \quad \delta_s \phi = 0, \quad (3.11) \]

where \( \phi \) is defined in (3.14) and \( H_i (i = 1, 2, \ldots, 7) \) are defined by

\[ H_i = (e\epsilon)K_i + 2\nu_i\bar{\epsilon}X_0 + s_i, \quad (3.12) \]

\[ s_i := \nu_i \left( \frac{1}{2} \sum_{P,Q=1}^9 F_{PQ}\Gamma^{PQ}\epsilon - 2 \sum_{m=5}^9 X_m\Gamma^m\bar{\epsilon} \right). \quad (3.13) \]

\( \delta_\phi \) denotes the gauge transformation with parameter \( \phi \) and \( \delta_{U(1)} \) is the \( U(1) \) transformation,

\[
\begin{align*}
\delta_{U(1)} X_{a'} &= -2\epsilon_{a'b'd}v^dX^{b'}, \\
\delta_{U(1)} X_{m'} &= 2(-\delta_{5m'}X_8 + \delta_{8m'}X_5 - \delta_{7m'}X_6 + \delta_{6m'}X_7), \\
\delta_{U(1)} Y_i &= 2(\delta_{i1}Y_4 + \delta_{i2}Y_3 - \delta_{i3}Y_2 - \delta_{i4}Y_1 + \delta_{i6}Y_7 - \delta_{i7}Y_6). 
\end{align*}
\quad (3.14)
\]

This transformation forms a diagonal \( U(1) \) subgroup of the \( SO(3) \times SO(6)_R \) symmetry in PWMM.

We also introduce the collective notation,

\[ X := \begin{pmatrix} X_{M'} \\ (e\epsilon)Y_i \end{pmatrix}, \quad X' := \begin{pmatrix} -i(e\epsilon)\Psi_{M'} \\ H_i \end{pmatrix}. \]

Then the supersymmetry can be written in a compact form as

\[ \delta_s X = X', \quad \delta_s X' = -i(\delta_\phi + \delta_{U(1)})X, \quad \delta_s \phi = 0. \quad (3.15) \]

### 3.2 Saddle point

We construct a supersymmetry exact term \( \delta_s V \). The Grassmannian functional \( V \) is defined by

\[ V = \Psi\overline{\delta_s \Psi}, \quad (3.16) \]

where

\[ \overline{\delta_s \Psi} = \frac{1}{2} F_{MN}\tilde{\Gamma}^{MN}\epsilon + \frac{1}{2} X_m\tilde{\Gamma}^{am}\nabla_a\epsilon - K^i\nu_i. \quad (3.17) \]

\(^7\)In the Lorentzian signature that we consider in this paper, it is a subgroup of \( SO(3) \times SO(5,1)_R \).
The bar stands for the Hermitian conjugate when $X_0$ and $K_i$’s are integrated over the imaginary axis and are regarded as anti-Hermitian matrices.

The functional $V$ can be expressed in terms of $\Psi_M$ and $\Upsilon_i$ defined in (3.10) as,

$$V = \left( D_M'(v^Q X_Q) + \delta_{U(1)} X_M' \right) \Psi_M + \bar{H}^i \Upsilon_i,$$

(3.18)

where $\bar{H}^i$ and $\bar{X}_Q$ are defined as the Hermitian conjugates of $H_i$ and $X_Q$, respectively. Namely, they are obtained by flipping signs of $X_0$ and $K_i$ in $H_i$ and $X_Q$.

The bosonic part of $\delta_s V$ is calculated to be

$$\delta_s V_{bos} = -e^\tau(D_1 X_0 + X_0 - e^{-\tau} K_5)^2 - e^{-\tau}(D_1 X_0 - X_0 + e^\tau K_5)^2 - 2c \sum_{a'} 4 (D_{a'} X_0)^2$$

$$- 2c \sum_{i \neq 5} (K^i)^2 + 2c(D_4 X_9)^2 + 2c[X_0, X_9]^2 + 2c \sum_{m' = 5} [X_0, X_{m'}]^2 + S$$

$$+ 4 \sum_{a = 1}^3 \left[ e^{-\tau} \left\{ F^a_{a4} - \frac{1}{2} D_a(e^\tau X_9) + F^a_{a+4,8} \right\}^2 + e^\tau \left\{ F^{-}_{a4} + \frac{1}{2} D_a(e^{-\tau} X_9) - F^{-}_{a+4,8} \right\}^2 \right],$$

(3.19)

where $c$ is just a shorthand notation, $c := \cosh \tau$. In the following, we also use $s := \sinh \tau$. $S$ is defined by

$$S = e^\tau(X_5 + D_1 X_5 + D_2 X_6 + D_3 X_7 + D_4 X_8 + e^{-\tau} F_{98})^2$$

$$+ e^{-\tau}(X_5 - D_1 X_5 - D_2 X_6 - D_3 X_7 + D_4 X_8 - e^\tau F_{98})^2$$

$$+ e^\tau(X_6 + D_1 X_6 - D_2 X_5 + D_3 X_8 - D_4 X_7 - e^{-\tau} F_{97})^2$$

$$+ e^{-\tau}(X_6 - D_1 X_6 + D_2 X_5 - D_3 X_8 - D_4 X_7 + e^\tau F_{97})^2$$

$$+ e^\tau(X_7 + D_1 X_7 - D_2 X_8 - D_3 X_5 + D_4 X_6 + e^{-\tau} F_{96})^2$$

$$+ e^{-\tau}(X_7 - D_1 X_7 + D_2 X_8 + D_3 X_5 + D_4 X_6 - e^\tau F_{96})^2$$

$$+ e^\tau(X_8 + D_1 X_8 + D_2 X_7 - D_3 X_6 - D_4 X_5 - e^{-\tau} F_{95})^2$$

$$+ e^{-\tau}(X_8 - D_1 X_8 - D_2 X_7 + D_3 X_6 - D_4 X_5 + e^\tau F_{95})^2.$$

(3.20)

The covariant derivatives $D_M$ in PWMM are defined in (2.19). $F^\pm_{ab}$ and $F^\pm_{m'n'}$ are selfdual and anti-selfdual part:

$$F^\pm_{ab} = \frac{1}{2}(F_{ab} \pm \frac{1}{2} \varepsilon_{abcd} F^{cd}), \quad F^\pm_{m'n'} = \frac{1}{2}(F_{m'n'} \pm \frac{1}{2} \varepsilon_{m'n'p'q'} F^{p'q'}).$$

(3.21)
Here $\varepsilon_{abcd}$ and $\varepsilon_{m' n' p' q'}$ are completely anti-symmetric tensors with $\varepsilon_{1234} = 1$ and $\varepsilon_{5678} = 1$, respectively. After the Wick rotation, $X_0 = iX_0^{(E)}$ and $K_i = iK_i^{(E)}$ for $i = 1, 2, \cdots, 7$, the bosonic part $\delta_s V|_{bos}$ is given by a sum of positive-definite terms.

Then the saddle point of $\delta_s V$ is determined by putting all the terms to be zero. If we ignore possible instanton configurations discussed below, the saddle point configuration (denoted by putting a hat on the fields, $\hat{\cdots}$) is given, in the temporal gauge $X_1 = 0$, by

$$
\hat{X}_0^{(E)} = \frac{M}{c}, \quad \hat{K}_5^{(E)} = \frac{M}{c^2}, \quad \hat{X}_{a'} = -2L_{a'}. (a' = 2, 3, 4)
$$

All the other fields are zero at the saddle point. Here $L_{a'}(a' = 2, 3, 4)$ are representation matrices of $SU(2)$ generators (2.22) and $M$ is a constant matrix satisfying $[L_{a'}, M] = 0$ for $a' = 2, 3, 4$. It is decomposed as

$$
M = \begin{pmatrix}
M_{-\Lambda/2} \otimes 1_{2j_{-\Lambda/2}+1} \\
& \ddots \\
& & M_s \otimes 1_{2j_s+1} \\
& & & \ddots \\
& & & & M_{\Lambda/2} \otimes 1_{2j_{\Lambda/2}+1}
\end{pmatrix}, \quad (3.23)
$$

where $M_s (s = -\Lambda/2, -\Lambda/2+1, \cdots, \Lambda/2)$ is an $N_s \times N_s$ constant matrix. Thus the saddle point is labeled by the representation of $SU(2)$ and $\Lambda + 1$ matrices $\{M_s\}$. One can take a gauge in which all $M_s$ are simultaneously diagonalized. We will work in this gauge in the following. The eigenvalues of $M_s$ are denoted by $m_{si}$, where $i = 1, 2, \cdots, N_s$.

The configurations (3.22) are obtained as follows. The first two in (3.22) can be obtained straightforwardly by equating the terms containing $X_0^{(E)}$ and $K_5^{(E)}$ in (3.19) with zero. Similarly, $K_i = 0$ ($i = 1, 2, 3, 4, 6, 7$) and $D_4X_9 = 0$ follow easily. By using the saddle point equations for $F_{ab}$ with $a, b = 1, 2, 3$, one can rewrite the Bianchi identity, $D_1F_{23} + D_2F_{31} + D_3F_{12} = 0$, to

$$\sum_{a=1}^{3} D_a^2(cX_9) + D_1(F_{67} - F_{58}) + D_2(F_{75} - F_{86}) + D_3(F_{56} - F_{78}) = 0. \quad (3.24)$$

The sum of the last three terms are calculated as

$$-i[D_1X_6 - D_2X_5 + D_3X_8, X_7] - i[-D_1X_7 + D_2X_8 + D_3X_5, X_6]$$

$$-i[D_1X_8 + D_2X_7 - D_3X_6, X_5] - i[-D_1X_5 - D_2X_6 - D_3X_7, X_8]$$
\begin{equation}
-i \sum_{m=5}^{8} [cF_{9m}, X_m].
\end{equation}

The equality follows if one uses the saddle point equations coming from $S$. Thus (3.24) becomes

\begin{equation}
\sum_{a=1}^{3} D_a^2(cX_9) - \sum_{m'=5}^{8} [X_{m'}, [X_{m'}, cX_9]] = 0.
\end{equation}

By multiplying (3.26) by $cX_9$, taking the trace and integrating over $\tau$, one obtains

\begin{equation}
\int_{-\infty}^{\infty} d\tau \text{Tr} \left[ \sum_{a=1}^{3} \{D_a(cX_9)\}^2 - \sum_{m'=5}^{8} [X_{m'}, cX_9]^2 \right] = 0.
\end{equation}

The surface term for the partial integration is dropped above. This is justified as follows. The field configurations which diverges at infinity, $\tau \to \pm \infty$, do not contribute to the path integral since the action is infinite with such configurations. Hence we impose that all fields are finite at both infinities, $\tau \to \pm \infty$, so that the action is finite. Under this boundary condition, $cX_9$ is not necessarily finite at infinity. However, if one assumes that $cX_9$ is divergent at infinity, there is no solution to the saddle point equations. Hence one can assume that $cX_9 \sim O(1)$ as $\tau \to \pm \infty$ at the saddle point. Within this assumption, the surface term is vanishing.

Since all the terms in (3.27) are positive-definite, each term should vanish. Thus in the temporal gauge, one finds that

\begin{equation}
X_9 = \frac{B_9}{c}, \quad [X_M, X_9] = 0, \quad (M = 2, \cdots, 8)
\end{equation}

where $B_9$ is a constant matrix. Similarly, for $m' = 5, \cdots, 8$, by combining (3.28) and the saddle point equations coming from $S$, one can obtain

\begin{equation}
\int d\tau \text{Tr} \left[ \sum_{a=1}^{3} (D_a X_{m'})^2 - \sum_{n'=5}^{8} [X_{n'}, [X_{n'}, X_m]]^2 \right] = 0.
\end{equation}

Applying the same argument as $X_9$ yields

\begin{equation}
X_{m'} = B_{m'}, \quad [X_M, X_{m'}] = 0, \quad (M = 2, 3, 5, 6, 7, 8)
\end{equation}

where $B_{m'}$ are constant matrices. By substituting (3.28) and (3.30) into the saddle point equations, (3.22) is obtained.
In addition to (3.22), one should also take into account the instanton configurations localizing at infinity \( \tau \to \pm \infty \). When \( \tau \) goes to infinity, some terms in (3.19) automatically vanish because of the coefficients \( e^{\pm \tau} \). Then the saddle point equations for the remaining terms in (3.19) are reduced to (anti-)self-dual equations. They are solved by the instanton solutions in PWMM [45,46], which interpolate different fuzzy sphere vacua. Since all fields should take the form of (3.22) for finite \( \tau \), these instantons should be localized at infinity \( \tau \to \pm \infty \). The evaluation of the contribution from the instanton configurations is beyond the scope of this paper and here we ignore the instantons.

### 3.3 Ghost fields

In order to make a gauge-fixing, we introduce ghost fields, \((C, C_0, \tilde{C}, \tilde{C}_0, b, b_0, a_0, \tilde{a}_0)\), which obey the following BRS transformations,

\[
\begin{align*}
\delta_B X &= [X, C], & \delta_B X' &= [X', C], \\
\delta_B C &= a_0 - C^2, & \delta_B \phi &= [\phi, C], \\
\delta_B \tilde{C} &= b, & \delta_B b &= [\tilde{C}, a_0], \\
\delta_B \tilde{a}_0 &= i\tilde{C}_0, & \delta_B \tilde{C}_0 &= -i[\tilde{a}_0, a_0], \\
\delta_B b_0 &= iC_0, & \delta_B C_0 &= -i[b_0, a_0], & \delta_B a_0 &= 0.
\end{align*}
\]

(3.31)

Our convention is that \((b, b_0, a_0, \tilde{a}_0)\) are bosonic and \((C, \tilde{C}, C_0, \tilde{C}_0)\) are fermionic. The latter anti-commutes with \(\Psi\) so that if \(X\) (or \(X'\)) denotes a fermionic field, the commutator in (3.31) shall express the anti-commutator, \(-\{X, C\}\) (or \(-\{X', C\}\)). The ghost fields with subscript 0 have only zero modes for both \(R\) direction and the fuzzy sphere direction and they eliminate the zero modes of the ghosts properly as we will see shortly. As in [2], \(a_0\) should be integrated over the imaginary axis. The square of \(\delta_B\) is a gauge transformation with parameter \(a_0\),

\[
\delta_B^2 = [\ , a_0].
\]

(3.32)

We define the action of the supersymmetry on the ghost fields as follows,

\[
\delta_s C = \phi, \quad \delta_s (\text{the other ghosts}) = 0.
\]

(3.33)
Then the combined operator $Q = \delta_s + \delta_B$ acts on the fields as,

\[
QX = X' + [X, C], \quad QX' = -i(\delta\phi + \delta U_1(1))X + [X', C], \\
QC = \phi + a_0 - C^2, \quad Q\phi = [\phi, C], \\
Q\tilde{C} = b, \quad Qb = [\tilde{C}, a_0], \\
Q\tilde{a}_0 = i\tilde{C} + \phi + a_0 - C^2, \quad Q\phi = [\tilde{a}_0, a_0], \\
Qb_0 = iC, \quad QC_0 = -i[b_0, a_0], \quad Qa_0 = 0.
\]  

(3.34)

$Q^2$ is given as the sum of the $U(1)$ transformation and the gauge transformation with parameter $a_0$,

\[
Q^2 = R, \quad R := -i\delta U_1(1) + [ , a_0].
\]  

(3.35)

The gauge-fixing action and the ghost action are introduced as a $Q$-exact form,

\[
V_{gh} = \text{Tr} \left[ \tilde{C} \left( iF + \frac{\xi_1}{2}b + ib_0 \right) + C \left( \tilde{a}_0 - \frac{\xi_2}{2}a_0 \right) \right], \\
S_{gh} = \int d\tau QV_{gh} = \int d\tau \text{Tr} \left[ b \left( iF + \frac{\xi_1}{2}b + ib_0 \right) - \tilde{C} \left( QF + \frac{\xi_1}{2}[\tilde{C}, a_0] - C \right) \\
+ (\phi + a_0 - C^2) \left( \tilde{a}_0 - \frac{\xi_2}{2}a_0 \right) + iC\tilde{C}_0 \right],
\]  

(3.36)

where $F$ denotes the gauge fixing condition. In the following computation, we adopt the following gauge-fixing condition for the theory expanded around the saddle point (3.22).

\[
F = \hat{D}_a \left( \frac{1}{\cosh \tau} X_a \right).
\]  

(3.37)

Here, the background covariant derivative $\hat{D}_a$ is defined by

\[
\hat{D}_a X := -i[\tilde{X}_a, X]
\]  

(3.38)

for $a = 1, 2, 3, 4$, where $\tilde{X}_1 = i\partial/\partial\tau$ and $\tilde{X}_a(a = 2, 3, 4)$ are given by the fuzzy sphere background (3.22). A similar gauge is taken in [32] and it is checked that this condition properly eliminates massless modes.

Since the theory does not depend on the parameters $\xi_1$ and $\xi_2$, we put $\xi_1 = \xi_2 = 0$ in (3.36). Then, the action is reduced to

\[
S_{gh} = \int d\tau \text{Tr} \left[ b (iF + ib_0) + \tilde{C}\hat{D}_a \left( \frac{1}{\cosh \tau} D_a C \right) + \tilde{C}C_0 \right]
\]
\[ +(\phi + a_0 - C^2)\tilde{a}_0 - iC\tilde{C}_0 - \tilde{C}\hat{D}_a \left( \frac{1}{\cosh \tau} \Psi \Gamma_\alpha \epsilon \right) \]. \quad (3.39)

It is easy to see that the last term does not contribute to any Feynman diagram, so that one can neglect it. By integrating \((b_0, \tilde{C}_0, C_0)\), the zero modes of \((b, C, \tilde{C})\) are eliminated. Integration over \(b\) produces the gauge fixing constraint \(F = 0\). After the Wick rotation of \(\phi\) and \(a_0\), the integration over \(\tilde{a}_0\) yields the identification \(a_0 = -\phi\). Thus, this action provides an ordinary ghost action bilinear in \((\tilde{C}, C)\), the gauge fixing condition and the identification \(a_0 = -\phi\).

### 3.4 One-loop determinant

Now we consider the \(Q\) transform of \(V + V_{gh}\) and perform the one-loop integration around the saddle point \((3.22)\). For this purpose, we make a redefinition of the fields as

\[ \tilde{X}' := X' + [X, C], \quad \tilde{\phi} := \phi + a_0 - C^2, \]

and divide all fields to four groups,

\[ Z_0 = (X_M', \tilde{a}_0, b_0), \quad Z_1 = (Y_i, C, \tilde{C}), \]

\[ Z_0' = (\tilde{\Psi}_M', \tilde{C}_0, C_0), \quad Z_1' = (\tilde{H}_i, \tilde{\phi}, b). \] \quad (3.41)

These groups form doublets under the \(Q\) transformation,

\[ QZ_i = Z_i', \quad QZ_i' = RZ_i, \quad (i = 0, 1) \] \quad (3.42)

where \(R\) is defined in \((3.35)\). We expand the full action given by \(S_{PW} - Q(V + V_{gh})\) around the saddle point configuration \((3.22)\): \(Z_i \rightarrow \tilde{Z}_i + Z_i\) and \(Z_i' \rightarrow \tilde{Z}_i' + Z_i'\). Then the quadratic term of the fluctuations in \(V + V_{gh}\), which is needed for the one-loop calculation, is schematically written as

\[ V^{(2)} = (Z_0', Z_1) \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} Z_0 \\ Z_1' \end{pmatrix}, \] \quad (3.43)

where \(D_{ij}(i, j = 0, 1)\) are certain linear differential operators. Then the quadratic part of the action is

\[ QV^{(2)} = (RZ_0, Z_1') \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} Z_0 \\ Z_1' \end{pmatrix} + (Z_0', Z_1) \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} Z_0' \\ RZ_1 \end{pmatrix}. \] \quad (3.44)
Hence, the one-loop integral produces

$$Z_{1\text{-loop}} = \left( \frac{\det_{VZ_1} R}{\det_{VZ_0} R} \right)^{\frac{1}{2}},$$  \hspace{1cm} (3.45)$$

where the determinants are taken in the functional spaces of fluctuations of $Z_1$ or $Z_0$, denoted by $V_{Z_1}$ or $V_{Z_0}$, respectively. Recall that we have adopted the boundary condition that the fields are finite at infinity so that the action is finite. This condition implies that when the fields $Z_1$ and $Z_0$ are expanded around the saddle point (3.22), the fluctuations should vanish at infinity since they are massive. Hence $V_{Z_1}$ and $V_{Z_0}$ are defined to be the linear span of field configurations vanishing at infinity.

Note that there exists a natural linear map $D_{10}$ from $V_{Z_0}$ to $V_{Z_1}$ which commutes with $R$. Then the determinants in (3.45) cancel between Im$D_{10} \subset V_{Z_1}$ and Im$D_{10}^* \subset V_{Z_0}$. Here, $D_{10}^*$ is the adjoint operator of $D_{10}$ which is obtained by the partial integration in the action (3.44). Hence, $Z_{1\text{-loop}}$ is reduced to

$$Z_{1\text{-loop}} = \left( \frac{\det_{\ker D_{10}} R}{\det_{\coker D_{10}} R} \right)^{\frac{1}{2}}.$$  \hspace{1cm} (3.46)$$

Since $R$ and $D_{10}$ commute, the kernel and the cokernel can be decomposed to a direct sum of the eigenspaces of $R$. From (3.35) and the identification $a_0 = -\phi = -2iM + 4L_4$, we see that eigenvalue $r_i$ of $R$ is written as the sum of eigenvalue of $[-2iM + 4L_4, ]$ and $U(1)$ charge. The decomposition is expressed as

$$\ker D_{10} = \bigoplus_i V_{r_i}, \quad \coker D_{10} = \bigoplus_i V_{r_i}^\prime,$$  \hspace{1cm} (3.47)$$

where $V_{r_i}$ and $V_{r_i}^\prime$ are the restrictions of the kernel and the cokernel, respectively, to the eigenspace of $R$ with eigenvalue $r_i$. The one-loop determinant is then written as

$$Z_{1\text{-loop}} = \prod_i r_i^{(\dim V_{r_i}^\prime - \dim V_{r_i})/2}.$$  \hspace{1cm} (3.48)$$

Thus, computing $Z_{1\text{-loop}}$ amounts to finding the index of $D_{10}$ for each eigenspace of $R$. Note that since our model is one dimensional matrix model, both $\ker D_{10}$ and $\coker D_{10}$ are finite dimensional. Hence $D_{10}$ is Fredholm and the index is well-defined.

In the following, we compute the dimensions of these spaces for the theory expanded around the saddle point (3.22). For this purpose, we first describe how to compute it
for a general class of linear differential operators on the real line $\mathbb{R}$. We consider the set of all $n$ dimensional vector valued smooth functions on $\mathbb{R}$ which vanish at infinity, $S := \{f : \mathbb{R} \to C^n | \lim_{\tau \to \pm \infty} f(\tau) = 0\}$. We introduce a linear differential operator $D$ acting on the vector space $S$ as
\[
Df(\tau) := \frac{\partial f}{\partial \tau}(\tau) + (A \cdot f)(\tau),
\]
where $f \in S$ and $A$ is a smooth function from $\mathbb{R}$ to the space of $n \times n$ complex matrices. The product between $f$ and $A$ is defined as usual, $(A \cdot f)_i(\tau) := A_{ij}(\tau)f_j(\tau)$. We assume that $A$ has definite limit values, $\lim_{\tau \to \pm \infty} A_{ij}(\tau) < \infty$ for any $i, j = 1, \cdots, n$, and $A(\tau)$ can be diagonalized for any $\tau \in \mathbb{R}$ as
\[
V^{-1}(\tau)A(\tau)V(\tau) = A_d(\tau) := \text{diag}(\lambda_1(\tau), \cdots, \lambda_n(\tau))
\]
by a certain $V \in \Gamma(E)$, where $E$ is $SL(n, C)$ bundle on $\mathbb{R}$ and $\Gamma(E)$ is the set of all smooth sections of $E$. Since $A(\tau)$ is constant at infinity, $\lim_{\tau \to \pm \infty} \lambda_i(\tau)$ are also constants for $1 \leq i \leq n$. Then $\lim_{\tau \to \pm \infty} V(\tau) < \infty$ are also constant matrices. Suppose that $k$ ($1 \leq k \leq n$) eigenvalues in (3.50) satisfy both
\[
\lim_{\tau \to \infty} \text{Re}\lambda_i(\tau) > 0 \quad \text{and} \quad \lim_{\tau \to -\infty} \text{Re}\lambda_i(\tau) < 0
\]
and the other $n - k$ do not. Then, the dimension of kernel of $D$ is given by the formula,
\[
\dim(\ker D) = k.
\]

One can show (3.52) as follows. Since $D$ is the gauge covariant derivative on $\mathbb{R}$, the gauge field $A$ can be transformed to any value by inhomogeneous gauge transformation. Consider such a transformation by $U \in \Gamma(E)$ which maps $A$ to the right-hand side of (3.50),
\[
U^{-1}AU + U^{-1}\partial U = A_d.
\]
Such $U$ can be formally written using a path ordering product. Then, the differential equation $Df = 0$ is solved by
\[
f(\tau) = U(\tau) \exp \left( - \int_0^\tau A_d(\tau')d\tau' \right) f_0,
\]
\footnote{It will not cause any confusion to use the letter $R$ both for the real line and $Q^2$ in (3.35).}
where $f_0$ is a constant vector. (3.54) has to vanish at infinity, for $f \in S$. Since $U$ should converge to $V$ at infinity, $U$ goes to a constant matrix. By multiplying the inverse of $U$ at infinity, the vanishing condition of (3.54) at infinity implies (3.52). Indeed, when $k$ of $\lambda_i$’s satisfy (3.51), $k$ components of $f_0$ can be nonzero keeping $f(\tau)$ vanishing at infinity. This means that the dimension of $\ker D$ is equal to $k$.

Then let us apply the above argument to the plane wave matrix model. The relevant part of the action, $Z_1 D_{10} Z_0$, is

$$
2s_i \Upsilon_i + i \tilde{C}(F + b_0) + C\tilde{a}_0
$$

$$
- \frac{i}{\epsilon \epsilon} \left( \delta_{U(1)}X_{m'} - 2i[\tilde{X}_{M'}, v^4 X_4 + v^6 X_9] - i[X_{M'}, -2i M + v^4 \tilde{X}_4] \right) [\tilde{X}_{M'}, C].
$$

(3.55)

Since the fields in the hypermultiplet, $(X_{m'}, \Upsilon_i)$ $(m' = 5, 8, 7, 8, i = 1, 2, 3, 4)$, decouple from the fields in the vector multiplet in (3.55), the index is decomposed to a sum of contributions from these two sectors.

**Hypermultiplet**

We first consider the hypermultiplet sector. We define complex scalar fields as

$$W_1 = X_5 + iX_8, \quad W_2 = X_6 + iX_7.$$  

(3.56)

One can read off the action of $D_{10}$ on these fields from (3.55). If $W_1, W_2 \in \ker D_{10}$, they satisfy

$$\partial W_1 + 2i[L_-, W_2] + \frac{s}{c}(W_1 + 2[L_4, W_1]) = 0,$$

$$\partial W_2 - 2i[L_+, W_1] + \frac{s}{c}(W_2 - 2[L_4, W_2]) = 0,$$

(3.57)

where $s = \sinh \tau$ and $c = \cosh \tau$. We first decompose $W_i(i = 1, 2)$ to block components $\{W^{(s, t)}_i | s, t = -\Lambda/2, \cdots, \Lambda/2 \}$ and then expand each block in terms of the fuzzy spherical harmonics as

$$W^{(s, t)}_i = \sum_{s' = |s - t|}^{J} \sum_{m = -J}^{J} W^{(s, t)}_{i, j, m} \otimes \hat{Y}^{(s, t)}_{j, m}.$$  

(3.58)

By substituting this expansion to (3.57), we obtain

$$\partial W^{(s, t)}_{1, j, m} + \frac{s}{c}(1 + 2m)W^{(s, t)}_{1, j, m} + 2i\delta_- W^{(s, t)}_{2, j, m+1} = 0,$$

31
\[ \partial W_{2Jm}^{(s,t)} + \frac{S}{c} (1 - 2m) W_{2Jm}^{(s,t)} - 2i \delta_+ W_{1Jm-1}^{(s,t)} = 0, \]  

(3.59)

where we have defined \( \delta_\pm = \sqrt{(J \pm m)(J \mp m + 1)} \). It is easy to check that (3.51) is satisfied only by \( W_{1Jm}^{(s,t)} \) and \( W_{2J-J}^{(s,t)} \). Indeed, these modes have eigenvalues \((2J + 1) \tanh \tau\) which satisfy (3.51). The equations for the other modes can be rewritten in the form of (3.49), where

\[ f = (W_{1Jm}^{(s,t)}, W_{2Jm+1}^{(s,t)})^T \]  

and

\[ A = \begin{pmatrix} \frac{s}{c}(2m + 1) & 2i \delta_- \\ -2i \delta_- & -\frac{s}{c}(2m + 1) \end{pmatrix} \]  

(3.60)

for \( m = -J, -J + 1, \ldots, J - 1 \). The eigenvalues of (3.60) do not satisfy (3.51). Thus, we find that only \( W_{1Jm}^{(s,t)} \) and \( W_{2J-J}^{(s,t)} \) and their complex conjugates contribute to the index.

Then, we consider the contribution from \( \{ \Upsilon_i, i = 1, 2, 3, 4 \} \) to the index. Introducing complex fields,

\[ \xi_1 = \Upsilon_1 + i \Upsilon_4, \quad \xi_2 = \Upsilon_3 + i \Upsilon_2, \]  

(3.61)

and expanding their block components in terms of the spherical harmonics as in (3.58), one obtains

\[ \partial \xi_{1Jm}^{(s,t)} + \frac{2sm}{c} \xi_{1Jm}^{(s,t)} + 2 \delta_+ \xi_{2Jm-1}^{(s,t)} = 0, \]  

\[ \partial \xi_{2Jm}^{(s,t)} - \frac{2sm}{c} \xi_{2Jm}^{(s,t)} + 2 \delta_- \xi_{1Jm+1}^{(s,t)} = 0, \]  

(3.62)

for \( \xi_1, \xi_2 \in \text{coker} D_{10} \). Since in this case, there is no eigenvalue satisfying (3.51), one finds that \( \{ \Upsilon_i, i = 1, 2, 3, 4 \} \) do not contribute to the index.

In summary, only \( W_{1Jm}^{(s,t)} \) and \( W_{2J-J}^{(s,t)} \) and their complex conjugates contribute to the index for the hypermultiplet. The eigenvalues of \( R \) for these fields are read off from

\[ RW_{1Jm}^{(s,t)} = 2W_{1Jm}^{(s,t)} + [\hat{\phi}, W_1^{(s,t)}] \]  

\[ = \sum_{J,m} 2 \left\{ (1 + 2m) W_{1Jm}^{(s,t)} + i (M_s W_{1Jm}^{(s,t)} - W_{1Jm}^{(s,t)} M_t) \right\} \otimes \hat{\Upsilon}_{Jm(j_s,j_t)} \]  

(3.63)

and so on. Thus, we find that the contribution to (3.48) from the hypermultiplet is given, up to an overall constant, by

\[ \prod_{s,t=-\frac{N}{2}}^{\frac{N}{2}} \prod_{j_s+j_t}^{N_s} \prod_{i=j}^{N_t} \prod_{j=1}^{1} \frac{1}{(2J + 1)^2 + (m_{s_i} - m_{t_j})^2}. \]  

(3.64)
Vector multiplet

We then compute contribution from the vector multiplet. We first calculate the dimension of $\ker D_{10}$. If the fields $\{X_M, \tilde{a}_0, b_0 | M = 1, 2, 3, 4, 9\}$ are in $\ker D_{10}$, they satisfy

\begin{align}
F + b_0 &= 0, \quad (3.65) \\
\tilde{a}_0 + 2 \left[ \hat{X}_{M'}, \frac{1}{\epsilon} \hat{X}_{M'}, v^4 X_4 + v^9 X_9 \right] + \left[ \hat{X}_{M'}, \frac{1}{\epsilon} X_{M'} \right], -2iM + v^4 \hat{X}_4 &= 0, \quad (3.66) \\
c(2X_4 - i[\hat{X}_2, X_3] + i[\hat{X}_3, X_2]) - s(\partial X_4 + i[\hat{X}_4, X_1]) - \partial X_9 &= 0, \quad (3.67) \\
c(\partial X_3 + i[\hat{X}_3, X_1]) - s(2X_3 + i[\hat{X}_2, X_4] - i[\hat{X}_4, X_2]) - i[\hat{X}_2, X_9] &= 0, \quad (3.68) \\
c(\partial X_2 + i[\hat{X}_2, X_1]) - s(2X_2 - i[\hat{X}_3, X_4] + i[\hat{X}_4, X_3]) + i[\hat{X}_3, X_9] &= 0. \quad (3.69)
\end{align}

Here we reduce the number of equations by partially solving the equations before applying the argument of eigenvalues. First, by taking the limit $\tau \to \pm \infty$ in (3.65), since $F \to 0$, one obtains $b_0 = 0$. Then,

\begin{align}
F = \left[ \hat{X}_a, \frac{1}{\cosh \tau} X_a \right] = 0, \quad (3.70)
\end{align}

for arbitrary $\tau \in \mathbb{R}$ follows again from (3.65). Similarly, $\tilde{a}_0 = 0$ follows from (3.66). By substituting (3.70) to (3.66), one obtains,

\begin{align}
-\partial \left( \frac{1}{c} \partial (X_4 - sX_9) \right) + \frac{4}{c} [L_{a'}, [L_{a'}, X_4 - sX_9]] &= 0. \quad (3.71)
\end{align}

From (3.71) we show $X_4 - sX_9 = 0$ as follows. (3.71) takes the form, $\partial^2 f - \frac{2}{c} \partial f - 4J(J + 1)f = 0$, where $f$ corresponds to $X_4 - sX_9$ and $J(J + 1)$ is the eigenvalue of $[L_{a'}, [L_{a'}, ]]$. Since $f/c$ should vanish at infinity,

\begin{align}
0 = \int d\tau \partial \left( \frac{1}{c^2} f \partial f \right) = \int dx \left[ \left( \frac{\partial f}{c} \right)^2 + \left( \frac{4J(J + 1) - 1}{c^2} + \frac{3}{2c^4} \right) f^2 \right]. \quad (3.72)
\end{align}

The right-hand side is a sum of positive definite terms except when $J = 0$. Hence $f = 0$ when $J \neq 0$. When $J = 0$, the original equation is $\partial ((\partial f)/c) = 0$ and integrating it under the boundary condition, $f/c \to 0$, yields $f = \text{constant}$. On the other hand, when $J = 0$, the commutator terms in (3.67) vanish. By solving this equation together with the conditions, $f = X_4 - sX_9 = \text{constant}$ and $X_4, X_9 \to 0$ ($\tau \to \pm \infty$), one obtains $X_4 = X_9 = 0$ for $J = 0$. In summary, one can put $X_4 - sX_9 = 0$ for any $J$. 
We introduce the complex scalar fields $X_\pm = X_2 \pm iX_3$ and eliminate $X_4$ by $X_4 = sX_9$. Then, the equations (3.65), (3.67), (3.68), (3.69) are written as

\[-i\partial X_1 + i\frac{s}{c}X_1 + [L_+, X_-] + [L_-, X_+] + 2s[L_4, X_9] = 0,

\[-[L_+, X_-] + [L_-, X_+] + sX_9 - c\partial X_9 + 2i\frac{s}{c}[L_4, X_1] = 0,

\[c(\partial X_+ - 2i[L_+, X_1]) - s(2X_+ - 2[L_4, X_+]) - 2c^2[L_+, X_9] = 0,

\[c(\partial X_- - 2i[L_-, X_1]) - s(2X_- + 2[L_4, X_-]) + 2c^2[L_-, X_9] = 0. \quad (3.73)\]

We make a redefinition for $X_9$ as $X_9' = cX_9$. Note that $X_9'$ does not necessarily vanish at infinity but there is no solution to (3.73) such that $X_9'$ is non-zero at infinity. So one can assume that $X_9' \to 0$ ($\tau \to \pm\infty$). We then decompose $X_\pm, X_1, X_9'$ into the block components and expand them by the fuzzy spherical harmonics as we have done in (3.58). For $f = (X_{Jm+1}^{+(s,t)}/\sqrt{2}, X_{Jm-1}^{-(s,t)}/\sqrt{2}, iX_{Jm}^{1(s,t)}, X_{Jm}^{9(s,t)})^T$ ($m = -J+1, -J+2, \cdots, J-1, J \geq 1$), the equations (3.73) are written in the form of (3.49), where $A$ is given by

\[
A = \begin{pmatrix}
\frac{2ms}{c} & 0 & -\sqrt{2}\delta_- & -\sqrt{2}\delta_-
\frac{1}{c} & -\frac{2ms}{c} & -\sqrt{2}\delta_+ & \sqrt{2}\delta_+
-\sqrt{2}\delta_- & -\sqrt{2}\delta_+ & \frac{1}{c} & -\frac{2ms}{c}
-\sqrt{2}\delta_- & \sqrt{2}\delta_+ & -\frac{2ms}{c} & -\frac{2s}{c}
\end{pmatrix}. \quad (3.74)
\]

Since $A$ is a real symmetric matrix, the eigenvalues of $A$ are real. By examining the determinant of $A$ for given $J$ and $m$, it turns out that $A$ does not have zero eigenvalues for any $\tau$. Therefore, the sign of each eigenvalue does not change as a function of $\tau$ and we conclude that there is no eigenvalue satisfying (3.51). When $m = J$ or $m = -J$, the equations (3.73) are closed with three fields $f = (X_{Jm+1}^{-(s,t)}/\sqrt{2}, iX_{Jm}^{1(s,t)}, X_{Jm}^{9(s,t)})^T$ or $f = (X_{Jm-1}^{+(s,t)}/\sqrt{2}, iX_{Jm}^{1(s,t)}, X_{Jm}^{9(s,t)})^T$. Similarly, we find that the eigenvalues of the $3 \times 3$ matrices do not satisfy (3.51). Hence, the bosonic fields in the vector multiplet do not contribute to the index.

Then let us consider the cokernel of $D_{10}$. The elements of coker$D_{10}$ in the vector multiplet, $(C, \dot{C}, \Upsilon_5, \Upsilon_6, \Upsilon_7)$, satisfy the following conditions,

\[-\frac{1}{c}\partial \dot{C} + \frac{1}{c}[iM - 2L_4, \partial C] - 8s[L_4, \Upsilon_5] - 8c[L_3, \Upsilon_6] + 8c[L_2, \Upsilon_7] = 0,

\[\frac{1}{c}[L_2, C] + \frac{1}{c}[L_2, [iM - 2L_4, C]] + 4ic[L_3, \Upsilon_5] - 4is[L_4, \Upsilon_6] - 2c\partial \Upsilon_7 - 6s\Upsilon_7 = 0,

\[\frac{1}{c}[L_3, \dot{C}] - \frac{1}{c}[L_3, [iM - 2L_4, C]] - 4ic[L_2, \Upsilon_5] - 4is[L_4, \Upsilon_7] + 2c\partial \Upsilon_6 + 6s\Upsilon_6 = 0,

\[\frac{1}{c}[L_4, C] + \frac{1}{c}[L_4, [iM - 2L_4, C]] - 8s[L_4, \Upsilon_5] - 8c[L_3, \Upsilon_6] + 8c[L_2, \Upsilon_7] = 0.\]
\[ \frac{1}{c}[L_4, \tilde{C}] + \partial \left( \frac{1}{c} \partial C \right) - 4 \frac{1}{c}[L_{a'}, [L_{a'}, C]] - \frac{1}{c}[L_4, [iM - 2L_4, C]] + 2s \partial \Upsilon_5 + 6c \Upsilon_5 \]
\[ + 4is[L_2, \Upsilon_6] + 4is[L_3, \Upsilon_7] = 0, \]
\[ - s\partial \left( \frac{1}{c} \partial C \right) + \frac{4s}{c}[L_{a'}, [L_{a'}, C]] + 2\partial \Upsilon_5 + 4i[L_2, \Upsilon_6] + 4i[L_3, \Upsilon_7] = 0. \] (3.75)

From the coefficients of \( \tilde{a}_0 \) and \( b_0 \) in (3.55), we also have
\[ \int_{-\infty}^{\infty} d\tau \tilde{C}^{(s,s)}_{00}(\tau) = \int_{-\infty}^{\infty} d\tau \tilde{C}^{(s,s)}_{00}(\tau) = 0, \] (3.76)
where the subscript 00 indicates the zero mode of the fuzzy sphere which exists only in the diagonal blocks. We redefine the fields as \( \tilde{C}' = (\tilde{C} - [iM - 2L_4, C])/(2\sqrt{2}c) \), \( C' = C/c \), \( \Upsilon'_5 = \sqrt{2} \Upsilon_5 \) and also introduce complex fields, \( \Upsilon_{\pm} = \Upsilon_6 \pm i \Upsilon_7 \). Note that these fields also vanish at infinity. We also introduce a new field \( d = \partial C' \) in order to make the equations first order. With these variables, (3.75) is rewritten as
\[ \partial C' - d = 0, \]
\[ \partial d + \frac{3s}{c}d + 2C' - 4[L_{a'}, [L_{a'}, C']] + \frac{2\sqrt{2}}{c^2}[L_4, \tilde{C}'] + \frac{3\sqrt{2}}{c^2} \Upsilon_5' = 0, \]
\[ \partial \Upsilon_+ - \sqrt{2}i[L_+, \tilde{C}'] - \sqrt{2}i[L_+, \Upsilon_5'] + \frac{3s}{c} \Upsilon_+ - \frac{2s}{c}[L_4, \Upsilon_+] = 0, \]
\[ \partial \Upsilon_- + \sqrt{2}i[L_-, \tilde{C}'] - \sqrt{2}i[L_-, \Upsilon_5'] + \frac{3s}{c} \Upsilon_- + \frac{2s}{c}[L_4, \Upsilon_-] = 0, \]
\[ \partial \tilde{C}' + \frac{2s}{c}[L_4, \tilde{C}'] - \frac{2s}{c}[L_4, \Upsilon_5'] - \sqrt{2}i([L_+, \Upsilon_-] - [L_-, \Upsilon_+]) = 0, \]
\[ \partial \Upsilon_5' + \frac{2s}{c}[L_4, \tilde{C}'] + \frac{3s}{c} \Upsilon_5 + \sqrt{2}i([L_+, \Upsilon_-] + [L_-, \Upsilon_+]) = 0. \] (3.77)

We decompose the fields to the block components and expand them by the harmonics. Then \( f = (C_{J_m}^{(s,t)}, d_{J_m}^{(s,t)}, \Upsilon^{(s,t)}_{J_m+1}, \Upsilon^{(s,t)}_{J_m-1}, \Upsilon^{(s,t)}_{J_m} \tilde{C}_{J_m}^{(s,t)})^T \) \( (m = -J + 1, -J + 2, \ldots, J - 1) \) satisfies (3.49) with
\[ A = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
\frac{3s}{c} & 2 - 4J(J + 1) & 0 & 0 & \frac{3\sqrt{2}}{c^2} & \frac{3\sqrt{2m}}{c^2} \\
0 & 0 & \frac{2s}{c}(1 - 2m) & 0 & -\sqrt{2}i\delta_- & -\sqrt{2}i\delta_- \\
0 & 0 & 0 & \frac{2s}{c}(1 + 2m) & -\sqrt{2}i\delta_+ & \sqrt{2}i\delta_+ \\
0 & 0 & \sqrt{2}i\delta_- & \sqrt{2}i\delta_+ & \frac{3s}{c} & \frac{2ms}{c} \\
0 & 0 & \sqrt{2}i\delta_- & \sqrt{2}i\delta_+ & \frac{2ms}{c} & \frac{2s}{c}
\end{pmatrix}. \] (3.78)

The eigenvalues of (3.78) do not satisfy (3.51) and hence, these modes do not contribute to the index. On the other hand, the mode \( f = (C_{J,J}^{(s,t)}, d_{J,J}^{(s,t)}, \Upsilon_{J,J-1}^{(s,t)}, \Upsilon_{J,J}^{(s,t)} \tilde{C}_{J,J}^{(s,t)})^T \)
satisfies (3.49), where \( A \) is given by a 5 \( \times \) 5 matrix obtained by eliminating the row and column for \( \Upsilon_j \) and putting \( m = J \) in (3.78). Then, we find that there is an eigenvalue which satisfies (3.51) while the other four do not. Therefore, together with the complex conjugate, \( f = (C^{(s,t)}_{j-j}, d^{(s,t)}_{j-j}, \Upsilon^{(s,t)}_{j-j} + 1, \varphi^{(s,t)}_{j-j}, \tilde{C}^{(s,t)}_{j-j})^T \), they contribute to the index. The eigenvalues of \( R \) for these modes are given by 
\[
R \sim 2(\pm 2J + i (m_{si} - m_{tj})).
\]
Note that this contribution is absent for \( J = 0 \) because of the constraint (3.76). Finally, it is easy to see that \( \Upsilon_{j-j}^{(s,t)} \) and \( \Upsilon_{jj}^{-} \) obey the equation \( \partial \Upsilon + (2J + 3) s \Upsilon = 0 \). Then (3.51) is satisfied for these modes, so that they contribute to the index. The value of \( R \) for these are 
\[
R \sim 2(\pm (2J + 2) + i (m_{si} - m_{tj})).
\]

In summary, we find that the determinant from the vector multiplet is given, up to an overall constants, by 
\[
\Lambda^{-1/2} \prod_{s,t=-\Lambda/2}^{\Lambda/2} \prod_{J=|j_s-j_t|} \prod_{J \neq 0} N_s N_t \prod_{i=1}^{N_s} \prod_{j=1}^{N_t} \left\{ (2J)^2 + (m_{si} - m_{tj})^2 \right\}^{1/2}
\]
\[
\times \prod_{s,t=-\Lambda/2}^{\Lambda/2} \prod_{J=|j_s-j_t|} \prod_{J \neq 0} N_s N_t \prod_{i=1}^{N_s} \prod_{j=1}^{N_t} \left\{ (2J + 2)^2 + (m_{si} - m_{tj})^2 \right\}^{1/2}.
\]  
(3.79)

Combined with (3.64) and the Vandermonde determinant which comes from the diagonalization of the moduli matrix \( M \), the total one-loop determinant is given, up to an overall constant, by 
\[
Z_{1-loop} = \prod_{s,t=-\Lambda/2}^{\Lambda/2} \prod_{J=|j_s-j_t|} \prod_{J \neq 0} N_s N_t \left[ \prod_{i=1}^{N_s} \prod_{j=1}^{N_t} \left\{ (2J + 2)^2 + (m_{si} - m_{tj})^2 \right\} \left\{ (2J)^2 + (m_{si} - m_{tj})^2 \right\} \right]^{1/2} \left\{ (2J + 1)^2 + (m_{si} - m_{tj})^2 \right\}^2.
\]  
(3.80)

By \( \prod' \) we mean that the second factor in the numerator with \( s = t, J = 0 \) and \( i = j \) is not included in the product.

### 3.5 Partition function and Wilson loop

The determinant (3.80) depends on the background around which the theory has been expanded. So it is labeled by a representation \( \mathcal{R} \) of \( SU(2) \). If we ignore the instanton part, the total partition function is written as (3.1), where \( Z_{\mathcal{R}} \) is given by, 
\[
Z_{\mathcal{R}} = \mathcal{C}_{\mathcal{R}} \int \prod_{s=-\Lambda/2}^{\Lambda/2} \prod_{i=1}^{N_s} dm_{si} Z_{1-loop}(\mathcal{R}, \{m_{si}\}) e^{-\frac{g^2}{4\pi} \sum_s \sum_i (2j_s + 1)m_{si}^2}. 
\]  
(3.81)
The coefficient $C_R$ is a constant which determines the relative normalization in the sum (3.1) and is given by

$$C_R = \prod_s \left( \frac{1}{2} \right)^{N_s^2} \cdot \frac{N_{PW}!}{\prod_s \{N_s!(2j_s + 1)!\}} \cdot \prod_s \frac{(2\pi)^{N_s(N_s+1)/2}}{\prod_{k=1}^{N_s} k!}. \quad (3.82)$$

The first factor is the overall constant of $Z_{1\text{-loop}}$ we neglected. The second factor is the number of ways to permute the eigenvalues of $L_4$ in the representation $R$ and is part of the gauge volume. The last factor is the product of the volume of $U(N_s)$, which arises from the diagonalization of $M_s$. We ignore an overall constant which does not depend on the representation $R$. The Gaussian factor in (3.81) is obtained by substituting the saddle point configuration (3.22) to the original action of PWMM.

(3.81) for each representation $R$ has a definite meaning. It describes the PWMM expanded around the fuzzy sphere background with representation $R$. Recall that the theories with $SU(2|4)$ symmetry are also realized as the theories around particular fuzzy sphere backgrounds in PWMM. Then the partition functions of these theories can be obtained from (3.81) through the relations in Fig. 1 as we will see in the next section.

The Wilson loop (2.47) in PWMM is invariant under the supersymmetry (3.2). Hence, the calculation of its vev is also reduced to the matrix integral through the localization. At the saddle point, the operator (2.47) is reduced to

$$\frac{1}{N_{PW}} \sum_{s=-N_s/2}^{N_s/2} \sum_{i=1}^{N_s} (2j_s + 1) e^{2\pi i m_{si}}, \quad (3.83)$$

where we used the fact that $M$ and $L_4$ commute and the eigenvalues of $L_4$ are integers or half-integers. The vev of the Wilson loop is then reduced to the average of (3.83) with respect to the matrix integral (3.1).

More generally, any operator in PWMM constructed only of $\phi$ is invariant under the supersymmetry with parameter (3.2) and hence its vev is reduced to a matrix integral.

In order to check our result, we perform the one-loop calculation of one-point function of the operator $Tr\phi^2(0) = Tr(X_4 + iX_0^{(E)})^2(0)$ in the case of the trivial background, based on two different method. One is from the original action of PWMM and the other is from the eigenvalue integral (3.81). We obtain the same results as shown in Appendix E.
4 Exact results for theories with $SU(2|4)$ symmetry

In this section, we utilize the relations in Fig. 1 to obtain exact results for SYM on $R \times S^2$ and SYM on $R \times S^3/Z_k$. The partition functions of these theories can be obtained from (3.81) through the relations (a) and (c) in Fig. 1. We also consider (c”) and test the large-$N$ reduction for $\mathcal{N} = 4$ SYM on $R \times S^3$.

4.1 SYM on $R \times S^2$

Recall that the theory on $R \times S^2$ has many nontrivial vacua in which gauge fields take the Dirac monopole configuration, (2.15). Each background is labeled by a set of monopole charges $\{q_s\}$ as well as their multiplicities $\{N_s\}$. As shown in Section 2.2.1, the theory on $R \times S^2$ around each background is realized from PWMM under the limit (2.25). The partition function of the theory on $R \times S^2$ around the background labeled by $\{(q_s, N_s) | s = -\Lambda/2, \ldots, \Lambda/2\}$ is then obtained from (3.81) by taking the limit (2.25);

$$Z_{R \times S^2}^{(q_s, N_s)} = \int \prod_{s=-\Lambda/2}^{\Lambda/2} \prod_{i=1}^{N_s} dm_{si} \prod_{s=-\Lambda/2}^{\Lambda/2} \prod_{i,j=1}^{N_s} \Delta(m_s)^2 \prod_{s=-\Lambda/2}^{\Lambda/2} \prod_{i,j=1}^{N_s} \left[ \frac{1}{2} \left( \frac{(m_{si} - m_{sj})}{1 + (m_{si} - m_{sj})^2} \right)^2 \right]^{1/2} \prod_{s,t=-\Lambda/2}^{\Lambda/2} \prod_{J=|q_s - q_t|}^{\infty} \prod_{J \neq 0}^{N_s} \prod_{j=1}^{N_t} \left[ \frac{1}{2} \left( \frac{(m_{si} - m_{sj})}{1 + (m_{si} - m_{sj})^2} \right)^2 \right]^{1/2} e^{-\frac{4\pi}{g^2 S^2} \sum_{s,i} m_{si}^2},$$

(4.1)

where we have dropped an overall constant and $\Delta(m_s) = \prod_{i<j} (m_{si} - m_{sj})$ is the Vandermonde determinant. One can see that the infinite product of $J$ is convergent. The full partition function is given as a sum over all (4.1) with $\{(q_s, N_s)\}$ satisfying $\sum_s N_s = N_{S^2}$.

Note that the commutative limit has been taken smoothly. This implies that the non-commutativity vanishes and does not affect to the partition function, unlike the UV/IR mixing on the Moyal plane.

For the trivial background given by $\Lambda = 0$ and $q_0 = 0$ in (2.15), (4.1) is simplified to

$$Z_{R \times S^2}^{t.h.} = \int \prod_i dm_i \prod_{i>j} \tanh^2 \left( \frac{\pi (m_i - m_j)}{2} \right) e^{-\frac{4\pi}{g^2 S^2} \sum_i m_i^2},$$

(4.2)

where $i, j$ run from 1 to $N_{S^2}$. 

38
The operator (3.83) is now reduced to

\[
\frac{1}{N_{S^2}} \sum_{s=-(\Lambda/2)}^{\Lambda/2} \sum_{i=1}^{N_s} e^{2\pi m_{si}}.
\]  

(4.3)

Through the relation (a) in Fig. 1, the vev of the above operator with respect to the matrix integral (4.1) is equal to the vev of (2.46) in SYM on \( R \times S^2 \) around the monopole background with \( \{(q_s, N_s)\} \).

4.2 \( \mathcal{N} = 4 \) SYM on \( R \times S^3 / \mathbb{Z}_k \)

Taylor’s T-duality

SYM on \( R \times S^3 / \mathbb{Z}_k \) is realized from PWMM through the relation (c) or (c’) reviewed in Section 2.2.3. We first apply the relation (c) which is based on Taylor’s T-duality. As explained in Section 2.2.3, \( U(N) \) SYM on \( R \times S^3 / \mathbb{Z}_k \) around trivial vacuum is obtained by expanding PWMM around the background (2.22) with \( 2j_s + 1 = n + k s \) and \( N_s = N \) and imposing the orbifolding condition on the fluctuations. Applying this to (3.81) yields,

\[
Z_{R \times S^3 / \mathbb{Z}_k}^{t,b.} = \int \prod_{i=1}^{N} dm_i \Delta(m) \prod_{i,j=1}^{N} \left[ \frac{1 + \left( \frac{m_i - m_j}{2} \right)^2}{1 + (m_i - m_j)^2} \right]^{1/2} \prod_{u=-\infty}^{\infty} \prod_{J=\left|ku/2\right|, J \neq 0}^{\infty} \prod_{i,j=1}^{N} \left[ \left\{ \frac{1 + \left( \frac{m_i - m_j}{2J+2} \right)^2}{1 + \left( \frac{m_i - m_j}{2J+1} \right)^2} \right\} \left\{ 1 + \left( \frac{m_i - m_j}{2J+1} \right)^2 \right\} \right]^{1/2} e^{-4\pi^2 \sum_{i=1}^{N} m_i^2}, \tag{4.4}
\]

where \( \Delta(m) = \prod_{i<j} (m_i - m_j) \) is the Vandermonde determinant and an over all constant is dropped. The product of \( u \) comes from the product of \( s \) and \( t \) in (3.81). Under the orbifolding condition, the blocks are labeled only by the difference, \( u = s - t \), so that the only one product of \( u \) is remaining in (4.4). The subscript \( s \) of \( m_{si} \) is also dropped for the same reason. The exponent is obtained by using (2.37).

By changing the order of the products of \( u \) and \( J \) as

\[
\prod_{u=-\infty}^{\infty} \prod_{J=\left|ku/2\right|, J \neq 0}^{\infty} = \prod_{J \in \mathbb{Z}/2}^{J} \prod_{u=-J}^{J} \prod_{u \in k \mathbb{Z}/2}^{u}, \tag{4.5}
\]

39
one can first take the product of $u$ in (4.4) for each $k$. For example, when $k = 2$, the partition function becomes

$$Z_{R \times S^3/Z_2}^{t.b.} = \int \prod_{i=1}^{N} dm_i \Delta(m)^2 \prod_{i,j=1}^{N} \left[ \frac{1 + \left(\frac{m_i - m_j}{2}\right)^2}{1 + (m_i - m_j)^2} \right]^{\frac{1}{2}} \prod_{J=1}^{\infty} \prod_{i,j=1}^{N} \left[ \left\{ 1 + \left(\frac{m_i - m_j}{2J + 2}\right)^2 \right\} \left\{ 1 + \left(\frac{m_i - m_j}{2J}\right)^2 \right\} \right]^{\frac{2J+1}{2}} e^{-\frac{4\pi^2}{g^2} \sum_{i=1}^{N} m_i^2}, \quad (4.6)$$

where, $J$ runs only over positive integers. One can see that the infinite product is convergent.

The theory on $R \times S^3/Z_k$ has nontrivial vacua labeled by a holonomy (2.10). The partition functions of such theories are also obtained from (3.81) by taking appropriate representations shown in Section 2.2.3.

The vev of the circular Wilson loop operator (2.43) in $R \times S^3/Z_k$ with the contour (2.42) is reduced to the vev of the following operator with respect to the matrix integral (4.4),

$$\frac{1}{N} \sum_{i=1}^{N} e^{2\pi m_i}. \quad (4.7)$$

This operator is obtained from (3.83) by dropping the $s$ dependence as above and using the formal expression $N_{PW} = Nn\Lambda$.

We then consider the case with $k = 1$. In this case, $\mathcal{N} = 4$ SYM on $R \times S^3$ has a unique vacuum and its partition function is obtained in the same way as (4.6). The partition function takes the same form as (4.6) except that $J$ runs over integers and half-integers starting from $1/2$. Then it is easy to see that the measure factors except the Vandermonde determinant completely cancel out. Thus, we obtain the Gaussian matrix model. This is consistent with the results for $\mathcal{N} = 4$ SYM obtained in [2, 26, 27].

Large-$N$ reduction

Alternatively, one can use the relation (c') in Fig. 1, which is based on the large-$N$ reduction, to obtain the partition function of the planar SYM on $R \times S^3/Z_k$. In particular, the theory around the trivial background is obtained from PWMM by taking the continuum
limit (2.39) in PWMM around the background (2.38). Applying this to (3.81), one can easily obtain the partition function.

In the following, we focus on the case of $k = 1$ to check the claim of the large-$N$ reduction. In this case, before one takes the continuum limit, the partition function is given by (3.81) with $\mathcal{R}$ given by (2.38) with $k = 1$;

\[
Z_{\text{planar}}^{R \times S^3} = \int_{s=-\Lambda/2}^{\Lambda/2} \prod_{i, j=1}^{N} dm_{si} \prod_{s,t=-\Lambda/2}^{\Lambda/2} \prod_{J=|j_s-j_t|}^{N} \left[ \frac{(2J+2)^2 + (m_{si} - m_{tj})^2}{(2J+1)^2 + (m_{si} - m_{tj})^2} \right] \frac{1}{2} e^{-\frac{2}{g_{PW}^2} \sum_{s,i} (n+s)m_{si}^2}, \tag{4.8}
\]

where $2j_s + 1 = n + s$. We show that in the continuum limit (2.39), this matrix integral is indeed equivalent to the Gaussian matrix model (2.45) of $\mathcal{N} = 4$ SYM. Since $n \gg s$ for $s = -\Lambda/2, \cdots, \Lambda/2$ in the continuum limit, the exponent in (4.8) goes to $-\frac{4\pi^2}{g^2} \sum_{s,i} m_{si}^2$, where the identification for the coupling constants in (2.38) is used. The coefficient in the exponent agrees with that in (2.45). We will see that the interactions between the modes with different $s$ in (4.8) are suppressed in the continuum limit and the model becomes a set of independent copies of the Gaussian matrix model.

We assume the 't Hooft limit in the following. Then the saddle point approximation is exact. We introduce the eigenvalue density for each $s$ as

\[
\rho^{[s]}(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - m_{si}). \tag{4.9}
\]

The saddle point equation for $\rho^{[s]}$ is given by

\[
0 = \frac{2}{\lambda_s} x - \sum_{t=-\Lambda/2}^{\Lambda/2} \sum_{J=|j_s-j_t|}^{j_s+j_t} \int dy p^{[l]}(y)(x - y) \left\{ \frac{1}{2(2J+2)^2 + (x - y)^2} \right. \\
+ \frac{1}{2(2J+1)^2 + (x - y)^2} - \left. \frac{2}{2(2J+1)^2 + (x - y)^2} \right\}, \tag{4.10}
\]

where $\lambda_s = g_{PW}^2 N/(n + s)$. One can rewrite (4.10) to

\[
0 = \frac{2}{\lambda_s} x - \int dy \frac{\rho^{[s]}(y)}{x - y} + 2 \sum_{t=-\Lambda/2}^{\Lambda/2} \sum_{J=|j_s-j_t|}^{j_s+j_t} f^{[l]}_J(x) \\
- \sum_{t=-\Lambda/2-1}^{\Lambda/2-1} \sum_{J=|j_s-j_t|}^{j_s+j_t} f^{[l+1]}_J(x) - \sum_{t=-\Lambda/2+1}^{\Lambda/2+1} \sum_{J=|j_s-j_t|}^{j_s+j_t} f^{[l-1]}_J(x), \tag{4.11}
\]
where $f^J_t(x)$ is defined by

$$f^J_t(x) = \int dy \rho^J_t(y) \frac{x - y}{(2J + 1)^2 + (x - y)^2},$$

for $J = 0, 1, 2, \cdots$ and $t = -\Lambda/2, \cdots, \Lambda/2$.

In the continuum limit (2.39), the saddle point equation (4.11) is solved by

$$\rho^s(x) = \hat{\rho}(x) := \frac{2}{\lambda} \sqrt{\lambda - x^2},$$

for any $s \in \mathbb{Z}$. Here $\lambda = g^2 N / (2\pi^2)$, which is the limit of $\lambda_s$ under the identification (2.39). The distribution of $\hat{\rho}$ is just the semicircle law and we find that the model consists of infinitely many copies of the Gaussian matrix model in the continuum limit.

One can see the equivalence for some physical observables. For example, the free energy of the reduced model, (4.8), divided by the multiplicity $\Lambda$ is equal to that of $\mathcal{N} = 4$ SYM,

$$\frac{F_r}{\Lambda} = F_{\mathcal{N}=4 \text{ SYM}},$$

in the continuum limit. The left-hand side can be computed by using (4.13) and the right-hand side is the free energy of the Gaussian matrix model in (2.45). Also the VEV of the circular Wilson loop (3.83) is calculated in (4.8) as

$$\frac{1}{\Lambda N} \sum_s \sum_i \langle e^{2\pi m_s i} \rangle = \frac{1}{\Lambda} \sum_s \int dx \rho^s(x) e^{2\pi x} = \int dx \hat{\rho}(x) e^{2\pi x},$$

where we have used the relation $N_{PW} \sim n \Lambda N$ which holds in the continuum limit. The right-hand side of (4.15) is nothing but the known result in $\mathcal{N} = 4$ SYM, (2.45).

In the above argument, we have ignored a cutoff effect. When $s$ is sufficiently close to the cutoff $\pm \Lambda/2$, $\rho^s$ should deviate from the semicircle law (4.13) since the last three terms in (4.11) do not vanish with (4.13). However, this deviation rapidly disappears when $s$ goes to a distance from the cutoffs. More precisely, when $|s| < \Lambda/2 - \mathcal{O}(\log \Lambda)$, the cutoff effect in (4.11) can be neglected. In fact, the cutoff effect is caused by the terms with $t = \mathcal{O}(\Lambda)$ in the last three terms in (4.11) and when $|s| < \Lambda/2 - \mathcal{O}(\log \Lambda)$ such effect is suppressed since the lower edge of $J$ is at least $\mathcal{O}(\log \Lambda)$. Hence the deviation from (4.13) appears only when the distance from $s$ to the cutoff is $\mathcal{O}(1)$. This means that the number of the deviating modes is $\mathcal{O}(1)$ and it is negligible compared with the total
number of the modes, $\Lambda + 1$, and then the other modes satisfying (4.13) are dominant in the continuum limit. Since, as we have seen above, an expectation value in the reduced model is written as an average over all the modes, contribution from the deviating modes are suppressed.

In [32], the large-$N$ equivalence for the circular Wilson loop was studied in the perturbation theory. Within the ladder approximation, it was shown that the vev of (2.47) agrees with (2.45) to all orders in the perturbative expansion. The above result provides a nonperturbative proof of the large-$N$ equivalence for the free energy and the circular Wilson loop operator.

5 Summary

In this paper, we used the localization technique to obtain the matrix integral (3.81), which is equivalent to the partition function of PWMM around the fuzzy sphere vacuum with the representation $\mathcal{R}$. We first constructed off-shell supersymmetries in PWMM and added a $Q$-exact term to the action. Then, the path integral is reduced to the one-loop integral around saddle points of the $Q$-exact term. Except for possible instanton effects, the saddle points are given by fuzzy spheres labeled by an $SU(2)$ representation. In the end, up to the instantons, the partition function is given by a sum of terms, each of which is labeled by an $SU(2)$ representation and given by a matrix integral. We also obtained the vev of $Q$-closed operators as the matrix integral. As a consistency check of our results, we performed one-loop computation in PWMM and found the exact agreement with the result obtained by using the localization. Although the instanton effects are not included in our computation, in the 't Hooft limit, where the instanton effects are negligible, our results are exact.

Using the relations (a) and (c) explained in Section 2.2.1 and 2.2.3, we obtained matrix integrals equivalent to the partition function of theories with $SU(2|4)$ symmetry, 2+1 SYM on $R \times S^2$ and $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$. The $SU(2|4)$ symmetric theories have many nontrivial vacua. The theory around each vacuum of these theories are realized by PWMM around a particular fuzzy sphere vacuum through (a) and (c). We applied

\footnote{Similar situations are found also in the large-$N$ reduced model for Chern-Simons theories on $S^3$ [38,40].}
these relations to (3.81) and obtained the matrix integral for $SU(2|4)$ symmetric theories. In the case of $\mathcal{N} = 4$ SYM on $R \times S^3$, we saw that our result correctly reproduces the Gaussian matrix model of $\mathcal{N} = 4$ SYM \cite{2, 26, 27}.

We also considered the relation (c') in Fig. 1. This is regarded as the large-$N$ reduction for theories on $R \times S^3/Z_k$. From the result of the localization, we obtained the partition function and the vev of the circular Wilson loop in the reduced model of SYM on $R \times S^3$. We found that the free energy and the circular Wilson loop agree between the reduced model and SYM on $R \times S^3$. Our result provides a non-perturbative proof of the large-$N$ equivalence for these observables.

It may be possible to compute the matrix integral (3.81) exactly. If not, at least, one can compute it numerically. It is interesting to compare these results with gravity duals. The gravity dual of (3.81) for each $\mathcal{R}$ is constructed in \cite{15}. It would be possible to study the gauge/gravity duality for the family of various different theories labelled by $\mathcal{R}$ in a unified manner.

The remaining task to obtain the full partition function of PWMM is to compute the instanton part, which we have not addressed in this paper. As noted in the last part of Section 3.2, the saddle point equations at the future and the past infinities reduce to anti-self-dual and self-dual equations, respectively. They are indeed the mass deformed Nahm equations \cite{47}. By examining the moduli space of these equations, it would be possible to obtain the instanton corrections to our results, which may shed light on the nature of M-theory. We hope to report on these issues in the near future.

**Acknowledgements**

The work of Y.A. is supported by the Grant-in-Aid for the Global COE Program “The Next Generation of Physics, Spun from Universality and Emergence” from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan. The work of G.I. T.O. and S.S. is supported in part by the JSPS Research Fellowship for Young Scientists.

**A Gamma matrices**

Our gamma matrices are the same as those in \cite{2}. 

44
The local Lorentz metric is “mostly plus”, $g_{MN} = \text{diag}(-1, 1, 1, \ldots, 1)$ ($M, N = 0, 1, \ldots, 9$). The ten-dimensional $32 \times 32$ gamma matrices $\gamma^M$ ($M = 0, 1, \ldots, 9$) obey

$$\gamma^{\{M} \gamma^{N\}} = g^{MN}. \quad (A.1)$$

The associated representation of $\text{Spin}(1, 9)$ can be decomposed into two irreducible representations by the chirality,

$$\gamma^{11} \equiv \gamma^{1} \cdots \gamma^{9} \gamma^{0}. \quad (A.2)$$

We decompose the ten-dimensional Dirac spinor as

$$(S^+ \begin{array}{c} 0 \\ S^- \end{array}). \quad (A.3)$$

Then, the gamma matrices $\gamma^M$ are expressed in the block form,

$$\gamma^M = \begin{pmatrix} 0 & \tilde{\Gamma}^M \\ \Gamma^M & 0 \end{pmatrix}. \quad (A.4)$$

We take $\Gamma^M, \tilde{\Gamma}^M$ to be symmetric;

$$(\Gamma^M)^T = \Gamma^M, \quad (\tilde{\Gamma}^M)^T = \tilde{\Gamma}^M. \quad (A.5)$$

We define $\gamma^{MN}, \Gamma^{MN},$ and $\tilde{\Gamma}^{MN}$ as

$$\gamma^{MN} \equiv \gamma^{[M} \gamma^{N]} = \begin{pmatrix} \tilde{\Gamma}^{[M} \Gamma^{N]} & 0 \\ 0 & \Gamma^{[M} \tilde{\Gamma}^{N]} \end{pmatrix} \equiv \begin{pmatrix} \Gamma^{MN} & 0 \\ 0 & \tilde{\Gamma}^{MN} \end{pmatrix}. \quad (A.6)$$

Then, we have

$$\tilde{\Gamma}^{(M} \Gamma^{N)} = \Gamma^{\{M} \tilde{\Gamma}^{N\}} = g^{MN}, \quad (A.7)$$

$$\Gamma^M \Gamma^{PQ} = 4g^{M[P} \Gamma^{Q]} + \tilde{\Gamma}^{PQ} \Gamma^M. \quad (A.8)$$

We write some useful identities:

$$\left(\Gamma^M\right)_{\alpha_1 \alpha_2} \left(\Gamma^M\right)_{\alpha_3 \alpha_4} = 0, \quad (A.9)$$

$$\left(\Gamma^M\right)_{\alpha \delta} \left(\Gamma^M\right)_{\gamma \beta} = -\frac{1}{2} \left(\Gamma^M\right)_{\alpha \beta} \left(\Gamma^M\right)_{\gamma \delta} + \frac{1}{24} \left(\Gamma^{MNP}\right)_{\alpha \beta} \left(\Gamma_{MNP}\right)_{\gamma \delta}, \quad (A.10)$$

$$\left(\tilde{\Gamma}^{MN}\right)_{\alpha} \left(\tilde{\Gamma}^{MN}\right)_{\gamma} = 4 \left(\Gamma^M\right)_{\alpha \gamma} \left(\tilde{\Gamma}^M\right)^{\beta \delta} - 2 \delta^{\beta}_{\alpha} \delta^{\delta}_{\gamma} - 8 \delta^{\beta}_{\alpha} \delta^{\delta}_{\gamma}, \quad (A.11)$$
where \( \alpha, \beta, \cdots \) are spinor indices. The first equality is so called “triality”, and the last two are Fierz identities.

Decomposing the indices \( M = 0, \cdots, 9 \) into \( a = 1, 2, 3, 4 \) and \( m = 0, \cdots, 9 \), we obtain the following identities

\[
\Gamma_{am} \tilde{\Gamma}^a = -4 \tilde{\Gamma}_m, \quad (A.12)
\]
\[
\Gamma^a \Gamma_{be} \tilde{\Gamma}_a = 0, \quad (A.13)
\]
\[
\Gamma^a \Gamma_{bm} \tilde{\Gamma}_a = 2 \tilde{\Gamma}_b, \quad (A.14)
\]
\[
\Gamma^a \Gamma_{mn} \tilde{\Gamma}_a = 4 \tilde{\Gamma}_{mn}. \quad (A.15)
\]

In the rest of this appendix, we write down the gamma matrices \( \Gamma^M \) and \( \tilde{\Gamma}^M \) explicitly.

\[
\Gamma^0 = \begin{pmatrix} 1_{8 \times 8} & 0 \\ 0 & 1_{8 \times 8} \end{pmatrix}, \quad \Gamma^9 = \begin{pmatrix} 1_{8 \times 8} & 0 \\ 0 & -1_{8 \times 8} \end{pmatrix}, \quad (A.16)
\]
\[
\Gamma^i = \begin{pmatrix} 0 & E_i^T \\ E_i & 0 \end{pmatrix} \quad (i = 1, \cdots, 8).
\]

The \( 8 \times 8 \) matrices \( E_i \) \((i = 1, \cdots, 8)\) are given by

\[
E_a = \begin{pmatrix} J_a & 0 \\ 0 & \bar{J}_a \end{pmatrix} \quad (a = 1, 2, 3, 4), \quad E_{m'} = \begin{pmatrix} 0 & -J_{m'}^T \\ J_{m'} & 0 \end{pmatrix} \quad (m = 5, 6, 7, 8). \quad (A.17)
\]

Finally, the \( 4 \times 4 \) matrices \( J_a, \bar{J}_a \) are given as follows;

\[
J_1 = 1_{4 \times 4}, \quad \bar{J}_1 = 1_{4 \times 4},
\]

\[
J_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (A.18)
\]

\[
\bar{J}_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \bar{J}_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \bar{J}_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (A.19)
\]

and the matrices \( J_m \) are given by

\[
J_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad J_6 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad (A.19)
\]
\[ J_7 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J_8 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \]

The matrices \( J_{a'} \) and \( \bar{J}_{a'} \) satisfy
\[ J_{a'} J_{b'} = -\delta_{a'b'} 1_4 + \varepsilon_{a'b'c'} J_{c'}, \quad \bar{J}_{a'} \bar{J}_{b'} = -\delta_{a'b'} 1_4 - \varepsilon_{a'b'c'} \bar{J}_{c'} \quad (a', b', c' = 2, 3, 4). \quad (A.20) \]

Note that, in this representation, we have
\[ \Gamma_{1234} = \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 = \begin{pmatrix} 1_{4 \times 4} & 0 & 0 & 0 \\ 0 & -1_{4 \times 4} & 0 & 0 \\ 0 & 0 & -1_{4 \times 4} & 0 \\ 0 & 0 & 0 & 1_{4 \times 4} \end{pmatrix}, \quad (A.21) \]
\[ \Gamma_{5678} = \Gamma^5 \Gamma^6 \Gamma^7 \Gamma^8 = \begin{pmatrix} 1_{4 \times 4} & 0 & 0 & 0 \\ 0 & -1_{4 \times 4} & 0 & 0 \\ 0 & 0 & 1_{4 \times 4} & 0 \\ 0 & 0 & 0 & -1_{4 \times 4} \end{pmatrix}. \]

### B Our convention for \( S^3 \)

In this appendix, we summarize our convention for \( S^3 \) with a unit radius (See also [19,28]). \( S^3 \) is viewed as the \( SU(2) \) group manifold. We parametrize an element of \( SU(2) \) in terms of the Euler angles as
\[ g = e^{-i\varphi J_4/2} e^{-i\theta J_3/2} e^{-i\psi J_4/2}, \quad (B.1) \]
where \( 0 \leq \theta \leq \pi, \ 0 \leq \varphi < 2\pi, \ 0 \leq \psi < 4\pi \) and \( J_{a'} (a' = 2, 3, 4) \) satisfies \( [J_{a'}, J_{b'}] = i\varepsilon_{a'b'c'} J_{c'} \). The periodicity for these angle variables is given by
\[ (\theta, \varphi, \psi) \sim (\theta, \varphi + 2\pi, \psi + 2\pi) \sim (\theta, \varphi, \psi + 4\pi). \quad (B.2) \]

The isometry of \( S^3 \) corresponds to the left and the right multiplications of \( SU(2) \) elements on \( g \). We construct the right-invariant 1-forms under the multiplications,
\[ dgg^{-1} = -ie^{a'} J_{a'}. \quad (B.3) \]

The explicit form of \( e^{a'} \) is given by
\[ e^2 = \frac{1}{2}(- \sin \varphi d\theta + \sin \theta \cos \varphi d\psi), \]

47
\[ e^3 = \frac{1}{2}(\cos \varphi d\theta + \sin \theta \sin \varphi d\psi), \]
\[ e^4 = \frac{1}{2}(d\varphi + \cos \theta d\psi). \]  
(B.4)

It is easy to see that \( e_{a'} \) satisfy the Maurer-Cartan equation,

\[ de^{a'} - \varepsilon_{a'bc} e^b \wedge e^c = 0. \]  
(B.5)

We take \( e^a \) as the vielbein in this paper. In this frame, the spin connection is simply given by \( \omega^{ab'} = \varepsilon^{ab'c'} e^c \). The metric is given by

\[ ds^2 = e^a e^{a'} = \frac{1}{4} \left( d\theta^2 + \sin^2 \theta d\varphi^2 + (d\psi + \cos \theta d\varphi)^2 \right). \]  
(B.6)

The Killing vectors \( L_{a'} \) dual to \( e^{a'} \) are given by

\[ L_{a'} = -i e^{\mu}_{a'} \partial_{\mu}, \]  
where \( \mu = \theta, \varphi, \psi \), and \( e^\mu_{a'} \) are inverse of \( e^{a'}_{\mu} \). The explicit form of the Killing vectors are

\[ L_2 = -i \left( -\sin \varphi \partial_{\theta} - \cot \theta \cos \varphi \partial_{\varphi} + \frac{\cos \varphi}{\sin \theta} \partial_{\psi} \right), \]
\[ L_3 = -i \left( \cos \varphi \partial_{\theta} - \cot \theta \sin \varphi \partial_{\varphi} + \frac{\sin \varphi}{\sin \theta} \partial_{\psi} \right), \]
\[ L_4 = -i \partial_{\varphi}. \]  
(B.8)

Because of the Maurer-Cartan equation (B.5), the Killing vectors satisfy the SU(2) algebra, \([L_{a'}, L_{b'}] = i \varepsilon_{a'b'c'} L_{c'}\).

## Monopole spherical harmonics

Here, we write down the monopole spherical harmonics \([19]\). One can regard \( S^3 \) as a \( U(1) \) bundle over \( S^2 = SU(2)/U(1) \). \( S^2 \) is parametrized by \( \theta \) and \( \varphi \) and covered with two local patches: the patch I defined by \( 0 \leq \theta < \pi \) and the patch II defined by \( 0 < \theta \leq \pi \). In the following expressions, the upper sign is taken in the patch I while the lower sign in the patch II. The element of \( SU(2) \) in (B.1) is decomposed as

\[ g = L \cdot h \text{ with } L = e^{-i\varphi J_3/2} e^{-i\theta J_3/2} e^{i\psi J_3/2} \text{ and } h = e^{-i(\psi \pm \varphi) J_3/2}. \]  
(C.1)
$L$ is a representative of $SU(2)/U(1)$, while $h$ represents the fiber $U(1)$. The fiber direction is parametrized by $y = \psi \pm \varphi$. Note that $L$ has no $\varphi$-dependence for $\theta = 0, \pi$. The zweibein of $S^2$ is given by the $a' = 2, 3$ components of the left-invariant 1-form, $-iL^{-1} dL = 2 e^{a'} J_{a'}/2 [48]$. It takes the form

\[
e^2 = \frac{1}{2}((\pm \sin \varphi d\theta + \sin \theta \cos \varphi d\varphi), \quad e^3 = \frac{1}{2}(-\cos \varphi d\theta \pm \sin \theta \sin \varphi d\varphi).
\]

This zweibein gives the standard metric of $S^2$ with the radius $1/2$:

\[
ds^2 = \frac{1}{4}(d\theta^2 + \sin^2 \theta \varphi^2). \tag{C.2}\]

Making a replacement $\partial_p \to -iq$ in (B.8) leads to the angular momentum operator in the presence of a monopole with magnetic charge $q$ at the origin [49]:

\[
L^{(q)}_2 = i\sin \varphi \partial_\theta + \cot \theta \cos \varphi \partial_\varphi - \frac{1}{\sin \theta} \cos \varphi, \\
L^{(q)}_3 = i(-\cos \varphi \partial_\theta + \cot \theta \sin \varphi \partial_\varphi) - \frac{1}{\sin \theta} \sin \varphi, \\
L^{(q)}_4 = -i\partial_\varphi \mp q, \tag{C.3}
\]

where $q$ is quantized as $q = 0, \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \ldots$, because $y$ is a periodic variable with the period $4\pi$. These operators act on the local sections on $S^2$ and satisfy the $SU(2)$ algebra $[L^{(q)}_{a'}, L^{(q)}_{b'}] = i\varepsilon_{a'b'c'} L^{(q)}_c$. Note that when $q = 0$, these operators are reduced to (B.8) with $\partial_\psi = 0$, which is the ordinary angular momentum operators on $S^2$. The $SU(2)$ acting on $g$ from left survives as the isometry of $S^2$. Note that in 2+1 SYM on $R \times S^2$ the isometry of $S^2$ is included in the $SU(2|4)$ symmetry as a subgroup.

The monopole spherical harmonics are the basis of local sections on $S^2$. They are given by

\[
\hat{Y}_{Jmq}(\Omega_2) = (-1)^{J-q} \sqrt{2J+1}(J-q)! e^{i\theta J_3} |Jm\rangle e^{i(\pm q+m)\varphi}. \tag{C.4}\]

Here, $J = |q|, |q| + 1, |q + 2|, \ldots$, $m = -J, -J + 1, \ldots, J - 1, J$. The existence of the lower bound of the angular momentum $J \geq |q|$ is due to the fact that the magnetic field produced by the monopole also has nonzero angular momentum. Note that the monopole harmonics with $q = 0$ do not transform on the overlap of two patches. They correspond to global sections (functions) on $S^2$ which are expressed by the ordinary spherical harmonics on $S^2$. The action of $L^{(q)}_{a'}$ on the monopole spherical harmonics is given by

\[
(L^{(q)})^2 \hat{Y}_{Jmq} = J(J + 1)\hat{Y}_{Jmq},
\]
\[ L_{\pm}^{(q)} \tilde{Y}_{Jmq} = \sqrt{(J \mp m)(J \pm m + 1)} \tilde{Y}_{Jm\pm1q}, \]
\[ L_{4}^{(q)} \tilde{Y}_{Jmq} = m \tilde{Y}_{Jmq}, \]  
(C.5)

where \( L_{\pm}^{(q)} \equiv L_{2}^{(q)} \pm i L_{3}^{(q)} \). The complex conjugates of the monopole spherical harmonics obeys the following relation,

\[ (\tilde{Y}_{Jmq})^* = (-1)^{m-q} \tilde{Y}_{J-m-q}. \]  
(C.6)

The monopole spherical harmonics are orthonormal to each other;

\[ \int \frac{d\Omega_2}{4\pi} (\tilde{Y}_{Jmq})^* \tilde{Y}_{J'm'q} = \delta_{JJ'} \delta_{mm'}. \]  
(C.7)

\section{D Fuzzy spherical harmonics}

In this appendix, we review the fuzzy spherical harmonics which form a basis of rectangular matrices [17,28].

Let us consider a \((2j_s+1) \times (2j_t+1)\) rectangular complex matrix, where \(j_s,j_t \in \mathbb{Z}_{\geq 0}/2\). Such a matrix \(M^{(s,t)}\) can be generally expanded as

\[ M^{(s,t)} = \sum_{m_s,m_t} M_{m_s,m_t} |j_s m_s\rangle \langle j_t m_t|, \]  
(D.1)

by using a basis \(\{|jm\}; m = -j, -j + 1, \cdots, j\rangle\) of the spin \(j\) representation space of \(SU(2)\) algebra. We define an operation which multiplies the representation matrices of the \(SU(2)\) generators from left and right:

\[ L_{a'} \circ M^{(s,t)} = \sum_{m_s,m_t} M_{m_s,m_t} (L_{a'}^{[j]} |j_s m_s\rangle \langle j_t m_t| - |j_s m_s\rangle \langle j_t m_t| L_{a'}^{[j]}), \]  
(D.2)

where \(L_{a'}^{[j]} (a' = 2, 3, 4)\) stands for the spin \(j\) representation matrix of the generator.

We can construct another basis of the rectangular matrices denoted by \(\{\tilde{Y}_{Jm(j_s,j_t)}\}\) such that they satisfy

\[ (L_{a'} \circ)^2 \tilde{Y}_{Jm(j_s,j_t)} = J(J + 1) \tilde{Y}_{Jm(j_s,j_t)}, \]
\[ L_{\pm} \circ \tilde{Y}_{Jm(j_s,j_t)} = \sqrt{(J \mp m)(J \pm m + 1)} \tilde{Y}_{Jm\pm1(j_s,j_t)}, \]
\[ L_{4} \circ \tilde{Y}_{Jm(j_s,j_t)} = m \tilde{Y}_{Jm(j_s,j_t)}. \]  
(D.3)
\( \hat{Y}_{Jm(j_s j_t)} \) are called scalar fuzzy spherical harmonics and defined by
\[
\hat{Y}_{Jm(j_s j_t)} = \sum_{m_s, m_t} (-)^{-j_s + m_t} C_{j_s m_s j_t m_t}^{Jm} |j_s m_s \rangle \langle j_t m_t |,
\]  
(D.4)
where \( C_{j_s m_s j_t m_t}^{Jm} \) are the Clebsch-Gordan coefficients. Their hermitian conjugates are given by
\[
(\hat{Y}_{Jm(j_s j_t)})^\dagger = (-)^{m - j_s - j_t} \hat{Y}_{J - m(j_t j_s)},
\]  
(D.5)
and they satisfy the orthogonality relation
\[
\text{tr} \left\{ (\hat{Y}_{Jm(j_s j_t)})^\dagger \hat{Y}_{J'm'(j_s j_t)} \right\} = \delta_{J, J'} \delta_{m, m'},
\]  
(D.6)

E  Perturbative check of our result for trivial background

We consider the following observable around the trivial background, \( \hat{X}_{a'} = 0 \), in the PWMM,
\[
\langle \text{Tr} \left( X_4 + iX_0^{(E)} \right)^2 (\tau = 0) \rangle.
\]  
(E.1)
This observable is \( Q \)-closed and so can be computed by the localization method. In this appendix, in order to illustrate the validity of the matrix integral (3.81), we will compute this observable perturbatively both from the original PWMM and the matrix integral (3.81). We will see that the two different computations agree completely up to the one-loop level.

One-loop calculation in PWMM

In the trivial background, the action of PWMM \( S = S_{\text{free}} + S_{\text{int}} \) is given by
\[
S_{\text{free}} = \frac{1}{g_{PW}^2} \int d\tau \text{Tr} \left[ -\frac{1}{2} (\partial_\tau X_{a'})^2 - 2X_{a'}^2 - \frac{i}{2} X_m^2 - \frac{i}{4} \Psi \Gamma^1 \partial_\tau \Psi - \frac{3i}{4} \Psi \Gamma^{234} \Psi \right],
\]  
(E.2)
\[
S_{\text{int}} = \frac{1}{g_{PW}^2} \int d\tau \text{Tr} \left[ i \varepsilon_{a'b'c'} X_{a'} [X_{b'}, X_{c'}] + \frac{1}{4} [X_{a'}, X_{b'}]^2 + \frac{1}{2} [X_{a'}, X_m]^2 + \frac{1}{4} [X_m, X_n]^2 \right.
\]
\[
- \frac{1}{2} \Psi \Gamma^M [X_M, \Psi].
\]  
(E.3)
Here, we have taken \( X_1 = 0 \) gauge, and \( M = 0, 1, \cdots, 9, a' = 2, 3, 4 \) and \( m = 5, \cdots, 9 \).
We can read off the Feynman rule of PWMM in momentum space. The propagators are given by

\[ \langle X_{a',ij}(p)X_{b',kl}(q) \rangle = 2\pi \delta(p + q)\delta_{a'b'}\delta_{il}\delta_{jk} \frac{g_{PW}^2}{p^2 + 4}, \]  

(E.4)

\[ \langle X_{m,ij}(p)X_{n,kl}(q) \rangle = 2\pi \delta(p + q)\delta_{mn}\delta_{il}\delta_{jk} \frac{g_{PW}^2}{p^2 + 1}, \]  

(E.5)

\[ \langle \Psi_{\alpha,ij}(p)\Psi_{\beta,kl}(q) \rangle = 2\pi \delta(p + q)\delta_{il}\delta_{jk} \frac{p\Gamma_1 + \frac{3i}{2}\Gamma_2^{\alpha\beta}}{p^2 + \frac{9}{4}} g_{PW}^2. \]  

(E.6)

Note that (E.5) with \( m = n = 0 \) is not the propagator of \( X_0^{(E)} \), but that of the wick rotated field \( X_0^{(E)} \).

We compute (E.1) up to the one-loop order. Note that the term \( \langle \text{Tr} X_4 X_4^{(E)} \rangle \) vanishes up to the one-loop level, we compute

\[ \langle \text{Tr} X_4 X_4 \rangle - \langle \text{Tr} X_0^{(E)} X_0^{(E)} \rangle. \]  

(E.7)

The tree level diagrams are easy to compute. For example,

\[ \langle \text{Tr} X_4 X_4(\tau = 0) \rangle \big|_{\text{tree}} = \int \frac{dp\, dq}{(2\pi)^2} 2\pi \delta(p + q) \frac{g_{PW}^2}{p^2 + 4} N^2 = \frac{g_{PW}^2 N^2}{4}. \]  

(E.8)

Similarly,

\[ \langle \text{Tr} X_0^{(E)} X_0^{(E)}(\tau = 0) \rangle \big|_{\text{tree}} = \frac{g_{PW}^2}{2} N^2. \]  

(E.9)

At the one-loop level, the diagrams shown in Fig. 2 and Fig. 3 contribute. For example, the vertices of the first diagram in Fig. 2 comes from the first terms in (E.3),

\[ \frac{i}{g_{PW}^2} \varepsilon_{a'b'c'} \text{Tr} X_{a'}[X_{b'}, X_{c'}] = \frac{6i}{g_{PW}^2} \text{Tr} X_4[X_2, X_3], \]  

(E.10)

and thus, the diagram can be evaluated as

\[ \frac{1}{2} \left( \frac{6i}{g_{PW}^2} \right)^2 \int \frac{dp_1\, dp_2}{(2\pi)^2} \int \frac{dq_1\cdots dq_6}{(2\pi)^6} (2\pi)^6 \delta(q_1 + q_2 + q_3)(2\pi)^6 \delta(q_4 + q_5 + q_6) \times \langle \text{Tr} X_4(p_1) X_4(p_2) \text{Tr} X_4(q_1)[X_2(q_2), X_3(q_3)] \text{Tr} X_4(q_4)[X_2(q_5), X_3(q_6)] \rangle_{\text{conn}} \]

\[ = \frac{1}{16} g_{PW}^4 (N^3 - N). \]  

(E.11)

The other diagrams can be evaluated in a similar manner (for the contribution of each diagram, see the captions of Fig. 2 and Fig. 3). The result is

\[ \langle \text{Tr} X_4 X_4 \rangle = \frac{g_{PW}^2 N^2}{4} + \left( \frac{1}{16} - \frac{1}{32} - \frac{3}{16} \right) g_{PW}^4 (N^3 - N) + \mathcal{O}(g_{PW}^6) \]  

(E.12)
Figure 2: The diagrams contributing to $\langle \text{Tr} X_4 X_4 \rangle$. The dotted line represents fermion loop. The vertices of the left diagram come from the first term in (E.3), which gives $\frac{1}{16} g_{PW}^4 (N^3 - N)$. The vertices of the middle diagram come from the second and third terms in (E.3), which give $-\frac{1}{32} g_{PW}^4 (N^3 - N)$ and $-\frac{3}{16} g_{PW}^4 (N^3 - N)$. The vertices of the right diagram come from the fifth term in (E.3). This diagram actually vanishes.

Figure 3: The diagrams contributing to $\langle \text{Tr} X_0^{(E)} X_0^{(E)} \rangle$. The vertices of the left diagram come from the third and the fourth terms in (E.3), which give $-\frac{3}{8} g_{PW}^4 (N^3 - N)$ and $-\frac{5}{4} g_{PW}^4 (N^3 - N)$. The vertices of the right diagram come from the fifth term in (E.3), which gives $\frac{5}{4} g_{PW}^4 (N^3 - N)$.

\[
\langle \text{Tr} X_0^{(E)} X_0^{(E)} \rangle = \frac{g_{PW}^2}{2} N^2 + \left( -\frac{3}{8} - \frac{5}{4} + \frac{5}{4} \right) g_{PW}^4 (N^3 - N) + \mathcal{O}(g_{PW}^6) \tag{E.13}
\]

Therefore, up to the one-loop order, we obtain

\[
\langle \text{Tr} X_4 X_4 \rangle - \langle \text{Tr} X_0^{(E)} X_0^{(E)} \rangle = -\frac{g_{PW}^2 N^2}{4} + \frac{7}{32} (N^3 - N) g_{PW}^4 + \mathcal{O}(g_{PW}^6). \tag{E.14}
\]

**One-loop calculation in matrix integral**

We can apply the localization method to compute the observable (E.11) around the trivial background in the PWMM. The saddle point configuration corresponding to the trivial background is given by $\hat{X}_{a'} = 0$ in (3.22). In this case, since $\hat{X}_0^{(E)}(\tau = 0) = M$, (E.11) is reduced to the following matrix integral,

\[
\langle \text{Tr} (iM)^2 \rangle = -\int \left( \prod_{i=1}^{N} dm_i \right) \sum_{i=1}^{N} (m_i)^2 Z_{1\text{-loop}}^{(\text{trivial})} \exp \left( -\frac{2}{g_{PW}^2} \sum_{i} (m_i)^2 \right), \tag{E.15}
\]

53
where $Z_{1-loop}^{(trivial)}$ is the determinant factor (3.80) for the trivial background;

$$Z_{1-loop}^{(trivial)} = \prod_{i<j} \frac{(4 + (m_i - m_j)^2)(m_i - m_j)^2}{(1 + (m_i - m_j)^2)^2}. \tag{E.16}$$

In order to perform a perturbative calculation, we express the above eigenvalue integral into a covariant form. Firstly, the factor $\prod_{i<j} (m_i - m_j)^2$ in (E.16) gives the correct measure of the hermitian matrix integral,

$$\int \left( \prod_{i=1}^{N} dm_i \right) \prod_{i<j} (m_i - m_j)^2 = \int dM. \tag{E.17}$$

For the other part of (E.16), we exponentiate it as

$$\prod_{i<j} \frac{4 + (m_i - m_j)^2}{(1 + (m_i - m_j)^2)^2} = \exp \left[ \sum_{i<j} \log(4 + (m_i - m_j)^2) - 2 \sum_{i<j} \log(1 + (m_i - m_j)^2) \right]. \tag{E.18}$$

The first term in the exponent can be written in terms of the matrix $M$ (up to the irrelevant constant $N(N - 1) \log 4$) as

$$\sum_{i<j} \log[1 + \frac{1}{4}(m_i - m_j)^2] = \frac{1}{2} \sum_{i,j} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{n \cdot 4^n} \sum_{r=0}^{2n} \binom{2n}{r} (m_i)^{2n-r} (-m_j)^r$$

$$= \frac{1}{2} \sum_{i,j} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{n \cdot 4^n} \sum_{r=0}^{2n} \binom{2n}{r} (-1)^r \text{Tr} M^{2n-r} \text{Tr} M^r. \tag{E.19}$$

Similarly, the second term in the exponent can also be written as

$$-2 \sum_{i<j} \log(1 + (m_i - m_j)^2) = \sum_{n=1}^{\infty} (-1)^n \frac{2n}{n} \sum_{r=0}^{2n} \binom{2n}{r} (-1)^r \text{Tr} M^{2n-r} \text{Tr} M^r. \tag{E.20}$$

Thus, we can express the eigenvalue integral as the following matrix model,

$$\int dM \ e^{S[M]}, \tag{E.21}$$

where the action $S[M]$ is

$$S = -\frac{2}{g_{PW}^2} \text{Tr} M^2 + \sum_{n=1}^{\infty} \sum_{r=0}^{2n} C_{n,r} \text{Tr} M^{2n-r} \text{Tr} M^r. \tag{E.22}$$

Here, the coefficients $C_{n,r}$ are given by

$$C_{n,r} = \frac{(-1)^{n+r}}{n} \binom{2n}{r} \left( 1 - \frac{1}{2 \cdot 4^n} \right). \tag{E.23}$$
The propagator is given by
\[ \langle M_{ij}M_{kl} \rangle = \frac{g_{PW}^2}{4} \delta_{il} \delta_{jk}. \] (E.24)

We compute the observable [E.15] up to the one-loop order. The tree level contribution is given by \(-\frac{g_{PW}^2 N^2}{4}\). Note that at the one-loop level, the relevant interactions in [E.22] are only the terms with \(n = 1\),
\[ (C_{1,0} + C_{1,2}) \text{Tr} 1_{N \times N} \text{Tr} M^2 + C_{1,1} \text{Tr} M \text{Tr} M = \frac{7}{4} (-N \text{Tr} M^2 + \text{Tr} M \text{Tr} M). \] (E.25)

Then we can easily find
\[ \langle \text{Tr} (iM)^2 \rangle = -\frac{g_{PW}^2 N^2}{4} + \frac{7}{32} (N^3 - N) g_{PW}^4 + \mathcal{O}(g_{PW}^6). \] (E.26)

This agrees completely with the result obtained from the original PWMM, [E.14].

References

[1] N. A. Nekrasov, Adv. Theor. Math. Phys. 7, 831 (2004) [hep-th/0206161].

[2] V. Pestun, arXiv:0712.2824 [hep-th].

[3] A. Kapustin, B. Willett and I. Yaakov, JHEP 1003, 089 (2010) arXiv:0909.4559 [hep-th]].

[4] N. Drukker, M. Marino and P. Putrov, Commun. Math. Phys. 306, 511 (2011) arXiv:1007.3837 [hep-th]].

[5] L. F. Alday, D. Gaiotto and Y. Tachikawa, Lett. Math. Phys. 91, 167 (2010) arXiv:0906.3219 [hep-th]].

[6] N. Wyllard, JHEP 0911, 002 (2009) arXiv:0907.2189 [hep-th]].

[7] J. Kallen and M. Zabzine, JHEP 1205, 125 (2012) arXiv:1202.1956 [hep-th]].

[8] K. Hosomichi, R. -K. Seong and S. Terashima, Nucl. Phys. B 865, 376 (2012) arXiv:1203.0371 [hep-th]].
[9] H. -C. Kim and S. Kim, arXiv:1206.6339 [hep-th].

[10] H. -C. Kim, J. Kim and S. Kim, arXiv:1211.0144 [hep-th].

[11] G. W. Moore, N. Nekrasov and S. Shatashvili, Commun. Math. Phys. 209, 77 (2000) hep-th/9803265.

[12] V. A. Kazakov, I. K. Kostov and N. A. Nekrasov, Nucl. Phys. B 557, 413 (1999) hep-th/9810035.

[13] D. E. Berenstein, J. M. Maldacena and H. S. Nastase, JHEP 0204, 013 (2002) arXiv:hep-th/0202021.

[14] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, Phys. Rev. D 55 (1997) 5112 arXiv:hep-th/9610043.

[15] H. Lin and J. M. Maldacena, Phys. Rev. D 74, 084014 (2006) hep-th/0509235.

[16] J. M. Maldacena, M. M. Sheikh-Jabbari and M. Van Raamsdonk, JHEP 0301, 038 (2003) hep-th/0211139.

[17] G. Ishiki, S. Shimasaki, Y. Takayama and A. Tsuchiya, JHEP 0611 (2006) 089 arXiv:hep-th/0610038.

[18] T. Ishii, G. Ishiki, S. Shimasaki and A. Tsuchiya, JHEP 0705 (2007) 014 arXiv:hep-th/0703021.

[19] T. Ishii, G. Ishiki, S. Shimasaki and A. Tsuchiya, Phys. Rev. D 77, 126015 (2008) arXiv:0802.2782 [hep-th].

[20] R. C. Myers, JHEP 9912, 022 (1999) hep-th/9910053.

[21] W. Taylor, Phys. Lett. B 394, 283 (1997) hep-th/9611042.

[22] T. Klose and J. Plefka, Nucl. Phys. B 679, 127 (2004) hep-th/0310232.

[23] T. Fischbacher, T. Klose and J. Plefka, JHEP 0502, 039 (2005) hep-th/0412331.

[24] A. Agarwal and D. Young, Phys. Rev. D 82, 045024 (2010) arXiv:1003.5547 [hep-th].
[25] H. Ling, A. R. Mohazab, H.-H. Shieh, G. van Anders and M. Van Raamsdonk, JHEP 0610, 018 (2006) [hep-th/0606014].

[26] J. K. Erickson, G. W. Semenoff and K. Zarembo, Nucl. Phys. B 582, 155 (2000) [hep-th/0003055].

[27] N. Drukker and D. J. Gross, J. Math. Phys. 42, 2896 (2001) [hep-th/0010274].

[28] T. Ishii, G. Ishiki, S. Shimasaki and A. Tsuchiya, Phys. Rev. D 78 (2008) 106001 [arXiv:0807.2352 [hep-th]].

[29] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200]; S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109]; E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

[30] N. Kim, T. Klose and J. Plefka, Nucl. Phys. B 671, 359 (2003) [hep-th/0306054].

[31] T. Eguchi and H. Kawai, Phys. Rev. Lett. 48, 1063 (1982).

[32] G. Ishiki, S. Shimasaki and A. Tsuchiya, JHEP 1111, 036 (2011) [arXiv:1106.5590 [hep-th]].

[33] Y. Kitazawa and K. Matsumoto, Phys. Rev. D 79, 065003 (2009) [arXiv:0811.0529 [hep-th]].

[34] G. Ishiki, S. W. Kim, J. Nishimura and A. Tsuchiya, Phys. Rev. Lett. 102, 111601 (2009) [arXiv:0810.2884 [hep-th]].

[35] G. Ishiki, S. W. Kim, J. Nishimura and A. Tsuchiya, JHEP 0909, 029 (2009) [arXiv:0907.1488 [hep-th]].

[36] M. Honda, G. Ishiki, J. Nishimura and A. Tsuchiya, PoS LAT2011, 244 (2011) [arXiv:1112.4274 [hep-lat]].

[37] M. Honda, G. Ishiki, S. W. Kim, J. Nishimura and A. Tsuchiya, PoS LATICE2010, 253 (2010) [arXiv:1011.3904 [hep-lat]].
[38] G. Ishiki, S. Shimasaki and A. Tsuchiya, Phys. Rev. D 80, 086004 (2009) [arXiv:0908.1711 [hep-th]].

[39] G. Ishiki, S. Shimasaki and A. Tsuchiya, Nucl. Phys. B 834, 423 (2010) [arXiv:1001.4917 [hep-th]].

[40] Y. Asano, G. Ishiki, T. Okada and S. Shimasaki, Phys. Rev. D 85, 106003 (2012) [arXiv:1203.0559 [hep-th]].

[41] M. Honda and Y. Yoshida, Nucl. Phys. B 865, 21 (2012) [arXiv:1203.1016 [hep-th]].

[42] D. Varshalovich, A. Moskalev and V. Khersonskii, Quantum Theory of Angular Momentum (World Scientific, Singapore, 1988).

[43] T. Ishii, G. Ishiki, K. Ohta, S. Shimasaki and A. Tsuchiya, Prog. Theor. Phys. 119, 863 (2008) [arXiv:0711.4235 [hep-th]].

[44] N. Berkovits, Phys. Lett. B 318, 104 (1993) [hep-th/9308128].

[45] J. -T. Yee and P. Yi, JHEP 0302, 040 (2003) [hep-th/0301120].

[46] H. Lin, Phys. Rev. D 74, 125013 (2006) [hep-th/0609186].

[47] C. Bachas, J. Hoppe and B. Pioline, JHEP 0107, 041 (2001) [hep-th/0007067].

[48] A. Salam and J. A. Strathdee, Annals Phys. 141, 316 (1982).

[49] T. T. Wu and C. N. Yang, Nucl. Phys. B 107, 365 (1976).