On the Secure Degrees-of-Freedom of the Multiple-Access-Channel

Ghadamali Bagherikaram, Abolfazl S. Motahari, Amir K. Khandani
Coding and Signal Transmission Laboratory, Department of Electrical and Computer Engineering,
University of Waterloo, Ontario, N2L 3G1, Emails: \{gbagheri,abolfazl,khandani\}@cst.uwaterloo.ca

Abstract

A K-user secure Gaussian Multiple-Access-Channel (MAC) with an external eavesdropper is considered in this paper. An achievable rate region is established for the secure discrete memoryless MAC. The secrecy sum capacity of the degraded Gaussian MIMO MAC is proven using Gaussian codebooks. For the non-degraded Gaussian MIMO MAC, an algorithm inspired by interference alignment technique is proposed to achieve the largest possible total Secure-Degrees-of-Freedom (S-DoF). When all the terminals are equipped with a single antenna, Gaussian codebooks have shown to be inefficient in providing a positive S-DoF. Instead, a novel secure coding scheme is proposed to achieve a positive S-DoF in the single antenna MAC. This scheme converts the single-antenna system into a multiple-dimension system with fractional dimensions. The achievability scheme is based on the alignment of signals into a small sub-space at the eavesdropper, and the simultaneous separation of the signals at the intended receiver. Tools from the field of Diophantine Approximation in number theory are used to analyze the probability of error in the coding scheme. It is proven that the total S-DoF of $\frac{K}{d_n}$ can be achieved for almost all channel gains. For the other channel gains, a multi-layer coding scheme is proposed to achieve a positive S-DoF. As a function of channel gains, therefore, the achievable S-DoF is discontinued.

I. INTRODUCTION

The notion of information theoretic secrecy in communication systems was first introduced by Shannon in [1]. The information theoretic secrecy requires that the received signal of the eavesdropper not provide any information about the transmitted messages. Following the pioneering works of Wyner [2] and Csiszar et. al. [3] which studied the wiretap channel, many multi-user channel models have been considered from a perfect secrecy point of view. The secrecy capacity rate has been established for some basic models, including the Gaussian wiretap channel [4], the MIMO wiretap channel [5]–[9], and the fading wiretap channel [10], [11]. Additionally, the secrecy capacity rate regions have been characterized successfully for some multi-user channels, including the MIMO Gaussian broadcast channel with confidential messages [12]–[16].

The secure Gaussian MAC with/without an external eavesdropper is introduced in [17]–[20]. The secure Gaussian MAC with an external eavesdropper consists of an ordinary Gaussian MAC and an external eavesdropper. The capacity region of this channel is still an open problem in the information theory field. For this channel, an achievable rate scheme based on Gaussian codebooks is proposed in [20], and the sum secrecy capacity of the degraded Gaussian channel is found in [18]. For some special cases, upper bounds, lower bounds, and some asymptotic results on the secrecy capacity exist; see for example [21]–[24]. For the achievable part, Shannon’s random coding argument has proven to be effective in these works.

The secure MAC generalizes the wiretap channel. In the wiretap channel, the direct coding scheme uses the framework of random coding, which is widely used in the analysis of multi-terminal source and channel coding problems. One approach to find achievable sum rates for the secure MAC is to extend the random coding solution to the secure multi-user. As we will show in this paper, this extension leads to a single-letter characterization for the secure rate region of the MAC. Our achievability, as usual, is based on the i.i.d random binning scheme.

On the other hand, it is shown that the random coding argument may be insufficient to prove capacity theorems for certain channels; instead, structure codes can be used to construct efficient channel codes for Gaussian channels. In reference [25], nested lattice codes are used to provide secrecy in two-user Gaussian channels. In [25] it is shown that structure codes can achieve a positive S-DoF in a two-user MAC. In particular, the achievability scheme of [25] provides an S-DoF of $\frac{1}{2}$ for a small category of channel gains and for the other categories, it provides a S-DoF of strictly less than $\frac{1}{2}$.

In reference [26], the concept of interference alignment is introduced and has illustrated its capability in achieving the full DoF of a class of two-user X channels. In reference [27], and [28], a novel coding scheme applicable in networks with single antenna nodes is proposed. This scheme converts a single antenna system into an equivalent Multiple Input Multiple Output (MIMO) system with fractional dimensions.

In this work we establish the secrecy sum capacity of the degraded Gaussian MIMO MAC using random binning of Gaussian codebooks. For the non-degraded channel, we present an algorithm inspiring the notion of signal alignment to achieve the largest S-DoF by using Gaussian codebooks. We then use the notion of real alignment of [27] to prove that for almost all...
channel gains in the secure $K$ user single-antenna Gaussian MAC, we can achieve the S-DoF of $\frac{K-1}{K}$. Here, our scheme uses structure codes instead of Gaussian codebooks. In the case of the channel gains for which the S-DoF of $\frac{K-1}{K}$ cannot be achieved, we propose a multi-layer coding scheme to achieve a positive S-DoF. The scheme of this work differs from that of [25], in the sense that our scheme achieves the S-DoF of $\frac{1}{2}$ for almost all channel gains, while reference [25] provides a scheme that achieves the S-DoF of $\frac{1}{2}$ (when $K = 2$) for specific channel gains, i.e., algebraic irrational gains. Therefore, we prove that the curve of S-DoF versus channel gains is almost certainly constant with many discontinuations.

The rest of the paper is organized as follows: Section II provides some background and preliminaries. In section III, we consider the secure Gaussian MIMO MAC and establish the secrecy sum rate for the degraded channel. In this section, we propose an algorithm to achieve the largest possible value of S-DoF for the non-degraded channel model. We present our results for the achievable S-DoF of the single-antenna MAC in section IV. Finally, section V concludes the paper.

II. PRELIMINARIES

Consider a secure $K$-user Gaussian MIMO Multiple-Access-Channel (MAC) as depicted in Fig. 1. In this confidential setting, each user $k$ ($k \in K = \{1, 2, ..., K\}$) wishes to send a message $W_k$ to the intended receiver in $n$ uses of the channel simultaneously, and prevent the eavesdropper from having any information about the messages. At a specific time, the signals received by the intended receiver and the eavesdropper is given by

\[
y = \sum_{k=1}^{K} H_k x_k + n_1
\]

\[
z = \sum_{k=1}^{K} H_{k,e} x_k + n_2
\]

where

- $x_k$ for $k = 1, 2, ..., K$ is a real input vector of size $M_k \times 1$ under an input average power constraint. We require that $Tr(Q_k) \leq P$, where $Q_k = E[x_k x_k^H]$. Here, the superscript $^H$ denotes the Hermitian transpose of a vector and $Tr(\cdot)$ denotes the Trace operator on the matrices.
- $y$ and $z$ are real output vectors which are received by the destination and the eavesdropper, respectively. These are vectors of size $N \times 1$ and $N_e \times 1$, respectively.
- $H_k$ and $H_{k,e}$ for $k = 1, 2, ..., K$ are fixed, real gain matrices which model the channel gains between the transmitters and the intended receiver, and the eavesdropper, respectively. These are matrices of size $N \times M_k$ and $N_e \times M_k$, respectively.

The channel state information is assumed to be known perfectly at all the transmitters and all receivers.

- $n_1$ and $n_2$ are real Gaussian random vectors with zero means and covariance matrices $N_1 = E[n_1 n_1^T] = I_N$ and $N_2 = E[n_2 n_2^T] = I_{N_e}$, respectively. Here, $I_M$ represents the identity matrix of size $M \times M$.

Let $x^n_k$, $y^n$ and $z^n$ denote the random channel inputs and random channel outputs matrices over a block of $n$ samples. Furthermore, let $n^n_1$, and $n^n_2$ denote the additive noises of the channels. Therefore, we have

\[
y^n = \sum_{k=1}^{K} H_k x^n_k + n^n_1
\]

\[
z^n = \sum_{k=1}^{K} H_{k,e} x^n_k + n^n_2
\]
Note that bold vectors are random while the matrices \( \mathbf{H}_k \) and \( \mathbf{H}_{k,e} \) are deterministic matrices for all \( k \in \mathcal{K} \). The columns of \( \mathbf{n}_1^n \) and \( \mathbf{n}_2^n \) are independent Gaussian random vectors with covariance matrices \( \mathbf{I}_k \) and \( \mathbf{I}_{N_e} \), respectively. In addition \( \mathbf{n}_1^n \) and \( \mathbf{n}_2^n \) are independent of \( \mathbf{x}_k^n \)'s and \( \mathbf{W}_k \)'s. A \( (2^{nR_1}, 2^{nR_2}, ..., 2^{nR_K}, n) \) secret code for the above channel consists of the following components:

1) \( K \) secret message sets \( \mathcal{W}_k = \{1, 2, ..., 2^{nR_k}\} \).

2) \( K \) stochastic encoding functions \( f_k(.) \) which map the secret messages to the transmitted symbols, i.e., \( f_k : w_k \rightarrow \mathbf{x}_k^n \) for each \( w_k \in \mathcal{W}_k \). At encoder \( k \), each codeword is designed according to the transmitter’s average power constraint \( P \).

3) A decoding function \( \phi(.) \) which maps the received symbols to estimate the messages: \( \phi(y^n) \rightarrow (W_1, ..., W_K) \). The reliability of the transmission is measured by the average probability of error, which is defined as the probability that the decoded messages are not equal to the transmitted messages; that is, 

\[
P_e^{(n)} = \frac{1}{\prod_{k=1}^{K} 2^{nR_k}} \sum_{(w_1, ..., w_K) \in \mathcal{W}_1 \times ... \times \mathcal{W}_K} P(\phi(y^n) \neq (w_1, ..., w_K)|(w_1, ..., w_K) \text{ is sent}).
\] (3)

The secrecy level is measured by the normalized equivocation defined as follows: The normalized equivocation for each subset of messages \( \mathcal{W}_S \) for \( S \subseteq \mathcal{K} \) is 

\[
\Delta_S = \frac{H(W_S|z^n)}{H(W_S)}.
\] (4)

The rate-equivocation tuple \( (R_1, ..., R_K, d) \) is said to be achievable for the Gaussian MIMO Multiple-Access-Channel with confidential messages, if for any \( \epsilon > 0 \), there exists a sequence of \( ((2^{nR_1}, ..., 2^{nR_K}), n) \) secret codes, such that for sufficiently large \( n \),

\[
P_e^{(n)} \leq \epsilon,
\] (5)

and

\[
\Delta_S \geq d - \epsilon, \quad \forall S \subseteq \mathcal{K}.
\] (6)

The perfect secrecy rate tuple \( (R_1, ..., R_K) \) is said to be achievable when \( d = 1 \). When all the transmitted messages are perfectly secure, we have

\[
\Delta_K \geq 1 - \epsilon,
\] (7)

or equivalently

\[
H(W_K|z^n) \geq H(W_K) - \epsilon H(W_K).
\] (8)

The normalized equivocation of each subset of messages can then be written as follows:

\[
H(W_S|z^n)^{(a)} = H(W_S, W_{S'}|z^n) - H(W_{S'}|W_S, z^n)
\]

\[
= H(W_S|z^n) - H(W_{S'}|W_S, z^n)
\]

\[
\geq H(W_K) - \epsilon H(W_K) - H(W_{S'}|W_S, z^n)
\]

\[
= H(W_S) + H(W_{S'}|W_S) - \epsilon H(W_K) - H(W_{S'}|W_S, z^n)
\]

\[
\geq H(W_S) - \epsilon H(W_K),
\]

where \( (a) \) and \( (c) \) follow from the chain rule, \( (b) \) follows from \( (7) \) and \( (d) \) follows from the fact that conditioning always decreases the amount of entropy. Therefore, the normalized equivocation of each subset of messages is

\[
\Delta_S \geq 1 - \epsilon',
\] (10)

where \( \epsilon' = \frac{H(W_K)}{H(W_S)} \epsilon \). Thus, when all of the \( K \) messages are perfectly secure then it is guaranteed that any subset of the messages becomes perfectly secure.

The total Secure Degrees-of-Freedom (S-DoF) of \( \eta \) is said to be achievable, if the rate-equivocation tuple \( (R_1, ..., R_K, d = 1) \) is achievable, and

\[
\eta = \lim_{P \to \infty} \sum_{k=1}^{K} \frac{R_k}{2 \log P}
\] (11)

III. SECURE DoF OF THE MULTIPLE-ANTENNA MULTIPLE-ACCESS-CHANNEL

In this section, we first present an achievable rate region for the secure discrete memoryless MAC. We then characterize the sum capacity of the degraded secure discrete memoryless and degraded Gaussian MIMO MAC. We present an achievable S-DoF of the non-degraded Gaussian MIMO MAC under the perfect secrecy constraint using Gaussian codebooks. In order to satisfy the perfect secrecy constraint, we use the random binning coding scheme to generate the codebooks. To maximize the achievable degrees of freedom, we adopt the signal alignment scheme to separate the signals at the intended receiver and simultaneously align the signals into a small subspace at the eavesdropper.
A. Discrete Memoryless MAC

In this subsection, we study the secure discrete MAC of \( P(y, z|x_1, ..., x_K) \) with \( K \) users and an external eavesdropper. The following theorems illustrate our results:

**Theorem 1:** For the perfectly secure discrete memoryless MAC of \( P(y, z|x_1, ..., x_K) \), the region of

\[
\left\{(R_1, ..., R_K) \mid \sum_{i \in S} R_i \leq I(U_S; Y|U_{S^c}), \sum_{k \in K} R_k \leq I(U_K; Y) - I(U_K; Z)^+, \forall S \subset K \right\}, \tag{12}
\]

for any distribution of \( P(u_1)P(u_2) ... P(u_K)P(x_1|u_1)P(x_2|u_2) ... P(x_K|u_K)P(y, z|x_1, ..., x_K) \), is achievable.

**Proof:** The proof is available in Appendix A. \[ \square \]

In this theorem \( [x]^+ \) denotes the positivity operator, i.e., \( [x]^+ = \max(x, 0) \). Reference \[20\] derived an achievable rate region with Gaussian codebooks and power control for the Gaussian secure MAC when all the transmitters and receivers are equipped with a single antenna. Theorem \[1\] however, gives an achievability secrecy rate region for the general discrete memoryless MAC. Our achievability rate region is also larger than the region of \[20\] in the special Gaussian channel case. Therefore, we have the following achievable sum rate for the secure discrete memoryless MAC:

**Corollary 1:** For the secure discrete memoryless MAC of \( P(y, z|x_1, ..., x_K) \), the following sum rate is achievable:

\[
R_{\text{sum}} = \max I(U_K; Y) - I(U_K; Z)^+ \tag{13}
\]

where the maximization is over all distributions \( P(u_1)P(u_2) ... P(u_K)P(x_1|u_1)P(x_2|u_2) ... P(x_K|u_K)P(y, z|x_1, ..., x_K) \) that satisfy the markov chain \( W_K \rightarrow U_K \rightarrow X_K \rightarrow YZ \).

B. Gaussian MIMO MAC

Consider the secure Gaussian MIMO MAC of \[1\] which can be re-written as follows:

\[
y = Hx + n_1 \tag{14}
\]

\[
z = H_e x + n_2
\]

where, \( H = [H_1, H_2, ..., H_K] \), \( H_e = [H_{1,e}, H_{2,e}, ..., H_{K,e}] \), and \( x = [x_1^+, x_2^+, ..., x_K^+]^+ \). Without loss of generality, assume that all nodes are equipped with the same number of antennas, i.e., \( M_k = N = N_e \) for all \( k \in K \). Note that when the channel gain matrices \( H_k \) and \( H_{k,e} \) are identity matrices we can determine that one channel output is degraded w.r.t. another by examining whether their noise covariances can be ordered correctly. In \[1\], however, all noise covariances are identity matrices and the receive vectors differ only in their channel gain matrices. Therefore, similar to \[20\], we use the following definition to determine a degradedness order:

**Definition 1:** A receive vector \( z = H_e x + n_2 \) is said to be degraded w.r.t. \( y = Hx + n_1 \) if there exists a matrix \( D \) such that \( DH = H_e \) and such that \( DD^\dagger \preceq I \). Alternatively, we say that \( H_e \) is degraded w.r.t. \( H \).

According to this definition, it is easy to see that \( y \) can be approximated by multiplying \( Dy \). The approximated channel has a different additive noise which is now given by \( Dn_1 \sim N(0, DD^\dagger) \) compared to the original channel. As this approximated channel has less noise \( (DD^\dagger \preceq I) \), however, it is clear that any message that can be decoded by the eavesdropper, can also be decoded by the intended receiver. In the other words \( W_K \rightarrow x \rightarrow y \rightarrow z \) forms a Markov chain.

**Theorem 2:** The secrecy sum capacity of the degraded Gaussian MIMO MAC is given by

\[
C_{\text{sum}} = \max_{Q_k, Q_k \succeq 0, Tr(Q_k) \leq P} \frac{1}{2} \log \left| I + \sum_{k \in K} H_k Q_k H_k^\dagger \right| - \frac{1}{2} \log \left| I + \sum_{k \in K} H_{k,e} Q_k H_{k,e}^\dagger \right|. \tag{15}
\]

**Proof:** We need to show that the secrecy sum capacity is as follows:

\[
C_{\text{sum}} = \frac{1}{2} \log \left| I + \sum_{k \in K} H_k Q_k H_k^\dagger \right| - \frac{1}{2} \log \left| I + \sum_{k \in K} H_{k,e} Q_k H_{k,e}^\dagger \right|. \tag{16}
\]

if the inputs are subject to the following covariance matrices constraints:

\[
K_{x_k} \preceq Q_k, \quad \forall k \in K, \tag{17}
\]

where \( K_{x_k} \) denotes the covariance matrix of \( x_k \). Theorem \[1\] then follows by maximization over all \( Q_1, Q_2, ..., Q_K \) that satisfy the power constraint, i.e., \( Tr(Q_k) \leq P \), for all \( k \in K \).

The achievability of this theorem follows from Theorem \[1\] by choosing \( U_k = x_k \sim N(0, Q_k) \). The converse proof is presented in Appendix B. \[ \square \]

According to \[69\], it is easy to show that:

**Corollary 2:** The total S-DoF for the degraded Gaussian MIMO MAC is \( \eta = 0 \).
The above algorithm can be followed when the users and the receivers are equipped with different numbers of antennae. The algorithm to choose the components with zero mean and unit variance, i.e., $\eta = k \sim N(0, I_K)$, it is easy to see that the following secrecy sum rate is achievable.

**Theorem 3:** For the Gaussian MIMO MAC, an achievable secrecy sum rate is given by

$$ R_{\text{Sum}} = \sum_{k \in K} R_k = \max_{Q: Q \succeq 0, Tr(Q) \leq P} \frac{1}{2} \log \left| I + \sum_{k \in K} H_k Q_k H_k^H \right| - \frac{1}{2} \log \left| I + \sum_{k \in K} H_{k,e} Q_k H_{k,e}^H \right|. $$

(18)

Note that in the above achievable scheme, we chose $K_{\text{se}} = Q_k$ which generally is not optimal. In general, solving the maximization problem of (18) is difficult. Reference [20], however, has solved this problem for a single antenna case and derived the optimal power control policy. As shown in [20] even for a single antenna case some users need to be silent and therefore those users can cooperatively help to jam the eavesdropper.

We study the S-DoF defined in (11) to analyze the behavior of the sum rate in high $SNR$. We design the following strategy scheme at the transmitters.

To achieve the largest value for S-DoF we need to separate the received signals at the legitimate receiver, such that each received signal has a different dimension in the signal space of the legitimate receiver. At the same time all the received signals at the eavesdropper need to be aligned in a minimal subspace of the signal space of the eavesdropper (see Fig. 2).

Let $x_k = F_k v_k$ where, $F_k$ is a pre-coding matrix such that $F_k F_k^H = Q_k$ and $v_k$ is a vector with i.i.d Gaussian components with zero mean and unit variance, i.e., $v_k = [v_{k1}, \ldots, v_{kJ}]$, such that $v_{kj} \sim N(0,1)$ for $j = 1, 2, \ldots, N$. Let $\psi_\eta = [0, 0, \ldots, 0, 1, 0, \ldots, 0]^T$ be a $N \times 1$ vector that all of its elements are zero except the $\eta$th element which is 1. Let $F_k = [f^1_k, f^2_k, \ldots, f^N_k]$, where $f^i_k$’s for $i = 1, 2, \ldots, N$ are $N \times 1$ vectors that represent the columns of $F_k$. We use the following algorithm to choose $f^i_k$’s:

- Assume that for users $k = 1, 2, \ldots, J$, the matrix $H_{k,e}$ has non-empty null space and the null space of $H_{k,e}$ for users $k = J+1, J+2, \ldots, K$ is empty. The users $k = 1, 2, \ldots, \min\{J, N\}$ choose $f^1_k$ such that $H_{k,e} f^1_k = 0$. Almost surly, we can assume that these vectors occupy separate dimensions at the legitimate receiver.
- If $J \geq N$, then all the $N$ dimensions at the legitimate receiver are full and $\eta = N$.
- If $J < N$, users $k = J+1, J+2, \ldots, \min\{K, N\}$ then create a vector $f^1_k$ such that $H_{k,e} f^1_k = \psi_1$. Theses vectors, therefore, are aligned at the eavesdropper in one dimension and almost surely occupy separate dimensions in the remaining subspace of the legitimate receiver.
- If $N \leq K$, at this step all the dimensions at the legitimate receiver are full and one dimension at the eavesdropper has a non-zero signal. Thus, $\eta = N - 1$.

Consider the general Gaussian MIMO MAC where it’s not necessarily degraded. According to Theorem 1 by choosing $U_k = x_k \sim N(0, Q_k)$, it is easy to see that the following secrecy sum rate is achievable.

**Corollary 3:** For the Gaussian MIMO MAC, an achievable secrecy sum rate is given by

$$ R_{\text{Sum}} = \sum_{k \in K} R_k = \max_{Q: Q \succeq 0, Tr(Q) \leq P} \frac{1}{2} \log \left| I + \sum_{k \in K} H_k Q_k H_k^H \right| - \frac{1}{2} \log \left| I + \sum_{k \in K} H_{k,e} Q_k H_{k,e}^H \right|. $$

(19)

The above algorithm can be followed when the users and the receivers are equipped with different numbers of antennae. The following theorem characterizes the maximum amount of the total S-DoF that can be achieved by Gaussian codebooks.

**Theorem 3:** For the Gaussian MIMO MAC, the following $\eta$ for S-DoF can almost surely be achieved for almost all channel gains by using Gaussian codebooks and under perfect secrecy constraint:

$$ \eta = \left\lfloor \min \left\{ \sum_{k \in K} M_k, N_e \right\} - r \right\rfloor^+, $$

where $0 \leq r \leq \min\{\sum_{k \in K} M_k, N_e\}$ depends on the channel gain matrix $H_e$. 

Fig. 2. Separation/Alignment of Signals at the Intended Receiver/Eavesdropper.
Note that in Theorem 3 it is emphasized that total $\min \left\{ \sum_{k \in K} M_k, N \right\} - r^+ \text{ S-DoF}$ is achievable for almost all channel gains. It means the set of all possible gains that the total amount of $\min \left\{ \sum_{k \in K} M_k, N \right\} - r^+ \text{ S-DoF}$ may not be achieved has the Lebesgue measure zero. In other words, if all the channel gains are drawn independently from a random distribution, then almost all of them satisfy properties required to achieve the total S-DoF, almost surely. The term “almost surely” means with a probability arbitrary close to 1.

Remark 1: In the achievability scheme of Theorem 3 all the transmitted signals are aligned into a $r$ dimensional subspace at the eavesdropper, and hence, impairs the ability of the eavesdropper to distinguish any of the secure messages efficiently. Now assume that the transmitters can cooperate with each other: we have a MIMO wiretap channel where the transmitter has $\sum_{k \in K} M_k$ antennas and the legitimate and eavesdropper have $N$ and $N_e$ antennas, respectively. The secrecy capacity of this channel is indeed an upper-bound for the secrecy sum capacity of the Gaussian MIMO MAC. As the capacity of a non-secret MIMO channel is an upper-bound for the secrecy sum capacity of the MIMO wiretap channel, we have the following upper-bound for S-DoF for the secure Gaussian MIMO MAC.

Lemma 1: For the Gaussian MIMO MAC, the maximum total achievable S-DoF under perfect secrecy constraint is given by

$$\eta_{\text{max}} = \min \left\{ \sum_{k \in K} M_k, N \right\}.$$  \hspace{1cm} (20)

The maximum penalty for the achievable S-DoF in Theorem 3 is therefore $r$, where, $0 \leq r \leq \min \{ \sum_{k \in K} M_k, N_e \}$. We note that an achievable S-DoF in the MIMO wiretap channel using zero-forcing beamforming is given as follows.

Lemma 2: In the MIMO wiretap channel, the following S-DoF is achievable, almost surely.

$$\eta = \min \left\{ \sum_{k \in K} M_k - r^+, N \right\},$$  \hspace{1cm} (21)

where $1 \leq r^+ \leq N_e$ depends on the channel gain matrix $H_e$.

Remark 2: When the transmitters and the intended receiver are equipped with a sufficiently large number of antenna while the eavesdropper is equipped with a limited number of antenna, then Gaussian codebooks provide a near optimum total of S-DoF for the Gaussian MIMO multiple-access-channel under perfect secrecy constraint. Note that when the transmitters and the receivers are equipped with a single antenna, i.e., $M_k = N_e = N = 1$, then the total achieved S-DoF is 0. It should be noted that this result comes from the lack of enough dimension for signal management at the receivers by using Gaussian codebooks. In our achievability scheme, nodes $K - 1$ send sequences from a codebook randomly generated in an i.i.d. fashion according to a Gaussian distribution. These are the worst noises from the eavesdroppers perspective if Gaussian i.i.d. signaling is used in $X_1$, see [31]. However, since the channel is fully connected, $K - 1$ are also the worst noises for the intended receiver. This effect causes the secrecy rate to saturate, leading to zero S-DoF. The following Theorem establishes an upper-bound for the total S-DoF.

IV. SECURE DOF OF THE SINGLE-ANTENNA MULTIPLE-ACCESS-CHANNEL

In this section we consider the secure multiple-access-channel of [4] when all the transmitters and the receivers have a single antenna, i.e., $M_k = N = N_e = 1$ for all $k \in K$. We showed in the previous section that Gaussian codebooks lead to zero total S-DoF. Here, we will provide a coding scheme based on integer codebooks and show that for almost all channel gains a positive total S-DoF is achievable, almost surely. The following theorem illustrates our results.

Theorem 4: For the Gaussian single antenna multiple-access-channel of [4] with $M_k = N = N_e = 1$, a total $\frac{K-1}{K}$ secure degrees-of-freedom can be achieved for almost all channel gains, almost surely.

Proof: When the transmitters and the receivers are equipped with a single antenna, then the channel model of [4] is equivalent as follows:

$$Y = \sum_{k=1}^{K} h_k X_k + \tilde{W}_1$$  \hspace{1cm} (22)

$$Z = \sum_{k=1}^{K} h_{k,e} X_k + \tilde{W}_2,$$
where $\tilde{W}_1 \sim \mathcal{N}(0, 1)$, $\tilde{W}_2 \sim \mathcal{N}(0, 1)$, and $E[X_k^2] \leq P$ for all $k \in \mathcal{K}$. Let us define $\tilde{X}_k \triangleq \frac{h_k}{\sqrt{\rho}} X_k$ and $\tilde{h}_k \triangleq \frac{h_k}{\sqrt{\rho}}$ and without loss of generality assume that $\tilde{h}_K = 1$, then the channel model is equivalent as follows:

$$Y = A \left[ \sum_{k=1}^{K-1} \tilde{h}_k \tilde{X}_k + \tilde{X}_K \right] + \tilde{W}_1$$

$$Z = A \sum_{k=1}^{K} \tilde{X}_k + \tilde{W}_2,$$

where, $A^2 E[X_k^2] \leq P \triangleq h_{k,e}^2 P$. In this model we say that the signals are aligned at the eavesdropper according to the following definition:

**Definition 2:** The signals $\tilde{X}_1$, $\tilde{X}_2$,...,$\tilde{X}_K$ are said to be aligned at a receiver if its received signal is a rational combination of them.

Note that, in $n$-dimensional Euclidean spaces ($n \geq 2$), two signals are aligned when they are received in the same direction at the receiver. In general, $m$ signals are aligned at a receiver if they span a subspace with dimension less than $m$. The above definition, however, generalizes the concept of alignment for the one-dimensional real numbers. Our coding scheme is based on integer codebooks, which means that $\tilde{X}_k \in \mathbb{Z}$ for all $k \in \mathcal{K}$. If some integer signals are aligned at a receiver, then their effect is similar to a single signal at high SNR regimes. This is due to the fact that rational numbers form a filed and therefore the sum of constellations from $\mathbb{Q}$ form a constellation in $\mathbb{Q}$ with an enlarged cardinality.

Before we present our achievable coding scheme, we need to define the rational dimension of a set of real numbers.

**Definition 3:** (Rational Dimension) The rational dimension of a set of real numbers $\{h_1, h_2, ..., h_{K-1}, h_K = 1\}$ is $M$ if there exists a set of real numbers $\{g_1, g_2, ..., g_M\}$ such that each $h_k$ can be represented as a rational combination of $g_i$s, i.e., $h_k = a_{k,1}g_1 + a_{k,2}g_2 + ... + a_{k,M}g_M$, where $a_{k,i} \in \mathbb{Q}$ for all $k \in \mathcal{K}$ and $i \in \mathcal{M}$.

In fact, the rational dimension of a set of channel gains is the effective dimension seen at the corresponding receiver. In particular, $\{h_1, h_2, ..., h_K\}$ are rationally independent if the rational dimension is $K$, i.e., none of the $h_k$ can be represented as the rational combination of other numbers.

Note that all of the channel gains $h_k$ are generated independently with a distribution. From the number theory, it is known that the set of all possible channel gains that are rationally independent have a Lebesgue measure 1. Therefore, we can assume that $\{h_1, h_2, ..., h_K\}$ are rationally independent, almost surely. Our achievability coding scheme is as follows:

1) **Encoding:** Each transmitter limits its input symbols to a finite set which is called the transmit constellation. Even though it has access to the continuum of real numbers, restriction to a finite set has the benefit of easy and feasible decoding at the intended receiver. The transmitter $k$ selects a constellation $\mathcal{V}_k$ to send message $W_k$. The constellation points are chosen from integer points, i.e., $\mathcal{V}_k \subset \mathbb{Z}$. We assume that $\mathcal{V}_k$ is a bounded set. Hence, there is a constant $Q_k$ such that $\mathcal{V}_k \subset [-Q_k, Q_k]$. The cardinality of $\mathcal{V}_k$ which limits the rate of message $W_k$ is denoted by $||\mathcal{V}_k||$.

Having formed the constellation, the transmitter $k$ constructs a random codebook for message $W_k$ with rate $R_k$. This can be accomplished by choosing a probability distribution on the input alphabets. The uniform distribution is the first candidate and it is selected for the sake of simplicity. Therefore, the stochastic encoder $k$ generates $2^n I(\tilde{x}_k;Y_k|\mathcal{V}_k=\mathcal{V}_k)$ independent and identically distributed sequences $\tilde{x}_k^n$ according to the distribution $P(\tilde{x}_k^n) = \prod_{i=1}^{n} P(\tilde{x}_{k,i})$, where $P(\tilde{x}_{k,i})$ denotes the probability distribution function of the uniformly distributed random variable $\tilde{x}_{k,i}$ over $\mathcal{V}_k$. Next, randomly distribute these sequences into $2^n R_k$ bins. Index each of the bins by $w_k \in \{1, 2, ..., 2^n R_k\}$.

For each user $k \in \mathcal{K}$, to send message $w_k$, the transmitter looks for a $\tilde{x}_k^n$ in bin $w_k$. The rates are such that there exist more than one $\tilde{x}_k^n$. The transmitter randomly chooses one of them and sends $x_k^n = A \tilde{x}_k^n$. The parameter $A$ controls the input power.

2) **Decoding:** At a specific time, the received signal at the legitimate receiver is as follows:

$$Y = A \left[ \tilde{h}_1 \tilde{x}_1 + \tilde{h}_2 \tilde{x}_2 + ... + \tilde{h}_{K-1} \tilde{x}_{K-1} + \tilde{x}_K \right] + \tilde{W}_1$$

The legitimate receiver passes the received signal $Y$ through a hard decoder. The hard decoder looks for a point $\tilde{Y}$ in the received constellation $\mathcal{V}_r = A \left[ \tilde{h}_1 \mathcal{V}_1 + \tilde{h}_2 \mathcal{V}_2 + ... + \tilde{h}_{K-1} \mathcal{V}_{K-1} + \mathcal{V}_K \right]$ which is the nearest point to the received signal $Y$. Therefore, the continuous channel changes to a discrete one in which the input symbols are taken from the transmit constellations $\mathcal{V}_k$ and the output symbols belong to the received constellation $\mathcal{V}_r$. $\tilde{h}_k$'s are rationally independent which means that the equation $A \left[ \tilde{h}_1 X_1 + \tilde{h}_2 X_2 + ... + \tilde{h}_{K-1} X_{K-1} + X_K \right] = 0$ has no rational solution. This property implies that any real number $v_r$ belonging to the constellation $\mathcal{V}_r$ is uniquely decomposable as $v_r = A \sum_{k=1}^{K} \tilde{h}_k \tilde{x}_k$. Note that if there exists another possible decomposition $\tilde{v}_r = A \sum_{k=1}^{K} \tilde{h}_k \tilde{x}_k$, then $\tilde{h}_k$'s have to be rationally-dependent which is a contradiction. We call this property as property $\Gamma$. This property in fact implies that if there is no additive noise in the channel, then the receiver can decode all the transmitted signals with zero error probability.
Remark 3: In a random environment it is easy to show that the set of channels gains which are rationally-dependent has a measure of zero with respect to the Lebesgue measure. Therefore, Property $\Gamma$ is almost surely satisfied.

3) Error Probability Analysis: Let $d_{\min}$ denote the minimum distance in the received constellation $V_r$. Having property $\Gamma$, the receiver can decode the transmitted signals. Let $V_t$ and $V_r$ be the transmitted and decoded symbols, respectively. The probability of error, i.e., $P_e = P(V_r \neq V_t)$, is bounded as follows:

$$P_e \leq Q\left(\frac{d_{\min}}{2}\right) \leq \exp(-\frac{d_{\min}^2}{8})$$

(25)

where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(-\frac{t^2}{2}) dt$. Note that finding $d_{\min}$ is not easy in general. Using Khintchine and Groshev theorems, however, it is possible to lower bound the minimum distance. Here we explain some background information for using the theorems of Khintchine and Groshev.

The field of Diophantine approximation in number theory deals with approximation of real numbers with rational numbers. The reader is referred to \cite{32, 33} and the references therein. The Khintchine theorem is one of the cornerstones in this field. This theorem provides a criteria for a given function $\psi : \mathbb{N} \to \mathbb{R}_+$ and real number $h$, such that $|p + \tilde{h} q| < \psi(|q|)$ has either infinitely many solutions or at most, finitely many solutions for $(p, q) \in \mathbb{Z}^2$. Let $\mathcal{A}(\psi)$ denote the set of real numbers such that $|p + \tilde{h} q| < \psi(|q|)$ has infinitely many solutions in integers. The theorem has two parts. The first part is the convergent part and states that if $\psi(|q|)$ is convergent, i.e.,

$$\sum_{q=1}^{\infty} \psi(q) < \infty$$

(26)

then $\mathcal{A}(\psi)$ has a measure of zero with respect to the Lebesgue measure. This part can be rephrased in a more convenient way, as follows. For almost all real numbers, $|p + \tilde{h} q| > \psi(|q|)$ holds for all $(p, q) \in \mathbb{Z}^2$ except for finitely many of them. Since the number of integers violating the inequality is finite, one can find a constant $c$ such that

$$|p + \tilde{h} q| > c \psi(|q|)$$

(27)

holds for all integers $p$ and $q$, almost surely. The divergent part of the theorem states that $\mathcal{A}(\psi)$ has the full measure, i.e. the set $\mathbb{R} - \mathcal{A}(\psi)$ has measure zero, provided that $\psi$ is decreasing and $\psi(|q|)$ is divergent, i.e.,

$$\sum_{q=1}^{\infty} \psi(q) = \infty.$$  

(28)

There is an extension to Khintchine theorem which regards the approximation of linear forms. Let $\tilde{h} = (\tilde{h}_1, \tilde{h}_2, ..., \tilde{h}_{K-1})$ and $q = (q_1, q_2, ..., q_{K-1})$ denote $(K-1)$-tuples in $\mathbb{R}^{K-1}$ and $\mathbb{Z}^{K-1}$, respectively. Let $\mathcal{A}_{K-1}(\psi)$ denote the set of $(K-1)$-tuple real numbers such that

$$|p + q_1 \tilde{h}_1 + q_2 \tilde{h}_2 + ... + q_{K-1} \tilde{h}_{K-1}| < \psi(|q|)$$

(29)

has infinitely many solutions for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^{K-1}$. Here, $|q|_{\infty}$ is the supreme norm of $q$ which is defined as $\max_k |q_k|$. The following theorem illustrates the Lebesque measure of the set $\mathcal{A}_{K-1}(\psi)$ \cite{1}.

**Theorem 5:** (Khintchine-Groshev) Let $\psi : \mathbb{N} \to \mathbb{R}_+$. Then, the set $\mathcal{A}_{K-1}(\psi)$ has measure zero provided that

$$\sum_{q=1}^{\infty} q^{K-2} \psi(q) < \infty$$

(30)

and has the full measure if

$$\sum_{q=1}^{\infty} q^{K-2} \psi(q) = \infty \quad \text{and} \quad \psi \quad \text{is monotonic}$$

(31)

In this paper, we are interested in the convergent part of the theorem. Moreover, given an arbitrary $\epsilon > 0$ the function $\psi(q) = \frac{1}{q^{K-1+\epsilon}}$ satisfies the condition of (30). In fact, the convergent part of the above theorem can be stated as follows. For almost all $K-1$-tuple real numbers $\tilde{h}$ there exists a constant $c$ such that

$$|p + q_1 \tilde{h}_1 + q_2 \tilde{h}_2 + ... + q_{K-1} \tilde{h}_{K-1}| > \frac{c}{(\max_k |q_k|)^{K-1+\epsilon}}$$

(32)

holds for all $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^{K-1}$. The Khintchine-Groshev theorem can be used to bound the minimum distance of points in the received constellation $V_r$. In fact, a point in the received constellation has a linear form of

$$v_r = A \left[ \tilde{h}_1 v_1 + \tilde{h}_2 v_2 + ... + \tilde{h}_{K-1} v_{K-1} + v_K \right],$$

(33)

Therefore, we can conclude that

$$d_{\min} > \frac{A c}{(\max_{k \in \{1,2,\ldots,K-1\}} Q_k)^{K-1+\epsilon}}.$$  

(34)
The probability of error in hard decoding, see (25), can be bounded as:

\[ P_e < \exp \left( - \frac{(Ac)^2}{8(\max_{k \in \{1,2,\ldots,K-1\}} Q_k)^{2K-2+2\epsilon}} \right) \]  

(35)

Let us assume that \( Q_k \) for all \( k \in \{1,2,\ldots,K-1\} \) is \( Q = [\frac{P}{4KQ}] \). Moreover, since \( E[\tilde{X}_k^2] \leq A^2 Q_k^2 \leq \bar{P} \), we can choose \( A = \bar{P}^{\frac{K-1+2\epsilon}{4KQ}} \). Substituting in (35) yields

\[ P_e < \exp\left( -\frac{c^2}{8} \bar{P}^c \right). \]  

(36)

Thus, \( P_e \to 0 \) when \( \bar{P} \to \infty \) or equivalently \( P \to \infty \).

4) Equivocation Calculation: Since the equivocation analysis of Theorem I is valid for any input distribution, therefore integer inputs satisfy the perfect secrecy constraint.

5) S-DoF Calculation: The maximum achievable sum rate is as follows:

\[ \sum_{k \in K} R_k = I(\tilde{X}_1,\tilde{X}_2,\ldots,\tilde{X}_K; Y) - I(\tilde{X}_1,\tilde{X}_2,\ldots,\tilde{X}_K; Z) \]  

(37)

\[ \geq H(\tilde{X}_1,\tilde{X}_2,\ldots,\tilde{X}_K|Z) - H(\tilde{X}_1,\tilde{X}_2,\ldots,\tilde{X}_K|Y) \]  

\[ \geq \sum_{k \in K} H(\tilde{X}_k) - H(\sum_{k \in K} \tilde{X}_k) - 1 - P_e \log \| \tilde{X} \| \]  

\[ \geq K \log(2Q + 1) - \log(2KQ + 1) - 1 - P_e \log \| \tilde{X} \|, \]

where (a) follows from Fano’s inequality, (b) follows from the fact that conditioning always decreases entropy, (c) follows from chain rule, and (d) follows from the fact that \( \tilde{X}_k \) has uniform distribution over \( \mathcal{V}_k = [-Q, Q] \). The S-DoF therefore can be computed as follows:

\[ \eta = \lim_{P \to \infty} \frac{\sum_{k \in K} R_k}{\frac{1}{2} \log P} = \frac{(K-1)(1-\epsilon)}{K + \epsilon} \]  

(38)

Since \( \epsilon \) can be arbitrarily small, then \( \eta = \frac{K-1}{K} \) is indeed achievable.

As we saw in the previous section, multiple-antennas (or equivalently, time-varying and/or frequency-selective channels) provide enough freedom, which allows us to choose appropriate signaling directions to separate between the messages at the intended receiver and at the same time pack the signals into a low dimensionality subspace at the eavesdropper. In contrary, it was commonly believed that time-invariant frequency flat single-antenna channels cannot provide any degrees-of-freedom. In Theorem, 4 however, we developed a machinery that transforms the single-antenna systems to a pseudo multiple-antenna system with some antennas. The number of available dimensions in the equivalent pseudo multiple-antenna systems is \( K \) when all of \( K \) channel gains between the transmitters and the intended receiver are rationally-independent (this condition is satisfied almost surely). The equivalent pseudo multiple-antenna system can simulate the behavior of a multiple-dimensional system (in time/frequency/space) and allows us to simultaneously separate the signals at the intended receiver and align them to the eavesdropper.

Note that in the MISOSE wire-tap channel (Multiple-Input Single-Output Single-Eavesdropper), when the channel realization is unique, we can achieve the optimum S-DoF of 1 through cooperation among transmitters. This Theorem clarifies the fact that, we lose the amount of \( \frac{1}{K} \) in S-DoF due to the lack of cooperation between the transmitters. We still gain through the possibility of signal alignment at the eavesdropper, however.

A. Rationally-Dependent Channel Gains: Multiple-layer coding

When the channel gains are rationally dependent, then a more sophisticated multiple-layer constellation design is required to achieve higher S-DoF. The reason is that some messages share the same dimension at the intended receiver and as a result, splitting them requires more structure in constellations. We propose a multiple-layer constellation that can not only be distinguished at the intended receiver but are also packed efficiently at the eavesdropper. This is accomplished by allowing
where $A$ controls the output power. When the channel gain $\tilde{h}_1$ is irrational then according to Theorem 4 the total S-DoF of $\frac{1}{2}$ is indeed achievable. For the rational channel gain $\tilde{h}_1 = \frac{m}{n}$, $n, m \in \mathbb{N}, m \neq 0$, however, the coding scheme of Theorem 4 fails and we need to use a multiple-layer coding scheme.

In multiple-layer coding scheme, we select the constellation points in the base $W \in \mathbb{N}$ as follows:
\begin{equation}
\nu(b) = \sum_{i=0}^{L-1} b_i W^i,
\end{equation}

where, $b_l$ for all $l \in \{0, 1, ..., L-1\}$ are independent random variables which take value from $\{0, 1, 2, ..., a-1\}$ with uniform probability distribution. $b$ represents the vector $b = \{b_0, b_1, ..., b_{L-1}\}$ and the parameter $a$ controls the number of constellation points. We assume that $a < W$ and therefore, all constellation points are distinct and the size of the constellations $|V_1| = |V_2| = a^L$. The maximum possible rate for each user is therefore bounded by $L \log a$.

At each transmitter a random codebook is generated by randomly choosing $b_l$ according to a uniform distribution. The signal transmitted by users 1 and 2 are $\tilde{X}_1 = v(b)$ and $\tilde{X}_2 = v(b')$, respectively. Note that the above multiple-layer constellation had a DC component and this component needs to be removed at the transmitters. The DC component, however, duplicated the achievable rates and has no effect on the S-DoF.

To calculate $A$, since $b_l$ and $b_j$ are independent for $l \neq j$, we have the following chain of inequalities:
\begin{equation}
A^2 E[\tilde{X}_1^2] = A^2 W^{2(L-1)} \sum_{l=0}^{L-1} E[b_l^2] W^{-2l} \leq A^2 W^{2(L-1)} \frac{(a-1)(2a-1)}{6} \sum_{l=0}^{\infty} W^{-2l} \leq A^2 W^{2(L-1)} \frac{a^2}{3} \frac{1}{1-W^2} \leq \frac{A^2 a^2 W^{2L}}{W^2 - 1}.
\end{equation}

Therefore, by choosing $A = \sqrt{\frac{(W^2 - 1)^p_{\alpha}}{aW^2}}$ the power constraint $A^2 E[\tilde{X}_1^2] \leq \tilde{P}$ is satisfied. The received constellation at the intended receiver and the eavesdropper can be written as follows, respectively:
\begin{equation}
Y = A \sum_{l=0}^{L-1} \left( nb_l + mb'_l \right) W^l + \tilde{W}_1
\end{equation}
\begin{equation}
Z = A \sum_{l=0}^{L-1} \left( b_l + b'_l \right) W^l + \tilde{W}_2.
\end{equation}

A point in the received constellation $V_r$ of the intended receiver can be represented as follows:
\begin{equation}
v_r(b, b') = A \sum_{l=0}^{L-1} \left( nb_l + mb'_l \right) W^l.
\end{equation}

Note that the received constellation needs to satisfy the property $\Gamma$, as the intended receiver needs to uniquely decode the transmitted signals. The following theorem characterizes the total achievable S-DoF.

**Theorem 6:** The following S-DoF is achievable for the two user single antenna MAC with rational channel gain $\tilde{h}_1 = \frac{m}{n}$:
\begin{equation}
\eta = \begin{cases} 
\log(n), & \text{if } 2n \geq m \\
\frac{\log(n(2n-1))}{\log(n(s+1))}, & \text{if } 2n < m \text{ and } m = 2s + 1 \\
\frac{\log(s)}{\log(2s-n)}, & \text{if } 2n < m \text{ and } m = 2s
\end{cases}
\end{equation}

a carry-over from different levels. In this subsection, for the sake of simplicity, we consider a two user-secure MAC. This channel is modeled as follows:
\begin{equation}
Y = A \left[ \tilde{h}_1 \tilde{X}_1 + \tilde{X}_2 \right] + \tilde{W}_1
\end{equation}
\begin{equation}
Z = A \left[ \tilde{X}_1 + \tilde{X}_2 \right] + \tilde{W}_2,
\end{equation}

where $A$ controls the output power. When the channel gain $\tilde{h}_1$ is irrational then according to Theorem 4 the total S-DoF of $\frac{1}{2}$ is indeed achievable. For the rational channel gain $\tilde{h}_1 = \frac{m}{n}$, $n, m \in \mathbb{N}, m \neq 0$, however, the coding scheme of Theorem 4 fails and we need to use a multiple-layer coding scheme.
TABLE I

| Case       | $n > m$ and $m = 2s + 1$ | $n$ | $n(2n - 1)$ | $s + 1$ | $(s + 1)/(2s + 1)$ | $2s^2 - n$ |
|------------|--------------------------|-----|-------------|---------|-------------------|-------------|
| Case 1     | $2n > m$                 |     |             |         |                   |             |
| Case 2     | $2n < m$ and $m = 2s + 1$|     |             |         |                   |             |

Proof: Let us first assume that the property $\Gamma$ is satisfied for given $W$ and $a$. It is easy to show that the minimum distance in the received constellation $\mathcal{V}_r$ is $d_{\text{min}} = \frac{a}{m}$. The probability of error is therefore bounded as follows:

$$P_e \leq \exp\left(-\frac{d_{\text{min}}^2}{8}\right) \quad (45)$$

$$= \exp\left(-\frac{(W^2 - 1)P}{8a^2m^2W^2L}\right).$$

Let us choose $L$ as

$$L = \left\lfloor \frac{\log(\tilde{P}^{0.5-\epsilon})}{\log(W)} \right\rfloor, \quad (46)$$

where $\epsilon > 0$ is an arbitrary small constant. Clearly, with this choice of $L$, $P_e \leq \exp(-\gamma\tilde{P}2\epsilon)$ where $\gamma$ is a constant. Thus, when SNR $\rightarrow \infty$, then $P_e \rightarrow 0$. Using (47), the S-DoF of the system can be derived as follows:

$$\eta = \lim_{P \rightarrow \infty} \frac{L \log(a)}{\frac{1}{2} \log P}$$

$$= \lim_{P \rightarrow \infty} \frac{\log(\tilde{P}^{0.5-\epsilon})}{\log(W)} \log(a)$$

$$= \left(1 - 2\epsilon\right) \frac{\log(a)}{\log(W)}.$$ 

Since $\epsilon$ is an arbitrary small constant, the total S-DoF of the system is

$$\eta = \frac{\log(a)}{\log(W)}, \quad (48)$$

This equation implies that to achieve the maximum possible $\eta$, we need to maximize $a$ and minimize $W$ with the constraint that the property $\Gamma$ is satisfied. Table I shows the choices of $a$ and $W$ for Theorem 6. To complete the proof we need to show that with the choices of Table I, the property $\Gamma$ is satisfied.

Lemma 3: The property $\Gamma$ holds for all the choices of Table I.

Proof: Please see Appendix C.

Note that this result implies that the total achievable S-DoF by using integer lattice codes is discontinuous with respect to the channel coefficients.

V. Conclusion

In this paper, we considered a $K$ user secure Gaussian MAC with an external eavesdropper. We proved an achievable rate region for the secure discrete memoryless MAC and thereafter we established the secrecy sum capacity of the degraded Gaussian MIMO MAC using Gaussian codebooks. For the non-degraded Gaussian MIMO MAC, we proposed an algorithm inspired by the interference alignment technique to achieve the largest possible total S-DoF. When all the terminals are equipped with single antenna, the Gaussian codebooks lead to zero S-DoF. Therefore, we proposed a novel secure coding scheme to achieve positive S-DoF in the single antenna MAC. This scheme converts the single-antenna system into a multiple-dimension system with fractional dimensions. The achievability scheme is based on the alignment of signals into a small sub-space at the eavesdropper, and the simultaneous separation of the signals at the intended receiver. We proved that total S-DoF of $\frac{K-1}{K}$ can be achieved for almost all channel gains which are rationally independent. For the rationally dependent channel gains, we illustrated the power of the multi-layer coding scheme, through an example channel, to achieve a positive S-DoF. As a function of channel gains, therefore, we showed that the achievable S-DoF is discontinues.
APPENDIX

A. Proof of Theorem 7

1) Codebook Generation: The structure of the encoder for user \( k \in \mathcal{K} \) is depicted in Fig. Fix \( P(u_k) \) and \( P(x_k|u_k) \). The stochastic encoder \( k \) generates \( 2^n \text{I}(U_k;Y(U_{k-1}Y)+\epsilon_k) \) independent and identically distributed sequences \( u^n_k \) according to the distribution \( P(u^n_k) = \prod_{i=1}^n P(u_{ki}) \). Next, randomly distribute these sequences into \( 2^nR_k \) bins. Index each of the bins by \( w_k \in \{1, 2, ..., 2^nR_k\} \).

2) Encoding: For each user \( k \in \mathcal{K} \), to send message \( w_k \), the transmitter looks for a \( u^n_k \) in bin \( w_k \). The rates are such that there exist more than one \( u^n_k \). The transmitter randomly chooses one of them and then generates \( x^n_k \) according to \( P(x^n_k|u^n_k) = \prod_{i=1}^n P(x_{ki}|u_{ki}) \) and sends it.

3) Decoding: The received signals at the legitimate receiver, \( y^n \), is the output of the channel \( P(y^n|x^n_k) = \prod_{i=1}^n P(y_{i}|x_{ki}) \).

The legitimate receiver looks for the unique sequence \( u^n_k \) such that \( (u^n_k, y^n) \) is jointly typical and declares the indices of the bins containing \( u^n_k \) as the messages received.

4) Error Probability Analysis: Since the region of (12) is a subset of the capacity region of the multiple-access-channel without secrecy constraint, then the error probability analysis is the same as (3) and omitted here.

5) Equivocation Calculation: To satisfy the perfect secrecy constraint, we need to prove the requirement of (7). From \( H(W_K|Z^n) \) we have

\[
H(W_K|Z^n) = H(W_K, Z^n) - H(Z^n)
\]

where (a) follows from Fano’s inequality, which states that for sufficiently large \( n \), \( H(U^n_K|W^n_K, Z^n) \leq h(P^n_{we}) + nP^n_{we}R_w \leq n\epsilon_n \). Here \( P^n_{we} \) denotes the wiretapper’s error probability of decoding \( u^n_k \) in the case that the bin numbers \( w_k \) are known to the eavesdropper and \( R_w = I(U_K; Z) \). Since the sum rate is small enough, then \( P^n_{we} \to 0 \) for sufficiently large \( n \). (b) follows from the following Markov chain: \( W_K \to U^n_K \to Z^n \). Hence, we have \( H(Z^n|W_K, U^n_K) = H(Z^n|U^n_K) \). (c) follows from the fact that \( H(W_K, U^n_K) \geq H(U^n_K) \). (d) follows from that fact that \( H(U^n_K; Y^n) \). (e) follows from the following lemma:

**Lemma 4:** Assume \( U^n_K, Y^n \) and \( Z^n \) are generated according to the achievability scheme of Theorem 1 then we have,

\[
\begin{align*}
&nI(U^n_K; Y) - n\delta_{1n} \leq I(U^n_K; Y^n) \leq nI(U^n_K; Y) + n\delta_{2n}, \\
nI(U^n_K; Z) - n\delta_{3n} \leq I(U^n_K; Z^n) \leq nI(U^n_K; Z) + n\delta_{4n},
\end{align*}
\]

where, \( \delta_{1n}, \delta_{2n}, \delta_{3n}, \delta_{4n} \to 0 \) when \( n \to \infty \).

**Proof:** We first prove (50). Let \( A^{(n)}(P_{U_K,Z}) \) denote the set of typical sequences \( (u^n_k, z^n) \) with respect to \( P_{U_K,Z} \), and

\[
\zeta = \begin{cases} 
1, & (u^n_k, z^n) \in A^{(n)}_1 \\
0, & \text{otherwise}
\end{cases}
\]

be the corresponding indicator function. We expand \( I(U^n_K; Z^n, \zeta) \) and \( I(U^n_K, \zeta; Z^n) \) as follow:

\[
\begin{align*}
I(U^n_K; Z^n, \zeta) &= I(U^n_K; Z^n) + I(U^n_K; \zeta|Z^n) \\
&= I(U^n_K; \zeta) + I(U^n_K; Z^n|\zeta),
\end{align*}
\]

\[
\begin{align*}
I(U^n_K, \zeta; Z^n) &= I(U^n_K; Z^n) + I(\zeta; Z^n|U^n_K) \\
&= I(\zeta; Z^n) + I(U^n_K; Z^n|\zeta).
\end{align*}
\]

Therefore, we have

\[
I(U^n_K; Z^n|\zeta) - I(U^n_K; \zeta|Z^n) \leq I(U^n_K; Z^n) \leq I(U^n_K; Z^n|\zeta) + I(\zeta; Z^n).
\]
Note that \( I(\zeta; Z^n) \leq H(\zeta) \leq 1 \) and \( I(U_K^n; \zeta Z^n) \leq H(\zeta | Z^n) \leq H(\zeta) \leq 1 \). Thus, the above inequality implies that
\[
\sum_{j=0}^{1} P(\zeta = j)I(U_K^n; Z^n | \zeta = j) - 1 \leq I(U_K^n; Z^n) \leq \sum_{j=0}^{1} P(\zeta = j)I(U_K^n; Z^n | \zeta = j) + 1. \tag{56}
\]
According to the joint typicality property, we have
\[
0 \leq P(\zeta = 1)I(U_K^n; Z^n | \zeta = 1) \leq nP((u_K^n, z^n) \in A^{(n)}(P_{U_K} Z)) \log \| Z \| \leq n\epsilon_n \log \| Z \|. \tag{57}
\]
Now consider the term \( P(\zeta = 0)I(U_K^n; Z^n | \zeta = 0) \). Following the sequence joint typicality properties, we have
\[
(1 - \epsilon_n)I(U_K^n; Z^n | \zeta = 0) \leq P(\zeta = 0)I(U_K^n; Z^n | \zeta = 0) \leq I(U_K^n; Z^n | \zeta = 0), \tag{58}
\]
where
\[
I(U_K^n; Z^n | \zeta = 0) = \sum_{(u_K^n, z^n) \in A^{(n)}} P(u_K^n, z^n) \log \frac{P(u_K^n, z^n)}{P(u_K^n)P(z^n)}. \tag{59}
\]
Since \( H(U_K, Z) - \epsilon_n \leq -\frac{1}{n} \log P(u_K^n, z^n) \leq H(U_K, Z) + \epsilon_n \), then we have,
\[
n[-H(U_K, Z) + H(U_K) + H(Z) - 3\epsilon_n] \leq I(U_K^n; Z^n | \zeta = 0) \leq n[-H(U_K, Z) + H(U_K) + H(Z) + 3\epsilon_n], \tag{60}
\]
or equivalently,
\[
n[I(U_K; Z) - 3\epsilon_n] \leq I(U_K^n; Z^n | \zeta = 0) \leq n[I(U_K; Z) + 3\epsilon_n]. \tag{61}
\]
By substituting (61) into (58) and then substituting (58) and (57) into (56), we get the desired result,
\[
nI(U_K; Z) - n\delta_{1n} \leq I(U_K^n; Z^n) \leq nI(U_K; Z) + n\delta_{2n}, \tag{62}
\]
where
\[
\delta_{1n} = \epsilon_n I(U_K; Z) + 3(1 - \epsilon_n)\epsilon_n + \frac{1}{n} \tag{63}
\]
\[
\delta_{2n} = 3\epsilon_n + \epsilon_n \log \| Z \| + \frac{1}{n}. \tag{64}
\]
Following the same steps, one can prove (51).

B. Proof of the Converse for Theorem 2

Before starting the proof, we first present some useful lemmas.

Lemma 5: The secrecy sum capacity of the Gaussian MIMO MAC is upper-bounded by
\[
C_{\text{sum}} \leq \max_{P(x_1) P(x_2) ... P(x_K)} I(x_1, x_2, ..., x_K; y | z), \tag{64}
\]
where maximization is over all distributions \( P(x_1) P(x_2) ... P(x_K) \) that satisfy the power constraint, i.e., \( Tr(x_i^4) \leq P \).

Proof: According to Fano’s inequality and the perfect secrecy constraint, we have
\[
n \sum_{k \in K} R_k \leq I(W_K; y^n) - I(W_K; z^n) \tag{65}
\]
\[
\leq I(W_K; y^n, z^n) - I(W_K; z^n) \tag{66}
\]
\[
\leq h(y^n | z^n) - h(y^n | W_K, z^n) \tag{67}
\]
\[
\leq h(y^n | z^n) - h(y^n | W_K, x_K^n, z^n) \tag{68}
\]
\[
\leq h(y^n | z^n) - h(y^n | x_K^n, z^n) \tag{69}
\]
\[
\leq h(y^n | z^n) - \sum_{i=1}^{n} h(y(i) | x_K(i), z(i)) \tag{70}
\]
\[
\leq \sum_{i=1}^{n} h(y(i) | z(i)) - h(y(i) | x_K(i), z(i)) \tag{71}
\]
\[
\leq n I(x_K; y | z, q) \tag{72}
\]
\[
\leq n I(x_K; y | z), \tag{73}
\]
where (a) and (b) follows from chain rule, (c) follows from the fact that conditioning decreases the differential entropy, (d) follows from the Markov chain $W_K \rightarrow (x^n_K, z^n) \rightarrow y^n$, (e) follows from the fact that the channel is memoryless, (f) is obtained by defining a time-sharing random variable $q$ that takes values uniformly over the index set $\{1, 2, \ldots, n\}$ and defining $(x_K, y, z)$ to be the tuple of random variables that conditioned on $q = i$, have the same joint distribution as $(x_K(i), y(i), z(i))$. Finally, (g) follows from the fact that $I(x_K; y|z)$ is concave in $P(x_1)\ldots P(x_K)$ (see, e.g.,[1],[9, Appendix I] for a proof), so that Jense's inequality can be applied.

**Lemma 6:** If $DH_k = H_{k,e}$ for all $k \in K$ and $DD_k \preceq I$, then the function

$$f(X_1, X_2, \ldots, X_K) = \frac{1}{2} \log \left| I + \sum_{k \in K} H_k X_k H_k^\dagger \right| - \frac{1}{2} \log \left| I + \sum_{k \in K} H_{k,e} X_k H_{k,e}^\dagger \right|$$

(66)

is a concave function of $(X_1, \ldots, X_K)$ for $X_k \succeq 0$ for all $k \in K$. Moreover, for $(X_1, \ldots, X_K)$ such that $X_k \succeq 0$ and $(\Delta_1, \ldots, \Delta_K)$ such that $\Delta_k \succeq 0$, we have

$$f(X_1, \ldots, X_K) \leq f(X_1 + \Delta_1, \ldots, X_K + \Delta_K).$$

(67)

**Proof:** Using the degradedness property of $DH_k = H_{k,e}$, the function $f(.)$ can be re-written as follows:

$$f(X_1, X_2, \ldots, X_K) = \frac{1}{2} \log \left| I + \sum_{k \in K} H_k X_k H_k^\dagger \right| - \frac{1}{2} \log \left| I + \sum_{k \in K} H_{k,e} X_k H_{k,e}^\dagger \right| \leq f(X_1 + \Delta_1, \ldots, X_K + \Delta_K).$$

(68)

$$= \frac{1}{2} \log \left| I + \sum_{k \in K} H_k X_k H_k^\dagger \right| - \frac{1}{2} \log \left| I + \sum_{k \in K} DH_k X_k H_k^\dagger DD_k \right|$$

(69)

where (a) follows from the fact that $|I + AB| = |I + BA|$. According to [23, Lemma II.3], this function is concave with regard to $I + \sum_{k \in K} H_k X_k H_k^\dagger$, and is therefore concave with regard to $(X_1, X_2, \ldots, X_K)$.

To prove the property of (67), note that for any $A \succeq 0$, $\Delta \succeq 0$ and $B > 0$, we have the following property [24,p.3942]:

$$\frac{|B|}{|A + B|} \leq \frac{|B + \Delta|}{|A + B + \Delta|}.$$  

(70)

We choose $\Delta = \sum_{k \in K} H_k \Delta_k H_k^\dagger$ and apply the above property to (69). We thus obtain,

$$f(X_1, X_2, \ldots, X_K) = \frac{1}{2} \log \left| I + \sum_{k \in K} H_k X_k H_k^\dagger \right| \leq \frac{1}{2} \log \left| \left( (D^\dagger D)^{-1} - I \right) + \left( I + \sum_{k \in K} H_k X_k H_k^\dagger \right) \right| \left| D^\dagger D \right|$$

(71)

$$= f(X_1 + \Delta_1, \ldots, X_K + \Delta_K).$$

To prove the converse part, we first start with Lemma 5 to bound the sum rate as follows:

$$\sum_{k \in K} R_k \leq I(x_K; y|z)$$

(72)

$$= h(y|z) - h(y|x_K, z)$$

$$= h(y|z) - h(n_1)$$

$$\leq \frac{1}{2} \log \left| I + \sum_{k \in K} H_k X_k H_k^\dagger \right| - \frac{1}{2} \log \left| I + \sum_{k \in K} H_{k,e} X_k H_{k,e}^\dagger \right|$$

(73)

$$\leq \frac{1}{2} \log \left| I + \sum_{k \in K} H_k Q_k H_k^\dagger \right| - \frac{1}{2} \log \left| I + \sum_{k \in K} H_{k,e} Q_k H_{k,e}^\dagger \right|,$$

where (a) follows from the fact that $h(y|z)$ is maximized by jointly Gaussian $y$ and $z$ for fixed covariance matrix $Q_{y,z}$ and (b) follows from the degradedness assumption and therefore concavity and monotonicity properties given in Lemma 5 and the fact that $K_{x_k} \preceq Q_k$. 
C. Proof of Lemma

We prove this lemma by induction on $L$. To show the lemma for $L = 0$, it is sufficient to prove that the equation

$$n(b_0 - \overline{b}_0) + m(b_0' - \overline{b}_0) = 0,$$

(73)

has no nontrivial solution when $b_0, b_0', \overline{b}_0, \overline{b}_0 \in \{0, 1, 2, ..., a - 1\}$. The necessary and sufficient conditions for the equation (73) are that $b_0 - \overline{b}_0$ is dividable by $n$ and $b_0 - \overline{b}_0$ is dividable by $n$. We show that one of these conditions does not hold for all choices of Table I.

Case 1: In this case $a = n$. Following the fact that $-(n - 1) \leq b_0' - \overline{b}_0 \leq n - 1$, it is easy to deduce that $n \nmid (b_0' - \overline{b}_0)$.

Case 2: In this case $a = s + 1$ where $m = 2s + 1$. Following the fact that $-(2s + 1) \leq b_0 - \overline{b}_0 \leq 2s + 1$, it is easy to deduce that $m \nmid b_0 - \overline{b}_0$.

Case 3: In this case $a = s$ where $m = 2s$. Following the fact that $-2s \leq b_0 - \overline{b}_0 \leq 2s$, it is easy to show that $m \nmid b_0 - \overline{b}_0$.

Now assume that property $\Gamma$ holds for $L - 1$. We need to show that the equation

$$\frac{A}{m} \sum_{l=0}^{L-1} \left( n(b_l - \overline{b}_l) + m(b_l' - \overline{b}_l) \right) W^l = 0,$$

(74)

has no nontrivial solution. Equivalently, this equation can be written as follows:

$$n(b_0 - \overline{b}_0) + m(b_0' - \overline{b}_0) = W \sum_{l=0}^{L-2} \left( n(b_{l+1} - \overline{b}_{l+1}) + m(b_{l+1}' - \overline{b}_{l+1}) \right) W^l.$$

(75)

We prove that the above equation has no nontrivial solution in two steps. First, assume that the right hand side of (75) is zero. This equation therefore reduces to

$$n(b_0 - \overline{b}_0) + m(b_0' - \overline{b}_0) = 0,$$

(76)

which we have already shown that has no nontrivial solution for all three cases.

Secondly, assume that the right side of (75) is non-zero. The equation (75) can therefore be written as follows:

$$n(b_0 - \overline{b}_0) + m(b_0' - \overline{b}_0) = cW,$$

(77)

where $c \in \mathbb{Z}$ and $c \neq 0$. We need to prove that equation (77) has no nontrivial solution for all three cases.

Case 1: In this case $W = n(2n - 1)$ and $n$ divides $n(b_0 - \overline{b}_0)$ and $cW$ as well. However, as $(m, n) = 1$ and $-(n - 1) \leq b_0 - \overline{b}_0 \leq n - 1$, the equation (77) has a solution iff $b_0 = \overline{b}_0$ which is a contradiction with the fact that $n|b_0 - \overline{b}_0| < n(n-1) < |c|n(2n-1) = |c|W$.

Case 2: In this case $W = (s + 1)(2s + 1)$, $n = s + 1$ and $m = 2s + 1$. Thus, $2s + 1$ divides both $m(b_0' - \overline{b}_0)$ and $cW$. Since $(2n, m = 2s + 1) = 1$ and $-2s \leq b_0 - \overline{b}_0 \leq 2s$, therefore, $2s + 1$ cannot divide $n(b_0 - \overline{b}_0)$. Hence, equation (77) has a solution iff $b_0 = \overline{b}_0$ which contradicts the fact that $m|b_0 - \overline{b}_0| < |c|W$.

Case 3: In this case $W = 2s^2 - n$, $a = s$, $2n < m$ and $m = 2s$. We have

$$|n(b_0 - \overline{b}_0) + m(b_0' - \overline{b}_0)| < m(b_0 - \overline{b}_0) + b_0' - \overline{b}_0| \leq 2m(a-1) = 4s(s-1) < 2W,$$

(78)

and therefore, it suffices to assume $c = 1$. Substituting $W = 2s^2 - n$ in (77), we have the following equation:

$$n(b_0 - \overline{b}_0 + 1) + 2s(b_0' - \overline{b}_0) = 2s^2.$$

(79)

Obviously, $2s$ divides $2s(b_0' - \overline{b}_0)$ and $2s^2$. However, since $(2s, n) = 1$ and $-(2s - 1) \leq b_0 - \overline{b}_0 \leq 2s - 1$, equation (79) has a solution iff $b_0 = \overline{b}_0 - 1$ which is impossible due to the fact $2s|b_0' - \overline{b}_0| < 2s^2$. This completes the proof.

References

[1] C. E. Shannon, “Communication Theory of Secrecy Systems”, Bell System Technical Journal, vol. 28, pp. 656-715, Oct. 1949.
[2] A. Wyner, “The Wire-tap Channel”, Bell System Technical Journal, vol. 54, pp. 1355-1387, 1975.
[3] I. Csiszar and J. Korner, “Broadcast Channels with Confidential Messages”, IEEE Trans. Inform. Theory, vol. 24, no. 3, pp. 339-348, May 1978.
[4] S. K. Leung-Yan-Cheong and M. E. Hellman, “Gaussian Wiretap Channel”, IEEE Trans. Inform. Theory, vol. 24, no. 4, pp. 451-456, July 1978.
[5] F. Oggier, B. Hassibi, “The MIMO Wiretap Channel”, Communications, Control and Signal Processing, 2008. ISCCSP 2008. 3rd International Symposium on, pp. 213-218, Mar. 2008.
[6] S. Shaheen, L. Nan and S. Ulukus, “Secrecy Capacity of the 2-2-1 Gaussian MIMO Wiretap Channel”, Communications, Control and Signal Processing, 2008. ISCCSP 2008. 3rd International Symposium on, pp. 207-212, Mar. 2008.
[7] A. Khisti, G. Wornell, A. Wiesel, and Y. Eldar, “On the Gaussian MIMO Wiretap Channel”, in Proc. IEEE Int. Symp. Information Theory (ISIT), Nice, France, Jun, 2007.
