GOURSAT’S \((n + 1)\)-WEBS
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Abstract. We consider the Goursat’s \((n + 1)\)-webs of codimension one of two kinds on an \(n\)-dimensional manifold. They are characterized by the specific closed form equations or by two special relations between components of the torsion tensor of the web. These relations allow us to establish a connection with solutions of two systems of nonlinear second-order PDEs investigated by Goursat in 1899. The integrability conditions of some distributions invariantly associated with both kinds of Goursat’s \((n + 1)\)-webs are also investigated.

1 The principal equations of \((n + 1)\)-webs
1. Let \(W(n+1,n,1)\) be an \((n + 1)\)-web defined on a differentiable manifold \(X^n\) of dimension \(n\) by \(n + 1\) foliations \(\lambda_\xi, \xi = 1, \ldots, n + 1\), of codimension one. Each foliation \(\lambda_\xi\) can be defined by the completely integrable system of Pfaffian equations

\[
\omega_\xi = 0, \quad \xi = 1, \ldots, n + 1.
\]

The 1-forms \(\omega_\alpha, \alpha = 1, \ldots, n\), define a co-frame in the tangent bundle \(T(X^n)\) and satisfy the following structure equations:

\[
d\omega_\alpha = \omega_\alpha \wedge \omega + \sum_{\beta \neq \alpha} a_{\alpha\beta} \omega_\alpha \wedge \omega_\beta,
\]

\[
d\omega = \sum_{\alpha, \beta} b_{\alpha\beta} \omega_\alpha \wedge \omega_\beta,
\]

where the quantities \(a_{\alpha\beta}\) and \(b_{\alpha\beta}\) are connected by certain relations (see [G 73] or [G 74] or [G 88], Section 1.2). We indicate some of these relations:

\[
\nabla a_{\alpha\beta} = \sum_{\gamma=1}^{n} \left( a_{\alpha\gamma\beta} + a_{\alpha\beta\gamma} + a_{\gamma\alpha\beta} \right) \omega_{\gamma}, \quad \alpha \neq \beta,
\]

\[
a_{\alpha\beta} = a_{\beta\alpha},
\]

\[
\sum_{\alpha, \beta=1}^{n} a_{\alpha\beta} = 0,
\]

\[
b_{\alpha\beta} = \frac{1}{2} \left( a_{\gamma\alpha\beta} - a_{\beta\alpha\gamma} \right), \quad \gamma \neq \alpha, \beta,
\]

\[
b_{\alpha\beta} = b_{\alpha\gamma} + b_{\gamma\beta},
\]

where \(\nabla a_{\alpha\beta} = d a_{\alpha\beta} - a_{\alpha\beta} \omega\). By (7), we have

\[
b_{\alpha\beta} = -b_{\beta\alpha}.
\]

The quantities \(a_{\alpha\beta}\) and \(b_{\alpha\beta}\) form tensor fields in the tangent bundle \(T(X^n)\) which are called respectively the torsion and curvature tensors of the web \(W(n + 1, n, 1)\).
2 Goursat’s \((n + 1)\)-webs of first kind

2. Suppose that \(n \geq 4\). Consider two pairs of three-dimensional distributions defined by the following Pfaffian equations:

\[ a_{13} \omega_3 + a_{14} \omega_4 = 0, \omega = 0, \quad (10) \]
\[ a_{23} \omega_3 + a_{24} \omega_4 = 0, \omega = 0, \quad (11) \]

and

\[ a_{13} \omega_1 + a_{23} \omega_2 = 0, \omega = 0, \quad (12) \]
\[ a_{14} \omega_1 + a_{24} \omega_2 = 0, \omega = 0, \quad (13) \]

where \(\sigma = 5, \ldots, n\).

In general, the distributions defined by each of equations (10)–(13) are not integrable.

It is easy to see that the distribution (10) and (11) (or (12) and (13)) coincide if and only if the torsion tensor \(a_{\alpha \beta}\) of the web \(W(n + 1, n, r)\) satisfies the following condition:

\[ a_{13} a_{24} - a_{14} a_{23} = 0. \quad (14) \]

One can also see that condition (14) is necessary and sufficient for the distribution defined by the system of equations (10) and (11) (or (12) and (13)) to be three-dimensional (for a general web \(W\) these distributions are two-dimensional).

We will call the webs \(W(n + 1, n, 1)\) satisfying condition (14) the \textit{Goursat webs of the first kind}. The reason for this name will be clear later.

**Lemma 1** The Pfaffian derivatives of the torsion tensor \(a_{\alpha \beta}\) of the Goursat web \(W(n + 1, n, 1)\) of the first kind and the components of the torsion tensor \(a_{\alpha \beta}\) itself satisfy the following relations:

\[ a_{24} a_{13} c + a_{13} a_{24} c - a_{23} a_{14} c - a_{14} a_{23} c = 0, \quad c = 1, 2, 3, 4. \quad (15) \]

**Proof.** In fact, differentiating condition (14), we find that

\[ a_{24} \nabla a_{13} c + a_{13} \nabla a_{24} c - a_{23} \nabla a_{14} c - a_{14} \nabla a_{23} c = 0. \quad (16) \]

By (4), (10), and (14) and by linear independence of the forms \(\omega_c, c = 1, 2, 3, 4\), equation (16) implies conditions (15).

**Theorem 2** For the Goursat webs \(W(n + 1, n, 1)\) of the first kind the two-dimensional distribution defined by equations (10) and (12) (or (11) and (13)) is integrable.

**Proof.** In fact, denote by \(\theta\) and \(\rho\) the left-hand sides of equations (10) and (12):

\[ \theta = a_{13} \omega_3 + a_{14} \omega_4, \quad \rho = a_{13} \omega_1 + a_{14} \omega_2. \]

Since \(W(n + 1, n, 1)\) is a Goursat web of the first kind, the components of its torsion tensor and their Pfaffian derivatives satisfy conditions (14) and (15). Applying these conditions, one can easily prove that

\[ \begin{cases} d\theta \equiv 0 \pmod{\theta, \rho}, \\ d\rho \equiv 0 \pmod{\theta, \rho}. \end{cases} \]
Thus, the two-dimensional distribution defined by the equations $\theta = 0, \rho = 0$ is integrable.

3. Suppose that in some domain $D \subset X^n$, a web $W(n+1,n,1)$ is defined by the closed form equations

$$x_{n+1} = F(x_1, \ldots, x_n), \quad \det \left( \frac{\partial F}{\partial x_\alpha} \right) \neq 0. \quad (17)$$

It is proved in [G 76] (see also [G 88], Section 4.1) that

$$a_{\alpha \beta} = \frac{F_{\alpha \beta}}{F_\alpha F_\beta}, \quad (18)$$

and that by (18), conditions (14) are equivalent to the following second-order nonlinear partial differential equation:

$$\frac{F_{13}}{F_{14}} = \frac{F_{14}}{F_{24}}. \quad (19)$$

Goursat [Go 99] considered such an equation for $n = 4, 5$. This is the reason we named webs satisfying condition (14) Goursat webs.

The following theorem follows from [Go 99].

**Theorem 3** For the Goursat web $W(n+1,n,1)$ of the first kind, closed form equation (17) takes the form

$$\begin{cases} x_{n+1} = \phi(x_1, x_2, x_3, \ldots, x_n, a) + \psi(x_3, x_4, x_5, \ldots, x_n, a), \\ \frac{\partial \phi}{\partial a} + \frac{\partial \psi}{\partial a} = 0, \end{cases} \quad (20)$$

where $\phi$ and $\psi$ are arbitrary functions of $n - 1$ variables each satisfying the second equation.

**Proof.** In fact, it is proved in [Go 99] that the general solution of equation (19) has the form (20). The only difference between [Go 99] and our considerations is that in [Go 99] $n = 4$ and $n = 5$ while we consider the general case $n \geq 4$. ■

3 Goursat’s $(n+1)$-webs of second kind

4. Suppose that $n \geq 5$. Consider the distribution $\Delta_2$ defined by the following equations:

$$\begin{cases} \omega + \omega + \omega = 0, \\ a_{13} \omega + a_{14} \omega + a_{15} \omega = 0, \\ a_{23} \omega + a_{24} \omega + a_{25} \omega = 0, \\ \omega = 0, \quad \sigma = 6, \ldots, n. \end{cases} \quad (21)$$

Note that equation (1) and the first equation of (21) implies that

$$\omega_1 + \omega_2 = 0. \quad (22)$$

Equations (21) and (22) imply that, in general, the distribution $\Delta_2$ is one-dimensional.
Consider also the distribution $\Delta_3$ defined by the equations

$$\begin{align*}
\omega_3 + a_{31} \omega_1 + a_{32} \omega_2 &= 0, \\
\omega_3 + a_{41} \omega_1 + a_{42} \omega_2 &= 0, \\
\omega_3 + a_{41} \omega_1 + a_{42} \omega_2 &= 0, \\
\omega &\equiv \sigma = 0, \quad \sigma = 6, \ldots, n.
\end{align*}$$

In general, the distribution $\Delta_3$ is two-dimensional, and the distributions defined by equations (21) and (23) are not integrable.

Note that the $3 \times 3$ matrices of coefficients of the first three equations of (21) and (23) are transposes of each other. It is easy to see that the distributions $\Delta_2$ and $\Delta_3$ defined by equations (21) and (23) are two- and three-dimensional, respectively, if and only if

$$\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix} = 0.$$  

(24)

This explains why we used the notation $\Delta_2$ and $\Delta_3$ for these two distributions: the lower index indicates the distribution dimension.

We will call the webs $W(n+1, n, 1)$ satisfying condition (24) the Goursat webs of the second kind. The reason for this name will be clear later.

It is easy to see that condition (24) is equivalent to the condition

$$\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix} + \begin{vmatrix}
a_{14} & a_{15} & a_{16} \\
a_{24} & a_{25} & a_{26} \\
a_{34} & a_{35} & a_{36}
\end{vmatrix} = 0,$$

or to the condition

$$a_\alpha (a_\beta - a_\gamma) + a_\beta (a_\gamma - a_\alpha) + a_\gamma (a_\alpha - a_\beta) = 0,$$

or to the condition

$$\frac{a_{p\alpha} - a_{q\alpha}}{a_{p\alpha}} = \frac{a_{n\beta} - a_{n\gamma}}{a_{n\beta}} \quad p, q = 1, 2; \ a, b, c = 3, 4, 5; \ p \neq q; \ a \neq b, c; \ b \neq c.$$

(27)

**Theorem 4** For the Goursat web $W(n+1, n, 1)$ of the second kind, closed form equation (17) take the form

$$x_{n+1} = \phi(x_1, x_2, x_3, \ldots, x_n, a, \psi(x_3, x_4, x_5, \ldots, x_n, a)),$$

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial \psi} \frac{\partial \psi}{\partial a} = 0,$$

where $\phi$ and $\psi$ are arbitrary functions of $n-1$ variables each, and $a$ is an arbitrary parameter.

**Proof.** In fact, by (18), conditions (24) are equivalent to the following second-order nonlinear partial differential equation:

$$\begin{vmatrix}
F_3 & F_4 & F_5 \\
F_{13} & F_{14} & F_{15} \\
F_{23} & F_{24} & F_{25}
\end{vmatrix} = 0.$$

(28)
It is proved in [Go 99] that the general solution of equation (29) has the form (28). The only difference between [Go 99] and our considerations is that in [Go 99] \( n = 5 \) while we consider the general case \( n \geq 5 \).

5. Let us find the differential consequences of conditions (27). First, we will prove the following lemma.

**Lemma 5** For the Goursat web \( W(n + 1, n, 1) \) of the second kind, the following identities hold:

\[
\begin{align*}
\alpha (a - a) + a (a - a) + & a (a - a) = 0, \\
\alpha^2 (a - a) + & \alpha^2 (a - a) + \alpha^2 (a - a) + \alpha (a - a) + a (a - a) + a (a - a) = 0,
\end{align*}
\]

where \( p, q = 1, 2; \ a, b, c = 3, 4, 5; \ p \neq q; \ a \neq b, c; \ b \neq c \).

**Proof.** The proof of (30) and (31) is straightforward and can be obtained by applying conditions (27) (or (25)) several times.

**Theorem 6** The components \( a_{\alpha \beta} \) of the torsion tensor and their Pfaffian derivatives \( a_{\alpha \beta \gamma} \) satisfy the following identities:

\[
\begin{align*}
m_1 &= 0, \quad m_2 = 0, \quad m_3 = 0, \quad \sigma = 6, \ldots, n, \\
m_4 &= C a_A + A_a, \\
m_5 &= B a_A + A_A.
\end{align*}
\]

where

\[
m_\alpha = \frac{(a - a) a}{13} + \frac{(a - a) a}{12} + \frac{(a - a) a}{11} + \frac{(a - a) a}{10} + \frac{(a - a) a}{9} + \frac{(a - a) a}{8} + \frac{(a - a) a}{7} + \frac{(a - a) a}{6} + \frac{(a - a) a}{5} + \frac{(a - a) a}{4} + \frac{(a - a) a}{3} + \frac{(a - a) a}{2} + \frac{(a - a) a}{1} + \frac{(a - a) a}{0},
\]

and

\[
A = \begin{vmatrix} a & a \\ a & 13 \\ 23 & 24 \end{vmatrix}, \quad B = \begin{vmatrix} a & a \\ a & 13 \\ 23 & 24 \end{vmatrix}, \quad C = \begin{vmatrix} a & a \\ a & 13 \\ 23 & 24 \end{vmatrix}, \quad A + B + C = 0.
\]

**Proof.** In fact, differentiating (27), we arrive at the following Pfaffian equation:

\[
\begin{align*}
\frac{(a - a) \nabla a}{12} + \frac{(a - a) \nabla a}{11} + \frac{(a - a) \nabla a}{10} + \frac{(a - a) \nabla a}{9} + \frac{(a - a) \nabla a}{8} + \frac{(a - a) \nabla a}{7} + \frac{(a - a) \nabla a}{6} + \frac{(a - a) \nabla a}{5} + \frac{(a - a) \nabla a}{4} + \frac{(a - a) \nabla a}{3} + \frac{(a - a) \nabla a}{2} + \frac{(a - a) \nabla a}{1} + \frac{(a - a) \nabla a}{0} = 0,
\end{align*}
\]

where \( \nabla a = d a - a \omega \). Substituting into (33) the values of \( \nabla a \) taken from equations (4), equating to 0 the coefficients in independent 1-forms \( \omega \), and simplifying the equations obtained by means of (30) and (31), we arrive at conditions (32).

6. For the Goursat web \( W(n + 1, n, 1) \) of the second kind, the second and the third equations of (21) are equivalent either to the equation

\[
(a - a) \omega + (a - a) \omega = 0
\]

or to the equation

\[
(a - a) \omega + (a - a) \omega = 0.
\]

By (27), equations (34) and (35) are equivalent. Each of these two equations along with equations \( \omega_\sigma = 0, \ \sigma = 6, \ldots, n \), defines a four-dimensional distribution \( \Delta_4 \). The following theorem gives the conditions of integrability of the distribution \( \Delta_4 \).
**Theorem 7** For the Goursat web \(W(n + 1, n, 1)\) of the second kind, the distribution \(\Delta_4\) is integrable if and only if the following three conditions hold:

\[
\begin{align*}
n_1 &= 0, \quad n_2 = 0, \quad n_3 = a \left[ (a - a) a + (a - a) a \right],
\end{align*}
\]

where

\[
\begin{align*}
n_h &= (a - a) a + (a - a) a + (a - a) a, \quad h = 1, 2, 3.
\end{align*}
\]

**Proof.** Taking the exterior derivative of equation (34), we obtain the following exterior quadratic equation:

\[
\begin{align*}
\left[ \nabla a - \nabla a + (a - a)(a \omega + a \omega + a \omega) \right] \wedge \omega \\
+ \left[ \nabla a - \nabla a + (a - a)(a \omega + a \omega + a \omega) \right] \wedge \omega &= 0.
\end{align*}
\]

Next, we use (34) to express the form \(\omega\) in terms of the form \(\omega\), substitute its value into equation (37), and equate to 0 the coefficients in the independent exterior quadratic products \(\omega \wedge \omega, \omega \wedge \omega, \) and \(\omega \wedge \omega\) (there are no other exterior quadratic products in the exterior quadratic equation). As a result, we obtain conditions (36). ■

**Remark** Note that the conditions of integrability of equation (35) have the form

\[
r_1 = 0, \quad r_2 = 0, \quad r_3 = a \left[ (a - a) a + (a - a) a \right],
\]

where

\[
r_h = (a - a) a + (a - a) a + (a - a) a, \quad h = 1, 2, 3.
\]

However, these conditions are not independent: they follow from the corresponding conditions (32) and (36). Moreover, a straightforward calculation shows that for each \(h = 1, 2, 3,\) any two of three systems (32), (36), and (38) imply the third one.

7. The two-dimensional distribution \(\Delta_2\) defined by equations (21) and (22) belongs to the four-dimensional distribution \(\Delta_4\). In general, \(\Delta_2\) is not integrable even if \(\Delta_4\) is integrable. However, for the Goursat web \(W(n + 1, n, 1)\) of the second kind, \(\Delta_2\) is integrable if \(\Delta_4\) is integrable.

**Theorem 8** If a web \(W(n + 1, n, 1)\) is the Goursat web of the second kind, then integrability of the distribution \(\Delta_4\) implies integrability of the distribution \(\Delta_2\).

**Proof.** First, we find the conditions of integrability of the distribution \(\Delta_2\). Note that exterior differentiation of the first equation of (21) and equation (22) leads to the identities. Thus, we must take the exterior derivative of the second or the third equation of (21). However, we will take the exterior derivative of equation (34) which is equivalent to each of them:

\[
\begin{align*}
\left[ \nabla a - \nabla a + (a - a)(a \omega + a \omega) \right] \wedge \omega \\
+ \left[ \nabla a - \nabla a + (a - a)(a \omega + a \omega) \right] \wedge \omega &= 0.
\end{align*}
\]

Using equations (21) and (22), we can express the form \(\omega\) in terms of \(\omega\) and the forms \(\omega\) and \(\omega\) in terms of the form \(\omega\). As a result, equation (39) will contain only one
The following proposition gives the conditions of integrability of the distribution $\Delta'$

\textbf{Theorem 9}  
For the Goursat web $W(n+1,n,1)$ of the second kind, the distribution $\Delta'_4$ is integrable if and only if the following three conditions hold:

\[ s_4 = -C \sigma_{34}, \quad s_5 = B a_{34}, \quad s_6 = C(a - a), \]  \hspace{1cm} (42)

where

\[ s_h = \frac{(a - a)(a - a) + (a - a)(a - a)(a - a)}{\omega}, \quad h = 3, 4, 5. \]

\textbf{Proof.} We take the exterior derivative of equation (41) considered for $a = 4, b = 3$:

\[ [\nabla a - \nabla a + (a - a)(a + a + a)] \wedge \omega + [\nabla a - \nabla a + (a - a)(a + a + a)] \wedge \omega = 0. \]  \hspace{1cm} (43)

Next, we use (41) to express the form $\omega$ in terms of the form $\omega$. Substitute its value into equation (43), and equate to 0 the coefficients in the independent exterior quadratic products $\omega \wedge \omega, \omega \wedge \omega$, and $\omega \wedge \omega$ (there are no other exterior quadratic products in the exterior quadratic equation). As a result, we obtain conditions (42). \hspace{1cm} \blacksquare

\textbf{Remark} Note that the conditions of integrability of equations (41) taken for $a = 5, b = 3$ or $a = 4, b = 5$, are obtained from the corresponding conditions (32) and (42).

\textbf{9.} The three-dimensional distribution $\Delta_3$ defined by equations (23) belongs to the four-dimensional distribution $\Delta'_4$. In general, $\Delta_3$ is not integrable even if $\Delta'_4$ is integrable. As opposed to what we had in Theorem 8, the following theorem shows that this is also true for the Goursat web $W(n+1,n,1)$ of the second kind.

\textbf{Theorem 10}  
For the Goursat web $W(n+1,n,1)$ of the second kind, the distribution $\Delta_3$ is integrable if and only if the following conditions hold:

\[ u_4 = 2A a, \quad u_5 = 2A a, \quad u_4 - u_4 = -A a, \quad u_5 - u_5 = A(a - a), \]  \hspace{1cm} (44)

where

\[ u_h = \frac{(a - a) a - (a - a) a}{\omega}, \quad v_h = \frac{(a - a) a - (a - a) a}{\omega}, \quad k = 4, 5. \]
Proof. In order to find the conditions of integrability of the distribution \( \Delta_3 \), we must take the exterior derivatives of only the first two equations of (23) since by (27) the third equation follows from the second one. We will take the exterior derivative of the first equation of (23) and equation (41) taken for \( a = 4 \), \( b = 3 \), which is equivalent to the second equation of (23).

Exterior differentiation of these two equations gives the following exterior quadratic equations:

\[
\begin{align*}
\nabla_{a_{13}} &- a_{13} (a_{12} \omega + a_{13} \omega + a_{14} \omega + a_{15} \omega) \wedge \omega = 0 \\
\nabla_{a_{23}} &- a_{23} (a_{21} \omega + a_{23} \omega + a_{24} \omega + a_{25} \omega) \wedge \omega = 0
\end{align*}
\]

(45)

and

\[
\begin{align*}
\nabla_{a_{14}} - \nabla_{a_{13}} &+ (a_{13} - a_{14}) (a_{12} \omega + a_{13} \omega + a_{14} \omega + a_{15} \omega) \wedge \omega = 0 \\
\n\nabla_{a_{24}} - \nabla_{a_{23}} &+ (a_{23} - a_{24}) (a_{21} \omega + a_{23} \omega + a_{24} \omega + a_{25} \omega) \wedge \omega = 0
\end{align*}
\]

(46)

Using equations (23) and (41), we can express the forms \( \omega_2 \) and \( \omega_3 \) in terms of the form \( \omega_1 \). As a result, equations (45) and (46) will contain only two independent exterior quadratic products \( \omega \wedge \omega \) and \( \omega \wedge \omega \) (there is no the form \( \omega \wedge \omega \) in (45) and (46)). Equating the coefficients in these products in equation (45) to 0, we obtain the first two conditions of (44); and equating the coefficients in these products in equation (46) to 0, we obtain the last two conditions of (44).

It is easy to see that conditions (42) do not imply conditions (44), i.e., integrability of \( \Delta'_4 \) does not imply integrability of \( \Delta_3 \).

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