Exact Scalar-Tensor Cosmological Models

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Scalar-tensor gravitational theories are important extensions of standard general relativity, which can explain both the initial inflationary evolution, as well as the late accelerating expansion of the Universe. In the present paper we investigate the cosmological solution of a scalar-tensor gravitational theory, in which the scalar field $\phi$ couples to the geometry via an arbitrary function $F(\phi)$. The kinetic energy of the scalar field as well as its self-interaction potential $V(\phi)$ are also included in the gravitational action. By using a standard mathematical procedure, the Lie group approach, and Noether symmetry techniques, we obtain several exact solutions of the gravitational field equations describing the time evolutions of a flat Friedman-Robertson-Walker Universe in the framework of the scalar-tensor gravity. The obtained solutions can describe both accelerating and decelerating phases during the cosmological expansion of the Universe.

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I. INTRODUCTION

There is strong observational and theoretical evidence that general relativity (GR) is not the complete, and final geometric theory describing the nature of the gravitational force. From the theoretical point of view, there are many shortcomings of GR in cosmology and quantum field theory. For example, it is known that the standard model of cosmology based on GR and on particle physics is unable to depict the gravitational force in extreme regimes, including the description of gravity on microscopic scales of the order of Planck length. In order to describe the very early expansion history of the universe accurately, one needs to quantize the geometry, in a way to connect it with quantum field theory. On the other hand, from the particle physics point of view, we must note the fact that the Dirac equation is extremely successful in unifying special relativity and quantum mechanics. Although the unification of GR and quantum field theory has gained some progresses, yet the unification of GR and of quantum mechanics still remains an open question.

On the other hand, observational evidence obtained on distant Type Ia Supernovae (SNeIa) between 1998 to 2010, suggested that the universe has undergone a late time accelerated expansion $\Omega$. Recently, the Cosmic Microwave Background temperature anisotropies measured by the Wilkinson Microwave Anisotropy Probe and the Planck satellites have been reported in $\Omega$, confirming that the total amount of baryonic matter in the Universe is very low. It is worth to notice that the study of the baryonic acoustic oscillations have also confirmed, and correlated the other cosmological observations $\Omega$.

In order to explain the accelerated expansion of the universe, one may introduce the idea of dark energy (DE), which from a mathematical point of view and in the context of GR represents a new contribution in the matter part of the Einstein’s gravitational field equation. The most popular candidate for dark energy is the cosmological constant $\Lambda$, which is introduced without modifying the geometric part of the Einstein field equations. On the other hand, dark energy can also be explained as a modification of the geometrical part of Einstein’s equation. Such an approach leads to the so called modified gravitational theories (see reviews in $\Omega$).

Among the most interesting modified gravity theories are the scalar-tensor theories of gravitation, which are an important generalization of standard GR theory. In the so-called Brans-Dicke theory, an additional scalar field $\phi$ besides the metric tensor $g_{\mu\nu}$ and a dimensionless coupling constant $\omega$ were introduced in order to describe the gravitational interaction $\Omega$. Brans-Dicke theory recovers the results of GR for large value of the coupling constant $\omega$, that is for $\omega > 500$. Similar formalisms were developed earlier by Jordan $\Omega$. Note that the history and developments of scalar-tensor theories (STT) can be found in $\Omega$.

The astrophysical and cosmological implications of the Brans-Dicke type theories have been extensively investigated. The dynamics of a causal bulk viscous cosmolog-
cial fluid filled flat homogeneous Universe in the framework of the BD theory was reported in [26]. Scalar-tensor theories are very promising because they can provide an explanation of the inflationary behavior in cosmology. Also the scalar field can be considered an additional degree of freedom of the gravitational interaction. Scalar-tensor theories has been used to model DE, because the scalar fields are a good candidate for phantom and quintessence fields [27, 28]. Furthermore, it is natural to couple the scalar field to the curvature after compactification of higher dimensional theories of gravity such as Kaluza-Klein and string theory, offering the possibility of linking fundamental scalar fields with the nature of the DE.

Very recently, Hojman symmetry approach was used to study the behavior of the flat Friedman-Robertson-Walker (FRW) universe in the context of STT [30]. The possibility of probing a yet unconstrained region of the parameter space of STTs is based on the fact that stability properties of highly compact neutron stars may radically differ from those in GR [32]. In the scalar-tensor model with Gauss-Bonnet and kinetic couplings the power law dark energy solution may be described by Higgs-type potential [33]. A covariant multipolar Mathisson-Papaetrou-Dixon type approach was used to derive the equations of motion in a systematic way for both Jordan and Einstein formulations of STT in [31]. The parametrized post-Newtonian (PPN) parameters $\gamma$ and $\beta$ for general STT was computed, and it was suggested that the PPN parameters $\gamma$ and $\beta$ given by the condition $|\gamma - 1| \sim |\beta - 1| \sim 10^{-6}$ may be detectable by a satellite that carries a clock with fractional frequency uncertainty $\Delta f/l \sim 10^{-16}$ in an eccentric orbit around the Earth [35].

A complete Noether symmetry analysis in the framework of scalar-tensor cosmology was reported in [30] (see also [37, 38] for other approaches). Scalar-tensor cosmology with $1/R$ curvature correction by Noether symmetry was discussed in [39]. The weak field limit of STT of gravitation was discussed in view of conformal transformations in [40], and a new reconstruction method of STT based on the use of conformal transformations was proposed. This method allows the derivation of a set of interesting exact cosmological solutions in BD gravity, as well as in other extensions of GR [41]. Recently, (An)isotropic models in the presence of variable gravitational and cosmological constants were studied in the context of scalar-tensor cosmology [42]. Generalized self-similar STT was investigated and some new exact self-similar solutions were obtained in [43], by using the Kantowski-Sach models. Exact solution of gravitational field equations describing the dynamics of the anisotropic universe with string fluid was presented in the framework of STT in [44]. Testing the feasibility of scalar-tensor gravity by scale dependent mass and coupling to matter was investigated in [45]. The phase space of Friedmann-Lemaître-Robertson-Walker models derived from STT in the presence of the non-minimal coupling $F(\phi) = \xi \phi^2$ and the effective potential $V(\phi) = \lambda \phi^n$ was studied in [46]. The Noether symmetries of a generalized scalar-tensor, Brans-Dicke type cosmological model, in which the explicit scalar field dependent couplings to the Ricci scalar, and to the scalar field kinetic energy, respectively, were considered, was investigated in [47]. The scalar field self-interaction potential into the gravitational action was also included. Three cosmological solutions describing the time evolution of a spatially flat Friedman-Robertson-Walker Universe were obtained, and the cosmological properties of the solutions were investigated in detail. The obtained models can describe a large variety of cosmological evolutions, including models that experience a smooth transition from a decelerating to an accelerating phase.

The purpose of the present paper is to consider a detailed mathematical analysis of the cosmological solutions of a special class of scalar-tensor type gravitational theories, in which in the gravitational action the Ricci scalar, the scalar field kinetic energy, and the scalar field self interaction potential are also considered. As a first step in our analysis we present several exact analytical solutions of the gravitational field equations, describing the time evolution of the flat FRW universe in the presence of the scalar field by using the standard procedure. Then two rigorous mathematical approaches, the Lie group approach, and the Noether symmetry techniques, respectively, are applied for the investigation of the gravitational field equations. Both approaches provide several classes of solutions of the gravitational field equations, allowing the determination of the cosmological scalar field and of its self-interaction potential.

The present paper is organized as follows. The scalar-tensor gravitational model and the cosmological field equations are presented in Section II. A set of simple analytical solutions of the gravitational field equations describing the time evolution of the flat FRW universe in the presence of the scalar field by using standard procedures are obtained in Section III. The cosmological field equations are investigated by using the Lie group approach in Section IV. The Noether symmetry techniques are applied in Section V. We conclude our results in Section VI.

II. SCALAR-TENSOR COSMOLOGY

In the present paper we are considering a gravitational model in which the Ricci scalar, describing pure gravity, couples in a non-standard way with a scalar field. For this model the gravitational action takes the form [30, 31]

$$S = \int d^4 x \sqrt{-g} \left[ F(\phi) R + \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right],$$

(1)

where $R$ is the Ricci scalar. The plus sign appears in the kinetic term of the action (1) corresponding to the
phantom phase \cite{48,49}. The arbitrary functions $F(\phi)$ and $V(\phi)$ are the coupling to the Ricci scalar and the potential of the scalar field $\phi$ respectively. In a spatially flat FRW space-time, with metric

$$ds^2 = dt^2 - a^2(t) (dx^2 + dy^2 + dz^2),$$

(2)

the Lagrangian density on the configuration space $(a, \phi)$ for non-minimal coupled scalar-tensor cosmology is

$$\mathcal{L} = 6F(\phi) a\dot{a}^2 + 6F'(\phi) a^2 \dot{\phi}^2 + \frac{1}{2} a^3 \ddot{\phi}^2 - a^3 V(\phi),$$

(3)

where the overdot and the prime denote the derivative with respect to the time and the scalar field $\phi$, respectively. The function $a(t)$ is the scale factor of the Universe, which gives the full information on the expansion history of the Universe. Now, the Euler-Lagrange equations can be obtained from Eq. \cite{30}, and are given by \cite{30}

$$2\ddot{H} + 3H^2 + \left(2H\dot{\phi} + \dot{\phi}^2 \right) \frac{F'}{F} + \left(\frac{F'}{2} - \frac{1}{4}\right) \frac{\dot{\phi}^2}{F} = -\frac{V}{2F},$$

(4)

$$\ddot{\phi} + 3H \dot{\phi} + 6 \left( \dot{H} + 2H^2 \right) \frac{F'}{F} + V' = 0,$$

(5)

corresponding to the Einstein-Friedmann equation, and to the Klein-Gordon equation for the FRW geometry, respectively. In Eqs. \cite{11} and \cite{12} we have defined the Hubble parameter $H$ as $H = \frac{\dot{a}}{a}$. The energy function corresponding to the Einstein (0,0) equation takes immediately the form

$$-6H \left(FH + 2\dot{\phi}^2 \right) - \frac{\dot{\phi}^2}{2} = V.$$  

(6)

By eliminating the potential $V$ from Eqs. \cite{11} and \cite{12}, we get the general differential equation given by

$$H' = \frac{F'}{2F} H + \frac{\dot{\phi}}{4F} \left(1 - 2F''\right) - \frac{\dot{\phi}^2}{2\dot{F}} F'.$$

(7)

By defining the arbitrary function $M(\phi)$ as $M(\phi) = \dot{\phi}$, with this transformation, thus Eq. \cite{13} becomes

$$H' = \frac{F'}{2F} H + \frac{M}{4F} \left(1 - 2F''\right) - \frac{F'}{2F} M'.$$

(8)

In the following, we shall present the analytical solutions of Eq. \cite{8}, which describe the dynamics of the flat FRW Universe in the presence of the scalar field coupled to the geometrical part of the action.

### III. ANALYTICAL SOLUTIONS OF THE GRAVITATIONAL FIELD EQUATIONS

In order to obtain a solution of the differential Eq. \cite{8}, we assume that the coupling function $F(\phi)$, the function $M(\phi)$ and the Hubble function $H(\phi)$ obey the relations

$$F(\phi) = \alpha_0 \phi^2,$$

(9)

$$M(\phi) = \alpha_1 \phi^s,$$

(10)

$$H(\phi) = \alpha_2 \phi^{s-1},$$

(11)

respectively, where $s$ and $\alpha_i$, $i = 0, 1, 2$ are arbitrary constants. By substituting Eqs. \cite{11,11,11} into Eq. \cite{8}, the latter yields the relation

$$\alpha_2 \left( s - 2 \right) = \alpha_1 \left( \frac{1}{4\alpha_0} - 1 - s \right).$$

(12)

By substituting Eqs. \cite{11,11,11} into Eq. \cite{8}, the latter gives the potential as

$$V(\phi) = - \left(6\alpha_0 \alpha_2^2 + 12\alpha_0 \alpha_1 \alpha_2 + \frac{1}{2} \alpha_1^2 \right) \phi^{2s}$$

$$= \alpha_1^2 \left(12\alpha_0 - 1\right) \frac{[3 + 4\alpha_0 (s + 1) (s - 5)]}{8\alpha_0(s - 2)^2} \phi^{2s}$$

(13)

or, equivalently,

$$V(\phi) = \frac{4\alpha_0 [11 + (2 - s) s + 12\alpha_0 (s - 3) (1 + s)] - 3}{[1 - 4\alpha_0 (1 + s)]^2} \phi^{2s},$$

(14)

where we have used Eq. \cite{12} in Eq. \cite{8}. In order not to have a vanishing potential $V(\phi)$, the coupling constant $\alpha_0$ must satisfy in this cosmological model the following relations

$$\alpha_0 \neq \frac{1}{12}, \alpha_0 \neq \frac{3}{4(s + 1) (5 - s)}.$$  

(15)
where we have denoted the arbitrary constant $\mu$ as $\phi_1 = (\alpha_1 (1-s))^{1/2}$. By inserting Eq. (17) into Eq. (11), thus the latter can be integrated to yield the scale factor as

$$a(t) = a_0 (t-t_0)^s, s \neq 1, 2, \quad (18)$$

where $a_0$ is an arbitrary constant of integration, and we have denoted the arbitrary constant $\mu$ as

$$\mu = \frac{\alpha_2}{\alpha_1 (1-s)} = \frac{1 + s - \frac{1}{\alpha_2}}{(s-2) (s-1)}. \quad (19)$$

If the constant $\alpha_0$ satisfies the relation $\alpha_0 = \frac{1}{4s(s+1)}$, then the FRW line element will reduce to the static case. Therefore we have completely obtained a cosmological solutions of the STT model, given by Eqs. (14), (17), (18), and describing the time evolution of the flat FRW universe in the presence of the scalar field $\phi$. In the next Section, we shall present the analytical solutions of the differential Eq. (7) by means of the Lie approach, subsequently leading to the complete solutions of the gravitational field equation describing the dynamics of the FRW universe in the presence of the scalar field in the framework of the STT.

### IV. STANDARD LIE GROUP APPROACH

It is known that a vector field $X$ defined as

$$X = \xi(x,y) \frac{\partial}{\partial x} + \eta(x,y) \frac{\partial}{\partial y}, \quad (20)$$

where $\xi(x,y)$ and $\eta(x,y)$ are two arbitrary functions is a symmetry of a differential equation, which is an invertible transformation that leaves it form-invariant. By applying the standard Lie procedure, one needs to solve the following over-determined system of the partial differential equations for the functions $\eta$ and $\xi$, based on the extended infinitesimal or prolonged transformations. Thus this approach allows us to determine the set of the symmetries admitted by the considered equation. By considering a general second order differential equation,

$$\phi'' = \psi(t, \phi, \phi'), \quad (21)$$

then we apply the standard procedures of Lie group analysis to Eq. (21). A vector field $X$ defined as

$$X = \xi(t, \phi) \frac{\partial}{\partial t} + \eta(t, \phi) \frac{\partial}{\partial \phi}, \quad (22)$$

is a symmetry of Eq. (21) if the arbitrary functions $\xi(x,y)$ and $\eta(x,y)$ satisfy the partial differential equation

$$-\xi f_t - \eta f_\phi + \eta_t (2\eta_\phi - \xi \eta_t) \phi' + (\eta_\phi - 2\xi_\phi) \phi'^2 - \xi_\phi \phi'^3 + (\eta \phi - 2\xi_\phi - 3\phi' \xi_\phi) f_\phi = 0. \quad (23)$$

The knowledge of one symmetry $X$ provides the form of a particular solution as an invariant of the operator $X$, i.e., a solution of the differential equation $\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)}$. This particular solution is known as an invariant solution (generalization of similarity solution). As a first step in our analysis of STT cosmologies, we rewrite Eq. (17) in the form

$$\phi' = \frac{1}{F'} \left[ \left( \frac{1}{2} - F'' \right) \phi'' - 2F\dot{H} + F' \dot{\phi} H \right]. \quad (24)$$

By applying the Lie group approach to Eq. (21), we immediately obtain the system of the partial differential equations

$$-2 F'^2 \xi'' + 2 F' (2 F'' - 1) \xi' = 0, \quad (25)$$

$$2 F'^2 \left( \eta'' - 2 \xi' + 2 H \xi' \right) + (2 F' F''' - 2 F'' + F''') \eta + F' (2 F'' - 1) \eta' = 0, \quad (26)$$

$$12 F' F' H \xi' + 2 F'^2 \left( 2 \eta'' - \xi' - \left( H \xi' + \dot{H} \xi \right) \right) + 2 F' (2 F'' - 1) \eta' = 0, \quad (27)$$

$$4 \left( F'^2 - F'' \right) \dot{H} \eta + 4 F F' \left( 2 \dot{H} \xi + \ddot{H} \xi - 4 \eta' \right) + 2 \dot{F} (\eta' - \eta \dot{H}) = 0. \quad (28)$$

In the following, we shall find two analytical solutions of Eq. (21) depicting the dynamics of the FRW universe, with the help of the system given by Eq. (25)–(28), subsequently leading to some classes of solutions of the gravitational field equations.
A. Solution 1

Firstly, by imposing the symmetry \([c, \phi]\) where \(c \in \mathbb{R}\), and using Eqs. \((25)-(28)\), we obtain the coupling function \(F(\phi)\), the Hubble parameter \(H(t)\) and the scale factor \(a(t)\) as

\[
F = m\phi^2, \quad H = \frac{a_1}{t}, \quad a(t) = t^{a_1},
\]
respectively where \(m, a_1 \in \mathbb{R}\). Now by inserting these results into Eq. \((24)\), we obtain the scalar field \(\phi(t)\) in the form

\[
\phi(t) = \phi_0 t^S (c_1 t^B - c_2)^\alpha,
\]
where we have defined the arbitrary constants \(S, B, \phi_0\) and \(\alpha\) as

\[
B = \sqrt{a_1^2 + \left(10 - \frac{1}{m}\right) a_1 + 1,} \quad 2S = \alpha \left(1 + a_1 - B\right),
\]
\[
\phi_0 = (B \alpha)^{\pm \alpha}, \quad \alpha = \frac{4m}{8m - 1},
\]
respectively, with \(c_1, c_2 \in \mathbb{R}\). Note that the invariant solution induced by the symmetry \([t, n\phi]\) yields the relation

\[
\phi = \phi_0 t^n,
\]
where we have defined the arbitrary constant \(n\) as \(2n = \alpha \left(1 + a_1 \pm B\right)\). One may find the exact value of constant \(n\) from Eq. \((24)\). For simplicity, we assume that \(c_1 = 0\), then from the field Eq. \((3)\), we obtain the potential \(V\) as

\[
V \sim t^n,
\]
the proportionality constant obtained from Eq. \((3)\) is complicated thus we shall not present its explicit form here, where we have denoted the arbitrary constant \(\gamma\) as

\[
\gamma = \frac{2}{8m - 1} \left[2m \left(a_1 - 3 - B\right) + 1\right].
\]

Note that the invariant solution is homothetic, since it verifies the relation obtained in the matter collineation approach, that is,

\[
L_H \left(F^{-1}(\phi) T_{\mu\nu}\right) = 0,
\]
where \(L_H\) is the homothetic vector field associated to the flat FRW metric, \(L\) is the Lie derivative, and the energy momentum tensor \((\phi) T_{\mu\nu}\) is defined as

\[
(\phi) T_{\mu\nu} = \phi_{, \mu} \phi_{, \nu} - \frac{1}{2} g_{\mu\nu} \phi_{, \sigma} \phi^{\sigma} + g_{\mu\nu} V.
\]

In view of the above equations we get the relations: \(F^{-1}\phi^2 = t^{-2}\) and \(F^{-1}V = t^{-2}\), which are verified for our invariant solution.

We have generated some plots for the scalar field \(\phi\) given by Eq. \((30)\) and the potential \(V\) given by Eq. \((19)\) with the help of Eq. \((29)\) for some numerical values of the constants \(c_1, c_2, m, a_1\) (see Fig. \((1)\)).

Therefore, for the numerical values corresponding to the red and black lines \(V < 0\) while for the blue line \(V\) starts in a negative region, but goes to a positive one. The parameters values for the magenta line give \(V > 0\). We have chosen two numerical values for the exponent of the scale factor, \(a_1 = 1/2\), corresponding to a positive deceleration parameter, \(q > 0\), and the value \(a_1 = 3/2\), indicating that the deceleration parameter is negative, \(q < 0\), that is, this choice represents an accelerating solution.

B. Solution 2

Again, by imposing the symmetry \([c, \phi]\) where \(c \in \mathbb{R}\), and using Eqs. \((25)-(28)\), we obtain the coupling function \(F(\phi)\), the Hubble parameter \(H(t)\) and the scale factor \(a(t)\) given by

\[
F(\phi) = m\phi^2, \quad H(t) = a_1, \quad a(t) = e^{a_1 t}.
\]
It is possible to obtain a more general expression for $F$ from Eqs. (24)-(28) as

$$F = m\dot{\phi}^2 - \frac{2e^{c_1c_2/2}}{c_1(c_1 - 4)}\phi^2 - c_1 + c_3.$$  

(38)

However, this expression for $F$ does not verify the field equations, and this reason we have chosen only the particular solution $F = m\dot{\phi}^2$. Then, by substituting these results into Eq. (24), thus we get the scalar field

$$\phi = \phi_0 \left\{ \frac{1}{ma} \left[ c_1 (8m - 1) e^{a t} + c_2 (m - 1) \right] \right\}^{\frac{1}{8m - 1}},$$

$$c_1, c_2 \in \mathbb{R}^-, \quad m < \frac{1}{8}, \quad \phi_0 = 4^{\frac{1}{8m - 1}},$$

(39)

while the invariant solution induced by the symmetry $[c, \dot{\phi}]$ yields the relation

$$\phi = \phi_0 e^{ct},$$

(40)

immediately by inserting Eqs. (37, 40) into Eq. (24) thus we obtain the constant $c$ given by $c = \frac{4m}{8m - 1}$.

For simplicity, assume that $c_2 = 0$, then from Eq. (39), we obtain the potential $V$ as

$$V \sim e^{\frac{8ma}{8m - 1} t},$$

(41)

again the proportionality constant obtained from Eq. (41) is complicated thus we shall not present its explicit form here.

We have generated some plots for the scalar field $\phi$ given by Eq. (49) and the potential $V$ given by Eq. (40) with the help of Eq. (37) for some numerical values of the constants $c_1, c_2, m$ and $a_1$ (see Fig. 2).

Therefore, for the parameters corresponding to the blue line $V$ starts in a negative region, but goes into the positive one. Red, magenta and black lines give $V > 0$. As before, we have chosen two numerical values for the exponent of the scale factor, $a_1 = 1/2$ and $a_1 = 3/2$, but the deceleration parameter is always negative, $q = -1 < 0$, that is, we have obtained an accelerating solution.

In the next Section, we shall present the analytical solutions of the differential Eq. (47) by using Noether symmetries, subsequently leading to some solutions of the gravitational field equation describing the dynamics of the FRW universe in the presence of the scalar field in the framework of the STT.

V. NOETHER SYMMETRIES

In order to obtain the solutions of the gravitational field equation through the Noether symmetries, we first consider that the Lagrangian $\mathcal{L}$ is defined as $\mathcal{L} = \mathcal{L} (a, \dot{a}, \phi, \dot{\phi})$, thus the configuration space $Q$ takes the form $Q = (a, \phi)$. Note that the vector field $X$ is defined as

$$X = \xi (t, a, \phi) \partial_t + \eta (t, a, \phi) \partial_a + \beta (t, a, \phi) \partial_\phi,$$

(42)

where we have denoted the arbitrary functions $\xi, \eta$ and $\beta$ as $\xi = \xi (t, a, \phi), \eta = \eta (t, a, \phi), \beta = \beta (t, a, \phi)$ respectively. Now the first prolongation formula for the vector field $X$ is given by

$$X^{[1]} = X + \dot{\eta} \partial_\phi + \ddot{\beta} \partial_\phi,$$

(43)

where we have introduced the following notations

$$\dot{\eta} = D_t \eta - \dot{a} D_t \xi, \quad \dot{\beta} = D_t \beta - \dot{\phi} D_t \xi, \quad D_t = \partial_t + \dot{a} \partial_a + \dot{\phi} \partial_\phi.$$  

(44)

It is worth to note that the vector field $X$ can generate a Noether symmetry if the Lagrangian $\mathcal{L}$ and the function $A$ defined as $A = A (t, a, \phi)$ satisfy the relation (see for instance $[50, 51, 52]$ and $[53]$)

$$X^{[1]} \mathcal{L} + D_t (\xi) \mathcal{L} = D_t A.$$  

(45)

For the specific case $A = 0$, the vector field $X$ generates the classical Noether symmetry. From the mathematical
point of view, the conserved quantity \( I \) can be obtained from

\[
I = \sum_i (\alpha_i - \xi \dot{q}_i) \frac{\partial L}{\partial \dot{q}_i} + \xi L - A, \tag{46}
\]

where \( \alpha_1 = \xi, \alpha_2 = \eta \) and \( \alpha_3 = \beta \). In view of Eqs. \( 3 \) and \( 45 \), we get a system of the partial differential equations

\[
\partial_t \xi = 0 = \partial_\phi \xi, \tag{47}
\]

\[
F \eta + F' a \beta + 2 F a \dot{a} \eta + F' a^2 \partial_a \beta - F a \partial_a \xi = 0, \tag{48}
\]

\[
\frac{3}{2} a^2 \eta + 6 F' a^2 \partial_\phi \eta + a^3 \beta \partial_\phi \beta - \frac{1}{2} a^3 \partial_\phi \xi = 0, \tag{49}
\]

\[
12 F' a \eta + 6 F'' a^2 \beta + 12 F a \partial_a \eta + 6 F' a^2 (\partial_a \beta - \partial_\phi \xi) + a^3 \partial_\phi \beta = 0, \tag{50}
\]

\[
2 F a \partial_\phi \eta + F' a^2 \partial_\phi \beta = \partial_\phi A, \tag{51}
\]

\[
6 F' a^2 \partial_\phi \eta + a^3 \partial_\phi \beta = \partial_\phi A, \tag{52}
\]

\[
-3 a^2 V \eta - a^3 V' \beta - a^3 V \partial_\phi \xi = \partial_\phi A. \tag{53}
\]

Furthermore, if the conserved quantity \( I \) vanishes, then from Eq. \( 58 \) we obtain the relation

\[
\phi_{\pm}(a) = \pm \sqrt{C_{1\pm} a^{-m/3}}, \tag{60}
\]

The invariant solution is given by \( \frac{da}{a} = -\frac{2d\phi}{3\phi} \), thus we get: \( \phi = a^{-3/2} \). By substituting this result together with \( F = m\phi^2 \) and \( V = V_0 \phi^2 \), for example, from Eq. \( 24 \), we obtain the relation \( \phi = e^{-K_1} \) where \( K \) is a constant that depends on \( a \). Therefore this invariant solution is the same as the invariant one obtained in the second solution through the Lie group method. Note that the Noether symmetries generate an algebra which is a subalgebra of the Lie algebra.

Next in order to obtain the general solutions of the gravitational field equations, we introduce two arbitrary functions \( w(a, \phi) \) and \( z(a, \phi) \) induced by the vector field \( X \) defined as

\[
w(a, \phi) = a^{3/2} \phi, \quad z(a, \phi) = -\frac{3}{2} \ln a + a^{3/2} \phi, \tag{61}
\]

respectively. For mathematical convenience, we rewrite Eqs. \( 61 \) in the forms

\[
a(w, z) = e^{\frac{3}{2}(w-z)}, \quad \phi(w, z) = w e^{z-w} \tag{62}.
\]

Now by inserting the relations \( F = m\phi^2 \), \( V = V_0 \phi^2 \), \( m = 3/32 \) and Eqs. \( 59 \) into Eq. \( 4 \), then the Lagrangian \( 3 \) takes the simpler form

\[
\mathcal{L} = \frac{a w}{4} (\dot{z} - \dot{w}) + \frac{w^2}{2} - V_0 w^2. \tag{63}
\]

Note that the variable \( z \) here is cyclic. Next using the Euler-Lagrangian equations, thus we obtain the arbitrary functions \( w(t) \) and \( z(t) \) given by

A. Solution 1 with \( A = 0 \)

With the help of Eqs. \( 17 \)–\( 33 \), we obtain the following results

\[
F = ma^2, \quad \eta = -\frac{2}{3} ac, \quad \beta = c_2 \phi, \quad V = V_0 a^2, \quad \xi = 0, \quad m, c_2, V_0 \in \mathbb{R}. \tag{54}
\]

The conserved quantity \( I \) is given by Eq. \( 40 \), where in case it reads

\[
I = \eta \frac{\partial L}{\partial \dot{a}} + \beta \frac{\partial L}{\partial \phi}. \tag{55}
\]

By substituting Eqs. \( 21 \)–\( 33 \) into Eq. \( 55 \), the latter yields the following differential equation

\[
I = c_2 \left[ 4m \phi^2 a^2 \dot{a} + (1 - 8m) a^3 \phi \right]. \tag{56}
\]

In order to find the scalar field \( \phi \) explicitly, thus we rewrite Eq. \( 56 \) as a Bernoulli differential equation for \( \phi^2 \) in the form

\[
\frac{d}{da} \phi^2 + \frac{8m}{8m - 1} a \phi^2 = \frac{2I}{c_2 (1 - 8m) a^3 \frac{da}{dt}}, \tag{57}
\]

Eq. \( 57 \) can be integrated to give the scalar field

\[
\phi_{\pm}(a, t) = \pm a^{\frac{3m}{2}} \left[ C_{1\pm} + \frac{2}{c_2 (1 - 8m)} \int a^{\frac{3m}{2}} dt \right], \tag{58}
\]

where \( C_{1\pm} \) are the arbitrary constants of integration. Consider that the conserved quantity \( I \) is a constant and \( m \) is 3/32, then from Eq. \( 58 \), we get the result

\[
\phi_{\pm}(a, t) = \pm a^{-\frac{3}{2}} \left[ C_{1\pm} + \frac{8I}{c_2 t} \right]. \tag{59}
\]
We have plotted the quantities $a$, $\phi$, $q$ defined as $q = \frac{1}{3} \phi - 1$, for different values of the constants $c$, $b$, $c_1$, $c_2$ and $V_0$. See Fig. 3.

\begin{equation}
\frac{\partial}{\partial \tau} + \tau = 0.
\end{equation}

We have obtained another parametrization by using the Lie group method (Solution 1), since it verifies the relations: $F^{-1}V = t^{-2}$, and $F^{-1}\phi^2 = t^{-2}$. Now using Eq. (10) and $c_1 = 1$, $c_2 = 0$, thus it is clear to write down the conserved quantity

\begin{equation}
I_1 = \left[ \frac{n+2}{3(n-2)} a - \dot{a} \right] \frac{\partial L}{\partial a} + \left( \frac{2}{2-n} - \dot{\phi} \right) \frac{\partial L}{\partial \phi} + tL - A,
\end{equation}

\begin{equation}
w(t) = \sqrt{ct + b}, \quad z(t) = -\ln(ct + b) + (ct + b)^{1/2} - 4V_0t^2 + c_1t + c_2.
\end{equation}

B. Solution 2 with $A = 0$

This solution has been already obtained one by Capozziello and the Ritis [31], with $A = 0$:

\begin{equation}
F = m\phi^2, \quad \eta = -c_2 K a^n, \quad \beta = c_2 a^b, \quad V = V_0 \phi^2.
\end{equation}

where

\begin{equation}
K = \frac{\sqrt{3}(2\sqrt{m} + \sqrt{12m - 1})}{6\sqrt{m}}, \quad n_1 = -\frac{1}{2} - \frac{\sqrt{3m}}{\sqrt{12m - 1}},
\end{equation}

\begin{equation}
n_2 = -1 + \frac{\sqrt{3}(8m - 1)}{4\sqrt{12m} - m}, \quad b_1 = -9 - \frac{2\sqrt{3m}}{\sqrt{12m - 1}},
\end{equation}

\begin{equation}
b_2 = \frac{\sqrt{3}(8m - 1)}{4\sqrt{m}\sqrt{12m - 1}}, \quad \lambda = 3 + \frac{\sqrt{3m}}{\sqrt{12m - 3}}.
\end{equation}

In [31] the authors studied in particular the case $m = \frac{3}{32}$. We have only obtained another parametrization by using a Lie point symmetry approach instead of the geometrical one followed from [31] (see also [13]).

C. Solution 3 with $A \neq 0$

A third set of solutions corresponds to the choices

\begin{equation}
F = m\phi^2, \quad V = \lambda \phi^n, \quad \xi = c_1 t + c_2,
\end{equation}

\begin{equation}
\eta = \frac{(n+2)c_1}{3(n-2)} a, \quad \beta = -\frac{2c_1}{n-2} \phi, \quad A = c_3.
\end{equation}

Since $n \neq 2$, it follows that

\begin{equation}
X = (c_1 t + c_2) \partial_t + \frac{(n+2)c_1}{3(n-2)} a \partial_a - \frac{2c_1}{n-2} \phi \partial_\phi,
\end{equation}

and therefore we obtain

\begin{equation}
X_1 = t \partial_t + \frac{n+2}{3(n-2)} a \partial_a - \frac{2}{n-2} \phi \partial_\phi, \quad X_2 = \partial_t.
\end{equation}

As one can observe easily, $X_1$ induces the following invariant solution,

\begin{equation}
a = a_0 t^{\frac{n+2}{3(n-2)}}, \quad \phi = \phi_0 t^{-\frac{2}{n-2}},
\end{equation}

which is the self-similar solution already obtained by using the Lie group method (Solution 1), since it verifies the relations: $F^{-1}V = t^{-2}$, and $F^{-1}\phi^2 = t^{-2}$. Now using Eq. (10) and $c_1 = 1$, $c_2 = 0$, thus it is clear to write down the conserved quantity

\begin{equation}
I_1 = \left[ \frac{n+2}{3(n-2)} a - \dot{a} \right] \frac{\partial L}{\partial a} + \left( \frac{2}{2-n} - \dot{\phi} \right) \frac{\partial L}{\partial \phi} + tL - A,
\end{equation}
for calculational convenience, we rewrite Eq. (71) in the form

\[ I_1 = \frac{1}{n-2} \left( \frac{n+2}{3} a \frac{\partial \mathcal{L}}{\partial a} - 2 \phi \frac{\partial \mathcal{L}}{\partial \phi} \right) + t \left( \mathcal{L} - \frac{\partial \mathcal{L}}{\partial a} - \dot{\phi} \frac{\partial \mathcal{L}}{\partial \phi} \right) - A. \]  

(72)

With the help of Eqs. (3) and (67), Eq. (72) takes the form

\[ I_1 = \frac{2ma^2 \phi}{n-2} \left[ 2(n-4) \phi \dot{a} + (2n+3) a \dot{\phi} \right] + t \left[ -6ma \dot{a} \left( \phi \dot{a} + 2a \dot{\phi} \right) - \frac{1}{2} a^3 \dot{\phi}^2 - a^3 \lambda \phi^3 - A \right]. \]  

(73)

Similarly, another conserved quantity \( I_2 \) takes the form

\[ I_2 = - \frac{a}{\dot{a}} \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial \mathcal{L}}{\partial \phi} + \mathcal{L} - A. \]  

(74)

In view of Eq. (3), thus we obtain the relation

\[ I_2 = -6ma \dot{a} \left( \phi \dot{a} + 2a \dot{\phi} \right) - \frac{1}{2} a^3 \dot{\phi}^2 - a^3 \lambda \phi^3 - A, \]  

(75)

explicitly. Note that from Eq. (5), we get

\[ -6ma \dot{a} \left( \phi \dot{a} + 2a \dot{\phi} \right) - \frac{1}{2} a^3 \dot{\phi}^2 - a^3 \lambda \phi^3 = 0, \]  

(76)

and therefore the relation \( I_2 = -A \) holds. Thus we get the result

\[ \dot{I}_1 = \frac{2ma^2 \phi}{n-2} \left[ 2(n-4) \phi \dot{a} + (2n+3) a \dot{\phi} \right], \]  

(77)

where we have denoted \( \dot{I}_1 = I_1 + A \). In order to solve Eq. (71) for the scalar field \( \phi \), thus we rewrite Eq. (71) as a Bernoulli differential equation for \( \phi^2 \) in the form

\[ \frac{d}{da} \phi^2 + \frac{4(n-4)}{2n+3} \frac{1}{a} \phi^2 = \frac{n-2}{(2n+3) m a^3} \frac{\dot{I}_1}{d}, \]  

(78)

Eq. (78) can be integrated to give the scalar field

\[ \phi_{\pm} (a, t) = \pm \sqrt{a^{2(n-4)} \int C_{2\pm} + \frac{n-2}{(2n+3) m} \int \dot{I}_1 a^{-\frac{2n+5}{(2n+3)}} dt}, \]  

(79)

where \( C_{2\pm} \) are the arbitrary constants of integration. Consider that the quantity \( \dot{I}_1 \) vanishes, that is \( I_1 = -A = -c_3 \) then from Eq. (79), we get the relation

\[ \phi_{\pm} (a) = \pm \sqrt{C_{2\pm} a^{\frac{2(n-4)}{2n+3}}}. \]  

(80)

For the special case \( n = 4 \), then the scalar field \( \phi \) becomes an arbitrary constant, thus from Eq. (4), we obtain the scale factor as

\[ a(t) = e^{\frac{\sqrt{-6m \lambda a C_{2\pm}(t-c_2)}}{6m}}. \]  

(81)

In order to have the realistic cosmological model, therefore we have chosen the relations \( c_1 < 0, \lambda > 0 \) and \( m > 0 \).

VI. CONCLUSIONS

Scalar fields play an important role in the explanation of the observational aspects of present day cosmology. In particular, scalar fields are responsible for the early inflationary expansion of the Universe, and they also represent a powerful candidate for the dark energy determining the recent accelerated expansion of the Universe. In the present work, by using the standard procedures, Lie group approach and Noether symmetry techniques, we have obtained several analytical solutions of the gravitational field equations describing the time evolutions of a flat Friedman-Robertson-Walker Universe in the presence of scalar fields.

With regard to the Noether symmetry method, in the present paper we have employed an approach, that allows us to obtain more symmetries thanks to the function \( A \) (a boundary term introduced into the action to keep the action invariant by \( X \)). An alternative method is the geometrical one (see [54] for an excellent review of this method). To obtain exact cosmological solutions in generalized gravity theories, which can be, for example, scalar-tensor theories with a non-minimally (or minimally) coupled scalar field, or higher-order theories (that is theories whose Lagrangian is not simply linear in Ricci scalar), one can use a method specifically developed to this purpose, which is called the Noether Symmetry Approach [54]. This method can be described as follows. If a (point-like) Lagrangian is given in terms of some variables (i.e., in the cosmological case, in terms of the scale factor of the Universe \( a \) and a scalar field \( \phi \)), one can search for a Noether symmetry of such a Lagrangian, which contains an a priori unspecified potential \( V \), or, in the non-minimal coupling case, an unspecified coupling \( F(\phi) \), and a potential \( V(\phi) \), respectively. Hence in this approach one relates the possible existence of the Noether symmetry (that is, the existence of a constant of motion) with the selection of specific forms of \( V(\phi) \), or of \( F(\phi) \) and \( V(\phi) \), respectively. In this way, the problem of solving the model is connected with the specification of \( V(\phi) \) (or \( F(\phi) \) and \( V(\phi) \)). The existence of symmetries selects couplings and potentials of physical interest as \( V(\phi) = \lambda \phi^4 \) or \( F(\phi) = k_0 \phi^2 \) [54], respectively, with \( \lambda \) and \( k_0 \) constants.

Hence by using Noether symmetry approaches a large number of cosmological models of physical interest can.
be found, and their properties can be tested with the astrophysical observations.

The solutions we have found show a large variety of cosmological behaviors, ranging from non-accelerating (decelerating) solutions to accelerating ones. The scalar field potential can be also obtained, and it is basically determined by symmetry and/or invariance considerations. Our solutions can also be interpreted as the background on which to compare the observational data from large-scale structure formation, from the Cosmic Microwave Background Radiation anisotropies [8], or on which to formulate a theory of cosmological perturbations in modified gravity models. For example, our solutions show that it is possible to control the cosmological background by a free parameter, so that it is possible to select interesting scales useful for large-cosmological-structure formation. Generally, the potentials have either a power law type dependence on the scalar field, like in the case of the first analytic solution, or their time dependence can be obtained explicitly. In this latter case we also have a large variety of behaviors, with both power law and exponential time evolution of V. Hence the results of the present study indicate that simple exact cosmological models can be obtained in the framework of scalar field cosmologies by using the powerful methods of Lie group analysis, and Noether symmetry approach. The cosmological implications of the present results will be investigated in detail in a future publication.

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