Statistical properties and decoherence of two-mode photon-subtracted squeezed vacuum

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We investigate the statistical properties of the photon-subtractions from the two-mode squeezed vacuum state and its decoherence in a thermal environment. It is found that the state can be considered as a squeezed two-variable Hermite polynomial excitation vacuum and the normalization of this state is the Jacobi polynomial of the squeezing parameter. The compact expression for Wigner function (WF) is also derived analytically by using the Weyl ordered operators' invariance under similar transformations. Especially, the nonclassicality is discussed in terms of the negativity of WF. The effect of decoherence on this state is then discussed by deriving the analytical time evolution results of WF. It is shown that the WF is always positive for any squeezing parameter and photon-subtraction number if the decay time exceeds an upper bound ($\kappa t > \frac{1}{2} \ln \frac{2\kappa}{\kappa + 2}$).

Key Words: photon-subtraction; nonclassicality; Wigner function; negativity; two-mode squeezed vacuum state
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I. INTRODUCTION

Entanglement is an important resource for quantum information \cite{1}. In a quantum optics laboratory, Gaussian states, being characteristic of Gaussian Wigner functions, have been generated but there is some limitation in using them for various tasks of quantum information processing \cite{2}. For example, in the first demonstration of continuous variables quantum teleportation (two-mode squeezed vacuum state as a quantum channel), the squeezing is low, thus the entanglement of the quantum channel is such low that the average fidelity of quantum teleportation is just more than the classical limits. In order to increase the quantum entanglement there have been suggestions and realizations to engineering the quantum state by subtracting or adding photons from/to a Gaussian field which are plausible ways to conditionally manipulate nonclassical state of optical field \cite{3,4,5,6,7,8}. In fact, such methods allowed the preparation and analysis of several states with negative Wigner functions, including one- and two-photon Fock states \cite{10,11,12,13}, delocalized single photons \cite{14,15}, and photon-subtracted squeezed vacuum states (PSSV), very similar to quantum superpositions of coherent states with small amplitudes (a Schrödinger kitten state \cite{16,17,18,19}) for single-mode case.

Recently, the two-mode PSSVs (TPSSVs) have been paid enough attention by both experimentalists and theoreticians \cite{10,11,20,21,22,23,24,25,26,27,28,29}. Olivares et al. \cite{20,21} considered the photon subtraction using on-off photodetectors and showed the improvement of quantum teleportation depending on various parameters involved. Then they further studied the nonlocality of photon subtraction state in the presence of noise \cite{22}. Kitagawa et al. \cite{23}, on the other hand, investigated the degree of entanglement for the TPSSV by using an on-off photodetector. Using operation with single photon counts, Ourjoumtsev et al. \cite{10,11} have demonstrated experimentally that the entanglement between Gaussian entangled states, can be increased by subtracting only one photon from two-mode squeezed vacuum states. The resulted state is a complex quantum state with a negative two-mode Wigner function. However, so far as we know, there is no report about the nonclassicality and decoherence of TPSSV for arbitrary number PSSV in literature before.

In this paper, we will explore theoretically the statistical properties and decoherence of arbitrary number TPSSV. This paper is arranged as follows: in Sect. II we introduce the TPSSV, denoted as $a_m^a b_n^b S_2(\lambda) \, |00\rangle$, where $S_2(\lambda)$ is two-mode squeezing operator with $\lambda$ being squeezing parameter and $m, n$ are the subtracted photon number from $S_2(\lambda) \, |00\rangle$ for mode $a$ and $b$, respectively. It is found that it is just a squeezed two-variable Hermite polynomial excitation on the vacuum state, and then the normalization factor for $a_m^a b_n^b S_2(\lambda) \, |00\rangle$ is derived, which turns out to be a Jacobi polynomial, a remarkable result. In Sec. III, the quantum statistical properties of the TPSSV, such as distribution of photon number, squeezing properties, cross-correlation function and antibunching, are calculated analytically and then be discussed in details. Especially, in Sec. IV, the explicit analytical expression of Wigner function (WF) of the TPSSV is derived by using the Weyl ordered operators' invariance under similar transformations, which is related to the two-variable Gaussian-Hermite polynomials, and then its nonclassicality is discussed in terms of the negativity of WF which implies the highly nonclassical properties of quantum states. Sec. V is devoted to studying the effect of the decoherence on the TPSSV in a thermal environment. The analytical expressions for the time-evolution of the state and its WF are derived, and the loss of nonclassicality is discussed in reference of the negativity of WF due to decoherence. We find that the WF for TPSSV has no chance to present negative value for all parameters $\lambda$ and...
we can reform Eq. (2) as

\[ |\lambda, m, n\rangle = a^m b^n S_2(\lambda) |00\rangle, \] (2)

where \(|\lambda, m, n\rangle\) is an un-normalization state. Noticing the transform relations,

\[ S_2^\dagger(\lambda)aS_2(\lambda) = a \cosh \lambda + b^\dagger \sinh \lambda, \]
\[ S_2^\dagger(\lambda)bS_2(\lambda) = b \cosh \lambda + a^\dagger \sinh \lambda, \] (3)

we can reform Eq. (2) as

\[ |\lambda, m, n\rangle = S_2(\lambda) S_2^\dagger(\lambda) a^m b^n S_2(\lambda) |00\rangle \]
\[ = S_2(\lambda) (a \cosh \lambda + b^\dagger \sinh \lambda)^m (b \cosh \lambda + a^\dagger \sinh \lambda)^n |00\rangle \]
\[ = S_2(\lambda) \sinh^{m+n} \lambda \sum_{l=0}^{m} \frac{m! \text{coth} \lambda b^{m-l} a^l}{l! (m-l)!} |00\rangle. \] (4)

Further note that \(a^n |0\rangle = \sqrt{n!} |n\rangle\) and \(a^\dagger |n\rangle = \frac{\sqrt{n}}{(n-l)!} a^{l-n} |0\rangle\), leading to \(a^l a^n |0\rangle = \frac{n!}{(n-l)!} a^{l-n} |0\rangle\), thus Eq. (4) can be re-expressed as

\[ |\lambda, m, n\rangle = S_2(\lambda) \sinh^{m+n} \lambda \sum_{l=0}^{\min(m,n)} \frac{m! \text{coth} \lambda b^{m-l} a^l |00\rangle}{l! (m-l)! (n-l)!} \]
\[ = \frac{\sinh^{(m+n)/2} \lambda}{(i\sqrt{2})^{n+m}} S_2(\lambda) \sum_{l=0}^{\min(m,n)} \frac{(-1)^l l! (i \sqrt{\text{tanh} \lambda b^l})^{m-l} (i \sqrt{\text{tanh} \lambda a^l})^{n-l}}{(n-l)! (m-l)!} |00\rangle \]
\[ = \frac{\sinh^{(m+n)/2} \lambda}{(i\sqrt{2})^{n+m}} S_2(\lambda) H_{m,n} \left( i \sqrt{\text{tanh} \lambda b^l}, i \sqrt{\text{tanh} \lambda a^l} \right) |00\rangle, \] (5)

where in the last step we have used the definition of the two variables Hermitian polynomials [33, 34], i.e.,

\[ H_{m,n}(\epsilon, \zeta) = \sum_{k=0}^{\min(m,n)} \frac{(-1)^k m! n! \epsilon^{m-k} \zeta^{n-k}}{k!(m-k)!(n-k)!}. \] (6)

From Eq. (5) one can see clearly that the TPSSV \(|\lambda, m, n\rangle\) is equivalent to a two-mode squeezed two-variable Hermite-excited vacuum state and exhibits the exchanging symmetry, namely, interchanging \(m \leftrightarrow n\) is equivalent to \(a^\dagger \leftrightarrow b^\dagger\).

It is clear that, when \(m = n = 0\), Eq. (5) just reduces to the two-mode squeezed vacuum state due to \(H_{0,0} = 1\); while for \(n \neq 0\) and \(m = 0\), noticing \(H_{0,n} (x,y) = y^n\), Eq. (5) becomes \((N_{\lambda,0,n}^{-1} = n! \sinh^{2n} \lambda, \) see Eq. (11) below) \(|\lambda, 0, n\rangle = S_2(\lambda) |n, 0\rangle\), which is just a squeezed number state, corresponding to a pure negative binomial state [35].
B. The normalization of $|\lambda, m, n\rangle$

To know the normalization factor $N_{\lambda,m,n}$ of $|\lambda, m, n\rangle$, let us first calculate the overlap $\langle \lambda, m + s, n + t |\lambda, m, n\rangle$. For this purpose, using the first equation in Eq. (5), one can express $|\lambda, m, n\rangle$ as

$$|\lambda, m, n\rangle = S_2(\lambda) \sum_{l=0}^{\min(m,n)} \frac{m!n! \sinh^{n+m} \lambda \coth^{l} \lambda}{l!(n-l)!l!(m-l)!} |n-l, m-l\rangle,$$

(7)

which leading to

$$\langle \lambda, m + s, n + t |\lambda, m, n\rangle = m! (n+s)! \delta_{s,t} \sinh^{2n+2m+2s} \lambda \sum_{l=0}^{\min(m,n)} \frac{(m+s)!n! \coth^{2l+s} \lambda}{l!(n-l)!l!(l+s)!},$$

(8)

where $\delta_{s,t}$ is the Kronecker delta function. Without losing the generality, supposing $m < n$ and comparing Eq. (8) with the standard expression of Jacobi polynomials [36], one can put Eq. (8) into the following form

$$P^{(n,\beta)}_m(x) = \left(\frac{x-1}{2}\right)^m \sum_{k=0}^{m} \binom{m+\alpha}{k} \binom{m+\beta}{m-k} \left(\frac{x+1}{x-1}\right)^k,$$

(9)

which is just related to Jacobi polynomials. In particular, when $s = t = 0$, the normalization constant $N_{m,n,\lambda}$ for the state $|\lambda, m, n\rangle$ is given by

$$N_{\lambda,m,n} = \langle \lambda, m, n |\lambda, m, n\rangle = m!n! \sinh^{2n} \lambda P^{(n-m,0)}_m(\cosh 2\lambda),$$

(10)

which is important for further studying analytically the statistical properties of the TPSSV. For the case $m = n$, it becomes Legendre polynomial of the squeezing parameter $\lambda$, because of $P^{(0,0)}_n(x) = P_n(x)$, $P_0(x) = 1$; while for $n \neq 0$ and $m = 0$, noticing that $P^{(n,0)}_0(x) = 1$ then $N_{\lambda,0,n} = n! \sinh^{2n} \lambda$. Therefore, the normalized TPSSV is

$$|\lambda, m, n\rangle \equiv \left[m!n! \sinh^{2n} \lambda P^{(n-m,0)}_m(\cosh 2\lambda)\right]^{-1/2} a^m b^n S_2(\lambda) |00\rangle.$$

(12)

From Eqs. (2) and (11) we can easily calculate the average photon number in TPSSV (denoting $\tau = \cosh 2\lambda$),

$$\langle a^\dagger a \rangle = N^2_{\lambda,m,n} \langle 00 |S_2^\dagger(\lambda)a^{i m+1}b^{i n}a^{n+1}b^n S_2(\lambda) |00\rangle$$

$$= (m + 1) \frac{P^{(n-m-1,0)}_{m+1}(\tau)}{P^{(n-m,0)}_m(\tau)},$$

(13)

$$\langle b^\dagger b \rangle = (n + 1) \sinh^{2n} \lambda \frac{P^{(n-m+1,0)}_m(\tau)}{P^{(n-m,0)}_m(\tau)}.$$  

(14)

In a similar way we have

$$\langle a^\dagger b^\dagger a b \rangle = (m + 1) (n + 1) \sinh^{2n} \lambda \frac{P^{(n-m,0)}_m(\tau)}{P^{(n-m,0)}_m(\tau)}.$$  

(15)

Thus the cross-correlation function $g^{(2)}_{12}$ can be obtained by [37,38,39]

$$g^{(2)}_{12}(\lambda) = \frac{\langle a^\dagger b^\dagger a b \rangle}{\langle a^\dagger a \rangle \langle b^\dagger b \rangle} = \frac{P^{(n-m,0)}_{m+1}(\tau)}{P^{(n-m-1,0)}_m(\tau)} \frac{P^{(n-m,0)}_m(\tau)}{P^{(n-m+1,0)}_m(\tau)},$$

(16)
and leads to

From Eqs. (13), (14) and (18) it then follows that

\[ Q = \left( \frac{Q_1 + Q_2}{\sqrt{2}} \right), \quad P = \left( \frac{P_1 + P_2}{\sqrt{2}} \right), \quad [Q, P] = 1, \]  

where \( Q_1 = (a + a^\dagger)/\sqrt{2}, \) \( P_1 = (a - a^\dagger)/(\sqrt{2}i), \) \( Q_2 = (b + b^\dagger)/\sqrt{2} \) and \( P_2 = (b - b^\dagger)/(\sqrt{2}i) \) are coordinate- and momentum- operator, respectively. Their variances are \( (\Delta Q)^2 = \langle Q^2 \rangle - \langle Q \rangle^2 \) and \( (\Delta P)^2 = \langle P^2 \rangle - \langle P \rangle^2. \) The phase amplifications satisfy the uncertainty relation of quantum mechanics \( \Delta Q \Delta P \geq \frac{1}{2}. \) By using Eqs. (10) and (11), it is easy to see that \( \langle a \rangle = \langle a^\dagger \rangle = \langle b \rangle = \langle b^\dagger \rangle = 0 \) and \( \langle a^2 \rangle = \langle a^2 \rangle = \langle b^2 \rangle = \langle b^2 \rangle = 0 \) as well as \( \langle ab \rangle = \langle a b \rangle = 0, \) which leads to \( \langle Q \rangle = 0, \langle P \rangle = 0. \) Moreover, using Eq. (10) one can see

\[ \langle a^\dagger b \rangle = \langle ab \rangle = \frac{n + 1}{2} \frac{P_{m}^{(n-m,0)}(\tau)}{P_{m}^{(n-m,0)}(\tau)} \sinh 2\lambda. \]  

From Eqs. (13), (14) and (18) it then follows that

\[
(\Delta Q)^2 = \frac{1}{2}(\langle a^\dagger a \rangle + \langle b^\dagger b \rangle + \langle ab \rangle + \langle a^\dagger b^\dagger \rangle + 1)
\]

\[
= \frac{1}{2P_{m}^{(n-m,0)}(\tau)}[(m + 1) P_{m+1}^{(n-m-1,0)}(\tau) + (n + 1) P_{m}^{(n-m+1,0)}(\tau) \sinh^2 \lambda
\]

\[
+ (n + 1) P_{m}^{(n-m,1)}(\tau) \sinh 2\lambda + P_{m}^{(n-m,0)}(\tau)],
\]  

and

\[
(\Delta P)^2 = \frac{1}{2}(\langle a a \rangle + \langle b b \rangle - \langle ab \rangle - \langle a^\dagger b^\dagger \rangle + 1)
\]

\[
= \frac{1}{2P_{m}^{(n-m,0)}(\tau)}[(m + 1) P_{m+1}^{(n-m-1,0)}(\tau) + (n + 1) P_{m}^{(n-m+1,0)}(\tau) \sinh^2 \lambda
\]

\[
- (n + 1) P_{m}^{(n-m,1)}(\tau) \sinh 2\lambda + P_{m}^{(n-m,0)}(\tau)].
\]  

FIG. 1: (Color online) Cross-correlation function between the two modes \( a \) and \( b \) as a function of \( \lambda \) for different parameters \( (m, n). \) The number 1,2,3,4,5,6 in (a) denote that \( (m, n) \) are equal to \((1,2), (3,4), (2,4), (6,8), (3,6)\) and \((7,10)\) respectively.

Actually, the cross-correlation between the two modes reflects correlation between photons in two different modes, which plays a key role in rendering many two-mode radiations nonclassically. In Fig. 1, we plot the graph of \( g_{12}^{(2)}(\lambda) \) as the function of \( \lambda \) for some different \((m, n)\) values. It is shown that \( g_{12}^{(2)}(\lambda) \) are always larger than unit, thus there exist correlations between the two modes. We emphasize that the WF has negative region for all \( \lambda, \) and thus the TPSSV is nonclassical. In our following work, we pay attention to the ideal TPSSV.

III. QUANTUM STATISTICAL PROPERTIES OF THE TPSSV

A. Squeezing properties

For a two-mode system, the optical quadrature phase amplitudes can be expressed as follows:

\[
Q = \frac{Q_1 + Q_2}{\sqrt{2}}, \quad P = \frac{P_1 + P_2}{\sqrt{2}}, \quad [Q, P] = 1,
\]  

where \( Q_1 = (a + a^\dagger)/\sqrt{2}, \) \( P_1 = (a - a^\dagger)/(\sqrt{2}i), \) \( Q_2 = (b + b^\dagger)/\sqrt{2} \) and \( P_2 = (b - b^\dagger)/(\sqrt{2}i) \) are coordinate- and momentum- operator, respectively. Their variances are \( (\Delta Q)^2 = \langle Q^2 \rangle - \langle Q \rangle^2 \) and \( (\Delta P)^2 = \langle P^2 \rangle - \langle P \rangle^2. \) The phase amplifications satisfy the uncertainty relation of quantum mechanics \( \Delta Q \Delta P \geq \frac{1}{2}. \)
Next, let us analyze some special cases. When \( m = n = 0 \), corresponding to the two-mode squeezed state, Eqs. (19) and (20) becomes, respectively, to

\[
(\Delta Q)^2|_{m=n=0} = \frac{1}{2} e^{2\lambda}, \quad (\Delta P)^2|_{m=n=0} = \frac{1}{2} e^{-2\lambda}, \quad \Delta Q \Delta P = \frac{1}{2},
\]

(21)

which is just the standard squeezing case; while for \( m = 0, n = 1 \), Eqs. (19) and (20) reduce to

\[
(\Delta Q)^2|_{m=0, n=1} = e^{2\lambda}, \quad (\Delta P)^2|_{m=0, n=1} = e^{-2\lambda}, \quad \Delta Q \Delta P = 1,
\]

(22)

from which one can see that the state \( ||\lambda, m, n|| \) is squeezed at the "p-direction" when \( e^{-2\lambda} < \frac{1}{2} \), i.e., \( \lambda > \frac{1}{2} \ln 2 \). In addition, when \( m = n = 1 \), in a similar way, one can get

\[
(\Delta Q)^2|_{m=n=1} = \frac{1}{2} e^{2\lambda} \left( 1 + \frac{2e^{2\lambda} - 2}{e^{2\lambda} + e^{-2\lambda}} \right),
\]

\[
(\Delta P)^2|_{m=n=1} = \frac{1}{2} \left[ 1 - \frac{e^{2\lambda} - 3e^{-2\lambda} (1 - e^{-2\lambda})}{e^{2\lambda} + e^{-2\lambda}} \right] + \frac{1}{2} < \frac{1}{2},
\]

(23)

which indicates that, for any squeezing parameter \( \lambda \), there always exist squeezing effect for state \( ||\lambda, 1, 1|| \) at the "p-direction".

In order to see clearly the fluctuations of \( (\Delta P)^2 \) with other parameters \( m, n \) values, the figures are plotted in Fig.2. From Fig.2(a) one can see that the fluctuations of \( (\Delta P)^2 \) are always less than \( \frac{1}{2} \) when \( m = n \), say, the state \( ||\lambda, m, m|| \) is always squeezed at the "p-direction"; for given \( m \) values, there exist the squeezing effect only when the squeezing parameter exceeds a certain threshold value that increases with the incrementation of \( n \) (see Fig.2(b)).

### B. Distribution of photon number

In order to obtain the photon number distribution of the TPSSV, we begin with evaluating the overlap between two-mode number state \( |n_a, n_b\rangle \) and \( ||\lambda, m, n|| \). Using Eq. (11) and the un-normalized coherent state \[ |\lambda, m, n\rangle = \exp \left( za^\dagger \right) |0\rangle \], leading to \( \langle n | = \frac{1}{\sqrt{n!}} \frac{\partial^n}{\partial^n z^*} \langle z | \rangle_{z^*=0} \), it is easy to see that

\[
\langle n_a, n_b | \lambda, m, n \rangle \\
= \text{sech} \lambda \langle n_a | a^m b^n e^{a^\dagger b^\dagger \tanh \lambda} |0\rangle \\
= \frac{(m + n_a)!}{\sqrt{n_a! n_b!}} \text{sech} \lambda \tanh^{m+n_a} \lambda \delta_{m+n_a, n+b},
\]

(24)

It is easy to follow that the photon number distribution of \( ||\lambda, m, n|| \), i.e.,

\[
P(n_a, n_b) = N_{\lambda, m, n}^{-1} |\langle n_a, n_b | \lambda, m, n \rangle|^2 \\
= \left[ (m + n_a)! \text{sech} \lambda \tanh^{m+n_a} \lambda \delta_{m+n_a, n+b} \right]^2 \\
= \frac{n_a! n_b! m! n! \sinh^{2n} \lambda P_m^{(m-n,0)}(\cosh 2\lambda)}{n_a! n_b! m! n! \sinh^{2n} \lambda P_m^{(m-n,0)}(\cosh 2\lambda)}.
\]

(25)
FIG. 3: (Color online) Photon number distribution $P(n_a, n_b)$ in the Fock space $(n_a, n_b)$ for some given $m = n$ values: (a) $m = n = 0$, $\lambda = 1$, (b) $m = n = 1$, $\lambda = 0.5$, (c) $m = n = 1$, $\lambda = 1$, (d) $m = 2$, $n = 5$, $\lambda = 1$.

From Eq. (25) one can see that there exists a constrained condition, $m + n_a = n + n_b$, for the photon number distribution (see Fig. 3). In particular, when $m = n = 0$, Eq. (25) becomes

$$P(n_a, n_b) = \begin{cases} \text{sech}^2 \lambda \tanh^{2n_a} \lambda, & n_a = n_b \\ 0, & n_a \neq n_b \end{cases},$$

which is just the photon number distribution (PND) of two-mode squeezed vacuum state.

In Fig. 3, we plot the distribution $P(n_a, n_b)$ in the Fock space $(n_a, n_b)$ for some given $m, n$ values and squeezing parameter $\lambda$. From Fig. 3 it is found that the PND is constrained by $m + n_a = n + n_b$, resulting from the paired-presence of photons in two-mode squeezed state. By subtracting photons, we have been able to move the peak from zero photons to nonzero photons (see Fig. 3 (a) and (c)). The position of peak depends on how many photons are annihilated and how much the state is squeezed initially. In addition, for example, the PND mainly shifts to the bigger number states and becomes more “flat” and “wide” with the increasing parameter $\lambda$ (see Fig. 3 (b) and (c)).

C. Antibunching effect of the TPSSV

Next we will discuss the antibunching for the TPSSV. The criterion for the existence of antibunching in two-mode radiation is given by [40]

$$R_{ab} = \frac{\langle a^1 a^2 \rangle + \langle b^1 b^2 \rangle}{2 \langle a^1 a^2 \rangle} - 1 < 0. \quad (27)$$

In a similar way to Eq. (13) we have

$$\langle a^1 a^2 \rangle = (m + 1)(m + 2) \frac{P^{(m+m-2,0)}(\tau)}{P^{(m,0)}(\tau)}. \quad (28)$$
for the state $|\lambda, m, n\rangle$, substituting Eqs. (15), (28) and (29) into Eq. (27), we can recast $R_{ab}$ to

$$
R_{ab} = \frac{(m+1)(m+2)P_m^{(n-m-2,0)}(\tau) + (n+1)(n+2)\sinh^4 \lambda P_m^{(n-m+2,0)}(\tau)}{2(m+1)(n+1)\sinh^2 \lambda P_m^{(n-m,0)}(\tau)} - 1.
$$  

(30)

In particular, when $m = n = 0$ (corresponding to two-mode squeezed vacuum state), Eq. (30) reduces to $R_{ab,m=n=0} = -\frac{5 - 3 \cosh 2\lambda}{6(1 + 2 \cosh 2\lambda)} \csch^3 \lambda$ may be less than zero with $\lambda > 0.549$ about.

IV. WIGNER FUNCTION OF THE TPSSV

The Wigner function (WF) [32, 41, 42] is a powerful tool to investigate the nonclassicality of optical fields. Its partial negativity implies the highly nonclassical properties of quantum states and is often used to describe the decoherence of quantum states, e.g., the excited coherent state in both photon-loss and thermal channels [43, 44], the single-photon subtracted squeezed vacuum (SPSSV) state in both amplitude decay and phase damping channels [6], and so on [11, 17, 45, 46, 47]. In this section, we derive the analytical expression of WF for the TPSSV. For this purpose, we first recall that the Weyl ordered form of single-mode Wigner operator [48, 49, 50],

$$
\Delta_1(\alpha) = \frac{1}{2} : \delta(\alpha - a) \delta(\alpha^* - a^\dagger) :,
$$  

(31)

where $\alpha = (q_1 + ip_1)/\sqrt{2}$ and the symbol $:\ :$ denotes Weyl ordering. The merit of Weyl ordering lies in the Weyl ordered operators' invariance under similar transformations proved in Ref. [48], which means

$$
S \cdot (\circ \circ \circ) \cdot S^{-1} = \cdot S (\circ \circ \circ) S^{-1} \cdot,
$$  

(32)

as if the "fence" $:\ :$ did not exist, so $S$ can pass through it.
Following this invariance and Eq. (33) we have
\[ S_2^{\dagger} (\lambda ) \Delta_1 (\alpha ) \Delta_2 (\beta ) S_2 (\lambda ) \]
\[ = \frac{1}{4} S_2^{\dagger} (\lambda ) \delta (\alpha - a) \delta (\alpha^* - a^*) \delta (\beta - b) \delta (\beta^* - b^*) \frac{1}{4} S_2 (\lambda ) \]
where \( m \) is the number of photons. The WF is always negative in phase space. Fig. 5 shows that the negative region becomes more visible as the value of \( |m| \) increases. We have set \( \bar{W} \) to be the case where \( |m| = 0 \).

In Figs. 5-7, the phase space Wigner distributions are depicted for several different parameter values \( m, n, \) and \( \lambda \). They are presented in Fig. 7, from which it is interesting to notice that there are around \( |m - n| = 1 \) wave valleys and \( |m - n| \) + 1 wave peaks.

V. DECOHERENCE OF TPSSV IN THERMAL ENVIRONMENTS

In this section, we next consider how this state evolves at the presence of thermal environment.
A. Model

When the TPSSV evolves in the thermal channel, the evolution of the density matrix can be described by the following master equation in the interaction picture [52]

$$\frac{d}{dt} \rho(t) = (L_1 + L_2) \rho(t),$$  

(36)

where

$$L_i \rho = \kappa (\bar{n} + 1) \left( 2a_i \rho a_i^\dagger - a_i^\dagger a_i \rho - \rho a_i^\dagger a_i \right) + \kappa \bar{n} \left( 2a_i^\dagger \rho a_i - a_i^\dagger a_i \rho - \rho a_i^\dagger a_i \right), \quad (a_1 = a, a_2 = b),$$  

(37)

and $\kappa$ represents the dissipative coefficient and $\bar{n}$ denotes the average thermal photon number of the environment. When $\bar{n} = 0$, Eq. (35) reduces to the master equation (ME) describing the photon-loss channel. The two thermal modes are assumed to have the same average energy and coupled with the channel in the same strength and have the same average thermal photon number $\bar{n}$. This assumption is reasonable as the two-mode of squeezed state are in the same frequency and temperature of the environment is normally the same [53, 54]. By introducing two entangled
state representations and using the technique of integration within an ordered product (IWOP) of operators, we can obtain the infinite operator-sum expression of density matrix in Eq. (36) (see Appendix B):

$$\rho(t) = \sum_{i,j,r,s=0}^{\infty} M_{i,j,r,s} \rho_0 M_{i,j,r,s}^\dagger$$

where $\rho_0$ denotes the density matrix at initial time, $M_{i,j,r,s}$ and $M_{i,j,r,s}^\dagger$ are Hermite conjugated operators (Kraus operator) with each other,

$$M_{i,j,r,s} = \frac{1}{\sqrt{nT} + 1} \sqrt{\frac{(T_1)^{l+s}}{r!s!l!j!}} a^l b^s e^{(a^\dagger b + b^\dagger a) \ln T_2} a^i b^j,$$

and we have set $T = 1 - e^{-2\kappa t}$, as well as

$$T_1 = \frac{\tilde{n} T}{nT + 1}, T_2 = \frac{e^{-\kappa t}}{nT + 1}, T_3 = \frac{(\tilde{n} + 1) T}{nT + 1}.$$ (40)

It is not difficult to prove the $M_{i,j,r,s}$ obeys the normalization condition $\sum_{i,j,r,s=0}^{\infty} M_{i,j,r,s}^\dagger M_{i,j,r,s} = 1$ by using the IWOP technique.

### B. Evolution of Wigner function

By using the thermal field dynamics theory [55, 56] and thermal entangled state representation, the time evolution of Wigner function at time $t$ to be given by the convolution of the Wigner function at initial time and those of two single-mode thermal state (see Appendix C), i.e.,

$$W(\alpha, \beta, t) = \frac{4}{(2\tilde{n} + 1)^2 T^2} \int \frac{d^2 \zeta d^2 \eta}{\pi^2} W(\zeta, \eta, 0) e^{-\frac{1}{2} \left| \alpha - \zeta e^{-\kappa t} \right|^2 + \left| \beta - \eta e^{-\kappa t} \right|^2}.$$

Eq. (41) is just the evolution formula of Wigner function of two-mode quantum state in thermal channel. Thus the WF at any time can be obtained by performing the integration when the initial WF is known.

In a similar way to deriving Eq. (34), substituting Eq. (34) into Eq. (41) and using the generating function of two-variable Hermite polynomials (A2), we finally obtain

$$W(\alpha, \beta, t) = \frac{N_{\lambda, m, n}^{-1} (E \sinh 2\lambda)^{m+n}}{\pi^2 2^{m+n} (2\tilde{n} + 1)^2 T^2 D} \frac{e^{-\frac{1}{2} \left| \alpha - \beta^* \right|^2}}{e^{-\frac{1}{2} \left| \alpha + \beta^* \right|^2}} \times \sum_{l=0}^{n} \sum_{k=0}^{m} \frac{[m!n!]^2}{l!k! [m-k]! [n-l]!} \left| H_{m-k,n-l} \left( \frac{G}{\sqrt{E}}, \frac{K}{\sqrt{E}} \right) \right|^2,$$

where we have set

$$C = \frac{e^{-2\kappa t}}{2\tilde{n} + 1}, D = (1 + Ce^{-2\lambda}) (1 + Ce^{2\lambda}),$$

$$E = \frac{e^{4\kappa t}}{D} (2\tilde{n}T + 1)^2 C^2, F = \frac{C^2 - 1}{D},$$

$$G = \frac{Ce^{\kappa t}}{D} (\bar{B} + B^* C), \bar{B} = i2\sqrt{\tanh \lambda} (\beta^* \cosh \lambda + \alpha \sinh \lambda),$$

$$K = \frac{Ce^{\kappa t}}{D} (\bar{A} + A^* C), \bar{A} = i2\sqrt{\tanh \lambda} (\alpha^* \cosh \lambda + \beta \sinh \lambda).$$ (43)

Eq. (42) is just the analytical expression of WF for the TPSSV in thermal channel. It is obvious that the WF loss its Gaussian property due to the presence of two-variable Hermite polynomials.
In particular, at the initial time \((t = 0)\), noting \(E \to 1\), \((2n + 1)^2 T^2 D \to 1\), \(\frac{a}{\sqrt{Z}} \to 1\) and \(\frac{c^2}{\delta} \to 1\), \(Z \to 0\) as well as \(K \to A^*, G \to B^*\), Eq.\(42\) just reduce to Eq.\(34\), i.e., the WF of the TPSSV. On the other hand, when \(\kappa t \to \infty\), noticing that \(C \to 0\), \(D \to 1\), \(E \to 1\), \(F \to -1\), and \(G/\sqrt{E} \to 0\), \(K/\sqrt{E} \to 0\), as well as \(H_{m,n} (0,0) = (-1)^m m! \delta_{m,n}\), as well as the definition of Jacobi polynomials in Eq.\(9\), then Eq.\(42\) becomes

\[
W(\alpha, \beta, \infty) = \frac{1}{\pi^2 (2n + 1)^2} e^{-2\kappa t |(\alpha|^2 + |\beta|^2|,}
\]

which is independent of photon-subtraction number \(m\) and \(n\) and corresponds to the product of two thermal states with mean thermal photon number \(\bar{n}\). This implies that the two-mode system reduces to two-mode thermal state after a long time interaction with the environment. Eq.\(44\) denotes a Gaussian distribution. Thus the thermal noise causes the absence of the partial negative of the WF if the decay time \(\kappa t\) exceeds a threshold value. In addition, for the case of \(m = n = 0\), corresponding to the case of two-mode squeezed vacuum, Eq.\(42\) just becomes

\[
W_{m=n=0}(\alpha, \beta, t) = \Omega^{-1} e^{-\frac{\pi}{2} (|\alpha|^2 + |\beta|^2 + \alpha \beta^* + \alpha^* \beta)}
\]

where \(\Omega = \pi^2 (2n + 1)^2 T^2 D\) is the normalization factor, \(\Omega = (2n + 1)^2 T^2 D\), \(\epsilon = 2 (2n + 1) T + e^{-2\kappa t} \cosh 2\lambda\), and \(\Omega = 2 e^{-2\kappa t} \sinh 2\lambda\). Eq.\(45\) is just the result in Eq.\(14\) of Ref.\(54\).

In Fig.8, the WFs of the TPSSV for \((m = 0, n = 1)\) are depicted in phase space with \(\lambda = 0.3\) and \(\bar{n} = 1\) for several different \(\kappa t\). It is easy to see that the negative region of WF gradually disappears as the time \(\kappa t\) increases. Actually, from Eq.\(43\) one can see that \(D > 0\) and \(E > 0\), so when \(F < 0\) leading to the following condition:

\[
\kappa t > \kappa t_c \equiv \frac{1}{2} \ln \left(\frac{2\bar{n} + 2}{2\bar{n} + 1}\right)^{-1}
\]

we know that the WF of TPSSV has no chance to be negative in the whole phase space when \(\kappa t\) exceeds a threshold value \(\kappa t_c\). Here we should point out that the effective threshold value of the decay time corresponding to the transition of the WF from partial negative to fully positive definite is dependent of \(m\) and \(n\). When \(\kappa t = \kappa t_c\), it then follows from Eq.\(42\) that

\[
W(\alpha, \beta, t_c) = \frac{\tanh^{m+n} \lambda \sech^2 \lambda}{4\pi^2 N_{m,n} \epsilon^{-4\kappa t_c}} e^{-2\kappa t_c |(\alpha|^2 + |\beta|^2 - (\alpha^* \beta + \alpha \beta^*) \tanh \lambda|} 
\times \left[H_{m,n}(i \sqrt{\tanh \lambda \beta^* e^{\kappa t_c}}, i \sqrt{\tanh \lambda \alpha^* e^{\kappa t_c}})\right]^2,
\]

which is an Hermite-Gaussian function and positive definite, as expected.

In Figs. 9 and 10, we have presented the time-evolution of WF in phase space for different \(\bar{n}\) and \(\lambda\), respectively. One can see clearly that the partial negativity of WF decreases gradually as \(\bar{n}\) (or \(\lambda\)) increases for a given time. This case is true for a given \(\bar{n}\) (or \(\kappa t\)) as the increase of \(\kappa t\) (or \(\bar{n}\)). The squeezing effect in one of quadratures is shown in Fig.10. In principle, by using the explicit expression of WF in Eq.\(42\), we can draw its distributions in phase space. For the case of \(m = 0, n = 2\), there are two negative regions of WF, which is different from the case of \(m = 0, n = 1\) (see Fig.11). The absolute value of the negative minimum of the WF decreases as \(\kappa t\) increases, which leads to the full absence of partial negative region.

VI. CONCLUSIONS

In summary, we have investigated the statistical properties of two-mode photon-subtracted squeezed vacuum state (TPSSV) and its decoherence in thermal channel with average thermal photon number \(\bar{n}\) and dissipative coefficient \(\kappa\). For arbitrary number TPSSV, we have for the first time calculated the normalization factor, which turns out to be a Jacobi polynomial of the squeezing parameter \(\lambda\), a remarkable result. We also show that the TPSSV can be treated as a squeezed two-variable Hermite polynomial excitation vacuum. Based on Jacobi polynomials’ behavior the statistical properties of the field, such as photon number distribution, squeezing properties, cross-correlation function and antibunching, are also derived analytically. Especially, the nonclassicality of TPSSV is discussed in terms of the negativity of WF after deriving the explicit expression of WF. Then the decoherence of TPSSV in thermal channel is also demonstrated according to the compact expression for the WF. The threshold value of the decay time corresponding to the transition of the WF from partial negative to completely positive is presented. It is found that the WF has no chance to present negative value for all parameters \(\lambda\) and any photon-subtraction number \((m,n)\) if \(\kappa t > \frac{1}{2} \ln \frac{2\bar{n}+2}{2\bar{n}+1}\) for TPSSV. The technique of integration within an ordered product of operators brings convenience in our derivation.
FIG. 8: (Color online) The time evolution of WF \((m = 0, n = 1)\) at \((q_1, q_2, 0, 0)\) phase space for \(\bar{n} = 1, \lambda = 0.3\). (a) \(\kappa t = 0.05\), (b) \(\kappa t = 0.1\), (c) \(\kappa t = 0.12\), (d) \(\kappa t = 0.2\).

FIG. 9: (Color online) The time evolution of WF \((m = 0, n = 1)\) in \((q_1, q_2, 0, 0)\) phase space for \(\bar{n} = 1\) and \(\kappa t = 0.05\) with (a) \(\bar{n} = 0\), (b) \(\bar{n} = 1\), (c) \(\bar{n} = 2\), (d) \(\bar{n} = 7\).

FIG. 10: (Color online) The time evolution of WF \((m = 0, n = 1)\) in \((q_1, q_2, 0, 0)\) phase space for \(\bar{n} = 1\), and \(\kappa t = 0.05\) with (a) \(\lambda = 0.03\), (b) \(\lambda = 0.5\), (c) \(\lambda = 0.8\), (d) \(\lambda = 1.2\).
Eqs. (5), and (33) the WF of TPSSV can be calculated as

\[ \kappa_t = 0 \]

Further noticing the generating function of two variables Hermitian polynomials,

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**Appendix A: Derivation of Wigner function Eq. (34) of TPSSV**

The definite of the WF of two-mode quantum state \(|\Psi\rangle\) is given by \( W(\alpha, \beta) = (\Psi| \Delta_1 (\alpha) \Delta_2 (\beta) | |\alpha, \beta\rangle \), thus by using Eqs. (5), and (33) the WF of TPSSV can be calculated as

\[
W(\alpha, \beta) = \langle \lambda, m, n | \Delta_1 (\alpha) \Delta_2 (\beta) | |\lambda, m, n\rangle \\
= \frac{\sinh^{n+m} 2\lambda}{2^{n+m} N_{\lambda, m, n}} \langle 00 | H_{m,n} \left( -i \sqrt{\tanh \lambda b}, -i \sqrt{\tanh \lambda a} \right) \Delta_1 (\bar{\alpha}) \otimes \Delta_2 (\bar{\beta}) H_{m,n} \left( i \sqrt{\tanh \lambda b}, i \sqrt{\tanh \lambda a} \right) | 00 \rangle \\
= \frac{\sinh^{n+m} 2\lambda}{2^{n+m} N_{\lambda, m, n}} \exp \left[ -|\alpha|^2 - |\beta|^2 \right] \int \frac{d^2 z_1 d^2 z_2}{\pi^2} e^{-|z_1|^2 - |z_2|^2 - 2(z_1 \bar{\alpha}^* - \bar{\alpha}z_1^*) - 2(z_2 \bar{\beta}^* - \bar{\beta}z_2^*)} \\
\times H_{m,n} \left( -i \sqrt{\tanh \lambda z_2}, -i \sqrt{\tanh \lambda z_1} \right) H_{m,n} \left( -i \sqrt{\tanh \lambda \bar{z}_2}, -i \sqrt{\tanh \lambda \bar{z}_1} \right) \\
\tag{A1}
\]

Further noticing the generating function of two variables Hermitian polynomials,

\[
H_{m,n} (\epsilon, \bar{\epsilon}) = \frac{\partial^{m+n}}{\partial \epsilon^m \partial \bar{\epsilon}^n} \exp \left[ -t\epsilon + \epsilon t' \right] |_{\epsilon = \bar{\epsilon} = 0} \tag{A2}
\]

Eq. (A1) can be further rewritten as

\[
W(\alpha, \beta) = \frac{\sinh^{n+m} 2\lambda}{2^{n+m} N_{\lambda, m, n}} e^{2|\alpha|^2 + 2|\beta|^2} \frac{\partial^{m+n}}{\partial t^m \partial \tau^n} \frac{\partial^{m+n}}{\partial t'^m \partial \tau'^n} e^{-tr - t'r'} \\
\times \int \frac{d^2 z_1}{\pi^2} e^{-|z_1|^2} \left( 2\bar{\alpha}^* - \sqrt{\tanh \lambda \tau^*} z_1 + \sqrt{\tanh \lambda \tau^*} \bar{z}_1 \right) |_{\tau = \bar{\tau} = 0} \\
\times \int \frac{d^2 z_2}{\pi^2} e^{-|z_2|^2} \left( 2\bar{\beta}^* - \sqrt{\tanh \lambda \tau} z_2 + \sqrt{\tanh \lambda \tau} \bar{z}_2 \right) |_{\tau = \bar{\tau} = 0} \\
= \frac{\sinh^{n+m} 2\lambda}{2^{n+m} N_{\lambda, m, n}} e^{-2|\alpha|^2 - 2|\beta|^2} \frac{\partial^{m+n}}{\partial \tau^m \partial \tau'^n} \frac{\partial^{m+n}}{\partial t'^m \partial \tau^n} \\
\times e^{-tr - t'r' + A^* \tau + B* t' + A \tau + Bt - (t'^r + \tau^r)} \tanh \lambda |_{\tau = \bar{\tau} = 0} \\
\tag{A3}
\]

where we have set

\[
B = -2i\bar{\beta} \sqrt{\tanh \lambda}, A = -2i\bar{\alpha} \sqrt{\tanh \lambda}, \tag{A4}
\]
and have used the following integration formula

\[
\int \frac{d^2 z}{\pi} e^{\zeta |z|^2 + \xi z^* z} = -\frac{1}{\xi} e^{-\frac{\zeta}{\xi}}, \text{Re}(\zeta) < 0.
\]  
(A5)

Expanding the exponential term exp \([- (t\tau' + \tau\tau') \tanh \lambda]\), and using Eq.(A2), we have

\[
W(\alpha, \beta) = \frac{\sinh^{n+m} 2\lambda}{2^{n+m}N_{\lambda,m,n}} e^{-2|\alpha|^2-2|\beta|^2} \sum_{l=0}^\infty \sum_{k=0}^\infty \frac{(-\tanh \lambda)^{l+k}}{l!k!} \times \frac{\partial^{l+k}}{\partial B^l \partial A^k} \frac{\partial^{l+k}}{\partial B^l' \partial A^k'} H_{m,n} (B, A) H_{m,n} (B^*, A^*). 
\]  
(A6)

Noticing the well-known differential relations of \(H_{m,n}(\epsilon, \varepsilon)\),

\[
\frac{\partial^{l+k}}{\partial \epsilon^l \partial \varepsilon^k} H_{m,n}(\epsilon, \varepsilon) = \frac{m!n!H_{m-l,n-k}(\epsilon, \varepsilon)}{(m-l)!(n-k)!},
\]  
(A7)

we can further recast Eq.(A6) to Eq.(34).

**Appendix B: Derivation of solution of Eq.(36)**

To solve the ME in Eq.(36), we first introduce two entangled state representations

\[
|\eta_a\rangle = \exp \left(-\frac{1}{2} |\eta_a|^2 + \eta_a a^\dagger - \eta^*_a \tilde{a}^\dagger + a^\dagger \tilde{a}\right) |0\rangle \text{,} \quad (B1)
\]

\[
|\eta_b\rangle = \exp \left(-\frac{1}{2} |\eta_b|^2 + \eta_b b^\dagger - \eta^*_b \tilde{b}^\dagger + b^\dagger \tilde{b}\right) |0\rangle ,
\]  
(B2)

which satisfy the following eigenvector equations, for instance,

\[
(a - \tilde{a}^\dagger) |\eta_a\rangle = \lambda a |\eta_a\rangle, \quad (a^\dagger - \tilde{a}) |\eta_a\rangle = \lambda^* a^\dagger |\eta_a\rangle, \\
(\eta_a | a^\dagger - \tilde{a}) = \eta^*_a |\eta_a\rangle, \quad (\eta_a | (a - \tilde{a}^\dagger)) = \eta_a |\eta_a\rangle.
\]  
(B3)

which imply operators \(a - \tilde{a}^\dagger\) and \((a^\dagger - \tilde{a})\) can be replaced by number \(\eta_a\) and \(\eta^*_a\), \([a - \tilde{a}^\dagger], (a^\dagger - \tilde{a})] = 0\). Operating two-side of Eq.(36) on the vector \(|I_a, I_b\rangle \equiv |\eta_a = 0\rangle \otimes |\eta_b = 0\rangle\), (denote \(|\rho(t)\rangle \equiv \rho (t) |I_a, I_b\rangle\)), and noticing the corresponding relation:

\[
|a| I_a, I_b\rangle = \tilde{a}^\dagger |I_a, I_b\rangle, \quad |a^\dagger | I_a, I_b\rangle = \tilde{a} |I_a, I_b\rangle, \\
|b| I_a, I_b\rangle = \tilde{b}^\dagger |I_a, I_b\rangle, \quad |b^\dagger | I_a, I_b\rangle = \tilde{b} |I_a, I_b\rangle.
\]  
(B4)

we can put Eq.(36) into the following form:

\[
\frac{d}{dt} |\rho(t)\rangle = [\kappa (\bar{n} + 1) (2a \tilde{a} - a^\dagger a - \tilde{a}^\dagger \tilde{a}) + \kappa \tilde{n} (2a^\dagger \tilde{a}^\dagger - a a^\dagger - \tilde{a} \tilde{a}^\dagger)] |\rho(t)\rangle.
\]  
(B5)

It’s formal solution is given by

\[
|\rho(t)\rangle = \exp \left[k \kappa (\bar{n} + 1) (2a \tilde{a} - a^\dagger a - \tilde{a}^\dagger \tilde{a}) + \kappa \tilde{n} (2a^\dagger \tilde{a}^\dagger - a a^\dagger - \tilde{a} \tilde{a}^\dagger)] \right] |\rho_0\rangle, 
\]  
(B6)
where $|\rho_0\rangle \equiv \rho_0 |I_a, I_b\rangle$. In order to solve Eq.(B6), noticing that, for example,

$$2a\ddot{a} - a^\dagger a - \ddot{a} a = -(a^\dagger - \ddot{a}) (a - \ddot{a}) + \dddot{a} a - \dddot{a} a^\dagger,$$

we have

$$|\rho(t)\rangle = \exp \left[ (a\ddot{a} - \dddot{a} a^\dagger + 1) \kappa t \right]$$

$$\times \exp \left[ \frac{2\dddot{a} + 1}{2} (1 - e^{2\kappa t}) (a^\dagger \dddot{a} - \dddot{a} a) \right]$$

$$\times \exp \left[ \left( \ddot{b}^\dagger - \ddot{b} b^\dagger + 1 \right) \kappa t \right]$$

$$\times \exp \left[ \frac{2\dddot{a} + 1}{2} (1 - e^{2\kappa t}) \left( b^\dagger \dddot{b} - \dddot{b} b^\dagger \right) \right] |\rho_0\rangle,$$

where we have used the identity operator,

$$\exp[\lambda(A + \sigma B)] = e^{\lambda A} \exp[\sigma B(1 - e^{-\lambda t})/\tau]$$

valid for $[A, B] = \tau B$.

Thus the element of $\rho(t)$ between $|\eta_a, \eta_b\rangle$ and $|I_a, I_b\rangle$ is

$$\langle \eta_a, \eta_b | \rho(t) | \eta_a, \eta_b \rangle = \exp \left[ -\frac{\dddot{a} + 1}{2} T (|\eta_a|^2 + |\eta_b|^2) \right] \langle \eta_a e^{-\kappa t}, \eta_b e^{-\kappa t} | \rho_0 \rangle,$$

from which one can see clearly the attenuation due to the presence of environment.

Further, using the completeness relation of $|\eta_a, \eta_b\rangle$, $\int \frac{d^2\eta_a d^2\eta_b}{\pi^4} |\eta_a, \eta_b\rangle \langle \eta_a, \eta_b| = 1$ and the IWOP technique [58, 59], we see

$$|\rho(t)\rangle = \frac{1}{(\bar{n} + 1)^2} \exp \left[ T_1 \left( a^\dagger \dddot{a} + b^\dagger \dddot{b} \right) \right]$$

$$\times \exp \left[ \left( a^\dagger a + b^\dagger b + \dddot{a} a + \dddot{b} b \right) \ln T_2 \right]$$

$$\times \exp \left[ T_3 \left( a\ddot{a} + b\ddot{b} \right) \right] |\rho_0 | I_a, I_b \rangle,$$

where $T_1$, $T_2$ and $T_3$ are defined in Eq.(40). Noticing Eq.(B4), we can reform Eq.(B10) as $\rho(t) = \sum_{\alpha, \beta, \gamma, \delta} M^\dagger_{\alpha, \beta, \gamma, \delta} M_{\alpha, \beta, \gamma, \delta}$, where $M^\dagger_{i,j,r,s}$ are defined in Eq.(39).

**Appendix C: Derivation of Eq.(41) by using thermo field dynamics and entangled state representation**

In this appendix, we shall derive the evolution formula of WF, i.e., the relation between the any time WF and the initial time WF. According to the definition of WF of density operator $\rho$: $W(\alpha) = \text{Tr} [\Delta(\alpha) \rho]$, where $\Delta(\alpha)$ is the single-mode Wigner operator, $\Delta(\alpha) = \frac{i}{\pi} D(2\alpha) (1)^{-1/2}$. By using $\langle \tilde{n} | \tilde{m} \rangle = \delta_{m,n}$ we can reform $W(\alpha)$ as [60]

$$W(\alpha) = \sum_{m,n} \langle m, \tilde{n} | \Delta(\alpha) | m, \tilde{n} \rangle = \frac{1}{\pi} \langle \xi = 2\alpha | \rho \rangle,$$

where $\langle \xi \rangle$ is the conjugate state of $|\eta\rangle$, whose overlap is $\langle \eta | \xi \rangle = \frac{1}{2} \exp \left[ \frac{1}{2} (\xi \eta^* - \xi^* \eta) \right] \cdot$ a Fourier transformation kernel. In a similar way, thus for two-mode quantum system, the WF is given by

$$W(\alpha, \beta) = \text{Tr} [\Delta_a(\alpha) \Delta_b(\beta) \rho] = \frac{1}{\pi^2} \langle \xi_{a=2\alpha}, \xi_{b=2\beta} | \rho \rangle.$$

Employing the above overlap relation, Eq.(C2) can be recast into the following form:

$$W(\alpha, \beta, t) = \int \frac{d^2\eta_a d^2\eta_b}{4\pi^4} e^ {-\frac{2\kappa t}{\pi} T \left( |\eta_a|^2 + |\eta_b|^2 \right) }$$

$$\times e^{\alpha^\dagger \eta_a - \alpha^* \eta_a + \beta^* \eta_b - \beta \eta_b} \langle \eta_a e^{-\kappa t}, \eta_b e^{-\kappa t} | \rho_0 \rangle$$

$$= \int \frac{d^2\xi_a d^2\xi_b}{2\pi^2} W(\xi_a, 0) \int \frac{d^2\eta_a d^2\eta_b}{4\pi^2} e^ {-\frac{2\kappa t}{\pi} T \left( |\eta_a|^2 + |\eta_b|^2 \right) }$$

$$\times e^{\alpha^\dagger \eta_a - \alpha^* \eta_a + \beta^* \eta_b - \beta \eta_b} \langle \eta_a e^{-\kappa t}, \eta_b e^{-\kappa t} | \xi_{a=2\xi}, \xi_{b=2\xi} \rangle.$$
Performing the integration in Eq.(C4) over $d^2 \eta_a d^2 \eta_b$ then we can obtain Eq.(41). Making variables replacement, $\frac{\alpha - \zeta e^{-\kappa t}}{\sqrt{T}} \rightarrow \zeta, \frac{\beta - \eta e^{-\kappa t}}{\sqrt{T}} \rightarrow \eta$, Eq.(41) can be reformulated as

$$W(\alpha, \beta, t) = 4e^{4\kappa t} \int d^2 \zeta d^2 \eta W^t(\zeta) W^t(\eta) \times W\left(e^{\kappa t} \left(\alpha - \sqrt{T}\zeta\right), e^{\kappa t} \left(\beta - \sqrt{T}\eta\right), 0\right)$$

(C5)

where $W^t(\zeta)$ is the Wigner function of thermal state with average thermal photon number $\bar{n}$: $W^t(\zeta) = \frac{1}{\pi^{2n+1}} e^{-\frac{2\xi^2}{\bar{n}+1}}$. Eq.(C5) is another expression of the evolution of WF and is actually agreement with that in Refs. [53, 54].

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