INFINITESIMAL INVARIANCE OF COMPLETELY RANDOM MEASURES FOR 2D EULER EQUATIONS

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Abstract. We consider suitable weak solutions of 2-dimensional Euler equations on bounded domains, and show that the class of completely random measures is infinitesimally invariant for the dynamics. Space regularity of samples of these random fields falls outside of the well-posedness regime of the PDE under consideration, so it is necessary to resort to stochastic integrals with respect to the candidate invariant measure in order to give a definition of the dynamics. Our findings generalize and unify previous results on Gaussian stationary solutions of Euler equations and point vortices dynamics. We also discuss difficulties arising when attempting to produce a solution flow for Euler’s equations preserving independently scattered random measures.

1. Introduction

“Completely random” or “independently scattered” measures (the precise definition of which we defer to subsequent sections) are random fields indexed by collections of measurable sets, whose restrictions to disjoint regions of the underlying space are stochastically independent. These objects are by now fundamental components of modern stochastic analysis: for instance, completely random measures appear in the stochastic integral representation of stationary random fields on manifolds [2, 42, 26] and Lévy processes [38], and they are the axis around which stochastic differential calculus is built, both in a continuum [30, 28] and discrete setting [23, 32].

The aim of this paper is to illustrate a distinguished use of completely random measures in the context of (deterministic) partial differential equations with random initial data: we will show how such random fields constitute natural candidates as invariant measures for 2-dimensional Euler’s equations, and illustrate how an appropriate notion of weak solution relying on stochastic integrals can be used to study the PDE dynamics preserving these random fields. Our approach involves the simultaneous use of Gaussian and discrete (more precisely, compound Poisson) random measures, and is meant to extend and unify earlier works on the matter, where Gaussian and discrete initial conditions are dealt with separately.

We observe that our study will bring to the forefront an important collection of random variables that are naturally attached to random measures, that is, multiple stochastic integrals [22, 32], that we will study in a dynamical setting. These objects have recently played a crucial role in the asymptotic analysis of local geometric quantities of random fields — see [26, 29] for some representative applications in this direction.

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1.1. 2D Euler’s Equations and Heuristics on Invariant Measures. We refer the reader to the classic monograph [25] for a detailed introduction to the topics discussed in the present section. Let $\mathcal{D} \subset \mathbb{R}^2$ be a simply connected, bounded open domain with smooth boundary. We consider 2-dimensional Euler’s equations on $\mathcal{D}$ in terms of the vorticity $\omega$,

\[
\begin{align*}
\partial_t \omega(t,x) + u(t,x) \cdot \nabla \omega(t,x) &= 0, \\
\omega(t,x) &= \nabla^\perp \cdot u(t,x), \quad \text{div } u(t,x) = 0, \\
u(t,x) \cdot \hat{n} &= 0, \\
\end{align*}
\]

where the velocity field $u$ is expressed in terms of $\omega$ by means of the Biot-Savart law [25, Section 1.2],

\[u = K * \omega, \quad K(x,y) = \nabla^\perp \ast G(x,y)\]

with $\nabla^\perp = (-\partial_2, \partial_1)$ and $G = (\Delta)^{-1}$ indicates the Green function of the Laplace’s operator with Dirichlet boundary conditions. Since $G$ has a logarithmic singularity, $K$ is a singular convolution kernel.

Equations Eq. (1.1) are in the form of a (nonlinear) transport equation, driven by the divergence-less vector field $u$, and the system is well-posed for initial data $\omega_0 \in L^\infty(\mathcal{D})$, see [21]. In this regime, if $\Phi_t : \mathcal{D} \to \mathcal{D}$ is the flow of $u$ at time $t$, the evolution of vorticity is thus given by $\omega_t \circ \Phi_t = \omega_0$. This fact provides a remarkable intuition for candidate invariant measures of Eq. (1.1), which we now present from a mostly heuristic perspective.

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and consider a random measure $M$, that is, $M$ a $\sigma$-additive map from the class $\mathcal{B}$ of Borel sets of $\mathcal{D}$ to the Banach space $L^1(\Omega, \mathcal{F}, \mathbb{P})$, satisfying the following:

**Hypothesis 1.** For any $A, B \in \mathcal{B}$,

a) if $A$ and $B$ are disjoint (up to a Lebesgue-negligible set), the random variables $M(A)$ and $M(B)$ are independent;

b) if $A$ and $B$ have the same Lebesgue measure, $|A| = |B|$, random variables $M(A)$ and $M(B)$ have the same distribution.

Under this assumption, since the flow $\Phi_t$ of the divergence-less vector field $u$ preserves the Lebesgue measure on $\mathcal{D}$, we infer that the image random measure $M_t(A) = M(\Phi_t^{-1}A)$ has the same law of $M$. Indeed, by Hypothesis 1 b), for fixed $A$ we have that $M(A)$ and $M_t(A)$ have the same law, and it is an easy exercise to show (by using the properties of $\Phi_t$) that the vectors $(M(A_1), \ldots, M(A_n))$ and $(M_t(A_1), \ldots, M_t(A_n))$ have the same distribution for every choice of measurable sets $A_1, \ldots, A_n$.

A natural heuristic claim is thus that a random vorticity distribution $\omega$, such that its (suitably defined) integrals $\int_A \omega(x) dx = M(A)$ are a random measure as above, is preserved by the dynamics, or in other words, its law is an invariant measure for Eq. (1.1). However, the above reasoning only makes sense if $\Phi_t$ is regular enough, and when vorticity is taken to be a random measure satisfying Hypothesis 1 its samples are not even functions, the velocity field is all but regular: in fact it is not trivial even to give a sensible meaning to the dynamics.

A weak formulation of Eq. (1.1) allowing solutions with poor space regularity has been known since the works of Delort and Schochet [13, 40], but it does not allow to define solutions starting from a given sample of a random field $M$ as above. The next fundamental step to be considered consists in interpreting integrals involving $\omega$ in the weak formulation as (multiple) stochastic integrals with respect to the random integrator $\omega(x) dx \sim M$. This point of view was first considered by Flandoli [14] in the case where $M$ is a Gaussian measure satisfying Hypothesis 1, that is, when $M$
is a multiple of a white noise; in that work it was also discussed how such a notion of solution is a sensible one, producing approximations of Gaussian solutions with smooth solutions and scaling limits of point vortices systems. Related works [19, 20] have analyzed more general Gaussian invariant measures, also in cases where the PDE dynamics include stochastic forcing.

1.2. Towards Equilibrium Flows. Our ideal goal is now to produce a solution flow of Euler’s equations Eq. (1.1) preserving a given completely random measure. In other words, given a completely random measure \( M \), we would like to accomplish the following tasks:

(a) identify \( M \) with a random element taking values in a suitable space \( \mathcal{X} \) of generalised functions, in such a way that the triple \( (\mathcal{X}, X, m) \) — where \( m \) and \( X \) are, respectively, the law of and the \( \sigma \)-field generated by \( M \) — is a probability space;

(b) provide a sensible notion of weak solutions of the PDE taking values in \( X \), exploiting the probability measure \( m \) defined on it;

(c) build a measure-preserving one-parameter group of transformations \( T_t : \mathcal{X} \rightarrow \mathcal{X}, t \in \mathbb{R}, \) whose trajectories can be identified with weak solutions of point (b).

Remark 1. We will simply write \((\mathcal{X}, m)\) to denote the triple \((\mathcal{X}, X, m)\) evoked above, tacitly assuming that the underlying \( \sigma \)-field is the one generated by \( M \).

Fully realising task (c) is an open problem even in the aforementioned purely Gaussian case, in which the existence of solutions has been shown [9, 14] only in the following sense: for almost any sample of Gaussian white noise \( W \), there exists a continuous map \( \omega \in C([0, T], \mathcal{S}'(\mathcal{D})) \), taking values in the space \( \mathcal{S}'(\mathcal{D}) \) of generalized functions on \( \mathcal{D} \), such that

- \( \omega_0 = W \) and, under the law of \( W \), the random element \( \omega_t \) has the law of \( W \), for all \( t \in (0, T] \);
- Euler’s equation is satisfied in a suitable weak form (detailed in Section 3).

This is clearly weaker compared to existence of a solution flow.

At present, building a solution flow (in the sense of (a)—(c)) in the even broader setting of completely random measures, is out of reach. In this contribution, we will describe a precise strategy for tackling this objective by the Koopman-von Neumann approach introduced in this fluid dynamics context by [5, 6, 3, 4]. Although we will fall short of fully realising our program (see Section 5 for a detailed discussion), our main contribution Theorem 14 represents a first, essential step.

Our way of attacking the matter is to follow the steps detailed in the next five points, of which only the first four will be completely dealt with in the present work.

1. We will first select a collection of random measures \( M \) satisfying Hypothesis 1, (the restriction being finiteness of moments) and prove in Lemma 5 that a suitable restriction of \( M \) lives a.s. in a subspace \( \mathcal{X} \) of generalised functions on \( \mathcal{D} \). We will then focus on the particular set of cylinder observables associated with the probability space \((\mathcal{X}, m)\).

2. In Section 3.1, we will then consider a suitable weak formulation of Euler’s equations Eq. (1.1), allowing solutions taking values in the phase space \( \mathcal{X} \): in fact, and more precisely, the notion of solution we put forward makes essential use of the probability measure \( m \), since the nonlinearity of the PDE is too singular to be defined for generic elements of \( \mathcal{X} \), and solutions are defined only almost surely with respect to \( m \), in terms of stochastic integrals.
(3) By Koopman’s lemma [15], we argue in Eq. (1.1) that a solution flow preserving \( m \) would induce a strongly continuous group \( U = e^{itA} \) of unitary transformations on \( L^2(\mathcal{X}, m) \), and in sight of the previous point we deduce the form that the generator \( A \) of such a group must take on cylinder observables.

(4) Our main result Theorem 14 is that the expression for \( A \) deduced in the previous point defines a symmetric operator on \( L^2(\mathcal{X}, m) \), and that it has at least a self-adjoint extension generating a strongly continuous group of unitary operators.

(5) As argued in Section 5, if one could prove that the latter group of operators preserve \( L^\infty(\mathcal{X}, m) \), a converse of Koopman’s lemma would imply that the group consists of Koopman operators of a measure-preserving flow on \( (\mathcal{X}, m) \), and because of the way we define \( A \) this would be a solution flow for Euler’s equation in the sense of (a)—(c).

As anticipated, Point (5) will not be completed in the present paper: should it be done, this would actually be the first uniqueness result for Euler’s dynamics in a very low regularity regime.

The rest of the paper is organized according to the list of items (1)-(5). In particular our main result, Theorem 14, is proved in Section 4, and in Section 5 we discuss more precisely point (5), together with a number of further remarks and generalizations.

To conclude the section, let us recall some relevant previous studies. Gaussian invariant measures of Eq. (1.1) were originally introduced and studied in a series of works by Albeverio and others [8, 9]: in fact, the idea of Gaussian and Poissonian invariant measures as special cases of the general, independently scattered one, dates back to [7]. The works [10, 19, 20, 17] concern relations, in the form of scaling limits, between Gaussian and Poissonian random fields preserved by Euler’s dynamics.

1.3. Notation. From now on, all random elements are assumed to be defined on a suitable probability space \((\Omega, \mathcal{F}, \mathbb{P})\), with \( \mathbb{E} \) denoting expectation with respect to \( \mathbb{P} \). The letter \( \mathcal{B} \) stands for the Borel \( \sigma \)-algebra of \( \mathcal{D} \), whereas \( dx \) is the Lebesgue measure. When convenient, we will also use the symbol \(|A|\) to indicate the Lebesgue measure of a given \( A \in \mathcal{B} \). We denote by \( \mathcal{S} = \mathcal{S}(\mathcal{D}) = C_c^{\infty}(\mathcal{D}) \) the space of smooth functions with compact support in \( \mathcal{D} \), and by \( \mathcal{S}' \) the dual space of distributions\(^1\) on \( \mathcal{D} \).

The symbol \( \omega \) will always be used for vorticity, never for elements of the probability space \( \Omega \) (playing a role exclusively in Section 2); in fact, we will not need to explicitly specify any dependence on the elements of \( \Omega \), in such a way that no ambiguity will arise.

2. Independently Scattered Random Measures and Stochastic Integrals

We begin with a review of concepts from probability theory. All results and facts of this section are standard: we refer to [22, 35, 36, 41] and, in particular, to [32] (whose setting is closest to ours).

**Definition 2.** A random measure on a measurable space \((\mathcal{D}, \mathcal{B})\) is a real-valued random field \( \{M(A) : A \in \mathcal{B}\} \), indexed by \( \mathcal{B} \) and satisfying the following property:

\(^1\)To avoid confusion, distribution will always refer to a linear continuous functional of smooth functions, and never to the law of a random variable.
for all sequences $A_1, A_2, \ldots \in \mathcal{B}$ of disjoint Borel sets,
\[
M \left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} M(A_n),
\]
where the series on the right-hand side converges in probability.

We now give an explicit definition of the random measures we consider in the present paper: as explained in Section 2.2, up to some additional integrability assumption they are actually the ones characterized by Hypothesis 1.

We assume that on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ the following two random measures are defined (see [32, Chapter 5], as well as [23, 28] for further details).

- A Gaussian white noise on $\mathcal{D}$, that is, a centered Gaussian field $W$ defined on $\mathcal{B}$ with covariance
  \[
  \mathbb{E} [W(A)W(B)] = |A \cap B|, \quad A, B \in \mathcal{B}.
  \]

- A compensated compound Poisson point process $P$ independent of $W$ given by
  \[
P(A) = \int_A \int \gamma \mathcal{N}(dx, d\gamma), \quad A \in \mathcal{B},
  \]
  where $\mathcal{N}$ is a compensated Poisson point process independent of $W$ on the product space $\mathcal{D} \times \mathbb{R}$, whose control measure is the product of the Lebesgue measure on $\mathcal{D}$ and of an atomless measure $\nu$ on $\mathbb{R}$ such that $\int_\mathbb{R} |\gamma| d\nu(\gamma) < \infty$.

**Remark 3.** The integrability properties imposed on $\nu$ imply that it is a $\sigma$-finite measure on $\mathbb{R}$, possibly giving infinite mass to neighborhoods of 0. It is a standard fact (see [23, Corollary 6.5]) that samples of $P$ can be identified (without changing probability space) with infinite sums of Dirac deltas $\sum_{i=1}^{\infty} \gamma_i \delta_{x_i}$, $x_i \in \mathcal{D}$, $\gamma_i \in \mathbb{R}$ with $\sum_{i=1}^{\infty} |\gamma_i| = \int_\mathbb{R} |\gamma| d\nu(\gamma) < \infty$. Notice that $\nu$ being atomless implies in particular that $\nu(\{0\}) = 0$, which makes this representation of samples unique, since $\gamma_i$ (almost) never take value 0. Finally, assuming that $P$ is compensated implies $\int_\mathbb{R} |\gamma| d\nu(\gamma) = 0$.

For $a \in \mathbb{R}$, $q \in [0, \infty)$, we introduce the random measure
\begin{equation}
M(A) := a|A| + \sqrt{q}W(A) + P(A), \quad A \in \mathcal{B},
\end{equation}
and write $N \sim [a, q, \nu]$, to indicate that another random measure $N$ has the same law of $M$, as defined in Eq. (2.1).

**Remark 4.** In applications, it is often natural to consider measures of the type of Eq. (2.1) with $P$ a non-compensated compound Poisson process. However, it is immediately seen that the compensation can always be forced into the definition of $M$ by suitably modifying the constant $a$.

In order to specify a formulation of Euler’s equation Eq. (1.1) that is meaningful when the (suitably defined) random mapping $A \mapsto \int_A \omega_t dx$ has the law of $A \mapsto M(A)$, it is convenient to regard the random measures introduced above as random variables taking values in distribution spaces. If $M \sim [a, q, \nu]$ and $f \in L^2(\mathcal{D})$, we can set
\begin{equation}
I_M^1(f) = a \int_\mathcal{D} f dx + \sqrt{q}W(f) + P(f),
\end{equation}
where $I_W^1(f)$ is the Itô integral of $f$ with respect to the white noise $W$ [30, 28], and
\[
I_P^1(f) = \int_\mathbb{R} \int_\mathbb{R} \gamma f(x) \mathcal{N}(dx, d\gamma)
\]
is the Poissonian integral of $f$ [23]. When $f$ is an indicator function of a Borel set $A$, Eq. (2.2) corresponds to Eq. (2.1).

Although $I_{1M}^2(f)$ is well-defined as a random variable for $f \in L^2(\mathcal{D})$, it turns out that restricting the set of integrands yields fixed samples of $M$ as distributions, i.e. generalized functions.

**Lemma 5.** Let $M \sim [a, q, \nu]$ be defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, $\mathbb{P}$-almost surely, the map $\mathcal{H}(\mathcal{D}) \ni f \mapsto I_{1M}^2(f)$ is a continuous linear functional; in other words, we can regard $M$ as a $\mathcal{H}$-valued random variable.

The support of the law $\mu$ of $M$ in $\mathcal{H}(\mathcal{D})$ is contained in a Polish subspace $\mathcal{X} \subset \mathcal{H}(\mathcal{D})$, so that $(\mathcal{X}, \mu)$ is a standard Borel probability space.

In the following, $H^2_0(\mathcal{D})$ is the usual Sobolev space of functions vanishing at the boundary of $\mathcal{D}$, that is the closure of $\mathcal{H}(\mathcal{D})$ under the norm $\|\cdot\|_{H^2_0(\mathcal{D})} = \|(1 + \Delta)^{k/2}\|_{L^2(\mathcal{D})}$ with $\Delta$ the Dirichlet Laplacian as above.

**Proof.** The constant part $adx$ and the Poissonian one are both elements of signed Radon measures on $\mathcal{D}$, $\mathcal{M}(\mathcal{D})$ (cf. Remark 3). We can thus focus on the Gaussian part. The following argument is standard, so we only sketch it: we refer to [37, Chapter 3] for a detailed discussion of representations and orthogonal expansions of Gaussian random fields.

It is a standard fact that the Itô integral $L^2(\mathcal{D}) \ni f \mapsto I_{1W}(f) \in L^2(\mathbb{P})$ defines an isometry of Hilbert spaces. It is then a general fact (see [12]) that a Gaussian measure (in our case $W$) whose reproducing kernel Hilbert space $H$ is Hilbert-Schmidt embedded in another Hilbert space $E$, can be identified as a Gaussian random element of $E$. We then conclude observing that the embedding $L^2(\mathcal{D}) \hookrightarrow H^2_0(\mathcal{D})' = H^{-2}(\mathcal{D})$ is Hilbert-Schmidt: this is easily checked by considering a Fourier basis diagonalizing the Dirichlet Laplacian, recalling that $(1 + \Delta)^2 : H^2_0(\mathcal{D}) \rightarrow L^2(\mathcal{D})$ is an isometry of Hilbert spaces and Weyl’s law in dimension 2.

The proof is concluded since $H^{-2}(\mathcal{D})$ is a Polish space and a topological subspace of $\mathcal{H}''(\mathcal{D})$, and since $\mathcal{M}(\mathcal{D}) \subset H^{-2}(\mathcal{D})$, again as topological vector spaces. \qed

In fact, regularity of $W$ discussed above might be improved with a finer Sobolev embedding (see [14, 17]). For our purposes, it suffices to identify $(\mathcal{X}, \mu)$ with a standard Borel probability space contained (topologically) in $\mathcal{H}(\mathcal{D})$, so to have both a well-behaved underlying probability space and a natural way to couple samples of $M$ and smooth functions.

### 2.1. Double Stochastic Integrals and Moments.

Having identified the random measure $M \sim [a, q, \nu]$ with a random distribution on $\mathcal{D}$, one can consider $M \otimes M \in \mathcal{H}(\mathcal{D}^2)$, its tensor product defined sample-by-sample. In particular, for $h \in \mathcal{H}(\mathcal{D}^2)$, we can consider the duality coupling $\langle h, M \otimes M \rangle$. By symmetry, coupling of $M \otimes M$ with $h(x, y)$ is the same as with $\tilde{h}(x, y) = (h(x, y) + h(y, x))/2$, so one can reduce the discussion to symmetric functions.

From the point of view of Probability Theory, this is a rather unnatural definition for double integrals with respect to $M$, and one typically prefers to use double stochastic integrals, which are classically defined as follows (see [32, Chapter 5] for a full account).

Let $M \sim [a, q, \nu]$ be defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any pair of disjoint sets $A, B \in \mathcal{B}$, we consider the mapping $(x, y) \mapsto h(x, y) = 1_A(x)1_B(y)$, $x, y \in \mathcal{D}^2$, and define the **double stochastic integral of $h$ with respect to $M$ as**

$$I^2_M(h) := 2M(A)M(B).$$
One canonically extends by linearity the previous definition of $I_M^2(h)$ to functions $h$ that are generic finite linear combinations of simple functions as above. For any such $h$ it is easy to verify that, writing explicitly $M = adx + \sqrt{W} + P$,

$$I_M^2(h) = I_M^2(\tilde{h})$$

$$I_M^2(h) = a^2 \int_D \tilde{h} d^2x + 2a I_M^1 \left( \int_D \tilde{h}(x, \cdot) dx \right) + q I_W^1(\tilde{h}) + 2\sqrt{q} I_W^1 I_P^1(\tilde{h}) + I_P^2(\tilde{h}),$$

where:

- $d^2x$ stands for the product Lebesgue measure on $\mathcal{D}^2$;
- $I_W^s$ and $I_P^s$ are, respectively, double Wiener-Itô integrals with respect to $W$ and $P$;
- the (slightly clumsy, but convenient for our further discussion) symbol $I_W^1 I_P^1$ denotes a mixed iterated stochastic integral of order 2, with respect to $W$ and $P$. Note that $I_W^1 I_P^1(\tilde{h}) = I_P^2 I_W^1(\tilde{h})$ because of symmetry. See e.g. [31] for a discussion of several distinguished properties of mixed iterated stochastic integrals.

The following result is standard, and the proof is therefore omitted.  

**Lemma 6.** The mapping $h \mapsto I_M^2(h)$ defined above extends by density to a continuous mapping $I_M^2 : L^2(D) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ between Hilbert spaces, such that (2.1) and (2.1) are preserved.

We refer to [41, Proposition 3.1] further details, as well as to [30, 28, 32] for a discussion of the purely Gaussian setting, and to [22, 23] for the Poissonian one.

Comparing definitions, one easily obtains the following relation between the two notions of double integrals discussed above: for $h \in \mathcal{S}(D^2)$,

$$\langle h, M \otimes M \rangle = I_M^2(h) + \int_D h(x, x) dx + \int_D \int_R r^2 \tilde{h}(x, x) \mathcal{N}(dx, dr).$$

We eventually observe that moments of stochastic integrals can be evaluated by means of usual formulae (see again [32]): for $f \in L^2(D)$ and $h \in L^2(D^2)$ it holds

$$\mathbb{E} [I_W^1(f)] = \mathbb{E} [I_P^1(f)] = \mathbb{E} [I_W^2(h)] = \mathbb{E} [I_P^2(h)] = 0,$$

$$\mathbb{E} [I_W^1(f)^2] = \mathbb{E} [I_P^1(f)^2] = \frac{1}{2} \mathbb{E} [I_W^2(h)^2] = \mathbb{E} [I_P^2(h)^2] = 2 \mathbb{E} [I_W^2(h)^2] = 2 \mathbb{E} [I_P^2(h)^2] = \left( \int_R x^2 dx \right)^2 \mathbb{E} [h]_{L^2(D)},$$

(recall that we are assuming that $\mathcal{N}$ is compensated).

### 2.2. Characterizing Random Measures

Before moving on to Euler’s dynamics, let us argue how the random measures we are considering are special cases of the objects satisfying Definition 2 and Hypothesis 1.

A random measure is called infinitely divisible if all random variables $M(A), A \in \mathcal{B}$, are infinitely divisible. This condition is clearly implied by Hypothesis 1 b) and these random measures are characterized in [35, Proposition 2.4] by means of their characteristic function, generalizing the Lévy-Khintchine representation for real-valued infinitely divisible random variables.

Random measures satisfying Hypothesis 1 a) are usually referred to as independently scattered or completely random. Assuming this condition and the requirement that all variables $M(A)$ have finite second moments, the aforementioned result of [35] on infinitely divisible random measures specializes to the following one: we refer to [33, Section 4] for a thorough discussion.
Proposition 7. Let $M$ be a random measure satisfying Hypothesis 1. Assume moreover that $\mathbb{E} [M(A)^2] < \infty$ for all $A \in \mathcal{B}$. Then the characteristic function of $M$ has the Lévy-Khintchine form

$$
\mathbb{E} \left[ e^{it M(A)} \right] = \exp \left( ita|A| - \frac{t^2}{2} q |A| + |A| \int_{\mathbb{R}} \left( e^{itx} - 1 - itx \right) d\nu(x) \right), \quad t \in \mathbb{R},
$$

where $a \in \mathbb{R}$, $q \in [0, \infty)$ and $\nu$ is an atomless measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} |x| \vee |x|^2 d\nu(x) < \infty$.

Conversely, given such a triple $a, q, \nu$, there exist a random measure satisfying Hypothesis 1 with the above characteristic function.

It is easily seen that random measures of the type (2.1) correspond to those characterized by Eq. (2.6). Moreover, given $M$ as in Proposition 7, it is possible to define Gaussian and compound Poisson processes $W, P$ as above such that $M = a + \sqrt{q} W + P$ without changing probability space, and such representation is unique: this is a particular case of [34, Theorem 4.5], and we also refer to [1, 39] for details on the topic.

3. Weak Solutions of 2-dimensional Euler Equations

Let us first consider a smooth solution $\omega(t, x)$ of Eq. (1.1). The weak (i.e. integral) formulation of Eq. (1.1) reads, for all test functions $\varphi \in \mathcal{S}(\mathcal{D})$,

$$
\int_{\mathcal{D}} \varphi(x) \omega_t(x) dx - \int_{\mathcal{D}} \varphi(x) \omega_0(x) dx = \int_0^t \int_{\mathcal{D}^2} K(x, y) \cdot \nabla \omega_s(x) \omega_s(y) dxdy.
$$

Since $G(x, y)$ is a symmetric function of its variables, $K(x, y)$ is in turn antisymmetric: the latter expression can thus be rewritten as

$$
\int_{\mathcal{D}} \varphi(x) \omega_t(x) dx - \int_{\mathcal{D}} \varphi(x) \omega_0(x) dx = \int_0^t \int_{\mathcal{D}^2} H_\varphi(x, y) \omega_s(x) \omega_s(y) dxdy,
$$

$$
H_\varphi(x, y) = \frac{1}{2} (\nabla \varphi(x) - \nabla \varphi(y)) \cdot K(x, y).
$$

The fundamental advantage of this second, symmetrized form, is that instead of the singular kernel $K$ it involves the new symmetric function $H_\varphi$ (depending on the test function), which is bounded and smooth outside the diagonal set $\{ x, y \in \mathcal{D} : x = y \}$, on which it has a jump discontinuity, regardless of the choice of $\varphi$. In fact, one can provide an explicit bound:

$$
\exists C > 0 : \forall \varphi \in \mathcal{S}(\mathcal{D}), \quad \| H_\varphi \|_\infty \leq C \| \varphi \|_{C^2(\mathcal{D})}.
$$

This is readily seen by Taylor expanding $\nabla \varphi$, keeping in mind that $|K(x, y)| = O(|x-y|^{-1})$ for $|x-y| \to 0$. Clearly, formulation Eq. (3.1) is equivalent to Eq. (1.1) for smooth solutions.

The double space integral of Eq. (3.1) can be defined for $\omega(t, x)$ whose space regularity is lower than the well-posedness regime $L^\infty(\mathcal{D})$. We refer to [13, 40] for the original application of this trick to Euler flows with low space regularity, and a more thorough discussion of the symmetrization argument outlined above.

Aiming to define solutions starting from a random initial datum whose law is a random field $M \sim [0, q, \nu]$, considering Eq. (3.1) sample-by-sample is still insufficient. This is essentially due to the Gaussian part: as first observed in [14], $W \otimes W \in \mathcal{S}(\mathcal{D}^2)$ can not be coupled with discontinuous functions, thus $H_\varphi$ is not regular enough to be coupled sample-by-sample with $W \otimes W$, due to the already mentioned discontinuity on the diagonal.

We thus need to resort to stochastic integrals in order to define solutions of Eq. (3.1) for which $\omega(t, \cdot)$ is a random field with the law of $M \sim [0, q, \nu]$. The idea
is simple: we interpret all space integrals of Eq. (3.1) as stochastic integrals with respect to the random vorticity \( \omega_t \). We refer once again to [14, 17] for a discussion on why this gives a sensible definition of weak solution, in particular obtaining this kind of weak solutions as limits of classical ones.

### 3.1. Weak Solutions and Stochastic Integrals

The following definitions are analogous to the ones given in the aforementioned [14, 17] in the purely Gaussian or Poissonian case: in fact they are a generalization. We write \((X, m)\) denoting the law of the restriction of \( M \sim [a, q, \nu] \) to \( \mathcal{S}(\mathcal{D}) \) on its support \( X \subset \mathcal{S}(\mathcal{D}) \).

**Definition 8.** Let \( I \subset \mathbb{R} \) be an interval and fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Consider a \( \mathcal{S}(\mathcal{D}) \)-valued stochastic processes \((\omega_t)_{t \in I}\) satisfying

1. the fixed-time marginals have law \( \omega_t \sim [a, q, \nu] \);
2. for any \( \varphi \in \mathcal{S}(\mathcal{D}) \), \( \mathbb{P}\)-almost surely, the maps

\[
I \ni t \mapsto I^1_{\omega_t}(\varphi), \quad I \ni t \mapsto I^2_{\omega_t}(H\varphi),
\]

are measurable.

We call \((\omega_t)_{t \in I}\) a weak solution of Euler’s equations if for all \( \varphi \in \mathcal{S}(\mathcal{D}) \), \( \mathbb{P}\)-almost surely, for all \( t, s \in I \),

\[
I^1_{\omega_t}(\varphi) - I^1_{\omega_s}(\varphi) = \int_s^t I^2_{\omega_r}(H\varphi)dr.
\]

Although we use the term “stochastic process”, solutions are actually deterministic in nature, randomness coming only from an initial datum distributed according to the invariant measure \( M \). No external noise is involved, and the whole forthcoming discussion will concern transformations of the standard Borel probability space \((X, m)\), acting both as probability and phase space (in the spirit of Statistical Mechanics).

**Remark 9.** Definition 8 actually implies some more properties of a weak solution \((\omega_t)_{t \in I}\). For instance, the processes of item (2) must also be, \( \mathbb{P}\)-almost surely, locally square-integrable in time: for a fixed \( \varphi \in \mathcal{S} \),

\[
\mathbb{E} \left[ \int_0^T I^2_{\omega_t}(\varphi)^2 dt \right] = \int_0^T \mathbb{E} \left[ I^1_{\omega_t}(\varphi)^2 \right] dt \leq CT \| \varphi \|^2_{L^2(\mathcal{D})},
\]

with \( C > 0 \) depending only on \( q, \nu \), and the analogue for \( I^2_{\omega_t}(H\varphi) \). Swapping integrals follows from the Fubini-Tonelli theorem.

**Remark 10.** Assume that \( M \) is purely Gaussian, i.e. a multiple of \( W \). The stochastic integral \( I^1_{\omega_t}(\varphi) \) appearing on the left-hand side of Eq. (2.3) is then an element of the first Wiener chaos of the Gaussian space associated to \( \omega_t \sim \sqrt{q}W \). On the right-hand side, however, it appears the double stochastic integral \( I^2_{\omega_t}(H\varphi) \), belonging to the second chaos of the Gaussian space associated to \( \omega_t \sim \sqrt{q}W \).

If \( \omega_t \) was a Gaussian process, that is its marginal distributions at different times are jointly Gaussian, this would make Eq. (3.3) an equation between non-trivial elements of different chaoses. It is thus important to notice that we only prescribed the law of \( \omega_t \) at a single fixed time, so there is no contradiction. In fact, this reasoning implies that a solution \((\omega_t)_{t \in I}\) with marginals \( \omega_t \sim [0, q, 0] \) is not a Gaussian process.

In the light of Definition 8, it is natural to consider the notion of solution flow for Euler’s equations on \((X, m)\).

**Definition 11.** A flow of weak solutions of Euler’s equations in the sense of Definition 8 is a one-parameter group \( T_t : (X, m) \to (X, m) \) of almost surely invertible,
measurable, measure-preserving transformations of the probability space \((\mathcal{X}, \mathcal{M})\) such that the stochastic process

\[ \omega_t = T_t \omega_0, \quad \omega_0 = M, \]

(defined on the probability space \((\mathcal{X}, \mathcal{M})\)) is a weak solution of Euler’s equation in the sense of Definition 8.

We already remarked in the Introduction that the existence of a flow in this sense remains an open question. As anticipated, we now change our point of view in favor of observables of \((\mathcal{X}, \mathcal{M})\), rather than looking at samples of \(\omega_t\).

3.2. The Koopman-Von Neumann Point of View. It is a general fact, known as Koopman’s Lemma, that measure preserving flows give rise to groups of unitary operators on the \(L^2\) space relative to the invariant measure. In fact, there is a one-to-one correspondence between groups of unitaries and measure-preserving flows, which is made precise in the following:

**Proposition 12.** Let \((\mathcal{X}, \mathcal{F}, \mu)\) be a standard Borel probability space, i.e. \(\mathcal{X}\) is a Polish space and \(\mathcal{F}\) the associated Borel \(\sigma\)-algebra. Consider maps

\[ \forall t \in \mathbb{R}, \quad T_t : \mathcal{X} \to \mathcal{X}, \quad U_t f = f \circ T_t, \quad f \in L^0(\mathcal{X}, \mathcal{F}, \mu). \]

The following are equivalent:

- \(T_t\) is a group of \(\mu\)-almost surely invertible, measurable, measure preserving maps, and \(t \to T_t\) is weakly measurable:
  \[ \forall A, B \in \mathcal{F}, \quad t \mapsto \mu(T_t^{-1} A \cap B) \text{ is measurable}; \]

- \(U_t\) defines a strongly continuous group of unitary operators on \(L^2(\mathcal{X}, \mathcal{F}, \mu)\), such that for all \(t \in \mathbb{R}\)
  \[ U_t 1 = 1, \quad \forall f \geq 0, \quad U_t f \geq 0. \]

We refer to [15] for the proof. In the latter, and from now on, \(L^p\) spaces are meant to be composed of complex-valued functions.

Since \((\mathcal{X}, \mathcal{M})\) is a standard Borel probability space thanks to Lemma 5, a solution flow in the sense of Definition 11 would thus correspond to a group of unitary operators on \(L^2(\mathcal{X}, \mathcal{M})\). The crucial feature we care about is Euler’s dynamics as prescribed in Definition 8, and in the operator setting this information is encoded in the infinitesimal generator.

Consider the class of cylinder observables

\[ \mathcal{C}(M) = \{ F(M) = f(I_M^1(\varphi_1), \ldots, I_M^1(\varphi_k)), \varphi_1, \ldots, \varphi_n \in \mathcal{D}, f \in C_\infty^0(\mathbb{R}^n, \mathbb{C}) \}. \]

The latter expression for \(F(M)\) clearly defines a measurable function of class \(L^\infty(\mathcal{X}, \mathcal{M})\).

**Lemma 13.** Let \(M \sim [a, q, \nu]\). The linear subspace \(\mathcal{C}(M) \subset L^2(\mathcal{X}, \mathcal{M})\) is dense.

The linear operator

\[ A F(M) = \sum_{k=1}^n \partial_\varphi f(I_M^1(\varphi_1), \ldots, I_M^1(\varphi_k))I_M^2(H_{\varphi_k}), \quad (3.4) \]

defined on \(F \in \mathcal{C}(M)\) takes value in \(L^2(\mathcal{X}, \mathcal{M})\).

If \(T_t\) is a solution flow in the sense of Definition 11, the infinitesimal generator of the strongly continuous group of Koopman unitary operators \(U_t F = F \circ T_t, \quad F \in L^2(\mathcal{X}, \mathcal{M}), \) takes the form (3.4) when restricted to \(\mathcal{C}(M)\).
Proof. Density of the class \( C(M) \) is proved by means of the following two facts: (1) the \( \sigma \)-field generated by random variables of the type \( I^1_{A_0}(\varphi) \) is the same as the \( \sigma \)-field generated by \( M \), and (2) the class \( C^\infty_b(\mathbb{R}^n, \mathbb{C}) \) contains arbitrary products of trigonometric functions. To conclude, one then classically exploits uniqueness of Fourier transforms — e.g. along the lines of [28, proof of Theorem 2.2.4].

The second statement is an easy consequence of the fact that double stochastic integrals in \( M \) have finite second moments.

As for the last statement, a process \( \omega_t \) given by a solution flow as in Definition 11 satisfies Eq. (3.3), so \( F(\omega_t), F \in C(M) \), is almost surely differentiable at almost all times (recall also Remark 9) and

\[
\frac{d}{dt} F(\omega_t) = \sum_{k=1}^{n} \partial_{\varphi_k} f(I^1_{A_0}(\varphi_1), \ldots, I^1_{A_0}(\varphi_n)) I^2_{A_0}(H_{\varphi_k}) = AF(\omega_t). \]

The following is the main result of the present contribution.

**Theorem 14.** For all observables \( F \in C(M) \) of the form above, it holds

\[
\int_X AF(M) d\mu(M) = 0.
\]

As a consequence, the densely defined operator \((A, C(M))\) is skew-symmetric.

In Section 5 we will detail how from this one can deduce the existence of self-adjoint extensions of \((A, C(M))\), and discuss related issues anticipated in the Introduction.

Before we move to the proof of Theorem 14, we digress on the particular case in which \( M \sim [0, 0, \nu] \), and \( \nu \) is a finite measure, which is both simpler and better behaved with respect to the general one, and it also has a particular relevance in fluid-dynamics.

### 3.3. Point Vortices

If \( M \sim [0, 0, \nu] \) with \( \nu \) a finite measure on \( \mathbb{R} \), then \( M \) has the following representation: letting \( \nu = c \sigma \) with \( \sigma \) a probability measure, there exist (on the same probability space on which \( M \) is defined) a Poisson process \( N_c \) of parameter \( c \), i.i.d. uniform variables \( x_1, x_2, \ldots \) on \( D \) and i.i.d. real variables \( \gamma_1, \gamma_2, \ldots \) with law \( \sigma \), all these variables being independent, such that \( M = \sum_{i=0}^{\infty} \gamma_i \delta_{x_i} \).

Euler’s evolution starting from an initial datum made of finitely many (weighted) Dirac’s deltas becomes a finite-dimensional system, in which positions \( x_i \) evolve but intensities \( \gamma_i \) do not: this is the so-called point vortex system, for which we recall a classical well-posedness result.

**Proposition 15.** Given any \( \gamma_1, \ldots, \gamma_N \in \mathbb{R}^* \), for almost all \((x_1, \ldots, x_N) \in D^N\) with respect to the \( N \)-fold product Lebesgue measure, there exist a unique, global-in-time solution of the singular ODE system

\[
\dot{x}_i(t) = \sum_{j \neq i} \gamma_j K(x_i(t), x_j(t)), \quad x_i(0) = x_i.
\]

Such solutions define a measurable, measure-preserving flow \( T_t : D^N \to D^N \).

The proof can be obtained with minor modifications to the one given in [24, Chapter 2] for the case of the disk \( D = \{ x \in \mathbb{R}^2 : |x| < 1 \} \), as discussed therein. The above dynamics is meant to describe the evolution of a planar, incompressible, inviscid fluid whose vorticity is concentrated at a finite number of points. These points, called vortices, move under according to the above ODE dynamics, each under the influence of the other ones, without self-interactions.

There is of course a deep connection between point vortex dynamics and Euler’s equations Eq. (1.1) in PDE form: the former can be obtained as a limit of \( L^\infty(D) \)
solutions of the latter, as detailed by a well-established part of the related literature (see the recent work [11] for complete references). Our interest for this result stems from the fact that the empirical measure of vortices, \( \omega_t = \sum_{i=1}^{N} \gamma_i \delta_{x_i} \), still satisfies Euler’s equations if these are considered in the symmetrized form Eq. (3.1), provided that we set \( H_x(x,x) = 0 \) for all \( x \), in order to encode the absence of self-interaction, see [25].

With that being said, it is clear that our study in the case \( M \sim [0,0,\nu] \), finite \( \nu \), essentially reduces to the point vortex problem. Indeed, each given sample of such \( M \) can be regarded as the empirical measure of a finite set of vortices, and the above observations immediately lead to:

**Corollary 16.** For almost all \( P = \sum_{i=1}^{N_c} \gamma_i \delta_{x_i} \) with respect to the law of \( P \sim [0,0,\nu] \), \( \nu \) finite, by Proposition 15 we can define

\[
T_t(P) = T_t\left( \sum_{i=1}^{N_c} \gamma_i \delta_{x_i} \right) = \sum_{i=1}^{N_c} \gamma_i \delta_{x_i(t)},
\]

with \( x_i(t) \) the solution of Eq. (3.5) having \( N = N_c \) and the \( \gamma_i \)'s prescribed by \( P \), starting from \( x_i(0) = x_i \). Such \( T_t \) is a solution flow for Euler’s dynamics preserving the random measure \( P \) in the sense of Definition 8.

**Proof.** The number of vortices and their intensities are invariant for the dynamics of Proposition 15. Hence, it suffices to condition on random variables \( N_c, \gamma_i \): \( x_i \)'s are then uniform i.i.d. random variables on \( D \), so the dynamics Eq. (3.5) preserves their law. \( \square \)

Proposition 15 implies that for almost all configurations of vortices the evolution is unique, because vortices never travel too close one to another; the ODE system Eq. (3.5) is driven by a vector field involving \( K \), which is singular only when two or more \( x_i \)'s coincide. This implies (as discussed in [6]) Markov uniqueness of extensions of the generator \( A \).

However, it is possible to exhibit configurations of vortices leading to collapse: we refer to [18] for details. How these phenomena relate to uniqueness in the extension of the generator \( A \) on cylinder function is an interesting problem, for which we refer to [6, 16] (see also Section 5 for further remarks on general \( M \sim [0,q,\nu] \)).

Finally, we stress that the case in which \( \nu \) is not a finite measure has a completely different behavior: in the general case \( M \sim [0,0,\nu] \), samples are infinite sums of Dirac’s deltas (with summable weights), so the problem has an infinite-dimensional nature, just as in the purely Gaussian case. Indeed, existence of a solution flow for general \( \nu \) remains an open problem.

## 4. Infinitesimal Invariance

We will first prove Theorem 14 separately in the purely Gaussian and Poissonian cases, by using techniques specific to those setting, namely Malliavin’s integration by parts and Mecke’s formula. The latter will also be used to deal with the interaction between Gaussian and Poissonian parts.

Infinitesimal invariance of the Gaussian white noise was first proved in [8, 9] on the 2-dimensional torus, and our computation generalizes theirs on arbitrary bounded domain. We will need to resort to Fourier series, and the following observation:

**Remark 17.** Since \( G(x,y) \) vanishes if either \( x \) or \( y \in \partial D \), \( \nabla_x G(x,y) \) is parallel to the outward normal \( \hat{n} \) of \( \partial D \). With this in mind, integration by parts shows that

\[
\forall \phi \in \mathcal{S}(D), \quad \int_D H_\phi(x,y)dx = 0.
\]
4.1. Fourier Decomposition and Gaussian Infinitesimal Invariance. Denote by \( \{ \phi_k \}_{k \in \mathbb{N}} \) a complete orthonormal basis of \( L^2(D) \) diagonalizing \(-\Delta\) with Dirichlet boundary conditions,
\[
-\Delta \phi_k = \lambda_k \phi_k, \quad \lambda_k \geq 0.
\]
In what follows, imposing that \( \lambda_k \)'s are ordered increasingly is not relevant. The forthcoming two Lemmata collect important computations we will rely on, as they concern expressions naturally appearing when considering the Fourier expansion of \( H_\phi \).

**Lemma 18.** For any eigenfunction \( \phi_k \) and \( y \in D \) it holds
\[
\int_D \phi_k(x) H_\phi(x,y) dx = 0.
\]

**Proof.** Integrating by parts, we have:
\[
2 \int_D \phi_k(x) H_\phi(x,y) dx = \int_D \phi_k(x) (\nabla \phi_k(x) - \nabla \phi_k(y)) \cdot K(x,y) dx
\]
\[
= \int_D \frac{1}{2} (\phi_k(x))^2 \cdot K(x,y) dx - \int_D \phi_k(x) K(x,y) dx
\]
\[
= -\frac{1}{2} \int_D \phi_k(x)^2 \div K(x,y) dx + \nabla \phi_k(y) \cdot \int_D \nabla^\perp \phi_k(x) G(x,y) dx,
\]
from which we conclude recalling that \( \div K = 0 \) and
\[
\int_D \nabla^\perp \phi_k(x) G(x,y) dx = \frac{1}{\lambda_k} \nabla^\perp \phi_k(y),
\]
the latter being orthogonal to \( \nabla \phi_k(y) \) at all points \( y \in D \).

**Lemma 19.** Define, for \( h, k, \ell \in \mathbb{N} \),
\[
C_{h,k,\ell} = \int_D \int_D H_\phi(x,y) \phi_k(x) \phi_h(y) dx dy.
\]
It holds that:
- if any two indices coincide, \( C_{h,k,\ell} = 0 \);
- for all \( h, k, \ell \in \mathbb{N} \), \( C_{h,k,\ell} = C_{k,h,\ell} \), that is: \( C \) is symmetric in the first two indices;
- for all \( h, k, \ell \in \mathbb{N} \),
\[
(4.1) \quad C_{h,k,\ell} + C_{k,h,\ell} + C_{k,\ell,h} = C_{h,\ell,k} + C_{k,h,\ell} + C_{\ell,k,h} = 0,
\]
that is, \( C_{h,k,\ell} \) is cyclic symmetric in its indices.

*En passant*, we recall that cyclic symmetric functions are one of the three symmetry classes partitioning the set of functions of three variables, the other classes being the symmetric and antisymmetric functions, [27].

**Proof.** Expanding the definition of \( H_\phi \), recalling that \( K(x,y) = \nabla^\perp_x G(x,y) \) and integrating by parts in \( x \), we obtain
\[
C_{h,k,\ell} = \frac{1}{2} \int_D \int_D (\nabla \phi_{\ell}(x) - \nabla \phi_{\ell}(y)) \nabla^\perp_x G(x,y) \phi_k(x) \phi_h(y) dx dy
\]
\[
= \left( \frac{1}{\lambda_h} - \frac{1}{\lambda_k} \right) \int_D \nabla \phi_{\ell}(x) \nabla^\perp \phi_k(x) \phi_h(y) dx dy.
\]
If any two indices coincide, the latter expression readily shows that \( C_{h,k,\ell} = 0 \): either the difference of eigenvalues is null or the integrand vanishes due to orthogonality.
Starting from the integral in \(dx\) only we just obtained to represent \(C_{h,k,\ell}\), integrating by parts we have

\[
C_{\ell,h,k} = \left( \frac{1}{\lambda_k} - \frac{1}{\lambda_h} \right) \int_D \nabla \phi_\ell(x) \nabla^\perp \phi_k(x) \phi_h(x) dx,
\]

\[
C_{k,\ell,h} = \left( \frac{1}{\lambda_k} - \frac{1}{\lambda_\ell} \right) \int_D \nabla \phi_\ell(x) \nabla^\perp \phi_k(x) \phi_h(x) dx,
\]

which suffices to prove that the right-hand side of Eq. (4.1) vanishes. Noticing that \(C_{h,k,\ell}\) is invariant under exchange of the first two indices, which is clear from the definition, concludes the proof of Eq. (4.1), since this symmetry transforms one member of the equation into the other.

Let us introduce a subclass of cylinder functions depending only on couplings with elements of the Fourier basis \(\{\phi_k\}_{k \in \mathbb{N}}\).

\[
C_{F}(W) = \{F(W) = f(W(\phi_1), \ldots, W(\phi_n)), n \in \mathbb{N}, f \in C^\infty_b(\mathbb{R}^n; \mathbb{C})\}.
\]

We first prove infinitesimal invariance on these observables, and then extend by density to the full \(\mathcal{C}(W)\).

**Lemma 20.** For all \(F \in \mathcal{C}(W)\) it holds

\[
\mathbb{E} \left[ AF(W) \right] = 0.
\]

**Proof.** Consider first \(F \in C_{F}(W)\): in this case we can apply directly Gaussian integration by parts,

\[
\mathbb{E} \left[ \sum_{k=1}^n \partial_k f(W(\phi_1), \ldots, W(\phi_k)) I^2_W (H_{\phi_k}) \right]
\]

\[
= \mathbb{E} \left[ \int_D D_x F(W) \left( \sum_{k=1}^n \phi_k(x) I^2_W (H_{\phi_k}) \right) dx \right]
\]

\[
= \mathbb{E} \left[ F(W) D^* \left( \sum_{k=1}^n \phi_k(x) I^2_W (H_{\phi_k}) \right) \right],
\]

denoting by \(D\) the Malliavin derivative and \(D^*\) its adjoint (see e.g. [30, Chapter 2]). Gaussian divergence in the right-hand side is given by

\[
D^* \left( \sum_{k=1}^n \phi_k(x) I^2_W (H_{\phi_k}) \right)
\]

\[
= \sum_{k=1}^n I^1_W (\phi_k) I^2_W (H_{\phi_k}) - 2 \sum_{k=1}^n \int_D \phi_k(x) I^1_W (H_{\phi_k}(x, \cdot)) dx.
\]

We now notice that

\[
\int_D \phi_k(x) I^1_W (H_{\phi_k}(x, \cdot)) dx = I^1_W \left( \int_D \phi_k(x) H_{\phi_k}(x, \cdot) dx \right) = 0
\]

thanks to **Lemma 18**, so the second part of the divergence vanishes. Moreover, if \(\mathcal{F}_n = \sigma \left( I^1_W (\phi_1), \ldots, I^1_W (\phi_n) \right)\) is the \(\sigma\)-algebra generated by standard Gaussian variables \(I^1_W (\phi_1), \ldots, I^1_W (\phi_n)\), \(F(W)\) is \(\mathcal{F}_n\)-measurable, so conditioning on \(\mathcal{F}_n\) we obtain

\[
\mathbb{E} \left[ AF(W) \right] = \mathbb{E} \left[ F(W) \sum_{k=1}^n I^1_W (\phi_k) \mathbb{E} \left[ I^2_W (H_{\phi_k}) \mid \mathcal{F}_n \right] \right].
\]
We conclude by proving that the inner sum in the last expression is almost surely null. Expanding the function of two variables $H_{\phi_i}(x, y)$ in Fourier basis, its coefficients are given by $C_{h, k, \ell}$ of Lemma 19, thus we now write

$$E \left[ I_W^1(H_{\phi_k}) \mid F_n \right] = E \left[ I_W^1 \left( \sum_{k, h, \ell \in \mathbb{N}} \phi_k(x) \phi_h(y) C_{h, k, \ell} \right) \mid F_n \right] = \sum_{n=1}^{\infty} W_n (h, k, \ell) E \left[ I_W^2(H_{\phi_k}) \mid F_n \right],$$

where the very last triple sum vanishes almost surely: indeed, each summand is the product of $I_W^1(\phi_h)I_W^1(\phi_k)C_{h, k, \ell}$, which is symmetric in the indices $h, k, \ell$, and $C_{h, k, \ell}$, which is cyclic symmetric by Lemma 19. This concludes the thesis for $F \in C_F(W)$.

Consider now, for $m \in \mathbb{N}$,

$$F(W) = f(I_W^1(\varphi_1), \ldots, I_W^1(\varphi_1)), \quad F_m(W) = f(I_W^1(\Pi_m \varphi_1), \ldots, I_W^1(\Pi_m \varphi_1)),$$

where $\Pi_m \varphi$ denotes the projection of $\varphi$ on the linear span of $\varphi_1, \ldots, \varphi_m$. We will also write $\Pi_m$ for the projection on the orthogonal space. We already proved the thesis for observables $F_m \in C_F(W)$, so the proof is concluded if we show that $AF - AF_m$ converges to 0 in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. Using the definition of $A$ we can expand

$$AF(W) - AF_m(W) = \sum_{k=1}^{\infty} \partial_k f(I_W^1(\varphi_1), \ldots) \left[ I_W^1(H_{\varphi_k}) - I_W^1(H_{\Pi_m \varphi_k}) \right]$$

$$+ \sum_{k=1}^{\infty} \left[ \partial_k f(I_W^1(\varphi_1), \ldots) - \partial_k f(I_W^1(\Pi_m \varphi_1), \ldots) \right] I_W^1(H_{\Pi_m \varphi_k})$$

$$:= (1) + (2),$$

and show that each of the two sums on the right-hand side go to 0 in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ as $m \to 0$. Since derivatives of $f$ are uniformly bounded, we have

$$E \left[ (1)^2 \right] \leq n \|f\|^2_{C^1} \sum_{k=1}^{\infty} E \left[ \left( I_W^2(H_{\varphi_k}) - I_W^2(H_{\Pi_m \varphi_k}) \right)^2 \right],$$

in which every summand vanishes as $m \to \infty$, since

$$E \left[ (I_W^2(H_{\varphi_k}) - I_W^2(H_{\Pi_m \varphi_k}))^2 \right] = E \left[ I_W^2(H_{\Pi_m \varphi_k})^2 \right] = \left\| H_{\Pi_m \varphi_k} \right\|^2_{L^2(D)} \leq |D|^2 \left\| H_{\Pi_m \varphi_k} \right\|^2_{C^1(D)} \leq C|D|^2 \left\| \Pi_m \varphi_k \right\|^2_{C^2(D)} \xrightarrow{m \to \infty} 0,$$

where the first step uses linearity of $H_{\varphi}$ in $\varphi$, the second uses Itô isometry, the fourth Eq. (3.2) and the conclusion follows since $\varphi$ is smooth (thus it is approximated by Fourier truncations in all Sobolev norms, and by Sobolev embedding in $C^k$ norms). As for the second sum, by Taylor expanding derivatives of $f$ we have

$$E \left[ (2)^2 \right] \leq \|f\|^2_{C^2} \sum_{k=1}^{\infty} E \left[ I_W^2(\Pi_m \varphi_k)I_W^1(H_{\Pi_m \varphi_k}) \right],$$

in which every summand vanishes as $m \to \infty$, by virtue of the Cauchy-Schwarz inequality and of the relations

$$E \left[ I_W^1(\Pi_m \varphi_k) \right] = \left\| \Pi_m \varphi_k \right\|^2_{L^2(D)} \xrightarrow{m \to \infty} 0,$$

$$E \left[ I_W^2(H_{\Pi_m \varphi_k}) \right] \leq C|D|^2 \left\| \Pi_m \varphi_k \right\|^2_{C^2(D)} \leq C|D|^2 \left\| \varphi_k \right\|^2_{C^2(D)}.$$
where the second estimate is deduced by rehearsing arguments similar to those given above. □

4.2. Poissonian Infinitesimal Invariance. We consider first invariance of a purely Poissonian random measure.

Lemma 21. If $P \sim [0, 0, \nu]$, then for all $F \in \mathcal{C}(P)$ it holds
\[
\mathbb{E}[AF(P)] = 0.
\]

Proof. Applying the bivariate Mecke’s formula [23, Theorem 4.4] applied to the Poisson point process $\mathcal{N}$ on $\mathbb{R} \times \mathcal{D}$ with intensity $d\nu(\xi) \otimes dx$, we infer that
\[
\begin{align*}
\mathbb{E} & \left[ \sum_{k=1}^{n} \partial_k f(I_P^1(\varphi_1), \ldots, I_P^1(\varphi_k))I_P^2(H_{\varphi_k}) \right] = \\
& = \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathcal{D}^2} \mathbb{E} \left[ \partial_k f(I_P^1(\varphi_1) + \xi \varphi_1(x) + \xi' \varphi(x'), \ldots) \right] \\
& \quad \cdot \xi'^2 H_{\varphi_k}(x, x') d\nu(\xi) d\nu(\xi') dx dx'.
\end{align*}
\]

Conclusion now follows from integration by parts on $\mathcal{D}$. Observe first that
\[
(4.2) \quad \nabla_x F(P + \xi \delta_x + \xi' \delta_{x'}) = \sum_{k=1}^{n} \partial_k f(I_P^1(\varphi_1) + \xi \varphi_1(x) + \xi' \varphi(x'), \ldots) \nabla \varphi_k(x),
\]

so combining the latter, the definition of $H_{\varphi}$ and Mecke’s formula above we obtain
\[
\begin{align*}
\mathbb{E} & \left[ \sum_{k=1}^{n} \partial_k f(I_P^1(\varphi_1), \ldots, I_P^1(\varphi_k))I_P^2(H_{\varphi_k}) \right] = \\
& = \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathcal{D}^2} \mathbb{E} \left[ \nabla_x F(P + \xi \delta_x + \xi' \delta_{x'}) - \nabla_{x'} F(P + \xi \delta_x + \xi' \delta_{x'}) \right] \\
& \quad \cdot K(x, x') d\nu(\xi) d\nu(\xi') dx dx'.
\end{align*}
\]

Integrating by parts in $x$ and $x'$ this last expression and recalling that $K(x, x')$ is divergence-less in both variables, the proof is concluded. □

Combining invariance of the Gaussian and Poissonian parts, we deduce infinitesimal invariance on cylinder functions.

Lemma 22. Given any $M \sim [a, q, \nu]$, for all $F \in \mathcal{C}(M)$ it holds
\[
\mathbb{E}[AF(M)] = 0.
\]

Proof. Assume first that $a = 0$. We distinguish contributions from the Gaussian and Poissonian parts, and exploit their independence. Starting from the definition of $A$ in Eq. (3.4) we expand the double stochastic integral according to the decomposition $M = \sqrt{\mathcal{W}} + P$,
\[
I_M^2(H_{\varphi}) = q I_W^2(H_{\varphi}) + 2 \sqrt{q} I_P^1 I_W^1(H_{\varphi}) + I_P^2(H_{\varphi}).
\]

By linearity and Lemmas 20 and 21 the thesis reduces to
\[
\mathbb{E}[AF(M)] = \mathbb{E} \left[ \sum_{k=1}^{n} \partial_k f(I_M^1(\varphi_1), \ldots, I_M^1(\varphi_k))I_M^2 I_W^1(H_{\varphi_k}) \right] = 0.
\]

In order to apply Gaussian integration by parts we resort again to Fourier decomposition and assume $F \in \mathcal{C}_F(M)$, that is $\varphi_k = \phi_n$ are elements of the Fourier basis. We denote in the following by $D_W$ and $D_W^*$ the Malliavin derivative and
Skorohod integral in the Gaussian space associated to $W$. For a fixed sample of the compound Poisson process $P$, the functions
\[ \sum_{k=1}^{n} \partial_k f(I_P^1(\phi_1) + \sqrt{q} I_W^1(\phi_1), \ldots, \phi_1(x), \sum_{k=1}^{n} I_P^1 I_W^1 (H_{\phi_k}) \phi_1(x), \]
are cylinder functions of $W$, that is they belong to $\mathcal{C}_F(W)$, thus by conditioning on $P$ and Gaussian integration by parts (applied as in Theorem 20) we obtain
\[ \mathbb{E} [AF(M)] = \mathbb{E} \left[ \sum_{k=1}^{n} \partial_k f(\sqrt{q} I_W^1(\phi_1) + I_P^1(\phi_1), \ldots, \sqrt{q} I_P^1 I_W^1 (H_{\phi_k}) \right] \]
\[ = \mathbb{E} \left[ F(M) \sqrt{q} D^* \left( \sum_{k=1}^{n} \phi_k I_P^1 I_W^1 (H_{\phi_k}) \right) \right]. \]
The Gaussian divergence in the latter is given by
\[ D^* \left( \sum_{k=1}^{n} \phi_k I_P^1 I_W^1 (H_{\phi_k}) \right) \]
\[ = \sum_{k=1}^{n} I_W^1(\phi_k) I_P^1 I_W^1 (H_{\phi_k}) - 2I_P^1 \left( \int_D \phi_k(x) H_{\phi_k}(x, \cdot) dx \right), \]
in which the deterministic integral vanishes because of Lemma 18, so
\[ \mathbb{E} [AF(M)] = \mathbb{E} \left[ F(M) \sum_{k=1}^{n} I_W^1(\phi_k) I_P^1 I_W^1 (H_{\phi_k}) \right]. \]
By expanding in Fourier series $H_{\phi_k}$ with symbols $C_{h,k,\ell}$ defined in Lemma 19, and conditioning on the $\sigma$-algebra generated by $I_P^1(\phi_1), \ldots I_W^1(\phi_n)$ and $I_P(\phi_1), \ldots I_P(\phi_n)$ similarly to what we did in the proof of Lemma 20, we arrive at
\[ \mathbb{E} [AF(M)] = \mathbb{E} \left[ F(M) \sum_{h,k,\ell=1}^{n} I_W^1(\phi_k) I_W^1 (\phi_k) I_P^1(\phi_h) C_{h,k,\ell} \right]. \]
We conclude once again with a symmetry argument relying on Lemma 19: since $C_{h,k,\ell}$ is invariant under exchange of $h, k$, we can symmetrize the two indices, so that the inner sum becomes
\[ \sum_{h,k,\ell=1}^{n} I_W^1(\phi_k) I_W^1 (\phi_k) I_P^1(\phi_h) C_{h,k,\ell} \]
\[ = \frac{1}{2} \sum_{h,k,\ell=1}^{n} I_W^1(\phi_k) \left( I_W^1(\phi_k) I_P^1(\phi_h) + I_P^1(\phi_k) I_W^1(\phi_h) \right) C_{h,k,\ell}, \]
and thus must vanish, since each summand is the product of the cyclic-symmetric $C_{h,k,\ell}$ and a symmetric function of the indices.

In order to remove the additional hypothesis $F \in \mathcal{C}_F(M)$ we can proceed by approximation as in Lemma 20. For $m \in \mathbb{N}$ one considers, as we did above, the Fourier truncation $\Pi_{m,\varphi}$ of a function $\varphi$, and shows that the following difference vanishes in $L^2(\Omega, \mathcal{F}, \mathbb{P})$:
\[ \sum_{h,k,\ell=1}^{n} \partial_k f(I_M^1(\varphi_1), \ldots, I_M^2 (H_{\varphi_k}) - \sum_{h,k,\ell=1}^{n} \partial_k f(I_W^1(\Pi_{m,\varphi_1}), \ldots, I_W^2 (H_{\Pi_{m,\varphi_k}}). \]
The proof actually proceeds in complete analogy with the one of Lemma 20, the only difference being that now mixed integrals appear in the estimates: this does
not create any additional difficulty since one only uses isometric relations, that are the same for multiple and iterated integrals.

Finally, we remove the additional hypothesis $a = 0$. This is easily done thanks to the cancellation observed in Remark 17: if $M = a + \sqrt{t}W + P$ it holds

$$I_M^2(H) = a^2 \int_{\mathbb{D}^2} H(x,y)dxdy + I_{\sqrt{t}W+P}^1\left( a \int_{\mathbb{D}} H(x,s)dx \right) + I_{\sqrt{t}W+P}^2(H)$$

so one can always reduce to the case $a = 0$ by considering the cylinder function of $M - a$ defined by $G(M - a) = F(M)$.

We are thus only left to prove the last claim of Theorem 14.

**Corollary 23.** For $M \sim [a,q,\nu]$, the linear operator $\mathcal{A}$ defined on cylinder functions $\mathcal{C}(M)$ by Eq. (3.4) is a skew-symmetric operator in $L^2(M)$.

**Proof.** It suffices to observe that $\mathcal{A}$ acts on $\mathcal{C}(M)$ as a derivation, that is we have the following Leibnitz rule: for $F, G \in \mathcal{C}(M)$ also $FG \in \mathcal{C}(M)$ and

$$\mathcal{A}(FG) = \mathcal{A}F \cdot G + F \cdot \mathcal{A}G,$$

as it follows from direct inspection of the definition. From Lemma 22 it then follows

$$\mathbb{E}[FAG] = \mathbb{E}[A(FG)] - \mathbb{E}[GAF] = -\mathbb{E}[GAF].$$

**Remark 24.** The argument of Corollary 23 clearly applies also to $(\mathcal{A}, \mathcal{C}_F(M))$, which is thus closable in $L^2_\mathbb{C}(\mathcal{X}, m)$. The part of Lemma 20 extending the thesis from $\mathcal{C}_F(W)$ to $\mathcal{C}(W)$ can then be understood as a proof that $\mathcal{C}(W)$ is contained in the domain of the closure of $(\mathcal{A}, \mathcal{C}_F(W))$.

5. **Concluding Remarks, Generalizations and Open Problems**

We now briefly comment on point (c) of the program outlined in Section 1.2, and conclude the paper with some remarks on generalization and possible further developments.

5.1. **Self-Adjoint Extensions, Uniqueness and Solution Flows.** By Corollary 23, the operator $(\mathcal{A}, \mathcal{C}(M))$ defined by Eq. (3.4) is closable in $L^2_\mathbb{C}(\mathcal{X}, m)$. We can extend $(\mathcal{A}, \mathcal{C}(M))$ to the generator of a group of unitaries on $L^2_\mathbb{C}(\mathcal{X}, m)$, by means of the following results from Functional Analysis.

**Proposition 25** (Stone). To each self-adjoint operator $(L, D(L))$ on a separable (complex) Hilbert space $H$ it is associated a unique strongly continuous one-parameter group of unitary operators $U_t : H \to H$, $t \in \mathbb{R}$ (usually denoted $U_t = e^{itL}$), such that

$$\forall f \in D(L), \quad \frac{d}{dt}U_tf = iLf.$$

**Proposition 26** (Von Neumann). Let $(L, D(L))$ be a densely defined symmetric operator on a separable (complex) Hilbert space $H$. Assume that there exists an antilinear, norm-preserving involution $J : H \to H$ mapping $D(L)$ into itself and commuting with $L$ on $D(L)$. Then $L$ has at least a self-adjoint extension.

These results concern self-adjoint extensions of symmetric operators, but in our setting $(\mathcal{A}, \mathcal{C}(M))$ is skew-symmetric, so in order to apply these results we need to consider $\tilde{A} = -iA$. The operator $\tilde{A}$ is now a densely defined, symmetric operator on $L^2_\mathbb{C}(\mathcal{X}, m)$, and it commutes with

$$L^2_\mathbb{C}(\mathcal{X}, m) \ni F \mapsto JF \in L^2_\mathbb{C}(\mathcal{X}, m), \quad JF(x) = F(-x), \ x \in \mathcal{X},$$
which satisfies the hypothesis of Proposition 26. We conclude that there exists a self-adjoint extension of $\tilde{A}$, and by Proposition 25 we can consider the associated group of unitaries $U_t : L^2(X, m) \rightarrow L^2(X, m)$. Here we encounter the first difficult open question (which remains open also in the purely Gaussian and Poissonian case):

**Open Question 27.** Is $(\tilde{A}, \mathcal{C}(M))$ essentially self-adjoint? Equivalently, is the closure of $(\tilde{A}, \mathcal{C}(M))$ already a self-adjoint operator?

It is immediate to observe that $U_t$ is unit-preserving: the constant 1 belongs to $\mathcal{C}(M)$ and $U_0 1 = 1 = U_0 1$, $\frac{d}{dt} U_t 1 = -i A 1 = 0$.

The only requirement left to satisfy the hypothesis of Proposition 12 is preservation of positivity.

**Open Question 28.** Are there self-adjoint extensions of $(\tilde{A}, \mathcal{C}(M))$ that generate a positivity preserving group of unitaries $U_t$?

Indeed, in that case, by Proposition 12, $U_t$ would be the Koopman group of a one-parameter group $T_t : (X, m) \rightarrow (X, m)$ of almost surely invertible, measurable, measure-preserving transformations, that is $U_t F = F \circ T_t$ for all $F \in L^2(X, m)$. Existence of a solution flow in the sense of Definition 11 would follow observing that transformations $T_t$ produce solutions of Euler’s dynamics in the sense of Definition 8, see Lemma 13.

### 5.2. Compact Surfaces Without Boundaries.

All the above arguments are easily repeated in the case where the domain $D$ is replaced by a compact surface without boundary, since the analytic (thus, functional analytic) setting is completely analogous. We refer to [25, 19] for Euler’s dynamics, both in the PDE case and point vortices model, on surfaces such as the 2-dimensional torus and sphere, in particular to the second reference for a comparison between those settings and the bounded domain case.

### 5.3. About Point Vortices Approximations.

It should be possible to prove existence of solutions in the sense of Definition 8 preserving a completely random measure $M \sim [a, q, \nu]$ as a scaling limit of point vortices systems, that is considering the empirical measure of the particle system described in Section 3.3, and taking a limit in which the number of vortices $N \rightarrow \infty$ and intensities of vortices are suitably rescaled. This is the strategy of [14, 19] in the purely Gaussian case, which we propose to investigate in our setting in future works.

One one hand, it might be possible to identify a suitably weak compactness criterion in spaces of distributions so that finiteness of moments of $\nu$ is not necessary for compactness estimates, leading to an even larger class of invariant measures. On the other hand, building of solutions by means of such approximations clearly does not imply any uniqueness result, and neither the flow property of Definition 11.

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