NON-POTENTIAL AND NON-RADIAL DIRICHLET SYSTEMS WITH MEAN CURVATURE OPERATOR IN MINKOWSKI SPACE

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Abstract. We deal with a multiparameter Dirichlet system having the form
\[
\begin{align*}
-M(u) &= \lambda_1 f_1(u, v), & \text{in } \Omega, \\
-M(v) &= \lambda_2 f_2(u, v), & \text{in } \Omega, \\
u|_{\partial \Omega} &= 0 = v|_{\partial \Omega},
\end{align*}
\]
where \(M\) stands for the mean curvature operator in Minkowski space
\[
M(u) = \text{div} \left( \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right),
\]
\(\Omega\) is a general bounded regular domain in \(\mathbb{R}^N\) and the continuous functions \(f_1, f_2\) satisfy some sign and quasi-monotonicity conditions. Among others, these type of nonlinearities, include the Lane-Emden ones. For such a system we show the existence of a hyperbola like curve which separates the first quadrant in two disjoint sets, an open one \(O_0\) and a closed one \(F\), such that the system has zero or at least one strictly positive solution, according to \((\lambda_1, \lambda_2) \in O_0\) or \((\lambda_1, \lambda_2) \in F\). Moreover, we show that inside of \(F\) there exists an infinite rectangle in which the parameters being, the system has at least two strictly positive solutions. Our approach relies on a lower and upper solutions method - which we develop here, together with topological degree type arguments. In a sense, our results extend to non-radial systems some recent existence/non-existence and multiplicity results obtained in the radial case.

1. Introduction. Let \(\Omega\) be a bounded domain in \(\mathbb{R}^N\) (\(N \geq 2\)) with boundary \(\partial \Omega\) of class \(C^2\) and \(f_1, f_2 : \Omega \times \mathbb{R}^2 \to \mathbb{R}\) be two mappings satisfying the \(L^\infty\)-Carathéodory condition, i.e.,

\((H_f)\) \((i)\) \(f_i(\cdot, u, v) : \Omega \to \mathbb{R} \ (i = 1, 2)\) are measurable for all \((u, v) \in \mathbb{R}^2\);

\((ii)\) \(f_i(x, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R} \ (i = 1, 2)\) are continuous on \(\mathbb{R}^2\) for a.e. \(x \in \Omega\);

\((iii)\) for each \(\rho > 0\) there is some \(\alpha_\rho \in L^\infty(\Omega)\) such that

\[
|f_i(x, u, v)| \leq \alpha_\rho(x) \ (i = 1, 2),
\]

for a.e. \(x \in \Omega\) and all \((u, v) \in \mathbb{R}^2\) with \(|(u, v)| \leq \rho\).

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where \(| \cdot |\) is the Euclidean norm in \(\mathbb{R}^2\).

In this paper we deal with non-potential systems of type

\[
\begin{align*}
-\mathcal{M}(u) &= f_1(x, u, v), \quad x \in \Omega, \\
-\mathcal{M}(v) &= f_2(x, u, v), \quad x \in \Omega, \\
u|_{\partial \Omega} &= 0 = v|_{\partial \Omega},
\end{align*}
\]

where \(\mathcal{M}\) stands for the mean curvature operator in Minkowski space:

\[
\mathcal{M}(u) = \text{div} \left( \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} \right).
\]

Problems involving operator \(\mathcal{M}\) are originated in differential geometry and in the theory of relativity, being related to maximal and constant mean curvature spacelike hypersurfaces (spacelike submanifolds of codimension one in the flat Minkowski space \(\mathbb{R}^{N+1}\) := \(\{(x, t) : x \in \mathbb{R}^N, t \in \mathbb{R}\}\) endowed with the Lorentzian metric \(\sum_{j=1}^{N} (dx_j)^2 - (dt)^2\), where \((x, t)\) are the canonical coordinates in \(\mathbb{R}^{N+1}\) having the property that the trace of the extrinsic curvature is zero, respectively constant. The importance of spacelike hypersurfaces in the study of different aspects (singularities, gravitational radiation, positivity of mass) from general relativity was emphasized in [10, 11, 21, 22, 23]. Also, their geometric interest is motivated by the fact that the spacelike hypersurfaces in the Minkowski space display nice Bernstein-type properties [1, 2, 8, 9].

In recent years, many papers were devoted to the study of Dirichlet problems for a single equation with operator \(\mathcal{M}\) in a general bounded domain (see e.g. [3, 5, 6, 13, 14, 15, 20]), while at our best knowledge, for systems with such an operator the study was recently initiated in [18]. Also, results for systems having a radial structure in a ball were obtained in [4, 16, 17]. So, using a variational approach, in the case of one parameter systems of type

\[
\begin{align*}
-\mathcal{M}(u) &= \lambda F_u(x, u, v), \quad x \in \Omega, \\
-\mathcal{M}(v) &= \lambda F_v(x, u, v), \quad x \in \Omega, \\
u|_{\partial \Omega} &= 0 = v|_{\partial \Omega},
\end{align*}
\]

it was proved in [18] that, under suitable assumptions on the potential \(F\), there exists a positive number \(\Lambda\) such that (2) has at least two non-trivial and non-negative solutions, for all \(\lambda > \Lambda\). In [16], for systems involving Lane-Emden type perturbations of the operator \(\mathcal{M}\) and having a variational structure,

\[
\begin{align*}
-\mathcal{M}(u) &= \lambda \mu(|x|)(p + 1)u^{p+1}, \quad \text{in } B_R, \\
-\mathcal{M}(v) &= \lambda \mu(|x|)(q + 1)v^{q+1}, \quad \text{in } B_R, \\
u|_{\partial \Omega} &= 0 = v|_{\partial \Omega},
\end{align*}
\]

where \(B_R = \{x \in \mathbb{R}^N : |x| < R\}\) \((R > 0, N \geq 2)\), the positive exponents \(p, q\) satisfy \(\max\{p, q\} > 1\) and the function \(\mu : [0, R] \to [0, \infty)\) is continuous and \(\mu(r) > 0\) for all \(r \in (0, R]\); it was shown that there exists \(\Lambda > 0\) such that (3) has zero, at least one or at least two positive solutions according to \(\lambda \in (0, \Lambda), \lambda = \Lambda\) or \(\lambda > \Lambda\). This result extends the corresponding one obtained in [7] in the case of a single equation. Then, in the recent paper [17] were considered non-potential radial systems having the form

\[
\begin{align*}
-\mathcal{M}(u) &= \lambda_1 \mu_1(|x|)u^{p_1+1}, \quad \text{in } B_R, \\
-\mathcal{M}(v) &= \lambda_2 \mu_2(|x|)v^{q_2+1}, \quad \text{in } B_R, \\
u|_{\partial \Omega} &= 0 = v|_{\partial \Omega},
\end{align*}
\]
where $\lambda_1, \lambda_2$ are two positive parameters, $p_1, p_2, q_1, q_2$ are positive exponents with $\min\{p_1, q_2\} > 1$ and the weight functions $\mu_1, \mu_2 : [0, R] \to [0, \infty)$ are assumed to be continuous with $\mu_1(r) > 0 < \mu_2(r)$ for all $r \in (0, R]$. Using some fixed point index estimations and lower and upper solutions method, it was proved the existence of a continuous curve $\Gamma$ splitting the first quadrant into two disjoint open sets $O_1$ and $O_2$ such that the system (4) has zero, at least one or at least two positive (radial) solutions according to $(\lambda_1, \lambda_2) \in O_1$, $(\lambda_1, \lambda_2) \in \Gamma$ or $(\lambda_1, \lambda_2) \in O_2$, respectively.

In view of the above it appears as being natural to consider also non-potential systems in a general bounded domain $\Omega \subset \mathbb{R}^N$. Since the solvability of (1) is always guaranteed without any additional assumptions on the functions $f_1$ and $f_2$ (see [18]), a main interest concerns the multiplicity and possible localization of solutions.

It is the goal of this work to study non-existence, existence and multiplicity of strictly positive solutions (see Definition 2.2) for autonomous systems having the form

$$
\begin{cases}
-\mathcal{M}(u) = \lambda_1 f_1(u, v), & \text{in } \Omega, \\
-\mathcal{M}(v) = \lambda_2 f_2(u, v), & \text{in } \Omega, \\
u|_{\partial\Omega} = 0 = v|_{\partial\Omega},
\end{cases}
$$

(5)

where the parameters $\lambda_1, \lambda_2$ are positive, under the following hypothesis:

$(\tilde{H})$ (i) $f_1, f_2 : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ depending on $(u, v)$ are continuous and quasi-monotone nondecreasing with respect to both $u$ and $v$;

(ii) there exist constants $c > 0$, $p_1, q_2 > 1$ and $q_1, p_2 > 0$ such that

$$
0 < f_1(u, v) \leq c u^{p_1} v^{q_1}, \\
0 < f_2(u, v) \leq c u^{p_2} v^{q_2},
$$

(6)

for all $u, v > 0$.

Notice that $(\tilde{H})$ is verified by Lane-Emden type nonlinearities, i.e. $f_1(u, v) = u^{p_1} v^{q_1}$ and $f_2(u, v) = u^{p_2} v^{q_2}$. In our main result (Theorem 4.6) we obtain the existence of a continuous curve which separates the first quadrant in two disjoint sets, an open one $O$ and a closed one $F$, such that (5) has zero or at least one strictly positive solution, according to $(\lambda_1, \lambda_2) \in O$ or $(\lambda_1, \lambda_2) \in F$. The set $O$ is adjacent to the coordinates axes $0\lambda_1$ and $0\lambda_2$ and the curve approaches asymptotically to two lines parallel to the axes $0\lambda_1$ and $0\lambda_2$. Moreover, we show that there exists $(\lambda_1^*, \lambda_2^*) \in F$ such that, for all $\lambda_1 > \lambda_1^*$ and $\lambda_2 > \lambda_2^*$, problem (5) has at least two strictly positive solutions.

Our approach relies on a lower and upper solutions method, which we develop for the general system (1) under some hypotheses of quasi-monotonicity on the nonlinearities $f_1$ and $f_2$. This allows us to depict multiplicity and localization informations about solutions. We note that our lower and upper solutions method - which is different from the one in [16, 17] used for radial systems, as well as the multiplicity and localization results we obtain for problem (1) (Propositions 3.3 - 3.5) are inspired by the corresponding ones proved for a single equation in [13, 14].

The rest of the paper is organized as follows. In Section 2 we present the preliminary setup including some definitions and results later used in our approach. Section 3 is devoted to the lower and upper solutions method where the general results concerning the existence, multiplicity and localization of solutions for system (1) are provided. The main non-existence, existence and multiplicity result for the two parameters system (5) is proved in Section 4.
2. Preliminaries. First, we list some definitions and notations which will be used throughout the paper. Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^N$ with boundary $\partial \mathcal{O}$ of class $C^2$. For two functions $u, v : \overline{\mathcal{O}} \to \mathbb{R}$, we write $u \leq v$ if $u(x) \leq v(x)$ for a.e. $x \in \mathcal{O}$, and $u < v$ if $u \leq v$ and $u(x) < v(x)$ in a subset of $\mathcal{O}$ having positive measure. For functions $u, v \in C^1(\overline{\mathcal{O}})$, we also write $u \ll v$ (in $\overline{\mathcal{O}}$) if $u(x) < v(x)$ for every $x \in \mathcal{O}$ and, if $u(x_0) = v(x_0)$ for some $x_0 \in \partial \mathcal{O}$, then $\frac{\partial u}{\partial \nu}(x_0) < \frac{\partial v}{\partial \nu}(x_0)$, where $\nu = \nu(x_0)$ stands for the unit outer normal to $\partial \mathcal{O}$ at $x_0$. We set $C^1_0(\overline{\mathcal{O}}) = \{ u \in C^1(\overline{\mathcal{O}}) : u = 0$ on $\partial \mathcal{O} \}$, $s^+ = \max\{s, 0\}$, resp. $s^- = \min\{s, 0\}$ ($s \in \mathbb{R}$) and $\| \cdot \|_{\infty}$ will denote the usual sup-norm on $L^\infty(\mathcal{O})$. For given $h \in L^\infty(\mathcal{O})$, by a solution of the Dirichlet problem

\[-M(u) = h \text{ in } \mathcal{O}, \quad u|_{\partial \mathcal{O}} = 0 \quad (7)\]

we mean a function $u_h \in W^{2,r}(\mathcal{O})$ for some $r > N$, with $\| \nabla u_h \|_{\infty} < 1$, which satisfies the equation a.e. in $\mathcal{O}$ and vanishes on $\partial \mathcal{O}$; notice that, since $W^{2,r}(\mathcal{O}) \subset C^1(\overline{\mathcal{O}})$, the evaluation at $\partial \mathcal{O}$ is understood in the usual pointwise sense.

We shall need the following Lemma which is proved in [15, Lemmas 2.1, 2.2 and 2.6].

Lemma 2.1. (i) If $h \in L^\infty(\mathcal{O})$, then problem (7) has an unique solution $u_h$ and $u_h \in W^{2,r}(\mathcal{O})$ for all $1 \leq r < \infty$.

(ii) Let $h \in L^\infty(\mathcal{O})$ be such that $h > 0$ in $\mathcal{O}$. If $u$ is a solution of (7), then $u \gg 0$.

(iii) Let $h_1, h_2 \in L^\infty(\mathcal{O})$ be with $h_1 \leq h_2$ and, for $i = 1, 2$, let $u_i \in W^{2,r}(\mathcal{O})$, for some $r > N$, be such that $\| \nabla u_i \|_{\infty} < 1$ and

\[-M(u_i) = h_i \text{ a.e. in } \mathcal{O}. \quad (8)\]

Then

\[u_1 \leq u_2 - \min_{\partial \mathcal{O}}(u_2 - u_1).\]

Concerning system (1), we adopt the following notion of solution.

Definition 2.2. By a solution of (1) we mean a couple of functions $(u, v) \in C^0[I](\overline{\Omega}) \times C^0[I](\overline{\Omega})$, such that $\| \nabla u \|_{\infty} < 1$, $\| \nabla v \|_{\infty} < 1$, which vanishes on $\partial \Omega$ and satisfies

\[
\int_{\Omega} \frac{\nabla u \cdot \nabla w}{\sqrt{1 - |\nabla u|^2}} \, dx = \int_{\Omega} f_1(x, u, v) w \, dx,
\]

\[
\int_{\Omega} \frac{\nabla v \cdot \nabla w}{\sqrt{1 - |\nabla v|^2}} \, dx = \int_{\Omega} f_2(x, u, v) w \, dx, \quad (8)
\]

for every $w \in W^{1,1}_0(\Omega)$. A solution $(u, v)$ of (1) is said to be positive if $u > 0$ and $v > 0$, respectively strictly positive if $u \gg 0$ and $v \gg 0$.

Remark 2.3. It is worth pointing out that the notion of solution in the sense of Definition 2.2 fits with the above one in the case of a single equation, as well as with the definition of a solution for a system in the sense of paper [18]. Precisely, one has that if $(u, v)$ is a solution of (1), then $u \in W^{2,r}(\Omega)$, $v \in W^{2,s}(\Omega)$, for all finite $r, s \geq 1$, it satisfies the equations a.e. in $\Omega$ and vanishes on $\partial \Omega$. This can be easily checked by following exactly the outline of the argument in [14, Remark 2].

Remark 2.4. An immediate consequence of Remark 2.3 is that any solution $(u, v)$ of (1) satisfies $\| u \|_{\infty} < d(\Omega)/2$ and $\| v \|_{\infty} < d(\Omega)/2$, where $d(\Omega)$ denotes the diameter of $\Omega$. 


By means of Lemma 2.1 (i) we can define the operator \( S : L^\infty(\Omega) \to W^{2,r}(\Omega) \subset C^1(\bar{\Omega}) \) (with some \( r > N \) fixed), which maps any \( h \in L^\infty(\Omega) \) into the unique solution \( S(h) := u_h \in W^{2,r}(\Omega) \) of problem (7). According to [18, Remark 4.4] (also see [15, Lemma 2.3]) one has that \( S : L^\infty(\Omega) \to C^1(\bar{\Omega}) \) is completely continuous. Also, from hypothesis (Hf) we have that the Nemytskii operators
\[
N_i(u, v) = f_i(\cdot, u(\cdot), v(\cdot)) \quad (i = 1, 2)
\]
are continuous from the product space \( C^1(\bar{\Omega}) \times C^1(\bar{\Omega}) \) to \( L^1(\Omega) \) and map bounded sets from \( C^1(\bar{\Omega}) \times C^1(\bar{\Omega}) \) into bounded sets in \( L^\infty(\Omega) \). From this, arguing as in the proof of [18, Theorem 4.5] we infer that the operator \( T : C^1(\bar{\Omega}) \times C^1(\bar{\Omega}) \to C^1(\bar{\Omega}) \times C^1(\bar{\Omega}) \) defined by
\[
T(u, v) = (S \circ N_1(u, v), S \circ N_2(u, v)), \quad \forall (u, v) \in C^1(\bar{\Omega}) \times C^1(\bar{\Omega}) \quad (9)
\]
is completely continuous. A couple of functions \((u, v) \in C^1(\bar{\Omega}) \times C^1(\bar{\Omega})\) is a solution of (1) iff it is a fixed point of the operator \( T \). Then, since the convex set
\[
U := \{(u, v) \in C^1_0(\bar{\Omega}) \times C^1_0(\bar{\Omega}) : \|u\|_{\infty} < 1, \|v\|_{\infty} < 1\}
\]
is bounded in \( C^1(\bar{\Omega}) \times C^1(\bar{\Omega}) \) and \( T(U) \subset U \), from Schauder’s theorem the operator \( T \) has a fixed point. Also, it is easy to see that the invariance under homotopy of the Leray-Schauder degree, yields \( d_{L.S}[I - T, U, 0] = 1 \).

3. Lower and upper solutions.

**Definition 3.1.** A lower solution of (1) is a couple of functions \((\alpha_u, \alpha_v) \in C^{0,1}(\bar{\Omega}) \times C^{0,1}(\bar{\Omega})\), such that \( \|\nabla \alpha_u\|_{\infty} < 1, \|\nabla \alpha_v\|_{\infty} < 1\) and satisfies
- for every \( w \in W^{1,1}_0(\Omega) \) with \( w \geq 0 \) in \( \Omega\),
  \[
  \int_{\Omega} \frac{\nabla \alpha_u \cdot \nabla w}{\sqrt{1 - |\nabla \alpha_u|^2}} \, dx \leq \int_{\Omega} f_1(x, \alpha_u, \alpha_v) w \, dx,
  \]
  \[
  \int_{\Omega} \frac{\nabla \alpha_v \cdot \nabla w}{\sqrt{1 - |\nabla \alpha_v|^2}} \, dx \leq \int_{\Omega} f_2(x, \alpha_u, \alpha_v) w \, dx;
  \]
- \( \alpha_u \leq 0, \alpha_v \leq 0 \) on \( \partial \Omega \).

We say that a lower solution \((\alpha_u, \alpha_v)\) of (1) is strict if every solution \((u, v)\) of (1) with \( u \geq \alpha_u, v \geq \alpha_v\) satisfies \( u \gg \alpha_u \) and \( v \gg \alpha_v \) in \( \bar{\Omega} \).

An upper solution is defined by reversing the above inequalities, i.e.,

**Definition 3.2.** An upper solution of (1) is a couple of functions \((\beta_u, \beta_v) \in C^{0,1}(\bar{\Omega}) \times C^{0,1}(\bar{\Omega})\), such that \( \|\nabla \beta_u\|_{\infty} < 1, \|\nabla \beta_v\|_{\infty} < 1\) and satisfies
- for every \( w \in W^{1,1}_0(\Omega) \) with \( w \geq 0 \) in \( \Omega\),
  \[
  \int_{\Omega} \frac{\nabla \beta_u \cdot \nabla w}{\sqrt{1 - |\nabla \beta_u|^2}} \, dx \geq \int_{\Omega} f_1(x, \beta_u, \beta_v) w \, dx,
  \]
  \[
  \int_{\Omega} \frac{\nabla \beta_v \cdot \nabla w}{\sqrt{1 - |\nabla \beta_v|^2}} \, dx \geq \int_{\Omega} f_2(x, \beta_u, \beta_v) w \, dx;
  \]
- \( \beta_u \geq 0, \beta_v \geq 0 \) on \( \partial \Omega \).

We say that an upper solution \((\beta_u, \beta_v)\) of (1) is strict if every solution \((u, v)\) of (1) with \( u \leq \beta_u, v \leq \beta_v\) satisfies \( u \ll \beta_u \) and \( v \ll \beta_v \) in \( \bar{\Omega} \).
The quasi-monotonicity of

Part 1. Existence of a solution

some ideas from [13, 14], we divide the proof in three parts.

Step 1. Assume

Proposition 3.3. Assume (Hf) and suppose that \( f_1(x,s,t) \) (resp. \( f_2(x,s,t) \)) is quasi-monotone nondecreasing with respect to \( t \) (resp. \( s \)). If (1) has a lower solution \((\alpha_u,\alpha_v)\) and an upper solution \((\beta_u,\beta_v)\) with \( \alpha_u \leq \beta_u, \alpha_v \leq \beta_v \), then problem (1) has solutions \((w^1_u,w^1_v), (w^2_u,w^2_v)\) with \( \alpha_u \leq w^1_u \leq w^2_u \leq \beta_u, \alpha_v \leq w^1_v \leq w^2_v \leq \beta_v \) such that every solution \((u,v)\) of (1) with \( \alpha_u \leq u \leq \beta_u \) and \( \alpha_v \leq v \leq \beta_v \) satisfies \( w^1_u \leq u \leq w^2_u \) and \( w^1_v \leq v \leq w^2_v \). Further, if both of \((\alpha_u,\alpha_v)\) and \((\beta_u,\beta_v)\) are strict, then

\[
d_{LS}[I-T,U,0] = 1,
\]  
(12)

where \( T \) is defined in (9) and

\[
U := \{(z_u,z_v) \in C^1_0(\Omega) \times C^1_0(\Omega) : \alpha_u \ll z_u \ll \beta_u, \alpha_v \ll z_v \ll \beta_v
\]
and \( \|\nabla z_u\|_\infty < 1, \|\nabla z_v\|_\infty < 1 \}.

Proof: The quasi-monotonicity of \( f_1 \) and \( f_2 \) will play a key role here. Adopting some ideas from [13, 14], we divide the proof in three parts.

Part 1. Existence of a solution \((u,v)\) of (1) with \( \alpha_u \leq u \leq \beta_u, \alpha_v \leq v \leq \beta_v \).

Step 1. Construction of a modified problem. We define the functions

\[
\Gamma_1(x,u,v) = f_1(x,\gamma_u(x,u),\gamma_v(x,v)),
\]

respectively

\[
\Gamma_2(x,u,v) = f_2(x,\gamma_u(x,u),\gamma_v(x,v)),
\]

where \( \gamma_u, \gamma_v : \Omega \times \mathbb{R} \to \mathbb{R} \) are given by

\[
\gamma_u(x,\xi) = \max\{\alpha_u(x),\min\{\xi,\beta_u(x)\}\}, \quad \gamma_v(x,\xi) = \max\{\alpha_v(x),\min\{\xi,\beta_v(x)\}\}.
\]

Then we consider the modified problem

\[
\begin{cases}
-\mathcal{M}(u) = \Gamma_1(x,u,v), & x \in \Omega, \\
-\mathcal{M}(v) = \Gamma_2(x,u,v), & x \in \Omega, \\
u|_{\partial \Omega} = 0 = v|_{\partial \Omega},
\end{cases}
\]  
(13)

which, being of type (1), we know that has a solution.

Step 2. Problem (1) has at least one solution \((u,v)\) with \( \alpha_u \leq u \leq \beta_u, \alpha_v \leq v \leq \beta_v \). This reduces to showing that every solution \((u,v)\) of problem (13) satisfies \( \alpha_u \leq u \leq \beta_u \) and \( \alpha_v \leq v \leq \beta_v \). We prove that \( \alpha_u \leq u \); in a similar way one obtains that \( \alpha_v \leq v \). Taking \( w = (u-\alpha_u)^- \in W_0^{1,1}(\Omega) \) in (8) with \( \Gamma_1 \) instead of \( f_1 \), and (10), one has

\[
\int_{\{u<\alpha_u\}} \frac{\nabla u \cdot \nabla (u-\alpha_u)}{\sqrt{1-|\nabla u|^2}} \, dx = \int_{\Omega} \frac{\nabla u \cdot \nabla (u-\alpha_u)^-}{\sqrt{1-|\nabla u|^2}} \, dx =
\]

\[
-\int_{\Omega} \Gamma_1(x,u,v)(u-\alpha_u)^- \, dx = \int_{\{u<\alpha_u\}} \Gamma_1(x,u,v)(u-\alpha_u) \, dx.
\]
and
\[- \int_{\{u < \alpha_u\}} \frac{\nabla \alpha_u \cdot \nabla (u - \alpha_u)}{\sqrt{1 - |\nabla \alpha_u|^2}} \, dx = \int_{\Omega} \frac{\nabla \alpha_u \cdot \nabla (u - \alpha_u)}{\sqrt{1 - |\nabla \alpha_u|^2}} \, dx \leq 0\]
\[
\int_{\{u < \alpha_u\}} f_1(x, \alpha_u, \alpha_v)(u - \alpha_u) \, dx = - \int_{\{u < \alpha_u\}} f_1(x, \alpha_u, \alpha_v)(u - \alpha_u) \, dx.
\]

Summing up we get
\[
\int_{\{u < \alpha_u\}} \left( \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} - \frac{\nabla \alpha_u}{\sqrt{1 - |\nabla \alpha_u|^2}} \right) \cdot \nabla (u - \alpha_u) \, dx \leq 0.
\]
and using that \( f_1(x, \alpha_u, \gamma_v(x, v)) \geq f_1(x, \alpha_u, \alpha_v) \), since \( f_1 \) is quasi-monotone non-decreasing with respect to the third argument, we infer that
\[
\int_{\{u < \alpha_u\}} \left( \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} - \frac{\nabla \alpha_u}{\sqrt{1 - |\nabla \alpha_u|^2}} \right) \cdot (\nabla u - \nabla \alpha_u) \, dx = 0.
\]
This and the strict monotonicity of the function \( y \mapsto y/\sqrt{1 - |y|^2} \) (\( y \in \mathbb{R}^N \) with \( |y| < 1 \)) yield
\[
\int_{\{u < \alpha_u\}} \left( \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} - \frac{\nabla \alpha_u}{\sqrt{1 - |\nabla \alpha_u|^2}} \right) \cdot (\nabla u - \nabla \alpha_u) \, dx = 0.
\]
Then, either \( \nabla (u - \alpha_u) = 0 \) in \( \{u < \alpha_u\} \) or the \( N \)-dimensional measure of the set \( \{u < \alpha_u\} \) is zero. In both cases we get \( \alpha_u = u \) and hence \( u \geq \alpha_u \). Using (11) and similar arguments as above we infer that \( u \leq \beta_u \) and \( v \leq \beta_v \).

**Part 2. Existence of extremal solutions.** We have that the fixed points of the operator \( T \) defined in (9) are precisely the solutions of problem (1). By the complete continuity of \( T \) the bounded set
\[\mathcal{S} = \{(u, v) \in C^1_0(\overline{\Omega}) \times C^1_0(\overline{\Omega}) : (u, v) = T(u, v) \text{ and } \alpha_u \leq u \leq \beta_u, \alpha_v \leq v \leq \beta_v\}\]
is compact. Also, \( \mathcal{S} \) is not empty (from Part 1).

**Step 1.** We prove the existence of some \((w_u^1, w_v^1) \in \mathcal{S} \) such that every \((u, v) \in \mathcal{S} \) satisfies \( w^1_u \leq u \) and \( w^1_v \leq v \). With this aim, for each \((u, v) \in \mathcal{S} \), we introduce the closed subset of \( \mathcal{S} \)
\[\mathcal{K}_{u,v} = \{(u, v) \in \mathcal{S} : u \leq u, \ v \leq v\}\]
and we show that the family \( \{\mathcal{K}_{u,v} : (u, v) \in \mathcal{S}\} \) has the finite intersection property. If \( k \in \mathbb{N}^+ \) and \((u_1, v_1), \ldots, (u_k, v_k) \in \mathcal{S} \), let \( u_0 = \min\{u_1, \ldots, u_k\} \), resp. \( v_0 = \min\{v_1, \ldots, v_k\} \). Then \( \alpha_u \leq u_0 \leq \beta_u, \alpha_v \leq v_0 \leq \beta_v \) and as in Part 1, for all \( j = 0, 1, \ldots, k \), we define the functions
\[\Gamma^j_{\alpha_u}(x, u, v) = f_1 \left( x, \gamma^j_{\alpha_u}(x, u), \gamma^j_{\alpha_v}(x, v) \right)\]
and
\[
\Gamma_i^j(x, u, v) = f_2\left(x, \gamma_i^j(x, u), \gamma_i^j(x, v)\right),
\]
where \(\gamma_i^j, \gamma_i^j : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}\) are given by
\[
\gamma_i^j(x, \xi) = \max\{\alpha_u(x), \min\{\xi, u_j(x)\}\}, \quad \gamma_i^j(x, \xi) = \max\{\alpha_v(x), \min\{\xi, v_j(x)\}\}.
\]

Next, we set
\[
\Gamma_i = \Gamma_i^0 - \sum_{j=1}^{k} |\Gamma_i^0 - \Gamma_i^j| \quad (i = 1, 2)
\]
and consider the modified problem
\[
\begin{aligned}
-\mathcal{M}(u) & = \Gamma_1(x, u, v), \quad x \in \Omega, \\
-\mathcal{M}(v) & = \Gamma_2(x, u, v), \quad x \in \Omega, \\
u|_{\partial \Omega} & = 0 = v|_{\partial \Omega}.
\end{aligned}
\]
(14)

As (14) is of type (1), we know that it has a solution \((z_1, z_2)\). We have to prove that \(\alpha_u \leq z_1 \leq u_0, \alpha_v \leq z_2 \leq v_0\). For each \(j = 1, \ldots, k\), since \(\alpha_u \leq u_j, \alpha_v \leq v_j\) and \(f_1, f_2\) are quasi-monotone nondecreasing, it is easy to check that
\[
\begin{aligned}
\Gamma_i^j(x, z_1(x), z_2(x)) & \leq \Gamma_i^j(x, u_j(x), v_j(x)) \quad \text{if} \quad z_1(x) > u_j(x), \\
\Gamma_i^j(x, z_1(x), z_2(x)) & \leq \Gamma_i^j(x, u_j(x), v_j(x)) \quad \text{if} \quad z_2(x) > v_j(x),
\end{aligned}
\]
for a.e. \(x \in \Omega\). Also, we have
\[
\int_{\Omega} \frac{\nabla z_i \cdot \nabla w}{\sqrt{1 - |\nabla z_i|^2}} \, dx = \int_{\Omega} \Gamma_i(x, z_1, z_2)w \, dx \quad (i = 1, 2),
\]
(16)
for every \(w \in W^{1,1}_0(\Omega)\) and, given any \(j = 1, \ldots, k\), we will show next that \(z_1 \leq u_j\).

Taking \(w = (z_1 - u_j)^+ \in W^{1,1}_0(\Omega)\) in (16) \((i = 1)\) and in (8) \((u, v)\) replaced by \((u_j, v_j))\) and using the first inequality in (15), we deduce that
\[
0 \leq \int_{\{z_1 > u_j\}} \left(\frac{\nabla z_1}{\sqrt{1 - |\nabla z_1|^2}} - \frac{\nabla u_j}{\sqrt{1 - |\nabla u_j|^2}}\right) \cdot \nabla (z_1 - u_j) \, dx
\]
\[
= \int_{\Omega} \left(\frac{\nabla z_1}{\sqrt{1 - |\nabla z_1|^2}} - \frac{\nabla u_j}{\sqrt{1 - |\nabla u_j|^2}}\right) \cdot \nabla (z_1 - u_j)^+ \, dx
\]
\[
= \int_{\Omega} (\Gamma_1(x, z_1, z_2) - f_1(x, u_j, v_j)) (z_1 - u_j)^+ \, dx
\]
\[
= \int_{\{z_1 > u_j\}} \left(\Gamma_1^0(x, z_1, z_2) - f_1(x, u_j, v_j) - |\Gamma_1^0(x, z_1, z_2) - \Gamma_1^j(x, z_1, z_2)| - \sum_{i=1, i \neq j}^{k} |\Gamma_i^0(x, z_1, z_2) - \Gamma_i^j(x, z_1, z_2)|\right) (z_1 - u_j) \, dx
\]
\[ \begin{align*}
\Gamma_i(x, u, v) &= f_i(x, \tau_u^i(x, u), \tau_v^i(x, v)) \quad (i = 1, 2) \\
\tau_u^i(x, \xi) &= \max\{u_j(x), \min\{\xi, \beta_u(x)\}\}, \quad \tau_v^i(x, \xi) = \max\{v_j(x), \min\{\xi, \beta_v(x)\}\}.
\end{align*} \]

Setting
\[ \Gamma_i = \Gamma_i^0 + \sum_{j=1}^k |\Gamma_j^i - \Gamma_i^j| \quad (i = 1, 2) \]
and following the outline of the proof of the previous step, we show that any solution \((z_1, z_2)\) of problem
\[ \begin{align*}
-M(u) &= \Gamma_1(x, u, v), \quad x \in \Omega, \\
-M(v) &= \Gamma_2(x, u, v), \quad x \in \Omega, \\
u|_{\partial \Omega} &= 0 = v|_{\partial \Omega}
\end{align*} \]
satisfies \( \alpha_u \leq u_0 \leq z_1 \leq \beta_u \), resp. \( \alpha_v \leq v_0 \leq z_2 \leq \beta_v \) and the family \( \{ \mathcal{K}^{-,+} : (u, v) \in \mathcal{S} \} \) has the finite intersection property. From this we easily deduce the existence of a maximal solution of (1) lying between \((\alpha_u, \alpha_v)\) and \((\beta_u, \beta_v)\).

**Part 3. Degree computation.** Let \( \overline{T} : C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \to C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \) be the fixed point operator associated with problem (13), defined as in (9) with \( N_i (i = 1, 2) \) replaced by

\[
\overline{N}_i(u, v) = \Gamma_i(\cdot, u(\cdot), v(\cdot)) \quad (i = 1, 2).
\]

We have that there exists a solution \((u, v)\) of (1) with \( \alpha_u \leq u \leq \beta_u \), \( \alpha_v \leq v \leq \beta_v \) and, since \((\alpha_u, \alpha_v)\) and \((\beta_u, \beta_v)\) are, respectively, a strict lower and a strict upper solution of (1), every such a solution satisfies \( \alpha_u \ll u \ll \beta_u \), resp. \( \alpha_v \ll v \ll \beta_v \), hence it belongs to \( \mathcal{U} \). Therefore, the set \( \mathcal{U} \) is a non-empty open bounded set in \( C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \) such that there is no fixed point either of \( T \) or of \( \overline{T} \) on \( \partial \mathcal{U} \). As \( T \) and \( \overline{T} \) coincide in \( \mathcal{U} \), one has

\[
d_{LS}(I - T, \mathcal{U}, 0) = d_{LS}(I - \overline{T}, \mathcal{U}, 0).
\]

Then, since \( d_{LS}(I - \overline{T}, \mathcal{U}, 0) = 1 \) and \( \overline{T} \) has no fixed point in \( \overline{\mathcal{U}} \setminus \mathcal{U} \), from the excision property of the degree, we obtain

\[
d_{LS}(I - \overline{T}, \mathcal{U}, 0) = d_{LS}(I - \overline{T}, U, 0) = 1,
\]

which completes the proof. \( \square \)

**Proposition 3.4.** Assume \((H_f)\) and suppose that \( f_1(x, s, t) \) (resp. \( f_2(x, s, t) \)) is quasi-monotone nondecreasing with respect to \( t \) (resp. \( s \)). If there exist a strict lower solution \((\alpha_u, \alpha_v)\) and a strict upper solution \((\beta_u, \beta_v)\) of (1) with \( \alpha_u \ll \beta_u \) or \( \alpha_v \ll \beta_v \), then problem (1) has at least three solutions \((u_1, v_1)\), \((u_2, v_2)\) and \((u_3, v_3)\) such that

\[
u_1 \ll \beta_u, \quad v_1 \ll \beta_v, \quad u_3 \gg \alpha_u, \quad v_3 \gg \alpha_v
\]

and one of the following holds:

\( \begin{align*}
(\text{i}) \quad & u_1 < u_2 < u_3, \quad v_1 < v_2 < v_3; \\
(\text{ii}) \quad & u_1 \leq u_2 < u_3, \quad v_1 < v_2 < v_3; \\
(\text{iii}) \quad & u_1 < u_2 \leq u_3, \quad v_1 \leq v_2 < v_3; \\
(\text{iv}) \quad & u_1 \leq u_2 \leq u_3, \quad v_1 < v_2 < v_3.
\end{align*} \)

**Proof.** The arguments are quite similar to those from the proof of Proposition 3.2 in [13] (also see [14, Proposition 2]). However, for the sake of completeness, we give a sketch of the proof below. Setting

\[
R = \max \left\{ \|\alpha_u\|_\infty, \|\alpha_v\|_\infty, \|\beta_u\|_\infty, \|\beta_v\|_\infty, \frac{d(\Omega)}{2} \right\},
\]

we define \( f^R_1, f^R_2 : \Omega \times \mathbb{R}^2 \to \mathbb{R} \) by

\[
f^R_1(x, s, t) = \begin{cases} 
 f_1(x, s, t), & \text{if } |s| \leq R, \\
 0, & \text{if } |s| \geq R + 1, \\
 (R + 1 - s)f_1(x, R, t), & \text{if } R < s < R + 1, \\
 (R + 1 + s)f_1(x, -R, t), & \text{if } -R - 1 < s < -R,
\end{cases}
\]

respectively

\[
f^R_2(x, s, t) = \begin{cases} 
 f_2(x, s, t), & \text{if } |t| \leq R, \\
 0, & \text{if } |t| \geq R + 1, \\
 (R + 1 - t)f_2(x, s, R), & \text{if } R < t < R + 1, \\
 (R + 1 + t)f_2(x, s, -R), & \text{if } -R - 1 < t < -R.
\end{cases}
\]
Notice that \( f^1_x, f^2_x \) satisfy hypothesis \((H_f)\) and \( f^1_x(x,s,t) \), resp. \( f^2_x(x,s,t) \) are quasi-monotone nondecreasing with respect to \( t \), resp. \( s \). Now we consider the modified problem

\[
\begin{align*}
-M(u) &= f^1(x,u,v), \quad x \in \Omega, \\
-M(v) &= f^2(x,u,v), \quad x \in \Omega, \\
|u|\partial \Omega &= 0 = |v|\partial \Omega.
\end{align*}
\]

Due to the choice of \( R \), Remark 2.4 implies that any solution of (19) is a solution of (1) and \((\alpha_u, \alpha_v), (\beta_u, \beta_v)\) are strict lower and upper solutions of (19). Also, denoting \( \bar{\alpha} = -R - 1 \) and \( \bar{\beta} = R + 1 \), clearly one has that \((\bar{\alpha}, \bar{\alpha})\), resp. \((\bar{\beta}, \bar{\beta})\) are strict lower resp. upper solutions of (19).

Next, we introduce the following open bounded subsets of \( C^1_0(\overline{\Omega}) \times C^1_0(\overline{\Omega}) \):

\[
\begin{align*}
\mathcal{U}^\alpha &= \{ (u,v) \in C^1_0(\overline{\Omega}) \times C^1_0(\overline{\Omega}) : \alpha \leq u \leq \beta_u, \quad \alpha \leq v \leq \beta_v \}, \\
\mathcal{U}^\beta &= \{ (u,v) \in C^1_0(\overline{\Omega}) \times C^1_0(\overline{\Omega}) : \alpha_u \leq u \leq \beta, \quad \alpha_v \leq v \leq \beta \}, \\
\mathcal{U}^{\bar{\alpha}} &= \{ (u,v) \in C^1_0(\overline{\Omega}) \times C^1_0(\overline{\Omega}) : \bar{\alpha} \leq u \leq \bar{\beta}, \quad \bar{\alpha} \leq v \leq \bar{\beta} \}, \\
\mathcal{U}^{\bar{\beta}} &= \{ (u,v) \in C^1_0(\overline{\Omega}) \times C^1_0(\overline{\Omega}) : \bar{\alpha} \leq u \leq \bar{\beta}, \quad \bar{\alpha} \leq v \leq \bar{\beta} \},
\end{align*}
\]

Note that \( \mathcal{U}^\alpha \subset \mathcal{U}^\beta, \mathcal{U}^{\bar{\alpha}} \subset \mathcal{U}^{\bar{\beta}} \) and since \( \alpha_u \not\leq \beta_u \) (or \( \alpha_v \not\leq \beta_v \)), \( \mathcal{U}^\alpha \cap \mathcal{U}^{\bar{\alpha}} = \emptyset \).

Let us consider the operator \( T_R : C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \to C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \) defined as in (9) with \( N_i \) (i = 1, 2) replaced by

\[ N_i^{R}(u,v) = f_i^{R}(\cdot,u(\cdot),v(\cdot)) \quad (i = 1, 2) \]

and since \((\alpha_u, \alpha_v), (\bar{\alpha}, \bar{\alpha})\) are strict lower solutions of (19), and \((\beta_u, \beta_v), (\bar{\beta}, \bar{\beta})\) are strict upper solutions of (19), one has

\[ 0 \not\in (I - T_R)(\partial \mathcal{U}_R^\alpha \cup \partial \mathcal{U}_R^{\bar{\alpha}} \cup \partial \mathcal{U}_R^\beta \cup \partial \mathcal{U}_R^{\bar{\beta}}). \]

Then, from Proposition 3.3, we infer

\[ d_{\text{LS}}[I - T_R, \mathcal{U}_R^\alpha, 0] = d_{\text{LS}}[I - T_R, \mathcal{U}_R^{\bar{\alpha}}, 0] = d_{\text{LS}}[I - T_R, \mathcal{U}_R^\beta, 0] = d_{\text{LS}}[I - T_R, \mathcal{U}_R^{\bar{\beta}}, 0] = 1, \]

which together with (20) and the additivity-excision property of the Leray-Schauder degree, imply (see [13, 14])

\[ d_{\text{LS}}[I - T_R, \mathcal{V}, 0] = -1, \]

where \( \mathcal{V} = \mathcal{U}_R^{\bar{\beta}} \setminus (\mathcal{U}_R^\alpha \cup \mathcal{U}_R^{\bar{\alpha}}) \). Since \( \mathcal{U}_R^\alpha \), \( \mathcal{U}_R^{\bar{\alpha}} \) and \( \mathcal{V} \) are pairwise disjoint, from the above degree calculations we infer that there exist three distinct fixed points \((u_1, v_1), (u_2, v_2)\) and \((u_3, v_3)\) of \( T_R \) with

\[(u_1, v_1) \in \mathcal{U}_R^\alpha, \quad (u_2, v_2) \in \mathcal{V}, \quad (u_3, v_3) \in \mathcal{U}_R^{\bar{\beta}}.\]

This means that (17) is fulfilled and using again the fact that \((\alpha_u, \alpha_v)\) and \((\beta_u, \beta_v)\) are, respectively, a strict lower and a strict upper solution of (19), one of the following holds:

\[
\begin{align*}
(i') \quad & \alpha_u \not\leq u_2, \quad u_2 \not\leq \beta_u; \\
(ii') \quad & \alpha_u \not\leq u_2, \quad v_2 \not\leq \beta_v; \\
(iii') \quad & \alpha_v \not\leq v_2, \quad u_2 \not\leq \beta_u; \\
(iv') \quad & \alpha_v \not\leq v_2, \quad v_2 \not\leq \beta_v.
\end{align*}
\]
Next, we observe that we can replace \((u_1, v_1)\) and \((u_3, v_3)\) with the minimal, resp. the maximal solution of \((19)\) lying between \((\bar{\alpha}, \bar{\alpha})\) and \((\beta, \beta)\) and \((17)\) still remains valid. Additionally, we have that

\[ u_1 \leq u_2 \leq u_3 \text{ and } v_1 \leq v_2 \leq v_3. \]

Then, it is easy to check that \((i') - (iv')\) imply \((i) - (iv)\). The conclusion follows from the fact that this solutions also solve problem \((1)\).

**Proposition 3.5.** Assume \((H_f)\) and that \(f_1(x, s, t)\) (resp. \(f_2(x, s, t)\)) is quasi-monotone nondecreasing with respect to \(t\) (resp. \(s)\). If there exists a lower solution \((\alpha_u, \alpha_v)\) of problem \((1)\), then it has at least one solution \((u, v)\) with \(u \geq \alpha_u\) and \(v \geq \alpha_v\).

**Proof.** We consider problem \((19)\) constructed in the proof of Proposition 3.4 with

\[ R = \max \left\{ \|\alpha_u\|_\infty, \|\alpha_v\|_\infty, \frac{d(\Omega)}{2} \right\} \]

instead of \(R\) given by \((18)\). From Remark 2.4 we see that any solution of \((19)\) is a solution of the original problem \((1)\). Setting \(\bar{\beta} = R + 1\) one has that \((\alpha_u, \alpha_v), (\bar{\beta}, \bar{\beta})\) are lower, respectively upper solutions for \((19)\), with \(\alpha_u \leq \bar{\beta} \) and \(\alpha_v \leq \bar{\beta}\). Then, Proposition 3.3 implies the existence of at least one solution of \((19)\) lying between \((\alpha_u, \alpha_v)\) and \((\bar{\beta}, \bar{\beta})\), which also is a solution for \((1)\). \(\square\)

4. Non-existence, existence, multiplicity and localization results. In this section, under hypothesis \((\tilde{H}_f)\), we study the existence and multiplicity of strictly positive solutions for the autonomous system \((5)\). In order to treat \((5)\) we introduce \(g_i(s, t) = f_i(s^+, t^+)\) \((s, t \in \mathbb{R}, \ i = 1, 2)\) and consider the modified system

\[
\begin{cases}
-M(u) = \lambda_1 g_1(u, v), & \text{in } \Omega, \\
-M(v) = \lambda_2 g_2(u, v), & \text{in } \Omega, \\
u|_{\partial \Omega} = 0 = v|_{\partial \Omega}.
\end{cases}
\]

\(21\)

**Remark 4.1.** (i) Notice that on account of \((\tilde{H}_f)\) (ii) one has that

\[ g_1(\xi, 0) = g_1(0, \xi) = 0 = g_2(\xi, 0) = g_2(0, \xi), \ \forall \ \xi \geq 0, \]

\(22\)

hence, problem \((21)\) always has the trivial solution. Also, each solution \((u, v)\) of \((21)\) satisfies \(u \geq 0 \leq v\) (see \([18, \text{Lemma 3.6}]\)) and it solves \((5)\).

(ii) If \((u, v)\) is a non-trivial solution of \((21)\) with some \(\lambda_1, \lambda_2 > 0\), then \((u, v)\) is strictly positive. Indeed, from \((22)\) and since \((u, v)\) is non-trivial we easily infer that \(uv > 0\). This ensures that \(u(x) > 0 \) and \(v(x) > 0\) for all \(x\) belonging to a set of positive measure. Then, on account of \((6)\) one has \(-M(u) > 0\) and \(-M(v) > 0\) in \(\Omega\) and Lemma 2.1 (ii) yields that \(u \gg 0\) and \(v \gg 0\).

**Lemma 4.2.** For any constants \(c_1 > 0, \ p > 1\) and any interval \([a, b] \subset \mathbb{R}\), the equation

\[
-w' \left( \frac{w'}{\sqrt{1-w'^2}} \right)' = c_1(w^+)^p
\]

\((23)\)

has a sequence \(\{w_k\}\) of solutions in \(C^2[a, b]\), such that \(\min_{[a, b]} w_k > 0 \) for every \(k\) and

\[ \lim_{k \to +\infty} \|w_k\|_\infty = 0. \]
Proof. This can be done as in the proof of [12, Theorem 2.5] by using a time-map argument. Here, we give an alternative proof. With this aim, let us consider the Cauchy problem
\[
\begin{cases}
-\left(\frac{w'/\sqrt{1-w'^2}}{1-w'^2}\right)' = c_1 (w^+)^p, & \text{in } [a, b], \\
w(a) = r, \quad w'(a) = 0,
\end{cases}
\]
with \( r > 0 \). From the proof of Proposition 2 in [7], one has that (24) has an unique solution \( w \) which is the unique fixed point of the compact operator
\[
T : C[a, b] \to C[a, b], \quad Tw(t) = r - \int_a^t \phi^{-1} \left( c_1 \int_a^s (w^+(\xi))^p \, d\xi \right) \, ds,
\]
where \( \phi^{-1}(y) = y/\sqrt{1+y^2}, \ y \in \mathbb{R} \). Note that the solution of (24) is strictly decreasing on \([a, b] \). Next, let \( \{r_k\} \) be a sequence of positive numbers such that \( r_k \searrow 0 \), as \( k \to +\infty \), and \( w_k \) be the solution of (24) with \( r = r_k \). If we assume, by contradiction, that there are infinitely many \( w_k \) which vanish in some \( R_k \in [a, b] \) then, relabeling the sequence \( \{w_k\} \) if necessary, we can write
\[
r_k = \int_a^{R_k} \phi^{-1} \left( c_1 \int_a^s (w_k^+(\xi))^p \, d\xi \right) \, ds.
\]
Using the mean value theorem, we get
\[
r_k = (R_k - a) \phi^{-1} \left( c_1 \int_a^{t_k} (w_k^+(\xi))^p \, d\xi \right),
\]
with some \( t_k \in [a, R_k] \). Then, as \( \|w_k^+\|_\infty = r_k \) we infer
\[
R_k - a = \frac{r_k}{\phi^{-1} \left( c_1 \int_a^{t_k} (w_k^+(\xi))^p \, d\xi \right)} \geq \frac{r_k}{\phi^{-1} (c_1 (R_k - a)r_k^p)} \geq \frac{r_k}{c_1 (R_k - a)r_k^p}.
\]
Hence,
\[
R_k - a \geq \frac{1}{\sqrt{c_1 r_k^{p-1}}} \to +\infty, \quad \text{as } k \to \infty,
\]
a contradiction. Consequently, there exists a sequence \( \{w_k\} \) in \( C^2[a, b] \) of solutions of (23), such that, for every \( k \), \( \min_{[a, b]} w_k > 0 \) and \( \lim_{k \to +\infty} \|w_k\|_\infty = \lim_{k \to +\infty} r_k = 0 \). □

Hereafter, for \( x \in \mathbb{R}^N \) and \( \rho > 0 \) we denote by \( B_\rho(x) \) the open ball centered at \( x \) of radius \( \rho \) and by \( \overline{B}_\rho(x) \) its closure.

Proposition 4.3. Under hypothesis \( (\tilde{H}_j) \), there exist \( \lambda_1^*, \lambda_2^* > 0 \) such that problem (5) has at least two strictly positive solutions, for each \( \lambda_1 > \lambda_1^* \) and \( \lambda_2 > \lambda_2^* \).

Proof. We shall apply Proposition 3.4. With this aim we divide the proof in two steps.

Step 1. Construction of a strict lower solution \( (\alpha_u, \alpha_v) \) of (21), with \( \alpha_u > 0 < \alpha_v \).
Let \( x_0 \in \Omega \) and \( \rho > 0 \) be such that \( \overline{B}_\rho(x_0) \subset \Omega \) and consider the problem
\[
\begin{cases}
-\mathcal{M}(u) = \lambda_1 g_1(u, v), & \text{in } B_\rho(x_0), \\
-\mathcal{M}(v) = \lambda_2 g_2(u, v), & \text{in } B_\rho(x_0), \\
u|_{\partial B_\rho(x_0)} = 0 = v|_{\partial B_\rho(x_0)},
\end{cases}
\]
Using Theorem 2.3 in [17], we infer that there exist $\lambda_1, \lambda_2 > 0$ such that, for each $\lambda_1 > \lambda_1^*$ and $\lambda_2 > \lambda_2^*$, problem (25) has a radial solution $(u^*, v^*) \in C^1(\overline{B}_p(x_0)) \times C^1(\overline{B}_p(x_0))$ with $u^*(x) > 0$ and $v^*(x) > 0$ for every $x \in B_p(x_0)$.

In order to construct a strict lower solution $(\alpha_u, \alpha_v)$ of (21), we fix $\lambda_1 > \lambda_1^*$, $\lambda_2 > \lambda_2^*$ and define the functions $\alpha_u, \alpha_v \in C^{0,1}(\overline{\Omega})$ by

$$
\alpha_u(x) = \begin{cases} 
  u^*(x), & \text{if } x \in B_p(x_0), \\
  0, & \text{if } x \in \overline{\Omega} \setminus \overline{B}_p(x_0), 
\end{cases} 
\alpha_v(x) = \begin{cases} 
  v^*(x), & \text{if } x \in B_p(x_0), \\
  0, & \text{if } x \in \overline{\Omega} \setminus \overline{B}_p(x_0). 
\end{cases}
$$

Let $w \in W_0^{1,1}(\Omega)$ with $w \geq 0$. From (22), using that the outer normal derivative of $u^*$ (resp. $v^*$) on $\partial B_p(x_0)$ is negative (Lemma 2.1 (ii)), we infer

$$
\int_\Omega \frac{\nabla \alpha_u \cdot \nabla w}{\sqrt{1 - |\nabla \alpha_u|^2}} \, dx = \int_{B_p(x_0)} \frac{\nabla \alpha_u \cdot \nabla w}{\sqrt{1 - |\nabla \alpha_u|^2}} \, dx = \int_{B_p(x_0)} \frac{\nabla u^* \cdot \nabla w}{\sqrt{1 - |\nabla u^*|^2}} \, dx
$$
$$
= \int_{\partial B_p(x_0)} \frac{\partial u^*/\partial w}{\sqrt{1 - |\nabla u^*|^2}} w \, ds - \int_{B_p(x_0)} \text{div} \left( \frac{\nabla u^*}{\sqrt{1 - |\nabla u^*|^2}} \right) w \, dx
$$
$$
\leq \lambda_1 \int_{B_p(x_0)} g_1(u^*, v^*) w \, dx = \lambda_1 \int_\Omega g_1(\alpha_u, \alpha_v) w \, dx
$$

and proceeding in a similar way for $\alpha_v$, we obtain (see (10)) that $(\alpha_u, \alpha_v)$ is a lower solution of (21). Next we show that it is strict. Let $(u, v)$ be a solution of (21) with $u \geq \alpha_u$ and $v \geq \alpha_v$. By (6) we infer that $-M(u) > 0$ and $-M(v) > 0$ in $\Omega$ and applying Lemma 2.1 (ii) we get that $u \gg 0$ and $v \gg 0$ in $\overline{\Omega}$. In particular, one gets $\min_{\overline{B}_p(x_0)} u > 0$ and $\min_{\overline{B}_p(x_0)} v > 0$. Since $\lambda_1 g_1(u, v) \geq \lambda_1 g_1(\alpha_u, \alpha_v) \geq \lambda_1 g_1(\alpha_u, \alpha_v)$ in $B_p(x_0)$ as $g_1$ is quasi-monotone nondecreasing with respect to both arguments, on account of the comparison result in Lemma 2.1 (iii) we obtain that $\alpha_u(x) \leq u(x) - \min_{\partial B_p(x_0)} u < u(x)$ for every $x \in B_p(x_0)$. From this and $u \gg 0$ we deduce that $u \gg \alpha_u$ in $\overline{\Omega}$. Analogously it is shown that $v \gg \alpha_v$ in $\overline{\Omega}$.

Step 2. Construction of a strict upper solution $(\beta_u, \beta_v)$ with $\beta_u > 0 < \beta_v$ such that $\alpha_u \not\leq \beta_u$ or $\alpha_v \not\leq \beta_v$. From Step 1 we have the existence of $(\alpha_u, \alpha_v)$, for all $\lambda_1 > \lambda_1^*$ and $\lambda_2 > \lambda_2^*$. To prove the existence of $(\beta_u, \beta_v)$, let $[a, b]$ be the projection of $\overline{\Omega}$ over the $x_1$-axis and define

$$
g_1^*(s) := \lambda_1 c(s^+)^{p_1} (d(\Omega))^{q_1} \quad \text{and} \quad g_2^*(s) := \lambda_2 c (d(\Omega))^{p_2} (s^+)^{q_2},$$

where $\lambda_1, \lambda_2$ are fixed with $\lambda_1 > \lambda_1^*$ and $\lambda_2 > \lambda_2^*$. From Lemma 4.2 it follows that the equations

$$
- \left( w'/\sqrt{1 - w'^2} \right)' = g_1^*(w) \quad \text{and} \quad - \left( z'/\sqrt{1 - z'^2} \right)' = g_2^*(z)
$$

have sequences of solutions $\{w_k\}$ resp. $\{z_k\}$, such that, for every $k$, $\min_{[a,b]} w_k > 0 < \min_{[a,b]} z_k$ and

$$
\lim_{k \to +\infty} \|w_k\|_\infty = 0 \quad \text{resp.} \quad \lim_{k \to +\infty} \|z_k\|_\infty = 0. \quad (26)
$$

For all $x \in \overline{\Omega}$ and each $k$, we set $\beta_u^k(x) = w_k(x_1)$ and $\beta_v^k(x) = z_k(x_1)$. Then, using (6), we get that, for all sufficiently large $k$, $(\beta_u^k, \beta_v^k)$ is an upper solution of (21) with $\min_{\overline{\Omega}} \beta_u^k > 0 < \min_{\overline{\Omega}} \beta_v^k$. Moreover, according to (26), we have $\alpha_u \not\leq \beta_u^k$ and $\alpha_v \not\leq \beta_v^k$ in $\overline{\Omega}$, provided that $k$ is sufficiently large. We finally verify that $(\beta_u^k, \beta_v^k)$ is strict. Let $k$ be such a sufficiently large index and $(u, v)$ be a solution of (21) with $u \leq \beta_u^k$ and $v \leq \beta_v^k$. From Remark 4.1 (i) one has that $u \geq 0 \leq v$. Since
\( \lambda_1 c^{p_1} v^{q_1} \leq \lambda_1 c (\beta_k^p)^p (d(\Omega))^{q_1} \), by Lemma 2.1 (iii) one gets \( u \leq \beta_k^p - \min_{\partial \Omega} \beta_k^p \), which gives \( u \ll \beta_k^p \). Similarly we obtain that \( v \ll \beta_k^p \).

Now, the conclusion follows from Proposition 3.4 and Remark 4.1 (iii). \( \square \)

Next, we introduce the set
\( \mathcal{L} := \{ (\lambda_1, \lambda_2) : \lambda_1, \lambda_2 > 0 \text{ and } (5) \text{ has at least one strictly positive solution} \} \), which by virtue of the above proposition is non-empty.

**Proposition 4.4.** Under hypothesis \((H_f)\), the following are true:

1. there exist \( \Lambda_1, \Lambda_2 > 0 \) such that \( \mathcal{L} \subset [\Lambda_1, +\infty) \times [\Lambda_2, +\infty) \) and \( (5) \) has only the trivial solution, for all \( (\lambda_1, \lambda_2) \in (0, +\infty)^2 \setminus ([\Lambda_1, +\infty) \times [\Lambda_2, +\infty)) \);
2. if \( (\overline{\lambda}_1, \overline{\lambda}_2) \in \mathcal{L} \), then \( [\overline{\lambda}_1, +\infty) \times [\overline{\lambda}_2, +\infty) \subset \mathcal{L} \).

**Proof.** (i) Let \( \lambda_1, \lambda_2 > 0 \) and \( (u, v) \) be a strictly positive solution of (21). Thus, for every \( w \in W^{1,1}_0(\Omega) \), one has
\[
\int_{\Omega} \frac{\nabla u \cdot \nabla w}{\sqrt{1 - |\nabla u|^2}} \, dx = \lambda_1 \int_{\Omega} g_1(u, v) w \, dx, \tag{27}
\]
respectively
\[
\int_{\Omega} \frac{\nabla v \cdot \nabla w}{\sqrt{1 - |\nabla v|^2}} \, dx = \lambda_2 \int_{\Omega} g_2(u, v) w \, dx. \tag{28}
\]
Next, we denote by \( \lambda_1^\Delta \) the first eigenvalue of \( -\Delta \) on \( H_0^1(\Omega) \), i.e.,
\[
\lambda_1^\Delta = \inf \left\{ \frac{\int_{\Omega} |\nabla z|^2}{\int_{\Omega} z^2} : z \in H_0^1(\Omega) \setminus \{0\} \right\}. \tag{29}
\]
Taking \( w = u (\gg 0) \) in (27) and from (29), (6) and Remark 2.4, one gets
\[
\lambda_1^\Delta \int_{\Omega} u^2 \, dx \leq \int_{\Omega} |\nabla u|^2 \, dx \leq \int_{\Omega} \frac{|\nabla u|^2}{\sqrt{1 - |\nabla u|^2}} \, dx = \lambda_1 \int_{\Omega} g_1(u, v) u \, dx \leq \lambda_1 c \int_{\Omega} u^{p_1 - 1} v^{q_1} u^2 \, dx < \lambda_1 c \left( \frac{d(\Omega)}{2} \right)^{p_1 + q_1 - 1} \int_{\Omega} u^2 \, dx.
\]
Hence, one has
\[
\lambda_1 > \frac{\lambda_1^\Delta}{c} \left( \frac{d(\Omega)}{2} \right)^{-p_1 - q_1 + 1} > 0.
\]
Analogously, by taking \( w = v \) in (28), we obtain
\[
\lambda_2 > \frac{\lambda_1^\Delta}{c} \left( \frac{d(\Omega)}{2} \right)^{-p_2 - q_2 + 1} > 0.
\]
Consider now the non-empty sets
\[
\mathcal{L}_1 := \{ \lambda_1 > 0 : \exists \lambda_2 > 0 \text{ such that } (\lambda_1, \lambda_2) \in \mathcal{L} \},
\]
\[
\mathcal{L}_2 := \{ \lambda_2 > 0 : \exists \lambda_1 > 0 \text{ such that } (\lambda_1, \lambda_2) \in \mathcal{L} \}
\]
and setting
\[
\Lambda_i := \inf \mathcal{L}_i \left( > \frac{\lambda_1^\Delta}{c} \left( \frac{d(\Omega)}{2} \right)^{-p_i - q_i + 1} \right) \quad (i = 1, 2), \tag{30}
\]
it follows that \( \mathcal{L} \subset [\Lambda_1, +\infty) \times [\Lambda_2, +\infty) \) and for all \( (\lambda_1, \lambda_2) \in (0, +\infty)^2 \setminus ([\Lambda_1, +\infty) \times [\Lambda_2, +\infty)) \),
problem (5) has only the trivial solution on account of Remark 4.1 (ii).

(ii) Let \((\lambda_1, \lambda_2) \in \overline{[\lambda_1, +\infty) \times [\lambda_2, +\infty]}\) be arbitrarily chosen and \((\tilde{\pi}, \tilde{\sigma})\) be a strictly positive solution for (21) with \(\lambda_1 = \tilde{\lambda}_1\) and \(\lambda_2 = \tilde{\lambda}_2\). Then, \((\tilde{\pi}, \tilde{\sigma})\) is a lower solution of (21) with \(\lambda_1 = \tilde{\lambda}_1\) and \(\lambda_2 = \tilde{\lambda}_2\). From Proposition 3.5 and the fact that \(\tilde{\pi} \gg 0\) and \(\tilde{\sigma} \gg 0\), we obtain \((\lambda_1, \lambda_2) \in \mathcal{L}\).

For all \(\theta \in (0, \pi/2)\), denoting by \(\tilde{\mathcal{L}}(\theta)\) the non-empty set
\[
\tilde{\mathcal{L}}(\theta) := \{ \lambda > 0 : (\lambda \cos \theta, \lambda \sin \theta) \in \mathcal{L} \}
\]
one has the following:

**Proposition 4.5.** Assume hypothesis \((\tilde{H}_f)\). There exists a continuous function \(\Lambda : (0, \pi/2) \rightarrow (0, \infty)\) such that
\[
\lim_{\theta \to \pi/2} \Lambda(\theta) \cos \theta - \Lambda_1 = 0 = \lim_{\theta \to 0} \Lambda(\theta) \sin \theta - \Lambda_2
\]
and, for all \(\theta \in (0, \pi/2)\), the following are true:

(i) \(\Lambda(\theta) \in \tilde{\mathcal{L}}(\theta)\);

(ii) system (5) has at least one strictly positive solution, for all \(\lambda_1 \geq \Lambda(\theta) \cos \theta\) and \(\lambda_2 \geq \Lambda(\theta) \sin \theta\).

**Proof.** Step 1. For every \(\theta \in (0, \pi/2)\), we define
\[
\Lambda(\theta) := \inf \tilde{\mathcal{L}}(\theta) (\ll +\infty).
\]
We first show (i). Let \(\{\lambda^k\} \subset \tilde{\mathcal{L}}(\theta)\) be a sequence converging to \(\Lambda(\theta)\) and \((u_k, v_k) \in W^{2,r}(\Omega) \times W^{2,s}(\Omega) (r, s > N)\) with \(u_k \gg 0, v_k \gg 0\) be such that
\[
||\nabla u_k||_\infty < 1, ||\nabla v_k||_\infty < 1\) and \((u_k, v_k) = T(u_k, v_k),\) where \(T\) is the operator defined as in (9) with \(N_i (i = 1, 2)\) replaced by
\[
N_{1, \lambda^k}(u, v) = \lambda^k \cos \theta g_1(u(\cdot), v(\cdot)) \text{ resp. } N_{2, \lambda^k}(u, v) = \lambda^k \sin \theta g_2(u(\cdot), v(\cdot)).
\]
By Remark 2.4 and (6), we have
\[
||N_{i, \lambda^k}(u_k, v_k)||_\infty \leq \lambda^k c \left( \frac{d(\Omega)}{2} \right)^{p_i + q_i} (i = 1, 2),
\]
and since \(S\) is completely continuous, from the writing
\[
(u_k, v_k) = (S \circ N_{1, \lambda^k}(u_k, v_k), S \circ N_{2, \lambda^k}(u_k, v_k))
\]
we infer that there exists \((u, v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})\) such that, passing eventually to a subsequence, \(\{u_k, v_k\}\) converges to some \((u, v)\) in \(C^1(\overline{\Omega}) \times C^1(\overline{\Omega})\). We have that \(\{N_{i, \lambda^k}(u_k, v_k)\}\) converges to \(\{N_{i, \Lambda(\theta)}(u, v)\}\) in \(C(\overline{\Omega}) (i = 1, 2)\). This ensures that \(\{S(N_{1, \lambda^k}(u_k, v_k))\}\) converges to \(S(N_{1, \Lambda(\theta)}(u, v))\) (i = 1, 2), which by (32), means that \((S(N_{1, \Lambda(\theta)}(u, v))), S(N_{2, \Lambda(\theta)}(u, v))) = (u, v)\). It remains to prove that \(u \gg 0\) and \(v \gg 0\). Indeed, since \(u_k \gg 0\), taking \(w = u_k\) in (27) and using (29), (6) and Remark 2.4, one gets
\[
\lambda_1^\Delta \int_\Omega u_k^2 \, dx \leq \int_\Omega |\nabla u_k|^2 \, dx \leq \int_\Omega \frac{|\nabla u_k|^2}{\sqrt{1 - |\nabla u_k|^2}} \, dx = \lambda^k \int_\Omega g_1(u, v) u_k \, dx
\]
\[
\leq \lambda^k c \int_\Omega u_k^{p_i - 1} u_k \, dx < \lambda^k c \left( \frac{d(\Omega)}{2} \right)^{p_i} ||u_k||_\infty^{p_i - 1} \int_\Omega u_2 \, dx,
\]
i.e.,
\[
||u_k||_\infty^{p_i - 1} > \frac{\lambda_1^\Delta}{c \lambda^k} \left( \frac{d(\Omega)}{2} \right)^{-q_i} > 0.
\]
By taking $k \to \infty$ the conclusion follows from Remark 4.1 (ii). Consequently, $\Lambda(\theta) \in \mathcal{L}(\theta)$.

To see that (ii) holds true, we apply Proposition 4.4 (ii).

Step 2. We first prove that $\Lambda$ is continuous at each $\theta_0 \in (0, \pi/2)$. Otherwise, we can find some $\varepsilon \in (0, \Lambda(\theta_0))$ such that for all sufficiently large $n \in \mathbb{N}$, there exists $\theta_n \in (\theta_0 - 1/n, \theta_0 + 1/n) \subset (0, \pi/2)$ with $\Lambda(\theta_n) \notin (\Lambda(\theta_0) - \varepsilon, \Lambda(\theta_0) + \varepsilon)$. If we suppose that $\Lambda(\theta_n) \geq \Lambda(\theta_0) + \varepsilon$ for infinitely many $n \in \mathbb{N}$, then passing eventually to a subsequence, still denoted by $\{\theta_n\}$, one has

$$\left(\Lambda(\theta_n) - \frac{\varepsilon}{2}\right) \cos \theta_n \geq \left(\Lambda(\theta_0) + \frac{\varepsilon}{2}\right) \cos \theta_n,$$

respectively,

$$\left(\Lambda(\theta_n) - \frac{\varepsilon}{2}\right) \sin \theta_n \geq \left(\Lambda(\theta_0) + \frac{\varepsilon}{2}\right) \sin \theta_n.$$

On the other hand, notice that there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, one has $\Lambda(\theta_n) + \varepsilon/2 \cos \theta_n > \Lambda(\theta_0) \cos \theta_0$ (resp. $(\Lambda(\theta_0) + \varepsilon/2) \sin \theta_n > \Lambda(\theta_0) \sin \theta_0$).

Therefore, for all $n \geq n_0$, we infer that

$$\left(\Lambda(\theta_n) - \frac{\varepsilon}{2}\right) \cos \theta_n > \Lambda(\theta_0) \cos \theta_0 \text{ resp. } \left(\Lambda(\theta_n) - \frac{\varepsilon}{2}\right) \sin \theta_n > \Lambda(\theta_0) \sin \theta_0,$$

which by (ii) (proved in Step 1), gives $\Lambda(\theta_n) - \varepsilon/2 \in \mathcal{L}(\theta_n)$, contradicting the definition of $\Lambda(\theta_n)$. Similarly, assuming that $\Lambda(\theta_n) \leq \Lambda(\theta_0) - \varepsilon$ for infinitely many $n \in \mathbb{N}$, we obtain again a contradiction.

Next, to obtain (31), let $\{\theta_n\} \subset (0, \pi/2)$ be a sequence with $\theta_n \to \pi/2$ as $n \to \infty$. We have to show that $\Lambda(\theta_n) \cos \theta_n \to \Lambda_1$, as $n \to \infty$. With this aim, it is sufficient to prove that any subsequence of $\{\theta_n\}$, still denoted by $\{\theta_n\}$, contains a subsequence $\{\theta_{n_k}\}$ with

$$\Lambda(\theta_{n_k}) \cos \theta_{n_k} \to \Lambda_1, \text{ when } k \to \infty.$$

Now, from (30) there exists a sequence $\{\lambda^k\} \subset \mathcal{L}_1$ with $\lambda^k \to \Lambda_1$, as $k \to \infty$. Since $\theta_n \to \pi/2$, using Proposition 4.4 (ii) by an inductive reasoning we can find a sequence $\{r_k\} \subset (0, \infty)$ and a subsequence $\{\theta_{n_k}\} \subset \{\theta_n\}$ which, for all $k \in \mathbb{N}$, satisfy

$$r_k \cos \theta_{n_k} = \lambda^k_1 \tag{33}$$

and

$$(r_k \cos \theta_{n_k}, r_k \sin \theta_{n_k}) \in \mathcal{L}.$$

Then, by the definition of the mapping $\Lambda$ one has $\Lambda(\theta_{n_k}) \leq r_k$ and hence, $\Lambda(\theta_{n_k}) \cos \theta_{n_k} \leq r_k \cos \theta_{n_k}$. As by (i) (proved in Step 1) one has $\Lambda(\theta_{n_k}) \cos \theta_{n_k} \in \mathcal{L}_1$, from (30) and (33), we get

$$\Lambda_1 \leq \Lambda(\theta_{n_k}) \cos \theta_{n_k} \leq r_k \cos \theta_{n_k} = \lambda^k_1 \to \Lambda_1, \text{ as } k \to \infty.$$

Similarly, it can be proved that $\Lambda(\theta_n) \sin \theta_n \to \Lambda_2$ when $\theta_n \to 0$ as $n \to \infty$, and the proof is complete.

\textbf{Theorem 4.6.} Under hypothesis $(\tilde{H}_f)$, there exist $\Lambda_1, \Lambda_2 > 0$ and a continuous function $\Lambda : (0, \pi/2) \to (0, +\infty)$, generating the curve

$$\begin{cases}
\lambda_1(\theta) = \Lambda(\theta) \cos \theta, \\
\lambda_2(\theta) = \Lambda(\theta) \sin \theta,
\end{cases} \quad \theta \in (0, \pi/2)$$

such that

(i) $\Gamma \subset [\Lambda_1, +\infty) \times [\Lambda_2, +\infty)$;
(ii) the following asymptotic behaviors hold
\[ \lim_{\theta \to 0} \lambda_1(\theta) = +\infty = \lim_{\theta \to \pi/2} \lambda_2(\theta), \]  
(34)
\[ \lim_{\theta \to \pi/2} \lambda_1(\theta) - \Lambda_1 = 0 = \lim_{\theta \to 0} \lambda_2(\theta) - \Lambda_2; \]  
(35)
(iii) \( \Gamma \) separates the first quadrant \((0, +\infty) \times (0, +\infty)\) in two disjoint sets, an open one \( \mathcal{O}_0 \) and a closed one \( \mathcal{F} \supset \Gamma \), such that problem (5) has zero or at least one strictly positive solution, according to \((\lambda_1, \lambda_2) \in \mathcal{O}_0\) or \((\lambda_1, \lambda_2) \in \mathcal{F}\). Moreover, there exists \((\lambda_1^*, \lambda_2^*) \in \mathcal{F}\) such that (5) has at least two strictly positive solutions, for all \( \lambda_1 > \lambda_1^* \) and \( \lambda_2 > \lambda_2^* \).

Proof. The existence of \( \lambda_1, \lambda_2 \) and of the continuous function \( \Lambda \) as well as statement (i) and (35) are proved in Proposition 4.4 (i) and Proposition 4.5. Equalities in (34) are immediate from the inequalities
\[ \Lambda(\theta) \geq \frac{\Lambda_1}{\cos \theta}. \]  
resp. \( \Lambda(\theta) \geq \frac{\Lambda_2}{\sin \theta} \).

Taking into account the definition of \( \Lambda(\theta) \) and using Propositions 4.3 and 4.5 we get the conclusion.

Example 4.7. Let \( p_1, q_2 > 1 \) and \( q_1, p_2 > 0 \). Then Theorem 4.6 applies with the following choices of \( f_1 \) and \( f_2 \) in problem (5):

(i) \( f_1(u, v) = u^{p_1} v^{q_1} \) and \( f_2(u, v) = v^{q_2} u^{p_2} - \) Lane-Emden type nonlinearities;

(ii) \( f_1(u, v) = u^{p_1} \ln(1 + v^{q_1}) \) and \( f_2(u, v) = v^{q_2} \ln(1 + u^{p_2}) \).

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