Combined \((q, h)\)-Deformation as a Nonlinear Map on \(U_q(sl(2))\)

B. Abdesselam\(^\dagger\), A. Chakrabarti\(^\ddagger\) and R. Chakrabarti\(^\dagger\)

\(^\dagger\)Centre de Physique Théorique, Ecole Polytechnique,
91128 Palaiseau Cedex, France.

\(^\ddagger\)Department of Theoretical Physics, University of Madras, Guindy campus,
Madras-600025, India.

Abstract

The generators \((J_\pm, J_0)\) of the algebra \(U_q(sl(2))\) is our starting point. An invertible nonlinear map involving, apart from \(q\), a second arbitrary complex parameter \(h\), defines a triplet \((\hat{X}, \hat{Y}, \hat{H})\). The latter set forms a closed algebra under commutation relations. The nonlinear algebra \(U_{q,h}(sl(2))\), thus generated, has two different limits. For \(q \to 1\), the Jordanian \(h\)-deformation \(U_h(sl(2))\) is obtained. For \(h \to 0\), the \(q\)-deformed algebra \(U_q(sl(2))\) is reproduced. From the nonlinear map, the irreducible representations of the doubly-deformed algebra \(U_{q,h}(sl(2))\) may be directly and explicitly obtained form the known representations of the algebra \(U_q(sl(2))\). Here we consider only generic values of \(q\).
1 Introduction

The standard $q$-deformation of the $sl(2)$ algebra is familiar enough. Here we treat this as known. The Jordanian $h$-deformation $U_h(sl(2))$ is relatively recent. Its various aspects are being studied [3-9] intensively. In a recent work [10] the present authors established a nonlinear invertible map between the generators of $U_h(sl(2))$ and the classical $sl(2)$ generators. The principal interest of the map in [10] was that it simply and immediately provided a construction of the irreducible representations of the algebra $U_h(sl(2))$. Along similar lines, a map was established in [11] relating the $h$-deformed 3-dimensional Euclidean algebra $U_h(e(3))$ and its classical partner $e(3)$.

Here, analogously, we consider an appropriate generalization of the map in [10]. Instead of the generators of the classical algebra $sl(2)$, we take the generators of the standard $q$-deformed algebra $U_q(SL(2))$ as our starting point; and, via a nonlinear map, introduce a second deformation parameter $h$. The generators $(\hat{X}, \hat{Y}, \hat{H})$, thus achieved, form a closed set upon commutation. We refer to this doubly deformed nonlinear algebra as $U_{q,h}(sl(2))$.

Such an algebra may be of interest for various reasons, leading to applications [12, 13]. The algebra $U_{q,h}(sl(2))$ has two distinct limits: $U_{q,h}(sl(2)) \rightarrow U_h(sl(2))$ as $q \rightarrow 1$; and $U_{q,h}(sl(2)) \rightarrow U_q(sl(2))$ as $h \rightarrow 0$. When both limits are imposed the classical algebra $sl(2)$ is, of course, reproduced. Parallel to our construction [10], here also the known representations of $U_q(sl(2))$ immediately yield, through the map, the representations of $U_{q,h}(sl(2))$. Here we consider only the generic values of $q$. Special features [14, 15] for $q$ being a root of unity will be explored elsewhere. The map to be presented below provides two induced coalgebraic structures for the doubly deformed algebra $U_{q,h}(sl(2))$. These induced structures reflect the underlying coalgebraic properties of $U_q(sl(2))$ and $U_h(sl(2))$ respectively.

2 The map

We first briefly review the map [10] between the $U_h(sl(2))$ and $sl(2)$ generators. The complications due to the presence of another deformation parameter $q(\neq 1)$ may then be easier to grasp. The generators $(J_\pm, J_0)$ of the algebra $sl(2)$ satisfy

\[ [J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_0 \] (1)

Introducing an arbitrary complex parameter $h$, we now define

\[ X = \frac{2}{h} \text{arctanh} \left( \frac{h J_+}{2} \right) = \frac{2}{h} \sum_{n \geq 0} \frac{(\frac{h}{2} J_+)^{2n+1}}{(2n+1)}, \]

\[ Y = (1 - (\frac{h}{2} J_+)^2)^{1/2} J_-(1 - (\frac{h}{2} J_+)^2)^{1/2}, \]

\[ H = J_0. \] (2)
It may be shown [10] that the triplet \((X, Y, H)\) satisfies the commutation relations [2] corresponding to the generators of the \(U_h(sl(2))\) algebra:

\[
[H, X] = \frac{1}{\hbar} \sinh \hbar X \tag{3}
\]

\[
[H, Y] = -\frac{1}{2} \left( Y(\cosh \hbar X) + (\cosh \hbar X)Y \right) \tag{4}
\]

\[
[X, Y] = 2H. \tag{5}
\]

The Casimir operator for the algebra \(U_h(sl(2))\), then, via the map, reduces to its expression corresponding to the classical algebra \(sl(2)\):

\[
C = \frac{1}{2\hbar} [(\sinh \hbar X)Y + Y(\sinh \hbar X)] + \frac{1}{4}(\sinh \hbar X)^2 + H^2
\]

\[
= \frac{1}{2}(J_+J_- + J_-J_+) + J_0^2. \tag{6}
\]

As demonstrated in [10], the map (2) now explicitly furnishes the irreducible representations of the \(U_h(sl(2))\) algebra.

We now proceed to obtain a \(q\)-deformed generalization of the map (2). The basic elements of our construction are the generators of the \(q\)-deformed algebra \(U_q(sl(2))\). These generators obey the following commutation rules

\[
[J_0, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = [2J_0], \tag{7}
\]

where

\[
[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{12}
\]

The coproduct structure for the generators reads

\[
\Delta(J_{\pm}) = J_{\pm} \otimes q^{J_0} + q^{-J_0} \otimes J_{\pm}, \quad \Delta(J_0) = J_0 \otimes 1 + 1 \otimes J_0. \tag{8}
\]

For later use, we here obtain the following identity

\[
[J_+^p, J_-] = \frac{[p]}{q - q^{-1}}(q^{J_0}J_+^{p-1}q^{J_0} - q^{-J_0}J_+^{p-1}q^{-J_0}). \tag{9}
\]

In order to get a closed algebra, we choose the following \(q\)-generalization of the map (2)

\[
\frac{\hbar \hat{X}}{2} = \sum_{n \geq 0} \alpha_n \left( \frac{\hbar}{2}J_+ \right)^{2n+1}, \tag{10}
\]

\[
\hat{Y} = (1 - \left( \frac{\hbar}{2}J_+ \right)^2)^{1/2}J_-(1 - \left( \frac{\hbar}{2}J_+ \right)^2)^{1/2}, \tag{11}
\]

\[
\hat{H} = J_0. \tag{12}
\]
where
\[ \alpha_n = \frac{1}{[2n + 1]} P_n(\xi), \quad \xi = \frac{q^2 + q^{-2}}{2} \] (13)

The purpose of introducing the Legendre polynomials \( P_n(\xi) \), will become evident as we proceed. It plays a crucial role. We note that in (10) (in space of the standard finite, \((2j + 1)\) dimensional representations) the convergence of the series is not a problem as \( J_+ \) is a nilpotent operator. The series in (10) may be looked as a particular \( q \)-generalization of the series for the \( \text{arctanh} \) function. The maps for the operators \( \hat{Y}, \hat{H} \) in (11,12) read formally the same as in (2), which may now be looked as \( q = 1 \) limit of the present maps. In order to invert the map, we postulate a series
\[ J_+ = \frac{2}{h} \sum_{n \geq 0} \beta_n (\frac{h}{2} \hat{X})^{2n+1}. \] (14)

Substituting (14) in (10) and comparing terms of identical powers on both sides, we obtain
\[ \beta_0 = 1, \quad \beta_n = - \sum_{m=1}^{n} \alpha_m Z_{n,m} \quad \text{for} \ n \geq 1 \] (15)

where
\[ Z_{n,n} = 1, \quad Z_{n,m} = \sum_{\text{partitions}} \zeta_{m,\{\nu_p\}} \prod_{p=1}^{n-m} \beta_b^{\nu_p}. \] (16)

The sum over “partitions” in the rhs of (16) maintains the combinatorial properties.
\[ \sum_{p=1}^{n-m} \nu_p = n - m, \quad \sum_{p=1}^{n-m} \nu_p \leq (2m + 1) \] (17)

The symmetry factor \( \zeta_{m,\{\nu_p\}} \) in the rhs of (16) reads
\[ \zeta_{m,\{\nu_p\}} = \frac{(2m + 1)!}{(2m + 1 - \sum_{p=1}^{n-m} \nu_p)! \prod_{b=1}^{n-m} (\nu_p)!} \] (18)

Thus, the inversion of the series (10) may be looked as a combinatorial problem; and, an arbitrary coefficient \( \beta_n \) in the series (14) may now be found recursively. Again, owing to the nilpotency of \( \hat{X} \), the convergence of the rhs in (14) readily follows. The first few coefficients are listed below:
\[ \beta_1 = -\alpha_1, \quad \beta_2 = -\alpha_2 + 3\alpha_1^2, \quad \beta_3 = -\alpha_3 + 8\alpha_2\alpha_1 - 12\alpha_1^3 \]
\[ \beta_4 = -\alpha_4 + 10\alpha_3\alpha_1 + 5\alpha_2^2 - 55\alpha_2\alpha_1^2 + 55\alpha_1^4, \]
\[ \beta_5 = -\alpha_5 + 12\alpha_4\alpha_1 + 12\alpha_3\alpha_2 - 78\alpha_3\alpha_1^2 - 78\alpha_2^2\alpha_1 + 364\alpha_2\alpha_1^3 - 273\alpha_1^5, \]
\[ \beta_6 = -\alpha_6 + 14\alpha_5\alpha_1 + 14\alpha_4\alpha_2 - 105\alpha_4\alpha_1^2 + 7\alpha_3^2 - 210\alpha_3\alpha_2\alpha_1 + 560\alpha_3\alpha_1^3 - 35\alpha_2^3 + 840\alpha_2\alpha_1^2 - 2380\alpha_2\alpha_1^4 + 1428\alpha_1^6. \] (19)
In the limit $q \to 1$ we have $\alpha_n \to \frac{1}{2n+1}$; and, consequently, it follows from the successive terms in (15) that

$$
\beta_n \to \frac{2^{2n}(2^{2n} - 1)}{(2n)!} B_{2n},
$$

(20)

where $B_{2n}$ are the Bernoulli numbers. Thus, the series (14) may be viewed as a particular $q$-generalization of the $\tanh$ function. In this sense, the series (10) and (14) may be expressed respectively as

$$
\frac{h\hat{X}}{2} = \text{arctanh}_q \left( \frac{h\hat{J}_+}{2} \right),
$$

(21)

and

$$
\frac{h\hat{J}_+}{2} = \text{tanh}_q \left( \frac{h\hat{X}}{2} \right),
$$

(22)

It is to be emphasized that we define the $q$-functions $\text{arctanh}_q$ and $\text{tanh}_q$ only by their series expressions in (10) and (14) respectively. (Other definitions can and do exist. But the foregoing ones are necessary for our purpose.)

We are now in a position to demonstrate the commutation relations for the generators $(\hat{X}, \hat{Y}, \hat{H})$ defined by the map (10, 11, 12). To this end, we obtain by employing the identity (9)

$$
[\hat{X}, \hat{J}_-] = \frac{1}{q - q^{-1}} \left[ q^{\hat{J}_0} F \left( \frac{h\hat{J}_+}{2} \right) q^{\hat{J}_0} - q^{-\hat{J}_0} F \left( \frac{h\hat{J}_+}{2} \right) q^{-\hat{J}_0} \right],
$$

(23)

where

$$
F(x) = \sum_{n \geq 0} P_n(\xi) x^{2n} = (1 - 2\xi x^2 + x^4)^{-1/2} = (1 - (qx)^2)^{-1/2} (1 - (q^{-1}x)^2)^{-1/2}
$$

(24)

Use of (23) and the identity

$$
q^{\hat{J}_0} f(\hat{J}_+) q^{-\hat{J}_0} = f(q^{\hat{J}_0} \hat{J}_+) = f(q^{\hat{J}_0} \hat{J}_+)
$$

(25)

leads to

$$
[\hat{X}, \hat{Y}] = [2\hat{H}].
$$

(26)

This is the direct $q$-deformation of the commutator (5). We may point out that the specific choice of the coefficients $\alpha_n$ in (13) is instrumental in ensuring (26), while a simple prescription for $\hat{Y}$ is maintained. The other commutators for the triplet $\hat{X}, \hat{Y}, \hat{H}$ may also be determined:

$$
[\hat{H}, \hat{X}] = \frac{2}{\hbar} \sum_{n \geq 0} ^{\frac{2n+1}{2n+1}} \frac{P_n(\xi)}{2n+1} \left( \frac{h\hat{J}_+}{2} \right)^{2n+1},
$$

(27)

$$
[\hat{H}, \hat{Y}] = -\frac{1}{2} \left( \frac{1 + \left( \frac{h}{2} \hat{J}_+ \right)^2}{1 - \left( \frac{h}{2} \hat{J}_+ \right)^2} \hat{Y} + \hat{Y} \frac{1 + \left( \frac{h}{2} \hat{J}_+ \right)^2}{1 - \left( \frac{h}{2} \hat{J}_+ \right)^2} \right)
$$

(28)
In the rhs of (27) and (28) the operator $\mathcal{J}_+$ stands for the series expression (14); and this leads to the closed algebra for the set $\{\dot{X}, \dot{Y}, \dot{H}\}$. It has been verified explicitly that the rhs of (26), (27) and (28) are indeed compatible with the Jacobi identity.

We now describe an alternate approach to the inversion of the series (10). Starting with a variant of (27), we obtain a closed expression for $\mathcal{J}_+$ in terms of $\dot{X}$ and $q^{\pm \dot{H}}$. To this end, we define

$$u = \frac{1}{q - q^{-1}}(q^{\dot{H}} \frac{\hbar}{2} \dot{X} q^{-\dot{H}} - q^{-\dot{H}} \frac{\hbar}{2} \dot{X} q^{\dot{H}}), \quad (29)$$

$$v = \frac{\hbar}{2} \mathcal{J}_+ \quad (30)$$

In the $q \to 1$ limit, we have

$$u \to \frac{\hbar}{2} [H, X] \quad (31)$$

$$= \frac{1}{2} \sinh \hbar X \quad (32)$$

and $v \to \frac{\hbar}{2} \mathcal{J}_+$. The equality in (31) follows from (3). In this limit the two operators $u$ and $v$ interrelate, via the map (2), as follows

$$v = \tanh \left(\frac{1}{2} \sinh^{-1} 2u\right) \quad (33)$$

$$= -\frac{1}{2u} + \sqrt{\frac{1}{4u^2} + 1}. \quad (34)$$

When $q \neq 1$, the "deformed" relationship between $u$ and $v$ may be observed as follows. Equation (7,10,12) yield

$$u = \sum_{n \geq 0} P_n(\xi) v^{2n+1} \quad (35)$$

$$= \frac{v}{\sqrt{(1 - (qv)^2)(1 - (q^{-1}v)^2)}}. \quad (36)$$

Solving $v$ in terms of $u$, we get

$$v^2 = \xi + \frac{1}{2u^2} \pm \sqrt{\left(\frac{\xi + \frac{1}{2u^2}}{2u^2}\right)^2 - 1} \quad (37)$$

Noticing that $u \to 0$ as $v \to 0$, indicates that we choose

$$v^2 = \xi + \frac{1}{2u^2} - \sqrt{\left(\frac{\xi + \frac{1}{2u^2}}{2u^2}\right)^2 - 1}. \quad (38)$$
In the limit $q \to 1$, the expression (33) is obtained. Equation (38), therefore, embodies the $q$-generalized closed form expression of the operator $J_+$ in terms of the operators $(\hat{H}, \hat{X})$.

To sum up, starting from the standard $U_q(sl(2))$ algebra, we constructed a triplet $(\hat{X}, \hat{Y}, \hat{H})$ that form a closed algebra under commutation relations. This defines a doubly deformed universal enveloping algebra $U_{q,h}(sl(2))$.

3 Representations of $U_{q,h}(sl(2))$

The action of the generators of the algebra $U_q(sl(2))$ on the standard basis $\{|jm\rangle| (2j + 1) \in N, -j \leq m \leq j \}$ are given by

$$J_\pm |jm\rangle = ([j \mp m]|j \pm m + 1\rangle)^{1/2} jm \pm 1 >, \quad (39)$$

$$q^{\pm \mathcal{J}_0} |jm\rangle = q^{\pm m} |jm\rangle. \quad (40)$$

Repeated actions of $J_+$ on the basis states may be expressed as

$$J_+^p |jm\rangle = \left(\frac{j - m}{j + m}!\frac{j + m + p}{j - m - p}!\right)^{1/2} |j + m + p\rangle \quad (41)$$

where $[n]! = \prod_{k=1}^{n}[k]$. Using (41) and the map (10), the action of the operator $\hat{X}$ on the basis states is obtained as

$$\hat{X}|jm\rangle = \sum_{k,l \geq 0} \left(\frac{h^2}{2}\right)^{2k} P_k(\xi) \left(\frac{j - m}{j + m}!\frac{j + m + 2k + 1}{j - m - 2k - 1}!\right)^{1/2} |j + m + 2k + 1\rangle \quad (42)$$

To obtain the action of $\hat{Y}$ on the basis states, we use the identity (9) and reexpress $\hat{Y}$ in the normal ordered form:

$$\hat{Y} = \sum_{k,l \geq 0} \left(-\frac{h^2}{4}\right)^{k+l} \left(\frac{1}{2}\right)^{k} \left(\frac{1}{2}\right)^{l} \left(\mathcal{J}_- \mathcal{J}_+^{2k+l} + [2k] \mathcal{J}_+^{2k+2l-1} [2\mathcal{J}_0 + 2k + 4l - 1]\right). \quad (43)$$

Using (41), it may be readily obtained

$$\hat{Y}|jm\rangle = \sum_{k,l \geq 0} \left(-\frac{h^2}{4}\right)^{k+l} \left(\frac{1}{2}\right)^{k} \left(\frac{1}{2}\right)^{l} \left\{ \left(\frac{[j + m + 2k + 2l]!|j - m - 2k - 2l + 1| |j - m|!\frac{j + m + 2k + 2l}{j + m}! |j - m - 2k - 2l|!}{|j + m|! |j - m - 2k - 2l|!}\right)^{1/2} \right\} |j + m + 2k + 2l - 1\rangle \quad (44)$$

$$+ [2k][2k + 4l + 2m - 1] \left(\frac{[j - m]!\frac{j + m + 2k + 2l - 1}{|j + m|! |j - m - 2k - 2l + 1|!}}{[j + m]! |j - m - 2k - 2l + 1|!}\right)^{1/2} |j + m + 2k + 2l - 1\rangle \quad (45)$$
The Casimir operator for the algebra $\mathcal{U}_q(sl(2))$

$$C_q = J_+J_- + [J_0][J_0 - 1]$$

$$= J_-J_+ + [J_0][J_0 + 1]$$

may be expressed in terms of the generators $(\hat{X}, \hat{Y}, \hat{H})$, leading to a $q$-deformation of (6). But the eigenvalues will, of course, be given by

$$C_q(jm) = [j][j + 1][jm].$$

This completes our construction of the irreducible representations of $\mathcal{U}_{q,h}(sl(2))$ for generic values of $q$.

4 Conclusion

We have discussed here nonlinear mappings starting from the algebras $sl(2)$ and $\mathcal{U}_q(sl(2))$, leading to the algebras $\mathcal{U}_h(sl(2))$ and $\mathcal{U}_{q,h}(sl(2))$ respectively. Elsewhere [13] the relation between $sl(2)$ and $\mathcal{U}_q(sl(2))$ was also exhibited as the limiting case of a class of nonlinear maps (where other references can be found). The maps leading from $sl(2)$ to $\mathcal{U}_q(sl(2))$ and $\mathcal{U}_h(sl(2))$ are complementary in the following sense:

(i). In the case of $q$-deformation the nonlinearity enters through the diagonalizable generator $J_0$.

(ii). For the $h$-deformation it enters via the nilpotent (non-diagonalizable) generator $J_+$. Finally both cases are combined in the passage to $\mathcal{U}_{q,h}(sl(2))$. In terms of the nonlinear maps, here we have followed the path

$$\mathcal{U}(sl(2)) \longrightarrow \mathcal{U}_q(sl(2)) \longrightarrow \mathcal{U}_{q,h}(sl(2)).$$

It is also possible to envisage the alternate route

$$\mathcal{U}(sl(2)) \longrightarrow \mathcal{U}_h(sl(2)) \longrightarrow \mathcal{U}_{q,h}(sl(2)).$$

This is less simple. One has to implement (in the rhs of (10) and (11)) the map [10, 16] relating (1) and (7) and then the inverse of the map (2). Thus, for example,

$$\hat{X} = \frac{2}{\hbar} \sum_{n \geq 0} \alpha_n \text{tanh}\left(\frac{\hbar X}{2}\right) \left(\frac{[\hat{J} + \frac{1}{2}]^2 - [H + \frac{1}{2}]^2}{(\hat{J} + \frac{1}{2})^2 - (H + \frac{1}{2})^2}\right)^{1/2} 2^{n+1}$$

where the operator $\hat{J}$ is here given in terms of $C(X,Y,H)$ of (6) as

$$\hat{J}(\hat{J} + 1) = C.$$
One obtains $\hat{Y}$ analogously. The following pattern of possibilities emerges:

\[
\begin{array}{c}
\mathcal{U}(sl(2)) \rightarrow \mathcal{U}_q(sl(2)) \\
\uparrow \quad \uparrow \\
\mathcal{U}_q(sl(2)) \rightarrow \mathcal{U}_q,h(sl(2))
\end{array}
\]

For following the small arrows, it is more simple at each stage to take the limits $(q \to 1$ and/or $h \to 0)$, instead of inverting the map. The problem of explicit construction of irreducible representations is quite efficiently solved at each higher stage via our maps.

An induced coalgebraic structure of the algebra $\mathcal{U}_{q,h}(sl(2))$ is immediately generated by the map $(10,11,12)$. The induced coproduct for the generators read (in terms of coproducts $(8)$)

\[
\Delta(\hat{X}) = \frac{2}{h} \sum_{n \geq 0} \alpha_n \left( \frac{h}{2} \Delta(J_+) \right)^{2n+1}
\]

\[
\Delta(\hat{Y}) = \left( 1 - \left( \frac{h}{2} \Delta(J_+) \right)^2 \right)^{1/2} \Delta(J_-) \left( 1 - \left( \frac{h}{2} \Delta(J_+) \right)^2 \right)^{1/2}
\]

\[
\Delta(\hat{H}) = \Delta(J_0)
\]

The full induced Hopf structure may be obtained.

An alternative induced Hopf structure is provided by the map indicated by (51) and the Hopf structure of $\mathcal{U}_h(sl(2))$ [2]. The noncommutative coproducts for $\mathcal{U}_h(sl(2))$ are

\[
\Delta(X) = X \otimes 1 + 1 \otimes X
\]

\[
\Delta(Y) = Y \otimes e^{hx} + e^{-hx} \otimes Y
\]

\[
\Delta(H) = H \otimes e^{hx} + e^{-hx} \otimes H
\]

Thus

\[
\Delta'(\hat{X}) = \frac{2}{h} \sum_{n \geq 0} \alpha_n \left( \tanh \left( \frac{h}{2} \Delta(X) \right) \right) \left( \frac{\Delta(J) + \frac{1}{2}(1 \otimes 1) - \Delta(H) + \frac{1}{2}(1 \otimes 1)}{(\Delta(J) + \frac{1}{2}(1 \otimes 1))^2 - (\Delta(H) + \frac{1}{2}(1 \otimes 1))^2} \right)^{1/2} 2n+1
\]

The corresponding coproducts $\Delta'(\hat{Y})$ and $\Delta'(\hat{H})$ (and also the counits and antipodes for all) can thus be written directly. Appropriate (explicitly given) inverse maps implemented in the rhs for $\Delta$ and $\Delta'$, complete de picture. The $\mathcal{R}$-matrices of $\mathcal{U}_{h}(sl(2))$ (see [10] and references
cited there) can be implemented here through $\Delta'$ (whereas the well-known $R$ matrices for the standard $q$-deformation correspond to $\Delta$). The “non-classical” automorphism of $\mathcal{U}_h(sl(2))$ can also be implemented via (51).

New features and possibilities arising for $q$ a root of unity remain to be studied. Another possibility is worth mentioning. It is known that at the level of one parameter deformations, the $q$ and $h$ deformations are essentially distinct. The $h$-deformation is a non-invertible singular limit of the $q$-deformation. Can the $(q,h)$-deformation may be obtained as a singular limit of a $(q,q')$-deformation, where $q$ and $q'$ play comparable roles in the algebra. In this context, we mention that a class of double deformations was presented in [13] with $\mathcal{U}_q(sl(2))$ as the starting point.

Recently we have studied elsewhere [17,18] other two parametric deformations of $sl(2)$ involving elliptic functions. The discussion in [18] concerning possibilities of applications has evident parallel features in the present case. Instead of more or less reproducing it we just mention that the difference of the role of the parameters ($(h,k)$ in [18] and $(q,h)$ here) can be of particular interest.

**Citations**

[1] Demidov E E, Manin Yu I, Mukhin E E and Zhadannovich D Z, Prog. Theo. Phys. Suppl. 102 (1990) 203.

[2] Ohn Ch., Lett. Math. Phys. 25 (1992) 85.

[3] Vladimirov A A, Mod. Phys. Lett. A8 (1993) 2573.

[4] Aghamohammdi A, Mod. Phys. Lett. A8 (1993) 2607.

[5] Aghamohammdi A, Khorrami M and Shariati A, J. Phys. A : Math. Gen. 28 (1995) L225.

[6] Ballesteros A, Herranz F J, Del Olmo M A and Santander M, J. Phys. A: Math. Gen. 28 (1995) 941.

[7] Ballesteros A, Herranz F J, Del Olmo M A, Perena C M and Santander M, J. Phys. A : Math. Gen. 28 (1995) 7713.

[8] Shariati M, Aghamohammdi A and Khorrami M, Mod. Phys. Lett. A11 (1996) 187.

[9] Parashar P, *Nonstandard Poincaré and Heisenberg Algebras*; preprint SISSA 85/96/FM.

[10] Abdesselam B, Chakrabarti A and Chakrabarti R, *Irreducible representations of Jordanian quantum algebra $\mathcal{U}_h(sl(2))$ via a nonlinear map*, (to be published in Mod. Phys. Lett).
[11] Abdesselam B, Chakrabarti A and Chakrabarti R, *On $U_h(sl(2)), U_h(e(3))$ and their representations*; (to be published in Int. Jour. Mod. phys).

[12] Rocek M, Phys. Lett. B255 (1991) 554.

[13] Abdesselam B, Beckers J, Chakrabarti A and Debergh N, J. Phys. A: Math. Gen. 29 (1996) 6729-6736.

[14] Roche P and Arnaudon D, Lett. Math. Phys. 17 (1989) 295.

[15] Arnaudon D and Chakrabarti A, Comm. Math. Phys. 139 (1991) 461.

[16] Curtright T L, Ghandour G I and Zachos C K, 1991 J. Math. Phys. 32 676.

[17] Chakrabarti A, *Induced Hopf structure and irreducible representations of an elliptic $U_{q,p}(sl(2))$ via a nonlinear map*, CPTH-9611476 ([q-alg/9611014]).

[18] Chakrabarti A, *A generalization of $U_h(sl(2))$ via Jacobian elliptic functions*, CPTH-9611477 ([q-alg/9611015]).