Latent Association Mining in Binary Data

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Abstract

We consider the problem of identifying groups of mutually associated variables in moderate or high dimensional data. In many cases, ordinary Pearson correlation provides useful information concerning the linear relationship between variables. However, for binary data, ordinary correlation may lose power and may lack interpretability. In this paper, we develop and investigate a new method called Latent Association Mining in Binary Data (LAMB). The LAMB method is built on the assumption that the binary observations represent a random thresholding of a latent continuous variable that may have a complex correlation structure. We consider a new measure of association, latent correlation, that is designed to assess association in the underlying continuous variable, without bias due to the mediating effects of the thresholding procedure. The full LAMB procedure makes use of iterative hypothesis testing to identify groups of latently correlated variables. LAMB is shown to improve power over existing methods in simulated settings, to be computationally efficient for large datasets, and to uncover new meaningful results from common real data types.

Keywords: correlation mining, frequent itemsets, association rules, data mining, association mining, binary observations

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1 Introduction

Correlation mining is a common problem in the analysis of high-dimensional data, the broad goal of which is to identify subsets of variables with large pairwise correlation. In this work we focused on correlation mining from binary data. There is an extensive literature on correlation mining methods for binary data, often in the context of clustering and community detection; see, e.g. Kriegel et al. (2009) for a representative overview. This paper specifically addresses the issue of mining for latent correlation. We introduce a new method, Latent Association Mining in Binary Data (LAMB), that is based on a flexible thresholding model, and an algorithmic framework built on statistical testing principles. Our method is computationally efficient, and is able to address datasets that may yield misleading results in conventional correlation mining analyses. We provide general theoretical support for the method in high and low dimensional settings.

Data in correlation mining is typically modeled as a random matrix $X \in \mathbb{R}^{n \times d}$, where $X_{ij}$ corresponds to the value of the $i$-th sample of variable $j$. Although continuous data is ubiquitous, in many familiar settings observations may be binary, i.e. $X_{ij} \in \{0, 1\}$. For example, in market basket data, $X_{ij}$ is equal to one only if a particular buyer $i$ purchased a particular item $j$. Correlation mining of such data can provide valuable information to researchers for the purposes of advertising and inventory control. However, popular measures of association, like the standard Pearson correlation, may not be appropriate for binary data as they are not parameters of the joint distribution.

As such, a wealth of metrics have been proposed to quantify relationships between binary variables (Choi et al., 2010). In the field of frequent itemset mining it is common for methods to screen for sets of items with high co-occurrence. Despite their diversity, methods of binary correlation mining and itemset mining generally adhere to the assumption that each row of the data matrix, $X_i = (X_{i1}, \ldots, X_{id})$ represents a direct sample of the $d$ variables (items) of interest. In some settings, however, it may be more reasonable to assume that the observed binary data represents incomplete information. That is, dichotomous observations may be derived from a function of some underlying and unobserved continuous data. In the example of market basket data, a buyer’s decision to purchase a product or not may depend on the buyer’s financial situation, the other products available, the layout of the store, and so on.

The LAMB method differs from existing binary correlation mining methods in that the target association measure is the latent association between variables, which is not directly observed. This idea is closely related to the tolerance distribution interpretation in classical generalized linear models (McCullagh and Nelder, 1989). In addition to its focus on latent correlation, LAMB also employs a novel search procedure first introduced in Wilson et al. (2014). This iterative procedure uses multiple testing principles to successively update a candidate variable set. The LAMB method identifies sets of variables that have strong mutual latent correlation.

Our development and analysis of the LAMB method are carried out in the context of
a simple thresholded latent variable model. Under the model, the sample $X_i \in \{0, 1\}^d$ is obtained by thresholding the components of an unobserved, continuous random vector $Z_i \in \mathbb{R}^d$. Moreover, the thresholds yielding $X_i$ from $Z_i$ are themselves random.

Treating binary data as thresholded continuous data is a common practice in regression models, see for example Qu et al. (1992), Antal (2007), Tan et al. (1999). In typical regression models it is assumed that binary observations derive from a much lower dimensional structure and high dimensional noise. For example, in multilevel modeling, latent data generally takes the form of a small number of fixed vectors, to which one adds Gaussian noise before thresholding to produce binary observations. By contrast, our model assumes that $X_i$ and $Z_i$ have the same dimension $d$ and concern the same quantities of interest. The latent model serves not to reduce dimension, but to create a framework where binary observations reflect incomplete information. The purpose of the randomized thresholding is to allow heterogeneity in sample behavior, for example, to model differences between high-volume buyers and low-volume buyers in market basket data. The threshold models employed in regression and similar settings generally account for random noise in the latent variables, but typically assume a fixed thresholding process. As we illustrate by example in Section 2, complications arise in correlation mining when sample heterogeneity is not incorporated into a model.

1.1 Related Work

Most existing methods for clustering or community detection can be applied to binary data; one need only specify an appropriate measure of dissimilarity, such as those provided in Choi et al. (2010). (See e.g. Zhang et al. (2008b)) A general framework and discussion of methods for clustering binary data can be found in Li and Li (2005); a survey of classic approaches is given in Neuhaus et al. (1991). Latent space regression models can be partially adapted to unsupervised clustering tasks, as discussed in Harpaz and Haralick (2007) and Dunson (2000).

Frequent Itemset Mining. Association mining in binary data is also common in the field of computer science, where it is known as frequent itemset mining or association rule mining. Frequent itemsets are roughly defined to be sets of items that are often bought together. In general, approaches to frequent itemset mining are non-stochastic; instead of modeling the data in a stochastic fashion, they proceed by screening datasets for sets of items whose support - or percentage of buyers who purchased the entire itemset - is above a certain threshold. The study of frequent itemsets and association rules arguably began with the work of Agrawal et al. (1996), which introduced the apriori algorithm. Subsequent methods have made algorithmic solutions and computational improvements to apriori. Some notable examples include eclat (Zaki et al. 1997a), MAFIA (Burdick et al. 2001), COBBLER (Pan et al. 2004), fp-close (Grahne and Zhu 2003), and CHARM (Zaki and Hsiao 2002). An excellent summary of early and recent work in frequent itemset mining can be found in Zaki et al. (1997b), Prabha et al. (2013), Zaki et al. (1999) and the references
therein. There are some exceptions to the non-stochastic nature of itemset mining. For example, Zhang et al. (2008a) estimates the probability of itemsets exceeding a specified frequency, rather than simply screening for itemsets exceeding a threshold. More complex, model-based approaches to data uncertainty can be found in Aggarwal et al. (2009) and Tong et al. (2012); instead of screening for high support, they screen for high expected support under a probability model.

Computationally, most frequent itemset mining methods can handle data that has a very large number of samples (transactions), but a moderate number (hundreds or fewer) of items. The latter restriction arises from the fact that most algorithms screening all possible itemsets. More recent work in itemset mining addresses the challenge of high dimensional data, in which the number of items studied may be very large (usually $10^4$ or more). As with itemset mining in small data, existing methods are primarily non-stochastic, and the research focus is algorithmic and computational alternatives to an exhaustive search over all possible itemsets. For important examples, see Liu et al. (2006), Sohrabi and Barforoush (2012), and Zhu et al. (2007). Unfortunately, public software is not readily available for large datasets, and foundational small-data methods like apriori and eclat are still the norm in analyses of market basket data.

Latent Semantic Analysis. Existing methods for correlation mining in textual data share many aspects of the LAMB approach. Text data is often noncontinuous, usually consisting of word or feature counts from a collection of documents. Much like regression models, methods of Latent Semantic Analysis (LSA) rely on a latent model to lower the underlying dimension of data features. Typically, LSA methods are based on the assumption that word counts are random realizations from a low dimensional structure of document topic and word meaning. (See Landauer et al. (1998) or Landauer (2006) and the references therein for an overview of LSA.) Unlike other areas of correlation mining, many LSA approaches incorporate sample heterogeneity. A notable example is the Term Frequency - Inverse Document Frequency, or TF-IDF (Ramos et al., 2003) approach, which standardizes textual data by adjusting observed word frequencies according to their uniqueness among documents. (A TF-IDF analysis is supplied alongside the LAMB results in Section 6.)

1.2 Organization

The remainder of the paper is organized as follows. The next section contains an example motivating the latent threshold model, and the latent correlation. A detailed discussion and formalization of our model assumptions is provided in Section 3. In Section 4, we outline the search procedure of our method, Latent Association Mining in Binary Data (LAMB), and in Section 5 we present the results of a comparative simulation study with related approaches. Sections 6 and 7 contain results from the application of the LAMB software to real world datasets in textual analysis and music recommendation respectively. Finally, Section 8 and 9 presents the estimation techniques and supporting theoretical results that
underly the LAMB method.

2 Motivating Examples

Although many association mining methods are applicable (or even tailored specifically) to binary data, most measures of association or dissimilarity treat observations as homogeneous. Statistical models commonly assume that samples are i.i.d. or approximately so, and binary association measures typically derive from raw counts over samples. In reality, the assumption of identically distributed or indistinguishable samples may not be reasonable. For example, in market basket data, it may be unrealistic to assume that all buyers tend to buy the same overall number of items. Variation in income, household size, shopping habits, etc. may effect the quantity of items that a particular buyer is inclined to purchase. The following toy dataset, consisting of 12 samples (buyers) and 14 items, illustrates problems that can arise when all samples equal treatment.

| Buyers | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---------|---|---|---|---|---|---|---|---|---|----|----|----|
| Item 1  | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| Item 2  | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| Item 3  | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Item 4  | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| Item 5  | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Item 6  | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| Item 7  | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Item 8  | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| Item 9  | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Item 10 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| Item 11 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| Item 12 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| Item 13 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| Item 14 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |

Figure 1: Toy Dataset

Consider the itemsets (Item 1, Item 2) and (Item 3, Item 4). These two sets show identical behavior up to a reordering of the samples. The two itemsets are equally correlated ($\rho_{12} = \rho_{34} = 0.667$) and equally far apart in $\ell_1$ and $\ell_2$ distance ($d_{12} = d_{34} = 1$ (out of 12)). Despite this, it is not clear that we should believe there is any association structure in Items 3 and 4 aside from an overall pattern in buyer behavior. Buyers 1-5 bought most available items, while buyers 8-12 bought very few items. The apparent association between Items
3 and 4 may simply reflect this pattern in buyers. Items 1 and 2, on the other hand, show similar buying patterns that can not be explained by buyer differences. Buyers who do not purchase many items overall still tend to purchase Items 1 and 2 together, which is a strong indicator of a true underlying relationship between these items.

Figure 2 illustrates the difference between standard linear correlation and our proposed sample estimate of latent correlation. Figure 2(a) is the sample correlation matrix for the toy dataset, for which we expect (Item 1, Item 2) and (Item 3, Item 4) to have the same values. Figure 2(b) is the estimated latent correlation matrix, calculated by the methods outlined in this paper, for the toy dataset. In Figure the set 2(b), (Item 1, Item 2) remains associated, but other associations are attenuated since they are not distinguishable from the overall pattern among the buyers.

As expected, when common association mining were applied to the toy dataset, both (Item 1, Item 2) and (Item 3, Item 4) were selected as associated pairs. The LAMB procedure, on the other hand, identifies only (Item 1, Item 2) as an associated variable subset.

2.1 Real Data: Grocery store transactions

The following simple application provides a real-data illustration of the difference between frequent itemset mining and our approach. The package arules (Hahsler et al., 2012) in R supplies software for several common frequent itemset mining and association mining methods. Also included in this package is a dataset from grocery store transactions, Groceries, intended as ideal data for exploring and testing methods. This dataset consists
of 9835 observed transactions over 169 available items for sale. Tables 1 and 2 show the results of applying the well-known eclat algorithm and the LAMB method respectively to the grocery store data.  

All three frequent sets in Table 1 contain whole milk, the most common item in the Groceries dataset. Intuitively, this makes sense, because the eclat algorithm seeks items that appear together in a large percentage of transactions. Thus items that are purchased more frequently are more likely to appear in frequent itemsets. The itemsets in Table 2 are more diverse, while still being readily interpretable in terms of real world grocery needs. For instance, Set 1 in Table 2 is easily recognizable as a ham and cheese sandwich, Set 5 contains drinks one might buy for a party, while Set 7 corresponds to baking staples.

Table 1: Results from eclat with support threshold = 0.05

| 1. whole milk, other vegetables |
| 2. whole milk, rolls/buns |
| 3. whole milk, yogurt |

Since eclat screens for itemsets with support above a certain threshold, we applied the method with many thresholds. Table 1 shows the results for a threshold that yielded a moderate number of reasonably-sized itemsets. The LAMB procedure is fully automatic and so the contents Table 2 are simply the direct output of the method.
Table 2: Results from LAMB

1. white bread, processed cheese, ham
2. canned beer, soda, shopping bags
3. pip fruit, tropical fruit
4. root vegetables, herbs, beef, other vegetables, pork, chicken
5. soda, bottled water, bottled beer, red/blush wine, canned beer
6. berries, whipped/sour cream
7. sugar, flour, baking powder
8. Instant food products, hamburger meat
9. waffles, chocolate, long life bakery product, specialty bar, candy, specialty chocolate, salty snack, chocolate marshmallow

3 Threshold model and latent correlation

We now formalize the model on which the LAMB procedure is built. Our method is based on a simple threshold model in which the observed binary variables reflect whether or not the coordinates of a latent random vector lie above or below a vector of thresholds. We assume that the threshold vector is independent of the latent values and, for notational convenience, specify thresholds as quantiles.

Definition 3.1. (Threshold Model) Let $Z = (Z_1, \ldots, Z_d)^t \in \mathbb{R}^d$ be a $d$-dimensional random vector, with distribution $\varphi$. We assume that each random variable $Z_j$ has a continuous CDF $F_j$, with associated quantile function $F_j^{-1}$. Let $\theta = (\theta_1, \ldots, \theta_d)^t \in (0,1)^d$ be a $d$-dimensional random vector, with distribution $\nu$, that is independent of $Z$. The observed random vector $X = (X_1, \ldots, X_d) \in \{0,1\}^d$ is defined coordinate-wise by

$$X_j = \mathbb{I}(Z_j \leq F_j^{-1}(\theta_j)) \quad j = 1, \ldots, d.$$  

(1)

In what follows we will express the threshold model in shorthand as $X = \mathbb{I}\{Z \leq F^{-1}(\theta)\}$.

In what follows we regard the available binary data as independent replicates of a fixed but unknown threshold model. In the context of itemset mining, the random vector $Z$
represents the intrinsic value of a set of \( d \) items available for purchase at a store. Individual variables \( Z_j \) may have different marginal distributions and means, reflecting the fact that some items are more common or more desirable than others. Variables in \( Z \) may also be highly dependent as the value of an item may be associated with the value of another item that is used at the same time or for related tasks (e.g., peanut butter and jelly, or cereal and milk). If i.i.d. samples of \( Z \) from many buyers were available, one could assess associations between the value of different items using the standard correlation. For example, a survey questionnaire or carefully designed behavioral experiment could access the underlying values \( Z \). However, such techniques are expensive and require experts to design and execute. In many cases, data on item valuation is easier to obtain, and more readily available, in the form of purchasing behavior, in which a single transaction is summarized by a binary vector \( X \in \{0, 1\}^d \), representing whether the buyer purchases each item or not. Thus it is necessary to use the observed binary vector \( X \) to understand the association structure in unobserved continuous vector \( Z \). With this objective in mind, we define the \textit{latent correlation}, a population quantity that serves as a surrogate for correlation in \( Z \).

**Definition 3.2. (Latent Correlation)** Let the binary vector \( X = \mathbb{I}\{Z \leq F^{-1}(\theta)\} \) be defined as in the Threshold Model 3.1. The latent correlation between \( X_j \) and \( X_k \) is defined by

\[
\psi(j, k) = \mathbb{E} \left[ \frac{(X_j - \theta_j)(X_k - \theta_k)}{\sqrt{\theta_j(1 - \theta_j)\theta_k(1 - \theta_k)}} \right].
\]

(2)

Here and in what follows all expectations are taken with respect to the joint distribution of \((X, \theta)\) inherited from the product distribution \( \varphi \otimes \nu \) on \((Z, \theta)\).

Note that \( \psi(j, k) = \mathbb{E}[\rho(X_j, X_k | \theta)] \) where \( \rho(X_j, X_k | \theta) \) is the conditional correlation of \( X_j \) and \( X_k \) given \( \theta \). The quantity \( \rho(X_j, X_k | \theta) \) reflects the correlation between \( X_j \) and \( X_k \) arising from association between the latent variables \( Z_j \) and \( Z_k \) with fixed thresholds \( \theta_j \) and \( \theta_k \). As such, \( \psi(j, k) \) does not capture correlation arising purely from dependence among the thresholds \( \theta_j \). In particular, \( \psi(j, k) = 0 \) if \( Z_j \) and \( Z_k \) are independent, regardless of the distribution of \( \theta \).

We remark that the latent correlation \( \psi(j, k) \) is not a formal parameter of the threshold model (Definition 3.1) as it does not determine the joint distribution of \( \theta \) and \( Z \). However, the latent correlation is meaningful as it reflects the average association between \( X_j \) and \( X_k \) in the model. Its use is analogous to the use of correlation for a non-normal random vector \( Z \): though it is not a parameter, it is a good measure of linear relationship between variables.

Section 2 provided an illustration of the importance of measuring latent correlation rather than standard correlation. Below we give an explicit example in which \( X \) has standard correlation induced by \( \theta \), despite independence in \( Z \).
Example 1. Let \( Z \sim N_d(u, I_d) \), where \( u \in \mathbb{R}^d \) is fixed and \( I_d \) is the \( d \times d \) identity matrix. Let \( \theta_1 = \ldots = \theta_d \), with

\[
\theta_j = \begin{cases} 
\epsilon & \text{with probability } 1/2, \\
1 - \epsilon & \text{with probability } 1/2 
\end{cases}
\]  

(3)

for some \( 0 < \epsilon < 1/2 \). Let \( X \) be as in Definition 3.1, i.e., \( X = I \{ Z \leq \Phi^{-1}(\theta) + u \} \), where \( \Phi(\cdot) \) is the standard normal CDF. Then, for any \( j \neq k \), \( Z_j \) is independent of \( Z_k \), but

\[
\text{cor}(X_j, X_k) > 0,
\]

since \( E(X_j X_k - E(X_j) E(X_k)) = \left( \frac{\epsilon^2}{2} + \frac{(1 - \epsilon)^2}{2} \right) - \left( \frac{1}{2} \right)^2 = \frac{1}{4} - \epsilon(1 - \epsilon) \).  

(4)

It is easy to see that as \( \epsilon \) approaches 0, the correlation between \( X_j \) and \( X_k \) gets arbitrarily close to 1. Intuitively, this dependence arises from the individual variables \( Z_j \) being simultaneously thresholded at either very large or very small values, so that \( X_j = X_k \) with high probability for any pair \( (j, k) \). Standard correlation indicates, correctly, that there is association between the components of \( X \). However, this association derives from \( \theta \) rather than from the latent structure in \( Z \). When \( Z \) contains the variables of interest, and \( \theta \) is ancillary to the analysis, standard correlation is misleading in this example. By contrast, the latent correlation of \( X_j \) and \( X_k \) is equal to 0 for any \( (j, k) \), which accurately reflects the underlying lack of association structure in \( Z \).

We remark here that one common measure of association between binary variables \( X_j \) and \( X_k \) is the odds ratio

\[
P(X_j = 1, X_k = 1) P(X_j = 0, X_k = 0) 
P(X_j = 0, X_k = 1) P(X_j = 1, X_k = 0).
\]

However, since the threshold model is reasonable for many practical problems, we decide to focus on the latent correlation which incorporates the model information and is more related to our goal to recover the correlation structure in \( Z \).

### 3.1 Properties of latent correlation

Latent correlation shares a number of the basic properties of standard correlation. For example \( |\psi(j, k)| \leq 1 \) (see Lemma 0.1) and \( \psi(j, k) \neq 0 \) implies dependence between \((Z_j, \theta_j)\) and \((Z_k, \theta_k)\). Although the latent correlation of \( X \) is, in most cases, not equal to the correlation of the latent vector \( Z \), there is a directional relationship when \( Z \) is Gaussian.

**Proposition 1.** Let \( X = I \{ Z \leq F^{-1}(\theta) \} \) as in Definition 3.1 where \( Z \sim N_d(0, \Sigma) \) with \( \Sigma_{jj} = \sigma^2 \) for all \( j \). Then, for any random vector \( \theta \),

\[
\text{sgn}(\psi(j, k)) = \text{sgn}(\Sigma_{jk}),
\]

(5)
i.e., the sign of the latent correlation between $X_i$ and $X_i$ is equal to the sign of the correlation between $Z_i$ and $Z_j$.

Remark: We note that the LAMB procedure described below does not rely on normality assumptions for $Z$.

4 Latent Association Mining in Binary Data (LAMB)

We now present the details of the LAMB search procedure. Given a data set, the goal of this procedure is to identify subsets of variables with large average pairwise latent correlation. We begin by specifying the targets, called coherent sets, that the algorithm is designed to identify. We then briefly discuss the details of multiple testing update process, including the statement of a central limit theorem that justifies our hypothesis testing approach.

4.1 Coherent Sets

Definition 3.2 provides a measure of pairwise association under the threshold model. However, the goal of Latent Correlation Mining is to identify subsets of correlated variables, rather than pairs. To this end we make the following definition.

Definition 4.1. (Average Latent Correlation) Given $j \in [d]$ and $A \subseteq [d]$ let

$$
\psi(j, A) := \frac{1}{|A|} \sum_{k \in A \setminus \{j\}} \psi(j, k)
$$

be the average latent correlation between $X_j$ and $\{X_k : k \in A\}$.

Definition 4.2. (Coherent Set) Let $\psi(\cdot, \cdot)$ be defined as in Definition 4.1. A subset $A \subseteq [d]$ with at least two elements is a coherent for latent correlation if

(i) $\psi(j, A) > 0$ for each $j \in A$, and

(ii) $\psi(j, A) \leq 0$ for each $j \notin A$.

A coherent set $A$ is minimal if no proper subset $B \subset A$ is a coherent set.

By definition a variable set is coherent if each element of the set has positive average latent correlation with the rest of the set, and no element outside the set has this property. The definition ensures that if we remove or add a single variable to coherent set then it is no longer coherent.

Like clusters of objects in exploratory data analysis, a coherent set of binary variables are mutually positively associated, analogous to a block of positive correlations in a covariance matrix. For binary data, coherence offers advantages over clustering in terms of
interpretability and power, as it is based directly on the binary threshold model and on probability statements from meaningful tests. Coherent sets can be identified in a computationally efficient fashion using a fast search procedure based on iterative testing. This procedure is detailed below.

4.2 Iterative search procedure

The set update process in the LAMB method uses multiple testing to iteratively update and refine a variable set $A$. The procedure runs as follows.

1. Choose an initial set $A_0 = \{j\}$ for some $j \subset [d]$.
2. Given $A_t$, apply a multiple testing procedure to the tests
   \[ H_0(j, A_t) : \psi(j, A_t) \leq 0 \text{ vs. } H_1(j, A_t) : \psi(j, A_t) > 0 \text{ for } j \in [d]. \]  
3. Let $A_{t+1}$ be the set of $j$ such that $H_0(j, A_t)$ is rejected by the multiple testing procedure.
4. Repeat steps (2) and (3) until $A_t = A_{t'} := A^*$ for some $t' < t$.
5. Output set $A^*$ if $A^*$ is non-empty.

If $t' = t - 1$, then the set $A^*$ is a fixed point of the search procedure, and further updates will not change $A^*$. Note that $A^* = \emptyset$ is a fixed point of the procedure that represents an unsuccessful search; non-empty fixed points are of primary interest. When $t' \neq t - 1$, the algorithm has reached a terminating cycle $A_t, \ldots, A_{t'}$ of two or more sets. There is a close relationship between nonempty fixed points and coherent variable sets. By definition, a fixed point $A^*$ of the search procedure has the property that

- $H_0(j, A^*)$ is rejected for all $j \in A$, and
- $H_0(j, A^*)$ is accepted for all $j \notin A$,

and therefore $A^*$ satisfies Definition 4.2 up to a level of statistical significance. As such, non-empty fixed points of the search procedure are natural estimates of coherent sets. When the algorithm cycles through two or more sets, the final set $A^*$ is output. In our experience with simulated and real data sets, cycling is rare.

In principle, any multiple testing procedure can be applied in Step 3 of the search process. A Bonferroni-type adjustment would guarantee family wise error control at each step, but would, in many cases, greatly reduce the sensitivity of the algorithm. The default implementation of LAMB uses the FDR-controlling procedure of Benjamini and Yekutieli (2001), which controls the expected false discovery rate even when the p-values of the hypotheses $H_0(j, A_t)$ are correlated.
In the LAMB method, the search procedure is run $d$ times, with initial sets $\{1\}, \ldots, \{d\}$ equal to a single variable. In practice, many (or even all) of these searches may degenerate to $A^* = \emptyset$, indicating no signal in the data. Other searches will result in overlapping or identical output sets. Multiple instances of the same set are presented as a single set. In cases where substantial overlap is present, a variety of heuristic methods may be employed to select a representative set from an overlapping group.

4.3 Hypothesis testing

The LAMB search procedure is based on the hypothesis tests in \(7\). To carry out these tests we construct an estimator for the latent correlation, and then appeal to a central limit theorem to calculate approximate p-values.

Recall that the observed data is assumed to be independent replicates of the threshold model \(1\). Suppose for the moment that the threshold vectors $\theta_1, \ldots, \theta_n$ are observed alongside the binary outcome vectors $X_1, \ldots, X_n$. In this case, a straightforward estimator for the latent correlation (Definition 3.2) is the corresponding sample average,

$$\hat{\psi}_n(j, k) = \frac{1}{n} \sum_{i=1}^{n} U_{ij} U_{ik} \quad \text{where} \quad U_{ij} := \frac{X_{ij} - \theta_{ij}}{\sqrt{\theta_{ij}(1 - \theta_{ij})}},$$

where $X_{ij}$ is the $j$-th variable in sample $X_i$, and $\theta_{ij}$ is defined similarly. Let $\hat{\psi}_n(j, A) = |A|^{-1} \sum_{k \in A} \hat{\psi}_n(j, k)$. We note that the sample quantities $\hat{\psi}_n(j, k)$ and $\hat{\psi}_n(j, A)$ are not guaranteed to fall between -1 and 1. However, under mild conditions, their values will converge to the interval $[-1, 1]$ as $n$ tends to infinity, see Proposition 4 in Appendix A.

In Theorem 1 of Section 8 we establish that, for a suitable variance estimator $\hat{\sigma}^2_n(j, A)$, the quantity $\sqrt{n} (\hat{\psi}_n(j, A)/\hat{\sigma}_n(j, A))$ is approximately standard normal. As large values of $\hat{\psi}_n(j, A)$ provide evidence for strong latent correlation, we define approximate p-values by

$$\text{pv}(j, A) = 1 - \Phi^{-1} \left( \frac{\hat{\psi}_n(j, A)}{\hat{\sigma}_n(j, A)} \right),$$

These p-values are then passed to a multiple testing procedure in order to perform the set update in Step 3 of the LAMB search procedure.

In practice, the thresholds $\theta_1, \ldots, \theta_n$ are not observed, but can be estimated from the binary vectors $X_1, \ldots, X_n$ under suitable assumptions on $\theta$. These assumptions, and our approach to estimation of $\theta$, are detailed in Section 9. The LAMB method uses the estimates of $\theta_j$ to define $\hat{\psi}_n(j, A)$ and $\hat{\sigma}_n(j, A)$, and the associated p-values $\text{pv}(j, A)$.
5 Simulation study

To establish the effectiveness of the LAMB method under controlled conditions, we applied LAMB and several common set detection approaches to an artificial dataset. We created this dataset in accordance with Definition 3.1, such that binary observations represented a thresholded version of a continuous dataset with nontrivial correlation structure. We then measured the success of the methods by comparing the output to the known coherent sets in the latent data.

The artificial data in our study was created via the following simulation procedure.

1. Fix a $d \times d$ correlation matrix $\Sigma$, such that off-diagonal values are equal to $\rho$ in the first $m \times m$ block and zero elsewhere.

2. Generate dataset $Z \in \mathbb{R}^{n \times d}$ by $n$ i.i.d draws from a multivariate normal distribution $Z \sim N_d(0, \Sigma)$.

3. Generate thresholds $\Theta \in \mathbb{R}^{n \times d}$ by $n$ i.i.d. draws from an exponential model, that is, $\theta = (\theta_1, \ldots, \theta_d)$ and
   \[
   \theta_j = 1 - \exp(-\tau \alpha_j),
   \]
   for random $\tau$ and fixed $(\alpha_1, \ldots, \alpha_d)$. This threshold form is further discussed in Section 9. For purposes of our simulation, we fixed each $\alpha_j$ from a uniform distribution, and we considered both random $\tau \sim \text{Expo}(1)$ and nonrandom $\tau = 1$.

4. Generate binary dataset $X \in \{0, 1\}^{n \times d}$ by threshold model $X_{ij} = \mathbb{I}(Z_{ij} \leq \Phi^{-1}(\theta_j))$.

By Proposition 1, the latent correlation of $X$ has the same sign as $\rho$, so when $\rho > 0$ in the above generative model, the subset $\{1, \ldots, m\}$ is a coherent set. By varying the values of $\rho$, we were able to study the effect of signal strength on performance of the LAMB method. Further, note that in this model, the randomness of $\Theta$ depends entirely on the parameter $\tau$. When $\tau$ is fixed at 1, rows of $\Theta$ (and therefore also $X$) are fully i.i.d. Most common set identification methods are designed to address this setting. However, when $\tau$ is taken to be random, rows of $X$ are conditionally non-identical. In this setting, we expect many existing methods to be misled, as illustrated in Section 2.

Our simulations also considered changes to parameters $n, d,$ and $m$. We found these results to be non-informative, in the sense that all methods responded similarly to changes. We omit these results here; in what follows, values are set to $n = 101, d = 1000$, and $m = 100$.

5.1 Methods compared

In addition to LAMB, we applied hierarchical clustering with four different measures of association. We selected a cutoff for the dendrogram based on our knowledge of the $m \times m$
latent correlated set, such that the selected cluster was as close to the correct size as possible. In a real data setting, a dendrogram cutoff selection method would need to be employed.

The measures of (dis)association included in our study were:

1. **L1 Dist**: The $\ell_1$ or “Manhattan” distance between sample vectors,

\[
  d_1(j, k) = \sum_{i=1}^{n} |X_{ij} - X_{ik}|
\]

(11)

2. **L2 Dist**: The $\ell_2$ or Euclidean distance between sample vectors.

\[
  d_2(j, k) = \left( \sum_{i=1}^{n} (X_{ij} - X_{ik})^2 \right)^{1/2}
\]

(12)

3. **Binary Dist**: A distance metric based on treating binary data as on/off bits and comparing the individual frequency of two variables to their joint frequency,

\[
  d_{\text{bin}}(j, k) = \frac{\left( \sum_{i=1}^{n} X_{ij} \right) \left( \sum_{i=1}^{n} X_{ik} \right)}{\left( \sum_{i=1}^{n} X_{ij} X_{ik} \right)}
\]

(13)

4. **Correlation Distance**: A transformation of the ordinary product-moment correlation between two sample vectors,

\[
  d_{\text{corr}}(j, k) = \sqrt{2(1 - r(X_{j}, X_{k}))}
\]

(14)

We also applied a correlation mining method (CM, adapted from the methods of Bodwin et al. (2015)) to the latent data matrix $Z$ as a performance benchmark. We expect this method to perform better than LAMB and others, as it is applied directly to the latent data before thresholding. It is included to better understand the ability of the methods to recover latent information after the thresholding procedure.

**Remark.** We do not include Frequent Itemset Mining in the simulation study, as these methods are not computationally feasible for higher dimensional datasets ($d > 100$), and do not come equipped with efficient procedures for selecting large sets ($m > 10$).

### 5.2 Results

Methods were compared using the false positive rate (FPR) and the true discovery rate (TDR). Let $B$ be the output variable set of a particular method, and let $A$ be the known latent $m \times m$ correlated set. Then,

\[
  \text{FPR} = \frac{|B \setminus A|}{|B|} \quad \text{and} \quad \text{TDR} = \frac{|A \setminus B|}{|A|}.
\]
Figure 3 shows the True Discovery Rate for all methods as a function of the strength of the true correlation (\(\rho\)) in the latent set. Figure 3(a) represents the data setting of interest, where \(\theta\) is taken to be random, while (b) corresponds to the classic setting of fully i.i.d. samples. It is clear from the superior performance of the latent CM approach that, as one would expect, thresholding continuous data greatly reduces the level at which signal can be detected. However, LAMB was able to reliably detect latent correlation at around \(\rho = 0.5\) (for the baseline parameter choices of \(n, d, m\)). All other methods are unreliable in this setting even for large values of \(\rho\), and only the clustering based on correlation detects signal at all. Figure 3(b) provides reassurance that LAMB is also applicable in classic settings. Even when \(\theta\) was nonrandom, LAMB outperformed competing methods. In particular, distance-based clustering (L1, L2 and Binary Dist) approaches could not detect signal in this setting, as they do not account for differences in mean behavior between variables (i.e., \(\alpha_j\)'s not all identical).

We conclude that under the thresholding model, LAMB is an effective tool for discovering latent correlated sets. Further, when thresholds have a hierarchical structure that induces randomness, LAMB was the only method of those studied able to distinguish true underlying correlation from association induced by thresholding.

6 Application: Wordsets in Shakespeare plays

The LAMB procedure is applicable to any binary dataset, and is particularly well-suited to data where samples may not be identically distributed. Word usage in documents presents an ideal data source for this paradigm. Text analysts are often interested in finding sets of words that appear together frequently (for an overview of relevant history, see e.g. Salton and McGill (1986)). However, we usually expect documents to vary enormously in length; thus, even if word choice is identical across documents, we expect to observe a non-identical distribution of word presence. By searching for latent correlation rather than standard correlation or frequent word sets, we are better able to extract word groups that are truly associated in a meaningful way, rather than simply appearing often together in longer documents.

We used the online database http://shakespeare.mit.edu/ to obtain the text of all known Shakespeare plays. We then created a binary dataset for the 1638 unique words that appeared in more than one play and that were used in at least one, but not all, of the 429 acts of Shakespeare’s twenty tragedies/histories. That is, a “1” in the data matrix indicated that a particular word appeared at least once in a particular act of a play.

In addition to the Coherent Mining Method, for comparative purposes we also applied a Text Frequency - Inverse Document Frequency (TF-IDF) clustering procedure to the Shakespeare data. Clusters were selected by performing hierarchical clustering on the TF-IDF data matrix by ordinary Euclidean distance. The dendrogram was cut at a height that yielded a similar number of clusters as the LAMB results, for comparison. (Clusters
Figure 3: True discovery rate (when false positive rate < 0.05) by signal latent correlation strength.
with more than 50 words were considered “background” and disregarded.)

The LAMB software identified 56 coherent word sets from this data. The TF-IDF approach identified 38 associated words sets. On the whole, in both cases these word sets have obvious semantic and/or linguistic themes. Table 3 displays five selected coherent word sets, and Table 4 displays four word sets from the TF-IDF clustering that roughly correspond to those in Table 3.

Table 3: Selected coherent word sets in Shakespearean tragedies

| 1. earth, heaven |
| 2. thousand, ten, twenty |
| 3. she, her, lady, madam, husband, wife, queen, woman, daughter, shes, marriage, me, tell, sister, herself, sweet |
| 4. hast, dost, art, thy, wilt, thee, thine, thou, death, shalt, canst, didst, ill, sweet, ah, hadst, if, thefthself, away, father, eyes, boy, villain, child, mine, mother, kill, wert, me, then, die, o, flesh, am, cheeks, leave, young, sight |
| 5. king, duke, majesty, lords, france, prince, grace, god, princely, unto, liege, sovereign, crown, english, french, highness, uncle, princes, arms, lord, gracious, subjects, cousin, soul, title, now, blood, fathers, then, until, queen, father, traitor, yield, son, right, royal, john, forward, brother, doth, presence, heir, war, sons, embrace, hath, hath, guilty |

Table 4: Selected word sets in Shakespearean tragedies clustered by TF-IDF adjusted distance

| 1. arm, arms, base, blood, body, day, doth, earth, eye, farewell, foul, hand, hands, head, heaven, mouth, myself, power, proud, royal, saint, soul, souls, sweet, tale, tongue |
| 2. five, hundred, knight, morrow, today |
| 3. beauty, fair, ladies |
| 4. dead, death, deed, didst, eyes, kill, killd, life, tender, wilt |
Set 1 in Table 3 is a typical two-word related pair, "earth, heaven". Many such pairs with obvious relationships were selected by both methods. Set 1 in Table 4 also joined “earth” and "heaven", but also included many other words in the set. The second set in both analyses captured a numerical relationship, and the third sets are clearly concerned with feminine words. Perhaps most compelling is Set 4 in Table 3 which is mostly marked by language rather than meaning - the words are almost entirely from Old English. Set 4 in Table 4 shares some of the same words, but is not obviously a linguistically joined word set. Finally, Set 5 in Table 3 represents an easily interpretable word set identified by LAMB, concerning royalty and titles, that has no equivalent in the TF-IDF results.

The results of LAMB on text data are encouraging for several reasons. First, the identified word sets have ready interpretations, suggesting that latent association is capturing thematic relationships that are distinct from surface-level associations. Second, we find that relationships underling the word sets may be semantic or linguistic. Word sets like “earth, heaven” are validating, but do not provide new linguistic information. However, the ability to extract word sets like Set 4 that have a deeper linguistic connection may have applications in the study of language structure.

The comparison between LAMB and the popular TF-IDF approach highlights other advantages of LAMB. The results of LAMB were similar to, and arguably more nuanced, than those of TF-IDF, even though TF-IDF had access to full word counts, rather than binary observations representing word presence or absence. Additionally, the LAMB algorithm allowed for overlapping word sets, and a selection process that did not require a choice of cut level for a dendrogram. It is worth noting that the hierarchical clustering approach requires the calculation of a full $1638 \times 1638$ matrix. In larger datasets, such as the one in Section 7, this approach would be computationally difficult.

7 Application: Similar Music Artists

Music streaming services such as Pandora, Last.fm, and Spotify offer users the opportunity to discover new musical artists based on existing preferences. These companies have developed complex algorithms for finding similar artists based on era, genre, user ratings, etc. The LAMB method provides a novel means of artist matching based on latent correlation. To preserve the directionality of a recommendation approach, instead of seeking coherent sets, we seek coherent neighborhoods, consisting of the set of all items that have positive latent correlation with a chosen target set A. That is, given a set A of preferred artists for an individual, we would like to recommend a neighborhood of similar artists around A. Such neighborhoods are easily estimated by performing only a single iterative step of the LAMB algorithm. By considering latent correlation, we were able to identify related artists (as measured by listener history) without skewing the results towards globally popular music or allowing differences in listener behavior to mask artist associations.

As an example of this approach, we analyzed a dataset provided by Celma (2010).
downloaded from the last.fm public API. The data consists of listening history for 1893 anonymized users, covering 17,632 unique artists. The data was converted to a binary matrix, where a 1 indicates that a particular user listened at least once to a particular artist. We then applied the single-step Latent Association Mining algorithm for each individual artist.

Two results from a coherent neighborhood analysis of the last.fm data are in Tables 5 and 6. We also include the top five user-chosen descriptive tags for each artist, to show the type of metadata that might alternatively be used to group artists. Interestingly, although the coherent neighborhoods tend to have clear themes, they do not directly represent the closest artists to the seed based on genre or musical style. For example, the coherent neighborhood in Table 5 for “Hannah Montana”, a fictional country star from a Disney TV show portrayed by Miley Cyrus, consisted of Cyrus herself and many other singers who got their start on Disney shows (Demi Lovato, Selena Gomez, Ashley Tisdale). Similarly, although many musicians produce similar music to Paul McCartney, the coherent neighborhood in Table 6 consists only of the Beatles and fellow Beatles members. This suggests that unsupervised grouping based on latent association may capture links between artists that are not apparent from subjective expert analysis of musical similarities.

### Table 5: Coherent neighborhood for “Hannah Montana”

| Artist | Top 5 Tags |
|--------|------------|
| Hannah Montana | love at first listen, pop rock, soundtrack, amazing, female vocalist |
| Miley Cyrus | <3, catchy, love at first listen, amazing, pop rock |
| Rihanna | rub, ballad, sexy, love, dance |
| Katy Perry | pop rock, <3, catchy, love, love at first listen |
| Britney Spears | catchy, female, sexy, amazing, dance |
| Ke$ha | love at first listen, dance, <3, pop, catchy |
| Lady Gaga | dance, female vocalist, love at first listen, catchy, sexy |
| Demi Lovato | love at first listen, <3, pop rock, catchy, female vocalist |
| Avril Lavigne | pop rock, canadian, pop punk, female, love at first listen |
| Taylor Swift | country, <3, catchy, love, amazing |
| Selena Gomez & the Scene | <3, pop rock, love at first listen, catchy, love |
| Ashley Tisdale | <3, catchy, pop rock, ballad, awesome |
| Hilary Duff | favorites, amazing, sexy, pop rock, dance |
| Christina Aguilera | ballad, sexy, soul, rub, amazing |
| Jonas Brothers | pop rock, <3, love, love at first listen, amazing |
| Beyoncé | rub, sexy, soul, ballad, female vocalist |
| Glee Cast | cover, love at first listen, love, catchy, soundtrack |

---

2Top tags were selected by the percent of times the tag appeared for the artists versus overall in the dataset. Tags were limited to top 100 most popular, to avoid single-artist or single-user tag strings, e.g. “David Bowie” or “Songs for my breakup with Maria.”
Table 6: Coherent neighborhood for “Paul McCartney”

| Artist           | Top 5 Tags                                                                 |
|------------------|---------------------------------------------------------------------------|
| Paul McCartney   | sad, classic rock, cool, british, beautiful                               |
| The Beatles      | 60s, classic rock, british, psychedelic, <3                              |
| George Harrison  | classic rock, 70s, singer-songwriter, sad, british                       |
| John Lennon      | classic rock, singer-songwriter, 70s, british, male vocalists            |

8 CLT for Sample Latent Correlation

As discussed in Section 4.3, the hypothesis tests of the LAMB procedure rely on approximate p-values for estimates of the latent correlation $\psi(j, A)$. These p-values are derived from a central limit theorem that we now present in detail. In what follows, let the $d$-dimensional random vectors $Z, \theta$, and $X = \mathbb{I}(Z \leq F(\theta))$ be as in Definition 3.1. Further, for each $j \in [d]$ let

$$U_j = \frac{X_j - \theta_j}{\sqrt{\theta_j(1 - \theta_j)}}$$

be the (conditionally) standardized version of $X_j$. For a subset $A \subseteq [d]$ let

$$U_A = \frac{1}{|A|} \sum_{j \in A} U_j,$$

be the average of $U_j$ over $j \in A$, and let

$$\Psi(A) := \frac{1}{|A|^2} \sum_{j,k \in A} \psi(j, k).$$

be the average pairwise latent correlation between the variables in $A$. The next two lemmas establish some basic properties of the quantities defined above.

**Lemma 0.1.** For each $j \in [d]$ and $A \subseteq [d],$

(i) $\mathbb{E}U_j = 0$ and $\mathbb{E}U_j^2 = 1,$

(ii) $\psi(j, k) = \mathbb{E}(U_j U_k)$ and $|\psi(j, k)| \leq 1,$

(iii) $\mathbb{E}(U_j U_A) = \psi(j, A)$ and $\mathbb{E}U_A^2 = \Psi(A).$

Moreover, if $Z_j$ is independent of $\{Z_k : k \in A\}$ then

(iv) $\mathbb{E}(U_j U_A) = 0$ and $\mathbb{E}(U_j^2 U_A^2) = \mathbb{E}U_A^2.$
Lemma 0.2. Let $U_j$ and $U_A$ be defined as in (15) and (16). Then

(i) $E[U_j^4] \leq E[\theta_j^{-1}(1 - \theta_j)^{-1}]$

(ii) If $Z_j$ is independent of $\{Z_k : k \in A\}$, then

$E[U_j^4 U_A^4] \leq |A|^{-1} \sum_{k \in A} E[\theta_j^{-1}(1 - \theta_j)^{-1}\theta_k^{-1}(1 - \theta_k)^{-1}]$.

With the above lemmas, we now establish a central limit theorem for latent correlation in the following asymptotic setting. For each $n \geq 1$ let $Z_1, \ldots, Z_n \sim \varphi_n$ and $\theta_1, \ldots, \theta_n \sim \nu_n$ be independent random vectors in $\mathbb{R}^{d_n}$, and let $X_i = I(Z_i \leq F_n^{-1}(\theta_i))$, $1 \leq i \leq n$, be the binary vectors generated under the threshold model. Note that the distributions $\varphi_n$ and $\nu_n$, as well as the dimension $d_n$, may depend on $n$. Let $\hat{\psi}(j,k)$ be the sample latent correlation between variables $X_j$ and $X_k$, defined in (8), and let $U_iA$ be defined as in (16). Then a natural estimate of $\psi(j,A) = E(U_j U_A)$ is the average

$$\hat{\psi}_n(j,A) := \frac{1}{n} \sum_{i=1}^n U_{ij} U_{iA}. \quad (18)$$

It follows from Lemma 0.1 that $E\hat{\psi}_n(j,A) = \psi(j,A)$. Furthermore, if $Z_j$ is independent of $\{Z_k : k \in A\}$ then $E\hat{\psi}_n(j,A) = \psi(j,A) = E(U_j U_A) = 0$ and

$$\text{var}(n^{1/2} \hat{\psi}_n(j,A)) = n \text{var}(U_j U_A) = E(U_j^2 U_A^2) = \Psi(A).$$

In particular,

$$\hat{\sigma}_n^2(j,A) := \frac{1}{n} \sum_{i=1}^n U_{ij}^2 U_{iA}^2. \quad (19)$$

is an unbiased estimate of the variance $\Psi(A)$ of $n^{1/2} \hat{\psi}_n(j,A)$.

We are now ready to state the central limit theorem for sample latent correlation.

Theorem 1. (Central Limit Theorem for Sample Latent Correlation) Fix $j$ and for each $n$ let $A_n \subset [d_n] \setminus \{j\}$ be an index set with cardinality $|A_n| = m_n$. Let $\psi_n(j,A_n)$ be the sample latent correlation of $j$ and $A_n$ under $(\varphi_n, \nu_n)$ and let $\hat{\sigma}_n^2(j,A_n)$ be defined as in (19). Assume that

(i) For each $n$, $Z_j$ is independent of $\{Z_k : k \in A_n\}$ under $\varphi_n$; and

(ii) $\Psi(A_n)^{-2} \left( \frac{1}{m_n} \sum_{k \in A_n} E\left[ \theta_j^{-1}\theta_k^{-1}(1 - \theta_j)^{-1}(1 - \theta_k)^{-1}\right] \right) = o(n)$,

Then,

$$\sqrt{n}\left( \frac{\hat{\psi}_n(j,A_n)}{\hat{\sigma}_n(j,A_n)} \right) \overset{d}{\rightarrow} \mathcal{N}(0,1) \text{ as } n \rightarrow \infty. \quad (20)$$
Proof. The theorem is a corollary of Slutsky’s theorem and the following two lemmas. □

Lemma 1.1. Under the conditions of Theorem [7]

\[
\sqrt{n} \left( \frac{\hat{\psi}_n(j, A_n)}{\Psi(A_n)^{1/2}} \right) \xrightarrow{d} \mathcal{N}(0, 1) \text{ as } n \to \infty. \tag{21}
\]

Lemma 1.2. Under the conditions of Theorem [7]

\[
\left| \frac{\tilde{\sigma}^2_n(j, A_n)}{\Psi(A_n)} - 1 \right| \xrightarrow{p} 0. \tag{22}
\]

The above results establish the basis of the LAMB algorithm, as described in Section 4.2 and 4.3.

9 Plug-in estimation of thresholds

9.1 Model for \( \Theta \)

The hypothesis testing approach requires that \( \Theta_{n \times d} = (\theta_1, \ldots, \theta_n)^t \) be known alongside \( X_{n \times d} = (X_1, \ldots, X_n)^t \). In practice, \( \theta \) are not typically observed. Our approach is to derive consistent estimators for \( \Theta \) from observations \( X \). We then treat these estimates as observed values and insert them directly into the sample latent correlation \( \theta \). Our positive results from simulation and applications, as well as the theoretical consistency of our estimators, suggests that this is a reasonable approximation.

In order to estimate \( \Theta \) from \( X \), we must reduce the effective dimension of \( \theta \). Our approach is to model \( \theta_j \) marginally as a function of a fixed positive parameter \( \alpha_j \) capturing the behavior of variable \( j \), and a positive random parameter \( \tau \). We impose a model that is a univariate analog to the Poisson factorization approach of Gopalan et al. (2014) and subsequent work, which models expected counts from random sample and variable parameters.

Definition 9.1. (Threshold Parameter Model) Fix \( \alpha = (\alpha_1, \ldots, \alpha_d) \in (0, 1)^d \). For each variable \( j \in [d] \), let

\[
\theta_j = 1 - \exp(-\tau \alpha_j), \tag{23}
\]

where \( \tau \sim \pi \) is a univariate random variable.

In other words, we assume that the dependence structure and randomness of \( \theta \) derives from a single shared random parameter \( \tau \). Differences between the marginal distributions of \( \theta_1, \ldots, \theta_d \) are then entirely captured by the fixed parameters \( \alpha = (\alpha_1, \ldots, \alpha_d) \). In the
buyer-item paradigm, we may interpret $\tau$ as the buying propensity of a particular individual randomly selected from the population, and $\alpha_j$ as a measure of the overall prevalence of a particular item.

Under this model, samples $(\theta_1, \ldots, \theta_n)$ then depend fully on samples from $\tau$, $(\tau_1, \ldots, \tau_n)$. Thus, the number of quantities to estimate is reduced from $(n \times d)$ parameters $\{\theta_{ij}\}$ to $(n + d)$ parameters $(\alpha_1, \ldots, \alpha_d, \tau_1, \ldots, \tau_n)$, which fully specify $\Theta$. We may therefore estimate $\Theta$ from $X$ under certain mild conditions. In particular, since in many practical cases $\theta_{ij}$’s are small, we focus on the case when their expected values tend towards 0.

9.2 Estimation Consistency

We first consider the consistency in estimating the expected value of $\theta_j \sim \nu_n$. Note that $E[X_{ij}] = E[E[X_{ij}|\theta_j]] = E[\theta_j]$. Therefore, unconditionally for each $j$, $X_{ij} \overset{iid}{\sim} \text{Bernoulli}(E[\theta_j])$ for $i = 1, \ldots, n$. A natural estimate of $E[\theta_j]$ is thus $X_j = \frac{1}{n} \sum_{i=1}^{n} X_{ij}$. We have the following theorem on the consistency of $X_j$ when $E[\theta_j]$’s are small but not too small.

Proposition 2. Consider the setting in Definition 3.1 and (23) where $\theta_j \sim \nu_n$. Suppose $\lim \sup_{n \to \infty} \frac{1}{n} \max_{1 \leq j \leq d_n} E^{-1}[\theta_j] = 0$, then

$$\left| \frac{X_j}{E[\theta_j]} - 1 \right| \overset{P}{\to} 0 \quad (24)$$

for every $j \in [d_n]$.

Proof. By Chebyshev’s inequality, for any $j$ and any $\epsilon > 0$,

$$\max_{1 \leq j \leq d_n} \mathbb{P}\left( \left| \frac{X_j}{E[\theta_j]} - 1 \right| > \epsilon \right) \leq \max_{1 \leq j \leq d_n} \frac{1}{n \epsilon^2} \left( \frac{1}{E[\theta_j]} - 1 \right) = o(1).$$

An immediate consequence of Proposition 2 is the estimation of $\alpha_j, j = 1, \ldots, d_n$. From (23), we see that $E[\theta_j] = g(\alpha_j) := \int_{\tau} \left( 1 - e^{t \alpha_j} \right) \pi(t) dt$. Therefore, if the distribution of $\tau \sim \pi$ leads to a function $g(\cdot)$ that is continuous and invertible, by the continuous mapping theorem, $g^{-1}(X_j)$ is consistent for $g^{-1}(E[\theta_j]) = \alpha_j$.

To estimate $\Theta$, we must also estimate the unobserved realized values of random variables $(\tau_1, \ldots, \tau_n)$, which we denote by $(\tau^0_1, \ldots, \tau^0_n)$. Consider the posterior distribution of $\tau_i$ given $X_i = (X_{i1}, \ldots, X_{id_n})^t$ and $\alpha_n$, which we denote by $\pi(\cdot | X_i, \alpha_n)$. A straightforward estimator for $\tau_i^0$ is the posterior mean,

$$E[\tau_i | X_i, \alpha_n] = \int_0^{\infty} t \pi(t | X_i, \alpha_n) dt. \quad (25)$$

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The following result guarantees consistency of the posterior mean. We appeal directly to Theorem 4.1 of Choi et al. (2008), which requires the following condition on the prior \( \pi \) for \( \tau \).

**Condition 2.1.** For each \( \delta > 0 \) there exist sets \( S_1, S_2, \ldots \) such that diameter of each set is less than \( \delta \), \( \bigcup_{k \geq 1} S_k = \mathbb{R}^+ \), and \( \sum_{k \geq 1} \sqrt{\pi(S_k)} < \infty \).

In essence, Condition 2.1 is a concentration condition for \( \pi \), guaranteeing that the measure is not too spread out over the range of \( \tau \).

**Theorem 2.** (Choi et al. (2008)) Suppose that Condition 2.1 holds and that \( \pi(\cdot | X_i, \alpha_n) \) is bounded. Then, for every \( \epsilon > 0 \),

\[
P \left( |E[\tau_i | X_i, \alpha] - \tau_i^0| > \epsilon \right) \to 0 \quad (26)
\]

as \( n \to \infty \), where the probability is taken over the measure of \( X_i \) from \((\varphi_n, \nu_n)\).

In practice \( \alpha_n \) is not known, so we instead estimate \( \tau_i^0 \) by plugging in consistent estimates \((\hat{\alpha}_1, \ldots, \hat{\alpha}_{dn})\), i.e.,

\[
\hat{\tau}_i = E[\tau_i | X_i, \hat{\alpha}_n] = \int_0^\infty t \pi(t | X_i, \hat{\alpha}_n) \, dt. \quad (27)
\]

Then, for every \( n \) and for \( i \in [n], j \in [dn] \), \( \theta_{ij} \) is estimated by \( \hat{\theta}_{ij} = 1 - e^{-\hat{\tau}_i \hat{\alpha}_j} \).

The following example provides a general family of models which satisfy the conditions in Theorem 1 and Proposition 2.

**Proposition 3.** Consider the Gamma(\( \zeta, \beta \)) prior distribution for \( \tau \). Density of \( \tau \) is

\[
\tau(x) = \frac{\beta^\zeta}{\Gamma(\beta)} x^{\zeta-1} \exp(-\beta x).
\]

Assume the following:

(a) \( \min_{1 \leq j \leq dn} n\alpha_j \to \infty \)

(b) \( \max_{1 \leq j \leq dn} \alpha_j \leq M < \infty \).

(c) Let \( c_n := \min_{1 \leq j \leq dn} \alpha_j \) and \( \rho_n := \lambda_{\min}(\Sigma(A_n)) \) be the minimal eigenvalue of \( \Sigma(A_n) := (\psi(j,k))_{j,k \in A_n} \). Assume \( m_n \rho_n \to 0 \).

Then the Gamma prior with \( \beta > 6M \) and \( \zeta \geq 3 \) satisfies

1. \( \Psi(A_n)^{-2} \left( \frac{1}{m_n} \sum_{k \in A_n} E \left[ \theta_j^{-1} \theta_k^{-1} (1 - \theta_j)^{-1} (1 - \theta_k)^{-1} \right] \right) = o(n) \) as in Theorem 1,

2. \( \limsup_{n \to \infty} \frac{1}{n} \max_{1 \leq j \leq dn} E^{-1}[\theta_j] = 0 \) as in Proposition 2.

**Remark:** The three additional assumptions in Proposition 3 are mild for practical use. For example, if \( \rho_n = O(1), c_n \gg \frac{1}{n^{1/2}} \) then condition (c) holds when \( m_n << n^{1/2} \).
9.3 Implementation

The consistent estimators derived in Proposition 2 and Theorem 2 rely on the parent distribution $\pi$, and thus the function $g(\cdot)$, being known. In practice, we generally do not know the distribution of $\tau$. Our approach is therefore to approximate the consistent estimators via an empirical distribution function. We replace the unknown prior $\pi$ with an empirical distribution function,

$$\hat{\pi}_n(t) = \begin{cases} \frac{1}{n} & \text{if } t \in \{\tau_1, \ldots, \tau_n\}, \\ 0 & \text{otherwise}, \end{cases}$$ \hspace{1cm} (28)

The estimation approach suggested by Proposition 2 and Theorem 2 is to maximize a posterior likelihood for $\tau$ under an appropriate moment-of-moments estimation of $\alpha$. With the empirical prior (28), we arrive at the following optimization problem.

$$(\hat{\tau}_i, \hat{\alpha}_i) = \arg \max_{(\tau, \alpha) \in \mathbb{R}^n} \prod_{j=1}^n (1 - \exp^{-\tau_i \alpha_j})^{X_{ij}} (\exp^{-\tau_i \alpha_j})(1-X_{ij})$$ \hspace{1cm} (29)

subject to $$X_j = \frac{1}{n} \sum_{i=1}^n (1 - e^{-\tau_i \alpha_j})$$ \hspace{1cm} (30)

Although these equations have no closed form solution, they can be computed to within arbitrary error by an ordinary MM algorithm (see, for example, Hunter and Lange (2004)). The results of Proposition 2 and Theorem 2 provide reassurance of the validity of these estimators, since they are an empirical approach to the consistent estimation in Proposition 2 and Theorem 2.

In the Itemset Mining software, we also supply an option to compute $\hat{\tau}_1, \ldots, \hat{\tau}_n$ and $\hat{\alpha}_n$ under an assumed gamma prior, as in Proposition 3. This option should only be used when there is a compelling reason to believe the prior $\pi$ is known to be gamma with a particular rate and shape. Despite the theoretical advantages of the estimators with known prior, we find the flexible empirical approach is more robust in real data settings, since $\pi$ is commonly unknown.

10 Conclusion

In this work, we have introduced the LAMB algorithm, a novel approach to identifying sets of associated variables from binary observations. The LAMB approach offers two major improvements on existing methods: First, it frames the analysis as a latent data mining problem, which prevents some common pitfalls in typical approaches to binary data; second, it exploits a new search procedure that builds from statistical testing principles. LAMB is demonstrated to outperform existing methods in artificial data, and to produce new insights
in real data, particularly in market basket type data. Further, we provide theoretical results and corresponding proofs to support the estimation and testing techniques used in the LAMB algorithm.

Public code for the LAMB method in R and Matlab will be available upon publication at github.com/kbodwin.

A Proofs and Derivations

Proof of Proposition 7. Let $U_j$ be defined as in (15), and recall that $\psi(j, k) = \mathbb{E}(U_j U_k)$. It suffices to show that with probability one

$$\text{sgn}(\mathbb{E}(U_j U_k | \theta)) = \text{sgn}(\Sigma_{jk}) \quad \text{for } 1 \leq j \neq k \leq d.$$ \hfill (31)

To begin, note that

$$\text{sgn}(\mathbb{E}(U_j U_k | \theta)) = \text{sgn}(\mathbb{E}[X_j X_k - \theta_j \theta_k | \theta])$$

and that the last expression in parentheses is equal to

$$\mathbb{P}(Z_j < \Phi^{-1}(\theta_j), Z_k < \Phi^{-1}(\theta_k) | \theta) - \mathbb{P}(Z_j = 1 | \theta) \mathbb{P}(Z_k = 1 | \theta).$$ \hfill (32)

As $Z$ is independent of $\theta$, it follows from Slepian’s Lemma (Slepian, 1962) that the sign of the difference in (32) is the same as the sign of $\Sigma_{jk} = \text{Cov}(Z_j, Z_k)$. This establishes (31) and completes the proof.

Proposition 4. If $\max_{1 \leq j \leq d_n} \mathbb{E}[\theta_j^{-1} - (1 - \theta_j)^{-1}] = o(n)$ then for any $\epsilon > 0$

$$\max_{j \in [d_n]} \max_{A \subseteq [d_n]} \mathbb{P}(|\hat{\psi}_n(j, A)| > 1 + \epsilon) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$ \hfill (33)

Proof. Fix $j$ and $A$ for the moment. As $\mathbb{E}\hat{\psi}_n(j, A) = \psi(j, A)$ and $|\psi(j, A)| \leq 1$, a routine argument shows that

$$\mathbb{P}(|\hat{\psi}_n(j, A)| > 1 + \epsilon) \leq \mathbb{P}(|\hat{\psi}_n(j, A) - \mathbb{E}\hat{\psi}_n(j, A)| > \epsilon)$$

and it suffices by Chebyshev’s inequality to show that $\text{var}(\hat{\psi}_n(j, A)) = o(1)$. By Jensen, $U_A^2 \leq |A|^{-1} \sum_{k \in A} U_k^2$, and it follows that

$$\text{var}(\hat{\psi}_n(j, A)) = \frac{1}{n} \text{var}(U_j U_A) \leq \frac{1}{n} \mathbb{E}(U_j^2 U_A^2) \leq \frac{1}{|A|} \frac{\mathbb{E}(U_j^2 U_k^2)}{n}.$$
Thus it is enough to show that $\max_{j,k} \mathbb{E}(U_j^2 U_k^2) = o(n)$. By the Cauchy-Schwarz inequality and part (i) of Lemma 0.2,

$$
\mathbb{E}(U_j^2 U_k^2) \leq \mathbb{E}(U_j^4)^{1/2} \mathbb{E}(U_k^4)^{1/2} \leq \mathbb{E}[\theta_j^{-1}(1 - \theta_j)^{-1}]^{1/2} \mathbb{E}[\theta_k^{-1}(1 - \theta_k)^{-1}]^{1/2}.
$$

(34)

Now the condition $\max_{j \leq d,n} \mathbb{E}[\theta_j^{-1}(1 - \theta_j)^{-1}] = o(n)$ completes the proof. \qed

**Proof of Lemma 0.1.**

The definition of $U_j$ ensures that $\mathbb{E}(U_j \mid \theta) = 0$ and $\mathbb{E}(U_j^2 \mid \theta) = 1$, and (i) follows. It is clear from the definitions that $\psi(j, k) = \mathbb{E}(U_j U_k)$; the bound on $|\psi(j, k)|$ follows from (i) and the Cauchy-Schwarz inequality. The relations (iii) follow from (ii) and the expressions

$$
\mathbb{E}(U_j U_A) = \frac{1}{|A|} \sum_{k \in A} \mathbb{E}(U_j U_k) \quad \text{and} \quad \mathbb{E} U_A^2 = \frac{1}{|A|^2} \sum_{k, \ell \in A} \mathbb{E}(U_k U_\ell).
$$

If $Z_j$ is independent of $\{Z_k : k \in A\}$ then $U_j$ is conditionally independent of $U_A$ given $\theta$ and therefore

$$
\mathbb{E}(U_j U_A) = \mathbb{E}[\mathbb{E}(U_j U_A \mid \theta)] = \mathbb{E}[\mathbb{E}(U_j \mid \theta) \mathbb{E}(U_A \mid \theta)] = 0.
$$

A similar conditioning argument shows that $\mathbb{E}(U_j^2 U_A^2) = \mathbb{E}(U_A^2)$ as $\mathbb{E}(U_j^2 \mid \theta) = 1$, which establishes (iv). \qed

**Proof of Lemma 0.2.**

Fix $j \in [d]$ for the moment and note that as $X_j$ is binary

$$
U_j^4 = X_j \left( \frac{1 - \theta_k}{\theta_j(1 - \theta_j)} \right)^4 + (1 - X_j) \left( \frac{-\theta_j}{\theta_j(1 - \theta_j)} \right)^4.
$$

Using the fact that $\mathbb{E}(X_j \mid \theta) = \theta_j$, we find that

$$
\mathbb{E}(U_j^4 \mid \theta) = \frac{(1 - \theta_j)^2}{\theta_j} + \frac{\theta_j^2}{1 - \theta_j} \leq \frac{1}{\theta_j} + \frac{1}{1 - \theta_j} = \frac{1}{\theta_j(1 - \theta_j)},
$$

which implies (i). Moreover, it follows from Jensen’s inequality that

$$
U_A^4 = \left( \frac{1}{|A|} \sum_{k \in A} U_k \right)^4 \leq \frac{1}{|A|} \sum_{k \in A} U_k^4
$$

and therefore, $\mathbb{E}(U_j^4 U_k^4) \leq |A|^{-1} \sum_{k \in A} \mathbb{E}(U_j^4 U_k^4)$. Conditioning on $\theta$ we find

$$
\begin{align*}
\mathbb{E}(U_j^4 U_k^4) &= \mathbb{E}\{\mathbb{E}(U_j^4 U_k^4 \mid \theta)\} = \mathbb{E}\{\mathbb{E}(U_j^4 \mid \theta) \mathbb{E}(U_k^4 \mid \theta)\} \\
&\leq \mathbb{E}[\theta_j^{-1}(1 - \theta_j)^{-1} \theta_k^{-1}(1 - \theta_k)^{-1}]
\end{align*}
$$

28
where the second equality follows from the assumed independence of \( Z_j \) and \( \{ Z_k : k \in A \} \), and the inequality follows from part (i) of the lemma.

Proof of Lemma 1.1

Note that \( \hat{\psi}_n(j,A_n) = n^{-1} \sum_{i=1}^{n} U_{ij} U_{i,A_n} \) is the average of i.i.d. random variables distributed as \( U_j U_{A_n} \), where \( U_j \) and \( U_{A_n} \) are defined in (15) and (16), respectively. It suffices to verify the Lindeberg-Feller conditions for the central limit theorem, which require that for every \( \epsilon > 0 \),

\[
\lim_{n \to \infty} \frac{\mathbb{E}\left[U_j^2 U_{A_n}^2 \mathbb{I}\{|U_j U_{A_n}| > \epsilon n^{1/2} \Psi(A_n)^{1/2}\}\right]}{\Psi(A_n)} = 0.
\]

Applying the Cauchy-Schwarz inequality, it is enough to show that

\[
\lim_{n \to \infty} \frac{\mathbb{E}(U_j^4 U_{A_n}^4)}{\Psi(A_n)^2} \frac{\mathbb{P}\left(|U_j U_{A_n}| > \epsilon n^{1/2} \Psi(A_n)^{1/2}\right)}{n \epsilon^2} = 0.
\] (35)

By Markov’s inequality and Lemma 0.1

\[
\mathbb{P}\left(|U_j U_{A_n}| > \epsilon n^{1/2} \Psi(A_n)^{1/2}\right) \leq \frac{\mathbb{E}[U_j^2 U_{A_n}^2]}{n \epsilon^2 \Psi(A_n)} = \frac{\Psi(A_n)}{n \epsilon^2} = \frac{1}{n \epsilon^2}.
\]

Part (ii) of Lemma 0.2 and Condition (ii) then ensure that (35) holds, which completes the proof.

Proof of Lemma 1.2

To reduce notation, let \( \hat{\sigma}_n^2 = \hat{\sigma}_n^2(j,A_n) \). Note that \( \mathbb{E}(\hat{\sigma}_n^2) = \mathbb{E}(U_j^2 U_{A_n}^2) = \Psi(A_n) \) by Lemma 0.1 so (22) is equivalent to

\[
\left| \frac{\hat{\sigma}_n^2 - \mathbb{E}(\hat{\sigma}_n^2)}{\Psi(A_n)} \right| \xrightarrow{p} 0,
\]

and it therefore suffices to show that \( \text{var}(\hat{\sigma}_n^2)/\Psi(A_n)^2 \to 0 \). It is clear that

\[
\text{var}(\hat{\sigma}_n^2) = \frac{1}{n} \text{var}(U_j^2 U_{A_n}^2) = \frac{1}{n} \left( \mathbb{E}(U_j^4 U_{A_n}^4) - \Psi(A_n)^2 \right),
\]

so it is enough to show that \( \mathbb{E}(U_j^4 U_{A_n}^4)/\Psi(A_n)^2 = o(n) \). This follows from Lemma 0.2.

Proof of Proposition 3

For (1), we have \( \theta_j = 1 - \exp(-\tau \alpha_j) \). Since \( \alpha_j > 0 \),

\[
\mathbb{E}[\theta_j] = 1 - \left( \frac{\beta}{\beta + \alpha_j} \right) \geq 1 - \left( \frac{\beta}{\beta + \alpha_j} \right) = \frac{\alpha_j}{\beta + \alpha_j}.
\]
Thus
\[
\frac{1}{n} \max_{1 \leq j \leq d_n} \mathbb{E}^{-1}[\theta_j] \leq \frac{1}{n} + \frac{\beta}{\min_{1 \leq j \leq d_n} n\alpha_j} \to 0.
\]

For (2), first notice \(\Psi(A_n) = \frac{1}{m_n} \sum (A_n)\). Thus by our notation,
\[
\Psi(A_n) = \frac{1}{m_n} \left( \frac{1}{\sqrt{m_n}} \right)^\prime \sum (A_n) \left( \frac{1}{\sqrt{m_n}} \right) \geq \rho_n / m_n.
\] (36)

Using Holder inequality,
\[
\mathbb{E}^3 \left[ \theta_j^{-1}(1 - \theta_j)^{-1}(1 - \theta_k)^{-1} \right] \leq \mathbb{E}[\theta_j^{-3}] \mathbb{E}[\theta_k^{-3}] \mathbb{E}(1 - \theta_j)^{-3} (1 - \theta_k)^{-3}
\] (37)

Now, by the moment generating function of Gamma(\(\zeta, \beta\)),
\[
\mathbb{E}(1 - \theta_j)^{-3} (1 - \theta_k)^{-3} = \mathbb{E}[\exp(3(\alpha_j + \alpha_k)\tau)] = \left( 1 - \frac{3(\alpha_j + \alpha_k)}{\beta} \right)^{-\zeta} \leq \left( 1 - \frac{6M}{\beta} \right)^{-\zeta}.
\] (38)

Also
\[
\mathbb{E}[\theta_j^{-3}] = \mathbb{E} \left( \frac{1}{1 - \exp(-\tau\alpha_j)} \right)^3 = \int_{x > 0} \left( \frac{1}{1 - \exp(-x\alpha_j)} \right)^3 \tau(x) \, dx
\]
\[
= \int_{\alpha, x \geq 1/2} \left( \frac{1}{1 - \exp(-x\alpha_j)} \right)^3 \tau(x) \, dx + \int_{\alpha, x < 1/2} \left( \frac{1}{1 - \exp(-x\alpha_j)} \right)^3 \tau(x) \, dx.
\]

We will use two facts \(\frac{1}{1 - e^{-y}}\) is a decreasing function of \(y\) and \(1 - \exp(-y) \geq y - y^2/2\) when \(y \geq 0\). Hence
\[
\mathbb{E}[\theta_j^{-3}] \leq \left( \frac{1}{1 - \exp(-1/2)} \right)^3 \int_{\alpha, x \geq 1/2} \tau(x) \, dx + \int_{\alpha, x < 1/2} \left( \frac{1}{x\alpha_j - (x\alpha_j)^2/2} \right)^3 \tau(x) \, dx.
\] (39)

The first term in (39) is clearly not bigger than \(\left( \frac{1}{1 - \exp(-1/2)} \right)^3\). For the second term in (39), when \(\zeta \geq 3\),
\[
\int_{\alpha, x < 1/2} \left( \frac{1}{x\alpha_j - (x\alpha_j)^2/2} \right)^3 \tau(x) \, dx \leq \int_{\alpha, x < 1/2} \left( \frac{1}{x\alpha_j(1 - x\alpha_j/2)} \right)^3 \tau(x) \, dx
\]
\[
\leq \frac{64}{27} \int_{\alpha, x < 1/2} \left( \frac{1}{x\alpha_j} \right)^3 \tau(x) \, dx
\]
\[
= \frac{1}{\alpha_j^3} \frac{64}{27} \Gamma(\beta) \int_{\alpha, x < 1/2} x^{\zeta-3-1} \exp(-\beta x) \, dx
\]
\[
\leq \frac{C(\beta, \zeta)}{\alpha_j^3}.
\] (40)
Now combining (37), (38), (39), (40) there is a constant \( \bar{C}(\beta, \zeta) \) such that,

\[
\mathbb{E} \left[ \theta_j^{-1} \theta_k^{-1} (1 - \theta_j)^{-1} (1 - \theta_k)^{-1} \right] \leq \frac{\bar{C}(\beta, \zeta)}{\alpha_j \alpha_k} \leq \frac{\bar{C}(\beta, \zeta)}{c_n^2}.
\]  

(41)

Now plugging this in (41) and using (36) we get

\[
\Psi(A_n)^{-2} \left( \frac{1}{m_n} \sum_{k \in A_n} \mathbb{E} \left[ \theta_j^{-1} \theta_k^{-1} (1 - \theta_j)^{-1} (1 - \theta_k)^{-1} \right] \right) \leq \frac{\bar{C}(\beta, \zeta)m_n}{c_n^2 \rho_n^2} = o(n).
\]

\( \square \)

B The full LAMB algorithm

1. **Estimation:** Compute \( \hat{\Theta} \), the matrix of estimates of means \( \theta_{ij} \), as in Section 9.3.

2. **Initialization:** Set \( A_0 = \{j\} \) for some \( j \in [d] \).

3. **Testing:**
   - Given \( A_t \), for each \( j \in [d] \), compute \( \hat{\psi}(j, A_t) \) and \( \hat{\sigma}(j, A_t) \) from \( \bar{X} \) and \( \hat{\Theta} \) as in Section 8.
   - Compute p-values \( \{p_1, \ldots, p_d\} \) as in (9).
   - Simultaneously test hypotheses
     
     \[
     H_0(j) : \psi(j, A_t) = 0 \quad \text{vs} \quad H_1(j) : \psi(j, A_t) > 0
     \]

     by applying the multiple testing procedure of Benjamini and Yekutieli (2001) to the set of p-values.

4. **Update:** Set \( A_{t+1} = \{ j : H_0(j) \text{ was rejected} \} \).

5. **Iteration:** Repeat steps 3 and 4 until \( A_t = A_{t'} := A^* \) for some \( t' < t \).

6. **Output:** If \( A^* \) is not empty, select it as an empirical coherent itemset.

7. **Repetition:** Repeat steps 2-5 as many times as desired, or for every initial \( j \in [d] \).
## C Additional last.fm results

Table 7: Coherent neighborhood for “Slayer”

| Artist          | Top 5 Tags                                                                 |
|-----------------|-----------------------------------------------------------------------------|
| Slayer          | thrash metal, heavy metal, metal, power metal, death metal                   |
| Iron Maiden     | heavy metal, metal, power metal, hard rock, seen live                       |
| Metallica       | thrash metal, heavy metal, metal, hard rock, awesome                         |
| Megadeth        | thrash metal, heavy metal, metal, cool, power metal                          |
| Motrhead        | heavy metal, hard rock, metal, thrash metal, uk                              |
| Black Sabbath   | thrash metal, heavy metal, hard rock, metal, classic rock, 70s               |
| Pantera         | heavy metal, hard rock, metal, classic rock, thrash metal                    |
| Judas Priest    | thrash metal, death metal, brazilian, heavy metal, metal                     |
| Sepultura       | thrash metal, metal, heavy metal, power metal, german                        |
| Kreator         | thrash metal, heavy metal, metal, cool, american                            |
| Anthrax         | hard rock, heavy metal, classic rock, 70s, metal                            |
| AC/DC           | melodic death metal, death metal, power metal, metal, gothic                 |
| Children of Bodom | death metal, progressive metal, melodic death metal, thrash metal, metal   |
| Death           | thrash metal, heavy metal, metal, 80s, rock                                 |
| Exodus          | 70s, classic rock, progressive rock, hard rock, blues                        |
| Led Zeppelin    | thrash metal, heavy metal, death metal, metal, seen live                    |
| Testament       | hard rock, progressive rock, classic rock, heavy metal, 70s                  |
| Deep Purple     |                                                                            |

Table 8: Coherent neighborhood for “Brandy”

| Artist          | Top 5 Tags                                                                 |
|-----------------|-----------------------------------------------------------------------------|
| Brandy          | ballad, rnb, sexy, soul, hip-hop                                            |
| Rihanna         | rnb, ballad, sexy, love, dance                                              |
| Mariah Carey    | rnb, soul, love, ballad, female                                             |
| Beyonce         | rnb, sexy, soul, ballad, female vocalist                                     |
| Christina Aguilera | ballad, sexy, soul, rnb, amazing                                        |
| The Pussycat Dolls | rnb, sexy, favorites, dance, pop                                            |
| Jennifer Lopez  | female, rnb, female vocalist, dance, sexy                                   |
| Ciara           | rnb, hip hop, hip-hop, sexy, amazing                                        |
| Janet Jackson   | rnb, female, sexy, soul, female vocalist                                     |
Table 9: Coherent neighborhood for “Creedence Clearwater Revival”

| Artist                      | Top 5 Tags                                                                 |
|-----------------------------|---------------------------------------------------------------------------|
| **Creedence Clearwater Revival** | 60s, classic rock, 70s, folk, blues                                       |
| Led Zeppelin                | 70s, classic rock, progressive rock, hard rock, blues                     |
| The Doors                   | psychedelic, 60s, classic rock, blues, rock                                |
| The Rolling Stones           | 60s, classic rock, blues, 70s, british                                    |
| The Beatles                  | 60s, classic rock, british, psychedelic, <3                               |
| Pink Floyd                   | progressive rock, psychedelic, classic rock, 70s, 60s                     |
| AC/DC                        | hard rock, heavy metal, classic rock, 70s, metal                          |
| Deep Purple                  | hard rock, progressive rock, classic rock, heavy metal, 70s               |
| Queen                        | classic rock, 70s, hard rock, 80s, progressive rock                       |
| Black Sabbath                | heavy metal, hard rock, metal, classic rock, 70s                          |
| The Who                      | 60s, classic rock, uk, hard rock, 70s                                     |
| Jimi Hendrix                 | blues, psychedelic, classic rock, 60s, funk                               |
References

Aggarwal, C. C., Li, Y., Wang, J., and Wang, J. (2009). Frequent pattern mining with uncertain data. In *Proceedings of the 15th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, KDD ’09, pages 29–38, New York, NY, USA. ACM.

Agrawal, R., Mannila, H., Srikant, R., Toivonen, H., Verkamo, A. I., et al. (1996). Fast discovery of association rules. *Advances in knowledge discovery and data mining*, 12(1):307–328.

Anderson, T. (1959). *An Introduction to Multivariate Statistical Analysis*. Wiley-Interscience.

Angus, J. E. (1994). The probability integral transform and related results. *SIAM Rev.*, 36(4):652–654.

Antal, T. (2007). On the latent regression model of item response theory. *ETS Research Report Series*, 2007(1):i–19.

Benjamini, Y. and Hochberg, Y. (1995). Controlling the false discovery rate: a practical and powerful approach to multiple testing. *Journal of the royal statistical society. Series B (Methodological)*, pages 289–300.

Benjamini, Y. and Yekutieli, D. (2001). The control of the false discovery rate in multiple testing under dependency. *Ann. Stat.*, 29(4):1165–1188.

Bishop, C. M. (2006). *Pattern Recognition and Machine Learning*. Springer Science and Business Media, LLC., Oxford, England.

Bodwin, K., Zhang, K., and Nobel, A. (2015). A testing-based approach to the discovery of differentially correlated variable sets. *arXiv preprint arXiv:1509.08124*.

Burdick, D., Calimlim, M., and Gehrke, J. (2001). Mafia: A maximal frequent itemset algorithm for transactional databases. In *Data Engineering, 2001. Proceedings. 17th International Conference on*, pages 443–452. IEEE.

Celma, O. (2010). *Music Recommendation and Discovery in the Long Tail*. Springer.

Choi, S.-S., Cha, S.-H., and Tappert, C. C. (2010). A survey of binary similarity and distance measures. *Journal of Systemics, Cybernetics and Informatics*, pages 43–48.

Choi, T., Ramamoorthi, R., et al. (2008). Remarks on consistency of posterior distributions. In *Pushing the limits of contemporary statistics: contributions in honor of Jayanta K. Ghosh*, pages 170–186. Institute of Mathematical Statistics.
Dunson, D. B. (2000). Bayesian latent variable models for clustered mixed outcomes. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 62(2):355–366.

Everitt, B. S., Landau, S., Leese, M., and Stahl, D. (2011). *Cluster Analysis*. Wiley Series in Probability and Statistics.

Gopalan, P., Ruiz, F. J., Ranganath, R., and Blei, D. M. (2014). Bayesian nonparametric poisson factorization for recommendation systems. In *AISTATS*, pages 275–283.

Grahne, G. and Zhu, J. (2003). Efficiently using prefix-trees in mining frequent itemsets. In *FIMI*, volume 90.

Hahsler, M., Buchta, C., Gruen, B., and Hornik, K. (2012). *arules: Mining Association Rules and Frequent Itemsets*. R package version 1.0-12.

Harpaz, R. and Haralick, R. (2007). Mining subspace correlations. *2007 IEEE Symposium on Computational Intelligence and Data Mining*.

Hastie, T. and Tibshirani, R. (1990). *Generalized additive models*. Wiley Online Library.

Hunter, D. R. and Lange, K. (2004). A tutorial on mm algorithms. *The American Statistician*, 58(1):30–37.

Jaccard, P. (1901). *Etude comparative de la distribution florale dans une portion des Alpes et du Jura*. Impr. Corbaz.

Kriegel, H.-P., Krger, P., and Zimek, A. (2009). Clustering high-dimensional data: A survey on subspace clustering, pattern-based clustering, and correlation clustering. *ACM Trans. on Knowledge Disc. from Data (TKDD)*, 3(1).

Landauer, T. K. (2006). Latent semantic analysis. *Encyclopedia of Cognitive Science*.

Landauer, T. K., Foltz, P. W., and Laham, D. (1998). An introduction to latent semantic analysis. *Discourse Processes*, 25(2-3):259–284.

Li, T. A unified view on clustering binary data. In *Machine Learning*, page 2006.

Li, T. (2005). A general model for clustering binary data. In *Proceedings of the Eleventh ACM SIGKDD International Conference on Knowledge Discovery in Data Mining*, KDD ’05, pages 188–197, New York, NY, USA. ACM.

Liu, H., Han, J., Xin, D., and Shao, Z. (2006). Mining frequent patterns from very high dimensional data: A top-down row enumeration approach. In *Proceedings of the 2006 SIAM International Conference on Data Mining*, pages 282–293. SIAM.
McCullagh, P. and Nelder, J. (1989). *Generalized Linear Models, Second Edition*. Chapman & Hall/CRC Monographs on Statistics & Applied Probability. Taylor & Francis.

Muirhead, R. J. (1982). *Aspects of Multivariate Statistical Theory*. John Wiley and Sons, Inc.

Neuhaus, J., Kalbfleisch, J., and Hauck, W. (1991). A comparison of cluster-specific and population-averaged approaches for analyzing correlated binary data. *International Statistical Review*, 59(1):25–35.

Pan, F., Tung, A. K., Cong, G., and Xu, X. (2004). Cobbler: combining column and row enumeration for closed pattern discovery. In *Scientific and Statistical Database Management, 2004. Proceedings. 16th International Conference on*, pages 21–30. IEEE.

Prabha, S., Shanmugapriya, S., and Duraiswamy, K. (2013). A survey on closed frequent pattern mining. *International Journal of Computer Applications*, 63(14):47–52.

Prasad, K. S. N. and Ramakrishna, P. S. (2011). Mining closed itemsets for coherent rules: An inference analysis approach. *Global Journal of Computer Science and Technology*, 11(19):1–6.

Qu, Y., Williams, G. W., Beck, G. J., and Medendorp, S. V. (1992). Latent variable models for clustered dichotomous data with multiple subclusters. *Biometrics*, 48(4):1095.

Ramos, J. et al. (2003). Using tf-idf to determine word relevance in document queries. In *Proceedings of the first instructional conference on machine learning*.

Salton, G. and McGill, M. J. (1986). *Introduction to Modern Information Retrieval*. McGraw-Hill, Inc., New York, NY, USA.

Slepian, D. (1962). The one-sided barrier problem for gaussian noise. *Bell Labs Technical Journal*, 41(2):463–501.

Sohrabi, M. K. and Barforoush, A. A. (2012). Efficient colossal pattern mining in high dimensional datasets. *Know.-Based Syst.*, 33:41–52.

Szekely, G. J., Rizzo, M. L., Bakirov, N. K., et al. (2007). Measuring and testing dependence by correlation of distances. *The Annals of Statistics*, 35(6):2769–2794.

Tan, M., Qu, Y., and Rao, J. S. (1999). Robustness of the latent variable model for correlated binary data. *Biometrics*, 55(1):258–263.

Tan, P.-N., Kumar, V., and Srivastava, J. (2002). Selecting the right interestingness measure for association patterns. In *Proceedings of the eighth ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 32–41. ACM.
Tong, Y., Chen, L., Cheng, Y., and Yu, P. S. (2012). Mining frequent itemsets over uncertain databases. *Proceedings of the VLDB Endowment*, 5(11):1650–1661.

Wilson, J. D., Wang, S., Mucha, P. J., Bhamidi, S., and Nobel, A. B. (2014). A testing based extraction algorithm for identifying significant communities in networks. *The Annals of Applied Statistics*, 8(3):1853–1891.

Zaki, M. J. et al. (1999). Parallel and distributed association mining: A survey. *IEEE concurrency*, 7(4):14–25.

Zaki, M. J. and Hsiao, C.-J. (2002). Charm: An efficient algorithm for closed itemset mining. In *Proceedings of the 2002 SIAM international conference on data mining*, pages 457–473. SIAM.

Zaki, M. J., Parthasarathy, S., Ogihara, M., and Li, W. (1997a). Parallel algorithms for discovery of association rules. *Data mining and knowledge discovery*, 1(4):343–373.

Zaki, M. J., Parthasarathy, S., Ogihara, M., Li, W., et al. (1997b). New algorithms for fast discovery of association rules. In *KDD*, volume 97, pages 283–286.

Zhang, K. (2017). Bet on independence. *pre-print*.

Zhang, Q., Li, F., and Yi, K. (2008a). Finding frequent items in probabilistic data. *Proceedings of the ACM SIGMOD International Conference on Management of Data*, pages 819–832.

Zhang, X., Pan, F., Wang, W., and Nobel, A. (2008b). Mining non-redundant high order correlations in binary data. *Proceedings of the VLDB Endowment*, 1(1):1178–1188.

Zhu, F., Yan, X., Han, J., Philip, S. Y., and Cheng, H. (2007). Mining colossal frequent patterns by core pattern fusion. In *Data Engineering, 2007. ICDE 2007. IEEE 23rd International Conference on*, pages 706–715. IEEE.