Ore and Goldie theorems for skew \( PBW \) extensions

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Abstract

Many rings and algebras arising in quantum mechanics can be interpreted as skew \( PBW \) (Poincaré-Birkhoff-Witt) extensions. Indeed, Weyl algebras, enveloping algebras of finite-dimensional Lie algebras (and its quantization), Artamonov quantum polynomials, diffusion algebras, Manin algebra of quantum matrices, among many others, are examples of skew \( PBW \) extensions. In this paper we extend the classical Ore and Goldie theorems, known for skew polynomial rings, to this wide class of non-commutative rings. As application, we prove the quantum version of the Gelfand-Kirillov conjecture for the skew quantum polynomials.

Key words and phrases. Ore’s theorem, Goldie’s theorem, skew polynomial rings, \( PBW \) extensions, quantum algebras, skew \( PBW \) extensions.

2010 Mathematics Subject Classification. Primary: 16U20, 16S80. Secondary: 16N60, 16S36.

1 Skew \( PBW \) extensions

The classical Ore’s theorem says that if \( R \) is a left Ore domain and \( R[x; \sigma, \delta] \) is the skew polynomial ring over \( R \), with \( \sigma \) injective, then \( R[x; \sigma, \delta] \) is also a left Ore domain, and hence has left total division ring of fractions (see [15] or also [7]). In this paper we generalize this result to skew \( PBW \) extensions, a wide class of non-commutative rings introduced in [12]. Skew \( PBW \) extensions include many rings and algebras arising in quantum mechanics such as the classical \( PBW \) extensions (see [1]), Weyl algebras, enveloping algebras of finite-dimensional Lie algebras (and its quantization), Artamonov quantum polynomials (see [2], [3]), diffusion algebras, Manin algebra of quantum matrices, among many others. A very long list of remarkable examples of skew \( PBW \) extensions is presented in [13], where some ring-theoretic properties have been investigated for this class of rings, for example, the global, Krull, Goldie and Gelfand-Kirillov dimensions were estimated. In the present paper we are interested in proving Ore and Goldie theorems for skew \( PBW \) extensions, generalizing this way two well known results.

In this section we recall the definition of skew \( PBW \) (Poincaré-Birkhoff-Witt) extensions defined firstly in [12], and we will review also some elementary properties about the polynomial interpretation of this kind of non-commutative rings. Two particular subclasses of these extensions are recalled also.

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Definition 1.1. Let $R$ and $A$ be rings. We say that $A$ is an skew PBW extension of $R$ (also called a $\sigma$–PBW extension of $R$) if the following conditions hold:

(i) $R \subseteq A$.

(ii) There exist finite elements $x_1, \ldots, x_n \in A$ such $A$ is a left $R$-free module with basis

$$\text{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}.$$ 

In this case it says also that $A$ is a left polynomial ring over $R$ with respect to $\{x_1, \ldots, x_n\}$ and $\text{Mon}(A)$ is the set of standard monomials of $A$. Moreover, $x_1^{\alpha_1} \cdots x_n^{\alpha_n} := 1 \in \text{Mon}(A)$.

(iii) For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R. \quad (1.1)$$

(iv) For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n. \quad (1.2)$$

Under these conditions we will write $A := \sigma(R)\langle x_1, \ldots, x_n \rangle$.

The following proposition justifies the notation and the alternative name given for the skew PBW extensions.

Proposition 1.2. Let $A$ be an skew PBW extension of $R$. Then, for every $1 \leq i \leq n$, there exists an injective ring endomorphism $\sigma_i : R \to R$ and a $\sigma_i$-derivation $\delta_i : R \to R$ such that

$$x_i r = \sigma_i(r) x_i + \delta_i(r),$$

for each $r \in R$.

Proof. See [12], Proposition 3. \qed

A particular case of skew PBW extension is when all derivations $\delta_i$ are zero. Another interesting case is when all $\sigma_i$ are bijective and the constants $c_{ij}$ are invertible. We recall the following definition (cf. [12]).

Definition 1.3. Let $A$ be an skew PBW extension.

(a) $A$ is quasi-commutative if the conditions (iii) and (iv) in Definition 1.1 are replaced by

(iii') For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that

$$x_i r = c_{i,r} x_i. \quad (1.3)$$

(iv') For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that

$$x_j x_i = c_{i,j} x_i x_j. \quad (1.4)$$

(b) $A$ is bijective if $\sigma_i$ is bijective for every $1 \leq i \leq n$ and $c_{ij}$ is invertible for any $1 \leq i < j \leq n$.

Some extra notation will be used in the paper.

Definition 1.4. Let $A$ be an skew PBW extension of $R$ with endomorphisms $\sigma_i$, $1 \leq i \leq n$, as in Proposition 1.2.
Proposition 1.7. Let $A$ be a skew PBW extension of a ring $R$. If $R$ is a domain, then $A$ is a domain.

Proof. See [13].

Remark 1.6. (i) We observe that if $A$ is quasi-commutative, then $p_{\alpha,r} = 0$ and $p_{\alpha,\beta} = 0$ for every $0 \neq r \in R$ and every $\alpha, \beta \in \mathbb{N}^n$.

(ii) If $A$ is bijective, then $c_{\alpha,\beta}$ is invertible for any $\alpha, \beta \in \mathbb{N}^n$.

(iii) In $\text{Mon}(A)$ we define

$$x^\alpha \succeq x^\beta \iff \begin{cases} x^\alpha = x^\beta \\
\text{or} \\
 x^\alpha \neq x^\beta \text{ but } |\alpha| > |\beta| \\
\text{or} \\
 x^\alpha \neq x^\beta, |\alpha| = |\beta| \text{ but } \exists i \text{ with } \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i.
\end{cases}$$

It is clear that this is a total order on $\text{Mon}(A)$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha > x^\beta$. Each element $f \in A$ can be represented in a unique way as $f = c_1x^{\alpha_1} + \cdots + c_tx^{\alpha_t}$, with $c_i \in R - \{0\}$, $1 \leq i \leq t$, and $x^{\alpha_1} > \cdots > x^{\alpha_t}$. We say that $x^{\alpha_1}$ is the leader monomial of $f$ and we write $\text{lm}(f) := x^{\alpha_1}$; $c_1$ is the leader coefficient of $f$, $\text{lc}(f) := c_1$, and $c_1x^{\alpha_1}$ is the leader term of $f$ denoted by $\text{lt}(f) := c_1x^{\alpha_1}$.

A natural and useful result that we will use later is the following property.

Proposition 1.7. Let $A$ be an skew PBW extension of a ring $R$. If $R$ is a domain, then $A$ is a domain.

Proof. See [13].

The next theorem characterizes the quasi-commutative skew PBW extensions.

Theorem 1.8. Let $A$ be a quasi-commutative skew PBW extension of a ring $R$. Then,

(i) $A$ is isomorphic to an iterated skew polynomial ring of endomorphism type, i.e.,
Proof. See \cite{13}. \hfill $\Box$

Theorem 1.9. Let $A$ be an arbitrary skew PBW extension of $R$. Then, $A$ is a filtered ring with filtration given by

$$F_m := \begin{cases} R & \text{if } m = 0 \\ \{f \in A \mid \deg(f) \leq m\} & \text{if } m \geq 1 \end{cases} \quad (1.7)$$

and the corresponding graded ring $\text{Gr}(A)$ is a quasi-commutative skew PBW extension of $R$. Moreover, if $A$ is bijective, then $\text{Gr}(A)$ is a quasi-commutative bijective skew PBW extension of $R$.

Proof. See \cite{13}. \hfill $\Box$

Theorem 1.10 (Hilbert Basis Theorem). Let $A$ be a bijective skew PBW extension of $R$. If $R$ is a left (right) Noetherian ring then $A$ is also a left (right) Noetherian ring.

Proof. \cite{13}. \hfill $\Box$

2 Preliminary lemmas

Let us recall first the non-commutative localization. If $R$ is a ring and $S$ is a multiplicative subset of $R$ (i.e., $1 \in S, 0 \not\in S, ss' \in S$, for $s, s' \in S$) then the left ring of fractions of $R$ exists if and only if two conditions hold: (i) given $a \in R$ and $s \in S$ such that $as = 0$, then there exists $s' \in S$ such that $s'a = 0$; (ii) (the left Ore condition) given $a \in R$ and $s \in S$ there exist $s' \in S$ and $a' \in R$ such that $s'a = a's$. When these conditions hold, the left ring of fractions of $R$ with respect to $S$ is denoted by $S^{-1}R$, and its elements are classes represented by fractions: two elements $\frac{a}{s}, \frac{b}{t}$ are equal if and only if there exist $c, d \in R$ such that $ca = db, cs = dt \in S$. The operations of $S^{-1}R$ are given by $\frac{a}{s} + \frac{b}{t} := \frac{ca + db}{st}$, where $u := cs = dt \in S$, for some $c, d \in R$ (the Ore’s condition applied to $s$ and $t$), and $\frac{a}{s} \cdot \frac{b}{t} := \frac{ab}{st}$, where $ua = ct$, for some $u \in S$ and $c \in R$ (the Ore’s condition applied to $a$ and $t$). In a similar way are defined the right rings of fractions. Note that any domain $R$ satisfies (i) with respect to any multiplicative subset $S$, and it is said that $R$ is a left Ore domain if $R$ satisfies (ii) with respect to $S := R - \{0\}$. The elements of the ring $R$ that are non-zero divisors are called regular and the set of regular elements of $R$ will denoted by $S_{0}(R)$.

In this second section we localize skew polynomial rings and skew PBW extensions by multiplicative subsets of the ring of coefficients. The basic results presented here will used in the other sections of the present paper. We start recalling a couple of well known facts.

Proposition 2.1. Let $\sigma$ be an automorphism of $R$ and $R[x; \sigma, \delta]$ the left skew polynomial ring. Then, the right skew polynomial ring $R[x; \sigma^{-1}, -\delta\sigma^{-1}]_{r}$ is isomorphic to $R[x; \sigma, \delta]$.

Proof. See \cite{14}. \hfill $\Box$

Proposition 2.2. Let $R$ be a ring and $S \subset R$ a multiplicative subset. If $Q := S^{-1}R$ exists, then any finite set $\{q_1, \ldots, q_n\}$ of elements of $Q$ posses a common denominator, i.e., there exist $r_1, \ldots, r_n \in R$ and $s \in S$ such that $q_i = \frac{r_i}{s}, 1 \leq i \leq n$.

Proof. See \cite{14}, Lemma 2.1.8. \hfill $\Box$

The first preliminary result is the following lemma, the first part of which is well known and can be found in \cite{5}.
Lemma 2.3. Let \( R \) be a ring and \( S \subset R \) a multiplicative subset.

(a) If \( S^{-1}R \) exists and \( \sigma(S) \subseteq S \), then

\[
S^{-1}(R[x; \sigma, \delta]) \cong (S^{-1}R)[x; \sigma, \delta],
\]

(2.1)

with

\[
\frac{a}{s} \mapsto \frac{\sigma(a)}{\sigma(s)}.
\]

(b) If \( RS^{-1} \) exists and \( \sigma \) is bijective with \( \sigma(S) = S \), then

\[
(R[x; \sigma, \delta])S^{-1} \cong (RS^{-1})[x; \delta, \tilde{\delta}],
\]

(2.2)

with

\[
\frac{a}{s} \mapsto \frac{\sigma(a)}{\sigma(s)} + \frac{\delta(a)}{s}.
\]

Proof. (a) The sketch of the proof can be found in [3, Chapter 8, Lemma 1.10 and Proposition 1.11.

(b) From Proposition 2.1, we have \( R[x; \sigma^{-1}, -\delta \sigma^{-1}]_d \cong R[x; \sigma, \delta] \). Let \( \theta := \sigma^{-1} \) and \( \gamma := -\delta \sigma^{-1} \), then \( R[x; \theta, \gamma]_d \cong R[x; \sigma, \delta] \), so \( (R[x; \theta, \gamma]_d)S^{-1} \cong (R[x; \sigma, \delta])S^{-1} \). Adapting the proof of [3], but for the right side (the inclusion \( \theta(S) \subset S \) is guaranteed by the condition \( \sigma(S) = S \), we obtain

\[
(R[x; \theta, \gamma]_d)S^{-1} \cong (RS^{-1})[x; \theta, \gamma], \text{ with } \overline{\theta}(\frac{a}{s}) := \frac{\theta(a)}{\theta(s)}, \overline{\gamma}(\frac{a}{s}) := -\frac{\gamma(a)}{\gamma(s)}.
\]

Hence, \( (R[x; \sigma, \delta])S^{-1} \cong (RS^{-1})[x; \tilde{\theta}(\tilde{\gamma})^{-1}] \). But note that \( \tilde{\theta}(\tilde{\gamma})^{-1} = \overline{\sigma} \) and \( -\gamma(\tilde{\theta})^{-1} = \overline{\delta} \), where \( \overline{\sigma}, \overline{\delta} \) are defined as in the statement of the theorem. In fact, if \( \overline{\theta}(\overline{\gamma})^{-1} = \frac{s}{t} \), then \( \frac{s}{t} = \frac{\theta(b)}{\theta(d)} = \frac{\sigma^{-1}(b)}{\sigma^{-1}(d)} \) and there exist \( c, d \in A \) such that \( ac = \sigma^{-1}(b)d \) and \( sc = \sigma^{-1}(t)d \in S \). From this we get \( \sigma(a)\sigma(c) = b\sigma(d) \) and \( \sigma(s)\sigma(c) = t\sigma(d) \in S \), i.e., \( \frac{\sigma(a)}{\sigma(s)} = \frac{b}{t} \). For the other equality we have \( -\gamma(\tilde{\theta})^{-1} = \overline{\gamma(\overline{\theta})} = -[\sigma(\gamma(\sigma))]/\sigma(s) + \delta(\sigma) = \overline{\delta}(\overline{\sigma}) \).

The previous lemma can be extended to iterated skew polynomial rings.

Corollary 2.4. Let \( R \) be a ring and \( A := R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n] \) the iterated skew polynomial ring. Let \( S \) be a multiplicative system of \( R \).

(a) If \( S^{-1}R \) exists and \( \sigma_i(S) \subseteq S \) for every \( 1 \leq i \leq n \), then

\[
S^{-1}A \cong (S^{-1}R)[x_1; \overline{\sigma_1}, \overline{\delta_1}] \cdots [x_n; \overline{\sigma_n}, \overline{\delta_n}],
\]

with

\[
\frac{a}{s} \mapsto \frac{\sigma_i(a)}{\sigma_i(s)}
\]

and

\[
\frac{a}{s} \mapsto \frac{\delta_i(a)}{\sigma_i(s)}
\]
(b) If \( RS^{-1} \) exists and \( \sigma_i \) is bijective with \( \sigma_i(S) = S \) for every \( 1 \leq i \leq n \), then

\[
AS^{-1} \cong (RS^{-1})[x_1; \tilde{\sigma}_1, \tilde{\delta}_1] \cdots [x_n; \tilde{\sigma}_n, \tilde{\delta}_n],
\]

with

\[
(RS^{-1})[x_1; \tilde{\sigma}_1, \tilde{\delta}_1] \cdots [x_{i-1}; \tilde{\sigma}_{i-1}, \tilde{\delta}_{i-1}] \xrightarrow{\tilde{\sigma}} (RS^{-1})[x_1; \tilde{\sigma}_1, \tilde{\delta}_1] \cdots [x_{i-1}; \tilde{\sigma}_{i-1}, \tilde{\delta}_{i-1}]
\]

\[
a \xrightarrow{s} \frac{\sigma_i(a)}{\sigma_i(s)}
\]

\[
(RS^{-1})[x_1; \tilde{\sigma}_1, \tilde{\delta}_1] \cdots [x_{i-1}; \tilde{\sigma}_{i-1}, \tilde{\delta}_{i-1}] \xrightarrow{\tilde{\delta}} (RS^{-1})[x_1; \tilde{\sigma}_1, \tilde{\delta}_1] \cdots [x_{i-1}; \tilde{\sigma}_{i-1}, \tilde{\delta}_{i-1}]
\]

\[
a \xrightarrow{s} -\frac{\sigma_i(a)}{\sigma_i(s)} \frac{\delta_i(a)}{s} + \frac{\delta_i(a)}{s}
\]

Proof. The part (a) of the corollary follows from Lemma 2.3 by iteration and observing that

\[
(S^{-1}R)[x_1; \overline{\sigma}_1, \overline{\delta}_1] \cdots [x_{i-1}; \overline{\sigma}_{i-1}, \overline{\delta}_{i-1}] \cong S^{-1} \langle R[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}] \rangle,
\]

thus any element of \((S^{-1}R)[x_1; \overline{\sigma}_1, \overline{\delta}_1] \cdots [x_{i-1}; \overline{\sigma}_{i-1}, \overline{\delta}_{i-1}]\) can be represented as a fraction \( \frac{a}{s} \), with \( a \in R[x_1; \sigma_1, \delta_1] \cdots [x_{i-1}; \sigma_{i-1}, \delta_{i-1}] \) and \( s \in S \). The same remark apply for the part (b).

Corollary 2.5. Let \( A := R[z_1; \sigma_1] \cdots [z_n; \sigma_n] \) be a quasi-commutative skew PBW extension of a ring \( R \) and let \( S \) be a multiplicative system of \( R \).

(a) If \( S^{-1}R \) exists and \( \sigma_i(S) \subseteq S \) for every \( 1 \leq i \leq n \), then

\[
S^{-1}A \cong (S^{-1}R)[z_1; \overline{\sigma}_1] \cdots [z_n; \overline{\sigma}_n].
\]

In particular, if \( A \) is bijective with \( \sigma_i(S) = S \) for every \( i \), then \( S^{-1}A \) is a quasi-commutative bijective skew PBW extension of \( S^{-1}R \).

(b) If \( RS^{-1} \) exists and \( A \) is bijective with \( \sigma_i(S) = S \) for every \( 1 \leq i \leq n \), then \( AS^{-1} \) is a quasi-commutative bijective skew PBW extension of \( RS^{-1} \) and

\[
AS^{-1} \cong (RS^{-1})[x_1; \tilde{\sigma}_1] \cdots [x_n; \tilde{\sigma}_n].
\]

Proof. This is a direct consequence of the previous corollary. Assuming that each \( \sigma_i \) is bijective and \( \sigma_i(S) = S \), for each \( 1 \leq i \leq n \), then each \( \overline{\sigma} \) is bijective. In fact, if \( \frac{\sigma_i(a)}{\sigma_i(s)} = 0 \), then there exist \( c, d \in R[x_1; \sigma_1] \cdots [x_{i-1}; \sigma_{i-1}] \) such that \( \sigma_i(a) = 0 \) and \( c \sigma_i(s) = d \in S \). Since \( \sigma_i \) is surjective and \( S = \sigma_i(S) \), there exist \( c' \in R[x_1; \sigma_1] \cdots [x_{i-1}; \sigma_{i-1}] \) and \( d' \in S \) such that \( \sigma_i(c') = c \) and \( \sigma_i(d') = d \), hence \( \sigma_i(c'a) = 0 \) and \( \sigma_i(c's) = \sigma_i(d') \), but \( \sigma_i \) is injective, then \( c'a = 0 \) and \( c's = d' \). This means that \( \frac{a}{s} = 0 \), i.e., \( \overline{\sigma} \) is injective. It is clear that \( \overline{\sigma} \) is surjective. Finally, if the constants \( c_{i,j} \) that define \( A \) are invertible (see Definition 1.3), then \( \frac{c_i}{c_j} \in S^{-1}A \) are also invertible.

For the part (b) the proof is analogous.

Now we consider arbitrary bijective skew PBW extensions and \( S \) a multiplicative subset of \( R \) consisting of regular elements, i.e., \( S \subseteq S_0(R) \). The next powerful lemma generalizes Lemma 14.2.7 of [14].

Lemma 2.6. Let \( R \) be a ring and \( A := \sigma(R)[x_1, \ldots, x_n] \) a bijective skew PBW extension of \( R \). Let \( S \subseteq S_0(R) \) a multiplicative subset of \( R \) such that \( \sigma_i(S) = S \), for every \( 1 \leq i \leq n \), where \( \sigma_i \) is defined by Proposition 1.2.
(a) If $S^{-1}R$ exists, then $S^{-1}A$ exists and it is a bijective skew PBW extension of $S^{-1}R$ with

$$S^{-1}A = \sigma(S^{-1}R)(x'_1, \ldots, x'_n),$$

where $x'_i := \frac{x_i}{s}$ and the system of constants of $S^{-1}R$ is given by \(c'_{i,j} := \frac{c_{i,j}}{s}, \quad c'_{i,r} := \frac{\sigma(r)}{\sigma_i(s)}, 1 \leq i, j \leq n\).

(b) If $RS^{-1}$ exists, then $AS^{-1}$ exists and it is a bijective skew PBW extension of $RS^{-1}$ with

$$AS^{-1} = \sigma(RS^{-1})(x''_1, \ldots, x''_n),$$

where $x''_i := \frac{x_i}{s}$ and the system of constants of $RS^{-1}$ is given by \(c''_{i,j} := \frac{c_{i,j}}{s}, \quad c''_{i,r} := \frac{\sigma(r)}{\sigma_i(s)}, 1 \leq i, j \leq n\).

**Proof.** We will use the notation given in Definition 1.4 and Remark 1.6.

(a) Let \(f \in A\) and \(s \in S\) such that \(fs = 0\). This implies that \(f = 0\) and hence \(sf = 0\). In fact, suppose that \(f \neq 0\), let \(lt(f) := cx^n, c \in R - \{0\}\) and \(x^n \in \text{Mon}(A)\). Then \(cx^n(s) = 0\), but since \(\sigma^n(s) \in S\), then \(c = 0\), a contradiction. Now, let again \(f \in A\) and \(s \in S\), we have to find \(u \in S\) and \(g \in A\) such that \(uf = gs\). If \(f = 0\) we take \(u = 1\) and \(g = 0\). Let \(f \neq 0\) and again \(lt(f) := cx^n\), then there exists \(u_1 \in S\) and \(r \in R\) such that \(u_1c = ra^\alpha(s)\). Consider \(u_1f - rx'^s\); if \(u_1f - rx'^s = 0\), then the Ore condition is satisfied. Let \(u_1f - rx'^s \neq 0\), then \(lm(u_1f - rx'^s) < lm(f)\). By induction on \(lm\), there exists \(u_2 \in S\) and \(g' \in A\) such that \(u_2(u_1f - rx'^s) = g's\). Thus, \(uf = gs\), with \(u := u_2u_1\) and \(g := u_2x'^s + g'\). This proves that \(S^{-1}A\) exists.

Let \(R' := S^{-1}R\) and \(A' := S^{-1}A\); from \(R \subseteq A\) we get that \(R' \hookrightarrow A'\). In fact, the correspondence \(\frac{\xi}{s} \mapsto \frac{\xi}{s}\) is an injective ring homomorphism since if \(\frac{\xi}{s} = \frac{\eta}{s}\) in \(A'\), then \(s^{-1}r = 0\) and hence \(r = 0\), i.e., \(\frac{\xi}{s} = \frac{\eta}{s}\) in \(R'\). We denote \(x'_i := \frac{x_i}{s}\) for every \(1 \leq i \leq n\); since \(S\) has not zero divisors and \(\text{Mon}\{x_1, \ldots, x_n\}\) is a left \(R\)-basis of \(A\), then \(A'\) is a free left \(R'\)-module with basis \(\text{Mon}\{x'_1, \ldots, x'_n\}\).

Let \(c'_{i,j} := \frac{c_{i,j}}{s}\), then \(c'_{i,j} \neq 0\) and \(x'_i x'_j - c'_{i,j} x'_j x'_i \in R' + R'x'_i + \cdots + R'x'_n, \) for every \(1 \leq i, j \leq n\).

The endomorphisms \(\sigma_i\) of \(R\) and the \(\sigma_i\)-derivations \(\delta_i\) that define \(A\) (Proposition 1.2) induce endomorphisms \(\sigma_1'\) of \(R'\) and \(\sigma_i\)-derivations \(\delta_i\) of \(R'\) (see Lemma 2.8). Since \(\sigma_i\) is bijective and \(\sigma_i(S) = S\) then each \(\sigma_i'\) is bijective (the proof is similar as in Corollary 2.8). We claim that \(x'_i' = \sigma_i'\frac{x_i}{s} + \delta_i(x)\).

Indeed,

$$\sigma_i'\frac{x_i}{s} + \delta_i(x) = \frac{\sigma(r)}{\sigma_i(s)} \frac{x_i}{s} + \frac{\delta_i(x)}{\sigma_i(s)} \frac{s}{s}$$

and

$$x_i' = \frac{c(x)r}{u} = x_i' = \frac{c(x)r}{u}, \quad \text{with} \quad u \in S, \quad c(x) \in A \quad \text{and} \quad u x_i = c(x)s.$$  

So deg\((c(x)) = 1\) and \(c(x)\) involves only \(x_i\), hence \(c(x) = c_1x_i + c_0\), where \(c_0, c_1 \in R\). From this we get the relations

$$u = c_1 \sigma_i(s), \quad c_1 \delta_i(s) + c_0 s = 0.$$  

Therefore,

$$x'_i = \frac{c(x)r}{u} c_1 \sigma_i(s) + c_0 c_1 \delta_i(s) + c_0 r$$

but \(\frac{c_0 \sigma_i(s)}{c_1 \sigma_i(s)} = \frac{c_0 \sigma_i(s)}{c_1 \sigma_i(s)} \) and \(- \frac{c_0 \delta_i(s)}{c_1 \sigma_i(s)} \) and \(- \frac{c_0 \sigma_i(s)}{c_1 \sigma_i(s)} \) and \(- \frac{c_0 \delta_i(s)}{c_1 \sigma_i(s)} \). This proved the claimed. Thus, given \(\frac{x}{s} \in R' - \{0\}\) there exists \(c_{i,r} := \sigma_i'\frac{x_i}{s} \in R' - \{0\}\) such that \(x'_i = c_{i,r} x'_i \in R'\). This completes the proof that \(S^{-1}A\) is an skew PBW extension of \(S^{-1}R\).

(b) Let \(f \in A\) and \(s \in S\) such that \(sf = 0\), then \(f = 0\) and \(fs = 0\). This proved the first condition for the existence of \(AS^{-1}\). Now, we have to find \(u \in S\) and \(g \in A\) such that \(fu = sg\). If \(f = 0\) we take \(u = 1\) and \(g := 0\). Let \(f \neq 0\), \(lt(f) := cx^n\); there exist \(u_1 \in S\) and \(r \in R\) such that \(cu_1 = sr\). Consider
\[ f \sigma^{-\alpha}(u_1) - srx^\alpha; \text{ if } f \sigma^{-\alpha}(u_1) = srx^\alpha, \text{ then the Ore condition is satisfied. Let } f \sigma^{-\alpha}(u_1) - srx^\alpha \neq 0, \text{ then } lm(f \sigma^{-\alpha}(u_1) - srx^\alpha) < lm(f). \] By induction on \( lm \), there exist \( u_2 \in S \) and \( g' \in A \) such that \((f \sigma^{-\alpha}(u_1) - srx^\alpha)u_2 = sg'\). Then \( fu = sg \), with \( u := \sigma^{-\alpha}(u_1)u_2 \) and \( g := rx^\alpha u_2 + g' \). This proves that \( AS^{-1} \) exists.

Let \( R'' := RS^{-1} \) and \( A'\prime := AS^{-1} \); from \( R \subseteq A \) we get that \( R'' \hookrightarrow A'\prime \). In fact, the correspondence \( \frac{r}{s} \mapsto \frac{r}{s} \) is an injective ring homomorphism since if \( \frac{r}{s} = \frac{r'}{s'} \) in \( A'\prime \), then \( rs^{-1} = 0 \) and hence \( r = 0 \), i.e., \( \frac{r}{s} = \frac{r}{s} \in R'' \).

We note \( x''_i \equiv x'_i \) in \( A'\prime \) for every \( 1 \leq i \leq n \). Let \( c''_{i,j} := \frac{c''_{i,j}}{1} \), then \( c''_{i,j} \neq 0 \) and \( x''_i x''_j = c''_{i,j} x''_i x''_j \in R'' + R'' x'' + \cdots + R'' x''_n \) for every \( 1 \leq i, j \leq n \).

The endomorphisms \( \sigma_i \) of \( R \) and the \( \sigma_i \)-derivations \( \delta_i \) that define \( A \) (see Proposition \[\ref{prop:1} \]) induce endomorphisms \( \overline{\sigma}_i \) of \( R'' \) and \( \overline{\sigma}_i \)-derivations \( \overline{\delta}_i \) of \( R'' \) (see Lemma \[\ref{lem:2} \]). Note that each \( \overline{\sigma}_i \) is bijective. We claim that \( x''_i x''_j = x'_i x'_j + \overline{\delta}_i(x'_j) \). Indeed,

\[
x''_i x''_j = x''_i + \overline{\delta}_i(x''_j).
\]

On the other hand,

\[
\overline{\delta}_i(x'_i) x''_j + \overline{\delta}_i(x'_j) = \frac{\sigma_i(x'_i x'_j + \delta_i(x'_j))}{\sigma_i(s)} - \frac{\sigma_i(x'_i) \delta_i(x'_j)}{\sigma_i(s)} + \frac{\delta_i(x'_j)}{s}.
\]

Thus, we must prove that

\[
\frac{\sigma_i(x'_i)}{\sigma_i(s)} = \frac{\sigma_i(x''_i)}{\sigma_i(s)} + \frac{\delta_i(x'_j)}{s}.
\]

Applying the right Ore condition to \( \sigma_i(s) \) and \( x_i u = \sigma_i(s) c(x) \), with \( u \in S \) and \( c(x) \in A \). As in the part (a), \( c(x) = cx_i + d \), with \( c, d \in R \), so \( \sigma_i(u) = \sigma_i(s) c \) and \( \delta_i(u) = \sigma_i(s) d \).

Thus,

\[
\frac{\sigma_i(x'_i)}{\sigma_i(s)} = \frac{\sigma_i(cx_i + d)}{\sigma_i(s)} = \frac{\sigma_i(cx_i)}{\sigma_i(s)} + \frac{\sigma_i(d)}{\sigma_i(s)}
\]

and hence

\[
\frac{\sigma_i(x'_i)}{\sigma_i(s)} = \frac{\sigma_i(cx_i)}{\sigma_i(s)} + \frac{\sigma_i(d)}{\sigma_i(s)} - \frac{\sigma_i(\delta_i)}{\sigma_i(s)}.
\]

but \( u = s \sigma_i^{-1}(c) \), so

\[
\frac{\sigma_i(x'_i)}{\sigma_i(s)} = \frac{\sigma_i(cx_i)}{\sigma_i(s)} = \frac{\sigma_i(s) \sigma_i^{-1}(c) - \delta_i(s \sigma_i^{-1}(c))}{\sigma_i(s)} = \frac{\sigma_i(s) \sigma_i^{-1}(c) - \delta_i(s \sigma_i^{-1}(c))}{\sigma_i(s)} = \frac{\sigma_i(s) \sigma_i^{-1}(c)}{s} - \frac{\delta_i(s \sigma_i^{-1}(c))}{\sigma_i(s)}.
\]

Hence, the problem is reduced to prove the equality

\[
\frac{\sigma_i(d)}{\sigma_i(s)} - \frac{\sigma_i(\delta_i)}{\sigma_i(s)} = \frac{\delta_i(s \sigma_i^{-1}(c))}{\sigma_i(s)},
\]

or equivalently, to prove

\[
\frac{\sigma_i(d)}{\sigma_i(s)} - \frac{\delta_i(s \sigma_i^{-1}(c))}{\sigma_i(s)} = \frac{\sigma_i(\delta_i)}{\sigma_i(s)}.
\]

Note that \( \delta_i(u) = \sigma_i(s) \delta_i(s \sigma_i^{-1}(c)) + \delta_i(s) \sigma_i^{-1}(c) = \sigma_i(s) d \), i.e., \( \delta_i(s) \sigma_i^{-1}(c) = \sigma_i(s)(d - \delta_i(s \sigma_i^{-1}(c))) \). But this relation indicates that

\[
\frac{\sigma_i(s) \delta_i(s)}{\sigma_i(s)} = \frac{\sigma_i(s)(d - \delta_i(s \sigma_i^{-1}(c)))}{\sigma_i(s)} = \frac{\sigma_i(d)}{\sigma_i(s)} - \frac{\delta_i(s \sigma_i^{-1}(c))}{\sigma_i(s)}.
\]
This proved the claimed. Thus, given $\frac{t}{u} \in R'^{n} - \{0\}$ there exists $c''_{i}(\frac{t}{u}) = \bar{c}_{i}(\frac{t}{u}) \in R''^{n} - \{0\}$ such that $x''_{i} \frac{t}{u} - c''_{i} \frac{t}{u} x''_{i} \in R''$. 

Now we will show that $A''$ is a free left $R'^{n}$-module with basis $\text{Mon}\{x''_{1}, \ldots, x''_{n}\}$. First note that $A''$ is generated by $\text{Mon}\{x''_{1}, \ldots, x''_{n}\}$. In fact, let $z \in A''$, then $z$ has the form $z = \frac{c_{1}x''_{1} + \cdots + c_{n}x''_{n}}{u}$, with $c_{i} \in R$, $x''_{i} \in \text{Mon}\{x_{1}, \ldots, x_{n}\}$, $1 \leq i \leq t$, and $s \in S$. It is enough to show that each summand $\frac{c_{i}x''_{i}}{u}$ is generated by $\text{Mon}\{x''_{1}, \ldots, x''_{n}\}$. But observe that $\frac{c_{i}x''_{i}}{u} = \frac{c_{i}}{u} \frac{t}{u} x''_{i} + \bar{c}_{i}(\frac{t}{u})$, where $\bar{c}_{i}(\frac{t}{u})$ is a left linear combination of elements of $\text{Mon}\{x''_{1}, \ldots, x''_{n}\}$ with coefficients in $R''$. Thus, $A''$ is left generated over $R''$ by $\text{Mon}\{x''_{1}, \ldots, x''_{n}\}$. Now let $\frac{t}{u}_{s}, \ldots, \frac{t}{u}_{s} \in R''$ and $x''_{i}, \ldots, x''_{i} \in \text{Mon}\{x''_{1}, \ldots, x''_{n}\}$ such that $\frac{t}{u}_{s}x''_{i} + \cdots + \frac{t}{u}_{s}x''_{i} = 0$. Taking common denominator (Proposition 2.2), and without lost of generality, we can write $\bar{c}_{i}(\frac{t}{u}) = 0$ since $\frac{t}{u}$ is a left (right) Ore domain, and hence, the left (right) total division ring of fractions. In particular, we will extend the Ore’s theorem to skew PBW extensions. A first elementary result is the following proposition.

**Proposition 3.1.** If $R$ is a left (right) Noetherian domain and $A$ is a bijective skew PBW extension of $R$, then $A$ is a left (right) Ore domain, and hence, the left (right) division ring of fractions of $A$ exists.

**Proof.** It is well known that left (right) Noetherian domains are left (right) Ore domains (see [2], Theorem 2.1.15). The result is consequence of Proposition 1.7 and Theorem 1.11. □

The main purpose of the present section is to replace the Noetherianity in Proposition 3.1 by the Ore condition. A preliminary result is needed.

**Proposition 3.2.** Let $B$ be a domain and $S$ a multiplicative subset of $B$ such that $S^{-1}B$ exists. Then, $B$ is left Ore domain if and only if $S^{-1}B$ is a left Ore domain. In such case

$Q_{l}(B) \cong Q_{l}(S^{-1}B)$.

The right side version of the proposition holds too.

**Proof.** (i) $\Rightarrow$): Note first that $S^{-1}B$ is a domain: let $\frac{a}{s}, \frac{b}{t} \in S^{-1}B$ such that $\frac{a}{s} \frac{b}{t} = \frac{0}{1}$. There exist $u \in S$ and $c \in B$ such that $ua = ct$ and $\frac{cd}{s} = \frac{0}{t}$. Hence, there exist $c', d' \in B$ such that $c'cb = 0$ and $c'bu = d'c$. Since $B$ is a domain $cb = 0$, then $b = 0$ or $c = 0$, and in this last case we get that $a = 0$. Thus, $\frac{a}{s} = 0$ or $\frac{b}{t} = 0$.

Let again $\frac{a}{s}, \frac{b}{t} \in S^{-1}B$ with $\frac{b}{t} \neq 0$, then $b \neq 0$ and there exist $p \neq 0$ and $q \in B$ such that $pa = qb$.

Then, $\frac{p}{s} \frac{a}{t} = \frac{p0}{1} = \frac{0}{1} = \frac{bt}{1}$, with $\frac{ta}{s} \neq 0$.

$\Leftarrow$): Let $a, b \in B$, $u \neq 0$, then $\frac{a}{s} \frac{u}{t} \in S^{-1}B$, with $\frac{a}{s} \neq 0$. There exist $\frac{c}{s}, \frac{d}{t} \in S^{-1}A$, with $\frac{c}{s} \neq 0$ such that $\frac{c}{s} \frac{d}{t} = \frac{0}{s}$, i.e., $\frac{ca}{s} = \frac{0}{s}$. There exist $c', d' \in B$ such that $c'ca = c'd'au$ and $c't = d's \in S$. Note that $c' \neq 0$ since $c' \neq 0$ and $c \neq 0$. □
(ii) The function
\[ \varphi : S^{-1}B \to Q_l(B) \]
\[ \frac{b}{s} \mapsto \frac{\varphi(b)}{s} \]
verify the conditions that define a left total ring of fractions, i.e., \( \varphi \) is an injective ring homomorphism, the non-zero elements of \( S^{-1}B \) are invertible in \( Q_l(B) \) and each element \( \frac{b}{s} \) of \( Q_l(B) \) can be written as \( \frac{b}{a} = \varphi(\frac{x}{y})^{-1}\varphi(\frac{x}{y}). \)

**Proposition 3.3.** If \( R \) is a left Ore domain and \( \sigma \) is injective, then \( R[x; \sigma, \delta] \) is a left Ore domain and

\[ Q_l(R(x; \sigma, \delta)) \cong Q_l(Q_l(R)[x; \sigma, \delta]), \]  
(3.1)

If \( R \) is a right Ore domain and \( \sigma \) is bijective, then \( R[x; \sigma, \delta] \) is a right Ore domain and

\[ Q_d(R[x; \sigma, \delta]) \cong Q_d(Q_d(R)[x; \sigma, \delta]). \]  
(3.2)

**Proof.** The conditions in (a) of Lemma 2.3 are trivially satisfied for \( S := R - \{0\} \). Thus, \( Q_l(R)[x; \sigma, \delta] \) is a well-defined skew polynomial ring over the division ring \( Q_l(R) \) and we have the isomorphism \( S^{-1}(R[x; \sigma, \delta]) \cong Q_l(R)[x; \sigma, \delta] \). Note that \( \sigma \) is injective, and hence \( Q_l(R)[x; \sigma, \delta] \) is a left Noetherian domain and therefore a left Ore domain. From this we get that \( S^{-1}(R[x; \sigma, \delta]) \) is a left Ore domain. From Proposition 3.2 \( R[x; \sigma, \delta] \) is a left Ore domain and \( Q_l(R[x; \sigma, \delta]) \cong Q_l(S^{-1}(R[x; \sigma, \delta])) \cong Q_l(Q_l(R)[x; \sigma, \delta]) \). This proves (3.1).

For the second statement note that if \( R \) is a right Ore domain, then the right skew polynomial ring is a right Ore domain. Therefore, Proposition 2.4 guarantees that if \( R \) is a right Ore domain, then \( R[x; \sigma, \delta] \) is a right Ore domain, and from (2.2) of Lemma 2.3 we get \( Q_d(R[x; \sigma, \delta]) \cong Q_d(Q_d(R)[x; \sigma, \delta]). \)

**Corollary 3.4.** Let \( R \) be a left Ore domain and \( A := R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n] \), with \( \sigma_i \) injective for every \( 1 \leq i \leq n \). Then, \( A \) is a left Ore domain and

\[ Q_l(A) \cong Q_l(Q_l(R)[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]), \]

If \( R \) is a right Ore domain and \( \sigma_i \) is bijective for every \( 1 \leq i \leq n \), then \( A \) is a right Ore domain and

\[ Q_d(A) \cong Q_d(Q_d(R)[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]). \]

**Proof.** The result follows from Proposition 3.3 by iteration.

**Theorem 3.5** (Ore’s theorem: quasi-commutative case). Let \( R \) be a left Ore domain and \( A := R[x_1; \sigma_1] \cdots [x_n; \sigma_n] \) be a quasi-commutative skew PBW extension of \( R \). Then \( A \) is a left Ore domain, and hence, \( A \) has left total division ring of fractions such that

\[ Q_l(A) \cong Q_l(Q_l(R)[x_1; \sigma_1] \cdots [x_n; \sigma_n]). \]

If \( R \) is a right Ore domain and \( \sigma_i \) is bijective for every \( 1 \leq i \leq n \), then \( A \) is a right Ore domain and

\[ Q_d(A) \cong Q_d(Q_d(R)[x_1; \sigma] \cdots [x_n; \sigma]). \]

**Proof.** This follows from Corollary 3.4 since for any skew PBW extension the endomorphisms \( \sigma \)'s are always injective, see Proposition 1.2.

Now we consider the previous theorem for bijective extensions, extending this way Proposition 3.1 to left (right) Ore domains.
Theorem 3.6 (Ore’s theorem: bijective case). Let \( A = \sigma(R)[x_1, \ldots, x_n] \) be a bijective skew PBW extension of a left Ore domain \( R \). Then \( A \) is also a left Ore domain, and hence, \( A \) has left total division ring of fractions such that

\[
Q_l(A) \cong Q_l(\sigma(Q_l(R))(x'_1, \ldots, x'_n)).
\]

If \( R \) is a right Ore domain, then \( A \) is also a right Ore domain, and hence, \( A \) has right total division ring of fractions such that

\[
Q_r(A) \cong Q_r(\sigma(Q_r(R))(x''_1, \ldots, x''_n)).
\]

**Proof.** With \( S := R - \{0\} \) in Lemma 2.6, \( S^{-1}A = \sigma(Q_l(R))(x'_1, \ldots, x'_n) \) is a left Ore domain. In fact, we have that \( Q_l(R) \) is a division ring, so from Theorem 1.10 and Proposition 1.7 we obtain that \( \sigma(Q_l(R))(x'_1, \ldots, x'_n) \) is a left Noetherian domain, and hence, a left Ore domain. From Proposition 3.2 we get that \( A \) is a left Ore domain and \( Q_l(A) \cong Q_l(S^{-1}A) \cong Q_l(\sigma(Q_l(R))(x'_1, \ldots, x'_n)) \). The proof for the right side is analogous. \( \square \)

4 Goldie’s theorem

Now we pass to study the second classical theorem that we want to prove for the skew PBW extensions. Goldie’s theorem says that a ring \( B \) has semisimple left (right) total rings of fractions if and only if \( B \) is semiprime and left (right) Goldie. In particular, \( B \) has simple left (right) Artinian left (right) total ring of fractions if and only if \( B \) is prime and left (right) Goldie (see [9]). In this section we study this result for skew PBW extensions.

The first remark for this problem is the following proposition.

**Proposition 4.1.** Let \( R \) be a prime left (right) Noetherian ring and let \( A \) be a bijective skew PBW extension of \( R \). Then \( A \) has left (right) total ring of fractions \( Q_l(A) \) which is simple and left (right) Artinian.

**Proof.** By Theorem 1.10 we know that \( A \) is left (right) Noetherian and hence left (right) Goldie. Now, observe that \( A \) is also a prime ring. In fact, it is well known that an skew polynomial ring of automorphism type over a prime ring is prime ([14], Theorem 1.2.9.), hence, from Theorems 1.8 and 1.9 we conclude that \( Gr(A) \) is a prime ring, whence, \( A \) is prime (see [14], Proposition 1.6.6). The assertion of the proposition follows from Goldie’s theorem. \( \square \)

Next we want to extend the previous proposition to the case when the ring \( R \) of coefficients is semiprime and left (right) Goldie. We will consider separately the quasi-commutative and bijective cases. We start recalling the following recent result that motivated us to investigate Goldie’s theorem for skew PBW extensions.

**Proposition 4.2.** Let \( R \) be a semiprime left Goldie ring and let \( \sigma \) be injective. Then, \( R[x; \sigma, \delta] \) is semiprime left Goldie, and hence, \( Q_l(R[x; \sigma, \delta]) \) exists and it is semisimple. If \( R \) is right Goldie and \( \sigma \) is bijective, then \( R[x; \sigma, \delta] \) is semiprime right Goldie, and hence, \( Q_r(R[x; \sigma, \delta]) \) exists and it is semisimple.

**Proof.** See [11], Theorem 3.8. For the second part we use also Proposition 2.1 \( \square \)

**Corollary 4.3.** Let \( R \) be a semiprime left Goldie ring and \( \sigma_i \) injective for every \( 1 \leq i \leq n \). Then, \( A := R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n] \) is semiprime left Goldie, and hence, \( Q_l(A) \) exists and it is semisimple. If \( R \) is right Goldie and every \( \sigma_i \) is bijective, then \( A \) is semiprime right Goldie, and hence, \( Q_r(A) \) exists and it is semisimple.

**Proof.** Direct consequence of the previous proposition by iteration. \( \square \)
Theorem 4.4 (Goldie's theorem: quasi-commutative case). Let $R$ be a semiprime left Goldie ring and $A$ a quasi-commutative skew PBW extension of $R$. Then, $A$ is semiprime left Goldie, and hence, $Q_l(A)$ exists and it is semisimple. If $R$ is right Goldie and every $\sigma_i$ is bijective, then $A$ is semiprime right Goldie, and hence, $Q_r(A)$ exists and it is semisimple.

Proof. This follows from Theorem 1.8 and the previous corollary.

Next we consider Goldie’s theorem for bijective extensions. Some preliminaries are needed. Recall that an element $x$ of a ring $B$ is left regular if $rx = 0$ implies that $r = 0$ for $r \in B$. We start considering rings for which the set of left regular elements coincides with the set of regular elements. One remarkable example of this class of rings are the semiprime left Goldie rings (see [14], Proposition 2.3.4). Similar statements are true for the right side.

Proposition 4.5. Let $B$ be a ring and $S \subseteq S_0(B)$ a multiplicative system of $B$ such that $S^{-1}B$ exists. Suppose that any left regular element of $S^{-1}B$ is regular, then the same holds for $B$. The right side version of the proposition is also true.

Proof. Let $a \in B$ be a left regular element, and let $b \in B$ such that $ab = 0$. Then $\frac{a}{b}$ is a left regular element of $S^{-1}B$. In fact, if $\frac{a}{b} \notin S$, then $\frac{ab}{b} = 0$, i.e., $\frac{a}{b} = 0$, but since $S$ has not zero divisors, then $ca = 0$. This implies that $c = 0$, i.e., $\frac{a}{u} = 0$. Now, from $ab = 0$ we get $\frac{a}{b} = 0$, and by the hypothesis, $\frac{b}{1} = 0$, i.e., $b = 0$.

Proposition 4.6. Let $B$ be a ring such that any left regular element is regular. Let $S \subseteq S_0(B)$ a multiplicative system of $B$ such that $S^{-1}B$ exists. Then,

(i) $Q_l(B)$ exists if and only if $Q_l(S^{-1}B)$ exists. In such case, $Q_l(B) \cong Q_l(S^{-1}B)$.

(ii) $B$ is semiprime left Goldie if and only if $S^{-1}B$ is semiprime left Goldie.

The right side version of the proposition holds.

Proof. (i) $\Rightarrow$): Let $\frac{a}{S} \in S^{-1}B$ and $\frac{b}{l} \in S_0(S^{-1}B)$. Note that $b \in S_0(B)$. In fact, if $bc = 0$ for some $c \in B$, then $\frac{b}{S} = 0$ and hence $\frac{a}{S} = 0$. Since $S$ has not zero divisors, then $c = 0$. On the other hand, if $db = 0$ for some $d \in B$, then $\frac{db}{S} = 0$, i.e., $\frac{a}{S} = 0$. Now, if $ab = 0$, then $\frac{a}{S} \in S^{-1}B$ such that $\frac{a}{S} = 0$, and hence, $d = 0$.

By the hypothesis, there exist $z \in S_0(B)$ and $z' \in S$ such that $za = z'b$. From this we obtain $\frac{z}{S} = \frac{z'}{l}$, but observe that $zS \subseteq S_0(B)$ and hence, $\frac{z}{S} \in S_0(S^{-1}B)$. In fact, we will show that if $u \in S_0(B)$, then $\frac{u}{S} \in S_0(S^{-1}B)$. Let $\frac{u}{S} \in S^{-1}B$ such that $\frac{u}{S} = 0$, then $\frac{u}{S} = 0$, so $\frac{u}{S} = 0$ and hence $p = 0$, i.e., $\frac{u}{S} = 0$. Now, let $\frac{u}{S} \in S^{-1}B$ such that $\frac{u}{S} = 0$. There exist $v \in S$ and $x \in B$ such that $vu = xv$ and $\frac{u}{S} = 0$, i.e., $xq = 0$. Note that $x$ is left regular since $vu$ is regular, then by the hypothesis $x$ is regular, and hence, $q = 0$, i.e., $\frac{u}{S} = 0$.

This proves that $Q_l(S^{-1}B)$ exists.

$\Leftarrow$): Let $a \in B$ and $u \in S_0(B)$, then $\frac{a}{S}, \frac{u}{S} \in S^{-1}B$ and, as above, $\frac{u}{S} \in S_0(S^{-1}B)$. By the hypothesis, there exist $\frac{x}{S}, \frac{z'}{l} \in S^{-1}B$ with $\frac{x}{S} \in S_0(S^{-1}B)$ such that $\frac{x}{S} \frac{a}{S} = \frac{z'}{l}$, i.e., $\frac{xa}{S} = \frac{z'}{l}$, so there exist $c, d \in B$ such that $cz/da = dzu$ and $cs' = ds \in S$. In order to complete the proof of the left Ore condition we have to show that $cz' \in S_0(B)$. If $xzc' = 0$ for some $c \in B$, then $xz \frac{c}{S} = 0$, i.e., $xzS \frac{c}{S} = 0$, so $xzS \frac{a}{S} = 0$, and hence $xzS \frac{a}{S} = 0$. This means $xcS' = 0$, so $x = 0$. Now, if $cz'p = 0$ for some $p \in B$, then $cz'p \frac{1}{S} \frac{1}{S} = 0 = cz'p \frac{1}{S}$, but since $cz' \in S$ we get that $\frac{cz'}{S} \in S_0(S^{-1}B)$, and hence $\frac{cz'}{S} \frac{p}{S} = 0$, and from this we obtain $\frac{p}{S} = 0$, i.e., $p = 0$. This proves that $Q_l(B)$ exists.
The function \( \varphi : S^{-1}B \to Q_l(B) \)
\[
\frac{b}{s} \mapsto \frac{b}{s}
\]
verify the four conditions that define a left total ring of fractions, i.e., (a) \( \varphi \) is a ring homomorphism. (b) \( S_0(S^{-1}B) \subseteq Q_l(B) \): in fact, let \( \frac{b}{s} \in S_0(S^{-1}B) \), then as we observed at the beginning of the proof, \( b \in S_0(B) \), and hence, \( \varphi\left(\frac{b}{s}\right) = \frac{b}{s} \) is invertible in \( Q_l(B) \) with inverse \( \frac{s}{b} \). (c) \( \frac{b}{s} \in \ker(\varphi) \) if and only if \( \frac{b \cdot s}{b} = 0 \) with \( \frac{b}{s} \in S_0(S^{-1}B) \): in fact, if \( \frac{b}{s} \in \ker(\varphi) \), then there exist \( c, d \in B \) such that \( cb = 0 \) and \( cs = d \), with \( d \in S_0(B) \), but this means that \( \frac{d}{s} \frac{b}{1} = 0 \), with \( \frac{d}{s} \in S_0(S^{-1}B) \). The converse is trivial. (d) each element \( \frac{b}{s} \) of \( Q_l(B) \) can be written as \( \frac{b}{s} = \varphi(\frac{d}{u})^{-1} \varphi(\frac{e}{v}) \).

(ii) This numeral is a direct consequence of (i) and the Goldie’s theorem.

**Proposition 4.7.** Let \( B \) be a positive filtered ring. If \( Gr(B) \) is semiprime, then \( B \) is semiprime.

**Proof.** Let \( I \) be a two-sided ideal of \( B \) such that \( I^2 = 0 \). Then, \( Gr(I)^2 = 0 \) and hence \( Gr(I) = 0 \). This implies that \( I = 0 \).

**Theorem 4.8** (Goldie’s theorem: bijective case). Let \( R \) be a semiprime left Goldie ring and \( A = \sigma(R)[x_1, \ldots , x_n] \) a bijective skew PBW extension of \( R \). Then, \( A \) is semiprime left Goldie, and hence, \( Q_l(A) \) exists and it is semisimple. The right side version of the theorem also holds.

**Proof.** By Goldie’s theorem, \( Q_l(R) = S_0(R)^{-1}R \) exists and is semisimple. Note that for every \( 1 \leq i \leq n \), \( \sigma_i(S_0(R)) = S_0(R) \). By Lemma 2.6, \( S_0(R)^{-1}A \) exists and it is a bijective extension of \( Q_l(R) \), i.e., \( S_0(R)^{-1}A = \sigma(Q_l(R)) \langle x_1, \ldots , x_n \rangle \). Since \( Q_l(R) \) is left Noetherian, then by Theorem 1.10, \( S_0(R)^{-1}A \) is left Noetherian, i.e., left Goldie. By Theorem 1.9, \( Gr(S_0(R)^{-1}A) = Gr(\sigma(Q_l(R)) \langle x_1, \ldots , x_n \rangle) \) is a quasi-commutative (and bijective) extension of the semiprime left Goldie ring \( Q_l(R) \), so by Theorem 4.4, \( Gr(S_0(R)^{-1}A) \) is semiprime (left Goldie). Proposition 4.7 says that \( S_0(R)^{-1}A \) is semiprime. In order to apply Proposition 4.6 and conclude the proof only rest to observe that \( S_0(R) \subseteq S_0(A) \) and the left regular elements of \( A \) coincide with \( S_0(A) \). The last statement can be justified in the following way: since \( S_0(R)^{-1}A \) is semiprime left Goldie, then the left regular elements of \( S_0(R)^{-1}A \) coincide with its regular elements, so by Proposition 4.6 the same is true for \( A \).

5 The quantum version of the Gelfand-Kirillov conjecture for skew quantum polynomials

As application of the results of the previous sections, we can prove a quantum version of the Gelfand-Kirillov conjecture for the ring of skew quantum polynomials. This class of rings were defined in [13], and represent a generalization of Artamonov’s quantum polynomials (see [2, 3]). They can be defined as a quasi-commutative bijective skew PBW extension of the \( r \)-multiparameter quantum torus, or also, as a localization of a quasi-commutative bijective skew PBW extension. We recall next its definition.

Let \( R \) be a ring with a fixed matrix of parameters \( \mathbf{q} := [q_{ij}] \in M_n(R) \), \( n \geq 2 \), such that \( q_{ii} = 1 = q_{ij}q_{ji} = q_{ji}q_{ij} \) for every \( 1 \leq i, j \leq n \), and suppose also that it is given a system \( \sigma_1, \ldots , \sigma_n \) of automorphisms of \( R \). The ring of skew quantum polynomials over \( R \), denoted by \( R_{\mathbf{q}, \sigma}[x_1^{\pm 1}, \ldots , x_r^{\pm 1}, x_{r+1}, \ldots , x_n] \), is defined as follows:

(i) \( R \subseteq R_{\mathbf{q}, \sigma}[x_1^{\pm 1}, \ldots , x_r^{\pm 1}, x_{r+1}, \ldots , x_n] \);

(ii) \( R_{\mathbf{q}, \sigma}[x_1^{\pm 1}, \ldots , x_r^{\pm 1}, x_{r+1}, \ldots , x_n] \) is a free left \( R \)-module with basis

\[
\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} | \alpha_i \in \mathbb{Z} \text{ for } 1 \leq i \leq r \text{ and } \alpha_i \in \mathbb{N} \text{ for } r+1 \leq i \leq n\};
\]
(iii) the variables $x_1, \ldots, x_n$ satisfy the defining relations

$$
x_i x_i^{-1} = 1 = x_i^{-1} x_i, \quad 1 \leq i \leq r,
$$

$$
x_j x_i = q_{ij} x_i x_j, \quad x_i r = \sigma_i(r) x_i, \quad r \in R, \quad 1 \leq i, j \leq n.
$$

When all automorphisms are trivial, we write $R_{q,\sigma}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n]$, and this ring is called the ring of quantum polynomials over $R$. If $R = \mathbb{k}$ is a field, then $\mathbb{k}_{q,\sigma}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n]$ is the algebra of skew quantum polynomials. For trivial automorphisms we get the algebra of quantum polynomials simply denoted by $\mathcal{O}_q$ (see [2]). When $r = 0$, $R_{q,\sigma}[x_1^{\pm 1}, \ldots, x_{r+1}, \ldots, x_n] = R_{q,\sigma}[x_1, \ldots, x_n]$ is the $n$-multiparametric skew quantum space over $R$, and when $r = n$, it coincides with $R_{q,\sigma}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, i.e., with the $n$-multiparametric skew quantum torus over $R$.

Note that $R_{q,\sigma}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n]$ can be viewed as a localization of the $n$-multiparametric skew quantum space, which, in turn, is an skew PBW extension. In fact, we have the quasi-commutative bijective skew PBW extension

$$
A := \sigma(R)(x_1, \ldots, x_n), \quad \text{with } x_i r = \sigma_i(r) x_i \text{ and } x_j x_i = q_{ij} x_i x_j, \quad 1 \leq i, j \leq n;
$$

observe that $A = R_{q,\sigma}[x_1, \ldots, x_n]$. If we set

$$
S := \{rx^\alpha \mid r \in R^*, x^\alpha \in \text{Mon}\{x_1, \ldots, x_r\}\},
$$

then $S$ is a multiplicative subset of $A$ and

$$
S^{-1} A \cong R_{q,\sigma}[x_1^{\pm 1}, \ldots, x_{r+1}^{\pm 1}, x_{r+2}, \ldots, x_n] \cong A S^{-1}.
$$

Before presenting our next result, let us first recall the classical Gelfand-Kirillov conjecture and some well known cases, classical and quantum, where the conjecture have positive answer. We start with the classical formulation.

(i) (Gelfand-Kirillov conjecture, [8]) Let $\mathcal{G}$ be an algebraic Lie algebra of finite dimension over a field $L$, with $\text{char}(L) = 0$. Then, there exist integers $n, k \geq 1$ such that

$$
Q(\mathcal{U}(\mathcal{G})) \cong Q(A_n(L[s_1, \ldots, s_k])),
$$

$\mathcal{U}(\mathcal{G})$ is the enveloping algebra of the Lie algebra $\mathcal{G}$ and $A_n(L[s_1, \ldots, s_k])$ is the Weyl algebra over the polynomial ring $L[s_1, \ldots, s_k]$.

(ii) (8, Lemma 7) Let $\mathcal{G}$ be the algebra of all $n \times n$ matrices over a field $L$, i.e., $\mathcal{G} = M_n(L)$, with $\text{char}(L) = 0$. Then, $\mathcal{G}$ is algebraic and (5.4) holds. The same is true if $\mathcal{G}$ is the algebra of matrices of null trace.

(iii) (8, Lemma 8) Let $\mathcal{G}$ be a finite dimensional nilpotent Lie algebra over a field $L$, with $\text{char}(L) = 0$. Then, $\mathcal{G}$ is algebraic and (5.3) holds.

(iv) ([10], Theorem 3.2) Let $\mathcal{G}$ be a finite dimensional solvable algebraic Lie algebra over the field $C$ of complex numbers. Then, $\mathcal{G}$ satisfies the conjecture (5.4).

Now we review some well known results about the analog quantum version of the Gelfand-Kirillov conjecture, where the Weyl algebra $A_n(L[s_1, \ldots, s_k])$ in (5.4) is replaced by a suitable $n$-multiparametric quantum space. $Z(B)$ will represent the center of the ring $B$.

(vi) ([11], Theorem 2.15) Let $U^+_m(sl_m)$ be the quantum enveloping algebra of the Lie algebra of strictly superior triangular matrices of size $m \times m$ over a field $L$. 

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Proposition 3.2 and Theorem 3.5 (or also using Theorem 3.6), if $Q$ is a domain, then 

\[ Q(U^+_q(sl_m)) \cong Q(K_q[x_1, \ldots, x_{2n^2}]), \]

where $K := Q(Z(U^+_q(sl_m)))$ and $q := [q_{ij}] \in M_{2n^2}(L)$, with $q_{ii} = 1 = q_{ij}q_{ji}$ for every $1 \leq i, j \leq 2n^2$.

(b) If $m = 2n$, then 

\[ Q(U^+_q(sl_m)) \cong Q(K_q[x_1, \ldots, x_{2(n-1)}]), \]

where $K := Q(Z(U^+_q(sl_m)))$ and $q := [q_{ij}] \in M_{2(n-1)}(L)$, with $q_{ii} = 1 = q_{ij}q_{ji}$ for every $1 \leq i, j \leq 2n(n-1)$.

(vii) (Main Theorem) Let $B$ be a pure $q$-solvable $C$-algebra. Then, $Q(B) \cong Q(Gr(B))$ and $Gr(B) \cong C_n[x_1, \ldots, x_n]$, where $C$ is a Noetherian commutative domain.

Let $L$ be a field and $B := L[x_1,x_2,\ldots,x_n]$ an iterated skew polynomial ring with some extra adequate conditions on $\sigma$'s and $\delta$'s. Then, there exits $q := [g_{i,j}] \in M_n(L)$ with $q_{ii} = 1 = q_{ij}q_{ji}$, for every $1 \leq i, j \leq n$, such that $Q(B) \cong Q(L_q[x_1, \ldots, x_n])$.

With the previous antecedents, our next result can be better understood.

**Corollary 5.1** (Gelfand-Kirillov conjecture for skew quantum polynomials). Let $R$ be a left (right) Ore domain. Then, 

\[ Q(R_q,\sigma[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n]) \cong Q(R_q,\sigma[x_1, \ldots, x_n]), \]

where $Q := Q(R)$.

**Proof.** In order to simplify the notation we write $Q_{a,n}^\ell(R) := R_q,\sigma[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n]$. If $R$ is a domain, then $Q_{a,n}^{\ell}(R)$ is also a domain (see Proposition 4.1, Proposition 4.2, and Proposition 3.2). Thus, from Proposition 3.2 and Theorem 3.3 (or also using Theorem 3.6), if $R$ is a left (right) Ore domain, then $Q_{a,n}^{\ell}(R)$ is a left (right) Ore domain, and hence $Q_{a,n}^{\ell}(R)$ has left (right) total division ring of fractions, $Q(Q_{a,n}^{\ell}(R)) \cong Q(A)$, with $A$ as in (3.2). Therefore, with the notation of the previous sections, we have 

\[ Q(Q_{a,n}^{\ell}(R)) \cong Q(A) \cong Q(\sigma(Q(R))(x_1^{'}, \ldots, x_n^{'}) \cong Q(Q_{a,n}^{\ell}(R)[x_1, \ldots, x_n]), \]

where $Q := Q(R)$ and we identify $x_i^{'} = \frac{x_i}{q} := x_i$ and $\sigma_i^{'} := \sigma_i$, $1 \leq i \leq n$. Thus, we have proved that the left (right) total rings of fractions of $Q_{a,n}^{\ell}(R)$ is the left (right) total ring of fractions of the $n$-multiparametric skew quantum space over $Q(R)$. \qed

As another application of the results of the previous sections, we conclude the paper with the Goldie's theorem for the skew quantum polynomials.

**Corollary 5.2.** Let $R$ be a semiprime left (right) Goldie ring, then $R_q,\sigma[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n]$ is also a semiprime left (right) Goldie ring.

**Proof.** From Theorem 4.1 (we can use also Theorem 4.2) we get that $A$ in (4.2) is a semiprime left (right) Goldie ring. In addition, note that the set $S$ in (5.3) satisfies the hypothesis of Proposition 4.6 in fact, since $A$ is semiprime left (right) Goldie, any left (right) regular element is regular; $S \subseteq S_0(A)$ since if $rx^a \in S$ and $p = c_1x^{\beta_1} + \cdots + c_lx^{\beta_l} \in A$ are such that $rx^{a}p = 0 = prx^{a}$, then $p = 0$ since $r$ and the constants $c_{\alpha, \beta}$ are invertible (Remark 1.6). Proposition 4.6 says that $R_q,\sigma[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n]$ is semiprime left (right) Goldie. \qed
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