DESCRIPTIVE CHARACTERIZATIONS OF THE INTEGRAL BY SEMINORMS

SOKOL KALIAJ, ZENEPE SHKOZA

Abstract. In this paper, we first define the concept of the limit average range of a function defined on [0, 1] and taking values in a Hausdorff locally convex topological vector space (locally convex space) X. Then, we present characterizations of the primitive \( F : [0, 1] \to X \) of an integrable by seminorms function \( f : [0, 1] \to X \) in terms of the limit average range of \( F \). It is shown that the limit average range characterizes the integral by seminorms better then the usual differential.

1. Introduction and Preliminaries

The integral by seminorms is an extension of Bochner integral to locally convex spaces. The following theorem gives necessary and sufficient conditions for a Banach space valued function \( F : [0, 1] \to X \) to be the primitive of a Bochner integrable function \( f : [0, 1] \to X \) in terms of the derivative of \( F \), c.f. Theorem 7.4.15 in [7].

**Theorem 1.1.** Let \( X \) be a Banach space and let \( f, F : [0; 1] \to X \) be functions. Then, the following are equivalent:

(i) \( f \) is Bochner integrable on \([0, 1] \) with the primitive \( F \), i.e.,

\[
\tilde{F}(I) = (B) \int_I f(t)d\lambda(t), \quad \text{for all } I \in \mathcal{I},
\]

where \( \mathcal{I} \) is the family of all non-degenerate closed subintervals of \([0, 1] \), \( \lambda \) is the Lebesgue measure on \([0, 1] \) and \( \tilde{F}([u, v]) = F(v) - F(u) \), for all \([u, v] \in \mathcal{I} \),

(ii) \( F \) is strongly absolutely continuous, \( F'(t) \) exists and \( F'(t) = f(t) \), at almost all \( t \in [0, 1] \).

In this paper, functions defined on \([0, 1] \) and taking values in a locally convex space \( X \) are considered. At first, the concept of the limit average range of a function \( F : [0, 1] \to X \) at a point \( t \in [0, 1] \) is defined. Then, we present characterizations of the primitive \( F : [0, 1] \to X \) of an integrable by seminorms function \( f : [0, 1] \to X \) in terms of the limit average range of \( F \), Theorem 2.2 and Theorem 2.3. We give an example of a function \( F : [0, 1] \to X \) which is the primitive of an integrable by seminorms function \( f : [0, 1] \to X \) and it is shown that \( F \) has the limit average range at all \( t \in (0, 1) \), but \( F \) is not differentiable at any \( t \in (0, 1) \), Example 2.4.

Let \( X \) be a locally convex space with the topology \( \tau \) and let \( \mathcal{P} \) be the family of all continuous seminorms on \( X \). For any \( p \in \mathcal{P} \), we denote by \( \tilde{X}_p \) the quotient vector space \( X/p^{-1}(0) \), by \( \phi_p : X \to \tilde{X}_p \) the canonical quotient map, by \((\tilde{X}_p, \tilde{p})\)
the quotient normed space and by \((\overline{X}_p, \overline{\| \cdot \|})\) the completion of \((\widetilde{X}_p, \widetilde{\| \cdot \|})\). For every \(p, q \in \mathcal{P}\) such that \(p \leq q\), we denote by \(\overline{g}_{pq} : \overline{X}_q \to \overline{X}_p\) the map defined as follows
\[
\overline{g}_{pq}(w_q) = u_p, \quad \text{for each } w_q \in \overline{X}_q,
\]
where \(u_p = \phi_p(x)\), for some vector \(x \in w_q\). By \(\overline{g}_{pq}\) the continuous linear extension of \(\overline{g}_{pq}\) to \(\overline{X}_q\) is denoted.

We now denote by
\[
\lim_{\leftarrow} \overline{g}_{pq} \overline{X}_q \quad (\lim_{\leftarrow} \overline{g}_{pq} \overline{X}_q)
\]
the projective limit of the family
\[
\{(\overline{X}_p, \overline{\| \cdot \|}) : p \in \mathcal{P}\} \quad (\{(\widetilde{X}_p, \widetilde{\| \cdot \|}) : p \in \mathcal{P}\})
\]
with respect to the family
\[
\{
\overline{g}_{pq} : p, q \in \mathcal{P}, p \leq q\} \quad (\{\overline{g}_{pq} : p, q \in \mathcal{P}, p \leq q\})
\]
c.f [6], p.52. By virtue of II.5.4 in [6], p.53, if \(X\) is a complete locally convex space, then we have
\[
(1.1) \quad X \equiv \lim_{\leftarrow} \overline{g}_{pq} \overline{X}_q \quad \text{and} \quad \lim_{\leftarrow} \overline{g}_{pq} \overline{X}_q = \lim_{\leftarrow} \overline{g}_{pq} \overline{X}_q,
\]
where the symbol ‘\(\equiv\)’ shows that the two spaces above are isomorphic.

Assume that a function \(F : [0, 1] \to X\) and a point \(t \in [0, 1]\) are given. We put
\[
\Delta F(t, h) = \frac{F(t + h) - F(t)}{h}, \quad (h \neq 0) \quad A_F(t, \delta) = \{\Delta F(t, h) : 0 < |h| < \delta\}
\]
and
\[
A^{(p)}_F(t) = \bigcap_{\delta > 0} \overline{A}^{(p)}_F(t, \delta) \quad A_F(t) = \bigcap_{p \in \mathcal{P}} A^{(p)}_F(t)
\]
where \(\overline{A}^{(p)}_F(t, \delta)\) is the closure of \(A_F(t, \delta)\) with respect to the seminorm \(p \in \mathcal{P}\). The set \(A_F(t)\) is said to be the average range of \(F\) at \(t\). Since
\[
\overline{A}_F(t, \delta) = \bigcap_{p \in \mathcal{P}} \overline{A}^{(p)}_F(t, \delta)
\]
we have also
\[
A_F(t) = \bigcap_{\delta > 0} \overline{A}_F(t, \delta).
\]
Define
\[
p\text{-diam}(W) = \sup\{p(x - y) : x, y \in W\} \quad (W \subset X, W \neq \emptyset),
\]
and \(p\text{-diam}(\emptyset) = 0\). We say that \(F\) has the limit average range at \(t \in [0, 1]\), if \(A_F(t)\) is a bounded set, and for each \(\varepsilon > 0\) and \(p \in \mathcal{P}\) there exists \(\delta_{\varepsilon, p} > 0\) such that
\[
p\text{-diam}(A_F(t, \delta_{\varepsilon, p})) < p\text{-diam}(A_F(t)) + \varepsilon.
\]
The function \(F\) is said to be differentiable at the point \(t\), if there exists a vector \(x \in X\) such that for each \(p \in \mathcal{P}\), we have
\[
\lim_{h \to 0} p(\Delta F(t, h) - x) = 0.
\]
By \(x = F'(t)\) the derivative of \(F\) at \(t\) is denoted.

We now recall the concept of the integral by seminorms, Definition 2.4 in [1].
Definition 1.2. A function $f : [0, 1] \to X$ is said to be integrable by seminorms if, for any $p \in \mathcal{P}$, there exists a sequence $(f_k^{(p)})$ of measurable simple functions and a subset $Z_p \subset [0, 1]$ with $\lambda(Z_p) = 0$, such that

(i) for all $t \in [0, 1] \setminus Z_p$, we have
\[
\lim_{k \to \infty} p\left(f_k^{(p)}(t) - f(t)\right) = 0,
\]
(ii) each function $p \circ (f_k^{(p)} - f)$ is Lebesgue integrable on $[0, 1]$ and
\[
\lim_{k \to \infty} \int_{[0,1]} p\left(f_k^{(p)}(t) - f(t)\right) d\lambda(t) = 0,
\]
(iii) for each $E \in \mathcal{L}$ there exists a vector $x_E \in X$ such that
\[
\lim_{k \to \infty} p\left(\int_E f_k^{(p)}(t) d\lambda(t) - x_E\right) = 0,
\]
where $\mathcal{L}$ is the family of all Lebesgue measurable subsets of $[0, 1]$.

The vector $x_E$ is said to be the integral by seminorms of $f$ over $E$, and we set
\[
x_E = \int_E f(t) d\lambda(t).
\]

This notion coincides with the Bochner integral in a Banach space. For more information about the integral by seminorms and the Bochner integral, we refer to [1], [2], [3], [5] and [7].

A function $F : [0, 1] \to X$ is said to be strongly absolutely continuous (sAC), if for each $p \in \mathcal{P}$, we have that $p \circ F$ is sAC, i.e., given $\varepsilon > 0$ there exists $\eta_{\varepsilon p} > 0$ such that for each finite collection $\{I_1, \ldots, I_n\}$ of pairwise nonoverlapping intervals in $\mathcal{I}$, we have
\[
\sum_{j=1}^n \lambda(I_j) < \eta_{\varepsilon p} \Rightarrow \sum_{j=1}^n p(F(I_j)) < \varepsilon.
\]

2. The Main Results

The main results are Theorem 2.2 and Theorem 2.3. Using Pettis’s Measurability Theorem, Theorem II.1.2 in [2], and Severini-Egorov-Theorem, Theorem III.6.12 in [3], it can be proved the following auxiliary lemma.

Lemma 2.1. Let $X$ be a Banach space with the norm $|| \cdot ||$ and let $f : [0, 1] \to X$ be a function. If $f$ is Bochner integrable on $[0, 1]$, then there exists a sequence $(f_k)$ of measurable simple functions such that
\[
f_k([0, 1]) \subset f([0, 1]), \quad \text{for each} \quad k \in \mathbb{N},
\]
and
\[
\lim_{k \to \infty} f_k(t) = f(t), \quad \text{at almost all} \quad t \in [0, 1].
\]
Moreover, $(f_k)$ converges to $f$ in $\lambda$-measure, i.e.,
\[
\lim_{k \to \infty} \lambda(\{t \in [0, 1] : ||f_k(t) - F(t)|| \geq \eta\}) = 0, \quad \text{for every} \quad \eta > 0.
\]

We now present a descriptive characterization of the integral by seminorms of a function taking values in a complete locally convex space.
Theorem 2.2. Let \( X \) be a complete locally convex space and let \( F : [0, 1] \to X \) be a function. If \( F \) is \( sAC \) and has the limit average range at almost all \( t \in [0, 1] \), then \( F \) is the primitive of an integrable by seminorms function \( f \), i.e.,

\[
\tilde{F}(I) = \int_I f(t) d\lambda(t), \quad \text{for all} \quad I \in \mathcal{I}.
\]

\[\text{(2.3)}\]

Proof. By hypothesis, there exists a subset \( Z \subset [0, 1] \) with \( \lambda(Z) = 0 \) such that \( F \) has the limit average range at all \( t \in [0, 1] \setminus Z \).

Claim 1. For any \( p \in \mathcal{P} \), the function \( \phi_p \circ F \) has the limit average range at all \( t \in [0, 1] \setminus Z \). To see this, assume that an arbitrary \( t \in [0, 1] \setminus Z \) and \( \varepsilon > 0 \) are given. Then, since \( F \) has the limit average range at \( t \), there exists \( \delta_{\varepsilon, p} > 0 \) such that

\[
p - \text{diam}(A_F(t, \delta_{\varepsilon, p})) < p - \text{diam}(A_F(t)) + \varepsilon.
\]

\[\text{(2.4)}\]

It is easy to see that

\[
x \in \overline{A_F^{(p)}(t, \delta)} \iff \phi_p(x) \in \overline{A_{\phi_p \circ F}^{(p)}(t, \delta)}
\]

and

\[
\overline{A_{\phi_p \circ F}^{(p)}(t, \delta)} \subset \overline{A_{\phi_p \circ F}^{(p)}(t, \delta)}
\]

and since

\[
A_{\phi_p \circ F}(t) = \bigcap_{\delta > 0} \overline{A_{\phi_p \circ F}^{(p)}(t, \delta)}
\]

it follows that

\[
\phi_p(A_F(t)) \subset \phi_p\left( \bigcap_{\delta > 0} \overline{A_F^{(p)}(t, \delta)} \right) = \bigcap_{\delta > 0} \overline{A_{\phi_p \circ F}(t, \delta)} = A_{\phi_p \circ F}(t).
\]

Hence

\[\text{(2.5)}\]

\[
p - \text{diam}(A_F(t)) = \overline{p} - \text{diam}[\phi_p(A_F(t))] \leq \overline{p} - \text{diam}(A_{\phi_p \circ F}(t)).
\]

The equality

\[
p - \text{diam}(A_F(t, \delta)) = \overline{p} - \text{diam}(A_{\phi_p \circ F}(t, \delta))
\]

together with \([2.4]\) and \([2.5]\) yields

\[
\overline{p} - \text{diam}(A_{\phi_p \circ F}(t, \delta_{\varepsilon, p})) < \overline{p} - \text{diam}(A_{\phi_p \circ F}(t)) + \varepsilon.
\]

This means that \( \phi_p \circ F \) has the limit average range at \( t \), and since \( t \) is arbitrary, \( \phi_p \circ F \) has the limit average range at all \( t \in [0, 1] \setminus Z \).

Let us now show that

\[
A_F(t) \neq \emptyset, \quad \text{at all} \quad t \in [0, 1] \setminus Z.
\]

Fix an arbitrary \( t \in [0, 1] \setminus Z \) and suppose that

\[\text{(2.6)}\]

\[
A_F(t) = \bigcap_{p \in \mathcal{P}} A_F^{(p)}(t) = \emptyset.
\]

Then, for any \( p \in \mathcal{P} \), by the definition of the limit average range, there is a decreasing sequence \( (\delta_n^p) \) of real numbers such that for each \( n \in \mathbb{N} \), we have

\[\text{(2.7)}\]

\[
0 < \delta_n^p < \frac{1}{n} \quad \text{and} \quad p - \text{diam}(A_F(t, \delta_n^p)) < p - \text{diam}(A_F(t)) + \frac{1}{n} = \frac{1}{n}.
\]
Define a sequence \((\Delta F(t, h^m_n))_n\) by choosing a vector \(\Delta F(t, h^m_n) \in A_F(t, \delta^m_n)\), for each \(n \in \mathbb{N}\). Note that
\[
P(\Delta F(t, h^m_n) - \Delta F(t, h^m_n)) < \frac{1}{n},
\]
whenever \(m > n\). This means that \((\Delta(\phi_p \circ F)(t, h^m_n))_n\) is a Cauchy sequence. Therefore, there exists \(\overline{w}_p \in X_p\) such that
\[
\lim_{n \to \infty} p[\Delta(\phi_p \circ F)(t, h^m_n) - \overline{w}_p] = 0.
\]

Claim 2. For each \(p \in \mathcal{P}\), we have that the equality
\[
A_{\phi_p \circ F}(t) = \{\overline{w}_p\}
\]
holds.

To see this, fix an arbitrary \(p \in \mathcal{P}\). Since
\[
A_{\phi_p \circ F}(t) \subset A_{\phi_p \circ F}(t, \delta^m_n), \text{ for all } n \in \mathbb{N},
\]
and
\[
\overline{p} - \text{diam}(A_{\phi_p \circ F}(t, \delta^m_n)) = \overline{p} - \text{diam}(A_{\phi_p \circ F}(t, \delta^m_n))
\]
\[
= \overline{p} - \text{diam}(A_{\phi_p \circ F}(t, \delta^m_n))
\]
\[
= p - \text{diam}(A_F(t, \delta^m_n)),
\]
from (2.7) we obtain
\[
\overline{p} - \text{diam}(A_{\phi_p \circ F}(t)) = 0.
\]
The equality (2.8) together with
\[
A_{\phi_p \circ F}(t) = \bigcap_{n=1}^{\infty} A_{\phi_p \circ F}(t, \frac{1}{n})
\]
yields that \(\overline{w}_p \in A_{\phi_p \circ F}(t)\) and, therefore by (2.10), it follows that \(A_{\phi_p \circ F}(t) = \{\overline{w}_p\}\).

Claim 3. There exists \(w \in X\) such that
\[
\phi_p(w) = \overline{w}_p, \text{ for all } p \in \mathcal{P}.
\]
Assume that \(p, q \in \mathcal{P}\) such that \(p \leq q\) are given. Since
\[
\lim_{n \to \infty} q[\Delta(\phi_q \circ F)(t, h^m_n) - \overline{w}_q] = 0
\]
we obtain
\[
\lim_{n \to \infty} p[\Delta(\phi_q \circ F)(t, h^m_n) - \overline{w}_q] = 0
\]
\[
= \lim_{n \to \infty} p[\overline{g}_{pq}(\Delta(\phi_q \circ F)(t, h^m_n)) - \overline{g}_{pq}(\overline{w}_q)]
\]
\[
= \lim_{n \to \infty} p[\overline{g}_{pq}(\Delta(\phi_q \circ F)(t, h^m_n)) - \overline{w}_q] = 0.
\]
This yields that \(\overline{g}_{pq}(\overline{w}_q) \in A_{\phi_q \circ F}(t)\) and, therefore from (2.9), we obtain
\[
\overline{g}_{pq}(\overline{w}_q) = \overline{w}_p.
\]
Hence, by (1.1), there exists \(w \in X\) such that (2.11) holds true.

By (2.9) and (2.11), we get
\[
\bigcap_{\delta>0} A_{\phi_p \circ F}(t, \delta) = \{\phi_p(w)\}, \text{ for all } p \in \mathcal{P},
\]
and since \( \phi_p(A_F^p(t, \delta)) = A_{\phi_p \circ F}^p(t, \delta) \), we obtain
\[
w \in \bigcap_{p \in P} A_F^{(p)}(t).
\]

But, this contradicts (2.6). Consequently, \( A_F(t) \neq \emptyset \), and since \( t \) is arbitrary, the last result holds for all \( t \in [0, 1] \setminus Z \). Then, we can choose a vector \( x_t \in A_F(t) \), for each \( t \in [0, 1] \setminus Z \), and define the function \( f : [0, 1] \to X \) as follows
\[
f(t) = \begin{cases} x_t & \text{if } t \in [0, 1] \setminus Z \\ 0 & \text{if } t \in Z \end{cases}
\]

We are going to prove that \( f \) is an integrable by seminorms function satisfying (2.3).

Since \( \phi_p \circ F \) is \( sAC \) and has the limit average range at almost all \( t \in [0, 1] \), by Lemma 2.4 in [4], \( \phi_p \circ F \) is differentiable almost everywhere on \( [0, 1] \), and since
\[
(\phi_p \circ f)(t) \in A_{\phi_p \circ F}(t), \quad \text{at almost all } t \in [0, 1],
\]
by Lemma 2.1 in [4], we obtain
\[
(\phi_p \circ F)'(t) = (\phi_p \circ f)(t), \quad \text{at almost all } t \in [0, 1].
\]

Hence, by Theorem 1.1, \( \phi_p \circ f \) is Bochner integrable with the primitive \( \phi_p \circ \tilde{F} \), i.e.,
\[
(\phi_p \circ \tilde{F})(I) = (B) \int_I (\phi_p \circ f)(t)d\lambda(t), \quad \text{for all } I \in \mathcal{I}.
\]

Since \( F \) is \( sAC \), there exists a unique countable additive \( \lambda \)-continuous vector measure \( \nu : \mathcal{L} \to X \) such that \( \tilde{F}(I) = \nu(I) \), for all \( I \in \mathcal{I} \). Hence, from (2.13), we obtain
\[
(\phi_p \circ \nu)(E) = (B) \int_E (\phi_p \circ f)(t)d\lambda(t), \quad \text{for all } E \in \mathcal{L}.
\]

Since \( \phi_p \circ f \) is Bochner integrable, by Lemma 2.1 there exists a sequence \( (f_k^{(p)}) \) of measurable simple functions such that
\[
(\phi_p \circ f_k^{(p)})([0, 1]) \subset (\phi_p \circ f)([0, 1]), \quad \text{for all } k \in \mathbb{N},
\]
\[
\lim_{k \to \infty} p\left(f_k^{(p)}(t) - f(t)\right) = \lim_{k \to \infty} \tilde{p}\left((\phi_p \circ f_k^{(p)})(t) - (\phi_p \circ f)(t)\right) = 0,
\]
at almost all \( t \in [0, 1] \), and the sequence \( (\phi_P \circ f_k^{(p)}) \) converges to \( \phi_p \circ f \) in \( \lambda \)-measure.

By Theorem II.1.2 in [2], the function \( \tilde{p}((\phi_p \circ f)(\cdot)) \) is Lebesgue integrable and, therefore it is bounded almost everywhere on \( [0, 1] \). Thus, there exists \( M > 0 \) such that
\[
p(f(t)) = \tilde{p}((\phi_p \circ f)(t)) \leq M, \quad \text{at almost all } t \in [0, 1].
\]

The last result together with (2.15) yields
\[
p(f_k^{(p)}(t)) = \tilde{p}((\phi_p \circ f_k^{(p)})(t)) \leq M, \quad \text{at almost all } t \in [0, 1],
\]
for every \( k \in \mathbb{N} \). Therefore, by Dominated Convergence Theorem, Theorem II.2.3 in [2], we obtain
\[
\lim_{k \to \infty} \tilde{p}\left(\int_E (\phi_p \circ f_k^{(p)})(t)d\lambda(t) - (B) \int_E (\phi_p \circ f)(t)d\lambda(t)\right) = 0,
\]
for each \( E \in \mathcal{L} \), and
\[
(2.18) \quad \lim_{k \to \infty} \int_{[0,1]} p \left( f_k^{(p)}(t) - f(t) \right) d\lambda(t) = 0.
\]
By virtue of (2.14) and (2.17), we obtain
\[
\lim_{k \to \infty} p \left( \int_{E} f_k^{(p)}(t) d\lambda(t) - \nu(E) \right) = 0.
\]
By Definition 1.2, the last equality together with (2.16) and (2.18) yields that \( f \) is integrable by seminorms and
\[
\tilde{F}(I) = \nu(I) = \int_{I} f(t) d\lambda(t), \quad \text{for each } I \in \mathcal{I},
\]
and this ends the proof. \( \square \)

The next theorem presents full descriptive characterizations of the integral by seminorms in a Frechet space.

**Theorem 2.3.** Let \( X \) be a Frechet space and let \( f, F : [0,1] \to X \) be functions. Assume that \((p_k)_{k \in \mathbb{N}}\) is a countable family of increasing seminorms on \( X \) so that the topology of \( X \) is generated by \((p_k)_{k \in \mathbb{N}}\). Then the following are equivalent:

\( i \) \( f \) is integrable by seminorms with the primitive \( F \), i.e.,
\[
\tilde{F}(I) = \int_{I} f(t) d\lambda(t), \quad \text{for all } I \in \mathcal{I},
\]

\( ii \) \( F \) is \( sAC \), \( F \) has the limit average range at almost all \( t \in [0,1] \) and
\[
f(t) \in A_F(t), \quad \text{almost everywhere on } [0,1].
\]

**Proof.** \( (i) \Rightarrow (ii) \) Assume that \( f \) is integrable by seminorms with the primitive \( F \). Then, each function \( \phi_{p_k} \circ f \) is Bochner integrable with the primitive \( \phi_{p_k} \circ \tilde{F} \), i.e.,
\[
(\phi_{p_k} \circ \tilde{F})(I) = (B) \int_{I} f(t) d\lambda(t), \quad \text{for all } I \in \mathcal{I}.
\]
Hence, by Theorem 1.1, \( \phi_{p_k} \circ F \) is \( sAC \) and there exists \( Z_k \subset [0,1] \) with \( \lambda(Z_k) = 0 \) such that \( (\phi_{p_k} \circ \tilde{F})(t) \) exists and
\[
(\phi_{p_k} \circ \tilde{F})(t) = (\phi_{p_k} \circ f)(t), \quad \text{at all } t \in [0,1] \setminus Z_k.
\]
Further, by Lemma 2.1 in [3], \( \phi_{p_k} \circ F \) has the limit average range at \( t \) and
\[
(2.19) \quad A_{\phi_{p_k} \circ F}(t) = \{(\phi_{p_k} \circ f)(t)\}, \quad \text{for all } t \in [0,1] \setminus Z_k.
\]
We now fix a point \( t \in [0,1] \setminus Z \), where \( Z = \cup_{k=1}^{+\infty} Z_k \) and \( \lambda(Z) = 0 \). We will prove that

\( a \) \( F \) has the limit average range at \( t \),

\( b \) \( f(t) \in A_F(t) \).

\( a \) Since \( \phi_{p_k} \circ F \) has the limit average range at \( t \), given \( \varepsilon > 0 \) there exists \( \delta_{\varepsilon}^{(k)} > 0 \) such that
\[
(2.20) \quad \overline{A}_{\delta_{\varepsilon}^{(k)}}(\phi_{p_k} \circ F(t)) < \overline{A}_{\delta_{\varepsilon}^{(k)}}(\phi_{p_k} \circ F(t)) + \varepsilon = \varepsilon.
\]
By inclusions \( A_F(t) \subset A_{p_k}^{(p_k)}(t) \) and \( \phi_{p_k}(A_{p_k}^{(p_k)}(t)) \subset A_{p_k} \circ F(t) \) we obtain
\[
p_k - diam(\phi_{p_k} \circ F(t)) = 0.
\]
Hence, the equality
\[ p_k - \text{diam}(A_F(t, \delta)) = \mathbb{P}_k - \text{diam}(A_{\phi_p \circ F}(t, \delta)), \]
together with (2.20) yields
\[ (2.21) \quad p_k - \text{diam}(A_F(t, \delta^{(k)})) < \varepsilon = p_k - \text{diam}(A_F(t)) + \varepsilon. \]
Thus, \( F \) has the limit average range at \( t \).

(b) At first, by the same manner as in the proof of Theorem 2.2, we can prove that
\[ A_F(t) \neq \emptyset. \]
Then, we may choose a vector \( x_t \in A_F(t) \). By (2.19), we obtain
\[ A_{\phi_p \circ F}(t) = \{(\phi_p \circ f)(t)\}, \text{ for all } k \in \mathbb{N}. \]
Hence
\[ \phi_p(x_t) = \phi_p(f(t)), \text{ for all } k \in \mathbb{N}, \]
and since \( X \) is Hausdorff, we infer \( x_t = f(t) \).

By virtue of Theorem 2.2 we obtain \((ii) \Rightarrow (i)\), and this ends the proof. \( \square \)

Finally, we give an example of a function \( F : [0, 1] \to X \) such that
\begin{itemize}
  \item \( F \) is the primitive of an integrable by seminorms function \( f \),
  \item \( F \) has the limit average range at all \( t \in (0, 1) \),
  \item \( F \) is not differentiable at any \( t \in (0, 1) \).
\end{itemize}

**Example 2.4.** Let \( X \) be the locally convex space \( \mathbb{R}^J \) with respect to the topology of pointwise convergence, where \( J = [0, 1] \). Thus, the family of seminorms \( (p_j)_{j \in J} \) defined as follows
\[ p_j(x) = |x(j)| \text{ for each } j \in J, x \in X, \]
determines the topology of \( X \). It easy to see that \( X \) is a complete locally convex space and each normed space \((X_{p_j}, \mathbb{P}_j)\) is isometrically isomorphic with \((\mathbb{R}, |\cdot|)\). Hence, for each \( j \in J \), we can consider
\[ X_{p_j} = X_{\mathbb{P}_j} = \mathbb{R} \quad \mathbb{P}_j = \mathbb{P}_j = p_j \]
and
\[ \phi_{p_j}(x) = x(j) \text{ for all } x \in X. \]

Let us now define the function \( F : [0, 1] \to X \) as follows
\[ F(t) = x_t, \quad x_t(\theta) = \begin{cases} \theta & \text{if } 0 \leq \theta \leq t \\ t & \text{if } t < \theta \leq 1 \end{cases} \text{ for all } t \in [0, 1]. \]

(a) The function \( F \) is sAC. To see this, fix an arbitrary \( j \in J \). Note that
\[ (\phi_{p_j} \circ F)(t) = \begin{cases} t & \text{if } 0 \leq t \leq j \\ j & \text{if } j < t \leq 1 \end{cases}. \]

Then, for each two points \( t', t'' \in [0, 1] \), we have
\[ p_j(F(t'')) - F(t') \leq |t'' - t'|. \]
This yields that \( \varphi_{p_j} \circ F \) is sAC, and since \( j \) is arbitrary, we infer that \( F \) is sAC.

(b) The function \( F \) has the limit average range at all \( t \in (0, 1) \). Fix an arbitrary \( t \in (0, 1), \delta > 0 \) and \( h > 0 \) such that
\[ 0 < h < \delta \quad \text{and} \quad t - h, t + h \in (0, 1). \]
Note that
\[ \Delta F(t, h)(\theta) = \frac{x_{t+h}(\theta) - x_t(\theta)}{h} = x^+_{t,h}(\theta) = \begin{cases} 0 & \text{if } 0 \leq \theta < t \\ \frac{\theta - t}{h} & \text{if } t \leq \theta < t + h \\ 1 & \text{if } t + h < \theta \leq 1 \end{cases} \]
and
\[ \Delta F(t, -h)(\theta) = \frac{x_{t-h}(\theta) - x_t(\theta)}{-h} = x^-_{t,h}(\theta) = \begin{cases} 0 & \text{if } 0 \leq \theta < t - h \\ \frac{\theta - (t-h)}{h} & \text{if } t - h < \theta < t \\ 1 & \text{if } \theta \leq 1 \end{cases} . \]

For each \( j \in J \), we have
\[ \lim_{n \to \infty} q_j \left( \Delta F \left( t, \frac{1}{n^2} \right) - x^+_t \right) = 0, \quad \lim_{n \to \infty} q_j \left( \Delta F \left( t, -\frac{1}{n^2} \right) - x^-_t \right) = 0, \]
where
\[ x^+_t(\theta) = \begin{cases} 0 & \text{if } 0 \leq \theta \leq t \\ 1 & \text{if } t < \theta \leq 1 \end{cases} \quad \text{and} \quad x^-_t(\theta) = \begin{cases} 0 & \text{if } 0 \leq \theta < t \\ 1 & \text{if } t \leq \theta \leq 1 \end{cases} . \]

Hence
\[ x^+_t, x^-_t \in A_F(q_j)(t) = \bigcap_{n=1}^\infty A_F(q_j) \left( t, \frac{1}{n} \right), \quad \text{for all } j \in J. \]

Therefore
\[ \{x^+_t, x^-_t\} \subset A_F(t). \]

We now assume that \( x \in A_F(t) \). Fix an arbitrary \( j \in J \) such that \( 0 \leq j < t \). Since
\[ x \in A_F(q_j)(t) = \bigcap_{n=1}^\infty A_F(q_j) \left( t, \frac{1}{n} \right), \]
there exists a sequence \((h_n)\) of real numbers such that
\[ 0 < \|h_n\| < \frac{1}{n} \quad \text{and} \quad \lim_{n \to \infty} q_j(\Delta F(t, h_n) - x) = 0. \]

From the fact that \( j < t \), it follows that there exists \( n_j^- \in \mathbb{N} \) such that for each \( n \in \mathbb{N} \), we have
\[ n \geq n_j^- \Rightarrow \Delta F(t, h_n)(j) = 0. \]

The last result together with \((2.23)\) yields \( x(j) = 0 \), and since \( j \) is arbitrary, \( x(j) = 0 \) for all \( 0 \leq j < t \). Similarly, we get that \( x(j) = 1 \) for all \( t < j \leq 1 \).

Let us now consider \( j = t \). By the same way as above there exists a sequence \((h_n)\) of real numbers such that \((2.23)\) is satisfied. There exists a subsequence \((h_{n_k})\) of \((h_n)\) such that \( h_{n_k} > 0 \) for all \( k \), or \( h_{n_k} < 0 \) for all \( k \). In the first case, we get \( x(t) = 0 \), and similarly \( x(t) = 1 \) for the second case. Thus, \( x \in \{x^+_t, x^-_t\} \), and we infer
\[ A_F(t) = \{x^+_t, x^-_t\}. \]

Note that if \( j \neq t \), then for \( 0 < \delta < |t - j| \) we have
\[ p_j - \text{diam}(A_F(t, \delta)) = 0 \quad \text{and} \quad p_j - \text{diam}(A_F(t)) = 0, \]
otherwise if \( j = t \), then for each \( \delta > 0 \), we have
\[ p_t - \text{diam}(A_F(t, \delta)) = 1 \quad \text{and} \quad p_t - \text{diam}(A_F(t)) = 1. \]

Therefore, \( F \) has the limit average range at \( t \), and since \( t \) is arbitrary we infer that \( F \) has the limit average range at all \( t \in (0, 1) \).
By virtue of Theorem 2.2, the functions $f_1, f_2 : [0,1] \to X$ defined as follows
\[ f_1(t) = x_t^+, \quad f_2(t) = x_t^-, \quad \text{for all} \quad t \in [0,1], \]
are integrable by seminorms with the primitive $F$.

(c) The function $F$ is not differentiable at any $t \in (0,1)$. Indeed, (2.22) yields immediately desired result.

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