EXACT SOLUTION AND INSTABILITY FOR GEOPHYSICAL WAVES AT ARBITRARY LATITUDE

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Abstract. We present an exact solution to the nonlinear governing equations in the \( \beta \)-plane approximation for geophysical waves propagating at arbitrary latitude on a zonal current. Such an exact solution is explicit in the Lagrangian framework and represents three-dimensional, nonlinear oceanic wave-current interactions. Based on the short-wavelength instability approach, we prove criteria for the hydrodynamical instability of such waves.

1. Introduction. The aim of this paper is to present an exact solution as well as criteria for its instability, to the \( \beta \)-plane approximation of the governing equations for geophysical fluid dynamics in a relatively narrow ocean strip at an arbitrary latitude. Geophysical fluid dynamics is the study of fluid motion where the Earth’s rotation plays a significant role, the Coriolis forces are incorporated into the governing Euler equations, and applies to a wide range of oceanic and atmospheric flows [15, 21, 49]. Geophysical fluid dynamics contains high complexity, which leads to an inherent mathematical intractability (see the discussions in [4], in which the mathematical intractability of fluid dynamics is illustrated even in the absence of geophysical considerations). In order to mitigate the complexity, it is natural and common to derive simpler approximate equations of Euler equations. There are two approximate models which have been typically employed in the oceanographic considerations. One is the \( \beta \)-plane approximation, which introduces a linear variation with latitude of the Coriolis parameter to allow for the variation of the Coriolis force from point to point. This approximation applies in regions within 5° latitude,
either side of the Equator (see [15, 21]). Modifications of the standard geophysical equatorial $\beta$-plane model equations which incorporate in addition centripetal forces, or a gravitational-correction term in the tangent plane approximation, were derived in [11, 25, 26]. Another approximate model is the $f$-plane approximation, which takes a constant Coriolis parameter into account and does not consider the latitudinal variations. This approximation has been applied to oceanic flows within a restricted meridional of approximately $2^\circ$ latitude, either side of the Equator [6, 10, 15]. An $f$-plane approximation at an arbitrary latitude was taken into account in [45]. In the paper at hand we will consider a $\beta$-plane approximation at an arbitrary latitude.

Exact solutions play an important role in the study of geophysical flows because many apparently intangible wave motions can be regarded as the perturbations of them, and relevant information about the dynamics of more complex flows can be extracted by controlling the perturbations. The approach pioneered by Gerstner [20] (for modern detailed descriptions see [3, 22]) of finding explicit exact solutions for gravity fluid flows within the Lagrangian framework [2], was extended to geophysical flows too (see the survey [27]). Gerstner-like three-dimensional solutions were obtained in the $f$-plane approximation at an arbitrary latitude in [45] and very recently in [14, 17], the internal wave in [38, 39], in the $\beta$-plane approximation in [5, 7, 8, 23] or in a modified $\beta$-plane approximation [25, 26]; for other studies, we refer the reader to [24, 30, 31, 32, 43, 44]. We mention that the above exact solutions fail to capture, for example, strong depth variations of the flows. A mixture of analytical and topological methods were applied in [46, 47, 48] to prove that the Lagrangian flow-map describing a number of three-dimensional geophysical exact solutions is a global diffeomorphism, thereby rigorously establishing that the flow description is globally dynamically possible. Very recently, [11, 12, 13, 36] investigate nonlinear three-dimensional models for flows with sufficient freedom and find, in the Eulerian framework, exact solutions that capture the most relevant geophysical features (see also the survey [37]).

Once an exact solution is available, the study of its stability and instability becomes an important issue. Instability occurs when the effect of some disturbance of the forces acting on the fluid grows as time progresses, and it is important for understanding the factors that might trigger the transition from the large-scale coherent structure represented by the exact solution to a more chaotic fluid motion. Although it is very important, from the viewpoint of mathematics, the study of the hydrodynamical stability or instability of a flow is difficult, because the fully nonlinear governing equations for fluid motion are highly intractable and there are only a handful of explicit exact solutions (see the discussions in [40]). A rigorous mathematical approach to the problem of stability for general three-dimensional inviscid incompressible flows is the short-wavelength method, which was developed independently by Bayly [1], Friedlander & Vishik [18] and Lifschitz & Hameiri [42]. For Gerstner-like geophysical surface waves in various settings, the short wavelength instability analysis is suitable and elegant; these solutions have been shown to be unstable when the travelling wave profiles are steep enough, the critical steepness being very close to $\frac{1}{3}$-instability results were established in [9, 16, 19, 28, 29, 33, 34] (see also the survey [35]).

The paper is organized as follows. In Section 2, we present the $\beta$-plane governing equations for geophysical waves at an arbitrary latitude in the presence of an underlying current. In Section 3, we obtain a Gerstner-like solution to this problem; our
results can be regarded as an extension of the exact solution for equatorial waves in the \( \beta \)-plane approximation. The geophysical effects and the constant uniform zonal current influence the dispersion relation of the waves. The waves propagate both eastward and westward. We mention that we do not deal with the meridional decay of solutions, and such consideration will limit the choice of wavespeeds which are admissible, in spite of the apparently general form of the dispersion relation. We make a detailed discussion of the situations encountered in the Northern Hemisphere and the Southern Hemisphere, for admissible following as well as adverse currents. Finally, in Section 4, we prove a wave-steepness instability criterion for these exact waves: if their steepness exceeds a specific threshold, very close to \( \frac{1}{3} \), then, certain localised small perturbations grow at an exponential rate and the flow is consequently unstable. The waves which travel from east to west are more prone to instability than those which travel from west to east. In particular, we deduce that an adverse current favours instability in the sense that the threshold on the steepness for the wave to be unstable is decreased compared to the case without current. Conversely, this threshold is increased by a following current.

2. The governing equations. We assume that the shape of the Earth is a perfect sphere of radius \( R = 6378 \) km which rotates with a constant rotational speed \( \Omega = 7.29 \times 10^{-5} \) rad/s round the polar axis towards east. A natural framework to describe our problem is a rotating one. We adopt a local “flat” coordinate system with the origin at a point on the Earth’s surface with latitude \( \phi \), \( -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2} \), the \( x \)-axis chosen horizontally due east, the \( y \)-axis horizontally due north (in the tangent plane) and the \( z \)-axis vertically upward (see Figure 1). In this coordinate system, \( \Omega = (0, \Omega \cos \phi, \Omega \sin \phi) \) is the rotation vector of the Earth along the polar axis. The Coriolis parameters are defined by

\[
 f := 2\Omega \sin \phi, \quad \hat{f} := 2\Omega \cos \phi.
\]

These parameters depend on the variable latitude \( \phi \). At the North Pole we have \( f = 2\Omega, \hat{f} = 0 \) and at the Equator \( f = 0, \hat{f} = 2\Omega \). The values \( f = \hat{f} = 10^{-4} \) s\(^{-1} \) are appropriate to 45\(^\circ\) latitude in the Northern Hemisphere, while \( f = -\hat{f} = -10^{-4} \) s\(^{-1} \) are appropriate to 45\(^\circ\) latitude in the Southern Hemisphere (See [21]). For surface
water waves propagating zonally in a relatively narrow ocean strip less than a few degrees of latitude wide, it is adequate to use the $f$- or $\beta$-plane approximations. Within the $f$-plane approximation the Coriolis parameters are treated as constants, but within the $\beta$-plane approximation $\hat{f}$ is constant and for $f$ a linear variation with the latitude is introduced, that is, $f + \beta y$, with

$$\beta = \frac{\hat{f}}{R} = \frac{2\Omega \cos \phi}{R},$$

at the fixed latitude $\phi$. In the regions close to the line of the Equator, $\beta = \frac{2\Omega}{R} = 2.28 \cdot 10^{-11} \text{m}^{-1}\text{s}^{-1}$.

In terms of the Cartesian coordinate system, the full governing equations for geophysical fluid dynamics in the $\beta$-plane approximation are the Euler equations

$$\begin{align*}
U_t + UU_x + VU_y + WU_z + \hat{f}W - (f + \beta y)V &= -\frac{1}{\rho}P_x, \\
V_t + UV_x + VV_y + WV_z + (f + \beta y)U &= -\frac{1}{\rho}P_y, \\
W_t + UW_x + VW_y + WW_z - \hat{f}U &= -\frac{1}{\rho}P_z - g,
\end{align*}$$

(1)

together with the equation of mass conservation

$$\rho_t + U\rho_x + V\rho_y + W\rho_z = 0,$$  \hspace{1cm} (2)

and with the condition of incompressibility

$$U_x + V_y + W_z = 0.$$ \hspace{1cm} (3)

Here $t$ is the time, $U = (U, V, W)$ is the fluid velocity, $P$ is the pressure, $g = 9.8\text{ms}^{-2}$ is the standard gravitational acceleration at the Earth’s surface and $\rho$ is the water’s density (which we take to be constant). Denoting the free surface by $\eta(x, y, t)$ and letting $P_{\text{atm}}$ be the constant atmospheric pressure, the relevant boundary conditions at the free surface are the kinematic boundary condition

$$W = \eta_t + U\eta_x + V\eta_y \quad \text{on} \quad z = \eta(x, y, t),$$

(4)

which implies that fluid particles on the free surface remain on the surface for all time, and the dynamic boundary condition

$$P = P_{\text{atm}} \quad \text{on} \quad z = \eta(x, y, t),$$

(5)

which decouples the water flow from the motion of the air above. Finally, we assume that the water is infinitely deep, with the flow converging rapidly with depth to a constant uniform zonal current of strength $c_0$, that is,

$$(U, V, W) \to (-c_0, 0, 0) \quad \text{as} \quad z \to -\infty.$$ \hspace{1cm} (6)

The direction of the current is connected with the dynamics of the exact solution that we present below.

3. **Exact solution.** We will use the Lagrangian framework for the exact solution. In the Lagrangian framework, the Eulerian coordinates of fluid particles $x = (x, y, z)$ at the time $t$ are expressed as functions of Lagrangian labelling variables $(q, s, r)$ which specify the fluid particle. We suppose that the position of a particle at time $t$ is given as

$$\begin{align*}
x &= q - c_0 t - \frac{1}{k} e^{k[r-m(s)]} \sin[k(q - ct)], \\
y &= s, \\
z &= r + \frac{1}{k} e^{k[r-m(s)]} \cos[k(q - ct)],
\end{align*}$$

(7)
in which \( k \) is the wave number, \( c \) is the wave speed and \( c_0 \) is the constant underlying current such that for \( cc_0 > 0 \) the current is adverse, while for \( cc_0 < 0 \) the current is following. For later considerations, we take, on physical grounds,

\[
|c_0| < \frac{g}{f}.
\]  
(8)

By remembering that \( \frac{g}{2f} \approx 6.7 \times 10^4 \text{ m/s} \), and for \( \frac{\pi}{2} \leq \phi \leq \frac{\pi}{2} \), we have \( \frac{g}{2f} < \frac{g}{f} \), this is indeed a plausible range for \( c_0 \).

We will prove that the system (7) defines an exact solution of the \( \beta \)-plane governing equations (1)-(6), where the travelling speed \( c \) and the function \( m \) depending on \( s \) are determined below. In (7), the Lagrangian labelling variables take the following values:

\[
q \in \mathbb{R}, \quad s \in [-s_0, s_0],
\]
and

\[
r \in (-\infty, r_0] \quad \text{such that} \quad r - m(s) \leq r_0 < 0,
\]  
(9)

to ensure that the flow has the appropriate decay properties.

For notational convenience, we set

\[
\xi = k[r - m(s)], \quad \theta = k(q - ct).
\]

The Jacobian matrix of the transformation (7) is given by

\[
J = \begin{pmatrix}
\frac{\partial x}{\partial q} & \frac{\partial y}{\partial q} & \frac{\partial z}{\partial q} \\
\frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\
\frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r}
\end{pmatrix} = \begin{pmatrix}
1 - e^{\xi} \cos \theta & 0 & -e^{\xi} \sin \theta \\
me^{\xi} \sin \theta & 1 & -me^{\xi} \cos \theta \\
-e^{\xi} \sin \theta & 0 & 1 + e^{\xi} \cos \theta
\end{pmatrix},
\]
therefore, its determinant is \( 1 - e^{2\xi} \), which is non-zero, and hence the transformation (7) is well defined. Furthermore, as the determinant of the Jacobian is time-independent, the flow defined by (7) is volume preserving and (3) holds.

3.1. The pressure functions. Let us now write the Euler equation (1) in the form

\[
\begin{aligned}
\frac{DU}{Dt} + \hat{f}W - (f + \beta y)V &= -\frac{1}{\rho}P_x, \\
\frac{DV}{Dt} + (f + \beta y)U &= -\frac{1}{\rho}P_y, \\
\frac{DW}{Dt} - \hat{f}U &= -\frac{1}{\rho}P_z - g,
\end{aligned}
\]  
(10)

where \( \frac{DU}{Dt} \) stands for the material derivative. From (7) we can compute the velocity and acceleration of a particle as

\[
\begin{aligned}
U &= \frac{Dx}{Dt} = -c_0 + ce^{\xi} \cos \theta, \\
V &= \frac{Dy}{Dt} = 0, \\
W &= \frac{Dz}{Dt} = ce^{\xi} \sin \theta,
\end{aligned}
\]  
(11)

and

\[
\begin{aligned}
\frac{DU}{Dt} &= kc^2 e^{\xi} \sin \theta, \\
\frac{DV}{Dt} &= 0, \\
\frac{DW}{Dt} &= -kc^2 e^{\xi} \cos \theta,
\end{aligned}
\]
respectively. We can therefore write (10) as

\[
\begin{aligned}
P_x &= -\rho(kc^2 e^{\xi} \sin \theta + \hat{f}ce^{\xi} \sin \theta), \\
P_y &= -\rho(f + \beta s)(-c_0 + ce^{\xi} \cos \theta), \\
P_z &= -\rho(-kc^2 e^{\xi} \cos \theta + \hat{f}c_0 - \hat{f}ce^{\xi} \cos \theta + g).
\end{aligned}
\]  
(12)
The change of variables
\[
\begin{pmatrix} P_q \\ P_s \\ P_r \end{pmatrix} = J \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}
\]
transforms (12) into
\[
\begin{cases} 
  P_q = -\rho(kc^2 + \hat{f}c - \hat{f}c_0 - g)e^\xi \sin \theta, \\
  P_s = -\rho(m_s(kc^2 + \hat{f}c)e^{2\xi} + M(s)e^\xi \cos \theta - (f + \beta s)c_0], \\
  P_r = -\rho[-(kc^2 + \hat{f}c)e^{2\xi} - (kc^2 + \hat{f}c - \hat{f}c_0 - g)e^\xi \cos \theta + \hat{f}c_0 + g],
\end{cases}
\]
where
\[M(s) = fc + c\beta s - \hat{f}c_0 m_s - gm_s.\]

Now we give some suitable conditions on the pressure function \(P\) such that (13) holds and (7) is indeed an exact solution of the governing equations (1)-(6). Since the condition (5) enforces a time independence in the pressure function at the surface, it is necessary to eliminate terms containing \(\theta\) in (13) by setting
\[kc^2 + \hat{f}c - \hat{f}c_0 - g = 0,\]
and
\[M(s) = fc + c\beta s - \hat{f}c_0 m_s - gm_s = 0.\]

It follows from (15) that we can choose
\[m(s) = \frac{c\beta}{2(f\hat{c}_0 + g)} s^2 + \frac{fc}{f\hat{c}_0 + g} s.\]
From (8), we have \(c_0 > -\frac{g}{\hat{f}}\), and since \(-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}\), we get
\[\hat{f}c_0 + g > 0.\]

With the constraints (14) and (15), we can solve the pressure function as
\[P(r, s) = \rho(\hat{f}c_0 + g)\left[\frac{e^{2\xi}}{2k} + \frac{c_0}{c} m(s) - r\right] + P_{\text{atm}} - \rho(\hat{f}c_0 + g)\left[\frac{e^{2k\rho}}{2k} - r_0\right].\]

The constant terms in (18) have been chosen to ensure the conditions (4) and (5) hold on the free surface, see the subsection 3.2 below.

In addition, the condition (14) leads us to the dispersion relation for the wave motion prescribed by (7). By regarding (14) as a quadratic in \(c\), if \(c_0 \neq c\), we get
\[c_{\pm} = \frac{-\hat{f} \pm \sqrt{\hat{f}^2 + 4k(\hat{f}c_0 + g)}}{2k},\]
which are well defined due to (17). The case \(c_0 = c\) gives us the classical \(c = \pm \sqrt{\frac{g}{\hat{f}}},\) that is, the geophysical effects have no influence on the dispersion relation of the wave. If \(c = c_+ > 0\), the wave travels from west to east and if \(c = c_- < 0\), the wave travels from east to west. We observe that for our solution (7), the Coriolis parameter \(f\) does not appear in the dispersion relation (14) and its solutions (19), but it appears in the expression (16) of the function \(m(s)\).

In the absence of an underlying current, for \(c_0 = 0\), (14), (19) and (16) reduce to
\[kc^2 + \hat{f}c - g = 0 \implies c_{\pm} = \frac{-\hat{f} \pm \sqrt{\hat{f}^2 + 4kg}}{2k},\]
\[m(s) = \frac{c\beta}{2g} s^2 + \frac{fc}{g} s.\]
For equatorial waves \( f = 0 \) and \( \dot{f} = 2\Omega \) we recover the result obtained in [23], [26], that is, the dispersion relation (14) becomes
\[
kc^2 + 2\Omega c - 2\Omega c_0 - g = 0,
\]
yielding
\[
c_{\pm} = -\Omega \pm \sqrt{\Omega^2 + k(2\Omega c_0 + g)},
\]
and the function \( m(s) \) becomes
\[
m(s) = \frac{c_0^2}{2(2\Omega c_0 + g)} s^2.
\]

3.2. The free-surface interface. We now focus on the expression (18) of the pressure function and the fulfilment of the boundary conditions (4) and (5). In Lagrangian variables, the kinematic boundary condition (4) holds if at each fixed latitude \( s \) the free surface is given by specifying a value of \( r \), the label \( q \) being the free parameter of the curve that represents the wave profile at this latitude. With (18) in view, this is achieved if we show that, at each fixed \( s \), there exists a unique solution \( r(s) \leq r_0 < 0 \) such that \( P(r(s), s) = P_{\text{atm}} \), which is equivalent to
\[
h(r(s), s) = \frac{e^{2kr_0}}{2k} - r_0,
\]
in which
\[
h(r, s) = \frac{e^{2c}}{2k} + \frac{c_0 m(s)}{c} r (16) = \frac{e^{2c}}{2k} + \frac{c_0^2}{2(fc_0 + g)} s^2 + \frac{fc_0}{fc_0 + g} s - r.
\]
For \( s = 0 \), the choice \( r(0) = r_0 \) works in (20). For the case \( s \neq 0 \), by condition (9), we have
\[
h_r(r, s) = e^{2k[r-m(s)]} - 1 < 0.
\]
Consequently, \( h \) is a monotonically decreasing function of \( r \). Since
\[
\lim_{r \to -\infty} h(r, s) = +\infty,
\]
the existence of a unique solution of (20), for fixed \( s \), is ensured by the following inequality
\[
\lim_{r \to r_0} h(r, s) = \frac{e^{2kr_0}}{2k} + \frac{c_0}{c} m(s) - r_0 < \frac{e^{2kr_0}}{2k} - r_0.
\]
The inequality (21) takes the form
\[
\frac{e^{2kr_0}}{2k} [e^{-2km(s)} - 1] + \frac{c_0}{c} m(s) < 0.
\]
We will look for the geophysical waves satisfying
\[
\begin{cases} 
  r'(s) < 0, & s > 0, \\
  r'(s) > 0, & s < 0.
\end{cases}
\]
Differentiating (20) with respect to \( s \), we obtain that
\[
r'(s) [e^{2k[r(s)-m(s)]} - 1] - m'(s) [e^{2k[r(s)-m(s)]} - \frac{c_0}{c}] = 0,
\]
and therefore we get
\[
r'(s) = \left[ \frac{c_0 - ce^{2k[r(s)-m(s)]}}{1 - e^{2k[r(s)-m(s)]}} \right] m'(s) \frac{1}{c}.
\]
Taking into account the expression (16) of $m(s)$ and the conditions (9), (14), the inequalities (23) become

$$\begin{cases} 
[c_0 - cc^2 k [r(s) - m(s)]] (\beta s + f) < 0, & s > 0, \\
[c_0 - cc^2 k [r(s) - m(s)]] (\beta s + f) > 0, & s < 0.
\end{cases} \tag{24}$$

From (23) and $r(0) = r_0$, it follows that $r(s) < r_0$ for $s \neq 0$. Since (9) holds and $m(0) = 0$, we obtain that, for $s \neq 0$,

$$m(s) > 0, \text{ or } m(s) < 0 \text{ close enough to zero.}$$

Let us see for which values of the uniform current $c_0$ the required restrictions (22) and (24) are satisfied. We will make a separate case discussion in the Northern Hemisphere and in the Southern Hemisphere.

I. Northern Hemisphere

For the Northern Hemisphere, we have that $\hat{f} > 0$, $f > 0$ and $\beta > 0$. We further distinguish the following cases.

Case N1. $m(s) > 0$ and $cc_0 < 0$, that is, the current is following.

In this case, (22) is obviously satisfied. Let us see what happens with the inequality (24). The condition $cc_0 < 0$ is obtained if:

(a) the solution $c = c_+$ > 0 and the uniform current $-\frac{f}{\beta} < c_0 < 0$.

Then, the term $c_0 - cc^2 k [r(s) - m(s)] < 0$ and from (16) and (17) it follows that $m(s) > 0$ if and only if $s > 0$ or $s < -\frac{2f}{\beta}$. Physically, the possibility $s < -\frac{2f}{\beta}$ can be excluded, and therefore $s > 0$. We thus get that inequality (24) is satisfied for all $s > 0$.

(b) the solution $c = c_- < 0$ and the uniform current $0 < c_0 < \frac{f}{\beta}$.

In this situation, $c_0 - cc^2 k [r(s) - m(s)] > 0$ and from (16) and (17) it follows that $m(s) > 0$ if and only if $-\frac{2f}{\beta} < s < 0$. Since we consider a relatively narrow ocean strip, we can restrict to $-\frac{f}{\beta} < s < 0$, where the inequality (24) holds.

Case N2. $m(s) > 0$ and $cc_0 > 0$, that is, the current is adverse.

The condition $cc_0 > 0$ is obtained if:

(a) the solution $c = c_+ > 0$ and the uniform current $0 < c_0 < \frac{f}{\beta}$.

Just as in the case N1(a), we can exclude $s < -\frac{2f}{\beta}$, and we get $s > 0$. The inequality (24) is reduced to

$$c_0 - cc^2 k [r(s) - m(s)] < 0, \quad s > 0. \tag{25}$$

This condition breaks down for large enough values of $s > 0$. We are interested in water waves propagating in a relatively narrow ocean strip. We show that (22) and (25) are satisfied for small values $s$ depending on the size of the current $c_0 > 0$.

We denote the left hand side in (22) by

$$A(s) := \frac{e^{2k r_0}}{2k} [e^{-2k m(s)} - 1] + \frac{c_0}{c} m(s) \tag{26}$$

We note that $A(0) = 0$. Thus, to ensure the validity of the inequality

$$A(s) < 0, \quad s > 0,$$
the necessary condition is

\[ A'(s) = \left[ c_0 - ce^{2k[r_0-m(s)]} \right] \frac{m'(s)}{c} = \left[ c_0 - ce^{2k[r_0-m(s)]} \right] \frac{\beta s + f}{f c_0 + g} < 0, \]

for \( s > 0 \) small enough, which is equivalent to

\[ c_0 - ce^{2k[r_0-m(s)]} < 0 \]  \hspace{1cm} (27)

for \( s > 0 \) small enough. Therefore, for a given \( 0 < c_0 < ce^{2kr_0} \), (27) holds for some \( s \in (0, s_1] \) and accordingly (25) holds for \( s \in (0, s_0] \) with \( s_0 < s_1 \).

(b) the solution \( c = c_- < 0 \) and the uniform current \(-\frac{f}{\beta} < c_0 < 0\).

In this case, \(-\frac{2f}{\beta} < s < 0\). Since we consider a relatively narrow ocean strip, we can restrict \(-\frac{f}{\beta} < s < 0\), and therefore \( \beta s + f > 0 \). The inequalities (22) and (24) become now

\[ A(s) < 0, \quad -\frac{f}{\beta} < s < 0 \]  \hspace{1cm} (28)

with \( A(s) \) defined in (26), and

\[ c_0 - ce^{2k[r(s)-m(s)]} > 0, \quad -\frac{f}{\beta} < s < 0, \]  \hspace{1cm} (29)

respectively. For water waves propagating in a relatively narrow ocean strip, in the same manner as in the case \textbf{N2(a)}, we can show that (26) and (29) are satisfied for small values \( s \) depending on the size of the current \( c_0 \). To ensure the validity of (28), the necessary condition is

\[ A'(s) > 0 \]

for \(-\frac{f}{\beta} < s < 0\) with \( |s| \) small enough, which is equivalent to

\[ c_0 - ce^{2k[r_0-m(s)]} > 0 \]  \hspace{1cm} (30)

for \(-\frac{f}{\beta} < s < 0\) with \( |s| \) small enough. Therefore, for a given \( ce^{2kr_0} < c_0 < 0\), (30) holds for \( s \in [-s_1, 0) \) and accordingly (29) holds for \( s \in [-s_0, 0) \) with \( s_0 < s_1 \).

**Case N3.** \( m(s) < 0 \) and \( cc_0 < 0 \), that is, the current is following.

In this case, we see easily that the inequality (22) can not be satisfied. Thus, this is a nonvalid case.

**Case N4.** \( m(s) < 0 \) and \( cc_0 > 0 \), that is, the current is adverse.

The condition \( cc_0 > 0 \) is obtained if:

(a) the solution \( c = c_+ > 0 \) and the uniform current \( 0 < c_0 < \frac{g}{f} \).

\( m(s) < 0 \) if and only if \(-\frac{2f}{\beta} < s < 0\). We can restrict \(-\frac{f}{\beta} < s < 0\), and therefore \( \beta s + f > 0 \). The inequalities (22) and (24) become now

\[ A(s) < 0, \quad -\frac{f}{\beta} < s < 0 \]  \hspace{1cm} (31)

with \( A(s) \) defined in (26), and

\[ c_0 - ce^{2k[r(s)-m(s)]} > 0, \quad -\frac{f}{\beta} < s < 0, \]  \hspace{1cm} (32)

respectively. To ensure the validity of (31),

\[ c_0 - ce^{2k[r_0-m(s)]} > 0 \]
for \(-\hat{f}/\beta < s < 0\) with \(|s|\) small enough, has to be satisfied. Hence, for a given \(ce^{2kr_0} < c_0 < \hat{f}/\beta\), this condition holds for \(s \in [-s_1, 0]\) and accordingly (32) holds for \(s \in [-s_0, 0]\) with \(s_0 < s_1\).

(b) the solution \(c = c_- < 0\) and the uniform current \(-\hat{f}/\beta < c_0 < 0\).

Similar arguments apply to this case too, and we get that for a given \(-\hat{f}/\beta < c_0 < ce^{2kr_0} < 0\), (22) and (24) hold for \(s \in (0, s_0]\) with \(s_0 > 0\) small enough.

II. Southern Hemisphere

For the Southern Hemisphere, we have \(\hat{f} > 0\), \(\beta > 0\) and \(\beta > 0\). The four cases above can be handled in much the same way, the only difference being the range for \(s\). Now \(m(s) > 0\) if and only if \(s < 0\) or \(s > -2\hat{f}/\beta\). The possibility \(s > -2\hat{f}/\beta\) can be excluded for physical reasons. Therefore \(s < 0\), which also yields \(\beta s + \hat{f} < 0\).

For \(m(s) < 0\), we get \(0 < s < -2\hat{f}/\beta\). This range can be restricted to \(0 < s < -\hat{f}/\beta\) for narrow ocean strips and again \(\beta s + \hat{f} < 0\). The conclusions in the four cases are:

Case S1. \(m(s) > 0\) and \(cc_0 < 0\), that is, the current is following.

In this case, (22) is obviously satisfied for all \(s \neq 0\), and if:

(a) the solution \(c = c_+ > 0\) and the uniform current \(-\hat{f}/\beta < c_0 < 0\), then, the inequality (24) is satisfied for all \(s < 0\).

(b) the solution \(c = c_- < 0\) and the uniform current \(0 < c_0 < \hat{f}/\beta\), then, the inequality (24) is satisfied for the strip \(0 < s < -\hat{f}/\beta\).

Case S2. \(m(s) > 0\) and \(cc_0 > 0\), that is, the current is adverse.

If:

(a) the solution \(c = c_+ > 0\) and the uniform current \(0 < c_0 < \hat{f}/\beta\), then, for a given \(0 < c_0 < ce^{2kr_0}\) the inequalities (22) and (24) hold for \(s \in [-s_0, 0]\) small enough.

(b) the solution \(c = c_- < 0\) and the uniform current \(-\hat{f}/\beta < c_0 < 0\), then, for a given \(ce^{2kr_0} < c_0 < 0\), the inequalities (22) and (24) hold for \(s \in (0, s_0]\) small enough.

Case S3. \(m(s) < 0\) and \(cc_0 < 0\), that is, the current is following.

Since the inequality (22) can not be satisfied, this is a nonvalid case.

Case S4. \(m(s) < 0\) and \(cc_0 > 0\), that is, the current is adverse.

If:

(a) the solution \(c = c_+ > 0\) and the uniform current \(0 < c_0 < \hat{f}/\beta\), then, for a given \(ce^{2kr_0} < c_0 < \hat{f}/\beta\), the inequalities (22) and (24) hold for some \(s \in (0, s_0]\) small enough.

(b) the solution \(c = c_- < 0\) and the uniform current \(-\hat{f}/\beta < c_0 < 0\), then, for a given \(-\hat{f}/\beta < c_0 < ce^{2kr_0}\), the inequalities (22) and (24) hold for some \(s \in [-s_0, 0]\) for some \(s_0 > 0\) small enough.

Summing up, we have got the following theorem:

Theorem 3.1. The fluid motion prescribed by (7) represents an exact solution of the governing equations (1)-(6) in:

I. Northern Hemisphere

• if the current \(c_0\) with \(|c_0| < \hat{f}/\beta\) is following, that is, \(cc_0 < 0\) and the function \(m(s)\) is positive; for flows with positive wave speed \(c = c_+ > 0\), the range for
the fixed latitude is \( s > 0 \), and for flows with negative wave speed \( c = c_- < 0 \), we are restricted to latitudes in the region \(-\frac{f}{2} < s < 0\).

- if the current \( c_0 \) with \( |c_0| > \frac{f}{2} \) is adverse, that is, \( cc_0 > 0 \) and the function \( m(s) \) is positive, for that values of the current \( |c_0| < |c|e^{2kr_0} \), or \( m(s) \) is negative, for that values of the current \( |c|e^{2kr_0} < |c_0| < \frac{f}{2} \); in this case, the fixed latitude \( s \) belongs to the interval \([-s_0, s_0]\), with \( s_0 > 0 \) small enough.

II. Southern Hemisphere

- if the current \( c_0 \) with \( |c_0| < \frac{f}{2} \) is following, that is, \( cc_0 < 0 \) and the function \( m(s) \) is positive; for flows with positive wave speed \( c = c_+ > 0 \), the range for the fixed latitude is \( s < 0 \), and for flows with negative wave speed \( c = c_- < 0 \), we are restricted to latitudes in the region \( 0 < s < \frac{f}{2} \).

- if the current \( c_0 \) with \( |c_0| < \frac{f}{2} \) is adverse, that is, \( cc_0 > 0 \) and the function \( m(s) \) is positive, for that values of the current \( |c_0| < |c|e^{2kr_0} \), or \( m(s) \) is negative, for that values of the current \( |c|e^{2kr_0} < |c_0| < \frac{f}{2} \); in this case, the fixed latitude \( s \) belongs to the interval \([-s_0, s_0]\), with \( s_0 > 0 \) small enough.

The free surface \( z = \eta(x, y, t) \) is prescribed at \( s = 0 \) by setting \( r = r_0 \) in (7), and for any other fixed latitude \( s \neq 0 \) in the intervals mentioned above, there exists a unique value \( r(s) < r_0 \) which implicitly prescribes the free surface \( z = \eta(x, s, t) \) by setting \( r = r(s) \) in (7).

Finally, we note that the form of the surface wave, for fixed values of \( s \) and \( t \), is an inverted trochoid (see [4, 5]).

3.3. The vorticity. The velocity gradient matrix is given by

\[
\nabla U = \frac{cke^\xi}{1 - e^{2\xi}} \begin{pmatrix}
-\sin \theta & m_\phi(e^\xi - \cos \theta) & -e^\xi + \cos \theta \\
0 & 0 & 0 \\
e^\xi + \cos \theta & -m_\phi \sin \theta & \sin \theta
\end{pmatrix}.
\] (33)

Thus, the vorticity \( \omega = (w_y - v_z, u_z - w_x, v_x - u_y) \) takes the form

\[
\omega = \left( -\frac{c^2k(\beta s + f)}{fc_0 + g} \right) e^\xi \sin \theta \quad \frac{2ke^{2\xi}}{1 - e^{2\xi}} - \frac{c^2k(\beta s + f)}{fc_0 + g} e^{2\xi} - \frac{e^2 \xi - \xi \cos \theta}{1 - e^{2\xi}} \right).
\] (34)

We can see that the vorticity (34) is three-dimensional away from the Equator, although the velocity field (11) is two-dimensional. Moreover, the first and third components in (34) depends explicitly on the latitude \( \phi \) and the underlying current \( c_0 \). The underlying current \( c_0 \) does not feature directly in the second component of the vorticity (34), accounting for the local spin in the \( y \)-direction, it plays an implicit role as determined by the dispersion relation (19) for the wave speed \( c \). We also observe that the sign of this component is opposed to propagation wave speed.

4. Instability. Since the stability issue concerns the evolution of small perturbations with time, it is reasonable to pursue its investigation within a linear framework by neglecting nonlinear terms arising from products of the perturbed quantities. Small perturbations \( u(t, x), p(t, x) \) from the geophysical flow \( (U(t, x), P(t, x)) \) (see (11), (18)) which satisfies the problem (1)-(6), are governed by the following linearized equations

\[
u_t + (U \cdot \nabla)u + (u \cdot \nabla)U + \mathcal{L}_{f, \beta} u = -\nabla p
\] (35)

\[
\nabla \cdot u = 0
\] (36)
with $\mathcal{L}_{f,\beta}$ given by
\[
\mathcal{L}_{f,\beta} := \begin{pmatrix}
0 & -(f + \beta y) & \dot{f} \\
(f + \beta y) & 0 & 0 \\
-\dot{f} & 0 & 0
\end{pmatrix}.
\]

We examine the evolution in time of the localised and rapidly-varying solutions of
the linearized system (35)-(36) in the wave packet form
\[
u(t, x) = \left[ A(t, x) + \epsilon A(t, x) \right] e^{i \Phi(t, x)} + \epsilon u_{rem}(t, x, \epsilon) \tag{37}
\]
\[
p(t, x) = \left[ B(t, x) + \epsilon B(t, x) \right] e^{i \Phi(t, x)} + \epsilon p_{rem}(t, x, \epsilon). \tag{38}
\]
with a sharply-peaked initial condition
\[
u_0 := \nu(0, x) = A(0, x) e^{i \Phi(0, x)} = A_0(x) e^{i \Phi_0(x)},
\]
where $\epsilon$ is a small parameter, $A, A$ are vector functions and $\Phi, B, B$ are scalar func-
tions. Plugging (37) and (38) into the linearized equations (35)-(36) and equating
the terms of the same order in $\epsilon$, we get for the leading order terms the relations
\[
A \cdot \nabla \Phi = 0 \quad \text{for all } t \geq 0,
\]
\[
B = 0,
\]
and the following coupled system, consisting of the eikonal equation for the wave
phase and the transport equation for the wave amplitude of the velocity,
\[
\begin{align*}
\Phi_t + U \cdot \nabla \Phi &= 0, \quad (39) \\
A_t + (U \cdot \nabla)A + (A \cdot \nabla)U + \mathcal{L}_{f,\beta} A &= \frac{\nabla \Phi \cdot [2(\nabla \cdot U) + \mathcal{L}_{f,\beta} A]}{\|\nabla \Phi\|^2} \nabla \Phi, \quad (40)
\end{align*}
\]
with the initial conditions
\[
\Phi(0, x) = \Phi_0(x), \quad A(0, x) = A_0(x), \quad \text{with} \quad A_0 \cdot \nabla \Phi_0 = 0.
\]
Since the next-order terms $A$ and $B$ depend only on $A$ and $\nabla \Phi$, and the remainder
terms $u_{rem}, p_{rem}$ are bounded, in an appropriate norm, at any time $t$ by functions
that can depend on $t$ but are independent of $\epsilon$, then, in (37), (38) these terms can
be made as small as we want by multiplication with $\epsilon$ and they are dominated by
the growth of the leading order terms. See, for example, [33] for details.

The system of partial differential equations (39)-(40) can be written as a system
of ordinary differential equations along the trajectories of the basic flow $U(t, x)$
(11). Consider the trajectory (7) passing through a point $x_0$:
\[
\frac{dx}{dt} = U(t, x), \quad x(0) = x_0,
\]
then, straightforward computations show that the system (39)-(40) can be rewritten
along the trajectories of the basic flow in the following form
\[
\frac{d\xi}{dt} = -(\nabla U)^T \xi, \quad (41)
\]
\[
\frac{dA}{dt} = -(A \cdot \nabla)U - \mathcal{L}_{f,\beta} A + \frac{\xi \cdot [2(\nabla \cdot U) + \mathcal{L}_{f,\beta} A]}{\|\xi\|^2} \xi, \quad (42)
\]
where $\xi := \nabla \Phi$ is the local wave vector, and $\nabla U$ is the velocity gradient matrix
(33). The initial conditions for the ODE system (41)-(42) are
\[
\xi(0) = \xi_0, \quad A(0) = A_0, \quad \text{with} \quad A_0 \cdot \xi_0 = 0.
\]
To prove the instability of the geophysical fluid flow (7), it is not necessary to investigate (41), (42) for all initial data. We only need to chose an initial disturbance which can lead to an exponentially growing amplitude \( \mathbf{A} \). Let us choose the longitudinal wave vector \( \mathbf{\xi}_0 = (0 \ 1 \ 0)^T \). For this initial condition, taking into account the expression (33) of the velocity gradient matrix, (41) yields

\[
\mathbf{\xi}(t) = (0 \ 1 \ 0)^T \quad \text{for all time} \quad t \geq 0.
\]

Hence, from (42), it follows that \( \mathbf{A} = (A_1, A_2, A_3) \) satisfies

\[
\begin{cases}
\dot{A}_1 = -\hat{f}A_3 + (f + \beta s)A_2 - \frac{ck\xi^3}{1 - e^{2\xi}}[-A_1 \sin \theta + A_2 m_s (e^\xi - \cos \theta) + A_3 (\cos \theta - e^\xi)], \\
A_2 = 0, \\
\dot{A}_3 = \hat{f}A_1 - \frac{ck \xi^3}{1 - e^{2\xi}} [A_1 (e^\xi + \cos \theta) - A_2 m_s \sin \theta + A_3 \sin \theta].
\end{cases}
\]

For the chosen initial vector \( \mathbf{\xi}_0 = (0 \ 1 \ 0)^T \) and the condition of orthogonality in (43), we must have \( A_2(0) = 0 \). Thus, the second equation in the above system yields

\[ A_2(t) = 0 \quad \text{for all} \quad t \geq 0. \]

Thus, the system reduces to the following two-dimensional system

\[
\begin{cases}
\dot{A}_1 = -\hat{f}A_3 - \frac{ck \xi^3}{1 - e^{2\xi}} [-A_1 \sin \theta + A_3 (\cos \theta - e^\xi)], \\
\dot{A}_3 = \hat{f}A_1 - \frac{ck \xi^3}{1 - e^{2\xi}} [A_1 (e^\xi + \cos \theta) + A_3 \sin \theta],
\end{cases}
\]

which can be written as

\[
\begin{pmatrix}
\dot{A}_1 \\
\dot{A}_3
\end{pmatrix} = M(t) \begin{pmatrix}
A_1 \\
A_3
\end{pmatrix},
\]

with

\[
M(t) = \begin{pmatrix}
\hat{f} - \frac{ck \xi^3}{1 - e^{2\xi}} \sin \theta & -\frac{ck \xi^3}{1 - e^{2\xi}} (e^\xi - \cos \theta) \\
\hat{f} - \frac{ck \xi^3}{1 - e^{2\xi}} (e^\xi + \cos \theta) & -\frac{ck \xi^3}{1 - e^{2\xi}} \sin \theta
\end{pmatrix}.
\]

By rotating the canonical Cartesian basis with the angle \( \alpha = \frac{kq}{ck} \) about the vector \( \mathbf{\xi}(t) \) from (44), the system (45) can be transformed to an autonomous linear system. We denote by \( P(t) \)

\[
P(t) = \begin{pmatrix}
\cos(kct/2) & \sin(kct/2) \\
-\sin(kct/2) & \cos(kct/2)
\end{pmatrix},
\]

the \( \alpha \)-rotation change-of-basis matrix, and by

\[
\begin{pmatrix}
\hat{A}_1 \\
\hat{A}_3
\end{pmatrix} := P^{-1}(t) \begin{pmatrix}
A_1 \\
A_3
\end{pmatrix}.
\]

In the rotating basis, the system (45) becomes the autonomous system

\[
\begin{pmatrix}
\dot{\hat{A}}_1 \\
\dot{\hat{A}}_3
\end{pmatrix} = D \begin{pmatrix}
\hat{A}_1 \\
\hat{A}_3
\end{pmatrix},
\]

where \( D \) the time-independent matrix

\[
D = \begin{pmatrix}
\frac{ck \xi^3}{1 - e^{2\xi}} \sin(kq) & -\hat{f} - \frac{ck \xi^3}{1 - e^{2\xi}} (e^\xi - \cos \theta) - \frac{ck \xi^3}{1 - e^{2\xi}} m_s \\
\hat{f} - \frac{ck \xi^3}{1 - e^{2\xi}} (e^\xi + \cos \theta) + \frac{ck \xi^3}{1 - e^{2\xi}} \sin(kq) & -\frac{ck \xi^3}{1 - e^{2\xi}} \sin(kq)
\end{pmatrix}.
\]

By (46), the solution to the non-autonomous system (45) is obtained by multiplying the rotation matrix \( P(t) \) with the solution to the autonomous system (47). The matrix \( P(t) \) being time periodic, the behaviour in time of the amplitude vector \( \mathbf{A} \) is determined by the eigenvalues of the matrix \( D \). The amplitude grows exponentially
with time if $D$ has a positive eigenvalue. The eigenvalues of $D$ satisfy the following equation

$$\lambda^2 = \frac{(2 \hat{f} + 3kc)^2 e^{2\xi} - (2 \hat{f} + kc)^2}{4(1 - e^{2\xi})}.$$ 

Therefore, taking into account (9), if

$$e^{\xi} > \frac{2 \hat{f} + kc}{2 \hat{f} + 3kc} \quad \Rightarrow \quad \frac{3 \hat{f} \pm \sqrt{\hat{f}^2 + 4k(\hat{f}c_0 + g)}}{\hat{f} \pm 3\sqrt{\hat{f}^2 + 4k(\hat{f}c_0 + g)}},$$

then, the amplitude $A$ increases unboundedly in time, the exponential growth rate being

$$\lambda = \frac{1}{2} \sqrt{\frac{(2 \hat{f} + 3kc)^2 e^{2\xi} - (2 \hat{f} + kc)^2}{1 - e^{2\xi}}}.$$ 

Due to (7), the steepness of the longitudinal wave profile, defined as the amplitude multiplied by the wave number, is $e^{\xi}$. We have proved the following wave-steepness instability criterion:

**Theorem 4.1.** At arbitrary latitude, if the geophysical waves (7) are sufficiently large - their steepness exceeds a specific threshold given by the right-hand side of (48) - then, certain localised small perturbations grow at an exponential rate and the flow is consequently unstable.

In the absence of the underlying current, for $c_0 = 0$, the right-hand side of (48) becomes

$$\frac{3 \hat{f} \pm \sqrt{\hat{f}^2 + 4kg}}{\hat{f} \pm 3\sqrt{\hat{f}^2 + 4kg}}.$$ 

(49)

Since $\hat{f} \ll \sqrt{\hat{f}^2 + 4kg}$, the threshold (49) is

$$\frac{3 \hat{f} + 1}{\sqrt{\hat{f}^2 + 4kg}} \geq \frac{1}{3} \quad \text{for} \quad c = c_+ > 0,$$

$$\frac{3 \hat{f} - 1}{\sqrt{\hat{f}^2 + 4kg}} \approx \frac{1}{3} \quad \text{for} \quad c = c_- < 0.$$ 

These considerations suggest that waves which travel from east to west are more prone to instability than those which travel from west to east.

In the presence of the current $c_0 \neq 0$, we get

$$\frac{3 \hat{f}}{\sqrt{\hat{f}^2 + 4k(fc_0 + g)}} \geq \frac{1}{3} \quad \text{for} \quad c = c_+ > 0,$$

$$\frac{3 \hat{f} - 1}{\sqrt{\hat{f}^2 + 4k(fc_0 + g)}} \approx \frac{1}{3} \quad \text{for} \quad c = c_- < 0.$$ 

In particular, we deduce that an adverse current with $cc_0 > 0$ favours instability in the sense that the threshold on the steepness for the wave to be unstable is decreased compared to the case without current. Conversely, this threshold is increased by a following current with $cc_0 < 0$. 

Let us also note that, for equatorial waves, $\hat{f} = 2\Omega$, we recover the result obtained in [19], that is, the right-hand side of (48) has the expression

$$\frac{3\Omega \pm \sqrt{\Omega^2 + k(2\Omega c_0 + g)}}{\Omega \pm 3\sqrt{\Omega^2 + k(\Omega c_0 + g)}}.$$ 

For waves near the North Pole, $\hat{f} = 0$, the right-hand side of (48) becomes $\frac{1}{3}$ and we recover the result [41] for Gerstner’s wave.

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**REFERENCES**

[1] B. J. Bayly, Three-dimensional instabilities in quasi-two-dimensional inviscid flows, in *Nonlinear Wave Interactions in Fluids*, edited by R. W. Miksad et al., 71–77, ASME, New York, 1987.

[2] A. Bennett, *Lagrangian Fluid Dynamics*, Cambridge University Press, Cambridge, 2006.

[3] A. Constantin, *On the deep water wave motion*, *J. Phys. A*, 34 (2001), 1405–1417.

[4] A. Constantin, *Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis*, CBMS-NSF Conference Series in Applied Mathematics, vol. 81, SIAM, Philadelphia, 2011.

[5] A. Constantin, *An exact solution for equatorially trapped waves*, *J. Geophys. Res.*, 117 (2012), C05029.

[6] A. Constantin, *On the modelling of equatorial waves*, *Geophys. Res. Lett.*, 39 (2012), L05602.

[7] A. Constantin, *Some three-dimensional nonlinear equatorial flows*, *J. Phys. Oceanogr.*, 43 (2013), 165–175.

[8] A. Constantin, *Some nonlinear, equatorially trapped, nonhydrostatic internal geophysical waves*, *J. Phys. Oceanogr.*, 44 (2014), 781–789.

[9] A. Constantin and P. Germain, *Instability of some equatorially trapped waves*, *J. Geophys. Res. Oceans*, 118 (2013), 2802–2810.

[10] A. Constantin and R. S. Johnson, *The dynamics of waves interacting with the Equatorial Undercurrent*, *Geophys. Astrophys. Fluid Dyn.*, 109 (2015), 311–358.

[11] A. Constantin and R. S. Johnson, *An exact, steady, purely azimuthal equatorial flow with a free surface*, *J. Phys. Oceanogr.*, 46 (2016), 1935–1945.

[12] A. Constantin and R. S. Johnson, *An exact, steady, purely azimuthal flow as a model for the Antarctic circumpolar current*, *J. Phys. Oceanogr.*, 46 (2016), 358503594.

[13] A. Constantin and R. S. Johnson, *A nonlinear, three-dimensional model for ocean flows, motivated by some observations of the pacific equatorial undercurrent and thermocline*, *Phys. Fluids*, 29 (2017), 056604.

[14] A. Constantin and S. G. Monismith, *Gerstner waves in the presence of mean currents and rotation*, *J. Fluid Mech.*, 820 (2017), 511–528.

[15] B. Cushman-Roisin and J. M. Beckers, *Introduction to Geophysical Fluid Dynamics: Physical and Numerical Aspects*, Academic, Waltham, Mass., 2011.

[16] L. Fan and H. Gao, *Instability of equatorial edge waves in the background flow*, *Proc. Amer. Math. Soc.*, 145 (2017), 765–778.

[17] B. J. Bayly, *Three-dimensional instabilities in quasi-two-dimensional inviscid flows, in Nonlinear Wave Interactions in Fluids*, edited by R. W. Miksad et al., 71–77, ASME, New York, 1987.

[18] S. Friedlander and M. M. Vishik, *Instability criteria for the flow of an inviscid incompressible fluid*, *Phys. Rev. Lett.*, 66 (1991), 2204–2206.

[19] F. Genoud and D. Henry, *Instability of equatorial water waves with an underlying current*, *J. Math. Fluid Mech.*, 16 (2014), 661–667.

[20] F. Gerstner, *Theorie der Wellen samt einer daraus abgeleiteten Theorie der Deichprofile*, *Ann. Phys.*, 2 (1809), 412–445.

[21] A. E. Gill, *Atmosphere-Ocean Dynamics*, Academic, 1982.

[22] D. Henry, *On Gerstner’s water wave*, *J. Nonlinear Math. Phys.*, 15 (2008), 87–95.

[23] D. Henry, *An exact solution for equatorial geophysical water waves with an underlying current*, *Eur. J. Mech. B Fluids*, 38 (2013), 18–21.
[24] D. Henry, Exact equatorial water waves in the $f$-plane, *Nonlinear Anal. Real World Appl.*, 28 (2016), 284–289.
[25] D. Henry, Equatorially trapped nonlinear water waves in a $\beta$-plane approximation with centripetal forces, *J. Fluid Mech.*, 804 (2016), R1, 11 pp.
[26] D. Henry, A modified equatorial $\beta$-plane approximation modelling nonlinear wave-current interactions, *J. Differential Equations*, 263 (2017), 2554–2566.
[27] D. Henry, On three-dimensional Gerstner-like equatorial water waves, *Philos. Trans. Roy. Soc. A*, 376 (2018), 20170088, 16 pp.
[28] D. Henry and H.-C. Hsu, Instability of Equatorial water waves in the $f$-plane, *Discrete Contin. Dyn. Syst.*, 35 (2015), 909–916.
[29] D. Henry and H.-C. Hsu, Instability of internal equatorial water waves, *J. Differential Equations*, 258 (2015), 1015–1024.
[30] H.-C. Hsu, An exact solution for equatorial waves, *Monatsh. Math.*, 176 (2015), 143–152.
[31] D. Ionescu-Kruse, An exact solution for geophysical edge waves in the $f$-plane approximation, *Nonlinear Anal. Real World Appl.*, 24 (2015), 190–195.
[32] D. Ionescu-Kruse, An exact solution for geophysical edge waves in the $\beta$-plane approximation, *J. Math. Fluid Mech.*, 17 (2015), 699–706.
[33] D. Ionescu-Kruse, Instability of equatorially trapped waves in stratified water, *Ann. Mat. Pura Appl.*, 195 (2016), 585–599.
[34] D. Ionescu-Kruse, Instability of Pollard’s exact solution for geophysical ocean flows, *Phys. Fluids*, 28 (2016), 086601.
[35] D. Ionescu-Kruse, On the short-wavelength stabilities of some geophysical flows, *Phil. Trans. R. Soc. A*, 376 (2017), 20170090, 21pp.
[36] D. Ionescu-Kruse, A three-dimensional autonomous nonlinear dynamical system modelling equatorial ocean flows, *J. Differential Equations*, 264 (2018), 4650–4668.
[37] R. S. Johnson, Application of the ideas and techniques of classical fluid mechanics to some problems in physical oceanography, *Philos. Trans. R. Soc. A*, 376 (2018), 20170092, 19 pp.
[38] M. Kluczek, Physical flow properties for Pollard-like internal water waves, *J. Math. Phys.*, 59 (2018), 123102, 12pp.
[39] M. Kluczek, Exact Pollard-like internal eater waves, *J. Nonlinear Math. Phys.*, 26 (2019), 135–146.
[40] H. Lamb, *Hydrodynamics*, Reprint of the 1932 sixth edition. With a foreword by R. A. Caflisch [Russel E. Caflisch]. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1993.
[41] S. Leblanc, Local stability of Gerstner’s waves, *J. Fluid Mech.*, 506 (2004), 245–254.
[42] A. Lifschitz and E. Hameiri, Local stability conditions in fluid mechanics, *Phys. Fluids*, 3 (1991), 2644–2651.
[43] A.-V. Matioc, An exact solution for geophysical equatorial edge waves over a sloping beach, *J. Phys. A*, 45 (2012), 365501, 10pp.
[44] A.-V. Matioc, Exact geophysical waves in stratified fluids, *Appl. Anal.*, 92 (2013), 2254–2261.
[45] R. T. Pollard, Surface waves with rotation: An exact solution, *J. Geophys. Res.*, 75 (1970), 5895–5898.
[46] A. Rodriguez-Sanjurjo, Global diffeomorphism of the Lagrangian flow-map for equatorially-trapped internal water waves, *Nonlinear Anal.*, 149 (2017), 156–164.
[47] A. Rodriguez-Sanjurjo, Global diffeomorphism of the Lagrangian flow-map for Pollard-like solutions, *Ann. Mat. Pura Appl.*, 197 (2018), 1787–1797.
[48] S. Sastre-Gomez, Global diffeomorphism of the Lagrangian flow-map defining equatorially trapped water waves, *Nonlinear Anal.*, 125 (2015), 725–731.
[49] G. K. Vallis, *Atmospheric and Oceanic Fluid Dynamics*, Cambridge University Press, 2006.

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