Limiting behaviors of the Brownian motions on hyperbolic spaces

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Abstract

Using the explicit representations of the Brownian motions on the hyperbolic spaces, we show that their almost sure convergence and the central limit theorems for the radial components as time tends to infinity are easily obtained. We also give a straightforward strategy to obtain the explicit expressions for the limit distributions or the Poisson kernels.

1. Introduction

Hyperbolic spaces are non-compact Riemannian symmetric spaces of rank one. By classification, we have four types of hyperbolic spaces: the real one $\mathbb{H}^n = SO_0(1,n)/SO(n)$, the complex one $\mathbb{H}^n = SU(1,n)/SU(n)$, the quaternionic one $Sp(1,n)/(Sp(1) \times Sp(n))$ and the Cayley hyperbolic plane. In this article we consider the limiting behaviors of the Brownian motions, that is, the diffusion processes generated by the Laplace-Beltrami operators on the first three types of the hyperbolic spaces.

The hyperbolic spaces have negative bounded curvatures. The Brownian motions on negatively curved manifolds have been studied in the connection of so-called the Liouville property by many authors and it is well known that the Brownian motions tends to infinity almost surely as time tends to infinity. See, e.g., Kifer [16]. Needless to say, the limit distributions are given by the Poisson kernels.

On the other hand, the Brownian motions on the Riemannian symmetric spaces of non-compact type have been also studied by several authors since the work by Malliavin-Malliavin [20]. Among them, we refer to Babillot [3], where

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a central limit theorem for the radial components of the Brownian motions has been shown.

The purpose of this article is to show that these properties are easily and directly shown for the Brownian motions on the hyperbolic spaces if we adopt the upper half space realizations of the hyperbolic spaces instead of the ball models. We can describe the same stories on the three types of spaces. At first, by solving the corresponding stochastic differential equations, we represent the Brownian motions in closed forms as Wiener functionals. Then, the almost sure convergence of them is readily seen from the representations. Moreover, by inserting the representations into the formulae for the distance functions, we can also show the central limit theorems for the radial components.

For the computations of the limiting distributions or the Poisson kernels, we need some results on the distributions of the random variables defined by the perpetual (infinite) integrals in time of the usual geometric Brownian motions with negative drifts. The auxiliary results are given in the appendix and, by using them, we compute the Fourier transforms of the limiting distributions and the inverse transforms in direct ways.

2. Real hyperbolic spaces

For \( n \geq 1 \), let \( \mathbb{H}^{n+1}_r \) be the upper half space in \( \mathbb{R}^{n+1} \),

\[
\mathbb{H}^{n+1}_r = \{ z = (x, y) = (x_1, \ldots, x_n, y); x \in \mathbb{R}^n, y > 0 \},
\]

endowed with the Riemannian metric \( ds^2 = y^{-2}(dx^2 + dy^2) \). The volume element is given by \( y^{-n-1}dx dy \), the distance function \( d(z, z') \) is given by

\[
\cosh(d(z, z')) = \frac{|x - x'|^2 + y^2 + (y')^2}{2yy'},
\]

in an obvious notation, where \( |x| \) is the Euclidean norm. The Laplace-Beltrami operator is written as

\[
\Delta_r = y^2 \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} + y^2 \frac{\partial^2}{\partial y^2} - (n - 1)y \frac{\partial}{\partial y}.
\]

For details of the fundamental objects on \( \mathbb{H}^{n+1}_r \), see, e.g., Davies [6].

We first show an explicit expression as Wiener functional of the Brownian motion on \( \mathbb{H}^{n+1}_r \) by solving the corresponding stochastic differential equation. Let \( (W^{(n+1)}, B^{(n+1)}, \mathbb{P}^{(n+1)}) \) be the \((n + 1)\)-dimensional standard Wiener space with the canonical filtration \( \{\mathcal{B}^{(n+1)}_t\}_{t \geq 0} \). Corresponding to the rectangular coordinate, we denote an element of \( W^{(n+1)} \) by

\[
(w(\cdot), B(\cdot)) \quad \text{or} \quad (w_1(\cdot), \ldots, w_n(\cdot), B(\cdot)),
\]

which is an \( \mathbb{R}^{n+1} \)-valued continuous function on \([0, \infty)\) with \( w_i(0) = B(0) = 0 \). Then the Brownian motion on \( \mathbb{H}^{n+1}_r \), the diffusion process with infinitesimal generator \( \Delta_r/2 \), is obtained by solving the following stochastic differential equation
defined on \((W^{(n+1)}, B^{(n+1)}, P^{(n+1)})\) (see [15]):

\[
\begin{align*}
&dX_i(t) = Y(t)dw_i(t), \\
&dY(t) = Y(t)dB(t) - \frac{n-1}{2}Y(t)dt.
\end{align*}
\]

The unique solution \(Z_z = \{(X(t, z), Y(t, z))\}_{t \geq 0}, z = (x, y),\) satisfying \(X(0) = x\) and \(Y(0) = y\) is given by

\[
\begin{align*}
X_i(t, z) &= x_i + \int_0^t y \exp(B_s^{(-\mu)})dw_i(s), \\
Y(t, z) &= y \exp(B_t^{(-\mu)}),
\end{align*}
\]

where \(B_s^{(-\mu)} = B(s) - \mu s\) and \(\mu = n/2\). \{Y(t, z)\} is a usual geometric Brownian motion with negative drift and it is easy to see that \(Z_z(t)\) converges to the boundary as \(t \to \infty\) almost surely.

Now we consider the exponential functional \(A_t^{(-\mu)}\) given by

\[
A_t^{(-\mu)} = \int_0^t \exp(2B_s^{(-\mu)})ds.
\]

Then it is easy to see the identity in law

\[
(X(t, z), Y(t, z)) \overset{\text{(law)}}{=} (x + yw(A_t^{(-\mu)}), y \exp(B_t^{(-\mu)}))
\]

for fixed \(t > 0\).

An explicit expression for the density of the distribution of \((A_t^{(-\mu)}, B_t^{(-\mu)})\) is known by Yor [27] and, by using it, Gruet [12] has shown an expression for the heat kernel of the semigroup generated by \(\Delta_r\). For the classical expression, see Davies [6]. We also refer to [2, 14, 21, 23] for related topics.

We combine the identity in law with formula (2.1). Then we get

\[
\cosh(d(Z(t, z), z)) \overset{\text{(law)}}{=} \frac{1}{2} \left\{ |yw(A_t^{(-\mu)})|^2 + 1 \right\} \exp(-B_t^{(-\mu)}) + \frac{1}{2} \exp(B_t^{(-\mu)}).
\]

Since \(A_t^{(-\mu)}\) converges as \(t \to \infty\) and \(\log(\cosh(u)) = u \cdot (1 + o(1))\) as \(u \to \infty\), we readily get the following central limit theorem.

**Theorem 2.1.** The probability distribution of \(\sqrt{t}^{-1}(d(Z(t, z), z) - nt/2)\) converges weakly as \(t \to \infty\) to the standard normal distribution.

Recall formula \(\Delta_r d(z_0, \cdot) = n \coth d(z_0, \cdot)\). Then, by Itô’s formula, we get

\[
\begin{align*}
d(Z_z(t), z) &= \sum_{i=1}^n \int_0^t \frac{1}{\sinh d(z, Z_z(s))} \frac{X_i^z(s) - x}{y}dw_i^z \\
&\quad + \int_0^t \frac{1}{\sinh d(z, Z_z(s))} \left( \frac{Y_z^z(s)}{y} - \cosh d(z, Z_z(s)) \right) dB(s) \\
&\quad + \frac{n}{2} \int \coth d(z, Z_z(s)) ds,
\end{align*}
\]
Next we recall Dufresne’s identity (Theorem A.1 in the Appendix) in law
\( A^{-\mu}_\infty \) (law) \((2\gamma_\mu)^{-1}\) for a Gamma random variable \( \gamma_\mu \) with parameter \( \mu \). Then, for a bounded continuous function \( \varphi \) on \( \mathbb{R}^n \), we obtain

\[
E[\varphi(X(t, z))] = E[\varphi(x + yw(\mathcal{A}_t^{[-n/2]}))] \rightarrow \\
\int_0^\infty \frac{1}{\Gamma(n/2)} t^{(n/2)-1} e^{-t} dt \int_{\mathbb{R}^n} \varphi(x + \eta) \frac{1}{(2\pi y^2/2t)^{n/2}} \exp\left(-\frac{|\eta|^2}{2y^2/2t}\right) d\eta
\]

\[
= \int_{\mathbb{R}^n} \varphi(\xi) d\xi \int_0^\infty \frac{1}{\Gamma(n/2)} \frac{1}{\pi^{n/2}y^n} t^{n-1} \exp\left(-\frac{y^2 + |\xi - x|^2}{y^2/2t}\right) dt
\]

\[
= \int_{\mathbb{R}^n} \varphi(\xi) p_{n+1}(\xi - x, y) d\xi,
\]

where

\[
p_{n+1}(\xi, y) = \frac{2^{n-1} \Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{y^n}{(y^2 + |\xi|^2)^{(n+1)/2}}, \quad \xi \in \mathbb{R}^n,
\]

and we have used the duplication formula for the Gamma function.

Hence we have proved the following.

**Theorem 2.2.** For any \((x, y) \in H^{n+1}_r\), \( X(t, z) \) converges almost surely as \( t \to \infty \) and the density of the limit distribution is given by the Poisson kernel \( p_{n+1}(\xi - x, y) \). In particular, when \( n = 1 \), the limit distribution is Cauchy.

We end this section with mentioning on the Poisson kernel in the Euclidean spaces and on the Fourier transforms. The Poisson kernel on \( \mathbb{R}^{n+1} \) of the hyper-plane \( \{y = 0\} \) is given by

\[
q_{n+1}(\xi, y) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{y^n}{(y^2 + |\xi|^2)^{(n+1)/2}}
\]

which is different from \( p_{n+1}(\xi, y) \) for \( n \geq 2 \). The Brownian motion on the hyperbolic plane \( H^2 \) is a time change of the 2-dimensional standard Brownian motion, and the Poisson kernels coincide.

It is well known that the Fourier transform of \( q_{n+1}(\xi, y) \) in \( \xi \) is the simple exponential function,

\[
\int_{\mathbb{R}^n} e^{\sqrt{-1} \langle \lambda, \xi \rangle} q_{n+1}(\xi, y) d\xi = e^{-y|\lambda|}.
\]

For the hyperbolic spaces, we can show, for example,

\[
\varphi_3(\lambda; y) \equiv \int_{\mathbb{R}^2} e^{\sqrt{-1} \langle \lambda, \xi \rangle} p_3(\xi, y) d\xi = y|\lambda| K_1(y|\lambda|)
\]

and

\[
\varphi_4(\lambda; y) \equiv \int_{\mathbb{R}^3} e^{\sqrt{-1} \langle \lambda, \xi \rangle} p_4(\xi, y) d\xi = (y|\lambda| + 1) e^{-y|\lambda|},
\]
where $K_1$ is the modified Bessel function. By virtue of the strong Markov property, we can easily show that the distribution of $X(\tau_a)$ for the first hitting time $\tau_a$ of the Brownian motion $\{Z_z(t, z)\}$ at the level $y = a, a > 0$ is determined by

$$E[\exp(\sqrt{-1}(\lambda, X(\tau_a)))] = e^{\sqrt{-1} \langle \lambda, x \rangle \varphi_n(\lambda; y) \varphi_n(\lambda; a)}, \lambda \in \mathbb{R}^n.$$  

3. Complex hyperbolic spaces

Let $\mathbb{H}_c^n, n \geq 2$, be the upper half space of $\mathbb{C}^n$ given by

$$\{z = (z_1, z_2, \ldots, z_n) = (z_1, \bar{z}) \in \mathbb{C}^n; h(z) \equiv \text{Im}(z_1) - |\bar{z}|^2 > 0\},$$

endowed with the Bergmann metric

$$ds^2 = -\sum_{j,k=1}^n \partial_{z_j} \partial_{\bar{z}_k} \log(h) dz_j d\bar{z}_k.$$

The unit ball $\{|z| < 1\}$ in $\mathbb{C}^n$ with the Bergmann metric

$$-\sum_{j,k=1}^n \partial_{z_j} \partial_{\bar{z}_k} \log(1 - |z|^2) dz_j d\bar{z}_k.$$

is isometric with $\mathbb{H}_c^n$. For details, we refer to [7, 8, 10, 24]. We should be aware of difference of conventions. The curvatures of these manifolds are bounded and negative, but are not constant (cf. p.190, [10]).

We change the first coordinate by $x_1 = \text{Re}(z_1)/2$ and $y = h(z)^{1/2}$. Then we have the same realization of the complex hyperbolic space $SU(1, n)/SU(n)$ as in Venkov [24]: if we write $z_k = x_k + \sqrt{-1} y_k, k = 2, \ldots, n$, the Riemannian metric is written as

$$ds^2 = \frac{1}{y^2} dy^2 + \frac{1}{y^2} \sum_{k=2}^n (dx_k^2 + dy_k^2) + \frac{1}{y^4} \left(dx_1 + \sum_{k=2}^n (x_k dy_k - y_k dx_k)\right)^2,$$

and the distance function $d(z, z')$ is given by

$$(\cosh(d(z, z')))^2 = \frac{(y')^2 + \Phi)^2 + 4\varphi^2}{4y'^2(y')^2},$$

where

$$\Phi = y^2 + |z' - \bar{z}|^2 \quad \text{and} \quad \varphi = x'_1 - x_1 + \sum_{k=2}^n (y'_k x_k - x'_k y_k).$$

The Laplace-Beltrami operator is written by

$$\Delta_c = y^4 \frac{\partial^2}{\partial x_1^2} + y^2 \frac{\partial^2}{\partial y^2} - (2n - 1)y \frac{\partial}{\partial y}$$

$$+ y^2 \sum_{k=2}^n \left\{ \left( \frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial x_1} \right)^2 + \left( \frac{\partial}{\partial y_k} - x_k \frac{\partial}{\partial x_1} \right)^2 \right\}.$$
Letting \((W^{(2n)}, B^{(2n)}, P^{(2n)})\) be the \((2n)\)-dimensional standard Wiener space and denoting an element of \(W^{(2n)}\) by 

\[
(B(\cdot), w_2(\cdot), w_3(\cdot), ..., w_{2n}(\cdot)) \quad \text{or} \quad (B(\cdot), w_2(\cdot), \tilde{w}(\cdot)),
\]

we can check that the Brownian motion \(\{Z(t)\}\) on \(H^n_c\) with \(Z(0) = (x_1, y, z_2, ..., z_n)\), \(z_k\) being identified with \((x_k, y_k)\), is given by

\[
\begin{align*}
X(t) &= x_1 + \int_0^t Y(s)^2 dW_2(s) + 2 \sum_{k=2}^n S_k(t) \\
Y(t) &= y \exp(B(t) - nt) \\
X_k(t) &= x_k + \int_0^t Y(s) dW_{2k-1}(s), \\
Y_k(t) &= y_k + \int_0^t Y(s) dW_{2k}, \quad k = 2, ..., n,
\end{align*}
\]

where we have used the trivial notations \(X(t), Y(t), X_k(t), Y_k(t)\) for the components of \(Z(t)\) and \(S_k(t)\) is the stochastic area enclosed by \(\{(X_k(s), Y_k(s))\}_{0 \leq s \leq t}\) and its chord,

\[
S_k(t) = \frac{1}{2} \int_0^t (Y_k(s) dX_k(s) - X_k(s) dY_k(s)).
\]

\(\{Y(t)\}\) is again a usual geometric Brownian motion with negative drift and \(Z(t)\) converges as \(t \to \infty\). Hence we easily obtain the following central limit theorem.

**Theorem 3.1.** For the Brownian motion \(\{Z(t)\}\) on the \(n\)-dimensional complex hyperbolic space, the probability law of \(\sqrt{\frac{t}{n}} (d(Z(t), Z(0)) - nt)\) converges weakly as \(t \to \infty\) to the standard normal distribution.

Next we compute the limiting distribution of \((X(t), \tilde{Z}(t))\) as \(t \to \infty\), where

\[
\tilde{Z}(t) = (X_2(t), Y_2(t), ..., X_n(t), Y_n(t)).
\]

If we consider the ball model, we obtain the Poisson kernels as the densities of the image measures of the uniform measure on the sphere by the isometries. However, since the same strategy works in the more complicated case of the quaternionic hyperbolic space whose geometry has not been well understood (see a recent work by Kim-Parker \[17\] and references cited therein), we give the following straightforward computations.

For this purpose we first fix \(t\) and consider the characteristic function. As in the previous section, we set

\[
\begin{align*}
A_t^{(-\mu)} &= \int_0^t e^{2B_t^{(-\mu)}} ds, \\
\tilde{A}_t^{(-\mu)} &= \int_0^t e^{4B_t^{(-\mu)}} ds,
\end{align*}
\]
\[ B_s^{(-\mu)} = B(s) - \mu s \] and \( \mu = n \). For the stochastic analysis on \( \mathbb{H}_c^n \) and \( \mathbb{H}_q^n \), we need to consider these two exponential functionals. Then, by the expression (1.23), it is easy to see that, for fixed \( t > 0 \), \( \langle X(t), Z(t) \rangle \) is identical in law with

\[
(x_1 + y^2w_2(A^{(-\mu)}_t) + y\phi(A^{(-\mu)}_t)) + 2y^2\sum \tilde{S}_k(A^{(-\mu)}_t), \tilde{z} + y\tilde{w}(A^{(-\mu)}_t),
\]

where \( \sum \) denotes the sum over \( k = 2, \ldots, n \), \( \phi(t) = \sum(y_kw_{2k-1}(t) - x_kw_{2k}(t)) \) and \( \tilde{S}_k(t) = \frac{1}{2} \int_0^t (w_{2k}(s)dw_{2k-1}(s) - w_{2k-1}(s)dw_{2k}(s)) \).

Hence we may write, for any bounded continuous function \( g \) on \( \mathbb{R}^{2(n-1)} \),

\[
E[e^{\sqrt{-T}pX(t)}g(Z(t))] = E\left[e^{\sqrt{-T}(x+y^2w_2(A^{(-\mu)}_t)+y\phi(A^{(-\mu)}_t))}g(\tilde{z} + y\tilde{w}(A^{(-\mu)}_t)) \times E\left[\prod_{k=2}^n e^{2\sqrt{-T}p^2(A^{(-\mu)}_t)}\right]|\{B(s)\}, \tilde{w}(A^{(\mu)}_t)\right].
\]

Then, applying the Lévy formula for the characteristic function of the stochastic area (cf. [13], p.473), we get

\[
E[e^{\sqrt{-T}pX(t)}g(\tilde{z} + Z(t))] = E\left[e^{\sqrt{-T}(x+y^2w_2(A^{(-\mu)}_t)+y\phi(A^{(-\mu)}_t))}g(\tilde{z} + y\tilde{w}(A^{(-\mu)}_t)) \times \left(\frac{py^2A^{(-\mu)}_t}{\sinh(py^2A^{(-\mu)}_t)}\right)^{n-1} \exp\left(1 - py^2A^{(-\mu)}_t \coth(py^2A^{(-\mu)}_t)\right)\right] \cdot \int_{\mathbb{R}^{2(n-1)}} \sum (y_k\xi_k - x_k\eta_k) \cdot \frac{p}{2\pi \sinh(py^2A^{(-\mu)}_t)} \cdot \left(\frac{1}{\cosh(py^2A^{(-\mu)}_t)}\right)^{n-1} e^{-p \coth(py^2A^{(-\mu)}_t)\zeta^2/2} d\zeta d\eta,
\]

where \( \zeta = (\xi, \eta) = (\xi_2, \eta_2, \ldots, \xi_n, \eta_n) \).

Now we put, for \( \mathbf{q} = (q_2, \ldots, q_n), \mathbf{r} = (r_2, \ldots, r_n) \in \mathbb{R}^{n-1} \),

\[
g(\zeta) = \exp(\sqrt{-1}(\langle \mathbf{q}, \xi \rangle + \langle \mathbf{r}, \eta \rangle)).
\]

Then, carrying out the Gaussian integral with respect to \( \xi \) and \( \eta \), we get

\[
E\left[\exp\left\{\sqrt{-1}(pX(t) + \sum(q_kX_k(t) + r_kY_k(t)))\right\}\right] = e^{\sqrt{-T}p} E\left[e^{-p^2y^4/2 \left(\frac{1}{\cosh(py^2A^{(-\mu)}_t)}\right)^{n-1} e^{-F \tanh(py^2A^{(-\mu)}_t)}}\right],
\]
where
\[ f = f(p, q, r) = px + \sum (q_kx_k + r_ky_k) \]
and
\[ F = F(p, q, r) = \sum \frac{(q_k + py_k)^2 + (r_k - px_k)^2}{2p}. \]

Now, letting \( t \to \infty \), we obtain the following.

**Proposition 3.2.** For any \( p \in \mathbb{R}, q, r \in \mathbb{R}^{n-1} \), one has

\[
(3.4) \quad \lim_{t \to \infty} E\left[ \exp\left\{ \sqrt{-1}(pX(t) + \sum (q_kX_k(t) + r_kY_k(t))) \right\} \right] = e^{\sqrt{-1}f} E\left[ e^{-p^2y^4A^-(-n)/2} \left( \frac{1}{\cosh(py^2A^-(-n))} \right)^{n-1} e^{-F \tanh(py^2A^-(-n))} \right].
\]

Denote the right hand side of (3.4) by \( I(p, q, r) \). By using the joint Laplace transform of \( A^-\left(\cdot\right) \) and \( \tilde{A}^-\left(\cdot\right) \) given by Corollary [A.5] in the appendix, we obtain

\[
I(p, q, r) = e^{\sqrt{-1}f} \int_0^\infty \left( \frac{1}{\cosh(py^2u)} \right)^{n-1} e^{-F \tanh(py^2u)} P\left( A^-\left(\cdot\right) \in du \right) du.
\]

Then, changing the variable, we see that \( I(p, q, r) \) is equal to

\[
e^{\sqrt{-1}f} \int_0^\infty \left( \frac{1}{\sinh(u)} \right)^{n+1} \left( \frac{1}{\cosh(u)} \right)^{n-1} e^{-py^2 \coth(u)/2} \times \exp\left(- \sum \frac{(q_k + py_k)^2 + (r_k - px_k)^2}{2p} \tanh(u) \right) du
\]

if \( p > 0 \) and to

\[
e^{\sqrt{-1}f} \int_0^\infty \left( \frac{1}{\sinh(u)} \right)^{n+1} \left( \frac{1}{\cosh(u)} \right)^{n-1} e^{py^2 \coth(u)/2} \times \exp\left( \sum \frac{(q_k + py_k)^2 + (r_k - px_k)^2}{2p} \tanh(u) \right) du
\]
if $p < 0$. From these expressions, we can take the Fourier inversion

$$f_n(x', z'; z) = \frac{1}{(2\pi)^{2n-1}} \int_{\mathbb{R}^{2n-1}} I(p, q, r) e^{-\sqrt{-1}(px' + \sum(q_kx'_k + ry'_k))} dp dq_2 \cdots dr_n.$$

For the integral with respect to $q_k$ when $p > 0$, we note as usual

$$-\frac{(q_k + ry_k)^2}{2p} \tanh(u) + \sqrt{-1}q_k(x_k - x'_k)$$

$$= -\frac{\tanh(u)}{2p} (q_k + ry_k - \sqrt{-1}p(x_k - x'_k) \coth(u))^2$$

$$- \sqrt{-1}ry_k(x_k - x'_k) - \frac{p}{2}(x_k - x'_k)^2 \coth(u).$$

We do the same computations also for the other variables and for $p < 0$. Then, after some manipulations, we obtain

$$f_n(x', z'; z) = \frac{2y^{2n}}{(4\pi)^n \Gamma(n)} \int_{\mathbb{R}^n} [p]^{2n-1} e^{\sqrt{-1}p^2} dp \int_0^\infty \left( \frac{1}{\sinh(u)} \right)^{2n} e^{-\Phi[p] \coth(u)/2} du$$

$$= \frac{2y^{2n}}{\pi^n \Gamma(n)} \int_0^\infty p^{2n-1} \cos(2\varphi p) dp \int_0^\infty \left( \frac{1}{\sinh(u)} \right)^{2n} e^{-\Phi[p] \coth(u)} du,$$

where we have made a simple change of variable for the second equality and $\varphi$ and $\Phi$ are given by (3.1).

For the second integral, we change the variable by $k = \coth(u)$ to obtain

$$f_n(x', z'; z) = \frac{2y^{2n}}{\pi^n \Gamma(n)} \int_0^\infty p^{2n-1} \cos(2\varphi p) dp \int_1^\infty e^{-\Phi[p] (k^2 - 1)^{n-1}} dk.$$

Now we recall the following integral representation of the modified Bessel function (cf. Lebedev [18] p.119 or [11] p.322)

$$(3.5) \quad K_\nu(z) = \frac{\sqrt{\pi}}{\Gamma(\nu + 1/2)} \left( \frac{z}{2} \right)^\nu \int_1^\infty e^{-zt} (t^2 - 1)^{\nu-1/2} dt, \quad \nu > 0.$$  

Then we obtain

$$f_n(x', z'; z) = \frac{2^{n+1}2^{2n}}{\pi^{n+1}2^{n} \Phi^{n-1/2}} \int_0^\infty p^{n-1/2} \cos(2\varphi p) K_{n-1/2}(\Phi p) dp$$

For the integral on the right hand side, we may apply the formulae

$$\int_0^\infty x^\lambda K_\mu(ax) \cos(bx) dx = 2^{\lambda-1}a^{-\lambda-1} \Gamma\left( \frac{\mu + \lambda + 1}{2} \right) \Gamma\left( \frac{1 + \lambda - \mu}{2} \right)$$

$$\times F\left( \frac{\mu + \lambda + 1}{2}, \frac{1 + \lambda - \mu}{2}; \frac{1}{2}; \frac{b^2}{a^2} \right).$$

(cf. [11] p.747) and $F(n, a, a; z) = (1 - z)^{-n}$. Then we obtain

$$(3.6) \quad f_n(x', z'; z) = \frac{2^{2n-1} \Gamma(n) y^{2n}}{\pi^n \Phi^{2n}} \frac{1}{\Gamma(n)} F\left( n, -\frac{4\varphi^2}{\Phi^2} \right) = \frac{2^{2n-1} \Gamma(n) y^{2n}}{\pi^n (4\varphi^2 + \Phi^2)^n}.$$
Theorem 3.3 (cf. [7]). For any \( z \in \mathbb{H}^n_c \), \((X(t), \tilde{Z}(t))\) converges almost surely as \( t \to \infty \) and the density of the limit distribution on \( \mathbb{R}^{2n-1} \) is the Poisson kernel given by (3.6).

4. Quaternionic hyperbolic spaces

For the quaternion hyperbolic space \( Sp(1,n)/(Sp(1) \times Sp(n)) \), \( n \geq 2 \), we follow the conventions in Venkov [24]. See also Helgason [13], Lohoué-Rychner [19], and Kim-Parker [17] for the basic properties. For \( n \geq 2 \), let \( \mathbb{H}^n_q \) be the upper half space in \( \mathbb{C}^{2n} \),

\[
\mathbb{H}^n_q = \{ z = (z_1, z_2, \ldots, z_{2n}) = (z_1, \bar{z}) \in \mathbb{C}^{2n}; \text{Im}(z_1) > 0 \},
\]

with the Riemannian metric

\[
ds^2 = \frac{dy^2}{y^2} + \frac{1}{y^2} \sum_{k=2}^{n}(dz_k d\bar{z}_k + dz_{n+k} d\bar{z}_{n+k})
\]

\[
+ \frac{1}{y^4} \left( dx_1 + \text{Im} \sum_{k=2}^{n}(\bar{z}_k dz_k + \bar{z}_{n+k} d\bar{z}_{n+k}) \right)^2
\]

\[
+ \frac{1}{y^4} \left| dz_{n+1} + \sum_{k=2}^{n}(z_{n+k} dz_k - z_k d\bar{z}_{n+k}) \right|^2,
\]

where \( z_1 = x_1 + \sqrt{-1}y \). We will write \( z_k = x_k + \sqrt{-1}y_k \) for \( k = 2, \ldots, 2n \). Note that the first and \((n+1)\)-th components, \( z_1 \) and \( z_{n+1} \) play special roles.

The volume element is \( y^{-4n-3} dx_1 dy \prod_{k=2}^{2n} dx_k dy_k \) and the distance function \( d(z, z') \) is given by

\[
(\cosh(d(z, z')))^2 = \frac{(y'^2 + \Phi)^2 + 4(\varphi_1^2 + \varphi_2^2 + \varphi_3^2)}{4y^2(y')^2},
\]

where

\[
\Phi = y'^2 + \sum_{k=2}^{n}(|z'_k - z_k|^2 + |z'_{n+k} - z_{n+k}|^2),
\]

\[
\varphi_1 = x'_1 - x_1 + \sum_{k=2}^{n}\left((y'_k x_k - x'_k y_k) + (y'_{n+k} x_{n+k} - x'_{n+k} y_{n+k})\right),
\]

\[
\varphi_2 = x'_{n+1} - x_{n+1} + \sum_{k=2}^{n}\left((x'_{k} x_{n+k} - x'_{n+k} x_k) + (y'_{n+k} y_k - y'_k y_{n+k})\right),
\]

\[
\varphi_3 = y'_{n+1} - y_{n+1} + \sum_{k=2}^{n}\left((x'_{k} y_{n+k} - y'_{n+k} x_k) + (y'_{k} x_{n+k} - x'_{n+k} y_k)\right).
\]

Note that \( \varphi_i \)’s do not depend on \( y \).
The Laplace-Beltrami operator $\Delta_q$ may be written in a convenient way as
\[
\Delta_q = y^4 \frac{\partial^2}{\partial x_1^2} + y^2 \frac{\partial^2}{\partial y^2} - (4n + 1)y \frac{\partial}{\partial y} + y^4 \left( \frac{\partial^2}{\partial x_{n+1}^2} + \frac{\partial^2}{\partial y_{n+1}^2} \right)
\]
\[
+ y^2 \sum_{k=2}^n \left[ \left( \frac{\partial}{\partial x_k} + y^k \frac{\partial}{\partial y} - x_{n+k} \frac{\partial}{\partial x_{n+1}} - y_{n+k} \frac{\partial}{\partial y_{n+1}} \right)^2 
\]
\[
+ \left( \frac{\partial}{\partial y_k} + y_k \frac{\partial}{\partial y} - x_{n+k} \frac{\partial}{\partial x_{n+1}} - y_{n+k} \frac{\partial}{\partial y_{n+1}} \right)^2 
\]
\[
+ \left( \frac{\partial}{\partial y_{n+k}} - x_{n+k} \frac{\partial}{\partial x_{n+1}} - y_k \frac{\partial}{\partial y_{n+1}} - x_k \frac{\partial}{\partial y_{n+1}} \right)^2 \right].
\]

(4.3)

Note that the coefficients of $\partial^2/\partial x_1 \partial x_{n+1}$, $\partial^2/\partial x_1 \partial y_{n+1}$, $\partial^2/\partial x_{n+1} \partial y_{n+1}$ are zero.

We can describe the same story as for the complex hyperbolic space if we consider
a $4 \times 4$ skew-symmetric matrix instead of 2-dimensional one.

At first we give an explicit expression for the Brownian motion, the diffusion process with generator $\Delta_q/2$, on $H^n_q$. Let $(W^{(4n)}, B^{(4n)}, P^{(4n)})$ be the $(4n)$-dimensional Wiener space and denote an element in $W^{(4n)}$ by
\[
(B_1(\cdot), B(\cdot), w_{2,1}(\cdot), w_{2,2}(\cdot), \ldots, w_{n,1}(\cdot), w_{n,2}(\cdot), B_2(\cdot), B_3(\cdot), w_{n+2,1}(\cdot), w_{n+2,2}(\cdot), \ldots, w_{2n,1}(\cdot), w_{2n,2}(\cdot)).
\]

Then we can check that the Brownian motion $(X(t), Y(t), Z(t))$ starting from $(x_1, y, \tilde{z})$ is given by
\[
X_1(t) = x_1 + \int_0^t Y(s)^2 dB_1(s)
\]
\[
+ \sum_{k=2}^n \int_0^t \left\{ Y_k(s) dX_k(s) - X_k(s) dY_k(s) + Y_{n+k}(s) dX_{n+k}(s) - X_{n+k}(s) dY_{n+k}(s) \right\},
\]
\[
Y(t) = y \exp(B(t) - (2n + 1)t),
\]
\[
X_k(t) = x_k + \int_0^t Y(s) dw_{k,1}(s),
\]
\[
Y_k(t) = y_k + \int_0^t Y(s) dw_{k,2}(s), \quad k = 2, \ldots, n,
\]
\[
X_{n+1}(t) = x_{n+1} + \int_0^t Y(s)^2 dB_2(s)
\]
\[
+ \sum_{k=2}^n \int_0^t \left\{ -X_{n+k}(s) dX_k(s) + X_k(s) dX_{n+k}(s) + Y_{n+k}(s) dY_k(s) - Y_k(s) dY_{n+k}(s) \right\},
\]
\[
Y_{n+1}(t) = y_{n+1} + \int_0^t Y(s)^2 dB_3(s)
\]
Then, from (4.1), it is easy to show the following central limit theorem.

**Theorem 4.1.** The probability law of \((d(Z(t), Z(0)) - (2n + 1)t)/\sqrt{t}\) converges weakly as \(t \to \infty\) to the standard normal distribution.

Next we show that \((X(t), \tilde{Z}(t))\) converges in law as \(t \to \infty\). To identify the limit distribution, we set

\[
f_n(x', \tilde{z}'; z) = \frac{2^{4n+1}\Gamma(2n)}{\pi^{2n}} \frac{y^{2(2n+1)}}{(\Phi^2 + 4(\varphi_1^2 + \varphi_2^2 + \varphi_3^2))^{2n+1}},
\]

where \(\Phi\) and \(\varphi_i\)'s are given by (4.2). \(f_n\) is the Poisson kernel of the boundary \(\partial H_q^0 = \{y = 0\}\).

**Theorem 4.2.** \((X_1(t), \tilde{Z}(t))\), valued in \(\mathbb{R} \times C^{2(n-1)}\), converges almost surely as \(t \to \infty\) and the density of the limit distribution is given by \(f_n(x', \tilde{z}'; z)\).

In the following we give a proof of Theorem 4.2. At first we consider the characteristic function of \((X_1(t), \tilde{Z}(t))\) for fixed \(t\). For convenience we put

\[
X^0_k(t) = X_k(t) - x_k = \int_0^t Y(s)dw_{k,1}(s), \quad Y^0_k(t) = Y_k(t) - y_k = \int_0^t Y(s)dw_{k,2}(s),
\]

and

\[
\theta_k = \begin{pmatrix}
x_k \\
y_k \\
x_{n+k} \\
y_{n+k}
\end{pmatrix}, \quad \Theta_k(t) = \begin{pmatrix}
X_k(t) \\
Y_k(t) \\
X_{n+k}(t) \\
Y_{n+k}(t)
\end{pmatrix}, \quad \Theta^0_k(t) = \begin{pmatrix}
X^0_k(t) \\
Y^0_k(t) \\
X^0_{n+k} \\
Y^0_{n+k}(t)
\end{pmatrix}.
\]

Moreover, \(\xi = \{\xi_1, \xi_2, \xi_3\} \in \mathbb{R}^3\), \(w_k = \{w_k(v_k, u_{n+k}, v_{n+k}) \in \mathbb{R}^4\), \(w = (w_2, ..., w_n)\), we set

\[
\Psi(t) = \xi_1 X_1(t) + \xi_2 X_{n+1}(t) + \xi_3 Y_{n+1}(t), \quad U_k(t) = \langle w_k, \Theta_k(t) \rangle, \quad U^0_k(t) = \langle w_k, \Theta^0_k(t) \rangle.
\]

We throughout denote by \(^tQ\) the transpose of a matrix \(Q\). Then the characteristic function is

\[
\varphi(t) = E\left[\exp\left\{\sqrt{-1}(\xi_1 X_1(t) + \xi_2 X_{n+1}(t) + \xi_3 Y_{n+1}(t))\right\}\right]
\]
\[ + \sqrt{-1} \sum (u_k X_k(t) + v_k Y_k(t) + u_{n+k} X_{n+k}(t) + v_{n+k} Y_{n+k}(t)) \]
\[ = E[\exp(\sqrt{-1} \Psi(t) + \sum U_k(t))], \]
where the summation is taken over \( k = 2, \ldots, n \).

To compute the characteristic function, we introduce a \( 4 \times 4 \) skew symmetric matrix \( \Xi \) given by
\[
\Xi = \begin{pmatrix}
0 & \xi_1 & -\xi_2 & -\xi_3 \\
-\xi_1 & 0 & -\xi_3 & \xi_2 \\
\xi_2 & \xi_3 & 0 & \xi_4 \\
\xi_3 & -\xi_2 & -\xi_1 & 0
\end{pmatrix}.
\]
Then we have
\[
\Psi(t) + \sum U_k(t) = \psi + \int_0^t Y(s)^2 (\xi_1 dB_1(s) + \xi_2 dB_2(s) + \xi_3 dB_3(s))
+ \sum \langle \Xi \theta_k + w_k, \Theta_k^0(t) \rangle + \sum \int_0^t \langle \Xi \theta_k^0(s), d\Theta_k^0(s) \rangle,
\]
where \( \psi = \xi_1 x_1 + \xi_2 x_{n+1} + \xi_3 y_{n+1} + \sum \langle w_k, \theta_k \rangle. \) Note that \( \{ \sum_{j=1}^3 \xi_j B_j(s) \} \) is identical in law with \( \{ \xi|B_1(s) \}, |\xi| = (\xi_1^2 + \xi_2^2 + \xi_3^2)^{1/2} \).

The eigenvalues of \( \Xi \) are \( \pm \sqrt{-1}|\xi| \) and the multiplicities are two. Moreover, there exists an orthogonal matrix \( Q \) such that \( ^tQ \Xi Q = K \) is of the standard form. We take
\[
Q = \begin{pmatrix}
0 & 0 & \xi_3 + \xi_2 \\
0 & \xi_1 & -\xi_2 \\
0 & \xi_1 & \xi_3 \xi_2
\end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix}
0 & -|\xi| & 0 & 0 \\
|\xi| & 0 & 0 & 0 \\
0 & 0 & 0 & -|\xi| \\
0 & 0 & |\xi| & 0
\end{pmatrix}.
\]
We also put \( \hat{w}_k = ^tQ w_k, \hat{\Theta}_k = ^tQ \theta_k \) and
\[
\hat{\Theta}_k^0(t) = (\hat{X}_k^0(t), \hat{Y}_k^0(t), \hat{X}_{n+k}^0(t), \hat{Y}_{n+k}^0(t)) = ^tQ \Theta_k^0(t).
\]
By the rotation invariance of the probability law of Brownian motions, we see that \( \{ \hat{\Theta}_k^0(s) \} \) is a simple time change of a \( 4 \)-dimensional standard Brownian motion.

Under these notations, we have
\[
\langle \Xi \theta_k + w_k, \Theta_k^0(t) \rangle = \langle K \hat{\theta}_k + \hat{w}_k, \hat{\Theta}_k^0(t) \rangle
\]
and
\[
\int_0^t \langle \Xi \theta_k(s), d\Theta_k^0(s) \rangle = \int_0^t \langle K \hat{\theta}_k(s), d\hat{\Theta}_k^0(s) \rangle
\]
\[
= |\xi| \int_0^t \{ \hat{X}_k^0(s)d\hat{Y}_k^0(s) - \hat{Y}_k^0(s)d\hat{X}_k^0(s)
+ \hat{X}_{n+k}^0(s)d\hat{Y}_{n+k}^0(s) - \hat{Y}_{n+k}^0(s)d\hat{X}_{n+k}^0(s) \}. \tag{4.4}
\]
As in the previous sections, we set

\[ A_t^{(-\mu)} = \int_0^t \exp(2B_s^{(-\mu)})ds \quad \text{and} \quad \tilde{A}_t^{(-\mu)} = \int_0^t \exp(4B_s^{(-\mu)}(s))ds, \]

\[ B_s^{(-\mu)} = B(s) - \mu s \quad \text{and} \quad \mu = 2n + 1. \]

Then, by taking the conditional expectation given \( \{Y(s)\} \) and \( \tilde{\Theta}_k^0(t), k = 2, \ldots, n \), and applying the Lévy formula, we obtain

\[ \varphi(t) = e^{\sqrt{-1}\psi} E \left[ \exp \left( -\frac{1}{2} \|\xi\|^2 y^4 \tilde{A}_t^{(-\mu)} + \sqrt{-1} \sum_{k=2}^{n} \langle K\tilde{\theta}_k + \tilde{\omega}_k, \tilde{\Theta}_k(t) \rangle \right) \right. \]

\[ \times \prod_{k=2}^{n} E \left[ \exp \left( \sqrt{-1} \int_0^t \langle \Xi\Theta_k^0(s), d\Theta_k^0(s) \rangle \right) \right] \left( \{Y(s), \tilde{\Theta}_k(t)\} \right) \]

\[ = e^{\sqrt{-1}\psi} E \left[ \exp \left( -\frac{1}{2} \|\xi\|^2 y^4 \tilde{A}_t^{(-\mu)} + \sqrt{-1} \sum_{k=2}^{n} \langle K\tilde{\theta}_k + \tilde{\omega}_k, y\tilde{W}_k(A_t^{(-\mu)}) \rangle \right) \right. \]

\[ \times \left( \frac{\|\xi\|^2 y^2 \tilde{A}_t^{(-\mu)} - \sinh(\|\xi\|^2 y^2 \tilde{A}_t^{(-\mu)})} \sinh(\|\xi\|^2 y^2 \tilde{A}_t^{(-\mu)}) \right)^{2(n-1)} \]

\[ \times \exp \left( \left( 1 - \|\xi\|^2 y^2 \tilde{A}_t^{(-\mu)} \right) \coth(\|\xi\|^2 y^2 \tilde{A}_t^{(-\mu)}) \right) \left( \tilde{W}(A_t^{(-\mu)}) \right) \|	ilde{W}(A_t^{(-\mu)})\|^2 \right) \]

\[ = e^{\sqrt{-1}\psi} E \left[ \exp \left( -\frac{1}{2} \|\xi\|^2 y^4 \tilde{A}_t^{(-\mu)} \right) \int_{\mathbb{R}^{4(n-1)}} \left( \frac{\|\xi\|}{2\pi \sinh(\|\xi\|^2 y^2 \tilde{A}_t^{(-\mu)})} \right)^{2(n-1)} \right. \]

\[ \times \exp \left( \sqrt{-1} \sum_{k=2}^{n} \left( \langle K\tilde{\theta}_k + \tilde{\omega}_k, \zeta_k \rangle - \frac{1}{2} \|\xi\| \coth(\|\xi\|^2 y^2 \tilde{A}_t^{(-\mu)}) \|\zeta_k\|^2 \right) \right) d\zeta_2 \cdots d\zeta_n \].

where \( \{\tilde{W}_2(s), ..., \tilde{W}_n(s)\} \) is a \( 4(n-1) \)-dimensional standard Brownian motion, independent of \( \{Y(s)\} \) or \( \{B(s)\} \). We carry out the Gaussian integral over \( \mathbb{R}^{4(n-1)} \) to obtain

\[ \varphi(t) \]

\[ = e^{\sqrt{-1}\psi} E \left[ \left( \frac{1}{\cosh(\|\xi\|^2 y^2 \tilde{A}_t^{(-\mu)})} \right)^{2(n-1)} e^{-\|\xi\|^2 y^4 \tilde{A}_t^{(-\mu)}/2 - F \tanh(\|\xi\|^2 y^2 \tilde{A}_t^{(-\mu)})/2\|\xi\|} \right], \]

where \( F = \sum \|K\tilde{\theta}_k + \tilde{\omega}_k\|^2 \). Hence, letting \( t \) tend to \( \infty \), we obtain

\[ \lim_{t \to \infty} \varphi(t) \]

\[ = e^{\sqrt{-1}\psi} E \left[ \left( \frac{1}{\cosh(\|\xi\|^2 y^2 \tilde{A}_\infty^{(-\mu)})} \right)^{2(n-1)} e^{-\|\xi\|^2 y^4 \tilde{A}_\infty^{(-\mu)}/2 - F \tanh(\|\xi\|^2 y^2 \tilde{A}_\infty^{(-\mu)})/2\|\xi\|} \right]. \]

Now, applying (A.9), we obtain the explicit expression for the Fourier transform of the Poisson kernel \( f_n \).

**Proposition 4.3.** Under the notations above, the Fourier transform of the limit distribution of \( (X(t), \tilde{Z}(t)) \) as \( t \to \infty \), is given by
\[
\Phi(\xi, w) = \frac{e^{\sqrt{-1}\psi((\xi|y^2)^{2n+1}}}{2^{2n+1}\Gamma(2n + 1)} \int_0^\infty \left( \frac{1}{\cosh(u)} \right)^{2(n-1)} \left( \frac{1}{\sinh(u)} \right)^{2(n+1)} \times \exp\left( -\frac{1}{2} |\xi y^2 \coth(u) - \frac{F}{2|\xi|} \tanh(u) \right) du.
\]

We invert the Fourier transform. That is, setting \( x' = (x_1', x_{n+1}', y_{n+1}')\), \( \theta'_k = t(x_k, y_k', x_{n+k}', y_{n+k}') \) and

\[
\psi' = \xi_1 x'_1 + \xi_2 x'_{n+1} + \xi_3 y'_{n+1} + \sum \langle w_k, \theta'_k \rangle,
\]

we compute

\[
\tilde{f}_n(x', \tilde{z}; z) = \frac{1}{(2\pi)^{4n-1}} \int_{\mathbb{R}^{4n-1}} e^{\sqrt{-1}(\psi - \psi')} d\xi dw
\]

\[
= \frac{y^{2(2n+1)}}{(2\pi)^{4n-1}2^{2n+1}\Gamma(2n + 1)} \int_0^\infty \left( \frac{1}{\cosh(u)} \right)^{2(n-1)} \left( \frac{1}{\sinh(u)} \right)^{2(n+1)} \times \exp\left( -\frac{1}{2} |\xi y^2 \coth(u) - \frac{\tan\h(u)}{2|\xi|} \sum |K\tilde{\theta}_k\tilde{w}_k|^2 \right) du.
\]

Recall the definitions, \( \tilde{w}_k = tQw_k \) and \( \tilde{\theta}_k = tQ\theta_k \). Then, changing the order of the integrations, we have

\[
\tilde{f}_n(x', \tilde{z}; z)
\]

\[
= \frac{y^{2(2n+1)}}{(4\pi)^{2n+1}\Gamma(2n + 1)} \int_0^\infty \left( \frac{1}{\cosh(u)} \right)^{2(n-1)} \left( \frac{1}{\sinh(u)} \right)^{2(n+1)} du
\]

\[
\times \int_{\mathbb{R}^3} e^{\sqrt{-1}(\xi \cdot x' - \xi \cdot x')} e^{-|\xi y^2 \coth(u)/2 |\xi|^{2n+1}} d\xi
\]

\[
\times \int_{\mathbb{R}^4(n-1)} e^{\sqrt{-1}(\xi \cdot \tilde{z}) - \xi \cdot \tanh(u) \sum |K\tilde{\theta}_k\tilde{w}_k|^2 / 2|\xi|} \prod_{k=2}^n dw_k.
\]

We can easily carry out the third Gaussian integral since \( Q \in O(4) \) and we obtain

\[
\tilde{f}_n(x', \tilde{z}; z) = \frac{y^{2(2n+1)}}{(4\pi)^{2n+1}\Gamma(2n + 1)} \int_0^\infty \left( \frac{1}{\sinh(u)} \right)^{4n} du
\]

\[
\times \int_{\mathbb{R}^3} e^{\sqrt{-1}(\xi \cdot x' + |\xi \cdot \phi(\tilde{\theta}, \tilde{\theta}')) - |\xi|\Phi \coth(u)/2 |\xi|^{4n-1}} d\xi,
\]

where \( \Phi = y^2 + \sum |\tilde{\theta}_k - \tilde{\theta}'_k|^2 = y^2 + \sum |\theta_k - \theta'_k|^2 \) and

\[
\phi(\tilde{\theta}, \tilde{\theta}') = (K\tilde{\theta}_k, \tilde{\theta}'_k) = \sum (\tilde{y}_k \tilde{x}_k - \tilde{x}_k \tilde{y}_k + \tilde{y}_{n+k} \tilde{x}_{n+k} - \tilde{x}_{n+k} \tilde{y}_{n+k}).
\]

For the right hand side, changing the variables by \( k = \coth(u) \), we first compute the integral in \( u \). Then, by using formula (3.5) again, we get

\[
\int_0^\infty \left( \frac{1}{\sinh(u)} \right)^{4n} e^{-|\xi|\Phi \coth(u)/2} du = \int_1^\infty (k^2 - 1)^{2n-1} e^{-|\xi|\Phi k/2} dk
\]

\[
= \frac{\Gamma(2n)}{\sqrt{\pi}} \left( \frac{4}{|\xi|\Phi} \right)^{2n-1/2} K_{2n-1/2} \left( \frac{|\xi|\phi}{2} \right).
\]
Moreover, by definitions, we see
\[ \langle \xi, x' - x \rangle + |\xi| \phi(\theta, \theta') = -\langle \xi, \varphi \rangle, \]
where \( \varphi = \langle \varphi_1, \varphi_2, \varphi_3 \rangle \) is given by (4.2).
Combining these identities, we obtain
\[
\bar{f}_n(x', z'; z) = \frac{y^{2(2n+1)}}{8(2n+1)\pi^{2n+3/2}} \int_{\mathbb{R}^3} e^{-\sqrt{-1}\langle \xi, \varphi \rangle} K_{2n-1/2} \left( \frac{\Phi}{2} |\xi| \right) |\xi|^{2n-1/2} d\xi
\]
Moreover, changing the variables by the spherical coordinate, we obtain
\[
\bar{f}_n(x', z'; z) = \frac{4\pi y^{2(2n+1)}}{8(2n+1)\pi^{2n+3/2}(|\varphi|)} \times \int_0^\infty r^{2n+1/2} K_{2n-1/2} \left( \frac{\Phi}{2} r \right) \sin(|\varphi| r) dr.
\]
For the integral on the right hand side, the following formula is available (cf. [11], p.747):
\[
\int_0^\infty x^\lambda K_\mu(ax) \sin(bx) dx = 2^\lambda a^{-\lambda-2} \times \Gamma \left( \frac{2 + \mu + \lambda}{2} \right) \Gamma \left( \frac{2 + \lambda - \mu}{2} \right) \times F \left( \frac{2 + \mu + \lambda}{2}, \frac{2 + \lambda - \mu}{2}; \frac{3}{2}; -\frac{b^2}{a^2} \right).
\]
In our case \( \lambda = 2n + 1/2, \mu = 2n - 1/2 \) and
\[
F \left( \frac{2 + \mu + \lambda}{2}, \frac{2 + \lambda - \mu}{2}; \frac{3}{2}; -\frac{b^2}{a^2} \right) = \left( 1 + \frac{b^2}{a^2} \right)^{-(2n+1)}.
\]
Hence we may apply this identity and we arrive at our result
\[
\bar{f}_n(x, z'; z) = \frac{2^{4n+1} \Gamma(2n) y^{2(2n+1)}}{\pi^{2n} (\Phi^2 + 4|\varphi|^2)^{2n+1}}.
\]

A. Perpetual integrals of geometric Brownian motion

In this appendix we consider two perpetual integrals of geometric Brownian motions. Let \( B = \{ B(t) \}_{t \geq 0} \) be a one-dimensional Brownian motion with \( B_0 = 0 \) defined on a probability space \( (\Omega, \mathcal{F}, P) \). For \( \mu > 0 \), we set \( B^{(-\mu)} = \{ B_t^{(-\mu)} \equiv B(t) - \mu t \} \), a Brownian motion with negative constant drift \(-\mu\). Then Dufresne’s perpetual integral is defined by
\[
A_{-\mu} = \int_0^\infty \exp(2B_s^{(-\mu)}) ds.
\]
We also consider another integral

\[ a_{\infty}^{(-\mu)} = \int_0^\infty \exp(B_s^{(-\mu)}) ds. \]

Then the following is known:

**Theorem A.1** (Dufresne [9]). Let \( \gamma_\mu \) be a gamma random variable whose density is given by \( (\Gamma(\mu))^{-1} x^{\mu-1} e^{-x} \). Then \( A_{\infty}^{(-\mu)} \) is distributed as \( 2(\gamma_\mu)^{-1} \) and, accordingly, \( a_{\infty}^{(-\mu)} \) (law) = \( 2(\gamma_2)^{-1} \).

**Remark A.1.** Several different proofs of this theorem are known. In particular, see Yor [26]. The density of the exponential functional \( A_t = \int_0^t \exp(2B_s) ds \) for fixed \( t \) has been obtained by Yor [27] and the joint distribution of \( (A_t, a_t) \) in an obvious notation has been studied in [1]. See also [22, 23, 25] for several results and applications of these perpetual integrals and exponential functionals. Recently Baudoin-O’Connell [5] has shown several formulae, including (A.2) below, for the exponential type Wiener and discussed their close relation to the theory of quantum Toda lattice.

What we need in Sections 4 and 5 is the following explicit expression for the conditional Laplace transform of \( A_{\infty}^{(-\mu)} \) given \( a_{\infty}^{(-\mu)} \), which was originally obtained by Yor [28]. We set

\[ f_1(v) = \frac{2^{2\mu}}{\Gamma(2\mu)} v^{-(2\mu+1)} e^{-2/v}, \quad v > 0, \]

which is the density of the random variable \( a_{\infty}^{(-\mu)} \) or \( 2/\gamma_{2\mu} \).

**Theorem A.2.** For \( \lambda > 0 \) and \( v > 0 \), it holds that

\[ E \left[ \exp \left( -\frac{1}{2} \lambda^2 A_{\infty}^{(-\mu)} \right) \bigg| A_{\infty}^{(-\mu)} = v \right] f_1(v) \]

\[ = \frac{1}{2\Gamma(2\mu)} \left( \frac{\lambda}{\sinh(\lambda v/2)} \right)^{2\mu+1} \exp \left( -\lambda \coth \left( \frac{\lambda v}{2} \right) \right). \]  

(A.2)

We have this nice result only for the particular choice of \( A_t \) and \( a_t \), that is, it is available only when the ratio of the coefficients in the exponential functionals is two.

We give another proof of the theorem for completeness. Note that by letting \( \lambda \) tend to 0 in (A.2), we obtain Theorem A.1.

For this purpose, we consider the Brownian motion \( \{B_t^{(\mu)} = B_t + \mu t\} \) with the opposite positive drift and set \( X_x(s) = x \exp(B_s^{(\mu)}) \), which defines a diffusion process with infinitesimal generator

\[ \frac{1}{2} x^2 \frac{d^2}{dx^2} + \left( \frac{1}{2} + \mu \right) x \frac{d}{dx}. \]
Letting $\tau_z$ be the first hitting time of $\{X_x(s)\}$ at $z$, we set for $\lambda > 0$ and $\kappa \in \mathbb{R}$

$$v_z(x) = E \left[ \exp \left( -\frac{\lambda^2}{2} \int_0^{\tau_z} X_x(s)^{-2} \, ds + \lambda \kappa \int_0^{\tau_z} X_x(s)^{-1} \, ds \right) \right].$$

In [22] we have considered the case of $\kappa = 0$ and showed that $v_z(x)$ may be represented by means of the modified Bessel function to give another proof of Theorem A.1. Following the same line, we first show a representation for $v_z(x)$ by means of the Whittaker function.

Let $W_{\kappa,\mu}$ be a Whittaker function: if $\mu - \kappa + 1/2 > 0$,

$$W_{\kappa,\mu}(z) = \frac{e^{-z/2} z^{\mu+1/2}}{\Gamma(\mu - \kappa + 1/2)} \int_0^\infty e^{-zt} t^{-\mu-1/2} (1 + t)^{\mu+\kappa-1/2} \, dt. \tag{A.3}$$

From this expression it is easy to see $\lim_{z \to \infty} W_{\kappa,\mu}(z) = 0$ when $|\kappa|$ is small. We also recall that $W_{\kappa,\mu}$ solves the equation

$$W''(z) + \left( -\frac{1}{4} + \frac{\kappa}{z} - \mu^2 - (1/4) \frac{1}{z^2} \right) W(z) = 0.$$

**Proposition A.3.** For $\mu > 0$, $\lambda > 0$ and $\kappa \in \mathbb{R}$, it holds that

$$v_z(x) = \left( \frac{x}{z} \right)^{\mu - 1/2} \frac{W_{\kappa,\mu}(2\lambda/x)}{W_{\kappa,\mu}(2\lambda/z)}. \tag{A.4}$$

**Proof.** We have only to consider the case of $\kappa < 0$. The general case can be shown from this case by analytic continuation in $\kappa$. Note that, if $\kappa < 0$, $v_z(x)$ is monotone increasing in $x(> z)$.

At first we note that $v_z(x)$ is a solution for

$$\frac{1}{2} x^2 v''(x) + \left( 1 + \frac{\mu}{2} \right) x v'(x) = \left( \frac{\lambda^2}{2x^2} - \frac{\lambda \kappa}{x} \right) v(x)$$

and satisfies

$$v_z(x) \bigg|_{x=z} = 1 \quad \text{and} \quad \lim_{x \downarrow 0} v_z(x) = 0. \tag{A.5}$$

We now change the variable by $\xi = \lambda/x$ and set

$$v_z(x) = \xi^{\mu - 1/2} \phi(\xi).$$

Then, by straightforward computations, we see that $\phi$ satisfies

$$\phi''(\xi) + \left( -1 + \frac{2\kappa}{\xi} - \mu^2 - (1/4) \frac{1}{\xi^2} \right) \phi(\xi) = 0.$$

By considering the boundary conditions (A.5), we can easily show

$$\phi(\xi) = \left( \frac{z}{\lambda} \right)^{\mu - 1} \frac{W_{\kappa,\mu}(2\xi)}{W_{\kappa,\mu}(2\lambda/z)}$$

and hence the result (A.4). \qed
Proposition A.4. For $\mu > 0, \lambda > 0$ and $\kappa \in \mathbb{R}$, it holds that
\[
(A.6) \quad E \left[ \exp \left( -\frac{1}{2} \lambda^2 A^{(-\mu)}_{\infty} + \lambda \kappa a^{(-\mu)}_{\infty} \right) \right] = \frac{\Gamma(\mu - \kappa + 1/2)}{\Gamma(2\mu)} (2\lambda)^{\mu-1/2} W_{\kappa,\mu}(2\lambda).
\]

Proof. By the symmetry of the probability law of Brownian motion, $\{ -B_t \} \overset{\text{law}}{=} \{ B_t \}$, we have
\[
\lim_{z \to \infty} v_z(1) = E \left[ \exp \left( -\frac{1}{2} \lambda^2 \int_0^\infty e^{-2B_t^\mu} \, ds + \lambda \kappa \int_0^\infty e^{-B_t^{(-\mu)}} \, ds \right) \right] = E \left[ \exp \left( -\frac{1}{2} \lambda^2 \int_0^\infty e^{2B_t^{(-\mu)}} \, ds + \lambda \kappa \int_0^\infty e^{B_t^{(-\mu)}} \, ds \right) \right].
\]

On the other hand, by using the fact on the Whittaker function
\[
(A.7) \quad W_{\kappa,\mu}(z) = \frac{\Gamma(2\mu)}{\Gamma(\mu - \kappa + 1/2)} z^{-\mu+1/2}(1 + o(1)) \quad \text{as} \quad z \downarrow 0,
\]
we see from the expression (A.4) that
\[
\lim_{z \to \infty} v_z(1) = \frac{\Gamma(\mu - \kappa + 1/2)}{\Gamma(2\mu)} (2\lambda)^{\mu-1/2} W_{\kappa,\mu}(2\lambda).
\]

Remark A.2. The asymptotic behavior (A.7) of $W_{\kappa,\mu}$ can be easily shown by the definition of the Whittaker functions.

Now we are in a position to complete our proof of Theorem A.2. By (A.3) and (A.6), we have
\[
E \left[ \exp \left( -\frac{1}{2} \lambda^2 A^{(-\mu)}_{\infty} + \lambda \kappa a^{(-\mu)}_{\infty} \right) \right] = \frac{e^{-\lambda}}{\Gamma(2\mu)} (2\lambda)^{2\mu} \int_0^\infty e^{-2\lambda t^{\mu-\kappa-1/2}} (1 + t)^{\mu+\kappa-1/2} \, dt.
\]

Now we change the variable by $e^{\lambda v} = 1 + t^{-1}$ or $t = (e^{\lambda v} - 1)^{-1}$. Then some elementary computations show that this integral is equal to
\[
\frac{\lambda^{2\mu+1}}{2\Gamma(2\mu)} \int_0^\infty e^{-\lambda \coth(\lambda v/2)} \frac{1}{\sinh(\lambda v/2)} \, 2^{\mu+1} e^{\lambda \kappa v} \, dv.
\]

This completes our proof since
\[
E \left[ \exp \left( -\frac{1}{2} \lambda^2 A^{(-\mu)}_{\infty} + \lambda \kappa a^{(-\mu)}_{\infty} \right) \right] = \int_0^\infty E \left[ \exp \left( -\frac{1}{2} \lambda^2 A^{(-\mu)}_{\infty} \right) \right| a^{(-\mu)}_{\infty} = v] f_1(v) e^{\lambda \kappa v} \, dv.
\]

□
Corollary A.5. Define another perpetual integral $\tilde{A}_\infty^{(-\mu)}$ by

$$A_\infty^{(-\mu)} = \int_0^\infty \exp(4B_s^{(-\mu)})ds$$

and let $f_2(v)$ be the density of $A_\infty^{(-\mu)}$ or $(2\gamma_\mu)^{-1}$. Then one has

$$E\left[ \exp\left( -\frac{1}{2} \lambda^2 \tilde{A}_\infty^{(-\mu)} \right) \bigg| A_\infty^{(-\mu)} = v \right] f_2(v) = \frac{1}{2^{\mu} \Gamma(\mu)} \left( \frac{\lambda}{\sinh(\lambda v)} \right)^{\mu+1} \exp\left( -\frac{\lambda}{2} \coth(\lambda v) \right).$$

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