A generalization of weight polynomials to matroids

Trygve Johnsen, Jan Roksvold, Hugues Verdure

Department of Mathematics, University of Tromsø, N-9037 Tromsø, Norway

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Abstract

Generalizing polynomials previously studied in the context of linear codes, we define weight polynomials and an enumerator for a matroid $M$. Our main result is that both of these polynomials are determined by Betti numbers associated to the Stanley-Reisner ideals of $M$ and so-called elongations of $M$. Also, we show that Betti tables of elongations of $M$ are partly determined by the Betti table of $M$. Generalizing a known result in coding theory, we show that the enumerator of a matroid is equivalent to its Tutte polynomial, and vice versa.

1 Introduction

For a linear $[n,k]$-code $C$ over $\mathbb{F}_q$, let $A_{C,j}$ denote the number of words of weight $j$ in $C$. The weight enumerator

$$W_C(X,Y) = \sum_{j=0}^{n} A_{C,j} X^{n-j} Y^j$$

$*$Corresponding author. E-mail address: jan.n.roksvold@uit.no
has important applications in the theory of error-correcting codes, where it amongst other things determines the probability of having an undetected error (see [6, Proposition 1.12]).

For \(q\) a power of \(q\), the set of all \(F_q\)-linear combinations of words of \(C\) is itself a linear code. This code is commonly referred to as the extension of \(C\) to \(F_q\), and is denoted \(C \otimes F_q \equiv F_Q\). In [6], it is found that the number \(A_{C,j}(Q)\) of words of weight \(j\) in \(C \otimes F_q \equiv F_Q\) can be expressed in terms of the initial code \(C\), as a polynomial in \(Q\). This leads them to the definition of an extended weight enumerator \(W_C(X,Y,Q)\) for \(C\), with the desired property

\[
W_C(X,Y,Q) = W_{C \otimes F_q \equiv F_Q}(X,Y).
\]

Also in [6], it is demonstrated that the extended weight enumerator of \(C\) is in fact equivalent to the Tutte polynomial of \(M(G)\), where \(M(G)\) is the vector matroid associated to a generator matrix \(G\) of \(C\).

In this article, our primary goal is to show that the polynomial \(A_{C,j}(Q)\) is determined by certain Betti numbers associated to \(M(H)\) and its so-called elongations, where \(H\) is a parity-check matrix of \(C\). It seemed natural to first generalize the polynomial \(A_{C,j}(Q)\) to a polynomial \(P_{M,j}(Z)\) defined for all matroids - not only those stemming from a linear code. This immediately leads to the definition of a more general matroidal enumerator

\[
W_M(X,Y,Z) = \sum_{j=0}^{n} P_{M,j}(Z)X^{n-j}Y^j,
\]

as well.

In light of results in [6] already mentioned, we expected \(W_M\) to be equivalent to the Tutte polynomial. This indeed turned out to be the case; after a small leap (Proposition 3), an analogous proof to the one found in [6] for linear codes, went through.

As can be seen in [3, p. 131], the Tutte polynomial of a matroid determines its higher weights. Thus we already know that the polynomials \(P_{M,j}\) must, at least indirectly, determine the higher weights of \(M\), as well. We shall see towards the end of this article that they do so very directly – in a simple and applicable way.

1.1 Structure of this paper

- Section 2 contains definitions and results used later on.
• In Section 3 we look at the number of codewords in the extension of a code $C$ over $\mathbb{F}_q$ – as a polynomial in $q^m$.

• In Section 4 we generalize the polynomial from Section 3 to matroids, and use these generalized weight polynomials to define a matroidal enumerator. We proceed to demonstrate that this enumerator is equivalent to the Tutte polynomial of $M$.

• In Section 5 we prove our main result: The generalized weight polynomials are determined by Betti numbers associated to minimal free resolutions of $M$ and elongations of $M$.

• In Section 6 we shall see a counterexample showing that the converse of our main result is not true; the generalized weight enumerators do not determine the Betti numbers of $M$.

• In Section 7 we show how the generalized weight polynomials determine the higher weight hierarchy of $M$.

2 Preliminaries

2.1 Linear codes and weight enumerators

A linear $[n,k]$-code $C$ over $\mathbb{F}_q$ is, by definition, a $k$-dimensional subspace of $\mathbb{F}_q^n$. The elements of this subspace are commonly referred to as words, and any $k \times n$ matrix whose rows form a basis for $C$ is referred to as a generator matrix. Thus a code will typically have several generator matrices.

The dual code is the orthogonal complement of $C$, and is denoted $C^\perp$. A parity-check matrix of $C$ is a $(n-k) \times n$-matrix with the property

$$Hx^T = 0 \iff x \in C.$$ 

It is easy to see that $H$ is a parity check matrix for $C$ if and only if $H$ is a generator matrix for $C^\perp$.

2.2 Puncturing and shortening a linear code

Let $C$ be a linear code of length $n$, and let $J \subseteq \{1 \ldots n\}$. 


**Definition 2.1.** The puncturing of \( C \) in \( J \) is the code obtained by eliminating the coordinates indexed by \( J \) from the words of \( C \).

**Definition 2.2.**
\[
C(J) = \{ w \in C : w_j = 0 \text{ for all } j \in J \}.
\]
Clearly, \( C(J) \) is itself a linear code.

**Definition 2.3.** The shortening of \( C \) in \( J \) is the puncturing of \( C(J) \) in \( J \).

### 2.3 Matroids

There are numerous equivalent ways of defining a matroid. We choose to give here the definition in terms of independent sets. For an introduction to matroid theory in general, we recommend e.g. [8].

**Definition 2.4.** A matroid \( M \) consists of a finite set \( E \) and a set \( I(M) \) of subsets of \( E \) such that:

- \( \emptyset \in I(M) \).
- If \( I_1 \in I(M) \) and \( I_2 \subseteq I_1 \), then \( I_2 \in I(M) \).
- If \( I_1, I_2 \in I(M) \) and \( |I_1| > |I_2| \), then there is a \( x \in I_1 \setminus I_2 \) such that \( I_2 \cup x \in I(M) \).

The elements of \( I(M) \) are referred to as the independent sets (of \( M \)). The bases of \( M \) are the independent sets that are not contained in any other independent set. In other words, the maximal independent sets. Conversely, given the bases of a matroid, we find the independent sets to be those sets that are contained in a basis. We denote the bases of \( M \) by \( B(M) \). It is a fundamental result that all bases of a matroid have the same cardinality.

The dual matroid \( M^* \) is the matroid on \( E \) whose bases are the complements of the bases of \( M \). Thus
\[
B(M^*) = \{ E \setminus B : B \in B(M) \}.
\]

**Definition 2.5.** For \( \sigma \subseteq E \), the rank function \( r_M \) and nullity function \( n_M \) are defined by

\[
r_M(\sigma) = \max \{ |I| : I \in I(M), I \subseteq \sigma \},
\]
and
\[
n_M(\sigma) = |\sigma| - r_M(\sigma).
\]
Whenever the matroid $M$ is clear from the context, we omit the subscript and write simply $r$ and $n$. Note that a subset $\sigma$ of $E$ is independent if and only if $n(\sigma) = 0$. The rank $r(M)$ of $M$ itself is defined as $r(M) = r(M,E)$.

We let $r^*$ and $n^*$, respectively, denote the rank- and nullity function of $M^*$, and point out that

$$r^*(\sigma) = |\sigma| + r(E \setminus \sigma) - r(E).$$

**Definition 2.6.** If $\sigma \subseteq E$, then $\{I \subseteq \sigma : I \in I(M)\}$ form the set of independent sets of a matroid $M|_\sigma$ on $\sigma$. We refer to $M|_\sigma$ as the restriction of $M$ to $\sigma$.

**Definition 2.7.** The higher weights $\{d_i\}$ of $M$ are defined by

$$d_i = \min\{|\sigma| : \sigma \subseteq E(M) \text{ and } n(\sigma) = i\}.$$ 

**Definition 2.8.** The Tutte polynomial of $M$ is defined by

$$t_M(X,Y) = \sum_{\sigma \subseteq E} (X - 1)^r(\sigma)(Y - 1)^{|\sigma| - r(\sigma)}.$$

It carries information on several invariants of $M$. For example $t_M(1,1)$ counts the number of bases of $M$, while $t_M(2,1)$ is the number of independent sets.

**Definition 2.9.** Let $f_i$ denote the number of independent sets of cardinality $i$. The reduced Euler characteristic $\chi(M)$ of $M$ is defined by

$$\chi(M) = -1 + f_1 - f_2 + \cdots + (-1)^{r(M) - 1}f_{r(M)}.$$ 

**Example 2.1** ($U(r,n)$). Let $E$ be a set with $|E| = n$. The set of all cardinality-$r$ subsets of $E$ form the set of bases for a matroid $U(r,n)$ on $E$. We refer to $U(r,n)$ as the uniform matroid of rank $r$ on an $n$-element set. Observe that $I \subseteq E$ is independent in $U(r,n)$ if and only if $|I| \leq r$.

Clearly, we have $d_i(U(r,n)) = r + i$, for $1 \leq i \leq n - r$. And it is equally clear that

$$\chi(U(r,n)) = \sum_{i=0}^{r} (-1)^{i+1} \binom{n}{i}.$$ 

As for the Tutte polynomial, note that for $\sigma \subseteq E$ with $|\sigma| < r$ we have $|\sigma| - r(\sigma) = 0$. While for those $\sigma$ with $|\sigma| > r$ we have $r(E) - r(\sigma) = 0$. For the $\binom{n}{r}$ subsets $\sigma$ with $|\sigma| = r$, both $|\sigma| - r(\sigma)$ and $r(E) - r(\sigma)$ is equal to 0. Thus

$$t_{U(r,n)}(X,Y) = \sum_{i=0}^{r-1} \binom{n}{i} (X - 1)^{r - i} + \binom{n}{r} + \sum_{i=r+1}^{n} \binom{n}{i} (Y - 1)^{i - r}.$$
2.4 From linear code to matroid

Let $A$ be an $m \times n$ matrix over some field $\mathbb{k}$. Let $E$ be the set of column labels of $A$. It is easy to verify that if we take as independent sets those subsets of $E$ that correspond to a set of $\mathbb{k}$-linearly independent columns, this constitutes a matroid on $E$. We refer to this as the vector matroid of $A$ and denote it $M(A)$.

Thus from a linear code $C$, with generator matrix $G$ and parity-check matrix $H$, there naturally corresponds two matroids: $M(G)$ and $M(H)$. Note that if $G$ and $G'$ are two generator matrices for $C$, then $M(G) = M(G')$. Same goes for parity-check matrices, of course. It therefore makes sense to speak of the matroid corresponding to a generator (or parity-check) matrix of $C$, and to write $M(G)$ and $M(H)$ without specifying $G$ or $H$. We shall mostly consider $M(H)$, but this is not very crucial since duality results abound and $M(H) = M(G)^\perp$.

Note that $r(M(G)) = \dim(C)$, while $r(M(H)) = \dim(C^\perp)$, and that $d_1(M(H))$ is equal to the minimum distance of $C$.

Example 2.2. Let $C$ be the $[7,4]$-code over $\mathbb{F}_5$ with parity-check matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & 3 & 3 & 3 & 4 \\ 0 & 1 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 4 & 4 & 4 & 4 \end{pmatrix}.$$

Then $M(H)$ will be a matroid on $E = \{1, \ldots, 7\}$. Since the columns $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$ form a maximal linearly independent set of columns – the set $\{1, 3, 6\}$ must be a basis for $M(H)$. Thus $r(M) = 3$ (which could also have been inferred from $H$ being a $7 \times 3$ matrix). The full set of bases are

$$B(M(H)) = \{ \{1, 3, 6\}, \{1, 3, 5\}, \{1, 2, 6\}, \{2, 3, 6\}, \{1, 2, 5\}, \{1, 5, 7\}, \{3, 6, 7\}, \{2, 4, 7\}, \{1, 4, 6\}, \{2, 3, 4\}, \{4, 6, 7\}, \{1, 2, 3\}, \{1, 2, 7\}, \{3, 4, 5\}, \{1, 6, 7\}, \{1, 4, 5\}, \{1, 2, 4\}, \{2, 3, 7\}, \{4, 5, 7\}, \{3, 5, 7\}, \{2, 6, 7\}, \{2, 5, 7\}, \{2, 3, 5\}, \{3, 4, 6\}\}.$$

2.5 The elongation of $M$ to rank $r(M) + i$

Let $M$ be a matroid on $E$, with $|E| = n$. 
\textbf{Definition 2.10.} For $0 \leq i \leq n - r(M)$, let $M_i$ be the matroid whose independent sets are $I(M)_i = \{ \sigma \in E : n(\sigma) \leq i \}$.

That $M_i$ is in fact a matroid can be seen in e.g. [8, p.25]. Note that $M_0 = M$, and that $B(M_{n-r(M)}) = \{ E \}$.

The following is straightforward:

\textbf{Proposition 1.} Let $r_i$ and $n_i$ denote, respectively, the rank function and the nullity function of $M_i$. Then, for $\sigma \subseteq E$, we have

$$r_i(\sigma) = \begin{cases} r(\sigma) + i, & n(\sigma) > i, \\ |\sigma|, & n(\sigma) \leq i. \end{cases} \quad (2)$$

And

$$n_i(\sigma) = \begin{cases} n(\sigma) - i, & n(\sigma) > i, \\ 0, & n(\sigma) \leq i. \end{cases} \quad (3)$$

By definition we have $r_i(M_i) = r_i(E)$. It thus follows from Proposition 1 that

$$r_i(M_i) = r(M) + i. \quad (4)$$

The matroid $M_i$ is commonly referred to as the \textit{elongation} of $M$ to rank $r(M) + i$.

If $\sigma \subseteq E$ then the rank function of $M|_{\sigma}$ is the restriction of $r_M$ to subsets of $\sigma$. We point out, for later use, that this implies

$$(M_i)|_{\sigma} = (M|_{\sigma})_i. \quad (5)$$

\subsection{2.6 The Stanley-Reisner ideal, Betti numbers, and the reduced chain complex}

Let $M$ be a matroid on $E$, with $|E| = n$ and $r(M) = k$. Let $\mathbb{k}$ be a field.

\textbf{Definition 2.11.} A \textit{circuit} of $M$ is a subset $C$ of $E$ with the property that $C$ is not itself independent, but $C \setminus x$ is independent for every $x \in C$.

In other words, the circuits of a matroid are the minimal dependent sets, while the independent sets are precisely those that do not contain a circuit.

For the following we label the elements of $E$, such that $E = \{ e_1, \ldots, e_n \}$. Let $S = \mathbb{k}[x_1, \ldots, x_n]$. 

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Definition 2.12 (Stanley-Reisner ideal). Let $I_M$ be the ideal in $S$ generated by monomials corresponding to circuits of $M$. That is, let

$$I_M = \langle x_{j_1}x_{j_2} \cdots x_{j_s} : \{e_{j_1}e_{j_2}, \ldots, e_{j_s}\} \text{ is a circuit of } M \rangle.$$ 

We refer to $I_M$ as the Stanley-Reisner ideal of $M$.

A complex

$$\cdots \longleftarrow X_{i-1} \overset{\phi_i}{\longleftarrow} X_i \overset{\phi_i}{\longleftarrow} \cdots$$

over $S$ is said to be minimal whenever $\text{im} \phi_i \subseteq \langle x_1, x_2, \ldots, x_n \rangle X_{i-1}$ for each $i$.

Definition 2.13. An $\mathbb{N}_0$-graded minimal free resolution of an $\mathbb{N}_0$-graded $S$-module $N$ is a minimal left complex

$$0 \leftarrow F_0 \overset{\phi_1}{\leftarrow} F_1 \overset{\phi_2}{\leftarrow} F_2 \leftarrow \cdots \overset{\phi_l}{\leftarrow} F_l \leftarrow 0 \quad (6)$$

where

$$F_i = \bigoplus_j S(-j)^{\beta_{i,j}},$$

which is exact everywhere except for in $F_0$, where $F_0 / \text{im} \phi_1 \cong N$. We also require the boundary maps $\phi_i$ to be degree-preserving.

If $N$ permits an $\mathbb{N}_0^n$-grading (as e.g. the Stanley-Reisner ideal does) we may form an $\mathbb{N}_0^n$-graded minimal free resolution. In that case

$$F_i = \bigoplus_{\sigma \in \mathbb{N}_0^n} S(-\sigma)^{\beta_{i,\sigma}},$$

while the definition remains otherwise unchanged. Observe that

$$\beta_{i,j} = \sum_{|\sigma|=j} \beta_{i,\sigma}.$$ 

Hilbert Syzygy Theorem states that the length $l$ of (6) is less than or equal to $n$. We shall here only be looking at minimal free resolutions of the Stanley-Reisner ideal $I_M$; these all have length $n - r(M) - 1$ (see e.g. [4, Corollary 3(b)]).

For an empty ideal, all Betti numbers are zero. This is for example always the case with $I_{M_{n-r(M)}}$ since $M_{n-r(M)}$ has no circuits.

For the following definition, we shall assume, without loss of generality, that $E = \{1, \ldots, n\}$.  

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Definition 2.14. Let \( I_i(M) \) denote the set consisting of those independent sets that have cardinality \( i \), and let \( \mathbb{k} I_i(M) \) be the free \( \mathbb{k} \)-vector space on \( I_i(M) \). The (reduced) chain complex of \( M \) over \( \mathbb{k} \) is the complex

\[
0 \leftarrow \mathbb{k} I_0(M) \leftarrow \mathbb{k} I_1(M) \leftarrow \cdots \leftarrow \mathbb{k} I_{i-1}(M) \leftarrow \mathbb{k} I_i(M) \leftarrow \mathbb{k} I_{i-1}(M) \leftarrow \cdots \leftarrow \mathbb{k} I_1(M) \leftarrow \mathbb{k} I_0(M) \leftarrow 0,
\]

where the boundary maps \( \delta_i \) are defined on bases as follows: With the natural ordering \( e_u < e_v \iff u < v \) on \( E \), set sign \((j, \sigma) = (-1)^{r-1} \) if \( j \) is the \( r \)th element of \( \sigma \subseteq E \), and let

\[
\delta_i(\sigma) = \sum_{j \in \sigma} \text{sign}(j, \sigma) \sigma \setminus j.
\]

Extending \( \delta_i \) \( \mathbb{k} \)-linearly, we obtain a \( \mathbb{k} \)-linear map from \( \mathbb{k} I_i(M) \) to \( \mathbb{k} I_{i-1}(M) \).

Definition 2.15. The \( i \)th reduced homology of \( M \) over \( \mathbb{k} \) is the vector space

\[
H_i(M; \mathbb{k}) = \ker(\delta_i) / \text{im}(\delta_{i+1}).
\]

In proving our main result (Theorem 5.1), we shall draw upon the following two results, the first of which is a concatenation of [1, Proposition 7.4.7 (i) and Proposition 7.8.1].

Theorem 2.1. Let \( H_i(M; \mathbb{k}) \) denote the \( i \)th homology of \( M \) over \( \mathbb{k} \). Then

\[
H_i(M; \mathbb{k}) = \begin{cases} \mathbb{k}(-1)^{r(M)} \chi(M), & i = r(M) - 1 \\ 0, & i \neq r(M) - 1. \end{cases}
\]

Theorem 2.2 (Hochster’s formula).

\[
\beta_{i-1,\sigma}(I_M) = \dim_{\mathbb{k}} H_{|\sigma|-i-1}(M_{|\sigma|}; \mathbb{k}).
\]

First, we would like to point out, for later use, that Theorems 2.1 and 2.2 combined imply

\[
\sum_{i=0}^{n} (-1)^i \beta_{i,\sigma} = (-1)^{n_M(\sigma)-1} \beta_{n_M(\sigma)-1,\sigma}. \tag{7}
\]

Secondly, it is immediate from Hochster’s formula that the Betti numbers associated to a \( (\mathbb{N}_0 \text{- or } \mathbb{N}_0^n \text{-graded}) \) minimal free resolution are unique, in that any other minimal free resolution must have the same Betti numbers. Furthermore, it was found in [1] that for a matroid \( M \), the dimension of \( H_i(M; \mathbb{k}) \) is in fact independent of \( \mathbb{k} \). Thus for matroids, the \( (\mathbb{N}_0 \text{- or } \mathbb{N}_0^n \text{-graded}) \) Betti numbers are not only unique, but independent of choice of field. We shall therefore largely omit any reference to, or specifying of, a particular field \( \mathbb{k} \) – throughout.
Example 2.3 (Continuation of Ex. 2.2). Since $M(H)$ has set of circuits
\[
\{\{1,2,6,7\}, \{5,6\}, \{2,3,6,7\}, \{1,2,3,5\}, \{1,3,7\}, \{1,4,7\}, \{1,2,3,6\}, \{2,4,6\}, \{2,3,5,7\}, \{3,4,7\}, \{1,2,5,7\}, \{1,3,4\}, \{2,4,5\}\}
\]
its Stanley-Reisner ideal is
\[
I_{M(H)} = \langle x_1x_2x_6x_7, x_5x_6, x_2x_3x_6x_7, x_1x_2x_3x_5, x_1x_4x_7, x_1x_2x_3x_6, \\
x_2x_4x_6, x_2x_3x_5x_7, x_3x_4x_7, x_1x_2x_5x_7, x_1x_3x_4, x_2x_4x_5, \\
x_2x_4x_6, x_2x_3x_5x_7, x_3x_4x_7, x_1x_2x_5x_7, x_1x_3x_4, x_2x_4x_5 \rangle.
\]
Using MAGMA ([2]), we find the $\mathbb{N}_0$-graded minimal free resolutions of $I_{M(H)}$ to be
\[
0 \leftarrow S(-2) \oplus S(-3)^6 \oplus S(-4)^6 \leftarrow S(-4)^5 \oplus S(-5)^{28} \leftarrow S(-6)^{31} \leftarrow S(-7)^{10} \leftarrow 0.
\]
Similarly, we find the $\mathbb{N}_0$-graded minimal free resolutions corresponding to elongations of $M$.

\[
\begin{align*}
I_{M(H)_1} &: \\
0 &\leftarrow S(-4)^2 \oplus S(-5)^{15} \leftarrow S(-6)^{29} \leftarrow S(-7)^{13} \leftarrow 0,
\end{align*}
\]
\[
\begin{align*}
I_{M(H)_2} &: \\
0 &\leftarrow S(-6)^7 \leftarrow S(-7)^6 \leftarrow 0,
\end{align*}
\]
\[
\begin{align*}
I_{M(H)_3} &: \\
0 &\leftarrow S(-7) \leftarrow 0,
\end{align*}
\]

3 Number of codewords of weight $j$

Let $C$ be a linear $[n,k]$-code over $\mathbb{F}_q$, with a generator matrix $G = [g_{i,j}]$ for $1 \leq i \leq k$, $1 \leq j \leq n$. Let $Q = q^m$ for some $m \in \mathbb{N}$.

Definition 3.1. For $0 \leq k \leq n$, let $A_{C,k}(Q)$ denote the number of words of weight $k$ in $C \otimes_{\mathbb{F}_q} \mathbb{F}_Q$.

Let $c_j$ denote column $j$ of $G$. If $a = (a_1, a_2, \ldots, a_k) \in \mathbb{F}_Q^k$, the codeword $a \cdot G$ has weight $n$ if and only if
\[
c_j^T \cdot a \neq 0
\]
for all $1 \leq j \leq n$. In other words, if we let $S_j(Q)$ denote \{$x \in \mathbb{F}_q^k : c_j^T \cdot x = 0$\}, corresponding to column $j$, we have that $a \cdot G$ has weight $n$ if and only if
\[
a \in \mathbb{F}_q^k \setminus (S_1(Q) \cup S_2(Q) \cup \cdots \cup S_n(Q)).
\] (8)

**Definition 3.2.** For $U = \{u_1, u_2, \ldots, u_s\} \subseteq \{1, \ldots, n\}$, let
\[
S_U(Q) = S_{u_1}(Q) \cap S_{u_2}(Q) \cap \cdots \cap S_{u_s}(Q).
\]

By the inclusion/exclusion-principle then, we see from (8) that
\[
A_{C,n}(Q) = Q^k - \sum_{|U|=1} |S_U(Q)| + \sum_{|U|=2} |S_U(Q)| + \cdots + (-1)^n \sum_{|U|=n} |S_U(Q)|.
\]

If $B_U = \begin{pmatrix} e_{u_1}^T \\ e_{u_2}^T \\ \vdots \\ e_{u_s}^T \end{pmatrix}$, then $|S_U(Q)| = Q^{\dim(\ker B_U)} = Q^{k - \dim(\col B_U)} = Q^{k - \dim(M(G)(U))}$, which according to (9) is equal to $Q^{n_{M(H)}(E \setminus U)}$. Since $Q^k = \sum_{|U|=0} |S_U(Q)|$, we conclude that
\[
A_{C,n}(Q) = \sum_{U \subseteq E} (-1)^{|U|} Q^{n_{M(H)}(E \setminus U)} = (-1)^n \sum_{\gamma \subseteq E} (-1)^{|\gamma|} Q^{n_{M(H)}(\gamma)}. \] (9)

**Definition 3.3.**
\[
a_{C,\sigma}(Q) = |\{w \in C \otimes_{\mathbb{F}_q} \mathbb{F}_q : \text{Supp}(w) = \sigma\}|.
\]

**Lemma 3.1.**
\[
a_{C,\sigma}(Q) = (-1)^{|\sigma|} \sum_{\gamma \subseteq \sigma} (-1)^{|\gamma|} Q^{n_{M(H)}(\gamma)}.
\]

**Proof.** Let $C_{\sigma}(Q)$ denote the shortening of $C \otimes_{\mathbb{F}_q} \mathbb{F}_q$ in $\{1 \ldots n\} \setminus \sigma$, and let $H|_{\sigma}$ be the restriction of $H$ to columns indexed by $\sigma$. Then $H|_{\sigma}$ is a parity-check matrix for $C_{\sigma}(Q)$.

Clearly $a_{C,\sigma}(Q) = a_{C_{\sigma},\sigma}(Q)$, and since $M(H)|_{\sigma} \cong M(H|_{\sigma})$ it follows by an argument similar to the one leading to (9) that
\[
a_{C,\sigma}(Q) = (-1)^{|\sigma|} \sum_{\gamma \subseteq \sigma} (-1)^{|\gamma|} Q^{n_{M(H)}(\gamma)}.
\]
The result follows, since $n_{M(H)}(\gamma) = n_{M(H)}(\gamma)$ for all $\gamma \subseteq \sigma$. 

\[\square\]
Proposition 2. For $1 \leq k \leq n$

$$A_{C,k}(Q) = (-1)^k \sum_{|\sigma|=k} \sum_{Y \subseteq \sigma} (-1)^{|Y|} Q^{p_{M(H)}(Y)}$$

Proof. This is clear from Lemma 3.1, since $A_{C,k}(Q) = \sum_{|\sigma|=k} a_{C,\sigma}(Q)$. \qed

In the following sections, we shall see what comes from generalizing the weight polynomials $A_{C,k}(Q)$ to matroids.

4 Generalized weight polynomials and a generalized enumerator

Looking back at Proposition 2 it is clear that the polynomial $A_{C}(Q)$ appearing there may equally well be defined for matroids in general – not only for those derived from a linear code.

For the remainder of this section, let $M$ be a matroid on $E$, with $|E| = n$.

4.1 GWP and the enumerator

Definition 4.1 (GWP). We define the polynomial $P_{M,j}(Z)$ by letting $P_{M,0}(Z) = 1$ and

$$P_{M,j}(Z) = (-1)^j \sum_{|\sigma|=j} \sum_{Y \subseteq \sigma} (-1)^{|Y|} Z^{m_{M}(Y)} \text{ for } 1 \leq j \leq n.$$ 

We shall refer to $P_{M,j}$ as the $j^{th}$ generalized weight polynomial, or just GWP, of $M$.

Analogous to how $A_{C,j}(Q)$ is used to define the extended weight enumerator $W_{C}(X,Y,Q)$ of a code $C$ (see [6]), we use the GWP to define the enumerator of $M$:

Definition 4.2 (Matroid enumerator). The enumerator $W_{M}$ of $M$ is

$$W_{M}(X,Y,Z) = \sum_{i=0}^{n} P_{M,i}(Z) X^{n-i} Y^{i}.$$
**Example 4.1.** Let \( \mathcal{V}^8 \) be the matroid on \( E = \{1, \ldots, 8\} \) with bases
\[
\{ \sigma \subseteq E : |\sigma| = 4 \} \setminus \{ \{1, 2, 3, 4\}, \{1, 2, 7, 8\}, \{3, 4, 5, 6\}, \{3, 4, 7, 8\}, \{5, 6, 7, 8\} \}.
\]
This is the well-known Vámos matroid. It is non-representable; that is, it is not the vector matroid of any matrix (and thus does not come from any code). Using MAGMA, we find the enumerator of \( \mathcal{V}^8 \) to be
\[
W_{\mathcal{V}^8}(X, Y, Z) = X^8 + 5X^4Y^4Z - 5X^4Y^4 + 36X^3Y^5Z - 36X^3Y^5 + 28X^2Y^6Z^2 \\
- 138X^2Y^6Z + 110X^2Y^6 + 8XY^7Z^3 - 56XY^7Z^2 + 148XY^7Z \\
- 100XY^7 + Y^8Z^4 - 8Y^8Z^3 + 28Y^8Z^2 - 51Y^8Z + 30Y^8.
\]

Observe that if \( C \) is a linear code with parity-check matrix \( H \) and extended weight enumerator \( W_C(X, Y, Q) \) (see e.g. [6]), then
\[
W_C(X, Y, Q) = W_{M(H)}(X, Y, Q).
\]

### 4.2 Equivalence to the Tutte polynomial

It was shown in [6] that for vector matroids derived from a code, the extended weight enumerator of the code determines the Tutte polynomial of the matroid – and vice versa. We shall see that this is still true when it comes to matroids and their enumerators, in general. Despite being generalizations of the ones found in [6], the proofs of Lemma 4.1 and Theorems 4.3 and 4.4 are given here as well – for the sake of completeness and readability.

**Proposition 3.**
\[
P_{M,i}(Z) = \sum_{j=n-i}^{n} (-1)^{i+j+n} \binom{j}{n-i} \sum_{|\gamma| = j} Z^{p_M(E \setminus \gamma)}.
\]
Proof.

\[ P_{M,i}(Z) = (-1)^i \sum_{|\sigma|=i, \gamma \subseteq \sigma} (-1)^{|\gamma|} Z^{nM}(\gamma) \]

\[ = (-1)^i \sum_{|\sigma|=i, \gamma \subseteq \sigma} \sum_{|E \setminus \gamma| \leq \sigma} (-1)^{|E \setminus \gamma|} Z^{nM}(E \setminus \gamma) \]

\[ = (-1)^i \sum_{|\sigma|=i, \gamma \subseteq \sigma} \sum_{|E \setminus \gamma| \leq \sigma} (-1)^{|E \setminus \gamma|} Z^{nM}(E \setminus \gamma) \]

\[ = (-1)^i \sum_{|\gamma| \geq n-i} \left( \sum_{E \setminus \gamma \subseteq \sigma, |E \setminus \gamma| \leq \gamma} (-1)^{|E \setminus \gamma|} Z^{nM}(E \setminus \gamma) \right) \]

\[ = (-1)^i \sum_{|\gamma| \geq n-i} \left( \sum_{j=0}^{n} \binom{n}{j} (-1)^{n+j} Z^{nM}(E \setminus \gamma) \right). \]

\[ \blacksquare \]

Proposition 3 above is what enables us to use basically the same technique as that employed in [6] for the proofs of Theorems 4.3 and 4.4.

Lemma 4.1.

\[ W_{M}(X, Y, Z) = \sum_{j=0}^{n} \sum_{|\gamma|=j} Z^{nM}(E \setminus \gamma) (X - Y)^j Y^{n-j}. \]
We shall also need a slight reformulation of the Tutte polynomial. For the remainder of this section, let \( k = r(M) \).

**Lemma 4.2.**

\[
W_M(X, Y, Z) = \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{|\gamma| = j} (-1)^{i+j+n} \binom{j}{n-i} Z^{n_M(E \setminus \gamma)} X^{n-i} Y^i 
\]

\[
= \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{|\gamma| = j} (-1)^{i+j+n} \binom{j}{n-i} Z^{n_M(E \setminus \gamma)} X^{n-i} Y^i 
\]

\[
= \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{|\gamma| = j} (-1)^{i+j+n} \binom{j}{n-i} Z^{n_M(E \setminus \gamma)} X^{n-i} Y^i 
\]

\[
= \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{|\gamma| = j} (-1)^{i+j+n} \binom{j}{n-i} Z^{n_M(E \setminus \gamma)} X^{n-i} Y^i 
\]

\[
= \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{|\gamma| = j} (-1)^{i+j+n} \binom{j}{n-i} Z^{n_M(E \setminus \gamma)} X^{n-i} Y^i 
\]

We shall also need a slight reformulation of the Tutte polynomial. For the remainder of this section, let \( k = r(M) \).

**Theorem 4.3.**

\[
W_M(X, Y, Z) = (X - Y)^{n-k} Y^k t_M \left( \frac{X}{Y}, X + (Z - 1)Y, X - Y \right). 
\]
\begin{proof}
By Lemma 4.2 we have
\[
(X - Y)^{n-k}Y^k t_M \left( \frac{X}{Y}, \frac{X + (Z - 1)Y}{X - Y} \right)
= (X - Y)^{n-k}Y^k \sum_{j=0}^{n} \sum_{|\gamma| = j} \left( \frac{X - Y}{Y} \right)^{n(E \setminus \gamma) - (n-k-j)} \left( \frac{ZY}{X - Y} \right)^{n(E \setminus \gamma)}
= \sum_{j=0}^{n} \sum_{|\gamma| = j} Z^{n(E \setminus \gamma)} Y^{n-k-j} (X - Y)^{-(n-k-j)} (X - Y)^{n-k} \quad \tag{\ref{lem:4.2}}
= \sum_{j=0}^{n} \sum_{|\gamma| = j} Z^{n(E \setminus \gamma)} (X - Y)^{j} Y^{n-j},
\]
which according to Lemma 4.1 is equal to $W_M(X, Y, Z)$. \qed
\end{proof}

\begin{thm}
\[ t_M(X, Y) = (X - 1)^{-(n-k)} X^n W_M(1, X^{-1}, (X - 1)(Y - 1)). \]
\end{thm}

\begin{proof}
By Lemma 4.1 we have
\[
(X - 1)^{-(n-k)} X^n W_M(1, X^{-1}, (X - 1)(Y - 1))
= (X - 1)^{-(n-k)} X^n \sum_{j=0}^{n} \sum_{|\gamma| = j} ((X - 1)(Y - 1))^{|\gamma|} \left( 1 - X^{-1} \right)^j X^{-(n-j)}
= \sum_{j=0}^{n} \sum_{|\gamma| = j} ((X - 1)(Y - 1))^{|\gamma|} X^{-j} (X - 1)^j X^{-(n-j)} (X - 1)^{-(n-k)} X^n
= \sum_{j=0}^{n} \sum_{|\gamma| = j} (Y - 1)^{|\gamma|} (X - 1)^{|\gamma|} X^{-j} (X - 1)^j X^{-(n-j)} (X - 1)^{-(n-k)} X^n
= \sum_{j=0}^{n} \sum_{|\gamma| = j} (Y - 1)^{|\gamma|} (X - 1)^{|\gamma|} X^{-j} (X - 1)^j X^{-(n-j)} (X - 1)^{-(n-k)} X^n,
\]
which according to Lemma 4.2 is equal to $t_M(X, Y)$. \qed
\end{proof}

\begin{exa}[Continuation of Ex. 4.1]
Having already found the weight enumerator of $V^8$, we infer from Theorem 4.4 that
\[
t_{V^8}(X, Y) = X^4 + 4X^3 + 10X^2 + 5XY + 15X + Y^4 + 4Y^3 + 10Y^2 + 15Y.
\]
\end{exa}
5 The GWP is determined by Betti numbers

Let $M$ denote a matroid of rank $k$ on an $n$-element ground set $E$. Recall from Section 2 that the $\mathbb{N}_0$- and $\mathbb{N}_0^r$-graded Betti numbers corresponding $I_M$ are independent of choice of field for our minimal free resolution. The only thing of importance, and thus our only assumption, is that the $\mathbb{N}_0$-graded (or $\mathbb{N}_0^r$-graded) minimal free resolution of $I_M$ is constructed with respect to the same field as is the reduced chain complex over $M$. We may therefore omit specifying a field.

**Theorem 5.1** (Main result). Let $\beta^{(l)}$ distinguish the Betti numbers of $M_l$ from those of $M$, and set $\beta^{(l)}_{i,j} = 0$ whenever $l \not\in [0, n-r(M)]$. For each $1 \leq j \leq n$ the coefficient of $Z^j$ in $P_{M,j}$ is equal to

$$\sum_{i=0}^{n} (-1)^i \left( \beta^{(i-1)}_{i,j} - \beta^{(i)}_{i,j} \right).$$

**Proof.** Let $s_{\sigma,j}$ denote the coefficient of $Z^j$ in $P_{M[\sigma],|\sigma|}$. Since

$$P_{M,j}(Z) = \sum_{|\sigma|=j} P_{M[\sigma],|\sigma|}(Z),$$

the coefficient of $Z^j$ in $P_{M,j}(Z)$ is $\sum_{|\sigma|=j} s_{\sigma,j}$. On the other hand, we have

$$s_{\sigma,j} = (-1)^{|\sigma|} \sum_{\gamma \subseteq \sigma, n_M(\gamma) = l} (-1)^{|\gamma|} = (-1)^{|\sigma|} \left[ \sum_{\gamma \subseteq \sigma, n_{M_l}(\gamma) = 0} (-1)^{|\gamma|} - \sum_{\gamma \subseteq \sigma, n_{M_{l-1}}(\gamma) = 0} (-1)^{|\gamma|} \right].$$

Applying Theorems 2.1 and 2.2 in combination with (5), we see that

$$(-1)^{|\sigma|} \sum_{\gamma \subseteq \sigma, n_{M_l}(\gamma) = 0} (-1)^{|\gamma|} = (-1)^{n_{M_l}(\sigma)} \dim H_{M_l,(\sigma)}(M_l(\sigma))$$

$$= (-1)^{n_{M_l}(\sigma)} \beta_{n_{M_l}(\sigma)-1,\sigma}(I_{M_l(\sigma)}),$$

which is equal to $(-1)^{n_{M_l}(\sigma)} \beta^{(l)}_{n_{M_l}(\sigma)-1,\sigma} - \beta^{(l-1)}_{n_{M_l}(\sigma)-1,\sigma}$ since, in general, $\beta_i(\Delta) = \beta_i(\Delta_{\sigma})$.

Thus

$$s_{\sigma,j} = (-1)^{n_{M_l}(\sigma)} \beta^{(l)}_{n_{M_l}(\sigma)-1,\sigma} - (-1)^{n_{M_{l-1}}(\sigma)} \beta^{(l-1)}_{n_{M_{l-1}}(\sigma)-1,\sigma}$$

$$= (-1)^{n_{M_{l-1}}(\sigma)-1} \beta^{(l-1)}_{n_{M_{l-1}}(\sigma)-1,\sigma} - (-1)^{n_{M_l}(\sigma)-1} \beta^{(l)}_{n_{M_l}(\sigma)-1,\sigma},$$

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which by (7) is equal to
\[
\sum_{i=0}^{n} (-1)^{i} \beta_{i,\sigma}^{(l-1)} - \sum_{i=0}^{n} (-1)^{i} \beta_{i,\sigma}^{(l)}.
\]

Consequently, the coefficient of \(Z^{l}\) in \(P_{M,j}(Z)\) is
\[
\sum_{|\sigma|=j} \left( \sum_{i=0}^{n} (-1)^{i} \left( \beta_{i,\sigma}^{(l-1)} - \beta_{i,\sigma}^{(l)} \right) \right) = \sum_{i=0}^{n} (-1)^{i} \left( \sum_{|\sigma|=j} \beta_{i,\sigma}^{(l-1)} - \sum_{|\sigma|=j} \beta_{i,\sigma}^{(l)} \right)
= \sum_{i=0}^{n} (-1)^{i} \left( \beta_{i,j}^{(l-1)} - \beta_{i,j}^{(l)} \right).
\]

**Example 5.1** (Continuation of Ex. 2.3). Let us calculate \(P_{M(H),5}(Z)\) using Theorem 5.1. Having already found the Betti numbers of \(M(H)\) and its elongations, we easily calculate
\[
P_{M(H),5}(Z) = (\beta_{0,5}^{(1)} - \beta_{0,5}^{(1)})Z^{2} + \left( (-\beta_{1,5} + \beta_{2,5} - \beta_{3,5}) - (\beta_{0,5}^{(1)} - \beta_{0,5}^{(1)}) \right)Z
- (-\beta_{1,5} + \beta_{2,5} - \beta_{3,5})
= (15 - 0)Z^{2} + ((-28 + 0 - 0) - (15 - 0))Z
- (-28 + 0 - 0).
\]

Continuing like this, we find the complete set of weight polynomials:
\[
P_{M(H),0}(Z) = 1
P_{M(H),1}(Z) = 0
P_{M(H),2}(Z) = Z - 1
P_{M(H),3}(Z) = 6Z - 6
P_{M(H),4}(Z) = 2Z^{2} - Z - 1
P_{M(H),5}(Z) = 15Z^{2} - 43Z + 28
P_{M(H),6}(Z) = 7Z^{3} - 36Z^{2} + 60Z - 31
P_{M(H),7}(Z) = Z^{4} - 7Z^{3} + 19Z^{2} - 23Z + 10.
\]

**Corollary 1.** Let \(1 \leq m \leq n\). With the convention \(\beta_{i,j}^{(l)}(M(H)) = 0\) whenever \(l \notin [0, n - r(M(H))]\), we have
\[
A_{C,m}(Q) = \sum_{l=0}^{n} \left( \sum_{i=0}^{n} (-1)^{i} \left( \beta_{i,m}^{(l-1)}(M(H)) - \beta_{i,m}^{(l)}(M(H)) \right) \right) Q^{l}.
\]
Proof. This is immediate from Theorem 5.1 since $A_{C,m}(Q) = P_{M(H),m}(Q)$ by Proposition 2.

In light of Corollary 1, the polynomials found in 5.1 when evaluated in $q^m$, determine the number of codewords of a given weight in $C \otimes \mathbb{F}_q$. Occasionally, the result of Corollary 1 can greatly simplify the task of calculating weight polynomials $A_{C,j}(Q)$ for a code $C$. This is for instance the case with MDS-codes:

Example 5.2. Let $C$ be an MDS $[n,k]$-code over $\mathbb{F}_q$, with parity check matrix $H$. It is well known that $M(H)$ is the uniform matroid $U(r,n)$, where $r = n - k$; which of course implies that $M(H)(l) = U(r+l,n)$. From e.g. [4, Example 3], we see that

$$\beta_{i,j}^{(l)}(M(H)) = \begin{cases} \binom{j-1}{r+l}, & i = j - l - r - 1, \\ 0, & \text{otherwise}. \end{cases}$$

We conclude from Corollary 1 that for $1 \leq j \leq n$, and $Q = q^m$, we have

$$A_{C,j}(Q) = \sum_{l=1}^{n} (-1)^{i+l+r} \binom{n}{j} \left( \binom{j-1}{r+l-1} + \binom{j-1}{r+l} \right) Q^l + (-1)^{j+r} \binom{n}{j} \binom{j-1}{r}.$$

5.1 Further results

The generalized weight polynomial of $M$ determines the generalized weight polynomial of $M_i$ for all $i \geq 1$.

Proposition 4. Let $k \geq 1$. If

$$P_{M_{k-1},j}(Z) = a_n Z^n + a_{n-1} Z^{n-1} + \cdots + a_1 Z + a_0,$$

then

$$P_{M_k,j}(Z) = a_n Z^{n-1} + a_{n-1} Z^{n-2} + \cdots + a_2 Z + (a_1 + a_0).$$
Proof. Let $s_{\sigma,l}^{(k)}$ denote the coefficient of $Z^l$ in $P_{M_{k,|\sigma|}}$. As noted in the proof of Theorem 5.1 then, the coefficient of $Z^l$ in $P_{M_{j,|\sigma|}}$ is $\sum_{|\sigma|=j} s_{\sigma,l}^{(k)}$, and

$$s_{\sigma,l}^{(k)} = (-1)^{|\sigma|} \sum_{n_{M_k}(\gamma)=l} (-1)^{|\gamma|}.$$

Assume first that $l \geq 1$. By Proposition 1 we have,

$$s_{\sigma,l}^{(k)} = (-1)^{|\sigma|} \sum_{n_{M_k}(\gamma)=l} (-1)^{|\gamma|}$$

$$= (-1)^{|\sigma|} \sum_{n_{M_{k-1}}(\gamma)=l+1} (-1)^{|\gamma|}$$

$$= s_{\sigma,l+1}^{(k-1)}.$$

Finally, by Proposition 1 again, we see that

$$s_{\sigma,0}^{(k)} = (-1)^{|\sigma|} \sum_{n(\gamma) \leq k} (-1)^{|\gamma|}$$

$$= (-1)^{|\sigma|} \sum_{n(\gamma) = k} (-1)^{|\gamma|} + (-1)^{|\sigma|} \sum_{n(\gamma) \leq k-1} (-1)^{|\gamma|}$$

$$= (-1)^{|\sigma|} \sum_{n_{M_{k-1}}(\gamma)=1} (-1)^{|\gamma|} + (-1)^{|\sigma|} \sum_{n_{M_{k}}(\gamma)=0} (-1)^{|\gamma|}$$

$$= s_{\sigma,1}^{(k-1)} + s_{\sigma,0}^{(k-1)},$$

and this concludes our proof. \qed

6 Concerning the converse

Having seen that the Betti numbers associated to the $M_i$s determine the polynomials $P_{M_{j,\sigma}}(Z)$, it is natural to ask whether the opposite is true. The answer to this is negative, as the following counterexample shows:
Example 6.1 (Continuation of Ex. 5.1). Let $N$ be the matroid on $\{1, \ldots, 7\}$ with bases

$$B(N) = \{\{1, 4, 7\}, \{1, 3, 6\}, \{1, 3, 5\}, \{1, 3, 4\}, \{2, 3, 6\}, \{3, 4, 7\}, \{1, 2, 5\}, \{1, 5, 7\},$$
$$\{3, 6, 7\}, \{2, 4, 7\}, \{3, 5, 6\}, \{2, 3, 4\}, \{1, 2, 3\}, \{1, 2, 7\}, \{1, 5, 6\}, \{3, 4, 5\},$$
$$\{1, 6, 7\}, \{1, 4, 5\}, \{2, 3, 7\}, \{2, 5, 6\}, \{2, 4, 5\}, \{3, 5, 7\}, \{2, 6, 7\}, \{2, 5, 7\}\}.$$

The Stanley-Reisner ideal of $N$ has minimal free resolution

$$0 \leftarrow S(-2) \oplus S(-3)^6 \oplus S(-4)^5 \leftarrow S(-4)^4 \oplus S(-5)^28 \leftarrow S(-6)^{31} \leftarrow S(-7)^{10} \leftarrow 0.$$

Comparing to the minimal free resolution of $I_{M(H)}$, we see that the Betti numbers are not the same. However, it is easy to see, using Proposition 5.1, that $I_N$ has the same generalized weight polynomials as $M(H)$.

Note that this is the “smallest” counterexample, in that there are no counterexamples for $n < 7$.

Moreover, knowing the Betti numbers of $M$ is in itself not enough to calculate $P_{M,j}$ – we need the Betti numbers derived from the other $M_j$'s as well:

Example 6.2. The matroids $M$ and $N$ on $\{1, \ldots, 8\}$ with bases

$$B(M) = \{\{1, 3, 4, 6, 7\}, \{1, 2, 3, 6, 8\}, \{1, 2, 3, 4, 8\}, \{1, 2, 3, 5, 8\}, \{1, 2, 5, 6, 8\}, \{1, 2, 3, 4, 7\},$$
$$\{1, 2, 3, 5, 7\}, \{1, 2, 5, 6, 7\}, \{1, 3, 4, 5, 7\}, \{1, 3, 4, 6, 8\}, \{1, 2, 4, 6, 8\}, \{1, 2, 4, 6, 7\},$$
$$\{1, 3, 4, 5, 8\}, \{1, 2, 4, 5, 7\}, \{1, 4, 5, 6, 7\}, \{1, 2, 3, 6, 7\}, \{1, 3, 5, 6, 7\}, \{1, 4, 5, 6, 8\},$$
$$\{1, 3, 5, 6, 8\}, \{1, 2, 4, 5, 8\}\}$$

and

$$B(N) = \{\{1, 3, 4, 6, 7\}, \{1, 2, 3, 4, 8\}, \{1, 2, 3, 5, 8\}, \{1, 2, 5, 6, 8\}, \{1, 2, 3, 4, 7\}, \{1, 2, 3, 5, 7\},$$
$$\{1, 2, 5, 6, 7\}, \{1, 3, 4, 5, 7\}, \{1, 3, 4, 6, 8\}, \{1, 2, 4, 6, 8\}, \{1, 2, 4, 6, 7\}, \{1, 3, 4, 5, 8\},$$
$$\{1, 2, 4, 5, 7\}, \{1, 3, 4, 5, 8\}, \{1, 2, 4, 5, 6\}, \{1, 3, 5, 6, 7\}, \{1, 2, 3, 5, 6\}, \{1, 2, 3, 4, 6\},$$
$$\{1, 3, 5, 6, 8\}, \{1, 2, 4, 5, 8\}\},$$

respectively, both have

$$0 \leftarrow S(-2) \oplus S(-4)^5 \leftarrow S(-5)^4 \oplus S(-6)^5 \leftarrow S(-7)^4 \leftarrow 0$$

as the minimal free resolution of their associated Stanley-Reisner ideal, while

$$P_{M,4}(Z) = Z^2 - 5Z + 4 \neq 2Z^2 - 6Z + 4 = P_{N,4}(Z).$$

Again this is the “smallest” counterexample.
Two non-isomorphic matroids may however have identical Betti numbers in all levels (the smallest example of which are a couple of rank-3 matroids on \(\{1, \ldots, 6\}\)).

### 7 The GWPs determines the weight hierarchy

As before, we let \(M\) denote a matroid on \(n\) elements. It follows from Theorems 4.3 and 4.4 that the Tutte- and generalized weight polynomials determine each other. Since it is well known that the Tutte polynomial of a matroid in turn determines the weight hierarchy \(\{d_i\}\) (see [3, p. 131]), we conclude that the generalized weight polynomials – at least indirectly – do so as well. In this Section we demonstrate that they do so in a direct, accessible, and easily applicable manner.

Let \(\beta^{(l)}\) distinguish the Betti numbers of \(M_l\) from those of \(M(= M_0)\).

**Lemma 7.1.** For \(i \geq 1\),

\[
\beta^{(l)}_{i,j} \neq 0 \iff \beta^{(l+1)}_{i-1,j} \neq 0.
\]

**Proof.** According to [4, Theorem 1], we have that

\[
\beta^{(l)}_{i,j} \neq 0 \iff \sigma \text{ is minimal with the property } n_\sigma = i + 1.
\]

Since \(\beta_{i,j} = \sum_{|\sigma| = j} \beta_{i,\sigma}\), we see that

\[
\beta^{(l)}_{i,j} \neq 0 \iff \exists \sigma \text{ such that } |\sigma| = j \text{ and } \sigma \text{ is minimal with the property } n_{l}(\sigma) = i + 1
\]

\[
\iff \exists \sigma \text{ such that } |\sigma| = j \text{ and } \sigma \text{ is minimal with the property } n_{l+1}(\sigma) = i
\]

\[
\iff \beta^{(l+1)}_{i-1,j} \neq 0. \quad \square
\]

In terms of Betti-tables, this implies that when it comes to zeros and non-zeros the Betti-table of \(M_{l+1}\) is equal to the table you get by deleting the first column from \(M_l\)'s table.

**Lemma 7.2.**

\[
d_i(M) = \min \{ j : \beta^{(i-1)}_{0,j} \neq 0 \}.
\]

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Proof. By [4, Theorem 2] we have
\[ d_i(M) = \min\{ j : \beta_{i-1,j} \neq 0 \}, \]
and the result thus follows immediately from Lemma 7.1.

Proposition 5.
\[ d_i(M) = \min\{ s : \deg P_{M,s} = i \}. \]

Proof. It follows from Lemma 7.2 and from minimality of the free resolutions that
\[ \beta_{u,v}^{(i-1)} = 0 \text{ for all } u \geq 0, v \leq u + d_i - 1(M). \]

Recall that the coefficient of \( Z_i \) in \( P_{M,s} \) is equal to
\[ \sum_{t=0}^{n} (-1)^{t+1} \left( \beta_{t,s}^{(i)} - \beta_{t,s}^{(i-1)} \right). \] (10)

We conclude both that this is equal to zero whenever \( s < d_i(M) \), and that the coefficient of \( Z_i \) in \( P_{M,d_i(M)} \) is precisely \( \beta_{0,d_i(M)}^{(i-1)} \neq 0 \).

The following is an immediate consequence of Lemmas 7.1 and 7.2.

Proposition 6.
\[ d_i(M_{l+1}) = d_{i+1}(M_l). \]

Example 7.1 (The simplex code \( S_2(3) \)). Let \( S_2(3) \) be the simplex code of dimension 3 over \( \mathbb{F}_2 \). This code has length \( n = 7 \). Let \( H \) be a parity-check matrix of \( S_2(3) \).

The higher weights of \( S_2(3) \) are \( (d_1,d_2,d_3) = (4,6,7) \), from which it follows by way of [5 Theorem 2] that the non-zero Betti numbers of \( I_{M(H)} \) are
\[ (\beta_{0,4},\beta_{1,6},\beta_{2,7}) = (7,14,8). \]

By Proposition 6 the higher weights of \( M_1 \) are \( (d_1,d_2) = (6,7) \), which implies that \( M(H)_1 \) must be the uniform matroid \( U(5,7) \). From [4 Example 3] then, we see that the only non-zero Betti numbers of \( I_{M(H)_1} \) are \( \beta_{0,6}(M(H)_1) = 7 \) and \( \beta_{1,7}(M(H)_1) = 6 \). As always, the \((n-r(M(H)) - 1)^{th}\) elongation \( M(H)_2 \) has \( \{1,\ldots,7\} \) as its only circuit, such that the only non-zero Betti-number associated to \( I_{M(H)_2} \) is \( \beta_{0,7}(M(H)_2) = 1 \).
Having found all Betti numbers from all elongations, we easily calculate the weight polynomials using Corollary 1:

\[
\begin{align*}
A_{\gamma_2(3),0}(Q) &= 1 \\
A_{\gamma_2(3),1}(Q) &= 0 \\
A_{\gamma_2(3),2}(Q) &= 0 \\
A_{\gamma_2(3),3}(Q) &= 0 \\
A_{\gamma_2(3),4}(Q) &= 7Q - 7 \\
A_{\gamma_2(3),5}(Q) &= 0 \\
A_{\gamma_2(3),6}(Q) &= 7Q^2 - 21Q + 14 \\
A_{\gamma_2(3),7}(Q) &= Q^3 - 7Q^2 + 14Q - 8
\end{align*}
\]

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