A functional central limit theorem for regenerative chains

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Abstract

Using the regenerative scheme of [8], we establish a functional central limit theorem (FCLT) for discrete time stochastic processes (chains) with summable memory decay. Furthermore, under stronger assumptions on the memory decay, we identify the limiting variance in terms of the process only. As applications, we define classes of binary autoregressive processes and power-law Ising chains for which the FCLT is fulfilled.

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1 Introduction and preliminaries

1.1 Introduction

Chains are discrete-time stochastic processes with infinite memory that are natural extensions of Markov chains when the associated process depends on its whole past. Such processes have been extensively studied (see e.g. Fernández and Maillard [11] and references therein), but surprisingly very few is known about limit theorems. In this paper, we partially fill this gap by establishing a functional central limit theorem.

Historically, the first central limit theorems for chains have been established, under strong ergodic assumptions, by Cohn (1966) [7], Ibragimov and Linnik (1971) [14] (see also Iosifescu and Grigorescu (1990) [13]). More recently, the empirical entropies of chains with exponential memory decay have been studied both in terms of their limit behavior (Gabrielli, Galves and Guiol (2003) [12]) and their large deviations (Chazottes and Gabrielli (2005) [3]).

Limit theorems such as LCLT and LIL have been broadly studied for Markovian chains (see e.g. Meyn and Tweedie [19]). Regeneration methods, introduced by Chung (1967) [5] and refined by Chen (1999) [4], have been used to divide the Markov chain into independent random blocks in order to derive such limit theorems. This constitute a motivating challenge to extend those techniques to the non-Markovian case.

In that paper, we prove a general FCLT (Theorem 2.1) for chains satisfying the regenerative scheme introduced in Comets, Fernández and Ferrari [8]. We give an explicit expression for
the associated limiting variance depending both on the original and regenerative processes. As a corollary (Corollary 2.2), we give a more tractable condition on the memory decay of the chain under which the FCLT is fulfilled. We also give a regime of the memory decay for which the limiting variance can be expressed in terms on the original process only. As applications, we derive FCLTs for autoregressive binary processes and Ising chains (Propositions 3.1 and 3.2).

The paper is organized as follows. In the rest of this section, we give some definitions and preliminaries. In Section 2 we state the main results. In Section 3 we introduce binary autoregressive processes and power-law Ising chains for which we give central limit theorems. Finally, Section 4 is devoted to the proofs.

1.2 Notation and preliminary definitions

We consider a measurable space \((E, \mathcal{E})\) where \(E\) is a finite alphabet and \(\mathcal{E}\) is the discrete \(\sigma\)-algebra. We denote \((\Omega, F)\) the associated product measurable space with \(\Omega = \mathbb{Z}^E\). For each \(\Lambda \subset \mathbb{Z}\) we denote \(\Omega_\Lambda = \mathbb{E}^\Lambda\) and \(\sigma_\Lambda\) for the restriction of a configuration \(\sigma \in \Omega\) to \(\Omega_\Lambda\), namely the family \((\sigma_\Lambda)_{\Lambda \in \mathbb{Z}} \in \mathbb{E}^\Lambda\). Also, \(F_\Lambda\) will denote the sub-\(\sigma\)-algebra of \(F\) generated by cylinders based on \(\Lambda\) (\(F_\Lambda\)-measurable functions are insensitive to configuration values outside \(\Lambda\)). When \(\Lambda\) is an interval, \(\Lambda = [k, n]\) with \(k, n \in \mathbb{Z}\) such that \(k \leq n\), we use the notation: \(\omega^n_k = \omega_{[k,n]} = \omega_k, \ldots, \omega_n, \Omega^n_k = \Omega_{[k,n]}\) and \(F^n_k = F_{[k,n]}\). For semi-intervals we denote also \(F_{\leq n} = F_{(-\infty,n]}\), etc. The concatenation notation \(\omega_\Lambda \sigma_\Delta\), where \(\Lambda \cap \Delta = \emptyset\), indicates the configuration on \(\Lambda \cup \Delta\) coinciding with \(\omega_i\) for \(i \in \Lambda\) and with \(\sigma_i\) for \(i \in \Delta\).

1.3 Chains

We start by briefly reviewing the well-known notions of chains in a shift-invariant setting. In this particular case, chains are also called \(g\)-measures (see [16]).

**Definition 1.1** A \(g\)-function \(g\) is a probability kernel \(g: \Omega_0 \times \Omega_{-\infty}^{-1} \rightarrow [0,1]\), i.e.,

\[
\sum_{\omega_0 \in \Omega_0} g(\omega_0 \mid \omega_{-\infty}^{-1}) = 1, \quad \omega_{-\infty}^{-1} \in \Omega_{-\infty}^{-1}.
\]

The \(g\)-function \(g\) is:

(i) **Continuous** if the function \(g(\omega_0 \mid \cdot)\) is continuous for each \(\omega_0 \in \Omega_0\), i.e., for all \(\epsilon > 0\), there exists \(n \geq 0\) so that

\[
|g(\omega_0 \mid \omega_{-\infty}^{-1}) - g(\sigma_0 \mid \sigma_{-\infty}^{-1})| < \epsilon
\]

for all \(\omega_0^{-\infty}, \sigma_0^{-\infty} \in \Omega_0^{-\infty}\) with \(\omega_0^{-n} = \sigma_0^{-n}\); (ii) **Bounded away from zero** if \(g(\omega_0 \mid \cdot) \geq c > 0\) for each \(\omega_0 \in \Omega_0\);

(iii) **Regular** if \(f\) is continuous and bounded away from zero.

**Definition 1.2** A probability measure \(\mathbb{P}\) on \((\Omega, F)\) is said to be **consistent** with a \(g\)-function \(g\) if \(\mathbb{P}\) is shift-invariant and

\[
\int h(\omega) g(x \mid \omega) \mathbb{P}(d\omega) = \int_{\{\omega_0 = x\}} h(\omega) \mathbb{P}(d\omega)
\]

(1.3)
for all \( x \in E \) and \( F_{\leq-1} \)-measurable function \( h \). The family of these measures will be denoted by \( \mathcal{G}(g) \) and for each \( \mathbb{P} \in \mathcal{G}(g) \), the process \((X_i)_{i \in \mathbb{Z}}\) on \((\Omega, \mathcal{F}, \mathbb{P})\) will be called a \( g \)-chain.

**Remark 1.3** In the consistency definition (1.3), \( \mathbb{P} \) needs only to be defined on \((\Omega_{\leq 0}, \mathcal{F}_{\leq 0})\). Because of its shift-invariance, \( \mathbb{P} \) can be extended in a unique way to \((\Omega, \mathcal{F})\). That’s why, without loss of generality, we can make no distinction between \( \mathbb{P} \) on \((\Omega_{\leq 0}, \mathcal{F}_{\leq 0})\) and its natural extension on \((\Omega, \mathcal{F})\).

### 1.4 Regeneration

The following regeneration result is due to Comets, Fernández and Ferrari [8] (see Theorem 4.1, Corollary 4.3 and Proposition 5.1). It will be the starting point of our analysis.

**Theorem 1.4 (Comets & al. (2002))** Let \( g \) be a regular \( g \)-function such that

\[
\prod_{k \geq 0} a_k > 0 \quad \text{with} \quad a_k = \inf_{\sigma_{-k} \in \Omega_{-k}} \sum_{\xi_0 \in E} \inf_{\omega_{-\infty} \in \Omega_{-\infty}} g(\xi_0 | \sigma_{-1}^{-1} \omega_{-k-1}^{-1})
\]

with the convention \( \sigma_0^{-1} = \emptyset \).

Then

(i) there exists a unique probability measure \( \mathbb{P} \) consistent with \( g \);

(ii) there exists a shift-invariant renewal process \((T_i)_{i \in \mathbb{Z}}\) with renewal distribution

\[
\mathbb{P}(T_{i+1} - T_i \geq M) = \rho_M, \quad M > 0, \ i \neq 0
\]

with \( \rho_M \) the probability of return to the origin at epoch \( M \) of the Markov chain on \( \mathbb{N} \cup \{0\} \) starting at time zero at the origin with transition probabilities

\[
\begin{cases}
  p(k, k+1) = a_k, \\
  p(k, 0) = 1 - a_k, \\
  p(k, j) = 0 \text{ otherwise}
\end{cases}
\]

and such that

\[
T_0 \leq 0 < T_1.
\]

(iii) the random blocks \( \{(X_j: T_i \leq j < T_{i+1})\}_{i \in \mathbb{Z}} \), where \((X_i)_{i \in \mathbb{Z}}\) on \((\Omega, \mathcal{F}, \mathbb{P})\) is the associated \( g \)-chain, are independent and, except for \( i = 0 \), identically distributed.

(iv) \( 1 \leq \mathbb{E}(T_2 - T_1) = \sum_{i=1}^{\infty} \rho_i < \infty \).

### 2 Main Results

We are now ready to state the main results of this paper.
Theorem 2.1 (FCLT) Let \((X_i)_{i \in \mathbb{Z}}\) be a regular \(g\)-chain satisfying the regeneration assumption (1.4) and \(f : E \to \mathbb{R}\) be a function such that

\[ \mathbb{E}(f(X_0)) = 0. \] (2.1)

Then,

\[ \frac{1}{\sqrt{n}} S_n := \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} f(X_i) \to \mathcal{N}(0, \sigma^2) \quad \text{in distribution} \] (2.2)

with

\[ 0 \leq \sigma^2 = \frac{\mathbb{E} \left( \left( \sum_{i=T_1}^{T_2-1} f(X_i) \right)^2 \right)}{\mathbb{E}(T_2 - T_1)} < \infty. \] (2.3)

Furthermore, if \(\mathbb{E}((T_2 - T_1)^2) < \infty\), then

\[ \sigma^2 = \mathbb{E}\left(f^2(X_0)\right) + 2 \sum_{i \geq 1} \mathbb{E}\left(f(X_0) f(X_i)\right) < \infty. \] (2.4)

Corollary 2.2 Let \((X_i)_{i \in \mathbb{Z}}\) be a regular \(g\)-chain such that there exist \(a \in \mathbb{R}, b \in [1, \infty), C > 0\) and \(K \geq 2\) such that for all \(k \geq K\)

\[ a_k \geq 1 - C \frac{(\log(k))^a}{k^b}. \] (2.5)

Let \(f : E \to \mathbb{R}\) satisfying (2.1).

(i) If \(a \in \mathbb{R}\) and \(b > 1\), or \(a < -1\) and \(b = 1\), then (2.2) is satisfied with \(\sigma^2\) defined by (2.3).

(ii) If \(a \in \mathbb{R}\) and \(b > 2\), or \(a < -1\) and \(b = 2\), then (2.2) is satisfied with \(\sigma^2\) defined by (2.4).

Remark 2.3 It is not easy to compare our results with previous ones based on mixing rates (see e.g. [7], [14] or [13]), because there is no general relationship between mixing rates and our continuity rates expressed in terms of \((a_k)_{k \geq 0}\). However, in the case of regular \(g\)-chains with exponential continuity decay, the mixing rates are also of exponential decay (see [2] or [14]), and therefore CLT results as those describe in [14], Chapter 18, are fulfilled.

3 Applications

3.1 Binary autoregressive processes

The binary version of autoregressive processes is mainly used in statistics and econometrics. It describes binary responses when covariates are historical values of the process (see McCullagh and Nelder (1989), Section 4.3, for more details).

In what follows, we consider the example that was introduced previously in [8]. For the alphabet \(E = \{-1, +1\}\), consider \(\theta_0\) a real number and \((\theta_k : k \geq 1)\) an absolutely summable real sequence. Let \(q : \mathbb{R} \to (0, 1)\) be a function strictly increasing and continuously differentiable. Assume that \(g(\cdot | \omega_{-\infty}^{-1})\) is the Bernoulli law on \([-1, +1]\) with parameter \(q(\theta_0 + \sum_{k \geq 1} \theta_k \omega_{-k})\), that is,

\[ g(+1 | \omega_{-\infty}^{-1}) = q\left(\theta_0 + \sum_{k \geq 1} \theta_k \omega_{-k}\right) = 1 - g(-1 | \omega_{-\infty}^{-1}). \] (3.1)
Denote
\[ r_k = \sum_{m>k} |\theta_m|, \quad k \geq 0. \quad (3.2) \]

**Proposition 3.1** Let \((X_i)_{i \in \mathbb{Z}}\) be a regular \(g\)-chain with \(g\) defined by (3.1) and
\[ f(x) = x - \mathbb{E}(X_0). \quad (3.3) \]

(i) If \(\sum_{k \geq 0} r_k^2 < \infty\), then \(|\mathcal{G}(g)| = 1\).

(ii) If \(\sum_{k \geq 0} r_k < \infty\), then the FCLT is satisfied with \(\sigma^2\) defined by (2.3) for \(f\) defined by (3.3).

(iii) If there exists \(K \geq 0\) such that \(r_k \leq C(\log(k))^a/k^b\), \(k \geq K\), with \(C > 0\), \((a \in \mathbb{R} \text{ and } b > 2)\) or \((a < -1 \text{ and } b = 2)\), then the FCLT is satisfied with \(\sigma^2\) defined by (2.4) for \(f\) defined by (3.3).

### 3.2 Power-law Ising chain

For the usual Ising (Gibbs) model, the central limit theorem is well known (see Newman [20]). In the chain context, since the relationship between one dimensional Gibbs measures and chains, discussed in [10], does not allow to easily interpret Gibbs results in a chain setting, the problem becomes relevant. In what follows, we give central limit theorem for power-law Ising chains.

For the alphabet \(E = \{-1, 1\}\), consider the power-law Ising chain defined by
\[ g(\omega_0 | \omega_{-\infty}^{-1}) = \frac{\exp \left[ - \sum_{k=-\infty}^{-1} \phi_k(\omega_0^{-\infty}) \right]}{\sum_{\sigma_0 \in E} \exp \left[ - \sum_{k=-\infty}^{-1} \phi_k(\omega_{-\infty}^{-1}\sigma_0) \right]} \quad (3.4) \]
with
\[ \phi_k(\omega_0^{-\infty}) = -\beta \frac{1}{|k|^p} \omega_0 \omega_k, \quad k \leq -1, \beta > 0, \ p > 1. \quad (3.5) \]

**Proposition 3.2** Let \((X_i)_{i \in \mathbb{Z}}\) be a regular \(g\)-chain with \(g\) defined by (3.4–3.5) and
\[ f(x) = x. \quad (3.6) \]

(i) If \(p > 3/2\), then \(|\mathcal{G}(g)| = 1\).

(ii) If \(p > 2\), then the FCLT is satisfied with \(\sigma^2\) defined by (2.3) for \(f\) defined by (3.6).

(iii) If \(p > 3\), then the FCLT is satisfied with \(\sigma^2\) defined by (2.4) for \(f\) defined by (3.6).
4  Proofs

4.1  Proof of Theorem 2.1

Take \((T_i)_{i \in \mathbb{Z}}\) as given by Theorem 1.4 (ii) and write

\[
S_n = \sum_{i=0}^{n-1} f(X_i) = \sum_{i=0}^{T_1-1} f(X_i) + \sum_{i=T_1}^{T_{i(n)}-1} f(X_i) + \sum_{i=T_{i(n)}}^{n-1} f(X_i)
\]  

with

\[
i(n) = \begin{cases} 
\max \{k \geq 1: T_k < n \} & \text{if } T_1 < n, \\
0 & \text{otherwise}. 
\end{cases}
\]

\[ (4.1) \]

Lemma 4.1  Under (1.4), both \(n^{-1/2} \sum_{i=0}^{T_1-1} f(X_i)\) and \(n^{-1/2} \sum_{i=T_{i(n)}}^{n-1} f(X_i)\) tend to zero in probability.

Proof. For any \(K > 0\), we obviously have

\[
P \left( \left| \sum_{i=0}^{T_1-1} f(X_i) \right| > K \sqrt{n} \right) \leq P \left( T_1 \sup_{x \in E} |f(x)| > K \sqrt{n} \right).
\]  

(4.3)

Therefore,

\[
\sup_{n \geq 1} P \left( \left| \sum_{i=0}^{T_1-1} f(X_i) \right| > K \sqrt{n} \right) \leq P(T_1 > M_K)
\]

with

\[
M_K = \frac{K}{M} \quad \text{and} \quad M = \sup_{x \in E} |f(x)|.
\]

(4.4)

On the other hand, by the shift-invariance of \((T_i)_{i \in \mathbb{Z}}\),

\[
P \left( \left| \sum_{i=T_{i(n)}}^{n-1} f(X_i) \right| > K \sqrt{n} \right) \leq P \left( \sum_{i=T_{i(n)}}^{T_{i(n)}+1} f(X_i) > K \sqrt{n} \right)
\]

\[
\leq P \left( (T_{i(n)}+1 - T_{i(n)}) \sup_{x \in E} |f(x)| > K \sqrt{n} \right)
\]

(4.6)

\[
= P \left( (T_1 - T_0) \sup_{x \in E} |f(x)| > K \sqrt{n} \right),
\]

where we used that \(i(0) = 0\). Therefore,

\[
\sup_{n \geq 1} P \left( \left| \sum_{i=T_{i(n)}}^{n-1} f(X_i) \right| > K \sqrt{n} \right) \leq P(T_1 - T_0 > M_K),
\]

(4.7)

where \(M_K\) is defined by (4.5). Noticing that, under condition (1.4), both \(P(T_1 > M_K)\) and \(P(T_1 - T_0 > M_K)\) tend to zero as \(K\) goes to infinity, we can conclude that both \(n^{-1/2} \sum_{i=0}^{T_1-1} f(X_i)\) and \(n^{-1/2} \sum_{i=T_{i(n)}}^{n-1} f(X_i)\) tend to zero in probability.
Proposition 4.2 Let \((X_i)_{i \in \mathbb{Z}}\) be a regular \(g\)-chain satisfying (1.4) and \(f: E \to \mathbb{R}\). Then the following statements are equivalent:

(i) \(S_n/\sqrt{n} \to \mathcal{N}(0, \sigma^2)\) in distribution for
\[
\sigma^2 = \frac{\mathbb{E}\left(\left(\sum_{i=T_1}^{T_2-1} f(X_i)\right)^2\right)}{\mathbb{E}(T_2 - T_1)},
\]
where \(T_1\) and \(T_2\) are defined in (1.3–1.4);

(ii) \((S_n/\sqrt{n})_{n \geq 0}\) is bounded in probability;

(iii) \(\mathbb{E}\left(\sum_{i=T_1}^{T_2-1} f(X_i)\right) = 0\) and \(\mathbb{E}\left(\left(\sum_{i=T_1}^{T_2-1} f(X_i)\right)^2\right) < \infty\).

Proof. The direction (i) \(\Rightarrow\) (ii) is trivial, so we only have to show (ii) \(\Rightarrow\) (iii) and (iii) \(\Rightarrow\) (i).

To prove (ii) \(\Rightarrow\) (iii), we first remark that equation (4.1), Lemma 4.1, and assertion (ii) imply that \(n^{-1/2} \sum_{i=T_1}^{T_2(n)-1} f(X_i)\) is bounded in probability. Then, by the converse of the central limit theorem for real i.i.d. sequences (see e.g. [18], Section 10.1), we must have (iii).

To prove (iii) \(\Rightarrow\) (i), we first see that
\[
\frac{i(n)}{n} \to \frac{1}{\mathbb{E}(T_2 - T_1)} \quad \text{a.s.,}
\]
which follows from Theorem 5.5.2 of [6] and the fact that \((T_{i+1} - T_i)_{i > 0}\) is an i.i.d. process. Let us denote
\[
\xi_k = \sum_{i=T_k}^{T_{k+1}-1} f(X_i) \quad \text{and} \quad e(n) = \left\lfloor \frac{n}{\mathbb{E}(T_2 - T_1)} \right\rfloor.
\]

Thanks to the Lemma 4.1 to prove (i), it is enough to show that
\[
\frac{1}{\sqrt{n}} \sum_{k=1}^{e(n)} \xi_k \to \mathcal{N}\left(0, \frac{\mathbb{E}(\xi_k^2)}{\mathbb{E}(T_2 - T_1)}\right) \quad \text{in law.}
\]

This follows from the standard central limit theorem result
\[
\frac{1}{\sqrt{n}} \sum_{k=1}^{e(n)} \xi_k \to \mathcal{N}\left(0, \frac{\mathbb{E}(\xi_k^2)}{\mathbb{E}(T_2 - T_1)}\right) \quad \text{in law,}
\]
and
\[
\frac{1}{\sqrt{n}} \left(\sum_{k=1}^{i(n)-1} \xi_k - \sum_{k=1}^{e(n)} \xi_k\right) \to 0 \quad \text{in probability.}
\]

To prove the latter, for any \(\epsilon > 0\), let
\[
K' = \left\lfloor \frac{\epsilon^3}{2(1 + \mathbb{E}(\xi_k^2))} \right\rfloor.
\]
First, remark that
\[
P\left( \left| \sum_{k=1}^{i(n)-1} \xi_k - \sum_{k=1}^{e(n)} \xi_k \right| \geq \sqrt{n\epsilon} \right) \leq 2P\left( \max_{j \leq nK'} \left| \sum_{k=1}^{j} \xi_k \right| \geq \sqrt{n\epsilon} \right) + P\left( \left| i(n) - 1 - e(n) \right| \geq nK' \right). \tag{4.15}
\]

Then, use the Kolmogorov’s maximal inequality, (iii) and (4.14), to get
\[
P\left( \max_{j \leq nK'} \left| \sum_{k=1}^{j} \xi_k \right| \geq \sqrt{n\epsilon} \right) \leq \frac{1}{n\epsilon^2} \text{Var} \left( \sum_{k=1}^{nK'} \xi_k \right) = \frac{K' \text{E}(\xi_k^2)}{\epsilon^2} \leq \frac{\epsilon}{2}. \tag{4.16}
\]

Since
\[
\lim_{n \to \infty} P\left( \left| i(n) - 1 - e(n) \right| \geq nK' \right) = 0, \quad \tag{4.17}
\]

it follows from (4.15–4.16) that
\[
\lim_{n \to \infty} P\left( \left| \sum_{k=1}^{i(n)-1} \xi_k - \sum_{k=1}^{e(n)} \xi_k \right| \geq \sqrt{n\epsilon} \right) < \epsilon. \tag{4.18}
\]

**Proof of Theorem 2.1** First, using the shift-invariance of \( P \), we see that for all \( n > 1 \)
\[
\frac{\text{E}(S_n^2)}{n} = \frac{1}{n} \left( \text{E}\left( \sum_{i=0}^{n-1} f^2(X_i) \right) + 2 \text{E}\left( \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} f(X_i)f(X_j) \right) \right)
\]
\[
= \text{E}\left( f^2(X_0) \right) + \frac{2}{n} \sum_{i=1}^{n-1} (n-i) \text{E}\left( f(X_0)f(X_i) \right). \tag{4.19}
\]

Then, under assumptions (1.4) and (2.1), Theorem 1 of [2] insures that
\[
\sum_{i \geq 1} \text{E}\left( f(X_0)f(X_i) \right) \leq C \sum_{i \geq 1} \rho_i < \infty, \tag{4.20}
\]

for some \( C > 0 \). Therefore, it follows from Kronecker’s lemma that
\[
\lim_{n \to \infty} \frac{\text{E}(S_n^2)}{n} = \text{E}\left( f^2(X_0) \right) + 2 \sum_{i \geq 1} \text{E}\left( f(X_0)f(X_i) \right) < \infty. \tag{4.21}
\]

In particular, Chebyshev's inequality implies that
\[
\frac{S_n}{\sqrt{n}} \quad n \geq 1
\]

is bounded in probability. Then Proposition 4.2 concludes the proof of (2.2–2.3).

To show (2.4), we use first Proposition 4.2 (iii) to get
\[
\text{E}\left( \left[ \sum_{k=1}^{i(n)+1} \xi_k \right]^2 \right) = \text{E}\left( \sum_{k=1}^{i(n)+1} \xi_k^2 \right) = \text{E}(\xi_k^2) \text{E}(i(n) + 1), \tag{4.23}
\]
where the rightmost equality follows from the Wald’s Equation and the fact that $i(n) + 1$ is a stopping time w.r.t. $(\xi_k)_{k \geq 1}$. Since

$$\lim_{n \to \infty} \frac{\mathbb{E}(i(n))}{n} = \frac{1}{\mathbb{E}(T_2 - T_1)} \quad (4.24)$$

(see Theorem 5.5.2 in [6]), (2.3) and (4.23) give

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left( \sum_{k=1}^{i(n) + 1} \xi_k \right)^2 = \sigma^2. \quad (4.25)$$

But, we have

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left( \max_{k \leq n} \xi_k^2 \right) = 0 \quad (4.26)$$

(see [5] p.90 for a proof) and therefore

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left( \xi_k^2 \right) = 0, \quad k \in \{i(n), i(n) + 1\}. \quad (4.27)$$

Hence, by (4.25) and (4.27), we finally have

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left( \sum_{k=1}^{i(n) - 1} \xi_k \right)^2 = \sigma^2. \quad (4.28)$$

Now, if we show that

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left( \sum_{k=1}^{T_1 - 1} f(X_k) \right)^2 = 0 \quad (4.29)$$

and

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left( \sum_{k=T_i(n)}^{n-1} f(X_k) \right)^2 = 0, \quad (4.30)$$

then, using Theorem 1.4 (iii), (4.1) and (4.28), we can conclude the proof of (2.4).

The proofs of (4.29)–(4.30) are similar, we will prove (4.30) only. To that aim, we first remark that

$$k \mathbb{P}(T_1 - T_0 \geq k) \to 0 \quad \text{as} \quad k \to \infty, \quad (4.31)$$

because $\mathbb{P}(T_1 - T_0 \geq k)$ is decreasing to 0 and

$$\sum_{k \geq 0} \mathbb{P}(T_1 - T_0 \geq k) = \mathbb{E}(T_1 - T_0) = \frac{\mathbb{E}([T_2 - T_1]^2)}{\mathbb{E}(T_2 - T_1)} < \infty, \quad (4.32)$$

where the rightmost equality uses that $\mathbb{P}(T_1 - T_0 = k) = k \mathbb{P}(T_2 - T_1 = k) / \mathbb{E}(T_2 - T_1)$ (see e.g. Lawler [17], Chapter 6, Section 6.2, equality (6.10)). Therefore, recalling (4.5) and using the
shift-invariance of $P$, we have

$$\frac{1}{n} \mathbb{E}\left(\left( \sum_{k=T_i(n)}^{n-1} f(X_k) \right)^2 \right) \leq \frac{M^2}{n} \mathbb{E}\left( |n - T_i(n)|^2 \right)$$

$$= \frac{M^2}{n} \sum_{k=1}^{n} k^2 \mathbb{P}(n - T_i(n) = k)$$

$$= \frac{M^2}{n} \sum_{k=1}^{n} (2k - 1) \mathbb{P}(n - T_i(n) \geq k)$$

$$\leq \frac{M^2}{n} \sum_{k=1}^{n} (2k - 1) \mathbb{P}(T_1 - T_0 \geq k),$$

which in view of (4.31), goes to zero as $n$ tends to infinity. \hfill \Box

### 4.2 Proof of Corollary 2.2

**Proof.** To prove (i), it suffices to see that when $(a_k)_{k \geq 0}$ satisfies (2.5), then $(a \in \mathbb{R}$ and $b > 1)$ or $(a < -1$ and $b = 1)$ if and only if $\sum_{k \geq 0} (1 - a_k) < \infty$ which is equivalent to $\prod_{k \geq 0} a_k > 0$. Therefore, applying Theorem 2.1 first part, we get the result.

To prove (ii), we first denote

$$\tilde{a}_k = 1 - C \frac{(\log(k))^a}{kb}$$

and the associated $\tilde{\rho}_k$, defined by the analogous of (1.6). Since $b > 1$, Proposition 5.5 (iv) in Fernández, Ferrari and Galves [9] gives that there exists some constant $C_1 > 0$ so that

$$\tilde{\rho}_k \leq C_1 (1 - \tilde{a}_k), \quad k \geq 0.$$  \hfill (4.35)

Therefore, using that $a_k \geq \tilde{a}_k$ implies $\rho_k \leq \tilde{\rho}_k$, we have

$$\mathbb{E}\left( (T_2 - T_1)^2 \right) = \sum_{k \geq 0} k \rho_k \leq C_1 \sum_{k \geq 0} k (1 - \tilde{a}_k),$$

which is finite if and only if $(a \in \mathbb{R}$ and $b > 2)$ or $(a < -1$ and $b = 2)$. Then, applying Theorem 2.1 second part, we get the result. \hfill \Box

### 4.3 Proof of Proposition 3.1

**Proof.** Define the variation by

$$\text{var}_k = \sup \left\{ |g(\omega_0 \mid \omega_{-k}^{-1}) - g(\sigma_0 \mid \sigma_{-k}^{-1})| : \omega_{-k}^{-1}, \sigma_{-k}^{-1} \in \Omega_{-k}^{-1}, \omega_{-k}^0 = \sigma_{-k}^0 \right\}, \quad k \geq 0.$$  \hfill (4.37)

Then, because $|E| = 2$, we have for any $k \geq 0$ (with the convention $\sigma_0^{-1} = \emptyset$)

$$a_k = \inf \left\{ g(1 \mid \omega_{-k}^{-1} \sigma_{-k}^{-1}) + g(-1 \mid \xi_{-k}^{-1} \sigma_{-k}^{-1}) : \sigma_{-k}^{-1} \in \Omega_{-k}^{-1}, \omega_{-k}^{-1}, \xi_{-k}^{-1} \in \Omega_{-k}^{-1} \right\}$$

$$= 1 - \sup \left\{ -g(1 \mid \omega_{-k}^{-1} \sigma_{-k}^{-1}) + g(1 \mid \xi_{-k}^{-1} \sigma_{-k}^{-1}) : \sigma_{-k}^{-1} \in \Omega_{-k}^{-1}, \omega_{-k}^{-1}, \xi_{-k}^{-1} \in \Omega_{-k}^{-1} \right\}$$

$$= 1 - \text{var}_k.$$  \hfill (4.38)
Therefore,
\[
a_k = 1 - \sup \left\{ q \left( \theta_0 + \sum_{j=1}^k \theta_j \sigma_j + r_k \right) - q \left( \theta_0 + \sum_{j=1}^k \theta_j \sigma_j - r_k \right) : \sigma_k^{-1} \in \Omega_k^{-1} \right\}. \tag{4.39}
\]

Because \( q \) is continuously differentiable on a compact set, there exists \( C > 0 \) such that
\[
\text{var}_k \leq Cr_k, \quad k \geq 0. \tag{4.40}
\]

To prove (i), it suffices to remark that
\[
\sum_{k \geq 0} \text{var}_k^2 < \infty, \tag{4.41}
\]
which is a tight uniqueness criteria in terms on the variation (see Johansson and Öberg \[15\] and Berger, Hoffman and Sidoravicius \[1\]).

To show (ii), we simply note that
\[
\mathbb{E}(f(X_0)) = \mathbb{E}(X_0 - \mathbb{E}(X_0)) = 0 \tag{4.42}
\]
and that
\[
\sum_{k \geq 0} r_k < \infty \implies \prod_{k \geq 0} a_k > 0. \tag{4.43}
\]

Finally, to prove part (iii), it suffices first to combine (4.38) and (4.40) to get
\[
a_k \geq 1 - Cr_k, \quad k \geq 0, \tag{4.44}
\]
and then to apply Corollary 2.2 (ii).

### 4.4 Proof of Proposition 3.2

**Proof.** We need the following well-known bound, whose proof we present for completeness is given in Appendix A (we follow the approach of \[21\] Lemma V.1.4).

**Lemma 4.3** Let \( g \) be a \( g \)-function satisfying (3.4) with \( \sum_{k=-\infty}^{-1} |\phi_k| < \infty \) and \( h \) be a \( F_0 \)-measurable function. Then, for any \( \omega_{-\infty}^{-1}, \sigma_{-\infty}^{-1} \in \Omega_{-\infty}^{-1} \),
\[
\left| \sum_{\omega_0 \in \mathcal{E}} h(\omega_0) \left( g(\omega_0 | \omega_{-\infty}^{-1}) - g(\omega_0 | \sigma_{-\infty}^{-1}) \right) \right|
\[
\leq \sup_{x \in \mathcal{E}} |h(x)| \sup_{\omega_0 \in \mathcal{E}} \left| \sum_{k=-\infty}^{-1} \phi_k(\omega_0) - \phi_k(\sigma_{-\infty}^{-1} \omega_0) \right|. \tag{4.45}
\]

Applying the previous lemma for \( h \equiv 1 \), we obtain that, for any \( \omega_{\leq i-1}, \sigma_{\leq i-1} \in \Omega_{\leq i-1} \),
\[
\left| g(\omega_0 | \omega_{-\infty}^{-1}) - g(\omega_0 | \sigma_{-\infty}^{-1}) \right| \leq |\beta| \sup_{\omega_0 \in \mathcal{E}} \left| \sum_{k=-\infty}^{-1} \frac{1}{|k|^p} (\omega_0 \omega_k - \omega_0 \sigma_k) \right|, \tag{4.46}
\]

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from which
\[ \text{var}_k \leq 2|\beta| \sum_{j=-\infty}^{-k-1} \frac{1}{|j|^p} \leq 2|\beta| \frac{1}{k^{p-1}}, \quad k \geq 1, \] (4.47)
is an immediate consequence.

To prove (i), it suffices to see that (4.47) with \( p > 3/2 \) implies the validity of (4.41).

To prove (ii) and (iii), we first remark that similarly to the gibbsian setting, it can be easily checked that \( E(X_0) = 0 \) and therefore under (3.6), (2.1) is fulfilled. Then, we combine (4.38) and (4.47) to get
\[ a_k \geq 1 - 2|\beta| \frac{1}{k^{p-1}}, \quad k \geq 1. \] (4.48)
Thus, the results are direct consequences of Corollary 2.2 (i) and (ii).

\[ \blacksquare \]

A Appendix

In this appendix we give the proof of Lemma 4.3.

Proof. For all \( \omega_{-\infty}, \sigma_{-\infty} \in \Omega_{-\infty} \) and \( 0 < \theta < 1 \), define \( \Gamma^\theta_{\omega,\sigma} : E \to (0, 1) \) by
\[ \Gamma^\theta_{\omega,\sigma} (\xi_0) = \frac{\exp \left[ \theta H_\omega (\xi_0) + (1 - \theta) H_\sigma (\xi_0) \right]}{\sum_{\eta_0 \in E} \exp \left[ \theta H_\omega (\eta_0) + (1 - \theta) H_\sigma (\eta_0) \right]} \quad \text{with} \quad H_\omega (\xi_0) = \sum_{k=-\infty}^{-1} \phi_k (\omega_{-\infty} \xi_0). \] (A.1)

Then, to prove (4.45), it suffices to see that
\[
\left| \sum_{\xi_0 \in E} h(\xi_0) \left( g(\xi_0 | \omega_{-\infty}) - g(\xi_0 | \sigma_{-\infty}) \right) \right|
\leq \int_0^1 \left| \frac{d}{d\theta} \left[ \sum_{\xi_0 \in E} h(\xi_0) \Gamma^\theta_{\omega,\sigma} (\xi_0) \right] \right| d\theta
= \int_0^1 \left| \sum_{\xi_0 \in E} h(\xi_0) \left( H_\omega (\xi_0) - H_\sigma (\xi_0) \right) \Gamma^\theta_{\omega,\sigma} (\xi_0) \right|
- \sum_{\xi_0 \in E} h(\xi_0) \Gamma^\theta_{\omega,\sigma} (\xi_0) \sum_{\eta_0 \in E} \left( H_\omega (\eta_0) - H_\sigma (\eta_0) \right) \Gamma^\theta_{\omega,\sigma} (\eta_0) \right| d\theta
\leq \| h \|_\infty \int_0^1 \left| \sum_{\xi_0 \in E} \left( H_\omega (\xi_0) - H_\sigma (\xi_0) \right) - \sum_{\eta_0 \in E} \left( H_\omega (\eta_0) - H_\sigma (\eta_0) \right) \Gamma^\theta_{\omega,\sigma} (\eta_0) \right| \Gamma^\theta_{\omega,\sigma} (\xi_0) d\theta
\leq \| h \|_\infty \sup_{\xi_0 \in E} \left| H_\omega (\xi_0) - H_\sigma (\xi_0) \right|. \] (A.2)

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