Thermal Casimir interactions for higher derivative field Lagrangians: generalized Brazovskii models

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Abstract
We examine the Casimir effect for free statistical field theories which have Hamiltonians with second order derivative terms. Examples of such Hamiltonians arise from models of non-local electrostatics, membranes with non-zero bending rigidities and field theories of the Brazovskii type that arise for polymer systems. The presence of a second derivative term means that new types of boundary conditions can be imposed, leading to a richer phenomenology of interaction phenomena. In addition zero modes can be generated that are not present in standard first derivative models, and it is these zero modes which give rise to long range Casimir forces. Two physically distinct cases are considered: (i) unconfined fields, usually considered for finite size embedded inclusions in an infinite fluctuating medium, here in a two plate geometry the fluctuating field exists both inside and outside the plates, (ii) confined fields, where the field is absent outside the slab confined between the two plates. We show how these two physically distinct cases are mathematically related and discuss a wide range of commonly applied boundary conditions. We concentrate our
analysis to the critical region where the underlying bulk Hamiltonian has zero modes and show that very exotic Casimir forces can arise, characterised by very long range effects and oscillatory behavior that can lead to strong metastability in the system.

Keywords: Casimir interaction, statistical physics, soft matter physics, Brazovskii model

1. Introduction

Besides electrodynamic field fluctuations and the ensuing Casimir–van der Waals interactions, which can be captured by field theories with actions that contain at most first order (spatial and temporal) derivatives, many more complex systems have actions which contain second order derivative terms [1]. In the quantum mechanics context higher derivative Lagrangians arise naturally when weak relativistic corrections to standard quantum mechanics are taken into account [2]. In soft matter physics, Hamiltonians with second order derivative terms are notably encountered in the context of stiff or semi-flexible polymers [3–8] that are constrained either in the embedding space or in the internal space coordinates due to the presence of adsorbing fluctuation quenchers. A natural context for higher derivative Hamiltonians is in soft (phospholipid) membranes [9–11]. Liquid crystal theory also provides an interesting playground for exploring the fluctuation effects in the context of higher derivative Lagrangians [12, 13]. In addition, recent developments in the theory of ionic liquids, where finite ion sizes are important, lead to mean field theories which introduce higher order derivatives than the usual second order Poisson–Boltzmann theory [14–17] and the analysis of the one loop correction to such theories will typically require higher order path integral formulations. Higher derivative theories are also actively discussed in relation to modified gravity and the dark energy puzzle [18]. Also, effective field theories with higher derivative actions and even non-local field actions also arise from the dynamics of lower derivative theories in the dynamical context when analysing the evolution of fluctuation forces toward their equilibrium values from the general non-equilibrium initial states [19, 20].

There have been several attempts to evaluate the field fluctuations or equivalently the field propagators in the case of higher derivative Lagrangians, most notably by Kleinert [21] and his results are given in one of the standard textbooks on path integration [22]. Some special cases of these results were established even before the developments of the full theory [4]. Recently the authors have established an alternative derivation of Kleinert’s results [23] based on a link between unconfined systems, which can be treated using Green’s function methods, and confined systems (which correspond to the standard path integral). The aim of this paper is to exploit these results to explore the Casimir interaction arising in both unconfined and confined geometries.

For both confined and unconfined systems we examine the interaction between two parallel surfaces for the Brazovskii model field Hamiltonian [24, 25] with various boundary conditions imposed at each surface. For instance the obvious boundary conditions are Dirichlet (D) and Neumann (N), however for a second derivative action we can also apply D and N conditions at the same surface—the so called strong anchoring boundary condition. We concentrate our study at the critical point where the model has a continuum of zero modes with wave vectors such that $|k| = q_0$, where $q_0$ is a parameter of the Brazovskii Hamiltonian. At the critical point we see that, for both confined and unconfined systems, the modes in planes parallel to the surfaces make qualitatively different contributions to the Casimir interaction disjoining pressure. The modes with $q < q_0$, where, $q = |q|$, can lead to oscillatory behavior, while giving the same
contribution for confined and unconfined systems. Furthermore the Casimir interaction generated by these modes is of a longer range than usually seen in the thermal Casimir effect for free fields, giving a contribution to the disjoining pressure which decays with an envelope of $1/h$ ($h$ being the distance between the plates) rather than the usual $1/h^3$ behavior. The contribution to the Casimir interaction coming from the modes with $q > q_0$ leads to a standard thermal Casimir disjoining pressure, which decays as $1/h^3$ and is independent of the value of $q_0$. Its amplitude and sign is a strong function of applied boundary conditions.

2. Basic field theory

We first describe the field theory in its bulk form which can be used to study unconfined systems and where the plates can be regarded as inclusions which impose constraints on the fluctuating order parameter. An approximation to the Landau–Ginzburg–Wilson Hamiltonian for diblock copolymer micro-phase separation is given by the Brazovskii model [27] where the Hamiltonian for the density-field fluctuations $\phi$ is given by

$$\beta H[\phi] = \frac{1}{2} \int_V d^3 r \left[ \nabla^2 \phi(r) + q_0^2 \phi(r)^2 + p_0^4 \phi(r)^2 \right],$$

(1)

with $V$ the bulk volume of the system. We see that in the limit $p_0 \to 0$ the system has zero-mode fluctuations corresponding to fluctuations with wave vectors $k$ such that $|k| = q_0$. These finite wavelength zero-mode fluctuations then lead to ordering of the system at the characteristic length scale $q_0^{-1}$ after phase transition. The modification of these zero and close to zero modes by the presence of boundaries should lead therefore to a strong, long range power law in plate separation, Casimir effect. The calculations performed here can be performed when the mass term $p_0$ is nonzero, the presence of this term will lead to exponentially decaying interactions. However as we are interested in the case where zero modes are present we will only give results for the massless case $p_0 = 0$.

We also note that if we had considered an analogous relativistic quantum theory for the field $\phi$ one would replace $\nabla^2$ by $\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$ and the non-zero Matsubara frequencies would contain terms of the form multiplied by $q_0^2$. Here we consider a classical field theory which corresponds to the zero frequency Matsubara term of such a quantum field theory.

Another bulk Hamiltonian that can be naturally written down from a Landau–Ginzburg–Wilson perspective is

$$\beta H'[\phi] = \frac{1}{2} \int_V d^3 r \left[ \nabla^2 \phi(r) \right]^2 - 2q_0^2 \left[ \nabla \phi(r) \right]^2 + (p_0^4 + q_0^4) \phi(r)^2,$$

(2)

which is clearly equivalent to the first Hamiltonian up to a surface term and trivially we see that

$$\beta H'[\phi] = \beta H[\phi] - q_0^2 \int_{\partial V} \phi(r) \nabla \phi(r) \cdot n dS,$$

(3)

where $\partial V$ denotes the surface of the system, $n$ is the normal to the surface and $dS$ the area element. This second model with $H'$ was the one studied by Uchida [8]. The two models above will clearly have the same bulk behavior but interactions between embedded surfaces will in general be modified by this surface term. However, in the case where either Dirichlet or Neumann boundary conditions are imposed at any of the surfaces and for periodic boundary conditions, the two models are clearly equivalent. In this paper we will carry out the analysis for the model given in equation (1), as the Hamiltonian is clearly positive for any field configurations, and can thus have zero modes but not strictly unstable ones.
In what follows we will introduce planar boundaries perpendicular to the direction $z$. If we write $\mathbf{r} = (z, \mathbf{x})^T$ and then express the field in terms of Fourier modes in the subspace $\mathbf{x}$, we find the equivalent Hamiltonian in terms of real fields $\phi(q, z)$ (using the fact that the original field $\phi(z, \mathbf{x})$ is real).

$$\beta H[\phi] = \sum_q \frac{1}{2} \int_{z_1}^{z_2} dz \left[ \hat{\phi}(z, q) \right]^2 + \frac{\omega_1^2(q) + \omega_2^2(q)}{2} \left[ \phi(z, q) \right]^2$$

where $\omega_1^2(q) = q^2 - q_0^2 + ip_0^2$ and $\omega_2^2(q) = q^2 - q_0^2 - ip_0^2$. (5)

In the above, $z_1$ denotes the left most point where the field is present and $z_2$ the right most point. We have written the Hamiltonian in a form familiar in polymer physics, the second derivative term being the polymer bending energy and the first derivative term being the stretching energy, while the last term corresponds to an overall confining quadratic potential. In the unconfined case we can take the limit $z_1 \to -\infty$ and $z_2 \to +\infty$. In this case one can set the surface term at $z = z_1$ and $z = z_2$ to zero, for instance by taking Dirichlet $\phi(z_1, q) = 0$ or Neumann $\partial_z \phi(z_1, q) = \phi(z_1, q) = 0$ boundary conditions. However, this does not affect the result as the plates are within the bulk and insensitive to this choice, provided that the correlation function of the field decays with distance.

A general Casimir setup with two plates, at $z = 0$ and $z = h$, can be analysed by finding the solution to the problem where the fields $\phi(z, q)$ and $\dot{\phi}(z, q)$ are constrained on both boundaries. At this point we should discuss the boundary conditions that are relevant for theories with second order derivatives. This point can be discussed from two perspectives. First one could consider the computation of the path integral in terms of an eigenfunction expansion. The eigenfunctions will clearly obey a fourth order differential equation and as such will have at total four boundary conditions, which includes the case where the field and its derivative can be fixed on each boundary. As there are four solutions the homogeneous differential equation to the Dirichlet and Neumann boundary conditions can be imposed simultaneously at a single point while having a non-zero eigenfunction.

However it is more enlightening to discuss the nature of the paths considered in the path integral. The presence of the $\ddot{\phi}$ term ensures that both $\phi$ and $\dot{\phi}$ are continuous (in the same way in which the $\dot{\phi}$ term ensures the continuity of Brownian paths $\phi$ for the usual Wiener measure) therefore imposing these values constitutes a genuine physical constraint. The reader can find a discussion of this point in [23] where the case of theories with an arbitrary, but finite, number of finite derivatives is discussed. As a consequence we see that fixing $\phi$, or indeed higher derivatives has no effect and is not a physical constraint.

To proceed in our analysis we need to compute the propagator

$$K(\phi, \dot{\phi}, \phi', \dot{\phi}', \omega_1, \omega_2, h) = \int d[\phi] \delta(\dot{\phi}(0) - \phi) \delta(\phi(0) - \phi) \delta(\phi(h) - \phi') \delta(\dot{\phi}(h) - \phi') \times \exp(-\beta H_0(\omega_1, \omega_2))$$

where

$$\beta H_0(\omega_1, \omega_2) = \frac{1}{2} \int_{z_1}^{z_2} dz \left[ \dot{\phi}(z)^2 + (\omega_1^2 + \omega_2^2)\phi(z)^2 + \omega_1^2 \omega_2^2 \phi(z)^2 \right].$$
We will denote with $K_U(\phi, \dot{\phi}, \dot{\phi}', \omega_1, \omega_2, h)$ the propagator for the unconfined systems where $z_1 \to -\infty$ and $z_2 \to \infty$, while we denote by $K_C(\phi, \dot{\phi}, \dot{\phi}', \omega_1, \omega_2, h)$ the confined system where $z_1 = 0$ and $z_2 = h$. Note that the surface term in equation (4) then enters additively in the exponent of the propagator equation (6), computed solely with the bulk Hamiltonian (7) (figure 1).

- In the confined case Kleinert [21] showed that

$$K_C(\phi, \dot{\phi}, \dot{\phi}', \omega_1, \omega_2, h) = \frac{(\omega_1 \omega_2)^{1/2} \left[(\omega_1^2 - \omega_2^2)^{1/2}\right]}{2\pi M^2} \exp \left(-\frac{1}{2} \left(\frac{\dot{\phi}'}{D}\right) \cdot S_D \left(\frac{\dot{\phi}'}{D}\right) \right)$$

Where here

$$M = (\omega_1^2 + \omega_2^2) s_1 s_2 - 2\omega_1 \omega_2 c_1 c_2 + 2\omega_1 \omega_2,$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (9)

$$S_D = \frac{1}{M} \left(\begin{array}{cc} \omega_1 \omega_2(\omega_1^2 - \omega_2^2) & \omega_1 \omega_2 \left(2\omega_1 \omega_2 s_1 s_2 - (\omega_1^2 + \omega_2^2) (c_1 c_2 - 1)\right) \\ \omega_1 \omega_2 \left(2\omega_1 \omega_2 s_1 s_2 - (\omega_1^2 + \omega_2^2) (c_1 c_2 - 1)\right) & \omega_1 \omega_2 \left(\omega_1^2 - \omega_2^2\right)(c_1 c_2 - 1) \end{array}\right) \hspace{1cm} \hspace{1cm} \hspace{1cm} (10)$$

and

$$S_C = \frac{\omega_1^2 - \omega_2^2}{M} \left(\begin{array}{cc} \omega_1 \omega_2(\omega_1 s_1 - \omega_2 s_2) & -\omega_1 \omega_2(c_1 - c_2) \\ \omega_1 \omega_2(c_1 - c_2) & \omega_1 s_2 - \omega_2 s_1 \end{array}\right), \hspace{1cm} \hspace{1cm} \hspace{1cm} (11)$$

and we have used the notation $s_i = \sinh(\omega_i h)$ and $c_i = \cosh(\omega_i h)$. This result was red-erived, in a very different way to [21], in [23] by exploiting a link with the unconfined case.

- In the unconfined case we showed in [23] that

$$K_U(\phi, \dot{\phi}, \dot{\phi}', \omega_1, \omega_2, h) = \frac{1}{\pi (\omega_1 \omega_2)^{1/2}} (\omega_1 + \omega_2) \exp \left(\frac{1}{2} (\omega_1 + \omega_2) h\right) K_C(\phi, \dot{\phi}, \dot{\phi}', \omega_1, \omega_2, h) \times \exp \left(-\frac{1}{2} \left(\frac{\dot{\phi}'}{D}\right) \cdot S_L \left(\frac{\dot{\phi}'}{D}\right) - \frac{1}{2} \left(\frac{\phi'}{D}\right) \right) \times S_L \left(\frac{\phi'}{-D}\right) \hspace{1cm} \hspace{1cm} \hspace{1cm} (12)$$

where here

$$S_L = \left(\begin{array}{cc} \omega_1 \omega_2(\omega_1 + \omega_2) & \omega_1 \omega_2 \\ \omega_1 \omega_2 & \omega_1 + \omega_2 \end{array}\right). \hspace{1cm} \hspace{1cm} \hspace{1cm} (13)$$

At this point it is interesting to compare the difference between the unconfined and confined path integrals. We see first of all that the presence of the external bulk field with respect to the confined case leads to a difference in the overall prefactors independent of $h$, this difference is
Therefore unimportant for the Casimir effect. Secondly, with respect to the unconfined case, the confined case has an additional factor $\exp(-\frac{1}{2}(\omega_1 + \omega_2)\hbar)$ which corresponds to an additional bulk free energy $\Delta F_{\mathbf{q}} = k_B T(\omega_1(q) + \omega_2(q))\hbar/2$ per mode $\mathbf{q}$, implying a negative pressure per mode $\mathbf{q}$. The appearance of the two modes can be best understood in terms of the eigenfunction expansion as there are two fundamental frequencies associated with the fourth order homogeneous differential equation obeyed by the eigenfunctions. This 

**doubling** generalises to higher order derivative actions [23].

This then gives an additional bulk free energy

$$F_b = \frac{k_B T h A}{2} \int \frac{d\mathbf{q}}{(2\pi)^2}(\omega_1(q) + \omega_2(q)).$$

where $A$ is the area of the plates. The corresponding pressure difference between the unconfined and confined cases is an excess bulk pressure due to the presence of the external bulk. We will see later that the bulk part of the pressure for the unconfined system actually turns out to be zero, which is normal as the bulk pressure of the interior is cancelled by that of the exterior as the volume occupied by the field is conserved upon changing the distance between the plates for the unconfined system. Thirdly and most importantly, we see that the effective surface energy is renormalized due to the presence of additional quadratic terms in the vector $(\phi, \dot{\phi}, \phi', \dot{\phi}')^T$.

Even though these terms do not depend explicitly on $h$, they are coupled to terms that do depend on $h$ and will thus lead to a modification of the Casimir pressure which we define as the $h$ dependent component of the pressure.

This means that effective Hamiltonians for the problems where the fields are confined and unconfined respectively (but with the same boundary conditions on the two surfaces) can be written as

$$\beta H_{\text{C}}[\phi] = \sum_{\mathbf{q}} \frac{1}{2} \int_0^h dz \left[ \dot{\phi}(z, \mathbf{q})^2 + (\omega_1^2(q) + \omega_2^2(q))\phi(z, \mathbf{q})^2 + \omega_1^2(q)\omega_2^2(q)\phi(z, \mathbf{q})^2 \right]$$

$$- \frac{(\omega_1^2(q) + \omega_2^2(q))}{2} \left[ \dot{\phi}(z, \mathbf{q}) \phi(z, \mathbf{q}) \right]_0^h$$
while
\[
\beta \mathcal{H}_U[\phi] = \sum_q \int_0^\beta \frac{d\zeta}{2} \left[ \dot{\phi}(\zeta, \mathbf{q})^2 + (\omega_1^2(q) + \omega_2^2(q))\dot{\phi}(\zeta, \mathbf{q})^2 + \omega_1^2(q)\omega_2^2(q)\phi(\zeta, \mathbf{q})^2 \right] \\
- (\omega_1(q) + \omega_2(q))\frac{\hbar}{2} \phi(0, \mathbf{q}) - \omega_1(q)\omega_2(q) \dot{\phi}(0, \mathbf{q}) \phi(0, \mathbf{q}) \\
+ \frac{1}{2} \phi(0, \mathbf{q})^2 + \omega_1(q)\omega_2(q)(\omega_1(q) + \omega_2(q)) \\
+ \frac{1}{2} \dot{\phi}(0, \mathbf{q})^2 + \phi(\hbar, \mathbf{q})^2)(\omega_1(q) + \omega_2(q)).
\]

This is particularly interesting as the effect of external bulk can be represented in terms of a purely surface term in addition to the confined Hamiltonian along with a bulk free energy term, that is to say
\[
\beta \Delta H_B = \beta \mathcal{H}_U[\phi] - \beta \mathcal{H}_C[\phi] = \sum_q (\omega_1(q) + \omega_2(q)) \left[ \frac{\hbar}{2} + \frac{1}{2}(\omega_1(q) + \omega_2(q)) \right] \phi(0, \mathbf{q})^2 + \omega_1(q)\omega_2(q)(\omega_1(q) + \omega_2(q)) \\
+ \frac{1}{2} \dot{\phi}(0, \mathbf{q})^2 + \phi(\hbar, \mathbf{q})^2)(\omega_1(q) + \omega_2(q)).
\]

However if one tries to write this equivalence in real space, the resulting surface energies (written as integrals over the surface) are non-local.

For the two cases, \( q > q_0 \) and \( q < q_0 \), we have \( \omega_1(q) = \sqrt{q^2 - q_0^2 + \frac{i\pi}{\hbar}} \) and \( \omega_1(q) = i\sqrt{q^2 - q_0^2} \), respectively, with \( \omega_2(q) = \omega_1(q) \), in both cases. At the critical point defined as \( p_0 = 0 \), one finds that \( \omega_1(q) = \omega_2(q) = \sqrt{q^2 - q_0^2} \) are both real for \( q > q_0 \), while \( \omega_1(q) = \omega_2(q) = i\sqrt{q^2 - q_0^2} \) is purely imaginary for \( q < q_0 \). This has a remarkable consequence in that for \( q < q_0 \) we have \( \omega_1(q) + \omega_2(q) = 0 \) and thus when \( q < q_0 \) the Hamiltonians \( \mathcal{H}_U \) and \( \mathcal{H}_C \) are identical.

We also see that when \( p_0 = 0 \), the bulk free energy only contains contributions from the modes with \( q > q_0 \) and thus identifying in a standard fashion \( \sum_q = (\pi/2\pi)^3 \int dq \), we find
\[
F_b = k_B \Theta A \int_{q > q_0} \frac{dq}{(2\pi)^2} \sqrt{q^2 - q_0^2},
\]
which needs to be regularized by using an ultra-violet cut-off. This divergence of the bulk free energy is a usual occurrence in Casimir type calculations, both for classical thermal and quantum systems, however in most cases the \( h \) dependent part of the pressure is finite. Note that the divergence is of the same as that for a first derivative theory apart from the doubling of modes.

3. Fluctuation induced interactions for confined fields

We start with the confined field setup, where the field permeates only the slab in the intersurface region and it is absent outside. The basic partition function for each mode, for fixed values \( \phi, \dot{\phi}, \phi', \dot{\phi}' \) of the field and its normal derivatives on the surface, for this system can be written,
using the form of the propagator in equation (8) and the expression (15) as

\[ K_C(\phi, \dot{\phi}, \dot{\phi}', \omega_1, \omega_2, h) = \frac{(\omega_1 \omega_2)^{1/2}[\omega_1^2 - \omega_2^2]^{1/2}}{2\pi M^{1/2}} \exp \left( -\frac{1}{2} \left( \frac{\phi'}{\phi} \right) \cdot S_{DR} \left( \frac{\phi'}{\phi} \right) \right) \]

\[ - \frac{1}{2} \left( \frac{\phi}{\phi} \right) \cdot P_{\text{DR}} P \left( \frac{\phi}{\phi} \right) + \left( \frac{\phi}{\phi} \right) \cdot S_{C} \left( \frac{\phi'}{\phi} \right), \]

where

\[ S_{\text{DR}} = S_D - \frac{1}{2} \left( \begin{array}{cc} 0 & \omega_1^2 + \omega_2^2 \\
\omega_1^2 + \omega_2^2 & 0 \end{array} \right), \]

incorporates the surface terms in equation (15) and for notational convenience we have introduced the matrix

\[ P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

The corresponding free energy can be written in terms of the partition function \( Z_C(q, h) \) as

\[ F(h) = -\frac{A k_B T}{(2\pi)^2} \int dq \ln (Z_C(q, h)), \]

and incorporates the effects of the boundary conditions of the system as will become clear as we proceed.

### 3.1. Strong anchoring boundary condition

The strong anchoring limit is defined in [8] as \( \phi(z, x) = \dot{\phi}(z, x) \equiv \partial_z \phi(z, x) = 0 \) at both surfaces. We also denote this by DN–DN boundary conditions. The vanishing of the surface terms means that the partition function for the mode \( q \) is simply given by

\[ Z_C(q, h) = K_C(0, 0, 0, 0, \omega_1, \omega_2, h) = \frac{(\omega_1 \omega_2)^{1/2}[\omega_1^2 - \omega_2^2]^{1/2}}{2\pi M^{1/2}} \]

and from this we obtain at the critical point, where \( p_0 = 0 \),

\[ Z_C(|q| > q_0, h) = \frac{q^2 - q_0^2}{\pi(\sinh^2(t) - t^2)^{1/2}}. \]

with \( h(q^2 - q_0^2)^{1/2} = t. \) In the case where \( q < q_0 \) the partition function is

\[ Z_C(|q| < q_0, h) = \frac{q_0^2 - q^2}{\pi(t^2 - \sin^2(t))^{1/2}}, \]

where in this region \( t = h(q_0^2 - q^2)^{1/2}. \) The total free energy is then evaluated as

\[ F(h) = -\frac{A k_B T}{(2\pi)^2} \left( \int_{q>q_0} dq \ln (Z_C(|q| > q_0, h)) \right. \]

\[ + \left. \int_{q<q_0} dq \ln (Z_C(|q| < q_0, h)) \right) = F_>(h) + F_<(h), \]
where we have used $F_>, F_<$ to represent the free energy contributions from modes with $q > q_0$ and $q < q_0$, respectively. Using this we find that, up to terms independent of $h$, one has

$$F(h) = \frac{A k_B T}{2(2\pi)^2} \left( \int_{q>q_0} dq \ ln(\sinh^2(t) - t^2) + \int_{q<q_0} dq \ ln(t^2 - \sin^2(t)) \right)$$

(27)

Because in the confined case no field exists outside the slab between two bounding surfaces, the free energy has a bulk term proportional to the volume coming from the divergent part in term $F_>(h)$, which we see from writing

$$F(h) = \frac{A k_B T}{2(2\pi)^2} \left( \int_{q>q_0} dq \ ln(\exp(-2t)[\sinh^2(t) - t^2]) + \int_{q<q_0} dq \ ln(t^2 - \sin^2(t)) \right) + F_b$$

(28)

where the first integral above is now convergent, and from which one finds (again dropping $h$ independent terms),

$$F(h) = \frac{A k_B T}{2(2\pi)^2} \left( \int_{q>q_0} dq \ ln(\exp(-2t)[\sinh^2(t) - t^2]) + \int_{q<q_0} dq \ ln(t^2 - \sin^2(t)) \right) + F_b$$

(29)

where the bulk free energy is given in equation (18). One thus finds

$$(30)$$

The Casimir disjoining pressure is then given by

$$\Pi = \frac{-\partial f(h)}{\partial h},$$

(31)

where

$$f(h) = \frac{F(h) - F_b}{A} = f_>(h) + f_< (h),$$

(32)

and we find

$$f(h) = \frac{k_B T}{2\pi h^2} \left( -1.71629 + \frac{1}{2} \int_0^{q_0h} dt \ ln(t^2 - \sin^2(t)) \right).$$

(33)

For $q > q_0$, the contribution to the disjoining pressure is

$$\Pi_>(h) = \frac{-\partial f_>(h)}{\partial h} = -\frac{k_B T}{2\pi h^3} \int_0^{q_0h} dt \frac{t^2 e^{-t} \sinh(t) + t^2 - t}{\sinh^2(t) - t^2},$$

(34)

while for $q < q_0$ the contribution to the disjoining pressure is

$$\Pi_<(h) = \frac{-\partial f_< (h)}{\partial h} = -\frac{k_B T}{2\pi h^3} \int_0^{q_0h} dt \frac{\sinh(t) \cos(t)}{t^2 - \sin^2(t)},$$

(35)
The total pressure can then be written as
\[
\Pi(h) = \Pi_\prec + \Pi_\succ = \frac{k_B T q_0^3}{2\pi} r(q_0 h),
\] (36)
with
\[
r(x) = \frac{1}{x^4} \left( \frac{4}{x^2} - \int_0^x \frac{dr}{r^2} \left[ \sin(t) \cos(t) - t \right] - 3.43258 \right). \tag{37}
\]

This is the same as the result obtained from the Hamiltonian used by Uchida [8] because, as pointed out above, for these boundary conditions the two models used by Uchida and here are equivalent. The dimensionless Casimir disjoining pressure equation (37) shows very little structure, even if the first term is non-monotonic with
\[
-x^2 \leq\int_0^x \frac{dr}{r^2} \left[ \sin(t) \cos(t) - t \right] \leq -0.5x^2 \tag{38}
\]
for the limits \( x \ll 1 \) and \( x \gg 1 \), displaying slight oscillations in between, however both the free energy and the pressure are monotonic and consequently the system exhibits no metastability.

3.2. Robin boundary conditions

Strong anchoring examined above corresponds to imposing two boundary conditions at each surface. One can just as well impose a single boundary condition at each surface. For instance, imposing Robin boundary conditions on both surfaces mean that for each mode,
\[
\left. \left( b_1 \frac{\partial \phi}{\partial z} \right) \right|_{z=0,z=h} = 0, \tag{39}
\]
In terms of the inward normal of the surfaces, these boundary conditions correspond to
\[
\phi(z, x) - b_1 \nabla \phi(z, x) \cdot n_1 = 0 \quad \text{and} \quad \phi(z, x) + b_2 \nabla \phi(z, x) \cdot n_2 = 0. \tag{40}
\]
In this case the partition function for a single mode \( q \) is given by
\[
Z_C(q, h) = \int d\phi \, d\phi' \, K_C(b_1 \phi, \phi, b_2 \phi', \phi, \omega_1, \omega_2, h), \tag{41}
\]
and using equation (19) this yields
\[
Z_C(q, h) = \frac{(\omega_1 \omega_2)^2_2 [(\omega_1^2 - \omega_2^2)^2_2]}{2\pi M^2} \int d\phi \, d\phi' \, \exp \left( -\frac{\dot{\phi}^2}{2} \mathbf{u} \cdot B_2 \mathbf{S}_{DR} B_2 \mathbf{u} \\
-\frac{\dot{\phi}'^2}{2} \mathbf{u} \cdot B_1 \mathbf{S}_{DR} B_1 \mathbf{u} + \dot{\phi} \dot{\phi}' \mathbf{u} \cdot B_2 \mathbf{S}_C B_2 \mathbf{u} \right), \tag{42}
\]
where we have introduced a matrix and a vector as
\[
B_i = \begin{pmatrix} b_i & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{43}
\]
Performing the integrations over \( \dot{\phi} \) and \( \dot{\phi}' \) we find
\[ Z_C(q, h) = \left( \frac{\omega_1 \omega_2}{M^2} \right)^{\frac{1}{2}} \left[ (\omega_1^2 - \omega_2^2) \right]^{\frac{1}{2}} \left[ (\mathbf{u} \cdot B_2 S_{DR} B_2 \mathbf{u} (\mathbf{u} \cdot B_1 P S_{DR} P B_1 \mathbf{u}) - (\mathbf{u} \cdot B_2 S_{C} B_1 \mathbf{u})^2 \right]^{\frac{1}{2}} \]  

(44)

The parameters \( b_i \) introduce additional length scales and the general expression equation (44) is thus extremely complicated for general \( b_1 \) and \( b_2 \), below we thus restrict ourselves to a restricted set of parameters.

We now first consider the symmetric case where \( b_1 = b_2 = b \). Here we obtain the remarkably simple result

\[ Z_C(q, h) = \left( \frac{\omega_1 \omega_2}{\sinh(\omega_1 h) \sinh(\omega_2 h)(b^2 \omega_1^2 - 1)(b^2 \omega_2^2 - 1)} \right)^{\frac{1}{2}}. \]  

(45)

One can verify that the argument in the square root is positive definite due to the fact that \( \omega_2 = \omega_1^2 \). We thus see that the \( h \) dependent part of the free energy is independent of \( h \), therefore for the whole range of \( b \), from \( b = 0 \) (D–D boundary conditions) through to \( b = \pm \infty \) (N–N boundary conditions) the Casimir pressure is universal. The effect of \( b \) only appears in a term that corresponds to a surface energy. The above factorization of the partition function can be understood in terms of a Fourier expansion of the second order path integral which can be shown to factorize into two first order integrals with the same boundary conditions. This holds because there is only one boundary condition at each surface.

At the critical point we find that the Casimir interaction free energy (the \( h \) dependent part) is given by

\[ F(h) = \frac{A k_B T}{2(2\pi)^2} \int_{q > q_0} dq \ln \left( \sinh^2 \left( h \sqrt{q^2 - q_0^2} \right) \right) + \int_{q < q_0} dq \ln \left( \sin^2 \left( h \sqrt{q_0^2 - q^2} \right) \right). \]  

(46)

Subtracting the bulk free energy and again introducing the variable \( t = h \sqrt{q_0^2 - q^2} \) for \( q < q_0 \) and \( t = h \sqrt{q^2 - q_0^2} \) for \( q > q_0 \), we obtain

\[ f(h) = \frac{k_B T}{2\pi h^2} \left( \int_0^\infty dt \ln(1 - \exp(-2t)) + \int_0^{h q_0} dt \ln |\sin t| \right) \]
\[ = \frac{k_B T}{2\pi h^2} \left( \frac{\zeta(3)}{4} + \int_0^{h q_0} dt \ln |\sin t| \right), \]  

(47)

where \( \zeta(s) \) is the Riemann zeta function \( \zeta(3) = 1.202 \). This result agrees with that of Uchida [8] who gave the case of Dirichlet boundary conditions only. The first term is simply twice the free energy, per unit area, due to the universal thermal (attractive) Casimir effect for massless scalar fields with Dirichlet–Dirichlet boundary conditions, and is, interestingly, independent of \( q_0 \). The second term is oscillatory and long-range with respect to the usual thermal Casimir effect.

The absolute value inside the logarithm function is a consequence of the squared term in equation (46). The second integral in the bracket is related to the two famous Euler integrals as well as the Clausen function. The interaction free energy is finite for any non-zero separation, has a monotonic (attractive) envelope, but contains an infinite sequence of asymmetric local minima and maxima, separated by an infinite derivative. In the constant pressure ensemble, the
Figure 2. Dimensionless Casimir free energy at the critical point, $f(h) \rightarrow f(h)/K_BTq_0^2$ (blue—lower curve), from equation (47) and dimensionless Casimir disjoining pressure $\Pi(h) \rightarrow \Pi(h)/K_BTq_0^2$ (red—upper curve) from equation (48) for the symmetric boundary conditions, both as functions of dimensionless separation $hq_0$. The stable branches of disjoining pressure are indicated by the solid curve, and the unstable by dashed curve. The thermodynamic state is at $h = 0$, but there also exist an infinite number of metastable states in the stable regions, where $\partial \Pi(h)/\partial h < 0$ (black—solid curve), bounded by spikes of infinite pressure. The envelope of the free energy indicates an underlying attractive Casimir interaction.

position of thermodynamic equilibrium will depend on the total applied bulk pressure $P_t$ of the system and $h$ is determined from the solution of the equation $\Pi(h) = P_t$.

Apart from the regular part scaling as $h^{-3}$ and $h^{-1}$ [28, 29], in the vicinity of $hq_0 = n\pi$, $n = 1, 2, 3, \ldots$, the fluctuation-induced interaction pressure becomes repulsive and diverges logarithmically,

$$\Pi(h) = -\frac{\partial f(h)}{\partial h} = \frac{2f(h)}{h} - \frac{k_BTq_0^2}{2\pi h} \ln |\sin hq_0|.$$ (48)

As a result of this the equation $\Pi(h) = P_t$ will have an infinite number of solutions for $P_t$ positive (a net applied pressure acting inward on the system) and even for a range of negative pressures as can be see from figure 2. The thermodynamic stable state is, however, given by $h = 0$, while there also exists an infinite number of metastable states in the stable regions where $\partial \Pi(h)/\partial h < 0$ as shown in figure 2.

In the limit where $b_1 = b \to \infty$ while $b_2 = 0$ we obtain the case of Dirichlet–Neumann boundary conditions. Explicitly we have

$$Z_C(q, h) = (b^2 \cosh(h\omega_1) \cosh(h\omega_2))^{-\frac{1}{4}},$$ (49)

and this factorisation can again be understood in terms of an eigenfunction expansion.

At the critical point we find that the $h$ dependent part of the free energy is given by

$$F(h) = \frac{Ak_BT}{2(2\pi)^2} \left( \int_{q > q_0} dq \ln \left( \cosh^2 \left( h\sqrt{q^2 - q_0^2} \right) \right) + \int_{q < q_0} dq \ln \left( \cos^2 \left( h\sqrt{q^2 - q_0^2} \right) \right) \right).$$ (50)
Figure 3. Dimensionless free energy at the critical point, $f(h) \rightarrow f(h)/\frac{k_B T q_0^2}{12\pi^2}$ (blue—lower curve), from equation (51) for Dirichlet–Neumann boundary conditions and the corresponding dimensionless disjoining pressure $\Pi(h) \rightarrow \Pi(h)/\frac{k_B T q_0^2}{12\pi^2}$ (red—upper curve) from equation (52), both as functions of dimensionless separation $h q_0$. The stable branches of disjoining pressure are again indicated by the solid curve, and the unstable by dashed curve. The thermodynamic stable state is at a finite $h$, but there exits an infinite number of metastable states in the stable regions, where $\partial \Pi(h)/\partial h < 0$ (black—solid curve), delimited by infinite pressure spikes. The envelope of the free energy indicates an underlying repulsive Casimir interaction.

and this leads to

$$f(h) = \frac{k_B T}{2\pi h^2} \left( \int_0^\infty dt \ln (1 + \exp(-2t)) + \int_0^{h q_0} dt \ln |\cos t| \right)$$

$$= \frac{k_B T}{2\pi h^2} \left( \frac{3\zeta(3)}{16} + \int_0^{h q_0} dt \ln |\cos t| \right).$$

(51)

From this we see that

$$\Pi(h) = \frac{2f(h)}{h} = \frac{k_B T q_0^2}{2\pi h} \ln \left(|\cos q_0 h|\right).$$

(52)

In contrast to the symmetric case given in equation (47), we see that the monotonic contribution from the modes $q > q_0$ is repulsive (and has the form of twice the Dirichlet–Neumann Casimir force for massless first order Hamiltonians). The minimum free energy $f(h)$ occurs at a finite separation $h$, which means one can achieve the stable state at a finite slab depth. The equation $\Pi(h) = P_t$ in this case has no solution for sufficiently negative $P_t$, as can be seen from figure 3, but again displays an infinite number of metastable states for $P_t$ positive with $\partial \Pi(h)/\partial h < 0$. The oscillatory part of the pressure diverges at $h q_0 = (2n + 1)\frac{\pi}{2}$, $n = 0, 1, 2, \ldots$, and leads to a quite exotic behavior as can be seen in figure 3. The envelope of the free energy is monotonic and overall repulsive. A behavior reminiscent of this, but without a diverging pressure, is found in the one dimensional Coulomb gas with charge regulation [26].

### 3.3. Some more exotic boundary conditions

Up to now we have considered the cases of strong anchoring, where all fields are constrained to zero, meaning two constraints at each surface, and Robin boundary conditions, where a
single constraint is imposed on each surface. Now we consider the case of strong anchoring on one surface (surface 2) and Robin boundary conditions on the other, written as \( \phi(0, x) = b_1 n \cdot \nabla \phi|_{z=0} \). Here we find

\[
Z_C(q, h) = \frac{(\omega_1 \omega_2)^4 \left[(\omega_1^2 - \omega_2^2)^2\right]^{\frac{1}{2}}}{2\pi M^2} \int dq \exp \left(-\frac{q^2}{2} u \cdot B_1 P_{S_{DR}P} B_1 u\right)
\]

\[
= \frac{(\omega_1 \omega_2)^4 \left[(\omega_1^2 - \omega_2^2)^2\right]^{\frac{1}{2}}}{(2\pi)^2 M^2 (u \cdot B_1 P_{S_{DR}P} B_1 u)^{\frac{1}{2}}}
\]

The case for arbitrary \( b_1 \) is rather complicated due to the extra length scale introduced, however for Dirichlet boundary conditions \( b_1 = 0 \) (DN–D) we find

\[
Z_C(q, h) = \left(\frac{\omega_1 \omega_2 (\omega_1^2 - \omega_2^2)}{2\pi [\cosh(\omega_1 h) \sinh(\omega_2 h) - \omega_2 \cosh(\omega_2 h) \sinh(\omega_1 h)]}\right)^{\frac{1}{2}}
\]

At the critical point the interaction free energy, i.e., the \( h \) dependent part, is given by

\[
F(h) = \frac{A k_B T}{2(2\pi)^2} \left[ \int_{q>q_0} dq \ln \left(\sinh(2\sqrt{q^2 - q_0^2 h}) - 2\sqrt{q^2 - q_0^2 h}\right)
+ \int_{q<q_0} dq \ln \left(2\sqrt{q_0^2 - q^2 h} - \sin(2\sqrt{q_0^2 - q^2 h})\right) \right].
\]

Subtracting the bulk free energy then yields

\[
f(h) = \frac{k_B T}{2\pi h^2} \left[ \frac{1}{2} \int_0^{\infty} dt \ln (1 - \exp(-4t) - 4t \exp(-2t)) + \frac{1}{2} \int_0^{\varphi h} dt \ln (2t \sin(2t)) \right]
= \frac{k_B T}{2\pi h^2} \left[ -0.83591 + \frac{1}{2} \int_0^{\varphi h} dt \ln (2t \sin(2t)) \right].
\]

As in the case of strong anchoring, we find that the modes with \( q > q_0 \) give a contribution to the free energy which is attractive with a non-universal amplitude. The disjoining pressure is then given by

\[
\Pi(h) = \frac{2 f(h)}{h} = \frac{k_B T q_0^2}{4\pi h^2} \ln (2hq_0 - \sin(2hq_0)).
\]
Figure 4. Dimensionless free energy at the critical point for strong anchoring on one surface and Dirichlet boundary condition on the other, \( f(h) \rightarrow f(h)/\frac{k_B T q_0^2}{2\pi^2 h} \), (blue—upper curve) from equation (56), and the corresponding dimensionless disjoining pressure \( \Pi(h) \rightarrow \Pi(h)/\frac{k_B T q_0^2}{2\pi^2 h} \) (red—lower curve) from equation (57), both as functions of dimensionless separation \( hq_0 \). The stable branches of disjoining pressure are again indicated by the solid curve, and the unstable by dashed curve. The thermodynamic stable state is at \( h = 0 \), but there exits also an infinite number of metastable states in the stable regions, where \( \partial \Pi(h)/\partial h < 0 \) (black—solid curve), accessible for negative \( P_t \). The envelope of the free energy indicates an underlying attractive Casimir interaction.

For Neumann boundary conditions, \( b_1 \rightarrow \infty \), we find

\[
Z_C(q, h) = \left( \frac{\omega_1^2 - \omega_2^2}{2\pi b_1^2} \sinh(\omega_1 h) \cosh(\omega_2 h) - \omega_1 \sinh(\omega_2 h) \cosh(\omega_1 h) \right)^{1/2}
\]

and here

\[
F(h) = \frac{Ak_B T}{2(2\pi)^2} \left( \int_{q > q_0} dq \ln \left( \sinh(2\sqrt{q_0^2 - q^2} h) + 2\sqrt{q_0^2 - q^2} \right) + \int_{q < q_0} dq \ln \left( 2\sqrt{q_0^2 - q^2} h + \sin(2\sqrt{q_0^2 - q^2} h) \right) \right).
\]

Upon subtracting the bulk energy we find

\[
f(h) = \frac{k_B T}{2\pi h^2} \left[ 0.406839 + \frac{1}{2} \int_{0}^{\infty} dt t \ln(2t + \sin(2t)) \right] \]

Here we see that the modes with \( q > q_0 \) give a repulsive Casimir free energy, with the standard \( 1/h^2 \) behaviour but again with a non universal amplitude. The disjoining pressure in this case is given in complete analogy with the previous cases by

\[
\Pi(h) = \frac{2f(h)}{h} - \frac{k_B T q_0^2}{4\pi h} \ln \left( 2hq_0 + \sin(2hq_0) \right)
\]
Figure 5. Dimensionless free energy at the critical point, $f(h) \to f(h)/k_B Tq_0^2$, (blue—upper curve), from equation (60) for strong anchoring on one surface and Neumann boundary condition on the other and the corresponding dimensionless disjoining pressure $\Pi(h) \to \Pi(h)/k_B Tq_0^3 (2\pi)$ (red—dashed) from equation (61), both as functions of dimensionless separation $hq_0$. The stable branches of disjoining pressure are again indicated by the solid curve, and the unstable by dashed curve. The thermodynamic stable state is now at a finite value of $hq_0$. The equation $\Pi(h) = P_t$ has a single solution for any positive $P_t$, but can have multiple solutions for $P_t$ small and negative, corresponding to an infinite number of metastable states in the stable regions, where $\partial \Pi(h)/\partial h < 0$ (black—solid curve). The envelope of the free energy indicates a non-monotonic Casimir interaction.

In this case we observe, see figure 5, that the free energy is dominated by the monotonically repulsive Casimir interaction stemming from the first term in equation (60) for small $h$, while for larger $h$ the second term takes over, leading to attractive Casimir interactions scaling approximately as $h^2$. The global minimum is achieved at a finite value of the separation $h$. The pressure, equation (61), exhibits an oscillatory component superimposed on a non-monotonic background and again shows an infinite sequence of regions in the metastable regime where $\partial \Pi(h)/\partial h < 0$ separated by unstable regions. The equation $\Pi(h) = P_t$ has a single solution for any positive $P_t$, while it can have multiple solutions for $P_t$ small and negative. The overall envelope of the free energy indicates a non-monotonic Casimir interaction.

Another possibility is that no boundary conditions are applied on a given surface. For example one can take strong anchoring boundary conditions on one surface and use free boundary conditions at the other. We note that for an unconfined system this choice of boundaries cannot lead to any interaction, as it corresponds to a single interface in the system. However for confined systems there can still be an interaction. In this case we actually find

$$Z_C(q, h) = \frac{(\omega_1 \omega_2)^{1/2} [((\omega_1^2 - \omega_2^2) ]^{1/2}}{M^2 [\det(S_{DR})]^2} = \left( 1 - \frac{M}{4\omega_1 \omega_2} \right)^{-1/2} \tag{62}$$

and in general we see that $Z_C(q, h)$ will depend on $h$ through $M$. However at the critical point $p_0 = 0$ we find that for all $q$, $Z_C(q, h) = 1$ and so there is no interaction between a surface with strong anchoring boundary conditions and another with completely free boundary conditions! In general one can show that at the critical point, if one surface has free boundary conditions, there is no interaction between the surfaces.
For confined systems, periodic and antiperiodic boundary conditions are also relevant. Indeed the most general periodic/antiperiodic boundary conditions of this type can be written as

\[
\begin{pmatrix}
\phi(h, x) \\
\dot{\phi}(h, x)
\end{pmatrix} = R \begin{pmatrix}
\phi(0, x) \\
\dot{\phi}(0, x)
\end{pmatrix},
\]

(63)

where the matrix \( R \) can take 4 distinct forms

\[
R(\sigma, \sigma') = \begin{pmatrix}
\sigma & 0 \\
0 & \sigma'
\end{pmatrix},
\]

(64)

where \( \sigma \) and \( \sigma' = \pm 1 \). The partition function can then be derived as

\[
Z_C(q, h) = \frac{(\omega_1 \omega_2)^{\frac{1}{2}} [(\omega_1^2 - \omega_2^2)^{\frac{1}{2}}]}{2\pi M^\frac{1}{2}} \times \int d\phi \, d\dot{\phi} \, \exp \left( -\frac{1}{2} \begin{pmatrix}
\phi \\
\dot{\phi}
\end{pmatrix} \cdot [RS_{DR}R + PS_{DR}P - 2S_{C}R] \begin{pmatrix}
\phi \\
\dot{\phi}
\end{pmatrix} \right)
\]

\[
= \frac{(\omega_1 \omega_2)^{\frac{1}{2}} [(\omega_1^2 - \omega_2^2)^{\frac{1}{2}}]}{M^\frac{1}{2} \, \det (RS_{DR}R + PS_{DR}P - S_{C}R - RS_{C}^{T})^{\frac{1}{2}}},
\]

(65)

and while carrying out the Gaussian integration we must use the symmetric part of the relevant matrix. The corresponding partition functions are obtained with an obvious notation as

\[
Z_C(q, h, ++) = \frac{1}{4 \sinh(\frac{h \omega_1}{2}) \, \sinh(\frac{h \omega_2}{2})},
\]

(66)

\[
Z_C(q, h, --) = \frac{1}{4 \cosh(\frac{h \omega_1}{2}) \, \cosh(\frac{h \omega_2}{2})},
\]

(67)

\[
Z_C(q, h, +-) = Z_C(q, --) = \left[ -\frac{\omega_1 \omega_2}{M} \right]^{\frac{1}{2}},
\]

(68)

wherefrom we can derive the interaction free energies as

\[
f(h, ++) = \frac{k_B T \, 2\pi h^2}{2} \left( -2\zeta(3) + 8 \int_0^{\frac{h \omega_0}{2}} dt \, t \ln |\sin t| \right)
\]

(69)

\[
f(h, --) = \frac{k_B T \, 2\pi h^2}{2} \left( \frac{3\zeta(3)}{2} + 8 \int_0^{\frac{h \omega_0}{2}} dt \, t \ln |\cos t| \right),
\]

(70)

and furthermore we see that \( f(h, +-) \) is identical to the strong anchoring result given in equation (33). Furthermore, the periodic case is identical to twice the symmetric case equation (47) and the antiperiodic case to twice the non symmetric case equation (51), but evaluated at \( h/2 \) as opposed to \( h \). We shall thus not dwell on the details of these results.

4. Unconfined fields

We now consider systems where the field exists both between and outside the plates. As we have seen in section 2 the confined and unconfined systems differ by surface terms and in all but the strong anchoring case they will be different.
Here the partition function for each mode, for fixed values $\phi$, $\dot{\phi}$, $\phi'$, $\dot{\phi}'$ of the field and its normal derivatives on the surface, is given using equations (8) and (12) as

$$Z_V(\phi, \dot{\phi}, \phi', \dot{\phi}'; \omega_1, \omega_2, h) = \frac{\omega_1 \omega_2 (\omega_1 + \omega_2) \left(\left(\frac{\omega_1^2}{\omega_2} - \frac{\omega_2^2}{\omega_1} \right)^2 \right)^{\frac{1}{4}}}{2\pi^2 M^2} \exp\left(\frac{1}{2} (\omega_1 + \omega_2) h \right)$$

$$\times \exp\left(-\frac{1}{2} \left(\frac{\phi'}{\phi} \right) \cdot S_{DR} \left(\frac{\phi'}{\phi} \right) - \frac{1}{2} \left(\frac{\phi}{\phi} \right) \cdot P_{DR} \left(\frac{\phi}{\phi} \right) \right)$$

$$+ \left(\frac{\phi}{\phi} \right) \cdot S_C \left(\frac{\phi}{\phi} \right),$$

where

$$S_{DR} = S_D + S_L.$$  

Using this, one can read off the results from the various cases investigated for the confined field problem.

The disjoining Casimir pressure for the strong anchoring case is identical to that for confined systems as the only difference is the bulk term and $h$ independent terms, which can be interpreted as surface free energies. In what follows there is no need to subtract the bulk pressure as it is automatically subtracted due to the fact that the field exists both outside and between the plates. Below we consider the same set of boundary conditions as considered above for confined fields. As the basic idea of the calculations are given in section 3, we do not give the full details in what follows.

### 4.1. Robin boundary conditions

Using the same notation as section 3 we find

$$Z_V(q, h) = \frac{\omega_1 \omega_2 (\omega_1 + \omega_2) \left(\left(\frac{\omega_1^2}{\omega_2} - \frac{\omega_2^2}{\omega_1} \right)^2 \right)^{\frac{1}{4}}}{\pi M^2} \exp\left(\frac{1}{2} (\omega_1 + \omega_2) h \right)$$

$$\times \left[ \left(\mathbf{u} \cdot B_2 S_{DR} B_2 \mathbf{u} \right) \left(\mathbf{u} \cdot B_1 P_{DR} B_1 \mathbf{u} \right) - \left(\mathbf{u} \cdot B_2 S_C B_2 \mathbf{u} \right) \right]^{\frac{1}{2}},$$

which for symmetric Robin boundary conditions, where $b_1 = b_2 = h$, yields

$$f(h) = \frac{k_b T}{2(2\pi)^2} \int d\mathbf{q} \ln \left(1 - \frac{e^{-2b_1 h}}{(\omega_1 - \omega_2)^2 (b_1^2 \omega_1 + 1)^2} \right).$$

At the critical point this leads to the Casimir interaction free energy

$$f(h) = \frac{k_b T}{2\pi h^2} \left( \frac{1}{2} \int_0^\infty dt \ln \left(1 - \exp(-2t) \left(1 - \frac{b^2 t^2}{h^2} - \frac{h^2}{b^2 t^2 + h^2} \right)^2 \right) + \int_0^{h_0} dt \ln |\sin t| \right).$$

We see that in general the contributions from the modes $q > q_0$ depends on the precise value of $b$, that introduces an additional length scale, and consequently the simple $1/h^2$ part of the
The dependence of the first integral in equation (74) on the ratio $h/b$. The two asymptotes show the value of the integral for D–D boundary conditions ($b = 0$) and N–N boundary conditions ($b = \infty$), respectively, see equations (76) and (77).

Casimir interaction is modified. In fact, the first integral in equation (74) depends only on the ratio $x = h/b$

$$g(x) = \frac{1}{2} \int_0^\infty dt \ t \ \ln \left( 1 - \exp(-2t) \left( 1 - \frac{t^2 - x^2}{t^2 + x^2} \right)^2 \right)$$

and its behavior is shown in figure 6.

However, the contribution from the modes $q < q_0$ is independent of $b$ and is exactly the same as that derived for the confined field, in agreement with the observation made in section 2 that at the critical point this must be the case in general due to equation (17). For $b = 0$, that is to say in the case of Dirichlet–Dirichlet (D–D) boundary conditions, we find

$$f(h) = \frac{k_B T}{2 \pi h^2} \left( \frac{1}{2} \int_0^\infty dt \ t \ \ln \left( 1 - \exp(-2t)(1 + t)^2 \right) + \int_0^{\pi b_0} dt \ t \ \ln |\sin t| \right)$$

$$= \frac{k_B T}{2 \pi h^2} \left( -0.81726 + \int_0^{\pi b_0} dt \ t \ \ln |\sin t| \right).$$

while for Neumann–Neumann (N–N) boundary conditions we have

$$f(h) = \frac{k_B T}{2 \pi h^2} \left( \frac{1}{2} \int_0^\infty dt \ t \ \ln \left( 1 - \exp(-2t)(1 - t)^2 \right) + \int_0^{\pi b_0} dt \ t \ \ln |\sin t| \right)$$

$$= \frac{k_B T}{2 \pi h^2} \left( -0.0680951 + \int_0^{\pi b_0} dt \ t \ \ln |\sin t| \right).$$

We thus see that the presence of external bulk has a strong influence on the amplitude of the Casimir interaction free energy generated by modes with $q > q_0$, but as predicted earlier has no influence on the modes with $q < q_0$. The salient features of the Casimir interaction of the type equations (76) and (77) have been analysed before, see the discussion of equations (47) and (51), and will not be repeated here.
Dirichlet–Neumann (D–N) boundary conditions are again obtained via the limit $b_1 = 0$, $b_2 \to \infty$ which yields

$$f(h) = \frac{k_B T}{2(2\pi)^2} \int dq \ln \left( 1 - \frac{\omega_1 \omega_2 (\exp(-\omega_1 h) - \exp(-\omega_2 h))^2}{(\omega_1 - \omega_2)^2} \right),$$

which at the critical point becomes

$$f(h) = \frac{k_B T}{2\pi h^2} \left( \frac{1}{2} \int_0^\infty dt \ln \left( 1 - t^2 \exp(-2t) \right) + \int_0^{b_2} dt \ln |\cos(t)| \right) = \frac{k_B T}{2\pi h^2} \left( -0.195371 + \int_0^{b_2} dt \ln |\cos(t)| \right).$$

Again we see exactly the same contribution from the modes $q < q_0$ as the case of Dirichlet–Neumann boundary conditions for confined systems in equation (51). However the contribution from modes with $q > q_0$ leads to an attractive $1/h^2$ interaction as opposed to the repulsive form seen in equation (51).

### 4.2. Some more exotic boundary conditions

Here the case where there is no boundary condition on any of the surfaces leads to no interaction. This is obviously correct, as in a nonconfined system removing the boundary conditions at a surface effectively removes that surface.

In the case of strong anchoring on one surface and Robin on the other, we find

$$Z_U(q, h) = \frac{(\omega_1 \omega_2)(\omega_1 + \omega_2)[(\omega_1^2 - \omega_2^2)^2]^{\frac{1}{2}} \exp(\frac{1}{4}(\omega_1 + \omega_2)h)}{2\pi^2 M^2} \times \int d\phi \exp \left( -\frac{\phi^2}{2} \mathbf{u} \cdot \mathbf{B}_1 \mathbf{P} \mathbf{S} \mathbf{D} \mathbf{R} \mathbf{P} \mathbf{B}_1 \mathbf{u} \right)$$

$$= \frac{(\omega_1 \omega_2)(\omega_1 + \omega_2)[(\omega_1^2 - \omega_2^2)^2]^{\frac{1}{2}} \exp(\frac{1}{4}(\omega_1 + \omega_2)h)}{\pi(2\pi)^2 M^2 (\mathbf{u} \cdot \mathbf{B}_1 \mathbf{P} \mathbf{S} \mathbf{D} \mathbf{R} \mathbf{P} \mathbf{B}_1 \mathbf{u})^{\frac{1}{2}}}.$$ (80)

In the limit of Dirichlet boundary conditions on the Robin surface (DN–D), and at the critical point we find

$$f(h) = \frac{k_B T}{2\pi h^2} \left( \frac{1}{2} \int_0^\infty dt \ln \left( 1 - \exp(-2t)(1 + 2t + t^2) \right) \right.$$

$$+ \frac{1}{2} \int_0^{b_2} dt \ln \left( 2t - \sin(2t) \right) \right)$$

$$= \frac{k_B T}{2\pi h^2} \left( -1.20552 + \frac{1}{2} \int_0^{b_2} dt \ln \left( 2t - \sin(2t) \right) \right).$$ (81)

In the limit of Neumann boundary conditions on the Robin surface (DN–N), and at the critical
Again, the Casimir interaction of the type equations (81) and (82) has been discussed in section 3.3.

5. Discussion

We have considered a second order derivative field theory of the general Brazovskii type, equation (1), in the presence of parallel plates, which modify the field fluctuations. The field theory has a critical point at $p_0 = 0$ and the bulk has a continuum of zero modes. We thus expect the presence of a critical thermal long range Casimir interaction at the critical point. Two distinct cases have been considered, the confined field case and the unconfined field case. These two cases differ by the presence of surface terms, which can strongly modify the form of the corresponding Casimir free energy, from which one derives the Casimir component of the disjoining pressure.

In all cases, at the critical point, we find that the Casimir free energy per unit area can be written as

$$f(q_0h) = \frac{q_0^2 k_B T}{2 \pi (q_0 h)^2} \left[ H + \frac{1}{2} \int_0^{q_0h} dt \ln(r(t)) \right].$$

(83)

Here $H$ is an effective amplitude or Hamaker coefficient [30], which depends on the boundary conditions and whether the system is confined or not. The modes generating this term are the modes with $q > q_0$. The second term is the one that can introduce oscillatory behavior. The function $r(t)$ can take several forms with $r(t) > 0$, but it can have an infinite number of zeros leading to an infinite number of metastable states. Remarkably, in contrast to what is found for the amplitude $H$, the form of $r(t)$ is independent of confinement.

To summarize our results, the value of $H$ and the form of $r(t)$ are given table 1, from where one can compare the effect of confinement in various cases. Note that the cases of periodic/antiperiodic boundary conditions are omitted as they are not natural in the context of unconfined systems.

The above summary highlights two important points

- The coefficient $H$ which determines the Casimir potential for small values of $h$, or equivalently small $q_0$, is always attractive for unconfined fields. For confined fields, the $q > q_0$ components are always attractive for symmetric boundary conditions. However, the non symmetric cases of DN–D and DN–N are attractive and repulsive, respectively, and D–N is repulsive.

- In systems with a single boundary condition on each surface, the component coming from modes $q < q_0$ give an infinite number (periodic) of points, where the disjoining pressure diverges, this leads to an infinite number of metastable states in the system. No such divergence occurs when three or more boundary conditions are imposed (DN–D, DN–N and DN–DN).
Table 1. Summary of the components of the Casimir free energy, equation (83), for various boundary conditions comparing confined and unconfined systems.

|          | $H_{\text{confined}}$ | $H_{\text{unconfined}}$ | $r(t)$ | $r(t)$ |
|----------|----------------------|-------------------------|--------|--------|
| DN–DN    | $-1.71629$           | $-1.71629$              | $t^2 - \sin^2(t)$ |        |
| D–D      | $-0.30051$           | $-0.81726$              | $\sin^2(t)$       |        |
| N–N      | $-0.30051$           | $-0.068095$             | $\sin^2(t)$       |        |
| D–N      | $0.225386$           | $-0.195371$             | $\cos^2(t)$       |        |
| DN–D     | $-0.83591$           | $-1.20552$              | $2t - \sin(2t)$   |        |
| DN–N     | $0.406839$           | $-0.266976$             | $2t + \sin(2t)$   |        |

The behavior of the critical Casimir interaction for the general Brazovskii-type confined field theories with an additional length scale given by $q_0$ is thus quite distinct from the critical Casimir interactions (CCI) [31] studied in great detail in the context of collective behavior of colloids [32, 33]. In fact the critical Casimir interactions in the Brazovskii-type field theories can be seen as a generalization of the CCI applicable to the case of semi-flexible polymers [3–8], soft membranes [9–11], ionic liquids [14–17] and liquid crystals [12, 13], where higher derivative Hamiltonians emerge naturally. The main difference between the CCI and the general Brazovskii-type confined field theories boils down to the fact that the modes parallel to the bounding surfaces contribute to the Casimir interaction disjoining pressure in a manner quite different to that in the CCI. The modes with $q < q_0$ can lead to oscillatory terms in the pressure, and exhibit longer range interactions than the CCI, giving a contribution to the disjoining pressure which decays with an inverse first power of the separation between the bounding surfaces, as opposed to the usual inverse third power characteristic of the CCI as well as the Lifshitz–van der Waals interactions [30]. This long range effect and the non-monotonic dependence of the disjoining pressure on the separation between bounding surfaces, creating a possibility of multiple metastable states, makes the fundamental study of critical Casimir interaction for the general Brazovskii-type confined field theories quite interesting.

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