TWO REMARKS ABOUT MAÑÉ’S CONJECTURE

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1. Introduction

In this note we consider an autonomous Tonelli Lagrangian \( L \) on a closed manifold \( M \), that is, a \( C^2 \) function \( L: TM \rightarrow \mathbb{R} \) such that \( L \) is fiberwise strictly convex and superlinear. Then the Euler-Lagrange equation associated with \( L \) defines a complete flow \( \phi_t \) on \( TM \). Define \( \mathcal{M}_{inv} \) to be the set of \( \Phi_t \)-invariant, compactly supported, Borel probability measures on \( TM \). Mather showed that the function (called action of the Lagrangian on measures)

\[
\mathcal{M}_{inv} \rightarrow \mathbb{R} \\
\mu \mapsto \int_{TM} Ld\mu
\]

is well defined and has a minimum. A measure achieving this minimum is called \( L \)-minimizing. The union, in \( TM \), of the support of all minimizing measures is called Mather set of \( L \), and denoted \( \mathcal{M}(L) \). It is compact and \( \phi_t \)-invariant. See [Mr91] and [F] for more background.

Observe that if \( f \) is a \( C^2 \) function on \( M \), \( L + f \) is also a Tonelli Lagrangian. Adding a function to a Lagrangian is called perturbing the Lagrangian by a potential. Following Mañé we say a property holds for a generic Lagrangian if, given any Lagrangian, the property holds for a generic perturbation by a potential. Mañé conjectured a generic description of the minimizing measures:

Conjecture 1.1 ([Mn97]). Let

- \( M \) be a closed manifold
- \( L \) be an autonomous Tonelli Lagrangian on \( TM \)
- \( \mathcal{O}_1(L) \) be the set of \( f \) in \( C^\infty(M) \) such that the Mather set of \( L + f \) consists of one periodic orbit.

Then the set \( \mathcal{O}_1(L) \) is residual in \( C^\infty(M) \).

In other words, for a generic Lagrangian, there exits a unique minimizing measure, and it is supported by a periodic orbit. A similar conjecture can be made replacing \( C^\infty(M) \) by \( C^k(M) \) for any \( k \geq 2 \).

Many more interesting invariant sets can be obtained by minimization than just the Mather set. If \( \omega \) is a closed one-form on \( M \), then \( L - \omega \) is a Tonelli Lagrangian, and it has the same Euler-Lagrange flow as \( L \). Its Mather set, however, is different in general. The Mather set of \( L - \omega \) only depends on the cohomology class \( c \) of \( \omega \), we denote it \( \mathcal{M}(L,c) \). It is often interesting to obtain information simultaneously on the Mather sets \( \mathcal{M}(L,c) \) for a large set of cohomology classes. Thus Mañé proposed the
Conjecture 1.2 ([Mn96]). If $L$ is a Tonelli Lagrangian on a manifold $M$, there exists a residual subset $O_2(L)$ of $C^\infty(M)$, such that for any $f$ in $O_2(L)$, there exists an open and dense subset $U(L, f)$ of $H^1(M, \mathbb{R})$ such that, for any $c$ in $U(L, f)$, the Mather set of $(L, c)$ consists of one periodic orbit.

Intuitively Conjecture 1.2 is weaker than Conjecture 1.1 because we allow a larger set of perturbations (potentials and closed one-forms instead of just potentials). However the requirement of an open dense set in Conjecture 1.2 makes it far from obvious. In section 2 we prove that Conjecture 1.1 contains Conjecture 1.2, using recent tools from Fathi’s weak KAM theory, the most prominent of which is the Aubry set $A(L)$. All we need to know about the Aubry set is that

- it consists of the Mather set, and (possibly) orbits homoclinic to the Mather set (see [F])
- when there is only one minimizing measure, the Aubry set is upper semi-continuous as a function of the Lagrangian, that is, for any neighborhood $V$ of $A(L)$ in $TM$, there exists a neighborhood $U$ of $L$ in the $C^2$ compact-open topology, such that for any $L_1$ in $U$, we have $A(L_1) \subset V$ (see [Be]).

We first prove that Conjecture 1.1 is equivalent to the apparently stronger

Conjecture 1.3. Let

- $M$ be a closed manifold
- $L$ be an autonomous Tonelli Lagrangian on $TM$
- $O_3(L)$ be the set of $f$ in $C^\infty(M)$ such that the Aubry set of $L + f$ consists of one, hyperbolic periodic orbit.

Then the set $O_3(L)$ is residual in $C^\infty(M)$.

Then we prove that Conjecture 1.3 contains the following, which obviously contains Conjecture 1.2:

Conjecture 1.4. If $L$ is a Tonelli Lagrangian on a manifold $M$, there exists a residual subset $O_4(L)$ of $C^\infty(M)$, such that for any $f$ in $O_4(L)$, there exists an open and dense subset $U(L, f)$ of $H^1(M, \mathbb{R})$ such that, for any $c$ in $U(L, f)$, the Aubry set of $(L + f, c)$ consists of one, hyperbolic periodic orbit.

Conjecture 1.4 is proved, in the case where the dimension of $M$ is two, in [M1] (after a sketch of a proof appeared in [Mt03]). The analogous statement for Lagrangians which depend periodically on time is proved, in the case where the dimension of $M$ is one, in [O09].

Conjecture 1.4 may be seen as an Aubry-Mather version of the Closing Lemma. This suggests that it should be true in the $C^2$ topology on Lagrangians and false in the $C^k$ topology for $k > 2$. If we want to prove the $C^k$ version of Conjecture 1.4 and we are lucky enough to have a sequence of periodic orbits $\gamma_n$ which approximate our Mather set, then the first idea that comes to mind is to perturb $L$ by a non-negative potential $f_n$ which vanishes only on $\gamma_n$. Then $\gamma_n$ is still an orbit of $L + f_n$. If we can find $f_n$ big enough for $\gamma_n$ to be $L + f_n$-minimizing, but small enough for the $C^k$-norm of $f_n$ to converge to zero, then we are done. In Section 3 we prove that this naive approach doesn’t work in the $C^k$-topology, for $k \geq 4$. Specifically,
we give an example of a Lagrangian $L$ on the two-torus, such that for any periodic orbit $\gamma$ of $L$, and any $C^3$ function $f$ on the two-torus, if $\gamma$ is $L + f$-minimizing, then the $C^4$ norm of $f$ is bounded below by a constant which only depends on $L$.

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2.

**Lemma 2.1.** Let
- $M$ be a closed manifold
- $L$ be an autonomous Tonelli Lagrangian on $TM$
- $\mathcal{O}_3(L)$ be the set of $f$ in $C^\infty(M)$ such that the Aubry set of $L + f$ consists of one, hyperbolic periodic orbit
- $\mathcal{O}_1(L)$ be the set of $f$ in $C^\infty(M)$ such that the Mather set of $L + f$ consists of one periodic orbit.

Then $\mathcal{O}_3(L)$ is open and dense in $\mathcal{O}_1(L)$.

**Proof.** We first prove that $\mathcal{O}_3(L)$ is open in $\mathcal{O}_1(L)$. Take $f \in \mathcal{O}_3(L)$. Replacing $L$ with $L + f$, we may assume $f = 0$. Let $\gamma$ be the hyperbolic periodic orbit which comprises $\mathcal{A}(L + f)$. By a classical property of hyperbolic periodic orbits, there exists a neighborhood $U_1$ of the zero function in $C^\infty(M)$, and a neighborhood $V$ of $\gamma$ in $TM$ such that for any $f \in U_1$, for any energy level $E$ of $L$, the only invariant set of the Euler-Lagrange flow of $L$ contained in $E \cap V$, if any, is a hyperbolic periodic orbit homotopic to $\gamma$.

Since $\mathcal{A}(L)$ is a periodic orbit, the quotient Aubry set $A$ has but one element. Thus by [199], there exists a neighborhood $U_2$ of the zero function in $C^\infty(M)$, such that for all $f \in U_2$, we have $\mathcal{A}(L + f) \subset V$. Therefore, for any $f \in U_1 \cap U_2$, the Aubry set $\mathcal{A}(L + f)$ consists of one, hyperbolic periodic orbit.

Now let us prove that $\mathcal{O}_3(L)$ is dense in $\mathcal{O}_1(L)$.

Take $f \in \mathcal{O}_1(L)$. Replacing $L$ with $L + f$, we may assume $f = 0$. Let $\gamma$ be the periodic orbit which comprises $\mathcal{M}(L)$. Now let us take a smooth function $g$ on $M$ such that $g$ vanishes on the projection to $M$ of $\gamma$ (which we again denote $\gamma$ for simplicity), and $\forall x \in M$, $g(x) \geq d(x, \gamma)^2$, where the distance is meant with respect to some Riemannian metric on $M$. Let $\lambda$ be any positive number. We will show that $\lambda g \in \mathcal{O}(h)$, which proves that $\mathcal{O}(h)$ is dense in $\mathcal{O}_1(L)$. Observe that $\gamma$ is a minimizing hyperbolic periodic orbit of the Euler-Lagrange flow of $L + \lambda g$ (see [199]). Furthermore, $\alpha_{L + \lambda g}(0) = \alpha_{L}(0)$, where $\alpha_{L}(0)$ is Mañé’s critical value for the Lagrangian $L$.

Adding a constant to $L$ if necessary, we assume $\alpha_{L}(0) = 0$. Recall that the Aubry set is the union of the Mather set and orbits homoclinic to the Mather set. Therefore, to prove that $\lambda g \in \mathcal{O}_1(h)$, it suffices to prove that the Aubry set $\mathcal{A}(L + \lambda g)$ does not contain any orbit homoclinic to $\gamma$.

Assume $\delta : \mathbb{R} \rightarrow M$ is an extremal of $L + \lambda g$, homoclinic to $\gamma$. Since $g(\delta(t)) > 0$ for all $t$, there exists $C > 0$ such that

$$\int_{-\infty}^{+\infty} g(\delta(t))dt \geq 2C.$$
Let $u$ be a weak KAM solution for $L$. We have, for any $s, t \in \mathbb{R}$, remembering that $\alpha_L(0) = 0$,
\[
\int_s^t L(\delta(t), \dot{\delta}(t))dt \geq u(\delta(t)) - u(\delta(s)).
\]
Since is homoclinic to $\gamma$ there exist two sequences $t_n$ and $s_n$ that converge to $+\infty$, such that $\delta(t_n)$ and $\delta(-s_n)$ converge to the same point $x$ on $\gamma$, so for $n$ large enough
\[
\int_{-s_n}^{t_n} L(\delta(t), \dot{\delta}(t))dt \geq -C.
\]
Therefore, for $n$ large enough,
\[
\int_{-s_n}^{t_n} (L + \lambda g)(\delta(t), \dot{\delta}(t))dt > C.
\]
On the other hand, since $\alpha_{L+\lambda g}(0) = 0$, if $\delta$ were contained in the projected Aubry set of $L + \lambda g$, we would have, denoting $u_\lambda$ a weak KAM solution for $L + \lambda g$,
\[
\int_{-s_n}^{t_n} (L + \lambda g)(\delta(t), \dot{\delta}(t))dt = u_\lambda(\delta(t)) - u_\lambda(\delta(s))
\]
which converges to zero because $\delta(t_n)$ and $\delta(-s_n)$ converge to the same point $x$ on $\gamma$. Therefore the Aubry set of $L + \lambda g$ consists of $\gamma$ alone, which proves that $\lambda g \in \mathcal{O}_3(L)$, and the Lemma.\hfill$\square$

Therefore $\mathcal{O}(L)$ is residual in $C^\infty(M)$ if and only if $\mathcal{O}_1(L)$ is. Therefore, Conjecture 1.1 is equivalent to Conjecture 1.3. Now we show that Conjecture 1.3 contains Conjecture 1.4, which obviously contains Conjecture 1.2.

Assume Conjecture 1.3 is true. Let $L$ be an autonomous Tonelli Lagrangian on a manifold $M$. Let $c_i, i \in \mathbb{N}$ be a countable dense subset of $H^1(M, \mathbb{R})$. Take, for every $i \in \mathbb{N}$, a closed one-form $\omega_i$ with cohomology $c_i$. Since Conjecture 1.3 is true for every Lagrangian $L - \omega_i$, for every $i \in \mathbb{N}$, there exists a residual subset $\mathcal{O}_i$ of $C^\infty(M)$ such that for every $f$ in $\mathcal{O}_i$, $\mathcal{A}(L + f, c_i)$ consists of one hyperbolic periodic orbit. Then the intersection, over $i \in \mathbb{N}$, of $\mathcal{O}_i$ is a residual subset $\mathcal{O}$ of $C^\infty(M)$. For every $f$ in $\mathcal{O}$, for every $i \in \mathbb{N}$, $\mathcal{A}(L + f, c)$ consists of one hyperbolic periodic orbit $\gamma_i$. As in the proof of Lemma 2.1 there exists a neighborhood $V_i$ of $c_i$ in $H^1(M, \mathbb{R})$, such that for any $c$ in $V_i$, the Aubry set $\mathcal{A}(L + f, c)$ consists of one hyperbolic periodic orbit homotopic to $\gamma_i$. The union, over $i \in \mathbb{N}$, of the $V_i$, is an open and dense subset $V$ of $H^1(M, \mathbb{R})$, and for any $c$ in $V$, $\mathcal{A}(L + f, c)$ consists of one hyperbolic periodic orbit, so Conjecture 1.4 is true.

3. An example

Let
- $r$ be a quadratic irrational number, for instance $\sqrt{2}$
- $p_0$ and $q_0$ be real numbers such that $p_0^2 + q_0^2 = 1$ and $p_0/q_0 = r$
- $\mathbb{T}^2$ be $\mathbb{R}^2/\mathbb{Z}^2$, endowed with canonical coordinates $(x, y)$
- $L$ be the Lagrangian on $\mathbb{T}^2$ defined by
  \[
  L(x, y, u, v) := \frac{u^2 + v^2}{2} - (p_0 u + q_0 v)
  \]
where $(u, v)$ are the tangent coordinates to $(x, y)$.
Assume that for some function $f$ on $T^2$, $L + f$ has a minimizing periodic orbit $\gamma$, and furthermore, $\gamma$ is an orbit of $L$, that is, it has the form $t \mapsto (pt, qt)$ for some real numbers $p$ and $q$. Then, if $T$ is the smallest period of $\gamma$, $(pT, qT) \in \mathbb{Z}^2$ and $pT, qT$ are mutually prime. Consider the map

$$ F: \mathbb{R} \to \mathbb{R} $$

$$ \lambda \mapsto \frac{1}{T} \int_0^T f(pt, qt + \lambda)dt. $$

Observe that $F$ is 1-periodic. We now prove that $F$ is $(pT)^{-1}$-periodic. Indeed, take $r, s$ in $\mathbb{Z}$ such that $pTr - qTs = 1$. Then for any $t$,

$$(pt, qt + \frac{1}{pT}) = (pt, qt + r - \frac{qs}{p})$$

$$ = \left( pt, q(t - \frac{s}{p}) \right) \mod \mathbb{Z}^2$$

$$ = \left( p(t - \frac{s}{p}) + s, q(t - \frac{s}{p}) \right) \mod \mathbb{Z}^2$$

$$ = \left( p(t - \frac{s}{p}), q(t - \frac{s}{p}) \right) \mod \mathbb{Z}^2$$

so

$$ F\left( \lambda + \frac{1}{pT} \right) = \frac{1}{T} \int_0^T f(pt, qt + \frac{1}{pT} + \lambda)dt$$

$$ = \frac{1}{T} \int_0^T f(p(t - \frac{s}{p}), q(t - \frac{s}{p}) + \lambda)dt = F(\lambda)$$

using the change of variable $t \mapsto t - s/p$ (and the fact that $F$ is 1-periodic).

Now we prove that

$$ \int_0^1 F(\lambda)d\lambda = \int_{T^2} f d\text{leb} $$

where $\text{leb}$ denotes the standard Lebesgue measure on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Indeed, let $\nu$ be the measure on $T^2$ defined by

$$ \int g(x, y)d\nu(x, y) := \int_0^1 d\lambda \left\{ \frac{1}{T} \int_0^T g(pt, qt + \lambda)dt \right\} $$

for any continuous function $g$ on $T^2$. We want to prove that $\nu$ is actually leb. First let us show that $\nu$ is invariant under translations. Let $(u, v)$ be any vector in $\mathbb{R}^2$. We have

$$ \int g(x + u, y + v)d\nu(x, y) = \int_0^1 d\lambda \left\{ \frac{1}{T} \int_0^T g(pt + u, qt + \lambda + v)dt \right\}$$

$$ = \int_0^1 d\lambda \left\{ \frac{1}{T} \int_0^T g(p(t + \frac{u}{p}), q(t + \frac{u}{p}) + \lambda - \frac{uq}{p})dt \right\}$$

$$ = \int_0^1 d\lambda \left\{ \frac{1}{T} \int_0^T g(pt, qt + \lambda - \frac{uq}{p})dt \right\}$$

$$ = \int_0^1 d\lambda \left\{ \frac{1}{T} \int_0^T g(pt, qt)dt \right\} = \int g(x, y)d\nu(x, y)$$
where we have used, in succession, the changes of variables \( t \mapsto t + u/p \) and \( \lambda \mapsto \lambda - uq/p \). So \( \nu \) is invariant under translations. Furthermore \( \int 1 d\nu = 1 \) so \( \nu \) is actually leb.

Now let us use the fact that \( \gamma \) is \( L+f \)-minimizing. Let \( \mu \) be the probability measure equidistributed along \( \gamma \). We have

\[
\int (L + f) \, d\mu = \frac{1}{T} \int_0^T dt \left\{ \frac{p^2 + q^2}{2} - (p_0p + q_0q) + f(pt, qt) \right\}
\]

\[
= \frac{p^2 + q^2}{2} - (p_0p + q_0q) + F(0).
\]

Let \( \mu_0 \) be the measure on \( T^2 \) defined by

\[
\int g(x, y, u, v) d\mu_0(x, y, u, v) := \int_{x=0}^{1} dx \int_{y=0}^{1} g(x, y, p_0, q_0) dy
\]

for any continuous function \( g \) on \( T^2 \). Observe that the measure \( \mu_0 \) is \( L \)-minimizing. In particular it is closed (see [FS04], Theorem 1.6). We have

\[
\int_{T^2} L d\mu_0 = -\frac{1}{2} \quad \text{and} \quad \int_{T^2} f d\mu_0 = \int_{T^2} f d\text{leb} = \int_0^1 F(\lambda) d\lambda
\]

that is,

\[
\int_0^1 F(\lambda) d\lambda - F(0) \geq \int_{T^2} L d\mu_0 - \int_{T^2} L d\mu_0
\]

\[
= \frac{p^2 + q^2}{2} - (p_0p + q_0q) + \frac{1}{2}
\]

\[
= \frac{1}{2} (p - p_0)^2 + \frac{1}{2} (q - q_0)^2.
\]

Now let us use the fact that \( r = p_0/q_0 \) is quadratic, and \( pT/qT = p/q \) is rational, so there exists a constant \( C_0 \) such that

\[
\left| \frac{p_0}{q_0} - \frac{p}{q} \right| \geq \frac{C_0}{(pT)^2}.
\]

Hence, setting \( C := \frac{C_0^2}{2} \),

\[
\frac{1}{2} (p - p_0)^2 + \frac{1}{2} (q - q_0)^2 \geq \frac{C}{(pT)^4}
\]

whence

\[
\int_0^1 F(\lambda) d\lambda - F(0) \geq \frac{C}{(pT)^4}.
\]

Therefore, since \( F \) is \((pT)^{-1}\)-periodic, there exists a \( \lambda_0 \in [0, (pT)^{-1}] \) such that

\[
F(\lambda_0) - F(0) \geq \frac{C}{(pT)^4}.
\]
Hence there exists a $\lambda_1 \in [0, (pT)^{-1}]$ such that $|F'(\lambda_1)| \geq C(pT)^{-3}$. On the other hand, since $F$ is $(pT)^{-1}$-periodic, there exists a $\lambda_2 \in [0, (pT)^{-1}]$ such that $F'(\lambda_2) = 0$. Thus $|F'(\lambda_1) - F'(\lambda_2)| \geq C(pT)^{-3}$, so there exists a $\lambda_3 \in [0, (pT)^{-1}]$ such that $|F''(\lambda_3)| \geq C(pT)^{-2}$. Iterating this process we show there exists a $\lambda \in [0, (pT)^{-1}]$ such that $|F^{(4)}(\lambda)| \geq C$; that is,

$$\left| \frac{1}{T} \int_0^T f^{(4)}(pt, qt + \lambda) dt \right| \geq C.$$

In particular the $C^4$-norm of $f$ is bounded below by $C$.

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