The Polar Decomposition of q-Deformed Boson Algebra

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Abstract

In this paper we used the finite Fourier transformation to obtain the polar decomposition of the q-deformed boson algebra with \( q \) a root of unity.

Since the q-deformed boson algebra was known [1,2], much work has been accomplished in this direction [3-8]. It is well known that this algebra has an infinite dimensional representation when the deformation parameter \( q \) is real, while it has a finite dimensional one when \( q \) is a root of unity.

In this paper we use the finite Fourier transformation to obtain the polar decomposition of the q-deformed boson algebra with \( q \) a root of unity. The polar decomposition of an operator means that the operator can be written as a unitary times a hermitian operator.

Now let us consider the q-deformed boson algebra [1,2]. It is defined as

\[
\begin{align*}
    aa^\dagger - qa^\dagger a &= q^{-N}, \\
    [N, a^\dagger] &= a^\dagger, \\
    [N, a] &= -a.
\end{align*}
\]

(1)
Then, the step operators and the number operator are defined as

\[ a = \sum_{n=0}^{\infty} \sqrt{|n|} |n-1\rangle \langle n|, \]

\[ a^\dagger = \sum_{n=0}^{\infty} \sqrt{|n+1|} |n+1\rangle \langle n|, \]

\[ N = \sum_{n=0}^{\infty} n |n\rangle \langle n|, \]

(2)

where the q-number \([x]\) is given by

\[ [x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \]

If we take \(q\) to be \(q = e^{i\frac{2\pi}{s+1}}\), we have \([s+1] = 0\), which implies

\[ a^\dagger |s \rangle = 0. \]

(3)

Thus, the spectrum becomes finite dimensional and the matrix representation of the step and the number operators are given by

\[ a = \sum_{n=0}^{s} \sqrt{|n|} |n-1\rangle \langle n|, \]

\[ a^\dagger = \sum_{n=0}^{s} \sqrt{|n+1|} |n+1\rangle \langle n|, \]

\[ N = \sum_{n=0}^{s} n |n\rangle \langle n|, \]

(4)

It can be easily checked that the step operators satisfy the nilpotency condition;

\[ a^{s+1} = (a^\dagger)^{s+1} = 0. \]

(5)

In this case the step operators can be decomposed into the following two pieces:
\[
a = \sqrt{\{g+1\}} h^\dagger = h^\dagger \sqrt{\{g\}},
\]
\[
a^\dagger = \sqrt{\{g\}} h = h \sqrt{\{g+1\}},
\] (6)

with \( g = q^N \) and

\[
h = \sum_{n=0}^{s-1} |n+1><n| = |1><0| + \cdots + |s><s-1|,
\]
\[
h^\dagger = \sum_{n=0}^{s-1} |n><n+1| = |0><1| + \cdots + |s-1><s|,
\] (7)

where we have used the symbols

\[
\{g\} = \frac{g - g^{-1}}{q - q^{-1}} = a^\dagger a = [N],
\]
\[
\{g+1\} = \frac{qq - q^{-1}g^{-1}}{q - q^{-1}} = aa^\dagger = [N+1].
\] (8)

Here we cannot call eq.(6) the polar decomposition because in the finite dimensional representation with \( q \) a root of unity the operator \( h \) and \( h^\dagger \) do not satisfy the unitary condition. From the definition of \( h \) and \( h^\dagger \), the commutation relations of \((h, h^\dagger)\) and \( g \) become

\[
gh = qhg,
\]
\[
gh^\dagger = q^{-1}h^\dagger g.
\] (9)

Since \( a \) and \( a^\dagger \) operator act on the basis of \((s+1)\)-dimensional Fock space \( \{|0>, \cdots , |s>\} \), the operator \( h \) and \( h^\dagger \) are not unitary any more. Instead \( h \) and \( h^\dagger \) satisfy

\[
hh^\dagger = 1 - |0><0|,
\]
\[
h^\dagger h = 1 - |s><s|.
\] (10)
Then it can be easily checked that the decomposition (6) satisfy the relation
\[ a^\dagger a = [N], \quad a a^\dagger = [N + 1]. \] (11)

From the definitin of \( g \) and \( h \) with \( q = e^{i 2\pi s+1} \), we have
\[ g^{s+1} = 1, \quad h^{s+1} = 0. \] (12)

Let us introduce the finite Fourier transformation
\[ F = \frac{1}{\sqrt{s+1}} \sum_{n,m=0}^{s} q^{mn} |m><m| \] (13)
satisfying the unitary condition
\[ FF^\dagger = F^\dagger F = 1. \]

Using this operator we have
\[ h = F g^{-1} F^\dagger - |0><s|, \]
\[ h^\dagger = F g F^\dagger - |s><0|. \] (14)

It can be easily checked that the eq.(14) satisfies the relation (9).

The finite Fourier transformation defines the phase state \(|\phi_m><m|\) and the phase state \(|\phi_0>,\cdots,|\phi_s>\) constitute an orthonormal set of states dual to that of the number states \(|0>,\cdots,|s>\):
\[ <\phi_m|\phi_m> = \delta_{mn}. \] (15)

Further we can define creation and annihilation operator acting in the phase state basis as follows;
\[ \tilde{a} = F a F^\dagger = \sum_{m=0}^{s} \sqrt{|m|} |\phi_{m-1}><m| \].
\[ \tilde{a}^\dagger = F a^\dagger F^\dagger = \sum_{m=0}^{s} \sqrt{[m+1]}|\phi_{m+1}><\phi_m|, \]
\[ \tilde{N} = F N F^\dagger = \sum_{m=0}^{s} m|\phi_m><\phi_m|. \] (16)

Then, we see that they satisfy
\[ \tilde{a} \tilde{a}^\dagger - q \tilde{a}^\dagger \tilde{a} = q^{-\tilde{N}} = H, \] (17)

where
\[ H = h + |0><s|. \]

At this stage we notice that \( H \) is a unitary operator although \( h \) is not unitary. Thus, the new operators \((H, H^\dagger)\) and \(g\) obey the following commutation relation
\[ gH = qHg, \quad gH^\dagger = q^{-1}H^\dagger g, \]
\[ H^{s+1} = 1. \] (18)

Therefore, we obtain the polar decomposition of \( \tilde{a} \) and \( \tilde{a}^\dagger \), which is given by
\[ \tilde{a} = \sqrt{\{H^\dagger + 1\}} g^{-1} = g^{-1} \sqrt{\{H^\dagger\}}, \]
\[ \tilde{a}^\dagger = \sqrt{\{H^\dagger\}} = g \sqrt{\{H^\dagger + 1\}}, \] (19)

where
\[ \{H^\dagger\} = \frac{H^\dagger - H}{q - q^{-1}}, \]
\[ \{H^\dagger + 1\} = \frac{qH^\dagger - q^{-1}H}{q - q^{-1}}. \] (20)

To conclude, we considered the finite dimensional representation of the q-deformed boson algebra when \( q \) is a root of unity. In this case the step operators of this algebra do not seem to have the polar decomposition (a unitary times a hermitian operator). But we used the finite Fourier transformation and the unitary transformation of the step and the number operators to obtain the polar decomposition of the step operator of the q-deformed boson algebra.
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