EVEN AND ODD TREES

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Abstract. In this paper we at first consider plane trees with the root vertex and a marked directed edge, outgoing from the root vertex. For such trees we introduce a new characteristic — the parity, using the bracket code. It turns out that the parity depends only on the root vertex (not on the marked edge). And in the case of an even number of vertices the parity does not depend on the root vertex also. Then we consider rotation groups of bipartite trees, study their properties and prove that in the case of even number of vertices rotation groups of even and odd trees are different.

1. Introduction

Let us consider a tree with \( n \) vertices embedded into the plane. One of vertices will be a root vertex. Such tree will be represented on the plane as "growing" downwards from the root vertex. The leftmost edge outgoing from the root vertex will be called the marked edge. Such trees will be called pr-trees (i.e. plane rooted trees). The root vertex will be the vertex of level 0, vertices adjacent to the root will be vertices of level 1, and so on.

Vertices of level 1 are root vertices of pr-trees (subtrees) of the original pr-tree. By a level one permutation we will understand any change of the order of level 1 vertices (and corresponding pr-trees).

We can describe pr-trees using proper bracket structures (bracket codes) with the number of opening brackets (and the number of closing brackets) equal to the number of vertices. A bracket structure is proper, if the number of opening brackets is not less, than the number of closing ones in any left segment of this structure. The bracket structure is constructed in the following way: beginning at root vertex, we make the counterclockwise walkabout (or simply the walk) around the tree, i.e. the tree is always on our left. The length of this walkabout (measured by edges) is \( 2(n - 1) \), where \( n \) is the number of vertices: we pass each edge twice in two opposite directions. When we meet a vertex for the first time we open bracket. When we meet a vertex for the last time we close the bracket. The first opening bracket and the last closing bracket correspond to the root vertex. Here is an example of this correspondence.

\[
\begin{align*}
\Rightarrow & \quad (()(())))
\end{align*}
\]

Figure 1

Definition 1.1. A pair "closing bracket before opening bracket (not necessary adjacent)" in a bracket structure is called an inversion.
Remark 1.1. There are 6 inversions in the bracket code of the tree in the Figure 1.

Definition 1.2. A pr-tree is called even, if the number of inversions in the corresponding bracket structure is even, and is called odd in the opposite case.

Remark 1.2. The tree in the Figure 1 is even.

The following properties of the parity will be proved.

- A level one permutation does not change the parity of a pr-tree (Proposition 2.1).
- If the number of vertices $n$ is even, then the parity does not depend on the choice of the root vertex (Proposition 2.2).

Remark 1.3. From propositions 2.1 and 2.2 it follows that if two pr-trees with even number of vertices are graph-isomorphic, then they have the same parity.

Let $A$, $B$ and $C$ be three vertices of a pr-tree (see Figure 2), such that: a) the degree of $B$ is one; b) during the walk we meet vertices in the sequence $A, B, A, C$ as in the figure below.

![Figure 2](image1.png)

Definition 1.3. A transposition is a translation of the vertex $B$ with the attached edge along one(!) edge of the tree (as in the walkabout) from vertex $A$ to the next vertex $C$ (as in the Figure 3).

![Figure 3](image2.png)

The transformation of the right tree in Figure 2 is analogous.

It will be proved (Proposition 3.1) that a transposition changes the parity.

Remark 1.4. The standard chain-tree is obviously even (see Figure 4).

![Figure 4](image3.png)
Thus, we can determine the parity of any pr-tree by counting the number of transpositions that transform our tree to the chain-tree.

**Definition 1.4.** A plane tree $T$ is a *sum* of two plane trees $T_1$ and $T_2$, if after deletion of some edge of $T$ we obtain the disjoint union of $T_1$ and $T_2$.

It will be proved (Proposition 4.1) that the sum of two trees with even number of vertices each is odd only when one of the summands is odd and another is even.

**Remark 1.5.** By representing a tree as a sum, we can simplify the determination of its parity.

Let us consider a tree $T$ with $m$ edges. The *passport* (see [1]) of $T$ is the list $[a_1, \ldots, a_m]$, where $a_i$, $i = 1, \ldots, m$, is the number of vertices of degree $i$.

A bipartite tree is a tree with vertices colored in two colors — black and white so, that adjacent vertices have different colors. The passport of a bipartite tree with $m$ edges is the list $[[a_1, \ldots, a_m], [b_1, \ldots, b_m]]$, where $a_i$, $i = 1, \ldots, m$, is the number of white vertices of degree $i$ and $b_i$, $i = 1, \ldots, m$, is the number of black vertices of degree $i$.

**Definition 1.5.** Let $T$ be a plane bipartite tree with $m$ edges, enumerated in some arbitrary way. Let $M = \{1, 2, \ldots, m\}$. The permutation $s_w : M \rightarrow M$ is defined in the following way: $s_w(i) = j$, if $i$-th and $j$-th edges are incident to the same white vertex $v$ and $j$-th edge is the next after $i$-th edge in the counterclockwise going around $v$. The permutation $s_b$ is defined in the same way, but with respect to black vertices. The rotation group (see [1]) $R(T)$ of $T$ is the subgroup of $S_m$, generated by permutations $s_w$ and $s_b$, i.e. $R(T) = \langle s_w, s_b \rangle \subset S_m$.

**Remark 1.6.** The product $s_w s_b$ is an $m$-cycle. Different numerations of edges give us conjugate rotation groups.

It will be proved (Propositions 5.1) that rotations groups of two plane trees with the same even number of vertices, but of different parities, are different.

**Definition 1.6.** A bipartite tree is called *clean*, if all its black vertices have degree two (see [1]).

For any given tree $T$, we can construct a bipartite clean tree $T^c$ by dividing each edge by black vertex into two edges (initial vertices now are white). Let $T$ and $U$ be two plane trees with $n$ vertices and $n$ is even. If $T$ and $U$ are of different parities, then rotation groups of $T^c$ and $U^c$ are different (Proposition 6.1).

In Supplement 1 we demonstrate that trees with different parities may be cospectral.

In Supplement 2 we introduce graph $G_n$ (for even $n$) — the "graph of trees" and study its planarity for small $n$.

## 2. Even and Odd Rooted Trees

**Proposition 2.1.** A level one permutation does not change the parity of a pr-tree $T$. 

Proof. Let $v_0$ be the root vertex of a pr-tree $T$, $v_1, \ldots, v_s$ be level one vertices (enumerated from left to the right) and $T_1, \ldots, T_s$ be the corresponding pr-subtrees (Figure 5).

Let a subtree $T_i$ has $k_i$ vertices and its bracket structure has $x_i$ inversions. Then the bracket structure of the tree $T$ has

$$\sum_i x_i + \sum_{i<j} k_i k_j$$

inversions. If we interchange subtrees $T_i$ and $T_{i+1}$, then the first sum will be the same and in the second sum the term $k_i k_{i+1}$ will be replaced by $k_{i+1}k_i$. Hence, the parity of a pr-tree does not depend on the choice of marked edge. \hfill \square

Proposition 2.2. If a pr-tree has an even number of vertices $n$ then its parity does not depend on the choice of root vertex.

Proof. Let $O$ be the root vertex and $OP$ be the marked edge. We make $P$ the new root vertex (and preserve the order of the walk). Then the tree will be transformed in the following way:

The corresponding change of bracket code is as follows:

$$\begin{align*}
( & ( \cdots ) & \cdots ) & \Rightarrow & ( \cdots ( \cdots ) ) \\
\uparrow A \uparrow B & \Rightarrow & A \uparrow B \uparrow \\
P & P & & & O & O
\end{align*}$$

If there are $k$ vertices in the forest $A$ and $s$ inversions in its bracket code, and if there are $l$ vertices in the forest $B$ and $t$ inversions in its bracket code, then the left and the right trees in Figure 6 have $s + kl + l + t$ and $s + k + kl + t$ inversions in their bracket codes, respectively. As $n = k + l + 2$, then numbers $s + kl + l + t$ and $s + k + kl + t$ have the same parity. \hfill \square
3. Transpositions

**Proposition 3.1.** A transposition changes the parity of a pr-tree.

*Proof.* We can see in the Figure 2 that there are two possibilities: either we meet $C$ for the first time in the walk around the tree, or we meet $A$ for the last time. The change of the bracket code (see Figure 3) in the first case is as follows:

$$\ldots \ ( \ ) \ ( \ldots \ \uparrow \ \uparrow \ \uparrow \ \Rightarrow \ \uparrow \ \uparrow \ \uparrow \ \ B \ \ B \ \ C \ \ C \ \ B \ \ B$$

and in the second —

$$\ldots \ ( \ ) \ \ldots \ \Rightarrow \ \uparrow \ \uparrow \ \uparrow \ \ B \ \ B \ \ A \ \ A \ \ B \ \ B$$

In both cases the number of inversions changes by 1. \qed

4. Sums

Here we will study the sum of two trees $T_1$ and $T_2$ (see Definition 1.4).

**Proposition 4.1.** If a tree $T_1$ has an even number of vertices $k$ and a tree $T_2$ also has an even number of vertices $l$, then their sum is odd only when one of summands is odd and another — even.

*Proof.* Let $T$ be the sum of $T_1$ and $T_2$ and let $e$ be the edge, connecting $T_1$ and $T_2$. $T$ has $k + l$ vertices — an even number, hence, by Proposition 2.2 we can choose any vertex as the root vertex. So the root vertex $O$ will be the vertex of $T_1$, incident $e$. By Proposition 2.1 any edge, incident to $O$, can be chosen as a marked edge, so the marked edge will be the next after $e$ in going counterclockwise around $O$ (i.e. in the walk around $T$ we at first make the walk around $T_1$).

The number of inversions is $s + t + kl$, where $s$ and $t$ are number of inversions in bracket codes of $T_1$ and $T_2$, respectively. It remains to note that $kl$ is even. \qed

5. Rotation group

Here we will consider trees with an even number of vertices $n$. $R(T)$ will be the rotation group of a tree $T$ (see Definition 1.5).

**Proposition 5.1.** Even and odd trees with $n$ vertices have different rotation groups.

*Proof.* If $T$ is a tree with $n$ vertices, then it has an odd number of edges $m = n - 1$. Let us consider any bipartite structure on $T$ (there are two of them). The permutation $t = s_w s_b$ is an $m$-cycle, hence it is an even permutation, thus, permutations $s_w$ and $s_b$ are both odd or both even. Let $U$ be a tree, obtained from $T$ by one transposition: a vertex of degree 1 with attached edge moves from white vertex to the nearest black one. The trees $T$ and $U$ have different parities by Proposition 3.1. After the transposition the number of even-length cycles in the cyclic presentation of $s_w$ changes by one and the number of even-length cycles in the cyclic presentation of $s_b$ is also changes by one. Thus, rotations groups of trees of one parity belong to the alternating group $A_m$ and rotation groups of trees of another parity do not belong to $A_m$. \qed
Conclusion If \( n \equiv 2 \mod 4 \), then rotation groups of even trees with \( n \) vertices belong to the alternating group. If \( n \equiv 0 \mod 4 \), then rotation groups of odd trees belong to the alternating group.

Proof. Let consider the chain-tree (see Figure 4) with \( n \) vertices. This tree is even. It has one white vertex of degree one, one black vertex of degree one, \((n-2)/2\) white vertices of degree 2 and \((n-2)/2\) black vertices of degree 2. Permutations \( s_w \) and \( s_b \) are even, if the number \((n-2)/2\) is even. It remains to note that By Remark 1.4 any tree can be transformed in the chain tree by a sequence of transpositions. □

Remark 5.1. Let \( T \) and \( U \) be trees with \( n \) vertices, one even and one odd. Then their passports as bipartite trees are different. Indeed, one tree has even number of white vertices of even degrees and even number of black vertices of even degrees and another has odd number of white vertices of even degrees and odd number of black vertices of even degrees.

6. Clean trees

Proposition 6.1. If two trees \( T \) and \( U \) with an even number \( n \) of vertices have different parities, then rotation groups of trees \( T^c \) and \( U^c \) are different.

Proof. Let us consider some enumeration of \( T^c \) edges. As the permutation \( t = s_w s_b \) is a \((2n-2)\)-cycle, then it is odd. As \( s_b \) is a product of \( n-1 \) — and odd number, of 2-cycles, then it is odd also, thus \( s_w \) is even.

The tree \( T \) has its own bipartite structure with black and white vertices (in \( T^c \) all these vertices are white). We will call them pw-vertices (previously white) and pb-vertices (previously black). Each edge of \( T \) is divided by black vertex in two edges of \( T^c \). Thus, half of \( T^c \) edges are incident to pw-vertices and half — to pb-vertices. Let \( N_w \) be the set of \( T^c \) edges, incident to pw-vertices, and \( N_b \) be the set of \( T^c \) edges, incident to pb-vertices. As the permutation \( s_w \) permutes separately elements of \( N_w \) and \( N_b \), then it generates two permutations: the permutation \( \sigma_w \) of the set \( N_w \) and the permutation \( \sigma_b \) of the set \( N_b \). As \( s_w = \sigma_w \sigma_b \), then permutations \( \sigma_w \) and \( \sigma_b \) are both even or both odd. A transposition on \( T \) changes the degree of a pw-vertex by one and the degree of a pb-vertex by one also, thus, it simultaneously changes parities of \( \sigma_w \) and \( \sigma_b \).

Let \( A_w (A_b) \) be the alternating group of permutations of the set \( N_w \) (\( N_b \)) and let \( G = A_w A_b \). Then \( s_w \) belongs to \( G \) for trees of one parity and does not belong to \( G \) for trees of another. As \( e \in N_w \) only when \( t(e) \in N_b \), then \( t^2 \in G \). As \( R(T^c) = \langle t, s_w \rangle \), then \( R(T^c) \subset tG \cup G \) for trees of one parity and \( R(T^c) \not\subset tG \cup G \) for trees of another. It remains to note that this result does not depend on the enumeration of \( T^c \) edges. □

Supplement 1. Characteristic polynomial

One can ask a natural question: can the spectrum of an odd tree be the same as the spectrum of an even tree? Two non-isomorphic trees with the same characteristic polynomial are called cospectral (see [2]).
We can easily find two cospectral trees with characteristic polynomial $x^8 - 7x^6 + 9x^4$

but both of them are even.

However, there exit two cospectral trees with twelve vertices with characteristic polynomial $x^{12} - 11x^{10} + 42x^8 - 66x^6 + 39x^4 - 6x^2$.

The tree on the left is even and the tree on the right is odd.

Supplement 2. Graph of trees

For an even $n$ we can define the graph $G_n$. The set of vertices of $G_n$ is the set of all pairwise non-isomorphic trees with $n$ vertices. Two vertices of $G_n$ are adjacent if a transposition transforms one of the corresponding trees into another. $G_n$ will be called the graph of trees. By Proposition 3.1 $G_n$ is bipartite.

For $n = 6, 8$ we can construct this graphs. The graph $G_6$ is quite simple.

The graph $G_8$ has 23 vertices and 37 edges. All pairwise non-isomorphic trees are presented below.
It turns out that the graph $G_8$ is planar:

However, the graph $G_{10}$ with 106 vertices and 238 edges is not planar. In order to demonstrate its non-planarity we will construct a subgraph, homeomorphic to the graph $K_{3,3}$ (see [3]), i.e. to the bipartite graph

1. Let us consider trees $a$, $b$ and $c$ up to isomorphism

   $a$: \[
   \begin{array}{c}
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \end{array}
   \]

   $b$: \[
   \begin{array}{c}
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \end{array}
   \]

   $c$: \[
   \begin{array}{c}
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \end{array}
   \]

Each of these trees can be transformed by one transposition into the tree $d$

   $d$: \[
   \begin{array}{c}
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \end{array}
   \]

Trees $a$ and $b$ can be transformed by one transposition into the tree $e$

   $e$: \[
   \begin{array}{c}
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \end{array}
   \]

but it takes three transpositions to transform the tree $c$ into $e$:

   \[
   \begin{array}{c}
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \end{array}
   \Rightarrow
   \begin{array}{c}
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \end{array}
   \Rightarrow
   \begin{array}{c}
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \end{array}
   \Rightarrow
   \begin{array}{c}
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \, \\
   \end{array}
   \]
The tree $a$ can be transformed by one transposition into the tree $f$:

$$f: \begin{array}{c}
\end{array}$$

It takes three transpositions to transform the tree $c$ into the tree $f$:

$$\begin{array}{c}
\end{array} \Rightarrow \begin{array}{c}
\end{array} \Rightarrow \begin{array}{c}
\end{array} \Rightarrow \begin{array}{c}
\end{array}$$

And it takes nine transpositions to transform the tree $b$ into the tree $f$:

$$\begin{array}{c}
\end{array} \Rightarrow \begin{array}{c}
\end{array} \Rightarrow \begin{array}{c}
\end{array} \Rightarrow \begin{array}{c}
\end{array} \Rightarrow \begin{array}{c}
\end{array} \Downarrow$$

$$\begin{array}{c}
\end{array} \Leftarrow \begin{array}{c}
\end{array} \Leftarrow \begin{array}{c}
\end{array} \Leftarrow \begin{array}{c}
\end{array} \Leftarrow \begin{array}{c}
\end{array}$$

So the subgraph of $G_{10}$ which has all above trees as vertices and whose edges correspond to above mentioned transpositions is homeomorphic to $K_{3,3}$.

**References**

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