Stable Separation and Super-Resolution of Mixture Models

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Abstract

We consider simultaneously identifying the membership and locations of point sources that are convolved with different band-limited point spread functions, from the observation of their superpositions. This problem arises in three-dimensional super-resolution single-molecule imaging, neural spike sorting, multi-user channel identification, among other applications. We propose a novel algorithm, based on convex programming, and establish its near-optimal performance guarantee for exact recovery in the noise-free setting by exploiting the spectral sparsity of the point source models as well as the incoherence between point spread functions. Furthermore, robustness of the recovery algorithm in the presence of bounded noise is also established. Numerical examples are provided to demonstrate the effectiveness of the proposed approach.

Keywords: super-resolution, parameter estimation, atomic norm minimization, mixture models

1 Introduction

In many emerging applications in applied science and engineering, the acquired signal at the sensor can be regarded as a noisy superposition of returns from multiple modalities, where the return from each modality is a band-limited observation of a point source signal captured through a low-pass point spread function, governed by either the underlying physical field or the system design. Mathematically, we consider the following parametric mixture model of the acquired signal, $y(t)$, given as

$$y(t) = \sum_{i=1}^{I} x_i(t) * g_i(t) + w(t) = \sum_{i=1}^{I} \left( \sum_{k=1}^{K_i} a_{ik} g_i(t - \tau_{ik}) \right) + w(t),$$  \hspace{1cm} (1)

where $*$ denotes the convolution operator, $w(t)$ is an additive noise, and $I$ is the total number of modalities. Moreover,

$$x_i(t) = \sum_{k=1}^{K_i} a_{ik} \delta(t - \tau_{ik})$$

is the point source signal observed from the $i$th modality, and $g_i(t)$ is the corresponding point spread function. For the $i$th modality, let $\tau_{ik} \in [0,1)$ and $a_{ik} \in \mathbb{C}$ be the location and the amplitude of the $k$th point source, $1 \leq k \leq K_i$, respectively, where the locations of point sources $\tau_{ik}$’s are continuous-valued and can lie anywhere in the parameter space, at nature’s will. The point source model can be used to model a variety of physical phenomena occurring in a wide range of practical problems, such as the activation pattern of fluorescence in single-molecule imaging [1], sparse channel impulse response in multi-path fading environments, the locations of pollution plants in urban areas, firing times of neurons, and many more.

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Our goal is to stably invert for the field parameters, i.e. the parameters of the point source models, of each modality from the acquired signal reflecting the ensemble behavior of all modalities, even in the presence of noise. This allows us to separate the contributions of each modality to the acquired signal. Moreover, typically we are interested in super resolution, i.e. resolving the parameters at a resolution much higher than the native resolution of the acquired signal, determined by the Rayleigh limit, or in other words, the reciprocal of the bandwidth of the point spread functions.

1.1 Motivating Applications

The mixture model (1) is motivated by the modeling and analysis of many practical problems, such as three-dimensional super-resolution single-molecule imaging [2, 3], spike sorting in neural recording [4, 5], multi-user multi-path channel identification [6, 7], and blind calibration of time-interleaved analog-to-digital converters [8, 9]. We describe several example applications below.

Three-dimensional super-resolution single-molecule imaging: By employing photoswitchable fluorescent molecules, the imaging process of single-molecule microscopies (Stochastic Optical Reconstruction Microscopy (STORM) [1] or Photo Activated Localization Microscopy (PALM) [10]) is divided into many frames, where in each frame, a sparse number of fluorophores (point sources) are randomly activated, localized at a resolution below the diffraction limit, and deactivated. The final image is thus obtained by superimposing the localization outcomes of all the frames. This principle can be extended to reconstruct a 3-D object from 2-D image frames, for example, by introducing a cylindrical lens to modulate the ellipticity of the point spread function based on the depth of the fluorescent object in 3-D STORM [2]. Therefore, the acquired image in each frame can be regarded as a superposition of returns from multiple depth layers, where the return from each layer corresponds to the convolution outcome of the fluorophores in that depth layer with the depth-dependent point spread function, as modeled in (1). The goal is thus to recover the locations and depth membership of each point source given the image frame.

Spike sorting for neural recording: Neurons in the brain communicate by firing action potentials, i.e. spikes, and it is possible to capture their communications through the use of a microelectrode, which records simultaneous activities of multiple neurons within a local neighborhood. Spike sorting [11], thus, refers to the grouping of spikes according to each neuron, from the recording of the microelectrode. Interestingly, it is possible to model the spike fired by each neuron with a characteristic shape [12]. The neural recording can thus be modeled as a superposition of returns from multiple neurons, as in (1), where the return from each neuron corresponds to the convolution of its characteristic spike shape with the sequence of its firing times. A similar problem also arises in DNA sequencing, please refer to [13].

Multi-path identification in random-access channels: In multi-user multiple access model [7], each active user transmits a signature waveform modulated via a signature sequence, which can be designed to optimize performance and the base station receives a superposition of returns from active users, as in (1), where the received signal from each active user corresponds to the convolution of its signature waveform with the unknown sparse multi-path channel from the user to the base station. The goal is to identify the set of active users, as well as their channel states, from the received signal at the base station.

1.2 Related Work and Our Contributions

There is an extensive research literature [14] on inverting (1) when there is only a single modality with \( I = 1 \), where conventional approaches for parameter estimation such as matched filtering, MUSIC [15], matrix pencil [16], to more recent approaches based on the trigonometric polynomial frame [17] or total variation minimization [18], can be applied. However, these approaches can not be applied directly when multiple modalities exist in the observed signal, due to the mutual interference. To the best of the authors’ knowledge, methods for inverting (1) with multiple modalities have been extremely limited. Sparse recovery algorithms have been proposed to estimate the mixture model in [19, 6, 7] with a discretized set of delays, but the performance may degenerate when the actual delays do not belong to the discrete grid [20]. Even when all the point sources indeed lie on the grid, existing work suggests that the sample complexity, or the bandwidth of the acquire signal, may have to grow logarithmically with the size of the grid, which is undesirable. More recently, [4, 5] have proposed heuristic sparse recovery algorithms to estimate the continuous-valued delays in
the mixture model for spike sorting, however no performance guarantees are available. Finally, an algebraic approach has been proposed in [8], but it is sensitive to noise due to the nature of the employed root-finding procedure and does not extend well to a large number of modalities due to the prohibitive sample complexity.

In this paper, we study the problem of super-resolving the mixture model (1) when there are two modalities, i.e. $I = 2$. The methodology in this paper can be extended straightforwardly to the analysis of the case $I > 2$ and is left for future work. We start by recognizing that in the Fourier domain, the observed signal can be regarded as a linear combination of two spectrally-sparse signals, each composed of a small number of distinct complex sinusoids. The atomic norm [21, 22] of spectrally-sparse signals is developed and proposed as an efficient convex optimization framework to motivate parsimonious structures [18, 21, 22, 23, 24] recently, which can be computed efficiently via semidefinite programming. We then separate and recover the two signals by motivating their spectral structures using atomic norm minimization, in addition to satisfying the observation constraints. The proposed algorithm, denoted by AtomicDemix, is reminiscent of the algorithms for sparse error correction [25], robust principal component analysis [26], demixing of sines and spikes [27, 24], and source separation [28], where one aims to separate two low-dimensional signals with incoherent structures via convex optimization.

The separation and identification of the two point source signals, using the proposed AtomicDemix algorithm, is made possible with two additional natural conditions. The first condition is that the point source signal of each modality satisfies a mild separation condition, such that the locations of the point sources are separated by at least four times the Rayleigh limit; this is the same separation condition required by Candès and Fernandez-Granda [18] even with $I = 1$ when applying total variation minimization for super-resolution. The second condition is that the point spread functions of different modalities have to be sufficiently incoherent, which is supplied in our theoretical analysis by assuming they are randomly generated from a uniform distribution on the complex unit circle. Define $K_{\max} = \max\{K_1, K_2\}$. Our main results are summarized as below:

- For the noise-free case, we demonstrate that, provided that the coefficients of the point sources have symmetric random signs, that is to say the signs of the coefficients of the point sources are randomly generated from a symmetric distribution on the complex unit circle, as soon as the number of measurements $M$, or equivalently, the bandwidth of the point spread functions, is on the order $M/\log M = O(K_{\max} \log(K_1 + K_2))$, AtomicDemix exactly recovers the point source model of each modality with high probability. Since at least an order of $O(K_1 + K_2)$ measurements is necessary, our sample complexity is near-optimal up to logarithmic factors. When the coefficients of the point sources have arbitrary signs, we establish a similar performance guarantee with a higher sample complexity, on the order of $M = O(K_{\max}^2 \log(K_1 + K_2))$.

- For the noisy case, when the coefficients of the point sources have arbitrary signs, under same conditions that guarantee exact recovery in the noise-free case, we establish that AtomicDemix is stable in the presence of possibly adversarial bounded noise.

- The point sources of each modality can be localized from the dual solution of the proposed algorithms, without estimating or knowing the model order a priori. Numerical examples are provided to corroborate the theoretical analysis, with comparisons against the standard Cramér-Rao Bound (CRB) for parameter estimation.

1.3 Organization and Notations

The rest of this paper is organized as follows. We specify the problem formulation and main results in Section 2. Numerical experiments are provided to corroborate the theoretical analysis in Section 3. Section 4 and Section 5 provide detailed proof procedures of our main results for the noise-free case and the noisy case, respectively. Finally, the paper is concluded in Section 6 with discussions on extensions and future work.

Throughout the paper, $(\cdot)^T$ and $(\cdot)^H$ denote the transpose and Hermitian transpose, respectively, and $(\cdot)$ denotes the (element-wise) conjugate of a complex scalar or vector. For a function $f(\tau)$ with variable $\tau$, we denote its first-order derivative and second-order derivative by $f'(\tau)$ and $f''(\tau)$, respectively. We also use $f^{(l)}(\tau)$ to represent its $l$th-order derivative. The quantity $\sqrt{-1}$ is denoted by $j$. Besides, we use $C$ with different superscripts and subscripts to represent constants, whose values may change from line to line.
2 Problem Formulation and Main Results

2.1 Observation Model

Due to hardware and physical limits, the resolution of the sensor suite is limited by the diffraction limit or Rayleigh limit, which heuristically is often referred to as half the width of the mainlobe of \( g_i(t) \)'s. Alternatively, in the frequency domain, we say \( g_i(t) \)'s are band-limited with cut-off frequency \( 2M \). Denote the discrete-time Fourier transform of \( g_i(t) \) as

\[
g_{i,n} = \int_{-\infty}^{\infty} g_i(t) e^{-j2\pi nt} dt, \tag{2}
\]

then \( g_{i,n} = 0 \) whenever \( n \notin \Omega_M = \{-2M, \ldots, 0, \ldots, 2M\} \). Taking the discrete-time Fourier transform of (1), the measurements can be represented as, in the Fourier domain,

\[
y_n = \sum_{i=1}^{I} g_{i,n} \cdot \left( \sum_{k=1}^{K_i} a_{ik} e^{-j2\pi n\tau_{ik}} \right) + w_n, \quad n \in \Omega_M, \tag{3}
\]

where the noise \( w_n \) is

\[
w_n = \int_{-\infty}^{\infty} w(t) e^{-j2\pi nt} dt, \quad n \in \Omega_M.
\]

When \( I = 2 \), the measurements (3) in the Fourier domain can be equivalently formulated as

\[
y_n = g_{1,n} \cdot \left( \sum_{k=1}^{K_1} a_{1k} e^{-j2\pi n\tau_{1k}} \right) + g_{2,n} \cdot \left( \sum_{k=1}^{K_2} a_{2k} e^{-j2\pi n\tau_{2k}} \right) + w_n, \quad n \in \Omega_M. \tag{4}
\]

The measurements \( y_n \)'s in (4) can be considered as a linear combination of two spectrally-sparse signals, with \( g_{i,n} \)'s determining the combination coefficients. In vector form, we have

\[
y = g_1 \odot x_1^* + g_2 \odot x_2^* + w, \tag{5}
\]

where \( y = [y_{-2M}, \ldots, y_0, \ldots, y_{2M}]^T \), \( w = [w_{-2M}, \ldots, w_0, \ldots, w_{2M}]^T \), \( g_i = [g_{i,-2M}, \ldots, g_{i,0}, \ldots, g_{i,2M}]^T \) for \( i = 1, 2 \), and \( \odot \) denotes the Hadamard element-wise product operator. Furthermore, let \( x_1^* \in \mathbb{C}^{4M+1} \) and \( x_2^* \in \mathbb{C}^{4M+1} \) denote two spectrally-sparse signals, each composed of a small number of distinct complex harmonics, represented as

\[
x_1^* = \sum_{k=1}^{K_1} a_{1k} c(\tau_{1k}), \quad \text{and} \quad x_2^* = \sum_{k=1}^{K_2} a_{2k} c(\tau_{2k}), \tag{6}
\]

where \( K_1 \) is the spectral sparsity of \( x_1^* \) and \( K_2 \) is the spectral sparsity of \( x_2^* \). The atom \( c(\tau) \) is defined as

\[
c(\tau) = \left[ e^{-j2\pi(-2M)\tau}, \ldots, 1, \ldots, e^{-j2\pi(2M)\tau} \right]^T,
\]

which corresponds to a point source at the location \( \tau \in [0, 1) \). Further denote the location set of point sources in \( x_1^* \) and \( x_2^* \) by \( \Upsilon_1 = \{\tau_{11}, \ldots, \tau_{1K_1}\} \) and \( \Upsilon_2 = \{\tau_{21}, \ldots, \tau_{2K_2}\} \), respectively. The goal is thus to recover \( \Upsilon_1 \) and \( \Upsilon_2 \), and their corresponding amplitudes, from the observation (5).

Intuitively, it is impossible to separate the two modalities if \( g_1 \) and \( g_2 \) are highly coherent. In this paper, we assume the entries of the point spread functions \( g_{i,n} \)'s are i.i.d. generated from a uniform distribution on the complex unit circle. This randomness assumption is reasonable when \( g_{i,n} \)'s can be designed, such as the spreading sequences in multi-user communications, and provides the incoherence between different modalities that is necessary for separation. Multiplying both sides of (4) with \( \bar{g}_{1,n} \), and with slight abuse of notation, (5) can be rewritten as

\[
y = x_1^* + g \odot x_2^* + w \in \mathbb{C}^{4M+1}, \tag{7}
\]

where \( g = [g_{-2M}, \ldots, g_0, \ldots, g_{2M}]^T \in \mathbb{C}^{4M+1} \) with \( g_n = g_{2,n}\bar{g}_{1,n} \) uniformly drawn from the unit complex circle. In the noisy case, we consider the scenario where \( w \) is bounded as \( \|w\|_2^2 \leq \sigma_w^2 \).
\[ \|x\|_A = \inf_{a_k \in C, \tau_k \in [0,1]} \left\{ \sum_k |a_k| \left| x = \sum_k a_k c(\tau_k) \right| \right\}, \]

which can be regarded as the tightest convex relaxation of counting the smallest number of atoms \( c(\tau) \) that is needed to represent a signal \( x \). Therefore, we seek to recover the signals \( x_1 \) and \( x_2 \) by motivating their spectral sparsity via minimizing the sum of their atomic norms, with respect to the observation constraint in the noise-free case where \( w = 0 \):

\[ \{ \hat{x}_1, \hat{x}_2 \} = \arg\min_{x_1, x_2} \| x_1 \|_A + \| x_2 \|_A, \quad \text{s.t.} \quad y = x_1 + g \odot x_2. \] (8)

In the noisy case, we propose a regularized atomic norm minimization algorithm as

\[ \{ \hat{x}_1, \hat{x}_2 \} = \arg\min_{x_1, x_2} \frac{1}{2} \| y - x_1 - g \odot x_2 \|_2^2 + \lambda_w (\| x_1 \|_A + \| x_2 \|_A), \] (9)

where \( \lambda_w \) is the regularization parameter to balance the data fitting term and the structural promoting term, to be determined later. The above algorithms are referred to as AtomicDemix. Interestingly, the atomic norm \( \| x \|_A \) can be equivalently characterized via semidefinite programming [23], therefore the proposed algorithms can be solved efficiently using off-the-shelf solvers.

### 2.3 Performance Guarantee in the Noise-free Case

Recall \( K_{\text{max}} = \max\{K_1, K_2\} \). Define the separation of the point source signal of the \( i \)th modality as

\[ \Delta_i = \min_{k \neq t} |\tau_{ik} - \tau_{it}|, \] (10)

which is understood as the wrapped-around distance on \([0,1)\), and the minimum separation of the point source signals of all modalities as \( \Delta = \min_i \Delta_i \). We have the following performance guarantee for the noise-free algorithm (8), whose proof is provided in Section 4.

**Theorem 2.1** (Noise-free Case). Assume that \( g_n = e^{2\pi \phi_n} \)’s are i.i.d. randomly generated from a uniform distribution on the complex unit circle with \( \phi_n \sim U(0,1) \), and that the minimum separation satisfies \( \Delta \geq 1/M \). Let \( \eta \in (0,1) \), then there exists a numerical constant \( C \) such that

\[ M \geq C \max \left\{ \log^2 \left( \frac{M(K_1 + K_2)}{\eta} \right), K_{\text{max}} \log \left( \frac{M(K_1 + K_2)}{\eta} \right), K_{\text{max}}^2 \log \left( \frac{K_1 + K_2}{\eta} \right) \right\} \] (11)

is sufficient to guarantee that \( x_1^* \) and \( x_2^* \) are the unique solutions of (8) with probability at least \( 1 - \eta \).

Moreover, if the signs of the coefficients \( a_{ik} \)’s are i.i.d. generated from a symmetric distribution on the complex unit circle, there exists a numerical constant \( C \) such that

\[ M \geq C \max \left\{ \log^2 \left( \frac{M(K_1 + K_2)}{\eta} \right), K_{\text{max}} \log \left( \frac{K_1 + K_2}{\eta} \right) \log \left( \frac{M(K_1 + K_2)}{\eta} \right) \right\} \] (12)

is sufficient to guarantee that \( x_1^* \) and \( x_2^* \) are the unique solutions of (8) with probability at least \( 1 - \eta \).

Theorem 2.1 provides two sample complexities depending on whether the signs of the coefficients \( a_{ik} \)’s are random. Given random signs of \( a_{ik} \)’s, Theorem 2.1 indicates that as soon as the number of measurements \( M \) is on the order \( M/\log M = O(K_{\text{max}} \log(K_1 + K_2)) \), AtomicDemix exactly recovers the point source models with high probability. This suggests that the performance of AtomicDemix is near-optimal in terms of the sample complexity as at least \( O(K_1 + K_2) \) measurements are necessary to identify the unknown parameters. Without requiring random signs of \( a_{ik} \)’s, the sample complexity is slightly higher, roughly dominated by the last term on the order of \( M = O(K_{\text{max}}^2 \log(K_1 + K_2)) \).
Remark 1. The separation condition \( \Delta \geq 1/M \) is a sufficient condition in Theorem 2.1 to guarantee accurate signal demixing, which is the same as the one required by Candès and Fernandez-Granda in [18] even with \( I = 1 \). Our results suggest that the separation condition to achieve super resolution in mixture models is no stronger than that required even in the single modality case, provided the point spread functions are incoherent enough. It is implied in [18, 29] that a reasonable separation is also necessary to guarantee stable super-resolution. Interestingly, no separation between point sources from different modalities is required, as long as their point spread functions are incoherent enough.

Remark 2. Theorem 2.1 assumes \( g_n \)'s are i.i.d. from a uniform distribution on the complex unit circle, which may be relaxed as long as \( g_n \)'s are independently drawn from a distribution satisfying \( \mathbb{E}[\bar{g}_n] = \mathbb{E}[\bar{g}_n^{-1}] = 0 \) and \( C_1 \leq |g_n| \leq C_2 \) for some constants \( 0 \leq C_1 \leq C_2 \). Both \( \text{sign} (a_{1k}) \) and \( \text{sign} (a_{2k}) \) are assumed randomly generated, which are reasonable in many applications.

Remark 3. Theorem 2.1 can also be extended into multi-dimensional point source models, following similar techniques in [30], where the same order of measurements shall be sufficient to localize the point sources under similar mild separation conditions. We leave this extension to interested readers.

2.4 Performance Guarantee in the Noisy Case

In the presence of bounded noise, AtomicDemix in (9) still stably recovers the point source signals, as established in the following theorem, whose proof is provided in Section 5.

**Theorem 2.2 (Noisy Case).** Let \( \lambda_w = C_w \sigma_w \sqrt{4M + 1} \), for some constant \( C_w > 1 \) large enough. Assume that \( g_n = e^{2\pi \phi_n} \)'s are i.i.d. randomly generated from a uniform distribution on the complex unit circle with \( \phi_n \sim U(0, 1) \), and that the minimum separation satisfies \( \Delta \geq 1/M \). Let \( \eta \in (0, 1) \), then as long as (11) holds for some constant \( C \), the solution to (9) satisfies

\[
\frac{1}{\sqrt{4M + 1}} (\|\hat{x}_1 - x_1^*\|_2 + \|\hat{x}_2 - x_2^*\|_2) \leq C_1 \sigma_w \sqrt{K_{\max}^3 \log M}, \tag{13}
\]

and

\[
\frac{1}{\sqrt{4M + 1}} \| (\hat{x}_1 + g \circ \hat{x}_2) - (x_1^* + g \circ x_2^*) \|_2 \leq C_2 \sigma_w \left( \frac{K_{\max}^3 \log M}{M} \right)^{1/4}, \tag{14}
\]

with probability at least \( 1 - \eta - C_3 (M^3 \log M)^{-1/2} \), where \( C_1, C_2 \) and \( C_3 \) are some constants.

Theorem 2.2 does not make any assumptions on the signs of the coefficients of point sources. It guarantees the stability for inversion in the presence of bounded noise, even when the noise is adversarially generated. When \( \sigma_w = 0 \), Theorem 2.2 degenerates to the noise-free case, providing a performance guarantee of AtomicDemix in accordance with Theorem 2.1 when the point sources have deterministic coefficients. The first bound (13) concerns signal reconstruction, which guarantees that one can stably separate \( \hat{x}_1 \) and \( \hat{x}_2 \) even in the presence of noise. The second bound (14) concerns denoising, which guarantees that AtomicDemix can output a denoised signal \( \hat{y} = \hat{x}_1 + g \circ \hat{x}_2 \) proportional to the noise level.

2.5 Localization via Dual Polynomials

With the demixing results \( \hat{x}_1 \) and \( \hat{x}_2 \), the source locations \( \tau_{ik} \)'s of each signal can be estimated accurately by MUSIC [15], ESPRIT [31], the Prony’s method [32] or other linear prediction methods. More interestingly, the source locations can be identified directly from the dual solutions of (8) and (9). The coefficients \( a_1 \) and \( a_2 \) can then be estimated by least-squares using the estimates of \( \tau_{ik} \)'s.

We first characterize the dual problem of (8) and (9). Define the inner product of two vectors as \( \langle p, x \rangle = x^H p \) and the real-valued inner product as \( \langle p, x \rangle_{\mathbb{R}} = \text{Re} (x^H p) \), where \( \text{Re}(\cdot) \) takes the real part of a complex scaler. The dual norm of \( \| \cdot \|_A \) can be represented as

\[
\| p \|_A = \sup_{\| x \|_A \leq 1} \langle p, x \rangle_{\mathbb{R}} = \sup_{\| x \|_A \leq 1} | \langle p, c(\tau) \rangle | = \sup_{\tau \in [0, 1)} \left| \sum_{n=-2M}^{2M} p_n e^{j2\pi n\tau} \right|,
\]
where \( \mathbf{p} = [p_{-2M}, \ldots, p_0, \ldots, p_{2M}]^T \). Then the dual problem of (8) can be written as
\[
\hat{\mathbf{p}} = \arg\max_{\mathbf{p}} \langle \mathbf{p}, \mathbf{y} \rangle_{\mathbb{R}}, \quad \text{s.t.} \quad \|\mathbf{p}\|_A^* \leq 1, \quad \|\hat{\mathbf{g}} \odot \mathbf{p}\|_A^* \leq 1,
\]
whose derivations can be found in Appendix B. Similarly, by standard Lagrangian calculation the dual problem of (9) can be obtained as
\[
\hat{\mathbf{p}} = \arg\max_{\mathbf{p}} \frac{1}{2} \left( \|\mathbf{y}\|_2^2 - \|\mathbf{y} - \lambda \mathbf{w} \mathbf{p}\|_2^2 \right), \quad \text{s.t.} \quad \|\mathbf{p}\|_A^* \leq 1, \quad \|\hat{\mathbf{g}} \odot \mathbf{p}\|_A^* \leq 1.
\]

Based on the definition of the dual norm, define the dual polynomials \( \hat{P}(\tau) \) and \( \hat{Q}(\tau) \) generated from the dual solutions of (15) or (16) as
\[
\hat{P}(\tau) = \sum_{n=-2M}^{2M} \hat{p}_n e^{j2\pi n \tau}, \quad \hat{Q}(\tau) = \sum_{n=-2M}^{2M} \hat{p}_n \bar{g}_n e^{j2\pi n \tau}.
\]

Then the source locations can be identified as
\[
\hat{\Upsilon}_1 = \{ \tau \in [0, 1) : |\hat{P}(\tau)| = 1 \}, \quad \text{and} \quad \hat{\Upsilon}_2 = \{ \tau \in [0, 1) : |\hat{Q}(\tau)| = 1 \}.
\]

For the noise-free case, it is straightforward to show that \( \Upsilon_1 \subseteq \hat{\Upsilon}_1 \) and \( \Upsilon_2 \subseteq \hat{\Upsilon}_2 \) whenever the optimal primal solution is \( \{x_1^*, x_2^*\} \) in Appendix C. Note however in general both \( \hat{\Upsilon}_1 \) and \( \hat{\Upsilon}_2 \) may contain spurious source locations. Interested readers can refer to relevant discussions in [23, Proposition 2.5] on when the dual polynomials return exact source locations, which also apply to our proposed algorithms with little modifications.

### 3 Numerical Examples

We carry out a series of numerical simulations to validate the performance of AtomicDemix in both noise-free and noisy cases under different parameter settings.

#### 3.1 Phase Transitions in the Noise-free Case

We first examine the phase transition as a function of \((K_1, K_2)\) for a fixed \(M\). We vary the spectral sparsity levels of the two modalities as \(K_1\) and \(K_2\). For each pair of \((K_1, K_2)\), we first randomly generate a pair of point sources \(\Upsilon_1 \) and \(\Upsilon_2\) that satisfy a separation condition \(\Delta \geq 1/(2M)\), with the coefficients of the point sources i.i.d. drawn from the complex standard Gaussian distribution. For each Monte Carlo trial, we then randomly generate the point spread functions \(g_n\)'s in the Fourier domain with i.i.d. entries drawn uniformly from the complex unit circle, and perform AtomicDemix by solving (8) using CVX [33]. The algorithm is considered successful when the normalized estimate error satisfies \(\sum_{i=1}^{2} \|\hat{x}_i - x_i^*\|_2 / \|x_i^*\|_2 \leq 10^{-4}\).

Fig. 1 shows the success rates of AtomicDemix over 20 Monte Carlo trials for each cell, when \(M = 8\) in (a) and \(M = 16\) in (b), respectively. Fig. 2 (a) shows the success rates of AtomicDemix with respect to \(M\) for different values of \(K_1 = K_2\), and Fig. 2 (b) shows the success rates of AtomicDemix with respect to \(K_1 = K_2\) for different values of \(M\).

#### 3.2 Point Source Recovery from Dual Polynomials

As described earlier, the locations of the point sources can be recovered from the dual solutions of the proposed algorithm. Fix \(M = 16\), \(K_1 = 4\) and \(K_2 = 3\). We randomly generate a pair of point sources that satisfy a separation condition \(\Delta \geq 1/(2M)\), with the coefficients of the point sources i.i.d. drawn from the complex standard Gaussian distribution. In the noise-free case, the amplitudes of the dual polynomials \(\hat{P}(\tau)\) and \(\hat{Q}(\tau)\) constructed from the solution of (15) are shown in Fig. 3 (a), superimposed on the ground truth, indicating the accurate recovery of the point sources.
We then consider the noisy case when the noise is composed of i.i.d. complex Gaussian entries $\mathcal{CN}(0,\sigma^2)$, and set $\lambda_w = \sigma \sqrt{(1 + \frac{1}{\log (4M + 1)})(4M + 1)(\log \alpha + \sqrt{2\log \alpha + 2 + \sqrt{2}})}$, where $\alpha = 8\pi(4M + 1)\log (4M + 1)$ based on the discussions in [34, 35] or $\lambda_w = \sigma \sqrt{4M + 1} \sqrt{1.2 \log (8\pi (4M + 1) \log (4M + 1))}$ for simplicity of use. The amplitudes of the dual polynomials $\hat{P}(\tau)$ and $\hat{Q}(\tau)$ are shown in Fig. 3 (b) and (c) for SNR = 16 dB and SNR = 5dB, respectively, where the Signal-to-Noise Ratio (SNR) is defined as $\text{SNR} = 10\log_{10} \left( \frac{\|x_1^\star g \circ x_2^\star\|_2^2}{\sigma^2} / \right) \text{dB}$. It is clear that the source locations can be estimated stably from the dual solutions, and the performance degenerates gracefully with the increase of the noise level.

### 3.3 Comparisons with CRB for Point Source Localization

We further examine the performance of (9) on estimating the locations of the point sources from noisy measurements by comparing it against the CRB. Specifically, consider the special case with a single point source for each modality, by letting $K_1 = K_2 = 1$. Denote the point source location in $x_1^\star$ and $x_2^\star$ by $\tau_1$ and $\tau_2$, respectively. We assume the corresponding amplitude of each point source is known and unity when
computing the CRB for estimating $\tau_1$ and $\tau_2$, which can be found as the diagonal entries of the inverse of the following Fisher information matrix:

\[ J(\tau_1, \tau_2) = \frac{8\pi^2}{\sigma^2} \left[ \frac{\sum_{n=-2M}^{2M} n^2}{\sum_{n=-2M}^{2M} n^2} Re \left( \sum_{n=-2M}^{2M} n^2 g_n e^{-j2\pi n(\tau_1-\tau_2)} \right) \right]. \]

For each SNR, we randomly generate 200 noise realizations and compute the average squared estimate error $(\hat{\tau}_i - \tau_i)^2$, where $\hat{\tau}_i$ is the dual solution of (9), $i = 1, 2$. Fig. 4 shows the average squared estimate error in comparison with the CRB with respect to SNR when $M = 10$ in (a) and $M = 16$ in (b). The performance of parameter estimation shows a similar “thresholding effect” [36] as for conventional spectrum estimation algorithms, where the average squared estimate error approaches the CRB as soon as SNR is large enough. Moreover, as we increase $M$, the threshold SNR becomes smaller. Characterizing the exact threshold SNR for AtomicDemix is an interesting future research topic.

Figure 3. Point source localization from dual polynomials (a) in absence of noise, (b) SNR = 16dB, and (c) SNR = 5dB, for $M = 16$, $K_1 = 4$ and $K_2 = 3$. 
4 Proof of Theorem 2.1

In this section, we proceed to prove Theorem 2.1. We first provide the optimality conditions using dual polynomials to certify the optimality of the solution of (8). Illuminated by [18, 23], where the dual polynomial is constructed using the squared Fejér’s kernel, we propose a construction of dual polynomials which are composed of a deterministic term and a random perturbation term induced by the interference between modalities. Finally, we show that the constructed dual polynomials satisfy the optimality conditions with high probability when the sample complexity $M$ is large enough.

4.1 Optimality Conditions using Dual Polynomials

We first certify the optimality of the primal problem (8) using the following proposition whose proof is in Appendix D.

**Proposition 1.** $(x^*_1, x^*_2)$ is the unique optimizer of (8) if there exists a vector $p = [p_{-2M}, \ldots, p_0, \ldots, p_{2M}]^T$ such that the dual polynomials $P(\tau)$ and $Q(\tau)$ constructed from it, represented as

$$
P(\tau) = \sum_{n=-2M}^{2M} p_n e^{j2\pi n \tau}, \quad Q(\tau) = \sum_{n=-2M}^{2M} p_n \tilde{g}_n e^{j2\pi n \tau}$$

(17)

satisfy

$$
\begin{align*}
P(\tau_k) &= \text{sign}(a_{1k}), \quad \forall \tau_k \in \Upsilon_1 \\
|P(\tau)| &< 1, \quad \forall \tau \notin \Upsilon_1 \\
Q(\tau_k) &= \text{sign}(a_{2k}), \quad \forall \tau_k \in \Upsilon_2 \\
|Q(\tau)| &< 1, \quad \forall \tau \notin \Upsilon_2
\end{align*}
$$

(18)

where the sign should be understood as the complex sign.

4.2 Constructing the Dual Certificate

Proposition 1 suggests that if we can find a vector $p$ to construct two dual polynomials $P(\tau)$ and $Q(\tau)$ in (17) that satisfy (18), AtomicDemix is guaranteed to recover the ground truth. Our construction is inspired...
by [18, 23], based on use of the squared Fejér’s kernel. However, since the two dual polynomials are coupled together, the construction is more involved.

Define the squared Fejér’s kernel [18] as

\[ K(\tau) = \frac{1}{M} \sum_{n=-2M}^{2M} s_n e^{i2\pi n\tau}, \]

where \( s_n = \frac{1}{4M} \sum_{i=\max\{-n+M,M\}}^{\min\{n+M,M\}} (1 - |\frac{i}{M}|) \left( 1 - |\frac{i}{M} - \frac{1}{2}| \right) \) with \(|s_n| \leq 1\). The value of \( K(\tau) \) is nonnegative, attaining the peak at \( \tau = 0 \) and decaying to zero rapidly with the increase of \(|\tau|\).

We define two functions \( K_\beta(\tau) \) and \( K_\gamma(\tau) \) respectively as

\[ K_\beta(\tau) = \frac{1}{M} \sum_{n=-2M}^{2M} s_n g_n e^{i2\pi n\tau}, \quad \text{and} \quad K_\gamma(\tau) = \frac{1}{M} \sum_{n=-2M}^{2M} s_n g_n e^{i2\pi n\tau}. \]

We then construct two polynomials \( P(\tau) \) and \( Q(\tau) \) as

\[ P(\tau) = \sum_{k=1}^{K_1} \alpha_{1k} K(\tau - \tau_{1k}) + \sum_{k=1}^{K_1} \beta_{1k} K'(\tau - \tau_{1k}) + \sum_{k=1}^{K_2} \alpha_{2k} K_\gamma(\tau - \tau_{2k}) + \sum_{k=1}^{K_2} \beta_{2k} K_\beta'(\tau - \tau_{2k}), \]

and

\[ Q(\tau) = \sum_{k=1}^{K_1} \alpha_{1k} K_\beta(\tau - \tau_{1k}) + \sum_{k=1}^{K_1} \beta_{1k} K_\beta'(\tau - \tau_{1k}) + \sum_{k=1}^{K_2} \alpha_{2k} K(\tau - \tau_{2k}) + \sum_{k=1}^{K_2} \beta_{2k} K'(\tau - \tau_{2k}), \]

where \( \tau_{1k} \in \mathcal{Y}_1 \) and \( \tau_{2k} \in \mathcal{Y}_2 \). It is straightforward to validate that there exists a corresponding vector \( \mathbf{p} \) such that (21) and (22) can be equivalently written in the form of (17). Set the coefficients \( \alpha_i = [\alpha_{i1}, \ldots, \alpha_{iK_i}]^T \), \( \beta_i = [\beta_{i1}, \ldots, \beta_{iK_i}]^T \), for \( i = 1, 2 \) by solving the following equations:

\[
\begin{align*}
P(\tau_{1k}) &= \text{sign}(a_{1k}), & \tau_{1k} &\in \mathcal{Y}_1, \\
P'(\tau_{1k}) &= 0, & \tau_{1k} &\in \mathcal{Y}_1, \\
Q(\tau_{2k}) &= \text{sign}(a_{2k}), & \tau_{2k} &\in \mathcal{Y}_2, \\
Q'(\tau_{2k}) &= 0, & \tau_{2k} &\in \mathcal{Y}_2.
\end{align*}
\]

The above setting, if exists, immediately satisfies the first and third conditions in (18). The rest of the proof is then to, under the condition of Theorem 2.1, guarantee that a solution of (23) exists with high probability, and moreover, when existing, the solution satisfies the second and forth conditions in (18) with high probability, therefore completing the proof.

**Example 1.** Before proceeding, we demonstrate the above dual polynomial construction by an example. Set \( M = 32 \). Let \( K_1 = 4 \) and \( K_2 = 6 \). We randomly generate the source locations \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \) each satisfying the separation \( \Delta \geq 1/M \). The amplitudes of the constructed \( P(\tau) \) and \( Q(\tau) \) are shown in Fig. 5, which indeed satisfy all the conditions in (18).

### 4.3 Invertibility of (23)

We first show that the solution of (23) exists with high probability in this subsection. Let

\[ u_i = [\text{sign}(a_{i1}), \ldots, \text{sign}(a_{iK_i})]^T, \]

for \( i = 1, 2 \). Rewrite (23) into a matrix form as

\[
\begin{bmatrix}
\begin{bmatrix}
W_{10} & W_{11} & W_{12} & W_{13} \\
\frac{1}{\sqrt{|K''(0)|}} & \frac{1}{\sqrt{|K''(0)|}} & \frac{1}{\sqrt{|K''(0)|}} & \frac{1}{\sqrt{|K''(0)|}} \\
W_{90} & W_{91} & W_{92} & W_{93} \\
\frac{1}{\sqrt{|K''(0)|}} & \frac{1}{\sqrt{|K''(0)|}} & \frac{1}{\sqrt{|K''(0)|}} & \frac{1}{\sqrt{|K''(0)|}} \\
W_{50} & W_{51} & W_{52} & W_{53} \\
\frac{1}{\sqrt{|K''(0)|}} & \frac{1}{\sqrt{|K''(0)|}} & \frac{1}{\sqrt{|K''(0)|}} & \frac{1}{\sqrt{|K''(0)|}} \\
W_{31} & W_{32} & W_{33} & W_{34} \\
\frac{1}{\sqrt{|K''(0)|}} & \frac{1}{\sqrt{|K''(0)|}} & \frac{1}{\sqrt{|K''(0)|}} & \frac{1}{\sqrt{|K''(0)|}} \\
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\beta_1 \\
\beta_2 \\
\end{bmatrix}
= \begin{bmatrix}
u_1 \\
u_2 \\
0 \\
0 \\
\end{bmatrix},
\end{bmatrix}
\]

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Figure 5. The absolute values of the constructed dual polynomials $|P(\tau)|$ and $|Q(\tau)|$ following (23) with respect to $\tau \in [0, 1)$.

where $K''(0)$ is a scaler, defined as

$$K''(0) = -\frac{4}{3}\pi^2 (M^2 - 1).$$

The entries of $W_{1i} \in \mathbb{C}^{K_i \times K_1}$, $W_{gi} \in \mathbb{C}^{K_i \times K_2}$, $W_{g1} \in \mathbb{C}^{K_2 \times K_1}$, and $W_{2i} \in \mathbb{C}^{K_2 \times K_2}$, $i = 0, 1, 2$, are specified respectively as

$$W_{1i}(l, k) = K^{(i)}(\tau_{1l} - \tau_{1k}), \quad W_{gi}(l, k) = K^{(i)}_g(\tau_{1l} - \tau_{2k}),\quad W_{g1}(l, k) = K^{(i)}_g(\tau_{2l} - \tau_{1k}), \quad W_{2i}(l, k) = K^{(i)}(\tau_{2l} - \tau_{2k}).$$

For simplicity, we further introduce the following notations:

$$W_1 = \begin{bmatrix} W_{10} & \frac{1}{\sqrt{|K''(0)|}}W_{11} \\ -\frac{1}{\sqrt{|K''(0)|}}W_{11} & W_{12} \end{bmatrix}, \quad W_g = \begin{bmatrix} W_{g0} & \frac{1}{\sqrt{|K''(0)|}}W_{g1} \\ -\frac{1}{\sqrt{|K''(0)|}}W_{g1} & W_{g2} \end{bmatrix}, \quad W_2 = \begin{bmatrix} W_{20} & \frac{1}{\sqrt{|K''(0)|}}W_{21} \\ -\frac{1}{\sqrt{|K''(0)|}}W_{21} & W_{22} \end{bmatrix},$$

and $W = \begin{bmatrix} W_1 & W_g \\ W_{\bar{g}} & W_2 \end{bmatrix}$. Moreover, we have $W_g = W_{\bar{g}}^H$. The diagonal blocks $W_i$ of $W$ are deterministic and well-conditioned if the separation $\Delta$ is not so small. This is formalized in the following proposition.

**Proposition 2.** [23, Proposition 4.1] Suppose $\Delta \geq 1/M$, then both $W_1$ and $W_2$ are invertible and satisfy the following

$$\|I - W_i\| \leq 0.3623,$$

$$\|W_i\| \leq 1.3623,$$

$$\|W_i^{-1}\| \leq 1.568,$$

for $i = 1, 2$, where $\|\cdot\|$ represents the matrix operator norm.

The off-diagonal block $W_g$ is a random matrix with respect to $g$, which can be written as

$$W_g = \frac{1}{M} \sum_{n=-2M}^{2M} s_ng_n e_1(n) e_2^H(n) = \sum_{n=-2M}^{2M} E_n,$$
where

$$e_1(n) = \begin{bmatrix} e^{2\pi n\tau_1} \\ e^{2\pi n\tau_2} \\ \vdots \\ e^{2\pi n\tau_{21K_1}} \end{bmatrix}, \quad e_2(n) = \begin{bmatrix} e^{2\pi n\tau_{21}} \\ e^{2\pi n\tau_{22}} \\ \vdots \\ e^{2\pi n\tau_{21K_1}} \end{bmatrix} \in \mathbb{C}^{2K_1}, \quad e_1(n) e_2^H(n) \in \mathbb{C}^{2K_2}, \quad (30)$$

and

$$E_n = \frac{1}{M} s_n g_n e_1(n) e_2^H(n) \quad (31)$$

is a zero-mean random matrix with $E[E_n] = \frac{1}{M} s_n E[g_n] e_1(n) e_2^H(n) = 0$ since $E[g_n] = E[e^{2\pi n\phi}] = 0$. We have $W_g$ is a sum of independent zero-mean random matrices with $E[W_g] = 0$. The following proposition establishes the spectral norm of $W_g$ is bounded with high probability, whose proof is given in Appendix E.

**Proposition 3.** Assume $M \geq 4$. Let $\delta \in (0, 0.6376)$ and $\eta \in (0, 1)$, then $P\{\|W_g\| \geq \delta\} \leq \eta$ provided that

$$M \geq \frac{46}{\delta^2} K_{\text{max}} \log \left( \frac{2(K_1 + K_2)}{\eta} \right). \quad (32)$$

Denote the event $E_\delta = \{\|W_g\| \leq \delta\}$, which holds with probability at least $1 - \eta$ if (32) holds, following Proposition 3. Assume $E_\delta$ holds for some $0 < \delta < 0.6376$ and $\Delta \geq 1/M$, then

$$\|I - W\| \leq \|I - \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}\| + \|\begin{bmatrix} 0 & W_g \\ W_\bar{g} & 0 \end{bmatrix}\| \leq \max_{i=1,2} \|I - W_i\| + \|W_g\| \leq 0.3623 + \delta < 1,$$

which yields that $W$ is invertible under $E_\delta$. Equivalently, under $E_\delta$ the solution to (23) exists. Write $W^{-1}$ as

$$W^{-1} = \begin{bmatrix} L_1 & R_1 & L_g & R_g \\ L_\bar{g} & R_\bar{g} & L_2 & R_2 \end{bmatrix},$$

where $L_i, R_i \in \mathbb{C}^{2K_i \times K_i}$ for $i = 1, 2$, $L_g, R_g \in \mathbb{C}^{2K_1 \times K_2}$ and $L_\bar{g}, R_\bar{g} \in \mathbb{C}^{2K_2 \times K_1}$. We can then invert (24) and obtain

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \sqrt{|K'(0)|} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad (33)$$

which gives

$$L_1 u_1 + L_g u_2 \quad \text{and} \quad L_\bar{g} u_1 + L_2 u_2.$$

### 4.4 Bounding the Dual Polynomials

The rest of the proof is then given (33), we need to verify that $|P(\tau)| < 1$, $\forall \tau \not\in T_1$ and similarly, $|Q(\tau)| < 1$, $\forall \tau \not\in T_2$. Since the expressions for $P(\tau)$ and $Q(\tau)$ are very similar, it is sufficient to only establish the above for $P(\tau)$.

Recall the form of $P(\tau)$ in (21), the $l$th derivative of $P(\tau)$ can be represented as

$$P^{(l)}(\tau) = \sum_{k=1}^{K_1} \alpha_{1k} K'(\tau - \tau_1) + \sum_{k=1}^{K_1} \beta_{1k} K'(\tau - \tau_1) + \sum_{k=1}^{K_2} \alpha_{2k} K_g(\tau - \tau_2) + \sum_{k=1}^{K_2} \beta_{2k} K_g(\tau - \tau_2), \quad (34)$$
which can be rewritten as
\[
\frac{1}{\sqrt{|K''(0)|}} P^{(l)}(\tau) = \sum_{k=1}^{K_1} \frac{\alpha_{1k}}{\sqrt{|K''(0)|}} K^{(l)}(\tau - \tau_{1k}) + \sum_{k=1}^{K_1} \frac{\beta_{1k}}{\sqrt{|K''(0)|}} \frac{1}{\sqrt{|K''(0)|}} K^{(l+1)}(\tau - \tau_{1k})
+ \sum_{k=1}^{K_2} \frac{\alpha_{2k}}{\sqrt{|K''(0)|}} K^{(l)}(\tau - \tau_{2k}) + \sum_{k=1}^{K_2} \frac{\beta_{2k}}{\sqrt{|K''(0)|}} \frac{1}{\sqrt{|K''(0)|}} K^{(l+1)}(\tau - \tau_{2k})
= v_{1l}^H(\tau) \left[ \frac{\alpha_1}{\sqrt{|K''(0)|}} \beta_1 \right] + v_{2l}^H(\tau) \left[ \frac{\alpha_2}{\sqrt{|K''(0)|}} \beta_2 \right],
\]
where
\[
\bar{v}_{1l}(\tau) = \frac{1}{\sqrt{|K''(0)|}} \begin{bmatrix}
K^{(l)}(\tau - \tau_{11}) \\
K^{(l)}(\tau - \tau_{12}) \\
\vdots \\
K^{(l)}(\tau - \tau_{1K_1})
\end{bmatrix}, \quad
\bar{v}_{2l}(\tau) = \frac{1}{\sqrt{|K''(0)|}} \begin{bmatrix}
K^{(l)}(\tau - \tau_{21}) \\
K^{(l)}(\tau - \tau_{22}) \\
\vdots \\
K^{(l)}(\tau - \tau_{2K_2})
\end{bmatrix},
\]
and $K''(0)$ is the scaler defined in (25). Using the forms of $K(\tau)$ and $K_g(\tau)$, we can rewrite the above as
\[
v_{1l}(\tau) = \frac{1}{M} \sum_{n=-2M}^{2M} s_n \left( \frac{-j2\pi n}{\sqrt{|K''(0)|}} \right)^t e^{-j2\pi n \tau} e_1(n),
\]
\[
v_{2l}(\tau) = \frac{1}{M} \sum_{n=-2M}^{2M} s_n g_n \left( \frac{-j2\pi n}{\sqrt{|K''(0)|}} \right)^t e^{-j2\pi n \tau} e_2(n),
\]
where $e_1(n)$ and $e_2(n)$ are defined in (30). Then
\[
\frac{1}{\sqrt{|K''(0)|}} P^{(l)}(\tau) \text{ can be rewritten as}
\]
\[
\frac{1}{\sqrt{|K''(0)|}} P^{(l)}(\tau) = v_{1l}^H(\tau) (L_1 u_1 + L_g u_2) + v_{2l}^H(\tau) (L_g u_1 + L_2 u_2)
= \langle u_1, L_1^H v_{1l}(\tau) \rangle + \langle u_2, L_g^H v_{1l}(\tau) \rangle + \langle u_1, L_g^H v_{2l}(\tau) \rangle + \langle u_2, L_2^H v_{2l}(\tau) \rangle,
\]
where (36) follows from (33). Let
\[
W_\mu = E[W] = \begin{bmatrix} E[W_1] & E[W_2] \end{bmatrix} = \begin{bmatrix} W_1 & 0 \\
0 & W_2 \end{bmatrix},
\]
and
\[
W_\mu^{-1} = \begin{bmatrix} W_1^{-1} & 0 \\
0 & W_2^{-1} \end{bmatrix} = \begin{bmatrix} L_{\mu_1} & R_{\mu_1} & 0 & 0 \\
0 & 0 & L_{\mu_2} & R_{\mu_2} \end{bmatrix},
\]
where $L_{\mu i} \in \mathbb{C}^{2K_i \times K_i}$ and $R_{\mu i} \in \mathbb{C}^{2K_i \times K_i}$, $i = 1, 2$. We can then further rewrite (37) as
\[
\frac{1}{\sqrt{|K''(0)|}} P^{(l)}(\tau) = \langle u_1, L_1^H v_{1l}(\tau) \rangle + \langle u_1, (L_1 - L_{\mu_1})^H v_{1l}(\tau) \rangle + \langle u_2, L_g^H v_{1l}(\tau) \rangle
+ \langle u_1, L_g^H v_{2l}(\tau) \rangle + \langle u_2, L_2^H v_{2l}(\tau) \rangle.
\]
Denote
\[
\frac{1}{\sqrt{|K''(0)|}} P_\mu^{(l)}(\tau) = \langle u_1, L_\mu^H v_{1l}(\tau) \rangle.
\]
Our proof proceeds in the following steps:
• **Step 1**: show that \( \frac{1}{\sqrt{|K^*(0)|}} P^{(i)}(\tau) \) is bounded around \( \frac{1}{\sqrt{|K^*(0)|}} P^{(i)}(\tau) \) for a set of grid points \( \mathcal{Y}_{grid} \);

• **Step 2**: show that \( \frac{1}{\sqrt{|K^*(0)|}} P^{(i)}(\tau) \) is uniformly bounded around \( \frac{1}{\sqrt{|K^*(0)|}} P^{(i)}(\tau) \) for all \( \tau \in [0,1) \);

• **Step 3**: finally, show that \( |P(\tau)| < 1, \forall \tau \notin \mathcal{Y}_1 \).

### 4.4.1 Proof of Step 1

Here the goal is to bound the last four residual terms in (38) with high probability on a set of uniform grid points \( \tau \in \mathcal{Y}_{grid} \) from \([0,1) \) whose size will be specified later. We first record the following useful lemma whose proof is given in Appendix F.

**Lemma 1.** Under the event \( \mathcal{E}_\delta \) for some \( \delta \in (0, 1/4] \), we have

\[
\|L_i\| \leq 2 \|W_{\mu}^{-1}\|, \quad \text{for} \quad i = 1, 2,
\]

\[
\|L_i - L_{\mu i}\| \leq 2 \|W_{\mu}^{-1}\|^2 \delta, \quad \text{for} \quad i = 1, 2,
\]

\[
\|L_0\| \leq 2 \|W_{\mu}^{-1}\|^2 \delta \leq 0.8 \|W_{\mu}^{-1}\|,
\]

\[
\|L_0\| \leq 2 \|W_{\mu}^{-1}\|^2 \delta \leq 0.8 \|W_{\mu}^{-1}\|.
\]

When the signs of the coefficients \( a_{ik} \)'s are arbitrary, the last four terms in (38) can be bounded by

\[
\left| \langle u_1, (L_1 - L_{\mu 1})^H v_{1l}(\tau) \rangle \right| \leq \|u_1\|_2 \|L_1 - L_{\mu 1}\|_2 \|v_{1l}(\tau)\|_2 \leq C_1 \sqrt{K_1} \delta,
\]

\[
\left| \langle u_2, L_2^H v_{1l}(\tau) \rangle \right| \leq \|u_2\|_2 \|L_2^H v_{1l}(\tau)\|_2 \leq C_2 \sqrt{K_2} \delta,
\]

\[
\left| \langle u_1, L_0^H v_{2l}(\tau) \rangle \right| \leq \|u_1\|_2 \|L_0^H v_{2l}(\tau)\|_2,
\]

\[
\left| \langle u_2, L_2^H v_{2l}(\tau) \rangle \right| \leq \|u_2\|_2 \|L_2^H v_{2l}(\tau)\|_2,
\]

where the last steps of the first two inequalities follow from Lemma 1, and \( \|v_{1l}(\tau)\|_2 \leq C \) for some numerical constant \( C \) [23, Lemma 4.9]. By setting \( \delta \) properly and we can obtain the bound on \( M \) using Proposition 3, Lemma 4.6 and Lemma 4.7 in [23]. When the signs of the coefficients \( a_{ik} \)'s are random, we can provide a tighter bound by applying the Hoeffding’s inequality, which follows similarly as the proof of [23, Lemma 4.8 and 4.9]. We have the following proposition.

**Proposition 4.** Suppose \( \Delta \geq 1/M \). There exists a numerical constant \( C \) such that

\[
M \geq C \max \left\{ \log^2 \left( \frac{\mathcal{Y}_{grid}}{\eta} \right), \frac{K_{max}^{\mathcal{Y}_{grid}}}{\epsilon^2} \log \left( \frac{\mathcal{Y}_{grid}}{\eta} \right), \frac{K_{max}^2}{\epsilon^2} \log \left( \frac{K_1 + K_2}{\eta} \right) \right\},
\]

or additionally, if the signs of the coefficients \( a_{ik} \)'s are i.i.d. generated from a symmetric distribution on the complex unit circle, there exists a numerical constant \( C \) such that

\[
M \geq C \max \left\{ \frac{1}{\epsilon^2} \log^2 \left( \frac{\mathcal{Y}_{grid}}{\eta} \right), \frac{1}{\epsilon^2} K_{max} \log \left( \frac{K_1 + K_2}{\eta} \right) \log \left( \frac{\mathcal{Y}_{grid}}{\eta} \right) \right\},
\]

where \( \mathcal{Y}_{grid} \) is the grid size, then we have

\[
\sup_{\tau_d \in \mathcal{Y}_{grid}} \left| \langle u_1, (L_1 - L_{\mu 1})^H v_{1l}(\tau_d) \rangle \right| < \epsilon, \quad l = 0, 1, 2, 3;
\]

\[
\sup_{\tau_d \in \mathcal{Y}_{grid}} \left| \langle u_2, L_2^H v_{1l}(\tau_d) \rangle \right| < \epsilon, \quad l = 0, 1, 2, 3;
\]

\[
\sup_{\tau_d \in \mathcal{Y}_{grid}} \left| \langle u_1, L_0^H v_{2l}(\tau_d) \rangle \right| < \epsilon, \quad l = 0, 1, 2, 3;
\]

\[
\sup_{\tau_d \in \mathcal{Y}_{grid}} \left| \langle u_2, L_2^H v_{2l}(\tau_d) \rangle \right| < \epsilon, \quad l = 0, 1, 2, 3,
\]

hold with probability at least \( 1 - 8\eta \).
Denote the event
\[ \mathcal{E}_1 = \left\{ \sup_{\tau_d \in \mathcal{Y}_{\text{grid}}} \left| \frac{1}{\sqrt{|K''(0)|}} P(\tau_d) - \frac{1}{\sqrt{|K''(0)|}} P_{\mu}(\tau_d) \right| \leq \frac{\epsilon}{3}, \ l = 0, 1, 2, 3 \right\}, \]
for some \( \epsilon > 0 \). Then by rescaling the constants, it is straightforward that \( \mathcal{E}_1 \) holds with probability at least \( 1 - \eta \) as soon as the conditions in Proposition 4 are met.

### 4.4.2 Proof of Step 2

We have shown that the differences between \( \frac{1}{\sqrt{|K''(0)|}} P(\tau) \) and \( \frac{1}{\sqrt{|K''(0)|}} P_{\mu}(\tau) \) are bounded on a finite grid. In this step we extend this statement to the continuous domain \( \tau \in [0, 1) \) by assigning the size of \( \mathcal{Y}_{\text{grid}} \) properly. This is given in the following proposition whose proof is given in Appendix G.

**Proposition 5.** Suppose \( \Delta \geq 1/M \). There exists a numerical constant \( C \) such that
\[
M \geq C \max \left\{ \log^{2} \left( \frac{M (K_1 + K_2)}{\epsilon \eta} \right), \frac{1}{\epsilon^2} K_{\max} \log \left( \frac{M (K_1 + K_2)}{\epsilon \eta} \right), \frac{1}{\epsilon^2} K_{\max}^{2} \log \left( \frac{K_1 + K_2}{\eta} \right) \right\},
\]
or additionally, if the signs of the coefficients \( a_{ik} \)'s are i.i.d. generated from a symmetric distribution on the complex unit circle, there exists a numerical constant \( C \) such that
\[
M \geq C \max \left\{ \log^{2} \left( \frac{M (K_1 + K_2)}{\epsilon \eta} \right), \frac{1}{\epsilon^2} K_{\max} \log \left( \frac{K_1 + K_2}{\eta} \right) \log \left( \frac{M (K_1 + K_2)}{\epsilon \eta} \right) \right\},
\]
then we have
\[
\mathbb{P} \left\{ \left| \frac{1}{\sqrt{|K''(0)|}} P(\tau) - \frac{1}{\sqrt{|K''(0)|}} P_{\mu}(\tau) \right| \leq \epsilon, \ \forall \tau \in [0, 1), \ l = 0, 1, 2, 3 \right\} \geq 1 - \eta.
\]

### 4.4.3 Proof of Step 3

This step follows essentially the same procedure as those in [23, Lemma 4.13 and 4.14], where we divide \([0, 1)\) into
\[
\mathcal{Y}_{\text{near}}^i = \cup_{k=1}^{K_i} [\tau_{ik} - \tau_s, \tau_{ik} + \tau_s], \quad \text{and} \quad \mathcal{Y}_{\text{far}} = [0, 1) \setminus \mathcal{Y}_{\text{near}},
\]
for \( i = 1, 2 \), where \( \tau_s = 8.245 \times 10^{-2}/M \). Then conditioned on the event in Proposition 5 one can bound \( |P(\tau)| < 1 \) in \( \mathcal{Y}_{\text{near}} \) \( \mathcal{Y}_1 \) and \( \mathcal{Y}_{\text{far}} \), respectively following straightforward calculus. We shall omit the details and refer interested readers to [23, Lemma 4.13 and 4.14]. We have the following proposition.

**Proposition 6.** Suppose \( \Delta \geq 1/M \). There exists a numerical constant \( C \) such that
\[
M \geq C \max \left\{ \log^{2} \left( \frac{M (K_1 + K_2)}{\eta} \right), K_{\max} \log \left( \frac{M (K_1 + K_2)}{\eta} \right), K_{\max}^{2} \log \left( \frac{K_1 + K_2}{\eta} \right) \right\},
\]
or additionally, if the signs of the coefficients \( a_{ik} \)'s are i.i.d. generated from a symmetric distribution on the complex unit circle, there exists a numerical constant \( C \) such that
\[
M \geq C \max \left\{ \log^{2} \left( \frac{M (K_1 + K_2)}{\eta} \right), K_{\max} \log \left( \frac{K_1 + K_2}{\eta} \right) \log \left( \frac{M (K_1 + K_2)}{\eta} \right) \right\},
\]
then we have
\[
|P(\tau)| \leq 1 - C_{p} M^{2} (\tau - \tau_{ik})^{2}, \quad \tau \in \mathcal{Y}_{\text{near}} \setminus \{\tau_{ik}\}, \quad k = 1, \ldots, K_1,
|P(\tau) - \text{sign}(a_{ik})| \leq C_{p} M^{2} (\tau - \tau_{ik})^{2}, \quad \tau \in \mathcal{Y}_{\text{near}} \setminus \{\tau_{ik}\}, \quad k = 1, \ldots, K_1,
|P(\tau)| \leq 1 - C''_{p} < 1, \quad \tau \in \mathcal{Y}_{\text{far}},
\]
with probability at least \( 1 - \eta \), where \( C_{p}, C'_{p} \) and \( C''_{p} \) are some positive numerical constants.
4.5 Finishing the Proof

The proof of Theorem 2.1 is now complete since we have established that \( P(\tau) \) and \( Q(\tau) \) constructed in (21) and (22) are indeed valid dual certificates under the condition of Theorem 2.1.

5 Proof of Theorem 2.2

We first provide a proposition on optimality conditions of (9), which is proved in Appendix H.

**Proposition 7.** \( \{\hat{x}_1, \hat{x}_2\} \) is the minimizer of (9) if and only if the following holds:

\[
\|y - (\hat{x}_1 + g \odot \hat{x}_2)\|_A^* \leq \lambda_w,
\]

\[
\|g \odot (y - (\hat{x}_1 + g \odot \hat{x}_2))\|_A^* \leq \lambda_w,
\]

\[
(y - (\hat{x}_1 + g \odot \hat{x}_2), x_1 + g \odot \hat{x}_2)_R = \lambda_w \|x_1\|_A + \lambda_w \|\hat{x}_2\|_A.
\]

Let \( e_1 = \hat{x}_1 - x_1^* \) and \( e_2 = \hat{x}_2 - x_2^* \). Moreover, let \( \nu_1 \) and \( \nu_2 \) be the corresponding representing measures [18, 37] of \( e_1 \) and \( e_2 \), respectively, which are given as

\[ e_1 = \int_0^1 c(\tau) \nu_1(\tau) d\tau, \quad e_2 = \int_0^1 c(\tau) \nu_2(\tau) d\tau. \]

Therefore, we have \( \|e_i\|_A = \|\nu_i\|_{TV}, \ i = 1, 2 \), where \( \|\cdot\|_{TV} \) is the total variation norm of the representing measure. Define

\[ I_{i,0}^k = \left| \int_{Y^i_{\text{near}}} \nu_i(\tau) d\tau \right|, \]

\[ I_{i,1}^k = (4M + 1) \left| \int_{Y^i_{\text{near}}} (\tau - \tau_{ik}) \nu_i(\tau) d\tau \right|, \]

\[ I_{i,2}^k = \frac{(4M + 1)^2}{2} \int_{Y^i_{\text{near}}} (\tau - \tau_{ik})^2 \nu_i(\tau) d\tau, \]

and \( I_{i,j} = \sum_{k=1}^{K_i} I_{i,j}^k \) for \( j = 0, 1, 2 \) and \( i = 1, 2 \), where \( Y^i_{\text{near}} \) and \( Y^i_{\text{far}} \) are defined in (39). We have the following proposition whose proof can be found in Appendix I.

**Proposition 8.** Assume the noise is bounded as \( \|w\|_2^2 \leq \sigma_w^2 \). Set \( \lambda_w = C_w \sigma_w \sqrt{4M + 1} \), for some constant \( C_w > 1 \) large enough, then we have

\[ \|e_1\|_2 + \|e_2\|_2 \leq \sqrt{4M + 1} \sum_{i=1}^{2} \left( \|P_{Y^i_{\text{near}}} (\nu_i)\|_{TV} + \sum_{j=0}^{2} I_{i,j} \right), \]

\[ \|e_1 + g \odot e_2\|_2 \leq \sqrt{2\lambda_w} \sum_{i=1}^{2} \left( \|P_{Y^i_{\text{near}}} (\nu_i)\|_{TV} + \sum_{j=0}^{2} I_{i,j} \right). \]

Hence the rest is to provide an upper bound on the term \( \sum_{i=1}^{2} \left( \|P_{Y^i_{\text{near}}} (\nu_i)\|_{TV} + \sum_{j=0}^{2} I_{i,j} \right) \). We have the following proposition to control the sum value of zeroth moment terms \( \sum_{i=1}^{2} I_{i,0} \) and the sum value of first moment terms \( \sum_{i=1}^{2} I_{i,1} \), whose proof is given in Appendix J.

**Proposition 9.** Under the conditions in Theorem 2.2, there exist some numerical constants \( C_0 \) and \( C_1 \), such that

\[ \sum_{i=1}^{2} I_{i,0} \leq C_0 \left( \lambda_w \sqrt{\frac{K_{\text{max}}^2 \log M}{M}} + \sum_{i=1}^{2} I_{i,2} + \sum_{i=1}^{2} \|P_{Y^i_{\text{near}}} (\nu_i)\|_{TV} \right), \]

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\[ \sum_{i=1}^{2} I_{i,1} \leq C_1 \left( \lambda_w \sqrt{\frac{K_{3 \max}^3 \log M}{M}} + \sum_{i=1}^{2} I_{i,2} + \sum_{i=1}^{2} \left\| P_{T_{ir}}(\nu_i) \right\|_{TV} \right), \]

for \( i = 1, 2 \), with high probability given in Theorem 2.2.

What remains is to bound \( \sum_{i=1}^{2} \left\| P_{T_{ir}}(\nu_i) \right\|_{TV} + \sum_{i=1}^{2} I_{i,2} \), which is given in the following proposition proved in Appendix K.

**Proposition 10.** Under the conditions in Theorem 2.2, there exists a numerical constant \( C \), such that

\[ \sum_{i=1}^{2} \left\| P_{T_{ir}}(\nu_i) \right\|_{TV} + \sum_{i=1}^{2} I_{i,2} \leq C \lambda_w \sqrt{\frac{K_{3 \max}^3 \log M}{M}} \]

holds with high probability given in Theorem 2.2.

Combining Propositions 8, 9 and 10, there exists some constant \( C \) such that

\[ \frac{1}{\sqrt{4M+1}} (\| e_1 \|_2 + \| e_2 \|_2) \leq C \frac{K_{3 \max}^3 \sqrt{\log M} \lambda_w}{\sqrt{M}} \leq C_1 \sigma_w \sqrt{K_{3 \max}^3 \log M}, \]

and

\[ \frac{1}{\sqrt{4M+1}} \| e_1 + g \odot e_2 \|_2 \leq \frac{1}{\sqrt{4M+1}} \sqrt{2 \lambda_w C \sqrt{K_{3 \max}^3 \log M} \lambda_w} \leq C_2 \sigma_w \left( \frac{K_{3 \max}^3 \log M}{M} \right)^{1/4}. \]

### 6 Conclusions

We propose a convex optimization method based on atomic norm minimization to super-resolve two point source models from the measurements of their superposition, where each point source signal is convolved with a different low-pass point spread function. It is demonstrated, with high probability, that the point source locations of each modality can be simultaneously determined perfectly in the noise-free setting, from a near-optimal number of measurements when each point source signal satisfies a mild separation condition, and the point spread functions are randomly generated in the frequency domain. The proposed algorithm is also robust in the presence of bounded noise.

Our algorithmic framework and the proof methodology can be extended straightforwardly to handle more than two modalities when all of the modalities obey the conditions set forth in the current paper. There are a few possible future research directions. In applications such as multi-user detection, only a small number of users are active out of all the possible users. It will then be of great interest to simultaneously identify a small set of active users as well as identify their corresponding point source signals. In addition, it will also be of interest to develop performance guarantees of the proposed algorithm under milder conditions of the point spread functions, for example when they are deterministic but weakly correlated.

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### References

[1] M. J. Rust, M. Bates, and X. Zhuang, “Sub-diffraction-limit imaging by stochastic optical reconstruction microscopy (storm),” *Nature methods*, vol. 3, no. 10, pp. 793–796, 2006.

[2] B. Huang, W. Wang, M. Bates, and X. Zhuang, “Three-dimensional super-resolution imaging by stochastic optical reconstruction microscopy,” *Science*, vol. 319, no. 5864, pp. 810–813, February 2008.
[3] J. Huang, M. Sun, K. Gumpper, Y. Chi, and J. Ma, “3D multifocus astigmatism and compressed sensing (3D macs) based superresolution reconstruction,” *Biomedical optics express*, vol. 6, no. 3, pp. 902–917, 2015.

[4] C. Ekanadham, D. Tranchina, and E. P. Simoncelli, “A unified framework and method for automatic neural spike identification,” *Journal of neuroscience methods*, vol. 222, pp. 47–55, 2014.

[5] K. C. Knudson, J. Yates, A. Huk, and J. W. Pillow, “Inferring sparse representations of continuous signals with continuous orthogonal matching pursuit,” in *Advances in Neural Information Processing Systems*, 2014, pp. 1215–1223.

[6] Y. Chi, Y. Xie, and R. Calderbank, “Compressive demodulation of mutually interfering signals,” *arXiv preprint arXiv:1303.3904*, 2013.

[7] L. Applebaum, W. U. Bajwa, M. F. Duarte, and R. Calderbank, “Asynchronous code-division random access using convex optimization,” *Physical Communication*, vol. 5, no. 2, pp. 129–147, 2012.

[8] Y. M. Lu and M. Vetterli, “Multichannel sampling with unknown gains and offsets: A fast reconstruction algorithm,” in *Proc. Allerton Conference on Communication, Control and Computing*, Monticello, 2010.

[9] Y. Li, Y. He, Y. Chi, and Y. M. Lu, “Blind calibration of multi-channel samplers using sparse recovery,” in *Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP)*, 2015 IEEE 6th International Workshop on. IEEE, 2015, pp. 33–36.

[10] E. Betzig, G. H. Patterson, R. Sougrat, O. W. Lindwasser, S. Olenych, J. S. Bonifacino, M. W. Davidson, J. Lippincott-Schwartz, and H. F. Hess, “Imaging intracellular fluorescent proteins at nanometer resolution,” *Science*, vol. 313, no. 5793, pp. 1642–1645, 2006.

[11] M. S. Lewicki, “A review of methods for spike sorting: the detection and classification of neural action potentials,” *Network: Computation in Neural Systems*, vol. 9, no. 4, pp. R53–R78, 1998.

[12] G. Gerstein and W. Clark, “Simultaneous studies of firing patterns in several neurons,” *Science*, vol. 143, no. 3612, pp. 1325–1327, 1964.

[13] L. Li and T. P. Speed, “Parametric deconvolution of positive spike trains,” *Annals of Statistics*, pp. 1279–1301, 2000.

[14] P. Stoica and R. L. Moses, *Introduction to spectral analysis*. Prentice hall Upper Saddle River, 1997, vol. 1.

[15] R. Schmidt, “Multiple emitter location and signal parameter estimation,” *IEEE Transactions on Antennas and Propagation*, vol. 34, no. 3, pp. 276–280, 1986.

[16] Y. Hua and T. K. Sarkar, “Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise,” *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 38, no. 5, pp. 814–824, may 1990.

[17] H. N. Mhaskar and J. Prestin, “On the detection of singularities of a periodic function,” *Advances in Computational Mathematics*, vol. 12, no. 2-3, pp. 95–131, 2000.

[18] E. J. Candès and C. Fernandez-Granda, “Towards a mathematical theory of super-resolution,” *Communications on Pure and Applied Mathematics*, vol. 67, no. 6, pp. 906–956, 2014.

[19] J. Romberg, “Multiple channel estimation using spectrally random probes,” in *SPIE Optical Engineering+ Applications*. International Society for Optics and Photonics, 2009, pp. 744606–744606.

[20] Y. Chi, L. Scharf, A. Pezeshki, and A. Calderbank, “Sensitivity to basis mismatch in compressed sensing,” *IEEE Transactions on Signal Processing*, vol. 59, no. 5, pp. 2182–2195, May 2011.
[21] V. Chandrasekaran, B. Recht, P. Parrilo, and A. Willsky, “The convex algebraic geometry of linear inverse problems,” 48th Annual Allerton Conference on Communication, Control, and Computing, pp. 699–703, 2010.

[22] V. Chandrasekaran, B. Recht, P. A. Parrilo, and A. S. Willsky, “The convex geometry of linear inverse problems,” Foundations of Computational mathematics, vol. 12, no. 6, pp. 805–849, 2012.

[23] G. Tang, B. N. Bhaskar, P. Shah, and B. Recht, “Compressed sensing off the grid,” Information Theory, IEEE Transactions on, vol. 59, no. 11, pp. 7465–7490, 2013.

[24] C. Fernandez-Granda, “Super-resolution of point sources via convex programming,” Information and Inference, p. iaw005, 2016.

[25] N. H. Nguyen and T. D. Tran, “Exact recoverability from dense corrupted observations via $\ell_1$-minimization,” IEEE transactions on information theory, vol. 59, no. 4, pp. 2017–2035, 2013.

[26] E. J. Candès, X. Li, Y. Ma, and J. Wright, “Robust principal component analysis?” Journal of ACM, vol. 58, no. 3, pp. 11:1–11:37, Jun 2011.

[27] Y. Chen and Y. Chi, “Robust spectral compressed sensing via structured matrix completion,” IEEE Transactions on Information Theory, vol. 60, no. 10, pp. 6576–6601, 2014.

[28] M. McCoy, V. Cevher, Q. Dinh, A. Asaei, and L. Baldassarre, “Convexity in source separation: Models, geometry, and algorithms,” Signal Processing Magazine, IEEE, vol. 31, no. 3, pp. 87–95, 2014.

[29] A. Moitra, “The threshold for super-resolution via extremal functions,” arXiv preprint arXiv:1408.1681, 2014.

[30] Y. Chi and Y. Chen, “Compressive two-dimensional harmonic retrieval via atomic norm minimization,” Signal Processing, IEEE Transactions on, vol. 63, no. 4, pp. 1030–1042, Feb 2015.

[31] R. Roy and T. Kailath, “Esprit-estimation of signal parameters via rotational invariance techniques,” Acoustics, Speech and Signal Processing, IEEE Transactions on, vol. 37, no. 7, pp. 984–995, 1989.

[32] R. Prony, “Essai experimental et analytique,” J. de l’Ecole Polytechnique (Paris), vol. 1, no. 2, pp. 24–76, 1795.

[33] M. Grant, S. Boyd, and Y. Ye, “Cvx: Matlab software for disciplined convex programming,” Online accessable: http://stanford. edu/˜ boyd/cvx, 2008.

[34] B. N. Bhaskar, G. Tang, and B. Recht, “Atomic norm denoising with applications to line spectral estimation,” Signal Processing, IEEE Transactions on, vol. 61, no. 23, pp. 5987–5999, 2013.

[35] Y. Li and Y. Chi, “Off-the-grid line spectrum denoising and estimation with multiple measurement vectors,” IEEE Transactions on Signal Processing, vol. 64, no. 5, pp. 1257–1269, 2016.

[36] D. Tufts, A. Kot, and R. Vaccaro, “The threshold effect in signal processing algorithms which use an estimated subspace,” SVD and Signal Processing II: Algorithms, Analysis and Applications, pp. 301–320, 1991.

[37] G. Tang, B. Bhaskar, and B. Recht, “Near minimax line spectral estimation,” Information Theory, IEEE Transactions on, vol. 61, no. 1, pp. 499–512, Jan 2015.

[38] J. A. Tropp, “User-friendly tail bounds for sums of random matrices,” Foundations of Computational Mathematics, vol. 12, no. 4, pp. 389–434, 2012.

[39] A. Schaeffer, “Inequalities of a. markoff and s. bernstein for polynomials and related functions,” Bull. Amer. Math. Soc, vol. 47, pp. 565–579, 1941.

[40] W. Hoeffding, “Probability inequalities for sums of bounded random variables,” Journal of the American statistical association, vol. 58, no. 301, pp. 13–30, 1963.

[41] E. J. Candès and C. Fernandez-Granda, “Super-resolution from noisy data,” Journal of Fourier Analysis and Applications, vol. 19, no. 6, pp. 1229–1254, 2013.
A Useful Lemmas

Lemma 2. [38, noncommutative Bernstein’s inequality] Let \( \{E_n\} \) be a finite sequence of independent, random matrices with dimensions \( d_1 \times d_2 \). Suppose that each random matrix satisfies
\[
\mathbb{E} \|E_n\| = 0, \quad \text{and} \quad \|E_n\| \leq R \quad \text{almost surely.}
\]

Define
\[
\sigma^2 = \max \left\{ \left\| \sum_n \mathbb{E} \left[ E_n E_n^H \right] \right\|, \left\| \sum_n \mathbb{E} \left[ E_n^H E_n \right] \right\| \right\}.
\]

Then for any \( t \geq 0 \),
\[
\mathbb{P} \left\{ \left\| \sum_n E_n \right\| \geq t \right\} \leq (d_1 + d_2) \cdot \exp \left( \frac{-t^2/2}{\sigma^2 + Rt/3} \right).
\]

Lemma 3. [39, Bernstein's polynomial inequality] Suppose \( F(z) \) is a polynomial of degree \( N \) with complex coefficients, then there exists
\[
\sup_{|z| \leq 1} |F'(z)| \leq N \cdot \sup_{|z| \leq 1} |F(z)|.
\]

Lemma 4. [40, Hoeffding’s inequality] Let the components of \( u \in \mathbb{C}^N \) be sampled i.i.d. from a symmetric distribution on the complex unit circle, \( w \in \mathbb{C}^N \), and \( t \) be a positive real number. Then
\[
\mathbb{P} \left\{ |\langle u, w \rangle| \geq t \right\} \leq 4e^{-\frac{t^2}{4|u|^2}}.
\]

B Proof of Dual Problem (15)

The Lagrangian function of (8) is given as
\[
L(x_1, x_2, p) = \|x_1\|_A + \|x_2\|_A + \langle p, y - x_1 - g \odot x_2 \rangle_R,
\]
whose infimum over \( x_1 \) and \( x_2 \) can be found as
\[
D(p) = \inf_{x_1, x_2} L(x_1, x_2, p)
= \inf_{x_1, x_2} \left\{ \|x_1\|_A - \langle p, x_1 \rangle_R + \|x_2\|_A - \langle p, g \odot x_2 \rangle_R + \langle p, y \rangle_R \right\}
= \inf_{x_1, x_2} \left\{ \|x_1\|_A - \langle p, x_1 \rangle_R + \|x_2\|_A - \langle g \odot p, x_2 \rangle_R + \langle p, y \rangle_R \right\}
= \inf_{x_1} \left\{ \|x_1\|_A - \langle p, x_1 \rangle_R \right\} + \inf_{x_2} \left\{ \|x_2\|_A - \langle g \odot p, x_2 \rangle_R + \langle p, y \rangle_R \right\}. \tag{42}
\]

Plugging into (42) the facts that
\[
\inf_{x_i} \left\{ \|x_i\|_A - \langle p, x_i \rangle_R \right\} = \begin{cases} 0, & \|p\|_A^* \leq 1 \\ -\infty, & \text{otherwise} \end{cases},
\]
for \( i = 1, 2 \), we can have the dual problem of (8) as given in (15).

C Proof of \( \Upsilon_1 \subseteq \hat{\Upsilon}_1 \) and \( \Upsilon_2 \subseteq \hat{\Upsilon}_2 \)

If \( \Upsilon_1 \setminus \hat{\Upsilon}_1 \neq \emptyset \) or \( \Upsilon_2 \setminus \hat{\Upsilon}_2 \neq \emptyset \), there exists \( |\dot{P}(\tau)| < 1 \) for \( \tau \in \Upsilon_1 \setminus \hat{\Upsilon}_1 \) or \( |\dot{Q}(\tau)| < 1 \) for \( \tau \in \Upsilon_2 \setminus \hat{\Upsilon}_2 \). Then we have
\[
\langle \dot{p}, y \rangle_R = \langle \dot{p}, x_1^* \rangle_R + \langle \dot{p}, g \odot x_2^* \rangle_R
= \langle \dot{p}, \sum_{k=1}^{K_1} a_{1k} e(\tau_{1k}) \rangle_R + \langle \dot{g} \odot \dot{p}, \sum_{k=1}^{K_2} a_{2k} e(\tau_{2k}) \rangle_R
\]
\[21\]
\[\begin{align*}
&= \sum_{\tau_{1k} \in \hat{\mathcal{Y}}_1 \cap \tilde{\mathcal{Y}}_1} \text{Re} \left( \tilde{a}_{1k} \hat{P} (\tau_{1k}) \right) + \sum_{\tau_{1k} \in \hat{\mathcal{Y}}_1 \setminus \tilde{\mathcal{Y}}_1} \text{Re} \left( \tilde{a}_{1k} \hat{P} (\tau_{1k}) \right) \\
&\quad + \sum_{\tau_{2k} \in \hat{\mathcal{Y}}_2 \cap \tilde{\mathcal{Y}}_2} \text{Re} \left( \tilde{a}_{2k} \hat{Q} (\tau_{2k}) \right) + \sum_{\tau_{2k} \in \hat{\mathcal{Y}}_2 \setminus \tilde{\mathcal{Y}}_2} \text{Re} \left( \tilde{a}_{2k} \hat{Q} (\tau_{2k}) \right) \\
&< \sum_{\tau_{1k} \in \hat{\mathcal{Y}}_1 \cap \tilde{\mathcal{Y}}_1} |a_{1k}| + \sum_{\tau_{1k} \in \hat{\mathcal{Y}}_1 \setminus \tilde{\mathcal{Y}}_1} |a_{1k}| + \sum_{\tau_{2k} \in \hat{\mathcal{Y}}_2 \cap \tilde{\mathcal{Y}}_2} |a_{2k}| + \sum_{\tau_{2k} \in \hat{\mathcal{Y}}_2 \setminus \tilde{\mathcal{Y}}_2} |a_{2k}|
\end{align*}\]

where the strict inequality violates strong duality. Therefore, \(\mathcal{Y}_1 \subseteq \hat{\mathcal{Y}}_1\) and \(\mathcal{Y}_2 \subseteq \hat{\mathcal{Y}}_2\).

**D Proof of Proposition 1**

*Proof.* Since

\[\|p\|_A^* = \sup_{\tau \in [0,1]} \left| \sum_{n=-2M}^{2M} p_n e^{j2\pi n \tau} \right| = \sup_{\tau \in [0,1]} |P(\tau)| \leq 1,\]

\[\|g \odot p\|_A^* = \sup_{\tau \in [0,1]} \left| \sum_{n=-2M}^{2M} \tilde{g}_n p_n e^{j2\pi n \tau} \right| = \sup_{\tau \in [0,1]} |Q(\tau)| \leq 1,\]

the vector \(p\) satisfying (18) is dual feasible. First,

\[\langle p, y \rangle_R = \langle p, x_1^* + g \odot x_2^* \rangle_R \]

\[= \langle p, x_1^* \rangle_R + \langle g \odot p, x_2^* \rangle_R \]

\[= \sum_{k=1}^{K_1} \text{Re} \left( \tilde{a}_{1k} \sum_{n=-2M}^{2M} p_n e^{j2\pi n \tau_{1k}} \right) + \sum_{k=1}^{K_2} \text{Re} \left( \tilde{a}_{2k} \sum_{n=-2M}^{2M} \tilde{g}_n p_n e^{j2\pi n \tau_{2k}} \right) \]

\[= \sum_{k=1}^{K_1} |a_{1k}| + \sum_{k=1}^{K_2} |a_{2k}| \geq \|x_1^*\|_A + \|x_2^*\|_A.\]

Also, we have

\[\langle p, y \rangle_R = \langle p, x_1^* \rangle_R + \langle g \odot p, x_2^* \rangle_R \leq \|p\|_A^* \|x_1^*\|_A + \|g \odot p\|_A^* \|x_2^*\|_A \leq \|x_1^*\|_A + \|x_2^*\|_A,\]

which gives \(\langle p, y \rangle_R = \|x_1^*\|_A + \|x_2^*\|_A\). This implies that \(p\) is a dual optimal solution of (15), and that \(x_1^*\) and \(x_2^*\) are the primal optimal solutions of (8).

Now validate the uniqueness of \(x_1^*\) and \(x_2^*\). Suppose there is a different optimal solution of (8), which can be written as \(\tilde{x}_i = \sum_{k=1}^{K_i} \tilde{a}_{ik} c(\tilde{\tau}_{ik})\), where \(\tilde{\mathcal{Y}}_i = \{\tilde{\tau}_{ik} | k = 1, \ldots, K_i\}\), and \(\|\tilde{x}_i\|_A = \sum_{k=1}^{K_i} |\tilde{a}_{ik}|\) for \(i = 1, 2\), and it satisfies \(y = \tilde{x}_1 + g \odot \tilde{x}_2\). If \(\tilde{\mathcal{Y}}_i = \mathcal{Y}_i\) for \(i = 1, 2\), we have \(\tilde{x}_1 = x_1^*\) and \(\tilde{x}_2 = x_2^*\) straightforwardly. We then consider the case when at least \(\tilde{\mathcal{Y}}_i \neq \mathcal{Y}_i\) for some \(i\). We have

\[\langle p, y \rangle_R = \langle p, \tilde{x}_1 + g \odot \tilde{x}_2 \rangle_R \]

\[= \langle p, \tilde{x}_1 \rangle_R + \langle g \odot p, \tilde{x}_2 \rangle_R \]

\[= \sum_{\tilde{\tau}_{1k} \in \tilde{\mathcal{Y}}_1 \cap \mathcal{Y}_1} \text{Re} \left( \tilde{\tilde{a}}_{1k} \sum_{n=-2M}^{2M} p_n e^{j2\pi n \tilde{\tau}_{1k}} \right) + \sum_{\tilde{\tau}_{1k} \in \tilde{\mathcal{Y}}_1 \setminus \mathcal{Y}_1} \text{Re} \left( \tilde{\tilde{a}}_{1k} \sum_{n=-2M}^{2M} p_n e^{j2\pi n \tilde{\tau}_{1k}} \right) \]

\[+ \sum_{\tilde{\tau}_{2k} \in \tilde{\mathcal{Y}}_2 \cap \mathcal{Y}_2} \text{Re} \left( \tilde{\tilde{a}}_{2k} \sum_{n=-2M}^{2M} \tilde{g}_n p_n e^{j2\pi n \tilde{\tau}_{2k}} \right) + \sum_{\tilde{\tau}_{2k} \in \tilde{\mathcal{Y}}_2 \setminus \mathcal{Y}_2} \text{Re} \left( \tilde{\tilde{a}}_{2k} \sum_{n=-2M}^{2M} \tilde{g}_n p_n e^{j2\pi n \tilde{\tau}_{2k}} \right) \]
To apply Lemma 2 to (29), we first bound $\|E_n\|$ as

$$\|E_n\| = \left\| \frac{1}{M} s_n g_n e_1 (n) e_2^H (n) \right\|
\leq \frac{1}{M} \left| s_n \right| \sqrt{K_1 + \frac{K_1}{K'' (0)}} (2\pi n)^2 \sqrt{K_2 + \frac{K_2}{K'' (0)}} (2\pi n)^2
\leq \frac{1}{M} \left( \max_{|n| \leq 2M} |s_n| \right) \sqrt{K_1 K_2} \left( 1 + \max_{|n| \leq 2M} \frac{(2\pi n)^2}{K'' (0)} \right)
\leq 14 \sqrt{K_1 K_2} \frac{M}{M} := R, \quad \text{for } M \geq 4,$$

where $\max_{|n| \leq 2M} |s_n| \leq 1$, and $\left( 1 + \max_{|n| \leq 2M} \frac{(2\pi n)^2}{K'' (0)} \right) = 1 + \frac{12M^2}{M^2} \leq 14$, for $M \geq 4$ [23].

Furthermore,

$$\left\| \sum_{n=-2M}^{2M} \mathbb{E} [E_n E_n^H] \right\|
= \left\| \sum_{n=-2M}^{2M} \mathbb{E} \left[ \frac{1}{M} s_n g_n e_1 (n) e_2^H (n) \cdot \frac{1}{M} s_n g_n e_2 (n) e_1^H (n) \right] \right\|
\leq \frac{1}{M} \sum_{n=-2M}^{2M} \frac{1}{M^2} s_n^2 K_2 \left( 1 + \frac{(2\pi n)^2}{K'' (0)} \right) e_1 (n) e_1^H (n)
\leq \frac{1}{M} K_2 \left( 1 + \max_{|n| \leq 2M} \frac{(2\pi n)^2}{K'' (0)} \right) \left( \max_{|n| \leq 2M} \left| s_n \right| \right) \left\| \sum_{n=-2M}^{2M} \frac{1}{M} s_n e_1 (n) e_1^H (n) \right\|
\leq \frac{14}{M} K_2 \left\| W_1 \right\| \leq 20 \frac{K_2}{M},$$

where the last inequality follows from (27). Similarly we can obtain $\left\| \sum_{n=-2M}^{2M} \mathbb{E} [E_n^H E_n] \right\| \leq 20 \frac{K_2}{M}$. Hence,

$$\sigma^2 = \max \left\{ \left\| \sum_n \mathbb{E} [E_n E_n^H] \right\|, \left\| \sum_n \mathbb{E} [E_n^H E_n] \right\| \right\} = \frac{20}{M} K_{\max}.$$

Apply Lemma 2, for $0 < \delta < 0.6376$, then we have

$$\mathbb{P} \{|W_\delta| \geq \delta\} \leq 2 \left( K_1 + K_2 \right) \exp \left( \frac{-\delta^2 / 2}{\frac{20}{M} K_{\max} + \frac{144}{3M} \sqrt{K_1 K_2}} \right)
\leq 2 \left( K_1 + K_2 \right) \exp \left( \frac{-\delta^2 M}{46 K_{\max}} \right) \leq \eta,$$

if $M \geq \frac{46}{\eta} K_{\max} \log \left( \frac{2(K_1 + K_2)}{\eta} \right)$.
F  Proof of Lemma 1

Proof. For both invertible $A$ and $B$ that satisfy $\|A - B\| \|B^{-1}\| \leq 1/2$, it has \cite{23}

$$\|A^{-1}\| \leq 2 \|B^{-1}\|,$$

and $$\|A^{-1} - B^{-1}\| \leq 2 \|B^{-1}\|^2 \|A - B\|.$$

Applying the above to $A = W$ and $B = W_\mu$, from (28), we have $\|W_\mu^{-1}\| \leq 1.568$. Under the event $\mathcal{E}_\delta$, $\|W - W_\mu\| = \|W_\mu\| \leq \delta$. Therefore as soon as $\delta \leq \frac{1}{4} \leq \frac{1}{2\|W_\mu^{-1}\|}$, we have

$$\|W_\mu^{-1}\| \leq 2 \|W_\mu^{-1}\|,$$

$$\|W^{-1} - W_\mu^{-1}\| \leq 2 \|W_\mu^{-1}\|^2 \|W - W_\mu\| \leq 2 \|W_\mu^{-1}\|^2 \delta.$$

Finally, because the operator norm of a matrix dominates that of its submatrices, we have

$$\|L_i - L_{\mu i}\| \leq 2 \|W_\mu^{-1}\|^2 \delta, \quad \text{for } i = 1, 2,$$

$$\|L_\theta\| \leq 2 \|W_\mu^{-1}\|^2 \delta,$$

$$\|L_\theta\| \leq 2 \|W_\mu^{-1}\|^2 \delta,$$

and $\|L_i\| \leq \|L_i - L_{\mu i}\| + \|L_{\mu i}\| \leq 2 \|W_\mu^{-1}\|^2 \delta + \|W_\mu^{-1}\| \leq 2 \|W_\mu^{-1}\|$ for $i = 1, 2$ where we have used $\|W_\mu^{-1}\| \leq 1.568$ and $\delta \leq 1/4$. \hfill \Box

G  Proof of Proposition 5

Proof. Conditioned on the event $\mathcal{E}_\delta$ with $\delta \in (0, 1/4]$, we have

$$\left| \frac{1}{\sqrt{|K^n(0)|}} P(l) (\tau) \right| \leq \left| \langle u_1, L_1^H v_{1l}(\tau) \rangle \right| + \left| \langle u_2, L_2^H v_{1l}(\tau) \rangle \right| + \left| \langle u_1, L_2^H v_{2l}(\tau) \rangle \right| + \left| \langle u_2, L_2^H v_{2l}(\tau) \rangle \right|$$

$$\leq \|u_1\|_2 \|L_1\| \|v_{1l}(\tau)\|_2 + \|u_2\|_2 \|L_2\| \|v_{1l}(\tau)\|_2 + \|u_1\|_2 \|L_2\| \|v_{2l}(\tau)\|_2 + \|u_2\|_2 \|L_2\| \|v_{2l}(\tau)\|_2$$

$$\leq \sqrt{K_1} \cdot 2 \|W_\mu^{-1}\| \cdot (4M + 1) \frac{1}{M} 4^{l+1} \sqrt{K_1} + \sqrt{K_2} \cdot 0.8 \|W_\mu^{-1}\| \cdot (4M + 1) \frac{1}{M} 4^{l+1} \sqrt{K_1} \cdot (4M + 1) \frac{1}{M} 4^{l+1} \sqrt{K_2}$$

$$\leq C (K_1 + K_2),$$

(44)

for some universal constant $C$. In (44), we applied Lemma 1, $\|u_i\|_2 = \sqrt{K_i}$, for $i = 1, 2$, and

$$\|v_{1l}(\tau)\|_2 = \left\| \frac{1}{M} \sum_{n=-2M}^{2M} s_n \left( \frac{-j2\pi n}{\sqrt{|K^n(0)|}} \right)^l e^{-j2\pi n \tau} e_1(n) \right\|_2$$

$$\leq \frac{1}{M} (4M + 1) \left( \max_{|n| \leq 2M} |s_n| \right) \left( \max_{|n| \leq 2M} \left| \frac{j2\pi n}{\sqrt{|K^n(0)|}} \right| \right)^l \left( \max_{|n| \leq 2M} |e_1(n)| \right)_2$$

$$\leq \frac{1}{M} (4M + 1) 4^l \sqrt{K_1} \max_{|n| \leq 2M} \left( 1 + \frac{(2\pi n)^2}{|K^n(0)|} \right)$$

$$\leq \frac{1}{M} (4M + 1) 4^l \sqrt{14K_1},$$

(45)

1This choice of $\delta$ is not unique but good enough for our purpose.

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and similarly
\[ ||v_{2\ell}(\tau)||_2 \leq \frac{1}{M} (4M + 1) 4^l \sqrt{14K_2}. \]

In (45) we have used \( \max_{|n| \leq 2M} |s_n| \leq 1, \) \( \max_{|n| \leq 2M} \left| \frac{j^{2\pi n}}{\sqrt{|K''(0)|}} \right| \leq 4, \) for \( M \geq 2 \) and \( \left( 1 + \max_{|n| \leq 2M} \left( \frac{2\pi n}{|K''(0)|} \right)^2 \right) \leq 14, \) for \( M \geq 4. \)

Using Lemma 3, we have
\[
\frac{1}{\sqrt{|K''(0)|}} P^{(\ell)}(\tau_a) - \frac{1}{\sqrt{|K''(0)|}} P^{(\ell)}(\tau_b) \leq \left| e^{2\pi \tau_a} - e^{2\pi \tau_b} \right| \sup_{\tau \in [0,1]} \left| \frac{\partial}{\partial e^{2\pi \tau}} \right| \left( P^{(\ell)} \left( e^{2\pi \tau} \right) \right)
\]
\[
\leq \left| e^{2\pi \tau_a} - e^{2\pi \tau_b} \right| \cdot 2M \sup_{\tau \in [0,1]} \left| \frac{1}{\sqrt{|K''(0)|}} P^{(\ell)}(\tau) \right|
\]
\[
\leq 4\pi |\tau_a - \tau_b| \cdot 2M \cdot C (K_1 + K_2).
\]

Note that similar bounds also hold for \( P^{(\ell)}(\tau). \) Conditioned on the event \( \mathcal{E}_\delta \cap \mathcal{E}_1 \) with \( \delta \in (0, 1/4], \) we have
\[
\left| \frac{1}{\sqrt{|K''(0)|}} P^{(\ell)}(\tau) - \frac{1}{\sqrt{|K''(0)|}} P^{(\ell)}(\tau_d) \right|
\]
\[
\leq \left| \frac{1}{\sqrt{|K''(0)|}} P^{(\ell)}(\tau) - \frac{1}{\sqrt{|K''(0)|}} P^{(\ell)}(\tau_d) \right| + \left| \frac{1}{\sqrt{|K''(0)|}} P^{(\ell)}(\tau_d) - \frac{1}{\sqrt{|K''(0)|}} P^{(\ell)}(\tau) \right|
\]
\[
\leq 4\pi |\tau - \tau_d| \cdot 2M \cdot C (K_1 + K_2) + \frac{\epsilon}{3} + 4\pi |\tau_d - \tau| \cdot 2M \cdot C (K_1 + K_2),
\]
for any \( \tau \in [0,1], \) where \( \tau_d \in \gamma_{\text{grid}}. \) By setting the grid size \( |\gamma_{\text{grid}}| = \left[ \frac{24\pi CM (K_1 + K_2)}{\epsilon} \right] \), we have \( |\tau_d - \tau| \leq \frac{24\pi CM (K_1 + K_2)}{24\pi CM (K_1 + K_2)}, \) which yields
\[
\left| \frac{1}{\sqrt{|K''(0)|}} P^{(\ell)}(\tau) - \frac{1}{\sqrt{|K''(0)|}} P^{(\ell)}(\tau) \right| \leq \epsilon.
\]

By plugging the grid size and modifying the condition on \( M, \) the proof is complete. \( \square \)

H Proof of Proposition 7

Proof. Denote \( f(x_1, x_2) = \frac{1}{2} \|y - x_1 - g \odot x_2\|^2 + \lambda_w (\|x_1\|_A + \|x_2\|_A) \) as the objective function of (9). Since \( \{\hat{x}_1, \hat{x}_2\} \) is the minimizer of (9), for all \( \alpha \in (0, 1] \) and all \( \{\hat{x}_1, \hat{x}_2\}, \) we have
\[
f(\alpha \hat{x}_1 + (1 - \alpha \hat{x}_1), \alpha \hat{x}_2 + (1 - \alpha \hat{x}_2)) \geq f(\hat{x}_1, \hat{x}_2).
\]

This is equivalent to the following
\[
\alpha_w^{-1} \lambda_w (\|\hat{x}_1 + \alpha_w (x_1 - \hat{x}_1)\|_A - \|\hat{x}_1\|_A) + \alpha_w^{-1} \lambda_w (\|\hat{x}_2 + \alpha_w (x_2 - \hat{x}_2)\|_A - \|\hat{x}_2\|_A)
\]
\[
\geq \langle y - (\hat{x}_1 + g \odot \hat{x}_2), (\hat{x}_1 - \hat{x}_1) + g \odot (\hat{x}_2 - \hat{x}_2) \rangle - \frac{1}{2} \alpha_w \|\hat{x}_1 - \hat{x}_1 + g \odot (\hat{x}_2 - \hat{x}_2)\|^2_2.
\]

As the atomic norm \( \|\cdot\|_A \) is convex, the following inequalities hold:
\[
\|\hat{x}_1\|_A - \|\hat{x}_1\|_A \geq \alpha_w^{-1} (\|\hat{x}_1 + \alpha_w (\hat{x}_1 - \hat{x}_1)\|_A - \|\hat{x}_1\|_A),
\]
\[
\|\hat{x}_2\|_A - \|\hat{x}_2\|_A \geq \alpha_w^{-1} (\|\hat{x}_2 + \alpha_w (\hat{x}_2 - \hat{x}_2)\|_A - \|\hat{x}_2\|_A),
\]
which can be plugged into the previous inequality to obtain
\[ \lambda_w (\|\hat{x}_1\|_A + \|\hat{x}_2\|_A - \|\hat{x}_1\|_A - \|\hat{x}_2\|_A) \]
\[ \geq \langle y - (\hat{x}_1 + g \circ \hat{x}_2), (\hat{x}_1 - \hat{x}_1) + g \circ (\hat{x}_2 - \hat{x}_2) \rangle \]
\[ - \frac{1}{2} \alpha_w \|\hat{x}_1 - \hat{x}_1 + g \circ (\hat{x}_2 - \hat{x}_2)\|^2. \]
Set \( \alpha_w \rightarrow 0 \), we can obtain that \( \{\hat{x}_1, \hat{x}_2\} \) is the minimizer of (9) only if for all \( \{\hat{x}_1, \hat{x}_2\} \), there exists
\[ \lambda_w (\|\hat{x}_1\|_A + \|\hat{x}_2\|_A - \|\hat{x}_1\|_A - \|\hat{x}_2\|_A) \geq \langle y - (\hat{x}_1 + g \circ \hat{x}_2), (\hat{x}_1 - \hat{x}_1) + g \circ (\hat{x}_2 - \hat{x}_2) \rangle. \quad (46) \]
On the other hand, if (46) holds for all \( \{\hat{x}_1, \hat{x}_2\} \), we have
\[ f(\hat{x}_1, \hat{x}_2) = \frac{1}{2} \|y - \hat{x}_1 - g \circ \hat{x}_2\|^2 + \lambda_w (\|\hat{x}_1\|_A + \|\hat{x}_2\|_A) \]
\[ = \frac{1}{2} \|y - \hat{x}_1 - g \circ \hat{x}_2 + \hat{x}_1 + g \circ \hat{x}_2 - \hat{x}_1 - g \circ \hat{x}_2\|^2 + \lambda_w (\|\hat{x}_1\|_A + \|\hat{x}_2\|_A) \]
\[ + \lambda_w (\|\hat{x}_1\|_A + \|\hat{x}_2\|_A - \|\hat{x}_1\|_A - \|\hat{x}_2\|_A) \]
\[ = \frac{1}{2} \|y - \hat{x}_1 - g \circ \hat{x}_2\|^2 + \lambda_w (\|\hat{x}_1\|_A + \|\hat{x}_2\|_A) + \frac{1}{2} \|\hat{x}_1 + g \circ \hat{x}_2 - \hat{x}_1 - g \circ \hat{x}_2\|^2 \]
\[ + \langle y - \hat{x}_1 - g \circ \hat{x}_2, \hat{x}_1 + g \circ \hat{x}_2 - \hat{x}_1 - g \circ \hat{x}_2 \rangle + \lambda_w (\|\hat{x}_1\|_A + \|\hat{x}_2\|_A - \|\hat{x}_1\|_A - \|\hat{x}_2\|_A) \]
\[ \geq f(\hat{x}_1, \hat{x}_2) + \frac{1}{2} \|\hat{x}_1 + g \circ \hat{x}_2 - \hat{x}_1 - g \circ \hat{x}_2\|^2 \]
\[ \geq f(\hat{x}_1, \hat{x}_2). \]
Therefore, (46) holds if and only if \( \{\hat{x}_1, \hat{x}_2\} \) is the minimizer of (9).
Furthermore, we can rewrite (46) by moving all the terms containing \( \{\hat{x}_1, \hat{x}_2\} \) onto one side as
\[ \lambda_w (\|\hat{x}_1\|_A + \|\hat{x}_2\|_A) - \langle y - (\hat{x}_1 + g \circ \hat{x}_2), \hat{x}_1 + g \circ \hat{x}_2 \rangle \]
\[ \leq \lambda_w \|\hat{x}_1\|_A - \langle y - (\hat{x}_1 + g \circ \hat{x}_2), \hat{x}_1 \rangle_R + \lambda_w \|\hat{x}_2\|_A - \langle y - (\hat{x}_1 + g \circ \hat{x}_2), g \circ \hat{x}_2 \rangle_R. \quad (47) \]
Since (47) holds for all \( \{\hat{x}_1, \hat{x}_2\} \), (47) still holds if taking infimum on the right-hand side with respect to \( \{\hat{x}_1, \hat{x}_2\} \). That is
\[ \lambda_w (\|\hat{x}_1\|_A + \|\hat{x}_2\|_A) - \langle y - (\hat{x}_1 + g \circ \hat{x}_2), \hat{x}_1 + g \circ \hat{x}_2 \rangle_R \]
\[ \leq \inf_{\hat{x}_1} \{\lambda_w \|\hat{x}_1\|_A - \langle y - (\hat{x}_1 + g \circ \hat{x}_2), \hat{x}_1 \rangle_R + \inf_{\hat{x}_2} \{\lambda_w \|\hat{x}_2\|_A - \langle y - (\hat{x}_1 + g \circ \hat{x}_2), g \circ \hat{x}_2 \rangle_R \}. \]
Plugging in the facts that
\[ \inf_{\hat{x}_i} \{\|\hat{x}_i\|_A - \langle y - (\hat{x}_1 + g \circ \hat{x}_2), \hat{x}_i \rangle_R \} = \begin{cases} 0, & \|y - (\hat{x}_1 + g \circ \hat{x}_2)\|_A \leq 1, \\ -\infty, & \text{otherwise} \end{cases} \]
we have
\[ \langle y - (\hat{x}_1 + g \circ \hat{x}_2), \hat{x}_1 + g \circ \hat{x}_2 \rangle_R \geq \lambda_w \|\hat{x}_1\|_A + \lambda_w \|\hat{x}_2\|_A, \]
as well as
\[ \|y - (\hat{x}_1 + g \circ \hat{x}_2)\|_A \leq \lambda_w, \text{ and } \|g \circ (y - (\hat{x}_1 + g \circ \hat{x}_2))\|_A \leq \lambda_w. \]

\section{Proof of Proposition 8}

\textbf{Proof.} We first record a useful lemma from [37].

\textbf{Lemma 5. [37, Lemma 1]} For any 2mth-order trigonometric polynomial \( X(\tau) = (x, c(\tau)) \), we have
\[ \int_{0}^{1} X(\tau) \nu_i(\tau) d\tau \leq \|x\|_A^* \left( \|P_A(\nu_i)\|_T + \sum_{j=0}^{2} I_{i,j} \right), \]
for \( i = 1, 2 \), where \( P_A(\nu_i) \) denote the projection of the measure \( \nu_i \) on the support set \( A \).
Setting $X(\tau) = \langle e_i, c(\tau) \rangle$ for $i = 1, 2$ in Lemma 5, we obtain

$$\|e_i\|_2^2 = \left\langle e_i, \int_0^1 c(\tau) \nu_i(d\tau) \right\rangle = \int_0^1 \langle e_i, c(\tau) \rangle \nu_i(d\tau)$$

$$\leq \|e_i\|_A \left( \|P_{T_{far}}(\nu_i)\|_{TV} + \sum_{j=0}^2 I_{i,j} \right)$$

$$\leq \sqrt{4M + 1}\|e_i\|_2^2 \left( \|P_{T_{far}}(\nu_i)\|_{TV} + \sum_{j=0}^2 I_{i,j} \right),$$

where we used the fact $\|e_i\|_A = \sup_{\tau \in [0,1]} |\langle e_i, c(\tau) \rangle|$ $\leq \|e_i\|_2 \|c(\tau)\|_2 = \sqrt{4M + 1}\|e_i\|_2$ following the Cauchy-Schwarz inequality. This yields the estimation error of $x_i^*$ in (40). For the denoising error, first notice that,

$$\|e_1 + g \circ e_2\|_A \leq \|y - x_1^* - g \circ x_2^*\|_A + \|y - \hat{x}_1 - g \circ \hat{x}_2\|_A$$

$$\leq \|w\|_A + \lambda_w$$

$$\leq 2\lambda_w,$$ 

(48)

(49)

where (48) follows from Proposition 7, and (49) follows from $\|w\|_A = \sup_{\tau \in [0,1]} |\langle w, c(\tau) \rangle| \leq \|w\|_2 \|c(\tau)\|_2 \leq \sigma_w \sqrt{4M + 1} = \lambda_w/C_w \leq \lambda_w$. Similarly, we have $\|\tilde{g} \circ w\|_A \leq \lambda_w/C_w$ and consequently $\|\tilde{g} \circ e_1 + e_2\|_A \leq 2\lambda_w$. Therefore, we have

$$\|e_1 + g \circ e_2\|_2^2 = \langle e_1 + g \circ e_2, e_1 \rangle + \langle \tilde{g} \circ e_1 + e_2, e_2 \rangle$$

$$= \langle e_1 + g \circ e_2, \int_0^1 c(\tau) \nu_1(d\tau) \rangle + \langle \tilde{g} \circ e_1 + e_2, \int_0^1 c(\tau) \nu_2(d\tau) \rangle$$

$$= \int_0^1 \langle e_1 + g \circ e_2, c(\tau) \rangle \nu_1(d\tau) + \int_0^1 \langle \tilde{g} \circ e_1 + e_2, c(\tau) \rangle \nu_2(d\tau)$$

$$\leq \|e_1 + g \circ e_2\|_A \left( \|P_{T_{far}}(\nu_1)\|_{TV} + \sum_{j=0}^2 I_{1,j} \right) + \|\tilde{g} \circ e_1 + e_2\|_A \left( \|P_{T_{far}}(\nu_2)\|_{TV} + \sum_{j=0}^2 I_{2,j} \right)$$

$$\leq 2\lambda_w \sum_{i=1}^2 \left( \|P_{T_{far}}(\nu_i)\|_{TV} + \sum_{j=0}^2 I_{i,j} \right),$$

where we used (49) in the last inequality. \hfill \qed

### J Proof of Proposition 9

We first construct a pair of trigonometric polynomials $P_1(\tau)$ and $Q_1(\tau)$ with the following properties whose proof can be found in Appendix L.

**Lemma 6.** Assume that $g_n = e^{j2\pi \phi_n}$’s are i.i.d. randomly generated from a uniform distribution on the complex unit circle with $\phi_n \sim \mathcal{U}[0,1]$. Provided that the separation $\Delta \geq 1/M$, there exists a numerical constant $C$ such that as soon as

$$M \geq C \max \left\{ \log^2 \left( \frac{M (K_1 + K_2)}{\eta} \right), K_{\max} \log \left( \frac{M (K_1 + K_2)}{\eta} \right), K_{\max}^2 \log \left( \frac{K_1 + K_2}{\eta} \right) \right\},$$

we can construct $P_1(\tau) = \sum_{n=-2M}^{2M} p_{1n} e^{j2\pi n \tau}$ and $Q_1(\tau) = \sum_{n=-2M}^{2M} p_{1n} \tilde{g}_n e^{j2\pi n \tau}$ that satisfy

$$|P_1(\tau) - \text{sign}(a_{1k})(\tau - \tau_{1k})| \leq C_p M (\tau - \tau_{1k})^2, \quad \tau \in \mathbb{Y}_{\text{near}}, \quad k = 1, \ldots, K_1,$$
\[ |P_1(\tau)| \leq \frac{C'_p}{M}, \quad \tau \in \mathcal{Y}_{\text{far}}, \]
\[ |Q_1(\tau) - \text{sign} (a_{2k}) (\tau - \tau_{2k})| \leq C_q M (\tau - \tau_{2k})^2, \quad \tau \in \mathcal{Y}_{\text{near}}, \quad k = 1, \ldots, K_2, \]
\[ |Q_1(\tau)| \leq \frac{C'_q}{M}, \quad \tau \in \mathcal{Y}_{\text{far}}, \]

with probability at least \(1 - \eta\), where \(C_p, C'_p, C_q\) and \(C'_q\) are numerical constants.

Furthermore, we derive the following useful lemma in Appendix M.

**Lemma 7.** For \(P(\tau)\) and \(Q(\tau)\) constructed in Proposition 6, and \(P_1(\tau)\) and \(Q_1(\tau)\) constructed in Lemma 6, there exist numerical constant \(C\) and \(C_1\) such that

\[ \left| \int_0^1 P(\tau) \nu_1 (d\tau) + \int_0^1 Q(\tau) \nu_2 (d\tau) \right| \leq C\lambda_w \sqrt{\frac{K^3_{\text{max}} \log M}{M}}, \quad (50) \]
\[ \left| \int_0^1 P_1(\tau) \nu_1 (d\tau) + \int_0^1 Q_1(\tau) \nu_2 (d\tau) \right| \leq C_1\lambda_w \sqrt{\frac{K^3_{\text{max}} \log M}{M^3}}, \quad (51) \]

with high probability given in Theorem 2.2.

**Proof.** Consider the polar form

\[ \left| \int_{\mathcal{Y}_{\text{near}}} \nu_i (d\tau) \right| = e^{-j\rho_{ik}} \int_{\mathcal{Y}_{\text{near}}} \nu_i (d\tau), \quad i = 1, 2, \]

then we can construct a pair of dual polynomials \(P(\tau)\) and \(Q(\tau)\) that interpolate a pair of point sources with \(\text{sign} (\delta_{ik}) = e^{-j\rho_{ik}}\), as in Proposition 6. Therefore, we have

\[ I_{1,0} = \sum_{k=1}^{K_1} \left| \int_{\mathcal{Y}_{\text{near}}} \nu_1 (d\tau) \right| \]
\[ = \sum_{k=1}^{K_1} \int_{\mathcal{Y}_{\text{near}}} P(\tau) \nu_1 (d\tau) + \sum_{k=1}^{K_1} \int_{\mathcal{Y}_{\text{near}}} (e^{-j\rho_{ik}} - P(\tau)) \nu_1 (d\tau) \]
\[ = \int_0^1 P(\tau) \nu_1 (d\tau) - \int_{\mathcal{Y}_{\text{far}}} P(\tau) \nu_1 (d\tau) + \sum_{k=1}^{K_1} \int_{\mathcal{Y}_{\text{near}}} (e^{-j\rho_{ik}} - P(\tau)) \nu_1 (d\tau). \]

Similarly,

\[ I_{2,0} = \int_0^1 Q(\tau) \nu_2 (d\tau) - \int_{\mathcal{Y}_{\text{far}}} Q(\tau) \nu_2 (d\tau) + \sum_{k=1}^{K_2} \int_{\mathcal{Y}_{\text{near}}} (e^{-j\rho_{2k}} - Q(\tau)) \nu_2 (d\tau). \]

Now consider their sum, then we have

\[ \sum_{i=1}^{2} I_{i,0} \leq \left| \int_0^1 P(\tau) \nu_1 (d\tau) + \int_0^1 Q(\tau) \nu_2 (d\tau) \right| + \sum_{i=1}^{2} \left| \mathcal{P}_{\mathcal{Y}_{\text{far}}} (\nu_i) \right|_{TV} \]
\[ + \sum_{k=1}^{K_1} \int_{\mathcal{Y}_{\text{near}}} C_p M^2 (\tau - \tau_{1k})^2 |\nu_1|(d\tau) + \sum_{k=1}^{K_2} \int_{\mathcal{Y}_{\text{near}}} C_q M^2 (\tau - \tau_{2k})^2 |\nu_2|(d\tau) \]
\[ \leq \left| \int_0^1 P(\tau) \nu_1 (d\tau) + \int_0^1 Q(\tau) \nu_2 (d\tau) \right| + \sum_{i=1}^{2} \left| \mathcal{P}_{\mathcal{Y}_{\text{far}}} (\nu_i) \right|_{TV} + C_2 \sum_{i=1}^{2} I_{i,2} \]
\[ \leq C\lambda_w \sqrt{\frac{K^3_{\text{max}} \log M}{M}} + \sum_{i=1}^{2} \left| \mathcal{P}_{\mathcal{Y}_{\text{far}}} (\nu_i) \right|_{TV} + C_2 \sum_{i=1}^{2} I_{i,2}, \quad (54) \]
where (52) follows from the triangle inequality and the properties of the dual polynomials in Proposition 6, (53) follows from the definition of $I_{i,2}$, and (54) follows from Lemma 7.

Then, we consider bounding $\sum_{i=1}^{2} I_{i,1}$ in a similar way. Again, consider the polar form

$$\left| \int_{Y_{\text{near}}^{1,k}} (\tau - \tau_{1k}) \nu_i (d\tau) \right| = e^{-j\rho_{1k}} \int_{Y_{\text{near}}^{1,k}} (\tau - \tau_{1k}) \nu_i (d\tau), \quad i = 1, 2,$$

then we can construct a pair of polynomials $P_1 (\tau)$ and $Q_1 (\tau)$ in the form of Lemma 6 by letting sign $(\hat{a}_{ik}) = e^{-j\rho_{1k}}$. Then we have

$$\frac{I_{1,1}}{4M + 1} = \sum_{k=1}^{K_1} \int_{Y_{\text{near}}^{1,k}} (\tau - \tau_{1k}) \nu_1 (d\tau)$$

$$= \sum_{k=1}^{K_1} \int_{Y_{\text{near}}^{1,k}} (e^{-j\rho_{1k}} (\tau - \tau_{1k}) - P_1 (\tau)) \nu_1 (d\tau) + \sum_{k=1}^{K_1} \int_{Y_{\text{near}}^{1,k}} P_1 (\tau) \nu_1 (d\tau)$$

$$= \sum_{k=1}^{K_1} \int_{Y_{\text{near}}^{1,k}} (e^{-j\rho_{1k}} (\tau - \tau_{1k}) - P_1 (\tau)) \nu_1 (d\tau) + \int_0^1 P_1 (\tau) \nu_1 (d\tau) - \int_{Y_{\text{far}}} P_1 (\tau) \nu_1 (d\tau),$$

and

$$\frac{I_{2,1}}{4M + 1} = \sum_{k=1}^{K_2} \int_{Y_{\text{near}}^{2,k}} (e^{-j\rho_{2k}} (\tau - \tau_{2k}) - Q_1 (\tau)) \nu_2 (d\tau) + \int_0^1 Q_1 (\tau) \nu_2 (d\tau) - \int_{Y_{\text{far}}} Q_1 (\tau) \nu_2 (d\tau).$$

Taking their sum, we have

$$\sum_{i=1}^{2} I_{i,1} \leq (4M + 1) \left( \left| \int_0^1 P_1 (\tau) \nu_1 (d\tau) + \int_0^1 Q_1 (\tau) \nu_2 (d\tau) \right| + \sum_{k=1}^{K_1} \int_{Y_{\text{near}}^{1,k}} |e^{-j\rho_{1k}} (\tau - \tau_{1k}) - P_1 (\tau)| \|\nu_1\| (d\tau) \right.$$

$$+ \sum_{k=1}^{K_2} \int_{Y_{\text{near}}^{2,k}} |e^{-j\rho_{2k}} (\tau - \tau_{2k}) - Q_1 (\tau)| \|\nu_2\| (d\tau) \bigg) + C_3 \sum_{i=1}^{2} \left\| P_{Y_{\text{far}}} (\nu_i) \right\|_{TV}$$

$$\leq C\lambda w \sqrt{K_{\text{max}}^3 \log M \over M} + C_2 \sum_{i=1}^{2} I_{i,2} + C_3 \sum_{i=1}^{2} \left\| P_{Y_{\text{far}}} (\nu_i) \right\|_{TV},$$

where the first inequality follows from the triangle inequality and Lemma 6, and the last inequality follows from Lemma 7, the definition of $I_{i,2}$ and Lemma 6.

K Proof of Proposition 10

Proof. Let $\hat{u}_i$ and $u_i^*$ denote the representing measure of $\hat{x}_i$ and $x_i^*$, then we have $\nu_i = \hat{u}_i - u_i^*, \quad i = 1, 2$. Since $\|u_i^*\|_{TV} = \|x_i^*\|_A$ and $\|\hat{u}_i\|_{TV} = \|\hat{x}_i\|_A$, from Proposition 7, we have

$$\|\hat{u}_1\|_{TV} + \|\hat{u}_2\|_{TV} = \|\hat{x}_1\|_A + \|\hat{x}_2\|_A$$

$$= \frac{1}{\lambda w} \langle y - (\hat{x}_1 + g \circ \hat{x}_2), e_1 + g \circ e_2 \rangle_R$$

$$+ \frac{1}{\lambda w} \langle y - (\hat{x}_1 + g \circ \hat{x}_2), x_1^* \rangle_R + \frac{1}{\lambda w} \langle y - (\hat{x}_1 + g \circ \hat{x}_2), g \circ x_1^* \rangle_R$$

$$\leq \frac{1}{\lambda w} \langle y - (\hat{x}_1 + g \circ \hat{x}_2), e_1 + g \circ e_2 \rangle_R + \|x_1^*\|_A + \|x_2^*\|_A$$

$$\leq \frac{1}{\lambda w} \langle x_1^* + g \circ x_2^* + w - (\hat{x}_1 + g \circ \hat{x}_2), e_1 + g \circ e_2 \rangle_R + \|x_1^*\|_A + \|x_2^*\|_A$$

$$= \frac{1}{\lambda w} \langle e_1 + g \circ e_2 \rangle_R^2 + \frac{1}{\lambda w} \langle w, e_1 + g \circ e_2 \rangle_R + \|u_1^*\|_{TV} + \|u_2^*\|_{TV}$$

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Then the last term in (55) can be bounded by
\[
|\langle w, e_1 + g \circ e_2 \rangle| \leq |\langle w, e_1 \rangle| + |\langle g \circ w, e_2 \rangle| \\
= \int_0^1 \langle w, c(\tau) \nu_1 (d\tau) \rangle + \int_0^1 \langle g \circ w, c(\tau) \nu_2 (d\tau) \rangle \\
\leq \|w\|_A \left( \|P_{Y_{1, \text{far}}}(\nu_1)\|_{TV} + \sum_{j=0}^2 I_{1,j} \right) + \|g \circ w\|^*_{A} \left( \|P_{Y_{2, \text{far}}}(\nu_2)\|_{TV} + \sum_{j=0}^2 I_{2,j} \right),
\]
where (56) follows from Lemma 5, and the last inequality (57) follows from \( \|w\|^*_{A} \leq \lambda_w/C_w \) and \( \|g \circ w\|^*_{A} \leq \lambda_w/C_w \). Moreover, since
\[
\|\mathbf{u}_i\|_{TV} = \|u_i^* + \nu_i\|_{TV} \geq \|u_i^*\|_{TV} - \|P_{Y_1}(\nu_i)\|_{TV} + \|P_{Y_1}(\nu_i)\|_{TV}, \quad i = 1, 2,
\]
plugging this and (57) into (55), we have
\[
\sum_{i=1}^2 \|P_{Y_{i, \text{far}}}(\nu_i)\|_{TV} - \sum_{i=1}^2 \|P_{Y_i}(\nu_i)\|_{TV} \leq \frac{1}{C_w} \sum_{i=1}^2 \left( \|P_{Y_{i, \text{far}}}(\nu_i)\|_{TV} + \sum_{j=0}^2 I_{i,j} \right).
\]

Set \(P(\tau)\) and \(Q(\tau)\) as a pair of polynomials that interpolate the conjugate sign of \(P_{Y_1}(\nu_1)\) and \(P_{Y_2}(\nu_2)\), respectively, whose existence is established in Proposition 6, then we have
\[
\sum_{i=1}^2 \|P_{Y_i}(\nu_i)\|_{TV} = \int_0^1 P(\tau)P_{Y_1}(\nu_1)(d\tau) + \int_0^1 Q(\tau)P_{Y_2}(\nu_2)(d\tau) \\
\leq \left| \int_0^1 P(\tau)\nu_1(\tau) d\tau \right| + \left| \int_0^1 Q(\tau)\nu_2(\tau) d\tau \right| + \left| \int_{Y_1} P(\tau)\nu_1(\tau) d\tau \right| + \left| \int_{Y_2} Q(\tau)\nu_2(\tau) d\tau \right|, 
\frac{K_3 \max \log M}{M} + \int_{Y_1} P(\tau)\nu_1(\tau) d\tau + \int_{Y_2} Q(\tau)\nu_2(\tau) d\tau,
\]
where the first term in (59) can be bounded using Lemma 7. For the second term in (59), according to the properties of \(P(\tau)\) established in Proposition 6, we have
\[
\left| \int_{Y_1} P(\tau)\nu_1(\tau) d\tau \right| = \left| \int_{Y_{\text{far}}} P(\tau)\nu_1(\tau) d\tau \right| + \left| \int_{Y_{\text{near}} \setminus \{\tau_1\}} P(\tau)\nu_1(\tau) d\tau \right| \\
\leq \sum_{k=1}^{K_1} \left| \int_{Y_{\text{near}} \setminus \{\tau_1\}} P(\tau)\nu_1(\tau) d\tau \right| + \left| \int_{Y_{\text{far}}} P(\tau)\nu_1(\tau) d\tau \right| \\
\leq \sum_{k=1}^{K_1} \left| \int_{Y_{\text{near}} \setminus \{\tau_1\}} |P(\tau)| |\nu_1| (d\tau) + (1 - C_b) \int_{Y_{\text{far}}} |\nu_1| (d\tau) \right| \\
\leq \sum_{k=1}^{K_1} \left| \int_{Y_{\text{near}} \setminus \{\tau_1\}} (1 - CM^2 (\tau - \tau_{1k})^2) |\nu_1| (d\tau) + (1 - C_b) \int_{Y_{\text{far}}} |\nu_1| (d\tau) \right| \\
= \sum_{k=1}^{K_1} \left| \int_{Y_{\text{near}} \setminus \{\tau_1\}} |\nu_1| (d\tau) + \int_{Y_{\text{far}}} |\nu_1| (d\tau) \right|
\]
Therefore, following Proposition 3, we have

\[
- C \sum_{k=1}^{K_1} \int_{\tau_{1,n,a} \setminus \{\tau_{1,k}\}} M^2 (\tau - \tau_{1,k})^2 |\nu_1| (d\tau) - C_b \int_{\tau_{1,a}} |\nu_1| (d\tau)
\leq \|P_{\bar{Y}^i} (\nu_i)\|_{TV} - C_a I_{1,2} - C_b \|P_{\bar{Y}^i} (\nu_i)\|_{TV},
\]

for some positive constants $C_a$ and $C_b$. A similar bound holds for the third term. Putting together, we have

\[
\sum_{i=1}^{2} \|P_{\bar{Y}^i} (\nu_i)\|_{TV} - \sum_{i=1}^{2} \|P_{\bar{Y}^i} (\nu_i)\|_{TV} \geq \sum_{i=1}^{2} \left( C_a I_{i,2} + C_b \|P_{\bar{Y}^i} (\nu_i)\|_{TV} \right) - C\lambda w \sqrt{\frac{K_{\max}^3 \log M}{M}},
\]

which combined with (58) yields:

\[
\frac{1}{C_w} \sum_{i=1}^{2} \left( \|P_{\bar{Y}^i} (\nu_i)\|_{TV} + \sum_{j=0}^{2} I_{i,j} \right) + C\lambda w \sqrt{\frac{K_{\max}^3 \log M}{M}} \geq \sum_{i=1}^{2} \left( C_a I_{i,2} + C_b \|P_{\bar{Y}^i} (\nu_i)\|_{TV} \right).
\]

The proof is finished by reorganizing terms and plugging in Proposition 9, for a large enough constant $C_w > 1$. \qed

L Proof of Lemma 6

Here we constructed the pair of polynomials $P_1 (\tau)$ and $Q_1 (\tau)$ using the same techniques as the ones in proof of Theorem 2.1. Recall the definitions of $K (\tau), K_g (\tau)$ and $K_g (\tau)$ in (19) and (20), and we construct two polynomials $P_1 (\tau)$ and $Q_1 (\tau)$ as

\[
P_1 (\tau) = K_{1,k} K (\tau - \tau_{1,k}) + \sum_{k=1}^{K_1} \psi_{1,k} K' (\tau - \tau_{1,k}) + \sum_{k=1}^{K_2} \theta_{2,k} K_g (\tau - \tau_{2,k}) + \sum_{k=1}^{K_2} \psi_{2,k} K' (\tau - \tau_{2,k}),
\]

and

\[
Q_1 (\tau) = \sum_{k=1}^{K_1} \theta_{1,k} K_g (\tau - \tau_{1,k}) + \sum_{k=1}^{K_1} \psi_{1,k} K' (\tau - \tau_{1,k}) + \sum_{k=1}^{K_2} \theta_{2,k} K (\tau - \tau_{2,k}) + \sum_{k=1}^{K_2} \psi_{2,k} K' (\tau - \tau_{2,k}),
\]

where $\tau_{1,k} \in \Upsilon_1$ and $\tau_{2,k} \in \Upsilon_2$. Set the coefficients $\theta_i = [\theta_{1,1}, \ldots, \theta_{1,K_1}]^T$, $\psi_i = [\psi_{1,1}, \ldots, \psi_{1,K_1}]^T$, for $i = 1, 2$ by solving the following set of equations

\[
\begin{align*}
P_1 (\tau_{1,k}) &= 0, & \tau_{1,k} &\in \Upsilon_1, \\
P'_1 (\tau_{1,k}) &= \text{sign} (a_{1,k}), & \tau_{1,k} &\in \Upsilon_1, \\
Q_1 (\tau_{2,k}) &= 0, & \tau_{2,k} &\in \Upsilon_2, \\
Q'_1 (\tau_{2,k}) &= \text{sign} (a_{2,k}), & \tau_{2,k} &\in \Upsilon_2,
\end{align*}
\]

which can be rewritten into a matrix form as

\[
\begin{bmatrix}
W_{10} & -\frac{1}{\sqrt{|K''(0)|}} W_{11} & W_{g0} & \frac{1}{\sqrt{|K''(0)|}} W_{g1} & \frac{\theta_1}{\sqrt{|K''(0)|}} \\
-\frac{1}{\sqrt{|K''(0)|}} W_{11} & W_{12} & \frac{1}{\sqrt{|K''(0)|}} W_{g2} & \frac{1}{\sqrt{|K''(0)|}} W_{g2} & \frac{\theta_2}{\sqrt{|K''(0)|}} \\
W_{g0} & -\frac{1}{\sqrt{|K''(0)|}} W_{g1} & W_{g1} & \frac{1}{\sqrt{|K''(0)|}} W_{20} & \frac{\theta_1}{\sqrt{|K''(0)|}} \\
-\frac{1}{\sqrt{|K''(0)|}} W_{g1} & W_{g2} & \frac{1}{\sqrt{|K''(0)|}} W_{21} & W_{21} & \frac{\theta_2}{\sqrt{|K''(0)|}} \\
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\frac{\theta_1}{\sqrt{|K''(0)|}} \\
\frac{\theta_2}{\sqrt{|K''(0)|}} \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
-\frac{1}{\sqrt{|K''(0)|}} u_1 \\
0 \\
-\frac{1}{\sqrt{|K''(0)|}} u_2
\end{bmatrix},
\]

whose left-hand side matrix is the same as that in (24), called $W$, where $K'' (0)$ is the scalar defined in (25). Therefore, following Proposition 3, under the event $\mathcal{E}_d$, $W$ is invertible, which gives

\[
\begin{bmatrix}
\frac{\theta_1}{\sqrt{|K''(0)|}} \\
\frac{\theta_2}{\sqrt{|K''(0)|}}
\end{bmatrix}
= -\frac{1}{\sqrt{|K''(0)|}} (R_1 u_1 + R_g u_2), \quad \text{and} \quad
\begin{bmatrix}
\frac{\theta_1}{\sqrt{|K''(0)|}} \\
\frac{\theta_2}{\sqrt{|K''(0)|}}
\end{bmatrix}
= -\frac{1}{\sqrt{|K''(0)|}} (R_3 u_1 + R_2 u_2).
\]
And further we know
\[ \frac{1}{\sqrt{|K''(0)|}} P^{(l)}(\tau) = - \frac{1}{\sqrt{|K''(0)|}} v_{H}^{\mu}(\tau) (R_{1} u_{1} + R_{3} u_{2}) - \frac{1}{\sqrt{|K''(0)|}} v_{l}^{\mu}(\tau) (R_{3} u_{1} + R_{2} u_{2}). \]

Under this choice, we will establish that \( P_{1}(\tau) \) satisfies the properties in Lemma 6, and \( Q_{1}(\tau) \) will follow similarly. Denote
\[ \frac{1}{\sqrt{|K''(0)|}} P^{(l)}_{\mu l}(\tau) = - \frac{1}{\sqrt{|K''(0)|}} (u_{1}, R_{\mu l} v_{H l}(\tau)), \]
then it is straightforward to obtain the following proposition to bound the distance between \( \frac{1}{\sqrt{|K''(0)|}} P^{(l)}_{1}(\tau) \) and \( \frac{1}{\sqrt{|K''(0)|}} P^{(l)}_{\mu l}(\tau) \), following essentially the same proof of Proposition 5.

**Lemma 8.** Suppose \( \Delta \geq 1/M \). There exists a numerical constant \( C \) such that
\[ M \geq C \max \left\{ \log^{2} \left( \frac{M (K_{1} + K_{2})}{\eta} \right), \frac{1}{\epsilon^{2} K_{\max} \log \left( \frac{M (K_{1} + K_{2})}{\eta} \right)}, \frac{1}{\epsilon^{2} K_{\max}^{2} \log \left( \frac{K_{1} + K_{2}}{\eta} \right)} \right\}, \]
then we have
\[ \mathbb{P} \left\{ \left| \frac{1}{\sqrt{|K''(0)|}} P^{(l)}_{1}(\tau) - \frac{1}{\sqrt{|K''(0)|}} P^{(l)}_{\mu l}(\tau) \right| \leq \frac{\epsilon}{4M + 1}, \forall \tau \in [0, 1], l = 0, 1, 2, 3 \right\} \geq 1 - \eta. \]

When \( \tau \in Y_{\text{far}}^{1/k} \), since \( |P_{\mu l}(\tau)| \leq \frac{C}{4M + 1} \) for some numerical constant \( C \) [41, Lemma 2.7], under the event in Lemma 8, we have
\[ |P_{1}(\tau)| \leq |P_{\mu l}(\tau)| + \frac{\epsilon}{M} \leq \frac{C_{1}}{M} \]
for some numerical constant \( C_{1} \). Next consider \( |P_{1}(\tau) - \text{sign}(a_{1k})(\tau - \tau_{1k})| \) when \( \tau \in Y_{\text{near}}^{1/k} \). Without loss of generality, assume \( \tau_{1k} = 0 \). Denote \( Z(\tau) = \text{sign}(a_{1k}) \tau - P_{1}(\tau) = Z_{R}(\tau) + jZ_{I}(\tau) \), where \( Z_{R}(\tau) \) and \( Z_{I}(\tau) \) are the real and the imaginary part of \( Z(\tau) \), respectively. Thus we have \( Z_{R}(0) = 0, Z_{I}(0) = 0 \), and \( Z_{R}'(0) = 0, Z_{I}'(0) = 0 \). Similarly define \( Z_{\mu}(\tau) = \text{sign}(a_{1k}) \tau - P_{\mu l}(\tau) = Z_{\mu R}(\tau) + jZ_{\mu I}(\tau) \), where \( Z_{\mu R}(\tau) \) and \( Z_{\mu I}(\tau) \) are the real and the imaginary part of \( Z_{\mu}(\tau) \), respectively. Since \( |Z_{\mu R}'(\tau)| \leq CM \) and \( |Z_{\mu I}'(\tau)| \leq CM \) for some constant \( C \) from the proof of Lemma 6.1 in [41], combining with Lemma 8, we can obtain \( |Z_{R}'(\tau)| \leq C_{P}M \) and \( |Z_{I}'(\tau)| \leq C_{P}M \) with numerical constant \( C_{P} \). Then we have
\[ |\text{sign}(a_{1k}) \tau - P_{1}(\tau)| = |Z(\tau)| \leq C_{P}M \tau^{2}. \]

**M Proof of Lemma 7**

We record the following lemma whose proof is given in Appendix N.

**Lemma 9.** Set \( M \geq 4 \). There exist numerical constants \( C_{1}, C_{2} \) and \( C_{3} \) such that we have \( |K_{g}(\tau)| \leq C_{1} \sqrt{\log M} \), and \( |K_{g}'(\tau)| \leq C_{2} \sqrt{\log M} \), with probability at least \( 1 - C_{3} (M^{3} \log M)^{-1/2} \).

**Proof.** Since \( P(\tau) = \langle p, c(\tau) \rangle \), and \( Q(\tau) = \langle g \odot p, c(\tau) \rangle \), we have
\[ \left| \int_{0}^{1} P(\tau) \nu_{1}(d\tau) + \int_{0}^{1} Q(\tau) \nu_{2}(d\tau) \right| = \left| \int_{0}^{1} \langle p, c(\tau) \rangle \nu_{1}(d\tau) + \int_{0}^{1} \langle g \odot p, c(\tau) \rangle \nu_{2}(d\tau) \right| \]
\[ = \langle p, \int_{0}^{1} c(\tau) \nu_{1}(d\tau) \rangle + \langle p, \int_{0}^{1} g \odot c(\tau) \nu_{2}(d\tau) \rangle \]
\[ = |\langle p, e_{1} + g \odot e_{2} \rangle| \]
\[ = |\langle P(\tau), E(\tau) \rangle| \]
\[ \leq \| P(\tau) \|_{1} \| e_{1} + g \odot e_{2} \|_{A}^{*}, \] (64)
where \( E(\tau) = (e_1 + g \odot e_2 \cdot c(\tau)) \) and \( \| P(\tau) \|_1 = \int_0^1 |P(\tau)| \, d\tau \). Here the penultimate step follows from Parseval’s identity, and the last inequality follows from Hölder’s inequality. Therefore, we need to bound \( \| P(\tau) \|_1 \). Recall

\[
\left[ \frac{\alpha_1}{\sqrt{|K''(0)|\beta_1}} \right] = L_1 u_1 + L_g u_2, \quad \text{and} \quad \left[ \frac{\alpha_2}{\sqrt{|K''(0)|\beta_2}} \right] = L_\beta u_1 + L_2 u_2
\]

in (33). Define

\[
\left[ \frac{\alpha_{\mu 1}}{\sqrt{|K''(0)|\beta_{\mu 1}}} \right] = L_{\mu 1} u_1, \quad \text{and} \quad \left[ \frac{\alpha_{\mu 2}}{\sqrt{|K''(0)|\beta_{\mu 2}}} \right] = L_{\mu 2} u_2.
\]

From [18, Lemma 2.2], we have \( \| \alpha_{\mu i} \|_\infty \leq C_\alpha \) and \( \| \beta_{\mu i} \|_\infty \leq \frac{C_\beta}{M} \) for some constants \( C_\alpha \) and \( C_\beta \), \( i = 1, 2 \). Under the event \( \mathcal{E}_3 \) for \( 0 < \delta \leq 1/4 \) in Lemma 1, we have

\[
\begin{align*}
\| \left[ \frac{\alpha_1}{\sqrt{|K''(0)|\beta_1}} \right] - \left[ \frac{\alpha_{\mu 1}}{\sqrt{|K''(0)|\beta_{\mu 1}}} \right] \|_\infty &\leq \|(L_1 - L_{\mu 1}) u_1\|_\infty + \|L_g u_2\|_\infty \\
&\leq \|(L_1 - L_{\mu 1}) u_1\|_2 + \|L_g u_2\|_2 \leq C_3 \sqrt{M}.
\end{align*}
\]

Therefore, we have \( \| \alpha_1 \|_\infty \leq C'_\alpha \sqrt{K_{\max}} \) and \( \| \beta_1 \|_\infty \leq \frac{C'_\beta}{M} \sqrt{K_{\max}} \) for some constants \( C'_\alpha \) and \( C'_\beta \). Similar bounds hold for \( \| \alpha_{\mu 2} \|_\infty \) and \( \| \beta_{\mu 2} \|_\infty \) as well. Then \( \| P(\tau) \|_1 \) can be bounded as follows:

\[
\begin{align*}
\| P(\tau) \|_1 &= \int_0^1 |P(\tau)| \, d\tau \\
&\leq K_1 \| \alpha_1 \|_\infty \int_0^1 |K(\tau)| \, d\tau + K_1 \| \beta_1 \|_\infty \int_0^1 |K'(\tau)| \, d\tau + K_2 \| \alpha_2 \|_\infty \int_0^1 |K_1(\tau)| \, d\tau + K_2 \| \beta_2 \|_\infty \int_0^1 |K_2(\tau)| \, d\tau \\
&\leq K_1 C'_\alpha \sqrt{K_{\max}} C M + K_1 C'_\beta \sqrt{K_{\max}} C + K_2 C'_\alpha \sqrt{K_{\max}} C_1 \sqrt{\log M \over M} + K_2 C'_\beta \sqrt{K_{\max}} C_2 \sqrt{M \log M} \\
&\leq C_p \sqrt{K_{\max} \log M},
\end{align*}
\]

where we used \( \int_0^1 |K(\tau)| \, d\tau \leq C_M \), \( \int_0^1 |K'(\tau)| \, d\tau \leq C \) from [37, Lemma 4], and \( |K_1(\tau)| \leq C_1 \sqrt{\log M \over M} \), \( |K_2(\tau)| \leq C_2 \sqrt{M \log M} \) from Lemma 9. Plugging this into (64) and combining (49), we have proved (50).

Next, we can write similarly that

\[
\left| \int_0^1 P_1(\tau) v_1(\tau) \, d\tau \right| + \left| \int_0^1 Q_1(\tau) v_2(\tau) \, d\tau \right| \leq \| P_1(\tau) \|_1 \| e_1 + g \odot e_2 \|_4, \quad (65)
\]

then it suffices to bound \( \| P_1(\tau) \|_1 \). Recall that

\[
\left[ \frac{\theta_1}{\sqrt{|K''(0)|\psi_1}} \right] = -\frac{1}{\sqrt{|K''(0)|}} (R_1 u_1 + R_g u_2), \quad \text{and} \quad \left[ \frac{\theta_2}{\sqrt{|K''(0)|\psi_2}} \right] = -\frac{1}{\sqrt{|K''(0)|}} (R_\beta u_1 + R_2 u_2)
\]

in Appendix L. Define

\[
\left[ \frac{\theta_{\mu 1}}{\sqrt{|K''(0)|\psi_{\mu 1}}} \right] = -\frac{1}{\sqrt{|K''(0)|}} R_{\mu 1} u_1, \quad \text{and} \quad \left[ \frac{\theta_{\mu 2}}{\sqrt{|K''(0)|\psi_{\mu 2}}} \right] = -\frac{1}{\sqrt{|K''(0)|}} R_{\mu 2} u_2.
\]

From [41, Lemma 2.7], we have \( \| \theta_{\mu i} \|_\infty \leq C_\theta M \) and \( \| \psi_{\mu i} \|_\infty \leq C_\psi M^2 \) for some constants \( C_\theta \) and \( C_\psi \), \( i = 1, 2 \). Following similar arguments as above, we have \( \| \theta_i \|_\infty \leq C_\theta \sqrt{K_{\max}} M \) and \( \| \psi_i \|_\infty \leq C_\psi \sqrt{K_{\max}} M^2 \), \( i = 1, 2 \). Hence \( \| P_1(\tau) \|_1 \) can be bounded as

\[
\| P_1(\tau) \|_1 = \int_0^1 |P_1(\tau)| \, d\tau
\]
\[ \begin{align*}
&\leq K_1 \| \theta_1 \|_\infty \int_0^1 |K(\tau)| \, d\tau + K_1 \| \psi_1 \|_\infty \int_0^1 |K'(\tau)| \, d\tau + K_2 \| \theta_2 \|_\infty \int_0^1 |K_g(\tau)| \, d\tau + K_2 \| \psi_2 \|_\infty \int_0^1 |K'_g(\tau)| \, d\tau \\
&\leq K_1 C'_0 \sqrt{K_{\max}} \frac{C}{M} + K_1 C'_0 \sqrt{K_{\max}} C + K_2 C'_0 \sqrt{K_{\max}} C_1 \sqrt{\log \frac{M}{M}} + K_2 C'_0 \sqrt{K_{\max}} C_2 \sqrt{M \log M} \\
&\leq C'_p \sqrt{\frac{K_{\max} \log M}{M^3}}.
\end{align*} \]

Plugging this into (65) and combining (49), we have proved (51). \qed

N Proof of Lemma 9

Proof. Suppose \( M \geq 4 \). For a fixed \( \tau \in [0, 1) \), applying the Hoeffding’s inequality in Lemma 4, we have

\[ \Pr \left\{ \left| K_g(\tau) \right| \geq \zeta \right\} = \Pr \left\{ \left| \frac{1}{M} \sum_{n=-2M}^{2M} s_n g_n e^{j2\pi n \tau} \right| \geq \zeta \right\} \leq 4 e^{-\frac{\zeta^2}{4 \sum_{n=-2M}^{2M} |s_n|^2}}, \]

where we used \( |s_n| \leq 1 \). Let \( \Psi_{\text{grid}} = \{ \tau_d \in [0, 1) \} \) be a uniform grid of \([0, 1)\) whose size will be determined later. As a result of the union bound, we have

\[ \Pr \left\{ \sup_{\tau_d \in \Psi_{\text{grid}}} \left| K_g(\tau_d) \right| \leq \zeta \right\} = 1 - 4 |\Psi_{\text{grid}}| e^{-\frac{\zeta^2}{M^2}}. \]

For any \( \tau_a, \tau_b \in [0, 1) \), following Lemma 3 we have

\[ |K_g(\tau_a) - K_g(\tau_b)| \leq e^{j2\pi \tau_a} - e^{j2\pi \tau_b} \sup_{\tau} \left| \frac{\partial K_g(\tau)}{\partial \tau} \right| \leq 4\pi |\tau_a - \tau_b| 2M \sup_{\tau} |K_g(\tau)| \leq 40\pi M |\tau_a - \tau_b|, \]

where the last inequality follows from \( |K_g(\tau)| \leq \frac{1}{M} \sqrt{\sum_{n=-2M}^{2M} s_n^2} \sqrt{\sum_{n=-2M}^{2M} |g_n e^{j2\pi n \tau}|^2} \leq \frac{4M+1}{M} \leq 5 \). By choosing the grid size such that for any \( \tau \in [0, 1) \), there exists a point \( \tau_d \in \Psi_{\text{grid}} \) satisfying \( 40\pi M |\tau - \tau_d| \leq \zeta \), which means we can set \( |\Psi_{\text{grid}}| = \left\lceil \frac{40\pi M}{\zeta} \right\rceil \). Consequently, for any \( \tau \in [0, 1) \), we have

\[ |K_g(\tau)| \leq |K_g(\tau) - K_g(\tau_d)| + |K_g(\tau_d)| \leq 40\pi M |\tau - \tau_d| + \zeta \leq 2\zeta, \]

with probability at least \( 1 - 4 |\Psi_{\text{grid}}| e^{-\frac{\zeta^2}{M^2}} \). Choose \( \zeta = \sqrt{\frac{51 \log M}{M}} \), then we have

\[ \Pr \left\{ |K_g(\tau)| \leq 2 \sqrt{\frac{51 \log M}{M}} \right\} \geq 1 - 71(M^3 \log M)^{-1/2}. \]

Next consider \( |K'_g(\tau)| \). For a fixed \( \tau \in [0, 1) \), applying the Hoeffding’s inequality in Lemma 4, we have

\[ \Pr \left\{ \left| K'_g(\tau) \right| \geq \zeta \right\} = \Pr \left\{ \left| \frac{1}{M} \sum_{n=-2M}^{2M} s_n g_n e^{j2\pi n \tau} (j2\pi n) \right| \geq \zeta \right\} \]

\[ \leq 4 e^{-\frac{\zeta^2}{4 \sum_{n=-2M}^{2M} |s_n|^2}}, \]

\[ \leq 4 e^{-\frac{\zeta^2}{M^2}}. \]

Set \( \Psi_{\text{grid}} = \{ \tau_d \in [0, 1) \} \) be a uniform grid of \([0, 1)\) whose size will be determined later. As a result of the union bound, we have

\[ \Pr \left\{ \sup_{\tau_d \in \Psi_{\text{grid}}} \left| K'_g(\tau_d) \right| \leq \zeta \right\} = 1 - 4 |\Psi_{\text{grid}}| e^{-\frac{\zeta^2}{M^2}}. \]

For any \( \tau_a, \tau_b \in [0, 1) \), following Lemma 3 we have

\[ |K'_g(\tau_a) - K'_g(\tau_b)| \leq 4\pi |\tau_a - \tau_b| 2M \sup_{\tau} \left| K'_g(\tau) \right| \leq 88\pi^2 M^2 |\tau_a - \tau_b|, \]
where in the last inequality we use $|K'_g(\tau)| \leq \frac{1}{M} \sqrt{\sum_{n=-2M}^{2M} s_n^2} \sqrt{\sum_{n=-2M}^{2M} |g_n e^{j2\pi n\tau} (j2\pi n)|^2} \leq 11 \pi M$. Hence, by choosing the grid size such that for any $\tau \in [0,1)$, there exists a point $\tau_d \in \mathcal{Y}_{\text{grid}}$ satisfying $88\pi^2 M^2 |\tau - \tau_d| \leq \zeta$, which gives $|\mathcal{Y}_{\text{grid}}| = \lceil \frac{88\pi^2 M^2}{\zeta} \rceil$. Then for any $\tau \in [0,1)$, we have

$$
|K'_g(\tau)| \leq |K'_g(\tau) - K'_g(\tau_d)| + |K'_g(\tau_d)| \leq 88\pi^2 M^2 |\tau - \tau_d| + \zeta \leq 2\zeta
$$

with probability at least $1 - 4|\mathcal{Y}_{\text{grid}}| e^{-\frac{\zeta^2}{2\pi^2 M}}$. Choosing $\zeta = \sqrt{963\pi M \log M}$ gives

$$
P\left(|K'_g(\tau)| \leq 2\sqrt{963\pi M \log M}\right) \geq 1 - 64(M^3 \log M)^{-1/2}.
$$