Existence results for singular elliptic problem involving a fractional $p$-Laplacian

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Abstract
In this article, the problem to be studied is the following

$$
\begin{cases}
(-\Delta)^s_p u \pm \beta \frac{|u|^{p-2}u}{|x|^s} = \lambda f(x,u) & \text{in } \Omega \\
u = 0 & \text{on } \mathbb{R}^N \setminus \Omega,
\end{cases}
(P\pm)
$$

$\Omega$ is a bounded regular domain in $\mathbb{R}^N (N \geq 2)$ containing the origin, $p > 1, s \in (0, 1), (N > ps), \lambda > 0, 0 < \beta < 1/c_H$ where $c_H$ is the best constant in the fractional Hardy inequality, $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfying a suitable growth condition and $(-\Delta)^s_p$ is the fractional $p$-Laplacian defined as

$$
(-\Delta)^s_p u(x) := 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} \, dy, \quad x \in \mathbb{R}^N,
$$

where $B_\varepsilon(x)$ is the open $\varepsilon$-ball of centre $x$ and radius $\varepsilon$. Using the critical point theory combining with the fractional Hardy inequality, we show that the problem $(P\pm)$ admits at least two distinct nontrivial weak solutions. For the problem $(P_-)$, we use the concentration-compactness principle for fractional Sobolev spaces to give a weak lower semicontinuity result and prove that problem $(P_-)$ admits at least one non-trivial weak solution.

Keywords Fractional calculus (primary) · Singular problem · Fractional $p$-Laplacian · Critical point · Variational method · Fractional Hardy inequality

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1 Introduction

Singular elliptic problems arise in several context like chemical heterogeneous catalysts, in the theory of pseudoplastic fluid, non-Newtonian fluid and in the study of relativistic matter in magnetic fluid. for more details, see [9, 10, 21].

On the other hand, recently, a great deal of works is devoted to the fractional nonlocal problem arising in many fields such as anomalous diffusion in plasma, flames propagation, geophysical fluid dynamics and American options in finances. See [7, 11] and the references therein.

In this work, we continue the study of the semilinear singular elliptic problem with the nonlocal operator theory. More precisely, we are interested by the fractional p-Laplacian operator which up to normalization functions may be defined as

\[
(-\Delta)^s_p u(x) := 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} \, dy, \quad x \in \mathbb{R}^N,
\]

where \(B_\varepsilon(x)\) is the open \(\varepsilon\)-ball of centre \(x\) and radius \(\varepsilon\). We refer to [8, 16] for more details.

In the last years, many works are devoted to the quasi-linear problem

\[
\begin{cases}
(-\Delta)^s_p u = f(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

(1.1)

For \(p = 2\), problem (1.1) reduces to the fractional Laplacian problem. We refer to the series of papers of Servadei and Valdinoci [23–25].

When the nonlinearity is discontinuous, the second author study problem (1.1) and prove the existence and multiplicity results for the following problem

\[
\begin{cases}
(-\Delta)^s u = f(u)H(u - \mu) & \text{in } \Omega \\
u = 0 & \text{on } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where \(H\) is the Heaviside function, \(f\) is a given function and \(\mu > 0\).

For \(p \neq 2\) and when \(f\) is regular nonlinearity, there is a large number of papers treated problem (1.1) using different techniques. we refer the reader to the papers [2–4, 8, 16, 20, 23, 27] and to the references therein.

When \(f\) is discontinuous nonlinearity with respect to \(u\), problem (1.1) was treated by the authors in [1]. More precisely, \(f\) is given by

\[
f(x, u) = m(x) \sum_{i=1}^n H(u - \mu_i),
\]
for some \( \mu_i > 0 \) verifying the condition
\[
\mu_1 < \mu_2 < \cdots < \mu_n, \quad \text{for } n \geq 1
\]
and \( m \in L^\infty(\Omega) \) changes sign.

The authors prove the existence and multiplicity result via the nonsmooth critical point theory.

In this paper, we are interested to study the existence of weak solutions for problem (1.1) with singular absorption term of the following type
\[
\begin{cases}
(-\Delta)^s_p u + \beta \frac{|u|^{p-2}u}{|x|^p} = \lambda f(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
and a diffusion term of the following type
\[
\begin{cases}
(-\Delta)^s_p u - \beta \frac{|u|^{p-2}u}{|x|^p} = \lambda f(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
where \( \lambda \) is a positive parameter, \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( p > 1 \), \( s \in (0, 1) \), \( (N > ps) \) containing the origin and with smooth boundary, \( 0 < \beta < 1/c_H \) where \( c_H \) is the best constant in the fractional Hardy inequality, \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function satisfying the following subcritical growth condition
\[
(H1) \quad |f(x, t)| \leq \alpha_1 + \alpha_2 |t|^{q-1}, \forall (x, t) \in \Omega \times \mathbb{R}
\]
for some non-negative constants \( \alpha_1, \alpha_2 \) and \( q \in ]1, p_s^* [ \), where the fractional critical exponent be defined as
\[
p_s^* := \begin{cases}
\frac{Np}{N-ps} & \text{if } N > ps \\
\infty & \text{if } N \leq ps.
\end{cases}
\]

In the local setting \( (s = 1) \), Khodabakhshi and al. [17] studied the existence of solutions to the problem (1.2) which was motivated by the work of Ferrara and Bisci [12]. They studied the existence of at least one non-trivial solution of the following elliptic problem
\[
\begin{cases}
-\Delta_p u = \mu \frac{|u|^{p-2}u}{|x|^p} + \lambda f(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where \( -\Delta_p u := \text{div} (|\nabla u|^{p-2}\nabla u) \) denotes the \( p \)-Laplacian, \( \lambda, \mu > 0 \).
Note also the work of Khodabakhshi and Hadjian in [18] when the authors prove the existence of three weak solutions of the following problem

\[
\begin{cases}
-\Delta_p u + \frac{|u|^{p-2}u}{|x|^p} = \lambda f(x, u) + \mu g(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{1.5}
\]

where \( f \) and \( g \) are Carathéodory functions.

Hence, in this paper, we consider the nonlocal counter part of problem (1.5) when \( \mu = 0 \) and the counter part of problem (1.4). Using a variational structure of our problems and based on a version of critical point theorem contained in [5, 22] (See respectively Theorem 1 and Theorem 2), we prove the existence of two weak solutions of problem (1.2) and we show that the problem (1.3) admits at least one nontrivial solution.

The paper is organized as follows. In Section 2, we recall some basic definitions. Section 3 is denoted to state and prove our main results and finally some useful comments related to problems (1.2) and (1.3) are given.

2 Preliminaries

In this section, we provide the variational setting for the problem (1.2), jointly with some preliminary results for the fractional \( p \)-Laplacian. Also, we recall certain definitions and essential results on the critical point theory.

2.1 Variational Formulation of the Problem.

Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain containing the origin and with smooth boundary \( \partial\Omega \), for \( p \in (1, \infty) \) and \( s \in (0, 1) \), we define the fractional Sobolev space

\[
W^{s, p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{N/p + s}} \in L^p(\mathbb{R}^{2N}) \right\},
\]

endowed with the norm

\[
\|u\|_{W^{s, p}(\mathbb{R}^N)} := \left( \|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{W^{s, p}(\mathbb{R}^N)} \right)^{\frac{1}{p}},
\]

where the following term is the Gagliardo semi-norm of \( u \) defined as

\[
[u]_{W^{s, p}(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy \right)^{\frac{1}{p}}.
\]
We shall work in the following closed linear subspace

\[ W_0 := \left\{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}, \]

which can be equivalently renormed by

\[ \| u \|_{W_0} := [u]_{W^{s,p}(\mathbb{R}^N)}. \]

Let \( 1 < p_s < N \), then there exist a positive constant \( c_H \) such that we present the fractional Hardy inequality, which says that

\[
\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{sp}} \, dx \leq c_H \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dxdy}{\| u \|_{W^{s,p}(\mathbb{R}^N)}}, \quad \forall u \in W_0.
\]

For more details, see [15].

Now, let \( C_c^\infty(\mathbb{R}^N) \) the space of functions on \( \mathbb{R}^N \) that are infinitely differentiable and have compact support contained in \( \mathbb{R}^N \). If we denote by \( D^{s,p}(\mathbb{R}^N) \) the closure of \( C_c^\infty(\mathbb{R}^N) \) with respect to the Gagliardo semi-norm \( [u]_{W^{s,p}(\mathbb{R}^N)} \), then for \( 0 \leq \alpha \leq sp \) and \( p_s^*(\alpha) = p(N - \alpha)/(N - sp) \leq p_s^*(0) = p_s^* \), we define the best fractional Hardy-Sobolev constant \( H_\alpha \) by

\[
H_\alpha = \inf_{u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\}} \frac{[u]_{W^{s,p}(\mathbb{R}^N)}^p}{\int_{\mathbb{R}^N} \frac{|u(x)|^{p_s^*(\alpha)}}{|x|^{s \alpha}} \, dx}, \quad (2.2)
\]

where \( p \leq p_s^* = \frac{Np}{N-sp} \) is the critical Sobolev exponent.

In particular, for \( \alpha = sp < N \) and \( p_s^*(sp) = p \leq p_s^*(0) = p_s^* \), \( H_{sp} \) is the best constant of fractional Hardy’s inequality (2.1), such that

\[
H_{sp} = \frac{1}{c_H} = \inf_{u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\}} \frac{[u]_{W^{s,p}(\mathbb{R}^N)}^p}{\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{sp}} \, dx}, \quad (2.3)
\]

Also, it is well-known that \( (W_0, \| \cdot \|_{W_0}) \) is a uniformly convex (i.e. reflexive) Banach space, continuously embedded into \( L^p(\Omega) \) for all \( p \in [1, p_s^*] \) and compactly injected in \( L^p(\Omega) \) for all \( p \in [1, p_s^*] \), there exist a \( c_p > 0 \) which is the best constant of the embedding, such that

\[
\| u \|_{L^p(\Omega)} \leq c_p \| u \|_{W_0}, \quad \forall u \in W_0. \quad (2.4)
\]

Next, let us denote by \( (W_0^*, \| \cdot \|_{W_0^*}) \) the dual space of \( (W_0, \| \cdot \|_{W_0}) \) and we define the nonlinear operators \( A_{p,m} : W_0 \to W_0^* \) as
\[ (A_{p,m}(u), v) = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} \, dx dy \]
\[ + m\beta \int_{\Omega} \frac{|u(x)|^{p-2}}{|x|^s} u(x)v(x) \, dx, \]

where \((\cdot, \cdot)\) denotes the inner product on \(W_0\) and \(m = 1\) or \(m = -1\).

**Lemma 1** Let for \(u, v \in W_0\) there exist a constant \(k \geq 1\), then the nonlinear operator \(A_{p,1}\) is well defined and verifies the following:

\[ (A_{p,1}(u), v) \leq k\|u\|_{W_0}^{p-1} \|v\|_{W_0}. \]

Moreover, the operator \(A_{p,-1}\) is well defined and verifies

\[ (A_{p,-1}(u), v) \leq \|u\|_{W_0}^{p-1} \|v\|_{W_0}. \]

**Proof** For all \(u, v \in W_0\), we have

\[ (A_{p,1}(u), v) = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} \, dx dy \]
\[ + \beta \int_{\Omega} \frac{|u(x)|^{p-2}}{|x|^s} u(x)v(x) \, dx \]
\[ \leq \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}|u(x) - u(y)||v(x) - v(y)|}{|x - y|^{(N+sp)(p-1)+1}} \, dx dy \]
\[ + \beta \int_{\Omega} \frac{|u(x)|^{p-2}}{|x|^s(p-1)} |u(x)||v(x)| \, dx \]
\[ \leq \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-1}|v(x) - v(y)|}{|x - y|^{(N+sp)(p-1)+1}} \, dx dy \]
\[ + \beta \int_{\Omega} \frac{|u(x)|^{p-1}}{|x|^s(p-1)} |v(x)| \, dx. \]

By the Holder inequality, we get

\[ (A_{p,1}(u), v) \leq \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx dy \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^p}{|x - y|^{N+sp}} \, dx dy \right)^{\frac{1}{p}} \]
\[ + \left( \beta \int_{\Omega} \frac{|u(x)|^p}{|x|^s} \, dx \right)^{\frac{p-1}{p}} \left( \beta \int_{\Omega} \frac{|v(x)|^p}{|x|^s} \, dx \right)^{\frac{1}{p}}. \]

For any \(\kappa \in (0, 1)\) and \(a, b, c, d > 0\), we use the following inequality

\[ a^{\kappa}c^{1-\kappa} + b^{\kappa}d^{1-\kappa} \leq (a + b)^{\kappa} (c + d)^{1-\kappa}, \]

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by setting \( \kappa = \frac{p-1}{p} \) and

\[
\begin{align*}
    a &= \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy, \\
    b &= \beta \int_{\Omega} \frac{|u(x)|^p}{|x|^{sp}} \, dx, \\
    c &= \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x-y|^{N+sp}} \, dx \, dy, \\
    d &= \beta \int_{\Omega} \frac{|v(x)|^p}{|x|^{sp}} \, dx.
\end{align*}
\]

We can deduce that

\[
\langle A_{p,1}(u), v \rangle \leq \left( \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy + \beta \int_{\Omega} \frac{|u(x)|^p}{|x|^{sp}} \, dx \right)^{\frac{p-1}{p}} \\
\times \left( \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x-y|^{N+sp}} \, dx \, dy + \beta \int_{\Omega} \frac{|v(x)|^p}{|x|^{sp}} \, dx \right)^{\frac{1}{p}}.
\]

Then, according to the fractional Hardy inequality (2.1) we get

\[
\langle A_{p,1}(u), v \rangle \leq \left( \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy + \beta c_H \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \right)^{\frac{p-1}{p}} \\
\times \left( \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x-y|^{N+sp}} \, dx \, dy + \beta c_H \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \right)^{\frac{1}{p}} \\
\leq (\beta c_H + 1) \|u\|_{W_0^{p-1}} \|v\|_{W_0} \\
\leq k \|u\|_{W_0^{p-1}} \|v\|_{W_0} \\
< +\infty,
\]

where \( k = (\beta c_H + 1) \) and \( k \geq 1 \).

Now, for the operator \( A_{p,-1} \), we have

\[
\begin{align*}
    \langle A_{p,-1}(u), v \rangle &= \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+sp}} \, dx \, dy \\
&\quad - \beta \int_{\Omega} \frac{|u(x)|^{p-2} u(x)v(x)}{|x|^{sp}} \, dx \\
&\leq \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+sp}} \, dx \, dy \\
&\leq \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-1}}{|x-y|^{(N+sp)(\frac{p-1}{p})}} \frac{|v(x) - v(y)|}{|x-y|^{\left(\frac{N+sp}{p}\right)}} \, dx \, dy.
\end{align*}
\]
Then, by Holder inequality we get

\[
\langle A_{p,-1}(u), v \rangle \leq \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dxdy \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^p}{|x - y|^{N+sp}} \, dxdy \right)^{\frac{1}{p}}
\leq \|u\|_{W_0}^{p-1} \|v\|_{W_0}
< +\infty.
\]

\[\Box\]

Initially, let us introduce the energy functionals \(E_{\lambda}^{p,m} : W_0 \to \mathbb{R}\) associated with problems (1.2) and (1.3) as

\[E_{\lambda}^{p,m}(u) := \Phi_{p,m}(u) - \lambda \Psi(u), \quad \forall u \in W_0,
\]

where

\[
\Phi_{p,m}(u) := \frac{1}{p} \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dxdy + \beta \int_{\Omega} \frac{|u(x)|^p}{|x|^{sp}} \, dx \right)
\]

and

\[
\Psi(u) := \int_{\Omega} F(x, u(x)) \, dx,
\]

where \(F(x, t) = \int_0^t f(x, s) \, ds\), for every \((x, t) \in \Omega \times \mathbb{R}\).

Moreover, from the fractional Hardy’s inequality (2.1), we have

\[
\Phi_{p,1}(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dxdy + \beta \int_{\Omega} \frac{|u(x)|^p}{|x|^{sp}} \, dx
\leq \frac{1}{p} \|u\|_{W_0}^p + \frac{\beta c_H}{p} \|u\|_{W_0}^p
\leq \left( \frac{\beta c_H + 1}{p} \right) \|u\|_{W_0}^p
\]

and

\[
\Phi_{p,-1}(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dxdy - \beta \int_{\Omega} \frac{|u(x)|^p}{|x|^{sp}} \, dx
\geq \frac{1}{p} \|u\|_{W_0}^p - \frac{\beta c_H}{p} \|u\|_{W_0}^p
\geq \left( 1 - \frac{\beta c_H}{p} \right) \|u\|_{W_0}^p.
\]
Also,
\[ \Phi_{p,1}(u) = \frac{1}{p} \|u\|_{W_0^p}^p + \frac{\beta}{p} \int_{\Omega} \frac{|u(x)|^p}{|x|^{sp}} \, dx \geq \frac{1}{p} \|u\|_{W_0^p}^p \]  
\text{(2.6)}

and
\[ \Phi_{p,-1}(u) = \frac{1}{p} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dxdy - \frac{\beta}{p} \int_{\Omega} \frac{|u(x)|^p}{|x|^{sp}} \, dx \]
\[ \leq \frac{1}{p} \|u\|_{W_0^p}^p. \]

It follows that
\[ \frac{1}{p} \|u\|_{W_0^p}^p \leq \Phi_{p,1}(u) \leq \left( \frac{\beta c_H + 1}{p} \right) \|u\|_{W_0^p}^p, \quad \forall u \in W_0 \]
equally,
\[ \left( \frac{1 - \beta c_H}{p} \right) \|u\|_{W_0^p}^p \leq \Phi_{p,-1}(u) \leq \frac{1}{p} \|u\|_{W_0^p}^p, \quad \forall u \in W_0. \]

So, \( \Phi_{p,m}(u) \) are well defined and coercive in \( W_0 \).

Next, we introduce some essential properties of the operator \( A_{p,1} \) which will be of significant use in our main result.

**Proposition 1** The nonlinear operator \( A_{p,1} \) has the following properties:

1) \( A_{p,1} : W_0 \rightarrow W_0^* \) is a continuous, bounded and strictly monotone operator, i.e: if \( \langle A_{p,1}(u) - A_{p,1}(v), u - v \rangle > 0, \forall u \neq v. \)

2) \( A_{p,1} \) is a mapping of type \((S+)\), i.e: if \( u_n \rightarrow u \) weakly in \( W_0 \) and
\[ \lim_{n \rightarrow \infty} \langle A_{p,1}(u_n) - A_{p,1}(u), u_n - u \rangle \leq 0, \text{ then } u_n \rightarrow u \text{ strongly in } W_0. \]

3) \( A_{p,1} : W_0 \rightarrow W_0^* \) is a homomorphism.

**Proof** 1) By Lemma 1, there exists a constant \( k \geq 1 \) such that
\[ |\langle A_{p,1}(u), v \rangle| \leq k \|u\|_{W_0^p}^{p-1} \|v\|_{W_0}, \forall u, v \in W_0, \]

from this inequality, it is obvious that \( A_{p,1} \) is continuous and bounded.

Now, by a direct computation
\[ \langle A_{p,1}(u) - A_{p,1}(v), u - v \rangle = \langle A(u), u - v \rangle - \langle A_{p,1}(v), u - v \rangle \]
\[ = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(u - v)(x) - (u - v)(y))}{|x - y|^{N+sp}} \, dxdy \]
\[ + \beta \int_{\Omega} \frac{|u(x)|^{p-2}}{|x|^{sp}} u(x)(u - v)(x) \, dx \]
\[-\int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))((u - v)(x) - (u - v)(y))}{|x - y|^{N+sp}} \, dx \, dy \]
\[-\beta \int_{\Omega} \frac{|v(x)|^{p-2}}{|x|^{sp}} v(x)(u - v)(x) \, dx \]
\[= \left[ \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} \left[ (u(x) - u(y)) - (v(x) - v(y)) \right] \, dx \, dy \right. \\
\left. - \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x - y|^{N+sp}} \left[ (u(x) - u(y)) - (v(x) - v(y)) \right] \, dx \, dy \right. \\
\left. + \beta \int_{\Omega} \left[ \frac{|u(x)|^{p-2}u(x)}{|x|^{sp}} - \frac{|v(x)|^{p-2}v(x)}{|x|^{sp}} \right] \left[ u(x) - v(x) \right] \, dx \right]
\[= I_1 + I_2, \]

where

\[I_1 = \int_{\mathbb{R}^{2N}} \left[ \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} - \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x - y|^{N+sp}} \right] \left[ (u(x) - u(y)) - (v(x) - v(y)) \right] \, dx \, dy \]

and

\[I_2 = \beta \int_{\Omega} \left[ \frac{|u(x)|^{p-2}u(x)}{|x|^{sp}} - \frac{|v(x)|^{p-2}v(x)}{|x|^{sp}} \right] \left[ u(x) - v(x) \right] \, dx. \]

According to Simon’s algebraic inequality (see [26, formula (2.2)]) which states that; for all \( \xi, \eta \in \mathbb{R}^N \), for \( p > 1 \) there exists a positive constant \( C_p \) such that

\[ \langle |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta \rangle \geq \begin{cases} 
C_p |\xi - \eta|^p & \text{if } p \geq 2, \\
C_p |\xi - \eta|^2 (|\xi|^p + |\eta|^{p-2})/p & \text{if } 1 < p < 2.
\end{cases} \quad (2.7) \]

Then, applying the inequality (2.7) on \( I_1 \) and \( I_2 \), such that

For \( I_1 \), we take \( \xi = u(x) - u(y) \) and \( \eta = v(x) - v(y) \).

and

for \( I_2 \), we take \( \xi = u(x) \) and \( \eta = v(x) \).

For all \( u, v \in W_0 \) with \( u \neq v \), if \( p \geq 2 \) we have

\[ I_1 \geq C_p \int_{\mathbb{R}^{2N}} \frac{|(u(x) - u(y)) - (v(x) - v(y))|^p}{|x - y|^{N+sp}} \, dx \, dy \]
and

\[ I_2 \geq C_p \beta \int_{\Omega} \frac{|(u(x) - v(x))|^p}{|x|^{sp}} \, dxdy. \]

Thus,

\[
(A_{p,1}(u) - A_{p,1}(v), u - v) = I_1 + I_2 \\
\geq C_p \left[ \int_{\mathbb{R}^{2N}} \frac{|(u(x) - u(y)) - (v(x) - v(y))|^p}{|x - y|^{N + sp}} \, dxdy \\
+ \beta \int_{\Omega} \frac{|(u(x) - v(x))|^p}{|x|^{sp}} \, dxdy \right] \\
= C_p p \Phi_{p,1}(u - v).
\]

Then, according to (2.6) we have

\[
(A_{p,1}(u) - A_{p,1}(v), u - v) \geq C_p \| u - v \|^p_{W_0} > 0, \quad p \geq 2. \quad (2.8)
\]

Thus, if \( 1 < p < 2 \), we have

\[
I_1 \geq C_p \int_{\mathbb{R}^{2N}} \frac{|(u(x) - u(y)) - (v(x) - v(y))|^2 \left( |u(x) - u(y)|^p + |v(x) - v(y)|^p \right)^{\frac{p-2}{p}}}{|x - y|^{N + sp}} \, dxdy
\]

and

\[
I_2 \geq C_p \beta \int_{\Omega} \frac{|(u(x) - v(x))|^2 \left( |u(x)|^p + |v(x)|^p \right)^{\frac{p-2}{p}}}{|x|^{sp}} \, dxdy.
\]

For all \( 1 \leq p \leq \infty \) and \( a, b \in \mathbb{R}^N \), we use the following inequality

\[ |a - b|^p \leq (|a| + |b|)^p \leq 2^{p-1} \left( |a|^p + |b|^p \right). \]

For \( I_1 \), we put \( a = u(x) - u(y) \) and \( b = v(x) - v(y) \), we get that

\[
I_1 \geq 2 \left( 1 - \frac{1}{p} \right) \frac{(1-p)(p-2)}{p} C_p \int_{\mathbb{R}^{2N}} \frac{|(u(x) - u(y)) - (v(x) - v(y))|^2 \left( |u(x) - u(y)| - (v(x) - v(y)) \right)^{p-2}}{|x - y|^{N + sp}} \, dxdy.
\]

Similarly, for \( I_2 \), we set \( a = u(x) \) and \( b = v(x) \), we have

\[
I_2 \geq 2 \left( 1 - \frac{1}{p} \right) \frac{(1-p)(p-2)}{p} C_p \beta \int_{\Omega} \frac{|(u(x) - v(x))|^2 \left( |u(x) - v(x)| \right)^{p-2}}{|x|^{sp}} \, dxdy.
\]
Then,

\[
\langle A_{p,1}(u) - A_{p,1}(v), u - v \rangle = I_1 + I_2 \geq C \left[ \int_{\mathbb{R}^{2N}} \frac{|(u(x) - u(y)) - (v(x) - v(y))|^2 |(u(x) - u(y)) - (v(x) - v(y))|^{p-2}}{|x - y|^{N+sp}} \, dx \, dy \right] + \beta \int_{\Omega} \frac{|(u(x) - v(x))|^2 |u(x) - v(x)|^{p-2}}{|x|^{sp}} \, dx \, dy
\]

\[
= C \left[ \int_{\mathbb{R}^{2N}} \frac{|(u(x) - u(y)) - (v(x) - v(y))|^{p}}{|x - y|^{N+sp}} \, dx \, dy + \beta \int_{\Omega} \frac{|u(x) - v(x)|^p}{|x|^{sp}} \, dx \, dy \right] = C p \Phi_{p,1}(u - v),
\]

where \( C = \frac{2(1-p)(p-2)}{p} C_p > 0 \) is a constant.

Consequently, according to (2.6) we will have

\[
\langle A_{p,1}(u) - A_{p,1}(v), u - v \rangle \geq C \|u - v\|^p_{\mathcal{W}_0} > 0, \quad 1 < p < 2. \tag{2.9}
\]

Which leads us to conclude that \( A \) is strictly monotone.

2) Since \( \mathcal{W}_0 \) is a reflexive Banach space, it is isometrically isomorphic to a locally uniformly convex space. So as it was already proved, weak convergence and norm convergence imply strong convergence. Therefore we only need to show that \( \|u_n\|_{\mathcal{W}_0} \to \|u\|_{\mathcal{W}_0} \).

Further, we have that if \( u_n \to u \) weakly in \( \mathcal{W}_0 \) and

\[
\lim \sup_{n \to \infty} \langle A_{p,1}(u_n) - A_{p,1}(u), u_n - u \rangle \leq 0.
\]

Then,

\[
\lim_{n \to +\infty} \langle A_{p,1}(u_n) - A_{p,1}(u), u_n - u \rangle = \lim_{n \to +\infty} \langle A_{p,1}(u_n), u_n - u \rangle - \langle A_{p,1}(u), u_n - u \rangle = 0.
\]

By combining (2.8) with (2.9) of 1), we obtain that \( u_n \to u \) strongly in \( \mathcal{W}_0 \) as \( n \to +\infty \) and this leads us to conclude that \( A_{p,1} \) is an application of type \((S_+)\).

3) By 1), we know that \( A_{p,1} \) are strictly monotone, which implicate that \( A_{p,1} \) is injective. Also, according to Lemma 1, we have

\[
\lim_{\|u\|_{\mathcal{W}_0} \to +\infty} \frac{\langle A_{p,1}(u), u \rangle}{\|u\|_{\mathcal{W}_0}^p} = \lim_{\|u\|_{\mathcal{W}_0} \to +\infty} \frac{\left( \|u\|_{\mathcal{W}_0}^p + \beta \int_{\Omega} \frac{|u(x)|^p}{|x|^{sp}} \, dx \right)}{\|u\|_{\mathcal{W}_0}}
\]

\[
= \lim_{\|u\|_{\mathcal{W}_0} \to +\infty} \|u\|_{\mathcal{W}_0}^{p-1} + \beta \lim_{\|u\|_{\mathcal{W}_0} \to +\infty} \|u\|_{\mathcal{W}_0}^{-1} \int_{\Omega} \frac{|u(x)|^p}{|x|^{sp}} \, dx.
\]
Hence, as $\beta > 0$ we have
\[
\lim_{\|u\|_{W^0} \to +\infty} \frac{\langle A_{p,1}(u), u \rangle}{\|u\|_{W^0}} = \lim_{\|u\|_{W^0} \to +\infty} \|u\|_{W^0}^{p-1} = +\infty.
\]

Thanks to $1 < p < \frac{N}{s}$, hence $A_{p,1}$ is coercive on $W_0$. Since $A_{p,1}$ is continuous and bounded by 1), then, by the Minty-Browder Theorem (see [28, Theorem 26.A]), we conclude that $A_{p,1}$ is a surjection.

Thus, $A_{p,1}$ has an inverse mapping $A_{-1,1} : W^*_0 \to W_0$. Therefore, the continuity of $A_{-1,1}$ is sufficient to ensure $A_{p,1}$ to be a homeomorphism.

Assume that $g_n, g \in W_0$ with $g_n \to g$ in $W_0$. Let $u_n = A_{-1,1}(g_n)$ and $u = A_{-1,1}(g)$. Then $A_{p,1}(u_n) = g_n$ and $A_{p,1}(u) = g$. Clearly, $\{u_n\}$ is bounded in $W_0$. Thus there exist $u_0 \in W_0$ and a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ such that $u_n \to u_0$ since $g_n \to g$, we have
\[
\lim_{n \to +\infty} \langle A_{p,1}(u_n) - A_{p,1}(u_0), u_n - u_0 \rangle = \lim_{n \to +\infty} \langle g_n, u_n - u_0 \rangle = 0.
\]

In view of $A_{p,1}$ satisfying the $(S_+)$ condition by 2), we get $u_n \to u_0$ in $W_0$. Moreover, $u = u_0$ a.e. in $\Omega$. Hence, $u_n \to u$ in $W_0$, so that $A_{-1,1}$ is continuous. \qed

Lastly, it is evident that our energy functionals $E_{\lambda}^{p,m}$ are well defined and of class $C^1$. The derivative of $E_{\lambda}^{p,m}$ is given by
\[
\langle (E_{\lambda}^{p,m})'(u), v \rangle = \langle \Phi_{p,m}'(u), v \rangle - \lambda \langle \Psi'(u), v \rangle, \quad \forall u, v \in W_0.
\]

**Definition 1** Fixing the real parameter $\lambda$, a function $u : \Omega \to \mathbb{R}$ is said to be a weak solution of the problem (1.2) and (1.3), if $u \in W_0$ and
\[
\langle \Phi_{p,m}'(u), v \rangle = \langle A_{p,m}(u), v \rangle = \lambda \langle \Psi'(u), v \rangle, \quad v \in W_0,
\]
where
\[
\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} (v(x) - v(y)) \, dx \, dy + m\beta \int_{\Omega} \frac{|u(x)|^{p-2}}{|x|^{sp}} u(x)v(x) \, dx = \lambda \int_{\Omega} f(x, u(x))v(x) \, dx,
\]
for every $v \in W_0$. Hence, the critical points of $E_{\lambda}^{p,m}$ are exactly the weak solutions of problems (1.2) and (1.3).
Definition 2 Let \( X \) be a real Banach space. A Gâteaux differentiable function \( E \) satisfies the Palais-Smale condition (in short (PS)-condition), if any sequence \( \{u_n\}_{n \in \mathbb{N}} \) such that

(a) \( \{E(u_n)\} \) is bounded, i.e. \( E(u_n) \leq d := \sup_{n \in \mathbb{N}} \{ E(u_n) \} \),

(b) \( \|E'(u_n)\|_{X^*}\to 0 \) as \( n \to +\infty \),

has a convergent subsequence.

Our main tools are the following critical point theorems.

Theorem 1 [5, Theorem 3.2.] Let \( X \) be a real Banach space and let \( \Phi, \Psi : X \to \mathbb{R} \) be two continuously Gâteaux differentiable functionals such that \( \Phi \) is bounded from below and \( \Phi(0) = \Psi(0) = 0 \). Fix \( r > 0 \) such that

\[
\sup_{\{\Phi(u) < r\}} \Psi(u) < +\infty
\]

and assume that, for each

\[
\lambda \in \left[0, \frac{r}{\sup_{\{\Phi(u) < r\}} \Psi(u)}\right],
\]

the functional \( E_\lambda := \Phi - \lambda \Psi \) satisfies the (PS)-condition and it is unbounded from below. Then, for each \( \lambda \in \left(0, \frac{r}{\sup_{\{\Phi(u) < r\}} \Psi(u)}\right) \), the functional \( E_\lambda \) admits two distinct critical points.

Theorem 2 [22, Theorem 2.5.] Let \( X \) be a real Banach space and let \( \Phi, \Psi : X \to \mathbb{R} \) be two continuously Gâteaux differentiable functionals such that \( \Phi \) is strongly continuous, sequentially weakly lower semicontinuous and coercive. Further, assume that \( \Psi \) is sequentially weakly upper semicontinuous. For every \( r > \inf_{X} \Phi \), put

\[
\varphi(r) := \inf_{u \in \Phi^{-1}(]-\infty, r[)} \left( \frac{\left(\sup_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)} \right).
\]

Then, for every \( r > \inf_{X} \Phi \) and every \( \lambda \in \left(0, 1/\varphi(r)\right) \), the restriction of \( E_\lambda := \Phi - \lambda \Psi \) to \( \Phi^{-1}(]-\infty, r[) \) admits a global minimum, which is a critical point (local minima) of \( E_\lambda \) in \( X \).

Now, we introduce the following variant of Concentration-Compactness principle (see [19]) established in [13]. This result is useful for the proof of the semicontinuity property of functionals.

Theorem 3 Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^N \), \( \alpha \in (0, sp] \) and \( p^*_s(\alpha) \in [p, p^*_s] \). Let \( \{u_n\}_{n \in \mathbb{N}} \subset W_0 \) be a weakly convergent sequence with weak limit \( u \). Then, there exist two finite positive measures \( \mu \) and \( \nu \) in \( \mathbb{R}^N \), such that the following convergence hold weakly in the sense of measures,

\[
\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+sp}} \, dy dx \to \mu \quad \tag{2.10}
\]
and
\[
\frac{|u_n(x)|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx \rightharpoonup \nu. \tag{2.11}
\]

Furthermore, there exist two nonnegative numbers \(\mu_0, \nu_0\) such that
\[
\nu = \frac{|u(x)|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx + \nu_0 \delta_0, \tag{2.12}
\]
\[
\mu \geq \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dy \, dx + \mu_0 \delta_0 \tag{2.13}
\]
and
\[
0 \leq H_\alpha \nu_0^{p/p^*_s(\alpha)} \leq \mu_0, \tag{2.14}
\]
where \(H_\alpha\) is the fractional Hardy-Sobolev constant given by (2.2) and \(\delta_0\) denotes the Dirac mass at 0.

The proof of this result can be found in [13, Theorem 1.1].

3 Main results

In this section, we establish the main results of this paper. The first result is the following

**Theorem 4** Let \(f : \Omega \times \mathbb{R} \to \mathbb{R}\) be a Carathéodory function such that condition \((H1)\) holds. Moreover, assume that

\((H2)\) there exist \(\theta > p\) and \(M > 0\) such that

\[0 < \theta F(x, t) \leq tf(x, t),\]

for each \(x \in \Omega\) and \(|t| \geq M\).

Then, for each \(\lambda \in ]0, \bar{\lambda}[\), the following initial case of problem (1.2)

\[
\begin{cases}
(-\Delta)^{\alpha}_{p} u + \beta \frac{|u|^{p-2} u}{|x|^{sp}} = \lambda f(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

admits at least two distinct weak solutions, where

\[
\bar{\lambda} := \frac{q}{q \alpha_1 c_1 p^\frac{1}{p} + \alpha_2 c_2^q p^\frac{q}{p}}.
\]
To achieve the proof of this result, we may split into a sequence of propositions the proof of Theorem 4.

**Proposition 2** Let for every $\lambda > 0$ the assumptions $(H1)$, $(H2)$ hold. Then, $E_{\lambda}^{p,1} = \Phi_{p,1} - \lambda \Psi$ satisfies the Palais-Smale condition.

**Proof** To prove that $E_{\lambda}^{p,1}$ satisfies the Palais-Smale condition for every $\lambda > 0$. Namely, we need to show that any sequence of Palais-Smale is bounded in $W_0$ and admits a convergent subsequence. We proceed by steps.

**Step 1.** The sequence $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $W_0$.

For $n$ large enough, by (a) in Definition 2, we have

$$E_{\lambda}^{p,1}(u_n) = \Phi_{p,1}(u_n) - \lambda \Psi(u_n) = \frac{1}{p} \left( \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N + sp}} \, dx dy + \beta \int_{\Omega} \frac{|u_n(x)|^p}{|x|^{sp}} \, dx \right)$$

$$- \lambda \int_{\Omega} F(x, u_n(x)) \, dx \leq d.$$

In the other hand, by $(H2)$ we have

$$E_{\lambda}^{p,1}(u_n) \geq \frac{1}{p} \|u_n\|_{W_0}^p - \frac{\lambda}{\theta} \int_{\Omega} f(x, u_n(x))u_n(x) \, dx$$

$$\geq \frac{1}{p} \|u_n\|_{W_0}^p - \frac{\lambda}{\theta} \int_{\Omega} f(x, u_n(x))u_n(x) \, dx + \frac{1}{\theta} \|u_n\|_{W_0}^p - \frac{1}{\theta} \|u_n\|_{W_0}^p$$

$$> \left( \frac{1}{p} - \frac{1}{\theta} \right) \|u_n\|_{W_0}^p + \frac{1}{\theta} \left( \|u_n\|_{W_0}^p - \lambda \int_{\Omega} f(x, u_n(x))u_n(x) \, dx \right)$$

$$\geq \left( \frac{1}{p} - \frac{1}{\theta} \right) \|u_n\|_{W_0}^p + \frac{1}{\theta} \left( E_{\lambda}^{p,1}(u_n) \right)' (u_n, u_n).$$

Due to (b) in Definition 2, we have $\theta > p > 1$ and $\epsilon \to 0$ such that

$$\| \left( E_{\lambda}^{p,1} \right)' (u_n) \|_{W_0^*} \leq \epsilon$$

and

$$- \frac{1}{\theta} \left( E_{\lambda}^{p,1} \right)' (u_n, u_n) \leq \| \left( E_{\lambda}^{p,1} \right)' (u_n) \|_{W_0} \|u_n\|_{W_0}$$

$$\leq \epsilon \|u_n\|_{W_0}.$$

Thus,

$$\left( \frac{1}{p} - \frac{1}{\theta} \right) \|u\|_{W_0}^p \leq E_{\lambda}^{p,1}(u_n) - \frac{1}{\theta} \left( E_{\lambda}^{p,1} \right)' (u_n, u_n) \leq d + \epsilon \|u_n\|_{W_0}.$$

It follows from this inequality that $\{u_n\}$ is bounded in $W_0$. 

\[ \square \] Springer
**Step 2.** The sequence \( \{u_n\}_{n \in \mathbb{N}} \) possesses a convergent subsequence.

By the Eberliq-Smulyan theorem (see [29, Theorem 21.D.]), passing to a subsequence if necessary, we can assume that \( u_n \rightarrow u \). Then, because of the compactness of \( \Psi' \) thanks to condition \((H1)\) and to the compact embedding \( W_0 \hookrightarrow L^q(\Omega) \), for every \( q \in [1, p^*_s) \) we have

\[
\Psi'(u_n) \longrightarrow \Psi'(u),
\]

Since

\[
\left( E^{p-1}_{\lambda} \right)'(u_n) = \Phi'_{p,1}(u_n) - \lambda \Psi'(u_n) \longrightarrow 0,
\]

then

\[
\Phi'_{p,1}(u_n) = \left( E^{p-1}_{\lambda} \right)'(u_n) + \lambda \Psi'(u_n) \longrightarrow 0 + \lambda \Psi'(u).
\]

Moreover, as \( \Phi'_{p,1} \) is a homeomorphism according to Proposition 1, then \( u_n \rightarrow u \) in \( W_0 \) and so \( E^{p-1}_{\lambda} \) satisfies the \((PS)\)-condition.

**Lemma 2** Assume that \( f \) satisfies \((H2)\). Then, there exists a positive \( c \) constant such that

\[
F(x, t) \geq c|t|^\theta, \quad \forall x \in \Omega, \quad |t| > M.
\]  

(3.1)

**Proof** We start by setting \( a(x) := \min_{|\xi|=M} F(x, \xi) \) and

\[
\varphi_t(s) := F(x, st), \quad \forall s > 0.
\]  

(3.2)

By the assumption \((H2)\), we have for every \( x \in \Omega \) and \( |t| > M \) such that

\[
0 < \theta \varphi_t(s) = \theta F(x, st) \leq st f(x, st) = s \varphi'_t(s), \quad \forall s > \frac{M}{|t|}.
\]

Therefore,

\[
\theta \varphi_t(s) \leq s \varphi'_t(s).
\]

Also,

\[
\int_{\frac{M}{|t|}}^{1} \theta \varphi_t(s) \frac{ds}{s} \leq \int_{\frac{M}{|t|}}^{1} \varphi'_t(s) \frac{ds}{s}.
\]
We see that,

\[
\int_{\frac{M}{|t|}}^{1} (\ln |s|^{\theta})' \, ds \leq \int_{\frac{M}{|t|}}^{1} (\ln |\varphi_{t}(s)|)' \, ds, \\
\left[ \ln |s|^{\theta} \right]_{\frac{M}{|t|}}^{1} \leq \left[ \ln |\varphi_{t}(s)| \right]_{\frac{M}{|t|}}^{1}, \\
- \ln \left[ \frac{M^{\gamma\theta}}{|t|} \right] \leq \ln |\varphi_{t}(1)| - \ln \left| \varphi_{t} \left( \frac{M}{|t|} \right) \right|.
\]

Then,

\[
\ln \varphi_{t} \left( \frac{M}{|t|} \right) - \ln \frac{M^{\theta}}{|t|^{\theta}} \leq \ln \varphi_{t}(1).
\]

We have

\[
- \ln \frac{M^{\theta}}{|t|^{\theta}} = - \left[ \ln M^{\theta} - \ln |t|^{\theta} \right] = \ln |t|^{\theta} - \ln M^{\theta} = \ln \frac{|t|^{\theta}}{M^{\theta}}.
\]

So,

\[
\ln \varphi_{t} \left( \frac{M}{|t|} \right) + \ln \frac{|t|^{\theta}}{M^{\theta}} \leq \ln \varphi_{t}(1), \\
e^{\ln \varphi_{t} \left( \frac{M}{|t|} \right) + \ln \frac{|t|^{\theta}}{M^{\theta}}} = e^{\ln \left[ \varphi_{t} \left( \frac{M}{|t|} \right) \frac{|t|^{\theta}}{M^{\theta}} \right]} \leq e^{\ln \varphi_{t}(1)}.
\]

Thus,

\[
\varphi_{t} \left( \frac{M}{|t|} \right) \frac{|t|^{\theta}}{M^{\theta}} \leq \varphi_{t}(1).
\]

Taking into account of (3.2), we obtain

\[
c|t|^{\theta} \leq a(x) \frac{|t|^{\theta}}{M^{\theta}} \leq F \left( x, \frac{M}{|t|} \right) \frac{|t|^{\theta}}{M^{\theta}} \leq F(x, t),
\]

where \( c \geq 0 \) is a constant. Thus, (3.1) is proved. □

Now, we prove the following Proposition 3 using the previous Lemma 2.

**Proposition 3** Let for every \( \lambda > 0 \) the assumption (H2) hold. Then, \( E_{\lambda}^{p,1} \) is unbounded from below.

**Proof** We fix \( u_{0} \in W_{0} \setminus \{0\} \) and for each \( t > 1 \), we have

\[
E_{\lambda}^{p,1}(tu_{0}) = \Phi_{p,1}(tu_{0}) - \lambda \Psi(tu_{0})
\]
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\[
\frac{t^p}{p} \left( \int_{\mathbb{R}^{2N}} \frac{|u_0(x) - u_0(y)|^p}{|x - y|^{N+sp}} \, dx dy + \beta \int_{\Omega} \frac{|u_0(x)|^p}{|x|^{sp}} \, dx \right) \\
- \lambda \int_{\Omega} F(x, u_0(x)) \, dx.
\]

So, from Lemma 2 and Hardy’s inequality (2.1) one has

\[
E_{p,1}^{-1}(tu_0) \leq \frac{t^p(\beta c_H + 1)}{p} \|u_0\|_{W_0^p} - \lambda \int_{\Omega} c |tu_0(x)|^\theta \, dx \\
\leq \frac{t^p(\beta c_H + 1)}{p} \|u_0\|_{W_0^p} - \lambda c t^\theta \int_{\Omega} |u_0(x)|^\theta \, dx \\
< 0.
\]

The assumption (H2) ensures that \( \theta > p \), this condition guarantees that

\[
E_{p,1}^{-1}(tu_0) \longrightarrow -\infty, \text{ as } t \rightarrow +\infty,
\]

which lead us to deduce that \( E_{p,1}^{-1} \) is unbounded from below. \( \square \)

**Proof of Theorem 4.** Setting \( \lambda \in ]0, \lambda[ \), we seek to apply Theorem 1 to problem (1.2) in the case \( r = 1 \) to the space \( X := W_0 \) and to the functionals

\[
\Phi_{p,1}(u) := \frac{1}{p} \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx dy + \beta \int_{\Omega} \frac{|u(x)|^p}{|x|^{sp}} \, dx \right)
\]

and

\[
\Psi(u) := \int_{\Omega} F(x, u(x)) \, dx.
\]

The functional \( \Phi_{p,1} \) is continuous and \( \Phi_{p,1}' : W_0 \to W_0^* \) is a homeomorphism according to Proposition 1. Moreover, thanks to condition (H1) and to the compact embedding \( W_0 \hookrightarrow L^q(\Omega) \), for every \( q \in [1, p_s^* \). Then, \( \Psi \) is continuous and has a compact derivative.

Via Proposition 2 and Proposition 3, we established that \( E_{p,1}^{-1} \) is unbounded from below and satisfies the Palais-Smale condition. Therefore, we can apply our main tool Theorem 1, and it only remains to prove that

\[
\lambda \in ]0, \bar{\lambda}[ \subseteq ]0, \frac{1}{\sup_{\Phi_{p,1}(u) < 1} \Psi(u)}[, \notag
\]

where \( \bar{\lambda} := \frac{q}{q \alpha_1 p_1^p + \alpha_2 q p_2^p \frac{q}{p}} \).

From (2.6), we have

\[
\frac{1}{p} \|u\|_{W_0^p}^p \leq \Phi_{p,1}(u) < r, \text{ such that } u \in \Phi_{p,1}^{-1}(]-\infty, r[).
\]
For $r = 1$, we get

$$\|u\|_{W_0}^p < p.$$  

Then, for each $u \in W_0$

$$\|u\|_{W_0} < \frac{p}{p^*},$$  

such that, $u \in \Phi^{-1}_{p,1}(]-\infty, 1[).$ (3.3)

Moreover, according to (H1), we have

$$|F(x, t)| = \left| \int_0^t f(x, s) \, ds \right| \leq \alpha_1 \left| \int_0^t s \, ds \right| + \alpha_2 \left| \int_0^t |s|^{q-1} \, ds \right| = \alpha_1 |t| + \frac{\alpha_2 |t|^q}{q}.$$  

Then

$$\Psi(u) = \int_\Omega |F(x, u(x))| \, dx \leq \int_\Omega \left[ \alpha_1 |u(x)| + \frac{\alpha_2 |u(x)|^q}{q} \right] \, dx \leq \alpha_1 \int_\Omega |u(x)| \, dx + \frac{\alpha_2}{q} \int_\Omega |u(x)|^q \, dx = \alpha_1 \|u\|_{L^1(\Omega)} + \frac{\alpha_2}{q} \|u\|_{L^q(\Omega)}^q.$$  

Using the compact embedding $W_0 \hookrightarrow L^q(\Omega)$ for every $q \in [1, p^*_s)$ and for each $u \in \Phi^{-1}_{p,1}(]-\infty, 1[)$, we have

$$\Psi(u) \leq \alpha_1 c_1 \|u\|_{W_0} + \frac{\alpha_2}{q} (c_q \|u\|_{W_0})^q.$$  

And by (3.3), we obtain

$$\Psi(u) < \alpha_1 c_1 p^{\frac{1}{p}} + \frac{\alpha_2}{q} c_q^{\frac{q}{p}} p^{\frac{q}{p}} = \frac{q \alpha_1 c_1 p^{\frac{1}{p}} + \alpha_2 c_q^{\frac{q}{p}} p^{\frac{q}{p}}}{q}.$$  

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Since \( \lambda \in ]0, \bar{\lambda}[, \) we get

\[
\sup_{\{u \in \Phi^{-1}_{p,1}(-\infty,1]\}} \Psi(u) < \frac{q \alpha_1 c_1 p^{\frac{1}{p}}} {q \alpha_1 c_1 p^{\frac{1}{p}} + \alpha_2 c_2 p^{\frac{q}{p}}} =: \frac{1}{\bar{\lambda}} < \frac{1}{\lambda},
\]

from, the latter one has

\[
0 < \lambda < \bar{\lambda} := \frac{q} {q \alpha_1 c_1 p^{\frac{1}{p}} + \alpha_2 c_2 p^{\frac{q}{p}}} < \frac{1} {\sup_{\{u \in \Phi^{-1}_{p,1}(1]\}} \Psi(u)}.
\]

Then,

\[
\lambda \in ]0, \bar{\lambda}[, \sup_{\{u \in \Phi^{-1}_{p,1}(1]\}} \Psi(u) \left[ 0, \frac{1} {\sup_{\{u \in \Phi^{-1}_{p,1}(1]\}} \Psi(u)} \right].
\]

Now that all hypotheses of Theorem 1 are verified. We conclude that for each \( \lambda \in ]0, \bar{\lambda}[, \) the functional \( E^{p,1}_\lambda \) admits two distinct critical points that are weak solutions of problem (1.2).

\( \square \)

**Remark 1** Remark that Theorem 4 confirms the existence of two positive weak solutions for problem (1.2), if the function \( f \) is positive and \( f(x, 0) \neq 0 \) in \( \Omega \).

The second result of this work is the following.

**Theorem 5** Let \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) be a function with \( f(x, 0) \neq 0 \) in \( \Omega \) satisfying condition (H1). For \( 0 < \beta < \frac{1}{c_H} \), there exists a positive number \( \Lambda \) given by

\[
\Lambda := \frac{\rho^{p-1}} {q \alpha_1 c_1 \left( \frac{p}{1-\beta c_H} \right)^{\frac{1}{p}} + \alpha_2 c_2 \left( \frac{p}{1-\beta c_H} \right)^{\frac{q}{p}} \rho^{q-1}},
\]

such that, for every \( \lambda \in ]0, \Lambda[, \) the following problem (1.3)

\[
\begin{cases}
  (-\Delta)^s u - \beta \frac{|u|^{p-2} u}{|x|^s} = \lambda f(x, u) & \text{in } \Omega \\
  u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

admits at least one non-trivial weak solution \( u_\lambda \in W_0 \). Moreover,

\[
\lim_{\lambda \to 0^+} \|u_\lambda\|_{W_0} = 0
\]

and the function \( g(\lambda) := E^{p-1}_{\lambda}(u_\lambda) \) is negative and strictly decreasing in \( ]0, \Lambda[ \).
To achieve the proof of this result, we need to prove the following lemmas:

**Lemma 3** Let $s \in (0, 1)$ and $N > ps$. Then, the functional $\Phi_{p,-1}$ is coercive and sequentially weakly lower semicontinuous on $W_0$, i.e:

$$\Phi_{p,-1}(u) \leq \liminf_{n \to +\infty} \Phi_{p,-1}(u_n) \text{ if } u_n \rightharpoonup u \text{ weakly in } W_0.$$  

**Proof** Using the fractional Hardy inequality (2.1), we obtain that for any $u \in W_0$,

$$\Phi_{p,-1}(u) = \frac{1}{p} \left( \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx dy - \beta \int_{\Omega} \frac{|u(x)|^p}{|x|^p} \, dx \right) \geq \left( \frac{1 - \beta c_H}{p} \right) \|u\|_{W_0}^p,$$

where $0 < \beta < 1/c_H$.

We conclude that

$$\Phi_{p,-1}(u) \rightharpoonup +\infty, \text{ as } \|u\|_{W_0} \to +\infty.$$

which means that $\Phi_{p,-1}(u)$ is coercive on $W_0$.

Now, by [14, Theorem 6], we know that $C_c^\infty(\Omega)$ is dense subset of $W_0$. Hence, using density arguments, to prove that $\Phi_{p,-1}$ is sequentially weakly lower semicontinuous on $W_0$, it is enough to show that the functional

$$\Phi_{p,-1}$$

is sequentially weakly lower semicontinuous on $C_c^\infty(\Omega)$.  \hspace{1cm} (3.4)

So, let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence in $C_c^\infty(\Omega)$ such that

$$u_n \rightharpoonup u \text{ weakly in } W_0, \text{ as } n \to +\infty. \hspace{1cm} (3.5)$$

Thus, according to Theorem 3, we have for $\alpha = sp$ and $p_s^*(sp) = p \leq p_s^*(0) = p_s^*$, so there exist two finite positive measures $\mu$ and $\nu$ in $\mathbb{R}^N$, and two positive numbers $\mu_0, \nu_0$ such that the following convergence hold weakly in the sense of measures,

$$\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+sp}} \, dy dx \rightharpoonup \mu \geq \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dy dx + \mu_0 \delta_0, \hspace{1cm} (3.6)$$

$$\frac{|u_n(x)|^p}{|x|^{sp}} \, dx \rightharpoonup \nu = \frac{|u(x)|^p}{|x|^{sp}} \, dx + \nu_0 \delta_0, \hspace{1cm} (3.7)$$

and finally, we know that the best constant of the fractional Hardy inequality $H_{sp}$ is given by (2.3), such that

$$0 \leq \nu_0 \leq c_H \mu_0, \hspace{1cm} (3.8)$$

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where $c_H$ is the fractional Hardy inequality constant (2.1) and $\delta_0$ denotes the Dirac mass at point 0.

From the continuity of the embedding $W_0 \hookrightarrow L^p(\Omega)$, for every $p \in [1, p^*_s]$, we have that

$$u_n \longrightarrow u \text{ strongly in } L^p(\Omega), \text{ as } n \rightarrow +\infty. \quad (3.9)$$

By, (3.6), (3.7), (3.8) and (3.9), we obtain

$$\liminf_{n \rightarrow +\infty} \Phi_{p,-1}(u_n) = \Phi_{p,-1}(u),$$

since $\beta < 1/c_H$. This leads us to deduce the statement stated in (3.4).

Now, let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence in $W_0$ satisfying the same condition (3.5). Then, using density arguments, we have for any $n \in \mathbb{N}$ there exists $\{u^j_n\}_{j \in \mathbb{N}} \subseteq C^\infty_c(\Omega)$ such that

$$u^j_n \longrightarrow u_n \text{ strongly in } W_0, \text{ as } j \rightarrow +\infty. \quad (3.10)$$

From (3.5) and (3.10), we have that for any $\varphi \in W_0$

$$\langle u^j_n - u, \varphi \rangle = \langle u^j_n - u_n, \varphi \rangle + \langle u_n - u, \varphi \rangle \rightarrow 0, \text{ as } n, j \rightarrow +\infty.$$

Then,

$$u^j_n \longrightarrow u \text{ weakly in } W_0, \text{ as } n, j \rightarrow +\infty. \quad (3.11)$$

Since $\{u^j_n\}_{j \in \mathbb{N}} \subseteq C^\infty_c(\Omega)$ and the statement (3.4) is satisfied, we deduce that

$$\liminf_{n,j \rightarrow +\infty} \Phi_{p,-1}(u^j_n) \geq \Phi_{p,-1}(u). \quad (3.12)$$
Moreover, by (3.10) it is easy to see that for any \( n \in \mathbb{N} \), we have
\[
\lim_{j \to +\infty} \Phi_{p,-1}(u_j^n) = \Phi_{p,-1}(u_n),
\]
so that, passing to \( \lim \inf \), we get
\[
\liminf_{n,j \to +\infty} \Phi_{p,-1}(u_j^n) = \liminf_{n \to +\infty} \lim_{j \to +\infty} \Phi_{p,-1}(u_j^n) = \liminf_{n \to +\infty} \Phi_{p,-1}(u_n) \tag{3.13}
\]
By (3.12) and (3.13) we get that
\[
\liminf_{n \to +\infty} \Phi_{p,-1}(u_n) \geq \Phi_{p,-1}(u).
\]
Therefore, \( \Phi_{p,-1} \) is sequentially weakly lower semicontinuous on \( W_0 \).

**Lemma 4** Let (H1) be satisfied. Then, the functional \( \Psi \) is sequentially weakly continuous.

**Proof** Let \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( W_0 \) and \( W_0 \) is a reflexive space. Then, up to a subsequence denoted by \( u_n \), there exists \( u \in W_0 \), such that
\[
\begin{align*}
    u_n &\rightharpoonup u \quad \text{weakly in } W_0, \\
    u_n &\to u \quad \text{strongly in } L^p(\Omega), \\
    u_n(x) &\to u(x) \quad \text{a.e. in } \Omega.
\end{align*}
\]
Now, by assumption (H1), we have that
\[
|F(x, t)| = \left| \int_0^t f(x, s) \, ds \right| \\ 
\leq \alpha_1 \int_0^t ds + \alpha_2 \left( \int_0^t |s|^{q-1} \, ds \right) \\ 
= \alpha_1 |t| + \frac{\alpha_2 |t|^q}{q}.
\]
Further, according to the compact embedding \( W_0 \hookrightarrow L^q(\Omega) \) for every \( q \in [1, p_\ast^*) \), we get
\[
|\Psi(u_n)| \leq \int_{\Omega} |F(x, u_n)| \, dx \leq \alpha_1 c_1 \|u_n\|_W_0 + \frac{\alpha_2}{q} \left( c_q \|u_n\|_W_0 \right)^q < +\infty,
\]
thus, we apply the Lebesgue dominated convergence Theorem, we obtain
\[
\lim_{n \to 0} \int_{\Omega} \Psi(u_n) \, dx = \int_{\Omega} \Psi(u) \, dx.
\]
Then, the functional \( \Psi \) is weakly semicontinuous. \( \square \)
Proof of Theorem 5 We set \( \lambda \in ]0, \Lambda[ \). In order to apply Theorem 2 to problem (1.3) with the space \( X = W_0 \) and to the functionals

\[
\Phi_{p,-1}(u) := \frac{1}{p} \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy - \beta \int_{\Omega} \frac{|u(x)|^p}{|x|^p} \, dx \right)
\]

and

\[
\Psi(u) := \int_{\Omega} F(x, u(x)) \, dx.
\]

In view of Lemma 1 and Lemma 3, the functional \( \Phi_{p,-1} \) is continuous, coercive and sequentially weakly lower semicontinuous, also its \( \inf_{u \in W_0} \Phi_{p,-1}(u) = 0 \). Moreover, the functional \( \Psi \) is continuous, has a compact derivative and is sequentially weakly continuous according to Lemma 4. We prove the theorem in the following steps.

Step 1. We start by proving that problem (1.3) admits at least one non-trivial weak solution \( u_{\lambda} \in W_0 \).

By (H1), we have

\[
|F(x, t)| \leq \alpha_1 |t| + \alpha_2 \frac{|t|^q}{q}, \quad \forall (x, t) \in \Omega \times \mathbb{R}.
\]  

(3.14)

Using the above inequality, we obtain

\[
\Psi(u) = \int_{\Omega} |F(x, u(x))| \, dx \\
\leq \int_{\Omega} \left[ \alpha_1 |u(x)| + \alpha_2 \frac{|u(x)|^q}{q} \right] \, dx \\
\leq \int_{\Omega} \alpha_1 |u(x)| \, dx + \alpha_2 \frac{|u(x)|^q}{q} \int_{\Omega} |u(x)|^q \, dx \\
= \alpha_1 \|u\|_{L^1(\Omega)} + \alpha_2 \frac{|u|^q}{q} \|u\|_{L^q(\Omega)}^q.
\]

And according to the compact embedding \( W_0 \hookrightarrow L^q(\Omega) \), for every \( q \in [1, p_s^\ast) \), we have

\[
\Psi(u) \leq \alpha_1 c_1 \|u\|_{W_0} + \alpha_2 \frac{|u|^q}{q} \left(c_q \|u\|_{W_0}^q\right)^q.
\]

(3.15)

On the other hand, we get from (2.5) that

\[
\|u\|_{W_0} < \left( \frac{pr}{1 - \beta c_H} \right)^{\frac{1}{p}}, \quad \forall u \in W_0, \quad \Phi_{p,-1}(u) < r.
\]  

(3.15)
Now, from (3.15), one has
\[
\Psi(u) < \alpha_1 c_1 \left( \frac{pr}{1 - \beta c_H} \right)^{\frac{1}{p}} + \alpha_2 c_1^q \left( \frac{pr}{1 - \beta c_H} \right)^{\frac{q}{p}},
\]
for every \( u \in W_0 \) such that \( \Phi_{p,-1}(u) < r \).

Then,
\[
\sup_{\{u \in \Phi_{p,-1}([-\infty,r])\}} \Psi(u) < \alpha_1 c_1 \left( \frac{p}{1 - \beta c_H} \right)^{\frac{1}{p}} r^{\frac{1}{p}} + \alpha_2 c_1^q \left( \frac{p}{1 - \beta c_H} \right)^{\frac{q}{p}} r^{\frac{q}{p}}.
\]

Hence, for every \( r \in ]0, +\infty[ \) we have
\[
\frac{\left( \sup_{\{u \in \Phi_{p,-1}([-\infty,r])\}} \Psi(u) \right)}{r} \leq \alpha_1 c_1 \left( \frac{p}{1 - \beta c_H} \right)^{\frac{1}{p}} r^{\frac{1-p}{p}} + \alpha_2 c_1^q \left( \frac{p}{1 - \beta c_H} \right)^{\frac{q}{p}} r^{\frac{q-p}{p}}.
\]

In particular, for \( r = \rho^p \) we have
\[
\frac{\left( \sup_{\{u \in \Phi_{p,-1}([-\infty,\rho^p])\}} \Psi(u) \right)}{\rho^p} \leq \alpha_1 c_1 \left( \frac{p}{1 - \beta c_H} \right)^{\frac{1}{p}} \rho^{1-p} + \alpha_2 c_1^q \left( \frac{p}{1 - \beta c_H} \right)^{\frac{q}{p}} \rho^{q-p}.
\]

(3.16)

Now, setting the function \( u_0 \in W_0 \) defined as follows
\[
u_0(x) = 0 \text{ for all } x \in \Omega, \text{ such that } u_0 \in \Phi_{p,-1}([-\infty, \rho^p]).\]

Also, we observe that
\[
\Phi_{p,-1}(u_0) = \frac{1}{p} \| u_0 \|_{W_0}^p + \frac{B}{p} \int \omega \frac{|u_0(x)|^p}{|x|^p} \omega \frac{|x|^p}{\omega} \omega = 0.
\]

(3.18)

and
\[
\Psi(u_0) = \int_{\Omega} F(x, u_0(x)) \omega \frac{|x|^p}{\omega} \omega = 0.
\]

(3.19)

Therefore, from (3.17), (3.18) and (3.19), we obtain that
\[
\varphi(\rho^p) := \inf_{u \in \Phi_{p,-1}([-\infty, \rho^p])} \left( \frac{\left( \sup_{\{v \in \Phi_{p,-1}([-\infty, \rho^p])\}} \Psi(v) \right) - \Psi(u)}{\rho^p - \Phi_{p,-1}(u)} \right) \leq \left( \frac{\left( \sup_{\{v \in \Phi_{p,-1}([-\infty, \rho^p])\}} \Psi(v) \right) - \Psi(u_0)}{\rho^p - \Phi_{p,-1}(u_0)} \right).
\]
Thus, by (3.16) we have

$$\varphi(\rho^p) \leq \alpha_1 c_1 \left( \frac{p}{1 - \beta c_H} \right)^{\frac{1}{p}} \rho^{1-p} + \alpha_2 \frac{c_q}{q} \left( \frac{p}{1 - \beta c_H} \right)^{\frac{q}{p}} \rho^{q-p}.$$

Furthermore, since $0 < \lambda < \Lambda$

$$\varphi(\rho^p) \leq \alpha_1 c_1 \left( \frac{p}{1 - \beta c_H} \right)^{\frac{1}{p}} \rho^{1-p} + \alpha_2 \frac{c_q}{q} \left( \frac{p}{1 - \beta c_H} \right)^{\frac{q}{p}} \rho^{q-p} =: \frac{1}{\Lambda(\rho)} < \frac{1}{\lambda}.$$ 

So,

$$0 < \lambda < \Lambda(\rho) := \frac{q \rho^{p-1}}{q \alpha_1 c_1 \left( \frac{p}{1 - \beta c_H} \right)^{\frac{1}{p}} + \alpha_2 \frac{c_q}{q} \left( \frac{p}{1 - \beta c_H} \right)^{\frac{q}{p}} \rho^{q-1}} \leq \frac{1}{\varphi(\rho^p)}.$$ 

Then,

$$\lambda \in ]0, \Lambda(\rho)[ \subseteq ]1, \frac{1}{\varphi(\rho^p)}[.$$ 

In conclusion, according to Theorem 2, there exists a critical point $u_\lambda \in \Phi^{-1}_{p,-1}([-\infty, \rho^p])$ for $E_{\lambda}^{p,-1}$ in $W_0$ which is a global minimum of the restriction $E_{\lambda}^{p,-1}$ to $\Phi^{-1}_{p,-1}([-\infty, \rho^p])$. Moreover, the function $u_\lambda \neq 0$ since $f(x, 0) \neq 0$ in $\Omega$.

**Step 2.** We show that $\lim_{\lambda \to 0^+} \|u_\lambda\|_{W_0} = 0$ and that the functional $g(\lambda) := E_{\lambda}^{p,-1}(u_\lambda)$ is negative and strictly decreasing in $]0, \Lambda(\rho)[$.

As $\Phi_{p,-1}$ is coercive, then $u_\lambda \in \Phi^{-1}_{p,-1}([-\infty, \rho^p])$ is bounded in $W_0$, that is to say

$$\|u_\lambda\|_{W_0} \leq K, \text{ for } K > 0.$$ 

Thus, due to the compactness of the operator $\Psi'$, there exists a constant $C > 0$ such that

$$\left| \langle \Psi'(u_\lambda), u_\lambda \rangle \right| \leq \|\Psi'(u_\lambda)\|_{W_0^*} \|u_\lambda\|_{W_0} < CK^2, \forall \lambda \in ]0, \Lambda(\rho)[.$$ 

(3.20)

On the other hand, since $u_\lambda$ is a critical point of $E_{\lambda}^{p,-1}$, then

$$\left( \left( E_{\lambda}^{p,-1} \right)' \right)(u_\lambda), u_\lambda) = 0.$$
which implies that
\[
\langle \Phi_{p,-1}'(\lambda), u_{\lambda} \rangle \quad \text{is an equilibrium.}
\]
So,
\[
p\Phi_{p,-1}(u_{\lambda}) = \langle \Phi_{p,-1}'(\lambda), u_{\lambda} \rangle = \lambda \langle \Psi'(u_{\lambda}), u_{\lambda} \rangle, \quad \forall \lambda \in ]0, \Lambda[. \tag{3.21}
\]
Therefore, by (3.20) and (3.21), we get
\[
\lim_{\lambda \to 0^+} p\Phi_{p,-1}(u_{\lambda}) = \lim_{\lambda \to 0^+} \lambda \langle \Psi'(u_{\lambda}), u_{\lambda} \rangle = 0, \quad \forall p < 1. \tag{3.22}
\]
Moreover, by (2.5) one has
\[
\|u_{\lambda}\|_{W_0}^p \leq \frac{p\Phi_{p,-1}(u_{\lambda})}{1 - \beta c_H}, \quad \forall \lambda \in ]0, \Lambda[. \tag{3.23}
\]
Then, we conclude by the conditions (3.22) and (3.23) that
\[
\lim_{\lambda \to 0^+} \|u_{\lambda}\|_{W_0} = 0.
\]
Furthermore, since the restriction $E_{\lambda}^{p,-1}$ to $\Phi_{p,-1}^{-1}([-\infty, 0])$ admits a global minimum, which is a local minimum of $E_{\lambda}^{p,-1}$ in $W_0$, the map $g(\lambda) := E_{\lambda}^{p,-1}(u_{\lambda})$ is negative in $]0, \Lambda[$, because $u_{\lambda} \neq 0$ and $E_{\lambda}^{p,-1}(0) = 0$.

Finally, we prove that the function $g(\lambda) := E_{\lambda}^{p,-1}(u_{\lambda})$ is negative and strictly decreasing in $]0, \Lambda[$, by observing that
\[
E_{\lambda}^{p,-1}(u) = \lambda \left( \frac{\Phi_{p,-1}(u)}{\lambda} - \Psi(u) \right).
\]
Now, we assume $u_{\lambda_1}, u_{\lambda_2} \in W_0$ are critical points of $E_{\lambda}^{p,-1}$, for every $\lambda_1, \lambda_2 \in ]0, \Lambda[$, with $\lambda_1 < \lambda_2$. Further, we set
\[
I_{\lambda_i} := \inf_{u \in \Phi_{p,-1}^{-1}([-\infty, 0])} \left( \frac{\Phi_{p,-1}(u)}{\lambda_i} - \Psi(u) \right) = \frac{1}{\lambda_i} E_{\lambda_i}^{p,-1}(u_{\lambda_i}), \quad i = 1, 2.
\]
Obviously, as mentioned earlier $I_{\lambda_i} < 0$ for $i = 1, 2$, and since $\lambda_1 < \lambda_2$, we have $I_{\lambda_2} \leq I_{\lambda_1}$. Therefore,
\[
E_{\lambda_2}^{p,-1}(u_{\lambda_2}) = \lambda_2 I_{\lambda_2} \leq \lambda_2 I_{\lambda_1} < \lambda_1 I_{\lambda_1} = E_{\lambda_1}^{p,-1}(u_{\lambda_1}).
\]
We conclude that, as $\lambda \in ]0, \Lambda[$, is arbitrary, the above conclusions are still true in $]0, \Lambda[$. Hence the proof is completed. \qed
Remark 2 To calculate the maximum of $\Lambda$, we need to look at its first derivative

$$\Lambda'(\rho) = -\left[ \frac{\alpha_1 c_1 \left( \frac{p}{1 - \beta c_H} \right)^{\frac{1}{p}} (1 - p)\rho^{-p} + \alpha_2 c_q^q \left( \frac{p}{1 - \beta c_H} \right)^{\frac{q}{p}} (q - p)\rho^{q-p-1}}{\alpha_1 c_1 \left( \frac{p}{1 - \beta c_H} \right)^{\frac{1}{p}} \rho^{1-p} + \alpha_2 c_q^q \left( \frac{p}{1 - \beta c_H} \right)^{\frac{q}{p}} \rho^{q-p}} \right].$$

Further, we set $\Lambda'(\rho)$ equal to zero and obtain

$$\alpha_1 c_1 \left( \frac{p}{1 - \beta c_H} \right)^{\frac{1}{p}} (1 - p)\rho^{-p} + \alpha_2 c_q^q \left( \frac{p}{1 - \beta c_H} \right)^{\frac{q}{p}} (q - p)\rho^{q-p-1} = 0.$$

Then,

$$\rho_{\text{max}} := \left( \frac{p}{1 - \beta c_H} \right)^{-\frac{1}{p}} \left[ q \frac{\alpha_1 c_1}{\alpha_2 c_q^q} \left( \frac{1 - p}{p - q} \right) \right]^{\frac{1}{q-1}}.$$

Which lead us to conclude that $\Lambda$ is defined as follows

$$\Lambda(\rho) = \begin{cases} +\infty & \text{if } 1 < q < p \\ \frac{1 - \beta c_H}{\alpha_2 c_q^q} & \text{if } q = p \\ \frac{q^{-1}}{\rho_{\text{max}}}^{p-1} & \text{if } q \in ]p, p^*[, \end{cases}$$

Notice from the latter that if $f$ satisfy the condition (H1) at infinity, with $q \in ]1, p[$. Then, from Theorem 5 we confirm that for each $\lambda > 0$, our problem (1.3) admits at least a non-trivial weak solution.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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