A nonconvex approach to low-rank and sparse matrix decomposition with application to video surveillance

Angang Cui, Jigen Peng, Haiyang Li and Changlong Wang

Abstract—In this paper, we develop a new nonconvex approach to the problem of low-rank and sparse matrix decomposition. In our nonconvex method, we replace the rank function and the $\ell_0$-norm of a given matrix with a non-convex fraction function on the singular values and the elements of the matrix respectively. An alternative direction method of multipliers algorithm is utilized to solve our nonconvex problem with the non-convex fraction function penalty. Numerical experiments on video surveillance show that our method performs very well in separating the moving objects from the static background.

Index Terms—Nonconvex fraction function, Low-rank and sparse matrix decomposition, Alternative direction method of multipliers algorithm, Video surveillance

I. INTRODUCTION

The problem of low-rank and sparse matrix decomposition considers to recover the essentially low-rank data matrix from a large data matrix with sparse errors. In other words, it aims to decompose a data matrix into a sum of two components, one having low rank, the other being sparse. The problem of low-rank and sparse matrix decomposition has been widely applied in automated background removal in video [1], text analysis [2], image alignment [3] and shadows and specularities removing in face images [4].

In mathematics, the problem of low-rank and sparse matrix decomposition for a large data matrix $M \in \mathbb{R}^{m \times n}$ can be defined as

$$M = L + S,$$

where $L \in \mathbb{R}^{m \times n}$ is the low-rank matrix, and $S \in \mathbb{R}^{m \times n}$ is the sparse error matrix. One can exactly recover the low rank matrix $L$ and sparse error matrix $S$ from $M = L + S$ by solving the following optimization problem:

$$\min_{L,S \in \mathbb{R}^{m \times n}} \text{rank}(L) + \lambda \|S\|_0, \text{ subject to } M = L + S,$$

where $\lambda > 0$ is a tuning parameter, $\text{rank}(L)$ represents the rank of matrix $L$, and $\|S\|_0$ is the $\ell_0$-norm of the matrix $S$,

which counts the number of nonzero elements. In general, however, problem (2) is a challenging nonconvex optimization problem and is known as NP-hard because of the discrete natures of the rank function $\text{rank}(L)$ and $\ell_0$-norm $\|S\|_0$.

Motivated by our previous results and inspired by the good performance of the nonconvex fraction function in compressed sensing [5] and affine matrix rank minimization problem [6], in this paper, we replace the rank function $\text{rank}(L)$ and the $\ell_0$-norm $\|S\|_0$ in problem (2) with the continuous promoting low rank non-convex function

$$P_a(\sigma(L)) = \sum_{i \in [n]} a(\sigma_i(L)), \quad n_1 = \min\{m, n\}$$

and the continuous promoting sparse non-convex function

$$P_a(S) = \sum_{i \in [m], j \in [n]} a(S_{i,j})$$

respectively, where $\sigma(L)$ is the vector of singular values of matrix $L$ ($\sigma_i(X)$ represents the $i$th largest singular value of matrix $L$ arranged in descending order), $S_{i,j}$ is the $i,j$th element of the matrix $S$ and the nonconvex function

$$a(t) = \frac{at}{|t| + 1}$$

is the fraction function for all $a \in (0, +\infty)$. With the change of parameter $a > 0$, we have

$$\lim_{a \to +\infty} a(t) = \lim_{a \to +\infty} \frac{at}{|t| + 1} = \begin{cases} 0, & \text{if } t = 0; \\ 1, & \text{if } t \neq 0. \end{cases}$$

Then, the function $P_a(\sigma(L))$ interpolates the rank of matrix $L$:

$$\lim_{a \to +\infty} P_a(\sigma(L)) = \lim_{a \to +\infty} \sum_{i \in [n]} a_i(\sigma_i(L)) = \text{rank}(L)$$

and the function $P_a(S)$ interpolates the $\ell_0$-norm of matrix $S$:

$$\lim_{a \to +\infty} P_a(S) = \lim_{a \to +\infty} \sum_{i \in [m], j \in [n]} a_i(S_{i,j}) = \|S\|_0.$$
we consider the following augmented Lagrange minimization problem of the nonconvex constrained problem (9):

$$
\min_{L,S \in \mathbb{R}^{m \times n}} P_a(\sigma(L)) + \lambda P_a(S) + \langle Y, M - L - S \rangle + \frac{\mu}{2} \| M - L - S \|_F^2
$$

(10)

where $Y \in \mathbb{R}^{m \times n}$ is the Lagrange multiplier matrix and the parameter $\mu > 0$ leading to an optimization problem that can be easily solved.

The rest of this paper is organized as follows. In Section II, we summarize some preliminary results that will be used in this paper. In Section III, we use an alternative direction method of multipliers (ADMM) algorithm to solve the problem (10). In Section IV, we present some numerical experiments on video surveillance to demonstrate the performances of our method. Finally, some conclusion remarks are presented in Section V.

II. PRELIMINARIES

In this section, we summarize some crucial results and definitions that will be used in this paper.

Lemma 1. (see [5], [6]) Define a function of $\beta \in \mathbb{R}$ as

$$
f_\lambda(\beta) = \frac{1}{2} (\beta - \gamma)^2 + \lambda \rho_\alpha(\beta)
$$

(11)

where $\gamma \in \mathbb{R}$, the proximal mapping of the function $f_\lambda$ can be described as

$$
\text{prox}_{f_\lambda}(\gamma) = \begin{cases} 
g_a,\lambda(\gamma), & \text{if } |\gamma| > t^*_a,\lambda; \\
0, & \text{if } |\gamma| \leq t^*_a,\lambda.
\end{cases}
$$

(12)

where $g_a,\lambda$ is defined as

$$
g_a,\lambda(\gamma) = \text{sign}(\gamma) \left( \frac{1}{3} \left( 1 + 2 \cos \left( \frac{2\gamma}{3} - \frac{\pi}{3} \right) \right) - 1 \right),
$$

(13)

and the threshold value satisfies

$$
t^*_a,\lambda = \begin{cases} 
\lambda a, & \text{if } \lambda \leq \frac{1}{2\sqrt{a}}; \\
\sqrt{2(a - 1)}, & \text{if } \lambda > \frac{1}{2\sqrt{a}}.
\end{cases}
$$

(14)

Definition 1. Let $X = U \text{Diag}(\sigma_i(X)) V^T$ be the singular value decomposition of matrix $X \in \mathbb{R}^{m \times n}$, we define the singular value thresholding operator $\mathcal{G}_\lambda$ on the matrix $X$ and nonconvex fraction function $\rho_\alpha$ as

$$
\mathcal{G}_\lambda(X) = U \text{Diag}(\text{prox}_{f_\lambda}(\sigma_i(X))) V^T
$$

(15)

where the proximal mapping operator $\text{prox}_{f_\lambda}$ is defined in Lemma 1.

Definition 2. For any matrix $X \in \mathbb{R}^{m \times n}$, we define the matrix thresholding operator $\mathcal{F}_\lambda$ on matrix $X$ and nonconvex fraction function $\rho_\alpha$ as

$$
\mathcal{F}_\lambda(X) = [\text{prox}_{f_\lambda}(X_{i,j})]
$$

(16)

where the proximal mapping operator $\text{prox}_{f_\lambda}$ is defined in Lemma 1.

The singular value thresholding operator $\mathcal{G}_\lambda$ defined in Definition 1 simply applies the proximal operator $\text{prox}_\lambda$ to the singular values of a matrix, and effectively shrinks them towards zero. It is clear that if many of the singular values of matrix $X$ are below the threshold value $t^*_a,\lambda$, the rank of $\mathcal{G}_\lambda(X)$ may be considerably lower than the rank of matrix $X$. Similarly, the matrix thresholding operator $\mathcal{F}_\lambda$ defined in Definition 2 simply applies the proximal operator $\text{prox}_\lambda$ to the elements of a matrix. If many of the elements of matrix $X$ are below the threshold value $t^*_a,\lambda$, the matrix thresholding operator $\mathcal{F}_\lambda$ effectively shrinks them towards zero, and the matrix $\mathcal{F}_\lambda(X)$ may be considerably a sparse matrix.

III. THE ALGORITHM FOR SOLVING THE PROBLEM (10)

In this section, we use an alternative direction method of multipliers (ADMM) algorithm [4] to solve the problem (10).

We proceed the ADMM algorithm by recognizing

$$
\min_{L \in \mathbb{R}^{m \times n}} P_a(\sigma(L)) + \lambda P_a(S) + \langle Y, M-L-S \rangle + \frac{\mu}{2} \| M-L-S \|_F^2
$$

with fixed $S \in \mathbb{R}^{m \times n}$, and

$$
\min_{S \in \mathbb{R}^{m \times n}} P_a(\sigma(L)) + \lambda P_a(S) + \langle Y, M-L-S \rangle + \frac{\mu}{2} \| M-L-S \|_F^2
$$

(17)

(18)

where $\mathcal{F}_\mu^{-1}$ is defined in Definition 2 and obtained by replacing $\lambda$ with $\mu^{-1}$ in $\mathcal{F}_\lambda$. Similarly, combining problem (17), Lemma 1 and Definition 1, it is easy to show that

$$
\arg \min_{L \in \mathbb{R}^{m \times n}} P_a(\sigma(L)) + \lambda P_a(S) + \langle Y, M-L-S \rangle + \frac{\mu}{2} \| M-L-S \|_F^2
$$

$$
= \mathcal{G}_{\mu^{-1}}(M-S+\mu^{-1}Y)
$$

(19)

where $\mathcal{G}_{\mu^{-1}}$ is defined in Definition 1 and obtained by replacing $\lambda$ with $\mu^{-1}$ in $\mathcal{G}_\lambda$. Similarly, combining problem (18), Lemma 1 and Definition 2, we can get that

$$
\arg \min_{S \in \mathbb{R}^{m \times n}} P_a(\sigma(L)) + \lambda P_a(S) + \langle Y, M-L-S \rangle + \frac{\mu}{2} \| M-L-S \|_F^2
$$

$$
= \mathcal{F}_{\mu^{-1}}(M-L+\mu^{-1}Y)
$$

(20)

where $\mathcal{F}_{\mu^{-1}}$ is defined in Definition 2 and obtained by replacing $\lambda$ with $\mu^{-1}$ in $\mathcal{F}_\lambda$. Finally, we update the Lagrange multiplier matrix $Y$ based on the residual $M-L-S$.

Algorithm 1: ADMM algorithm for solving the problem (10)

Initialize: $L^0 = Y^0 = 0$, $\lambda > 0$, $\mu > 0$ and $\varepsilon > 0$.

While not converged do

$$
L_{k+1} = \mathcal{G}_{\mu^{-1}}(M-S_{k}^{-1} - \mu^{-1}Y_{k}^{-1});
$$

$$
S_{k+1} = \mathcal{F}_{\mu^{-1}}(M-L_{k+1}^{-1} + \mu^{-1}Y_{k+1}^{-1});
$$

$$
Y_{k+1} = Y_{k}^{-1} + \mu(M-L_{k+1}^{-1} - S_{k+1}^{-1});
$$

End while

Output: $L^*$, $S^*$.

The ADMM algorithm for solving the problem (10) can be summarized in Algorithm 1. It is necessary to emphasize that, in this paper, we cannot prove the convergence of the Algorithm 1, and we would like to treat it as our future work.
Fig. 1. The decomposition result for one frame of video surveillance with \( a = 1 \) in iteration \( k = 1 \). Top: one frame from the video, \( M \). Second row: the low-rank background, \( L^1 \). Third row: the sparse component contains the moving objects, \( S^1 \). Bottom: the residual \( M - L^1 - S^1 \).

Fig. 2. The decomposition result for one frame of video surveillance with \( a = 1 \) in iteration \( k = 10 \). Top: one frame from the video, \( M \). Second row: the low-rank background, \( L^{10} \). Third row: the sparse component contains the moving objects, \( S^{10} \). Bottom: the residual \( M - L^{10} - S^{10} \).
Fig. 3. The decomposition result for one frame of video surveillance with $\alpha = 0.1$ in iteration $k = 1$. Top: one frame from the video, $M$. Second row: the low-rank background, $L^1$. Third row: the sparse component contains the moving objects, $S^1$. Bottom: the residual $M - L^1 - S^1$.

Fig. 4. The decomposition result for one frame of video surveillance with $\alpha = 0.1$ in iteration $k = 10$. Top: one frame from the video, $M$. Second row: the low-rank background, $L^{10}$. Third row: the sparse component contains the moving objects, $S^{10}$. Bottom: the residual $M - L^{10} - S^{10}$.
IV. Numerical experiments

In this section, we present some numerical experiments on video surveillance to demonstrate the performances of our method. The video surveillance is a natural candidate for the problem of low-rank and sparse matrix decomposition due to the correlation between frames and the sparsity of the foreground variations. In this paper, the numerical experiments are all conducted on a personal computer (3.40GHz, 16.0GB RAM) with MATLAB R2015b.

We consider a busy scene video introduced in [7]. This video has a relatively static background, but significant foreground variations. We generate the first 200 frames, and the size of each is 144 × 176, and the size of our resulting matrix $M$ is $25344 \times 200$. We decompose matrix $M$ into a low-rank term and a sparse term by Algorithm 1. In our numerical experiments, we set the parameters $\lambda = \frac{1}{12}, \mu = 2\|M\|_2^2$, and consider the cases of $a \in \{0.1, 0.3, 0.5, 1, 2, 5, 20\}$. We measure the relative error $\frac{\|M - L_k^* - S^*\|_F}{\|M\|_F}$ versus the iteration number of the Algorithm 1 for different values of $a$. The optimal $L^*$ should be of rank 1 because of the static of background. Then $M - L^* = S^*$ will consist of solely objects that are moving. In our numerical experiments, the sparsity of optimal $S^*$ is set to $3.1 \times 10^5$. The results are given in Fig. 5, with a few cases shown in Figs. 1, 2, 3 and 4.

In this paper, based on the nonconvex fraction function, we presented a nonconvex optimization problem for the low-rank and sparse matrix decomposition. The ADMM algorithm is utilized to solve our nonconvex optimization problem, and the numerical results on video surveillance show that our method performs very well in separating the moving objects from the static background.

There are some interesting future work. First, the convergence of our algorithm is not proved in this paper, and we would like to treat it as our future work. Second, the numerical results on video surveillance suggest that our method could very well separate the moving objects and the static background in one iteration. It is possible to provide some support in theory under some conditions.

Fig. 5 compares the performances of the Algorithm 1 for the video surveillance with different values of $a$, and the graphs presented in Fig.1 show the relative error $\frac{\|M - L_k^* - S^*\|_F}{\|M\|_F}$ in separating the low-rank component $L$ and the sparse component $S$. As we can see, our algorithm has the best performance in iteration $k = 1$ when the parameter $a$ is set to 0.1, with $a = 0.3$ as the second and $a = 20$ as the last.

It is interesting to mention that, as the iteration number exceeds 2, the parameter $a = 1$ performs the best in our numerical experiments, with $a = 2$ as the second and $a = 0.1$ as the last.

V. Conclusion

References

[1] J. Wright, A. Ganesh, S. Rao and Y. Ma, Robust principal component analysis: Exact recovery of corrupted low-rank matrices by convex optimization. In Proceeding of the Advances in Neural Information Processing Systems Conference, NIPS 2009, 2080–2088 (2009)

[2] K. Min, Z. Zhang, J. Wright and Y. Ma, Decomposing background topics from keywords by principal component pursuit. In Proceedings of the 19th ACM international conference on Information and knowledge management, 269–278 (2010)

[3] Y. Peng, A. Ganesh, J. Wright, W. Xu and Y. Ma, RASL: robust alignment by sparse and low-rank decomposition for linearly correlated images. IEEE Transactions on Pattern Analysis and Machine Intelligence, 34, 2233–2246 (2012)

[4] E. J. Candes, X. Li, Y. Ma and J. Wright, Robust Principal Component Analysis?. Journal of the ACM, 58, Journal of the ACM (2011)

[5] H. Li, Q. Zhang, A. Cui and J. Peng, Minimization of fraction function penalty in compressed sensing. arXiv preprint arXiv:1705.06048 (2017)

[6] A. Cui, J. Peng, H. Li, C. Zhang and Y. Yu, Affine matrix rank minimization problem via non-convex fraction function penalty. Journal of Computational and Applied Mathematics, 336, 353–374 (2018)

[7] L. Li, W. Huang, I. Yu-Hua Gu, and Q. Tian, Statistical modeling of complex backgrounds for foreground object detection. IEEE Transaction on Image Processing, 13(11), 1459–1472 (2004)