This is a summary of the proof by G.E. Coxson [1] that P-matrix recognition is co-NP-complete. The result follows by a reduction from the MAX CUT problem using results of S. Poljak and J. Rohn [5].

1 Considered problems

Our main interest is the complexity of deciding whether an input matrix is a P-matrix. A P-matrix is a square matrix $M \in \mathbb{R}^{n \times n}$ such that all its principal minors are positive. Such matrices were first studied by Fiedler and Pták [2].

**P-MATRIX**

**Instance:** A square matrix $M \in \mathbb{Q}^{n\times n}$.

**Question:** Are all the principal minors of $M$ positive?

To start with, we use a well-known combinatorial problem.

**SIMPLE MAX CUT**

**Instance:** A graph $G = (V, E)$, a positive integer $K$.

**Question:** Is there a partition of the vertex set $V$ into sets $V_1$ and $V_2$ such that the number of edges with one end in $V_1$ and the other end in $V_2$ is at least $K$?

Garey, Johnson and Stockmeyer [4] showed that the SIMPLE MAX CUT problem is NP-complete.

The reduction from SIMPLE MAX CUT to P-MATRIX uses two intermediate steps. The first of them is the computation of the $r$-norm of a matrix.
For an arbitrary matrix $A \in \mathbb{R}^{n \times n}$, let
\[
r(A) = \max \left\{ z^T Ay : z, y \in \{-1, 1\}^n \right\}.
\]

**Remark.** The function $r$ is a matrix norm.

**Proof.** For an arbitrary square matrix $A$, we have $r(A) \geq 0$ because $z^T Ay = -(z^T Ay)$. Moreover if $r(A) = 0$, then $z^T Ay = 0$ for all choices of $z, y \in \{-1, 1\}^n$, hence $A = 0$. If $k \in \mathbb{R}$, then $z^T (kA) = k \cdot z^T Ay$, so $r(kA) = |k| \cdot r(A)$.

Let $A, B \in \mathbb{R}^{n \times n}$. Then
\[
r(A + B) = \max \{ z^T (A+B)y : z, y \in \{-1, 1\}^n \} = \max \{ z^T Ay + z^T By : y, z \in \{-1, 1\}^n \} \\
\leq \max \{ z^T Ay : y, z \in \{-1, 1\}^n \} + \max \{ z^T By : y, z \in \{-1, 1\}^n \} \\
= r(A) + r(B).
\]

Thus $r$ is also subadditive. \qed

The decision problem corresponding to $r$-norm computation is defined as follows.

**MATRIX R-NORM**

**Instance:** A matrix $A \in \mathbb{Q}^{n \times n}$ and a rational number $K$.

**Question:** Is $r(A) \geq K$?

For the last of the decision problems considered here, we need the notion of matrix interval. If $A_-$ and $A_+$ are $n \times n$ real matrices such that $A_- \leq A_+$ (that is, for each $r$ and $s$ we have $(A_-)_{r,s} \leq (A_+)_{r,s}$), then the matrix interval $[A_-, A_+]$ is the set of all matrices $A$ satisfying $A_- \leq A \leq A_+$.

A matrix interval is singular if it contains a singular matrix; otherwise it is non-singular.

The decision problem we consider consists in testing whether a given matrix interval is singular. We will see that this is a computationally hard problem even when the difference $A_+ - A_-$ has rank 1.

**RK1-MATRIX-INTERVAL SINGULARITY**

**Instance:** A non-singular matrix $A \in \mathbb{Q}^{n \times n}$ and a non-negative matrix $\Delta \in \mathbb{Q}^{n \times n}$ of rank 1.

**Question:** Is the matrix interval $[A - \Delta, A + \Delta]$ singular?

The rest of this exposition contains three polynomial reductions of these problems, ultimately proving that P-MATRIX is co-NP-complete.

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*This object is usually called an interval matrix. Since it is actually an interval and not a matrix, I beg the reader to pardon my decision to call it an uncommon but appropriate name.*
2 Reduction from SIMPLE MAX CUT to MATRIX R-NORM

Let $G = (V, E)$ be an undirected graph with $n = |V|$ and let $\ell = 2|E| + 1$. If $A(G)$ is the adjacency matrix of $G$, define $A = \ell \cdot I_n - A(G)$. Thus

$$A_{u,v} = \begin{cases} 
\ell & \text{if } u = v, \\
-1 & \text{if } uv \in E, \\
0 & \text{otherwise.}
\end{cases}$$

Observe that for $y, z \in \{-1, 1\}^n$ we have $z^T Ay \leq y^T Ay$ because of the choice of $\ell$. Hence $r(A) = y^T Ay$ for some $y \in \{-1, 1\}^n$.

Let $S \subseteq V$ be defined by $S = \{u : y_u = 1\}$ and let $m'$ be the number of edges of $G$ with one end in $S$ and the other end in $V \setminus S$. In this way, $m'$ is the size of the cut defined by $S$ and $V \setminus S$.

Then

$$y^T Ay = n\ell + 4m' - 2|E|$$

and therefore there is a cut in $G$ of size at least $K$ if and only if $r(A) \geq n\ell - 2|E| + 4K$.

The described reduction (by Poljak and Rohn [5]) establishes the hardness of computing the $r$-norm.

**Theorem 1.** MATRIX R-NORM is NP-complete, even if input is restricted to non-singular matrices.

**Proof.** It follows from the reduction above that MATRIX R-NORM is NP-hard. Observe that by the choice of $\ell$ the matrix $A$ in the reduction is strictly diagonally dominant and thus non-singular.

A non-deterministic Turing machine can guess the values of $y, z \in \{-1, 1\}^n$ and check in polynomial time that $z^T Ay \geq K$, so the problem is in the class NP. \qed

3 Reduction from MATRIX R-NORM to RK1-MATRIX-INTERVAL SINGULARITY

For a matrix $A \in \mathbb{R}^{n \times n}$ define

$$\rho_0(A) = \max\{|\lambda| : \lambda \text{ is a real eigenvalue of } A\}$$

and set $\rho_0(A) = 0$ if $A$ has no real eigenvalue.

Further for a vector $y \in \mathbb{R}^n$ define $D(y)$ to be the diagonal $n \times n$ matrix with diagonal vector $y$.

The following fact was proved by Rohn [6].

**Lemma 2.** Let $A$ be a real non-singular $n \times n$ matrix and let $\Delta$ be a real non-negative $n \times n$ matrix. Then the matrix interval $[A - \Delta, A + \Delta]$ is singular if and only if $\rho_0(A^{-1} D(y) \Delta D(z)) \geq 1$ for some $y, z \in \{-1, 1\}^n$. 

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Proof. For \( y, z \in \{-1, 1\}^n \) let \( \Delta_{y,z} \) denote the matrix \( D(y) \Delta D(z) \).

First suppose that \( A^{-1} \Delta_{y,z} \) has a real eigenvalue \( \lambda \) such that \( |\lambda| \geq 1 \) and \( A^{-1} \Delta_{y,z} x = \lambda x \) for some \( y, z \in \{-1, 1\}^n \) and a non-zero vector \( x \). Then

\[
(1 - \frac{1}{\lambda} A^{-1} \Delta_{y,z}) x = 0,
\]

\[
(A - \frac{1}{\lambda} \Delta_{y,z}) x = 0.
\]

Hence \( A - (1/\lambda) \Delta_{y,z} \) is a singular matrix in the interval \([A - \Delta, A + \Delta]\) because

\[
\left| \frac{1}{\lambda} \Delta_{y,z} \right| = \left| \frac{1}{\lambda} D(y) \Delta D(z) \right| \leq \Delta.
\]

Therefore the interval \([A - \Delta, A + \Delta]\) is singular.

To prove the converse, suppose that \( B \) is a singular matrix, \( B \in [A - \Delta, A + \Delta] \). Let \( x \) be a non-zero vector for which \( Bx = 0 \).

For \( i = 1, 2, \ldots, n \) set

\[
t_i = \frac{(Ax)_i}{(\Delta|x|)_i}.
\]

We claim that \( t \in [0,1]^n \). Indeed, \( |Ax| = |(A - B)x| \leq \Delta|x| \) because \( Bx = 0 \) and \( B \in [A - \Delta, A + \Delta] \).

Moreover, set \( z = \text{sgn} x \). Then \( D(z)x = |x| \) and

\[
(A - \Delta_{t,z})x = Ax - D(t)\Delta D(z)x = Ax - D(t)\Delta|x| = 0
\]

by the definition of \( t \). Thus the matrix \( A - \Delta_{t,z} \) is a singular matrix in the interval \([A - \Delta, A + \Delta]\).

Define \( \psi(s) = \det(A - \Delta_{s,z}) \). The function \( \psi \) is affine in each of the variables \( s_1, \ldots, s_n \). Since \( \psi(t) = \det(A - \Delta_{t,z}) = 0 \), either there exists \( y \in \{-1, 1\}^n \) such that \( \det(A - \Delta_{y,z}) = 0 \), or there exist \( y, y' \in \{-1, 1\}^n \) such that \( \det(A - \Delta_{y,z}) \cdot \det(A - \Delta_{y',z}) < 0 \).

In the latter case, without loss of generality we may assume that \( \det A \cdot \det(A - \Delta_{y,z}) < 0 \). The function \( \phi \) defined by \( \phi(\alpha) = \det(A - \alpha \Delta_{y,z}) \) is continuous and \( \phi(0)\phi(1) < 0 \), so \( \phi \) has a root in \((0,1)\).

In either case, there exist \( y \in \{-1, 1\}^n \) and \( \alpha \in (0,1) \) such that \( \det(A - \alpha \Delta_{y,z}) = 0 \). Then

\[
\det \left( \frac{1}{\alpha} A - \Delta_{y,z} \right) = 0,
\]

\[
\det \left( \frac{1}{\alpha} I - A^{-1} \Delta_{y,z} \right) = 0,
\]

hence \( \frac{1}{\alpha} \) is a real eigenvalue of the matrix \( A^{-1} D(y) \Delta D(z) \) and \( \frac{1}{\alpha} \geq 1 \), as we were supposed to prove.

This lemma provides a useful connection between singularity of matrix intervals and a parameter \( \rho_0 \) dependent on the two matrices \( A, \Delta \) that define the interval. Next we establish a connection between \( \rho_0 \) and the \( r \)-norm of matrices.

From now on let \( 1 \) be the all-one vector \((1,1,\ldots,1) \in \mathbb{R}^n \) and let \( J = 1 \cdot 1^T \) be the all-one \( n \times n \) matrix.
Lemma 3. Let $A \in \mathbb{R}^{n \times n}$ be a non-singular matrix, let $\alpha$ be a positive real number and let $\Delta = \alpha J$. Then

$$\max \{\rho_0(AD(y)\Delta D(z)) : y, z \in \{-1, 1\}^n\} = \alpha \cdot r(A).$$

Proof. First observe that $D(y)\Delta D(z) = \alpha \cdot D(y) I \cdot I^T D(z) = \alpha \cdot yz^T$ for arbitrary $y, z \in \{-1, 1\}^n$. If $\lambda$ is a non-zero real eigenvalue of $\alpha \cdot Ayz^T$ and $x$ is a non-zero vector such that

$$\alpha \cdot Ayz^T x = \lambda x \neq 0,$$

then $z^T x \neq 0$ and

$$\alpha \cdot z^T Ayz^T x = \lambda \cdot z^T x,$$

$$\alpha \cdot z^T Ay = \lambda.$$

Thus $\rho_0(AD(y)\Delta D(z)) = \alpha \cdot |z^T Ay|$. Hence

$$\max \{\rho_0(AD(y)\Delta D(z)) : y, z \in \{-1, 1\}^n\}$$

$$= \alpha \cdot \max \{|z^T Ay| : y, z \in \{-1, 1\}^n\} = \alpha \cdot r(A).$$

Now everything is set for Poljak and Rohn’s reduction [5].

Theorem 4. Let $A \in \mathbb{R}^{n \times n}$ be a non-singular matrix, let $K$ be a positive real number and let $\Delta = (1/K) \cdot J$. Then $r(A) \geq K$ if and only if the matrix interval $[A^{-1} - \Delta, A^{-1} + \Delta]$ is singular.

Proof. By Lemma 2 the matrix interval $[A^{-1} - \Delta, A^{-1} + \Delta]$ is singular if and only if $\rho_0(AD(y)\Delta D(z)) \geq 1$ for some $y, z \in \{-1, 1\}^n$. By Lemma 3 $\rho_0(AD(y)\Delta D(z)) \geq 1$ for some $y, z \in \{-1, 1\}^n$ if and only if $r(A) \geq K$.

Corollary 5. RK1-MATRIX-INTERVAL SINGULARITY is NP-hard.

Remark. Poljak and Rohn [5] show that RK1-MATRIX-INTERVAL SINGULARITY belongs to the class NP by proving the existence of a singular matrix in every singular matrix interval, with a polynomial bound on the size of all entries of that matrix.

4 Reduction from RK1-MATRIX-INTERVAL SINGULARITY to P-MATRIX

The described reduction is by Coxson [1].
Let \( A, \Delta \in \mathbb{R}^{n \times n} \). Consider the matrix interval \([A, A + \Delta]\). Let \( \Delta^{i,j} \) be the matrix whose element in the \( i \)th row and \( j \)th column is \( \Delta_{i,j} \) and which has zeros elsewhere. Then each matrix \( M \) in the interval \([A, A + \Delta]\) can be uniquely expressed as

\[
M = A + \sum_{i,j=1}^{n} p_{i,j} \Delta^{i,j},
\]

where \( p_{i,j} \in [0, 1] \) for all values of \( i,j \).

Each matrix \( \Delta^{i,j} \) is a rank-1 matrix (even if \( \Delta \) has higher rank), and so \( \Delta^{i,j} = r_{i,j} s_{i,j}^T \) for some vectors \( r_{i,j}, s_{i,j} \in \mathbb{R}^n \). We can actually take \( r_{i,j} \) to be \( \Delta_{i,j} \) in its \( i \)th entry and zero elsewhere, and \( s_{i,j} \) to be \( 1 \) in its \( j \)th entry and zero elsewhere.

Now let \( R \) be the matrix whose columns are all the \( n^2 \) vectors \( r_{i,j} \) and let \( S \) be the matrix whose columns are all the \( n^2 \) vectors \( s_{i,j} \). Thus \( \Delta = RS^T \). Moreover, if \( p \in \mathbb{R}^{n^2} \) is the vector formed by the numbers \( p_{i,j} \), we can write (1) as

\[
M = A + RD(p)S^T.
\]

Suppose that \( A \) is non-singular. Then the matrix interval \([A, A + \Delta]\) is non-singular if and only if

\[
det(A + RD(p)S^T) = det(A) det(I_n + A^{-1}RD(p)S^T) \neq 0
\]

for each vector \( p \in [0, 1]^{n^2} \).

Supposing that the matrix \( A \) is non-singular, inequality (2) holds if and only if

\[
det(I_n + A^{-1}RD(p)S^T) \neq 0.
\]

In this way we have proved that for a non-singular matrix \( A \), singularity of the matrix interval \([A, A + \Delta]\) is equivalent to the existence of a vector \( p \in [0, 1]^{n^2} \) that does not satisfy inequality (3). Since the expression in (3) is a multi-affine function of \( p \), we can actually derive another condition.

**Lemma 6.** Let \( \psi(p) = det(I_n + A^{-1}RD(p)S^T) \). Then inequality (3) holds for each \( p \in [0, 1]^{n^2} \) if and only if \( \psi(p) > 0 \) for each \( p \in [0, 1]^{n^2} \).

**Proof.** First observe that \( \psi(p) = det(I_n + A^{-1}RD(p)S^T) \) is a multi-affine function of \( p \), that is, for each \( i \) we have \( \psi(p) = c_1 + c_2 p_i \), where \( c_1, c_2 \) depend on \( i \) and \( p_j \) for \( j \neq i \).

We claim that any multi-affine function \( \phi: [0, 1]^k \to \mathbb{R} \) is non-zero on the whole domain if and only if its values on the vertices \( \{0, 1\}^k \) have all the same sign. Assuming this claim holds, we notice that \( \psi(0) = det I_n = 1 > 0 \), so \( \psi \) is non-zero on \([0, 1]^{n^2}\) if and only if it is positive on \([0, 1]^k\).

To prove the claim, first suppose that \( \phi \) is non-zero on \([0, 1]^k\) but there are two vertices \( u, v \in \{0, 1\}^k \) such that \( \phi(u) < 0 \) and \( \phi(v) > 0 \). Following the path along the edges of \([0, 1]\), we will find two vertices \( u', v' \in \{0, 1\} \) that differ in exactly one coordinate and such that \( \phi(u') < 0 \) and \( \phi(v') > 0 \). Without loss of generality we may assume that \( u'_1 = 0 \) and \( v'_1 = 1 \), while \( u'_i = v'_i \) for \( i \geq 2 \). Let \( x \in [0, 1]^k \) be defined by \( x_i = \phi(u')/(\phi(u') - \phi(v')) \) and \( x_i = u'_i \) for \( i \geq 2 \). Then \( \phi(x) = 0 \), a contradiction.

Conversely, if \( \phi \) is positive (negative) on all the vertices, it is easy to prove by induction on face dimension that \( \phi \) is positive (negative) in every internal point of each face. \( \square \)
Lemma 6 together with the discussion that precedes it imply the following characterisation.

**Lemma 7.** Let $A$ be a non-singular matrix and let $R, S$ be defined as above. Then the matrix interval $[A, A + \Delta]$ is singular if and only if

$$\det(I_n + A^{-1}RD(p)S^T) \leq 0$$

for some $p \in \{0, 1\}^{n^2}$. \qed

In order to get $D(p)$ from the middle of the product to the beginning, we use the following lemma, whose proof we present in the Appendix.

**Lemma 8.** Let $F \in \mathbb{R}^{k \times n}$ and $G \in \mathbb{R}^{n \times k}$. Then $\det(I_k + FG) = \det(I_n + GF)$. \qed

This fact can be exploited to prove the following equivalence.

**Theorem 9.** Let $A$ be a non-singular matrix and let $R, S$ be defined as in Lemma 7. Then the matrix interval $[A, A + \Delta]$ is singular if and only if the matrix $M = I_{n^2} + S^T A^{-1} R$ is not a P-matrix. 

**Proof.** Because of Lemma 8

$$\psi(p) = \det(I_{n^2} + A^{-1}RD(p)S^T) = \det(I_{n^2} + D(p)S^T A^{-1} R).$$

If $p \in \{0, 1\}^{n^2}$ and $p \neq 0$, the expression $\det(I_{n^2} + D(p)S^T A^{-1} R)$ is equal to the principal minor of the matrix $M$ obtained by selecting exactly those rows and columns that correspond to the 1-entries of the vector $p$. Thus $\psi(p)$ is non-positive for some $p \in \{0, 1\}^{n^2}$ if and only if the matrix $M$ is not a P-matrix.

The proof is now completed by applying Lemma 7. \qed

**Corollary 10.** The problem $P$-MATRIX is co-NP-complete.

**Proof.** NP-hardness follows from Corollary 5 and Theorem 9. The problem belongs to co-NP because after guessing the rows and columns, the corresponding principal minor, which certifies the negative answer, can be computed in polynomial time. \qed

**Appendix: Proof of Lemma 8**

One of the basic facts about determinants is that adding a multiple of a row to another row does not change the determinant. The following lemma (Theorem 3 in Section 2.5 of Gantmacher’s book [3]) is a block version of this fact. Even though it holds for matrices with an arbitrary number of blocks, we state it just for $2 \times 2$ blocks. This variant is sufficient for the proof of Lemma 8.
Lemma 11. Let $A \in \mathbb{R}^{m \times n}$ be a matrix with block structure

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$$

and let $X \in \mathbb{R}^{m_1 \times m_2}$, $Y \in \mathbb{R}^{n_1 \times n_2}$. Then

$$\det A = \det \begin{pmatrix} A_{1,1} + XA_{2,1} & A_{1,2} + XA_{2,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} = \det \begin{pmatrix} A_{1,1} & A_{1,2} + A_{1,1}Y \\ A_{2,1} & A_{2,2} + A_{2,1}Y \end{pmatrix}.$$ 

Proof. Since

$$\begin{pmatrix} A_{1,1} + XA_{2,1} & A_{1,2} + XA_{2,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} = \begin{pmatrix} I_{m_1} & X \\ 0 & I_{m_2} \end{pmatrix} A,$$

we have

$$\det \begin{pmatrix} A_{1,1} + XA_{2,1} & A_{1,2} + XA_{2,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} = \det \begin{pmatrix} I_{m_1} & X \\ 0 & I_{m_2} \end{pmatrix} \cdot \det A = \det A.$$

Similarly

$$\det \begin{pmatrix} A_{1,1} & A_{1,2} + A_{1,1}Y \\ A_{2,1} & A_{2,2} + A_{2,1}Y \end{pmatrix} = \det A \cdot \det \begin{pmatrix} I_{n_1} & Y \\ 0 & I_{n_2} \end{pmatrix} = \det A.$$

Finally comes the proof of Lemma 8.

Proof of Lemma 8. Applying Lemma 11 twice, we get

$$\det(I_k + FG) = \det \begin{pmatrix} I_k + FG & 0 \\ G & I_n \end{pmatrix} \overset{(*)}{=} \det \begin{pmatrix} I_k & -F \\ G & I_n \end{pmatrix} \overset{(*)}{=} \det \begin{pmatrix} I_k & 0 \\ G & I_n + GF \end{pmatrix} = \det(I_n + GF).$$

Here ($*$) follows by applying Lemma 11 to rows with $X = F$ and ($*$) follows by applying it to columns with $Y = F$.

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