Abstract. Discontinuous Galerkin (DG) methods are considered for solving a plate contact problem, which is a 4th-order elliptic variational inequality of second kind. Numerous $C^0$ DG schemes for the Kirchhoff plate bending problem are extended to the variational inequality. Properties of the DG methods, such as consistency and stability, are studied, and optimal order error estimates are derived. A numerical example is presented to show the performance of the DG methods; the numerical convergence orders confirm the theoretical prediction.

Keywords. Variational inequality of 4th-order, discontinuous Galerkin method, plate frictional contact problem, error estimation

AMS Classification. 65N30, 49J40

1 Introduction

In this paper, we introduce and analyze some $C^0$ discontinuous Galerkin (DG) methods for a model 4th-order elliptic variational inequality of second kind. The model variational inequality arises in the study of a frictional contact problem for Kirchhoff plates.

1.1 Discontinuous Galerkin methods

Discontinuous Galerkin methods are an important family of nonconforming finite element methods for solving partial differential equations. We refer to [11] for a historical account about DG methods. Discontinuous Galerkin methods use piecewise smooth yet globally less smooth functions to approximate problem solutions, and relate the information between two
neighboring elements by numerical traces. The practical interest in DG methods is due to their flexibility in mesh design and adaptivity, in that they allow elements of arbitrary shapes, irregular meshes with hanging nodes, and the discretionary local shape function spaces. In addition, the increase of the locality in discretization enhances the degree of parallelizability.

There are basically two approaches to construct DG methods for linear elliptic boundary value problems. The first approach is through the choice of an appropriate bilinear form that contains penalty terms to penalize jumps across neighboring elements to make the scheme stable. The second approach is based on choosing appropriate numerical fluxes to make the method consistent, conservative and stable. In [1] and [2], Arnold, Brezzi, Cockburn, and Marini provided a unified error analysis of DG methods for linear elliptic boundary value problems of 2nd-order and succeeded in building a bridge between these two families, establishing a framework to understand their properties, differences and the connections between them. In [23], numerous DG methods were extended for solving elliptic variational inequalities of 2nd-order, and a priori error estimates were established, which are of optimal order for linear elements. In [24], five discontinuous Galerkin schemes with linear elements for solving the Signorini problem were studied, and optimal convergence order was proved. The ideas presented in [24] were extended to solve a quasistatic contact problem in [25].

In this paper, we study DG methods to solve an elliptic variational inequality of 4th-order for the Kirchhoff plates. It is difficult to construct stable DG methods for such problems because of the higher order in differentiation and of the inequality form. The major known DG methods for the biharmonic equation in the literature are primal DG methods, namely variations of interior penalty (IP) methods ([4, 5, 7, 13, 18, 19, 20, 22]). Fully discontinuous IP methods, which cover meshes with hanging nodes and locally varying polynomial degrees, thus ideally suited for hp-adaptivity, were investigated systematically in [18, 19, 20, 22] for biharmonic problems. In [13], a $C^0$ IP formulation was introduced for Kirchhoff plates and quasi-optimal error estimates were obtained for smooth solutions. Unlike fully discontinuous Galerkin methods, $C^0$ type DG methods do not “double” the degrees of freedom at element boundaries. A rigorous error analysis was presented in [7] for the $C^0$ IP method under weak regularity assumption on the solution. A weakness of this method is that the penalty parameter can not be precisely quantified a priori, and it must be chosen suitably large to guarantee stability. However, a large penalty parameter has a negative impact on accuracy. Based on this observation, a $C^0$ DG (CDG) method was introduced in [27], where the stability condition can be precisely quantified. In [17], a consistent and stable CDG method, called the LCDG method, was derived for the Kirchhoff plate bending problem. The LCDG method can be viewed as an extension of the LDG method studied in [9, 10]. We will extend these three methods and additionally propose two more CDG methods to solve the 4th-order elliptic variational inequality of second kind. For 4th-order elliptic variational inequalities of first kind, some DG methods were developed in [26]; however, no error estimates were
derived. In [8], a quadratic $C^0$ IP method for Kirchhoff plates problem with the displacement obstacle was studied, and errors in the energy norm and the $L^\infty$ norm are given by $O(h^\alpha)$, where $0.5 < \alpha \leq 1$.

### 1.2 Kirchhoff plate bending problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary $\Gamma$. The boundary value problem of a clamped Kirchhoff plate under a given scaled vertical load $f \in L^2(\Omega)$ is (cf. [21])

\[
\begin{cases}
\sigma = - (1 - \kappa) \nabla^2 u - \kappa \text{tr}(\nabla^2 u) I & \text{in } \Omega, \\
- \nabla \cdot (\nabla \cdot \sigma) = f & \text{in } \Omega, \\
u = \partial_\nu u = 0 & \text{on } \Gamma,
\end{cases}
\]

(1.1)

where $\kappa \in (0, 0.5)$ denotes the Poisson ratio of an elastic thin plate occupying the region $\Omega$ and $\nu$ stands for the unit outward normal vector on $\Gamma$. $I$ is the identity matrix of order 2 and tr is the trace operation on matrices. Here, $\nabla$ is the usual nabla operator, and we denote the Hessian of $v$ by $\nabla^2 v$, i.e.,

\[
\nabla^2 v := \nabla(\nabla v) = \nabla((\partial_1 v, \partial_2 v)^t) = \begin{pmatrix} \partial_{11} v & \partial_{12} v \\ \partial_{21} v & \partial_{22} v \end{pmatrix}.
\]

Note that the first equation in (1.1) can be rewritten as

\[
\frac{1}{1 - \kappa} \sigma - \frac{\kappa}{1 - \kappa^2} \text{tr}(\sigma) I = - \nabla^2 u.
\]

(1.2)

For a vector-valued function $v = (v_1, v_2)^t$ and a matrix-valued function $\sigma = (\sigma_{ij})_{2 \times 2}$, we define their divergence by

\[
\nabla \cdot v := v_{1,1} + v_{2,2}, \quad \nabla \cdot \sigma := (\sigma_{11,1} + \sigma_{21,2}, \sigma_{12,1} + \sigma_{22,2})^t.
\]

We denote the normal and tangential components of a vector $v$ on the boundary by $v_\nu = v \cdot \nu$ and $v_\tau = v - v_\nu \nu$. Similarly, for a tensor $\sigma$, we define its normal component $\sigma_\nu = \sigma v \cdot \nu$ and tangential component $\sigma_\tau = \sigma - \sigma_\nu \nu$. We have the decomposition formula

\[
(\sigma v) \cdot v = (\sigma_\nu v + \sigma_\tau) \cdot (v_\nu \nu + v_\tau) = \sigma_\nu v_\nu + \sigma_\tau \cdot v_\tau.
\]

For two matrices $\tau$ and $\sigma$, their double dot inner product and corresponding norm are $\sigma : \tau = \sum_{i,j=1}^2 \sigma_{ij} \tau_{ij}$ and $|\sigma| = (\tau : \tau)^{1/2}$.

The following result is very useful for the analysis of DG methods, which can be verified directly through integration by parts.
Lemma 1.1 Let $D$ be a bounded domain with a Lipschitz boundary $\partial D$. For a symmetric matrix-valued function $\tau$ and a scalar function $v$, the following two identities hold

$$
\int_D v \nabla \cdot (\nabla \cdot \tau) \, dx = \int_D \nabla^2 v : \tau \, dx - \int_{\partial D} \nabla v \cdot (\tau n) \, ds + \int_{\partial D} v n \cdot (\nabla \cdot \tau) \, ds,
$$

$$
\int_D \nabla^2 v : \tau \, dx = -\int_D \nabla v \cdot (\nabla \cdot \tau) \, dx + \int_{\partial D} \nabla v \cdot (\tau n) \, ds,
$$

whenever the terms appearing on both sides of the above identities make sense. Here $n$ is the unit outward normal to $\partial D$.

Multiplying the second equation in (1.1) by a test function $v \in H^2_0(\Omega)$ and noticing $v = \partial_\nu v = 0$, we get the following equation by Lemma 1.1

$$
-\int_\Omega \sigma : \nabla^2 v \, dx = \int_\Omega fv \, dx. 
$$

(1.3)

By the definition of $\sigma$ and (1.3), the weak formulation of problem (1.1) can be written as

Find $u \in H^2_0(\Omega) : a(u, v) = (f, v) \, \forall \, v \in H^2_0(\Omega)$

(1.4)

where the bilinear form is

$$
a(u, v) = \int_\Omega [\Delta u \Delta v + (1 - \kappa) (2 \partial_{12} u \partial_{12} v - \partial_{11} u \partial_{22} v - \partial_{22} u \partial_{11} v)] \, dx,
$$

(1.5)

and the linear form is

$$
(f, v) = \int_\Omega f v \, dx.
$$

In this paper, we consider a plate frictional contact problem, which is a 4th-order elliptic variational inequality (EVI) of second kind ([12]). The Lipschitz continuous boundary $\Gamma$ of the domain $\Omega$ is decomposed into three parts: $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ with $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ relatively open and mutually disjoint such that $\text{meas}(\Gamma_1) > 0$. Then the plate frictional contact problem we consider is:

Find $u \in V : a(u, v - u) + j(v) - j(u) \geq (f, v - u) \, \forall \, v \in V$

(1.6)

Here,

$$
V = \{ v \in H^2(\Omega) : v = \partial_\nu v = 0 \text{ on } \Gamma_1 \},
$$

$$
j(v) = \int_{\Gamma_3} g |v| \, ds.
$$
This variational inequality describes a simply supported plate. The plate is clamped on the boundary $\Gamma_1$:
\begin{equation}
v = \partial_n v = 0 \text{ on } \Gamma_1,
\end{equation}
is free on $\Gamma_2$, and is in frictional contact on $\Gamma_3$ with a rigid foundation; $g$ can be interpreted as a frictional bound. Applying the standard theory on elliptic variational inequalities (e.g., [3, 14]), we know the problem (1.6) has a unique solution $u \in V$.

Let
\begin{equation}
\Lambda = \{ \lambda \in L^\infty(\Gamma_3) : |\lambda| \leq 1 \text{ a.e. on } \Gamma_3 \}.
\end{equation}
We have the following result ([16]).

**Theorem 1.2** A function $u \in V$ is a solution of (1.6) if and only if there is a $\lambda \in \Lambda$ such that
\begin{equation}
a(u, v) + \int_{\Gamma_3} g \lambda v ds = (f, v) \quad \forall v \in V,
\end{equation}
\begin{equation}
\lambda u = |u| \quad \text{a.e. on } \Gamma_3.
\end{equation}

Throughout the paper, we assume the solution of the problem (1.6) has the regularity $u \in H^3(\Omega)$. The regularity result $u \in H^3(\Omega)$ is shown for solutions of some variational inequalities of 4th-order ([15, pp. 323–327]). In error analysis of numerical solutions for the problem (1.6), we need to take advantage of pointwise relations satisfied by the solution $u$.

Note that $\sigma$ is defined by the first equation of (1.1). Then $\sigma \in H^1(\Omega)^{2x2}$. We rewrite (1.6) as
\begin{equation}
\int_{\Omega} [-\sigma : \nabla^2(v-u) - f(v-u)] dx + \int_{\Gamma_3} g|v| ds - \int_{\Gamma_3} g|u| ds \geq 0 \quad \forall v \in V.
\end{equation}
Take $v = u \pm \varphi$ for any $\varphi \in C_0^\infty(\Omega)$ to obtain
\begin{equation}
-\int_{\Omega} \sigma : \nabla^2 \varphi dx = \int_{\Omega} f \varphi dx \quad \forall \varphi \in C_0^\infty(\Omega).
\end{equation}

Thus,
\begin{equation}
-\nabla \cdot (\nabla \cdot \sigma) = f \quad \text{in the sense of distribution.}
\end{equation}
Since $f \in L^2(\Omega)$, we deduce that $\nabla \cdot (\nabla \cdot \sigma) \in L^2(\Omega)$ and
\begin{equation}
-\nabla \cdot (\nabla \cdot \sigma) = f \quad \text{a.e. in } \Omega.
\end{equation}
Since $\nabla \cdot \sigma \in L^2(\Omega)^2$ and $\nabla \cdot (\nabla \cdot \sigma) \in L^2(\Omega)$, we can define $(\nabla \cdot \sigma) \cdot \nu \in H^{-1/2}(\Gamma)$ and it satisfies the relation

$$
\langle (\nabla \cdot \sigma) \cdot \nu, v \rangle_{1/2,\Gamma} = \int_{\Omega} \nabla \cdot (\nabla \cdot \sigma) v \, dx + \int_{\Gamma} (\nabla \cdot \sigma) \cdot \nabla v \, dx \quad \forall v \in H^1(\Omega).
$$

(1.11)

Therefore, for any $v \in H^2(\Omega)$,

$$
- \int_{\Omega} \nabla \cdot (\nabla \cdot \sigma) v \, dx = \int_{\Omega} (\nabla \cdot \sigma) \cdot \nabla v \, dx - \langle (\nabla \cdot \sigma) \cdot \nu, v \rangle_{1/2,\Gamma}
$$

$$
= - \int_{\Omega} \sigma : \nabla^2 v \, dx + \int_{\Gamma} (\sigma \nu) \cdot \nabla v \, ds - \langle (\nabla \cdot \sigma) \cdot \nu, v \rangle_{1/2,\Gamma},
$$

i.e.,

$$
a(u, v) = \int_{\Omega} f v \, dx - \int_{\Gamma} (\sigma \nu) \cdot \nabla v \, ds + \langle (\nabla \cdot \sigma) \cdot \nu, v \rangle_{1/2,\Gamma} \quad \forall v \in H^2(\Omega).
$$

Recalling the equation (1.8), we then have for any $v \in V$,

$$
- \int_{\Gamma} (\sigma \nu) \cdot \nabla v \, ds + \langle (\nabla \cdot \sigma) \cdot \nu, v \rangle_{1/2,\Gamma} + \int_{\Gamma_3} g \lambda v \, ds = 0.
$$

(1.12)

Let $\sigma_\nu$ and $\sigma_\tau$ be the normal and tangential components of the vector $\sigma \nu$ on $\Gamma$. In (1.12), taking $v \in V$ such that $v = 0$ on $\Gamma$ and $\partial_\nu v$ arbitrary on $\Gamma_2 \cup \Gamma_3$, we have

$$
\sigma_\nu = 0 \quad \text{a.e. on } \Gamma_2 \cup \Gamma_3
$$

(1.13)

Then from (1.12) we get

$$
- \int_{\Gamma_2 \cup \Gamma_3} \sigma_\nu \partial_\tau v \, ds + \langle (\nabla \cdot \sigma) \cdot \nu, v \rangle_{1/2,\Gamma} + \int_{\Gamma_3} g \lambda v \, ds = 0 \quad \forall v \in V.
$$

(1.14)

Note that the closure of $V$ in $H^1(\Omega)$ is

$$
H^1_{\Gamma_1}(\Omega) = \{ v \in H^1(\Omega) : v = 0 \text{ a.e. on } \Gamma_1 \}.
$$

Denote

$$
\tilde{H}^1_{\Gamma_1}(\Omega) = \{ v \in H^1_{\Gamma_1}(\Omega) : \partial_\tau v \in L^2(\Gamma) \}.
$$

Then from (1.14), we conclude that

$$
- \int_{\Gamma_2 \cup \Gamma_3} \sigma_\tau \partial_\tau v \, ds + \langle (\nabla \cdot \sigma) \cdot \nu, v \rangle_{1/2,\Gamma} + \int_{\Gamma_3} g \lambda v \, ds = 0 \quad \forall v \in \tilde{H}^1_{\Gamma_1}(\Omega).
$$

(1.15)

The rest of the paper is organized as follows. In Section 2, we present some notations, introduce some $C^0$ discontinuous Galerkin methods for solving the Kirchhoff plate bending problem and extend them to solve the elliptic variational inequality of 4th-order. In Section 3, consistency of the CDG methods, boundedness and stability of the bilinear forms are presented. A priori error analysis for these CDG methods is established in Section 4. In the final section, we report simulation results from a numerical example.
2 DG methods for Kirchhoff plate problem

2.1 Notations

Here we introduce some notations to be used later. For a given function space $B$, let $(B)^{2\times 2} := \{\tau \in (B)^{2\times 2} : \tau^t = \tau\}$. Given a bounded set $D \subset \mathbb{R}^2$ and a positive integer $m$, $H^m(D)$ is the usual Sobolev space with the corresponding norm $\| \cdot \|_{m,D}$ and semi-norm $| \cdot |_{m,D}$, which are abbreviated by $\| \cdot \|_m$ and $| \cdot |_m$, respectively, when $D$ is chosen as $\Omega$. $\| \cdot \|_D$ is the norm of the Lebesgue space $L^2(D)$. We assume $\Omega$ is a polygonal domain and denote by $\{T_h\}_h$ a family of triangulations of $\Omega$, with the minimal angle condition satisfied. Denote $h_K = \text{diam}(K)$ and $h = \max \{h_K : K \in T_h\}$. For a triangulation $T_h$, let $E_h$ be the set of all the element edges, $E^b_h$ the set of all the element edges that lie on the boundary $\Gamma$, $E^i_h := E_h \setminus E^b_h$ the set of all interior edges, and $E^0_h \subset E_h$ the set of all the edges that do not lie on $\Gamma_2$ or $\Gamma_3$. For any $e \in E_h$, denote by $h_e$ its length. Related to the triangulation $T_h$, let

$$\Sigma := \left\{ \tau \in (L^2(\Omega))^{2\times 2}_s : \tau_{ij}|_K \in H^1(K) \ \forall \ K \in T_h, \ i, j = 1, 2 \right\},$$

$$V := \left\{ v \in H^1_{\Gamma_1}(\Omega) : v|_K \in H^2(K) \ \forall \ K \in T_h \right\}.$$

The corresponding finite element spaces are

$$\Sigma_h := \left\{ \tau_h \in (L^2(\Omega))^{2\times 2}_s : \tau_{ij}|_K \in P_l(K) \ \forall \ K \in T_h, \ i, j = 1, 2 \right\},$$

$$V_h := \left\{ v_h \in H^1_{\Gamma_1}(\Omega) : v_h|_K \in P_2(K) \ \forall \ K \in T_h \right\}.$$

Here, for a triangle $K \in T_h$, $P_l(K)$ ($l = 0, 1$) and $P_2(K)$ are the polynomial spaces on $K$ of degrees $l$ and 2, respectively. Note that we have the following property

$$\nabla^2_h V_h \subset \Sigma_h, \quad \frac{1}{1 - \kappa} \Sigma_h - \frac{\kappa}{1 - \kappa^2} (\text{tr} \Sigma_h) I \subset \Sigma_h,$$

where $\nabla^2_h V_h|_K := \nabla^2(V_h|_K)$ for any $K \in T_h$. $\nabla^2_h v$ is defined by the relation $\nabla^2_h v = \nabla^2 v$ on any element $K \in T_h$.

For a function $v \in L^2(\Omega)$ with $v|_K \in H^m(K)$ for all $K \in T_h$, define the broken norm and seminorm by

$$\|v\|_{m,h} = \left( \sum_{K \in T_h} \|v|_K^2 \right)^{1/2}, \quad |v|_{m,h} = \left( \sum_{K \in T_h} |v|_K^2 \right)^{1/2}.$$

The above symbols are used in a similar manner when $v$ is a vector or matrix-valued function. Throughout this paper, $C$ denotes a generic positive constant independent of $h$ and other parameters, which may take different values at different occurrences. To avoid writing these
constants repeatedly, we use “\(x \lesssim y\)” to mean that “\(x \leq Cy\)”.

For two vectors \(u\) and \(v\), \(u \otimes v\) is a matrix with \(u_i v_j\) as its \((i, j)\)-th component.

Consider two elements \(K^+\) and \(K^-\) with a common edge \(e \in E_i\) and let \(n^+\) and \(n^-\) be their outward unit normals on \(e\). For a scalar-valued function \(v\), denote its restriction on \(K^\pm\) by \(v|_{K^\pm}\). Similarly, for a matrix-valued function \(\tau\), write \(\tau^\pm = \tau|_{K^\pm}\). Then we define averages and jumps on \(e \in E_i\) as follows:

\[
\{v\} = \frac{1}{2}(v^+ + v^-), \quad [v] = v^+ n^+ + v^- n^-,
\]
\[
\{\nabla v\} = \frac{1}{2}(\nabla v^+ + \nabla v^-), \quad [\nabla v] = \nabla v^+ \cdot n^+ + \nabla v^- \cdot n^-,
\]
\[
\{\tau\} = \frac{1}{2}(\tau^+ + \tau^-), \quad [\tau] = \tau^+ n^+ + \tau^- n^-.
\]

For \(e \in E_b\), the above definitions need to be modified:

\[
\{v\} = v, \quad [v] = v \nu,
\]
\[
\{\nabla v\} = \nabla v, \quad [\nabla v] = \nabla v \cdot \nu,
\]
\[
\{\tau\} = \tau, \quad [\tau] = \tau \nu.
\]

The jump \([\cdot]\) of the vector \(\nabla v\) is

\[
[\nabla v] = \frac{1}{2}(\nabla v^+ \otimes n^+ + n^+ \otimes \nabla v^+ + \nabla v^- \otimes n^- + n^- \otimes \nabla v^-) \quad \text{on } e \in E_i,
\]
\[
[\nabla v] = \frac{1}{2}(\nabla v \otimes \nu + \nu \otimes \nabla v) \quad \text{on } e \in E_b.
\]

Define a global lifting operator \(r_0 : (L^2(E_0^i))_{s}^{2 \times 2} \rightarrow \Sigma_h\) by

\[
\int_{\Omega} r_0(\phi) : \tau dx = -\int_{E_0^i} \phi : \{\tau\} ds \quad \forall \tau \in \Sigma_h, \phi \in (L^2(E_0^i))_{s}^{2 \times 2}. \quad (2.2)
\]

Moreover, for each \(e \in E_h\), introduce a local lifting operator \(r_e : (L^2(e))_{s}^{2 \times 2} \rightarrow \Sigma_h\) by

\[
\int_{\Omega} r_e(\phi) : \tau dx = -\int_{e} \phi : \{\tau\} ds \quad \forall \tau \in \Sigma_h, \phi \in (L^2(e))_{s}^{2 \times 2}. \quad (2.3)
\]

It is easy to check that the following identity holds

\[
r_0(\phi) = \sum_{e \in E_0^i} r_e(\phi|_e) \quad \forall \phi \in (L^2(E_0^i))_{s}^{2 \times 2},
\]

so we have

\[
\|r_0(\phi)\|^2 = \|\sum_{e \in E_0^i} r_e(\phi|_e)\|^2 \leq 3 \sum_{e \in E_0^i} \|r_e(\phi|_e)\|^2. \quad (2.4)
\]
2.2 Discontinuous Galerkin formulations

In [26], a general primal formulation of CDG methods was presented for a 4th-order elliptic variational inequality of first kind. The process of deriving CDG schemes for 4th-order elliptic equations can also be found in [17]. Based on the discussions in [26] and [17], we introduce five CDG methods for the problem (1.6) as follows: Find $u_h \in V_h$ such that

$$B_h(u_h, v_h - u_h) + j(v_h) - j(u_h) \geq (f, v_h - u_h) \quad \forall v_h \in V_h,$$  \hspace{1cm} (2.5)

where the bilinear form $B_h(w, v) = B_{1,h}^{(j)}(w, v)$ with $j = 1, \cdots, 5$, and $B_{1,h}^{(j)}(w, v)$ are given next.

The method with $j = 1$ is a $C^0$ interior penalty (IP) method, and the bilinear form is

$$B_{1,h}^{(1)}(u_h, v_h) = \int_\Omega (1 - \kappa) \nabla_h^2 u_h : \nabla^2 v_h \, dx + \int_\Omega \kappa \text{tr} (\nabla^2 u_h) \text{tr} (\nabla^2 v_h) \, dx
- \int_{\mathcal{E}_h^0} \left[ \nabla u_h \right] : \left( (1 - \kappa) \left\{ \nabla^2 v_h \right\} + \kappa \text{tr} \left( \left\{ \nabla^2 v_h \right\} I \right) \right) \, ds
- \int_{\mathcal{E}_h^0} \left[ \nabla v_h \right] : \left( (1 - \kappa) \left\{ \nabla^2 u_h \right\} + \kappa \text{tr} \left( \left\{ \nabla^2 u_h \right\} I \right) \right) \, ds
+ \int_{\mathcal{E}_h^0} \eta h^{-1}_e \left[ \nabla u_h \right] : \left[ \nabla v_h \right] \, ds.$$  \hspace{1cm} (2.6)

Here $\eta$ is a function, defined to be a constant $\eta_e$ on each $e \in \mathcal{E}_h^0$, with $\left\{ \eta_e \right\}_{e \in \mathcal{E}_h^0}$ having a uniform positive bound from above and below. For a compact formulation, we can use lifting operator $r_0$ (cf. (2.2)) to get

$$B_{2,h}^{(1)}(u_h, v_h) = \int_\Omega (1 - \kappa) \nabla_h^2 u_h : \left( \nabla_h^2 v_h + r_0(\left[ \nabla v_h \right]) \right) \, dx
+ \int_\Omega \kappa \text{tr} (\nabla^2 u_h) \text{tr} (\nabla^2 v_h + r_0(\left[ \nabla v_h \right])) \, dx
+ \int_\Omega r_0(\left[ \nabla u_h \right]) : \left( (1 - \kappa) \nabla^2 v_h + \kappa \text{tr} (\nabla^2 v_h) I \right) \, dx
+ \int_{\mathcal{E}_h^0} \eta h^{-1}_e \left[ \nabla u_h \right] : \left[ \nabla v_h \right] \, ds.$$  \hspace{1cm} (2.7)

A similar $C^0$ IP method was studied in [7].

The two formulas (2.6) and (2.7) are equivalent on the finite element spaces $V_h$, so either form can be used to compute the finite element solution $u_h$. In this paper, we give a priori error estimates strictly based on the first formula $B_{1,h}^{(1)}$. Because of the equivalence of these two formulations on $V_h$, we will prove the stability for the second formula $B_{2,h}^{(1)}$ on $V_h$, which
ensures the stability for the first formulation $B_{1,h}^{(1)}$ on $V_h$. This comment is valid for the other CDG methods introduced next.

Motivated by related DG methods for the second order elliptic problem, we can define the $C^0$ non-symmetric interior penalty (NIPG) formulation,

$$B_{1,h}^{(2)}(u_h, v_h) = \int_{\Omega} (1 - \kappa) \nabla_h^2 u_h : \nabla_h^2 v_h \, dx + \int_{\Omega} \kappa \text{tr} (\nabla_h^2 u_h) \text{tr} (\nabla_h^2 v_h) \, dx$$

$$+ \int_{e_h^0} [\nabla u_h] : ( (1 - \kappa) \{\nabla_h^2 v_h \} + \kappa \text{tr} (\{\nabla_h^2 v_h \}) I ) \, ds$$

$$- \int_{e_h^0} [\nabla v_h] : ( (1 - \kappa) \{\nabla_h^2 u_h \} + \kappa \text{tr} (\{\nabla_h^2 u_h \}) I ) \, ds$$

$$+ \int_{e_h^0} \eta h_{c,h}^{-1} [\nabla u_h] : [\nabla v_h] \, ds,$$

(2.8)

or equivalently,

$$B_{2,h}^{(2)}(u_h, v_h) = \int_{\Omega} (1 - \kappa) \nabla_h^2 u_h : \nabla_h^2 v_h + r_0([\nabla v_h]) \, dx$$

$$+ \int_{\Omega} \kappa \text{tr} (\nabla_h^2 u_h) \text{tr} (\nabla_h^2 v_h + r_0([\nabla v_h])) \, dx$$

$$- \int_{\Omega} r_0 ([\nabla u_h]) : ( (1 - \kappa) \nabla_h^2 v_h + \kappa \text{tr} (\nabla_h^2 v_h) I ) \, dx$$

$$+ \int_{e_h^0} \eta h_{c,h}^{-1} [\nabla u_h] : [\nabla v_h] \, ds.$$

(2.9)

The CDG method with $j = 3$ has the bilinear form

$$B_{1,h}^{(3)}(u_h, v_h) = \int_{\Omega} (1 - \kappa) \nabla_h^2 u_h : \nabla_h^2 v_h \, dx + \int_{\Omega} \kappa \text{tr} (\nabla_h^2 u_h) \text{tr} (\nabla_h^2 v_h) \, dx$$

$$- \int_{e_h^0} [\nabla u_h] : ( (1 - \kappa) \{\nabla_h^2 v_h \} + \kappa \text{tr} (\{\nabla_h^2 v_h \}) I ) \, ds$$

$$- \int_{e_h^0} [\nabla v_h] : ( (1 - \kappa) \{\nabla_h^2 u_h \} + \kappa \text{tr} (\{\nabla_h^2 u_h \}) I ) \, ds$$

$$+ \int_{\Omega} r_0 ([\nabla v_h]) : ( (1 - \kappa) r_0([\nabla u_h]) + \kappa \text{tr}(r_0([\nabla u_h])) I ) \, dx$$

$$+ \sum_{e \in e_h^0} \int_{\Omega} \eta ((1 - \kappa) r_e([\nabla u_h]) : r_e([\nabla v_h]) + \kappa \text{tr}(r_e([\nabla u_h])) tr(r_e([\nabla v_h]))) \, dx,$$
or equivalently,

\[
B_{2,h}^{(3)}(u_h, v_h) = \int_{\Omega} (1 - \kappa) \left( \nabla_h^2 u_h + \mathbf{r}_0([\nabla u_h]) \right) : \left( \nabla_h^2 v_h + \mathbf{r}_0([\nabla v_h]) \right) \, dx \\
+ \int_{\Omega} \kappa \text{tr} \left( \nabla_h^2 u_h + \mathbf{r}_0([\nabla u_h]) \right) \text{tr} \left( \nabla_h^2 v_h + \mathbf{r}_0([\nabla v_h]) \right) \, dx \\
+ \sum_{e \in E_h^0} \int_{e} \eta \left( (1 - \kappa) \mathbf{r}_e([\nabla u_h]) : \mathbf{r}_e([\nabla v_h]) + \kappa \text{tr}(\mathbf{r}_e([\nabla u_h])) \text{tr}(\mathbf{r}_e([\nabla v_h])) \right) \, ds,
\]

(2.10)

which is the CDG formulation proposed in [27].

The bilinear form of the CDG scheme with \( j = 4 \) is

\[
B_{1,h}^{(4)}(u_h, v_h) = \int_{\Omega} (1 - \kappa) \nabla_h^2 u_h : \nabla_h^2 v_h \, dx + \int_{\Omega} \kappa \text{tr} \left( \nabla_h^2 u_h \right) \text{tr} \left( \nabla_h^2 v_h \right) \, dx \\
- \int_{e_h^0} \nabla u_h : ((1 - \kappa)\{\nabla_h^2 v_h\} + \kappa \text{tr} \{\nabla_h^2 v_h\} \mathbf{I}) \, ds \\
- \int_{e_h^0} \nabla v_h : ((1 - \kappa)\{\nabla_h^2 u_h\} + \kappa \text{tr} \{\nabla_h^2 u_h\} \mathbf{I}) \, ds \\
+ \sum_{e \in E_h^0} \int_{e} \eta \left( (1 - \kappa) \mathbf{r}_e([\nabla u_h]) : \mathbf{r}_e([\nabla v_h]) + \kappa \text{tr}(\mathbf{r}_e([\nabla u_h])) \text{tr}(\mathbf{r}_e([\nabla v_h])) \right) \, ds,
\]

(2.11)

or equivalently,

\[
B_{2,h}^{(4)}(u_h, v_h) = \int_{\Omega} (1 - \kappa) \nabla_h^2 u_h : \left( \nabla_h^2 v_h + \mathbf{r}_0([\nabla v_h]) \right) \, dx \\
+ \int_{\Omega} \kappa \text{tr} \left( \nabla_h^2 u_h \right) \text{tr} \left( \nabla_h^2 v_h + \mathbf{r}_0([\nabla v_h]) \right) \, dx \\
+ \int_{\Omega} \mathbf{r}_0([\nabla v_h]) : ((1 - \kappa)\nabla_h^2 v_h + \kappa \text{tr} \nabla_h^2 v_h \mathbf{I}) \, dx \\
+ \sum_{e \in E_h^0} \int_{e} \eta \left( (1 - \kappa) \mathbf{r}_e([\nabla u_h]) : \mathbf{r}_e([\nabla v_h]) + \kappa \text{tr}(\mathbf{r}_e([\nabla u_h])) \text{tr}(\mathbf{r}_e([\nabla v_h])) \right) \, ds,
\]

(2.12)

which is the CDG formulation extended from the DG method of [6] for elliptic problems of second order.
For the LCDG method (17), the bilinear form is

\[ B_{1,h}^{(5)}(u_h, v_h) := \int_{\Omega} (1 - \kappa) \nabla_h^2 u_h : \nabla_h^2 v_h \, dx + \int_{\Omega} \kappa \text{tr} (\nabla_h^2 u_h) \text{tr} (\nabla_h^2 v_h) \, dx \]

\[ - \int_{e_h^i} [\nabla u_h] : ((1 - \kappa) [\nabla_h^2 v_h] + \kappa \text{tr} \{ [\nabla_h^2 v_h] \} I) \, ds \]

\[ - \int_{e_h^i} [\nabla v_h] : ((1 - \kappa) [\nabla_h^2 u_h] + \kappa \text{tr} \{ [\nabla_h^2 u_h] \} I) \, ds \]

\[ + \int_{\Omega} r_0([\nabla u_h]) : (1 - \kappa) r_0([\nabla u_h]) + \kappa \text{tr} (r_0([\nabla u_h])) I \, dx \]

\[ + \int_{e_h^i} \eta h^{-1} [\nabla u_h] : [\nabla v_h] \, ds, \]  \hspace{1cm} (2.13)

or equivalently,

\[ B_{2,h}^{(5)}(u_h, v_h) := \int_{\Omega} (1 - \kappa) (\nabla_h^2 u_h + r_0([\nabla u_h])) : (\nabla_h^2 v_h + r_0([\nabla v_h])) \, dx \]

\[ + \int_{\Omega} \kappa \text{tr} (\nabla_h^2 u_h + r_0([\nabla u_h])) \text{tr} (\nabla_h^2 v_h + r_0([\nabla v_h])) \, dx \]

\[ + \int_{e_h^i} \eta h^{-1} [\nabla u_h] : [\nabla v_h] \, ds. \]  \hspace{1cm} (2.14)

### 3 Consistency, boundedness and stability

First, we address the consistency of the methods (2.3).

**Lemma 3.1** For the solution of the problem (1.6), assume \( u \in H^3(\Omega) \). Then for all the five CDG methods with \( B_h(w, v) = B_{1,h}^{(5)}(w, v) \), \( 1 \leq j \leq 5 \), we have

\[ B_h(u, v_h - u) \geq (f, v_h - u) \quad \forall v_h \in V_h. \]

**Proof.** Noting \([\nabla u] = 0\) on each edge \( e \in E_h^i\), we use (1.2) to get

\[ B_h(u, v_h - u) = \int_{\Omega} (1 - \kappa) \nabla^2 u : \nabla^2 (v_h - u) \, dx + \int_{\Omega} \kappa \text{tr} (\nabla^2 u) \text{tr} (\nabla^2 (v_h - u)) \, dx \]

\[ - \int_{e_h^i} [\nabla (v_h - u)] : ((1 - \kappa) \nabla^2 u + \kappa \text{tr} (\nabla^2 u) I) \, ds \]

\[ = - \sum_{K \in T_h} \int_{K} \sigma : \nabla^2_h (v_h - u) \, dx + \int_{e_h^i} [\nabla (v_h - u)] : \sigma \, ds. \]
Using Lemma 1.1 and noticing $[\sigma] = 0$ on each edge $e \in \mathcal{E}_h^i$, we have

$$\sum_{K \in \mathcal{T}_h} \int_K \sigma : \nabla_h^2 (v_h - u) \, dx = - \sum_{K \in \mathcal{T}_h} \int_K \nabla (v_h - u) \cdot (\nabla \cdot \sigma) \, dx$$

$$+ \sum_{K \in \mathcal{T}_h} \int_{\partial K} \nabla (v_h - u) \cdot (\sigma n_K) \, ds$$

$$= - \int_\Omega \nabla (v_h - u) \cdot (\nabla \cdot \sigma) \, dx + \int_{\mathcal{E}_h} \left[ \nabla (v_h - u) \right] : \sigma \, ds.$$  

Combining the above two equations, we obtain

$$B_h(u, v_h - u) = \int_\Omega \nabla (v_h - u) \cdot (\nabla \cdot \sigma) \, dx - \int_{\Gamma_2 \cup \Gamma_3} \left[ \nabla (v_h - u) \right] : \sigma \, ds$$

$$= \int_\Omega \nabla (v_h - u) \cdot (\nabla \cdot \sigma) \, dx - \int_{\Gamma_2 \cup \Gamma_3} \sigma \, \partial_\tau (v_h - u) \, ds$$

$$= - \int_\Omega \nabla \cdot (\nabla \cdot \sigma)(v_h - u) \, dx + \langle (\nabla \cdot \sigma)(v_h - u), 1/2\Gamma \rangle$$

$$- \int_{\Gamma_2 \cup \Gamma_3} \sigma \, \partial_\tau (v_h - u) \, ds.$$  

Here, the second equation comes from the relation (1.13), and the last equation holds by (1.11).

We apply the relation (1.15), Lemma 1.1, (1.9) and (1.10) to obtain

$$B_h(u, v_h - u) = - \int_\Omega \nabla \cdot (\nabla \cdot \sigma)(v_h - u) \, dx - \int_{\Gamma_3} g \lambda v_h \, ds + \int_{\Gamma_3} g \lambda u \, ds$$

$$= \int_\Omega f (v_h - u) \, dx - \int_{\Gamma_3} g \lambda v_h \, ds + \int_{\Gamma_3} g |v_h| \, ds$$

$$\geq \int_\Omega f (v_h - u) \, dx - \int_{\Gamma_3} g |v_h| \, ds + \int_{\Gamma_3} g |u| \, ds.$$  

So the stated result holds.  

Let $V(h) := V_h + V \cap H^3(\Omega)$ and define two mesh-dependent energy norms by

$$|v|_*^2 := |v|_{2,h}^2 + \sum_{e \in \mathcal{E}_h^0} h_e^{-1} \| \nabla v \|_{0,e}^2, \quad ||v||^2 := |v|_*^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |v|_{3,K}^2, \quad v \in V(h).$$

To show these formulas define norms, we only need prove that $|v|_* = 0$ and $v \in V(h)$ imply $v = 0$. From $|v|_{2,h} = 0$, we have $v|_K \in P_1(K)$ and so $\nabla v$ is piecewise constant. Let $e$ be the common edge of two neighboring elements $K^+$ and $K^-$. From $\| \nabla v \|_{0,e} = 0$, we obtain
\((\nabla v)^+ = (\nabla v)^-\). Thus, \(\nabla v\) is constant in \(\Omega\) and so \(v \in P_1(\Omega)\). Since \(v = 0\) on \(\Gamma_1\), we conclude that \(v = 0\) in \(\Omega\).

Before presenting boundedness and stability results of the bilinear forms, we give a useful estimate for the lifting operator \(r_e\).

**Lemma 3.2** There exist two positive constants \(C_1 \leq C_2\) such that for any \(v \in V(h)\) and \(e \in \mathcal{E}_h^0\),

\[
C_1 h_e^{-1} \|\nabla v\|_{0,e}^2 \leq \|r_e(\nabla v)\|_{0,h}^2 \leq C_2 h_e^{-1} \|\nabla v\|_{0,e}^2.
\]

**Proof.** The second inequality was proved in [17]. For \(v \in V \cap H^3(\Omega)\), \(\|\nabla v\| = 0\) on \(e \in \mathcal{E}_h^0\). So we only need to consider the case \(v \in V_h\). By the formula between (4.4) and (4.5) in [2], we know

\[
h_e^{-1} \|\phi\|_{0,e}^2 \lesssim \|r_e^*(\phi)\|_{0,\Omega}^2 \lesssim h_e^{-1} \|\phi\|_{0,e}^2 \quad \forall \phi \in [P_1(e)]^2,
\]

where the lifting operator \(r_e^*: (L^2(e))^2 \to W_h\) is defined by

\[
\int_{\Omega} r_e^*(v) \cdot w_h \, dx = - \int_{e} v \cdot \{w_h\} \, ds, \quad \forall w_h \in W_h.
\]

Here, \(W_h := \left\{ w_h \in (L^2(\Omega))^2 : w_h|_K \in P_i(K), \ \forall K \in \mathcal{T}_h, i = 1,2 \right\}\).

For two matrix-valued functions \(\phi = (\phi_{ij})_{2 \times 2}\) and \(\tau = (\tau_{ij})_{2 \times 2}\), let \(\phi_1 = (\phi_{11}, \phi_{21})^t\), \(\phi_2 = (\phi_{12}, \phi_{22})^t\), \(\tau_1 = (\tau_{11}, \tau_{21})^t\), \(\tau_2 = (\tau_{12}, \tau_{22})^t\), so that \(\phi = (\phi_1, \phi_2)\), \(\tau = (\tau_1, \tau_2)\). Then

\[
\int_{\Omega} r_e(\phi) : \tau \, dx = - \int_{e} \phi : \{\tau\} \, ds = - \int_{e} \phi_1 : \{\tau_1\} \, ds - \int_{e} \phi_2 : \{\tau_2\} \, ds
\]

\[
= \int_{\Omega} r_e^*(\phi_1) : \tau_1 \, dx + \int_{\Omega} r_e^*(\phi_2) : \tau_2 \, dx = \int_{\Omega} (r_e^*(\phi_1), r_e^*(\phi_2)) : \tau \, dx,
\]

for all \(\tau \in \mathcal{O}_h\). So \(r_e(\phi) = (r_e^*(\phi_1), r_e^*(\phi_2))\), \(\|r_e(\phi)\|_{0,\Omega}^2 = \|r_e^*(\phi_1)\|_{0,\Omega}^2 + \|r_e^*(\phi_2)\|_{0,\Omega}^2\), and

\[
h_e^{-1} \|\phi\|_{0,e}^2 = h_e^{-1} (\|\phi_1\|_{0,e}^2 + \|\phi_2\|_{0,e}^2) \lesssim \|r_e^*(\phi_1)\|_{0,\Omega}^2 + \|r_e^*(\phi_2)\|_{0,\Omega}^2 = \|r_e(\phi)\|_{0,\Omega}^2.
\]

Let \(\phi = \nabla v\), then the first inequality follows. \(\blacksquare\)

From (3.1) and (2.4), we have

\[
\|r_0(\nabla v)\|_{0,h}^2 = \sum_{e \in \mathcal{E}_h^0} \|r_e(\nabla v)\|_{0,h}^2 \leq 3C_2 \sum_{e \in \mathcal{E}_h^0} h_e^{-1} \|\nabla v\|_{0,e}^2.
\]

For the boundedness of the primal forms \(B_{1,h}^{(j)}\) with \(j = 1, \cdots, 5\), first notice that \(\|\text{tr}(\tau)\|_{0,h} \lesssim \|\tau\|_{0,h}\). By the Cauchy-Schwarz inequality and Lemma 3.2, we get the fol-
following inequalities:

\[
\int_{\Omega} \nabla_h^2 w : \nabla_h^2 v \, dx \leq |w|_{2,h} |v|_{2,h}, \tag{3.3}
\]

\[
\int_{\Omega} r_0(\nabla w) : r_0(\nabla v) \, dx \lesssim \left( \sum_{e \in T_h^0} h_e^{-1} \|\nabla w\|_{0,e}^2 \right)^{1/2} \left( \sum_{e \in T_h^0} h_e^{-1} \|\nabla v\|_{0,e}^2 \right)^{1/2}, \tag{3.4}
\]

\[
\int_{T_h^0} \eta h_e^{-1} \|\nabla w\| \, ds \leq \sup_{e \in T_h^0} \eta_e \left( \sum_{e \in T_h^0} h_e^{-1} \|\nabla w\|_{0,e}^2 \right)^{1/2} \left( \sum_{e \in T_h^0} h_e^{-1} \|\nabla v\|_{0,e}^2 \right)^{1/2}, \tag{3.5}
\]

\[
\sum_{e \in T_h^0} \int_{\Omega} r_e(\nabla w) : r_e(\nabla v) \, dx \lesssim \sup_{e \in T_h^0} \eta_e \left( \sum_{e \in T_h^0} h_e^{-1} \|\nabla w\|_{0,e}^2 \right)^{1/2} \left( \sum_{e \in T_h^0} h_e^{-1} \|\nabla v\|_{0,e}^2 \right)^{1/2}. \tag{3.6}
\]

Using the trace inequality \(|\nabla^2 v|_{0,e} \leq h_e^{-1} |v|_{2,K} + h_e |v|_{3,K}^2\) with \(e\) an edge of \(K\), we have

\[
\int_{T_h^0} \|\nabla w\| : \{\nabla_h^2 v\} \, ds = \sum_{e \in T_h^0} \int_{e} \|\nabla w\| : \{\nabla_h^2 v\} \, ds
\]

\[
\lesssim \left( \sum_{e \in T_h^0} h_e^{-1} \|\nabla w\|_{0,e}^2 \right)^{1/2} \left( \sum_{e \in T_h^0} h_e \|\{\nabla_h^2 v\}_{0,e}^2 \right)^{1/2}
\]

\[
\lesssim \left( \sum_{e \in T_h^0} h_e^{-1} \|\nabla w\|_{0,e}^2 \right)^{1/2} \left( \sum_{K \in T_h} \left(|v|_{2,K}^2 + h_K^2 |v|_{3,K}^2 \right) \right)^{1/2}. \tag{3.7}
\]

The inequalities (3.3) and (3.7) are needed by all bilinear forms. For the CDG methods with the bilinear form \(B_{1,h}^{(j)}\), \(j = 1, 2, 5\), the inequality (3.5) is needed. The inequality (3.4) is needed by the formulas \(B_{1,h}^{(j)}\) with \(j = 3, 5\). The methods with the bilinear forms \(B_{1,h}^{(j)}\), \(j = 3, 4\), need the inequality (3.6). Then we have the following result.

**Lemma 3.3 (Boundedness)** Let \(B_h = B_{1,h}^{(j)}\) with \(j = 1, \ldots, 5\). Then

\[
B_h(w, v) \lesssim \|w\| \|v\| \quad \forall (w, v) \in V(h) \times V(h). \tag{3.8}
\]

For stability over \(V_h\), note that \(\|v\| = |v|_\ast\) for any \(v \in V_h\). Formulations \(B_{1,h}^{(j)}\) and \(B_{2,h}^{(j)}\) are equivalent on \(V_h\), so we just need to prove the stability for \(B_{2,h}^{(j)}\) based on \(|\cdot|_\ast\). We use
the Cauchy-Schwarz inequality and Lemma 3.2 to get

\[
B_{2,h}^{(1)}(v, v) = (1 - \kappa) \int_{\Omega} \nabla_h^2 v : \nabla_h^2 v \, dx + \kappa \int_{\Omega} (\text{tr}(\nabla_h^2 v))^2 \, dx + 2(1 - \kappa) \int_{\Omega} \nabla_h^2 v : r_0([\nabla v]) \, dx \\
+ 2\kappa \int_{\Omega} \text{tr}(\nabla_h^2 v) \, tr(r_0([\nabla v])) \, dx + \int_{e_h^0} \eta h_e^{-1} ||[\nabla v]||^2 \, ds \\
\geq (1 - \kappa)|v|_{2,h}^2 + \kappa \|\Delta_h v\|^2_{0,h} - d_{0,h} - 2(1 - \kappa) \int_{\Omega} \nabla_h^2 v : r_0([\nabla v]) \, dx \\
+ 2\kappa \int_{\Omega} \nabla_h^2 v \, tr(r_0([\nabla v])) \, dx + \int_{e_h^0} \eta h_e^{-1} ||[\nabla v]||^2 \, ds
\]

where \(0 < \epsilon < 1\) is a constant and \(C_2\) is the generic positive constant in (3.3). Therefore, stability is valid for the \(C^0\) IP method when

\[
\min_{e \in e_h^0} \eta_e > 3(1 - \kappa)C_2 + 6C_2\kappa = 3(1 + \kappa)C_2.
\]

Next,

\[
B_{2,h}^{(2)}(v, v) = \int_{\Omega} (1 - \kappa) \nabla_h^2 v : \nabla_h^2 v \, dx + \int_{\Omega} \kappa (\text{tr}(\nabla_h^2 v))^2 \, dx + \int_{e_h^0} \eta h_e^{-1} ||[\nabla v]||^2 \, ds \\
\geq (1 - \kappa)|v|_{2,h}^2 + \eta_0 \sum_{e \in e_h^0} h_e^{-1} ||[\nabla v]||^2_{0,e}.
\]

So stability is valid for the \(C^0\) NIPG method for any \(\eta_0 > 0\). This property is the reason why the method with the bilinear form \(B_{2,h}^{(2)}\) is useful even though \(B_{2,h}^{(2)}\) is not symmetric.

\[
B_{2,h}^{(4)}(v, v) \geq (1 - \kappa)|v|_{2,h}^2 + \kappa \|\Delta_h v\|^2_{0,h} + 2(1 - \kappa) \int_{\Omega} \nabla_h^2 v : r_0([\nabla v]) \, dx \\
+ 2\kappa \int_{\Omega} \Delta_h v \, tr(r_0([\nabla v])) \, dx + \eta_0 \sum_{e \in e_h^0} ((1 - \kappa)||r_e([\nabla v])||^2_{0,h} + \kappa||tr(r_e([\nabla v]))||^2_{0,h}) \\
\geq (1 - \kappa)|v|_{2,h}^2 + \kappa \|\Delta_h v\|^2_{0,h} - d_{0,h} - 2(1 - \kappa) \int_{\Omega} \nabla_h^2 v : r_0([\nabla v]) \, dx \\
+ 2\kappa \int_{\Omega} \nabla_h^2 v \, tr(r_0([\nabla v])) \, dx + \eta_0 \sum_{e \in e_h^0} h_e^{-1} ||[\nabla v]||^2_{0,e} + \eta_0 C_1(1 - \kappa) \sum_{e \in e_h^0} ||tr(r_e([\nabla v]))||^2_{0,h} \\
\geq (1 - \epsilon)(1 - \kappa)|v|_{2,h}^2 + (1 - \kappa) \left( \eta_0 C_1 - \frac{3C_2}{\epsilon} \right) \sum_{e \in e_h^0} h_e^{-1} ||[\nabla v]||^2_{0,e} \\
+ (\eta_0 \kappa - 3\kappa) \sum_{e \in e_h^0} ||tr(r_e([\nabla v]))||^2_{0,h}.
\]
Since $C_2 > C_1$, $\eta_0 > 3$ is guaranteed from $\eta_0 > 3C_2/C_1$. Thus, stability is valid for this CDG formulation when $\eta_0 > 3C_2/C_1$. For the method of Wells-Dung corresponding to the form $B_{2,h}^{(3)}$ and the LCDG method corresponding to the form $B_{2,h}^{(5)}$, stability can be analyzed by a similar argument (cf. [27] and [17], respectively), with $\eta_0 > 0$.

Summarizing, we have shown the following result.

**Lemma 3.4 (Stability)** Let $B_h = B_{2,h}^{(j)}$ with $j = 1, \ldots, 5$. Assume

$$\min_{e \in E_h^0} \eta_e > 3(1 + \kappa)C_2$$

for $j = 1$ and

$$\min_{e \in E_h^0} \eta_e > 3C_2/C_1$$

for $j = 4$, with $C_1$ and $C_2$ the constants in the inequality (3.1). Then,

$$\|v\|_h^2 \lesssim B_h(v,v) \quad \forall v \in V_h. \quad (3.9)$$

We further conclude that the stability is also valid for $B_{1,h}^{(j)}$ with $j = 1, \ldots, 5$.

## 4 Error analysis

We turn to an error estimation for the CDG methods. Write the error as

$$e = u - u_h = (u - u_I) + (u_I - u_h),$$

where $u_I \in V_h$ is the usual continuous piecewise quadratic interpolant of the exact solution $u$. Using the scaling argument and the trace theorem, we have the following result.

**Lemma 4.1** For all $v \in H^3(K)$ on $K \in T_h$,

$$\|v - v_I\|_K + h_K |v - v_I|_{1,K} + h_K^2 |v - v_I|_{2,K} \lesssim h_K^3 |v|_{3,K},$$

$$\|\nabla (v - v_I)\|_{0,\partial K} \lesssim h_K^{3/2} |v|_{3,K}.$$

As a consequence of Lemma 4.1, we obtain the estimate

$$\|u - u_I\|_h \lesssim h |u|_{3,\Omega} \quad (4.1)$$

Now, we are ready to derive a priori error estimates of the CDG methods when they are applied to solve the 4th-order elliptic variational inequality (1.6).
Theorem 4.2 Assume the solution of the problem (1.6) satisfies \( u \in H^3(\Omega) \) and the assumptions in Lemma 3.4 hold. Let \( B_h = B_h^{(j)} \) with \( j = 1, \ldots, 5 \), and \( u_h \in K_h \) be the solution of (2.5). Then we have the optimal order error estimate

\[
\| u - u_h \| \lesssim h \| u \|_{3, \Omega}. \tag{4.2}
\]

Proof. Recall the boundedness and stability of the bilinear form \( B_h \). We have

\[
\| u_I - u_h \|^2 \lesssim B_h(u_I - u_h, u_I - u_h) \equiv T_1 + T_2, \tag{4.3}
\]

where

\[
T_1 = B_h(u_I - u, u_I - u_h), \\
T_2 = B_h(u - u_h, u_I - u_h).
\]

We bound \( T_1 \) as follows:

\[
T_1 \lesssim \| u_I - u \| \| u_I - u_h \| \lesssim \epsilon \| u_I - u_h \|^2 + \frac{1}{4\epsilon} \| u_I - u \|^2, \tag{4.4}
\]

where \( \epsilon > 0 \) is an arbitrarily small number.

Since \( \| \nabla u \| = 0 \) on \( e \in E_h \), we use the definition (1.2) to obtain

\[
B_h(u, u_I - u_h) = \int_{\Omega} (1 - \kappa) \nabla^2 u : \nabla_h^2 (u_I - u_h) \, dx + \int_{\Omega} \kappa \text{tr} (\nabla^2 u) \text{tr} (\nabla_h^2 (u_I - u_h)) \, dx
- \int_{E_h^0} \nabla (u_I - u_h) : ((1 - \kappa) \nabla^2 u + \kappa \text{tr} (\nabla^2 u) I) \, ds
= - \sum_{K \in \mathcal{T}_h} \int_{K} \sigma : \nabla^2 (u_I - u_h) \, dx + \int_{E_h^0} \nabla (u_I - u_h) : \sigma \, ds.
\]

Noting \( [\sigma] = 0 \) on \( e \in E_h \), we get by Lemma 1.1

\[
\sum_{K \in \mathcal{T}_h} \int_{K} \sigma : \nabla^2 (u_I - u_h) \, dx = - \sum_{K \in \mathcal{T}_h} \int_{K} \nabla (u_I - u_h) \cdot (\nabla \cdot \sigma) \, dx
+ \sum_{K \in \mathcal{T}_h} \int_{\partial K} \nabla (u_I - u_h) \cdot (\sigma n_K) \, ds
= - \int_{\Omega} \nabla (u_I - u_h) \cdot (\nabla \cdot \sigma) \, dx + \int_{E_h} \nabla (u_I - u_h) : \sigma \, ds.
\]
Thus,

\[
B_h(u, u_I - u_h) = \int_\Omega \nabla (u_I - u_h) \cdot (\nabla \cdot \sigma) \, dx - \int_{\Gamma_{2} \cup \Gamma_{3}} \llbracket \nabla (u_I - u_h) \rrbracket \cdot \sigma \, ds
\]

\[
= \int_\Omega \nabla (u_I - u_h) \cdot (\nabla \cdot \sigma) \, dx - \int_{\Gamma_{2} \cup \Gamma_{3}} \sigma \partial_t (u_I - u_h) \, ds
\]

\[
= - \int_\Omega \nabla \cdot (\nabla \cdot \sigma) (u_I - u_h) \, dx + \langle (\nabla \cdot \sigma) \cdot \mathbf{v}, u_I - u_h \rangle_{1/2, \Gamma}
\]

\[
- \int_{\Gamma_{2} \cup \Gamma_{3}} \sigma \partial_t (u_I - u_h) \, ds.
\]

By (1.15) and (1.10), we have

\[
B_h(u, u_I - u_h) = - \int_\Omega \nabla \cdot (\nabla \cdot \sigma) (u_I - u_h) \, dx - \int_{\Gamma_{3}} g\lambda u_I \, ds + \int_{\Gamma_{3}} g\lambda u_h \, ds
\]

\[
= \int_\Omega f(u_I - u_h) \, dx - \int_{\Gamma_{3}} g\lambda u_I \, ds + \int_{\Gamma_{3}} g\lambda u_h \, ds.
\] (4.5)

Let \(v_h = u_I\) in (2.5),

\[
B_h(u_I, u_I - u_h) + j(u_I) - j(u_h) \geq (f, u_I - u_h).
\] (4.6)

Combining (4.6) and (4.5), and with the use of (1.9), we can bound \(T_2 = B_h(u - u_h, u_I - u_h)\) as follows:

\[
T_2 \leq - \int_{\Gamma_{3}} g\lambda u_I \, ds + \int_{\Gamma_{3}} g\lambda u_h \, ds + j(u_I) - j(u_h)
\]

\[
= \int_{\Gamma_{3}} g(|u_I| - \lambda u_I) \, ds + \int_{\Gamma_{3}} g(\lambda u_h - |u_h|) \, ds
\]

\[
\leq \int_{\Gamma_{3}} g(|u_I| - \lambda u_I) \, ds = \int_{\Gamma_{3}} g(|u_I| - |u| + \lambda u - \lambda u_I) \, ds
\]

\[
\leq 2 \int_{\Gamma_{3}} g|u_I - u| \, ds \leq 2\|g\|_{0, \Gamma_{3}}\|u_I - u\|_{0, \Gamma_{3}}.
\]

Hence,

\[
T_2 \lesssim h^2\|u\|_{3, \Omega}
\] (4.7)

Combining (4.3), (4.4), and (4.7), and applying Lemma 4.1, we have

\[
\|u_I - u_h\|^2 \lesssim h^2\|u\|^2_{3, \Omega}.
\] (4.8)

Finally, from the triangle inequality \(||u - u_h|| \leq ||u - u_I|| + ||u_I - u_h||\), (4.1) and (4.8), we obtain the error bound. \(\blacksquare\)
5 Numerical Results

In this section, we present a numerical example with the five CDG schemes studied in solving the elliptic variational inequality (1.6). Let $\Omega = (-1, 1) \times (-1, 1)$, $\kappa = 0.3$. A generic point in $\Omega$ is denoted as $x = (x, y)^T$. The Dirichlet boundary is $\Gamma_1 = (-1, 1) \times \{1\}$, and the free boundary is $\Gamma_2 = \{(-1) \times (-1, 1) \cup \{(1) \times (-1, 1)\}$. On the friction boundary $\Gamma_3 = (-1, 1) \times \{-1\}$, we choose $g = 1$. The right hand side function is $f(x) = 24(1 - x^2)^2 + 24(1 - y^2)^2 + 32(3x^2 - 1)(3y^2 - 1)$.

For a discretization of the variational inequality (1.6), we use uniform triangulations $\{T_h\}$ of the region $\Omega$, and define the finite element spaces to be

$$
V_h := \{v_h \in H^1(\Gamma_1) : v_h|_K \in P_2(K) \forall K \in T_h\},
$$

$$
\Sigma_h := \{\tau_h \in (L^2(\Omega))^2 : \tau_h,ij|_K \in P_1(K) \forall K \in T_h, i,j = 1,2\}.
$$

Any function $v^h \in V_h$ can be expressed as

$$
v^h(x) = \sum_i v_i \phi_i(x),
$$

where $v_i = v^h(x_i)$, $\{x_i\}$ are the nodal points, and $\{\phi_i\}$ are the standard nodal basis functions of the space $V_h$. The basis functions satisfy the relation $\phi_i(x_j) = \delta_{ij}$, $\delta_{ij}$ being the Kronecher delta. The functional $j(\cdot)$ is approximated through numerical integration:

$$
j^h(v^h) = S_n^{\Gamma_3}(g |v^h|) = \sum w_j g(x_j) \sum v_i \phi_i(x_j) = \sum |w_j g(x_j) v_j|,
$$

where the summations extend to all the finite element nodes on $\Gamma_3$, and $S_n^{\Gamma_3}$ denotes the composite Simpson’s rule using these finite element nodes. Then the discrete problem is

$$
\min_{u^h \in V_h} \frac{1}{2} a(u^h, u^h) + j^h(u^h) - (f, u^h). \quad (5.1)
$$

The matrix/vector form of the discrete optimization problem is

$$
\min_u \frac{1}{2} u^T A u + \|B u\|_{\ell_1} - u^T f, \quad (5.2)
$$

where $u = (u_i)^T$, $A = (a(\phi_i, \phi_j))$, $B = (w_i g(x_i) \delta_{ij})$, and $f = ((f, \phi_j))^T$.

To solve the discrete problem (5.2), we use the following primal-dual fixed point iteration Algorithm II proposed in [28]. Here for a given function $F$ of a vector variable $x$, the proximal operator $\text{prox}_F$ is defined as

$$
\text{prox}_F(x) = \arg \min_y F(y) + \frac{1}{2} \|x - y\|^2.
$$

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Algorithm 1 Primal Dual Fixed Point Algorithm

Initialize $u_0$ and $v_0$, set parameters $\lambda \in (0, \frac{1}{\lambda_{\text{max}}(BB^T)})$, $\gamma \in (0, \frac{2}{\|A\|_2})$

for $i = 1, 2, 3, \ldots$ do
\begin{align*}
    u_{k+\frac{1}{2}} &= u_k - \gamma(Au_k - f), \\
    v_{k+1} &= (I - \text{prox}_{\gamma \frac{1}{\lambda} \|\cdot\|_1})(Bu_{k+\frac{1}{2}} + (I - \lambda BB^T)v_k), \\
    u_{k+1} &= u_{k+\frac{1}{2}} - \lambda B^T v_{k+1}
\end{align*}
end for

For $F = \frac{2}{\lambda} \| \cdot \|_1$, the proximal operator has the explicit form (applied to each component of the vector variable):
\[
    \text{prox}_{\frac{2}{\lambda} \| \cdot \|_1} x = \text{sgn}(x) \max\left( |x| - \frac{\gamma}{\lambda}, 0 \right) = \text{sgn}(x) \left( |x| - \frac{\gamma}{\lambda} \right)_+.
\]

Tables 1–5 provide numerical solution errors in the energy norm $\| \cdot \|$ and $H^1(\Omega)$ seminorm for the five DG methods discussed in this paper. Since the true solution of the variational inequality (1.6) is not known, we use the numerical solution corresponding to the meshsize $h = 1/64$ as the true solution in computing the errors. We observe that the numerical convergence orders in the energy norm are around one, agreeing with the theoretical error estimate (4.2). We note that the numerical convergence orders in the $H^1(\Omega)$-seminorm are also close to one.

Table 1: Error for $C^0$ IP method (2.7)

| $h$ | $\|u - u_h\|$ | $\|u - u_h\|_{H^1(\Omega)}$ |
|-----|----------------|-----------------|
|     | $\eta = 1$ | $\eta = 10$ | $\eta = 100$ | $\eta = 1$ | $\eta = 10$ | $\eta = 100$ |
| 1/2 | 5.1859  | 4.2973  | 3.1164  | 0.9367 | 0.8126 | 0.5635 |
| 1/4 | 3.3677  | 2.6726  | 1.5923  | 0.5105 | 0.4135 | 0.3133 |
| 1/8 | 1.8625  | 1.4407  | 0.8122  | 0.2691 | 0.2049 | 0.1580 |
| 1/16 | 0.8601 | 0.7652 | 0.4422 | 0.1355 | 0.1059 | 0.0801 |
Table 2: Error for NIPG method (2.9)

| $h$  | $\|u - u_h\|$ | $u - u_h|_{H^1(\Omega)}$ |
|------|--------------|-----------------|
|      | $\eta = 1$ | $\eta = 10$ | $\eta = 100$ | $\eta = 1$ | $\eta = 10$ | $\eta = 100$ |
| 1/2  | 5.5411      | 4.4659         | 3.1178        | 1.0000      | 0.8250      | 0.6638        |
| 1/4  | 3.6029      | 2.7708         | 1.6071        | 0.5699      | 0.4233      | 0.3607        |
| 1/8  | 1.9137      | 1.5359         | 0.7961        | 0.2829      | 0.2246      | 0.1853        |
| 1/16 | 0.9485      | 0.7594         | 0.3929        | 0.1491      | 0.1144      | 0.0934        |

Table 3: Error for Wells-Dung DG formulation (2.10)

| $h$  | $\|u - u_h\|$ | $u - u_h|_{H^1(\Omega)}$ |
|------|--------------|-----------------|
|      | $\eta = 1$ | $\eta = 10$ | $\eta = 100$ | $\eta = 1$ | $\eta = 10$ | $\eta = 100$ |
| 1/2  | 4.4617      | 3.4573         | 2.0932        | 0.8062      | 0.7650      | 0.3785        |
| 1/4  | 2.8185      | 2.2331         | 1.3618        | 0.4301      | 0.3885      | 0.2486        |
| 1/8  | 1.4545      | 1.1473         | 0.6794        | 0.2131      | 0.2035      | 0.1253        |
| 1/16 | 0.7322      | 0.6270         | 0.3832        | 0.1085      | 0.1036      | 0.0645        |

Table 4: Error for Baker-DG formulation (2.12)

| $h$  | $\|u - u_h\|$ | $u - u_h|_{H^1(\Omega)}$ |
|------|--------------|-----------------|
|      | $\eta = 1$ | $\eta = 10$ | $\eta = 100$ | $\eta = 1$ | $\eta = 10$ | $\eta = 100$ |
| 1/2  | 4.8538      | 4.1180         | 2.0977        | 0.8771      | 0.8360      | 0.3793        |
| 1/4  | 2.8524      | 2.4987         | 1.3632        | 0.4662      | 0.4625      | 0.2489        |
| 1/8  | 1.5067      | 1.2842         | 0.6765        | 0.2447      | 0.2409      | 0.1255        |
| 1/16 | 0.7629      | 0.6747         | 0.3842        | 0.1259      | 0.1212      | 0.0657        |
| \( h \) | \( \| u - u_h \| \) | \( |u - u_h|_{H^1(\Omega)} \) |
|---|---|---|
| \( \eta = 1 \) | \( \eta = 10 \) | \( \eta = 100 \) | \( \eta = 1 \) | \( \eta = 10 \) | \( \eta = 100 \) |
| 1/2 | 4.6407 | 4.2599 | 2.5863 | 0.8384 | 0.7696 | 0.5579 |
| 1/4 | 2.8265 | 2.2147 | 1.6213 | 0.4546 | 0.4111 | 0.2863 |
| 1/8 | 1.5011 | 1.2460 | 0.8669 | 0.2311 | 0.2301 | 0.1453 |
| 1/16 | 0.7517 | 0.6341 | 0.4705 | 0.1189 | 0.1186 | 0.0812 |

References

[1] D.N. Arnold, F. Brezzi, B. Cockburn, and L.D. Marini, Discontinuous Galerkin methods for elliptic problems, in *Discontinuous Galerkin Methods. Theory, Computation and Applications*, B. Cockburn, G.E. Karniadakis, and C.-W. Shu, eds., Lecture Notes in Comput. Sci. Engrg. 11, Springer-Verlag, New York, 2000, 89–101.

[2] D.N. Arnold, F. Brezzi, B. Cockburn, and L.D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, *SIAM J. Numer. Anal.* 39 (2002), 1749–1779.

[3] K. Atkinson and W. Han, *Theoretical Numerical Analysis: A Functional Analysis Framework*, third edition, Springer-Verlag, New York, Texts in Applied Mathematics, Volume 39, 2009.

[4] I. Babuška and M. Zlámal, Nonconforming elements in the finite element method with penalty, *SIAM J. Numer. Anal.* 10 (1973), 863–875.

[5] G.A. Baker, Finite element methods for elliptic equations using nonconforming elements, *Math. Comp.* 31 (1977), 44–59.

[6] F. Bassi, S. Rebay, G. Mariotti, S. Pedinotti, and M. Savini, A high-order accurate discontinuous finite element method for inviscid and viscous turbomachinery flows, in *Proceedings of 2nd European Conference on Turbomachinery, Fluid Dynamics and Thermodynamics*, R. Decuyper and G. Dibelius, eds., Technologisch Instituut, Antwerpen, Belgium, 1997, 99–108.

[7] S.C. Brenner and L. Sung, \( C^0 \) interior penalty methods for fourth order elliptic boundary value problems on polygonal domains, *J. Sci. Comput.* 22/23 (2005), 83–118.
[8] S.C. Brenner, L. Sung, H. Zhang, and Y. Zhang, A quadratic $C^0$ interior penalty method for the displacement obstacle problem of clamped Kirchhoff plates, *SIAM J. Numerical Analysis* 50 (2012), 3329–3350.

[9] P. Castillo, B. Cockburn, I. Perugia, and D. Schötzau, An a priori error analysis of the local discontinuous Galerkin method for elliptic problems, *SIAM J. Numer. Anal.* 18 (2000), 1676–1706.

[10] B. Cockburn, Discontinuous Galerkin methods, *ZAMM Z. Angew. Math. Mech.* 83 (2003), 731–754.

[11] B. Cockburn, G.E. Karniadakis, and C.-W. Shu, eds, *Discontinuous Galerkin Methods. Theory, Computation and Applications*, Lecture Notes in Comput. Sci. Engrg. 11, Springer-Verlag, New York, 2000.

[12] G. Duvaut and J.-L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.

[13] G. Engel, K. Garikipati, T. Hughes, M. Larson, L. Mazzei, and R. Taylor, Continuous/discontinuous finite element approximations of fourth-order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity, *Comput. Methods Appl. Mech. Engrg.* 191 (2002), 3669–3750.

[14] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer-Verlag, New York, 1984.

[15] R. Glowinski, J.-L. Lions and R. Trémolières, *Numerical Analysis of Variational Inequalities*, North-Holland, New York, 1981.

[16] W. Han and L.H. Wang, Non-conforming finite element analysis for a plate contact problem, *SIAM Journal on Numerical Analysis* 40 (2002), 1683–1697.

[17] J. Huang, X. Huang, and W. Han, A new $C^0$ discontinuous Galerkin method for Kirchhoff plates, *Comput. Methods Appl. Mech. Engrg.* 199 (2010), 1446–1454.

[18] I. Mozolevski and P. R. Bösing, Sharp expressions for the stabilization parameters in symmetric interior-penalty discontinuous Galerkin finite element approximations of fourth-order elliptic problems, *Comput. Methods Appl. Math.* 7 (2007), 365–375.

[19] I. Mozolevski and E. Süli, A priori error analysis for the $hp$-version of the discontinuous Galerkin finite element method for the biharmonic equation, *Comput. Methods Appl. Math.* 3 (2003), 596–607.
[20] I. Mozolevski, E. Süli, and P.R. Bösing, \emph{hp}-version a priori error analysis of interior penalty discontinuous Galerkin finite element approximations to the biharmonic equation, \emph{J. Sci. Comput.} 30 (2007), 465–491.

[21] J.N. Reddy, \emph{Theory and Analysis of Elastic Plates and Shells}, second edition, CRC Press, New York, 2007.

[22] E. Süli and I. Mozolevski, \emph{hp}-version interior penalty DGFEMs for the biharmonic equation, \emph{Comput. Methods Appl. Mech. Engrg.} 196 (2007), 1851–1863.

[23] F. Wang, W. Han, and X. Cheng, Discontinuous Galerkin methods for solving elliptic variational inequalities, \emph{SIAM Journal on Numerical Analysis} 48 (2010), 708–733.

[24] F. Wang, W. Han, and X. Cheng, Discontinuous Galerkin methods for solving Signorini problem, \emph{IMA J. Numer. Anal.} 31 (2011), 1754–1772.

[25] F. Wang, W. Han, and X. Cheng, Discontinuous Galerkin methods for solving a quasistatic contact problem, \emph{Numer. Math.} 126 (2014), 771–800.

[26] F. Wang, W. Han, J. Huang, and T. Zhang, Discontinuous Galerkin methods for an elliptic variational inequality of 4th-order, submitted.

[27] G.N. Wells and N.T. Dung, A \emph{C^0} discontinuous Galerkin formulation for Kirchhoff plates, \emph{Comput. Methods Appl. Mech. Engrg.} 196 (2007), 3370–3380.

[28] X. Zhang, M. Burger and S. Osher, A unified primal-dual algorithm framework based on Bregman iteration, \emph{J. Sci. Comput.} 46 (2011), 20–46.