ON THE FINITE-TIME BHAT-BERNSTEIN FEEDBACKS FOR
THE STRINGS CONNECTED BY POINT MASS

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Abstract. In this article, the problem of finite-time stabilization of two strings connected by point mass is discussed. We use the so-called Riemann coordinates to convert the study system into four transport equations coupled with the dynamic of the charge. We act by Bhat-Bernstein feedbacks in various positions (two extremities, the point mass and one of boundaries, only on the point mass,...) and we show that in some cases the nature of the stability depends sensitively on the physical parameters of the system.

1. Introduction. There has been great interest in the topic of control and stabilization of hybrid systems in which the dynamics of elastic systems and possibly rigid structures are coupled. These problems are investigated by a number of authors, e.g. [30, 31, 42, 43].

As an example of hybrid systems, we treat in this paper the system modeling a vibrating string involving an interior point mass. The controllability issues was considered in [9, 10, 18], while the stabilization problem was the topic of [18, 29, 32]. In [18], stabilizing feedback laws are established to put the system in the equilibrium configuration. The analysis requires the energy estimate in the good asymmetric functional space, and authors have shown that the energy decays exponentially if the velocity feedbacks are applied at both ends. Moreover, they have shown that acting by one boundary feedback, every solution converges asymptotically to equilibrium point. Authors in [32] improved the result obtained by Hansen and Zuazua [18], by showing that the energy to the right of the mass decays like $c/t$ provided that the initial data belongs to an invariant subspace of the semigroup operators that generates the solution of the system.

In this article, it seems an attractive idea to study the same problem in the case of finite-time stabilization. This property is called sometimes super stability [47], and means that the solution of the dynamical system vanishes in finite-time. For further background and motivation, the reader is referred to [5, 6, 7, 11, 1, 17, 19, 20, 21, 22, 24, 25, 26, 33, 35, 36, 38], and references therein.

It is worth noting that in finite dimension, the finite-time stabilization means the design of feedback controller which renders the closed loop system Lyapunov stable.

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and all trajectories converge to the origin in finite-time. In the most papers cited below, e.g. [3, 5, 6, 17, 19, 20, 21, 22, 24, 36], the finite-time stability requires the construction of Lyapunov function $V$ satisfying the differential inequality $\dot{V} + c V^\alpha \leq 0$ where $\alpha \in (0, 1)$ and $c > 0$. For infinite-dimensional systems, the above inequality renders the finite-time stabilization task more difficult, and it is used in few papers as [8] for the problem of Schrödinger equation with damping and [39] for the heat and wave equations by using sliding mode control.

However, due to the specificity of this strong stability, a great effort has been done recently to reach the finite-time stability of some partial differential equations. Thus, it is interesting to cite different approaches used to solve these problems in infinite-dimensional systems:

- **The spectral analysis**, is used to prove the super stability in [47] for a star-shaped network. Moreover, this problem was generalized in [45] for the case of tree-shaped network.

- **The backstepping approach**, is considered in [13] to solve the finite-time stability of the heat equation with variable coefficients in 1-D. Again, in [23] and by the same aspect, authors proved that $3 \times 3$ hyperbolic systems with one positive and two negative transport speeds can be stabilized in finite-time by boundary feedbacks applied on states corresponding to the negative velocities.

- **The geometric homogeneity approach**, is extended by authors [41] for some evolution equations in Banach space. This result combined the asymptotic stability with the negative degree of homogeneity to get the finite-time stability (Theorem well known in finite-dimensional). Therefore, this property is applied on the classical uncontrolled examples as the KdV, Saint-Venant and fast-diffusion equations. For control systems in Hilbert space, authors constructed in [40] a universal homogeneous feedback. As application the examples of controlled heat equation, respectively the wave equation are treated.

- **The explicit approach**, is used in [38] to establish the finite-time boundary stabilization of a 1-D first order quasilinear hyperbolic system of diagonal form. In [1], authors showed that for any tree-shaped network of strings with transparent boundary conditions at all external nodes, the finite-time stabilization is achieved under a particular choice of the damping term $\alpha u_t$ at each internal node $n$ connecting $k$ edges.

In addition, the finite-time stabilization of Kinematic waves that exhibit time switching between boundary conditions is proved in [4] and illustrated through the examples of SMB chromatography and the Ramp-metering control for road traffic network.

In this paper, we focus our attention on the issue of finite-time stabilization of a vibrating string involving an interior point mass. To the best of our knowledge, the finite-time stabilization for this hybrid system is not solved in the literature.

In fact, the presence of the dynamic of interior point mass in the strings generates undesirable oscillations of the system. This renders the finite-time stability of this network difficult, and feedbacks developed in [1] cannot be applied.

To get around the drawback of the charge mass, we act by appropriate feedback laws vanishing the oscillations of the system in finite-time.

Since our system is an example of hyperbolic systems, then Riemann invariants considered in [2, 46, 14] are used to convert our system to four transport equations coupled with the dynamic of the charge.
At this step, finite-time homogeneous Hölderian feedback laws are developed for the transformed system. In particular, this finite-time stabilizability is obtained when acting by two feedbacks in various positions (in the boundaries, in the charge mass and or in both one boundary and charge mass). Hence, it is remarked that the nature of stability depends sensitively on the physical parameters of the system when boundary feedbacks are chosen.

This paper is structured as follows. A preliminary results that we need is given in Section 2. The Section 3 is devoted to describe the system under investigation and its transformation in Riemann coordinates. The analysis of the finite-time stabilization with different cases are presented in Section 4. Some comments and Conclusion are the subject of Section 5. Finally, an Appendix dealing with some clarifications of the notion of finite-time stabilization and the well-posedness of closed-loop system is given.

2. Preliminary results.

2.1. The transport equation revisited. Let $T > 0$. The transport equation is the control system described by the following linear partial differential equation

$$
y_t + a y_x = 0, \quad (x, t) \in (x_l, x_r) \times (0, T),
$$

$$
y(x_l, t) = u_t, \quad t \in (0, T),
$$

$$
y(x, 0) = y_0(x), \quad x \in (x_l, x_r),
$$

where, at time $t$, the state is $y(., t) : (x_l, x_r) \to \mathbb{R}$; the control is $u(t) \in \mathbb{R}$ and the positif constant $a$ is the velocity of propagation.

It was proved in [12, section 2.1] that for every $y_0 \in L^2(x_l, x_r)$ and $u \in L^2(0, T)$ the Cauchy problem (1)-(3) admits a unique weak solution in $L^2((x_l, x_r) \times (0, T))$. This result is recalled in the following theorem.

**Theorem 2.1.** ([12, Theorem 2.4, pp. 27]) For every $y_0 \in L^2(x_l, x_r)$ and $u \in L^2(0, T)$, the Cauchy problem (1)-(3) has a unique solution. Moreover, this solution satisfies

$$
||y(., t)||_{L^2(x_l, x_r)} \leq ||y_0||_{L^2(x_l, x_r)} + ||u||_{L^2(0, T)}. \quad \text{(4)}
$$

In addition, it is known, see e.g. [16, 44] that this solution can be found explicitly thanks to characteristic method which reduces the study of the transport equation to an ordinary differential equation along the characteristics of the transport operator. Hence the solution of (1)-(3) can be expressed as follows

$$
y(x, t) = \begin{cases} 
y_0(x - ta) & \text{if } x - ta \geq x_l, \\
u(t - \frac{x-x_l}{a}) & \text{if } x - ta < x_l.
\end{cases} \quad \text{(5)}
$$

For transport equation with negative velocity, the control is applied at the right extremity $x_r$. Then, instead of (2), we have

$$
y(x_r, t) = v(t),
$$

and the explicit solution of the transport equation is given by

$$
y(x, t) = \begin{cases} 
y_0(x + ta) & \text{if } x + ta \leq x_r, \\
v(t + \frac{x-x_r}{a}) & \text{if } x + ta > x_r.
\end{cases} \quad \text{(6)}
$$
2.2. Finite-time stabilization definition. Let be the dynamical controlled system
\[ \dot{y} = f(y, u), \quad y \in \mathcal{Y}, \ u \in \mathcal{U}, \]
where \( f : \mathcal{Y} \times \mathcal{U} \to \mathcal{Y} \) is a smooth function, with \( \mathcal{Y} \) and \( \mathcal{U} \) are Hilbert spaces denote respectively the state space and the control space. Assume that the control system (7) is well-posed. We have the following definition.

**Definition 2.2.** The control system (7) is said to be finite-time stabilizable if there exists an admissible control \( y \mapsto u(y) \in \mathcal{U} \) with \( u(0) = 0 \) and the closed loop system \( \dot{y} = f(y, u(y)) \) is finite-time stable. More precisely.

(a) The origin of \( \dot{y} = f(y, u(y)) \) is Lyapunov stable,
(b) there exists \( r > 0 \) and \( T = T(y(0)) > 0 \), called the settling-time function such that if \( |y(0)| < r \), then \( y(t) = 0 \) for every \( t \geq T \).

For example, the control system (1)-(3) can be finite-time stabilizable by taking \( u \equiv 0 \), clearly this null feedback provides \( y(x, t) = 0 \) \( \forall t \geq T/a \).

The next lemma will be used in the proof of main results.

**Lemma 2.3.** ([28]) Consider the system
\[
\begin{cases}
\dot{x} = f_1(x), \\
\dot{y} = f_2(x, y),
\end{cases}
\]
where \( f_1(0) = 0, \ f_2(0, 0) = 0, \) and for \( i = 1, 2, \ f_i \) are smooth on their domains. If \( x = 0 \) is finite-time stable and \( \dot{y} = f_2(0, y) \) is asymptotically stable then \( (0, 0) \) is asymptotically stable for (8).

Moreover, if \( x = 0 \) is finite-time stable and \( (0, 0) \) is asymptotically stable for (8), then \( \dot{y} = f_2(0, y) \) is asymptotically stable.

For instance, the system
\[
\begin{cases}
\dot{x}_1 = -\text{sgn}(x_1)|x_1|^\alpha, \quad \alpha \in (0, 1), \\
\dot{x}_2 = -x_2^3 \cos^{1/3}(x_1),
\end{cases}
\]
is asymptotically stable.

3. The problem of vibrating strings connected by a point charge.

3.1. Description of the problem. The vibrating string system under study can be modeled as an hybrid system. In fact, it can be considered as two separate strings in which one end of each string is attached to a common point mass (see e.g. [9, 10, 18, 32]). The first and second string occupies respectively \( \Omega_1 = (-\ell_1, 0) \subset \mathbb{R} \) and \( \Omega_2 = (0, \ell_2) \subset \mathbb{R} \), where \( \ell_1 \) and \( \ell_2 \) are positive.

We suppose that both strings are homogeneous. The deformation of the first and second string at abscissa \( x \) and time \( t \) will be described respectively by the functions \( u \) and \( v \) where each one is solution of a one dimensional wave equation associated to each interval between the boundary and the point mass;

\[
\begin{align*}
\rho_1 u_{tt} &= \sigma_1 u_{xx}, & x &\in \Omega_1, \ t > 0, \\
\rho_2 v_{tt} &= \sigma_2 v_{xx}, & x &\in \Omega_2, \ t > 0.
\end{align*}
\]
The physical parameters $\rho_i > 0$ and $\sigma_i > 0 \ (i = 1, 2)$ represent, respectively, the density and the tension of each string.

The position $z(t)$ of the mass $M$, attached to the string at $x = 0$, satisfies the continuity relation
\[ u(0, t) = v(0, t) = z(t), \quad t > 0, \] (11)
and the second order linear ordinary differential equation describing the motion of the mass is
\[ Mz_{tt}(t) + \sigma_1 u_x(0, t) - \sigma_2 v_x(0, t) = 0, \quad t > 0. \] (12)

The initial conditions are
\[ (u(x, 0), u_t(x, 0)) = (u^0(x), u^1(x)), \quad x \in \Omega_1, \] (13)
\[ (v(x, 0), v_t(x, 0)) = (v^0(x), v^1(x)), \quad x \in \Omega_2, \] (14)
\[ (z(0), z_t(0)) = (z^0, z^1). \] (15)

For boundaries, we consider Dirichlet boundary conditions given in free regime as
\[ u(-\ell_1, t) = v(\ell_2, t) = 0 \quad t > 0, \] (16)
and in the presence of controls $p(t)$ and $q(t)$ by
\[ \begin{cases}
    u(-\ell_1, t) = p(t), & t > 0, \\
    v(\ell_2, t) = q(t), & t > 0.
\end{cases} \] (17)

3.2. Existence, uniqueness and regularity of solutions. In this part, we recall some results given in [18] concerning the existence, uniqueness and regularity of solutions for the string-mass system in the absence and presence of controls.

Define the Hilbert space $\mathcal{H} := \mathcal{H}_1 \times \mathcal{H}_2$, where
\[ \mathcal{H}_1 = \{(u, v, z) \in H^1_{(-\ell_1)}(\Omega_1) \times H^2_{(\ell_2)}(\Omega_2) : u(0) = v(0) = z\}, \]
\[ \mathcal{H}_2 = L^2(\Omega_1) \times L^2(\Omega_2) \times \mathbb{R}. \]

In terms of $y := (u, v, z, u_t, v_t, z_t)^t$, system (9)-(15) can be written as
\[ \frac{dy}{dt} = Ay, \quad y(0) = y^0 = (u^0, v^0, z^0, u^1, v^1, z^1)^t, \]
where $^t$ denotes the transposition.
It is shown in [18] that $A$ is skew-adjoint and $m$-dissipative on $\mathcal{H}$. Then the Lumer Phillips Theorem stated in [37] can be used to deduce that $A$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $\mathcal{H}$. Therefore we have the following existence and uniqueness result.

**Proposition 1.** (18)

1) For every $y^0 = (u^0, v^0, z^0, u^1, v^1, z^1) \in \mathcal{H}$ and every $T > 0$, there exists a unique solution of (9)-(16) in the class

$$(u, v, z) \in C((0, T); \mathcal{H}_1) \cap C^1((0, T); \mathcal{H}_2)$$

2) If $y^0 \in D(A) = \{y \in \mathcal{H} : u \in H^2(\Omega_1), v \in H^2(\Omega_2), (\dot{u}, \dot{v}, \dot{z}) \in \mathcal{H}_2\}$ then the corresponding solution satisfies the following additional regularity

$$(u, v, z) \in C((0, T); \mathcal{H}_1) \cap C^1((0, T); \mathcal{H}_2).$$

The next result states the existence and uniqueness of weak solutions when $L^2(0, T)$ Dirichlet controls (17) are introduced at the extreme points.

**Proposition 2.** For every $p, q \in L^2(0, T)$, $(u^0, v^0, z^0) \in \mathcal{H}_2$ and $(u^1, v^1, z^1)$ belonging to the dual space $\mathcal{H}_3 = \left(H^1_{(-\ell_1)}(\Omega_1)\right)' \times \left(H^1_{(\ell_2)}(\Omega_2)\right)' \times \mathbb{R}$, there exists a solution (in the sense of transposition) of (9)-(15), (17) in the class

$$(u, v, z) \in C((0, T); \mathcal{H}_2)$$

$$(u_t, v_t, z_t) \in C^1((0, T); \mathcal{H}_3).$$

The regularity of solutions where the initial data belong to a space where the regularity is not the same in each of strings was also studied and proved in [18] for both uncontrolled and controlled case.

### 3.3. System energy.

The energy of any physical system plays an important role in determining the asymptotic behavior of such system. For the string-mass system, this energy is given in [18] by

$$E_M(t) = \frac{1}{\ell_1} \int_{-\ell_1}^{0} \rho_1 |u_t(x, t)|^2 + \sigma_1 |u_x(x, t)|^2 \, dx$$

$$+ \frac{1}{\ell_2} \int_{0}^{\ell_2} \rho_2 |v_t(x, t)|^2 + \sigma_2 |v_x(x, t)|^2 \, dx$$

$$+ \frac{M}{2} |z_t(t)|^2.$$  

In order to build boundary or internal feedbacks such that the energy of these two vibrating strings connected by a point mass disappear in finite-time, we need to transform our system in Riemann invariant coordinates. This is the subject of the next subsection.

### 3.4. Riemann invariant transformation.

Our idea for the finite-time stabilization of this hybrid system is to pass through the explicit approach which is based on the characteristic method that provides the explicit solution. For that we write (9)-(10) as a first order hyperbolic system. Then, we use the so-called Riemann invariants [15] and we write our system in the new coordinates. Hence, we get four
transport equations coupled with the dynamic of the charge. More precisely, for the wave equation (9), these invariants are given by

\[ u_1 = \partial_t u - \sqrt{\frac{\sigma_1}{\rho_1}} \partial_x u, \]
\[ u_2 = \partial_t u + \sqrt{\frac{\sigma_1}{\rho_1}} \partial_x u. \]

Then, the equation (9) is equivalent to the following two transport equations

\[ \partial_t u_1 + \sqrt{\frac{\sigma_1}{\rho_1}} \partial_x u_1 = 0, \quad -\ell_1 \leq x \leq 0, \]
\[ \partial_t u_2 - \sqrt{\frac{\sigma_1}{\rho_1}} \partial_x u_2 = 0, \quad -\ell_1 \leq x \leq 0. \]

While the wave equation (10) is equivalent to

\[ \partial_t v_1 + \sqrt{\frac{\sigma_2}{\rho_2}} \partial_x v_1 = 0, \quad 0 \leq x \leq \ell_2, \]
\[ \partial_t v_2 - \sqrt{\frac{\sigma_2}{\rho_2}} \partial_x v_2 = 0, \quad 0 \leq x \leq \ell_2, \]

with \( v_1 \) and \( v_2 \) are the Riemann invariants given as follows

\[ v_1 = \partial_t v - \sqrt{\frac{\sigma_2}{\rho_2}} \partial_x v, \]
\[ v_2 = \partial_t v + \sqrt{\frac{\sigma_2}{\rho_2}} \partial_x v. \]

The main difficulty in this work follows from the point mass whose dynamic can change sensitively the topological nature (stability) of the solutions. Let us rewrite the dynamic of the point mass in terms of Riemann invariants. The equation (11) giving the continuity of the vibrating string at the point mass is

\[ 2z_t(t) = u_1(0, t) + u_2(0, t) = v_1(0, t) + v_2(0, t), \quad t > 0, \]

while the conservation equation (12) is converted to

\[ 2M z_{tt}(t) + \sqrt{\sigma_1 \rho_1} (u_2(0, t) - u_1(0, t)) - \sqrt{\sigma_2 \rho_2} (v_2(0, t) - v_1(0, t)) = 0, \quad t > 0. \]

For simplicity, we denote

\[ \lambda = \sqrt{\frac{\sigma_1}{\rho_1}} \quad \text{and} \quad \mu = \sqrt{\frac{\sigma_2}{\rho_2}}. \]

Again, the initial conditions (13)-(14) are given in function of Riemann coordinates as follows

\[ u_1^0(x) = u_1^1(x) - \lambda u_2^0(x), \]
\[ u_2^0(x) = u_1^1(x) + \lambda u_2^0(x), \]
\[ v_1^0(x) = v_1^1(x) - \mu v_2^0(x), \]
\[ v_2^0(x) = v_1^1(x) + \mu v_2^0(x). \]

and the initial condition (15) for the dynamic of the point mass is the same

\[ (z(0), z_t(0)) = (z^0, z^1). \]
The energy (20) in terms of Riemann invariants is

\[
E_R(t) = \frac{1}{4}\rho_1 \int_{-\ell_1}^{0} (u_1^2(x, t) + u_2^2(x, t))dx + \frac{1}{4}\rho_2 \int_{0}^{\ell_2} (u_1^2(x, t) + u_2^2(x, t))dx + \frac{M}{2}|z_t(t)|^2.
\]

4. Finite-time stabilization of the transformed system. Recall that our objective is to construct boundary or internal feedbacks vanishing the solutions in finite-time. However, the system trajectories can undergo some form of explosion before reaching this settling time. In order to avoid this undesirable phenomenon, we should consider the following assumptions:

For \(i = 1, 2\), there exists \(\delta > 0\) such that

\[|u_i(-\ell_1, 0)| < \delta, \quad \text{and} \quad |v_i(\ell_2, 0)| < \delta.\]

However before proceeding the analysis, it is interesting to recall the notions of partial finite-time (respectively, partial asymptotic) stabilization of the considered system. This is given in Appendix A. Then, we discuss different cases providing the dissipation of the system energy in finite-time, and we begin with.

4.1. Case of two boundary feedbacks. For the study of boundary finite-time stabilization, we come back to the system (23)-(33) that we complete with the so-called Bhat-Bernstein feedbacks [6]

\[\frac{du_1}{dt}(-\ell_1, t) = -k \text{sgn}(u_1(-\ell_1, t))|u_1(-\ell_1, t)|^\gamma, \quad (35)\]
\[\frac{dv_2}{dt}(\ell_2, t) = -k \text{sgn}(v_2(\ell_2, t))|v_2(\ell_2, t)|^\gamma, \quad (36)\]

where \((k, \gamma) \in (0, \infty) \times (0, 1)\) and “\(\text{sgn}\)” is the sign function defined by

\[\text{sgn}(x) = \frac{x}{|x|}, \quad x \neq 0 \quad \text{and} \quad \text{sgn}(0) = 0.\]

Let

\[T^* = \frac{1}{k(1 - \gamma)} \max \left(|u_1^0(-\ell_1)|, |v_2^0(\ell_2)|\right)^{1-\gamma}.\]

The next lemma presents the existence and uniqueness of solution for the closed-loop system (23)-(33), (35)-(36), and the proof is given in Appendix B.

**Lemma 4.1.** The closed-loop system (23)-(33) subject to controllers (35)-(36) is well-posed.

Now, we are ready to present the main result of boundary finite-time stabilization.

**Theorem 4.2.** Under the boundary feedback laws (35)-(36), system (23)-(33) is partially finite-time stable. More precisely:

\[u_1(., t) = v_2(., t) = 0 \quad \forall t \geq T^*,\]

and if
\textbf{a)} \( \frac{\rho_2}{\rho_1} = \frac{\sigma_1}{\sigma_2} \), then

\[ u_2(., t) = v_1(., t) = 0 \quad \forall \ t \geq T^*, \]

and there exists \( c \in \mathbb{R} \), such that

\[ z(t) = c, \quad \forall \ t \geq T^*. \]

\textbf{b)} \( \frac{\rho_2}{\rho_1} < \frac{\sigma_1}{\sigma_2} \), then

\[ \lim_{t \to +\infty} (u_2, v_1)(., t) = (0, 0) \text{ and } z(t) \text{ converges.} \]

\textbf{Proof.} To show the partial finite-time stabilization of our transformed system, we divide the proof in two steps.

\textbf{Step 1.} The first task consists on determining the settling time for which Riemann invariants should be vanished.

By an integration of the differential equation (35), a simple calculation (see e.g. [7, 27]) gives

\[ u_1(-\ell_1, t) = \begin{cases} 
\text{sgn}(u_1^0(-\ell_1))(|u_1^0(-\ell_1)|^{1-\gamma} - k(1-\gamma)t)^{\frac{1}{1-\gamma}}, & \text{if } 0 \leq t \leq T_{u_1}, \\
0, & \text{if } t \geq T_{u_1},
\end{cases} \]  

where

\[ T_{u_1} = \frac{|u_1^0(-\ell_1)|^{1-\gamma}}{k(1-\gamma)}. \]

Then, using (5)

\[ u_1(x, t) = \begin{cases} 
 u_1(-\ell_1, t - \frac{x + \ell_1}{\lambda}), & \text{if } x - \lambda t < -\ell_1, \\
 u_1^0(x - \lambda t), & \text{if } x - \lambda t \geq -\ell_1.
\end{cases} \]

Combine (37) and (38) we get

\[ u_1(x, t) = 0, \quad \forall \ t - \frac{x + \ell_1}{\lambda} \geq T_{u_1}, \]

since \( \frac{x + \ell_1}{\lambda} > 0 \), we deduce that

\[ u_1(x, t) = 0 \quad \forall \ t \geq T_{u_1}. \]

Let us now move to the feedback (36) applied at the extremity \( \ell_2 \), which gives with respect to time

\[ v_2(\ell_2, t) = \begin{cases} 
\text{sgn}(v_2^0(\ell_2))(|v_2^0(\ell_2)|^{1-\gamma} - k(1-\gamma)t)^{\frac{1}{1-\gamma}}, & \text{if } 0 \leq t \leq T_{v_2}, \\
0, & \text{if } t \geq T_{v_2},
\end{cases} \]

where

\[ T_{v_2} = \frac{|v_2^0(\ell_2)|^{1-\gamma}}{k(1-\gamma)}, \]
then, \(v_2\) solves
\[
\partial_t v_2 - \mu \partial_x v_2 = 0, \quad 0 \leq x \leq \ell_2,
\]
\[
v_2(\ell_2, t) = 0, \quad t \geq T_{v_2},
\]
\[
v_2(., 0) = v_2^0(x).
\]
Similarly, from (6)
\[
v_2(x, t) = \begin{cases} v_2(\ell_2, t + \frac{x - \ell_2}{\mu}), & \text{if } x + \mu t > \ell_2, \\ v_2^0(x + \mu t), & \text{if } x + \mu t \leq \ell_2. \end{cases} \tag{40}
\]
Combined with (39), this yields
\[
v_2(x, t) = 0, \quad \forall t + \frac{x - \ell_2}{\mu} \geq T_{v_2},
\]

since \(\frac{x - \ell_2}{\mu} < 0\), we deduce that
\[
v_2(x, t) = 0 \quad \forall t \geq T_{v_2}.
\]

If we consider
\[
T^* = \max(T_{u_1}, T_{v_2}),
\]
then we get
\[
u_1(x, t) = v_2(x, t) = 0 \quad \forall t \geq T^*. \tag{41}
\]

**Step 2.** This step consists on the study of the dynamic of the system after the settling time \(T^*\). From (27)-(28) and (41), the motion of the point mass at \(x = 0\) is modified as follows
\[
2z_{tt}(t) = u_2(0, t) = v_1(0, t), \quad \forall t \geq T^*, \tag{42}
\]
\[
2Mz_{tt}(t) + \sqrt{\rho_1 \sigma_1} u_2(0, t) - \sqrt{\rho_2 \sigma_2} v_1(0, t) = 0, \quad \forall t \geq T^*. \tag{43}
\]
Clearly, for \(t\) is large enough, (43) becomes
\[
Mz_{tt}(t) + (\sqrt{\rho_1 \sigma_1} - \sqrt{\rho_2 \sigma_2}) z_t(t) = 0, \quad \forall t \geq T^*. \tag{44}
\]
The solution of the equation (44) depends on the discriminant
\[
\Delta = (\sqrt{\rho_1 \sigma_1} - \sqrt{\rho_2 \sigma_2})^2.
\]
Two cases are present
\[
\Delta = 0 \quad \text{or} \quad \Delta > 0.
\]

**a) Case 1.**
\[
\Delta = 0, \quad \text{i.e.} \quad \frac{\rho_2}{\rho_1} = \frac{\sigma_1}{\sigma_2}, \tag{45}
\]
then, the solution of (44) is constant in finite-time
\[
z(t) = q, \quad \forall t \geq T^*,
\]
with \(q\) is a constant, and therefore we get
\[
z(t) := z_t(t) = 0, \quad \forall t \geq T^*.
\]
Returning to (42), we deduce that
\[
u_2(0, t) = 0 \quad \forall t \geq T^*, \tag{46}
\]
\[
v_1(0, t) = 0 \quad \forall t \geq T^*. \tag{47}
\]
Then, $u_2$ solves
\[
\begin{align*}
\partial_t u_2 - \lambda \partial_x u_2 &= 0, \quad -\ell_1 \leq x \leq 0, \\
u_2(0, t) &= 0, \quad \forall t \geq T^*, \\
u_2(., 0) &= u_0^2(x).
\end{align*}
\]
From (6), the explicit solution of $u_2$ is given by
\[
u_2(x, t) = \begin{cases} u_2(0, t + \frac{x}{\lambda}), & \text{if } x + \lambda t > 0, \\
u_0^2(x + \lambda t), & \text{if } x + \lambda t \leq 0. \end{cases}
\tag{48}
\]
Combined with (46), yields
\[
u_2(x, t) = 0, \quad \forall t + \frac{x}{\lambda} \geq T^*.
\tag{49}
\]
Again, from (47) $v_1$ solves
\[
\begin{align*}
\partial_t v_1 + \mu \partial_x v_1 &= 0, \quad 0 < x < \ell_2, \\
v_1(0, t) &= 0, \quad \forall t \geq T^*, \\
v_1(., 0) &= v_0^1(x).
\end{align*}
\]
The explicit solution is
\[
v_1(x, t) = \begin{cases} v_1(0, t - \frac{x}{\mu}), & \text{if } x - \mu t < 0, \\
v_0^1(x - \mu t), & \text{if } x - \mu t \geq 0. \end{cases}
\tag{50}
\]
and clearly
\[
v_1(x, t) = 0 \quad \forall t - \frac{x}{\mu} \geq T^*,
\]
since $\frac{x}{\mu} > 0$, we deduce that
\[
v_1(x, t) = 0 \quad \forall t \geq T^*.
\tag{51}
\]
Gathering together (41), (49), (51), we infer that the energy (34) is dissipated. Moreover
\[
E_R(t) = 0 \quad \forall t \geq T^*.
\]
Thus, under the condition (45), the closed loop string-mass system is finite-time stable.

b) Case 2.

\[
\Delta > 0, \quad \text{i.e.} \quad \frac{\rho_2}{\rho_1} \leq \frac{\sigma_1}{\sigma_2},
\tag{52}
\]
the solution of (44) is expressed as follows
\[
z(t) = q_1 \exp\left(-\sqrt{\frac{\rho_1 \sigma_1 - \sqrt{\rho_2 \sigma_2}}{M}} t\right) + q_2, \quad \forall t \geq T^*,
\tag{53}
\]
where $q_1$ and $q_2$ are two constants. Then
\[
z(t) \rightarrow q_2, \quad \text{as } t \rightarrow +\infty.
\]
By taking the time derivative of the solution $z(.)$ in (53), we get

$$\dot{z}(t) = z_t(t) \to 0, \quad \text{as } t \to +\infty.$$  

Now, using (42) we deduce that

$$u_2(0, t) \to 0, \quad \text{as } t \to +\infty, \quad (54)$$

$$v_1(0, t) \to 0, \quad \text{as } t \to +\infty. \quad (55)$$

In addition, by the same aspect we show that

$$u_2(x, t) \to 0 \text{ as } t \to +\infty, \quad (56)$$

$$v_1(x, t) \to 0 \text{ as } t \to +\infty. \quad (57)$$

The finite-time stability (41) of $u_1$ and $v_2$ leads to the asymptotic stability (56)-(57) of $u_2$ and $v_1$. Hence, hypothesis of Lemma 2.3 are satisfied, and we conclude that under the condition (52) the string-mass system is asymptotically stable.

This completes the proof. \(\square\)

**Remark 1.** Clearly, if $\frac{\rho_2}{\rho_1} > \frac{\sigma_1}{\sigma_2}$, then $z$ diverges.

### 4.2. Feedbacks at the point mass.

For this case, three strategies are developed.

**Strategy 1:** We will act from one side while the other extremity will be free from any control. For example, assuming that the end $-\ell_1$ will be acted and $\ell_2$ be free. We have then this boundary condition at the extremity $\ell_2$

$$v_1(\ell_2, t) = -v_2(\ell_2, t), \quad t \geq 0, \quad (58)$$

with the following feedback law at the extremity $-\ell_1$

$$\frac{du_1(-\ell_1, t)}{dt} = -k \text{sgn}(u_1(-\ell_1, t))|u_1(-\ell_1, t)|^\gamma, \quad (59)$$

with $(k, \gamma) \in (0, \infty) \times (0, 1)$.

However, this will be insufficient to reach the finite-time stabilization of the string-mass system. For that, another feedback law should be added and applied at the point mass. Then, we act by the following Bhat-Bernstein feedback form

$$\frac{dz(t)}{dt} = -k \text{sgn}(z(t))|z(t)|^\gamma. \quad (60)$$

Consider the system (23)-(33) that we complete with (58)-(60).

Let

$$T^* = \frac{1}{(1-\gamma)k} \max \left( |u_1^0(-\ell_1)|, |z^0| \right) ^{1-\gamma}.$$  

The next theorem states the main result of this part.

**Theorem 4.3.** Under the feedback laws (59)-(60) and satisfying the boundary condition (58), the system (23)-(33) is finite-time stable. Furthermore, the solution satisfies for $i = 1, 2$

$$u_i(., t) = v_i(., t) = z(t) = 0, \quad \forall t \geq T^*.$$  

**Proof.** It is similar to those of Theorem 4.2.

By an integration over time of the feedback (60), we get

$$z(t) = \begin{cases} 
\text{sgn}(z^0)(|z^0|^{1-\gamma} - (1-\gamma)kt)^{\frac{1}{1-\gamma}}, & \text{if } 0 \leq t \leq T_z, \\
0, & \text{if } t \geq T_z,
\end{cases}$$
with
\[ T_z = \frac{|z^0|^{1-\gamma}}{(1-\gamma)k}, \]
clearly that
\[ z(t) = 0, \quad \forall \, t \geq T_z. \] (61)
Gathering together (27)-(28) and (61), we infer that
\[ u_1(0, t) + u_2(0, t) = 0, \quad \forall \, t \geq T_z, \] (62)
\[ v_1(0, t) + v_2(0, t) = 0, \quad \forall \, t \geq T_z, \] (63)
\[ u_2(0, t) = -\sqrt{\frac{\sigma_2 \rho_2}{\sigma_1 \rho_1}} v_1(0, t), \quad \forall \, t \geq T_z. \] (64)
It is already shown that under the feedback law (59) we have
\[ u_1(x, t) = 0, \quad \forall \, t \geq T_{u_1} = \frac{|u_1^0(-\ell_1)|^{1-\gamma}}{k(1-\gamma)}. \]

Thus
\[ u_1(x, t) = z(t) = 0, \quad \forall \, t \geq T^* := \max(T_{u_1}, T_z). \] (65)
Combined with (62), we have
\[ u_2(0, t) = 0, \quad \forall \, t \geq T^*. \]
Using (48), we get
\[ u_2(x, t) = 0, \quad \forall \, t + \frac{x}{\lambda} \geq T^*, \]
since \( \frac{x}{\lambda} < 0 \), we deduce that
\[ v_2(x, t) = 0, \quad \forall \, t \geq T^*. \] (66)
Returning to the relation (64) between \( u_2 \) and \( v_1 \), it is easy to see that
\[ v_1(0, t) = 0, \quad \forall \, t \geq T^*. \]
Again, by the explicit expression (50), we infer that
\[ v_1(x, t) = 0, \quad \forall \, t - \frac{x}{\mu} \geq T^*, \]
since \( \frac{x}{\mu} > 0 \), then
\[ v_1(x, t) = 0, \quad \forall \, t \geq T^*. \] (67)
Gathering together (58) and (40), we have
\[ v_2(x, t) = 0, \quad \forall \, t + \frac{x - \ell_2}{\mu} \geq T^*. \]
since \( \frac{x - \ell_2}{\mu} < 0 \), then
\[ v_2(x, t) = 0 \quad \forall \, t \geq T^*. \] (68)
Finally, from (65)-(68), we conclude that the energy (34) of the system satisfies
\[ E_R(t) = 0, \quad \forall \, t \geq T^*. \]
Hence, the system is finite-time stable with a settling time \( T^* \). \( \square \)
Strategy 2: In this strategy, we will act at the point mass and one of the extremities. We can complete the system (23)-(33) either with the finite-time stabilizing feedback laws

\[
\begin{align*}
\frac{d}{dt} u_1(-\ell_1, t) &= -k \text{sgn}(u_1(-\ell_1, t))|u_1(-\ell_1, t)|^\gamma, \\
\frac{d}{dt} v_1(0, t) &= -k \text{sgn}(v_1(0, t))|v_1(0, t)|^\gamma,
\end{align*}
\]

(69)

and the boundary condition

\[v_1(\ell_2, t) = -v_2(\ell_2, t),\]

(70)

or with the feedback laws

\[
\begin{align*}
\frac{d}{dt} v_2(\ell_2, t) &= -k \text{sgn}(v_2(\ell_2, t))|v_2(\ell_2, t)|^\gamma, \\
\frac{d}{dt} u_2(0, t) &= -k \text{sgn}(u_2(0, t))|u_2(0, t)|^\gamma,
\end{align*}
\]

(71)

and the boundary

\[u_1(-\ell_1, t) = -u_2(-\ell_1, t).\]

(72)

Then, we have

**Corollary 1.** Under the feedback laws (69) and boundary condition (70), system (23)-(33) is partial finite-time stable with settling time

\[T^\ast = \frac{1}{(1-\gamma)k} \max \left(|u_1^0(-\ell_1)|, |v_1^0(0)|\right)^{1-\gamma}.\]

The same conclusion is obtained if we apply feedbacks (71) and boundary condition (72) with settling time

\[T^\ast = \frac{1}{(1-\gamma)k} \max \left(|v_2^0(\ell_2)|, |u_2^0(0)|\right)^{1-\gamma}.\]

Moreover, for \(i = 1, 2\)

\[u_i(., t) = v_i(., t) = 0, \quad \forall t \geq T^\ast,\]

and there exists \(c \in \mathbb{R}\), such that

\[z(t) = c, \quad \forall t \geq T^\ast.\]

**Proof.** We will give the proof only for the first point.

Thanks to the proof of Theorem 4.3 we have

\[u_1(x, t) = 0 \quad \forall t \geq T_{u_1}.\]

In addition, the solution \(v_1\) at position \(x = 0\) is

\[v_1(0, t) = \begin{cases} 
\text{sgn}(v_1^0(0))(|v_1^0(0)|^{1-\gamma} - (1-\gamma)kt)^{\frac{1}{1-\gamma}} & \text{if } 0 \leq t \leq T_{v_1}, \\
0, & \text{if } t \geq T_{v_1},
\end{cases}\]

(73)

with

\[T_{v_1} = \frac{|v_1^0(0)|^{1-\gamma}}{(1-\gamma)k}.\]
Combining (50) with (73), we infer that
\[ v_1(x, t) = 0 \quad \forall t \geq T_v. \]
Then,
\[ u_1(x, t) = v_1(x, t) = 0 \quad \forall t \geq T^* := \max(T_1, T_2). \] (74)
Using (70) together with (74) and (40), we obtain
\[ v_2(x, t) = 0 \quad \forall t \geq T^*. \] (75)
By the continuity condition (27) and the explicit formula (48) we get
\[ u_2(x, t) = 0 \quad \forall t \geq T^*, \]
\[ z_r(t) = 0 \quad \forall t \geq T^*. \]
Hence, we infer the existence of some constant \( c \in \mathbb{R} \) such that
\[ z(t) = c \quad \forall t \geq T^*. \] (77)
Gathering together (74)-(77), we deduce that the energy is dissipated in finite-time. Moreover
\[ E_R(t) = 0 \quad \forall t \geq T^*. \]

**Strategy 3:** In this case, the finite-time stabilization will be reached when acting only at \( x = 0 \) while the two extremities \(-\ell_1, \ell_2\) of the mass-string are free. This is traduced by the boundary conditions
\[ u_1(-\ell_1, t) = -u_2(-\ell_1, t), \]
\[ v_1(\ell_2, t) = -v_2(\ell_2, t). \] (79)
Now, we consider system (23)-(33) under the free boundary conditions (78)-(79) and the feedback law at the point mass
\[ \frac{d}{dt}z(t) = -k \text{sgn}(z(t))|z(t)|^\gamma. \] (60)

**Remark 2.** Applying just the feedback law (60) is not sufficient. It is necessary to add an other feedback at the point mass \( M \).

We deal with internal controllers. So, with (60) we complete the string-mass system by the feedback
\[ \frac{d}{dt}u_2(0, t) = -k \text{sgn}(u_2(0, t))|u_2(0, t)|^\gamma, \] (80)
or
\[ \frac{d}{dt}v_1(0, t) = -k \text{sgn}(v_1(0, t))|v_1(0, t)|^\gamma, \] (81)
where \((k, \gamma) \in (0, \infty) \times (0, 1)\).

The next theorem states the finite-time stabilization result for the case when internal feedbacks are applied.

**Theorem 4.4.** The system (23)-(33) subject to boundary conditions (78)-(79) and controllers (60), (80) is finite-time stable with settling time
\[ T^* = \frac{1}{(1-\gamma)k} \max\left(|z_0^0|, |u_2^0(0)|\right)^{1-\gamma}. \]
The same result is obtained if the controller (80) is replaced by (81). In this case, the settling time is

\[ T^* = \frac{1}{(1 - \gamma)k} \max \left( |z^0|, |v_1^0(0)| \right)^{1-\gamma}. \]

Furthermore, for \( i = 1, 2 \),

\[ u_i(., t) = v_i(., t) = z(t) = 0, \quad \forall t \geq T^*. \]

**Proof.** We deal with the choice of feedback laws (60) and (81). The proof is similar for the choice of the feedback (60) and (80).

From the proof of Theorem 4.3, the feedback (60) provides (62)-(64), and from the proof of Corollary 1 we have

\[ v_1(x, t) = 0, \quad \forall t \geq T_{v_1} = \frac{|v_1^0(0)|^{1-\gamma}}{(1 - \gamma)k}. \]

Then

\[ z(t) = v_1(x, t) = 0, \quad \forall t \geq T^* := \max(T_z, T_{v_1}). \quad (82) \]

Now, using (64) and (48), we get

\[ u_2(x, t) = 0, \quad \forall t \geq T^*. \quad (83) \]

From the boundary condition (78) combined with (38), we have

\[ u_1(x, t) = 0, \quad \forall t \geq T^*. \quad (84) \]

Similarly, from the boundary (79) combined with (82) and (40), we get

\[ v_2(x, t) = 0, \quad \forall t \geq T^*. \quad (85) \]

Consequently, from the above analysis, the energy (34) of the system is dissipated in finite-time and therefore the closed loop system describing the vibrating string is finite-time stable by the feedback laws (60) and (81).

The finite-time stabilization by internal feedbacks can be reached even when eliminating the feedback (60).

Let

\[ T^* = \frac{1}{(1 - \gamma)k} \max \left( |u_2^0(0)|, |v_1^0(0)| \right)^{1-\gamma}. \]

This is confirmed by the following result.

**Corollary 2.** Under the feedback laws (80)-(81), the system (23)-(33) satisfying the boundary condition (78)-(79) is partial finite-time stable.

Moreover, for \( i = 1, 2 \)

\[ u_i(., t) = v_i(., t) = 0, \quad \forall t \geq T^*, \]

and for some number \( c \in \mathbb{R} \)

\[ z(t) = c, \quad \forall t \geq T^*. \]
4.3. Case of acting only at one of strings. In this part, we prove that the finite-time stabilization is reached when acting at one of strings while the other string supposed to be free. Assume that the function \( z \) is free (i.e. no feedbacks acting at the position \( x = 0 \)).

Two choices are present.

i) Acting at the first string \( (x \in \Omega_1) \) by the two feedbacks

\[
\frac{d}{dt} u_1(-\ell_1, t) = -k \text{sgn}(u_1(-\ell_1, t))|u_1(-\ell_1, t)|^\gamma, \\
\frac{d}{dt} u_2(0, t) = -k \text{sgn}(u_2(0, t))|u_2(0, t)|^\gamma,
\]

with the free boundary condition

\[
v_1(\ell_2, t) = -v_2(\ell_2, t),
\]

ii) Acting at the second string \( (x \in \Omega_2) \) by the two feedbacks

\[
\frac{d}{dt} v_1(0, t) = -k \text{sgn}(v_1(0, t))|v_1(0, t)|^\gamma, \\
\frac{d}{dt} v_2(\ell_2, t) = -k \text{sgn}(v_2(\ell_2, t))|v_2(\ell_2, t)|^\gamma,
\]

with the boundary condition

\[
u_1(-\ell_1, t) = -u_2(-\ell_1, t).
\]

The same result is obtained for i) and ii). Assume that the first string is selected, and let

\[
T^* = \frac{1}{(1 - \gamma)k} \max \left( |u_1^0(-\ell_1)|, |u_2^0(0)| \right)^{1-\gamma}.
\]

Then we have the following result.

**Corollary 3.** Acting with the feedback laws (35) and (80), system (23)-(33) subject to boundary condition (79) is partial finite-time stable.

More precisely, we have for \( i = 1, 2 \)

\[
u_i(\cdot, t) = v_i(\cdot, t) = 0, \quad \forall t \geq T^*.
\]

and there exists \( c \in \mathbb{R} \), such that

\[
z(t) = c, \quad \forall t \geq T^*.
\]

**Proof.** Again by the same principle, the result is easily established.

5. Comments and conclusion.

5.1. Comparison between results. Let us return to the initial system (9)-(15) to see what happens to the solutions \((u,v)\) of the wave equations in each case. Firstly, recall that these solutions are expressed in terms of Riemann invariants as follows

\[
\begin{align*}
\partial_t u &= \frac{1}{2}(u_1 + u_2), \\
\partial_x u &= \frac{1}{2\lambda}(u_2 - u_1),
\end{align*}
\]

(86)
and
\[
\begin{aligned}
\partial_t v &= \frac{1}{2}(v_1 + v_2), \\
\partial_x v &= \frac{1}{2\mu}(v_2 - v_1).
\end{aligned}
\] (87)

Then, we have the following discussion:

**Case 1. Acting at the two boundaries**

The result is given by Theorem 4.2 where the finite-time convergence of \( u_1 \) and \( v_2 \) is assured

\[ u_1(x, t) = v_2(x, t) = 0 \quad \forall \ t \geq T^*, \]

then, the nature of stability of the system depends on the following two conditions:

**a)** If \( \frac{\rho_2}{\rho_1} = \frac{\sigma_1}{\sigma_2} \), we have

\[ u_2(x, t) = v_1(x, t) = 0 \quad \forall \ t \geq T^*, \]

and

\[ z(t) = q, \quad \forall \ t \geq T^*, \] with \( q \) is a constant.

Thus, from (86)-(87) we deduce the existence of some constants \( c_1, c_2 \in \mathbb{R} \) depending on the initial conditions such that

\[ u(x, t) = c_1 \quad \text{and} \quad v(x, t) = c_2. \]

Using the continuity relation (11), we conclude that

\[ c_1 = c_2 = q. \]

Hence,

\[ u(x, t) = v(x, t) = z(t) = q \quad \forall \ t \geq T^*. \]

**b)** If \( \frac{\rho_2}{\rho_1} < \frac{\sigma_1}{\sigma_2} \) we have

\[ u_2(x, t) \to 0 \quad \text{as} \ t \to +\infty, \]

\[ v_1(x, t) \to 0 \quad \text{as} \ t \to +\infty. \]

Then, from (86)-(87), we deduce the existence of some constants \( c_1, c_2 \in \mathbb{R} \) depending on the initial conditions such that

\[
\begin{aligned}
u(x, t) &\to c_1 \quad \text{as} \ t \to +\infty, \\
v(x, t) &\to c_2 \quad \text{as} \ t \to +\infty.
\end{aligned}
\]

Since there exists a constant \( q_2 \) such that

\[ z(t) \to q_2 \quad \text{as} \ t \to +\infty, \]

again by (11), we deduce that

\[ c_1 = c_2 = q_2. \]

Thus,

\( (u, v, z) \to (q_2, q_2, q_2) \quad \text{as} \ t \to +\infty. \)

**Case 2. Acting at the charge and one of boundaries**

We have two results of finite-time stabilization:
1) Theorem 4.3 provides for $i = 1, 2$

\[ u_i(x, t) = v_i(x, t) = z(t) = 0 \quad \forall t \geq T^*. \]

Again, by (86)-(87) combined with the continuity relation (11), we deduce that

\[ u(x, t) = v(x, t) = z(t) = 0 \quad \forall t \geq T^*. \]

2) Corollary 1 provides the existence of $c \in \mathbb{R}$ such that

\[ z(t) = c \quad \forall t \geq T^*, \]

and for $i = 1, 2$

\[ u_i(x, t) = v_i(x, t) = 0 \quad \forall t \geq T^*. \]

Similarly, from (86)-(87) and (11) we get

\[ u(x, t) = v(x, t) = z(t) = c \quad \forall t \geq T^*. \]

Case 3. Acting only at the point mass

At this case, using (86)-(87) and the continuity relation (11) combined with the result of

1) Theorem 4.4, we get

\[ u(x, t) = v(x, t) = z(t) = 0 \quad \forall t \geq T^*. \]

2) Corollary 2, we have

\[ u(x, t) = v(x, t) = z(t) = c \quad \forall t \geq T^*, \]

with $c \in \mathbb{R}$.

Case 4. Acting at one of strings

The result is given by Corollary 3. Similarly to other cases, we prove easily that there exists $c \in \mathbb{R}$ depending on the initial conditions such that

\[ u(x, t) = v(x, t) = z(t) = c \quad \forall t \geq T^*. \]

Remark 3. Instead of using homogeneous continuous stabilizing feedbacks, we can act by discontinuous feedback laws. In this case, solutions of the ordinary differential equation are defined in Filippov sense [36]. For example, we choose $u_2(0, t)$ such that

\[ \frac{d}{dt} u_2(0, t) = -k \text{sgn}(u_2(0, t)), \quad k \in (0, \infty), \]

and in order to avoid the trouble of discontinuity at $x = 0$, we assume that for $t = 0$ and $\eta > 0$ small enough

\[ |u_2^0(0)| < \eta, \]

which provides that

\[ \int_0^\infty |u_2(0, t)|^2 dt < \infty. \]

5.2. Conclusion and future work. In this paper, under the feedbacks that we have developed, we arrived to avoid the bursting and the oscillations of strings by ensuring its stability either complete or partial in finite-time. Different cases are investigated and we have shown that the presence of the point mass is crucial in the stabilization in finite-time of these strings.

Several questions under consideration, for example:

Can we extend our results for a network of $N$ strings connected with $N - 1$ masses?
Appendix A. Partial stabilization.

In this Appendix we will give the definitions of partial finite-time (respectively, partial asymptotic) stabilization of system (23)-(33).

Definition A.1. The system (23)-(33) is said to be finite-time partially stabilizable if there exists an “artificial” continuous feedback law, for example as defined in (35-36), such that if

\[ \exists \delta > 0 \left| u_i(-\ell_1, 0) \right| + \left| v_i(\ell_2, 0) \right| + \left| z^0 \right| < \delta, \quad i = 1, 2, \]

then there exists \( T > 0 \) (function of initial data) such that

\[ u_i(., t) = v_i(., t) = 0 \quad \forall t \geq T. \]
and \( z(t) \) is constant for \( t \) is large enough.

Definition A.2. The system (23)-(33) is said to be partially asymptotically stabilizable if there exists an “artificial” continuous feedback law such that if

\[ \exists \delta > 0 \left| u_i(-\ell_1, 0) \right| + \left| v_i(\ell_2, 0) \right| + \left| z^0 \right| < \delta, \quad i = 1, 2, \]

then

i) \( (u_i, v_i)(., t), z(t) \) is Lyapunov stable,

ii) \( \lim_{t \to +\infty} (u_i, v_i)(., t) = (0, 0) \) and \( z(t) \) converges.

Appendix B. Well-posedness.

Proof of Lemma 4.1. For the proof of this lemma, we will move to use the wave equations coordinates combined with the charge dynamic in closed-loop. Therefore, we begin with the transformation of our feedbacks (by Riemann invariants) in function of the wave equations solutions.

From (21) and (29), system (37) becomes

\[ \partial_t u_i(-\ell_1, t) = \begin{cases} \lambda \partial_x u_i(-\ell_1, t) + g(u_0^i(-\ell_1), t), & \text{if } 0 < t \leq T^*, \\ \lambda \partial_x v_i(-\ell_1, t), & \text{if } t \geq T^*, \end{cases} \]

where \( g \) is the continuous function defined by

\[ g(u_0^i(-\ell_1), t) = sgn(u_0^i(-\ell_1)) \left( \left| u_0^i(-\ell_1) \right|^{1-\gamma} - k(1-\gamma)t \right)^{\frac{1}{1-\gamma}}, \]

with

\[ u_0^i(-\ell_1) = u^i(-\ell_1) - \lambda u_0^x(-\ell_1). \]

In addition, using the relation

\[ v_2 = \partial_t v + \sqrt{\frac{\sigma_2}{\rho_2}} \partial_x v \]
combined with (32), then system (39) becomes

\[ \partial_t v_i(\ell_2, t) = \begin{cases} -\mu \partial_x v_i(\ell_2, t) + h(v_0^i(\ell_2), t), & \text{if } 0 < t \leq T^*, \\ -\mu \partial_x v_i(\ell_2, t), & \text{if } t \geq T^*, \end{cases} \]

where \( h \) is the following continuous function

\[ h(v_0^i(\ell_2), t) = sgn(v_0^i(\ell_2)) \left( \left| v_2^0(\ell_2) \right|^{1-\gamma} - k(1-\gamma)t \right)^{\frac{1}{1-\gamma}}, \]

with

\[ v_2^0(\ell_2) = v^i(\ell_2) + \mu v_0^i(\ell_2). \]
Integrate (88), respectively (89) over $[0, T^*]$, we get
\[
    u(-\ell_1, t) = \lambda \int_0^t \partial_x u(-\ell_1, s) ds + \int_0^t g(u_1^0(-\ell_1), s) ds,
\]
and
\[
    v(\ell_2, t) = \lambda \int_0^t \partial_x v(\ell_2, s) ds + \int_0^t h(v_1^0(-\ell_1), s) ds,
\]
then, for $t = 0$, we have
\[
    u(-\ell_1, 0) = v(\ell_2, 0) = 0.
\]
Clearly, the action of our finite-time controllers leads to the system (9)-(15), (88)-(89) which is similar to that presented in [18, pp. 30]. Hence, with judicious choice of initial data $y^0 = (u^0, v^0, z^0, u^1, v^1, z^1)^T \in H^1(\Omega_1) \times H^1(\Omega_2) \times \mathbb{R} \times L^2(\Omega_1) \times L^2(\Omega_2) \times \mathbb{R}$ such that $u^0(0) = v^0(0) = z^0$, and by the standard semigroup theory [18], the problem (9)-(15) satisfying (88) and (89) is well-posed.

**Remark 4.** By the same way, we prove that the string-mass system is well-posed for several choices of feedback laws applied along this paper.

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