A New Graded Algebra Structure on Differential Polynomials: Level Grading and its Application to the Classification of Scalar Evolution Equations in 1 + 1 Dimension

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Abstract

We define a new grading, that we call the “level grading”, on the algebra of polynomials generated by the derivatives \( u_{k+i} = \partial^{k+i}u / \partial x^{k+i} \) over the ring \( K^{(k)} \) of \( C^\infty \) functions of \( u, u_1, \ldots, u_k \). This grading has the property that the total derivative and the integration by parts with respect to \( x \) are filtered algebra maps. In addition, if \( u \) satisfies an evolution equation \( u_t = F[u] \) and \( F \) is a level homogeneous differential polynomial, then the total derivative with respect to \( t \), \( D_t \), is also a filtered algebra map. Furthermore if \( \rho \) is level homogeneous over \( K^{(k)} \), then the top level part of \( D_t \rho \) depends on \( u_k \) only. This property allows to determine the dependency of \( F[u] \) on \( u_k \) from the top level part of the conserved density conditions. We apply this structure to the classification of “level homogeneous” scalar evolution equations and we obtain the top level parts of integrable evolution equations of “KdV-type”, admitting an unbroken sequence of conserved densities at orders \( m = 5, 7, 9, 11, 13, 15 \).
1 Introduction

The classification of evolution equations has been a long standing problem in the literature on evolution equations. The existence of higher symmetries, an infinite sequence of conserved densities, of a recursion operator, the Painleve property of reduced equations have been proposed as integrability tests. Among these, we follow the “formal symmetry” method of Mikhailov-Shabat-Sokolov (MSS) [3], which is based on the remark that if the evolution equation admits a recursion operator than, its expansion as a pseudo-differential should satisfy an operator equation. The solvability of the coefficients of this pseudo-differential operator in the class of local functions necessitates that certain quantities be conserved. These quantities are called “canonical densities” and their existence is proposed as an integrability test [3].

In [3], a preliminary classification of third order equations has been obtained by the formal symmetry method. The classification given in [3] asserts the existence of a three classes of candidates for evolution equations, one class being quasi-linear, two of these being “essentially nonlinear”.

We recall that the KdV hierarchy consists of the symmetries of the third order KdV equation at every odd order. The KdV hierarchy is characterized by the existence of conserved densities that are quadratic in the highest derivative at any order. At the fifth order, there are two basic hierarchies that start at this order, i.e., that are not symmetries of third order equations. These are the Sawada-Kotera and Kaup hierarchies that are derived from a third order Lax operator. Their symmetries give integrable equations at odd orders that are not divisible by 3. Similarly, every third order conserved density is trivial. The equations that are related to these three hierarchies via Miura type transformations may have quite different appearance and form a large list.

In the following years, the search for new hierarchies of integrable equations starting at higher orders turned out to be fruitless; the situation was clarified by Wang and Sanders, who proved that scale homogeneous scalar integrable evolution equations of orders greater than or equal to seven are symmetries of lower order equations [1]. In subsequent papers these results were extended to the cases where negative powers are involved [2] but the case where $F$ is arbitrary remained open.

The general case where the functional form of $F$ is arbitrary was studied in the references [6] and [9]. The first result in this direction has been obtained in [6], where the canonical densities $\rho^{(i)}$, $i = 1, 2, 3$ were computed for evolution equations of arbitrary order $m$. It was first proved that, up to total derivatives, higher order ($m \geq 7$) conserved densities are at most quadratic in the highest derivative. Then assuming that an evolution equation $u_t = F(x, t, u, \ldots, u_m)$ admits a conserved density $\rho^{(1)} = Pu^2_{m+1} + Qu_{m+1} + R$, where $P, Q, R$ are functions independent of $u_{m+1}$ it has been shown that for $m \geq 7$, $PF_{mm} = 0$ [6]. Finally, it was shown that the coefficient $P$ in the canonical density $\rho^{(1)}$, has the form $P = F_m$ [6], hence it was concluded that evolution equations of order $m \geq 7$ that admit the canonical density $\rho^{(1)}$ are quasi-linear. In the proofs of these results, the remarkable was the fact that the explicit form of the conserved densities
was needed only at the last stage, to prove that $P F_{mm} = 0$ implies $F_{mm} = 0$. Furthermore the derivations used the dependency of $F$ and $P$ on $u_m$ only and the functions $Q$ and $R$ never appeared in the computations.

In [9] the same scheme was applied to the quasilinear equations and it was proved that if the canonical densities $\rho^{(i)}$, $i = 1, 2, 3$ are conserved then the evolution equation has to be polynomial in the derivatives $u_{m-1}$ and $u_{m-2}$. In the derivation of these results, in many places we had first assumed the existence of a “generic” conserved density of a specific form to obtain a polynomiality result, then we have shown that there is in fact a canonical density of the required form. In this work also, remarkably, all polynomiality results involved the dependencies of the unknown functions on the top order derivatives, i.e., on $u_{m-1}$, if $u_t = A(x,t,u,\ldots,u_{m-1})u_m + B(x,t,u,\ldots,u_{m-1})$ and so on.

These observations above lead to the definition of a graded algebra structure [8] which will be the main subject of this paper.

The classification of 5th order, constant separant evolution equations is given in [3]. The non-constant separant case is studied by the MSS method in [10] where “KdV-like” equations are defined to be the ones that admit an unbroken sequence of conserved densities at all orders. Although the classification is not complete, it has been shown that the non-constant separant KdV-type equations are of the form $u_t = a_5 u_5 + Bu_4^2 + Cu_4 + G$, where $B$, $C$ and $G$ are polynomial in $a$ and $u_3$ and a new exact solution is given [10]. This class of equations as well as the ones that we present in Section 4, are expected to belong to the hierarchy of essentially nonlinear equations of the third order, given by Eqn.(3.3.9) of [3].

Recall that eventhough all quasilinear third order equations are shown to be either linearizable or transformable to the Korteweg-deVries (KdV) equation, the Krichever-Novikov equation is possibly an exception [4],[11],[12].

In the present paper, we shall first introduce the grading scheme mentioned above and prove its main properties, namely its invariance under integrations by parts and the “dependency of the top level on the top derivative”. We shall give the top level parts of the candidates for integrable equations at orders $m = 7, 9, 11, 13, 15$. An explicit form for integrable evolution equations of order $m = 7, 9$ and a closed form for orders $m = 11, 13, 15$ where the explicit form of the coefficient $B$ is given.

The notation and terminology is reviewed in Section 2 and the level grading is introduced in Section 3. In Section 4 we give the applications of level homogeneity to the classification of evolution equations of order $m \geq 7$. Results and discussions are given in Section 5.
2 Notation and terminology

2.1 Notation

Let \( u = u(x,t) \). A function \( \varphi \) of \( x, t, u \) and the derivatives of \( u \) up to a fixed but finite order, denoted by \( \varphi[u] \), will be called a “differential function” [5]. We shall assume that \( \varphi \) has partial derivatives of all orders. For notational convenience, we shall denote indices by subscripts or superscripts in parenthesis such as in \( \alpha(i) \) or \( \rho(i) \) and reserve subscripts without parentheses for partial derivatives, i.e., for \( u = u(x,t) \),

\[
\begin{align*}
  u_0 &= u, & u_t &= \frac{\partial u}{\partial t}, & u_x &= \frac{\partial u}{\partial x}, & u_k &= \frac{\partial^k u}{\partial x^k},
\end{align*}
\]

and for \( \varphi = \varphi(x,t,u,u_1,\ldots,u_n) \),

\[
\begin{align*}
  \varphi_t &= \frac{\partial \varphi}{\partial t}, & \varphi_x &= \frac{\partial \varphi}{\partial x}, & \varphi_k &= \frac{\partial \varphi}{\partial u_k}.
\end{align*}
\]

If \( \varphi \) is a differential function, the total derivative with respect to \( x \) is denoted by \( D\varphi \) and it is given by

\[
D\varphi = \sum_{i=0}^{n} \varphi_i u_{i+1} + \varphi_x. \tag{1}
\]

Higher order derivatives can be computed by applying the binomial formula as given below,

\[
D^k\varphi = \sum_{i=0}^{n} \left[ \sum_{j=0}^{k-1} \binom{k-1}{j} \left( D^j \varphi_i \right) u_{i+k-j} \right] + D^{k-1}\varphi_x. \tag{2}
\]

If \( u_t = F[u] \), then the total derivative of \( \varphi \) with respect to \( t \) is given by

\[
D_t\varphi = \sum_{i=0}^{n} \varphi_i D^i F + \varphi_t. \tag{3}
\]

The “order” of a differential function \( \varphi[u] \), denoted by \( \text{ord}(\varphi) = n \) is the order of the highest derivative of \( u \) present in \( \varphi[u] \). The total derivative with respect to \( x \) increases the order by one. From the expression of the total derivative with respect to \( t \) given by (3) it can be seen that if \( u \) satisfies an evolution equation of order \( m \), \( D_t \) increases the order by \( m \).

Equalities up to total derivatives with respect to \( x \) will be denoted by \( \cong \), i.e.,

\[
\varphi \cong \psi \quad \text{if and only if} \quad \varphi = \psi + D\eta.
\]

The effect of the integration by parts on monomials is described as follows. Note that if a monomial is non-linear in its highest derivative we cannot integrate by parts and reduce the order. Let \( k < p_1 < p_2 < \ldots < p_l < s - 1 \) and \( \varphi \) be a function of \( x, t, u, u_1, \ldots, u_k \). Then

\[
\varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l} u_s \cong -D \left( \varphi u_{p_1}^{a_1} \ldots u_{p_l}^{a_l} \right) u_{s-1},
\]

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\[ \varphi u^{a_1}_{p_1} \ldots u^{a_p}_{p_1} u^{s-1}_{s} \cong -\frac{1}{p+1} D \left( \varphi u^{a_1}_{p_1} \ldots u^{a_p}_{p_1} \right) u^{p+1}_{s-1}. \]

The integrations by parts are repeated successively until one encounters a “non-integrable monomial” of the following form:

\[ u^{a_1}_{p_1} \ldots u^{a_p}_{p_1} u^{s}_s, \quad p > 1. \]

The order of a differential monomial is not invariant under integration by parts, but we will show in the next section that its level decreases by one under integration by parts [8]. This will be the rationale and the main advantage of using the level grading.

3 The Ring of Polynomials and “level-grading”

The scaling symmetry and scale homogeneity are well known properties of polynomial integrable equations. We recall that scaling symmetry is the invariance of an equation under the transformation \( u \rightarrow \lambda^a u, \quad x \rightarrow \lambda^{-1} x, \quad t \rightarrow \lambda^{-b} t. \) If \( a = 0, \) then scale invariant quantities may be non-polynomial and the scaling weight is just the order of differentiation. In the early stages of our investigations we have noticed that if \( F \) is a function of the derivatives of \( u \) up to order say \( k, \) and we differentiate \( F \) say \( j \) times, the resulting expressions are polynomial in \( u_{k+1}, \ldots, u_{k+j}. \) Furthermore, the sum of the order of differentiations exceeding \( k \) has some type of invariance. This remark led us to the definition of “level grading” as a generalization of the scaling symmetry for the case \( a = 0, \) as a graded algebra structure.

We consider the ring of functions of \( x, t, u, \ldots, u_k \) the modules generated by the derivatives \( u_{k+1}, \ldots. \) This set up is given a graded algebra structure as described below.

Let \( M \) be an algebra over a ring \( K. \) If we can write \( M = \oplus_{i \in N} M_i, \) as a direct sum of its submodules \( M_i, \) with the property that \( M_i M_j \subseteq M_{i+j}, \) then we have a “graded algebra” structure on \( M. \) For example if \( K = R \) and \( M \) is the algebra of polynomials in \( x \) and \( y, \) then the \( M_i \)'s may be chosen as the submodule consisting of homogeneous polynomials of degree \( i. \) In the same example, we may also consider the submodules consisting of polynomials of degree \( i \) (not necessarily homogeneous) that we denote by \( M^i. \) Then \( M^1 \) is the direct sum of the submodules \( M_j, j \) ranging from zero to \( i. \) It follows that the full algebra \( M \) can be written as a sum of the submodules \( M_i, \) but the sum is no more direct. This structure is called a “filtered algebra”. The formal definitions are given below.

**Definition 3.1:** Let \( K \) be a ring and \( M \) be an algebra over \( K \) and \( M_i \) be submodules of \( M. \) The decomposition of \( M \) to a direct sum of submodules: \( M = \oplus_{i \in N} M_i \) and \( M_i M_j \subseteq M_{i+j} \) is called a graded algebra structure on \( M. \) Given a graded algebra \( M, \) we can obtain an associated “filtered algebra” \( \tilde{M} = M, \) [7], by defining \( M = \sum_{i \in N} \tilde{M}_i \) where \( \tilde{M}_i = \oplus_{j=0}^i M_j. \)

If the algebra \( M \) is characterized by a set of generators, then the submodules \( M_i \) can also be characterized similarly. If \( K^{(k)} \) be the ring of \( C^\infty \) functions of \( x, t, u, \ldots, u_k, \) and \( M^{(k)} \) is
the polynomial algebra over $K^{(k)}$ generated by the set

$$S^{(k)} = \{u_{k+1}, u_{k+2}, \ldots\},$$

a monomial in $M^{(k)}$ is a product of a finite number of elements of $S^{(k)}$. We define the “level above $k$” of a monomial as follows:

**Definition 3.2**: Let $\mu = u_{k+j_1}^{a_1} u_{k+j_2}^{a_2} \ldots u_{k+j_n}^{a_n}$ be a monomial in $M^{(k)}$. The level of $\mu$ above $k$, is defined by

$$\text{lev}_k(\mu) = a_1 j_1 + a_2 j_2 + \ldots + a_n j_n.$$ 

The level of the differential operator $D$ is defined to be 1. The level of a pseudo-differential operator is thus $\text{lev}_k(\varphi D^j) = \text{lev}_k(\varphi) + j$.

It can be seen that for any two monomials $\mu$ and $\tilde{\mu}$,

$$\text{lev}_k(\mu \tilde{\mu}) = \text{lev}_k(\mu) + \text{lev}_k(\tilde{\mu}).$$

The “level above $k$” gives a graded algebra structure to $M^{(k)}$. Monomials of a fixed level $p$ form a free module $M_p^{(k)}$ over $K^{(k)}$ and we denote its set of generators by $S_p^{(k)}$. If $a$ is a polynomial in $\tilde{M}_p^{(k)}$, $a = \sum_{p \geq 0} a_p$ where $a_p \in M_p^{(k)}$, $a_p$ is called the homogeneous component of $a$ of level $p$ above $k$.

**Definition 3.3** Let $a$ be a polynomial in $\tilde{M}_p^{(k)}$. The image of a polynomial $a$ under the natural projection

$$\pi : \tilde{M}_p^{(k)} \to M_p^{(k)}$$

denoted by $\pi(a)$ is called the “top level part of $a$”.

We will now present certain results that demonstrate the importance of the level grading. These will be the proofs that partial derivatives with respect to $u_i$, total derivatives with respect to $x$, total derivatives with respect to $t$, the integration by parts hence the conserved density conditions are filtered algebra maps.

**Proposition 3.4** If $\varphi$ is level homogeneous of level $p$ above $k$ and if $\frac{\partial \varphi}{\partial u_{k+j}} \neq 0$, then the partial derivative of $\varphi$ with respect to $u_{k+j}$ has level $p - j$, for $j \geq 0$.

**Proof**: Let $\varphi \in M_p^{(k)}$ and $\varphi$ be level homogeneous above $k$ of level $p$. Then $\varphi$ is a linear combination of monomials of level $p$; $M_p^{(k)} = \langle u_{k+p}, u_{k+p-1} u_{k+1}, u_{k+p-2} u_{k+2}, u_{k+p-2} u_{k+1}^2, \ldots \rangle$. Clearly if $\frac{\partial \varphi}{\partial u_{k+j}} \neq 0$, the effect of differentiation with respect to $\frac{\partial \varphi}{\partial u_{k+j}}$ decreases the level by $j$.

**Proposition 3.5** The total derivative with respect to $x$, $D$ is a filtered algebra map $\tilde{M}_p^{(k)} \to \tilde{M}_{p+1}^{(k)}$.

**Proof**: Clearly it is sufficient to consider the effect of $D$ of a product of a function $\varphi$ in $K^{(k)}$ and a monomial of level $p$ above $k$. The effect of $D$ on a monomial increases the level by 1. On
the other hand $D\phi = \frac{\partial \phi}{\partial u_k} u_{k+1} + \ldots$. In particular, the level $p + 1$ part depend only on $\varphi$ and its derivative with respect to $u_k$. It follows that $D$ is a filtered algebra map. \hfill \square

We now study the effect of integration by parts. The subset of the generating set $S_p^{(k)}$ of the module $M_p^{(k)}$, consisting of the monomials that are nonlinear in the highest derivative and the submodule that it generates are denoted respectively by $\bar{S}_p^{(k)}$ and $\bar{M}_p^{(k)}$. If a monomial is nonlinear in its highest derivative it cannot be integrated. If it is linear, one can proceed with the integrations by parts until a term that is nonlinear in its highest derivative is encountered. By virtue of the propositions above, these operations will be filtered algebra maps.

**Proposition 3.6** Let $\alpha$ be a polynomial in $\bar{M}_p^{(k)}$. Then

$$\int \alpha = \beta - \int \gamma$$

where $\beta$ belongs to $\bar{M}_{p-1}^{(k)}$ and $\gamma$ belongs to $\bar{M}_p^{(k)}$.

**Proof:** Let

$$\mu = u_{k+1}^{i_1} u_{k+i_2}^{i_2} \ldots u_{k+i_j}^{i_j}, \quad i_1 > i_2 > \ldots > i_j \quad i_1 a_1 + i_2 a_2 + \ldots + i_j a_j = p$$

We have the following three mutually exclusive cases.

**i.** When $a_1 > 1$, the monomial is not a total derivative and $\mu \in \bar{S}_p^{(k)}$. We cannot proceed with integration by parts.

**ii.** When $a = 1$ the term $\varphi \mu$ where $\varphi \in K^{(k)}$ can be integrated. For $i_2 < i_1 - 1$

$$\int \varphi \mu = \int \varphi u_{k+1}^{a_1} u_{k+i_2}^{a_2} \ldots u_{k+i_j}^{a_j} = \varphi u_{k+1}^{a_1} u_{k+i_2}^{a_2} \ldots u_{k+i_j}^{a_j} - \int u_{k+1}^{a_1} D \left( \varphi u_{k+i_2}^{a_2} \ldots u_{k+i_j}^{a_j} \right).$$

**iii.** When $a = 1$ but $i_2 = i_1 - 1$ then

$$\int \varphi \mu = \int \varphi u_{k+1}^{a_1} u_{k+i_2}^{a_2} u_{k+i_3}^{a_3} \ldots u_{k+i_j}^{a_j} = \frac{u_{k+i_2}^{a_2+1}}{a_2 + 1} \frac{u_{k+i_3}^{a_3}}{a_3 + 1} \ldots \frac{u_{k+i_j}^{a_j}}{a_j + 1} - \int \frac{u_{k+i_2}^{a_2+1}}{a_2 + 1} D \left( \varphi u_{k+i_3}^{a_3} \ldots u_{k+i_j}^{a_j} \right).$$

In (i) and (ii), the level of the term that has been integrated decreases by 1 while the terms under the integral sign have levels $p$ or lower. \hfill \square

We will now give an example that illustrates the effect of total derivatives and integration by parts.

**Example 3.7** Let

$$R = \varphi u_8 + \psi u_7 u_6 + \eta u_6^3,$$

where $\varphi, \psi, \eta \in K^{(5)}$ be a polynomial in $M_3^{(5)}$. It can easily be seen that $DR$ is a sum of polynomials in $M_4^{(5)}$ and $M_3^{(5)}$. 

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dependency of the coefficients on $u$.

Corollary 3.9

If $R_p q$ $D_k$ most $p$ $\int = \rho u$ $\eta$ then clearly, $D_t \rho$ is of order $n + m$. A similar result holds for level grading.

Proposition 3.8 Let $u_t = F[u]$, where $F$ is a differential polynomial of order $m$ and of level $q$ above the base level $k$. Then $D_t$ is a filtered algebra map $\tilde{M}^{(k)}_p \to \tilde{M}^{(k)}_{p+q+k}$.

Proof: Let $\rho$ be a differential polynomial of order $n$ and of level $p$ above the base level $k$. Then

$$D_t \rho = \rho_t + \sum_{i=0}^{k} \frac{\partial \rho}{\partial u_i} D^i F + \sum_{j=1}^{n-k} \frac{\partial \rho}{\partial u_{k+j}} D^{k+j} F.$$ 

Note that $\rho_t$ has level at most $p$. Similarly, the level of $\frac{\partial \rho}{\partial u_i}$ for $i \leq k$, is at most $p$ hence each of the terms in the first summation are of levels at most $p + q + i$, and the sum has level at most $p + q + k$. In the second summation, $\frac{\partial \rho}{\partial u_{k+j}}$ has level $p - j$, hence the level of $\frac{\partial \rho}{\partial u_{k+j}} D^{k+j} F$ is $(p - j) + (k + j) + q = p + q + k$. □

Corollary 3.9 If $F$ is quasi-linear, and $m = k + q$, then $D_t$ increases the level by $m$. □

We will now prove a very useful proposition stating that the top level depends only on the dependency of the coefficients on $u_k$. 

$$DR = \frac{\varphi u_9 + (\varphi_5 + \psi) u_8 u_6 + \psi u_7^2 + (\psi_5 + 3 \eta) u_7 u_6^2 + \eta_5 u_6^4}{M^{(5)}_4}$$

$$+ \frac{(\varphi_4 u_5 + \ldots + \varphi_x) u_8 + (\psi_4 u_5 + \ldots + \psi_x) u_7 u_6 + (\eta_4 u_5 + \ldots + \eta_x) u_6^3}{M^{(5)}_3}$$

Note that the projection to $M^{(5)}_4$ depends only on the derivatives with respect to $u_5$. We later prove that this holds in general.

For convenience, we define the operator $D_0$ to denote the part of $D \phi$ on lower order derivatives by, $D_\phi = \varphi_k u_{k+1} + D_0 \phi$ as a sum of level 1 and level 0 terms. It follows that $D^2 \phi = \varphi_k u_{k+2} + \varphi_{kk} u_{k+1}^2 + (D_0 \phi) k u_{k+1} + D_0 (D_0 \phi)$ is a sum of level 2, level 1 and level 0 terms. The integration by parts of $R$ gives:

$$\int R dx = \underbrace{\varphi u_7 + \frac{1}{2} (\psi - \varphi_5) u_6^2 - D_0 \psi u_6}_{M^{(5)}_2}$$

$$+ \int \left[ \frac{1}{2} \varphi_{55} - \frac{1}{2} \psi_5 - \eta \right] u_6^3 + \left[ \frac{1}{2} D_0 \varphi_5 - \frac{1}{2} D_0 \psi + (D_0 \varphi)_5 \right] u_6^2 + D_0 (D_0 \varphi) u_6 \right] dx.$$
Proposition 3.10 Let \( \rho \) be a differential polynomial in \( \tilde{M}_p^{(k)} \). Then the projection \( \pi(D^j \rho) \) depends only on the dependency of the coefficients in \( \rho \) on \( u_k \).

Proof. Let \( \rho = \sum_i \varphi_i P_i \) where \( \varphi_i \in K^{(k)} \) and \( P_i \in M_p^{(k)} \). Without loss of generality \( \rho = \varphi P \) and \( P = u_1^{a_1} \ldots u_m^{a_m} \). Here \( \text{lev}(\rho) = p \).

\[
D \rho = D\varphi P + \varphi DP = \left[ \varphi_x + \sum_{i=0}^{k-1} \frac{\partial \varphi}{\partial u_i} u_{i+1} + \frac{\partial \varphi}{\partial u_k} u_{k+1} \right] P + \varphi DP
\]

(4)

It follows that the projection \( \pi(D^j \rho) \) is independent of \( \frac{\partial \varphi}{\partial u_j} \) for \( j < k \) and independent of \( \frac{\partial \varphi}{\partial u_x} \).

It follows that in the conserved density computations, if \( \rho \) and \( F[u] \) are level homogeneous, then \( \rho_t \) up to total derivatives is also level homogeneous. This is a very important and useful result.

4 Application of “level homogeneity” structure on the classification problem

In this section we apply the “level homogeneity” structure to the classification of scalar evolution equations of orders \( m \geq 7 \). In [9] we have shown that if \( F \) is integrable in the sense of admitting a formal symmetry, then it is of the form

\[
F = u_t = a^m u_m + Bu_{m-1}u_{m-2} + \ldots
\]

where \( a, B, C, G, H \) and \( K \) are functions of \( x, t, u, u_i, i \leq m-3 \), i.e., it is level homogeneous above level \( m - 3 \).

Here we start from this form of \( F \) which is level homogeneous above level \( m - 3 \). We shall assume that the conserved densities \( \rho^{(-1)}, \rho^{(1)} \) and \( \rho^{(3)} \) are non-trivial. The cases where either of these are trivial will be dealt with elsewhere. We characterize such equations as “KdV-like”. It is well known that the canonical densities of even order are trivial, hence we give the definition as below.

Definition 4.1 An evolution equation \( u_t = F[u] \) is called “KdV-like” if its sequence of odd numbered canonical densities is nontrivial.

When we substitute the form of \( F \) given above in the canonical conserved densities \( \rho_c^{(i)} \) and we integrate by parts we can see that the canonical densities are of the form given below
and we use the subscript \( c \) to denote canonical quantities. Alternatively, if haven’t the explicit expression of the canonical densities we could assume that the evolution equation admits an infinite sequence of level homogeneous conserved densities that we call “generic conserved densities”. In fact, from a computational point of view, it is preferable to use the generic conserved densities and compare with the explicit form of the canonical densities whenever necessary. The generic quantities are labeled by the excess of the order of the highest derivative above the base level. If \( k \) be the base level, \( m \) the order and \( F_m = A = a^m \), then the generic form of the conserved densities are as follows:

\[
\begin{align*}
\rho^{(-1)}_c &= \rho^{(0)} = A^{-1/m} = a^{-1} \\
\pi(\rho^{(1)}_c) &\cong \rho^{(1)} = P^{(1)} u_{k+1} \\
\pi(\rho^{(3)}_c) &\cong \rho^{(2)} = P^{(2)} u_{k+2} + Q^{(2)} u_{k+1}^4. 
\end{align*}
\] (5)

**Remark 4.2** Recall that conserved densities can be given up to total derivatives. Thus a generic conserved density of of order \( k + j \) is a polynomial in the monomials \( M_{2j}^{(k)} \), as given in Appendix B.

We will outline below the steps leading to the classification of the top level parts of the integrable equations of odd orders \( m = 7, \ldots, 15 \) for scalar evolution equations admitting the (nontrivial) canonical conserved densities \( \rho^{(1)}_c, \rho^{(2)}_c, \rho^{(3)}_c \). In particular the nontriviality of \( \rho^{(3)}_c \) will be crucial.

**Step 1.** \( k = m - 3, \ m = 7, 9, 11, 13, 15 \). We begin our computations for \( k = m - 3 \), for \( m = 7, 9, 11, 13, 15 \). In [9] it has been shown that any \( m \) integrable evolution equations are of the form

\[
F = u_t = a^{k+3} u_{k+3} + B u_{k+2} u_{k+1} + C u_{k+1}^3, \quad m \geq 7.
\] (6)

For each order, we compute the conserved density conditions, integrate by parts and collect the top level terms. The solutions of these equations give that all the coefficients \( B, \ C \) are functions of \( a \) and the derivatives of \( a \) with respect to \( u_k \) of various orders and \( a \) is independent of \( u_k \). This implies that \( F \) is level homogeneous over \( K^{(m-4)} \).

**Step 2.** \( k = m - 4, \ m = 7, 9, 11, 13, 15 \). For \( k = m - 4 \) the generic form of the evolution equation is:

\[
F = u_t = a^{k+4} u_{k+4} + B u_{k+3} u_{k+1} + C u_{k+2}^2 + G u_{k+2} u_{k+1}^2 + H u_{k+1}^4, \quad m \geq 7,
\] (7)

where the coefficients depend on \( u_i \) for \( i \leq m - 4 \). The top level parts of the conserved densities have the same form. Computing the conserved density conditions and integrating by parts we obtain systems of equations. For \( m = 7 \), we find that \( a \) satisfies the third order differential equation

\[
a_{333} - 9 a_{33} a_{3}^{-1} + 12 a_{3}^{3} a^{-2}
\]

and the classification of the 7th order equations is not pursued further. For \( m > 7 \), we have \( a_{m-4} = 0 \) and it turns out that \( F \) is level homogeneous over \( K^{(m-5)} \). 10
Step 3. $k = m - 5$, $m = 9, 11, 13, 15$. For $k = m - 5$ the generic form of the evolution equation is:

$$F = u_t = a^{k+5} u_{k+5} + B u_{k+4} u_{k+1} + C u_{k+4} u_{k+2} + E u_{k+4} u_{k+1} + G u_{k+3}^2 + H u_{k+3} u_{k+2} u_{k+1} + K u_{k+1}^5, \ m \geq 9$$

(8)

The conserved density conditions imply that $F$ is level homogeneous over $K^{(m-6)}$, for $m = 9, 11, 13, 15$.

Step 4. $k = m - 6$, $m = 9, 11, 13, 15$. For $k = m - 6$ the generic form of the evolution equation is:

$$F = a^{k+6} u_{k+6} + B u_{k+5} u_{k+1} + C u_{k+4} u_{k+2} + E u_{k+4} u_{k+1} + G u_{k+3}^2 + H u_{k+3} u_{k+2} u_{k+1} + K u_{k+1}^5$$

(9)

where $F$ satisfies the equation above (4). We note that the expression of $F$ given above is a linear combination of the monomials in the generating set of $M_6^{(k)}$, as given in Appendix A. For $m = 9$, surprisingly we find that $a$ satisfies the same equation as above (4), and for $m > 9$ we find that $a_{m-6} = 0$, and it follows that $F$ is level homogeneous over $K^{(m-7)}$.

Step 5. $k = m - 7$, $m = 11, 13, 15$. At this step, $F$ is a linear combination of the monomials in the generating set of $M_7^{(k)}$, given in Appendix A and we omit the explicit expression here. The conserved density conditions imply that $F$ is level homogeneous over $K^{(m-8)}$.

Step 6. $k = m - 8$, $m = 11, 13, 15$. $F$ is now a linear combination of the monomials in the generating set of $M_8^{(k)}$, given in Appendix A. The conserved density conditions imply that for $m = 11$, $a$ satisfies the equation above (4) and for $m > 11$, $a_{m-8} = 0$. It follows that for $m > 11$, $F$ is level homogeneous over $K^{(m-9)}$.

Step 7. $k = m - 9$, $m = 13, 15$. $F$ is a linear combination of the monomials in the generating set of $M_9^{(k)}$, given in Appendix A. The conserved density conditions imply that $a_{m-9} = 0$ and $F$ is level homogeneous over $K^{(m-10)}$.

Step 8. $k = m - 10$, $m = 13, 15$. $F$ is a linear combination of the monomials in the generating set of $M_{10}^{(k)}$, given in Appendix A. For $m = 13$, $a$ satisfies that equation above (4) while for $m > 11$, $a_{m-10} = 0$ and $F$ is level homogeneous over $K^{(m-11)}$.

Step 9. $k = m - 11$, $m = 15$. $F$ is a linear combination of the monomials in the generating set of $M_{11}^{(k)}$, given in Appendix A. The conserved density conditions imply that $a_{m-11} = 0$ and $F$ is level homogeneous over $K^{(m-12)}$.

Step 10. $k = m - 12$, $m = 15$. $F$ is a linear combination of the monomials in the generating set of $M_{12}^{(k)}$, given in Appendix A. The conserved density conditions imply that $a$ satisfies the equation above (4).

It is a remarkable fact that at all orders $m \geq 7$, the separant $a$ satisfies the following equation.
\[ a_{333} - 9a_{33}a^{-1} + 12a^3a^{-2} = 0. \quad (10) \]

Using the substitution \( a = Z^{-1/2} \) in the equation above, we obtain \( Z_{333} = 0 \), hence

\[ a = \left( \alpha u_3^2 + \beta u_3 + \gamma \right)^{-1/2}, \quad (11) \]

where \( \alpha, \beta, \gamma \) are functions of \( x, t, u, u_1 \) and \( u_2 \) in general. Here as we are interested in the top level form of the equations, the dependencies on these derivatives are irrelevant. We give below certain expressions that are useful for a controlled substitution in the conserved density conditions.

### 4.1 Classification of scalar evolution equations of order \( m = 7 \)

Generally for coefficients that depend on lower orders we will use capital letters. We summarize this computations in two parts. First our base level is \( m - 3 = 4 \). We work with scalar evolution equations of order \( m = 7 \),

\[ u_t = a^7u_7 + Bu_5u_6 + Cu_3^3 \quad (12) \]

and the generic conserved densities \( \rho^{(0)}, \rho^{(1)}, \rho^{(2)} \) given in (5) where the coefficients \( a, B, C \) and \( P^{(1)}, P^{(2)}, Q^{(2)} \) depend on \( u_4 \).

We get that all the coefficients \( B, C, P^{(1)}, \ldots \) are functions of \( a \), and the derivatives of \( a \) with respect to \( u_4 \) of various orders. Finally we get that the derivative of \( a \) with respect to \( u_4 \) is zero,

\[ a_4 = 0, \]

which implies that all the coefficients vanishes. Then we reduce the base level \( u_4 \) by one.

In the second part our base level is \( m - 4 = 3 \). We work with scalar evolution equations of order \( m = 7 \),

\[ u_t = a^7u_7 + Bu_6u_4 + Cu_5^2 + Gu_5u_4^2 + Hu_4^4 \quad (13) \]

and the generic conserved densities \( \rho^{(0)}, \rho^{(1)}, \rho^{(2)} \) given in (5) where the coefficients \( a, B, C, G, H \) and \( P^{(1)}, P^{(2)}, Q^{(2)} \) depend on \( u_3 \).

In this step we obtain, \( P^{(1)} = P^{(10)}a^5 \), \( P^{(2)} = P^{(20)}a^7 \) and

\[ u_t = a^7u_7 + 14a_3a_6u_6u_4 + \frac{21}{2}a_3a^6u_5^2 + a^5\left( \frac{35}{2}a_{33}a + 63a_3^2 \right)u_5u_4^2 \]
\[ + a_4a_3\left( \frac{399}{8}a_{33}a - \frac{21}{4}a_3^2 \right)u_4^4. \quad (14) \]

When the conserved densities are:

\[ \rho^{(1)} = P^{(10)}a^5u_4^2 \]
\[ \rho^{(2)} = P^{(20)}a^7u_4^2 + P^{(20)}a^5\left( -\frac{7}{4}a_{33}a + \frac{7}{2}a_3^2 \right)u_4^4 \quad (15) \]
Thus seventh order integrable scalar evolution equations have the following form:

\[
    u_t = a^7u_7 - 7a^9z_3u_4u_6 - \frac{21}{4}a^9z_3u_5^2 \\
    + \left( -\frac{231}{8}a^2p + 98\alpha \right) a^9u_4^2u_5 \\
    + \left( \frac{1155}{64}a^2p - \frac{189}{4}\alpha \right) a^{11}z_3u_4^4.
\]
(16)

Where \( a \) satisfies the equations given in (10), (11).

### 4.2 Classification of scalar evolution equations of order \( m = 9 \)

In this section we compute 9th order evolution equations in 4 steps. First we work with

\[
    u_t = a^9u_9 + Bu_7u_8 + Cu_7^3
\]
(17)

and the generic conserved densities \( \rho^{(0)}, \rho^{(1)}, \rho^{(2)} \) given in (5) where the coefficients \( a, B, C, \) and \( P^{(1)}, P^{(2)}, Q^{(2)} \) depend on \( u_6 \).

We get that all the coefficients \( B, C, P^{(1)}, ... \) are functions of \( a \) and the derivatives of \( a \) with respect to \( u_6 \) of various orders. Finally we get that the derivative of \( a \) with respect to \( u_6 \) is zero,

\[
    a_6 = 0,
\]
which means that all the coefficients vanishes. Then we reduce the base order \( u_6 \) by one.

In the second step we work with scalar evolution equations of order \( m = 9 \),

\[
    u_t = a^9u_9 + Bu_8u_6 + Cu_7^2 + Gu_7u_6^2 + Hu_6^4
\]
(18)

and the generic conserved densities \( \rho^{(0)}, \rho^{(1)}, \rho^{(2)} \) given in (5) where the coefficients \( a, B, C, G, H \) and \( P^{(1)}, P^{(2)}, Q^{(2)} \) depend on \( u_5 \).

In this step also we get that all the coefficients \( B, C, P^{(1)}, ... \) are functions of \( a \), and the derivatives of \( a \) with respect to \( u_5 \) of various orders. Finally we get that the derivative of \( a \) with respect to \( u_5 \) is zero,

\[
    a_5 = 0,
\]
which means that all the coefficients vanishes. Then we reduce the base order \( u_5 \) by one.

In the third step we obtain the similar results where all the coefficients \( B, C, P^{(1)}, ... \) are functions of \( a \), and the derivatives of \( a \) with respect to \( u_4 \) of various orders and

\[
    a_4 = 0.
\]

The scalar evolution equations of order \( m = 9 \), that we work with, in this step, is

\[
    u_t = a^9u_9 + Bu_8u_5 + Cu_7u_6 + Gu_7u_5^2 + Hu_6^2u_5 + Ku_5^3 + Lu_5^3
\]
(19)
and the generic conserved densities $\rho^{(0)}, \rho^{(1)}, \rho^{(2)}$ are given in (5) where the coefficients $a, B, C, G, H, K, L$ and $P^{(1)}, P^{(2)}, Q^{(2)}$ depend on $u_4$. Since we get that the derivative of $a$ with respect to $u_4$ is zero, and all the coefficients vanishes, we reduce the base order $u_4$ by one.

Scalar evolution equations of order $m = 9$, that we use in the last step computations have the following form.

$$u_t = a^9 u_9 + B u_8 u_4 + C u_7 u_5 + G u_7 u_4^2 + H u_6 + K u_6 u_5 u_4 + L u_6 u_3^2 + M u_5^3 + N u_5 u_4^2 + P u_5 u_4 + Q u_4^3$$

(20)

and the generic conserved densities $\rho^{(0)}, \rho^{(1)}, \rho^{(2)}$ given in (5) where the coefficients $a, B, C, G, H, K, L, M, N, P, Q$ and $P^{(1)}, P^{(2)}, Q^{(2)}$ depend on $u_3$.

In this step we get $P^{(1)} = P^{(10)} a^5, P^{(2)} = P^{(20)} a^7$.

The conserved densities are the same as order $m = 7$ given in (15).

Thus ninth order integrable scalar evolution equations have the following form:

$$u_t = a^9 u_9 - \frac{27}{2} a^{11} z_3 u_8 u_4 - \frac{57}{2} a^{11} z_3 u_7 u_5 + a^{11} \left( -\frac{825}{8} a^2 p + 360 \alpha \right) u_7 u_4^2$$

$$- \frac{69}{4} a^{11} z_3 u_5^2 + a^{11} \left( -\frac{419}{4} a^2 p + 1230 \alpha \right) u_6 u_5 u_4$$

$$+ a^{13} z_3 \left( \frac{2145}{32} a^2 p - 1485 \alpha \right) u_6 u_4^3 + a^{11} \left( -\frac{671}{8} a^2 p + 290 \alpha \right) u_5^3$$

$$+ a^{13} z_3 \left( \frac{35607}{32} a^2 p - 6105 \alpha \right) u_5 u_4^2 + a^{13} \left( \frac{255255}{128} a^4 p^2 - \frac{94809}{8} a^2 \alpha p + 16335 \alpha^2 \right) u_4^4$$

$$+ a^{15} z_3 \left( -\frac{425925}{512} a^4 p^2 + \frac{135135}{32} a^2 \alpha p - \frac{19305}{4} \alpha^2 \right) u_4^6. \quad (21)$$

Where $a$ satisfies the equations given in (10), (11).

### 4.3 Classification of scalar evolution equations of order $m = 11, 13, 15$

In this section we give the final form of equations of scalar integrable evolution equations of order $m = 11, 13, 15$ that are computed in 6, 8, 10 steps respectively.

The conserved densities used are the same as order $m = 7$ given in (15). For each equation of order $m = 11, 13, 15$, $a$ satisfies the equations given in (10), (11).

Scalar integrable evolution equations of order $m = 11$ has the form of:

$$u_t = a^{11} u_{11} + B_0 u_{10} u_4 + B_1 u_9 u_5 + B_2 u_9 u_4^2 + B_3 u_8 u_6 + B_4 u_8 u_5 u_4 + B_5 u_8 u_4^3$$

$$+ B_6 u_7 + B_7 u_7 u_6 u_4 + B_8 u_7 u_5 u_4^2 + B_9 u_7 u_5 u_4 + B_{10} u_7 u_4 + B_{11} u_6 u_5 u_4$$

$$+ B_{12} u_6 u_4^2 + B_{13} u_6 u_4^2 + B_{14} u_6 u_5 u_4^2 + B_{15} u_6 u_4^3 + B_{16} u_5^4 + B_{17} u_5 u_4^2$$

$$+ B_{18} u_5 u_4^2 + B_{19} u_5 u_4^2 + B_{20} u_4^8$$

(22)

where $B_0 = -22 a^{13} z_3$.

Scalar integrable evolution equations of order $m = 13$ has the form of:

$$u_t = a^{13} u_{13} + B_0 u_{12} u_4 + B_1 u_{11} u_5 + B_2 u_{11} u_4^2 + B_3 u_{10} u_6 + B_4 u_{10} u_5 u_4 + B_5 u_{10} u_4^3$$

$$+ B_6 u_9 u_5 + B_7 u_9 u_4^2 + B_8 u_8 u_6 + B_9 u_8 u_5 u_4 + B_{10} u_8 u_4^2 + B_{11} u_7 u_5 u_4$$

$$+ B_{12} u_7 u_4^2 + B_{13} u_7 u_4^2 + B_{14} u_7 u_5 u_4^2 + B_{15} u_7 u_4^3 + B_{16} u_6^4 + B_{17} u_6 u_4^2$$

$$+ B_{18} u_6 u_4^2 + B_{19} u_6 u_4^2 + B_{20} u_4^{10}.$$
where $B_0 = -\frac{65}{2}a^{15}z_3$.

Scalar integrable evolution equations of order $m = 15$ has the form of:

$$u_t = a^{15}u_{15} + B_0u_{14}u_4 + B_1u_{13}u_5 + B_2u_{12}u_6 + B_3u_{11}u_7 + B_4u_{10}u_{6} + B_5u_{12}u_3$$

$$+ B_6u_{11}u_7 + B_7u_{11}u_6 + B_8u_{11}u_5 + B_9u_{11}u_4 + B_{10}u_{11}u_3 + B_{11}u_{10}u_8$$

$$+ B_{12}u_{10}u_7 + u_4 + B_{13}u_{10}u_6 + B_{14}u_{10}u_5 + B_{15}u_{10}u_4 + B_{16}u_{10}u_3$$

$$+ B_{17}u_{10}u_5 + u_5 + B_{18}u_5 + B_{19}u_9u_8 + B_{20}u_9u_7 + B_{21}u_9u_6 + B_{22}u_9u_5$$

$$+ B_{23}u_9u_4 + B_{24}u_9u_3 + B_{25}u_9u_2 + B_{26}u_9u_1 + B_{27}u_9u_0 + B_{28}u_9$$

$$+ B_{29}u_8u_5 + B_{30}u_8u_4 + B_{31}u_8u_3 + B_{32}u_8u_2 + B_{33}u_8u_1 + B_{34}u_8u_0$$

$$+ B_{35}u_8u_0 + B_{36}u_8u_0 + B_{37}u_8u_0 + B_{38}u_8u_0 + B_{39}u_8u_0 + B_{40}u_8u_0$$

$$+ B_{41}u_8u_0 + B_{42}u_8u_0 + B_{43}u_8u_0 + B_{44}u_8u_0 + B_{45}u_8u_0 + B_{46}u_8u_0 + B_{47}u_8u_0$$

$$+ B_{48}u_8u_0 + B_{49}u_8u_0 + B_{50}u_8u_0 + B_{51}u_8u_0 + B_{52}u_8u_0$$

$$+ B_{53}u_8u_0 + B_{54}u_8u_0 + B_{55}u_8u_0 + B_{56}u_8u_0 + B_{57}u_8u_0 + B_{58}u_8u_0$$

$$+ B_{59}u_8u_0 + B_{60}u_8u_0 + B_{61}u_8u_0 + B_{62}u_8u_0 + B_{63}u_8u_0 + B_{64}u_8u_0$$

$$+ B_{65}u_8u_0 + B_{66}u_8u_0 + B_{67}u_8u_0 + B_{68}u_8u_0 + B_{69}u_8u_0 + B_{70}u_8u_0$$

$$+ B_{71}u_8u_0 + B_{72}u_8u_0 + B_{73}u_8u_0 + B_{74}u_8u_0 + B_{75}u_8u_0$$

(24)

where $B_0 = -45a^{17}z_3$.

5 Results and Discussion

In this study, we introduced a new grading structure, that we call the “level grading”. We applied this structure on the algebra of polynomials generated by the derivatives $u_k$ over the coefficient ring $K^{(k)}$ of $C^\infty$ functions of $u_i$, $i = 0, 1, 2, \ldots, k$, where $k$ is denoted as the base level. We prove that this grading structure has the property that the total derivative with respect to $x$ and the integration by parts are filtered algebra maps. We also prove that, if $u$ satisfies an evolution equation $u_t = F[u]$ and $F$ is a level homogeneous differential polynomial, then the total derivative with respect to $t$ is also a filtered algebra map, and the conserved density conditions are level homogeneous and their top level part is independent of $u_j$ for $j < k$. We applied this “level homogeneity” property on the classification of integrable scalar
evolution equations of order $m \geq 7$. We give explicit formulas for order $m = 7, 9$ and give the formulas for order $m = 11, 13, 15$ in a closed form with the explicit form of the coefficients $B_0$. We observed that, at all orders, $a$ satisfies the same equations given in (10). The occurrence of the same form for the separant $a$ suggests strongly that these equations belong to a hierarchy. The same form of $a$ has occurred in the classification of fifth order equation [10], where it has been noted that these equations would be intrinsically related to the class of fully nonlinear third order equations [1],

$$u_t = F = (\alpha u_3^2 + \beta u_3 + \gamma)^{-1/2} (2\alpha u_3 + \beta) + \delta,$$

In this equation when we compute $\frac{\partial F}{\partial u_3}$ we find that

$$a = \left(\frac{\partial F}{\partial u_3}\right)^{1/3} = \frac{1}{2} (\alpha u_3^2 + \beta u_3 + \gamma)^{-1/2} (4\alpha\gamma - \beta^2)$$

This result suggests that the equations for which we have determined the top level part belong probably to a hierarchy starting at the fully nonlinear third order equation and the hierarchy is possibly generated by a second order recursion operator.
Appendix A

The submodules $M_i^{(k)}$ and their generating monomials where: $i = 1, 2, 3, ..., 13$ and $k = m - 3, m - 4, \ldots, 3$ used in classification of $m = 7th, m = 9th, m = 11th$ order evolution equations:

Submodules with base level $k$

\[ M_1^{(k)} = \langle u_{k+1} \rangle \]
\[ M_2^{(k)} = \langle u_{k+2}, u_{k+1}^2 \rangle \]
\[ M_3^{(k)} = \langle u_{k+3}, u_{k+2}u_{k+1}, u_{k+1}^3 \rangle \]
\[ M_4^{(k)} = \langle u_{k+4}, u_{k+3}u_{k+1}, u_{k+2}^2, u_{k+2}u_{k+1}^2, u_{k+1}^4 \rangle \]
\[ M_5^{(k)} = \langle u_{k+5}, u_{k+4}u_{k+1}, u_{k+3}u_{k+2}, u_{k+3}u_{k+1}^2, u_{k+2}^2u_{k+1} + u_{k+2}u_{k+1}^3, u_{k+1}^5 \rangle \]
\[ M_6^{(k)} = \langle u_{k+6}, u_{k+5}u_{k+1}, u_{k+4}u_{k+2} + u_{k+4}u_{k+1}^2, u_{k+3}^2, u_{k+3}u_{k+2}u_{k+1} + u_{k+3}u_{k+1}^3, u_{k+2}^2 + u_{k+2}u_{k+1}^2, u_{k+2}u_{k+1}^4 + u_{k+1}^6 \rangle \]
\[ M_7^{(k)} = \langle u_{k+7}, u_{k+6}u_{k+1} + u_{k+5}u_{k+2} + u_{k+5}u_{k+1}^2, u_{k+4}u_{k+3}u_{k+1} + u_{k+4}u_{k+1}^3, u_{k+4}u_{k+1}^4 \rangle \]
\[ M_8^{(k)} = \langle u_{k+8}, u_{k+7}u_{k+1} + u_{k+6}u_{k+2} + u_{k+5}u_{k+3}u_{k+1} + u_{k+5}u_{k+1}^3, u_{k+4}u_{k+2}u_{k+1} + u_{k+4}u_{k+1}^2, u_{k+4}u_{k+1} \rangle \]
\[ M_9^{(k)} = \langle u_{k+9}, u_{k+8}u_{k+1} + u_{k+7}u_{k+2} + u_{k+7}u_{k+1}^2, u_{k+6}u_{k+1} + u_{k+6}u_{k+1}^2, u_{k+5}u_{k+1} + u_{k+5}u_{k+1}^2, u_{k+4}u_{k+2}u_{k+1} + u_{k+4}u_{k+1}^3, u_{k+4}u_{k+1}^4 \rangle \]
\[ M_{10}^{(k)} = \langle u_{k+10}, u_{k+9}u_{k+1} + u_{k+8}u_{k+2} + u_{k+8}u_{k+1}^2, u_{k+7}u_{k+1} + u_{k+7}u_{k+1}^2, u_{k+6}u_{k+1} + u_{k+6}u_{k+1}^2, u_{k+5}u_{k+1} + u_{k+5}u_{k+1}^2, u_{k+4}u_{k+2}u_{k+1} + u_{k+4}u_{k+1}^3, u_{k+4}u_{k+1}^4 \rangle \]
\( M_{11}^{(k)} = \left( u_{k+11}, u_{k+10}u_{k+1}, u_{k+9}u_{k+2}, u_{k+9}u_{k+1} \right) \) 
\( M_{12}^{(k)} = \left( u_{k+12}, u_{k+11}u_{k+1}, u_{k+10}u_{k+2}, u_{k+10}u_{k+1} \right) \) 
\( M_{13}^{(k)} = \left( u_{k+13}, u_{k+12}u_{k+1}, u_{k+11}u_{k+2}, u_{k+11}u_{k+1} \right) \)
u_k + 7u_k + 6, u_k + 7u_k + 5u_{k+1}, u_k + 7u_k + 4u_{k+2}, u_k + 7u_k + 4u_{k+1}^2, u_k + 7u_k + 3u_{k+1} + u_k + 7u_k + 3u_{k+2} + 2u_{k+1} + u_k + 7u_k + 3u_{k+1}^3,

u_k + 7u_k + 2, u_k + 7u_k + 2u_{k+1}, u_k + 7u_k + 2u_{k+1}^2, u_k + 7u_k + 6u_{k+1} + u_k + 6u_k + 5u_{k+2} + 2u_{k+1} + u_k + 6u_k + 5u_{k+1}^2,

u_k + 6u_k + 4u_{k+3}, u_k + 6u_k + 4u_{k+2} + 2u_{k+1} + u_k + 6u_k + 4u_{k+1}^3,

u_k + 6u_k + 3u_{k+1}, u_k + 6u_k + 3u_{k+2}^2, u_k + 6u_k + 3u_{k+2} + 2u_{k+1} + u_k + 6u_k + 3u_{k+1}^4,

u_k + 6u_k + 2u_{k+1}, u_k + 6u_k + 2u_{k+1}^2, u_k + 6u_k + 2u_{k+1}^3, u_k + 6u_k + 2u_{k+1}^4 + 1,

u_k^2 + 5u_k + 3, u_k^2 + 5u_k + 2u_{k+1}^2, u_k^2 + 5u_k + 2u_{k+1}^3, u_k + 5u_k + 4u_{k+1}^4 + 1,

u_k^2 + 5u_k + 4u_{k+1} + u_k + 5u_k + 4u_{k+1}^2 + 2u_{k+1} + u_k + 5u_k + 4u_{k+1}^4 + 1,

u_k + u_k + 2u_{k+1}, u_k + u_k + 2u_{k+1}^2, u_k + u_k + 2u_{k+1}^3, u_k + u_k + 2u_{k+1}^4 + 1,

u_k + u_k + 3u_{k+1} + u_k + u_k + 3u_{k+1}^2 + 2u_{k+1} + u_k + u_k + 3u_{k+1}^4 + 1,

u_k + 4u_{k+1}, u_k + 4u_{k+1}^2, u_k + 4u_{k+1}^3, u_k + 4u_{k+1}^4 + 1,

u_k + 4u_{k+1} + 2u_{k+1}, u_k + 4u_{k+1}^2 + 2u_{k+1}^2, u_k + 4u_{k+1}^3 + 2u_{k+1}^3, u_k + 4u_{k+1}^4 + 1,

u_k + 3u_{k+1} + u_k + 3u_{k+1}^2 + 2u_{k+1} + u_k + 3u_{k+1}^4 + 1,

u_k + 3u_{k+2} + u_k + 3u_{k+2}^2 + 2u_{k+1} + u_k + 3u_{k+2}^3 + 2u_{k+1} + u_k + 3u_{k+2}^5 + 1,

u_k + 3u_{k+2} + u_k + 3u_{k+2} + 2u_{k+1} + u_k + 3u_{k+2}^3 + 2u_{k+1} + u_k + 3u_{k+2}^5 + 1,

u_k + 3u_{k+2} + u_k + 3u_{k+2} + 2u_{k+1} + u_k + 3u_{k+2}^3 + 2u_{k+1} + u_k + 3u_{k+2}^5 + 1,

u_k + 3u_{k+2} + u_k + 3u_{k+2} + 2u_{k+1} + u_k + 3u_{k+2}^3 + 2u_{k+1} + u_k + 3u_{k+2}^5 + 1,

u_k + 3u_{k+2} + u_k + 3u_{k+2} + 2u_{k+1} + u_k + 3u_{k+2}^3 + 2u_{k+1} + u_k + 3u_{k+2}^5 + 1,

u_k^6 + 2u_{k+1}, u_k^6 + 2u_{k+1}^2, u_k^6 + 2u_{k+1}^3, u_k^6 + 2u_{k+1}^4 + 1,

Appendix B

The quotient submodules $M_t^{(k)}$ and their generating monomials (that are not total derivatives), where $k = m - 3, m - 4, \ldots, 3$ and $t = 1, 2, 3, \ldots, 11, 13$ used in classification of $m = 7th, m = 9th, m = 11th$ order evolution equations:

**Quotient Submodules with base level $k$**

$M_1^{(k)} = \langle \emptyset \rangle$

$M_2^{(k)} = \langle u_{k+1}^2 \rangle$

$M_3^{(k)} = \langle u_{k+1}^3 \rangle$

$M_4^{(k)} = \langle u_{k+2}, u_{k+1}^4 \rangle$

$M_5^{(k)} = \langle u_{k+2}^5, u_{k+1}^5 \rangle$

$M_6^{(k)} = \langle u_{k+3}^5, u_{k+2}^5, u_{k+1}^6 \rangle$

$M_7^{(k)} = \langle u_{k+3}^5 u_{k+1}^7, u_{k+2}^5 u_{k+1}^7, u_{k+1}^7 \rangle$
\[
M_k^{(k)} = \left\langle u_{k+4}, u_{k+3}^2 u_{k+2}, u_{k+3}^3 u_{k+1}, u_{k+2}^4, u_{k+2}^2 u_{k+1}^2, u_{k+2}^2 u_{k+1}, u_{k+1} \right\rangle \\
M_9^{(k)} = \left\langle u_{k+4}^4 u_{k+1}, u_{k+3}^3 u_{k+2} u_{k+1}^2, u_{k+2}^4 u_{k+1}^3, u_{k+2}^3 u_{k+1}^2, u_{k+2}^2 u_{k+1}, u_{k+1}^9 \right\rangle \\
M_{10}^{(k)} = \left\langle u_{k+4}^2 u_{k+3}^2 u_{k+2}, u_{k+3}^3 u_{k+1}^2, u_{k+2}^4 u_{k+1}, u_{k+2}^3 u_{k+1}, u_{k+2}^2 u_{k+1}^2, u_{k+2} u_{k+1}^2, u_{k+1}^8 \right\rangle \\
M_{11}^{(k)} = \left\langle u_{k+4}^2 u_{k+3}^2 u_{k+2} u_{k+1}, u_{k+3}^3 u_{k+2} u_{k+1}, u_{k+2}^4 u_{k+1}^2, u_{k+2}^3 u_{k+1}^2, u_{k+2}^2 u_{k+1}, u_{k+1}^{11} \right\rangle \\
M_{12}^{(k)} = \left\langle u_{k+4}^2 u_{k+3}^2 u_{k+2} u_{k+1}, u_{k+3}^3 u_{k+2} u_{k+1}, u_{k+2}^4 u_{k+1}^2, u_{k+2}^3 u_{k+1}^2, u_{k+2}^2 u_{k+1}, u_{k+1}^{10} \right\rangle \\
M_{13}^{(k)} = \left\langle u_{k+4}^2 u_{k+3}^2 u_{k+2} u_{k+1}, u_{k+3}^3 u_{k+2} u_{k+1}, u_{k+2}^4 u_{k+1}^2, u_{k+2}^3 u_{k+1}^2, u_{k+2}^2 u_{k+1}, u_{k+1}^{12} \right\rangle \\
\]
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