Average kissing numbers
for non-congruent sphere packings

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1 Introduction

Let $P$ be a packing of $n$ (round) balls in $\mathbb{R}^3$. (A packing of round balls, also known as a sphere packing, is a collection of round balls with disjoint interiors.) The balls may have different radii. The average kissing number of $P$ is defined as $k(P) = 2m/n$, where $m$ is the number of tangencies between balls in the packing. Let

$$k = \sup\{k(P) | P \text{ is a finite packing of balls in } \mathbb{R}^3\}.$$ 

**Theorem 1**

$$12.566 \approx 666/53 \leq k < 8 + 4\sqrt{3} \approx 14.928.$$ 

(The appearance of the number of the beast in the lower bound is purely coincidental.)

The supremal average kissing number $k$ is defined in any dimension, as are $k_c$, the supremal average kissing number for congruent ball packing, and $k_s$, the maximal kissing number for a single ball surrounded by congruent balls with disjoint interiors. (Clearly, $k_c \leq k$ and $k_c \leq k_s$.) It is interesting that $k$ is always finite, because a large ball can be surrounded by many small balls in a non-congruent ball packing. Nevertheless, a simple argument presented below shows that $k \leq 2k_s$ in every dimension, and clearly $k_s$ is always finite. In two dimensions, an Euler characteristic argument shows that $k \leq 6$, but it is also well-known that $k_s = k_c = 6$. One might therefore conjecture that $k = k_c$ always, or at least in dimensions such as 2, 3, 8, and 24 (and conjecturally several others) in which $k_s = k_c$. Surprisingly, in three dimensions, $k > 12$ even though $k_s = k_c = 12$.

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Remark 1 No packing $P$ achieves the supremum $k = k(P)$, because if $P'$ is a translate of $P$ that meets $P$ in only one point, then $k(P \cup P') > k(P)$.

Let $P = (P_v, v \in V)$ be a packing, where $V$ is some indexing set. The nerve of $P$ is a combinatorial object that encodes the combinatorics of the packing. It is the (abstract) graph $G = (V, E)$ on $V$, where an edge $\{u, w\}$ appears in $E$ precisely when $P_u$ and $P_w$ intersect. If $P$ is a packing of round disks in the plane, then it is easy to see that $G$ is a planar graph. Conversely, the circle packing theorem [3], states that every finite planar graph is the nerve of some disk packing in the plane. This non-trivial theorem has received much attention lately, mostly because of its surprising relation with complex analysis. (Compare references [7], [5], and [8].)

Since the nerves of planar disk packings are understood, it is natural to ask for a description of all graphs that are nerves of ball packings in $\mathbb{R}^3$. In lieu of a complete characterization, which is probably intractable, Theorem 1 gives a necessary condition on such graphs:

$$2|E| < (8 + 4\sqrt{3})|V|.$$  

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2 The upper bound

Theorem 2 If $P$ is a finite ball packing in $\mathbb{R}^3$, then $k(P) < 8 + 4\sqrt{3}$.

As a warm-up, we will show that $k(P) \leq 24$. Let $E$ be the set of unordered pairs of balls in $P$ that kiss. Let $r(B)$ be the radius of a ball $B \in P$. By a famous result [6], [4], it is impossible for more than 12 unit balls with disjoint interiors to kiss a unit ball $B$. If $C$ kisses $B$ and $r(C) > 1 = r(B)$, then $C$ contains a (unique) unit ball that kisses $B$. Thus, in a packing, $B$ cannot kiss more than 12 balls at least as large as $B$. Consider a function $f : E \to P$ that assigns to $\{B, C\} \in E$ the smaller of the balls $B$ and $C$, or either if they are the same size. Since $f$ is at most 12 to 1, $|E| \leq 12|P|$. Consequently, $k(P) = 2|E|/|P| \leq 24$.

The proof of Theorem 2 is a refinement of this argument.

Proof: In addition to the above notation, we let $E(B)$ denote the set of $C \in P$ such that $\{B, C\} \in E$.

Let $\rho > 1$ be a constant to be determined below. For each ball $B \in P$, let $S(B)$ be the concentric spherical shell with radius $\rho r(B)$. For each $B, C \in P$, define

$$a(B, C) = \frac{\text{area}(C \cap S(B))}{\text{area}(S(B))}. \quad (1)$$

Since the interiors of the balls in $P$ are disjoint, for any $B$,

$$1 \geq \sum_{C \in P} a(B, C) \geq \sum_{C \in E(B)} a(B, C). \quad (2)$$

Summing over $B$,

$$|P| \geq \sum_{\{B, C\} \in E} (a(B, C) + a(C, B)). \quad (3)$$

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We will obtain a lower bound on $a(B, C) + a(C, B)$ for two kissing balls $B$ and $C$. Suppose that $B$ intersects $S(C)$ and $C$ intersects $S(B)$, as shown in Figure 1. Let $b$ and $c$ be the centers of $B$ and $C$. Let $q$ be a point on the relative boundary in $S(B)$ of the spherical disk $C \cap S(B)$. Clearly,

$$d(b, c) = r(B) + r(C)$$

$$d(b, q) = \rho r(B)$$

$$d(c, q) = r(C),$$

where $d(x, y)$ is the distance from $x$ to $y$. Let $\theta = \angle cbq$ be the angular radius of $C \cap S(B)$. By the law of cosines,

$$\cos \theta = \frac{(r(B) + r(C))^2 + (\rho r(B))^2 - r(C)^2}{2(r(B) + r(C))\rho r(B)} = \frac{r(B) + \rho^2 r(B) + 2r(C)}{2\rho(r(B) + r(C))}. \quad (4)$$

Also,

$$\text{area}(C \cap S(B)) = \frac{1 - \cos \theta}{2} \text{area}(S(B)). \quad (5)$$

Combining equations (4), (4) and (5),

$$a(B, C) = \frac{1}{2} - \frac{r(B) + \rho^2 r(B) + 2r(C)}{4\rho(r(B) + r(C))}. \quad (6)$$

Switching $B$ and $C$ and adding,

$$a(B, C) + a(C, B) = 1 - \frac{3 + \rho^2}{4\rho}. \quad (7)$$

Isn’t it remarkable that $a(B, C) + a(C, B)$ does not depend on $r(B)$ and $r(C)$? We now choose $\rho = \sqrt{3}$ to maximize the right side of equation (7). Then $a(B, C) + a(C, B) = 1 - \frac{\sqrt{3}}{2}$, under the assumption that $S(B) \cap C$ and $S(C) \cap B$ are non-empty. If $S(B) \cap C = \emptyset$, \[ \]
As a result, \( a(B, C) + a(C, B) \geq 1 - \frac{\sqrt{3}}{2} \) in the general case. Applying this inequality to inequality (3) yields \( |P| \geq |E| \left(1 - \frac{\sqrt{3}}{2}\right) \), which gives

\[
k(P) = 2|E|/|P| \leq 8 + 4\sqrt{3}.
\]

In conclusion, \( k \leq 8 + 4\sqrt{3} \). By Remark 1, \( k(P) < k \), establishing Theorem 2. □

**Remark 2** In fact, \( k < 8+4\sqrt{3} \). Let \( B \in P \). Since each ball \( C \in E(B) \) that intersects \( S(B) \) must have \( r(C) \geq (\rho - 1)r(B)/2 \), there is a finite bound for the number of balls \( C \in E(B) \) such that \( a(B, C) > 0 \). Therefore there is some \( \alpha < 1 \) (depending on \( \rho \) but not \( P \)) such that

\[
\sum_{C \in E(B)} a(B, C) \leq \alpha.
\]

Using this inequality in place of inequality (2) in the above proof would multiply the upper bound by a factor of \( \alpha \). A good estimate for \( \alpha \) would consequently strengthen Theorem 2.

### 3 The lower bound

**Theorem 3** There exists a sequence of finite packings \( \{P_n\} \) with

\[
\lim_{n \to \infty} k(P_n) = 666/53.
\]

Observe that all questions about nerves of ball packings and average kissing numbers are invariant under sphere-preserving transformations such as stereographic projection from the 3-sphere \( S^3 \) to \( \mathbb{R}^3 \) and inversion in a sphere.

There exists a packing \( D \) in \( S^3 \) of 120 congruent spherical balls such that each ball kisses exactly 12 others \( \Box \), or 720 kissing points in total. The existence of \( D \) already implies that \( k(P) > 12 \) for some packing \( P \), because by Remark 1, \( k > k(D) = 12 \).

The proof of Theorem 3 is a refinement of this construction. **Proof:** We give an explicit description of \( D \). Let \( S^3 \) be the unit 3-sphere in \( \mathbb{R}^4 \) and let \( \tau = \frac{1 + \sqrt{5}}{2} \) be the golden ratio. Choose the centers of the balls of \( D \) to be the points in the orbits of \( \frac{1}{2}(\tau, 1, \frac{1}{\tau}, 0), \frac{1}{2}(1, \frac{1}{\tau}, 1, 1) \), and \( (1, 0, 0, 0) \) under change of sign of any coordinate and even permutations of coordinates. The radius of each ball is 18°. We will need the following four properties of \( D \), which can be verified using the explicit description or by other means: The 12 balls that kiss a given ball have an icosahedral arrangement with 30 mutual kissing points, the centers of two kissing balls of \( D \) are 36° apart, the centers of two next-nearest balls of \( D \) are 60° apart, and \( D \) is self-antipodal. (If \( X \) is a point, set of points, or set of set of points in \( S^3 \), the antipode of \( X \) is given by negating all coordinates in \( \mathbb{R}^4 \) and is denoted \( -X \).)

Let \( B_0 \in D \) be a ball with center \( b \) and let \( P_0 = D \setminus \{B_0, -B_0\} \). The packing \( P_0 \) has 720 - 24 = 696 kissing points and 118 balls. Let \( R \) be the set of 12 balls in \( D \) that kiss \( B_0 \),
and let $S$ be the unique sphere centered at $b$ which contains the 30 kissing points between the balls in $R$. Let $I_S : S^3 \to S^3$ be inversion in the sphere $S$. Observe that $S$ meets the boundary of each $B \in R$ orthogonally in a circle (because, by symmetry, it is orthogonal to the boundary at each kissing point), and therefore each $B \in R$ is invariant under $I_S$. Let $\sigma : S^3 \mapsto S^3$ be the map $\sigma(p) = I_S(-p)$. This map $\sigma$ contracts $S^3 \setminus \{-b\}$ towards $b$, sends $-S$ to $S$, and preserves spheres. Because $I_S$ leaves each $B \in R$ invariant, $\sigma$ sends $-R$ to $R$. For each $n > 0$, let

$$P_n = P_{n-1} \cup \sigma^n(P_0).$$

We claim that the sphere $S$ does not intersect any ball in $P_0 \setminus R$. Assuming this claim, the packing $Q = P_0 \setminus (R \cup -R)$ lies between $-S$ and $S$, and $\sigma^n(Q)$ is separated from $\sigma^{n+1}(Q)$ by $\sigma^n(S)$. Therefore each $P_n$ consists of an alternation of layers

$$-R, Q, \sigma(-R) = R, \sigma(Q), \sigma^2(-R), \sigma^2(Q), \ldots, \sigma^n(-R)$$

such that each layer only intersects the two neighboring layers and intersects only in kissing points. In particular, each $P_n$ is a packing. Moreover, $P_{n+1}$ has $118 - 12 = 106$ more balls and $696 - 30 = 666$ more kissing points than $P_n$ does. Therefore

$$\lim_{n \to \infty} k(P_n) = 2 \frac{666}{106} = 66 \frac{6}{53}.$$

It remains to check the claim. Let $B_1, B_2$ be two kissing balls in $R$. Let $b_1$ and $b_2$ be their centers and let $p$ be their kissing point. Evidently the angular radius of $S$ is $\angle b_0 p$. Using the inclusion $S^3 \subset \mathbb{R}^4$ and the notation of vector calculus,

$$b_1 \cdot b_2 = b \cdot b_1 = b \cdot b_2 = \frac{\tau}{2},$$

$$b \cdot b = b_1 \cdot b_1 = b_2 \cdot b_2 = 1,$$

$$p = \frac{b_1 + b_2}{|b_1 + b_2|},$$

$$\angle b_0 p = \cos^{-1}\left(\frac{b \cdot (b_1 + b_2)}{|b_1 + b_2|}\right) = \cos^{-1}\left(\sqrt{\frac{2 + \tau}{5}}\right) \approx 31.717^\circ.$$ 

On the other hand, the center of a ball in $P_0$ which is not in $R$ is at least $60^\circ$ away from $b$, and therefore the closest point of any such ball is at least $42^\circ$ away from $b$. Thus, $S$ does not intersect any such ball. $\square$

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