PI-groups and PI-representations of groups

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Abstract

It is well known that many famous Burnside-type problems have positive solutions for PI-groups and PI-algebras. In the present article we also consider various Burnside-type problems for PI-groups and PI-representations of groups.

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1 Introduction

We start with a short historical background. The General Burnside problem asks: Is a torsion group locally finite? In 1964 E.S. Golod obtained negative solution of this problem: he constructed infinite finitely generated residual finite torsion groups. His result follows from the theorem of Golod-Shafarevich [1]. However, C. Procesi [13] and A. Tokarenko [15] obtained a positive solution of the Burnside Problem for PI-groups: every periodic PI-group is locally finite.

The General Burnside problem gives rise to numerous questions of Burnside-type which have positive answers in the case of PI-groups and PI-algebras. Remind basic definitions.

Let K be a field. Let A be an associative PI-algebra with unit over K, that is an algebra satisfying some non-trivial polynomial identity.

Definition 1.1 A group G is called a PI-group if there is an injective homomorphism \( \rho : G \to A^* \) of the group G to the group \( A^* \) of invertible elements of the PI-algebra A.
Definition 1.2 An element $g$ of a group $G$ is called a nil-element if for every $x \in G$ there is $n = n(g,x)$ such that $[[x,g],\ldots,g_n] = 1$.

Definition 1.3 A group $G$ is called a nil-group if every its element is a nil-element.

Definition 1.4 A group $G$ is called Engel if it satisfies the identity $[[x,y],\ldots,y_n] = 1$ for some $n$.

Every locally nilpotent group is a nil-group, but the opposite statement in general is not true: the theorem of Golod-Shafarevich [1] gives a negative counterexample. For the case of $PI$-groups there is a positive solution: every nil-$PI$-group (and then Engel group) is locally nilpotent (see [8], [10]).

Moreover for Engel groups there is a long-standing conjecture:

Conjecture 1.5 An Engel group is not necessary locally nilpotent.

This problem is still open: it is not known whether there exists a non-locally nilpotent Engel group.

For algebraic algebras the following Burnside-type problem posted by A.Kurosh is well-known. Remind the necessary definitions.

Definition 1.6 An algebra $A$ is called algebraic if every element $a \in A$ satisfies some polynomial identity $f(x) = 0$.

Definition 1.7 An algebra $A$ is called locally finite if every finite set of elements of the algebra $A$ generates a finite dimensional subalgebra.

Every locally finite algebra is an algebraic algebra. The problem of Kurosh asks: Is every algebraic algebra locally finite? In general it is not true, but I. Kaplansky and A. Shirshov [3] give positive answer in the case of $PI$-algebras: every algebraic $PI$-algebra is locally finite.

We consider a variant of Kurosh’s problem for groups. Below are the necessary definitions.

Definition 1.8 An element $g$ of a group $G$ is called an algebraic element if for every $x \in G$ the subgroup generated by the all elements of the form $[[x,g],\ldots,g_n]$, $n \in \mathbb{N}$ is finitely generated.

Definition 1.9 A group $G$ is called algebraic if every its element is an algebraic element.

Definition 1.10 A group $G$ is called locally Noetherian if every its finitely generated subgroup is Noetherian.

It is obvious that every locally Noetherian group is algebraic. We consider the question when the inverse statement is true. In Section 2 we show that there is the positive solution of this problem for $PI$-groups.

Theorem 1.11 Every algebraic $PI$-group is locally Noetherian.
In fact we will prove that every finitely generated algebraic $PI$-group is a Hirsch (polycyclic-by-finite) group.

The other our main result deals with $PI$-representations of groups.

Let $K$ be a field, $V$ be a $K$-module. Let $G$ be a group and let $(V, G)$ be a representation of the group $G$ in the $K$-module $V$. So we have a homomorphism $\rho$ of the group $G$ to the group of automorphisms of the $K$-module $V$:

$$\rho : G \rightarrow \text{Aut}_K V, \ g \mapsto g^\rho.$$

We can say that the group $G$ acts on the module $V$ by the rule:

$$(v, g) \mapsto v \circ g = g^\rho(v),$$

for all $v \in V$, $g \in G$. Note that the group algebra $KF$ also acts on the module $V$.

Let $F$ be a free group of countable rank with free generators $x_1, x_2, \ldots$, and let $KF$ be the group algebra of $F$. Let $u(x_1, \ldots, x_n)$ be an element of $KF$.

**Definition 1.12** We say that a representation $(V, G)$ satisfies an identity $y \circ u(x_1, \ldots, x_n) \equiv 0$ if for all $v \in V$ and all $g_i \in G$ we have $v \circ u(g_1, \ldots, g_n) = 0$.

**Definition 1.13** An element $g \in G$ is called a unipotent element of a representation $(V, G)$ if there is $n = n(g)$ such that $x \circ (g - 1)^n \equiv 0$ for every $x \in V$.

**Definition 1.14** A representation $(V, G)$ is called a unipotent representation if every $g \in G$ is a unipotent element of $(V, G)$.

**Definition 1.15** A representation $(V, G)$ is called a unitriangular if it satisfies the identity $x \circ (y_1 - 1) \ldots (y_n - 1) \equiv 0$.

**Definition 1.16** A representation $(V, G)$ is called a locally unitriangular if for every finitely generated subgroup $H$ of the group $G$ the subrepresentation $(V, H)$ is unitriangular.

Every locally unitriangular representation is unipotent. There is the following Burnside-type problem for unipotent representations:

**Problem 1.17** Is every unipotent representation locally unitriangular?

In general, it is not true. We prove (Section 3) the positive solution of this problem for $PI$-representations of groups.

Let $(V, G)$ be a representation of the group $G$ in the $K$-module $V$. Let $\overline{G} = G/\text{Ker}(V, G)$ and $(\overline{V}, \overline{G})$ be the faithful representation corresponding to the representation $(V, G)$.

**Definition 1.18** A representation $(V, G)$ is called a $PI$-representation if the linear span $\langle \overline{G} \rangle$ of the group $\overline{G}$ in the algebra $\text{End} V$ is a $PI$-algebra.

We have the following Theorem (see Section 3)

**Theorem 1.19** Every unipotent $PI$-representation is locally unitriangular.

In the frameworks of studying the unipotent representations we also consider the following problem:
**Problem 1.20** Is there a unipotent radical for a representation of a group?

It is well-known that every group has a locally nilpotent radical (Hirsch-Plotkin radical) that is unique maximal normal locally nilpotent subgroup \[2, 11\], but it is not true for locally solvable radical, since there exist groups without locally solvable radical (see results of G. Baumslag, L. Kovach, B. Neumann, V. Mikaelian). However S. Pikhtilkov proved that every PI-group has a locally solvable radical \[7\].

Let \((V, G)\) be a representation of the group \(G\) in the module \(V\).

**Definition 1.21** The unique maximal normal subgroup \(H\) of the group \(G\), such that the subrepresentation \((V, H)\) is locally unitriangular, is called a locally unitriangular radical of the representation \((V, G)\).

**Definition 1.22** The unique maximal normal subgroup \(H\) of the group \(G\), such that the subrepresentation \((V, H)\) is unitriangular, is called a unipotent radical of the representation \((V, G)\).

It is known \[9\] that for every representation of a group there is a locally unitriangular radical. We have the following

**Theorem 1.23** There exists a unipotent radical for a PI-representation and it coincides with the locally unitriangular radical.

which is an immediate corollary of Theorem \[10\].

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## 2 Algebraic PI-groups

In this section we will prove the theorem \[11\] which states that every algebraic PI-group is locally Noetherian.

**Proof of the theorem** Let \(G\) be an algebraic finitely generated PI-group. Since \(G\) is a PI-group then there exists a PI-algebra \(A\) such that the group \(G\) is a subgroup of the group of invertible elements of the algebra \(A\). Let \(R(A)\) be the Levitzky radical of the algebra \(A\). Consider the algebra \(A/R(A)\). The group \(G\) acts on \(A/R(G)\) and the kernel of this action is \(H = G \cap (1 + R(A))\), moreover the group \(H\) is a locally nilpotent subgroup of the group \(G\) \[10\].

Now consider the group \(\overline{G} = G/H\). It is known \[3\] that there is an embedding \(A/R(A) \to M_n(K)\), where \(M_n(K)\) is the matrix algebra of dimension \(n\) and \(K\) is a commutative ring with unit which is a Cartesian sum of fields. So the group \(\overline{G}\) is a subgroup of the group \(GL_n(K)\) of invertible elements of the algebra \(M_n(K)\).

Since the group \(G\) is finitely generated algebraic group then the group \(\overline{G}\) is also finitely generated and algebraic. Let \(\overline{G} = \langle \overline{g}_1, \ldots, \overline{g}_m \rangle\), \(\overline{g}_i \in GL_n(K)\), \(i = 1, 2, \ldots, m\). Let \(\overline{g}_i = (\alpha^i_{st})\) and let \(S\) be a set of alleles \(\alpha^i_{st}\) such that \(\alpha^i_{st} \in K\), \(i = 1, 2, \ldots, m\) and \(s, t = 1, 2, \ldots, n\). Note that the set \(S\) is finite.

Let \(K_0\) be a subring of the ring \(K\) generated by the set \(S\). Since the ring \(K_0\) is a finitely generated commutative ring then it is Noetherian. From the
Theorem of Lasker [16] follows that the ring $K_0$ has a finitely many prime ideals $I_\alpha$ such that $\bigcap_\alpha I_\alpha = 0$ and $K_0/I_\alpha$ are fields. Then using the Remak's theorem we have that $K_0$ is a Cartesian sum of the fields $K_0/I_\alpha$.

Let $M_n(K_0)$ be the algebra of matrices over $K_0$. Then $\mathcal{G} \subset M_n(K_0)$ and $M_n(K_0)$ is a subalgebra of the algebra $\bigoplus_\alpha M_n(K_0/I_\alpha)$, where $\alpha \in \mathbb{N}$ and $\alpha < \infty$.

So the group $\mathcal{G}$ is embedded into a direct product of the finite number of groups of matrices over fields:

$$\mathcal{G} \rightarrow \prod_\alpha GL_n(K_0/I_\alpha), \ \alpha < \infty.$$  

According to the well known Tits alternative [14] a finitely generated linear group either contains a non-abelian free group or has a solvable subgroup of finite index.

Since the group $\mathcal{G}$ is algebraic then every its subgroup is algebraic, but a free group is not algebraic. So the group $\mathcal{G}$ contains a solvable subgroup $\mathcal{G}_0$ of finite index. It was proved that every locally soluble algebraic group is locally Noetherian and consequently locally polycyclic (see [9]). Thus the subgroup $\mathcal{G}_0$ is finitely generated and locally polycyclic, so it is a polycyclic group. Moreover since $\mathcal{G}_0$ is a subgroup of a finite index in $\mathcal{G}$ then the group $\mathcal{G}$ is a Hirsch group.

Remind that $\mathcal{G} = G/H$ and $H$ is a locally nilpotent subgroup of the group $G$. If $\mathcal{G}_0 = G_0/H$, then the group $G_0$ is an extension of the locally nilpotent group $H$ by the solvable group $\mathcal{G}_0$ and, hence, $G_0$ is algebraic. So the group $G_0$ is locally Noetherian and consequently it is a polycyclic group. Moreover $G_0$ is a subgroup of a finite index in $G$ since $\mathcal{G}_0$ is a subgroup of finite index in $\mathcal{G}$. So in the group $G$ there exists the polycyclic subgroup $G_0$ of finite index and then $G$ is a Hirsch group. But every Hirsch group is Noetherian. Thus $G$ is a Noetherian group.

Consequently every non-finitely generated algebraic $PI$-group is locally Noetherian. Theorem is proved. \[\Box\]

## 3 Unipotent $PI$-representations of groups

In this section we will prove the theorem [19]. This theorem is a generalization of the well-known theorem of Kolchin [20] which states:

**Theorem 3.1** Let $G$ be a linear group, $G \leq GL_n(K)$, $K$ is a field. Let $(K^n, G)$ be an unipotent representation of the group $G$. Then the representation $(K^n, G)$ is unipotent. \[\Box\]

Note that the representation $(K^n, G)$ from this theorem is a $PI$-representation since the group $G$ is embedded into the algebra $M_n(K)$ which satisfies the standard identity of Amitsur-Levitzky.

To prove the theorem [19] we need two lemmas.

**Lemma 3.2** Let $(V, G)$ be a representation of a group $G$ and let $G$ is generated by the set $M$. Let $\hat{h} = (h_1, \ldots, h_n)$ be a sequence from the set $M^n$. If every $\hat{h}$ satisfies the equation $x \circ (h_1 - 1) \cdots (h_n - 1) = 0$ for all $x \in V$ then the representation $(V, G)$ satisfies the identity $x \circ (y_1 - 1) \cdots (y_n - 1) \equiv 0$. 

Lemma is proved. □

The group $G$ such that the group $G$ acts trivially on the factors.

We prove the lemma by induction on $n$. Let $n = 1$ then for every $h \in M$ we have

$$x \circ (h - 1) = 0 \text{ or } x \circ h = x, \text{ for all } x \in V.$$

For all $h_1, h_2$ from the set $M$ we have

$$x \circ (h_1 h_2) = (x \circ h_1) \circ h_2 = x \circ h_2 = x,$$

then the group $G$ acts trivially on $V$ and we have the following series:

$$0 = V_1 \subset V_0 = V.$$

Let now the statement of the lemma is hold for all positive integer less or equal $(n - 1)$. Let $(h_1, \ldots, h_n)$ be an arbitrary sequence in $M^n$ and let for every $x \in V$ we have

$$x \circ (h_1 - 1) \ldots (h_n - 1) = 0.$$

Let $V_{n-1}$ be a linear span of all elements of the form $x \circ (h_1 - 1) \ldots (h_{n-1} - 1)$. The element $(h_n - 1)$ annihilates the submodule $V_{n-1}$, since this element annihilates all generators of $V_{n-1}$ and for every $v \in V_{n-1}$ we have $v \circ (h_n - 1) = 0$. Since $h_n$ is an arbitrary element in $M$ then we have $v \circ (g - 1) = 0$ for all $g \in G$ and for all $v \in V_{n-1}$. So the group $G$ acts trivially on the module $V_{n-1}$.

Consider the representation $(V/V_{n-1}, G)$. For all $h_1, \ldots, h_{n-1} \in M$ and for all $x \in V$ we have

$$x \circ (g_1 - 1) \ldots (g_{n-1} - 1) = 0.$$

Using the inductive assumption we have that the representation $(V/V_{n-1}, G)$ satisfies the identity

$$x \circ (y_1 - 1) \ldots (y_{n-1} - 1) \equiv 0.$$

It means that there is the following series of submodules of the module $V/V_{n-1}$:

$$0 = V_{n-1}/V_{n-1} \subset V_{n-2}/V_{n-1} \subset \ldots \subset V_1/V_{n-1} \subset V/V_{n-1},$$

such that the group $G$ acts trivially in the factors.

Now consider the following series of submodules of the module $V$:

$$0 = V_n \subset V_{n-1} \subset \ldots \subset V_1 \subset V.$$

The group $G$ acts trivially in the factors of this series. Thus the representation $(V, G)$ satisfies the identity

$$x \circ (y_1 - 1) \ldots (y_{n-1} - 1) \equiv 0.$$

Lemma is proved. □

Let $(V, G)$ be a PI-representation. Let $(V, \overline{G})$ be a faithful representation corresponding to the representation $(V, G)$ and let $A = \langle \overline{G} \rangle$ be the linear span of the group $\overline{G}$ in the algebra $\text{End } V$. Note that $A$ is a PI-algebra. We can consider the regular action of the group $\overline{G}$ in the algebra $A$ assuming that the group $\overline{G}$ is embedded in $A$ and so we have the representation $(A, \overline{G})$ of the group $\overline{G}$.
Lemma 3.3 Let $A_1$ be a nilpotent ideal of the algebra $A$ and let the representation $(A/A_1, \overline{G})$ satisfy the identity $x \circ (y_1 - 1) \ldots (y_n - 1) \equiv 0$. Then the representations $(A, \overline{G}), (V, \overline{G})$ and $(V, G)$ are unitriangular.

Proof. Let $\overline{g}_1, \ldots, \overline{g}_n$ be arbitrary elements of the group $\overline{G}$. Since the representation $(A/A_1, \overline{G})$ satisfies the identity $x \circ (y_1 - 1) \ldots (y_n - 1) \equiv 0$ then the element $(\overline{g}_1 - 1) \ldots (\overline{g}_n - 1)$ belongs to the ideal $A_1$.

Let $A_1$ is nilpotent ideal of class nilpotency $m$ and let the elements
\[
(\overline{g}_{11} - 1)(\overline{g}_{12} - 1) \ldots (\overline{g}_{1n} - 1),
(\overline{g}_{21} - 1)(\overline{g}_{22} - 1) \ldots (\overline{g}_{2n} - 1),
\ldots
(\overline{g}_{m1} - 1)(\overline{g}_{m2} - 1) \ldots (\overline{g}_{mn} - 1)
\]
belong to the ideal $A_1$, where $\overline{g}_{11}, \ldots, \overline{g}_{mn}$ are arbitrary elements of the group $\overline{G}$. The product of these elements is equal to zero:
\[
(\overline{g}_{11} - 1) \ldots (\overline{g}_{mn} - 1) = 0.
\]

Then the representations $(A, \overline{G})$ and $(V, \overline{G})$ satisfy the identity
\[
x \circ (y_{11} - 1) \ldots (y_{mn} - 1) \equiv 0.
\]

So the representations $(A, \overline{G})$ and $(V, \overline{G})$ are unitriangular.

For varieties of representations of groups we have the following invariant description [12]: A class of representations of groups forms a variety of representations of groups if and only it is closed under subrepresentations, homomorphic images, Cartesian products and saturations.

Remind that a class of representations of groups $\mathfrak{X}$ is closed under the saturation if the following condition is true: if a representation $(V, H)$ lies in $\mathfrak{X}$ then all representations $(V, G)$ such that $(V, H)$ is a right epimorphic image of the representation $(V, G)$ also belong to $\mathfrak{X}$.

Thus the representation $(V, G)$ is also unitriangular. □

Remark. For a PI-representation we have the following properties: if a PI-representation $(V, G)$ satisfies the identity
\[
x \circ (y_1 - 1) \ldots (y_n - 1) \equiv 0,
\]
then the representation $(A, \overline{G})$ also satisfies this identity.

Indeed, let $(V, G)$ be a PI-representation, let $(V, \overline{G})$ be a faithful representation corresponding to $(V, G)$ and let $A = \overline{G}$. So we have also the faithful representation $(V, A)$ of the algebra $A$ in the module $V$.

Let the PI-representation $(V, G)$ satisfies the identity
\[
x \circ (y_1 - 1) \ldots (y_n - 1) \equiv 0.
\]

Then the faithful representation $(V, \overline{G})$ also satisfies this identity. So for every $x \in V$ the element $(\overline{g}_1 - 1) \ldots (\overline{g}_n - 1) \in K\overline{G}$ acts as zero. Since the representation $(V, A)$ is faithful we have $(\overline{g}_1 - 1) \ldots (\overline{g}_n - 1) = 0$. So for every $a \in A$ we have
\[
a \cdot (\overline{g}_1 - 1) \ldots (\overline{g}_n - 1) = 0,
\]
and the regular representation \((A, \mathcal{G})\) satisfies the identity
\[
x \circ (y_1 - 1) \ldots (y_n - 1) \equiv 0.
\]

In a similar manner, we can show that if a \(PI\)-representation \((V, G)\) satisfies the identity
\[
x \circ (g - 1)^n = 0,
\]
then the representation \((A, \mathcal{G})\) also satisfies this identity. Thus, if \(g\) is a unipotent element of the representation \((V, G)\) then it is a unipotent element of the representation \((A, \mathcal{G})\).

Now we can prove the theorem 1.19.

**Proof of the theorem.** Let \((V, G)\) be a unipotent \(PI\)-representation, let \((V, \mathcal{G})\) be the faithful representation corresponding to the representation \((V, G)\) and let \(A = \langle \mathcal{G} \rangle\) be a linear span of the group \(\mathcal{G}\) in the algebra \text{End} \(V\). We can note that the regular representation \((A, \mathcal{G})\) also is unipotent.

Assume that the group \(G\) is finitely generated and let \(G = \text{gr}(M)\), where \(M\) is a finite set. We will prove that the representation \((V, \mathcal{G})\) is unitriangular and hence the representation \((V, G)\) is also unitriangular [12].

From a general structure theory [3] it follows that the \(PI\)-algebra \(A\) has a series of ideals
\[
0 = A_0 \subset A_1 \subset A_2 \subset A,
\]
where \(A_1\) is a sum of nilpotent ideals of \(A\), \(A_2\) is the Levitzky radical of \(A\), and the group \(\mathcal{G}\) acts on the factors \(A/A_1\) and \(A/A_2\).

We also have a series of subgroups of the group \(\mathcal{G}\):
\[
\mathcal{T} = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{G},
\]
where \(\mathcal{G}_1\) is a kernel of the action \(\mathcal{G}\) in \(A/A_1\) and \(\mathcal{G}_2\) is a kernel of the action \(\mathcal{G}\) in \(A/A_2\).

Then, From a general theory [3] it also follows that there is an injective homomorphism
\[
\tau : A/A_2 \rightarrow M_n(R),
\]
where \(M_n(R)\) is an algebra of matrices over a commutative ring \(R\) with unit and the given ring \(K\) is contained in \(R\) (see [3]). We can consider the algebra \(M_n(R)\) as an algebra over \(K\), so the homomorphism \(\tau\) is a homomorphism of algebras over \(K\). Moreover we have the representation \((R^n, M_n(R))\).

The representation \((A/A_2, \mathcal{G}/\mathcal{G}_2)\) is unipotent as a homomorphic image of the unipotent representation \((A, \mathcal{G})\), and an element \(\mathcal{g} \in \mathcal{G}\) is a unipotent element of the representation \((A, \mathcal{G})\) if the element \((\mathcal{g} - 1)\) is a nilpotent element of the algebra \(A\).

A group \((\mathcal{G}/\mathcal{G}_2)^{\tau}\) is a subgroup of the group \(GL_n(R)\) of all invertible elements of the algebra \(M_n(R)\). For each element \(\mathcal{g} \in \mathcal{G}/\mathcal{G}_2\) the element \((\mathcal{g} - 1)\) from \(A/A_2\) is nilpotent since the representation \((A/A_2, \mathcal{G}/\mathcal{G}_2)\) is unipotent. The image of this element in the algebra \(M_n(R)\) is also nilpotent. Thus the representation \((R^n, (\mathcal{G}/\mathcal{G}_2)^{\tau})\) is a unipotent representation. According to the Kolchin’s theorem \((R^n, (\mathcal{G}/\mathcal{G}_2)^{\tau})\) is a unitriangular representation.

Well known theorem of Kaloujnine [4] states that if a representation \((V, \mathcal{G})\) of a group \(\mathcal{G}\) is faithful and \(n\)-unitriangular then the group \(\mathcal{G}\) is a nilpotent group of the nilpotency class \((n - 1)\).
Consider the representation \((R^n, \overline{G/G_2})\). This representation is faithful since the regular representation \((A/A_2, \overline{G/G_2})\) and the representation \((R^n, A/A_2)\) are faithful. Moreover the representation \((R^n, (\overline{G/G_2})^r)\) is unitriangular and therefore the representation \((R^n, \overline{G/G_2})\) is also unitriangular. So the group \(\overline{G/G_2}\) is nilpotent.

Since the representation \((A/A_2, \overline{G/G_2})\) is unipotent and the group \(\overline{G/G_2}\) is finitely generated nilpotent group then from \([9]\) it follows that this representation is unitriangular. So the representation \((A/A_1, \overline{G})\) is also unitriangular.

Let the representation \((A/A_1, \overline{G})\) satisfy the identity
\[ x \circ (y_1 - 1) \ldots (y_m - 1) \equiv 0. \]

Remind that we consider the finitely generated group \(\overline{G}\) with a generating set \(M\). Let \(\hat{g} = (\overline{g}_1, \ldots, \overline{g}_m)\) be a sequence of elements from the set \(M^m\). Then the element \((\overline{g}_1 - 1) \ldots (\overline{g}_m - 1)\) lies in \(A_1\). Since the set of different sequences \(\hat{g}\) is finite then there is a nilpotent ideal \(A'_1 \subset A_1\) such that for every \(\hat{g}\) we have \((\overline{g}_1 - 1) \ldots (\overline{g}_m - 1) \in A'_1\). Since \(A'_1\) is an ideal of the algebra \(A\) then it is closed under the regular action of the group \(\overline{G}\) and we can consider the representation \((A/A'_1, \overline{G})\). According to the lemma \([3,3]\) the representation \((A/A'_1, \overline{G})\) satisfies the identity
\[ x \circ (y_1 - 1) \ldots (y_m - 1) \equiv 0. \]

So this representation is unitriangular. Using the lemma \([3,3]\) we conclude that the representation \((V, G)\) is also unitriangular.

For a non-finitely generated group \(G\) we have that the representation \((V, G)\) is locally unitriangular. Theorem is proved. \(\square\)

Theorem \([1,19]\) implies Theorem \([1,23]\) which states that there exists a unipotent radical for a PI-representation and it is coincides with the locally unitriangular radical as a straightforward corollary.

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