On Weighted Simplicial Homology

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Abstract We develop a framework for computing the homology of weighted simplicial complexes with coefficients in a discrete valuation ring. A weighted simplicial complex, $(X, v)$, introduced by Dawson [Cah. Topol. Géom. Différ. Catég. 31 (1990), pp. 229–243], is a simplicial complex, $X$, together with an integer-valued function, $v$, assigning weights to simplices, such that the weight of any of faces are monotonously increasing. In addition, weighted homology, $H^v_n(X)$, features a new boundary operator, $\partial^v_n$. In difference to Dawson, our approach is centered at a natural homomorphism $\theta$ of weighted chain complexes. The key object is $H^v_n(X/\theta)$, the weighted homology of a quotient of chain complexes induced by $\theta$, appearing in a long exact sequence linking weighted homologies with different weights. We shall construct bases for the kernel and image of the weighted boundary map, identifying $n$-simplices as either $\kappa_n$- or $\mu_n$-vertices. Long exact sequences of weighted homology groups and the bases, allow us to prove a structure theorem for the weighted simplicial homology with coefficients in a ring of formal power series $R = F[[\pi]]$, where $F$ is a field. Relative to simplicial homology new torsion arises and we shall show that the torsion modules are connected to a pairing between distinguished $\kappa_n$ and $\mu_{n+1}$ simplices.

Keywords simplicial homology · weighted homology · exact sequence · primary module · bijection

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1 Introduction

Topology aside, the concept of simplicial complexes is of central importance in a variety of fields including data analysis and biology. Many real world data-sets exhibit a simplicial structure [10,11,9] and indeed have been organized as such [3,14,7]. While the arising simplicial complexes can straightforwardly be studied via topological data analysis (TDA) [19,3,15], a prevalent feature of data-sets is the presence of additional simplex-specific data [6].

Dawson introduced in 1990 [4] the concept of a weighted simplicial complex as a simplicial complex equipped with a function \( v: X \to \mathbb{R} \) such that for simplices \( \sigma, \tau \in X \) with \( \sigma \subseteq \tau \), we have \( v(\sigma) | v(\tau) \). Dawson focused on establishing the Eilenberg-Steenrod axioms based on a weighted version of the Mayer-Vietoris sequence and provided a category-theory centered treatment. The key difference between standard and weighted simplicial complexes lies in the weighted boundary operator that incorporates the weight-function \( v \)

\[
d^n_v(\sigma) = \sum_{i=0}^{n} \frac{v(\sigma)}{v(\hat{\sigma}_i)} (-1)^i \hat{\sigma}_i,
\]

where \( \sigma \) is an \( n \)-simplex and \( \hat{\sigma}_i \) denotes the \( i \)-th face of \( \sigma \). By assumption \( v(\hat{\sigma}_i) | v(\sigma) \), whence \( d^n_v \) is a well-defined boundary map.

Subsequent contributions of Ren et al. [12] were more application focused, where an extension of Dawsons framework to a persistent homology of weighted simplicial complexes was presented, followed by [18], where weighted Laplacians were introduced.

Bura et al. [1] studied the homology of certain weighted simplicial complexes with coefficients in discrete valuation rings, arising from the intersections of loops of a pair of RNA secondary structures [2]. [1] connected weighted simplicial homology with simplicial homology via short exact sequences and a certain chain maps \( \theta \). These chain maps originated from the inflation map defined in [1] that allowed to compute the first weighted homology group.

To illustrate how weighted complexes naturally arise and reflecting on [4] and [1], we shall have a closer look at research collaboration networks. These exhibit a simplicial complex structure as follows: researchers are considered vertices, and a \( n \)-simplex in-between \( n + 1 \) researchers appears if those researchers appeared together as authors on a paper (by themselves or among others), see Fig. 1.

However, important features cannot be expressed via the simplicial structure alone, as, for instance, the citation number of a simplex, i.e. the number of citations the \( n + 1 \) authors appeared on together. For each simplex, this integer constitutes a weight and, by construction, the weight of a face of a simplex is larger than or equal to its weight. The weight of a face, however, does not necessarily divide the weight of its simplex and as a result the weighted homology theory put forward by [4,12] is not immediately applicable. To incorporate this type of integer-valued weights, arising in a plethora of real-world data, we follow [1] and work with homology with coefficients in discrete valuation rings.
Definition 1 A weighted simplicial complex is a pair \((X, \omega)\) consisting of a simplicial complex \(X\) and a non-negative integer function \(\omega : X \to \mathbb{N}\) satisfying
\[
\sigma \subseteq \tau \implies \omega(\sigma) \geq \omega(\tau),
\]
for simplices \(\sigma, \tau \in X\).

Given an integral domain \(R\) with \(\pi \in R \setminus \{0\}\), \((X, \omega)\) naturally induces a weight function \(v\) by setting \(v(\sigma) = \pi^{\omega(\sigma)}\) and taking the reciprocal of the coefficients in \(d^n_v\) produces the weighted boundary map \(\partial^n_v : C_n(X, R) \to C_{n-1}(X, R)\),
\[
\partial^n_v(\sigma) = \sum_{i=0}^{n} \frac{v(\hat{\sigma}_i)}{v(\sigma)} \cdot (-1)^i \hat{\sigma}_i = \sum_{i=0}^{n} \pi^{\omega(\hat{\sigma}_i) - \omega(\sigma)} \cdot (-1)^i \hat{\sigma}_i,
\]
where the weighted chain complex \(C_n(X, R)\) is the free \(R\)-module generated by all \(n\)-simplices of \(X\).

The weighted homology \(H^n_v(X)\) of \((X, \omega)\) is then the \(R\)-module \(H^n_v(X) = \ker \partial^n_v / \text{Im} \partial^n_{v+1}\). Clearly, weighted homology is a generalization of the standard simplicial homology, since the weighted homology of a complex having constant weighting is isomorphic to its simplicial homology. Furthermore, it is straightforward to see that, over discrete valuation rings, weighted complexes defined via \(d^n_v\) and \(\partial^n_v\) produce equivalent homology theories.

In this paper, we establish an exact sequence relating simplicial and weighted simplicial complexes. The main result of this paper is a structure theorem for the weighted simplicial homology with coefficients in a ring \(R\) of formal power series over a field \(\mathbb{F}\), i.e., \(R = \mathbb{F}[\![\pi]\!]\).

To this end, we utilize the chain map \(\theta\), which produces homomorphisms between weighted homology groups with respect to different weights on the same simplicial complex. The \(\theta\)-map generalizes the inflation map employed to relate simplicial and weighted homology in case of bi-structures \([1]\). In case \(R = \mathbb{Q}[\![\pi]\!]\), we prove that \(\theta\) is an injective mapping from simplicial homology with integer coefficients to the weighted homology over \(R\) if and only if the integral simplicial homology has no torsion. The \(\theta\)-map gives rise to new homology groups, \(H^n_v(X/\theta)\), constructed via quotients of chain complexes. We establish a long exact sequence linking weighted homologies having two different weights, connected via \(H^n_v(X/\theta)\). Here \(H^n_v(X/\theta)\) is a weighted analogue of the relative homology of a pair and our long exact sequence is a weighted analogue of the long exact sequence for a pair. In case of \(R = \mathbb{F}[\![\pi]\!]\), we proceed by constructing distinguished bases for the kernel \(H^n_v(X^n)\) and the image \(\partial^n_v(X)\) of the weighted boundary map. Such bases do not exist in homology with integer coefficients and split the set of \(n\)-simplices into \(\kappa_n\)- and \(\mu_n\)-simplices. We provide an algorithm producing \(n\)-cycles \(\hat{\beta}_\kappa\) each of which containing exactly one distinguished \(\kappa_n\)-simplex such that \(\{\hat{\beta}_\kappa \mid \kappa_n\}\) forms a basis of \(H^n_v(X^n)\) and the set of \(\mu_n\)-simplices forms a basis of \(\partial^n_v(X)\). We show that the coefficients of \(\hat{\beta}_\kappa\) can be reduced to \(\mathbb{F}\), which in turn, using Nakayama’s Lemma, facilitates the efficient computation of weighted homology modules \([5]\). We are
Fig. 1 Weighted simplicial complex, \((X, \omega)\), of a research collaboration network composed by filled (gray) and empty (white) triangles. Suppose \(A, B, C, D\) represent four authors that have not appeared as co-authors on any papers, however, \(\{A, B, C\}\) or \(\{A, C, D\}\) have written papers together. Suppose \(\{A, B, C\}\) has been cited twice, while \(\{A, C, D\}\) has been cited once, i.e., \(\omega(ABC) = 2\) and \(\omega(ACD) = 1\). Furthermore, any pair appears as authors on some paper, such that the respective citation numbers are given by \(\omega(AB) = 3, \omega(BC) = 4, \omega(AC) = 5, \omega(CD) = 6, \omega(AD) = 7, \omega(BD) = 8\). Furthermore, suppose each individual author has been cited 100 times. Then the first simplicial homology group of the complex is given by \(H_1(X) \cong \mathbb{Z}\) and the first weighted homology group is given by \(H_v^1(X) \cong R \oplus R/\langle \pi \rangle \oplus R/\langle \pi^4 \rangle\). Note that the free submodule of \(H_v^1(X)\) satisfies \(\text{rnk} H_v^1(X) = \text{rnk} H_1(X)\) and the torsion is determined by the differences of citation numbers between pairs of simplices \(AB, ABC\) and \(AC, ACD\).

then in position to prove the structure theorem for the weighted simplicial homology. Specifically, we shall prove that the rank of the weighted simplicial homology equals that of the simplicial homology with coefficients in \(R\), and provide a combinatorial interpretation for the torsion of weighted homology. We show that there exists a pairing between \(\kappa_n\) - and \(\mu_{n+1}\)-simplices of dimension \(n\) and \((n + 1)\), such that the torsion modules stem from primary ideals determined by the difference of weights of each respective pair.

We finally present a case study, where we apply the structure theorem to RNA bi-structures \[1\]. This produces a different, short proof for the weighted homology of the loop complex of an RNA bi-secondary structure \[1\].

The paper is organized as follows: in Section \[2\] we show that \(\theta: H_n(X) \rightarrow H_v^n(X)\) is injective if and only if \(H_n(X)\) has no torsion and establish a long exact sequence for weighted homologies having different weights. In Section \[3\] we construct the \(\kappa_n\) - and \(\mu_n\)-basis for the kernel and image of the weighted boundary map, \(\partial_v^n\). In Section \[4\] we prove the structure theorem for weighted homology and in Section \[5\] we apply our results to RNA bi-structures.

2 First properties of weighted homology

Given weighted complexes \((X, v')\) and \((X, v)\), we define \(\theta_n^{v', v}: C_n(X, R) \rightarrow C_n(X, R)\) by \(\theta_n^{v', v}(\sigma) = \frac{v(\sigma)}{v'(\sigma)} \sigma\). By abuse of notation we shall write \(\theta_n = \theta_n^{v', v}\).
Lemma 1  Let $\theta_n : C_n(X, R) \rightarrow C_n(X, R)$, $\theta_n(\sigma) = \frac{v(\sigma)}{v(\sigma)} \sigma$, then we have the commutative diagram

$$
\cdots \rightarrow C_n(X, R) \xrightarrow{\partial_{n+1}} C_{n-1}(X, R) \xrightarrow{\theta_n} C_n(X, R) \xrightarrow{\partial_n} C_{n-1}(X, R) \xrightarrow{\theta_{n-1}} \cdots
$$

Proof  Clearly,

$$
\theta_{n-1} \circ \partial_n (v) = \sum_{i=0}^{n} (-1)^i \frac{v(\sigma_i)}{v(\sigma)} \partial_i (v) = \sum_{i=0}^{n} (-1)^i \frac{\tau(\sigma_i)}{v(\sigma)} \partial_i (\sigma_i) = \partial_n \circ \theta_{n-1}(v).
$$

Since the $\theta_n$ are chain maps, they induce homomorphisms

Lemma 2  The chain maps $\theta_n$ induce natural homomorphisms

$$
\hat{\theta_n} : H_n^v(X) \rightarrow H_n^v(X), \quad \hat{\theta_n}(\sum_j a_j \sigma_j + \text{Im} \partial^v_{n+1}) = \theta_n(\sum_j a_j \sigma_j) + \text{Im} \partial^v_{n+1}.
$$

The next proposition is straightforward to verify:

Proposition 1  Suppose we have two chain complexes such that $i^v \circ \theta_{n,A} = \theta_{n,B} \circ i^v$ and $j^v \circ \theta_{n,B} = \theta_{n,C} \circ j^v$, i.e. we have the commutative diagram

$$
0 \rightarrow C_n(A, R) \xrightarrow{i^v} C_n(B, R) \xrightarrow{j^v} C_n(C, R) \xrightarrow{0}
$$

Then we have the commutative diagram of long exact homology sequences

$$
\cdots \rightarrow H_n^v(B) \rightarrow H_n^v(C) \rightarrow H_n^v(A) \rightarrow H_{n-1}^v(B) \rightarrow \cdots
$$

and in particular $\hat{\theta}_{n-1,A} \circ \delta_n^v = \delta_n^v \circ \hat{\theta}_{n,A}$ as well as $\hat{\theta}_{n,C} \circ j^v = j^v \circ \hat{\theta}_{n,B}$.

Each simplicial complex can be equipped with a constant weight by setting $v(\sigma) = 1_R$ (the multiplicative identity of $R$) for any $\sigma \in X$. Accordingly, we obtain the chain map $\theta_n : C_n(X) \rightarrow C_n(X, R)$ given by $\theta_n(v(\sigma)) = v(\sigma) \sigma$, and the induced homomorphism $\hat{\theta}_n : H_n(X) \rightarrow H_n^v(X)$ between the simplicial homology and the weighted homology.
Then the following assertions are equivalent:

(a) $\theta_n$ induces the short exact sequence

$$0 \rightarrow H_n(X) \xrightarrow{\theta_n} H_n^v(X)$$

(b) $H_n(X)$ has no torsion.

Proof (a) $\Rightarrow$ (b): we show that if $H_n(X)$ has torsion, then $\theta_n$ is not injective. Suppose there exists some nontrivial $\sum_i a_i \sigma_i + \text{Im} \partial_{n+1}$ such that $q(\sum_i a_i \sigma_i + \text{Im} \partial_{n+1}) = 0$. Then $q(\sum_i a_i \sigma_i) = \partial_{n+1} \sum_j z_j \tau_j$ is equivalent to $\sum_i v(\sigma_i) a_i \sigma_i = \partial_{n+1}^v \sum_j \frac{z_j}{q} v(\tau_j) \tau_j$. Consequently,

$$\bar{\theta}_n(\sum_i a_i \sigma_i + \text{Im} \partial_{n+1}) = \theta_n(\sum_i a_i \sigma_i) + \text{Im} \partial_{n+1} = 0.$$  

(b) $\Rightarrow$ (a): let $\sum_i a_i \sigma_i + \text{Im} \partial_{n+1} \in H_n(X)$, where $a_i \in \mathbb{Z}$ and $\sigma_i \in C_n(X)$. 

Claim. Suppose $\theta_n(\sum_i a_i \sigma_i) = \partial_{n+1}^v (\sum_j q_j \tau_j)$.

To prove the Claim, let $z = \sum_{j \in I} b_j \tau_j$, where $z \in C_{n+1}(X, R)$. We compute

$$\partial_{n+1}^v(z) = \sum_{j,k} b_j (-1)^k \frac{v(\tau_{j,k})}{v(\tau_j)} \bar{\tau}_{j,k} = \sum_i \left[ \sum_{\sigma_i \subset \tau_j} c_{i,j} b_j \frac{v(\sigma_i)}{v(\tau_j)} \right] \sigma_i.$$  

Then

$$\sum_h a_h v(\sigma_h) \sigma_h = \sum_h \left[ \sum_{\sigma_h \subset \tau_j} c_{h,j} b_j \frac{v(\sigma_h)}{v(\tau_j)} \right] \sigma_h, \quad (1)$$

where $\{\sigma_h\}$ is the set of faces of the set of simplices $\{\tau_j\}$ and $a_h = 0$ for $h \not\in I$. We write $v(\tau_j) = \pi^{m_j}$ and $b_j = \sum_n x_{j,n} \pi^n$, where $x_{j,n} \in \mathbb{Q}$ and reformulate eq. (1) via power series

$$\sum_h a_h \sigma_h = \sum_h \left[ \sum_{\sigma_h \subset \tau_j} c_{h,j} x_{j,n} \pi^{n-m_j} \right] \sigma_h, \quad (2)$$

Eq. (2) implies that $r_j = x_{j,m_j} \pi^{m_j}$ has the property

$$\theta_n (\sum_i a_i \sigma_i) = \partial_{n+1}^v (\sum_j b_j \tau_j) = \partial_{n+1}^v (\sum_j r_j \tau_j)$$

By construction, any $r_j \equiv 0 \mod \pi^{m_j}$ which implies

$$\partial_{n+1}^v (\sum_j r_j \tau_j) = \partial_{n+1}^v (\theta_{n+1} (\sum_j x_{j,m_j} \tau_j))$$
and setting $q_j = x_{j,m_j}$ the Claim follows.

Consequently $\zeta = \sum_j q_j \tau_j$ has the property $\theta_n(\sum_i a_i \sigma_i) = (\partial_{n+1}^\nu \circ \theta_{n+1})(\zeta)$. Let $q$ denote the smallest common multiple of the denominators of the $q_j$. Then $q \cdot \zeta$ has integer coefficients and we have

$$(\partial_{n+1}^\nu \circ \theta_{n+1})(q \cdot \zeta) = \theta_n(q \cdot \sum_i a_i \sigma_i).$$

In view of $\partial_{n+1}^\nu \circ \theta_{n+1} = \theta_n \circ \partial_{n+1}$, we derive

$$\theta_n(q \cdot \sum_i a_i \sigma_i) = \partial_{n+1}^\nu \circ \theta_{n+1}(q \cdot \zeta) = \theta_n \circ \partial_{n+1}(q \cdot \zeta).$$

Since $\theta_n : C_n(X) \to C_n(X, R)$ is injective on $n$-chains, this implies $q \cdot \sum_i a_i \sigma_i = \partial_{n+1}(q \cdot \zeta)$, i.e. $q \cdot \sum_i a_i \sigma_i$ is a boundary in $H_n(X)$. By construction, $q \cdot (\sum_i a_i \sigma_i + \text{Im} \partial_{n+1}) = q \cdot \sum_i a_i \sigma_i + \text{Im} \partial_{n+1} = 0 + \text{Im} \partial_{n+1}$, whence $q \cdot (\sum_i a_i \sigma_i + \text{Im} \partial_{n+1} = 0$. Since $H_n(X)$ has no torsion this implies $\sum_i a_i \sigma_i + \text{Im} \partial_{n+1} = 0$, i.e. $\sum_i a_i \sigma_i$ is a boundary and thus trivial in $H_n(X)$ and the proof of the theorem is complete.

Clearly, $\theta_n^\nu(C_n(X, R)) \subset C_n(X, R)$ and denoting the quotient module by $C_n(X/\theta_n^\nu) = C_n(X, R)/\theta_n^\nu(C_n(X, R))$ we have the following commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & C_n(X, R) & \stackrel{\partial_n}{\longrightarrow} & C_{n-1}(X, R) & \stackrel{\theta_n^\nu}{\longrightarrow} & C_n(X/\theta_n^\nu) & \longrightarrow & 0 \\
0 & \longrightarrow & C_{n-1}(X, R) & \stackrel{\theta_n^\nu}{\longrightarrow} & C_n(X/\theta_n^\nu) & \longrightarrow & 0 \\
\end{array}
$$

We shall write $\theta_n$ instead of $\theta_n^\nu$, and $C_n(X/\theta)$ instead of $C_n(X/\theta_n^\nu)$.

Let $H_n^\nu(X/\theta)$ denote the homology with respect to the chain complex $(C_n(X/\theta), \partial_n^\nu)_n$.

**Theorem 2** (a) Let $(X, \nu')$ and $(X, v)$ be weighted complexes with coefficients in an integral domain $R$. Then we have the long exact homology sequence

$$
\cdots \longrightarrow H_{n+1}^\nu(X/\theta) \stackrel{\delta_n^\nu}{\longrightarrow} H_n^\nu(X) \stackrel{\theta_n}{\longrightarrow} H_n^\nu(X/\theta) \stackrel{\partial_n^\nu}{\longrightarrow} H_n^\nu(X) \stackrel{j}{\longrightarrow} H_n^\nu(X/\theta) \stackrel{\delta_n^\nu}{\longrightarrow} H_{n-1}^\nu(X) \longrightarrow \cdots
$$

(b) Suppose $R = \mathbb{F}[[\pi]]$, where $\mathbb{F}$ is a field and $\nu'(\sigma) = 1_R$ for any $\sigma$, i.e., $H_n^\nu(X) \cong H_n(X, R)$. Then the long sequence splits into the exact sequences

$$
\begin{array}{ccccccc}
0 & \longrightarrow & H_n(X, R) & \stackrel{\theta_n}{\longrightarrow} & H_n^\nu(X) & \stackrel{j}{\longrightarrow} & H_n^\nu(X/\theta) & \longrightarrow & 0.
\end{array}
$$
Proof. We consider the commutative diagram of eq. [3]. To define the boundary map $\delta_n^v : H_n^v(X/\theta) \to H_{n-1}^v(X)$, let $c \in C_n(X/\theta)$ be a cycle. Then $c = j(b)$ for some $b \in C_n(X,R)$. Since $j(\partial_n^v(b)) = \partial_n^v(j(b)) = \partial_n^v(c) = 0$, we have $\partial_n^v(b) \in \text{Ker}(j)$. Thus $\partial_n^v(a) = \theta_{n-1}(a)$ for some $a \in C_{n-1}(X)$, since $\text{Ker}(j) = \text{Im}(\theta_{n-1})$. Furthermore $\partial_n^v(a) = 0$ since

$$\theta_{n-2}(\theta_{n-1}(a)) = \partial_n^v(\theta_{n-1}(a)) = \partial_n^v(\partial_n^v(b)) = 0$$

and $\theta_{n-2}$ is injective. We define $\delta_n^v : H_n^v(X/\theta) \to H_{n-1}^v(X)$ by sending the homology class of $c$ to the homology class of $a$, $\delta_n^v[c] = [a]$. This is well-defined: the element $a$ is uniquely determined by $\partial_n^v(b)$, since $\theta_n$ is injective. A different choice $b'$ for $b$ produces $j(b') = j(b)$, whence $b' - b \in \text{Ker}(j) = \text{Im}(\theta_n^v)$. Thus $b' - b = \theta_n(a')$ for some $a'$ and $b = b + \theta_n(a')$. Replacing $b$ by $b + \theta_n(a')$ means to change $a$ to the homologous element $a + \partial_n^v(a')$:

$$\theta_{n-1}(a + \partial_n^v(a')) = \theta_{n-1}(a) + \theta_{n-1}(\partial_n^v(a')) = \partial_n^v b + \theta_n^v(\theta_n(a')) = \partial_n^v(b + \theta_n(a')).$$

A different choice of $c$ within its homology class, i.e. $c + \partial_{n+1}^v(c')$ has the following effect: since $c' = j(b')$ for some $b'$, we then have

$$c + \partial_{n+1}^v(c') = c + j(\partial_{n+1}^v(b')) = c + j(\partial_n^v(b')) = j(b + \partial_n^v(b')),$$

whence $b$ is replaced by $b + \partial_n^v(b')$, which leaves $\partial_n^v(b)$ and $\theta_{n-1}(a)$ and therefore also $a$ unchanged.

As for exactness, we observe first $\text{Im}(\delta_n^v) \subset \text{Ker}(\theta_{n-1})$. By construction, $\theta_{n-1}(a) = \partial_n^v(b)$ and $\delta_n^v(b)$ is by definition trivial in $H_{n-1}^v(X)$. Secondly, $\text{Ker}(\theta_{n-1}) \subset \text{Im}(\delta_n^v)$ holds. Given a cycle $a \in H_{n-1}^v(X)$ such that $\theta_{n-1}(a) = \partial_n^v(b)$ for some $b \in H_n^v(X)$. Consider $j(b)$, we immediately observe that $j(b)$ is a cycle, since $\partial_n^v(j(b)) = j(\partial_n^v(b)) = j(\theta_{n-1}(a)) = 0$, and by construction $\delta_n^v([j(b)]) = [a]$, whence $\text{Ker}(\theta_{n-1}) \subset \text{Im}(\delta_n^v)$ follows.

Therefore we obtain the long exact sequence

$$\begin{array}{cccccc}
\to & H_{n+1}^v(X/\theta) & \xrightarrow{\delta_{n+1}^v} & H_n^v(X) & \xrightarrow{\theta_n} & H_n^v(X/\theta) & \xrightarrow{\delta_n^v} & H_{n-1}^v(X) & \to
\end{array}$$

Claim. We have the short exact sequence

$$\begin{array}{c}
0 \to H_n(X,R) \xrightarrow{\theta_n} H_n^v(X) \\
\xrightarrow{\delta_n} H_{n-1}^v(X)
\end{array}.$$

We first observe that, if $\theta_n(\sum_{i \in I} a_i \sigma_i) = \partial_{n+1}^v(z)$, then

$$\theta_n(\sum_{i \in I} a_i \sigma_i) = \partial_{n+1}^v(\theta_{n+1}(\sum_{j \in J} r_j \tau_j)),$$

where $r_j \in R$. Let $z = \sum_{j \in J} b_j \tau_j$, $z \in C_{n+1}(X,R)$. Then

$$\sum_h a_h \nu(\sigma_h) \sigma_h = \sum_h \left[ \sum_{\sigma_h \subset \tau_j} c_{h,j} \frac{\nu(\sigma_h)}{\nu(\tau_j)} \right] \sigma_h,$$  \hspace{1cm} (4)
where \( \{ \sigma_h \} \) is the set of faces of the set of simplices \( \{ \tau_j \} \) and \( a_h = 0 \) for \( h \notin I \).

We write \( v(\tau_j) = \pi^{m_j} \), \( a_h = \sum_n y_{h,n} \pi^n \) and \( b_j = \sum_n x_{j,n} \pi^n \), where \( x_{j,n} \in \mathbb{F} \).

Rewriting eq. (4) via power series we obtain

\[
\sum_h \left[ \sum_n y_{h,n} \pi^n \right] \sigma_h = \sum_h \left[ \sum_{\sigma_h \subset \tau_j} \sum_n c_{h,j} x_{j,n} \pi^{n-m_j} \right] \sigma_h \tag{5}
\]

and eq. (5) implies that \( r_j = \sum_{n \geq m_j} x_{j,n} \pi^{m_j} \) has the property

\[
\theta_n(\sum_i a_i \sigma_i) = \partial_{n+1}^v(\sum_j b_j \tau_j) = \partial_{n+1}^v(\sum_j r_j \tau_j).
\]

Furthermore by construction, for any \( r_j \) holds \( r_j \equiv 0 \) mod \( \pi^{m_j} \) i.e., \( r_j = r'_j \pi^{m_j} \), whence

\[
\partial_{n+1}^v(\sum_j r_j \tau_j) = \partial_{n+1}^v(\theta_{n+1}(\sum_j r'_j \tau_j)).
\]

As a result we obtain the equality of \( n \)-chains with coefficients in \( R \):

\[
\theta_n(\sum_i a_i \sigma_i) = \partial_{n+1}^v(\sum_j b_j \tau_j) = \partial_{n+1}^v(\theta_{n+1}(\sum_j r'_j \tau_j)) = \theta_n \circ \partial_{n+1}(\sum_j r'_j \tau_j).
\]

Consequently we derive \( \sum_i a_i \sigma_i = \partial_{n+1}(\sum_j r'_j \tau_j) \), i.e., \( \sum_i a_i \sigma_i \) is a boundary in \( H_n(X,R) \).

As a result the connecting homomorphisms, \( \delta_{n+1}^v \), are trivial, whence the long exact sequence splits into the exact sequences

\[
0 \to H_n(X,R) \overset{\theta_n}{\to} H_n^v(X) \overset{j}{\to} H_n^v(X/\theta) \to 0.
\]

**Corollary 1** We have the exact sequence

\[
0 \to H_n(X^n,R) \overset{\theta_n}{\to} H_n^v(X^n) \overset{j}{\to} H_n^v(X^n/\theta) \to 0,
\]

where \( X^n \) denotes the \( n \)-skeleton of \( X \).

### 3 Some combinatorics

**Lemma 3** Let \((X,v)\) be a weighted complex with coefficients in \( R = \mathbb{F}[[\pi]] \). Then we have the short exact sequence of \( R \)-modules

\[
0 \to \pi H_n(X,R) \overset{\theta_n}{\to} H_n(X,R) \overset{\rho}{\to} H_n(X,\mathbb{F}) \to 0, \tag{6}
\]

where the homomorphism \( \rho \) is induced by \( \rho \), which maps a formal power series \( r \in R \) to its constant term \( \bar{r} \).
Proof We first show $\text{Ker}(\hat{\rho}) \subset \pi H_n(X, R)$. Suppose $\hat{\rho}(\sum_j r_j \tau_j + \text{Im} \, \partial_{n+1}) = \sum_j r_j \tau_j + \text{Im} \, \partial_{n+1} = 0$. Then there exists some $\sum_h \hat{a}_h \mu_h \in C_{n+1}(X, F)$, producing the equality of $n$-chains $\sum_j \hat{r}_j \tau_j = \hat{\partial}_{n+1}(\sum_h \hat{a}_h \mu_h) = \sum_j \hat{x}_j \tau_j = 0$, where each coefficient, $\hat{x}_j = 0$. Clearly

$$\sum_j r_j \tau_j + \text{Im} \, \partial_{n+1} = \left[ \sum_j r_j \tau_j - \partial_{n+1, R}(\sum_h \hat{a}_h \mu_h) \right] + \text{Im} \, \partial_{n+1} \in \pi H_n(X, R),$$

from which $\text{Ker}(\hat{\rho}) \subset \pi H_n(X, R)$ follows. It remains to observe $\pi H_n(X, R) \subset \text{Ker}(\hat{\rho})$, which is immediate.

Remark. While $H_n(X, \mathbb{F})$ is free as an $F$-module, $H_n(X, F)$ is not a free $R$-module. In fact, by Lemma 3 we can derive that, as an $R$-module, $H_n(X, \mathbb{F})$ is full torsion and is composed of $m$ copies of $R/(\pi)$, where $m = \text{rk} H_n(X, R)$. Accordingly, the short exact sequence \[\mathfrak{9}\] is not split exact.

Theorem 3 Let $(X, v)$ be a weighted complex with coefficients in $R = \mathbb{F}[\{\pi\}]$. Then there exists a subset of $n$-simplices, $K \subseteq \{ \sigma \ | \ \sigma \in C_n(X, R) \}$ and a distinguished $K$-basis of $H_n^w(X^n)$, $\bar{\mathfrak{Y}}_K^w = \{ \bar{\beta}_\kappa \ | \ \kappa \in K \}$, such that the following holds

(i) any $K$-set has the same cardinality and $M = \mathbb{C}K$ is a basis of $\partial_n^w(C_n(X, R))$,

(ii) each $\bar{\beta}_\kappa \in \bar{\mathfrak{Y}}_K^w$ contains a unique, distinguished simplex $\kappa \in K$, having coefficient one,

$$\hat{\beta}_\kappa = \sum r_\ell \mu_\ell + \kappa, \text{ where } r_\ell \text{ are monomials satisfying } \deg v(\mu_\ell) = \deg r_\ell v(\kappa),$$

(iii) let $\theta_\kappa(\hat{\beta}_\kappa) = v(\kappa)\bar{\beta}_\kappa$, then $\mathfrak{Y}_K^w = \{ \beta_\kappa \ | \ \kappa \in K \}$ is a basis of $H_n(X^n, R)$,

(iv) let $\gamma_\kappa = \hat{\rho}(\hat{\beta}_\kappa)$, then $\{ \gamma_\kappa \ | \ \kappa \in K \}$ is a basis of $H_n(X^n, \mathbb{F})$ and $H_n(X^n, R)$.

Proof We construct $\bar{\mathfrak{Y}}_K^w$ recursively via the following procedure: set $M_0 = \emptyset$ and $S_0 = \{ \sigma \ | \ \sigma \in C_n(X, R) \}$. Label the simplices $\sigma_i$ arbitrary and examine them one by one, producing recursively the sequence $(M_i, S_i)$, where $M_1 = M_0 \cup \{ \mu_1 \ | \ \mu_1 = \sigma_1 \}$ and $S_1 = S_0 \setminus \{ \sigma_1 \}$, i.e. we remove $\sigma_1$ from $S_0$, relabel as $\mu_1$ and add to $M_0 = \emptyset$.

Having constructed $(M_m, S_m)$ we proceed by examining $\sigma_{m+1}$. We set $S_{m+1} = S_m \setminus \{ \sigma_{m+1} \}$ and given the equation

$$\partial_n^w(\sum_{\ell} r_\ell \mu_\ell + r_{m+1} \sigma_{m+1}) = 0, \quad (7)$$

distinguish two scenarios. In case there exists no nontrivial solution of $r_\ell, r_{m+1} \in R$, we set $M_{m+1} = M_m \cup \{ \mu_{m+1} = \sigma_{m+1} \}$. Otherwise, clearing the gcd of $r_\ell$ and $r_{m+1}$, we either have $\sigma_{m+1}$ has coefficient one or some $\mu_\ell$ does. In the former case we set $M_{m+1} = M_m$ and in the latter

$$M_{m+1} = (M_m \setminus \{ \mu_\ell \}) \cup \{ \mu_{m+1} = \sigma_{m+1} \}, \quad S_{m+1} = S_m \setminus \{ \sigma_{m+1} \}.$$
Accordingly we either add a new $\mu$-simplex or replace a previously added $\mu$-simplex, while step by step examining all $n$-simplices. In this process we have $\partial^n_\mu(M_m) \subset \partial^n_\mu(M_{m+1})$, since a $\mu$-simplex replaced in $M_m$ is by construction a linear combination of $M_{m+1}$ $\mu$-simplices.

The procedure terminates in case of $S_t = \emptyset$ and all simplices have been examined. $M_t$ is by construction a basis of $\partial^n_\mu(C_n(X,R))$ inducing the bipartition into the set of $\mu$-simplices, $M$, and the complimentary set of $\kappa$-simplices, $K$. Since any $\partial^n_\mu(C_n(X,R))$-basis has the same size, all $K$-sets have the same cardinality.

For each $\kappa$ there exist unique coefficients $r_\ell \in R$, such that $\hat{\beta}_\kappa = \sum r_\ell \mu_\ell + \kappa$ is a $H^n_\mu(X^n)$-cycle and the $\hat{\beta}_\kappa$-cycles are linearly independent: $0 = \sum \lambda_\kappa \hat{\beta}_\kappa$ implies $\lambda_\kappa = 0$ for all $\kappa$, since the simplex $\kappa$ appears uniquely in $\hat{\beta}_\kappa$.

**Claim 1.** $\mathcal{B}^\mu_\kappa = \{\hat{\beta}_\kappa \mid \kappa \in K\}$ is a basis of $H^n_\mu(X^n)$.

Let $c = \sum_h a_h \sigma_h$ be a $H^n_\mu(X^n)$-cycle. By construction, $c$ contains at least one $\kappa$-simplex. We prove by induction on the number of distinct $\kappa$-simplices contained in $c$ that $c = \sum_\kappa \lambda_\kappa \hat{\beta}_\kappa$. In case of the induction basis $c$ contains exactly one $\kappa$-simplex, $\kappa_0$. Then $c$ contains the summand $r_{\kappa_0} \kappa_0$ and exclusively $\mu$-simplices, otherwise. Clearly, $c - r_{\kappa_0} \kappa_0 = c'$ is a cycle containing only $\mu$-simplices which is, by construction, trivial, whence $c = r_{\kappa_0} \kappa_0$. For the induction step assume $c$ contains $(m + 1)$ $\kappa$-simplices, $\kappa_1, \ldots, \kappa_{m+1}$. Suppose $c$ has the summand $r_{\kappa_{m+1}} \kappa_{m+1}$. Then $c - r_{\kappa_{m+1}} \kappa_{m+1}$ is a cycle that contains exactly $m$ $\kappa$-simplices since $\hat{\beta}_\kappa$ contains, besides $\kappa_{m+1}$, only $\mu$-simplices. By induction hypothesis we then have $c - r_{\kappa_{m+1}} \kappa_{m+1} = \sum_{i=1}^m r_i \kappa_i$ and Claim 1 follows.

**Claim 2.** For each $\hat{\beta}_\kappa = \sum r_\ell \mu_\ell + \kappa$, there exist monomials $r_\ell$ satisfying $\deg v(\mu_\ell) = \deg(rv(\kappa))$ for any $\ell$.

As a $H^n_\mu(X^n)$-cycle, $\hat{\beta}_\kappa$ satisfies $\partial^n_\mu(\hat{\beta}_\kappa) = \partial^n_\mu(\sum r_\ell \mu_\ell + \kappa) = 0$. For any $\beta_\kappa$-face $\sigma$, we derive

$$\sum_{\sigma \subset \mu_\ell} c_{\ell} \frac{r_\ell}{v(\mu_\ell)} + c_\kappa \frac{1}{v(\kappa)} = 0, \quad \text{for } \sigma \subset \kappa,$$

$$\sum_{\sigma \subset \mu_\ell} c_{\ell} \frac{r_\ell}{v(\mu_\ell)} = 0, \quad \text{for } \sigma \not\subset \kappa,$$

where $c_{\ell}$ and $c_\kappa$ are $\pm 1$. We write $v(\mu_\ell) = \pi^{\omega(\mu_\ell)}$, $v(\kappa) = \pi^{\omega(\kappa)}$ and $r_\ell = \sum_n x_{\ell,n} \pi^n$, where $x_{\ell,n} \in \mathbb{F}$. Rewriting the equations we obtain

$$\sum_n \sum_{\sigma \subset \mu_\ell} c_{\ell} x_{\ell,n} \pi^{n-\omega(\mu_\ell)} + c_\kappa \pi^{-\omega(\kappa)} = 0 \quad \text{for } \sigma \subset \kappa,$$

$$\sum_n \sum_{\sigma \subset \mu_\ell} c_{\ell} x_{\ell,n} \pi^{n-\omega(\mu_\ell)} = 0 \quad \text{for } \sigma \not\subset \kappa.$$

In particular, taking $[\pi^{-\omega(\kappa)}]$-terms, we derive

$$\sum_{\sigma \subset \mu_\ell} c_{\ell} x_{\ell,\omega(\mu_\ell)} - \omega(\kappa) + c_\kappa = 0 \quad \text{for } \sigma \subset \kappa,$$

$$\sum_{\sigma \subset \mu_\ell} c_{\ell} x_{\ell,\omega(\mu_\ell)} - \omega(\kappa) = 0 \quad \text{for } \sigma \not\subset \kappa.$$
Let \( m_{\ell} = x_{\ell \omega(\mu_\ell) - \omega(\kappa)} \) be the monomials obtained by taking \( [\pi^{\omega(\mu_\ell) - \omega(\kappa)}] \) terms of \( r_\ell \). Then \( \beta_\kappa = \sum m_{\ell} \mu_\ell + \kappa \) is by construction a \( H_n(X^n) \)-cycle, and therefore \( \beta_\kappa = \hat{\beta}_\kappa \) since \( \beta_\kappa \) is unique. Accordingly, \( r_\ell = m_\ell \), i.e., \( r_\ell \) are monomials satisfying \( \deg v(\mu_\ell) = \deg(r_\ell v(\kappa)) \).

**Claim 3.** \( \mathfrak{B}_K = \{ \beta_\kappa \mid \kappa \in K \} \) is a basis of \( H_n(X^n, R) \), and \( \{ \gamma_\kappa \mid \kappa \in K \} \) is a basis of \( H_n(X^n, \mathbb{F}) \) and \( H_n(X^n, R) \).

By definition, \( \beta_\kappa = \theta_n^{-1}(v(\kappa) \beta_\kappa) = \sum r(v(\kappa))_i \mu_\ell + \kappa \). Since \( r_\ell \) satisfy \( \deg v(\mu_\ell) = \deg(r_\ell v(\kappa)) \) by Claim 2, \( \beta_\kappa \) is well-defined. Note that \( \sum \lambda_i \beta_i = 0 \) implies \( 0 = \sum \lambda_i \theta_n(v_i) \beta_i \), and hence \( \lambda_i v(\kappa) = 0 \) for all \( i \), from which \( \lambda_i = 0 \) follows.

To prove \( \{ \beta_\kappa \mid \kappa \in K \} \) generates \( H_n(X^n, R) \), we observe that \( \kappa \) retains coefficient one in \( \beta_\kappa \). In view of this we proceed as in Claim 1 by induction on the number of distinct \( \kappa \)-edges contained in a \( H_n(X^n, R) \)-cycle.

Analogously we can show, using Lemma 3 that \( \{ \hat{\rho}(\beta_\kappa) \mid \kappa \in K \} \) is a basis of \( H_n(X^n, \mathbb{F}) \), observing that \( \kappa \) appears exclusively in \( \rho(\beta_\kappa) \) having coefficient one. Lemma 3 and Nakayama’s Lemma 4 imply that \( \{ \hat{\rho}(\beta_\kappa) \mid \kappa \in K \} \) is also a basis of \( H_n(X, R) \), whence Claim 3.

Therefore \( \mathfrak{B}_K = \{ \hat{\beta}_\kappa \mid \kappa \in K \} \) is a basis of \( H_n(X^n) \) satisfying (i)-(vi) and the proof of Theorem 3 is complete.

**Remark.** (a) The \( K \)-bases of \( H_n(X^n) \), \( \{ \hat{\beta}_\kappa \mid \kappa \in K \} \), depend on \( \mathbb{F} \), since \( \mathbb{F} \) factors into whether or not eq. 4 has a nontrivial solution in \( R = \mathbb{F}[\pi] \).

(b) The above proof can be generalized to the case where \( R \) is a ring of polynomials over a field, i.e., \( R = \mathbb{F}[\pi] \).

(c) In case \( R \) is a discrete valuation ring, whose uniformizer \( \pi \) is algebraic, we can construct the \( K \)-basis \( \{ \hat{\beta}_\kappa \mid \kappa \in K \} \) as in Theorem 3; however, in general the basis does not satisfy properties (ii)-(vi).

**Corollary 2** Let \( \mathfrak{B}_K \) be a \( K \)-basis of \( H_n(X^n) \). Then

\[
H_n(X^n/\theta) \cong \bigoplus_{\kappa \in K} R/(v(\kappa)).
\]

**Proof** The projection \( \rho: C_n(X, R) \to \oplus_\sigma R/v(\sigma) \), given by \( \sum a_\sigma \mapsto \sum (a_\sigma + v(\sigma)) \) has kernel \( \theta_n(C_n(X, R)) \) and consequently \( C_n(X, R)/\theta_n(C_n(X, R)) \cong \oplus_\sigma R/v(\sigma) \). Since \( v(\kappa) \beta_\kappa = \theta_n(\beta_\kappa) \), each \( \beta_\kappa \) generates a cyclic \( H_n(X^n/\theta) \) submodule isomorphic to \( R/(v(\kappa)) \), from which the Corollary follows.

4 The main theorem

**Lemma 4** Let \( (X, \nu) \) be a weighted complex with coefficients in \( R = \mathbb{F}[\pi] \). Given \( \nu \), we consider the sequence of weight functions \((v_0, v_1, \ldots, v_t = \nu)\).

\[1\] Let \( M \) be a finitely generated module over a local ring \( R \) with maximal ideal \( m \). Then every minimal set of generators of \( M \) is obtained from the lifting of some basis of \( M/mM \).
defined by \( v_r(\sigma) = v(\sigma) \) for \( \dim(\sigma) \leq r \) and \( v_r(\sigma) = 1 \), otherwise. Then there exist the exact sequences

\[
0 \longrightarrow H_n(X, R) \xrightarrow{\bar{\eta}_n^r} H_n^v(X) \xrightarrow{j} \oplus_\kappa R/(v(\kappa)) \longrightarrow 0 \tag{9}
\]

\[
0 \longrightarrow \oplus_\mu R/(v(\mu)) \longrightarrow H_{n-1}^v(X) \xrightarrow{\delta_{n-1}^v} H_{n-1}^\nu(X) \longrightarrow 0, \tag{10}
\]

where \( \bar{\eta}_n^r \) is induced from

\[
\eta_n^r(\sigma) = \theta_{n-1}^r v_r(\sigma) = \begin{cases} v(\sigma) & \text{if } \dim(\sigma) = r \\ \sigma & \text{otherwise.} \end{cases}
\]

**Proof** By construction of \( v_n \), the quotient \( C_\ell(X, R)/\eta_n^\ell(C_\ell(X, R)) \) is only non-trivial for \( \ell = n \), in which case \( C_n(X, R)/\eta_n^\ell(C_n(X, R)) \cong \oplus_\mu R/(v(\sigma)) \), where the summation is over the set of all \( n \)-simplices. Consequently, the boundary maps \( \partial_n^v \) and \( \partial_{n+1}^v \) are trivial, whence

\[
H_\ell^v(X/\eta_n^\ell) \cong \begin{cases} \oplus_\mu R/(v(\sigma)) & \text{for } \ell = n \\ 0 & \text{for } \ell \neq n. \end{cases}
\]

The long homology sequence of Theorem\(2\) then becomes the five term exact sequence

\[
0 \longrightarrow H_{n-1}^v(X) \xrightarrow{\eta_n^r} H_n^v(X) \xrightarrow{j} H_n^v(X/\eta_n^\ell) \xrightarrow{\partial_n^v} H_{n-1}^v(X) \xrightarrow{\delta_{n-1}^v} H_{n-1}^\nu(X) \longrightarrow 0
\]

where \( H_{n-1}^v(X) = H_n(X, R) \), since all \( v_{n-1} \)-weights of \( n \)- and \((n+1)\)-simplices are one. By exactness at \( H_n^v(X/\eta_n^\ell) \cong \oplus_\mu R/(v(\sigma)) \) and \( H_{n-1}^v(X) \), we have

\[
\text{Im } j = \ker \partial_n^v \text{ and } \text{Im } \partial_n^v = \ker \bar{\eta}_{n-1}^n. \quad \text{Since } \eta_{n-1}^n |_{C_{n-1}(X, R)} = \text{id and } \partial_{n-1}^v = \partial_n^v \circ \eta_{n}^n, \text{ we have}
\]

\[
\text{Im } \partial_n^v / \text{Im } \partial_{n-1}^v = \oplus_\mu (\partial_n^{v_n}(\mu))/v(\mu) \partial_n^v(\mu) \cong \oplus_\mu R/(v(\mu))
\]

and the sequence

\[
0 \longrightarrow \oplus_\mu R/(v(\mu)) \xrightarrow{\partial_n^v} H_{n-1}^v(X) \xrightarrow{\partial_{n-1}^v} H_{n-1}^\nu(X) \longrightarrow 0
\]

is exact. Since \( H_{n-1}^v(X/n-1) = H_{n-1}^\nu(X/n-1) \) we have \( H_{n-1}^\nu(X/n) / \text{Im } \partial_{n-1}^v \cong H_{n-1}^\nu(X) \) which provides an interpretation of \( \ker \bar{\eta}_{n-1}^n \), via

\[
0 \longrightarrow \oplus_\mu R/(v(\mu)) \xrightarrow{\partial_n^v} H_{n-1}^v(X) \xrightarrow{\partial_{n-1}^v=\text{proj}} H_{n-1}^\nu(X) \longrightarrow 0.
\]

\( v \) bipartitions the set of \( n \)-simplices into \( \kappa \)- and \( \mu \)-simplices and using the exactness at \( H_n^v(X/\eta_n^\ell) \cong \oplus_\kappa R/(v(\kappa)) \), we obtain

\[
0 \longrightarrow H_n(X, R) \xrightarrow{\bar{\eta}_n^r} H_n^v(X) \xrightarrow{j} \oplus_\kappa R/(v(\kappa)) \longrightarrow 0.
\]
Proof By the general structure theorem of finitely generated modules over pids, we have

where \( \{ \kappa_n^0, \ldots, \kappa_n^n \} \cup \{ \kappa_{n+1}^0, \ldots, \kappa_n^n \} = K \) is a distinguished bipartition of the \( \kappa \)-simplices of dimension \( n \). Furthermore, \( \text{rnk}_R(H^v_n(X)) = \text{rnk}_F(H_n(X,F)) \) and

where \( \alpha \in S_{p-q} \) establishes a pairing between \( \kappa_n^v \)- and \( \mu_{\alpha(1)}^{n+1} \)-simplices of dimension \( n \) and \( (n + 1) \), respectively.

Let \( \varphi_n^0 = \eta^{n+1} \circ \eta_n \), we note that \( \varphi_n^0 = \theta_n \) since both maps coincide on \( n \)- and \( (n + 1) \)-simplices.

Claim 1. We have the exact sequence

By Theorem 2 we have the long exact sequence of homology groups

\[
\begin{align*}
H^\psi_{n+1}(X/\varphi_n) &\xrightarrow{\partial_n^{n+1}} H^\psi_{n}(X) \\
&\xrightarrow{\varphi_n} H^\psi_{n-1}(X) \\
&\xrightarrow{\partial_n^{n-1}} H^\psi_{n-2}(X/\varphi_n)
\end{align*}
\]
By construction \( \varphi_n^\tau = \bar{\theta}_n, H_{n+1}^\tau(X) \cong H_n(X, R), H_{n+1}^\tau(X) = H_n^\tau(X) \) and \( H_{n+1}^\tau(X/\varphi^n) = 0 \). In view of \( C_{n+1}(X/\varphi^n) \cong \bigoplus_{v^\mu+1} R/(v(\mu+1)), \) where the direct sum ranges over all \((n+1)\)-simplices, \( v^\tau \), we have

\[
\bar{\delta}_{n+1}^\tau(C_{n+1}(X/\varphi^n)) = \bar{\delta}_{n+1}^\tau(\bigoplus_{\mu+1} R/(v(\mu+1))) \cong \bigoplus_{\mu+1} R/(v(\mu+1)),
\]

where the summation ranges over all \( \mu+1 \)-simplices which form a basis of \( \bar{\delta}_{n+1}^\tau(X) \). Since \( C_{n-1}(X/\varphi^n) = 0 \) we obtain \( \bar{\delta}_{n+1}^\tau: C_n(X/\varphi^n) \rightarrow 0 \), where \( C_n(X/\varphi^n) \cong \bigoplus_{v^\mu} R/(v(\mu+1)) \). Using \( \bar{\delta}_{n+1}^\tau \circ \delta_{n+1}^\tau = 0 \), we derive

\[
H_{n+1}^\tau(X/\varphi^n) \cong \left( \bigoplus_{v^n} R/(v(\kappa^n)) / \bar{\delta}_{n+1}^\tau(\bigoplus_{\mu+1} R/(v(\mu+1))) \right) \oplus \left( \bigoplus_{v^n} R/(v(\mu^n)) \right).
\]

By Lemma 4 we have the exact sequence

\[
0 \rightarrow \bigoplus_{\mu^n} R/(v(\mu^n)) \rightarrow H_{n+1}^\tau(X) \rightarrow \bar{\delta}_{n+1}^\tau(\bigoplus_{\mu+1} R/(v(\mu+1))) \rightarrow 0,
\]

which combined with exactness of eq. (12) at \( H_{n+1}^\tau(X/\varphi^n) \) and eq. (13) gives rise to the exact sequence of Claim 1:

\[
0 \rightarrow H_n(X, R) \xrightarrow{\delta_n} H_n^\tau(X) \xrightarrow{j} \bigoplus_{v^n} R/(v(\kappa^n))(\bigoplus_{\mu+1} R/(v(\mu+1))) \rightarrow 0
\]

and Claim 1 follows.

We proceed by dissecting the exact sequence of Claim 1 into the free and torsion modules.

**Claim 2.** We have the exact sequence

\[
0 \rightarrow H_n(X, R) \xrightarrow{\delta_n} F_n^\kappa(X) \xrightarrow{j} \bigoplus_{s=1}^q R/(v(\kappa_s^n)) \rightarrow 0,
\]

and \( \text{rk}_R(H_n(X, \mathcal{F})) = \text{rk}_R(H_n(X, \mathcal{F})). \)

In view of \( \delta_n(H_n(X, R)) \subset F_n^\kappa(X) \) and Theorem 2 we have

\[
0 \rightarrow H_n(X, R) \xrightarrow{\delta_n} F_n^\kappa(X).
\]

Furthermore, by Theorem 2 and Corollary 2

\[
0 \rightarrow H_n(X^n, R) \xrightarrow{\delta_n} H_n^\tau(X^n) \xrightarrow{j} \bigoplus_{v^n} R/(v(\kappa^n)) \rightarrow 0.
\]

By restriction, \( j \) induces the surjective homomorphism \( j_{|\delta_n^\tau}: \delta_n^\tau \rightarrow \bigoplus_{s=1}^q R/(v(\kappa_s^n)) \) and

\[
0 \rightarrow H_n(X, R) \xrightarrow{\delta_n} F_n^\kappa(X) \xrightarrow{j_{|\delta_n^\tau}} \bigoplus_{s=1}^q R/(v(\kappa_s^n)) \rightarrow 0.
\]
Since $\bigoplus_{s=1}^{q} R/(v(\kappa^n_s))$ is full torsion, the exact sequence implies \( \text{rnk}_R(H^v_{n+1}(X)) = \text{rnk}_R(H_n(X, R)) \). Combining with \( \text{rnk}_F(H_n(X, F)) = \text{rnk}_R(H_n(X, R)) \) derived by Lemma 3, we have \( \text{rnk}_R(H^v_{n+1}(X)) = \text{rnk}_F(H_n(X, F)) \), whence Claim 2.

Claim 3. We have

\[
T^v_n(X) \cong \bigoplus_{s=q+1}^{[K]} \frac{R/(v(\kappa^n_s))}{(v(\mu^{n+1}_s))}.
\]

We consider the homomorphism embedding \( \text{Im} \partial^v_{n+1} \) into \( \Sigma^v_n \). Since \( R \) is pid, there exists a \( \Sigma^v_n \)-basis, \( \mathfrak{B}_1 = \{ i_{q+1}, \ldots, i_p \} \) and a \( \Sigma^v_{n+1} \)-basis \( \mathfrak{B}_0 = \{ x_s \cdot t_s | s = q + 1, \ldots, p \} \), where \( x_s \in R \) represent the invariant factors.

Claim 3 follows from two observations that put these bases into context with Corollary 1 and Corollary 2. First, since \( \varphi^n_s \) elevates \( n \)- as well as \( (n+1) \)-simplicies to their \( v \)-weight, we have

\[
\text{Im} \partial^v_{n+1}/\partial_n \text{Im} \partial_{n+1} = \partial^v_{n+1} (C_{n+1}(X/\varphi^n)) \cong \bigoplus_{s=q+1}^{p} \frac{R/(v(\mu^{n+1}_s))}{(v(\mu^{n+1}_s))}.
\]

Secondly, using \( H^v_n(X^n) \cong \Sigma^v_n \oplus \hat{\mathfrak{B}}^v_n \) and the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H_n(X^n, R) & \xrightarrow{\theta_n} & \hat{\mathfrak{B}}^v_n & \oplus & j & \bigoplus_{s=1}^{q} R/(v(\kappa^n_s)) & \rightarrow & 0 \\
\downarrow{id} & & \downarrow{\text{inj}} & & \downarrow{\text{inj}} & & \downarrow{\text{inj}} & & \\
0 & \rightarrow & H_n(X^n, R) & \xrightarrow{\theta_n} & H^v_n(X^n) & \oplus & \bigoplus_{s=q+1}^{p} R/(v(\kappa^n_s)) & \rightarrow & 0 \\
\downarrow{\text{inj}} & & \downarrow{\text{inj}} & & \downarrow{\text{inj}} & & \downarrow{\text{inj}} & & \\
0 & \rightarrow & \text{Im} \partial_{n+1} & \xrightarrow{\text{res} \theta_n} & \Sigma^v_n & \oplus & \text{res} j & \bigoplus_{s=q+1}^{p} R/(v(\kappa^n_s)) & \rightarrow & 0
\end{array}
\]

we arrive at

\[
\Sigma^v_n(X)/\partial_n \text{Im} \partial_{n+1} \cong \bigoplus_{s=q+1}^{p} \frac{R/(v(\kappa^n_s))}{(v(\mu^{n+1}_s))}.
\]

In order to see how the \( \kappa^n_s \) and \( \mu^{n+1}_s \) align, we consider the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{Im} \partial_{n+1} & \xrightarrow{\text{res} \theta_n} & \Sigma^v_n & \oplus & \text{res} j & \bigoplus_{s=q+1}^{p} R/(v(\kappa^n_s)) & \rightarrow & 0 \\
\downarrow{id} & & \downarrow{\psi} & & \downarrow{\psi} & & \downarrow{\text{inj}} & & \\
0 & \rightarrow & \text{Im} \partial_{n+1} & \xrightarrow{\text{res} \theta_n} & \text{Im} \partial^v_{n+1} & \oplus & \text{res} j & \bigoplus_{s=q+1}^{p} R/(v(\mu^{n+1}_s)) & \rightarrow & 0
\end{array}
\]

where we extend \( \psi(t_s) = x_s \cdot t_s \) linearly to an \( R \)-module homomorphism \( \psi \). Choosing the \( \Sigma^v_n \)- and \( \partial^v_{n+1} \)-bases \( \mathfrak{B}_1 \) and \( \mathfrak{B}_0 \), respectively, we have \( \Sigma^v_n(X)/\partial_n \text{Im} \partial_{n+1} \cong \sum_s (t_s + \theta_n (\text{Im} \partial_{n+1})) \) as well as \( \text{Im} \partial^v_{n+1} / \partial_n \text{Im} \partial_{n+1} \cong \sum_s (s \cdot t_s + \theta_n (\text{Im} \partial_{n+1})) \). Since \( R \) is a discrete valuation ring, \( (t_s + \theta_n (\text{Im} \partial_{n+1})) \)
and \( t_s + \theta_n(\text{Im} \partial_{n+1}) \) are primary modules and as such indecomposable, whence for each \( q + 1 \leq s \leq p \)

\[
(t_s + \theta_n(\text{Im} \partial_{n+1})) \cong R/(v(\kappa^n_{s_i})) \quad \text{and} \quad (t_s + \theta_n(\text{Im} \partial_{n+1})) \cong R/(v(\mu_{n+1}^{s_1})).
\]

By the commutativity of the right square,

\[
\bar{v}(R/(v(\kappa^n_{s_i}))) \cong \bar{v}(t_s + \theta_n(\text{Im} \partial_{n+1})) = (t_s + \theta_n(\text{Im} \partial_{n+1})) \cong R/(v(\mu_{n+1}^{s_1})).
\]

Thus there exists some permutation \( \alpha \) that pairs \( \kappa^n_{s_i} \) with \( \mu_{n+1}^{s_1} \) with \( \alpha(s_1) = s_2 \) such that

\[
\bar{v}(R/(v(\kappa^n_{s_i}))) \cong R/(v(\mu_{n+1}^{s_1})),
\]

and as a result we arrive at

\[
\mathcal{T}^v_n(X)/\text{Im} \partial_{n+1}^v \cong \left[ \mathcal{T}^v_n(X)/\theta_n(\text{Im} \partial_{n+1}) \right]/\left[ \text{Im} \partial_{n+1}^v/\theta_n(\text{Im} \partial_{n+1}) \right]
\]

\[
\cong \bigoplus_{s=q+1}^{p} \left[ R/(v(\kappa^n_{s_i})) \right]/\left[ R/(v(\mu_{n+1}^{s_1})) \right]
\]

\[
\cong \bigoplus_{s=q+1}^{p} R/(v(\kappa^n_s)/v(\mu_{n+1}^{\alpha(s)})).
\]

**Remark.** In view of the structure theorem, let us revisit the weighted simplicial complex \((X,v)\) depicted in Figure 1. Based on Theorem 3 we compute the \(K\)-basis of \(H_1^v(X)\) given by \(\mathcal{B}_K^v = \{ \hat{\beta}_{AC}, \hat{\beta}_{CB}, \hat{\beta}_{BA} \} \) with \(K = \{AC, CB, BA\}\), where

\[
\hat{\beta}_{AC} = AC + \pi CD + \pi^2 DA
\]

\[
\hat{\beta}_{CB} = CB + \pi^2 BD + \pi^2 DC
\]

\[
\hat{\beta}_{BA} = BA + \pi^2 AD + \pi^5 DB.
\]

The \(\mu^2\)-simplices are given by \(ABC, ACD\) and thus \(\partial_2^v(X) = \{ \partial_2^v(ABC), \partial_2^v(ACD) \}\).

By Theorem 4 we derive a partition \(K = \{BA, AC\} \cup \{CB\}\) and a pairing \(\alpha : \{BA, AC\} \rightarrow \{ABC, ACD\}\) with \(\alpha(BA) = ABC, \alpha(AC) = ACD\). Then the torsion of the first weighted homology \(H_1^v(X)\) is given by

\[
T_1^v \cong R/(\pi^2(AB) - \omega(AC)) \bigoplus R/(\pi^2(AC) - \omega(AD)) \cong R/(\pi) \oplus R/(\pi^4).
\]

Since \(\text{rnk} \ H_1^v(X) = \text{rnk} \ H_1(X, R) = 1\), we obtain \(H_1^v(X) \cong R \oplus R/(\pi) \oplus R/(\pi^4)\).
5 Case study: RNA bi-structures

RNA is a biomolecule that folds into a helical configuration of its sequence by forming base pairs. The most prominent class of coarse-grained structures are the RNA secondary structures [16,13]. A secondary structure can be uniquely decomposed into loops and the free energy of a structure is calculated as the sum of the energy of its individual loops [20].

A bi-structure \((S, T)\) is a pair of secondary structures \(S\) and \(T\) over the same backbone. We represent a bi-structure as a diagram on a horizontal backbone with the \(S\)-arcs drawn in the upper and the \(T\)-arcs drawn in the lower half plane. Two arcs \((i,j)\) and \((k,l)\) are crossing if \(i < k < j < l\). Crossing induces an equivalence relation for which nontrivial equivalence classes are called crossing components. A vertex \(k\) is covered by an arc \((i,j)\) if \(i \leq k \leq j\) and there exists no other arc \((p,q)\) such that \(i < p < k < q < j\). A loop is the set of vertices covered by an arc \((i,j)\), in particular, the exterior loop is given by the set of vertices covered by an artificial rainbow arc connecting the first and last vertices. The loop complex, \(K(S, T)\), is the nerve formed by \(S\)-loops and \(T\)-loops of a bi-structure \((S, T)\). The loop complex \(X = K(S, T)\) can be augmented by assigning a weight to each simplex of \(X\), where the weight encodes the cardinality of intersections of loops in the simplex, see Fig. 2.

\[ \omega(1) = 9, \omega(2) = 10, \omega(3) = 3, \omega(4) = 12, \omega(5) = 8, \]
\[ \omega([1, 2]) = \omega([2, 3]) = \omega([4, 5]) = 2, \omega([1, 4]) = 7, \omega([1, 5]) = 3, \omega([2, 4]) = 6, \omega([2, 5]) = 6, \omega([3, 5]) = 3 \]
\[ \omega([1, 2, 4]) = \omega([1, 2, 5]) = \omega([1, 4, 5]) = \omega([2, 4, 5]) = 1, \omega([2, 3, 5]) = 2. \]

\[ H \] computed the weighted homology for the loop complex of RNA bi-structures. In particular, \[ H \] showed that the weighted simplicial complex of an arbitrary bi-structure can be transformed via Whitehead moves [17] to a complex, which does not contain any 3-simplices or 2-simplices having weight greater than 1. Referring to such complexes as lean, the following holds:

**Theorem 5** \[ H \] Let \((X, v)\) be a lean, weighted loop complex of a bi-structure \((S, T)\), where \(v(\sigma) = \pi \omega(\sigma)\) is given by the size of the intersection of loops. Let
Let $K$ be a $K$-basis of $H^v_{1,R}(X^1)$ and $M = \mathcal{C}K = \{\mu_s\}$ be a basis of $\partial^v_{i,R}(X)$. Then

$$H^v_2(X) \cong R^C$$
$$H^v_1(X) \cong \bigoplus_{\kappa \in K} R/(\pi^\omega(\kappa) - 1)$$
$$H^v_0(X) \cong R \oplus \bigoplus_{\mu_{\alpha(s)} \in M} R/(\pi^\omega(\mu_{\alpha(s)}) - \omega(\mu_{\alpha(s)})),$$

where $C$ denotes the number of crossing components in $(S,T)$, $v_s$ is a 0-simplex of $X$ and the pairing $(v_s, \mu_{\alpha(s)})$ between 0-simplices $v_s$ and 1-simplices $\mu_{\alpha(s)} \in M$ is given by Theorem 4.

This result can be derived from our structure theorem as follows:

**Proof** For simplicial homology with integer coefficients, [1] proved that the loop complex of a bi-structure satisfy $H_2(X) = \mathbb{Z}^C$, $H_1(X) = 0$ and $H_0(X) = \mathbb{Z}$. Combing with $\text{rnk}_{R}(H^v_n(X)) = \text{rnk}_{\mathbb{F}}(H_n(X,\mathbb{F}))$ by Theorem 4 we have $\text{rnk}_{R}(H^v_2(X)) = C$, $\text{rnk}_{R}(H^v_1(X)) = 0$ and $\text{rnk}_{R}(H^v_0(X)) = 1$.

Since the lean complex $X$ contains no 3-simplices, $H^v_2(X)$ is free, whence $H^v_2(X) \cong R^C$.

Let $\{ \delta \ | \ \delta \in \Delta \}$ denote the set of 2-simplices in $X$. Since $\text{rnk}_{R}(H^v_1(X)) = 0$, Theorem 4 shows there exists a bijection $p$ between the set $K$ of 1-simplices $\kappa$ and the set of 2-simplices, i.e., the pairings $(\kappa_i, \delta_{p(i)})$ for each $\kappa_i \in K$. Since each 2-simplex in a lean complex has weight 1, i.e. $\nu(\delta) = \pi$, we have $\frac{v(\kappa_i)}{v(\delta_{p(i)})} = \pi^{\omega(\kappa_i)} - 1$. Theorem 4 establishes that $H^v_1(X) \cong \bigoplus_{\kappa \in K} R/(\pi^\omega(\kappa) - 1)$.

Similarly, Theorem 4 provides the pairing $\alpha$ between 0-simplices $v_s$ and 1-simplices $\mu_{\alpha(s)} \in M$, i.e., $(v_s, \mu_{\alpha(s)})$. Consequently, the torsion of $H^v_0(X)$ is given by $T^v_0 \cong \bigoplus_{\mu_{\alpha(s)} \in M} R/(\pi^\omega(\mu_{\alpha(s)}) - \omega(\mu_{\alpha(s)}))$, completing the proof.

**Remark.** We can extend the above analysis to $\tau$-structures [8], which can be viewed as RNA-RNA interaction structures and generalize bi-structures.

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