APPLICATION OF LOCALIZATION TO THE MULTIVARIATE MOMENT PROBLEM II

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Abstract. The paper is a sequel to the paper [5] by the same author. A new criterion is presented for a PSD linear map $L : \mathbb{R}[x] \rightarrow \mathbb{R}$ to correspond to a positive Borel measure on $\mathbb{R}^n$. The criterion is stronger than Nussbaum’s criterion in [6] and is similar in nature to Schmüdgen’s criterion in [5] [7]. It is also explained how the criterion allows one to understand the support of the associated measure in terms of the non-negativity of $L$ on a quadratic module of $\mathbb{R}[x]$. This latter result extends a result of Lasserre in [3]. The techniques employed are the same localization techniques employed already in [4] and [5], specifically one works in the localization of $\mathbb{R}[x]$ at $p = \prod_{i=1}^n (1 + x_i^2)$ or $p' = \prod_{i=1}^{n-1} (1 + x_i^2)$.

This paper is a sequel to the earlier paper [5]. We present a couple of interesting and illuminating results which were inadvertently overlooked when [5] was written; see Theorems 0.1 and 0.5 below. Theorem 0.1 extends an old result of Nussbaum in [6]. See Theorem 0.3 below for a statement of Nussbaum’s result. The density condition (0.1) appearing in Theorem 0.1 is weaker than the Carleman condition (0.2) appearing in Nussbaum’s result. Theorem 0.5 shows how condition (0.1) allows one to read off information about the support of the measure from the non-negativity of the linear functional on a quadratic module. This illustrates how natural condition (0.1) is. Theorem 0.5 extends a result of Lasserre in [3].

We recall some terminology and notation from [4] and [5]. For an $\mathbb{R}$-algebra $A$ (commutative with 1), a quadratic module of $A$ is a subset $M$ of $A$ such that $1 \in M$, $M + M \subseteq M$ and $f^2 M \subseteq M$ for all $f \in A$. $\sum A^2$ denotes the set of all (finite) sums of squares of $A$. $\sum A^2$ is the unique smallest quadratic module of $A$. A linear map $L : A \rightarrow \mathbb{R}$ is said to be PSD (positive semidefinite) if $L(f^2) \geq 0$ for all $f \in A$, equivalently, if $L(\sum A^2) \subseteq [0, \infty)$. Define $\mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n]$, $\mathbb{C}[x] := \mathbb{C}[x_1, \ldots, x_n]$. If $\mu$ is a positive Borel measure on $\mathbb{R}^n$ having finite moments, i.e., $\int x^k d\mu$ is well-defined and finite for all monomials $x^k := x_1^{k_1} \ldots x_n^{k_n}$, $k_j \geq 0$, $j = 1, \ldots, n$, the PSD linear map $L_\mu : \mathbb{R}[x] \rightarrow \mathbb{R}$ is defined by $L_\mu(f) = \int f d\mu$. If

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\(\nu\) is another positive Borel measure on \(\mathbb{R}^n\) having finite moments then we write \(\mu \sim \nu\) is indicate that \(\mu\) and \(\nu\) have the same moments, i.e., \(L_\mu = L_\nu\). We say \(\mu\) is determinate if \(\mu \sim \nu \implies \mu = \nu\).

**Theorem 0.1.** Suppose \(L: \mathbb{R}[x] \to \mathbb{R}\) is linear and PSD and, for \(j = 1, \ldots, n-1\),

\[
(0.1) \exists \text{ a sequence } \{q_{jk}\}_{k=1}^\infty \text{ in } \mathbb{C}[x] \text{ such that } \lim_{k \to \infty} L([1 - (1 + x_j^2)q_{jk}\overline{q_{jk}}]) = 0.
\]

Then there exists a positive Borel measure \(\mu\) on \(\mathbb{R}^n\) such that \(L = L_\mu\). If condition \((0.1)\) holds also for \(j = n\) then the measure is determinate.

**Proof.** Extend \(L\) to \(\mathbb{C}[x]\) in the obvious way, i.e., \(L(f_1 + if_2) := L(f_1) + iL(f_2)\). Define \((f, g) := L(f\overline{g})\), \(|f| := \sqrt{(f, f)}\). According to [5, Corollary 4.8] to prove the existence assertion it suffices to show that \(\forall g \in \mathbb{C}[x]\) and \(\forall j = 1, \ldots, n-1,\)

\[
\lim_{k \to \infty} L(g(1 - (1 + x_j^2)q_{jk}\overline{q_{jk}})) = 0.
\]

This is immediate from condition \((0.1)\), using the Cauchy-Schwartz inequality. According to [5, Corollary 2.7], to show uniqueness it suffices to show \(\forall j = 1, \ldots, n\) \exists a sequence \(\{p_{jk}\}_{k=1}^\infty\) in \(\mathbb{C}[x]\) such that

\[
\lim_{k \to \infty} L([1 - (x_j - i)p_{jk}]^2) = 0.
\]

Uniqueness follows from this criterion, taking \(p_{jk} := (x_j + i)q_{jk}\overline{q_{jk}}\).

We remark that [5, Theorem 4.9] is a consequence of Theorem 0.1. This is immediate from the following:

**Lemma 0.2.** Suppose \(L: \mathbb{R}[x] \to \mathbb{R}\) is linear and PSD. Suppose \(\{q_{jk}\}_{k=1}^\infty\) is a sequence of polynomials in \(\mathbb{C}[x]\). Then

\[
\lim_{k \to \infty} L([1 - (x_j - i)q_{jk}]^4) = 0 \iff \lim_{k \to \infty} L([1 - (1 + x_j^2)q_{jk}\overline{q_{jk}}]^2) = 0.
\]

**Proof.** Let \(Q_k := 1 - (x_j - i)q_{jk}\). Thus

\[
1 - (1 + x_j^2)q_{jk}\overline{q_{jk}} = 1 - (1 - Q_k)(1 - \overline{Q_k}) = Q_k + \overline{Q_k} - Q_k\overline{Q_k}.
\]

We are assuming \(\|Q_k\overline{Q_k}\| \to 0\) as \(k \to \infty\) and we want to show \(\|Q_k + \overline{Q_k} - Q_k\overline{Q_k}\| \to 0\) as \(k \to \infty\). Applying the Cauchy-Schwartz inequality and the triangle inequality we obtain

\[
\|Q_k\|^2 = \|\overline{Q_k}\|^2 = (Q_k\overline{Q_k}, 1) \leq \|Q_k\overline{Q_k}\| \cdot \|1\| \quad \text{and}
\]

\[
\|Q_k + \overline{Q_k} - Q_k\overline{Q_k}\| \leq \|Q_k\| + \|\overline{Q_k}\| + \|Q_k\overline{Q_k}\| \leq 2\sqrt{\|Q_k\overline{Q_k}\| \cdot \|1\| + \|Q_k\overline{Q_k}\|}.
\]

At this point the result is clear.

The following result of Nussbaum [6, Theorem 4.11] can also be seen as a consequence of Theorem 0.1.
Theorem 0.3 (Nussbaum). Suppose \( L : \mathbb{R}[x] \to \mathbb{R} \) is linear and PSD and, for \( j = 1, \ldots, n-1 \), the Carleman condition

\[
\sum_{k=1}^{\infty} \frac{1}{2^k L(x_j^{2k})} = \infty
\]

holds. Then there exists a positive Borel measure \( \mu \) on \( \mathbb{R}^n \) such that \( L = L_\mu \). If condition (0.2) holds also for \( j = n \) then the measure is determinate.

Proof. Argue as in [5, Theorem 4.10]. Let \( \mu_j \) be the positive Borel measure on \( \mathbb{R} \) such that \( L|_{\mathbb{R}[x_j]} = L \mu_j \). According to [1, Théorème 3], the Carleman condition (0.2) implies that \( C[x_j] \) is dense in the Lebesgue space \( L^s(\mu_j) \) for all \( s \in [1, \infty) \). Fix \( s > 4 \). Thus \( \exists q_{jk} \in C[x_j] \) such that \( \lim_{k \to \infty} \|q_{jk} - \frac{1}{x_j - i}\|_{s,\mu_j} = 0 \). An easy application of Hölder’s inequality (taking \( p = \frac{s}{4}, q = \frac{s}{s-4} \)) shows that

\[
L(|1 - (x_j - i)q_{jk}|^4) = \int |q_{jk} - \frac{1}{x_j - i}|^4 |x_j - i|^4 d\mu_j
\leq [\|q_{jk} - \frac{1}{x_j - i}\|_{s,\mu_j} \cdot \|x - i\|_{s,\mu_j}]^4
\]

so \( \lim_{k \to \infty} L(|1 - (x_j - i)q_{jk}|^4) = 0 \). The result follows now, by Lemma 0.2 and Theorem 0.1. \( \square \)

The reader should compare Theorems 0.1 and 0.3 with the following result of Schmüdgen [5, Theorem 4.11] [7, Proposition 1], which, according to Fuglede [2, p. 62], is an unpublished result of J.P.R. Christensen, 1981.

Theorem 0.4 (Schmüdgen). Suppose \( L : \mathbb{R}[x] \to \mathbb{R} \) is linear and PSD. Fix a positive Borel measure \( \mu_j \) on \( \mathbb{R} \) such that \( L|_{\mathbb{R}[x_j]} = L \mu_j \) and suppose for \( j = 1, \ldots, n-1 \) that \( C[x_j] \) is dense in \( L^4(\mu_j) \), i.e.,

\[
(0.3) \quad \exists \text{ a sequence } \{q_{jk}\}_{k=1}^\infty \text{ in } C[x_j] \text{ such that } \lim_{k \to \infty} \|q_{jk} - \frac{1}{x_j - i}\|_{4,\mu_j} = 0.
\]

Then there exists a positive Borel measure \( \mu \) on \( \mathbb{R}^n \) such that \( L = L_\mu \). If condition (0.3) holds also for \( j = n \) then the measure is determinate.

By considering products of measures of the sort considered by Sodin in [8], one sees that Theorem 0.1 and Theorem 0.4 are both strictly stronger than Nussbaum’s result. But it is not clear, to the author at least, how Theorems 0.1 and 0.4 are related. In particular, it is not clear that either result implies the other.

We turn now to the problem of describing the support of \( \mu \). By definition, the support of \( \mu \) is the smallest closed set \( K \) of \( \mathbb{R}^n \) satisfying \( \mu(\mathbb{R}^n \setminus K) = 0 \). We recall
additional notation from [4] and [5]. If $M$ is a quadratic module of an $\mathbb{R}$-algebra $A$, define

$$X_M := \{ \alpha : A \to \mathbb{R} \mid \alpha \text{ is an } \mathbb{R} \text{-algebra homomorphism, } \alpha(M) \subseteq [0, \infty) \}.$$ 

If $M = \sum A^2 + I$, where $I$ is an ideal of $A$, the condition $\alpha(M) \subseteq [0, \infty)$ is equivalent to the condition $\alpha(I) = \{0\}$. $\mathbb{R}[x]_p$ denotes the localization of $\mathbb{R}[x]$ at $p$ where $p := \prod_{j=1}^n (1 + x_j^2)$. If $A$ is $\mathbb{R}[x]$ or $\mathbb{R}[x]_p$, then algebra homomorphisms $\alpha : A \to \mathbb{R}$ are identified with points of $\mathbb{R}^n$ via the map $\alpha \mapsto (\alpha(x_1), \ldots, \alpha(x_n))$ and $X_M$ is identified with the set $\{ \mathbf{a} \in \mathbb{R}^n \mid g(\mathbf{a}) \geq 0 \ \forall \ \mathbf{a} \in M \}$.

**Theorem 0.5.** Suppose $L : \mathbb{R}[x] \to \mathbb{R}$ is a PSD linear map satisfying (0.7) for $j = 1, \ldots, n$ and $g \in \mathbb{R}[x]$ is such that $L(gh^2) \geq 0 \ \forall \ h \in \mathbb{R}[x]$. Then the support of the associated positive Borel measure $\mu$ is contained in the set $\{ \mathbf{a} \in \mathbb{R}^n \mid g(\mathbf{a}) \geq 0 \}$.

See [3] Theorem 2.2 for an earlier version of this result.

**Proof.** Denote by $L : \mathbb{R}[x]_p \to \mathbb{R}$ the PSD linear extension of $L$ defined by $L(f) := \int f \, d\nu \ \forall \ f \in \mathbb{R}[x]_p$. We claim that $L(gh^2) \geq 0 \ \forall \ h \in \mathbb{R}[x]_p$ (so, in particular, $L(gh^2) \geq 0 \ \forall \ h \in \mathbb{R}[x]_p$). The proof is by induction of the number of factors of the form $x_j \pm i$, $j = 1, \ldots, n$ appearing in the denominator of $h$. Suppose $x_j \pm i$ appears in the denominator of $h$. Note that $(x_j \pm i)h_{qjk}$ has fewer factors $x_j \pm i$ appearing in the denominator, so, by induction, $L(g(1 + x_j^2)h_{qjk} q_{jk}) \geq 0$. Applying the Cauchy-Schwartz inequality, we see that $L(g(1 + x_j^2)h_{qjk} \overline{q_{jk}}) \to 0$ as $k \to \infty$. It follows that $L(g(1 + x_j^2)h_{qjk} \overline{q_{jk}}) \to L(gh \overline{h})$ as $k \to \infty$, so $L(gh \overline{h}) \geq 0$. This proves the claim. Denote by $Q$ the quadratic module of $\mathbb{R}[x]_p$ generated by $g$, i.e., $Q := \sum \mathbb{R}[x]^2 + \sum \mathbb{R}[x] q_{jk} q_{jk}$. It follows from the claim together with the fact that $L$ is PSD on $\mathbb{R}[x]_p$ that $L(Q) \subseteq [0, \infty)$. By [4] Corollary 3.4 there exists a positive Borel measure $\nu$ on $X_Q = \{ \mathbf{a} \in \mathbb{R}^n \mid g(\mathbf{a}) \geq 0 \}$ such that $L(f) = \int f \, d\nu \ \forall \ f \in \mathbb{R}[x]_p$. Uniqueness of $\nu$ implies $\mu = \nu$. \hfill \Box

**Corollary 0.6.** If $L$ satisfies condition (0.7) for $j = 1, \ldots, n$ and $L(M) \subseteq [0, \infty)$ for some quadratic module $M$ of $\mathbb{R}[x]$ then the support of the associated positive Borel measure $\mu$ is contained in the set $X_M = \{ \mathbf{a} \in \mathbb{R}^n \mid g(\mathbf{a}) \geq 0 \ \forall \ \mathbf{a} \in M \}$.

**Remark 0.7.**

(1) The quadratic module $M$ is not required to be finitely generated, although this seems to be the most interesting case.

(2) For a quadratic module of the form $M = \sum \mathbb{R}[x]^2 + I$, $I$ an ideal of $\mathbb{R}[x]$, one can weaken the hypothesis. It is no longer necessary to assume that $L$ satisfies condition (0.7) for $j = 1, \ldots, n$ but only that $L = L_\mu$. This is more or less clear.
By the Cauchy-Schwartz inequality, for $g \in \mathbb{R}[x],$

$$L(gh) = 0 \forall h \in \mathbb{R}[x] \iff L(g^2) = 0 \iff L(gh) = 0 \forall h \in \mathbb{R}[x]_p.$$ 

Also, in this case, $X_M = \mathcal{Z}(I) = \{a \in \mathbb{R}^n \mid g(a) = 0 \forall g \in I\}.$

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