SHARPLY O-MINIMAL STRUCTURES AND SHARP CELL
DECOMPOSITION

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Abstract. Sharply o-minimal structures (denoted *o-minimal) are a strict subclass of the o-minimal structures, aimed at capturing some finer features of structures arising from algebraic geometry and Hodge theory. Sharp o-minimality associates to each definable set a pair of integers known as format and degree, similar to the ambient dimension and degree in the algebraic case; gives bounds on the growth of these quantities under the logical operations; and allows one to control the geometric complexity of a set in terms of its format and degree. These axioms have significant implications on arithmetic properties of definable sets – for example, *o-minimality was recently used by the authors to settle Wilkie’s conjecture on rational points in $\mathbb{R}^{\exp}$-definable sets.

In this paper we develop some basic theory of sharply o-minimal structures. We introduce the notions of reduction and equivalence on the class of *o-minimal structures. We give three variants of the definition of *o-minimality, of increasing strength, and show that they all agree up to reduction. We also consider the problem of “sharp cell decomposition”, i.e. cell decomposition with good control on the number of the cells and their formats and degrees. We show that every *o-minimal structure can be reduced to one admitting sharp cell decomposition, and use this to prove bounds on the Betti numbers of definable sets in terms of format and degree.

1. Introduction

1.1. Notation. For positive integers $m \geq n$, let $\pi_n^m : \mathbb{R}^m \to \mathbb{R}^n$ be the standard projection to the first $n$ coordinates. Usually $m$ is clear from the context and is omitted from the notation. We sometimes restrict $\pi_n^m$ to $I^n$ where $I := (0,1)$, without changing the notation. For a set $X \subset \mathbb{R}^\ell$ we denote $X^c$ to be the complement of $X$ and $\partial X$ to be the frontier of $X$, that is, $\partial X := \overline{X} \setminus X$.

1.2. About this paper. Sharply o-minimal structures (see Section 1.3) are o-minimal structures with a double filtration on the collection of all definable sets by “degree” and “format” (called FD-filtration) satisfying some natural axioms. They were introduced by the first two authors in [4], and used by the authors in [5] to prove Wilkie’s conjecture [8, Conjecture 1.11]. It turns out, that with sharp cellular decomposition (or *CD for short, see Section 1.6) one can significantly strengthen the axiom system for sharply o-minimal structures. We will therefore refer to the sharply o-minimal structures defined in [5, 4] by weakly-sharp, or W*o-minimal, and rename sharply o-minimal structures to be structures satisfying these stronger axioms, see Section 1.3. Sharp cellular decomposition is crucial.

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for the application of sharply o-minimal structures, for instance, all of the results of [5] are proved under the assumption of #CD.

It is not currently known whether every sharply o-minimal structure has #CD or not. In this paper we prove a somewhat weaker result sufficient for all applications. Namely, given a *o-minimal structure \((\mathcal{S}, \Omega)\) we define a new FD-filtration \(\Omega^*\) of \(\mathcal{S}\) such that \(\Omega\) is reducible to \(\Omega^*\) (see Section 1.3) and such that \((\mathcal{S}, \Omega^*)\) is *o-minimal structure with #CD. This construction is inspired by the one developed in [3], where the authors produce an FD-filtration \(\Omega^*\) on \(\mathbb{R}^{\text{Pfaff}}\) which is weakly-sharp with #CD, out of an FD-filtration \(\Omega\) introduced by Gabrielov and Vorobjov (see [1] or section 1.4.3) that doesn’t satisfy these properties. The FD-filtration \(\Omega^*\) on the structure \(\mathbb{R}^{\text{Pfaff}}\) was the one used in [5].

The main claim above holds with much weaker assumptions on \((\mathcal{S}, \Omega)\). Thus we introduce presharp structures, denoted \(P^\#\text{o-minimal}\), and prove that a \(P^\#\text{o-minimal}\) structure \((\mathcal{S}, \Omega)\) can be reduced to a *o-minimal structure \((\mathcal{S}, \Omega^*)\) with #CD. In particular, it is sufficient to construct a presharp structure in order to obtain a *o-minimal structure. As an application of our main results, in the last section we discuss sharp triangulations of sets definable in sharply o-minimal structures, and prove a bound on their Betti numbers.

1.3. Sharply o-minimal structures. Let \(\mathcal{S}\) be an o-minimal expansion of the real field. We introduce the notion of an FD-filtration, a filtration on the collection of all definable sets in \(\mathcal{S}\) by two natural numbers, called format (denoted \(F\)) and degree (denoted \(D\)).

**Definition 1.1** (FD-filtrations). We say that \(\Omega = \{\Omega_{F,D}\}_{F,D \in \mathbb{N}}\) is an FD-filtration on \(\mathcal{S}\) if

1. every \(\Omega_{F,D}\) is a collection of definable sets,
2. \(\Omega_{F,D} \subseteq \Omega_{F+1,D} \cap \Omega_{F,D+1}\) for every \(F, D\), and
3. every definable set belongs to \(\Omega_{F,D}\) for some \(F, D\).

If \(X \in \Omega_{F,D}\) we say that \(X\) has format \(F\) and degree \(D\). We say that a definable function \(f : X \to Y\) has format \(F\) and degree \(D\) if its graph \(\Gamma_f\) is in \(\Omega_{F,D}\).

We introduce the notion of reduction and equivalence of FD-filtrations.

**Definition 1.2** (Reduction of FD-filtrations). Let \(\Omega, \Omega'\) be two FD-filtrations on a structure \(\mathcal{S}\). We say that \(\Omega\) is reducible to \(\Omega'\) and write \(\Omega \leq \Omega'\) if there exists a function \(a : \mathbb{N} \to \mathbb{N}\), and for every \(F, D \in \mathbb{N}\) a polynomial \(P_F\) with positive coefficients such that

\[
\forall F, D \in \mathbb{N}, \quad \Omega_{F,D} \subseteq \Omega'_{a(F),P_F(D)}.
\]

We say that \(\Omega, \Omega'\) are equivalent if \(\Omega \leq \Omega'\) and \(\Omega' \leq \Omega\).

**Remark 1.3.** The following is the main idea of the definition of reduction. Let \(\mathcal{S}\) be an o-minimal expansion of \(\mathbb{R}\), and let \(\Omega \leq \Omega'\) be two FD-filtrations on \(\mathcal{S}\). Let

\[
F(X_1, \ldots, X_m, f_1, \ldots, f_n)
\]

be a function which associates a real number to every tuple of \(m\) definable sets and \(n\) definable functions. Suppose that it is known, that if the sets \(X_j\) and functions \(f_i\) are in \(\Omega'_{F,D}\), then \(F(X_1, \ldots, X_m, f_1, \ldots, f_n)\) is bounded by \(\text{poly}_F(D)\). Then the same is true for \(\Omega\) (albeit with a larger bound).
We will now define *o-minimal (resp. W*o-minimal, P*o-minimal) structure.

**Definition 1.4 (Presharp structures).** A presharp o-minimal structure, denoted by P*o-minimal, is a pair $\Sigma = (S, \Omega)$ where $S$ is an o-minimal expansion of $\mathbb{R}$ and $\Omega = \{\Omega_{F,D}\}_{F,D \in \mathbb{N}}$ is an FD-filtration, such that for every $F \in \mathbb{N}$ there exists a polynomial $P_F$ with positive coefficients such that the following axioms are satisfied.

1. If $A \in \Omega_{F,D}$, then:
   1. (P1) If $A \subset \mathbb{R}$, it has at most $P_F(D)$ connected components,
   2. (P2) If $A \subset \mathbb{R}^\ell$ then $F \geq \ell$,
   3. (P3) If $A \subset \mathbb{R}^\ell$ then $\pi_{\ell-1}(A), A^c, A \times \mathbb{R}, \mathbb{R} \times A$ are in $\Omega_{F+1,D}$. If $A_1, A_2 \subset \mathbb{R}^\ell$ with $A_i \in \Omega_{F_i,D_i}$ and $F := \max\{F_1, F_2\}$, $D := D_1 + D_2$, then
   4. (P4) $A_1 \cup A_2 \in \Omega_{F+1,D}$,
   5. (P5) $A_1 \cap A_2 \in \Omega_{F+1,D}$.

2. If $P \in \mathbb{R}[x_1, \ldots, x_\ell]$, then
   6. (P6) $\{P = 0\} \in \Omega_{\ell, \deg P}$.

**Definition 1.5.** A presharp o-minimal structure $\Sigma = (S, \Omega)$ is called weakly-sharp (denoted W*o-minimal) if, instead of (P4-P5), a stronger property holds: for every $A_1, \ldots, A_k \subset \mathbb{R}^\ell$ with $A_i \in \Omega_{F_i,D_i}$ and $F := \max\{F_i\}$, $D := \sum D_i$,

1. (W4) $\cup_i A_i \in \Omega_{F,D}$,
2. (W5) $\cap_i A_i \in \Omega_{F+1,D}$.

For $j \in \{1, 2, 3, 6\}$ we set $(W_j) = (P_j)$ for convenience.

**Definition 1.6.** A W*o-minimal structure is called *o-minimal if (W3) is replaced by a stronger axiom:

1. (W3) If $A \subset \mathbb{R}^\ell$ and $A \in \Omega_{F,D}$, then $\mathbb{R} \times A, A \times \mathbb{R}$ are in $\Omega_{F+1,D}$, while $\pi_{\ell-1}(A), A^c$ are in $\Omega_{F,D}$,

and in addition in (W5) the format is not raised, i.e (W5) is replaced by

1. (W5) $\cap_i A_i \in \Omega_{F,D}$.

under the same notation for $A_i$ as in (W5). For $j \in \{1, 2, 4, 6\}$, we set $(W_j) = (W_j)$ for convenience.

**Remark 1.7.** Given an o-minimal expansion $S$ of $\mathbb{R}$, we will sometimes refer to a an FD-filtration as being sharp (resp. weakly-sharp, presharp) instead of the pair $(S, \Omega)$ for brevity.

**Remark 1.8.** We recall once again that in [5, 4], W*o-minimal structures were called #o-minimal structures.

We now state a simplified version of our main result.

**Theorem 1.9.** Let $(S, \Omega)$ be a presharp structure. Then there exists a filtration $\Omega^*$ such that $\Omega \leq \Omega^*$, and moreover $(S, \Omega^*)$ is a *o-minimal structure with #CD.
Thus the current state of affairs can be summarized in the following diagram, where the arrow represents reduction, and the equality represents equivalence.

\[
P^\#\text{o-minimal} \cup P^\#\text{o-minimal + #CD} \supset W^\#\text{o-minimal} \cup W^\#\text{o-minimal + #CD} \supset \#\text{o-minimal} \cup \#\text{o-minimal + #CD}
\]

**Remark 1.10.** Given a collection \( \{A_\alpha \subset \mathbb{R}^{L_\alpha}\} \) of sets generating a structure \( S \), and associated formats and degrees \( F_\alpha, D_\alpha \) satisfying \( F_\alpha \geq L_\alpha \), one can consider the minimal FD-filtration \( \Omega \) satisfying all axioms of \#o-minimal structures (resp. W\#o-minimal and P\#o-minimal) except for the first axiom. We call it the FD-filtration sharply (resp. weakly-sharply or presharply) generated by \( \{(A_\alpha, F_\alpha, D_\alpha)\} \). The pair \((S, \Omega)\) will be \#o-minimal (resp. W\#o-minimal or P\#o-minimal) if and only if the axiom (1) is satisfied. In fact, one can repeat this construction even if for every \( A_\alpha \) there is an associated subset \( P_{A_\alpha} \subset \mathbb{N}^2 \) of format-degree pairs \((F, D)\) such that \( A_\alpha \in \Omega_{F,D} \). Therefore given any FD-filtration \( \Omega \), one can consider the FD-filtration sharply (resp. weakly-sharply or presharply) generated by \( \Omega \).

### 1.4. Examples and non-examples of \#o-minimal structures.

We briefly recall below the three main examples, and refer the reader to [4] for details.

1.4.1. **The semi-algebraic structure** \( \mathbb{R}_{\text{alg}} \). Let \( \Omega'_{F,D} \) be the collection of sets \( X \) in \( \mathbb{R}^\ell \) which are presentable as a union of finitely many basic sets

\[
\{P_1 = \cdots = P_k = 0, Q_1, \ldots, Q_s > 0\}, \quad P_j, Q_j \in \mathbb{R}[x_1, \ldots, x_\ell],
\]

such that the sum of the degrees of all \( P \) and \( Q \) over all these basic sets defining \( X \) is \( D \). The filtration \( \Omega' \) is not \#o-minimal. However, the filtration \( \Omega \) sharply generated by \( \Omega' \) is \#o-minimal. This is a non-trivial fact that follows from effective cell decomposition in the semialgebraic category, see [9].

1.4.2. **The analytic structure** \( \mathbb{R}_{\text{an}} \). Not surprisingly, \( \mathbb{R}_{\text{an}} \) is not sharply o-minimal with respect to any FD-filtration. In fact, it is not even presharp, but by Theorem 1.9 it is sufficient to show that it is not sharp. Assume the contrary. Let \( \omega_1 = 1 \) and \( \omega_{n+1} = 2^{\omega_n} \), and let \( \Gamma = \{y = f(z)\} \subset \mathbb{C}^2 \) denote the graph of the holomorphic function \( f(z) = \sum_{j=1}^{\infty} z^{\omega_j} \) restricted to the disc of radius 1/2 (this function is definable in \( \mathbb{R}_{\text{an}} \)). By axioms of sharpness, the number of points in

\[
\Gamma \cap \{y = \epsilon_n + \sum_{j=1}^{n} z^{\omega_j}\},
\]

should be bounded by \( P(\omega_n) \), where \( P \) is some polynomial defined by the format and degree of \( \Gamma \). However, by basic complex analysis, if \( \epsilon_n \) is small enough, then the number of points in this set is at least \( \omega_{n+1} = 2^{\omega_n} \), giving a contradiction for large \( n \).
1.4.3. Pfaffian structures. Let $B \subset \mathbb{R}^\ell$ be an open box. A sequence $f_1, \ldots, f_m : B \to \mathbb{R}$ of real-analytic functions is called a Pfaffian chain if they satisfy a triangular system of algebraic differential equations of the form

\[(5) \quad \frac{\partial f_i}{\partial x_j} = P_{ij}(x_1, \ldots, x_\ell, f_1, \ldots, f_i), \quad \forall i, j,\]

where $P_{ij}$ are polynomials. The Pfaffian chain is called restricted if $B$ is bounded and $f_1, \ldots, f_m$ extend as real analytic functions to a neighborhood of $\bar{B}$. A Pfaffian function $f$ is a polynomial $Q(x_1, \ldots, x_\ell, f_1, \ldots, f_m)$. The degree of $f$ is defined to be the degree of $Q$.

We denote the structure generated by the Pfaffian functions by $\mathbb{R}_{\text{Paff}}$, and its restricted analog by $\mathbb{R}_{\text{rPaff}}$.

Gabrielov and Vorobjov [1] defined an FD-filtration $\Omega$ on $\mathbb{R}_{\text{rPaff}}$, which is not known to be sharp (or even presharp). Roughly speaking, in $\Omega$, the format of a semipfaffian set $X \subset \mathbb{R}^k$ is the maximum among $k$ and the length of the Pfaffian chains defining the Pfaffian functions appearing in a representation of $X$ as a finite union of basic sets, and the degree of $X$ is the sum of the degrees of all Pfaffian functions and the polynomials $P_{ij}$ in all Pfaffian chains appearing in the same representation. If $Y$ is a projection of $X$, the format and degree of the subpfaffian set $Y$ were defined to be those of $X$. While Gabrielov and Vorobjov were able to obtain bounds on the sum of the Betti numbers of a semipfaffian set $X$ which are polynomial in the degree of $X$, they were not able to obtain the same bounds for subpfaffian sets in full generality. The main, and crucial, difficulty is that if $A \in \Omega_{\mathcal{F},D}$ then it is only known that $A^c \in \Omega_{\text{poly}_{\mathcal{F}}(D),\text{poly}_{\mathcal{F}}(D)}$, i.e. the format of $A^c$ depends also on the degree of $A$.

In [3], Binyamini and Vorobjov introduce a new notion of degree for subpfaffian sets, with which they do achieve polynomial bounds on the Betti numbers of subpfaffian sets. Essentially, they introduce an FD-filtration $\Omega^*$, based on $\Omega$, that makes $\mathbb{R}_{\text{rPaff}}$ into a $\mathcal{W}^\#o$-minimal structure with $\#CD$, and conjecturally makes $\mathbb{R}_{\text{Paff}}$ into a $\mathcal{W}^\#o$-minimal structure with $\#CD$ as well. As $\Omega \leq \Omega^*$, they obtain bounds on the sum of the Betti numbers of subpfaffian sets which are polynomial in the degree in the sense of [1]. As mentioned above, our construction generalizes the construction of $\Omega^*$ from [3] to the settings of presharp o-minimal structures.

1.4.4. A remark on notation. The symbol $O_a(1)$ denotes a specific universally fixed function (possibly different at each occurrence) $a \mapsto C(a)$, where $C(a)$ is positive. We will make the non-restrictive (for our purposes) assumption that if $a = (a_1, \ldots, a_t)$ ranges over $\mathbb{N}^t$, then $C(a) \geq \max\{a_1, \ldots, a_t\}$, and moreover that $C(a)$ is (weakly) monotone increasing with respect to every variable $a_i$. The symbol $\text{poly}_a(b)$ denotes a polynomial $P_a(b)$ in $b$ with positive coefficients, where $a \mapsto P_a$ is a specific universally fixed function. Thus, rather then representing classes of functions like ordinary asymptotic notations, these symbols are simply stand-ins for specific constants and polynomials we do not keep track of. Additionally, we will write $O_{\mathcal{F}}(1)$, $\text{poly}_{\mathcal{F}}(D)$ (perhaps with extra variables) as short hand for
$O_{F,\Sigma}(1)$, $\text{poly}_{F,\Sigma}(D)$ (perhaps with extra variables) respectively. In some of our results, such as Proposition 1.13, this is redundant as there is no dependence on $\Sigma$.

1.5. Format and degree of first order formulae. Let $\Sigma = (S, \Omega)$ be a $\sharp$o-minimal structure, and let $\mathcal{L}$ be the language with atomic predicates of the form $(x \in X)$ for every definable set $X$, and with neither constants nor function symbols. We will assume that the variables of $\mathcal{L}$ are linearly ordered, so that if a formula $\psi$ has $n$ free variables, then it defines a set in $\mathbb{R}^n$. The goal of this subsection is to filter the formulae in $\mathcal{L}$ by format and degree, such that if a formula $\psi$ has format $\mathcal{F}$ and degree $D$, then the set it defines is in $\Omega_{O_{F}(1),\text{poly}_{F}(D)}$.

Definition 1.11 (Format and degree of formulae). Let $\psi$ be a formula. Suppose that there are $n$ different variables appearing in $\psi$ (either free or quantified), and let $X_j \in \Omega_{F_j,D_j}$ be the sets appearing in the atomic predicates of $\psi$. Denote $F = \max F_j$ and $D := \sum D_j$, then we say that $\psi$ has format $\max\{F,n\}$ and degree $D$.

In this text we will also need a notion of P-format, and it is natural to consider a notion of W-format as well. Since in the presharp or weakly-sharp case, geometric operations such as intersections and projections may increase the format, unlike the format of a formula $\psi$, the P-format and W-format of $\psi$ can’t be defined just in terms of its atoms with Proposition 1.13 in mind. Rather, the P-format and W-format will depend on the parse-tree of $\psi$.

Definition 1.12 (P-format). Let $\psi$ be a formula, and $d$ be the depth of the parse-tree of $\psi$. Then the P-format of $\psi$ is defined to be $\max\{F,d\}$, where $F$ is the format of $\psi$.

For the definition of W-format, one needs to consider a different kind of parse-tree (one which is not necessarily binary), and moreover disregard vertices of the tree that are associated with disjunction. Since we won’t actually need the W-$\#o$-minimal case of the following proposition in this paper, we omit a formal definition of W-format. The following proposition is clear from the definition. In fact, axiom $(\#1)$ is not even needed.

Proposition 1.13. Let $(S, \Omega)$ be a $\#o$-minimal structure (resp. $P\#o$-minimal, $W\#o$-minimal), and let $\psi$ be a formula of format (resp. $P$-format, $W$-format) $\mathcal{F}$ and degree $D$, then the set that $\psi$ defines is in $\Omega_{O_{F}(1),\text{poly}_{F}(D)}$.

1.6. Sharp cellular decomposition, $\Omega^\ast$. We recall the notion of a cell. A cell $C \subset \mathbb{R}$ is either a point or an open interval (possibly infinite). A cell $C \subset \mathbb{R}^{\ell+1}$ is either the graph of a definable continuous function $f : X' \to \mathbb{R}$ where $X' \subset \mathbb{R}^{\ell}$ is a cell, or the set

\begin{equation}
\{(x, y) \in X' \times \mathbb{R} | f(x) < y < g(x)\},
\end{equation}

bounded between two graphs of definable continuous functions $f, g : X' \to \mathbb{R}$ satisfying $f < g$ on $X'$. One can also take $f \equiv -\infty$ or $g \equiv \infty$ or both in this definition.

We say that a cell $C \subset \mathbb{R}^{\ell}$ is compatible with $X \subset \mathbb{R}^{\ell}$ if either $C \subset X$, or $C \cap X = \emptyset$.

Definition 1.14 (Cylindrical cellular decomposition). A cylindrical cell decomposition of $\mathbb{R}$ is any decomposition of $\mathbb{R}$ into disjoint cells. A decomposition $\{C_j\}$ of $\mathbb{R}^{\ell+1}$ into
disjoint cells is a cylindrical cell decomposition if the collection \( \{ \pi_\ell(C_j) \} \) is a cylindrical cell decomposition of \( \mathbb{R}^\ell \).

For brevity from now on we will use "cell decomposition" for "cylindrical cell decomposition". The following Cell decomposition theorem is the fundamental central result of the classical theory of o-minimal structures. We say that a cell decomposition is compatible with a set \( X \) if every cell from the decomposition is compatible with \( X \).

**Theorem 1.15 (Cell decomposition).** Let \( X_1, \ldots, X_k \subset \mathbb{R}^\ell \) be definable sets. Then there is a cell decomposition of \( \mathbb{R}^\ell \) whose cells are compatible with \( X_1, \ldots, X_k \).

**Proof.** See [6, Section 2.11]. □

We are finally in position to define sharp cellular decompositions.

**Definition 1.16.** Let \( \mathcal{S} \) be an o-minimal expansion of \( \mathbb{R} \) and \( \Omega \) be an FD-filtration. We say that \((\mathcal{S}, \Omega)\) has sharp cellular decomposition (or \#CD for short) if for every collection \( X_j \subset \mathbb{R}^\ell \) of \( k \) sets of format \( \mathcal{F} \) and degree \( D \), there exists a cell decomposition of \( \mathbb{R}^\ell \) into \( \text{poly}_{\mathcal{F}}(D, k) \) cells of format \( O_{\mathcal{F}}(1) \) and degree \( \text{poly}_{\mathcal{F}}(D) \).

The following proposition justifies the introduction of the stronger notion of sharp o-minimality as opposed to the previous notion of \( \#W \#\)o-minimality. Its proof can be found in Section 5.

**Proposition 1.17.** Let \((\mathcal{S}, \Omega)\) be a \( \#W \#\)o-minimal structure with \#CD. Then there exists a filtration \( \Omega' \) which is equivalent to \( \Omega \), such that \((\mathcal{S}, \Omega')\) is \#o-minimal with \#CD.

**Remark 1.18.** Following the proof of cellular decomposition as it appears in [6], it is possible to recover a cellular decomposition where the formats of the cells are polynomial in the degree. The most basic difficulty can be illustrated by the following example. Let \( X \subset \mathbb{R}^2 \) be a curve of degree \( D \), then in general it seems impossible to "select" the middle element of a fiber of \( \pi_1|_X \). Indeed, the the fiber over \( x \in \mathbb{R} \) is of size \( \sim D \), so to pick the middle element \( y_{D/2} \) one needs the following formula:

\[
\exists y_1 \ldots \exists y_{D/2} \ldots \exists y_D (y_1 < \cdots < y_D) \land ((x, y_1) \in X, \ldots, (x, y_D) \in X),
\]

where the hat signifies that the variable \( y_{D/2} \) is not quantified. This formula has format \( \geq D \). The authors suspect that in general not every \#o-minimal structure has \#CD. This means, for example, that in a general \#o-minimal structure, if \( X \) is in \( \Omega_{\mathcal{F}, D} \) and \( X^\circ \) is a connected component of \( X \), then we can only prove that \( X^\circ \in \Omega_{\text{poly}_{\mathcal{F}}(D), \text{poly}_{\mathcal{F}}(D)} \). The introduction of \( \Omega^* \) below is meant to circumvent this problem. The idea is, that in \( \Omega^* \) we force the connected component \( X^\circ \) to be in \( \Omega_{\text{poly}_{\mathcal{F}}(1), \text{poly}_{\mathcal{F}}(D)} \), and that with this condition we are able to show \#CD.

Inspired by [3], given a \( \#P \#\)o-minimal structure \((\mathcal{S}, \Omega)\), we will define a new filtration \( \Omega^* \) such that \( \Omega \leq \Omega^* \) and \( \Omega^* \) is equivalent to a sharp filtration with \#CD.

**Definition 1.19 (\#Format and \#Degree).** Let \( \mathcal{S} \) be an o-minimal expansion of \( \mathbb{R} \), and let \( \Omega \) be an FD-filtration on \( \mathcal{S} \). Let \( X \subset \mathbb{R}^\ell \) be definable. We say that \( X \in \Omega^*_{\mathcal{F}, D} \) if there
exists a representation of $X$ as a finite union $X = \bigcup_{\alpha} \pi_{\ell}^R (X^\circ_{\alpha})$, where for every $\alpha$ we have $X_\alpha \subset \mathbb{R}^\ell$ is in $\Omega_{\mathcal{F}_\alpha,D_\alpha}$, the set $X^\circ_{\alpha}$ is a connected component of $X_\alpha$, and $\mathcal{F} = \max_{\alpha} \{ \mathcal{F}_\alpha \}$, $D = \sum_{\alpha} D_\alpha$. If $X \in \Omega^*_{\mathcal{F},D}$ we say that $X$ has *-format $\mathcal{F}$ and *-degree $D$.

The following theorem (Theorem 1.20) is a straightforward generalization of the main result of [3], the proof going through in the general presharp case verbatim, with Proposition 2.1, Proposition 4.1, and Proposition 3.2 replacing [3, Fact 17], [3, Lemma 18], and [3, Fact 13] respectively. A sketch of the proof of Theorem 1.20 is provided in Section 5.

**Theorem 1.20.** Let $(\mathcal{S}, \Omega)$ be a P#o-minimal structure and let $X_1, \ldots, X_k \subset \mathbb{R}^\ell$ be definable sets of *-format $\mathcal{F}$ and *-degree $D$. Then there exists a cell decomposition of $\mathbb{R}^\ell$ compatible with $X_1, \ldots, X_k$, where each cell has *-format $O_{\mathcal{F}}(1)$, *-degree $\text{poly}_{\mathcal{F}}(D)$, and the number of cells is bounded by $\text{poly}_{\mathcal{F}}(k,D)$.

In particular one can show that $\Omega^*$ is equivalent to a weakly-sharp FD-filtration on $\mathcal{S}$ with *CD, and by Proposition 1.17 we conclude that $\Omega^*$ is equivalent to a *o-minimal filtration on $\mathcal{S}$ with *CD. Thus we obtain the following proposition, a more precise version of our main result, Theorem 1.9. A detailed proof of Proposition 1.21 is in Section 5.

**Proposition 1.21.** Let $(\mathcal{S}, \Omega)$ be a P#o-minimal structure. Then $\Omega^*$ is equivalent to a filtration making $\mathcal{S}$ into a *o-minimal structure with *CD.

**Remark 1.22.** In the spirit of Remark 1.3, the above proposition essentially means that one may always assume *CD when working with a *o-minimal structure.

Note that in a P#o-minimal structure, one always has $\Omega \leq \Omega^*$, for if $X \in \Omega_{\mathcal{F},D}$ and $X = \bigcup_{i=1}^N X_i$ where $X_i$ are the connected components of $X$, then by definition $X$ has *-format $\mathcal{F}$ and *-degree $N \cdot D$, but we know from Proposition 2.1 that $N \leq \text{poly}_{\mathcal{F}}(D)$. The following is an easy consequence of Theorem 1.20.

**Proposition 1.23.** Let $(\mathcal{S}, \Omega)$ be a *o-minimal structure. Then it has sharp cellular decomposition if and only if $\Omega^* \leq \Omega$. In particular, $\Omega^*$ is equivalent to $(\Omega^*)^*$.

**Proof.** If $\Omega^* \leq \Omega$ then $\Omega, \Omega^*$ are equivalent and since $\Omega^*$ has *CD, so does $\Omega$. Assume now that $(\mathcal{S}, \Omega)$ is a *o-minimal structure with *CD. Then it follows from *CD that if $X \in \Omega_{\mathcal{F},D}$ and $X^\circ \subset X$ is a connected component, then $X^\circ \in \Omega_{O_{\mathcal{F}}(1),\text{poly}_{\mathcal{F}}(D)}$. It now follows immediately from the definition that $\Omega^* \leq \Omega$. $\square$

**Remark 1.24** (Effectivity). A *o-minimal (resp. W#o-minimal,P#o-minimal) structure is effective if the polynomial $P_{\mathcal{F}}(D)$ from axiom (*1) (resp. (W1),(P1)) is a primitive recursive function of $\mathcal{F}$. Similarly, a reduction $\Omega \leq \Omega'$ is effective if $a(\mathcal{F})$ and $P_{\mathcal{F}}(D)$ from Definition 1.2 are primitive recursive functions of $\mathcal{F}$. Assuming effectivity, all the reductions constructed in this paper are effective, and all appearances of $O_{\mathcal{F}}(1),\text{poly}_{\mathcal{F}}(D)$ (perhaps with dependence on other variables such as $r$) are primitive recursive functions of $\mathcal{F}$ (and the other variables).
1.7. Structure of this paper. In Section 2 we show that in a presharp structure a set of format $\mathcal{F}$ and degree $D$ has at most $\text{poly}_\mathcal{F}(D)$ connected components, even if it is not contained in $\mathbb{R}$. In Section 3 we discuss derivatives of functions definable in presharp structures and prove a sharp stratification result. We also review the notion of sharp derivatives, first introduced in [5]. In Section 4 we prove a sharp version of definable choice. Note that while a sharp version of the definable choice was proved in [5], it was only under the assumption of $\#\text{CD}$, and in this paper we prove it without that assumption, for the results of, Section 2, Section 3 and Section 4 are needed to prove Theorem 1.20. In Section 5 we provide a sketch of a proof for Theorem 1.20, and we prove Proposition 1.17 and Proposition 1.21. As an application of the main result, in Section 6 we show that a $\#\text{o-minimal}$ structure with $\#\text{CD}$ has sharp triangulation, and deduce bounds on the Betti numbers of a definable set in any presharp structure.

2. Bound on connected components of definable sets in $\mathbf{P}^{\#\text{o-minimal}}$ structures

Fix a presharp structure $(\mathcal{S}, \Omega)$.

Proposition 2.1. Let $X \subset \mathbb{R}^\ell$ be a set of format $\mathcal{F}$ and degree $D$. Then $X$ has at most $\text{poly}_\mathcal{F}(D)$ connected components.

We will need the following lemma, which is a standard exercise in o-minimality.

Lemma 2.2. For a functional $f \in (\mathbb{R}^\ell)^\dagger$ and a definable set $X \subset \mathbb{R}^\ell$ denote $X_f$ to be the set of local maxima of $f|_X$. If $\dim X > 0$ then there exists a functional $f$ such that $\dim X_f < \dim X$.

Proof of Proposition 2.1. Suppose first that $X \subset D(R)$, where $D(R)$ is an open disk of radius $R$ around the origin. Let $X = X_1 \cup \cdots \cup X_N$ be the decomposition of $X$ into its connected components. The proof is by induction on $\dim X$. The idea is to use Lemma 2.2 and the induction hypothesis, but the problem is that the $X_i$ may have intersecting closures.

If $\dim X = 0$, then there exists a linear functional $f : \mathbb{R}^\ell \to \mathbb{R}$ such that $f|_X$ is injective. Thus $\dim H_0(X; \mathbb{R}) = \dim H_0(f(X); \mathbb{R})$, but $f(X)$ has format $O_\mathcal{F}(1)$ and degree $\text{poly}_\mathcal{F}(D)$, so according to (P1) we have $\dim H_0(f(X); \mathbb{R}) \leq \text{poly}_\mathcal{F}(D)$.

Now suppose $\dim X > 0$. Let $\epsilon > 0$ and consider $X_\epsilon := \{x \in X \mid d(x, \partial X) < \epsilon\}$ and $Y := X \setminus X_\epsilon$. Clearly $Y$ has format $O_\mathcal{F}(1)$ and degree $\text{poly}_\mathcal{F}(D)$. We claim that for $\epsilon$ small enough, $Y$ is a union of $N$ sets with disjoint closures. Define $Y_i = X_i \setminus X_\epsilon$, and to ensure that the $Y_i$ are nonempty, choose $\epsilon < \min \sup_{i} d(x, \partial X)$.

Let us prove that the $Y_i$ have disjoint closures. Say $x \in \overline{Y_i} \cap \overline{Y_j}$, and so $x \in \overline{X}$. If $x \notin X$, then $x \in \partial X$, but $d(\overline{Y_i}, \partial X) \geq \epsilon$, contradicting $x \in \overline{Y_i}$. So $x \in X$, say $x \in X_k$, but since $x \in \overline{X_i}$ we conclude that $X_i \cup X_k$ is connected. This forces $k = i$, and by repeating the argument that $k = j$. Of course this shows that $Y$ has at least $N$ components.
Since \( \dim Y > 0 \), by Lemma 2.2 there exists a functional \( f \) such that \( \dim (\overline{Y})_f < \dim Y \). Certainly, \((\overline{Y})_f \) has format \( O_F(1) \) and degree \( \text{poly}_F(D) \). Moreover, since \( Y \) is bounded, \((\overline{Y})_f \) meets every \( Y_i \), and since the \( Y_i \) are disjoint we conclude \( N \leq \dim H_0((\overline{Y})_f; \mathbb{R}) \).

As \( \dim (\overline{Y})_f < \dim Y \leq \dim X \), by induction \( N \leq \dim H_0((\overline{Y})_f; \mathbb{R}) \leq \text{poly}_F(D) \).

To end the proof, note that while we assume that \( X \subset D(R) \), our bounds on \( \dim H_0(X; \mathbb{R}) \) does not depend on \( R \). Assume now that \( X \) is not bounded. Since homology commutes with direct limit, we see that \( H_0(X; \mathbb{R}) = \lim \dim H_0(X \cap D(R); \mathbb{R}) \), and so \( \dim H_0(X; \mathbb{R}) \leq \lim \sup \dim H_0(X \cap D(R); \mathbb{R}) \leq \text{poly}_F(D) \).

### 3. Stratification

Let \( r \) be a positive integer, and fix a presharp structure \((\mathcal{S}, \Omega)\), then we have the following.

**Proposition 3.1.** Let \( f : \mathbb{R}^\ell \to \mathbb{R}^k \) be a definable map of format \( F \) and degree \( D \). Then \( f \) is \( C^r \) outside a definable set \( V \) of codimension \( \geq 1 \) of format \( O_{F,r}(1) \) and degree \( \text{poly}_F(D, k, r) \).

**Proof.** We prove it for \( r = 1 \) and leave the general case for the reader. Fix \( 1 \leq i \leq k \), then the set \( A_i = \{ x \in \mathbb{R}^\ell \mid f_i \text{ is differentiable at } x \} \) can be given by the following formula:

\[
\exists L_i \forall \epsilon > 0 \exists \delta > 0 \forall y \left( |y - x| < \delta \rightarrow |f_i(y) - f_i(x) - L_i \cdot (y - x)| < \epsilon |y - x| \right),
\]

where \( L_i \) should be understood as a tuple of \( \ell \) variables. Thus, by Proposition 1.13 \( A_i \) has format \( O_F(1) \) and degree \( \text{poly}_F(D) \). Therefore \( \cap_i A_i \) has format \( O_{F,k}(1) \leq O_F(1) \) and degree \( \text{poly}_F(D, k) \). By \( \omega \)-minimality, \( \mathbb{R}^\ell \setminus \cap_i A_i \) has codimension \( \geq 1 \), and thus \( V := \mathbb{R}^\ell \setminus \cap_i A_i \) has codimension \( \geq 1 \).

Given a positive integer \( r \) and a set \( X \subset \mathbb{R}^\ell \), it is always possible to stratify \( X \) in the following sense.

**Proposition 3.2.** Let \( X \subset \mathbb{R}^\ell \) be of format \( F \) and degree \( D \). Then there exists a stratification of \( X = X_1 \cup \cdots \cup X_s \) where \( s = \dim X \) and each \( X_i \in \Omega_{O_{F,r}(1), \text{poly}_F(D, r)} \) is a \( C^r \) smooth embedded submanifold (possibly disconnected) of \( \mathbb{R}^\ell \).

**Proof.** While a proof of this proposition appears in [5], it is instructive to repeat it here. The proof is by induction on dimension. Let \( X_{\text{reg}} \subset X \) be the set of points near which \( X \) is a \( C^r \) manifold. By an argument similar to that of Proposition 3.1, \( X_{\text{reg}} \) has format \( O_{F,r}(1) \) and degree \( \text{poly}_F(D, r) \), and in the case of sharp derivatives, format \( O_F(1) \) and degree \( \text{poly}_F(D, r) \). By \( \omega \)-minimality dimension \( \dim X \setminus X_{\text{reg}} < \dim X \), so we can apply the induction hypothesis on \( X \setminus X_{\text{reg}} \), and we are done.

**Remark 3.3.** In [2] a similar result more compatible with cellular decomposition is proved. In particular, we may assume that the \( X_i \) form a cellular decomposition of \( X \), albeit then \( s = \text{poly}_F(D) \) instead of \( s = \dim X \).
Similarly it follows that the derivatives of \( f \), where defined, have format \( O_{\mathcal{F},r}(1) \) and degree \( \text{poly}_{\mathcal{F}}(D,r) \). The fact that the format of \( f^{(r)} \) depends on \( r \) is very restrictive in general applications, and so far it seems generally unavoidable, even in \#o-minimal structures. In structures like \( \mathbb{R}_{\text{alg}} \) and \( \mathbb{R}_{\text{Pfaff}} \) however, the format of the derivatives is independent of \( r \). We therefore recall the notion of \textit{sharp derivatives}, first introduced in [5].

\textbf{Definition 3.4.} Let \( \mathcal{S} \) be an o-minimal structure and \( \Omega \) be an FD filtration. We say that \((\mathcal{S},\Omega)\) has \textit{sharp derivatives} if for every \( \mathcal{F} \in \mathbb{N} \) there are

\begin{equation}
\alpha_{\mathcal{F}} \in \mathbb{N}, \quad b_{\mathcal{F}} \in \mathbb{N}[D,k]
\end{equation}

such that the following holds. Given a definable \( f : \mathbb{R}^n \to \mathbb{R} \) with \( f \in \Omega_{\mathcal{F},D} \), we have for every \( \alpha \in \mathbb{Z}^{\geq 0} \)

\begin{equation}
f^{(\alpha)} \in \Omega_{\alpha_{\mathcal{F}},b_{\mathcal{F}}(D,|\alpha|)}.
\end{equation}

Where by \( f^{(\alpha)} \) we mean that \( f \) is restricted to the locus where it is in \( C^{[\alpha]} \).

In Proposition 3.2 above, if the structure has sharp derivatives, then the strata \( X_i \) can be taken to have format \( O_{\mathcal{F}}(1) \) and degree \( \text{poly}_{\mathcal{F}}(D,r) \).

\textbf{4. Sharp definable choice}

Fix a presharp structure. We prove the following form of sharp definable choice. Our proof is completely inspired by the construction in [6].

\textbf{Proposition 4.1.} Let \( \{X_\lambda \subset \mathbb{R}^\ell\}_{\lambda \in \Lambda} \) be a definable family whose elements are nonempty, such that the format of the total space \( X_\Lambda := \{(\lambda, x) | x \in X_\lambda, \lambda \in \Lambda\} \) is \( \mathcal{F} \) and its degree is \( D \). Then there exists a definable map \( g : \Lambda \to \mathbb{R}^\ell \) of format \( O_{\mathcal{F}}(1) \) and degree \( \text{poly}_{\mathcal{F}}(D) \) such that \( g(\lambda) \in X_\lambda \) for all \( \lambda \in \Lambda \).

\textit{Proof.} We prove this by induction on \( \ell \). First we show the induction step. We use the induction hypothesis on the family \( \pi_{\ell-1}(X_\lambda) \subset \mathbb{R}^{\ell-1} \) to obtain a map \( g_1 : \Lambda \to \mathbb{R}^{\ell-1} \) with \( g_1(\lambda) \in X_\lambda \) for every \( \lambda \in \Lambda \). Since \( g_1 \) has format \( O_{\mathcal{F}}(1) \) and degree \( \text{poly}_{\mathcal{F}}(D) \), the total space of the family \( \{x \in \mathbb{R} : (g_1(\lambda), x) \in X_\lambda\}_{\lambda \in \Lambda} \) has format \( O_{\mathcal{F}}(1) \) and degree \( \text{poly}_{\mathcal{F}}(D) \). Finally, we use the induction hypothesis on this family, producing a map \( g_2 : \Lambda \to \mathbb{R} \) of format \( O_{\mathcal{F}}(1) \) and degree \( \text{poly}_{\mathcal{F}}(D) \). Then the map \( (g_1,g_2) \) satisfies the requirements of the proposition.

Finally, we show the statement for \( \ell = 1 \). For \( X \subset \mathbb{R} \), define \( a(X) := \inf X \) and \( b(X) := \sup \{ x : (a(X), x) \subset X \} \). We decompose \( \Lambda \) into the following sets, and define \( g \) on them:

\begin{align*}
A &= \{ \lambda : a(X_\lambda) = -\infty, \ b(X_\lambda) = \infty \}, \quad g(\lambda) = 0, \\
B &= \{ \lambda : a(X_\lambda) = -\infty, \ b(X_\lambda) \in \mathbb{R} \}, \quad g(\lambda) = b(X_\lambda) - 1, \\
C &= \{ \lambda : a(X_\lambda) \in \mathbb{R}, \ b(X_\lambda) = \infty \}, \quad g(\lambda) = a(X_\lambda) + 1, \\
D &= \{ \lambda : a(X_\lambda), b(X_\lambda) \in \mathbb{R} \}, \quad g(\lambda) = \frac{a(X_\lambda) + b(X_\lambda)}{2}
\end{align*}
It remains to check that $A, B, C, D$, as well as the functions $a(X_\lambda), b(X_\lambda)$ are of format $O_F(1)$ and degree $\text{poly}_F(D)$. We will only check this for $B, a(X_\lambda)$, and leave the rest for the reader. $B$ can be described by the formula

\[(\forall M \exists x \in X_\lambda \ x < M) \land (\exists N \forall x \in X_\lambda \ x < N),\]

thus the format and degree of $B$ is bounded by Proposition 1.13. As for $a(X_\lambda)$, we bound its format and degree on $C$ (the same works for $D$). The graph of $a(X_\lambda)$ on $C$ is given by the set

\[\{(\lambda, a) \in C \times R : (\forall x \in X_\lambda \ a < x) \land (\forall \epsilon > 0 \exists z \in X_\lambda \ a + \epsilon > z)\}\]

Thus, just as before, we can bound the format and degree of $a(X_\lambda)$. □

5. Proof of the main results

5.1. Sketch of the proof of Theorem 1.20. We will explain the main ideas and the main steps of the proof, while trying to avoid as many technical details as possible. The interested reader should consult [3]. We may replace $R$ by $I$, using the definable homeomorphism $\frac{x-1/2}{x-x^2} : I \to R$. The following proposition is the key ingredient in the proof.

**Proposition 5.1.** Let $\{X_\alpha\}$ be a collection of $N$ definable sets of format $F$ and degree $D$ in $I^\ell$. Let $n \leq \ell$ be a positive integer, then there exists a cylindrical decomposition of $I^n$ of size $\text{poly}_F(D, N)$, compatible with the collection $\{\pi_n(X_\alpha)\}$, and whose cells have $*$-format $O_F(1)$ and $*$-degree $\text{poly}_F(D)$.

Let us show how Theorem 1.20 follows from Proposition 5.1. Given $X_1, \ldots, X_k \subset I^n$ in $\Omega^*_F,D$, each $X_i$ is union of projections of connected components of sets $X_{\alpha,i}$. We may suppose without loss of generality that the $X_{\alpha,i}$ are contained in the same ambient space $I^\ell$. Now we can use Proposition 5.1 on all the sets $X_{\alpha,i}$, producing a cellular decomposition compatible with them. Then it must also be compatible with their connected components, and the projections of the cells to $I^n$ will be compatible with the sets $X_i$.

We next turn to prove Proposition 5.1. The proof is by lexicographic induction on $(n, k)$ where $k := \max \{\dim (\pi_{n-1}(X_\alpha))\}$. For a collection $\Pi$ of subsets of $I^\ell$, let $\bigcup \Pi$ be the union of the sets in $\Pi$, and for a positive integer $n \leq \ell$ denote $\pi_n(\Pi) := \{\pi_n(X) : X \in \Pi\}$. For a positive integer $t$, we denote by $\Pi_{<t}$ and $\Pi_t$ to be the collection of sets in $\Pi$ whose dimension is smaller than $t$ and whose dimension is equal to $t$, respectively.

**Step 1:** We stratify the sets $X_\alpha$ as in Proposition 3.2, so that we may assume that the collection $\Pi := \{X_\alpha\}$ consists of smooth pure dimensional manifolds, and for a technical reason we also assume that $\Pi$ is closed under taking frontiers. We may also assume that $\pi_n|_{X_\alpha}$ has constant rank $\dim \pi_n(X_\alpha)$, and that $\pi_{n-1}|_{X_\alpha}$ has constant rank $k$. The first step is to remove the singularities of $\bigcup \pi_{n-1}(X_\alpha)$ by removing sets of dimension $< k$. More specifically, we will define a collection $B$ of closed definable sets of bounded $*$-format and $*$-degree such that $\dim \bigcup \pi_{n-1}(B) < k$. The idea is that $\Pi$ is well behaved above $\bigcup \pi_{n-1}(\Pi) \setminus \bigcup \pi_{n-1}(B)$, so that a cell decomposition can be explicitly constructed, and
over $\bigcup \pi_{n-1}(\mathcal{B})$ we will simply construct cells by induction. We begin by putting the closures of the sets in $\Pi_{<k}$ into $\mathcal{B}$.

By sharp definable choice, we may assume that $\bigcup \pi_{n-1}(\Pi_{k+1}) \subset \bigcup \pi_{n-1}(\Pi_{<k} \cup \Pi_k)$, and so for the purpose of this step we may ignore $\Pi_{k+1}$. Let us analyze $Y := \bigcup \pi_{n-1}(\Pi_k)$. As $\pi_{n-1}|_{X_\alpha}$ has constant rank $k$, $Y$ can be thought as an immersed manifold with self-intersections, so if a point $p \in Y$ is not smooth then one of the following must hold.

1. There are two sets $X_\alpha, X_\beta \in \Pi_k$ and two points $p_\alpha \in X_\alpha$, $p_\beta \in X_\beta$ such that $\pi_{n-1}(p_\alpha) = \pi_{n-1}(p_\beta) = p$, but the projections of the germs $(X_\alpha, p_\alpha)$, $(X_\beta, p_\beta)$ to $I^{n-1}$ are different.
2. The point $p$ is in $\bigcup \pi_{n-1}(\{\partial X|X \in \Pi_k\})$.

The second case is handled by induction - since frontier reduces dimension, we may add the frontiers of the sets in $\Pi_{\leq k}$ to $\mathcal{B}$. Let us now explain how to deal with the first case. Consider the following set,

$$X_{\alpha,\beta} := \{x_1 = y_1, \ldots, x_{n-1} = y_{n-1}|(x, y) \in X_\alpha \times X_\beta\} \subset I^\ell \times I^\ell,$$

which in particular contains the point $(p_\alpha, p_\beta)$. Let us stratify $X_{\alpha,\beta}$. We claim that $(p_\alpha, p_\beta)$ must lie in a stratum of dimension $< k$. Indeed, first notice that $X_{\alpha,\beta}$ has discrete fibers over $\pi_{n-1}(X_\alpha)$ (under projection on the coordinates $x_1, \ldots, x_{n-1}$), and therefore its strata are of dimension at most $k$. Moreover, $(p_\alpha, p_\beta)$ cannot lie in a stratum of $Y$ dimension $k$, since then the germ of $Y$ at $(p_\alpha, p_\beta)$ is diffeomorphically mapped to both the germ of $\pi_{n-1}(X_\alpha)$ and the germ of $\pi_{n-1}(X_\beta)$ at $p$, which were assumed to be different. A contradiction.

We now add the strata of $X_{\alpha,\beta}$ of dimension $< k$ into $\mathcal{B}$. Note crucially, that to remove the singularities of $\bigcup \pi_{n-1}(X_\alpha)$ we only had to intersect two sets per singular point (Of course, while a germ at a singular point $p$ can be a union of more than two projections of germs of sets $X_\alpha \in \Pi_k$, it only takes two of them to remove $p$). Thus to get the required bounds on the number and complexity of the strata it is enough to assume that the structure was presharp.

We finish this step by adding some more sets to $\mathcal{B}$ for future use. For any $X_\alpha, X_\beta \in \Pi_k$ consider

$$Z_{\alpha,\beta} := \{x_n = y_n| (x, y) \in X_{\alpha,\beta}\}.$$

We stratify $Z_{\alpha,\beta}$ and add the strata of dimension $< k$, as well as the frontiers of the $k$ dimensional strata to $\mathcal{B}$.

**Step 2:** We go on to construct the cylindrical decomposition. We will use the induction hypothesis to construct a cylindrical decomposition of $I^{\ell-1}$ compatible with $\pi_{n-1}(\Pi)$ and $\pi_{n-1}(\mathcal{B})$. Since $\pi_{n-1}(\mathcal{B})$ has dimension $< k$, if $\mathcal{C}$ is a cell contained in $\pi_{n-1}(\mathcal{B})$, we construct the required cells over $\mathcal{C}$ by induction. Now let $\mathcal{C}$ be a cell contained in $G := \bigcup \pi_{n-1}(\Pi)\setminus \Sigma$. We need to construct cells over $\mathcal{C}$ compatible with $\pi_n(\Pi)$, and it is not hard to see that it suffices for the cells to be compatible with $\pi_{n}(\Pi_k)$ in order to be compatible with $\pi_n(\Pi)$.

We claim that the set $\cup_{\alpha} \pi_n(X_\alpha) \cap (\mathcal{C} \times I)$ is a union of graph cells $\{(x, y): x \in \mathcal{C}, y = s(x)\}$ over $\mathcal{C}$. Indeed, let $s_\alpha, s_\beta: \mathcal{C} \to I$ be two sections of $(\pi_n(X_\alpha)) \cap (\mathcal{C} \times I)$, $(\pi_n(X_\beta)) \cap (\mathcal{C} \times I)$, respectively. If neither of $s_\alpha < s_\beta, s_\alpha = s_\beta, s_\alpha > s_\beta$ holds globally over $\mathcal{C}$, then there exists a point $x$ such that $s_\alpha(x) = s_\beta(x)$, but $s_\alpha, s_\beta$ are not equal in a neighborhood
of \( x \). Then \( (x, s_\alpha(x)) \) is in a strata of \( Z_{\alpha,\beta} \) of dimension \( < k \), or it belongs to a frontier of \( k \)-dimensional strata of \( Z_{\alpha,\beta} \), a contradiction. Again, note crucially that we have achieved this while working with at most two sets \( X_\alpha, X_\beta \) at any time, so the construction holds through in the presharp case.

Now, let \( s_1 < \cdots < s_q \) be all the sections of all the sets \( \pi_n(X_\alpha) \) over \( C \). Then the cells are just given by

\[
\{(x,y) \in C \times I | 0 < y < s_1(x)\}, \{(x,y) \in C \times I | y = s_1(x)\}, \ldots, \{(x,y) \in C \times I | s_q(x) < y < 1\}.
\]

Finally, with some technicalities one can bound the *-format and *-degree of these cells. Note once more, that each cell is defined by at most two sections, so the construction provides the required bounds under the assumption of \( \text{P}^\#_o \)-minimality.

5.2. **Structure Trees.** In this section we define the notion of a structure tree. It is similar to a parse-tree for a first order formula, but the operations allowed are in correspondence to the operations in the axioms of \( \text{o-minimality} \), rather then to the standard operations for first-order formulae. Let us fix some notation. A **rooted tree** is a pair \((T,r)\) where \( T \) is a tree and \( r \) is a vertex of \( T \). If \( v \) is a vertex of \( T \), we consider the neighbors \( w \) of \( v \) such that \( d(w,r) = d(v,r) + 1 \), and refer to such neighbors as **children**.

We will now define structure trees, and for future use we will also define **slanted** structure trees.

**Definition 5.2** (Structure trees). A structure tree is a rooted tree \((T,r)\) where for every vertex \( v \) of \( T \) there is an associated definable set \( T_v \), such that for every vertex \( v \) with children \( v_1, \ldots, v_k \), the following holds:

1. The sets \( T_{v_j} \) for \( j = 1, \ldots, k \) are contained in the same \( \mathbb{R}^\ell \).
2. If \( k > 1 \), then \( T_v \) is either the union of, or the intersection of, the sets \( T_{v_1}, \ldots, T_{v_k} \).
3. If \( k = 1 \), then \( T_v \) is one of the following: \( \pi_{\ell-1}(T_{v_1}), (T_{v_1})^C, T_{v_1} \times \mathbb{R} \).

A slanted structure tree is defined similarly to a structure tree, the only difference is that in item (3) above the operation \( \mathbb{R} \times T_{v_1} \) is also allowed.

**Definition 5.3** (\( \Omega \)-format of structure trees). Let \( \Omega \) be any FD-filtration. We define \( \Omega \)-format of structure trees by induction on the depth. If \( T \) has a single vertex \( r \) and the associated set \( T_r \) is in \( \Omega_{F,D} \), then the \( \Omega \)-format of \( T \) is defined to be \( F \).

Let \((T,r)\) be a structure tree, let \( v_1, \ldots, v_k \) be the children of \( r \), and denote by \( T_1, \ldots, T_k \) the subtrees defined by them. Suppose that \( T_i \) has \( \Omega \)-format \( F_i \), then:

1. If \( k = 1 \) and \( T_r = T_{v_1} \times \mathbb{R} \) (or \( \mathbb{R} \times T_{v_1} \) in the slanted case), then the \( \Omega \)-format of \( T \) is \( \max\{F_i\} + 1 \).
2. In any other case, the \( \Omega \)-format of \( T \) is \( \max\{F_i\} \).

If \( A_j \in \Omega_{F,j,D_j} \) are the sets associated to the leaves of \( T \), then the degree of \( T \) is defined to be \( \sum D_j \).

The following proposition is the key ingredient in the proof of Proposition 1.17.
Proposition 5.4. Let $\langle S, \Omega \rangle$ be a $W^*o$-minimal structure with $\#CD$, and let $(T, r)$ be a structure tree of $\Omega$-format $\mathcal{F}$ and degree $D$. Then $T_r \in \Theta_{\mathcal{F}(1), \text{poly}_{\mathcal{F}}(D)}$.

Proof. Let $A_i \in \mathbb{R}^m$ be the sets associated to the leaves of $T$. Denote $m := \max \{e_i\}$, and for a set $X \subset \mathbb{R}^m$ and an integer $\ell$ we denote $P_{\ell}(X) := \pi_{\min(e_i, m)}(X) \times \mathbb{R}^{\max(\ell - m, 0)}$.

Let $C_1, \ldots, C_N$ be a cellular decomposition of $\mathbb{R}^m$ compatible with the sets $A_i \times \mathbb{R}^{m - e_i}$. We claim that for every vertex $v$ of $T$, if the associated set $S_v$ is a subset of $\mathbb{R}^\ell$, then $P_{\ell}(C_1), \ldots, P_{\ell}(C_N)$ is a cellular decomposition of $\mathbb{R}^\ell$ compatible with $S_v$.

We prove this by descending induction on the distance from $v$ to $r$. If $v$ is a leaf then the claim is clear by definition. Now let $v$ be any vertex, and let $v_1, \ldots, v_k$ be its children.

Then by the definition of structure trees, one of the following holds.

1. If $k > 1$, then $T_v$ is either the union of, or the intersection of, the sets $T_{v_1}, \ldots, T_{v_k}$, but by the inductive hypothesis $P_{\ell}(C_1), \ldots, P_{\ell}(C_N)$ are compatible with $T_{v_1}, \ldots, T_{v_k}$, and thus they are compatible with $T_v$.
2. If $k = 1$, then a straightforward check shows the same conclusion.

Since we assumed $\langle S, \Omega \rangle$ has $\#CD$, the cells $C_i$ can be chosen to have format $O_\mathcal{F}(1)$, degree $\text{poly}_{\mathcal{F}}(D)$ and their number $N$ can be chosen to be $\text{poly}_{\mathcal{F}}(D)$. Assume that $S_r \subset \mathbb{R}^n$, then $\mathcal{F} \geq n$, and the cells $P_{\ell}(C_i)$ have format $O_{\mathcal{F}, r}(1) \leq O_{\mathcal{F}}(1)$ and degree $\text{poly}_{\mathcal{F}}(D)$, so $T_r$ has format $O_{\mathcal{F}}(1)$ and degree $\text{poly}_{\mathcal{F}}(D)$ as well. \hfill $\square$

5.3. Proof of Proposition 1.17. Let $\langle S, \Omega \rangle$ be $W^*o$-minimal with $\#CD$, we define an FD-filtration $\Omega'$ in the following way. The set $X \subset \mathbb{R}^n$ is in $\Omega_{\mathcal{F}, D}'$ if there exists a structure tree $(T, r)$ of $\Omega$-format $\mathcal{F}$ and degree $D$ such that $T_r = X$. It is clear that $\langle S, \Omega' \rangle$ satisfies the axioms of $o$-minimal structures, except for (*1). Perhaps the only other unclear point is the second part of axiom (*2), i.e. that if $A \in \Omega_{\mathcal{F}, D}'$ then $\mathbb{R} \times A \in \Omega_{\mathcal{F}, D+1}'$, but this follows from the simple observation that if $(T, r)$ is any structure tree, one can replace all the associated sets $T_v$ by $\mathbb{R} \times T_v$ and obtain a new structure tree of $\Omega$-format greater by 1 than the $\Omega$-format of $T$.

It is also clear that $\Omega \subset \Omega'$, since if $X$ is in $\Omega_{\mathcal{F}, D}'$, then $T$ can be taken to be a single vertex with associated set $X$. By Proposition 5.4, we have $\Omega' \leq \Omega$, thus we simultaneously obtain the following.

1. The FD-filtration $\Omega'$ satisfies (*1), or in other words, $\langle S, \Omega' \rangle$ is $o$-minimal.
2. The filtrations $\Omega, \Omega'$ are equivalent, therefore $\langle S, \Omega' \rangle$ has $\#CD$.

\hfill $\square$

Before turning to the proof of Proposition 1.21, we need the following lemma on the normalization of format.

Lemma 5.5. Let $S$ be an o-minimal expansion of $\mathbb{R}$, and let $\Omega$ be an FD-filtration such that $\langle S, \Omega \rangle$ satisfies the axioms (*j) (resp. $(W_j, (P_j))$ for $j = 1, \ldots, 6$, where every occurrence of $\mathcal{F} + 1$ is replaced by $O_{\mathcal{F}}(1)$. Then $\Omega$ is equivalent to a filtration $\Omega'$ such that $\langle S, \Omega' \rangle$ is $o$-minimal (resp. $W^*o$-minimal, $P^*o$-minimal).

Proof. We will prove this in the $o$-minimal case, the proof being similar (but somewhat more complicated) in the $W^*o$-minimal and $P^*o$-minimal case. We will denote by $C(F)$
the appearance of $O_F(1)$ in axiom (2). Let $\Omega'$ be the FD-filtration sharply generated by $\Omega$. It is sufficient to show that $\Omega' \leq \Omega$. Let $X \subset \mathbb{R}^n$ be in $\Omega'_{\mathcal{F}, D}$, then there exists a slanted structure tree $(T, r)$ of $\Omega$-format $\mathcal{F}$ and degree $D$, such that $T_r = X$. In particular it follows that $T_r \in \Omega_{\mathcal{F}^F(D)}$, where by $C^F$ we mean the composition of $C$ with itself $F$ times, which is sufficient.

As mentioned above, the proof in the $W^\#o$-minimal and $P^\#o$-minimal case is somewhat more involved. It requires a notion of “weak” and “pre” $\Omega$-format for structure trees. The same conclusion holds - the weak\ pre $\Omega$-format of sets in $\Omega'_{\mathcal{F}, D}$ is bounded by

$$\max_{[\mathcal{F}] \to \{C\}} \left\{ C_F(\ldots (C_1(\mathcal{F})) \ldots) \right\},$$

where the set $\{C\}$ is the set all functions $C(\mathcal{F})$ replacing appearances of $\mathcal{F} + 1$ in the axioms that $\Omega$ satisfies, and the maximum is taken over all sequences of length $\mathcal{F}$ in this set.

5.4. **Proof of Proposition 1.21.** Let $(\mathcal{S}, \Omega)$ be $P^\#o$-minimal. By definition of $\Omega^*$, the pair $(\mathcal{S}, \Omega^*)$ satisfies axiom (W4). Moreover, according to Theorem 1.20, the pair $(\mathcal{S}, \Omega^*)$ satisfies the axioms (Wj) for $j \in \{1, 2, 3, 5, 6\}$, where every appearance of $\mathcal{F} + 1$ is replaced by $O_F(1)$. By Lemma 5.5 we can define an FD-filtration $\Omega'$ which is equivalent to $\Omega^*$, and such that $(\mathcal{S}, \Omega')$ is $W^\#o$-minimal with $\#CD$. We finish by applying Proposition 1.17. □

**6. Sharp Triangulation**

In [3], a sharp triangulation theorem for $\mathbb{R}_r\text{Pfaff}$ is deduced from $\#CD$ and the ordinary proof of triangulation in o-minimality as it appears in [7]. The deduction is simple; one need only verify that the formulas describing the operation in [7] have format $O_F(1)$ and degree $\text{poly}_F(D)$. Thus we have the following. Fix a $\#o$-minimal structure with sharp cellular decomposition.

**Theorem 6.1.** Let $X \subset I^\ell$ be closed and definable, $X_1, \ldots, X_k \subset X$ definable subsets such that all these sets have format $\mathcal{F}$ and degree $D$. Then there exists a simplicial complex $K$ of size $\text{poly}_F(D)$ with vertices in $\mathbb{Q}^\ell$ and a definable homeomorphism $\Phi : |K| \to X$ of format $O_F(1)$ and degree $\text{poly}_F(D)$, such that each $X_i$ is a union of images of simplices.

We immediately conclude the following.

**Theorem 6.2** (Bound on sum of Betti numbers). Let $\Sigma$ be a presharp structure. If $X \in \Omega_{\mathcal{F}, D}$ is compact, then the sum of the Betti numbers of $X$ is bounded by $\text{poly}_F(D)$.

**Proof.** The theorem follows from Theorem 6.1 and Remark 1.3. □

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