Reciprocity Relations for Summations of Squares of Floor Functions and Fractional Parts of Fractions

Damanvir Singh Binner
Department of Mathematics
Simon Fraser University
Burnaby, BC V5A 1S6
Canada
dbinner@sfu.ca

Abstract

Given positive coprime integers $a$ and $b$ and a natural number $h$, we obtain reciprocity relations which can be used to quickly evaluate summations like $\sum_{i=1}^{h} \{ \frac{ib}{a} \}^2$ and $\sum_{i=1}^{h} [\frac{ib}{a}]^2$, where $[x]$ and $\{x\}$ denote the floor function and the fractional part of $x$, respectively.

1 Introduction

We introduce the following notation.

- $T_1(a, b; h) := \sum_{i=1}^{h} \{ \frac{ib}{a} \}^2$.
- $T_2(a, b; h) := \sum_{i=1}^{h} i \lfloor \frac{ib}{a} \rfloor$.
- $T_3(a, b; h) := \sum_{i=1}^{h} \lfloor \frac{ib}{a} \rfloor^2$.

We can reformulate these sums as follows. Let $q_i$ and $r_i$ denote the quotient and remainder when $ib$ is divided by $a$. Then

$$T_1(a, b; h) = \frac{1}{a^2} \sum_{i=1}^{h} r_i^2,$$

$$T_2(a, b; h) = \sum_{i=1}^{h} iq_i,$$

$$T_3(a, b; h) = \sum_{i=1}^{h} q_i^2.$$
Note that summations like $\sum_{i=1}^{h} ir_i$ and $\sum_{i=1}^{h} q_i r_i$ can be easily expressed in terms of these sums using the division algorithm. We remark in passing that in 2020, the present author described a reciprocity relation which can be used to quickly calculate $\sum_{i=1}^{h} q_i$ and $\sum_{i=1}^{h} r_i$ (see [III, Lemma 7]). This reciprocity relationship is also described in Theorem 3 below.

In Section 2, we derive reciprocity relations for $T_1(a, b; h)$. Using these, we then obtain a reciprocity relation for $T_2(a, b; h)$ in Section 3. These reciprocity relations help us to easily calculate $T_1(a, b; h)$ and $T_2(a, b; h)$. In Section 4, we show that $T_1(a, b; h)$ and $T_2(a, b; h)$ can be calculated in $O(\log t)$ and $O((\log t)^2)$ steps, where $t = \max(a, b)$ and by a step, we mean a basic arithmetic operation on the bits of $a$ and $b$. Further we show that $T_3(a, b; h)$ can be easily calculated using the values of $T_1(a, b; h)$ and $T_2(a, b; h)$. In Sections 2.2 and 3.2, we demonstrate our formulas for an example. Let $q_i$ and $r_i$ denote the quotients and remainders when $2732i$ is divided by 8411. By performing only a few steps, we show that

$$\sum_{i=1}^{1221} r_i^2 = 28850219593,$$

$$\sum_{i=1}^{1221} i q_i = 196956430,$$

$$\sum_{i=1}^{1221} q_i^2 = 63853169.$$

We require three main results. The first one is the following well-known result of Sylvester.

**Theorem 1** (Sylvester (1882)). If $a$ and $b$ are positive coprime numbers, the number of natural numbers that cannot be expressed in the form $ax + by$ for nonnegative integers $x$ and $y$ is equal to $\frac{(a-1)(b-1)}{2}$.

This result can be found in [3]. Moreover, Sylvester posed this as a recreational problem, and Curran [4] published a short proof based on generating functions.

Let $NR(a, b)$ denotes the set of nonnegative integers nonrepresentable in terms of $a$ and $b$. That is, $NR(a, b)$ is the set of nonnegative integers $n$ that cannot be expressed in the form $ax + by$. Then, by Theorem 1 $|NR(a, b)| = \frac{(a-1)(b-1)}{2}$. In 1993, Brown and Shiue [2] discovered the sum $S(a, b)$ of natural numbers that cannot be expressed in the form $ax + by$.

**Theorem 2** (Brown and Shiue (1993)). For positive coprime numbers $a$ and $b$,

$$S(a, b) := \sum_{n \in NR(a, b)} n = \frac{1}{12}(a-1)(b-1)(2ab - a - b - 1).$$

For various calculations involved in our examples, we need the following reciprocity relationship proved by the present author in 2020.

**Theorem 3** (Binner(2020)). Let $a$, $b$, $d$, and $K$ be positive integers such that $b < a$, $d < a$, $\gcd(a, b) = 1$, and $K = \left\lfloor \frac{bd}{a} \right\rfloor$. Then

$$\sum_{i=1}^{d} \left\lfloor \frac{ib}{a} \right\rfloor + \sum_{i=1}^{K} \left\lfloor \frac{ia}{b} \right\rfloor = dK.$$
2 An algorithm for $T_1(a, b; h)$

In this section, we derive reciprocity relations which can be used to calculate $T_1(a, b; h)$.

2.1 Reciprocity relation

Define

\[ S(a, b; h) := \left(\frac{a}{2}\right) T_1(a, b; h) + \left(\frac{a}{2} + 1\right) \sum_{i=1}^{h} \left\lfloor \frac{ib}{a} \right\rfloor. \]

We describe reciprocity relations for $S(a, b; h)$. This leads to a method to quickly calculate $T_1(a, b; h)$ because $\sum_{i=1}^{h} \left\lfloor \frac{ib}{a} \right\rfloor$ can be easily calculated using the algorithm described in [1, Section 2.3]. We need some more notation.

- $n_0$ is the remainder obtained upon dividing $-b(h + 1)$ by $a$.
- $n := ab - a + n_0$. Note that $ab - a \leq n < ab$.
- $H := n_1 - 1$, where $n_1$ is the remainder when $-na^{-1}$ is divided by $b$.

Our approach is to calculate the number of nonnegative integer solutions $(x, y, z, u)$ of the equation $ax + by + z + u = n$ in two different ways. First, we use Theorems 1 and 2 to find the number of solutions of this equation.

**Lemma 4.** The number of nonnegative integer solutions of the equation $ax + by + z + u = n$ is given by

\[
\frac{(n + 1)(n + 2)}{2} + \frac{(a - 1)(b - 1)}{12} (2ab - a - b - 6n - 7).
\]

**Proof.** It is well-known that the equation $ax + by = n$ has either 0 or 1 solutions if $n < ab$ (see [5, Lemma 2 and Lemma 4]). We view the equation $ax + by + z + u = n$ as the pair of equations $ax + by = i$ and $z + u = n - i$, as $i$ varies from 0 to $n$. Note that the former equation has a solution only if $i \notin NR(a, b)$. Then the required number of solutions of the equation $ax + by + z + u = n$ is given by

\[
\sum_{\substack{i=0, \ldots, n \notin NR(a, b)}} (n + 1 - i).
\]

Using Theorems 1 and 2 and simplifying, we get that
\[
\sum_{\substack{i=0, \\
i \in NR(a,b)}}^n (n+1-i) \\
= \sum_{i=0}^n (n+1-i) - \sum_{i \in NR(a,b)} (n+1-i) \\
= \frac{(n+1)(n+2)}{2} - (n+1)|NR(a,b)| + \sum_{i \in NR(a,b)} i \\
= \frac{(n+1)(n+2)}{2} - (n+1)\frac{(a-1)(b-1)}{2} + \frac{1}{12}(a-1)(b-1)(2ab-a-b-1) \\
= \frac{(n+1)(n+2)}{2} + \frac{(a-1)(b-1)}{12}(2ab-a-b-6n-7). \\
\]

Next, we find the number of solutions of this equation using the method of generating functions. Though our method is similar in spirit to the proof of [1, Theorem 5], there are several key differences and we provide all the details here for the sake of completeness. We require some more notation.

\[
\alpha(a,b) := \frac{ab(a+b-2)}{2}, \\
\beta(a,b) := \frac{ab(a-1)(b-1)}{2} + \frac{ab((a-1)(a-2) + (b-1)(b-2))}{3}, \\
\gamma(a,b) := \frac{2\alpha^2(a,b) - ab\beta(a,b)}{2(ab)^3}, \\
\eta_1(a,b,h) := (h + H + 1) + n\gamma(a,b) + \frac{n(n+3)}{2} \left(\frac{a+b-2}{2ab}\right) + \frac{n^3 + 6n^2 + 11n}{6ab} \\
+ \frac{(h+1)(a-1)(a-5)}{12a} + \frac{(H+1)(b-1)(b-5)}{12b} - \frac{bh(h+1)(a+2)}{4a} - \frac{aH(H+1)(b+2)}{4b}. \\
\]

**Lemma 5.** Let \(N\) denote the number of nonnegative integer solutions of the equation \(ax + by + z + u = n\). Then

\[
N = S(a,b;h) + S(b,a;H) + \eta_1(a,b,h). \\
\]

**Proof.** By elementary theory of generating functions, we know that \(N\) is equal to the coefficient of \(x^n\) in

\[
\frac{1}{(1-x^a)(1-x^b)(1-x)^2}. \\
\]

Let \(\zeta_m = e^{2\pi i m}\). We know that

\[
(1-x^a)(1-x^b)(1-x)^2 = (1-x)^4 \prod_{k=1}^{a-1} (1-\zeta_a^{-k}x) \prod_{k=1}^{b-1} (1-\zeta_b^{-k}x). \\
\]
Since \(a\) and \(b\) are coprime, \(1 - \zeta_a^{-k}x\) and \(1 - \zeta_b^{-k}x\) are distinct for all values of \(k\). Thus, we obtain the partial fraction decomposition

\[
\frac{1}{(1-x^a)(1-x^b)(1-x)^2} = \frac{d_1}{1-x} + \frac{d_2}{(1-x)^2} + \frac{d_3}{(1-x)^3} + \frac{d_4}{(1-x)^4} + \sum_{k=1}^{a-1} \frac{A_k}{1 - \zeta_a^{-k}x} + \sum_{k=1}^{b-1} \frac{B_k}{1 - \zeta_b^{-k}x}.
\]

(1)

On comparing the coefficients of \(x^n\) on both sides of (1), we find

\[
N = d_1 + (n+1)d_2 + \frac{(n+2)(n+1)}{2}d_3 + \frac{(n+3)(n+2)(n+1)}{6}d_4 + \sum_{k=1}^{a-1} A_k \zeta_a^{-nk} + \sum_{k=1}^{b-1} B_k \zeta_b^{-nk}.
\]

(2)

If we substitute \(x = 0\) in (1), we get

\[
1 = d_1 + d_2 + d_3 + d_4 + \sum_{k=1}^{a-1} A_k + \sum_{k=1}^{b-1} B_k.
\]

(3)

Upon subtracting (3) from (2), we get

\[
N - 1 = nd_2 + \frac{n(n+3)}{2}d_3 + \frac{n^3 + 6n^2 + 11n}{6}d_4 - \sum_{k=1}^{a-1} A_k (1 - \zeta_a^{-nk}) - \sum_{k=1}^{b-1} B_k (1 - \zeta_b^{-nk}).
\]

(4)

The usual procedure for finding coefficients of a partial fraction expansion gives the following equations.

\[
\begin{align*}
d_4 &= \frac{1}{ab}, \\
d_3 &= \frac{a+b-2}{2ab}, \\
d_2 &= \gamma(a, b), \\
A_k &= \frac{1}{a(1 - \zeta_a^{-bk})(1 - \zeta_a^{-k})^2}, \\
B_k &= \frac{1}{b(1 - \zeta_b^{-ck})(1 - \zeta_b^{-k})^2}.
\end{align*}
\]

Substituting these back into (4), we have

\[
N = 1 + n\gamma(a, b) + \frac{n(n+3)}{2} \left( \frac{a+b-2}{2ab} \right) + \frac{n^3 + 6n^2 + 11n}{6ab} - \left( \frac{S_1}{a} + \frac{S_2}{b} \right),
\]

(5)

where

\[
S_1 = \sum_{k=1}^{a-1} \frac{1 - \zeta_a^{-nk}}{(1 - \zeta_a^{-bk})(1 - \zeta_a^{-k})^2}
\]
and

\[ S_2 = \sum_{k=1}^{b-1} \frac{1 - \zeta^{-nk}}{(1 - \zeta_b^k)(1 - \zeta_b^k)^2}. \]

Next, we find \( S_1 \) and \( S_2 \). By definition of \( n \), we have \( n \equiv -b(h + 1) \pmod{a} \), so \( \zeta^{-nk} = \zeta_a^{b(h+1)k} \), and thus,

\[
S_1 = \sum_{k=1}^{a-1} \frac{1 - \zeta_a^{b(h+1)k}}{(1 - \zeta_a^k)(1 - \zeta_a^k)^2} = \sum_{k=1}^{a-1} \sum_{j=0}^{h} \frac{\zeta_a^{j} k}{(1 - \zeta_a^k)^2} - \sum_{k=1}^{a-1} \sum_{j=0}^{h} \frac{1 - \zeta_a^{j} k}{(1 - \zeta_a^k)^2}. \quad (6)
\]

Note that for each \( 1 \leq k \leq (a - 1) \), \( \frac{1}{1 - \zeta_a^k} \) satisfies \( (1 - \frac{1}{x})^a = 1 \). That is, for each \( 1 \leq k \leq (a - 1) \), \( \frac{1}{1 - \zeta_a^k} \) is a root of the equation

\[ ax^{a-1} - \left( \frac{a}{2} \right) x^{a-2} + \left( \frac{a}{3} \right) x^{a-3} - \cdots = 0. \]

From there, it is easy to see that

\[ \sum_{k=1}^{a-1} \frac{1}{(1 - \zeta_a^k)^2} = -\frac{(a - 1)(a - 5)}{12}, \]

and thus, changing the order of summations yields

\[
\sum_{k=1}^{a-1} \sum_{j=0}^{h} \frac{1}{(1 - \zeta_a^k)^2} = -\frac{(h + 1)(a - 1)(a - 5)}{12}. \quad (7)
\]

Moreover,
\[
\sum_{k=1}^{a-1} \sum_{j=0}^{h} \frac{1 - \zeta_a^{jb k}}{(1 - \zeta_a^{k})^2} = \sum_{k=1}^{a-1} \sum_{j=1}^{h} \frac{1 - \zeta_a^{jb k}}{(1 - \zeta_a^{k})^2}
\]
\[
= \sum_{k=1}^{a-1} \sum_{j=1}^{h} \sum_{l=0}^{b_j - 1} \frac{\zeta_a^{k l}}{1 - \zeta_a^{k}}
\]
\[
= \sum_{k=1}^{a-1} \sum_{j=1}^{h} \sum_{l=0}^{b_j - 1} \frac{1}{1 - \zeta_a^{k}} - \sum_{k=1}^{a-1} \sum_{j=1}^{h} \sum_{l=1}^{b_j - 1} \frac{1 - \zeta_a^{k l}}{1 - \zeta_a^{k}}
\]
\[
= \sum_{j=1}^{h} \sum_{l=0}^{b_j - 1} \frac{1}{1 - \zeta_a^{k}} - \sum_{k=1}^{a-1} \sum_{j=1}^{h} \sum_{l=1}^{b_j - 1} \sum_{m=0}^{l-1} \zeta_a^{m k}
\]
\[
= \frac{(a - 1)bh(h + 1)}{4} - \sum_{k=0}^{a-1} \sum_{j=1}^{h} \sum_{l=1}^{b_j - 1} \sum_{m=0}^{l-1} \zeta_a^{m k} + \frac{b^2 h(h + 1)(2h + 1)}{12} - \frac{bh(h + 1)}{4}
\]
\[
= \frac{(a - 2)bh(h + 1)}{4} - \sum_{k=0}^{a-1} \sum_{j=1}^{h} \sum_{l=1}^{b_j - 1} \sum_{m=0}^{l-1} \zeta_a^{m k} + \frac{b^2 h(h + 1)(2h + 1)}{12}.
\]

We know that \(\sum_{k=0}^{a-1} \zeta_a^{m k} \neq 0\) only if \(a\) divides \(m\), and in that case, the sum is \(a\). Therefore,

\[
\sum_{k=0}^{a-1} \sum_{j=1}^{h} \sum_{l=1}^{b_j - 1} \sum_{m=0}^{l-1} \zeta_a^{m k} = \sum_{j=1}^{h} \sum_{l=1}^{b_j - 1} \sum_{m=0}^{l-1} \zeta_a^{m k}
\]
\[
= a \sum_{j=1}^{h} \sum_{l=1}^{b_j - 1} \left( \left\lfloor \frac{l - 1}{a} \right\rfloor + 1 \right)
\]
\[
= a \sum_{j=1}^{h} \sum_{l=1}^{b_j - 1} \left\lfloor \frac{l - 1}{a} \right\rfloor + abh(h + 1) - ah.
\]  

Next, note that \(\left\lfloor \frac{l - 1}{a} \right\rfloor = \left\lfloor \frac{l}{a} \right\rfloor\) unless \(a\) divides \(l\). Therefore,

\[
\sum_{j=1}^{h} \sum_{l=1}^{b_j - 1} \left\lfloor \frac{l - 1}{a} \right\rfloor = \sum_{j=1}^{h} \sum_{l=1}^{b_j - 1} \left\lfloor \frac{l}{a} \right\rfloor - \sum_{j=1}^{h} \left\lfloor \frac{b_j - 1}{a} \right\rfloor
\]
\[
= \sum_{j=1}^{h} \sum_{l=1}^{b_j} \left\lfloor \frac{l}{a} \right\rfloor - \sum_{j=1}^{h} \left\lfloor \frac{b_j}{a} \right\rfloor - 2 \sum_{j=1}^{h} \left\lfloor \frac{b_j}{a} \right\rfloor.
\]
Finally, note that for any $1 \leq j \leq h$,
\[
\sum_{l=1}^{bj} \left\lfloor \frac{l}{a} \right\rfloor = a \left( 1 + 2 + \cdots + \left( \left\lfloor \frac{bj}{a} \right\rfloor - 1 \right) \right) + \left\lfloor \frac{bj}{a} \right\rfloor \left( bj - a \left\lfloor \frac{bj}{a} \right\rfloor + 1 \right)
\]
\[
= \left( bj \left\lfloor \frac{bj}{a} \right\rfloor - a \left( \frac{bj}{a} \right)^2 \right) - \left( \frac{a}{2} - 1 \right) \left\lfloor \frac{bj}{a} \right\rfloor
\]
\[
= \frac{a}{2} \left( bj - \left\{ \frac{bj}{a} \right\} \right) - \left( \frac{a}{2} - 1 \right) \left\lfloor \frac{bj}{a} \right\rfloor
\]
\[
= \frac{a}{2} \left( \left( \frac{bj}{a} \right)^2 - \left\{ \frac{bj}{a} \right\}^2 \right) - \left( \frac{a}{2} - 1 \right) \left\lfloor \frac{bj}{a} \right\rfloor.
\]
(11)

Therefore, by (10) and (11),
\[
\sum_{j=1}^{h} \sum_{l=1}^{bj-1} \left\lfloor \frac{l-1}{a} \right\rfloor = \frac{b^2 h (h+1)(2h+1)}{12a} - \frac{a}{2} \left( \sum_{j=1}^{h} \left\{ \frac{bj}{a} \right\}^2 \right) - \left( \frac{a}{2} + 1 \right) \sum_{j=1}^{h} \left\lfloor \frac{bj}{a} \right\rfloor
\]
\[
= \frac{b^2 h (h+1)(2h+1)}{12a} - \frac{S(a, b; h)}{a}.
\]
(12)

From (6), (7), (8), (9) and (12), we get that
\[
S_1 = -\frac{(h+1)(a-1)(a-5)}{12} + \frac{bh(h+1)(a+2)}{4} - a \sum_{j=1}^{h} \left\lfloor \frac{bj}{a} \right\rfloor - ah - aS(a, b; h).
\]
(13)

Symmetrically, we get
\[
S_2 = -\frac{(H+1)(b-1)(b-5)}{12} + \frac{aH(H+1)(b+2)}{4} - b \sum_{j=1}^{H} \left\lfloor \frac{aj}{b} \right\rfloor - bH - bS(b, a; H).
\]
(14)

The result now follows from (13), (13) and (14).

Using Lemma 4 and Lemma 5, we get the following reciprocity relation for $S(a, b; h)$. For brevity of notation, define
\[
\eta_2(a, b, h) := \frac{(n+1)(n+2)}{2} + \frac{(a-1)(b-1)(2ab - a - b - 6n - 7)}{12} - \eta_1(a, b, h).
\]

**Theorem 6.** For given positive coprime integers $a$ and $b$, and a given natural number $h$, $S(a, b; h)$ satisfies the following reciprocity relationship:

\[
S(a, b; h) + S(b, a; H) = \eta_2(a, b, h).
\]
Next, we describe our algorithm for calculating $S(a, b; h)$.

1. Suppose $a > b$. We express $S(a, b; h)$ in terms of $S(b, a; H)$ using Theorem 6.

2. Suppose $b \geq a$. Then, $b = aq + r$ for some $q \geq 1$ and $r < a$. Then, it is easy to observe that
   \[ S(a, b; h) = S(a, r; h) + \frac{qh(h+1)(a+2)}{4}. \]  
   (15)

3. We keep repeating Steps 1 and 2 until we are done.

2.2 An example

Suppose we want to calculate the value of $T_1(8411, 2732; 1221)$, that is
\[
\sum_{i=1}^{1221} \left\{ \frac{2732i}{8411} \right\}^2.
\]

First, we evaluate $S(8411, 2732; 1221)$ using the above algorithm. Set $a = 8411$, $b = 2732$ and $h = 1221$ in Theorem 6 to get
\[
S(8411, 2732; 1221) + S(2732, 8411; 2335) = 5521952154451967 + 441901.
\]

Using (15), we get
\[
S(2732, 8411; 2335) = S(2732, 215; 2335) + 11184575280.
\]

Using Theorem 6 we get
\[
S(2732, 215; 2335) + S(215, 2732; 31) = \frac{43105956866071}{146845}.
\]

Using (15), we get
\[
S(215, 2732; 31) = S(215, 152; 31) + 645792.
\]

Using Theorem 6, we get
\[
S(215, 152; 31) + S(152, 215; 129) = \frac{62027530983}{65360}.
\]

Using (15), we get
\[
S(152, 215; 129) = S(152, 63; 129) + 645645.
\]

Using Theorem 6, we get
\[
S(152, 63; 129) + S(63, 152; 9) = \frac{1719655381}{6384}.
\]

Using (15), we get
\[
S(63, 152; 9) = S(63, 26; 9) + 2925.
\]
Using Theorem 6, we get

\[ S(63, 26; 9) + S(26, 63; 21) = \frac{9093619}{1092}. \]  

(24)

Using (15), we get

\[ S(26, 63; 21) = S(26, 11; 21) + 6468. \]  

(25)

Using Theorem 6, we get

\[ S(26, 11; 21) + S(11, 26; 1) = \frac{757997}{572}. \]  

(26)

Finally, it is easy to see that

\[ S(11, 26; 1) = \frac{151}{11}. \]  

(27)

From (16) to (27), we get that

\[ S(8411, 2732; 1221) = \frac{658946167630}{647}. \]

That is,

\[ \left( \frac{8411}{2} \right)^{\sum_{i=1}^{1221} \left\lfloor \frac{2732i}{8411} \right\rfloor^2} + \left( \frac{8413}{2} \right)^{\sum_{i=1}^{1221} \left\lfloor \frac{2732i}{8411} \right\rfloor} = \frac{658946167630}{647}. \]  

(28)

The summation \( \sum_{i=1}^{1221} \left\lfloor \frac{2732i}{8411} \right\rfloor \) can be easily calculated using the algorithm described in [1, Section 2.3]. However, we provide all the details here for the sake of completeness.

In order to solve the first sum, we apply Theorem 3 to get

\[ \sum_{i=1}^{1221} \left\lfloor \frac{2732i}{8411} \right\rfloor = 483516 - \sum_{i=1}^{396} \left\lfloor \frac{8411i}{2732} \right\rfloor. \]  

(29)

Then, by the division algorithm,

\[ \sum_{i=1}^{396} \left\lfloor \frac{8411i}{2732} \right\rfloor = \sum_{i=1}^{396} \left( 3i + \left\lfloor \frac{215i}{2732} \right\rfloor \right) \]

\[ = 235818 + \sum_{i=1}^{396} \left\lfloor \frac{215i}{2732} \right\rfloor. \]

(30)

Repeated applications of Theorem 3 followed by the division algorithm, give the following equations.

\[ \sum_{i=1}^{396} \left\lfloor \frac{215i}{2732} \right\rfloor = 12276 - \sum_{i=1}^{31} \left\lfloor \frac{2732i}{215} \right\rfloor \]

\[ = 6324 - \sum_{i=1}^{31} \left\lfloor \frac{152i}{215} \right\rfloor, \]  

(31)
\[
\sum_{i=1}^{31} \left\lfloor \frac{152i}{215} \right\rfloor = 651 - \sum_{i=1}^{21} \left\lfloor \frac{215i}{152} \right\rfloor = 420 - \sum_{i=1}^{21} \left\lfloor \frac{63i}{152} \right\rfloor, \tag{32}
\]
\[
\sum_{i=1}^{21} \left\lfloor \frac{63i}{152} \right\rfloor = 168 - \sum_{i=1}^{8} \left\lfloor \frac{152i}{63} \right\rfloor = 96 - \sum_{i=1}^{8} \left\lfloor \frac{26i}{63} \right\rfloor, \tag{33}
\]
\[
\sum_{i=1}^{8} \left\lfloor \frac{26i}{63} \right\rfloor = 24 - \sum_{i=1}^{3} \left\lfloor \frac{63i}{26} \right\rfloor = 12 - \sum_{i=1}^{3} \left\lfloor \frac{11i}{26} \right\rfloor, \tag{34}
\]
and
\[
\sum_{i=1}^{3} \left\lfloor \frac{11i}{26} \right\rfloor = 3 - \sum_{i=1}^{1} \left\lfloor \frac{26i}{11} \right\rfloor = 1. \tag{35}
\]

From (29) to (35), we get
\[
\sum_{i=1}^{1221} \left\lfloor \frac{2732i}{8411} \right\rfloor = 241709. \tag{36}
\]

From (28) and (36), we get that
\[
T_1(8411, 2732; 1221) = \sum_{i=1}^{1221} \left\{ \frac{2732i}{8411} \right\}^2 = \frac{2219247661}{5441917}. \tag{37}
\]

Multiplying both sides of this equation by \(8411^2\), the above statement is equivalent to
\[
\sum_{i=1}^{1221} r_i^2 = 28850219593,
\]
where \(r_i\) is the remainder when \(2732i\) is divided by \(8411\).

### 3 An algorithm for \(T_2(a, b; h)\) and \(T_3(a, b; h)\)

Recall our notation from Section 1.

- \(T_1(a, b; h) = \sum_{i=1}^{h} \left\{ \frac{i}{a} \right\}^2\).
- \(T_2(a, b; h) = \sum_{i=1}^{h} i \left\lfloor \frac{b}{a} \right\rfloor\).
\[ T_3(a, b; h) = \sum_{i=1}^{h} \left\lfloor \frac{ib}{a} \right\rfloor^2. \]

Note that
\[
T_3(a, b; h) - T_1(a, b; h) = \sum_{i=1}^{h} \frac{ib}{a} \left( \left\lfloor \frac{ib}{a} \right\rfloor - \left\{ \frac{ib}{a} \right\} \right)
= \sum_{i=1}^{h} \frac{ib}{a} \left( 2 \left\lfloor \frac{ib}{a} \right\rfloor - \frac{ib}{a} \right)
= \frac{2b^2}{a} T_2(a, b; h) - \frac{b^2 h(h+1)(2h+1)}{6a^2}.
\]

Thus, we get the following relationship between \( T_1(a, b; h), T_2(a, b; h) \) and \( T_3(a, b; h) \).

\[
T_3(a, b; h) = T_1(a, b; h) + \frac{2b^2}{a} T_2(a, b; h) - \frac{b^2 h(h+1)(2h+1)}{6a^2}. \tag{38}
\]

### 3.1 Reciprocity relation for \( T_2(a, b; h) \)

Next, we use another method to calculate \( T_3(a, b; h) \). We generalize the ideas in the proof of Theorem 3 described in [1]. For the sake of completeness, we provide all the details here.

Let \( h' \) denote the quantity \( \left\lfloor \frac{bh}{a} \right\rfloor \). Then,

\[
T_3(a, b; h) = \sum_{t=1}^{h'} t^2 n_t,
\]

where \( n_t \) is the number of \( i \) such that \( 1 \leq i \leq h \) and \( \left\lfloor \frac{ib}{a} \right\rfloor = t \). Clearly, if \( t < h' \), then

\[
n_t = \left\lfloor \frac{(t+1)a}{b} \right\rfloor - \left\lfloor \frac{ta}{b} \right\rfloor;
\]

if \( t = h' \), then

\[
n_t = h - \left\lfloor \frac{h'a}{b} \right\rfloor.
\]

Therefore,

\[
\sum_{i=1}^{h} \left\lfloor \frac{ib}{a} \right\rfloor^2 = \sum_{t=1}^{h' - 1} \left( \left\lfloor \frac{(t+1)a}{b} \right\rfloor - \left\lfloor \frac{ta}{b} \right\rfloor \right) t^2 + \left( h - \left\lfloor \frac{h'a}{b} \right\rfloor \right) h'^2
= \sum_{t=1}^{h' - 1} \left( t^2 \left\lfloor \frac{(t+1)a}{b} \right\rfloor - t^2 \left\lfloor \frac{ta}{b} \right\rfloor \right) - \sum_{t=1}^{h' - 1} (2t - 1) \left\lfloor \frac{ta}{b} \right\rfloor + \left( h - \left\lfloor \frac{h'a}{b} \right\rfloor \right) h'^2
= (h' - 1)^2 \left\lfloor \frac{h'a}{b} \right\rfloor - \sum_{t=1}^{h' - 1} (2t - 1) \left\lfloor \frac{ta}{b} \right\rfloor + hh'^2 - h'^2 \left\lfloor \frac{h'a}{b} \right\rfloor
= hh'^2 - \sum_{t=1}^{h' - 1} (2t - 1) \left\lfloor \frac{ta}{b} \right\rfloor
+ \sum_{t=1}^{h'} \left\lfloor \frac{ta}{b} \right\rfloor.
\]
Thus, we obtain the following relation:

$$T_3(a, b; h) = hh'^2 - 2T_2(b, a; h') + \sum_{t=1}^{h'} \left\lfloor \frac{ta}{b} \right\rfloor .$$  \hspace{1cm} (39)

Using (38) and (39), we get the following reciprocity relation for $T_2(a, b; h)$:

$$T_2(a, b; h) + \frac{a}{b}T_2(b, a; h') = \frac{ah'^2}{2b} + \frac{a}{2b} \left( \sum_{t=1}^{h'} \left\lfloor \frac{ta}{b} \right\rfloor \right) - \frac{a}{2b} T_1(a, b; h) + \frac{bh(h + 1)(2h + 1)}{12a}. \hspace{1cm} (40)$$

We describe our algorithm for calculating $T_2(a, b; h)$. The quantity $T_3(a, b; h)$ can then be easily obtained from $T_1(a, b; h)$ and $T_2(a, b; h)$ using (38). Our algorithm for $T_2(a, b; h)$ is as follows:

1. Suppose $a > b$. We express $T_2(a, b; h)$ in terms of $T_2(b, a; h')$ using (40). Note that the expression involves the terms $T_1(a, b; h)$ and $\sum_{t=1}^{h'} \left\lfloor \frac{ta}{b} \right\rfloor$. The former can be calculated using the algorithm in Section 2 and the latter can be calculated using Theorem 3 as described in the algorithm in [1, Section 2.3].

2. Suppose $b \geq a$. Then, $b = aq + r$ for some $q \geq 1$ and $r < a$. Then, it is easy to observe that

$$T_2(a, b; h) = T_2(a, r; h) + \frac{qh(h + 1)(2h + 1)}{6}. \hspace{1cm} (41)$$

3. We keep repeating Steps 1 and 2 until we are done.

### 3.2 An example

We return to our example $a = 8411$, $b = 2732$ and $h = 1221$. Using our algorithm for $T_1(a, b; h)$ in Section 2 and the algorithm for $\sum_{i=1}^{h'} \left\lfloor \frac{ai}{b} \right\rfloor$ in [1, Section 2.3], we easily obtain

$$T_1(8411, 2732; 1221) = \frac{2219247661}{5441917},$$

$$\sum_{i=1}^{396} \left\lfloor \frac{8411i}{2732} \right\rfloor = 241807.$$

Then using (40),

$$T_2(8411, 2732; 1221) + \frac{8411}{2732}T_2(2732, 8411; 396) = \frac{1075804292917}{2732}. \hspace{1cm} (42)$$

From (41), we get

$$T_2(2732, 8411; 396) = T_2(2732, 215; 396) + 62334558. \hspace{1cm} (43)$$
Using our algorithm for $T_1(a, b; h)$ in Section 2 and the algorithm for $\sum_{i=1}^{h'} \left\lfloor \frac{ia}{b} \right\rfloor$ in [1, Section 2.3], we easily obtain

\[ T_1(2732, 215; 396) = \frac{489539849}{3731912}, \]
\[ \sum_{i=1}^{31} \left\lfloor \frac{2732i}{215} \right\rfloor = 6287. \]

Then using (40),

\[ T_2(2732, 215; 396) + \frac{2732}{215}T_2(215, 2732; 31) = \frac{704030131}{215}. \] (44)

From (41), we get

\[ T_2(215, 2732; 31) = T_2(215, 152; 31) + 124992. \] (45)

Using our algorithm for $T_1(a, b; h)$ in Section 2 and the algorithm for $\sum_{i=1}^{h'} \left\lfloor \frac{ia}{b} \right\rfloor$ in [1, Section 2.3], we easily obtain

\[ T_1(215, 152; 31) = \frac{483579}{46225}, \]
\[ \sum_{i=1}^{21} \left\lfloor \frac{215i}{152} \right\rfloor = 316. \]

Then using (40),

\[ T_2(215, 152; 31) + \frac{215}{152}T_2(152, 215; 21) = \frac{515533}{38}. \] (46)

From (41), we get

\[ T_2(152, 215; 21) = T_2(152, 63; 21) + 3311. \] (47)

Using our algorithm for $T_1(a, b; h)$ in Section 2 and the algorithm for $\sum_{i=1}^{h'} \left\lfloor \frac{ia}{b} \right\rfloor$ in [1, Section 2.3], we easily obtain

\[ T_1(152, 63; 21) = \frac{164511}{23104}, \]
\[ \sum_{i=1}^{8} \left\lfloor \frac{152i}{63} \right\rfloor = 83. \]

Then using (40),

\[ T_2(152, 63; 21) + \frac{152}{63}T_2(63, 152; 8) = \frac{151139}{63}. \] (48)

From (41), we get

\[ T_2(63, 152; 8) = T_2(63, 26; 8) + 408. \] (49)

Using our algorithm for $T_1(a, b; h)$ in Section 2 and the algorithm for $\sum_{i=1}^{h'} \left\lfloor \frac{ia}{b} \right\rfloor$ in [1, Section 2.3], we easily obtain

\[ T_1(63, 26; 8) = \frac{3233}{1323}, \]
\[ \sum_{i=1}^{3} \left\lfloor \frac{63i}{26} \right\rfloor = 13. \]
Then using (40),
\[ T_2(63, 26; 8) + \frac{63}{26} T_2(26, 63; 3) = \frac{3695}{26}. \]  
(50)
From (41), we get
\[ T_2(26, 63; 3) = T_2(26, 11; 3) + 28. \]  
(51)
Using our algorithm for \( T_1(a, b; h) \) in Section 2 and the algorithm for \( \sum_{i=1}^{h'} \left\lfloor \frac{ia}{b} \right\rfloor \) in [1, Section 2.3], we easily obtain
\[ T_1(26, 11; 3) = \frac{327}{388}, \]
\[ \sum_{i=1}^{1} \left\lfloor \frac{26i}{11} \right\rfloor = 2. \]
Then using (40),
\[ T_2(26, 11; 3) + \frac{26}{11} T_2(11, 26; 1) = \frac{85}{11}. \]  
(52)
From (41), we get
\[ T_2(11, 26; 1) = T_2(11, 4; 1) + 2. \]  
(53)
It is easy to see that
\[ T_2(11, 4; 1) = 0. \]  
(54)
From (42) to (51), it follows that
\[ T_2(8411, 2732; 1221) = \sum_{i=1}^{1221} \left\lfloor \frac{2732i}{8411} \right\rfloor = 196956430. \]  
(55)
Finally, we use (38) to calculate \( T_3(8411, 2732; 1211) \) from the values of \( T_1(8411, 2732; 1211) \) and \( T_1(8411, 2732; 1211) \) obtained in (37) and (55), respectively.
\[ T_3(8411, 2732; 1221) = T_1(8411, 2732; 1221) + \frac{5464}{8411} T_2(8411, 2732; 1221) - \frac{348800520350128}{5441917} \]
\[ = \frac{2219247661}{5441917} + \frac{5464}{8411} \times 196956430 - \frac{348800520350128}{5441917} \]
\[ = 63853169. \]
That is,
\[ T_3(8411, 2732; 1221) = \sum_{i=1}^{1221} \left\lfloor \frac{2732i}{8411} \right\rfloor^2 = 63853169. \]

4 Efficiency of the algorithms

We compare the reciprocity relation in Theorem 6 with that in Theorem 3. The analysis in [1, Section 2.5] shows that \( S(a, b; h) \) can be calculated in \( O(\log t) \) steps where \( t = \max(a, b) \).
The quantity $\sum_{i=1}^{h} \left\lfloor \frac{ib}{a} \right\rfloor$ can also be calculated in $O(\log t)$ steps, as described in [1, Section 2.5]. Therefore, $T_1(a, b; h) = \sum_{i=1}^{h} \left\{ \frac{ib}{a} \right\}^2$ can be calculated in $O(\log t)$ steps.

Consider the reciprocity relation for $T_2(a, b; h)$ in [10]. Note that this is similar to the ones above except that in each step, we need to calculate $T_1(a, b; h)$ and $\sum_{i=1}^{h'} \left\lfloor \frac{ia}{b} \right\rfloor$, both of which require $O(\log t)$ steps. Thus, in order to calculate $T_2(a, b; h)$, we need to apply the reciprocity relation $O(\log t)$ times and each time, we need to perform $O(\log t)$ steps. Hence, the number of steps required for calculating $T_2(a, b; h) = \sum_{i=1}^{h} i \left\lfloor \frac{ib}{a} \right\rfloor^2$ is $O((\log t)^2)$.

The quantity $T_3(a, b; h)$ can be obtained from $T_1(a, b; h)$ and $T_2(a, b; h)$ using (38). Therefore, the number of steps required for calculating $T_3(a, b; h) = \sum_{i=1}^{h} \left\lfloor \frac{ib}{a} \right\rfloor^2$ is also $O((\log t)^2)$.

5 Acknowledgements

I want to thank the Maths Department at SFU for providing me various awards and fellowships which help me conduct my research.

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