On suspensions, and conjugacy of hyperbolic automorphisms (and of a few more).

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Abstract

We remark that the conjugacy problem for pairs of hyperbolic automorphisms of a finitely presented group (typically a free group) is decidable. The solution that we propose uses the isomorphism problem for the suspensions, and the study of their automorphism group. In the last part of the paper we attempt to extend our strategy to some cases of relatively hyperbolic automorphisms of a free group.

Let $F$ be a finitely presented group, $\text{Aut}(F)$ be its automorphism group, and $\text{Out}(F) = \text{Aut}(F)/\text{Inn}(F)$ be its outer automorphism group.

A way to consider the conjugacy problem in $\text{Aut}(F)$, or $\text{Out}(F)$, is to relate it to an isomorphism problem on semi-direct products of $F$ with $\mathbb{Z}$.

Given two semi-direct products, $F \rtimes_\alpha \langle t \rangle$ and $F \rtimes_\beta \langle t' \rangle$, their structural automorphisms $\alpha$ and $\beta$ are conjugated in $\text{Aut}(F)$ if and only if there is an isomorphism $F \rtimes_\alpha \langle t \rangle \rightarrow F \rtimes_\beta \langle t' \rangle$ sending $F$ on $F$, and $t$ on $t'$. They are conjugated in $\text{Out}(F)$ if and only if there is an isomorphism sending $F$ on $F$, and $t$ in $t'F$.

By analogy with topology and dynamical systems, we wish to call such semi-direct products suspensions of $F$, in which $F$ is the fiber and $t$ is the choice of a transverse direction, and $tF$ is the choice of a transverse orientation. The conjugacy problem in $\text{Out}(F)$ can be expressed as the problem of determining whether suspensions are fiber-and-orientation-preserving isomorphic.

In the first part of this paper we carry out this approach for automorphisms of finitely presented groups producing word-hyperbolic suspensions.

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Consider for instance $F = F_n$, a free group of finite rank $n$. In that case, a solution to the conjugacy problem in $\text{Out}(F_n)$ was announced by Lustig \cite{Lu1, Lu2}. However, it might still be desirable to find short complete solutions for specific classes of elements in $\text{Out}(F_n)$. For instance, consider the class of atoroidal automorphisms: those whose powers do not preserve any conjugacy class beside $\{1\}$. Since Brinkmann proved in \cite{Br1} that an automorphism produces an hyperbolic suspension if and only if it is atoroidal, there is a conceptually simple (slightly brutal) way to algorithmically check whether two given automorphisms are indeed atoroidal (look for a preserved conjugacy class, and simultaneously look for a certificate of hyperbolicity of the semi-direct product), and if they are, our main result will allow to decide whether they are conjugate in $\text{Out} F_n$.

For hyperbolic groups, the isomorphism problem is solved \cite{Sel, DGr, DGu2}. In several examples, the solution available can settle the conjugacy problem. Take two pseudo-Anosov diffeomorphisms of a hyperbolic surface. The mapping tori are closed hyperbolic 3-manifold, hence hyperbolic and rigid. Sela’s solution to the isomorphism problem of their fundamental groups provides all conjugacy classes of isomorphisms (there are finitely many), and from that point, it is possible to check whether one of them preserves the fiber.

For automorphisms of a free group, the analogous situation is when the two automorphisms are atoroidal, fully irreducible (with irreducible powers), and for their conjugacy problem, see \cite{Lo, Lu3, Sel}, the later using the same strategy as above. However, there are atoroidal automorphisms for which the suspension, though hyperbolic, is not rigid. In \cite{Br2} Brinkmann gave several examples with different behaviors. In particular, the solution to the isomorphism problem of hyperbolic groups will not reveal all isomorphisms between suspensions, and since the fibers are exponentially distorted in the suspensions, the usual rational tools (see \cite{D, DGu1}) do not work for solving the isomorphism problem with such a preservation constraint. One can thus merely detect the existence of one isomorphism (say $\iota$), but for investigating the existence of an isomorphism with the aforementioned properties, one is led to consider an orbit problem of the automorphism group of $F \rtimes \langle t \rangle$: decide whether an automorphism sends $\iota(F)$ on $F$ and $\iota(t)$ on $t' \ (\text{or in } t'F)$.

Orbits problems are not necessarily easier, especially if the group acting is large and complicated. In \cite{BMV}, for instance, Bogopolski, Martino and Ventura propose a subgroup of $\text{GL}(4, \mathbb{Z})$ whose orbit problem on $\mathbb{Z}^4$ is

\footnote{In the sense that the exposition is short; in this paper we ostensibly ignore algorithmic complexity, and to some extend conceptual complexity hidden in the tools that are used.}
undecidable.

In the first part of this paper we prove that, if $F$ is finitely generated and $F \rtimes \langle t \rangle$ hyperbolic, then $\text{Out}(F \rtimes \langle t \rangle)$ contains a finite index abelian subgroup, whose action on $H_1(F \rtimes \langle t \rangle)$ is generated by transvections. This allows us to prove that the specific orbit problem above is solvable in that case, by reducing it to a system of linear Diophantine equations, read in $H_1(F \rtimes \langle t \rangle)$. This is explained in 2.1.

These are thus the key steps to produce what we see as a picturesque way of solving the conjugacy problem for automorphisms of finitely presented groups with hyperbolic suspension (Theorem 3.2).

The proof that $\text{Out}(F \rtimes \langle t \rangle)$ is virtually abelian is done by considering the canonical JSJ decomposition of the hyperbolic group $F \rtimes \langle t \rangle$. It suffices to show that this graph-of-groups decomposition does not contain any surface vertex group. We cannot resist to sketch the proof (that will be detailed in 2.2) and that takes roots in the way Brinkmann produces his examples in [Br2]. Consider the tree of the JSJ decomposition $T$, and $\mathbb{X}$ the graph of group quotient of $T$ by $G = F \rtimes \langle t \rangle$. Since $F$ is normal in $G$, $T$ is a minimal tree for $F$, and $\mathbb{Y} = F \setminus T$ is a graph-of-group decomposition of $F$ whose underlying graph is its own core, and since its genus is bounded by the rank of $F$, it is finite. It follows that every vertex group (resp. edge group) in $\mathbb{X}$ is the suspension of a vertex group (resp. edge group) in $\mathbb{Y}$: lift the vertex in $T$, where its $\langle t \rangle$-orbit passes twice on a pre-image of a vertex in $\mathbb{Y}$, thus yielding the suspension. Since edge groups in $\mathbb{X}$ are cyclic, edge groups in $\mathbb{Y}$ must be trivial. Therefore $\mathbb{Y}$ is a free decomposition of $F$, and its vertex groups are of finite type. Going back to $\mathbb{X}$ again, vertex groups of $\mathbb{X}$ are suspensions of infinite, finitely generated groups, hence cannot be free nor surface groups, because finitely generated normal subgroups of free groups (or surface groups) are of finite index or trivial.

To present these arguments, the formalism of automorphisms of graph of groups is to be recalled, in a rather precise way in order to be useful. The confident reader may skip this part (section 1), and only retain that the small modular group is the group of automorphisms generated by Dehn twists on edges of a graph of groups.

In the last part of this paper we attempt to extend this strategy to the case of automorphisms of free groups producing relatively hyperbolic suspensions, but ultimately have to restrict our study to those whose suspension does not split over a parabolic subgroup. I also have to concede that this last part (section 4) uses three results un published at the time of writing ([GLu], [T] and [DGu4], the later in a crucial way). The main result there is Proposition 4.11 stating that there is an (explicit) algorithm
that given two automorphisms $\phi_1, \phi_2$ of a free group $F$ terminates if both produce proper relatively hyperbolic suspensions (relative to suspensions of polynomial subgroups), without parabolic splitting, and indicates whether they are conjugated in $\text{Out}(F)$.

1 Preliminary on automorphisms of graphs-of-groups

1.1 Trees and splittings

We fix our formalism for graphs and graph of groups. This material is classical, and can be found in Serre’s book [Ser]. A graph is a tuple $X = (V, E, \bar{e}, o, t)$ where $V$ is a set (of vertices), $E$ is a set (of oriented edges) and $\bar{e} : E \to E$, $o : E \to V$, $t : E \to V$ verify $(\circ \bar{e}) = \text{Id}_E$ and $(o \circ \bar{e}) = t$. A graph-of-groups $\mathfrak{X}$ consists of a graph $X$, for each vertex $v$ of $X$, a group $\Gamma_v$, for each unoriented edge $\{e, \bar{e}\}$ of $X$, a group $\Gamma_{\{e, \bar{e}\}}$ (but we will write $\Gamma_e = \Gamma_{\bar{e}}$ for it), and for each oriented edge $e$ of $X$, a injective homomorphism $i_e : \Gamma_e \to \Gamma_{o(e)}$, where $o(e)$ is the origin vertex of the oriented edge $e$.

The Bass group $B(\mathfrak{X})$ is

$$B(\mathfrak{X}) = \langle \bigcup_{v \in V} \Gamma_v \cup E \mid \forall e \in E, \forall g \in \Gamma_e, \bar{e} = e^{-1}, e^{-1}i_e(g)e = i_{\bar{e}}(g) \rangle.$$  

An element $g_0e_1g_1e_2 \ldots g_ne_ng_{n+1}$ is a path element from $o(e_1)$ to $t(e_n)$ if $g_{i-1} \in o(e_i)$ and $g_i \in t(e_i)$ for all $i$.

The fundamental group $\pi_1(\mathfrak{X}, v_0)$ of the graph-of-groups $\mathfrak{X}$ at the vertex $v_0$, is the subgroup of the Bass group consisting of path elements from $v_0$ to $v_0$.

Choose a maximal subtree $\tau$ of the graph $X$, and consider $\pi_1(\mathfrak{X}, \tau) = B(\mathfrak{X})/\langle \langle e \in \tau \rangle \rangle$. Then the quotient map $B(\mathfrak{X}) \to \pi_1(\mathfrak{X}, \tau)$ is, in restriction to $\pi_1(\mathfrak{X}, v_0)$, an isomorphism.

A $G$-tree is a simplicial tree with a simplicial action of $G$ without inversion. It is minimal if there is no proper invariant subtree. It is reduced if the stabiliser of any edge is a proper subgroup of the stabiliser of any adjacent vertex.

The quotient of a $G$-tree by $G$ is naturally marked by the family of conjugacy classes of stabilizers of vertices and edges, and inherits a structure of a graph-of-groups (whose fundamental group is isomorphic to $G$).

The Bass-Serre tree of a graph-of-groups is its universal covering in the sense of graphs of groups. It is a $\pi_1(\mathfrak{X}, v_0)$-tree.

A collapse of a $G$-tree $T$ is a $G$-tree $S$ with an equivariant map from $T$ to $S$ that sends each edge on an edge or a vertex and no pair of edges of $T$
in different orbits are sent in the same edge (but they may be sent on the same vertex). A collapse of an edge $e$ in a graph-of-group decomposition is a collapse of the corresponding Bass-Serre tree in which only the edges in the orbit of a preimage of $e$ are mapped to a vertex.

Given a group $G$, and a class of groups $C$, a splitting of $G$ over groups in $C$ is an isomorphism between $G$ and the fundamental group of a graph of groups whose edge groups are in $C$. Equivalently, a splitting of $G$ can be thought of as an action of $G$ on a tree, with edge stabilizers in $C$. A splitting is called reduced if, in the tree, there is no edge whose stabilizer equals that of an adjacent vertex.

1.2 Automorphisms, and the small modular group $\text{Mod}_X$

Let $X$ and $X'$ be two graphs-of-groups. An isomorphism of graphs-of-groups $\Phi : X \to X'$ is a tuple $(\Phi_X, (\phi_v), (\phi_e), (\gamma_e))$ such that

- $\Phi_X : X \to X'$ is an isomorphism of the underlying graphs,
- for all vertex $v$, $\phi_v : \Gamma_v \to \Gamma_{\Phi_X(v)}$ is an isomorphism, for all edge $e$, $\phi_e = \phi_e : \Gamma_e \to \Gamma_{\Phi_X(e)}$ is an isomorphism,
- and for each edge $e$, $\gamma_e \in \Gamma_{\Phi_X(t(e))}$ satisfying, for $v = t(e)$,
  $$\phi_v \circ i_e = \text{ad}_{\gamma_e} \circ i_{\Phi_X(e)} \circ \phi_e,$$
  for $(\text{ad}_{\gamma_e} : x \mapsto \gamma_e^{-1}x\gamma_e)$ the inner automorphism of $\Gamma_{\Phi_X(t(e))}$ defined by the conjugacy by $\gamma_e$.

The last point is the commutation of the following diagram (for each edge $e$):

$$
\begin{array}{ccc}
\Gamma_v & \xrightarrow{\phi_v} & \Gamma_{\Phi_X(v)} \\
\uparrow & & \uparrow \\
\Gamma_e & \xrightarrow{\phi_e} & \Gamma_{\Phi_X(e)}
\end{array}
$$

(1)

When $X = X'$ the isomorphisms can be composed in a natural way (see [Bal 2.11]: $(\Phi_X, (\phi_v), (\phi_e), (\gamma_e)) \circ (\Psi_X, (\psi_v), (\psi_e), (\eta_e)) = (\Phi_X \circ \Psi_X, (\phi_{\Psi_X(v)} \circ \psi_v), (\phi_{\Psi_X(e)} \circ \psi_e), (\phi_{\Psi_X(t(e))} \circ \psi_e), (\phi_{\Psi_X(t(e))} \circ \psi_e), (\phi_{\Psi_X(t(e))} \circ \psi_e))$), thus providing the automorphism group of the graph-of-group $X$, denote by $\delta\text{Aut}(X)$.

This group $\delta\text{Aut}(X)$ naturally maps into the automorphism group of the Bass group $\mathcal{B}(X)$ in the following way: for all edge $e$, and all automorphism $\Phi = (\Phi_X, (\phi_v), (\phi_e), (\gamma_e)) \in \delta\text{Aut}(X)$, one has $\Phi(\epsilon) = \gamma_e^{-1} \Phi_X(\epsilon) \gamma_e$, and $\Phi|_{\Gamma_v} = \phi_v$. One can check that, for any $\Phi$, the relations of the Bass group are preserved, and that the thus induced morphism is bijective.
Remark 1.1. For this argument, see [Ba 2.1, 2.2], but notice that Bass chose to let conjugations act on the left, while we chose to let them act on the right (as in [DGu 2]). This difference yields a few harmless inversions in the formulae, (actually the attentive reader may have spotted them already in the relations of the Bass group, [Ba 1.5 (1.2)]). and the only risk here is to mix both (incompatible) choices.

Each automorphism in $\delta \text{Aut}(X)$ sends path elements to path elements, hence naturally provides an outer-automorphism of $\pi_1(X,v_0)$ (and a genuine automorphism if $\Phi(v_0) = v_0$), and we will often implicitly make this identification.

Let us define the small modular group of $X$, denoted by $\text{Mod}_X$, to be the subgroup of $\delta \text{Aut}(X)$ consisting of elements of the form $(\text{Id}_X, (\phi_v), (\text{Id}_{\gamma_v}), (\gamma_v))$, for $\phi_v \in \text{Inn}(\Gamma_v)$ inner automorphisms. One can check, using the composition rule, that this forms a subgroup of $\delta \text{Aut}(X)$.

If we note $\phi_v = \text{ad}_{\gamma_v}$, the compatibility condition (1) imposes that $\gamma_v \gamma_{e^{-1}} \in Z_{\Gamma_{\epsilon(e)}}(i_e(\Gamma_e))$ for all edge $e$.

If one chooses a generating set $S_e$ for each group $Z_{\Gamma_{\epsilon(e)}}(i_e(\Gamma_e))$, then the small modular group is generated by the union of two collections of elements. First, the collection of oriented Dehn twists $\{D_{\epsilon,\gamma}, \epsilon \in E, \gamma \in S_{\epsilon}\}$, defined by $D_{\epsilon,\gamma} = (\text{Id}_X, (\text{Id}_v), (\text{Id}_{\gamma}), (x_{\epsilon}))$ for $x_{\epsilon} = \gamma$ and $x_{\epsilon} = 1$ for all $\epsilon \neq \epsilon$. Second, the collection of inert twists, $\{(\text{Id}_X, (\text{ad}_{\gamma_v}), (\text{Id}_{\gamma}), (\gamma_v = \gamma_{\epsilon(e)}))\}$ (note that it is the same family of elements involved as $(\gamma_v)$ and in defining the $(\phi_v)$, and that this defines an element of $\text{Mod}_X$).

The inert twist are not so interesting. They correspond to changing some choice of fundamental domain in the Bass-Serre tree. In particular:

Lemma 1.2. Any inert twist vanishes in $\text{Out}(\pi_1(X,v_0))$.

Proof. Consider an inert twist $\Phi$. After conjugation over the whole group, we may assume that $\phi_{v_0} = \text{Id}_{\gamma_{v_0}}$. Then take a path element in the Bass group, $g_0 \epsilon_0 \ldots g_n \epsilon_n g_{n+1}$ that is a loop from $v_0$ to $v_0$. Each $g_i$ is in $\Gamma_{\epsilon(e)} = \Gamma_{\epsilon(e+1)}$. Thus, $\Phi(g_i) = \text{ad}_{\gamma_{\epsilon(i)}}(g_i)$, and $\Phi(\epsilon_i) = \gamma_{i-1}^{-1} \epsilon_i \gamma_i$. Concatenation makes everything collapse, and $\Phi(w) = w$. \qed

We record how oriented Dehn twists are realised as automorphisms of $\pi_1(X,v_0)$. The following is an immediate consequence of the definitions.

Lemma 1.3. Let $\epsilon \in E$ be an oriented edge of $X$, and $\gamma \in Z_{\Gamma_{\epsilon(e)}}(i(\Gamma_{\epsilon}))$.

Let $\epsilon$ be an oriented edge different from $\epsilon$ and $\epsilon$.

The oriented Dehn twist $D_{\epsilon,\gamma}$, seen as an automorphism of $B(X)$, is such that $D_{\epsilon,\gamma}(\epsilon) = \epsilon \gamma$, $D_{\epsilon,\gamma}(\epsilon) = \gamma^{-1} \epsilon$ and $D_{\epsilon,\gamma}(\epsilon) = \epsilon$. 

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Moreover, for all vertex \( v \) in \( X \), and \( \gamma_v \in \Gamma_v \), \( D_{\epsilon, \gamma}(\gamma_v) = \gamma_v \).

One should be nonetheless cautious with the interpretation of \( D_{\epsilon, \gamma} \) as an automorphism of \( \pi_1(X, v_0) \), since the identification of a preferred copy of \( \Gamma_v \) in \( \pi_1(X, v_0) \) is subject to a choice of path from \( v_0 \) to \( v \). If this path contains a (or several) copy of the edge \( \epsilon \), then \( D_{\epsilon, \gamma} \) is actually conjugating, on the right, \( \Gamma_v \) by \( \gamma^{-1} \) (or a power of it).

2 The \( \text{Aut}(G) \)-orbit of the fiber of a suspension

2.1 An orbit problem for the small modular group

In this section, we discuss an orbit problem for \( \operatorname{Mod}_X \) in \( H_1(G) \), the abelianisation of \( G \). For that, we note that \( \operatorname{Mod}_X \) naturally maps into \( \text{Aut}(H_1(G)) \simeq GL(H_1(G)) \). We will denote by \( \overline{\gamma} : G \to H_1(G) \) the abelianisation map. Let us also choose \( E^+ \subset E \) a set of representatives of unoriented edges.

Proposition 2.1. Let \( G = \pi_1(X, v_0) \). The image of \( \operatorname{Mod}_X \) in \( GL(H_1(G)) \) is abelian, generated by transvections.

Proof. Since inert twists vanish in \( \text{Out}(G) \), the images of oriented Dehn twists, generate the image of \( \operatorname{Mod}_X \) in \( GL(H_1(G)) \). It suffice to show that oriented Dehn twists induce transvections on \( H_1(G) \) that commute.

Any element \( \gamma \) of \( \pi_1(X, v_0) \) has an expression as a normal form coming from the ambient Bass group: \( \gamma = g_0 e_0 g_1 e_1 \cdots e_m g_{m+1} \) where for all \( i, e_i \in E \) and \( g_i, g_{i+1} \in \Gamma_{t(e_i)}, g_0, g_{m+1} \in \Gamma_{v_0} \).

The normal form of \( \gamma \) turns, in \( H_1(G) \), into \( \bar{\gamma} = g_0 \cdots g_{m+1} \times \prod_{e \in E^+} e^n(\gamma, e) \) where \( n(\gamma, e) \) is the number of occurrences of \( e \) in \( (e_0, \ldots, e_m) \) minus the number of occurrences of \( \bar{e} \).

If \( D_{\epsilon, h} \) is the Dehn twist of \( h \) on \( \epsilon \), the induced automorphism on \( H_1(G) \) is denoted by \( \overline{D_{\epsilon, h}} \). From the expression of \( D_{\epsilon, h}(\epsilon) \) in 1.3, \( \overline{D_{\epsilon, h}} \) is the following transvection

\[
\overline{D_{\epsilon, h}}(\bar{\gamma}) = (g_0 g_1 \cdots g_{m+1}) \times \left( \prod_{e \in E^+} e^n(\gamma, e) \right) \times \bar{h}^n(\gamma, e) = \bar{\gamma} \times \bar{h}^n(\gamma, e). \tag{*}
\]

Since \( h \in \Gamma_{t(\epsilon)} \), the element \( h^n(\gamma, e) \) is in a vertex group, and it is fixed (if seen in the Bass group) or conjugated (if seen in \( G \)) by all oriented Dehn twists on edges, thus oriented Dehn twists on edges commute in the abelianisation. \( \square \)
In fact, if $Z_{\Gamma t}(i_e(\Gamma_e))$ is abelian (which is the case if $G$ torsion free hyperbolic), then $\text{Mod}_X$ is abelian itself.

We keep the same notations. Note that the group $H_1(G)/\bar{F}$ is infinite cyclic, by assumption, and is generated by the image of $\bar{t}$. Define $\gamma: G \to H_1(G)/\bar{F}$ and for all $\gamma \in G$, define $\delta(\gamma)$ to be the unique integer such that $\bar{\gamma} = \bar{\bar{t}} \delta(\gamma)$.

**Proposition 2.2.** Let $G$ be a finitely generated group that can be expressed as a semi-direct product $F \rtimes \langle t \rangle$.

Given a splitting $X$ of $G$, and for each $e \in E$, a generating set $S_e \subset \Gamma_t(\Gamma_e)$ (the centralizer of the attached edge group in vertex group), and a family $(\gamma_j)_{0 \leq j \leq j_0}$ of elements of $G$, one can decide whether there is an element $\eta \in \text{Mod}_X$ whose image $\bar{\eta}$ in $\text{Aut}(H_1(G))$ sends $\bar{\gamma}_j$ in $\bar{F}$ for all $j < j_0$ and $\bar{\gamma}_{j_0}$ inside $\bar{t} \bar{F}$.

More precisely, there is such an element $\eta$ if and only if the explicit Diophantine linear system of equations

$$\forall j, \sum_{e \in E^+, s_e \in S_e} r_{s_e} n(\gamma_j, e) \delta(s_e) = -\delta(\gamma_j) + \text{dirac}_{j = j_0}$$

(with unknowns $r_{s_e}$) has a solution.

**Proof.** By $(\ast)$ (from the proof of Proposition 2.1), $\delta(D_{\epsilon, h}(\gamma)) = \delta(\gamma) + n(\gamma, \epsilon) \delta(h)$ and by induction, and the fact that $h$ is fixed by all Dehn twists,

$$\delta(D_{\epsilon_1, h_1} D_{\epsilon_2, h_2} \cdots D_{\epsilon_k, h_k}(\gamma)) = \delta(\gamma) + \sum_{i=1}^{k} n(\gamma, \epsilon_i) \delta(h_i).$$

Assume that there exists $\eta \in \text{Mod}_X$ such that $\delta(\eta(\gamma_j)) = 0$ for $j < j_0$ and $= 1$ for $j = j_0$. Since oriented Dehn twists generate the image of $\text{Mod}_X$ in $\text{GL}(H_1(G))$, this element $\eta$ can be chosen as a product of oriented Dehn twists $D_{\epsilon_1, h_1} D_{\epsilon_2, h_2} \cdots D_{\epsilon_k, h_k}$, which by commutation in $\text{Aut}(H_1(G))$ can be chosen to be

$$\prod_{e \in E^+, s_e \in S_e} D_{\epsilon, s_e}^{\bar{s}_e}.$$

Therefore by the previous equation, $\forall e \in E, \forall s_e \in S_e, \exists r_{s_e} \in \mathbb{Z}$

$$\forall j, \sum_{e \in E^+, s_e \in S_e} r_{s_e} n(\gamma_j, e) \delta(s_e) = -\delta(\gamma_j) + \text{dirac}_{j = j_0}$$

(in which $\text{dirac}_{j = j_0}$ yields 0 if $j \neq j_0$ and 1 otherwise).
Conversely, if the system of equations
\[
\forall j, \sum_{e \in E^+, s_e \in S_e} r_{s_e} \delta(\gamma_j, e) + \text{dirac}_{j=j_0} = -\delta(\gamma_j) \tag{1}
\]
has a solution (in unknowns \(r_{s_e}\)), then \(\delta(\prod_{e \in E^+, s_e \in S_e} D_{s_e}^{\ell_e}(\gamma_j)) = 0\) for \(j < j_0\) and = 1 for \(j = j_0\).

We have thus reduced the orbit problem to an equivalent problem of satisfaction of a system of linear Diophantine equations, that are explicitly computable. The problem is therefore solvable, by classical technics of linear algebra.

If we had the same statement with \(\text{Aut}(G)\) replacing \(\text{Mod}_X\), we would be very close to our conclusion. What we will show is that \(\text{Mod}_X\) has finite index image in \(\text{Out}(G)\).

### 2.2 No surface vertex group in splittings of suspensions

Our aim is Proposition 2.9, that states that in any splitting of a suspension, there is no vertex whose group which is a surface group with boundary, where the boundary subgroups are the adjacent edge groups.

More precisely, if \(T\) is a \(G\)-tree, we say that a vertex stabilizer \(H\) is a hanging surface group if it is the fundamental group of a non-elementary compact surface with boundary components, and the adjacent edge groups are exactly the subgroups of the boundary components. We say that it is a hanging bounded Fuchsian group if there is a finite normal subgroup \(K \triangleleft H\) such that \(H/K\) is isomorphic to the fundamental group of a non-elementary compact 2-orbifold with boundary, by an isomorphism sending the images of the adjacent edge stabilizers on the boundary subgroups of the orbifold group. We will show that a splitting of a infinitely many ended finitely generated group has no hanging bounded Fuchsian vertex group.

Let \(T\) be the Bass-Serre \(G\)-tree of a reduced splitting of \(G\). Let us introduce \(Y = F\setminus T\) and \(X = G\setminus T\). Both graphs provide respectively graph-of-groups decompositions \(\Upsilon\) and \(\Xi\) of \(F\) and of \(G\).

**Lemma 2.3.** \(Y\) is a finite graph, and the \(G\) action on \(T\) factorises through a \(\langle \ell \rangle\)-action on \(Y\).

*Proof.* Since \(F\) is normal in \(G\), its minimal subtree in \(T\) is \(G\)-invariant, therefore it is \(T\) itself. In other words, \(Y\) is its own core. As for its genus, it it is finite, since it bounds from below the rank of \(F\) (the fundamental group of \(\Upsilon\) at a vertex \(v_0\) projects on the fundamental group of the underlying
graph $Y$ at $v_0$). It follows that $Y$ is a finite graph, and it is endowed with an action of $\langle \bar{t} \rangle = G/F$ for which the quotient is $X$.

**Lemma 2.4.** Given any vertex in $T$, its stabilizer is a suspension of a subgroup of $F$, which is a vertex group in the decomposition given by $Y$.

**Proof.** Let $\nu$ be such a vertex, and consider $v$ its image in $X$ and $\tilde{v}$ its image in $Y$. Because $Y$ is finite, there is a smallest $n > 0$ such that $\bar{t}^n \tilde{v}$ is $\tilde{v}$. Lifting in $T$, we obtain the existence of $\gamma \in F$ such that $\gamma t^n \nu = \nu$. Therefore, if $(F)_\nu$ is the stabilizer of $\nu$ in $F$, then it is normalized by $\gamma t^n$ and $(F)_\nu \rtimes \langle \gamma t^n \rangle$ fixes $\nu$.

We claim that $(F)_\nu \rtimes \langle \gamma t^n \rangle$ is the stabilizer in $G$ of $\nu$.

If there is another element in it, it is not in $F$ by definition of $(F)_\nu$, and therefore it is some $\eta t^m$, $\eta \in F, m \neq 0$. By minimality of $n$, $n$ divides $m$, and if $\gamma' t^m = (\gamma t^n)^k$, then $\gamma' t^m \nu = \eta t^m \nu$, and $t^{-m} \gamma' \eta t^m$ fixes $\nu$ and is in $F$ by normality of $F$. Thus it is in $(F)_\nu$, and $\eta t^m \in (F)_\nu \rtimes \langle \gamma t^n \rangle$. This ensures the claim.

**Lemma 2.5.** An edge stabilizer in $T$ is a suspension of a subgroup of $F$.

**Proof.** Subdivide such an edge (and all edges in its orbit, equivariantly) by inserting a vertex on its midpoint, and apply the previous lemma to this vertex.

Up to now we have not used any assumption on the nature of edge groups. But now two remarks are of interest, and directly follow from the previous lemmas.

**Lemma 2.6.** A reduced $G$-tree $T$ never has a finite edge stabilizer.

If $T$ has cyclic edge stabilizers in $G$, then $Y$ is a free splitting of $F$.

If $T$ has virtually cyclic edge stabilizers in $G$, then $Y$ is a splitting of $F$ over finite subgroups.

**Lemma 2.7.** Let $X$ be a graph-of-groups decomposition of $F \rtimes \langle t \rangle$ with virtually cyclic edge groups. Then no vertex group of $X$ is free non-abelian, or even infinitely many ended.

**Proof.** By Lemma 2.6, $T$, as an $F$-tree, is the tree of a virtually free splitting of $F$. In particular, since $F$ is finitely generated, each vertex stabilizer in $F$ is finitely generated. It follows from Lemma 2.4 that a vertex stabilizer of $T$ in $G$ contains an infinite index normal subgroup, which is as we saw, of finite type.

Such a group cannot be free. It cannot be infinitely many ended neither (for the same reason actually, that we recall in the following lemma).
Lemma 2.8. Let $H$ be a finitely generated group with infinitely many ends, then $H$ has no finitely generated infinite-index normal subgroup.

Proof. By Stallings’ theorem, $H$ is the fundamental group of a finite graph-of-groups with finite edge groups (with at least one edge). Let $T$ be the associated Bass-Serre tree, and $N$ be a normal subgroup of $H$. Then, the tree $T$ is minimal for $N$, hence $T/N$ equals its own core. If $N$ is finitely generated, $T/N$ is finite. Moreover, the action of $H$ on $T$ factorises through $T/N$, and if $N$ has infinite index in $H$, there is an edge $\tilde{e}$ in $T/N$ fixed by infinitely many different elements of $H$, $\{h_i, i \in I\}$, all in different $N$-coset. Let $e$ its image in $T/F$ and $\tilde{e}$ a choice of lift in $T$. There are $n_i \in N$ such that for all $i$, $h_in_i$ fix $e$. By finiteness of stabilizers, infinitely many of the elements $h_in_i$ are equal, contradicting that the $h_i$ were in different $N$-cosets. 

Proposition 2.9. Let $F$ a finitely generated group, and $G = F \rtimes \langle t \rangle$ a suspension.

Given any graph of group decomposition $\mathcal{X}$ of $G$, no vertex group of $\mathcal{X}$ is a hanging surface group, or a hanging bounded Fuchsian group.

Proof. Assume the contrary: let $\Gamma_{v_0}$ be an alleged hanging bounded Fuchsian group, which is, in particular, virtually free (a finite cover of the underlying 2-orbifold with boundary is a surface with boundary, hence has free fundamental group). Because it is hanging, all the neighboring edges of $v_0$ carry virtually cyclic groups. Collapse all other edges in $\mathcal{X}$ in order to get $\mathcal{X}'$, whose edges are virtually cyclic. The image $\nu_0'$ of $v_0$ carry the same group, since no adjacent edge has been collapsed, and this group is virtually free, by assumption. Apply Lemma 2.7 to $\mathcal{X}'$ to get the contradiction. 

2.3 The $\text{Aut}(G)$-orbit of the fiber in the hyperbolic case.

Let us recall that the canonical $\mathcal{Z}_{\text{max}}$-JSJ decomposition of a one-ended hyperbolic group is a certain finite splitting $\mathcal{X}$ of $G$ over certain virtually cyclic subgroups (maximal with infinite center), such that every automorphism of $G$ induces an automorphism of graph of groups of $\mathcal{X}$ (see [DG12, section 4.4]). In other word, the natural map $\delta \text{Aut}(\mathcal{X}) \to \text{Out}(G)$ is surjective.

Remark 2.10. The choice to use the rather technical $\mathcal{Z}_{\text{max}}$-JSJ splitting instead of the more natural “virtually-cyclic” JSJ-splitting, is only suggested
by our ability to algorithmically compute this decomposition. In principle, we could work with the classical JSJ splitting as well in the same way.

Let $X$ is a graph-of-group, $v$ a vertex therein, and $\Gamma_v$ the vertex group. The choice of an order on the oriented edges adjacent to $v$, and of a generating set of the edge groups, endows $\Gamma_v$ with a marked peripheral structure, that is the tuple of conjugacy classes of the images of these generating sets by the attaching maps. We denote by $T$ this tuple, and $\text{Out}_m(\Gamma_v, T)$ the subgroup of $\text{Out}(\Gamma_v)$ preserving $T$ (see also \cite{DG2}). In the following, the choices of order and generating sets are implicit, and done a priori.

The following is a typical feature of a JSJ decomposition of a hyperbolic group, whose proof, in this specific setting, is essentially contained in \cite{DG2}.

**Lemma 2.11.** Let $G$ be a one-ended hyperbolic group. Let $(\Gamma_v, T)$ be a vertex group of the $\mathbb{Z}_{\text{max}}$-JSJ decomposition $X$ of $G$, with the marked peripheral structure induced by $X$ (and some choice of finite generating sets of edge groups). If $\text{Out}_m(\Gamma_v, T)$ is infinite, then $G$ admits a splitting with a hanging bounded Fuchsian vertex group.

**Proof.** By \cite{DG2} Prop. 3.1, such a vertex group $\Gamma_v$ must have a further compatible splitting over a maximal virtually cyclic group with infinite center, which allows to use \cite{DG2} Prop 4.17 to ensure that $(\Gamma_v, T)$ is a so-called hanging orbisocket, which by definition \cite{DG2} Def. 4.15 allows to refine $X$ in order to get a splitting of $G$ whose one vertex group is a hanging bounded Fuchsian group. \hfill $\square$

**Proposition 2.12.** Let $F$ a finitely presented group, and $G = F \ast \langle t \rangle$ a suspension that is assumed to be hyperbolic.

Then, the image in $\text{Out}(G)$ of the small modular group of the $\mathbb{Z}_{\text{max}}$-JSJ decomposition of $G$ has finite index in $\text{Out}(G)$. Moreover, one can compute a set of right-coset representatives of $\text{Mod}_X$ in $\text{Out}(G)$ (in the form of automorphisms of $G$).

**Proof.** Since $\delta \text{Aut}(X)$ surjects on $\text{Out}(G)$ in the case of the $\mathbb{Z}_{\text{max}}$-JSJ decomposition, it suffices to show that the small modular group has finite index in $\delta \text{Aut}X$ and that coset representatives can be computed in $\delta \text{Aut}X$.

Once again, this is essentially done in \cite{DG2} (and probably in other places).

First, the splitting $X$ can be effectively computed \cite{DG2} Prop. 6.3].
Consider the following three maps. First $q_X : \delta \text{Aut}(X) \to \text{Aut}(X)$ where $\text{Aut}(X)$ is the automorphism group of the underlying finite graph $X$. Second, the natural map $q_E : \ker q_X \to \prod_{e \in E} \text{Aut}(\Gamma_e)$. Third, the natural map $q_V : \ker q_E \to \prod_{v \in V} \text{Out}_m(\Gamma_v, T)$, where $T$ is the marked peripheral structure induced by the ambient graph-of-group $X$.

The group $\ker q_V$ is the small modular group. The two first maps have finite image. By Lemma 2.11 and Proposition 2.9, the map $q_V$ also have finite image.

Therefore the small modular group has finite index in $\delta \text{Aut}(X)$. In order to reconstruct coset representatives of the small modular group in $\delta \text{Aut}(X)$, it is enough to find coset representatives for the kernel of each of these maps.

In order to compute coset representatives of $\ker q_E$ in $\delta \text{Aut}(X)$, we can make the finite list of all graph automorphisms of $X$ for which $\Gamma_{\Phi_X}(e) \cong \Gamma_e$. Let $\Phi_X$ be any of them. We can make the list of all isomorphisms $\Gamma_{\Phi_X}(e) \cong \Gamma_e$ (there are finitely many such isomorphisms since these groups are virtually cyclic). Then we can apply [DGu2, Prop. 2.28] in order to reveal whether this automorphism $\Phi_X$ has a preimage by $q_X$.

In order to compute coset representatives of $\ker q_E$ in $\ker q_X$, we consider a collection of automorphisms of edge groups, and apply [DGu2, Prop. 2.28] in order to reveal whether this collection has a preimage by $q_E$.

Finally, in order to compute coset representatives of $\ker q_V$ in $\ker q_E$, one can make the list of all elements in $\text{Out}(\Gamma_v, T)$ expressend as automorphisms (by [DGu2, Coro. 3.5], we can enumerate all of them) and for each choice of them (for each $v$), check whether the collection defines a graph-of-group automorphism by solving the simultaneous conjugacy problem that allows the diagram [1] to commute.

**Corollary 2.13.** Let $F$ be finitely presented, and $G = F \rtimes \langle t \rangle$ a suspension that is assumed to be hyperbolic. Given a splitting $X$ of $G$, generating sets for the centralizers of adjacent edge groups in vertex groups, and a family $(\gamma_j)_{0 \leq j < j_0}$ of elements of $G$, one can decide whether there is an element $\eta \in \text{Aut}(G)$ whose image $\bar{\eta}$ in $\text{Aut}(H_1(G))$ sends $\bar{\gamma}_j$ in $\bar{F}$ for all $j < j_0$ and $\bar{\gamma}_{j_0}$ inside $\bar{t}F$.

**Proof.** First we take note that $\eta \in \text{Aut}(G)$ satisfies the conclusion if and only if any other automorphism in the same class in $\text{Out}(F)$ satisfies it. Thus, let $\alpha_1, \ldots, \alpha_k$ be the right-coset representatives of $\text{Mod}_X$ in $\text{Out}(G)$ computed by Proposition 2.12 in the form of automorphisms. For each $i$, we compute, for each $j$, $\alpha_i(\gamma_j)$, and we use Proposition 2.2 in order to decide
whether there is $\eta_0 \in \text{Mod}_X$ such that $\overline{\eta_0(\alpha_i(\gamma_j))} \in \bar{F}$ for all $j < j_0$ and $\overline{\eta_0(\alpha_i(\gamma_{j_0}))} \in \bar{t}\bar{F}$.

If there exists an index $i$ such that the answer is positive, then $\overline{\eta_0 \circ \alpha_i}$ sends $\overline{\gamma_j}$ in $\bar{F}$ for all $j < j_0$ and $\overline{\gamma_{j_0}}$ inside $\bar{t}\bar{F}$.

If for all $i$ the answer is negative, then no automorphism of $G$ satisfies this property.

\[\square\]

3 Conjugacy and suspensions

The following observations elaborate on some well known point of view, and, as stated in the introduction, is our angle of attack of the conjugacy problem.

Lemma 3.1. Let $\phi_1$ and $\phi_2$ be two automorphisms of $F$. The following assertions are equivalent.

1. $\phi_1$ and $\phi_2$ are conjugate in $\text{Out}(F)$;

2. there is an isomorphism between their suspensions that preserves the fiber (in both directions) and the orientation;

3. there is an isomorphism between their suspensions that preserves the orientation and sends the fiber inside the fiber;

4. there is an isomorphism between their suspensions whose factorisation through the abelianisations preserves the orientation, and sends the image of the fiber inside the image of the fiber.

Proof. Of course, 2 implies 3 which implies 1. Since the derived subgroups of the suspensions are contained in the fibers, one obtains easily that 1 implies 3. Let us prove that 3 implies 2. Let $\alpha$ the isomorphism given by 3. Then $\alpha(F)$ is normal in $(F \rtimes_{\phi_2} \langle t \rangle)$ and the quotient is infinite cyclic. Thus the image of the fiber $F$ is trivial in this quotient (because the further quotient by this image is also infinite cyclic). It follows that $\alpha(F) = F$.

Let us prove that 1 is equivalent to 2. Assume that $\Psi : F \rtimes_{\phi_1} \langle t \rangle \to F \rtimes_{\phi_2} \langle t' \rangle$ sends $F$ to $F$, and $t$ to $f_0t'$. Then write $\psi$ for the restriction of $\Psi$ to $F$.

In $F \rtimes_{\phi_1} \langle t \rangle$, for all $f \in F$, one has $t^{-1}ft = \phi_1(f)$. Passing through $\Psi$, one gets (in $F \rtimes_{\phi_2} \langle t' \rangle$) $t'^{-1}f_0^{-1}\psi(f)f_0t' = \psi(\phi_1(f))$, that is $\phi_2 \circ \text{ad}_{f_0} \circ \psi = \psi \circ \phi_1$.

Thus, the classes of $\phi_1$ and $\phi_2$ are conjugate in $\text{Out}(F)$ and furthermore, if $f_0 = 1$, $\phi_1$ and $\phi_2$ are conjugate in $\text{Aut}(F)$.
Conversely, if $\phi_1 = \psi^{-1} \circ \text{ad} f_0 \circ \phi_2 \circ \psi$ for some $\psi$ in $\text{Aut}(F)$, one can extend $\psi$ to $\tilde{\Psi} : F \ast \langle t \rangle \to F \rtimes_{\phi_2} \langle t \rangle$ by setting $\tilde{\Psi}(t) = f_0 t \in F \rtimes_{\phi_2} \langle t \rangle$. The relation of the semi-direct product by $\phi_1$ vanishes in the image, thus $\tilde{\Psi}$ factorises through $F \rtimes_{\phi_1} \langle t \rangle$ producing a bijective morphism.

**Theorem 3.2.** There is an algorithm that, given $F$ a finitely presented group, and two automorphisms $\phi, \phi'$ of $F$ such that the suspensions are word-hyperbolic, decides whether $\phi$ and $\phi'$ are conjugated in $\text{Out}(F)$.

**Proof.** By Lemma 3.1, it suffices to decide whether the associated semi-direct products of $F$ with $\mathbb{Z}$ with structural automorphisms $\phi$ and $\phi'$ are isomorphic by an isomorphism satisfying characterisation 4 in Lemma 3.1.

Let $F'$ be another copy of $F$, with same presentation. We read $\phi'$ as an automorphism of $F'$. Let us denote by $G$ and $G'$ the groups of the suspensions of $F$ and $F'$ by the given $\phi$ and $\phi'$ respectively. Provided with a presentation of $F$, we have presentations of $G$ and $G'$.

By the main result of [DGn2], we can decide whether there is an isomorphism between $G$ and $G'$. If there is none, we are done. If there is one, say $\psi : G \to G'$, any other isomorphism is in the orbit of $\psi$ by $\text{Aut}(G')$. Let $\{f_i, i = 1, \ldots, j_0 - 1\}$ be a generating set of $F$. We apply our solution to the orbit problem 2.13 to the elements $\gamma_i = \psi(f_i)$ for $i < j_0$, and $\gamma_{j_0} = \psi(t)$. By definition, the answer to this orbit problem is positive if and only if there is an automorphism $\eta$ such that $\eta \circ \psi$ satisfies characterisation 4 in Lemma 3.1. Since all isomorphisms $G \to G'$ are of this form (i.e. $\eta \circ \psi$ for some automorphism $\eta$), this decides whether there is an isomorphism satisfying the assertion 4 of Lemma 3.1 hence, whether $\phi$ and $\phi'$ are conjugated in $\text{Out}(F)$.

**Corollary 3.3.** Let $F$ be a free group. The conjugacy problem in $\text{Out}(F)$ restricted to atoroidal automorphisms is solvable.

**4 Beyond hyperbolicity (to relative hyperbolicity)**

In view of [DG1], [DGn3] (and [DT]), it is natural to ask whether one can approach the conjugacy problem of larger classes of automorphism, namely
producing proper relatively hyperbolic suspensions.

**Definition 4.1.** Let $\phi \in \text{Aut}(F)$, and $F_1, \ldots, F_k$ finitely generated proper subgroups of $F$. We say that the automorphism $\phi$ is hyperbolic relative to $\{F_1, \ldots, F_k\}$ if there exists integers $m_1, \ldots, m_k > 0$ and elements $f_1, \ldots, f_k \in F$ such that, for all $i$, $t^{m_i} f_i$ normalises $F_i$, and such that the group $(F \rtimes \langle t \rangle)$ is hyperbolic relative to $\{(F_i \rtimes \langle t^{m_i} f_i \rangle), i = 1, \ldots, k\}$.

The case of automorphism of free groups in this case is particularly interesting, since according to [GLu], all non-polynomial automorphisms of free groups produce interesting relatively hyperbolic suspensions.

In this section, we explain what can be done, and we put in evidence some inherent difficulties.

Unfortunately, the arguments presented below rely on a certain number of currently unpublished results, so I would like to make this reliance clear. First, there is the main result of Gautero and Lustig paper [GLu]. This is used twice; to produce examples to which the results might apply (so, in some sense, as a motivation), and to compute explicitly the polynomial subgroups (actually, this is to certify that a polynomially growing automorphism is polynomially growing). Then there is the splitting computation of Touikan [T]. And finally, there is the solution of the isomorphism problem of some rigid relatively hyperbolic groups, by Guirardel and myself, [DGu4].

From now on $F$ is a free group (even though it could be interesting to consider also a free product of nice groups).

One says that a subgroup $F_0$ of $F$ is polynomial for a given automorphism $\phi$ if all conjugacy class of elements in $F_0$ has polynomial growth under iterates of $\phi$ (more explicitly, that means that for all $\gamma \in F_0$, the length of a cyclically reduced representative of $\phi^n(\gamma)$ is bounded above by a polynomial). We say that an automorphism is polynomial if $F$ itself is polynomial.

For any automorphism of a free group, there is a finite family of finitely generated polynomial subgroups of the free group such that all polynomial subgroups are conjugated into one of them ([GLu Prop. 3.2]).

The following is an important preliminary (it is possible that it has been well known to specialists, but I did not find any reference.)

**Proposition 4.2.** There is an algorithm that, provided with a free group $F$ and an automorphism $\phi$, terminates and indicates whether $F$ is polynomial for $\phi$.

**Proof.** First, we will give a procedure certifying that an automorphism is polynomial, and then a procedure certifying that an automorphism is of exponential growth.
By [BFH1] Coro 5.7.6, if \( \phi \) is a polynomially growing automorphism, then there is \( n \) such that \( \phi^n \) is unipotent in \( GL(H_1(F)) \). It is then sufficient to devise a procedure certifying whether a unipotent automorphism is polynomially growing. Then we use (a weak aspect of) Theorem [BFH2, 3.11]: if the automorphism is polynomially growing, there exists a topological representative \( \tau : G \to G \) of \( \phi \) on a graph \( G \), and a filtration of \( G \), \( \{v\} = G_0 \subset \cdots \subset G_n = G \) such that any edge \( e \) in \( G_i \setminus G_{i-1} \) is sent on a path \( ec \) where \( c \) is a path in \( G_{i-1} \). Also, if such a representative exists, then, for every edge \( e \in G_i \), the length of \( \phi^n(e) \) can be bounded by a polynomial in \( n \) depending only on \( i \) (this can be seen by induction; it is obvious for \( i = 0 \) or \( 1 \) since \( G_0 \) contains no edge, and if it is true for \( i - 1 \), let \( P_{(i-1)}(k) \) the corresponding polynomial, and for \( e \in G_i \), with \( \phi(e) = ec \), we can write \( \phi^n(e) = e\phi(c)\phi(e)^2 \cdots \phi(e)^{n-1} \), and the total length is bounded by \( \sum_{k \leq n-1} P_{(i-1)}(k)^{|c|} \), hence by \( \sum_{k \leq n-1} P_{(i-1)}(k)^M \) for \( M = \max\{|\phi(e)|, e \in G_i\} \), which is polynomial in \( n \). Thus, \( \phi \) is polynomial if and only if there is such a topological representative. This can be certified by enumeration of topological representatives, since the condition used is easily algorithmically checked.

We now need a procedure that produces a certificate that \( \phi \) is not polynomially growing when it is the case.

For that, we’ll use that \( \phi \) is not polynomially growing if and only if the suspension has a proper relative hyperbolic structure. One direction of this equivalence (\( \implies \)) is the content of [GLu] (actually [GLu] describes the relative hyperbolic structure). We present now an argument for the other direction. If the suspension is a proper relatively hyperbolic group (with at least one hyperbolic element), then the fiber cannot be a parabolic group (maximal parabolic groups are their own normalisers). Therefore, there is necessarily an hyperbolic element \( f_h \) in the fiber \( F \). It follows that there exists \( f_0 \in F \) such that \( t' = tf_0 \) is hyperbolic (if \( t \) happened to be parabolic, \( tf_h^n \) is eventually hyperbolic). The relative distance of \( F \) and \( t_kF \) grows therefore linearly, and by exponential divergence, the shortest path in \( t_kF \) from \( t_k \) to \( f_ht_k \) is exponential in \( k \). This makes \( f_h \) an exponentially growing element for the automorphism \( \phi \circ \text{ad}_{f_0} \). This means that this automorphism cannot be polynomially growing (as it is visible on the topological representative \( \tau \) of a polynomially growing automorphism that no element can be exponentially growing). Therefore \( \phi \) has an exponentially growing conjugacy class in \( F \).

By [DGu3], and enumeration of the proper subgroups of \( F \), if there exists a proper relative hyperbolic structure, one can eventually find it and thus a certificate that the automorphism is not polynomially growing.
Proposition 4.3. Let $F$ be a free group. There is an explicit algorithm that, given $\phi \in \text{Aut}(F)$ expressed on a basis of $F$, terminates and produces the basis of a collection $F_1, \ldots, F_k$ of maximal polynomial subgroups of $F$ for $\phi$, and computes minimal exponents $m_i > 0$ and elements $f_i$ so that $t^{m_i}f_i$ normalises $F_i$ (see definition 4.1).

Proof. Note that, there is a unique relative hyperbolic structure for $F \rtimes_\phi \langle t \rangle$ whose parabolic groups are suspensions of polynomial subgroups. Indeed, the polynomial subgroups must be parabolic, by the observation made in the proof of 4.2. Moreover, by [GLu], $F \rtimes_\phi \langle t \rangle$ is indeed relatively hyperbolic to the subgroups that we need to compute.

By enumeration, one can find subgroups of $F$ that are normalised by some element $t^{m}f$, and using Proposition 4.2, we may certify whether this is a suspension of a polynomial automorphism. For any collection of such subgroups, we may use [DGu3] in order to certify that $F \rtimes_\phi \langle t \rangle$ is relatively hyperbolic. When this happens, the algorithm is done.

Let us say that $\phi \in \text{Aut}(F)$ is relatively hyperbolic with no parabolic splitting (RH-noPS for short) if it is properly hyperbolic relative to a collection of polynomial subgroups of $F$, and the suspension $F \rtimes_\phi \langle t \rangle$ has no reduced peripheral splitting over a subgroup of a parabolic subgroup, except the trivial one (a splitting is peripheral if, in the tree, all parabolic subgroups are elliptic).

Let us say that $\phi \in \text{Aut}(F)$ is relatively hyperbolic with no elementary splitting (RH-noES for short) if it is properly hyperbolic relative to a collection of polynomial subgroups of $F$, and the suspension $F \rtimes_\phi \langle t \rangle$ has no reduced peripheral splitting over a cyclic or parabolic subgroup, except the trivial one.

An unsatisfying aspect of this work is that I am unable to provide an algorithm certifying whether an element of $\text{Aut}(F)$ is RH-noPS. But if it is, then we can do something.

4.1 Conjugacy of two relatively hyperbolic automorphisms without elementary splitting

Proposition 4.4. Let $F$ be a free group. There is an (explicit) algorithm that, given two automorphisms, $\phi_1, \phi_2$, terminates if both $\phi_i$ are RH-noPS, and provides
• either an isomorphism \( F \rtimes_{\phi_1} \langle t \rangle \to F \rtimes_{\phi_2} \langle t \rangle \) preserving fiber, and orientation;
• or a certificate that \( F \rtimes_{\phi_1} \langle t \rangle \) and \( F \rtimes_{\phi_2} \langle t \rangle \) are not isomorphic by an isomorphism preserving fiber, orientation;
• or a reduced peripheral splitting of either \( F \rtimes_{\phi_i} \langle t \rangle \) over a cyclic subgroup.

The algorithm in question may terminate even if one of the \( \phi_i \) is not RH-noPS. It never lies though.

The following application is immediate, given Lemma 3.1.

**Corollary 4.5.** The conjugacy problem for RH-noES elements of \( \text{Out}(F) \) is solvable: there is an algorithm that given two automorphisms that are RH-noES, decides whether or not they are conjugated.

Let us now prove Proposition 4.4

**Proof.** First, by Proposition 4.3 and [GLu], we may assume that we know explicitly both relative hyperbolic structures of \( G_i = F \rtimes_{\phi_i} \langle t \rangle \) with presentations of the parabolic subgroups. We can, by enumeration, find presentations of these subgroups as suspensions of subgroups of \( F \).

In parallel, we then perform the three following searches (so-called procedures, below).

The first procedure is the enumeration of morphisms \( G_1 \to G_2 \to G_1 \). It stops when mutually inverse isomorphisms preserving orientation and sending the fiber into the fiber are found.

The second procedure is as follows. Let us write \( P_{1,j} = F_{1,j} \rtimes \langle r_j \rangle, j = 1, \ldots, k \) the maximal parabolic subgroups of \( G_1 \), with \( P_{1,j} \cap F = F_{1,j} \) (recall that we have explicit presentations of the groups \( P_{1,j} \) as such suspensions). For incrementing integers \( m \), we compute \( N_{1,m,j} \) the intersection of all subgroups of \( F_{1,j} \) of index \( \leq m \). Then we try to certify, using [P], that \( \bar{G}_1^{(m)} = G_1/\langle\langle \cup_j N_{1,m,j} \rangle \rangle \) is hyperbolic.

We will denote by \( K_{m,1} \) the kernel \( \langle\langle \cup_j N_{1,m,j} \rangle \rangle \) in \( G_1 \).

Similarly, we compute \( \bar{G}_2^{(m)} \) and check that it is hyperbolic. Since \( P_{1,j}/N_{1,m,j} \) is virtually cyclic, by virtue of the Dehn Filling theorem [O, Thm. 1.1], for \( m \) large enough these groups are indeed hyperbolic. So this step of the second procedure will eventually provide groups \( \bar{G}_i^{(m)}, (i = 1, 2) \), that are certified hyperbolic. Since, for all \( m \), \( \cup_j N_{1,m,j} \) is contained in \( F \) which is normal in \( G_1 \). Hence, the whole group \( K_{m,1} \) is contained in \( F \), and
$\bar{G}_1^{(m)}$ is naturally a suspension $\bar{G}_1^{(m)} = (F/K_{m,1}) \rtimes \langle \bar{i} \rangle$. The second procedure then calls the algorithm of Theorem 3.2 in order to decide whether $\bar{G}_1^{(m)}$ and $\bar{G}_2^{(m)}$ are isomorphic by a fiber and orientation preserving isomorphism. This is done in parallel for all incrementing $m$ for which the groups are certified hyperbolic. The second procedure stops if an integer $m$ is found so that $\bar{G}_1^{(m)}$ and $\bar{G}_2^{(m)}$ are not isomorphic (by a fiber and orientation preserving isomorphism).

The third procedure is as follows. For both $i = 1, 2$, one enumerates presentations of $G_i = F \rtimes_{\phi_i} \langle t \rangle$ by Tietze transformations, and for each one exhibiting a splitting of $G_i$ over a cyclic subgroup, as an amalgamation or an HNN extension, we check whether the splitting is non-trivial (it suffices to check that, in the case of an amalgamation, both factors have a generator that does not commute with the cyclic subgroup, and in the case of an HNN extension, that the stable letter is non-trivial). If we discover a non-trivial cyclic splitting in which parabolic groups are conjugated in vertex groups, this third procedure stops, and outputs the splitting.

Now that we described the three procedures, we discuss the implication of their termination.

If the first procedure terminates, then by Lemma 3.1, there exists a fiber-and-orientation preserving isomorphism, and the two given automorphisms of $F$ are conjugated in $\text{Out}(F)$. If the second procedure terminates, there cannot exist any isomorphism $G_1 \to G_2$ preserving fiber and orientation (it would preserve the class of parabolic subgroups, characterised by being polynomial, and hence pass to the characteristic quotients). If the third procedure terminates, we have found a reduced splitting of either $F \rtimes_{\phi_i} \langle t \rangle$ over a cyclic subgroup.

Now we need to show that there is always at least one procedure that terminates, i.e. the following lemma.

**Lemma 4.6.** Assume that $G_1$ and $G_2$ are RH-noPS. If the third procedure and the second procedure never terminate, then the first procedure terminates.

This Lemma is actually a consequence of a result obtained by the author in collaboration with Vincent Guirardel.

**Proof.** We assume that for all $m$ large enough, $\phi_m : \bar{G}_1^{(m)} \to \bar{G}_2^{(m)}$ is a fiber and orientation preserving isomorphism. Eventually each $\phi_m$ must send $\bar{P}_j$ on some conjugate of $\bar{Q}_\ell$ since $\bar{P}_j$ contains a large finite normal subgroup, and the only finite subgroups of $\bar{G}_2^{(m)}$ are conjugated to the $\bar{Q}_\ell$. Then, Theorem 20
[DGK14] 5.1] states that either $G_1$ or $G_2$ has a peripheral splitting over an elementary subgroup (which must be parabolic if we assume that the third procedure does not terminate, hence the contradiction with the assumption of the lemma), or there is an isomorphism $\phi : G_1 \to G_2$ that commutes with infinitely many $\phi_m$, up to composition with a conjugation in $G_2^{(m)}$. In this second circumstance, $\phi$ has to preserve the fiber and orientation, since the $\phi_m$ do, and the kernels are co-final.

There is a slightly stronger version of Proposition 4.4 that we will need for the next part.

Given a suspension $F \rtimes \langle t \rangle$, a transverse cyclic peripheral structure is a tuple of elements of the form $(t^{k_j}f_j)_{i=1,...,r}$, for $k_j \neq 0$ and $f_j \in F$. The norm of an element being the minimal word length of a conjugate of this element, we define the norm of a cyclic peripheral structure to be the maximum of the norms of its elements.

A fiber-and-orientation preserving isomorphism between suspensions equipped with such structures is said to preserve the structure if it sends the conjugacy classes of the first exactly on the conjugacy classes of the second.

The following Proposition is, as we said, similar to Proposition 4.4. The difference is in the presence of the transverse cyclic peripheral structure (a minor difference) but also in the fact that we had ambitionned to get the full list of fiber-and-orientation preserving isomorphisms. This ambition is not realised unfortunately, but enough is granted for the application in the next part.

**Proposition 4.7.** Let $F$ be a free group. There is an (explicit) algorithm that, given two automorphisms, $\phi_1, \phi_2$, and two transverse cyclic peripheral structures $\mathcal{P}_1, \mathcal{P}_2$ of $F \rtimes \phi_1 \langle t \rangle$ and $F \rtimes \phi_2 \langle t \rangle$ respectively, terminates if both $\phi_i$ are RH-noPS, and provides

1. either a list of isomorphisms $F \rtimes \phi_1 \langle t \rangle \to F \rtimes \phi_2 \langle t \rangle$ preserving fiber, orientation, and transverse cyclic peripheral structure, and an integer $m$ such that for each $p \in \mathcal{P}_2$, the centralizer of $p$ in $G_2^{(m)} = G_2/K_m$ is the image of the centralizer of $p$ in $G_2$ and such that, for any other such isomorphism $\psi$, there is one, $\phi$, in the list, an element $g \in G_2$, such that for all $h \in G_1$, there is $z_h \in K_m$ for which $\psi(h^g) = \phi(h)z_h$.

2. or a certificate that $F \rtimes \phi_1 \langle t \rangle$ and $F \rtimes \phi_2 \langle t \rangle$ are not isomorphic by an isomorphism preserving fiber, orientation, and transverse cyclic peripheral structure;
3. or a reduced peripheral splitting of either $F \rtimes_{\phi_1} \langle t \rangle$ over a cyclic subgroup.

The first point means that the list contains all isomorphisms up to conjugacy in $G_2$ and multiplication by a large element of $F$.

**Proof.** As in the proof of Proposition 4.4, we use three procedures.

The first procedure is the enumeration of morphisms $G_1 \to G_2 \to G_1$. This procedures has an incrementing list $\mathcal{L}$, which is empty at the beginning. Every time mutually inverse isomorphisms preserving orientation and sending the fiber into the fiber are found, such that the isomorphism $G_1 \to G_2$ is not conjugated to any item of the list $\mathcal{L}$, the procedure stores $G_1 \to G_2$ into $\mathcal{L}$. We precise below when this first procedure is set to stop.

The second one slightly differs from Proposition 4.4. We still compute $\tilde{G}_1^{(m)}$ and $\tilde{G}_2^{(m)}$ (and the images of the transversal peripheral structure in them), and try to certify that they are hyperbolic and that the assumption of point 1, on the centralizers, is satisfied (which happens if $m$ is large enough, by Lemma 4.12). Let us call this “certification $\alpha$ for $m$”. When this is the case, using [DGu2, Coro. 3.4], we check whether these groups are rigid (in the sense that they have no compatible splitting over a virtually cyclic group with infinite center) and if they are we proceed and compute by [DGu2, Coro. 3.5] the complete list of isomorphisms $\tilde{G}_1^{(m)} \to \tilde{G}_2^{(m)}$ up to conjugacy in $G_2^{(m)}$. Once this is done, we check which one of them are fiber-and-orientation preserving, and we record them in a list $\mathcal{L}_m$.

The second procedure is set to stop if either an $m$ is found so that there is no fiber-and-orientation preserving isomorphisms $\tilde{G}_1^{(m)} \to \tilde{G}_2^{(m)}$ that preserves the transversal peripheral structure.

The first procedure (which was run in parallel with the second) is set to stop if a list $\mathcal{L}$ is found and an integer $m$ is found so that the following three conditions are satisfied. First, “certification $\alpha$” for $m$ is done. Second, $\mathcal{L}_m$ is completely computed, and third, the currently computed list $\mathcal{L}$ surjects on $\mathcal{L}_m$, by the natural quotient map.

The third procedure looks for a reduced peripheral splitting over a cyclic subgroup (it is the same as in Proposition 4.4).

Observe that, if the first procedure stops, we have in $\mathcal{L}$ a list of isomorphisms $F \rtimes_{\phi_1} \langle t \rangle \to F \rtimes_{\phi_2} \langle t \rangle$ preserving fiber (by Lemma 3.1, Corollary 3.3), orientation, and transverse cyclic peripheral structure, such that any other such isomorphism differs from one in the list by a conjugation, and the multiplication by elements in the fiber (in the sense of 4.7-(1)). Observe also that if the second procedure stops, we have (as in 4.4) a certificate that
$F \rtimes_{\phi_1} \langle t \rangle$ and $F \rtimes_{\phi_2} \langle t \rangle$ are not isomorphic by an isomorphism preserving fiber, orientation, and transverse cyclic peripheral structure.

Again, we can conclude by the following lemma.

**Lemma 4.8.** Assume that $G_1$ and $G_2$ are RH-noPS, with transverse cyclic peripheral structures. If the third procedure and the second procedure never terminate, then the first procedure terminates (i.e. there is $m$ as in (4.7) such that any isomorphism preserving fiber, orientation, and peripheral structure $G_1^{(m)} \to G_2^{(m)}$ is the image of a such isomorphism $G_1 \to G_2$).

**Proof.** Assume that for all $m$, there is an isomorphism preserving fiber, orientation, and peripheral structure $\psi_m : G_1^{(m)} \to G_2^{(m)}$ that is not the image of any isomorphism $G_1 \to G_2$. Since the groups $G_1$ and $G_2$ are assumed not to split over an elementary subgroup, we may use Theorem [DGu4, 5.1] to extract a converging subsequence of the isomorphisms $\psi_m$, contradicting the absence of $\psi_\infty : G_1 \to G_2$ commuting with some $\psi_m$ (after conjugation).

\[\square\]

### 4.2 Conjugacy of two relatively hyperbolic automorphisms without parabolic splitting

The class of RH-noPS automorphisms, is larger than that of RH-noES. We can treat it as well, but it requires a little care.

**Proposition 4.9.** Let $X_1, X_2$ be graph-of-groups decompositions of $F \rtimes_{\phi_1} \langle t \rangle$ and $F \rtimes_{\phi_2} \langle t \rangle$ respectively, over cyclic subgroups.

Assume that $\Phi_X : X_1 \to X_2$ is an isomorphism of the underlying graphs of the decompositions, and that $m$ is an integer, and, for all vertex $v \in X_1^{(0)}$, $L_v$ is a list of isomorphisms, that are all as in (4.7).

The following are equivalent.

1. There is an isomorphism $\Phi : \pi_1(X_1, \tau) \to \pi_1(X_2, \Phi_X(\tau))$ that is fiber and orientation preserving, that induces a graph-of-groups isomorphism, and that induces $\Phi_X$ at the level of graphs.

2. There is $\Phi_0 = (\Phi_X, (\phi_v), (\phi_e), (\gamma_e))$ an isomorphism of graphs of groups, such that $\phi_v$ is fiber-and-orientation preserving, and such that the linear diophantine equation (2) has a solution.
3. There is $\Phi'_0 = (\Phi_X, (\phi'_v), (\phi_e), (\gamma'_e))$ an isomorphism of graphs of groups, such that $\phi'_v \in \mathcal{L}_v$, and such that the linear diophantine equation (2) has a solution.

Proof. The first point implies the second: by Lemma 2.4, all vertex groups are suspensions of their intersections with $F$, therefore if $\Phi$ preserves the fiber and orientation, so do all $\phi_v$, and the equation (2) admits an obvious solution (the null solution).

The second point implies the first, because, by Proposition 2.2, the system (2) has a solution if and only if there is a graph-of-group automorphism of $\pi_1(X_2, \Phi_X(\tau))$ that sends $\Phi_0(F)$ exactly on $F$, and preserves orientation.

The third point obviously implies the second one.

We need to show that the second point implies the third one. So, $\Phi'_0 = (\Phi_X, (\phi'_v), (\phi_e), (\gamma'_e))$ is an isomorphism of graphs of groups, as in the second point.

By assumption on $\mathcal{L}_v$, the given isomorphisms $\phi_v$ differ from isomorphisms in $\mathcal{L}_v$ by conjugation (in the target), and multiplication by elements of $F$. By composing with inert twists (vanishing in $\text{Out}(F \rtimes \phi_2(\ell))$), we can assume that each $\phi_v$ has same image as an element of $\mathcal{L}_v$ in the Dehn Filling reduction $\Gamma^{(m)}_v \to \Gamma^{(m)}_{\phi_X(v)}$.

**Lemma 4.10.** There is an isomorphism of graphs of groups $\Phi_0 = (\Phi_X, (\phi'_v), (\phi_e), (\gamma'_e))$ such that $\phi'_v \in \mathcal{L}_v$ has same image as $\phi_v$ in the Dehn Filling reduction $\Gamma^{(m)}_v \to \Gamma^{(m)}_{\phi_X(v)}$, and such that for all edge $e$, $(\gamma'_e)^{-1}\gamma_e \in F$, and also $\gamma_e(\gamma'_e)^{-1} \in F$, and $(\gamma'_e)\gamma_e^{-1} \in F$.

Proof. We thus construct $\Phi'_0$ using these elements $\phi'_v \in \mathcal{L}_v$. The morphisms $\phi_e$ are given by the marking of the cyclic edge groups. We need to check that there exists elements $\gamma'_e$ completing the collection into an isomorphism of graph-of-groups, but this is actually the condition that, for $v = o(e)$, $\phi'_v$ preserve the peripheral structure of the adjacent cyclic edge groups. Note that one can choose the $\gamma'_e$ up to a multiplication on the left by an element of $\Gamma_{o(e)}$ centralising $i_e(\Gamma_e)$.

Once such elements $\gamma'_e$ are chosen, $\Phi'_0$ is defined. Recall that on a Bass generator $e \in X^{(1)} \setminus \tau$, $\Phi'_0(e) = (\gamma'_e)^{-1}\Phi_X(e)(\gamma'_e)$. We need to compute how it differs from $\Phi_0$ on Bass generators.

Let us call $c_e$ the marked generator of the edge group $\Gamma_e$. To make notations readable, we will still write $c_e$ for $i_e(c_e)$. 

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One has $c_{\Phi_X}(e) = \phi_v(c_e)^{\gamma_e}$ by Bass diagram $[1]$. By virtue of $\phi'_v$ preserving the peripheral structure (for $v = o(e)$) this is also $(\phi'_v(c_e))^{h_e} \gamma_e$, for some $h_e \in \Gamma_{\phi_X(v)}$ which can be chosen up to left multiplication by an element centralising $\phi'_v(c_e)$. By Bass diagram (for $\phi'_v$) this is $((c_{\Phi_X(e)}(\gamma'_e)^{-1})h_e) \gamma_e$. It follows that $(\gamma'_e)^{-1}h_e \gamma_e$ centralises $c_{\Phi_X(e)}$, and lies in $\Gamma_{\phi_X(v)}$, for $v = o(e)$.

Recall that $\gamma'_e$ can be chosen up to a multiplication on the left by an element of $\Gamma_{o(e)}$ centralising $i_e(\Gamma_e)$. By a right choice of the collection of $\gamma'_e$ (or, in different words, by the right application of Dehn twists), we may assume that $(\gamma'_e)^{-1}h_e \gamma_e = 1$.

By virtue of $\phi'_v$ coinciding with $\phi_v$ in the Dehn filling $\overline{\gamma}_v^m$, the image $\overline{h}_e$ of $h_e$ actually centralises $\phi'_v(e)$. By assumption on $m$ (see [4.7] - [1]), the centralizers of the transverse peripheral structure in $\overline{G}_2$ are the images of the centralizers in $G_2$, so this makes $h_e = z_e f_e$ for $z_e$ centralising $\phi'_v(c_e)$ and $f_e$ in $F$. Thus, we may choose it so that $z_e = 1$, hence $h_e \in F$. Finally, since $F$ is normal, and $(\gamma'_e)^{-1}h_e \gamma_e = 1$, it follows that $(\gamma'_e)^{-1} \gamma_e = 1$. The two other relations are obtained respectively by conjugating by $\gamma_e^{-1}$ ($F$ is normal) and taking the inverse of the later.

Finally, we can compare the images of the Bass generator $e$, namely $\Phi_0(e)$ to $\Phi'_0(e)$. Recall that $\Phi_0(e) = \gamma_e^{-1} \phi_X(e) \gamma_e$ and $\Phi'_0(e) = (\gamma'_e)^{-1} \phi_X(e) \gamma'_e$. The difference is therefore $\Phi_0(e)^{-1} \Phi_0(e) = (\gamma'_e)^{-1} \phi_X(e)^{-1} (\gamma'_e \gamma_e)^{-1} \phi_X(e) \gamma_e$ which can also be written $\Phi_0(e)^{-1} \Phi_0(e) = (\gamma'_e)^{-1} (\gamma'_e \gamma_e)^{-1} \phi_X(e) \gamma_e$ and slightly less naturally, $\Phi'_0(e)^{-1} \Phi_0(e) = (\gamma_e \gamma'_e)^{-1} (\gamma'_e \gamma_e)^{-1} \phi_X(e) \gamma_e$.

Since we established that $\gamma_e \gamma'_e = 1$ in $F$, $\gamma'_e \gamma_e = 1$ in $F$ and $F$ is normal, $\Phi'_0(e)^{-1} \Phi_0(e) \in F$.

We may now finish the argument and show that the system of equations (2) for $\Phi'_0$ has a solution.

Consider an element $f$ of the given basis of $F$. Write the normal form in $\pi_1(X_1, \tau)$ as $f = g_0 e_1, g_1, \ldots, e_n g_{n+1}$. The normal form of its image by $\Phi_0$ in $\pi_1(X_2, \Phi_X(\tau))$ is thus

$$\Phi_0(f) = \phi_{v_0}(g_0) \phi_X(e_0) \ldots \phi_X(e_n) \phi_{v_{n+1}}(g_{n+1}).$$

Since the $\phi_v$ differ from This equals

$$\Phi_0(f) = \phi'_v(g_0) f_0 \phi_X(e_0) f_{e_0, r} \ldots \phi_{v_n}(g_n) f_n \phi_X(e_n) f_{e_n, r} \phi_{v_{n+1}}(g_{n+1}) f_{n+1}$$
or written differently

$$\Phi_0(f) = \left( \prod_{i=0}^n \phi'_v(g_i) f_i \times (f_{e_i, r} \phi_X(e_i) f_{e_i, r}) \right) \phi_{v_{n+1}}(g_{n+1}) f_{n+1}.$$
(the \( f_i \) depend on \( v_i, g_i \)).

In \( H_1(F \times_{\phi_2} \langle t \rangle) \) this normal form turns into

\[
\Phi_0(f) = \left( \prod_{i=0}^{n+1} \phi'_{v_i}(g_i) \right) \times \left( \prod_{i=0}^{n} \phi_X(e_i) \right) \times (f_{\text{tot}}),
\]

where \( f_{\text{tot}} \) is in the image of \( F \) in the \( H_1 \). Whether a composition of Dehn twists (that are transvections in \( H_1 \)) pushes this element into \( F \) does not depend on \( f_{\text{tot}} \). Therefore the equation (2) has a solution for \( \Phi'_0 \) if it has a solution for \( \Phi_0 \).

\[
\square
\]

**Proposition 4.11.** Let \( F \) be a free group. There is an (explicit) algorithm that given two automorphisms \( \phi_1, \phi_2 \) of \( F \) terminates if both are RH-noPS, and indicates whether they are conjugated in \( \text{Out}(F) \).

**Proof.** The computation of the relative hyperbolicity structure was done in 4.3 and the \( Z_{\text{max}} \)-JSJ splitting of both suspensions can be computed using 4.4 (actually one can use Theorem C in that paper, with the fact that the absence of peripheral splitting is assumed, in order to certify whether a proposed splitting is maximal, and once such a splitting is found, by Proposition 2.9 it is necessarily a \( Z_{\text{max}} \)-JSJ splitting). The situation reduces to the case where an isomorphism of underlying graph of the \( Z_{\text{max}} \)-JSJ decomposition is chosen, and we need to decide whether there is an isomorphism of graph of groups (inducing that isomorphism of underlying graphs) that preserve the fiber, and the orientation. We also choose a maximal subtree \( \tau \) of the underlying graph \( X \) so that all edges outside this subtree correspond to Bass generators in the fundamental group of the graph of group.

Let us write \( X_1, X_2 \) the \( Z_{\text{max}} \)-JSJ decompositions of \( G_1 = F \times_{\phi_1} \langle t \rangle \) and \( G_2 = F \times_{\phi_2} \langle t \rangle \), and we identify the underlying graphs, according to the choice of isomorphism above (the algorithm has to treat all possible such isomorphisms of graphs in parallel). We write \( \Gamma_{v,i} \) for the vertex group of \( v \) in \( X_i \).

We remark that any elementary vertex group (or edge group) is cyclic, and transversal to the fiber (Lemma 2.5), hence the orientation of the suspension provides a canonical marking of each edge group. For each non-elementary vertex group, equipped with the thus marked cyclic transverse peripheral structure of its adjacent edge groups, it is possible, thanks to Proposition 4.7 to decide whether there is an isomorphism \( \Gamma_{v,1} \rightarrow \Gamma_{v,2} \), preserving fiber, orientation, and the (cyclic, transverse) peripheral structure of its adjacent edge groups, thus marked.

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If for some vertex there is no such isomorphism, then there cannot be any fiber-and-orientation preserving isomorphism between $G_1$ and $G_2$, inducing this graph isomorphism (hence $\phi_1$ and $\phi_2$ are not conjugated, in view of Lemma 3.1). If, on the contrary, for all such vertices, there exists such an isomorphism, then Proposition 4.7 actually provides a list $L_v$ of such isomorphisms, that satisfies (4.7-(1)), for all vertex $v \in X(0)$. Then, for any choice $(\Phi_v, v \in X(0))$ in $\prod_v L_v$, one can extend this collection into an isomorphism of graph-of-groups $\Phi$, by choosing appropriately the images of the Bass generators to be $b_e, e \in X \setminus \tau$ (so that they conjugate the edge group in their origin vertex to the edge group in their target vertex). Let us write $(\Phi_s, s \in \prod_v L_v)$ the collection thus obtained.

Using 2.2 we can decide whether, given $\Phi_s$ for some $s \in \prod_v L_v$, there is an automorphism of graph-of-groups, in the orbit of $\Phi_s$ under the small modular group of $G_2$, that is fiber-and-orientation preserving. If there is one, then we may stop and declare, in view of Lemma 3.1 that $\phi_1$ and $\phi_2$ are conjugated.

Assume then that there is none in the orbits of all $\Phi_s, s \in \prod_v L_v$. By Proposition 4.9 (2 $\Rightarrow$ 3'), there is no isomorphism of graph-of-groups preserving fiber and orientation. We are done.

4.3 A lemma on Dehn fillings

In this paragraph we prove a Lemma that we used above. We refer the reader to the setting of [DGO, §7.]

Lemma 4.12. Let $(G, P)$ be a relatively hyperbolic group, and $c \in G$, a hyperbolic element.

There exists $m_0$ such that for all $m > m_0$, if $h$ is such that $\bar{h}$ centralises $\bar{c}$ in $\bar{G}^{(m)}$, then, there is $z \in K_m$, and $h'$ centralizing $c$ such that $h = h'z$.

The bound on $m$ will be explicit (but we do not need this particular aspect), though probably not optimal.

Proof. Consider a hyperbolic space $X_0$ associated to $(G, P)$, upon which $G$ acts as a geometrically finite group, and for convenience, let us choose it to be a cusped-space as defined by Groves and Manning (see [GM, §3]). By assumption $c$ is hyperbolic in $X$, let $\rho$ be a quasi-axis, and $\|c\|$ the translation length of $c$ on this axis. We choose $m_0$ such that $N_{j,m_0} \setminus \{1\}$ does not intersect the ball of $P_j$ of radius $10\sinh(r_U) \times 2^{100\delta + \|c\|}$.

Consider $H_0$ the 2-separated system of horoball from the construction of $X_0$. And in this system, the system of horoballs at depth $(50\delta + 20\|c\|)$,
which we call $H'_0$. This way, $\rho$, which has a fundamental domain for $c$ of length $\|c\|$, and intersects $X_0 \setminus \mathcal{H}_0$, does not get $50\delta$-close to an horoball of $H'_0$. Rescale the space $(X_0, \mathcal{H}_0)$ into $(X, \mathcal{H})$ so that it is now $\delta_u$-hyperbolic for the constant prescribed in [DGOP] §7. We then consider the parabolic cone-off, as defined in [DGOP], §7, for this pair, and for a radius of cone $\geq 50\delta_u$. Observe that the axis of $c$ (which we still denote by $\rho$) does not get $50\delta_u$ close to an apex.

Observe also that, for the chosen $m$, any (non-trivial) element of $N_{j,m}$ translate on the corresponding horosphere of $H'_0$ by a distance of at least $10 \sinh(r_U)$ (measured in the graph distance of the horosphere). Therefore, for the chosen $m$, $K_m$ is the group of a very rotating family at the apices of the parabolic cone-off.

Let $x_0 \in \rho$. The segment $[x_0, cx_0]$ is contained in $\rho$, and does not get $50\delta_u$ close to an apex.

Consider the segment $[x_0, hx_0]$ and for all apex $a$ on it, define the two points $a_-$ and $a_+$ on $[x_0, a]$ and $[a, hx_0]$ (subsegments of $[x_0, hx_0]$) at distance $27\delta_u$ from $a$ (they exist since the cones have much larger radius than $27\delta_u$, and $x_0$ is not in a cone).

By multiplying $h$ by elements of $K_m$, we may assume that $d(a_-, a_+) = d(a_-, (\text{Fix}(a) \cap K_m)a_+)$, for every apex $a$ in the segment $[x_0, hx_0]$.

By assumption, $hch^{-1}c^{-1} \in K_m$, and we can assume that it is non trivial (otherwise there is nothing to prove). By [DGOP], Lemma 5.10, the segment $[hc x_0, ch x_0]$ contains an apex $a_0$ and a $5\delta_u$-shortening pair.

Hyperbolicity in the pentagon $(x_0, hx_0, hc x_0, ch x_0)$, together with the absence of apices in $[x_0, cx_0]$, and in its image by $h$, $[hx_0, hc x_0]$, shows that at least one segment among $[x_0, hx_0]$ and $[cx_0, ch x_0]$ must contain $a_0$. Assume that only one of them contains $a_0$, and let us say that it is $[x_0, hx_0]$ (if it is the other, the argument is identical). Hyperbolicity forces the $5\delta_u$-shortening pair of $[hc x_0, ch x_0]$ at $a_0$ to be $5\delta$-close to $a_-$ and $a_+$. Therefore some element of $K_m \cap \text{Fix}(a_0)$ takes $a_+$ to a point at distance at most $20\delta$ from $a_-$, thus contradicting the above minimality condition.

It follows that both $[x_0, hx_0]$ and $[cx_0, ch x_0]$ (which is its image by $c$) must contain $a_0$.

But then, the image of $a_0$ by $c$ is at distance at most $\|c\|$ from $a_0$. By separation of apices, it must then be $a_0$, and $c$ fixes an apex, contrarily to our assumption.

\[ \square \]
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