Canceling effects in higher-order Hardy-Sobolev inequalities

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Abstract

A classical first-order Hardy-Sobolev inequality in Euclidean domains, involving weighted
norms depending on powers of the distance function from their boundary, is known to hold
for every, but one, value of the power. We show that, by contrast, the missing power is
admissible in a suitable counterpart for higher-order Sobolev norms. Our result com-
plements and extends contributions by Castro and Wang [CW], and Castro, Dávila and
Wang [CDW1, CDW2], where a surprising canceling phenomenon underlying the relevant
inequalities was discovered in the special case of functions with derivatives in $L^1$.

1 Introduction and main results

Weighted Sobolev inequalities, namely Sobolev inequalities for norms on open sets $\Omega$ of $\mathbb{R}^n$
equipped with measures having densities – the weights – with respect to the Lebesgue measure,
have been extensively investigated, mainly in connection with the theory of degenerate partial
differential equations. The literature in this area is very rich. Let us just mention that various
characterizations of the weights supporting the relevant inequalities are available, such as
those depending on global integrability properties of the weights [MS], on the growth of their
integrals on balls [Ad1, Ad2], on their membership to Muckenhoupt $A_p$-classes [HKM, OK],
on associated capacities [Ma], on their rearrangements [CEG].

The most popular and widely exploited inequalities of this kind are presumably the so
called Hardy-Sobolev inequalities, whose weights are just powers of the distance function
from the boundary $\partial \Omega$. This function is known to inherit regularity properties of $\partial \Omega$ in a
sufficiently narrow neighborhood of the latter. Let us call $d : \Omega \to (0, \infty)$ a function which
agrees with the distance function in such neighborhood of $\partial \Omega$, and enjoys the same regularity
properties, but in the whole of $\Omega$.

Given $p \in [1, \infty]$ and $\alpha \in \mathbb{R}$, we denote by $L^p(\Omega, d^\alpha)$ the weighted Lebesgue space
equipped with the norm defined as

$$
\|u\|_{L^p(\Omega, d^\alpha)} = \left( \int_{\Omega} |u(x)|^p d(x)^\alpha \, dx \right)^{\frac{1}{p}}
$$

Mathematics Subject Classification: 46E35, 46E30.
Keywords: Hardy inequality, higher-order Sobolev spaces, distance function.
for a measurable function \( u \) in \( \Omega \). Moreover, if \( m \in \mathbb{N} \), the notation \( W^{m,p}(\Omega, d^\alpha) \) is adopted for the associated Sobolev space of \( m \)-times weakly differentiable functions \( u \) in \( \Omega \) endowed with the norm
\[
\|u\|_{W^{m,p}(\Omega, d^\alpha)} = \sum_{j=0}^{m} \|\nabla^j u\|_{L^p(\Omega, d^\alpha)},
\]
where \( \nabla^j u \) stands for the vector of all derivatives of \( u \) of order \( j \). We also simply denote \( \nabla^1 u \) by \( \nabla u \); also, \( \nabla^0 u \) stands for \( u \). The notation \( W^{0,m}_0(\Omega, d^\alpha) \) is devoted to the closure of \( C_0^\infty(\Omega) \) in \( W^{m,p}(\Omega, d^\alpha) \).

A classical Hardy-Sobolev inequality asserts that if \( \Omega \) is a bounded Lipschitz domain, and
\[
\alpha \neq p - 1,
\]
then there exists a constant \( C \) such that
\[
\|u\|_{L^p(\Omega, d^\alpha)} \leq C\|u\|_{W^{1,p}(\Omega, d^\alpha)} \quad (1.1)
\]
for every \( u \in W^{1,p}_0(\Omega, d^\alpha) \) [Ku, Theorem 8.4]. On the other hand, inequality (1.1) fails for the critical value \( \alpha = p - 1 \).

The main purpose of the present paper is to show that, this notwithstanding, suitable higher-order versions of inequality (1.1), which cannot just be obtained from (1.1) via iteration, do hold even when \( \alpha = p - 1 \).

A prototypical second-order inequality may help to grasp the spirit of our results. Assume that \( \Omega \) has a smooth boundary, so that \( d \) is also smooth in a neighborhood of \( \partial \Omega \). Let \( u \in W^{2,p}_0(\Omega, d^{p-1}) \). A standard property of the distance function ensures that \(|\nabla d| = 1\) in a neighborhood of \( \partial \Omega \), whence
\[
\left| \nabla \left( \frac{u}{d} \right) \right| = \left| \frac{\nabla u}{d} - \frac{u}{d} \frac{\nabla d}{d^2} \right| \leq \frac{|\nabla u|}{d} + \frac{|u|}{d^2}, \quad \text{a.e. in the same neighborhood.}
\]
Inequality (1.1) cannot be exploited to infer that the functions \( \frac{\nabla u}{d} \) and \( \frac{u}{d^2} \) belong to \( L^p(\Omega, d^{p-1}) \). In fact, membership to \( L^p(\Omega, d^{p-1}) \) of neither of these functions is guaranteed under the sole assumption that \( u \in W^{2,p}_0(\Omega, d^{p-1}) \) (this can be verified, for instance, by taking \( \Omega = (0, 1) \), and considering functions \( u(x) \) decaying like \( x \log^{-\alpha}(\frac{1}{x}) \) as \( x \to 0^+ \), with \( \alpha \in (0, \frac{1}{p}) \)). Nevertheless, we show that the inequality
\[
\left\| \frac{u}{d} \right\|_{W^{1,p}(\Omega, d^{p-1})} \leq C\|u\|_{W^{2,p}(\Omega, d^{p-1})}, \quad (1.2)
\]
holds for some constant \( C \), and for every \( u \in W^{2,p}_0(\Omega, d^{p-1}) \). This is possible thanks to a canceling effect which allows the leftmost side of (1.2) to have stronger integrability properties than each addend on its rightmost side.

In the case when \( p = 1 \), such a striking phenomenon has been elucidated in remarkable contributions, by which ours is inspired, of Castro and Wang [CW], for \( n = 1 \), and of Castro, Dávila and Wang [CDW1, CDW2], for any \( n \geq 1 \). In this case, non-weighted Lebesgue and Sobolev norms appear in (1.1) and (1.3), and in their higher-order counterparts from [CDW2].

The arbitrary-order version of inequality (1.3) to be established asserts that, if \( 1 \leq p < \infty \), and \( k, m \in \mathbb{N} \), with \( m \geq 2 \), and \( 1 \leq k \leq m - 1 \), then there exists a constant \( C \) such that
\[
\left\| \frac{u}{d^{m-k}} \right\|_{W^{k,p}(\Omega, d^{p-1})} \leq C\|u\|_{W^{m,p}(\Omega, d^{p-1})}. \quad (1.4)
\]
for every \( u \in W_0^{m,p}(\Omega, d^{p-1}) \).

Inequality (1.4) is in turn a special instance of our most general result, stated in the following theorem, where Sobolev type spaces associated with different Lebesgue norms and distance weights are allowed on the two sides of the relevant inequality.

**Theorem 1.1.** Let \( \Omega \) be a bounded open set with a smooth boundary in \( \mathbb{R}^n \), \( n \geq 1 \), and let \( k, m \in \mathbb{N} \), \( m \geq 2 \), and \( 1 \leq k \leq m - 1 \). Assume that \( 1 \leq p \leq q < \infty \), and

\[
\frac{1}{q} \geq \frac{n - p(m - k)}{np}.
\]

Let

\[
r \geq \frac{q(n - 1) - p(n - q)}{p}.
\]

Then, there exists a constant \( C \) such that

\[
\left\| \frac{u}{d^{m-k}} \right\|_{W^{k,q}(\Omega,d^r)} \leq C \left\| u \right\|_{W^{m,p}(\Omega,d^{p-1})}
\]

for every \( u \in W_0^{m,p}(\Omega, d^{p-1}) \).

**Remark 1.2.** Conditions (1.5) and (1.6) in Theorem 1.1 are sharp, as shown in Propositions 3.2 and 3.3, Section 3. The assumption that \( k \geq 1 \) is also sharp, since inequality (1.7) breaks down for \( k = 0 \), as pointed out in the discussion above.

**Remark 1.3.** As already mentioned, the case when \( p = q = 1 \) and \( r = 0 \) in Theorem 1.1 is the object of [CDW2]. Its one-dimensional version was earlier proved in [CW], where a higher-order inequality for \( p = q > 1 \) is also established. However, that higher-order inequality does not correspond to the critical missing case of (1.1), and can be derived through a repeated application of the latter.

Our proof of Theorem 1.1 combines a flattening argument for \( \partial \Omega \), which was introduced in [CDW2] and involves highly original tricks, with Whitney type decomposition techniques exploited in [Ho1, Ho2], and with one-dimensional Hardy type inequalities. In comparison with the proofs of [CDW2], additional difficulties arise, due to the presence of weights and of possibly different norms on the two sides of the inequalities under consideration. In particular, an iterative argument relying upon the use of a fundamental second-order inequality as in [CDW2] is not possible. This calls for a different, direct proof, which requires a careful combinatorial analysis of the mutual canceling of partial derivatives of trial functions.

## 2 Inequalities in the half space \( \mathbb{R}^n_+ \)

This section is devoted to a Hardy-Sobolev inequality in the half-space, contained in Theorem 2.1 below. This is a key step, of independent interest, towards the proof of Theorem 1.1.

**Theorem 2.1.** Let \( k, m \in \mathbb{N} \), \( m \geq 2 \), and \( 1 \leq k \leq m - 1 \). Assume that \( 1 \leq p \leq q < \infty \),

\[
\frac{1}{p} - \frac{1}{q} = \frac{\beta - \alpha}{n}, \quad \alpha < k + \frac{p - 1}{p}, \quad \text{and} \quad \alpha \leq \beta \leq \alpha + (m - k).
\]

Then, there exists a constant \( C \) such that

\[
\left( \int_{\mathbb{R}^n_+} x^{\beta q} \left| \nabla^k \left( \frac{u(x)}{x^{m-k}} \right) \right|^q \, dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^n_+} x^{\alpha p} \left| \nabla^m u \right|^p \, dx \right)^{\frac{1}{p}}
\]

for every \( u \in C_0^\infty(\mathbb{R}^n_+) \).
The proof of Theorem 2.1 rests upon several lemmas. The first one is a one-dimensional version of its, in the special case when \( p = q \) and \( \alpha = \beta \).

**Lemma 2.2.** Let \( k, m \in \mathbb{N}, m \geq 2 \) and \( 1 \leq k \leq m - 1 \). Assume that \( 1 \leq p < \infty \) and \( \alpha < k + \frac{p-1}{p} \). Then there exists a constant \( C \) such that

\[
\left( \int_0^\infty r^{\alpha p} \left| \frac{d^k}{dr^k} \left( \frac{f(r)}{r^{m-k}} \right) \right|^p dr \right)^{\frac{1}{p}} \leq C \left( \int_0^\infty r^{\alpha p} \left| \frac{d^m}{dr^m} (r) \right|^p dr \right)^{\frac{1}{p}}
\]

for every \( f \in C_0^\infty (0, \infty) \).

**Proof.** An application of Taylor’s formula, with remainder term in integral form, tells us that

\[
\frac{d^k}{dr^k} \left( \frac{f(r)}{r^{m-k}} \right) = \frac{1}{(m-k-1)!} \int_0^r \frac{d^m}{ds^m} f(s) \left( 1 - \frac{s}{r} \right)^{m-k-1} \left( \frac{s}{r} \right)^{k-1} \frac{s}{r^p} ds \quad \text{for } r > 0,
\]

see [CW] Proof of Theorem 1.2. Hence, since \( (1 - \frac{s}{r})^{m-k-1} \leq 1 \) if \( 0 \leq s \leq r \), one has that

\[
\frac{d^k}{dr^k} \left( \frac{f(r)}{r^{m-k}} \right) \leq \frac{1}{r^{(k+1)(m-k-1)}} \int_0^r s \left| \frac{d^m}{ds^m}(s) \right| ds \quad \text{for } r > 0.
\]

Owing to the assumption that \( \alpha < k + \frac{p-1}{p} \), inequalities (2.3) and (2.4), combined with a classical one-dimensional Hardy inequality [Ku, Theorem 5.1], ensure that

\[
\left( \int_0^\infty r^{\alpha p} \left| \frac{d^k}{dr^k} \left( \frac{f(r)}{r^{m-k}} \right) \right|^p dr \right)^{\frac{1}{p}} \leq \frac{1}{(m-k-1)!} \left( \int_0^\infty r^{\alpha(k-1)p} \left( \int_0^r s \left| \frac{d^m}{ds^m}(s) \right| ds \right)^p dr \right)^{\frac{1}{p}}
\]

\[
\leq C \left( \int_0^\infty r^{\alpha p} \left| \frac{d^m}{dr^m} (r) \right|^p dr \right)^{\frac{1}{p}}
\]

for some constant \( C \).

The following result is a kind of combinatorial identity. In its statement, we use the abridged notation

\( [k] = \{1, 2, \ldots, k\} \).

**Lemma 2.3.** Let \( k \in \mathbb{N} \), and let \( a_1, a_2, \ldots, a_k \in \mathbb{R} \). Given \( I \subset [k] \), define \( a_0 = 0 \), and \( a_I = \sum_{i \in I} a_i \) if \( I \neq \emptyset \). Then

\[
\sum_{l=0}^{k} (-1)^{k-l} \sum_{I \subset [k], \#I = l} (a_I + s)^k \quad \text{for } s \in \mathbb{R},
\]

where \( \#I \) denotes the cardinality of \( I \).

**Proof.** Define \( \varphi_k : \mathbb{R} \to \mathbb{R} \) as

\[
\varphi_k(s) = \sum_{j=0}^{k} (-1)^{k-j} \sum_{I \subset [k], \#I = k-j} (a_I + s)^k \quad \text{for } s \in \mathbb{R}.
\]

Equation (2.5) can thus be rewritten as

\[
\varphi_k(s) = k! \prod_{i=1}^{k} a_i \quad \text{for } s \in \mathbb{R}.
\]
We establish equation (2.6) by induction. The case when $k = 1$ is trivial. Assume that (2.6) holds for $k - 1$, for some $k \geq 2$. We begin by proving that $\varphi_k$ is constant. The following chain holds:

$$
\varphi_k(s) = \sum_{j=0}^{k} (-1)^{k-j} \left( \sum_{I \subseteq [k], I \neq \{j\}} (a_I + s)^k + \sum_{I \subseteq [k], I \neq \{j\}} (a_I + s)^{k-j} \right)
$$

$$
= \sum_{j=0}^{k-1} (-1)^{k-j} \left( \sum_{I \subseteq [k], I \neq \{j\}} (a_I + s)^k + \sum_{I \subseteq [k], I \neq \{j\}} (a_I + s)^{k-j} \right)
$$

$$
= -\sum_{j=0}^{k-1} (-1)^{(k-1)-j} \sum_{I \subseteq [k], I \neq \{j\}} (a_I + s)^{k-1} + \sum_{j=0}^{k-1} (-1)^{(k-1)-j} \sum_{I \subseteq [k], I \neq \{j\}} (a_I + s)^{k-1}
$$

Hence, by the induction assumption,

$$
\frac{d\varphi_k}{ds}(s) = -k \sum_{j=0}^{k-1} (-1)^{(k-1)-j} \sum_{I \subseteq [k], I \neq \{j\}} (a_I + s)^{k-1} + k \sum_{j=0}^{k-1} (-1)^{(k-1)-j} \sum_{I \subseteq [k], I \neq \{j\}} (a_I + s)^{k-1}
$$

$$
= -k \varphi_{k-1}(s) + k \varphi_{k-1}(a_k + s) = -k(k-1) \prod_{i=1}^{k-1} a_i + k(k-1) \prod_{i=1}^{k-1} a_i = 0 \quad \text{for } s \in \mathbb{R}.
$$

Therefore $\varphi_k$ is constant. In particular,

(2.7) \quad \varphi_k(s) = \varphi_k(0) \quad \text{for } s \in \mathbb{R}.

It remains to show that

(2.8) \quad \varphi_k(0) = k! \prod_{i=1}^{k} a_i.

To this purpose, observe that, by the induction assumption,

(2.9) \quad k \int_0^{a_k} \varphi_{k-1}(s) ds = k(k-1) \prod_{i=1}^{k-1} a_i \int_0^{a_k} ds = k! \prod_{i=1}^{k} a_i.
On the other hand, the very definition of $\varphi_{k-1}$ ensures that

$$k \int_0^{a_k} \varphi_{k-1}(s) \, ds = \int_0^{a_k} \sum_{j=0}^{k-1} (-1)^{k-1-j} \sum_{I \subseteq [k-1]} \sum_{\#I = j} k(a_I + s)^{k-1} \, ds$$

$$= \sum_{j=0}^{k-1} (-1)^{k-1-j} \sum_{I \subseteq [k-1]} \sum_{\#I = j} (a_I + a_k)^k - a_I^k$$

$$= \sum_{j=0}^{k-1} (-1)^{k-1-j} \sum_{I \subseteq [k-1]} \sum_{\#I = j} (a_I + a_k)^k - \sum_{j=0}^{k-1} (-1)^{(k-1)-j} \sum_{I \subseteq [k-1]} a_I^k$$

$$= \sum_{j=1}^{k} (-1)^{k-j} \sum_{I \subseteq [k-1]} \sum_{\#I = j} (a_I + a_k)^k + \sum_{j=0}^{k-1} (-1)^{k-j} \sum_{I \subseteq [k-1]} a_I^k$$

$$= \sum_{j=0}^{k} (-1)^{k-j} \left( \sum_{I \subseteq [k]} a_I^k + \sum_{I \subseteq [k]} a_I^k \right) = \sum_{j=0}^{k} (-1)^{k-j} \sum_{I \subseteq [k]} a_I^k = \varphi_k(0).$$

Hence,

$$k \int_0^{a_k} \varphi_{k-1}(s) \, ds = \varphi_k(0).$$

Equation (2.8) follows from (2.9) and (2.10). \( \square \)

The next lemma is concerned with the special case when $\alpha = \beta$, and hence $p = q$, in Theorem 2.1.

**Lemma 2.4.** Let $k, m \in \mathbb{N}$, $m \geq 2$ and $1 \leq k \leq m - 1$. Assume that $1 \leq p < \infty$ and $\alpha < k + \frac{p-1}{p}$. Then there exists a constant $C$ such that

$$\left( \int_{\mathbb{R}^n_+} x_n^{\alpha p} \left| \nabla^k \left( \frac{u(x)}{x_n^{m-k}} \right) \right|^p \, dx \right)^{\frac{1}{p}} \leq C \left( \int_{\mathbb{R}^n_+} x_n^{\alpha p} |\nabla^m u|^p \, dx \right)^{\frac{1}{p}}$$

for every $u \in C_0^\infty(\mathbb{R}^n_+)$.\[2.11\]

**Proof.** The case when $n = 1$ is the object of Lemma 2.2. We may thus assume that $n \geq 2$. Set $\bar{x} = (x_1, \ldots, x_{n-1})$, so that $x = (\bar{x}, x_n)$ for $x \in \mathbb{R}^n$. Fubini’s theorem and Lemma 2.2 imply that

$$\left( \int_{\mathbb{R}^n_+} x_n^{\alpha p} \left| \frac{\partial^k}{\partial x_n^k} \left( \frac{u(x)}{x_n^{m-k}} \right) \right|^p \, dx \right)^{\frac{1}{p}} = \left( \int_{\mathbb{R}^{n-1}_+} \int_0^\infty x_n^{\alpha p} \left| \frac{\partial^k}{\partial x_n^k} \left( \frac{u(\bar{x}, x_n)}{x_n^{m-k}} \right) \right|^p \, dx_n d\bar{x} \right)^{\frac{1}{p}}$$

$$\leq C \left( \int_{\mathbb{R}^{n-1}_+} \int_0^\infty x_n^{\alpha p} \left| \frac{\partial^m u}{\partial x_n^m} (\bar{x}, x_n) \right|^p \, dx_n d\bar{x} \right)^{\frac{1}{p}}$$

$$= C \left( \int_{\mathbb{R}^n_+} x_n^{\alpha p} \left| \frac{\partial^m u}{\partial x_n^m} (x) \right|^p \, dx \right)^{\frac{1}{p}},$$

where $C$ denotes the constant appearing in inequality (2.2). This establishes inequality (2.11) with $\nabla^k$ replaced by the sole derivative $\frac{\partial^k}{\partial x_n^k}$ on its left-hand side.
Our next task is to extend this inequality to arbitrary \( k \)-th order derivatives. For ease of notation, let us set

\[
\partial_{\ell} = \frac{\partial}{\partial x_{\ell}}
\]

for \( \ell = 1, \ldots, n \). Moreover, given \( k \) indices \( j_i \in \{1, \cdots, n\} \), with \( i = 1, \ldots, k \), we define

\[
\partial^k = \prod_{i=1}^{k} \partial_{j_i}.
\]

Lemma 2.3 implies that

\[
k! \prod_{i=1}^{k} \partial_{j_i} = \sum_{l=0}^{k} (-1)^{k-l} \sum_{i \in [k], \#i = l} \left( \sum_{j_i \in I} \partial_{j_i} + \partial_n \right)^k.
\]

Now fix \( I \subset [k] \). Then, there exist \( h_l \in \{0, 1, \cdots, n\} \) with \( l = 1, \ldots, n \) and \( \sum_{l=1}^{n} h_l = \#I \), such that

\[
\sum_{i \in I} \partial_{j_i} + \partial_n = \sum_{l=1}^{n} h_l \partial_l + \partial_n.
\]

Let \( \Psi : \mathbb{R}^n \to \mathbb{R}^n \) be the bijective linear map defined as

\[
\Psi(y) = y + y_n \sum_{l=1}^{n} h_l e_l \quad \text{for} \quad y \in \mathbb{R}^n,
\]

where \( \{e_1, \ldots, e_n\} \) is the canonical basis in \( \mathbb{R}^n \). Note that \( \Psi(\mathbb{R}_+^n) = \mathbb{R}_+^n \), and \( \Psi(\partial \mathbb{R}_+^n) = \partial \mathbb{R}_+^n \). Consider the change of variables

\[
x = \Psi(y) \quad \text{for} \quad y \in \mathbb{R}^n.
\]

Since its Jacobian determinant equals 1 + \( h_n \),

\[
\int_{\mathbb{R}_+^n} x_n^{\alpha} \left( \sum_{l=1}^{n} h_l \partial_l + \partial_n \right)^{m} \left( \frac{u(x)}{x_n^{m-k}} \right)^p \, dx = (1 + h_n) \int_{\mathbb{R}_+^n} y_n^{\alpha} \left( \sum_{l=1}^{n} h_l \partial_l + \partial_n \right)^{m} \left( \frac{u(x)}{x_n^{m-k}} \right)^p |_{x=\Psi(y)} \, dy,
\]

and

\[
\int_{\mathbb{R}_+^n} x_n^{\alpha} \left( \sum_{l=1}^{n} h_l \partial_l + \partial_n \right)^{m} u(x)^p \, dx = (1 + h_n) \int_{\mathbb{R}_+^n} y_n^{\alpha} \left[ \left( \sum_{l=1}^{n} h_l \partial_l + \partial_n \right)^{m} u(x) \right]_{x=\Psi(y)}^p \, dy
\]

for \( u \in C_0^\infty(\mathbb{R}_+^n) \). Now define \( v : \mathbb{R}_+^n \to \mathbb{R}_+^n \) as \( v(y) = u(\Psi(y)) \) for \( y \in \mathbb{R}_+^n \). We have that \( v \in C_0^\infty(\mathbb{R}_+^n) \). By the chain rule for derivatives and the definition of \( \Psi \),

\[
\frac{\partial}{\partial y_n} \left( \frac{v(y)}{y_n^{m-k}} \right) = \frac{\partial}{\partial y_n} \left( (1 + h_n)^{m-k} \left( \frac{u(x)}{x_n^{m-k}} \right)^p \right)_{x=\Psi(y)}
\]

\[
= (1 + h_n)^{m-k} \left[ \left( \sum_{l=1}^{n} h_l \partial_l + \partial_n \right)^{m} \left( \frac{u(x)}{x_n^{m-k}} \right)^p \right]_{x=\Psi(y)} \quad \text{for} \quad y \in \mathbb{R}_+^n.
\]
Iterating equation (2.16) yields
\begin{equation}
(2.17) \quad \frac{\partial^k}{\partial y_n^k} \left( \frac{v(y)}{y_n^{m-k}} \right) = (1 + h_n)^{m-k} \left[ \left( \sum_{l=1}^{n} h_l \partial_l + \partial_n \right)^k \left( \frac{u(x)}{x_n^{m-k}} \right) \right]_{x=\psi(y)}
\end{equation}
for \( y \in \mathbb{R}^n_+ \).

Analogously,
\begin{equation}
(2.18) \quad \frac{\partial^m v}{\partial y_n^m}(y) = \left[ \left( \sum_{l=1}^{n} h_l \partial_l + \partial_n \right)^m u(x) \right]_{x=\psi(y)}
\end{equation}
for \( y \in \mathbb{R}^n_+ \).

Coupling (2.17) with (2.12) tells us that
\begin{equation}
(2.19) \quad \int_{\mathbb{R}^n_+} y_n^{\alpha p} \left( \sum_{l=1}^{n} h_l \partial_l + \partial_n \right)^k \left( \frac{u(x)}{x_n^{m-k}} \right) \left| \frac{\partial^k}{\partial y_n^k} \left( \frac{v(y)}{y_n^{m-k}} \right) \right|^p dy
\end{equation}
\begin{equation}
= (1 + h_n)^{k-m} \left( \int_{\mathbb{R}^n_+} y_n^{\alpha p} \left| \frac{\partial^k}{\partial y_n^k} \left( \frac{v(y)}{y_n^{m-k}} \right) \right|^p dy \right)^{\frac{1}{p}} \leq C \left( \int_{\mathbb{R}^n_+} y_n^{\alpha p} \left| \frac{\partial^m v}{\partial y_n^m} \right|^p dy \right)^{\frac{1}{p}}.
\end{equation}

Hence, via (2.14), (2.15), and (2.18),
\begin{equation}
(2.20) \quad \left( \int_{\mathbb{R}^n_+} x_n^{\alpha q} \left| \sum_{i \in I} \partial_{j_i} + \partial_n \right|^k \left( \frac{u(x)}{x_n^{m-k}} \right) \left| \frac{\partial^k}{\partial x_n^k} \left( \frac{v(x)}{x_n^{m-k}} \right) \right|^p dx \right)^{\frac{1}{p}} \leq C \left( \int_{\mathbb{R}^n_+} x_n^{\alpha q} \left| \nabla^m u \right|^p dx \right)^{\frac{1}{p}}
\end{equation}
for some constant \( C > 0 \). Thanks to the arbitrariness of \( I \subset [k] \), inequality (2.11) follows.

Our last intermediate step consists in an estimate for the left-hand side of inequality (2.2), involving \( k \)-th order derivatives of \( u(x)x_n^{k-m} \), in terms of the \( k \)-th and \((k+1)\)-th order derivatives of the same expression, but with different weights. Note that no condition on the trace of \( u \) on \( \partial \mathbb{R}^n_+ \) is required here.

**Lemma 2.5.** Let \( k, m \in \mathbb{N} \), \( m \geq 2 \) and \( 1 \leq k \leq m - 1 \). Assume that \( 1 \leq p \leq q < \infty \), \( \frac{1}{p} - \frac{1}{q} = \frac{\beta - \alpha}{n} \), and \( \alpha \leq \beta \leq \alpha + 1 \). Then, there exists a constant \( C \) such that
\begin{equation}
(2.21) \quad \left( \int_{\mathbb{R}^n_+} x_n^{\beta q} \left| \nabla^k \left( \frac{u(x)}{x_n^{m-k}} \right) \right|^q dx \right)^{\frac{1}{q}}
\end{equation}
\begin{equation}
\leq C \left( \int_{\mathbb{R}^n_+} x_n^{(\alpha+1)p} \left| \nabla^{k+1} \left( \frac{u(x)}{x_n^{m-k}} \right) \right|^p dx + x_n^{\alpha p} \left| \nabla^k \left( \frac{u(x)}{x_n^{m-k}} \right) \right|^p dx \right)^{\frac{1}{p}}
\end{equation}
for every \( u \in C^\infty(\mathbb{R}^n_+) \).

**Proof.** By the same argument as in Lemma 2.4, it suffices to prove inequality (2.21) with \( \nabla^k \) replaced with just \( \frac{\partial^k}{\partial x_n^k} \) on the left-hand side. Let us set, for simplicity of notation,
\begin{equation}
\partial_n^k = \frac{\partial^k}{\partial x_n^k}.
\end{equation}

Assume first that \( 1 \leq p < n \). Let \( \{B_j\}_{j \in \mathbb{N}} \) be a covering of \( \mathbb{R}^n_+ \) as in \cite[Lemma 4.1]{Ho1}, namely a family of balls \( B_j \) with radius \( r_j \), and centers in \( \mathbb{R}^n_+ \), such that:
(i) \( \mathbb{R}_+^n \subset \bigcup_{j=1}^\infty B_j \);

(ii) there exist positive constants \( C' \) and \( C'' \) such that \( C' x_n \leq r_j \leq C'' x_n \) for every \( j \in \mathbb{N} \) and \( x \in B_j \);

(iii) there exists a positive constant \( C \) such that \( \# \{ i \in \mathbb{N} : B_j \cap B_i \neq \emptyset \} \leq C \) for every \( j \in \mathbb{N} \).

Let \( \{ \phi_j \} \) be a partition of unity subordinate to this covering, namely a sequence of nonnegative functions \( \phi_i \in C_0^\infty(B_j) \) such that \( \sum_j \phi_j = 1 \). The functions \( \phi_j \) can be chosen in such a way that \( |\nabla \phi_j| \leq C/r_j \) for some constant \( C \), and for every \( j \in \mathbb{N} \). Set \( p^* = \frac{np}{n-p} \), the Sobolev conjugate of \( p \). Since \( \beta \leq \alpha + 1 \), we have that \( q \leq p^* \). Owing to the standard Sobolev inequality for compactly supported functions, applied in each ball \( B_j \), the following chain holds:

\[
\left( \int_{\mathbb{R}_+^n} x_n^{\beta q} \left| \partial_n^k \left( \frac{u(x)}{x_n^{m-k}} \right) \right|^q dx \right)^{\frac{1}{q}} \\
\leq \sum_{j \in \mathbb{N}} \left( \int_{B_j} x_n^{\beta q} \left| \phi_j \partial_n^k \left( \frac{u(x)}{x_n^{m-k}} \right) \right|^q dx \right)^{\frac{1}{q}} \\
\leq C_1 \sum_{j \in \mathbb{N}} r_j^\beta \left( \int_{B_j} \left| \phi_j \partial_n^k \left( \frac{u(x)}{x_n^{m-k}} \right) \right|^q dx \right)^{\frac{1}{q}} \\
\leq C_2 \sum_{j \in \mathbb{N}} r_j^\beta \left( \int_{B_j} \left| \phi_j \partial_n^k \left( \frac{u(x)}{x_n^{m-k}} \right) \right|^{p^*} dx \right)^{\frac{1}{p^*}} \left( \int_{B_j} 1 dx \right)^{\frac{1}{q}} \\
\leq C_3 \sum_{j \in \mathbb{N}} r_j^{1+\alpha} \left( \int_{B_j} \left| \nabla \left( \phi_j \partial_n^k \left( \frac{u(x)}{x_n^{m-k}} \right) \right) \right|^p dx \right)^{\frac{1}{p}} \\
\leq C_4 \left( \int_{\mathbb{R}_+^n} x_n^{\alpha p} \left| \partial_n^k \left( \frac{u(x)}{x_n^{m-k}} \right) \right|^p dx \right)^{\frac{1}{p}} + C_4 \left( \int_{\mathbb{R}_+^n} x_n^{(\alpha+1)p} \left| \nabla \left( \frac{u(x)}{x_n^{m-k}} \right) \right|^p dx \right)^{\frac{1}{p}},
\]

for suitable constants \( C_i, i = 1, \ldots, 4 \), for every \( u \in C^\infty(\mathbb{R}_+^n) \). This establishes inequality \( (2.21) \) in the case when \( 1 \leq p < n \). Note that, in fact, this argument proves \( (2.21) \) for every \((k+1)\)-times weakly differentiable function \( u \) in \( \mathbb{R}_+^n \) making the right-hand side finite.

Assume next that \( n \leq p < \infty \). Let \( p_1, q_1, \alpha_1, \beta_1, r \) be such that

\[
\max \left\{ 1, n \left( 1 - \frac{p}{q} \right) \right\} \leq p_1 < n, \quad r = \frac{p}{p_1}, \quad q_1 = \frac{q}{r}, \quad \alpha_1 = r \alpha, \quad \beta_1 = r \beta.
\]

Hence,

\[
\begin{cases}
  r p_1 = p, & r q_1 = q, & \alpha_1 p_1 = \alpha p, & \beta_1 q_1 = \beta q, & p_1 (\alpha_1 + r) = p (\alpha + 1), \\
  \frac{r}{n} \left( \beta - \alpha \right) = \frac{r}{n} (\frac{\alpha - \alpha_1}{\alpha_1 - \beta_1}) = \frac{r}{p_1} - \frac{r}{q_1}, \\
  0 \leq \beta_1 - \alpha_1 = n \left( \frac{1}{p_1} - \frac{1}{q_1} \right) = n \left( \frac{1}{p} - \frac{1}{q} \right) \leq 1.
\end{cases}
\]

Given any function \( u \in C^\infty(\mathbb{R}_+^n) \), define \( w : \mathbb{R}_+^n \rightarrow \mathbb{R} \) as

\[
w(\overline{x}, x_n) = x_n^{m-k} \int_0^{x_n} \int_0^{x_{n-1}} \cdots \int_0^{x_1} \left| \partial_n^k \left( \frac{u(\overline{x}, s)}{s^{m-k}} \right) \right|^r ds ds_1 \cdots ds_{k-1}
\]
for \((\bar{x}, x_n) \in \mathbb{R}^n_+\). Note that the function \(w\) is \(k+1\)-times weakly differentiable. An application of inequality (2.22) to \(w\), with \(\alpha, \beta, p, q\) replaced by \(\alpha_1, \beta_1, p_1, q_1\), yields

\[
(\int_{\mathbb{R}^n_+} x_n^{\beta_1q_1} \left| \partial_n^k \left( \frac{w(x)}{x_n^{m-k}} \right) \right|^{q_1} dx)^{\frac{1}{q_1}} \leq C \left( \int_{\mathbb{R}^n_+} x_n^{(\alpha_1+1)p_1} \left| \nabla \partial_n^k \left( \frac{w(x)}{x_n^{m-k}} \right) \right|^{p_1} dx \right)^{\frac{1}{p_1}} + C \left( \int_{\mathbb{R}^n_+} x_n^{\alpha_1p_1} \left| \partial_n^k \left( \frac{w(x)}{x_n^{m-k}} \right) \right|^{p_1} dx \right)^{\frac{1}{p_1}}.
\]

By the very definition of \(w\),

\[
(\int_{\mathbb{R}^n_+} x_n^{\beta_1q_1} \left| \partial_n^k \left( \frac{w(x)}{x_n^{m-k}} \right) \right|^{q_1} dx)^{\frac{1}{q_1}} = \left( \int_{\mathbb{R}^n_+} x_n^{\beta_1q_1} \left| \partial_n^k \left( \frac{u(x)}{x_n^{m-k}} \right) \right|^{q_1} dx \right)^{\frac{1}{q_1}}.
\]

Moreover,

\[
\left| \nabla \partial_n^k \left( \frac{w(x)}{x_n^{m-k}} \right) \right| = \left| \nabla \partial_n^k \left( \frac{u(x)}{x_n^{m-k}} \right) \right| = \left| \nabla \partial_n^k \left( \frac{u(x)}{x_n^{m-k}} \right) \right|^{\frac{r-1}{p}}
\]

for \(x \in \mathbb{R}^n_+\). From equation (2.26) and Young’s inequality one can infer that

\[
(\int_{\mathbb{R}^n_+} x_n^{(\alpha_1+1)p_1} \left| \nabla \partial_n^k \left( \frac{w(x)}{x_n^{m-k}} \right) \right|^{p_1} dx)^{\frac{1}{p_1}} \leq C \left( \int_{\mathbb{R}^n_+} x_n^{\alpha_1p_1} \left| \partial_n^k \left( \frac{u(x)}{x_n^{m-k}} \right) \right|^{p_1} dx \right)^{\frac{1}{p_1}} + C \left( \int_{\mathbb{R}^n_+} x_n^{(\alpha_1+r)p_1} \left| \nabla \partial_n^k \left( \frac{u(x)}{x_n^{m-k}} \right) \right|^{p_1} dx \right)^{\frac{1}{p_1}},
\]

for some constant \(C\). Combining (2.24), (2.25), and (2.27), we have, owing to the first line in (2.23),

\[
\left( \int_{\mathbb{R}^n_+} x_n^{\beta q} \left| \partial_n^k \left( \frac{u(x)}{x_n^{m-k}} \right) \right|^{q} dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^n_+} x_n^{\alpha p} \left| \partial_n^k \left( \frac{u(x)}{x_n^{m-k}} \right) \right|^{p} dx \right)^{\frac{1}{p}} + C \left( \int_{\mathbb{R}^n_+} x_n^{(\alpha_1+1)p} \left| \nabla \partial_n^k \left( \frac{u(x)}{x_n^{m-k}} \right) \right|^{p} dx \right)^{\frac{1}{p}},
\]

namely (2.21). The proof is complete.

We are now in a position to accomplish the proof of Theorem 2.1.

**Proof of Theorem 2.1** Let \(p, q, \alpha, \beta\) be as in the statement. For \(i = 0, \cdots, m-k\), define \(p_i, \alpha_i\) as:

\[
\alpha_i = \frac{(m-k-i)\beta + i\alpha}{m-k}, \quad \frac{1}{p_i} = \frac{1}{m-k} \left( \frac{i}{p} + \frac{m-k-i}{q} \right).
\]

One can verify that

\[
1 \leq p_{i+1} \leq p_i < \infty, \quad \frac{1}{p_{i+1}} - \frac{1}{p_i} = \frac{\alpha_i - \alpha_{i+1}}{n}, \quad \alpha_i - \alpha_{i+1} = \frac{\beta - \alpha}{m-k} \leq 1,
\]

for every \(i = 0, \cdots, m-k\), and

\[
\alpha_0 = \beta, \quad \alpha_{m-k} = \alpha, \quad p_0 = q, \quad p_{m-k} = p.
\]
An iterated application of Lemma 2.5 with \( \alpha = \alpha_{i+1}, \beta = \alpha_i, \ q = p_i, \ p = p_{i+1} \), yields:

\[
(2.29) \quad \left( \int_{\mathbb{R}^n_+} x^{\beta q} \left| \nabla^k \left( \frac{u(x)}{x^{m-k}} \right) \right|^q \, dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^n_+} x^{(\alpha_1+1)p_1} \left| \nabla^{k+1} \left( \frac{u(x)}{x^{m-k}} \right) \right|^{p_1} \, dx \right)^{\frac{1}{p_1}} + \left( \int_{\mathbb{R}^n_+} x^{\alpha_1 p_1} \left| \nabla^k \left( \frac{u(x)}{x^{m-k}} \right) \right| \, dx \right)^{\frac{1}{p_1}}
\]

\[
\leq C \sum_{j=0}^{\frac{m-k}{2}} \left( \int_{\mathbb{R}^n_+} x^{(\alpha_j+1)p_{2j}} \left| \nabla^{k+j} \left( \frac{u(x)}{x^{m-k}} \right) \right|^{p_{2j}} \, dx \right)^{\frac{1}{p_{2j}}},
\]

for every \( u \in C^\infty_0 (\mathbb{R}^n_+) \). If we show that

\[
(2.30) \quad \left| \nabla^{k+j} \left( \frac{u(x)}{x^{m-k}} \right) \right| \leq C \sum_{i=k}^{k+j} \nabla^i \left( \frac{u(x)}{x^{m-i}} \right) \quad \text{for} \quad x \in \mathbb{R}^n_+,
\]

for some constant \( C > 0 \), and for any such function \( u \), then the conclusion will follow via Lemma 2.1 and inequality (2.29).

In order to prove (2.30), let us denote by \( \partial^i \) any partial derivative of order \( i \) with respect to the variables \( x_1, \ldots, x_{n-1} \). Then any derivative in \( \nabla^{k+j} \) can be written as \( \partial^{k+j-l} \partial^l \) for some \( l = 0, 1, \ldots, k+j \). Inequality (2.30) trivially holds if \( l = 0 \). Assume now that \( l \geq 1 \). Let us preliminarily observe that

\[
(2.31) \quad \partial^l \left( \frac{u(x)}{x^{m-k}} \right) = \frac{1}{x_n} \left[ \partial^l \left( \frac{u(x)}{x^{m-k-1}} \right) - l \partial^{l-1} \left( \frac{u(x)}{x^{m-k}} \right) \right] \quad \text{for} \quad x \in \mathbb{R}^n_+.
\]

Equation (2.31) can be verified by induction on \( l \). The case when \( l = 1 \) is easy, since

\[
\partial^1 \left( \frac{u(x)}{x^{m-k}} \right) = \frac{1}{x_n} \left[ \partial^1 \left( \frac{u(x)}{x^{m-k-1}} \right) - \frac{u(x)}{x^{m-k}} \right] \quad \text{for} \quad x \in \mathbb{R}^n_+.
\]

Assume next that (2.31) holds with \( l \) replaced by \( l - 1 \), for some \( l \geq 2 \). Computations show that

\[
\partial^l \left[ x_n \partial^{l-1}_{x_n} \left( \frac{u(x)}{x^{m-k}} \right) \right] = \partial^{l-1}_{x_n} \left( \frac{u(x)}{x^{m-k-1}} \right) + x_n \partial^l \left( \frac{u(x)}{x^{m-k}} \right) \quad \text{for} \quad x \in \mathbb{R}^n_+.
\]

On the other hand, the induction assumption ensures that

\[
\partial^l \left[ x_n \partial^{l-1}_{x_n} \left( \frac{u(x)}{x^{m-k}} \right) \right] = \partial^l \left( \frac{u(x)}{x^{m-k-1}} \right) - (l-1) \partial^{l-1}_{x_n} \left( \frac{u(x)}{x^{m-k}} \right) \quad \text{for} \quad x \in \mathbb{R}^n_+.
\]

Equation (2.31) is thus established for every \( l \geq 1 \).

Iterating equation (2.31) \( j \) times tells us that

\[
\partial^j \left( \frac{u(x)}{x^{m-k}} \right) = \frac{1}{x_n} \sum_{i=\max(0,l-j)}^{l} c_i \partial^i \left( \frac{u(x)}{x^{m-k+l-i}} \right) \quad \text{for} \quad x \in \mathbb{R}^n_+,
\]
for some constants $c_i \in \mathbb{R}$. Therefore,

$$
\left| \partial^{i+k-l} \partial_n \left( \frac{u(x)}{x_{m-k}} \right) \right| \leq C \frac{1}{x_n} \sum_{i=\max\{0,l-t\}}^j \left| \partial^{i+k-l} \partial_n \left( \frac{u(x)}{x_{m-k+l-i-j}} \right) \right|
$$

\begin{align*}
&\leq \frac{C}{x_n} \sum_{h=k}^{k+j} \left| \nabla^h \left( \frac{u(x)}{x_{m-h}} \right) \right| \quad \text{for } x \in \mathbb{R}^n,
\end{align*}

for some constant $C > 0$. This establishes inequality (2.30). \qed

3 Proof of the main result

Given a bounded smooth open set $\Omega$ with smooth boundary in $\mathbb{R}^n$, with $n \geq 1$, we make use of an orthogonal coordinate system which, in a sense, rectifies $\partial \Omega$ in a suitable neighborhood inside $\Omega$. By the latter expression, we mean a subset $\Omega_{\varepsilon}$ of $\Omega$ of the form

$$(3.1) \quad \Omega_{\varepsilon} = \{ x \in \Omega : d(x) < \varepsilon \}$$

for some $\varepsilon > 0$. Let $\varepsilon_0$ be small enough for $d$ to agree, in $\Omega_{\varepsilon_0}$, with the distance function from $\partial \Omega$. It is well known that $\varepsilon_0$ can be chosen so small that, for every $x \in \Omega_{\varepsilon_0}$, there exists a unique $y_x \in \partial \Omega$ fulfilling

$$(3.2) \quad x = y_x + d(x) \nu(y_x),$$

where $\nu$ denotes the inward unit normal vector on $\partial \Omega$.

Since $\partial \Omega$ is smooth, for every $x_0 \in \partial \Omega$ there exist an open neighborhood $U(x_0)$ of $x_0$ on $\partial \Omega$, a radius $r_0 > 0$, and a smooth diffeomorphism

$$\overline{\Phi} : B_{r_0}^{n-1}(0) \to U(x_0).$$

Next, define $\Phi : B_{r_0}^{n-1}(0) \times (0, \varepsilon_0) \to \mathbb{R}^n$ as

$$\Phi(y) = \overline{\Phi}(\overline{y}) + y_n \nu(\overline{\Phi}(\overline{y})) \quad \text{for } y \in B_{r_0}^{n-1}(0) \times (0, \varepsilon_0),$$

where $y = (y, y_n)$, and $\overline{y} = (y_1, \ldots, y_{n-1})$. By (3.2),

$$y_n = d(\Phi(y)) \quad \text{for } y \in B_{r_0}^{n-1}(0) \times (0, \varepsilon_0).$$

On setting

$$(3.4) \quad \mathcal{N}(x_0) = \{ x \in \Omega_{\varepsilon_0} : y_x \in U(x_0) \},$$

one can prove that the map $\Phi : B_{r_0}^{n-1}(0) \times (0, \varepsilon_0) \to \mathcal{N}(x_0)$ is a smooth diffeomorphism. In particular,

$$\mathcal{N}(x_0) = \Phi(B_{r_0}^{n-1}(0) \times (0, \varepsilon_0)).$$

As a consequence, there exists a positive constant $C$ such that

$$(3.5) \quad \frac{1}{C} \int_{B_{r_0}^{n-1}(0)} \int_0^{\varepsilon_0} |f(\Phi(y))| dy_n d\overline{y} \leq \int_{\mathcal{N}(x_0)} |f(x)| dx \leq C \int_{B_{r_0}^{n-1}(0)} \int_0^{\varepsilon_0} |f(\Phi(y))| dy_n d\overline{y}$$

for every function $f \in L^1(\mathcal{N}(x_0))$.

In preparation for the proof of Theorem 1.1, we establish the following local version.
Lemma 3.1. Let \( \Omega, p, q, r, m, k \) be as in the statement of Theorem 1.1. Given any point \( x_0 \in \partial \Omega \), let \( \mathcal{N}(x_0) \) be defined as in (3.4). Then there exists a constant \( C \) such that

\[
(3.6) \quad \left( \int_{\mathcal{N}(x_0)} d(x)^r \left| \nabla^k \left( \frac{u(x)}{d(x)^{m-k}} \right) \right|^q dx \right)^{\frac{1}{q}} \leq C \sum_{l=1}^{m} \left( \int_{\mathcal{N}(x_0)} d(x)^{p-1} |\nabla^l u|^p dx \right)^{\frac{1}{p}}
\]

for every \( u \in C_0^\infty(\mathcal{N}(x_0)) \).

Proof. Fix any function \( u \in C_0^\infty(\mathcal{N}(x_0)) \). Owing to inequality (3.5),

\[
(3.7) \quad \left( \int_{\mathcal{N}(x_0)} d(x)^r \left| \nabla^k \left( \frac{u(x)}{d(x)^{m-k}} \right) \right|^q dx \right)^{\frac{1}{q}} \leq C \left( \int_{B_{r_0}^{n-1}(0)} \int_0^{\varepsilon_0} y_n^r \left| \nabla^k \left( \frac{u(x)}{d(x)^{m-k}} \right) \right|_{x=\Phi(y)} \mid \nabla^l \left( \frac{v(y)}{\delta(y)^{m-k}} \right) \mid^{q} d\tilde{y}dy_n \right)^{\frac{1}{q}}
\]

for some constant \( C > 0 \). Define

\[
v(y) = u(\Phi(y)) \quad \text{and} \quad \delta(y) = d(\Phi(y)) \quad \text{for} \ y \in B_{r_0}^{n-1}(0) \times (0, \varepsilon_0).
\]

Observe that, by (3.5), \( \delta(y) = \gamma_n \) for \( y \in B_{r_0}^{n-1}(0) \times (0, \varepsilon_0) \). By the chain rule for derivatives,

\[
(3.8) \quad \left| \nabla^k \left( \frac{u(x)}{d(x)^{m-k}} \right) \right|_{x=\Phi(y)} \leq C \sum_{l=1}^{k} \left| \nabla^l \left( \frac{v(y)}{\delta(y)^{m-k}} \right) \right| \quad \text{for} \ y \in B_{r_0}^{n-1}(0) \times (0, \varepsilon_0),
\]

for some constant \( C \). Inequality (3.8) implies that

\[
(3.9) \quad \left( \int_{B_{r_0}^{n-1}(0)} \int_0^{\varepsilon_0} y_n^r \left| \nabla^k \left( \frac{u(x)}{d(x)^{m-k}} \right) \right|_{x=\Phi(y)} \mid \nabla^l \left( \frac{v(y)}{\delta(y)^{m-k}} \right) \mid^{q} d\tilde{y}dy_n \right)^{\frac{1}{q}} \leq C \sum_{l=1}^{k} \left( \int_{B_{r_0}^{n-1}(0)} \int_0^{\varepsilon_0} y_n^r \left| \nabla^l \left( \frac{v(y)}{\delta(y)^{m-k}} \right) \mid^{q} d\tilde{y}dy_n \right)^{\frac{1}{q}}
\]

It follows from Theorem 2.1, applied with \( m \) replaced with \( m - k + l \), and with \( k = l, \alpha = \frac{p-1}{p} \), and \( \beta = \frac{p}{q} - \frac{q}{q} + \frac{p-1}{p} \), that

\[
(3.10) \quad \left( \int_{B_{r_0}^{n-1}(0)} \int_0^{\varepsilon_0} y_n^r \left| \nabla^l \left( \frac{v(y)}{\delta(y)^{m-k}} \right) \mid^{q} d\tilde{y}dy_n \right)^{\frac{1}{q}} \leq C \left( \int_{B_{r_0}^{n-1}(0)} \int_0^{\varepsilon_0} y_n^{p-1} \left| \nabla^{m-k+l} v(y) \mid^{p} d\tilde{y}dy_n \right)^{\frac{1}{p}}
\]

for some positive constant \( C \). Observe that the first inequality holds owing to condition (1.6). The chain rule again ensures that

\[
|\nabla^{m-k+l} v(y)| \leq C \sum_{j=1}^{m-k+l} |\nabla^j u(x)|_{x=\Phi(y)} \quad \text{for} \ y \in B_{r_0}^{n-1}(0) \times (0, \varepsilon_0).
\]
Hence, by (3.5) and (3.3),
\[
\sum_{l=1}^{k} \left( \int_{B_{0}^{-1}(0)} \int_{0}^{\varepsilon_{0}} y_{n}^{-1} |\nabla^{m-k+l} v(y)|^{p} dy \right)^{\frac{1}{p}} \leq C \sum_{j=1}^{m} \left( \int_{\mathcal{N}(x_{0})} d(x)^{p-1} |\nabla^{j} u(x)|^{p} dx \right)^{\frac{1}{p}}
\]
for some constant C. The conclusion follows from inequalities (3.7), (3.9), (3.10), and (3.11).

\[\Box\]

**Proof of Theorem 1.1.** Without loss of generality, we can assume that \( u \in C_{0}^{\infty}(\Omega) \). Since \( \partial \Omega \) is compact and \( \{ \mathcal{U}(x) : x \in \partial \Omega \} \) is an open covering of \( \partial \Omega \), there exist \( N \in \mathbb{N} \) and \( \{ x_{l} \}_{l=1}^{N} \subset \partial \Omega \) such that \( \Omega_{\varepsilon_{0}} = \bigcup_{l=1}^{N} \mathcal{N}(x_{l}) \), where \( \Omega_{\varepsilon_{0}} \) and \( \mathcal{N}(x_{l}) \) are defined as in (3.1) and (3.4), and \( \varepsilon_{0} \) is chosen in such a way that (3.2) holds. Let \( \{ \phi_{l} \}_{l=0}^{N} \) be a partition of unity of functions \( \phi_{l} \subset C_{0}^{\infty}(\mathbb{R}^{n}) \) such that

(i) \( 0 \leq \phi_{l} \leq 1 \) for \( l = 0, \ldots, N \), and \( \sum_{l=0}^{N} \phi(x) = 1 \) for \( x \in \Omega \);

(ii) \( \text{supp} \phi_{l} \cap \Omega \subset \mathcal{N}(x_{l}) \) for \( l = 1, \ldots, N \);

(iii) \( \text{supp} \phi_{0} \subset \Omega \).

Set \( u_{l} = \phi_{l} u \). Then
\[
\int_{\Omega} d(x)^{r} \left| \nabla^{k} \left( \frac{u(x)}{d(x)^{m-k}} \right) \right|^{q} dx \leq C \sum_{l=1}^{N} \left( \int_{\mathcal{N}(x_{l})} d(x)^{r} \left| \nabla^{k} \left( \frac{u_{l}(x)}{d(x)^{m-k}} \right) \right|^{q} dx \right)^{\frac{1}{q}} + \left( \int_{\text{supp} \phi_{0}} d(x)^{r} \left| \nabla^{k} \left( \frac{u_{0}(x)}{d(x)^{m-k}} \right) \right|^{q} dx \right)^{\frac{1}{q}}.
\]

Observe that there exists a positive constant \( C \) such that \( C \leq d(x) \leq 1/C > 0 \), and \( |\nabla^{h} d(x)(x)| \leq C \) for \( h = 1, \ldots, m \) and for \( x \in \text{supp} \phi_{0} \). Thus, owing to assumption (1.5), the standard Sobolev inequality ensures that
\[
\int_{\text{supp} \phi_{0}} d(x)^{r} \left| \nabla^{k} \left( \frac{u_{0}(x)}{d(x)^{m-k}} \right) \right|^{q} dx \leq C \sum_{j=0}^{m} \left( \int_{\text{supp} \phi_{0}} d(x)^{p-1} |\nabla^{j} u_{0}|^{p} dx \right)^{\frac{1}{p}}
\]
for some constant C. On the other hand, Lemma 3.1 tells us that
\[
\int_{\mathcal{N}(x_{l})} d(x)^{r} \left| \nabla^{k} \left( \frac{u_{l}(x)}{d(x)^{m-k}} \right) \right|^{q} dx \leq C \sum_{j=1}^{m} \left( \int_{\mathcal{N}(x_{l})} d(x)^{p-1} |\nabla^{j} u_{l}|^{p} dx \right)^{\frac{1}{p}}
\]
for \( l = 1, \ldots, N \). Inequalities (3.12), (3.13), and (3.14) yield
\[
\int_{\Omega} d(x)^{r} \left| \nabla^{k} \left( \frac{u(x)}{d(x)^{m-k}} \right) \right|^{q} dx \leq C \sum_{l=1}^{N} \sum_{j=1}^{m} \left( \int_{\mathcal{N}(x_{l})} d(x)^{p-1} |\nabla^{j} u_{l}|^{p} dx \right)^{\frac{1}{p}} + C \sum_{j=0}^{m} \left( \int_{\text{supp} \phi_{0}} d(x)^{p-1} |\nabla^{j} u_{0}|^{p} dx \right)^{\frac{1}{p}} + C' \sum_{j=0}^{m} \left( \int_{\Omega} d(x)^{p-1} |\nabla^{j} u|^{p} dx \right)^{\frac{1}{p}}.
\]
It remains to show that there exists a constant $C$ such that
\[
(3.16) \quad \left( \int_\Omega d(x)^r \left| \nabla h \left( \frac{u(x)}{d(x)^{m-k}} \right) \right|^q \, dx \right)^{\frac{1}{q}} \leq C \sum_{j=0}^m \left( \int_\Omega d(x)^{p-1} |\nabla_j u|^p \, dx \right)^{\frac{1}{p}}
\]
for $h = 0, \ldots, k - 1$ as well. To this purpose, we apply a Hardy-Sobolev inequality from [Ho1, Theorem 3], which tells us what follows. Assume that $1 \leq p \leq q < \infty$ and $\alpha, \beta \in \mathbb{R}$ fulfil the conditions:
\[
(3.17) \quad -1/q < \beta \leq \alpha, \quad m - k + 1 - \alpha + \beta < \frac{n}{p} \leq m - k + 1 - \alpha + \beta + \frac{n}{q}.
\]
Then there exists a constant $C > 0$ such that
\[
(3.18) \quad \|u\|_{W^{k-1,q}(\Omega,d^{\beta q})} \leq C \|u\|_{W^{m,p}(\Omega,d^{\alpha p})}
\]
for every $u \in W^{m,p}(\Omega,d^{\alpha p})$. Choose
\[
\beta = \frac{r}{q},
\]
and $\alpha$ such that
\[
(3.19) \quad \max \left\{ \frac{p-1}{p} + m - k, \frac{r}{q} + m - k + 1, \frac{r}{q} \right\} < \alpha \leq m - k + 1 + \frac{r}{q} - \frac{n}{p} + \frac{n}{q}, \quad \alpha \notin \mathbb{N} - \{1/p\}.
\]
Conditions (3.17) are fulfilled with such a choice of $\alpha$ and $\beta$, which is possible thanks to assumptions (1.5) and (1.6). Consequently,
\[
\left\| \frac{u}{d^{m-k}} \right\|_{W^{k-1,q}(\Omega,d^{\beta q})} \leq C \left\| \frac{u}{d^{m-k}} \right\|_{W^{m,p}(\Omega,d^{\alpha p})} \leq C \sum_{j=0}^m \sum_{i=0}^j \left\| \nabla_i u \right\|_{d^{m-k+j-i}} \leq C \sum_{j=0}^m \sum_{i=0}^j \left\| \nabla_i u \right\|_{L^p(\Omega,d^{\alpha p})}
\]
for every $u \in C_0^\infty(\Omega)$. Since $\alpha \notin \mathbb{N} - \frac{1}{p}$, we have that $p(\alpha - (m - k + j - i)) \neq -1$ if $0 \leq j \leq m$, $0 \leq i \leq j$. Hence, by a standard Hardy-Sobolev embedding [Ku, Equation (8.37)],
\[
\sum_{i=0}^j \left\| \nabla_i u \right\|_{L^p(\Omega,d^{\alpha p})} \leq C \sum_{i=0}^j \left\| \nabla_i u \right\|_{L^p(\Omega,d^{\alpha p})}
\]
for some constant $C > 0$, and for every function $u \in C_0^\infty(\Omega)$. Inasmuch as the function $d$ is bounded in $\Omega$ and, by (3.19), $\frac{p-1}{p} + m - k < \alpha$, one hence deduces that
\[
\left\| \nabla_j u \right\|_{L^p(\Omega,d^{\alpha p})} \leq C \left\| \nabla_j u \right\|_{L^p(\Omega,d^{\alpha - 1})}.
\]
Altogether, inequality (3.16) follows. The proof is complete. \qed

We conclude by demonstrating the sharpness of Theorem 1.1. Let us begin with condition (1.5).

**Proposition 3.2.** Let $n, k, m \in \mathbb{N}$, $n, m \geq 2$, and $1 \leq k \leq m - 1$. Assume that $1 \leq p \leq q < \infty$ and
\[
(3.20) \quad \frac{1}{q} < \frac{n - p(m - k)}{np}.
\]
Then inequality (1.7) fails in any (smooth) bounded open set $\Omega \subset \mathbb{R}^n$. 

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\textbf{Proof.} We may assume, without loss of generality, that $0 \in \Omega$. Suppose that inequality (1.7) holds for every $u \in C_0^{\infty}(\Omega)$, with $m,k,p,q,r$ as in the statement. Fix any such function $u$, and, for $\lambda > 1$, consider the function $u_\lambda$ in $\Omega$ defined as

\begin{equation}
(3.21) \quad u_\lambda(x) = u(\lambda x) \quad \text{for} \; x \in \Omega.
\end{equation}

Hence, in particular,

\begin{equation}
(3.22) \quad \left( \int_{\Omega} \left| \nabla^k \frac{u_\lambda(x)}{d(x)^{m-k}} \right|^q dx \right)^{\frac{1}{q}} \leq C \sum_{j=0}^{m} \left( \int_{\Omega} \left| \nabla^j u(x) \right|^p dx \right)^{\frac{1}{p}}
\end{equation}

for $\lambda > 1$. Inequality (3.22) can be rewritten, via a change of variable, as

\begin{equation}
(3.23) \quad \lambda^{k-n} \left( \int_{\Omega} \left| \nabla^k \left( \frac{u(x)}{d(x/\lambda)^{m-k}} \right) \right|^q dx \right)^{\frac{1}{q}} \leq C \sum_{j=0}^{m} \lambda^{j-n} \left( \int_{\Omega} \left| \nabla^j u(x) \right|^p dx \right)^{\frac{1}{p}}
\end{equation}

for $\lambda > 1$. Since $\lim_{\lambda \to \infty} d(x/\lambda) = d(0)$ and $\lim_{\lambda \to \infty} \nabla^j(d(x/\lambda)) = 0$ for $j = 1, \ldots, m$, uniformly in $x \in \Omega$, all the integrals in (3.23) converge to a finite limit as $\lambda \to \infty$. Hence, passing to the limit as $\lambda \to \infty$ in (3.23) leads to a contradiction, inasmuch as inequality (3.20) is in force. \hfill \Box

The optimality of assumption (1.6) in Theorem 1.1 is the object of our last result.

**Proposition 3.3.** Let $n,k,m \in \mathbb{N}$, $m \geq 2$, and $1 \leq k \leq m-1$. Assume that $1 \leq p \leq q < \infty$, and

\begin{equation}
(3.24) \quad r < \frac{q}{p}(n-1) - n + q.
\end{equation}

Then there exists a smooth bounded open set $\Omega \subset \mathbb{R}^n$ such that inequality (1.7) fails.

**Proof.** Assume first that $n \geq 2$. Let $B_{r^{-1}}^n(0)$ denote the ball in $\mathbb{R}^{n-1}$, centered at 0, with radius $\rho$. Let $\Omega$ be a smooth bounded open set in $\mathbb{R}^n$ such that $B_{r^{-1}}^n(0) \times (0,1) \subset \Omega$, $B_{r^{-1}}^n(0) \times \{0\} \subset \partial \Omega$, and

\[ d(x) = x_n \quad \text{for} \; x \in B_{r^{-1}}^n(0) \times (0,1). \]

For instance, the set $\Omega$ can just be obtained by smoothing the cylinder $B_{r^{-1}}^n(0) \times (0,2)$. Let $u$ be any smooth function in $\mathbb{R}^n$, compactly supported in $B_{r^{-1}}^n(0) \times (0,1)$. Hence, in particular, (the restriction of) $u$ to $\Omega$ belongs to $C_0^{\infty}(\Omega)$. Given any $\lambda > 1$, consider the function $u_\lambda : \Omega \to \mathbb{R}$ defined as in (3.21). Since we are assuming that $\lambda > 1$, we have that $u_\lambda \in C_0^{\infty}(B_{r^{-1}}^n(0) \times (0,1)) \subset C_0^{\infty}(\Omega)$. Suppose that inequality (1.7) holds for every $u \in C_0^{\infty}(\Omega)$, with $m,k,p,q,r$ as in the statement. The choice of $u_\lambda$ as a trial function in (1.7) implies that

\begin{equation}
(3.25) \quad \left( \int_{B_{r^{-1}}^n(0) \times (0,1)} x_n^r \left| \frac{\partial^k}{\partial x_n^k} \left( \frac{u_\lambda(x)}{x_n^{m-k}} \right) \right|^q dx \right)^{\frac{1}{q}} \leq C \sum_{j=0}^{m} \left( \int_{B_{r^{-1}}^n(0) \times (0,1)} x_n^{p-1} \left| \nabla^j u_\lambda(x) \right|^p dx \right)^{\frac{1}{p}}
\end{equation}

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for $\lambda > 1$. Hence, via a change of variable,

$$
\lambda^{-\frac{r}{q} + \frac{m-n}{q}} \left( \int_{B_1^{n-1}(0) \times (0,1)} x_n^r \left| \frac{\partial^k u(x)}{\partial x^k_n} \right|^q \frac{1}{x_n^m} \right)^{\frac{1}{q}}
$$

$$
\leq C \sum_{j=0}^m \lambda^{j-1+\frac{1}{p}-\frac{n}{p}} \left( \int_{B_1^{n-1}(0) \times (0,1)} x_n^{p-1} \left| \nabla^j u(x) \right|^p \right)^{\frac{1}{p}}
$$

for $\lambda > 1$. Letting $\lambda \to \infty$ in (3.26) yields a contradiction, under assumption (3.24). When $n = 1$, the same argument applies on replacing $B_1^{n-1}(0) \times (0,1) \subset \mathbb{R}^n$ with the interval $(0,1)$. 

**Acknowledgments.** The authors are grateful to Yoshinori Yamasaki for some helpful discussions.

This research was initiated during a visit of the second named author at the Department of Mathematics and Informatics “U.Dini” of the University of Florence, in the fall-winter semester 2013-2014. He wishes to thank the members of the Department for their kind hospitality.

This work was partly funded by: Research project of MIUR (Italian Ministry of Education, University and Research) Prin 2012 “Elliptic and parabolic partial differential equations: geometric aspects, related inequalities, and applications” (grant number 2012TC7588); GNAMPA of the Italian INdAM (National Institute of High Mathematics); JSPS KAKENHI (grant number 25220702).

**References**

[Ad1] D.R. Adams, Traces of potentials arising from translation invariant operators, *Ann. Sc. Norm. Super. Pisa* 25 (1971), 203–217.

[Ad2] D.R. Adams, A trace inequality for generalized potentials, *Studia Math.* 48 (1973), 99–105.

[CW] H. Castro & H. Wang, A Hardy type inequality for $W^{m,1}(0,1)$ functions, *Calc. Var.* 34 (2010), 525–531.

[CDW1] H. Castro, J. Dávila & H. Wang, A Hardy type inequality for $W^{2,1}_0(\Omega)$ functions, *C. R. Math. Acad. Sci. Paris* 349 (2011), 765–767.

[CDW2] H. Castro, J. Dávila & H. Wang, A Hardy type inequality for $W^{m,1}_0(\Omega)$ functions, *J. Eur. Math. Soc.* 15 (2013), 145–155.

[CEG] A. Cianchi, D.E. Edmunds & P.Gurka, On weighted Poincaré inequalities, *Math. Nachr.* 180 (1996), 15-41.

[HKM] J. Heinonen, T. Kilpeläinen & O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford University Press, New York, 1993.

[Ku] A. Kufner, *Weighted Sobolev spaces*, Teubner-Texte zur Mathematik, Leipzig, 1980.

[Ho1] T. Horiuchi, The imbedding theorems for weighted Sobolev spaces, *J. Math. Kyoto Univ.* 29 (1989), 365–403.
[Ho2] T. Horiuchi, The imbedding theorems for weighted Sobolev spaces. II, *Bull. Fac. Sci. Ibaraki Univ. Ser. A* 23 (1991), 11–37.

[Ma] V.G. Maz’ya, *Sobolev spaces with applications to elliptic partial differential equations*, Springer, Berlin, 2011.

[MS] M.K.V. Murthy & G. Stampacchia, Boundary value problems for some degenerate-elliptic operators, *Ann. Mat. Pura Appl.* 80 (1968), 1–122.

[OK] B. Opic & A. Kufner, *Hardy-type inequalities*, Longman Scientific & Technical, New York, 1990.