Some aspects of the cosmological conformal equivalence between “Jordan Frame” and “Einstein Frame”

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Abstract

The conformal equivalence between Jordan frame and Einstein frame can be used in order to search for exact solutions in general theories of gravity in which scalar fields are minimally or nonminimally coupled with geometry. In the cosmological arena a relevant role is played by the time parameter in which dynamics is described. In this paper we discuss such issues considering also if cosmological Noether symmetries in the “point–like” Lagrangian are conformally preserved.

Through this analysis and through also a careful analysis of the cosmological parameters Ω and Λ, it is possible to contribute to the discussion on which is the physical system.

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1 Introduction

Alternative theories of gravity has been formulated and investigated in different context. The Brans–Dicke approach [1], closely related to the Jordan approach [2], has been carried on in the context of a generalization of the Mach’s principle; in that approach the Einstein theory of gravitation is modified by introducing a scalar field with a non standard coupling with gravity, i.e. the gravitational coupling turns out to be no longer constant. Later on more general couplings have been considered, and the compatibility of such approaches with the different formulations of the Equivalence Principle have been considered [3] [4] [5] [6] [7].

A generalization of the standard gravity comes also from quantum field theories on curved space–times; in such context we find the so called higher order gravitational theories [8] [9] [10].

In all these approaches, the problem of reducing both these two kinds of more general theories in Einstein standard form has been extensively treated; one can see that, through a “Legendre” transformation on the metric, higher order theories, under suitable regularity conditions on the Lagrangian, take the form of the Einstein one in which a scalar field (or more than one) is the source of the gravitational field (see for example [8] [12] [14]); on the other side, it has been studied the equivalence between models with $G$–variable with the Einstein standard gravity through a suitable conformal transformation (see [8] [11]).

In this paper we analyse, through an appropriately defined conformal transformation, the problem of the equivalence between the non minimally coupled (NMC) theories and the Einstein gravity for scalar–tensor theories in absence of ordinary matter. First, we will do it in the general context and then in the cosmological case, that is, we will study the conformal invariance with the hypotheses of homogeneity and isotropy. In such case we also consider the case in which ordinary matter is present beside the scalar field and we do some consideration on the problem of which is the “physical system” between the two conformally equivalent systems [8] [12] [13].

Furthermore, we analyse the relation between the conformal equivalence and the existence of a Noether symmetry in the $(a, \phi)$–space seen as configuration space (i.e. in the minisuperspace), where the cosmological “point–like” Lagrangian is defined (we will better clarify the meaning of such expression in our forthcoming considerations); of course such Lagrangian density comes from the general field Lagrangian density once homogeneity and isotropy are assumed [15].

We conclude discussing some examples of physical interest.
2 Conformally equivalent theories

In four dimension, the most general action involving gravity nonminimally coupled with one scalar field is

\[ A = \int d^4x \sqrt{-g} \left[ F(\phi)R + \frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi) \right] \] (1)

where \( R \) is the Ricci scalar, \( V(\phi) \) and \( F(\phi) \) are generic functions describing respectively the potential for the field \( \phi \) and the coupling of \( \phi \) with the gravity; the metric signature is \((+ - - -)\). We use Planck units.

The variation with respect to \( g_{\mu\nu} \) gives rise to the field equations

\[ F(\phi)G_{\mu\nu} = -\frac{1}{2}T_{\mu\nu} - g_{\mu\nu}\Box_{\Gamma}F(\phi) + F(\phi)g_{\mu\nu} \] (2)

which are the generalized Einstein equations; here \( \Box_{\Gamma} \) is the d’Alembert operator with respect to the connection \( \Gamma \);

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \] (3)

is the Einstein tensor, and

\[ T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{\alpha} + g_{\mu\nu}V(\phi) \] (4)

is the energy–momentum tensor relative to the scalar field.

The variation with respect to \( \phi \) provides the Klein–Gordon equation

\[ \Box_{\Gamma}\phi - RF_{\phi}(\phi) + V_{\phi}(\phi) = 0 \] (5)

where \( F_{\phi} = dF(\phi)/d\phi \), \( V_{\phi} = dV(\phi)/d\phi \). This last equation is equivalent to the Bianchi contracted identity (see [15]).

Let us consider now a conformal transformation on the metric \( g_{\mu\nu} \) [16], that is

\[ \bar{g}_{\mu\nu} = e^{2\omega}g_{\mu\nu} \] (6)

in which \( e^{2\omega} \) is the conformal factor. Under this transformation the connection, the Riemann and Ricci tensors, and the Ricci scalar transform in the corresponding way [16], so that the Lagrangian density in (1) becomes

\[ \sqrt{-g}(FR + \frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V) = \sqrt{-\bar{g}}e^{-2\omega}(F\bar{R} - 6F\Box_{\bar{\Gamma}}\omega + \]

\[ -6F\omega_{;\alpha}\omega^{;\alpha} + \frac{1}{2}\bar{g}^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - e^{-2\omega}V) \] (7)

in which \( \bar{R} \), \( \bar{\Gamma} \) and \( \Box_{\bar{\Gamma}} \) are respectively the Ricci scalar and the connection relative to the metric \( \bar{g}_{\mu\nu} \), and the d’Alembert operator relative to the connection \( \bar{\Gamma} \). If we require the theory in the metric \( \bar{g}_{\mu\nu} \) to appear as a standard Einstein theory, we get at once that the
conformal factor has to be related to $F$, that is (see also [4] for a review of particular cases)

$$e^{2\omega} = -2F.$$  \hspace{1cm} (8)

We see that $F$ must be negative. Using this relation, the Lagrangian density (7) becomes

$$\sqrt{-g} \left( FR + \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V \right) = \sqrt{-g} \left( -\frac{1}{2} \bar{R} + 3 \Box \omega + \frac{3F_{,\phi}^2 - F}{4F^2} \phi_{,\alpha} \phi^{,\alpha} - \frac{V}{4F^2} \right),$$  \hspace{1cm} (9)

Introducing a new scalar field $\bar{\phi}$ and the potential $\bar{V}$, respectively, defined by

$$\bar{\phi}_{,\alpha} = \sqrt{\frac{3F_{,\phi}^2 - F}{2F^2}} \phi_{,\alpha}, \quad \bar{V}(\bar{\phi}(\phi)) = \frac{V(\phi)}{4F(\phi)}$$  \hspace{1cm} (10)

we get

$$\sqrt{-g} \left( FR + \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V \right) = \sqrt{-g} \left( -\frac{1}{2} \bar{R} + \frac{1}{2} \bar{\phi}_{,\alpha} \bar{\phi}^{,\alpha} - \bar{V} \right),$$  \hspace{1cm} (11)

which is the usual Einstein–Hilbert Lagrangian density plus the standard Lagrangian density relative to the scalar field $\bar{\phi}$ (see [13]). (We have not considered the divergence–type term appearing in the Lagrangian (11); we will return on this point in our forthcoming considerations). Therefore, any nonminimally coupled theory, in absence of ordinary matter, is conformally equivalent to an Einstein theory, being the conformal transformation and the potential opportunely defined by (8) and (10) (see also [5]). The converse is also true: for a given $F(\phi)$, such that $3F_{,\phi}^2 - F > 0$, we can transform a standard Einstein theory into a NMC theory. This means that, in principle, if we are able to solve the field equations in the framework of the Einstein theory in presence of a scalar field with a given potential, we should be able to get the solutions for the class of nonminimally coupled theories, assigned by the coupling $F(\phi)$, via the conformal transformation defined by the (8) the only constraint being the second of (10). This is exactly what we are going to discuss in the cosmological context in cases in which the potentials as well as the couplings are relevant from the point of view of the fundamental physics.

Following the standard terminology, we denote here as “Einstein frame” the framework of the Einstein theory, also indicated as minimally coupled theory and as “Jordan frame” the framework of the nonminimally coupled theory [4].

There are some remarks to do with respect to (9) and (10): first we want to stress that the “new” scalar field as defined in (10) is given in differential form in terms of the “old” one and its integration can be not trivial; the second remark concerns the divergence appearing in (9). The transformed Lagrangian density obtained from (7) imposing (8) contains a divergence term, in which appears not only the metric but also its derivative, through the connection $\bar{\Gamma}$. Therefore the equivalence of this total Lagrangian density to the Einstein–Hilbert Lagrangian density plus scalar field is not trivial. To check that they are actually equivalent, let us perform the conformal transformation (4) on the field...
Eqs. (2), obtaining
\[
\bar{G}_{\mu\nu} = \left( -\frac{1}{2} - \frac{F_{\phi\phi}}{F} + \frac{2\omega\phi F_{\phi}}{F} - 2\omega_{\phi}\right) \phi_{\mu\nu} + \\
+ \left( \frac{1}{4F} - \frac{F_{\phi\phi}}{F} + \frac{\omega_0 F_{\phi}}{F} - \omega_0^2 + 2\omega_{\phi}\right) \bar{g}_{\mu\nu} \phi_{,\alpha} \phi_{,\alpha}^a + \left( -\frac{F_{\phi}}{F} + 2\omega_{\phi}\right) \bar{g}_{\mu\nu} \Box \phi + \right.
\]
\[
+ \left( \frac{F_{\phi}}{F} - 2\omega_{\phi}\right) (\nabla_{\Gamma})_\mu (\nabla_{\Gamma})_\nu \phi - \frac{1}{2F} e^{-2\omega} \bar{g}_{\mu\nu} V \quad (12)
\]
in which \((\nabla_{\Gamma})_\mu\) is the covariant derivative with respect to \(x^\mu\) relative to the connection \(\Gamma\). We see, from (12), that if \(\omega\) satisfies the relation
\[
\frac{F_{\phi}}{F} - 2\omega_{\phi} = 0 \quad (13)
\]
Eqs. (12) can be rewritten as
\[
\bar{G}_{\mu\nu} = \frac{3F_{\phi}}{2F^2} \phi_{,\mu} \phi_{,\nu} - \bar{g}_{\mu\nu} \frac{3F_{\phi}}{2F^2} \phi_{,\alpha} \phi_{,\alpha}^a - \bar{g}_{\mu\nu} \frac{e^{-2\omega}}{2F} V \quad (14)
\]
Then, performing on \(\phi\) the transformation given by (10) and on \(V\) the transformation
\[
W(\tilde{\phi}(\phi)) = -\frac{e^{-2\omega(F)}}{2F} V \quad (15)
\]
in which \(\omega(F)\) satisfies (13), Eq. (14) becomes
\[
\bar{G}_{\mu\nu} = \tilde{\phi}_{,\mu} \tilde{\phi}_{,\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \tilde{\phi}_{,\alpha} \tilde{\phi}_{,\alpha}^a - \bar{g}_{\mu\nu} W, \quad (16)
\]
which correspond to the Einstein equations in presence of a scalar field \(\tilde{\phi}\) with potential \(W\). The expression for \(\omega(F)\) is easily obtained from (13), that is
\[
\omega = \frac{1}{2} \ln F + \omega_0 \quad (17)
\]
in which \(\omega_0\) is the integration constant. The potential \(W\) takes the form
\[
W = -\frac{V}{2\alpha F}. \quad (18)
\]
Comparing (18) with the second of (10), we see that, fixing \(\alpha = -2\), the definition of \(W\) coincides with that one of \(\tilde{V}\). We have then the full compatibility with the Lagrangian approach obtaining for \(\omega\) the same relation as (8); in this sense we have verified the full equivalence between the NMC and the Einstein–Hilbert Lagrangian density plus scalar field.

Our final remark regards the relations (10): actually, from (9) the relation between \(\phi_{,\alpha}\) and \(\bar{\phi}_{,\alpha}\) present a \(\pm\) sign in front of the square root, which corresponds to have the same or opposite sign in the derivative of \(\phi\) and \(\bar{\phi}\) with respect to \(x^\alpha\). What follows is independent of such sign; we will choose then the positive one.
The cosmological case

Let us assume now that the spacetime manifold is described by a FRW metric, that is we consider homogeneous and isotropic cosmology. Then the Ricci scalar, has the expression $R = -6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} \right)$, in which the dot means the derivative with respect to time and $\kappa$ is the curvature constant. The Lagrangian density (1) takes the form

$$L_t = 6F(\phi)a\dot{a}^2 + 6F(\phi)a^2\dot{\phi} - 6F(\phi)a\kappa + \frac{1}{2}a^3\dot{\phi}^2 - a^3V(\phi).$$

Expression (19) can be seen as a “point–like” Lagrangian on the configuration space $(a, \phi$) (in this way the meaning of the expression we used in the introduction is clarified). With the subscript $t$, we mean that the time–coordinate considered is the universal time $t$: this remark is important for the forthcoming discussion. The Euler–Lagrange equations relative to (19) are then

$$\begin{cases}
\frac{2a\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{2F(\phi)a\dot{\phi}}{F} + \frac{F(\phi)\ddot{\phi}}{F} + \frac{\kappa}{a^2} + \frac{F(\phi)a\kappa}{F} + \frac{V}{2F} = 0 \\
\dot{\phi} + \frac{3a\dot{\phi}}{a} + \frac{6F(\phi)a^2}{a^2} + \frac{6F(\phi)a\dot{\phi}}{a^2} + \frac{6F(\phi)a^2\kappa}{a^2} + V = 0
\end{cases}$$

which correspond to the (generalized) second order Einstein equation and the Klein–Gordon equation in the FRW case. The energy function relative to (19) is

$$E_t = \frac{\partial L_t}{\partial \dot{a}} \dot{a} + \frac{\partial L_t}{\partial \dot{\phi}} \dot{\phi} - L_t =$$

$$= 6Fa\dot{a}^2 + 6F(\phi)a^2\dot{\phi} + 6F(\phi)a\kappa + \frac{1}{2}a^3\dot{\phi}^2 + a^3V,$$

and we see that the first order generalized Einstein equation is equivalent to

$$E_t = 0.$$
transformation from the Jordan frame to the Einstein frame in the cosmological case is given by

\[
\begin{align*}
\ddot{a} &= \sqrt{-2F(\phi)}a \\
\frac{d\dot{\phi}}{dt} &= \sqrt{\frac{3F_\phi^2 - F}{2F^2}} \frac{d\phi}{dt} \\
\ddot{\bar{t}} &= \sqrt{-2F(\phi)} dt.
\end{align*}
\] (23)

From the first and the third of (23) we have that, on the Jordan–frame solutions \(a(t), \phi(t)\), we obtain \(\bar{a}\) as a function of \(\bar{t}\) only; indeed the important thing is the fact that the equations for \(\bar{a}\) we will obtain are the standard Einstein equations. The second of (23) corresponds to relation (10) under the given assumption of homogeneity and isotropy.

Under transformation (23) we have that

\[
\frac{1}{\sqrt{-2F}} L_t = \frac{1}{\sqrt{-2F}} \left( 6F a \ddot{a}^2 + 6F_\phi a^2 \dot{a} \dot{\phi} - 6F a \kappa + \frac{1}{2} a^3 \dot{\phi}^2 - a^3 V \right) =
\] (24)

\[
= -3\dot{a} \ddot{a}^2 + 3\kappa \dot{a} + \frac{1}{2} a^3 \dot{\phi}^2 - \dot{a}^3 \bar{V}(\bar{\phi}) = \bar{L}_t
\]

in which the dot over barred quantities means the derivative with respect to \(\bar{t}\); \(L_t\) is given by (19) and \(\bar{L}_t\) coincides with the “point–like” Lagrangian obtained from the Einstein–Hilbert action plus a scalar field under the assumption of homogeneity and isotropy. In this way the invariance of the homogeneous and isotropic action under (23) is insured, being \(L_t\) and \(\bar{L}_t\) connected by the (24). The same correspondence as (24) exists between the energy function \(E_t\) and \(\bar{E}_t\), that is, there is correspondence between the two first order Einstein equations in the two frames. We focus now our attention on the way the Euler–Lagrange equations transform under (23). The Euler–Lagrange equations relative to (24) are the usual second order Einstein equation and Klein–Gordon equation

\[
\begin{align*}
\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} + \frac{1}{2} \frac{\dot{\phi}^2}{2} - \bar{V} &= 0 \\
\frac{\dot{\phi}}{a} + \frac{3\dot{a} \dot{\phi}}{a^2} + \bar{V}_\phi &= 0.
\end{align*}
\] (25)

Under (23) it is straighforward to verify that they become

\[
\begin{align*}
\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{2F_\phi \dot{a} \dot{\phi}}{Fa} + \frac{F_\phi \dot{\phi}}{F} + \frac{\kappa}{a^2} + \frac{F_{\phi\phi} \dot{\phi}^2}{F} - \frac{\dot{\phi}^2}{4F} + \frac{V}{2F} &= 0 \\
\dot{\phi} + \frac{3\dot{a} \dot{\phi}}{a} + \frac{6F_\phi F_{\phi\phi} - F_{\phi} \dot{\phi}^2}{3F_\phi^2 - F} + \frac{2F_\phi V}{3F_\phi^2 - F} - \frac{FV_\phi}{3F_\phi^2 - F} &= 0
\end{align*}
\] (26)
which do not coincide with the Euler–Lagrange equations given by (20). Using the first of (20), the second of (26) can be written as

\[
\frac{F - 3F_\phi^2}{F} \ddot{\phi} + \frac{3(F - 3F_\phi^2)}{F} \frac{\dot{a}}{a} \dot{\phi} + \frac{F_\phi - 6F_\phi F_\phi}{F} \frac{\dot{\phi}^2}{2} + \frac{F_\phi \dot{\phi}^2}{4F} - \frac{2F_\phi V}{F} + V_\phi + \\
\frac{3F_\phi \dot{a}^2}{a^2} + \frac{3F_\phi \kappa}{a^2} + \frac{3F_\phi^2 \dot{a} \dot{\phi}}{a} = 0
\]

which becomes, taking into account (21)

\[
\frac{F - 3F_\phi^2}{F} \ddot{\phi} + \frac{3(F - 3F_\phi^2)}{F} \frac{\dot{a}}{a} \dot{\phi} + \frac{1}{F} \frac{d}{d\phi} \left( F - 3F_\phi^2 \right) \frac{\dot{\phi}^2}{2} + \frac{F_\phi \dot{\phi}^2}{4F} - \frac{2F_\phi V}{F} + V_\phi + \\
\frac{F_\phi}{2a^3} E_\ell = 0.
\]

Comparing (28) with the second of (26) we see that they coincide if \( F - 3F_\phi^2 \neq 0 \) and \( E_\ell = 0 \). The quantity \( F - 3F_\phi^2 \) is proportional to the Hessian determinant of \( L_t \) with respect to \( (\dot{a}, \dot{\phi}) \); we want this Hessian different from zero in order to avoid pathologies in the dynamics [17], while \( E_\ell = 0 \) corresponds to the first order Einstein equation. It seems that, under the assumption of homogeneity and isotropy and the request of having the metric expressed in the universal time in both the Einstein and Jordan frame, we have conformal equivalence between the Euler–Lagrange Eqs. (20) and (25) only on the (cosmological) solutions. Actually, if we look more carefully to this problem, we notice that, making the hypotheses of homogeneity and isotropy on the field Eqs. (2) and (5), we get the generalized Einstein equations of first and second order, and the Klein–Gordon equation. On the other side, the Euler–Lagrange equations relative to (19) are just the second order Einstein equation and the Klein–Gordon equation, whereas the first order Einstein equation is obtained from \( E_\ell = 0 \). Of course the same happens in the Einstein frame. Therefore it is natural to expect that the full conformal equivalence in the “point–like” formulation is verified taking into consideration \( E_\ell = 0 \).

It is possible to see more clearly at the problem of the cosmological conformal equivalence, formulated in the context of the “point–like” Lagrangian, if we use as time–coordinate the conformal time \( \eta \), connected to the universal time \( t \) by the usual relation

\[
a^2(\eta) d\eta^2 = dt^2.
\]

We can see that the use of \( \eta \) makes much easier the treatment of all the problems we have discussed till now.

The crucial point is the following: given the form of the FRW line element expressed in conformal time \( \eta \) one does not face the problem of redefining time after performing a conformal transformation, since in this case, the expansion parameter appears in front of
all the terms of the line element. From this point of view, the conformal transformation which connects Einstein and Jordan frame is given by

\[
\begin{align*}
\bar{a} &= \sqrt{-2F(\phi)a} \\
\frac{d\bar{\phi}}{d\eta} &= \sqrt{\frac{3F_\phi^2 - F}{2F^2}} \frac{d\phi}{d\eta}
\end{align*}
\]

(30)

where \(a, \phi, \bar{a}, \bar{\phi}\) are assumed as functions of \(\eta\).

The Einstein–Hilbert “point–like” Lagrangian is given by

\[
\mathcal{L}_\eta = -3\bar{a}'^2 + 3\kappa a^2 + \frac{1}{2}a^2 \bar{\phi}'^2 - a^4 V(\bar{\phi})
\]

in which the prime means the derivative with respect to \(\eta\), and the subscript \(\eta\) means that the time-coordinate considered is the conformal time. Under transformation (30), it becomes

\[
\mathcal{L}_\eta = -3\bar{a}'^2 + 3\kappa a^2 + \frac{1}{2}a^2 \bar{\phi}'^2 - a^4 V(\bar{\phi}) = 6F(\phi)a' + 6F_\phi(\phi)aa' \phi' - 6F(\phi)\kappa a^2 + \frac{1}{2}a^2 \phi'^2 - a^4 V(\phi) = L_\eta
\]

(32)

which corresponds to the “point–like” Lagrangian obtained from the Lagrangian density in (1) under the hypotheses of homogeneity and isotropy, using the conformal time as time coordinate.

This means that the Euler–Lagrange equations relative to (31), which coincides with the second order Einstein equation and the Klein–Gordon equation in conformal time, correspond to the Euler–Lagrange equations relative to (32), under the transformation (30). Moreover, the energy function \(E_\eta\) relative to (29) corresponds to the energy function \(E_\eta\) relative to (32), so that there is correspondence between the first order Einstein equations. Furthermore, in order to have full coherence between the two formulations, it is easy to verify that, both in the Jordan frame and in the Einstein frame, the Euler–Lagrange equations, written using the conformal time, correspond to the Euler–Lagrange equations written using the universal time except for terms in the energy function; for it one gets the relation

\[
E_\eta = aE_t
\]

(33)

which holds in both the frames; thus the first order Einstein equation is preserved under the transformation from \(\eta\) to \(t\) and there is full equivalence between the two formulations. We want to point out that for the two Lagrangians \(L_\eta\) and \(L_t\) the same relation as (33) holds; this remark is useful for forthcoming considerations.

4 The presence of ordinary matter

So far we have analysed the general conformal equivalence and the cosmological conformal equivalence between Einstein frame and Jordan frame in presence of a scalar field.
What happens when ordinary matter is present (see [3])? We focus our attention on the cosmological case.

The standard Einstein (cosmological) “point–like” Lagrangian (when noninteracting scalar field and ordinary matter are present) is given by

$$\mathcal{L}_{\text{tot}} = \mathcal{L}_{\bar{t}} + \mathcal{L}_{\text{mat}}$$ (34)

in which $\mathcal{L}_{\bar{t}}$ is given by (24) and $\mathcal{L}_{\text{mat}}$ is the Lagrangian relative to matter. Using the contracted Bianchi identity, it can be seen that $\mathcal{L}_{\text{mat}}$ can be written as [19]

$$\mathcal{L}_{\text{mat}} = -D\bar{a}^{3(1-\gamma)}$$ (35)

where $D$ is connected with the total amount of matter. In writing (34) and (35) we have chosen the universal time as time–coordinate. Under the transformation (23) we have, beside relation (24), that (35) corresponds to

$$\mathcal{L}_{\text{mat}} = (\sqrt{-2F})^{3(1-\gamma)} L_{\text{mat}}$$ (36)

where, analogously to (35)

$$L_{\text{mat}} = D\bar{a}^{3(1-\gamma)}. \tag{37}$$

Then we have that, under (23), (34) becomes

$$\frac{1}{\sqrt{-2F}} L_{\text{tot}}^{(1)} = \frac{1}{\sqrt{-2F}} [L_{\bar{t}} + (\sqrt{-2F})^{4-3\gamma} L_{\text{mat}}] \tag{38}$$

in which we have defined the total “point–like” Lagrangian after the conformal transformation as

$$L_{\text{tot}}^{(1)} = L_{\bar{t}} + (\sqrt{-2F})^{4-3\gamma} L_{\text{mat}}, \tag{39}$$

(cfr. (24)); the transformation of $\mathcal{L}_{\text{tot}}$ under (23) has to be written following the expression (38) and consequently the “point–like” Lagrangian $L_{\text{tot}}^{(1)}$, has to be defined as in (39). The use of the superscript (1) for $L_{\text{tot}}$ will be clarified in a moment. The factor $\frac{1}{\sqrt{-2F}}$ in evidence out of the square bracket, is introduced in order to preserve the invariance of the reduced action under transformation (23), since that factor is also the one which appears in the time–coordinate transformation in (23).

The Lagrangian (33) could be then assumed to describe a cosmological NMC–model with a scalar field and ordinary matter as gravitational sources. By the way, we see that, unless $\gamma = \frac{4}{3}$, the standard matter Lagrangian term is coupled with the scalar field in a way which depends on the coupling $F$. Such coupling between the matter and the scalar field is an effect of the transformation, therefore depending on the coupling $F$. This is one way to look at the problem, but we can also proceed in a different way to determine the Lagrangian in presence of matter. We can consider as the total “point–like” Lagrangian

$$L_{\text{tot}}^{(2)} = L_{\bar{t}} + L_{\text{mat}}. \tag{40}$$
That is, we take the “point–like” Lagrangian of the NMC theories, given by (19), and add up to it the standard matter term defined in (37). It is clear now why we have introduced the notation $L_{\text{tot}}^{(1)}$ and $L_{\text{tot}}^{(2)}$. Of course the full theory described by (40) is by no mean conformally equivalent to that one described by (34). Also, the transformation does not give rise to any coupling between matter and scalar field. One could just point out that the matter term defined by (37) has been obtained in the context of the Einstein frame and in this sense it could be not legitimate using it in a NMC theory.

The problem of the physical system, also connected with the formulation of the Equivalence Principle \cite{20}, has been already discussed and it is well known, in particular in the case of the higher order theories \cite{11} \cite{12} \cite{13} \cite{14} (and references quoted therein), but also in the context of NMC theories \cite{3} \cite{5}. In the case we are considering, the problem still concerns the choice of the physical system but from another point of view, since the Lagrangians (39) and (40) are not connected by a conformal transformation. The problem concerns which is the Lagrangian to describe, in the Jordan frame, a cosmological model with a scalar field plus ordinary matter, between the lagrangian $L^{(1)}_{\text{tot}}$ and $L^{(2)}_{\text{tot}}$, once we assume that the physical system is that one of a NMC theory (some authors consider the Jordan frame as the physical one, see for examples \cite{21}, while in \cite{22} the Einstein frame is the physical one).

This sort of ambiguity can be clarified in the general context of the field theory, introduced in Sec. 2. We focus our attention on the contracted Bianchi identity. From the point of view of the field equations, the choice of the Lagrangian $L^{(2)}_{\text{tot}}$ to describe the gravitational field with a scalar field and ordinary matter (non interacting with the scalar field) as sources in the Jordan frame corresponds to write the field equations as

$$F(\phi)G_{\mu\nu} = -\frac{1}{2}T_{\mu\nu} - g_{\mu\nu} \nabla F(\phi) + F(\phi)_{;\mu\nu} + T_{(\text{mat}) \, \mu\nu}$$  \hspace{1cm} (41)

which is obtained just adding up the ordinary matter as a further source term to the field equations in the NMC case given by (2). Performing the covariant divergence of both sides and taking into account of the expression of $T(\phi)_{\mu\nu}$ given by (1), we get

$$F_{;\mu}G^{\nu} + \frac{1}{2} \phi_{;\mu} \phi_{;\nu} + \frac{1}{2} \delta_{\mu}^{\nu} V_{\phi_{;\nu}} + \delta_{\mu}^{\nu} (\nabla F)_{;\nu} - F_{;\mu}^{\nu} = T_{(\text{mat}) \, \mu\nu}$$  \hspace{1cm} (42)

which can be written as

$$F_{\mu}R_{\nu} - \frac{1}{2} RF_{;\mu} + \frac{1}{2} \phi_{;\mu} (\nabla_{\Gamma} \phi + V_{\phi}) + (\nabla_{\Gamma} F)_{;\mu} - F_{;\mu}^{\nu} = T_{(\text{mat}) \, \mu\nu}$$  \hspace{1cm} (43)

where we have taken into account Eq. (3). The last two terms of the lefthand side of (43) give

$$(g^{\alpha\beta} F_{;\alpha\beta})_{;\mu} - F_{;\mu}^{\nu} = -g^{\alpha\beta} R^{\lambda}_{\beta\alpha\nu} F_{;\lambda}$$

$$= -F_{;\lambda} R^{\lambda}_{\mu}.$$  \hspace{1cm} (44)
Eq. (43), taking into account Eq. (44) gives then an interesting relation, that is

$$\frac{1}{2} \phi_{,\mu} (\square \phi + V_\phi - RF_\phi) = T_{(\text{mat}) \; \mu ; \nu}.$$  \hspace{1cm} (45)

Comparing relation (45) with Eq. (5) we see that the lefthand side coincides with the lefthand side of the Klein–Gordon equation; it means that the continuity equation for the ordinary matter holds, that is

$$T_{(\text{mat}) \; \nu ; \nu} = 0$$  \hspace{1cm} (46)

in which $T_{(\text{mat}) \; \mu ; \nu}$ is the energy–momentum tensor of the matter, relative to the NMC metric $g_{\mu \nu}$. Thus, it means that, choosing the Jordan frame as the physical frame and equations (41) as field equations, the conservation of matter is relative to the physical metric $g_{\mu \nu}$. In this sense the legitimate way to describe scalar field plus ordinary matter in the Jordan frame is the one given by the field equations (41). The corresponding action, in the cosmological case, corresponds to choose $L_{tot}^{(2)}$ as Lagrangian [11] [12] [13] [14] (and ref. quoted therein).

We can only say that such considerations could be a hint for further developments in the context of the Jordan frame (for a totally different point of view see [3]).

The problem can be further analysed from the point of view of the energy density parameter $\Omega$. We can see in fact that the presence of the coupling gives some contributions to $\Omega$. Let us consider the first order Einstein equation relative to the total lagrangian $L_{tot}^{(2)}$ in the Jordan frame in presence of matter

$$\frac{a^2}{a^2} + \frac{F_{\phi} \dot{\phi}}{Fa} + \frac{\kappa}{a^2} + \frac{1}{2} \frac{\dot{\phi}^2}{6F} + \frac{V}{6F} + \frac{D}{6Fa^{3\gamma}} = 0$$  \hspace{1cm} (47)

which can be seen as obtained from the standard first order Einstein equation after the conformal transformation (23), having just added ordinary matter. The last term on the lefthand side being just the effective energy density relative to matter; the factor $-\frac{1}{2F}$ represents the effective coupling. Taking into account the definition of the Hubble parameter $H$, (17) can be rewritten as

$$H^2 + \frac{\dot{F} H}{F} + \frac{1}{6F} \left( \frac{1}{2} \dot{\phi}^2 + V \right) + \frac{D}{6Fa^{3\gamma}} = 0$$  \hspace{1cm} (48)

(we consider the case $\kappa = 0$). We get then

$$- \frac{\dot{F}}{FH} - \frac{1}{6H^2 F} \left( \frac{1}{2} \dot{\phi}^2 + V \right) - \frac{D}{6H^2 Fa^{3\gamma}} = 1$$  \hspace{1cm} (49)

after dividing by $H^2$. As usual, the righthand side is the total energy density parameter $\Omega_{tot}$ which is equal to 1, having assumed $\kappa = 0$. The last term on the lefthand side represents the effective contribution to the density parameter due to the matter, $\Omega_{\text{mat}}$ =
while the term in parentheses represents the effective energy density contribution due to the scalar field, $\Omega_\phi = -\frac{1}{6H^2F}\left(\frac{1}{2}\dot{\phi}^2 + V\right)$; the first term is connected with the variation of the coupling, $\Omega_{coup} = -\frac{F}{H^2F}$. That is, there is a contribution to the energy density parameter due to the presence of the nonminimal coupling. This term, for what we have said, can be seen as coming from the conformal transformation considered. The parameter $\Omega$ is an observable quantity (the present value of $\Omega_{tot}$ is assumed to be equal to 1) thus, in principle, from its analysis one could be able to infer whether the physical frame is the Jordan or the Einstein frame (see also [12]).

A final remark we would like to do concerns the case $\gamma = \frac{4}{3}$. As we have said, performing the transformation (23) on (34) no coupling between the scalar field and the matter is induced if the matter is a radiative perfect fluid: this seems to be quite reasonable, since the particles which constitute a radiative fluid have zero mass.

5 Conformal transformations and Noether symmetries

We want to analyse now the compatibility between the conformal transformation we have considered so far and the presence of Noether symmetries in the “point–like” Lagrangian in the configuration space $(a, \phi)$, i.e. in the cosmological case. Some of the authors, in previous papers (see for examples [17] [23]) have developed a method to find exact cosmological solutions relative either to purely scalar–tensor Lagrangians or to scalar–tensor Lagrangians with ordinary matter, both in MC theories, when the Lagrangian is given by (34), and in NMC theories, having taken (40) as Lagrangian.

Now we want to analyse the problem whether the conformal transformation connecting Einstein and Jordan frame preserves the presence of a Noether symmetry. Since the existence of a Noether symmetry implies the existence of a vector field $X$ along which $L_X L = 0$, this happens if the Lie derivative of the Lagrangian along a vector field is preserved. We can see that the Lie derivative is preserved under the conformal transformation considered, but only in absence of ordinary matter. It turns out to be quite simple to be verified if we choose as time–coordinate the conformal time. As we have seen, we have that in absence of matter, using the time $\eta$ the “point–like” Lagrangian in the Einstein and Jordan frame given by (31) and (32) respectively, correspond to each other under the conformal transformation given by (30). The second of (30), in principle, can be integrated, so that its finite form together with the first of (30) can represent a “coordinate transformation” on the configuration space $(a, \phi)$. Thus, a given lift–vector field of the form [24]

$$X_\eta = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \phi} + \alpha' \frac{\partial}{\partial a'} + \beta' \frac{\partial}{\partial \phi'}$$

in which $\alpha = \alpha(a, \phi), \beta = \beta(a, \phi)$ corresponds under this transformation to the lift–vector field on the configuration space $(\bar{a}, \bar{\phi})$

$$\bar{X}_\eta = \bar{\alpha} \frac{\partial}{\partial \bar{a}} + \bar{\beta} \frac{\partial}{\partial \bar{\phi}} + \bar{\alpha}' \frac{\partial}{\partial \bar{a}'} + \bar{\beta}' \frac{\partial}{\partial \bar{\phi}'}$$

12
in which $\bar{\alpha} = \bar{\alpha}(\bar{a}, \bar{\phi}), \bar{\beta} = \bar{\beta}(\bar{a}, \bar{\phi})$ are connected to $\alpha = \alpha(a, \phi), \beta = \beta(a, \phi)$ through the Jacobian matrix relative to the “coordinate transformation” defined by (30). We remind that the prime means the derivative with respect to the time $\eta$. The Lie derivative of $L_\eta$ along the vector field $X_\eta$ corresponds then to the Lie derivative of $\bar{L}_\eta$ along $\bar{X}_\eta$.

$$\mathcal{L}_{X_\eta} L_\eta = \mathcal{L}_{\bar{X}_\eta} \bar{L}_\eta.$$  (52)

Therefore, if $X_\eta$ is a Noether vector field relative to $L_\eta$ one has

$$\mathcal{L}_{X_\eta} L_\eta = 0$$  (53)

and, from (52), $\bar{X}_\eta$ is a Noether vector field relative to $\bar{L}_\eta$.

We have seen till now that the choice of $\eta$ as time–coordinate is convenient from a formal point of view, but, as we have already remarked, in order to analyse the phenomenology relative to a given model and to obtain then quantities comparable with the observational data, the appropriate choice of time–coordinate is the universal time $t$. The problem with the universal time is that it is not preserved by the conformal transformation, as we have pointed out in Sec. 2, thus the conformal transformation we consider does not take simply the form of a “coordinate transformation” on the phase space $(a, \phi)$, then its compatibility with the presence of a Noether symmetry cannot be easily verified. Of course it must hold also under such choice of time–coordinate.

We decide not to verify such compatibility directly. Rather, we analyse how does the Lie derivative $\mathcal{L}_{X_\eta} L_\eta$ in the Jordan frame is transformed under the time transformation (29) which connects $t$ with $\eta$.

The explicit expression of $\mathcal{L}_{X_\eta} L_\eta$ is given by

$$\mathcal{L}_{X_\eta} L_\eta = 6 \left[ 2F \frac{\partial \alpha}{\partial a} + \left( \beta + a \frac{\partial \beta}{\partial a} \right) F_\phi \right] a^2 + a \left[ \alpha + 6F_\phi \frac{\partial \alpha}{\partial \phi} + a \frac{\partial \beta}{\partial \phi} \right] \phi^2 +$$

$$+ 6 \left[ a\beta F_{\phi\phi} + \left( \alpha + a \frac{\partial \alpha}{\partial a} + a \frac{\partial \beta}{\partial \phi} \right) F_\phi + 2F \frac{\partial \alpha}{\partial \phi} + \frac{a^2 \partial \beta}{6} \right] a \dot{\phi} +$$

$$- a^3 (4aV + a\beta V_\phi) - 6a(2F \alpha + F_\phi a \beta) \kappa$$  (54)

in which we have taken into account that

$$\alpha' = \frac{\partial \alpha}{\partial a} a' + \frac{\partial \alpha}{\partial \phi} \dot{\phi}; \quad \beta' = \frac{\partial \beta}{\partial a} a' + \frac{\partial \beta}{\partial \phi} \dot{\phi};$$  (55)

(54) under the transformation (29) becomes

$$\mathcal{L}_{X_\eta} L_\eta = 6 \left[ 2F \frac{\partial \alpha}{\partial a} + \left( \beta + a \frac{\partial \beta}{\partial a} \right) F_\phi \right] a^2 \dot{a}^2 + a \left[ \alpha + 6F_\phi \frac{\partial \alpha}{\partial \phi} + a \frac{\partial \beta}{\partial \phi} \right] a^2 \dot{\phi}^2 +$$

$$+ 6 \left[ a\beta F_{\phi\phi} + \left( \alpha + a \frac{\partial \alpha}{\partial a} + a \frac{\partial \beta}{\partial \phi} \right) F_\phi + 2F \frac{\partial \alpha}{\partial \phi} + \frac{a^2 \partial \beta}{6} \right] a^2 \dot{a} \dot{\phi} +$$

$$- a^3 (4aV + a\beta V_\phi) - 6a(2F \alpha + F_\phi a \beta) \kappa$$  (56)
which can be written as
\[
\mathcal{L}_{X_t} L_\eta = 6a \left[ \alpha F + 2a F \frac{\partial \alpha}{\partial a} + a \left( \beta + a \frac{\partial \beta}{\partial a} \right) F_\phi \right] \dot{a}^2 + a^2 \left[ \frac{3}{2} \alpha + 6F_\phi \frac{\partial \alpha}{\partial \phi} + a \frac{\partial \beta}{\partial \phi} \right] \dot{\phi}^2 + \\
+6a^2 \left[ a \beta F_\phi + \left( 2 \alpha + a \frac{\partial \alpha}{\partial a} + a \frac{\partial \beta}{\partial \phi} \right) F_\phi + 2F \frac{\partial \alpha}{\partial \phi} + \frac{a^2 \partial \beta}{6} \frac{\partial \phi}{\partial a} \right] \dot{a} \dot{\phi} + \\
-a^3 (3\alpha V + a \beta V_\phi) - 6a (F \alpha + F_\phi a \beta) \kappa + \\
-6a F a \dot{a}^2 - \frac{1}{2} a \alpha^3 \phi^2 - 6a F_\phi a^2 \dot{\phi} - 6a F a \kappa - \alpha a^3 V.
\]
(57)

The Lie derivative of \( L_t \) given by (54) along a lift–vector field of the form
\[
X_t = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \phi} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{\phi}}
\]
(58)
is given by
\[
\mathcal{L}_{X_t} L_t = 6a \left[ \alpha F + 2a F \frac{\partial \alpha}{\partial a} + a \left( \beta + a \frac{\partial \beta}{\partial a} \right) F_\phi \right] \dot{a}^2 + a^2 \left[ \frac{3}{2} \alpha + 6F_\phi \frac{\partial \alpha}{\partial \phi} + a \frac{\partial \beta}{\partial \phi} \right] \dot{\phi}^2 + \\
+6a \left[ a \beta F_\phi + \left( 2 \alpha + a \frac{\partial \alpha}{\partial a} + a \frac{\partial \beta}{\partial \phi} \right) F_\phi + 2F \frac{\partial \alpha}{\partial \phi} + \frac{a^2 \partial \beta}{6} \frac{\partial \phi}{\partial a} \right] \dot{a} \dot{\phi} + \\
-a^2 (3\alpha V + a \beta V_\phi) - 6(F \alpha + F_\phi a \beta) \kappa
\]
(59)
in which we have taken into account that
\[
\dot{\alpha} = \frac{\partial \alpha}{\partial a} \dot{a} + \frac{\partial \alpha}{\partial \phi} \dot{\phi}; \quad \dot{\beta} = \frac{\partial \beta}{\partial a} \dot{a} + \frac{\partial \beta}{\partial \phi} \dot{\phi}.
\]
(60)

We remind that the dot means the derivative with respect to \( t \).

Comparing (57) with (59) and taking into account the expression of \( E_t \) given by (21) we obtain that, under the transformation (29) the Lie derivative \( \mathcal{L}_{X_t} L_\eta \) becomes
\[
\mathcal{L}_{X_t} L_\eta = a \mathcal{L}_{X_t} L_t - (\mathcal{L}_{X_t} a) E_t,
\]
(61)
being \( \mathcal{L}_{X_t} a = \alpha \).

It can be seen that the same relation as (61) holds in the Einstein frame, that is
\[
\mathcal{L}_{\tilde{X}_t} \tilde{L}_\eta = a \mathcal{L}_{\tilde{X}_t} \tilde{L}_t - (\mathcal{L}_{\tilde{X}_t} \tilde{a}) \tilde{E}_t,
\]
(62)
with quite obvious meaning of \( X_t \).
This implies that, if $X_\eta$ is a Noether vector field relative to $L_\eta$, that is, if (53) holds, the corresponding vector field $X_t$ is such that
\[ \mathcal{L}_{X_t} L_t - \frac{\mathcal{L}_{X_t} a}{a} E_t = 0. \] (63)

It means also that, when the universal time is taken as time-coordinate, the conformal transformation preserves the expression given by the righthand side of (61) and not the Lie derivative along a given vector field $X_t$. Relation (63) represents a more general way to express the presence of a first integral for the Lagrangian $L_t$; associated to (63) we have the conserved quantity (26)
\[ -E_t \int \frac{\mathcal{L}_{X_t}}{a} dt + \frac{\partial L_t}{\partial \dot{a}} \alpha + \frac{\partial L_t}{\partial \dot{\phi}} \beta = \text{const} \] (64)
which, of course, holds on the solutions of the Euler–Lagrange equations. The vector field $X_t$ verifying (63) can thus be seen as a generalized Noether vector field and the conformal transformation (23) preserves this generalized symmetry. That is, if $X_t$ is a Noether vector field, in the sense of (63), relative to $L_t$ then $\tilde{X}_t$ is a Noether vector field relative to $\tilde{L}_\tilde{t}$ in the same sense, that is
\[ \mathcal{L}_{\tilde{X}_t} \tilde{L}_\tilde{t} - \frac{\mathcal{L}_{\tilde{X}_t} \tilde{a}}{\tilde{a}} \tilde{E}_\tilde{t} = 0. \] (65)

In terms of the conformal time, the first integral relative to (53) for the Lagrangian $L_\eta$ is given by
\[ \frac{\partial L_\eta}{\partial a} \alpha + \frac{\partial L_\eta}{\partial \phi} \beta = \text{const}. \] (66)

We see that the expression (66) corresponds to (64) under the transformation (23), except for a term in the energy function. In fact (66) explicitly written is
\[ (12F a' + 6F_\phi a' \phi')\alpha + (6F_\phi a a' + a^2 \phi')\beta = \text{const}, \] (67)
while (64) is
\[ -E_t \int \frac{\alpha}{a} dt + (12F a\dot{a} + 6F_\phi a^2 \dot{\phi})\alpha + (6F_\phi a^2 \dot{a} + a^3 \dot{\phi})\beta = \text{const}. \] (68)

Taking into account (53), we have that (68), under (29), becomes
\[ -\frac{E_\eta}{a} \int \alpha \, d\eta + (12F a' + 6F_\phi a' \phi')\alpha + (6F_\phi a a' + a\phi')\beta = \text{const} \] (69)
which coincides with (67) except for the term in $E_\eta$.

Therefore there is equivalence between the two formulations except for the term in $E_\eta$, coherently with what we have said at the end of Sec. 2.
As already said, some of the authors have formulated the existence of a Noether vector field imposing
\[ \mathcal{L}_{X_t} L_t = 0 \] (70)
using the universal time as time-coordinate; condition (70) after the analysis we have done till now, turns out to be less general than (65). By the way, condition (70) has the interesting property that it implies the possibility to define some new coordinates on the configuration space \((a, \phi)\), such that the Lagrangian has a cyclic coordinate \([15, 24]\), reducing in this way the Euler–Lagrange equations. In fact, one can always define new coordinate, say \((z, w)\), in the configuration space of the Lagrangian, such that the lift–vector field assumes the form \(X_t = \frac{\partial}{\partial z}\), so that one has \(\mathcal{L}_{X_t} L = \frac{\partial L}{\partial z}\); in case (70) holds one has then that \(z\) is cyclic. In the generalized case we are considering, it is no longer possible to get this behavior, since \(\mathcal{L}_{X_t} L_t \neq 0\) and consequently \(z\) is no longer cyclic. In this case one has to use the first integral (64) together with the relation on the energy function to reduce the Euler–Lagrange equations, that is, one has the system of Eqs. (20), Eq. (22) and Eq. (64).

This problem corresponds in the Einstein frame to the system of equations (23), the equation analogue to (22), \(\mathcal{E}_{\bar{t}} = 0\), and the equation analogue to (64),
\[ -\mathcal{E}_{\bar{t}} \int \frac{\mathcal{L}_{\bar{X}_t}}{\bar{a}} d\bar{t} + \frac{\partial \mathcal{L}_{\bar{t}}}{\partial \ddot{\bar{a}}} \ddot{\bar{a}} + \frac{\partial \mathcal{L}_{\bar{t}}}{\partial \dot{\bar{\phi}}} \dot{\bar{\phi}} = \text{const.} \] (71)
Thus, finding the solutions of some cosmological model using the presence of a Noether symmetry (and therefore fixing the class of model compatible with it) in the Einstein frame, one gets via the conformal transformation (as given by (23)) the solutions to the class of models in the Jordan frame corresponding to the one given in the Einstein frame through the second of (10). We are going to give some significant examples in the following section.

6 Examples

i) Let us consider a quite easily solvable model in the Einstein frame. We consider the cosmological model with a scalar field, a constant potential and zero curvature. The Lagrangian is given by
\[ \mathcal{L}_t = -3\ddot{a} \dot{a}^2 + \frac{1}{2} \ddot{\phi}^2 - \dot{\phi}^2 \Lambda; \] (72)
the Euler–Lagrange equations and the zero energy function condition are given by
\[ \begin{cases} \frac{2 \ddot{a}}{\dot{a}} + \frac{\dot{a}^2}{\dot{a}} + \frac{1}{2} \dot{\phi}^2 - \dot{\phi}^2 - \Lambda = 0 \\ \frac{\dot{\phi}}{\dot{a}} + \frac{3 \ddot{\phi}}{\dot{a}} = 0 \end{cases} \] (73)
\[
\frac{\dot{a}^2}{a^2} - \frac{1}{3} \left( \frac{1}{2} \dot{\phi}^2 + \Lambda \right) = 0. \tag{74}
\]

The solutions are (see also [27])

\[
\begin{align*}
\bar{a} &= \left[ c_1 e^{\sqrt{3} \Lambda t} - \frac{\dot{\phi}_0}{8 \Lambda c_1^2} e^{-\sqrt{3} \Lambda t} \right]^{\frac{1}{3}} \\
\bar{\phi} &= \bar{\phi}_0 + \sqrt{\frac{2}{3}} \ln \frac{1 - \frac{\dot{\phi}_0}{2 c_1 \sqrt{2 \Lambda} e^{-\sqrt{3} \Lambda t}}}{1 + \frac{\dot{\phi}_0}{2 c_1 \sqrt{2 \Lambda} e^{-\sqrt{3} \Lambda t}}} \tag{75}
\end{align*}
\]

Of course only three constants of integration appear in the solution, since Eq. (74) corresponds to a constraint on the value of the first integral \(E_\dot{t}\). We have that, in the limit of \(\bar{t} \to +\infty\), the behavior of \(\bar{a}\) is exponential with characteristic time given by \(\sqrt{\frac{\Lambda}{3}}\), as we would expect (see also [18]), and \(\bar{\phi}\) goes to a constant.

Looking at the second of (74) we have that such a model in the Einstein frame corresponds in the, Jordan frame, to the class of models with (arbitrarily given) coupling \(F\) and potential \(V\) connected by the relation

\[
\frac{V}{4 F^2} = \Lambda, \tag{76}
\]

the solution of which can be obtained from (75) via the transformation (23). We can thus fix the potential \(V\) and obtain from (76) the corresponding coupling. This can be used as a method to find the solutions of NMC models with given potentials, the coupling being determined by (76). We consider, as an example, the case

\[
V = \lambda \phi^4, \quad \lambda > 0 \tag{77}
\]

which correspond to a “chaotic inflationary” potential [28]. The corresponding coupling is quadratic in \(\phi\)

\[
F = k_0 \phi^2 \tag{78}
\]

in which

\[
k_0 = -\frac{1}{2} \sqrt{\frac{\lambda}{\Lambda}} \tag{79}
\]
Substituting (78) into (23) we get

\[
\begin{cases}
  a = \frac{\ddot{a}}{\phi \sqrt{-2k_0}} \\
  d\phi = \phi \frac{2k_0}{12k_0 - 1} d\phi \\
  dt = \frac{d\bar{t}}{\phi \sqrt{-2k_0}}.
\end{cases}
\]  

(80)

As we see from these relations, it has to be \( k_0 < 0 \). Integrating the second of (80) we have \( \phi \) in terms of \( \bar{\phi} \)

\[
\phi = \alpha_0 e^{\sqrt{\frac{2k_0}{12k_0 - 1}} \bar{\phi}}.
\]  

(81)

Substituting (81) in the first of (80) and taking into account the second of (75), we have the solutions \( a \) and \( \phi \) as functions of \( \bar{t} \)

\[
\begin{cases}
  \phi = \phi_0 \left[ \frac{1 - \frac{\bar{\phi}_0}{2c_1 \sqrt{2\Lambda}} e^{-\sqrt{3\Lambda} t}}{1 + \frac{\bar{\phi}_0}{2c_1 \sqrt{2\Lambda}} e^{-\sqrt{3\Lambda} t}} \right] \sqrt{\frac{4k_0}{3(12k_0 - 1)}} \\
  a = \frac{1}{\phi_0 \sqrt{-2k_0}} \left[ c_1 e^{\sqrt{3\Lambda} t} - \frac{\bar{\phi}_0}{8\Lambda c_1^2} e^{-\sqrt{3\Lambda} t} \right] \frac{1}{\frac{1 - \frac{\bar{\phi}_0}{2c_1 \sqrt{2\Lambda}} e^{-\sqrt{3\Lambda} t}}{1 + \frac{\bar{\phi}_0}{2c_1 \sqrt{2\Lambda}} e^{-\sqrt{3\Lambda} t}}} \sqrt{\frac{4k_0}{3(12k_0 - 1)}} \right]^4
\end{cases}
\]  

(82)

in which \( \phi_0 = \alpha_0 e^{\sqrt{\frac{2k_0}{12k_0 - 1}} \bar{\phi}_0} \). Substituting (81) in the third of (80), taking into account of (73), we get

\[
dt = \frac{d\bar{t}}{\phi_0 \sqrt{-2k_0}} \left[ 1 + \frac{\bar{\phi}_0}{2c_1 \sqrt{2\Lambda}} e^{-\sqrt{3\Lambda} t} \right] \sqrt{\frac{4k_0}{3(12k_0 - 1)}} \right].
\]  

(83)

We can obtain \( \bar{t} \) as a function of \( t \) integrating (83) and then considering the inverse function; (83) could be easily integrated if the exponent \( \sqrt{\frac{4k_0}{3(12k_0 - 1)}} \) would be equal to \( \pm 1 \), but this corresponds to a value of \( k_0 = \frac{3}{32} \) which is positive and thus it turns out to be not acceptable. In general, (83) is not of easy solution. We can analyse its asymptotic
behavior, obtaining

\[ \frac{dt}{dt} \xrightarrow{t \to +\infty} \frac{1}{\phi_0 \sqrt{-2k_0}} \]  

(84)

that is, asymptotically

\[ t - t_0 \simeq \frac{\bar{t}}{\phi_0 \sqrt{-2k_0}}. \]  

(85)

Substituting (85) in the asymptotic expression of (82), we obtain the asymptotic behavior of the solutions (since from (84) one has \( t \xrightarrow{t \to +\infty} +\infty \))

\[ \begin{align*}
    a & \simeq \frac{c_1^{1/3}}{\phi_0 \sqrt{-2k_0}} e^{\phi_0 \sqrt{-2k_0} \frac{1}{3} (t - t_0)} \\
    \phi & \simeq \phi_0.
\end{align*} \]  

(86)

Thus we have that, asymptotically, \( a(t) \) is exponential as it had to be (cfr. [13]), and \( \phi(t) \) is constant; the coupling \( F \) is asymptotically constant too, so that, fixing the arbitrary constant of integration to obtain the finite transformation of \( \bar{a}, \tilde{\phi} \) (that is, fixing the units, see [12]), once \( k_0 \) is fixed, it is possible to recover asymptotically the Einstein gravity.

As a remark we would like to notice that the asymptotic expression (86) of \( a(t) \) and \( \phi(t) \) are solutions of the Einstein equations and Klein–Gordon equation with zero curvature and \( F \) given by (77), (78). They have not been obtained as solutions of the asymptotic limits of these equations. It means then that they are, in any case, particular solutions of the given NMC–model.

\textbf{ii)} Another interesting case is the Ginzburg–Landau potential

\[ V = \lambda(\phi^2 - \mu^2)^2, \quad \lambda > 0. \]  

(87)

The corresponding coupling is given by

\[ F = k_0(\phi^2 - \mu^2) \]  

(88)

in which \( k_0 \) is given by (79) when \( \phi^2 > \mu^2 \) while is given by (79) with opposite sign when \( \phi^2 < \mu^2 \), in order to have \( F < 0 \). With this coupling the corresponding conformal transformation turns out to be singular for \( \phi^2 = \mu^2 \), thus with this method it is not possible to solve this model for \( \phi \) equal to the Ginzburg–Landau mass \( \mu \).

In this case, it is not so straightforward to get the explicit function \( \phi = \phi(\tilde{\phi}) \) as in the previous case, since one should perform and then obtain the inverse of the integral

\[ \tilde{\phi} - \bar{\phi}_0 = \int \left[ \frac{3}{4} \sqrt{\frac{\lambda}{\phi^2} + \frac{1}{2}(\phi^2 - \mu^2)} \right]^{1/2} d\phi. \]  

(89)

It is not so difficult to integrate (89) [29], the difficulty raises in finding the inverse, which is needed to obtain \( t \) in terms of \( \bar{t} \). By the way, analysing the integrand of (89), that is
we can say that, except for $\phi^2 = \mu^2$, the function $\bar{\phi} = \bar{\phi}(\phi)$ is invertible; in particular, since asymptotically $\bar{\phi}$ is constant, so is $\bar{\phi}$. Thus it is possible to carry out a reasoning analogous to the previous one, concluding that asymptotically the behavior of $a(t)$ is exponential and that of $\phi(t)$ is constant.

**iii)** We want to consider now the $V = \lambda\phi^4$ case from the point of view of the Noether symmetries. We want to show that, in the context of generalized Noether symmetries the NMC–model with quartic potential and negative quadratic coupling admits a Noether symmetry, while such a result has been not found in the previous analysis of Noether symmetries (see [15] [17]).

We have seen that the corresponding case in the Einstein frame is that one of constant potential. The system of equations for the Noether vector field obtained from (65) is given by

$$
\begin{align*}
\frac{\partial \bar{\alpha}}{\partial \bar{a}} &= 0 \\
\bar{\alpha} + \bar{a} \frac{\partial \bar{\beta}}{\partial \bar{\phi}} &= 0 \\
6 \frac{\partial \bar{\alpha}}{\partial \bar{\phi}} - \bar{a}^2 \frac{\partial \bar{\beta}}{\partial \bar{a}} &= 0 \\
4\alpha \bar{V} + \bar{a} \bar{\beta} \bar{V}_{\bar{\phi}} &= 0.
\end{align*}
$$

(90)

Substituting $\bar{V} = \Lambda$ in the fourth of (90) one gets $\bar{\alpha} = 0$; from the second one gets $\bar{\beta} = \text{const}$; the first and the third turn out to be identically verified. It is immediate to see that the Lagrangian (72) presents a Noether symmetry, since it does not depend on $\bar{\phi}$; being, in this particular case, $L_{X_i} \bar{a} = \bar{\alpha} = 0$, this is compatible, for what we have already said, with the presence of a cyclic coordinate in the Lagrangian. Performing the conformal transformation given by (80) on the Noether vector field

$$
\begin{align*}
\bar{\alpha} &= 0 \\
\bar{\beta} &= \bar{\beta}_0.
\end{align*}
$$

(91)

where $\bar{\beta}_0$ is arbitrary, we obtain

$$
\begin{align*}
\alpha &= -a \sqrt{\frac{2k_0}{12k_0 - 1}} \bar{\beta}_0 \\
\beta &= \phi \sqrt{\frac{2k_0}{12k_0 - 1}} \bar{\beta}_0.
\end{align*}
$$

(92)

As it has been shown in the previous section, (92) is a Noether vector field relative to the corresponding Lagrangian in the Jordan frame, with potential given by (77) and coupling given by (78). It is easy to verify that (63) holds.
iv) There is another interesting case we would like to quote in the context of inflationary models, as last example, \( i.e. \)

\[
V = \lambda \phi^2, \quad \lambda > 0; \quad F = k_0 \phi^2, \quad k_0 < 0
\] (93)

in the Jordan frame. Since the coupling is the same as the previous case (cfr. (78)), the relative conformal transformation is given by (80). To obtain the corresponding potential in the Einstein frame we have to substitute (81) in the relation

\[
\bar{V}(\bar{\phi}) = \frac{\lambda}{4k_0^2 \phi^2(\phi)}
\] (94)

that is

\[
\bar{V}(\bar{\phi}) = \frac{\lambda}{4k_0^2 \phi^2} e^{-2\sqrt{-2F} \frac{\bar{\phi}}{12k_0}}.
\] (95)

We see that this case corresponds in the Einstein frame to the case of the exponential potential, for which a particular solution is the power–law inflation \([30]\) \([31]\). The choice of the sign that Lucchin and Matarrese do in \([30]\) corresponds to the choice of the sign we have done at the end of Sec. 1.

Finally, we would like to make a general remark connected with all the examples we have treated, concerning the relation between the Hubble parameter in the Einstein and in the Jordan frame. Such relation is given by

\[
\bar{H} = \frac{\dot{a}}{a} = \frac{1}{(-2F)} \left( -\frac{\dot{F}}{\sqrt{-2F}} + \sqrt{-2F} \frac{\dot{a}}{a} \right) = \frac{\dot{F}}{2F \sqrt{-2F}} + \frac{H}{\sqrt{-2F}}
\] (96)

in which we have used (23) and the definition of the Hubble parameter. Relation (96) is quite useful to make some considerations on the asymptotic behavior of the Hubble parameter (see also \([18]\)): for examples, if we require an asymptotic de Sitter–behavior in both the Einstein and Jordan frame, that is, we require \(\bar{H} \overset{t \rightarrow \pm \infty}{\rightarrow} \bar{C} \) and \(H \overset{t \rightarrow \pm \infty}{\rightarrow} C \) where \(\bar{C} \) and \(C \) are constants, from (96) we obtain a differential equation for the coupling \(F\) as a function of \(t \) \((t \gg 0)\), given by

\[
\dot{F} + 2CF - 2\bar{C}F \sqrt{-2F} = 0.
\] (97)

Its solution is

\[
F = \frac{C^2}{2C^2} \left[ \frac{1}{1 - F_0 e^{\bar{C}t}} - 1 \right]^2
\] (98)

in which \(F_0\) is the integration constant; this is the time–behavior that \(F\) has to assume on the solution \(\phi(t)\), in order to have a de Sitter asymptotical behavior in both frames. We see from (98) that, asymptotically, we recover the standard gravity.
7 Conclusions

We have analysed the conformal equivalence between Jordan frame and Einstein frame for general coupling functions and potentials, and we have seen that any NMC theory assumes the form of the Einstein theory with a scalar field as source of the gravitational field, provided the metric undergoes the conformal transformation defined by (8) and the scalar field and the potential are transformed according to (10). We see, from these transformations, that the scalar field, although being scalar with respect to the coordinate transformations of the space–time by definition, is not conformally scalar.

We have considered such equivalence more carefully in the cosmological case, and we have seen that the conformal transformation in this case takes the form given by (23), in which also the time–coordinate is transformed. The transformation of the time-coordinate turns out to be necessary if one requires the time–coordinate in the Einstein frame to be the universal time as well. We have seen that in case one chooses the conformal time as time–coordinate the transformation defined by (8) reduces to the form (30), which can be seen as a “coordinate transformation” on the configuration space \((a, \phi)\); with such a choice of time–coordinate the conformal equivalence between Jordan and Einstein frame in the cosmological case turns out to be very simple to verify.

The situation changes when ordinary matter is considered besides the scalar field: the conformal equivalence in this case is broken. We have analysed the possible descriptions of NMC theories in presence of ordinary matter and we have seen how (46) could be taken as hint in the definition of the appropriate Lagrangian, in agreement with the current discussions about the compatibility of NMC theories and the different formulations of the Equivalence Principle.

We have also seen that if a Noether symmetry is present in the “point–like” cosmological Lagrangian, this is preserved by the conformal transformation which connects Jordan and Einstein frames. This has been formally formulated in the conformal time through relation (53), since in this case the problem of the redefining the time–coordinate does not exist. We have then analysed the problem in the universal time and we have seen that the Noether symmetry is preserved under the generalized form (63), which implies as well the presence of a first integral for the corresponding Lagrangian.

We have thus analysed some aspects of the conformal equivalence between Jordan frame and Einstein frame, in particular in the cosmological case. Moreover we have generalized and improved a method of solution of cosmological NMC–models, having shown that the conformal transformation considered preserves a Noether symmetry present in the “point–like” Lagrangian, in the sense of (53). The forthcoming steps will be to investigate more deeply the implications of that: it would be interesting to classify the classes of NMC theories which are solvable by this method, and also to understand if it is possible to characterize the inflationary solutions in such context, on one side; to apply this method and to analyse the phenomenology of the models with couplings and potentials of physical interest, which can be solved, on the other side. This can be done also in connection with the problem of an opportune redefinition of the “cosmological
“constant”, which in this context can be time–dependent ([18] see also [32]).

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