Veneziano like amplitude as a test for AdS/QCD models

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Abstract

The high energy asymptotics of QCD correlation functions is often used as a test for bottom-up holographic models. Since QCD is not strongly coupled in the ultraviolet domain, such a test may look questionable. We propose that the sum over resonance poles emerging in correlators of a bottom-up model should reproduce the structure of a Veneziano like amplitude at zero momentum transfer assuming equivalence of spin and radial states in the latter. This requires a five-dimensional background that suppresses the ultraviolet part in the effective action of a model. We give examples of emerging low-energy holographic models.

1 Introduction

The ideas of the gauge/gravity correspondence from string theory [1,2] have found interesting applications to real phenomenology of the strong interactions. One example is given by the bottom-up holographic models, called also AdS/QCD models (the idea originated in Ref. [3]; many references on this activity are contained, e.g., in [4]). The spectroscopy of hadrons is most successfully described within the soft wall model introduced in Ref. [5]. The bottom-up approach provides a tentative semiclassical approximation to planar QCD [6].

Unfortunately, this method is not free of problems at the present stage. One of the problems is that QCD correlators are calculated in the deep UV domain where QCD is weakly coupled, hence, its gravity dual should be strongly coupled in this domain. For this reason, validity of the semiclassical approximation becomes questionable. A straightforward possibility for improving self-consistency of the approach consists in introducing the UV cutoff [7,8]. Such a cutoff corresponds to the scale of onset of perturbative regime in QCD, above which the gravity dual becomes strongly coupled. The cutoff affects both masses and residues of resonances (since the large-$N_c$ limit is implicit in the approach, the correlators are saturated by
poles corresponding to the exchange of an infinite number of narrow resonances \cite{6}). Usually one assumes that the spectrum of light mesons is linear in masses squared, as predicted by hadron string models, and the available data, in many cases, approximately agrees with this assumption \cite{9} while the experimental information on residues (related to decay constants) is very scarce except for some ground states. Given a Regge like behavior of poles, the residues can be fine tuned such that the high energy asymptotics of a correlator under consideration is reproduced (see, e.g., \cite{10} and references therein). As pointed out above, there is no strong reason to expect that the QCD correlators obtained via the holographic duality should satisfy the correct high energy asymptotics. But then what may one use instead as a test for calculated correlators at low energies? In the given Letter, we propose a possible answer.

2 Veneziano like amplitudes and resonances

Consider, for instance, the elastic $\pi\pi$-scattering. This process is usually described by some kind of resonance exchange. In the large-$N_c$ limit of QCD \cite{6}, the meson resonances become narrow and one expects the appearance of an infinite number of resonances with growing masses. At vanishing momentum transfer, the amplitude of $\pi\pi$-scattering will be proportional to the infinite sum over poles of $s$-channel resonances. On the other hand, a very similar sum emerges in the resonance representation of two-point correlation functions. This observation suggests that the low-energy amplitudes at zero momentum transfer and the two-point correlators could be directly related. In QCD, low energies mean the strong coupling regime. The correlators of the strongly coupled gauge theory in the large-$N_c$ limit may be (hopefully) obtained within the holographic approach. This, in turn, implies that the latter might even yield directly the scattering amplitudes at zero momentum transfer. To motivate this suggestion further, we remind the reader the form of the Veneziano amplitudes.

A successful ansatz for the low-energy scattering amplitudes was proposed by Veneziano in the sixties. In particular, the most satisfactory Veneziano like amplitude for the $\pi\pi$ scattering is represented by the following combination of gamma functions \cite{11}

$$A(s,t) \sim \frac{\Gamma(1-\alpha_s)\Gamma(1-\alpha_t)}{\Gamma(1-\alpha_s-\alpha_t)}; \quad \alpha_s = \alpha_0 + \alpha's, \quad (1)$$

with permutations of Mandelstam variables $s$, $t$ and $u$ that satisfy the relation $s + t + u = 4m_\pi^2$. We will consider the chiral limit, $m_\pi = 0$. The Adler
self-consistency condition (disappearance of the amplitude at vanishing momentum) leads then to the condition that \( \alpha_0 = \frac{1}{2} \) given by the pole \( \Gamma(0) \) of denominator in (1). Thus, the amplitude under consideration at \( t = 0 \) reads

\[
A(s, t = 0) \sim \frac{\Gamma \left( \frac{1}{2} - \alpha's \right)}{\Gamma(-\alpha's)}.
\]  

(2)

In the original Veneziano like amplitudes (1), the poles of the amplitudes correspond to states with increasing spins \( J = \alpha_s \) and the daughter trajectories appear via extensions consisting in adding to (1) terms of the form

\[
\Delta A(s, t) \sim \frac{\Gamma(k' - \alpha_s)\Gamma(m' - \alpha_s)}{\Gamma(l' - \alpha_s - \alpha_t)},
\]  

(3)

where \( k', m' = 1, 2, \ldots \) and \( l' \leq k' + m' \). At vanishing momentum transfer and \( l' = 1 \), the generalization of the amplitude (2) reads

\[
A^{(k)}(s, t = 0) \sim \frac{\Gamma \left( k + \frac{1}{2} - \alpha's \right)}{\Gamma(-\alpha's)}, \quad k = 0, 1, 2, \ldots ,
\]  

(4)

while for \( l' > 1 \) the generalization is

\[
A^{(k,l)}(s, t = 0) \sim \frac{\Gamma \left( k + \frac{1}{2} - \alpha's \right)}{\Gamma(l - \alpha's)}, \quad l < k.
\]  

(5)

The amplitude (5), however, does not satisfy the Adler self-consistency condition at \( l > 0 \). We will obtain (5) for the sake of completeness.

In the case of \( \pi\pi \)-scattering, the poles of amplitudes correspond to states having the quantum numbers of the \( \pi\pi \)-system, i.e. the spin-parities \( J^P = 0^{++}, 1^{--}, 2^{++}, \ldots \). The position of these poles dictates a linear in masses squared spectrum of such states. The spectrum possesses a remarkable feature: It behaves as \( m^2 \sim J + n \), where \( J \) is the spin and \( n \) is the radial number (the state on the \( n \)-th daughter trajectory at a fixed spin). From the string point of view, this means that the energy of oscillatory and rotational motions of a string coincide — such a physical picture holds for the Nambu-Goto strings which emerged from the Veneziano model. The same spectral behaviour is reproduced by the soft wall model [5]. Let us assume that the amplitudes of creation of the \( J \)-th and \( n \)-th states also coincide, i.e., if we mark each state as \( (J, n) \), the states \( (J, n) \) and \( (n, J) \) are completely interchangeable in the amplitude. Then the poles in the amplitude (2) (spin excitations at \( n = 0 \)) may be interpreted as radial excitations with \( J = 0 \). They must be the same as the states saturating the two-point correlation
function of scalar currents in the large-$N_c$ limit,

$$\langle j_S(x)j_S(0) \rangle \sim \int d^4xe^{ipx} \sum_{n=0}^{\infty} \frac{Z_n}{p^2 - m_n^2}. \quad (6)$$

It is plausible to assume that the intermediate states in the amplitude (2) and in the correlator (6) appear with equal probabilities (up to a normalization factor). This assumption provides a possibility to apply the holographic approach to the calculation of the scattering amplitudes.

Thus, our task is to reproduce the expressions (2), (4), and (5) within a five-dimensional setup using the holographic method.

### 3 A holographic model for the amplitude

We first remind the reader about some basic hypotheses behind the AdS/QCD models. According to AdS/CFT prescription [2], each operator $O(x)$ of a 4D gauge theory corresponds to a field $\Phi(x, z)$ of the 5D dual theory, with the boundary value $\Phi(x, \varepsilon)$ being identified with the source $\Phi_0(x)$ for the operator $O(x)$. The generating functional for the correlation functions of the 4D theory is then given by the action of the 5D dual theory via the relation

$$W_{4D}[\Phi_0(x)] = S_{5D}[\Phi(x, \varepsilon)] \quad \text{at} \quad \Phi(x, \varepsilon) = \Phi_0(x). \quad (7)$$

Thus, if action of the 5D dual theory is known, the $n$-point correlation functions of the 4D gauge theory can be obtained by calculating the $n$-th functional derivative, $\Pi_n \sim \frac{\delta^n}{\delta \Phi_0^n} S_{5D}$. It should be emphasized that the relation (7) was not derived; it represents rather a hypothesis that has passed many tests.

The AdS/QCD models are based on a bold assumption that the holographic prescriptions may be directly applied to QCD for finding a QCD dual theory that would describe real QCD in the strong coupling regime. At present, it is not clear at all how to justify this assumption because QCD is very different from the $\mathcal{N} = 4$ SYM theory in the original Maldacena’s proposal [1]. One can try, however, to take a practical viewpoint: Use the holographic prescriptions for building effective models and check how good they are. The ensuing five-dimensional models proved to represent an interesting and compact language allowing to describe the phenomenology of QCD in the large-$N_c$ limit (short reviews are contained in [4]).

The original holographic duality [2] was conjectured for the case of anti-de Sitter (AdS) bulk space as the isometries of AdS$_5$ are equivalent to the 4D conformal symmetry. The metric of the AdS$_5$ space can be parametrized
\[ ds^2 = \frac{R^2}{y^2} (dx_\mu^2 - dy^2), \] (8)

where \( 0 \leq y < \infty \) and \( R \) denotes the AdS radius. Since we want to use a known holographic dictionary, it is preferable to keep the AdS part of the 5D geometry. Our first step is to find a 5D background that leads to the expression (2).

We propose the following ansatz for the 5D effective action,

\[ S_{5D,\text{eff}} = \int d^4x \sqrt{g} \left( \frac{y}{R} \right)^3 e^{-\lambda^2 y^2 (\partial_M \Phi)^2}, \] (9)

where the AdS metric (8) is implied (\( M = 0, 1, 2, 3, 4 \)). Here the dilaton like background providing the mass scale \( \lambda \) is inspired by the soft wall AdS/QCD model [5]. The scalar field \( \Phi(x, y) \) is chosen as the simplest illustrative example. We set \( m_5 = 0 \) for the 5D mass, the case \( m_5 \neq 0 \) will be discussed below.

We perform now the standard steps of the bottom-up AdS/QCD models. The equation of motion for the 4D Fourier transform \( \Phi(p, y) = \int d^4x e^{ipx} \Phi(p, y) \) is

\[ \partial_y (e^{-\lambda^2 y^2 \partial_y \Phi}) + e^{-\lambda^2 y^2} p^2 \Phi = 0. \] (10)

Evaluating the action (9) on the solution leaves the boundary term

\[ S_{5D,\text{eff}} = \int d^4x (\Phi \partial_y \Phi)_{y \to 0}. \] (11)

Requiring that \( \Phi(p, y) = \phi(p, y)\Phi_0(p) \), where \( \phi(p, 0) = 1 \), we interpret \( \Phi_0(p) \) as the Fourier transform of the source of a scalar current corresponding to the field \( \Phi(x, y) \). The solution for \( \phi(p, y) \) bounded as \( y \to \infty \) reads

\[ \phi(p, y) = \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{1}{2} - \frac{p^2}{4\lambda^2} \right) U \left( -\frac{p^2}{4\lambda^2}; \frac{1}{2}, \lambda^2 y^2 \right), \] (12)

where \( U \) denotes the Tricomi confluent hypergeometric function.

Differentiating twice with respect to the source \( \Phi_0 \) in (11) and making use of the expansion

\[ \phi(p, y)_{y \to 0} = 1 - 2 \frac{\Gamma \left( \frac{1}{2} - \frac{p^2}{4\lambda^2} \right)}{\Gamma \left( -\frac{p^2}{4\lambda^2} \right)} \lambda y + O(\lambda^2 y^2), \] (13)

we arrive at our final result,

\[ A(p^2) \sim \frac{\Gamma \left( \frac{1}{2} - \frac{p^2}{4\lambda^2} \right)}{\Gamma \left( -\frac{p^2}{4\lambda^2} \right)}. \] (14)
that is identical to (2) if we identify $p^2 = s$ and $\alpha = (4\lambda^2)^{-1}$. The AdS radius $R$ enters the overall factor in (12). In principle, given a concrete normalization factor in the relation (2), it can be used for a precise matching at $p^2 = 0$.

The amplitude (14) has poles located at

$$p_n^2 = 4\lambda^2 (n + 1/2); \quad n = 0, 1, 2, \ldots ,$$

which correspond to physical masses. Similarly to the soft wall models [5], the same mass spectrum can be also obtained by finding the eigenvalues of Eq. (10) with the boundary condition $\Phi\big|_{y=0} = 0$. The corresponding eigenfunctions are given by the Hermite polynomials, $\Phi_n(y) \sim \sqrt{\frac{2}{\pi \lambda}} \frac{(-1)^n e^{-\lambda^2/4}}{\sqrt{n!}} H_n\left(\frac{\lambda y}{\sqrt{2}}\right)$.

It is easy to find a 5D background that leads to the generalization (4):

$$S^{(k)}_{5D,\text{eff}} = \int d^4x dy \sqrt{g} \left(\frac{y}{R}\right)^{3-2k} e^{-\lambda^2 y^2} (\partial_M \Phi_{(k)})^2, \quad k = 0, 1, 2, \ldots$$

Repeating the same steps as before we first obtain the normalized solution of the corresponding equation of motion,

$$\phi_{(k)}(p, y) = \frac{\Gamma\left(k + \frac{1}{2} - \frac{p^2}{4\lambda^2}\right)}{\Gamma\left(k + \frac{1}{2}\right)} U\left(-\frac{p^2}{4\lambda^2}, \frac{1}{2} - k, \lambda^2 y^2\right),$$

and expand this solution at $y = 0$,

$$\phi_{(k)}(p, y)_{y \to 0} = 1 + \sum_{i=1}^{k} C_i y^{2i} \prod_{j=0}^{i-1} \left(j - \frac{p^2}{4\lambda^2}\right)$$

$$+ (\lambda y)^{2k} \left[ \frac{\Gamma\left(-k - \frac{1}{2}\right)}{\Gamma\left(k + \frac{1}{2}\right)} \frac{\Gamma\left(k + \frac{1}{2} - \frac{p^2}{4\lambda^2}\right)}{\Gamma\left(-\frac{p^2}{4\lambda^2}\right)} \lambda y + \mathcal{O}\left((\lambda y)^3\right) \right].$$

Here the second term yields polynomial in $y$ contributions ($C_i$ are some numerical factors). The generalization of the boundary term (11) is

$$S^{(k)}_{5D,\text{eff}} = \int d^4x \left(\frac{\Phi_{(k)} \partial_y \Phi_{(k)}}{y^{2k}}\right)_{y \to 0}.$$  

Now we should differentiate (19) twice with respect to the source. The contributions coming from the second term in (18) lead to infinities in the final answer. Since these contributions are polynomial in $p^2$ they represent contact terms which can be subtracted. Alternatively, one may $k + 1$ times
differentiate in $p^2$ — those terms irrelevant for physics will not survive. An analogous situation appears when calculating the correlators of higher-spin currents in the soft wall model. It is interesting to note that the contact terms meet the Adler self-consistency condition (disappearance at $p^2 = 0$). After subtracting we have

$$A^{(k)}(p^2) \sim \frac{\Gamma \left(-k - \frac{1}{2}\right) \Gamma \left(k + \frac{1}{2} - \frac{p^2}{4\lambda^2}\right)}{\Gamma \left(k + \frac{1}{2}\right) \Gamma \left(-\frac{p^2}{4\lambda^2}\right)}$$

which proves our statement. We intentionally kept the general factor in (20) that indicates a strong suppression of contributions with large $k$.

To obtain the generalization of the amplitude for higher $l$, the expression (5), we must introduce the mass term $-m^2\Phi^2(k)$ into the action (16). It is tantamount to the replacement $p^2 \to p^2 - m^2$ for the 4D momentum squared. Setting $m^2 = 4\lambda^2l$, we get the expression

$$A^{(k,l)}(p^2) \sim \frac{\Gamma \left(-k - \frac{1}{2}\right) \Gamma \left(k + l + \frac{1}{2} - \frac{p^2}{4\lambda^2}\right)}{\Gamma \left(k + \frac{1}{2}\right) \Gamma \left(l - \frac{p^2}{4\lambda^2}\right)},$$

which has the form of (5). Thus, we see that requirement of the 5D field $\Phi$ being massless is equivalent to the Adler self-consistency condition, $l = 0$.

4 Concluding remarks

In comparison to the soft wall model [5], we have added an extra factor to the 5D background that effectively suppresses the weight of the high energy part of the 5D action. Speaking more concretely, the integrand of the action of the soft wall model diverges at $y \to 0$ while we made this part finite. It is obvious that if field $\Phi$ is a vector, the same purpose is achieved if we replace $(\frac{y}{R})^3$ in the action (9) by $(\frac{y}{R})^3$.

A question appears as to how we should interpret the additional fields $\Phi^{(k)}$ which result in the contributions (20)? We propose the following analogy. The way of inclusion of the gauge higher spin fields, $J > 1$, put forward in the soft wall model [5] is equivalent to considering them as coupled to the background $-\frac{\lambda^2\Phi^2}{y^{2J-2}}$ after contracting the Lorentz indices in the AdS space [8]. The higher spin fields correspond to the QCD operators of higher dimension, $\Delta = 2 + J$. This analogy suggests that the fields $\Phi^{(k)}$ correspond to operators of higher dimension, $\Delta = 3 + 2k$. The contribution of these operators to physical quantities is expected to be suppressed since they have higher twist.
The expression (20) proposes an interesting estimate for the rate of this suppression.

In summary, we have proposed a bottom-up holographic model in which the high energy part is effectively suppressed by a 5D background. It was conjectured that the two-point correlators in such a low-energy model should give a structure of poles of a Veneziano like amplitude at vanishing momentum transfer. This requirement allows us to fix the background. The Adler self-consistency condition turns out to require a massless field from the 5D side. We hope that our observations may open the doors for a holographic description of Veneziano like amplitudes.

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