P-ADIC HEIGHTS AND P-ADIC HODGE THEORY

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ABSTRACT. Using the theory of \((\varphi, \Gamma)\)-modules and the formalism of Selmer complexes we construct the \(p\)-adic height pairing for \(p\)-adic representations with coefficients in an affinoid algebra over \(\mathbb{Q}_p\). For \(p\)-adic representations that are potentially semistable at \(p\), we relate our construction to universal norms and compare it to the \(p\)-adic height pairing of Nekovář.

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INTRODUCTION

0.1. Selmer complexes.

0.1.1. Let $F$ be a number field. We denote by $S_F$ and $S_\infty$ the set of non-archimedean and archimedean places of $F$ respectively. Fix a prime number $p$ and denote by $S_p$ the set of places $q$ above $p$.

Let $S$ be a finite set of non-archimedean places of $F$ containing $S_p$. To simplify notation, set $\Sigma_p = S \setminus S_p$. We denote by $G_{F,S}$ the Galois group of the maximal algebraic extension of $F$ unramified outside $S \cup S_\infty$. For each $q \in S$ we denote by $F_q$ the completion of $F$ with respect to $q$ and by $G_{F_q}$ the absolute Galois group of $F_q$ which we identify with the decomposition group at $q$. We will write $I_q$ for the inertia subgroup of $G_{F_q}$ and $\text{Fr}_q$ for the relative Frobenius over $F_q$.

If $G$ is a topological group and $M$ is a topological $G$-module, we denote by $C^\bullet(G,M)$ the complex of continuous cochains of $G$ with coefficients in $M$.

0.1.2. Let $T$ be a continuous representation of $G_{F,S}$ with coefficients in a complete local noetherian ring $A$ with a finite residue field of characteristic $p$. A local condition at $q \in S$ is a morphism of complexes

$$g_q : U_q^\bullet(T) \to C^\bullet(G_{F_q}, T).$$
To each collection $U^\bullet(T) = (U_q^\bullet(T), g_q)_{q \in S}$ of local conditions one can associate the following diagram

$$\begin{array}{ccc}
C^\bullet(G_{F,S}, T) & \longrightarrow & \bigoplus_{q \in S} C^\bullet(G_{F_q}, T) \\
\bigoplus U_q^\bullet(T), & & \bigoplus_{q \in S} U_q^\bullet(T),
\end{array}$$

where the upper row is the restriction map. The Selmer complex associated to the local conditions $U^\bullet(T)$ is defined as the mapping cone

$$S^\bullet(T, U^\bullet(T)) = \text{cone} \left( C^\bullet(G_{F,S}, T) \oplus \bigoplus_{q \in S} U_q^\bullet(T) \longrightarrow \bigoplus_{q \in S} C^\bullet(G_{F_q}, T) \right)[−1].$$

This notion was introduced by Nekovář in [34], where the machinery of Selmer complexes was developed in full generality.

0.1.3. The most important example of local conditions is provided by Greenberg’s local conditions. If $q \in S$, we will denote by $T_q$ the restriction of $T$ on $G_{F_q}$. Fix, for each $q \in S_p$, a subrepresentation $M_q$ of $T_q$ and define

$$U_q^\bullet(T) = C^\bullet(G_{F_q}, M_q) \quad q \in S_p.$$

For $q \in \Sigma_p$ we consider the unramified local conditions

$$U_q^\bullet(T) = C^\bullet_{ur}(T_q) = \left[ T_q \xrightarrow{\text{Fr}_q^{-1}} T_q \right],$$

where the terms are placed in degrees 0 and 1. To simplify notation, we will write $S^\bullet(T, M)$ for the Selmer complex associated to these conditions and $R\Gamma(T, M)$ for the corresponding object of the derived category. Let $T^\ast(1)$ be the Tate dual of $T$ equipped with Greenberg local conditions $N = (N_q)_{q \in S_p}$ such that $M$ and $N$ are orthogonal to each other under the canonical duality $T \times T^\ast(1) \to A$. In this case, the general construction of cup products for cones gives a pairing

$$\cup : R\Gamma(T, M) \otimes^L_A R\Gamma(T^\ast(1), N) \to A[−3]$$

(see [34], Section 6.3). Nekovář constructed the $p$-adic height pairing

$$h^{\text{sel}} : R\Gamma(T, M) \otimes^L_A R\Gamma(T^\ast(1), N) \to A[−2]$$

as the composition of $\cup$ with the Bockstein map $[1] \beta_{T,M} : R\Gamma(T, M) \to R\Gamma(T, M)[1]$: $h^{\text{sel}}(x, y) = \beta_{T,M}(x) \cup y$.

\[\text{See [34], Section 11.1 or Section 3.2 below for the definition of the Bockstein map.}\]
Passing to cohomology groups \( H^i(T, M) = R^i \Gamma(T, M) \), we obtain a pairing
\[
h^\text{sel}_1 : H^1(T, M) \otimes_A H^1(T^* (1), N) \to A.
\]

0.1.4. The relationship of these constructions to the traditional treatments is the following. Let \( A = \mathcal{O}_E \) be the ring of integers of a local field \( E / \mathbb{Q}_p \) and let \( T \) be a Galois stable \( \mathcal{O}_E \)-lattice of a \( p \)-adic Galois representation \( V \) with coefficients in \( E \). Assume that \( V \) is semistable at all \( q \in S_p \). We say that \( V \) satisfies the Panchishkin condition at \( p \) if, for each \( q \in S_p \), there exists a subrepresentation \( V_q^+ \subset V_q \) such that all Hodge–Tate weights \(^2\) of \( V_q/V_q^+ \) are \( \geq 0 \). Set \( T_q^+ = T \cap V_q^+ \), \( T^+ = (T_q^+)_{q \in S_p} \), and consider the associated Selmer complex \( R \Gamma(T, T^+) \). The first cohomology group \( H^1(T, T^+) \) of \( R \Gamma(T, T^+) \) is very close to the Selmer group defined by Greenberg \([22],[24]\) and therefore to the Bloch–Kato Selmer group \([17]\). It can be shown (see \([34]\), Chapter 11), that, under some mild conditions, the pairing \( h^\text{sel}_1 \) coincides with the \( p \)-adic height pairing constructed in \([33]\) and \([36]\) using universal norms (see also \([39]\)). Note, that Nekovář’s construction has many advantages over the classical definitions. In particular, it allows to study the variation of the \( p \)-adic heights in ordinary families of \( p \)-adic representations (see \([34]\), Section 0.16 and Chapter 11 for further discussion).

0.2. Selmer complexes and \((\varphi, \Gamma)\)-modules.

0.2.1. In this paper we consider Selmer complexes with local conditions coming from the theory of \((\varphi, \Gamma)\)-modules and extend a part of Nekovář’s machinery to this situation. Let now \( A \) be a \( \mathbb{Q}_p \)-affinoid algebra and let \( V \) be a \( p \)-adic representation of \( G_{F, S} \) with coefficients in \( A \). In \([37]\), Pottharst studied Selmer complexes associated to the diagrams of the form (1) in this context. We will consider a slightly more general situation because, for the local conditions \( U_q^*(V) \) that we have in mind, the maps \( g_q : U_q^*(V) \to C^*(G_{F_q}, V) \) are not defined on the level of complexes but only in the derived category of \( A \)-modules.

For each \( q \in S_p \) we denote by \( \Gamma_q \) the Galois group of the cyclotomic \( p \)-extension of \( F_q \). As before, we denote by \( V_q \) the restriction of \( V \) on the decomposition group at \( q \). The theory of \((\varphi, \Gamma)\)-modules associates to \( V_q \) a finitely generated projective module \( D^\dag_{\text{rig}, A}(V) \) over the Robba ring \( \mathcal{R}_{F_q, A} \) equipped with a semilinear Frobenius map \( \varphi \) and a continuous action of \( \Gamma_q \) which commute to each other (see \([18],[13],[16],[29]\)). In \([30]\), Kedlaya, Pottharst and Xiao extended the results of Liu \([31]\) about the cohomology of \((\varphi, \Gamma)\)-modules to the relative case. Their results play a key role in this paper.

\(^2\)We call Hodge–Tate weights the jumps of the Hodge–Tate filtration on the associated de Rham module.
Namely, to each \((\varphi, \Gamma_q)\)-module \(D\) over \(\mathcal{B}_{F_q, A}\) one can associate the Fontaine–Herr complex \(C^\bullet_{\varphi, \gamma_q}(D)\) of \(D\). The cohomology \(H^*(D)\) of \(D\) is defined as the cohomology of \(C^\bullet_{\varphi, \gamma_q}(D)\). If \(D = D^\dagger_{\text{rig}, A}(V)\), there exist isomorphisms \(H^*(D^\dagger_{\text{rig}, A}(V)) \cong H^*(F_q, V)\), but the complexes \(C^\bullet_{\varphi, \gamma_q}(D^\dagger_{\text{rig}, A}(V))\) and \(C^\bullet(G_{F_q}, V_q)\) are not quasi-isomorphic. A simple argument allows us to construct a complex \(K^\bullet(V_q)\) together with quasi-isomorphisms \(\xi_q : C^\bullet(G_{F_q}, V) \to K^\bullet(V_q)\) and \(\alpha_q : C^\bullet_{\varphi, \gamma_q}(D^\dagger_{\text{rig}, A}(V_q)) \to K^\bullet(V_q)\). For each \(q \in S_p\), we choose a \((\varphi, \Gamma_q)\)-submodule \(D_q\) of \(D^\dagger_{\text{rig}, A}(V_q)\) that is a \(\mathcal{B}_{F_q, A}\)-module direct summand of \(D^\dagger_{\text{rig}, A}(V_q)\) and set \(D = (D_q)_{q \in S_p}\). Set

\[
K^\bullet(V) = \left( \bigoplus_{q \in \Sigma_p} C^\bullet(G_{F_q}, V) \right) \oplus \left( \bigoplus_{q \in S_p} K^\bullet(V_q) \right)
\]

and

\[
U^\bullet_q(V, D) = \begin{cases} 
C^\bullet_{\varphi, \gamma_q}(D_q), & \text{if } q \in S_p, \\
C^\bullet_{ur}(V_q), & \text{if } q \in \Sigma_p.
\end{cases}
\]

For each \(q \in S_p\), we have morphisms

\[
f_q : C^\bullet(G_{F_S}, V) \xrightarrow{\text{res}_q} C^\bullet(G_{F_q}, V) \xrightarrow{\xi_q} K^\bullet(V_q),
\]

\[
g_q : U^\bullet_q(V, D) \to C^\bullet_{\varphi, \gamma_q}(D^\dagger_{\text{rig}, A}(V_q)) \xrightarrow{\alpha_q} K^\bullet(V_q).
\]

If \(q \in \Sigma_p\), we define the maps \(f_q : C^\bullet(G_{F_S}, V) \to C^\bullet(G_{F_q}, V)\) and \(g_q : C^\bullet_{ur}(V_q) \to C^\bullet(G_{F_q}, V)\) exactly as in the case of Greenberg local conditions.

Consider the diagram

\[
\begin{aligned}
C^\bullet(G_{F_S}, V) & \xrightarrow{\bigoplus_{q \in S} f_q} K^\bullet(V) \\
\oplus U^\bullet_q(V, D) & \xrightarrow{\oplus g_q} \bigoplus_{q \in S} U^\bullet_q(V, D).
\end{aligned}
\]

We denote by \(S^\bullet(V, D)\) the Selmer complex associated to this diagram and by \(\text{R} \Gamma(V, D)\) the corresponding object in the derived category of \(A\)-modules.

\footnote{This complex was first introduced in \cite{5}.}
0.3. $p$-adic height pairings.

0.3.1. Let $V^*(1)$ be the Tate dual of $V$. We equip $V^*(1)$ with orthogonal local conditions $\mathbf{D}^\perp$ setting

$$\mathbf{D}^\perp_q = \text{Hom}_{\mathbb{F}_q, A}(\mathbf{D}^*_{\text{rig}, A}(V_q)/\mathbf{D}_q, \mathbb{F}_q.A), \quad q \in S_p.$$  

The general machinery gives us a cup product pairing

$$\cup_{V, \mathbf{D}} : \mathbb{R} \Gamma(V, \mathbf{D}) \otimes L_A \mathbb{R} \Gamma(V^*(1), \mathbf{D}^\perp) \to A[-3].$$

This allows us to construct the $p$-adic height pairing exactly in the same way as in the case of Greenberg local conditions.

**Definition.** The $p$-adic height pairing associated to the data $(V, \mathbf{D})$ is defined as the morphism

$$h^\text{sel}_{V, \mathbf{D}} : \mathbb{R} \Gamma(V, \mathbf{D}) \otimes L_A \mathbb{R} \Gamma(V^*(1), \mathbf{D}^\perp) \xrightarrow{\delta_{V, \mathbf{D}}} \mathbb{R} \Gamma(V, \mathbf{D})[-1] \otimes L_A \mathbb{R} \Gamma(V^*(1), \mathbf{D}^\perp) \xrightarrow{\cup_{V, \mathbf{D}}} A[-2],$$

where $\delta_{V, \mathbf{D}}$ denotes the Bockstein map.

The height pairing $h^\text{sel}_{V, \mathbf{D}, M}$ induces a pairing on cohomology groups

$$h^\text{sel}_{V, \mathbf{D}, 1} : H^1(V, \mathbf{D}) \times H^1(V^*(1), \mathbf{D}^\perp) \to A.$$

Applying the machinery of Selmer complexes, we obtain the following result (see Theorem 3.2.4 below).

**Theorem I.** We have a commutative diagram

$$\begin{array}{ccc}
\mathbb{R} \Gamma(V, \mathbf{D}) \otimes L_A \mathbb{R} \Gamma(V^*(1), \mathbf{D}^\perp) & \xrightarrow{h^\text{sel}_{V, \mathbf{D}}} & A[-2] \\
| s_{12} \downarrow & & \downarrow = \\
\mathbb{R} \Gamma(V^*(1), \mathbf{D}^\perp) \otimes L_A \mathbb{R} \Gamma(V, \mathbf{D}) & \xrightarrow{h^\text{sel}_{V, \mathbf{D}, 1}} & A[-2],
\end{array}$$

where $s_{12}(a \otimes b) = -(1)^{\text{deg}(a)\text{deg}(b)} b \otimes a$. In particular, the pairing $h^\text{sel}_{V, \mathbf{D}, 1}$ is skew symmetric.

0.3.2. Assume that $A = E$, where $E$ is a finite extension of $\mathbb{Q}_p$. For each family $\mathbf{D} = (\mathbf{D}_q)_{q \in S_p}$ satisfying the conditions N1-2) of Section 5 we construct a pairing

$$h^\text{norm}_{V, \mathbf{D}} : H^1(V, \mathbf{D}) \times H^1(V^*(1), \mathbf{D}^\perp) \to E,$$

which can be seen as a direct generalization of the $p$-adic height pairing, constructed for representations satisfying the Panchishkin condition using universal norms. The following theorem generalizes Theorem 11.3.9 of [34] (see Theorem 5.2.2 below).
Theorem II. Let $V$ be a $p$-adic representation of $G_{F,S}$ with coefficients in a finite extension $E$ of $\mathbb{Q}_p$. Assume that the family $D = (D_q)_{q \in S_p}$ satisfies the conditions N1-2). Then

$$h_{V,D}^\text{norm} = h_{V,D,1}^\text{sel}.$$ 

0.3.3. We denote by $D_{\text{dR}}$, $D_{\text{st}}$ and $D_{\text{st}}$ Fontaine’s classical functors. Let $V$ be a $p$-adic representation with coefficients in $E/\mathbb{Q}_p$. Assume that the restriction of $V$ on $G_{F_q}$ is potentially semistable for all $q \in S_p$, and that $V$ satisfies the following condition

**S)** $D_{\text{crys}}(V)^{\varphi=1} = D_{\text{crys}}(V^*(1))^{\varphi=1} = 0$, $\forall q \in S_p$.

For each $q \in S_p$ we fix a splitting $w_q : D_{\text{dR}}(V_q) \to D_{\text{dR}}(V_q)/F^0D_{\text{dR}}(V_q)$ of the canonical projection $D_{\text{dR}}(V_q) \to D_{\text{dR}}(V_q)/\text{Fil}^0D_{\text{dR}}(V_q)$ and set $w = (w_q)_{q \in S_p}$. In this situation, Nekovář [33] constructed a $p$-adic height pairing

$$h_{V,w}^\text{Hodge} : H^1_f(V) \times H^1_f(V^*(1)) \to E$$

on the Bloch–Kato Selmer groups of $V$ and $V^*(1)$, which is defined using the Bloch–Kato exponential map and depends on the choice of splittings $w$.

Let $q \in S_p$, and let $L$ be a finite extension of $F_q$ such that $V_q$ is semistable over $L$. The semistable module $D_{\text{st}/L}(V_q)$ is a finite dimensional vector space over the maximal unramified subextension $L_0$ of $L$, equipped with a Frobenius $\varphi$, a monodromy $N$, and an action of $G_{L/F_q} = \text{Gal}(L/F_q)$.

**Definition.** Let $q \in S_p$. We say that a $(\varphi,N,G_{L/F_q})$-submodule $D_q$ of $D_{\text{st}/L}(V_q)$ is a splitting submodule if

$$D_{\text{dR}/L}(V_q) = D_{q,L} \oplus \text{Fil}^0D_{\text{dR}/L}(V_q), \quad D_{q,L} = D_q \otimes_{L_0} L$$

as $L$-vector spaces.

It is easy to see, that each splitting submodule $D_q$ defines a splitting of the Hodge filtration of $D_{\text{dR}}(V)$, which we denote by $w_{D,q}$. For each family $D = (D_q)_{q \in S_p}$ of splitting submodules we construct a pairing

$$h_{V,D}^{\text{spl}} : H^1_f(V) \times H^1_f(V^*(1)) \to E$$

using the theory of $(\varphi,\Gamma)$-modules and prove that

$$h_{V,D}^{\text{spl}} = h_{V,w_D}^\text{Hodge}$$

(see Proposition 6.2.3). Let $D_q$ denote the $(\varphi,\Gamma_q)$-submodule of $D_{\text{rig}}^\varphi(V_q)$ associated to $D_q$ by Berger [10] and let $D = (D_q)_{q \in S_p}$. In the following theorem we compare this pairing with previous constructions (see Theorem 6.3.4 and Corollary 6.3.5).
Theorem III. Assume that \((V, D)\) satisfies the conditions S and N2. Then

i) \(H^1(V, D) = H^1_f(V)\) and \(H^1(V^*(1), D^\perp) = H^1_f(V^*(1))\);

ii) The height pairings \(h^\text{sel}_{V, D, 1}\), \(h^\text{norm}_{V, D}\) and \(h^\text{spl}_{V, D}\) coincide.

0.3.4. If \(F = Q\), we can slightly relax the conditions on \((V, D)\) in Theorem III. We replace these conditions by the conditions F1-2) of Section 7.1, which reflect the conjectural behavior of \(V\) at \(p\) in the presence of trivial zeros \([4], [6]\). We show, that under these conditions we have a canonically splitting exact sequence

\[0 \rightarrow H^0(D') \rightarrow H^1(V, D) \xrightarrow{\text{spl}_{V, f}} H^1_f(V) \rightarrow 0,\]

where \(D' = D^\perp_{\text{rig}}(V_p)/D\). By modifying previous constructions, one can define a pairing

\[h^\text{norm}_{V, D} : H^1_f(V) \times H^1_f(V^*(1)) \rightarrow E\]

The following result is a simplified form of Theorem 7.2.2 of Section 7, which can be seen as a generalization of Theorem 11.4.6 of [34].

Theorem IV. Let \(V\) be a \(p\)-adic representation of \(G_{Q,S}\) that is potentially semistable at \(p\) and satisfies the conditions F1-2). Then

i) \(h^\text{norm}_{V, D} = h^\text{spl}_{V, D}\);

ii) For all \(x \in H^1_f(V)\) and \(y \in H^1_f(V^*(1))\) we have

\[h^\text{sel}_{V, D}(\text{spl}_{V, f}(x), \text{spl}_{V^*(1), f}(y)) = h^\text{norm}_{V, D}(x, y).\]

0.4. The organization of this paper. This paper is very technical by the nature, and in Sections 1-2 we assemble necessary preliminaries. In Section 1 we recall the formalism of cup products for cones following [34] (see also [35]), and prove some auxiliary cohomological results. In Section 2 we review the theory of \((\varphi, \Gamma)\)-modules and define the complex \(K^*(V_q)\). The reader familiar with \((\varphi, \Gamma)\)-modules can skip these sections on first reading. In Section 3.1 we construct Selmer complexes \(R\Gamma(V, D)\) and study their first properties. The \(p\)-adic height pairing \(h^\text{sel}_{V, D}\) is defined in Section 3.2. Theorem I (Theorem 3.2.4 of Section 3.2) follows directly from the definition of \(h^\text{sel}_{V, D}\) and the machinery of Section 1. Preliminary results about splitting submodules are assembled in Section 4. In Sections 5-7 we construct the pairings \(h^\text{norm}_{V, D}\) and \(h^\text{spl}_{V, D}\) and prove Theorems II, III and IV.

0.5. Remarks. 1) In the forthcoming joint paper with K. Büyükboduk [7], we prove the Rubin style formula for our height pairing and apply it to the study of extra-zeros of \(p\)-adic \(L\)-functions.

2) During the preparation of this paper, I learned from L. Xiao of an independent work in progress of Rufe Ren on this subject.
1. Complexes and Products

1.1. The complex $T^\bullet(A^\bullet)$.  

1.1.1. If $R$ is a commutative ring, we write $\mathcal{K}(R)$ for the category of complexes of $R$-modules and $\mathcal{K}_{\text{perf}}(R)$ for the subcategory of $\mathcal{K}(R)$ consisting of complexes $C^\bullet = (C^n, d^n)$ such that $H^n(C^\bullet)$ are finitely generated over $R$ for all $n \in \mathbb{Z}$. We write $\mathcal{D}(R)$ and $\mathcal{D}_{\text{perf}}(R)$ for the corresponding derived categories and denote by $[\cdot]: \mathcal{K}(R) \to \mathcal{D}(R)$, $(\ast \in \{\emptyset, \text{ft}\})$ the obvious functors. We will also consider the subcategories $\mathcal{K}_{\text{perf}}^{[a,b]}(R)$, $(a \leq b)$ consisting of objects of $\mathcal{K}_{\text{perf}}(R)$ whose cohomologies are concentrated in degrees $[a, b]$.

A perfect complex of $R$-modules is one of the form

$$0 \to P_a \to P_{a+1} \to \ldots \to P_b \to 0,$$

where each $P_i$ is a finitely generated projective $R$-module. If $R$ is noetherian, we denote by $\mathcal{H}_{\text{perf}}^{[a,b]}(R)$ the full subcategory of $\mathcal{D}_{\text{perf}}(R)$ consisting of objects quasi-isomorphic to perfect complexes concentrated in degrees $[a, b]$.

If $C^\bullet = (C^n, d^n)_{n\in\mathbb{Z}}$ is a complex of $R$-modules and $m \in \mathbb{Z}$, we will denote by $C^\bullet[m]$ the complex defined by $C^\bullet[m] = C^{n+m}$ and $d^n_{C^\bullet[m]}(x) = (-1)^m d^n_{C^\bullet}(x)$. We will often write $d^n$ or just simply $d$ instead $d^n_{C^\bullet}$. For each $m$, the truncation $\tau_{\geq m} C^\bullet$ of $C^\bullet$ is the complex

$$0 \to \text{coker}(d^{m-1}) \to C^{m+1} \to C^{m+2} \to \ldots.$$

Therefore

$$H^i(\tau_{\geq m} C^\bullet) = \begin{cases} 0, & \text{if } i < m, \\ H^i(C^\bullet), & \text{if } i \geq m. \end{cases}$$

The tensor product $A^\bullet \otimes B^\bullet$ of two complexes $A^\bullet$ and $B^\bullet$ is defined by

$$(A^\bullet \otimes B^\bullet)^n = \bigoplus_{i \in \mathbb{Z}} (A^i \otimes B^{n-i}),$$

d$(a_i \otimes b_{n-i}) = dx_i \otimes y_{n-i} + (-1)^i a_i \otimes b_{n-i}$, $a_i \in A^i$, $b_{n-i} \in B^{n-i}$.

We denote by $s_{12} : A^\bullet \otimes B^\bullet \to B^\bullet \otimes A^\bullet$ the transposition

$$s_{12}(a_n \otimes b_m) = (-1)^{nm} b_m \otimes a_n, \quad a_n \in A^n, \quad b_m \in B^m.$$

It is easy to check that $s_{12}$ is a morphism of complexes. We will also consider the map $s_{12}^* : A^\bullet \otimes B^\bullet \to B^\bullet \otimes A^\bullet$ given by

$$s_{12}^*(a_n \otimes b_m) = b_m \otimes a_n,$$

which is not a morphism of complexes in general.

Recall that a homotopy $h : f \rightsquigarrow g$ between two morphisms $f, g : A^\bullet \to B^\bullet$ is a family of maps $h = (h^n : A^{n+1} \to B^n)$ such that $dh + hd = g - f$. We will sometimes write $h$ instead $h^n$. A second order homotopy $H : h \rightsquigarrow k$ between
homotopies \( h, k : f \rightsquigarrow g \) is a collection of maps \( H = (H^n : A^{n+2} \to B^n) \) such that \( Hd - dH = k - h \).

If \( f_i : A^i_1 \to B^i_1 \) (\( i = 1, 2 \)) and \( g_i : A^i_2 \to B^i_2 \) (\( i = 1, 2 \)) are morphisms of complexes and \( h : f_1 \rightsquigarrow f_2 \) and \( k : g_1 \rightsquigarrow g_2 \) are homotopies between them, then the formula

(2) \[(h \otimes k)_1(x_n \otimes y_m) = h(x_n) \otimes g_1(y_m) + (-1)^n f_2(x_n) \otimes k(y_m),\]

where \( x_n \in A^n_1, y_m \in A^n_2 \), defines a homotopy \( (h \otimes k)_1 : f_1 \otimes g_1 \rightsquigarrow f_2 \otimes g_2 \).

1.1.2. For the content of this subsection we refer the reader to [41], §3.1.

If \( f : A^\bullet \to B^\bullet \) is a morphism of complexes, the cone of \( f \) is defined to be the complex

\( \text{cone}(f) = A^\bullet[1] \oplus B^\bullet, \)

with differentials

\[ d^n(a_{n+1}, b_n) = (-d^{n+1}(a_{n+1}), f(a_{n+1}) + d^n(b_n)). \]

We have a canonical distinguished triangle

\[ A^\bullet \xrightarrow{f} B^\bullet \to \text{cone}(f) \to A^\bullet[1]. \]

We say that a diagram of complexes of the form

(3) \[
\begin{array}{ccc}
A^\bullet & \xrightarrow{f_1} & B^\bullet \\
\downarrow{\alpha_1} & & \downarrow{\alpha_2} \\
A_2^\bullet & \xrightarrow{f_2} & B_2^\bullet
\end{array}
\]

is commutative up to homotopy, if there exists a homotopy

\( h : \alpha_2 \circ f_1 \rightsquigarrow f_2 \circ \alpha_1. \)

In this case, the formula

\[ c(\alpha_1, \alpha_2, h)^n(a_{n+1}, b_n) = (\alpha_1(a_{n+1}), \alpha_2(b_n) + h^n(a_n)) \]

defines a morphism of complexes

(4) \[ c(\alpha_1, \alpha_2, h) : \text{cone}(f_1) \to \text{cone}(f_2). \]
Assume that, in addition to (3), we have a diagram

\[
\begin{array}{ccc}
A_1 \xrightarrow{f_1} & B_1 \\
\alpha' \downarrow & & \alpha' \\
A_2 \xrightarrow{f_2} & B_2
\end{array}
\]

together with homotopies

\[
k_1 : \alpha_1 \sim \alpha'_1
\]
\[
k_2 : \alpha_2 \sim \alpha'_2
\]

and a second order homotopy

\[
H : f_2 \circ k_1 + h' \sim k_2 \circ f_1 + h.
\]

Then the map

\[
(a_{n+1}, b_n) \mapsto (-k_1(a_{n+1}), k_2(b_n) + H(a_{n+1}))
\]

defines a homotopy \( c(\alpha_1, \alpha_2, h) \sim c(\alpha'_1, \alpha'_2, h') \).

1.1.3. In the remainder of this section \( R \) is a commutative ring and all complexes are complexes of \( R \)-modules. Let \( A^\bullet = (A^n, d^n) \) be a complex equipped with a morphism \( \varphi : A^\bullet \to A^\bullet \). By definition, the total complex

\[
T^\bullet(A^\bullet) = \text{Tot}(A^\bullet \xrightarrow{\varphi^{-1}} A^\bullet).
\]

is given by \( T^n(A^\bullet) = A^{n-1} \oplus A^n \) with differentials

\[
d^n(a_{n-1}, a_n) = (d^{n-1}a_{n-1} + (-1)^n(\varphi - 1)a_n, d^n a_n), \quad (a_{n-1}, a_n) \in T^n(A^\bullet).
\]

If \( A^\bullet \) and \( B^\bullet \) are two complexes equipped with morphisms \( \varphi : A^\bullet \to A^\bullet \) and \( \psi : B^\bullet \to B^\bullet \), and if \( \alpha : A^\bullet \to B^\bullet \) is a morphism such that \( \alpha \circ \varphi = \psi \circ \alpha \), then \( \alpha \) induces a morphism \( T(\alpha) : T^\bullet(A^\bullet) \to T^\bullet(B^\bullet) \). We will often write \( \alpha \) instead \( T(\alpha) \) to simplify notation.

**Lemma 1.1.4.** Let \( A^\bullet \) and \( B^\bullet \) be two complexes equipped with morphisms \( \varphi : A^\bullet \to A^\bullet \) and \( \psi : B^\bullet \to B^\bullet \), and let \( \alpha_i : A^\bullet \to B^\bullet \) \((i = 1, 2)\) be two morphisms such that

\[
\alpha_i \circ \varphi = \psi \circ \alpha_i \quad i = 1, 2.
\]

If \( h : \alpha_1 \sim \alpha_2 \) is a homotopy between \( \alpha_1 \) and \( \alpha_2 \) such that \( h \circ \varphi = \psi \circ h \), then the collection of maps \( h_T = (h^n_T : T^{n+1}(A^\bullet) \to T^n(B^\bullet)) \)
defined by \( h^n_T(a_n, a_{n+1}) = (h(a_n), h(a_{n+1})) \) is a homotopy between \( T(\alpha_1) \) and \( T(\alpha_2) \).
Proof. The proof of this lemma is a direct computation and is omitted here.

In the remainder of this subsection we will consider triples $(A_i^*, A_i^*, A_i^*)$ of complexes of $R$-modules equipped with the following structures

A1) Morphisms $\varphi_i : A_i^* \to A_i^*$ ($i = 1, 2, 3$).

A2) A morphism $\cup_A : A_1^* \otimes A_2^* \to A_3^*$ which satisfies

$$\cup_A \circ (\varphi_1 \otimes \varphi_2) = \varphi_3 \circ \cup_A.$$  

Proposition 1.1.5. Assume that a triple $(A_i^*, \varphi_i)$ $(1 \leq i \leq 3)$ satisfies the conditions A1-2). Then the map

$$\cup_A^T : T^*(A_1^*) \otimes T^*(A_2^*) \to T^*(A_3^*)$$

given by

$$(x_{n-1}, x_n) \cup_A^T (y_{m-1}, y_m) = (x_n \cup_A y_{m-1} + (-1)^m x_{n-1} \cup_A \varphi_2(y_m), x_n \cup_A y_m),$$

is a morphism of complexes.

Proof. This proposition is well known to the experts (compare, for example, to [35] Proposition 3.1). It follows from a direct computation which we recall for the convenience of the reader. Let $(x_{n-1}, x_n) \in T^n(A_1^*)$ and $(y_{m-1}, y_m) \in T^m(A_2^*)$. Then

$$d((x_{n-1}, x_n) \cup_A^T (y_{m-1}, y_m)) =$$

$$= d(x_n \cup_A y_{m-1} + (-1)^m x_{n-1} \cup_A \varphi_2(y_m), x_n \cup_A y_m) = (z_{n+m}, z_{n+m+1}),$$

where

$$z_{n+m} = dx_n \cup_A y_{m-1} + (-1)^n x_n \cup_A dy_{m-1} + (-1)^m dx_{n-1} \cup_A \varphi_2(y_m) +$$

$$+ (-1)^{m+n-1} x_{n-1} \cup_A d(\varphi_2(y_m)) + (-1)^{n+m}(\varphi_3 - 1)(x_n \cup_A y_m)$$

and $z_{n+m+1} = d(x_n \cup_A y_m)$. On the other hand

$$\cup_A^T \circ d((x_{n-1}, x_n) \otimes (y_{m-1}, y_m)) =$$

$$= \cup_A^T \circ ((dx_{n-1} + (-1)^n(\varphi_1 - 1)x_n, dx_n) \otimes (y_{m-1}, y_m)) +$$

$$+ (-1)^n \cup_A^T \circ ((x_{n-1}, x_n) \otimes (dy_{m-1} + (-1)^m(\varphi_2 - 1)y_m, dy_m)) =$$

$$= (u_{n+m}, u_{n+m+1}),$$

where

$$u_{n+m} = dx_n \cup_A y_{m-1} + (-1)^m(dx_{n-1} + (-1)^n(\varphi_1 - 1)x_n) \cup \varphi_2(y_m) +$$

$$+ (-1)^n x_n \cup (dy_{m-1} + (-1)^m(\varphi_2 - 1)y_m) + (-1)^{n+m-1} x_{n-1} \cup \varphi_2(dy_m),$$

and
and \( u_{n+m+1} = dx_n \cup_A y_m + (-1)^n x_n \cup_A dy_m \). Now the proposition follows from the formula
\[
d(x_n \cup_A y_m) = dx_n \cup_A y_m + (-1)^n x_n \cup_A dy_m
\]
and the assumption A2) that reads \( \varphi_1(x_n) \cup_A \varphi_2(y_m) = \varphi_3(x_n \cup_A y_m) \). □

**Proposition 1.1.6.** Let \((A^•_1, \varphi_i)\) and \((B^•_1, \psi_i)\) \((1 \leq i \leq 3)\) be two triples of complexes that satisfy the conditions A1-2). Assume that they are equipped with morphisms
\[
\alpha_i : A^•_i \to B^•_i,
\]
such that \( \alpha_i \circ \varphi_i = \psi_i \circ \alpha_i \) for all \( 1 \leq i \leq 3 \). Assume, in addition, that in the diagram

\[
\begin{array}{ccc}
A^•_1 \otimes A^•_2 & \xrightarrow{\cup_A} & A^•_3 \\
\downarrow \alpha_1 \otimes \alpha_2 & & \downarrow \alpha_3 \\
B^•_1 \otimes B^•_2 & \xrightarrow{\cup_B} & B^•_3
\end{array}
\]

there exists a homotopy
\[
h : \alpha_3 \circ \cup_A \rightsquigarrow \cup_B \circ (\alpha_1 \otimes \alpha_2).
\]
such that \( h \circ (\varphi_1 \otimes \varphi_2) = \psi_3 \circ h \). Then the collection \( h_T \) of maps
\[
h^k_T : \bigoplus_{m+n=k+1} (T^n(A^•_1) \otimes T^m(A^•_2)) \to T^k(B^•_3)
\]
defined by
\[
h^k_T((x_{n-1}, x_n) \otimes (y_{m-1} \otimes y_m)) = (h(x_n \otimes y_{m-1}) + (-1)^m h(x_{n-1} \otimes \varphi_2(y_m)), h(x_n \otimes y_m)).
\]
provides a homotopy \( h_T : \alpha_3 \circ \cup^T_A \rightsquigarrow \cup^T_B \circ (\alpha_1 \otimes \alpha_2) : 
\[
\begin{array}{ccc}
T^•(A^•_1) \otimes T^•(A^•_2) & \xrightarrow{\cup_A^T} & T^•(A^•_3) \\
\downarrow \alpha_1 \otimes \alpha_2 & & \downarrow \alpha_3 \\
T^•(B^•_1) \otimes T^•(B^•_2) & \xrightarrow{\cup_B^T} & T^•(B^•_3)
\end{array}
\]
Proof. Again, the proof is a routine computation. Let \((x_{n-1}, x_n) \in T^n(A^*_1)\) and \((y_{m-1}, y_m) \in T^m(A^*_2)\). We have
\[
d((x_{n-1}, x_n) \otimes (y_{m-1}, y_m)) = (dx_{n-1} + (-1)^n(\varphi_1 - 1)x_n, dx_n) \otimes (y_{m-1} - 1, y_m) +
+ (-1)^n(x_{n-1}, x_n) \otimes (dy_{m-1} + (-1)^m(\varphi_2 - 1)y_m, dy_m),
\]
and therefore
\[
h_T \circ d((x_{n-1}, x_n) \otimes (y_{m-1}, y_m)) = (a, b),
\]
where
\[
a = h(dx_n \otimes y_{m-1}) + (-1)^n h((dx_{n-1} + (-1)^n(\varphi_1 - 1)x_n) \otimes \varphi_2(y_m)) +
+ (-1)^n(h(x_n \otimes (dy_{m-1} + (-1)^m(\varphi_2 - 1)y_m)) +
+ (-1)^{n+m-1}h(x_{n-1} \otimes \varphi_2(dy_m)) =
= h \circ d(x_n \otimes y_{m-1}) + (-1)^m h \circ d(x_{n-1} \otimes \varphi_2(y_m)) +
+ (-1)^{n+m}(\psi_3 - 1) h(x_n \otimes y_m)
\]
and
\[
b = h(dx_n \otimes y_m) + (-1)^n h(x_n \otimes dy_m) = h \circ d(x_n \otimes y_m).
\]

On the other hand
\[
d \circ h_T((x_{n-1}, x_n) \otimes (y_{m-1}, y_m)) =
= d(h(x_n \otimes y_{m-1}) + (-1)^m h(x_{n-1} \otimes \varphi_2(y_m)), h(x_n \otimes y_m)) =
= (d \circ h(x_n \otimes y_{m-1}) + (-1)^m d \circ h(x_{n-1} \otimes \varphi_2(y_m)) +
+ (-1)^{n+m-1}(\psi_3 - 1) h(x_n \otimes y_m), d \circ h(x_n \otimes y_m)).
\]
Thus
\[
(h_T d + dh_T)((x_{n-1}, x_n) \otimes (y_{m-1}, y_m)) =
= ((h d + dh)(x_n \otimes y_{m-1}) + (-1)^m(h d + dh)(x_{n-1} \otimes \varphi_2(y_m)),
(h d + dh)(x_n \otimes y_m)) =
= ((\alpha_1(x_n) \cup_B \alpha_2(y_{m-1}) - \alpha_3(x_n \cup_A y_{m-1})) +
(-1)^m(\alpha_1(x_{n-1}) \cup_B \varphi_2(\alpha_2(y_m)) - \alpha_3(x_{n-1} \cup_A \varphi_2(y_m)),
\alpha_1(x_n) \cup_B \alpha_2(y_m) - \alpha_3(x_n \cup_A y_m)) =
= ((\cup_T^B \circ (\alpha_1 \otimes \alpha_2) - \alpha_3 \circ \cup_T^A)((x_{n-1}, x_n) \otimes (y_{m-1}, y_m)).
\]
and the proposition is proved. \(\square\)

Proposition 1.1.7. Let \(A^*_i\) \((1 \leq i \leq 4)\) be four complexes equipped with morphisms \(\varphi_i : A^*_1 \rightarrow A^*_i\) and such that
a) The triples \((A^*_1, A^*_2, A^*_3)\) and \((A^*_1, A^*_2, A^*_4)\) satisfy A1-2).
b) The complexes $A^*_i (i = 1, 2)$ are equipped with morphisms $\mathcal{F}_i : A^*_i \rightarrow A^*_i$ which commute with morphisms $\mathcal{F}_i$:

$$\mathcal{F}_i \circ \phi_i = \phi_i \circ \mathcal{F}_i, \quad i = 1, 2.$$

c) There exists a morphism $\mathcal{F}_{34} : A^*_3 \rightarrow A^*_4$ such that

$$\mathcal{F}_{34} \circ \phi_3 = \phi_4 \circ \mathcal{F}_{34}.$$

d) The diagram

\[
\begin{array}{c}
A^*_1 \otimes A^*_2 \xrightarrow{\cup_A} A^*_3 \\
\downarrow s_{12} \circ (\mathcal{F}_1 \otimes \mathcal{F}_2) \quad \quad \quad \mathcal{F}_{34} \\
A^*_2 \otimes A^*_1 \xrightarrow{\cup_A} A^*_4
\end{array}
\]

commutes.

Let $\mathcal{F}_i : T^*(A^*_i) \rightarrow T^*(A^*_i) (i = 1, 2)$ and $\mathcal{F}_{34} : T^*(A^*_3) \rightarrow T^*(A^*_4)$ be the morphisms (which we denote again by the same letter) defined by

$$\mathcal{F}_1(x_{n-1}, x_n) = (\mathcal{F}_1(x_{n-1}), \mathcal{F}_1(x_n)), \quad \mathcal{F}_{34}(x_{n-1}, x_n) = (\mathcal{F}_{34}(x_{n-1}), \mathcal{F}_{34}(x_n)).$$

Then in the diagram

\[
\begin{array}{cc}
T^*(A^*_1) \otimes T^*(A^*_2) \xrightarrow{\cup_A^T} T^*(A^*_3) \\
\downarrow s_{12} \circ (\mathcal{F}_1 \otimes \mathcal{F}_2) \quad \quad \quad \mathcal{F}_{34} \\
T^*(A^*_2) \otimes T^*(A^*_1) \xrightarrow{\cup_A^T} T^*(A^*_4)
\end{array}
\]

the maps $\mathcal{F}_{34} \circ \cup_A^T$ and $\cup_A^T \circ s_{12} \circ (\mathcal{F}_1 \otimes \mathcal{F}_2)$ are homotopic.

**Proof.** Let $(x_{n-1}, x_n) \in T^n(A^*_1)$ and $(y_{m-1}, y_m) \in T^m(A^*_2)$. Then

$$(6) \quad \mathcal{F}_{34}(x_{n-1}, x_n) \cup_A^T (y_{m-1}, y_m) =$$

$$= (\mathcal{F}_{34}(x_n \cup_A y_{m-1}), (-1)^m \mathcal{F}_{34}(x_{n-1} \cup_A \varphi_2(y_m)), \mathcal{F}_{34}(x_n \cup_A y_m))$$

and

$$(7) \quad \cup_A^T \circ s_{12} \circ (\mathcal{F}_1 \otimes \mathcal{F}_2)((x_{n-1}, x_n) \otimes (y_{m-1}, y_m)) =$$

$$= (-1)^m \mathcal{F}_2(y_{m-1}, y_m) \cup_A \mathcal{F}_1(x_{n-1}, x_n) =$$

$$= (-1)^m \mathcal{F}_2(y_{m-1} \cup_A \mathcal{F}_1(x_{n-1}) + (-1)^n \mathcal{F}_2(y_{m-1}) \cup_A \varphi_1(\mathcal{F}_1(x_n)),$$

$$\mathcal{F}_2(y_m) \cup_A \mathcal{F}_1(x_n)) =$$

$$= ((-1)^m \mathcal{F}_{34}(x_{n-1} \cup_A y_{m}) + \mathcal{F}_{34}(\varphi_1(x_n) \cup_A y_{m-1}), \mathcal{F}_{34}(x_n \cup_A y_m)).$$
Define
\[ h^k_{\mathcal{F}} : \bigoplus_{m+n=k+1} (T^n(A_1^\bullet) \otimes T^m(A_2^\bullet)) \to T^k(A_3^\bullet), \]
by
\[ (8) \quad h^k_{\mathcal{F}}((x_{n-1}, x_n) \otimes (y_{m-1} \otimes y_m)) = (-1)^{n-1}(\mathcal{F}_{34}(x_{n-1} \cup_A y_{m-1}), 0). \]
Then
\[ (9) \quad dh_{\mathcal{F}}((x_{n-1}, x_n) \otimes (y_{m-1} \otimes y_m)) = \]
\[ = ((-1)^{n-1} \mathcal{F}_{34}(d(x_{n-1} \cup_A y_{m-1}) + (-1)^{n-1}x_{n-1} \cup_A dy_{m-1}), 0) = \]
\[ = (\mathcal{F}_{34}(x_{n-1} \cup_A d(y_{m-1})), 0), \]
and
\[ (10) \quad h_{\mathcal{F}}d((x_{n-1}, x_n) \otimes (y_{m-1} \otimes y_m)) = \]
\[ = h_{\mathcal{F}}((d_{n-1}+(-1)^{n}(\varphi_1-1)x_n, dx_{n}) \otimes (y_{m-1}, y_m) + \]
\[ = ((-1)^{n} \mathcal{F}_{34}(d_{n-1} \cup_A y_{m-1}), 0), \]
From (8)-(10) it follows that
\[ \cup_A^T \circ s_{12} \circ (\mathcal{F}_1 \otimes \mathcal{F}_2) - \mathcal{F}_{34} \circ \cup_A^T = dh_{\mathcal{F}} + h_{\mathcal{F}}d \]
and the proposition is proved.

1.2. Products.

1.2.1. In this subsection we review the construction of products for cones following Nekovář [34] and Nizioł [35]. We will work with the following data

P1) Diagrams
\[ A_i^\bullet \xrightarrow{f_i} C_i^\bullet \xleftarrow{g_i} B_i^\bullet, \quad i = 1, 2, 3, \]
where \( A_i^\bullet, B_i^\bullet \) and \( C_i^\bullet \) are complexes of \( R \)-modules.

P2) Morphisms
\[ \cup_A : A_1^\bullet \otimes A_2^\bullet \to A_3^\bullet, \]
\[ \cup_B : B_1^\bullet \otimes B_2^\bullet \to B_3^\bullet, \]
\[ \cup_C : C_1^\bullet \otimes C_2^\bullet \to C_3^\bullet. \]
**P3)** A pair of homotopies \( h = (h_f, h_g) \)

\[
\begin{align*}
  h_f : & \cup C \circ (f_1 \otimes f_2) \leadsto f_3 \circ A, \\
  h_g : & \cup C \circ (g_1 \otimes g_2) \leadsto g_3 \circ B.
\end{align*}
\]

Define

\[
E_i^* = \text{cone} \left( A_i^o \oplus B_i^o \xrightarrow{f_i-g_i} C_i^o \right) [-1].
\]

Thus

\[
E_i^n = A_i^n \oplus B_i^n \oplus C_i^{n-1}
\]

with \( d(a_n, b_n, c_{n-1}) = (da_n, db_n, -f_i(a_n) + g_i(b_n) - dc_{n-1}) \).

**Proposition 1.2.2.**  

i) Given the data **P1-3)**, for each \( r \in R \) the formula

\[
(a_n, b_n, c_{n-1}) \cup_{r,h} (a'_m, b'_m, c'_{m-1}) = \\
(a_n \cup A'_m, b_n \cup B'_m, c_{n-1} \cup C (rf_2(a'_m) + (1-r)g_2(b'_m))) + \\
(-1)^n((1-r)f_1(a_n) + rg_1(b_n)) \cup C c'_{m-1} - (h_f (a_n \otimes a'_m) - h_g (b_n \otimes b'_m)))
\]

defines a morphism in \( \mathcal{X} (R) \)

\[
\cup_{r,h} : E_1^o \otimes E_2^o \rightarrow E_3^o.
\]

ii) If \( r_1, r_2 \in R \), then the map

\[
k : E_1^o \otimes E_2^o \rightarrow E_3^o [-1],
\]

given by

\[
k((a_n, b_n, c_{n-1}) \otimes (a'_m, b'_m, c'_{m-1})) = (0, 0, (-1)^n (r_1 - r_2)c_{n-1} \cup C c'_{m-1})
\]

for all \( (a_n, b_n, c_{n-1}) \in E_1^o \) and \( (a'_m, b'_m, c'_{m-1}) \in E_2^o \), defines a homotopy

\[
k : \cup_{r_1,h} \leadsto \cup_{r_2,h}.
\]

iii) If \( h' = (h'_f, h'_g) \) is another pair of homotopies as in **P3)**, and if \( \alpha : h_f \leadsto h'_f \) and \( \beta : h_g \leadsto h'_g \) is a pair of second order homotopies, then the map

\[
s : E_1^o \otimes E_2^o \rightarrow E_3^o [-1],
\]

\[
s((a_n, b_n, c_{n-1}) \otimes (a'_m, b'_m, c'_{m-1})) = (0, 0, \alpha(a_n \otimes a'_m) - \beta(b_n, b'_m))
\]

defines a homotopy \( s : \cup_{r,h} \leadsto \cup_{r,h'} \).

**Proof.** See [35], Proposition 3.1.
1.2.3. Assume that, in addition to P1-3), we are given the following data

T1) Morphisms of complexes

\[ \mathcal{T}_A : A_i^* \to A_i^*, \]
\[ \mathcal{T}_B : B_i^* \to B_i^*, \]
\[ \mathcal{T}_C : C_i^* \to C_i^*, \]

for \( i = 1, 2, 3. \)

T2) Morphisms of complexes

\[ \cup'_A : A_2^* \otimes A_1^* \to A_3^*, \]
\[ \cup'_B : B_2^* \otimes B_1^* \to B_3^*, \]
\[ \cup'_C : C_2^* \otimes C_1^* \to C_3^*. \]

T3) A pair of homotopies \( h' = (h'_f, h'_g) \)

\[ h'_f : \cup'_C \circ (f_2 \otimes f_1) \Rightarrow f_3 \circ \cup'_A, \]
\[ h'_g : \cup'_C \circ (g_2 \otimes g_1) \Rightarrow g_3 \circ \cup'_B. \]

T4) Homotopies

\[ U_i : f_i \circ \mathcal{T}_A \Rightarrow \mathcal{T}_C \circ f_i, \]
\[ V_i : g_i \circ \mathcal{T}_B \Rightarrow \mathcal{T}_C \circ g_i, \]

for \( i = 1, 2, 3. \)

T5) Homotopies

\[ t_A : \cup'_A \circ s_{12} \circ (\mathcal{T}_A \otimes \mathcal{T}_A) \Rightarrow \mathcal{T}_A \circ \cup_A, \]
\[ t_B : \cup'_B \circ s_{12} \circ (\mathcal{T}_B \otimes \mathcal{T}_B) \Rightarrow \mathcal{T}_B \circ \cup_B, \]
\[ t_C : \cup'_C \circ s_{12} \circ (\mathcal{T}_C \otimes \mathcal{T}_C) \Rightarrow \mathcal{T}_C \circ \cup_C. \]
A second order homotopy $H_f$ trivializing the boundary of the cube

$$dH_f - H_fd = -t_C \circ (f_1 \otimes f_2) - T_C \circ h_f + U_3 \circ \cup_A + f_3 \circ t_A + h_f' \circ (s_{12} \circ (T_A \otimes T_A)) - (U_C \circ s_{12}) \circ (U_1 \otimes U_2)_1.$$

In this formula, $(U_1 \otimes U_2)_1$ denotes the homotopy defined by (2).
i.e. a system $H_g = (H^i_g)_{i \in \mathbb{Z}}$ of maps $H^i_g : (B_1 \otimes B_2)^i \to C^{i-2}_3$ such that

$$dH_g - H_{gd} = -t_C \circ (g_1 \otimes g_2) - \mathcal{T}_C \circ h_g + V_3 \circ \cup_B +$$

$$+ g_3 \circ t_B + h'_g \circ (s_{12} \circ (\mathcal{T}_B \otimes \mathcal{T}_B)) - (\cup_C \circ s_{12}) \circ (V_1 \otimes V_2)_1.$$

**Proposition 1.2.4.** i) Given the data P1-3) and T1-7), the formula

$$\mathcal{T}_i(a_n, b_n, c_{n-1}) = (\mathcal{T}_A(a_n), \mathcal{T}_B(b_n), \mathcal{T}_C(c_{n-1}) + U_i(a_n) - V_i(b_n))$$

defines morphisms of complexes

$$\mathcal{T}_i : E^\bullet_i \to E^\bullet_i, \quad i = 1, 2, 3$$

such that, for any $r \in R$, the diagram

$$\begin{array}{ccc}
E^\bullet_1 \otimes E^\bullet_2 & \xrightarrow{\cup_{r,h}} & E^\bullet_3 \\
\downarrow s_{12} \circ (\mathcal{T}_1 \otimes \mathcal{T}_2) & & \downarrow \mathcal{T}_3 \\
E^\bullet_2 \otimes E^\bullet_1 & \xrightarrow{\cup'_{1-r,h'}} & E^\bullet_3 \\
\end{array}$$

commutes up to homotopy.

**Proof.** See [34], Proposition 1.3.6.

\[\square\]

**1.2.5. Bockstein maps.** Assume that, in addition to P1-3), we are given the following data

- **B1)** Morphisms of complexes

  $$\beta_{Z,i} : Z^\bullet_i \to Z^\bullet_i[1], \quad Z^\bullet_i = A^\bullet_i, B^\bullet_i, C^\bullet_i, \quad i = 1, 2.$$  

- **B2)** Homotopies

  $$u_i : f_i[1] \circ \beta_{A,i} \rightsquigarrow \beta_{C,i} \circ f_i,$$

  $$v_i : g_i[1] \circ \beta_{B,i} \rightsquigarrow \beta_{C,i} \circ g_i$$

for $i = 1, 2$.

- **B3)** Homotopies

  $$h_Z : \cup_Z[1] \circ (\text{id} \otimes \beta_{Z,2}) \rightsquigarrow \cup_Z[1] \circ (\beta_{Z,1} \otimes \text{id}),$$

for $Z^\bullet = A^\bullet, B^\bullet, C^\bullet$.  

B4) A second order homotopy trivializing the boundary of the following diagram

\[
\begin{array}{c}
A_1^* \otimes A_2^* \xrightarrow{\beta_{A,1} \otimes \text{id}} A_1^*[1] \otimes A_2^* \\
\downarrow f_1 \otimes f_2 & f_1[1] \otimes f_2 \downarrow \\
A_1^* \otimes A_2^*[1] \xrightarrow{f_1 \otimes f_2[1]} C_1^* \otimes C_2^* \\
\downarrow f_1 \otimes v_2 & f_1 \otimes v_2 \downarrow \\
C_1^* \otimes C_2^*[1] \xrightarrow{v_1 \otimes f_2} C_1^* \otimes C_2^* \\
\downarrow v_1 \otimes v_2 & v_1 \otimes v_2 \downarrow \\
C_1^* \otimes C_2^*[1] \xrightarrow{g_3[1]} C_3^*[1].
\end{array}
\]

B5) A second order homotopy trivializing the boundary of the cube

\[
\begin{array}{c}
B_1^* \otimes B_2^* \xrightarrow{\beta_{B,1} \otimes \text{id}} B_1^*[1] \otimes B_2^* \\
\downarrow g_1 \otimes g_2 & g_1[1] \otimes g_2 \downarrow \\
B_1^* \otimes B_2^*[1] \xrightarrow{g_1 \otimes g_2[1]} C_1^* \otimes C_2^* \\
\downarrow g_1 \otimes v_2 & g_1 \otimes v_2 \downarrow \\
C_1^* \otimes C_2^*[1] \xrightarrow{g_3[1]} C_3^*[1].
\end{array}
\]

Proposition 1.2.6. i) Given the data P1-3) and B1-5), the formula

\[\beta_{E,i}(a_n, b_n, c_{n-1}) = (\beta_{A,i}(a_n), \beta_{B,i}(b_n), -\beta_{C,i}(c_{n-1}) - u_i(a_n) + v_i(b_n))\]
defines a morphism of complexes

\[ \beta_{E,i} : E_i^\bullet \rightarrow E_i^\bullet[1] \]

such that for any \( r \in R \) the diagram

\[
\begin{array}{ccc}
E_1^\bullet \otimes E_2^\bullet & \xrightarrow{\beta_{E,1} \otimes \text{id}} & E_1^\bullet[1] \otimes E_2^\bullet \\
\downarrow \text{id} \otimes \beta_{E,2} & & \downarrow \cup_{r,h}[1] \\
E_1^\bullet \otimes E_2^\bullet[1] & \xrightarrow{\cup_{r,h}[1]} & E_3^\bullet[1]
\end{array}
\]

is commutative up to homotopy.

ii) Given the data \( \text{P1-3), T1-7) and B1-5) \), for each \( r \in R \) the diagram

\[
\begin{array}{ccc}
E_1^\bullet \otimes E_2^\bullet & \xrightarrow{\beta_{E,1} \otimes \text{id}} & E_1^\bullet[1] \otimes E_2^\bullet \\
\downarrow s_1 & & \downarrow \cup_{r,h}[1] \\
E_2^\bullet \otimes E_1^\bullet & \xrightarrow{\beta_{E,2} \otimes \text{id}} & E_2^\bullet[1] \otimes E_1^\bullet \\
\downarrow \beta_{E,2} & & \downarrow \beta_2[1] \otimes \beta_1 \\
E_2^\bullet[1] \otimes E_1^\bullet & \xrightarrow{\cup_{r,h}[1]} & E_3^\bullet[1]
\end{array}
\]

is commutative up to a homotopy.

Proof: See [34], Propositions 1.3.9 and 1.3.10. \( \square \)

2. COHOMOLOGY OF \( (\varphi, \Gamma_K) \)-MODULES

2.1. \( (\varphi, \Gamma_K) \)-modules.

2.1.1. Throughout this section, \( K \) denotes a finite extension of \( \mathbb{Q}_p \). Let \( k_K \)
be the residue field of \( K \), \( O_K \) its ring of integers and \( K_0 \) the maximal unramified subfield of \( K \). We denote by \( K_0^{ur} \) the maximal unramified extension of \( K_0 \) and by \( \sigma \) the absolute Frobenius acting on \( K_0^{ur} \). Fix an algebraic closure \( \overline{K} \) of \( K \) and set \( G_K = \text{Gal}(\overline{K}/K) \). Let \( C_p \) be the \( p \)-adic completion of \( \overline{K} \). We denote by \( v_p : C_p \rightarrow \mathbb{R} \cup \{\infty\} \) the \( p \)-adic valuation on \( C_p \) normalized so that \( v_p(p) = 1 \) and set \( |x|_p = p^{-v_p(x)} \). Write \( A(r,1) \) for the \( p \)-adic annulus

\[ A(r,1) = \{ x \in C_p \mid r \leq |x|_p < 1 \}. \]

Fix a system of primitive \( p^n \)-th roots of unity \( \varepsilon = (\zeta_{p^n})_{n \geq 0} \) such that \( \zeta_{p^{n+1}} = \zeta_{p^n}^{p^n} \) for all \( n \geq 0 \). Let \( K^{\text{cyc}} = \bigcup_{n=0}^\infty K(\zeta_{p^n}) \), \( H_K = \text{Gal}(K^{\text{cyc}}/K) \), \( \Gamma_K = \text{Gal}(K^{\text{cyc}}/K) \) and let \( \chi_K : \Gamma_K \rightarrow \mathbb{Z}_p^* \) denote the cyclotomic character.

Recall the constructions of some of Fontaine’s rings of \( p \)-adic periods. Define

\[ \widehat{E}^+ = \varprojlim_{x_i \rightarrow x_p^p} O_{C_p}/pO_{C_p} = \{ x = (x_0,x_1,\ldots,x_n,\ldots) \mid x_{ip}^p = x_i, \ \forall i \in \mathbb{N} \}. \]
Let \( x = (x_0, x_1, \ldots) \in \tilde{E}^+ \). For each \( n \), choose a lift \( \tilde{x}_n \in O_{\mathbb{C}_p} \) of \( x_n \). Then, for all \( m \geq 0 \), the sequence \( x_{m+n}^{sp} \) converges to \( x^{(m)} = \lim_{n \to \infty} x_{m+n}^{sp} \in O_{\mathbb{C}_p} \), which does not depend on the choice of lifts. The ring \( \tilde{E}^+ \), equipped with the valuation \( v_E(x) = v_p(x^{(0)}) \), is a complete local ring of characteristic \( p \) with residue field \( \bar{k}_K \). Moreover, it is integrally closed in its field of fractions \( \tilde{E} = \text{Fr}(\tilde{E}^+) \).

Let \( A = W(\tilde{E}) \) be the ring of Witt vectors with coefficients in \( \tilde{E} \). Denote by \( [\cdot]: \tilde{E} \to W(\tilde{E}) \) the Teichmüller lift. Each \( u = (u_0, u_1, \ldots) \in \tilde{A} \) can be written in the form

\[
u = \sum_{n=0}^{\infty} [u_n p^n].
\]

Set \( \pi = [\varepsilon] - 1 \), \( A_{Q_p}^+ = \mathbb{Z}_p[[\pi]] \) and denote by \( A_{Q_p} \) the \( p \)-adic completion of \( A_{Q_p}^+ [1/\pi] \) in \( \tilde{A} \).

Let \( B = \tilde{A} [1/p] \), \( B_{Q_p} = A_{Q_p} [1/p] \) and let \( B \) denote the completion of the maximal unramified subextension of \( B_{Q_p} \) in \( \tilde{B} \). All these rings are endowed with natural actions of the Galois group \( G_K \) and the Frobenius operator \( \varphi \), and we set \( B_K = B^{\text{H}_K} \). Note that

\[
\gamma(\pi) = (1 + \pi)^{\text{Fr}}(\tau) - 1, \quad \gamma \in \Gamma_K,
\]

\[
\varphi(\pi) = (1 + \pi)^p - 1.
\]

For any \( r > 0 \) define

\[
\tilde{B}^{+, r} = \left\{ x \in \tilde{B} \mid \lim_{k \to +\infty} \left( v_E(x_k) + \frac{pr}{p - 1} k \right) = +\infty \right\}.
\]

Set \( B^{+, r} = B \cap \tilde{B}^{+, r} \), \( B_{K}^{+, r} = B_K \cap \tilde{B}^{+, r} \), \( B^{+, r} = \bigcup_{r > 0} B^{+, r} \) and \( B_{K}^{+, r} = \bigcup_{r > 0} B_{K}^{+, r} \).

Let \( L \) denote the maximal unramified subextension of \( K^{\text{cyc}}/\mathbb{Q}_p \) and let \( e_K = [K^{\text{cyc}} : L^{\text{cyc}}] \). It can be shown (see \[13\]) that there exists \( r_K \geq 0 \) and \( \pi_K \in B_{K}^{+, r_K} \) such that for all \( r \geq r_K \) the ring \( B_{K}^{+, r} \) has the following explicit description

\[
B_{K}^{+, r} = \left\{ f(\pi_K) = \sum_{k \in \mathbb{Z}} a_k \pi_K^k \mid a_k \in L \text{ and } f \text{ is holomorphic and bounded on } A(p^{-1/ekr}, 1) \right\}.
\]

Note that, if \( K/\mathbb{Q}_p \) is unramified, \( L = K_0 \) and one can take \( \pi_K = \pi \).
Define
\[ B_{\text{rig}, K}^{\dagger, r} = \left\{ f(\pi_K) = \sum_{k \in \mathbb{Z}} a_k \pi_K^k \mid a_k \in L \text{ and } f \text{ is holomorphic on } A(p^{-1/eK}, 1) \right\}. \]

The rings \( B_{K}^{\dagger, r} \) and \( B_{\text{rig}, K}^{\dagger, r} \) are stable under \( \Gamma_K \), and the Frobenius \( \varphi \) sends \( B_{K}^{\dagger, r} \) into \( B_{\text{rig}, K}^{\dagger, r} \) and \( B_{\text{rig}, K}^{\dagger, r} \) into \( B_{\text{rig}, K}^{\dagger, r} \). The ring
\[ \mathcal{R}_K = \bigcup_{r \geq r_K} B_{\text{rig}, K}^{\dagger, r} \]
is isomorphic to the Robba ring over \( L \). Note that it is stable under \( \Gamma_K \) and \( \varphi \). As usual, we set
\[ t = \log(1 + \pi) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\pi^n}{n} \in \mathbb{Q}_p. \]
Note that \( \varphi(t) = pt \) and \( \gamma(t) = \chi_K(\gamma)t, \gamma \in \Gamma_K \).

To simplify notation, for each \( r \geq r_K \) we set \( \mathcal{R}_K^{(r)} = B_{\text{rig}, K}^{\dagger, r} \). The ring \( \mathcal{R}_K^{(r)} \) is equipped with a canonical Fréchet topology (see \([8]\)). Let \( A \) be an affinoid algebra over \( \mathbb{Q}_p \).

Define
\[ \mathcal{R}_K^{(r)} A = A \otimes_{\mathbb{Q}_p} \mathcal{R}_K^{(r)}, \quad \mathcal{R}_K A = \bigcup_{r \geq r_K} \mathcal{R}_K^{(r)} A. \]

If the field \( K \) is clear from the context, we will often write \( \mathcal{R}_K^{(r)} A \) instead \( \mathcal{R}_K^{(r)} A \) and \( \mathcal{R}_K A \) instead \( \mathcal{R}_K A \).

**Definition.**

i) A \((\varphi, \Gamma_K)\)-module over \( \mathcal{R}_K^{(r)} \) is a finitely generated projective \( \mathcal{R}_A^{(r)} \)-module \( D^{(r)} \) equipped with the following structures:

a) A \( \varphi \)-semilinear map
\[ D^{(r)} \to D^{(r)} \otimes_{\mathcal{R}_A^{(r)}} \mathcal{R}_A^{(pr)} \]
such that the induced linear map
\[ \varphi^*: D^{(r)} \otimes_{\mathcal{R}_A^{(r)}} \mathcal{R}_A^{(pr)} \to D^{(r)} \otimes_{\mathcal{R}_A^{(r)}} \mathcal{R}_A^{(pr)} \]
is an isomorphism of \( \mathcal{R}_A^{(pr)} \)-modules;

b) A semilinear continuous action of \( \Gamma_K \) on \( D^{(r)} \).

ii) \( D \) is a \((\varphi, \Gamma_K)\)-module over \( \mathcal{R}_A \) if \( D = D^{(r)} \otimes_{\mathcal{R}_A^{(r)}} \mathcal{R}_A \) for some \((\varphi, \Gamma_K)\)-module \( D^{(r)} \) over \( \mathcal{R}_A^{(r)} \), with \( r \geq r_K \).
If $D$ is a $(\phi, \Gamma_K)$-module over $\mathcal{R}_A$, we write $D^* = \text{Hom}_{\mathcal{R}_A}(D, A)$ for the dual $(\phi, \Gamma)$-module. Let $M^{\phi, \Gamma}_{\mathcal{R}_A}$ denote the $\otimes$-category of $(\phi, \Gamma_K)$-modules over $\mathcal{R}_A$.

2.1.2. A $p$-adic representation of $G_K$ with coefficients in an affinoid $\mathbb{Q}_p$-algebra $A$ is a finitely generated projective $A$-module equipped with a continuous $A$-linear action of $G_K$. Note that, as $A$ is a noetherian ring, a finitely generated $A$-module is projective if and only if it is flat. Let $\text{Rep}_A(G_K)$ denote the $\otimes$-category of $p$-adic representations with coefficients in $A$. The relationship between $p$-adic representations and $(\phi, \Gamma_K)$-modules first appeared in the pioneering paper of Fontaine [18]. The key result of this theory is the following theorem.

**Theorem 2.1.3 (Fontaine, Cherbonnier–Colmez, Kedlaya).** Let $A$ be an affinoid algebra over $\mathbb{Q}_p$.

i) There exists a fully faithful functor

$$D_{\text{rig}, A}^*: \text{Rep}_A(G_K) \rightarrow M^{\phi, \Gamma}_{\mathcal{R}_A},$$

which commutes with base change. More precisely, let $\mathcal{X} = \text{Spm}(A)$. For each $x \in \mathcal{X}$, denote by $m_x$ the maximal ideal of $A$ associated to $x$ and set $E_x = A/m_x$. If $V$ (resp. $D$) is an object of $\text{Rep}_A(G_{\mathbb{Q}_p})$ (resp. of $M^{\phi, \Gamma}_{\mathcal{R}_A}$), set $V_x = V \otimes_A E_x$ (resp. $D_x = D \otimes_A E_x$). Then the diagram

$$
\begin{array}{ccc}
\text{Rep}_A(G_{\mathbb{Q}_p}) & \xrightarrow{D_{\text{rig}, A}^*} & M^{\phi, \Gamma}_{\mathcal{R}_A} \\
\downarrow \otimes_{E_x} & & \downarrow \otimes_{E_x} \\
\text{Rep}_{E_x}(G_{\mathbb{Q}_p}) & \xrightarrow{D_{\text{rig}, E_x}^*} & M^{\phi, \Gamma}_{\mathcal{R}_E}
\end{array}
$$

commutes, i.e., $D_{\text{rig}, A}^*(V)_x \simeq D_{\text{rig}, E_x}^*(V_x)$.

ii) If $E$ is a finite extension of $\mathbb{Q}_p$, then the essential image of $D_{\text{rig}, E}^*$ is the subcategory of $(\phi, \Gamma_K)$-modules of slope 0 in the sense of Kedlaya [28].

**Proof.** This follows from the main results of [18], [13] and [28]. See also [16].

2.1.4. **Remark.** Note that in general the essential image of $D_{\text{rig}, A}^*$ does not coincide with the subcategory of étale modules. See [11] [30], [25] for further discussion.
2.2. Relation to the p-adic Hodge theory.

2.2.1. In [18], Fontaine proposed to classify the p-adic representations arising in the p-adic Hodge theory in terms of \((\varphi, \Gamma_K)\)-modules (Fontaine’s program). More precisely, the problem is to recover classical Fontaine’s \(\varphi\)-modules with a semilinear action of \(G_K\) with coefficients in \(E\) and by \(\varphi\)-semilinear bijective operator \(\varphi : M \rightarrow M\).

The complete solution was obtained by Berger in [8], [10]. His theory also allowed him to prove that each de Rham representation is potentially semistable. In this subsection, we review some of results of Berger. See also [15] for introduction and relation to the theory of \(p\)-adic differential equations. Let \(E\) be a fixed finite extension of \(\mathbb{Q}_p\).

**Definition.**

i) A filtered module over \(K\) with coefficients in \(E\) is a free \(K \otimes \mathbb{Q}_p\)-module \(M\) of finite rank equipped with a decreasing exhaustive filtration \((\text{Fil}^i M)_{i \in \mathbb{Z}}\). We denote by \(MF_{K,E}^\dagger\) the \(\otimes\)-category of such modules.

ii) A filtered \((\varphi, N)\)-module over \(K\) with coefficients in \(E\) is a free \(K_0 \otimes \mathbb{Q}_p\)-module \(M\) of finite rank equipped with the following structures:

a) An exhaustive decreasing filtration \((\text{Fil}^i M_K)_{i \in \mathbb{Z}}\) on \(M_K = M \otimes_{K_0} K\);

b) A \(\sigma\)-semilinear bijective operator \(\varphi : M \rightarrow M\);

c) A \(K_0 \otimes \mathbb{Q}_p\)-linear operator \(N\) such that \(N \varphi = p \varphi N\).

iii) A filtered \(\varphi\)-module over \(K\) with coefficients in \(E\) is a filtered \((\varphi, N)\)-module such that \(N = 0\).

We denote by \(MF_{K,E}^{\varphi,N}\) the \(\otimes\)-category of filtered \((\varphi, N)\)-modules over \(K\) with coefficients in \(E\) and by \(MF_{K,E}^{\varphi}\) the category of filtered \(\varphi\)-modules.

iv) If \(L/K\) is a finite Galois extension and \(G_{L/K} = \text{Gal}(L/K)\), then a filtered \((\varphi, N, G_{L/K})\)-module is a filtered \((\varphi, N)\)-module \(M\) over \(L\) equipped with a semilinear action of \(G_{L/K}\) such that the filtration \((\text{Fil}^i M_L)_{i \in \mathbb{Z}}\) is stable under the action of \(G_{L/K}\).

v) We say that \(M\) is a filtered \((\varphi, N, G_K)\)-module if \(M = K_0^{ur} \otimes_{L_0} M'\), where \(M'\) is a filtered \((\varphi, N, G_{L/K})\)-module for some \(L/K\). We denote by \(MF_{K,E}^{\varphi,N,G_K}\) the \(\otimes\)-category of \((\varphi, N, G_K)\)-modules.

Let \(K^{\text{cyc}}((t))\) denote the ring of formal Laurent power series with coefficients in \(K^{\text{cyc}}\) equipped with the filtration \(\text{Fil}^i K^{\text{cyc}}((t)) = t^i K^{\text{cyc}}[[t]]\) and the action of \(\Gamma_K\) given by

\[
\gamma \left( \sum_{k \in \mathbb{Z}} a_k t^k \right) = \sum_{k \in \mathbb{Z}} \gamma(a_k) \chi_k(\gamma) t^k, \quad \gamma \in \Gamma_K.
\]

The ring \(\mathcal{O}_{K,E}\) can not be naturally embedded in \(E \otimes_{\mathbb{Q}_p} K^{\text{cyc}}((t))\), but for any \(r > r_K\) there exists a \(\Gamma_K\)-equivariant embedding \(i_n : \mathcal{O}_{K,E}^{(r)} \rightarrow E \otimes_{\mathbb{Q}_p} \mathbb{Q}_p\).
$K^{\text{cyc}}(t)$ which sends $\pi$ to $\zeta_{p^n}e^{t/p^n} - 1$. Let $D$ be a $(\varphi, \Gamma_K)$-module over $\mathcal{R}_{K,E}$ and let $D = D^{(r)} \otimes_{\mathcal{R}_{K,E}} \mathcal{R}_{K,E}$ for some $r \geq r_K$. Then

$$D_{\text{dR}/K}(D) = \left( E \otimes Q_p K^{\text{cyc}}((t)) \otimes_{i_n} D^{(r)} \right)^{\Gamma_K}$$

is a free $E \otimes Q_p K$-module of finite rank equipped with a decreasing filtration

$$\text{Fil}^i D_{\text{dR}/K}(D) = \left( E \otimes Q_p \text{Fil}^i K^{\text{cyc}}((t)) \otimes_{i_n} D^{(r)} \right)^{\Gamma_K},$$

which does not depend on the choice of $r$ and $n$.

Let $\mathcal{R}_{K,E}[\log \pi]$ denote the ring of power series in variable $\log \pi$ with coefficients in $\mathcal{R}_{K,E}$. Extend the actions of $\varphi$ and $\Gamma_K$ to $\mathcal{R}_{K,E}[\log \pi]$ setting

$$\varphi(\log \pi) = p \log \pi + \log \left( \frac{\varphi(\pi)}{\pi^p} \right),$$

$$\gamma(\log \pi) = \log \pi + \log \left( \frac{\gamma(\pi)}{\pi} \right), \quad \gamma \in \Gamma_K.$$

( Note that $\log(\varphi(\pi)/\pi^p)$ and $\log(\tau(\pi)/\pi)$ converge in $\mathcal{R}_{K,E}$.) Define a monodromy operator $N : \mathcal{R}_{K,E}[\log \pi] \to \mathcal{R}_{K,E}[\log \pi]$ by

$$N = - \left( 1 - \frac{1}{p} \right)^{-1} \frac{d}{d \log \pi}.$$

For any $(\varphi, \Gamma_K)$-module $D$ define

$$D_{\text{st}/K}(D) = (D \otimes_{\mathcal{R}_{K,E}} \mathcal{R}_{K,E}[\log \pi, 1/t])^{\Gamma_K}, \quad t = \log(1 + \pi),$$

$$D_{\text{cris}/K}(D) = D_{\text{st}}(D)^{N=0} = (D[1/t])^{\Gamma_K}.$$

Then $D_{\text{st}}(D)$ is a free $E \otimes Q_p K_0$-module of finite rank equipped with natural actions of $\varphi$ and $N$ such that $N\varphi = p \varphi N$. Moreover, it is equipped with a canonical exhaustive decreasing filtration induced by the embeddings $i_n$. If $L/K$ is a finite extension and $D$ is a $(\varphi, \Gamma_K)$-module, the tensor product $D_L = \mathcal{R}_{L,E} \otimes_{\mathcal{R}_{K,E}} D$ has a natural structure of a $(\varphi, \Gamma_L)$-module, and we define

$$D_{\text{pst}/K}(D) = \lim_{L/K} D_{\text{st}/L}(D_L).$$
Then \( \mathcal{D}_{pst/K}(D) \) is a \( K^tr \)-vector space equipped with natural actions of \( \varphi \) and \( N \) and a discrete action of \( G_K \). Therefore, we have four functors

\[
\begin{align*}
\mathcal{D}_{dR/K} : M_{\mathcal{A}_{K,E}}^{\varphi, \Gamma} & \rightarrow MF_{K,E}, \\
\mathcal{D}_{st/K} : M_{\mathcal{A}_{K,E}}^{\varphi, \Gamma} & \rightarrow MF_{K,E}^{\varphi, N}, \\
\mathcal{D}_{pst/K} : M_{\mathcal{A}_{K,E}}^{\varphi, \Gamma} & \rightarrow MF_{K,E}^{\varphi, N, G_K}, \\
\mathcal{D}_{cris/K} : M_{\mathcal{A}_{K,E}}^{\varphi, \Gamma} & \rightarrow MF_{K,E}^{\varphi}.
\end{align*}
\]

If the field \( K \) is fixed and understood from context, we will omit it and simply write \( \mathcal{D}_{dR}, \mathcal{D}_{st}, \mathcal{D}_{pst} \) and \( \mathcal{D}_{cris} \).

**Theorem 2.2.2 (Berger).** Let \( V \) be a \( p \)-adic representation of \( G_K \). Then

\[
\mathcal{D}_{*/K}(V) \cong \mathcal{D}_{*/K}(V), \quad * \in \{ dR, st, pst, cris \}.
\]

**Proof.** See [8]. \( \Box \)

For any \( (\varphi, \Gamma_K) \)-module over \( \mathcal{A}_{K,E} \) one has

\[
\text{rk}_{K_0} \mathcal{D}_{cris/K}(D) \leq \text{rk}_{K_0} \mathcal{D}_{st/K}(D) \leq \text{rk}_{K_0} \mathcal{D}_{dR/K}(D) \leq \text{rk}_{K_0} \mathcal{D}_{cris/K}(D).
\]

**Definition.** One says that \( D \) is de Rham (resp. semistable, resp. potentially semistable, resp. crystalline) if

\[
\text{rk}_{K_0} \mathcal{D}_{cris/K}(D) = \text{rk}_{K_0} \mathcal{D}_{dR/K}(D) = \text{rk}_{K_0} \mathcal{D}_{st/K}(D) = \text{rk}_{K_0} \mathcal{D}_{pst/K}(D) = \text{rk}_{K_0} \mathcal{D}_{cris/K}(D).
\]

Let \( M_{\mathcal{A}_{E, st}}^{\varphi, \Gamma}, M_{\mathcal{A}_{E, pst}}^{\varphi, \Gamma} \) and \( M_{\mathcal{A}_{E, cris}}^{\varphi, \Gamma} \) denote the categories of semistable, potentially semistable and crystalline \( (\varphi, \Gamma) \)-modules respectively. If \( D \) is de Rham, the jumps of the filtration \( \text{Fil}^i \mathcal{D}_{dR}(D) \) will be called the Hodge–Tate weights of \( D \).

**Theorem 2.2.3 (Berger).** i) The functors

\[
\begin{align*}
\mathcal{D}_{st} : M_{\mathcal{A}_{K,E, st}}^{\varphi, \Gamma} & \rightarrow MF_{K,E}^{\varphi, N}, \\
\mathcal{D}_{pst} : M_{\mathcal{A}_{K,E, pst}}^{\varphi, \Gamma} & \rightarrow MF_{K,E}^{\varphi, N, G_K}, \\
\mathcal{D}_{cris} : M_{\mathcal{A}_{K,E, cris}}^{\varphi, \Gamma} & \rightarrow MF_{K,E}^{\varphi}
\end{align*}
\]

are equivalences of \( \otimes \)-categories.

ii) Let \( D \) be a \( (\varphi, \Gamma_K) \)-module. Then \( D \) is potentially semistable if and only if \( D \) is de Rham.

**Proof.** These results are proved in [10]. See Theorem A, Theorem III.2.4 and Theorem V.2.3 of op. cit.. \( \Box \)
2.3. The complex $C_{\varphi, K}(D)$.

2.3.1. We keep previous notation and conventions. Set $\Delta_K = \text{Gal}(K(\zeta_p)/K)$. Then $\Gamma_K = \Delta_K \times \Gamma^0_K$, where $\Gamma^0_K$ is a pro-$p$-group isomorphic to $\mathbb{Z}_p$. Fix a topological generator $\gamma_K$ of $\Gamma_K$. For each $(\varphi, \Gamma_K)$-module $D$ over $\mathcal{R}_A = \mathcal{R}_{K, A}$ define

$$C_{\gamma_K}^\bullet(D) : D^{\Delta_K} \xrightarrow{\gamma_K^{-1}} D^{\Delta_K},$$

where the first term is placed in degree 0. If $D'$ and $D''$ are two $(\varphi, \Gamma_K)$-modules, we will denote by

$$\cup_{\gamma} : C_{\gamma_K}^\bullet(D') \otimes C_{\gamma_K}^\bullet(D'') \to C_{\gamma_K}^\bullet(D' \otimes D'')$$

the bilinear map

$$\cup_{\gamma}(x_n \otimes y_m) = \begin{cases} x_n \otimes \gamma_K^n(y_m) & \text{if } x_n \in C_{\gamma_K}^n(D'), y_m \in C_{\gamma_K}^m(D''), \\ 0 & \text{if } n + m \geq 2. \end{cases}$$

Consider the total complex

$$C_{\varphi, \gamma_K}^\bullet(D) = \text{Tot} \left( C_{\gamma_K}^\bullet(D) \xrightarrow{\varphi^{-1}} C_{\gamma_K}^\bullet(D) \right).$$

More explicitly,

$$C_{\varphi, \gamma_K}^\bullet(D) : 0 \to D^{\Delta_K} \xrightarrow{d_0} D^{\Delta_K} \oplus D^{\Delta_K} \xrightarrow{d_1} D^{\Delta_K} \to 0,$$

where $d_0(x) = ((\varphi - 1)x, (\gamma_K - 1)x)$ and $d_1(x, y) = (\gamma_K - 1)x - (\varphi - 1)y$. Note that $C_{\varphi, \gamma_K}^\bullet(D)$ coincides with the complex of Fontaine–Herr (see [26], [27], [31]). We will write $H^*(D)$ for the cohomology of $C_{\varphi, \gamma_K}^\bullet(D)$. If $D'$ and $D''$ are two $(\varphi, \Gamma_K)$-modules, the cup product $\cup_{\gamma}$ induces, by Proposition 1.1.5, a bilinear map

$$\cup_{\varphi, \gamma} : C_{\varphi, \gamma_K}^\bullet(D') \otimes C_{\varphi, \gamma_K}^\bullet(D'') \to C_{\varphi, \gamma_K}^\bullet(D' \otimes D'').$$

Explicitly

$$\cup_{\varphi, \gamma}(x_n \otimes y_m) = (x_n \cup \gamma y_{m-1} + (-1)^m x_{n-1} \cup \gamma \varphi(y_m), x_n \cup \gamma y_m).$$
There exists a canonical isomorphism in
\[(12) \quad R\Gamma(K, D') \otimes C_{\varphi, K}^m(D') \oplus C_{\varphi, K}^m(D') = C_{\varphi, K}^m(D') \oplus C_{\varphi, K}^m(D'). \]
An easy computation gives the following formulas
\[
\begin{align*}
C_{\varphi, K}^0(D') &\otimes C_{\varphi, K}^0(D') \to C_{\varphi, K}^0(D' \otimes D'), \\
x_0 \otimes y_0 &\mapsto x_0 \otimes y_0,
\end{align*}
\[
\begin{align*}
C_{\varphi, K}^0(D') &\otimes C_{\varphi, K}^1(D') \to C_{\varphi, K}^1(D' \otimes D'), \\
x_0 \otimes (y_0, y_1) &\mapsto (x_0 \otimes y_0, x_0 \otimes y_1),
\end{align*}
\[
\begin{align*}
C_{\varphi, K}^1(D') &\otimes C_{\varphi, K}^0(D') \to C_{\varphi, K}^1(D' \otimes D'), \\
(x_0, x_1) \otimes y_0 &\mapsto (x_0 \otimes \varphi(y_0), x_1 \otimes \gamma_K(y_0)),
\end{align*}
\[
\begin{align*}
C_{\varphi, K}^1(D') &\otimes C_{\varphi, K}^1(D') \to C_{\varphi, K}^2(D' \otimes D'), \\
(x_0, x_1) \otimes (y_0, y_1) &\mapsto (x_1 \otimes \gamma_K(y_0) - x_0 \otimes \varphi(y_1)),
\end{align*}
\[
\begin{align*}
C_{\varphi, K}^0(D') &\otimes C_{\varphi, K}^2(D') \to C_{\varphi, K}^2(D' \otimes D'), \\
x_0 \otimes y_1 &\mapsto x_0 \otimes y_1,
\end{align*}
\[
\begin{align*}
C_{\varphi, K}^2(D') &\otimes C_{\varphi, K}^0(D') \to C_{\varphi, K}^2(D' \otimes D'), \\
x_1 \otimes y_0 &\mapsto x_1 \otimes \gamma_K(\varphi(y_1)).
\end{align*}
\]
Here the zero components are omitted.

2.3.2. For each \((\varphi, \Gamma_K)\)-module \(D\) we denote by
\[
R\Gamma(K, D) = \left[ C_{\varphi, K}^* (D) \right]
\]
the corresponding object of the derived category \(\mathcal{D}(A)\). The cohomology of
\(D\) is defined by
\[
H^i(D) = R^i\Gamma(K, D) = H^i(C_{\varphi, K}^*(D)), \quad i \geq 0.
\]
There exists a canonical isomorphism in \(\mathcal{D}(A)\)
\[
\text{TR}_K : \tau_{\geq 2} R\Gamma(K, \mathcal{R}_A(\chi_K)) \simeq A[-2]
\]
(see \([27], [31], [30]\)). Therefore, for each \(D\) we have morphisms
\[
(12) \quad R\Gamma(K, D) \otimes A \xrightarrow{\tau_{\geq 2}} R\Gamma(K, D \otimes \mathcal{R}_A(\chi_K)) \xrightarrow{\text{duality}} R\Gamma(K, \mathcal{R}_A(\chi_K)) \rightarrow \tau_{\geq 2} R\Gamma(K, \mathcal{R}_A(\chi_K)) \simeq A[-2].
\]

**Theorem 2.3.3 (Kedlaya–Pottharst–Xiao).** Let \(D\) be a \((\varphi, \Gamma_K)\)-module
over \(\mathcal{R}_{K,A}\), where \(A\) is an affinoid algebra.

i) Finiteness. We have \(R\Gamma(K, D) \in \mathcal{D}^{[0,2]}_{\text{perf}}(A)\).
ii) Euler–Poincaré characteristic formula. We have
\[ \sum_{i=0}^{2} (-1)^i \text{rk}_A H^i(D) = -[K : \mathbb{Q}_p] \text{rk}_{\mathbb{Q}_p}(D). \]

iii) Duality. The morphism (12) induces an isomorphism
\[ R\Gamma(K, D^*) \simeq R\text{Hom}_A(R\Gamma(K, D), A). \]
In particular, we have cohomological pairings
\[ \cup : H^i(D) \otimes H^{2-i}(D^*(\chi_K)) \to H^2(\mathcal{R}_A(\chi_K)) \simeq A, \quad i \in \{0, 1, 2\}. \]

iv) Comparison with Galois cohomology. Let \( V \) is a \( p \)-adic representation of \( G_K \) with coefficients in \( A \). There exist canonical (up to the choice of \( \gamma_K \)) and functorial isomorphisms
\[ H^i(K, V) \sim H^i(D^\dagger_{\text{rig}}(V)) \]
which are compatible with cup-products. In particular, we have a commutative diagram
\[ \begin{array}{ccc}
H^2(\mathcal{R}_A(\chi_K)) & \xrightarrow{\text{TR}_K} & A \\
\downarrow & & \downarrow \\
H^2(K, A(\chi_K)) & \xrightarrow{\text{inv}_K} & A,
\end{array} \]
where \( \text{inv}_K \) is the canonical isomorphism of the local class field theory.

Proof. See Theorem 4.4.5 of [30] and Theorem 2.8 of [37]. □

2.3.4. Remark. The explicit construction of the isomorphism \( \text{TR}_K \) is given in [27] and [3], Theorem 2.2.6.

2.4. The complex \( K^*(V) \).

2.4.1. In this section, we give the derived version of isomorphisms
\[ H^i(K, V) \sim H^i(D^\dagger_{\text{rig}}(V)) \] (of Theorem 2.3.3 iv). We write \( C^*_{\varphi, \gamma_K}(V) \) instead of \( C^*_{\varphi, \gamma_K}(D^\dagger_{\text{rig}}(V)) \) to simplify notation. Let \( K \) be a finite extension of \( \mathbb{Q}_p \). If \( M \) is a topological \( G_K \)-module, we denote by \( C^*(G_K, M) \) the complex of continuous cochains with coefficients in \( M \). Let \( V \) be a \( p \)-adic representation of \( G_K \) with coefficients in an affinoid algebra \( A \). Then
\[ C^*(G_K, V) \in \mathcal{D}^{0,2}_{\text{ft}}(A) \]
and for the associated object \( R\Gamma(K, V) \) of the derived category
\[ R\Gamma(K, V) = [C^*(G_K, V)] \in \mathcal{D}^{0,2}_{\text{perf}}(A) \]
(see [37], Theorem 1.1).
The continuous Galois cohomology of $G_K$ with coefficients in $V$ is defined by

$$H^*(K,V) = H^*(C^*(G_K,V)).$$

In [8], Berger constructed, for each $r \geq r_K$, a ring $\tilde{B}_{\text{rig}}^{+,r}$ which is the completion of $B_{\text{rig}}^{+,r}$ with respect to Frechet topology. Set $\tilde{B}_{\text{rig},A}^{+,r} = \tilde{B}_{\text{rig}}^{+,r} \otimes \mathbb{Q}_p A$ and $\tilde{B}_{\text{rig},A}^{+,r} = \bigcup_{r \geq r_K} \tilde{B}_{\text{rig},A}^{+,r}$. For each $r \geq r_K$ we have an exact sequence

$$0 \to \mathbb{Q}_p \to \tilde{B}_{\text{rig},A}^{+,r} \xrightarrow{\varphi^{-1}} \tilde{B}_{\text{rig},A}^{+,r} \to 0$$

(see [9], Lemma I.7). Since the completed tensor product by an orthonormalizable Banach space is exact in the category of Frechet spaces (see, for example, [1], proof of Lemma 3.9), the sequence

$$0 \to A \to \tilde{B}_{\text{rig},A}^{+,r} \xrightarrow{\varphi^{-1}} \tilde{B}_{\text{rig},A}^{+,r} \to 0.$$

is also exact. Passing to the direct limit we obtain an exact sequence

$$(13) \quad 0 \to A \to \tilde{B}_{\text{rig},A}^{+,r} \xrightarrow{\varphi^{-1}} \tilde{B}_{\text{rig},A}^{+,r} \to 0.$$

Set $V_{\text{rig}}^+ = V \otimes_A \tilde{B}_{\text{rig},A}^+$ and consider the complex $C^*(G_K,V_{\text{rig}}^+)$. Then (13) induces an exact sequence

$$0 \to C^*(G_K,V) \to C^*(G_K,V_{\text{rig}}^+) \xrightarrow{\varphi^{-1}} C^*(G_K,V_{\text{rig}}^+) \to 0.$$

Define

$$K^*(V) = T^*(C^*(G_K,V_{\text{rig}}^+)) = \text{Tot} \left( C^*(G_K,V_{\text{rig}}^+) \xrightarrow{\varphi^{-1}} C^*(G_K,V_{\text{rig}}^+) \right).$$

Consider the map

$$\alpha_V : C^*_{\gamma_K}(V) \to C^*(G_K,V_{\text{rig}}^+)$$

defined by

$$\begin{cases} 
\alpha_V(x_0) = x_0, & x_0 \in C^0_{\gamma_K}(V), \\
\alpha_V(x_i)(g) = \frac{\gamma_K^{\chi(g)} - 1}{\gamma_K - 1}(x_i), & x_i \in C^i_{\gamma_K}(V),
\end{cases}$$

where $g \in G_K$ and $\gamma_K^{\chi(g)} = g|_{\Gamma^{\gamma_K}}$. It is easy to check that $\alpha_V$ is a morphism of complexes which commutes with $\varphi$. By fonctoriality, we obtain a morphism (which we denote again by $\alpha_V$):

$$\alpha_V : C^*_{\gamma_K}(V) \to K^*(V).$$
Proposition 2.4.2. The map \( \alpha_V : C^*_\varphi,\gamma_K(V) \rightarrow K^*(V) \) and the map
\[
\xi_V : C^*(G_K, V) \rightarrow K^*(V),
\]
\[
x_n \mapsto (0, x_n), \quad x_n \in C^n(G_K, V)
\]
are quasi-isomorphisms and we have a diagram
\[
\begin{array}{ccc}
C^*(G_K, V) & \xrightarrow{\xi_V} & K^*(V) \\
\downarrow{\cong} & & \downarrow{\cong} \\
C^*_\varphi,\gamma_K(V) & \xrightarrow{\alpha_V} & K^*(V)
\end{array}
\]

Proof. This is Proposition 9 of [6]. \(\square\)

2.4.3. If \( M \) and \( N \) are two Galois modules, the cup-product
\[
\cup_c : C^*(M) \otimes C^*(M) \rightarrow C^*(M \otimes N)
\]
defined by
\[
(x_n \cup_c y_m)(g_1, g_2, \ldots, g_{n+m}) = x_n(g_1, \ldots, g_n) \otimes (g_1 g_2 \cdots g_n) y_m(g_{n+1}, \ldots, g_{n+m}),
\]
where \( x_n \in C^n(G_K, M) \) and \( y_m \in C^m(G_K, N) \), is a morphism of complexes. Let \( V \) and \( U \) be two Galois representations of \( G_K \). Applying Proposition 1.1.5 to the complexes \( C^*(G_K, V^\dagger_{\text{rig}}) \) and \( C^*(G_K, U^\dagger_{\text{rig}}) \) we obtain a morphism
\[
\cup_K : K^*(V) \otimes K^*(U) \rightarrow K^*(V \otimes U).
\]

The following proposition will not be used in the remainder of this paper, but we state it here for completeness.

Proposition 2.4.4. In the diagram
\[
\begin{array}{ccc}
C^*_\varphi,\gamma_K(V) \otimes C^*_\varphi,\gamma_K(U) & \xrightarrow{\cup_{\varphi,\gamma_K}} & C^*_\varphi,\gamma(V \otimes U) \\
\downarrow{\alpha_V \otimes \alpha_U} & & \downarrow{h_{\varphi,\gamma}} \\
K^*(V) \otimes K^*(U) & \xrightarrow{\cup_K} & K^*(V \otimes U)
\end{array}
\]
the maps \( \alpha_V \otimes U \circ \cup_{\varphi,\gamma} \) and \( \cup_K \circ (\alpha_V \otimes \alpha_U) \) are homotopic.

We need the following lemma.
Lemma 2.4.5. For any $x \in C^1_{\mathfrak{K}}(V)$, $y \in C^1_{\mathfrak{K}}(U)$, let $c_{x,y} \in C^1(\Gamma^0_K, D^\rig_{\mathfrak{K}}(V \otimes U))$ denote the 1-cochain defined by

\begin{equation}
(14) \quad c_{x,y}(\gamma^i_K) = \sum_{i=0}^{n-1} \gamma^i_K(x) \otimes \left( \frac{\gamma^i_K - \gamma^{i+1}_K}{\gamma_K - 1} \right)(y), \quad \text{if } n \neq 0, 1,
\end{equation}

and $c_{x,y}(1) = c_{x,y}(\gamma_K) = 0$. Then

i) For each $x \in C^1_{\mathfrak{K}}(V)$ and $y \in C^0_{\mathfrak{K}}(U)$

$$c_{x,(\gamma-1)y}(\gamma_K) = \alpha_V(x) \cup_c \alpha_U(y) - \alpha_{V \otimes U}(x \cup y).$$

ii) If $x \in C^0_{\mathfrak{K}}(V)$ and $y \in C^1_{\mathfrak{K}}(U)$ then

$$c_{(\gamma-1)x,y}(\gamma_K) = \alpha_{V \otimes U}(x \cup y) - \alpha_V(x) \cup_c \alpha_U(y).$$

iii) One has

$$d^1(c_{x,y}) = -\alpha_V(x) \cup_c \alpha_U(y).$$

Proof of the lemma. i) Note that $\Gamma^0_K$ is the profinite completion of the cyclic group $\langle \gamma_K \rangle$, and an easy computation shows that the map $c_{x,y}$, defined on $\langle \gamma_K \rangle$ by (14), extends by continuity to a unique cochain on $\Gamma^0_K$ which we denote again by $c_{x,y}$.

For any natural $n \neq 0, 1$ one has

$$c_{x,(\gamma-1)y}(\gamma_K) = \sum_{i=0}^{n-1} \gamma^i_K(x) \otimes (\gamma^i_K - \gamma^{i+1}_K)(y) =$$

$$= \sum_{i=0}^{n-1} \gamma^i_K(x) \otimes \gamma^i_K(y) - \sum_{i=0}^{n-1} \gamma^i_K(x) \otimes \gamma^{i+1}_K(y) =$$

$$= \frac{\gamma^i_K - 1}{\gamma_K - 1} (x \otimes \gamma^i_K(y)) - \frac{\gamma^{i+1}_K - 1}{\gamma_K - 1} (x \otimes y) =$$

$$= (g_U(x) \cup g_U(y))(\gamma^i_K) - (g_{V \otimes U}(x \cup y))(\gamma^{i+1}_K).$$

By continuity, this implies that $c_{x,(\gamma-1)y} = \alpha_V(x) \cup_c \alpha_U(y) - \alpha_{V \otimes U}(x \cup y)$, and i) is proved.

ii) An easy induction proves the formula

\begin{equation}
(15) \quad \sum_{i=0}^{m} \gamma^i_K(\gamma_K - 1)(x) \otimes \frac{\gamma^{i+1}_K - 1}{\gamma_K - 1}(y) =
\end{equation}

$$= \gamma^{m+1}_K(x) \otimes \frac{\gamma^{m+1}_K - 1}{\gamma_K - 1}(y) - \frac{\gamma^{m+1}_K - 1}{\gamma_K - 1}(x \otimes y).$$
Therefore
\[ c(\gamma_{k-1})_{x,y}(\gamma_k) = \sum_{i=0}^{n-1} (\gamma_{k}^{i+1} - \gamma_k^i)(x) \otimes \frac{\gamma_k^n - \gamma_k^i}{\gamma_k - 1}(y) = \]
\[ = \sum_{i=0}^{n-1} (\gamma_{k}^{i+1} - \gamma_k^i)(x) \otimes \frac{\gamma_k^n - 1}{\gamma_k - 1}(y) - \sum_{i=0}^{n-1} (\gamma_k^n - 1)(x) \otimes \frac{\gamma_k^n - 1}{\gamma_k - 1}(y) = \]
\[ = (\gamma_k^n - 1)(x) \otimes \frac{\gamma_k^n - 1}{\gamma_k - 1}(y) + (\gamma_k^n - 1)(x) \otimes \gamma_k^n - \gamma_k^i)(x) \otimes \frac{\gamma_k^n - 1}{\gamma_k - 1}(y) = \]
\[ = \frac{\gamma_k^n - 1}{\gamma_k - 1}(x \otimes y) - x \otimes \frac{\gamma_k^n - 1}{\gamma_k - 1}(y) = \]
\[ = (\alpha_V \otimes U(x \cup y))(\gamma_k^n) - (\alpha_V(x) \cup \alpha_U(y))(\gamma_k^n), \]
and ii) is proved.

iii) One has
\[ d^1c_{x,y}(\gamma_k^n, \gamma_k^m) = \gamma_k^n c_{x,y}(\gamma_k^m) - c_{x,y}(\gamma_k^{n+m}) + c_{x,y}(\gamma_k^n) = \]
\[ = \sum_{i=0}^{m-1} \gamma_k^{n+i}(x) \otimes \frac{\gamma_k^n - \gamma_k^i}{\gamma_k - 1}(y) - \]
\[ - \sum_{i=0}^{n+m-1} \gamma_k^i(x) \otimes \frac{\gamma_k^n - \gamma_k^i}{\gamma_k - 1}(y) + \sum_{i=0}^{n-1} \gamma_k^i(x) \otimes \frac{\gamma_k^n - \gamma_k^i}{\gamma_k - 1}(y) = \]
\[ = - \sum_{i=0}^{n+m-1} \gamma_k^i(x) \otimes \frac{\gamma_k^n - \gamma_k^i}{\gamma_k - 1}(y) - \gamma_k^n - \gamma_k^i)(x) \otimes \frac{\gamma_k^n - 1}{\gamma_k - 1}(y) = \]
\[ = - (\alpha_V(x) \cup \alpha_U(y))(\gamma_k^n, \gamma_k^m). \]

By continuity, \( d^1c_{x,y} = -\alpha_V(x) \cup \alpha_U(y) \), and the lemma is proved.

\( \square \)

\textbf{Proof of Proposition 2.4.4} Let
\[ h_\gamma : C_{\gamma_k}^*(V \otimes C_{\gamma_k}^*(U) \to C^*(G_k, V_{\text{rig}}^1 \otimes U_{\text{rig}}^1)[-1] \]
be the map defined by
\[ h_\gamma(x, y) = \begin{cases} -c_{x,y} & \text{if } x \in C_{\gamma_k}^1(V), y \in C_{\gamma_k}^1(U), \\ 0 & \text{elsewhere}. \end{cases} \]

From Lemma 2.4.5 it follows that \( h_\gamma \) defines a homotopy \( h_\gamma : \alpha_V \otimes U \cup \gamma \sim \cup \alpha_u \). By Proposition 1.1.6 \( h_\gamma \) induces a homotopy \( h_{\phi, \gamma} : \alpha_V \otimes U \cup \phi, \gamma \sim \cup \alpha_{u} \otimes \alpha_{u} \). The proposition is proved.
2.5. Transpositions.

2.5.1. Let $M$ be a continuous $G_K$-module. The complex $C^\bullet(G_K,M)$ is equipped with a transposition

$$\mathcal{T}_{V,c} : C^\bullet(G_K,M) \to C^\bullet(G_K,M)$$

which is defined by

$$\mathcal{T}_{V,c}(x_n)(g_1,g_2,\ldots,g_n) = (-1)^{n(n+1)/2}g_1g_2\cdots g_n(x_n(g_1^{-1},\ldots,g_n^{-1})),$$

(see [34], Section 3.4.5.1). We will often write $\mathcal{T}_c$ instead $\mathcal{T}_{V,c}$. The map $\mathcal{T}_c$ satisfies the following properties:

a) $\mathcal{T}_c$ is an involution, i.e. $\mathcal{T}_c^2 = \text{id}$.

b) $\mathcal{T}_c$ is functorially homotopic to the identity map.

c) Let $s_{12} : C^\bullet(G_K,M \otimes N) \to C^\bullet(C_K,N \otimes M)$ denote the map induced by the involution $M \otimes N \to N \otimes M$ given by $x \otimes y \mapsto y \otimes x$ (see Section 1.1.1).

Set $\mathcal{T}_{12} = \mathcal{T}_c \circ s_{12}$. Then for all $x_n \in C^n(G_K,M)$ and $y_m \in C^m(G_K,N)$ one has

$$\mathcal{T}_{12}(x_n \cup y_m) = (-1)^{nm}(\mathcal{T}_c y_m) \cup (\mathcal{T}_c x_n),$$

i.e. the diagram

$$\begin{array}{ccc}
C^\bullet(G_K,M) \otimes C^\bullet(G_K,N) & \xrightarrow{\cup_c} & C^\bullet(G_K,M \otimes N) \\
\downarrow^{s_{12}} & & \downarrow^{\mathcal{T}_{12}} \\
C^\bullet(G_K,N) \otimes C^\bullet(G_K,M) & \xrightarrow{\cup_c} & C^\bullet(G_K,N \otimes M)
\end{array}$$

commutes ([34], Section 3.4.5.3).

2.5.2. There exists a homotopy

$$a = (a^n) : \text{id} \sim \mathcal{T}_c$$

which is functorial in $M$ ([34], Section 3.4.5.5). We remark, that from the discussion in op. cit. it follows, that one can take $a$ such that $a^0 = a^1 = 0$.

2.5.3. Let $V$ be a $p$-adic representation of $G_K$. We denote by $\mathcal{T}_{K(V)}$, or simply by $\mathcal{T}_K$, the transposition induced on the complex $K^\bullet(V)$ by $\mathcal{T}_c$, thus

$$\mathcal{T}_{K(V)}(x_{n-1},x_n) = (\mathcal{T}_c(x_{n-1}),\mathcal{T}_c(x_n)).$$
From Proposition 1.1.7 it follows that in the diagram
\[
\begin{array}{ccc}
K^\bullet(V) \otimes K^\bullet(U) & \xrightarrow{\cup_K} & K^\bullet(V \otimes U) \\
\downarrow s_{12} \circ (T_K(V) \otimes T_K(U)) & & \downarrow \xi_V \\
K^\bullet(U) \otimes K^\bullet(V) & \xrightarrow{\cup_K} & K^\bullet(U \otimes V)
\end{array}
\]
the morphisms $T_K(V) \otimes s_{12} \circ \cup_K$ and $\cup_K \circ s_{12} \circ (T_K(V) \otimes T_K(U))$ are homotopic.

**Proposition 2.5.4.** i) The diagram
\[
\begin{array}{ccc}
\mathcal{C}^\bullet(G_K, V) & \xrightarrow{\xi_V} & K^\bullet(V) \\
\downarrow \mathcal{F}_c & & \downarrow \mathcal{F}_K(V) \\
\mathcal{C}^\bullet(G_K, V) & \xrightarrow{\xi_V} & K^\bullet(V)
\end{array}
\]
is commutative. The map $a_{K(V)} = (a, a)$ defines a homotopy $a_{K(V)} : \text{id}_{K(V)} \sim \mathcal{F}_K(V)$ such that $a_{K(V)} \circ \xi_V = \xi_V \circ a$.

ii) We have a commutative diagram
\[
\begin{array}{ccc}
\mathcal{C}^\bullet_{\phi, W_K}(V) & \xrightarrow{\alpha_V} & K^\bullet(V) \\
\downarrow \text{id} & & \downarrow \mathcal{F}_K \\
\mathcal{C}^\bullet_{\phi, W_K}(V) & \xrightarrow{\alpha_V} & K^\bullet(V)
\end{array}
\]
If $a : \text{id} \sim \mathcal{F}_c$ is a homotopy such that $a^0 = a^1 = 0$, then $a_{K(V)} \circ \alpha_V = 0$.

**Proof.** i) The first assertion follows from Lemma 1.1.6.

ii) If $x_1 \in \mathcal{C}^1_{\phi, W_K}(V)$ then $\alpha_V(x_1) \in \mathcal{C}^\bullet(G_K, V_{\text{rig}})$ satisfies
\[
\mathcal{F}_c(\alpha_V(x_1))(g) = -g(\alpha_V(x_1)(g^{-1})) =
\]
\[
= -\gamma_K^{\kappa(g)} \left( \frac{\gamma_K^{\kappa(g)} - 1}{\gamma_K - 1} \right)(x_1) = \frac{\gamma_K^{\kappa(g)} - 1}{\gamma_K - 1}(x_1) = (\alpha_V(x_1))(g).
\]
Thus $\mathcal{F}_c \circ \alpha_V = \alpha_V$. By functoriality, $\mathcal{F}_K \circ \alpha_V = \alpha_V$. Finally, the identity $a_{K(V)} \circ \alpha_V = 0$ follows directly from the definition of $\xi_V$ and the assumption that $a^0 = a^1 = 0$.  

\[\square\]
2.6. The Bockstein map.

2.6.1. Consider the group algebra $A[\Gamma_K^0]$ of $\Gamma_K^0$ over $A$. Let $t : A[\Gamma_K^0] \rightarrow A[\Gamma_K^0]$ denote the $A$-linear involution given by $t(\gamma) = \gamma^{-1}$, $\gamma \in \Gamma_K^0$. We equip $A[\Gamma_K^0]$ with the following structures:

a) The natural Galois action given by $g(x) = \bar{g}x$, where $g \in G_K$, $x \in A[\Gamma_K^0]$ and $\bar{g}$ is the image of $g$ under canonical projection of $G_K \rightarrow \Gamma_K^0$.

b) The $A[\Gamma_K^0]$-module structure $A[\Gamma_K^0]^\dagger$ given by the involution $t$, namely $\lambda(x) = t(\lambda)x$ for $\lambda \in A[\Gamma_K^0], x \in A[\Gamma_K^0]^\dagger$.

Let $J_K$ denote the kernel of the augmentation map $A[\Gamma_K^0] \rightarrow A$. Then the element

$\tilde{X} = \log^{-1}(\chi_K(\gamma))(\gamma - 1) \pmod{J_K^2} \in J_K/J_K^2$

does not depend on the choice of $\gamma \in \Gamma_K^0$ and we have an isomorphism of $A$-modules

$\theta_K : A \rightarrow J_K^2/J_K^2, \quad \theta_K(a) = a\tilde{X}$.

The action of $G_K$ on the quotient $\tilde{A}_K^1 = A[\Gamma_K^0]^\dagger/J_K^2$ is given by

$g(1) = 1 + \log(\chi_K(g))\tilde{X}, \quad g \in G_K$.

We have an exact sequence of $G_K$-modules

(19) \hspace{1cm} 0 \rightarrow A \xrightarrow{\theta_K} \tilde{A}_K^1 \rightarrow A \rightarrow 0.

Let $V$ be a $p$-adic representation of $G_K$ with coefficients in $A$. Set $\bar{V}_K = V \otimes_A \tilde{A}_K^1$. Then the sequence (19) induces an exact sequence of $p$-adic representations

$0 \rightarrow V \rightarrow \bar{V}_K \rightarrow V \rightarrow 0$.

Therefore, we have an exact sequence of complexes

$0 \rightarrow C^\bullet(G_K, V) \rightarrow C^\bullet(G_K, \bar{V}_K) \rightarrow C^\bullet(G_K, V) \rightarrow 0$

which gives a distinguished triangle

(20) \hspace{1cm} \mathbf{R} \Gamma(K, V) \rightarrow \mathbf{R} \Gamma(K, \bar{V}_K) \rightarrow \mathbf{R} \Gamma(K, V) \rightarrow \mathbf{R} \Gamma(K, V)[1].

The map $s : A \rightarrow A_1$ that sends $a$ to $a \pmod{J_K^2}$ induces a canonical non-equivariant section $s_V : V \rightarrow \bar{V}_K$ of the projection $\bar{V}_K \rightarrow V$. Define a morphism $\beta_{V,C} : C^\bullet(G_K, V) \rightarrow C^\bullet(G_K, V)[1]$ by

$\beta_{V,C}(x_n) = \frac{1}{\chi}(d \circ s_V - s_V \circ d)(x_n), \quad x_n \in C^\bullet(G_K, V)$.

We will write $\beta_c$ instead $\beta_{V,C}$ if the representation $V$ is clear from the context.
Proposition 2.6.2. i) The distinguished triangle (20) can be represented by the following distinguished triangle of complexes

\[ C^\bullet(G_K, V) \rightarrow C^\bullet(G_K, \tilde{V}_K) \rightarrow C^\bullet(G_K, V) \xrightarrow{\beta_{V,c}} C^\bullet(G_K, V)[1]. \]

ii) For any \( x_n \in C^n(G_K, V) \) one has

\[ \beta_{V,c}(x_n) = -\log \chi_K \cup c x_n. \]

Proof. See [34], Lemma 11.2.3. \( \square \)

2.6.3. We will prove analogs of this proposition for the complexes \( C^\bullet_{\phi, \gamma_K}(D) \) and \( K^\bullet(V) \). Let \( D \) be a \((\phi, \Gamma_K)\)-module with coefficients in \( A \). Set \( D = D \otimes_A \tilde{A}_K \). The splitting \( s \) induces a splitting of the exact sequence

\[ 0 \rightarrow D \rightarrow \tilde{D} \xrightarrow{sd} D \rightarrow 0 \]

which we denote by \( s_D \). Define

\[ \beta_D : C^\bullet_{\phi, \gamma_K}(D) \rightarrow C^\bullet_{\phi, \gamma_K}(D)[1], \]

\[ \beta_D(x) = \frac{1}{X}(d \circ s_D - s_D \circ d)(x), \quad x \in C^n_{\phi, \gamma_K}(D). \]

Proposition 2.6.4. i) The map \( \beta_D \) induces the connecting maps \( H^n(D) \rightarrow H^{n+1}(D) \) of the long cohomology sequence associated to the short exact sequence (27).

ii) For any \( x \in C^n_{\phi, \gamma_K}(D) \) one has

\[ \beta_D(x) = -(0, \log \chi_K(\gamma_K)) \cup_{\phi, \gamma} x, \]

where \((0, \log \chi_K(\gamma_K)) \in C^1_{\phi, \gamma_K}(\mathbb{Q}_p(0))\).

Proof. The first assertion follows directly from the definition of the connecting map. Now, let \( x = (x_{n-1}, x_n) \in C^n_{\phi, \gamma_K}(D) \). Then

\[ (ds_D - s_D d)(x) = \]

\[ = d(x_{n-1} \otimes 1, x_n \otimes 1) - s_D((\gamma_K - 1)x_{n-1} + (-1)^n(\phi - 1)x_n, (\gamma_K - 1)x_n) = \]

\[ = (\gamma_K(x_{n-1}) \otimes \gamma_K - x_{n-1} \otimes 1 + (-1)^n(\phi - 1)x_n \otimes 1, \gamma_K(x_n) \otimes \gamma_K - x_n \otimes 1) - \]

\[ - ((\gamma_K - 1)(x_{n-1}) \otimes 1 + (-1)^n(\phi - 1)x_n \otimes 1, (\gamma_K - 1)(x_n) \otimes 1) = \]

\[ = (\gamma_K(x_{n-1}) \otimes (\gamma_K - 1), \gamma_K(x_n) \otimes (\gamma_K - 1)). \]

From \( \gamma_K = 1 + X \log \chi_K(\gamma_K) \) it follows that \( \gamma_K^{-1} - 1 \equiv -X \log \chi_K(\gamma_K) \) (mod \( J_K^2 \)) and we obtain

\[ \beta_D(x) = \frac{1}{X}((\gamma_K(x_{n-1}), \gamma_K(x_n)) \otimes (\gamma_K - 1)) = \]

\[ = -\log \chi_K(\gamma_K)(\gamma_K(x_{n-1}), \gamma_K(x_n)) \in C^{n+1}_{\phi, \gamma_K}(D). \]
Proposition 2.6.5. Again, the splitting of (23) and we have a distinguished triangle of complexes

\[ 0 \to C^\bullet(G_K, V_{\text{rig}}^\dagger) \to C^\bullet(G_K, \tilde{V}_K) \to C^\bullet(G_K, V_{\text{rig}}^\dagger) \to 0, \]

induces an exact sequence

\[ 0 \to K^\bullet(V) \to K^\bullet(\tilde{V}_K) \to K^\bullet(V) \to 0. \tag{23} \]

Again, the splitting \( s_V : V \to \tilde{V}_K \) induces a splitting \( s_k : K^\bullet(V) \to K^\bullet(\tilde{V}_K) \) of (23) and we have a distinguished triangle of complexes

\[ K^\bullet(V) \to K^\bullet(\tilde{V}) \to K^\bullet(V) \xrightarrow{\beta_k(V)} K^\bullet(V)[1]. \]

We will often write \( \beta_k \) instead of \( \beta_k(V) \).

**Proposition 2.6.5.** i) One has

\[ \beta_k(x) = -(0, \log \chi_K(x)) \cup_K x, \quad x \in K^n(V). \]

ii) The following diagrams commute

\[
\begin{array}{ccc}
C^\bullet(G_K, V) & \xrightarrow{\xi_V} & C^\bullet(G_K, V)[1] \\
\downarrow{\beta_k} & & \downarrow{\xi_V[1]} \\
K^\bullet(V) & \xrightarrow{\beta_k} & K^\bullet(V)[1]
\end{array}
\quad
\begin{array}{ccc}
C^\bullet, \gamma_K(V) & \xrightarrow{\beta_{k, \gamma_K}(V)} & C^\bullet, \gamma_K(V)[1] \\
\downarrow{\gamma_k} & & \downarrow{\gamma_k[1]} \\
K^\bullet(V) & \xrightarrow{\beta_k} & K^\bullet(V)[1]
\end{array}
\]

**Proof.** i) The proof is a routine computation. Let \( x = (x_{n-1}, x_n) \in K^n(V) \), where \( x_{n-1} \in C^{n-1}(G_K, V_{\text{rig}}^\dagger), x_n \in C^n(G_K, V_{\text{rig}}^\dagger) \). Since \( s_K \) commutes with \( \varphi \) one has

\[ (ds_k - s_Kd)x = ((ds_V - s_Vd)x_{n-1}, (ds_V - s_Vd)x_n). \]

On the other hand,

\[ ((ds_V - s_Vd)x_{n-1})(g_1, g_2, \ldots, g_n) = g_1x_{n-1}(g_2, \ldots, g_n) \otimes (\tilde{g}_1 - 1), \]

where \( \tilde{g}_1 \) denote the image of \( g_1 \in G_K \) in \( \Gamma_K \). As in the proof of Proposition 2.6.4 we can write \( \tilde{g}_1 - 1 \pmod{J_K^2} = X \log \chi_K(g_1) \). Therefore

\[ (d \circ s_V - s_V \circ d)x_{n-1}(g_1, g_2, \ldots, g_n) = \log \chi_K(g_1)g_1x_{n-1}(g_2, \ldots, g_n) \otimes X. \]

and

\[ (d \circ s_V - s_V \circ d)x_n(g_1, g_2, \ldots, g_n, g_{n+1}) = \log \chi_K(g_1)g_1x_{n-1}(g_2, \ldots, g_n, g_{n+1}) \otimes X. \]
Since $t(g_1 - 1) = -\bar{X} \log \chi_K(g_1)$, we have
\[
\beta_K(x)(g_1, \ldots, g_n) = \frac{1}{X}(d \circ s_K - s_K \circ d)x(g_1, g_2, \ldots, g_n) = -\log \chi_K(g_1)(g_1, \ldots, g_n, g_1x(g_1, g_1, \ldots, g_n, g_1))
\]
On the other hand, $(0, \log \chi_K) \cup (x_{n-1}, x_n) = (z_n, z_{n+1})$, where
\[
z_i(g_1, g_2, \ldots, g_i) = \log \chi_K(g_1)g_1x_i(g_2, \ldots, g_i), \quad i = n, n + 1,
\]
and i) is proved.

ii) The second statement follows from the compatibility of the Bockstein morphisms $\beta_c, \beta_{\mathcal{D}_{Iw}(V)}$ and $\beta_K$ with the maps $\alpha_V$ and $\beta_V$. This can be also proved using i) and Propositions 2.6.2 and 2.6.4.

\[\square\]

2.7. Iwasawa cohomology.

2.7.1. We keep previous notation and conventions. Set $K_\infty = (K^{cyc})^A$, where $A_K = \text{Gal}(K(\zeta_p)/K)$. Then $\text{Gal}(K_\infty/K) \simeq \Gamma^0_K$ and we denote by $K_n$ the unique subextension of $K_\infty$ of degree $[K_n : K] = p^n$. Let $E$ be a finite extension of $\mathbb{Q}_p$ and let $\mathcal{O}_E$ be its ring of integers. We denote by $\Lambda_{\mathcal{O}_E} = \mathcal{O}_E[[\Gamma^0_K]]$ the Iwasawa algebra of $\Gamma^0_K$ with coefficients in $\mathcal{O}_E$. The choice of a generator $\gamma_K$ of $\Gamma^0_K$ fixes an isomorphism $\Lambda_{\mathcal{O}_E} \simeq \mathcal{O}_E[[X]]$ such that $\gamma_K \mapsto X + 1$. Let $\mathcal{H}_E$ denote the algebra of formal power series $f(X) \in E[[X]]$ that converge on the open unit disk $A(0, 1) = \{x \in \mathbb{C}_p \mid |x|_p < 1\}$ and let
\[
\mathcal{H}_E(\Gamma^0_K) = \{f(\gamma_K - 1) \mid f(X) \in \mathcal{H}_E\}.
\]
We consider $\Lambda_{\mathcal{O}_E}$ as a subring of $\mathcal{H}_E(\Gamma^0_K)$. The involution $\iota : \Lambda_{\mathcal{O}_E} \to \Lambda_{\mathcal{O}_E}$ extends to $\mathcal{H}_E(\Gamma^0_K)$. Let $\Lambda^1_{\mathcal{O}_E}$ (resp. $\mathcal{H}_E(\Gamma^0_K)$) denote $\Lambda_{\mathcal{O}_E}$ (resp. $\mathcal{H}_E(\Gamma^0_K)$) equipped with the $\Lambda_{\mathcal{O}_E}$-module (resp. $\mathcal{H}_E(\Gamma^0_K)$-module) structure given by $\alpha \ast \lambda = \iota(\alpha)\lambda$.

Let $V$ be a $p$-adic representation of $G_K$ with coefficients in $E$. Fix a $\mathcal{O}_E$-lattice $T$ of $V$ stable under the action of $G_K$ and set $\text{Ind}_{K_n/K}(T) = T \otimes_{\mathcal{O}_E} \Lambda^1_{\mathcal{O}_E}$. We equip $\text{Ind}_{K_n/K}(T)$ with the following structures:

a) The diagonal action of $G_K$, namely $g(x \otimes \lambda) = g(x) \otimes \bar{g}\lambda$, for all $g \in G_K$ and $x \otimes \lambda \in \text{Ind}_{K_n/K}(T)$;

b) The structure of $\Lambda_{\mathcal{O}_E}$-module given by $\alpha(x \otimes \lambda) = x \otimes \alpha t(\alpha)$ for all $\alpha \in \Lambda_{\mathcal{O}_E}$ and $x \otimes \lambda \in \text{Ind}_{K_n/K}(T)$.

Let $R\Gamma_{Iw}(K, T)$ denote the class of the complex $C^\bullet(G_K, \text{Ind}_{K_n/K}(T))$ in the derived category $\mathcal{D}(\Lambda_{\mathcal{O}_E})$ of $\Lambda_{\mathcal{O}_E}$-modules. The augmentation map $\Lambda_{\mathcal{O}_E} \to \mathcal{O}_E$ induces an isomorphism
\[
R\Gamma_{Iw}(K, T) \otimes_{\Lambda_{\mathcal{O}_E}} \mathcal{O}_E \simeq R\Gamma(K, T).
\]
We write \( H^i\l_{\text{tw}}(K, T) = \mathcal{R}^i\Gamma_{\text{tw}}(K, T) \) for the cohomology of \( \mathcal{R}\Gamma_{\text{tw}}(K, T) \).
From Shapiro’s lemma it follows that
\[
H^i\l_{\text{tw}}(K, T) = \lim_{\text{cores}} H^i(K_\bullet, T)
\]  
(see, for example, [34], Sections 8.1-8.3).

We review the Iwasawa cohomology of \((\phi, \Gamma_K)\)-modules (see [14] and [30]). The map \( \phi : \mathcal{B}_{\text{rig}, K}^{r, pr} \to \mathcal{B}_{\text{rig}, K}^{r, pr} \) equips \( \mathcal{B}_{\text{rig}, K}^{r, pr} \) with the structure of a free \( \phi : \mathcal{B}_{\text{rig}, K}^{r, pr} \)-module of rank \( p \). Define
\[
\psi : \mathcal{B}_{\text{rig}, K}^{r, pr} \to \mathcal{B}_{\text{rig}, K}^{r, pr}, \quad \psi(x) = \frac{1}{p} \phi^{-1} \circ \text{Tr}_{\mathcal{B}_{\text{rig}, K}^{r, pr}/\phi(\mathcal{B}_{\text{rig}, K}^{r, pr})}(x).
\]
Since \( \mathcal{A}_{K, \mathbb{Q}_p} = \bigcup_{r \neq r_k} \mathcal{B}_{\text{rig}, K}^{r, pr} \), the operator \( \psi \) extends by linearity to an operator \( \psi : \mathcal{A}_{K, E} \to \mathcal{A}_{K, E} \) such that \( \psi \circ \phi = \text{id} \).

Let \( \mathcal{D} \) be a \((\phi, \Gamma_\mathcal{E})\)-module over \( \mathcal{A}_{K, E} = \mathcal{A}_K \otimes_{\mathbb{Q}_p} \mathcal{E} \). If \( e_1, e_2, \ldots, e_d \) is a base of \( \mathcal{D} \) over \( \mathcal{A}_{K, E} \), then \( \phi(e_1), \phi(e_2), \ldots, \phi(e_d) \) is again a base of \( \mathcal{D} \), and we define
\[
\psi : \mathcal{D} \to \mathcal{D},
\]
\[
\psi \left( \sum_{i=1}^{d} a_i \phi(e_i) \right) = \sum_{i=1}^{d} \psi(a_i) e_i.
\]
The action of \( \Gamma_\mathcal{E} \) on \( \mathcal{D}^{\Delta K} \) extends to a natural action of \( \mathcal{H}_E(\Gamma_\mathcal{E}) \) and we consider the complex of \( \mathcal{H}_E(\Gamma_\mathcal{E}) \)-modules
\[
C_{\text{tw}}^\bullet(\mathcal{D}) : \mathcal{D}^{\Delta K} \xrightarrow{\psi^{-1}} \mathcal{D}^{\Delta K},
\]
where the terms are concentrated in degrees 1 and 2. Let \( \mathcal{R}\Gamma_{\text{tw}}(\mathcal{D}) = [C_{\text{tw}}^\bullet(\mathcal{D})] \) denote the class of \( C_{\text{tw}}^\bullet(\mathcal{D}) \) in the derived category \( \mathcal{D}(\mathcal{H}_E(\Gamma_\mathcal{E})) \). We also consider the complex \( C_{\phi, \gamma_K}(\text{Ind}_{K_{\text{ur}}/K}(\mathcal{D})) \), where \( \text{Ind}_{K_{\text{ur}}/K}(\mathcal{D}) = \mathcal{D} \otimes_{\mathcal{E}} \mathcal{H}_E(\Gamma_\mathcal{E})^{-1} \), and set \( \mathcal{R}\Gamma(K, \text{Ind}_{K_{\text{ur}}/K}(\mathcal{D})) = \left[ C_{\phi, \gamma_K}(\text{Ind}_{K_{\text{ur}}/K}(\mathcal{D})) \right] \).

**Theorem 2.7.2.** Let \( \mathcal{D} \) be a \((\phi, \Gamma_\mathcal{E})\)-module over \( \mathcal{A}_{K, E} \). Then

i) The complexes \( C_{\text{tw}}^\bullet(\mathcal{D}) \) and \( C_{\phi, \gamma_K}(\mathcal{D}) \) are quasi-isomorphic and therefore
\[
\mathcal{R}\Gamma_{\text{tw}}(\mathcal{D}) \simeq \mathcal{R}\Gamma(K, \text{Ind}_{K_{\text{ur}}/K}(\mathcal{D})).
\]

ii) The cohomology groups \( H^i_{\text{tw}}(\mathcal{D}) = \mathcal{R}^i\Gamma_{\text{tw}}(\mathcal{D}) \) are finitely-generated \( \mathcal{H}_E(\Gamma_\mathcal{E}) \)-modules. Moreover, \( \text{rk}_{\mathcal{H}_E(\Gamma_\mathcal{E})} H^i_{\text{tw}}(\mathcal{D}) = [K : \mathbb{Q}_p] \text{rk}_{\mathcal{A}_{K, E}} \mathcal{D} \) and \( H^1_{\text{tw}}(\mathcal{D})_{\text{tor}} \) and \( H^2_{\text{tw}}(\mathcal{D}) \) are finite-dimensional \( \mathcal{E} \)-vector spaces.

iii) We have an isomorphism
\[
C_{\phi, \gamma_K}(\text{Ind}_{K_{\text{ur}}/K}(\mathcal{D})) \otimes_{\mathcal{H}_E(\Gamma_\mathcal{E})} \mathcal{E} \xrightarrow{\sim} C_{\phi, \gamma_K}(\mathcal{D})
\]
which induces the Hochschild–Serre exact sequences

\[ 0 \to H^i_{Iw}(D)_{\Gamma_K^0} \to H^i(D) \to H^{i+1}_{Iw}(D)^{\Gamma_K^0} \to 0. \]

iv) Let \( \omega = \text{cone } [\mathcal{X}_E(\Gamma_K^0) \to \mathcal{X}_E(\Gamma_K^0)/\mathcal{X}_E(\Gamma_K^0)[-1]] \), where \( \mathcal{X}_E(\Gamma_K^0) \)

is the field of fractions of \( \mathcal{X}_E(\Gamma_K^0) \). Then the functor \( \mathcal{D} = \text{Hom}_{\mathcal{X}_E(\Gamma_K^0)}(-, \omega) \)

furnishes a duality

\[ \mathcal{D}R\Gamma_{Iw}(D) \simeq R\Gamma_{Iw}(D^*(\chi_K))^{\phi=1}[2]. \]

v) If \( V \) is a \( p \)-adic representation of \( G_K \), then there are canonical and functorial isomorphisms

\[ R\Gamma_{Iw}(K, T) \otimes_{\mathcal{X}_E} \mathcal{H}_E(\Gamma_K^0) \simeq R\Gamma(K, T \otimes_{\mathcal{O}_E} \mathcal{H}_E(\Gamma_K^0)^{\phi=1}) \simeq \]

\[ \simeq R\Gamma(K, \text{Ind}_{K_{cyc}}/K(D^*_\text{rig}(V))). \]

Proof. See [38], Theorem 2.6.

2.7.3. We will need the following lemma.

Lemma 2.7.4. Let \( E \) be a finite extension of \( \mathbb{Q}_p \) and let \( D \) be a potentially semistable \( (\phi, \Gamma_K) \)-module over \( \mathcal{B}_{K,E} \). Then

i) \( H^1_{Iw}(D)_{\text{tor}} \simeq (D^\Delta_K)^{\phi=1} \).

ii) Assume that

\[ \mathcal{D}_{\text{pst}}(D^*(\chi_K))^{\phi=p^i} = 0, \quad \forall i \in \mathbb{Z}. \]

Then \( H^2_{Iw}(D) = 0. \)

Proof. i) Consider the exact sequence

\[ 0 \to D^{\phi=1} \to D^{\psi=1} \xrightarrow{\phi-1} D^{\psi=0}. \]

Since \( (D^\Delta_K)^{\psi=1} \simeq H^1_{Iw}(D) \) and \( D^{\psi=0} \) is \( \mathcal{H}_E(\Gamma_K^0) \)-torsion free ([30], Theorem 3.1.1), \( H^1_{Iw}(D)_{\text{tor}} \subset (D^\Delta_K)^{\phi=1} \). On the other hand, \( D^{\phi=1} \) is a finitely dimensional \( E \)-vector space (see, for example [30], Lemma 4.3.5) and therefore is \( \mathcal{H}_E(\Gamma_K^0) \)-torsion. This proves the first statement.

ii) By Theorem [2.7.2 iv], \( H^2_{Iw}(D) \) and \( H^1_{Iw}(D^*(\chi_K))_{\text{tor}} \) are dual to each other and it is enough to show that \( D^*(\chi_K)^{\phi=1} = 0. \) Since \( \dim_E D^*(\chi_K)^{\phi=1} < \infty \), there exists \( r \) such that \( D^*(\chi_K)^{\phi=1} \subset D^*(\chi_K)^{(r)} \), and for \( n \gg 0 \) the map \( i_n = \phi^{-n} : \mathcal{B}_{K,E}^{(r)} \to E \otimes_{\mathbb{Q}_p} K_{\text{cyc}}[[t]] \) gives an injection

\[ D^*(\chi_K)^{\phi=1} \to D^*(\chi_K)^{(r)} \otimes i_n (E \otimes_{\mathbb{Q}_p} K_{\text{cyc}}[[t]]) \]

\[ \simeq \text{Fil}^0 (\mathcal{D}_{\text{dr}}(D^*(\chi_K)) \otimes K \otimes K_{\text{cyc}}((t))). \]
Looking at the action of $\Gamma_K$ on $\operatorname{Fil}^0(D_\st(D^*(\chi_K)) \otimes \ell \otimes K^{cyc}((t)))$ and using the fact that $D^*(\chi_K)^{\varphi=1}$ is finite-dimensional over $E$, it is easy to prove, that there exists a finite extension $L/K$ such that $D^*(\chi_K)^{\varphi=1}$, viewed as $G_L$-module, is isomorphic to a finite direct sum of modules $Q_p(i)$, $i \in \mathbb{Z}$. Therefore

$$D^*(\chi_K)^{\varphi=1} \simeq (D^*(\chi_K)^{\varphi=1} \otimes Q_p Q_p(-i))^{\Gamma_L} \otimes Q_p Q_p(i)$$

as $G_L$-modules. Since

$$(D^*(\chi_K)^{\varphi=1} \otimes Q_p Q_p(-i))^{\Gamma_L} \subset (D^*(\chi_K) \otimes \mathcal{O}_{K,E} \mathcal{O}_{L,E}[1/\ell, \ell_\gamma])^{\varphi=p^{-i}\Gamma_L} = \mathcal{O}_{\st}^{L}(D^*(\chi_K))^{\varphi=p^{-i}} = 0,$$

we obtain that $D^*(\chi_K)^{\varphi=1} = 0$, and the lemma is proved. \hfill \Box

2.8. The group $H^1_f(D)$.

2.8.1. Let $D$ be a potentially semistable $(\varphi, \Gamma_K)$-module over $\mathcal{O}_{K,E}$, where $E$ is a finite extension of $\mathbb{Q}_p$. As usual, we have the isomorphism

$$H^1_f(D) \simeq \operatorname{Ext}^1_{\mathcal{O}_{K,E}}(\mathcal{O}_{K,E}, D)$$

which associates to each cocycle $x = (a, b) \in C^1_{\varphi, \gamma_K}(D)$ the extension

$$0 \to D \to D_x \to \mathcal{O}_{K,E} \to 0$$

such that $D_x = D \oplus \mathcal{O}_{K,E} e$ with $\varphi(e) = e + a$ and $\gamma_K(e) = e + b$. We say that $[x] = \text{class}(x) \in H^1(D)$ is crystalline if

$$\text{rk}_{\mathcal{O}_{K_0}}(\mathcal{O}_{\cris}(D_x)) = \text{rk}_{\mathcal{O}_{K_0}}(\mathcal{O}_{\cris}(D)) + 1$$

and define

$$H^1_f(D) = \{ [x] \in H^1(D) \mid \text{cl}(x) \text{ is crystalline} \}.$$

**Proposition 2.8.2.** Let $D$ be a potentially semistable $(\varphi, \Gamma_K)$-module over $\mathcal{O}_{K,E}$. Then

i) $H^0(D) = \operatorname{Fil}^0(\mathcal{O}_{\pst}(D))^{\varphi=1,N=0, G_K}$ and $H^1_f(D)$ is an $E$-subspace of $H^1(D)$ of dimension

$$\dim_E H^1_f(D) = \dim_E \mathcal{O}_{\st}(D) - \dim_E \operatorname{Fil}^0 \mathcal{O}_{\st}(D) + \dim_E H^0(D).$$

ii) There exists an exact sequence

$$0 \to H^0(D) \to \mathcal{O}_{\cris}(D) \xrightarrow{(\rho, 1 - \varphi)} t_\ell(D) \oplus \mathcal{O}_{\cris}(D) \to H^1_f(D) \to 0,$$

where $t_\ell(D) = \mathcal{O}_{\st}(D)/\operatorname{Fil}^0 \mathcal{O}_{\st}(D)$.

iii) $H^1_f(D^*(\chi_K))$ is the orthogonal complement to $H^1_f(D)$ under the duality $H^1(D) \times H^1(D^*(\chi_K)) \to E$. 

iv) Let
\[ 0 \to D_1 \to D \to D_2 \to 0 \]
be an exact sequence of potentially semistable \((\varphi, \Gamma_K)\)-modules. Assume that one of the following conditions holds
a) \(D\) is crystalline;
b) \(\text{Im}(H^0(D_2) \to H^1(D_1)) \subset H^1_f(D_1)\).
Then one has an exact sequence
\[ 0 \to H^0(D_1) \to H^0(D) \to H^0(D_2) \to H^1_f(D_1) \to H^1(D) \to H^1_f(D_2) \to 0. \]

**Proof.** This proposition is proved in [4], Proposition 1.4.4, and Corollaries 1.4.6 and 1.4.10. For another approach to \(H^1_f(D)\) and an alternative proof see [32], Section 2.

**2.8.3.** In this subsection we assume that \(K = \mathbb{Q}_p\). We review the computation of the cohomology of some isoclinic \((\varphi, \Gamma_{\mathbb{Q}_p})\)-modules given in [4]. To simplify notation, we write \(\chi_p\) and \(\Gamma_{\mathbb{Q}_p}^0\) instead \(\chi_{\mathbb{Q}_p}\) and \(\Gamma_{\mathbb{Q}_p}^0\) respectively.

**Proposition 2.8.4.** Let \(D\) be a semistable \((\varphi, \Gamma_{\mathbb{Q}_p})\)-module of rank \(d\) over \(\mathcal{R}_{\mathbb{Q}_p, E}\) such that \(\mathcal{R}_{\text{st}}(D)^{\varphi = 1} = \mathcal{R}_{\text{st}}(D)\) and \(\text{Fil}^0\mathcal{R}_{\text{st}}(D) = \mathcal{R}_{\text{st}}(D)\). Then
i) \(D\) is crystalline and \(H^0(D) = \mathcal{D}_{\text{cris}}(D)\).
i) One has \(\dim_{\mathbb{E}} H^0(D) = d\), \(\dim_{\mathbb{E}} H^1(D) = 2d\) and \(H^2(D) = 0\).
i) The map
\[ i_D : \mathcal{D}_{\text{cris}}(D) \oplus \mathcal{D}_{\text{cris}}(D) \to H^1(D), \]
\[ i_D = \text{cl}(-x, \log \chi_p) \]
is an isomorphism of \(E\)-vector spaces. Let \(i_{D,f}\) and \(i_{D,c}\) denote the restrictions of \(i_D\) on the first and the second summand respectively. Then \(\text{Im}(i_{D,f}) = H^1_f(D)\) and we have a decomposition
\[ H^1(D) = H^1_f(D) \oplus H^1_c(D), \]
where \(H^1_c(D) = \text{Im}(i_{D,c})\).

iv) Let \(D^*(\chi_p)\) be the Tate dual of \(D\). Then
\[ \mathcal{D}_{\text{cris}}(D^*(\chi_p))^{\varphi = p^{-1}} = \mathcal{D}_{\text{cris}}(D^*(\chi_p)) \]
and \(\text{Fil}^0\mathcal{D}_{\text{cris}}(D^*(\chi_p)) = 0\). In particular, \(H^0(D^*(\chi_p)) = 0\). Let
\[ [\cdot, \cdot]_D : \mathcal{D}_{\text{cris}}(D^*(\chi_p)) \times \mathcal{D}_{\text{cris}}(D) \to E \]
denote the canonical duality. Define a morphism
\[ i_{D^*(\chi_p)} : \mathcal{D}_{\text{cris}}(D^*(\chi_p)) \oplus \mathcal{D}_{\text{cris}}(D^*(\chi_p)) \to H^1(D^*(\chi_p)) \]
by
\[ i_{D^*(\chi_p)}(\alpha, \beta) \cup i_D(x, y) = [\beta, x]_D - [\alpha, y]_D \]
and denote by \( \text{Im}(i_{D^*(\chi_p),f}) \) and \( \text{Im}(i_{D^*(\chi_p),c}) \) the restrictions of \( i_{D^*(\chi_p)} \) on the first and the second summand respectively. Then \( \text{Im}(i_{D^*(\chi_p),f}) = H^1_f(D^*(\chi_p)) \) and again we have
\[
H^1(D^*(\chi_p)) = H^1_f(D^*(\chi_p)) \oplus H^1_c(D^*(\chi_p)),
\]
where \( H^1_c(D^*(\chi_p)) = \text{Im}(i_{D^*(\chi_p),c}) \).

**Proof.** See [4], Proposition 1.5.9 and Section 1.5.10. \( \square \)

We also need the following result.

**Proposition 2.8.5.** Let \( D \) be a crystalline \((\varphi, \Gamma_{Q_p})\)-module over \( \mathcal{R}_{Q_p,E} \) such that \( \mathcal{D}_{\text{cris}}(D)^{\varphi=p^{-1}} = \mathcal{D}_{\text{cris}}(D) \) and \( \text{Fil}^0 \mathcal{D}_{\text{cris}}(D) = 0 \). Then
\[
H^1_{\text{tw}}(D)_{\Gamma_p^0} = H^1_c(D).
\]

**Proof.** See [6], Proposition 4. \( \square \)

### 3. \( p \)-adic height pairings I: Selmer complexes

#### 3.1. Selmer complexes.

**3.1.1.** In this section we construct \( p \)-adic height pairings using Nekovář’s formalism of Selmer complexes. Let \( F \) be a number field. We denote by \( S_f \) (resp. \( S_\infty \)) the set of all non-archimedean (resp. archimedean) absolute values on \( F \). Fix a prime number \( p \) and a finite subset \( S \subseteq S_f \) containing the set \( S_p \) of all \( q \in S_f \) such that \( q \mid p \). We will write \( \Sigma_p \) for the complement of \( S_p \) in \( S \). Let \( G_{F,S} \) denote the Galois group of the maximal algebraic extension of \( F \) unramified outside \( S \cup S_\infty \). For each \( q \in S \), we fix a decomposition group at \( q \) which we identify with \( G_{F_q} \), and set \( \Gamma_q = \Gamma_{F_q} \). Fix a generator \( \gamma_q \in \Gamma_q^0 \).

**3.1.2.** Let \( V \) be a \( p \)-adic representation of \( G_{F,S} \) with coefficients in a \( Q_p \)-affinoid algebra \( A \). We will write \( V_q \) for the restriction of \( V \) on the decomposition group at \( q \). For each \( q \in S_p \), we fix a \((\varphi, \Gamma_q)\)-submodule \( D_q \) of \( D^+_\text{rig}(V_q) \) that is a \( \mathcal{R}_{F_q,A} \)-module direct summand of \( D^+_\text{rig,A}(V_q) \). Set \( D = (D_q)_{q \in S_p} \) and define
\[
U^*_q(V,D) = \begin{cases} C^*_{\varphi,q}(D_q), & \text{if } q \in S_p, \\ C^*_{\text{ur}}(V_q), & \text{if } q \in \Sigma_p, \end{cases}
\]
where
\[
C^*_{\text{ur}}(V_q) : V_q^{I_q} \xrightarrow{\text{Fr}_q^{-1}} V_q^{I_q}, \quad q \in \Sigma_p,
\]
and the terms are concentrated in degrees 0 and 1. Note that, if \( q \in S_p \), we have \([U^*_q(V,D)] = R\Gamma(F_q,D_q) \in \mathcal{D}_{\text{perf}}^{(0,2)}(A) \) by Theorem 2.3.3 i). On the
other hand, if \( q \in \Sigma_p \), then in general the complex \( U_q^\bullet(V, D) \) is not quasi-isomorphic to a perfect complex of \( A \)-modules.

First assume that \( q \in \Sigma_p \). Then we have a canonical morphism

\[
g_q : U_q^\bullet(V, D) \to C^\bullet(G_{Fq}, V)
\]

defined by

\[
g_q(x_0) = x_0, \quad \text{if } x_0 \in U_q^0(V, D),
\]

\[
g_q(x_1)(\text{Fr}_q) = x_1, \quad \text{if } x_1 \in U_q^1(V, D)
\]

and the restriction map

\[
f_q = \text{res}_q : C^\bullet(G_{F,S}, V) \to C^\bullet(G_{Fq}, V).
\]

Now assume that \( q \in S_p \). The inclusion \( D_q \subset D_{\text{rig}}(V_q) \) induces a morphism

\[
U_q^\bullet(V, D) = C_{\varphi, \gamma}(D_q) \to C_{\varphi, \gamma}(V_q).
\]

We denote by

\[
g_q : U_q^\bullet(V, D) \to K^\bullet(V_q), \quad q \mid p
\]

the composition of this morphism with the quasi-isomorphism \( \alpha_{V_q} : C^\bullet(G_{F,S}, V) \simeq K^\bullet(V_q) \) constructed in Section 2.4 and by

\[
f_q : C^\bullet(G_{F,S}, V) \to K^\bullet(V_q), \quad q \mid p
\]

the composition of the restriction map \( \text{res}_q : C^\bullet(G_{F,S}, V) \to C^\bullet(G_{Fq}, V) \) with the quasi-isomorphism \( \xi_{V_q} : C^\bullet(G_{Fq}, V) \to K^\bullet(V_q) \) (see Proposition 2.4.2).

Set

\[
K^\bullet(V) = \left( \bigoplus_{q \in \Sigma_p} C^\bullet(G_{Fq}, V) \right) \oplus \left( \bigoplus_{q \in S_p} K^\bullet(V_q) \right)
\]

and \( U^\bullet(V, D) = \bigoplus_{q \in S} U_q^\bullet(V, D) \).

Recall that

\[
C^\bullet(G_{F,S}, V) \in \mathcal{H}_{\text{ft}}^{[0,3]}(A)
\]

and the associated object of the derived category

\[
R\Gamma_S(V) := [C^\bullet(G_{F,S}, V)] \in \mathcal{D}_{\text{perf}}^{[0,3]}(A)
\]

(see [34] and [37]). Therefore, we have a diagram in \( \mathcal{H}_{\text{ft}}^{[0,3]}(A) \)

\[
C^\bullet(G_{F,S}, V) \xrightarrow{f} K^\bullet(V) \xrightarrow{g} U^\bullet(V, D),
\]
where $f = (f_q)_{q \in S}$ and $g = (g_q)_{q \in S}$, and the corresponding diagram in $\mathcal{D}_{\text{ft}}^{[0,3]}(A)$

$$
\begin{align*}
\text{R} \Gamma_S(V) &\longrightarrow \bigoplus_{q \in S} \text{R} \Gamma(F_q, V) \\
\bigoplus_{q \in S} \text{R} \Gamma(F_q, V) &\longrightarrow \text{R} \Gamma(V)
\end{align*}
$$

where $\text{R} \Gamma(F_q, V) = [U^*(V, D)]$. The associated Selmer complex is defined as

$$
S^*(V, D) = \text{cone} \left[ C^*(G_{F,S}, V) \oplus U^*(V, D) \xrightarrow{f-g} K^*(V) \right] [-1].
$$

We set $\text{R} \Gamma(V, D) := [S^*(V, D)]$ and write $H^*(V, D)$ for the cohomology of $S^*(V, D)$. From the definition, it follows directly that $S^*(V, D) \in \mathcal{X}_{\text{ft}}^{[0,3]}(A)$ and if, in addition, $[U_q^*(V, D)] \in \mathcal{D}_{\text{perf}}^{[0,1]}(A)$ for all $q \in \Sigma_p$, then $\text{R} \Gamma(V, D) \in \mathcal{D}_{\text{perf}}^{[0,3]}(A)$.

3.1.3. The previous construction can be slightly generalized. Fix a finite subset $\Sigma \subset \Sigma_p$ and, for each $q \in \Sigma$, a locally direct summand $M_q$ of the $A$-module $V_q$ stable under the action of $G_{F_q}$. Let $M = (M_q)_{q \in \Sigma}$. Define

$$
U_q^*(V, D, M) = \begin{cases} 
C^*_{\varphi, q}(D_q), & \text{if } q \in \Sigma_p, \\
C^*_{\text{ur}}(V_q), & \text{if } q \in \Sigma_p \setminus \Sigma, \\
C^* (G_{F_q}, M_q), & \text{if } q \in \Sigma.
\end{cases}
$$

We denote by $S^*(V, D, M)$ the associated Selmer complex and set $\text{R} \Gamma(V, D, M) := [S^*(V, D, M)]$. If $M_q = 0$ for all $q \in \Sigma$, we write $S^*(V, D, \Sigma)$ and $\text{R} \Gamma(V, D, \Sigma)$ for $S^*(V, D, M)$ and $\text{R} \Gamma(V, D, M)$ respectively. Note that $\text{R} \Gamma(V, D, \Sigma_p) \in \mathcal{D}_{\text{perf}}^{[0,3]}(A)$.

3.1.4. We construct cup products for our Selmer complexes $\text{R} \Gamma(V, D, M)$. Consider the dual representation $V^*(1)$ of $V$. We equip $V^*(1)$ with the dual local conditions setting

$$
D_q = \text{Hom}_{\mathcal{R}_A}(D_{\text{rig}}^+(V_q^*(1))/D_q, \mathcal{R}_A), \quad \forall q \in \Sigma_p,
$$

$$
M_q = \text{Hom}_A(V_q/M_q, A), \quad \forall q \in \Sigma,
$$

and denote by $f_q^\perp$ and $g_q^\perp$ the morphisms $\{24, 27\}$ associated to $(V^*(1), D_q^\perp, M_q^\perp)$. Consider the following data
1) The complexes $A_1^\bullet = C^\bullet(G_{F,S}, V)$, $B_1^\bullet = U^\bullet(V, D, M)$, and $C_1^\bullet = K^\bullet(V)$ equipped with the morphisms $f_1 = (f_q)_{q \in S} : A_1^\bullet \to C_1^\bullet$ and $g_1 = \oplus g_q : B_1^\bullet \to C_1^\bullet$.

2) The complexes $A_2^\bullet = C^\bullet(G_{F,S}, V^*(1))$, $B_2^\bullet = U^\bullet(V^*(1), D^{⊥}, M^{⊥})$, and $C_2^\bullet = K^\bullet(V^*(1))$ equipped with the morphisms $f_2 = (f_q)_{q \in S} : A_2^\bullet \to C_2^\bullet$ and $g_2 = \oplus g_q : B_2^\bullet \to C_2^\bullet$.

3) The complexes $A_3^\bullet = \tau_{\geq 2} C^\bullet(G_{F,S}, A(1))$, $B_3^\bullet = 0$ and $C_3^\bullet = \tau_{\geq 2} K^\bullet(A(1))$ equipped with the map $f_3 : A_3^\bullet \to C_3^\bullet$ given by

$$\tau_{\geq 2} C^\bullet(G_{F,S}, A(1)) \xrightarrow{\text{(res)}} \bigoplus_q \tau_{\geq 2} C^\bullet(G_{F,q}, A(1)) \to \tau_{\geq 2} K^\bullet(A(1))$$

and the zero map $g_3 : B_3^\bullet \to C_3^\bullet$.

4) The cup product $\cup_A : A_1^\bullet \otimes A_2^\bullet \to A_3^\bullet$ defined as the composition

$$\cup_A : C^\bullet(G_{F,S}, V) \otimes C^\bullet(G_{F,S}, V^*(1)) \xleftarrow{\cup} C^\bullet(G_{F,S}, V \otimes V^*(1)) \to C^\bullet(G_{F,S}, A^*(1)) \to \tau_{\geq 2} C^\bullet(G_{F,S}, A^*(1)),$$

5) The zero cup product $\cup_B : B_1^\bullet \otimes B_2^\bullet \to B_3^\bullet$.

6) The cup product $\cup_C : C_1^\bullet \otimes C_2^\bullet \to C_3^\bullet$ defined as the composition

$$K^\bullet(V) \otimes K^\bullet(V^*(1)) \xleftarrow{\cup} K^\bullet(V \otimes V^*(1)) \to K^\bullet(A(1)) \to \tau_{\geq 2} K^\bullet(A(1)).$$

7) The zero maps $h_f : A_1^\bullet \otimes A_2^\bullet \to C_3^\bullet[-1]$ and $h_g : B_1^\bullet \otimes B_2^\bullet \to C_3^\bullet[-1]$.

**Theorem 3.1.5.**

1) There exists a canonical, up to homotopy, quasi-isomorphism $r_S : E_r^\bullet \to A[-2]$.

2) The data 1-7) above satisfy the conditions P1-3) of Section 1.2 and therefore define, for each $a \in A$ and each quasi-isomorphism $r_S$, the cup product

$$\cup_{a,r_S} : S^\bullet(V, D, M) \otimes_A S^\bullet(V^*(1), D^{⊥}, M^{⊥}) \to A[-3].$$

3) The homotopy class of $\cup_{a,r_S}$ does not depend on the choice of $r \in A$ and, therefore, defines a pairing

$$\cup_{V, D, M} : \mathbf{R} \Gamma(V, D, M) \otimes_A \mathbf{R} \Gamma(V^*(1), D^{⊥}, M^{⊥}) \to A[-3].$$

**Proof.**

i) We repeat verbatim the argument of [34], Section 5.4.1. For each $q \in S$, let $i_q$ denote the composition of the canonical isomorphism $A \simeq H^2(F_q, A(1))$ of the local class field theory with the morphism
\[ \tau_{\geq 2}C^*(G_{F_q}, A(1)) \to K^*(A_q(1)) \]. Then we have a commutative diagram

\[
\begin{array}{ccc}
\tau_{\geq 2}C^*(G_{F,S}, A(1)) & \xrightarrow{(\text{res}_{q})_{q}} & \bigoplus_{q \in S} \tau_{\geq 2}K^*(A_q(1)) \\
& j \downarrow & \downarrow i_S \\
\bigoplus_{q \in S} A[-2] & \xrightarrow{\Sigma} & A[-2],
\end{array}
\]

where \( i_s = j \circ i_{q_0} \) for some fixed \( q_0 \in S \) and \( \Sigma \) denotes the summation over \( q \in S \). By global class field theory, \( i_s \) is a quasi-isomorphism and, because \( A[-2] \) is concentrated in degree 2, there exists a homotopy inverse \( r_s \) of \( i_s \) which is unique up to homotopy.

ii) We only need to show that the condition \( \textbf{P3} \) holds in our case. Note that \( U_A = U_c, U_B = 0 \) and \( U_C = U_K \). From the definition of \( U_K \) it follows immediately that

\[ \tag{29} U_K \circ (f_1 \otimes f_2) = f_3 \circ U_c. \]

If \( q \in S_p \) (resp. if \( q \in \Sigma \)), from the orthogonality of \( D_q^\perp \) and \( D_q \) (resp. from the orthogonality of \( M_q \) and \( M_q^\perp \)) it follows that \( U_K \circ (g_q \otimes g_q^\perp) = 0 \). If \( q \in \Sigma_p \setminus \Sigma \), we have \( U_c \circ (g_q \otimes g_q^\perp) = 0 \) by \[34\], Lemma 7.6.4. Since \( g_3 \circ U_B = 0 \), this gives

\[ \tag{30} U_c \circ (g_1 \otimes g_2) = g_3 \circ U_B = 0. \]

The equations (29) and (30) show that \( \textbf{P3} \) holds with \( h_f = h_g = 0 \). We define \( U_{a_S} \) as the composition of the cup product constructed in Proposition 1.2.2 with \( r_S \). The rest of the theorem follows from Proposition 1.2.2.

3.1.6. Remark. Consider the case \( \Sigma = \emptyset \). The cup product \( U_{V,D} \) induces a map

\[ \tag{31} \text{RH}(V^*(1), D^\perp) \to \text{RHom}_A(\text{RH}(V, D), A)[-3], \]

which, in general, is not an isomorphism. For each \( q \in S \) define

\[ \tilde{U}_q^*(V, D) = \text{cone} \left( U_q^*(V, D) \xrightarrow{\delta_q} K^*(V_q) \right) [-1] \]

and \( \tilde{\text{RH}}(F_q, V, D) = \left[ \tilde{U}_q^*(V, D) \right] \). Then the cup product \( K^*(V_q) \otimes K^*(V_q^*(1)) \to A[-2] \) induces a pairing

\[ \tilde{U}_q^*(V, D) \otimes U_q^*(V^*(1), D^\perp) \to A[-2] \]

which gives rise to a morphism

\[ \tag{32} \text{RH}(F_q, V^*(1), D^\perp) \to \text{RHom}_A(\tilde{\text{RH}}(F_q, V, D), A)[-2]. \]
Repeating the arguments of [34] (see the proofs of Proposition 6.3.3 and Theorem 6.3.4 of op. cit.) it is easy to show that if $R\Gamma(F_q, V, D)$ and $R\Gamma(F_q, V^+(1), D^\perp)$ are perfect and (32) holds for all $q \in S$, then (31) is an isomorphism. First assume that $q \in S_p$. The complexes $R\Gamma(F_q, D)$ and $R\Gamma(F_q, D^\perp)$ are perfect by Theorem [2.3.3]. Consider the tautological exact sequence

$$0 \to D_q \to D^\dagger_{\text{rig}}(V_q) \to \tilde{D}_q \to 0,$$

where $\tilde{D}_q = D^\dagger_{\text{rig}}(V_q)/D_q$. Applying the functor $R\Gamma(F_q, -)$ to this sequence, we obtain an exact sequence

$$0 \to R\Gamma(F_q, D_q) \to R\Gamma(F_q, D^\dagger_{\text{rig}}(V_q)) \to R\Gamma(F_q, \tilde{D}_q) \to 0,$$

and therefore in this case $R\Gamma(F_q, V, D) \simeq R\Gamma(F_q, \tilde{D}_q)$. From the definition of $D^\perp_q$ we have $D^\perp_q \simeq \tilde{D}_q^*(\chi)$. Using Theorem [2.3.3] we obtain

$$R\Gamma(F_q, D^\perp_q) \simeq R\Gamma(F_q, \tilde{D}_q^*(\chi)) \simeq \text{RHom}_A(R\Gamma(F_q, \tilde{D}_q), A)[-2] \simeq \text{RHom}_A(R\Gamma(F_q, V, D), A)[-2],$$

and therefore (32) holds automatically for all $q \in S_p$.

If $q \in \Sigma_p$, the complexes $R\Gamma(F_q, V, D)$ and $R\Gamma(F_q, V^+(1), D^\perp)$ are not perfect in general and, in addition, (32) does not always hold. However, (32) holds in many important cases, in particular if $A$ is a finite extension of $\mathbb{Q}_p$. 
3.1.7. Equip the complexes $A_i^*$, $B_i^*$ and $C_i^*$ with the transpositions given by

$\mathcal{I}_1 = \mathcal{R}_{V,c}$,

$\mathcal{I}_2 = \bigoplus_{q \in \Sigma_p} \mathcal{I}_1(q) \bigoplus \bigoplus_{q \in \Sigma} \mathcal{I}_{K}(V(q)) \bigoplus \bigoplus_{q \in \Sigma} \mathcal{I}_{V,q,c}$,

$\mathcal{I}_3 = \bigoplus_{q \in \Sigma_p} \mathcal{I}_1(q) \bigoplus \bigoplus_{q \in \Sigma} \mathcal{I}_{K}(V(q)) \bigoplus \bigoplus_{q \in \Sigma} \mathcal{I}_{V,q,c}$,

$\mathcal{I}_4 = \bigoplus_{q \in \Sigma_p} \mathcal{I}_1(q) \bigoplus \bigoplus_{q \in \Sigma} \mathcal{I}_{K}(V(q)) \bigoplus \bigoplus_{q \in \Sigma} \mathcal{I}_{V,q,c}$.

(33) $\mathcal{I}_5 = \bigoplus_{q \in \Sigma_p} \mathcal{I}_1(q) \bigoplus \bigoplus_{q \in \Sigma} \mathcal{I}_{K}(V(q)) \bigoplus \bigoplus_{q \in \Sigma} \mathcal{I}_{V,q,c}$.

Theorem 3.1.8. i) The data (33) satisfy the conditions T1-7) of Section 1.2.

ii) We have a commutative diagram

$$
\begin{array}{ccc}
\text{RG}(V,D,M) \otimes_{A} \text{RG}(V^* (1), D^-, M^-) & \xrightarrow{\cup_{V,D}} & A[-3] \\
\downarrow_{s_{12}} & & \\
\text{RG}(V^*(1), D^-, M^-) \otimes_{A} \text{RG}(V,D,M) & \xrightarrow{\cup_{V,D,-1}} & A[-3].
\end{array}
$$

Proof. i) We check the conditions T3-7) taking $A' = A_c$, $B' = 0$ and $C' = K$. From (29) and (30) it follows that T3 holds if we take $h'_f = h'_g = 0$. To check the condition T4) we remark that, by Proposition 2.5.4(i) we have $f_i \circ \mathcal{R}_A = \mathcal{R}_C \circ f_i$ and we can take $U_i = 0$. The existence of a homotopy $V_i$ follows from Proposition 2.5.4 ii) and (34), Proposition 7.7.3. Note that again we can set $V_i = 0$.

We prove the existence of homotopies $t_A$, $t_B$ and $t_C$ satisfying T5). From the commutativity of the diagram (16), it follows that $\cup_{c} \circ s_{12} \circ (\mathcal{R}_A \otimes \mathcal{R}_B) = \mathcal{R}_A \circ \cup_{c}$ and we can take $t_A = 0$. Since $\cup_{c} = \cup_{B} = 0$, we can take $t_B = 0$. We construct $t_C$ as a system of homotopies $(t_{C,q})_{q \in S}$ such that $t_{C,q} : \cup_{c} \circ s_{12} \circ (\mathcal{R}_{V,q,c} \otimes \mathcal{R}(V(1)_q,c)) \sim \mathcal{R}_{V(1)_q,c} \circ \cup_{c}$ for $q \in \Sigma_p$ and $t_{C,q} : \cup_{V(1)_q,c} \circ \cup_{c}$ for $q \in S_p$. As before, from (16) it follows that
for $q \in \Sigma_p$ one can take $t_{C,q} = 0$. If $q \in S_p$, by Proposition 1.1.7 we can set
(34) $t_{C,q}((x_{n-1}, x_n) \otimes (y_{m-1} \otimes y_m)) = (-1)^n(\mathcal{T}_{A(q,c)}(x_{n-1} \cup_c y_{m-1}), 0)$
for $(x_{n-1}, x_n) \in K^n(V_q)$ and $(y_{m-1}, y_m) \in K^m(V^*(1)_q)$ (see (3)). This proves T5. From (34) it follows that $t_C \circ (f_1 \otimes f_2) = 0$ and it is easy to see that T6 and T7 hold if we take $H_f = H_g = 0$.

ii) For each Galois module $X$, we denote by $a_X : \text{id} \sim \mathcal{T}_X$ the homotopy (17). Recall that we can take $a_X$ such that $a_X^0 = a_X^1 = 0$. Consider the following homotopies

$$k_{A_1} = a_V \cdot \text{id} \sim_{\mathcal{T}_A^*},$$

for $A_1^*$,

$$k_{B_1} = \left( \bigoplus_{q \in S_p \cup \Sigma_p \setminus \Sigma} 0_{U_q(V,D,M)} \bigoplus \left( \bigoplus_{q \in \Sigma} a_{M_q} \right) \right) \cdot \text{id} \sim_{\mathcal{T}_B^*},$$

on $B_1^*$.

(35)

$$k_{C_1} = \left( \bigoplus_{q \in S_p} a_{K(q)} \bigoplus \left( \bigoplus_{q \in \Sigma_p} a_{V_q} \right) \right) \cdot \text{id} \sim_{\mathcal{T}_C^*},$$

on $C_1^*$.

We will denote by $k_{A_2}, k_{B_2}, k_{C_2}$ the homotopies on $A_2^*, B_2^*$ and $C_2^*$ defined by the analogous formulas. From Proposition 2.5.4 ii) it follows that

$$f \circ k_{A_1} = k_{C_1} \circ f, \quad f^\perp \circ k_{A_2} = k_{C_2} \circ f^\perp,$$

$$g \circ k_{B_1} = k_{C_1} \circ g, \quad g^\perp \circ k_{B_2} = k_{C_2} \circ g^\perp.$$

By (4), these data induce transpositions $\mathcal{T}_V^\text{sel}$ and $\mathcal{T}_{V^*(1)}^\text{sel}$ on $S^*(V,D,M)$ and $S^*(V^*(1), D^\perp, M^\perp)$, and the formula (5) of Subsection 1.1.2 defines homotopies $\mathcal{T}_{V}^\text{sel} : \text{id} \sim \mathcal{T}_V^\text{sel}$ and $\mathcal{T}_{V^*(1)}^\text{sel} : \text{id} \sim \mathcal{T}_{V^*(1)}^\text{sel}$. By Proposition 1.2.4, the following diagram commutes up to homotopy:

$$S^*(V,D,M) \otimes_A S^*(V^*(1), D^\perp, M^\perp) \xrightarrow{\cup_{a,r_j}} A[-3]$$

$$\downarrow_{\cup_{1-a,r_j}(\mathcal{T}_V^\text{sel} \otimes \mathcal{T}_{V^*(1)}^\text{sel})}$$

$$S^*(V^*(1), D^\perp, M^\perp) \otimes_A S^*(V,D,M) \xrightarrow{\cup_{1-a,r_j}} A[-3].$$

Now the theorem follows from the fact that the map $(k_V^\text{sel} \otimes k_{V^*(1)}^\text{sel})_1$, given by (2), furnishes a homotopy between id and $\mathcal{T}_V^\text{sel} \otimes \mathcal{T}_{V^*(1)}^\text{sel}$. \qed

3.2. $p$-adic height pairings.

3.2.1. We keep notation and conventions of the previous subsection. Let $F^{\text{cyc}} = \bigcup_{n=1}^{\infty} F(\zeta_p^n)$ denote the cyclotomic $p$-extension of $F$. The Galois group $\Gamma_F = \text{Gal}(F^{\text{cyc}}/F)$ decomposes into the direct sum $\Gamma_F = \Delta_F \times \Gamma_F^0$ of the group $\Delta_F = \text{Gal}(F(\zeta_p)/F)$ and a $p$-procyclic group $\Gamma_F^0$. We denote by $\chi$ :
\[ \Gamma_F \to \mathbb{Z}_p^* \text{ the cyclotomic character and by } \chi_0 \text{ the restriction of } \chi \text{ on } \Gamma_q, \text{ } q \in S. \] As in Section 2.6 we equip the group algebra \( A[\Gamma_0^q] \) with the involution \( t : A[\Gamma_0^q] \to A[\Gamma_0^q] \) such that \( t(\gamma) = \gamma^{-1}, \gamma \in \Gamma_0^q \). Fix a generator \( \gamma_F \) of \( \Gamma_0^q \).

Set \( \tilde{A}_F^1 = A[\Gamma_0^q]/(J_F^2) \), where \( J_F \) is the augmentation ideal of \( A[\Gamma_0^q] \). We have an exact sequence

\begin{equation}
0 \to \tilde{A}_F^1 \to A \xrightarrow{\theta_F} A \to 0,
\end{equation}

where \( \theta_F(a) = a\tilde{X} \) and \( \tilde{X} = \log^{-1}(\chi(\gamma_F))(\gamma - 1) \) does not depend on the choice of \( \gamma_F \in \Gamma_0^q \). For each \( p \)-adic representation \( V \) with coefficients in \( A \), \( (36) \) induces an exact sequence

\begin{equation}
0 \to V \to \tilde{V}_F \to V \to 0,
\end{equation}

where \( \tilde{V}_F = \tilde{A}_F^1 \otimes_AV \). As in Section 2.6 passing to continuous Galois cohomology, we obtain a distinguished triangle

\[ C^\bullet(G_{F,S},V) \to C^\bullet(G_{F,S},\tilde{V}_F) \to C^\bullet(G_{F,S},V) \xrightarrow{\beta_{v,c}} C^\bullet(G_{F,S},V)[1]. \]

For each \( q \in S \), the inclusion \( \Gamma_0^q \to \Gamma_0^q \) induces a commutative diagram

\[
\begin{array}{c}
0 \\ \downarrow = \\
\end{array}
\quad
\begin{array}{c}
V_q \\ \downarrow = \\
V_q \\
\end{array}
\quad
\begin{array}{c}
\theta_q \\ \downarrow = \\
\theta_q \\
\end{array}
\quad
\begin{array}{c}
\tilde{V}_F_q \\ \downarrow = \\
V_q \\
0.
\end{array}
\]

where the vertical middle arrow is an isomorphism by the five lemma. Taking into account Proposition 2.6.2 we see that the exact sequence \( (37) \) induces a distinguished triangle

\[ C^\bullet(G_{F,q},V) \to C^\bullet(G_{F,q},\tilde{V}_F) \to C^\bullet(G_{F,q},V) \xrightarrow{\beta_{v,c}} C^\bullet(G_{F,q},V)[1]. \]

where \( \beta_{v,c}(x) = -\log \chi_q \cup x \).

Let \( D_q \) be a \( (\varphi, \Gamma_q) \)-submodule of \( D_{\text{rig}}^1(V_q) \) and let \( \tilde{D}_{F,q} = \tilde{A}_F^1 \otimes AD_q \). As in Section 2.6 we have an exact sequence

\begin{equation}
0 \to D_q \to \tilde{D}_{F,q} \to D_q \to 0
\end{equation}

which sits in the diagram

\[
\begin{array}{c}
0 \\ \downarrow = \\
D_q \\ \downarrow = \\
D_q \\
\end{array}
\quad
\begin{array}{c}
\theta_q \\ \downarrow = \\
\theta_q \\
\downarrow = \\
\theta_q \\
\end{array}
\quad
\begin{array}{c}
\tilde{D}_q \\ \downarrow = \\
\tilde{D}_{F,q} \\
D_q \\
0.
\end{array}
\]
Taking into account Proposition 2.6.4, we obtain that (38) induces the distinguished triangle

\[ C_{\varphi, \gamma_q}^* (D_q) \rightarrow C_{\varphi, \gamma_q}^* (\tilde{D}_{F,q}) \rightarrow C_{\varphi, \gamma_q}^* (D_q) \xrightarrow{\beta_{D_q}} C_{\varphi, \gamma_q}^* (D_q)[1], \]

where \( \beta_{D_q} (x) = -(0, \log \chi_q (\gamma_q)) \cup x \). Finally, replacing in the exact sequence (23) \( \tilde{V} \) by \( \tilde{V}_F \), and taking into account Proposition 2.6.5 we obtain the distinguished triangle

\[ K^* (V_q) \rightarrow K^* ((\tilde{V}_F)_q) \rightarrow K^* (V_q) \xrightarrow{\beta_{K(V_q)}} K^* (V_q)[1], \]

where \( \beta_{K(V_q)} (x) = -(0, \log \chi_q) \cup x \).

If \( q \in \Sigma_p \), we construct the Bockstein map for \( U_q^* (V, D, M) \) following [34], Section 11.2.4. Namely, if \( q \in \Sigma \), then \( U_q^* (V, D, M) = C^* (G_{F_q}, M_q) \) and the exact sequence

\[ 0 \rightarrow M_q \rightarrow \tilde{M}_{F,q} \rightarrow M_q \rightarrow 0 \tag{39} \]

gives rise to a map \( \beta_{M,q,c} : C^* (G_{F_q}, M_q) \rightarrow C^* (G_{F_q}, M_q) \). If \( q \in \Sigma_p \setminus \Sigma \), then \( (\tilde{V}_F)^q = V^q \otimes \tilde{A}_F^1 \) and we denote by \( s : V^q \rightarrow (\tilde{V}_F)^q \) the section given by \( s(x) = x \otimes 1 \). There exists a distinguished triangle

\[ C_{ur}^* (V_q) \rightarrow C_{ur}^* ((\tilde{V}_F)_q) \rightarrow C_{ur}^* (V_q) \xrightarrow{\beta_{V_q,ur}} C_{ur}^* (V_q)[1], \]

where \( \beta_{V_q,ur} : C_{ur}^0 (V_q) \rightarrow C_{ur}^1 (V_q) \) is given by

\[ \beta_{V_q,ur}^F (x) = \frac{1}{X} (ds - sd)(x) = -\log \chi_q (Fr_q) x. \]
Proposition 3.2.2. In addition to (33), equip the complexes \( A_i^* \), \( B_i^* \) and \( C_i^* \) \((1 \leq i \leq 3)\) with the Bockstein maps given by

\[
\beta_{A_1} = \beta_{V,c},
\]

\[
\beta_{B_1} = \left( \bigoplus_{q \in S_p} \beta_{d_q} \right) \oplus \left( \bigoplus_{q \in \Sigma_p} \beta_{V,q,ur} \right),
\]

\[
\beta_{C_1} = \left( \bigoplus_{q \in S_p} \beta_{K(V_q)} \right) \oplus \left( \bigoplus_{q \in \Sigma_p} \beta_{V,q,c} \right),
\]

\[
\beta_{A_2} = \beta_{V^*(1),c},
\]

\[
\beta_{B_2} = \left( \bigoplus_{q \in S_p} \beta_{d_q^+} \right) \oplus \left( \bigoplus_{q \in \Sigma_p} \beta_{V,q^+(1),ur} \right),
\]

\[
\beta_{C_2} = \left( \bigoplus_{q \in S_p} \beta_{K(V_q^+(1))} \right) \oplus \left( \bigoplus_{q \in \Sigma_p} \beta_{V,q^+(1),c} \right),
\]

\[
\beta_{A_3} = \beta_{A(1),c},
\]

\[
\beta_{B_3} = 0,
\]

\[
\beta_{C_3} = \left( \bigoplus_{q \in S_p} \beta_{K(A(1)q)} \right) \oplus \left( \bigoplus_{q \in \Sigma_p} \beta_{A(1)q,c} \right).
\]

Then these data satisfy the conditions B1-5) of Section 1.2.

Proof. We check B2-5) for our Bockstein maps. For each \( q \in \Sigma_p \), Nekovář constructed homotopies

\[
v_{V,q} : \quad g_q \circ \beta_{V,q,ur} \rightsquigarrow \beta_{V,q,c} \circ g_q,
\]

\[
v_{V^*(1),q} : \quad g_q^\perp \circ \beta_{V^+(1),ur} \rightsquigarrow \beta_{V^+(1),c} \circ g_q^\perp.
\]

From Proposition 2.6.5 ii) it follows that for all \( q \in S_p \)

\[
g_q \circ \beta_{d_q} = \beta_{K(V_q)} \circ g_q,
\]

\[
g_q^\perp \circ \beta_{d_q} = \beta_{K(V_q^+(1))} \circ g_q^\perp.
\]

Set \( v_{V,q} = v_{V^*(1),q} = 0 \) for all \( q \in S_p \). Then the condition B2) holds for \( u_i = 0 \) and \( v_i = (v_{i,q})_{q \in S} \).

In B3), we can set \( h_B = 0 \) because \( \cup_B = 0 \). The existence of a homotopy \( h_A \) between \( \cup_A^1 \circ (\text{id} \otimes \beta_{A,2}) \) and \( \cup_A^1 \circ (\beta_{A,1} \otimes \text{id}) \) is proved in (34), Section 11.2.6 and the same method allows to construct \( h_C \). Namely, we construct a system \( h_C = (h_{C,q})_{q \in S} \) of homotopies such that \( h_{C,q} : \cup_c^1 \circ (\text{id} \otimes \beta_{V_q^+(1),c}) \rightsquigarrow \cup_c^1 \circ (\beta_{V_q^+(1),c} \otimes \text{id}) \) for \( q \in S_p \) and \( h_{C,a} : \cup_{K}^1 \circ (\text{id} \otimes \beta_{K(V_q)} \otimes \text{id}) \) for \( q \in S_p \). For \( q \in \Sigma_p \), the construction of
Proposition 1.1.7 (see also (18)) it follows that the diagram
\[
\begin{array}{ccc}
K^\bullet(V_q) \otimes_A K^\bullet(V_q^*)(1) & \xrightarrow{id} & K^\bullet(V_q) \otimes_A K^\bullet(V_q^*)(1) \\
\downarrow & & \downarrow \\
K^\bullet(V_q) \otimes_A A \otimes_A K^\bullet(V_q^*)(1) & \xrightarrow{id} & A \otimes_A K^\bullet(V_q) \otimes_A K^\bullet(V_q^*)(1) \\
\end{array}
\]
has \(\beta_{K(V_q)}(x) = -(0, \log \mathbb{Z}_q) \cup K x\). Consider the following diagram, where \(z_q = (0, \log \mathbb{Z}_q)\)
\[(40)\]
\[
\begin{array}{ccc}
K^\bullet(V_q) \otimes_A K^\bullet(V_q^*)(1) & \xrightarrow{id} & K^\bullet(V_q) \otimes_A K^\bullet(V_q^*)(1) \\
\downarrow & & \downarrow \\
K^\bullet(V_q) \otimes_A A \otimes_A K^\bullet(V_q^*)(1) & \xrightarrow{id} & A \otimes_A K^\bullet(V_q) \otimes_A K^\bullet(V_q^*)(1) \\
\end{array}
\]
The first, second and fourth squares of this diagram are commutative. From Proposition 1.1.7 (see also (18)) it follows that the diagram
\[
\begin{array}{ccc}
K^\bullet(V_q) \otimes K^\bullet(A)[1] & \xrightarrow{s_{12} \otimes id} & K^\bullet(A)[1] \otimes K^\bullet(V_q) \\
\downarrow & & \downarrow \\
K^\bullet(V_q)[1] \otimes K^\bullet(V_q^*) & \xrightarrow{id} & K^\bullet(V_q)[1] \otimes K^\bullet(V_q^*) \\
\end{array}
\]
is commutative up to some homotopy \(k_1 : \mathcal{T}_K \otimes K \rightsquigarrow \cup_K \circ s_{12} \circ (\mathcal{T}_K \otimes \mathcal{T}_K)\). Since \(\mathcal{T}_K^2 = \text{id}\), we have a homotopy
\[\mathcal{T}_K \circ k_1 : \cup_K \rightsquigarrow \mathcal{T}_K \circ \cup_K \circ s_{12} \circ (\mathcal{T}_K \otimes \mathcal{T}_K).\]
By [34], Section 3.4.5.5 (see also Section 2.5.2), for any topological \(G_{F_q}\)-module \(M\) there exists a functorial homotopy \(a : \text{id} \rightsquigarrow \mathcal{T}\). By Proposition 2.5.4 \(a\) induces a homotopy between \(\text{id} : K^\bullet(V_q) \rightarrow K^\bullet(V_q)\) and \(\mathcal{T}_K : K^\bullet(V_q) \rightarrow K^\bullet(V_q)\) which we denote by \(a_K\). Let \((a_K \otimes a_K) : \text{id} \rightsquigarrow \mathcal{T}_K \otimes \mathcal{T}_K\) denote the homotopy between the maps \(\text{id}\) and \(\mathcal{T}_K \otimes \mathcal{T}_K : K^\bullet(V_q) \otimes K^\bullet(Q_p)[1] \rightarrow K^\bullet(V_q) \otimes K^\bullet(Q_p)[1]\) given by (2). Then
\[
d(a_K \circ \mathcal{T}_K \circ \cup_K \circ s_{12} \circ (\mathcal{T}_K \otimes \mathcal{T}_K)) + (a_K \circ \mathcal{T}_K \circ \cup_K \circ s_{12} \circ (\mathcal{T}_K \otimes \mathcal{T}_K)) d = \]
\[= (\mathcal{T}_K - \text{id}) \circ \cup_K \circ s_{12} \circ (\mathcal{T}_K \otimes \mathcal{T}_K),\]
and
\[ d(\cup_K \circ s_{12} \circ (a_K \otimes a_K)_1) + (\cup_K \circ s_{12} \circ (a_K \otimes a_K)_1) = \]
\[ = \cup_K \circ s_{12} \circ (\mathcal{T}_K \otimes \mathcal{T}_K - \text{id}). \]

Therefore the formula
\[ (41) \quad k_2 = a_K \circ \mathcal{T}_K \circ \cup_K \circ s_{12} \circ (\mathcal{T}_K \otimes \mathcal{T}_K) + \cup_K \circ s_{12} \circ (a_K \otimes a_K)_1 \]
defines a homotopy
\[ k_2 : \cup_K \circ s_{12} \rightsquigarrow \mathcal{T}_K \circ \cup_K \circ s_{12} \circ (\mathcal{T}_K \otimes \mathcal{T}_K). \]
Then \( k_{C,q} = \mathcal{T}_K \circ k_1 - k_2 \) defines a homotopy
\[ k_{C,q} : \cup_K \rightsquigarrow \cup_K \circ s_{12} \]
and we proved that the third square of the diagram (40) commutes up to a homotopy. We define the homotopy
\[ h_{C,q} : \cup_K[1] \circ (\text{id} \otimes \beta_{K(V_q^*(1)),c}) \rightsquigarrow \cup_K[1] \circ (\beta_{K(V_q)} \otimes \text{id}) \]
by
\[ (42) \quad h_{C,q} = \cup_K \circ (k_{C,q} \otimes \text{id}) \circ (\text{id} \otimes (-z_q) \otimes \text{id}). \]
This proves B3).

Since \( u_1 = u_2 = h_f = 0 \), the condition B4) reads
\[ (43) \quad dK_f - K_fd = -h_C \circ (f_1 \otimes f_2) + f_3[1] \circ h_A \]
for some second order homotopy \( K_f \). It is proved in [34], Section 11.2.6, that if \( q \in \Sigma_p \), then
\[ (44) \quad h_{C,q} \circ (f_1 \otimes f_2) = \text{res}_q \circ h_A. \]
Assume that \( q \in S_p \). Recall (see [34], Section 11.2.6) that the homotopy \( h_A \) is given by
\[ (45) \quad h_A = \cup_c \circ (k_A \otimes \text{id}) \circ (\text{id} \otimes (z) \otimes \text{id}), \]
where \( z = \log \chi \) and
\[ (46) \quad k_A = -a \circ (\cup_c \circ s_{12} \circ (\mathcal{T}_c \otimes \mathcal{T}_c)) - (\mathcal{T}_c \circ \cup_c \circ s_{12}) \circ (a \otimes a)_1. \]
From [8], it follows that for all \( x \in C^n(G_F,S,V) \) and \( y \in C^n(G_F,S,V^*(1)) \) we have
\[ (47) \quad (k_1 \otimes \text{id}) \circ (\text{id} \otimes (-z_q) \otimes \text{id}) \circ (f_1 \otimes f_2)(x \otimes y) = \]
\[ = (k_1 \otimes \text{id}) \circ ((0, -\log \chi_q) \otimes (0, x_q) \otimes (0, y_q)) = \]
\[ = k_1((0, -\log \chi_q) \otimes (0, x_q)) \otimes (0, y_q) = 0, \]
where \( x_q = \text{res}_q(x), y_q = \text{res}_q(y) \). On the other hand, comparing (41) and (46) we see that

\[
(48) \quad (k_2 \otimes \text{id}) \circ (\text{id} \otimes (\varepsilon_q) \otimes \text{id}) \circ (f_1 \otimes f_2)(x \otimes y) = \\
= k_2((0, -\log x_q) \otimes (0, x_q)) \otimes (0, y_q) = \\
= -(0, \text{res}_q(k_A(-\varepsilon \otimes x))) \otimes (0, y_q).
\]

From (47), (48), (42) and (46) we obtain that

\[
(49) \quad h_{C,q} \circ (f_1 \otimes f_2)(x \otimes y) = \\
= (0, \text{res}_q(k_A(-\varepsilon \otimes x))) \cup_K (0, y_q) = \\
= (0, \text{res}_q(k_A(-\varepsilon \otimes x)) \cup_{\text{c}} y) = (0, \text{res}_q(h_A(x \otimes y))).
\]

From (49) and (44) it follows that \( h_C \circ (f_1 \otimes f_2) = f_3[1] \circ h_A \) and therefore we can set \( K_f = 0 \) in (43). Thus, B4 is proved.

It remains to check B5). Since \( v_1 = v_2 = h_g = 0 \), this condition reads

\[
(50) \quad dK_g - K_g d = -h_C \circ (g_1 \otimes g_2) + \cup_{C[1]} \circ (v_1 \otimes g_2) - \cup_{C[1]} \circ (g_1 \otimes v_2)
\]

for some second order homotopy \( K_g \). Write \( K_g = (K_{g,q})_{q \in S} \). For \( q \in \Sigma_p \), Nekovář proved that the \( q \)-component of the right hand side of (50) is equal to zero. For \( q \in S_p \), we have \( v_{1,q} = v_{2,q} = 0 \) and \( h_{C,v} \circ (g_1 \otimes g_2) = 0 \) because of orthogonality of \( D \) and \( D^\perp \), and again we can set \( K_{g,q} = 0 \). To sum up, the condition (50) holds for \( K_g = 0 \). The proposition is proved. \( \square \)

### 3.2.3.

The exact sequences (37), (38) and (39) give rise to a distinguished triangle

\[
\text{R} \Gamma(V, D, M) \to \text{R} \Gamma(\tilde{V}_F, \tilde{D}_F, \tilde{M}_F) \to \text{R} \Gamma(V, D, M) \xrightarrow{\delta_{V,D,M}} \text{R} \Gamma(V, D, M)[1]
\]

**Definition.** The \( p \)-adic height pairing associated to the data \((V, D, M)\) is defined as the morphism

\[
h_{V,D,M}^{\text{sel}} : \text{R} \Gamma(V, D, M) \otimes_A L \text{R} \Gamma(V^*(1), D^\perp, M^\perp) \xrightarrow{\delta_{V,D,M}} \\
\to \text{R} \Gamma(V, D, M)[1] \otimes_A L \text{R} \Gamma(V^*(1), D^\perp, M^\perp) \xrightarrow{\cup_{V,D,M}} A[-2],
\]

where \( \cup_{V,D,M} \) is the pairing (28).

The height pairing \( h_{V,D,M}^{\text{sel}} \) induces a pairing

\[
(51) \quad h_{V,D,M,1}^{\text{sel}} : H^1(V, D, M) \otimes A H^1(V^*(1), D^\perp, M^\perp) \to A.
\]
Theorem 3.2.4. The diagram

\[
\begin{array}{ccc}
\mathcal R\Gamma(V, D, M) \otimes_A^L \mathcal R\Gamma(V^*(1), D^\perp, M^\perp) & \xrightarrow{h_{V, D, M}^{\text{rel}}} & A[-2] \\
\downarrow_s & & \downarrow_s \\
\mathcal R\Gamma(V^*(1), D^\perp, M^\perp) \otimes_A^L \mathcal R\Gamma(V, D, M) & \xrightarrow{h_{V^*(1), D^\perp, M^\perp}^{\text{rel}}} & A[-2]
\end{array}
\]

is commutative. In particular, the pairing \(h_{V, D, 1}^{\text{rel}}\) is skew-symmetric.

Proof. From Propositions \[1.2.6\] and \[3.2.2\] it follows, that the diagram

\[
\begin{array}{ccc}
S^\bullet(V, D, M) \otimes_A S^\bullet(V^*(1), D^\perp, M^\perp) & \xrightarrow{h_{V, D, M}^{\text{rel}}} & E_3 \\
\downarrow_s \circ (\mathcal S^\text{sel} \otimes \mathcal S^\text{sel}_{V^*(1)}) & & \downarrow_s \\
S^\bullet(V^*(1), D^\perp, M^\perp) \otimes_A S^\bullet(V, D, M) & \xrightarrow{h_{V^*(1), D^\perp, M^\perp}^{\text{rel}}} & E_3
\end{array}
\]

is commutative up to homotopy. Now the theorem follows from the fact, that \((\mathcal S^\text{sel} \otimes \mathcal S^\text{sel}_{V^*(1)})\) is homotopic to the identity map (see the proof of Theorem \[3.1.8\]). \(\square\)

4. Splitting submodules

4.1. Splitting submodules.

4.1.1. Let \(K\) be a finite extension of \(Q_p\), and let \(V\) be a potentially semistable representation of \(G_K\) with coefficients in a finite extension \(E\) of \(Q_p\). For each finite extension \(L/K\) we set \(D_{*/L}(V) = (B_*/V)^{G_L}\), where \(* \in \{\text{cris}, \text{st}, \text{dR}\}\) and write \(D_*(V) = D_{*/K}(V)\) if \(L = K\). We will use the same convention for the functors \(\mathcal S^\text{sel}_{*/L}\).

Fix a finite Galois extension \(L/K\) such that the restriction of \(V\) on \(G_L\) is semistable. Then \(D_{\text{st}/L}(V)\) is a filtered \((\varphi, N, G_{L/K})\)-module and \(D_{\text{dR}/L}(V) = D_{\text{dR}/L}(V) \otimes_{L_0} L\). An \((\varphi, N, G_{L/K})\)-submodule of \(D_{\text{st}/L}(V)\) is a \(L_0\)-subspace \(D\) of \(D_{\text{st}/L}(V)\) stable under the action of \(\varphi, N\) and \(G_{L/K}\).

Definition. We say that a \((\varphi, N, G_{L/K})\)-submodule \(D\) of \(D_{\text{st}/L}(V)\) is a splitting submodule if

\[
D_{\text{dR}/L}(V) = D_L \oplus \text{Fil}^0 D_{\text{dR}/L}(V), \quad D_L = D \otimes_{L_0} L
\]
as \(L\)-vector spaces.

In Subsections 4.1-4.2 we will always assume that \(V\) satisfies the following condition:

S) \(D_{\text{cris}}(V)^{\varphi=1} = D_{\text{cris}}(V^*(1))^{\varphi=1} = 0\).
If $D$ is a splitting submodule, we denote by $D$ the $(\varphi, \Gamma_K)$-submodule of $D_{\text{rig}}^+(V)$ associated to $D$ by Theorem 2.2.3. The natural embedding $D \rightarrow D_{\text{rig}}^+(V)$ induces a map $H^1(D) \rightarrow H^1(D_{\text{rig}}^+(V)) \cong H^1(K, V)$. Passing to duals, we obtain a map $H^1(K, V^*(1)) \rightarrow H^1(D^*(\chi))$.

**Proposition 4.1.2.** Assume that $V$ satisfies the condition $S)$. Let $D$ be a splitting submodule. Then

i) $H^1_f(K, V^*(1)) \rightarrow H^1_f(D^*(\chi))$ is the zero map.

ii) $\text{Im}(H^1(D) \rightarrow H^1(K, V)) = H^1_f(K, V)$ and the map $H^1_f(D) \rightarrow H^1_f(K, V)$ is an isomorphism.

iii) If, in addition, Fil$^0(D_{st/L}(V)/D)_{\varphi=1, N=0, G_{L/K}} = 0$, then $H^1(D) = H^1_f(K, V)$.

**Proof.** i) By Proposition 2.8.2 we have a commutative diagram

\[
\begin{array}{ccc}
H^1_{\text{cris}}(V^*(1)) & \longrightarrow & H^1_{\text{cris}}(D^*(\chi)) \\
\downarrow & & \downarrow \\
H^1_f(K, V^*(1)) & \longrightarrow & H^1_f(D^*(\chi))
\end{array}
\]

where we set

\[
H^1_{\text{cris}}(V^*(1)) = \text{coker} \left( D_{\text{cris}}(V) \xrightarrow{(1-\varphi, \text{pr})} D_{\text{cris}}(V) \oplus t_V(K) \right)
\]

and

\[
H^1_{\text{cris}}(D^*(\chi)) = \text{coker} \left( D_{\text{cris}}(D^*(\chi)) \xrightarrow{(1-\varphi, \text{pr})} D_{\text{cris}}(D^*(\chi)) \oplus t_{D^*(\chi)}(K) \right)
\]

to simplify notation.

Since $D_{\text{cris}}(V^*(1))_{\varphi=1} = 0$, the map $1 - \varphi : D_{\text{cris}}(V^*(1)) \rightarrow D_{\text{cris}}(V^*(1))$ is an isomorphism and $H^1_{\text{cris}}(V^*(1)) = t_{V^*(1)}(K)$. On the other hand, all Hodge–Tate weights of $D^*(\chi)$ are $\geq 0$ and $t_{D^*(\chi)}(K) = 0$. Hence $H^1_{\text{cris}}(D^*(\chi)) = \text{coker}(1 - \varphi \mid D_{\text{cris}}(D^*(\chi)))$ and the upper map in (52) is zero because it is induced by the canonical projection of $t_{V^*(1)}(K)$ on $t_{D^*(\chi)}(K)$. This proves i).

Now we prove ii). Using i) together with the orthogonality property of $H^1_f$ we obtain that the map

\[
\text{Hom}_E(H^1(K, V)/H^1_f(K, V), E) \rightarrow \text{Hom}_E(H^1(D)/H^1_f(D), E),
\]
induced by $H^1(D) \to H^1(K,V)$, is zero. This implies that the image of $H^1(D)$ is $H^1(K,V)$ is contained in $H^1_f(K,V)$. Finally one has a diagram

$$
\begin{array}{ccc}
H^1_{\text{cris}}(D) & \longrightarrow & H^1_{\text{cris}}(V) \\
\downarrow & & \downarrow \\
H^1_f(D) & \longrightarrow & H^1_f(K,V).
\end{array}
$$

From S) it follows that the top arrow can be identified with the natural map $t_D(K) \to t_V(K)$ which is an isomorphism by the definition of a splitting submodule.

iii) Taking into account ii), we only need to prove that the natural map $H^1(D) \to H^1(K,V)$ is injective. This follows from the exact sequence

$$0 \to D \to D^+_{\text{rig}}(V) \to D' \to 0, \quad D' = D^+_{\text{rig}}(V)/D$$

and the fact that $H^0(D') = \text{Fil}^0(D_{\text{st}/L}(V)/D)^{\varphi=1,N=0,G_{L/K} = 0}$ (see Proposition 2.8.2 i)).

### 4.2. The canonical splitting.

#### 4.2.1. Let

$$y : 0 \to V^*(1) \to Y_y \to E \to 0$$

be an extension of $E$ by $V^*(1)$.

Passing to $(\varphi, \Gamma_K)$-modules, we obtain an extension

$$0 \to D^+_{\text{rig}}(V^*(1)) \to D^+_{\text{rig}}(Y_y) \to \mathcal{R}_{K,E} \to 0.$$

By duality, we have exact sequences

$$0 \to E(1) \to Y^*_y(1) \to V \to 0$$

and

$$0 \to \mathcal{R}_{K,E}(\chi) \to D^+_{\text{rig}}(V^*(1)) \to D^+_{\text{rig}}(V) \to 0.$$

We denote by $[y]$ the class of $y$ in $\text{Ext}^1_{E[G_K]}(E,V^*(1)) \simeq H^1(K,V^*(1))$. Assume that $y$ is crystalline, i.e. that $[y] \in H^1_f(K,V^*(1))$. Let $D$ be a splitting submodule of $D_{\text{st}/L}(V)$. Consider the commutative diagram

$$
\begin{array}{cccccc}
y : & 0 & \longrightarrow & D^+_{\text{rig}}(V^*(1)) & \longrightarrow & D^+_{\text{rig}}(Y_y) & \longrightarrow & \mathcal{R}_{K,E} & \longrightarrow & 0 \\
\text{pr}(y) : & 0 & \longrightarrow & D^*(\chi) & \longrightarrow & D^*_y(\chi) & \longrightarrow & \mathcal{R}_{K,E} & \longrightarrow & 0
\end{array}
$$
where $D_y$ is the inverse image of $D$ in $D_{\text{rig}}^\dagger(Y^*_y(1))$. The class of $\text{pr}(y)$ in $H^1(D^*(\chi))$ is the image of $[y]$ under the map

$$\text{Ext}^1(\mathcal{R}_{K,E}, D_{\text{rig}}^\dagger(V^*(1))) \to \text{Ext}^1(\mathcal{R}_{K,E}, D^*(\chi))$$

which coincides with the map

$$H^1(K, V^*(1)) = H^1(D_{\text{rig}}^\dagger(V^*(1))) \to H^1(D^*(\chi))$$

after the identification of $\text{Ext}^1(\mathcal{R}_{K,E}, -)$ with the cohomology group $\text{Ext}^1(-)$. Since we are assuming that $[y] \in H^1(K, V^*(1))$, by Proposition 4.1.2, we obtain that $[\text{pr}(y)] = 0$. Thus the sequence $\text{pr}(y)$ splits.

4.2.2. We will construct a canonical splitting of $\text{pr}(y)$ using the idea of Nekovář [33]. Since $\dim_E D_{\text{cris}}(Y_y) = \dim_E D_{\text{cris}}(V^*(1)) + 1$, the sequence

$$0 \to D_{\text{cris}}(V^*(1)) \to D_{\text{cris}}(Y_y) \to D_{\text{cris}}(E) \to 0$$

is exact by the dimension argument. From $D_{\text{cris}}(V^*(1))^{\varphi=1} = 0$ and the snake lemma follows that $D_{\text{cris}}(Y_y)^{\varphi=1} = D_{\text{cris}}(E)$ and we obtain a canonical $\varphi$-equivariant morphism of $K_0$-vector spaces $D_{\text{cris}}(E) \to D_{\text{cris}}(Y_y)$. By linearity, this map extends to a $(\varphi, N, G_{L/K})$-equivariant morphism of $L_0$-vector spaces $D_{\text{st}/L}(E) \to D_{\text{st}/L}(Y_y)$. Therefore we have a canonical splitting

$$D_{\text{st}/L}(Y_y) \cong D_{\text{st}/L}(V^*(1)) \oplus D_{\text{st}/L}(E)$$

of the sequence

$$0 \to D_{\text{st}/L}(V^*(1)) \to D_{\text{st}/L}(Y_y) \to D_{\text{st}/L}(E) \to 0$$

in the category of $(\varphi, N, G_{L/K})$-modules. This splitting induces a $(\varphi, N, G_{L/K})$-equivariant isomorphism

$$\mathcal{D}_{\text{st}/L}(D^*_y(\chi)) \cong \mathcal{D}_{\text{st}/L}(D^*(\chi)) \oplus \mathcal{D}_{\text{st}/L}(E).$$

Moreover, since all Hodge–Tate weights of $D^*(\chi)$ are positive, we have $\text{Fil}^i \mathcal{D}_{\text{dR}/L}(D^*_y(\chi)) \cong \text{Fil}^i \mathcal{D}_{\text{dR}/L}(D^*(\chi)) \oplus \text{Fil}^i \mathcal{D}_{\text{dR}/L}(E)$ and therefore the isomorphism

$$\mathcal{D}_{\text{dR}/L}(D^*_y(\chi)) \cong \mathcal{D}_{\text{dR}/L}(D^*(\chi)) \oplus \mathcal{D}_{\text{dR}/L}(E)$$

is compatible with filtrations. Thus, we obtain that (53) is an isomorphism in the category of filtered $(\varphi, N, G_{L/K})$-modules. This gives a canonical splitting

$$\text{pr}(y) : \quad 0 \longrightarrow D^*(\chi) \longrightarrow D^*_y(\chi) \xrightarrow{\cong} \mathcal{R}_{K,E} \longrightarrow 0$$
of the extension \( \text{pr}(y) \). Passing to duals, we obtain a splitting

\[
0 \longrightarrow \mathcal{R}_{K,E}(\chi) \longrightarrow \mathbf{D}_y \overset{s_{\mathbf{D},y}}{\longrightarrow} \mathbf{D} \longrightarrow 0. \tag{54}
\]

Taking cohomology, we get a splitting

\[
0 \longrightarrow H^1_f(K, E(1)) \longrightarrow H^1_f(D_y) \overset{s_y}{\longrightarrow} H^1_f(D) \longrightarrow 0. \tag{55}
\]

Our constructions can be summarized in the diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & H^1_f(K, E(1)) & \longrightarrow & H^1_f(D_y) & \longrightarrow & H^1_f(D) & \longrightarrow & 0 \\
\downarrow & & \downarrow & \simeq & \downarrow & \simeq \\
0 & \longrightarrow & H^1_f(K, E(1)) & \longrightarrow & H^1_f(K, Y^*_y(1)) & \longrightarrow & H^1_f(K, V) & \longrightarrow & 0.
\end{array}
\]

Here the vertical maps are isomorphisms by Proposition 4.1.2 and the five lemma.

**4.2.3. Remark.** Assume that \( H^0(D^*(\chi)) = 0 \). Then each crystalline extension of \( D \) by \( \mathcal{R}_K(\chi) \) splits uniquely. This follows from Proposition 2.8.2 i) which implies that \( H^1_f(D^*(\chi)) = 0 \) and from the fact that various splittings are parametrized by \( H^0(D^*(\chi)) \).

**4.3. Filtration associated to a splitting submodule.**

**4.3.1.** In this subsection we assume that \( K = \mathbb{Q}_p \). We review the construction of the canonical filtration on \( \mathbf{D}_{\text{st}/L}(V) \) associated to a splitting submodule \( D \). Note that this construction is the direct generalization of the filtration constructed by Greenberg [23] in the ordinary case. See [4] for more detail.

Let \( V \) be a potentially semistable representation of \( G_{\mathbb{Q}_p} \) with coefficients in a finite extension \( E \) of \( \mathbb{Q}_p \). As before, we fix a finite Galois extension \( L/\mathbb{Q}_p \) such that \( V \) is semistable over \( L \) and denote by \( \mathbf{D}_{\text{st}/L}(V) \) the semistable module of the restriction of \( V \) on \( G_L \). Let \( D \subset \mathbf{D}_{\text{st}/L}(V) \) be a splitting submodule. Set \( D' = \mathbf{D}_{\text{st}/L}(V)/D \). Then \( \text{Fil}^0 D' = D' \) and we define \( M_1 = (D')^{\phi=1,N=0,G_L/\mathbb{Q}_p} \). For the dual filtered \( (\phi,N,G_L/\mathbb{Q}_p) \)-module \( D^* = \text{Hom}_{L_0\otimes E}(D, \mathbf{D}_{\text{st}/L}(E(1))) \) we have \( \text{Fil}^0 D^* = D^* \) and we define \( M_0 = \left( (D^*)^{\phi=1,N=0,G_L/\mathbb{Q}_p} \right)^* \). We have canonical projections \( \text{pr}_{D'} : \mathbf{D}_{\text{st}/L}(V) \rightarrow \)
Since

By construction, the structure of modules

The exact sequence (56) induces the coboundary map $\delta_0 : H^0(M_1) \to H^1(M_0)$.

Passing to cohomology in the dual exact sequence

By Theorem 2.2.3, the filtration $(F_i D_{\text{st}/L}(V))_{i = -2}^2$ induces a filtration

on the $(\phi, \Gamma_{\mathbb{Q}_p})$-module $D_{\text{rig}}^\dagger (V)$ such that

Note that $D = F_0 D_{\text{rig}}^\dagger (V)$. We also set $M_0 = F_0 D_{\text{rig}}^\dagger (V)/F_{-1} D_{\text{rig}}^\dagger (V)$, $M_1 = F_1 D_{\text{rig}}^\dagger (V)/F_0 D_{\text{rig}}^\dagger (V)$ and $W = F_1 D_{\text{rig}}^\dagger (V)/F_{-1} D_{\text{rig}}^\dagger (V)$. We have a tautological exact sequence

(56) \[ 0 \to M_0 \xrightarrow{\alpha} W \xrightarrow{\beta} M_1 \to 0. \]

By construction, $M_0$ and $M_1$ are crystalline $(\phi, \Gamma_{\mathbb{Q}_p})$-modules such that

Since

the structure of modules $M_0$ and $M_1$ is given by Proposition 2.8.4. In particular, we have canonical decompositions

$H^1(M_0) \overset{(pr_f, pr_c)}{\cong} H_f^1(M_0) \oplus H_c^1(M_0)$, $H^1(M_1) \overset{(pr_f, pr_c)}{\cong} H_f^1(M_1) \oplus H_c^1(M_1)$.

The exact sequence (56) induces the coboundary map $\delta_0 : H^0(M_1) \to H^1(M_0)$.
we obtain the coboundary map $\delta_0^*: H^0(M_0^*(\chi)) \to H^1(M_1^*(\chi))$.

4.3.2. In the remainder of this subsection we assume that $(V, D)$ satisfies the following conditions:

**F1)** For all $i \in \mathbb{Z}$
\[ \mathcal{D}_{pst}(D_{\text{rig}}^+(V)/F_1 D_{\text{rig}}^+(V))^p = \mathcal{D}_{pst}(F_{-1} D_{\text{rig}}^+(V))^p = 0. \]

**F2)** The composed maps
\[ H^0(M_1) \xrightarrow{\delta_0} H^1(M_0) \xrightarrow{\text{pr}_c} H^1_c(M_0), \]
\[ H^0(M_1) \xrightarrow{\delta_0} H^1(M_0) \xrightarrow{\text{pr}_f} H^1_f(M_0), \]
where the second arrows denote canonical projections, are isomorphisms.

4.3.3. Remarks. 1) It is easy to see, that the filtration on $D_{\text{rig}}^+(V^*(1))$ associated to the dual splitting submodule $D^\perp = \text{Hom}(D_{\text{st/L}}(V)/D, D_{\text{st/L}}(E(1)))$, is dual to the filtration $F_i D_{\text{st}}(V)$. In particular,
\[ F_{-1} D_{\text{rig}}^+(V^*(1))^*(\chi) \simeq D_{\text{rig}}^+(V)/F_1 D_{\text{rig}}^+(V), \]
\[ D_{\text{rig}}^+(V^*(1))/F_1 D_{\text{rig}}^+(V^*(1)) \simeq (F_{-1} D_{\text{rig}}^+(V))^*(\chi), \]
and the sequence (56) for $(V^*(1), D^\perp)$ coincides with (57). In particular, the conditions **F1-2)** hold for $(V, D)$ if and only if they hold for $(V^*(1), D^\perp)$.

2) The condition **F1)** implies that
\[ H^0(D_{\text{st/L}}(V))/F_1 D_{\text{st/L}}(V) = H^0(F_{-1} D_{\text{rig}}^+(V)) = 0. \]

3) If $V$ is semistable over $\mathbb{Q}_p$, and the linear map $\varphi: D_{\text{st}}(V) \to D_{\text{st}}(V)$ is semisimple at 1 and $p^{-1}$, it is easy to see that
\[ F_i D_{\text{st}}(V) = \begin{cases} (1 - p^{-1} \varphi^{-1}) D + N(D^{\varphi^{-1}}) & \text{if } i = -1, \\ D & \text{if } i = 0, \\ D + D_{\text{st}}(V)^{\varphi=1} \cap N^{-1}(D^{\varphi=p^{-1}}) & \text{if } i = 1. \end{cases} \]
In this form, the filtration $F_i D_{\text{st}}(V)$ was constructed in [4].

We summarize some properties of the filtration $F_i D_{\text{rig}}^+(V)$.

**Proposition 4.3.4.** Let $D$ be a regular submodule of $D_{\text{st}}(V)$ such that $(V, D)$ satisfies the conditions **F1-2)**. Then

i) $\text{rk}(M_0) = \text{rk}(M_1)$.

ii) $H^0(W) = 0$. 


iii) The representation $V$ satisfies $S)$, namely

$$D_{\text{cris}}(V)^{\varphi=1} = D_{\text{cris}}(V^+(1))^{\varphi=1} = 0.$$ 

iv) One has $H^1_f(F_{-1}D^\dagger_{\text{rig}}(V)) = H^1(F_{-1}D^\dagger_{\text{rig}}(V))$ and $H^1_f(F_1D^\dagger_{\text{rig}}(V)) = H^1_f(\mathbb{Q}_p, V)$.

v) We have exact sequences

\begin{equation}
0 \to H^0(M_1) \to H^1(M_0) \to H^1_f(W) \to 0
\end{equation}

and

\begin{equation}
0 \to H^0(M_1) \to H^1(D) \to H^1_f(\mathbb{Q}_p, V) \to 0.
\end{equation}

Proof. i) From $F2)$ and the fact that $\dim_E H^0(M_1) = \text{rk}(M_1)$ and $\dim_E H^1_f(M_0) = \text{rk}(M_0)$ (see Proposition 2.8.4) we obtain that $\text{rk}(M_0) = \text{rk}(M_1)$.

ii) By Proposition 2.8.4 iv), $H^0(M_0) = 0$, and we have an exact sequence

$$0 \to H^0(W) \to H^0(M_1) \xrightarrow{\delta_0} H^1(M_0).$$

By $F2)$, the map $\delta_0$ is injective and therefore $H^0(W) = 0$. Applying the same argument to the dual exact sequence, we obtain that $H^0(W^*(\chi)) = 0$.

iii) First prove that $\mathcal{D}_{\text{cris}}(W) = \mathcal{D}_{\text{cris}}(M_0)$. The exact sequence (56) gives an exact sequence

$$0 \to \mathcal{D}_{\text{cris}}(M_0) \xrightarrow{\alpha} \mathcal{D}_{\text{cris}}(W) \xrightarrow{\beta} \mathcal{D}_{\text{cris}}(M_1)$$

and we have immediately the inclusion $\mathcal{D}_{\text{cris}}(M_0) \subset \mathcal{D}_{\text{cris}}(W)$. Thus, it is enough to check that $\dim_E \mathcal{D}_{\text{cris}}(W) = \dim_E \mathcal{D}_{\text{cris}}(M_0)$. Assume that $\dim_E \mathcal{D}_{\text{cris}}(W) > \dim_E \mathcal{D}_{\text{cris}}(M_0)$. Then there exists $x \in \mathcal{D}_{\text{cris}}(W)$ such that $m = \beta(x) \neq 0$. Since $\varphi$ acts trivially on $\mathcal{D}_{\text{cris}}(M_1) = M_1^{\Gamma_{\mathbb{Q}_p}}$, $\mathcal{R}_{\mathbb{Q}_p,E}m$ is a $(\varphi, \Gamma_{\mathbb{Q}_p})$-submodule of $M_1$, and there exists a submodule $X \subset W$ which sits in the following commutative diagram with exact rows

\begin{equation}
0 \to M_0 \to X \to \mathcal{R}_{\mathbb{Q}_p,E}m \to 0
\end{equation}

and

\begin{equation}
0 \to M_0 \to W \to M_1 \to 0.
\end{equation}
Since \( \mathcal{D}_{\text{cris}}(W) = (W[1/\ell])^{\Gamma_{Q_p}} \), there exists \( n \geq 0 \) such that \( \ell^n x \in X \), and therefore \( x \in \mathcal{D}_{\text{cris}}(X) \). This implies that \( X \) is crystalline, and by Proposition 2.8.2 iv) we have a commutative diagram

\[
\begin{align*}
Em \rightarrow H^1_f(M_0) \\
\downarrow \downarrow \\
H^0(M_1) \rightarrow \delta_0 \rightarrow H^1(M_0).
\end{align*}
\]

Thus, \( \text{Im}(\delta_0) \cap H^1_f(M_0) \neq \{0\} \) and the condition F2) is violated. This proves that \( \mathcal{D}_{\text{cris}}(W) = \mathcal{D}_{\text{cris}}(M_0) \).

Now we can finish the proof of iii). Taking invariants, we have \( \mathcal{D}_{\text{cris}}(W)^{\phi=1} = \mathcal{D}_{\text{cris}}(M_0)^{\phi=1} = 0 \). By F1)

\[ \mathcal{D}_{\text{cris}}(F_1 \mathbb{D}^\dagger_{\text{rig}}(V))^{\phi=1} = \mathcal{D}_{\text{cris}}(\mathbb{D}^\dagger_{\text{rig}}(V)/F_1 \mathbb{D}^\dagger_{\text{rig}}(V))^{\phi=1} = 0, \]

and, applying the functor \( \mathcal{D}_{\text{cris}}(-)^{\phi=1} \) to the exact sequences

\[
\begin{align*}
0 & \rightarrow F_1 \mathbb{D}^\dagger_{\text{rig}}(V) \rightarrow \mathbb{D}^\dagger_{\text{rig}}(V) \rightarrow \mathbb{D}^\dagger_{\text{rig}}(V)/F_1 \mathbb{D}^\dagger_{\text{rig}}(V), \\
0 & \rightarrow F_1 \mathbb{D}^\dagger_{\text{rig}}(V) \rightarrow F_1 \mathbb{D}^\dagger_{\text{rig}}(V) \rightarrow W \rightarrow 0,
\end{align*}
\]

we obtain that \( \mathbb{D}_{\text{cris}}(V)^{\phi=1} \subset \mathcal{D}_{\text{cris}}(W)^{\phi=1} = 0 \). The same argument shows that \( \mathbb{D}_{\text{cris}}(V^{(1)})^{\phi=1} = 0 \).

iv) By F1) together with Proposition 2.8.2 and the Euler–Poincaré characteristic formula, we have

\[
\dim_E H^1_f(F_1 \mathbb{D}^\dagger_{\text{rig}}(V)) = \dim_E H^1_f(F_1 \mathbb{D}^\dagger_{\text{rig}}(V)) = \dim_E H^0((F_1 \mathbb{D}^\dagger_{\text{rig}}(V))^{\phi=1}(\chi)) = 0,
\]

and therefore \( H^1_f(F_1 \mathbb{D}^\dagger_{\text{rig}}(V)) = H^1_f(F_1 \mathbb{D}^\dagger_{\text{rig}}(V)) \). Since \( H^0(\mathbb{D}^\dagger_{\text{rig}}(V)/F_1 \mathbb{D}^\dagger_{\text{rig}}(V)) = 0 \), the exact sequence

\[
0 \rightarrow F_1 \mathbb{D}^\dagger_{\text{rig}}(V) \rightarrow \mathbb{D}^\dagger_{\text{rig}}(V) \rightarrow \mathbb{D}^\dagger_{\text{rig}}(V)/F_1 \mathbb{D}^\dagger_{\text{rig}}(V) \rightarrow 0
\]

induces, by Proposition 2.8.2 iv), an exact sequence

\[
0 \rightarrow H^1_f(F_1 \mathbb{D}^\dagger_{\text{rig}}(V)) \rightarrow H^1_f(\mathbb{D}^\dagger_{\text{rig}}(V)) \rightarrow H^1_f(\mathbb{D}^\dagger_{\text{rig}}(V)/F_1 \mathbb{D}^\dagger_{\text{rig}}(V)) \rightarrow 0
\]

On the other hand, since

\[ \mathcal{D}_{\text{cris}}(\mathbb{D}^\dagger_{\text{rig}}(V)/F_1 \mathbb{D}^\dagger_{\text{rig}}(V)) = \text{Fil}^0 \mathcal{D}_{\text{cris}}(\mathbb{D}^\dagger_{\text{rig}}(V)/F_1 \mathbb{D}^\dagger_{\text{rig}}(V)), \]

by Proposition 2.8.2 i) we have

\[
\dim_H H^1_f(\mathbb{D}^\dagger_{\text{rig}}(V)/F_1 \mathbb{D}^\dagger_{\text{rig}}(V)) = \dim_E H^0(\mathbb{D}^\dagger_{\text{rig}}(V)/F_1 \mathbb{D}^\dagger_{\text{rig}}(V)) = 0,
\]

and therefore \( H^1_f(F_1 \mathbb{D}^\dagger_{\text{rig}}(V)) = H^1_f(\mathbb{D}^\dagger_{\text{rig}}(V)) = H^1_f(Q_p, V). \)
v) To prove the exacteness of (58), we only need to show that the image of the map \( \alpha : H^1(M_0) \to H^1(W) \), induced by the exact sequence (56), coincides with \( H^1_f(W) \). By F2), \( \text{Im}(\delta_0) \cap H^1_f(M_0) = \{0\} \), and therefore the map \( H^1_f(M_0) \to H^1_f(W) \) is injective. Set \( e = \text{rk}(M_0) = \text{rk}(M_1) \). Since

\[
\dim E H^1_f(W) = \dim E t_W(Q_p) - H^0(W) = e = \dim E H^1_f(M_0),
\]

we obtain that \( H^1_f(M_0) = H^1_f(W) \). On the other hand, the exact sequence

\[
0 \to H^0(M_1) \xrightarrow{\delta_0} H^1(M_0) \xrightarrow{\alpha} H^1(W)
\]

shows that \( \dim E \text{Im}(\alpha) = \dim E H^1_f(M_0) - \dim E H^0(M_1) = e = \dim E H^1_f(M_0) \). Therefore \( \text{Im}(\alpha) = H^1_f(M_0) = H^1_f(W) \), and the exactness of (58) is proved.

Since \( H^0(W) = 0 \) and \( H^1_f(F^{-1}D^\dagger_{\text{rig}}(V)) = H^1_f(F^{-1}D^\dagger_{\text{rig}}(V)) \), by Proposition 2.8.2 iv) we have an exact sequence

\[
0 \to H^1_f(F^{-1}D^\dagger_{\text{rig}}(V)) \to H^1_f(F_1D^\dagger_{\text{rig}}(V)) \to H^1_f(W) \to 0,
\]

which shows that \( H^1_f(F_1D^\dagger_{\text{rig}}(V)) \) is the inverse image of \( H^1_f(W) \) under the map \( H^1_f(F_1D^\dagger_{\text{rig}}(V)) \to H^1_f(W) \). Therefore we have the following commutative diagram with exact rows

\[
\begin{array}{ccccccc}
0 & \to & H^0(M_1) & \xrightarrow{=} & H^0(M_1) & \xrightarrow{} & 0 \\
0 & \to & H^1(F_{-1}D^\dagger_{\text{rig}}(V)) & \xrightarrow{} & H^1(D) & \xrightarrow{} & H^1(M_0) & \xrightarrow{} & 0 \\
0 & \to & H^1(F_{-1}D^\dagger_{\text{rig}}(V)) & \xrightarrow{=} & H^1(F_1D^\dagger_{\text{rig}}(V)) & \xrightarrow{=} & H^1_f(W) & \xrightarrow{} & 0 \\
0 & \to & 0 & \xrightarrow{} & 0 & \xrightarrow{} & 0.
\end{array}
\]

Since the right column of this diagram is exact, the five lemma gives the exactness of the middle column. Now the exactness of (59) follows from the fact that \( H^1_f(F_1D^\dagger_{\text{rig}}(V)) = H^1_f(Q_p, V) \) by iv).

\[\Box\]

4.3.5. The tautological exact sequence

\[
0 \to D \to D^\dagger_{\text{rig}}(V) \to D' \to 0
\]
induces the coboundary map
\[ \partial_0 : H^0(D') \to H^1(D), \]
which is injective because \( H^0(Q_p, V) = 0 \) by Proposition 4.3.4 ii).

**Proposition 4.3.6.** Let \( V \) be a \( p \)-adic representation of \( G_{Q_p} \) which satisfies the conditions F1-2). Then
\[ H^1(D) = H^1_{Iw}(D)_{\Gamma_0} \oplus \partial_0 \left( H^0(D) \right). \]

**Proof.** Since \( \mathcal{D}_{p^i} \left( \left( \frac{F_i D_{\text{rig}}^\dagger (V)}{} \right)^* \right)^{\varphi = p^j} = 0 \) for all \( i \in \mathbb{Z} \), by Lemma 2.7.4 we have \( H^2_{Iw}(F_{-1} D_{\text{rig}}^\dagger (V)) = 0 \). Then the tautological exact sequence
\[ 0 \to F_{-1} D_{\text{rig}}^\dagger (V) \to D \to M_0 \to 0 \]
induces an exact sequence
\[ 0 \to H^1_{Iw}(F_{-1} D_{\text{rig}}^\dagger (V)) \to H^1_{Iw}(D) \to H^1_{Iw}(M_0) \to 0. \]

Since \( H^1_{Iw}(M_0)_{\Gamma_0}^{Q_p} = H^0(M_0) = 0 \) by Proposition 4.3.4 the snake lemma gives an exact sequence
\[ 0 \to H^1_{Iw}(F_{-1} D_{\text{rig}}^\dagger (V))_{\Gamma_0}^{Q_p} \to H^1_{Iw}(D)_{\Gamma_0}^{Q_p} \to H^1_{Iw}(M_0)_{\Gamma_0}^{Q_p} \to 0. \]

The Hochschild–Serre exact sequence
\[ 0 \to H^1_{Iw}(F_{-1} D_{\text{rig}}^\dagger (V))_{\Gamma_0}^{Q_p} \to H^1(F_{-1} D_{\text{rig}}^\dagger (V)) \to H^1_{Iw}(F_{-1} D_{\text{rig}}^\dagger (V))_{\Gamma_0}^{Q_p} \to 0 \]
together with the fact that
\[ \dim_E H^2_{Iw}(F_{-1} D_{\text{rig}}^\dagger (V))_{\Gamma_0}^{Q_p} = \dim_E H^2_{Iw}(F_{-1} D_{\text{rig}}^\dagger (V))_{\Gamma_0}^{Q_p} = \]
\[ = \dim_E H^0((F_{-1} D_{\text{rig}}^\dagger (V))^*(\chi)) = 0 \]
implies that \( H^1_{Iw}(F_{-1} D_{\text{rig}}^\dagger (V))_{\Gamma_0}^{Q_p} = H^1(F_{-1} D_{\text{rig}}^\dagger (V)) \). On the other hand, \( H^1_{Iw}(M_0)_{\Gamma_0}^{Q_p} = H^1_c(M_0) \) by Proposition 2.8.5. Therefore, the sequence (60) reads
\[ 0 \to H^1(F_{-1} D_{\text{rig}}^\dagger (V)) \to H^1_{Iw}(D)_{\Gamma_0}^{Q_p} \to H^1_c(M_0) \to 0 \]
and we have a commutative diagram
\[ \begin{array}{cccccc}
0 & \to & H^1(F_{-1} D_{\text{rig}}^\dagger (V)) & \to & H^1_{Iw}(D)_{\Gamma_0}^{Q_p} & \to & H^1_c(M_0) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^1(F_{-1} D_{\text{rig}}^\dagger (V)) & \to & H^1(D) & \to & H^1(M_0) & \to & 0.
\end{array} \]
Since $H^0(D^\dagger_{\text{rig}}(V)/F_1D^\dagger_{\text{rig}}(V)) = 0$, the exact sequence
\begin{align*}
0 \to M_1 \to D' \to D^\dagger_{\text{rig}}(V)/F_1D^\dagger_{\text{rig}}(V) \to 0
\end{align*}
gives $H^0(M_1) = H^0(D')$ and we have a commutative diagram
\begin{align*}
H^0(D') \xrightarrow{\delta_0} H^1(D)
\end{align*}
Finally, from F2) it follows that $H^1_c(M_0) \cap \delta_0(H^0(M_1)) = \{0\}$, and the dimension argument shows that
\begin{align*}
H^1(M_0) = H^1_c(M_0) \oplus \delta_0(H^0(M_1)).
\end{align*}
Now, the proposition follows from (63) and the diagrams (61) and (62). □

5. $p$-adic Height Pairings II: Universal Norms

5.1. The pairing $h_{V,D}^{\text{norm}}$.

5.1.1. In this section, we construct the pairing $h_{V,D}^{\text{norm}}$, which is a direct generalization of the pairing constructed in [33], Section 6 and [36]. Let $V$ is a $p$-adic representation of $G_{F,S}$ with coefficients in a finite extension $E$ of $Q_p$. We fix a system $D = (D_q)_{q \in S_p}$ of submodules $D_q \subset D^\dagger_{\text{rig}}(V_q)$ and denote by $D^\perp = (D^\perp_q)_{q \in S_p}$ the orthogonal complement of $D$. We have tautological exact sequences
\begin{align*}
0 \to D_q \to D^\dagger_{\text{rig}}(V_q) \to D'_q \to 0, \quad q \in S_p,
\end{align*}
where $D'_q = D^\dagger_{\text{rig}}(V_q)/D_q$. Passing to duals, we have exact sequences
\begin{align*}
0 \to (D^\dagger_q)^*(\chi_q) \to D^\dagger_{\text{rig}}(V^*(1)) \to D^*_{\text{rig}}(\chi_q) \to 0,
\end{align*}
where $(D^\dagger_q)^*(\chi_q) = D^\perp_q$. If the contrary is not explicitly stated, in this section we will assume that the following conditions hold

N1) $H^0(F_q, V) = H^0(F_q, V^*(1)) = 0$ for all $q \in S_p$;

N2) $H^0(D^\dagger_q)/H^0(D^\dagger_q)^*(\chi_q) = 0$ for all $q \in S_p$.

From N2), it follows immediately that $H^1(D_q)$ injects into $H^1(F_q, V)$. By our definition of Selmer complexes we have
\begin{align*}
H^1(V, D) \simeq \ker \left( H^1_D(V) \to \bigoplus_{q \in \Sigma_p} H^1(F_q, V) \right) \oplus \left( \bigoplus_{v \in S_p} H^1(F_q, V) \right).
\end{align*}
and the same formula holds for \( V^*(1) \) if we replace \( D_q \) by \( D_q^\perp \). Using this isomorphism, we identify each cohomology class \([x, (x_q^+) \in H^1(V, D) \) with the corresponding class \([x] \in H^1_\chi(V) \).

**5.1.2.** Let \( [y] \in H^1(V^*(1), D^\perp) \) and let \( Y_y \) be the associated extention

\[
0 \to V^*(1) \to Y_y \to E \to 0.
\]

Passing to duals, we have an exact sequence

\[
0 \to E(1) \to Y_y^* (1) \to V \to 0.
\]

For each \( q \in S_p \), this sequence induces an exact sequence of \((\varphi, \Gamma_q)\) modules

\[
0 \to \mathcal{D}_{F_q, E}(\chi_q) \to D_{\text{rig}}^*(Y_y^*(1)) \to D_{\text{rig}}^*(V_q) \to 0.
\]

Consider the commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & H^1(F_q, E(1)) & \to & H^1(D_{q,y}) & \xrightarrow{\pi_{D,q}} & H^1(D_q) & \xrightarrow{\delta_{D,q}^1} & H^2(F_q, E(1)) \\
 & & & & & & (\gamma_q) & \downarrow \gamma_q & \downarrow \gamma_q & \downarrow \\
& & & & & & H^1(F_q, E(1)) & \to & H^1(F_q, Y_y^*(1)) & \xrightarrow{\pi_q} & H^1(F_q, V) & \xrightarrow{\delta_{\chi, q}^1} & H^2(F_q, E(1)) \\
 & & & & & & & \uparrow \text{res}_q & \uparrow \text{res}_q & \uparrow \text{res}_q \\
0 & \to & H^1_\chi(E(1)) & \to & H^1_\chi(Y_y^*(1)) & \xrightarrow{\pi} & H^1_\chi(V) & \xrightarrow{\delta_{\chi}^1} & H^2_\chi(E(1)),
\end{array}
\]

where \( D_{q,y} \) denotes the inverse image of \( D_q \) in \( D_{\text{rig}}^*(V_q) \).

In the following lemma we do not assume that the condition N2) holds.

**Lemma 5.1.3.** Assume that \( V \) is a p-adic representation satisfying the condition N1). Let \([x] = [(x_q^+) \in H^1(V, D) \) and let \( x_q = \text{res}_q(x) \). Then

i) If \( q \nmid p \), then \( H^1_f(F_q, E(1)) = 0 \) and

\[
H^1_f(F_q, Y_y^*(1)) \simeq H^1_f(F_q, V).
\]

ii) For each \( q \in S_p \) one has \( \delta_{\chi, q}^1([x_q]) = \delta_{D,q}^1([x_q^+]) = 0 \).

iii) \( \delta_{\chi}^1([x]) = 0 \).

iv) The sequence

\[
0 \to H^1(E(1), \mathcal{R}(\chi)) \to H^1(Y_y^*(1), D_y) \to H^1(V, D) \to 0,
\]

where \( \mathcal{R}(\chi) = (\mathcal{R}_{F_q, E}(\chi_q))_{q \in S_p} \), is exact.

**Proof.** i) If \( q \nmid p \), then \( E(1) \) is unramified at \( q \), \( H^0(F_q^{ur}/F_q, E(1)) = 0 \) and

\[
H^1_f(F_q, E(1)) = H^1(F_q^{ur}/F_q, E(1)) = E(1)/(F_q - 1)E(1) = 0.
\]
Since \([y]\) is unramified at \(q\), the sequence
\[
0 \to E(1) \to Y_y^*(1)^{I_q} \to V^{I_q} \to 0
\]
is exact. Passing to the associated long exact cohomology sequence of \(\text{Gal}(F^\text{ur}_q / F_q)\) and taking into account that
\[
H^1(F^\text{ur}_q / F_q, E(1)) = H^2(F^\text{ur}_q / F_q, E(1)) = 0
\]
we obtain that \(H^1(F^\text{ur}_q / F_q, Y_y^*(1)^{I_q}) \cong H^1(F^\text{ur}_q / F_q, V^{I_q}).\) This proves i).

ii) For each \(q \in S_p\) we have \(g_q([x_q^+]) = [x_q].\) From the orthogonality of \(D_q\) and \(D_q^\perp\) it follows that
\[
\delta_D^1(x_q^+) = -x_q^+ \cup y_q^+ = 0.
\]
Therefore, \(\delta_D^1_q([x_q]) = \delta_D^1_q([x_q^+]) = 0\) for each \(q \in S_p.\)

iii) Let \(q \in \Sigma_p.\) Since \([x_q] \in H^1_f(F_q, V)\) from i) it follows that again \(\delta_{V,q}([x_q]) = 0.\) As the localization map
\[
H^2_S(E(1)) \to \bigoplus_{v \in S} H^2(F_q, E(1))
\]
is injective, we obtain that \(\delta_D^1(x) = 0.\)

iv) First prove the surjectivity of \(\pi : H^1(Y_y^*(1), D_y) \to H^1(V, D).\) For each \(q \in \Sigma_p\) we denote by
\[
s_{y,q} : H^1_f(F_q, Y_y^*(1)) \cong H^1_f(F_q, V)
\]
the inverse of the isomorphism i). Let \([x] \in H^1(V, D).\) By ii), \(\delta_D^1([x]) = 0,\)
and there exists \([a] \in H^1_S(Y_y^*(1))\) such that \(\pi([a]) = [x].\) Let \(q \in \Sigma_p.\) Since
\([x_q^+] \in H^1_f(F_q, V),\) there exists \([b_q^+] \in H^1(F_q, E(1))\) such that
\[
[a_q^+] = s_{y,q}([x_q^+]) + [b_q^+].
\]
The localization map \(H^1_S(E(1)) \to \bigoplus_{q \in \Sigma_p} H^1(F_q, E(1))\) is surjective, and there
exists \([b] \in H^1_S(E(1))\) such that \([b_q] = \text{res}_q([b_q]) = [b_q^+]\) for each \(q \in \Sigma_p.\)
Then \([x] = [a] - [b] \in H^1(Y_y^*(1), D_y)\) and satisfies \(\pi([x]) = [x].\) Thus, the map \(\pi\) is surjective.

Finally, from i) we have
\[
H^1(E(1), \mathcal{R}(\chi)) = \ker \left( H^1_S(E(1)) \to \bigoplus_{q \in \Sigma_p} H^1(F_q, E(1)) \right),
\]
and it is easy to see that \(H^1(E(1), \mathcal{R}(\chi))\) coincides with the kernel of \(\pi.\)
The lemma is proved. \(\square\)
5.1.4. Let \( \log_p : \mathbb{Q}_p^* \to \mathbb{Q}_p \) denote the \( p \)-adic logarithm normalized by \( \log_p(p) = 0 \). For each finite place \( q \) we define an homomorphism \( \ell_q : \mathbb{Q}_p^* \to \mathbb{Q}_p \) by
\[
\ell_q(x) = \begin{cases} 
\log_p(N_{F_q/Q}(x)), & \text{if } q \mid p, \\
\log_p |x|_q, & \text{if } q \nmid p,
\end{cases}
\]
where \( N_{F_q/Q} \) denotes the norm map. By linearity, \( \ell_q \) can be extended to a map \( \ell_q : \mathbb{F}_q^* \otimes \mathbb{Z}_p \mathcal{E} \to \mathcal{E} \), and the isomorphism \( \mathbb{F}_q^* \otimes \mathbb{Z}_p \mathcal{E} \cong H^1(F_q, E(1)) \) allows to consider \( \ell_q \) as a map \( H^1(F_q, E(1)) \to \mathcal{E} \) which we denote again by \( \ell_q \).

From the product formula
\[
|N_F/Q(x)|_\infty \prod_{q \in S_f} |x|_q = 1
\]
and the fact that \( N_F/Q(x) = \prod_{q \nmid p} N_{F_q/Q_p}(x) \) it follows that
\[
(64) \quad \sum_{q \mid p} \ell_q(x) = 1, \quad \forall x \in \mathbb{F}_q^*.
\]

We set \( \Lambda_{F_q} = \mathcal{O}_E[[\Gamma^0_v]] \) and \( \Lambda_{F_q,E} = \Lambda_{F_q}[1/p] \).

**Lemma 5.1.5.** Let \( V \) be a \( p \)-adic representation of \( G_{F,S} \) that satisfies **N1-2** and let \( [y] \in H^1(V^*(1), \mathcal{D}^\perp) \). For each \( q \in S_p \), the following diagram is commutative with exact rows and columns
\[
\begin{array}{ccccccc}
0 & \to & \mathcal{H}(\Gamma^0_v) \otimes \Lambda_{F_q,E} & \to & H^1_{Iw}(F_q, E(1)) & \to & H^1(F_q, E(1)) & \to & \ell_q & \to & \mathcal{E} & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
& & H^1_{Iw}(\mathcal{D}_{q,y}) & \to & H^1_{Iw}(\mathcal{D}_{q,y}) & \to & H^1(\mathcal{D}_{q,y}) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
& & H^1_{Iw}(\mathcal{D}_q) & \to & H^1(\mathcal{D}_q) & \to & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]
Proof. The exactness of the left column is clear. The exactness of the right column follows from the fact that the diagram

\[ H^2(F_{q,n}, E(1)) \xrightarrow{\text{inv}_{v,n}} E \]

is commutative, and therefore

\[ H^2_{Iw}(F_q, E(1)) \simeq H^2(F_q, E(1)) \simeq E. \]

The diagram (65) is clearly commutative. Now, we prove that the projection map \( H^1_{Iw}(D_q) \rightarrow H^1(D_q) \) is surjective. We have an exact sequence

\[ 0 \rightarrow H^1_{Iw}(D_q)_{\Gamma_q^0} \rightarrow H^1(D_q) \rightarrow H^2_{Iw}(D_q)_{\Gamma_q^0} \rightarrow 0, \]

and therefore it is enough to show that \( H^2_{Iw}(D_q)_{\Gamma_q^0} = 0 \). Consider the exact sequence

\[ 0 \rightarrow H^2_{Iw}(D_q)_{\Gamma_q^0} \rightarrow H^2_{Iw}(D_q) \xrightarrow{\gamma^{-1}} H^2_{Iw}(D_q) \rightarrow H^2_{Iw}(D_q)_{\Gamma_q^0} \rightarrow 0. \]

Since \( H^2_{Iw}(D_q) \) is a finite-dimensional \( E \)-vector space, we have

\[ \dim_E H^2_{Iw}(D_q)_{\Gamma_q^0} = \dim_E H^2_{Iw}(D_q)_{\Gamma_q} = \dim_E H^2(D_q) = \dim_E H^0(D_q^*(\chi)) = 0. \]

Thus, the map \( H^1_{Iw}(D_q) \rightarrow H^1(D_q) \) is surjective. To prove the exactness of the first row, we remark that the sequence

\[ H^1_{Iw}(F_q, E(1)) \rightarrow H^1(F_q, E(1)) \xrightarrow{\ell_q} E \]

is known to be exact (see, for example, [34], Section 11.3.5), and that the image of the projection \( \mathcal{H}(\Gamma_q^0) \otimes_{\Lambda, q, E} H^1_{Iw}(F_q, E(1)) \rightarrow H^1(F_q, E(1)) \) coincides with the image of the projection \( H^1_{Iw}(F_q, E(1)) \rightarrow H^1(F_q, E(1)). \)

5.1.6. By Lemma 5.1.5, for each \( q \in S_p \) we have the following commutative diagram with exact rows, where the map \( \operatorname{pr}_q \) is surjective

(66)
Let \( [x] \in H^1(V, D) \). By Lemma [5.1.3](ii), for each \( q \in S_p \) we have \( \delta_p,q([x^+]) = 0 \), and therefore there exists \( [x^+_{q,y}] \in H^1_{lw}(D_{q,y}) \) such that \( \text{pr}_{q,y} \circ \pi_{D,q}([x^+_{q,y}]) = [x^+_q] \). By Lemma [5.1.3](iv), there exists a lift \( \tilde{x} \) of \( [x] \in H^1(Y_y^+(1), D_y) \) of \( [x] \). For each \( v \in S_p \) we set

\[
[u_q] = g_{q,y} \circ \text{pr}_{q,y}([x^+_{q,y}]) - \tilde{x}_q, \tag{67}
\]

where \( \tilde{x}_q = \text{res}_q(\tilde{x}) \). Then \( \pi_q([u_q]) = 0 \), and therefore \( [u_q] \in H^1(F_q, E(1)) \).

**Definition.** Let \( V \) be a \( p \)-adic representation of \( G_{F,S} \) equipped with a family \( D = (D_q)_{q \in S_p} \) of \((\varphi, \Gamma_q)\)-modules satisfying the conditions N1-2. The \( p \)-adic height pairing \( h^\text{norm}_{V,D} \) associated to these data is defined to be the map

\[
h^\text{norm}_{V,D} : H^1(V, D) \times H^1(V^+ (1), D^\perp) \to E, \quad h^\text{norm}_{V,D} ([x], [y]) = \sum_{q \in S_p} \ell_q([u_q]).
\]

**5.1.7. Remarks.** 1) If \( \tilde{x} \) is another lift of \([x]\), then from (64) and the fact that \( \tilde{x}_q = \tilde{x}_q = s_y,q([x_q]) \) for all \( q \in S_p \), it follows that the definition of \( h^\text{norm}_{V,D}([x], [y]) \) does not depend on the choice of the lift \( \tilde{x}_q \).

2) It is not indispensable to take \( [\tilde{x}] \) in \( H^1(Y_y^+(1), D_y) \). If \( [\tilde{x}] \in H^1_3(Y_y^+(1)) \) is such that \( \pi([\tilde{x}]) = [x] \), we can again define \([u_q]\) by (67). For \( q \in \Sigma_p \) we set

\[
[u_q] = g_{q,y} \circ s_y,q([x^+_q]) - \tilde{x}_q,
\]

where \( s_y,q : H^1_1(F_q, V_y^+(1)) \to H^1_1(F_q, Y_y^+(1)) \) denotes the isomorphism from Lemma [6.1.2](i). Note that again \([u_q] \in H^1(F_q, E(1)) \). Then

\[
h^\text{norm}_{V,D} ([x], [y]) = \sum_{q \in \Sigma} \ell_q([u_q]).
\]

3) The map \( h^\text{norm}_{V,D} \) is bilinear. This can be shown directly, but follows from Theorem [5.2.2](ii) below.

**5.2. Comparision with \( h^\text{sel}_{V,D} \).**

**5.2.1.** In this subsection we compare \( h^\text{norm}_{V,D} \) with the \( p \)-adic height pairing constructed in Subsection 3.2. We take \( \Sigma = \emptyset \) and denote by

\[
h^\text{sel}_{V, D_{1}} : H^1(V, D) \times H^1(V^+(1), D^\perp) \to E
\]

the associated height pairing (51).
Theorem 5.2.2. Let $V$ be a $p$-adic representation of $G_{F,S}$ with coefficients in a finite extension $E$ of $\mathbb{Q}_p$. Assume that the family $D = (D_q)_{q \in S_p}$ satisfies the conditions N1-2). Then $h_{v,D}^{\text{norm}}$ is a bilinear map and

$$h_{v,D}^{\text{norm}} = h_{v,D,1}^{\text{sel}}.$$  

Proof. The proof repeats the arguments of [34], Sections 11.3.9-11.3.12, where this statement is proved in the case of $p$-adic height pairings arising from Greenberg’s local conditions. We remark that in this case our definition of $h_{v,D}^{\text{norm}}$ differs from Nekovář’s $h_{\pi}^{\text{norm}}$ by a sign.

Let $[x] \in H^1(V, D)$ and $[y] \in H^1(V^*(1), D^\perp)$. We use the notation of Section 3.1 and denote by $f_q$ and $g_q$ the morphisms defined by (24–27). As before, to simplify notation we set $x_q = f_q(x)$ and $y_q = f_q^\perp(y)$. We represent $[x]$ and $[y]$ by cocycles $x^{\text{sel}} = (x, (x_q^+), (\lambda_q)) \in S^1(V, D)$ and $y^{\text{sel}} = (y, (y_q^+), (\mu_q)) \in S^1(V^*(1), D^\perp)$, where

$$x \in C^1(G_{F,S}, V), \quad x_q^+ \in U^1_q(V, D), \quad \lambda_q \in K^0(V_q),$$

$$y \in C^1(G_{F,S}, V^*(1)), \quad y_q^+ \in U^1_q(V^*(1), D^\perp), \quad \mu_q \in K^0(V_q^*(1))$$

and

$$dx = 0, \quad dy = 0,$$

$$d x_q^+ = 0, \quad d y_q^+ = 0,$$

$$g_q(x_q^+) = f_q(x) + d\lambda_q, \quad g_q^+(y_q^+) = f_q^\perp(x) + d\mu_q, \quad \forall q \in S.$$

By Propositions 2.6.2, 2.6.4 and 2.6.5 we have

$$\beta_{v,D}(x^{\text{sel}}) = (-z \cup x, (-w_q \cup x^+_q), (z_q \cup \lambda_q)) \in S^2(V, D),$$

where

$$z = \log \chi \in C^1(G_{F,S}, E(0)),$$

$$w_q = \begin{cases} 0, & \text{if } q \in \Sigma_p, \\ (0, \log \chi_q(\gamma_q)) \in C^1_{\phi, \gamma_q}(E(0)), & \text{if } q \in S_p, \end{cases}$$

$$z_q = \begin{cases} \log \chi_q \in C^1(G_{F,q}, E(0)), & \text{if } q \in \Sigma_p, \\ (0, \log \chi_q) \in K^1(E(0)_q), & \text{if } q \in S_p. \end{cases}$$

Lemma 5.2.3. Let $[x] \in H^1(Y^*_y(1), D_y)$ be a lift of an element $[x] \in H^1(V, D)$. Represent $[x]$ and $[x^+_q]$ in $H^1(D_{q,y})$ for $q \in S_p$ (respectively $[x^+_q] \in H^1(F_q, Y^*_y(1))$ for $q \in \Sigma_p$) by cocycles $\tilde{x} \in C^1(G_{F,S}, Y^*_y(1))$ and $\tilde{x}^+_q \in C^1_{\phi, \gamma_q}(D_{q,y})$ (respectively by $\tilde{x}^+_q \in C^1_{ur}(Y^*_y(1)_q)$). Since $g_q(\tilde{x}^+_q) = f_q(\tilde{x}) + d\tilde{\lambda}_q$ for some $\tilde{\lambda}_q \in$
\( K^0(Y^*_1(1)_q) \), we obtain a cocycle \( \tilde{x}^{\text{sel}} = (\tilde{x}, (\tilde{\lambda}_q), (\tilde{\eta}_q)) \in S^1(Y^*_1(1), D_q) \). Then

\[ \beta_{Y^*_1(1), D_q}(\tilde{x}^{\text{sel}}) = (\tilde{\alpha}, (\tilde{b}_q), (\tilde{c}_q)) \in S^2(Y^*_1(1), D_q), \]

where

\[ \tilde{b}_q = \begin{cases} 0, & \text{if } q \in \Sigma_p, \\
\omega_q \cup u_q \in \mathcal{C}^2_{\phi, \gamma_q}((E(1)_q), & \text{if } q \in S_p \end{cases}, \]

and \( u_q \) is a representative of the cohomology class \( (67) \).

**Proof.** The proof is analogous to the proof of Lemma 11.3.10 of [34]. Since the complex \( C^*_{ur}(Y^*_1(1)_v) \) is concentrated in degrees 0 and 1, it is clear that \( \tilde{b}_q = 0 \) for \( q \in \Sigma_p \). For each \( q \in S_p \), we consider the algebra \( \mathcal{H}_E(\Gamma^0_F) = \{ f(\gamma_F - 1) \mid f(X) \in \mathcal{H}_E \} \) and define \( \text{Ind}_{F_{\infty}/F}(D_q) = D_q \otimes_E \mathcal{H}_E(\Gamma^0_F) \). The natural inclusion \( \mathcal{H}_E(\Gamma^0_q) \subset \mathcal{H}_E(\Gamma^0_F) \) allows us to consider \( \text{Ind}_{F_{\infty}/F}(D_q) \) as a \( (\phi, \Gamma^0_q) \)-module over the ring \( \mathcal{H}_{F_q, \mathcal{H}_E} = \lim_{\mathcal{H}_{\text{rig}, F_q}} \mathcal{H}_E(\Gamma^0_q) \). Let \( J_{\mathcal{H}} \) denote the augmentation ideal of \( \mathcal{H}_E(\Gamma^0_F) \). The obvious commutative diagram

\[
\begin{array}{ccc}
\mathcal{H}_E(\Gamma^0_F) & \xrightarrow{\text{mod } \Gamma^0_q} & E[\Gamma^0_F / \Gamma^0_q] \\
\downarrow \text{mod } J_{\mathcal{H}} & & \downarrow \\
\tilde{A}_F & \longrightarrow & E,
\end{array}
\]

where the right vertical morphism is the augmentation map, induces a commutative diagram

\[
\begin{array}{ccc}
H^1(\text{Ind}_{F_{\infty}/F}(D_q)) & \longrightarrow & H^1(D_q \otimes_E E[\Gamma^0_F / \Gamma^0_q]) \\
\downarrow & & \downarrow \\
H^1(\tilde{A}_{F,q}) & \longrightarrow & H^1(D_q).
\end{array}
\]

Since \( \Gamma^0_q \) acts trivially of \( E[\Gamma^0_F / \Gamma^0_q] \), the group \( H^1(D_q \otimes_E E[\Gamma^0_F / \Gamma^0_q]) \) is isomorphic to the direct sum of \( (\Gamma^0_F : \Gamma^0_q) \) copies of \( H^1(D_q) \) and the right vertical map of (69) is surjective. Since \( \mathcal{H}_E(\Gamma^0_F) \) is a free \( \mathcal{H}_E(\Gamma^0_q) \)-module of rank \( [\Gamma^0_F : \Gamma^0_q] \), the group \( H^1(\text{Ind}_{F_{\infty}/F}(D_q)) \) is isomorphic to the direct sum of \( [\Gamma^0_F : \Gamma^0_q] \)-copies of \( H^1_{\text{rig}}(D_q) \). Since the projection \( \text{pr}^0_q : H^1_{\text{rig}}(D_q) \to H^1(D_q) \) is surjective by Proposition 5.1.5, the upper horizontal map of (69) is also
surjective. Therefore the projection $H^1(\text{Ind}_{F_q/F}(D_q)) \to H^1(D_q)$ is surjective and we have the following analog of the diagram (66)

\[
H^1(\text{Ind}_{F_q/F}(D_{q,y})) \xrightarrow{\pi_{D_q,y}^F} H^1(\text{Ind}_{F_q/F}(D_q)) \xrightarrow{\pi_{D_q}^F} \cdots
\]

From the above discussion it follows that $\hat{\mathcal{C}}_{\varphi,y}(\text{Ind}_{F_q/F}(D_{q,y}))$ is isomorphic to the direct sum of $[\Gamma_F^q : \Gamma_q^q]$ copies of $\mathcal{C}_{\varphi,y}(\text{Ind}_{F_q/F}(D_{q,y}))$. Therefore there exists a cocycle $\tilde{x}_{q,y} \in \mathcal{C}_{\varphi,y}(\text{Ind}_{F_q/F}(D_{q,y}))$ such that $pr_{q,y}^F(\tilde{x}_{q,y}) = pr_{q,y}(x_{q,y})$. Since the map

\[
\mathbf{C}_{\varphi,y}(\text{Ind}_{F_q/F}(D_{q,y})) \to \mathbf{C}_{\varphi,y}(D_{q,y})
\]

factors through $\mathbf{C}_{\varphi,y}(\tilde{D}_{F,q,y})$, where $\tilde{D}_{F,q,y} = D_{q,y} \otimes \tilde{A}_F$, from the distinguished triangle

\[
\mathbf{C}_{\varphi,y}(D_{q,y}) \to \mathbf{C}_{\varphi,y}(\tilde{D}_{F,q,y}) \to \mathbf{C}_{\varphi,y}(D_{q,y}) \xrightarrow{\beta_{D_{q,y}}} \mathbf{C}_{\varphi,y}(D_{q,y})[1]
\]

it follows that $\beta_{D_{q,y}}(pr_{q,y}^F(\tilde{x}_{q,y})) = 0$. Thus,

\[
\hat{b}_q = \beta_{D_{q,y}}(\tilde{x}_{q,y}^+) = \beta_{D_{q,y}}(\tilde{x}_{q,y}^+ - pr_{q,y}(x_{q,y}^w) + pr_{q,y}^F(\tilde{x}_{q,y})) = -\beta_{D_{q,y}}(u_q) + \beta_{D_{q,y}}(pr_{q,y}^F(\tilde{x}_{q,y})) = -\beta_{D_{q,y}}(u_q) = w_q \cup u_q.
\]

The lemma is proved.

For each $q \in S_p$, we have the canonical isomorphism of local class field theory

\[
\text{inv}_q : H^2(F_q, E(1)) \xrightarrow{\sim} E.
\]

Let $\kappa_q : F_q^* \to H^1(F_q, E(1))$ denote the Kummer map. Then

\[
\text{inv}_q(\log x_{q,y}) = \log_{F_q}(N_{F_q/Q_p}(x)) = \ell_q(x)
\]

([40], Chapitre 14, see also [2], Corollaire 1.1.3).

**Lemma 5.2.4.** Assume that $\beta_{\varphi,D}(x_{q,y}) \in H^2(V, D)$ is represented by a 2-cocycle $e = (a, (b_q), (c_q))$ of the form $e = \pi(\tilde{e})$, where

\[
\tilde{e} = (\tilde{a}, (\tilde{b}_q), (\tilde{c}_q)) \in S^2(Y^*(1), D_y)
\]
is also a 2-cocycle and $\pi : S^2(Y_\infty^*(1), D_y) \to S^2(V, D)$ denotes the canonical projection. Then

$$[\beta_{V, D}(x^{\text{sel}}) \cup y^{\text{sel}}] = \sum_{q \in S_p} \text{inv}_q(g_q(\hat{b}_q) \cup f_q^\perp(\alpha_y) + g_q(b_q) \cup \mu_q),$$

where $\alpha_y \in C^0(G_{F, S}, Y_y)$ is an element that maps to $1 \in C^0(G_{F, S}, E) = E$ and satisfies $d\alpha_y = y$. If, in addition,

$$\hat{b}_q \in C^2_{\phi, \gamma_q}(E(1)_q), \quad \forall q \in S_p,$$

then

$$[\beta_{V, D}(x^{\text{sel}}) \cup y^{\text{sel}}] = \sum_{q \in S_p} \text{inv}_q(g_q(\hat{b}_q)).$$

**Proof.** The proof of this lemma is purely formal and follows verbatim the proof of [34], Lemma 11.3.11. □

Now we can proof Theorem 5.2.2. Combining Lemma 5.2.3 and Lemma 5.2.4 we have

$$h_{V, D, 1}^{\text{sel}}([x], [y]) = [\beta_{V, D}(x^{\text{sel}}) \cup y^{\text{sel}}] = \sum_{q \in S_p} \text{inv}_q(g_q(\hat{b}_q)) =$$

$$= \sum_{q \in S_p} \text{inv}_q(g_q(w_q \cup u_q)) = \sum_{q \in S_p} \ell_q(u_q) = h_{V, D}^{\text{norm}}([x], [y]).$$

□

### 6. **p-Adic Height Pairings III: Splitting of Local Extensions**

#### 6.1. The pairing $h_{V, D}^{\text{spl}}$

**6.1.1.** Let $F$ be a finite extension of $\mathbb{Q}$. We keep notation of Sections 3-5. In particular, we fix a finite set $S$ of places of $F$ such that $S_p \subset S$ and denote by $G_{F, S}$ the Galois group of the maximal algebraic extension of $F$ which is unramified outside $S \cup S_\infty$. For each topological $G_{F, S}$-module $M$, we write $H^*_S(M)$ for the continuous cohomology of $G_{F, S}$ with coefficients in $M$.

Let $V$ be a $p$-adic representation of $G_{F, S}$ with coefficients in a finite extension $E/\mathbb{Q}_p$ which is potentially semistable at all $q \mid p$. Following Bloch and Kato, for each $q \in S$ we define the subgroup $H^1_f(F_q, V)$ of $H^1(F_q, V)$ by

$$H^1_f(F_q, V) = \begin{cases} \ker(H^1(F_q, V) \to H^1(F_q, V \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{cris}})) & \text{if } q \mid p, \\ \ker(H^1(F_q, V) \to H^1(F^ur_q, V)) & \text{if } q \nmid p. \end{cases}$$
The Bloch–Kato Selmer group of $V$ is defined as
\[ H^1_f(V) = \ker \left( H^1_S(V) \to \bigoplus_{q \in S} H^1_{f, q}(V) \right). \]

In this section, we assume that, for all $q \in S_p$, the representation $V_q$ satisfies the condition S) of Section 4, namely that
\[ S) \quad D_{\text{cris}}(V_q)^{\varphi = 1} = D_{\text{cris}}(V_q^*(1))^{\varphi = 1} = 0 \text{ for all } q \in S_p. \]

For each $q | p$, we fix a splitting $(\varphi, N, G_{F_q})$-submodule $D_q$ of $D_{\text{pst}}(V_q)$ (see Section 4.1). We will associate to these data a pairing
\[ h_{\text{spl}, V, D} : H^1_f(V) \times H^1_f(V^*(1)) \to E \]
and compare it with the height pairing constructed in Section 4 of [33] using the exponential map and splitting of the Hodge filtration.

Let $[y] \in H^1_f(V^*(1))$. Fix a representative $y \in C^1(G_{F, S}, V^*(1))$ of $y$ and consider the corresponding extension of Galois representations
\[ (72) \quad 0 \to V^*(1) \to Y_y \to E \to 0. \]
Passing to duals, we obtain an extension
\[ 0 \to E(1) \to Y_y^*(1) \to V \to 0. \]

From S) it follows that $H^0_S(V) = 0$, and the associated long exact sequence of global Galois cohomology reads
\[ 0 \to H^1_S(E(1)) \to H^1_S(Y_y^*(1)) \to H^1_S(V) \to \delta^1_{V, q} : H^2_{\text{cris}}(F_q, E(1)) \to \ldots. \]
Also, for each place $q \in S$ we have the long exact sequence of local Galois cohomology
\[ H^0(F_q, V) \to H^1(F_q, E(1)) \to H^1(F_q, Y_y^*(1)) \to \]
\[ \to H^1(F_q, V) \to H^2(F_q, E(1)) \to \ldots. \]

The following results, which can be seen as an analog of Lemma 5.1.3 are well known but we recall them for the reader’s convenience.

**Lemma 6.1.2.** Let $V$ be a $p$-adic representation of $G_{F, S}$ that is potentially semistable at all $q \in S_p$ and satisfies the condition S). Assume that $[y] \in H^1_f(V^*(1))$. Then
i) $\delta^1_{V, q}([x]) = 0$ for all $x \in H^1_f(V)$;
ii) There exists an exact sequence
\[ 0 \to H^1_f(E(1)) \to H^1_f(Y_y^*(1)) \to H^1_f(V) \to 0. \]
Proof. i) For any $x \in C^1(G_{F,S}, V)$, let $x_q = \text{res}_q(x) \in C^1(G_{F_q}, V)$ denote the localization of $x$ at $q$. If $[x] \in H^1_f(V)$, then for each $q$ one has $\delta^1_{V,q}([x_q]) = -[x_q] \cup [y_q] = 0$ because $H^1_f(F_q, V)$ and $H^1_f(F_q, V^*)$ are orthogonal to each other under the cup product. Since the map $H^2_S(E(1)) \to \bigoplus_{q \in S} H^2_f(F_q, E(1))$ is injective and the localization commutes with cup products, this shows that $\delta^1_{V}(x) = 0$.

ii) This is a particular case of [21], Proposition II, 2.2.3.

\[ \square \]

6.1.3. Let $[x] \in H^1_f(V)$ and $[y] \in H^1_f(V^*(1))$. In Section 4.2, for each $q \in S_p$ we constructed the canonical splitting (55) which sits in the diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_f(F_q, E(1)) & \longrightarrow & H^1_f(D_q, V) & \longrightarrow & H^1_f(D_q) & \longrightarrow & 0 \\
0 & \longrightarrow & H^1_f(F_q, E(1)) & \longrightarrow & H^1_f(F_q, Y^*(1)) & \longrightarrow & H^1_f(F_q, V) & \longrightarrow & 0.
\end{array}
\]

By Lemma (6.1.2ii), we can lift $[x] \in H^1_f(V)$ to an element $\hat{x} \in H^1_f(Y^*(1))$. Let $\hat{x}_q = \text{res}_q(\hat{x}) \in H^1_f(F_q, Y^*(1))$. If $q \in S_p$, we denote by $[\hat{x}_q^+]$ the unique element of $H^1_f(D_q)$ such that $g_q([\hat{x}_q^+]) = [x_q]$.

**Definition.** The $p$-adic height pairing associated to splitting submodules $D = (D_q)_{q \in S}$ is defined to be the map

\[ h^{\text{spl}}_{V,D} : H^1_f(V) \times H^1_f(V^*(1)) \to E \]

given by

\[ h^{\text{spl}}_{V,D}([x],[y]) = \sum_{q \in S_p} \ell_q \left( g_{q,y} \circ s_{y,q}(\hat{x}_q^+) - [x_q] \right). \]

6.1.4. **Remarks.** 1) For each $q \in \Sigma_p$, denote by $s_{y,q} : H^1_f(F_q, V) \to H^1_f(F_q, Y^*(1))$ the isomorphism constructed in Lemma 5.1.3 i) and by $g_q : H^1_f(F_q, V) \hookrightarrow H^1_f(F_q, V)$ and $g_{q,y} : H^1_f(F_q, Y^*(1)) \hookrightarrow H^1_f(F_q, Y^*(1))$ the canonical embeddings. Let $[\hat{x}_q^+] \in H^1_f(F_q, V)$ be the unique element such that $g_q([\hat{x}_q^+]) = [x_q]$.

From the product formula (64) it follows, that $h^{\text{spl}}_{V,D}$ can be defined by

\[ h^{\text{spl}}_{V,D}([x],[y]) = \sum_{q \in S} \ell_q \left( g_{q,y} \circ s_{y,q}(\hat{x}_q^+) - [x_q] \right), \]

where $\hat{x} \in H^1_f(V)$ is an arbitrary lift of $[x]$. 
2) The pairing $h^\text{spl}_{V,D}$ is a bilinear skew-symmetric map. This can be shown directly, but follows from the interpretation of $h^\text{spl}_{V,D}$ in terms of Nekovář’s height pairing (see Proposition 6.2.3 below).

6.2. Comparison with Nekovář’s height pairing.

6.2.1. We relate the pairing $h^\text{spl}_{V,D}$ to the $p$-adic height pairing constructed by Nekovář in [33], Section 4. First recall Nekovář’s construction. If $[y] \in H^1_f(V^*(1))$, the extension (72) is crystalline at all $q \in S_p$, and therefore the sequence

$$0 \to \text{D}_{\text{cris}}(V_q^*(1)) \to \text{D}_{\text{cris}}(Y_{y,q}) \to \text{D}_{\text{cris}}(E(0)q) \to 0$$

is exact. Since $\text{D}_{\text{cris}}(V_q^*(1))^{\varphi=1} = 0$, we have an isomorphism of vector spaces

$$\text{D}_{\text{cris}}(E(0)q) \cong \text{D}_{\text{cris}}(Y_{y,q})^{\varphi=1},$$

which can be extended by linearity to a map $\text{D}_{\text{dR}}(E(0)q) \to \text{D}_{\text{dR}}(Y_{y,q})$. Passing to duals, we obtain a $F_q$-linear map $\text{D}_{\text{dR}}(Y_{y,q}^*(1)) \to \text{D}_{\text{dR}}(E(1)q)$ which defines a splitting $s_{\text{dR},q}$ of the exact sequence

$$0 \longrightarrow \text{D}_{\text{dR}}(E(1)q) \longrightarrow \text{D}_{\text{dR}}(Y_{y,q}^*(1)) \xrightarrow{s_{\text{dR},q}} \text{D}_{\text{dR}}(V_q) \longrightarrow 0.$$

Fix a splitting $w_q : \text{D}_{\text{dR}}(V_q)/\text{Fil}^0\text{D}_{\text{dR}}(V_q) \to \text{D}_{\text{dR}}(V_q)$ of the canonical projection

(73) $\text{pr}_{\text{dR},V_q} : \text{D}_{\text{dR}}(V_q) \to \text{D}_{\text{dR}}(V_q)/\text{Fil}^0\text{D}_{\text{dR}}(V_q)$.

We have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_f(F_q, E(1)) & \longrightarrow & H^1_f(F_q, Y_{y,q}^*(1)) & \xrightarrow{s_{y,q}^w} & H^1_f(F_q, V) & \longrightarrow & 0 \\
\exp_{y,q}^\ast(1) & \sim & \text{D}_{\text{dR}}(Y_{y,q}^*(1)) & \xrightarrow{\text{D}_{\text{dR}}(V_q)} & \text{D}_{\text{dR}}(V_q) \\
\text{pr}_{\text{dR},V_q}^\ast(1) & \sim & \text{Fil}^0\text{D}_{\text{dR}}(Y_{y,q}^*(1)) & \xrightarrow{\text{Fil}^0\text{D}_{\text{dR}}(V_q)} & \text{Fil}^0\text{D}_{\text{dR}}(V_q) \\
\text{D}_{\text{dR}}(Y_{y,q}^*(1)) & \xrightarrow{s_{\text{dR},q}^w} & \text{D}_{\text{dR}}(V_q) & \xrightarrow{w_q} & \text{D}_{\text{dR}}(V_q).
\end{array}
$$

Then the map $s_{y,q}^w : H^1_f(F_q, V) \to H^1_f(F_q, Y_{y,q}^*(1))$ defined by

$$s_{y,q}^w = \exp_{y,q}^\ast(1) \circ \text{pr}_{\text{dR},V_q}^\ast(1) \circ s_{\text{dR},q} \circ w_q \circ \exp_{V_q}^{-1}$$

gives a splitting of the top row of the diagram, which depends only on the choice of $w_q$ and $[y]$. 
Definition (Nekovář). The p-adic height pairing associated to a family \( w = (w_q)_{q \in S_p} \) of splitting \( w_q \) of the projections (73) is defined to be the map
\[
h_{V,w}^{\text{Hodge}} : H^1_f(V) \times H^1_f(V^*(1)) \to E
\]
given by
\[
h_{V,w}^{\text{Hodge}}([x],[y]) = \sum_{q \mid p} \ell_q \left( s^w_{y,q}([x_q]) - \hat{[x_q]} \right),
\]
where \( \hat{[x]} \in H^1_f(Y_q^*(1)) \) is a lift of \([x] \in H^1_f(V)\) and \( \hat{[x_q]} \) denotes its localization at \( q \).

In [33], it is proved that \( h_{V,w}^{\text{Hodge}} \) is a \( E \)-bilinear map.

6.2.2. Now, let \( D_q \) be a splitting submodule of \( D_{st/L}(V_q) \). We have
\[
D_{\text{dR}/L}(V_q) = D_{q,L} \oplus \text{Fil}^0 D_{\text{dR}/L}(V_q), \quad D_{q,L} = D_q \otimes_{L_0} L.
\]
Set \( D_q,F_q = (D_{q,L})^{G_{F_q}} \). Since the decomposition (74) is compatible with the Galois action, taking galois invariants we have
\[
D_{\text{dR}}(V_q) = D_{q,F_q} \oplus \text{Fil}^0 D_{\text{dR}}(V_q).
\]
This decomposition defines a splitting of the projection (73) which we will denote by \( w_{D,q} \).

Proposition 6.2.3. Let \( V \) be a p-adic representation of \( G_{F,S} \) such that for each \( q \in S_p \) the restriction of \( V \) on the decomposition group at \( q \) is potentially semistable and satisfies the condition \( S \). Let \( (D_q)_{q \in S_p} \) be a family of splitting submodules and let \( w_D = (w_{D,q})_{q \in S_p} \) be the associated system of splittings. Then
\[
h_{V,D}^{\text{spl}} = h_{V,w_D}^{\text{Hodge}}.
\]

We need the following auxiliary result. As before, we denote by \( D_q \) the \((\varphi,\Gamma_q)\)-module associated to \( D_q \).

Lemma 6.2.4. The following diagram
\[
D_{\text{dR}}(D_q) \xrightarrow{s_{D,q,y}} D_{\text{dR}}(D_{q,y})
\]
\[
D_{\text{dR}}(V_q) \xrightarrow{s_{D,q,y}} D_{\text{dR}}(Y_{y,q}^*(1)),
\]
where the vertical maps are induced by the canonical inclusions of corresponding \((\varphi,\Gamma_q)\)-modules and \( s_{D,q,y} \) is the map induced by the splitting (54), is commutative.

Proof of the lemma. The proof is an easy exercise and is omitted here. \( \square \)
Proof of Proposition \[6.2.3\] From the functoriality of the exponential map and Proposition \[4.1.2\] it follows that the diagram

\[
\begin{array}{ccc}
\mathcal{D}_{dR}(D_q) & \xrightarrow{\exp D_q} & H^1_f(D_q) \\
\downarrow & & \downarrow \\
D_{dR}(V_q)/\Fil^0 D_{dR}(V_q) & \xrightarrow{\exp V_q} & H^1_f(F_q, V)
\end{array}
\]

is commutative. The same holds if we replace \(V_q\) and \(D_q\) by \(Y^*_{y,q}(1)\) and \(D_{q,y}\) respectively. Consider the diagram

\[
\begin{array}{ccc}
\mathcal{D}_{dR}(D_q) & \xrightarrow{s_{D_q,y}} & \mathcal{D}_{dR}(D_{q,y}) \\
\downarrow & & \downarrow \\
D_{dR}(V_q) & \xrightarrow{s_{D_q,q}} & D_{dR}(Y^*_{y,q}(1)) \\
\Fil^0 D_{dR}(V_q) & \xrightarrow{s_{D_{q,y},q}} & \Fil^0 D_{dR}(Y^*_{y,q}(1)) \\
\displaystyle \lim_{\rightarrow \downarrow} w_{D,q} & \xrightarrow{\lim_{\rightarrow \downarrow} s_{D_{q,y},q}} & D_{dR}(Y^*_{y,q}(1))
\end{array}
\]

From the definition of \(w_{D,q}\), it follows that the composition of vertical maps in the left (resp. right) column is induced by the inclusion \(D_q \subset D^\dag_{\text{rig}}(V_q)\) (resp. by \(D_{q,y} \subset D^\dag_{\text{rig}}(Y^*_{y,q}(1))\)) and therefore the diagram \(76\) is commutative by Lemma \[6.2.4\]. From the commutativity of \(75\) and \(76\) and the definition of \(s_{y,q}\) and \(s_{w_{y,q}}\), it follows now that \(s_{y,q} = s_{w_{y,q}}\) for all \(q \in S_p\), and the proposition is proved. \(\square\)

6.3. Comparison with \(h_{V,D}^{\text{norm}}\).

6.3.1. In this section, we compare the pairing \(h_{V,D}^{\text{spl}}\) with the pairing \(h_{V,D}^{\text{norm}}\) constructed in Section \[5\]. Let \(V\) be a \(p\)-adic representation of \(G_{F,S}\) that is potentially semistable at all \(q \in S_p\). Fix a system \((D_{q})_{q \in S_p}\) of splitting submodules and denote by \((D_{q})_{q \in S_p}\) the system of \((\phi, \Gamma_q)\)-submodules of \(D^\dag_{\text{rig}}(V_q)\) associated to \((D_{q})_{q \in S_p}\) by Theorem \[2.2.3\]. We will assume, that \((V,D)\) satisfies the condition \(S\) of Section \[6.1\] and the condition \(N2\) of Section \[5.1\]. Note, that \(S\) implies \(N1\). We also remark, that from Proposition \[2.8.2\] and the fact that the Hodge–Tate weights of \(D_{st/L}(V_q)/D_q\) and \(D_{st/L}(V_q^*(1))/D^\dag_{\text{rig}}\) are positive, it follows that, under our assumptions, \(N2\) is equivalent to the following condition
For each \( q \in S_p \),
\[
(D_{\text{st}/L}(V_q)/D_q)^{\varphi=1,N=0,G_L/F_q} = \left( D_{\text{st}/L}(V^*_q(1))/D_q^\perp\right)^{\varphi=1,N=0,G_L/F_q} = 0,
\]
where \( L \) is a finite extension of \( F_q \) such that \( V_q \) (respectively \( V^*_q(1) \)) is semistable over \( L \).

### 6.3.2
The following statement is known (see [37] and [6]), but we prove it here for completeness.

**Proposition 6.3.3.** Assume that \( V \) is a \( p \)-adic representation satisfying the conditions \( S \) and \( N2' \). Then

1. \( H^1_f(F_q,V) = H^1_f(D_q) = H^1(D_q) \) and \( H^1_f(F_q,V^*(1)) = H^1_f(D_q^\perp) = H^1(D_q^\perp) \) for all \( q \in S_p \).
2. \( H^1_s(V) \cong H^1(V,D) \) and \( H^1_f(V^*(1)) \cong H^1(V^*(1),D^\perp) \).

**Proof.**

i) The first statement follows from \( N2 \) and Proposition 4.1.2 iii).

ii) Note that by i)
\[
\mathbb{R}^1\Gamma(F_q,V,D) = \begin{cases} 
H^1_f(F_q,V), & \text{if } q \in \Sigma_p, \\
H^1(D_q), & \text{if } q \in S_p.
\end{cases}
\]

By definition, the group \( H^1(V,D) \) is the kernel of the morphism
\[
H^1_s(V) \bigoplus \left( \bigoplus_{q \in \Sigma_p} H^1_f(F_q,V) \right) \bigoplus \left( \bigoplus_{q \in S_p} H^1(D_q) \right) \to \bigoplus_{q \in S} H^1_f(F_q,V)
\]
given by
\[
([x],[y],q) \mapsto ([x] - g_q(y_q))_q \in S, \quad [x_q] = \text{res}_q([x]),
\]
where \( g_q \) denotes the canonical inclusion \( H^1_f(F_q,V) \to H^1(F_q,V) \) if \( q \in \Sigma_p \) and the map \( H^1(D_q) \to H^1(F_q,V) \) if \( q \in S_p \). In the both cases, \( g_q \) is injective and, in addition, for each \( q \in S_p \) we have \( H^1(D_q) = H^1_f(F_q,V) \) by i). This implies that \( H^1(V,D) = H^1_f(V) \). The same argument shows that \( H^1(V^*(1),D^\perp) = H^1_f(V^*(1)) \). \( \square \)

**Theorem 6.3.4.** Let \( V \) be a \( p \)-adic representation such that \( V_q \) is potentially semistable for each \( q \in S_p \), and let \( (D_q)_{q \in S_p} \) be a family of splitting submodules. Assume that \( (V,D) \) satisfies the conditions \( S \) and \( N2 \). Then
\[
h_{V,D}^{\text{norm}} = h_{V,D}^{\text{spl}}.
\]

**Proof.** First note, that in our case the element \( [x_q^+], \) defined in Section 6.1.3, coincides with \( [x_q^+] \). Comparing the definitions of \( h_{V,D}^{\text{norm}} \) and \( h_{V,D}^{\text{spl}} \) we see that
it is enough to show that \( \ell_q \left( \text{pr}_{q,y}([x_{q,Iw}]) - \text{s}_{q,y}([x_{q}^{+}]) \right) = 0 \) for all \( q \in S_p \). The splitting \( s_{y,q} \) of the exact sequence

\[
0 \to H^1_f(F_q, E(1)) \to H^1_f(D_{q,y}) \to H^1(D_q) \to 0
\]

(see (55)) gives an isomorphism

\[
H^1_{lw}(D_{q,y})_{\Gamma_q} \cong H^1_{lw}(D_q)_{\Gamma_q} \oplus H^1_{lw}(\mathcal{D}_F, E(\chi))_{\Gamma_q} \cong H^1(D_q) \oplus H^1_{lw}(F_q, E(1))_{\Gamma_q}.
\]

Since \( \pi_{D,q} \left( \text{pr}_{q,y}([x_{q,Iw}]) - \text{s}_{q,y}([x_{q}^{+}]) \right) = 0 \), from this decomposition it follows that

\[
\text{pr}_{q,y}([w_q]) - \text{s}_{q,y}([x_{q}^{+}]) \in H^1_{lw}(F_q, E(1))_{\Gamma_q} = \ker(\ell_q),
\]

and the theorem is proved.

\[\square\]

**Corollary 6.3.5.** If \((V,D)\) satisfies the conditions \( S) \) and \( N2)\), then the height pairings \( h^\text{sel}_{V,D,1}, h^\text{norm}_{V,D}\) and \( h^\text{pl}_{V,D}\) coincide.

**Proof.** This follows from Theorems 5.2.2 and 6.3.4. \[\square\]

## 7. \( p \)-adic Height Pairings IV: Extended Selmer Groups

### 7.1. Extended Selmer groups.

**7.1.1.** Let \( F = \mathbb{Q} \). Let \( V \) be a \( p \)-adic representation of \( G_{\mathbb{Q},S} \) that is potentially semistable at \( p \). We fix a splitting submodule \( D \) of \( V \). In Section 4.3, we associated to \( D \) a canonical filtration \( \left( F_iD^\dagger_{\text{rig}}(V) \right)_{-2 \leq i \leq 2} \). Recall that \( F_0D^\dagger_{\text{rig}}(V) = D \). We maintain the notation of Section 4.3 and set \( M_0 = D/F_{-1}D^\dagger_{\text{rig}}(V), M_1 = F_1D^\dagger_{\text{rig}}(V)/D \) and \( W = F_1D^\dagger_{\text{rig}}(V)/F_{-1}D^\dagger_{\text{rig}}(V) \). The exact sequence

\[
0 \to M_0 \to W \to M_1 \to 0
\]

induces the coboundary map \( \delta_0 : H^0(M_1) \to H^1(M_0) \). In this section, we generalize the construction of the height pairing \( h^\text{norm}_{V,D} \) to the case when \( V \) satisfies the conditions \( F1-2) \) of Subsection 4.3, namely

**F1)** For all \( i \in \mathbb{Z} \)

\[
\mathcal{D}_{\text{psl}}(D^\dagger_{\text{rig}}(V)/F_1D^\dagger_{\text{rig}}(V))^\varphi = p' = 0.
\]

**F2)** The composed maps

\[
\delta_{0,c} : H^0(M_1) \xrightarrow{\delta_0} H^1(M_0) \xrightarrow{\text{pr}_{c}} H^1_c(M_0),
\]

\[
\delta_{0,f} : H^0(M_1) \xrightarrow{\delta_0} H^1(M_0) \xrightarrow{\text{pr}_{f}} H^1_f(M_0),
\]

where the second arrows denote the canonical projections, are isomorphisms. Note, that if \( V \) satisfies the conditions \( N1-2) \) of Section 5 we have \( M_0 = M_1 = 0 \).
Lemma 7.1.3. For all $x \in H^1_f(M_0)$ and $y \in H^1_f(M^*_1(\chi))$ we have
\[
\langle x, y \rangle_{D, f} = -[i^{-1}_{M^*_1(\chi), f}(y), \delta_{0, f}(x)]M_1,
\]
where $[\ , \ ]_{M_0} : \mathcal{D}_{\text{cris}}(M^*_1(\chi)) \times \mathcal{D}_{\text{cris}}(M_1) \to E$ denotes the canonical duality and $i^{-1}_{M^*_1(\chi), f} : \mathcal{D}_{\text{cris}}(M^*_1(\chi)) \to H^1_f(M^*_1(\chi))$ is the isomorphism constructed in Proposition 2.8.4.

Proof. Recall that for each $z \in H^1(\mathcal{H}_{Q_p, E}(\chi))$ we have $\text{inv}_p(w_p \cup z) = \langle z \rangle$, where $w_p = (0, \log \chi(y_{Q_p}))$. Therefore, using Proposition 2.8.4, we obtain
\[
\langle x, y \rangle_{D, f} = \langle z \rangle = \text{inv}_p(w_p \cup \delta_{0, f}(x) \cup y) = -\text{inv}_p(i_{M_1, c}(\delta_{0, f}(x) \cup y)) = -\text{inv}_p(i_{M_1, c}(\delta_{0, f}(x)) \cup i^*_{M^*_1(\chi), f}(y)) = -\langle i^{-1}_{M^*_1(\chi), f}(y), \delta_{0, f}(x) \rangle_{M_1}.
\]

\[\square\]

7.1.4. Let $\rho_{D, f}$ and $\rho_{D, c}$ denote the composed maps
\[
\rho_{D, f} : H^1(D) \to H^1(M_0) \xrightarrow{pr_f} H^1_f(M_0),
\]
\[
\rho_{D, c} : H^1(D) \to H^1(M_0) \xrightarrow{pr_c} H^1_c(M_0).
\]

Note that $H^0(M_1) = H^0(D^*)$.

Proposition 7.1.5. Let $V$ be a $p$-adic representation of $G_{Q, S}$ which is potentially semistable at $p$. Assume that the restriction of $V$ on the decomposition group at $p$ satisfies the conditions F1-2) of Section 4.3. Then

i) There exists an exact sequence
\[
0 \to H^0(D^*) \to H^1(V, D) \to H^1_f(V) \to 0.
\]

ii) The map
\[
s_{V, D} : H^1(V, D) \to H^0(D^*),
\]
\[
[(x, (x^+_q), (\lambda_q))] \mapsto \delta_{0, c} \circ \rho_{D, c}([x^+_p])
\]
defines a canonical splitting of (77).

Proof. The first statement is proved in [6], Proposition 11. It follows directly from the definition of Selmer complexes and the exact sequence (59). The second statement follows immediately from the definition of \( \text{spl}_{V,D} \).

From Proposition 7.1.5 it follows that we have a canonical decomposition

\[
H^1(V, D) \simeq H^1_f(V) \oplus H^0(D')
\]

and we denote by \( j_{V,D} : H^1_f(V) \to H^1(V, D) \) the resulting injection.

7.2. The pairing \( h_{\text{norm}}^{V,D} \) for extended Selmer groups.

7.2.1. We keep previous notation and conventions. Let \( [y] \in H^1_f(V^*(1)) \) and let \( Y_y \) denote the associated extension (72). As before, we denote by \( D_y \) the inverse image of \( D \) in \( D^+_{\text{rig}}(Y_y^*(1)_p) \). By Proposition 4.3.4 iii), the representation \( V_p \) satisfies the condition \( S \), and therefore the exact sequence (54) have a canonical splitting \( s_{D,y} \). Thus, we have the following diagram which can be seen as an analog of the diagram (65) in our situation:

\[
\begin{array}{cccccccc}
0 & \to & \mathcal{H}(\Gamma_{Q_p}^0) \otimes_{\Lambda_{E,Q_p}} H^1_{1w}(Q_p,E(1)) & \to & H^1(Q_p,E(1)) & \to & E & \to & 0 \\
& & \downarrow & & \downarrow & & \to & & \\
& & H^1_{1w}(D_y) & \to & H^1(D_y) & \leftarrow & H^0(D') & \to & 0 \\
& & \downarrow & & \downarrow & & \to & & \\
& & H^1_{1w}(D) & \to & H^1(D) & \leftarrow & H^0(D') & \to & 0 \\
& & 0 & & 0 & & 0 & & 0
\end{array}
\]
In the diagram (66), the maps \( g_v \) and \( g_{v,y} \) are no more injective and we consider the following diagram with exact rows and columns

\[
\begin{array}{ccc}
0 & \\ & \downarrow & \downarrow \\
H^0(D') & = & H^0(D') \\
& \downarrow \partial_0 & \downarrow \partial_0 \\
0 & \longrightarrow & H^1(Q_p, E(1)) \\
& \downarrow g_{p,y} & \downarrow g_p \\
0 & \longrightarrow & H^1(Q_p, Y^*_y(1)) \\
& \downarrow \pi_p & \downarrow \pi_p \\
& \longrightarrow & H^1(Q_p, V).
\end{array}
\]

By Proposition 4.3.4 v), \( \text{Im}(g_p) = H^1_f(Q_p, V) \). Let \( [x] \in H^1_f(V) \) and let

\[ [(x, \tilde{x}_q^+), (\tilde{\lambda}_q)] = j_{V,D}([x]). \]

Then \( [\tilde{x}_p^+] \) is the unique element of \( H^1_f(D) \) such that \( g_p(\tilde{x}_p^+) = [x_p] \). Let

\[
\widehat{x} \in \ker \left( H^1_S(Y^*_y(1)) \to H^1(Q_{Q_p}, Y^*_y(1)) \right)
\]

be an arbitrary lift of \([x]\). (Note, that by Lemma 6.1.2 we can even take \( \widehat{x} \in H^1_f(Y^*_y(1)). \).

By the five lemma there exists a unique \( \widehat{x}_p^+ \in H^1(D_y) \) such that \( g_{p,y}(\widehat{x}_p^+) = f_p([\widehat{x}]) \) and \( \pi_p([\widehat{x}_p^+]) = [\widehat{x}_p^+] \). On the other hand, from Proposition 4.3.6 it follows that there exist \( [x^{\text{lw}}_{p,y}] \in H^1_{\text{lw}}(D_y) \) and \( [t_p] \in H^0(D') \) such that

\[
(\text{78}) \quad [\tilde{x}_p^+] + \partial_0([t_p]) = \text{pr}_D \circ \pi_{D'} \left( [x^{\text{lw}}_{p,y}] \right).
\]

Set

\[
(\text{79}) \quad [u_p] = \text{pr}_{D,y} \left( [x^{\text{lw}}_{p,y}] \right) - \partial_0(t_p) - \tilde{x}_p^+.
\]

Then \( [u_p] \in H^1(Q_p, E(1)) \).

**Definition.** Let \( V \) be a \( p \)-adic representation satisfying the conditions F1-2). We define the height pairing

\[
h^{\text{norm}}_{V,D} : H^1_f(V) \times H^1_f(V^*(1)) \to E
\]

by

\[
h^{\text{norm}}_{V,D}([x], [y]) = \ell \cdot Q_p([u_p]).
\]

It is easy to see that \( h^{\text{norm}}_{V,D}([x], [y]) \) does not depend on the choice of the lift \( [x^{\text{lw}}_{p,y}] \). The following result generalizes Theorem 11.4.6 of [34].
Theorem 7.2.2. Let \( V \) be a p-adic representation of \( G_{Q,S} \) that is potentially semistable at \( p \) and satisfies the conditions F1-2. Then

i) \( h^\text{norm}_{V,D} = h^\text{spl}_{V,D} \);

ii) For all \([x^\text{sel}] = [(x, (x^+_q)), (\lambda_q)] \in H^1(V, D)\) and \([y^\text{sel}] = [(y, (y^+_q)), (\mu_q)] \in H^1(V^+(1), D^\perp)\) we have

\[
h^\text{sel}_{V,D}([x^\text{sel}], [y^\text{sel}]) = h^\text{norm}_{V,D}([x], [y]) + \left< \rho_{D,f}([x^+_p]), \rho_{D^+,f}([y^+_p]) \right>_{D,f}.
\]

Proof. i) Recall that in the definition of \( h^\text{norm}_{V,D} \) we can take \([x] \in H^1_f(Y^+_y(1))\).

Comparing the definitions of \( h^\text{norm}_{V,D} \) and \( h^\text{spl}_{V,D} \), we see that it is enough to prove that

\[
[u_p] - (s_{x,p}([\bar{x}^+_p]) - [\bar{x}_p]) \in \ker(\ell_{Q,p}),
\]

where \([u_p]\) is defined by (79) and \(s_{x,p}\) denotes the splitting (55). Since the restriction of \(g_{p,y}\) on \(H^1(Q_p,E(1))\) is the identity map, we have

\[
[u_p] = g_{p,y}([u_p]) = g_{p,y}([x^\text{lw}_{p,y}]) - [\bar{x}_p],
\]

and it is enough to check that

\[
(80) \quad g_{p,y}([x^\text{lw}_{p,y}]) - g_{p,y} \circ s_{x,p}([\bar{x}^+_p]) \in \ker(\ell_{Q,p}).
\]

First remark that the canonical splitting (54) induces splittings \(s^\text{lw}_{p,y}\) and \(s_{p,y}\) in the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H^1_{Iw}(\mathcal{R}_{Q_p,E}(\chi)) & \rightarrow & H^1_{Iw}(D_y) & \rightarrow & H^1_{Iw}(D) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^1(Q_p,E(1)) & \rightarrow & H^1(D_y) & \rightarrow & H^1(D) & \rightarrow & 0.
\end{array}
\]

Write \([x^\text{lw}_{p,y}]\) in the form

\[
[x^\text{lw}_{p,y}] = s^\text{lw}_{p,y}(a^\text{lw}) + b^\text{lw}, \quad a^\text{lw} \in H^1_{Iw}(D), \quad b^\text{lw} \in H^1_{Iw}(\mathcal{R}_{Q_p,E}(\chi)).
\]

By the definition of \([x^\text{lw}_{p,y}]\), we have

\[
\text{pr}_{D,y}([x^\text{lw}_{p,y}]) = s_{x,y}(a) + b,
\]

where \(b \in \ker(\ell_{Q,p}) = H^1(Q_p,E(1))_{\Gamma_{Q_p}}\) and

\[
a = \partial([t_p]) + s_{y,p}([\bar{x}^+_p]) \in H^1(D).
\]

Since \(g_{p,y}(s_{x,p}(\partial([t_p]))) = 0\), we have

\[
g_{p,y}(\text{pr}_{D,y}([x^\text{lw}_{p,y}])) = b + g_{p,y}(s_{y,p}(a)) = b + g_{p,y}(s_{y,p}([\bar{x}^+_p])),
\]

and (80) is checked.
ii) The proof is the same as that of Theorem 11.4.6 of [34] with obvious modifications. First note that $j_{Y,D}(x^\text{sel}) = (x, (\hat{x}_q^+, (\hat{\lambda}_q)))$, where

$$\hat{x}_p^+ = x_p^+ - \partial_0 \circ \left( \delta^{-1}_{0,c} \circ \rho_{D,c}([x_p^+]) \right).$$

Set

$$[t_p] = -\delta^{-1}_{0,f} \circ \rho_{D,f}([x_p^+]).$$

Then

$$\rho_{D,f}([x_p^+]) + \rho_{D,f}(\partial_0([t_p])) = \rho_{D,f}([x_p^+]) + \delta_{0,f}([t_p]) = 0.$$ 

Thus, the image of $[x_p^+] + \partial_0([t_p])$ under the projection $H^1(D) \to H^1(M_0)$ lies in $H^1_c(M_0)$. Now, from the diagram (61) we obtain that $[x_p^+] + \partial_0([t_p]) \in H^1_{lw}(D_{Iw}^{\dagger}(1))$, and there exists $x_{p,y}^\text{lw} \in H^1_{Iw}(D_y)$ which satisfies (78) with $[t_p]$ given by (81).

It is not difficult to check that the statement of Lemma [5.2.3] holds if we replace $\hat{b}_v$ by

$$\hat{b}_p = w_p \cup (u_p + \partial_0(t_p)).$$

Since $g_{p,y}(\partial_0([t_p])) = 0$, there exists $\tilde{t}_p \in D_{Iw}^{\dagger}(1)$ such that $\tilde{t}_p \mapsto t_p$ under the projection $D_{Iw}^{\dagger}(1) \to D_y$ and

$$g_{p,y}(\partial_0(t_0)) = d_0(t_p) = ((\varphi - 1)(\tilde{t}_p), (\gamma_{Q_p} - 1)(\tilde{t}_p)).$$

Therefore

$$g_{p,y}(\hat{b}_p) = w_p \cup u_p + w_p \cup g_{p,y}(d_0\tilde{t}_p),$$

$$g_p(b_p) \cup \mu_p = w_p \cup g_p(d\tilde{t}_p) \cup \mu_p.$$ 

Since the projection of $\gamma$ on $E$ is 1, we have $w_p \cup u_p \cup f_p^\dagger(\gamma) = w_p \cup u_p$. Thus,

$$\text{(82) \hspace{1cm} } \text{inv}_p(g_{p,y}(\hat{b}_p) \cup f_p^\dagger(\gamma) + g_p(b_p) \cup \mu_p) =$$

$$\ell_{Q_p}(u_p) + \text{inv}_p(w_p \cup g_{p,y}(d\tilde{t}_p) \cup f_p^\dagger(\gamma) + w_p \cup g_p(\pi_p(d\tilde{t}_p)) \cup \mu_p) =$$

$$\ell_{Q_p}(u_p) - \text{inv}_p(w_p \cup g_{p,y}(\tilde{t}_p) \cup d f_p^\dagger(\gamma) + w_p \cup g_p(\tilde{t}_p) \cup d \mu_p) =$$

$$\ell_{Q_p}(u_p) - \text{inv}_p(w_p \cup \tilde{t}_p \cup (f_p^\dagger(y) + d \mu_p)) =$$

$$\ell_{Q_p}(u_p) - \text{inv}_p(w_p \cup \tilde{t}_p \cup g_p(y_p^+)) = \ell_{Q_p}(u_p) - \text{inv}_p(w_p \cup t_p \cup g_p(y_p^+)) =$$

$$\ell_{Q_p}(u_p) - \ell_{Q_p}(t_p \cup g_p(y_p^+)).$$
Now we remark that \( \ell_{Q_p}(t_p \cup g_p(y_p^+)) = \ell_{Q_p}(t_p \cup \rho_D(-f(y_p^+))) \) and, taking into account (81), we have

\[
(83) \quad \ell_{Q_p}(t_p \cup g_p(y_p^+)) = -\ell_{Q_p}(\delta_{0,f}^{-1} \circ \rho_D, f([x_p^+]) \cup \rho_D(f([y_p^+]))) = -\left\langle \rho_D, f([x_p^+]^+), \rho_D(-f(y_p^+)) \right\rangle_{D,f}.
\]

The first formula of Lemma 5.2.4 still holds in our case. From (82), (83) and the definition of \( h_{V,D}^{\text{sel}} \) it follows that

\[
h_{V,D}^{\text{sel}}([x_f], [y_f]) = \text{inv}_p(g_{p,3}(\hat{b}_p) \cup f_p^+([\hat{s}]) + g_p(b_p) \cup \mu_p) = \\
\ell_{Q_p}(u_p) + \left\langle \rho_D, f([x_p^+]) \right\rangle_{D,f} = \\
h_{V,D}^{\text{norm}}([x], [y]) + \left\langle \rho_D, f([x_p^+]) \right\rangle_{D,f}
\]

and the theorem is proved. \( \square \)

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