Energy-momentum operators with eigenfunctions localized along an axis
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Abstract The momentum operator \( \mathbf{p} = -i \nabla \) has radial component \( \tilde{p} \equiv -i \hat{r} \left( \frac{1}{r} \partial_r r \right) \). We show that \( \tilde{p} \) is the space part of a 4-vector operator, the zero component of which is a positive operator. Their eigenfunctions are localized along an axis through the origin. The solutions of the evolution equation \( i \partial_t \psi = \tilde{p}^0 \psi \) are waves along the propagation axis. Lorentz transformations of these waves yield the aberration and Doppler shift. We briefly consider spin-half and spin-one representations.

I. Introduction

Our everyday experience of photons is that they are localized along their propagation axis, whereas in the usual formalism a photon of definite momentum (an eigenvalue of \( \mathbf{p} \)) is a plane wave spread over all space. The operator \( \mathbf{p} = -i \nabla \) can be split into a radial component \( \tilde{p} \) and an angular (transverse) component due to the identity
\[
-i \nabla = -i \frac{\hat{r}}{r} \partial_r r - \frac{1}{2} \frac{1}{r} (\hat{r} \times \mathbf{L} - \mathbf{L} \times \hat{r}) \equiv \tilde{p} + \check{p}
\]
with \( \hat{r} \equiv r/r \) and \( \mathbf{L} \) the angular momentum operator. We note that \( \tilde{p} \psi(r) \) is undefined at the origin. In the appendix we show that continuity in \( r \psi(r) \) on opposite sides of the origin, i.e. that
\[
r \psi(\epsilon r) - r \psi(-\epsilon r) \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 0
\]
is a sufficient condition for \( \tilde{p} \) to be a symmetric operator with respect to the usual inner product space
\[
\langle \psi_1(r) | \psi_2(r) \rangle = \int d^3r \int_0^\infty d\rho \rho^2 \frac{d\Omega}{4\pi} \psi_1^*(\rho, \phi \psi_2(\rho, \phi).
\]

The eigenfunction of \( \tilde{p} \) with eigenvalue \( \mathbf{k} \) is
\[
u_k = \frac{1}{\sqrt{2\pi kr}} \left[ \delta(\hat{r}, \hat{k}) \exp(ikr) + \delta(-\hat{r}, \hat{k}) \exp(-ikr) \right]
\]

\[
= \frac{1}{\sqrt{2\pi kr}} \left[ \delta(\hat{r}, \hat{k}) \right] \exp(i\mathbf{k} \cdot \mathbf{r})
\]
where \( \hat{k} \equiv \mathbf{k}/|\mathbf{k}| \), \( k \equiv |\mathbf{k}| \) and \( \delta(\hat{r}, \hat{k}) \) is the delta function on the unit sphere (for this notation see the appendix). The equivalence between (5a) and (5b) may be seen by writing
\[
\delta(\hat{r}, \hat{k}) \exp(ikr) = \delta(\hat{r}, \hat{k}) \exp(i(\hat{r} \cdot \mathbf{r})r) = \delta(\hat{r}, \hat{k}) \exp(i\mathbf{k} \cdot \hat{r}) = \delta(\hat{r}, \hat{k}) \exp(i\mathbf{k} \cdot \mathbf{r}),
\]
and similarly for \( \delta(-\hat{r}, \hat{k}) \). Then from (5a)
\[ p u_k = \frac{1}{\sqrt{2\pi kr}} [\hat{r} k \delta(\hat{r}, \hat{k}) \exp(i kr) - \hat{r} k \delta(-\hat{r}, \hat{k}) \exp(-i kr)] \]

\[ = \frac{1}{\sqrt{2\pi kr}} [\hat{k} k \delta(\hat{r}, \hat{k}) \exp(i kr) + \hat{k} k \delta(-\hat{r}, \hat{k}) \exp(-i kr)] = k u_k. \] (6)

The state \( u_k \) is localized along the entire \( k \) axis \( r = \lambda \hat{k} \) \((-\infty < \lambda < \infty)\), and the density \( (u_k^* u_k) \) is evenly distributed along the \( k \) axis as

\[ \int_{r=a}^{b} (u_k^* u_k) \, d^3r = (b - a)/\pi k^2. \]

Note that

\[ \mathcal{P} u_k = u_k^* = u_{(k \to -k)} \]

where \( \mathcal{P} \) is the parity operator.

The \( u_k \) satisfy the orthogonality and completeness relations

\[ \int d^3r \, u_k^*(r) \, u_{k'}(r) = \delta(k - k'), \quad \int d^3k \, u_k^*(r) \, u_{k'}(r) = \delta(r - r') \] (7)

which are verified in the appendix. As the Lorentz invariant measure is \( d^3k/k \) rather than \( d^3k \), this suggests defining the inner product spaces

\[ \langle \phi_1(k) | \phi_2(k) \rangle_{1/k} \equiv \int \frac{d^3k}{k} \phi_1^*(k) \phi_2(k), \] (8)

\[ \langle \psi_1(r) | \psi_2(r) \rangle_{1/r} \equiv \int \frac{d^3r}{r} \psi_1^*(r) \psi_2(r), \] (9)

and the basis state

\[ w_k = \sqrt{kr} u_k = \frac{1}{\sqrt{2\pi kr}} [\delta(\hat{r}, \hat{k}) \exp(i kr) + \delta(-\hat{r}, \hat{k}) \exp(-i kr)] \]

\[ = \frac{1}{\sqrt{2\pi kr}} [\delta(\hat{r}, \hat{k}) + \delta(-\hat{r}, \hat{k})] \exp(i k \cdot r), \] (10a)

\[ = \frac{1}{\sqrt{2\pi kr}} [\delta(\hat{r}, \hat{k}) + \delta(-\hat{r}, \hat{k})] \exp(i k \cdot r), \] (10b)

which is an eigenfunction of the operator

\[ \overline{p} \equiv - i \hat{r} \left( \frac{1}{\sqrt{r}} \partial_r \sqrt{r} \right) = - i \hat{r} \partial_r + \frac{1}{2r} \] (11)

which is symmetric with respect to the \( 1/r \) inner product space \( (9) \). From now on we will be concerned with the operator \( \overline{p} \) and its eigenfunctions \( w_k \) (rather than with \( \tilde{p}, u_k \)) because \( w_k \) transforms as a scalar under Lorentz transformations (see below Sec. 3). From \( (7) \)

\[ \langle w_k(r) | w_{k'}(r) \rangle_{1/r} = k \delta(k - k'), \quad \langle w_k(r) | w_{k'}(r') \rangle_{1/k} = r \delta(r - r'). \] (12)

Given a \( \psi(r) \) which is a superposition of various \( w_k \)

\[ \psi(r) = \int \frac{d^3k}{k} \phi(k) \, w_k \equiv \langle w_k^* | \phi(k) \rangle_{1/k} \] (13)

then the orthogonality relation \( (12) \) enables one to project out the distribution \( \phi(k) \) as

\[ \langle w_k | \psi(r) \rangle_{1/r} = \langle w_k | \int \frac{d^3k'}{k'} \phi(k') \, w_{k'} \rangle_{1/r} = \phi(k). \]
Expanding out the quantity \( \langle w_k | \psi(r) \rangle_{1/r} \), we show that this is a Fourier transform of \( \psi(r) \) as well as of \( \psi(-r) \) along the line \( \mathbf{r} = \lambda \hat{\mathbf{k}} \), \((-\infty < \lambda < \infty)\), as
\[
\langle w_k | \psi(r) \rangle_{1/r} = \phi(k) \equiv \int \frac{d^3r}{r} \psi(r) w_k^*(r)
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_0^\infty dr \int_\Omega d\Omega \psi(\hat{\mathbf{r}}) \left[ \frac{1}{\sqrt{2\pi kr}} [\delta(\hat{\mathbf{r}}, \hat{\mathbf{k}}) \exp(-ikr) + \delta(-\hat{\mathbf{r}}, \hat{\mathbf{k}}) \exp(ikr)] \right]
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_0^\infty dr \sqrt{\frac{r}{k}} \left[ \psi(\hat{\mathbf{k}}) \exp(-ikr) + \psi(-\hat{\mathbf{k}}) \exp(ikr) \right].
\]
It is more transparent here to write \( \mathbf{k} \) in spherical coordinates: with \( \mathbf{k} = (k, \theta_k, \phi_k) \) then the above is
\[
\phi(k, \theta_k, \phi_k) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dr \sqrt{\frac{r}{k}} \left[ \psi(r, \theta_k, \phi_k) \exp(-ikr) + \psi(r, \pi - \theta_k, \phi_k + \pi) \exp(ikr) \right]
\]
\[
= \frac{1}{2\sqrt{k}} \left[ (\mathcal{F}_c - i \mathcal{F}_s) + (\mathcal{F}_c + i \mathcal{F}_s) \mathcal{P} \right] \sqrt{r} \psi(r, \theta_k, \phi_k)
\]
\[
= \hat{U} \psi(r, \theta_k, \phi_k)
\]
where \( \mathcal{P} \) is the parity operator and \( \mathcal{F}_c, \mathcal{F}_s \) are the Fourier cosine, sine transforms defined by
\[
\mathcal{F}_c f(r, \theta, \phi) = g_c(k, \theta, \phi) \equiv \sqrt{\frac{2}{\pi}} \int_0^\infty f(r, \theta, \phi) \cos(rk) \, dr,
\]
\[
\mathcal{F}_s f(r, \theta, \phi) = g_s(k, \theta, \phi) \equiv \sqrt{\frac{2}{\pi}} \int_0^\infty f(r, \theta, \phi) \sin(rk) \, dr.
\]
The operator of (14)
\[
\mathcal{U} \equiv \frac{1}{2\sqrt{k}} \left[ (\mathcal{F}_c - i \mathcal{F}_s) + (\mathcal{F}_c + i \mathcal{F}_s) \mathcal{P} \right] \sqrt{r}
\]
defines a unitary mapping from \( \psi(r) \) to \( \phi(k) \) whose inverse is
\[
\hat{\mathcal{U}} \equiv \frac{1}{2\sqrt{r}} \left[ (\mathcal{F}_c + i \mathcal{F}_s) + (\mathcal{F}_c - i \mathcal{F}_s) \mathcal{P} \right] \sqrt{k}
\]
where the \( \tilde{\mathcal{F}}_c, \tilde{\mathcal{F}}_s \) are as in (15) but with \( r, k \) interchanged. It is straightforward to check that \( \hat{\mathcal{U}} \mathcal{U} = 1 \), with the aid of \( \mathcal{F}_c \mathcal{F}_c = \tilde{\mathcal{F}}_s \mathcal{F}_s = \mathcal{P} \mathcal{P} = 1 \). We emphasize that \( \mathcal{U}, \hat{\mathcal{U}} \) are in a sense one dimensional transform operators: \( \mathcal{U} \) maps \( \psi(r) \) on any axis through the origin onto \( \phi(k) \) on the same axis.

As \( \mathbf{p} w_k = k w_k \), and \( \mathbf{k} \) is the space part of the 4-vector \( k^\Lambda \equiv (k, \mathbf{k}) \), we look for the operator \( \mathbf{p}^0 \) such that \( \mathbf{p}^0 w_k = k w_k \). (One can see by inspection of (10a) that \( w_k \) is not an eigenfunction of the “radial momentum operator” \( p_r \equiv -ir^{-1/2} \partial_r r^{1/2} \) which anyway is not a Hermitian operator [1]). The operator \( \mathbf{p}^0 \) is multiplication by \( k \) in momentum space, i.e.
\[
\mathbf{p}^0 = \hat{\mathcal{U}} k \mathcal{U}
\]
so that \( \mathbf{p}^0 \) is a positive operator. To simplify (18) we need the following identities [2]
\[
\tilde{\mathcal{F}}_c \mathcal{F}_c = \tilde{\mathcal{F}}_s \mathcal{F}_s = 1, \quad \tilde{\mathcal{F}}_s \mathcal{F}_c = -\mathcal{H}_e, \quad \tilde{\mathcal{F}}_c \mathcal{F}_s = \mathcal{H}_o
\]
where \( \mathcal{H}_e, \mathcal{H}_o \) are the Hilbert transforms of even, odd functions defined by
\[
\mathcal{H}_e f(r, \theta, \phi) = -\frac{2}{\pi} \int_0^\infty f(t, \theta, \phi) \frac{dt}{t^2 - r^2} dt, \quad \mathcal{H}_o f(r) = -\frac{2}{\pi} \int_0^\infty \frac{f(\lambda r)}{1 - \lambda^2} d\lambda,
\]
\[
\mathcal{H}_e f(r, \theta, \phi) = -\frac{2}{\pi} \int_0^\infty t f(t, \theta, \phi) \frac{dt}{t^2 - r^2} dt, \quad \mathcal{H}_o f(r) = -\frac{2}{\pi} \int_0^\infty \frac{\lambda f(\lambda r)}{1 - \lambda^2} d\lambda.
\]
We will write
\[ F_{\pm} \equiv F_c \pm i F_s \]
then from (19) there follows the further identities
\[
\begin{align*}
\mathcal{F}_+ \mathcal{F}_+ &= -i (\mathcal{H}_c - \mathcal{H}_o), & \mathcal{F}_- \mathcal{F}_- &= i (\mathcal{H}_c - \mathcal{H}_o), \\
\mathcal{F}_+ \mathcal{F}_- &= 2 - i (\mathcal{H}_c + \mathcal{H}_o), & \mathcal{F}_- \mathcal{F}_+ &= 2 + i (\mathcal{H}_c + \mathcal{H}_o),
\end{align*}
\]
also
\[
\mathcal{F}_\pm k = \mp i \partial_r \mathcal{F}_\pm, \quad k \mathcal{F}_\pm = \pm i \mathcal{F}_\pm \partial_r. \tag{21}
\]
Returning to \( \mathcal{P}^0 = \hat{U} k \mathcal{U} \) we have
\[
\begin{align*}
\mathcal{P}^0 &= \hat{U} k \mathcal{U} = \frac{1}{4\sqrt{r}} \left( \mathcal{F}_+ + \mathcal{F}_- \mathcal{P} \right) k \left( \mathcal{F}_- + \mathcal{F}_+ \mathcal{P} \right) \sqrt{r} \\
&= i \frac{1}{4\sqrt{r}} \partial_r \left( -\mathcal{F}_+ + \mathcal{F}_- \mathcal{P} \right) \left( \mathcal{F}_- + \mathcal{F}_+ \mathcal{P} \right) \sqrt{r} \\
&= i \frac{1}{4\sqrt{r}} \partial_r \left[ (-\mathcal{F}_+ \mathcal{F}_- + \mathcal{F}_- \mathcal{F}_+) + \left( \mathcal{F}_- \mathcal{F}_- - \mathcal{F}_+ \mathcal{F}_+ \right) \mathcal{P} \right] \sqrt{r} \\
&= -\frac{1}{2\sqrt{r}} \partial_r \left[ (\mathcal{H}_c + \mathcal{H}_o) + (\mathcal{H}_c - \mathcal{H}_o) \mathcal{P} \right] \sqrt{r} = -\frac{1}{\sqrt{r}} \partial_r \mathcal{H}_+ \sqrt{r} \tag{22}
\end{align*}
\]
where
\[
\begin{align*}
\mathcal{H}_+ f(r) &\equiv \frac{1}{2} \left[ (\mathcal{H}_c + \mathcal{H}_o) + (\mathcal{H}_c - \mathcal{H}_o) \mathcal{P} \right] f(r) \tag{23a} \\
&= \frac{1}{\pi} \int_{0}^{\infty} \left( \frac{f(\lambda r)}{\lambda - 1} - \frac{f(-\lambda r)}{1 + \lambda} \right) d\lambda \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\lambda r)}{\lambda - 1} d\lambda \tag{23b}
\end{align*}
\]
which is the Hilbert transform of \( f(r) \) along the axis \( \lambda r \), \((-\infty < \lambda < \infty)\).

We now establish that \( \mathcal{P}^0 w_k \equiv -\frac{1}{\sqrt{r}} \partial_r, \mathcal{H}_+ \sqrt{r} w_k = k w_k \). For simplicity we will choose the particular case when \( k = (0,0,k), \) \( k > 0 \). Then
\[
w_{k^3} = \frac{1}{\sqrt{2\pi kr}} [ \delta(\hat{r}^3 - 1) + \delta(-\hat{r}^3 - 1) ] \exp(i k z)
\]
and
\[
\begin{align*}
\mathcal{H}_+ \exp(i k z) &= +i \exp(i k z) \quad z > 0 \\
&\quad -i \exp(i k z) \quad z < 0 \\
\frac{1}{\sqrt{r}} \mathcal{H}_+ \sqrt{r} w_{k^3} &= \frac{i}{\sqrt{2\pi kr}} [ \delta(\hat{r}^3 - 1) - \delta(-\hat{r}^3 - 1) ] \exp(i k z) \\
- \frac{1}{\sqrt{r}} \partial_r \mathcal{H}_+ \sqrt{r} w_{k^3} &= \frac{1}{\sqrt{2\pi kr}} k \hat{r}^3 [ \delta(\hat{r}^3 - 1) - \delta(-\hat{r}^3 - 1) ] \exp(i k z) \\
&= \frac{1}{\sqrt{2\pi kr}} k [ \delta(\hat{r}^3 - 1) + \delta(-\hat{r}^3 - 1) ] \exp(i k z) = k w_{k^3}.
\end{align*}
\]
The operator components \( \partial_r, \mathcal{H}_+ \) of \( \mathcal{P}^0 \) do not commute. If in the working out of (22) we take \( k \) to the right instead of to the left, we arrive at
\[
\begin{align*}
\mathcal{P}^0 &= \hat{U} k \mathcal{U} = -\frac{1}{\sqrt{r}} \left[ (\mathcal{H}_c + \mathcal{H}_o) - (\mathcal{H}_c - \mathcal{H}_o) \mathcal{P} \right] \partial_r \sqrt{r} \\
&= -\frac{1}{\sqrt{r}} \mathcal{H}_- \partial_r \sqrt{r} \tag{24}
\end{align*}
\]
where

\[ \mathcal{H} f(r) \equiv \frac{1}{2} \left[ (\mathcal{H}_e + \mathcal{H}_o) - (\mathcal{H}_e - \mathcal{H}_o) \mathcal{P} \right] f(r) \]

\[ = \frac{1}{\pi} \int_{-\infty}^{\infty} \text{sgn}(\lambda) \frac{f(\lambda r)}{\lambda - 1} \, d\lambda. \quad (25a) \]

As

\[ \mathcal{H}_- \mathcal{H}_+ = \mathcal{H}_+ \mathcal{H}_- = -1, \]

which can be verified their definitions (23a), (25a) together with \( \mathcal{H}_e \mathcal{H}_o = \mathcal{H}_o \mathcal{H}_e = -1 \), then

\[ \hat{p}^2 = \frac{1}{\sqrt{r}} \partial_r \mathcal{H}_+ \partial_r \mathcal{H}_- \sqrt{r} = -\frac{1}{\sqrt{r}} \partial^2_r \sqrt{r} = \hat{p}^2 \]

as necessary. Also with the adjoint relations \( \frac{1}{\sqrt{r}} \mathcal{H}_o \sqrt{r} \hat{p} = -\frac{1}{\sqrt{r}} \mathcal{H}_c \sqrt{r} \)

then from (23a), (25a)

\[ \left( \frac{1}{\sqrt{r}} \mathcal{H}_\pm \sqrt{r} \right)^\dagger = -\frac{1}{\sqrt{r}} \mathcal{H}_\mp \sqrt{r}. \]

2. Wave equations

We can write a wave equation substituting \( \hat{p} \) for the usual \( p \):

\[ [(i\partial_t)^2 - \hat{p}^2] \psi = 0 \]

\[ [-\partial^2_t + r^{-1/2} \partial^2_r r^{1/2}] \psi = 0 \]

(26)

which is essentially the wave equation \( -\Box \psi \equiv [-\partial^2_t + \frac{1}{r} \partial^2_r r - \frac{1}{r} L^2] \psi = 0 \) with the operator component \( \frac{1}{r} L^2 \) excluded. This modified wave equation (26) has solutions \( \psi_k, \psi_k^* \) which are eigenfunctions of \( \hat{p} \):

\[ \psi_k = \exp(-ikt) w_k, \quad \psi_k^* = \exp(ikt) w_k^*. \]

(27)

Both the states \( \psi_k, \psi_k^* \) are waves localized along the propagation axis \( r = \lambda \hat{k} \) \((-\infty < \lambda < \infty\) and proceeding in the \( +\hat{k} \) direction, and accords with our everyday experience of photons being localized along their propagation axis. If \( \psi \) satisfies (26) then there is a conserved density

\[ \sigma = i (\psi^* \partial_t \psi - (\partial_t \psi^*) \psi) \]

However this density is indefinite.

We will from now consider the first order evolution equation

\[ i \partial_t \psi = \hat{p}^0 \psi \equiv -r^{-1/2} \partial_r \mathcal{H}_+ r^{1/2} \psi. \]

(28)

Now only the positive energy eigenstates \( \psi_k \) of (27) are solutions and the negative energy components \( \psi_k^* \) are excluded. Consider the non-negative density \( \rho \)

\[ \rho = \frac{1}{r} [\psi^* \psi + (r^{-1/2} \mathcal{H}_+ r^{1/2} \psi^*) (r^{-1/2} \mathcal{H}_+ r^{1/2} \psi)] \]

then if \( \psi \) is a solution of (28)

\[ \partial_t \rho = \frac{1}{r} \partial_t \psi^* \psi + \frac{1}{r} (r^{-1/2} \mathcal{H}_+ r^{1/2} \psi^*) (r^{-1/2} \mathcal{H}_+ r^{1/2} \partial_t \psi) + \text{CC} \]

\[ = -i \left( r^{-1/2} \partial_r \mathcal{H}_+ r^{1/2} \psi^* \right) \psi + i \left( r^{-1/2} \mathcal{H}_+ r^{1/2} \psi^* \right) (r^{-1/2} \mathcal{H}_+ \mathcal{H}_- \partial_t r^{1/2} \psi) + \text{CC} \]

\[ = -i (r^{-1} \partial_r \mathcal{H}_+ r^{1/2} \psi^*) (r^{-1/2} \psi) - i (r^{-1} \mathcal{H}_+ r^{1/2} \psi^*) (r^{-1} \partial_r r^{1/2} \psi) + \text{CC} \]

\[ = \nabla \cdot \mathbf{\hat{p}} \left[ -i (r^{-1} \mathcal{H}_+ r^{1/2} \psi^*) (r^{-1/2} \psi) + \text{CC} \right] = -\nabla \cdot \mathbf{J} \]
where CC stands for the complex conjugate terms and we have used the operator identity

\[(r^{-1} \partial_r r A) B + A(r^{-1} \partial_r r B) = \nabla \cdot [\hat{r} A B].\]

Substituting \(\psi = w_k\) into the above expressions for the density and current, both the density and current are evenly distributed along the \(k\) axis \(r = \lambda \hat{k} (-\infty < \lambda < \infty)\), and the current is uniformly in the \(+\hat{k}\) direction.

3. Lorentz transformations

The momentum space transformation is straightforward. The \(k^\lambda = (k, k) = (k, k^1, k^2, k^3)\) transforms in the usual way as a 4-vector, that is under a finite boost of velocity \(v\) in the \(z\) direction then

\[k'^1 = k^1, \quad k'^2 = k^2, \quad k'^3 = \gamma k^3 - \gamma v k^3, \quad k' = \gamma k - \gamma v k^3, \quad (29)\]

and

\[k'^1 = \tilde{k}/(1 - v \tilde{k}), \quad k'^2 = \tilde{k}/(1 - v \tilde{k}), \quad \tilde{k}' = (\tilde{k}' - v)/[(1 - v \tilde{k})] \quad (30)\]

with \(\gamma = (1 - v^2)^{-1/2}\). The last of (30) is the well known aberration formula, as putting \(\hat{k}^3 = -\cos \theta\) (light observed from the direction \(\hat{k}\) has momentum \(\propto -\hat{k}\)) then

\[\cos \theta' = (\cos \theta + v)/(1 + v \cos \theta).\]

The basis state \(\psi_k = \exp(-ik t) w_k\) transforms as a scalar in momentum space, i.e.

\[(\psi_k')' = (\psi_k)_{k \rightarrow k'} = \frac{1}{\sqrt{2\pi k'}} \left[ \delta(\hat{r}, \hat{k}) + \delta(\hat{r}, \hat{k}') \right] \exp(-ik' t + ik' \cdot r) \quad (31)\]

where \(k'\) is given by (29). From (31) can be inferred the frequency change from the exponential term, and also the change of the propagation direction \(\hat{k}'\) which agrees with the aberration formula.

The \(k\) transformation is generated by the boost and rotation operators in momentum space which are

\[N_k = i k (\partial/\partial k), \quad L_k = -i (k \times \partial/\partial k) \quad (32)\]

then (29) is \(k' = \exp(v N^3) k\). In configuration space the boost transformation will be complicated due to the fact that \(\bar{p}^0\) is a non-local operator, so there is no simple transformation of the configuration space \(r\) coordinates corresponding to (29). The Lorentz boost generator \(N\) must satisfy

\[[N, \bar{p}^0] = i \bar{p}, \quad [N^a, \bar{p}^b] = i \delta^{ab} \bar{p}^0 \quad (33)\]

which implies that \(N\) as well as \(\bar{p}^0\) is a non-local (integral) operator. The operators generating the Lorentz transformations in configuration space are

\[N = (r^{-1/2} \mathcal{H}_- r^{1/2}) (r \nabla - 2 \hat{r} \partial_r r) = (r \nabla - 2 \hat{r} \partial_r r) (r^{-1/2} \mathcal{H}_+ r^{1/2}) \quad (34)\]

\[L = -i \, r \times \nabla \]
which are Hermitian with respect to the $1/r$ inner product space (9). That the $N$ defined above satisfy the relations (33), and that the Lorentz group relation $[N^a, N^b] = -ie^{abc}L^c$ is also satisfied we leave to the appendix. An infinitesimal boost $\delta v$ causes a change in the wavefunction $\psi \rightarrow (1 + i \delta v \cdot N) \psi$. In the appendix we calculate $N w_k$ and show that the combined transformation $N w_k + N_k w_k = 0$, which means that $w_k$ is a Lorentz scalar.

This completes our investigation of the spin-zero operators $\vec{p}^0, \vec{p}, N, L$ which together obey the Poincaré group commutation relations.

4. Spin-$\frac{1}{2}$ and spin-1 representations

Massless spin-$\frac{1}{2}$ and spin-1 Hamiltonians can be constructed which are differential operators, in contrast to the non-local $\vec{p}^0$ Hamiltonian for the spin-zero case. The spin-$\frac{1}{2}$ massless (neutrino) Hamiltonian equation

$$i \partial_t \Psi = \sigma \cdot \vec{p} \Psi = -i (\sigma \cdot \hat{r}) \left( \partial_r + \frac{1}{2r} \right) \Psi$$

has as eigensolutions either column of the $2 \times 2$ matrix $e^{\pm i k t} (\sigma \cdot k) w_k$. Formally one can also construct a spin-$\frac{1}{2}$ Dirac-like equation with mass $i \partial_t \Psi = (\alpha \cdot \vec{p} + \beta m) \Psi$, which has as eigensolutions a column of the $4 \times 4$ matrix $[\exp(-i(m^2 + k^2)^{1/2} t)(\alpha \cdot k + \beta m) w_k]$, however in the rest state when $k = 0$ then $w_k$ is undefined.

The source-free electromagnetic field is a massless spin-1 field. Defining the complex field $\mathbf{F} \equiv \mathbf{E} + i \mathbf{B}$, then the source-free Maxwell’s equations are

$$[\partial_t + i \nabla \times] \mathbf{F} = 0, \quad \nabla \cdot \mathbf{F} = 0.$$ 

which we will replace by

$$[\partial_t - \vec{p} \times] \mathbf{F} = 0, \quad \vec{p} \cdot \mathbf{F} = 0.$$ 

A solution of (36) is

$$\mathbf{F} = e^{-ikt} (w_{k3}, iw_{k3}, 0)$$

which is a wave localized along the $z$ axis proceeding in the $+z$ direction.

5. Outlook

The attraction of the $\vec{p}^\lambda$ operators is that their eigenfunctions conform with our experience of photons: that they appear to be localized along their propagation axis. On the other hand the chosen origin of coordinates plays a more significant role in this formalism - indeed a shifted origin (or observer) may not register the localized wave at all. In the usual case the exponentiated momentum operator $\{\exp(a \cdot p)\}$ has a clear geometrical role: in shifting the coordinates by $a$. By contrast the meaning of the operator $\{\exp(a \cdot \vec{p})\}$ is obscure.

Mathematically the the $\vec{p}^\lambda$ operators are interesting because they demand continuity through the origin, in contrast to the usual radial operators whose use is limited apart from describing sources or sinks at the origin.
Appendix

That the operator $\hat{p}$ is symmetric with respect to the inner product (4)

Expanding the inner product

$$\langle \phi(r) \mid \hat{p}\psi(r) \rangle \equiv \int d\Omega \int_0^\infty dr \, r^2 \, \phi^*(r) \left[ -i \hat{r} r^{-1} \partial_r \psi(r) \right]$$

$$= \int d\Omega \int_0^\infty dr \, r^2 \left[ -i \hat{r} r^{-1} \partial_r \phi(r) \right]^* \psi(r) - i \int d\Omega \hat{f} \int_0^\infty (r \phi^*(r)) \left( r \psi(r) \right)$$

$$= \langle \hat{p}\phi(r) \mid \psi(r) \rangle - i \int d\Omega \hat{f} \left[ (r \phi^*(r)) \left( r \psi(r) \right) \right]_{r \to 0}$$

(A1)

assuming the $(r \phi), (r \psi)$ tend to zero sufficiently fast at infinity. But with the condition (3)

$$r \psi(r) - r \psi(-r) \to 0 \quad \text{as} \quad \epsilon \to 0$$

then the last term on the RHS of (A1) is zero, as the contributions form opposite sides of the origin cancel. Thus $\hat{p}$ is symmetric. Note that it is the presence of the $\hat{r}$ within the $\hat{p}$ operator that leads to symmetry, by contrast the operator $p_r = -i r^{-1} \partial_r r$ has the more onerous requirement for symmetry that $r \psi(r) \to 0 \quad \text{as} \quad r \to 0$, which is the underlying reason why $p_r$ is not Hermitian (as discussed by Hellwig p176 [3]).

The delta function on the unit sphere

In spherical coordinates

$$\hat{r} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

$$\hat{k} = (\sin \theta_k \cos \phi_k, \sin \theta_k \sin \phi_k, \cos \theta_k)$$

then

$$\delta(\hat{r}, \hat{k}) \equiv \delta(\cos \theta - \cos \theta_k) \, \delta(\phi - \phi_k)$$

$$\delta(-\hat{r}, \hat{k}) = \delta(\cos \theta + \cos \theta_k) \, \delta(\phi + \pi - \phi_k).$$

(A2)

(A3)

Orthogonality and completeness of the $u_k$

Expanding out (7a)

$$\int d^3r \, u_k^* (r) \, u_{k'} (r) = \int d\Omega \int_0^\infty \, dr \, \frac{1}{2 \pi k k'} \left[ \delta(-\hat{r}, \hat{k}) \exp(ik r) + \delta(\hat{r}, \hat{k}) \exp(-ik r) \right] \times$$

$$\left[ \delta(-\hat{r}, \hat{k'}) \exp(-ik' r) + \delta(\hat{r}, \hat{k'}) \exp(ik' r) \right]$$

$$= \frac{1}{2 kk'} \int d\Omega \left[ \delta(-\hat{r}, \hat{k}) \delta(-\hat{r}, \hat{k'}) + \delta(\hat{r}, \hat{k}) \delta(\hat{r}, \hat{k'}) \right] \delta(k - k')$$

$$= \frac{1}{kk'} \delta(k, k') \, \delta(k - k') \equiv \delta(k - k').$$

(A4)

The boost operator $N$ of (34)

First consider the operator

$$N' \equiv -i(r \nabla - 2 \hat{r} \partial_r r)$$

and defining

$$s^\lambda = (-\frac{1}{r}, \frac{\hat{r}}{r})$$

(A5)

(A6)
the following commutation relations may be directly verified

\[ [N', s^0] = i s, \quad [N'^a, s^b] = i \delta^{ab} s^0. \]  

(A7)

Also the operator \(-i \partial_r r \) commutes with \( N' \) so that

\[ t^\lambda = (i \frac{1}{\sqrt{r}} \partial_r \sqrt{r}, -i \, \sqrt{r} \partial_r \sqrt{r}) = (i \frac{1}{\sqrt{r}} \partial_r \sqrt{r}, \bar{p}) \]  

(A8)

also obeys (A7) with \( t^\lambda \) instead of \( s^\lambda \). Finally we multiply \( t^0 \) on the right by \((i \frac{1}{\sqrt{r}} H_+ \sqrt{r}) \) and \( N' \) on the left by \((i \frac{1}{\sqrt{r}} H_- \sqrt{r}) \) to make the new operators \( \bar{p}', N \), which recalling \( H_+ H_- = -1 \) also satisfy the relations corresponding to (A7), i.e.

\[
\begin{align*}
N &= (\frac{1}{\sqrt{r}} H_- \sqrt{r}) (r \nabla - 2 \hat{r} \partial_r r) = (r \nabla - 2 \hat{r} \partial_r r) (\frac{1}{\sqrt{r}} H_+ \sqrt{r}) \\
\bar{p}' &= -\frac{1}{\sqrt{r}} \partial_r H_+ \sqrt{r} = -\frac{1}{\sqrt{r}} H_- \partial_r \sqrt{r}, \quad \bar{p} = -i \hat{r} \frac{1}{\sqrt{r}} \partial_r \sqrt{r}.
\end{align*}
\]  

(A9)

Finally the fact that

\[ [N'^a, N'^b] = -i \epsilon^{abc} L^c \]  

(A10)

follows from \([N'^a, N'^b] = -i \epsilon^{abc} L^c \) where \( L = -i r \times \nabla \).

To show that \( N w_k + N_k w_k = 0 \)

We will consider the boost operators \( N^3, N^3_k \) of (32), (34), and recalling \( w_k \) from (10) we have in spherical coordinates

\[
\begin{align*}
N^3 &= i \cos \theta_k k \partial_k - i \sin \theta_k \partial_{\theta_k} \\
N^3_k &= -\cos \theta r \partial_r - \sin \theta \partial_{\theta} \left( \frac{1}{\sqrt{r}} H_+ \sqrt{r} \right) \\
w_k &= \frac{1}{2 \sqrt{\pi kr}} \left[ \delta (\cos \theta - \cos \theta_k) \delta (\phi - \phi_k) \right] \exp (ikr) \\
&\quad + \delta (\cos \theta + \cos \theta_k) \delta (\phi + \pi - \phi_k) \exp (-ikr) \\
\left( \frac{1}{\sqrt{r}} H_+ \sqrt{r} \right) w_k &= \frac{i}{2 \sqrt{\pi kr}} \left[ \delta (\cos \theta - \cos \theta_k) \delta (\phi - \phi_k) \right] \exp (ikr) \\
&\quad - \delta (\cos \theta + \cos \theta_k) \delta (\phi + \pi - \phi_k) \exp (-ikr).
\end{align*}
\]

Then

\[
\begin{align*}
N^3 w_k &= + i \cos \theta_k \delta (\cos \theta - \cos \theta_k) \delta (\phi - \phi_k) k \partial_k \left[ \frac{\exp (ikr)}{2 \sqrt{\pi kr}} \right] \\
&\quad - i \frac{\exp (ikr)}{2 \sqrt{\pi kr}} \delta (\phi - \phi_k) \sin \theta_k \partial_{\theta_k} \left[ \delta (\cos \theta - \cos \theta_k) \right] \\
&\quad + i \cos \theta_k \delta (\cos \theta + \cos \theta_k) \delta (\phi + \pi - \phi_k) k \partial_k \left[ \frac{\exp (-ikr)}{2 \sqrt{\pi kr}} \right] \\
&\quad - i \frac{\exp (-ikr)}{2 \sqrt{\pi kr}} \delta (\phi + \pi - \phi_k) \sin \theta_k \partial_{\theta_k} \left[ \delta (\cos \theta + \cos \theta_k) \right], \quad (A11)
\end{align*}
\]

and

\[
\begin{align*}
N^3 w_k &= - i \cos \theta \delta (\cos \theta - \cos \theta_k) \delta (\phi - \phi_k) r \partial_r \left[ \frac{\exp (ikr)}{2 \sqrt{\pi kr}} \right] \\
&\quad - i \frac{\exp (ikr)}{2 \sqrt{\pi kr}} \delta (\phi - \phi_k) \sin \theta \partial_{\theta} \left[ \delta (\cos \theta - \cos \theta_k) \right] \\
&\quad + i \cos \theta \delta (\cos \theta + \cos \theta_k) \delta (\phi + \pi - \phi_k) r \partial_r \left[ \frac{\exp (-ikr)}{2 \sqrt{\pi kr}} \right] \\
&\quad + i \frac{\exp (-ikr)}{2 \sqrt{\pi kr}} \delta (\phi + \pi - \phi_k) \sin \theta \partial_{\theta} \left[ \delta (\cos \theta + \cos \theta_k) \right]. \quad (A12)
\end{align*}
\]

Now if we add (A12) and (A12) together then the components cancel in order.
References

[1] Liboff R. L. et al, “On the radial momentum operator,” Am. J. Phys. 41, 976-980 (1973)

[2] Rooney P. G., “On the ranges of certain fractional integrals,” Can. J. Math. 24, 1198-1216 (1972)

[3] Hellwig G., Differential operators of mathematical physics (Addison-Wesley, Reading, 1964)