Quantum Big Bang without fine-tuning in a toy-model

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Abstract

The question of possible physics before Big Bang (or after Big Crunch) is addressed via a schematic non-covariant simulation of the loss of observability of the Universe. Our model is drastically simplified by the reduction of its degrees of freedom to the mere finite number. The Hilbert space of states is then allowed time-dependent and singular at the critical time $t = t_c$. This option circumvents several traditional theoretical difficulties in a way illustrated via solvable examples. In particular, the unitary evolution of our toy-model quantum Universe is shown interruptible, without any fine-tuning, at the instant of its bang or collapse $t = t_c$. 
1 Introduction

The apparently purely philosophical question of “what did exist before the Big Bang?” has recently changed its status. Its numerous recent innovative and non-speculative treatments may be sampled, e.g., by the Penrose’s deep theoretical analysis of possible physics before Big Bang [1] or by the Gurzadyan’s and Penrose’s proposal of the existence of cyclically recurring “aeons” before Big Bang, with potentially measurable (i.e., in principle, falsifiable!) consequences. Naturally, the topic involves also the parallel question of possible scenarios of the evolution of the Universe after the Big Crunch, i.e., at $t > t_{\text{final}}$ [2].

One of the main difficulties encountered in similar considerations can be seen in the fact that our current knowledge of the laws of nature is not too well adapted to the description of the Universe near the Big Bang (i.e., schematically, in a short interval of times $t \approx t_c = t_{\text{initial}}$) or, if you wish, near the Big Crunch (i.e., at $t \approx t_c = t_{\text{final}}$). At the same time, the picture offered by the classical theory of general relativity seems compatible with the schematic, simplified but still intuitively acceptable scenario in which the existence of the critical Big-Bang/Big-Crunch (BBC) instant $t = t_c$ may be visualized as the time-dependence of any $N$-plet of the spatial grid-point coordinates $g_j(t), j = 1, 2, \ldots, N$ (or of their, in principle, measured distances in a suitable frame) with the complete-confluence property

$$
\lim_{t \to t_c} g_j(t) = g_c, \quad j = 1, 2, \ldots, N. 
$$

The key difficulties emerge when one tries to make this picture compatible with the requirements of quantum theory. In this context, Penrose [1] emphasized that whenever one tries to “quantize” the picture treating the grid points $g_j(t)$ (or any other measurable data) as eigenvalues of an ad hoc self-adjoint operator $O = O^\dagger$ in Hilbert space $\mathcal{H}$, one encounters the well-known fine-tuning problem. Indeed, near $t = t_c$ it becomes extremely difficult to suppress, by the fine-tuning of parameters, the generic and well known property of the eigenvalues of any self-adjoint $O = O^\dagger$ which tend to avoid their crossings near any point of potential degeneracy.

The recent proposal of a conformal cyclic cosmology [3] may be perceived as one of the possible ways out of this quantum-theoretical trap. One simply admits that the $t = t_c$ degeneracy (1) remains avoided and that the avoided-crossing nature of the Big Bang must leave its traces, e.g., in the emergence of certain concentric circles in the cosmic microwave background measured by the Wilkinson Microwave Background Probe.

In our present paper we intend to join the discussion by showing that even in the framework of the entirely standard quantum theory the alternative assumption of the unavoidable degeneracy of eigenvalues at the Big Bang [as required, say, by Eq. (1)] need not necessarily require any low-probability fine-tuning.
The conceptual core of such a message may be traced back to our recently proposed extension of the quantum-theoretical perspective (cf. paper [4] or more detailed exposition [5]) which does not modify any “first principles” of quantum theory. One merely decides to work with the manifestly time-dependent representation of the “standard” physical Hilbert space of states, $\mathcal{H} = \mathcal{H}^{(S)}(t)$, which may simply cease to exist at $t = t_c$.

The latter option is to be shown here to enlarge the number of free parameters in the corresponding quantum models of dynamics in such a manner that one can satisfy the degeneracy constraints of the form (1) without any true difficulties. In addition, an optimal balance may be also achieved between the “classical” and “quantum” input information about the dynamics of the model.

For the sake of simplicity of presentation of the idea just an elementary illustrative phenomenological quantum model of Sec. 2 will be considered. In particular, no time re-parametrization invariance will be implemented to lead to an analog of the Weeler-DeWitt equation. In this way, in particular, the initial/final time moments will stay finite rather than transferred into conformal infinities.

The detailed analysis or our model will enable us to demonstrate that the BBC-like degeneracies of eigenvalues need not necessarily induce any enhanced sensitivity to perturbations nor the need of any particular fine-tuning. Thus, in our schematic model the quantum Universe may become strictly unobservable both before $t = t_{\text{initial}}$ and after $t = t_{\text{final}}$.

The technical essence of our message will lie in the recommended use of adiabatically time-dependent inner products in the Hilbert space of quantum theory (cf. Sec. 3). In the main body of the paper our quantum description of the BBC phenomenon will be illustrated via several non-numerical, exactly solvable examples (cf. Secs. 4 and 5 and Appendix A). In the subsequent discussion in Sec. 6 we shall emphasize that in the close vicinity of the critical BBC times $t = t_{\text{initial/final}} = t_c$ the role of the (adiabatic) time-dependence of the Hilbert space proves crucial.

## 2 The model

For methodical purposes several drastic mathematical simplifications of the overall physical scenario will be accepted. Firstly, we shall start building the quantum states of our schematic Universe inside Hilbert space $\mathcal{H}^{(\text{friendly})}$ of a finite dimension $N < \infty$. Secondly, we shall consider quantum theory of pure states only (i.e., no statistical physics). Thirdly, we shall follow some preliminary considerations by Bila [6] and treat the time-evolution of wave functions $|\psi(t)\rangle$ as adiabatic, circumventing thereby several technical complications as listed and discussed in [4].

Last but not least, we shall accept here a very pragmatic attitude towards the (up to now, unresolved) theoretical conflict between quantum theory
and general relativity. In this conflict we shall never leave the standard textbook quantum mechanics in its cryptohermitian or three-Hilbert-space (THS) recent reformulations as summarized, e.g., in our compact review [5]. We believe that for the time scales chosen as extremely short, this constraint (leading, e.g., to the manifest violation of the covariance requirements) may still represent a more or less safe territory of valid and consistent theoretical considerations admitting subsequent amendments, in principle at least.

For our present purposes the quantized generator of the time evolution (i.e., our toy-model Hamiltonian operator $H$) will be chosen in the following, extremely schematic and purely kinetic real and symmetric $N$-by-$N$-matrix time-independent and force-free form

$$H = H^{(N)} = \begin{bmatrix}
2 & -1 & 0 & \ldots & 0 & 0 \\
-1 & 2 & -1 & \ddots & \vdots & \vdots \\
0 & -1 & 2 & \ddots & 0 & 0 \\
0 & 0 & \ddots & \ddots & -1 & 0 \\
\vdots & \vdots & \ddots & -1 & 2 & -1 \\
0 & 0 & \ldots & 0 & -1 & 2
\end{bmatrix}. \quad (2)$$

For the questions we are going to ask (and concerning, e.g., the observability nature of the “eligible histories” of our schematic “Universe” near its BBC singularities) this operator itself even cannot be interpreted as directly related to the existence of these singularities. The reason is that precisely the very dynamical source of the emergence of these singularities lies already beyond the above-selected quantum-mechanical short-times scope and methodical range of our present message.

In the resulting picture of reality near the critical time $t = t_c$ all the information about the physics of the BBC dynamics will be assumed given in advance (say, from the purely external sources offered by the cosmological model-building and/or by non-quantum general relativity). We shall only work here with an empty-space phenomenological model of the collapsing Universe near $t = t_c$.

The spatial or geometric structure of the collapse will lie in the center of our interest. Four our purposes it will be characterized by the measurability and/or measurements of a finite sample $g_1(t)$, $g_2(t)$, \ldots, $g_N(t)$ of the $N$ spatial grid points at a classical, continuous time. These representative grid-point real coordinates will be treated as eigenvalues of a certain pre-determined general-matrix operator of the most essential observable

$$\hat{G} = \hat{G}^{(N)}(t) = \begin{bmatrix}
\gamma_{11} & \gamma_{12} & \ldots & \gamma_{1N} \\
\gamma_{21} & \gamma_{22} & \ldots & \gamma_{2N} \\
\ldots & \ldots & \ldots & \ldots \\
\gamma_{N1} & \gamma_{N2} & \ldots & \gamma_{NN}
\end{bmatrix}. \quad (3)$$

After Big Bang and before Big Crunch, the natural requirement of observability of the Universe forces us to impose $N$ conditions of reality of the
spectrum of this operator (i.e., in our toy model, of this matrix),

$$\text{Im } g_j(t) = 0, \quad j = 1, 2, \ldots, N, \quad t_{\text{initial}} \leq t \leq t_{\text{final}}. \quad (4)$$

Optionally, we might also add another, complementary requirement guaranteeing either the partial or the complete non-measurability of the space before Big Bang or after Big Crunch,

$$\text{Im } g_j(t) \neq 0, \quad j = 1, 2, \ldots, N_{\text{BBC}}, \quad t \notin [t_{\text{initial}}, t_{\text{final}}], \quad N_{\text{BBC}} \leq N. \quad (5)$$

In this language the BBC phenomenon itself will be simulated just by the $N - 1$ conditions of a complete confluence of the $N$-plet of eigenvalues

$$\lim_{t \rightarrow t_c} g_j(t) = g_N(t_c), \quad j = 1, 2, \ldots, N - 1 \quad (6)$$

which would guarantee also the complete single-point geometrical collapse of our toy-model Universe at the critical time.

Let us re-emphasize that we shall solely speak here about the privileged (viz., time-evolution) boosts generated by the quantum mechanical Hamiltonian operators $H$ and considered just along certain very short intervals of the time which will be assumed measured by the classical clocks. Naturally, such a decision (motivated, first of all, by the technical feasibility of at least some quantitative considerations) will force us to leave many important (and, up to these days, open) questions entirely aside.

Due to these assumptions we shall be able to keep working with the naive, non-covariant Schrödinger time-evolution equation. Naturally, we shall be unable to estimate the extent of the modifications of this picture after some future (and, of course, theoretically necessary) transition to the less scale-restrictive scenarios based on some suitable general-relativistic covariance requirements (sampled, e.g., by their well known incorporation [7] by Bryce DeWitt).

Our present key message will be restricted, therefore, to the constructive demonstration that in a very close vicinity of the BBC regime the language of quantum mechanics admits the complete (or, in alternative models, partial) loss of the measurability of the geometry of the (collapsing) space before the Big Bang and/or after the Big Crunch. In this sense, we do not see any theoretical necessity of the existence of any measurable Universe (or, alternatively, of a measurable Universe with the same number of dimensions), say, before the Big Bang.

This being said, we should add, as early as possible, that our present model is really too schematic for any cosmology-related and/or prediction-making purposes. In particular, the quantum-mechanics-based demonstration of the possibility of the (partial or complete) complexification of the coordinates (say, after the Big Crunch) certainly does not exclude their subsequent return to reality (say, in the cyclic form proposed in the very interesting recent preprint [3]).
3 The method

In a way inspired by the so called $\mathcal{P}\mathcal{T}$—symmetric quantum mechanics [8] the key to the resolution of the above-mentioned Penrose’s paradox of incompatibility of the assumption of Hermiticity of observables with the existence of the critical BBC times $t_c$ will be sought here in the omission of the former, overrestrictive assumption. In other words, we shall broaden the class of the admissible operators of geometry (3) and admit that

$$\hat{G}(t) \neq \hat{G}^\dagger(t) \quad \text{in} \quad \mathcal{H}^{(\text{friendly})}.$$  

(7)

One must emphasize here that this relation must not be read as a non-Hermiticity of $\hat{G}(t)$. Its true meaning is much simpler: Equation (7) will be understood as a mere consequence of our re-classification of the original time-independent representation $\mathcal{H}^{(\text{friendly})} \neq \mathcal{H}^{(\text{friendly})}(t)$ of the Hilbert space of states as overrestrictive and manifestly unphysical.

The necessary mathematics underlying such a change of perspective has been offered in Refs. [9]. The main idea is that the naive choice of Hilbert space $\mathcal{H}^{(\text{friendly})}$ is being replaced by a more flexible option. In it, the inner product is being determined via operator $\Theta = \Theta^\dagger > 0$ called metric (i.e., “Hilbert-space” metric, certainly different from the much more common Riemann-space-metric function $g_{\mu\nu}$).

Naturally, such a decision leads to the new form of Hermitian conjugation (marked, conveniently, by a double-cross superscript $\dagger$) and, hence, to the new, unitarily inequivalent Hilbert space $\mathcal{H}^{(\text{true})}$ which is declared physical. Any pre-selected non-Hermitian operator acting in $\mathcal{H}^{(\text{friendly})}$ and possessing real spectrum may be then reinterpreted as the “cryptohermitian” [10] operator of an observable quantity, i.e., as an operator which becomes self-adjoint in the amended, physical Hilbert space $\mathcal{H}^{(\text{true})}$.

A realistic BBC phenomenology may be built on this background. In the simplest arrangement of the theory the transition from trivial metric $\Theta = I := \Theta^{(\text{Dirac})}$ to nontrivial metric $\Theta = \Theta(t) > 0$ will in fact represent, in our present considerations, the only difference between $\mathcal{H}^{(\text{friendly})}$ and $\mathcal{H}^{(\text{true})}$. Nevertheless, it is necessary to emphasize that in contrast to the usual applications of transition from $\mathcal{H}^{(\text{friendly})}$ to $\mathcal{H}^{(\text{true})}$ dealing with single observable (usually, with the Hamiltonian), our present model will require the simultaneous guarantee of cryptohermiticity of both our observables $H$ and $\hat{G}$.

In the light of property (7) of the latter operator one really cannot choose $\Theta = \Theta^{(\text{Dirac})}$ so that the Hermiticity $H = H^\dagger$ of our toy Hamiltonian in $\mathcal{H}^{(\text{friendly})}$ is in fact irrelevant. The metric must be constructed which would make both our operators of observables self-adjoint, yielding

$$H = H^\dagger := \Theta^{-1} H^\dagger \Theta \equiv \Theta^{-1} H \Theta,$$

(8)

as well as

$$\hat{G}(t) = \hat{G}^\dagger(t) := \Theta^{-1} \hat{G}^\dagger(t) \Theta.$$

(9)
From the point of view of physics the additional model-building freedom offered by Eqs. (7), (8) and (9) opens a way towards the construction of metrics which could vary with time, $\Theta = \Theta(t)$. In this manner many “no-go” consequences of the restrictive formal framework provided by the ill-chosen space $\mathcal{H}^{(\text{friendly})}$ may be circumvented [4].

The former constraint (8) appears much easier to satisfy because our Hamiltonian itself remains time-independent. As long as this operator is represented by the real and symmetric $N$-dimensional matrix (2), the most natural representation of the metric can be provided by polynomial formula

$$\Theta(t) = a(t) I + b(t) H + c(t) H^2 + \ldots + z(t) H^{N-1}$$

(10)

containing $N$ unknown real-function coefficients. Such an ansatz may be inserted in Eq. (9) yielding the ultimate set of algebraic constraints expressed in terms of modified commutators $[A, B] := AB - B^\dagger A$,

$$a(t) [I, \hat{G}(t)] + b(t) [H, \hat{G}(t)] + c(t) [H^2, \hat{G}(t)] + \ldots + z(t) [H^{N-1}, \hat{G}(t)] = 0.$$  

(11)

We may summarize that the variability of the adiabatically time-dependent real matrices (3) carrying the input dynamical information and containing $N^2$ independent matrix elements $\gamma_{ij}(t)$ is only restricted by the $N(N-1)/2$ metric-compatibility conditions (11), by the $N$ spectral-reality (i.e., Universe-observability) conditions (4) and by the $N-1$ complete-degeneracy conditions (6) imposed at $t = t_c$. This means that at least the $(N-1)(N-2)/2$-plet of input parameters remains arbitrary. No particular fine-tuning will be needed at $N \geq 3$, therefore.

Naturally, the domain of variability of the input parameters is not arbitrary since one must guarantee the invertibility and positive definiteness of the metric as well as its compatibility with the concrete pre-selected Hamiltonian $H$. Via Eq. (11) these conditions further restrict the variability of parameters in $\hat{G}(t)$ and in $\Theta(t)$ to a certain domain $\mathcal{D}^{(\text{physical})}(t)$. Although the exhaustive specification of this time-dependent domain is difficult in general, it is usually sufficient and not so difficult to find its nonempty time-independent subdomain $\mathcal{D}^{(\text{practical})}$. Better insight in the latter restrictions may be gained via the detailed inspection of the model at the lowest dimensions $N$.

4 Metrics $\Theta^{(N)}$

4.1 Grid dimension $N = 2$

The Hamiltonian as well as the metric are elementary at $N = 2$,

$$H^{(2)} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \Theta^{(2)}(t) = \begin{pmatrix} a(t) + 2b(t) & -b(t) \\ -b(t) & a(t) + 2b(t) \end{pmatrix}.$$  

(12)
The eigenvalues $\theta_\pm(t) = a(t) + 2b(t) \pm b(t)$ of the metric are easily evaluated. The positivity of the metric (i.e., of all of its eigenvalues) imposes just the single constraint at $N = 2$, viz., $a(t) > \max(-b(t), -3b(t))$. Inside this interval the standard probabilistic interpretation of our $N = 2$ quantum Universe is guaranteed.

The detailed dynamics of the model must be deduced (typically, via the principle of correspondence) from the classical theory of gravity. In our approach this information is carried solely by the operator of geometry (3). Even its $N = 2$ realization illustrates quite well the idea. We even do not need the fully general matrix for this purpose. One of its elements may certainly be fixed by the convenient location of the BBC limiting coordinate in the origin, $g_c(t_e) = 0$. The resulting reduced three-parametric matrix

$$\hat{G}^{(2)}(t) = \begin{pmatrix} -r(t) & -v(t) \\ u(t) & r(t) \end{pmatrix}$$

(with, say, positive $r(t) > 0$) has the two eigenvalues

$$g^{(2)}_\pm(t) = \pm \sqrt{r^2(t) - u(t)v(t)}$$

for which it is easy to find the boundary between the observable and non-observable regimes. After a re-parametrization

$$u(t) = \frac{1}{2} \varrho(t) e^{\mu(t)}, \quad v(t) = \frac{1}{2} \varrho(t) e^{-\mu(t)}$$

we may recall Eq. (14) and conclude that irrespectively of the variation of the “inessential” exponent $\mu(t)$ the system will behave as unobservable at $\varrho(t) < -2r(t)$, observable at $-2r(t) \leq \varrho(t) \leq 2r(t)$ and unobservable again at $\varrho(t) > 2r(t)$.

In the physical interval of $t \in (t_{\text{initial}}, t_{\text{final}})$, i.e., for $\varrho(t)/[2r(t)] \in (-1, 1)$, i.e., during all the existence of our $N = 2$ toy quantum Universe, the probabilistic interpretation of its admissible states $|\psi^{(2)}\rangle \in \mathcal{H}^{(\text{true})}(t)$ will be fully determined by the metric $\Theta^{(2)}(t) > 0$. Conditions (11) of the compatibility of this metric with the geometry specified by the input operator $\hat{G}(t)$ degenerate to the single constraint at $N = 2$,

$$2b(t)r(t) + u(t)a(t) + 2b(t)u(t) + v(t)a(t) + 2b(t)v(t) = 0.$$ 

In its light, up to an irrelevant overall factor the resulting metric of the model becomes unique and solely defined in terms of the (variable) matrix elements of $\hat{G}^{(2)}(t)$,

$$\Theta^{(2)}(t) = \begin{pmatrix} 2r(t) & u(t) + v(t) \\ u(t) + v(t) & 2r(t) \end{pmatrix}.$$ 

Eigenvalues $2r(t) \pm (u(t) + v(t))$ of this matrix must be both positive so that we must keep $-2r(t) < u(t) + v(t) < 2r(t)$ for $t - t_{\text{initial}}$ small and positive as well as for $t - t_{\text{final}}$ small and negative.
We may fix another redundant degree of freedom by putting \( r(t) = 1/2 \). Then the third parameter \( \varrho(t) \) acquires the role of a “new time”, with the simplest, linear exemplification \( \varrho_{\text{lin}}(t) = t/\alpha \). This reduction will lead to the BBC identifications \( t_{\text{initial}} = -\alpha \), \( t_{\text{final}} = +\alpha \). In the resulting model any spatial measurement before Big Bang as well as after Big Crunch will only admit the purely imaginary results \( g_{\pm}^{(2)}(t) \). Parameter \( \mu(t) \) remains freely variable controlling the probabilistic interpretation of the Universe in the following three distinct dynamical regimes (cf. also Fig. 1):

1. for the times before Big Bang and after Big Crunch, i.e., in the unphysical domain with \(|\varrho| > 1\) (marked by symbol \( \Omega \) in Fig. 1) the spatial-point eigenvalues (14) stay purely imaginary; the whole toy-model Universe remains unobservable;

2. in the intermediate domain with \( 1 > |\varrho| > 1/\cosh \mu \) (marked by symbol \( \Sigma \) in Fig. 1) the spatial-point eigenvalues get real but the toy-model Universe still cannot be given the probabilistic interpretation. The only candidate (15) for the metric in \( \mathcal{H}^{(\text{true})} \) remains indefinite or, in other words, no positive-definite metric becomes simultaneously compatible with input Hamiltonian (2) and with input quantized geometry (14);

3. in the remaining and fully physical domain with \(|\varrho| < 1/\cosh \mu \) (marked by symbol \( \Phi \) in Fig. 1), both the given Hamiltonian \( H \) and the given geometry \( \hat{G}(t) \) become self-adjoint in \( \mathcal{H}^{(\text{true})}(t) \); the spatial-point eigenvalues stay real (= observable) while formula (15) defines the unique, positive definite and adiabatically time-dependent metric.

On this background one has to impose the last, BBC-degeneracy condition (4) at \( t = t_c \) reconfirming the expectations that there are no free parameters at \( N = 2 \) in general. Indeed, Fig. 1 shows that and why the BBC phenomenon may only be consistently quantized at \( \mu(t_c) = 0 \), i.e., just for the input geometry \( \hat{G}^{(2)}(t) \) characterized by the vanishing asymmetry-parameter at \( t = t_c \).
This is our first physics-mimicking observation which may also be perceived as an encouragement of systematic study of the $N > 2$ models containing some variable parameters even at $t = t_c$. A parallel, purely mathematical encouragement may be found in Ref. [11]. There, in different context, a very specific generalization of our $\mu(t) = 0$ model (denoted by symbol $H^{(2)}$ in loc. cit.) has been found tractable by non-numerical means at all dimensions. Unfortunately, the number of free parameters in these models proves too low for our present purposes.

4.2 Grid dimension $N = 3$

Hamiltonian (2) with $N = 3$ possesses the three positive time-independent eigenenergies $\varepsilon_0 = 2 - 2^{1/2}$, $\varepsilon_1 = 2$ and $\varepsilon_2 = 2 + 2^{1/2}$. In combination with its square

$$H^2 = \begin{bmatrix} 5 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 5 \end{bmatrix}. \quad (16)$$

the insertion converts Eq. (10) into the three-parametric ansatz for the metric,

$$\Theta^{(3)} = \begin{bmatrix} a + 2b + 5c & -b - 4c & c \\ -b - 4c & a + 2b + 6c & -b - 4c \\ c & -b - 4c & a + 2b + 5c \end{bmatrix}. \quad (17)$$

All of the eigenvalues of the latter matrix, viz., the three quantities $\theta_- = a + 2b + 6c - 2^{1/2}(b + 4c)$, $\theta_0 = a + 2b + 4c$ and $\theta_+ = a + 2b + 6c + 2^{1/2}(b + 4c)$ must be positive. This requirement specifies the boundary of the domain of parameters $D^{(physical)}$ in which the real and symmetric matrix $\Theta^{(3)}$ may be treated as one of admissible metrics in $H^{(true)}$.

The reparametrization of $a = -2b - 4c + \sqrt{2}\omega$ with $\omega = \omega(a) > 0$ reduces the definition of $D^{(physical)}$ to the elementary inequality $b < 2\sqrt{2}\omega$ and constraint

$$\frac{-\omega + b}{4 + \sqrt{2}} < c < \frac{\omega - b}{4 - \sqrt{2}}.$$

Inside these intervals we have to select parameters which make the metric compatible with the operator $\hat{G}^{(3)}$. The not entirely general, four-parametric classical-input-simulating choice of the latter operator, viz.,

$$\hat{G}^{(3)} = \begin{bmatrix} -r & -u & -v \\ u & 0 & w \\ v & w & r \end{bmatrix}. \quad (18)$$

leads to the solvable secular equation for the observable grid points $g$,

$$-g^3 + (-u^2 - w^2 + r^2 - u^2)g + u^2r - rw^2 = 0. \quad (19)$$
In parallel, at \( N = 3 \) the condition of “hidden” Hermiticity of operator (18) (i.e., Eq. (11)) degenerates to the triplet of relations between the matrix elements of \( \Theta^{(3)} \) and \( \hat{G}^{(3)} \),

\[
rb + 4 cr + 2 ur + 4 ub + 11 cu - vb - 4 cv - cw = 0 \quad (20)
\]

\[
-2 cr - ub - 4 cu + 2 va + 4 vb + 10 cv - wb - 4 cw = 0 \quad (21)
\]

\[
-cu + 2 wa + 4 wb + 11 cw - vb - 4 cv + rb + 4 cr = 0 . \quad (22)
\]

The last line ceases to be linearly independent at \( w = u \). The reduced problem becomes easily solved in closed form,

\[
a = c \left( \frac{r^2 + 4 ur + 6 u^2 + 2 vv - 3 v^2}{2 u^2 + vr - v^2} \right), \quad w = u , \quad (23)
\]

\[
b = -2 c \left( \frac{ur + 4 u^2 + 2 vr - 2 v^2}{2 u^2 + vr - v^2} \right), \quad w = u . \quad (24)
\]

Another simplification of the solution with \( c = 1 \) and with \( v = 0 \), i.e., with the tridiagonal input matrix \( \hat{G}^{(3)} \) reads

\[
a = \frac{6 u^2 + r^2 + 4 ru}{2 u^2} , \quad w = u , \quad v = 0 , \quad c = 1 , \quad (25)
\]

\[
b = - \frac{r + 4 u}{u} , \quad w = u , \quad v = 0 , \quad c = 1 . \quad (26)
\]

After the latter reduction the triplet of the grid-point roots of secular Eq. (19) becomes particularly transparent,

\[
g_0 = 0 , \quad g_\pm = \pm \sqrt{r^2 - 2 u^2} . \quad (27)
\]

For the time-independent particular choice of \( r = \sqrt{2} \) the BBC spatial singularity at \( g_c = 0 \) is reached in the limit of \( u \to u_c = \pm 1 \). For this reason we may treat \( u \) as the updated, rescaled time-variable at \( N = 3 \).

The climax of the story is that the completion of the construction of the probabilistic model, i.e., the search for a non-empty domain \( D^{(practical)} \) of positivity of the metric remains non-numerical. The appropriate insertions imply that all of the eigenvalues of the metric candidate \( \Theta^{(3)} \) remain positive for the one-parametric subfamily with fixed \( v = 0 \), fixed \( r = \sqrt{2} \) and with the variable “time” \( w = u \) constrained to one of the following two half-infinite intervals,

\[
u < - \frac{1}{1 + \omega/\sqrt{2}} , \quad u > \frac{1}{1 + \omega/\sqrt{2}} . \quad (28)
\]

As long as we have \( \omega > 0 \), the first one of these intervals safely contains the instant \( u_{\text{initial}} = -1 \) of Big Bang while the second interval contains the Big-Crunch time \( u_{\text{final}} = +1 \).

In comparison with the preceding \( N = 2 \) model, its updated \( N = 3 \) descendant preserves the schematic pattern of the parametrization of the
operator of geometry as well as of its combination with Hamiltonian $H^{(3)}$. A new qualitative feature emerges since at $N = 2$ the two input observables already determined the admissible metric completely. At $N = 3$ one of the parameters [viz., $\omega = \omega(t)$] remains variable and may be adjusted to some additional phenomenological requirements (the deeper discussion of this problem of ambiguity of $\Theta$ as presented in Ref. [9] should be consulted in this context).

5 Evolution in time near $t = t_c$

The study of our family of toy models at higher numbers of grid points $N$ would require the use of the standard numerical and computer-assisted tools of linear algebra. The quick growth of the number $N^2$ of available free parameters in the input geometry matrix $\hat{G}$ would make such a study unnecessarily extensive. Thus, a concrete phenomenological motivation narrowing the choice of the input matrices $\hat{G}^{(N)}(t)$ would be welcome.

In our present methodical considerations we may only try to separate the set of the matrix elements $\gamma_{jk}$ into its “important” and “less essential” subsets. One of the methods of such a reduction of the input information is provided by the possibility of the elementary-rotation reduction of a general finite-dimensional matrix to its “canonical” Hessenberg form [12]. In this sense let us now admit just the special, tridiagonal form of matrices $\hat{G}^{(N)}$ containing $3N - 2$ “most important” real parameters.

We expect that due to the tridiagonal structure of our toy Hamiltonian (2) the number of independent items in the metric-compatibility condition (11) will be much lower than predicted by our original upper estimate $N(N - 1)/2$ based on the mere antisymmetry of the general matrix expression. In such a reduced setting the $N-$plet of constraints (4) of the necessary spectral reality as well as the BBC degeneracy condition (6) will play a much more decisive role, indeed. Nevertheless, we believe that the use of the tridiagonal matrices $\hat{G}^{(N)}$ will still leave some of their parameters unrestricted so that, from the point of view of physics, no unstable fine-tuning will be required even after such a drastic simplification of the underlying mathematics.

Our final sample of solvable examples may clarify this point.

5.1 $N = 4$ model and BBC degeneracy at $t = t_c$

The use of the simplest four-parametric toy model with $s > r > 0$ in

$$\hat{G}^{(4)} = \begin{bmatrix} -s & -u & 0 & 0 \\ u & -r & -p & 0 \\ 0 & p & r & -u \\ 0 & 0 & u & s \end{bmatrix}$$

(29)
preserves the exact solvability of the secular equation,
\[ g^4 + (-s^2 + 2u^2 + p^2 - r^2) g^2 + r^2 s^2 + 2 s r u^2 - p^2 s^2 + u^4 = 0. \] (30)
For \( t \in (t_{\text{initial}}, t_{\text{final}}) \) all of its roots given by the standard elementary formulae must be real. Thus, not only that the first coefficient in Eq. (30) must be non-positive, i.e.,
\[ r^2 + s^2 \geq 2 u^2 + p^2 \] (31)
but also we must demand that
\[ (rs + u^2)^2 \geq p^2 s^2. \] (32)
The third requirement must guarantee the non-negativity of the discriminant of our quadratic equation,
\[ s^4 - 4 s^2 u^2 + 2 p^2 s^2 - 2 r^2 s^2 + 4 p^2 u^2 - 4 r^2 u^2 + p^4 - 2 p^2 r^2 + r^4 - 8 s r u^2 \geq 0. \] (33)
This relation may be further simplified as follows,
\[ (p^2 + s^2 - r^2)^2 \geq 4 u^2 [(s + r)^2 - p^2]. \] (34)
The BBC phenomenon will be characterized by the quadruple confluence of the real roots \( g_k \) which is only possible when \( g_c = 0 \). Then, constraint (34) becomes redundant and we get two conditions at \( t = t_c \), viz.,
\[ s_c^2 + r_c^2 = 2 u_c^2 + p_c^2 \] (35)
and
\[ (r_c s_c + u_c^2)^2 = p_c^2 s_c^2. \] (36)
The elimination of \( p_c^2 \) defined by the former relation (35) leaves us with the three real BBC parameters constrained by the single equation
\[ 2 r_c s_c u_c^2 + u_c^4 + 2 s_c^2 u_c^2 - s_c^4 = 0. \] (37)
Most easily we may keep \( s_c \) and \( u_c \) as two freely variable parameters and eliminate
\[ r_c = r_c(s_c, u_c) = -s_c - \frac{s_c}{2} \left[ \frac{u_c^2}{s_c^2} - \frac{s_c^2}{u_c^2} \right]. \] (38)
This means that using Eq. (36) we have to define
\[ p_c = p_{\pm c}(s_c, u_c) = \pm [r_c(s_c, u_c) + u_c^2/s_c]. \] (39)
In place of independent variable \( u_c \) an alternative real parameter \( \varrho \) may be used in a reparametrization
\[ u_c = u_{\pm c}(s_c, \varrho) = \pm s_c e^{-\varrho}. \] (40)
This finally simplifies the form of the quantity
\[ r_c = r_c(s, \varrho) = s_c [-1 + \sinh 2 \varrho]. \] (41)
We may conclude that the existence of the BBC phenomenon in our \( N = 4 \) model with \( s > r > 0 \) will be guaranteed whenever the variability of \( \varrho \) is restricted to the interval where \( \sinh 2 \varrho \in (1, 2) \).
5.2 Evolution near $t = t_c$

Let us now return to Ref. [13] where we analyzed the properties of a four-dimensional matrix which coincides with our geometry operator (29) at the constant sample values of $r(t) = 1$ and $s(t) = 3$ corresponding to the special and, incidentally, BBC-compatible value of $\sinh 2 \varrho = 4/3$. In different context, a very specific time-dependence of the remaining two variable matrix elements has been postulated there,

$$u(t) = -\sqrt{3} - 3t - 3B t^2, \quad p(t) = -\sqrt{4} - 4t - 4A t^2. \quad (42)$$

This form of time-dependence of the system serves our present purposes well. Once we choose $A = B = -1/2$ we obtain the standard global BBC scenario in which the observable Universe exists strictly inside the whole interval of times $t \in (t_{\text{initial}}, t_{\text{final}})$ with $t_{\text{initial}} = 0$ and $t_{\text{final}} = 2$. As we already explained, however, without a deeper insight into the (presumably, covariantly described) dynamics of similar systems, the explicit time-dependence of the observable quadruplet of grid points as given by Eq. (42) only keeps its good physical meaning in some very short intervals of the “classical” continuous times near $t_{\text{initial}}$ or $t_{\text{final}}$.

The same comment applies also in the case of the alternative choice of $A = B = +1/2$ which leads to the permanently expanding Universe. Within the framework of our toy model the size of this “Universe” is just an asymptotically linear function of time.

We may conclude that the available menu of qualitative physical predictions remains sufficiently sensitive to the variations of our dynamical “input” assumptions. The bad news is that the choice of the mere two parameters (42) in the input $\hat{G}^{(4)}(t)$ where $r(t) = 1$ and $s(t) = 3$ (i.e., the lack of necessary parameters rendering Eq. (11) valid) already makes the resulting metric $\Theta^{(4)}(t)$ either incompatible with the Hamiltonian $H^{(4)}$ of Eq. (2) or, alternatively, compatible with this Hamiltonian at a single, BBC-incompatible time $t = t_{\text{fixed}} \neq t_c$.

This is a not too essential cloud which has its silver linen since the corresponding lengthy calculations (which we omit here) reveal that the distance between $t_{\text{fixed}}$ and $t_c$ proves unexpectedly small (in fact, of the order of $10^{-2}$ in our units). This indicates that any amended (i.e., necessarily, more-parametric) BBC-compatible input matrix $\hat{G}^{(4)}(t)$ will be not too different from its imperfect but still sufficiently transparent present illustrative $r_c = 1$ solvable example where we followed Ref. [13] and choose $\sinh 2 \varrho = 4/3$.

6 Comments and summary

In our paper we detected a gap in the argumentation denying the compatibility of the BBC phenomena with quantum mechanics. Our main assertion was that after an appropriate amendment of the representation of states the
Big Bang/Crunch (BBC) phenomenon may remain fully compatible with the very standard textbook quantum theory known from traditional textbooks [14].

In our non-covariant, purely quantum-mechanical toy-model simulation of an exploding or collapsing Universe an entirely elementary Hamiltonian $H = H^\dagger$ was complemented by a less trivial though still highly schematic cryptohermitian observable $\hat{G} \neq \hat{G}^\dagger$ which was required to represent a time-dependent spatial geometry near a hypothetical Big Bang/Big Crunch singularity.

An alternative operation of Hermitian conjugation has been introduced serving as an ad hoc definition of an amended, physical Hilbert space of states $\mathcal{H}^{(true)}$. This enabled us to keep both the observables $H$ and $\hat{G}$ self-adjoint strictly inside a finite interval of time $t \in (t_{\text{initial}}, t_{\text{final}})$. Beyond its boundaries (i.e., before Big Bang or after Big Crunch) the eigenvalues of $\hat{G}$ were allowed to get complex so that the Universe ceased to be observable.

We argued that the presented form of a purely quantum-mechanical collapse of our toy-model Universe at $t = t_{\text{initial/final}}$ was in fact mediated by the introduction of the “true” or “self-consistent” manifestly time-dependent Hermitian-conjugation operation. In such a setting particular attention has been paid to the ambiguity of the choice of the inner product as discussed in Refs. [9, 4].

Many questions have been skipped as inessential for our present, predominantly methodical purposes. Naturally, these questions will re-emerge immediately in any phenomenologically oriented considerations in which

- (a) a more specific form of the adiabatically time-dependent input matrix elements $\gamma_{jk}(t)$ of the operator $\hat{G}$ would be deduced from the classical general relativity theory, say, on the basis of some suitable version of the principle of correspondence;
- (b) a number of other observables (say, $\hat{F}_1, \hat{F}_2, \ldots$) would be introduced as reflecting, say, the presence of some matter fields;
- (c) the dimension $N$ which characterizes the discretization approximation would be sent to its infinite, continuous-space limit;
- (d) a realistic, three-dimensional measurable space would be considered;
- (e) at least an approximate version of the Lorentz special-relativistic covariance of kinematics would be taken into account, etc.

Our present discrete odd–$N$ model might be also interpreted as allowing the existence of a zero-dimensional observable Universe before Big Bang and/or after Big Crunch. Thus, in a more realistic three-dimensional Universe one could proceed in the highly speculative spirit of Refs. [1, 2] and conjecture
that our Universe might just change its dimension during Big Bang and/or Big Crunch.

Up to similar exceptions we tried here to avoid all of the speculative considerations. Instead, we presented just a few purely formal arguments based on the analysis of a few elementary models. Our results may be briefly characterized as a demonstration of tractability of the quantization of systems which seem to exhibit a “catastrophic”, BBC-resembling time-evolution behavior in their classical models. The main sources of our proposed systematic approach to quantization of such systems may most briefly be summarized as lying in the following four assumptions of

- (i) the availability of some external, non-quantum information about the system exemplified here by the expected knowledge of the “input” matrices $\hat{G} = G(t)$) plus $H \neq H(t)$ and also, perhaps, $\hat{F}_1(t), \hat{F}_2(t), \ldots$;

- (ii) the availability of some theoretical background for decisions, say, between the admissibility [15] and inadmissibility [16] of a fundamental length in the model;

- (iii) the feasibility of calculations; as long as we decided to admit non-trivial metrics $\Theta(t) \neq I$, this apparently purely formal requirement proves of paramount importance as limiting, e.g., the range of practical applicability of perturbation expansions [17] or of the Moyal-bracket recipes [18] etc;

- (iv) the feasibility of making the metric $\Theta$ compatible with two and more cryptohermitian observables; up to now there existed not too many constructions of this type [19]; even in our present paper we considered just $H \neq H(t)$. Moreover, we did not dare to move beyond the mere adiabatic dynamical regime.

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Appendix A. BBC degeneracy at $N = 5$

The four-parametric ansatz with $s \geq r \geq 0$,

$$\hat{G}^{(5)} = \begin{bmatrix}
-s & -u & 0 & 0 & 0 \\
u & -r & -p & 0 & 0 \\
0 & p & 0 & -p & 0 \\
0 & 0 & p & r & -u \\
0 & 0 & 0 & u & s
\end{bmatrix}$$  \hspace{1cm} (43)

leads to the secular equation

$$-g^5 + (s^2 - 2u^2 - 2p^2 + r^2)g^3 + (-r^2 s^2 - 2u^2rs + 2p^2 s^2 - 2p^2 u^2 - u^4)g = 0$$

with one root $g_0 = 0$ and four roots $g_{\pm,\pm}$ given by the standard formulae.

At $t = t_c$ we may apply the same sequence of manipulations as used at $N = 4$ leading to the modified pair of BBC-degeneracy constraints

$$s_c^2 + r_c^2 = 2u_c^2 + 2p_c^2$$  \hspace{1cm} (45)

and

$$(r_c s_c + u_c^2)^2 = 2p_c^2 (s_c^2 - u_c^2) .$$  \hspace{1cm} (46)

We have to keep $u_c^2 \leq s_c^2$, i.e.,

$$u_c = u_{\pm c}(s_c, \varrho) = \pm s_c e^{-\varrho}, \quad \varrho \geq 0 .$$  \hspace{1cm} (47)

The elimination of the “redundant” $p_c^2$ defined by relation (45) leads to the single constraint

$$u_c^2 (r_c + s_c)^2 = (s_c^2 - u_c^2)^2$$  \hspace{1cm} (48)

which provides the two alternative definitions of the second dependent parameter

$$r_c = r_{\pm c} = -s_c \pm s_c \left( \frac{s_c}{u_c} - \frac{u_c}{s_c} \right).$$  \hspace{1cm} (49)

As long as we decided to require that $s_c \geq r_c \geq 0$, we arrive at the unique prescription

$$r_c = r_{c}(s_c, \varrho) = s_c [-1 + \sinh \varrho], \quad 1 \leq \sinh \varrho \leq 2 .$$  \hspace{1cm} (50)

Once more we may recall Ref. [13] and find there the special case of our five-dimensional geometry-operator matrix (29) with $r(t) = 2$ and $s(t) = 4$, i.e., with $\sinh \varrho = 3/2$. Borrowing again the specific time-dependence of the matrix elements from the same reference,

$$u(t) = -\sqrt{4 - 4t - 4Bt^2}, \quad p(t) = -\sqrt{6 - 6t - 6At^2}, \quad t_c = 0 .$$  \hspace{1cm} (51)
we obtain the time-dependent \( N = 5 \) spectrum \( \{g_k(t)\} \) resembling the \( N = 4 \) pattern.

The news are that in a way typical for the odd dimensions \( N \) one of the roots (viz., the time-independent \( g_0(t) = 0 \)) remains real at all times. Once we choose \( A = B = -1/2 \) we obtain the global BBC scenario in which the observable Universe exists for all times but it is one-dimensional for \( t \in (t_{\text{initial}}, t_{\text{final}}) \) and zero-dimensional for \( t \notin [t_{\text{initial}}, t_{\text{final}}] \) (we have \( t_{\text{initial}} = 0 \) and \( t_{\text{final}} = 2 \) in our units).

Within our present restricted perspective provided by the mere standard quantum mechanics the complexification of the eigenvalues is precisely what is meant by the words “non-observable”. A good textbook illustration is provided by the Coulomb field in the Dirac equation in the superstrong-coupling regime where the sudden complexification of the energies mimics the moment of the sudden emergence of the many-body physics via the suddenly opened channel admitting the creation of particle-antiparticle pairs [20]. A thinking by analogy could equally well apply in the present model where one measures the coordinates and where their complexification might be also interpreted as mimicking a decrease of the dimension of the space in a continuous limit \( N \to \infty \). Of course, the decrease of the dimension is not the only possibility. One could even develop many other *ad hoc* toy models supporting alternative scenarios supporting, say, a complete survival of the observability (without any change of dimension - this would be typical, say, for the cyclic cosmologies of Ref. [3]) or even an increase of the dimension (which could even lead, in cosmology, to certain truly speculative darwinistic-sounding concepts [21]).