Onsager algebra and algebraic generalization of Jordan-Wigner transformation

Kazuhiko Minami

Graduate School of Mathematics, Nagoya University,
Furo-cho, Chikusa-ku, Nagoya, Aichi, 464-8602, JAPAN.

Abstract

Recently, an algebraic generalization of the Jordan-Wigner transformation was introduced and applied to one- and two-dimensional systems. This transformation is composed of the interactions $\eta_i$ that appear in the Hamiltonian $H = \sum_{i=1}^{N} J_i \eta_i$, where $J_i$ are coupling constants. In this short note, it is derived that operators that are composed of $\eta_i$, or its $n$-state clock generalizations, satisfy the Dolan-Grady condition and hence obey the Onsager algebra which was introduced in the original solution of the rectangular Ising model and appears in some integrable models.

Keywords: lattice spin system, Jordan-Wigner transformation, Onsager algebra, exact solution, integrable system
e-mail:minami@math.nagoya-u.ac.jp
tel.+81-52-789-5578, fax:+81-52-789-2829

1 Introduction

When Onsager solved in [1] the square-lattice Ising model, the basic structure of his derivation is an algebraic structure which is now called the Onsager algebra. He showed that the representation of the original Hamiltonian can be reduced into direct products of two-dimensional representations. This decomposition intrinsically suggests the structure of the free fermion system, although he did not at all used the word 'fermion' in his paper. Later, Kaufman [2] rederived the partition function of the model, with the use of generators of the Clifford algebra, and later Schultz, Mattis and Lieb [3] rederived the free energy through a direct transformation to the free fermion system. The transformations in both [2] and [3] were the Jordan-Wigner transformation.

In 1982, Dolan and Grady [4] constructed an infinite number of conserved charges for a self-dual Hamiltonian that satisfy a condition, which is now called the Dolan-Grady condition. Later, von Gehlen and Rittenberg [5] introduced a $n$-state chiral Potts model with specific coupling constants. They derived that this model is integrable in the sense that it satisfies the Dolan-Grady condition, and hence there exist an infinite number of conserved charges, and also they numerically showed that this model exhibits an Ising-like spectrum. This model was also investigated in [6] and called "superintegrable", because it obeys the Onsager algebra, in addition to showing the structure of commuting transfer matrices.
It was pointed out in [7] that specific operators appearing in [4] satisfy the defining relations of the Onsager algebra. Davies derived [8] that a pair of operators recursively generate the Onsager algebra provided that they satisfy two symmetric Dolan-Grady relations; self-duality of the Hamiltonian is not needed in this argument.

Let us summarize the following progresses. The irreducible representations of finite-dimensional Onsager algebra were obtained, and the general form of the eigenvalues of the associated Hamiltonians \( A_0 + kA_1 \), where \( k \) is the coupling constant, were determined in [9], and subsequently in [10]. Lie algebraic structure of the Onsager algebra was investigated in [11]. The Onsager algebra is a subalgebra of \( \mathfrak{sl}_2 \) [8][9][12], related with a class of Yang-Baxter algebras [13]. Integrable lattice models were derived based on the Onsager algebra and an extension of the Onsager algebra was also considered in [14]. Higher rank generalizations of the Dolan-Grady relations and Onsager algebras have been also investigated starting from [15][16] and more recently in [12][13]. The completely inhomogeneous transverse Ising chain was considered in [15]. A contracted case of the Dolan-Grady condition and related spin models were considered in [17][18]. A q-deformed analogue of Onsager’s symmetry was introduced in [19]. It was shown in [20] that the homogeneous XXZ open spin chain with integrable boundary conditions can be built from the generators of the q-Onsager algebra, and eigenstates were investigated in the thermodynamic limit in [21], its transfer matrix was diagonalized in [22], for a review see also [23]. The Onsager symmetry appears in a kind of \( n \)-state clock chains [24] whose \( Z_n \) symmetry is enhanced to U(1). It was derived that the Hamiltonian in [24] with additional terms exhibits [25] the scar states. Motivated by the results in [24], the spin-1/2 XXZ chain at root of unity was investigated [26], and the existence of the Onsager symmetry at root of unity was conjectured [27]. Dynamics of models where the Hamiltonian is an element of the Onsager algebra were also investigated in [28].

There exists another progress on solvable models. Recently, an algebraic generalization of the Jordan-Wigner transformation was introduced [29]. This formula can be summarized as follows: Consider a series of operators \( \{ \eta_j \} \) \((j = 1, 2, \ldots, M)\) that satisfy the following commutation relations

\[
\eta_j \eta_{j+1} = -\eta_{j+1} \eta_j, \quad \eta_j \eta_k = \eta_k \eta_j \quad (|j - k| > 1), \quad \eta_j^2 = 1. \tag{1}
\]

Then, the Hamiltonian \(-\beta \mathcal{H} = \sum_{j=1}^{M} K_j \eta_j\) is mapped to the free fermion system by the following transformation:

\[
\varphi_j = \frac{1}{\sqrt{2}} e^{\frac{i\pi}{4}(j-1)} \eta_0 \eta_1 \eta_2 \cdots \eta_j \quad (0 \leq j \leq M - 1), \tag{2}
\]

where \( \eta_0 \) is an initial operator satisfying \( \eta_0^2 = -1 \), \( \eta_0 \eta_1 = -\eta_1 \eta_0 \), and \( \eta_0 \eta_k = \eta_k \eta_0 \) \((2 \leq k \leq M)\). Then we obtain \((-2i)\varphi_j \varphi_{j+1} = \eta_{j+1}\) and \(\{\varphi_j, \varphi_k\} = \varphi_j \varphi_k + \varphi_k \varphi_j = \delta_{jk}\) for all \(j, k\). Hence, the Hamiltonian is written as a sum of two-body products of fermion operators \( \varphi_j \).
The transformation (2) is generated from \( \{ \eta_j \} \), and only the commutation relation (1) is needed to obtain the free energy. When we consider the transverse Ising chain, i.e. \( \eta_{2j-1} = \sigma_j^z \) and \( \eta_{2j} = \sigma_j^x \sigma_j^y + 1 \), (2) reduces to the original Jordan-Wigner transformation. In other cases, we obtain other transformations that diagonalize the Hamiltonian.

This fermionization formula was applied to one-dimensional quantum spin chains [30][31], and the honeycomb-lattice Kitaev model with the Wen-Toric code interactions [32]. The key idea of this transformation was developed into a graph-theoretic treatment [33], in which the transformations of operators are expressed as modifications of graphs, and the kernel of its adjacency matrices corresponds to conserved quantities of the system. The condition (1) was independently considered to introduce models that can be mapped to the free fermion system, and investigated in terms of the graph theory [34] [35].

In this short note, we extend the Onsager’s result and show that there exist an infinite number of interactions that satisfy the Dolan-Grady condition, and hence exist an infinite number of realizations of (quotients of) the Onsager algebra. We also consider the operators that satisfy the conditions

\[
\eta_j \eta_{j+1} = \omega \eta_{j+1} \eta_j, \quad \eta_j \eta_k = \eta_k \eta_j \quad (|j - k| > 1),
\]

\[
\eta_j^n = 1, \quad \omega = e^{i \frac{2 \pi}{n}},
\]

and show an infinite number of interactions satisfying the Dolan-Grady condition. Note that the replacement \( \omega \mapsto \omega^{-1} \) corresponds to the inversion of the indices \( j \mapsto M - j + 1 \).

Throughout this short note, the cyclic boundary condition

\[
\eta_{M+j} = \eta_j,
\]

where \( M \) is the number of operators, is assumed. The conditions (1) or (3) together with the cyclic boundary condition (4) yield the Theorem 1-3 and Corollary 1.

When \( n = 2 \), the condition (3) reduces to (1). Operators that satisfy (3) were considered in [36], and also investigated in [37] with \( n = 3 \) concerning the incommensurate phase, and considered with arbitrary integer \( n \) in [5]. The Hamiltonian in [5] was later obtained from the transfer matrix of the two-dimensional chiral Potts model [38] [39]. The Baxter’s clock chain [40] [41] can also be written in terms of the operators that satisfy (3), and can be rewritten, through the Fradkin-Kadanoff transformation [36], in terms of the parafermions, which are \( Z_n \) generalizations of Majorana fermions. It is easily shown that the Fradkin-Kadanoff transformation can be obtained from the formula (2). It is now known that the Baxter’s clock chain can be regarded as ‘free’ parafermions [42] [43]. Generalizations of the relation (3) were investigated in [43], and in [44]-[45], and the corresponding Hamiltonians were shown to have an Ising-like spectrum and be integrable.

In Theorem 1-3 and Corollary 1, we will show that operators composed of \( \eta_j \)’s satisfy the Dolan-Grady condition. Only the algebraic relations are needed.
to derive the results, and hence any operators that satisfy (1) or (3) generate operators that satisfy the Dolan-Grady condition, and hence generate a quotient of the Onsager algebra. In Table 1 and Theorem 5, we show explicit examples of operators that satisfy (3), including the interactions of the transverse Ising chain, and so-called the superintegrable chiral Potts model. At last, we will comment on a fact that an infinite number of models with inhomogeneous interactions also become integrable.

2 Onsager algebra and Theorems

Let us consider series of operators \( \{A_j\} \) and \( \{G_k\} \), where \( j \in \mathbb{Z} \). The Onsager algebra is a Lie algebra defined via the relations

\[
[A_j, A_k] = 4G_{j-k}, \quad [G_m, A_l] = 2A_{l+m} - 2A_{l-m}, \quad [G_j, G_k] = 0.
\] (5)

It is known that a pair of operators, \( A_0 \) and \( A_1 \), recursively generate all the \( A_j \) and \( G_k \) in (5) provided that they satisfy the relations

\[
[A_0[A_0[A_0,A_1]]] = C[A_0,A_1],
\] (6)

\[
[A_1[A_1[A_1,A_0]]] = C[A_1,A_0],
\] (7)

where \( C \) is a constant. We call (6) and (7) the Dolan-Grady condition.

When we consider a Hamiltonian

\[
\mathcal{H} = A_0 + kA_1,
\] (8)

where \( k \) is a constant, it can be derived that \( \mathcal{H} \) belongs to an infinite family of mutually commuting operators, i.e. \( \mathcal{H} \) is integrable. For irreducible finite dimensional representations of the Onsager algebra, the general form of the eigenvalue was obtained in [9] as

\[
(\alpha + \beta k) + \sum_{j=1}^{n} 4m_j \sqrt{1 + k^2 + 2k \cos \theta_j}, \quad m_j = 0, \pm 1, \pm 2, \ldots, \pm s_j,
\]

where \( \alpha \) and \( \beta \) are constants, and \( s_j \) are positive integers.

Onsager introduced, for the purpose to solve the rectangular Ising model, the transfer matrix which is expressed by the operators

\[
A_0 = \sum_{j=1}^{N} \sigma_j^x, \quad A_1 = \sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^z.
\] (9)

These \( A_0 \) and \( A_1 \) satisfy the Dolan-Grady condition, and in this case (8) is the Hamiltonian of the transverse Ising chain [46].

4
Theorem 1 Let us introduce
\[ A_0 = \sum_{j=1}^{N} \eta_{2j-1}, \quad A_1 = \sum_{j=1}^{N} \eta_{2j}, \]  
where \( N \geq 2 \), and \( \eta_j \) satisfy (1) and (4). Then \( A_0 \) and \( A_1 \) satisfy the Dolan-Grady condition (6) and (7) with \( C = 16 \).

Direct calculations yield Theorem 1. We can also convince
\[ A_2 = \sum_{j=1}^{N} \eta_{2j} \eta_{2j+1} \eta_{2j+2}, \quad A_3 = \sum_{j=1}^{N} \eta_{2j} \eta_{2j+1} \eta_{2j+2} \eta_{2j+3} \eta_{2j+4}, \]
\[ A_{-1} = \sum_{j=1}^{N} \eta_{2j-3} \eta_{2j-2} \eta_{2j-1}, \quad A_{-2} = \sum_{j=1}^{N} \eta_{2j-5} \eta_{2j-4} \eta_{2j-3} \eta_{2j-2} \eta_{2j-1}, \]
\[ G_0 = 0, \quad G_1 = \frac{1}{2} \sum_{j=1}^{N} (\eta_{2j} \eta_{2j+1} - \eta_{2j-1} \eta_{2j}), \]
\[ G_2 = \frac{1}{2} \sum_{j=1}^{N} (\eta_{2j} \eta_{2j+1} \eta_{2j+2} \eta_{2j+3} - \eta_{2j-1} \eta_{2j} \eta_{2j+1} \eta_{2j+2}), \]
\[ G_3 = \frac{1}{2} \sum_{j=1}^{N} (\eta_{2j} \eta_{2j+1} \eta_{2j+2} \eta_{2j+3} \eta_{2j+4} \eta_{2j+5} - \eta_{2j-1} \eta_{2j} \eta_{2j+1} \eta_{2j+2} \eta_{2j+3} \eta_{2j+4}), \]
and generally
\[ A_l = \sum_{j=1}^{N} \eta_{2j} \eta_{2j+1} \cdots \eta_{2j+2l-2}, \quad A_{-l} = \sum_{j=1}^{N} \eta_{2j-2l+1} \eta_{2j-2l+2} \cdots \eta_{2j-1}, \]
and
\[ G_l = \frac{1}{2} \sum_{j=1}^{N} (\eta_{2j} \eta_{2j+1} \eta_{2j+2} \cdots \eta_{2j+2l-1} - \eta_{2j-1} \eta_{2j} \eta_{2j+1} \cdots \eta_{2j+2l-2}). \]

It is straightforward to derive that \( A_l \) and \( G_l \) satisfy \( A_{l+2N} = A_l \) and \( G_{l+2N} = G_l \). The relation \( A_{l+2N} = A_l \) is the closure relation considered by Davies in [8]. This means that the algebra generated from \( A_0 \) and \( A_1 \) in Theorem 1 is a quotient of the Onsager algebra, i.e. an Onsager algebra with the restriction \( A_{l+2N} = A_l \).

Theorem 1 shows existence of an infinite number of Hamiltonians that are expressed by (8) and hence governed by the Onsager algebra, because we know examples of operators that satisfy (1), such as the interactions of the transverse Ising chain: \( \eta_{2j-1}^{(1)} = \sigma_z^j \) and \( \eta_{2j}^{(1)} = \sigma_x^j \sigma_x^{j+1} \), those of the Kitaev chain: \( \eta_{2j-1}^{(2)} = \sigma_{2j-1}^x \sigma_{2j}^x \) and \( \eta_{2j}^{(2)} = \sigma_{2j}^y \sigma_{2j+1}^y \), those of the cluster model: \( \eta_{2j-1}^{(3)} = \sigma_{2j-1}^x \sigma_{2j}^x \)
\[ \sigma_x^{2j-1} \sigma_x^{2j+1} \text{ and } \eta^{(3)}_{2j} = \sigma_x^{2j+1} 1_{2j+1} \sigma_x^{2j+2}, \text{ and other infinite number of interactions listed in Table 1 and Table 2 given in [30].} \]

We will consider Theorem 1 in the cases where the condition (3) is satisfied. Examples of operators that satisfy (3) are shown in Table 1, where operators \( \eta_j \) form one or several series of operators; operators in each series satisfy (3), and operators from different series commute with each other. The operators \( Z, X \) and \( Y \) in Table 1 are defined, with \( \omega^n = 1 \), as

\[
Z = \begin{pmatrix}
1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\
0 & \omega & \omega^2 & \cdots & \omega^{n-2} \\
0 & 0 & \omega & \cdots & \omega^{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 1 \\
\omega & \omega^2 & \cdots & \cdots & 0
\end{pmatrix},
\]

\[ X = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}, \]

\[ Y = \omega^{-\frac{1}{2}(n-1)} \begin{pmatrix}
1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\
0 & \omega & \omega^2 & \cdots & \omega^{n-2} \\
0 & 0 & \omega & \cdots & \omega^{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 1 \\
\omega & \omega^2 & \cdots & \cdots & 0
\end{pmatrix}, \quad (11)
\]

and \( Z_j, X_j \) and \( Y_j \) are defined as

\[ Q_j = 1 \otimes \cdots \otimes Q \otimes 1 \otimes \cdots \otimes 1, \quad Q = Z, X, Y. \quad (12) \]

The operators \( Z, X, \) and \( Y \) satisfy \( ZX = \omega XZ, XY = \omega YX, \) and \( YZ = \omega YZ. \)

Then we can derive the following corollary.

**Corollary 1** Let us consider

\[ A_0 = \sum_{j=1}^{N} \eta^{2j-1}, \quad A_1 = \sum_{j=1}^{N} \eta^{2j}, \quad (13) \]

where \( N \geq 2, \) and \( n \) is even, and \( k = l = n/2, \) and \( \eta_j \) satisfy (3) and (4). Then \( A_0 \) and \( A_1 \) satisfy the Dolan-Grady condition (6) and (7) with \( C = 16 \) when \( n/2 \) is odd, and \([A_0, A_1] = 0 \) when \( n/2 \) is even.

Proof: Let \( \zeta_j = \eta^{2j}_j, \) then \( \zeta^2_j = 1. \) We find \( \zeta_j \zeta_{j+1} = \eta^{2j}_j \eta^{2j+1}_j = \omega^{k_2} \eta^{k}_j \eta^{k+1}_j = \omega^{k_2} \zeta_{j+1} \zeta_j, \) where \( \omega^{k_2} = e^{i \frac{2\pi k_2}{n}} = e^{i \pi k} = (-1)^k. \) If \( k \) is even, then \([A_0, A_1] = 0. \)

If \( k \) is odd, the operators \( A_0 \) and \( A_1 \) written in terms of \( \zeta_j \) satisfy the Dolan-Grady condition (6) and (7) since \( \zeta_j \zeta_{j+1} = -\zeta_{j+1} \zeta_j \) and \( \zeta^2_j = 1. \)

When \( n = 2, \) then \( A_0 = \sum_{j=1}^{N} \eta_{2j-1} + A_1 = \sum_{j=1}^{N} \eta_{2j}, \) and thus Theorem 1 is obtained from Corollary 1.

In Corollary 1, the relation \( \eta_j \eta_{j+1} = \omega \eta_{j+1} \eta_j \) is assumed. When we assume \( \eta_j \eta_{j+1} = \omega_j \eta_{j+1} \eta_j, \) where \( \omega_j \) depends on \( j \) and equals to \( \omega \) or \( \omega^{-1} \), we obtain
Theorem 2 Let us consider

\[
A_0 = \sum_{j=1}^{N} (\eta_{2j-1}^k - \eta_{2j-1}^{n-k}), \quad A_1 = \sum_{j=1}^{N} (\eta_{2j}^k - \eta_{2j}^{n-k}),
\]

(14)

where \( N \geq 2 \), and \( k = n/3 \) is an integer that satisfy \( 1 \leq k \leq n - 1 \), and \( \eta_j \) satisfy (3) and (4). Then \( A_0 \) and \( A_1 \) satisfy the Dolan-Grady condition (6) and (7) when \( C = -27 \) when \( k = 3m - 1, 3m - 2 \) \((m \in \mathbb{N})\), and \([A_0, A_1] = 0 \) when \( k = 3m \).

It is easy to show

\[
[\eta_{2j-1}^k, \eta_{2j}^k] = \eta_{2j-1}^k \eta_{2j}^k - \eta_{2j}^k \eta_{2j-1}^k = (1 - \omega^{-k}) \eta_{2j-1}^k \eta_{2j}^k, \quad k = 1, \ldots, n, \quad j = 1, \ldots, N,
\]

(15)

The inner derivatives in terms of \( \eta_{2j+1}^k \), operated to \( \eta_{2j}^k \), are therefore equivalent to the multiplications of \( (1 - \omega^{\mp kl}) \eta_{2j+1}^k \) from the left. Then we will prove the Theorem.

Proof: It is easy to show

\[
\sum_{i=1}^{N} [\eta_{2i-1}^k, \eta_{2j}^k] = \left([\eta_{2j-1}^k, \eta_{2j}^k] + [\eta_{2j+1}^k, \eta_{2j}^k]\right) = \left((1 - \omega^{-k}) \eta_{2j-1}^k + (1 - \omega^{k}) \eta_{2j+1}^k\right) \eta_{2j}^k = a_{11j} \eta_{2j}^k,
\]

(16)

\[
\sum_{i=1}^{N} [\eta_{2i-1}^{n-k}, \eta_{2j}^k] = \left([\eta_{2j-1}^{n-k}, \eta_{2j}^k] + [\eta_{2j+1}^{n-k}, \eta_{2j}^k]\right) = \left((1 - \omega^{k}) \eta_{2j-1}^{n-k} + (1 - \omega^{-k}) \eta_{2j+1}^{n-k}\right) \eta_{2j}^k = a_{21j} \eta_{2j}^k,
\]

(17)

where

\[
a_{11j} = z \eta_{2j-1}^k + \bar{z} \eta_{2j+1}^k, \quad a_{21j} = \bar{z} \eta_{2j-1}^{n-k} + z \eta_{2j+1}^{n-k},
\]

\[
z = 1 - \omega^{-k^2}, \quad \bar{z} = 1 - \omega^{k^2}.
\]

(18)

Then we obtain

\[
[A_0, \eta_{2j}^k] = \left((z \eta_{2j-1}^k + \bar{z} \eta_{2j+1}^k) - (\bar{z} \eta_{2j-1}^{n-k} + z \eta_{2j+1}^{n-k})\right) \eta_{2j}^k = (a_{11j} - a_{21j}) \eta_{2j}^k,
\]

(19)
Similarly, we obtain
\[
[A_0, \eta_{2j}^{n-k}] = (2z\eta_{2j-1}^{k} + z\eta_{2j+1}^{k}) - (z\eta_{2j-1}^{n-k} + z\eta_{2j+1}^{n-k}) \right) \eta_{2j}^{n-k},
\] (20)
where
\[
a_{12j} = z\eta_{2j-1}^{k} + z\eta_{2j+1}^{k}, \quad a_{22j} = z\eta_{2j-1}^{n-k} + z\eta_{2j+1}^{n-k}. \]
(21)
Since \(\eta_j\)'s with odd \(j\) commute with each other, we obtain
\[
[A_0, [A_0, [A_0, \eta_{2j}^{k}]]] = (a_{11j} - a_{21j})^{3} \eta_{2j}^{k}, \]
(22)
and
\[
[A_0, [A_0, \eta_{2j}^{n-k}]] = (a_{12j} - a_{22j})^{3} \eta_{2j}^{n-k}. \]
(23)
We find the following terms appear in (22)
\[
a_{11j}^{3} = z^{3} \eta_{2j-1}^{3k} + 3z^{2}z\eta_{2j-1}^{2k} \eta_{2j+1}^{k} + z^{3} \eta_{2j-1}^{k} \eta_{2j+1}^{2k} + z^{3} \eta_{2j-1}^{2k} \eta_{2j+1}^{k}, \]
\[
a_{21j}^{3} = z^{3} \eta_{2j-1}^{2k} + 3z^{2}z\eta_{2j-1}^{k} \eta_{2j+1}^{2k} + z^{3} \eta_{2j-1}^{k} \eta_{2j+1}^{2k} \eta_{2j+1}^{2k} + z^{3} \eta_{2j-1}^{2k} \eta_{2j+1}^{k}, \]
(24)
and
\[
a_{11j}^{2} a_{21j} = 3z^{2}z^{2} \eta_{2j-1}^{k} + 3z^{2}z^{2} \eta_{2j-1}^{k} \eta_{2j+1}^{2k} + z^{3} \eta_{2j-1}^{2k} \eta_{2j+1}^{k}, \]
\[
a_{11j}^{2} a_{21j}^{2} = 3z^{2}z^{2} \eta_{2j-1}^{k} + 3z^{2}z^{2} \eta_{2j-1}^{2k} \eta_{2j+1}^{k} + z^{3} \eta_{2j-1}^{2k} \eta_{2j+1}^{k}. \]
(25)
(26)
The assumption \(n = 3k\) yields \(\eta_{3j-1}^{3k} = 1, \eta_{3j+1}^{3k} = 1, \) and hence \(a_{11j}^{3} - a_{21j}^{3} = 0.\)
From \(3k = n\), we find \(\omega^{3k} = (\omega^{n})^{k} = 1\) and \(z^{3} + z^{3} = 0, \) then the last two terms in the right-hand side of (25), and also those of (26), cancel each other, respectively. Together with (19), we obtain
\[
[A_0, [A_0, [A_0, \eta_{2j}^{k}]]] = -9z\bar{z}[A_0, \eta_{2j}^{k}], \]
(27)
Similarly we find
\[
[A_0, [A_0, [A_0, \eta_{2j}^{n-k}]]] = -9z\bar{z}[A_0, \eta_{2j}^{n-k}], \]
(28)
When \(k = 3m, \) then \(\omega^{k} = \omega^{3m} = -1, \ z = 0, \) and from (19) and (20) we find \([A_0, A_1] = 0.\) When \(k = 3m - 1, \) then \(\omega^{k} = (\omega^{3})^{m} \omega^{k} = (\omega^{n})^{m} (\omega^{k})^{-1} = 1 \cdot (e^{i\frac{2\pi}{3}k})^{-1} = e^{-i\frac{2\pi}{3}}, \) and when \(k = 3m - 2, \) then \(\omega^{k} = (\omega^{3})^{m} \omega^{2k} = (\omega^{n})^{m} (\omega^{k})^{-2} = 1 \cdot (e^{i\frac{2\pi}{3}k})^{-2} = e^{i\frac{2\pi}{3}}.\) We thus obtain \(z \bar{z} = 3, \) \([A_0, A_1] \neq 0, \) and the first part of the Dolan-Grady condition (6) is satisfied with \(C = -9z\bar{z}.\) The second part of the Dolan-Grady condition (7) is obtained by the shift of indices \(2j \rightarrow 2j + 1.\)
Let $\zeta_j = \eta_j^k$ $(k = n/3)$. Then we find $\zeta_j \zeta_{j+1} = \eta_j^k \eta_{j+1}^k = \omega^{k^2} \eta_{j+1}^k \eta_j^k = \omega^{k^2} \zeta_{j+1} \zeta_j$. When $k = 3m$, then $\omega^{k^2} = 1$, and therefore $\zeta_j \zeta_{j+1} = \zeta_{j+1} \zeta_j$, and we obtain $[A_0, A_1] = 0$. When $k = 3m - 2$, then $\omega^{k^2} = e^{i \frac{2\pi}{3}}$, and we find $\zeta_j \zeta_{j+1} = e^{i \frac{2\pi}{3}} \zeta_{j+1} \zeta_j$. When $k = 3m - 1$, then $\omega^{k^2} = e^{-i \frac{2\pi}{3}}$, and $\zeta_j \zeta_{j+1} = e^{i \frac{2\pi}{3}} \zeta_{j+1} \zeta_j$, where $\zeta_j = \zeta_{2N-j+1}$. The case with $\omega = e^{i \frac{2\pi}{3}}$ was already considered in [47] though the derivation is different.

With the choice of the operators $\zeta_{2j-1} = X_j$ and $\zeta_{2j} = Z_j Z_{j+1}^1$, the Hamiltonian $A_0 + k A_1$ with $n = 3$ can be written as $\mathcal{H}_B - \mathcal{H}_B^1$, where $\mathcal{H}_B$ is the Baxter’s clock model [40][41].

**Theorem 3** Let us consider

$$A_0 = \sum_{j=1}^{N} \sum_{k=1}^{n-1} \frac{\eta_{2j-1}^k}{1 - \omega^{-k}}, \quad A_1 = \sum_{j=1}^{N} \sum_{k=1}^{n-1} \frac{\eta_{2j}^k}{1 - \omega^{-k}},$$

(29)

where $N \geq 2$, and $\eta_j$ together with $\omega$ satisfy (3), and the cyclic boundary condition (4) is assumed. Then $[A_0, A_1] \neq 0$, and $A_0, A_1$ satisfy the Dolan-Grady condition (6) and (7) with $C = n^2$.

For the purpose to prove this Theorem, we use the formula [48][5]

$$\sum_{i=1}^{n-1} \omega^{\frac{(m-1)i}{2}} = \frac{1}{2} (m - n), \quad \sum_{i=1}^{n-1} \omega^{\frac{m+1+i-1}{2}} = -\frac{1}{2} (m - n),$$

(30)

$$n = 2, 3, 4, 5, \ldots, \quad m = 1, 3, 5, \ldots$$

$$m \leq n \quad \text{and} \quad \omega = e^{i \frac{2\pi}{3}}.$$

The first formula is equivalent to (2.16) of [5], and the second is obtained from the first.

**Proof:**

$$[A_0, \eta_{2j}^k] = [\sum_{i=1}^{N} \sum_{l=1}^{n-1} \frac{\eta_{2l+1}^k}{1 - \omega^{-l}}, \eta_{2j}^k]$$

$$= \sum_{i=1}^{n-1} \frac{1}{1 - \omega^{-l}} [\sum_{i=1}^{N} \eta_{2i+1}^l, \eta_{2j}^k]$$

$$= \sum_{i=1}^{n-1} \frac{1}{1 - \omega^{-l}} \left((1 - \omega^{-kl}) \eta_{2j-1}^l + (1 - \omega^{-kl}) \eta_{2j+1}^l\right) \eta_{2j}^k$$

$$= \left(\sum_{l=1}^{n-1} c_l (kl) \eta_{2j-1}^l + \sum_{l=1}^{n-1} c_l (-kl) \eta_{2j+1}^l\right) \eta_{2j}^k,$$

(31)

where

$$c_l(m) = \frac{1 - \omega^{-m}}{1 - \omega^{-l}}.$$

(32)
The Dolan-Grady relation (6) is satisfied if
\[
\left( \Delta_k(\eta_{2j-1}, \eta_{2j+1})^3 - C \Delta_k(\eta_{2j-1}, \eta_{2j+1}) \right) \eta_{2j}^k = 0,
\]
where
\[
\Delta_k(x, y) = \Delta_k(x) + \Delta_{-k}(y), \quad \Delta_k(x) = \sum_{l=1}^{n-1} c_l(k)x^l.
\]
For the purpose to derive (33), it is sufficient to show that
\[
\Delta_k(x, y)^3 - n^2 \Delta_k(x, y) = 0
\]
as a polynomial, with the condition \(x^n = 1\) and \(y^n = 1\). For the purpose to show (35), it is sufficient to show that (35) is satisfied with independent numbers \(x, y = 1, \omega, \omega^2, \ldots, \omega^{n-1}\), where \(\omega = e^{i\frac{2\pi}{n}}\) (that satisfy \(\omega^n = 1\)). It is straightforward to show, with the use of (30), which is valid when \(m < n\), that \(\Delta_k(\omega^s) = -k\) \((k = 1, 2, \ldots, s)\), \(\Delta_k(\omega^s) = -(n - k)\) \((k = s + 1, \ldots, n - 1)\), and \(\Delta_{-k}(\omega^s) = -(n - k)\) \((k = 1, 2, \ldots, s)\), \(\Delta_{-k}(\omega^s) = k\) \((k = s + 1, \ldots, n - 1)\), and thus \(\Delta_k(x, y)\) takes \(n\) or \(-n\), which yields (33) with \(C = n^2\) and \([A_0, A_1] \neq 0\). Similarly we obtain (7)

With the choice of the operators \(\eta_{2j-1} = Z_j\) and \(\eta_{2j} = X_jX_j^{-1}\), the Hamiltonian \(A_0 + kA_1\) results in that of the superintegrable chiral Potts chain [5]. Note that in [5], the Dolan-Grady condition was derived with the use of the explicit matrix representation. In our derivation, Theorem 4 is proved using only the algebraic relations, and thus valid for all operators which satisfy (3).

The operators \(A_0\) and \(A_1\) in Theorem 1 satisfy a secular equation and the generated algebra is a quotient of the Onsager algebra. It is an important problem to find the secular equations in the cases of Theorem 2 and 3, and specify the possible form of the eigenvalue for all cases.

We can find in Table 1 a list of operators that satisfy (3). The next Theorem shows that once we find an set of operators satisfying (3), we can generate other sets of operators that satisfy (3).

**Theorem 4** Let us consider the case \(\eta_j = \prod_{k=1}^N (X_k^{x_{jk}} Z_k^{z_{jk}})\), where \(x_{jk}\) and \(z_{jk}\) are non-negative integers. Let \(\varphi_X\) and \(\varphi_Z\) be transformations defined by
\[
\varphi_Z : X_k \mapsto X_k, \quad Z_k \mapsto Z_k^{-1},
\]
\[
\varphi_X : X_k \mapsto X_k^{-1}, \quad Z_k \mapsto Z_k
\]
for all \(k\), and \(\varphi_{XZ} = \varphi_X \circ \varphi_Z\). Assume that the operators \(\{\eta_j\} = \{\eta_{2j-1}\} \cup \{\eta_{2j}\}\) satisfy the condition (3), then the set of operators
\[
\{\varphi_Z(\eta_{2j-1})\} \cup \{\varphi_Z(\eta_{2j})\}, \quad \{\varphi_X(\eta_{2j-1})\} \cup \{\varphi_X(\eta_{2j})\},
\]
\[
\{\varphi_{XZ}(\eta_{2j-1})\} \cup \{\eta_{2j}\}, \quad \{\eta_{2j-1}\} \cup \{\varphi_{XZ}(\eta_{2j})\},
\]
\[
\{\varphi_{XZ}(\eta_{2j-1})\} \cup \{\varphi_{XZ}(\eta_{2j})\}
\]
also satisfy the condition (3).

Proof: The first condition in (3), \( \eta_j \eta_{j+1} = \omega \eta_{j+1} \eta_j \) or \( \eta_j \eta_{j+1} = \omega^{-1} \eta_{j+1} \eta_j \), is written as

\[
\sum_{k=1}^{N} (z_{jk} x_{j+k} - x_{j+k} z_{j+k}) = 1 \quad \text{or} \quad -1. \tag{38}
\]

The second condition in (3), \( \eta_i \eta_j = \eta_j \eta_i \) (\(|i - j| > 1\)), is written as

\[
\sum_{k=1}^{N} (z_{ik} x_{jk} - x_{ik} z_{jk}) = 0 \quad |i - j| > 1. \tag{39}
\]

The transformations (36) result in

\[
\varphi_Z : z_{jk} \mapsto -z_{jk}, \quad \varphi_X : x_{jk} \mapsto -x_{jk}
\]

for all \( k \). Then it is easy to convince that the conditions (38) and (39) are also satisfied after the transformations (37).

Uglov and Ivanov [15] considered a generalization of the original Onsager algebra. They considered a Hamiltonian of the form

\[
-\beta \mathcal{H} = \sum_{j=1}^{N} K_j e_j \quad (N \geq 3)
\]

and derived that if the operators \( e_j \) satisfy the relation

\[
[e_i, [e_i, e_j]] = e_j \quad (|i - j| = 1),
\]

\[
[e_i, e_j] = 0 \quad (|i - j| > 1), \tag{42}
\]

then there exists an infinite family of integrals \( \{I_m\} \), where \( I_0 = \mathcal{H} \) and \( [I_m, I_n] = 0 \) (\( m, n \geq 1 \)).

We would like to note that \( \frac{1}{2} \tilde{\eta}^k \) (\( k = n/2 = \text{odd} \)) satisfy the condition (42). A list of operators \( \eta_j \) with \( n = 2 \) can be found in Table.1 of [30], where one of the simplest example is

\[
e_{2j-1} = \frac{1}{2} \sigma^j, \quad e_{2j} = \frac{1}{2} \sigma^j \sigma^{j+1}, \tag{43}
\]

which are the interactions of the transverse Ising chain [49]-[54]. The transverse Ising chains with random interactions and fields have been investigated in [55]-[63]. Here we have to note that two-dimensional Ising models are equivalent to one-dimensional quantum chains [64] including random cases [65]. We can find, for example, the cluster models with random next-nearest-neighbor interactions are integrable. They cannot be diagonalized, even in the case of uniform interactions, through the standard Jordan-Wigner transformation, and the algebraic generalization (2) is needed to diagonalize them [31].

The author would like to thank Pascal Baseilhac for his valuable comments. This work was supported by JSPS KAKENHI Grant No. JP19K03668.
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Table 1: Examples of operators from which series of operators that satisfy (3) can be obtained. Here, for example, i) $XX^{n-1}$ denotes $X_jX_{j+1}$, ii) $XX\ldots X$ denotes $I_j$ or $X_i$, or $X_jX_{j+1}$ or $\prod_{k=j}^{j+m}X_k$ ($m \geq 2$), iii) $XX\ldots X$ denotes $X_j$ or $X_jX_{j+1}$ or $\prod_{k=j}^{j+m}X_k$ ($m \geq 2$). When we consider $XX^{n-1}$ and $ZZ$, let $\eta_1^{(1)} = X_jX_{j+1}$, $\eta_2^{(1)} = Z_{j+1}Z_{j+2}$, and $\eta_3^{(1)} = X_jX_{j+1}X_{j+2}$, $\eta_4^{(1)} = Z_{j+2}Z_{j+3}$. Then the series of operators $\{\eta_1^{(1)}\}$ satisfy (3), and $\{\eta_j^{(2)}\}$ satisfy (3), and $\eta_j^{(1)}$ and $\eta_k^{(2)}$ commute with each other for all $j$ and $k$.

| $XX^{n-1}$ | $ZZ$ |
|------------|-------|
| $XX^{n-1}$ | $Z\ldots Z\ldots Z$ |
| $XX^{n-1}$ | $X\ldots XZ\ldots Z\ldots ZX\ldots X$ |
| $XX$ | $X\ldots XZZ^{-1}X\ldots X$ |
| $XX\ldots X\ldots X$ | $X\ldots XZZ^{-1}X\ldots X$ |
| $YY^{n-1}$ | $Y\ldots YZ\ldots Z\ldots ZY\ldots Y$ |
| $Y\ldots Y\ldots Y$ | $Y\ldots YZZ^{-1}Y\ldots Y$ |
| $X1X^{n-1}$ | $ZZ$ |
| $X1X^{n-1}$ | $XZZX$ |
| $X1\ldots 1X^{n-1}$ | $XZZX\ldots X$ |
| $Z1Z^{n-1}$ | $XZZX$ |
| $Z\ldots Z$ | $XZZX^{n-1}$ |
| $Z\ldots Z\ldots ZX^{n-1}$ | $XZZX^{n-1}$ |
| $XXZXX^{n-1}$ | $XZZX$ |
| $XX^{n-1}ZXX^{n-1}$ | $XZZX^{n-1}$ |
| $XXZ^{n-1}XX^{n-1}$ | $XZZX^{n-1}X$ |
| $XXXZ^{n-1}X^{n-1}$ | $XZZX^{n-1}$ |
| $XXZ\ldots X\ldots X^{n-1}$ | $XZZX^{n-1}$ |
| $XX\ldots X\ldots X^{n-1}$ | $XZZX^{n-1}$ |
| $ZXX^{n-1}X^{n-1}$ | $XZZX^{n-1}$ |
| $XXZ^{n-1}X^{n-1}$ | $XZZX^{n-1}$ |
| $XXZ^{n-1}X^{n-1}$ | $XZZX^{n-1}$ |
| $XZ\ldots Z\ldots ZX^{n-1}$ | $XZZX^{n-1}$ |
| $XZXX^{n-1}$ | $XZZX^{n-1}$ |
| $XZ\ldots Z\ldots ZX^{n-1}$ | $XZZX^{n-1}$ |
| $XZ\ldots Z\ldots ZX^{n-1}$ | $XZZX^{n-1}$ |
| $XZXX^{n-1}$ | $XZZX^{n-1}$ |
| $XZ\ldots Z\ldots ZX^{n-1}$ | $XZZX^{n-1}$ |
| $XZ\ldots Z\ldots ZX^{n-1}$ | $XZZX^{n-1}$ |
| $XZ\ldots Z\ldots ZX^{n-1}$ | $XZZX^{n-1}$ |
| $XZ\ldots Z\ldots ZX^{n-1}$ | $XZZX^{n-1}$ |
| $XZ\ldots Z\ldots ZX^{n-1}$ | $XZZX^{n-1}$ |
| $XZ\ldots Z\ldots ZX^{n-1}$ | $XZZX^{n-1}$ |

