Abstract

In this paper, we show that if we decompose a polygon into two smaller polygons, then by comparing the number of extremal vertices in the original polygon versus the sum of the two smaller polygons, we can gain at most two globally extremal vertices in the smaller polygons, as well as at most two locally extremal vertices. We then will derive two discrete Four-Vertex Theorems from our results.

1. Introduction

There are many notions of extremality in polygons, the earliest appeared circa 1813 in [2]. Recently, a very natural type of extremality was introduced in [5], one which very consistently adhered to that of curvature in the smooth case. A closely related global analogue had already appeared much earlier, and it as well has a smooth and discrete interpretation. While it is debatable to whom we attribute this discrete global notion of extremality, closely related ideas are present in [1].

In this paper, we will expound on these two types of extremality by providing a few observations and facts to build intuition. We will then discuss the notion of decomposing a polygon and investigate how this impacts our two types of extremality. We then derive fresh results relating the number of extremal vertices of the larger polygon versus the two smaller polygons of decomposition. While our results will be relevant geometrically on their own, we will observe that they are closely tied to two discrete Four-Vertex Theorems pertaining to our two types of extremality, which follow almost immediately from our stronger results.

We note that we will skip proofs of the more simple results. All results in this paper are considered with much more detail in [4].

2. Global and Local Extremality

We denote by $P$ a polygonal curve, which is a simple piecewise linear curve with vertices $V_1, V_2, ..., V_n$. When we speak of a closed polygonal curve, we will refer to it as a polygon. Also, we restrict our consideration simply to the planar case and all indices will be taken modulo the number of vertices of the polygonal curve. The following definition was coined in [3]:

**Definition 2.1.** We say that a polygonal curve is generic if the maximal number of vertices that lie on a circle is three and no three vertices are collinear.

Observe that all regular polygons are not generic.
**Definition 2.2.** Let $C_{ijk}$ be a circle passing through any three vertices $V_i, V_j, V_k$ of a polygonal curve. We say that $C_{ijk}$ is empty if it contains no other vertices of the polygonal curve in its interior, and we say that it is full if it contains all of the other vertices of the polygonal curve in its interior.

For simplicity, we will denote a circle passing through consecutive vertices $V_{i-1}, V_i$ and $V_{i+1}$ by $C_i$.

**Definition 2.3.** We call a full or empty circle $C_i$ an extremal circle. We refer to the corresponding vertex $V_i$ as a globally extremal vertex.

Some of our results will use triangulation arguments. Consider all of the empty circles passing through any three distinct points of a polygon. In [3] B. Delaunay shows that the triangles formed by each of the three points corresponding to an empty circle form a triangulation of the polygon $P$. This triangulation is called a Delaunay triangulation.

Analogously, if we assume convexity on our polygon and consider the full circles passing through any given three points, the triangles given by each of the three points corresponding to a full circle also form a triangulation. This triangulation is commonly known as the Anti-Delaunay triangulation.

**Definition 2.4.** A vertex $V_i$ is said to be positive if the left angle with respect to orientation, $\angle V_{i-1}V_iV_{i+1}$, is at most $\pi$. Otherwise, it is said to be negative.

**Definition 2.5 (Discrete Curvature).** Assume that a vertex $V_i$ is positive. We say that the curvature of the vertex $V_i$ is greater than the curvature at $V_{i+1}$ ($V_i \succ V_{i+1}$) if the vertex $V_{i+1}$ is positive and $V_{i+2}$ lies outside the circle $C_i$ or if the vertex $V_{i+1}$ is negative and $V_{i+2}$ lies inside the circle $C_i$.

By switching the word “inside” with the word “outside” in the above definition (and vice-versa), we obtain that $V_i \prec V_{i+1}$, or that the curvature at $V_i$ is less than the curvature at $V_{i+1}$. In the case that the vertex $V_i$ is negative, simply switch the word “greater” with the word “less”, and the word “outside” by the word “inside”.

**Definition 2.6.** A vertex $V_i$ of a polygonal line $P$ is locally extremal if $V_{i-1} \prec V_i \succ V_{i+1}$ or $V_{i-1} \succ V_i \prec V_{i+1}$.

**Remark 2.1.** If we assume convexity on our polygon and observe the definition of locally extremal vertices closely, we simply are considering the position of the vertices $V_{i-2}$ and $V_{i+2}$ with respect to the circle $C_i$. Our vertex $V_i$ will be locally extremal if and only if both vertices $V_{i-1}$ and $V_{i+2}$ lie inside or outside the circle $C_i$.

When defining global extremality, we discussed empty and full extremal circles. If a circle $C_i$ is empty, then we say that the corresponding vertex $V_i$ is maximal. If $C_i$ is full, then we say $V_i$ is minimal. Analogously for locally extremal vertices, we call a vertex maximal if $V_{i-1} \prec V_i \succ V_{i+1}$ and minimal if $V_{i-1} \succ V_i \prec V_{i+1}$.

We denote the number of globally maximal-extremal vertices of a polygonal curve $P$ by $s_{-}(P)$ and globally minimal-extremal vertices by $s_{+}(P)$ to be consistent with [1]. For locally extremal vertices, we will attribute the notation $l_{-}(P)$ and $l_{+}(P)$, respectively.

**Proposition 2.1.** Let $P$ be a generic convex polygon. Then
\[ l_{+}(P) = l_{-}(P). \]
Remark 2.2. The proof of this fact immediately follows by carefully observing the definition of locally extremal vertices. Note that it was very important for us to include the assumption that our polygon is generic, since this eliminates the possibility of having two extremal vertices adjacent to each other. Also, it is easy to see that the equality \( s_+(P) = s_-(P) \) does not hold. In fact, we cannot form any relationship between globally maximal-extremal and globally minimal-extremal vertices.

Proposition 2.2. Let \( P \) be a generic convex polygon. If \( V_i \) is a globally extremal vertex, then \( V_i \) is a locally extremal vertex.

This result follows immediately from the observation made in Remark 2.1.

Proposition 2.3. Let \( P \) be a generic convex quadrilateral. Then \( P \) has four globally extremal and locally extremal vertices.

Proof. For globally extremal vertices, we apply a Delaunay triangulation to \( P \), which immediately yields two globally maximal-extremal vertices. We then apply an Anti-Delaunay triangulation to \( P \), which yields two minimal-extremal vertices. Proposition 2.2 then yields the result for locally extremal vertices.

While the following proposition is technical yet quite obvious, it will be a vital proposition that will be used frequently to prove our main results.

Proposition 2.4. Let \( A, B, C \) and \( X \) be four points in the plane in a generic arrangement, \( C_B \) be the corresponding circle passing through \( A, B \) and \( C \), and let \( C_A \) be the circle passing through the points \( X, A \) and \( B \). We denote by \( \tilde{C}_A \) and \( \tilde{C}_B \) the open discs bounded by \( C_A \) and \( C_B \), respectively. Denote by \( H_{AB}^+ \) the half-plane formed by the infinite line \( AB \) containing the point \( C \) and by \( H_{AB}^- \) the half-plane formed by the infinite line \( AB \) not containing the point \( C \). If \( X \) lies in \( \tilde{C}_B \cap H_{AB}^+ \), then \( C \) lies in \( H_{AB}^+ \setminus \tilde{C}_A \). If \( X \) lies in \( H_{AB}^+ \setminus \tilde{C}_B \), then \( C \) lies in \( \tilde{C}_A \). Analogously, if \( X \) lies in \( \tilde{C}_B \cap H_{AB}^- \), then \( C \) lies in \( \tilde{C}_A \). If \( X \) lies in \( H_{AB}^- \setminus \tilde{C}_B \), then \( C \) lies in \( H_{AB}^- \setminus \tilde{C}_A \).

Proof. The proof is a simple verification of the the situation restricted around the origin and solving corresponding systems of equations.

3. Globally Extremal Vertices and Decomposition of Polygons

Definition 3.1. We say an edge or diagonal of a polygon is Delaunay if there exists an empty circle passing through the corresponding vertices of that edge or diagonal. If there exists a full circle passing through the vertices of this edge or diagonal, then we say the edge or diagonal is Anti-Delaunay.

Remark 3.1. Note that a triangulation of a polygon where every edge and diagonal is Delaunay is a Delaunay Triangulation. Similarly, if every edge and diagonal of a triangulation is Anti-Delaunay, then we have an Anti-Delaunay triangulation.

So what exactly does it mean to decompose a polygon? Here the notion of decomposing a polygon will simply be the cutting of a polygon \( P \) by passing a line segment through any two vertices so that the line segment lies in the interior of the polygon. We will call this line segment a diagonal. Also, we denote the two new polygons formed by a decomposition by \( P_1 \) and \( P_2 \) and require that they each have at least four vertices. By this last requirement it automatically follows that \( P \) must have at least six vertices to successfully perform a decomposition.
Theorem 3.1. Let $P$ be a generic convex polygon with six or more vertices and $P_1$ and $P_2$ be the resulting polygons of a decomposition of $P$. Assume that the diagonal of this decomposition is Delaunay. Then

$$s_-(P) \geq s_-(P_1) + s_-(P_2) - 2.$$  

Analogously, if the diagonal is Anti-Delaunay, then

$$s_+(P) \geq s_+(P_1) + s_+(P_2) - 2.$$ 

Proof. We begin by applying a Delaunay triangulation to $P$, $P_1$ and $P_2$. Noticing that the diagonal is Delaunay for $P_1$ and $P_2$, as well as $P$ by assumption, we obtain our first inequality. For the second inequality we mimic this argument, instead applying an Anti-Delaunay triangulation. \hfill \square 

It turns out that from the above result, we can derive a very nice geometric corollary. First, we need two small lemmas.

Lemma 3.1. Let $P$ be a convex polygon with seven or more vertices and let $T(P)$ be a triangulation of $P$. Then, there exists a diagonal of our triangulation such that, if we apply a decomposition of $P$ using this diagonal, then both $P_1$ and $P_2$ have four or more vertices.

This result is clear, and follows immediately by an induction argument on the number of vertices.

Remark 3.2. It is obvious that this result does not hold if $n = 6$. In fact, it is easy to find a convex polygon whose Delaunay Triangulation does not satisfy Lemma 3.1, hence the need for one more lemma.

Lemma 3.2. Let $P$ be a generic convex polygon with six vertices and let $P_1$ and $P_2$ be the resulting polygons of a decomposition. Then

$$s_-(P) \geq s_-(P_1) + s_-(P_2) - 2$$

and

$$s_+(P) \geq s_+(P_1) + s_+(P_2) - 2.$$ 

Proof. Since we have no guarantee that our diagonal is Delaunay, we cannot mimic the proof of Theorem 3.1. We observe that, since $P$ is generic, $P_1$ and $P_2$ are generic as well. Moreover, $P_1$ and $P_2$ are quadrilaterals. By applying Proposition 2.3 to $P_1$ and $P_2$, we prove our assertion. \hfill \square 

Corollary 3.1 (The Global Four-Vertex Theorem). Let $P$ be a generic convex polygon with six or more vertices. Then

$$s_+(P) + s_-(P) \geq 4.$$ 

Proof. We will prove the result by induction on the number of vertices of $P$. We first consider the base case $n = 6$, noticing if we apply a decomposition to $P$, then $P_1$ and $P_2$ are both quadrilaterals. By Proposition 2.3 we obtain that $P_1$ and $P_2$ each have four globally extremal vertices. It follows from Lemma 3.2 that $P$ has four globally extremal vertices.

We now consider the case where $n \geq 7$. We begin by applying a Delaunay triangulation to $P$. By Lemma 3.1 it follows that there exists a diagonal $d$ such that when we decompose $P$ by this diagonal, $P_1$ and $P_2$ each have four or more vertices. Since our diagonal corresponds to a Delaunay triangulation, it follows that $d$ is Delaunay. Since $P_1$ and $P_2$ have less vertices than $P$, we apply the inductive assumption to obtain $s_-(P_1) \geq 2$ and $s_-(P_2) \geq 2$. Applying this to Theorem 3.1 we obtain $s_-(P) \geq 2$. An analogous argument using an Anti-Delaunay triangulation and Theorem 3.1 yields $s_+(P) \geq 2$. So $s_+(P) + s_-(P) \geq 4$, proving the assertion. \hfill \square
4. Locally Extremal Vertices and Decomposition of Polygons

When considering locally extremal vertices, it is easy to see that the only vertices affected by a decomposition of a polygon will be the vertices on the diagonal of decomposition and the neighboring vertices.

![Figure 1][1]

This means that we have a total of six vertices impacted by a decomposition, leading us to a feasible case-by-case analysis. Before proving our main result, we need a few lemmas.

**Lemma 4.1.** Let $P$ be a generic convex polygon and $P_1$ and $P_2$ the resulting polygons of a decomposition. Denote the vertices of the diagonal by $B$ and $D$, the neighboring vertex of $B$ in $P_1$ by $A$, and the neighboring vertex of $B$ in $P_2$ by $C$. Assume that $A$ is locally maximal-extremal in $P_1$ but not in $P$, and that $C$ is locally maximal-extremal for $P_2$ but not in $P$. Then, $B$ is a locally maximal-extremal vertex for $P$.

**Proof.** Let $X$ be the neighbor of $A$ in $P_1$ and $Y$ be the neighbor of $C$ in $P_2$. Denote the circle passing through vertices $A$, $B$ and $C$ by $C_B$, the circle passing through vertices $X$, $A$ and $B$ by $C_A$, and the circle passing through vertices $B$, $C$ and $Y$ by $C_C$. Since $A$ is not maximal-extremal in $P$, it follows that $A$ lies inside the circle $C_C$. By Proposition 2.4, it follows that $Y$ lies outside of the circle $C_B$. Since $C$ is not maximal-extremal in $P$, it follows that $C$ lies inside the circle $C_A$. By Proposition 2.4 it follows that $X$ lies outside of the circle $C_B$. The following figure illustrates the situation:
Since both $X$ and $Y$ lie outside of the circle $C_B$, $B$ is maximal-extremal in $P$.

**Lemma 4.2.** Let $P$ be a generic convex polygon and $P_1$ and $P_2$ the resulting polygons of a decomposition. Denote the vertices of the diagonal by $B$ and $D$, the neighboring vertex of $B$ in $P_1$ by $A$, and the neighboring vertex of $B$ in $P_2$ by $C$. Assume that $A$ is locally maximal-extremal in $P_1$ but not in $P$, and that $B$ is locally maximal-extremal in $P_2$. Then, $B$ is locally maximal-extremal in $P$.

**Proof.** For simplicity, consider the following figure, which will illustrate our configuration of points and circles:

Let $X$ be the neighbor of $A$ in $P_1$ and $Y$ be the neighbor of $C$ in $P_2$. Denote by $C_A$ the circle
passing through vertices $X$, $A$, and $B$. Since $A$ is maximal-extremal in $P_1$, it follows that $D$ lies outside of the circle $C_A$. Since $A$ is not maximal-extremal in $P$, it follows that $C$ must lie inside the circle $C_A$. Now, denote the circle passing through vertices $A$, $B$, and $C$ by $C_B$. Our goal is to show that vertices $X$ and $Y$ lie outside of the circle $C_B$.

A quick application of Proposition 2.4 to points $X$, $C$, $A$ and $B$ yields that $X$ lies outside of $C_B$, so we need to show that $Y$ lies outside of the circle $C_B$. Denote by $C_B'$ the circle passing through the points $C$, $B$ and $D$. We will show that if $Y$ lies outside of $C_B'$, then it lies outside of $C_B$. To do this, we first must show that $A$ lies inside the circle $C_B'$.

Consider the circles $C_A$ and $C_B'$. These circles intersect at two points, point $B$ and some other point, say $Z$. Since $D$ lies outside of the circle $C_A$, it follows by an application of Proposition 2.4 to points $A$, $D$, $B$ and $Z$ that $A$ lies inside the circle $C_B'$.

Lastly, consider the circles $C_B$ and $C_B'$. These two circles intersect at the points $B$ and $C$. Since $A$ lies inside the circle $C_B'$, it follows from applying Proposition 2.4 to points $A$, $D$, $B$ and $C$ that $D$ lies outside of the circle $C_B$.

Now, since $B$ is maximal-extremal in $P_2$, it follows that $Y$ lies outside of $C_B'$. By our above observation, it follows immediately from Proposition 2.4 that $Y$ lies outside of $C_B$. Since points $X$ and $Y$ both lie outside of the circle $C_B$, it follows that $B$ is maximal-extremal in $P$. \hfill $\square$

**Lemma 4.3.** Let $P$ be a generic convex polygon and $P_1$ and $P_2$ the resulting polygons of a decomposition. Denote the vertices of the diagonal by $B$ and $D$, the neighboring vertex of $B$ in $P_1$ by $A$, and the neighboring vertex of $B$ in $P_2$ by $C$. Assume that $A$ is locally maximal-extremal for $P_1$ and $D$ is locally maximal-extremal for both $P_1$ and $P_2$, but not for $P$. Then $A$ is locally maximal-extremal for $P$.

**Proof.** Let $X$ be the neighbor of $A$ in $P_1$, $E$ be the neighbor of $D$ in $P_1$, and $F$ be the neighbor of $D$ in $P_2$. Denote by $C_{D_1}$ the circle passing through vertices $B$, $D$ and $E$, by $C_{D_2}$ the circle passing through vertices $B$, $E$ and $F$, and by $C_A$ the circle passing through vertices $X$, $A$ and $B$. The following figure illustrates our configuration:

![Figure 4](image-url)
Our goal is to show that vertex $C$ lies outside of the circle $C_A$. We will do this by showing that if $C$ lies outside the circle $C_D$, then it also lies outside of circle $C_A$. Since $A$ is maximal-extremal in $P_1$, it follows that $D$ lies outside of $C_A$. Since $D$ is maximal-extremal in $P_1$, it follows that $A$ lies outside of circle $C_D$. By a similar argument used in the previous lemma, it follows that if $C$ lies outside of $C_D$ then it lies outside of $C_A$. The following figure illustrates this situation:

It remains to show that $C$ lies outside of $C_{D_1}$. Consider the circles $C_{D_1}$ and $C_D$. Since $D$ is maximal-extremal in $P_2$, it follows that $C$ lies outside of the circle $C_{D_2}$. If we show that $C$ also lies outside of $C_{D_1}$, then we are done. To do this, we will heavily use the fact that $D$ is not maximal-extremal in $P$. We will show that if $E$ lies inside the circle $C_{D_2}$ or if $F$ lies in $C_{D_1}$, then $D$ is maximal-extremal in $P$, contradicting our assumption.

It is enough just to check this for $E$. Denote the circle passing through vertices $E$, $D$ and $F$ by $C_F$. If $E$ lies inside the circle $C_D$, then applying Proposition 2.4 to points $E$, $D$, $F$ and $B$ yields that $B$ lies outside of the circle $C_D$. Similarly, it follows by Proposition 2.4 that $F$ lies inside the circle $C_{D_1}$.

Now denote by $E'$ the neighbor of $E$ and by $F'$ the neighbor of $F$. The following figure illustrates the situation:
Since $D$ is maximal-extremal in $P_1$, it follows that $E'$ lies outside of the circle $C_{D_1}$. Similarly, since $D$ is maximal-extremal in $P_2$, it follows that $F'$ lies outside the circle $C_{D_2}$. Now, recall that $B$ lies outside of the circle $C_D$. Proposition 2.4 applied to points $E'$, $B$, $D$ and $D$ tells us that $E'$ lies outside of $C_D$. A similar argument yields that $F'$ also lies outside of $C_D$. So, we obtain that $D$ is maximal-extremal in $P$, a contradiction.

So now we know that $E$ must lie outside of the circle $C_{D_2}$. Proposition 2.4 applied to points $F$, $E$, $B$ and $D$ now tells us that $F$ lies outside of the circle $C_{D_1}$. So, if $C$ were to lie outside of circle $C_{D_2}$, then it would also lie outside of the circle $C_{D_1}$. But earlier we showed that if $C$ would lie outside of circle $C_{D_1}$, then $C$ would lie outside of the circle $C_A$. Indeed, by assumption, $C$ lies outside of $C_{D_2}$ and hence outside of $C_A$. Since $A$ is maximal-extremal in $P_1$, it also follows that $X$ lies outside of the circle $C_A$. Therefore $A$ is maximal-extremal in $P$. \hfill \Box

**Theorem 4.1.** Let $P$ be a generic convex polygon with at least 6 vertices and let $P_1$ and $P_2$ be the resulting polygons of a decomposition. Then

$$l_-(P) \geq l_-(P_1) + l_-(P_2) - 2.$$ 

**Proof.** We note that only six vertices are affected by a decomposition from the local point of view: the vertices of the diagonal and the neighbors of those vertices. So, we eliminate the cases which violate our inequality. It is easy to check that by the symmetry of our cases, we only need to check three:

*Case 1:* We gain two maximal-extremal vertices in $P_1$, as well as $P_2$, but none of the six vertices are maximal-extremal in $P$.

*Case 2:* We gain two maximal-extremal vertices in $P_1$ and gain two maximal-extremal vertex in $P_2$, and one of the six vertices is maximal-extremal in $P$.

*Case 3:* We gain two maximal-extremal vertices in $P_1$ and gain one maximal-extremal vertex in $P_2$, and none of the six vertices is maximal-extremal in $P$. 

Figure 6
By checking the possible configurations of vertices in each of the cases, we see that each case admits a configuration which is deemed not feasible by one of the three preceding lemmas.

**Corollary 4.1** (The Local Four-Vertex Theorem). Let $P$ be a generic convex polygon with at least six vertices. Then

$$l_+(P) + l_-(P) \geq 4.$$  

**Proof.** We apply induction on the number of vertices of $P$. For the case where $n = 6$, we know that if we apply a decomposition to $P$, then both $P_1$ and $P_2$ will be quadrilaterals. Proposition 2.3 yields that $l_-(P_1) = l_-(P_2) = 2$. Applying this to Theorem 4.1 completes the proof for this case.

Now, assume that $n \geq 7$. We now apply induction to the smaller polygons $P_1$ and $P_2$ to obtain that $l_-(P_1) \geq 2$ and $l_-(P_2) \geq 2$. We now apply this to Theorem 4.1 to obtain that $l_-(P) \geq 2$. By Proposition 2.1, we obtain that $l_+(P) \geq 2$. Therefore $l_+(P) + l_-(P) \geq 4$, proving the assertion.

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