A Stochastic LQR Model for Child Order Placement in Algorithmic Trading *

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Abstract

Modern Algorithmic Trading (“Algo”) allows institutional investors and traders to liquidate or establish big security positions in a fully automated or low-touch manner. Most existing academic or industrial Algos focus on how to “slice” a big parent order into smaller child orders over a given time horizon. Few models rigorously tackle the actual placement of these child orders. Instead, placement is mostly done with a combination of empirical signals and heuristic decision processes. A self-contained, realistic, and fully functional Child Order Placement (COP) model may never exist due to all the inherent complexities, e.g., fragmentation due to multiple venues, dynamics of limit order books, lit vs. dark liquidity, different trading sessions and rules. In this paper, we propose a reductionism COP model that focuses exclusively on the interplay between placing passive limit orders and sniping using aggressive takeout orders. The dynamic programming model assumes the form of a stochastic linear-quadratic regulator (LQR) and allows closed-form solutions under the backward Bellman equations. Explored in detail are model assumptions and general settings, the choice of state and control variables and the cost functions, and the derivation of the closed-form solutions.

Keywords: child order placement, dynamic programming, LQR, delay cost, spread cost, impact cost, Poisson hits, passive, aggressive, Bellman equation, optimal policy, positive matrix.

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The following terms are frequently used throughout the paper.

| term     | meaning                                                                 |
|----------|-------------------------------------------------------------------------|
| Algo     | an automated trading process based on algorithms                       |
| Profile  | an intraday time series derived from history, e.g., for volumes         |
| IS       | implementation shortfall - a popular optimization-based Algo            |
| TWAP     | time-weighted average price - a standard profile-based Algo             |
| VWAP     | volume-weighted average price - a standard profile-based Algo           |
| COP      | child order placement                                                  |
| DP       | dynamic programming                                                    |
| LQR      | linear quadratic regulator with linear transitions & quadratic costs   |
| (N)BBO   | (national) best bid and offer of a venue or market system               |
| LOB      | limit order book, with buy/sell orders displayed                        |
| SOR      | smart order router/routing                                             |
| FX       | foreign exchanges                                                      |
| ADV      | average daily volume                                                   |
| Passive Touch | best bid for buy and best offer for sell                              |
| Aggressive Touch | best offer for buy and best bid for sell                               |
| Near Touch | same as Passive Touch                                                 |
| Far Touch   | same as Aggressive Touch                                              |
| Takeout or Sniping | aggressive market order at the far touch                             |
| TMV      | true market value                                                      |
| Positive Matrix | a symmetric real matrix with positive eigenvalues                  |

The following symbols have been consistently used in the paper.

| symbol     | meaning                                                                 |
|------------|-------------------------------------------------------------------------|
| $t_n$      | a discrete action time in dynamic programming                           |
| $X_n^\pm$  | the right/left limit of a quantity $X$ at $t_n$, via $t_n \pm \varepsilon$ |
| $HS$       | half spread between the BBO                                            |
| $Mid$      | mid price between the BBO                                              |
| $N(\lambda)$ | Poisson distribution with rate $\lambda$                             |
| $q_n$      | outstanding positions right before $t_n$; a state variable            |
| $\lambda_n$ | Poisson hitting rate right before $t_n$; a state variable           |
| $u_n$      | aggressive market orders at $t_n$; the control variable               |
| $\eta$    | market impact of aggressive orders on passive fills                    |
| $\gamma_n$ | delay cost penalty at $t_n$; also expressing risk aversion            |
| $x_n$      | state variable penalty right before $t_n$; $x_n = (q_n, \lambda_n)$   |
| $V_n(x_n)$ | the value function at $t_n$ in dynamic programming                    |
1 Introduction to Child Order Placement (COP)

Small retail orders can be filled by simple vanilla market or limit orders. There is no need for automated algorithmic trading (or “Algos”) involving sophisticated strategies. Algos are primarily designed to execute sizable institutional orders from portfolio managers or traders in various funds or broker-dealers. In the current work, the term “Algos” is restricted to the automated execution service provided to either external or internal clients by the buy side, sell side, or specialized execution agents.

A typical single-name Algo is stratified to at least two distinguishable layers, which we shall refer to as “macro” and “micro” layers or (sub-)Algos.

(A) At the macro layer, a big parent order is sliced into smaller child orders over a series of time buckets, e.g., 5-minute intervals, which usually depend on the liquidity profiles of a security.

(B) The micro layer handles the execution of the resulted child orders. The actual implementation and architecture could vary significantly among broker-dealers and execution agents, and constitute into the very proprietary core of all Algos.

Most well-known Algos in the industry are named after the macro layers, e.g., TWAP, VWAP, or Implementation Shortfall (IS). These macro Algos are either configured based on historical benchmarks or optimized under proper utility objectives. The optimization techniques involve either static or dynamic frameworks, as in these sample works [1, 3, 4, 6, 7, 9, 10, 11]. Overall, the macro Algos are built upon the macro behaviors of the targeted securities, including for instance, the historical profiles of volumes, volatilities, spreads, etc, as in our earlier works in [9, 10, 11]. They do not act upon the real-time micro structure signals such as the dynamics of limit order books (LOB).

The complexities of these Algos mainly reside within the micro layers. The actual real-time placement of orders on various venues is implemented at this layer, and different Algo providers may take very different approaches. For example, two offerings of the same VWAP Algo could differ significantly in terms of architecture and logic.

In general, a micro Algo must handle actions like the following:

(a) a dynamic decision flows for the placement actions and monitoring of their status,

(b) allocation among different order types offered by all accessible venues, and

(c) real-time routing to all accessible liquidity venues, including both lit and dark venues.

In particular, the last component often assumes its own identity in most broker-dealer Algo offerings, and is called the SOR - Smart Order Router. SORs are vital for some liquid asset classes with highly fragmented markets, e.g., the common stocks in the USA.

In terms of modeling techniques, macro Algos are either configured or scheduled using benchmark profiles such as TWAP or VWAP, or optimized using proper utility functions (e.g., the mean-variance framework of Almgren-Chriss [2]; also see Shen et al. [9, 10, 11]). Modeling of micro Algos is much more challenging due to the aforementioned multiple tasks. The main complexities inherent to market microstructures include venues, sessions, order types, and their optimal real-time management. SOR is part of this grand effort and probably the most well known or actively marketed by Algo providers. But the SOR alone is only the last segment of the placement stream, and is actually irrelevant for single-venue securities such as commodities futures or some FX products.

In this paper, we attempt to develop a COP model for a single-venue security. Hence SOR is out of the scope. Instead, the primary focus is on how to dynamically place and manage aggressive market orders and passive limit orders. The two order types compete in the following manner.
(i) Aggressive market orders get filled fast and hence help accomplish order completion, but at the cost of paying a half spread (with respect to the market mid price) and information leakage that could harm trailing orders.

(ii) Passive limit orders gain a half spread if filled, but are subject to fill uncertainty and an extra "chasing" cost if the market drifts away. In general, they jeopardize order completion.

A COP Algo in this context must optimally manage the interplay between the two order types, in order to achieve the two primary objectives:

(a) lower total cost for filled orders, and

(b) a higher completion rate.

The current work rolls out as follows. After making some reductionism assumptions, we first build the COP model based on stochastic dynamic programming (SDP). The solution to the resultant stochastic linear-quadratic regulator (LQR) problem is then worked out via the associated Bellman equations. Theoretical assumptions or practical implications are always discussed in detail along the way. The main result is summarized into Theorem 1 in Section 4.

We believe this is the first self-contained and mathematically rigorous COP model that has a closed-form solution.

2 Basic Model Settings and Scopes

2.1 From Macro to Micro

Throughout the rest, in order to gain a more tangible sense of all the model settings, we assume a concrete working parent order with the following attributes.

(i) It is a “buy” order for a common stock, say.

(ii) The total quantity is $Q = 100,000$ shares.

(iii) The average daily volume (ADV) is 2,000,000 shares, based on a monthly rolling window.

(iv) The client prefers the order to be completed during the horizon from 12:00 pm to 3:00 pm EST in the US equity market (which is however consolidated into a single-venue market to avoid SOR).

Of course none of these example order details actually puts restrictions on the proposed COP model. For example, the model is applicable to other liquid asset classes such as futures and rates.

At the macro level, Algos such as TWAP, VWAP and IS "slice" the parent order into smaller child orders over a series of time buckets. For illustration, assume that such an Algo works with 5-minute time buckets. Then the target 3-hour execution horizon requested by the client is split into 36 time buckets. Further assume that this macro Algo decides to allocate 2,500 shares or 25 lots to the specific time bucket [1:00pm, 1:05pm]. In practice these time knots can all be randomized for anti-gaming.

If the time buckets of the macro Algo are relatively big, e.g., 5 minutes, a micro-layer scheduler can be further designed to schedule the allocated 2,500 shares, say, over finer micro bins. At this layer, sophisticated optimization may be spared in order to save time. In general, an equal partition, i.e., allocation based on TWAP, can be applied. Take the above working example for instance. The 25 lots allocated for the macro time bucket [1:00pm, 1:05pm] can be further split
to 5 lots over each micro bins of 1 minute, i.e., [1:00pm, 1:01pm], [1:01pm, 1:02pm], etc. Again randomization of time knots should also be applied.

The introduction of finer micro time bins is necessary for most scenarios. This is because the macro time buckets must last long enough so that bucket signals are statistically meaningful. Take the VWAP or IS Algo for example. These macro Algos all depend on the temporal profiles of security volumes, volatilities, or spreads (e.g., Shen et al. [9, 10, 11]). When the macro buckets are too brief, the profile values based on historical averaging or filtering would be too noisy and result in unreliable Algo scheduling at the macro layer. In general, bucket size should be optimized or adapted to the liquidity profiles of a given security.

The current work takes a reductionism approach and attempts to develop a self-contained rigorous COP model at the micro bin level. That is, the model handles the actual execution of individual child (or grandchild) orders over micro bins, e.g., 5 lots over [1:00pm, 1:01pm] for the above running example. It is a fully automated model based on the framework of stochastic linear-quadratic regulators (LQR) and allows closed-form solutions.

2.2 Reductionism and Value of the Current Work

In our previous two works on macro Algos:

- Shen [9] on a generic pre-trade macro Algo based on static quadratic programming in Hilbert spaces, and
- Shen [10] on a real-time adaptive macro Algo based on dynamic programming that integrates the VWAP and IS Algos,

the proposed Algo models can actually be implemented in execution houses after proper model calibration is performed using their proprietary trading data.

Hence it is important to point out at the outset that the current micro Algo is more a theoretical model in the spirit of reductionism. As explained earlier, it is almost impossible to have a single self-contained model to comprehensively handle the entire micro-layer execution. For instance, it is nontrivial to handle unexpected intraday trading halts or participation in various auctions. The main reductionism of the proposed model involves the following aspects.

(a) It is restricted to single-venue executions and does not involve SOR modeling.

(b) It deals with neither lit vs. dark venues nor complex order types (e.g., icebergs or mid-pegging).

(c) It only handles the two most basic order types - limit and market orders.

Notice that in professional trading one almost never sends a “naked” market order. Hence by market orders we mean more precisely marketable limit orders whose limits cross the far touch.

Despite the reductionism, the value of the current work can be summarized as follows.

- To the academic community, to our best knowledge this is the first rigorous COP model at the micro layer, which is self-contained and allows closed-form solutions. It opens the door to more sophisticated or realistic COP models in the future.

- To the Algo practitioners on Wall Street, the model does reveal the intriguing dependency and competition among different key Algo components: aggressive sniping, passive waiting, execution cost, information leakage, market impact, and the requirement of completion. The modeling techniques here can always be tweaked to facilitate existing COP processes.
2.3 The Placement Problem to be Modelled

Recall earlier in this section, after macro bucketing and micro binning, one ends up with a COP problem like the following:

- to buy 5 lots over a micro bin \([1:00\text{pm}, 1:01\text{pm}]\),
- or more generally, to buy \(q\) lots over a micro bin \([T_0, T_1]\) with a brief duration \(\Delta T = T_1 - T_0\) of 30 or 60 seconds or so.

The COP problem modelled herein is formulated as follows. A given micro bin \([T_0, T_1]\) is partitioned into \(N\) action times:

\[
t_0 = T_0 < t_1 < \ldots < t_{N-1} < t_N = T_1.
\]

In practice, one could choose periodic knots, say, \(\tau = 5\) or 10 seconds, and

\[
t_{n+1} = t_n + \tau, \quad n = 0, 1, \ldots, N - 1.
\]

Actual implementation could also have them randomized for anti-gaming. Since the micro bin duration \(\Delta T = T_1 - T_0 = T_N - t_0\) is brief, it is also assumed that the BBO, i.e., the best bid and offer, remain unchanged over the given micro bin. (This is introduced merely for convenience. In reality, the BBO can change realistically in the current model as long as one assumes that the benchmark market price is the moving mid price and that the limit order placement is cancelled and replaced whenever the BBO move so that it is effectively pegged to the BBO.)

Following the running example introduced earlier, we shall always work with a buy order - to buy \(q_0\) lots over a micro bin \([T_0, T_1]\). For a buy order, we shall introduce the following concepts:

- the passive or near touch - the best bid of the venue, and
- the aggressive or far touch - the best offer of the venue.

For a sell order the other way around holds. Furthermore, let \(q_n\) denote the remaining lots right before \(t \in [t_0, t_N]\). To simplify notation, we introduce the following convention: for any observable, variable or parameter \(X\),

\[
X_n := X_n^- = \lim_{\varepsilon \to 0^+} X_{t-\varepsilon}, \quad \text{and} \quad X_n^+ = \lim_{\varepsilon \to 0^+} X_{t+\varepsilon}.
\]

For a continuous \(X\) there is no difference among the three. For the proposed COP model, a sniping action (or a control in the context of dynamic programming) will take place right at a given action time \(t_n\), and hence they could differ.

**Attention - We have defaulted \(X_n\) to \(X_n^-\) to simplify notation and equation lines.**

The proposed COP model adopts the following action plan.

(i) At any time \(t\), as long as \(q_n > 0\), a single-lot limit order will be placed at the passive touch.

(ii) Whenever such a limit order is filled at \(t\) and the remainder \(q_t^+ = q_t - 1\) is still positive, a new single-lot limit order will be immediately placed at the passive touch.

(iii) At each action time \(t_n\) from

\[
t_0, \ldots, t_n, \ldots, t_{N-1},
\]

if \(q_n\) (which represents \(q_n^-\)) still has some positive lots to trade, the COP Algo has an option to send a single-lot market order at the far touch. We shall nickname this action by “single-lot sniping” or simply “sniping.” The term “Sniper” or “Sniping” has been popularly used in the Algo world, e.g., the Sniper Algo of Credit Suisse in their 2007 article of Reuters (with an active URL link in PDF).
The major three characteristics of this target COP problem are: order completion, passive waiting, and aggressive sniping, which are elaborated as follows.

**The Constraint on Completion** Order completion is usually enforced at the macro layer. It is either explicitly formulated into optimization as a constraint (e.g., for the IS Algo) or enforced through hard scheduling (e.g., for TWAP/VWAP Algos). For micro-layer execution, e.g., executing 5 lots within a micro bin of 30 or 60 seconds, completion can be soft or delayed, in order to better dance with the liquidity waves in the market. The unfilled can be handed over to the next micro bin, and so on so forth. The macro Algo on the top usually deploys a dedicated schedule “keeper” to enforce the schedules.

**Passive Limit Order at the Near Touch** Recall that for convenience, the BBO have been assumed invariant over a brief micro bin \([T_0, T_1]\). Let

\[
HS = \frac{1}{2}(BestOfferPrice - BestBidPrice)
\]

denote the half spread, and

\[
Mid = \frac{1}{2}(BestOfferPrice + BestBidPrice)
\]

the unbiased mid price that represents the true market value (TMV) at the moment. Compared with other more sophisticated averaging schemes, this is usually called the simple mid. Once filled, a limit order saves a half spread \(HS\) compared with the TMV. However, if left unfilled by the target end time, limit orders can jeopardize order completion. A reasonable COP model must be able to reflect this tradeoff.

**Aggressive Sniping at the Far Touch** Market or marketable orders take out liquidity from the top of the opposite LOB, i.e., the far touch. At the micro layer, market orders are always kept small, e.g., a couple of lots on average for major exchanges in US. Hence the proposed model always assumes that such small orders are filled instantaneously. Market orders fill fast and help achieve completion, but at the cost of a half spread \(HS\). Furthermore, aggressive market orders could also leak information and result in opposite market participants or market makers biasing their perceived TMV towards the aggressive touch. As a result, they become less willing to take out “our” limit orders posted at the passive touch. A reasonable COP model must be able to demonstrate such tradeoffs as well.

### 3 The COP Model Based on Dynamic Programming

#### 3.1 Model and Process Assumptions

We now introduce the basic assumptions for the model and process.

**Assumptions on Limit Order Placement** For the placement of limit orders, the following basic assumptions are made.

(A) The size of the limit order is always a single lot.

(B) A single-lot limit order is placed initially at \(t_0\).
(C) Afterwards, whenever the limit order is taken out by an opposite market order at time $t$, a new single-lot limit order is immediately replenished as long as the remaining position $q_n^{-} = q_n - 1$ is still positive.

(D) For any time interval of duration $\Delta t$, as long as “we” do not snipe at the aggressive touch within the interval, the number $W$ of single-lot limit orders being hit is subject to a Poisson distribution $N(\lambda \Delta t)$ with some rate $\lambda$. That is

$$\text{Prob}(W = n) = e^{-\lambda \Delta t} \frac{(\lambda \Delta t)^n}{n!}, \quad n = 0, 1, \ldots$$

(1)

It is also assumed that once being hit the entire lot is taken. Poisson distribution or process is not unfamiliar in the context of Algo trading and limit order books [8].

Assumptions on the Aggressive Market Orders For sniping using aggressive market orders, e.g., sending a marketable order $u_n$ at time $t_n$ at the far touch, the following assumptions are made.

(a) The ideal goal is to set $u_n$ to either a single lot or zero (i.e., no sniping), so that information leakage and market impact can be curbed. In reality, to facilitate a tractable dynamic programming (DP) formulation with closed-form solutions, the single lot constraint is not explicitly imposed. The cost function designed later will naturally encourage $u_n$ to stay small.

(b) It is also assumed that once an aggressive market order $u_n$ is sniped, the entire order will be filled. Since the cost function in general keeps $u_n$ small, this assumption holds naturally for most liquid securities.

(c) Since during the short duration of a micro bin, the BBO are assumed to be invariant, we adopt the following model for information leakage caused by an aggressive sniping at time $t$. Once $u_n$ lots are taken out (by “us”) at the far touch, other market participants, including in particular market makers, will update their belief on the TMV and bias it towards the far touch. As a result, either fewer opposite participants are willing to snipe at the passive touch or market makers will also cancel their more passive inside limit orders and replace with new ones at the passive touch.

The latter will congest the queue at the passive touch. Hence heuristically both will reduce the chance of “our” limit orders being hit by the market.

Quantitatively, we assume that the Poisson hitting rate introduced in Eqn. (1) will be negatively impacted by the following linear form:

$$\lambda_n^+ = \lambda_n - \eta u_n,$$

(2)

after $u_n$ lots are sniped and filled at time $t_n$. Here $\eta > 0$ is a model parameter that calibrates the rate of information leakage or market impact caused by aggressive sniping. In general, it should depend on the liquidity profile of a given security.

For instance, assume $\lambda = 5.0$ lots per minute, and $\eta = 0.5$ per lot per minute. Then a sniping of $u_n = 2$ lots at some time $t_n$ will reduce the Poisson hitting rate according to:

$$\lambda_n^+ = \lambda_n - \eta u_n = 5.0 - 0.5 \times 2 = 4.0.$$

The impact parameter $\eta$ can be calibrated or estimated using experimental orders that are designed specifically for this purpose.
3.2 State and Control Variables, and State Transition

To develop the dynamic programming (DP) COP model, we first define the state variables and their transition.

There are two state variables, \( q_n \) and \( \lambda_n \), or organized into a state vector \( \mathbf{x}_n = (q_n, \lambda_n)^T \), where the superscript \( T \) denotes transposition of vectors or matrices.

- State variable \( q_n \) denotes the outstanding lots still needed to be traded right before \( t_n \). The symbol is equivalent to \( q_n^- \) but with the minus superscript omitted, as set up earlier.
- State variable \( \lambda_n \) denotes the Poisson hitting rate right before \( t_n \). It represents the rate the opposite aggressive orders hit “our” limit orders. It changes whenever “we” snipe a market order \( u_n \) at the far touch, due to information leakage and its digestion by other market participants.

The control variable is the aggressive takeout order \( u_n \) that “we” snipe at \( t_n \) at the far touch. Furthermore, let \( W_n \) denote the Poisson random number of “our” limit orders being hit by the opposite aggressive orders. Then from \( t_n \) to \( t_{n+1} \) (or more precisely \( t_{n+1}^- \)), the following state transition equations hold.

\[
q_{n+1} = q_n - u_n - W_n, \\
\lambda_{n+1} = \lambda_n - \eta u_n.
\]

In the vector form, define
\[
\mathbf{a} = (1, \eta)^T, \quad \mathbf{z}_n = (W_n, 0)^T.
\]

Then the state vector transits as follows:
\[
\mathbf{x}_{n+1} = \mathbf{x}_n - \mathbf{a}u_n - \mathbf{z}_n.
\]

3.3 The Stage Cost

All variables are assumed to be continuous, as normally done in the Algo literature. (In reality trades are mostly in whole shares or lots.)

Following all the previous preparation, the COP problem is formulated as a DP problem with controls taken at one of the following action times:

\[ t_n \in \{t_0, t_1, \ldots, t_{N-1}\}. \]

No action is taken at the terminal time knot \( t_N \). At each \( t_n \), we first define the initial candidate \( j_n^{(1)} \) for the stage cost, which is still subject to revision later on:

\[
j_n^{(1)}(u_n, W_n \mid \mathbf{x}_n) = \gamma_n q_n^2 + u_n - W_n.
\]

It is explained as follows.

(a) The first term \( \gamma_n q_n^2 \) favors fast execution so that \( q_n \)'s quickly touch down to zero. It appears in earlier works for both static and dynamic macro Algos, e.g., Algren-Chriss [2], Hora [6], and Shen [9, 10], just to name a few. The penalty coefficient \( \gamma_n \)'s are the control parameters to penalize trading delays. In general, \( \gamma_n \) can be set in proportion to the real-time variance \( \sigma_n^2 \) of the security, i.e., in the form of \( \gamma_n = \tilde{\gamma}_n \sigma_n^2 \). In addition, \( \gamma_n \) should increase monotonically to facilitate order completion. For instance, \( \gamma_N = +\infty \) would enforce hard completion: \( q_N = 0 \).
(b) \( W_n \) is the number of single-lot limit orders being taken out by opposite market orders at “our” passive touch. Limit orders save a half-spread \( HS \). We incorporate the scaling of \( HS \) into \( \gamma_n \) so that \(-HS \cdot W_n\) can be simplified to \(-W_n\). Following the general Poisson setting in Eqn. (1), we assume more specifically that \( W_n \) is subject to the Poisson distribution \( N(\lambda_n + n \Delta t_n) \) with

\[
\lambda_n^+ = \lambda_n - \eta u_n, \quad \text{and} \quad \Delta t_n = t_{n+1} - t_n.
\]

(c) \( u_n \) is the number of lots that “we” snipe at the aggressive touch at the action time \( t_n \). It pays the cost of a half spread, i.e., \( HS \cdot u_n \). Since \( HS \) is scaled into \( \gamma_n \), it is simply expressed as \( u_n \) in the stage cost.

For a buy order, \( u_n \) is preferably nonnegative. In order to facilitate a close-form solution, however, we do not explicitly impose the constraint of \( u_n \geq 0 \). When \( u_n < 0 \), we shall interpret it as an aggressive sell order of size \(-u_n\) at the current passive touch. If this is the case, the update position will increase:

\[
q_n^+ = q_n - u_n > q_n.
\]

The first term \( \gamma_n q_n^2 \) will discourage such opposite trades as long as the risk aversion weights \( \gamma_n \)’s are not negligible.

On the other hand, compared with the market mid price, selling at the passive touch also incurs a cost of a half spread, i.e, \( HS \cdot (-u_n) \). Hence when \( u_n \) is allowed to be signed, the stage cost in Eqn. (6) should at least be revised to:

\[
j_n^{(2)}(u_n, W_n | x_n) = \gamma_n q_n^2 + |u_n| - W_n.
\]

Since we intend to design a DP model with a closed-form solution, the absolute value is further revised to a squared form:

\[
j_n(u_n, W_n | x_n) = \gamma_n q_n^2 + u_n^2 - W_n, \quad (7)
\]

which is the final stage cost adopted for the current model.

When \( u_n \) stays close to a single lot, \( u_n^2 \simeq |u_n| \). For \(|u_n| > 1\), this quadratic form penalizes big sizes even heavier than the linear form. It favors smaller aggressive order sizes as a result.

Inspired by the \( \Gamma \)-convergence theory and its application in multi-phase variational problems [5], one could also introduce the double-well cost function:

\[
\frac{u_n^2(1-u_n)^2}{\varepsilon}, \quad \text{with} \quad 0 < \varepsilon \ll 1.
\]

This will softly enforce the binary sniping behavior - either no action with \( u_n = 0 \) or sniping with a single lot \( u_n = 1 \). However, such high order non-convex costs can completely thwart the effort of designing a DP model with a unique and close-form solution.

**Summary.** The stage cost model in Eqn. (7) and the state transition model in Eqn. (5) define the proposed stochastic dynamic programming model for child order placement (COP).

### 4 Bellman Equations and Optimal Solutions

#### 4.1 Value Functions \( V(x) \)

At any “current” action time \( t_n \) with state variable \( x_n = (q_n, \lambda_n)^T \), for any choice of policy or action sequence \( u_n = (u_n, \ldots, u_{N-1}) \) at \((t_n, \ldots, t_{N-1})\), let \( W_n = (W_n, \ldots, W_{N-1}) \) denote the resulting
Poisson hits on “our” passive limit orders during individual intervals \([t_k, t_{k+1})\)’s. Then the future cost is given by

\[
J_n(u_n, W_n \mid x_n) = J_n(u_n, \ldots, u_{N-1}; W_n, \ldots, W_{N-1} \mid x_n) \\
= j_n(u_n, W_n \mid x_n) + \ldots + j_{N-1}(u_{N-1}, W_{N-1} \mid x_{N-1}) + j_N(x_N) \\
= j_n(u_n, W_n \mid x_n) + J_{n+1}(u_{n+1}, W_{n+1} \mid x_{n+1})
\]

(8)

Here \(j_N(x_N)\) denotes the terminal cost at \(t_N\). Define the value function by:

\[
V_n(x_n) = \inf_{u_n} E_{W_n} [J_n(u_n, W_n \mid x_n)],
\]

(9)

ranging over all state-driven policies in the form of \(u_k = \phi_k(x_k)\), \(k = n, \ldots, N - 1\). Then we have the Bellman equation at each action time \(t_n\):

\[
V_n(x_n) = \inf_{u_n} E_{W_n} [j_n(u_n, W_n \mid x_n)] + E_{W_n} [V_{n+1}(x_{n+1})].
\]

(10)

The terminal cost at \(t_N\) is defined to be

\[
V_N(x_N) = j_N(x_N) = \gamma_N q_N^2.
\]

(11)

The other two terms are dropped out since the end of the given micro bin is reached.

Next, we shall work out first the solution for the last period \([t_{N-1}, t_N)\) to gain some tangible knowledge, and then the general solution in the framework of stochastic LQR.

### 4.2 Solution at the Last Period

At action time \(t_{N-1}\) for the last period \([t_{N-1}, t_N)\), the Bellman equation reads:

\[
V_{N-1}(x_{N-1}) = \inf_{u_{N-1}} \gamma_{N-1} q_{N-1}^2 + u_{N-1}^2 - \lambda_{N-1}^+ \Delta t_{N-1} + \gamma_{N} E_{W_{N-1}} [(q_{N-1} - u_{N-1} - W_{N-1})^2].
\]

For clarity in calculation, we drop the subscript \(N - 1\) so that it becomes

\[
V(x) = \inf_u f(u \mid x) = \inf_u \gamma q^2 + u^2 - \lambda^+ \Delta t + \gamma_{N} E_W (q - u - W)^2,
\]

with the current state vector \(x = (q, \lambda)^T\) which is known, and \(\lambda^+ = \lambda - \eta u\). The problem becomes the minimization of a single-variate function \(f(u \mid x)\) given \(x\).

Since \(E[W] = \lambda^+ \Delta t\), and \(E[X^2] = \text{Var}(X) + E[X]^2\) for a generic random variable \(X\), one has

\[
E(q - u - W)^2 = \text{Var}(q - u - W) + (q - u - \lambda^+ \Delta t)^2 \\
= \text{Var}(W) + (q - u - \lambda^+ \Delta t)^2 \\
= \lambda^+ \Delta t + (q - u - \lambda^+ \Delta t)^2.
\]

As a result, the derivative of \(f\) is:

\[
\frac{df}{du} = 2u + (\gamma_{N-1}) \Delta t \frac{d \lambda^+}{du} - 2 \gamma_{N} (q - u - \lambda^+ \Delta t)(1 + \Delta t \frac{d \lambda^+}{du}) \\
= 2(1 + \gamma_{N}(1 - \eta \Delta t)^2) u - \eta \Delta t(\gamma_{N} - 1) - 2 \gamma_{N}(1 - \eta \Delta t)(q - \lambda \Delta t).
\]

At the optimal \(u^*_n\), the derivative vanishes. Hence the optimal aggressive trading is given by:

\[
u^*_n = \alpha^*_n + \beta^*_n (q - \lambda \Delta t), \quad \text{with} \quad \alpha^*_n = \frac{\gamma_{N}(1 - \eta \Delta t)}{2(1 + \gamma_{N}(1 - \eta \Delta t)^2)}, \quad \text{and} \quad \beta^*_n = \frac{\gamma_{N}(1 - \eta \Delta t)}{1 + \gamma_{N}(1 - \eta \Delta t)^2}
\]

(12)
In particular, the optimal policy \( u^* = \phi_*(x) = \phi_*(q, \lambda) \) is a linear function of the state variable. Consider the asymptotic case when \( \gamma_N = +\infty \). Then one has

\[
\alpha_* = \frac{\eta \Delta t}{2(1 - \eta \Delta t)^2}, \quad \text{and} \quad \beta_* = \frac{1}{1 - \eta \Delta t}.
\]

In particular, for a highly liquid security so that \( \eta \approx 0 \), the optimal policy is simply

\[
u^* = \alpha_* + \beta_*(q - \lambda \Delta t) \approx q - \lambda \Delta t.
\]

That is, on the last action time \( t_{N-1} \), the proposed COP Algo will trade the expected remaining lots that cannot be filled by the passive limit orders \( W \) (since \( \text{E}[W] = \lambda^+ \Delta t \approx \lambda \Delta t \) when \( \eta \approx 0 \)).

This certainly makes business sense.

In terms of the value function at \( t_{N-1} \), one then has

\[
V(x) = f(u_* | x) = \gamma q^2 + \alpha_*^2 + (\gamma_N - 1)(\lambda \Delta t - \eta \Delta t u_*) + \gamma_N(q - \lambda \Delta t - (1 - \eta \Delta t)u_*)^2
\]

\[
= \gamma q^2 + c_1(q - \lambda \Delta t)^2 + c_2(q - \lambda \Delta t) + c_3,
\]

where

\[
c_1 = \beta_*^2 + \gamma_N(1 - (1 - \eta \Delta t)\beta_*)^2 = \frac{\gamma N}{1 + \gamma N(1 - \eta \Delta t)^2} > 0.
\]

Hence the value function for the last period can be written in the canonical quadratic form as:

\[
V(x) = x^T P x + b^T x + c,
\]

where \( P \) must be a positive definite matrix. This is because by the quadratic portion of \( V(x) \),

\[
\gamma q^2 + c_1(q - \lambda \Delta t)^2 = 0 \Rightarrow q = 0, \lambda = 0.
\]

Next we show that this is not accidental.

### 4.3 General Solution to the Stochastic LQR

In general, for \( n = 0, 1, \ldots, N - 2 \), assume that the value function at \( n + 1 \) is in the quadratic form:

\[
V_{n+1}(x_{n+1}) = x_{n+1}^T P_{n+1} x_{n+1} + b_{n+1}^T x_{n+1} + c_{n+1},
\]

where \( P_{n+1} \) is positive definite. We now show that this implies that

\[
V_n(x_n) = x_n^T P_n x_n + b_n^T x_n + c_n,
\]

where \( P_n \) is also positive definite.

Furthermore, we show that the optimal action is given in the linear form:

\[
u_n = \phi_n(x_n) = \alpha_n + \beta_n^T x_n.
\]

The objective is to derive \( \alpha_n, \beta_n, P_n, b_n \) and \( c_n \) from \( P_{n+1}, b_{n+1} \) and \( c_{n+1} \) recursively.

For clarity, we drop the subscript \( n \) so that for any variable or parameter \( X \), we use instead

\[
X_n \rightarrow X, \quad \text{and} \quad X_{n+1} \rightarrow X_1.
\]
In particular, the value function at \( n + 1 \) now assumes the cleaner form:

\[
V_1(x_1) = x_1^T P_1 x_1 + b_1^T x_1 + c_1. \tag{16}
\]

Also the state transition equation becomes:

\[
x_1 = x - au - z, \quad \text{with} \quad a = (1, \eta)^T, \quad z = (W, 0)^T,
\]

where \( \eta \) is a model parameter or constant that represents the market impact or information leakage and \( W \) is the Poisson random hits on “our” passive limit orders over \([t_n, t_{n+1})\).

Then the value function at \( t_n \) is given by:

\[
V(x) = \min_u f(u \mid x) = \min_u \gamma q^2 + u^2 - \lambda^+ \Delta t + E_W V_1(x_1).
\]

Let \( p_{11} \) denote the \((1,1)\)-element of \( P_1 \), and \( z_E = (\lambda^+ \Delta t, 0)^T = E[z] \). Then

\[
E_W V_1(x_1) = V_1(E_W x_1) + E_W (z - z_E)^T P_1 (z - z_E) = V_1(x - au - z_E) + p_{11} \text{Var}(W)
\]

\[
= V_1(x - au - z_E) + p_{11} \lambda^+ \Delta t.
\]

We now Define

\[
L = \begin{pmatrix} 0 & \Delta t \\ 0 & 0 \end{pmatrix}, \quad J = I_2 - L, \quad \text{and} \quad h = \begin{pmatrix} 1 - \eta \Delta t \\ \eta \end{pmatrix}, \tag{17}
\]

where \( I_2 \) denote the 2 by 2 identity matrix. Then

\[
z_E = \begin{pmatrix} \lambda \Delta t - \eta \Delta tu \\ 0 \end{pmatrix} = Lx - \begin{pmatrix} \eta \Delta t \\ 0 \end{pmatrix} u,
\]

\[
x - au - z_E = Jx - hu.
\]

Hence we have, with \( l_{11} = 1 - p_{11} \),

\[
f(u) = f(u \mid x) = \gamma q^2 + u^2 - l_{11} \lambda^+ \Delta t + V_1(Jx - hu). \tag{18}
\]

Assume that

\[
f(u) = f(0) - Bu + Au^2. \quad \text{Then,} \quad f'(u) = 2Au - B. \tag{19}
\]

On the other hand, direct differentiation gives

\[
f'(u) = 2u + l_{11} \eta \Delta t - h^T \nabla V_1(Jx - hu)
\]

\[
= 2(1 + h^T P_1 h)u - (h^T b_1 - l_{11} \eta \Delta t) - 2h^T P_1 Jx.
\]

Hence we have

\[
A = 1 + h^T P_1 h, \quad \text{and} \quad B = (h^T b_1 - l_{11} \eta \Delta t) + 2h^T P_1 Jx. \tag{20}
\]

Therefore, the optimal policy at \( t_{n-1} \) is given by

\[
u_* = \frac{B}{2A} = \alpha_* + \beta_*^T x, \quad \text{with}
\]

\[
\alpha_* = \frac{h^T b_1 - l_{11} \eta \Delta t}{2(1 + h^T P_1 h)}, \quad \text{and} \quad \beta_*^T = \frac{h^T P_1 J}{1 + h^T P_1 h}. \tag{21}
\]
Next we derive the associated value function \( V(x) = f(u_* \mid x) \). For convenience, for any positive definite matrix \( Q \) and any real vector \( x \) of the same dimension, we define the Q-stretched Euclidean norm by:

\[
\|x\|^2_Q = x^T Q x.
\]

Then the coefficient \( A \) is simply \( 1 + \|h\|^2_{P_1} \).

Since \( 2Au_* = B \), one has \( 2Au_*^2 = Bu_* \). Hence by Eqn. (18),

\[
V(x) = f(0) - Bu_* + Au_*^2
\]

\[
= f(0) - Bu_* + Au_*^2
\]

\[
= \gamma q^2 - l_{11} \lambda \Delta t + V_1(Jx) - A(\alpha_* + \beta_*^T x)^2
\]

\[
= \gamma q^2 - l_{11} \lambda \Delta t + (x^T J^T P_1 J x + b_1^T J x + c_1) - (1 + \|h\|^2_{P_1})(\alpha_* + \beta_*^T x)^2
\]

\[
= \gamma q^2 + x^T (J^T P_1 J x) - (1 + \|h\|^2_{P_1})x^T \beta_* \beta_*^T x
\]

\[
- l_{11} \lambda \Delta t + b_1^T J x - 2(1 + \|h\|^2_{P_1})\alpha_* \beta_*^T x
\]

\[
+ c_1 - (1 + \|h\|^2_{P_1})\alpha_*^2
\]

\[
= x^T P x + b^T x + c,
\]

where the quadratic parameters are given by

\[
P = \gamma \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + J^T Q J,
\]

with \( Q = P_1 - \frac{P_1 hh^T P_1}{1 + \|h\|^2_{P_1}} \),

\[
b^T = b_1^T J - l_{11} \Delta t (0, 1) - 2(1 + \|h\|^2_{P_1})\alpha_* \beta_*^T,
\]

\[
c = c_1 - (1 + \|h\|^2_{P_1})\alpha_*^2.
\]

(22)

where \( l_{11} = 1 - P_1(1, 1) \) and \( \alpha_* , \beta_* \) are given as in Eqn. (21).

We now show that \( P \) is positive definite.

**Lemma 1** If \( P_1 \) is positive definite, so must be \( P \).

**Proof.** Since \( J = I_2 - L \) is non-singular as in Eqn (17), it suffices to show that \( Q \) is positive definite.

For any non-zero vector \( v \in \mathbb{R}^2 \), previously we have used the notation \( \|v\|_{P_1} \) to denote the \( P_1 \)-stretched Euclidean distance. More generally, we use

\[
\langle v, h \rangle_{P_1} := v^T P_1 h
\]

to denote the \( P_1 \)-stretched inner product. By the Cauchy-Schwarz Theorem, one has

\[
\langle v, h \rangle_{P_1} \leq \|v\|_{P_1} \cdot \|h\|_{P_1}.
\]
Then back to $Q$, one has

$$
v^T Q v = v^T \left( P_1 - \frac{P_1 h h^T P_1}{1 + \|h\|_F^2} \right) v
$$

$$
= \|v\|_{P_1}^2 - \frac{\langle v, h \rangle_{P_1}^2}{1 + \|h\|_{P_1}^2}
$$

$$
= \|v\|_{P_1}^2 + \|v\|_{P_1}^2 \|h\|_{P_1}^2 - \langle v, h \rangle_{P_1}^2
$$

$$
\ge \frac{\|v\|_{P_1}^2}{1 + \|h\|_{P_1}^2} > 0.
$$

Since this holds for any non-zero vector $v$, $Q$ and hence $P$ must be positive definite.

We have thus established the following theorem, with $J_n$ and $h_n$ defined as in Eqn. (17):

$$
J_n = \begin{pmatrix} 1 & -\Delta t_n \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad h_n = \begin{pmatrix} 1 - \eta \Delta t_n \\ \eta \end{pmatrix}.
$$

They are constant for equal partitioning when $\Delta t_n = t_{n+1} - t_n$’s are all the same.

**Theorem 1** Let $V_N(x_N) = \gamma_N q_N^2$ be the terminal cost at the ending time $t_N$. Then at each action time $t_n$ with $n < N$, there exist a positive definite $2 \times 2$ matrix $P_n$, a $2 \times 1$ vector $b_n$, a scalar $c_n$, such that the value function $V_n$ is given by:

$$
V_n(x_n) = x_n^T P_n x + b_n^T x_n + c_n = (q_n, \lambda_n)P_n \begin{pmatrix} q_n \\ \lambda_n \end{pmatrix} + b_n^T \begin{pmatrix} q_n \\ \lambda_n \end{pmatrix} + c_n.
$$

(23)

The optimal policy $u_n$ is given by the linear form using parameters at $t_{n+1}$:

$$
u_n^* = \phi_n(x_n) = \alpha_n^* + x_n^T \beta_n^*,$n

where $P_{n+1}(1,1)$ denotes the $(1,1)$-element of $P_{n+1}$. Furthermore, the structure of the value functions also cascades backwards as follows:

$$P_n = \gamma_n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + J_n^T \left( P_{n+1} - \frac{P_{n+1}h_n h_n^T P_{n+1}}{1 + h_n^T P_{n+1} h_n} \right) J_n,$n

$$b_n = J_n^T b_{n+1} - (1 - P_{n+1}(1,1))\Delta t_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 2(1 + h_n^T P_{n+1} h_n)\alpha_n^* \beta_n^*,$n

$$c_n = c_{n+1} - (1 + h_n^T P_{n+1} h_n)(\alpha_n^*)^2,$n

with $n = N - 1, \ldots, 1, 0$, and terminal values $P_N = \text{diag}(\gamma_N, 0), b_N = 0$ and $c_N = 0$. 

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5 Conclusion and Disclaimers

We conclude the current work with the following comments and disclaimers.

(1) The proposed dynamic programming COP Algo had not been the internal or external product of any execution houses where the author worked previously. Any potential industrial conflict or suspected proprietary trespass should be promptly directed to the attention of the author, together with necessary evidences.

(2) In the spirit of reductionism and the pursuit of a dynamic programming COP model with closed-form solutions, the current model does not address other important execution or implementation details, including fragmented venues in the national market system (NMS), different trading sessions and rules, various order types, lit vs. dark, and so on.

(3) The current work focuses exclusively on the dynamic interplay between aggressive takeout orders and passive limit orders. The price improvement or cost is represented by a half spread. Information leakage or the market impact of aggressive orders is reflected in the reduction of Poisson hitting rates on passive limit orders.

(4) Like some earlier DP macro Algos, risk aversion is implemented by the delay cost in the stage cost model. It facilitates soft completion of a given child order over its designated micro time bin. Hard completion or catchup is usually implemented at the macro layer.

(5) If the results here are to be integrated into an existing COP program in an execution house, a practitioner should apply some heuristic but necessary overlays. For instance, $u^*_n \leq 0$ can be interpreted as no sniping, while $u^*_n > 0$ should also be capped by the outstanding position $q_n$.

(6) Overall, the author wishes that the current model could inspire more similar and rigorous works that can improve the heuristic decision trees prevailing in the COP processes in the contemporary Algo industry.

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This work was completed when the pandemic Covid-19 was sweeping through the entire globe mercilessly. Under tremendous mental pressure living in the epicenter of New York, the author is extremely grateful to his family, friends, and colleagues, as well as thousands of courageous and selfless medical professionals, policemen and policewomen, and fire fighters of this great city.

The pandemic has actually brought the people in the city and around the globe much closer and more united, as my 9-year old observes from her numerous Zoom online classes and chats, as well as all the touching stories around the world on fighting against the virus. Beyond A.I. or automated trading “robots” as the current work has covered, the pandemic has grounded all of us to the very core meaning of human beings and human societies. At the end of this darkest storm there will be a brightest rainbow — so colorful, refreshing, and full of new hopes.
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