Basic \textit{N}-Body Modelling of the Evolution of Globular Clusters. I. Time Scaling

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Abstract

We consider the use of $N$-body simulations for studying the evolution of rich star clusters (i.e. globular clusters). The dynamical processes included in this study are restricted to gravitational (point-mass) interactions, the steady tidal field of a galaxy, and instantaneous mass loss resulting from stellar evolution. With evolution driven by these mechanisms, it is known that clusters fall roughly into two broad classes: those that dissipate promptly in the tidal field, as a result of mass loss, and those that survive long enough for their evolution to become dominated by two-body relaxation.

The time scales of the processes we consider scale in different ways with the number of stars in the simulation, and the main aim of the paper is to suggest how the scaling of a simulation should be done so that the results are representative of the evolution of a “real” cluster. We investigate three different ways of scaling time. One of these is appropriate to the first type of cluster, i.e. those that dissipate rapidly, and similarly a second scaling is appropriate only to the second (relaxation-dominated) type. We also develop a hybrid scaling which is a satisfactory compromise for both types of cluster. Finally we present evidence that the widely used Fokker-Planck method produces models which are in good agreement with $N$-body models of those clusters which are relaxation dominated, at least for $N$-body models with several thousand particles, but that the Fokker-Planck models evolve too fast for clusters which dissipate promptly.

Key words: gravitation – methods: numerical – celestial mechanics, stellar dynamics – globular clusters: general
1. Introduction

This paper is concerned with techniques for studying the dynamical evolution of a globular star cluster. This is a complicated problem: a complete description would be impossible without a detailed understanding of the way in which stars interact non-gravitationally, and how the internal evolution of the stars interacts with their dynamical behaviour. This is illustrated by the emerging study of the behaviour of binaries in star clusters (Milone and Mermilliod 1996). Nevertheless there is no astrophysical problem which admits of a complete description, and much may be learned by the study of more tractable, idealised models, provided that the goal of understanding the behaviour of real clusters is always kept in mind.

To a very good approximation the internal dynamics of a globular star cluster is an $N$-body problem. The dominant forces which govern the internal motions (i.e. motions relative to the centre of mass of a cluster) are the gravitational forces of the stars themselves and the tidal field of the parent galaxy. In addition, the masses of the stars are constant for long periods of time. Nevertheless, stars lose significant amounts of mass from about the time at which they complete their main sequence evolution, and it is known from simplified models (e.g. Applegate 1986, Chernoff & Shapiro 1987) that the effects on the cluster can be devastating. It is fair to say that this is the principal known channel through which the internal evolution of the stars can influence the gross dynamics of a cluster.

We have now stated the three main processes whose effects we explore in this paper: (i) mutual gravitational interactions of the stars; (ii) the tidal field of the galaxy; (iii) mass loss by stellar evolution. Clearly much has been omitted. For example, nothing has been said about the effects of primordial gas, whose early dissipation may well be sufficient to unbind a young cluster (e.g. Lada et al. 1984, Goodwin 1997). Nevertheless, as such studies show, the influence of this matter may well almost cease very early on in the evolution of a cluster, i.e. by the time the most massive stars have evolved. Another omission is the influence of primordial binaries (e.g. Hut et al. 1992). Their effect is more persistent, and indeed they may well be crucial for long periods in the later evolution of a cluster, probably determining such factors as the size of the core (Goodman & Hut 1989, Gao et al. 1991, Vesperini & Chernoff 1994). Nevertheless, up to the point of core collapse (the time scale for which is one of the features on which we concentrate in this paper), primordial binaries are thought to behave roughly as a species of rather massive stars, i.e. with a mass equal to the sum of the masses of the components (Heggie & Aarseth 1992). Up to this point, then, it is not expected that they have a very important effect on the gross dynamics.

The simplified problem that we have stated can be tackled by more than one technique. At present the most popular is the Fokker-Planck model, which treats a cluster of stars rather like a gas, the gravitational encounters between the stars being modelled in much the same way as collisions in an ideal gas. This technique is idealised in a number of ways: often it is assumed (i) that the cluster and tidal field are spherically symmetric, (ii) that the distribution of stellar velocities is isotropic, (iii) that the evolutionary time scale is long compared with the time scale on which stars orbit within the cluster, and (iv) that small-angle scattering encounters are dominant. Techniques for dispensing with most of these assumptions are known (e.g. (i) Goodman 1983a, Einsel & Spurzem 1998;
Apart from Fokker-Planck techniques, another potential contender is the $N$-body method (e.g. Aarseth 1985). This has no difficulty with any of issues (i)-(iv) raised above. Indeed there are important ways in which $N$-body techniques become easier if the models are made more realistic. One example is the effect of introducing a spectrum of stellar masses. Multi-mass Fokker-Planck models are more time-consuming than models in which all stars have the same mass, whereas $N$-body models are easier because the time scale for evolution becomes shorter.

In general $N$-body models are more free of simplifying assumptions than other techniques. Their great drawback is that it is still impractical to model systems with the correct $N$, because of the computing time required. On a general-purpose supercomputer a 10,000-body model requires of order 2 months to pass core collapse (with equal masses; Spurzem & Aarseth 1996), while even the fastest special-purpose computer (the GRAPE-4 in Tokyo) takes of order 3 months for about 32,000 particles (Makino 1996; the calculation was actually performed with only one quarter of the full configuration of the hardware). These are small simulations compared with a median globular cluster, which may contain of order $10^6$ stars. Therefore the main problem in studying the evolution of globular clusters with $N$-body models is knowing how the results depend on $N$.

In this paper we have two aims. One aim is to show that the results of $N$-body simulations can indeed be scaled reliably to systems of the size of globular clusters. We do so primarily by comparing simulations with different $N$, the scaling being determined in accordance with theoretical considerations. Our second aim, however, is to demonstrate how the resulting evolution compares with that obtained with Fokker-Planck techniques incorporating the same physical mechanisms.

To some extent both questions have already been studied by Fukushige and Heggie (1995), who compared the results of $N$-body and Fokker-Planck calculations, and varied the number of particles in the simulations until the results were nearly independent of $N$. That paper imposed on itself two limitations, however. First, the paper was concerned only with clusters in which relaxation effects were unimportant: the models dissolved in the tidal field before there was any significant mass segregation. The second limitation, which derives from the first, was that the scaling of the models to real star clusters was uncomplicated, essentially because it was unnecessary to scale the results so that relaxation effects in the models would closely mimic those in a real cluster.

In the present paper we remove these limitations, and extend the problem by considering how to model those clusters in which relaxation is important, which in effect means those in which there is relatively little mass loss by stellar evolution, and/or the initial configuration is sufficiently centrally concentrated that relaxation effects can occur sufficiently
quickly.

Here is the plan of the paper. In the next section we introduce the physical mechanisms which drive the evolution, summarise our choice of initial conditions, outline some of the hardware and software aspects of our simulations, and discuss the central issue of time scaling. Section three describes the results, which show that clusters whose evolution is driven by tidal overflow, two-body relaxation, and mass-loss from stellar evolution, may indeed be simulated with relatively modest values of \( N \). We also show that the results of simulations with a few thousand particles are remarkably consistent with those already found using Fokker-Planck techniques, at least for those clusters which are not so seriously affected by the third mechanism (i.e. mass loss by stellar evolution) that they disrupt in a time much less than a Hubble time. In the last section of the paper we summarise our conclusions, and also draw attention to some astrophysically interesting features of the evolution, such as the anisotropy of velocities.

A companion paper (Vesperini & Heggie 1997) applies some of the techniques of the present paper to discuss the evolution of the mass function. Finally, the question of how to scale \( N \)-body models so that the statistics of stellar collisions in evolving globular clusters may be studied is the subject of a further paper (Aarseth & Heggie 1998).

2. Technique

2.1 Initial and Boundary Conditions

As has been stated, one of our aims has been to compare the results of \( N \)-body simulations with those of Fokker-Planck calculations. Fortunately there exists an excellent and instructive set of such calculations described by Chernoff & Weinberg (1990), and so we have adopted their initial and boundary conditions (with appropriate modifications described below), even where these differ from those that might now be considered more appropriate for the study of galactic globular clusters (e.g. the minimum mass). In one or two cases we have carried out a few \( N \)-body simulations to observe the effect of changing these parameters, but much more detailed results along these lines will be found in the paper by Vesperini & Heggie (1997).

Chernoff & Weinberg studied models of clusters in circular orbit about a galaxy whose gravitational field is taken to be that of a point mass. The circular speed is \( v_g = 220 \text{km/s} \). In our \( N \)-body simulations we have adopted the corresponding equations

\[
\begin{align*}
\ddot{x} - 2\omega \dot{y} - 3\omega^2 x &= F_x \\
\ddot{y} + 2\omega \dot{x} &= F_y \\
\ddot{z} + \omega^2 z &= F_z
\end{align*}
\]

(e.g. Chandrasekhar 1942), where the coordinates of a star are taken relative to a rotating frame whose origin rotates about the galaxy in a circular orbit of angular velocity \( \omega \); the \( x \)- and \( y \)-axes point in the direction away from the galactic centre and in the direction of galactic rotation, respectively; and \( \mathbf{F} \) is the acceleration due to other cluster members. Note that Fokker-Planck codes usually impose a cutoff, rather than a tidal field as in the \( N \)-body simulations, though the evidence is that this makes little difference to the rate at which mass is lost (Giersz & Heggie 1997).
For initial conditions Chernoff & Weinberg (1990) used King models (King 1966) with scaled central potential $W_0 = 1, 3$ and 7. These models have also been used in the present study, but with $W_0 = 3, 5$ and 7. The reason for ignoring the models of lowest concentration is that Chernoff & Weinberg found that they always disrupted in much less than a Hubble time, a result that has already been confirmed (with substantial quantitative differences) with N-body models by Fukushige & Heggie (1995). Chernoff & Weinberg found that, depending on the initial mass function, clusters with $W_0 = 3$ and 7 could survive until the present day, and we have added the case $W_0 = 5$ to help reveal trends in the results. In planning our programme of work we regarded the results of Fukushige & Heggie as providing a definitive N-body answer to the evolution of those clusters which dissolve quickly, without surviving long enough for the effects of relaxation to become important. Therefore we have concentrated on those models whose long-term fate is determined in an important way by relaxation effects, and was left open by Fukushige & Heggie.

We have followed Chernoff & Weinberg in the choice of initial mass function, which is a power law $dN \propto m^{-\alpha} dm$ in the range $0.4M_\odot < m < 15M_\odot$, with $\alpha = 1.5, 2.5$ or 3.5. We have also followed their prescription for mass loss through stellar evolution, which is assumed to take place instantaneously at the end of main sequence evolution. The time and amount of mass lost are determined by linear interpolation in a table. At the time of its formation a remnant is assumed to have exactly the same velocity as its progenitor; in making this assumption we are again following Chernoff & Weinberg, though it is likely to be in serious error for neutron stars, only a small fraction of which would probably be retained (Drukier 1996). There are no primordial binary stars in our models, though these will be included in the next paper in this series (Aarseth & Heggie 1998).

Let us consider next the number of parameters in this problem. There are two dimensionless parameters, $W_0$ and $\alpha$, which we retain, and two dimensional parameters, which are the initial mass of the cluster, $M(0)$, and its galactocentric distance, $R_g$. As shown by Chernoff & Weinberg, clusters with the same relaxation time exhibit the same evolution, if the assumptions of the orbit-averaged Fokker-Planck model are valid. Therefore the evolution should depend on $M(0)$ and $R_g$ only through the parameter $F$ introduced by Chernoff & Weinberg, as this is a measure of the relaxation time. We follow Chernoff & Weinberg in presenting results for four distinct values of $F$, corresponding to the four “families” that they studied. In Sec.3.2 we check whether different models within the same family evolve similarly.

2.2 Practical considerations

In carrying out our N-body models we have employed a version of the code NBODY4 (Aarseth 1985), which may be briefly described as follows. It is a direct summation code with a Hermite integrator (Makino & Aarseth 1992) and a binary hierarchy of time-steps (i.e. “block” time steps). It employs various mechanisms for the treatment of compact subsystems, including two-body and chain regularizations (Mikkola & Aarseth 1993). The inner binary of a hierarchical triple system is treated by a novel technique for rescaling of time (Mikkola & Aarseth 1996). The main feature of other advanced codes which NBODY4 lacks is the distinction between irregular and regular forces (Ahmad & Cohen 1973), i.e. those due to near neighbours and those due to the remainder of the system.

The evaluation of forces and force derivatives (which are also required in the Hermite
integrator) has been carried out with the aid of a special-purpose computer. This is a single GRAPE-4 board containing 48 HARP processors (cf. Makino et al. 1993), which also performs one or two other compute-intensive tasks, such as finding neighbours and potentials. The board connects to a standard Dec Alphastation 3000/700 via a HARP control board and interface. The performance of the board is summarised in Table 1; most of the results in this table are discussed in §3 below, but it also gives the CPU time for a variety of models. It can be seen that most models take less than one day, which is a reasonable target for a programme in which many models must be computed in order to explore an adequate range of parameters.

It is interesting to compare the timing with that of the full-size GRAPE-4 in Tokyo (Makino et al. 1997). They timed equal-mass King models with $W_0 = 3$ using a single cluster of nine processor boards, and we assume that our models with the same $W_0$ and the steepest mass function ($\alpha = 3.5$) are the most comparable. We find that our single board (with associated host) takes about 0.97 minutes for a single $N$-body time unit (Heggie & Mathieu 1986) if $N = 4096$, and this appears to be only about twice as long as on the nine boards in a single cluster in Tokyo. For $N = 8192$ the corresponding time for our single board is about 2.42 minutes, which exceeds the result of Makino et al. (on a cluster of nine boards) by a factor of a little more than two. In fact our timings are quite insensitive to the slope of the mass function.

The fact that our system performs at almost half the speed of one whose peak speed is nine times higher can be interpreted in terms of efficiency, i.e the fraction of peak speed actually achieved. As shown by Makino et al. (their Fig.17) the efficiency of their hardware is about 8% when $N = 8192$ (for a certain model). It may be expected that a smaller system, though slower, would be more efficient. Indeed, we estimate the efficiency of our hardware at about 25% for the same $N$, and this largely accounts for the relative timings.

Now we turn to some practical aspects in the setting up of the initial $N$-body model. Using the units introduced by King (1966), the desired King model was first constructed by numerical integration of Poisson’s equation, which also yields the corresponding value of the tidal radius $r_t$. Then an $N$-body realisation of the model was constructed, also in King’s units. This $N$-body model was then scaled to standard $N$-body units (Heggie & Mathieu 1986), in which the total mass of the cluster is 1, the constant of gravitation is 1 and the initial energy is $-1/4$. Henceforth quantities expressed in these units will be distinguished by *. The scaling was carried out so that this $N$-body model, regarded as an isolated model, is in virial equilibrium. The theoretical tidal radius of the model in King units was scaled in the same way, giving a value $r_t^*$ for the tidal radius of the $N$-body model in $N$-body units. Now in these units it follows from eqs.(1) that the theoretical tidal radius is also given by

$$r_t^* = (3\omega^*)^{-1/3},$$

and so the value of $\omega^*$ is determined. Note that the resulting initial model is not quite in virial equilibrium, because of the contribution to virial balance of the terms involving $\omega$ in eqs.(1) (cf. Chandrasekhar 1942). This could be rectified (Heggie & Ramamani 1995), but the neglected contribution is no more than a few percent, and is not noticeable in comparison with statistical fluctuations in the virial ratio.
2.3 Scaling

We now consider how the model so constructed is to be scaled in such a way that it represents the evolution of a “real” star cluster. As we shall see, the most delicate aspect of this question is how to scale the time from the $N$-body units of the simulation to some astrophysical unit of time such as $10^6$ yr. One of the principal roles of this scaling is that it determines the points in the simulation at which mass loss by stellar evolution takes place. In this subsection we first state two scalings which have an obvious physical basis, and then present a somewhat novel hybrid, which is designed to combine their advantages.

2.3.1 Fixed Scalings

The simplest choice is scaling by the crossing time, i.e. we write

$$t = t^* \frac{t_{cr}}{t_{cr}^*},$$

(3)

where $t_{cr}$ denotes the crossing time of the real cluster (in Myr, for example), and $t_{cr}^*$ denotes that of the $N$-body model (in $N$-body units). This choice guarantees that the period of radial motion of a particle in the simulation scales to the period of radial motion of a star in the cluster. It may be shown readily that it also ensures that the angular velocity $\omega^*$, defined by eq.(2), scales correctly to the orbital angular velocity of the cluster, i.e. to $\omega = v_g/R_g$. Thus eq.(3) is equivalent to assuming that $t = t^* U_t$, where

$$U_t = \omega^*/\omega.$$  

(4)

In general, scaling by the crossing time is appropriate at times when mass loss is important, because simple arguments show (Hills 1980) that the effect of impulsive mass loss, i.e. mass loss taking place on a time scale much smaller than the crossing time, is quantitatively very different from that of adiabatic mass loss.

Scaling by the crossing time, eq.(3), does not ensure that the number of relaxation times that elapse in the two systems correspond correctly, because the relaxation time is an $N$-dependent multiple of the crossing time, and the number of particles in a practical simulation is much smaller than that in the globular clusters of interest. A modified time scaling, which ensures that relaxation proceeds at the correct rate, is

$$t = t_{rh} \frac{t^*}{t_{rh}^*},$$

(5)

where $t_{rh}$ is the half-mass relaxation time in the real cluster, and $t_{rh}^*$ is the corresponding quantity for the $N$-body model in $N$-body units. (Any other measure of relaxation times, such as the central value, would be equally suitable.)

2.3.2 Variable Scaling

Scaling by the crossing time, i.e. eq.(3), was used by Fukushige & Heggie (1995) to simulate the early evolution of globular clusters, but this meant that these authors were unable to determine the long-term fate of the clusters which survived this early phase, i.e. those with steep initial mass functions and/or high initial concentration, for the evolution
of such clusters becomes relaxation-dominated. It is important to be able to simulate such clusters, for if the initial conditions of the galactic globular clusters bear any resemblance to those of the models surveyed by Chernoff & Weinberg, it is these long-lived models which are relevant to the evolutionary history of the clusters which exist today. Ideally, therefore, one should devise a strategy which combines the advantages of both kinds of scaling, i.e. eq.(3) for a cluster which dissolves promptly, and eq.(5) for a long-lived cluster, at least during the long phase of its evolution which is dominated by relaxation. In order to decide which scaling to adopt, and when, we may compare the time scales for the two dominant processes, i.e. mass-loss and relaxation.

The time scale for mass-loss by stellar evolution may be defined as

\[ t_M = -M/(dM/dt), \]  

where the derivative on the right indicates only mass loss arising from this mechanism (cf. eq.[8] and Fig.1). We denote the time scale of two-body relaxation by \( t_r \), and defer a precise definition until later in this section. Then our strategy suggests that we adopt eq.(5), i.e. scaling by the relaxation time, if \( t_M > t_r \). We must be more careful, however, before concluding that we use eq.(3) if \( t_M < t_r \), because this scaling does not correctly scale the relaxation time of the cluster to that of the model. Therefore a situation could arise in which \( t_r^* < t_M^* \), even though \( t_r > t_M \). For this reason we introduce an intermediate regime in which we compromise as best we can, by adopting a scaling which ensures that \( t_r^* = t_M^* \). It provides a smooth transition from the scaling of eq.(3) to that of eq.(5) as the mass loss from stellar evolution gradually slows down.

At this point we summarise the variable scaling that is adopted in the bulk of the simulations described in this paper. Because we must switch between one scaling and another we will not usually be able to express \( t \) as a simple multiple of \( t^* \), and instead must consider scalings in differential form. Thus we have

\[ \frac{dt}{dt^*} = \begin{cases} \frac{t_m}{t_m^*} & \text{if } t_M \leq t_r, \\ t_M^* & \text{if } t_r^* \leq t_M \leq t_r, \\ \frac{t_r^*}{t_r} & \text{if } t_M \geq t_r. \end{cases} \]  

Now we must consider how this may be implemented in an N-body calculation. We assume that, at any time, there is a power-law mass function with \( N(m)dm \propto m^{-\alpha}dm \) for \( m_1 > m > m_0 \). Then the total mass is \( M \propto m_1^{2-\alpha} - m_0^{2-\alpha} \) and if we assume (for this purpose only) that each star loses all its mass at the end of its evolution, then it is easy to show that

\[ t_M = \left(1 - \left(\frac{m_0}{m_1}\right)^{2-\alpha}\right) \frac{\alpha - 2}{\alpha - 2} \frac{d\log m_1}{d\log t} t. \]
This is easily computed if the current value of \( m_1 \) is known (i.e. the mass of a star which is just reaching the end of its evolution) and if we assume that \( \alpha \) retains its initial value; and the value \( \frac{d\log m_1}{d\log t} \) is easily computed from the table of lifetime versus mass. Fig.1 shows the result for several values of \( \alpha \).

In eq.(7) the value of \( t_M \) must be compared with \( t_r \), and we must adopt a suitable definition of the relaxation time which allows us to determine whether the evolution of a cluster is dominated by relaxation or mass-loss. This is not straightforward. The evolution of the core of a high-concentration cluster may be dominated by relaxation even though mass-loss causes the half-mass radius to expand. The issue is further complicated by the mass-dependence of mass segregation, which depends in turn on the evolving mass function. Despite these complications we have adopted a simplistic criterion based on a result of Spitzer (1987, his eq.[4-1]) for values at the half-mass radius. We write it in the form

\[
\frac{t_r}{t_{cr}} = \frac{N}{11 \ln(\gamma N)}.
\]

The current value of \( N \) may be obtained from the current value of \( N^* \) and the initial values \( N(0) \) and \( N^*(0) \), where \( N(0) = M(0)/\bar{m}(0) \), if we assume that \( N/N^* \) is constant, and this is correct provided that the model correctly simulates the evolution of the real cluster.

The remaining quantities in eq.(7) are easily calculated. Thus \( t_{cr} = U_t t_{cr}^* \), where \( U_t \) is given by eq.(4), and \( t_{cr}^* \) may be computed from the conventional definition \( t_{cr}^* = M^*5/2/(2|E^*|)^{3/2} \), in which \( E^* \) is the energy of the model in \( N \)-body units. The ratio \( t_r^*/t_{cr}^* \) is obtained in analogy with eq.(9).

We close with a few practical details. First, we adopt a relatively low value of \( \gamma \simeq 0.01 \), based on the work of Giersz & Heggie (1997). Second, as \( N^* \) can become very small, the factor \( \ln \gamma N^* \) in these expressions was modified to \( \ln(1 + \gamma N^*) \). When \( N^* = 200 \), for example, the value of \( dt/dt^* \) actually used is too large by a factor of about 1.6. Finally, for astrophysical reasons it was deemed sufficient to terminate each simulation by the time \( t = 20 \) Gyr.

3 Numerical Results

3.1 Tests of the scaling strategy

3.1.1 \( N \)-Dependence of Scaling

In the previous section we have introduced three ways of scaling the time in a small \( N \)-body simulation to that of a typical globular cluster. We refer to these respectively as \( t^*-\)scaling (equation 3), \( t_r-\)scaling (equation 5) and \( \text{variable scaling} \) (equation 7). The most straightforward test of any of these is the comparison of \( N \)-body simulations carried out with different values of \( N^* \). This is the purpose of the present subsection. In §3.2 we go on to carry out a comparison between the \( N \)-body results and those of Fokker-Planck models.

Fig.2 shows an example of the same model, specified by the information in the title, computed with different \( N^* \), and using \( t^*-\)scaling (equation 5). The uppermost curve in each plot shows the tidal radius, and the remainder are 0.1%, 1%, 10% and 50% Lagrangian radii, as defined in the caption. From a comparison of the two plots it is gratifying to note
that core collapse ends at very nearly the same time (about 10 Gyr). Towards the end of the simulations, however, the larger of the two (Fig. 2b, $N^* = 8192$) loses mass at a relatively larger rate. This can be seen from the uppermost curve (the tidal radius), as $r_t \propto M^{1/3}$. This is an example of a general trend which will be discussed in connection with Table 1 below.

Table 1 shows a more extensive set of results, though these models were computed with variable scaling (equation 7), and each model is now summarised by only a small number of data. These data include the time at which the model either dissipated or was terminated, $t_{\text{max}}$, the time at the end of core collapse, $t_{\text{cc}}$, and the mass and half-mass radius (in N-body units) at this time and at 15 Gyr. Many of the models, however, do not reach core collapse, and of these models most also disappear before 15 Gyr.

For fixed $W_0$ and $\alpha$, the results from different $N^*$ are in qualitative agreement. Quantitatively, a systematic result emerges, which is that the larger model ($N^* = 8192$) tends to lose mass somewhat faster than the smaller model (cf. also Fig. 2). To quantify this we have determined the time (in Gyr) at which the value of $M^*$ in the smaller model takes the value at 15 Gyr in the larger model. The resulting five values range from 15.3 to 17.4 Gyr, with a mean of 16.6. This gives a quantitative estimate of about 10% for the order of magnitude of the discrepancy in the rate of loss of mass.

3.1.2 Comparison of Different Scalings

Though much was made in §2.3 about the desirability of variable scaling, in practice the differences between results produced by the different scaling methods are not usually very large, at least for initial particle numbers as large as $N^* = 8192$. The basis for this assessment is the data in Table 2, which compares different types of scaling for the same clusters. Results for $t_r$-scaling have been taken from Fukushige & Heggie (1995).

Let us first concentrate on those clusters which dissipate promptly, either during or immediately following the period of heavy mass loss by stellar evolution. These are the three models in this table for which Fukushige & Heggie obtained a definitive result, viz. $W_0 = 3$, $\alpha = 1.5$ and 2.5, and $W_0 = 5$, $\alpha = 1.5$. For such clusters the dissolution time estimated by $t_r$-scaling is systematically too large. The results from variable scaling are better, and we include here the case $W_0 = 3$, $\alpha = 1.5$, even though the lifetime given by variable scaling is considerably less than that of Fukushige & Heggie; their Fig. 6 suggests that statistical differences between models will lead to differences of order 0.03 Gyr, and that the result stated in Table 2 (i.e. 0.11 Gyr) is near the upper end of the range of values.

Though variable scaling gives a better result for $W_0 = 3$, $\alpha = 2.5$ than $t_r$-scaling, it is still poor. Inspection of Fig. 7 in Fukushige & Heggie (1995) suggests that their result is quite robust in this case. Though data is not given in Table 2, we have found that, for smaller particle numbers, the disagreement is still larger, as expected.

The model that we have been discussing ($W_0 = 3$, $\alpha = 2.5$) is expected to be difficult to simulate, because it lies close to the border between those clusters which dissipate promptly (without significant relaxation) and those that can survive for several Gyr. Another problematic case of this kind is $W_0 = 7$, $\alpha = 1.5$. Fukushige & Heggie stopped their calculation at 1.6 Gyr, when the mass of this cluster was approximately $M^* = 0.05$ (their Fig. 1), whereas our model with variable scaling dissolved at 1.2 Gyr. It seems likely that the model of Fukushige & Heggie, despite the softening they used, is significantly affected
by two-body relaxation. This is the longest of their simulations (to $t^* = 3000$), and their Fig.8 shows that even a shorter run (to $t^* = 1000$, though with different model parameters) may be significantly affected by two-body relaxation. If so, the resulting core evolution could make the cluster too robust to the effects of mass loss, and so prolong its life.

For the remaining five sets of models presented in Table 2 (which, as it happens, are those that survive to 15Gyr) the results are more straightforward. For these models the work of Fukushige & Heggie gives only a lower limit, and can be ignored. The results from variable- and $t_r$-scaling are quite consistent, the only noticeable general trend being that the models with $t_r$-scaling evolve more slowly. The one exception to this trend is the core collapse time for $W_0 = 7$, $\alpha = 2.5$, but this quantity is known to be subject to considerable statistical fluctuation (Spurzem & Aarseth 1996).

3.2 Comparison with Fokker-Planck Results

In this subsection we consider the second main purpose of the investigation, which is to compare $N$-body results with Fokker-Planck results for the same cluster parameters, partly as a consistency check, and partly as a check on the validity of the Fokker-Planck model, which is likely to remain a standard tool for the investigation of the dynamical evolution of globular clusters for some time to come.

The basic information we present is given in Tables 3 and 4, which compare a number of data for our $N$-body models and a subset of the Fokker-Planck models of Chernoff & Weinberg (1990): we omit their models in which $W_0 = 1$ initially. Note also that our models with $W_0 = 5$ are omitted here because this was not one of the values which Chernoff & Weinberg included. All $N$-body models use variable scaling (eq.(7)).

A factor to be borne in mind in interpreting the Fokker-Planck data is that the time $t_{\text{max}}$ does not correspond to the time at which the model loses all its mass. As Chernoff & Weinberg explain, there comes a point at which the method fails to find a new structure corresponding to the evolving distribution function, and this is interpreted as indicating the imminent rapid dissolution of the model.

The caption in Table 3 (and those in Table 4, to be discussed below) refers to one of the “families” adopted by Chernoff & Weinberg (cf.§2.1). For the orbit-averaged technique used by Chernoff & Weinberg, the evolution is the same for all members of one family, given the initial value of $W_0$ and the initial mass function. Thus a low-mass cluster at large galactocentric radius will have the same predicted evolution as a cluster of high mass at an appropriate small radius. Comparison of the models in Table 3 illustrates this scaling in the $N$-body data, along with a comparison with the Fokker-Planck data. In each triplet of models the two $N$-body models belong to the same family of Chernoff & Weinberg (family 1), but at different galactocentric radii. To avoid complications with the residual $N$-dependence of the results of the $N$-body models (cf.§3.1.1), all models in this Table share the same initial value of $N^*$, taken to be 4096. The data of Table 3 shows yet again that the models divide into two types: those that dissolve in much less than a Hubble time, and those in which the $N$-body models survive to at least 20Gyr.

Let us consider first the rapidly dissolving models, i.e. those with $(W_0, \alpha) = (3, 1.5), (3, 2.5)$ and $(7, 1.5)$. Though the $N$-body models indicate systematically longer lifetimes at larger radii, the evidence is in fact not significant. We have already argued (in discussing Table 2) that the difference in the lifetimes of the two models with $W_0 = 3$, $\alpha = 1.5$, and $W_0 = 7$, $\alpha = 2.5$ is not significant.
would probably be within statistical fluctuations even for simulations with $N^* = 8192$. A similar argument probably applies to the $N$-body models with $W_0 = 7$, $\alpha = 1.5$, though for somewhat more subtle reasons. These models rapidly lose most of their mass, retaining only about 20% by 0.5Gyr. The subsequent evolution is therefore subject to the sizeable fluctuations between different $N$-body models with rather small $N^*$.

Of the three models which dissolve rapidly, the Fokker-Planck and $N$-body results are consistent only for the model we have just discussed, i.e. $W_0 = 7$, $\alpha = 1.5$. The other two models ($W_0 = 3$, $\alpha = 1.5$ and 2.5) confirm the finding of Fukushige & Heggie (1995) that the Fokker-Planck model underestimates the true lifetime by a factor of order 10 in some cases. As they suggested, a plausible reason for this is that the Fokker-Planck models necessarily assume that $t_M$, the time scale on which the cluster loses mass, much exceeds $t_{cr}$, and that the lifetime is underestimated when this assumption is not valid.

Now we turn to the more straightforward comparison of the three models in Table 3 which survive for at least a Hubble time. The $N$-body models illustrate rather well the fact that the evolution is similar for clusters within the same family but at different $R_g$. The results also agree rather well with the Fokker-Planck data. Though it might seem that the Fokker-Planck models evolve somewhat faster than these $N$-body models, it must be recalled (cf. discussion of Table 1) that larger $N$-body models would also do so.

Table 4 is the main vehicle for examining the relation between the $N$-body data and the Fokker-Planck data. Here all models are placed at about 4kpc, as the evidence of Table 3 suggests that the results will be independent of $R_g$ within a given family, at least for long-lasting models. In fact we have omitted those models that dissolve rapidly, as an adequate comparison has been provided by Fukushige & Heggie.

The data on core collapse in this Table is rather meagre and adds little to what was learned from Table 3, except that the models for Family 1 have larger $N^*$. It confirms that there appear to be no systematic differences in either the time or the total mass at core collapse.

Now let us consider the mass at 15Gyr. The absence of this data in two of the Fokker-Planck models of Family 1 is due to the fact that Chernoff & Weinberg stopped their calculations at the time of core collapse, which occurs before 15Gyr for these models. For the one remaining model in this family, and all models in the other families, there is evidence of some systematic differences in the mass at 15Gyr. In particular, the disagreement is most pronounced for the models with $W_0 = 3$ and $\alpha = 3.5$, especially for the first two families, i.e. those in which two-body relaxation is fastest. The Fokker-Planck models yield substantially smaller masses at 15Gyr. If we assume that the mass decreases approximately linearly with time, the differences correspond to differences in the lifetime which are in the range of 13–23%. The comparisons in Table 2, however, suggest that the disagreement is well within the residual $N^*$-dependence of the $N$-body results, and has the expected sign: the mass at 15Gyr decreases slightly with increasing $N^*$, and the Fokker-Planck results are also smaller than the $N$-body results. These remarks apply to the models with $W_0 = 3$ and $\alpha = 3.5$; for other models the discrepancy in the mass at 15Gyr is much more modest.
4. Discussion and Conclusions

4.1 Anisotropy

Though the main purposes of this paper are to investigate the scaling of $N$-body models and to compare results with Fokker-Planck data, it is a pity to leave the $N$-body models without drawing attention to some other interesting results. One of these is the evolution of the mass function and the proportion of degenerate stars, and we refer interested readers to Vesperini & Heggie (1997), where detailed and comprehensive results are presented (though with a more realistic lower limit to the initial mass function, and disk shocking).

Here we concentrate on the anisotropy of the models, which Chernoff & Weinberg did not address, as they used an isotropic Fokker-Planck model. For the purposes of this discussion we adopt the usual parameter $\beta = 1 - \langle v^2_t \rangle / \langle v^2_r \rangle$, where $v^2_t$ is half of the square of the transverse component of velocity of one star and $v^2_r$ is the radial component, i.e. in the direction away from the density centre. Only stars inside the tidal radius are included in the averaging, which is not mass-weighted. Initially there is a short period in which $\beta > 0$, and this is attributable to the radial outward drift of stars during the early phase of rapid mass loss. Thereafter the sense of the anisotropy reverses, i.e. $\beta < 0$, and there is a slight predominance of transverse velocities. Nevertheless the anisotropy is small, values of order $\beta \approx -0.1$ being typical.

In an isolated model, $\beta$ would become positive (e.g. Giersz & Heggie 1996), and the negative values in our model are attributable to the tidal field. This has two effects: in the first place it alters the angular momentum of the stars, as it is not spherically symmetric (cf.eq.[1]); and, secondly, it preferentially removes stars on nearly radial orbits, as these are most likely to escape. We attribute the negative values of $\beta$ to the second effect mainly, as similar negative values of $\beta$ are observed in models (not otherwise discussed in this paper) in which the tidal field of eq.(1) is replaced by a spherical cutoff, as in most implementations of the Fokker-Planck model.

The smallness of the values of $|\beta|$ helps to justify the use of an isotropic model such as the Fokker-Planck model of Chernoff & Weinberg. On the other hand, our result also has a variety of implications for the interpretation of observational data. There are a few clusters for which careful study of proper motions requires the presence of significant radial anisotropy, i.e. $\beta > 0$ (e.g. Cudworth 1979 for the cluster M3). There is a larger sample of clusters where careful dynamical modelling (using multi-mass anisotropic King-Michie models) again requires the presence of radial anisotropy (e.g. Meylan & Mayor 1991). Here the requirement of substantial anisotropy is less direct, as the models which are fitted are constrained by theoretical considerations (e.g. the choice of distribution function), and in principle it is possible that an incorrect choice of distribution function is compensated by the anisotropy when fitting to the observational data. Nevertheless, if the evidence for anisotropy is accepted, the contradiction with the anisotropy found in our models may only indicate that the initial conditions adopted in our survey differ too widely from those appropriate to the observed clusters. For example, all our models are initially tidally limited: the edge of the model is placed initially at the tidal radius. It is possible that initially much more compact models would develop strongly positive anisotropy. They would behave much more like isolated models until, after sufficient expansion in the post-
collapse phase of evolution, their development might more closely resemble that of the models we have studied. Interestingly, it was concluded by Phinney (1993), on quite different grounds, that only an initially very condensed cluster would evolve into an object resembling the cluster M15 at the present day.

4.2 Conclusions

Now we close with a summary of the main conclusions of this paper. It has been concerned with the dynamical evolution, over a time up to 20Gyr, of a set of models for the globular star clusters of the Galaxy. The evolutionary mechanisms considered here include all gravitational interactions between the members of a cluster, the tidal field of the Galaxy, and instantaneous mass loss at the end of main sequence evolution. We do not include the effects of finite stellar radii and collisions, or primordial binaries. The initial mass function is a power law between masses of 15 and \(0.4 M_\odot\), with index \(\alpha\). The initial model is a King model, whose concentration is specified by the dimensionless central potential \(W_0\).

In qualitative agreement with the results of Chernoff & Weinberg (1990) we find that the fate of the clusters divides them into two classes: those that disrupt relatively rapidly (in at most one or two Gyr) as a result of mass loss by stellar evolution and tidal stripping, without being significantly affected by two-body relaxation; and those that survive this initial phase of heavy mass loss and evolve towards core collapse. For clusters of initial mass about \(1.5 \times 10^5 M_\odot\) and galactocentric radius of about 4kpc, clusters which disrupt are those with \(W_0 = 3\) and \(\alpha \lesssim 2.5\), and those with \(W_0 = 5\) or 7 and \(\alpha \lesssim 1.5\). In general those with larger \(W_0\) and/or larger \(\alpha\) enter the phase in which their evolution is dominated by relaxation.

For those clusters which disrupt promptly, a comparison between \(N\)-body and Fokker-Planck models was carried out by Fukushige & Heggie (1995). In this paper we have concentrated on the longer-lived systems which become relaxation-dominated. For some parameter values the Fokker-Planck and \(N\)-body results agree remarkably well; the time of core collapse of models with \(W_0 = 7\) and \(\alpha \gtrsim 2.5\) agree as well as could be expected, bearing in mind the fact that there will be different collapse times even for different \(N\)-body realisations with the same initial parameters. More systematic differences, of order 10%, arise for models which lie closer to the borderline between early disruption and relaxation-dominated evolution, e.g. the model with \(W_0 = 3\) and \(\alpha = 3.5\). For these models several lines of evidence (e.g. the mass at the present day, taken here as 15Gyr) imply that the Fokker-Planck models evolve somewhat more rapidly than the \(N\)-body models. The sense of the disagreement is consistent with the conclusion of Fukushige & Heggie for the clusters which disrupt quickly, though the size of the disagreement is much smaller for the clusters which do not dissolve rapidly. Indeed the residual \(N^*_\) dependence of the rate of escape from the \(N\)-body models may explain part or all of the discrepancy.

Besides a comparison with Fokker-Planck models, the other main purpose of this paper was an investigation of the way in which the evolution of small \(N\)-body models may be scaled (in time) to that of globular clusters, where the number of stars is much larger. We have considered three possible scalings: (i) a scaling which ensures that the crossing time of the \(N\)-body model scales correctly to that of a real cluster; (ii) a scaling which ensures that the model relaxes at the same rate as the real cluster; and (iii) a hybrid, or
variable, scaling, which progresses from the first type of scaling to the second as the mass loss from stellar evolution slows down.

The first type of scaling was applied by Fukushige & Heggie. It successfully allows the scaling of $N$-body data for clusters which dissipate promptly without significant relaxation. The model relaxes too quickly, however, and so the scaling of the simulation is no longer valid as soon as relaxation in the model becomes significant. The second type of scaling produces results which appear to be correct for clusters which survive the initial phase of mass loss and enter the relaxation dominated phase of evolution. Unless the number of particles in the simulation is quite large, however, it can produce misleading results for those clusters which dissipate promptly.

The third (variable) scaling was devised to incorporate the advantages of both fixed types of scaling. It produces qualitatively correct results for both kinds of clusters, i.e. those that dissipate rapidly and those that become relaxation-dominated. For clusters of the latter type its quantitative results are indistinguishable from those obtained by scaling of type (ii). It is the most successful scaling method for clusters on the borderline between rapid destruction and long-term survival, e.g. King models with $W_0 = 7$ and $\alpha = 1.5$ (for our chosen initial range of stellar masses).

We recommend variable scaling as a means for using small $N$-body simulations to model the evolution of globular star clusters. Its advantage is that it can be used (with some care) to model the evolution of clusters which dissipate promptly as well as those with long lifetimes. Though this might not seem an important property for the galactic globular cluster system, it is relevant if $N$-body models are to be used for studying the evolution of other globular cluster systems, as in the Magellanic Clouds, which contain both young and old clusters.

The results obtained in this paper are based on models which incorporate only a subset of the many dynamical and evolutionary processes which are needed in a full $N$-body description of a globular cluster (e.g. Aarseth 1996), but the problem of how to scale models cannot be avoided as long as the number of particles in a feasible $N$-body simulation is much smaller than in a real globular star cluster. In a further paper (Aarseth & Heggie 1998) we shall consider how the radii of the stars may be scaled in order to simulate stellar collisions.

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### Table 1

N-Dependence of Models with Variable Scaling

| $W_0$ | $\alpha$ | $R_g$ kpc | $N^*$ | $t_{cpu}$ hr | $t^*_{max}$ Gyr | $t_{max}$ Gyr | $t_{cc}$ Gyr | $M^*_cc$ | $r^*_h,cc$ | $M^*_15$ | $r^*_h,15$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 1.5 | 3.7 | 4096 | 0.3 | 39 | 0.075 | - | - | - | - | - |
| 3 | 1.5 | 3.7 | 8192 | 0.8 | 37 | 0.071 | - | - | - | - | - |
| 3 | 2.5 | 4.0 | 4096 | 1.0 | 169 | 3.0 | - | - | - | - | - |
| 3 | 2.5 | 4.0 | 8192 | 3.3 | 215 | 2.1 | - | - | - | - | - |
| 3 | 3.5 | 4.1 | 4096 | 3.3 | 432 | 20 | - | - | - | 0.390 | 0.69 |
| 3 | 3.5 | 4.1 | 8192 | 12 | 669 | 18.5$^+$ | - | - | - | 0.355 | 0.67 |
| 5 | 1.5 | 3.7 | 4096 | 0.6 | 123 | 0.29 | - | - | - | - | - |
| 5 | 1.5 | 3.7 | 8192 | 1.7 | 144 | 0.23 | - | - | - | - | - |
| 5 | 2.5 | 4.0 | 4096 | 12$^{[4]}$ | 1161 | 16.7 | 13 | 0.086 | 0.48 | 0.0385 | 0.42 |
| 5 | 2.5 | 4.0 | 8192 | 13 | 1789 | 16.6 | 13.5 | 0.067 | 0.40 | 0.032 | 0.32 |
| 5 | 3.5 | 4.1 | 4096 | 7.0 | 675 | 20 | - | - | - | 0.515 | 0.69 |
| 5 | 3.5 | 4.1 | 8192 $\approx$ 27 | 1130 | 20 | - | - | - | 0.477 | 0.67 |
| 7 | 1.5 | 3.7 | 4096 | 1.3 | 438 | 1.2 | - | - | - | - | - |
| 7 | 1.5 | 3.7 | 8192 | 4.2 | 658 | 1.2 | - | - | - | - | - |
| 7 | 2.5 | 4.0 | 4096 | 10 | 2351 | 20 | 13 | 0.21 | 0.83 | 0.148 | 0.76 |
| 7 | 2.5 | 4.0 | 8192 | 33 | 3825 | 18.7 | 11 | 0.210 | 0.77 | 0.089 | 0.60 |
| 7 | 3.5 | 4.1 | 4096 | 22 | 1368 | 20 | 10.4 | 0.606 | 0.94 | 0.476 | 0.92 |
| 7 | 3.5 | 4.1 | 8192 | 85 | 2277 | 20 | 9.2 | 0.617 | 0.89 | 0.435 | 0.94 |

**Notes**

1. An asterisk in the top line denotes a quantity in $N$-body units in an $N$-body model. The values of $W_0$, $\alpha$ and $N^*$ are initial values. Quantities at core collapse are denoted by the subscript cc, and at 15Gyr by a subscript 15. The half-mass radius (of all particles inside the tidal radius) is denoted by $r_h$.

2. The simulations end at $t_{max}$; where this value is less than 20, the calculation stopped with less than 10 bound particles, except that a plus indicates that it ended with a software or hardware error.

3. In all these models the initial mass is $M = 1.49 \times 10^5 M_\odot$.

4. This model was mainly computed without the HARP board.
Table 2
Comparison of Different Time Scalings

| $W_0$ | $\alpha$ | $R_g$ kpc | Scaling | $t_{max}$ Gyr | $t_{cc}$ Gyr | $M^{*}_{cc}$ | $r^{*}_{h,cc}$ | $M^{*}_{15}$ | $r^{*}_{h,15}$ |
|-------|-------|---------|---------|-------------|-------------|-------------|-------------|-------------|-------------|
| 3     | 1.5   | 3.7     | $t_c$   | 0.11        | –           | –           | –           | –           | –           |
| 3     | 1.5   | 3.7     | var.    | 0.071       | –           | –           | –           | –           | –           |
| 3     | 1.5   | 3.7     | $t_r$   | 0.14        | –           | –           | –           | –           | –           |
| 3     | 2.5   | 4.0     | $t_c$   | 0.91        | –           | –           | –           | –           | –           |
| 3     | 2.5   | 4.0     | var.    | 2.1         | –           | –           | –           | –           | –           |
| 3     | 2.5   | 4.0     | $t_r$   | 2.9         | –           | –           | –           | –           | –           |
| 3     | 3.5   | 4.1     | $t_c$   | > 3.9       | –           | –           | –           | –           | –           |
| 3     | 3.5   | 4.1     | var.    | 18.5+       | –           | –           | –           | 0.355       | 0.67        |
| 3     | 3.5   | 4.1     | $t_r$   | 19+         | –           | –           | –           | 0.370       | 0.67        |
| 5     | 1.5   | 3.7     | $t_c$   | 0.24        | –           | –           | –           | –           | –           |
| 5     | 1.5   | 3.7     | var.    | 0.23        | –           | –           | –           | –           | –           |
| 5     | 1.5   | 3.7     | $t_r$   | 0.42        | –           | –           | –           | –           | –           |
| 5     | 2.5   | 4.0     | $t_c$   | > 2.3       | –           | –           | –           | –           | –           |
| 5     | 2.5   | 4.0     | var.    | 16.6 13.5   | 0.067 0.40  | 0.032 0.32  | –           | –           | –           |
| 5     | 2.5   | 4.0     | $t_r$   | 18.6 15     | 0.0847 0.42 | 0.0847 0.42 | –           | –           | –           |
| 5     | 3.5   | 4.1     | $t_c$   | > 2.4       | –           | –           | –           | –           | –           |
| 5     | 3.5   | 4.1     | var.    | 20          | –           | –           | –           | 0.477       | 0.67        |
| 5     | 3.5   | 4.1     | $t_r$   | 20          | –           | –           | –           | 0.480       | 0.65        |
| 7     | 1.5   | 3.7     | $t_c$   | > 1.6       | –           | –           | –           | –           | –           |
| 7     | 1.5   | 3.7     | var.    | 1.2         | –           | –           | –           | –           | –           |
| 7     | 1.5   | 3.7     | $t_r$   | 4.8 4.0     | 0.0069 0.30 | –           | –           | –           | –           |
| 7     | 2.5   | 4.0     | $t_c$   | > 1.1       | –           | –           | –           | –           | –           |
| 7     | 2.5   | 4.0     | var.    | 18.7 11     | 0.210 0.77  | 0.089 0.60  | –           | –           | –           |
| 7     | 2.5   | 4.0     | $t_r$   | 20 10.0     | 0.2666 0.75 | 0.1113 0.67 | –           | –           | –           |
| 7     | 3.5   | 4.1     | $t_c$   | > 1.2       | –           | –           | –           | –           | –           |
| 7     | 3.5   | 4.1     | var.    | 20 9.2      | 0.617 0.89  | 0.435 0.94  | –           | –           | –           |
| 7     | 3.5   | 4.1     | $t_r$   | 20 10       | 0.594 0.89  | 0.438 0.89  | –           | –           | –           |

Notes
1. Most of the notation is defined in the notes to Table 1.
2. Under the heading “Scaling”, $t_c$ denotes scaling by the crossing time, eq.(3); $t_r$ denotes scaling by the relaxation time, eq.(5); and “var.” denotes variable scaling, eq.(7). Results for $t_c$-scaling are taken from Fukushima & Heggie (1995), and for several models the lifetime is only a lower limit.
3. In all these models the initial mass is $M = 1.49 \times 10^5 M_\odot$. All runs had $N^* = 8192$ particles initially.
Table 3
Comparison of Fokker-Planck and $N$-body results for Family1

| $W_0$ | $\alpha$ | Model | $t_{cc}$ | $t_{max}$ | $M_{cc}^*$ | $M_{15}^*$ |
|-------|---------|-------|--------|---------|---------|---------|
|       |         |       |  Gyr   |  Gyr    |         |         |
| 3     | 1.5     | 3.7   | –      | 0.075   | –       | –       |
| 3     | 1.5     | 9.2   | –      | 0.12    | –       | –       |
| 3     | 1.5     | FP    | –      | 0.014   | –       | –       |
| 3     | 2.5     | 4.0   | –      | 3.0     | –       | –       |
| 3     | 2.5     | 10.0  | –      | 3.4     | –       | –       |
| 3     | 2.5     | FP    | –      | 0.28    | –       | –       |
| 3     | 3.5     | 4.1   | –      | 20      | –       | 0.390   |
| 3     | 3.5     | 10.4  | –      | 20      | –       | 0.419   |
| 3     | 3.5     | FP    | 21.5   | –       | 0.078   | 0.23    |
| 7     | 1.5     | 3.7   | –      | 1.2     | –       | –       |
| 7     | 1.5     | 9.2   | –      | 3.2     | –       | –       |
| 7     | 1.5     | FP    | –      | 1.0     | –       | –       |
| 7     | 2.5     | 4.0   | 13     | 20      | 0.21    | 0.148   |
| 7     | 2.5     | 10.0  | 11     | 20      | 0.257   | 0.161   |
| 7     | 2.5     | FP    | 9.6    | –       | 0.26    | –       |
| 7     | 3.5     | 4.1   | 10.4   | 20      | 0.606   | 0.476   |
| 7     | 3.5     | 10.4  | 10.5   | 20      | 0.62    | 0.494   |
| 7     | 3.5     | FP    | 10.5   | –       | 0.57    | –       |

Notes
1. Most of the notation is defined in the notes to Table 1.
2. Under the heading “Model”, a number indicates results of an $N$-body model (with $N^* = 4096$ initially) at the stated galactocentric distance (in kpc). “FP” denotes a Fokker-Planck model of Chernoff & Weinberg (1990). Their models were stopped at the point of dissolution of the cluster (in which case we give the corresponding time as $t_{max}$) or at the end of core collapse (at time $t_{cc}$), whichever is earlier.
3. The initial cluster mass is $5.45 \times 10^4 M_\odot$ at around 10kpc, and $1.49 \times 10^5 M_\odot$ at around 4kpc. All $N$-body models use variable scaling (eq.[7]).
Table 4
Comparison with Fokker-Planck Results

| $W_0$ | $\alpha$ | $R_g$ (kpc) | Model | $t_{cc}$ (Gyr) | $M^*_c$ | $M^*_{15}$ |
|-------|--------|-------------|-------|----------------|--------|------------|
| Family 1: $M(0) = 1.49 \times 10^5 M_\odot$ |
| 3     | 3.5    | 4.1         | 8192  | > 18.5         | –      | 0.355      |
| 3     | 3.5    | 4.1         | FP    | 21.5           | 0.078  | 0.23       |
| 7     | 1.5    | 3.7         | 8192  | (1.2D)         | –      | –          |
| 7     | 1.5    | 3.7         | FP    | (1.0D)         | (0.022)| –          |
| 7     | 2.5    | 4.0         | 8192  | 11             | 0.210  | 0.089      |
| 7     | 2.5    | 4.0         | FP    | 9.6            | 0.26   | –          |
| 7     | 3.5    | 4.1         | 8192  | 9.2            | 0.617  | 0.435      |
| 7     | 3.5    | 4.1         | FP    | 10.5           | 0.57   | –          |
| Family 2: $M(0) = 4.33 \times 10^5 M_\odot$ |
| 3     | 3.5    | 4.1         | 4096  | > 20           | –      | 0.589      |
| 3     | 3.5    | 4.1         | FP    | 44.4           | 0.035  | 0.48       |
| 7     | 1.5    | 3.7         | 4096  | (2.0D)         | –      | –          |
| 7     | 1.5    | 3.7         | FP    | (3.0D)         | (0.0033)| –        |
| 7     | 2.5    | 4.0         | 4096  | > 20           | –      | 0.331      |
| 7     | 2.5    | 4.0         | FP    | 22.5           | 0.26   | 0.35       |
| 7     | 3.5    | 4.1         | 4096  | > 20           | –      | 0.698      |
| 7     | 3.5    | 4.1         | FP    | 31.1           | 0.51   | 0.68       |
| Family 3: $M(0) = 7.65 \times 10^5 M_\odot$ |
| 3     | 3.5    | 4.1         | 4096  | > 20           | –      | 0.633      |
| 3     | 3.5    | 4.1         | FP    | (42.3D)        | (0.085)| 0.58       |
| 7     | 1.5    | 3.7         | 4096  | (3.1D)         | –      | –          |
| 7     | 1.5    | 3.7         | FP    | (4.2D)         | (0.0080)| –        |
| 7     | 2.5    | 4.0         | 4096  | > 20           | –      | 0.374      |
| 7     | 2.5    | 4.0         | FP    | 35.5           | 0.26   | 0.40       |
| 7     | 3.5    | 4.1         | 4096  | > 20           | –      | 0.741      |
| 7     | 3.5    | 4.1         | FP    | 51.3           | 0.48   | 0.71       |
| Family 4: $M(0) = 2.17 \times 10^6 M_\odot$ |
| 3     | 3.5    | 4.1         | 4096  | > 20           | –      | 0.692      |
| 3     | 3.5    | 4.1         | FP    | (43.5D)        | (0.28)| 0.64       |
| 7     | 1.5    | 3.7         | 4096  | (5.0D)         | –      | –          |
| 7     | 1.5    | 3.7         | FP    | (5.9D)         | (0.023)| –          |
| 7     | 2.5    | 4.0         | 4096  | > 20           | –      | 0.443      |
| 7     | 2.5    | 4.0         | FP    | 83.1           | 0.25   | 0.45       |
| 7     | 3.5    | 4.1         | 4096  | > 20           | –      | 0.783      |
| 7     | 3.5    | 4.1         | FP    | 131.3          | 0.49   | 0.77       |

22
Notes
1. Most of the notation is explained in the notes to the previous tables.
2. A number in the column “Model” indicates that the results are for an $N$-body model with the initial value of $N^*$ as stated.
3. Though the time of core collapse, $t_{cc}$, is given as “$> 20$” for those models which stopped at 20Gyr without collapsing, it is not possible to state whether these models would reach core collapse before dissipation. Where the value in this column is enclosed in brackets, with a “D”, the model dissipated at the time stated, without previously undergoing core collapse; for Fokker-Planck models the mass at this time is entered in parentheses in the next column.
Figure Captions

Fig.1 Time scale of mass loss (eq.(8)) for mass functions with minimum mass $0.4M_\odot$ and mass function index $\alpha = 1.5$, 2.5 and 3.5 (where Salpeter is 2.35). The piecewise linear nature of the curves arises from low-order interpolation in the table of evolution time as a function of initial stellar mass (Chernoff & Weinberg 1990).

Fig.2 Comparison of results for the same initial and boundary conditions, but with different initial values of $N^*$: (a) $N^* = 4096$, (b) $N^* = 8192$. The tidal radius and four Lagrangian radii (corresponding to the innermost 0.1%, 1%, 10% and 50% of the mass within the tidal radius, measured from the density centre) are plotted against time.
(a) $W_0 = \gamma$, $\alpha = 2.5$, $R_g = 4\, \text{kpc}$, $M(0) = 1.49 \times 10^5 M_\odot$, $N^* = 4096$
