Upper bound of critical values for contact processes on open clusters of bond percolation

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Abstract
In this paper we are concerned with contact processes on open clusters of bond percolation in $\mathbb{Z}^d$. We give the definitions of the two critical values for the process to survive in the annealed and quenched cases. We show that the two critical values are equal with probability one and are not larger than $[1+o(1)]/(2dp)^{-1}$ for sufficiently large $d$, where $p$ is the probability of edge 'open'. In a weak sense, we show that $1/(2dp)$ is also a critical point for our process.

Keywords: Contact process, percolation, critical value, random walk.

1 Introduction
In this paper we are concerned with contact processes on open clusters of bond percolation in $\mathbb{Z}^d$. We denoted by $E_d$ the set of edges of $\mathbb{Z}^d$. Let $\{X_e\}_{e \in E_d}$ be i. i. d. random variables such that

$$P(X_e = 1) = 1 - P(X_e = 0) = p \in (0, 1).$$

If $X_e = 1$, then the edge $e$ is called 'open' and is remained on $\mathbb{Z}^d$. If $X_e = 0$, then $e$ is called 'closed' and is deleted. After deleting all the closed edges, we obtain a random graph $G$ depending on $\{X_e\}_{e \in E_d}$. The above process is called 'bond percolation', which is introduced in [1] by Broadbent and Hammersley. For any two vertices $x, y \in G$, if there is an open edge connecting $x$ and $y$, then we say that $x$ and $y$ are neighbors, which is denoted by $x \sim y$.

The contact process $\{\eta_t\}_{t \geq 0}$ on $G$ with state space $\{0, 1\}^G$ is a spin system

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(See Chapter 3 of [11]) with flip rates given by

\[
c(x, \eta) = \begin{cases} 
1 & \text{if } \eta(x) = 1 \\
\lambda \sum_{y : y \sim x} \eta(y) & \text{if } \eta(x) = 0
\end{cases} \quad (1.1)
\]

for any \((\eta, x) \in \{0, 1\}^G \times G\), where \(\lambda > 0\) is a parameter called ‘infection rate’.

Intuitively, the contact process describes the spread of an infectious disease. Each vertex represents an individual which may be infected by the disease. 1 and 0 represent the states ‘infected’ and ‘healthy’ respectively. An infected vertex waits for an exponential time with rate 1 to recover. A healthy vertex is infected at a rate in proportion to his infected neighbors.

The main topic we concerned with in this paper is the estimation of the critical value, the definition of which will be given in the next section. We are inspired a lot by related results about contact processes on other graphs. In Chapter 6 of [11], Liggett shows that the critical value \(\lambda_c\) for the contact process on \(Z\) is not larger than 2. He improves this result in [12] by proving that \(\lambda_c \leq 1.92\). In [4], Griffeath shows that \(\lambda_c \approx 1/(2d)\) for contact processes on \(Z^d\). In [13], Pemantle studies the contact process on regular tree \(T_n\). He shows that the first critical value for the process to survive is asymptotically equal to \(1/n\) and the second critical value for a fixed vertex to be infected infinite times is \(O(n^{-1/2})\). In [2], Chatterjee and Durrett show that contact processes on random graphs with power law degree distributions have critical value 0, which points out that the estimation of critical value given by the non-rigorous mean field approach is incorrect. In [15], Peterson shows that the contact process on the complete graph with random vertex-dependent infection rates has critical value inversely proportional to the second moment of the weight of a vertex, which is consistent with the mean field analysis.

We are also inspired a lot by references about the critical probability of percolation model. The critical probability is the minimum of the edge open probability such that the percolation model has an infinite open cluster. In [4], Harris shows that the critical probability \(p_c\) for bond percolation on \(Z^2\) has lower bound 1/2. Kesten improves Harris’ result in [9]. He shows that \(p_c = 1/2\), which is the most important result about critical probability since then. In [10], Kesten shows that \(p_c(d) \approx 1/(2d)\) for percolation on \(Z^d\) as the dimension \(d\) grows to infinity. The references about critical probability are so many that we can not list all of them in this paper. A detailed introduction of this field is in Chapter 1 of [5].

We are inspired by Chen and Yao to consider the contact process on random graph generated in the percolation model. In [3], Chen and Yao show that the complete convergence theorem holds for contact processes on open clusters of \(Z^d \times Z^+\).
2 Main results

We introduce some notations before giving the definition of critical value and giving our main results. For a given graph $G$, we denote by $P_G^\lambda$ the probability measure of the contact process $\{\eta_t\}_{t \geq 0}$ on $G$ with infection rate $\lambda$. We denote by $E_G^\lambda$ the expectation operator with respect to $P_G^\lambda$. We assume that the random variables $\{X_e\}_{e \in E_d}$ are defined on the product measurable space $(\{0,1\}^{E_d}, \mathcal{F}_d, \mathbb{P}_d)$ (see Section 1.3 of [5]), where $\mathbb{P}_d = \prod_{e \in E_d} \mu_e$ and $\mu_e$ is a measure on $\{0,1\}$ such that $\mu_e(1) = 1 - \mu_e(0) = p$ for any $e \in E_d$. We denote by $E_d,p$ the expectation operator with respect to $\mathbb{P}_d$.

For any $\omega \in \{0,1\}^{E_d}$, we denote by $G(\omega)$ the random graph generated in the percolation model depending on $\{X_e(\omega)\}_{e \in E_d}$. For fixed $\omega$, $P_G^\lambda(\omega)$ is called the quenched measure. We define $P_{\lambda,d,p}(\cdot) = E_{d,p}[P_G^\lambda(\omega)(\cdot)]$, which is called the annealed measure. We denote by $E_{\lambda,d,p}$ the expectation operator with respect to $P_{\lambda,d,p}$.

We write $\eta_t$ as $\eta_t^A$ when

$\{x : \eta_0(x) = 1\} = A$.

When all the vertices are infected at $t = 0$, we omit the superscript of $\eta_t$. So in later sections, $\eta_t$ refers to $\eta_t^{Z_d}$.

By basic coupling of spin systems (see Chapter 3 of [1]), $P_G^\lambda(\eta_t(x) = 1)$ is decreasing with $t$ and

$P_{\lambda_1}(\eta_t(x) = 1) \geq P_{\lambda_2}(\eta_t(x) = 1)$

for any $\lambda_1 > \lambda_2$.

Therefore, the following two definitions of critical values are reasonable.

$\tilde{\lambda}_c(G) = \sup \{\lambda : \forall x \in G, \lim_{t \to +\infty} P_G^\lambda(\eta_t(x) = 1) = 0\}$ \hspace{1cm} (2.1)

and

$\lambda_c(d,p) = \sup \{\lambda : \lim_{t \to +\infty} P_{\lambda,d,p}(\eta_t(0) = 1) = 0\}$, \hspace{1cm} (2.2)

where $0$ is the origin of $Z_d$.

According to (2.1) and (2.2), when $\lambda < \tilde{\lambda}_c(G)(\lambda < \lambda_c(d,p))$, $\eta_t$ converges weakly under the quenched (annealed) measure $P_G^\lambda (P_{\lambda,d,p})$ to the state that all the vertices are healthy. Therefore, $\tilde{\lambda}_c(G)$ and $\lambda_c(d,p)$ are critical values for the process to survive in the quenched and annealed cases.
The difference between the two definitions in (2.1) and (2.2) is due to the fact that our model is symmetric under the annealed measure but not symmetric under the quenched measure.

The following two theorems about the estimations of critical values are our main results. The first theorem shows that the two critical values in the annealed and quenched cases are equal with probability one.

**Theorem 2.1.** For any \( d \geq 1 \) and \( p \in (0, 1] \), there exists \( A_{d,p} \in F_d \) such that

\[
\mathbb{P}_{d,p}(A_{d,p}) = 1
\]

and

\[
\tilde{\lambda}_c(G(\omega)) = \lambda_c(d, p)
\]

(2.3)

for any \( \omega \in A_{d,p} \).

The next theorem is about the asymptotic behavior of \( \lambda_c(d, p) \) as \( d \) grows to infinity.

**Theorem 2.2.** For any \( p \in (0, 1] \),

\[
\limsup_{d \to +\infty} 2dp\lambda_c(d, p) \leq 1.
\]

(2.4)

(2.4) shows that \( \lambda_c(d, p) \leq [1 + o(1)](2dp)^{-1} \) for sufficiently large \( d \). We guess that

\[
\lim_{d \to +\infty} 2dp\lambda_c(d, p) = 1
\]

but we have not find an approach to prove \( \liminf_{d \to +\infty} 2dp\lambda_c(d, p) \geq 1 \) yet.

We can show that \( 1/(2dp) \) is a critical point for our process in a weak sense, which is the following theorem.

**Theorem 2.3.** Let

\[
\mu(\lambda, d, p) = \lim_{t \to +\infty} P_{\lambda, d, p}(\eta_t(0) = 1).
\]

Then, for \( \gamma > 1 \), there exists \( A > 0 \), and \( B > 0 \) which do not depend on \( \gamma \) such that,

\[
\liminf_{d \to +\infty} \mu(\frac{\gamma}{2dp}, d, p) \geq \frac{\gamma(\gamma - 1)}{A(\gamma - 1)(\gamma + 1) + B} > 0.
\]

(2.5)

For \( \gamma < 1 \),

\[
\lim_{d \to +\infty} \mu(\frac{\gamma}{2dp}, d, p) = 0.
\]

(2.6)

Our guess that

\[
\lim_{d \to +\infty} 2dp\lambda_c(d, p) = 1
\]
will be true if one can improve (2.6) to a stronger conclusion that \( \mu(\gamma, d, p) = 0 \) for fixed \( \gamma < 1 \) and sufficiently large \( d \), we hope readers can find a way and teach us.

The proofs of Theorem 2.2 and (2.5) will be given in Section 3, which is the main task of this paper. The proof of (2.6) will be given in Section 4.

At the end of this section, we give the proof of Theorem 2.1.

Proof of Theorem 2.1. For any \( x \in \mathbb{Z}^d \) and \( e \in E_d \) connecting \( y, z \), we denote by \( x + e \) the edge connecting \( y + x, z + x \). For any \( x \in \mathbb{Z}^d \), we define \( T_x : \{0, 1\}^{E_d} \to \{0, 1\}^{E_d} \) as

\[
T_x(\omega)(e) = \omega(x + e)
\]

for any \( \omega \in \{0, 1\}^{E_d} \) and \( e \in E_d \). It is obviously that

\[
\hat{\lambda}_c(G(\omega)) = \hat{\lambda}_c(G(T_x(\omega)))
\]

for any \( x \in \mathbb{Z}^d \) and \( \omega \in \{0, 1\}^{E_d} \).

Therefore, according to the ergodic theory of i.i.d. measures, there exist \( A_{d,p} \in \mathcal{F}_d \) and \( \hat{\lambda}_c(d, p) \geq 0 \) such that

\[
\mathbb{P}_{d,p}(A_{d,p}) = 1
\]

and

\[
\hat{\lambda}_c(G(\omega)) = \hat{\lambda}_c(d, p)
\]

for any \( \omega \in A_{d,p} \).

Now we only need to show that \( \lambda_c(d, p) = \hat{\lambda}_c(d, p) \). For any \( \lambda < \hat{\lambda}_c(d, p) \),

\[
\lim_{t \to +\infty} P_{\lambda, d, p}(\eta_t(0) = 1) = \mathbb{E}_{d,p} \left[ \lim_{t \to +\infty} P_{\lambda}^{G(\omega)}(\eta_t(0) = 1) \right]
\]

\[
= \mathbb{E}_{d,p} \left[ \lim_{t \to +\infty} P_{\lambda}^{G(\omega)}(\eta_t(0) = 1) \right]
\]

\[
= 0.
\]

Therefore \( \lambda \leq \lambda_c(d, p) \). As a result,

\[
\hat{\lambda}_c(d, p) \leq \lambda_c(d, p).
\] (2.7)

For any \( \lambda < \lambda_c(d, p) \),

\[
0 = \lim_{t \to +\infty} P_{\lambda, d, p}(\eta_t(x) = 1)
\]

\[
= \mathbb{E}_{d,p} \left[ \lim_{t \to +\infty} P_{\lambda}^{G(\omega)}(\eta_t(x) = 1) \right]
\]

for each \( x \in \mathbb{Z}^d \).

Therefore, for each \( x \in \mathbb{Z}^d \),

\[
\lim_{t \to +\infty} P_{\lambda}^{G(\omega)}(\eta_t(x) = 1) = 0
\]
with probability one.

Because there are countable vertices on $\mathbb{Z}^d$, there exists $B_{d,p} \in \mathcal{F}_d$ such that $\mathbb{P}_{d,p}(B_{d,p}) = 1$ and for any $\omega \in B_{d,p}$,

$$\lim_{t \to +\infty} P^G(\omega)(\eta_t(x) = 1) = 0, \ \forall x \in \mathbb{Z}^d.$$ 

Therefore,

$$\lambda \leq \tilde{\lambda}_c(G(\omega))$$

for any $\omega \in B_{d,p}$.

$A_{d,p} \cap B_{d,p} \neq \emptyset$ since $\mathbb{P}_{d,p}(A_{d,p} \cap B_{d,p}) = 1$. We choose $\omega_0 \in A_{d,p} \cap B_{d,p}$, then

$$\lambda \leq \tilde{\lambda}_c(G(\omega_0)) = \tilde{\lambda}_c(d,p).$$

As a result,

$$\lambda_c(d,p) \leq \tilde{\lambda}_c(d,p). \quad (2.8)$$

Theorem 2.1 follows (2.7) and (2.8).

\[ \square \]

3 The proofs of Theorem 2.2 and (2.5)

Our proof of Theorem 2.2 and (2.5) is inspired a lot by the approach introduced in [10] by Kesten.

In our proof, we firstly show in Lemma 3.2 that the estimation of upper bound of the annealed critical value is connected with a random walk on $\mathbb{Z}^d$ which is not time-homogeneous. In the second step, we find $\lambda$ which satisfies the condition of Lemma 3.2 to complete the proof.

3.1 Upper bound and a time-nonhomogeneous random walk

In this subsection, we will introduce a time-nonhomogeneous random walk on $\mathbb{Z}^d$ and give the connection between the upper bound of the annealed critical value and this random walk, which is Lemma 3.3 at the end of this subsection.

We need introduce some notations first. For each $e \in E_d$, let $\{X_e(i)\}_{i=1}^{+\infty}$ be independent identically distributed random variables such that $X_e(1) = X_e$ which is introduced in Section 1. Furthermore, we assume that $\{\{X_e(i)\}_{i=1}^{+\infty}\}_{e \in E_d}$ are independent. For each $n \geq 1$, we define

$$L_n = \{y \in \bigoplus_{j=1}^n \mathbb{Z}^d : \|y_1\| = 1, \|y_{j+1} - y_j\| = 1 \text{ for } 1 \leq j \leq n - 1\}. \quad (3.1)$$
where
\[\|x\| = \sum_{l=1}^{d} |x_l|\]
is the \(l_1\) norm. For later use, we define \(y_0 = 0\).

Please notice the difference between \(\|x - y\| = 1\) and \(x \sim y\). We write \(x \sim y\) when and only when the edge connecting \(x\) and \(y\) is open in the percolation model.

For later use, for any \(A \subseteq Z^d\), we define
\[A_t^A = \{x \in Z^d : \eta_t^A(x) = 1\}\]
as the set of infected vertices at \(t\). According to the self-duality of contact processes (see Part one of [13]),
\[P^G_{\lambda}(\eta_t(0) = 1) = P^G_{\lambda}(A_t^0 \neq \emptyset)\]
and
\[P_{\lambda,d,p}(\eta_t(0) = 1) = E_{d,p}[P^G_{\lambda}(\eta_t(0) = 1)]\]
\[= E_{d,p}[P^G_{\lambda}(A_t^0 \neq \emptyset)] = P_{\lambda,d,p}(A_t^0 \neq \emptyset).\]

Therefore,
\[\mu(\lambda, d, p) = \lim_{t \to +\infty} P_{\lambda,d,p}(\eta_t(0) = 1) = P_{\lambda,d,p}(\forall t > 0, A_t^0 \neq \emptyset). \quad (3.2)\]

To control \(\{\forall t > 0, A_t^0 \neq \emptyset\}\) from below, we utilize the graphic representation of contact processes introduced in [7] by Harris.

We erect an time line on each \(x \in Z^d\) to form the graph \(Z^d \times [0, +\infty)\). For each \(x \in Z^d\) and \(\|y - x\| = 1\), let \(\{T_x(i)\}_{i=1}^{+\infty}\) be i.i.d. exponential times with rate one and let \(\{U_{(x,y)}(i)\}_{i=1}^{+\infty}\) be i.i.d. exponential times with rate \(\lambda\). We assume that all these exponential times are independent and are independent with \(\{X_x(i)\}_{i=1}^{+\infty}\) for \(x \in Z^d\), \(\|y - x\| = 1\) and each integer \(l \geq 1\), we write a \(\delta\) at \((x, \sum_{j=1}^{l} T_x(j)) \in Z^d \times [0, +\infty)\) and write an arrow from \((x, \sum_{j=1}^{l} U_{(x,y)}(j))\) to \((y, \sum_{j=1}^{l} U_{x,y}(j))\).

For \(y \in L_n\), we say that \(y\) is a special infection path with length \(n\) when the following four conditions holds.

(a) There exist \(0 < t_1 < t_2 < \ldots < t_n\) such that there is an arrow from \((y_{i-1}, t_i)\) to \((y_i, t_i)\) for \(1 \leq i \leq n\).
(b) For \(1 \leq i \leq n\), there is no \(\delta\) at \((y_{i-1}, s)\) for \(s \in [t_{i-1}, t_i]\).
(c) For \(1 \leq i \leq n\), there is no arrow from \((y_{i-1}, s)\) to \((y_i, s)\) for \(s \in [t_{i-1}, t_i]\).
(d) For each \(e \in E_d\), if \(y\) goes through \(e\) for \(k \geq 1\) times, then \(X_e(j) = 1\) for \(1 \leq j \leq k\).
It is easy to see that if \( y \) is a special infection path, then \( y_n \in \bigcup_{t \geq 0} A^t \).

We denote by \( I_y \) the event that \( y \) is a special infection path with length \( n \). According to the Markov property,

\[
P(I_y) = p^n \left[ P(U(x,y)(1) < T_x(1)) \right]^n = \left( \frac{\lambda p}{\lambda + 1} \right)^n.
\] (3.3)

For \( y, y' \in L_n \), we need to estimate the value of \( P(I_y \cap I_{y'}) \). For this purpose, we define

\[
K_1(y, y') = \{ 0 \leq i \leq n - 1 : \text{there exists } 0 \leq j \leq n \\
\text{such that } y_j = y'_i \text{ and } y_{j+1} = y'_{i+1} \}. \tag{3.4}
\]

and

\[
K_2(y, y') = \{ 0 \leq i \leq n : i \notin K_1(y, y') \text{ and } \\
\text{there exists } 0 \leq j \leq n \text{ such that } y_j = y'_i \}. \tag{3.5}
\]

Please notice that \( K_1, K_2 \) are not symmetric for \( y, y' \).

For \( i \in K_2(y, y') \), we define

\[
C(y, y')(i) = \sum_{j=0}^{n} 1_{\{y_j = y'_i\}} \tag{3.6}
\]

as the times \( y \) visits \( y'_i \).

Then we have the following lemma of the upper bound of \( P(I_y \cap I_{y'}) \).

**Lemma 3.1.**

\[
P(I_y \cap I_{y'}) \leq \left( \frac{\lambda p}{1 + \lambda} \right)^{2n - |K_1(y, y')|} \exp \left\{ \sum_{i \in K_2(y, y')} C(y, y')(i) \log 2 \right\}. \tag{3.7}
\]

We put the proof of Lemma 3.1 in the appendix.

The following definitions is inspired by the approach introduced in [10]. For sufficiently large \( d \), let

\[
N = \left\lfloor \frac{\log d}{2 \log \log d} \right\rfloor.
\]

We assume that \( d \) is so large that \( N > 10 \).

We use \( N[i] \) to denote \( i \) is a multiple of \( N \).

For \( 1 \leq i \leq d \), let

\[
ed_i = (0, \ldots, 0, 1, 0, \ldots, 0).
\]

Then we define \( J \subseteq \bigoplus_{i=0}^{+\infty} Z^d \) as follows. For any \( y \in \bigoplus_{i=0}^{+\infty} Z^d \), \( y \in J \) if and only if the following three conditions holds.
(a) \( y_0 = 0 \).
(b) For any \( i \geq 1 \), if \( N | i \), then \( y_i - y_{i-1} \in \{ e_j : d - \lfloor \frac{d}{N} \rfloor < j \leq d \} \).
(c) For any \( i \geq 1 \), if \( N \not| i \), then \( y_i - y_{i-1} \in \{ \pm e_j : 1 \leq j \leq d - \lfloor \frac{d}{N} \rfloor \} \).

For any \( y \in J \) and \( n \geq 1 \), we define
\[
y|_n = (y_1, y_2, \ldots, y_n) \in L_n.
\]

For \( n \geq 1 \), we define
\[
J_n = \{ x \in L_n : \text{there exists } y \in J \text{ such that } y|_n = x \}.
\]

We consider the following random walk \( \{ S_n \}_{n=0}^{+\infty} \) on \( Z^d \). \( S_0 = 0 \). For \( n \geq 1 \), if \( N | n \), then
\[
P(S_n - S_{n-1} = e_j) = \frac{1}{\lfloor \frac{d}{N} \rfloor}
\]
for \( d - \lfloor \frac{d}{N} \rfloor < j \leq d \).
If \( N \not| n \), then
\[
P(S_n - S_{n-1} = e_j) = P(S_n - S_{n-1} = -e_j) = \frac{1}{2(d - \lfloor \frac{d}{N} \rfloor)}
\]
for \( 1 \leq j \leq d - \lfloor \frac{d}{N} \rfloor \).
We define
\[
S = (S_0, S_1, \ldots, S_n, \ldots)
\]
as the trajectory of \( \{ S_n \}_{n=0}^{+\infty} \). Then \( S \in J \) and \( S|_n \in J_n \) for \( n \geq 1 \).
Let \( S' \) be a independent copy of \( S \). Similar with (3.4) and (3.5), we define
\[
K_1(S, S') = \{ i \geq 0 : \text{there exists } j \geq 0 \text{ such that } S_j = S'_i \text{ and } S_{j+1} = S'_{i+1} \}
\]
(3.8)
and
\[
K_2(S, S') = \{ i \geq 0 : i \not\in K_1(S, S') \text{ and there exists } j \geq 0 \text{ such that } S_j = S'_i \}
\]
(3.9)
The following lemma is crucial for us to give upper bound of \( \lambda_c(d, p) \).

**Lemma 3.2.** If \( \lambda > 0 \) makes
\[
E \left[ 2^{N|K_2(S, S')|} \left( \frac{1 + \lambda}{\lambda p} \right)^{|K_1(S, S')|} \right] < +\infty,
\]
then
\[
\lambda \geq \lambda_c(d, p).
\]

9
Proof. For any $y \in J_{kN}, k \geq 1$,
\[
\sum_{j=d-\lfloor \frac{d}{N} \rfloor +1}^{d} y_{kN}(j) = k. \tag{3.11}
\]
If $y \in J_{kN}$ is a special infection path, then in the sense of coupling,
\[
y_{kN} \in \bigcup_{t \geq 0} A^0_t. \tag{3.12}
\]
By (3.11) and (3.12),
\[
\{ \forall t, A^0_t \neq \emptyset \} = \{ \lfloor \bigcup_{t \geq 0} A^0_t \rfloor = +\infty \} \supseteq \bigcap_{k=1}^{+\infty} \bigcup_{y \in J_{kN}} I_y. \tag{3.13}
\]
By (3.2) and (3.13),
\[
\mu(\lambda, d, p) \geq \lim_{k \to +\infty} P\left( \bigcup_{y \in J_{kN}} I_y \right). \tag{3.14}
\]
For $y \in J_{kN}$, we define random variable
\[
\chi_y = \begin{cases} 1 & \text{on } I_y, \\ 0 & \text{on the complement of } I_y. \end{cases}
\]
Then, according to (3.13),
\[
P\left( \bigcup_{y \in J_{kN}} I_y \right) = P\left( \sum_{y \in J_{kN}} \chi_y > 0 \right) \geq \frac{\left[ E\left( \sum_{y \in J_{kN}} \chi_y \right) \right]^2}{E\left( \sum_{y \in J_{kN}} \chi_y \right)^2} = \frac{|J_{kN}|^2 P^2(I_y)}{\sum_{y \in J_{kN}} \sum_{y' \in J_{kN}} P(I_y \cap I_{y'})} = \frac{P^2(I_y)}{P(I_{S|kN} \cap I_{S'|kN})} = \frac{(\lambda p)^{2kN}}{P(I_{S|kN} \cap I_{S'|kN})}. \tag{3.15}
\]
By Lemma 3.1,
\[
P(I_{S|kN} \cap I_{S'|kN}) \leq E\left[ \frac{\lambda p}{1 + \lambda} \right]^{2kN-|K_1(S_{kN}, S'|_{kN})|} \exp\left\{ \sum_{j \in K_2(S_{kN}, S'|_{kN})} \sum_{j \in K_2(S_{kN}, S'|_{kN})} C(S_{kN}, S'|_{kN})(j) \log 2 \right\}. \tag{3.16}
\]
For any $y \in J$ and $l \geq 1$,
\[ \sum_{j=d-\lfloor \frac{d}{N} \rfloor +1}^{d} y_i(j) = l \]
if and only if
\[ lN \leq i < (l+1)N - 1. \]
Hence, for any $j \geq 0$, $S$ visits $S'_j$ for at most $N$ times.

As a result,
\[ C_{(S|_{kN},S'|_{kN})}(j) \leq N \]
for any $j \in K_2(S|_{kN},S'|_{kN})$.

By (3.16) and (3.17),
\[ P(I_{S|_{kN}} \cap I_{S'|_{kN}}) \leq E[(\lambda p)^{2kN-|K_1(S|_{kN},S'|_{kN})|}2^{K_2(S|_{kN},S'|_{kN})}] . \]  (3.18)

By (3.16) and (3.18),
\[ P\left( \bigcup_{y \in J_{kN}} I_y \right) \geq \frac{1}{E[(\lambda p)^{K_1(S|_{kN},S'|_{kN})}2^{K_2(S|_{kN},S'|_{kN})}]}. \]  (3.19)

According to (3.14), (3.19) and the fact that
\[ \lim_{k \to +\infty} K_1(S|_{kN},S'|_{kN}) = K_1(S,S') \]
and
\[ \lim_{k \to +\infty} K_2(S|_{kN},S'|_{kN}) = K_2(S,S'), \]
\[ \lim_{l \to +\infty} P_{\lambda,d,p}(\eta_l(0) = 1) = \mu(\lambda,d,p) \geq \frac{1}{E[(\lambda p)^{K_1(S,S')}2^{K_2(S,S')}]} > 0 \]  (3.20)
when $\lambda$ makes
\[ E\left[2^{N(K_2(S,S'))} \left(\frac{1 + \lambda}{\lambda p}\right)^{|K_1(S,S')|}\right] < +\infty. \]

\[ \square \]

### 3.2 $\lambda$ which satisfies (3.10) and the proof of Theorem 2.2

According to Lemma 3.2, we need to find $\lambda$ which satisfies (3.10). For simple random walk $\{W_n\}_{n=0}^{+\infty}$ on $Z^m$ with $W_0 = 0$, the following estimations proved by Kesten in [8] will be repeatedly used in our approach to find proper $\lambda$.

\[ \sup_{x \in Z^m} P(W_{2n+1} = x) \leq \sup_{x \in Z^m} P(W_{2n} = x) = P(W_{2n} = 0). \]  (3.21)
\[ \sum_{n=1}^{\infty} P(W_{2n} = 0) \leq \frac{1}{2m}(1 + \frac{M}{m}). \quad (3.22) \]

\[ \sum_{n=2}^{\infty} P(W_{2n} = 0) \leq \frac{M}{m^2}. \quad (3.23) \]

\[ \sup_{x \in \mathbb{Z}^m} P(W_1 = x) = \frac{1}{2m}, \quad \sup_{x \in \mathbb{Z}^m, x \neq 0} P(W_2 = x) \leq \frac{M}{m^2}. \quad (3.24) \]

\[ \sup_{x \in \mathbb{Z}^m} P(W_3 = x) \leq \frac{M}{m}. \quad (3.25) \]

From (3.22) to (3.25), \( M \) is a constant which does not depend on the dimension \( m \).

For any \( k \geq 0 \), we say that \([kN, (k+1)N - 1]\) is a good interval when \( S_i \neq S_j \) for any \( kN \leq i \neq j \leq (k+1)N - 1 \). Otherwise, we say that \([kN, (k+1)N - 1]\) is a bad interval.

We introduce the definition of special integers in \( \{0,1,2,\ldots\} \). For a good interval \([kN, (k+1)N - 1]\), if \( i \in [kN, (k+1)N - 1] \cap K_1(S,S') \), then we say that \( i \) is a special integers. For a bad interval \([lN, (l+1)N - 1]\), if \([lN, (l+1)N - 1] \cap K_1(S,S') = \emptyset \), then there are no special integers in \([lN, lN+1, \ldots, (l+1)N - 1] \). If \([lN, (l+1)N - 1] \cap K_1(S,S') \neq \emptyset \), then we define

\[ j = \sup\{lN \leq i \leq (l+1)N - 1 : i \in K_1(S,S')\} \]

as the unique special integers in \([lN, (l+1)N - 1] \). Please note that there may be more than one special integers in a good interval.

We denote by \( \rho \) the total number of special integers. For \( j \geq 1 \), if \( \rho \geq j \), then we denote by \( t(j) \) the \( j \)th special integer. For later use, we define \( t(0) = -1 \).

For any \( j \in K_1(S,S') \) and \( \bar{j} = \inf\{k > j : k \in K_1(S,S')\} \), we define

\[ C(j) = \sum_{i=j+1}^{\bar{j}-1} 1_{\{i \in K_2(S,S')\}}. \]

For \( i \geq 1 \) and \( i \leq \rho \), we define \( L(i) \) as follows.

\[ L(i) = \begin{cases} 
(\frac{\lambda+1}{\lambda p})^{2NC(t(\rho))} & \text{if } i = \rho, \\
(\frac{\lambda+1}{\lambda p})^{2NC(t(i))} & \text{if } i < \rho \text{ and } t(i + 1) \text{ belongs to a good interval}, \\
t^{(i+1)-1} \prod_{l=t(i)}^{t(i+1)-1} (\frac{\lambda+1}{\lambda p})^{2NC(l)} 1_{\{i \in K_1(S,S')\}} & \text{if } i < \rho \text{ and } t(i + 1) \text{ belongs to a bad interval}.
\end{cases} \quad (3.26) \]
Furthermore, we define

\[
L(0) = \begin{cases}
2^{N|K_2(S,S')|} & \text{if } \rho = 0, \\
1 & \text{if } t(1) = 0, \\
\exp \left\{ N \left[ \sum_{j=0}^{t(1)-1} 1_{j \in K_2(S,S')} \right] \log 2 \right\} & \text{if } t(1) > 0 \text{ and } t(1) \text{ belongs to a good interval,} \\
\exp \left\{ N \left[ \inf_{K_1(S,S')} \sum_{j=0}^{t(1)-1} 1_{j \in K_2(S,S')} \right] \log 2 \right\} \prod_{l=\inf_{K_1(S,S')}}^{t(1)-1} \left( \frac{\lambda+1}{\lambda p} \right)^{2^{NC(l)}} & \text{if } t(1) > 0 \text{ and } t(1) \text{ belongs to a bad interval.}
\end{cases}
\]

(3.27)

It is easy to see that

\[
E \left[ \left( \frac{\lambda+1}{\lambda p} \right)^{|K_1(S,S')|} 2^{N|K_2(S,S')|} \right] = E \left[ \prod_{0 \leq j \leq \rho} L(j) \right].
\]

Then, according to Lemma 3.2,

\[
\lambda \geq \lambda_c
\]

when \( \lambda \) makes

\[
E \left[ \prod_{0 \leq j \leq \rho} L(j) \right] < +\infty.
\]

(3.28)

To find \( \lambda \) which satisfies (3.28), we divide the special integers into four groups. For \( i \geq 1 \), we say that the special integer \( t(i) \) is of type 1 if \( t(i) \) belongs to a good interval, \( t(i) - t(i-1) = 1 \) and \( t(i) + 1 \) is not a multiple of \( N \).

We say that \( t(i) \) is of type 2 if \( t(i) \) belongs to a good interval, \( t(i) - t(i-1) = 1 \) and \( t(i) + 1 \) is a multiple of \( N \).

We say that \( t(i) \) is of type 3 if \( t(i) \) belongs to a good interval and \( t(i) - t(i-1) \geq 2 \).

We say that \( t(i) \) is of type 4 if \( t(i) \) belongs to a bad interval.

We denote by \( \mathcal{F}_j \) the \( \sigma \)-algebra generated by \( t(j) \), \( \{ S_n', n \leq t(j) + 1 \} \) and \( \{ S_n, n \leq N \left\lceil \frac{t(j)+1}{N} \right\rceil \} \). Now, our main task is to estimate

\[
E \left[ L(j); \{ \rho \geq j+1, t(j+1) \text{ is of type } l \} \bigg| \mathcal{F}_j \right]
\]

(3.29)

on the event

\( \{ \rho \geq j, t(j) \text{ is of type } m \} \)

for \( 1 \leq m, l \leq 4 \).

We denote the expression in (3.29) by \( E(j, m, l) \). The following lemma is important for us to estimate \( E(j, m, l) \).
Lemma 3.3. For \( j \geq 0 \), let
\[
G_j = \sigma(S_n, n \geq 0) \lor \sigma(S_n', n \leq j).
\]
Then,
\[
P\left( \exists i \geq j + 1 \text{ such that } i \in K_1(S,S') \cup K_2(S,S') \mid G_j \right) \leq \frac{M_1 N}{d} \quad (3.30)
\]
on the event \( \{ j \in K_1(S,S') \cup K_2(S,S') \} \) for sufficiently large \( d \), where \( M_1 \) is a constant which does not depend on \( d \).

Proof. For integers \( i, r \geq 0 \), we say that \( i \) and \( r \) are in the same interval with length \( N - 1 \) if there exists \( k \geq 1 \) such that \( kN \leq i, r \leq (k + 1)N - 1 \). We denote by \( i \sim r \) that \( i \) and \( r \) are in the same interval.

If \( i \) and \( r \) are not in the same interval, then
\[
\sum_{l=0}^{d} S'_i(l) \neq \sum_{l=0}^{d} S_r(l).
\]
As a result, if \( S'_i = S_r \) for some \( i, r \), then \( i \sim r \).

Therefore, the left side of (3.30) equals
\[
P(\exists i \geq j + 1 \text{ such that } S'_i = S_r \text{ for some } i \sim r \mid G_j). \quad (3.31)
\]
For \( x \in \mathbb{Z}^d \), we define
\[
\hat{x} = (x(1), x(2), \ldots, x(d - \lfloor \frac{d}{N} \rfloor)) \in \mathbb{Z}^{d - \lfloor \frac{d}{N} \rfloor}
\]
and
\[
\bar{x} = (x(d - \lfloor \frac{d}{N} \rfloor + 1), x(d - \lfloor \frac{d}{N} \rfloor + 2), \ldots, x(d)) \in \mathbb{Z}^{\lfloor \frac{d}{N} \rfloor}.
\]
By (3.31),
\[
P(\exists i \geq j + 1 \text{ such that } i \in K_1(S,S') \cup K_2(S,S') \mid G_j)
\leq P\left( \exists j + 1 \leq i \leq j + 4 \text{ such that } S'_i = S_r \text{ for some } i \mid G_j \right)
\leq P\left( \exists i \geq j + 1 \text{ such that } S'_i = S_r \text{ for some } i \mid G_j \right)
\leq \sum_{i=j+1}^{j+4} P(\hat{S}_i = S_r \text{ for some } i \mid G_j). \quad (3.32)
\]
For the first part of the right-side of (3.32),
\[
P\left( \exists j + 1 \leq i \leq j + 4 \text{ such that } S'_i = S_r \text{ for some } i \mid G_j \right)
\leq \sum_{i=j+1}^{j+4} P\left( S'_i = S_r \text{ for some } i \mid G_j \right). \quad (3.33)
\]
For $j + 1 \leq i \leq j + 4$, if there exists $k \geq 1$ such that $j < kN \leq i$, then

$$P\left( S'_i = S_r \text{ for some } r \sim i \mid G_j \right)$$

$$\leq P\left( \tilde{S}'_{kN} = \tilde{S}_{kN} \mid G_j \right)$$

$$= P\left( \tilde{S}'_{kN} - \tilde{S}'_{kN-1} = \tilde{S}_{kN} - \tilde{S}'_{kN-1} \mid G_j \right)$$

$$= P\left( \tilde{S}'_{kN} - \tilde{S}'_{kN-1} = \tilde{S}_{kN} - \tilde{S}'_{(k-1)N} \mid G_j \right) = \frac{1}{\left\lfloor \frac{d}{N} \right\rfloor}. \quad (3.34)$$

If there are no multiple of $N$ in $(j, i]$, then

$$P\left( S'_i = S_r \text{ for some } r \sim i \mid G_j \right)$$

$$\leq P\left( \tilde{S}'_i - \tilde{S}'_j = \tilde{S}_r - \tilde{S}'_j \text{ for some } r \sim i \mid G_j \right)$$

$$\leq N \sup_{x \in \mathbb{Z}^d - \left\lfloor \frac{d}{N} \right\rfloor} P\left( \tilde{S}'_i - \tilde{S}'_j = x \right). \quad (3.35)$$

Let $\{W_n\}_{n=0}^{+\infty}$ be simple random walk on $\mathbb{Z}^d - \left\lfloor \frac{d}{N} \right\rfloor$ with $W_0 = 0$, then $\tilde{S}'_i - \tilde{S}'_j$ has the same distribution as $W_{i-j}$.

Therefore, by (3.33), (3.34) and (3.36),

$$P\left( \exists j + 1 \leq i \leq j + 4 \text{ such that } S'_i = S_r \text{ for some } r \sim i \mid G_j \right) \leq \frac{M_2N}{d}, \quad (3.37)$$

where $M_2$ does not depend on $d$.

For $i > j$, we define

$$u(i, j) = \sup\{k : kN \leq i\} - \inf\{l : lN > j\} + 1$$

as the number of integers which are multiple of $N$ in $(j, i]$.

Then for $i > j + 4$,

$$i - j - u(i, j) \geq 4 \quad (3.38)$$

and

$$P\left( S'_i = S_r \text{ for some } r \sim i \mid G_j \right)$$

$$\leq P\left( W_{i-j-u(i, j)} = \tilde{S}_r - \tilde{S}'_j \text{ for some } r \sim i \mid G_j \right)$$

$$\leq N \sup_{x \in \mathbb{Z}^d - \left\lfloor \frac{d}{N} \right\rfloor} P\left( W_{i-j-u(i, j)} = x \right), \quad (3.39)$$

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where \( \{W_n\}_{n=0}^{\infty} \) is independent of \( S \) and \( S' \).

For \( i > i' \), \( i-j-u(i,j) = i'-j-u(i',j) \) if and only if
\[
i' + 1 = i = kN
\]  
(3.40)

for some \( k \geq 1 \).

By (3.39), (3.37) and (3.40),
\[
P(\exists \ i > j + 4 \text{ such that } S'_i = S_r \text{ for some } r \sim i \mid G_j) \
\leq \sum_{i=j+4}^{\infty} P(S'_i = S_r \text{ for some } r \sim i \mid G_j) \
\leq N \sum_{i=j+4}^{\infty} \sup_{x \in \mathbb{Z}^d - \lfloor \frac{d}{N} \rfloor} P(W_{i-j-u(i,j)} = x) \
\leq 2N \sum_{i=4}^{\infty} \sup_{x \in \mathbb{Z}^d - \lfloor \frac{d}{N} \rfloor} P(W_i = x). \quad (3.41)
\]

Then by (3.21), (3.23) and (3.41),
\[
P(\exists i > j + 4 \text{ such that } S'_i = S_r \text{ for some } r \sim i \mid G_j) \leq \frac{4NM}{(d - \lfloor \frac{d}{N} \rfloor)^2}. \quad (3.42)
\]

Lemma 3.3 follows (3.32), (3.37) and (3.42).

\[\Box\]

\textbf{Lemma 3.4.} For any \( i \geq 1 \),
\[
E(i, m, 1) \leq \left( \frac{1 + \lambda}{\lambda p} \right) \frac{1}{2(d - \lfloor \frac{d}{N} \rfloor)} \quad (3.43)
\]

for \( 1 \leq m \leq 4 \).

\textbf{Proof.} When \( t(i+1) - t(i) = 1 \), then \( L(i) = \frac{\lambda + 1}{\lambda p} \). When \( t(i+1) \) is of type one, then \( t(i+1) \) belongs to a good interval \([IN, (l+1)N-1] \), and there exists a unique \( r \in [IN, (l+1)N-1] \) such that \( S'_t(i+1) = S_r \) and \( S'_t(i+1) = S_{r+1} \). Therefore, \( S'_{t(i+1)} - S_{t(i+1)} \) has only one choice which is \( S_{r+1} - S_r \). Since \( N \lfloor t(i+1) + 1 \rfloor, \hat{S}'_{t(i+1)} - \hat{S}'_{t(i+1)} \) has the same distribution as \( W_1 \).

As a result,
\[
E(i, m, 1) \leq \left( \frac{\lambda + 1}{\lambda p} \right) \sup_{x \in \mathbb{Z}^d - \lfloor \frac{d}{N} \rfloor} P(W_1 = x) = \left( \frac{1 + \lambda}{\lambda p} \right) \frac{1}{2(d - \lfloor \frac{d}{N} \rfloor)}
\]

for \( i \geq 1 \) and \( 1 \leq m \leq 4 \). \[\Box\]
Lemma 3.5. For any $i \geq 1$,

$$E(i, m, 2) \leq \begin{cases} 
\left(\frac{1+\lambda}{\lambda p}\right) \frac{M_3 N}{d} & \text{if } m = 1, 3, 4, \\
0 & \text{if } m = 2
\end{cases} \quad (3.44)$$

for sufficiently large $d$, where $M_3$ does not depend on $d$.

Proof. When $t(i)$ is of type 2 and $t(i+1) - t(i) = 1$, then $t(i+1) = kN$ for some $k$ and $t(i+1) + 1 = kN + 1$ which is not a multiple of $N$. Hence the case of $m = 2$ in (3.44) holds.

The proof of cases of $m = 1, 3, 4$ are similar as the proof of Lemma 3.4. The only difference is that when $t(i+1)$ is of type 2, the distribution of $S'_{t(i+1)+1} - S'_{t(i+1)}$ is

$$P(S'_{t(i+1)+1} - S'_{t(i+1)} = \epsilon_q) = \frac{1}{d \left\lfloor \frac{d}{N} \right\rfloor}$$

for $d - \left\lfloor \frac{d}{N} \right\rfloor + 1 \leq q \leq d$.

Therefore,

$$E(i, m, 2) \leq \left(\frac{1+\lambda}{\lambda p}\right) \frac{1}{d \left\lfloor \frac{d}{N} \right\rfloor} \leq \left(\frac{1+\lambda}{\lambda p}\right) \frac{M_3 N}{d}$$

for $m = 1, 3, 4$.

\qed

Lemma 3.6. For $i \geq 1$,

$$E(i, m, 3) \leq \frac{(1+\lambda)}{\lambda p} \frac{M_4 N^{2N}}{d^2} \quad (3.45)$$

for $1 \leq m \leq 4$ and sufficiently large $d$, where $M_4$ does not depend on $d$.

Proof. When $t(i+1)$ is of type 3, then $t(i+1) = \inf\{k > t(i) : k \in K_1(S, S')\}$ and

$$L(i) = \left(\frac{\lambda + 1}{\lambda p}\right) 2^{C(t(i))}.$$ 

When $t(i+1)$ is of type 3, then $t(i+1) + 1 \in K_1(S, S') \cup K_2(S, S')$ but $t(i+1) - t(i) \geq 2$. So $t(i) + 1 \in K_2(S, S')$ and $C(t(i)) \geq 1$.

For any $q \geq 1$, if $C(t(i)) = q$, $\rho \geq i + 1$ and $t(i+1)$ is of type 3, then there exists $t(i) + 1 = l_1 < l_2 < \ldots < l_q < l_{q+1}$ such that $l_k \in K_2(S, S')$ for $1 \leq k \leq q$, $l_{q+1} \in K_1(S, S')$ and $l_{q+1} + 1 \in K_1(S, S') \cup K_2(S, S')$.

Therefore,

$$\{t(i+1) \text{ is of type 3, } C(t(i)) = q\} \subseteq \left\{t(i) + 1 \in K_1(S, S') \cup K_2(S, S'), \exists t(i) + 1 < l_2 < l_3 < \ldots < l_{q+1} < l_{q+2} \text{ such that } l_k \in K_1(S, S') \cup K_2(S, S') \text{ for } 2 \leq k \leq q + 2\}.$$  (3.46)
By strong Markov property, Lemma 3.3 and (3.46),

\[ P(t(i + 1) \text{ is of type } 3, C(t(i)) = q | \mathcal{F}_i) \]

\[ \leq P(\exists l_2 > t(i) + 1 \text{ such that } l_2 \in K_1(S, S') \bigcup K_2(S, S') | t(i) + 1 \in K_1(S, S') \bigcup K_2(S, S'), \mathcal{F}_i) \times \]

\[ \prod_{k=2}^{q+1} P(\exists l_{k+1} > l_k \text{ such that } l_{k+1} \in K_1(S, S') \bigcup K_2(S, S') | t(i) + 1 \in K_1(S, S') \bigcup K_2(S, S')) \mid t(i) + 1 \in K_1(S, S') \bigcup K_2(S, S'), \mathcal{F}_i) \]

\[ \leq \left( \frac{M_1 N}{d} \right)^q \left( \frac{M_1 N}{d} \right)^{q+1}. \]  \hspace{1cm} (3.47)

By (3.47),

\[ E(t, m, 3) = \sum_{q=1}^{+\infty} \left( \frac{\lambda + 1}{\lambda p} \right) 2^{Nq} P(C(t(i)) = q, t(i + 1) \text{ is of type } 3 | \mathcal{F}_i) \]

\[ \leq \sum_{q=1}^{+\infty} \left( \frac{\lambda + 1}{\lambda p} \right) 2^{Nq} \left( \frac{M_1 N}{d} \right)^{q+1} \]

\[ = \left( \frac{\lambda + 1}{\lambda p} \right) \frac{M_1^2 N^2 2^N}{d^2 - d M_1 N^2} \]

\[ \leq \left( \frac{\lambda + 1}{\lambda p} \right) \frac{M_4 N^2 2^N}{d^2} \]  \hspace{1cm} (3.48)

for sufficiently large \( d \), where \( M_4 \) does not depend on \( d \).

For \( i \in K_1(S, S') \), we define

\[ \sigma(i) = \inf \{ k > i : k \in K_1(S, S') \} . \]

**Lemma 3.7.** On the event

\[ \{ t(j) \text{ belongs to } [kN, (k + 1)N - 1] \}, \]

\[ E(2^{NC(t(j))}; \sigma(t(j)) \text{ belongs to a bad interval } [lN, (l + 1)N - 1] \]

\[ \text{for some } l > k | \mathcal{F}_j) \leq \frac{M_6 N^2 2^N}{d^2} \]  \hspace{1cm} (3.49)

for sufficiently large \( d \), where \( M_6 \) does not depend on \( d \).
Proof. For any \( l \geq 1 \),

\[
P([lN, (l + 1)N - 1] \text{ is bad})
\]

\[
= P(\exists \, t \leq (l + 1)N - 1 \text{ such that } S_t = S_{lN})
\]

\[
\leq \sum_{s = lN}^{(l+1)N-1} P(\exists \, t > s \text{ such that } \hat{S}_t - \hat{S}_s = 0)
\]

\[
= NP(W_m = 0 \text{ for some } m > 0),
\]

where \( \{W_n\}_{n \geq 1} \) is simple random walk on \( \mathbb{Z}^{d-\lfloor \frac{d}{N} \rfloor} \) with \( W_0 = 0 \).

According to (3.22),

\[
P(W_m = 0 \text{ for some } m > 0) \leq \sum_{m=1}^{+\infty} P(W_{2m} = 0)
\]

\[
\leq \frac{1}{2(d - \lfloor \frac{d}{N} \rfloor)} \left( 1 + \frac{M}{d - \lfloor \frac{d}{N} \rfloor} \right) \leq \frac{M_7}{d}.
\]

Therefore,

\[
P([lN, (l + 1)N - 1] \text{ is bad}) \leq \frac{NM_7}{d} \tag{3.50}
\]

for sufficiently large \( d \), where \( M_7 \) does not depend on \( d \).

\[ t(j) + 1 \in K_1(S, S') \cup K_2(S, S'). \]

Hence, if \( \sigma(t(j)) - t(j) \geq 2 \), then \( t(j) + 1 \in K_2(S, S') \) and \( C(t(j)) \geq 1 \). As a result, if \( C(t(j)) = 0 \) and \( \sigma(t(j)) \) belongs to \( [lN, (l + 1)N - 1] \) for \( l > k \), then \( t(j) = (k+1)N - 1 \), \( \sigma(t(j)) = t(j) + 1 = (k+1)N \) and there exists \( r \in [(k+1)N, (k + 2)N - 1] \) such that \( S'_{(k+1)N+1} = S_r \).

Therefore, by (3.24) and (3.50),

\[
P(C(t(j)) = 0, \sigma(t(j)) \text{ belongs to a bad interval } [lN, (l + 1)N - 1], \text{ for some } l > k | \mathcal{F}_j)
\]

\[
\leq \sum_{r = (k+1)N}^{(k+2)N-1} P(S'_{(k+1)N+1} = S_r, [(k+1)N, (k + 2)N - 1] \text{ is bad} | \mathcal{F}_j)
\]

\[
\leq N \sup_{x \in \mathbb{Z}^{d-\lfloor \frac{d}{N} \rfloor}} P(W_1 = x) P([(k+1)N, (k + 2)N - 1] \text{ is bad})
\]

\[
\leq M_8 N^2 \frac{1}{d} \frac{1}{2(d - \lfloor \frac{d}{N} \rfloor)}
\]

\[
\leq \frac{M_8 N^2}{d^2} \tag{3.51}
\]

for sufficiently large \( d \), where \( M_8 \) does not depend on \( d \). Please note that we use the fact that \( \{[(k+1)N, (k + 2)N - 1] \text{ is bad} \} \) is independent with \( \sigma(S_n, S'_n, n \leq (k+1)N) \).
If \( \text{C}(t(j)) = q \geq 1 \) and \( \sigma(t(j)) \leq +\infty \), then there exists \( t(i) + 1 = l_1 < l_2 < \ldots < l_q < l_{q+1} \) such that \( l_j \in K_2(S, S') \) for \( 1 \leq j \leq q \), \( l_{q+1} \in K_1(S, S') \) and \( l_{q+1} + 1 \in K_1(S, S') \cup K_2(S, S') \). According to a similar analysis with that in the proof of Lemma 3.6:

\[
\Pr(\text{C}(t(j)) = q, \sigma(t(j)) < +\infty | F_j) \leq \left( \frac{M_1N}{d} \right)^{q+1}, \tag{3.52}
\]

where \( M_1 \) is the same as that in Lemma 3.3.

By (3.51) and (3.52),

\[
E(2^{NC(t(j))}; \sigma(t(j))) \text{ belongs to a bad interval } [lN, (l+1)N - 1]
\]
for some \( l > k|F_j| \)

\[
\begin{align*}
&\leq \frac{M_8N^2}{d^2} + \sum_{q=1}^{+\infty} 2^{Nq} \left( \frac{M_1N}{d} \right)^{q+1} \\
&= \frac{M_8N^2}{d^2} + \frac{M_1^2N^22^N}{d^2} - dM_1N2^N \\
&\leq \frac{M_6N^22^N}{d^2} \tag{3.53}
\end{align*}
\]

for sufficiently large \( d \), where \( M_6 \) does not depend on \( d \).

\( \square \)

**Lemma 3.8.** Let \( \lambda = \frac{\gamma}{2dN} \) for \( \gamma > 1 \). Then,

\[
E(i, m, 4) \leq \frac{M_{10}M_0(\gamma)N^{N+1}}{d} \tag{3.54}
\]

for \( i \geq 1, 1 \leq m \leq 4 \) and sufficiently large \( d \), where \( M_{10} \) does not depend on \( d \) and \( M_0(\gamma) \) depends only on \( \gamma \).

**Proof.** For any \( l \geq 0 \), we define \( u(l) = \sup\{k : kN \leq l\} \).

For \( i \leq \rho \), if \( \rho \geq i+1 \) and \( t(i+1) \) is of type 4, then \( \sigma(t(i)) \) belongs to a bad interval \( [kN, (k+1)N - 1] \) for \( k > u(t(i)) \).

We define \( s_1 = \sigma(t(i)) \) and \( s_m = \sigma(s_{m-1}) \) for \( m \geq 2 \). Since an interval \( [kN, (k+1)N - 1] \) has \( N \) integers, there exists \( q \leq N \) such that \( u(s_q) = u(s_1) \) and \( u(s_{q+1}) > u(s_q) \). In other words, \( t(i+1) = s_q \) when \( t(i+1) \) is of type 4.
Therefore, by Markov property,

\[
E(i, m, 4) = \sum_{q=1}^{N} E(L(i); t(i) + 1) = s_q, s_1 \text{ belongs to a bad interval } [kN, (k+1)N - 1] \text{ for some } k > u(t(i))|F_i)
\]

\[
\leq \sum_{q=1}^{N} \left( \frac{\lambda + 1}{\lambda p} \right)^q E(2^{NC(t(i))}; s_1 \text{ belongs to a bad interval } [kN, (k+1)N - 1] \text{ for some } k > u(t(i))|F_i)
\]

\[
\times \prod_{j=2}^{q} E(2^{NC(s_{j-1})}; s_j \sim s_1 | s_{j-1} \sim s_1, u(s_1) > u(t(i)), [u(s_1)N, (u(s_1) + 1)N - 1] \text{ is bad}, F_i). \quad (3.55)
\]

According to Lemma 3.7,

\[
E(2^{NC(t(i))}; s_1 \text{ belongs to a bad interval } [kN, (k+1)N - 1] \text{ for some } k > u(t(i))|F_i) \leq \frac{M_6N^22^N}{d^2}. \quad (3.56)
\]

According to a similar analysis with that in the proof of Lemma 3.6

\[
P(C(s_{j-1}) = m, s_j < +\infty | s_{j-1} \sim s_1, u(s_1) > u(t(i)), [u(s_1)N, (u(s_1) + 1)N - 1] \text{ is bad}, F_i) \leq \left( \frac{M_1N}{d} \right)^{m+1} \quad (3.57)
\]

for \(2 \leq j \leq N\) and \(m \geq 0\).

By (3.57),

\[
E(2^{NC(s_{j-1})}; s_j \sim s_1 | s_{j-1} \sim s_1, u(s_1) > u(t(i)), [u(s_1)N, (u(s_1) + 1)N - 1] \text{ is bad}, F_i)
\]

\[
\leq \sum_{m=0}^{\infty} 2^{Nm} \left( \frac{M_1N}{d} \right)^{m+1}
\]

\[
= \frac{M_1N}{d - M_1N2^N} \leq \frac{M_{11}N}{d} \quad (3.58)
\]

for sufficiently large \(d\), where \(M_{11}\) does not depend on \(d\).
By (3.55), (3.56), (3.58) and $\lambda = \frac{\gamma}{2d}$,

$$E(i, m, l) \leq \sum_{q=1}^{N} \left( \frac{\lambda + 1}{\lambda p} \right) q \frac{M_6 N^2 2^N}{d^2} \left( \frac{M_{11} N}{d} \right)^{q-1}$$

$$\leq \sum_{q=1}^{N} \left( \frac{3d}{\gamma} \right) q \frac{M_6 N^2 2^N}{d^2} \left( \frac{M_{11} N}{d} \right)^{q-1}$$

$$= \frac{3M_6 N^2 2^N \left( \frac{3M_{11} N}{\gamma} \right)^{N+1} - 1}{dr} \frac{3M_{11} N}{\gamma} - 1$$

$$\leq \frac{M_{12} N^{N+1} 16^N M_{11}^N}{d\gamma^{N+1}} \quad (3.59)$$

for sufficiently large $d$, where $M_{11}$ and $M_{12}$ do not depend on $d$.

Let $M_{10} = \frac{M_{12}}{6M_{11}}$ and $M_9(\gamma) = \frac{6M_{11}}{\gamma}$, then (3.54) follows (3.59). \[\Box\]

For $i \geq 1$ and $1 \leq m \leq 4$, we use $E(i, m, +\infty)$ to denote

$$E(L(i); \rho = i | F_i)$$

on the event

$$\{ \rho \geq i, t(i) \text{ is of type } m \}.$$ 

**Lemma 3.9.**

$$E(i, m, +\infty) \leq M_{13} \left( \frac{1 + \lambda}{\lambda p} \right)^N 2^N \quad (3.60)$$

for $i \geq 1$, $1 \leq m \leq 4$ and sufficiently large $d$, where $M_{13}$ does not depend on $d$.

**Proof.** If $\rho = i$ and $C(t(i)) = q$, then there exists $t(i) + 1 = l_1 < l_2 < \ldots < l_q$ such that $l_k \in K_2(S, S')$ for $1 \leq k \leq q$. Therefore,

$$\{ C(t(i)) = q, \rho = i \} \subseteq \{ \exists \ t(i) + 1 < l_2 < \ldots < l_q$$

$$\text{such that } l_k \in K_1(S, S') \bigcup K_2(S, S') \text{ for } 2 \leq k \leq q \}.$$

(3.61)

According to a similar analysis with that in the proof of Lemma (3.6) and (3.61),

$$P(C(t(i)) = q, \rho = i | F_i) \leq \left( \frac{M_{11} N}{d} \right)^{q-1} \quad (3.62)$$

for $q \geq 1$. 

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By (3.62),
\[
E(i, m, +\infty) = \frac{1 + \lambda}{\lambda p} \sum_{q=1}^{+\infty} 2^{Nq} P(C(t(i)) = q, \rho = i | \mathcal{F}_i)
\]
\[
\leq (\frac{1 + \lambda}{\lambda p}) \sum_{q=1}^{+\infty} 2^{Nq} (\frac{M_1 N}{d})^{q-1}
\]
\[
= \frac{1 + \lambda}{\lambda p} \frac{2^N}{1 - \frac{M_1 N 2^N}{d}}
\]
\[
\leq M_{13} \frac{1 + \lambda}{\lambda p} 2^N
\] (3.63)
for sufficiently large \(d\), where \(M_{13}\) does not depend on \(d\).

\[\square\]

For \(1 \leq l \leq 4\), we define
\[E(0, l) = E(L(0); \rho \geq 1, t(1) \text{ is of type } l)\]
and
\[E(0, +\infty) = E(L(0); \rho = 0).\]

**Lemma 3.10.**
\[
E(0, 1) \leq \frac{1}{2(d - \lfloor \frac{d}{N} \rfloor)}. \quad (3.64)
\]
\[
E(0, 2) = 0. \quad (3.65)
\]
\[
E(0, 3) \leq \frac{M_4 N^2 2^N}{d^2}; \quad (3.66)
\]
where \(M_4\) is the same as that in Lemma 3.6.

Let \(\lambda = \frac{\gamma}{2dp}\), then
\[
E(0, 4) \leq \frac{M_{10} [M_9(\gamma) N]^{N+1} \gamma p}{d(\gamma + 2dp)}, \quad (3.67)
\]
where \(M_{10}\) and \(M_9(\gamma)\) are the same as that in Lemma 3.8.

\[
E(0, +\infty) \leq M_{13} 2^N, \quad (3.68)
\]
where \(M_{13}\) is the same as that in the Lemma 3.9.

**Proof.** The proof of (3.6-1) is similar with the proof of Lemma 3.4 where 0 plays the same role as \(t(i)\) in the proof of Lemma 3.6. We omit the details.

When \(t(1) = 1 = t(0) + 1\), then \(t(1) + 1 = 2\) is not a multiple of \(N\). Hence \(P(t(1) \text{ is of type } 2) = 0\) and (3.65) holds.
The proof of (3.66) is similar with the proof of Lemma 3.6 where 0 = t(−1) + 1 plays the same role as t(1) + 1 in the proof of Lemma 3.6. We omit the details.

The proof of (3.67) is similar with the proof of Lemma 3.8 where inf {k : k ∈ K1(S, S′)} = σ(t(−1)) plays the same role as s1 = σ(t(1)) in the proof of Lemma 3.8. We omit the details.

The proof of (3.68) is similar with the proof of Lemma 3.9 where 0 = t(−1) + 1 plays the same role as t(1) + 1 in the proof of Lemma 3.9. We omit the details.

For γ > 1, we define 4 × 4 matrix Γγ as

$$
\Gamma_{\gamma} = \begin{pmatrix}
\frac{2dp+\gamma}{2\gamma p(d-\left\lfloor \frac{p}{d} \right\rfloor)} & M_3(2dp+\gamma)N & M_4N^22^N(2dp+\gamma) & M_{10}[M_6(\gamma)N]^{N+1} \\
\frac{2dp+\gamma}{2\gamma p(d-\left\lfloor \frac{p}{d} \right\rfloor)} & 0 & M_4N^22^N(2dp+\gamma) & M_{10}[M_6(\gamma)N]^{N+1} \\
\frac{2dp+\gamma}{2\gamma p(d-\left\lfloor \frac{p}{d} \right\rfloor)} & M_3(2dp+\gamma)N & M_4N^22^N(2dp+\gamma) & M_{10}[M_6(\gamma)N]^{N+1} \\
\frac{2dp+\gamma}{2\gamma p(d-\left\lfloor \frac{p}{d} \right\rfloor)} & M_3(2dp+\gamma)N & M_4N^22^N(2dp+\gamma) & M_{10}[M_6(\gamma)N]^{N+1}
\end{pmatrix}.
$$

(3.69)

We multiply the first column of Γ(γ) by $N^{\frac{3}{N-3}}$ and the third and fourth columns by $N^3$, then we divide the second column by $N^3$. Then we get a 4 × 4 matrix $\Delta_{\gamma}$ which equals

$$
\begin{pmatrix}
\frac{(2dp+\gamma)N}{2\gamma p(d-\left\lfloor \frac{p}{d} \right\rfloor)} & M_3(2dp+\gamma)N & M_4N^22^N(2dp+\gamma) & M_{10}[M_6(\gamma)N]^{N+4} \\
\frac{(2dp+\gamma)N}{2\gamma p(d-\left\lfloor \frac{p}{d} \right\rfloor)} & 0 & M_4N^22^N(2dp+\gamma) & M_{10}[M_6(\gamma)N]^{N+4} \\
\frac{(2dp+\gamma)N}{2\gamma p(d-\left\lfloor \frac{p}{d} \right\rfloor)} & M_3(2dp+\gamma)N & M_4N^22^N(2dp+\gamma) & M_{10}[M_6(\gamma)N]^{N+4} \\
\frac{(2dp+\gamma)N}{2\gamma p(d-\left\lfloor \frac{p}{d} \right\rfloor)} & M_3(2dp+\gamma)N & M_4N^22^N(2dp+\gamma) & M_{10}[M_6(\gamma)N]^{N+4}
\end{pmatrix}.
$$

(3.70)

For γ > 1, we define

$$
v_{\gamma} = \sup_{1 \leq i \leq 4} \sum_{j=1}^{4} \Delta_{\gamma}(i, j)
$$

(3.71)

$$
= \frac{(2dp+\gamma)N}{2\gamma p(d-\left\lfloor \frac{p}{d} \right\rfloor)^{\frac{3}{N-3}}} + \frac{M_3(2dp+\gamma)}{d\gamma pN^2} + \frac{M_4N^22^N(2dp+\gamma)}{d^2\gamma p} + \frac{M_{10}[M_6(\gamma)N]^{N+4}}{d},
$$

then we have the following lemma.

**Lemma 3.11.** When $\lambda = \frac{3}{2dp}$, for any $k \geq 1$ and $1 \leq i \leq 4$,

$$
E\left( \prod_{j=1}^{k} L(j); \rho \geq k + 1 | F_1 \right) \leq v_{\gamma}^k
$$

(3.72)
on the event \( \{ \rho \geq 1, t(1) \text{ is of type } i \} \).

**Proof.** According to Lemma 3.4, 3.5, 3.6, 3.8 and Markov property,

\[
E(\prod_{j=1}^{k} L(j); \rho \geq k + 1 | \mathcal{F}_1) \leq \sum_{q=1}^{4} \Gamma^k_i(\rho, q)
\]  

(3.73)

on the event \( \{ \rho \geq 1, t(1) \text{ is of type } i \} \) when \( \lambda = \frac{\gamma}{2dp} \).

If \( t(j) \) is of type 2 for some \( j \geq 1 \), then either \( t(k) \) is of type 1 for \( j - N + 1 \leq k < j \) or one \( t(k) \) for \( j - N + 1 \leq k < j \) is of type 3 or 4 and \( t(l) \) is of type 1 for \( k < l < j \). As a result, we can replace \( \Gamma^k_\gamma \) in the right of (3.73) by \( \Delta^k_\gamma \).

Therefore,

\[
E(\prod_{j=1}^{k} L(j); \rho \geq k + 1 | \mathcal{F}_1) \leq \sum_{q=1}^{4} \Delta^k_i(\rho, q) \leq v^k_\gamma.
\]

(3.74)

After all these prepared works, now we give the proof of Theorem 2.2.

**Proof of Theorem 2.2.** We write \( v^\gamma \) as \( v^r(d) \) when the dimension \( d \) should be distinguished.

\[
E \prod_{0 \leq j \leq \rho} L(j) = E \prod_{j=0}^{+\infty} L(j) \mathbf{1}_{\rho=+\infty} + \sum_{k=0}^{+\infty} E \prod_{l=0}^{k} L(l) \mathbf{1}_{\rho=l}.
\]

(3.75)

Assume that \( \lambda = \frac{\gamma}{2dp} \) for \( \gamma > 1 \). Then according to Lemma 3.10,

\[
E(0, i) \leq \frac{M_{14}}{d} < 1
\]

(3.76)

for \( 1 \leq i \leq 4 \) and sufficiently large \( d \), where \( M_{14} \) does not depend on \( d \).

By (3.76), Lemma 3.11 and Markov property,

\[
E \prod_{j=0}^{+\infty} L(j) \mathbf{1}_{\rho=+\infty} = \lim_{k \to +\infty} E \prod_{j=0}^{k} L(j) \mathbf{1}_{\rho=k+1} = \lim_{k \to +\infty} \sum_{q=1}^{4} E[L(0)E(\prod_{j=1}^{k} L(j); \rho \geq k + 1 | \mathcal{F}_1); \rho \geq 1, t(1) \text{ is of type } q] \leq \lim_{k \to +\infty} \frac{4M_{14}}{d} v^k_\gamma = \frac{4M_{14}}{d} v^+\infty
\]

(3.77)

\[25\]
for sufficiently large $d$.

By Lemma 3.10,

$$EL(0)1_{\rho=0} \leq M_{13}2^N.$$  \hspace{1cm} (3.78)

By (3.76), Lemma 3.9 and Lemma 3.12 for $k \geq 1$,

$$E \prod_{l=0}^{k-1} L(l)1_{\rho=k}$$

$$= E \left[ \prod_{j=0}^{k-1} L(j)1_{\rho=k} \right]$$

$$= \sum_{q=1}^{4} E\left[ L(0)E\left( \prod_{j=1}^{k-1} L(j) ; \rho \geq k | F_l \right) E(L(\rho) ; \rho = k | F_k) ; t(1) \text{ is of type } q \right]$$

$$\leq \frac{4M_{13}M_{14}}{d} v_{\gamma}^{k-1} \left( \frac{2dp + \gamma}{\gamma p} \right) 2^N$$ \hspace{1cm} (3.79)

for sufficiently large $d$.

By (3.75), (3.77), (3.78) and (3.79), when $\lambda = \frac{\gamma}{2dp}$,

$$E \prod_{0 \leq j \leq \rho} L(j) \leq \frac{4M_{14}}{d} v_{\gamma}^{+\infty} + M_{13}2^N \left[ 1 + \frac{4M_{14}(2dp + \gamma)}{d\gamma p} \sum_{k=1}^{+\infty} \frac{v_{k-1}}{\gamma} \right]$$ \hspace{1cm} (3.80)

for sufficiently large $d$.

By direct calculation,

$$\lim_{d \rightarrow +\infty} v_{\gamma}(d) = \frac{1}{\gamma} < 1$$ \hspace{1cm} (3.81)

when $\gamma > 1$.

Therefore by (3.80) and (3.81), for sufficiently large $d$,

$$E \prod_{0 \leq j \leq \rho} L(j) < +\infty$$ \hspace{1cm} (3.82)

when $\lambda = \frac{\gamma}{2dp}$ for $\gamma > 1$.

We have shown in Lemma 3.2 that $\lambda \geq \lambda_c(d, p)$ when $\lambda$ makes

$$E \prod_{0 \leq j \leq \rho} L(j) < +\infty.$$

As a result, when $\gamma > 1$,

$$\lambda_c(d, p) \leq \frac{\gamma}{2dp}$$ \hspace{1cm} (3.83)

for sufficiently large $d$ and therefore,

$$\limsup_{d \rightarrow +\infty} 2dp\lambda_c(d, p) \leq \gamma.$$ \hspace{1cm} (3.84)
Let $\gamma$ in (3.84) converge to 1, then
\[
\limsup_{d \to +\infty} 2dp\lambda_c(d, p) \leq 1.
\] (3.85)

3.3 The proof of (2.5)

In this subsection we will give the proof of (2.5). When $\rho < +\infty$, we modify the definition of $L(\rho)$. We define
\[
F(\rho) = N[C(t(\rho)) - 1] + C_{(S,S')} t(\rho) + 1
\]
when $1 \leq \rho < +\infty$ and
\[
F(\rho) = N[C(0) - 1] + C_{(S,S')} (0)
\]
when $\rho = 0$, where
\[
C_{(S,S')}(i) = \sum_{j=0}^{+\infty} 1\{s_j = s_i\}
\]
as what we have defined in (3.6).

In this subsection, we give up the former definition of $L(\rho)$. We redefine
\[
L(\rho) = \left(\frac{\lambda + 1}{\lambda p}\right)^2 F(\rho)
\] (3.86)
when $1 \leq \rho \leq +\infty$ and
\[
L(0) = 2F(\rho)
\] (3.87)
when $\rho = 0$.

**Lemma 3.12.** Under the new definition of $L(\rho)$ in (3.86) and (3.87),
\[
\mu(\lambda, d, p) \geq \frac{1}{E \prod_{j \leq \rho} L(j)}
\]
still holds.

The proof of Lemma 3.12 is nearly the same as that of Lemma 3.2. The only difference is that in the proof of Lemma 3.12 we do not enlarge $C_{(S,S')}(j)$ to $N$ when $j = t(\rho) + 1$. We omit the details.

Under the new definition of $L(\rho)$, we still use $E(i, m, +\infty)$ to denote
\[
E(L(\rho); \rho = i|F_i)
\] (3.88)
on the event
\[
\{\rho \geq i, t(i) \text{ is of type } m\}.
\]
The following lemma is crucial for us to prove (2.5).
Lemma 3.13. For $i \geq 1$ and $\lambda = \frac{\gamma}{2dp}$,

\[
E(i, m, +\infty) \leq \begin{cases} 
\frac{M_{15}(2dp+\gamma)}{\lambda p} & \text{if } m = 1, 2, \\
\frac{M_{13}(2dp+\gamma)}{\gamma p} & \text{if } m = 3, 4 
\end{cases}
\] (3.89)

for sufficiently large $d$, where $M_{15}$ does not depend on $d$ and $M_{13}$ is the same as that in Lemma 3.9.

Proof. The case of $m = 3, 4$ follows Lemma 3.9, since $F(q) \leq NC(t(q))$. When $m = 1$, then $t(i)$ is of type 1 and hence $t(i) + 1$ is in a good interval. Therefore,

\[
C_{(S,S')}(t(i) + 1) = 1.
\] (3.90)

Then a similar analysis with that of (3.63) shows that,

\[
E(i, 1, +\infty) = \frac{\lambda + 1}{\lambda p} \sum_{q=1}^{+\infty} 2^{1+N(q-1)} P\left( C(t(i)) = q, \rho = i \mid F_i \right)
\]

\[
\leq \frac{\lambda + 1}{\lambda p} \sum_{q=1}^{+\infty} 2^{1+N(q-1)} \left( \frac{M_1N}{d} \right)^{q-1}
\]

\[
= \left( \frac{1 + \lambda}{\lambda p} \right) \frac{2}{1 - \frac{M_1N^2}{d}}
\]

\[
\leq \frac{M_{16}(2dp+\gamma)}{\gamma p}
\] (3.91)

when $\lambda = \frac{\gamma}{2dp}$.

When $m = 2$, then $t(i) + 1$ is a multiple of $N$. If $\rho = i$ and $[t(i) + 1, t(i) + N]$ is good, then $F(q) = 1 + N[C(t(i)) - 1]$. As a result,

\[E(i, 2, +\infty)\]

\[= E\left( L(\rho); \rho = i, [t(i) + 1, t(i) + N] \text{ is good} \mid F_i \right)
\]

\[+ E\left( L(\rho); \rho = i, [t(i) + 1, t(i) + N] \text{ is bad} \mid F_i \right)
\]

\[\leq \frac{1 + \lambda}{\lambda p} E\left[ E\left( 2^{1+N[C(t(i))-1]} \mid F_i, S_n, n \geq 0 \right); [t(i) + 1, t(i) + N] \text{ is good} \mid F_i \right]
\]

\[+ \frac{1 + \lambda}{\lambda p} E\left[ E\left( 2^{NC(t(i))} \mid F_i, S_n, n \geq 0 \right); [t(i) + 1, t(i) + N] \text{ is bad} \mid F_i \right].
\] (3.92)

Since $t(i) + 1$ is a multiple of $N$, \{ $[t(i) + 1, t(i) + N]$ is bad \} is independent with $F_i$. Then by (3.30),

\[
P\left( [t(i) + 1, t(i) + N] \text{ is bad} \mid F_i \right) \leq \frac{M_7N}{d}
\] (3.93)

for sufficiently large $d$. 

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A similar analysis with that of (3.63) shows that

\[
E(2^{1+N[C(t(i))^{-1}]|\mathcal{F}_i, S_n, n \geq 0}) \leq \sum_{q=1}^{+\infty} 2^{1+N(q-1)} \left( \frac{M_1 N}{d} \right)^{q-1} = \frac{2}{1 - \frac{M_1 N^2}{d}}
\]  

(3.94)

and

\[
E(2^{NC(t(i))}|\mathcal{F}_i, S_n, n \geq 0) \leq \sum_{q=1}^{+\infty} 2^{Nq} \left( \frac{M_1 N}{d} \right)^{q-1} = \frac{2^N}{1 - \frac{M_1 N^2}{d}}.
\]  

(3.95)

By (3.92), (3.93), (3.94) and (3.95), when \(\lambda = \gamma \frac{2}{dp}\),

\[
E(i, 2, +\infty) \leq \frac{2dp + \gamma}{\gamma p} \left[ \frac{2}{1 - \frac{M_1 N^2}{d}} + \frac{2N}{1 - \frac{M_1 N^2}{d}} M_7 N \right] \leq \frac{M_{17}(2dp + \gamma)}{\gamma p}
\]  

(3.96)

for sufficiently large \(d\), where \(M_{17}\) does not dependent on \(d\).

Let \(M_{15} = M_{16} \lor M_{17}\), then the case of \(m = 1, 2\) holds.

We still use \(E(0, +\infty)\) to denote \(E(L(0); \rho = 0)\) under the new definition of \(L(\rho)\). Then we have the following lemma.

**Lemma 3.14.**

\[
E(0, +\infty) \leq M_{15}
\]  

(3.97)

for sufficiently large \(d\), where \(M_{15}\) is the same as that in Lemma 3.13.

**Proof.** The proof of (3.97) is similar with the proof of the case \(m = 2\) of Lemma 3.13 where \(0\) plays the same role as \(t(i) + 1\) in that proof. We omit the details.

\[\square\]

Now we give the proof of (2.5).

**Proof of (2.5).** For \(\gamma > 1\) and \(\lambda = \gamma \frac{2}{2dp}\), let

\[
\phi_{\gamma} = \begin{pmatrix}
\frac{(2dp+\gamma)N^2}{\gamma p} & M_4(2dp+\gamma) & M_4N^2(2dp+\gamma) & M_{10}[M_6(\gamma)N]^{N+4} \\
2d+p(d-[\frac{2}{\gamma}]) & \frac{M_4N^2}{\gamma p} & \frac{M_4N^2(2dp+\gamma)}{\gamma p} & \frac{M_{10}[M_6(\gamma)N]^{N+4}}{\gamma p}
\end{pmatrix},
\]

\[
\kappa_{\gamma} = \begin{pmatrix}
\frac{1}{2d-[\frac{2}{\gamma}]} & 0 & M_{15}(2dp+\gamma) & M_{13}2^N(2dp+\gamma) \\
M_{15}(2dp+\gamma) & \frac{M_{10}[M_6(\gamma)N]^{N+4}}{\gamma p} & \frac{M_{13}2^N(2dp+\gamma)}{\gamma p} & \frac{M_{13}2^N(2dp+\gamma)}{\gamma p}
\end{pmatrix},
\]

(3.98)

and

\[
\varsigma_{\gamma} = \phi_{\gamma}^T \kappa_{\gamma}^T, \quad q_{\gamma} = \kappa_{\gamma}^T \phi_{\gamma}^T.
\]

(3.99)
By direct calculation,
\[
\lim_{d \to +\infty} \frac{\varsigma(d)}{d} = \frac{2M_{15}}{\gamma^2}, \quad \lim_{d \to +\infty} \varrho(d) = \frac{M_{15}}{\gamma}.
\] (3.100)

According to (3.74) and Lemma 3.13 on the event \( \{ t(1) \) is of type \( i \} \),
\[
E(\prod_{j=1}^{k+1} L(j); \rho = k + 1 | F_1) \\
\leq \sum_{q=1}^{4} \Delta_{\gamma}^2(i, q) \psi_{\gamma}(1, q) = \sum_{i=1}^{4} \Delta_{\gamma}^{k-1}(i, l) \Delta_{\gamma}(l, q) \psi_{\gamma}(1, q) \\
\leq \sum_{l=1}^{4} \Delta_{\gamma}^{k-1}(i, l) \varsigma_{\gamma} \leq \psi_{\gamma}^{k-1} \varsigma_{\gamma}
\] (3.101)

for \( k \geq 1 \).

Therefore, by (3.70) and (3.101),
\[
E(\prod_{0 \leq i \leq \rho} L(i); \rho = k) \leq \sum_{i=1}^{4} E(0, i) \psi_{\gamma}^{k-2} \varsigma_{\gamma} \\
\leq \frac{4M_{14} \psi_{\gamma}^{k-2} \varsigma_{\gamma}}{d}
\] (3.102)

for \( k \geq 2 \). By Lemma 3.10 and Lemma 3.13
\[
E[L(0)L(1); \rho = 1] \leq \sum_{i=1}^{4} E(0, i) \psi_{\gamma}(1, i) \\
\leq \sum_{i=1}^{4} \kappa_{\gamma}(1, i) \psi_{\gamma}(1, i) = \varrho_{\gamma}.
\] (3.103)

By Lemma 3.13
\[
E[L(0); \rho = 0] = E(0, +\infty) \leq M_{15}.
\] (3.104)

Then by (3.102), (3.103) and (3.104),
\[
E(\prod_{i \leq \rho} L(i); \rho \leq +\infty) \leq M_{15} + \varrho_{\gamma} + \sum_{k=2}^{+\infty} \frac{4M_{14} \psi_{\gamma}^{k-2} \varsigma_{\gamma}}{d} \\
= M_{15} + \varrho_{\gamma} + \frac{4M_{14} \varsigma_{\gamma}}{d(1 - \psi_{\gamma})}
\] (3.105)

By (3.77) and (3.105),
\[
E(\prod_{j \leq \rho} L(j)) \leq M_{15} + \varrho_{\gamma} + \frac{4M_{14} \varsigma_{\gamma}}{d(1 - \psi_{\gamma})} + \frac{4M_{14} \varsigma_{\gamma}}{d \psi_{\gamma}^{+\infty}}
\] (3.106)
By (3.81), (3.100) and (3.106),

\[
\limsup_{d \to +\infty} E \left[ \prod_{j \leq \rho} L(j) \right] \leq M_{15} + \frac{M_{15}}{\gamma} + \frac{8M_{14}M_{15}}{\gamma(\gamma - 1)}.
\] (3.107)

Let \( A = M_{15} \) and \( B = 8M_{14}M_{15} \), then (2.5) follows Lemma 3.12 and (3.107).

\[\square\]

4 The proof of (2.6)

To prove (2.6), we utilize the binary contact path process introduced in [4] as a tool. The binary contact path process \( \{\zeta_t\}_{t \geq 0} \) on a graph \( G \) is with state space \( \{0, 1, 2, \ldots\}^G \), which means that each vertex of \( G \) takes a value from the set of nonnegative integers.

\( \{\zeta_t\}_{t \geq 0} \) evolves as follows. For any \( x \in G \), \( \zeta_t(x) \) flips to 0 at rate one. For any \( y \) that \( y \sim x \), \( \zeta_t(x) \) flips to \( \zeta_t(x) + \zeta_t(y) \) at rate \( \lambda > 0 \). So, the generator \( \Omega \) of \( \{\zeta_t\}_{t \geq 0} \) is given by

\[
\Omega f(\zeta) = \sum_{x \in G} \left[ f(\zeta^{x,0}) - f(\zeta) \right] + \lambda \sum_{x \in \mathbb{Z}^d} \sum_{y \sim x} \left[ f(\zeta^{x,\zeta(x)+\zeta(y)}) - f(\zeta) \right]
\] (4.1)

for any \( \zeta \in \{0, 1, 2, \ldots\}^G \) and \( f \in C(\{0, 1, 2, \ldots\}^G) \) properly fast decaying, where

\[
\zeta^{x,m}(y) = \begin{cases} 
\zeta(y) & \text{if } y \neq x, \\
 m & \text{if } y = x
\end{cases}
\]

for any \( (x, m) \in G \times \{0, 1, 2, \ldots\} \).

We assume that \( \zeta_0(x) = 1 \) for any \( x \in G \). Then it is easy to see that the contact process \( \{\eta_t\}_{t \geq 0} \) on \( G \) with

\( \{x : \eta_0(x) = 1\} = G \)

can be coupled with \( \{\zeta_t\}_{t \geq 0} \) as follows.

\[
\eta_t(x) = \begin{cases} 
1 & \text{if } \zeta_t(x) \geq 1, \\
0 & \text{if } \zeta_t(x) = 0
\end{cases}
\] (4.2)

for any \( x \in G \).

Therefore, by (4.2),

\[
P_{\lambda,d,p}(\eta(0) = 1) = \mathbb{E}_{d,p} \left[ P_{\lambda}^{G(\omega)}(\eta_0(0) = 1) \right] = \mathbb{E}_{d,p} \left[ P_{\lambda}^{G(\omega)}(\zeta_t(0) \geq 1) \right] \leq \mathbb{E}_{d,p} \left[ E_{\lambda}^{G(\omega)}(\zeta_t(0)) \right] = E_{\lambda,d,p}(\zeta_t(0)).
\] (4.3)
Now we give the proof of (2.6).

Proof of (2.6). According to the generator $\Omega$ of $\zeta_t$ given in [11] and Theorem 1.27 of Chapter 9 of [11],

$$
\frac{d}{dt} E_\lambda^{G(\omega)} \zeta_t(x) = -E_\lambda^{G(\omega)} \zeta_t(x) + \lambda \sum_{y: y \sim x} E_\lambda^{G(\omega)} \zeta_t(y)
$$

$$
= -E_\lambda^{G(\omega)} \zeta_t(x) + \lambda \sum_{y: \|y-x\|=1} 1_{\{y \sim x\}} E_\lambda^{G(\omega)} \zeta_t(y)
$$

(4.4)

for any $x \in \mathbb{Z}^d$ and $\omega \in \{0, 1\}^{E_d}$, where

$$
\|y - x\| = \sum_{i=1}^d |y_i - x_i|
$$

is the $l_1$ norm.

By (4.4) and classic theory of ODE,

$$
E_\lambda^{G(\omega)} \zeta_t = e^{t(A_d - I_d)} \zeta_0,
$$

(4.5)

where $A_d$ is a $\mathbb{Z}^d \times \mathbb{Z}^d$ matrix such that

$$
A_d(x, y) = \begin{cases} 
\lambda & \text{if } x \sim y, \\
0 & \text{else}
\end{cases}
$$

(4.6)

and $I_d$ is the $\mathbb{Z}_d \times \mathbb{Z}_d$ identity matrix.

Therefore,

$$
E_\lambda^{G(\omega)} \zeta_t(0) = e^{-t} \sum_{x \in \mathbb{Z}^d} e^{t A_d}(0, x)
$$

$$
= e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} \left[ \sum_{x \in \mathbb{Z}^d} A_d^n(0, x) \right]
$$

and

$$
E_{\lambda, d, p} \zeta_t(0) = \mathbb{E}_{d, p} \left[ E_\lambda^{G(\omega)} \zeta_t(0) \right]
$$

$$
= e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} \left[ \mathbb{E}_{d, p} \left[ \sum_{x \in \mathbb{Z}^d} A_d^n(0, x) \right] \right].
$$

(4.7)

For any $n \geq 1$, $L_n$ is the same as that in (3.1). We define

$$
R_n = \{y \in L_n : 0 \neq y_i \neq y_j \neq 0 \text{ for any } 1 \leq i \neq j \leq n\}.
$$
According to the definition of $A_d$ in (4.6), it is easy to see that
\[
\mathbb{E}_{d,p} \left[ \sum_{x \in \mathbb{Z}^d} A_d^n(0, x) \right] = \lambda^n \mathbb{E}_{d,p} \left[ \sum_{y \in L_n} 1_{\{y_1 \sim 0\}} \prod_{j=1}^{n-1} 1_{\{y_{j+1} \sim y_j\}} \right] \\
= \lambda^n \sum_{y \in L_n} \mathbb{P}_{d,p}(y_1 \sim 0, y_j \sim y_{j+1} \text{ for } 1 \leq j \leq n-1).
\]

(4.8)

For $y \in R_n$, all the edges in the path with respect to $y$ are different with each other. Therefore,
\[
\sum_{y \in R_n} \mathbb{P}_{d,p}(y_1 \sim 0, y_j \sim y_{j+1} \text{ for } 1 \leq j \leq n-1) = p^n|R_n| \\
\leq p^n|L_n| = (2dp)^n. \tag{4.9}
\]

It is easy to see that
\[
|L_n \setminus R_n| \leq \binom{n+1}{2}(2d)^{n-1} \leq n^2(2d)^{n-1}.
\]

Then,
\[
\sum_{y \in L_n \setminus R_n} \mathbb{P}_{d,p}(y_1 \sim 0, y_j \sim y_{j+1} \text{ for } 1 \leq j \leq n-1) \leq \mathbb{P}_{d,p}(y_1 \sim 0)|L_n \setminus R_n| \\
\leq pn^2(2d)^{n-1}. \tag{4.10}
\]

By (4.8), (4.9) and (4.10),
\[
\mathbb{E}_{d,p} \left[ \sum_{x \in \mathbb{Z}^d} A_d^n(0, x) \right] \leq \lambda^n \left[(2dp)^n + pn^2(2d)^{n-1}\right]. \tag{4.11}
\]

By (4.7) and (4.11),
\[
E_{\lambda,d,p} \zeta_t(0) \leq e^{(2dp-1)t} + pt\lambda(1 + 2dt\lambda)e^{(2dp-1)t} \tag{4.12}
\]

for any $t > 0$.

For $\gamma < 1$, let
\[
u = \frac{\log d}{2\left(\frac{\gamma}{p} - 1\right) \vee 1}.
\]

Then, by (4.3) and (4.12),
\[
\mu(d, p) \leq \mathbb{P}_{d,p}(\eta_n(0) = 1) \\
\leq \mathbb{E}_{d,p} \zeta_n(0) \\
\leq \exp \left\{ \frac{(\gamma - 1) \log d}{2\left(\frac{\gamma}{p} - 1\right) \vee 1} \right\} + \frac{\gamma \log d(1 + \frac{\gamma \log d}{2p})}{4d^2 \left(\frac{\gamma}{p} - 1\right) \vee 1} \\
\rightarrow 0
\]

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for $\gamma < 1$ as $d$ grows to infinity. \hfill \qed

Since (2.5) is proven in Section 3, the whole proof of Theorem 2.3 is complete.

A Appendix

A.1 The proof of Lemma 3.1

For any $y \in L_n$ and $1 \leq j \leq n$, we define $y|_j = (y_1, y_2, \ldots, y_j) \in L_j$.

We denote by $I_{y|_j}$ the event that $y|_j$ is a special infection path with length $j$.

Then the following lemma is crucial for us to prove Lemma 3.1.

Lemma A.1. For any $y, y' \in L_n$, if $0 \not\in K_1(y, y')$, then

$$P(I_{y'}|I_y) \leq \left(\frac{\lambda p}{1 + \lambda}\right) \exp\{C_{y, y'}(0) \log 2\}. \quad (A.1)$$

Proof. We utilize the principle of mathematical induction.

If $C_{y, y'}(0) = 1$, then

$$P(I_{y'}|I_y) = pP(U_{(0, y')}^{(1)} < T_0^{(1)}|U_{(0, y)}^{(1)} < T_0^{(1)}) = \frac{2\lambda p}{1 + 2\lambda} \leq \frac{2\lambda p}{1 + \lambda}.$$

Then (A.1) holds.

Now we assume that (A.1) holds for any $y, y'$ such that $0 \not\in K_1(y, y')$ and $C_{y, y'}(0) = m$.

Assume that $y, y' \in L_n$ satisfies that $0 \not\in K_1(y, y')$ and $C_{y, y'}(0) = m + 1$.

Conditioned on $I_y$, we denote by $t_j, 1 \leq j \leq n$ the moment that the there is an arrow from $(y_{j-1}, t_j)$ to $(y_j, t_j)$ as that in the condition (a) of the definition of special infection paths. Let

$$\tau_0 = \inf\{j \geq 1 : y_j = 0\}.$$

Then, by Markov property,

$$P(I_{y'}|I_y) = \begin{cases} pP(U_{(0, y')}^{(1)} < T_0^{(1)}|I_y) \\ = pP(U_{(0, y')}^{(1)} < T_0^{(1)} \land t_{\tau_0}|I_y) + pP(t_{\tau_0} < U_{(0, y')}^{(1)} < T_0^{(1)}|I_y) \\ = pP(U_{(0, y')}^{(1)} < T_0^{(1)} \land t_{\tau_0}|U_{(0, y)}^{(1)}(1) < T_0^{(1)}) \\ + pP(U_{(0, y')}^{(1)}(1) - t_{\tau_0} < T_0^{(1)} - t_{\tau_0}|I_y, U_{(0, y')}^{(1)}(1) > t_{\tau_0}, U_{(0, y)}^{(1)}(1) > t_{\tau_0}, T_0^{(1)} > t_{\tau_0})P(U_{(0, y')}^{(1)}(1) > t_{\tau_0}, T_0^{(1)} > t_{\tau_0}|U_{(0, y)}^{(1)}(1) < T_0^{(1)}). \quad (A.2) \end{cases}$$
From $t = t\tau_0$, the special infection path $y$ will visit 0 for $m + 1 - 1 = m$ times. Therefore, by Markov property and our assumption,

$$P(U_{(0,y')}_{1}^{(1)}(1) - t\tau_0 < T_0 - t\tau_0|I_y, U_{(0,y')}_{1}^{(1)}(1) > t\tau_0, T_0(1) > t\tau_0) \leq 2^m \left( \frac{\lambda}{1 + \lambda} \right).$$  

(A.3)

Please note that in (A.3) there is no $p$ in the right side because in the left side we do not care whether the edge connecting 0 and $y'_1$ is open.

Let the first part of (A.2) times $2^m$, then by (A.3),

$$P(I_y'|I_y) \leq 2^m p \left[ P(U_{(0,y')}_{1}^{(1)}(1) < T_0(1) \land t\tau_0|U_{(0,y')}_{1}^{(1)}(1) < T_0(1)) 
+ \left( \frac{\lambda}{1 + \lambda} \right) P(U_{(0,y')}_{1}^{(1)}(1) > t\tau_0, T_0(1) > t\tau_0|U_{(0,y')}_{1}^{(1)}(1) < T_0(1)) \right]$$

(A.4)

Since the exponential distribution has no memory,

$$\left( \frac{\lambda}{1 + \lambda} \right) = P(U_{(0,y')}_{1}^{(1)}(1) < T_0(1)|U_{(0,y')}_{1}^{(1)}(1) > t\tau_0, T_0(1) > t\tau_0).$$

Therefore, the right side of (A.4) equals

$$2^m p P(U_{(0,y')}_{1}^{(1)}(1) < T_0(1)|U_{(0,y')}_{1}^{(1)}(1) < T_0(1)) = 2^m p \frac{2\lambda}{1 + 2\lambda} \leq \frac{2^{m+1}\lambda p}{1 + \lambda}.$$  

As a result, (A.1) holds for $y, y'$ such that $0 \not\in K_1(y, y')$ and $C_{y,y'}(0) = m + 1$.

According to the principle of mathematical induction, (A.1) holds for any $y, y'$ such that $0 \not\in K_1(y, y')$.

Now we give the proof of Lemma 3.1.

The proof of Lemma 3.1. By (3.3),

$$P(I_y \cap I_{y'}) = P(I_{y'}|I_y) P(I_y) = P(I_{y'}|I_y) \left( \frac{\lambda p}{1 + \lambda} \right)^n.$$  

(A.5)

According to Markov property,

$$P(I_{y'}|I_y) = \prod_{j=0}^{n-1} P(I_{y'_j+1}|I_y \cap I_{y'_j}).$$  

(A.6)

For $j \not\in K_1(y, y') \cup K_2(y, y')$,

$$P(I_{y'_j+1}|I_y \cap I_{y'_j}) = P(I_{y'_j+1}|I_{y'_j}) = p P(U_{(y'_j, y'_j+1)}^{(1)}(1) < T_{y'_j}(1)) = \frac{\lambda p}{1 + \lambda}.$$  

(A.7)
For \( j \in K_1(y, y') \),

\[
P(I_{y'_{j+1}} I_y \bigcap I_{y'_{j}}) \leq 1. \quad \text{(A.8)}
\]

For \( j \in K_2(y, y') \) and \( C_{(y, y')}(j) = m \), let

\[
k_1 = \inf\{ i \geq 0 : y_i = y'_j \}
\]

and

\[
k_l = \inf\{ i > k_{l-1} : y_i = y'_j \}
\]

for \( 2 \leq l \leq m \).

Conditioned on \( I_{y'_{j+1}} \), we denote by \( t'_j \) the moment that there is an arrow from \((y'_{j-1}, t'_j)\) to \((y'_j, t'_j)\) as that in the condition (a) of the definition of special infection path. \( \{t_l, 1 \leq l \leq n\} \) are the same as that in the proof of Lemma A.1.

Conditioned on \( t'_j > t_{k_{n+1}} \), by Markov property,

\[
P(I_{y'_{j+1}} | I_y \bigcap I_{y'_{j}}) = pP(U(y'_j, y'_{j+1})(1) < T_{y'_j}(1)) = \frac{p\lambda}{1+\lambda}. \quad \text{(A.9)}
\]

Conditioned on \( t_{k_i} < t'_j < t_{k_{i+1}} \) for some \( i \leq m \), by Markov property and Lemma A.1

\[
P(I_{y'_{j+1}} | I_y \bigcap I_{y'_{j}}) \leq \left( \frac{\lambda p}{1+\lambda} \right)^{2^{m+1-i}}. \quad \text{(A.10)}
\]

since \( y \) will visit \( y'_j \) for \( m - (i - 1) \) times after \( t_{k_i} \).

Conditioned on \( t_{k_i} < t'_j < t_{k_{i+1}} \) for some \( i \leq m \), we define

\[
T = \inf\{ s > t'_j : \text{There is a } \delta \text{ at } (y'_j, s) \}.
\]

Then, by Markov property and Lemma A.1

\[
P(I_{y'_{j+1}} | I_y \bigcap I_{y'_{j}})
\]

\[
= pP(t'_j + 1 - t'_j \leq T - t'_j | I_y \bigcap I_{y'_{j}})
\]

\[
= pP(t'_j + 1 - t'_j \leq (T - t'_j) \wedge (t_{k_i} - t'_j))
\]

\[
+ pP(t'_j + 1 - t'_j > t_{k_i} - t'_j, T - t'_j > t_{k_i} - t'_j) P(t'_j + 1 - t'_j \leq T - t'_j | I_y \bigcap I_{y'_{j}}),
\]

\[
t'_j + 1 - t'_j > t_{k_i} - t'_j, T - t'_j > t_{k_i} - t'_j)
\]

\[
\leq \frac{\lambda p}{1+\lambda} \left[ 1 - e^{-\frac{(\lambda + 1)(t_{k_i} - t'_j)}} \right] + \left( \frac{\lambda p}{1+\lambda} \right) e^{-\frac{(\lambda + 1)(t_{k_i} - t'_j)2^{m-i+1}}}
\]

\[
\leq \left( \frac{\lambda p}{1+\lambda} \right)^{2^{m-i+1}}. \quad \text{(A.11)}
\]

By A.9, A.10, A.11 and total probability theorem,

\[
P(I_{y'_{j+1}} | I_y \bigcap I_{y'_{j}}) \leq \left( \frac{\lambda p}{1+\lambda} \right) \exp\{ C_{(y, y')}(j) \log 2 \} \quad \text{(A.12)}
\]
for \( j \in K_2(y, y') \).

By (A.6), (A.7), (A.8) and (A.12),

\[
P(I_{y'}|I_y) \leq \left( \frac{\lambda p}{1 + \lambda} \right)^{n - |K_1(y, y')|} \exp \left\{ \sum_{j \in K_2(y, y')} C(y, y')(j) \log 2 \right\}. \tag{A.13}
\]

Then \((3.7)\) follows (A.5) and (A.13).

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