ANTISYMMETRIC PARAMODULAR FORMS OF WEIGHTS 2 AND 3

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Abstract. We define an algebraic set in 23 dimensional projective space whose \( \mathbb{Q} \)-rational points correspond to meromorphic, antisymmetric, paramodular Borcherds products. We know two lines inside this algebraic set. Some rational points on these lines give holomorphic Borcherds products and thus construct examples of Siegel modular forms on degree two paramodular groups. Weight 3 examples provide antisymmetric canonical differential forms on Siegel modular threefolds. Weight 2 is the minimal weight and these examples, via the Paramodular Conjecture, give evidence for the modularity of some rank one abelian surfaces defined over \( \mathbb{Q} \).

1. Introduction

This article formulates the construction of meromorphic Borcherds products as a diophantine problem. We discover two infinite families of solutions and produce interesting examples of holomorphic antisymmetric paramodular forms of weights 2 and 3. The first author constructed paramodular cusp forms of low weight and used them to study the geometry of the moduli spaces of abelian surfaces and Kummer surfaces \[7, 8, 9\]. These paramodular forms were constructed by lifting and so were symmetric. In \[9\], with geometric applications in mind, the first author posed the problem of constructing antisymmetric paramodular cusp forms of low weight.

Further motivation to construct paramodular cusp forms of low weight comes from questions of modularity. Certain cusp forms of weight 2 and paramodular level \( N \) conjecturally correspond to abelian surfaces of conductor \( N \) defined over \( \mathbb{Q} \) whose endomorphisms defined over \( \mathbb{Q} \) are trivial, as made precise in the Paramodular Conjecture of Brumer and Kramer \[4\]. Also, nonlift paramodular cusp forms of weight 3 conjecturally occur as cohomology classes in \( H^5(\Gamma_0(N); \mathbb{C}) \), as studied by Ash, Gunnells and McConnell \[1\].

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In this article, we use Borcherds products to construct examples of antisymmetric paramodular cusp forms of weights 2 and 3 with applications to both geometry and modularity. Weight two is the minimal weight, and weight three is the canonical weight. Our method relies heavily on the \textit{theta blocks} introduced by Gritsenko, Skoruppa and Zagier in [16]. We were led to these constructions by the Paramodular Conjecture and so we devote some space to it here.

**Paramodular Conjecture** (A. Brumer, K. Kramer, 2010.)

Let $N \in \mathbb{N}$. There is a bijection between:

1. isogeny classes of abelian surfaces $\mathcal{A}$ defined over $\mathbb{Q}$ with conductor $N$ and $\text{End}_{\mathbb{Q}}(\mathcal{A}) = \mathbb{Z}$, and
2. lines of Hecke eigenforms $f \in S_2(K(N))^\text{new}$ that are not Gritsenko lifts from $J_{2,N}^\text{cusp}$ and whose eigenvalues are rational.

In this correspondence, $L(\mathcal{A}, s, \text{Hasse-Weil}) = L(f, s, \text{spin})$.

In [19], intensive computations proved that, for primes $p < 600$, the space $S_2(K(p))$ is spanned by Gritsenko lifts with the possible exceptions of $p \in \{277, 349, 353, 389, 461, 523, 587\}$. These computations are consistent with the Paramodular Conjecture in two ways. First, Brumer and Kramer proved, among other results [4], that for primes $p < 600$ no abelian surfaces over $\mathbb{Q}$ with conductor $p$ can exist apart from the $p$ on this list. Second, an abelian surface defined over $\mathbb{Q}$ with conductor $p$ is known for every prime on this list. In fact, for $p = 587$ two isogeny classes are known, one $\mathcal{A}_{587}^+$ with even rank zero and one $\mathcal{A}_{587}^-$ with odd rank one. Still, on the automorphic side, there remains the problem of giving a construction to prove the existence of weight two paramodular cusp forms that are not Gritsenko lifts. Only in one case, $p = 277$, was the existence of a weight two paramodular nonlift eigenform proven. This article constructs a weight two nonlift paramodular eigenform in $S_2(K(587))^-$ and thereby produces additional evidence for the Paramodular Conjecture. This construction is our motivating example, see section 4.

We also construct a nonlift paramodular eigenform in $S_3(K(167))^+$, which experimentally seems to be the first weight three form all of whose paramodular Atkin-Lehner signs are $+1$. This form induces a canonical differential form on a moduli space of polarized Kummer surfaces, $K(167)^+ \backslash \mathcal{H}_2$, thereby proving that it is not rational or unirational. The final section gives a complete list of the examples we found by solving diophantine equations.

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2. Background

We follow [6] for the theory of Siegel modular forms. For a group $\Gamma$ commensurable with $\Gamma_n = \text{Sp}_n(\mathbb{Z})$, let $M_k(\Gamma, \chi)$ denote the $\mathbb{C}$-vector space of Siegel modular forms of weight $k$ and character $\chi$, and $S_k(\Gamma, \chi)$ the subspace of cusp forms. The basic factor of automorphy is defined by $\mu((C, D), \Omega) = \det(C\Omega + D)$. The space of meromorphic functions transforming by $\mu^k \chi$ for $\Gamma$ is denoted by $M_{\text{mero}}^k(\Gamma, \chi)$. We will be concerned with degree $n = 2$ and the paramodular group of paramodular level $N$:

$$K(N) = \left( \begin{array}{cccc} * & N* & * & * \\ * & * & * & * \\ * & * & * & * \\ N & N & N & * \end{array} \right) \cap \text{Sp}_2(\mathbb{Q}), \quad * \in \mathbb{Z}.$$ 

For any natural number $N > 1$, the paramodular group $K(N)$ has a normalizing involution $\mu_N$ given by $\mu_N = \left( F_N^* 0 \right)$, where $F_N = \gamma_N \left( \begin{array}{cc} 0 & 1 \\ -N & 0 \end{array} \right)$ is the Fricke involution, and we also use the group $K(N)^+$ generated by $K(N)$ and $\mu_N$. We have the direct sum $M_k(K(N)) = M_k(K(N))^+ \oplus M_k(K(N))^-$ into plus and minus $\mu_N$-eigenspaces. This involution and the Witt map can be used to prove that $M_2(K(p)) = S_2(K(p))$ when $p$ is prime, see [17]. We can also consider the group $K(N)^*$, generated by $K(N)$ and all the paramodular Atkin-Lehner involutions, see [11]; like $\mu_N$, these have an elliptic Atkin-Lehner involution in the lower right and the transpose inverse in the upper left. The moduli space of Kummer surfaces with polarization type $(1, N)$ is isomorphic to $K(N)^*/\mathcal{H}_2$. One can use the paramodular Atkin-Lehner operators, the Witt map, and the cusp structure of $K(N)$ to show that $M_2(K(N)) = S_2(K(N))$ for squarefree $N$.

We define Jacobi forms with reference to the subgroup

$$P_{2,1}(\mathbb{Z}) = \left( \begin{array}{cccc} * & 0 & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{array} \right) \cap \text{Sp}_2(\mathbb{Z}), \quad * \in \mathbb{Z}.$$ 

The group $P_{2,1}(\mathbb{Z})$ is generated by a Heisenberg group $H(\mathbb{Z})$, a copy of $\text{SL}_2(\mathbb{Z})$, and by $\{I_4, -I_4\}$. The general element $h$ of $H(\mathbb{Z})$ and $\sigma$
of $SL_2(\mathbb{Z})$ have the form
\[
h = \begin{pmatrix} 1 & 0 & 0 & v \\ \lambda & 1 & v & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \sigma = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]
for $\lambda, v, \kappa \in \mathbb{Z}$, and for $(\frac{a}{c}, \frac{b}{d}) \in SL_2(\mathbb{Z})$. A character $\nu_H : H(\mathbb{Z}) \to \{\pm 1\}$ is given by $\nu_H(h) = (-1)^{\lambda v + \lambda + v + \kappa}$ and extends to a character on $P_{2,1}(\mathbb{Z})$ trivial on $SL_2(\mathbb{Z})$. Similarly, the multiplier $\epsilon$ of the Dedekind Eta function extends to a multiplier on $P_{2,1}(\mathbb{Z})$ trivial on $H(\mathbb{Z})$.

The following definition of Jacobi forms, see [15], is equivalent to the usual one [5]. For $a, b, 2k, 2m \in \mathbb{Z}$, consider holomorphic $\phi : H_1 \times \mathbb{C} \to \mathbb{C}$, such that the modified function $\tilde{\phi} : H_2 \to \mathbb{C}$, given by $\tilde{\phi}(\tau, z) = \phi(\tau, z)e^{(m \omega)}$, transforms by the factor of automorphy $\mu^k e^{a \cdot \nu_H \cdot b}$ for $P_{2,1}(\mathbb{Z})$. We always select holomorphic branches of roots that are positive on the purely imaginary elements of the Siegel half space. We necessarily have $2k \equiv a \pmod{2}, 2m \equiv b \pmod{2}$, and $m \geq 0$ for nontrivial $\phi$. Such $\phi$ have Fourier expansions $\phi(\tau, z) = \sum_{n, r \in \mathbb{Q}} c(n, r; \phi) q^n \zeta^r$, for $q = e(\tau)$ and $\zeta = e(z)$. We write $\phi \in J_{k, m}^{w,h} (e^{a \cdot \nu_H})$ if, additionally, the support of $\phi$ has $n$ bounded from below, and call such forms weakly holomorphic. We write $\phi \in J_{k, m}^{weak} (e^{a \cdot \nu_H})$ if the support of $\phi$ satisfies $n \geq 0$; $\phi \in J_{k, m} (e^{a \cdot \nu_H})$ if $4mn - r^2 \geq 0$; $\phi \in J_{k, m}^{cusp} (e^{a \cdot \nu_H})$ if $4mn - r^2 > 0$. We also use the notation $J_{k, m}^{cusp} (\nu) = \{ \phi \in J_{k, m}^{cusp} : \ord_q \phi \geq \nu \}$ and denote the elements of $J_{k, m}^{w,h}$ whose Fourier coefficients are integral by $J_{k, m}^{w,h} (\mathbb{Z})$.

For basic examples of Jacobi forms we make use of the Dedekind Eta function $\eta(\tau) = q^{\frac{1}{24}} \prod_{n \in \mathbb{N}} (1 - q^n)$ and the odd Jacobi theta function:
\[
\vartheta(\tau, z) = q^{\frac{1}{2}} \left( \zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}} \right) \prod_{j \in \mathbb{N}} (1 - q^j \zeta)(1 - q^j \zeta^{-1})(1 - q^j).
\]
We have $\vartheta \in J_{k, 0}^{cusp} (e^{3 \cdot \nu_H})$, $\eta \in J_{k, 0}^{cusp} (\epsilon)$ and $\vartheta_\ell \in J_{k, \ell}^{cusp} (e^{3 \cdot \nu_H})$, where $\vartheta_\ell(\tau, z) = \vartheta(\tau, \ell z)$ and $\ell \in \mathbb{N}$, compare [15].

The Fourier-Jacobi expansion of a paramodular form is an important connection between Jacobi and paramodular forms. For $\Omega = \left( \begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix} \right) \in H_2$, and $f \in M_k (K(N))^\ell$, write the Fourier-Jacobi expansion of $f$ as
\[
f(\Omega) = \sum_{m=0}^{\infty} \phi_{Nm}(\tau, z)e(Nm \omega).
\]
Each Fourier-Jacobi coefficient is a Jacobi form, \( \phi_{N_m} \in J_{k,N_m} \), and these Fourier coefficients satisfy the involution condition:

\[
    c(n, r; \phi_{N_m}) = (-1)^k \epsilon c(m, r; \phi_{N_n}).
\]

Paramodular forms with \((-1)^k \epsilon = +1\) are called symmetric, those with \((-1)^k \epsilon = -1\) are called antisymmetric.

We let \( e^a \times v^b_H : K(N)^+ \to e(\frac{1}{12}Z) \) denote the unique character, if it exists, whose restriction to \( P_{2,1}(Z) \) is \( e^a v^b_H \), and whose value on \( \mu_N \) is 1. For \( a, b \in Z \), the character exists precisely when there is a \( j \in Z \) such that \( a \equiv j \frac{24}{\gcd(2N,12)} \mod 24 \) and \( b \equiv j \frac{2N}{\gcd(2N,12)} \mod 2 \), see \([12]\).

Let \( \phi \in J_{k,t}^{w.h.} \) be a weakly holomorphic Jacobi form. Recall the level raising Hecke operators \( V_m : J_{k,t}^{w.h.} \to J_{k,tm}^{w.h.} \) from \([3]\), page 41. These operators have the following action on Fourier coefficients:

\[
    c(n, r; \phi|V_m) = \sum_{d \in \mathbb{N}: d|(n,r,m)} d^{k-1}c \left( \frac{nm}{d^2}, \frac{r}{d}; \phi \right).
\]

Given any \( \phi \in J_{k,t}^{w.h.} \), we may consider the following series:

\[
    \operatorname{Grit}(\phi)(\tau, z) = \delta(k)c(0, 0; \phi)G_k(\tau) + \sum_{m \in \mathbb{N}} (\phi|V_m)(\tau, z)e(mt\omega)
\]

where \( \delta(k) = 1 \) for even \( k \geq 4 \) and \( \delta(k) = 0 \) for all other \( k \), and \( G_k(\tau) = \frac{1}{2}\zeta(1-k) + \Sigma_{n \geq 1} \sigma_{k-1}(n)e(\tau) \) is the Eisenstein series of weight \( k \).

**Theorem 2.1.** \([11], [8]\) For \( \phi \in J_{k,t} \), the series \( \operatorname{Grit}(\phi) \) converges on \( \mathcal{H}_2 \) and defines a holomorphic function \( \operatorname{Grit}(\phi) : \mathcal{H}_2 \to \mathbb{C} \) that is an element of \( M_k(K(t))^\epsilon \) with \( \epsilon = (-1)^k \). This is a cusp form if \( \phi \in J_{k,t}^{\text{cusp}} \).

The Gritsenko lift, \( \operatorname{Grit}(\phi) \), of the Jacobi form \( \phi \) defines a linear map \( \operatorname{Grit} : J_{k,t} \to M_k(K(t))^\epsilon \). Gritsenko lifts are hence symmetric. A different type of lifting construction is due to Borcherds (see \([3]\)) via his theory of multivariable infinite products on orthogonal groups. The divisor of a Borcherds Product is supported on rational quadratic divisors. In the case of the Siegel upper half plane of degree two, these rational quadratic divisors are the Humbert modular surfaces.

**Definition 2.2.** Let \( N \in \mathbb{N} \). For \( n_o, r_o, m_o \in \mathbb{Z} \) with \( m_o \geq 0 \) and \( \gcd(n_o, r_o, m_o) = 1 \), set \( T_o = \left( \frac{n_o}{r_o/2} \frac{r_o/2}{N m_o} \right) \) when \( \det(T_o) < 0 \). We call

\[
    \text{Hum}(T_o) = K(N)^+ \{ \Omega \in \mathcal{H}_2 : \text{tr}(\Omega T_o) = 0 \} \subseteq K(N)^+ \setminus \mathcal{H}_2.
\]

a Humbert surface.

A Humbert surface \( \text{Hum}(T_o) \) only depends upon two pieces of data \([11]\): the discriminant \( D = r_o^2 - 4Nm_on_o \) and \( r_o \mod 2N \).
Theorem 2.3. ([14], [15], [10]) Let $N, N_o \in \mathbb{N}$. Let $\Psi \in J_{N_o}^{w,h}$ be a weakly holomorphic Jacobi form with Fourier expansion

$$\Psi(\tau, z) = \sum_{n,r \in \mathbb{Z}} c(n, r) q^n \zeta^r$$

and $c(n, r) \in \mathbb{Z}$ for $4Nn - r^2 \leq 0$. Then we have $c(n, r) \in \mathbb{Z}$ for all $n, r \in \mathbb{Z}$. We set

$$24A = \sum_{\ell \in \mathbb{Z}} c(0, \ell); \quad 2B = \sum_{\ell \in \mathbb{N}} \ell c(0, \ell); \quad 4C = \sum_{\ell \in \mathbb{Z}} \ell^2 c(0, \ell);$$

$$D_0 = \sum_{n \in \mathbb{Z}: n < 0} \sigma_0(-n)c(n, 0); \quad k = \frac{1}{2} c(0, 0); \quad \chi = \epsilon^{24A} \times v_H^{2B}.$$ 

There is a function $Borch(\Psi) \in M^\text{mero}(K(N), \chi)$ whose divisor in $K(N)^+ \setminus \mathcal{H}_2$ consists of Humbert surfaces $\text{Hum}(T_o)$ for $T_o = \left( \frac{n_o}{r_o}, \frac{n_o}{2N_m_o} \right)$ with $\gcd(n_o, r_o, m_o) = 1$ and $m_o \geq 0$. The multiplicity of $Borch(\Psi)$ on $\text{Hum}(T_o)$ is $\sum_{n \in \mathbb{N}} c(n^2n_o, m_o, nr_o)$. We have,

$$Borch(\Psi)(F_N^{'}, \Omega F_N) = (-1)^{k+D_0} Borch(\Psi)(\Omega), \text{ for } \Omega \in \mathcal{H}_2.$$ 

For sufficiently large $\lambda$, for $\Omega = (\tau, z) \in \mathcal{H}_2$ and $q = e(\tau)$, $\zeta = e(z)$, $\xi = e(\omega)$, the following product converges on $\{\Omega \in \mathcal{H}_2: \text{Im } \Omega > \lambda I_2\}$:

$$Borch(\Psi)(\Omega) = q^A \zeta^B \xi^C \prod_{n,r,m \in \mathbb{Z}; m \geq 0, \text{ if } m = 0 \text{ then } n \geq 0} (1 - q^n \zeta^r \xi^{Nm})^{c(nm,r)}$$

and is on $\{\Omega \in \mathcal{H}_2: \text{Im } \Omega > \lambda I_2\}$ a rearrangement of

$$(2) \quad Borch(\Psi) = \left( \eta^{c(0,0)} \prod_{\ell \in \mathbb{N}} \left( \vartheta_\ell/\eta \right)^{c(0,\ell)} \right) \exp \left( - \text{Grit}(\Psi) \right).$$

3. Theta Blocks and Inflation

In the previous section, theta blocks occurred naturally as the leading Fourier-Jacobi coefficient of a Borcherds product, see [10] for a full exposition. A theta block is a meromorphic function $TB_f : \mathcal{H} \times \mathbb{C} \rightarrow \mathbb{C}$,

$$TB_f = \eta^{f(0)} \prod_{\ell \in \mathbb{N}} \left( \vartheta_\ell/\eta \right)^{f(\ell)},$$

for some even function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ of finite support. We see that $TB_f \in J_{k,m}^{\text{mero}}(\chi)$, where the weight $k$, index $m$ and character $\chi$ are
given by
\[ k = \frac{1}{2} f(0); \quad 4m = \sum_{\ell \in \mathbb{Z}} \ell^2 f(\ell); \]
\[ \chi = \epsilon_{24A} v_{2B}; \quad 24A = \sum_{\ell \in \mathbb{Z}} f(\ell); \quad 2B = \sum_{\ell \in \mathbb{N}} f(\ell). \]

We call the Laurent polynomial
\[ \sum_{\ell \in \mathbb{Z}} f(\ell) \zeta^\ell = f(0) + \sum_{\ell \in \mathbb{N}} f(\ell) (\zeta^\ell + \zeta^{-\ell}) \]
the germ of the theta block \( TB_f \). Note that the germ determines the theta block. By the shape of a theta block we simply mean the number of factors of \( \eta \) and of \( \vartheta \) in the numerator and denominator. To test whether a weakly holomorphic theta block is a Jacobi form, we need to see whether the order function \[ \text{ord}(TB_f; x) = k_{12} + \frac{1}{2} \sum_{\ell \in \mathbb{N}} f(\ell) \bar{B}_2(\ell x) \]
is nonnegative for \( 0 \leq x \leq 1 \). If \( \text{ord}(TB_f; x) > 0 \) then \( TB_f \in J_{k,m}^{\text{cusp}}(\chi) \).

Here, \( \bar{B}_2(x) = B_2(x - \lceil x \rceil) \) is the periodic extension of the second Bernoulli polynomial \( B_2(x) = x^2 - x + \frac{1}{6} \). If we multiply the defining infinite products together, we obtain
\[ TB_f(\tau, z) = q^A \prod_{\ell \in \mathbb{N}} (\zeta^{\ell/2} - \zeta^{-\ell/2})^{f(\ell)} \prod_{n \in \mathbb{N}} (1 - q^n \zeta^\ell)^{f(\ell)} \]
\[ = q^A \zeta^B \prod_{n, \ell \in \mathbb{Z}; n \geq 0 \text{ and } \text{if } n = 0 \text{ then } \ell < 0} (1 - q^n \zeta^\ell)^{f(\ell)}. \]

We set \( \text{BB}_f(\zeta) = \prod_{\ell \in \mathbb{N}} (\zeta^{\ell/2} - \zeta^{-\ell/2})^{f(\ell)} \) and affectionately refer to this as the baby theta block. The weight and the baby theta block determine the theta block.

When all the \( f(\ell) \) are nonnegative, then the theta functions are all in the numerator and we say that \( TB_f \) is a theta block without theta denominator. Theta blocks without theta denominator are automatically weakly holomorphic. In computations involving theta blocks without theta denominator it is often more convenient to describe theta blocks in terms of lists. Let \( T = T_f \) be the list consisting of \( f(1) \) 1s followed by \( f(2) \) 2s, \( f(3) \) 3s, etc. and let \( L = (-T) : [\bar{0}, \bar{0}, \ldots, \bar{0}] : T \) be defined as the concatenation of three lists. When we are dealing with lists we
have $24A = |L|$ and we write

$$TB_k(T)(\tau, z) = \eta^{2k} \prod_{d \in T} (\vartheta_d/\eta)$$

$$= q^A \prod_{d \in T} (\zeta^{d/2} - \zeta^{-d/2}) \prod_{n \in \mathbb{N}} \prod_{e \in L} (1 - q^n \zeta^e)$$

and write the baby theta block $BTB(T) = \prod_{d \in T} (\zeta^{d/2} - \zeta^{-d/2})$.

In order to construct paramodular Borcherds products we need a quotient $\psi$ of a Jacobi form by a theta block of the same weight. This weight zero $\psi$ should have integral Fourier coefficients and be weakly holomorphic as opposed to properly meromorphic. We now discuss two methods to guarantee that both these properties hold. The first method works for theta blocks without theta denominator and can fail otherwise. The second method is more general and does allow the theta block to have a theta denominator.

**Method 1.** Let $\phi$ be a theta block without theta denominator. The quotient $\phi|\mathcal{V}_\ell \phi$ has integral Fourier coefficients and is weakly holomorphic, see [17], Corollary 6.2, for the case $\ell = 2$.

**Method 2.** Suppose that $BTB_f(\zeta)$ divides $BTB_g(\zeta)$ in $\mathbb{Z}[\zeta^{1/2}, \zeta^{-1/2}]$, then the weight zero Jacobi form $\psi = \frac{TB_g}{TB_f}$ has integral Fourier coefficients and is weakly holomorphic. When we have this divisibility, we call $TB_g$ an inflation of $TB_f$.

As a special case of method two, suppose that the theta block is $\phi = TB_k(T)$ for some list $T$ and weight $k$. Now take another list $E$ of the same length with each entry of $T$, in some ordering, dividing the corresponding entry of $E$; in this case, we clearly have that $BTB(T)$ divides $BTB(E)$ in the ring $\mathbb{Z}[\zeta^{1/2}, \zeta^{-1/2}]$ and we call the theta block $TB_k(E)$ a strict inflation of the theta block $TB_k(T)$.

### 4. A Motivating Example

In this section we use the theory of Borcherds products to construct a paramodular eigenform $f \in S_2(K(587))^-$ that conjecturally shows the modularity of the isogeny class of rank one abelian surfaces defined over $\mathbb{Q}$ of prime conductor $p = 587$, the isogeny class represented by the Jacobian $A_{587}^-$ of $y^2 + (x^3 + x + 1)y = -x^3 - x^2$. In [19] it is proven that $\dim S_2(K(587))^- \leq 1$ and that $L(A_{587}^-, s, \text{Hasse-Weil})$ and $L(f, s, \text{spin})$ share the same Euler factors at 2 and 3, if $f$ can be proved to exist. The following analogy with the elliptic case led to the hope that $f$ is a
Borcherds product. The first odd rank elliptic curve over $\mathbb{Q}$ has conductor 37 and the associated modular form in $S_2(\Gamma_0(37))$ corresponds to a Jacobi form in $J_{2,37}^{\text{cusp}}$. From [16] we know that $J_{2,37}^{\text{cusp}}$ is generated by the theta block $\Theta B_2(1, 1, 1, 2, 2, 2, 3, 3, 4, 4, 5)$ and thus is given by an infinite product. The smallest known prime conductor of an odd rank abelian surface over $\mathbb{Q}$ is 587, see [4]. Our Borcherds product construction of $f$ proves the analogous result that the generator $f$ of $S_2(\mathcal{K}(587))$ is a multivariable infinite product. This Borcherds product representation of the eigenform $f$ will assist the computation of its eigenvalues.

We describe the experimental process that led to the rigorous construction of $f$ as a Borcherds product. Both formula (2) and the involution condition (1) are important tools in the experimental process. Suppose that we have an antisymmetric Borcherds product $f \in S_k(\mathcal{K}(p))$ with Fourier-Jacobi expansion

$$f = \phi_r e(2p\omega) + \phi_{2r} e(2p\omega) + \cdots$$

The antisymmetry of $f$ implies that $\phi_r \in J_{k,p}^{\text{cusp}}(2)$ because $c(1, r; \phi_r) = -c(1, r; \phi_p)$. If $f$ is a Borcherds product then $\phi_r$ is a theta block; indeed, $J_{2,587}^{\text{cusp}}(2)$ is spanned by the single theta block

$$\phi_{587} = \Theta B_2(1, 1, 2, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 8, 8, 9, 10, 11, 12, 13, 14).$$

As a Borcherds product, $f$ is determined by its first two Fourier-Jacobi coefficients $\phi_r$ and $\phi_{2r}$. If we write $\phi_{2r} = -\phi_r | V_2 + \Xi$, we claim this forces $\Xi \in J_{k,2r}^{\text{cusp}}(2)$. The involution condition tells us the $q^1$ and $q^2$ coefficients of $\Xi$. In general

$$c(n, r; \Xi) = c(n, r; \phi_r | V_2) + c(n, r; \phi_{2r}) = c(2n, r; \phi_r) + 2^{k-1} c(\frac{n}{2}, \frac{r}{2}; \phi_r) + c(n, r; \phi_{2r}).$$

The $q^1$-coefficient of $\Xi$ vanishes by antisymmetry

$$c(1, r; \Xi) = c(2, r; \phi_r) + c(1, r; \phi_{2r}) = -c(1, r; \phi_r) + c(1, r; \phi_{2r}) = 0,$$

so that $\Xi \in J_{k,2r}^{\text{cusp}}(2)$ as claimed. The $q^2$-coefficient of $\Xi$ is the $q^4$-coefficient of $\phi_r$:

$$c(2, r; \Xi) = c(4, r; \phi_r) + 2^{k-1} c(1, \frac{r}{2}; \phi_r) + c(2, r; \phi_{2r}) = c(4, r; \phi_r).$$

A direct computation reveals that the $q^4$-coefficient of $\phi_{587}$ is a baby theta block: $\text{Coeff}(q^4, \phi_{587}) = \text{Coeff}(q^2, \Xi) = \Theta B_2(1, 10, 2, 2, 18, 3, 3, 4, 4, 15, 5, 6, 6, 7, 8, 16, 9, 10, 22, 12, 13, 14).$

So we set

$$\Xi = \Theta B_2(1, 10, 2, 2, 18, 3, 3, 4, 4, 15, 5, 6, 6, 7, 8, 16, 9, 10, 22, 12, 13, 14)$$
and check that \( \Xi \in J_{2,1174}^{\text{cusp}}(2) \). The list for this theta block \( \Xi \) has been written so that each entry is visibly an integral multiple of the corresponding entry in the theta block \( \phi_{587} \); thus \( \Xi \) is a strict inflation of \( \phi_{587} \).

We now combine our two methods for constructing weight zero weakly holomorphic Jacobi forms with integral Fourier coefficients and define

\[
\psi = \frac{\phi_{587}|V_2 - \Xi}{\phi_{587}} \in J_{0,587}^{\text{w,h.}}(\mathbb{Z}).
\]

Many Fourier coefficients of \( \psi = \sum_{n,r} c(n,r;\psi)q^n \zeta^r \) can be seen at [20] and we give the singular part of \( \psi \) here, up to \( q^{-p/4-j} = q^{147} \):

\[
\psi_{\text{sing}} = \frac{1}{q} + 4 + \zeta^{-14} + \zeta^{-13} + \zeta^{-12} + \zeta^{-11} + \zeta^{-10} + \zeta^{-9} + 2\zeta^{-8} + \zeta^{-7} + 2\zeta^{-6} + 2\zeta^{-5} + 2\zeta^{-4} + 2\zeta^{-3} + 3\zeta^{-2} + 2\zeta^{-1} + 2\zeta + 3\zeta^2 + 2\zeta^3 + 2\zeta^4 + 2\zeta^5 + 2\zeta^6 + 7 + 2\zeta^8 + \zeta^9 + 10 + \zeta^{11} + \zeta^{12} + \zeta^{13} + \zeta^{14} + q \left( \zeta^{-50} + 2\zeta^{-49} + 2\zeta^{-49} + 2\zeta^5 \right) + q^2 \left( \zeta^{-69} + \zeta^69 \right) + q^3 \left( \zeta^{-85} + 2\zeta^{-84} + 2\zeta^{-84} + \zeta^{85} \right) + q^4 \left( \zeta^{-98} + 2\zeta^{-97} + 2\zeta^{-97} + \zeta^{98} \right) + q^5 \left( \zeta^{-109} + \zeta^{109} \right) + q^6 \left( \zeta^{-119} + \zeta^{119} \right) + q^{11} \left( \zeta^{-161} + \zeta^{161} \right) + q^{12} \left( 2\zeta^{-168} + 2\zeta^{-168} \right) + q^{13} \left( 2\zeta^{-175} + 2\zeta^{-175} \right) + q^{15} \left( \zeta^{-188} + \zeta^{188} \right) + q^{16} \left( 2\zeta^{-194} + 2\zeta^{-194} \right) + q^{17} \left( \zeta^{-200} + \zeta^{200} \right) + q^{27} \left( \zeta^{-252} + \zeta^{252} \right) + q^{29} \left( 3\zeta^{-261} + 3\zeta^{-261} \right) + q^{31} \left( \zeta^{-270} + \zeta^{270} \right) + q^{35} \left( \zeta^{-287} + \zeta^{287} \right) + q^{37} \left( \zeta^{-295} + \zeta^{295} \right) + q^{67} \left( \zeta^{-397} + \zeta^{397} \right) + q^{74} \left( \zeta^{-417} + \zeta^{417} \right) + q^{78} \left( 2\zeta^{-428} + 2\zeta^{-428} \right) + q^{79} \left( \zeta^{-431} + \zeta^{431} \right) + q^{85} \left( \zeta^{-447} + \zeta^{447} \right) + q^{87} \left( 2\zeta^{-452} + 2\zeta^{-452} \right) + q^{94} \left( \zeta^{-470} + \zeta^{470} \right) + q^{101} \left( 2\zeta^{-487} + 2\zeta^{487} \right) + q^{106} \left( \zeta^{-499} + \zeta^{499} \right) + q^{109} \left( \zeta^{-506} + \zeta^{506} \right) + q^{116} \left( \zeta^{-522} + \zeta^{522} \right) + q^{126} \left( 2\zeta^{-544} + 2\zeta^{544} \right) + q^{133} \left( \zeta^{-559} + \zeta^{559} \right) + q^{134} \left( \zeta^{-561} + \zeta^{561} \right) + O(q^{148}).
\]

The Borcherds product is holomorphic because these singular Fourier coefficients are positive. We compute \( A = 2, B = 68, C = 587, \) and \( D_0 = 1 \), so that \( f = \text{Borch} (\psi) \in M_2 (K(587))^- = S_2(K(587))^- \) as desired. We have the infinite product expansion

\[
f(\tau, \omega) = q^2 \zeta^{68} \zeta^{587} \prod_{n,r,m \in \mathbb{Z}: m \geq 0, \text{if } m = 0 \text{ then } n \geq 0 \text{ and if } m = n = 0 \text{ then } r < 0} (1 - q^n \zeta^r \zeta^{587m}) c(nm, r; \psi)\]
5. First Version of Main Result

We formalize the considerations that made the example of the previous section work, and isolate a condition that guarantees the existence of an antisymmetric, meromorphic, paramodular Borcherds product. This is the first version of our main result.

Theorem 5.1. Fix $m, \ell \in \mathbb{N}$ with $k = 24 - \ell \geq 0$, and fix $c \in \mathbb{N}$. Assume that $d \in \mathbb{N}$ satisfies

\begin{equation}
\prod_{j=1}^{\ell} \frac{\zeta_{\frac{1}{2}} c_j d_j - \zeta_{\frac{1}{2}} c_j d_j}{\zeta_{\frac{1}{2}} d_j - \zeta_{\frac{1}{2}} d_j} = \sigma_2(d) + (2k - 1)\sigma_1(d) + 2k^2 - 3k + \ell.
\end{equation}

in $\mathbb{Z}[\zeta^{1/2}, \zeta^{-1/2}]$, where $r_j = \frac{\zeta^{d_j} + \zeta^{-d_j}}{2}$, and the two symmetric functions are given by $\sigma_2(d) = \sum_{1 \leq i < j \leq \ell} r_i r_j$ and $\sigma_1(d) = \sum_{1 \leq i \leq \ell} r_i$.

Then there exists a meromorphic, antisymmetric Borcherds product $\text{Borch}(\psi) = \tilde{\phi} \exp (-\text{Grit}(\psi)) \in M_{k}^{\text{mero}} (K(N))^\ell$, where $\epsilon = (-1)^{k+1}$, $N = \frac{1}{2} \sum_{j=1}^{\ell} d_j^2 \in \mathbb{N}$, $\phi = TB_k(d_1, \ldots, d_\ell)$, and $\Xi = TB_k(c_1 d_1, \ldots, c_\ell d_\ell)$, and

$$\psi = \frac{\phi|V_2 - m\Xi}{\phi} \in J_{0,N}^{w.h.}(\mathbb{Z}).$$

Lemma 5.2. With the assumptions of Theorem 5.1, we have

$$m \prod_{j=1}^{\ell} c_j = 1080; \quad \sum_{j=1}^{\ell} c_j^2 d_j^2 = 2 \sum_{j=1}^{\ell} d_j^2; \quad \sum_{j=1}^{\ell} d_j^2 \in 2\mathbb{Z}.$$ 

Proof. We let $\zeta \to 1$ in equation (3); so $r_j \to 2$, $\sigma_1(d) \to 2\ell$, and $\sigma_2(d) \to 4\binom{\ell}{2}$. This gives us

$$m \prod_{j=1}^{\ell} c_j = 4 \binom{\ell}{2} + (2k - 1)2\ell + 2k^2 - 3k + \ell$$

$$= 2(\ell + k)^2 - 3(\ell + k) = 2(24)^2 - 3(24) = 1080.$$ 

Both sides of equation (3) are zero if we differentiate once with respect to $\zeta$ and let $\zeta \to 1$. We differentiate twice with respect to $\zeta$ and let $\zeta \to 1$. The left hand side gives us

$$\left( m \prod_{j=1}^{\ell} c_j \right) \sum_{j=1}^{\ell} \frac{1}{12} d_j^2 (c_j^2 - 1) = 90 \sum_{j=1}^{\ell} d_j^2 (c_j^2 - 1).$$
The right hand side gives us
\[ \sigma_2(d''|_{\zeta=1} + \sigma_1(d'')|_{\zeta=1} = 4(\ell - 1)\sum_{j=1}^{\ell} d_j^2 + (2k - 1)2 \sum_{j=1}^{\ell} d_j^2 = 90 \sum_{j=1}^{\ell} d_j^2. \]

Thus we have \( \sum_{j=1}^{\ell} c_j^2 d_j^2 = 2 \sum_{j=1}^{\ell} d_j^2 \) as asserted. For the final assertion, we note that
\[
m \prod_{j=1}^{\ell} \frac{\zeta^{\frac{1}{2}c_j d_j} - \zeta^{-\frac{1}{2}c_j d_j}}{\zeta^{\frac{1}{2}d_j} - \zeta^{-\frac{1}{2}d_j}} = m \zeta^{\sum_{j=1}^{\ell} \frac{1}{2}d_j(1-c_j)} \prod_{j=1}^{\ell} \left( \sum_{i=0}^{c_j-1} \zeta^{i d_j} \right). \]

From our assumption that this equals \( \sigma_2(d) + (2k-1)\sigma_1(d) + 2k^2 - 3k + \ell \), which is in \( \mathbb{Z}[\zeta, \zeta^{-1}] \), we obtain \( \sum_{j=1}^{\ell} d_j(1-c_j) \in 2\mathbb{Z} \) and \( \sum_{j=1}^{\ell} d_j \equiv \sum_{j=1}^{\ell} c_j d_j \mod 2 \). Thus we have
\[
\sum_{j=1}^{\ell} d_j^2 \equiv \sum_{j=1}^{\ell} c_j d_j \equiv \sum_{j=1}^{\ell} c_j^2 d_j^2 = 2 \sum_{j=1}^{\ell} d_j^2 \equiv 0 \mod 2. \quad \square
\]

**Proof of Theorem 5.1.** By the Lemma, \( N = \frac{1}{2} \sum_{j=1}^{\ell} d_j^2 \) is integral. From the shape of the theta block we have \( \phi \in J_{\kappa, N}^{\text{weak}}(2) \). We expand \( \phi \) out to order \( q^4 \).

\[ \phi = \text{BTB}(d)q^2 \left( 1 - (2k + \sigma_1(d))q \right) + \left( \sigma_2(d) + (2k-1)\sigma_1(d) + 2k^2 - 3k + \ell \right) q^2 + O(q^3) \]

The action of \( V_2 \) gives us the expansion of \( \phi | V_2 \in J_{\kappa, 2N}^{\text{weak}} \).

\[ \phi | V_2 = \text{BTB}(d)q \left( 1 + \left( \sigma_2(d) + (2k-1)\sigma_1(d) + 2k^2 - 3k + \ell \right) q + O(q^2) \right) \]

Thus, by the first method, we have \( \frac{\phi | V_2}{\phi} \in J_{0, N}^{\text{weak}}(\mathbb{Z}) \) and
\[ \frac{\phi | V_2}{\phi} = \frac{1}{q} + \left( \sigma_2(d) + 2k \sigma_1(d) + 2k^2 - k + \ell \right) + O(q). \]

We turn our attention to the theta block \( \Xi = \text{TB}_k(c * d) \), whose index is \( \frac{1}{2} \sum_{j=1}^{\ell} c_j^2 d_j^2 = \sum_{j=1}^{\ell} d_j^2 = 2N \) by Lemma 5.2. From the shape of \( \Xi \) we have \( \Xi \in J_{\kappa, 2N}^{\text{weak}}(2) \) and
\[ \Xi = \text{BTB}(c * d)q^2 \left( 1 + O(q) \right). \]

The theta block \( \Xi \) is a strict inflation of \( \phi \), so that \( \text{BTB}(d) \) divides \( \text{BTB}(c * d) \) in \( \mathbb{Z}[\zeta, \zeta^{-1}] \) and \( \Xi/\phi \in J_{0, N}^{\text{weak}}(\mathbb{Z}) \). The \( q \)-expansion begins
\[ \frac{\Xi}{\phi} = \frac{\text{BTB}(c * d)}{\text{BTB}(d)} \left( 1 + O(q) \right). \]
The next step is the crucial one. Equation (3) says exactly that
\[ m \frac{\text{BTB}(c \ast d)}{\text{BTB}(d)} = \sigma_2(d) + (2k - 1)\sigma_1(d) + 2k^2 - 3k + \ell. \]

The \( q \)-expansion of \( \psi = (\phi|V_2 - m\Xi)/\phi \in J_{0,w}^{h \phi}(\mathbb{Z}) \) is thus
\[
\psi = \frac{1}{q} + (\sigma_2(d) + 2k\sigma_1(d) + 2k^2 - k + \ell) - (\sigma_2(d) + (2k - 1)\sigma_1(d) + 2k^2 - 3k + \ell) + O(q)
\]
\[
= \frac{1}{q} + \sigma_1(d) + 2k + O(q).
\]

The paramodular form \( \text{Borch}(\psi) \) exists by Theorem 2.3. The \( q^0 \)-term of \( \psi \) is the germ of the theta block \( \phi \), therefore the weight of \( \text{Borch}(\psi) \) is \( k \),
\[
A = \frac{1}{24}(2k + 2\ell) = 2, \quad B = \frac{1}{2} \sum_{j=1}^{\ell} d_j, \quad \text{and} \quad C = \frac{1}{2} \sum_{j=1}^{\ell} d_j^2 = N.
\]
We also compute \( D_0 = \sum_{n \in \mathbb{Z}, n < 0} \sigma_1(-n)c(n, 0; \psi) = \sigma_1(1)c(-1, 0; \psi) = 1 \). The character of \( \text{Borch}(\psi) \) is trivial because \( A \) and \( B \) are integral. \( \text{Borch}(\psi) \) is antisymmetric because \( D_0 \) is odd, implying that \( (-1)^k\epsilon = -1 \). This verifies that \( \text{Borch}(\psi) \in M_k^{\text{mero}}(K(N))^\epsilon \).

6. Second Version of Main Result

We reformulate equation (3) as a diophantine equation and give a second version of our main result. We want the variables \( d_j \) to satisfy polynomial equations so that our meromorphic Borcherds products correspond to integral points on an algebraic set. We first reduce equation (3) to the special case where \( \ell = 24, k = 0, \) and \( m = 1 \):
\[
(4) \quad \prod_{j=1}^{24} \frac{\zeta^{\frac{1}{2}c_j d_j} - \zeta^{-\frac{1}{2}c_j d_j}}{\zeta^{\frac{1}{2}d_j} - \zeta^{-\frac{1}{2}d_j}} = \sigma_2(d) - \sigma_1(d) + 24,
\]
except that we now allow \( d \in \mathbb{Z}^{24} \) and interpret \( \frac{\zeta^{\frac{1}{2}c_j d_j} - \zeta^{-\frac{1}{2}c_j d_j}}{\zeta^{\frac{1}{2}d_j} - \zeta^{-\frac{1}{2}d_j}} \) as its limiting value \( c_j \) when \( d_j = 0 \).

Fix \( c \in \mathbb{N}^{24} \). Suppose that \( d \in \mathbb{Z}^{24} \) satisfies equation (4) and has \( k \) zero entries. For \( k + \ell = 24 \), let \( \bar{d} \in \mathbb{N}^{\ell} \) be given by the nonzero \( |d_j| \) in some order. Let \( \bar{c} \in \mathbb{N}^{\ell} \) be the corresponding entries of \( c \). If we set
m = \prod_{j:d_j=0} c_j \text{ then equation (4) becomes }
\begin{align*}
  m \prod_{j=1}^{\ell} \frac{\zeta^{\frac{1}{2}c_j d_j} - \zeta^{-\frac{1}{2}c_j d_j}}{\zeta^{\frac{1}{2}d_j} - \zeta^{-\frac{1}{2}d_j}} &= 4 \left( \frac{k}{2} \right) + 2 \left( \frac{k}{1} \right) \sigma_1(d) - \sigma_2(d) - (2k + \sigma_1(d)) + 24 \\
  &= \sigma_2(d) + (2k - 1)\sigma_1(d) + 2k^2 - 3k + \ell,
\end{align*}
which is the general form of equation (3) for \( \bar{c} \in \mathbb{N}^\ell \) with respect to \( c \in \mathbb{N}^\ell \). Thus, by allowing \( d \) to have zero entries we may take \( d \in \mathbb{Z}^{24} \) as the general case. Next, we substitute \( \zeta = e^{iz} \) in equation (4) and reformulate equation (4) as a diophantine problem.

**Lemma 6.1.** Fix \( c \in \mathbb{N}^{24} \) with \( \prod_{j=1}^{24} c_j = 1080 \). Then
\[
\{d \in \mathbb{Z}^{24} : \text{equation (4) holds}\} = \{d \in \mathbb{Z}^{24} : \text{equation (5) holds}\}
\]
where equation (5) means the equality of formal power series
\[
\exp \left( \sum_{n \in \mathbb{N}} \frac{(-1)^n}{(2n)!} \zeta(1 - 2n) \sum_{j=1}^{24} (1 - c_j^{2n})d_j^{2n}z^{2n} \right) = 1 + \frac{1}{540} \sum_{n \in \mathbb{N}} \frac{(-1)^n}{(2n)!} \left( \sum_{1 \leq i < j \leq 24} [(d_i + d_j)^{2n} + (d_i - d_j)^{2n}] - \sum_{j=1}^{24} d_j^{2n} \right) z^{2n}.
\]

**Proof.** Setting \( \zeta = e^{iz} \), we have \( r_j = \zeta^{d_j} + \zeta^{-d_j} = 2 \cos(d_jz) \); also, \( r_ir_j = 4 \cos(d_iz) \cos(d_jz) = 2 \cos((d_i + d_j)z) + 2 \cos((d_i - d_j)z) \). It is straightforward to check that
\[
\sigma_2(d) - \sigma_1(d) + 24 = 1080 + 2 \sum_{n \in \mathbb{N}} \frac{(-1)^n}{(2n)!} \left( \sum_{1 \leq i < j \leq 24} [(d_i + d_j)^{2n} + (d_i - d_j)^{2n}] - \sum_{j=1}^{24} d_j^{2n} \right) z^{2n},
\]
an equation whose series converge for all \( z \in \mathbb{C} \).

If we separate out the factors in equation (4) with \( d_j = 0 \), we obtain
\[
\prod_{j=1}^{24} \frac{\zeta^{\frac{1}{2}c_j d_j} - \zeta^{-\frac{1}{2}c_j d_j}}{\zeta^{\frac{1}{2}d_j} - \zeta^{-\frac{1}{2}d_j}} = \frac{\prod_{j=1}^{24} c_j}{\prod_{j:d_j \neq 0} \frac{\sin(\frac{1}{2}c_j d_j z)}{\frac{1}{2}c_j d_j z}} \prod_{j:d_j \neq 0} \frac{\sin(\frac{1}{2}d_j z)}{\frac{1}{2}d_j z}.
\]
The following series converges for \( |z| < \pi \).
\[
\ln(\frac{\sin(z)}{z}) = \sum_{n=1}^{\infty} \ln \left( 1 - \frac{z^2}{\pi^2 n^2} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!} \zeta(1 - 2n)(2z)^{2n},
\]
where \( \zeta(1 - 2n) \) denotes the value of the Riemann zeta function. Using this we have

\[
\prod_{j=1}^{24} \frac{\zeta^{\frac{1}{2}c_jd_j} - \zeta^{-\frac{1}{2}c_jd_j}}{\zeta^{\frac{1}{2}d_j} - \zeta^{-\frac{1}{2}d_j}} = \left( \prod_{j=1}^{24} c_j \right) \exp \left( \sum_{j:d_j \neq 0} \ln \left( \frac{\sin(\frac{1}{2}c_jd_jz)}{\frac{1}{2}c_jd_jz} \right) - \ln \left( \frac{\sin(\frac{1}{2}d_jz)}{\frac{1}{2}d_jz} \right) \right) 
\]

\[
= 1080 \exp \left( \sum_{n \in \mathbb{N}} \frac{(-1)^n}{(2n)!} \zeta(1 - 2n) \sum_{j=1}^{24} (d_j^{2n} - c_j^{2n}d_j^{2n})z^{2n} \right). 
\]

Reinserting the \( d_j \) that are zero does not change the value of the last sum. The equality of formal series asserted by equation (5) converges at least for \( |z| < \pi \), and so equation (5) is equivalent to the equality of Laurent polynomials in equation (4). \( \square \)

**Definition 6.2.** Take \( c \in \mathbb{N}^{24} \) with \( \prod_{j=1}^{24} c_j = 1080 \). Define the following algebraic set

\[
A_c = \{ d \in \mathbb{C}^{24} : \text{equation (5) holds} \}. 
\]

Note that \( A_c \subseteq \mathbb{C}^{24} \) is defined by a countable set of homogeneous polynomials, one for each positive even degree. The algebraic set \( A_c \) is invariant under the group that changes the sign of each entry and under the permutations of the 24 indices that fix \( c \). The first few of these equations are:

\[
\sum_{j=1}^{24} c_j^2d_j^2 = 2 \sum_{j=1}^{24} d_j^2. \quad (z^2 \text{ term}) 
\]

\[
\sum_{j=1}^{24} 46d_j^4 + 6c_j^4d_j^4 = \sum_{i,j=1}^{24} [15(c_i^2 - 1)(c_j^2 - 1) - 8]d_i^2d_j^2. \quad (z^4 \text{ term}) 
\]

\[
\sum_{j=1}^{24} (128 - 16c_j^2)d_j^6 + \sum_{i,j=1}^{24} (224 + 42(1 - c_i^2)(1 - c_j^4)) d_i^2d_j^4 + \sum_{i,j,k=1}^{24} 35(1 - c_i^2)(1 - c_j^2)(1 - c_k^2)d_i^2d_j^2d_k^2 = 0. \quad (z^6 \text{ term}) 
\]

The second version of the main result is formulated in terms of the algebraic set \( A_c \).
Theorem 6.3. Take $c \in \mathbb{N}^{24}$ with $\prod_{j=1}^{24} c_j = 1080$. Every nontrivial integral point $d \in A_c$ corresponds to an antisymmetric meromorphic paramodular Borcherds product as follows. By taking the absolute value of each entry, we may assume that $d \in A_c$ has nonnegative entries. Let $k$ be the number of zero entries in $d$, and set $\epsilon = (-1)^{k+1}$. The number $N = \frac{1}{2} \sum_{j=1}^{24} d_j^2$ is integral. Set $m = \prod_{j:d_j=0} c_j$. We have \( \text{Borch}(\psi) = \tilde{\phi} \exp (- \text{Grit}(\psi)) \in M^\text{mero}(K(N))^\epsilon, \)
where $\psi = \frac{\tilde{\phi} |V_2 - m\Xi|}{\phi} \in \mathcal{J}_{0,N}^w(\mathbb{Z})$, $\phi = \eta^{2k} \prod_{j:d_j\neq0} (\vartheta_d/\eta) \in \mathcal{J}_{k,N}^{\text{weak}}(2)$, and $\Xi = \eta^{2k} \prod_{j:d_j\neq0} (\vartheta_d/\eta) \in \mathcal{J}_{k,2N}^{\text{weak}}(2)$.

7. Two infinite families of integral points

For each $c \in \mathbb{N}^{24}$ with $\prod_{j=1}^{24} c_j = 1080$, we would like to know the decomposition of the projective algebraic set $[A_c \setminus \{0\}] \subseteq \mathbb{P}^{23}(\mathbb{C})$ into irreducible components. We do not know this decomposition. However, we wrote a program that, given an integral point $d \in A_c$ searches for linear spaces defined over $\mathbb{Q}$ that contain $d$ and also lie in $A_c$. Two infinite families were found in this way for the choice

$c = [5, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] \in \mathbb{N}^{24}.$

For this choice of $c$, $[A_c \setminus \{0\}]$ contains two projective lines $\mathcal{F}_1$ and $\mathcal{F}_2$.

$\mathcal{F}_1 = \{\beta, 2\beta, \beta + \alpha, \beta - \alpha, 2\beta + \alpha, 2\alpha, \alpha, 4\beta, 3\beta + 2\alpha, 3\beta + \alpha,$
$3\beta, 3\beta - \alpha, 2\beta + 2\alpha, 2\beta, 2\beta - \alpha, 2\beta - 2\alpha, \beta + 2\alpha, \beta + \alpha, \beta,$
$\beta - \alpha, \beta - 2\alpha, \alpha, 0, 0 : \alpha, \beta \in \mathbb{Z}\}$

$\mathcal{F}_2 = \{\beta, \alpha + \beta, \alpha, - \beta, \alpha + 2\beta, 2\alpha, \alpha + 2\beta, 2\alpha, 2\alpha + 2\beta, 2\alpha + \beta,$
$2\alpha - \beta, 2\alpha - 2\beta, \alpha + 3\beta, \alpha + \beta, \alpha, 4\beta, \alpha - \beta, 3\beta, 2\beta, \alpha - 3\beta,$
$\beta, \beta, 0, 0 : \alpha, \beta \in \mathbb{Z}\}$

Thus the countable number of homogeneous polynomials are consistent for this choice of inflation vector $c$. The authors know of no direct argument that makes this consistency clear. In summary, we have two infinite families of meromorphic antisymmetric Borcherds products with weights bounded by $k \leq 23$. It is interesting to note that the original weight two example for $N = 587$ has a different inflation vector $c$ and is not on either of these families.

8. Examples

We are especially interested in holomorphic Borcherds products. A direct search through the two infinite families found the holomorphic
antisymmetric paramodular Borcherds products listed in Table 1. We now explain how to read Table 1.

Fix $c = [5, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$. For the 24 integers given by $F_1(\alpha, \beta)$ or $F_2(\alpha, \beta)$, let $d \in \mathbb{Z}^{24}$ be the vector determined by their absolute values in the given ordering. Let $k$ be the number of zero entries in $d$, and let and $j_1, \ldots, j_\ell$ be the indices of the nonzero entries. For $N = \frac{1}{2} \sum_{j=1}^{24} d_j^2$, and $m = \prod_{j:d_j=0} c_j$, define

$$\phi = \text{TB}_k(d_{j_1}, \ldots, d_{j_\ell}) \in J^{\text{weak}}_{k,N}; \quad \Xi = \text{TB}_k(c_{j_1}d_{j_1}, \ldots, c_{j_\ell}d_{j_\ell}) \in J^{\text{weak}}_{k,2N};$$

$$\psi = \frac{\phi|V_2 - m\Xi}{\phi} \in J^{\text{w.h.}}_{0,N}(\mathbb{Z}).$$

Table 1 gives antisymmetric $\text{Borch}(\psi) \in S_k(K(N))^\epsilon$ for $\epsilon = (-1)^{k+1}$. Note that $\phi$ is the leading Fourier-Jacobi coefficient of $\text{Borch}(\psi)$.

Table 1. Antisymmetric Borcherds products in $S_k(K(N))^\epsilon$.

| $k$ | $N$ | $m$ | $(\alpha, \beta)$ for $F_1$ | $(\alpha, \beta)$ for $F_2$ | $\epsilon$ |
|-----|-----|-----|-----------------------------|-----------------------------|-----------|
| 2   | 587 | 1   | (1, 4) or (-5, 3)           |                             | -1        |
| 2   | 713 | 1   | (5, 3)                      |                             | -1        |
| 2   | 893 | 1   | (2, 1)                      |                             | +1        |
| 3   | 122 | 1   | (1, 2)                      |                             | +1        |
| 3   | 167 | 1   | (-2, 1)                     | (3, 1)                      | +1        |
| 3   | 173 | 1   | (-3, 2)                     |                             | +1        |
| 3   | 197 | 1   | (0, 1)                      |                             | +1        |
| 3   | 213 | 1   | (0, 1)                      |                             | +1        |
| 3   | 285 | 1   | (0, 1)                      |                             | +1        |
| 5   | 38  | 3   |                             | (0, 1)                      | +1        |
| 5   | 42  | 4   |                             | (0, 1)                      | +1        |
| 5   | 53  | 3   | (-1, 1)                     | (1, 1)                      | +1        |
| 5   | 65  | 3   | (1, 1)                      |                             | +1        |
| 8   | 17  | 15  |                             | (1, 0)                      | -1        |
| 9   | 15  | 10  |                             | (1, 0)                      | +1        |
We make some concluding remarks about these examples. Like 587, \(N = 713\) and 893 conjecturally show the modularity of known abelian surfaces defined over \(\mathbb{Q}\) of rank one and conductor \(N\). According to [4], equations of hyperelliptic curves whose Jacobians give these abelian surfaces are 
\[
y^2 = x^6 - 2x^5 + x^4 + 2x^3 + 2x^2 - 4x + 1 \text{ for } N = 713 \\
y^2 = x^6 - 2x^4 - 2x^3 - 3x^2 - 2x + 1 \text{ for } N = 893.
\]

The Siegel modular threefold \(K(t) \setminus \mathbb{H}_2\) is the moduli space of \((1,t)\)-polarized abelian surfaces because \(K(t)\) is isomorphic to the integral symplectic group of the symplectic form with elementary divisors \((1,t)\).

The paramodular group \(K(t)\) has the maximal extension \(K(t)^*\) in \(\text{Sp}_2(\mathbb{R})\) of order \(2^{\nu(t)}\) where \(\nu(t)\) is the number of prime divisors of \(t\), see [11]. In [11, Theorem 1.5] it was proved that the modular variety \(K(t)^* \setminus \mathbb{H}_2\) can be considered as the moduli space of Kummer surfaces associated to \((1,t)\)-polarized abelian surfaces. It was noted in [8] that the moduli space of polarized abelian surfaces might have trivial geometric genus only for twenty exceptional polarizations

\[t = 1, \ldots, 12, 14, 15, 16, 18, 20, 24, 30, 36,\]

It is now known [2] that \(\dim S_3(K(t)) = 0\) for these \(t\). For the moduli spaces of polarized Kummer surfaces we expect a rather long list of exceptional polarizations. One result in this direction is that for \(t = 21\) the space \(K(t)^* \setminus \mathbb{H}_2\) is uniruled, see [13].

Using our method we can construct the first canonical differential forms on \(K(t)^* \setminus \mathbb{H}_2\). According to the Freitag criterion (see [6, Hilfsatz 3.2.1]) for any smooth compactification \(\overline{K(t)^*} \setminus \mathbb{H}_2\), we have

\[h^{3,0}(\overline{K(t)^*} \setminus \mathbb{H}_2) = \dim S_3(K(t)^*)\]

where \(S_3(K(t)^*)\) denotes the space of antisymmetric cusp forms having all Atkin-Lehner signs equal to +1. We know \(\dim S_3(K(t)^*) = 0\) for \(t \leq 40\), see [2].

For weight three, the first antisymmetric form we know of is in \(S_3(K(122))^+\) but it is an oldform, in the sense of [21], and comes from a newform in \(S_3(K(61))^-\). As such, the oldform has Atkin-Lehner signs of \(-1\) at both 2 and 61. The first nontrivial \(S_3(K(t)^*)\) that we know of is for the prime \(t = 167\). The cases 173, 197, 213, and 285 also have this property, with 213 having +1 Atkin-Lehner signs at 3 and 71, and with 285 having +1 Atkin-Lehner signs at 3, 5, and 19.

**Corollary 8.1.** The moduli space \(K(t)^* \setminus \mathbb{H}_2\) of Kummer surfaces associated to \((1,t)\)-polarized abelian surfaces has positive geometric genus
if \( t = 167, 173, 197, 213, \) and 285. In particular, \( H^3(K(t)^*, \mathbb{C}) \) is nontrivial for these \( t \).

Contributions to the cohomology \( H^5(\Gamma_0(N), \mathbb{C}) \) studied by Ash, Gunnells and McConnell can also be seen in Table 4 of [1] for the primes \( N = 167, 173 \) and 197.

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