Deviation bound for non-causal machine learning

Rémy Garnier
Raphaël Langhendries

April 2020

1 Introduction

Concentration inequalities have been widely used in machine learning theory. Model selection techniques, for instance, relies heavily on concentration inequality [Massart, 2007]. They have also been used for high dimensional procedures [Bickel et al., 2009, Alquier et al., 2020] or for studying different machine learning framework, such as time series prediction [Kuznetsov and Mohri, 2015], online machine learning [Sanchez-Perez, 2015] or classification problems [Freund et al., 2004]. Many concentration inequalities has been proposed for different framework and different hypothesis. An interested reader may read [Boucheron et al., 2013] for an overview on stationary concentration inequalities. However, independance hypothesis [Dedecker and Fan, 2013] and stationarity hypothesis [Alquier et al., 2019] has been relaxed multiple times. In particular, many ways has been proposed to handle dependant models, such as mixing time (), contraction condition () and spectral conditions ().

Bidirectional machine learning models have been recently an important topic of research. These models use the past and the future of a time series to predict the present [Schuster and Paliwal, 1997]. These have many applications, in particular in term of speech processing and language recognition [Huang et al., 2015]. In particular, many applications have been developed around the BERT model [Kenton and Toutanova, 2019]. Using bidirectional models in the case of speech recognition make some sense, because the distribution of probability for a word in a given place in the text depend not only from previous words, but also from future words.

This imply that using standard concentration inequalities on such models is not an easy task, because of the non-causal interaction in such models. In terms of times series, the present values depends on both the past and the future of the times series . In this paper, we want to propose a framework to analyse such models. More precisely, we want to adapt classical machine learning analysis techniques in the case of non-causal random fields. Non-causality seems a good way to treat several type of model, including bidirectional models.

In the section 2 we recall a presentation of non-causal random fields proposed in [Doukhan and Truquet, 2007]. In the section 5.1 we provide an approximation of the model depending only on a finite number of independent variables. Then, in section 5 we establish concentration and deviation inequalities for non-causal random fields. Finally, we provide some example of application of this inequalities.

2 Model

2.1 Probabilistic Model

Let \((\Omega, \mathcal{F}, \mathbb{P})\) a probability space and \((\mathcal{X}, \mathcal{B}(\mathcal{X}))\) a measurable space.

We will study the random field

\[
(X_t)_{t \in \mathbb{Z}^\kappa} \text{ with } \kappa \in \mathbb{N}
\]

Along all this paper, the following \(m\)-norm \(\|\cdot\|_m\) will be used:

\[
\|X\|_m = (\mathbb{E}\|X\|^m)^{\frac{1}{m}} \text{ with } \|\cdot\| \text{ is norm on } \mathcal{X}
\]

We suppose that the random field \((X_t)_{t \in \mathbb{Z}^\kappa} \text{ with } \kappa \in \mathbb{N}\) is stationary with stationary distribution \(\mu_X\).
2.1.1 Functional equation

We introduce \( \delta = (\delta_1, \ldots, \delta_n) \in \mathbb{N}^n \) and \( B \subset \mathbb{Z}^n \) such that

\[
\mathcal{B} = \mathcal{B}(\delta) = \bigotimes_{i=1}^{n} [-\delta_i, \delta_i]
\]

Thus \( B \) is a \( \kappa \)-orthotope with widths \( 2\delta_i + 1 \)

Let \( (\varepsilon_t)_{t \in \mathbb{Z}^n} \) be a sequence of independent and identically distributed random variables on the measurable space \( (E, \mathcal{B}(E)) \). We denote \( \xi = (\varepsilon_t)_{t \in \mathbb{Z}^n} \) the random field of iid variables \( \varepsilon \).

Additionally, we define \( \mu_\varepsilon \) the distribution of \( \varepsilon_0 \) and \( \mu = \otimes_{t \in \mathbb{Z}^n} \mu_\varepsilon \), the convolution of this distribution on \( E^{\mathbb{Z}^n} \).

It is the distribution of \( \xi = (\varepsilon_t)_{t \in \mathbb{Z}^n} \).

We assume it exists a function \( F \) such that random fields \((X_t)_{t \in \mathbb{Z}^n} \) and \( \xi \) are solution of the equation:

\[
\forall t \in \mathbb{Z}^n, X_t = F((X_{t+s})_{s \in \mathcal{B}}, \varepsilon_t)
\]

we now state the following hypothesis on the \( F \) function:

**Contraction hypothesis** (H\(_1\)): It exists \((\lambda_t)_{t \in \mathcal{B}}\) and \( \eta \in [0, 1] \) such that :

- Noting \( \rho = \sum_{t \in \mathcal{B}} \lambda_t \), they verify:
  \[
  \sum_{t \in \mathcal{B}} \lambda_t + \eta = \rho + \eta < 1
  \]

- For all \((Y_t)_{t \in \mathbb{Z}^n}, (Y'_t)_{t \in \mathbb{Z}^n}\) with distribution \( \mu_X \) and \( \varepsilon_t, \varepsilon'_t \) with distribution \( \mu_\varepsilon \):
  \[
  \forall t \in \mathbb{Z}^n, \|F((Y_{t+s})_{s \in \mathcal{B}}, \varepsilon_t) - F((Y'_{t+s})_{s \in \mathcal{B}}, \varepsilon'_t)\|_m \leq \sum_{s \in \mathcal{B}} [\lambda_t \|Y_{t+s} - Y'_{t+s}\|_m] + \eta \|\varepsilon_t - \varepsilon'_t\|_m
  \]

**Remark 1.** This contraction hypothesis is similar to usual contraction hypothesis in causal learning. For instance, [Dedecker and Fan, 2015] use a similar hypothesis on the random variables \( X_t \), but doesn’t assume the contractivity regarding the innovation. [Alquier et al., 2019] has a much closer hypothesis, because it also assumes the contractivity regarding the innovation.

We need to make some hypothesis on the innovation

**Innovation moment hypothesis** (H\(_2\)) Let \( \varepsilon, \varepsilon' \) independant and \( \mu_\varepsilon \) distributed. For a given \( m \), the quantity \( \|\varepsilon - \varepsilon'\|_m \) only depends on \( \mu_\varepsilon \) and \( m \). It is thereafter denoted \( \mathbb{V}_m \). We make the hypothesis that it exists \( V_\infty \) such that :

\[
\forall m \in \mathbb{N}, \mathbb{V}_m \leq V_\infty
\]

This assumption is immediately satisfied if \( \varepsilon \) is gaussian or subgaussian.

2.1.2 Example

**Non-causal auto-regressive model** The simplest possible non-causal model using this framework is the bidirectional auto-regressive model. In this case \( \kappa = 1 \), and we have \( \alpha_{-1}, \alpha_1, \beta \) such that :

\[
F(X_{t-1}, X_{t+1}, \varepsilon_t) = \alpha_{-1} X_{t-1} + \alpha_1 X_{t+1} + \beta \varepsilon_t
\]

Where the \( \varepsilon_t \) are a Gaussian white noise of variance \( \sigma^2 \). In this setting, the condition (H\(_1\)) is satisfied if \( \alpha_{-1} + \alpha_1 + \beta < 1 \).

**Remark 2.** In this case, it suffices to have \( \alpha_{-1} + \alpha_1 \leq \rho < 1 \). indeed, we can consider a white noise with a different variable \( \varepsilon'_t = \frac{2\beta}{1-\rho} \varepsilon_t \) and apply the results with this noise instead of \( \varepsilon_t \). This does however increases the variance of the noise, and subsequently the moment of the noise.

**Bidirectional RNN** Bi-directional recurrent neural network (BRNN) has been used in Natural Language Processing. Applications are ranging from Translation to part of speech tagging and speech recognition.

Here, we present the formal version of a single-layer bidirectional neural network with a white noise \( \varepsilon_t \):

\[
F(X_t, \varepsilon_t) = f(AX_t) + \beta \varepsilon_t
\]

Where \( A \) is a \( p \times 2k \) matrix, \( f \) is an activation function, and \( X_t \) is the \( 2k \) vector \((X_{t-k}, \ldots, X_{t-1}, X_{t+1}, \ldots, X_{t+k})\).
We suppose that the activation function is 1-Lipschitz. This is the case for most activation functions (sigmoid, tanh, RELU, softmax). We suppose also that there is an operator norm $\|\|_\text{op}$ associated with the norm $\|\|_\text{op}$.

With this condition, the contraction condition $(H_1)$ is verified as soon as:

$$\|A\|_\text{op} + \beta < 1$$

### 2.2 Setting for concentration inequality

#### 2.2.1 Lipschitz separable function and the $S_I$ statistic

Along this paper, we focus on a function $\Phi : \mathcal{X}^\mathcal{B} \mapsto \mathbb{R}$ define on a small neighborhood $\mathcal{B}$.

$$\mathcal{B} = \mathcal{B}(\delta) = \bigotimes_{i=1}^\kappa [-\delta_i, \delta_i].$$

We suppose that we have a Lipschitz separable function $\Phi : \mathcal{X}^\mathcal{B} \mapsto \mathbb{R}$. For a subset $I$ of index, we define a statistic $S_I$:

$$S_I = \sum_{s \in I} \Phi((X_{s+t})_{t \in \mathcal{B}})$$

We want to control the deviation of $S_I$, that is the difference between $S_I$ and $\mathbb{E}[S_I]$.

**Lipschitz separability hypothesis $(H_3)$**: We make the hypothesis that $\Phi$ is Lipschitz separable. That is the following relation holds:

For all variable $(U_t), (V_t)$ with distribution $\mu_X$

$$\|\Phi((U_t)_{t \in \mathcal{B}}) - \Phi((V_t)_{t \in \mathcal{B}})\|_m \leq \sum_{t \in \mathcal{B}} \|U_t - V_t\|_m$$

This hypothesis is close to the condition proposed by [Dedecker and Fan, 2015] in the causal dependant case. However it does not need to holds for every possible values on $\mathcal{X}$, but only in probability.

#### 2.2.2 Learning settings

A typical use of this setting is the prediction problem, where the objective is to predict $X_t$ using its neighbors on $\kappa$ dimensional lattice $(X_t)_{t \in \mathcal{B}}$. In this case, we define a model $\hat{f}$ use for prediction.

$$\hat{x}_s = \hat{f}((x_{s+t})_{t \in \mathcal{B}})$$

We also introduce a cost function $c : \mathcal{X}^2 \mapsto \mathbb{R}$. Then our function $\Phi$ correspond to the cost of $\hat{f}$:

$$\Phi((X_{s+t})_{t \in \mathcal{B}}) = c(\hat{f}((X_{s+t})_{t \in \mathcal{B}}), X_s)$$

We note that $\mathcal{B}$ and $\mathcal{B}$ may be different. Indeed in practise, we may want to try simpler model or even not know at all $\mathcal{B}$. On figure 1 we sum up the learning scheme we use for $s \in I$ the validation set. We use all the $X$ indexed by something in the red ortotope $\mathcal{B} + t$ to predict $X_t$. It also implies that the set of values $X$ we know (the grey set) is bigger than the the set of index $I$ we can use for training (the blue set).

In the prediction problem, we want to control the gap between the theoretical risk of an estimator $\hat{f}$, denoted $R(\hat{f})$:

$$R(\hat{f}) = \mathbb{E}[c(\hat{f}((X_{s+t})_{t \in \mathcal{B}}), X_s)]$$

and its empirical risk:

$$\mathcal{R}_{\text{emp}}(\hat{f}) = \frac{1}{n} \sum_{s \in I} c(\hat{f}((X_{s+t})_{t \in \mathcal{B}}), X_s)$$

(5)
3 Main Result

3.1 Main result

Concentration inequalities has been proposed for a large number of settings. The simplest one, in the i.i.d. case is the McDiarmid inequality, proposed in [McDiarmid, 1989].

**Theorem 1 (iid McDiarmid).** Let $X_1, \ldots, X_n$ be iid random variables in $\mathcal{X}$ and $\Phi : \mathcal{X}^n \mapsto \mathbb{R}$. We suppose it exists $c_1, \ldots, c_n$ such that

$$\forall i \in [1, n], \forall (x_1, \ldots, x_i, x'_i, \ldots, x_n), |\Phi(x_1, \ldots, x_i, \ldots, x_n) - \Phi(x_1, \ldots, x'_i, \ldots, x_n)| \leq c_i$$

(6)

Then:

$$\mathbb{P}(\Phi(X_1, \ldots, X_n) - \mathbb{E}[\Phi] \geq \varepsilon) \leq \exp\left(\frac{-2\varepsilon^2}{\sum_{i=1}^{n} c_i^2}\right)$$

It has been widely used in statistical learning, notably for studying the consistency of different estimator. [Boucheron et al., 2013] gives some example of such application.

In this paper, we want to prove a McDiarmid inequality in the non-causal setting defined in Section 2. In order to do so, we provide an approximation to $\mathcal{S}_T$ depending only on a finite number of independent variables. Other types of concentration inequalities (Bernstein, von Bahr-Esseen) could have been proven in the same context using the same inequalities, but the McDiarmid inequality seems to us the most used inequality in practical applications.

Here is a simplified version of the proposed inequality:

**Theorem 2.** Assume $(H_1), (H_2), (H_3)$. Suppose also that $\mathcal{B} \subset \mathcal{B}$ (for simplicity). There is $A, B, C$ positive constant that do not depend on $n, \rho$ or $\kappa$ such that for $\varepsilon \geq C \text{Card}(\mathcal{B})\rho$ :

$$\mathbb{P}(|\mathcal{S}_T - \mathbb{E}[\mathcal{S}_T]| \geq \varepsilon) \leq 2 \exp\left(\frac{-2(\varepsilon - C \text{Card}(\mathcal{B})\rho)^2}{n \text{Card}(\mathcal{B})^6 \left(A\rho^2 + \frac{B \ln(n)\varepsilon^2}{\max(1, \ln(n)\varepsilon^2)}\right)}\right)$$

(7)

Due to the non-causality and the dependence of our data, we get an extra term in the denominator which increases as $O(n \ln n)$ instead of $O(n)$. This extra term is strongly affected by the dimension $\kappa$ of the problem. However, in most practical cases, $\kappa = 1$ or 2 so it is not really a problem.
Another important factor is the memory of the random field. This factor is sum up in the Card($\mathcal{B}$) term. The more variables appears in the function $F$, the harder it is to estimate this function. For a simple Markov chain, this term would be equal to 1.

The condition $\varepsilon \geq C\text{Card}(\mathcal{B}) \rho$ is not restrictive for applications as we will show in Section TODO.

### 3.2 Comparison with other McDiarmid inequalities

Other variants of the McDiarmid inequality has been proposed in different context. Here we compare our inequality with some find in other context.

The first one we want to compare is the McDiarmid inequality provided in [Dedecker and Fan, 2015]. The inequation proposed by the author apply to a dependant Markov Chain, with contraction condition similar to ($H_1$) for a Lipschitz separable function $\Phi$.

They obtain the following inequality:

**Theorem 3** (dependant McDiarmid, simplified version of 3.6 in [Dedecker and Fan, 2015]). Let $(\varepsilon_i)$ be a sequence of i.i.d. random variables. Let $F$ be a function, such that, there is a $\rho < 1$, such that for all $x, x'$:

$$\mathbb{E}[\|F(x, \varepsilon_1) - F(x', \varepsilon_1)\|] \leq \rho \|x - x'\|$$  \hspace{1cm} (8)

Let $(X_t)_{t \geq 1}$ be a Markov chain defined as $X_t = F(X_{t-1}, \varepsilon_t)$. We suppose that, for $1 \leq i \leq n$ the coupling moment $(M_i)_{1 \leq i \leq n} = \|X_i - X_i'\infty$ are bounded by a constant $M$. In this case, for a Lipschitz separable function $\Phi$ and $\varepsilon$ small enough:

$$\mathbb{P}(\Phi(X_1, \ldots, X_n) - \mathbb{E}[\Phi(X_1, \ldots, X_n)] > \varepsilon) \leq \exp\left(\frac{-2(1-\rho)^2\varepsilon^2}{nM^2}\right)$$  \hspace{1cm} (9)

This theorem uses a similar hypothesis on $g$, but applied it to a much more restrictive causal setting. They therefore get a better McDiarmid bound without the extra term in $O(\ln n)$ at the denominator.

We have found very few other settings for which it has been possible to establish concentration inequalities. The most notable is the random field settings described in [Chazottes et al., 2007]. We don’t have the space to describe it detail, so let’s emphasize the main similarities and differences with our own setting.

In this article, the authors consider a random field $(X_t)_{t \in \mathbb{Z}^d}$ indexed by $\mathbb{Z}^d$ taking its values in a finite set $\mathcal{X}$. They define a coupling matrix $D_{\mathcal{X}}$ using an approach described in [Lindvall, 2002] for sequences of random variables, but extend it using an enumeration $\Gamma$ of the set of index $\mathbb{Z}^d$.

Their main hypothesis is that the coupling matrix is bounded, i.e., there exist a matrix $\bar{D}$ with a bounded norm in $L_2(\mathbb{Z}^d)$ such that for all $x, y \in \mathbb{Z}^d$, $D_{x, y} := \sup_{\mathcal{X}} D_{x, y}$. This hypothesis leads to the following theorem.

**Theorem 4** (Theorem 5 of [Chazottes et al., 2007]). Assume that there is a finite $\delta\Phi$ such that, for all:

$$\forall i \in [1, n], \forall (x_1, \ldots, x_n, x'_i), |\Phi(x_1, \ldots, x_i, \ldots, x_n) - \Phi(x_1, \ldots, x'_i, \ldots, x_n)| \leq \delta\Phi$$  \hspace{1cm} (10)

We have the following inequality:

$$\mathbb{P}(\Phi(X_1, \ldots, X_n) - \mathbb{E}[\Phi(X_1, \ldots, X_n)] > \varepsilon) \leq \exp\left(\frac{-2\varepsilon^2}{\delta\Phi^2 \|D\|_{L_2}^2}\right)$$  \hspace{1cm} (11)

There is four main difference:

- Their settings is a finite settings, which can be used for classification, whereas we consider a setting more suited for regression.
- They use a global setting, which seems hard to verify in practice, whereas we only use local information to make our estimation.
- They made a strong difference bound hypothesis (10) on $\Phi$, whereas we make the stronger Lipschitz separability hypothesis ($H_2$).
- The uniform decay of the coupling matrix is far more restrictive than our contraction hypothesis ($H_1$).

1They also propose relax this hypothesis on the coupling matrix to obtain a McDiarmid inequality. To this end, they introduces instead a low temperature Ising hypothesis which seems very specific to a finite model, and therefore hard to compare.
4 Non-causal fixed point

In this section, we prove the existence, study and approximate functions that map the innovation \((\varepsilon_t)_{t \in \mathbb{Z}^n}\) to the non-causal field \((X_t)_{t \in \mathbb{Z}^n}\).

4.1 Existence of the fixed point

We denote \(L^m(\mu)\) be the \(\mu\)-measurable functions.

\[ L^m(\mu) = \{ H : \mathcal{X}^\mathbb{Z}^n \rightarrow \mathcal{X} / \| H(\varepsilon) \|_m \leq \infty \} \]

We note \(\| \cdot \|_{\mu, m}\) the semi-norm in \(L^m(\mu)\) such that \(\| H \|_{\mu, m} = E[\| H(\xi) \|_m] \) with \(\xi\) random variable with \(\mu\) distribution. and \(L^m(\mu)\) the quotient space with respect to the kernel of \(\| \cdot \|_m\).

\[ L^m(\mu) = L^m(\mu)/\{ f \in L^m(\mu) / f = 0 \ \mu \ \text{a.e.} \} \]

Thus \(\| \cdot \|_m\) is a norm for \(L^m(\mu)\) and according to Riesz-Fischer theorem \((L^m(\mu), \| \cdot \|_m)\) is a Banach space.

In [Doukhan and Truquet, 2007], theorem 1 make use of the Picard-Banach theorem to prove the existence of a function \(H\) in \((L^m(\mu), \| \cdot \|_m)\) such that: \(\forall t \in \mathbb{Z}^n, X_t = H((\varepsilon_{t+s})_{s \in \mathbb{Z}^n})\).

In this section, we propose to apply fixed-point theorem in a sub-space of \((L^m(\mu), \| \cdot \|_m)\). In this way, we will obtain a property for the function \(H\).

4.2 Notations

We introduce here some useful notations.

- \(\forall s \in \mathbb{Z}^n, \theta_s : \mathbb{Z}^n \rightarrow \mathbb{Z}^n\) denotes the shift operator. ie: \(\forall s \in \mathbb{Z}^n, \theta_s((\varepsilon_t)_{t \in \mathbb{Z}^n}) = (\varepsilon_{s+t})_{t \in \mathbb{Z}^n}\)
- We also denote \(\forall s \in \mathbb{Z}^n, \xi_s = \theta_s(\xi) = \theta_s((\varepsilon_t)_{t \in \mathbb{Z}^n}) = (\varepsilon_{s+t})_{t \in \mathbb{Z}^n}\)
- We note \(\xi'_s = (\xi'_t)_{t \in \mathbb{Z}^n}\) where \(\xi'_t\) are random variables \(\Omega \rightarrow E\)
- We will use the following norm on random field: \(\| \xi - \xi' \|_m = \sup_{t \in \mathbb{Z}^n} \| \xi_t - \xi'_t \|_m\)

4.3 Sub-space \(H_m(\mu_\varepsilon)\)

Now, we define the sub-space \(H_m(\mu_\varepsilon)\) including the Bernoulli shift \(H\) under consideration.

Definition 1. We denote \(H_m(\mu_\varepsilon) \subset L^m(\mu)\) the set of function \(H\) verifying:

\[ \forall s \in \mathbb{Z}^n, H_{\mu_\varepsilon} = \prod_{\mathbb{Z}^n} \mu_\varepsilon, \]

\[ \| H(\xi) - H(\xi') \|_m \leq \| \xi - \xi' \|_m \] \quad (12)

We want to show the following theorem

Theorem 5. Assume \((H_1)\). There is a unique function \(H \in H_m(\mu)\) such that:

\[ \forall t \in \mathbb{Z}^n, X_t = H((\varepsilon_{t+s})_{s \in \mathbb{Z}^n}) \]

and

\[ \forall t \in \mathbb{Z}^n, X_t = F((X_{t+s})_{s \in \mathbb{B}}, \varepsilon_t) \]

This theorem is a direct application of the Banach-Picard fixed point theorem to the following functional:

\[
\phi \left\{ \begin{array}{c}
H_m(\mu) \\
H
\end{array} \rightarrow \begin{array}{c}
L^m(\mu) \\
F(\pi_0((\varepsilon_t)_{t \in \mathbb{Z}^n}))
\end{array}, \text{where } \pi_0((\varepsilon_t)_{t \in \mathbb{Z}^n}) = \varepsilon_0
\right\}
\]

Our goal is now to apply the fixed point theorem to \(\phi\) in \(H_m(\mu_\varepsilon)\). Let’s check the hypothesis of the fixed point theorem

Lemma 1. Assume \((H_1)\). \(H_m(\mu_\varepsilon), \| \cdot \|_m\) is a sub-space of \((L^m(\mu)\) verifying the following properties
We introduce the following notations:

- \((\mathbb{H}_m(\mu_\varepsilon), \|\cdot\|_m)\) is a complete space
- \((\mathbb{H}_m(\mu_\varepsilon), \|\cdot\|_m)\) is stable by \(\phi\)
- \(\phi\) is a contracting mapping on \((\mathbb{H}_m(\mu_\varepsilon), \|\cdot\|_m)\)

Remark 3. \((H_1)\) is fully requested to prove the stability. In particular, we need \(F\) to be contractive with respect to both \((X_t)_{t \in \mathbb{B}}\) and the innovation \(\varepsilon\) (and not only just \((X_t)_{t \in \mathbb{B}}\))

We postponed the rather technical proof of this lemma to the appendix.

Remark 4. For all \(\mathbb{H}_{m+1}(\mu_\varepsilon) \subset \mathbb{H}_m(\mu_\varepsilon)\), therefore the fixed point doesn't depend on \(m\).

5 Approximation of the statistic \(\tilde{S}_{T}[d]\)

5.1 Approximation of the fixed point

From Lemma 3, we get an expression of each \(X_t\) according to an infinite number of iid random variables (the random field \(\xi\)). In order to stand concentration inequalities, we would like to get \(X_t\) according to a finite number of iid variables.

The purpose of this sequence is to find a finite approximation of the field \(\xi\) and bound the error made using this approximation.

5.1.1 Hint

The idea is to approximate each \(\xi_t = (\varepsilon_{t+s})_{s \in \mathbb{Z}^n}\) by another random field \(\tilde{\xi}\) composed of the same random variable \(\varepsilon_s\) for \(s\) in a finite neighbourhood around \(t\) and fill remaining space with another random variable draw from the law \(\mu_\varepsilon\).

Doing this way, each \(\tilde{\xi}_s\) depends on a finite number of random variable (those with index in the finite neighbourhood and the random variable used for ”filling”).

5.1.2 Notations

We introduce the following notations:

- \(\forall (d, s) \in \mathbb{N} \times \mathbb{Z}^n, \mathcal{V}(d\delta, s) = \{(t_1, \ldots, t_n) / -d \times \delta_i \leq t_i - s_i \leq d \times \delta_i\}\)
- \(\mathcal{V}(d\delta, s) = \mathcal{B}(d\delta) - s\) is also a \(\kappa\)-orthotope, it is a dilatation by \(d\) of the \(\kappa\)-orthotope \(\mathcal{B}\) centred on \(s\).

- \(\forall d \in \mathbb{N}, \forall s \in \mathbb{Z}^n, \tilde{\xi}_s[d] = (\tilde{\varepsilon}_s[t])_{t \in \mathbb{Z}^n}\) such that \(\forall t \in \mathbb{Z}^n, \tilde{\varepsilon}_s[t] = \begin{cases} \varepsilon_t, & \text{if } t \in \mathcal{V}(d\delta, s), \\ \bar{\varepsilon}^s, & \text{else.} \end{cases}\)

\(\tilde{\xi}_s[d]\) is an approximation of the random field \(\xi_s\) with equal \(\varepsilon_t\) on the finite \(\kappa\)-orthotope \(\mathcal{V}(d\delta, s)\). Outside of \(\mathcal{V}(d\delta, s)\), \(\varepsilon_t = \bar{\varepsilon}^s\) which is a random variable with law \(\mu_\varepsilon\) independent from \(\xi_s\).

for each \(s \in \mathbb{Z}^n\), we build a different approximation \(\tilde{\xi}[d]_s\). Moreover, we choose any ”filling” \(\bar{\varepsilon}^s\) pairwise independent.

5.1.3 Approximation Lemma

Furthermore, for \(s \in \mathbb{Z}^n\) and \(d \in \mathbb{N}\), \(\tilde{\xi}_s[d]\) is still drawn from the distribution \(\mu_\varepsilon\). Thus the we are in position to use the property \(\mathbb{P}_2\) of \(H\).

Lemma 2. Let \(\gamma, \gamma'\) independent and \(\mu_\varepsilon\) distributed. Assume \((H_1)\), then, for \(t \in \mathbb{Z}^n\) and \(d \in \mathbb{N}\):

\[\|H(\xi_t) - H(\tilde{\xi}_t[d])\|_m \leq \|\gamma - \gamma'\|_m.\]
Proof. Indeed, applying (12), we get:

$$\forall t \in \mathbb{Z}^c, \forall d \in \mathbb{N}, \|H(\xi_t) - H(\tilde{\xi}_t[d])\|_m \leq \|\xi_t - \tilde{\xi}_t[d]\|_m$$

Since all $\varepsilon$ in $\xi_t$ and $\tilde{\varepsilon}_t[d]$ have the same distribution $\mu_{\varepsilon}$,

$$\|\xi_t - \tilde{\xi}_t[d]\|_m = \sup_{s \in \mathbb{Z}^c} \|\varepsilon_s - \tilde{\varepsilon}_s[d]\|_m = \|\gamma - \gamma'\|_m$$

with $\gamma$ and $\gamma'$ independent random variables with distribution $\mu_{\varepsilon}$.

Lemma 2 gives a first bound for the error due to the approximation $\tilde{\varepsilon}_t[d]$. However, using the fact that $H$ is a fixed point for $\phi$ and the hypothesis $(H_1)$, we improve the bound of the lemma 2 with an exponential decay w.r.t. $d$.

**Lemma 3.** Assume that $(H_1)$ holds, then:

$$\forall t \in \mathbb{Z}^c, \forall d \in \mathbb{N}, \|H(\xi_t) - H(\tilde{\xi}_t[d])\|_m \leq \rho^{d+1} \times \forall_m$$

(13)

The rather complex proof is postponed in the appendix.

5.1.4 Approximation of $S_I$ : $\tilde{S}_I[d]$ statistic

Earlier, we define the statistic $S_I$, which depend on a the random variables $(X_t)_{t \in I}$, therefore of an infinite number of innovation $(\varepsilon_t)$. We approximate this statistic using the $\tilde{S}_I[d]$ statistic, using approximation fields $\tilde{\varepsilon}_t[d]$.

$$S_I = \sum_{s \in I} \Phi((H(\xi_{t+s}))_{t \in B})$$

(14)

$$\tilde{S}_I[d] = \sum_{s \in I} \Phi((H(\tilde{\xi}_{t+s}))_{t \in B})$$

(15)

Applying approximation lemma, we can bound the difference between $S_I$ and $\tilde{S}_I[d]$.

**Corollary 1** (Moment inequality for the approximation error). Assume $(H_1)$ and $(H_3)$. Denoting $n = \text{Card}(I)$, $n_B = \text{Card}(B)$, we have:

$$\mathbb{E}\left[|S_I - \tilde{S}_I[d]|^m\right] \leq (nn_B\rho^{d+1}\forall_m)^m$$

(16)

Moreover, if $(H_2)$ holds, we have almost surely:

$$|S_I - \tilde{S}_I[d]| \leq nn_B\rho^{d+1}\forall$$

**Proof.** Using (14), (15) and (H3):

$$\|S_I - \tilde{S}_I[d]\|_m \leq \sum_{s \in I} \|\Phi((H(\xi_{t+s}))_{t \in B}) - \Phi((H(\tilde{\xi}_{t+s}))_{t \in B})\|_m$$

$$\leq \sum_{s \in I} \sum_{t \in B} \|H(\xi_{t+s}) - H(\tilde{\xi}_{t+s})\|_m$$

Then, using (13):

$$\mathbb{E}\left[|S_I - \tilde{S}_I[d]|^m\right] \leq (nn_B\rho^{d+1}\forall_m)^m$$

(17)

This proves the equation (16). Next, we use Cramer-Chernoff method. For $u, t > 0$:
\[ \mathbb{P}\left[|S_I - \tilde{S}_I^{[d]}| > t\right] = \mathbb{P}\left[\exp(u|S_I - \tilde{S}_I^{[d]}|) > \exp(ut)\right] \leq \exp(-ut) \mathbb{E}\left[\exp(u|S_I - \tilde{S}_I^{[d]}|)\right] \]

Using (17) and (H2):

\[ \mathbb{E}\left[\exp(u|S_I - \tilde{S}_I^{[d]}|)\right] \leq \sum_{m=0}^{\infty} \frac{u^m}{m!} \mathbb{E}\left[|S_I - \tilde{S}_I^{[d]}|^m\right] \leq \sum_{m=0}^{\infty} \frac{u^m}{m!} (nn_B \rho^{d+1} \nu_m)^m \leq \exp(unn_B \rho^{d+1} \nu_{\infty}) \]

Therefore:

\[ \mathbb{P}\left[|S_I - \tilde{S}_I^{[d]}| > t\right] \leq \exp\left(u(nn_B \rho^{d+1} \nu_{\infty} - t)\right) \]

If \( t \geq nn_B \rho^{d+1} \nu_{\infty} \), then, as \( \lim_{u \to +\infty} \exp\left(u(nn_B \rho^{d+1} \nu_{\infty} - t)\right) = 0 \). Therefore, we have almost surely:

\[ |S_I - \tilde{S}_I^{[d]}| \leq nn_B \rho^{d+1} \nu_{\infty} \]

\[ \Box \]

### 5.2 Deviation inequality for \( \tilde{S}_I^{[d]} \)

In this subsection, we will establish a concentration inequality for \( \tilde{S}_I^{[d]} \). In order to do so, we will apply the McDiarmid inequality to the innovation \( \varepsilon_t \). First, we need to recall and add some notations:

#### 5.2.1 Notations

- \( n = \text{Card}(I) \) the number of variable appearing in \( \psi \).
- \( n_B = \text{Card}(B) \), the number of random variable in the \( \kappa \)-orthotope \( B \)
- \( n_B = \text{Card}(B(\delta)) = \text{Card}(\tilde{B}) \) the number of component of \( \psi \)
- \( n_d = \text{Card}(\mathcal{V}(d\delta, t)) \) the number of random marginal variables \( \varepsilon_t \) of \( \xi \) used to approximate \( X_t \).
- \( N_1 = \text{Card}(\bigcup_{I \subseteq I} \mathcal{V}(\delta, t)) \)
- \( N_2 = \text{Card}(\bigcup_{I \subseteq I} \mathcal{V}(d\delta, t)) \) number of random variables \( \varepsilon_t \) occurring in (15).

There are some relations between these numbers.

- \( n \leq N_1 \leq nn_B \). This higher bound is reached when the above union involves pairwise disjoint set. However, this situation is a pathological case. For training purpose, training set tends to be connexe.
- \( N_1 \leq N_2 \leq N_1 n_d \), but we can make the same remark.

Besides the component \( \varepsilon \) of \( \xi \), "filling" variables \( \tilde{\varepsilon}^a \) also occur in the approximation \( \tilde{S}_I^{[d]} \).

#### 5.2.2 McDiarmid theorem for \( S_I \)

In this section, we will show the following theorem:
Theorem 6 (Probabilistic Deviation Bound). Assume (H_1), (H_2) and (H_3). We define the following function: With:

\[
\Upsilon_{κ,ρ}(d) = \begin{cases} 
(κ - 1)! \ln(ρ^{-1})^κ \left(1 - ρ^{d+1} \sum_{i=0}^{κ-1} \frac{(d + 1) \ln(ρ^{-1})^i}{i!}\right), & \text{if } d < \left\lfloor \frac{κ-1}{\ln(ρ^{-1})} \right\rfloor \\
(κ - 1)! \ln(ρ^{-1})^κ \left(1 - ρ^{d} \sum_{i=0}^{κ-1} \frac{(d \ln(ρ^{-1}))^i}{i!}\right) + \left(\frac{κ - 1}{\ln(ρ^{-1})}\right)^{κ-1}, & \text{if } d = \left\lfloor \frac{κ-1}{\ln(ρ^{-1})} \right\rfloor \\
(κ - 1)! \ln(ρ^{-1})^κ \left(1 - ρ^{d} \sum_{i=0}^{κ-1} \frac{(d \ln(ρ^{-1}))^i}{i!}\right) + \left(\frac{κ - 1}{\ln(ρ^{-1})}\right)^{κ-1}, & \text{if } d > \left\lfloor \frac{κ-1}{\ln(ρ^{-1})} \right\rfloor 
\end{cases}
\]

Then, the following function holds:

\[
P\left(\left|\tilde{S}_{Z}[d] - \mathbb{E}[\tilde{S}_{Z}[d]]\right| \geq ε\right) \leq 2 \exp \left(-\frac{2ε^2}{(n_B N)^2 (N_1 (n_B ρ^{d+1})^2 + N_2 (n_B κ \Upsilon_{κ,ρ}(d))^2)}\right)
\]

(18)

This theorem is an immediate application of McDiarmid inequality (lemma [11]) to \(\tilde{S}_Z[d]\). In order to apply the lemma [11] we need to verify the strong difference bound \([11]\). Usually, it requires introduce a new random field \(\varepsilon'\) almost identical to \(\varepsilon\) to the previous random field \(\varepsilon\). In our case, there is a main differences with the usual application of McDiarmid inequality.

First, we cannot consider our random field \(ξ\), because it depends on a infinite number of variables. Instead, we consider \(\tilde{ξ}^{[d]}\) the approximation of our random field, which only depend on a finite number of marginal innovation \(ξ_t\).

There are two types of random variables that we use in the McDiarmid theorem:

- The marginal variables \(ξ_t\) of \(ξ\) used in \(\tilde{ξ}^{[d]}\)
- The "filling" variables \(\varepsilon_s\).

Let’s formalize this ideas:

Definition 2. Let the function \(ψ\) be the enumeration of all different \(ξ\)

\[
ψ: Z^n × Z^n × N → N
\]

\[
(\sigma, t, d) \mapsto i
\]

Using this enumeration, we distinguish the set of indexes \(R\) corresponding to the marginal variables \(ξ_t\) of \(ξ\):

\[
ψ^{-1}(R) = \{(σ, t, d), t ∈ V(\tilde{d}, s)\}
\]

It holds:

\[
ψ(σ, t, d) ∈ R ⇔ \tilde{ξ}_s = ξ_t ⇔ t ∈ V(\tilde{d}, s) ⇔ \tilde{ξ}_s^{[d]} = ξ_t
\]

\[
ψ(σ, t, d) /∈ R ⇔ \tilde{ξ}_s^{[d]} = ξ_s
\]

Let’s define a new random field \(\tilde{ξ}_s^{[d,i]}\), where we have changed the random variable corresponding to the index \(i = ψ(σ, t, d)\).

Definition 3.

\[
\tilde{ξ}_s^{[d,i]} = (\tilde{ξ}_s^{[d,i]})_{t ∈ Z^n}
\]

Where

\[
\tilde{ξ}_s^{[d,i]} = \begin{cases} 
\tilde{ξ}_s^{[d]} & \text{if } ψ(σ, t, d) \neq i \\
ξ_s & \text{else}
\end{cases}
\]

\(\tilde{ξ}_s^{[d,i]}\) is the random field where the epsilon \(i\) is replaced by an independent copy draw from the law \(μ_ε\).
Definition 4. Now, we can define $\tilde{S}_T^{[d,i]}$ the statistic $S_T^{[d]}$ where one epsilon is changed

$$\tilde{S}_T^{[d,i]} = \sum_{s \in \mathcal{I}} \Phi \left( (H(\tilde{\xi}_{s+i}))_{t \in B} \right)$$

$$= \sum_{s \in \mathcal{I}} \Phi \left( (H(\tilde{\xi}_{s+i}))_{t \in B+s} \right)$$

with $\forall s \in \mathbb{Z}^c, B + s = \{ r \in \mathbb{Z}^c / r = s + t \text{ and } t \in B \}$ the neighbourhood $B$ centred on $s$.

To use McDiarmid inequality, we need to bound the difference between $\tilde{S}_T^{[d,i]}$ and $\tilde{S}_T^{[d]}$.

Let first stand some intermediate lemma.

Lemma 4. Assume $(H_1)$, Let $d \in \mathbb{N}, i \in \mathbb{Z}^c$. We suppose that $i \notin \mathcal{R}$. Then there is a unique $r \in \mathbb{Z}^c$, such for all $t \in \mathbb{Z}^c \setminus \mathcal{V}(d\delta, r)$, there $\psi(r, t, d) = i$ is It holds:

- If $r \neq s$, $\|H(\tilde{\xi}_{s+i}) - H(\xi_{s+i})\|_m = 0$
- Else $r = s$, $\|H(\tilde{\xi}_{s+i}) - H(\xi_{s+i})\|_m \leq \rho^{d+1} \forall m$

Proof. If $r \neq s$ then $\tilde{\xi}_{s+i} = \xi_{s+i}$

- We get this result using the same demonstration as lemma (??)

Lemma 5. Assume $(H_1)$, Let $d \in \mathbb{N}, i \in \mathbb{Z}^c$. We suppose that $i \in \mathcal{R}$. It holds:

- If $i \notin \mathcal{V}(d\delta, s)$ then $\|H(\tilde{\xi}_{s+i}) - H(\xi_{s+i})\|_m = 0$
- Else $i \in \mathcal{V}(d\delta, s)$.

As $\forall c \in [0, d - 1], \mathcal{V}(c\delta, s) \subset \mathcal{V}((c+1)\delta, s)$ then $\exists! c \in [0, d], i \in \mathcal{V}(c\delta, s)$ and $i \notin \mathcal{V}((c-1)\delta, s)$. It holds:

$$\|H(\tilde{\xi}_{s+i}) - H(\xi_{s+i})\|_m \leq \rho^c \forall m$$

(19)

Proof. To demonstrate this lemma, we apply several times $\mathcal{B}$ in a similar way as $\mathcal{B}$

- If $i \notin \mathcal{V}(d\delta, s)$, then $\tilde{\xi}_{s+i} = \xi_{s+i}$
- Else $\exists! c \in [0, d] / i \in \mathcal{V}(c\delta, s)$ and $i \notin \mathcal{V}((c-1)\delta, s)$

$$\|H(\tilde{\xi}_{s+i}) - H(\xi_{s+i})\|_m \leq \sum_{t \in B} \lambda_t \|H(\tilde{\xi}_{s+i+1})_{t \in B} - H(\tilde{\xi}_{s+i+1})_{t \in B}\|_m$$

If $i = s$, applying $\mathcal{B}$ we get: $\|H(\tilde{\xi}_{s+i}) - H(\xi_{s+i})\|_m \leq \eta \|\tilde{\varepsilon}_{s+i} - \varepsilon_{s+i}\|_m \leq \eta \forall m \leq \mathcal{V}_m$

If $i \neq s$, we can still apply $\mathcal{B}$:

$$\|H(\tilde{\xi}_{s+i}) - H(\xi_{s+i})\|_m \leq \lambda_t \sum_{t \in B} \lambda_t \|H(\tilde{\xi}_{s+i+1})_{t \in B} - H(\tilde{\xi}_{s+i+1})_{t \in B}\|_m$$

We notice that $\forall (t_1, \ldots, t_i) \in \mathcal{B}$, $\left( s + \sum_{i=1}^{t_i} \right) \in \mathcal{V}(l\delta, s)$. Then we can apply $\mathcal{B}$ c times. Therefore we get:

$$\|H(\tilde{\xi}_{s+i}) - H(\xi_{s+i})\|_m \leq \sum_{t \in B} \lambda_t \sum_{t_2 \in B} \lambda_{t_2} \cdots \sum_{t_{i-1} \in B} \lambda_{t_{i-1}} \|H(\tilde{\xi}_{s+i+\sum_{j=1}^{t_j}})_{(t_1, \ldots, t_i)} - H(\xi_{s+i+\sum_{j=1}^{t_j}})_{(t_1, \ldots, t_i)}\|_m$$

$$\leq \mathcal{V}_m \sum_{t \in B} \cdots \sum_{t \in B} \prod_{j=1}^{c} \lambda_{t_j} = \mathcal{V}_m \rho^c$$

Lemma 6. Assume $(H_1)$ and $(H_3)$. It holds:
• If $i \notin \mathcal{R}$, 
  \[ \mathbb{E} \left[ \left| \tilde{S}_I^d | S_{\tilde{I}} \right|^2 \right] \leq \left( n_B \delta^{2d+1} \nu \right)^m \]

• Else $i \in \mathcal{R}$, 
  \[ \mathbb{E} \left[ \left| \tilde{S}_I^d - \tilde{S}_{\tilde{I}}^d \right|^2 \right] \leq \left( n_B \delta^{2d+1} \nu \sum_{c=1}^{d} \epsilon_{r} \rho^c \right)^m \]

Proof. We first suppose that $i \notin \mathcal{R}$

• Either $s \notin \mathcal{B} + r$:
  \[ \Phi \left( (H(\xi_t^{d,i}))_{t \in \mathcal{B} + r} \right) = \Phi \left( (H(\xi_t^{d,i}))_{t \in \mathcal{B} + r} \right) \]

• Either $s \in \mathcal{B} + r$, so using (12) and hypothesis (H_4) on $\Phi$:
  \[ \| \Phi \left( (H(\xi_t^{d,i}))_{t \in \mathcal{B} + r} \right) - \Phi \left( (H(\xi_t^{d,i}))_{t \in \mathcal{B} + r} \right) \| \leq \sum_{t \in \mathcal{B} + r} \| H(\xi_t^{d,i}) - H(\xi_t^{d,i}) \| \leq n_f \rho^{d+1} \nu \]

Since $s \in \mathcal{B} + r \iff r \in \mathcal{B} + s$, there is at most $n_f$ different $r$ such that:

\[ \Phi \left( (H(\xi_t^{d,i}))_{t \in \mathcal{B} + r} \right) \neq \Phi \left( (H(\xi_t^{d,i}))_{t \in \mathcal{B} + r} \right) \]

Consequently:

\[ \mathbb{E} \left[ \left| \tilde{S}_I^d - \tilde{S}_{\tilde{I}}^d \right|^2 \right] \leq \sum_{r \in \mathcal{I}} \| H(\xi_t^{d,i}) - H(\xi_t^{d,i}) \| \leq n_f^{\delta+1} \nu \]

We now suppose that $i \in \mathcal{R}$

In this case, there is a unique $s \in \mathcal{Z}^n$ such that $\varepsilon_s \in \xi$ has been replaced by $\varepsilon'_s$ in $\tilde{S}_I^{d,i}$

And for all $r \in \mathcal{Z}^n$ we have the following properties:

\[ \tilde{\xi}_r^{d,i} \neq \tilde{\xi}_r^{d,i} \iff s \in \mathcal{V}(d\delta, r) \iff r \in \mathcal{V}(d\delta, s) \]

\[ \mathbb{E} \left[ \left| \tilde{S}_I^d - \tilde{S}_{\tilde{I}}^d \right|^2 \right] \leq \sum_{r \in \mathcal{I}} \| H(\xi_t^{d,i}) - H(\xi_t^{d,i}) \| \leq n_f^{\delta+1} \nu \]

Only the field $\tilde{\xi}_r^{d,i}$ with $r \in \mathcal{V}(d\delta, s)$ are impacted by the replacement of the $\varepsilon$ number $i$. Thus, at worst, each $r \in \mathcal{V}(d\delta, s)$ appears $n_B$ times in the sum (with $n_B = \text{Card}(\mathcal{B}) + 1$). Therefore, we get:

\[ \mathbb{E} \left[ \left| \tilde{S}_I^d - \tilde{S}_{\tilde{I}}^d \right|^2 \right] \leq n_B \| H(\xi_t^{d,i}) - H(\xi_t^{d,i}) \| \]

However:

\[ \mathcal{V}(d\delta, s) = \bigcup_{c=1}^{d} \mathcal{V}(c\delta, s) \setminus \mathcal{V}(c-1, s) \text{ with } \mathcal{V}(0, s) = s \text{ and } \forall r_1 \neq r_2, (\mathcal{V}(r_1\delta, s) \setminus \mathcal{V}((r_1 - 1)\delta, s)) \cap (\mathcal{V}(r_2\delta, s) \setminus \mathcal{V}((r_2 - 1)\delta, s)) = \emptyset \]
Lemma 7. If

\[ \| \tilde{S}^{[d,i]}_{x} - \tilde{S}^{[d]}_{x} \|_{m} \leq n_{\mathcal{B}} \sum_{c=1}^{d} \left( \sum_{r \in \mathcal{V}(c\delta,s) \setminus \mathcal{V}((c-1)\delta,s)} \| H(\xi^{[d,i]}_{r}) - H(\xi^{[d]}_{r}) \|_{m} \right) \]

Applying inequality (19). It holds:

\[ \| \tilde{S}^{[d,i]}_{x} - \tilde{S}^{[d]}_{x} \|_{m} \leq n_{\mathcal{B}} \sum_{c=1}^{d} \text{Card}(r \in \mathcal{V}(c\delta,s) \setminus \mathcal{V}((c-1)\delta,s)) \times \rho^{\epsilon} \times \mathcal{V}_{m} \]

\( \mathcal{V}(c,s) \) is the \( \kappa \)-orthotope \( \mathcal{V}(c,s) = \{(t_{1}, \ldots, t_{n})/ -c \times \delta_{1} \leq t_{i} \leq c \times \delta_{1}\} \). Then:

\[ \text{Card}(\mathcal{V}(c,s)) = \prod_{i=1}^{\kappa} (2c\delta_{i} + 1) \]

Consequently:

\[ \text{Card}(r \in \mathcal{V}(c\delta,s) \setminus \mathcal{V}((c-1)\delta,s)) = \text{Card}(\mathcal{V}(c\delta,s)) - \text{Card}(\mathcal{V}((c-1)\delta,s)) = \prod_{i=1}^{\kappa} (2c\delta_{i} + 1) - \prod_{i=1}^{\kappa} (2(c-1)\delta_{i} + 1) \]

\[ \leq c^{\kappa} \prod_{i=1}^{\kappa} \left( 2\delta_{i} + \frac{1}{c} \right) - (c-1)^{\kappa} \prod_{i=1}^{\kappa} \left( 2\delta_{i} + \frac{1}{c-1} \right) \]

\[ \leq \prod_{i=1}^{\kappa} \left( 2\delta_{i} + \frac{1}{c-1} \right) (c^{\kappa} - (c-1)^{\kappa}) \]

\[ \leq \prod_{i=1}^{\kappa} (2\delta_{i} + 1) (c^{\kappa} - (c-1)^{\kappa}) \]

\[ \leq n_{\mathcal{B}} (c^{\kappa} - (c-1)^{\kappa}) \leq n_{\mathcal{B}} \kappa c^{\kappa-1} \text{ with } n_{\mathcal{B}} = \text{Card}(\mathcal{B}) \]

Thus, we finally get:

\[ \| \tilde{S}^{[d,i]}_{x} - \tilde{S}^{[d]}_{x} \|_{m} \leq n_{\mathcal{B}} \sum_{c=1}^{d} n_{\mathcal{B}} \kappa c^{\kappa-1} \rho^{\epsilon} \mathcal{V}_{m} \]

(22)

We are looking for a simpler bound for (22). We propose the following one:

**Lemma 7.** If \( i \in \mathcal{R} \):

\[ \mathbb{E} \left[ \| \tilde{S}^{[d,i]}_{x} - \tilde{S}^{[d]}_{x} \|_{m} \right] \leq (n_{\mathcal{B}} n_{\mathcal{B}} \kappa \mathcal{V}_{m} \Upsilon_{\kappa,\rho}(d))^{m} \]

(23)

With:

\[ \Upsilon_{\kappa,\rho}(d) = \begin{cases} 
\frac{(\kappa - 1)!}{\ln(\rho^{-1})^{\kappa}} \left( 1 - \rho^{d+1} \sum_{i=0}^{\kappa-1} \frac{(d+1)\ln(\rho^{-1})^{i}}{i!} \right) & \text{if } d < \left[ \frac{\kappa-1}{\ln(\rho^{-1})} \right] \\
\frac{(\kappa - 1)!}{\ln(\rho^{-1})^{\kappa}} \left( 1 - \rho^{d+1} \sum_{i=0}^{\kappa-1} \frac{\ln(\rho^{-1})^{i}}{i!} \right) & \text{if } d = \left[ \frac{\kappa-1}{\ln(\rho^{-1})} \right] \\
\frac{(\kappa - 1)!}{\ln(\rho^{-1})^{\kappa}} \left( 1 - \rho^{d} \sum_{i=0}^{\kappa-1} \frac{(d\ln(\rho^{-1})^{i})}{i!} \right) + \left( \frac{\kappa - 1}{\ln(\rho^{-1})} \right)^{\kappa-1} & \text{if } d > \left[ \frac{\kappa-1}{\ln(\rho^{-1})} \right] 
\end{cases} \]

And

\[ \forall d \in \mathbb{N}, \Upsilon_{\kappa,\rho}(d) \leq \nu_{\kappa,\rho} = \frac{(\kappa - 1)!}{\ln(\rho^{-1})^{\kappa}} + \left( \frac{\kappa - 1}{\ln(\rho^{-1})} \right)^{\kappa-1} \text{ where } \nu_{\kappa,\rho} \text{ is a constant independent from } d \]

(24)

The proof of this lemma is postponed in the appendix. Let now apply Cramer-Chernoff method to bound \( |\tilde{S}^{[d,i]}_{x} - \tilde{S}^{[d]}_{x} | \).
Lemma 8. Assume (H1), (H2) and (H3). It holds:

- If $i \notin R$, we have almost surely:
  \[ |\tilde{S}^{[d,i]} - \bar{S}^{[d]} | \leq n_B^2 \rho^{d+1} \nu_\infty \]

- Else $i \in R$, we have almost surely
  \[ |\tilde{S}^{[d,i]} - \bar{S}^{[d]} | \leq n_B n_B \kappa' \nu_\infty \Upsilon_{\kappa,\rho}(d) \]

Proof. We suppose that $i \notin R$ (point 1). The proof for point 2 is similar.
Using Cramer-Chernoff methods, we get:
\[
\forall \lambda \geq 0, \Pr \left( |\tilde{S}^{[d,i]} - \bar{S}^{[d]} | \geq \varepsilon \right) \leq \exp(-\lambda \varepsilon) \mathbb{E} \left[ \exp \left( \lambda |\tilde{S}^{[d,i]} - \bar{S}^{[d]} | \right) \right]
\]
\[
\mathbb{E} \left[ \exp \left( \lambda |\tilde{S}^{[d,i]} - \bar{S}^{[d]} | \right) \right] \text{ corresponds to the moment generating function, therefore we get:}
\]
\[
\mathbb{E} \left[ \exp \left( \lambda |\tilde{S}^{[d,i]} - \bar{S}^{[d]} | \right) \right] = \sum_{i \in \mathbb{N}} \mathbb{E} \left[ |\tilde{S}^{[d,i]} - \bar{S}^{[d]} | \right] \frac{\lambda^i}{i!}
\]

Lemma 8 hence:
\[
\mathbb{E} \left[ \exp \left( \lambda |\tilde{S}^{[d,i]} - \bar{S}^{[d]} | \right) \right] = \sum_{i \in \mathbb{N}} (n_B^2 \rho^{d+1} \nu_i)^i \frac{\lambda^i}{i!}
\]
\[
= \exp \left( \lambda n_B^2 \rho^{d+1} \nu_\infty \right) \text{ with } \nu_\infty = \max_{i \in \mathbb{N}} \nu_i
\]
Thus, for $\lambda \geq 0$:
\[
\Pr \left( |\tilde{S}^{[d,i]} - \bar{S}^{[d]} | \geq \varepsilon \right) \leq \exp \left( -\lambda (\varepsilon - n_B^2 \rho^{d+1} \nu_\infty) \right)
\]
Specifically, if $\varepsilon > n_B^2 \rho^{d+1} \nu_\infty$, we can choose $\lambda \to +\infty$ implying $\exp \left( -\lambda (\varepsilon - n_B^2 \rho^{d+1} \nu_\infty) \right) \to 0$
Consequently, if $\varepsilon > n_B^2 \rho^{d+1} \nu_\infty$:
\[
\Pr \left( |\tilde{S}^{[d,i]} - \bar{S}^{[d]} | \geq \varepsilon \right) = 0
\]

We can now stand McDiarmid inequality for $\tilde{S}^{[d]}$ and prove Theorem 6.
Proof. Using McDiarmid inequality and Lemma 8
\[
\Pr \left( |\tilde{S}^{[d]} - \mathbb{E} \left[ \tilde{S}^{[d]} \right] | \geq \varepsilon \right) \leq 2 \exp \left( -\frac{\varepsilon^2}{n_B^2 \rho^{d+1} \nu_\infty^2 + n_B n_B \kappa' \nu_\infty \Upsilon_{\kappa,\rho}(d)^2} \right)
\]
\[
\leq 2 \exp \left( -\frac{2 \varepsilon^2}{n_B \nu_\infty^2 (n_B \rho^{d+1})^2 + n_B n_B \kappa' \nu_\infty \Upsilon_{\kappa,\rho}(d)^2} \right)
\]

6 Deviation inequality for $S_I$

6.1 Concentration
In this section, we will establish a deviation inequality for $S_I$.

Theorem 7. Let $d \in \mathbb{N}$. Assume (H1), (H2), (H3) and $\varepsilon \geq 2nn_B \rho^{d+1} \nu_\infty$
\[
\Pr[|S_I - \mathbb{E}[S_I]| \geq \varepsilon] \leq 2 \exp \left( \frac{-2(\varepsilon - 2nn_B \rho^{d+1} \nu_\infty)^2}{(n_B \nu_\infty)^2 (n_B \rho^{d+1})^2 + n_B n_B \kappa' \nu_\infty \Upsilon_{\kappa,\rho}(d)^2} \right)
\]
Proof. Using the corollary 1:

\[ |S_I - E[S_I]| \leq |\hat{S}_I^{[d]} - E[\hat{S}_I^{[d]}]| + 2nn_B\rho^{d+1}V_\infty \]

Then applying previous theorem 6 with \( \varepsilon' = \varepsilon - 2nn_B\rho^{d+1}V_\infty \) yields the result

In the previous theorem, \( N_2 \) is the number of random variables \( \varepsilon \) occurring in the statistic \( \hat{S}_I^{[d]} \). \( N_2 \) highly depends on the geometry of the validation set indexed by \( I \). That’s why we bound it by \( N_1n_d \) even if this bound correspond to a worst case situation (when all \( V(d\delta, t) \) are disjoint sets).

Moreover, in statistical learning, the statistic \( \hat{S}_I^{[d]} \) is an empirical mean. In the corollary below we make explicit the \( \frac{1}{n} \) that appears.

**Corollary 2.** Let \( d \in \mathbb{N} \). Assume \((H_1),(H_2),(H_3)\) and \( \varepsilon \geq \frac{2\rho^dV_\infty}{n} \)

\[
\mathbb{P}\left[ \left| \frac{S_I - E[S_I]}{n} \right| \geq \frac{\varepsilon}{n} \right] \leq 2\exp\left( \frac{-2n^2(\varepsilon - 2n_B\rho^{d+1}V_\infty)^2}{(n_BV_\infty)^2N_1 \left( (n_B\rho^{d+1})^2 + d^k n_B^3 (\kappa\Upsilon_{\kappa,\rho}(d))^2 \right) } \right) 
\]

**Proof.** We use \( N_2 \leq N_1n_d \). Then:

\[
n_d = \text{Card}(V(d\delta, t)) = \prod_{i=1}^{\kappa} (2d\delta_i + 1) \leq d^\kappa \prod_{i=1}^{\kappa} \left( 2\delta_i + \frac{1}{d} \right) \leq d^\kappa n_B
\]

Moreover, using the corollary 1, we have:

\[
\frac{|S_I - E[S_I]|}{n} \leq |\hat{S}_I^{[d]} - E[\hat{S}_I^{[d]}]| + 2nn_B\rho^{d+1}V_\infty
\]

Therefore:

\[
\mathbb{P}\left[ \left| \frac{S_I - E[S_I]}{n} \right| \geq \frac{\varepsilon}{n} \right] \leq 2\exp\left( \frac{-2n^2(\varepsilon - 2n_B\rho^{d+1}V_\infty)^2}{(n_BV_\infty)^2N_1 \left( (n_B\rho^{d+1})^2 + d^k n_B^3 (\kappa\Upsilon_{\kappa,\rho}(d))^2 \right) } \right) \]

\[
\iff \mathbb{P}\left[ \left| \frac{S_I - E[S_I]}{n} \right| \geq \varepsilon \right] \leq 2\exp\left( \frac{-2n^2(\varepsilon - 2n_B\rho^{d+1}V_\infty)^2}{(n_BV_\infty)^2N_1 \left( (n_B\rho^{d+1})^2 + d^k n_B^3 (\kappa\Upsilon_{\kappa,\rho}(d))^2 \right) } \right) \]

\( d \) is a parameter that can be choose. We suggest to set \( d = \ln(n)^{\frac{1}{d'}} \), it leads to the following corollary.

**Corollary 3.** Assume \((H_1),(H_2),(H_3)\) and \( \varepsilon \geq \frac{2\rho^dV_\infty}{n} \)

\[
\mathbb{P}\left[ \left| \frac{S_I - E[S_I]}{n} \right| \geq \varepsilon \right] \leq 2\exp\left( \frac{-2n^2(\varepsilon - 2n_B\rho^{d+1}V_\infty)^2}{(n_BV_\infty)^2N_1 \left( (n_B\rho)^2 + \ln(n)n_B^3 (\kappa\Upsilon_{\kappa,\rho}(\ln(n)^{\frac{1}{d'}}))^2 \right) } \right) 
\]

**Proof.** Setting \( d = \ln(n)^{\frac{1}{d'}} \), we get:

\( d^k = \ln(n) \) and \( \rho^d = \exp\left( \ln(\rho)\ln(n)^{\frac{1}{d'}} \right) \leq 1 \) (because \( \rho < 1 \))

**Remark 5.** In the previous corollary, quantities \( n_B, n_\kappa, V_\infty, \kappa \) and \( \rho \) are constants and the function \( \Upsilon_{\kappa,\rho}(\ln(n)^{\frac{1}{d'}}) \) can be bounded by \( \nu_{\kappa,\rho} \).

\( N_1 \) is the number of random variables \( X_i \) in the validation set. Thus \( N_1 \geq n \) and in the worst case \( N_1 = n_Bn \).
References

[Alquier et al., 2020] Alquier, P., Bertin, K., Doukhan, P., and Garnier, R. (2020). High-dimensional var with low-rank transition. *Statistics and Computing*, pages 1–15.

[Alquier et al., 2019] Alquier, P., Doukhan, P., and Fan, X. (2019). Exponential inequalities for nonstationary markov chains. *Dependence Modeling*, 7(1):150–168.

[Bickel et al., 2009] Bickel, P. J., Ritov, Y., Tsybakov, A. B., et al. (2009). Simultaneous analysis of lasso and dantzig selector. *The Annals of Statistics*, 37(4):1705–1732.

[Boucheron et al., 2013] Boucheron, S., Lugosi, G., and Massart, P. (2013). *Concentration inequalities: A nonasymptotic theory of independence*. Oxford university press.

[Chazottes et al., 2007] Chazottes, J.-R., Collet, P., Külske, C., and Redig, F. (2007). Concentration inequalities for random fields via coupling. *Probability Theory and Related Fields*, 137(1-2):201–225.

[Dedecker and Fan, 2015] Dedecker, J. and Fan, X. (2015). Deviation inequalities for separately Lipschitz functionalsof iterated random functions. *Stochastic Processes and their Applications*, 125(1):60–90.

[Doukhan and Truquet, 2007] Doukhan, P. and Truquet, L. (2007). A fixed point approach to model random fields. *Alea*, 3:111–132.

[Freund et al., 2004] Freund, Y., Mansour, Y., Schapire, R. E., et al. (2004). Generalization bounds for averaged classifiers. *The annals of statistics*, 32(4):1698–1722.

[Huang et al., 2015] Huang, Z., Xu, W., and Yu, K. (2015). Bidirectional lstm-crf models for sequence tagging. *arXiv preprint arXiv:1508.01991*.

[Kenton and Toutanova, 2019] Kenton, J. D. M.-W. C. and Toutanova, L. K. (2019). Bert: Pre-training of deep bidirectional transformers for language understanding. In *Proceedings of NAACL-HLT*, pages 4171–4186.

[Kuznetsov and Mohri, 2015] Kuznetsov, V. and Mohri, M. (2015). Learning theory and algorithms for forecasting non-stationary time series. In *Advances in neural information processing systems*, pages 541–549.

[Lindvall, 2002] Lindvall, T. (2002). *Lectures on the coupling method*. Courier Corporation.

[Massart, 2007] Massart, P. (2007). *Concentration inequalities and model selection*, volume 6. Springer.

[McDiarmid, 1989] McDiarmid, C. (1989). *On the method of bounded differences*, page 148–188. London Mathematical Society Lecture Note Series. Cambridge University Press.

[Sanchez-Perez, 2015] Sanchez-Perez, A. (2015). Time series prediction via aggregation: an oracle bound including numerical cost. In *Modeling and Stochastic Learning for Forecasting in High Dimensions*, pages 243–265. Springer.

[Schuster and Paliwal, 1997] Schuster, M. and Paliwal, K. K. (1997). Bidirectional recurrent neural networks. *IEEE transactions on Signal Processing*, 45(11):2673–2681.

A Proof of lemma

Proof.

- **Completeness:** We first prove that $(\mathbb{H}_m(\mu), \| \cdot \|_m)$ is complete.
  
  We already know that $\mathbb{H}_m(\mu)$ is Banach space. Then every Cauchy sequence $(H_i)_{i \in \mathbb{N}}$ of $(\mathbb{H}_m(\mu), \| \cdot \|_m)$ admit a limit $H_i$ in $\mathbb{H}_m(\mu)$. We only need to show that $H_i \in (\mathbb{H}_m(\mu), \| \cdot \|_m)$.
  
  Let $\xi'$ random field with distribution $\mu_{\xi'}$ and $i \in \mathbb{N}^*$. 

  \[
  \| H_i(\xi) - H_i(\xi') \|_m = \| H_i(\xi) - H_i(\xi) + H_i(\xi) - H_i(\xi') + H_i(\xi') - H_i(\xi') \|_m \\
  \leq \| H_i(\xi) - H_i(\xi) \|_m + \| H_i(\xi) - H_i(\xi') \|_m + \| H_i(\xi') - H_i(\xi') \|_m \\
  \leq \| H_i(\xi) - H_i(\xi) \|_m + \| \xi - \xi' \|_m + \| H_i(\xi') - H_i(\xi') \|_m
  \]

  using (12)
with \( k \) have the same distribution \( \mu_\xi \) than \( \xi \), thus \( \lim_{i \to \infty} \| H_i(\xi^{'}) - H_i(\xi^{'}) \|_m = 0 \). Hence take the limit for \( i \to \infty \) in the previous equation raise:

\[
\| H_i(\xi) - H_i(\xi^{'}) \|_m \leq \| \xi - \xi^{' \|_m
\]

Consequently, \( H_i \) belongs to \( \mathbb{H}_m(\mu_\xi, \| . \|_m) \) and point 1 is demonstrated

- **Stability by \( \phi \):**
  
  Let \( H \in \mathbb{H}_m(\mu) \), then
  
  \[
  \| \phi(H)(\xi) - \phi(H)(\xi^{'}) \| \leq \| F(H \circ \theta_t(\xi))_{t \in \mathcal{B}, \varepsilon} - F(H \circ \theta_t(\xi^{'})_{t \in \mathcal{B}, \varepsilon'}) \|_m \\
  \leq \sum_{t \in \mathcal{B}} \lambda_t \| H \circ \theta_t(\xi) - H \circ \theta_t(\xi^{'}) \|_m + \eta \| \varepsilon - \varepsilon' \|_m \\
  \leq \sum_{t \in \mathcal{B}} \lambda_t \| \theta_t(\xi) - \theta_t(\xi^{'}) \|_m + \eta \| \varepsilon - \varepsilon' \|_m \\
  \leq \sum_{t \in \mathcal{B}} \lambda_t \| \xi - \xi^{' \|_m + \eta \| (\varepsilon_t)_{t \in \mathbb{Z}_\varepsilon} - (\varepsilon'_t)_{t \in \mathbb{Z}_\varepsilon} \|_m \\
  \leq \| \xi - \xi^{' \|_m
\]

So we get the stability by \( \phi \)

- **Contractivity:**
  
  Let \( (H, H') \in \mathbb{H}_m(\mu_\xi)^2 \)

  \[
  \| \phi(H)(\xi) - \phi(H')(\xi) \|_m \leq \| F[(H \circ \theta_t)_{t \in \mathcal{B}}(\xi), \varepsilon] - F[(H' \circ \theta_t)_{t \in \mathcal{B}}(\xi), \varepsilon] \|_m \\
  \leq \sum_{t \in \mathcal{B}} \lambda_t \| (H \circ \theta_t)_{t \in \mathcal{B}}(\xi) - (H' \circ \theta_t)_{t \in \mathcal{B}}(\xi) \|_m \\
  \leq \sum_{t \in \mathcal{B}} \lambda_t \| H(\xi) - H'(\xi) \|_m \text{ because the } \varepsilon_t \text{ are iid} \\
  \leq \rho \times \| H(\xi) - H'(\xi) \|_m \\
  \leq \| H(\xi) - H'(\xi) \|_m \text{ since } \rho < 1
\]

\[ \Box \]

### B Proof of lemma\(^3\)

**Proof.** We introduce some notations:

- \( B \) is a finite subset of \( \mathbb{Z}_\varepsilon^* \), thus it exists an enumeration of \( B \) that we denote \( \psi : B \to [1, |B|] \) (\(|B|\) is the number of element in \( B \))

- For \( k = (k_1, \ldots, k_{|B|}) \) we denote \( \binom{n}{k} \) the multinomial coefficient \( \binom{n}{k} = n! / \prod_{i=1}^{[B]} k_i! \)

- We define the function \( \Gamma \)

\[
\Gamma : \mathbb{Z}^{[B]} \to \mathbb{Z}_\varepsilon^* \\
k \mapsto \sum_{i \in [1, |B|]} k_i \psi^{-1}(i)
\]

Let \( d \in \mathbb{N}, t \in \mathbb{Z}_\varepsilon^* \). Let show by induction on \( i \in [0, d+1] \) that :

\[
\| H(\xi_i) - H(\xi^{'})^{[d]} \|_m \leq \sum_{|k|=i} \binom{i}{k} \prod_{j=1}^{[B]} \lambda_j^{k_j} \| H \circ \theta_{\Gamma(k)}(\xi_i) - H \circ \theta_{\Gamma(k)}(\xi^{'}) \|_m
\]

with \( k = (k_1, \ldots, k_{|B|}) \) and \( |k| = \sum_{i=1}^{[B]} k_i \)

\[ (28) \]

17
For $i = 0$, the relation (28) holds. Now, let suppose that (28) holds for some $i \in [0, D]$. Let $k \in \mathbb{Z}[B]$ with $|k| = i$. Then, we have:

$$
\| H \circ \theta_{\Gamma(k)}(\xi_t) - H \circ \theta_{\Gamma(k)}(\tilde{\xi}_t^{[d]}) \|_m = \| F \left[ \left( H \circ \theta_{\Gamma(k)+s}(\xi_t) \right)_{s \in B}, e_{\Gamma(k)+t} \right] - F \left[ \left( H \circ \theta_{\Gamma(k)+s}(\tilde{\xi}_t^{[d]}) \right)_{s \in B}, e_{\Gamma(k)+t} \right] \|_m
$$

Moreover $\Gamma(k) = \sum_{i \in [1, |B|]} k_i \psi^{-1}(i)$ with $\forall i \in [1, |B|], \psi^{-1}(i) = (t_1^i, \ldots, t_k^i)$.

And

$$
\forall i \in [1, |B|], \forall j \in [1, \kappa], -\delta_j < t_j^i < \delta_j \Rightarrow \forall j \in [1, \kappa] - d \times \delta_j < \sum_{i \in [1, |B|]} k_i t_j^i < d \times \delta_j \\
$$

because $\psi^{-1}(i) \in B$

$$
\sum_{i \in [1, |B|]} k_i = i \leq d
$$

Consequently $(t + \Gamma(k)) \in \mathcal{V}(d\delta, t)$ thus $\tilde{\xi}_t^{[d]} + \Gamma(k) = \xi_{t+\Gamma(k)}$

Then, using equation (29) and applying hypothesis (H$_1$):

$$
\| H \circ \theta_{\Gamma(k)}(\xi_t) - H \circ \theta_{\Gamma(k)}(\tilde{\xi}_t^{[d]}) \|_m \leq \sum_{s \in B} \lambda_s \| H \circ \theta_{\Gamma(k)+s}(\xi_t) - H \circ \theta_{\Gamma(k)+s}(\tilde{\xi}_t^{[d]}) \|_m
$$

$$
= \sum_{i=1}^{|B|} \lambda_{\psi^{-1}(i)} \| H \circ \theta_{\Gamma(k)+\psi^{-1}(i)}(\xi_t) - H \circ \theta_{\Gamma(k)+\psi^{-1}(i)}(\tilde{\xi}_t^{[d]}) \|_m
$$

Inserting this result in the equation (28) raises:

$$
\| H(\xi_t) - H(\tilde{\xi}_t^{[d]}) \|_m \leq \sum_{|k|=i} \binom{i}{k} \left( \prod_{j=1}^{|B|} \lambda_{k_j^+}^j \right) \sum_{|\nu|=1} |B| \lambda_{\psi^{-1}(j)} \| H \circ \theta_{\Gamma(k^+\nu)}(\xi_t) - H \circ \theta_{\Gamma(k^+\nu)}(\tilde{\xi}_t^{[d]}) \|_m
$$

$$
= \sum_{|k|=i} \binom{i}{k} \sum_{\nu=1}^{|B|} \prod_{j=1}^{|B|} \lambda_{k_j^+}^j \lambda_{\psi^{-1}(j)} \| H \circ \theta_{\Gamma(k^+\nu)}(\xi_t) - H \circ \theta_{\Gamma(k^+\nu)}(\tilde{\xi}_t^{[d]}) \|_m
$$

with $k^+_{\nu} = (k_1, \ldots, k_{\nu-1}, k_{\nu} + 1, k_{\nu+1}, \ldots, k_{|B|})$ and $\phi(k^+_{\nu}) = \prod_{j=1}^{|B|} \lambda_{\nu}^j$.

Let $|k| = i + 1$, then it exists $|B|$ different $(k', \nu) \in \mathbb{Z}' \times [1, B]$ such that: $k = k'^\nu$ (one by component of the vector $k$).

We denote $k^-_{\nu} = (k_1, \ldots, k_{\nu-1}, k_{\nu} - 1, k_{\nu+1}, \ldots, k_{|B|})$

Then:

$$
\sum_{|k|=i+1} \binom{i+1}{k} \sum_{\nu=1}^{|B|} \phi(k^+_{\nu}) \| H \circ \theta_{\Gamma(k^+\nu)}(\xi_t) - H \circ \theta_{\Gamma(k^+\nu)}(\tilde{\xi}_t^{[d]}) \|_m
$$

$$
= \sum_{|k|=i+1} \binom{i+1}{k} \sum_{\nu=1}^{|B|} \phi(k^+_{\nu}) \| H \circ \theta_{\Gamma(k)}(\xi_t) - H \circ \theta_{\Gamma(k)}(\tilde{\xi}_t^{[d]}) \|_m
$$

$$
= \sum_{|k|=i+1} \binom{i+1}{k} \phi(k) \| H \circ \theta_{\Gamma(k)}(\xi_t) - H \circ \theta_{\Gamma(k)}(\tilde{\xi}_t^{[d]}) \|_m
$$
which corresponds to the equation (28) for $i + 1$. Consequently, by induction (28) holds for $i \in [0, d + 1]$
In particular, for $d + 1$ we get

$$\|H(\xi_t) - H(\hat{\xi}_t)\|_m \leq \sum_{|k|=d+1}^{|B|} \left( \sum_{|k|=d+1}^{|B|} \left( \prod_{j=1}^{|B|} \lambda^{k_j}_{\psi-1(j)} \right) \|H \circ \theta_{\Gamma(k)}(\xi_t) - H \circ \theta_{\Gamma(k)}(\hat{\xi}_t)^{\bar{d}}\|_m \right)
\leq \sum_{|k|=d+1}^{|B|} \left( \prod_{j=1}^{|B|} \lambda^{k_j}_{\psi-1(j)} \right) \|V_m\|$$

using lemma (2)

and therefore

$$\|H(\xi_t) - H(\hat{\xi}_t)\|_m \leq \left( \sum_{i=1}^{|B|} \lambda^{\psi-1(i)} \right)^{d+1} \|V_m\|$$

using Newton binomia

\[ \square \]

**C \; Proof of lemma 7**

*Proof.* Let $p \in \mathbb{N}$ and $(a, b) \in \mathbb{R}^2$

We first compute $I_p = \int_a^b t^p \rho^t dt$

$$I_p = \int_a^b t^p \rho^t dt = \left[ t^p \rho^t \ln(\rho) \right]_a^b + \frac{p}{\ln(\rho-1)} \int_a^b t^{p-1} \rho^t dt = \frac{a^p \rho^a - b^p \rho^b}{\ln(\rho-1)} + \frac{p}{\ln(\rho-1)} I_{p-1}$$

And

$$I_0 = \int_a^b \rho^t dt = \frac{\rho^a - \rho^b}{\ln(\rho-1)}$$

By induction, we get:

$$I_p = \sum_{i=0}^p \frac{a^i \rho^a - b^i \rho^b}{\ln(\rho-1)^p} \times \frac{p!}{i!} = \frac{p!}{i!} \sum_{i=0}^p \frac{(a^i \rho^a - b^i \rho^b) \ln(\rho-1)^i}{i!}$$

(30)

Let $f(t) = t^p \rho^t$; we have $f'(t) = t^{p-1} \rho^t (p - t \ln(\rho-1))$. Then $f'(t) = 0 \iff t = \frac{p}{\ln(\rho-1)}$. Thus $f$ is increasing on $[0, \frac{p}{\ln(\rho-1)}]$ and decreasing on $[\frac{p}{\ln(\rho-1)}, +\infty]$

Applying this to our case,

- if $d < \left\lfloor \frac{\kappa-1}{\ln(\rho-1)} \right\rfloor$, Then: $\sum_{c=1}^d c^{\kappa-1} \rho^c \leq \int_0^{d+1} t^{\kappa-1} \rho^t dt$

- if $d = \left\lfloor \frac{\kappa-1}{\ln(\rho-1)} \right\rfloor$, Then: $\sum_{c=1}^d c^{\kappa-1} \rho^c \leq \int_0^{\frac{\kappa-1}{\ln(\rho-1)}} t^{\kappa-1} \rho^t dt + \left( \frac{\kappa-1}{\ln(\rho-1)} \right)^{\kappa-1}$

- if $d > \left\lfloor \frac{\kappa-1}{\ln(\rho-1)} \right\rfloor$, Then: $\sum_{c=1}^d c^{\kappa-1} \rho^c \leq \int_0^{\frac{\kappa-1}{\ln(\rho-1)}} t^{\kappa-1} \rho^t dt + \left( \frac{\kappa-1}{\ln(\rho-1)} \right)^{\kappa-1} + \int_{\frac{\kappa-1}{\ln(\rho-1)}}^{d} t^{\kappa-1} \rho^t dt$

Using (30), we get:

- $\int_0^{d+1} t^{\kappa-1} \rho^t dt \leq \left( \frac{\kappa-1}{\ln(\rho-1)} \right) \left( 1 - \rho^{d+1} \sum_{i=0}^{\kappa-1} \frac{(d+1) \ln(\rho-1)^i}{i!} \right)$

- $\int_{\frac{\kappa-1}{\ln(\rho-1)}}^{d} t^{\kappa-1} \rho^t dt \leq \left( \frac{\kappa-1}{\ln(\rho-1)} \right) \left( \rho \sum_{i=0}^{\kappa-1} \frac{\left( \frac{\kappa-1}{\ln(\rho-1)} \right) \ln(\rho-1)^i}{i!} - \rho^d \sum_{i=0}^{\kappa-1} \frac{(d \ln(\rho-1))^i}{i!} \right)$
Thus defining the function $\Upsilon_{\kappa, \rho}$:

$$\Upsilon_{\kappa, \rho} : \mathbb{N} \mapsto \mathbb{R}$$

$$\Upsilon_{\kappa, \rho}(d) = \begin{cases} 
\frac{(\kappa - 1)!}{\ln(\rho^{-1})^\kappa} \left(1 - \rho^{d+1} \sum_{i=0}^{\kappa-1} \frac{(d+1)\ln(\rho^{-1})^i}{i!}\right) & \text{if } d < \left\lfloor \frac{\kappa - 1}{\ln(\rho^{-1})} \right\rfloor \\
\frac{(\kappa - 1)!}{\ln(\rho^{-1})^\kappa} \left(1 - \rho^{d} \sum_{i=0}^{\kappa-1} \frac{\left(\frac{\kappa - 1}{\ln(\rho^{-1})}\right)^i}{i!}\right) + \left(\frac{\kappa - 1}{\ln(\rho^{-1})}\right)^{\kappa-1} & \text{if } d = \left\lfloor \frac{\kappa - 1}{\ln(\rho^{-1})} \right\rfloor \\
\frac{(\kappa - 1)!}{\ln(\rho^{-1})^\kappa} \left(1 - \rho^{d} \sum_{i=0}^{\kappa-1} \frac{(d\ln(\rho^{-1}))^i}{i!}\right) + \left(\frac{\kappa - 1}{\ln(\rho^{-1})}\right)^{\kappa-1} & \text{if } d > \left\lfloor \frac{\kappa - 1}{\ln(\rho^{-1})} \right\rfloor
\end{cases}$$

it holds:

$$\forall t \in \mathbb{N}, \sum_{c=1}^{d} c^{\kappa-1} \rho^c \leq \Upsilon_{\kappa, \rho}(d)$$