Topological Magneto-Electric Effect Decay

D. A. Pesin and A. H. MacDonald

Department of Physics, University of Texas at Austin, Austin TX 78712, USA

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We address the influence of realistic disorder on the effective magnetic monopole that is induced near the surface of an ideal topological insulator (TI) by azimuthal currents which flow in response to a suddenly introduced external electric charge. We show that when the longitudinal conductivity \( \sigma_{xx} = \frac{ge^2}{h} \) is accounted for, the apparent position of a magnetic monopole initially retreats from the TI surface at speed \( v_M = \alpha cg \), where \( \alpha \) is the fine structure constant and \( c \) is the speed of light. For the particular case of TI surface states described by a massive Dirac model, we further find that the temperature \( T = 0 \) Hall currents vanish once the surface charge has been redistributed to screen the external potential.

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Introduction—When a time-reversal-symmetry breaking perturbation opens a gap in the surface state spectrum of a three-dimensional topological insulator (TI)\(^1\)\(^,\)\(^2\), surface Hall currents and orbital magnetism are induced by electrical perturbations. This magneto-electric coupling effect can be attractively described\(^3\) by adding a \( E \cdot B \) term to the electromagnetic Lagrangian. The duality of the resulting axion electrodynamics model\(^4\) leads to a curious topological magneto-electric effect\(^5–7\) in which an electric charge placed above the TI surface induces Hall currents and associated orbital magnetization that appears to emanate from a magnetic monopole below the surface.

In this paper we show that a non-zero TI surface state longitudinal conductivity \( \sigma_{xx} = \frac{ge^2}{h} \), an omnipresent experimental reality that is not captured by the axion electrodynamics model, qualitatively alters the topological magneto-electric effect. We find that when the external charge is placed more than a screening length \( \lambda \) from the surface, the monopole moves away with velocity \( v_M = \alpha cg \). In the long-time limit the screened external potential becomes static. In this case we find that the orbital magnetization response depends on details of the surface state electronic structure, and that it vanishes in the particular case of a two-dimensional massive Dirac model with temperature \( T = 0 \) and a Fermi level position outside the gap.

Macroscopic Theory—We assume here that the TI surface has a well defined surface Hall conductivity and diffusion constant; this assumption can fail for very well developed quantum Hall effects. We first consider the limit in which the separation \( d \) between the external charge and the TI surface is larger than the screening length \( \lambda \). We introduce an external charge \( Qe \) located a distance \( d \) from the TI surface; since we wish to treat this object as a source of macroscopic inhomogeneity rather than as a contribution to the disorder potential we imagine that \( Q \gg 1 \) and that \( d \) is longer than microscopic lengths. Currents flow in the TI surface in response to the electric fields from the external charge and the screening charges that accumulate in the TI surface layer. Working in two-dimensional momentum space and assuming that the total electric field changes sufficiently slowly with time, we use the continuity equation to conclude that

\[
\frac{\partial n_{q}^{2D}}{\partial t} = -2\pi \sigma_{xx}Q (Q \exp(-qd) + n_{q}^{2D}).
\]

(1)

where \( d \) is the distance from the surface to the external positive charge \( Qe \) and \( n_{q}^{2D} \) is the Fourier transform of the induced surface state density. In Eq. (1) we neglect the diffusion current, which is permissible at long distances as we show below. If we assume that the external charge is introduced suddenly at time \( t = 0 \) and that the two-dimensional (2D) density evolves in time in accordance with Eq. (1) we find that

\[
n_{q}^{2D} = -Q \exp(-qd)(1 - \exp(-qv_M t))
\]

(2)

and that the total potential from external and screening charges is

\[
\phi_{tot}(q,t) = \frac{2\pi eQ}{q} \exp(-q(d + v_M t)).
\]

(3)

Here the monopole velocity \( v_M = 2\pi \sigma_{xx} = \alpha cg \) is large unless the dissipative conductivity is much smaller than the quantum unit of conductance, i.e. unless the quantum Hall effect on the TI surface is very well developed. The potential at time \( t \), which controls the instantaneous Hall currents and hence the instantaneous magnetization is identical to that from a external charge that is located at vertical position \( d \), but at vertical position \( d + v_M t \). As shown elsewhere\(^5\), because of the magneto-electric duality of axion electrodynamics, these Hall currents give rise to a magnetization that is identical to that produced by a magnetic monopole located at a distance \( d + v_M t \) below the TI surface. Currents flow until macroscopic electric fields vanish. The topological magneto-electric effect is therefore purely transient in the \( d \gg \lambda \) limit.
Screening in the Quantum Hall Regime—This result can be extended by including the diffusion contribution to the surface current:

\[
\frac{\partial n_{2D}^2}{\partial t} = -2\pi \sigma_{xx} q (Qe^{-q_d} + n_q^{2D}) - D_F q^2 n_q^{2D}. \tag{4}
\]

The longitudinal conductivity, \( \sigma_{xx} \), is related to the diffusion coefficient via the usual Einstein relation \( \sigma_{xx} = (\partial n/\partial \mu) e^2 D_F \). Solving Eq. (4), we obtain the final expression for the total electric potential on the surface:

\[
\phi_{tot}(q, t) = 2\pi e Q e^{-q_d} \left( \frac{1 + (q\lambda_{TF})^{-1} e^{-(D_F q^2 + 2\pi \sigma_{xx} q)t}}{q + \lambda_{TF}^{-1}} \right),
\]

where \( \lambda_{TF}^{-1} = 2\pi \sigma_{xx}/D_F = 2\pi \nu_F e^2 \) is the screening wavevector and \( \lambda_{TF} \) the screening length. The longitudinal currents vanish for \( t \to \infty \) due to the Einstein-relation cancellation between drift and diffusion contributions. The total potential for \( t \to \infty \) reduces to the standard result for Thomas-Fermi screening in 2D. Because \( \partial n/\partial \mu \) becomes extremely small when the quantum Hall effect is well developed, \( \lambda \) can be much larger than typical microscopic length scales.

Since the external potential remains large for \( t \to \infty \) at length scales smaller than \( \lambda \), there will be a macroscopic orbital magnetic response to the screened potential if the contributions to the Hall current from the screened electric field and from the induced density inhomogeneities do not cancel. Is there an Einstein relation for Hall currents? Below we use a quantum kinetic theory to answer this question microscopically. We conclude that the answer is no in general. Both drift and diffusion-type terms do appear. The contribution to the Hall current from density inhomogeneities can be understood as being due to non-uniform internal magnetic moment \( \mathbf{\hat{z}} \) densities. Moreover, for the two-dimensional massive Dirac equation that is normally used to model TI surface states, the drift and diffusion Hall currents do cancel in the clean limit.

Microscopic Theory—In the presence of an external potential the surface states of a 3D strong topological insulators can be described\(^{[1, 2]} \) approximately by a 2D massive Dirac Hamiltonian:

\[
H_{MD} = \int d^2 \Psi^\dagger \left( B_p \mathbf{\hat{r}} \cdot \mathbf{\hat{p}} + e \phi_{ext} + U_{\text{dis}} \right) \Psi. \tag{6}
\]

Here \( B_p = (vp_x, vp_y, \Delta) \) is a \( p \)-dependent effective Zeeman field which acts on spinful surface electrons. With this choice for \( B_p \), the Pauli matrices correspond to spins rotated by \( \pi/2 \) around the \( \hat{z} \) axis, which we have taken to be normal to the surface. The mass term \( \Delta \) breaks time-reversal-symmetry and is normally thought of as arising from proximity exchange coupling to an insulating ferromagnet. For definiteness and without loss of generality, we take \( \Delta > 0 \). \( U_{\text{dis}} \) describes an atomic scale disorder potential which we take to be created by short-range impurities with concentration \( n_{\text{imp}} \): \( U_{\text{dis}} = \sum_i u\delta(\mathbf{r} - \mathbf{r}_i) \).

From now on we work in the system of units with \( \hbar = 1 \).

In order to address the transport properties of this model, we use a quasiclassical kinetic equation for the electron density matrix, \( \hat{f} \), which takes the form

\[
\partial_t \hat{f}_p + \frac{1}{2} \left\{ \partial_p (B_p \cdot \mathbf{\hat{r}}), \partial_t \hat{f}_p \right\} + i [B_p \cdot \mathbf{\hat{z}}, \hat{f}_p] + e E_{tot} \cdot \partial_p \hat{f}_p^{\text{eq}} = \hat{I}_{\text{tot}}. \tag{7}
\]

In the above equation \( E_{tot} \) is the total electric field including both external and induced potential contributions, and \( \hat{I}_{\text{tot}} \) is the collision integral.\(^{[9]} \) We allow for an imperfect quantum Hall effect by considering the case in which carriers are present in at least one of the bands due either to doping or to finite temperature.

The distribution function can be decomposed into scalar and vector pieces, \( \hat{f}_p = n_p + \mathbf{\hat{r}} \cdot \mathbf{f}_p \), and the vector \( \mathbf{f}_p \) further separated into contributions parallel and perpendicular to \( B_p \): \( \mathbf{f}_p^{\parallel} \mathbf{B}_p \) and \( \mathbf{f}_p^{\perp} \). (\( \mathbf{B}_p \) is a unit vector in the direction of \( \mathbf{B}_p \)). In this parameterization of the density matrix \( n_p \) and \( f_p^{\parallel} \) specify valence and conduction band occupation numbers and \( f_p^{\perp} \) interband coherence. The kinetic equation for the full density matrix can be separated into a set of equations for these components.

The model’s intraband response is entirely standard\(^{[10]} \), except that scattering on the Fermi surface is influenced by the inner product of the momentum-dependent conduction band states. For the conduction band we find that...
where \( v_p = v^2 p / B_p \) is the band velocity appropriate for the conduction band of Hamiltonian \( \hat{H}_0 \). It follows that the longitudinal conductivity, \( \sigma_{xx} \), is related to the diffusion coefficient via the usual Einstein relation \( \sigma_{xx} = \nu_F e^2 D_F \), where \( \nu_F = B_p / 2\pi v^2 \) is the density of states at the Fermi level. The absence of a longitudinal current in equilibrium, assumed in the macroscopic theory, then follows from the cancelation between the second (diffusion) and third (drift) terms of the left-hand-side of Eq. (8), when \( f^c \) is replaced by its equilibrium Fermi function value. The diffusion coefficient \( D_F = \nu_F^2 \tau_{tr} / 2 \) with

\[
\tau_{tr}^{-1} = \frac{n_{imp} u^2}{4 v^2} \frac{v^2 p^2 + 4\Delta^2}{\sqrt{v^2 p^2 + \Delta^2}}.
\]

\( \text{Hall response} \) – We have seen above that even in the presence of screening there is a residual radially symmetric electric potential at the surface for \( t \to \infty \). The purpose of the following calculation is to determine whether or not that potential can drive an azimuthal Hall current which contributes to the orbital magnetization. The naive guess that one just has to multiply the screened electric field with the intrinsic Hall conductivity to find the current fails because gradients in the density of carriers, all of which generally carry intrinsic magnetic moments\[8, 11\], also yield an azimuthal current. The additional contribution can cancel the azimuthal electric field response either completely, as it does in the longitudinal case, or partially.

Since the response we seek to evaluate includes the time-reversal-symmetry broken system’s anomalous Hall effect, we should include side-jump and skew scattering contributions\[12\] to describe it fully in the presence of impurities. Since these are dependent on impurity scattering, we can obtain this by considering the leading quasiclassical corrections to Eq. (8). In the case of a uniform electric field, the quasiclassical kinetic equation for conduction band electrons has been derived in Refs. \[13, 14\]. This equation generalizes Eq. (8) to include an anomalous distribution generation term coming from the collision integral, and beyond-Born-approximation skew scattering amplitudes. Since we are interested here in response to a non-uniform static electric field, we need to generalize the quasiclassical Boltzmann equation of Refs. \[13, 14\] to the non-uniform case by adding a drift term, \( \tilde{v}_d \partial_t f^c \), just like the one in Eq. (8), but with \( \tilde{v}_d \) now including not only the band velocity, but also anomalous and side-jump corrections. It is then a simple matter to see that all electric-field drive terms vanish in that equation in local equilibrium. Therefore, side-jump and skew scattering contributions need not be considered and the entire Hall response comes from the intrinsic contribution.

The intrinsic contribution should be obtained from the equation for \( f^i_p \). Importantly, since we need not consider the side jump contribution, we can simply drop the contribution to the collision integral for \( f^i_p \) coming from \( f^\parallel_p \), since the latter gives a contribution to side-jump processes only. \[15\] Further, for a sufficiently clean surface, such that \( B_p \tau_{tr} \gg 1 \), we can also neglect the collisional relaxation of \( f^i_p \) as compared to the precession term, coming from the commutator on the left hand side of Eq. (7). The general expression for the static limit of \( f^i_p \) is thus obtained simply by isolating the inter-band terms on the left hand side Eq. (7). We obtain

\[
2B_p f^i_p = (\nabla n_p \partial_B) B_p \times b_p + (e \varepsilon F_\text{tot} \partial_B) f_p \times b_p.
\]

The second term on the right hand side of the above equation leads to the standard intrinsic contribution to the Hall conductivity due to the interband coherence created by the electric field. The first term on the right hand side of Eq. (10) is the response to the equilibrium density inhomogeneities. Its contribution to the current can be seen to equal the curl of the internal magnetic moment \[8\] density of quasiparticles, which is nonuniform in space. The right hand side of Eq. (10) does not necessarily vanish when equilibrium values are used for \( n_p = (f^0_p + f^\parallel_p) / 2 \) and \( f_p = (f^0_p - f^\parallel_p) / 2 \) \( (f^0_p = n_F (\pm B_p - \mu + e\phi\text{tot})) \). This property contrasts with Eq. (8), in which both left and right hand sides vanish under the local equilibrium ansatz.

Substituting the equilibrium values gives \( \nabla n_p = -e \varepsilon F_\text{tot} (\partial_B n_F \frac{\partial B_p}{\partial B_p} - \partial_B (-B_p - \mu) \frac{\partial B_p}{\partial B_p}) / 2 \) and \( f_p = b_p (n_F (\pm B_p - \mu) - n_F (-B_p - \mu)) / 2 \). Substituting these expressions in Eq. (10) and taking the local direction of the electric field to be along the \( \hat{x} \) axis we obtain:

\[
f^i_p = -\frac{1}{4} e \varepsilon F_\text{tot} (\partial_B n_F) \frac{\partial B_p}{B_p} \sum_{\nu = \pm} \frac{\partial}{\partial B_p} \left( n_F (\nu B_p - \mu) \right).
\]

This result will recover the usual intrinsic anomalous Hall conductivity when the derivative acts on the \( B_p \) factor only. When the derivative acts on both factors we obtain
When an external charge is placed near the surface of an ideal TI with weak time-reversal symmetry breaking, it induces an azimuthal current that produces the same magnetic field as would be produced by a magnetic monopole located below the TI surface. The axion electrodynamics model of TI magneto-electric and magneto-optical properties elegantly captures this intriguing property. In this paper we have examined how the Hall current response is altered by the samples imperfections which always results in a finite longitudinal conductivity $\sigma_{xx} = g(c^2/h)$. Systems with a finite $\sigma_{xx}$ are not fully described by the axion electrodynamics model so we develop our theory directly in terms of surface state electronic properties. We find that the apparent monopole position moves away from the TI surface with a velocity $v_M = \omega g$. Since graphene based two-dimensional electron systems, which are similar to TI surface states, can have $g$ values $\sim 10^{-7}$ or smaller when time-reversal symmetry is broken by an external magnetic field, there is a reasonable hope that it will be possible to obtain TI samples in which $v_M$ is small enough to enable observations in which $\sigma_{xx}$ plays no role and the axion electrodynamics model is applicable. There is a considerable recent experimental effort in this direction.

In the long-time limit after the external charge screening process has been completed, we find that the azimuthal current response has two contributions, one proportional to the Hall conductivity and treated previously by Zang and Nagaosa and one proportional to an external potential induced change in the internal magnetization of the surface states. For the particular case of a massive Dirac model the two contributions cancel exactly in the clean $T = 0$ limit in the presence of a Fermi surface. We obtain this result using a quasiclassical kinetic equation approach, which may not be reliable near band edges due to both quantum and non-linear screening effects, but nevertheless starkly demonstrates the distinction between azimuthal current and Hall conductivity responses. The special properties of the massive Dirac model are related to its well known unusual orbital magnetization properties in the uniform system limit. In general the magnetic flux induced by an electron charge near a time-reversal symmetry broken TI surface is dependent on the $|P|$-dependence of the exchange potential $\Delta_P$ and disorder effects, and not simply on surface’s Hall conductivity.

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