On Manin–Schechtman orders related to directed graphs

Vladimir I. Danilov, Alexander V. Karzanov and Gleb A. Koshevoy

Dedicated to Yurii Ivanovich Manin on the occasion of his 85-th birthday.

Abstract

As a generalization of weak Bruhat orders on permutations, in 1989 Manin and Schechtman introduced the notion of a higher Bruhat order on the \(d\)-element subsets of a set \([n] = \{1, 2, \ldots, n\}\). Among other results in this field, they proved that the set of such orders for \(n, d\) fixed, endowed with natural local transformations, constitutes a poset with one minimal and one maximal elements. In this paper we consider a wider model, involving the so-called convex order on certain path systems in an acyclic directed graph, introduce local transformations, or flips, on such orders and prove that the resulting structure gives a poset with one minimal and one maximal elements as well, yielding a generalization of the above-mentioned classical result.

Keywords: weak Bruhat order, higher Bruhat order, cyclic zonotope, cubillage

1 Introduction

Let \(n, d\) be integers with \(n \geq d > 0\). Studying higher simplex equations (Zamolodchikov equations), Manin and Schechtman \cite{4} introduced an important sort of linear (total) orders on the collection \(\binom{[n]}{d}\) of \(d\)-element subsets of the set \([n] = \{1, 2, \ldots, n\}\). They called an order \(\prec\) on \(\binom{[n]}{d}\) admissible if for any \((d + 1)\)-element subset \(P = (p_1 < \cdots < p_{d+1})\) (a packet), the family \(\mathcal{F}(P)\) consisting of \(d\)-element subsets occurring in \(P\) is ordered by \(\prec\) either lexicographically (namely, \(P - \{p_{d+1}\} \prec P - \{p_d\} \prec \cdots \prec P - \{p_1\}\)) or anti-lexicographically. Two admissible orders are said to be equivalent if their restrictions to each packet coincide; the equivalence classes of admissible orders were called in \cite{4} higher Bruhat orders (HBOs) and their set was denoted by \(B(n, d)\). (When \(d = 1\), this turns into classical weak Bruhat orders on permutations.)
Subsequently, Kapranov and Voevodsky [3] and Ziegler [6] gave a nice geometric interpretation for HBOs, by constructing a bijection between $B(n, d)$ and the set of cubillages on a cyclic zonotope $Z(n, d)$, where a cubillage (the term introduced in [3]) is a fine subdivision of $Z(n, d)$ into parallelopetopes. Among a variety of combinatorial and geometric results on HBOs and related cubillages, one, originally appeared in [4], is of especial interest for us. It says that $B(n, d)$ endowed with local operations constitutes a poset with one minimal and one maximal elements.

A local operation, or a flip, applied to an admissible order $\prec$ on $\left[\binom{n}{d}\right]$ chooses an appropriate packet and changes the order $\prec$ within its family, either from “lex” to “anti-lex”, or conversely; in the former (latter) case the flip is called raising (resp. lowering). This is viewed as a far generalization of usual transpositions in permutations (appeared when $d = 1$), and a remarkable result is that for any admissible order, one can arrange a series of raising (lowering) flips so as to reach an admissible order having no packet with a “lex” (resp. “anti-lex”) content.

In this paper, we extend that result to a larger type of orders, which live in a directed graph rather than $[n]$. A source of our model is based on the observation that one can interpret the ordered set $[n]$ as the directed path $R$ with $n$ vertices, and a set $S \in \left[\binom{n}{d}\right]$ as a tuple formed by $d - 1$ consecutive subpaths in $R$. In our general setting, we deal with an arbitrary finite acyclic directed graph $G = (V, E)$ without multiple edges, and for $d > 0$, consider a linear order $\prec$ on the set $P_{d-1}(G)$ of tuples, called $(d-1)$-corteges, formed by $d - 1$ consecutive paths in $G$. An order $\prec$ is called convex (of degree $d - 1$) if for each $(d-1)$-cortege $p$, the $(d-1)$-corteges contained in $p$ are ordered by $\prec$ in either lexicographic or anti-lexicographic manner. In a special case, when $G$ is a path itself, a convex order turns into an HBO.

For the set $O_{d-1}(G)$ of (equivalence classes of) convex orders of degree $d-1$ in $G$, we define flips that transform one order $\prec$ into another by changing $\prec$ within one $(d-1)$-cortege. Our main result (Theorem 2.1) is that $O_{d-1}(G)$ endowed with the flips constitutes a poset with one minimal and one maximal elements, thus generalizing the above-mentioned result on flips in $B(n, d)$. Our method of proof involves a series of results and constructions from the theory of cubillages of cyclic zonotopes (for descriptions in detail, see [1]).

This paper is organized as follows. Section 2 contains basic definitions and formulates the main result (Theorem 2.1) on the flip structure of $O_{d-1}(G)$ for $d \geq 2$. Section 3 is devoted to a review of a scope of constructions and results on cubillages that we use in the proof of Theorem 2.1 given in Section 4. Finally, in Section 5 we give two results on relations between convex orders in neighboring dimensions, namely, between $O_{d-1}(G)$ and $O_d(G)$ (which extends a relationship of elements of $B(n, d + 1)$ and maximal chains in $B(n, d)$).

2 Definitions and the main theorem

We consider a finite digraph (directed graph) $G = (V, E)$ with vertex set $V$ and edge set $E$, and assume that $G$ is acyclic (has no directed cycle) and contains no multiple edges.

A (directed) path in $G$ is a sequence $p = (v_0, e_1, v_1, \ldots, e_k, v_k)$, where $v_0, \ldots, v_k$ are
vertices, \(e_1, \ldots, e_k\) are edges, and each \(e_i\) leaves \(v_{i-1}\) and enters \(v_i\) (or goes from \(v_{i-1}\) to \(v_i\)). We may denote an edge \(e_i\) as \(v_{i-1}v_i\), and the path \(p\) as \(v_0v_1 \ldots v_k\), and call \(v_0\) the tail, and \(v_k\) the head of \(p\). We refer to an (inclusionwise) maximal path in \(G\) as a route. The set of tails (heads) of routes is just the set of vertices of \(G\) having no entering (resp. leaving) edges.

For paths \(p = u_0 \ldots u_k\) and \(q = v_0 \ldots v_{\ell}\), if the head \(u_k\) of \(p\) coincides with the tail \(v_0\) of \(q\), then we say that the pair \((p, q)\) is a tandem and denote the concatenation \(u_0 \ldots u_kv_1 \ldots v_{\ell}\) as \(p \ast q\).

Let \(P(G)\) denote the set of nontrivial paths in \(G\), i.e., those having at least two vertices. The next notion is important to us.

**Definition.** A linear (total) order \(\prec\) on \(P(G)\) is called convex if any tandem \((p_1, p_2)\) in \(G\) satisfies either
\[
p_1 \prec p_1 \ast p_2 < p_2 \quad (2.1)
\]
or
\[
p_2 \prec p_1 \ast p_2 < p_1. \quad (2.2)
\]

Such an order is closely related to the so-called natural order defined on the tiles of a rhombus tiling on a zonogon. We, however, are going to consider a more general construction which will admit an interpretation via cubillages of cyclic zonotopes. (Relations to rhombus tilings and zonogons will be explained in the next section.)

Let \(d\) be an integer \(\geq 2\). We say that a tuple \((p_1, \ldots, p_d)\) of paths in \(P(G)\) is a \(d\)-cortego if for \(i = 1, \ldots, d-1\), the head of \(p_i\) coincides with the tail of \(p_{i+1}\) (this is a tandem when \(d = 2\)). The set of \(d\)-corteges is denoted by \(P_d(G)\). We associate to \(p = (p_1, \ldots, p_d) \in P_d(G)\) the sequence \(S(p)\) of \((d-1)\)-corteges \(s^p_1, \ldots, s^p_{d+1}\), where
\[
s^p_1 := (p_2, \ldots, p_d);
\]
\[
s^p_i := (p_1, \ldots, p_{i-2}, p_{i-1} \ast p_i, p_{i+1}, \ldots, p_d), \quad i = 2, \ldots, d; \quad (2.3)
\]
\[
s^p_{d+1} := (p_1, \ldots, p_{d-1}).
\]

**Definition.** (Extending the previous definition.) A linear order on \(P_{d-1}(G)\) is called convex if for any \(d\)-cortego \(p \in P_d(G)\), either
\[
s^p_{d+1} \prec s^p_d \prec \cdots \prec s^p_1 \quad (2.4)
\]
or
\[
s^p_1 \prec s^p_2 \prec \cdots \prec s^p_{d+1}. \quad (2.5)
\]

In particular, when \(d = 2\) and \(p\) is a tandem \((p_1, p_2)\), we have \(S(p) = (p_2, p_1 \ast p_2, p_1)\), (2.4) turns into (2.1), and (2.5) into (2.2).

For some reasons that will be clearer later, we refer to relation (2.4) (and the cortego \(p\)) as being of standard type, while (2.5) of anti-standard type. Relative to a convex order, certain corteges will play an especial role.

**Definitions.** Let us say that two convex orders \(\prec\) and \(\prec'\) on \(P_{d-1}(G)\) are equivalent if each \(d\)-cortego has the same type for both \(\prec\) and \(\prec'\). For a convex order \(\prec\) and \(p, q \in P_{d-1}(G)\), we write \(p \prec q\) and say that \(p\) stably precedes \(q\) if \(p \prec' q\) holds for any convex order \(\prec'\) equivalent to \(\prec\). A \(d\)-cortego \(p \in P_d(G)\) of standard (anti-standard) type is called dense if there exists no \(r \in P_{d-1}(G)\) which stably separates two neighboring
terms in (2.4) (resp. (2.5)), i.e., such that \( s_i^p \prec \cdot r \prec \cdot s_i^p \) (resp. \( s_i^p \prec \cdot r \prec \cdot s_{i+1}^p \)) for all \( i = 1, \ldots, d \). If a dense \( d \)-cortege \( p \) is subject to (2.4), then the replacement of all order relations in it by the ones as in (2.5) is called the raising flip on \( p \); the converse transformation, from (2.5) to (2.4), is called the lowering flip on \( p \).

An important fact is that when dealing with a convex order, the raising (lowering) flip applied to a dense \( d \)-cortege results in a convex order again. This follows from two simple observations: (a) for a \( d \)-cortege \( p \), any two terms in the set \( S(p) \) determine the remaining terms (and therefore, two different \( d \)-corteges have at most one term in common); and (b) if \( p \) is dense and \( r \in P_{d-1}(G) - S(p) \) is smaller (greater) than one term of \( S(p) \), then \( r \) is smaller (resp. greater) than each term of \( S(p) \). Therefore, the flip on \( p \) does not change the order relations within any other \( d \)-cortege.

Thus, each application of the raising (lowering) flip transforms the current convex order into another one and decreases (resp. increases) the number of \( d \)-corteges of standard type by exactly one.

Form the directed graph \( O_d(G) \) whose vertices are the equivalence classes \( \sigma \) of convex orders on \( P_{d-1}(G) \) and whose edges are the pairs \((\sigma, \sigma')\) such that some (equivalently, any) representative \( \prec' \) of \( \sigma' \) is obtained by a raising flip from some representative \( \prec \) of \( \sigma \). Our main result in this paper is the following

**Theorem 2.1** Let \( G \) be a finite acyclic digraph without multiple edges and let \( d \) be an integer \( \geq 2 \). Then \( O_d(G) \) is acyclic and has one minimal (zero-indegree) and one maximal (zero-outdegree) vertex, where the former (latter) is represented by a convex order on \( P_{d-1}(G) \) having all \( d \)-corteges of standard (resp. anti-standard) type. Equivalently, for a convex order \( \prec \) on \( P_{d-1}(G) \), if the set of \( d \)-corteges of standard (anti-standard) type for \( \prec \) is nonempty, then \( \prec \) admits at least one raising (resp. lowering) flip (where the flip involves a dense \( d \)-cortege).

We will prove this theorem in Section 4 using a machinery of cubillages and related objects reviewed in the next section.

**Corollary 2.2** The poset determined by \( O_d(G) \) is ranged.

---

*Example.* To illustrate Theorem 2.1 let \( G \) be the graph drawn in the left fragment of Fig. 1. Here there are two routes: \( R = (1, p_1, 2, p_2, 3, q, 4) \) and \( R' = (1, p_1, 2, p_2, 3, r, 4') \).
Let $d = 2$. Then the 2-corteges in $G$ are $(p_1, p_2)$, $(p_1, p_2 \ast q)$, $(p_1 \ast p_2, q)$, $(p_2, q)$, $(p_1, p_2 \ast r)$, $(p_1 \ast p_2, r)$, $(p_2, r)$. Assign a linear order $\prec$ on $\mathcal{P}(G)$ satisfying:

\[
\begin{align*}
p_2 & \prec p_1 \ast p_2 < p_1 < R < p_2 \ast q < q \quad \text{(within $R$)}, \\
p_2 & < p_2 \ast r < R' < p_1 \quad \text{and} \quad p_1 \ast p_2 < R' < r \quad \text{(within $R'$)},
\end{align*}
\]

and arbitrary on the other pairs. One can check that $\prec$ is convex, the tandems $(p_1, p_2)$ and $(p_1, p_2 \ast r)$ are anti-standard, and the remaining ones are standard. The tandem $(p_1, p_2)$ is not dense (since $p_2$ and $p_1$ are stably separated by $R'$). In its turn, $(p_1, p_2 \ast r)$ is dense, and under the lowering flip, it becomes of standard type (satisfying $r \prec R' \prec p_1 \ast p_2$). In the resulting convex order, the tandem $(p_1, p_2)$ is already dense, and making the lowering flip on it, we obtain a minimal order in $\mathcal{O}_2(G)$. The middle and right fragments in the picture illustrate the rhombus tilings associated with the restrictions of $\prec$ to $R$ and $R'$; such a correspondence will be explained in Sect. 3.3.

### 3 Cubillages and their relations to convex orders on routes

In this section we give additional definitions and review some known facts on cubillages that will be used later (for details, see [1]). Also we explain relationships between cubillages, convex orders on routes, and Manin-Schechtman orders.

#### 3.1 Cubillages

Let $n, d$ be integers with $n \geq d > 1$. A cyclic configuration of size $n$ and dimension $d$ is meant to be an ordered set $\Xi$ of $n$ vectors $\xi_i = (\xi_i(1), \ldots, \xi_i(d)) \in \mathbb{R}^d$, $i = 1, \ldots, n$, satisfying

(3.1) (a) $\xi_i(1) = 1$ for each $i$, and

(b) for the $d \times n$ matrix $A$ formed by $\xi_1, \ldots, \xi_n$ as columns (in this order), any flag minor of $A$ is positive.

A typical sample of such a $\Xi$ is generated by the Veronese curve; namely, take reals $t_1 < t_2 < \cdots < t_n$ and assign $\xi_i := \xi_i(t_i)$, where $\xi(t) = (1, t, t^2, \ldots, t^{d-1})$.

**Definitions.** The zonotope $Z = Z(\Xi)$ generated by $\Xi$ is the Minkowski sum of line segments $[0, \xi_1], \ldots, [0, \xi_n]$. A cubillage is a subdivision $Q$ of $Z$ into $d$-dimensional parallelotopes such that any two either are disjoint or share a face, and each face of the boundary of $Z$ is contained in some of these parallelotopes. For brevity, we refer to these parallelotopes as cubes.

(The term “cubillage” appeared in [3]; this is also known as “fine zonotopal tiling” in the literature.) When $n, d$ are fixed, the choice of one or another cyclic configuration $\Xi$ (subject to (3.1)) does not matter in essence, and we unify notation $Z(n, d)$ for $Z(\Xi)$, referring to it as the cyclic zonotope of dimension $d$ having $n$ colors.

Each subset $X \subseteq [n]$ naturally corresponds to the point $\sum_{i \in X} \xi_i$ in $Z(n, d)$. (We may assume, w.l.o.g., that all 0,1-combinations of vectors $\xi_i$ are different.)

Depending on the context, we may think of a cubillage $Q$ on $Z(n, d)$ in two ways: either as a set of $d$-dimensional cubes (and write $C \in Q$ for a cube $C$) or as a polyhedral complex. The 0-, 1-, and $(d - 1)$-dimensional faces of $Q$ are called vertices, edges, and
facets, respectively. Each vertex is identified with a subset of \([n]\). In turn, each edge \(e\) is a parallel translation of some segment \([0, \xi_i]\); we say that \(e\) has color \(i\). A face \(F\) of \(Q\) can be denoted as \((X|T)\), where \(X \subset [n]\) is the bottommost vertex, and \(T \subset [n]\) is the set of colors of edges called the type of \(F\) (note that \(X \cap T = \emptyset\) always holds). An important fact is that

\[(3.2)\] a cubillage \(Q\) in \(Z(n, d)\) has exactly \(\binom{n}{d}\) cubes, and the types \(T\) of all cubes \((X|T)\) in \(Q\) are different.

Every cubillage \(Q\) can be restored from its vertex set \(V_Q\) (since \((X|T)\) is shown to be a face of \(Q\) if and only if \(V_Q\) contains the vertices of the form \(X \cup S\) for all \(S \subseteq T\)).

3.2 Membranes, pies, contractions, expansions, and tunnels.

Certain subcomplexes in a cubillage are of importance to us. To define them, consider

\[
\text{the projection } \pi : \mathbb{R}^d \to \mathbb{R}^{d-1} \text{ given by } x = (x(1), \ldots, x(d)) \mapsto (x(1), \ldots, x(d-1)) =: \pi(x) \text{ for } x \in \mathbb{R}^d. \]

By (3.1), if a set \(\Xi\) of vectors \(\xi_1, \ldots, \xi_n\) forms a cyclic configuration in \(\mathbb{R}^d\), then the set \(\Xi'\) of their projections \(\xi'_i := \pi(\xi_i), i = 1, \ldots, n\), is a cyclic configuration in \(\mathbb{R}^{d-1}\). So \(Z(\Xi') = \pi(Z(\Xi))\), and we may liberally say that \(\pi\) projects the zonotope \(Z(n, d)\) onto \(Z(n, d-1)\).

For a closed subset \(U\) of points in \(Z(n, d)\), let \(U^{\text{fr}} (U^{\text{rear}})\) denote the subset of \(U\) “seen” in the direction of the last, \(d\)-th, coordinate vector \(e_d\) (resp. \(-e_d\)), i.e., formed by the points \(x \in U\) such that there is no \(y \in U\) with \(\pi(y) = \pi(x)\) and \(y(d) < x(d)\) (resp. \(y(d) > x(d)\)). We call \(U^{\text{fr}}\) the front side, and \(U^{\text{rear}}\) the rear side of \(U\).

Membranes. A membrane of a cubillage \(Q\) on \(Z(n, d)\) is a subcomplex \(M\) of \(Q\) such that \(\pi\) homeomorphically projects \(M\) (regarded as a subset of \(\mathbb{R}^d\)) on \(Z(n, d-1)\).

Then each facet of \(Q\) occurring in \(M\) is projected to a cube of dimension \(d-1\) in \(Z(n, d-1)\) and these cubes form a cubillage \(Q'\) on \(Z(n, d-1)\), denoted as \(\pi(M)\).

Sometimes it is useful to deal with a membrane \(M\) in the zonotope \(Z = Z(n, d)\) without specifying a cubillage on \(Z\) to which \(M\) belongs. In this case, \(M\) is meant to be a \((d-1)\)-dimensional polyhedral complex lying in \(Z\) whose vertex set consists of subsets of \([n]\) (regarded as points in \(\mathbb{R}^d\)) and corresponds to the vertex set of some cubillage \(Q'\) on \(Z(n, d-1)\). Equivalently, the projection \(\pi\) gives an isomorphism between \(M\) and \(Q'\). We call such an \(M\) an (abstract) membrane in \(Z\) and denote it as \(M_Q\). Both notions of membranes are “consistent” since one shows that for any (abstract) membrane \(M\) in \(Z\), there exists a cubillage on \(Z\) containing \(M\) (see e.g. [4 Sect. 6]).

Two membranes in \(Z\) are of an especial interest. These are the front side \(Z^{\text{fr}}\) and the rear side \(Z^{\text{rear}}\) of \(Z\). Their projections \(\pi(Z^{\text{fr}})\) and \(\pi(Z^{\text{rear}})\) (regarded as complexes) are called the standard and anti-standard cubillages on \(Z(n, d-1)\), respectively. Such cubillages in dimension 2 (viz. rhombus tilings) with \(n = 4\) are drawn in Fig. 2.

In particular, if \(n = d\), then \(Z\) is nothing else than the \(d\)-dimensional cube \((\emptyset \mid [d])\), and there are exactly two membranes in \(Z\), namely, \(Z^{\text{fr}}\) and \(Z^{\text{rear}}\).

A membrane \(M\) separates \(Z\) into two (closed) parts \(Z^+_M\) and \(Z^-_M\), where \(M\) is the rear side of the former and the front side of the latter (so the former contains \(Z^{\text{fr}}\) and the latter does \(Z^{\text{rear}}\)). Accordingly, a cubillage \(Q\) containing \(M\) is separated into two subcubillages \(Q^+_M\) and \(Q^-_M\) lying in \(Z^-_M\) and \(Z^+_M\) respectively.
Expansions. Given a membrane $M$ in a cubillage $Q$ on $Z = Z(n, d)$, one can apply an expansion operation which transforms $Q$ into a cubillage $Q'$ on $Z(n+1, d)$. To do so, we add one more “color” $\alpha$ which is regarded as either larger than $n$ (denoted as $n < \alpha$) or smaller than 1 (denoted as $\alpha < 1$). Accordingly the current cyclic configuration $\Xi$ is extended by adding a new vector $\xi_{\alpha}$; this gives a new cyclic configuration $\Xi'$ (subject to $\Xi_i \simeq \Xi_{i+1}$) of the form either $(\xi_1, \ldots, \xi_n, \xi_{\alpha})$ or $(\xi_\alpha, \xi_1, \ldots, \xi_n)$. When $\alpha > n$, the part $Z^n_M$ of $Z$ is shifted by $\xi_{\alpha}$ and the gap between $M$ and $M + \xi_{\alpha}$ is filled by cubes of the form $F + [0, \xi_{\alpha}] = (X|T) \cup \alpha$, where $F = (X|T)$ ranges over all facets of $Q$ contained in $M$. In turn, when $\alpha < 1$, we move by $\xi_{\alpha}$ the part $Z^n_M$, filling the gap between $M$ and $M + \xi_{\alpha}$ in a similar way. In both cases, the resulting complex is a correct cubillage on $Z(\Xi') \simeq Z(n+1, d)$, called the extension of $Q$ by color $\alpha$ using the membrane $M$.

(See e.g. [1, Sect. 4].)

Contraction of pies. Consider a cubillage $Q$ on $Z = Z(n, d)$ and fix a color $i \in [n]$. The set (or union) of $d$-dimensional cubes whose types contain color $i$ is called the $i$-pie in $Q$ and denoted by $\Pi_i = \Pi_{Q, i}$. Each cube $C = (X|T) \in \Pi_i$ is viewed as $F + [0, \xi_i]$, where $F$ is the facet $(X|T - i)$; let $F_i$ be the union of these facets. This $F_i$ is a subcomplex of dimension $d - 1$ in $Q$ whose projection in the direction $\xi_i$ is injective, and the removal from $Z$ the “interior” of $\Pi_i$ (formed by all points except for those in $F$ and $F + \xi_i$) results in the set consisting of two connected components $Z_i^-$ and $Z_i^+$, where the former contains $F_i$ and the latter does $F_i + \xi_i$. Moreover, shifting the part $Z_i^+$ by $-\xi_i$ gives the zonotope $Z(\Xi - \xi_i) \simeq Z(n-1, d)$ filled by the cubillage obtained from $Q - \Pi_i$ by shifting the cubes in $Z_i^+$ by $-\xi_i$. We say that this cubillage is obtained from $Q$ by contracting $\Pi_i$ (or color $i$), and denote it as $Q/i$. (See e.g. [1] Sects. 3,4.)

(For disjoint subsets $A$ and $\{a, \ldots, b\}$ of $[n]$, we may use the abbreviated notation $Aa \ldots b$ for $A \cup \{a, \ldots, b\}$, and write $A - c$ for $A - \{c\}$ when $c \in A$.)

Tunnels. For a subset $S \subseteq [n]$ of colors of size $|S| = d - 1$, the $S$-tunnel of a cubillage $Q$ on $Z = Z(n, d)$ is the set $T(S)$ of cubes $(X|T)$ of $Q$ whose type $T$ includes $S$ (i.e., $T = S \cup i$ where $i$ ranges over the colors in $[n] - S$). In fact, $T(S)$ is shown to form a sequence $C_1, C_2, \ldots, C_k$ (with $k = n - d + 1$) such that $C^{fr}_1$ ($C^{rear}_k$) lies in the front side $Z^{fr}$ (resp. rear side $Z^{rear}$) of $Z$, and $C^{rear}_i = C^{fr}_{i+1}$ for $i = 1, \ldots, k - 1$.

3.3 Natural order on cubes and inversions. For two cubes $C, C'$ of a cubillage $Q$, if the rear side $C^{rear}$ of $C$ and the front side $C'^{fr}$ of $C'$ share a facet, then we say that $C$ immediately precedes $C'$. A known fact (see e.g. [1] Sect. 9) is that

![Figure 2: left: standard tiling; right: anti-standard tiling](image-url)
• the directed graph whose vertices are the cubes of a cubillage $Q$ and whose edges are the pairs $(C, C')$ such that $C$ immediately precedes $C'$ is acyclic.

This determines a partial order on the set of cubes of $Q$, called the natural (or shadow) order on $Q$; we denote it as $(Q, \preceq)$, or $\preceq_Q$.

One important application of this order is as follows. For a membrane $M$ of $Q$ and two cubes $C, C' \in Q$ such that $C$ immediately precedes $C'$, one easily shows that if $C' \in Q^+_M$ (i.e. $C'$ "lies before" $M$), then so does $C$. This implies that $Q^+_M$ is an ideal of $\preceq_Q$, and (see e.g. [1, Sect. 11]):

(3.3) the set $\mathcal{M}(Q)$ of membranes of a cubillage $Q$ on $Z = Z(n, d)$ is a distributive lattice in which for $M, M' \in \mathcal{M}(Q)$, the membranes $M \land M'$ and $M \lor M'$ satisfy

$$Q^-_{M \land M'} = Q^-_M \cap Q^-_{M'},$$

and

$$Q^-_{M \lor M'} = Q^-_M \cap Q^-_{M'};$$

the minimal and maximal elements of this lattice are the membranes $Z^{\text{fr}}$ and $Z^{\text{rear}}$, respectively.

Another important fact is that for an (abstract) membrane $M$ in $Z = Z(n, d)$, the set of types $T$ of the cubes $C = (X \mid T)$ in $Q^+_M$ does not depend on the cubillage $Q$ on $Z$ that contains $M$ (see e.g. [3, 6]). This subset of $\binom{[n]}{d}$ is called the set of inversions of $M$ and denoted by $\text{Inv}(M)$. Under the projection $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ as above, the membrane $M$ turns into a cubillage $Q'$ on the zonotope $Z(n, d-1)$, and we define the set Inv($Q'$) of inversions of $Q' = \pi(M)$ to be equal to $\text{Inv}(M)$ as well.

Ziegler found a criterion on a collection of $(d+1)$-element subsets of $[n]$ that can be represented as $\text{Inv}(Q)$ for some cubillage $Q$ on $Z(n, d)$, namely:

(3.4) [1] $C \subseteq \binom{[n]}{d+1}$ is realized as the set of inversions of a cubillage on $Z(n, d)$ if and only if for any $(d+2)$-element subset $P \in \binom{[n]}{d+2}$ and for the family $\mathcal{F}(P)$ consisting of $(d+1)$-element subsets of $P$ which are ranged lexicographically, the intersection $C \cap \mathcal{F}(P)$ forms a beginning or ending part of $\mathcal{F}(P)$.

3.4 Capsids. For a cubillage $Q$ on $Z = Z(n, d)$ and colors $p_1 < \cdots < p_{d+1}$ in $[n]$, let $T = \{p_1, \ldots, p_{d+1}\}$ and consider the set of cubes $C_i \in Q$ $(i = 1, \ldots, d+1)$ whose type is equal to $T - p_i$. It turns out that these cubes are comparable under the natural order on $Q$ (defined in Sect. [1, 13]) and, moreover, exactly two cases are possible:

(3.5) either (i) $C_{d+1} \prec C_d \prec \cdots < C_1$, or (ii) $C_1 < C_2 < \cdots < C_{d+1}$

(see e.g. [1, Sect. 10]). This follows from two facts: (a) the relations in (3.5) are valid when no two cubes among $C_1, \ldots, C_{d+1}$ are separated by any pie in $Q$; and (b) if $\Pi$ is an $i$-pie in $Q$ separating cubes $C, C' \in Q$ and if in the cubillage obtained from $Q$ by contracting $\Pi$ (the images of) these cubes are ordered as $C \prec C'$, then the same relation is true in the original $Q$. (Property (a) follows by considering the $(d+1)$ dimensional cube $D$ with edge colors $p_1, \ldots, p_{d+1}$. This cube has exactly two membranes, namely, $D^{\text{fr}}$ and $D^{\text{rear}}$. Under the projection $\mathbb{R}^{d+1} \to \mathbb{R}^d$, the facets of $D^{\text{fr}}$ turn into cubes $C_1, \ldots, C_{d+1}$ as in (i) of (3.3), whereas the facets of $D^{\text{rear}}$ into those in (ii).)

By terminology of [1], the set (or union) $\mathcal{D}$ of cubes $C_1, \ldots, C_{d+1}$ as above is called a capsid in $Q$. In case (i) (resp. (ii)), we say that $\mathcal{D}$ has the standard filling (resp. the anti-standard filling). When no two cubes of $\mathcal{D}$ are separated by a pie, the capsid $\mathcal{D}$ is called dense; otherwise it is called loose.
Definition. Suppose that a cubillage \( Q \) on \( Z = Z(n, d) \) contains a dense capsid \( \mathcal{D} \) having the standard filling \( \mathcal{D}^{\text{st}} \) (anti-standard filling \( \mathcal{D}^{\text{ant}} \)). The replacement of \( \mathcal{D}^{\text{st}} \) by \( \mathcal{D}^{\text{ant}} \) (resp. \( \mathcal{D}^{\text{ant}} \) by \( \mathcal{D}^{\text{st}} \)) is called the raising (resp. lowering) flip in \( Q \) using \( \mathcal{D} \).

(The resulting set of cubes is again a cubillage on \( Z \).)

The following property is of importance (cf. [1, Th. D.1]):

\[ (3.6) \text{the directed graph } \Gamma_{n,d} \text{ whose vertices are cubillages in } Z(n, d) \text{ and whose edges are the pairs } (Q, Q') \text{ such that } Q' \text{ is obtained from } Q \text{ by a raising flip using some dense capsid is acyclic and has one minimal and one maximal vertices, which are the standard cubillage } Q_{n,d}^{\text{st}} \text{ and the anti-standard cubillage } Q_{n,d}^{\text{ant}} \text{ on } Z(n, d), \text{ respectively.} \]

As a consequence, \( \Gamma_{n,d} \) determines a poset with the minimal element \( Q_{n,d}^{\text{st}} \) and the maximal element \( Q_{n,d}^{\text{ant}} \).

3.5 Relation to convex and Manin-Schechtman orders. Consider a route (maximal path) \( R = v_1 v_2 \ldots v_n \) in an acyclic finite digraph \( G \). It is convenient for us to identify a vertex \( v_i \) with “color” \( i \); so the sequence of vertices of \( R \) is 1, 2, \ldots, \( n \).

We associate with a \( k \)-cortege \( p = (p_1, \ldots, p_k) \) of nontrivial paths \( p_i \in \mathcal{P}(R) \) in \( R \) the \((k+1)\)-tuple \( T(p) \) of their endvertices \( t_1 < t_2 < \cdots < t_{k+1} \), where

\[
    t_1 = \text{tail}(p_1), \quad t_i = \text{tail}(p_{i-1}) = \text{head}(p_i), \quad i = 2, \ldots, k, \quad \text{and} \quad t_{k+1} = \text{head}(p_k). \tag{3.7}
\]

Then \( p \mapsto T(p) \) gives a bijection between \( \mathcal{P}_k(R) \) and \( \binom{[n]}{k+1} \).

Let \( d \) be an integer \( \geq 2 \). Then for a \((d-1)\)-cortege \( p \), the \( d \)-element set \( T(p) \) can be regarded as the type (set of colors) of an “abstract” \( d \)-dimensional cube in the zonotope \( Z = Z(n, d) \) (whose generating vectors correspond to the vertices of \( R \)).

Now consider a \( d \)-cortege \( p = (p_1, \ldots, p_d) \) in \( R \) and let \( S(p) = (s^p_1, \ldots, s^p_{d+1}) \) be the sequence of \((d-1)\)-corteges defined in (2.3). Then \( T(p) \) is the \((d+1)\)-element set of colors (viz. endvertices) occurring in \( S(p) \). Moreover, one can see that for \( i = 1, \ldots, d+1 \), the set \( T(s^p_i) \) consists of all colors from \( T(p) \) except for \( i \), namely, this is \( 1 \ldots (i-1)(i+1) \ldots (d+1) \). Therefore, the \( d \)-tuples

\[
    T(s^p_{d+1}), T(s^p_d), \ldots, T(s^p_1) \tag{3.8}
\]

follow lexicographically.

A binary relation \( \prec \) on \( \mathcal{P}_{d-1}(R) \) induces, in a natural way, a one on the tuples in \( \binom{[n]}{d} \), by setting \( T(p) \prec^{\circ} T(p') \) if \( p, p' \in \mathcal{P}_{d-1}(R) \) and \( p \prec p' \). This leads to an equivalent definition of convex orders when dealing with a route, namely:

\[ (3.9) \text{A linear order } \prec \text{ on } \mathcal{P}_{d-1}(R) \text{ is convex if and only if the induced order } \prec^{\circ} \text{ on } \binom{[n]}{d} \text{ is such that for any } d \text{-cortege } p \in \mathcal{P}_d(R), \text{ the corresponding sequence in } (3.8) \text{ is ordered by } \prec^{\circ} \text{ either lexicographically or anti-lexicographically.} \]

This in turn is equivalent to saying that a linear order on \( \mathcal{P}_{d-1}(R) \) is convex if and only if the induced linear order \( \prec^{\circ} \) on \( \binom{[n]}{d} \) satisfies Manin-Schechtman’s condition from [4] (where the term a higher Bruhat order of degree \((n, d)\) appeared), namely: for each “packet” \( P \in \binom{[n]}{d+1} \), its “family” \( \mathcal{F}(P) \) is ordered by \( \prec^{\circ} \) either lexicographically
or anti-lexicographically. (Here \( \mathcal{F}(P) \) is the sequence formed by the \( d \)-element subsets of \( P \) which are intrinsically ranged lexicographically.)

Kapranov and Voevodsky \[3\] and Ziegler \[6\] gave a nice geometric interpretation of Manin-Schechtman’s concept. More precisely, the equivalence classes of Bruhat orders of degree \((n, d)\) turn out to be bijective to the cubillages on \( \mathbb{Z}^n(n, d) \), and under this bijection, the sets of inversions for the cubillage and the order are the same. (An inversion of a cubillage is defined in Sect 3.3, and an inversion for an order \( \prec \) is meant to be a packet \( P \in \left( \begin{array}{c} n \\ d + 1 \end{array} \right) \) for which the family \( \mathcal{F}(P) \) is ordered by \( \prec \) anti-lexicographically; two orders are equivalent if their sets of inversions coincide.)

This correspondence together with (3.9) implies the following key property which will be used in what follows:

(3.10) for a route \( R \) with \( n \) vertices, the equivalence classes of convex orders \( \prec \) on \( \mathcal{P}_{d-1}(R) \) are bijective to the cubillages \( Q \) on \( \mathbb{Z}^n(n, d) \), and under this bijection, the inversions for \( Q \) correspond to the \( d \)-corteges \( p \) of anti-standard type (i.e., satisfying (2.5)) in \( R \).

The relationship between convex orders and cubillages is illustrated in Fig. 1 drawn in Sect 2. Here the bold rhombi correspond to the paths \( p_2, p_1 \ast p_2, p_1 \) (related to the 2-cortege \( (p_1, p_2) \)) in the rhombus tiling \( T \) of the middle fragment, and to the paths \( p_2 \ast r, R', p_1 \) (related to the 2-cortege \( (p_1, p_2 \ast r) \)) in the rhombus tiling \( T' \) of the right fragment. Note that the capsid with the anti-standard filling that is formed by the former three rhombi is dense in \( T \) and loose in \( T' \). The arrows indicate the natural orders on the rhombi in \( T \) and \( T' \); this corresponds to the convex order \( \prec \) on the paths in \( G \).

4 Proof of Theorem 2.1

Let \( \prec \) be a convex order on \( \mathcal{P}_{d-1}(G) \), and suppose that there is a \( d \)-cortege \( p \in \mathcal{P}_d(G) \) of anti-standard type for \( \prec \). We have to show that \( \prec \) admits a lowering flip (using some dense \( d \)-cortege). [A proof that \( \prec \) admits a raising flip when the set of \( d \)-corteges of standard type is nonempty is symmetric.]

Fix a route \( R \) in \( G \) containing \( p \). As is explained in Sect. 2.5, the restriction of \( \prec \) to \( R \) corresponds to a cubillage \( Q = Q^{R,\prec} \) on \( \mathbb{Z}(n, d) \) (where \( n \) is the number of vertices of \( R \)). Then by reasonings in Sect. 3.3, \( Q \) has a dense capsid \( \mathcal{D} \) having the anti-standard filling, or an anti-capsid, to say for short. Note that if the cubillage \( Q' = Q^{R',\prec} \) for another route \( R' \) in \( G \) meets a cube of \( \mathcal{D} \), then \( Q' \) contains either all cubes of \( \mathcal{D} \) or exactly one cube from \( \mathcal{D} \) (taking into account that any two \((d-1)\)-corteges within a \( d \)-cortege determine this \( d \)-cortege).

Therefore, if \( R \) is the only route of \( G \) such that \( Q^{R,\prec} \) contains \( \mathcal{D} \), then one can make a lowering flip for \((G, \prec)\), and we are done.

So we may assume that the set \( \mathcal{S}(\mathcal{D}) \) of cubillages \( Q^{R',\prec} \) containing all cubes of \( \mathcal{D} \) is of cardinality at least two. A nice case arises when \( \mathcal{D} \) is dense in all cubillages of \( \mathcal{S}(\mathcal{D}) \), for then the lowering flip using \( \mathcal{D} \) is well-defined in \( G \). (Note that \( \mathcal{D} \) has the anti-standard filling everywhere.)
Thus, we come to the situation when there is some cubillage in $S(\mathcal{D})$ for which $\mathcal{D}$ is loose (while $\mathcal{D}$ is dense in $Q$). Let $C_1 \prec \cdots \prec C_{d+1}$ be the sequence of cubes in $\mathcal{D}$ as in (3.5)(ii) (hereinafter we identify $\prec$ and $\prec^o$). Our goal is to show that

\begin{equation}
\text{(4.1)} \quad \text{for a cubillage } Q' \in S(\mathcal{D}) \text{ in which } \mathcal{D} \text{ is loose, there exists an anti-capsid } \mathcal{B} \text{ in } Q' \text{ such that } C_1 \preceq H_1, H_{d+1} \preceq C_{d+1} \text{ and } \{H_1, H_{d+1}\} \neq \{C_1, C_{d+1}\}, \text{ where } H_1 \prec \cdots \prec H_{d+1} \text{ is the sequence of cubes in } \mathcal{B}.
\end{equation}

Subject to validity of (4.1), the proof of Theorem 2.1 can be finished as follows. If there is a route $R''$ such that the cubillage $Q'' = Q R'', \prec^o$ contains $\mathcal{B}$ as above and this $\mathcal{B}$ is loose in $Q''$, then by (4.1) applied to $(Q'', \mathcal{B})$, there exists an anti-capsid $\mathcal{B}'$ in $Q''$ with $H_1 \preceq H'_1, H'_{d+1} \preceq H_{d+1}$ and $\{H'_1, H'_{d+1}\} \neq \{H_1, H_{d+1}\}$, where $H'_1 \prec \cdots \prec H'_{d+1}$ is the sequence of cubes in $\mathcal{B}'$. And so on. Due to the order inequalities, in the process we never return to a capsid that has been met before. Therefore, the process is finite, and we eventually obtain an anti-capsid which is dense in all cubillages (determined by routes in $G$) containing it, whence the result follows.

In order to show (4.1), we use the following observation. For $Q$ and $\mathcal{D}$ as above, let $a$ and $b$ be the minimal and maximal colors occurring in $\mathcal{D}$, respectively. Then

\begin{equation}
\text{(4.2)} \quad \text{if } \mathcal{D} \text{ is dense in } Q \text{ and loose in } Q', \text{ and if } \Pi \text{ is an } i\text{-pie in } Q' \text{ separating } \mathcal{D}, \text{ then the color } i \text{ is not in the segment } [a, b] \text{ (where pies are defined in Sect. 3.2)}.
\end{equation}

Indeed, let $\widehat{Q}$ and $\widehat{Q'}$ be the restrictions of $Q$ and $Q'$ (respectively) to the color set $[a, b]$, in the sense that they are obtained from $Q$ and $Q'$ by contracting all $j$-pies with $j \notin [a, b]$ (see Sect. 3.2). Since both $\widehat{Q}$ and $\widehat{Q'}$ correspond to the same path in $G$ (which connects the vertices $a$ and $b$ in $G$), we obtain $\widehat{Q} = \widehat{Q'}$. Then $\mathcal{D}$ is dense in $\widehat{Q'}$. This implies that no $i$-pie with $i \in [a, b]$ can separate $\mathcal{D}$ in $Q'$, as required.

In view of (4.2), the desired assertion (4.1) will follow from the next two propositions. They deal with a $d$-dimensional zonotope $Z$ with $d + 2$ colors, which for convenience we denote as $T = (1 < 2 < \cdots < d + 1)$ and $\alpha$. We consider a cubillage $Q$ on $Z$ such that the contraction of the pie $\Pi_\alpha$ in $Q$ results in the cubillage which is the dense anti-capsid $\mathcal{D}$ of type $T$. Let $C_1, \ldots, C_{d+1}$ be the cubes in $\mathcal{D}$, where each $C_i$ has type $T - i$. We know (see (3.5)(ii)) that under the natural (or “shadow”) order $\prec$, these cubes are ordered as

\begin{equation}
C_1 \prec C_2 \prec \cdots \prec C_{d+1}.
\end{equation}

Let $C'_i$ denote the image of the cube $C_i$ in $Q'$ (having the same type $T - i$). By explanations in Sect. 3.2, the set $\mathcal{D}' = \{C'_1, \ldots, C'_{d+1}\}$ forms an anti-capsid in $Q$ in which the cubes have a similar order

\begin{equation}
C'_1 \prec C'_2 \prec \cdots \prec C'_{d+1}.
\end{equation}

(We use notation with primes to differ objects in $Q'$ from their counterparts in $Q$.) Assume that $\mathcal{D}'$ is loose.

**Proposition 4.1** Suppose that color $\alpha$ is greater than $d + 1$. Then there exists $q \in T$ such that the set of cubes of $Q$ whose colors are in $(T - q) \cup \{\alpha\}$ forms a (dense or loose) anti-capsid $\mathcal{B}$ satisfying

\begin{equation}
C'_1 \prec C \quad \text{and} \quad D = C'_{d+1},
\end{equation}

where $C$ and $D$ are the minimal and maximal cubes in $\mathcal{B}$, respectively.
Proof In Manin-Schechtman’s packet $P = (1 < 2 < \cdots < d + 1 < \alpha)$, consider the family $\mathcal{F}(P)$ consisting of the $(d + 1)$-element subsets ranged lexicographically:

$$12 \cdots (d + 1), \ 12 \cdots d\alpha, \ 12 \cdots (d - 1)(d + 1)\alpha, \ \ldots, \ 23 \cdots (d + 1)\alpha \quad (4.5)$$

(for definitions, see Sect. 3.4). Each term in this sequence is the type of a capsid in $Q$. By Ziegler’s theorem (see (3.4)), the anti-capsids are related to a beginning or ending part in (4.5). Furthermore, the first term is just the type $T$ of $\mathfrak{D}$, and there must be at least one more anti-capsid in $Q$ (since $Q$ has a loose anti-capsid, namely, $\mathfrak{D}'$, which implies the existence of a dense anti-capsid in $Q$). Therefore, the second term $T' = 12 \cdots d\alpha$ in (4.5) must be the type of a (dense or loose) anti-capsid $\mathfrak{B}$ in $Q$ as well. We assert that $\mathfrak{B}$ (along with $q := d + 1$) satisfies (4.4).

To show this, note that $Q$ is obtained from $\mathfrak{D}$ by the expansion operation adding color $\alpha$ w.r.t. some membrane $M$ in $\mathfrak{D}$ (regarded as a cubillage in the zonotope with colors $1, \ldots, d + 1$). (Conversely, $M$ is the subcomplex obtained by contracting the pie $\Pi_\alpha$ in $Q$.) For explanations, see Sect. 3.2. The membrane $M$ divides the sequence (4.3) into two nonempty parts of the form $\{C_1, \ldots, C_j\}$ and $\{C_{j+1}, \ldots, C_{d+1}\}$ for some $1 < j \leq d$.

Under the expansion, the cubes in the first part do not move ($C_1' = C_i$ for $i \leq j$), whereas those in the second are shifted along the generating vector corresponding to $\alpha$. (Note that the cubes in $Q$ can have only three possible levels: lower, upper and middle, i.e., the cube contains the bottommost, topmost, and none of these vertices of $Z$, respectively).

The anti-capsid $\mathfrak{B}$ (having type $12 \cdots d\alpha$) is formed by the cubes $H_1 < H_2 < \cdots < H_{d+1}$ of types $(2 \cdots d\alpha), (13 \cdots d\alpha), \ldots, (12 \cdots d)$, respectively. Since both $C_{d+1}'$ and $H_{d+1}$ have the same type, namely, $12 \cdots d$, they must coincide.

Now let us compare $H_1$ and $C_1'$. Both cubes have facets of type $2 \cdots d$; they belong to the tunnel $\tau$ of this type (for definition, see Sect. 3.2). The cube $C_1$ in $\mathfrak{D}$ has two facets of type $2 \cdots d$, of which one, $F$, say, lies on the front side $\mathfrak{D}^\text{fr}$ of $\mathfrak{D}$ (which follows from the fact that there is no cube in $\mathfrak{D}$ “before” $C_1$, by (4.3)). Then $F$ lies on the front side $Z^\text{fr}$ of $Z$ as well (since $C_1$ lies “before” the membrane $M$). Therefore, $C_1'$ is the first cube in the tunnel $\tau$. This implies that $C_1' < H_1$, and we obtain (4.4) with $C = H_1$ and $D = H_{d+1}$, as required.

**Proposition 4.2** Suppose that color $\alpha$ is smaller than color 1. Then there exists $q \in T$ such that the set of cubes of $Q$ having the colors in $(T - q) \cup \{\alpha\}$ forms an anti-capsid $\mathfrak{B}$ satisfying

$$C_1' = C \quad \text{and} \quad D < C_{d+1}', \quad (4.6)$$

where $C$ and $D$ are the minimal and maximal cubes in $\mathfrak{B}$, respectively.

To prove this, reverse the colors, obtaining the order $d + 1 < d \cdots < 1 < \alpha$, and make the corresponding mirror reflection of $Q$. Then the result follows from the previous proposition.

Now we finish the proof of assertion (4.1) (yielding Theorem 2.1) by using induction on the number $k(Q')$ of pies separating the capsid $\mathfrak{D}$ figured in this assertion, as follows. By (4.2), each of these pies has color outside $[a, b]$. When $k(Q') = 1$, the assertion is immediate from Propositions 4.1 and 4.1. And when $k(Q') \geq 2$, contract one pie
separating $\mathcal{D}$. This gives a cubillage $Q''$ containing $\mathcal{D}$ for which $k(Q'') = k(Q') - 1$ (for details concerning contractions of pies, see Sect 3.2). By induction, $Q''$ has an anti-capsid $\mathcal{B}$ with $C_1 \preceq H_1$, $H_{d+1} \preceq C_{d+1}$ and $\{H_1, H_{d+1}\} \neq \{C_1, C_{d+1}\}$ (where $H_1 < \cdots < H_{d+1}$ are the cubes in $\mathcal{B}$), and its image in $Q'$ is as required in (4.1).

This completes the proof of Theorem 2.1.

**Remark.** When $d = 2$, one can give a simpler proof of Theorem 2.1. More precisely, in this case we deal with a rhombus tiling $T$ on the zonogon $Z(n, 2)$, and consider a loose “anti-capsid” $\mathcal{D}$ in $T$ formed by three rhombi $C_3, C_2, C_1$ of types $ab, ac, bc$. If the colors $a, b, c$ are subject to $a < b < c$, then the natural order on the rhombi is viewed as $C_1 \preceq C_2 \preceq C_3$. Take in $T$ the pies (“strips”) $\Pi_a, \Pi_b, \Pi_c$, and consider the subtiling $I(a, b, c)$ of $T$ surrounded by the parts of these pies connecting $C_1, C_2, C_3$ (including these parts as well), called the island for $a, b, c$. One easily shows that any rhombus $\rho$ in $I(a, b, c)$ satisfies $C_1 \preceq \rho \preceq C_3$. We assert that there exists another anti-capsid $\mathcal{D}'$ whose island is strictly included in $I(a, b, c)$ (whence Theorem 2.1 for $d = 2$ easily follows). To see this, consider an arbitrary pie $\Pi = \Pi_d$ with $d \neq a, b, c$ which meets $I(a, b, c)$. This $\Pi$ intersects two pies among $\Pi_a, \Pi_b, \Pi_c$ and separates some rhombus $C_i$ from the other two rhombi in $\mathcal{D}$. Then the capsid $\mathcal{D}'$ using the color $d$ and the pair of colors in $C_i$ is as required. Three possible cases of $\Pi$ are illustrated in Fig. 3.

![Figure 3: Three variants of pies $\Pi$ separating rhombi of types $bc, ac, ab$ for $a < b < c$.](image)

It should be noted that a similar method for $d \geq 3$ does not work.

## 5 Relations between convex orders in neighboring dimensions

Strictly speaking, in the proof of Theorem 2.1 we default assumed that the set of convex orders on $\mathcal{P}_{d-1}(G)$ is nonempty (as before, $G = (V, E)$ is a finite acyclic digraph and $d \geq 2$). This can be shown by an explicit construction, as follows.

We first label the vertices $v \in V$ by different integers $\ell(v)$ from 1 through $n := |V|$ so that $\ell(u) < \ell(v)$ hold for each edge $(u, v) \in E$ (such a labeling is called a topological order; it exists and can be found in $O(|V| + |E|)$ time, see e.g. [5, Sect. 6.3]). Now order the $(d - 1)$-corteges in $G$ lexicographically relative to $\ell$. Namely, associate with a $(d - 1)$-corteg $p$ the corresponding sequence of endvertices (“colors”) $T(p)$ defined as in (3.7), and for $p, q \in \mathcal{P}_{d-1}(G)$, denote $p \prec q$ if $\ell(T(p))$ is lexicographically less than $\ell(T(q))$ (where $\ell(T(p'))$ means the $d$-tuple of labels for the elements of $T(p')$).
that there may exist several different \((d - 1)\)-corteges \(p\) that have equal types \(T(p)\) and, therefore, equal tuples \(\ell(T(p))\) (such corteges lie in different routes). We assign an arbitrary linear order within each collection of such corteges (if any).

It is an easy exercise to check that for any \(d\)-cortège \(p \in \mathcal{P}_d(G)\), the sequence \(S(p)\) of \((d - 1)\)-corteges \(s_{r_1}^p, \ldots, s_{r_{d+1}}^p\) (as in \((2.3)\)) obeys the relations in \((2.3)\). Therefore, \(\prec\) is a convex order in which each \(d\)-cortège is of standard type. This implies that \(\mathcal{O}_{d-1}(G)\) is non-null, and \(\prec\) represents the minimal class \(\zeta_{\text{min}}\) in it.

Note that starting with a labeling \(\ell\) satisfying \(\ell(u) > \ell(v)\) for each edge \((u, v) \in E\), we would generate a convex order with all \(d\)-corteges of anti-standard type, which represents the maximal class \(\zeta_{\text{max}}\) in \(\mathcal{O}_{d-1}(G)\).

Once a minimal convex order \(\prec\) (representing \(\zeta_{\text{min}}\) in \(\mathcal{O}_{d-1}(G)\)) is available, we can use Theorem \(\text{2.1}\) to construct, step by step, a sequence (maximal chain) \(\mathcal{C}\) of convex orders \(\prec = \prec_0, \prec_1, \ldots, \prec_N\), where \(\prec_i\) is obtained from \(\prec_{i-1}\) by a raising flip (using a dense \(d\)-cortège that can be explicitly found), and where \(\prec_N\) represents the maximal class \(\zeta_{\text{max}}\). All such maximal chains \(\mathcal{C}\) have the same length \(N = |\mathcal{P}_d(G)|\). Moreover, they enable us to bridge \(\mathcal{O}_{d-1}(G)\) and \(\mathcal{O}_d(G)\) (in spirit of a description of \(B(n, d + 1)\) via “maximal chains” in \(B(n, d)\)).

**Theorem 5.1** Each maximal chain \(\mathcal{C} = (\prec_0, \prec_1, \ldots, \prec_N)\) of convex orders on \(\mathcal{P}_{d-1}(G)\) determines a convex order on \(\mathcal{P}_d(G)\).

(Details of determining such an order will be seen from the proof below.)

**Proof** For \(i = 1, \ldots, N\), let \(p_i\) be the dense \(d\)-cortège involved in the raising flip \(\prec_{i-1} \mapsto \prec_i\), and \(T_i\) the type (viz. vertex/color set) for \(p_i\). For a route \(R = (V_R, E_R)\) in \(G\), let \(T^R_i\) be the sequence of indices \(i\) such that \(p_i\) is contained in \(R\), and denote by \(T^R\) the corresponding sequence of types \(T_i\) for \(i \in T^R\).

We know (cf. \((3.10)\)) that the restriction to \(\mathcal{P}_{d-1}(R)\) of each order \(\prec_i\) determines a cubillage \(Q_i\) on the \(d\)-dimensional zonotope with the color set \(V_R\). In its turn, by explanations in Sects. \(\text{3.2}\) and \(\text{3.3}\) \(Q_i\) determines a membrane \(M = M_i\) on the corresponding zonotope \(Z \simeq Z(|V_R|, d + 1)\), and the set Inv\((M_i)\) of inversions of \(M_i\) is formed by the types of those \(d\)-corteges that lie in \(R\) and are anti-standard relative to \(\prec_i\). It follows that Inv\((M_i)\) consists of \(T_j\) \(\in T^R\) with \(j \leq i\).

As a result, letting \(T^R = (\alpha(1), \ldots, \alpha(k))\), the sequence \(M^R = (Z_{\text{fr}} = M_{\alpha(0)}, M_{\alpha(1)}, \ldots, M_{\alpha(k)})\) of membranes in \(Z\) possesses the properties that

\[ Z_{\text{fr}} = Z_{M_{\alpha(0)}} \subset Z_{M_{\alpha(1)}} \subset \cdots \subset Z_{M_{\alpha(k)}} = Z, \]

and that for \(i = 1, \ldots, k\), the gap between \(Z_{M_{\alpha(i-1)}}^-\) and \(Z_{M_{\alpha(i)}}^-\) is filled by one cube \(C_{\alpha(i)} = C_{\alpha(i)}^R\) of type \(T_{\alpha(i)}\). These cubes constitute a cubillage on \(Z\), denoted as \(Q^R\).

Obviously, for the natural (“shadow”) order \(\prec_{Q^R}\) on the cubes of \(Q^R\) (see Sect. \(\text{3.3}\) if \(C_{\alpha(i)} <_{Q^R} C_{\alpha(j)}\), then there is a membrane \(M = M_{\alpha(r)} \in M^R\) separating these cubes. This implies the important property that

\[ \text{(5.1) if } C_{\alpha(i)} <_{Q^R} C_{\alpha(j)} \text{ and if } M = M_{\alpha(r)} \text{ is a membrane in } M^R \text{ such that } C_{\alpha(i)} \in Q^-_{M}, \text{ and } C_{\alpha(j)} \in Q^+_{M}, \text{ then } \alpha(i) \leq \alpha(r) < \alpha(j). \]
Using this, we now establish the desired convex order on $\mathcal{P}_d(G)$ as follows. If a $d$-cortege $p_i (i = 1, \ldots, N)$ occurs in two (or more) routes, say, $R$ and $R'$, then we identify the cubes $C_i^R$ and $C_i^{R'}$, denoting it as $C_i$. This gives a bijection between the $d$-corteges and the cubes, namely, $p_i \mapsto C_i$. Note that the natural orders in cubillages as above are agreeable, in the sense that if $C_i, C_j$ occurs in both $Q^R, Q^{R'}$ and if $C_i \prec Q^R C_j$, then (in view of $i < j$, by (5.1)) either $C_i, C_j$ are incomparable in $Q^R$ or $C_i \prec Q^{R'} C_j$.

The above reasonings show that combining the orders $\prec Q^R$ over the routes $R$ in $G$, we obtain a structure of binary relations on the whole set of cubes which does not admit directed cycles, and therefore, it can be extended to a linear order. This induces a linear order on $\mathcal{P}_d(G)$ as required in the theorem. (The convexity of this order is provided by the fact that each $(d + 1)$-cortege determines a $(d + 2)$-capsid in some cubillage $Q^R$, yielding a relation as in (3.5), with $d + 2$ rather than $d + 1$.)

A converse relation between $\mathcal{O}_d(G)$ and $\mathcal{O}_{d-1}(G)$ takes place as well.

**Theorem 5.2** Any convex order on $\mathcal{P}_d(G)$ can be represented via a maximal chain of convex orders on $\mathcal{P}_{d-1}(G)$.

**Proof** Consider a convex order $\prec'$ on $\mathcal{P}_d(G)$. For a route $R = (V_R, E_R)$ in $G$, let $Q^R$ denote the cubillage on $Z^R \simeq Z(|V_R|, d + 1)$ corresponding to the restriction of $\prec'$ to $\mathcal{P}_d(G)$.

The desired chain of convex orders on $\mathcal{P}_{d-1}(G)$ is constructed in the reverse order, starting with a maximal element $\prec N$, where $N := |\mathcal{P}_d(G)|$. At a current step, we handle a convex order $\prec_i$ on $\mathcal{P}_{d-1}(G)$ satisfying the condition that

\[(5.2)\] for each route $R$ in $G$, the (abstract) membrane $M = M_i^R$ in $Z^R$ determined by

the restriction of $\prec_i$ to $\mathcal{P}_{d-1}(R)$ is a membrane in $Q^R$.

In other words, $M$ separates the sets of cubes of $Q^R$ whose types belong to $\text{Inv}(M)$ and to $\text{(d+1)-Inv}(M)$, see Sects. 3.2 and 3.3. In particular, the membrane $M^R_N$ is the rear side of $Z^R$.

Now we construct $\prec_{i-1}$ as follows. Choose a route $R$ and a maximal cube $C$ in the part (subcomplex) $(Q^R)_M$ of $Q^R$ lying between $(Z^R)_M$ and $M = M^R_R$. (In other words, type($C$) $\in \text{Inv}(M)$ and $C$ is “pressed” to $M$, in the sense that the rear side of $C$ is entirely contained in $M$.) One may assume that such $R, C$ exist (otherwise $\prec_i$ is already a minimal convex order on $\mathcal{P}_{d-1}(G)$). Let $p^0$ be the $d$-cortege corresponding to this cube $C = C^0$. Suppose that there is another route $R'$ containing $p^0$ such that its corresponding cube (a “copy” of $C^0$) is not maximal in the part $(Q^{R'})_{M'}$, where $M' := M_i^{R'}$. Then we choose in this part a maximal cube $C^1$ greater than $C^0$ (this $C^1$ is “pressed” to $M'$). Let $p^1$ be the $d$-cortege corresponding to $C^1$. If there is a route $R''$ containing $p^1$ for which (a “copy” of) $C^1$ is not maximal in the part of $Q^{R''}$ before the corresponding membrane, then we choose a maximal cube $C^2$ greater than $C^1$ there. And so on. By the construction, the arising $d$-corteges are monotone increasing:

$p^0 \prec' p^1 \prec' p^2 \prec' \cdots$.

Therefore, the process terminates with a cortege $p^k$ such that for all routes containing $p^k$, the corresponding cube $C^k$ is maximal (“pressed” to the corresponding membranes). This is equivalent to the density of $p^k$. Then, by Theorem 2.1 we can apply to $\prec_i$ the lowering flip involving $p^k$. This just gives the desired convex order $\prec_{i-1}$
satisfying (5.2). (Here if $Q^R$ contains cube $C^k$ corresponding to $p^k$, then the membrane for $\prec_{i-1}$ is obtained from the one for $\prec_i$ by replacing the rear side of $C^k$ by its front side.) This completes the proof of the theorem.

**Remark.** Symmetrically, we can start from a minimal convex order $\prec_0$ and form a required chain for $\prec'$ step by step, by transforming a current convex order $\prec_i$ into the next convex order $\prec_{i+1}$. Also the following property can be concluded from our reasonings. Let us say that a set $\mathcal{I} \subseteq \mathcal{P}_d(G)$ is an ideal for $\prec'$ if $q \in \mathcal{I}$ and $p \prec' q$ imply $p \in \mathcal{I}$. A convex order $\prec$ on $\mathcal{P}_{d-1}(G)$ is called compatible with $\mathcal{I}$ if the set of $d$-cortages of anti-standard type for $\prec$ is exactly $\mathcal{I}$. Then the following holds: if a convex order $\prec$ on $\mathcal{P}_{d-1}(G)$ is compatible with some ideal for $\prec'$, then $\prec$ can be included into some maximal chain representing $\prec'$.

**References**

[1] V.I. Danilov, A.V. Karzanov and G.A. Koshevoy, Cubillages of cyclic zonotopes, *Uspekhi Matematicheskikh Nauk* 74 (6) (2019) 55–118, in Russian. (English Translation in *Russian Math. Surveys* 74 (6) (2019) 1013–1074.)

[2] P. Galashin and A. Postnikov, Purity and separation for oriented matroids, ArXiv:1708.01329[math.CO], 2017.

[3] M.M. Kapranov and V.A. Voevodsky, Combinatorial-geometric aspects of polycategory theory: pasting schemes and higher Bruhat orders, *Cahiers de topologie et geometrie differentielle categoriques* 32 (1) (1991) 11–28.

[4] Yu. Manin and V. Schechtman, Arrangements of hyperplanes, higher braid groups and higher Bruhat orders, in: *Algebraic Number Theory – in Honour of K. Iwasawa*, Advances Studies in Pure Math. 17, Academic Press, NY, 1989, pp. 289–308.

[5] A. Schrijver, *Combinatorial Optimization*, Vol. A, Springer, 2003.

[6] G.M. Ziegler, Higher Bruhat orders and cyclic hyperplane arrangements, *Topology* 32 (2) (1993) 259–279.