Asymptotic Analysis of Feynman Diagrams and their Maximal Cuts

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Abstract

The ASPIRE program, which is based on the Landau singularities and the method of power geometry to unveil the regions required for the evaluation of a given Feynman diagram asymptotically in a given limit, also allows for the evaluation of scaling coming from the top facets. In this work, we relate the scaling having equal components of the top facets of the Newton polytope to the maximal cut of given Feynman integrals. We have therefore connected two independent approaches to the analysis of Feynman diagrams.

1 Introduction

The present work is a sequel to ref. [1] which presents a novel approach to the Method of Regions [2–7] (MoR) based on the analysis of Landau equations associated with given Feynman diagrams. The algorithm also allows us to compute the scalings of ‘top facets’ which in this work are related in some cases to the maximal cuts of these Feynman diagrams, thereby allowing us to study generalized unitarity in a novel manner to be further explained below.

The description of elementary particle physics through perturbative quantum field theory has been very successful. One expresses the field theoretical amplitudes as an expansion in Feynman integrals. The calculation of Feynman integrals with various scales is very difficult. One needs to, very often, calculate higher order loop corrections to these multi-scale Feynman integrals, in order for having better predictions for the field theoretical observables.

The Method of Regions [2–7] is one of the useful methods for the evaluation of the multi-scale Feynman integrals. This method uses the hierarchies between various scales of the problem to construct a small expansion parameter, performs Taylor expansion in each of the regions and evaluates the integral in each of the regions. The final result consists of the sum of the contributions coming from all the regions.

MoR had been successfully applied in many examples. The foundation and generalization of MoR had been discussed in ref. [8]. Very recent progress towards the proof of MoR can be found in Ref. [9], where Lee-Pomeransky representation of Feynman integrals [10] had been used to describe MoR. In another recent work [11], MoR had been employed in a systematic way to evaluate two-loop non-planar diagram appearing in the Higgs pair production cross-section at the next-to-leading order.

The identification of the regions for a multi-scale multi-loop Feynman integral in a given limit is a non-trivial task. The automatic identification of the regions based on geometrical approach can be found in Ref. [12]. The program had been named as ASY. The potential and Glauber regions were undetected in the first version. This issue had been fixed in Ref. [13]. ASY had been implemented inside FIESTA [14] to reveal the regions and numerically evaluate the expansion of the given integral with certain accuracy.

The mathematica program ASPIRE [1] is based on an alternative formalism, which also
unveils the regions associated with a given multi-scale Feynman integral in a given limit. The construction begins with the finding of the sum of Symanzik polynomials of first and second kind. One then finds the Gröbner basis of the Landau equations. By mapping the Gröbner basis elements to the origin, co-ordinate axes, co-ordinate planes, one obtains a set of linear transformations. All the transformations are applied to the sum of the Symanzik polynomials, which are then analyzed within the framework “Power Geometry” [15]. For all of the obtained polynomials, one finds the support of the corresponding polynomials. The convex hull of the support then gives the Newton polytope. The normal vector for each of the facets of the polytopes, one obtains the desired set of scalings.

While analyzing a given Feynman diagram within the framework ASPIRE, two types of facets of the Newton polytopes had been considered. For a given sum(polynomial), the bottom facet of the Newton polytope is defined to be the facet for which the points other than the vertices of that facet lie above that facet. On the other hand, top facet is the opposite case. The mathematical definitions of bottom and top facet are given in the appendix A.2 and also we give a detailed description of this discussion for a one loop vertex diagram in section 2.3.

The scalings from the bottom facets with the consideration of small expansion parameter lead to the well known case of “Regions” [1,12,13]. In this work, we explore the complimentary case i.e. we consider top facet scaling with the freedom of choosing the expansion parameter to be large. The set of Landau equations [16,17] for a given Feynman integral while combined with power geometry implies that the scaling with all the components to be equal from top facet corresponds to the case of maximal cut of the given integral. We explore the correspondence between the parametric integrals constructed based on the scalings(with all the components to be equal) of the top facets and the maximal cut for a given Feynman diagram.

The discontinuity due to the Landau singularities is given in terms of cut Feynman diagrams, by replacing the Feynman propagators by delta functions [18]. A Feynman diagram is said to be maximally cut when all of its propagators are replaced by Dirac-δ functions i.e. all the internal lines are put on-shell.

The cut Feynman integrals had been studied in a series of recent works [19–24]. These studies show various mathematical structures of the cut Feynman integrals. In [22,27], some conjectures on the relation of these cut integrals with co-products of multiple polylogarithms in Hopf algebra gives an interesting way to compute original Feynman Integrals without doing actual integration, but evaluating the comparatively easier cut integral and using Hopf algebra. This method actually relies on the possibility of expressing the original Feynman integral and the cut Feynman integral in terms of multiple polylogarithms. In [23,24], cut Feynman integrals had been evaluated in a systematic approach using Baikov-Lee representation.

In this paper we use the method of residues [19] to evaluate the cut integrals. The main idea is the equivalence of evaluating the original Feynman integral with cut propagators replaced by Dirac-δ functions and evaluating the integral of the residue of original Feynman integral at the singularities due to cut propagators. The evaluation of the residues involves deforming the integration contour to include the poles or singularities in Leray’s multivariate residue calculus. Right now the method of residues has been worked out only on one loop Feynman integrals. The extension for more than one loop case is a future research work. We use directly the results from literature for the one loop cases that we study and for the two loop case we solve the problem in two parts, i.e. evaluating the results for the one loop case and then applying it to solve the two loop problem by directly using the Dirac-δ functions inside the integral.

The organization of this paper is the following:

In section 2 we review the basics of power geometry and discuss the method to obtain the asymptotic solution of a given finite algebraic sum. For a generic Feynman integral, the Feynman parametric form of the integral in terms of Symanzik polynomials has been discussed in section 2.2. In section 2.3 we present brief description of the mathematica program ASPIRE. In section 2.4 we discuss correspondence of the top facet scaling with equal components to the
maximal cut of the corresponding Feynman diagram. Two one loop diagrams and one two loop non-planar diagram have been analyzed in section 3. We conclude in section 4.

2 Formalism

In this section, we review the frameworks, which have been considered during the analysis for obtaining the connection between the scaling (with all the components to be equal) of top facets and the maximal cut Feynman integrals. The framework ASPIRE uses power geometry [15] to find the Regions, required for the evaluation of Feynman diagrams by expanding asymptotically in each of the Regions. We start this section with the basic definitions used in power geometry and the way to get the asymptotic solutions for a given sum (polynomial), analyzed in the framework power geometry.

2.1 Power geometry and the asymptotic solutions for a given sum

Let us consider a finite sum

\[ g(Q) = \sum g_R Q^R, \]

where \( Q = (\alpha_1, \alpha_2, \ldots, \alpha_n), R = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n \) and \( g_R \) are the constant coefficients. By \( Q^R \), we mean the terms \( \alpha_1^{r_1} \alpha_2^{r_2} \cdots \alpha_n^{r_n} \).

Below we give few definitions which are necessary, when one deals with the method of Power Geometry.

- **Support of the sum:**
  The set of vector exponents, \( R = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n \) is called the support \( S(g) \) of the given sum \( g(Q) = \sum g_R Q^R \).

- **Newton Polytope:**
  The convex hull of the support is called the Newton Polytope. It consists of generalized facets \( \Gamma^d_j \), where \( d \) is the dimension of the facets and the label \( j \) stands for the \( j \)-th facet. For our case, we always consider \( d = 2 \).

- **Truncated sum:**
  Each of the generalized facets \( \Gamma^d_j \) corresponds to a sum \( \tilde{g}^d_j = \sum g_R Q^R, \) where \( R \in \Gamma^d_j \cap s(g) \). \( \tilde{g}^d_j \) is called the truncated sum.

- **Normal cone:**
  We consider the dual space, \( \mathbb{R}^*_n \) to the space \( \mathbb{R}^n \). We define the scalar product \( c_j = \langle R, S \rangle \), where \( R \in \mathbb{R}^n \) and \( S \in \mathbb{R}^*_n \). The set of all points \( S \) for which \( c_j \) becomes maximum for all the points \( R \in \Gamma^d_j \), is called the normal cone of the generalized facet \( \Gamma^d_j \). In our case, as we deal with \( d = 2 \), we consider only the outward normal vector to each of the facets.

- **Cone of the problem:**
  The set of points, \( S \in \mathbb{R}^*_n \) such that the curves of the form of the Equ.(1) that fill the space \( (\alpha_1, \alpha_2, \cdots, \alpha_n) \), to be studied is called the cone of the problem.

**Theorem:**

If curve

\[
\alpha_1 = a_1 x^{s_1}(1 + o(1)), \\
\alpha_2 = a_2 x^{s_2}(1 + o(1)), \\
\vdots \\
\alpha_n = a_n x^{s_n}(1 + o(1)),
\]

(1)
where \(a_i\) and \(s_i\) are constants, lie in the set \(G\) as \(x \to \infty\) and the vector \(\{s_1, s_2, \ldots, s_n\} \in U_d\), then the first approximation \(\alpha_1 = a_1 x^{s_1}, \alpha_2 = a_2 x^{s_2}, \ldots, \alpha_n = a_n x^{s_n}\) of Eq.(1) satisfies the truncated sum \(g_d^j = 0\).

One wishes to obtain the set \(G = \{Q : g(Q) = 0\}\) near singular points \(Q = Q_0\), or singular curves \(C\), or singular surfaces \(S\) consisting of the singular points. Below we discuss the steps for obtaining the solution set \(g\) for each of the facets of the Newton polytope:

1. Certain transformations \(Q(\alpha_1, \alpha_2, \ldots, \alpha_n) \to Q'(\alpha'_1, \alpha'_2, \ldots, \alpha'_n)\) need to be performed in order for mapping the singular points, singular curves, and singular surfaces to the origin, co-ordinate axes, and co-ordinate planes respectively.

2. Find \(g(Q')\) and the corresponding support \(S(g)\).

3. Obtain the Newton polytope for \(g(Q')\) and the outward normal vectors \(\{s_1, s_2, \ldots, s_n\}\) for each of the facets.

4. The above theorem then gives us the desired solution set \(G\) at the leading order.

We see that each of the facets of the Newton polytope, resulting from a given sum or polynomial, corresponds to an asymptotic solution according to the above theorem.

### 2.2 Parametric representation of Feynman integrals

For the sake of completeness, we, here, briefly discuss parametric representation [10, 28–30] of a generic Feynman diagram.

Consider a Feynman diagram having \(L\) loop momenta \((l_1, l_2, \ldots, l_L)\), \(E\) external momenta \((p_1, p_2, \ldots, p_E)\) in the generic form,

\[
I(n_1, n_2, \ldots, n_m) = (e^{\gamma E} \mu^{2L})^L \prod_{i=1}^{L} \frac{dD_{l_i}}{i \pi \frac{i}{2}} \prod_{a=1}^{m} D_{\alpha}^{-n_a},
\]

where \(D_{\alpha} = A_{ij} l_i l_j + 2B_{ik} l_i p_k + C_{\alpha}\) are the given set of propagators. \(A, B\) are respectively \(L \times L, L \times E\) matrices and \(C\) are constants. The parameter \(\mu\) is arbitrary having mass dimension 1. We put \(\mu = 1\) throughout our calculations.

One can express Eq.(2) in the following form,

\[
I(n_1, n_2, \ldots, n_m) = (e^{\gamma E})^L \left( \prod_{a} D_{\alpha}^{-n_\alpha} \right) \prod_{a} d\alpha z_\alpha z_\alpha^{-n_\alpha} \delta(1 - \sum_{a} z_\alpha)
\]

\[
\times \frac{\mathcal{F}(\frac{L_\mu}{2} - (n_1 + n_2 + \ldots + n_m))}{\mathcal{U}(\frac{L_\mu}{2} - (n_1 + n_2 + \ldots + n_m))}
\]

\(\mathcal{U}\) and \(\mathcal{F}\) are the Symanzik polynomials, of degree \(L\) and \(L + 1\) respectively.

In this work, we use the Parametric representation for a generic Feynman diagram to construct the integrals based on certain scalings, coming from the top facets of the Newton polytopes.

### 2.3 The mathematica program - ASPIRE

The mathematica program “ASPIRE” had been developed to isolate the regions associated with multi-scale, multiloop Feynman diagram in a given kinematic limit. The formalism of ASPIRE is based on the consideration of singularities of the given Feynman integral and the associated
Landau equations and analysis of the sum of the Symanzik polynomials of first and second kind using the power geometry.

The program ASPIRE has the following steps:

1. For a given multi-scale Feynman integral in a given limit, find the Symanzik polynomials $U, \mathcal{F}$.

2. Find the Gröbner basis of the Landau equations $\{F, \frac{\partial F}{\partial \alpha_i}\}$, where $\alpha_i$ are the Alpha parameters.

3. Map the Gröbner basis elements to origin, co-ordinate axes, coordinate planes via linear transformations.

4. Construct $\mathcal{G} = U + \mathcal{F}$ polynomials under the consideration of the obtained linear transformations, as mentioned in the previous step.

5. Find the support of each of the $\mathcal{G}$ polynomials.

6. Find the convex hull of the obtained support. Thus one obtains the Newton polytopes.

7. Look for the normal vectors corresponding to each of the facets of the Newton polytopes.

8. The set of the components of the valid normal vectors then gives the set of desired regions.

If for a given sum, one constructs Newton polytope with the vector exponents $\vec{r}$, and $\vec{v}$ is the outward normal vector to the facets of the polytope, then bottom facets of the Newton polytope are those facets which satisfy the following conditions,

$$\begin{align}
\vec{r}.\vec{v} &= c \quad \text{for the points on the facets.} \\
\vec{r}.\vec{v} &> c \quad \text{for the points which lie above the facets.} 
\end{align}$$

(4)

The top facets of the Newton polytope are defined as,

$$\begin{align}
\vec{r}.\vec{v} &= c \quad \text{for the points on the facets.} \\
\vec{r}.\vec{v} &< c \quad \text{for the points which lie below the facets.} 
\end{align}$$

(5)

It is important to note that we consider the expansion parameter $x$ to be small (i.e. $x \to 0$) while we consider the analysis for finding the scaling from the bottom facets of the Newton polytope. In the case of top facets, we choose the expansion parameter to be large (i.e. $x \to \infty$).

It is well known that the scaling coming from bottom facets are the regions, which are required for the asymptotic expansion of the Feynman integrals in the given limit.

Below we consider a one loop vertex diagram considered in Ref. [2] as an example to demonstrate the above discussion:

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{diagram1.png}
\caption{A one loop vertex diagram. The internal solid lines have mass m and the wavy line is massless.}
\end{figure}
Fig. 1 corresponds to the following Feynman integral,

\[ I(q^2, m^2) = \int \frac{d^Dk}{i\pi^{D/2}} \frac{1}{\left( (k + \frac{q}{2})^2 - m^2 \right) \left( (k - \frac{q}{2})^2 - m^2 \right) (k - p)^2} \]  

(6)

In this case, we have \( q = p_1 + p_2 \), \( p = \frac{p_1 + p_2}{2} \) and two kinematic invariants \( q^2 \) and \( m^2 \). We construct expansion parameter \( x = m^2 - \frac{q^2}{4} \) to expand the integral in terms, which have certain power in \( x \).

We find the Symanzik polynomials \( U, F \) using the mathematica code UF.m [31] with the following command,

\[ \text{UF}\left[k\right], \left\{ -((k + \frac{q}{2})^2 - m^2), -(k - \frac{q}{2})^2 - (k - p)^2 \right\}, \left\{ q^2 \rightarrow qq, pq \rightarrow 0, p^2 \rightarrow x, m^2 \rightarrow x + \frac{q q}{4} \right\} \],

(7)

which gives,

\[ U = \alpha_1 + \alpha_2 + \alpha_3 \]  

(8)

\[ F = \frac{qq}{4} \alpha_1^2 - \frac{1}{2} qqq \alpha_1 \alpha_2 + \frac{qq}{4} \alpha_2^2 + x \alpha_1^2 + 2 x \alpha_2 \alpha_1 + x \alpha_2^2, \]  

(9)

where \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are the alpha parameters.

We now find the Landau equations, encoding the location of singularities of the integral,

\[ F = 0, \]  

(10)

\[ \frac{\partial F}{\partial \alpha_i} = 0, \quad \text{where} \quad i = 1, 2, 3. \]  

(11)

The Gröbner basis of the Landau equations are, 

\[ \{ qq x \alpha_2, x (\alpha_1 + \alpha_2), qq (\alpha_1 - \alpha_2) \} \]

We map the Gröbner basis elements to the origin, co-ordinate axes with the following transformations,

\[ T1 = \{ \alpha_1 \rightarrow \alpha_1, \alpha_2 \rightarrow \alpha_2, \alpha_3 \rightarrow \alpha_3 \} \]

(12)

\[ T2 = \{ \alpha_1 \rightarrow \alpha_1 + \frac{\alpha_2}{2}, \alpha_2 \rightarrow \frac{\alpha_2}{2}, \alpha_3 \rightarrow \alpha_3 \} \]

(13)

\[ T3 = \{ \alpha_1 \rightarrow \frac{\alpha_1}{2}, \alpha_2 \rightarrow \frac{\alpha_1}{2} + \alpha_2, \alpha_3 \rightarrow \alpha_3 \} \]

(14)

In this example, we discuss the analysis with the transformation \( T1 \) only. Analysis with the other two transformations (\( T2 \) and \( T3 \)) can be found in the ancillary file \text{Vertex.nb}.

\[ G = \alpha_1 + \frac{qq}{4} \alpha_1^2 + x \alpha_1^2 + \alpha_2 - \frac{1}{2} qqq \alpha_1 \alpha_2 + 2 x \alpha_1 \alpha_2 + \frac{qq}{4} \alpha_2^2 + x \alpha_2^2 + \alpha_3 \]

(15)

We compute the support of \( G \) by extracting the vector exponents of each of the terms,

\[ S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]  

(16)
The co-ordinates of the points are considered in \((x, \alpha_1, \alpha_2, \alpha_3)\)-space. We assign label for each of the points of the support \(S\) as \(\{1(0, 1, 0, 0), 2(0, 2, 0, 0), 3(1, 2, 0, 0), 4(0, 0, 1, 0), 5(0, 1, 1, 0), 6(1, 1, 1, 0), 7(0, 0, 2, 0), 8(1, 0, 2, 0), 9(0, 0, 0, 1)\}\). The convex hull of the points of \(S\) gives the facets of the Newton polytope,

\[
NP = \begin{pmatrix}
1 & 2 & 3 & 9 \\
1 & 2 & 7 & 8 \\
1 & 2 & 8 & 3 \\
1 & 2 & 9 & 7 \\
1 & 3 & 8 & 9 \\
1 & 4 & 7 & 9 \\
1 & 4 & 8 & 7 \\
1 & 4 & 9 & 8 \\
2 & 3 & 9 & 8 \\
2 & 7 & 8 & 9 \\
4 & 7 & 9 & 8 \\
\end{pmatrix}
\]  

(17)

We now find the normal vector for each of the facets of the Newton polytope with the following considerations:

1. The component of the normal vector corresponding to the \(x\)-axis (i.e. zeroth component) should be non-zero.

2. The facets which satisfy \(\text{Equ.}\ 4\) have been labelled as \(\text{"surf} \to -1\) (i.e. bottom facets) and the facets which satisfy \(\text{Equ.}\ 5\) have been labelled as \(\text{"surf} \to 1\) (i.e. top facets).

3. One obtains “Null” when the zeroth component of the normal vector is zero.

We obtain the following normal vectors corresponding to the facets of \(NP\),

\[
\begin{pmatrix}
\text{Null} \\
\text{Null} \\
\text{Null} \\
\{v(1) \to 0, v(2) \to 0, v(3) \to 0, c \to 0, \text{surf} \to -1\} \\
\{v(1) \to -1, v(2) \to -1, v(3) \to -1, c \to -1, \text{surf} \to 1\} \\
\{v(1) \to 0, v(2) \to 0, v(3) \to 0, c \to 0, \text{surf} \to -1\} \\
\text{Null} \\
\{v(1) \to -1, v(2) \to -1, v(3) \to -1, c \to -1, \text{surf} \to 1\} \\
\text{Null} \\
\text{Null} \\
\text{Null} \\
\end{pmatrix}
\]  

(18)

We see that with transformation \(T1\), only one region \(\{0, 0, 0\}\) is isolated. With other two transformations \(T2, T3\), two other regions \(\{1/2, 0, 0\}\) and \(\{0, 1/2, 0\}\) are recovered.

There is one more scaling \(\{-1, -1, -1\}\), which comes from the top facets of the Newton polytope. In this paper, we construct the parametric integral using \(\text{Equ.}\ 8\) for the top facet scaling having equal components i.e. \(\{-1, -1, -1, \cdots, -1\}\) and find the correspondence of the top facet scaling having equal components to the maximal cut of a given Feynman diagram.
2.4 Top facet scaling with equal components and the maximal cut Feynman diagram

Consider a generic Feynman integral,

$$I(m_i^2, p_i^2) = \int \prod_i \frac{d^D k_i}{i \pi^{D/2}} \prod_j \frac{1}{(q_j^2 - m_j^2)^{n_j}},$$

(19)

where \(m_i\) is the mass of i-th internal line, and \(p_i\) are the external momenta. The momenta \(q_i\) are the linear combination of loop momenta \(k_i\) and the external momenta \(p_i\). In Feynman parametric form, Eqn. 19 can be written as,

$$I(m_i^2, p_i^2) = \int \prod_j d\alpha_j \prod_i \frac{d^D k_i}{i \pi^{D/2}} \frac{\delta(1 - \sum_j \alpha_j)}{(\sum_j \alpha_j(q_j^2 - m_j^2))^{\sum_j n_j}}$$

(20)

The Landau singularities are given by,

$$\sum_j \alpha_j(q_j^2 - m_j^2) = 0$$

(21)

Each of the facets of the Newton polytope corresponds to an asymptotic solution in the alpha parameter space according to Bruno’s theorem. We choose the expansion parameter, \(x \to \infty\) for the top facets. This means we are moving far away from the origin.

The scalings \(\{s_1, s_2, \ldots, s_j\}\) coming from the facets of the polytopes imply the asymptotic solutions of the form \(\{\alpha_1 \sim x^{s_1}, \alpha_2 \sim x^{s_2}, \ldots, \alpha_i \sim x^{s_i}\}\). The scalings can be given a constant shift. If \(\vec{S} = \{s_1, s_2, \ldots, s_j\}\) is a scaling coming from one of the facets of the Newton polytope, then \(\vec{S}' = \vec{S} + \vec{A} = \{s_1 + a, s_2 + a, \ldots, s_j + a\}\) corresponds to the same scaling \(\vec{S}\).

The expansion parameter \(x\) being large for the top facets, the top facet scaling with equal components essentially gives,

$$\alpha_j \neq 0, \text{ for all } j$$

(22)

Thus, for the top facet scaling having equal components, one has

$$q_j^2 - m_j^2 = 0, \text{ for all } j$$

(23)

This is the on-shell condition for the all the internal lines of the given diagram and hence the case of the maximal cut for the given diagram.

This analysis motivates us to express the maximal cut diagram in terms of the integrals constructed from the scaling(with equal components) of the top facet of the Newton polytope.

2.5 Brief description of the method of evaluation of cut Feynman diagrams

In this section, we give a brief review of the recent works for the evaluation of cut Feynman diagrams. We use this method for all of our calculations regarding cuts of the Feynman diagrams.

We start with Leray’s Multivariate Residues which states that an integrand (differential form of weight \(n\)) which is of the form given by

$$\omega = \frac{ds}{s^n} \wedge \psi + \theta,$$

(24)

has residues defined by

$$\text{Res}_S[\omega] = \psi|_S.$$
and the following equation holds

$$\int_{\delta \sigma} \omega = 2\pi i \int_{\sigma} \text{Res}[\omega].$$  \tag{26}$$

where $\wedge$ is the generalization of cross product in higher dimensions, $\psi$ is a differential form of weight $n-1$, $s$ is equivalent to the propagator, $S$ is the singularity zone, $\sigma \subset S$ and $\delta \sigma$ is the set of points which form a small circle around every point on $\sigma$ but not belonging to $S$ called as the "Tubular neighbourhood" or "Leray coboundary" which "wraps around" $\sigma$.

One loop Feynman Integrals can be written as

$$I_n = \int \omega_n^D,$$  \tag{27}$$

where the integrand is of the form

$$\omega_n^D = \frac{\epsilon^{\gamma \epsilon \epsilon} e^{-\epsilon^c}}{i\pi^{D/2}} \frac{d^D k}{D_1 \ldots D_n},$$  \tag{28}$$

with $D_j = (k - q_j)^2 - m_j^2 + i\delta$. In order to get the Residue of the Feynman Integral we have to write the integrand equ.(28) in the form of equ.(24). This can be achieved by a Jacobian transformation from $k$ to $D_j$ after which the integrand can be written in a form

$$\omega_n^D = \frac{2^{-c} \epsilon^{\gamma \epsilon \epsilon}}{\sqrt{\mu^c H_C}} \left( \mu' \frac{H_C}{\text{Gram}_C} \right)^{(D-c)/2} \frac{d\Omega_{D-c}}{i\pi^{D/2}} \left( \prod_{j \notin C} \frac{1}{D_j} \right) \left( \prod_{j \in C} \frac{dD_j}{D_j} \right),$$  \tag{29}$$

where the factor $\mu' = (+1)/(-1)$ accounts for the Euclidean/Minkowski space respectively, $C$ is the set of cut propagators and $c$ is the total number of cut propagators, $d\Omega$ is the angular part of the differential $d^D k$ in the remaining $D-c$ dimensions, $H_C$ and $\text{Gram}_C$ are the gram determinants given by

$$H_C = \text{det}((q_i - q_j) \cdot (q_j - q_i))_{i,j \in C \setminus \{\ast\}}, \quad \text{Gram}_C = \text{det}((q_i - k) \cdot (q_j - k))_{i,j \in C \setminus \{\ast\}}$$  \tag{30}$$

with $\{\ast\} \in C$. Thus using Equ.(25) gives

$$\text{Res}_C[\omega_n^D] = 2^{-c} \epsilon^{\gamma \epsilon \epsilon} \frac{d\Omega_{D-c}}{i\pi^{D/2}} \left[ \frac{1}{\sqrt{\mu^c H_C}} \left( \mu' \frac{H_C}{\text{Gram}_C} \right)^{(D-c)/2} \left( \prod_{j \notin C} \frac{1}{D_j} \right) \right]_C,$$  \tag{31}$$

where the notation $[.]_C$ indicates that the expression inside square brackets should be evaluated on the locus where the cut propagators vanish.

As discussed earlier, the integral of the residue is actually equivalent to the cut integral and hence we can write the cut integral corresponding to equ.(27) as

$$C_c I_n = 2^{-c} \frac{(2\pi i)^{c/2} \epsilon^{\gamma \epsilon \epsilon}}{\sqrt{\mu^c Y_C}} \left( \mu' \frac{Y_C}{\text{Gram}_C} \right)^{(D-c)/2} \int_{\delta \sigma} \frac{d\Omega_{D-c}}{i\pi^{D/2}} \left[ \prod_{j \notin C} \frac{1}{(k - q_j)^2 - m_j^2} \right]_C,$$  \tag{32}$$

where

$$Y_C = \text{det} \left( \frac{1}{2}(-(q_i - q_j)^2 + m_i^2 + m_j^2) \right)_{i,j \in C}.$$

$$3 \text{ \ Examples}$$

In this section, we evaluate the parametric integral for the top facet scaling with equal components, and find their correspondence to the maximal cut for a two point one loop diagram, a three point one loop diagram, and a non-planar two loop diagram.
3.1 Two point one loop diagram

We consider the following integral in dimension $D = 4 - 2\varepsilon$,

$$I(q^2, m^2) = \int \frac{d^D k}{i\pi^{D/2} (k^2 - m^2)((k - q)^2 - m^2)},$$

where $q$ is the external momentum and $m$ is the mass of both internal lines. The expansion parameter is $x = m^2 - \frac{q^2}{4}$.

![Figure 2: A two point one loop diagram.](image)

The symanzik polynomials are

$$U = \alpha_1 + \alpha_2$$

$$F = \frac{1}{4} q^2 \alpha_1^2 + \frac{1}{4} q^2 \alpha_2^2 - \frac{1}{2} q^2 \alpha_1 \alpha_2 + x \alpha_1^2 + x \alpha_2^2 + 2 x \alpha_1 \alpha_2,$$

where $\alpha_1$ and $\alpha_2$ are the alpha parameters.

Using ASPIRE, we find that the above diagram has only one top facet scaling $\{-1, -1\}$.

3.1.1 Parametric integral from the top facet scaling $\{-1, -1\}$:

We compute the Symanzik polynomials using the top facet scaling $\{-1, -1\}$,

$$U = \alpha_1 + \alpha_2,$$

$$F = x(\alpha_1 + \alpha_2)^2,$$

The integral for the scaling $\{-1, -1\}$ is obtained by substituting the expressions of $U, F$ (Eq. 37 and 38) in Eq. 3,

$$I^{(-1,-1)} = \Gamma(2 - \frac{D}{2}) \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \frac{\delta(1 - \alpha_1 - \alpha_2)(\alpha_1 + \alpha_2)^2 - D}{(x(\alpha_1 + \alpha_2)^2)^{2 - \frac{D}{2}}}$$

$$= \Gamma(\varepsilon) \left(m^2 - \frac{q^2}{4}\right)^{-\varepsilon}$$

3.1.2 The maximal cut integral

![Figure 3: The maximal cut of two point one loop diagram.](image)
The maximal-cut (MC) for this diagram is obtained by putting both of the two internal lines simultaneously to be on-shell, i.e. we substitute a delta function for both of the propagators. Thus,

\[ I^{MC} = \int \frac{d^D k}{i\pi^{D/2}} \delta(k^2 - m^2) \delta((k - q)^2 - m^2) \] (40)

In Eqn. (32) the quantity inside the square bracket is unity because there are no propagators which are not cut for this case and we have \( c = n = 2 \) with \( D = 4 - 2\epsilon \) as usual, thus the maximal cut for the figure (2),

\[ I^{MC} \sim \frac{1}{\sqrt{\mu^2 Y_C}} \left( \frac{\mu^2 Y_C}{\text{Gram}_C} \right)^{1-\epsilon} \int \frac{d\Omega_2-2\epsilon}{i\pi^{2-\epsilon}} \] (41)

Using Eqn. (30) and Eqn. (33) for this case we have,

\[ \text{Gram}_C = |q^2| = q^2 \]

and

\[ Y_C = \left| \frac{m^2}{2} \right| = q^2 \]

Also the angular part of the integration is given by\(^1\)

\[ \int d\Omega_2-2\epsilon = \frac{2\pi^{(3-2\epsilon)/2}}{\Gamma((3-2\epsilon)/2)} \]

Thus from Eqn. (41), after using the duplication formula of gamma function\(^2\) we obtain the final result for the maximal cut,

\[ I^{MC} \sim \frac{(4m^2 - q^2)^{1-\epsilon}}{\sqrt{q^2(4m^2 - q^2)}} \frac{\pi^{1-\epsilon} \Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)} \] (42)

3.1.3 Correlation between \( I^{(-1,-1)} \) and \( I^{MC} \)

We obtain the following relation\(^3\):

\[ I^{MC} \sim \frac{(4\pi)^{-\epsilon} \Gamma(2-\epsilon)}{\Gamma(\epsilon)\Gamma(3-2\epsilon)} \left( \frac{4m^2 - q^2}{q^2} \right)^{\frac{1}{2}} I^{(-1,-1)} \] (43)

3.2 A one loop scalar triangular diagram

We consider the triangular diagram in the limit \( p_1^2 = 0, \ p_2^2 = 0 \) and \( 2p_1.p_2 = Q^2 \). The integral in this limit is given by,

\[ I(q^2, m^2, D) = \int d^D k \frac{1}{(k^2 - 2p_1.k)(k^2 - 2p_2.k)(k^2 - m^2)} \] (44)

The expansion parameter is \( x = \frac{m^2}{Q^2} \).

---

\(^1\)This formula is according to the convention followed in [19] which is stated in Eqn. 38.

\(^2\)Gamma function duplication formula:

\[ \Gamma(2n) = \frac{1}{\sqrt{\pi}} \frac{\pi^{2n-1} \Gamma(n) \Gamma(n + \frac{1}{2})}{n!} \]

\(^3\)We neglect the constant prefactors of the integrals as we are interested in relations of the form \( I^{MC} \sim f(m^2, q^2, \epsilon) I^{(-1,-1)} \).
The Symanzik polynomials in this given limit are,

\[ U = \alpha_1 + \alpha_2 + \alpha_3 \] (45)

\[ F = x\alpha_3^2 + x\alpha_1\alpha_3 + x\alpha_2\alpha_3 + Q^2\alpha_1\alpha_2 \] (46)

The top facet scalings, obtained from ASPIRE are \{-1, -1, -1\} and \{0, 0, -1\}.

3.2.1 The integral for the scaling \{-1, -1, -1\}

The Symanzik polynomials for the scaling \{-1, -1, -1\} in the limit \(m^2 \gg Q^2\) are

\[ U = \alpha_1 + \alpha_2 + \alpha_3 \] (47)

\[ F = x\alpha_3(\alpha_1 + \alpha_2 + \alpha_3) \] (48)

The integral is given by,

\[ I^{(-1,-1,-1)} = \Gamma(3 - \frac{D}{2}) \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3 \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) \frac{x\alpha_3(\alpha_1 + \alpha_2 + \alpha_3)}{(\alpha_1 + \alpha_2 + \alpha_3)^{D-3}} \]

\[ = \Gamma(3 - \frac{D}{2}) \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 \frac{x^D - 3}{D-3} \]

\[ = \Gamma(1 + \epsilon) \left( \frac{m^2}{Q^2} \right)^{1-\epsilon} (Q^2)^{1-\epsilon} \frac{1}{\epsilon(\epsilon - 1)} \] (49)

While looking for the Symanzik polynomials \(U, F\), we consider \(m^2 \to x\). But as our expansion parameter is \(\frac{m^2}{Q^2}\). So in order for writing the result 49 in terms of expansion parameter with correct consideration, we substitute \(x = \frac{m^2}{Q^2} \times Q^2\).

3.2.2 The maximal cut

In the maximal cut all the propagators are replaced by Dirac Delta function and hence the cut integral is given by

\[ I^{MC}_{\text{Triangle}} = \int \frac{d^D k}{i\pi^\frac{D}{2}} \delta(k^2 - 2p_1.k)\delta(k^2 - 2p_2.k)\delta(k^2 - m^2) \] (50)
Figure 5: Maximal cut of the triangular diagram.

As in the previous case here also we have the angular part trivial to solve with $c = n = 3$, $D = 4 - 2\epsilon$ and

$$Y_C = \begin{vmatrix} 0 & -\frac{Q^2}{2} & \frac{m^2}{2} \\ \frac{m^2}{2} & 0 & \frac{m^2}{2} \\ \frac{m^2}{2} & \frac{m^2}{2} & 0 \end{vmatrix} = -\frac{m^2 Q^2 (Q^2 + m^2)}{4}$$

and

$$\text{Gram}_C = \begin{vmatrix} 0 & -\frac{Q^2}{2} \\ -\frac{Q^2}{2} & 0 \end{vmatrix} = -\frac{(Q^2)^2}{4}$$

Thus using Eqn. (32) we obtain the expression for the maximal cut of this diagram,

$$I^{MC}_{\text{Triangle}} \sim \frac{1}{\Gamma(1-\epsilon)} \sqrt{\frac{1}{m^2 Q^2 (m^2 + Q^2)}} \left(-\frac{4m^2 (Q^2 + m^2)}{Q^2}\right)^{-\epsilon}$$

(51)

Here we have used the following result for the angular integration:

$$\int d\Omega_{1-2\epsilon} = \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}$$

(52)

3.2.3 Correlation between top and cut integrals

We neglect the irrelevant constants in the expressions of $I^{(-1,-1,-1)}$ and $I^{MC}_{\text{Triangle}}$. For this diagram, we find the following,

$$I^{MC}_{\text{Triangle}} \sim \frac{\epsilon(1-\epsilon)}{\Gamma(1-\epsilon)\Gamma(1+\epsilon)} \frac{m}{Q^2} \left(\frac{m^2 + Q^2}{Q^2}\right)^{-\frac{1}{2}-\epsilon} I^{(-1,-1,-1)}$$

(53)

3.3 A non-planar two loop diagram

Let us consider a non-planar two loop triangular diagram\textsuperscript{6}. This diagram had been considered in Ref. \[23,26\] with $p_1^2 = 0$, $p_2^2 \neq 0$. The integral is defined to be the following,

$$I(q^2, m^2, D) = \int \frac{d^D k_1}{i \pi^{D/2}} \frac{d^D k_2}{i \pi^{D/2}} \frac{1}{(k_1 - p_1)^2((k_2 - p_1)^2 - m^2)(k_1 + p_2)^2((k_1 - k_2 + p_2)^2 - m^2)} \times \frac{1}{((k_1 - k_2)^2 - m^2)(k_2^2 - m^2)}$$

(54)

We consider the limit $p_1^2 = 0$, $p_2^2 = 0$, $2p_1 \cdot p_2 = q^2$ and construct the expansion parameter $x = \frac{m^2}{q^2}$. 

13
Figure 6: A non-planar two loop diagram.

The Symanzik polynomials are,

\[ U = \alpha_1 \alpha_{2456} + \alpha_2 \alpha_{345} + \alpha_3 \alpha_{45} + \alpha_{345} \alpha_6, \]  
(55)

\[ F = x \alpha_{2456} (\alpha_3 \alpha_{45} + \alpha_2 (\alpha_3 + \alpha_{45}) + \alpha_6 (\alpha_3 + \alpha_{45}) + \alpha_1 \alpha_{2456}) - q^2 (\alpha_2 \alpha_3 \alpha_5 + \alpha_1 (\alpha_6 x_{34} + \alpha_3 (\alpha_2 + \alpha_{45}))), \]  
(56)

where \( \alpha_{ijk\ldots} = \alpha_i + \alpha_j + \alpha_k + \cdots. \)

The integral has two top facet scalings \( \{-1, -1, -1, -1, -1\} \) and \( \{0, -1, 0, -1, -1\} \) for this given limit.

We evaluate the integral for the scaling \( \{-1, -1, -1, -1, -1\} \) only.

### 3.3.1 The integral for the scaling \( \{-1, -1, -1, -1, -1\} \)

The Symanzik polynomials are given by,

\[ U = \alpha_1 \alpha_{2456} + \alpha_2 \alpha_{345} + \alpha_3 \alpha_{45} + \alpha_{345} \alpha_6, \]  
(57)

\[ F = x \alpha_{2456} (\alpha_1 \alpha_{2456} + \alpha_2 \alpha_{345} + \alpha_3 \alpha_{456} + \alpha_4 \alpha_{56}) , \]  
(58)

In [11], four point functions in the high energy limit have been calculated in a systematic way using MoR. While calculating the integrals using MoR, there are regions for which one cannot just use the dimensional regularization, extra analytic regulators [6, 33] are necessary to regularize the contributions from those regions. After obtaining the regions, the parametric integrals have been calculated using the following representation,

\[ I_{\text{parametric}} = \int \mathcal{D}^n \alpha \ U^{-D/2} \ e^{-\frac{F}{2}}, \]  
(59)

where the integral measure is given by,

\[ \int \mathcal{D}^n \alpha \equiv \prod_{i=1}^{n} \int_0^\infty d\alpha_i \ \frac{\alpha_i^{\delta_i}}{\Gamma(1+\delta_i)}, \]  
with the consideration of the analytic regulators \( \delta_i. \)  
(60)

We consider \( \delta_i \to 0 \) while evaluating the parametric integral for the obtained scaling.

Thus, we construct the parametric integral for the scaling \( \{-1, -1, -1, -1, -1\}, \)

\[ I_{\{-1,-1,-1,-1,-1\}} = \int_0^\infty \prod_{i=1}^{6} d\alpha_i \ U^{-D/2} \ e^{-\frac{F}{2}} = \int_0^\infty \prod_{i=1}^{6} d\alpha_i (\alpha_{13} \alpha_{2456} + \alpha_{26} \alpha_{45})^{-D/2} e^{-x \alpha_{2456}} \]  
(61)

We make the following change of variables,

\[ \alpha_1 \to z_1 \tilde{z}_3, \alpha_2 \to \tilde{z}_2 \tilde{z}_4 \tilde{z}_5, \alpha_3 \to \tilde{z}_1 (1 - \tilde{z}_3), \alpha_4 \to \tilde{z}_2 (1 - \tilde{z}_4) (1 - \tilde{z}_6), \alpha_5 \to \tilde{z}_2 (1 - \tilde{z}_4) \tilde{z}_6, \]  
\[ \alpha_6 \to \tilde{z}_2 \tilde{z}_4 (1 - \tilde{z}_3) \]  
(62)
The Jacobian of the above transformations is \( z_1 z_2^3 (1 - z_4) z_4 \). The limits of the new integration variables are the following:

\[
z_1 \in [0, \infty), \quad z_2 \in [0, \infty], \quad z_3 \in [0, 1], \quad z_4 \in [0, 1], \quad z_5 \in [0, 1], \quad \text{and} \quad z_6 \in [0, 1]
\]

We get,

\[
I^{-1,-1,-1,-1,-1,-1} = \int_0^{\infty} dz_1 \int_0^{\infty} dz_2 \int_0^1 dz_4 z_1 z_2^3 (1 - z_4) z_4 \{ z_2 (1 - z_4) z_4 + z_1 \}^{-D/2} e^{-xz_2}
\]

(63)

We perform the \( z_1 \)-integral with the help of the following formula,

\[
\int_0^{\infty} dz \ z^{n_1}(a + z)^{n_2} = \frac{\alpha^{1+n_1+n_2}\Gamma(n_1 + 1)\Gamma(-n_1 - n_2)}{\Gamma(-n_2)}
\]

(64)

Thus, we obtain,

\[
I^{-1,-1,-1,-1,-1,-1} = \frac{\Gamma(2)\Gamma(D/2 - 2)}{\Gamma(D/2)} \int_0^{\infty} dz_2 z_2^5 - D e^{-xz_2} \int_0^1 dz_4 z_4^{3-D/2}(1 - z_4)^{3-D/2}
\]

(65)

In \( D = 4 - 2\xi \),

\[
I^{-1,-1,-1,-1,-1,-1} = \frac{\Gamma(-\xi)\Gamma(2 + 2\xi)\Gamma^2(2 + \xi)}{\Gamma(2 - \xi)\Gamma(4 + 2\xi)} \left( \frac{m^2}{q^2} \right)^{-2-2\xi} (q^2)^{-2-2\xi}
\]

(66)

### 3.4 The maximal cut

The maximal cut integral is given by

\[
I_{\text{nonplanar}}^{MC} = \int \frac{d^D k_1}{i\pi^{D/2}} \frac{d^D k_2}{i\pi^{D/2}} \delta(k_1^2) \delta((q - k_1)^2) \delta((p_2 - k_2)^2 - m^2) \delta((q - k_1 - k_2)^2 - m^2) \times \delta((k_1 + k_2 - p_2)^2 - m^2) \delta(k_2^2 - m^2)
\]

(67)

We can evaluate this integral by first using the maximal cut of the box integral involving \( k_2 \) as

\[\text{Figure 7: The maximal cut of the non-planar two loop diagram.}\]

the loop variable and then doing remaining integration of \( k_1 \) variable with the acquired result: \( I_{\text{nonplanar}}^{MC} \)

\[
I_{\text{nonplanar}}^{MC} = \int \frac{d^D k_1}{i\pi^{D/2}} \delta(k_1^2) \delta((q - k_1)^2) \int \frac{d^D k_2}{i\pi^{D/2}} \delta((p_2 - k_2)^2 - m^2) \delta((q - k_1 - k_2)^2 - m^2) \times \delta((k_1 + k_2 - p_2)^2 - m^2) \delta(k_2^2 - m^2)
\]

\[
\times \delta((k_1 + k_2 - p_2)^2 - m^2) \delta(k_2^2 - m^2)
\]

(68)

\[^4\text{We thank Sumit Banik for the independent check of the analytic expression for this parametric integral using suitable form of the Method of Brackets.}\]
We evaluate the expression inside the square bracket first which is equal to the maximal cut for a massive box diagram using Eq. (32). This gives,

$$I_{\text{nonplanar}}^{MC} = \int \frac{d^Dk_1}{i\pi D/2} \delta(k_1^2) \delta((q - k_1)^2) \Gamma(1 - \epsilon) \frac{1}{\Gamma(1 - 2\epsilon)} \sqrt{Y_C} \left( \frac{Y_C}{\text{Gram}_C} \right)^{-\epsilon}, \quad (69)$$

where,

$$\text{Gram}_C = \begin{vmatrix} (k-q)/(k-q) & (q-k)/(p_2-k) & (q-k)/(p_2-k) \\ (q-k)/(p_2-k) & (p_2-k)/(p_2-k) & (p_2-p_2-k) \\ (q-k)/(p_2-k) & (p_2-p_2-k) & p_2^2 \end{vmatrix}$$

and

$$Y_C = \begin{vmatrix} m^2 & m^2 & m^2 \\ m^2 & m^2 & m^2 \\ m^2 & m^2 & m^2 \end{vmatrix} \begin{vmatrix} m^2 - \frac{1}{2}(k-q)(k-q) & m^2 - \frac{1}{2}(k_2-p_2)(k-p_2) & m^2 - \frac{1}{2}(k-p_2)(k-p_2) \\ m^2 - \frac{1}{2}(k-p_2)(k-p_2) & m^2 - \frac{1}{2}(k-p_2)(k-p_2) & m^2 - \frac{1}{2}(k-p_2)(k-p_2) \\ m^2 - \frac{1}{2}(k-p_2)(k-p_2) & m^2 - \frac{1}{2}(k-p_2)(k-p_2) & m^2 - \frac{1}{2}(k-p_2)(k-p_2) \end{vmatrix} \quad (70)$$

Now to carry out this integral, without loss of generality we can select our frame and parametrize the loop momentum as follows:

$$q = \sqrt{q^2}(1, 0, \mathbf{0}_{D-2}), p_2 = \sqrt{p_2^2}(\alpha, \sqrt{\alpha^2 - 1}, \mathbf{0}_{D-2}), k_1 = (k_{10}, |k_1| \cos \theta, |k_1| \sin \theta \ 1_{D-2}) \quad (71)$$

where $\theta \in [0, \pi]$ and $|k| > 0$, and $1_{D-2}$ ranges over unit vectors in the dimensions transverse to $q$ and $p_2$. Momentum conservation fixes the value of $\alpha$ in terms of the momentum invariants to be

$$\alpha = \frac{q^2 - p_2^2}{2 \sqrt{q^2} \sqrt{p_2^2}}. \quad (72)$$

In D dimensions, we have

$$d^Dk_1 = dk_{10} \, |k_1|^{D-2} \, d|k_1| \, d\phi \, \sin \theta_1 \, d\theta_1 \, \sin^2 \theta_2 \, d\theta_2 \ldots \sin^{D-3} \theta_{D-3} \, d\theta_{D-3} \quad (73)$$

Thus in $D = 4 - 2\epsilon$ dimensions after doing the $\phi$ integration in the remaining $D - 2$ dimensions we get

$$d^Dk_1 \, \delta(k_1^2) = dk_{10} \, d|k_1| \, d \cos \theta \ \delta(k_{10}^2 - |k_1|^2) \ \frac{2\pi^{1-\epsilon}}{\Gamma(1 - \epsilon)} |k_1|^{2-2\epsilon} (\sin \theta)^{-2\epsilon}. \quad (74)$$

Thus Eq. (69) becomes:

$$I_{\text{nonplanar}}^{MC} = \frac{2}{i\pi} \int_0^\infty \frac{d|k_{10}|}{|k_{10}|} \int_0^1 \frac{d \cos \theta}{(1 - \epsilon)} \delta(k_{10}^2 - |k_1|^2) \frac{|k_1|^{2-2\epsilon} (\sin \theta)^{-2\epsilon}}{\Gamma(1 - 2\epsilon)} \ 
\times \delta(q^2 - 2k_{10}^2 q^2 + q^2)  \sqrt{Y_C} \left( \frac{Y_C}{\text{Gram}_C} \right)^{-\epsilon} \quad (75)$$

Now the integrations in $k_{10}$ and $|k_1|$ are trivial owing to the existence of the delta functions $^5$. Thus after performing these integrations and enforcing the condition $p_1^2 = p_2^2 = 0$ we get

$$I_{\text{nonplanar}}^{MC} = \frac{2\pi^{1-\epsilon} \epsilon}{\Gamma(1 - 2\epsilon)} (\sqrt{q^2})^{-3-2\epsilon} \int_{-1}^1 d \cos \theta \left( m^2 + \frac{q^2}{16} (1 - \cos^2 \theta) \right)^{-\frac{1}{2}-\epsilon} (1 - \cos^2 \theta)^{-\frac{1}{2}-\epsilon} \quad (76)$$

Performing the change of variables

$$\cos \theta = u \quad (77)$$

$^5$Here we have enforced the condition $k_{10} > 0$ for evaluating the delta function (see for reference Eq.(3.6) in [22])
we get,

\[ I_{MC}^{\text{nonplanar}} = \frac{4\pi e^{\gamma \epsilon}}{\Gamma(1 - 2\epsilon)} (\sqrt{q^2})^{-3 - 2\epsilon} \int_0^1 du \left( m^2 + \frac{q^2}{16} (1 - u^2) \right)^{-\frac{1}{2} - \epsilon} (1 - u^2)^{-\frac{1}{2} - \epsilon} \]  

(78)

We finally obtain,

\[ I_{MC}^{\text{nonplanar}} = \frac{4\pi e^{\gamma \epsilon}}{\Gamma(1 - 2\epsilon)} (\sqrt{q^2})^{-3 - 2\epsilon} \frac{\Gamma(1/2)\Gamma(1/2 - \epsilon)}{\Gamma(1 - \epsilon)} (m^2 + \frac{q^2}{16})^{-\frac{1}{2} - \epsilon} \times 2F_1(1/2 + \epsilon, 1/2; 1 - \epsilon; \frac{q^2}{q^2 + 16m^2}) \]  

(79)

3.5 Correlation between \( I\{−1,−1,−1,−1,−1\} \) and the maximal cut

For the non-planar diagram 6 in the given limit, we obtain the following relation:

\[ I_{MC}^{\text{nonplanar}} \sim \frac{\Gamma(1/2)\Gamma(1/2 - \epsilon)\Gamma(2 - \epsilon)\Gamma(4 + 2\epsilon)}{\Gamma(1 - \epsilon)\Gamma(1 - 2\epsilon)\Gamma(-\epsilon)\Gamma(2 + 2\epsilon)\Gamma^2(2 + \epsilon)} \left( \frac{m^2}{q^2} \right)^{\frac{3}{2} + \epsilon} \times \]  

\[ \left( 1 + \frac{q^2}{16m^2} \right)^{-\frac{1}{2} - \epsilon} 2F_1 \left( 1/2 + \epsilon, 1/2; 1 - \epsilon; \frac{q^2}{q^2 + 16m^2} \right) I\{−1,−1,−1,−1,−1\} \]  

(80)

4 Discussion and conclusion

We have considered given multi-scale Feynman diagrams in a given limit and obtained the scalings required for the asymptotic expansion of the diagram. The exploration here which is based on Landau equations allows to go beyond the bottom facet results. Furthermore ASY and ASPIRE were concerned with the unveiling the regions. Here Landau equations permit us to explore the consequences of the asymptotic analysis of the Feynman graphs combined with the corresponding maximal cuts.

A two point one loop diagram, a one loop triangular diagram and a two loop non-planar diagram have been studied. For these examples, we have found that the integral constructed based on the top facet scaling(with equal components) of the Newton polytope has the correspondence with the maximal cut of the corresponding Feynman diagrams, having the following form in \( D = 4 - 2\epsilon \),

\[ I_{MC} \sim f(m_i^2, Q_j^2, \epsilon) I\{−1,−1,−1,−1,−1\}, \]  

(81)

where \( m_i \) are the masses of the internal lines and \( Q_j \) are the external momenta.

The limit of expansion parameter tending to infinity is equivalent to the asymptotic expansion of a given Feynman diagram in the large mass expansion. There are prescriptions in the literature to deal with the large mass expansion in the language of expansion by sub-graphs [6,7]. Implementation of such prescription in the framework ASPIRE is a topic of future investigation.

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A Appendix

A.1 Brief description of the ancillary files

| File name              | Description                                                |
|------------------------|------------------------------------------------------------|
| OneLoopVertex.nb       | A one loop vertex integral has been analyzed.              |
| TwoPointOneLoop.nb     | A two point one loop diagram has been analyzed.            |
| ScalarTriangle.nb      | A scalar triangular diagram has been analyzed.             |
| TwoLoopNonPlanar.nb    | A non-planar two loop triangle diagram has been analyzed.  |

A.2 Comparison of the scales obtained using ASPIRE and ASY

In this section, we summarize the technical aspects of our consideration for the bottom and top facets of the Newton polytope obtained from the sum of the Symanzik polynomials with suitable linear transformations.

The bottom facets are those facets of the Newton polytope for which

\[ \vec{r} \cdot \vec{v} = c, \text{ for the points } \vec{r} \text{ lying on the facets,} \]

\[ \vec{r} \cdot \vec{v} > c, \text{ for the points } \vec{r} \text{ lying above the facets}, \]

where \( \vec{r} \) are the vector exponents of the terms of a given sum for the construction of the Newton polytope and \( \vec{v} \) are the normal vectors corresponding to the facets of the Newton polytope.

For bottom facets, we consider the limit \( x = \frac{m^2}{q^2} \to 0 \) (i.e. \( m^2 \ll q^2 \)) and \( q^2 \to 1 \). This is the well-known case of “Regions”.

The top facets are those facets of the Newton polytope for which

\[ \vec{r} \cdot \vec{v} = c, \text{ for the points } \vec{r} \text{ lying on the facets.} \]

\[ \vec{r} \cdot \vec{v} < c, \text{ for the points } \vec{r} \text{ lying below the facets.} \]

For the case of top facets, we utilize the freedom of considering the other possibility to take the expansion parameter \( x = \frac{m^2}{q^2} \to \infty \) (i.e. \( m^2 \gg q^2 \)) and we do not impose the constraint \( q^2 \to 1 \) while computing the Symanzik polynomials. This corresponds to the expansion of the Feynman graphs in the large mass limit.

It is trivial to see that the limit \( x = \frac{m^2}{q^2} \to 0 \) is equivalent to the limit \( x = \frac{q^2}{m^2} \to \infty \) and vice versa. This implies one can transform the bottom facets into top facets with the transformed limits and vice versa.
We here present the explicit comparison of the scaling coming from the bottom and top facets using ASPIRE and ASY for the given examples.

| Diagrams            | ASPIRE | ASY |
|---------------------|--------|-----|
|                     | Bottom facet | Top facet | Bottom facet | Top facet |
| Two point one loop  | \{0,0\}, \{-1/2,-1\}, \{-1,-1/2\} | \{-1,-1\} | \{0,0\}, \{0,1/2\}, \{0,-1/2\} | \{0,0\} |
| One loop triangle   | \{0,0,0\}, \{-1,0,-1\}, \{0,-1,-1\} | \{-1,-1,-1\} | \{0,0,0\}, \{0,0,0\}, \{0,-1,0\} | \{0,0,0\}, \{0,0,0\}, \{0,-1,0\} |
| Two loop non-planar | \{0,-1,0,0,-1,0\}, \{-1,-1,0,0,-1,0\}, \{0,0,0,0,0,0\}, \{0,-1,0,-1,0,-1\}, \{0,0,0,0,0,0\}, \{0,-1,1,-1,0,-1\}, \{0,0,0,0,0,0\}, \{0,0,1,0,0,0\}, \{0,0,1,1,0,0\}, | \{0,0,0,0,0,0\}, \{0,-1,0,-1,0,-1\} |  |

It immediately turns out that the scalings for the bottom and top facets as obtained from the package ASY and ASPIRE match exactly for the given examples.

### A.3 Hypergeometric Function $2F_1(a, b; c; x)$

The Hypergeometric function $2F_1(a, b; c; x)$ is given by,

$$2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{x^n}{n!},$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol. In the integral representation,

$$2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-xu)^{-a} du,$$

where $\text{Re}(b), \text{Re}(c) > 0$.

### A.4 Angular Integration

According to the convention followed in Equ.(A.1) in [19] which states that

$$d^Dk^E = d^{D-1}k_\parallel d^{D-c+1}k_\perp = \frac{1}{2}d^{D-1}k_\parallel d\Omega_{D-c}(k_\perp^{2(D-c+1)/2}d^2k_\perp$$

where $k_\parallel$ and $k_\perp$ are the parallel and perpendicular components to the set of cut propagators, the angular part of the integration is given by

$$\int d\Omega_D = \frac{2\pi^{(D+1)/2}}{\Gamma((D+1)/2)}$$

instead of

$$\int d\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

19
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