WEIGHTED VECTOR-VALUED FUNCTIONS AND THE ε-PRODUCT

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Abstract. We introduce a new class \( \mathcal{FA}(\Omega, E) \) of spaces of weighted functions on a set \( \Omega \) with values in a locally convex Hausdorff space \( E \) which covers many classical spaces of vector-valued functions like continuous, smooth, holomorphic or harmonic functions. Then we exploit the construction of \( \mathcal{FA}(\Omega, E) \) to derive sufficient conditions such that \( \mathcal{FA}(\Omega, E) \) can be linearised, i.e. that \( \mathcal{FA}(\Omega, E) \) is topologically isomorphic to the \( \varepsilon \)-product \( \mathcal{FA}(\Omega) \varepsilon E \) where \( \mathcal{FA}(\Omega) := \mathcal{FA}(\Omega, k) \) and \( k \) is the scalar field of \( E \).

1. Introduction

This work is dedicated to a classical topic, namely, the linearisation of spaces of weighted vector-valued functions. The setting we are interested in is the following. Let \( \mathcal{FA}(\Omega) \) be a locally convex Hausdorff space of functions from a non-empty set \( \Omega \) to a field \( K \) whose topology is generated by a family \( V \) of weight functions on \( \Omega \) and \( E \) be a locally convex Hausdorff space. The \( \varepsilon \)-product of \( \mathcal{FA}(\Omega) \) and \( E \) is defined as the space of linear continuous operators \( \mathcal{FA}(\Omega) \varepsilon E := L_e(\mathcal{FA}(\Omega)' k, E) \) equipped with the topology of uniform convergence on equicontinuous subsets of \( \mathcal{FA}(\Omega)' \) which itself is equipped with the topology of uniform convergence on absolutely convex compact subsets of \( \mathcal{FA}(\Omega) \). Suppose that there is a locally convex Hausdorff space \( \mathcal{FA}(\Omega, E) \) of \( E \)-valued functions on \( \Omega \) such that the map \( S : \mathcal{FA}(\Omega) \varepsilon E \rightarrow \mathcal{FA}(\Omega, E) \), \( u \mapsto [x \mapsto u(\delta_x)] \), is well-defined where \( \delta_x, x \in \Omega \), is the point-evaluation functional. The main question we want to answer reads as follows. When is \( \mathcal{FA}(\Omega) \varepsilon E \) a linearisation of \( \mathcal{FA}(\Omega, E) \), i.e. when is \( S \) a topological isomorphism?

In [2], [3] and [4] Bierstedt treats the space \( \mathcal{CV}(\Omega, E) \) of continuous functions on a completely regular Hausdorff space \( \Omega \) weighted with a Nachbin-family \( V \) and its topological subspace \( \mathcal{CV}_0(\Omega, E) \) of functions that vanish at infinity in the weighted topology. He derives sufficient conditions on \( \Omega, V \) and \( E \) such that the answer to our question is affirmative, i.e. \( S \) is a topological isomorphism. Schwartz answers this question for several spaces of weighted \( k \)-times continuously partially differentiable on \( \mathbb{R}^d \) like the Schwartz space in [36] and [37] for quasi-complete \( E \) with regard to vector-valued distributions. Grothendieck treats the question in [18], mainly for nuclear \( \mathcal{FA}(\Omega) \) and complete \( E \). In [24], [25] and [26] Komatsu gives a positive answer for ultradifferentiable functions of Beurling or Roumieux type and sequentially complete \( E \) with regard to vector-valued ultradistributions. For the space of \( k \)-times continuously partially differentiable functions on open subsets \( \Omega \) of infinite dimensional spaces equipped with the topology of uniform convergence...
of all partial derivatives up to order $k$ on compact subsets of $\Omega$ sufficient conditions for an affirmative answer are deduced by Meise in [32]. For holomorphic functions on open subsets of infinite dimensional spaces a positive answer is given in [11] by Dineen, Bonet, Frerick and Jordà show in [3] that $S$ is a topological isomorphism for certain closed subsheafs of the sheaf $C^\infty(\Omega, E)$ of smooth functions on an open subset $\Omega \subset \mathbb{R}^d$ with the topology of uniform convergence of all partial derivatives on compact subsets of $\Omega$ and locally complete $E$ which, in particular, covers the spaces of harmonic and holomorphic functions.

In [8], [10] and [17] linearisation is used to derive results on extensions of vector-valued functions and weak-strong principles. Another application of linearisation is within the field of partial differential equations. Let $P(\partial)$ be an elliptic linear partial differential operator with constant coefficients and $C^\infty(\Omega) := C^\infty(\Omega, \mathbb{K})$. Then

$$P(\partial): C^\infty(\Omega) \to C^\infty(\Omega)$$

is surjective by [20, Corollary 10.6.8, p. 43] and [20, Corollary 10.8.2, p. 51]. Due to [23, Satz 10.24, p. 255], the nuclearity of $C^\infty(\Omega, E)$ within the field of partial differential equations. Let $E$ for Fréchet spaces and of Bonet and Domański for PLS-spaces we even have that $\Omega$ holds via $\partial M$, $\partial E$ and when do they fulfil our sufficient conditions. We close this work with many

2. Notation and Preliminaries

We equip the spaces $\mathbb{R}^d$, $d \in \mathbb{N}$, and $\mathbb{C}$ with the usual Euclidean norm $|\cdot|$. denote by $B_r(x) := \{ w \in \mathbb{R}^d \mid |w - x| < r \}$ the ball around $x \in \mathbb{R}^d$ with radius $r > 0$. Furthermore, for a subset $M$ of a topological space $X$ we denote the closure of $M$ by $\overline{M}$ and the boundary of $M$ by $\partial M$. For a subset $M$ of a vector space $X$ we denote by $\text{ch}(M)$ the circled hull, by $\text{cx}(M)$ the convex hull and by $\text{ax}(M)$ the absolutely convex hull of $M$. If $X$ is a topological vector space, we write $\text{max}(M)$
for the closure of $acx(M)$ in $X$.

By $E$ we always denote a non-trivial locally convex Hausdorff space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ equipped with a directed fundamental system of seminorms $(p_\alpha)_{\alpha \in A}$ and, in short, we write $E$ is an lcHs. If $E = \mathbb{K}$, then we set $(p_\alpha)_{\alpha \in A} := \{\cdot\}$.

For details on the theory of locally convex spaces see [15], [22] or [33]. By $X^\Omega$ we denote the set of maps from a non-empty set $\Omega$ to a non-empty set $X$, by $\chi_K$ we mean the characteristic function of $K \subset \Omega$, by $C(\Omega, X)$ the space of continuous functions from a topological space $\Omega$ to a topological space $X$ and by $L(F, E)$ the space of continuous linear operators from $F$ to $E$ where $F$ and $E$ are locally convex Hausdorff spaces. If $E = \mathbb{K}$, we just write $F^\prime := L(F, \mathbb{K})$ for the dual space and $G^\circ$ for the polar set of $G \subset F$. If $F$ and $E$ are (linearly) topologically isomorphic, we write $F \equiv E$. We denote by $L_\epsilon(F, E)$ the space $L(F, E)$ equipped with the locally convex topology $t$ of uniform convergence on the finite subsets of $F$ if $t = \sigma$, on the absolutely convex, compact subsets of $F$ if $t = \kappa$, on the absolutely convex, $\sigma(F, F^\prime)$-compact subsets of $F$ if $t = \tau$, on the precompact (totally bounded) subsets of $F$ if $t = \gamma$ and on the bounded subsets of $F$ if $t = b$. We use the symbols $\epsilon(F^\prime, F)$ for the correspondence topology on $F^\prime$ and $\epsilon(F)$ for corresponding bornology on $F$. The so-called $\epsilon$-product of Schwartz is defined by

$$F \epsilon E := L_\epsilon(F^\prime, E)$$

(1)

where $L(F^\prime, E)$ is equipped with the topology of uniform convergence on equicontinuous subsets of $F^\prime$. This definition of the $\epsilon$-product coincides with the original one by Schwartz [37, Chap. I, §1, Définition, p. 18]. It is symmetric which means that $F \epsilon E \equiv E \epsilon F$. In the literature the definition of the $\epsilon$-product is sometimes done the other way around, i.e. $E \epsilon F$ is defined by the right-hand side of (1) but due to the symmetry these definitions are equivalent and for our purpose the given definition is more suitable. If we replace $F^\prime$ by $F^\prime_\epsilon$, we obtain Grothendieck’s definition of the $\epsilon$-product and we remark that the two $\epsilon$-products coincide if $F$ is quasi-complete because then $F^\prime_\epsilon = F^\prime_\kappa$ holds. However, we stick to Schwartz’ definition. For more information on the theory of $\epsilon$-products see [22] and [23]. Further, for a disk $D \subset F$, i.e. a bounded, absolutely convex set, the vector space $F_D := \bigcup_{n \in \mathbb{N}} nD$ becomes a normed space if it is equipped with gauge functional of $D$ as a norm (see [22, p. 151]). The space $F$ is called locally complete if $F_D$ is a Banach space for every closed disk $D \subset F$ (see [22, 10.2.1 Proposition, p. 197]).

3. The $\epsilon$-PRODUCT FOR WEIGHTED FUNCTION SPACES

In this section we introduce the space $\mathcal{F}V(\Omega, E)$ of weighted $E$-valued functions on $\Omega$ as the section of domains and kernels in $E^\Omega$ of linear operators $T^E_m$ equipped with a generalised version of a weighted graph topology. This space is the role model for many function spaces and as an example for these operators we can think of the partial derivative operators. Then we treat the question whether we can identify $\mathcal{F}V(\Omega, E)$ with $\mathcal{F}V(\Omega, E) \epsilon E$ topologically. This is deeply connected with the interplay of the pair of operators $(T^E_m, T^E_m)$ with the map $S$ from the introduction (see Definition 3.14). In our main theorem we give sufficient conditions such that $\mathcal{F}V(\Omega, E) \equiv \mathcal{F}V(\Omega, E) \epsilon E$ holds (see Theorem 3.14).

We begin with the definition of a family of weight functions which we want to use to define a kind of weighted graph topology.

3.1. Definition (weight function). Let $\Omega$, $J$, $L$ be non-empty sets and $(M_l)_{l \in L}$ a family of non-empty sets. We call $\nu := (\nu_{j, l, m}: j \in J, l \in L, m \in M_l)$ a family of weight functions on $\Omega$ if $\nu_{j, l, m}: \Omega \to [0, \infty)$ for every $m \in M_l$, $j \in J$ and $l \in L$ and

$$\forall \ x \in \Omega, \ l \in L \ \exists \ j \in J \ \forall \ m \in M_l : \ 0 < \nu_{j, l, m}(x).$$

(2)
Now, the spaces we want to consider are built up in the following way.

3.2. **Definition.** Let \( V := (\nu_{j,l,m})_{j,l\in L} \) be a family of weight functions on \( \Omega \) and \( M_{\text{top}} := \bigcup_{m\in M} M_m \). Let \( M_0 \) and \( M_r \) be sets, \( M_{\text{top}}, M_0 \) and \( M_r \) be pairwise disjoint and \( M := M_{\text{top}} \cup M_0 \cup M_r \). Let \( (\omega_m)_{m\in M} \) be a family of non-empty sets such that \( \Omega \subset \omega_m \) for every \( m \in M_{\text{top}} \) and \( T_m^E, E^\Omega \supset \ker T_m^E \) is a linear map for every \( m \in M \). We define the space of intersections

\[
W_M(\Omega, E) := \left( \bigcap_{m\in M} \ker T_m^E \right) \cap \left( \bigcap_{m\in M_0} \text{dom} T_m^E \right)
\]

as well as

\[
\mathcal{F}V(\Omega, E) := \{ f \in W_M(\Omega, E) \mid \forall j, l \in L, \alpha \in A : |f|_{j,l,\alpha} < \infty \}
\]

where

\[
|f|_{j,l,\alpha} := \sup_{x \in \Omega} p_{\alpha}(T_m^E(f)(x))\nu_{j,l,m}(x).
\]

Further, we write \( \mathcal{F}V(\Omega) := \mathcal{F}V(\Omega, \mathbb{K}) \). If we want to emphasise dependencies, we write \( \mathcal{M}(\mathcal{F}V) \) or \( \mathcal{M}(E) \) instead of \( \mathcal{M} \) and the same for \( M_{\text{top}}, M_0 \) and \( M_r \).

The space \( \mathcal{F}V(\Omega, E) \) is a locally convex Hausdorff space due to condition (2). Since it is easier to work with a directed family of seminorms and the (continuity of the) point evaluation functionals \( \delta_x : \mathcal{F}V(\Omega) \to \mathbb{K}, f \mapsto f(x) \), for \( x \in \Omega \) play a big role, we make the following definition.

3.3. **Definition** (\( \text{dom}-\) and \( T_m^E, E^\Omega \)). We call \( \mathcal{F}V(\Omega, E) \) a \( \text{dom}-\) space if the system of seminorms \( (|f|_{j,l,\alpha})_{j,l\in L,\alpha\in A} \) is directed and, in addition, \( \delta_x \in \mathcal{F}V(\Omega)^\prime \) for every \( x \in \Omega \) if \( E = \mathbb{K} \). We define the point evaluation of \( T_m^E \) by \( T_m^E : \text{dom} T_m^E \to E \), \( T_m^E(f) := T_m^E(f)(x) \), for \( m \in M \) and \( x \in \omega_m \).

3.4. **Remark.** It is easy to see that the system of seminorms \( (|f|_{j,l,\alpha})_{j,l\in L,\alpha\in A} \) is directed if

\[
\forall j_1, j_2 \in J, l_1, l_2 \in L \exists j_3 \in J, l_3 \in L, C > 0 \forall i \in \{1, 2\}, m \in M_i : (M_i \cup M_{l_2}) \subset M_{j_3} \quad \text{and} \quad \nu_{j, l, m} \leq C \nu_{j_3, l_3, m}
\]

since the system \( (p_{\alpha})_{\alpha\in A} \) of \( E \) is already directed. If there is \( m \in M_{\text{top}} \) such that \( T_m^E = \text{id}_{\text{dom} T_m^E} \), then \( \delta_x \in \mathcal{F}V(\Omega)^\prime \) for all \( x \in \Omega \) by (2).

The next lemma describes the topology of \( \mathcal{F}V(\Omega) \in E \) in terms of the operators \( T_m^E \) with \( m \in M_{\text{top}} \) and is a preparation to consider \( \mathcal{F}V(\Omega) \in E \) as a topological subspace of \( \mathcal{F}V(\Omega, E) \) under certain conditions.

3.5. **Lemma.** Let \( \mathcal{F}V(\Omega) \) be a \( \text{dom}-\) space. Then the following holds.

a) \( T_m^E \in \mathcal{F}V(\Omega)^\prime \) for all \( x \in \Omega \) and \( m \in M_{\text{top}} \).

b) The topology of \( \mathcal{F}V(\Omega) \in E \) is given by the system of seminorms defined by

\[
\|u\|_{j,l,\alpha} := \sup_{m \in M_l} p_{\alpha}(u(T_m^E))\nu_{j,l,m}(x), \quad u \in \mathcal{F}V(\Omega) \in E,
\]

for \( j \in J, l \in L \) and \( \alpha \in A \).

**Proof.**

a) For \( x \in \Omega \) and \( m \in M_{\text{top}} \) there exist \( l \in L \) and \( j \in J \) such that \( m \in M_l \) and \( \nu_{j,l,m}(x) > 0 \) by (2) implying for every \( f \in \mathcal{F}V(\Omega) \) that

\[
|T_m^E(f)| = \frac{1}{\nu_{j,l,m}(x)}|T_m^E(f)(x)| \nu_{j,l,m}(x) \leq \frac{1}{\nu_{j,l,m}(x)} |f|_{j,l}.
\]
b) First, we set $D_{j,l} := \{T^k_{m,x}(x) | x \in \Omega, m \in M_j \}$ and $B_{j,l} := \{f \in \mathcal{F}\mathcal{V}(\Omega) \mid |f|_{j,l} \leq 1 \}$ for every $j \in J$ and $l \in L$. We claim that $\mathcal{a}(D_{j,l})$ is dense in the polar $B_{j,l}^o$ with respect to $\kappa(\mathcal{F}\mathcal{V}(\Omega), \mathcal{F}\mathcal{V}(\Omega))$. The observation

$$
D_{j,l}^o = (T^k_{m,x}(x))_{x \in \Omega, m \in M_j}^o
= \{f \in \mathcal{F}\mathcal{V}(\Omega) \mid \forall x \in \Omega, m \in M_j ; |T^k_{m}(f)(x)|_{j,l,m}(x) \leq 1 \}
= \{f \in \mathcal{F}\mathcal{V}(\Omega) \mid |f|_{j,l} \leq 1 \} = B_{j,l},
$$

yields

$$
\mathcal{a}(D_{j,l}) = (\mathcal{a}(D_{j,l}))^o = B_{j,l}^o
$$

by the bipolar theorem. By [22, 8.4, p. 152, 8.5, p. 156-157] the system of seminorms defined by

$$
q_{j,l,\alpha}(u) := \sup_{y \in B_{j,l}^o} p_{\alpha}(u(y)), \quad u \in \mathcal{F}\mathcal{V}(\Omega) \mathcal{c} E,
$$

for $j \in J$, $l \in L$ and $\alpha \in \mathfrak{A}$ gives the topology on $\mathcal{F}\mathcal{V}(\Omega) \mathcal{c} E$ (here it is used that the system of seminorms $(| \cdot |_{j,l})$ of $\mathcal{F}\mathcal{V}(\Omega)$ is directed). We may replace $B_{j,l}^o$ by a $\kappa(\mathcal{F}\mathcal{V}(\Omega), \mathcal{F}\mathcal{V}(\Omega))$-dense subset as every $u \in \mathcal{F}\mathcal{V}(\Omega) \mathcal{c} E$ is continuous on $B_{j,l}^o$. Therefore we obtain

$$
q_{j,l,\alpha}(u) = \sup\{ p_{\alpha}(u(y)) \mid y \in \mathcal{a}(D_{j,l}) \}.
$$

For $y \in \mathcal{a}(D_{j,l})$ there are $n \in \mathbb{N}$, $\lambda_k \in \mathbb{K}$, $m_k \in M_j$, $x_k \in \Omega$, $1 \leq k \leq n$, with $\sum_{k=1}^n |\lambda_k| = 1$ such that $y = \sum_{k=1}^n \lambda_k T^k_{m_k,x_k} \nu_{j,l,m_k}(x_k)$. Then we have for every $u \in \mathcal{F}\mathcal{V}(\Omega) \mathcal{c} E$

$$
p_{\alpha}(u(y)) \leq \sum_{k=1}^n |\lambda_k| p_{\alpha}(u(T^k_{m_k,x_k})) \nu_{j,l,m_k}(x_k) \leq \|u\|_{j,l,\alpha}
$$

thus $q_{j,l,\alpha}(u) \leq \|u\|_{j,l,\alpha}$. On the other hand, we derive

$$
q_{j,l,\alpha}(u) \geq \sup_{y \in B_{j,l}^o} p_{\alpha}(u(y)) = \sup_{x \in M_j} p_{\alpha}(u(T^k_{m,x})) \nu_{j,l,m}(x) = \|u\|_{j,l,\alpha}.
$$

\[
\square
\]

For the lcHs $E$ over $\mathbb{K}$ we want to define a natural $E$-valued version of a dom-space $\mathcal{F}\mathcal{V}(\Omega) = \mathcal{F}\mathcal{V}(\Omega, \mathbb{K})$. Let $\mathcal{M} := M_{\text{top}} \cup M_{\text{r}} \cup M_{\text{r}}$ be the index set associated to $\mathcal{F}\mathcal{V}(\Omega)$. The natural $E$-valued version of $\mathcal{F}\mathcal{V}(\Omega)$ should be a dom-space $\mathcal{F}\mathcal{V}(\Omega, E)$ such that the three parts of its index set coincide with the corresponding parts of $\mathcal{M}$ and there is a canonical relation between the families $(T^k_{m})$ and $(T^k_{m})$. This canonical relation will be explained in terms of their interplay with the map

$$
S: \mathcal{F}\mathcal{V}(\Omega) \mathcal{c} E \rightarrow E^{\Omega}, \quad u \mapsto [x \mapsto u(\delta_x)].
$$

3.6. Definition (defining, consistent). Let $\mathcal{F}\mathcal{V}(\Omega)$ and $\mathcal{F}\mathcal{V}(\Omega, E)$ be dom-spaces such that $M_{\text{top}}(\mathbb{K}) = M_{\text{top}}(E)$, $M_{\text{r}}(\mathbb{K}) = M_{\text{r}}(E)$ and $M_{\text{r}}(\mathbb{K}) = M_{\text{r}}(E)$. Let $\mathcal{M} := \mathcal{M}(\mathbb{K}) = \mathcal{M}(E)$.

a) We call $(T^k_{m}, T^k_{m})_{m \in \mathcal{M}}$ a defining family for $\mathcal{F}\mathcal{V}(\Omega, E)$, in short, $(\mathcal{F}\mathcal{V}, E)$, b) We call $(T^k_{m}, T^k_{m})_{m \in \mathcal{M}}$ consistent if we have for every $u \in \mathcal{F}\mathcal{V}(\Omega) \mathcal{c} E$, $m \in \mathcal{M}$ and $x \in \omega_m$:

(i) $S(u) \in \text{dom} T^E_m$ and $T^k_{m,x} \in \mathcal{F}\mathcal{V}(\Omega)'$,

(ii) $(T^E_m S(u))(x) = u(T^k_{m,x})$.

c) Let $\mathcal{N} \subset \mathcal{M}$. We call $(T^E_m, T^k_{m})_{m \in \mathcal{N}}$ a consistent subfamily, if (i) and (ii) are fulfilled for every $m \in \mathcal{N}$. 

More precisely, $T_{m,x}^\mathbb{K}$ in (i) and (ii) means the restriction of $T_{m,x}^\mathbb{K}$ to $\mathcal{FV}(\Omega)$. Consistency is our measure whether we consider a space $\mathcal{FV}(\Omega,E)$ as a natural $E$-valued version of a space $\mathcal{FV}(\Omega)$ of scalar-valued functions.

3.7. Theorem. Let $(T_m^E, T_m^\mathbb{K})_{m \in \mathcal{M}}$ be a consistent family for $(\mathcal{FV}, E)$. Then the map $S: \mathcal{FV}(\Omega) \ni E \to \mathcal{FV}(\Omega,E)$ is a topological isomorphism into.

Proof. First, we show that $S(\mathcal{FV}(\Omega)\in E) \subset \mathcal{FV}(\Omega,E)$. Let $u \in \mathcal{FV}(\Omega)\in E$. Due to the consistency of $(T_m^E, T_m^\mathbb{K})_{m \in \mathcal{M}}$ we have $S(u) \in \text{dom } T_m^E$ and

$$(T_m^E S(u))(x) = u(T_{m,x}^\mathbb{K}), \quad m \in \mathcal{M}, \ x \in \omega_m.$$ 

For $m \in \mathcal{M}_0$ we get $T_{m,x}^\mathbb{K} = 0$ on $\mathcal{FV}(\Omega)$ for all $x \in \omega_m$ and thus

$$(T_m^E S(u))(x) = u(T_{m,x}^\mathbb{K}) = u(0) = 0, \quad x \in \omega_m.$$ 

Hence $S(u) \in \ker T_m^E$ for every $m \in \mathcal{M}_0$. Furthermore, we get by Lemma 3.5 b) for every $j \in J$, $l \in L$ and $\alpha \in \mathfrak{A}$

$$|S(u)|_{j,l,\alpha} = \sup_{x \in \Omega} p_{\alpha}(T_m^E(S(u))(x)) \nu_{j,l,m}(x) = \|u\|_{j,l,\alpha} < \infty$$

implying $S(u) \in \mathcal{FV}(\Omega,E)$ and the continuity of $S$. Moreover, we deduce from (3) that $S$ is injective and that the inverse of $S$ on the range of $S$ is also continuous. □

3.8. Remark. If $J$, $L$ and $\mathfrak{A}$ are countable, then $S$ is an isometry with respect to the induced metrics on $\mathcal{FV}(\Omega,E)$ and $\mathcal{FV}(\Omega)\ni E$ by (3).

The basic idea for Theorem 3.7 was derived from analysing the proof of an analogous statement for weighted continuous functions by Bierstedt [3, 4.2 Lemma, 4.3 Folgerung, p. 199-200] and [4, 2.1 Satz, p. 137]. Now, we try to answer the natural question. When is $S$ surjective? A weaker concept to define a natural $E$-valued version of $\mathcal{FV}(\Omega)$ will help us to answer the question. Let $\mathcal{FV}(\Omega)$ be a $\mathbb{K}$-space. We define the vector space of $E$-valued weak $\mathcal{FV}$-functions by

$$\mathcal{FV}(\Omega,E)_\sigma := \{ f: \Omega \to E' \mid \forall e' \in E' : e' \circ f \in \mathcal{FV}(\Omega) \}.$$ 

Moreover, for $f \in \mathcal{FV}(\Omega,E)_\sigma$ we define the linear map

$$R_f: E' \to \mathcal{FV}(\Omega), \ R_f(e') := e' \circ f,$$

and the dual map

$$R_f^\sharp: \mathcal{FV}(\Omega)' \to E'^* , \ f' \mapsto [e' \mapsto f'(R_f(e'))],$$

where $E'^*$ is the algebraic dual of $E'$. Furthermore, we set

$$\mathcal{FV}(\Omega,E)_\sigma := \{ f \in \mathcal{FV}(\Omega,E)_\sigma \mid \forall \alpha \in \mathfrak{A} : R_f(B_{\alpha}^E) \text{ relatively compact in } \mathcal{FV}(\Omega) \}$$

where $B_{\alpha} := \{ x \in E \mid p_{\alpha}(x) < 1 \}$ for $\alpha \in \mathfrak{A}$. Next, we give a sufficient condition for the inclusion $\mathcal{FV}(\Omega,E) \subset \mathcal{FV}(\Omega,E)_\sigma$ by means of the family $(T_m^E, T_m^\mathbb{K})_{m \in \mathcal{M}}$.

3.9. Definition (strong). Let $(T_m^E, T_m^\mathbb{K})_{m \in \mathcal{M}}$ be a defining family for $(\mathcal{FV}, E)$. We call $(T_m^E, T_m^\mathbb{K})_{m \in \mathcal{M}}$ strong if the following is valid for every $e' \in E'$, $f \in \mathcal{FV}(\Omega,E)$ and $m \in \mathcal{M}$:

$$\begin{align*}
(i) \quad & e' \circ f \in \text{dom } T_m^\mathbb{K}, \\
(ii) \quad & T_m^\mathbb{K}(e' \circ f) = e' \circ T_m^E(f) \text{ on } \omega_m.
\end{align*}$$

Let $\mathcal{N} \subset \mathcal{M}$. We call $(T_m^E, T_m^\mathbb{K})_{m \in \mathcal{N}}$ a strong subfamily if (i) and (ii) are fulfilled for every $m \in \mathcal{N}$. 

3.10. Lemma. If \((T^E_m, T^K_m)_{m \in M}\) is a strong family for \((\mathcal{F}V, E)\), then \(\mathcal{F}V(\Omega, E) \subset \mathcal{F}V(\Omega, E)_{\sigma}\) and
\[
\sup_{e \in B_{\varepsilon}^0} |R_f(e')|_{j,l} = |f|_{j,l,\alpha} = \sup_{x \in N_{j,l}(f)} p_\alpha(x)
\]
for every \(f \in \mathcal{F}V(\Omega, E)\), \(j \in J\), \(l \in L\) and \(\alpha \in \mathfrak{A}\) where
\[
N_{j,l}(f) := \{T^E_m(f)(x) \mid x \in \Omega, m \in M_0\}.
\]
Proof. Let \(f \in \mathcal{F}V(\Omega, E)\). Since \((T^E_m, T^K_m)_{m \in M}\) is a strong family, we have \(e' \circ f \in \text{dom} T^K_m\) for every \(m \in M\) and \(e' \in E'\). Further, we obtain for every \(m \in M_0\)
\[
T^K_m(e' \circ f) = e' \circ T^K_m(f) = 0
\]
because \(f \in \ker T^K_m\). Moreover, we have
\[
\sup_{e \in B_{\varepsilon}^0} |R_f(e')|_{j,l} = |e' \circ f|_{j,l} = \sup_{x \in \Omega} |T^K_m(e' \circ f)(x)|_{j,l,m}(x)
\]
\[
= \sup_{x \in \Omega} |e'(T^K_m(f)(x))|_{j,l,m}(x) = \sup_{x \in N_{j,l}(f)} |e'(x)|
\]
for every \(j \in J\) and \(l \in L\). Further, we observe that
\[
\sup_{e \in B_{\varepsilon}^0} |R_f(e')|_{j,l} = |f|_{j,l,\alpha} = \sup_{x \in N_{j,l}(f)} p_\alpha(x) < \infty
\]
for every \(j \in J\), \(l \in L\) and \(\alpha \in \mathfrak{A}\) where the first equality holds due to [33] Proposition 22.14, p. 256]. In particular, we obtain that \(N_{j,l}(f)\) is bounded in \(E\) and thus weakly bounded implying that the right-hand side of \((\text{4})\) is finite. Hence we conclude \(f \in \mathcal{F}V(\Omega, E)_{\sigma}\).

Now, we phrase some sufficient conditions for \(\mathcal{F}V(\Omega, E) \subset \mathcal{F}V(\Omega, E)_{\kappa}\) to hold which is one of the key points regarding the surjectivity of \(S\).

3.11. Lemma. If \((T^E_m, T^K_m)_{m \in M}\) is a strong family for \((\mathcal{F}V, E)\) and one of the following conditions is fulfilled, then \(\mathcal{F}V(\Omega, E) \subset \mathcal{F}V(\Omega, E)_{\kappa}\).

a) \(\mathcal{F}V(\Omega)\) is a semi-Montel space.
b) \(E\) is a semi-Montel or Schwartz space.
c) \(\pi \in X, a \in \mathfrak{A}\) such that \(\mathcal{F}V(\Omega, E)\) vanish at infinity in the weighted topology with respect to \((\pi, \mathfrak{R})\), i.e. every \(f \in \mathcal{F}V(\Omega, E)\) fulfils
\[
\forall \varepsilon > 0, \exists \gamma(E) : N_{j,l}(f) \subset K.
\]
\(\exists \gamma(E)\) is a semi-Montel or Schwartz space.
d) There are a set \(X\), a family \(\mathfrak{R}\) of sets and a map \(\pi : \Omega \times M_{\text{top}} \rightarrow X\) such that \(\bigcup_{K \in \mathfrak{R}} K \subset X\) and the functions of \(\mathcal{F}V(\Omega, E)\) vanish at infinity in the weighted topology with respect to \((\pi, \mathfrak{R})\), i.e. every \(f \in \mathcal{F}V(\Omega, E)\) fulfils:
\[
\forall \varepsilon > 0, j \in J, l \in L, \exists K \in \mathfrak{R}:
\]
\(\forall x \in \Omega, m \in M_{l} : (x, m) \in K\) \(\in \gamma(E)\)
\[
(i) \sup_{x \in \Omega, m \in M_{l}} p_\alpha(T^E_m(f)(x))_{j,l,m}(x) < \varepsilon
\]
(ii) \(N_{j,l}(f) := \{T^E_m(f)(x)_{j,l,m}(x) \mid x \in \Omega, m \in M_{l} : (x, m) \in K\} \in \gamma(E)\)

Proof. Let \(f \in \mathcal{F}V(\Omega, E)\). By virtue of Lemma 3.10 we already have \(f \in \mathcal{F}V(\Omega, E)_{\sigma}\).

a) For every \(j \in J\), \(l \in L\) and \(\alpha \in \mathfrak{A}\) we derive from
\[
\sup_{e \in B_{\varepsilon}^0} |R_f(e')|_{j,l} = |f|_{j,l,\alpha} < \infty
\]
that \(R_f(B_{\varepsilon}^0)\) is bounded and thus relatively compact in the semi-Montel space \(\mathcal{F}V(\Omega)\).
b) It follows from \((\text{5})\) that \(R_f \in L(E'_\alpha, \mathcal{F}V(\Omega))\). Further, the polar \(B_{\alpha}^0\) is relatively compact in \(E'_\alpha\) for every \(\alpha \in \mathfrak{A}\) by the Alaoglu-Bourbaki theorem. The continuity of \(R_f\) implies that \(R_f(B_{\alpha}^0)\) is relatively compact as well.
c) Let \( j \in J \) and \( l \in L \). The set \( K := N_{j,l}(f) \) is bounded in \( E \) by \( [10] \). If \( E \) is semi-Montel or Schwartz, we deduce that \( K \) is already precompact in \( E \) since it is relatively compact if \( E \) is semi-Montel resp. by \([22, 10.4.3 \text{ Corollary, p. 202}] \) if \( E \) is Schwartz. Hence the statement follows from b).

d) We show that the set \( N_{j,l}(f) \) is precompact in \( E \) for every \( f \in \mathcal{FV}(\Omega, E) \), \( j \in J \) and \( l \in L \). Let \( V \) be a 0-neighborhood in \( E \). Then there are \( \alpha \in \mathbb{R} \) and \( \varepsilon > 0 \) such that \( B_{\varepsilon,\alpha} \subset V \) where \( B_{\varepsilon,\alpha} := \{ x \in E \mid p_{\alpha}(x) < \varepsilon \} \). Due to \([5] \) there is \( K \in \mathcal{R} \) such that the set

\[
N_{\pi K,j,l}(f) := \{ T_m^E(f)(x) \nu_{j,l,m}(x) \mid x \in \Omega, \ m \in M_l, \ \pi(x,m) \notin K \}
\]

is contained in \( B_{\varepsilon,\alpha} \). Further, the precompactness of \( N_{\pi K,j,l}(f) \) implies that there exists a finite set \( P \subset E \) such that \( N_{\pi K,j,l}(f) \subset P + V \). Hence we conclude

\[
N_{j,l}(f) = (N_{\pi K,j,l}(f) \cup N_{\pi K,j,l}(f)) \subset (B_{\varepsilon,\alpha} \cup (P + V)) \subset (V \cup (P + V)) = (P \cup \{0\}) + V
\]

which means that \( N_{j,l}(f) \) is precompact proving the statement by b).

\[ \square \]

Concerning d), in all examples we consider later on we have to assume that \( \mathcal{R} \) is closed under taking finite unions (see Proposition \([12] \). The most common case is that \( \mathcal{R} \) consists of the compact subsets of \( \Omega \) and \( \pi \) is the projection on \( X := \Omega \). But we consider other examples in Example \([60] \) as well.

3.12. Remark. Let \( \mathcal{FV}(\Omega, E) \) be a dom-space, \( \Omega \) be a topological Hausdorff space, \( M_l \) be finite for every \( l \in L \), every \( \nu \in \mathcal{V} \) be bounded on the compact subsets of \( \Omega \), every \( f \in \mathcal{FV}(\Omega, E) \) fulfill \([10] \) with \( \mathcal{R} := \{ K \subset \Omega \mid K \text{ compact} \} \) and \( \pi \) be the projection on \( X := \Omega \). If \( T_m^E(f) \in C(\Omega, E) \) for every \( f \in \mathcal{FV}(\Omega, E) \) and \( m \in M_{\text{top}} \), then \( N_{\pi K,j,l}(f) \) is precompact in \( E \) for every \( f \in \mathcal{FV}(\Omega, E) \), \( K \in \mathcal{R} \), \( j \in J \) and \( l \in L \).

Proof. Let \( f \in \mathcal{FV}(\Omega, E) \), \( K \in \mathcal{R} \), \( j \in J \) and \( l \in L \). Writing

\[
N_{\pi K,j,l}(f) = \bigcup_{m \in M_l} T_m^E(f)(K) \nu_{j,l,m}(K),
\]

we see that we only have to prove that the sets \( T_m^E(f)(K) \nu_{j,l,m}(K) \) are precompact since \( N_{\pi K,j,l}(f) \) is a finite union of these sets. But this is a consequence of the proof of \([2, \S 1, 16, \text{ Lemma, p. 15}] \).

Let us turn to sufficient conditions for \( \mathcal{FV}(\Omega, E) \equiv \mathcal{FV}(\Omega) \circ E \). For the lcHs \( E \) we denote by \( J: E \rightarrow E^* \) the canonical injection.

3.13. Properties. Let \( (T_m^E, T_m^X)_{m \in M} \) be a strong family for \( (\mathcal{FV}, E) \). Define the following conditions:

- a) \( E \) is complete.
- b) \( E \) is quasi-complete and for every \( f \in \mathcal{FV}(\Omega, E) \) and \( f' \in \mathcal{FV}(\Omega)' \) there is a bounded net \( (f'_\tau)_{\tau \in \mathcal{T}} \) in \( \mathcal{FV}(\Omega)' \) converging to \( f' \) in \( \mathcal{FV}(\Omega)' \), such that \( R^E(f'_\tau) \in J(E) \) for every \( \tau \in \mathcal{T} \).
- c) \( E \) is sequentially complete and for every \( f \in \mathcal{FV}(\Omega, E) \) and \( f' \in \mathcal{FV}(\Omega)' \) there is a sequence \( (f'_n)_{n \in \mathbb{N}} \) in \( \mathcal{FV}(\Omega)' \) converging to \( f' \) in \( \mathcal{FV}(\Omega)' \), such that \( R^E(f'_n) \in J(E) \) for every \( n \in \mathbb{N} \).
- d) \( \forall \ f \in \mathcal{FV}(\Omega, E) \), \( j \in J \), \( l \in L \) \( \exists \ K \in \tau(E) : N_{j,l}(f) \subset K \).
3.14. Theorem. Let $(T^E_m, T^\mathbb{C}_m)_{m \in \mathcal{M}}$ be a consistent family for $(\mathcal{F}V, E)$ and let $\mathcal{F}V(\Omega, E) \subset \mathcal{F}V(\Omega, E)_{\kappa}$. If one of the Properties 2.13 is fulfilled, then $\mathcal{F}V(\Omega, E) \cong \mathcal{F}V(\Omega) \in E$ via $S$. The inverse of $S$ is given by the map

$$R^t : \mathcal{F}V(\Omega, E) \rightarrow \mathcal{F}V(\Omega) \in E, \quad f \mapsto J^{-1} \circ R^f.$$ 

Proof. Due to Theorem 5.4, we only have to show that $S$ is surjective. We equip $\mathcal{J}(E)$ with the system of seminorms given by

$$p_{B^\alpha}(\mathcal{J}(x)) := \sup_{e' \in B^\alpha_n} |\mathcal{J}(x)(e')| = p_\alpha(x), \quad x \in E, \quad (7)$$

for every $\alpha \in \mathfrak{A}$. Let $f \in \mathcal{F}V(\Omega, E)$. We consider the dual map $R^f$ and claim that $R^f \in L(\mathcal{F}V(\Omega)_{\kappa}, \mathcal{J}(E))$. Indeed, we have

$$p_{B^\alpha}(R^f(y)) = \sup_{e' \in B^\alpha_n} |y(R^f(e'))| = \sup_{x \in R^f(B^\alpha_n)} |y(x)| \leq \sup_{x \in K_\alpha} |y(x)| \quad (8)$$

where $K_\alpha := R^f(B^\alpha_n)$. Since $\mathcal{F}V(\Omega, E) \subset \mathcal{F}V(\Omega, E)_{\kappa}$, the set $R^f(B^\alpha_n)$ is absolutely convex and relatively compact implying that $K_\alpha$ is absolutely convex and compact in $\mathcal{F}V(\Omega)$ by [22, 6.2.1 Proposition, p. 103]. Further, we have for all $e' \in E'$ and $x \in \Omega$

$$R^f(f)(e') = \delta_x(e' \circ f) = e'(f(x)) = J(f(x))(e')$$

and thus $R^f(f)(e') \in \mathcal{J}(E)$.

a) Let $E$ be complete and $f' \in \mathcal{F}V(\Omega)'$. Since the span of $\{\delta_x | x \in \Omega\}$ is dense in $\mathcal{F}(\Omega)_{\kappa}'$ by the bipolar theorem, there is a net $(f'_\tau)_\tau$ of the form $f'_\tau = \sum_{k=1}^{n_\tau} a_{k,\tau} \delta_{x_{k,\tau}}$ converging to $f'$ in $\mathcal{F}V(\Omega)'_{\kappa}$. As

$$R^f(f'_\tau) = J(\sum_{k=1}^{n_\tau} a_{k,\tau} f(x_{k,\tau})) \in \mathcal{J}(E)$$

and

$$p_{B^\alpha}(R^f(f'_\tau) - R^f(f')) \leq \sup_{x \in K_\alpha} |(f'_\tau - f')(x)| \rightarrow 0, \quad (9)$$

for all $\alpha \in \mathfrak{A}$, we gain that $(R^f(f'_\tau))_\tau$ is a Cauchy net in the complete space $\mathcal{J}(E)$. Hence it has a limit $g \in \mathcal{J}(E)$ which coincides with $R^f(f')$ since

$$p_{B^\alpha}(g - R^f(f')) \leq p_{B^\alpha}(g - R^f(f'_\tau)) + p_{B^\alpha}(R^f(f'_\tau) - R^f(f')) \leq p_{B^\alpha}(g - R^f(f'_\tau)) + \sup_{x \in K_\alpha} |(f'_\tau - f')(x)| \rightarrow 0.$$

We conclude that $R^f(f') \in \mathcal{J}(E)$ for every $f' \in \mathcal{F}V(\Omega)'$.

b) Let Property 3.13 b) hold and $f' \in \mathcal{F}V(\Omega)'$. Then there is a bounded net $(f'_\tau)_{\tau \in \mathcal{T}}$ in $\mathcal{F}V(\Omega)'_{\kappa}$ converging to $f'$ in $\mathcal{F}V(\Omega)'_{\kappa}$ such that $R^f(f'_\tau) \in \mathcal{J}(E)$ for every $\tau \in \mathcal{T}$. Due to (8) we obtain that $(R^f(f'_\tau))_\tau$ is a bounded Cauchy net in the quasi-complete space $\mathcal{J}(E)$ converging to $R^f(f') \in \mathcal{J}(E)$.

c) Let Property 3.13 c) hold and $f' \in \mathcal{F}V(\Omega)'$. Then there is a sequence $(f'_n)_{n \in \mathbb{N}}$ in $\mathcal{F}V(\Omega)'$ converging to $f'$ in $\mathcal{F}V(\Omega)'_{\kappa}$ such that $R^f(f'_n) \in \mathcal{J}(E)$ for every $n \in \mathbb{N}$. Again (8) implies that $(R^f(f'_n))_n$ is a Cauchy sequence in the sequentially complete space $\mathcal{J}(E)$ which converges to $R^f(f') \in \mathcal{J}(E)$.

d) Let Property 3.13 d) be fulfilled. Let $f \in \mathcal{F}V(\Omega, E)$ and $e' \in E'$. For every $f' \in \mathcal{F}V(\Omega)'$ there are $j \in J$, $l \in L$ and $C > 0$ such that

$$|R^f(f')(e')(x)| \leq C|\mathcal{R}_j(e')(x)|_{j,l} = C \sup_{x \in N_{j,l}(f')} |e'(x)|$$
because \((T^E_m, T^K_m)_{m \in M}\) is a strong family. Since there is \(K \in \tau(E)\) such that \(N_{j,i}(f) \subset K\), we have

\[
|R^0_j(f')(e')| \leq C \sup_{x \in K} |e'(x)|
\]

implying \(R^0_j(f') \in (E'_r)^* = J(E)\) by the Mackey-Arens theorem.

Therefore we obtain that \(R^0_j \in L(\mathcal{F}(\Omega)_r, J(E))\). So we get for all \(\alpha \in A\) and \(y \in \mathcal{F}(\Omega)'\)

\[
p_\alpha((J^{-1} \circ R^0_j)(y)) = p_{\beta^n}(\mathcal{J}((J^{-1} \circ R^0_j)(y))) = p_{\beta^n}(R^0_j(y)) \leq \sup_{x \in x} |y(x)|.
\]

This implies \(J^{-1} \circ R^0_j \in L(\mathcal{F}(\Omega)'_\varepsilon, E) = \mathcal{F}(\Omega)E\) (as vector spaces) and we gain

\[
S(J^{-1} \circ R^0_j)(x) = J^{-1}(R^0_j(\delta_x)) = J^{-1}(\mathcal{J}(f(x))) = f(x)
\]

for every \(x \in \Omega\). Thus \(S(J^{-1} \circ R^0_j) = f\) proving the surjectivity of \(S\).

In particular, we get the following corollaries as special cases of Theorem 3.14

### Corollary 3.15
Let \(\mathcal{F}(\Omega)\) be semi-Montel, \(E\) complete and \((T^E_m, T^K_m)_{m \in M}\) a strong, consistent family for \((\mathcal{F}(\Omega), E)\). Then \(\mathcal{F}(\Omega)E \cong \mathcal{F}(\Omega)E\).

**Proof.** Follows from Lemma 3.11 a) and Theorem 3.14 with Property 3.13 a). □

### Corollary 3.16
Let \(E\) be semi-Montel and \((T^E_m, T^K_m)_{m \in M}\) a strong, consistent family for \((\mathcal{F}(\Omega), E)\). Then \(\mathcal{F}(\Omega, E) \cong \mathcal{F}(\Omega)E\).

**Proof.** We observe that \(\overline{\text{conv}}(N_{j,i}(f))\) is absolutely convex and compact in the semi-Montel space \(E\) by [22, 6.2.1 Proposition, p. 103] and [22, 6.7.1 Proposition, p. 112] for every \(f \in \mathcal{F}(\Omega, E), j \in J\) and \(l \in L\). Our statement follows from Lemma 3.11 c) and Theorem 3.14 with Property 3.13 d). □

### Corollary 3.17
Let \(E\) be quasi-complete, \((T^E_m, T^K_m)_{m \in M}\) a strong, consistent family for \((\mathcal{F}(\Omega), E)\) and the conditions of Remark 3.12 be fulfilled. Then \(\mathcal{F}(\Omega) \cong \mathcal{F}(\Omega)E\).

**Proof.** Let \(f \in \mathcal{F}(\Omega, E)\). The set \(N_{\overline{\text{conv}}K_{j,l}}(f)\) is precompact in \(E\) due to Remark 3.12 for every \(K \in A, j \in J\) and \(l \in L\). It follows from the proof of Lemma 3.11 d) that \(N_{j,i}(f)\) is precompact in \(E\). Since \(E\) is quasi-complete, \(N_{j,i}(f)\) is relatively compact as well by [22, 3.5.3 Proposition, p. 65]. This implies that \(K = \overline{\text{conv}}(N_{j,i}(f))\) is absolutely convex and compact by [41, 9-2-10 Example, p. 134] because \(E\) is quasi-complete. Our statement follows from Lemma 3.11 d) and Theorem 3.14 with Property 3.13 d). □

We close this section by phrasing some sufficient conditions in Proposition 3.19 such that \(\mathcal{F}(\Omega, E) \cong \mathcal{F}(\Omega)E\) passes on to topological subspaces which will simplify our proofs when considering subspaces.

### Remark 3.18
a) If \((T^E_m, T^K_m)_{m \in M}\) is a consistent family for \((\mathcal{F}(\Omega), E)\) and \(\mathcal{G}(\Omega)\) a locally convex Hausdorff space of functions from \(\Omega\) to \(K\) such that the inclusion \(\mathcal{G}(\Omega) \subset \mathcal{F}(\Omega)\) holds topologically, then the conditions (i) and (ii) of the consistency-Definition 3.6 are satisfied for every \(u \in \mathcal{G}(\Omega)E\).

b) If \((T^E_m, T^K_m)_{m \in M}\) is a strong family for \((\mathcal{F}(\Omega), E)\), \(\mathcal{G}(\Omega, E)\) is a vector space of functions from \(\Omega\) to \(E\) such that \(\mathcal{G}(\Omega, E) \subset \mathcal{F}(\Omega, E)\) as vector spaces, then the conditions (i) and (ii) of the strength-Definition 3.9 are satisfied for every \(f \in \mathcal{G}(\Omega, E)\).
Proof. We start with a). Since $\mathcal{FV}(\Omega)$ is a dom-space and $\mathcal{G}(\Omega) \subset \mathcal{FV}(\Omega)$ holds topologically, we obtain that $\delta \in \mathcal{G}(\Omega)'$ for every $x \in \Omega$. Furthermore, every compact subset $K \subset \mathcal{G}(\Omega)$ is also compact in $\mathcal{FV}(\Omega)$ implying the continuous embedding $\mathcal{FV}(\Omega) \hookrightarrow \mathcal{G}(\Omega)$, and addition, the restriction of every equicontinuous subset $\mathcal{FV}(\Omega)'$ to $\mathcal{G}(\Omega)'$ is an equiuniform embedding of the embedding $\mathcal{G}(\Omega) \subset \mathcal{FV}(\Omega)$. Hence we observe that the restriction $u_{|\mathcal{FV}(\Omega)'} \in \mathcal{FV}(\Omega)' \subset \mathcal{FV}(\Omega)$ for every $u \in \mathcal{G}(\Omega) \subset \mathcal{FV}(\Omega)$.

3.19. Proposition. Let $(T_{\alpha}^E, T_{\alpha}^S)_{\alpha \in \mathcal{M}}$ resp. $(T_{\alpha}^E, T_{\alpha}^S)_{\alpha \in \mathcal{M}}$ be defining families for $(\mathcal{FV}, E)$ resp. $(\mathcal{G}, E)$ and $\mathcal{M}_{\alpha}(\mathcal{FV}) = \mathcal{M}_{\alpha}(\mathcal{G})$. Let one of the Properties $(\mathcal{G}, \mathcal{F})$ a) or d) be fulfilled for $(\mathcal{FV}, E)$. Then

$$\mathcal{G}(\Omega, E) \equiv \mathcal{G}(\Omega) \subset \mathcal{G}(\Omega)$$

is valid if $(T_{\alpha}^E, T_{\alpha}^S)_{\alpha \in \mathcal{M}}$ is a consistent family for $(\mathcal{FV}, E)$ and one of the following conditions is satisfied:

(i) $(\mathcal{FV}, E)$ fulfills the conditions of Lemma 3.11 b), c) or d) and the subfamily $(T_{\alpha}^E, T_{\alpha}^S)_{\alpha \in \mathcal{M}}$ is strong and consistent for $(\mathcal{G}, E)$.

(ii) $(\mathcal{FV}, E)$ fulfills the conditions of Lemma 3.11 a), $\mathcal{G}(\Omega) \subset \mathcal{FV}(\Omega)$ and the subfamily $(T_{\alpha}^E, T_{\alpha}^S)_{\alpha \in \mathcal{M}}$ is strong and consistent for $(\mathcal{G}, E)$.

Proof. By Remark 3.18 $(T_{\alpha}^E, T_{\alpha}^S)_{\alpha \in \mathcal{M}}$ is a strong, consistent family for $(\mathcal{G}, E)$. Further, we get $\mathcal{G}(\Omega, E) \subset \mathcal{G}(\Omega) \subset \mathcal{G}(\Omega)$ from Lemma 3.11 b), c) resp. d) in case (i) because $\mathcal{G}(\Omega, E) \subset \mathcal{FV}(\Omega, E)$ and from Lemma 3.11 a) in case (ii) because $\mathcal{G}(\Omega) \subset \mathcal{FV}(\Omega)$ and closed subspaces of semi-Montel spaces are semi-Montel again. If one of the Properties 3.11 a) or d) is fulfilled for $(\mathcal{FV}, E)$, then it is also valid for $(\mathcal{G}, E)$ due to the inclusion $\mathcal{G}(\Omega, E) \subset \mathcal{FV}(\Omega, E)$. Hence Theorem 3.14 yields the statement.

4. Strong and consistent families

This section is dedicated to the properties of functions which can be described by defining families and answering the question when these defining families are strong and consistent. This is done in a quite general way so that we are not tied to certain spaces and have to redo our argumentation if we consider the same subfamily $(T_{\alpha}^E, T_{\alpha}^S)$ for two different spaces of functions. Among the properties of functions that can be described by strong, consistent families are vanishing at infinity by Proposition 4.2, continuity by Proposition 4.3, Cauchy continuity by Proposition 4.6, uniform continuity by Proposition 4.8, continuous extendability by Proposition 4.10 and purely algebraic properties of a function like linearity by Proposition 4.11. We collect these properties in propositions and in follow-up lemmas we handle properties which can be described by compositions of defining operators $T_{\alpha}^E \circ T_{\alpha}^S$ like continuous differentiability. We start with the properties we want to describe.

4.1. Hypothesis. Let $\mathcal{FV}(\Omega, E)$ be a dom-space and one of the following (families of) operators belong to $(T_{\alpha}^S)_{\alpha \in \mathcal{M}}$. 

\[ S(u)(x) = u_\delta) = u_{\mathcal{FV}(\Omega)'}(\Delta x) = S(u_{\mathcal{FV}(\Omega)})'(x) \]

for every $x \in \Omega$. Thus we have $S(u) = S(u_{\mathcal{FV}(\Omega)})$ and $u_{\mathcal{FV}(\Omega)'} \in \mathcal{FV}(\Omega) \subset \mathcal{FV}(\Omega)$ for every $u \in \mathcal{G}(\Omega) \subset \mathcal{FV}(\Omega)$. Therefore the conditions (i) and (ii) of the consistency-Definition are satisfied for every $u \in \mathcal{G}(\Omega) \subset \mathcal{FV}(\Omega)$ if $(T_{\alpha}^E, T_{\alpha}^S)_{\alpha \in \mathcal{M}}$ is a consistent family for $(\mathcal{FV}, E)$. Let us turn to b). If $(T_{\alpha}^E, T_{\alpha}^S)_{\alpha \in \mathcal{M}}$ is a strong family for $(\mathcal{FV}, E)$, then the conditions (i) and (ii) of the strength-Definition are satisfied for every $f \in \mathcal{G}(\Omega, E)$ as well because $\mathcal{G}(\Omega, E) \subset \mathcal{FV}(\Omega, E)$.

Proof. By Remark 3.18 $(T_{\alpha}^E, T_{\alpha}^S)_{\alpha \in \mathcal{M}}$ is a strong, consistent family for $(\mathcal{G}, E)$. Further, we get $\mathcal{G}(\Omega, E) \subset \mathcal{G}(\Omega, E)$ from Lemma 3.11 b), c) resp. d) in case (i) because $\mathcal{G}(\Omega, E) \subset \mathcal{FV}(\Omega, E)$ and from Lemma 3.11 a) in case (ii) because $\mathcal{G}(\Omega) \subset \mathcal{FV}(\Omega)$ and closed subspaces of semi-Montel spaces are semi-Montel again. If one of the Properties 3.11 a) or d) is fulfilled for $(\mathcal{FV}, E)$, then it is also valid for $(\mathcal{G}, E)$ due to the inclusion $\mathcal{G}(\Omega, E) \subset \mathcal{FV}(\Omega, E)$. Hence Theorem 3.14 yields the statement. 

\[ \mathcal{G}(\Omega, E) \equiv \mathcal{G}(\Omega) \subset \mathcal{G}(\Omega) \]

is valid if $(T_{\alpha}^E, T_{\alpha}^S)_{\alpha \in \mathcal{M}}$ is a consistent family for $(\mathcal{FV}, E)$ and one of the following conditions is satisfied:

(i) $(\mathcal{FV}, E)$ fulfills the conditions of Lemma 3.11 b), c) or d) and the subfamily $(T_{\alpha}^E, T_{\alpha}^S)_{\alpha \in \mathcal{M}}$ is strong and consistent for $(\mathcal{G}, E)$.

(ii) $(\mathcal{FV}, E)$ fulfills the conditions of Lemma 3.11 a), $\mathcal{G}(\Omega) \subset \mathcal{FV}(\Omega)$ and the subfamily $(T_{\alpha}^E, T_{\alpha}^S)_{\alpha \in \mathcal{M}}$ is strong and consistent for $(\mathcal{G}, E)$. 

Proof. By Remark 3.18 $(T_{\alpha}^E, T_{\alpha}^S)_{\alpha \in \mathcal{M}}$ is a strong, consistent family for $(\mathcal{G}, E)$. Further, we get $\mathcal{G}(\Omega, E) \subset \mathcal{G}(\Omega, E)$ from Lemma 3.11 b), c) resp. d) in case (i) because $\mathcal{G}(\Omega, E) \subset \mathcal{FV}(\Omega, E)$ and from Lemma 3.11 a) in case (ii) because $\mathcal{G}(\Omega) \subset \mathcal{FV}(\Omega)$ and closed subspaces of semi-Montel spaces are semi-Montel again. If one of the Properties 3.11 a) or d) is fulfilled for $(\mathcal{FV}, E)$, then it is also valid for $(\mathcal{G}, E)$ due to the inclusion $\mathcal{G}(\Omega, E) \subset \mathcal{FV}(\Omega, E)$. Hence Theorem 3.14 yields the statement.

\[ \mathcal{G}(\Omega, E) \equiv \mathcal{G}(\Omega) \subset \mathcal{G}(\Omega) \]
a) vanishing at infinity w.r.t. to \((\pi, \mathcal{R})\): Let \(\infty \in \mathcal{M}_r\) and
\[
T^E_{\infty}|\{f \in E^\Omega \mid f \text{ fulfills } (\square) \} \to E^{(1)}, \ T^E_{\infty}(f)(1) = 0.
\]
b) continuity: Let \(c \in \mathcal{M}_{\top} \cup \mathcal{M}_r, \Omega\) be a topological Hausdorff space and
\[
T^E_c: C(\Omega, E) \to E^\Omega, \ T^E_c(f)(x) := f(x).
\]
c) Cauchy continuity: Let \(cc \in \mathcal{M}_r, \Omega\) be a metric space and
\[
CC(\Omega, E) := \{f \in E^\Omega \mid \forall \text{ Cauchy seq. } (x_n)_{n\in\mathbb{N}} \subset \Omega : (f(x_n))_{n\in\mathbb{N}} \text{ Cauchy seq. in } E\}
\]
and
\[
T^E_{cc}: CC(\Omega, E) \to E^{(1)}, \ T^E_{cc}(f)(1) = 0.
\]
d) uniform continuity: Let \(\mathcal{M}_{uc} \subset \mathcal{M}_0, (\Omega, d)\) be a metric space and
\[
\mathcal{M}_{uc} = \{(z, x) \in \Omega \times \Omega \mid \lim_{n \to \infty} d(z_n, x_n) = 0\}.
\]
For \((z, x) \in \mathcal{M}_{uc}\) set
\[
dom T^E_{(z, x)} := \{f \in E^\Omega \mid \lim_{n \to \infty} (f(z_n) - f(x_n)) \text{ exists in } E\}
\]
and
\[
T^E_{(z, x)}|\dom T^E_{(z, x)} \to E^{(1)}, \ T^E_{(z, x)}(f)(1) = \lim_{n \to \infty} (f(z_n) - f(x_n)).
\]
e) continuous extendability: Let \(ext \in \mathcal{M}_r \cup \mathcal{M}_{\top}, E\) be a metric space, \(\Omega \subset X\) and
\[
dom T^E_{ext} := \{f \in C(\Omega, E) \mid \forall x \in \partial \Omega, (x_n)_{n\in\mathbb{N}} \subset \Omega, x_n \to x : \lim_{n} f(x_n) \text{ exists in } E \text{ independent of } (x_n)\}
\]
plus
\[
T^E_{\pi} \circ T^E_{ext} \to E^{\pi}, \ T^E_{ext}(f)(x) := \begin{cases} \lim_{n \to \infty} f(x_n), & x \in \partial \Omega, (x_n) \subset \Omega, x_n \to x, \\ f(x), & x \in \Omega. \end{cases}
\]
f) differentiability on a subset: Let \(X\) be a vector space over the field \(K_1 = \mathbb{R}\) or \(\mathbb{C}\) and \(\omega \subset \Omega \subset X\). Let \(v \in X\) be such that for every \(x \in \omega\) there is \(\varepsilon > 0\) with \(x + h \cdot v \in \omega\) for all \(h \in K_1\) with \(0 < |h| < \varepsilon\). Let \(\partial_0 \in \mathcal{M}_{\top} \cup \mathcal{M}_r\) and set
\[
T^E_{\partial_0}(f)(x) := \lim_{h \to 0} \frac{f(x + h \cdot v) - f(x)}{h}
\]
for \(f \in E^\Omega\) if existing in \(E\). Define
\[
dom T^E_{\partial_0} := \{f \in E^\Omega \mid \forall x \in \omega : T^E_{\partial_0}(f)(x) \in \mathbb{E}\}
\]
plus
\[
T^E_{\partial_0}|\dom T^E_{\partial_0} \to E^\omega, \ x \mapsto T^E_{\partial_0}(f)(x).
\]
g) vanishing on a subset: Let \(\omega \subset \Omega, \omega_0 \in \mathcal{M}_0\) and \(T^E_{\omega_0}: E^\Omega \to E^\omega, T^E_{\omega_0}(f)(x) := f(x)\).
h) additivity: Let \(a \in \mathcal{M}_0, \Omega\) be a vector space and set
\[
T^E_a: E^\Omega \to E^{\Omega^2}, \ T^E_a(f)(x, y) := f(x + y) - f(x) - f(y).
\]
i) homogeneity: Let \(h \in \mathcal{M}_0, \Omega\) be a vector space and set
\[
T^E_h: E^\Omega \to E^{K \times \Omega}, \ T^E_h(f)(\lambda, x) := f(\lambda x) - \lambda f(x).
\]
In b)-i) it is easily checked that the pair \(T^E_{m_1}, T^E_{m_2}\) with \(m \in \{c, cc, ext, \partial_0, \omega_0, a, h\}\) and the subfamily \(T^E_{m_1, m_2}\) are strong subfamilies due to simple calculations and the linearity and (uniform) continuity of every \(e' \in E^t\). Therefore we turn our attention to the question of consistency and in a) to strength as well.

4.2. Proposition (vanishing at infinity w.r.t. to \((\pi, \mathcal{R})\)). If \((T^E_{m_1}, T^E_{m_2})\) is a strong resp. consistent subfamily for \((\mathcal{F}V, E)\) and \(\mathcal{R}\) is closed under taking finite unions, then \((T^E_{\infty}, T^E_{\infty})\) is a strong resp. consistent subfamily for \((\mathcal{F}V, E)\).
Proof. First, we consider consistency. We set \( B_{j, l} := \{ f \in \mathcal{F}_{\Omega}^{\prime} \mid |f|_{j, l} \leq 1 \} \) for \( j \in J \) and \( l \in L \). Let \( u \in \mathcal{F}_{\Omega}(\varepsilon) \in E \). The topologies \( \sigma(\mathcal{F}_{\Omega}^{\prime}, \mathcal{F}_{\Omega}(\varepsilon)) \) and \( \kappa(\mathcal{F}_{\Omega}^{\prime}, \mathcal{F}_{\Omega}(\varepsilon)) \) coincide on the equicontinuous set \( B_{j, l}^{\varepsilon} \) and we deduce that the restriction of \( u \) to \( B_{j, l}^{\varepsilon} \) is \( \sigma(\mathcal{F}_{\Omega}^{\prime}, \mathcal{F}_{\Omega}(\varepsilon)) \)-continuous.

Let \( \varepsilon > 0 \), \( j \in J \), \( l \in L \), \( \alpha \in \mathfrak{A} \) and set \( B_{\alpha, \varepsilon} := \{ x \in E \mid p_{\alpha}(x) < \varepsilon \} \). Then there are a finite set \( N \subset \mathcal{F}_{\Omega}^{\prime} \) and \( \eta > 0 \) such that \( u(f') \in B_{\alpha, \varepsilon} \) for all \( f' \in V_{N, \eta} \) where

\[
V_{N, \eta} := \{ f' \in \mathcal{F}_{\Omega}(\varepsilon) \mid \sup_{f \in N} |f'(f)| < \eta \} \cap B_{j, l}^{\varepsilon},
\]

because the restriction of \( u \) to \( B_{j, l}^{\varepsilon} \) is \( \sigma(\mathcal{F}_{\Omega}^{\prime}, \mathcal{F}_{\Omega}(\varepsilon)) \)-continuous. Since \( N \subset \mathcal{F}_{\Omega}^{\prime} \) is finite and \( \mathfrak{A} \) closed under taking finite unions, there is \( K \in \mathfrak{A} \) such that

\[
\sup_{x \in \Omega, m \in M} |T_{m}^{K}(f)(x)| \leq \eta
\]

for every \( f \in N \). It follows from (10) and (the proof of) Lemma 3.5 b) that

\[
D_{\pi \in K, j, l} := \{ T_{m}^{K}(f) \mid x \in \Omega, m \in M_{l}, \pi(x, m) \notin K \} \subset V_{N, \eta}
\]

and thus \( u(D_{\pi \in K, j, l}) \subset B_{\alpha, \varepsilon} \). Therefore we have

\[
\sup_{x \in \Omega, m \in M_{l}} p_{\alpha}(T_{m}^{E}(S_{\alpha}))(x) \leq \varepsilon
\]

if \( (T_{m}^{E}, T_{m}^{K})_{\pi \in K} \) is a consistent subfamily for \( \mathcal{F}_{\Omega}^{\prime} \). Hence we derive \( S(\alpha) \in \text{dom} T_{m}^{E} \) and the other conditions for the consistency of \( (T_{m}^{E}, T_{m}^{K}) \) are obviously fulfilled. Let us consider strength. Let \( (T_{m}^{E}, T_{m}^{K})_{\pi \in K} \) be a strong subfamily for \( \mathcal{F}_{\Omega}^{\prime} \), \( \varepsilon > 0 \), \( f \in \mathcal{F}_{\Omega}(\varepsilon) \) and \( e' \in E^{\prime} \). Then there exist \( \alpha \in \mathfrak{A} \) and \( C > 0 \) such that \( |e'(x)| \leq C p_{\alpha}(x) \) for every \( x \in E \). For \( j \in J \) and \( l \in L \) there is \( K \in \mathfrak{A} \) such that

\[
\sup_{x \in \Omega, m \in M_{l}} p_{\alpha}(T_{m}^{E}(f)(x)) \leq \varepsilon
\]

Using that \( (T_{m}^{E}, T_{m}^{K})_{\pi \in K} \) is a strong subfamily for \( \mathcal{F}_{\Omega}^{\prime} \), it follows that

\[
\sup_{x \in \Omega, m \in M_{l}} |T_{m}^{E}(e'(f))(x)| \leq \varepsilon
\]

yielding to \( e'(f) \in \text{dom} T_{m}^{K} \). Furthermore, \( T_{m}^{K}(e'(f))(1) = 0 = (e'(f))(1) \). \( \square \)

The ‘consistency’-part of the proof above adapts an idea in the proof of [3, 4.4 Theorem, p. 199-200] where \( (T_{m}^{E}, T_{m}^{K})_{\pi \in K} = (T_{e}, T_{c}) \) which is a special case of our proposition.

4.3. Proposition (continuity). \( (T_{E}^{c}, T_{c}^{K}) \) is a strong and consistent subfamily for \( \mathcal{F}_{\Omega}^{\prime} \) if \( \delta: \Omega \to \mathcal{F}_{\Omega}(\varepsilon) \), \( x \mapsto \delta_{x} \), is continuous.

Proof. Let \( u \in \mathcal{F}_{\Omega}(\varepsilon) \in E \). Since \( S(u) = u \circ \delta \) and \( \delta \) is continuous, we obtain that \( S(u) \in \mathcal{C}(\varepsilon) \). Further, we have \( T_{c}^{K} = \delta_{x} \in \mathcal{F}_{\Omega}(\varepsilon) \) and \( T_{c}^{E} = S(u)(x) = S(u)(x) = u(T_{c}^{K}(x)) \) for \( x \in \Omega \). \( \square \)

Now, we tackle the problem of the continuity of \( \delta: \Omega \to \mathcal{F}_{\Omega}(\varepsilon) \) in the proposition above and phrase our solution in a way such that it can be applied to show the consistency of the subfamily describing the continuity of partial derivatives as well.

We recall that a topological space \( \Omega \) is called completely regular (Tychonoff or \( T_{3\frac{1}{2}} \)-space) if for any non-empty closed subset \( F \subset \Omega \) and \( x \in \Omega \setminus F \) there is \( f \in \mathcal{C}(\Omega, [0, 1]) \) such that \( f(x) = 0 \) and \( f(z) = 1 \) for all \( z \in F \) (see [21] Definition 11.1, p. 180]). Examples of completely regular spaces are uniformizable, particularly metrisable, spaces by [21], Proposition 11.5, p. 181] and locally convex Hausdorff spaces by
Proposition 3.27, p. 95]. A completely regular space $\Omega$ is a $k_2$-space if for any completely regular space $Y$ and any map $f: \Omega \to Y$, whose restriction to each compact $K \subset \Omega$ is continuous, the map is already continuous on $\Omega$ (see [10], (2.3.7) Proposition, p. 22). Examples of $k_2$-spaces are completely regular $k$-spaces by [13], 3.3.21 Theorem, p. 152]. A topological space $\Omega$ is called a $k$-space (compactly generated space) if it satisfies the following condition: $A \subset \Omega$ is closed if and only if $A \cap K$ is closed in $K$ for every compact $K \subset \Omega$. Every locally compact Hausdorff space is a completely regular $k$-space. Further, every sequential Hausdorff space is a $k$-space by [13], 3.3.20 Theorem, p. 152], in particular, every first-countable Hausdorff space. Thus metrisable spaces are completely regular $k$-spaces. Moreover, the strong dual of a Fréchet-Montel space (DFM-space) is a completely regular $k$-space by [27], 4.11 Theorem, p. 39].

We denote by $C(\Omega)$ the space of scalar-valued bounded, continuous functions on $\Omega$ with the topology of uniform convergence on compact subsets and by $C_0(\Omega)$ the space of scalar-valued bounded, continuous functions on $\Omega$ with the topology of uniform convergence on $\Omega$.

4.4. Lemma. Let $FV(\Omega)$ be a dom-space, $\Omega$ a topological Hausdorff space and $m \in \mathcal{M}_{top}$ with $T^\infty_m(FV(\Omega)) \subset C(\Omega)$. Then the map $\delta \circ T^\infty_m : \Omega \to FV(\Omega)'_c$ is continuous in each of the subsequent cases:

(i) $\Omega$ is a $k_2$-space and $T^\infty_m : FV(\Omega) \to C(\Omega)_c$ is continuous.

(ii) $T^\infty_m : FV(\Omega) \to C_0(\Omega)$ is continuous.

Proof. The map $\delta \circ T^\infty_m : \Omega \to FV(\Omega)'_c$ is well-defined by Lemma 3.5 a) and we claim that it is continuous. If $x \in \Omega$ and $(x_\tau)_{\tau \in T}$ is a net in $\Omega$ converging to $x$, then

$$\delta_{x_\tau} \circ T^\infty_m(f) = T^\infty_m(f(x_\tau)) \to T^\infty_m(f)(x) = \delta_x \circ T^\infty_m(f)$$

for every $f \in FV(\Omega)$ as $T^\infty_m(f)$ is continuous on $\Omega$ which proves our claim.

(i) Let $K \subset \Omega$ be compact. Then there are $j \in J$, $l \in L$ and $C > 0$ such that

$$\sup_{x \in K} |T^\infty_m(f)(x)| \leq C |f|_{j,l}$$

for every $f \in FV(\Omega)$. This means that $\{T^\infty_m(x) \mid x \in K\}$ is equicontinuous in $FV(\Omega)'$. The topologies $\sigma(FV(\Omega)', FV(\Omega))$ and $\gamma(FV(\Omega)', FV(\Omega))$ coincide on equicontinuous subsets of $FV(\Omega)'$ implying that the restriction $(\delta \circ T^\infty_m)_{|K} : K \to FV(\Omega)'_c$ is continuous by our first claim. As $\delta \circ T^\infty_m : \Omega \to FV(\Omega)'_c$ is continuous on every compact subset of the $k_2$-space $\Omega$, it follows that $\delta \circ T^\infty_m : \Omega \to FV(\Omega)'$ is continuous.

(ii) There are $j \in J$, $l \in L$ and $C > 0$ such that

$$\sup_{x \in \Omega} |T^\infty_{m,x}(f)(x)| \leq C |f|_{j,l}$$

for every $f \in FV(\Omega)$. This means that $\{T^\infty_{m,x} \mid x \in \Omega\}$ is equicontinuous in $FV(\Omega)'$ yielding to the statement like before.

The preceding lemma is just a modification of [3], 4.1 Lemma, p. 198] where $FV(\Omega) = CV(\Omega)$, the space of Nachbin-weighted continuous functions, and $T^\infty_m = \text{id}_\Omega$. Next, we consider the special case of continuous, linear operators. Let $(F, t)$ be a locally convex Hausdorff space with topology $t$ and $F'$ the dual with respect to $t$. Due to the Mackey-Arens theorem $F = (F')'$ holds algebraically and thus $\delta : F \to (F')'_c$ induces a locally convex topology $\gamma$ on $F$. This topology fulfills $t \subseteq \gamma \subseteq \tau(F, F')$. In particular, if $F$ is a Mackey space, i.e. $t = \tau(F, F')$, then $t = \gamma$ (see [57] Chap. I, §1, p. 17] where the topology $\gamma$ is called $\gamma$).
4.5. Remark. Let \( \Omega \) be a locally convex Hausdorff space.

(i) The map \( \delta : \Omega \to (\Omega'_\varsigma)_\varsigma \) is continuous if \( \Omega \) has the topology \( \varsigma \), in particular, if \( \Omega \) is quasi-barrelled or bornological.

(ii) The map \( \delta : \Omega \to (\Omega'_\varsigma)_\varsigma \) is continuous if \( \Omega \) is normed or semi-reflexive and metrisable.

Proof. Part (i) follows directly from the definition of \( \varsigma \). Further, if \( \Omega \) is quasi-barrelled, then it has the Mackey-topology by [34, Observation 4.1.5 (a), p. 96], and, if \( \Omega \) is bornological, then it is quasi-barrelled by [34, Observation 6.1.2 (c), p. 167]. Let us turn to part (ii). Let \( (x_n) \) be a sequence in \( \Omega \) converging to \( x \in \Omega \).

We observe that \( (\delta_{x_n}) \) converges to \( \delta_x \) in \( (\Omega'_\varsigma)_\varsigma \). If \( \Omega \) is normed or a semi-reflexive, metrisable space, then \( \Omega'_\varsigma \) is barrelled since it is a Banach space resp. by [22, 11.4.1 Proposition, p. 227]. The Banach-Steinhaus theorem yields our result. \( \square \)

Next, we turn to the problem of describing Cauchy continuity by strong and consistent families.

4.6. Proposition (Cauchy continuity). \( (T^{E}_cc, T^{cc}_cc) \) is a strong and consistent subfamily for \( (\mathcal{F}V, E) \) if \( (\delta_{x_n}) \) is a Cauchy sequence in \( \mathcal{F}V(\Omega)'_\varsigma \) for every Cauchy sequence \( (x_n) \) in \( \Omega \).

Proof. Let \( u \in \mathcal{F}V(\Omega) \in E \) and \( (\delta_{x_n}) \) be a Cauchy sequence in \( \mathcal{F}V(\Omega)'_\varsigma \). Then \( (S(u)(x_n)) \) is a Cauchy sequence in \( E \) since \( u \) is uniformly continuous and \( u(\delta_{x_n}) = S(u)(x_n) \). Hence we conclude \( S(u) \in \text{dom } T^{E}_cc \). The remaining part is obvious. \( \square \)

We write \( \mathcal{C}\mathcal{C}(\Omega)_\varsigma \), resp. \( \mathcal{C}\mathcal{C}_b(\Omega) \) for the space of scalar-valued Cauchy continuous functions equipped with the topology of uniform convergence on precompact sets resp. the space of scalar-valued bounded Cauchy continuous functions equipped with the topology of uniform convergence on \( \Omega \).

4.7. Lemma. Let \( \mathcal{F}V(\Omega) \) be a dom-space, \( \Omega \) a metric space and \( m \in \mathcal{M}_{\top} \) with \( T^{cc}_m(\mathcal{F}V(\Omega)) \subset \mathcal{C}\mathcal{C}(\Omega) \). Then the sequence \( (T^{cc}_m(x_n)) \) is Cauchy in \( \mathcal{F}V(\Omega)'_\varsigma \) for every Cauchy sequence \( (x_n) \) in \( \Omega \) in each of the subsequent cases:

(i) \( T^{cc}_m(\mathcal{F}V(\Omega)) \to \mathcal{C}\mathcal{C}(\Omega)_\varsigma \) is continuous.

(ii) \( T^{cc}_m(\mathcal{F}V(\Omega)) \to \mathcal{C}\mathcal{C}_b(\Omega) \) is continuous.

Proof. \( (T^{cc}_m(x_n)) \) is a sequence in \( \mathcal{F}V(\Omega)'_\varsigma \) by Lemma 3.5 a). Moreover, we have \( T^{cc}_m(f) = T^{cc}_m(f)(x_n) \) for every \( f \in \mathcal{F}V(\Omega) \) which implies that \( (T^{cc}_m(f)) \) is a Cauchy sequence in \( K \) because \( T^{cc}_m(f) \in \mathcal{C}\mathcal{C}(\Omega) \) by assumption. Since \( K \) is complete, it has a unique limit \( f := \lim_{\tau \to \infty} T^{cc}_m(x_n)(f) \) defining a linear functional in \( f \).

(i) The set \( N := \{ x_n | n \in \mathbb{N} \} \) is precompact in \( \Omega \) since Cauchy sequences are precompact. Hence there are \( j \in J \), \( l \in L \) and \( C > 0 \) such that

\[
\sup_{n \in \mathbb{N}} |T^{cc}_m(x_n)(f)| = \sup_{x \in N} |T^{cc}_m(f)(x)| \leq C |f|_{j,l}
\]

for every \( f \in \mathcal{F}V(\Omega) \). Therefore the set \( \{ T^{cc}_m(x_n) | n \in \mathbb{N} \} \) is equicontinuous in \( \mathcal{F}V(\Omega)'_\varsigma \) which implies that \( T \in \mathcal{F}V(\Omega)' \) and the convergence of \( (T^{cc}_m(x_n)) \) to \( T \) in \( \mathcal{F}V(\Omega)'_\varsigma \) due to the observation in the beginning and the fact that \( \gamma(\mathcal{F}V(\Omega)'_\varsigma, \mathcal{F}V(\Omega)) \) and \( \sigma(\mathcal{F}V(\Omega)'_\varsigma, \mathcal{F}V(\Omega)) \) coincide on equicontinuous sets. In particular, \( (T^{cc}_m(x_n)) \) is a Cauchy sequence in \( \mathcal{F}V(\Omega)'_\varsigma \).

(ii) There exist \( j \in J \), \( l \in L \) and \( C > 0 \) such that

\[
\sup_{n \in \mathbb{N}} |T^{cc}_m(x_n)(f)| \leq \sup_{x \in N} |T^{cc}_m(f)(x)| \leq C |f|_{j,l}
\]

for every \( f \in \mathcal{F}V(\Omega) \). Therefore the set \( \{ T^{cc}_m(x_n) | n \in \mathbb{N} \} \) is equicontinuous in \( \mathcal{F}V(\Omega)'_\varsigma \) and we conclude that \( (T^{cc}_m(x_n)) \) is a Cauchy sequence in \( \mathcal{F}V(\Omega)'_\varsigma \) like in (i).
The subsequent proposition and lemma handle the same question as before but now for uniform continuity.

4.8. **Proposition** (uniform continuity). \((T^{E}_{(z,x)},T^{X}_{(z,x)})_{(z,x)\in \mathcal{M}_{uc}}\) is a strong and consistent subfamily for \((\mathcal{F}V,E)\) if \((\delta_{za} - \delta_{za})_{n\in \mathbb{N}}\) converges to 0 in \(\mathcal{F}V(\Omega)\) for every \((z,x)\in \mathcal{M}_{uc}\).

**Proof.** Let \(u \in \mathcal{F}V(\Omega)\in E\) and \((z,x)\in \mathcal{M}_{uc}.\) If \((\delta_{za} - \delta_{za})\) converges to 0 in \(\mathcal{G}V(\Omega)\), then \((S(u)(z_n) - S(u)(x_n))\) converges to 0 in \(E\) since \(u\) is uniformly continuous and \(u(\delta_{za} - \delta_{za}) = S(u)(z_n) - S(u)(x_n)\). Hence we conclude \(S(u)\in \ker T^{E}_{(z,x)}\) and

\[
T^{E}_{(z,x)}(S(u))(1) = \lim_{n \to \infty} u(\delta_{za} - \delta_{za}) = u(\lim_{n \to \infty} \delta_{za} - \delta_{za}) = u(T^{X}_{(z,x),1}).
\]

We denote by \(\mathcal{UC}(\Omega)\) the space of scalar-valued uniformly continuous functions on a metric space \(\Omega\). We mean by \(\mathcal{BUC}(\Omega)\) the space of scalar-valued, uniformly continuous functions equipped with the topology of uniform convergence on \(\Omega\).

4.9. **Lemma.** Let \(\mathcal{F}V(\Omega)\) be a domain-space, \((\Omega, d)\) a metric space and \(m \in \mathcal{M}_{top}\) with \(T^{K}_{m}(\mathcal{F}V(\Omega)) \subset \mathcal{UC}(\Omega)\). Then the sequence \((T^{K}_{m,za} - T^{K}_{m,xa})\) converges to 0 in \(\mathcal{F}V(\Omega)\) for every \((z,x)\in \mathcal{M}_{uc}\) if the map \(T^{K}_{m}\colon \mathcal{F}V(\Omega) \to \mathcal{BUC}(\Omega)\) is continuous.

**Proof.** \((T^{K}_{m,za} - T^{K}_{m,xa})\) is a sequence in \(\mathcal{F}V(\Omega)\) by Lemma 3.5 a). Moreover, we have

\[
(T^{K}_{m,za} - T^{K}_{m,xa})(f) = T^{K}_{m}(f)(z_n) - T^{K}_{m}(f)(x_n)
\]

for every \(f \in \mathcal{F}V(\Omega)\) which implies that \((T^{K}_{m,za} - T^{K}_{m,xa})\) converges to 0 in \(\mathcal{F}V(\Omega)\) because \(T^{K}_{m}(f)\in \mathcal{UC}(\Omega)\). There exist \(j \in J, l \in L\) and \(C > 0\) such that

\[
\sup_{n \in \mathbb{N}} |(T^{K}_{m,za} - T^{K}_{m,xa})(f)| \leq 2 \sup_{x \in \Omega} |T^{K}_{m}(f)(x)| \leq 2C |f|_{j,l}
\]

for every \(f \in \mathcal{F}V(\Omega)\). Therefore the set \(\{T^{K}_{m,za} - T^{K}_{m,xa} \mid n \in \mathbb{N}\}\) is equicontinuous in \(\mathcal{F}V(\Omega)\) and we conclude the statement like before.

4.10. **Proposition** (continuous extendability). \((T^{E}_{ext}, T^{X}_{ext})\) is a strong and consistent subfamily for \((\mathcal{F}V,E)\) if \(\delta: \Omega \to \mathcal{F}V(\Omega)\), \(x \mapsto \delta_{x}\), is continuous and \(\mathcal{F}V(\Omega)\) is barrelled.

**Proof.** Let \(u \in \mathcal{F}V(\Omega)\in E\). From Proposition 4.3 we deduce \(S(u)\in \mathcal{G}V(\Omega)\). Let \(x \in \partial \Omega\) and \((x_n)\) be a sequence in \(\Omega\) converging to \(x\). Then we have

\[
\delta_{x}(f) = f(x_n) \to T^{K}_{ext}(f)(x) = T^{X}_{ext,x}(f)
\]

for every \(f \in \mathcal{F}V(\Omega)\) which implies \(T^{K}_{ext,x} \in \mathcal{F}V(\Omega)\) and \(\delta_{xa} \to T^{X}_{ext,x} \in \mathcal{F}V(\Omega)\), and thus in \(\mathcal{F}V(\Omega)\) by the Banach-Steinhaus theorem. Hence we conclude

\[
u(T^{X}_{ext,x}) = \lim_{n \to \infty} u(\delta_{x}) = \lim_{n \to \infty} S(u)(x_n) = T^{E}_{ext}(S(u))(x).
\]

4.11. **Lemma.** Let \(\mathcal{F}V(\Omega)\) be a domain-space, \(\Omega \subset X\), \(X\) a metric space and \(m \in \mathcal{M}_{top}\) with \(T^{K}_{m}(\mathcal{F}V(\Omega)\) \subset \text{dom } T^{K}_{ext}.\) The sequence \((T^{m,x}_{m,za})\) converges to \(\delta_{x} \circ (T^{x}_{ext} \circ T^{m}_{m,za})\) in \(\mathcal{F}V(\Omega)\) for \(x \in \partial \Omega\) and every sequence \((x_n)\) in \(\Omega\) with \(x_n \to x\) if \(\mathcal{F}V(\Omega)\) is barrelled.
Proof. \((T^K_{m,x_n})\) is a sequence in \(\mathcal{FV}(\Omega)'\) by Lemma 5.3(a). Furthermore, we have

\[
T^K_{m,x_n}(f) - \delta_x \circ (T^K_{ext} \circ T^K_m)(f) = (\delta_{x_n} - T^K_{ext})(T^K_m(f))
\]

for every \(f \in \mathcal{FV}(\Omega)\) which implies that \((T^K_{m,x_n})\) converges to \(\delta_x \circ (T^K_{ext} \circ T^K_m)\) pointwise in \(f\) because \(T^K_{ext}(f) \in \text{dom} T^K_{ext}\) by assumption. As a consequence of the Banach-Steinhaus theorem we get \(\delta_x \circ (T^K_{ext} \circ T^K_m) \in \mathcal{FV}(\Omega)'\) and the convergence in \(\mathcal{FV}(\Omega)'\).

Let us turn to differentiability.

4.12. Proposition (differentiability on a subset). \((T^E_{\partial_\epsilon}, T^K_{\partial_\epsilon})\) is a strong and consistent subfamily for \((\mathcal{FV}, E)\) if \(\frac{1}{h}(\delta_{x+h\cdot v} - \delta_x)\) converges to \(T^K_{\partial_\epsilon} (\partial_x f)\) in \(\mathcal{FV}(\Omega)'\) for every \(x \in \omega\) as \(h \to 0\).

Proof. Let \(u \in \mathcal{FV}(\Omega)\epsilon E\) and \(x \in \omega\). Then

\[
u(T^K_{\partial_\epsilon}) = \lim_{h \to 0} \frac{1}{h}u(\delta_{x+h\cdot v} - \delta_x) = \lim_{h \to 0} \frac{1}{h}(S(u)(x + h \cdot v) - S(u)(x)) = T^E_{\partial_\epsilon}(S(u))(x).
\]

4.13. Lemma. Let \(\mathcal{FV}(\Omega)\) be a born-space, \(X\) a vector space over \(\mathbb{R}\) or \(\mathbb{C}\) and \(\omega \in \Omega \subset X\). Let \(v \in X\) and \(m \in \mathcal{M}_{gap}\) such that for every \(x \in \omega\) there is \(\epsilon > 0\) with \(x + h \cdot v \in \omega\) for all \(0 < |h| < \epsilon\) and \(T^K_{\partial_\epsilon}(\mathcal{FV}(\Omega)) \subset \text{dom} \ T^K_{\partial_\epsilon}\). Then \(\frac{1}{h}(T^K_{m,x+h\cdot v} - T^K_{m,x})\) converges to \(\delta_x \circ (T^K_{\partial_\epsilon} \circ T^K_m)\) in \(\mathcal{FV}(\Omega)'\), as \(h\) tends to \(0\) for every \(x \in \omega\) if \(\mathcal{FV}(\Omega)\) is barrelled.

Proof. Due to Lemma 5.3(a) we have \(\frac{1}{h}(T^K_{m,x+h\cdot v} - T^K_{m,x}) \in \mathcal{FV}(\Omega)'\) for all \(x \in \omega\) and \(0 < |h| < \epsilon\). Furthermore, we know that \(\frac{1}{h}(T^K_{m,x+h\cdot v} - T^K_{m,x})(f)\) converges to

\[
T^K_{\partial_\epsilon}(T^K_m(f))(x) = \delta_x \circ (T^K_{\partial_\epsilon} \circ T^K_m)(f)
\]

for every \(f \in \mathcal{FV}(\Omega)\) and \(x \in \omega\) as \(h \to 0\). The Banach-Steinhaus theorem yields the statement.

Our last proposition of this section is immediate.

4.14. Proposition (vanishing on a subset, additivity, homogeneity), \((T^E_{\omega_0}, T^K_{\omega_0}), (T^E_a, T^K_a)\) and \((T^E_h, T^K_h)\) are strong and consistent subfamilies for \((\mathcal{FV}, E)\).

5. Examples

In our last section we treat many examples of spaces \(\mathcal{FV}(\Omega, E)\) of weighted functions on a set \(\Omega\) with values in a locally convex Hausdorff space \(E\) over the field \(\mathbb{K}\). Applying the results of the preceding sections, we give conditions on \(E\) such that

\[
\mathcal{FV}(\Omega, E) \equiv \mathcal{FV}(\Omega)\epsilon E
\]

holds. For this purpose we recapitulate some definitions which are connected to different types of completeness of \(E\). Let us recall the following definition from [11] 9.2-8 Definition, p. 134 and [14] p. 259. A locally convex Hausdorff space is said to have the [metric] convex compactness property ([metric] ccp) if the closure of the absolutely convex hull of every [metrisable] compact set is compact. Sometimes this condition is phrased with the term convex hull instead of absolutely convex hull but these definitions coincide. Indeed, the first definition implies the second since every convex hull of a set \(A \subset E\) is contained in its absolutely convex hull. On the other hand, we have \(\text{acx}(A) = \text{cx}(\text{ch}(A))\) by [22] 6.1.4 Proposition, p. 103 and the circled hull \(\text{ch}(A)\) of a [metrisable] compact set \(A\) is compact by [33], Chap. I, 5.2, p. 26 [and metrisable by 13, Chap. IX, §2.10, Proposition 17, p. 159] since \(\mathbb{D} \times A\) is metrisable and \(\text{ch}(A) = M_E(\mathbb{D} \times A)\) where \(M_E: \mathbb{K} \times E \to E\) is the continuous
scalar multiplication and $\mathbb{D}$ the open unit disc which yields the other implication. In particular, every locally convex Hausdorff space with ccp has obviously metric ccp, every quasi-complete locally convex Hausdorff space has ccp by [11, 9-2-10 Example, p. 134], every sequentially complete locally convex Hausdorff space has metric ccp by [23, A.1.7 Proposition (ii), p. 364] and every locally convex Hausdorff space with metric cpp is locally complete by [10, Remark 4.1, p. 267]. All these implications are strict. The second by [11, 9-2-10 Example, p. 134] and the others by [10, Remark 4.1, p. 267]. For more details on the [metric] convex compactness property and local completeness see [10] and [8].

Furthermore, every complete locally convex Hausdorff space is quasi-complete, every quasi-complete space is sequentially complete and every sequentially complete space is locally complete and all these implications are also strict. The first two by [22, p. 58] and the third by [34, 5.1.8 Corollary, p. 153] and [34, 5.1.12 Example, p. 154]. Moreover, a locally convex Hausdorff space is locally complete if and only if it is convenient by [27, 2.14 Theorem, p. 20]. In addition, we remark that every semi-Montel space is semi-reflexive by [22, 11.5.1 Proposition, p. 230] and every semi-reflexive locally convex Hausdorff space is quasi-complete by [35, Chap. IV, 5.5, Corollary 1, p. 144] and these implications are strict as well. Summarizing, we have the following diagram of strict implications:

\[
\begin{array}{ccc}
\text{semi-Montel} & \Rightarrow & \text{semi-reflexive} \\
\downarrow & & \\
\text{complete} & \Rightarrow & \text{quasi-complete} \\
\downarrow & & \downarrow \quad \Rightarrow \\
\text{ccp} & \Rightarrow & \text{metric ccp}
\end{array}
\]

We start with the simplest example of all. Let $\Omega$ be a non-empty set and equip the space $E^\Omega$ with the topology of pointwise convergence, i.e. the locally convex topology given by the seminorms

\[|f|_{K,\alpha} := \sup_{x \in K} p_{\alpha}(f(x)) \chi_K(x), \quad f \in E^\Omega,\]

for finite $K \subset \Omega$ and $\alpha \in \mathfrak{A}$. To prove $E^{\Omega_0} \cong K^{\Omega_0} \varepsilon E$ for complete $E$ is given as an exercise in [23, Aufgabe 10.5, p. 259] which we generalise now.

5.1. Example. Let $\Omega$ be a non-empty set and $E$ an lcHs. Then $E^\Omega \cong K^{\Omega_0} \varepsilon E$.

Proof. The strength and consistency of the defining family $(id_{E^\Omega}, id_{\Omega})$ is obvious. Let $f \in E^\Omega$, $K \subset \Omega$ be finite and set $N_K(f) := f(\Omega) \chi_K(\Omega)$. Then we have $N_K(f) = f(K) \cup \{0\}$ if $K \neq \Omega$ and $N_K(f) = f(K)$ if $K = \Omega$. Thus $N_K(f)$ is finite, hence compact, $N_K(f) \subset \operatorname{acx}(f(K))$ and $\operatorname{acx}(f(K))$ is a subset of the finite dimensional subspace $\operatorname{span}(f(K))$ of $E$. It follows that $\operatorname{acx}(f(K))$ is compact by [22, 6.7.4 Proposition, p. 113] implying $E^\Omega \subset E^\Omega$ by Lemma 3.11 b) and our statement by virtue of Theorem 3.13 with Property 3.13 d).

The space of càdlàg functions on a set $\Omega \subset \mathbb{R}$ with values in an lcHs $E$ is defined by

\[D(\Omega, E) := \{f \in E^\Omega \mid \forall x \in \Omega : \lim_{w \downarrow x} f(w) = f(x) \text{ and } \lim_{w \uparrow x} f(w) \text{ exists}\}.

5.2. Proposition. Let $\Omega \subset \mathbb{R}$, $K \subset \Omega$ be compact and $E$ an lcHs. Then $f(K)$ is precompact for every $f \in D(\Omega, E)$.

Proof. Let $f \in D(\Omega, E)$, $\alpha \in \mathfrak{A}$ and $\varepsilon > 0$. We set $f_x := \lim_{w \downarrow x} f(w)$,

\[B_r(x) = \{w \in \mathbb{R} \mid |w - x| < r\} \quad \text{and} \quad B_{\varepsilon,\alpha}(y) := \{w \in E \mid p_{\alpha}(w - y) < \varepsilon\}.\]
for every $x \in \Omega$, $y \in E$ and $r > 0$. Let $x \in \Omega$. Then there is $r_x > 0$ such that $p_n(f(w) - f_x) < \varepsilon$ for all $w \in B_{r_x}(x) \cap (-\infty, x) \cap \Omega$. Further, there is $r_x > 0$ such that $p_n(f(w) - f(x)) < \varepsilon$ for all $w \in B_{r_x}(x) \cap [x, \infty) \cap \Omega$. Choosing $r_x := \min(r_x, r_x')$ and setting $V_x := B_{r_x}(x) \cap \Omega$, we have $f(w) \in (B_{r_x}(f_x) \cup B_{r_x}(f(x)))$ for all $w \in V_x$. The sets $V_x$ are open in $\Omega$ with respect to the topology induced by $\mathbb{R}$ and $K \subset \bigcup_{x \in K} V_x$. Since $K$ is compact, there are $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in K$ such that $K \subset \bigcup_{i=1}^n V_{x_i}$. Hence we get

$$f(K) \subset \bigcup_{i=1}^n f(V_{x_i}) \subset \bigcup_{i=1}^n (B_{\varepsilon, \alpha}(f_{x_i}) \cup B_{\varepsilon, \alpha}(f(x_i)))$$

which means that $f(K)$ is precompact. $\square$

Due to the preceding proposition the maps given by

$$|f|_{K, \alpha} := \sup_{x \in \Omega} p_n(f(x)) \chi_K(x), \quad f \in D(\Omega, E),$$

for compact $K \subset \Omega$ and $\alpha \in \mathfrak{A}$ form a system of seminorms inducing a locally convex topology on $D(\Omega, E)$. Further, $D(\Omega, E) = \text{dom} \; T_{rc}^E \cap \text{dom} \; T_{ll}^E$ where the right-continuity is described by

$$\text{dom} \; T_{rc}^E := \{ f \in E' \mid \forall x \in \Omega, (x_n) \subset \Omega, x_n \searrow x : \lim_{n \to \infty} f(x_n) = f(x) \}$$

and $T_{rc}^E : E' \to \Omega, T_{rc}^E(f)(x) := f(x)$, and having limits from the left is described by

$$\text{dom} \; T_{ll}^E := \{ f \in E' \mid \forall x \in \Omega, (x_n) \subset \Omega, x_n \nearrow x : \lim_{n \to \infty} f(x_n) \text{ ex. in } E \text{ ind. of } (x_n) \}$$

and $T_{ll}^E : E' \to \Omega, T_{ll}^E(f)(x) := \lim_{n \to \infty} f(x_n)$, $(x_n) \subset \Omega, x_n \nearrow x$. $\mathbb{E}$

5.3. Example. Let $\Omega \subset \mathbb{R}$ be locally compact and $E$ an lcHs. If $E$ is quasi-complete, then $D(\Omega) \subset E \equiv D(\Omega, E)$. $\mathbb{E}$

Proof. First, we show that the defining family $(T_{rc}^E, T_{ll}^E)_{n \in \mathbb{R}, \alpha \in \Omega}$ for $(D, E)$ is strong and consistent. The strength is a consequence of a simple calculation, so we only prove the consistency explicitly. Let $x \in \Omega$, $(x_n)$ be a sequence in $\Omega$ such that $x_n \searrow x$ resp. $x_n \nearrow x$. We have

$$\delta_{x_n}(f) = f(x_n) - f(x) = T_{rc}^E(f), \quad x_n \searrow x,$$

and

$$\delta_{x_n}(f) = f(x_n) - \lim_{n \to \infty} f(x_n) = T_{ll}^E(f), \quad x_n \nearrow x,$$

for every $f \in D(\Omega)$ which implies that $(\delta_{x_n})$ converges to $T_{rc}^E$ if $x_n \searrow x$ and to $T_{ll}^E$ if $x_n \nearrow x$ in $D(\Omega)$. Since $\Omega$ is locally compact, there are a compact neighbourhood $U(x) \subset \Omega$ of $x$ and $n_0 \in \mathbb{N}$ such that $x_n \in U(x)$ for all $n \geq n_0$. Hence we deduce

$$\sup_{n \geq n_0} |\delta_{x_n}(f)| \leq |f|_{U(x)}$$

for every $f \in D(\Omega)$. Therefore the set $\{ \delta_{x_n} \mid n \geq n_0 \}$ is equicontinuous in $D(\Omega)'$ and $u(\delta_{x_n})$ converges to $T_{rc}^E$ if $x_n \searrow x$ and to $T_{ll}^E$ if $x_n \nearrow x$ in $D(\Omega)$. From

$$\lim_{n \to \infty} u(\delta_{x_n}) = u(T_{rc}^E) = u(\delta) = T_{rc}^E(S(u))(x), \quad x_n \searrow x,$$

and

$$u(T_{ll}^E) = \lim_{n \to \infty} u(\delta_{x_n}) = T_{ll}^E(S(u))(x), \quad x_n \nearrow x,$$

for every $u \in D(\Omega) \subset E$ follows the consistency. Second, let $f \in D(\Omega, E)$, $K \subset \Omega$ be compact and set $N_K(f) := f(\Omega) \chi_K(\Omega)$. We observe that $N_K(f) = f(K) \cup \{ 0 \}$ if $K \not= \Omega$ and $N_K(f) = f(K)$ if $K = \Omega$. Thus we deduce that $N_K(f)$ is precompact in $E$ for every $f \in D(\Omega, E)$ and every compact $K \subset \Omega$ by Proposition 5.2 and we obtain $D(\Omega, E) \subset D(\Omega, E)_K$ by virtue of Lemma 3.11 b. The quasi-completeness
of $E$ yields that $N_K(f)$ is relatively compact by \[22, 3.5.3\] Proposition, p. 65] and that $\text{acx}(N_K(f))$ is absolutely convex and compact. We derive our statement from Theorem \[5.14\] with Property \[5.13\] d).

Let us consider one of the most classical examples next, namely, the space $\mathcal{C}(\Omega, E)$ of continuous functions on a $k_\mathbb{R}$-space $\Omega$ with values in an lcHs $E$ equipped with the topology of uniform convergence on compact subsets of $\Omega$, i.e. we choose the family of weights $W$ given by $\nu_K := \chi_K$ for compact $K \subset \Omega$. In \[4, 2.4\] Theorem (2), p. 138-139] Bierstedt proved that $\mathcal{C}(\Omega, E) \cong \mathcal{C}(\Omega) \varepsilon E$ if $E$ is quasi-complete which we improve now.

5.4. Example. Let $\Omega$ be a [metrisable] $k_\mathbb{R}$-space and $E$ an lcHs. If $E$ has [metric] ccp, then $\mathcal{C}(\Omega, E) \cong \mathcal{C}(\Omega) \varepsilon E$.

Proof. First, we observe that the defining family $(T_{E}^\mathcal{E}, T_{f}^{\mathcal{E}})$ for $(\mathcal{C}(\Omega, E), E)$ is strong and consistent by Proposition \[4.3\] and Lemma \[4.1\] b)(i). Let $f \in \mathcal{C}(\Omega, E), K \subset \Omega$ be compact and set $N_K(f) = f(\Omega)\nu_K$. Then $N_K(f) = f(K) \cup \{0\}$ if $K \neq \Omega$ and $N_K(f) = f(\Omega)$ if $K = \Omega$ which yields that $N_K(f)$ is compact in $E$. If $\Omega$ is even metrisable, then $f(K)$ is also metrisable by \[1,\] Chap. IX, \$2.10\, Proposition 17, p. 159] and thus the finite union $N_K(f)$ as well by \[33,\] Theorem 1, p. 361] since the compact set $N_K(f)$ is collectionwise normal and locally countably compact by \[13,\] 5.1.18 Theorem, p. 305]. Further, $\text{acx}(N_K(f))$ is absolutely convex and compact in $E$ if $E$ has ccp resp. if $\Omega$ is metrisable and $E$ has metric ccp. Thus we deduce $\mathcal{C}(\Omega, E) \subset \mathcal{C}(\Omega, E)_c$ by Lemma \[3.11\] b). We conclude that $\mathcal{C}(\Omega, E) \cong \mathcal{C}(\Omega) \varepsilon E$ if $E$ has ccp resp. if $\Omega$ is metrisable and $E$ has metric ccp by Theorem \[3.14\] with Property \[3.13\] d). \[\square\]

We proceed to spaces of distributions. Let us denote by $\mathcal{D}(U)$ the linear subspace of the space $\mathcal{C}_0^\infty(U, \mathbb{K})$ of smooth functions consisting of all functions with compact support in an open subset $U \subset \mathbb{R}^d$ which is equipped with its usual inductive limit topology. A distribution $f \in L(D(U), E)$ with an lcHs $E$ and $U = \mathbb{R}^d$ or $U = \mathbb{R}^d - \{0\}$ is called homogeneous of degree $\lambda \in \mathbb{C}$ if

$$\langle f, \varphi \rangle = t^{\lambda}\langle f, \varphi_t \rangle, \quad \varphi \in D(U), \quad t > 0,$$

where $\varphi_t(x) := t^\lambda \varphi(tx)$ for $x \in U$ and $(\cdot, \cdot)$ denotes the canonical pairing (see \[12,\] Definition 3.2.2, p. 74]). By $L^{\mathcal{D}}(D(U), E)$ we mean the space of all distributions which are homogeneous of degree $\lambda$ and set $D^{\mathcal{D}}(D(U), E) := L^{\mathcal{D}}(D(U), E)$. It is easily seen that $g \in L(D(U), E)$ is homogeneous of degree $\lambda \in \mathbb{C}$ if and only if $g$ is in the kernel of the linear operator

$$T_{\lambda,h}^{\mathcal{D}}; L(D(U), E) \rightarrow E^{\mathbb{R}\times \mathcal{D}(U)}, \quad T_{\lambda,h}^E(f)(t, \varphi) := \langle f, \varphi_t \varphi \rangle,$$

5.5. Example. Let $\lambda \in \mathbb{C}$, $U = \mathbb{R}^d$ or $U = \mathbb{R}^d - \{0\}$ and $E$ be lcHs. Then $L^{\mathcal{D}}(D(U), E) \cong D^{\mathcal{D}}(D(U), E)$. \[\square\]

Proof. The defining family $(T_{E}^{E}_{\alpha}, T_{E}^{\mathcal{E}}_{\beta})_{\alpha \in \{a,b\}}$ for $(D(U))_{\alpha}^{\mathcal{E}}(E)$ is strong and consistent by Proposition \[4.3\] for continuity in combination with Remark \[13\] (i) since $\mathcal{D}(U)$ is a Montel space and by Proposition \[4.14\] b)(i) for linearity. Let $f \in L_0(\mathcal{D}(U), E), B \subset \mathcal{D}(U)$ be bounded and set $N_B(f) := f(\mathcal{D}(U))\chi_B(\mathcal{D}(U)) = f(B) \cup \{0\}$. We observe that $N_B(f) \subset f(\text{acx}(B)) = K$ as $f$ is linear. From $\mathcal{D}(U)$ being Montel, \[22, 6.2.1\, Proposition, p. 103\] and \[22, 6.7.1\, Proposition, p. 112\] follows that the set $\text{acx}(B)$ is absolutely convex and compact and thus $K$ as well. Therefore $L_0(\mathcal{D}(U), E) \subset L_0(\mathcal{D}(U), E)_c$ by Lemma \[5.17\] b) and $L_0(\mathcal{D}(U), E) \cong L_0(\mathcal{D}(U), E)_{\varepsilon E}$ by Theorem \[5.14\] with Property \[5.13\] d). Adding $(T_{E}^{E}_{\alpha}, T_{E}^{\mathcal{E}}_{\beta})$ to the defining family of $(D(U))_{\alpha}^{\mathcal{E}}(E)$ and $\{\lambda,h\}$ to $\{a,b\}$ (new $\mathcal{M}_0$) we get the defining family of $(D(U))_{\lambda,h}^{\mathcal{D}}(E)$. A simple calculation shows that the subfamily $(T_{E}^{E}_{\lambda,h}, T_{E}^{\mathcal{E}}_{\lambda,h})$ is strong and consistent. Applying Proposition \[5.19\] (i) proves our statement. \[\square\]
5.6. Example. Let \( \Omega \) be a normed or semi-reflexive, metrisable lcs and \( E \) a semi-Montel space. Then \( L_b(\Omega, E) \cong \Omega'_b \hookrightarrow E \).

Proof. The defining family \((T^E_m, T^E_m)_{m \in \mathrm{id}(\Omega)}\) for \( (L_b, E) \) is strong and consistent by Proposition 4.4 for linearity implying our statement by Corollary 3.16. \( \square \)

If \( E \) and \( \Omega \) are normed spaces, then \( L_b(\Omega, E) \) is just the space of bounded linear operators with the operator norm and \( \Omega'_b \hookrightarrow E \cong K(\Omega, E) \) by Satz 10.4, p. 235 where \( K(\Omega, E) \) is the space of compact, linear operators from \( \Omega \) to \( E \). Hence we cannot omit the condition that \( E \) is a semi-Montel space in general.

We turn to Cauchy continuous functions. Let \( \Omega \) be a metric space, \( E \) an lcHs and

\[
\mathcal{CC}(\Omega, E) := \{ f \in E^\Omega \mid f \text{ is Cauchy continuous} \}
\]

be equipped with the system of seminorms given by

\[
|f|_{K, \alpha} := \sup_{x \in K} p_\alpha(f(x)) \chi_K(x), \quad f \in \mathcal{CC}(\Omega, E),
\]

for \( K \subset \Omega \) precompact and \( \alpha \in \mathfrak{A} \). Further, let \( T^E_{id} := \text{id}_{E^\Omega} \).

5.7. Example. Let \( \Omega \) be a metric space and \( E \) an lcHs. If \( E \) is a Fréchet or a semi-Montel space, then \( \mathcal{CC}(\Omega, E) \cong \mathcal{CC}(\Omega) \hookrightarrow E \).

Proof. The defining family \((T^E_m, T^E_m)_{m \in \mathrm{id}(\Omega)}\) for \( (\mathcal{CC}, E) \) is strong and consistent by Proposition 4.4 with Lemma 4.7 (i) for Cauchy continuity. First, we consider the case that \( E \) is a Fréchet space. Let \( f \in \mathcal{CC}(\Omega, E) \), \( K \subset \Omega \) be precompact and set \( N_K(f) := f(\Omega) \chi_K(\Omega) \). Then \( N_K(f) = f(K) \cup \{0\} \) if \( K \neq \Omega \) and \( N_K(f) = f(K) \) if \( K = \Omega \). The set \( f(K) \) is precompact in the metrisable space \( E \) by Proposition 4.11, p. 576. Thus we obtain \( \mathcal{CC}(\Omega, E) \subset \mathcal{CC}(\Omega, E)_\kappa \) by virtue of Lemma 5.11 b).

Since \( E \) is complete, the first part of the statement follows from Theorem 3.14 with Property 3.13 a). If \( E \) is a semi-Montel space, then it is a consequence of Corollary 3.16. \( \square \)

Let \( (\Omega, d) \) be a metric space, \( E \) an lcHs and

\[
BUC(\Omega, E) := \{ f \in E^\Omega \mid f \text{ uniformly continuous and bounded} \}
\]

be equipped with the system of seminorms given by

\[
|f|_\alpha := \sup_{x \in \Omega} p_\alpha(f(x)), \quad f \in BUC(\Omega, E),
\]

for \( \alpha \in \mathfrak{A} \) and let \( T^E_{id} := \text{id}_{E^\Omega} \).

5.8. Example. Let \( (\Omega, d) \) be a metric space and \( E \) an lcHs. If \( E \) is a semi-Montel space, then \( BUC(\Omega, E) \cong BUC(\Omega) \hookrightarrow E \).

Proof. The defining family \((T^E_m, T^E_m)_{m \in \mathrm{id}(\Omega)}\) for \( (BUC, E) \) is strong and consistent by Proposition 4.8 with Lemma 4.9 for uniform continuity yielding our statement by Corollary 3.16. \( \square \)

Let \( (\Omega, d) \) be a metric space, \( z \in \Omega \), \( E \) an lcHs, \( 0 < \gamma \leq 1 \) and define the space of \( E \)-valued \( \gamma \)-Hölder continuous functions on \( \Omega \) that vanish at \( z \) by

\[
C_z^{[\gamma]}(\Omega, E) := \{ f \in E^\Omega \mid f(z) = 0 \text{ and } |f|_\alpha < \infty \forall \alpha \in \mathfrak{A} \}
\]

where

\[
|f|_\alpha := \sup_{x,w \in \Omega} \frac{p_\alpha(f(x) - f(w))}{d(x,w)^\gamma}.
\]
The topological subspace $C^{[\gamma]}_{\infty, 0}(\Omega, E)$ of $\gamma$-Hölder continuous functions that vanish at infinity consists of all $f \in C^{[\gamma]}_{\infty, 0}(\Omega, E)$ such that for all $\varepsilon > 0$ there is $\delta > 0$ with

$$\sup_{0 < d(x, w) < \delta} \frac{p_{\alpha}(f(x) - f(w))}{d(x, w)^{\gamma}} < \varepsilon.$$ 

Further, we set $T^{E}_{w} : E^{\Omega} \to \Omega', T^{E}_{w}(f)(x) := f(x) - f(w)$, and

$$\nu_{w} : \Omega \to [0, \infty), \quad \nu_{w}(x) := \begin{cases} \frac{1}{d(x, w)}, & x \neq w, \\ 1, & x = w, \end{cases}$$

for every $w \in \Omega$. Then we have for every $\alpha \in \mathfrak{A}$ that

$$|f|_{\alpha} = \sup_{x, w \in \Omega} p_{\alpha}(T^{E}_{w}(f)(x)) \nu_{w}(x), \quad f \in C^{[\gamma]}_{\infty, 0}(\Omega, E).$$

### 5.9. Example

Let $(\Omega, d)$ be a metric space, $z \in \Omega$, $E$ be an lchS and $0 < \gamma \leq 1$. Then

a) $C^{[\gamma]}(\Omega, E) \equiv C^{[\gamma]}(\Omega, \varepsilon E$ if $E$ is a semi-Montel space.

b) $C^{[\gamma]}_{\infty, 0}(\Omega, E) \equiv C^{[\gamma]}_{\infty, 0}(\Omega, E$ if $\Omega$ is precompact and $E$ quasi-complete.

**Proof.** Let us start with a). From Proposition 4.13 for vanishing at $z$ and a simple calculation follows that $(T^{E}_{m}, T^{E}_{m})_{m (\varepsilon E}{\infty})$ is a strong and consistent family for $(C^{[\gamma]}_{z, 0}, E)$. This proves part a) by Corollary 3.10. Concerning part b), let the family $\mathfrak{R} := \{(x, w) \in \Omega^{2} \mid d(x, w) \geq \delta\} \mid \delta > 0\}$. $\mathcal{M}_{\text{top}} := m := \Omega$ and $\pi : \Omega \times \Omega \to \Omega^{2}$ be the identity. Then the defining family $(T^{E}_{m}, T^{E}_{m})_{m (\varepsilon E}{\infty})$ is strong and consistent by Proposition 4.2 for vanishing at infinity w.r.t. $(\pi, \mathfrak{R})$. Let $f \in C^{[\gamma]}_{\infty, 0}(\Omega, E)$ and $K_{\delta} := \{(x, w) \in \Omega^{2} \mid d(x, w) \geq \delta\}$ for $\delta > 0$. Setting

$$N_{\pi \in K_{\delta}, 1}(f) := \{T^{E}_{w}(f)(x) \nu_{w}(x) \mid (x, w) \in K_{\delta}\} = \{\frac{f(x) - f(w)}{d(x, w)^{\gamma}} \mid (x, w) \in K_{\delta}\},$$

we have

$$N_{\pi \in K_{\delta}, 1}(f) \equiv \delta^{-\gamma} \{c(f(x) - f(w)) \mid (x, w) \in \Omega, \|c\| \leq 1\}$$

$$= \delta^{-\gamma} c(f(\Omega) - f(\Omega)).$$

The set $f(\Omega)$ is precompact because $\Omega$ is precompact and the $\gamma$-Hölder continuous function $f$ is uniformly continuous. It follows that the linear combination $f(\Omega) - f(\Omega)$ is precompact and the circled hull of a precompact set is still precompact by [35], Chap. I, 5.1, p. 25. Therefore $N_{\pi \in K_{\delta}, 1}(f)$ is precompact for every $\delta > 0$ connoting the precompactness of

$$N_{1}(f) := \{T^{E}_{w}(f)(x) \nu_{w}(x) \mid (x, w) \in \Omega \times \Omega\}$$

by the proof of Lemma 3.11(d). It follows that $N_{1}(f)$ is relatively compact by [22] 3.5.3 Proposition, p. 65 and $K := \max(N_{1}(f))$ is absolutely convex and compact if $E$ is quasi-complete and thus has ccpp. Hence statement b) is a consequence of Lemma 3.11(d) and Theorem 4.4 with Property 3.13(d). \(\square\)

Now, we consider spaces of weighted continuously partially differentiable functions and present the counterpart for differentiable functions to Bierstedt’s results [4, 2.4 Theorem, p. 138-139] and [4, 2.12 Satz, p. 141] for the space $C^\infty(\Omega, E)$ of continuous functions from a completely regular Hausdorff space $\Omega$ to an lchS $E$ equipped with a weighted topology given by a Nachbin-family $V$ of weights and its topological subspace $CV_{0}(\Omega, E)$ of functions which vanish at infinity in the weighted topology. We recall the following. A function $f : \Omega \to E$ on an open set $\Omega \subset \mathbb{R}^{d}$ to
an lcHs \( E \) is called continuously partially differentiable \((f \in C^1)\) if for the \( n \)-th unit vector \( e_n \in \mathbb{R}^d \) the limit
\[
\left( \frac{\partial}{\partial x_n} \right)^E f(x) := \lim_{h \to 0} \frac{f(x + h e_n) - f(x)}{h}
\]
even in \( E \) for every \( x \in \Omega \) and \( \left( \frac{\partial}{\partial x_n} \right)^E f \) is continuous on \( \Omega \) \((\left( \frac{\partial}{\partial x_n} \right)^E f \in C^0)\) for every \( 1 \leq n \leq d \). For \( k \in \mathbb{N} \) a function \( f \) is said to be \( k \)-times continuously partially differentiable \((f \in C^k)\) if \( f \in C^1 \) and all its first partial derivatives are \( C^{k-1} \). A function \( f \) is called infinitely continuously partially differentiable \((f \in C'^\infty)\) if \( f \) is \( C^k \) for every \( k \in \mathbb{N} \). For \( k \in \mathbb{N}_\infty := \mathbb{N} \cup \{ \infty \} \) the functions \( f : \Omega \to E \) which are \( C^k \) form a linear space which is denoted by \( C^k(\Omega, E) \). Let \( \text{Sym}_d \) be the set of all permutations of the set \( \{1, \ldots, d\} \). For \( \beta \in \mathbb{N}_0^d \) with \( |\beta| := \sum_{d} \beta_n \leq k \) and a function \( f : \Omega \to E \) on an open set \( \Omega \subset \mathbb{R}^d \) \( \text{to an lcHs} \( E \) \) we set \( \left( \frac{\partial^{|\beta|}}{\partial x_\beta} \right)^E f := f \) if \( \beta_n = 0 \), and
\[
\left( \frac{\partial^{|\beta|}}{\partial x_\beta} \right)^E f(x) = \left( \frac{\partial}{\partial x_\beta} \right)^E f(x)
\]
if \( \beta_n \neq 0 \) \( \text{and} \) the right-hand side exists in \( E \) for every \( x \in \Omega \). Further, we define
\[
\sigma(\partial^{|\beta|})^E f(x) := \left( \frac{\partial^{|\beta|}}{\partial x_\beta} \right)^E f(x) := \left( \frac{\partial^{|\beta|}}{\partial x_\beta} \right)^E f(x)
\]
if the right-hand side exists in \( E \) for every \( x \in \Omega \). If \( \sigma = \text{id} \), we write \( (\partial^{|\beta|})^E f(x) := \sigma(\partial^{|\beta|})^E f(x) \). Then we clearly have
\[
C^k(\Omega, E) = \bigcap_{(\sigma, \beta) \in \text{Sym}_d \times (\mathbb{N}_0^d)^{\mathbb{R}^d} \setminus \{ (\text{id}, \beta) \}} \text{dom}(\sigma(\partial^{|\beta|})^E)_e
\]
with
\[
\text{dom}(\sigma(\partial^{|\beta|})^E)_e := \{ f : \Omega \to E \mid \forall x \in \Omega : \sigma(\partial^{|\beta|})^E f(x) \in E \text{ and } \sigma(\partial^{|\beta|})^E f \in C(\Omega, E) \}
\]
and
\[
\sigma(\partial^{|\beta|})^E : \text{dom}(\sigma(\partial^{|\beta|})^E) \to E^\Omega, \sigma(\partial^{|\beta|})^E(f) := \sigma(\partial^{|\beta|})^E f.
\]
For \( k \in \mathbb{N}_\infty \) we set \( \{k\} := \{0, \ldots, k\} \) if \( k < \infty \) \( \text{and} \) \( \{k\} := \mathbb{N}_0 \) if \( k = \infty \) \( \text{and} \) let \( \mathcal{V}^k := \{(\nu_{j,l,\beta})_{\beta \in \mathbb{N}_0^d, |\beta| \leq |\gamma|} : j \leq |\gamma| \leq k\} \) be a family of weights on an open set \( \Omega \subset \mathbb{R}^d \) \( \text{which is} \) directed, i.e.
\[
\forall j_1, j_2 \in J, l_1, l_2 \in \{k\} \exists j_3 \in J, l_3 \in \{k\}, l_3 \geq \max(l_1, l_2), C > 0 \forall i \in \{1, 2\}, |\beta| \leq l_i : \nu_{j_i, l_i, \beta} \leq C \nu_{j_3, l_3, \beta}
\]
as well as
\[
\forall x \in \Omega, l \in \{k\} \exists j \in J \forall \beta \in \mathbb{N}_0^d, |\beta| \leq l : 0 < \nu_{j,l,\beta}(x)
\]
(see Definition 3.3 and Remark 3.4). For \( k \in \mathbb{N}_\infty \) \( \text{and} \) a directed family \( \mathcal{V}^k := \{(\nu_{j,l,\beta})_{\beta \in \mathbb{N}_0^d, |\beta| \leq |\gamma|} : j \leq |\gamma| \leq k\} \) \( \text{of weights} \) on an open set \( \Omega \subset \mathbb{R}^d \) \( \text{we define the space of} \) weighted \( k \)-times continuously partially differentiable functions with values in an lcHs \( E \)
\[
C\mathcal{V}^k(\Omega, E) := \{ f \in C^k(\Omega, E) \mid \forall j \in J, l \in \{k\}, \alpha \in \mathcal{A} : |f|_{j,l,\alpha} < \infty \}
\]
where
\[
|f|_{j,l,\alpha} := \sup_{2 \notin \Omega, |\beta| \leq l} p_\alpha((\partial^{|\beta|})^E f(x)) \nu_{j,l,\beta}(x).
\]
We define the topological subspace of \( C\mathcal{V}^k(\Omega, E) \) consisting of the functions that vanish with all their derivatives when weighted at infinity by
\[
C\mathcal{V}_0^k(\Omega, E) := \{ f \in C\mathcal{V}^k(\Omega, E) \mid \forall j \in J, l \in \{k\}, \alpha \in \mathcal{A}, \varepsilon > 0 \}
\[ \exists K \in \Omega \text{ compact : } |f|_{\Omega \setminus K, j, \lambda, \alpha} < \epsilon \]

where

\[ |f|_{\Omega \setminus K, j, \lambda, \alpha} := \sup_{x \in \Omega \setminus K,\beta \in M} |p_n((\partial^\beta)^E f(x))|_{j,\lambda,\alpha} \]

The following property for a family of directed weights allows us to use Lemma \[ \text{Lemma 4.4} \]

(i). A directed family of weights \( \nu^k \) is called locally bounded away from zero on \( \Omega \) if

\[ \forall K \subset \Omega \text{ compact, } l \in \{k\} \exists j \in J \forall \beta \in M_0^k, |\beta| \leq l : \inf_{x \in K} \nu_{j,\lambda,\alpha}(x) > 0. \]

5.10. Example. Let \( E \) be an lcHs, \( k \in \mathbb{N}_0 \), \( \nu^k \) be a family of weights which is locally bounded away from zero on an open set \( \Omega \subset \mathbb{R}^d \).

a) Let \( \mathcal{M} := \text{Sym}_d \times \{ \gamma \in \mathbb{N}_0^d | |\gamma| \leq k \} \). The family \( (\sigma(\partial^\beta)^E, \sigma(\partial^\beta)^K_{\sigma})_{\beta \in \mathcal{M}} \) is a strong and consistent family for \( (CV^k, E) \) resp. subfamily for \( (CV_0^k, E) \) if \( CV^k(\Omega) \) resp. \( CV_0^k(\Omega) \) is barrelled.

b) \( CV^k(\Omega, E) \equiv CV^k(\Omega);E \) if \( E \) is a semi-Montel space and \( CV^k(\Omega) \) barrelled.

c) \( CV^k(\Omega, E) \equiv CV_0^k(\Omega);E \) if \( E \) is complete and \( CV^k(\Omega) \) a Montel space.

d) \( CV_0^k(\Omega, E) \equiv CV_0^k(\Omega);E \) if \( E \) is quasi-complete and \( CV^k(\Omega) \) barrelled.

Proof. We start with the proof of part a). To prove strength is quite simple and thus we concentrate on consistency which we prove by induction. Let \( \sigma \in \text{Sym}_d \). For \( \beta \in \mathbb{N}_0^d \) with \( |\beta| = 0 \) the consistency of the subfamily \( (\sigma(\partial^\beta)^E, \sigma(\partial^\beta)^K_{\sigma})_{\beta \in M_l} \) follows from Proposition \[ \text{Lemma 4.3} \]

(i) for continuity since \( \nu^k \) is locally bounded away from zero. Let \( l \in \{k - 1\} \) such that \( (\sigma(\partial^\beta)^E, \sigma(\partial^\beta)^K_{\sigma})_{\beta \in M_l} \) with \( M_l := \{ \beta \in \mathbb{N}_0^d | |\beta| \leq l \} \) is a consistent subfamily. Let \( \beta \in \mathbb{N}_0^d \) with \( |\beta| = l + 1 \). Choose \( m := \min \{1 \leq n \leq d | \beta_{\sigma(m)} \neq 0\} \) and define \( \beta \in \mathbb{N}_0^d \) with \( |\beta| = l \) by \( \beta_{\sigma(m)} := \beta_{\sigma(m)} \) for \( n \neq m \) and \( \beta_{\sigma(m)} := \beta_{\sigma(m)} - 1 \). Then we have

\[ \sigma(\partial^\beta)^E f = (\partial^\beta)^E \sigma(\partial^\beta)^E f \]

for every \( f \in CV^k(\Omega, E) \) resp. \( CV_0^k(\Omega, E) \). The barrelledness of \( CV^k(\Omega) \) resp. \( CV_0^k(\Omega) \) yields that \( (\sigma(\partial^\beta)^E, \sigma(\partial^\beta)^K_{\sigma})_{\beta \in M_l} \) converges to \( \sigma(\partial^\beta)^E \) by Lemma \[ \text{Lemma 4.3} \]

in \( CV^k(\Omega)^{\ast} \), resp. \( CV_0^k(\Omega)^{\ast} \) for every \( x \in \Omega \). Therefore we derive from \( \sigma(\partial^\beta)^E \) that

\[ u(\delta_x \circ \sigma(\partial^\beta)^E f) = \lim_{h \to 0} h \sigma(\partial^\beta)^E S(u)(x + h \cdot e_{\sigma(m)}) \]

for every \( u \in CV^k(\Omega);E \) and \( x \in \Omega \). Due to \( \nu^k \) being locally bounded away from zero and \( \sigma(\partial^\beta)^E = (\partial^\beta)^E \) (Schwarz’ theorem) on \( CV^k(\Omega) \) resp. \( CV_0^k(\Omega) \) we get that \( \sigma(\partial^\beta)^E \) is a continuous map from \( CV^k(\Omega) \) to \( CW(\Omega) \). From Lemma \[ \text{Lemma 4.3} \]

(i) we conclude the consistency of \( (\sigma(\partial^\beta)^E, \sigma(\partial^\beta)^K_{\sigma})_{\beta \in M_l+1} \). The proof of part b) and c) is this all we need by Corollary \[ \text{Corollary 3.16} \]

and Corollary \[ \text{Corollary 3.17} \]. Let us turn to part d). Let \( \mathcal{A} \) be the family of compact subsets of \( \Omega \) and \( \pi: \Omega \times \{ \gamma \in \mathbb{N}_0^d | |\gamma| \leq k \} \to \Omega \) be the projection on \( \Omega \). The subfamily \( T_{\infty}^E, \gamma \) for vanishing at infinity w.r.t. \( (\pi, R) \) is strong and consistent by Proposition \[ \text{Proposition 4.2} \]. Thus Corollary \[ \text{Corollary 3.17} \] implies statement d). \[ \square \]
The spaces $\mathcal{C}^k(\Omega)$ and $\mathcal{C}^k_0(\Omega)$ are Fréchet spaces and thus barrelled if $J$ is countable by $[29, 3.4 Prop. p. 5]$. In $[29, 5.2 Th. p. 18]$ the question is answered when they have the approximation property. To illustrate Example 5.10 d) we consider an example which was already known to Schwartz $[36, Prop. 9, p. 108, Théorème 1, p. 111]$. For an lcHs $E$ the Schwartz space is defined by

$$S(\mathbb{R}^d, E) = \{ f \in \mathcal{C}^\infty(\mathbb{R}^d, E) \ | \ \forall l \in \mathbb{N}_0, \alpha \in \mathfrak{A} : |f|_{l, \alpha} < \infty \}$$

where

$$|f|_{l, \alpha} = \sup_{x \in \mathbb{R}^d} p_{\alpha}((\partial^\beta)^E f(x))(1 + |x|^2)^{-l/2}.$$

5.11. Example. If $E$ is a quasi-complete lcHs, then $S(\mathbb{R}^d, E) \equiv S(\mathbb{R}^d)E$.

Proof. First, we note that $S(\mathbb{R}^d)$ is a Fréchet space and hence barrelled. We define $\mathcal{V}^\infty$ as the family of weights given by $\nu_{l, \beta}(x) = (1 + |x|^2)^{-l/2}$, $x \in \Omega$, for all $l \in \mathbb{N}_0$ and $|\beta| \leq l$ which gives $S(\mathbb{R}^d, E) = \mathcal{C}^{\infty}(\mathbb{R}^d, E)$. Observing that for every $l \in \mathbb{N}$ and $\varepsilon > 0$ there is $r > 0$ such that

$$(1 + |x|^2)^{-l/2} = (1 + |x|^2)^{-l/2} < \varepsilon$$

for all $x \in \mathbb{R}_r(0)$, we deduce

$$|f|_{l, \alpha} = \sup_{x \in \mathbb{R}_r(0)} p_{\alpha}((\partial^\beta)^E f(x))(1 + |x|^2)^{-l/2} \leq \varepsilon |f|_{l, \alpha}$$

for every $f \in S(\mathbb{R}^d, E)$ and $\alpha \in \mathfrak{A}$ which proves $S(\mathbb{R}^d, E) = \mathcal{C}^{\infty}(\mathbb{R}^d, E)$.

Like in Example 5.9 we can improve Example 5.10 d) if we consider the usual topology on $\mathcal{C}^k(\Omega, E)$, namely, the one generated by the family of weights $W^k$ given by $\nu_{K,l,\beta}(x) = \chi_K(x)$, $x \in \Omega$, for $K \subset \Omega$ compact, $l \in (k)$ and $|\beta| \leq l$. That $\mathcal{C}W^k(\Omega, E) \equiv \mathcal{C}W^k(\Omega)E$ for $k \in \mathbb{N}$ and quasi-complete $E$ is already mentioned in $[23, (9), p. 236]$ (without a proof) and that $\mathcal{C}W^\infty(\Omega, E) \equiv \mathcal{C}W^\infty(\Omega)E$ holds for locally complete $E$ in $[1, p. 228]$. Our technique allows us to generalise the first result and to get back the second result.

5.12. Example. Let $E$ be an lcHs, $k \in \mathbb{N}_\infty$ and $\Omega \subset \mathbb{R}^d$ be open. Then $\mathcal{C}W^k(\Omega, E) \equiv \mathcal{C}W^k(\Omega)E$ if $k < \infty$ and $E$ has metric ccp or if $k = \infty$ and $E$ is locally complete.

Proof. We already know that the defining family is strong and consistent by Example 5.10 a). Let $f \in \mathcal{C}W^k(\Omega, E), K \subset \Omega$ be compact, $M_l := \{ \beta \in \mathbb{N}_0^d \ | \ |\beta| \leq l \}$ for $l \in (\Omega) \cup \{0\}$ and set

$$N_{K,l}(f) := \{ (\partial^\beta)^E f(x)_{\nu_{K,l,\beta}(x)} \ | \ x \in \Omega, \beta \in M_l \} = \{0\} \cup \{ (\partial^\beta)^E f(K) \}.$$ 

$N_{K,l}(f)$ is compact since it is a finite union of compact sets. Furthermore, the compact sets $\{0\}$ and $(\partial^\beta)^E f(K)$ are metrisable by $[8, Chap. IX, §2.10, Prop. 17, p. 159]$ and thus their finite union $N_{K,l}(f)$ is metrisable as well by $[32, Th. 1, p. 361]$ since the compact set $N_{K,l}(f)$ is collectionwise normal and locally countably compact by $[13, 5.1.18 Th. p. 305]$. Due to Lemma 3.11 b) we obtain $\mathcal{C}W^k(\Omega, E) \subset \mathcal{C}W^k(\Omega, E)_\kappa$ for any lcHs $E$. If $E$ has metric ccp, then the set $\text{ac}(N_{K,l}(f))$ is absolutely convex and compact. Thus Theorem 6.14 with Property 5.13 d) settles the case for $k < \infty$. If $k = \infty$ and $E$ is locally complete, we observe that $K_\beta = \text{ac}(\text{(\partial^\beta)^E f(K)})$ for $f \in \mathcal{C}W^\infty(\Omega, E)$ is absolutely convex and compact by $[8, Prop. 2, p. 354]$. Then we have

$$N_{K,l}(f) \subset \text{ac}(\bigcup_{|\beta| \leq l} K_\beta)$$
and the set on the right-hand side is absolutely convex and compact by \[22\text{, 6.7.3 Proposition, p. 113}\]. Again, the statement follows from Theorem \[14\text{ with Property 6.14 d}].

In the context of differentiability on infinite dimensional spaces the preceding example remains true for an open subset \(\Omega\) of a Fréchet space or DFM-space and quasi-complete \(E\) by \[22\text{, 3.2 Corollary, p. 286}\]. Like here this can be generalised to \(E\) with \([\text{metric}]\) cpc. Let us consider kernels of linear partial differential operators next. Let \(E\) be an lcHs, \(k \in \N\), and \(\Omega \subset \R^d\). Let \(n \in \N\), \(\beta_m \in \N^d\) with \(|\beta_m| \in (k)\) and \(a_m : \Omega \rightarrow \K\) for \(1 \leq m \leq n\). We set \(\text{dom} \, P(\partial)^E \coloneqq \bigcap_{m=1}^n \text{dom}(\partial^{\beta_m})^E\) and

\[
P(\partial)^E \colon \text{dom} \, P(\partial)^E \rightarrow E_\Omega, \quad P(\partial)^E(f)(x) := \sum_{m=1}^n a_m(x) (\partial^{\beta_m})^E(f)(x).
\]

For a directed family of weights \(\mathcal{V}^k\) on \(\Omega\) we define the topological subspaces of \(\mathcal{C} V^k(\Omega, E)\) given by \(\mathcal{C} V^k_{P(\partial)}(\Omega, E) \coloneqq \{ f \in \mathcal{C} V^k(\Omega, E) \mid f \in \ker P(\partial)^E \}\) and \(\mathcal{C} V^k_{0, P(\partial)}(\Omega, E) \coloneqq \{ f \in \mathcal{C} V^k(\Omega, E) \mid f \in \ker P(\partial)^E \}\).

5.13. Example. Let \(E\) be an lcHs, \(k \in \N\), \(\mathcal{V}^k\) be a family of weights which is locally bounded away from zero on an open set \(\Omega \subset \R^d\).

a) \(\mathcal{C} V^k_{P(\partial)}(\Omega, E) \equiv \mathcal{C} V^k_{P(\partial)}(\Omega)\in E\) if \(E\) is a semi-Montel space and \(\mathcal{C} V^k(\Omega)\) barreled.

b) \(\mathcal{C} V^k_{P(\partial)}(\Omega, E) \equiv \mathcal{C} V^k_{P(\partial)}(\Omega)\in E\) if \(E\) is complete, \(\mathcal{C} V^k(\Omega)\) a Montel space and \(\mathcal{C} V^k_{P(\partial)}(\Omega)\) closed in \(\mathcal{C} V^k(\Omega)\).

c) \(\mathcal{C} V^k_{0, P(\partial)}(\Omega, E) \equiv \mathcal{C} V^k_{0, P(\partial)}(\Omega)\in E\) if \(E\) is quasi-complete and \(\mathcal{C} V^k(\Omega)\) barreled.

d) \(\mathcal{C} V^k_{P(\partial)}(\Omega, E) \equiv \mathcal{C} V^k_{P(\partial)}(\Omega)\in E\) if \(k < \infty\) and \(E\) has metric cpc or if \(k = \infty\) and \(E\) is locally complete.

Proof. Due to (the proofs of) Example 5.10 and Example 5.12 and Proposition 8.19 we only need to show that the subfamily \((P(\partial)^E, \mathcal{C} V^k_{P(\partial)})\) is strong and consistent.
The strength is obvious and the consistency follows from

\[
P(\partial)^E(S(u))(x) = \sum_{m=1}^{n} a_m(x) (\partial^{\beta_m})^E(S(u))(x) = u(\sum_{m=1}^{n} a_m(x)(\partial^{\beta_m})^E) = u(\ker P(\partial)^E), \quad \forall u \in \mathcal{C} V^k_{P(\partial)}(\Omega)\in E.
\]

for every \(u \in \mathcal{C} V^k_{P(\partial)}(\Omega)\in E\) resp. \(u \in \mathcal{C} V^k_{0, P(\partial)}(\Omega)\in E\).

A special case of example d) is already known to be a consequence of \[3\text{, Theorem 9, p. 232}\], namely, if \(k = \infty\) and \(P(\partial)\) is hypoelliptic with constant coefficients. In particular, this covers the space of holomorphic functions and the space of harmonic functions. The special case of example b) of holomorphic functions with exponential growth on strips is handled in \[28\text{, 3.11 Theorem, p. 31}\]. Holomorphy on infinite dimensional spaces is treated in \[11\text{ Corollary 6.35, p. 332-333}\] where \(\mathcal{V} = \mathcal{V}, \Omega\) is an open subset of a locally convex Hausdorff \(k\)-space and \(E\) a quasi-complete locally convex Hausdorff space, both over \(\mathcal{C}\), which can be generalised to \(E\) with \([\text{metric}]\) cpc in a similar way. Now, we direct our attention to spaces of continuously partially differentiable functions on an open bounded set such that all derivatives can be continuously extended to the boundary. Let \(E\) be an lcHs, \(k \in \N\), and \(\Omega \subset \R^d\) open and bounded. The space \(\mathcal{C}^k(\Omega, E)\) is given by

\[
\mathcal{C}^k(\Omega, E) := \{ f \in \mathcal{C}^k(\Omega, E) \mid (\partial^\beta)^E f \text{ cont. extendable on } \Omega \text{ for all } \beta \in \N^d \}
\]
and equipped with the system of seminorms given by
\[ |f|_{\alpha} := \sup_{x \in \Omega} p_{\alpha}(\partial_{\beta}^E f(x)), \quad f \in C^k(\Omega, E), \]
for \( \alpha \in \mathfrak{A} \) if \( k < \infty \) and by
\[ |f|_{l, \alpha} := \sup_{x \in \Omega} p_{\alpha}(\partial_{\beta}^E f(x)), \quad f \in C^\infty(\Omega, E), \]
for \( l \in \mathbb{N}_0 \) and \( \alpha \in \mathfrak{A} \) if \( k = \infty \). For \( \beta \in \mathbb{N}_0^d \), \( |\beta| \in (k) \), we set
\[ \text{dom}(\partial_{\beta}^E)_{\text{ext}} := \{ f \in \text{dom}(\partial_{\beta}^E) | (\partial_{\beta}^E) f \in \text{dom} T_{\text{ext}}^E \}, \]
and \( (\partial_{\beta}^E)_{\text{ext}} := T_{\text{ext}}^E \circ (\partial_{\beta}^E)_c \).

5.14. **Example.** Let \( E \) be an lcHs, \( k \in \mathbb{N}_0 \) and \( \Omega \subset \mathbb{R}^d \) open and bounded. Then \( C^k(\Omega, E) \cong C^k(\overline{\Omega}) \) if \( E \) has metric ccp.

**Proof.** The defining family is the union of the families \( (\sigma(\partial_{\beta}^E)_c)_{\sigma(\partial_{\beta}^E)_c} \) with \( \mathcal{M} := \text{Sym}_{M} \times \{ \gamma \in \mathbb{N}_0^d | |\gamma| \leq k \} \) for continuous partial differentiability on \( \Omega \) and \( (\sigma(\partial_{\beta}^E)_c)_{\text{ext}} \) for continuous extendability of the partial derivatives on \( \partial \Omega \). The strength and consistency of the first subfamily for continuous partial differentiability follows like in Example 5.10 since \( C^k(\overline{\Omega}) \) is a Banach space if \( k < \infty \) and a Fréchet space if \( k = \infty \). The metric strength and consistency of the second subfamily follows from Proposition 5.10 and \( \delta \alpha \in (\sigma(\partial_{\beta}^E)_c)_{\text{ext}} \) for all \( \alpha \in C^k(\overline{\Omega}) \) if \( |\beta| \in (k) \) by the consistency of the first subfamily. Let \( f \in C^k(\Omega, E) \), \( l \in \mathbb{N}_0 \) and set \( M := \{ \beta \in \mathbb{N}_0^d | |\beta| \leq k \} \) if \( k < \infty \) and \( M := \{ \beta \in \mathbb{N}_0^d | |\beta| \leq l \} \) if \( k = \infty \). We denote by \( f_{\beta} \) the continuous extension of \( (\partial_{\beta}^E) f \) on the compact metrisable set \( \overline{\Omega} \). The set
\[ N_M(f) := \{ (\partial_{\beta}^E) f(x) | x \in \Omega, \beta \in M \} \subset \bigcup_{\beta \in M} f_{\beta}(\overline{\Omega}) \]
is relatively compact and metrisable since it is a subset of a finite union of the compact metrisable sets \( f_{\beta}(\overline{\Omega}) \) like in Example 5.12. Due to Lemma 5.11 b) and Theorem 8.14 with Property 8.13 d) we obtain our statement if \( E \) has metric ccp. \( \square \)

We close our last section with spaces of ultradifferentiable functions. Let \( E \) be an lcHs, \( \Omega \subset \mathbb{R}^d \) open, \( \mathfrak{R} := \{ K \subset \Omega | K \text{ compact} \} \) and \( \{ M_p \}_{p \in \mathbb{N}_0} \) be a sequence of positive numbers. The space \( \mathcal{E}^{(M_p)}(\Omega, E) \) of ultradifferentiable functions of class \( (M_p) \) of Beurling-type is defined as
\[ \mathcal{E}^{(M_p)}(\Omega, E) := \{ f \in C^\infty(\Omega, E) | \forall (K, h) \in J_1, \alpha \in \mathfrak{A} : |f|_{(K, h), \alpha} < \infty \} \]
where \( J_1 := \mathfrak{R} \times \mathbb{R}_{>0} \) and
\[ |f|_{(K, h), \alpha} := \sup_{x \in \Omega} p_{\alpha}(\partial_{\beta}^E f(x)) \chi_K(x) \frac{1}{h^{\beta_0} |M|_{\beta}}, \]
The space \( \mathcal{E}^{(M_p)}(\Omega, E) \) of ultradifferentiable functions of class \( \{ M_p \} \) of Roumieu-type is defined as
\[ \mathcal{E}^{(M_p)}(\Omega, E) := \{ f \in C^\infty(\Omega, E) | \forall (K, H) \in J_2, \alpha \in \mathfrak{A} : |f|_{(K, H), \alpha} < \infty \} \]
where \( J_2 := \mathfrak{R} \times \{ H = (H_n)_{n \in \mathbb{N}} | \exists (h_k)_{k \in \mathbb{N}}, h_k > 0, h_k \not\rightarrow \infty \forall n \in \mathbb{N} : H_n = h_1 \cdots h_n \} \)
and

$$|f|_{(K,H),\alpha} := \sup_{x \in \Omega} \left( \int_0^1 \left( (\partial^\beta)^E f(x) \right) \chi_K(x) \frac{1}{H\beta} \right)$$

(see [26, Proposition 3.5, p. 675]). Further, we recall the following conditions of Komatsu for the sequence $\{M_p\}_{p \in \mathbb{N}_0}$ (see [24, p. 26] and [26, p. 653]):

(M.0) \( M_0 = M_1 = 1 \),

(M.1) \( \forall p \in \mathbb{N} : M_p^2 \leq M_{p-1}M_{p+1} \),

(M.2) \( \exists A, C > 0 \forall p \in \mathbb{N}_0 : M_{p+1} \leq AC^{p+1}M_p \),

(M.3) \( \sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty \).

5.15. Example. Let $E$ be an lcHs, $\Omega \subset \mathbb{R}^d$ open and $\{M_p\}_{p \in \mathbb{N}_0}$ be a sequence of positive numbers. Then

a) $\mathcal{E}^{(M_p)}(\Omega, E) \cong \mathcal{E}^{(M_p)}(\Omega)\mathcal{E} E$ if $E$ is complete or semi-Montel,

b) $\mathcal{E}^{(M_p)}(\Omega, E) \cong \mathcal{E}^{(M_p)}(\Omega)\mathcal{E} E$ if $E$ is complete or semi-Montel and in both cases $\{M_p\}_{p \in \mathbb{N}_0}$ fulfills (M.1) and (M.3)’,

c) $\mathcal{E}^{(M_p)}(\Omega, E) \cong \mathcal{E}^{(M_p)}(\Omega)\mathcal{E} E$ and $\mathcal{E}^{(M_p)}(\Omega, E) \cong \mathcal{E}^{(M_p)}(\Omega)\mathcal{E} E$ if $E$ is sequentially complete and $\{M_p\}_{p \in \mathbb{N}_0}$ fulfills (M.0), (M.1), (M.2)’ and (M.3)’.

Proof. The defining family in a)-c) is $\{\sigma(\partial^\beta)^E, \sigma(\partial^\beta)^E\}_{(\beta,\sigma)\in M}$ with $M := \text{Sym}_{d} \times [0,1]$ for continuous partial differentiability on $\Omega$. Its strength and consistency follow like in Example 5.10 since $\mathcal{E}^{(M_p)}(\Omega)$ is a Fréchet-Schwartz space in a) and b) by [24, Theorem 2.6, p. 44] whereas $\mathcal{E}^{(M_p)}(\Omega)$ is a Montel space in b) and c) by [24, Theorem 5.12, p. 65-66]. Hence we have $\mathcal{E}^{(M_p)}(\Omega)E \subset \mathcal{E}^{(M_p)}(\Omega,E)_\kappa$ and $\mathcal{E}^{(M_p)}(\Omega,E) \subset \mathcal{E}^{(M_p)}(\Omega,E)_\kappa$ by Lemma 3.11(a) for any lcHs $E$. Statements a) and b) follow from Corollary 5.15 and Corollary 5.16. Let us turn to c). We claim that Property 3.13(c) is fulfilled. Let $f' \in \mathcal{E}^{(M_p)}(\Omega)'E'$. Due to [24, Proposition 3.7, p. 677] there is a sequence $\{f_n\}_{n \in \mathbb{N}}$ in the space $\mathcal{D}^{(M_p)}(\Omega)$ resp. $\mathcal{D}^{(M_p)}(\Omega)$ of ultradifferentiable functions of class $(M_p)$ of Beurling-type resp. $(M_p)$ of Roumieux-type with compact support which converges to $f'$ in $\mathcal{E}^{(M_p)}(\Omega)'E'$ resp. $\mathcal{E}^{(M_p)}(\Omega)'E'$. Let $f \in \mathcal{E}^{(M_p)}(\Omega,E)'E$. We observe that for every $\epsilon' \in E'$

$$|R_{\epsilon'}^f(f_n)(\epsilon')| = \int_{\Omega} f_n(x)\epsilon'(f(x))dx \leq \lambda(\text{supp}(f_n)) \sup_{y \in N_n(f)} |\epsilon'(y)|$$

where $\lambda$ is the Lebesgue measure, supp$(f_n)$ is the support of $f_n$ and $N_n(f) := \{f_n(x) f(x) \mid x \in \text{supp}(f_n)\}$. The set $N_n(f)$ is compact and metrisable by [3, Chap. IX, §2.10, Proposition 17, p. 159] and thus the closure of its absolutely convex hull is compact in $E$ as the sequentially complete space $E$ has metric ccp. We conclude that $R_{\epsilon'}^f(f_n) \in (E_n)' = \mathcal{J}(E)$ for every $n \in \mathbb{N}$. Therefore Property 3.13(c) is fulfilled implying statement c) for sequentially complete $E$ by Theorem 3.14. \qed

The results a) and b) in the our last example are new whereas c) is already proved in [26, Theorem 3.10, p. 678] in a different way. We included c) to demonstrate an application of Property 3.13(c).

References

[1] G. Beer and S. Levi. Strong uniform continuity. J. Math. Anal. Appl., 350(2):568–589, 2009.

[2] K.-D. Bierstedt. Gewichtete Räume stetiger vektorwertiger Funktionen und das injektive Tensorprodukt. PhD thesis, Johannes-Gutenberg Universität Mainz, Mainz, 1971.
[3] K.-D. Bierstedt. Gewichtete Räume stetiger vektorwertiger Funktionen und das injektive Tensorprodukt. I. *J. Reine Angew. Math.*, 259:186–210, 1973.

[4] K.-D. Bierstedt. Gewichtete Räume stetiger vektorwertiger Funktionen und das injektive Tensorprodukt. II. *J. Reine Angew. Math.*, 260:133–146, 1973.

[5] V. I. Bogachev. *Gaussian Measures*. Math. Surveys Monogr. 62. AMS, Providence, 1998.

[6] J. Bonet and P. Domanski. The splitting of exact sequences of PLS-spaces and smooth dependence of solutions of linear partial differential equations. *Adv. Math.*, 217:561–585, 2008.

[7] J. Bonet, L. Frerick, and E. Jordá. Extension of vector-valued holomorphic and harmonic functions. *Studia Math.*, 183(3):225–248, 2007.

[8] J. Bonet, E. Jordá, and M. Maestre. Vector-valued meromorphic functions. *Arch. Math. (Basel)*, 79(5):353–359, Nov 2002.

[9] N. Bourbaki. *General Topology, Part 2*. Elem. Math. Addison-Wesley, Reading, 1966.

[10] H. Buchwalter. Topologies et compactologies. *Publ. Dép. Math., Lyon*, 6(2):1–74, 1969.

[11] S. Dineen. *Complex Analysis in Locally Convex Spaces*. Math. Stud. 57. North-Holland, Amsterdam, 1981.

[12] P. Domanski and M. Langenbruch. Vector Valued Hyperfunctions and Boundary Values of Vector Valued Harmonic and Holomorphic Functions. *Publ. Res. Inst. Math. Sci.*, 44:1097–1142, 2008.

[13] R. Engelking. *General topology*. Sigma Series Pure Math. 6. Heldermann, Berlin, 1989.

[14] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler. *Banach Space Theory: The Basis for Linear and Nonlinear Analysis*. CMS Books Math. Springer, New York, 2011.

[15] K. Floret and J. Wloka. *Einführung in die Theorie der lokalkonvexen Räume*. Lecture Notes in Math. 56. Springer, Berlin, 1968.

[16] L. Frerick and E. Jordá. Extension of vector-valued functions. *Bull. Belg. Math. Soc. Simon Stevin*, 14:499–507, 2007.

[17] L. Frerick, E. Jordá, and J. Wengenroth. Extension of bounded vector-valued functions. *Math. Nachr.*, 282(5):690–696, 2009.

[18] A. Grothendieck. *Produits tensoriels topologiques et espaces nucléaires*. Mem. Amer. Math. Soc. 16. AMS, Providence, 4th edition, 1966.

[19] L. Hörmander. *The Analysis of linear partial differential operators I*. Classics Math. Springer, Berlin, 2nd edition, 1990.

[20] L. Hörmander. *The Analysis of linear partial differential operators II*. Classics Math. Springer, Berlin, 2nd edition, 1990.

[21] I. M. James. *Topologies and Uniformities*. Springer Undergr. Math. Ser. Springer, London, 1999.

[22] H. Jarchow. *Locally Convex Spaces*. Math. Leitfäden. Teubner, Stuttgart, 1981.

[23] W. Kaballo. *Aufbaukurs Funktionalanalysis und Operatortheorie*. Springer, Berlin, 2014.

[24] H. Komatsu. Ultradistributions, I, Structure theorems and a characterization. *J. Fac. Sci. Univ. Tokyo, Sect. IA*, Math. 20:25–105, 1973.

[25] H. Komatsu. Ultradistributions, II, The kernel theorem and ultradistributions with support in a submanifold. *J. Fac. Sci. Univ. Tokyo, Sect. IA*, Math. 24:607–628, 1977.

[26] H. Komatsu. Ultradistributions, III, Vector valued ultradistributions and the theory of kernels. *J. Fac. Sci. Univ. Tokyo, Sect. IA*, Math. 29:653–718, 1982.
[27] A. Kriegl and P. W. Michor. *The Convenient Setting of Global Analysis*. Math. Surveys Monogr. 53. AMS, Providence, 1997.

[28] K. Kruse. *Vector-valued Fourier hyperfunctions*. PhD thesis, Universität Oldenburg, Oldenburg, 2014.

[29] K. Kruse. The approximation property for spaces of weighted differentiable functions, 2018. arxiv preprint [https://arxiv.org/abs/1806.02926](https://arxiv.org/abs/1806.02926).

[30] K. Kruse. Extension of weighted vector-valued functions, 2018. arxiv preprint [https://arxiv.org/abs/1808.05182](https://arxiv.org/abs/1808.05182).

[31] K. Kruse. Series representations in spaces of vector-valued functions, 2018. arxiv preprint [https://arxiv.org/abs/1806.01889](https://arxiv.org/abs/1806.01889).

[32] R. Meise. Spaces of differentiable functions and the approximation theory. *Approximation Theory and Functional Analysis*, 35:263–307, 1979.

[33] R. Meise and D. Vogt. *Introduction to Functional Analysis*. Clarendon Press, Oxford, 1997.

[34] P. Pérez Carreras and J. Bonet. *Barrelled locally convex spaces*. Math. Stud. 131. North-Holland, Amsterdam, 1987.

[35] H. H. Schaefer. *Topological vector spaces*. Grad. Texts in Math. Springer, Berlin, 1971.

[36] L. Schwartz. Espaces de fonctions différentiables à valeurs vectorielles. *J. Analyse Math.*, 4:88–148, 1955.

[37] L. Schwartz. Théorie des distributions à valeurs vectorielles. I. *Annales de l’institut Fourier*, 7:1–142, 1957.

[38] A. H. Stone. Metrisability of unions of spaces. *Proc. Amer. Math. Soc.*, 10(3):361–366, 1959.

[39] D. Vogt. On the solvability of $P(D)f=g$ for vector valued functions. *RIMS Kokyuroku*, 508:168–181, 1983.

[40] J. Voigt. On the convex compactness property for the strong operator topology. *Note Math.*, XII:259–269, 1992.

[41] A. Wilansky. *Modern Methods in Topological Vector Spaces*. McGraw-Hill, New York, 1978.

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