Online learning with multiple kernels has gained increasing interests in recent years and found many applications. For classification tasks, Online Multiple Kernel Classification (OMKC), which learns a kernel based classifier by seeking the optimal linear combination of a pool of single kernel classifiers in an online fashion, achieves superior accuracy and enjoys great flexibility compared with traditional single-kernel classifiers. Despite being studied extensively, existing OMKC algorithms suffer from high computational cost due to their unbounded numbers of support vectors. To overcome this drawback, we present a novel framework of Budget Online Multiple Kernel Learning (BOMKL) and propose a new Sparse Passive Aggressive learning to perform effective budget online learning. Specifically, we adopt a simple yet effective Bernoulli sampling to decide if an incoming instance should be added to the current set of support vectors. By limiting the number of support vectors, our method can significantly accelerate OMKC while maintaining satisfactory accuracy that is comparable to that of the existing OMKC algorithms. We theoretically prove that our new method achieves an optimal regret bound in expectation, and empirically found that the proposed algorithm outperforms various OMKC algorithms and can easily scale up to large datasets.
One straightforward approach to make online multiple kernel learning scalable is to apply the existing budget online learning algorithms to replace the current online learner for each kernel, and allocate a uniform budget for each kernel due to the lack of prior on the quality of individual kernels. Such a naive approach is simple and easy to implement, but falls short in many aspects. First of all, some efficient budget online learning algorithms are too simple to achieve satisfactory accuracy (e.g., RBP (Cavallanti et al., 2007)), while some other algorithms, despite being more effective, are often computationally very intensive for multiple kernel learning (e.g., Projectron (Orabona et al., 2009)). Second, because of the varied performances of different kernel functions in practice, using a simple uniform budget allocation is clearly not optimal as one of key successful principles in online the multiple kernel learning process is to focus more in learning with the better quality kernels.

In this paper, we investigate a novel framework of Budget Online Multiple Kernel Classification (BOMKC) and propose a new algorithm termed online Sparse Passive Aggressive learning (SPA) by extending the popular online Passive Aggressive (PA) technique (Crammer et al., 2006). The key idea of the proposed method is to explore a simple yet effective stochastic sampling strategy for adding SV’s, where the probability of an incoming training instance to become a SV is determined by two factors: (i) the loss suffered by the classifier on this instance; and (ii) the historical accuracy of this individual kernel classifier. By limiting the number of SV’s, the proposed BOMKC with SPA significantly accelerates the learning process of OMKC while maintaining satisfactory accuracy which is fairly comparable to that of existing non-budget OMKC algorithms. We theoretically prove that the proposed new algorithm not only bounds the number of SV’s but also achieves an optimal regret bound in expectation. Finally, we conduct an extensive set of empirical studies which shows that the proposed algorithm outperforms a variety of existing OMKC algorithms and easily scales to large-scale datasets with million instances.

The rest of this paper is organized as follows. Section 2 formulates the problem of Budget OMKC learning and presents the SPA algorithm for BOMKC. Section 3 gives theoretical analysis of the proposed algorithm and achieves the optimal mistake bound in expectation. Section 4 presents the results of our empirical studies, and finally Section 5 concludes this paper.

2 Budget Online Multiple Kernel Learning

In this section, we first formulate the problem setting for Online Multiple Kernel Classification (OMKC), and then present the proposed Budget Online Multiple Kernel Classification (BOMKC) with Sparse Passive Aggressive (SPA) learning for classification tasks.

2.1 Problem Setting and Preliminaries

In a typical binary classification task, our goal is to learn a function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) from a sequence of training examples \( \{(x_1, y_1), \ldots, (x_T, y_T)\} \), where the feature vector \( x_t \in \mathcal{X} \subseteq \mathbb{R}^d \) and the class label \( y_t \in \mathcal{Y} = \{+1, -1\} \). We use \( \hat{y} = \text{sign}(f(x)) \) to predict the class label, and \( |f(x)| \) to measure the classification confidence.

Consider a collection of \( m \) kernel functions \( \mathcal{K} = \{\kappa_i : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}, i = 1, \ldots, m\} \). Each can be a predefined parametric or nonparametric function. MKC aims to learn a kernel-based prediction model by identifying the best linear combination of the \( m \) kernels whose weights are denoted by \( \theta = (\theta_1, \ldots, \theta_m) \). The learning task can be cast into the following optimization:

\[
\min_{\theta \in \Delta} \min_{f \in \mathcal{H}_{\mathcal{K}(\theta)}} \frac{1}{2} \left\| f \right\|^2_{\mathcal{H}_{\mathcal{K}(\theta)}} + C \sum_{t=1}^{T} \ell(f(x_t), y_t)
\]

where \( \Delta = \{\theta \in \mathbb{R}^m : \theta^T 1_m = 1\} \), \( \mathcal{K}(\theta)(\cdot, \cdot) = \sum_{i=1}^{m} \theta_i \kappa_i(\cdot, \cdot) \), and \( \ell(f(x_t), y_t) \) is a convex loss function that penalizes the deviation of estimating \( f(x_t) \) from observed labels \( y_t \). To simplify the discussion, we denote \( \ell_t(f) = \ell(f(x_t), y_t) \) throughout this paper.

The above convex optimization problem of regular batch MKL have been solved by different optimization schemes (Gonen & Alpaydın, 2011). Despite being studied extensively, it still suffers from the common drawbacks of batch learning methods, i.e., not scalable to large-scale applications, expensive in retraining cost for sequential data and not able to adapt to fast-evolving patterns.
To address the challenges faced by batch MKC methods, several algorithms attempt to solve the MKC problem in an online manner (Hoi et al., 2013; Sahoo et al., 2014) whose updating scheme usually consists of two steps. First learn a set of effective single kernel classifiers $f_i^t \in \mathcal{H}_{\kappa_i}, i = 1, \ldots, m$ using existing online kernel learning algorithms (such as kernel Passive Aggressive and kernel Perceptron). Second, learn an effective classifier $f_t(x)$ by combining these single kernel classifiers:

$$f_t(x) = \sum_{i=1}^{m} \theta_i^t f_i^t(x)$$

(1)

where $\theta_i^t \in [0, 1]$ is the combination weight of the classifier with respect to $\kappa_i$ at time $t$. The weights can be updated during the learning process by adopting the Hedge algorithm.

Compared with batch learning methods, these Online Multiple Kernel Classification algorithms enjoy better scalability to large-scale applications and better applicability to sequential data. Unfortunately, whenever a new instance is misclassified, it will be added to the SV set. The unbounded number of SV’s will eventually lead to memory explosion, making it difficult to scale up in practice.

2.2 Sparse Passive Aggressive Learning for Online Multiple Kernel Learning

To overcome the critical limitation of existing OMKC algorithms, we aim to explore novel techniques to build scalable and efficient OMKC algorithms. Similar to the existing OMKC algorithms, the proposed algorithm adopts a two-step updating scheme: (i) update the single kernel classifiers individually; (ii) update the weights for combining the classifiers. In the following, we will present the details of our proposed SPA algorithm.

Our update strategy of each single kernel classifier is generalized from the PA algorithm (Crammer et al., 2006). At the $t$-th step, the online hypothesis will be updated:

$$f_{t+1} = \min_{f \in \mathcal{H}_{\kappa_i}} \frac{1}{2} \| f - f_t \|^2_{\mathcal{H}_{\kappa_i}} + \eta \ell_i(f)$$

where $\eta > 0$ and $\ell_i(f) = [1 - y_i f(x_i)]_+$ is the hinge loss. This optimization involves two objectives: the first is to keep the new function close to the old one, while the second is to minimize the loss of the new function on the current example.

To limit the number of SV’s in each single kernel classifier, we propose a simple yet effective sampling rule which decides if an incoming instance should be a support vector by performing a Bernoulli trial as follows:

$$\Pr(Z_i^t = 1) = \rho_i, \quad \rho_i = \frac{\min(\alpha, \ell_i(f_i^t))}{\beta}$$

where $Z_i^t \in \{0, 1\}$ is a random variable such that $Z_i^t = 1$ indicates that a new SV should be added to update the classifier at the $t$-th step, and $\beta \geq \alpha > 0$ are parameters to adjust the ratio of support vectors with some given budget. The above sampling rule has two key concerns: (i) The probability of making the $t$-th step update is always less than $\alpha/\beta$, which avoids assigning too high probability on a noisy instance and guarantees that the ratio of support vectors to total received instances is always bounded in expectation. (ii) An example suffering higher loss has a higher probability of being assigned to the support vector set, which maximizes the learning accuracy by adding informative support vectors or equivalently avoid making unnecessary updates.

After obtaining the random variable $Z_i^t$, we will need to develop an effective strategy for updating the classifier. Following the PA learning principle, we propose the following updating method:

$$f_{t+1} = \min_{f \in \mathcal{H}_{\kappa_i}} P_i^t(f) := \frac{1}{2} \| f - f_t \|^2_{\mathcal{H}_{\kappa_i}} + \frac{Z_i^t}{\rho_i} \eta \ell_i(f)$$

(2)

Note that when $\rho_i = 0$, then $Z_i = 0$ and we set $Z_i/\rho_i = 0$. We adopt the above update since it is an unbiased estimation of that of PA update. We can derive a closed-form solution as following.

$$f_{t+1}(\cdot) = f_t(\cdot) + \tau_i y_t \kappa(x_t, \cdot), \quad \tau_i = \min \left( \frac{\eta Z_i^t}{\rho_i}, \frac{\ell_i(f_t)}{\kappa(x_t, x_t)} \right)$$

The remaining problem is to learn the appropriate combination weights for the set of single kernel classifiers. The simplest way is a uniform combination, i.e. $\theta^t = \frac{1}{m}$, which will be used in our initialization step. Hedge algorithm is used to update the weights of each classifier.

In addition, to focus on the best kernel, a larger update probability is assigned to a component classifier with higher historical accuracy, which is reflected by the current combination weight. Finally, we summarize the proposed Sparse Passive Aggressive algorithm in Algorithm[1].
SPA algorithm satisfies the following inequality for any larger chances of adding SV’s leads to smaller loss. Furthermore, for bound is less tight than the above case, which is consistent with the analysis before that too large expected regret bound is proportional to the ratio involves large number of noisy instances and might be harmful.

Let classifiers achieves almost the same performance as that of the best online single kernel classifier. In this section, we will provide detailed theoretical analysis to the loss bound of our proposed SPA algorithm. We denote \(\eta\) parameters:

\[
\text{Kernels: } \kappa_i(x, x') : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}, \quad i = 1, \ldots, m; \quad \text{Aggressiveness parameter } \eta > 0, \text{ and parameters } \beta \geq \alpha > 0; \text{ Discount parameter } \gamma \in (0, 1) \text{ and smoothing parameters } \delta_i \in (0, 1).
\]

Initialization: \(f_1^*, w_1^i = \theta_1^i = \frac{1}{m}, i = 1, \ldots, m\) for \(t = 1, 2, \ldots, T\) do

1. Receive an instance: \(x_t\) and predict \(\hat{y}_t = \text{sign}\left(\sum_{i=1}^{m} \theta_t^i \cdot f_t^i(x_t)\right)\)
2. Receive the class label: \(y_t\)
3. For \(i = 1, 2, \ldots, m\) do
   - Compute \(p_t^i = (1 - \delta_i) \frac{w_t^i}{\max_j w_t^j} + \delta_i\); Bernoulli Sampling \(c_t^i \in \{0, 1\}\) by \(\Pr(c_t^i = 1) = p_t^i\)
   - Compute \(\rho_t^i = \min(\alpha, \ell_t(c_t^i))\); Sample \(Z_t^i \in \{0, 1\}\) by: \(\Pr(Z_t^i = 1) = \rho_t^i\)
   - Update the classifier as Proposition 1
   - Update weight \(w_t^i = w_t^i \cdot \ell_t(c_t^i)\)
   - Scale the weights \(\theta_{t+1}^i = \frac{w_{t+1}^i}{\sum_{j=1}^{m} w_{t+1}^j}, i = 1, \ldots, m\)

3 Theoretical Analysis

In this section, we will provide detailed theoretical analysis to the loss bound of our proposed SPA algorithm. We first propose a lemma that bounds the regret of a single kernel classifier learnt by the SPA update strategy and then utilize this lemma for the final analysis of our proposed SPA algorithm for OMKC problem. We denote \(f_* = \arg \min_{f \in \mathcal{H}_\kappa} \sum_{t=1}^{T} \ell_t(f)\) as the optimal classifier in \(\mathcal{H}_\kappa\) space with the assumption of the foresight to all the instances.

**Lemma 1** Let \((x_1, y_1), \ldots, (x_T, y_T)\) be a sequence of examples where \(x_t \in \mathbb{R}^d, y_t \in \mathcal{Y} = \{-1, +1\}\) for all \(t\). If we assume \(\kappa(x, x) = 1\) and the hinge loss function \(\ell_t(\cdot)\) is 1-Lipschitz, then for any \(\beta \geq \alpha > 0\), and \(\eta > 0\), when using the proposed SPA update strategy to generate a sequence of single kernel classifiers in space \(\kappa\), we have

\[
\mathbb{E}\left[\sum_{t=1}^{T} (\ell_t(f_t) - \ell_t(f_*))\right] < \frac{1}{2\eta} \|f_*\|_{\mathcal{H}_\kappa} + \frac{\eta\beta}{\min(\alpha, \sqrt{\beta\eta})} T
\]

where \(\eta\) is the aggressiveness parameter. Setting \(\eta = \|f_*\|_{\mathcal{H}_\kappa} \sqrt{\frac{2}{\beta T}}\) and \(\alpha^3 \leq \frac{\beta}{T}\), we have

\[
\mathbb{E}\left[\sum_{t=1}^{T} (\ell_t(f_t) - \ell_t(f_*))\right] < \|f_*\|_{\mathcal{H}_\kappa} \sqrt{\frac{2 \beta T}{\alpha}}
\]

**Remark 1.** The theorem indicates that the expected regret of any single classifier in \(\mathcal{H}_\kappa\) by the SPA algorithm can be bounded by \(\|f_*\|_{\mathcal{H}_\kappa} \sqrt{\frac{2 \beta T}{\alpha}}\) in expectation. In practice, \(\beta/\alpha\) is usually a small constant. Thus the proposed algorithm achieves a strong regret bound in expectation.

**Remark 2.** The expected regret has a close relationship with the two parameters \(\alpha\) and \(\beta\). As indicated above, to avoid assigning too high probability on a noisy instance, the parameter \(\alpha\) should not be too large. Assuming \(\alpha \leq \sqrt{\beta \eta}\) (which is accessible in the practical parameter setting), the expected regret bound is proportional to the ratio \(\beta/\alpha\). This is consistent with the intuition that larger chances of adding SV’s leads to smaller loss. Further more, for \(\alpha > \sqrt{\beta \eta}\), the expected regret bound is less tight than the above case, which is consistent with the analysis before that too large \(\alpha\) involves large number of noisy instances and might be harmful.

We have demonstrated that the loss of each single kernel online classifier is bounded by the loss of its batch counterpart and an online regret term \(\sqrt{T}\). While in reality, the performance of single kernel classifiers varies significantly and we have no foresight to the winner. In the following, we will show that the final multiple kernel classifier in our proposed algorithm based on single kernel SPA classifiers achieves almost the same performance as that of the best online single kernel classifier.

**Theorem 1** Let \((x_1, y_1), \ldots, (x_T, y_T)\) be a sequence of examples where \(x_t \in \mathbb{R}^d, y_t \in \mathcal{Y} = \{-1, +1\}\) for all \(t\). If we assume \(\kappa_i(x, x) = 1\) and the hinge loss function \(\ell_t(\cdot)\) is 1-Lipschitz, then for any \(\beta \geq \alpha > 0\), and \(\eta > 0, \gamma \in (0, 1)\), the multiple kernel classifier generated by our proposed SPA algorithm satisfies the following inequality:

\[
\text{Algorithm 1 Sparse Passive Aggressive learning for Budget OMKC “BOMKC(SPA)”}
\]

**INPUT:** Kernels: \(k_i(\cdot, \cdot) : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}, \quad i = 1, \ldots, m\); Aggressiveness parameter \(\eta > 0\), and parameters \(\beta \geq \alpha > 0\); Discount parameter \(\gamma \in (0, 1)\) and smoothing parameters \(\delta_i \in (0, 1)\).

**Initialization:** \(f_1^* = 0, w_1^i = \theta_1^i = \frac{1}{m}, i = 1, \ldots, m\) for \(t = 1, 2, \ldots, T\) do

1. Receive an instance: \(x_t\) and predict \(\hat{y}_t = \text{sign}\left(\sum_{i=1}^{m} \theta_t^i \cdot f_t^i(x_t)\right)\)
2. Receive the class label: \(y_t\)
3. For \(i = 1, 2, \ldots, m\) do
   - Compute \(p_t^i = (1 - \delta_i) \frac{w_t^i}{\max_j w_t^j} + \delta_i\); Bernoulli Sampling \(c_t^i \in \{0, 1\}\) by \(\Pr(c_t^i = 1) = p_t^i\)
   - Compute \(\rho_t^i = \min(\alpha, \ell_t(c_t^i))\); Sample \(Z_t^i \in \{0, 1\}\) by: \(\Pr(Z_t^i = 1) = \rho_t^i\)
   - Update the classifier as Proposition 1
   - Update weight \(w_t^i = w_t^i \cdot \ell_t(c_t^i)\)
   - Scale the weights \(\theta_{t+1}^i = \frac{w_{t+1}^i}{\sum_{j=1}^{m} w_{t+1}^j}, i = 1, \ldots, m\)
\[
\mathbb{E}\left[\sum_{t=1}^{T} \ell_t(f_t)\right] \leq \min_i \left( \sum_{t=1}^{T} \ell_t(f_t^i) + \frac{1}{2\sqrt{\beta}} \|f_t^i\|_{\mathcal{H}_i}^2 \right) \mu \ln(1/\gamma) \frac{1 - \gamma}{1 - \gamma} + \frac{\mu \eta \beta \ln(1/\gamma)}{\min(\alpha, \sqrt{\beta \eta})(1 - \gamma)\delta} T + \frac{\mu \ln m}{1 - \gamma}
\]

where \(\eta\) is the aggressiveness parameter, \(\mu\) is a constant that only depends on \(\gamma\) and the upper bound of \(\ell_t\). Setting \(\eta = \|f_t^*\|_{\mathcal{H}_i} \sqrt{\frac{\alpha}{2\beta T}}\) and \(\alpha^3 \leq \frac{\beta}{2T} \|f_t^*\|_{\mathcal{H}_i}^2\), we have

\[
\mathbb{E}\left[\sum_{t=1}^{T} \ell_t(f_t)\right] \leq \frac{\mu \ln m}{1 - \gamma} + \frac{\mu \ln(1/\gamma)}{1 - \gamma} \min_i \left\{ \sum_{t=1}^{T} \ell_t(f_t^i) + \frac{1}{\delta^2} \|f_t^i\|_{\mathcal{H}_i} \sqrt{2\beta T/\alpha} \right\}
\]

Remark. This theorem suggests that even without any foresight of which kernel would achieve the highest accuracy, the performance of our proposed multiple kernel classifier is still comparable to the best single kernel classifier. Compared with the optimal batch classifier with foresight to all the training instances, the regret of our proposed algorithm is \(O(\sqrt{T})\).

Next, we will bound the number of SV’s of each single kernel classifier \(f_t^j\) in expectation.

**Theorem 2** Let \((x_1, y_1), \ldots, (x_T, y_T)\) be a sequence of examples where \(x_i \in \mathbb{R}^d\), \(y_i \in \mathcal{Y} = \{-1, +1\}\) for all \(t\). If we assume \(\kappa(x, x) = 1\) and the hinge loss function \(\ell_t(\cdot)\) is 1-Lipschitz, \(\beta \geq \alpha > 0\), and \(\eta > 0\), for any \(i = 1, \ldots, m\) the proposed SPA algorithm satisfies the following inequality

\[
\mathbb{E}\left[\sum_{t=1}^{T} Z_t^i\right] \leq \min \left\{ \frac{\alpha}{\beta} T, \frac{1}{\beta^2} \sum_{t=1}^{T} \ell_t(f_t^i) + \frac{1}{2\eta^2} \|f_t^i\|_{\mathcal{H}_i}^2 + \frac{\eta \beta}{\min(\alpha, \sqrt{\beta \eta})} T \right\}
\]

Especially, when \(\eta = \|f_t^i\|_{\mathcal{H}_i} \sqrt{\frac{\alpha}{2\beta T}}\) and \(\alpha^3 \leq \frac{\beta}{2T} \|f_t^i\|_{\mathcal{H}_i}^2\), we have

\[
\mathbb{E}\left[\sum_{t=1}^{T} Z_t^i\right] \leq \min \left\{ \frac{\alpha}{\beta} T, \frac{1}{\beta^2} \sum_{t=1}^{T} \ell_t(f_t^i) + \|f_t^i\|_{\mathcal{H}_i} \sqrt{2\beta T/\alpha} \right\}
\]

Remark. First, this theorem indicates the expected number of support vectors is less than \(\alpha T/\beta\).

Thus, by setting \(\beta \geq \alpha T/B (1 < B \leq T)\), we guarantee the expected number of support vectors of the final classifier is bounded by a budget \(B\). Second, this theorem also implies that, by setting \(\beta \geq \left[\sum_{t=1}^{T} \ell_t(f_t^i) + \|f_t^i\|_{\mathcal{H}_i} \sqrt{2\beta T/\alpha}\right] / B (1 < B \leq T)\), the expected number of support vectors is always less than \(B\), no matter what is the value of \(\alpha\). In practice, as the Hedge algorithm trends to converge to a single best kernel, the total number of support vectors of all classifiers is only slightly larger than that used by the best classifier.

4 Experiments

In this section, we conduct extensive experiments to evaluate the empirical performance of the proposed SPA multiple kernel algorithm for online binary classification tasks.

4.1 Experimental Testbed

All datasets used in our experiments are commonly used benchmark datasets for binary classification. These datasets are chosen fairly randomly to cover a variety of different sizes with the instances number varies from 1000 to 1,000,000. There details are shown in the Appendix.

4.2 Kernels

In our experiments, we examine BOMKC by exploring a set of 16 predefined kernels, including 3 polynomial kernels \(\kappa(x_i, x_j) = (x_i^\top x_j)^p\) with the degree parameter \(p = 1, 2, 3\); 13 Gaussian kernels \(\kappa(x_i, x_j) = \exp(-\frac{\|x_i - x_j\|^2}{2\sigma^2})\) with the kernel width parameter \(\sigma = [2^{-6}, 2^{-5}, \ldots, 2^6]\).

4.3 Compared Algorithms

First, we include an important baseline algorithm showing the best performance that could be achieved by a single kernel classifier assuming the foresight of optimal choice of the kernel function.
before the training instances arrives. We search for the best single kernel classifier from the set of our predefined 16 kernels using one random permutation of all the training examples and then apply the “Perceptron” algorithm [Rosenblatt, 1958] with the best kernel function.

Our second group of compared algorithms are the Online Multiple Kernel Classification algorithms [Hoi et al., 2013] which achieved state-of-the-art performance on many benchmark datasets. Three variants of this algorithm are included:

- “OMKC(U)”: the OMKC algorithm with a naive uniform combination;
- “OMKC(DD)”: the OMKC algorithm with deterministic combination and update;
- “OMKC(SD)”: OMKC with stochastic update and deterministic combination;

Finally, to test the efficiency and effectiveness of our budget strategy, we also compare with multiple kernel classification algorithms whose component classifiers are updated by budget kernel learning algorithms including:

- “RBP”: the Random Budget Perceptron algorithm [Cavallanti et al., 2007];
- “Forgetron”: the Forgetron algorithm that discards the oldest SV [Dekel et al., 2005];
- “BOGD”: the Budget Online Gradient Descent algorithm [Zhao et al., 2012];
- “BPAS”: the Budget Passive-aggressive algorithm (simple) [Wang & Vucetic, 2010].

Multiple Kernel classification is a more challenging problem compared to single kernel classification, which requires highly efficient component classifiers. Consequently, we did not compare with slow budget algorithm such as Projectron.

4.4 Parameter Settings

To make a fair comparison, we adopt the same experimental setup for all the algorithms. The weight discount parameter $\gamma$ is fixed to 0.99 for all multiple kernel algorithms on all datasets. The smoothing parameter $\delta$ for all stochastic update algorithms is fixed to 0.001. The learning rate parameters in all algorithms (SPA, the BOGD and BPAS) are all fixed to 0.1. For the proposed SPA algorithm, we simply fix the random sampling parameter $\alpha = 1$ and $\beta = 3$ for the first 8 datasets and will discuss the parameter sensitivity later. For a fair comparison between the budget algorithms, we set a uniform limit $B$ to the SV size for in all the component classifiers so that the total number of SV’s used by all component classifiers is nearly equal to the SV size of the SPA algorithm. To test the scalability of our proposed algorithm, we also include the experiment on million scale dataset. We set the $\beta = 300$ for efficiency.

All experiments were repeated 10 times on different random permutations of instances and all the results were obtained by averaging over 10 runs. All the algorithms were implemented in C++ on a Windows machine with 3.2 GHz CPU. We report the online mistake rates along the training process, the total number of SV’s used by all component single kernel classifiers and the running time.

4.5 Evaluation of Online Learning Performance with Comparison to Non-Budget OMKL Algorithms

The first experiment is to evaluate the performance of SPA for binary classification tasks with comparison to non-budget OMKL algorithms. We did not report the results on the 3 largest datasets since it is nearly impossible for non-budget algorithms to process large scale data in limited time and memory. Table summarizes the experimental results. We can draw the following observations.

First of all, we compare the accuracy of the three algorithms with deterministic update and full SV’s (Perceptron, OMKC(U), OMKC(DD)). Obviously, the mistake rate of OMKC(DD) is usually much lower than that of OMKC(U), which indicates that the OMKC(DD) algorithm is able to learn the best combination to build an effective multiple kernel classifier. Further, to our surprise, even with the unrealistic assumption that we have foresight to all the instances and can choose the best kernel before learning, the Perceptron algorithm still can not achieve the lowest error rate. We conjecture that there might be two explanations to this observation. First, our optimal kernel is searched in one random permutation, which might not be the optimal kernel for other permutations of instances, while OMKC(DD) always learns the best combination of the kernel functions. Second, in some datasets, no single kernel function has significant advantage over others, while an optimal weighted combination might perform better than any single kernel classifier. This further validates the significance of studying multiple kernel learning algorithms.
Table 1: Evaluation of Online Multiple Kernel Classification on small-scale and medium-scale datasets.

| Algorithm    | german Mistake (%) | #SV’s | Time (s) | madelon Mistake (%) | #SV’s | Time (s) |
|--------------|--------------------|-------|----------|--------------------|-------|----------|
| Perceptron(*) | 32.07 ± 0.90       | 320.7 ± 9.0 | 0.031    | 48.55 ± 1.57       | 973.0 ± 31.45 | 0.81     |
| OMKC(U)      | 35.37 ± 1.17       | 6912.4 ± 87.7 | 0.388    | 50.00 ± 0.00       | 26498.2 ± 92.62 | 35.27    |
| OMKC(DD)     | 31.05 ± 1.13       | 6912.4 ± 87.7 | 0.383    | 41.15 ± 0.50       | 26498.2 ± 92.62 | 35.46    |
| OMKC(SD)     | 34.80 ± 1.25       | 5724.5 ± 128.6 | 0.341    | 50.00 ± 0.00       | 13872.3 ± 150.61 | 20.67    |
| BOMKC(SPA)   | 30.19 ± 0.29       | 1688.1 ± 90.7 | 0.120    | 36.62 ± 0.61       | 8685.2 ± 93.5 | 11.7     |

Second, we find that although using fewer SV’s, the stochastic update algorithms OMKC(SD) can even achieve lower mistake rate compared with deterministic update OMKC(DD), which is consistent with the previous observations (Hoi et al., 2013). This indicates that not all SV’s are essential for an accurate classifier. This supports our main claim that when the added SV’s are wisely selected, the accuracy may not decrease much while the efficiency can be significantly upgraded.

Third, when comparing the time cost, we find that the non-budget OMKC algorithms are significantly slower than our proposed SPA algorithm, which is more serious in larger datasets. Actually, it is even impossible for non-budget algorithms to complete the learning process in realistic time and space in the three largest datasets. This is due to two reasons: first, similar to the problem faced by single kernel methods, the prediction cost grows rapidly with unlimited number of SV’s added; second, there is a big pool of kernel classifiers need to be updated and combined. This observation further validates the importance of studying efficient and scalable online multiple kernel methods.

Finally, we found that apart from the obvious advantage in time cost, our proposed SPA algorithm achieves the highest accuracy in all datasets even when adopting only a small portion of SV’s. This validates the effectiveness of our proposed algorithm in wisely selecting the most informative SV’s.

4.6 EVALUATION OF ONLINE LEARNING PERFORMANCE WITH COMPARISON TO DIFFERENT BUDGET OMKL ALGORITHMS

The next experiment is to test the accuracy and efficiency of our proposed SPA algorithm with comparison to other budget maintenance strategies. Table 1 summarizes the experimental results.

From the results, it is obvious that our SPA algorithm achieves the best accuracy among all the budget algorithms in most of the cases. In addition, the time cost of the proposed SPA algorithm is always the lowest. This is due to the advantage in our algorithm design: SPA can find the optimal kernel and concentrate the effort in this kernel. Thus, poor kernels will receive few SV’s and the prediction time is greatly reduced. When adopting the same number of SV’s as other budget algorithms, instead of paying equal attention to all components, SPA focuses on the best kernel and thus achieves the highest accuracy.

4.7 PARAMETER SENSITIVITY OF α AND β

The proposed SPA algorithm has two critical parameters α and β which could considerably affect the accuracy, support vector size and time cost. Our second experiment is to examine how different parameters of α and β affects the learning performance so as to give insights for how to choose
First of all, when \( \beta \) is fixed, increasing \( \alpha \) generally results in (i) larger support vector size, (ii) higher time cost, but (iii) better classification accuracy, especially when \( \alpha \) is small. However, when \( \alpha \) is large enough (e.g., \( \alpha > 1.5 \)), increasing \( \alpha \) has very minor impact to the performance. This is because the number of instances whose hinge loss above \( \alpha \) is relatively small. We also note that the accuracy decreases slightly when \( \alpha \) is too large. This might be because some (potentially noisy) instances with large loss are given a high chance of being assigned as SV’s, which may harm the classifier due to noise. Thus, on this dataset (“a9a”), it is easy to find a good \( \alpha \) in the range of [1,2].

Second, when \( \alpha \) is fixed, increasing \( \beta \) will result in (i) smaller support vector size, (ii) smaller time cost, but (iii) worse classification accuracy. On one hand, \( \beta \) cannot be too small as it will lead to too
many support vectors and thus suffer very high time cost. On the other hand, \( \beta \) cannot be too large as it will considerably decrease the classification accuracy. We shall choose \( \beta \) that yields a sufficiently accurate classifier while minimizing the support vector size and training time cost. For example, for this particular dataset, choosing \( \beta \) in the range of [3,6] achieves a good trade-off between accuracy and efficiency/sparsity.

5 Conclusions

This paper proposed a new framework of Budget Online Budget Multiple Kernel Learning with a novel Sparse Passive Aggressive learning algorithm. In contrast to the existing Online Multiple Kernel Classification (OMKC) algorithms that usually suffer from extremely high time cost due to their unbounded numbers of support vectors in the online learning process, our proposed algorithm adopts a simple but effective SV sampling strategy, making it applicable to large scale applications. We theoretically demonstrated that the SPA algorithm enjoyed an optimal regret bound in expectation. In addition, the experimental results with comparison to both the existing non-budget OMKL algorithms and budget OMKL algorithms showed that the proposed method achieved a significant speedup and yielded more accurate classifiers than existing methods, validating the efficacy, efficiency and scalability of the proposed technique.

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Supplementary Material

Appendix A: Details of the Datasets Used in Experiments

Table 3 summarizes details of the binary classification datasets used in our experiments. All of them are commonly used benchmark datasets and are publicly available from LIBSVM\(1\), UCI\(2\) and KDDDCUP competition site. These datasets are chosen fairly randomly to cover a variety of different sizes.

Table 3: Statistics of binary classification datasets in our experiments (\(T\) denotes the number of instances and \(d\) denotes the number of dimensions).

|               | GERMAN | SVMguide3 | MADISON | MAGIC04 | A9A | KDDCUP | IJCNN1 | CODRNA | SUSY |
|---------------|--------|-----------|---------|---------|-----|--------|--------|--------|------|
| \(T\)         | 1000   | 1,243     | 2,000   | 19,020  | 48,842 | 102,294 | 49,990 | 271,617 | 1,000,000 |
| \(d\)         | 24     | 21        | 500     | 10      | 123  | 117    | 22     | 8      | 18   |

Note that the original SUSY dataset has more than 5-million instances, from which we randomly sampled one fifth to form our dataset.

Appendix B: Proof for Lemma 1

Proof: Firstly, the \(P_t(f)\) defined in the equality

\[
f_t+1 = \min_{f \in \mathcal{H}_n} P_t(f) := \frac{1}{2} \| f - f_t \|^2_{\mathcal{H}_n} + \frac{Z_t}{\rho_t} \eta f_t(f)
\]

is 1-strongly convex. Further, \(f_t+1\) is the optimal solution of \(\min_f P_t(f)\), we thus have the following inequality according to the definition of strongly convex

\[
\frac{1}{2} \| f - f_t \|^2_{\mathcal{H}_n} + \frac{1}{\rho_t} Z_t \eta f_t(f) \geq \frac{1}{2} \| f_{t+1} - f_t \|^2_{\mathcal{H}_n} + \frac{1}{\rho_t} Z_t \eta f_t(f_{t+1}) + \frac{1}{2} \| f - f_{t+1} \|^2_{\mathcal{H}_n}
\]

where the inequality used \(\nabla P_t(f_{t+1}) = 0\). After rearranging the above inequality, we get

\[
\frac{1}{\rho_t} Z_t \eta f_t(f_{t+1}) - \frac{1}{\rho_t} Z_t \eta f_t(f) \leq \frac{1}{2} \| f - f_t \|^2_{\mathcal{H}_n} - \frac{1}{2} \| f - f_{t+1} \|^2_{\mathcal{H}_n} - \frac{1}{2} \| f_{t+1} - f_t \|^2_{\mathcal{H}_n}
\]

Secondly, since \(\ell_t(f)\) is 1-Lipshitz with respect to \(f\)

\[
\ell_t(f) - \ell_t(f_{t+1}) \leq \| f_t - f_{t+1} \|_{\mathcal{H}_n}.
\]

Combining the above two inequalities, we get

\[
\frac{1}{\rho_t} Z_t \eta [\ell_t(f_t) - \ell_t(f)] \leq \frac{1}{2} \| f - f_t \|^2_{\mathcal{H}_n} - \frac{1}{2} \| f - f_{t+1} \|^2_{\mathcal{H}_n} - \frac{1}{\rho_t} Z_t \eta \| f_t - f_{t+1} \|_{\mathcal{H}_n}
\]

Summing the above inequalities over all \(t\) leads to

\[
\sum_{t=1}^T \frac{1}{\rho_t} Z_t \eta [\ell_t(f_t) - \ell_t(f)] \leq \frac{1}{2} \| f - f_1 \|^2_{\mathcal{H}_n} + \sum_{t=1}^T \left[ -\frac{1}{2} \| f_{t+1} - f_t \|^2_{\mathcal{H}_n} + \frac{1}{\rho_t} Z_t \eta \| f_t - f_{t+1} \|_{\mathcal{H}_n} \right]
\]

We now take expectation on the left side. Note, by definition of the algorithm, \(\mathbb{E}_t Z_t = \rho_t\), where we used \(\mathbb{E}_t\) to indicate conditional expectation give all the random variables \(Z_1, \ldots, Z_{t-1}\). Assuming \(\rho_t > 0\), we have

\[
\mathbb{E} \left[ \frac{1}{\rho_t} Z_t \eta [\ell_t(f_t) - \ell_t(f)] \right] = \mathbb{E} \left[ \frac{1}{\rho_t} \mathbb{E}_t Z_t \eta [\ell_t(f_t) - \ell_t(f)] \right] = \eta \mathbb{E} [\ell_t(f_t) - \ell_t(f)]
\]

\[\text{(3)}\]

Note that in some iterations, \(\rho_t = 0\), in that case, we have \(\ell_t(f_t) = 0\), thus:

\[
\eta [\ell_t(f_t) - \ell_t(f)] \leq 0
\]

\[\text{(4)}\]

1\url{http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/}

2\url{http://archive.ics.uci.edu/ml/}
As mentioned before, \( \rho_t = 0 \) indicates \( Z_t = 0 \) and \( Z_t/\rho_t = 0 \), we get
\[
\frac{1}{\rho_t} Z_t \eta [\ell_t(f_t) - \ell_t(f)] = 0
\]  
Combining (3), (4) and (5) and summarizing over all \( t \) leads to
\[
\eta \mathbb{E} \sum_{t=1}^{T} [\ell_t(f_t) - \ell_t(f)] \leq \mathbb{E} \sum_{t=1}^{T} \frac{1}{\rho_t} Z_t \eta [\ell_t(f_t) - \ell_t(f)]
\]
We now take expectation on the right side of (3)
\[
\mathbb{E} \left[ \frac{1}{2} \| f - f_t \|_{\mathcal{H}}^2 \right] + \mathbb{E} \left[ \sum_{t=1}^{T} \left[ -\frac{1}{2} \| f_{t+1} - f_t \|_{\mathcal{H}}^2 + \frac{1}{\rho_t} Z_t \eta \| f_t - f_{t+1} \|_{\mathcal{H}} \right] \right]
\leq \frac{1}{2} \| f \|_{\mathcal{H}}^2 + \sum_{t=1}^{T} \mathbb{E} \left[ -\frac{1}{2} \tau_t^2 + \frac{1}{\rho_t} Z_t \eta \tau_t \right]
\]  
Given all the random variables \( Z_1, \ldots, Z_{t-1} \), we now calculate the conditional expectation of the variable \( M_t = -\frac{1}{2} \tau_t^2 + \frac{1}{\rho_t} Z_t \eta \tau_t \). In probability \( \rho_t, Z_t = 1 \) and \( \tau_t = \tau_t' = \min(\frac{2}{\eta}, \ell_t(f_t)) \). We have
\[
M_t (Z_t = 1) = -\frac{1}{2} \tau_t^2 + \frac{1}{\rho_t} \eta \tau_t'.
\]  
And in probability \( 1 - \rho_t, Z_t = 0 \) and \( \tau_t = 0 \). We have \( M_t (Z_t = 0) = 0 \). Considering the two cases, the conditional expectation is:
\[
\mathbb{E}_t [M_t] = \rho_t M_t (Z_t = 1) + (1 - \rho_t) M_t (Z_t = 0) = \rho_t \left[ -\frac{1}{2} \tau_t^2 + \frac{1}{\rho_t} \eta \tau_t' \right] < \eta \tau_t'
\]  
In the case when \( \alpha \leq \ell_t \) and \( \rho_t = \frac{\alpha}{\beta} \), \( \tau_t' = \min(\frac{2}{\eta}, \ell_t(f_t)) \leq \frac{2}{\eta} \), thus \( \eta \tau_t' \leq \frac{2}{\eta} \).
And in the case \( \alpha > \ell_t \) and \( \rho_t = \frac{\alpha}{\beta} \), \( \tau_t' = \min(\frac{2}{\eta}, \ell_t(f_t)) \leq \sqrt{\eta} \beta \). Thus, \( \eta \tau_t' \leq \frac{2}{\eta} \sqrt{\eta} \beta \).
Considering both of the cases leads to
\[
\mathbb{E}_t [M_t] < \frac{\eta \beta}{\min(\alpha, \sqrt{\beta} \eta)}
\]
Summing the above inequality over all \( t \) and combining with (6), we get
\[
\eta \mathbb{E} \sum_{t=1}^{T} [\ell_t(f_t) - \ell_t(f)] < \frac{1}{2} \| f \|^2_{\mathcal{H}} + \frac{\eta^2 \beta}{\min(\alpha, \sqrt{\beta} \eta)} T
\]
Setting \( f = f_\star \), and multiplying the above inequality with \( 1/\eta \) will conclude the lemma.

APPENDIX: PROOF FOR THEOREM 1

Proof: In the following proof, we first generalize the loss bound of the Hedge algorithm (Freund & Schapire [1995]) to a different situation where 1) stochastic update and stochastic combination are adopted and 2) the hinge loss function we adopt is bounded \( \ell_1^\epsilon(f) \in [0, L] \) and \( L \geq 1 \). Using the convexity, we have
\[
\gamma \ell_t^\epsilon \leq 1 - \frac{1 - \gamma}{\mu} \ell_t^\epsilon
\]
where \( \mu > 1 \) satisfies the equality when \( \ell_t^\epsilon = L \) and this constant only depends on \( L \) and \( \gamma \). We then get
\[
\sum_{i=1}^{m} w_i^\epsilon \ell_t^\epsilon \leq \sum_{i=1}^{m} w_i^\epsilon \left( 1 - \frac{1 - \gamma}{\mu} \ell_t^\epsilon \right) = \left( \sum_{i=1}^{m} w_i^\epsilon \right) \left( 1 - \frac{1 - \gamma}{\mu} \sum_{i=1}^{m} \theta_i^\epsilon \ell_t^\epsilon \right)
\]
Applying the above formula repeatedly for \( t = 1, \ldots, T \) and using \( 1 + x \leq e^x \) for all real number \( x \) yield,
\[
\sum_{i=1}^{m} w_i^\epsilon (1 + x) \leq \exp \left( \frac{1 - \gamma}{\mu} \sum_{i=1}^{m} \theta_i^\epsilon \ell_t^\epsilon \right)
\]
we may write the above as following,
\[
\sum_{t=1}^{T} \sum_{i=1}^{m} \theta_i^t \ell_i^t \leq -\frac{\mu \ln (\sum_{i=1}^{m} w_{T+1}^i)}{1 - \gamma}
\]  
(7)

Obviously, all the weights are positive, thus
\[
\sum_{t=1}^{m} w_{T+1}^i \geq w_{T+1}^i \sum_{t=1}^{T} \ell_i^t
\]
Plugging this into (7) and replace \(w_i^1\) with \(\frac{1}{m}\) yields
\[
\sum_{t=1}^{T} \sum_{i=1}^{m} \theta_i^t \ell_i^t \leq -\frac{\mu \ln (w_i^1 \gamma \sum_{t=1}^{T} \ell_i^t)}{1 - \gamma}
\]
(8)

Using the convexity of loss function, we have
\[
\sum_{t=1}^{T} \ell_t(f_t) = \sum_{t=1}^{T} \ell_t(\sum_{i=1}^{m} \theta_i^t f_i^t) \leq \sum_{t=1}^{T} \sum_{i=1}^{m} \theta_i^t \ell_i^t
\]
Thus
\[
\sum_{t=1}^{T} \ell_t(f_t) \leq \frac{\mu \ln m - \mu \ln \gamma \sum_{t=1}^{T} \ell_i^t}{1 - \gamma}
\]  
(9)

Now we need to bound the accumulate loss of a single classifier, \(\sum_{t=1}^{T} \ell_i^t\). In Lemma 1 we have proven the regret bound of one single kernel classifier learnt by SPA assuming \(E[Z_t] = \rho_t\). Here, we slightly modify this assumption for our proposed SPA multiple kernel classifier, i.e. \(E[Z_t] = \rho_t \cdot p_i^t\), where \(\delta < p_i^t < 1\). Consequently, the conclusion of Lemma 1 becomes,
\[
\delta E[\sum_{t=1}^{T} (\ell_t(f_t) - \ell_t(f_*))] < \frac{1}{2\eta} \|f_*\|_{H_{\alpha}}^2 + \frac{\eta \beta}{\min(\alpha, \sqrt{\beta \eta})} T
\]
Combining with (9) concludes this proof.

APPENDIX D: PROOF FOR THEOREM 2

Proof: Since \(E[Z_t] = \rho_t\), where \(E_t\) is the conditional expectation, we have
\[
E[\sum_{t=1}^{T} Z_t] = E[\sum_{t=1}^{T} E_t Z_t] = E[\sum_{t=1}^{T} \rho_t] = E[\sum_{t=1}^{T} \min (\frac{\alpha}{\beta}, \frac{\ell_t(f_t)}{\beta})] \leq \min (\frac{\alpha}{\beta}) \sum_{t=1}^{T} \frac{1}{\beta} E[\sum_{t=1}^{T} \ell_t(f_t))]
\]
\[
\leq \min \left\{ \frac{\alpha}{\beta} T \cdot \frac{1}{\beta} \sum_{t=1}^{T} \ell_t(f_*) + \frac{1}{2\eta} \|f_*\|_{H_{\alpha}}^2 + \frac{\eta \beta}{\min(\alpha, \sqrt{\beta \eta})} T \right\}
\]
which concludes the first part of the theorem. The second part of the theorem is trivial to be derived.