METASTABILITY FOR A NON-REVERSIBLE DYNAMICS: THE EVOLUTION OF THE CONDENSATE IN TOTALLY ASymmetric ZERO RANGE PROCESSES

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Abstract. Let $\mathbb{T}_L = \mathbb{Z}/L\mathbb{Z}$ be the one-dimensional torus with $L$ points. For $\alpha > 0$, let $g : \mathbb{N} \to \mathbb{R}_+$ be given by $g(0) = 0$, $g(1) = 1$, $g(k) = [k/(k - 1)]^\alpha$, $k \geq 2$. Consider the totally asymmetric zero range process on $\mathbb{T}_L$ in which a particle jumps from a site $x$, occupied by $k$ particles, to the site $x + 1$ at rate $g(k)$. Let $N$ stand for the total number of particles. In the stationary state, if $\alpha > 1$, as $N \uparrow \infty$, all particles but a finite number accumulate on one single site. We show in this article that in the time scale $N^{1+\alpha}$ the site which concentrates almost all particles evolves as a random walk on $\mathbb{T}_L$ whose transition rates are proportional to the capacities of the underlying random walk, extending to the asymmetric case the results obtained in [5] for reversible zero-range processes on finite sets.

1. Introduction

Metastability is a relevant dynamical phenomenon in the framework of non-equilibrium statistical mechanics, which occur in the vicinities of first order phase transitions. We refer to the monograph [21] for an overview of the literature.

Recently, [11] after [9, 10, 14] proposed a new approach to metastability for reversible dynamics based on potential theory. They applied this method in [5] to prove the metastable behavior of the condensate in sticky reversible zero range processes evolving on finite sets and in [6, 7] to examine the metastability of reversible Markov processes evolving on fixed finite sets. These methods were also used in [18, 19] to investigate the scaling limits of trap models.

More recently, we extended in [15] the potential theory of reversible dynamics to the non-reversible context by proving a Dirichlet principle for Markov chains on countable state spaces. In contrast with the reversible case, the formula for the capacity involves a double variational problem, and it wasn’t clear from this additional difficulty if such principle could be of any utility.

In this article, we use this Dirichlet principle for non-reversible dynamics to prove the metastable behavior of the condensate in sticky totally asymmetric zero range processes evolving on a fixed one-dimensional torus. This is, to our knowledge, the first proof of a metastable behavior of a non-reversible dynamics.

The first main message we want to convey is that the variational formula (2.6) for the capacity between two sets for non-reversible dynamics should be understood as an infimum over functions $H$ which satisfy certain boundary conditions and which solve the equation $S H = L^* F$ for functions $F$ which satisfy similar boundary conditions. Here, $L^*$ stands for the adjoint of the generator of the Markov process.

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and $S$ for its symmetric part. In this sense, it is similar to the known variational formula for reversible processes and one can use similar techniques to estimate the capacities. We illustrate this assertion by examining the metastable behavior of the condensate for asymmetric zero range dynamics.

**Condensation.** We conclude this introduction with a few words on condensation. The stationary states of sticky zero range processes exhibit a very peculiar structure called condensation in the physics literature. Mathematically, this means that under the stationary state, above a certain critical density a macroscopic number of particles concentrate on a single site [20, 17, 16, 12, 13, 1, 2, 3]. This phenomenon has been observed and investigated in shaken granular systems, growing and rewiring networks, traffic flows and wealth condensation in macroeconomics [11].

Once the presence of a condensate at the stationary state has been established, one is tempted to investigate its time evolution. This has been done in [5] for reversible dynamics, where the authors prove that on a certain time scale the position of the condensate evolves as a random walk with jump rates proportional to the capacities of the underlying random walks. This surprising fact is also observed in the asymmetric regime as shown in Theorem 2.2 below.

### 2. Notation and results

Denote by $\mathbb{T}_L$ the one dimensional discrete torus with $L$ sites and let $E = \mathbb{N}^{\mathbb{T}_L}$ be the set of configurations on $\mathbb{T}_L$. The configurations are denoted by the Greek letters $\eta$ and $\xi$. In particular, $\eta_x, x \in \mathbb{T}_L$, represents the number of particles at site $x$ for the configuration $\eta$.

Fix a real number $\alpha > 0$, define $a(n) = n^\alpha, n \geq 1$, and set $a(0) = 1$. Let us also define $g: \mathbb{N} \to \mathbb{R}_+$, $g(0) = 0$ and $g(n) = \frac{a(n)}{a(n-1)}, n \geq 1$, in such a way that $\prod_{i=1}^n g(i) = a(n), n \geq 1$, and that $\{g(n) : n \geq 2\}$ is a strictly decreasing sequence converging to 1 as $n \uparrow \infty$.

For each pair of sites $x, y \in \mathbb{T}_L$ and configuration $\eta \in E$ such that $\eta_x > 0$, denote by $\sigma^{x,y} \eta$ the configuration obtained from $\eta$ by moving a particle from $x$ to $y$:

$$ (\sigma^{x,y} \eta)_z = \begin{cases} 
\eta_x - 1 & \text{for } z = x \\
\eta_y + 1 & \text{for } z = y \\
\eta_z & \text{otherwise}.
\end{cases} $$

Denote by $\{\eta(t) : t \geq 0\}$ the Markov process on $E$ whose generator $L$ acts on functions $F: E \to \mathbb{R}$ as

$$ (LF)(\eta) = \sum_{x \in \mathbb{T}_L} g(\eta_x) \{F(\sigma^{x,x+1} \eta) - F(\eta)\} . \tag{2.1} $$

This process is known as the totally asymmetric zero range process with jump rate $g(\cdot)$.

**First order phase transition.** Let $Z(\varphi), \varphi > 0,$ be the partition function

$$ Z(\varphi) = \sum_{n \geq 0} \frac{\varphi^n}{a(n)} . $$
For $\alpha > 0$, the radius of convergence of this series is clearly equal to 1. A simple computation shows that for each $\varphi < 1$, $\varphi \leq 1$ if $\alpha > 1$, the product measure $\nu_{\varphi}$ on $E$ with marginals given by

$$
\nu_{\varphi}\{\eta : \eta(x) = k\} = \frac{1}{Z(\varphi) a(k)} \varphi^k, \quad x \in \mathbb{T}_L, \ k \geq 0,
$$

is a stationary measure.

Denote by $R(\varphi)$ the average density of particles under the measure $\nu_{\varphi}$:

$$
R(\varphi) := E_{\nu_{\varphi}}[\eta_0] = \frac{\varphi Z'(\varphi)}{Z(\varphi)}.
$$

It is easy to show that $R(0) = 0$ and that $R$ is strictly increasing since $R'(\varphi) = \varphi^{-1}\text{Var}_{\nu_{\varphi}}[\eta_0]$. There are three different regimes. For $\alpha \leq 1$, $Z(\varphi)$ increases to $\infty$ as $\varphi$ converges $1$. In particular, for each density $\rho \in [0, \infty)$, there exists a stationary measure $\nu_{\varphi}$ whose average density is $\rho$. For $1 < \alpha \leq 2$, $Z(\varphi)$ increases to $Z(1) < \infty$ as $\varphi$ converges $1$, but $Z'(\varphi)$ increases to $\infty$ as $\varphi \uparrow 1$. In this case also for each density $\rho \in [0, \infty)$, there exists a stationary measure $\nu_{\varphi}$ whose average density is $\rho$. In contrast, for $\alpha > 2$, $Z(\varphi)$ and $Z'(\varphi)$ converge to finite values as $\varphi \uparrow 1$, and we have a phase transition. Only for densities $\rho$ in the interval $[0, R(1)]$ there are stationary measures $\nu_{\varphi}$ with average density $\rho$. In fact, in [13] we proved that for fixed $L$ and for $\alpha > 2$, if we denote by $N$ the total number of particles and if we let $N \uparrow \infty$, all but a finite number of particles concentrate on one site, a phenomena called condensation and observed also in the thermodynamical limit as $L \uparrow \infty$ together with $N$ in such a way that the density $N/L$ converges to $\rho > R(1)$, [20 17 16 12 1].

**Stationary states.** For $N \geq 1$, denote by $E_N$ the set of configurations with $N$ particles:

$$
E_N = \{\eta \in E : \sum_{x \in \mathbb{T}_L} \eta_x = N\}.
$$

Since the dynamics conserves the total number of particles, the sets $E_N$, $N \geq 1$, are the irreducible classes of the Markov process $\eta(t)$. It will be convenient to represent the zero-range process on $E_N$ as a random walk on the simplex $\{(i_1, \ldots, i_{L-1}) : i_k \geq 0, i_1 + \cdots + i_{L-1} = N\}$.

Let $\mu_N$ be the probability measure on $E_N$ obtained from $\nu_{\varphi}$ by conditioning on the total number of particles being equal to $N$: $\mu_N(\eta) = \nu_{\varphi}(\eta|\sum_{0 \leq x < L} \eta_x = N)$. The measure $\mu_N$ does not depend on the parameter $\varphi$ and a calculation shows that

$$
\mu_N(\eta) = \frac{N^\alpha}{Z_N} \frac{1}{a(\eta)} := \frac{N^\alpha}{Z_N} \prod_{x \in \mathbb{T}_L} \frac{1}{a(\eta_x)}, \quad \eta \in E_N,
$$

where $Z_N$ is the normalizing constant

$$
Z_N = N^\alpha \sum_{\zeta \in E_N} \frac{1}{a(\zeta)}. \quad (2.2)
$$

By [5 Proposition 2.1],

$$
\lim_{N \to \infty} Z_N = L \Gamma(\alpha)^{L-1}, \quad \text{where } \Gamma(\alpha) := \sum_{j \geq 0} \frac{1}{a(j)}. \quad (2.3)
$$

An elementary computation shows that $\mu_N$ is the stationary state of the zero range process with generator $\mathcal{L}$ restricted to $E_N$. More precisely, let $\mathcal{L}^*$ be the
adjoint of the generator $L$ in $L^2(\mu_N)$. On can check that $L^*$ is the generator of the totally asymmetric zero range process in which particles jump to the left instead of jumping to the right:

$$(L^*F)(\eta) = \sum_{x \in \mathbb{Z}_+} g(\eta_x) \left\{ F(\sigma^x \cdot \eta) - F(\eta) \right\}.$$ 

Denote by $\langle \cdot, \cdot \rangle_{\mu_N}$ the scalar product in $L^2(\mu_N)$. A change of variables gives that

$$\langle LF, G \rangle_{\mu_N} = \langle F, L^*G \rangle_{\mu_N}$$

for every function $F, G : E_N \to \mathbb{R}$. In particular, taking $G = 1$, as $L^*1 = 0$, $\mu_N$ is the stationary state for the process restricted to $E_N$.

**Capacities.** Denote by $\{\eta^*(t) : t \geq 0\}$ the Markov process on $E$ whose generator is $L^*$. We shall refer to $\eta^*(t)$ as the adjoint or the time reversed process.

For a subset $A$ of $E$, denote by $H_A$ (resp. $H_A^+$) the hitting (resp. return) time of a set $A$:

$$H_A := \inf \{ s > 0 : \eta(s) \in A \},$$

$$H_A^+ := \inf \{ t > 0 : \eta(t) \in A, \eta(s) \neq \eta(0) \text{ for some } 0 < s < t \}.$$

When the set $A$ is a singleton $\{\xi\}$, we denote $H_{\{\xi\}}, H_{\{\xi\}}^+$ by $H_\xi$, $H_\xi^+$, respectively.

For each $\eta \in E$, let $P_\eta$ stand for the probability on the path space of right continuous trajectories with left limits, $D(\mathbb{R}^+, E)$, induced by the zero range process $\{\eta(t) : t \geq 0\}$ starting from $\eta \in E$. Expectation with respect to $P_\eta$ is denoted by $\mathbb{E}_\eta$.

Similarly, we denote by $P^*_\eta$, $E^*_\eta$ the probability and the expectation on $D(\mathbb{R}^+, E)$ induced by the time reversed process $\{\eta^*(t) : t \geq 0\}$ starting from $\eta \in E$.

For two disjoint subsets $A, B$ of $E_N$, denote by $V_{A,B}$, $V^*_{A,B}$ the equilibrium potentials defined by

$$V_{A,B}(\eta) = P_\eta[H_A < H_B], \quad V^*_{A,B}(\eta) = P^*_\eta[H_A < H_B], \quad \eta \in E_N. \quad (2.4)$$

Denote by $S$ the symmetric part of the generator $L$: $S = (1/2)(L + L^*)$, and by $D_N$ the Dirichlet form associated to the generator $L$. An elementary computation shows that

$$(SF)(\eta) = \frac{1}{2} \sum_{x \in \mathbb{Z}_+} \sum_{y = -1, 1} g(\eta_x) \left\{ F(\sigma^x \cdot \eta) - F(\eta) \right\}$$

and that

$$D_N(F) = \langle F, (-S)F \rangle_{\mu_N} = \frac{1}{2} \sum_{x \in \mathbb{Z}_+} \sum_{\eta \in E_N} \mu_N(\eta) g(\eta_x) \left\{ F(\sigma^x \cdot \eta) - F(\eta) \right\}^2,$$

for every $F : E_N \to \mathbb{R}$.

For two disjoint subsets $A, B$ of $E_N$, let $C(A, B)$ be the set of functions $h : E_N \to \mathbb{R}$ which are constant over $A$ and constant over $B$, with possibly different values at $A$ and $B$. Let $C_{1,0}(A, B)$ be the subset of functions in $C(A, B)$ equal to 1 on $A$ and 0 on $B$.

Let $\text{cap}_N(A, B)$ be the capacity between two disjoint subsets $A, B$ of $E_N$, defined in [15] Definition 1.1, and recall from [15] Theorem 1.4] the variational formula for the capacity:

$$\text{cap}_N(A, B) = \inf \sup_H \left\{ 2\langle L^*F, H \rangle_{\mu_N} - \langle H, (S)H \rangle_{\mu_N} \right\}, \quad (2.6)$$
where the supremum is carried over all functions $H$ in $C(A, B)$, and where the infimum is carried over all functions $F \in C_{1,0}(A, B)$. When the set $A$ is a singleton, $A = \{x\}$, we denote the capacity $\text{cap}_N(A, B)$ by $\text{cap}_N(x, B)$.

We have shown in [15] that the function $F_{A,B}$ which solves the variational problem for the capacity is equal to $(1/2)\{V_{A,B} + V_{A,B}^*\}$, where $V_{A,B}$, $V_{A,B}^*$ are the harmonic functions defined in (2.3), and that $\text{cap}_N(A, B) = D(V_{A,B})$.

Consider the continuous time totally asymmetric random walk $\{X(t) | t \geq 0\}$ on $\mathbb{T}_L$ jumping to the right with rate one. The stationary measure is the uniform measure. Denote by $\text{cap}(A, B)$ the capacity between two disjoint sets $A, B$ of $\mathbb{T}_L$. One can compute the capacity between two sites $x \neq y \in \mathbb{T}_L$ recalling the observation made in the previous paragraph. Clearly, $V_{x,y}$ is the indicator of the set $\{y + 1, \ldots, x\}$ and $V_{x,y}^*$ is the indicator of the set $\{x, \ldots, y - 1\}$. Hence, the solution $F_{x,y}$ of the variational problem (2.9) is given by $F_{x,y}(z) = \delta_{x,z} + (1/2)1\{z \notin \{x, y\}\}$, and $\text{cap}(x,y) = D(V_{x,y}) = L^{-1}$ is independent of $x, y$.

**Tunneling.** Fix a sequence $\{\ell_N : N \geq 1\}$ such that $1 \ll \ell_N \ll N$:

$$\lim_{N \to \infty} \ell_N = \infty \quad \text{and} \quad \lim_{N \to \infty} \ell_N/N = 0.$$  

(2.7)

For $x$ in $\mathbb{T}_L$, let

$$E^x_N := \{\eta \in E_N : \eta_x \geq N - \ell_N\}.$$  

Obviously, $E^x_N \neq \emptyset$ for all $x \in \mathbb{T}_L$ and every $N$ large enough.

Condition $\ell_N/N \to 0$ is required to guarantee that on each set $E^x_N$ the proportion of particles at $x \in \mathbb{T}_L$, i.e. $\eta_x/N$, is almost one. As a consequence, for $N$ sufficiently large, the subsets $E^x_N$, $x \in \mathbb{T}_L$, are pairwise disjoint. From now on, we assume that $N$ is large enough so that the partition

$$E_N = E_N \cup \Delta_N := \left( \bigcup_{x \in \mathbb{T}_L} E^x_N \right) \cup \Delta_N$$  

(2.8)

is well defined, where $\Delta_N$ is the set of configurations which do not belong to any set $E^x_N$, $x \in \mathbb{T}_L$.

The assumptions that $\ell_N \uparrow \infty$ are sufficient to prove that $\mu_N(\Delta_N) \to 0$, as we shall see in Section 7 and to deduce the limit of the capacities stated in Theorem 2.1 below. We shall need, however, further restrictions on the growth of $\ell_N$ to prove the tunneling behaviour of the zero range processes presented in Theorem 2.2 below.

To state the first main result of this article, for any nonempty subset $A$ of $\mathbb{T}_L$, let $E_N(A) = \cup_{x \in A} E^x_N$, and let

$$I_\alpha := \int_0^1 u^\alpha (1 - u)^\alpha \, du.$$  

(2.9)

**Theorem 2.1.** Assume that $\alpha > 3$ and consider a sequence $\{\ell_N : N \geq 1\}$ satisfying (2.7). Then, for all proper subset $A$ of $\mathbb{T}_L$,

$$\lim_{N \to \infty} N^{1 + \alpha} \text{cap}_N\left( E_N(A), E_N(A^c) \right) = \frac{1}{\Gamma(\alpha) I_\alpha} \sum_{x,y \in A, y \notin A} \text{cap}(x,y).$$

We have seen above that $\text{cap}(x,y) = L^{-1}$ for all $x, y \in \mathbb{T}_L$, $x \neq y$. Therefore, under the assumption of the previous theorem,

$$\lim_{N \to \infty} N^{1 + \alpha} \text{cap}_N\left( E_N(A), E_N(A^c) \right) = \frac{1}{\Gamma(\alpha) I_\alpha} \frac{|A|(L - |A|)}{L}.$$
where $|A|$ stands for the cardinality of the set $A$.

The second main result of this article states that the zero range process exhibits a metastable behavior. Fix a nonempty subset $A$ of $E_N$. For each $t \geq 0$, let $T_t^A$ be the time spent by the zero range process $\{\eta(t) : t \geq 0\}$ on the set $A$ in the time interval $[0, t]$:

$$T_t^A := \int_0^t 1\{\eta(s) \in A\} \, ds,$$

and let $S_t^A$ be the generalized inverse of $T_t^A$:

$$S_t^A := \sup\{s \geq 0 : T_s^A \leq t\}.$$

It is well known that the process $\{\eta^A(t) : t \geq 0\}$ defined by $\eta^A(t) = \eta(S_t^A)$ is a strong Markov process with state space $A$ [1]. This Markov process is called the trace of the Markov process $\{\eta(t) : t \geq 0\}$ on $A$.

Consider the trace of $\{\eta(t) : t \geq 0\}$ on $E_N$, referred to as $\eta^{E_N}(t)$. Let $\Psi_N : E_N \mapsto T_L$ be given by

$$\Psi_N(\eta) = \sum_{x \in T_L} x 1\{\eta \in E_N\}$$

and let $X_N^\eta := \Psi_N(\eta^{E_N}(t))$.

We prove in Theorem 2.2 below that the speeded up non-Markovian process $\{X_N^\eta(t) : t \geq 0\}$ converges to the random walk $\{X_t : t \geq 0\}$ on $T_L$ whose generator $L$ is given by

$$(Lf)(x) = \frac{L}{I(\alpha) I_\alpha} \sum_{y \in T_L} \text{cap}(x, y) \{f(y) - f(x)\} = \frac{1}{I(\alpha) I_\alpha} \sum_{y \in T_L} \{f(y) - f(x)\}. \tag{2.10}$$

For $x$ in $T_L$, denote by $P_x$ the probability measure on the path space $D(\mathbb{R}_+, T_L)$ induced by the random walk $\{X_t : t \geq 0\}$ starting from $x$.

**Theorem 2.2.** Assume that $\alpha > 3$ and that $1 < \ell_N \ll N$, where $\gamma = (1 + \alpha)/(1 + \alpha(L - 1))$. Then, for each $x \in T_L$,

1. **(M1) We have**

$$\lim_{N \to \infty} \inf_{\eta, \xi \in E_N} P^N_\eta[H(\xi) < H(\eta^{E_N}(T_L \setminus \{x\}))] = 1;$$

2. **(M2) For any sequence $\xi_N \in E_N$, $N \geq 1$, the law of the stochastic process $\{X_N^\eta(t) : t \geq 0\}$ under $P^N_\eta$ converges to $P_x$ as $N \uparrow \infty$;**

3. **(M3) For every $T > 0$,**

$$\lim_{N \to \infty} \sup_{\eta \in E_N} E^N_\eta \left[ \int_0^T 1\{\eta(sN^{\alpha+1}) \in \Delta_N\} \, ds \right] = 0.$$

The assumption that $\ell_N \ll N^\gamma$ is needed to prove assumption (H1) of metastability stated in Section 4. It should be possible to relax the assumption that $\alpha > 3$ if one tackles carefully Step 3 of the proof of Proposition 5.1 but our purpose here is not to give the optimal conditions for Theorem 2.2. Our main point is to show how to estimate capacities in the non-reversible case where these capacities are given by a double variational formula. We claim that the variational problem appearing in the definition (2.6) of the capacity $\text{cap}(A, B)$ has to be understood as the variational problem

$$\inf_H D(H).$$
where the sum of two configurations \( \eta \) Denote by \( \langle L, H \rangle \) the sector condition with constant \( 4 \). We hope that the proof of Proposition 5.1 will clarify this affirmation and will convince the reader of its correctness.

For the same reasons, we concentrated on the totally asymmetric case, where the computations are simpler. An analogous result should hold for asymmetric dynamics since the main tool pervading all the argument is a sector condition which holds in all asymmetric cases.

According to the terminology introduced in [4], Theorem 2.2 states that the sequence of zero range processes \( \{ \eta(t) : t \geq 0 \} \) exhibits a tunneling behaviour on the time-scale \( N^{\alpha+1} \) with metastates given by \( \{ \mathcal{E}_N^x : x \in \mathbb{T}_L \} \) and limit given by the random walk \( \{ X_t : t \geq 0 \} \).

The asymptotic evolution of the condensate is reversible even though the original dynamics is not. It does not coincide, however, with the asymptotic dynamics of the condensate in the reversible case where particles jump to the left and to the right neighbors with equal probability 1/2 [5].

Property (M3) states that, outside a time set of order smaller than \( N^{\alpha+1} \), one of the sites in \( \mathbb{T}_L \) is occupied by at least \( N - \ell \) particles. Property (M2) describes the time-evolution on the scale \( N^{\alpha+1} \) of the condensate. It evolves asymptotically as a Markov process on \( \mathbb{T}_L \) which jumps from a site \( x \) to \( y \) at a rate proportional to the capacity \( \text{cap}(x, y) \) of the underlying random walk. Property (M1) guarantees that the process starting in a metastate \( \mathcal{E}_N^x \) thermalizes therein before reaching any other metastate.

3. SECTOR CONDITION

It has been proved in [15] that we may estimate the capacity of a non-reversible process with the capacity of the reversible version of the process if a sector condition is in force. The first result of this section establishes a sector condition for the totally asymmetric zero range process.

**Lemma 3.1.** The zero range process with generator \( L \) defined in (2.1) satisfies a sector condition with constant \( 4L^2 \): For every pair of functions \( F, H : E_N \to \mathbb{R} \),

\[
\langle LF, H \rangle_{\mu_N}^2 \leq 4L^2 D_N(F) D_N(H),
\]

**Proof.** Denote by \( \vartheta_z, z \in \mathbb{T}_L \), the configuration of \( E_1 \) with one particle at \( z \in \mathbb{T}_L \), where the sum of two configurations \( \eta, \xi \) is performed by summing each component: \( (\eta + \xi)(x) = \eta(x) + \xi(x) \).

Fix two functions \( F, H : E_N \to \mathbb{R} \). By the the change of variables \( \xi = \eta - \vartheta_x \), \( \langle LF, H \rangle_{\mu_N} \) can be written as

\[
\frac{W_{N-1}}{W_N} \sum_{\xi \in E_{N-1}} \mu_{N-1}(\xi) \sum_{x \in \mathbb{T}_L} [F(\xi + \vartheta_{x+1}) - F(\xi + \vartheta_{x})] H(\xi + \vartheta_{x}),
\]

where \( W_N = Z_N / N^\alpha \). Fix \( \xi \in E_{N-1} \) and consider the sum

\[
\sum_{x \in \mathbb{T}_L} [F(\xi + \vartheta_{x+1}) - F(\xi + \vartheta_{x})] H(\xi + \vartheta_{x}),
\]

which can be rewritten as

\[
\sum_{x \in \mathbb{T}_L} \left\{ F(\xi + \vartheta_{x+1}) - F(\xi + \vartheta_{x}) \right\} \left\{ H(\xi + \vartheta_{x}) - \frac{1}{L} \sum_{z \in \mathbb{T}_L} H(\xi + \vartheta_{z}) \right\}.
\]
Since $2ab \leq \gamma a^2 + \gamma^{-1}b^2$, $\gamma > 0$, by Schwarz inequality, this expression is less than or equal to

$$\frac{\gamma}{2} \sum_{x \in T_L} [F(\xi + d_{x+1}) - F(\xi + d_x)]^2$$

$$+ \frac{1}{2\gamma} \sum_{x \in T_L} \frac{1}{L} \sum_{z \in T_L} \left\{ \sum_{z_i \in \Gamma(x,z)} H(\xi + d_{z_{i+1}}) - H(\xi + d_{z_i}) \right\}^2,$$

where $\Gamma(x,z)$ stands for a path $(x = z_0, \ldots, z_m = z)$ from $x$ to $z$ such that $|z_i - z_{i+1}| = 1$ for $0 \leq i < m$. Since we may find paths whose length are less than or equal to $L$, we may bound the second sum using Schwarz inequality. After a change in the order of summation this term becomes

$$\frac{1}{2\gamma} \sum_{w \in T_L} \left\{ H(\xi + d_w) - H(\xi + d_{w+1}) \right\}^2 \sum_{x, z \in T_L},$$

where the second sum is carried over all states $x, z \in T_L$ whose path $\Gamma(x,z)$ passes through the bond $(w, w+1)$. This sum is clearly less than or equal to

$$\frac{L^2}{2\gamma} \sum_{w \in T_L} \left\{ H(\xi + d_{w+1}) - H(\xi + d_w) \right\}^2.$$

Up to this point we proved that $(\mathcal{L}F, H)_{\mu_N}$ is absolutely bounded by

$$\frac{\gamma}{2} \frac{W_{N-1}}{W_N} \sum_{\xi \in E_{N-1}} \mu_{N-1}(\xi) \sum_{x \in T_L} [F(\xi + d_{x+1}) - F(\xi + d_x)]^2$$

$$+ \frac{L^2}{2\gamma} \frac{W_{N-1}}{W_N} \sum_{\xi \in E_{N-1}} \mu_{N-1}(\xi) \sum_{x \in T_L} \left\{ H(\xi + d_{x+1}) - H(\xi + d_x) \right\}^2.$$

After a change of variables, we bound this expression by

$$\gamma \langle (-\mathcal{L})F, H \rangle_{\mu_N} + \frac{L^2}{\gamma} \langle (-\mathcal{L})H, H \rangle_{\mu_N}.$$ 

To conclude the proof it remains to optimize over $\gamma$. \hfill \square

Denote by $\text{cap}_N^*(\mathcal{A}, \mathcal{B})$ the capacity between two disjoint subsets $\mathcal{A}, \mathcal{B}$ of $E_N$ with respect to the reversible zero range process with generator $\mathcal{S}$ given by (2.5):

$$\text{cap}_N^*(\mathcal{A}, \mathcal{B}) = \inf_{F} D_N(F),$$

where the infimum is carried over all functions $F$ which are equal to 1 at $\mathcal{A}$ and 0 at $\mathcal{B}$. The next result follows from [15] Lemma 2.5 and 2.6, and Lemma 3.1 above.

**Lemma 3.2.** For every subsets $\mathcal{A}, \mathcal{B}$ of $E_N$, $\mathcal{A} \cap \mathcal{B} = \emptyset$,

$$\text{cap}_N^*(\mathcal{A}, \mathcal{B}) \leq \text{cap}_N(\mathcal{A}, \mathcal{B}) \leq 4L^2 \text{cap}_N^*(\mathcal{A}, \mathcal{B}).$$

Denote by $\text{cap}^*(x, y)$, $x \neq y \in T_L$, the capacity between $x$ and $y$ for the nearest-neighbor symmetric random walk on $T_L$, which jumps with rate 1/2 to the right and rate 1/2 to the left. For a proper subset $\mathcal{A}$ of $T_L$, let

$$d_\alpha(\mathcal{A}, \mathcal{A}^c) = \frac{1}{\Gamma(\alpha)} I_\alpha \sum_{x \in \mathcal{A}, y \in \mathcal{A}^c} \text{cap}^*(x, y). \quad (3.2)$$
Lemma 3.3. Fix a proper subset $A$ of $\mathbb{T}_L$. Then,
\[
\mathcal{C}_\alpha(A, A^c) \leq \liminf_{N \to \infty} N^{1+\alpha} \text{cap}_N(\mathcal{E}_N(A), \mathcal{E}_N(A^c))
\] 
\[
\leq \limsup_{N \to \infty} N^{1+\alpha} \text{cap}_N(\mathcal{E}_N(A), \mathcal{E}_N(A^c)) \leq 4 L^2 \mathcal{C}_\alpha(A, A^c).
\]

Proof. By [5, Theorem 2.1], for every proper subset $A$ of $\mathbb{T}_L$,
\[
\lim_{N \to \infty} N^{1+\alpha} \text{cap}_N(\mathcal{E}_N(A), \mathcal{E}_N(A^c)) = \mathcal{C}_\alpha(A, A^c).
\] (3.3)

The result follows now from Lemma 3.2.

4. Upper Bound

We prove in this section the upper bound for the capacity. The following decomposition of the generator $\mathcal{L}$ as the sum of cycle generators will prove to be most helpful. Fix a configuration $\xi \in E_{N-1}$ and denote by $\mathcal{L}_\xi$ the generator on $E_N$ given by
\[
(\mathcal{L}_\xi f)(\eta) = \sum_{x \in \mathbb{T}_L} 1\{\eta = \xi + \sigma_x\} g(\eta(x)) [f(\sigma^{x,x+1} \eta) - f(\eta)] .
\]

Note that the generator $\mathcal{L}$ restricted to $E_N$ may be written as
\[
\mathcal{L} = \sum_{\xi \in E_{N-1}} \mathcal{L}_\xi
\]
and that the measure $\mu_N$ is stationary for each generator $\mathcal{L}_\xi$. Moreover, for any pair of functions $f, h : E_N \to \mathbb{R}$,
\[
(f, \mathcal{L}_\xi h)_{\mu_N} = \frac{N^\alpha}{Z_N} \frac{1}{a(\xi)} \sum_{x=1}^L f(\xi + \sigma_x) \{h(\xi + \sigma_{x+1}) - h(\xi + \sigma_x)\} ,
\]
(4.1)

In particular, the adjoint of $\mathcal{L}_\xi$ in $L^2(\mu_N)$, denoted by $\mathcal{L}_\xi^*$, is given by
\[
(\mathcal{L}_\xi^* f)(\eta) = \sum_{x \in \mathbb{T}_L} 1\{\eta = \xi + \sigma_x\} g(\eta(x)) [f(\sigma^{x,x-1} \eta) - f(\eta)] ,
\]
and the Dirichlet form $D_\xi$ associated to the generator $\mathcal{L}_\xi$ is given by
\[
D_\xi(f) := (f, -\mathcal{L}_\xi f)_{\mu_N} = \frac{N^\alpha}{2Z_N} \frac{1}{a(\xi)} \sum_{x=1}^L \{f(\xi + \sigma_{x+1}) - f(\xi + \sigma_x)\}^2 .
\] (4.2)

Proposition 4.1. Consider a sequence $\{\ell_N : N \geq 1\}$ satisfying (2.7). Fix a proper subset $A$ of $\mathbb{T}_L$. Then,
\[
\limsup_{N \to \infty} N^{1+\alpha} \text{cap}_N(\mathcal{E}_N(A), \mathcal{E}_N(A^c)) \leq \frac{1}{\Gamma(\alpha)} \ell_N \sum_{x \in A, y \in A} \text{cap}(x, y) .
\]

Proof. Fix a subset $A$ of $\mathbb{T}_L$. For $N \geq 1$, $x \in \mathbb{T}_L$ and a subset $C$ of $\mathbb{T}_L$, let
\[
\mathcal{D}^x_N := \{\eta \in E_N : \eta_x \geq N - 3\ell_N\} , \quad \mathcal{D}_N(C) := \bigcup_{x \in C} \mathcal{D}^x_N ,
\]
so that $\mathcal{E}_N \subset \mathcal{D}^x_N$, $\mathcal{E}_N(C) \subset \mathcal{D}_N(C)$. Therefore, by [13, Lemma 2.2], $\text{cap}_N(\mathcal{E}_N(A), \mathcal{E}_N(A^c)) \leq \text{cap}_N(\mathcal{D}_N(A), \mathcal{D}_N(A^c))$. In particular, to prove Proposition 4.1 it is
enough to exhibit a function \( F_A \) in \( C_{1,0}(\mathcal{D}_N(A), \mathcal{D}_N(A^c)) \) such that

\[
\limsup_{N \to \infty} N^{1+\alpha} \sup_{h \in C(\mathcal{D}_N(A), \mathcal{D}_N(A^c))} \left\{ 2(F_A, \mathcal{L}h)_{\mu_N} - D_N(h) \right\} \\
\leq \frac{1}{\Gamma(\alpha)} I_a \sum_{x \in A, y \notin A} \text{cap}(x, y).
\]  

(4.3)

The definition of the function \( F_A \) requires some notation. Fix an arbitrary \( 0 < \epsilon \ll 1 \) and let \( \mathbb{W} = \mathbb{W}_x : [0,1] \to [0,1] \) be the smooth function given by

\[
\mathbb{W}(t) := \frac{1}{I_a} \int_0^{\phi(t)} u^\alpha (1-u)\alpha \, du,
\]

where \( I_a \) is the constant defined in (4.4) and \( \phi : [0,1] \to [0,1] \) is a smooth non-decreasing function such that \( \phi(t) + \phi(1-t) = 1 \) for every \( t \in [0,1] \) and \( \phi(s) = 0 \) \( \forall s \in [0,3\epsilon] \). It can be easily checked that

\[
\mathbb{W}(t) + \mathbb{W}(1-t) = 1, \quad \forall t \in [0,1],
\]

(4.4)

and that \( \mathbb{W}|_{[0,3\epsilon]} \equiv 0 \) and \( \mathbb{W}|_{[1-3\epsilon,1]} \equiv 1 \).

Let \( D \subset \mathbb{R}^L \) be the compact subset

\[
D := \{ u \in \mathbb{R}_+^L : \sum_{x \in T_L} u_x = 1 \}.
\]

For each pair of sites \( x \neq y \in T_L \) and \( \epsilon > 0 \), consider the subsets of \( D \)

\[
R_\epsilon^x := \{ u \in D : u_x \leq \epsilon \} \quad \text{and} \quad L_\epsilon^{xy} := \{ u \in D : u_x + u_y \geq 1 - \epsilon \}.
\]

(4.5)

Clearly \( L_\epsilon^{xy} = L_\epsilon^{yx} \) for any \( x, y \in T_L \).

Fix \( x \) in \( T_L \) and define \( W_x : D \to [0,1] \) as follows. First define a function \( \hat{W}_x \) on the set \( \bigcup_{y \neq x} L_\epsilon^{xy} \cup R_\epsilon^x \) by

\[
\hat{W}_x(u) = \begin{cases} 
(1/2) \left\{ \mathbb{W}(u_x) + [1 - \mathbb{W}(u_y)] \right\} & \text{for } u \in L_\epsilon^{xy}, y \neq x, \\
0 & \text{if } u \in R_\epsilon^x.
\end{cases}
\]

Note that \( \hat{W}_x \) is well defined because \( \hat{W}_x(u) = 1 \) for \( u \in L_\epsilon^{xy} \cap L_\epsilon^{xz}, y \neq z \), and \( \hat{W}_x(u) = 0 \) for \( u \in R_\epsilon^x \cap L_\epsilon^{xz}, y \neq z \). Let \( W_x : D \to [0,1] \) be a Lipschitz continuous function which coincides with \( \hat{W}_x \) on \( \bigcup_{y \neq x} L_\epsilon^{xy} \cup R_\epsilon^x \).

Let \( F_x : E_N \to \mathbb{R} \) be given by

\[
F_x(\eta) := W_x(\eta/N),
\]

(4.6)

where each \( \eta/N \) is thought of as a point in \( D \). It follows from the definition of \( W_x \) that

\[
F_x(\eta) = (1/2) \left\{ \mathbb{W}(\eta_x/N) + [1 - \mathbb{W}(\eta_y/N)] \right\} \quad \text{for } \eta/N \in L_\epsilon^{xy},
\]

\[
F_x \equiv 1 \quad \text{on } \{ \eta \in E_N : \eta_x \geq (1-\epsilon)N \}
\]

(4.7)

\[
F_x \equiv 0 \quad \text{on } \{ \eta \in E_N : \eta_x \leq \epsilon N \}.
\]

Moreover, since \( W_x \) is Lipschitz continuous, there exists a finite constant \( C_\epsilon \), which depends only on \( \epsilon \), such that

\[
\max_{x \in T_L} \max_{\eta \in E_N} |F_x(\sigma^x \epsilon^{x+1} \eta) - F_x(\eta)| \leq \frac{C_\epsilon}{N}.
\]

(4.8)
Recall that we fixed a nonempty subset $A \subseteq \mathbb{T}_L$. Define the function $F_A : E_N \to \mathbb{R}$ as

$$F_A(\eta) := \sum_{x \in A} F_x(\eta).$$

The function $F_A$ is our candidate to estimate the left hand side of (4.3).

It follows from (4.7) that if $\eta \in \mathcal{D}^y_N$ for some $x \in \mathbb{T}_L$ then for $N$ large enough

$$F_A(\eta) = 1\{x \in A\} = F_A(\sigma^xw\eta),$$

for every $z, w \in \mathbb{T}_L$ and every $N$ large enough. In particular,

$$F_A \in C_{\ell,0}\left(\mathcal{D}_N(A), \mathcal{D}_N(A^c)\right).$$

It remains to prove (4.3). For $N \geq 1$ and $x, y \in \mathbb{T}_L, x \neq y$, let

$$\mathcal{T}^y_N := \{\eta \in E_N : \eta_x + \eta_y \geq N - \ell_N\}.$$ 

Clearly, $\mathcal{T}^y_N = \mathcal{T}^x_N, x \neq y \in \mathbb{T}_L$, and, for every $N$ large enough, $\eta/N$ belongs to $\mathcal{T}^y_N$ if $\eta$ belongs to $\mathcal{T}^y_N$. Moreover, for $N$ sufficiently large,

$$\mathcal{T}^y_N \cap \mathcal{T}^z_N \neq \emptyset \text{ if and only if } \{x, y\} \cap \{z, w\} \neq \emptyset;$$

$$\mathcal{T}^y_N \cap \mathcal{T}^z_N \subseteq \{\eta \in E_N : \eta_x \geq N - 2\ell_N\}, \quad y, z \neq x.$$  (4.9)

Let $\mathcal{R}_N = E_N \setminus \{\mathcal{T}^y_N\}$, and let $\mathcal{L}_R, \mathcal{L}_{x,y}, x \neq y \in \mathbb{T}_L$, be the generators on $E_N$ given by

$$\mathcal{L}_{x,y} = \sum_{\xi \in \mathcal{T}^y_N \setminus \mathcal{T}^x_N} \mathcal{L}_\xi, \quad \mathcal{L}_R = \sum_{\xi \in \mathcal{R}_{N-1}} \mathcal{L}_\xi.$$

Note that $N$ has been replaced by $N - 1$ so that each configuration $\xi$ in this formula belongs to $E_{N-1}$. Even though the generators $\mathcal{L}_{x,y}$ and $\mathcal{L}_{x,z}$ have common factors $\mathcal{L}_\xi$, in view of (4.9), for a function $f$ constant on each set $\mathcal{D}^w_N, w \in \mathbb{T}_L$,

$$\mathcal{L}f = \sum_{y \neq z} \mathcal{L}_{y,z}f + \mathcal{L}_Rf.$$ 

The first sum is carried over all pairs of sites $\{y, z\}$, each pair appearing only once. In particular, for functions $f, h$ in $C(\mathcal{D}^x_N, \mathcal{D}^y_N(A^c))$,

$$\langle f, \mathcal{L}h \rangle_{\mu_N} = \sum_{y \neq z} \langle f, \mathcal{L}_{y,z}h \rangle_{\mu_N} + \langle f, \mathcal{L}_Rh \rangle_{\mu_N}.$$ 

Therefore,

$$\sup_{h \in C(\mathcal{D}^x_N, \mathcal{D}^y_N(A^c))} \left\{2\langle f_A, \mathcal{L}h \rangle_{\mu_N} - \langle h, (-\mathcal{L})h \rangle_{\mu_N}\right\}$$

$$\leq \left(\sum_{\eta \in A} \sup_{h \in C(\mathcal{D}^y_N, \mathcal{D}^x_N(A^c))} \left\{2\langle f_A, \mathcal{L}_{y,z}h \rangle_{\mu_N} - \langle h, (-\mathcal{L}_{y,z})h \rangle_{\mu_N}\right\}\right. + \sum_{y \in A, z \neq y \in A^c} \sup_{h \in C(\mathcal{D}^y_N, \mathcal{D}^x_N)} \left\{2\langle f_A, \mathcal{L}_{y,z}h \rangle_{\mu_N} - \langle h, (-\mathcal{L}_{y,z})h \rangle_{\mu_N}\right\} + \left.\sup_{h \in C(\mathcal{D}^y_N, \mathcal{D}^x_N)} \left\{2\langle f_A, \mathcal{L}_Rh \rangle_{\mu_N} - \langle h, (-\mathcal{L}_R)h \rangle_{\mu_N}\right\}\right\}. $$  (4.10)

In view of (4.3), to complete the proof of Proposition 4.1, it remains to show that the limsup of the right hand side multiplied by $N^{1+\alpha}$ is bounded by the right hand side of (4.3). We estimate separately each piece of this decomposition.

We start with the first term on the right hand side. If $\eta/N$ belongs to $\mathcal{T}^y_N$ for some $x, y$ in $A, x \neq y$, by (4.7), $F_A(\eta) = F_x(\eta) + F_y(\eta) = 1$. Similarly, if $\eta/N$
We claim that 
\[ F_A(\sigma^{zw}\eta) = F_A(\eta) = 1 \] for all \( \eta \in \bigcup_{x,y \in A} \mathcal{F}^{xy}_N \) and \( z, w \in \mathcal{T}_L \).

Therefore, if \( y, z \in A \), or if \( y, z \in A^c \), \( \mathcal{L}^*_y, z F_A = 0 \), where \( \mathcal{L}^*_y, z \) is the adjoint of \( \mathcal{L}_{y, z} \) in \( L^2(\mu_N) \), so that
\[ \sup_h \{ 2(F_A, \mathcal{L}_{y, z}h)_{\mu_N} - \langle h, (-\mathcal{L}_{y, z})h \rangle_{\mu_N} \} = 0. \]

Consider now the second term on the right hand side of (4.10). Fix \( y \in A, z \notin A \). We claim that
\[ \sup_{h \in C(\mathcal{D}^{1,N}_0, \mathcal{D}^{1,N})} \{ 2(F_A, \mathcal{L}_{y, z}h)_{\mu_N} - \langle h, (-\mathcal{L}_{y, z})h \rangle_{\mu_N} \} \leq \frac{N^\alpha}{Z_N} \sum_{\xi \in \mathcal{J}^{y,z}_N} \frac{1}{a(\xi)} \left[ \mathcal{W}(\xi_y + 1)/N - \mathcal{W}(\xi_y/N) \right]^2. \] (4.11)

Indeed, in view of (4.11) we have that
\[ 2(\mathcal{L}^*_y, z F_A, h)_{\mu_N} = \frac{2N^\alpha}{Z_N} \sum_{\xi \in \mathcal{J}^{y,z}_N} \frac{1}{a(\xi)} \sum_{x=1}^L h(\xi + \delta_x) \{ F_A(\xi + \delta_{x-1}) - F_A(\xi + \delta_x) \}. \]

Since for any configuration \( \eta \) which can be written as \( \xi + \delta_w \) for some \( \xi \in \mathcal{J}^{y,z}_N \), \( w \in \mathcal{T}_L \), \( F_A(\eta) = F_y(\eta) = (1/2)\{ \mathcal{W}(\eta_y/N) + [1 - \mathcal{W}(\eta_z/N)] \} \) the sum over \( x \) becomes
\[ (1/2)\{ h(\xi + \delta_{y+1}) - h(\xi + \delta_y) \} \{ \mathcal{W}(\xi_y + 1)/N - \mathcal{W}(\xi_y/N) \} \]
\[ - (1/2)\{ h(\xi + \delta_{z+1}) - h(\xi + \delta_z) \} \{ \mathcal{W}(\xi_z + 1)/N - \mathcal{W}(\xi_z/N) \}. \]

Hence, by Schwarz inequality, \( 2(\mathcal{L}^*_y, z F_A, h)_{\mu_N} \) is absolutely bounded by
\[ \frac{N^\alpha}{2Z_N} \sum_{\xi \in \mathcal{J}^{y,z}_N} \frac{1}{a(\xi)} \left[ \mathcal{W}(\xi_y + 1)/N - \mathcal{W}(\xi_y/N) \right]^2 + \left[ \mathcal{W}(\xi_z + 1)/N - \mathcal{W}(\xi_z) \right]^2 \]
\[ + \frac{N^\alpha}{2Z_N} \sum_{\xi \in \mathcal{J}^{y,z}_N} \frac{1}{a(\xi)} \left[ h(\xi + \delta_{y+1}) - h(\xi + \delta_y) \right]^2 + \left[ h(\xi + \delta_{z+1}) - h(\xi + \delta_z) \right]^2 \].

By (4.12), the second term is bounded above by \( \sum_{\xi \in \mathcal{J}^{y,z}_N} \langle h, (-\mathcal{L}_x)h \rangle_{\mu_N} = \langle h, (-\mathcal{L}_{y, z})h \rangle_{\mu_N} \). On the other hand, since the set \( \mathcal{J}^{y,z}_N \) is symmetric in \( y \) and \( z \), the first line coincides with the right hand side of (4.11), which concludes the proof of this claim.

It remains to examine the last term of (4.10). We claim that
\[ \lim_{N \to \infty} \sup_h \{ 2(F_A, \mathcal{L}_R h)_{\mu_N} - \langle h, (-\mathcal{L}_R)h \rangle_{\mu_N} \} = 0. \] (4.12)

Indeed, by the strong sector condition the supremum on the left hand side of this identity is bounded by \( C_0 \langle F_A, (-\mathcal{L}_R)F_A \rangle_{\mu_N} \) for some finite constant \( C_0 \) depending only on \( L \). By definition of \( \mathcal{L}_R \), this expression is equal to
\[ C_0 \sum_{\xi \in \mathcal{J}^{xy}_N} \langle F_A, (-\mathcal{L}_x)F_A \rangle_{\mu_N} \leq \frac{C_0}{N^{\alpha+1} \ell_N^{\alpha-1}}, \]
where the last estimate follows from Lemma 4.2 below.

Up to this point we proved that the left hand side of (4.3) is bounded above by

\[
\limsup_{N \to \infty} \frac{N^{1+2\alpha}}{Z_N} \sum_{y \in A, z \notin A} \sum_{\xi \in \mathbb{Z}_N^{y+1}} \frac{1}{a(\xi)} \left[ \mathbb{W}(\xi_y + 1/N - \mathbb{W}(\xi_y/N) \right]^2.
\]

Proposition 2.1 in [5], the explicit expression of \( \mathbb{W}_\epsilon \), and a simple computation permits to show that this expression converges, as \( \epsilon \downarrow 0 \), to

\[
\frac{1}{\Gamma(\alpha) I_\alpha} \sum_{x \in A, y \notin A} \text{cap}(x, y) .
\]

This concludes the proof of Proposition 4.1.

We close this section with an estimate used above.

**Lemma 4.2.** For every \( x \in T_L \) and every \( N \) large enough,

\[
\frac{N^\alpha}{Z_N} \sum_{\xi \in \mathbb{R}^{N-1}} \frac{1}{a(\xi)} \sum_{z=1}^L \{ F_x(\xi + d_{z+1}) - F_x(\xi + d_z) \}^2 \leq \frac{C_\epsilon}{N^\alpha+1}.
\]

The proof of this lemma is similar to the one of Lemma 5.2 in [5] and therefore omitted.

5. Lower bound

**Proposition 5.1.** Suppose that \( \alpha > 3 \). Let \( \{ \ell_N : N \geq 1 \} \) be a sequence satisfying (2.7) and let \( A \) be a proper subset of \( T_L \). Then,

\[
\liminf_{N \to \infty} N^{1+\alpha} \text{cap}_N(\mathcal{E}_N(A), \mathcal{E}_N(A^c)) \geq \frac{1}{\Gamma(\alpha) I_\alpha} \sum_{x \in A, y \notin A} \text{cap}(x, y) .
\]

The proof of this statement relies on three observations. On the one hand, on any strip \( \{ \eta \in E_N : \eta/N \in L_{xy}^\epsilon \}, x \neq y \), where \( L_{xy}^\epsilon \) has been defined in (4.5), \( \eta_{x+1} + \cdots + \eta_y \) behaves essentially as a birth and death process with birth rate \( g(\eta_x) \) and death rate \( g(\eta_y) \approx g(N - \eta_x) \). This means that on each strip \( L_{xy}^\epsilon \) the variable \( \eta_{x+1} + \cdots + \eta_y \) evolves as a symmetric zero-range process on two sites whose metastable behavior is easy to determine [8].

Secondly, as we said just after the statement of Theorem 2.2, the variational formula (2.6) for the capacity has to be understood as an infimum over a class of functions \( H \) satisfying certain relations. The main object in this formula is \( H \) and not \( F \) as one may think.

Finally, we shall use in the argument the monotonicity of the capacity stated in [15, Lemma 2.2]: if \( A \subset A' \) and \( B \subset B' \),

\[
\text{cap}(A, B) \leq \text{cap}(A', B') .
\]

These three observations lead to the following approach for the proof of the lower bound. For a fixed function \( f \), we first consider the variational problem (2.6) with the generators \( \mathcal{L} \) and \( \mathcal{S} \) restricted to tubes contained in the strips \( L_{xy}^\epsilon \). For this problem we optimize over functions \( h = h(\eta_{x+1} + \cdots + \eta_y) \) to obtain a lower bound in terms of the Dirichlet form of \( h \) with respect to a symmetric zero-range dynamics over two sites. We then estimate the remaining piece of the original variational problem by extending the function \( h \) defined on the union of strips to the entire
space and by bounding its Dirichlet form on this space. By \( \text{[5.1]} \), we are allowed during this procedure to reduce the sets \( \mathcal{E}_N(A) \) and \( \mathcal{E}_N(A^c) \) whenever necessary.

This extension procedure is not difficult if \( 
\ell_N \gg N^{(1+\alpha)/(2\alpha-2)} \)
and is more demanding when this bound does not hold. We recommend to the reader to assume below that \( L = 3 \) in which case the dynamics can be viewed as a random walk on the simplex \( \{ (i, j) : i \geq 0, j \geq 0, i + j \leq N \} \).

**Proof of Proposition 5.1** Fix a subset \( A \) of \( \mathbb{T}_L \) and a sequence of functions \( f_N \) in \( C_{1,0}(\mathcal{E}_N(A), \mathcal{E}_N(A^c)) \). We have to show that

\[
\liminf_{N \to \infty} N^{1+\alpha} \sup_h \left\{ 2 \langle f_N, \mathcal{L}h \rangle_{\mu_N} - \langle h, (-\mathcal{S})h \rangle_{\mu_N} \right\} \geq \frac{1}{\Gamma(\alpha) I_\alpha} \frac{|A| (L - |A|)}{L},
\]

where the supremum is carried over functions \( h \) in \( C(\mathcal{E}_N(A), \mathcal{E}_N(A^c)) \).

Since we know from \([15, \text{Theorem 2.4}]\) that the function \( F \) which solves the variational problem \([2.10]\) is \( F = (1/2)\{V_{A, B} + V_{A, B}^*\} \), we may restrict our attention to functions \( f_N \) which possess certain properties. We may assume, for instance, that \( f_N \) is non-negative and bounded above by 1. Lemma \([3.3]\) permits also to assume that

\[
N^{1+\alpha} D_N(f_N) \leq 8L^2 \mathcal{E}_\alpha(A, A^c)
\]

for \( N \) large enough, where \( \mathcal{E}_\alpha(A, B) \) has been introduced in \([2.2]\). Indeed, by Lemma \([3.3]\) \( N^{1+\alpha} \text{cap}_N(\mathcal{E}_N(A), \mathcal{E}_N(B)) \leq 8L^2 \mathcal{E}_\alpha(A, B) \) for \( N \) sufficiently large. On the other hand, taking \( h = -f_N \) we get that

\[
\sup_h \left\{ 2 \langle f_N, \mathcal{L}h \rangle_{\mu_N} - \langle h, (-\mathcal{S})h \rangle_{\mu_N} \right\} \geq D_N(f_N),
\]

which proves the claim.

To fix ideas, consider the case where \( A \) is the singleton \( \{0\} \). Note, incidentally, that the proof of the metastability of a sequence of Markov processes presented in \([4]\) requires only Theorem \([2.1]\) for sets \( A \) which are either singletons or pairs. It will be clear from the proof, however, that the general case where \( A \) is any subset of \( \mathbb{T}_L \) can be treated similarly.

The proof is divided in three steps. We first examine the expression inside braces in \([5.2]\) along tiny strips in the space of configurations. Fix a non-negative function \( f \) in \( C_{1,0}(\mathcal{E}^0_N, \mathcal{E}_N(\{0\}^c)) \) bounded by 1 and such that

\[
N^{1+\alpha} D_N(f) \leq 8L^2 \mathcal{E}_\alpha(\{0\}, \{0\}^c).
\]

**Step 1: The main contributions.** Fix a sequence \( \{ k_N : N \geq 1 \}, 1 \ll k_N \ll \ell_N \). For \( x \neq y \), consider the strips

\[
\mathcal{J}_{x, y} = \{ \zeta \in E_{N-1} : \zeta_x \leq k_N, z \neq x, y \}, \quad \mathcal{J}_{x, y}^+ = \{ \zeta + \partial \zeta : \zeta \in \mathcal{J}_{x, y}, z \in \mathbb{T}_L \},
\]

and let

\[
\mathcal{J}_x = \bigcup_{y \neq x} \mathcal{J}_{x, y}, \quad \mathcal{J}_x^+ = \bigcup_{y \neq x} \mathcal{J}_{x, y}^+.
\]

We claim that there exists a function \( h : \mathcal{J}_0^+ \to [0, 1] \), \( h(\eta) = H(\eta_1 + \cdots + \eta_x) \) on \( \mathcal{J}_{0, x}^+ \), such that

\[
N^{1+\alpha} \sum_{\zeta \in \mathcal{J}_0^+} \left\{ 2 \langle f_N, \mathcal{L}\zeta h \rangle_{\mu_N} - \langle h, (\mathcal{L}\zeta^)h \rangle_{\mu_N} \right\} \geq \frac{L - 1}{L \Gamma(\alpha) I_\alpha} - o_N(1),
\]

where \( o_N(1) \) represents a constant which vanishes as \( N \uparrow \infty \).
Fix \( x \neq 0 \) and consider a function \( h(\eta) = H(\eta_1 + \cdots + \eta_x) \) on \( \mathbb{J}_{0,x}^+ \). We first compute
\[
\sum_{\zeta \in \mathbb{J}_{0,x}} \langle f, \mathcal{L}_\zeta h \rangle_{\nu_N} .
\]
(5.5)
In view of the definition of the generator \( \mathcal{L}_\zeta \) and the property of \( h \), this expression is equal to
\[
\frac{N^n}{Z_N} \sum_{\zeta \in \mathbb{J}_{0,x}} \frac{1}{a(\zeta)} \{ f(\zeta + \delta y) - f(\zeta + \delta x) \} \{ H(\zeta_{(0x)} + 1) - H(\zeta_{(0x)}) \} ,
\]
where \( \zeta_{(0x)} = \zeta_1 + \cdots + \zeta_x \). Decompose this sum according to the possible values of \( \zeta_{(0x)} \) to rewrite it as
\[
\frac{N^n}{Z_N} \sum_{i=0}^{N-1} \frac{1}{a(\zeta)} \{ f(\zeta + \delta y) - f(\zeta + \delta x) \} .
\]
Since \( k_N \ll \ell_N \) and since \( f \) is equal to 1 on \( E_N(\{0\}) \), all terms in the second sum vanish if \( i \leq Lk_N \). In particular, since \( \zeta_{(0x)} = i \), the second sum is carried over all configurations \( \zeta \) such that \( \zeta_y \leq k_N \), \( y = 1, \ldots, x - 1, x + 1, \ldots, L - 1 \), and \( \zeta_x = i - \sum y \leq k_N \). This observation will be used several times below.

Let \( \bar{N} = N - 1 \) and recall that \( \sum_{y \in \mathbb{Z}_+} \xi y = \bar{N} \). Denote by \( \xi \) the configuration \( \zeta \) without the coordinates \( \zeta_0 \) and \( \zeta_x \). \( \xi = (\zeta_1, \ldots, \zeta_{x-1}, \zeta_{x+1}, \ldots, \zeta_{L-1}) \) and let \( M_1 = \sum_{1 \leq y \leq x - 1} \xi y \), \( M_2 = \sum_{x + 1 \leq y \leq L - 1} \xi y \). With this notation, we may rewrite the second sum appearing in the previous displayed formula as
\[
\sum_{\xi \in \mathbb{R}_N} \frac{f(\bar{N} - i - M_2 + 1, i - M_1, \xi) - f(\bar{N} - i - M_2, i + 1 - M_1, \xi)}{a(N - i - M_2)a(i - M_1)a(\xi)} ,
\]
where \( \mathbb{R}_N = \{ \xi \in \mathbb{N}^{\ell_N - 2} : \xi y \leq k_N \} \), and where we changed the order of the coordinates of \( \zeta \) starting with the coordinates \( \zeta_0 \) and \( \zeta_x \).

Let \( F : \{0, \ldots, N\} \to \mathbb{R}_+ \) be given by
\[
F(i) = \sum_{\xi \in \mathbb{R}_N} \frac{1}{a(\xi)} f(\bar{N} - i - M_2 + 1, i - M_1, \xi) ,
\]
(5.6)
so that for \( N \) sufficiently large the \( \text{R}_N \) is equal to
\[
\frac{N^n}{Z_N} \sum_{i=Lk_N}^{N-1} \frac{1}{a(N - i)a(i)} [H(i + 1) - H(i)] [F(i) - F(i + 1)] + R_N ,
\]
(5.7)
where the remainder \( R_N \) is given by
\[
R_N = \frac{N^n}{Z_N} \sum_{i=Lk_N}^{N-1} \frac{1}{a(N - i - M_2)a(i - M_1)} \times \sum_{\xi \in \mathbb{R}_N} \left\{ \frac{1}{a(N - i - M_2)a(i - M_1)} \right\} ,
\]
where \( (\nabla_{0,x} f)(\bar{N} - i - M_2, i - M_1, \xi) = f(\bar{N} - M_2, i + 1 - M_1, \xi) - f(\bar{N} - i - M_2, i - M_1, \xi) \).
We are now in a position to define \( h \) on an \( \mathcal{J}^{+}_{0,x} \). In view of (5.6), (5.7), for \( \eta \in \mathcal{J}^{+}_{0,x} \) set

\[
h(\eta) = \left( \sum_{\xi \in \mathbb{R}_{N}} \frac{1}{a(\xi)} \right)^{-1} \sum_{\xi \in \mathbb{R}_{N}} \frac{1}{a(\xi)} f(N - \eta(0x) - M_2, \eta(0x) - M_1, \xi) ,
\]

where we inverted again the order of the coordinates starting with \( \eta \), \( \eta_x \) and where \( M_1 = M_1(\xi) = \sum_{1 \leq y < x} \xi_y, M_2 = M_2(\xi) = \sum_{x < y < L} \xi_y \). Defined in this way \( h \) is clearly a function of \( \eta(0,x) \), \( h(\eta) = H(\eta(0,x)) \), \( H(i) \) is equal to 1 for \( i \leq \ell_{N}/2 \), and equal to 0 for \( i \geq N - (\ell_{N}/2) \), and

\[
\left( \sum_{\xi \in \mathbb{R}_{N}} \frac{1}{a(\xi)} \right) H(i) = - \sum_{\xi \in \mathbb{R}_{N}} \frac{1}{a(\xi)} f(\bar{N} - i - M_2 + 1, i - M_1, \xi) = -F(i) .
\]

Therefore, the first term of (5.7) becomes

\[
\frac{N^\alpha}{Z_N} \left( \sum_{j=0}^{k_N} \frac{1}{a(j)} \right)^{(L-2)} \sum_{i=Lk_N}^{N-1} \frac{1}{a(N-i)a(i)} [H(i+1) - H(i)]^2 .
\]

A similar computation shows that for \( h \) given by (5.8),

\[
\sum_{\xi \in \mathcal{J}_{0,x}} \langle h, (-L_\xi)h \rangle_{\mu_N} = \frac{N^\alpha}{Z_N} \sum_{i=Lk_N}^{N-1} [H(i+1) - H(i)]^2 \sum_{\xi \in \mathbb{R}_{N}} \frac{1}{a(\xi)a(i-M_1)a(\bar{N}-i-M_2)} .
\]

Since \( H \) is constant at distance \( \ell_{N}/2 \) from 0 and from \( N \), and since \( k_N \ll \ell_N \), \( a(i-M_1) \geq a(i)(1-o_N(1)) \), \( a(\bar{N}-i-M_2) \geq a(\bar{N}-i)(1-o_N(1)) \). Therefore, the previous expression is bounded above by

\[
(1+o_N(1)) \left( \sum_{j=0}^{k_N} \frac{1}{a(j)} \right)^{(L-2)} \frac{N^\alpha}{Z_N} \sum_{i=Lk_N}^{N-1} \frac{1}{a(i)a(N-i)} [H(i+1) - H(i)]^2 .
\]

Up to this point we proved that defining \( h \) by (5.8) on the strip \( \mathcal{J}^{+}_{0,x} \), we have that

\[
\sum_{\xi \in \mathcal{J}_{0,x}} \left\{ 2 \langle f, L_\xi h \rangle_{\mu_N} - \langle h, (-L_\xi)h \rangle_{\mu_N} \right\} \geq \left( 1-o_N(1) \right) \left( \sum_{j=0}^{k_N} \frac{1}{a(j)} \right)^{(L-2)} \frac{N^\alpha}{Z_N} \sum_{i=Lk_N}^{N-1} \frac{1}{a(i)a(N-i)} [H(i+1) - H(i)]^2 + R_N .
\]

By (2.3), \( Z_N(\sum_{0 \leq j \leq k_N} a(j)^{-1})^{-(L-2)} \) converges to \( L\Gamma(\alpha) \). Due to the boundary conditions of \( H \) at 0 and \( N \), \( N^{1+2\alpha} \sum_{i} a(i)a(\bar{N}-i)^{-1}[H(i+1) - H(i)]^2 \) is bounded below by an expression which converges to \( I_\alpha^{-1} \) as \( N \uparrow \infty \). We claim that \( R_N \) vanishes as \( N \uparrow \infty \) and that

\[
N^{1+2\alpha} \sum_{i=Lk_N}^{N-1} \frac{1}{a(i)a(\bar{N}-i)} [H(i+1) - H(i)]^2 \leq C_0 \quad \text{(5.9)}
\]
for some finite constant $C_0$ independent of $N$. Assuming that these two claims are
correct, defining $h$ by (5.8) on the strip $\mathcal{J}_{0,x}$, we have that

$$N^{1+2\alpha} \sum_{\zeta \in \mathcal{J}_{0,x}} \left\{ 2 \langle f, \mathcal{L}_\zeta h \rangle_{\mu_N} - \langle h, (-\mathcal{L}_\zeta) h \rangle_{\mu_N} \right\} \geq \frac{1}{L \Gamma(\alpha) \Gamma_0} - o_N(1).$$

The proof that $R_N$ vanishes is similar to the one of (5.9) and relies on the
fact that $k_N \ll \ell_N$. We present the arguments which lead to (5.9) and leave to
the reader the details of the other assertion. By definition of $H$ and by Schwarz
inequality, the left hand side of (5.9) is bounded by

$$\sum_{i=1}^{N-1} \frac{N^{1+2\alpha}}{a(i)a(N-i)} \sum_{\xi \in \mathcal{R}_N} \frac{1}{a(\xi)} \{ f(N-i-1-M_2, i+1-M_1, \xi) - f(N-i-M_2, i-M_1, \xi) \}^2$$

because $\{ \sum_{\xi \in \mathcal{R}_N} a(\xi)^{-1} \}^{-1} \leq 1$. The change of variables
$i = \xi(0_{\mathcal{J}_{0}})$ permits to rewrite this sum as

$$\sum_{\xi \in \mathcal{J}_{0,x}} \frac{N^{1+2\alpha}}{a(\xi)} \prod_{y \neq 0, x} \frac{1}{a(\xi)} \{ f(\xi + d_x) - f(\xi + d_0) \}^2.$$ 

Since $a(\xi(0_{\mathcal{J}_{0}})) \geq a(\xi x)$ and $a(\hat{N} - \xi(0_{\mathcal{J}_{0}})) \geq a(\hat{N} - \sum_{y \neq 0} \xi y) = a(\xi x)$, the previous
expression is bounded by

$$\sum_{\xi \in \mathcal{J}_{0,x}} \frac{N^{1+2\alpha}}{a(\xi)} \{ f(\xi + d_x) - f(\xi + d_0) \}^2.$$ 

By Schwarz inequality this expression is less than or equal to $2LN^{1+\alpha} \sum_{\xi \in \mathcal{J}_{0,x}} \langle f, \mathcal{L}_\zeta f \rangle_{\mu_N}$ which is bounded by $C_0$ in view of (5.8). This proves (5.9) and assertion
(5.4).

Unfortunately, estimate (5.4) does not settle the question on the strips $\mathcal{J}_{0,x}^+$. The
alert reader certainly noticed that $h$ does not belong to $C(\mathcal{E}_N^0, \mathcal{E}_N(\{0\}^c))$ and some
surgery is necessary. The simplest way to overcome this difficulty seems to be to
modify the sets $\mathcal{E}_N^0, \mathcal{E}_N(\{0\}^c)$ and to use (5.11).

Two configurations $\eta, \xi$ belonging to the strip $\mathcal{J}_{0,x}^+$ are said to be equivalent,
$\eta \equiv \xi$, if $\eta(0_{\mathcal{J}_{0}}) = \xi(0_{\mathcal{J}_{0}})$. For $y \in \mathcal{T}_L$, let

$$\hat{\mathcal{E}}_N^y = \left\{ \eta \in \mathcal{E}_N: \xi \in \mathcal{E}_N^y \text{ for all } \xi \equiv \eta \right\}.$$ 

Clearly, $\mathcal{E}_N^y \subset \mathcal{E}_N^y$ and both sets differ at the boundary by a set with at most
$O(k_N^{L-1})$ elements.

Let $\mathcal{E}_N = \hat{\mathcal{E}}_N, \hat{\mathcal{E}}_N = \cup_{x \neq 0} \hat{\mathcal{E}}_N^x$. By (5.1), $\text{cap}(\mathcal{E}_N^0, \mathcal{E}_N(\{0\}^c)) \geq \text{cap}(\hat{\mathcal{E}}_N, \hat{\mathcal{E}}_N)$. Moreover, if $f$ is a function in $C_{1,0}(\hat{\mathcal{E}}_N, \hat{\mathcal{E}}_N)$ the above construction produces a
function $h: \mathcal{J}_{0}^+ \rightarrow [0,1]$ which also belongs to $C_{1,0}(\hat{\mathcal{E}}_N, \hat{\mathcal{E}}_N)$ and such that $h(\eta) = H(\eta(0_{\mathcal{J}_{0}}))$ on $\mathcal{J}_{0,x}^+$ and

$$N^{1+\alpha} \sum_{\xi \in \mathcal{J}_{0}} \left\{ 2 \langle f, \mathcal{L}_\zeta h \rangle_{\mu_N} - \langle h, (-\mathcal{L}_\zeta) h \rangle_{\mu_N} \right\} \geq \frac{L-1}{L \Gamma(\alpha) \Gamma_0} - o_N(1).$$

(5.10)

On the strips $\mathcal{J}_{y,x}^+, x, y \neq 0, x \neq y$, we set $h$ to be identically equal to 0. With
this definition, $h$ still belongs to $C_{1,0}(\hat{\mathcal{E}}_N, \hat{\mathcal{E}}_N)$ and the previous inequality remains
in force with the set $\mathcal{J}_{0}$ replaced by $\mathcal{J} = \cup_{x \in \mathcal{T}_L} \mathcal{J}_{x}$. Note that configurations which
do not belong to this latter set have at least three sites with at least \( k_N \) particles in each.

**Step 2. Extending the function** \( h \). Denote by \( \mathcal{L}_B, \mathcal{L}_C \) the generators \( \mathcal{L}_B = \sum_{\zeta \in \mathcal{J}} \mathcal{L}_\zeta, \mathcal{L}_C = \sum_{\zeta \notin \mathcal{J}} \mathcal{L}_\zeta \) so that \( \mathcal{L} = \mathcal{L}_B + \mathcal{L}_C \).

In view of (2.2) and (2.3), the last sum is bounded by

\[
\lim_{N \to \infty} N^{1+\alpha} \langle h, (-\mathcal{L}_C)h \rangle_{\mu_N} = 0. \tag{5.11}
\]

Indeed, the supremum appearing in (5.12) is certainly bounded below by

\[
\left\{ 2 \langle f, \mathcal{L}_B h \rangle_{\mu_N} - \langle h, (-\mathcal{L}_B)h \rangle_{\mu_N} \right\} +\left\{ 2 \langle f, \mathcal{L}_C h \rangle_{\mu_N} - \langle h, (-\mathcal{L}_C)h \rangle_{\mu_N} \right\}.
\]

We have shown that the first term multiplied by \( N^{1+\alpha} \) is bounded below by \((L - 1)/(L\Gamma(\alpha)I_\alpha) - o_N(1)\). On the other hand, since \( \langle f, (-\mathcal{L}_C)h \rangle_{\mu_N} \leq D_N(f) \), by the sector condition (3.1), the second term is bounded below by

\[-4L D_N(f) h^{1/2} \langle h, (-\mathcal{L}_C)h \rangle_{\mu_N}^{1/2} - \langle h, (-\mathcal{L}_C)h \rangle_{\mu_N}.\]

By the a-priori estimate (5.3) and by (5.11), this expression multiplied by \( N^{1+\alpha} \) vanishes as \( N \uparrow \infty \). This proves (5.2).

To conclude the proof of the proposition it remains to show the validity of (5.12). We start with a simple remark. Denote by \( \mathcal{U}_{N,m} \), \( 1 \leq m \leq N \), the set of configurations in \( E_N \) with at least three sites occupied by at least \( m \) particles, \( \mathcal{U}_{N,m} = \{ \eta \in E_N : \exists x_1, x_2, x_3, \eta_{x_i} \geq m, x_i \neq x_j \} \). We claim that

\[
\mu_N(\mathcal{U}_{N,m}) \leq \frac{C_0}{m^{2(\alpha - 1)}} \tag{5.12}
\]

for some finite constant \( C_0 \) independent of \( N \) and \( m \).

To prove (5.12), by symmetry, it is enough to estimate the measure of the set \( \{ \eta \in E_N : \eta_0, \eta_1, \eta_2 \geq m \} \). By definition of \( \mu_N \) and in view of (2.2), the measure of this set is bounded by

\[
C_0 N^\alpha \sum_{\eta \in E_N} \frac{1}{a(\eta)}
\]

for some finite constant \( C_0 \) independent of \( N, m \) and whose value may change from line to line. By definition of \( \mu_N \) again, we may rewrite the previous sum as

\[
C_0 N^\alpha \sum_{M=3m}^N \frac{1}{a(\eta_0) a(\eta_1) a(\eta_2) \cdots} \sum_{\eta_{M-1} = m} \frac{1}{a(\eta_{M-1})}
\]

In view of (2.2) and (2.3), the last sum is bounded by \( C_0 (N - M)^{-\alpha} \). The previous expression is thus less than or equal to

\[
C_0 \sum_{M=3m}^N \frac{N^\alpha}{M^\alpha(N - M)^\alpha} \mu_3,M(\eta_0, \eta_1, \eta_2 \geq m),
\]

where \( \mu_3,M \) stands for the stationary measure over three sites with \( M \) particles. It remains to show that

\[
\mu_3,M(\eta_0, \eta_1, \eta_2 \geq m) \leq \frac{C_0}{m^{2(\alpha - 1)}}
\]

uniformly over \( M \geq 3m \), which is elementary.
Let $h$ be an extension to $E_N$ of the function $h$ defined in (5.8) and assume that $h$ is absolutely bounded by one. By (5.12),

$$\sum_{\zeta \in U_{N,1,m}} (\mathbf{h}, (-\mathcal{L}_\zeta)\mathbf{h})_{\mu_N} \leq \frac{C_0}{m^{2(\alpha-1)}}$$

(5.13)

for some finite constant $C_0$. This concludes the proof of the proposition for $\alpha > 3$ and $\ell_N \gg N^{(1+\alpha)/(2\alpha-2)}$. The condition $\alpha > 3$ is needed to ensure that $(1+\alpha)/(2\alpha-2) < 1$. To prove the claim note that under these hypotheses we may choose $k_N \ll \ell_N$ such that $k_N \gg N^{(1+\alpha)/(2\alpha-2)}$. Defining $h$ by $\mathbf{g}_0^+$ and extending its definition to the whole space $E_N$ in an arbitrary way which keeps the function non-negative and bounded by one and which respects the values imposed on the sets $\tilde{E}_N$, $\hat{E}_N$, we obtain a function $\mathbf{h}: E_N \to [0,1]$ in $C_{1,0}(\tilde{E}_N, \hat{E}_N)$ such that

$$(\mathbf{h}, (-\mathcal{L}_\zeta)\mathbf{h})_{\mu_N} \leq \sum_{\zeta \in U_{N,1,m}} (\mathbf{h}, (-\mathcal{L}_\zeta)\mathbf{h})_{\mu_N}$$

because $\mathbf{g}_0^+ \subset U_{N,1,k_N} \cup \tilde{E}_N \cup \hat{E}_N$. There is an abuse of notation in the last relation but the meaning is clear. If a configuration $\zeta$ belongs to $\mathbf{g}_0^+$ and does not belong to $U_{N,1,k_N}$ then it belongs to the deep interior of one of the wells $\tilde{E}_N^\zeta$ where $h$ is constant and $\mathcal{L}_\zeta h = 0$. It follows from the previous estimate and from (5.13) that

$$N^{1+\alpha}(\mathbf{h}, (-\mathcal{L}_\zeta)\mathbf{h})_{\mu_N} \leq \frac{C_0N^{1+\alpha}}{k^{2(\alpha-1)}}$$

(5.14)

for some finite constant $C_0$. By definition of the sequence $k_N$, this expression vanishes as $N \uparrow \infty$, which proves (5.11) and the proposition.

**Step 3. Filling the gap.** Fix $0 < \epsilon < 1/3$ and denote by $\mathcal{W}_\epsilon$ the set of configurations in which at most two sites are occupied by more than $\epsilon N$ particles,

$$\mathcal{W}_\epsilon = \cup_{x < y} \{ \eta \in E_N : \eta_x \leq \epsilon N, \ z \neq x, y \}. $$

Since $\alpha > 3$, by (5.12) we need only to define $h$ on the set $\mathcal{W}_\epsilon$.

For $x < y$, denote by $H_{x,y}$ the functions $H: \{0, \ldots, N\} \to [0,1]$ introduced in Step 1 to define $h$ on the strips $\mathbf{g}_0^+$. In particular, $H_{x,y} = 0$ if $x, y \neq 0$. The function $h$ will be defined on $\mathcal{W}_\epsilon \setminus \mathbf{g}_0^+$ as a convex combination of these functions. Let $\Sigma_L$ be the simplex $\Sigma_L = \{ (\theta_0, \ldots, \theta_{L-1}) : \theta_i \geq 0, \sum_i \theta_i = 1 \}$ and let $h: \mathcal{W}_\epsilon \to [0,1]$ be given by

$$h(\eta) = \sum_{x < y} \theta_{x,y}(\eta/N)H_{x,y}(\eta_y),$$

(5.14)

where the function $\theta: \Sigma_L \to \Sigma_{L(L-1)/2}$ fulfills the following conditions:

- Each component is Lipschitz continuous,
- On the set $\{ \eta : \eta_x \geq (1-3\epsilon)N, \ \sum_{y \neq x} \theta_{x,y}(\eta) = 1, \ \theta_{x,w} = \theta_{w,v} \}$,
- On the set $\{ \eta \in E_N : \eta_x \leq \epsilon N, \ z \neq x, y, \ \eta_x, \eta_y \geq 2\epsilon N \}, \ \theta_{x,y}(\eta) = 1$.

There is a slight inaccuracy in the definition of $h$ above. The correct definition of $h$ involves a convex combination of the values of $h$ at the boundary of $\mathbf{g}_0^+$. In (5.14), we didn’t use the values of $h$ at the boundary of $\mathbf{g}_0^+$ but the values in the deep interior of the strips $\mathbf{g}_{x,y}$ to avoid complicated formulae. To clarify this remark, note that for $L = 3\ h_{0,1}(i, k_N + 1) = H_{0,1}(i)$ but it is not true that $h_{0,2}(k_N + 1, j) = H_{0,2}(j)$, we have instead $h_{0,2}(k_N + 1, j) = H_{0,2}(j + k_N + 1)$. Hence, to be absolutely rigorous, instead of $H_{0,2}(j)$ we should have $h_{0,2}(k_N + 1, j)$ in (5.14).
We present the proof for \( L = 3 \) to keep notation simple. In this case we represent the dynamics as a random walk in the simplex \( \{ (i, j) : i, j \geq 0, i + j \leq N \} \), where the first coordinate stands for the variable \( \eta_1 \) and the second one for the variable \( \eta_2 \). We omit from the notation the dependence on the variable \( \eta_0 \), writing a function \( f : E_N \to \mathbb{R} \) simply as \( f(\eta_1, \eta_2) \).

We turn to the proof of condition (6.12). Consider the subset \( \mathcal{W}_\epsilon^2 \) of \( \mathcal{W}_\epsilon \) consisting of all configurations \( \eta \) such that \( \eta_2 \leq \epsilon N \), \( \mathcal{W}_\epsilon^2 = \{ \eta \in \mathcal{W}_\epsilon : \eta_2 \leq \epsilon N \} \). The other cases are treated similarly. The set \( \mathcal{W}_\epsilon^2 \) can be decomposed in three different regions, \( \{ \eta : \eta_1 \leq 2\epsilon N \} \), \( \{ \eta : \eta_0 \leq 2\epsilon N \} \) and \( \{ \eta : \eta_1, \eta_0 \geq 2\epsilon N \} \). The Dirichlet form is estimated in the same way in the first two sets and is easier to estimate in the latter. We concentrated, therefore, in the first set. Assume that \( \eta_2 \leq \epsilon N \), \( \eta_1 \leq 2\epsilon N \).

For \( L = 3 \) with the simplex representation, the Dirichlet form has three types of terms. Jumps from \((i, j)\) to \((i + 1, j)\), which corresponds to jumps of particles from site 0 to site 1, jumps from \((j, i)\) to \((j + 1, i)\), and jumps from \((i + 1, j)\) to \((i, j + 1)\). We estimate the contributions to the total Dirichlet form of the first one. The second one is handled similarly and the third one can be decomposed as a jump from \((i + 1, j)\) to \((i, j)\) and then one from \((i, j)\) to \((i, j + 1)\).

On the set \( \eta_2 \leq \epsilon N \), \( \eta_1 \leq 2\epsilon N \), \( \theta_{0,1} + \theta_{0,2} = 1 \) and the difference \( h(i+1,j) - h(i,j) \) can be written as

\[
[\theta_{0,1}(i+1,j) - \theta_{0,1}(i,j)] [H_{0,1}(i+1) - H_{0,2}(j)] + \theta_{0,1}(i,j) [H_{0,1}(i+1) - H_{0,1}(i)] .
\]

Since \( \theta \) is Lipschitz continuous and since the difference \( H_{0,1}(i+1) - H_{0,2}(j) \) can be written as \( [H_{0,1}(i+1) - 1] + [1 - H_{0,2}(j)] \), the square of the previous expression is less than or equal to

\[
\frac{C_0}{N^2} \left\{ [H_{0,1}(i+1) - 1]^2 + [H_{0,2}(j) - 1]^2 \right\} + C_0 [H_{0,1}(i+1) - H_{0,1}(i)]^2 .
\]

The contribution to the total Dirichlet form of the jumps from \((i, j)\) to \((i + 1, j)\) on the set \( \eta_2 \leq \epsilon N \), \( \eta_1 \leq 2\epsilon N \) is bounded by

\[
C_0 N^{\alpha} \frac{2\epsilon N}{N^2 k_N} \sum_{i = k_N}^{2\epsilon N} \frac{1}{a(i) a(N - i) a(N - i - j)} [h(i + 1, j) - h(i, j)]^2
\]

for some finite constant \( C_0 \). By the previous bound for the square of the difference \( h(i + 1, j) - h(i, j) \), this expression is less than or equal to the sum of three terms. We estimate the first one. The second one is handled in a similar way and the third one is simpler.

The first term is given by

\[
\frac{C_0}{N^2 k_N^\alpha} \sum_{i = k_N}^{2\epsilon N} \frac{1}{a(i) a(N - i)} [H_{0,1}(i+1) - 1]^2
\]

Recall that \( H_{0,1}(i) = 1 \) for \( i \leq \ell_N - k_N \). Replacing 1 by \( H_{0,1}(\ell_N - k_N) \), writing the difference \( H_{0,1}(i+1) - H_{0,1}(\ell_N - k_N) \) as a telescopic sum and applying Schwarz inequality, we bound the last expression by

\[
\frac{C_0}{k_N^\alpha - N^2} \sum_{i = \ell_N - k_N}^{2\epsilon N} \frac{1}{a(i) a(N - i)} \sum_{j = \ell_N - k_N}^{i} [H_{0,1}(j) - H_{0,1}(1)]^2 \frac{1}{a(j)} .
\]
The factor \( w^{\alpha+1} \) came from the sum \( \sum_j a(j) \). It remains to change the order of summations to bound the previous sum by

\[
\frac{C_0 N^\alpha}{k_N^{\alpha-1} N^2} \sum_{j=\ell_N-k_N}^{2\ell_N} \frac{1}{a(j) a(N-j)} \left[ H_{0,1}(j+1) - H_{0,1}(j) \right]^2 \sum_{i=j}^{2\ell_N} i \leq \frac{C_0(\epsilon)}{k_N^{\alpha-1} N^{1+\alpha}},
\]

where the last inequality follows from (6.2). If one recalls that the Dirichlet form is multiplied by \( N^{1+\alpha} \), the contribution to the global Dirichlet from of the piece we just estimated is bounded by \( k_N^{-(\alpha-1)} \). This concludes the proof of the proposition. \( \square \)

### 6. A variational problem for the mean jump rate

Denote by \( R_N^e(\cdot, \cdot) \) the jump rates of the trace process \( \{\eta^e_N(t) : t \geq 0\} \) defined in Section 2. For \( x \neq y \) in \( T_L \), let

\[
R_N^e(x, y) := \frac{1}{\mu_N(x_N)} \sum_{\eta \in x_N^e, \xi \in y_N^e} \mu_N(\eta) R_N^e(\eta, \xi).
\]

We have shown in [8] that in contrast with the reversible case, there is no formula in the nonreversible context relating the mean rates \( r_N(x_N, y_N) \) to the capacities. The mean rates are instead defined implicitly through a class of variational problems examined in this section.

Fix \( x \in T_L \), let \( \bar{E}_N^x = \cup_{y \neq x} E_N^y \) and consider the variational problem

\[
\inf_{h} \sup_{f} \left\{ 2(f, Lh)_{\mu_N} - \langle h, (-S)h \rangle_{\mu_N} \right\}
\]

where the infimum is carried over all functions \( f \) in \( C_{1,0}(\bar{E}_N^x, \bar{E}_N^x) \) and the supremum over all functions \( h \) in \( C(\bar{E}_N^x, \bar{E}_N^x) \).

**Remark 6.1.** We learn from the proof of Proposition 4.1 that in the variational problem (6.2) we may restrict the supremum to functions \( f \) of the form (4.7). More exactly, if we denote by \( a_N \) the expression in (6.2) and by \( b_N \) the expression in (6.2) when the infimum is carried over functions \( f \) of the form (4.7), \( a_N \leq b_N \leq a_N + c_N \), where \( N^{1+\alpha} c_N \) vanishes as \( N \uparrow \infty \).

We learn from the proof of Proposition 5.1 that we may restrict the supremum in (6.2) to functions \( h \) which on the strips \( J_{y,z}^e \), \( z \neq y \in T_L \), depend on \( \eta_{y+1} + \cdots + \eta_z \), \( h(\eta) = U(\eta_{y+1} + \cdots + \eta_z) \) for some \( U : \mathbb{N} \rightarrow \mathbb{R}_+ \), and which are equal to 1, 0 on the sets \( E_N^x, \bar{E}_N^x \), respectively.

For three pairwise disjoint subsets \( A_1, A_2, A_3 \) of \( E_N \) and three numbers \( a_1, a_2, a_3 \) in \([0, 1] \), denote by \( G_{A_1, A_2, A_3}^{a_1, a_2, a_3} \) the functional defined on functions \( f : E_N \rightarrow \mathbb{R} \) by

\[
G_{A_1, A_2, A_3}^{a_1, a_2, a_3}(f) = \sup_{h} \left\{ 2(f, Lh)_{\mu_N} - \langle h, (-S)h \rangle_{\mu_N} \right\},
\]

where the supremum is carried over all functions \( h : E_N \rightarrow \mathbb{R} \), which are equal to \( a_i \) on \( A_i \), \( 1 \leq i \leq 3 \). When we require \( h \) to be only constant on the set \( A_i \), we replace \( a_i \) by \( * \). Moreover, when \( A_1 = E_N^x, A_2 = E_N^y, x \neq y \in T_L, A_3 = \cup_{z \neq x, y} E_N^z \), we denote \( G_{*, *, 0}^{x, y, 0}, G_{0,1,0,2}^{A_1, A_2, A_3} \) by \( G_{x, y}, G_{x, y} \), respectively.
Proposition 6.2. Suppose that \( \alpha > 3 \). Let \( \{ \ell_N : N \geq 1 \} \) be a sequence satisfying (2.7) and fix \( x \neq y \in \mathbb{T}_L \). Then,
\[
\lim_{N \to \infty} N^{1+\alpha} \inf_f G^{x,y}(f) = \frac{1}{\Gamma(\alpha) I_\alpha} \frac{L - 2}{L - 1},
\]
where the infimum is carried over all functions \( f \) which are equal to 1 on \( E_N^x \), 0 on \( \cup_{z \neq x,y} E_N^y \), and which are constant on \( E_N^y \).

The proof of this proposition is analogous to the one of Theorem 2.1 and left to the reader. In the proof of the upper bound we set the value \( \beta = 1/(L - 1) \) for \( f \) on the set \( E_N^x \), and define in (4.6) \( F_y(\eta) \) by \( \beta W_{\eta}(\eta/N) \) instead of \( W_{\eta}(\eta/N) \). The proof of the lower bound is carried as the proof of Proposition 5.1.

The optimal value at the set \( E_N^y \) for the function \( f \) is \( 1/(L - 1) \). More precisely,
\[
\lim_{N \to \infty} N^{1+\alpha} \inf_f G^{x,y}(f) = \frac{1}{\Gamma(\alpha) I_\alpha} \left\{ (L - 1)(1 + \gamma^2) - 2\gamma \right\},
\]
if the infimum is carried over functions \( f \) whose value at \( E_N^y \) is \( \gamma_N \to \gamma \in [0, 1] \).

If the function \( h \) appearing in the variational formula (6.3) does not coincide with \( f \) on \( E_N^x \) or on \( E_N^y \) the left hand side in (6.4) is strictly smaller than the right hand side.

Lemma 6.3. Consider sequences \( \{ \gamma_N : N \geq 1 \} \), \( \{ \beta_N : N \geq 1 \} \) such that \( \gamma_N, \beta_N \) does not converge to \( (L - 1)^{-1}, 1 \). Then,
\[
\limsup_{N \to \infty} N^{1+\alpha} \inf_f G^{x,y}_{\beta_N,\gamma_N}(f) < \frac{1}{\Gamma(\alpha) I_\alpha} \frac{L - 2}{L - 1},
\]
where the infimum is carried over functions \( f \) which are equal to 1, \( (L - 1)^{-1}, 0 \) on \( E_N^x, E_N^y, \cup_{z \neq x,y} E_N^y \), respectively.

Proof. To prove the lemma we may restrict the infimum to functions \( f \) which are of the form \( f = F_x + (L - 1)^{-1} F_y \), where \( F_z \) has been introduced in (4.6). To fix ideas, consider a function \( h \) whose value at \( E_N^x \) is \( \beta_N \) for a sequence \( \beta_N \) which does not converge to 1. In view of the proof of Proposition 4.1, where the difference \( 2(f, \mathcal{L} h)_{\mu_N} - \langle h, (-\mathcal{S}) h \rangle_{\mu_N} \) is analyzed separately on each strip \( J_{N,\mu}^{x,z} \), it is enough to show that
\[
\limsup_{N \to \infty} N^{1+\alpha} \sup_h \left\{ 2\langle F_A, \mathcal{L}_{x,z} h \rangle_{\mu_N} - \langle h, (-\mathcal{L}_{x,z}) h \rangle_{\mu_N} \right\} < \frac{1}{\Gamma(\alpha) I_\alpha} \frac{L - 2}{L - 1},
\]
for some \( z \neq x, y \). It will be more convenient here to define the set \( J_{N,\mu}^{x,z} \) as the set of configurations of \( E_N \) such that \( \eta_w \leq k_N \) for all \( w \neq x, z \), where \( k_N \) is a sequence such that \( 1 \ll k_N \ll \ell_N \).

Repeating the computations below (4.11), we write the linear term as
\[
\frac{N^\alpha}{Z_N} \sum_{\xi \in J_{N,\alpha}^{x,z}} \frac{1}{a(\xi)} \{ h(\xi + \mathcal{O}_{x+1}) - h(\xi + \mathcal{O}_x) \} \{ \mathbb{W}([\xi_x + 1]/N) - \mathbb{W}(\xi_x/N) \}
\]
and
\[
- \frac{N^\alpha}{Z_N} \sum_{\xi \in J_{N,\alpha}^{y,z}} \frac{1}{a(\xi)} \{ h(\xi + \mathcal{O}_{z+1}) - h(\xi + \mathcal{O}_z) \} \{ \mathbb{W}([\xi_z + 1]/N) - \mathbb{W}(\xi_z/N) \}.
\]
We estimate the first term, the second one is handled similarly. We may rewrite the first sum as
\[ N^\alpha \sum_{k=0}^{N-1} \frac{1}{\mathbb{W}(k+1/N) - \mathbb{W}(k/N)} \sum_{\ell \in B_k} \frac{1}{a(\ell)} \left\{ h(\ell + \ell_{x+1}) - h(\ell + \ell_x) \right\}, \]
where the second sum is carried over all configurations \( \ell \) in \( \mathcal{H}_{N-1}^x \) such that \( \ell_x = k \). By definition of \( \mathbb{W} \), we may in fact restrict the first sum to the interval \( \{\ell/N, \ldots, N-\ell/N\} \). Add and subtract the expression \( \mathcal{M}_N / N^{1+\alpha} \) to the sum over \( B_k \), where \( \mathcal{M} \) is an arbitrary constant. The telescopic sum of \( \mathbb{W}(k+1/N) - \mathbb{W}(k/N) \) gives \( \mathcal{M}_N / N^{1+\alpha} \) by definition of \( \mathbb{W} \). The other term is estimated using Schwarz inequality. After performing these steps, we obtain that the previous expression is less than or equal to
\[ \frac{N^\alpha \Gamma(\alpha)^{L-2}}{2Z_N} \sum_{k=0}^{N-1} \frac{1}{a(k)a(N-k)} \left\{ \mathbb{W}(k+1/N) - \mathbb{W}(k/N) \right\}^2 + \frac{\mathcal{M}}{N^{1+\alpha}} \]
\[ + \frac{2N^\alpha}{2Z_N} \sum_{k=N/2}^{N-\ell_N} \frac{a(k)a(\ell_N-k)}{\Gamma(\alpha)^{L-2}} \left\{ \sum_{\ell \in B_k} \frac{1}{a(\ell)} \left[ h(\ell + \ell_{x+1}) - h(\ell + \ell_x) \right] - \frac{\mathcal{M}_N}{N^{1+2\alpha}} \right\}^2, \]
where \( \ell_N = N-1 \). By definition of the function \( \mathbb{W} \), the first term is equal to \( (\sigma_N / 2\Theta_\alpha)N^{-1+\alpha} \), where here and below \( \sigma_N \) is a sequence which converges to 1 as \( N \to \infty \) and then \( \epsilon \downarrow 0 \), and \( \Theta_\alpha = L\Gamma(\alpha) \).

Expand the square in the second line. By (2.3), the simpler quadratic term is equal to \( \sigma_N \Theta_\alpha \mathcal{M}_N^2 / 2N^{1+\alpha} \). By Schwarz inequality, the second quadratic term is bounded by
\[ \frac{N^\alpha}{2Z_N} \sum_{k=N/2}^{N-\ell_N} \sum_{\ell \in B_k} \frac{1}{a(\ell)} \left[ h(\ell + \ell_{x+1}) - h(\ell + \ell_x) \right]^2 \sum_{\ell \in B_k} \frac{a(k)a(\ell_N-k)}{a(\ell)\Gamma(\alpha)^{L-2}}. \quad (6.5) \]
In the last sum, the factor \( a(k) \) in the numerator cancels with a similar one appearing in the denominator and \( a(N-k)/a(N-k-N_{w\neq x,z} \xi_w) \) is bounded by \( 1 + o_N(1) \) because \( N-k \gg \sum_{w\neq x,z} \xi_w \). By definition of \( \Gamma(\alpha) \), we conclude that this expression is thus bounded by
\[ \frac{N^\alpha(1+o_N(1))}{2Z_N} \sum_{k=N/2}^{N-\ell_N} \sum_{\ell \in B_k} \frac{1}{a(\ell)} \left[ h(\ell + \ell_{x+1}) - h(\ell + \ell_x) \right]^2 \]
\[ = \frac{N^\alpha(1+o_N(1))}{2Z_N} \sum_{\ell \in \mathcal{H}_{N-1}^x} \frac{1}{a(\ell)} \left[ h(\ell + \ell_{x+1}) - h(\ell + \ell_x) \right]^2. \]
It remains to estimate the linear term. It is equal to
\[ - \frac{\mathcal{M}}{N^{1+\alpha}} \sum_{k=N/2}^{N-\ell_N} \frac{1}{\Gamma(\alpha)^{L-2}} \sum_{\ell} \frac{1}{a(\ell)} \left[ h(k+1,\ell) - h(k,\ell) \right] \left\{ (\ell_N-k) - 1 \right\} \]
\[ - \frac{\mathcal{M}}{N^{1+\alpha}} \sum_{k=N/2}^{N-\ell_N} \left[ H(k+1) - H(k) \right], \]
where \( H(k) = \Gamma(\alpha)^{-(L-2)} \sum_{\ell} a(\ell)^{-1} h(k,\ell) \), the sum in \( \ell \) is carried over all configurations \( (\ell_1, \ldots, \ell_{L-2}) \) such that \( \ell_z \leq k_N \).
such that $\xi_x = k$, $\xi_y = \bar{N} - k - |\zeta|$, $\xi_w = \zeta_w$, $w \neq x, z$, and $|\zeta| = \sum_z \zeta_z$. By the boundary conditions satisfied by $h$, the second sum is equal to $-2M\beta_N\sigma_N/N^{1+\alpha}$.

By Schwarz inequality, the first one is bounded by

$$\frac{o_N(1)N^\alpha}{Z_N} \sum_{\xi \in \mathbb{Z}^d} \frac{1}{a(\xi)} |h(\xi + \vartheta_{x+1}) - h(\xi + \vartheta_x)|^2$$

$$+ \frac{Z_N \mathcal{M}}{o_N(1)N^{2+\delta_N}} \sum_{k=\ell_N/2}^{N-\ell_N} \sum_{k=\ell_N/2}^{\zeta} \frac{a(k)}{a(\xi)} \frac{\alpha}{\Gamma(\alpha)^2(L-2)} \{ \frac{a(\bar{N} - k)}{a(\bar{N} - k - |\xi| )} - 1 \}^2.$$ 

Since $|\zeta| \ll N - k$, we may choose the expression $o_N(1)$ appearing in this formula to decrease slowly enough for $\{|a(\bar{N} - k)/a(\bar{N} - k - |\zeta| )| - 1\}^2/o_N(1)$ to vanish as $N \uparrow \infty$. With that choice the second line of the previous formula becomes bounded by $o_N(1)\mathcal{M}^2\Theta_\alpha N^{-(1+\alpha)} = o_N(1)\mathcal{M}^2 N^{-(1+\alpha)}$.

Up to this point we showed that the first part of the linear term $2\langle F_A, L_{x,z}h \rangle_{\mu_N}$ is bounded by

$$\frac{1}{N^{1+\alpha}} \left\{ \frac{\sigma_N}{\Theta_\alpha} + \mathcal{M}(1 - \beta_N\sigma_N) + \frac{\Theta_\alpha\sigma_N}{2} \mathcal{M}^2 \right\}$$

$$+ \frac{N^\alpha(1 + o_N(1))}{2Z_N} \sum_{\xi \in \mathbb{Z}^d} \frac{1}{a(\xi)} |h(\xi + \vartheta_{x+1}) - h(\xi + \vartheta_x)|^2.$$ 

In the formula above (6.6), if we apply the inequality $2ab \leq Ra^2 + R^{-1}b^2$, $R > 0$, instead of the inequality $2ab \leq a^2 + b^2$, we may replace the term $1 + o_N(1)$ appearing in the second line by 1, without changing the first line since there is already $\sigma_N$ multiplying $(2\Theta_\alpha)^{-1}$.

Estimating the second piece of $2\langle F_A, L_{x,z}h \rangle_{\mu_N}$ in the same way, we get that $2\langle F_A, L_{x,z}h \rangle_{\mu_N}$ is bounded by

$$\frac{1}{N^{1+\alpha}} \left( \frac{\sigma_N}{\Theta_\alpha} + 2\mathcal{M}(1 - \beta_N\sigma_N) + \Theta_\alpha\sigma_N \mathcal{M}^2 \right)$$

$$+ \frac{N^\alpha}{2Z_N} \sum_{\xi \in \mathbb{Z}^d} \frac{1}{a(\xi)} |h(\xi + \vartheta_{x+1}) - h(\xi + \vartheta_x)|^2$$

$$\leq \frac{N^\alpha}{2Z_N} \sum_{\xi \in \mathbb{Z}^d} \frac{1}{a(\xi)} |h(\xi + \vartheta_{x+1}) - h(\xi + \vartheta_x)|^2.$$ 

for any constant $\mathcal{M}$. In view of (1.2), the sum of the last two lines is bounded by the Dirichlet form $\langle h, (-L_{x,z})h \rangle_{\mu_N}$. Replacing $\mathcal{M}$ by its optimal value $(\beta_N\sigma_N - 1)/\sigma_N\Theta_\alpha$, we obtain that

$$N^{1+\alpha} \sup_h \left\{ 2\langle F_A, L_{x,z}h \rangle_{\mu_N} - \langle h, (-L_{x,z})h \rangle_{\mu_N} \right\} \leq \frac{\sigma_N}{\Theta_\alpha} - \frac{(1 - \beta_N\sigma_N)^2}{\Theta_\alpha\sigma_N}.$$ 

This proves the lemma. \qed

It follows from Lemma 6.3 that the value at $E_N^y$ of the optimal function $h$ for the variational problem (6.2), (6.3) is asymptotically equal to $1/(L - 1)$. Hence, by \cite{S} Proposition 3.2,

$$\lim_{N \to \infty} \frac{r_N(E_N^y, E_N^y)}{r_N(E_N^y, \bar{E}_N^y)} = \frac{1}{L - 1}.$$
By equation (5.8) in [8], \( r_N(\mathcal{E}_N^y, \check{\mathcal{E}}_N^y) = \text{cap}(\mathcal{E}_N^y, \check{\mathcal{E}}_N^y)/\mu_N(\mathcal{E}_N^y) \). Hence, in view of Theorem 2.1 and the fact that \( \lim_{N \to \infty} \mu_N(\mathcal{E}_N^y) = L^{-1} \), \( N^{1+\alpha} r_N(\mathcal{E}_N^y, \check{\mathcal{E}}_N^y) \) converges to \((L - 1)/[\Gamma(\alpha)]\) so that

\[
\lim_{N \to \infty} N^{1+\alpha} r_N(\mathcal{E}_N^y, \mathcal{E}_N^x) = \frac{1}{\Gamma(\alpha)} I_\alpha. \quad (6.6)
\]

7. Proof of Theorem 2.2

In [3], we reduced the proof of the metastability of reversible Markov processes on countable sets to the verification of three conditions, denoted by (H0), (H1) and (H2). The same result holds for general Markov processes on countable state spaces [8].

Recall the definition of the set \( \Delta_N \) introduced in (2.3). Condition (H2), which requires that

\[
\lim_{N \to \infty} \frac{\mu_N(\Delta_N)}{\mu_N(\mathcal{E}_N^y)} = 0, \quad \forall x \in \mathcal{T}_L, \quad (H2)
\]

follows from the fact that \( \mu_N(\mathcal{E}_N^y) \) converges to \( 1/L \) for every \( x \in \mathcal{T}_L \). This last assertion is a consequence of the symmetry of the sets \( \mathcal{E}_N^x \) and of equation (3.2) in [5].

For each \( x \in \mathcal{T}_L \), let \( \xi_N^x \in E_N \) be the configuration with \( N \) particles at \( x \) and let \( \check{\mathcal{E}}_N^x \) represent the set \( \mathcal{E}_N(\mathcal{T}_L \setminus \{x\}) \). The second condition requires that

\[
\lim_{N \to \infty} \sup_{\eta \in \check{\mathcal{E}}_N^x} \frac{\text{cap}_N(\mathcal{E}_N^y, \check{\mathcal{E}}_N^y)}{\text{cap}_N(\eta, \xi_N^x)} = 0, \quad \forall x \in \mathcal{T}_L. \quad (H1)
\]

Recall from Section 3 that we denote by \( \text{cap}_N^x \) the capacity with respect to the reversible zero range process. By Lemma 3.2 the previous supremum is bounded

\[
\sup_{\eta \in \check{\mathcal{E}}_N^x} \frac{4L^2 \text{cap}_N^x(\mathcal{E}_N^y, \check{\mathcal{E}}_N^y)}{\text{cap}_N^x(\eta, \xi_N^x)}. \quad (7.1)
\]

We have shown in [5, Section 6] that \( \text{cap}_N^x(\eta, \xi_N^x) \geq C_0 \ell_N^{-(L-1)\alpha+1} \) for some positive constant \( C_0 \) independent of \( N \). A simple upper bound is given by the following argument. Let \( \eta \) be a configuration with \( O(\ell_N) \) particles at each site \( y \neq x \). By definition,

\[
\text{cap}_N^x(\eta, \xi_N^x) = \mu_N(\eta) \mathbb{P}_\eta[H_{\xi^x_N} < H_y^+] \leq \mu_N(\eta) \leq C_0 \ell_N^{-(L-1)\alpha}.
\]

In this formula, \( \mathbb{P}_\eta \) stands for the probability on the path space \( D(\mathbb{R}_+, E) \) induced by the Markov process with generator \( S \) starting from \( \eta \). This computations shows that \( \ell_N^{-(L-1)\alpha} \) should be the correct order and that the lower bounded obtained in [3] is not far away from the correct order. In any cases, by (3.3) if \( \ell_N^{(L-1)\alpha+1} \ll N^{1+\alpha} \), (7.1) vanishes as \( N \uparrow \infty \), proving (H1).

Finally, condition (H0) imposes the average rates \( r_N(\mathcal{E}_N^x, \mathcal{E}_N^y) \) of the trace process defined in (6.1) to converge:

\[
\lim_{N \to \infty} N^{1+\alpha} r_N(\mathcal{E}_N^x, \mathcal{E}_N^y) = r(x, y), \quad \forall x, y \in \mathcal{T}_L, x \neq y. \quad (H0)
\]

This property has been proved in (6.6).

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REFERENCES

[1] I. Armendáriz, M. Loulakis: Thermodynamic limit for the invariant measures in supercritical zero range processes. Probab. Theory Related Fields 145, 175–188 (2009).
[2] I. Armendáriz, M. Loulakis: Conditional Distribution of Heavy Tailed Random Variables on Large Deviations of their Sum. Stoch. Proc. Appl. 121, 1138–1147 (2011).
[3] I. Armendáriz, S. Großkinsky, M. Loulakis. Zero range condensation at criticality. arXiv:0912.1793 (2012).
[4] J. Beltrán, C. Landim: Tunneling and metastability of continuous time Markov chains. J. Stat. Phys. 140, 1065–1114 (2010).
[5] J. Beltrán, C. Landim: Metastability of reversible condensed zero range processes on a finite set. Probab. Th. Rel. Fields 152, 781–807 (2012).
[6] J. Beltrán, C. Landim: Metastability of reversible finite state Markov processes. Stoch. Proc. Appl. 121, 1633–1677 (2011).
[7] J. Beltrán, C. Landim: Tunneling of the Kawasaki dynamics at low temperatures in two dimensions. arXiv:1109.2776 (2011).
[8] J. Beltrán, C. Landim: Tunneling and metastability of continuous time Markov chains II, the nonreversible case. In preparation.
[9] A. Bovier, M. Eckhoff, V. Gayrard, M. Klein. Metastability in stochastic dynamics of disordered mean field models. Probab. Theory Relat. Fields 119, 99–161 (2001).
[10] J. Beltrán, C. Landim, V. Gayrard, M. Klein. Metastability and low lying spectra in reversible Markov chains. Commun. Math. Phys. 228, 219–255 (2002).
[11] M. R. Evans, T. Hanney: Nonequilibrium statistical mechanics of the zero-range process and related models. J. Phys. A 38(19), R195–R240 (2005).
[12] M. R. Evans, S. N. Majumdar, R. K. P. Zia: Canonical analysis of condensation in factorised steady states. J. Stat. Phys. 123, 357–390 (2006).
[13] P. A. Ferrari, C. Landim, V. V. Sisko. Condensation for a fixed number of independent random variables. J. Stat. Phys. 128, 1153–1158 (2007).
[14] A. Gaudilliére. Condenser physics applied to Markov chains: A brief introduction to potential theory. Online available at http://arxiv.org/abs/0901.3053.
[15] A. Gaudilliére, C. Landim: Potential theory for nonreversible Markov chains and applications. arXiv:1111.2445 (2011).
[16] C. Godrèche, J. M. Luck: Dynamics of the condensate in zero-range processes. J. Phys. A 38, 7215–7237 (2005).
[17] S. Großkinsky, G. M. Schütz, H. Spohn. Condensation in the zero range process: stationary and dynamical properties. J. Statist. Phys. 113, 389–410 (2003).
[18] M. Jara, C. Landim, A. Teixeira; Quenched scaling limits of trap models. Ann. Probab. 39, 176–223 (2011).
[19] M. Jara, C. Landim, A. Teixeira; Quenched scaling limits of trap models in random graphs. preprint (2012).
[20] I. Jeon, P. March, B. Pittel: Size of the largest cluster under zero-range invariant measures. Ann. Probab. 28, 1162–1194 (2000).
[21] E. Olivieri and M. E. Vares. Large deviations and metastability. Encyclopedia of Mathematics and its Applications, vol. 100. Cambridge University Press, Cambridge, 2005.

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