FINE AND HYPERFINE STRUCTURE OF
P-LEVELS IN MUONIC HYDROGEN

A. P. Martynenko
Samara State University, Pavlov Street 1, Samara 443011, Russia

Corrections of orders $\alpha^5$ and $\alpha^6$ are calculated in the fine structure interval $\Delta E^{fs} = E(2P_{3/2}) - E(2P_{1/2})$ and in the hyperfine structure of the energy levels $2P_{1/2}$ and $2P_{3/2}$ in muonic hydrogen. The obtained numerical values $\Delta E^{fs} = 8352.08$ $\mu$eV, $\Delta \tilde{E}^{hfs}(2P_{1/2}) = 7964.36$ $\mu$eV, $\Delta \tilde{E}^{hfs}(2P_{3/2}) = 3392.59$ $\mu$eV can be considered as a reliable estimate for the comparison with corresponding experimental data and for the extraction of the experimental value of the Lamb shift $(2P - 2S)$ in muonic hydrogen.

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I. INTRODUCTION

Simple atoms play an important role in the check of quantum electrodynamics (QED), the bound state theory and precise determination of fundamental physical constants (the fine structure constant, the lepton and proton masses, the Rydberg constant, the proton charge radius, etc) [1, 2]. On the one hand the essential progress achieved in the last years in a more precise determination of fundamental physical parameters [3] is closely related with the growth of the theoretical accuracy in the calculation of the fine and hyperfine structure of the energy levels of hydrogenic atoms, the lepton anomalous magnetic moments (AMM) and, on the other hand with the perfection of experimental methods in the investigation of atomic spectra of the hydrogen, muonium, positronium [1, 2, 4]. Light muonic atoms [5, 6, 7] (muonic hydrogen ($\mu p$), muonic deuterium, ions of muonic helium etc.) are distinguished among simple atoms because of the strong influence of the vacuum polarization (VP) effects [8, 9], the recoil effects [10], the nuclear structure and polarizability effects [11, 12, 13, 14, 15, 16, 17] on the structure of their energy levels. So, the comparison of the theoretical value of the Lamb shift $(2P - 2S)$ in muonic hydrogen with experimental data from the Paul Sherrer Institute (PSI) [18, 19] will make it possible to obtain a more precise value of the proton charge radius and to check the quantum electrodynamics with the accuracy $10^{-7}$. Considering that the energy interval $(2^5P_{3/2} - 2^3S_{1/2})$ is investigated at PSI with an accuracy 30 ppm, it is important to calculate precisely not only the Lamb shift value, but also the fine and hyperfine structure of the $S$-wave and $P$-wave energy levels in the atom ($\mu p$).

The energy levels of light muonic atoms were theoretically studied many years ago in [20, 21, 22, 23, 24] both on the basis of the Dirac relativistic equation and the nonrelativistic

* E-mail: mart@ssu.samara.ru
approach, taking into account corrections by the perturbation theory (PT). In these papers (see also references [23, 26]) the basic contributions to the fine and hyperfine structure of the \( P \)-energy levels in muonic hydrogen were evaluated with the accuracy 0.001 meV. The structure of \( S \)-wave states in \((\mu p)\) was studied in [21, 28, 29, 30] accounting corrections of order \( \alpha^5 \) and \( \alpha^6 \). In this work we continue the investigation of the energy spectrum of \((\mu p)\) in the \( P \)-wave part. The aim of the present study is to calculate such contributions of order \( \alpha^5 \) and \( \alpha^6 \) both in the fine and hyperfine structure of the energy states \( 2P_{1/2}, 2P_{3/2} \), which are connected with the electron vacuum polarization, the recoil effects, the muon anomalous magnetic moment and the relativistic corrections. The role of all these effects is crucial to attain the desirable accuracy. Our goal also consists in the refinement of the earlier performed calculations in [21, 23] and in the derivation of the reliable numerical estimate for the structure of \( P \)-wave levels in the atom \((\mu p)\), which can be used for the comparison with experimental data. Modern numerical values of fundamental physical constants are taken from Ref.[3]: the electron mass \( m_e = 0.510998918(44) \cdot 10^{-3} \) GeV, the muon mass \( m_\mu = 0.1056583692(94) \) GeV, the fine structure constant \( \alpha^{-1} = 137.0359911(46) \), the proton mass \( m_p = 0.938272029(80) \) GeV, the proton anomalous magnetic moment \( \kappa = 1.792847351(28) \), the muon anomalous magnetic moment \( a_\mu = 1.16591981(62) \cdot 10^{-3} \).

II. FINE STRUCTURE OF \( P \)-WAVE STATES

Our approach to the investigation of the energy spectrum of muonic hydrogen is based on the use of quasipotential method in quantum electrodynamics [31, 32], where the two-particle bound state is described by the Schrödinger equation. The basic contribution to the muon and proton interaction operator is determined by the Breit Hamiltonian [33]:

\[
H = \frac{p^2}{2\mu} - \frac{Z\alpha}{r} - \frac{p^4}{8m_1^3} - \frac{p^4}{8m_2^3} + \frac{\pi Z\alpha}{2} \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) - \frac{Z\alpha}{2m_1m_2r} \left( p^2 + \frac{r(rp)p}{r^2} \right) + \Delta V^{fs}(r) + \Delta V^{hfs}(r),
\]

where \( m_1, m_2 \) are the muon and proton masses, \( \mu = m_1m_2/(m_1 + m_2) \) is the reduced mass, \( \Delta V^{fs} \) is the muon spin-orbit interaction:

\[
\Delta V^{fs}(r) = \frac{Z\alpha}{4m_1^2r^3} \left[ 1 + \frac{2m_1}{m_2} + 2a_\mu \left( 1 + \frac{m_1}{m_2} \right) \right] (L\sigma_1),
\]

\( \Delta V^{hfs} \) is the proton spin-orbit interaction and the interaction of the muon and proton spins. The leading order \((Z\alpha)^4\) contribution to the fine structure is determined by the operator \( \Delta V^{fs} \). As it follows from the Eq.(2), \( \Delta V^{fs} \) includes also the recoil effects (the Barker-Glover correction [10]) and the muon anomalous magnetic moment \( a_\mu \) correction. Let us write the fine structure interval for the atom \((\mu p)\) in the form:

\[
\Delta E^{fs} = E(2P_{3/2}) - E(2P_{1/2}) = \frac{\mu^3(Z\alpha)^4}{32m_1^3} \left[ 1 + \frac{2m_1}{m_2} + 2a_\mu \left( 1 + \frac{m_1}{m_2} \right) \right] + \frac{5m_1(Z\alpha)^6}{256} - \frac{m_2^2(Z\alpha)^6}{64m_2} + \frac{\alpha(Z\alpha)^6\mu^3}{32\pi m_1^3} \left[ \ln \frac{\mu(Z\alpha)^2}{m_1} + \frac{1}{5} \right] + \alpha(Z\alpha)^4 A_{VP} + \alpha^2(Z\alpha)^4 B_{VP}.
\]
FIG. 1: The contribution of order $\alpha(Z\alpha)^4$ to the fine and hyperfine structure. The dashed line corresponds the Coulomb interaction. The wave line corresponds the fine or hyperfine interaction. $\tilde{G}$ is the reduced Coulomb Green’s function.

This equation contains the relativistic correction of order $(Z\alpha)^6$, which can be calculated on the basis of the Dirac equation [1, 34], the correction of order $\alpha(Z\alpha)^6$ enhanced by the $\ln(Z\alpha)\ [35, 36, 37]$, a number of terms of fifth and sixth order over $\alpha$ which are determined by the effects of the vacuum polarization [8, 9]. The relativistic recoil effects of order $m_1(Z\alpha)^6/m_2$ in the energy spectra of hydrogenic atoms were investigated in Refs. [1, 34, 38, 39, 40]. In the fine splitting (3) they were calculated in [34, 40]. Additional corrections of the same order were obtained in [41]. They don’t depend on the muon total momentum $j$ and give the contribution only to the Lamb shift. The contributions to the coefficients $A_{VP}$ and $B_{VP}$ are specified by the first and second order perturbation theory. Numerical values of the terms in the expression (3) presented in the analytical form, are depicted in Table I with the accuracy $0.001 \mu eV$. Such precision is related to the numerical values of a number of contributions to the energy spectrum obtained below from the potentials containing VP effects. The fine structure interval (3) in the energy spectrum of electronic hydrogen is considered for a long time as a basic test of quantum electrodynamics [34, 42, 43].

The interaction operator which gives the contribution to the coefficient $A_{VP}$, is represented by the Feynman diagrams in Fig. 1. The vacuum polarization effect leads to the modification both the Coulomb interaction and the spin-orbit interaction in expressions (1), (2):

$$\Delta V_{VP}(r) = \frac{\alpha}{3\pi} \int_1^\infty \rho(s)ds \left( -\frac{Z\alpha}{r} \right) e^{-2m_e sr},$$

$$\Delta V_{fs}^{c}(r) = \frac{\alpha}{3\pi} \int_1^\infty \rho(s)ds \left( 1 + \frac{m_1}{2m_2} + 2a_\mu \left( 1 + \frac{m_1}{m_2} \right) \right) e^{-2m_e sr}(1+2m_e sr)(L\sigma_1),$$

where the spectral function $\rho(s) = \sqrt{s^2 - 1}(2s^2 + 1)/s^4$, $m_e$ is the electron mass. The diagram (a) in Fig. 1 gives the first order perturbation theory contribution to the fine structure of order $\alpha(Z\alpha)^4$. Averaging the potential (2) over the wave functions of the $2P$ - state

$$\psi_{2P}(r) = \frac{1}{2\sqrt{6}} W r e^{-\frac{W}{2r}} Y_{1m}(\theta, \phi), \quad W = \mu Z\alpha,$$

we obtain the following contribution to the interval (3):

$$\Delta E_1^{fs} = \frac{\mu^3(Z\alpha)^4}{32m_1^2} \left[ 1 + \frac{m_1}{2m_2} + 2a_\mu \left( 1 + \frac{m_1}{m_2} \right) \right] \times$$
\[ \times \frac{\alpha}{3\pi} \int_1^\infty \rho(s)ds \int_0^\infty xdx e^{-x(1+\frac{2m_ee}{W})} \left(1 + \frac{2m_ee}{W}x\right) = 3.046 \, \mu eV. \]

Notice that the higher order corrections \( \alpha^2(Z\alpha)^4 \) entering the \( a_\mu \) are taken into account in this expression as well as the recoil effects. The same order contribution \( \alpha(Z\alpha)^4 \) can be obtained in the second order perturbation theory (see the diagram (b) in Fig.1). In the second order perturbation theory the energy spectrum is determined by the reduced Coulomb Green’s function whose partial expansion has the form \[44\]:

\[ \tilde{G}_n(r, r') = \sum_{l,m} \tilde{g}_{nl}(r, r')Y_{lm}(\mathbf{n})Y_{lm}(\mathbf{n'}). \]  

The radial function can be represented in the form of the Sturm expansion in the Laguerre polynomials. For the 2\( P \)-state this function is the following:

\[ \tilde{g}_{21}(r, r') = -2\mu^2 \alpha z' e^{-\frac{\mu z'}{\Delta}} \left\{ \sum_{m=3}^{\infty} \frac{L_{m+1}^3(z)L_{m+1}^3(z')}{(m-2)(m-1)m(m+1)} - \frac{1}{5184} \left[ z^4(-120+90z'-18z'^2+z'^3)+ 
\right. 
\right. 
\]

\[ +z^3(3960-3270z'+756z'^2-57z'^3+z'^4)-18z^2(2160-1920z'+486z'^2-42z'^3+z'^4)+ 
\]

\[ +30z(4440-4230z'+1152z'^2-109z'^3+3z'^4) \right\}, \]

where \( z = \mu Z\alpha r \), \( L_n^m \) are the Laguerre polynomials defined by the relation:

\[ L_n^m(x) = \frac{e^x x^{-m}}{n!} \left( \frac{d}{dx} \right)^n (e^{-x}x^{n+m}) . \]

To use the expression (9) in specific calculations \[27, 29, 30\] we have to integrate over \( z \), \( z' \) and after that to sum over \( m \). Another representation for the \( G_{2P}(r, r') \) was obtained in Ref. [23] by summing over \( m \) in (9):

\[ G_{2P}(r, r') = -\frac{\mu^2(Z\alpha)}{36z^2 z'^2} \left( \frac{3}{4\pi} \ln' \right) e^{-\frac{\mu z'}{\Delta}} g(z, z'), \]

\[ g(z, z') = 24z_3 + 36z_3z_2 + 36z_3z_2 + 24z_3 + 36z_2z_3 + 36z_2z_3 + 49z_3z_3 - 
\]

\[ -3z_3z_3 - 12z_3(2+z_2+z_2)z_3 - 3z_3z_3 + 12z_3z_3 [ -2C + Ei(z_2 - \ln(z_-) - \ln(z_+)] , \]

where \( z_- = \min(z, z') \), \( z_+ = \max(z, z') \), \( C = 0.577216... \) is the Euler constant, \( z = W\alpha r \). Using Eqs. (11) and (12) we transform the second part of the correction (the first part is given by Eq.(7)) of order \( \alpha(Z\alpha)^4 \) to the fine structure appearing in the second order perturbation theory as follows:

\[ \Delta E_2^{fs} = -\frac{\alpha(Z\alpha)^4 \mu^3}{3456\pi m_1 m_2} \left[ 1 + 2a_\mu + (1 + a_\mu) \frac{2m_1}{m_2} \right] \times \]

\[ \times \int_1^\infty \rho(s)ds \int_0^\infty dxe^{-x(1+\frac{2m_ee}{W})} \int_0^\infty \frac{dx'}{x'^2} e^{-x'} g(x, x') = 1.928 \, \mu eV. \]

Let us consider the two-loop vacuum polarization contributions in the one-photon interaction shown in Fig.2. They give the corrections to the fine splitting of \( P \)-levels of order \( \alpha^2(Z\alpha)^4 \).
FIG. 2: Effects of two-loop electron vacuum polarization in the one-photon interaction.

To find the interaction operator in the momentum representation for the amplitude (a) in Fig.2 it is necessary to make the double change

$$\frac{1}{k^2} \rightarrow \frac{\alpha}{3\pi} \int_1^\infty ds \frac{\sqrt{s^2 - 1(2s^2 + 1)}}{s^4(k^2 + 4m_e^2s^2)}$$  \hspace{1cm} (14)

in the photon propagator. Then in the coordinate representation the interaction operator takes the form:

$$\Delta V^{fs}_{V_{P-P}}(r) = \frac{Z\alpha}{r^3} \left[ \frac{1 + 2a_\mu}{4m_1^2} + \frac{1 + a_\mu}{2m_1m_2} \right] (L\sigma_1) \times$$

\begin{align*}
&\times \left( \frac{\alpha}{3\pi} \right)^2 \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \left[ \frac{1}{(\xi^2 - \eta^2)^2} \right] \\
&\times \left[ \xi^2 (1 + \frac{2m_\mu}{W}) e^{-x(1+\frac{2m_\mu}{W})} - \eta^2 (1 + \frac{2m_\mu}{W}) e^{-x(1+\frac{2m_\mu}{W})} \right] = 0.002 \mu eV.
\end{align*}

Averaging (15) over wave functions (6), we obtain the correction to the interval (3):

$$\Delta E^{fs}_3 = \frac{\mu^3\alpha^2(Z\alpha)^4}{72\pi^2} \left[ \frac{1 + 2a_\mu}{4m_1^2} + \frac{1 + a_\mu}{2m_1m_2} \right] \int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta)d\eta \left[ \frac{1}{(\xi^2 - \eta^2)^2} \right] \times$$

\begin{align*}
&\times \int_0^\infty xdx \left[ \xi^2 (1 + \frac{2m_\mu}{W}) e^{-x(1+\frac{2m_\mu}{W})} - \eta^2 (1 + \frac{2m_\mu}{W}) e^{-x(1+\frac{2m_\mu}{W})} \right] = 0.002 \mu eV.
\end{align*}

The two-loop vacuum polarization amplitudes in the diagrams (b,c) of Fig.2 can be calculated using the modification of the photon propagator of the different type [45]:

$$\frac{1}{k^2} \rightarrow \frac{2}{3} \left( \frac{\alpha}{\pi} \right)^2 \int_0^1 \frac{f(v)dv}{4m_e^2 + k^2(1-v^2)},$$  \hspace{1cm} (17)

$$f(v) = v \left\{ (3-v^2)(1+v^2) \left[ Li_2 \left( -\frac{1-v}{1+v} \right) + 2Li_2 \left( \frac{1-v}{1+v} \right) + \frac{3}{2} \ln \frac{1+v}{1-v} \ln \frac{1+v}{2} - \ln \frac{1+v}{1-v} \ln v \right] + \right. \right.$$

$$\left. \left[ \frac{11}{16}(3-v^2)(1+v^2) + \frac{v^2}{4} \right] \ln \frac{1+v}{1-v} + \left[ \frac{3}{2} v(3-v^2) \ln \frac{1-v^2}{4} - 2v(3-v^2) \ln v \right] + \frac{3}{8} v(5-3v^2) \right\}. \hspace{1cm} (18)$$
To obtain the numerical value of the two-loop contribution in this case it is convenient to use the coordinate representation in which the potential corresponding to the amplitudes (b,c) in Fig.2 has the form:

$$\Delta V_{2\text{-loop}, VP}(r) = \frac{2\alpha^2(Z\alpha)}{3\pi^2 r^3} \left[ 1 + \frac{a_\mu}{2m_1 m_2} \right] \int_0^1 \frac{f(v)dv}{1-v^2} e^{\frac{2m_e}{\sqrt{1-v^2}}} \left( 1 + \frac{2m_e r}{\sqrt{1-v^2}} \right) \left( L \sigma_1 \right).$$

(19)

Its contribution to the fine splitting $(2P_{3/2} - 2P_{1/2})$ in muonic hydrogen can be written in the integral form:

$$\Delta E_{fs}^{C} = \frac{\mu^3 \alpha^2(Z\alpha)^4}{12\pi^2} \left[ 1 + \frac{a_\mu}{4m_1^2} + \frac{1}{2m_1 m_2} \right] \times$$

$$\times \int_0^\infty \frac{f(v)dv}{1-v^2} e^{-x} \left( 1 + \frac{2m_e}{W \sqrt{1-v^2}} \right) \left( 1 + \frac{2m_e}{W \sqrt{1-v^2}} x \right) = 0.021 \mu eV.$$

(20)

FIG. 3: Effects of two-loop electron vacuum polarization in the second order perturbation theory.

The dashed line corresponds to the Coulomb interaction. The wave line corresponds to the fine or hyperfine interaction. $\tilde{G}$ is the reduced Coulomb Green's function.

Two-loop vacuum polarization contributions in the second order perturbation theory shown in Fig.3, have the same order $\alpha^2(Z\alpha)^4$. For their calculation we can use earlier obtained relations (2), (4), (5), (11), and the modified Coulomb potential because of the two-loop vacuum polarization [29, 30]:

$$\Delta V_{V P}^C = \left( \frac{\alpha}{\pi} \right)^2 \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \left( -\frac{Z\alpha}{r} \right) \frac{1}{\xi^2 - \eta^2} \left( \xi^2 e^{-2m_e \xi r} - \eta^2 e^{-2m_e \eta r} \right),$$

(21)

$$\Delta V_{2\text{-loop}, VP}^C = -\frac{2Z\alpha}{3r} \left( \frac{\alpha}{\pi} \right)^2 \int_0^1 \frac{f(v)dv}{1-v^2} e^{-\frac{2m_e r}{\sqrt{1-v^2}}}.$$  

(22)

The amplitude (a) in Fig.3 gives the following correction of order $\alpha^2(Z\alpha)^4$ to the fine splitting:

$$\Delta E_{fs}^{C} = \frac{\mu^3 \alpha^2(Z\alpha)^4}{1296\pi^2} \left[ 1 + \frac{a_\mu}{2m_1 m_2} + \frac{1}{4m_1^2} \right] \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \times$$

$$\times \int_0^\infty dx e^{-x} \left( 1 + \frac{2m_e x}{W} \right) e^{-x} \left( \frac{1}{W} \right) g(x, x') = 0.002 \mu eV.$$

(23)
Two other contributions from the amplitudes (b), (c) in Fig.3 have the similar integral structure. Their numerical values are included in Table I. The summary result for the fine splitting $\Delta E^{fs}$ in ($\mu p$) is presented here also. It accounts the numerous earlier performed calculations discussed in the review article [1] and new corrections obtained in this work.

### TABLE I: Fine structure of $P$-wave energy levels in muonic hydrogen.

| Contribution to fine splitting $\Delta E^{fs}$ | Numerical value of the contribution in $\mu eV$ | Reference, equation |
|---------------------------------------------|-----------------------------------------------|---------------------|
| Contribution of order $(Z\alpha)^4$ $\frac{\mu^4(Z\alpha)^4}{32m_1^2} \left(1 + \frac{m_1}{2m_2}\right)$ | 8329.150 | [21, 23],(3) |
| Muon AMM contribution $\frac{\mu^4(Z\alpha)^4}{16m_1^2} a_{\mu} \left(1 + \frac{m_1}{m_2}\right)$ | 17.637 | [21, 23],(3) |
| Contribution of order $(Z\alpha)^6$: $\frac{5m_1(Z\alpha)^4}{64m_1^2}$ | 0.312 | (3),[34, 40] |
| Contribution of order $(Z\alpha)^6 m_1/m_2$: $-\frac{m_1^2(Z\alpha)^6}{64m_1^2}$ | -0.028 | (3),[34, 40] |
| Contribution of order $\alpha(Z\alpha)^4$ in the first order PT $\langle \Delta V^{fs}_{VP} \rangle$ | 3.046 | (7), [21, 23] |
| Contribution of order $\alpha(Z\alpha)^4$ in the second order PT $\langle \Delta V^{fs}_{VP} \cdot \tilde{G} \cdot \Delta V^{fs}_{VP} \rangle$ | 1.928 | (13) |
| Contribution of order $\alpha(Z\alpha)^6$ $\frac{\alpha(Z\alpha)^6 \mu^5}{32\pi m_1^3} \left[ \ln \frac{\mu(Z\alpha)^2}{m_1} + \frac{1}{5} \right]$ | -0.008 | [35, 36],[1] |
| VP Contribution in the second order PT of order $\alpha^2(Z\alpha)^4$ $\langle \Delta V^{fs}_{VP} \cdot \tilde{G} \cdot \Delta V^{fs}_{VP} \rangle$ | 0.002 | (23) |
| VP Contribution from $1\gamma$ interaction of order $\alpha^2(Z\alpha)^4$ $\langle \Delta V^{fs}_{VP-VP} \rangle$ | 0.002 | (16) |
| VP Contribution from $1\gamma$ interaction of order $\alpha^2(Z\alpha)^4$ $\langle \Delta V^{fs}_{2-loop,VP} \rangle$ | 0.021 | (20) |
| VP Contribution in the second order PT of order $\alpha^2(Z\alpha)^4$ $\langle \Delta V^{fs}_{2-loop,VP} \cdot \tilde{G} \cdot \Delta V^{fs}_{VP} \rangle$ | -0.002 | (21) |
| VP Contribution in the second order PT of order $\alpha^2(Z\alpha)^4$ $\langle \Delta V^{fs}_{2-loop,VP} \cdot \tilde{G} \cdot \Delta V^{fs}_{VP} \rangle$ | 0.022 | (22) |
| Summary contribution | 8352.082 | |

### III. HYPERFINE STRUCTURE OF THE ENERGY LEVELS $2P_{1/2}$ $2P_{3/2}$

The leading order contribution to the hyperfine splitting of the energy levels $2P_{1/2}$ and $2P_{3/2}$ in muonic hydrogen of order $(Z\alpha)^4$ is determined by the following potential (the
hyperfine part of the Breit potential)\cite{33}:

\[
\Delta V_{B}^{hfs}(r) = \frac{Z\alpha(1+\kappa)}{2m_1m_2r^3}\left[1 + \frac{m_1(1+2\kappa)}{2m_2(1+\kappa)}\right](L\sigma_2) - \frac{Z\alpha(1+\kappa)}{4m_1m_2r^3}\left[(\sigma_1\sigma_2) - 3(\sigma_1n)(\sigma_2n)\right],
\]

(24)

where \(n = r/r\), \(\kappa\) is the proton anomalous magnetic moment. The operator (24) does not commute with the operator of the muon total angular momentum \(J = L + \frac{1}{2}\sigma_1\). So, the energy levels \(2P_{1/2}\) and \(2P_{3/2}\) are mixed and the hyperfine structure of \(P\)-wave levels is more complicated.

To calculate the diagonal matrix elements \(\langle 2P_{1/2}\Delta V_{B}^{hfs}|2P_{1/2}\rangle\) and \(\langle 2P_{3/2}\Delta V_{B}^{hfs}|2P_{3/2}\rangle\) we can use the following replacement for the operators \(s_1s_2\) (\(Ls_2\)) containing the nuclear spin:

\[
s_1 \rightarrow J\frac{(s_1J)}{J^2}, \quad L \rightarrow J\frac{(LJ)}{J^2},
\]

(25)

where \((s_1J), (LJ)\) are eigenvalues of corresponding operators between the states with equal orbital momentum \(l\). Moreover, the angle averaging in the second term of (24) can be carried out by means of the relation \cite{46}:

\[
\langle \delta_{ij} - 3n_in_j \rangle = -\frac{1}{5}(4\delta_{ij} - 3L_iL_j - 3L_jL_i).
\]

(26)

The diagonal matrix elements have the following general structure:

\[
\Delta E^{hfs}(P_{1/2}) = \langle P_{1/2}|\Delta V_{B}^{hfs}|P_{1/2}\rangle =
\]

(27)

\[
\begin{align*}
&= E_F \left[\frac{1}{3} + \frac{a_\mu}{6} + \frac{m_1(1+2\kappa)}{2m_2(1+\kappa)} + \frac{m_1^3}{\mu^3}A_{rel}^{1/2}(Z\alpha)^2 + 2A_{V\mu}\alpha + B_{V\mu}\alpha^2\right], \\
&\Delta E^{hfs}(P_{3/2}) = \langle P_{3/2}|\Delta V_{B}^{hfs}|P_{3/2}\rangle = \\
&= E_F \left[\frac{2}{15} - \frac{a_\mu}{30} + \frac{m_1(1+2\kappa)}{2m_2(1+\kappa)} + \frac{m_1^3}{\mu^3}A_{rel}^{3/2}(Z\alpha)^2 + 2A_{V\mu}\alpha + B_{V\mu}\alpha^2\right],
\end{align*}
\]

(28)

where \(E_F = (Z\alpha)^4\mu^3(1+\kappa)/3m_1m_2\) is the Fermi energy for the level with the \(n = 2\). The calculation of the relativistic corrections \(A_{rel}^{1/2}, A_{rel}^{3/2}\) in this formalism includes the study of two-photon, three-photon exchange diagrams and the second order perturbation theory contributions with the Breit Hamiltonian (1), (2), (24). More simple approach to their calculation is based on the Dirac relativistic equation \cite{37, 47}. In this case the hyperfine splitting potential has the form:

\[
\Delta V_{D}^{hfs} = e\mu\frac{|r \times \alpha|}{r^3},
\]

(29)

and gives the following contributions to the hyperfine structure:

\[
\Delta E_{rel}^{hfs}(2P_{1/2}) = \frac{4(Z\alpha)(1+\kappa)}{m_2}R_{1/2},
\]

(30)

\[
\Delta E_{rel}^{hfs}(2P_{3/2}) = -\frac{16(Z\alpha)(1+\kappa)}{15m_2}R_{3/2},
\]

\[
\Delta E_{rel}^{hfs}(2P_{3/2}) = -\frac{16(Z\alpha)(1+\kappa)}{15m_2}R_{3/2},
\]
where the nuclear magnetic moment $\mu = g_N \mu_N \mathbf{s}_2$ ($\mu_N = e/2m_2$). The typical radial integrals $R_k = \int_0^\infty g_k f_k dr$ for this case are determined by the Dirac wave functions $f_k, g_k$ of the states $2P_{1/2}, 2P_{3/2}$. Accounting their exact form we obtain the following values of the relativistic corrections to the hyperfine structure of the $P$-wave level:

$$A_{rel}^{1/2} = \frac{47}{72}, \quad A_{rel}^{3/2} = \frac{7}{180}.$$  

These values of the coefficients coincide with the analytical calculation of the contribution of order $m_e^2(Z\alpha)^6/m_2$ in the hyperfine structure of the $P$-levels of hydrogen atom for the $n = 2$, carried out in Ref. [40].

The fifth order contribution over $\alpha$ appears in the hyperfine splitting as well as in the fine structure due to the electron vacuum polarization (see diagrams (a), (b) in Fig.1). The modification of the hyperfine part of the Breit potential from the vacuum polarization is described by the following relation (the substitution (14) is used) [23]:

$$\Delta V_{P}^{hfs}(r) = \frac{Z\alpha(1 + \kappa)}{2m_1m_2r^3} \left[ 1 + \frac{m_1(1 + 2\kappa)}{2m_2(1 + \kappa)} \right] \left( L\sigma_2 \right) \int_1^{\infty} \rho(s) ds e^{-2m_e s r} (1 + 2m_e s r) - \right. \right. \left. \left. \left. \frac{Z\alpha(1 + \kappa)(1 + a_{\mu})}{4m_1m_2r^3} \right] \int_1^{\infty} \rho(s) ds e^{-2m_e s r} \left[ 4m_e^2 s^2 r^2 (\sigma_1 \sigma_2 - (\sigma_1 n)(\sigma_2 n)) + (1 + 2m_e s r) (\sigma_1 \sigma_2 - 3(\sigma_1 n)(\sigma_2 n)) \right]. \right.$$  

For the subsequent transformations of the diagonal matrix elements of the operator (32) we use the relation for the angle average (26) and the similar expression for the first term in the square brackets of Eq.(32):

$$\langle \delta_{ij} - n_i n_j \rangle = \frac{1}{3} [2\delta_{ij} + L_i L_j + L_j L_i].$$  

Then the contributions of the vacuum polarization in the first and second orders PT can be written as follows:

$$\Delta E_{1}^{hfs}(P_{1/2}) = E_F \frac{\alpha}{18\pi} \int_1^{\infty} \rho(s) ds \int_0^{\infty} x dx e^{-x(1 + 2m_e s x)} \times \left[ \left( 1 + \frac{m_1(1 + 2\kappa)}{2m_2(1 + \kappa)} \right) \left( 1 + \frac{2m_e s}{W} x \right) + (1 + a_{\mu}) \left( \frac{2m_e^2 s^2 x^2}{W^2} + 1 + \frac{2m_e s x}{W} \right) \right] = 3.830 \ \mu eV,$$

$$\Delta E_{1}^{hfs}(P_{3/2}) = E_F \frac{\alpha}{18\pi} \int_1^{\infty} \rho(s) ds \int_0^{\infty} x dx e^{-x(1 + 2m_e s x)} \times \left[ \left( 1 + \frac{m_1(1 + 2\kappa)}{2m_2(1 + \kappa)} \right) \left( 1 + \frac{2m_e s}{W} x \right) - \frac{1}{5} \left( 1 + a_{\mu} \right) \left( \frac{2m_e^2 s^2 x^2}{W^2} + 1 + \frac{2m_e s x}{W} \right) \right] = 0.497 \ \mu eV,$$

$$\Delta E_{2}^{hfs}(P_{1/2}) = E_F \frac{\alpha}{324\pi} \int_1^{\infty} \rho(s) ds \int_0^{\infty} dx e^{-x(1 + 2m_e s x)} \times \int_0^{\infty} dx' x^2 e^{-x' g(x, x')} \left[ 2 + \frac{m_1(1 + 2\kappa)}{2m_2(1 + \kappa)} + a_{\mu} \right] = 1.838 \ \mu eV,$$

$$\Delta E_{2}^{hfs}(P_{3/2}) = E_F \frac{\alpha}{324\pi} \int_1^{\infty} \rho(s) ds \int_0^{\infty} dx e^{-x(1 + 2m_e s x)} \times \int_0^{\infty} dx' x^2 e^{-x' g(x, x')} \left[ (1 + a_{\mu}) \left( 1 + \frac{2m_e s x}{W} \right) \right] = 1.838 \ \mu eV,$$
\[
\times \int_0^\infty \frac{dx'}{x'^2} e^{-x'} g(x,x') \left[ \frac{4}{5} + \frac{m_1(1+2\kappa)}{2m_2(1+\kappa)} - \frac{a_\mu}{5} \right] = 1.278 \ \mu eV.
\]

It should be noted that the sums of the corrections (34), (36) and (35), (37) in the terms \(A_{V'P}^{1/2}\alpha\), \(A_{V'P}^{3/2}\alpha\) are equal 0.00025 and 0.000056 correspondingly. They are slightly different from the results 0.00022 and 0.00008 obtained in [23]. Neglecting the recoil effects and muon anomalous magnetic moment in the expressions (34) and (35) we obtain the results which coincide with the calculation based on the Dirac equation with the potential:

\[
\Delta V_{V'P,D}^{hfs}(r) = e\mu \frac{r \times \alpha}{r^3} \frac{\alpha}{3\pi} \int_1^\infty \rho(s)(1 + 2m_e sr)e^{-2m_e sr} ds. \tag{38}
\]

Numerical values of the contributions \(A_{V'P}^{1/2}\alpha\) and \(A_{V'P}^{3/2}\alpha\) in this case are equal 0.00019 and 0.00002 correspondingly.

| TABLE II: Hyperfine structure of \(P\)-wave energy levels in muonic hydrogen. |
|--------------------------------------------------|-------------------------------|-------------------------------|----------------|
| Contribution to hyperfine splitting              | Numerical value of the contribution in \(\Delta E^{hfs}(2P_{1/2})\), \(\mu eV\) | Numerical value of the contribution in \(\Delta E^{hfs}(2P_{3/2})\), \(\mu eV\) | Reference, equation |
| Contribution of order \((Z\alpha)^4\)              | 7953.195                      | 3392.112                      | (27),(28),(21, 23) |
| Muon AMM contribution                             | 4.432                         | -0.886                        | (27),(28),(21, 23) |
| Contribution of order \((Z\alpha)^6\)             | 1.092                         | 0.065                         | (31),(40)         |
| Contribution of order \(\alpha(Z\alpha)^4\) in the first order PT \(\langle \Delta V_{V'P}^{hfs} \rangle\) | 3.830                         | 0.497                         | (34),(35),(21, 23) |
| Contribution of order \(\alpha(Z\alpha)^4\) in the second order PT \(\langle \Delta V_{V'P}^{hfs} \cdot \tilde{G} \cdot \Delta V_{V'P}^{hfs} \rangle\) | 1.838                         | 0.783                         | (36),(37)         |
| VP Contribution in the second order PT of order \(\alpha^2(Z\alpha)^4\) \(\langle \Delta V_{V'P}^{C} \cdot \tilde{G} \cdot \Delta V_{V'P}^{hfs} \rangle\) | 0.002                         | 0.002                         | (4),(32)          |
| VP Contribution of 1\(\gamma\) interaction of order \(\alpha^2(Z\alpha)^4\) \(\langle \Delta V_{V'P}^{hfs} \rangle\) | 0.003                         | 0.00004                       | (39)              |
| VP Contribution of 1\(\gamma\) interaction of order \(\alpha^2(Z\alpha)^4\) \(\langle \Delta V_{V'P}^{hfs} \rangle\) | 0.026                         | 0.008                         | (40)              |
| VP Contribution in the second order PT of order \(\alpha^2(Z\alpha)^4\) \(\langle \Delta V_{V'P}^{C} \cdot \tilde{G} \cdot \Delta V_{V'P}^{hfs} \rangle\) | -0.001                        | -0.0003                       | (11),(21),(24)    |
| VP Contribution in the second order PT of order \(\alpha^2(Z\alpha)^4\) \(\langle \Delta V_{2_{loop,V'P}}^{C} \cdot \tilde{G} \cdot \Delta V_{V'P}^{hfs} \rangle\) | 0.017                         | 0.007                         | (11),(22),(24)    |
| Summary contribution                              | 7964.364                      | 3392.588                      |                 |

Two-loop vacuum polarization corrections in the hyperfine part of the potential for the
\( l \neq 0 \) obtained by means of Eqs.(14), (17) have the integral representation:

\[
\Delta V_{V-P}^{\text{hfs}}(r) = \frac{Z\alpha(1 + \kappa)}{2m_1m_2r^3} \left( \frac{\alpha}{\pi} \right)^2 \int_1^{\infty} \rho(\xi)d\xi \int_1^{\infty} \rho(\eta)d\eta \frac{1}{\xi^2 - \eta^2} \times \\
\times \left[ 1 + \frac{m_1(1 + 2\kappa)}{2m_2(1 + \kappa)} \right] (L\sigma_2) \left[ \xi^2(1 + 2m_\epsilon\xi r)e^{-2m_\epsilon\xi r} - \eta^2(1 + 2m_\epsilon\eta r)e^{-2m_\epsilon\eta r} \right] - \\
- \frac{1}{2}a_\mu \left[ (\sigma_1\sigma_2 - 3(\sigma_1n)(\sigma_2n)) \left( \xi^2(1 + 2m_\epsilon\xi r)e^{-2m_\epsilon\xi r} - \eta^2(1 + 2m_\epsilon\eta r)e^{-2m_\epsilon\eta r} \right) + \\
+ 4m_\epsilon^2r^2 (\sigma_1\sigma_2 - 3(\sigma_1n)(\sigma_2n)) \left( \xi^4e^{-2m_\epsilon\xi r} - \eta^4e^{-2m_\epsilon\eta r} \right) \right],
\]

\[
\Delta V_{2-\text{loop},V-P}^{\text{hfs}}(r) = \frac{Z\alpha(1 + \kappa)\frac{2}{3}}{2m_1m_2r^3} \left( \frac{\alpha}{\pi} \right)^2 \int_0^{1/v^2} \frac{f(v)dv}{1-v^2} e^{-\frac{2m_\epsilon r}{\sqrt{1-v^2}}} \times \\
\times \left[ 1 + \frac{m_1(1 + 2\kappa)}{2m_2(1 + \kappa)} \right] \left( 1 + \frac{2m_\epsilon r}{\sqrt{1-v^2}} \right) (L\sigma_2) - \frac{1}{2}a_\mu \times \\
\times \left[ \frac{4m_\epsilon^2r^2}{1-v^2} (\sigma_1\sigma_2 - (\sigma_1n)(\sigma_2n)) + \left( 1 + \frac{2m_\epsilon r}{\sqrt{1-v^2}} \right) (\sigma_1\sigma_2 - 3(\sigma_1n)(\sigma_2n)) \right].
\]

Omitting further details of the calculation of their expectation values which can be performed analogously (20) and (23), we represent in Table II numerical results of the contributions of the potentials (39), (40). Another part of two-loop corrections of order \( \alpha^2(Z\alpha)^4 \) to the hyperfine structure in the second order PT is shown in Fig.3. We also included numerical results obtained by means of these amplitudes in Table II.

Nonadiagonal matrix element has an important role to attain the high accuracy of the calculation of the \( P \)-wave levels in muonic hydrogen. We present its general structure as follows:

\[
\gamma = \langle \tilde{3}P_{3/2} | \Delta V^{\text{hfs}} | \tilde{3}P_{3/2} \rangle = E_F \left( -\frac{\sqrt{2}}{48} \right) \left[ 1 - a_\mu + \frac{m_1(1 + 2\kappa)}{m_2(1 + \kappa)} + \frac{m_3^3}{\mu^3} C_{\text{rel}}(Z\alpha)^2 + C_{V-P}\alpha \right],
\]

(41)

where for the simplicity we have restricted by the fifth order terms over \( \alpha \) in the vacuum polarization effects, the fifth and higher order terms in the muon anomalous magnetic moment, the relativistic effects of order \( (Z\alpha)^6 \) and the recoil effects. First three terms in the right part of Eq. (41) appear from the potential (24). In the Dirac theory the relativistic corrections are determined by nonadiagonal radial integral:

\[
R_{\frac{3}{2}} = \int_0^{\infty} \left( g_{\frac{3}{2}}(r)f_{\frac{3}{2}}(r) + g_{\frac{3}{2}}(r)f_{\frac{3}{2}}(r) \right) dr.
\]

(42)

After the integration in (42) which can be performed using explicit form of the wave functions \( f_{1/2,3/2}(r), g_{1/2,3/2}(r) \), we obtain the coefficient \( C_{\text{rel}} = 9/16 \). To calculate the vacuum polarization effects we substitute the potential (32) into \( \gamma \). Then we have to calculate the matrix elements of the following operators:

\[
T_1 = (L\sigma_2), \quad T_2 = [\sigma_1\sigma_2 - 3(\sigma_1n)(\sigma_2n)], \quad T_3 = [\sigma_1\sigma_2 - (\sigma_1n)(\sigma_2n)].
\]

(43)
FIG. 4: The structure of $S$-wave and $P$-wave energy levels in muonic hydrogen for the $n = 2$.

After the angle averaging in Eq.(43) by means of Eqs.(26) and (33) these matrix elements can be expressed in terms of the $6j$ - symbols:

$$\langle T_3 \rangle = -\langle T_2 \rangle = -\langle T_1 \rangle = -6j \hat{j}' \left\{ \begin{array}{c} l \ 1 \\ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ j \\ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ j' \end{array} \right\} = \frac{2\sqrt{2}}{3} , \tag{44}$$

where the value of the total momentum $F = 1$ ($\mathbf{F} = \mathbf{s}_2 + \mathbf{J}$), $l = 1$, $\hat{j} = \sqrt{2j + 1}$, $\hat{j}' = \sqrt{2j' + 1}$, and the numerical values of the $6j$ - symbols are taken from [48]. As a result the vacuum polarization contributions to the nondiagonal matrix element (41) in the first and
second order PT have the form:

\[
\gamma_1 = \langle 3P_{1/2} | \Delta V_{VP}^{hfs} | 3P_{3/2}\rangle = E_F \left(-\frac{\sqrt{2}}{72}\right) \frac{\alpha}{\pi} \int \rho(s) ds \int_0^\infty xdx e^{-x(1 + \frac{2m_s s}{W})} \times (45)
\]

\[
\times \left[ \left(1 + \frac{m_1(1 + 2\kappa)}{2m_2(1 + \kappa)} \right) \left(1 + \frac{2m_e s}{W}x \right) - \frac{1 + a_\mu}{2} \left(1 + \frac{2m_e s}{W}x - \frac{4m^2 e^2 s^2}{W^2 x^2} \right) \right] = -0.617 \mu eV,
\]

\[
\gamma_2 = \langle 3P_{1/2} | \Delta V_{VP}^C \cdot \tilde{G} \cdot \Delta V_{B}^{hfs} | 3P_{3/2}\rangle = E_F \left(-\frac{\sqrt{2}}{2592}\right) \frac{\alpha}{\pi} \left[1 + \frac{m_1(1 + 2\kappa)}{m_2(1 + \kappa)} - a_\mu \right] \times (46)
\]

\[
\times \int \rho(s) ds \int_0^\infty xdx e^{-x(1 + \frac{2m_s s}{W})} \int_0^\infty \frac{dx'}{x'^2} e^{-x'} g(x, x') = -0.184 \mu eV.
\]

Putting \( a_\mu = 0, m_1/m_2 = 0 \) in Eq.(45), we obtain the value \(-0.889 \mu eV \) which coincides with the calculation of this matrix element in the Dirac theory with the potential (38). Summary numerical value of the matrix element (41) is equal \( \gamma = -796.192 \mu eV \). It leads to the shift of the energy levels \( 2^3P_{3/2} \) and \( 2^3P_{1/2} \) by the value \( \delta = 144.560 \mu eV \) as shown in Fig.4.

IV. SUMMARY AND CONCLUSION

In this work we calculate the QED effects in the fine and hyperfine structure of the \( 2P_{1/2}, 2P_{3/2} \) energy levels in muonic hydrogen. The electron vacuum polarization contributions of orders \( \alpha^5, \alpha^6 \) with the recoil corrections, the relativistic effects of order \( \alpha^6 \) are considered. Numerical values of the contributions are listed in Tables I and II. We give the references on the papers devoted to the leading order calculation of the structure of \( P \)-wave levels. For the comparison of the obtained results with the earlier performed calculations we used the review article [1] containing the modern status of the investigations in the physics of relativistic energy spectra of simple atoms.

Let us summarize the basic particularities of the calculation performed above.

1. Special attention in our investigation has been concentrated on the vacuum polarization effects. For this purpose we obtain the terms of the interaction operator in muonic hydrogen which contain the one-loop and two-loop vacuum polarization corrections.

2. At each order over \( \alpha \) we retain the recoil effects in the terms \( \sim m_1/m_2 \). The experimental values of the muon and proton anomalous magnetic moments are used [2].

3. The calculation of the relativistic corrections to the diagonal and nondiagonal matrix elements both for the fine and hyperfine structure intervals is performed by means of the Dirac equation. In the second order perturbation theory we use the compact representation for the reduced Coulomb Green’s function obtained in Ref. [23].

Total numerical values for the fine structure interval \( \Delta E^{fs} \) (3) and hyperfine structure splittings of \( 2P_{1/2}, 2P_{3/2} \) states are presented in Tables I,II. Accounting also our calculation of the mixing between states \( 2^3P_{3/2} \) and \( 2^3P_{1/2} \) (the correction \( \delta \)), we obtain the change of the hyperfine splittings on the \( \delta = 144.560 \mu eV \): \( \Delta E^{hfs}(2P_{1/2}) = \Delta E^{hfs}(2P_{3/2}) - \delta = 7819.804 \mu eV, \Delta E^{hfs}(2P_{3/2}) = \Delta E^{hfs}(2P_{3/2}) - \delta = 3248.028 \mu eV \). The theoretical error of the obtained results is determined by the contributions of higher order and amounts up to \( 10^{-6} \). The results of this work improve the previous calculations in [21, 23, 26] because of the investigation the effects of order \( \alpha^6 \) and can be considered as a reliable estimate for the
fine and hyperfine structure intervals for the $P$- levels in muonic hydrogen. These results are important for the experiment at PSI [18, 19]. The disposition of the $P$-wave energy levels, shown in Fig.4, is determined by the following numerical values: $-5973.27 \, \mu eV \, (2^1P_{1/2})$, $1846.53 \, \mu eV \, (2^3P_{1/2})$, $6376.28 \, \mu eV \, (2^3P_{3/2})$, $9624.30 \, \mu eV \, (2^5P_{3/2})$. Accounting the Lamb shift value $(2P - 2S)$ in $(\mu p)$, obtained in Refs. [25, 26], the hyperfine splitting of the $2S$-state from [29] and the results of present study we obtain the value of the energy interval in muonic hydrogen $\Delta E(2^5P_{3/2} - 2^3S_{1/2}) = 205975.6 \, \mu eV$.

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