On Weak Tail Domination of Random Vectors

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Abstract
Motivated by a question of Krzysztof Oleszkiewicz we study a notion of weak tail domination of random vectors. We show that if the dominating random variable is sufficiently regular weak tail domination implies strong tail domination. In particular positive answer to Oleszkiewicz question would follow from the so-called Bernoulli conjecture.

Introduction. This note is motivated by the following problem about Rademacher series, posed by Krzysztof Oleszkiewicz (private communication): Problem. Suppose that \((\varepsilon_i)\) is a Rademacher sequence (i.e. sequence of independent symmetric ±1 r.v.’s) and \(x_i, y_i\) are vectors in some Banach space \(F\) such that the series \(\sum x_i \varepsilon_i\) and \(\sum y_i \varepsilon_i\) are a.s. convergent and

\[
\forall x^* \in F^* \forall t > 0 \quad P\left( \left| x^* \left( \sum x_i \varepsilon_i \right) \right| \geq t \right) \leq P\left( \left| x^* \left( \sum y_i \varepsilon_i \right) \right| \geq t \right).
\]

Does it imply that

\[
E \left\| \sum x_i \varepsilon_i \right\| \leq L E \left\| \sum y_i \varepsilon_i \right\|,
\]

for some universal constant \(L < \infty\)?

Motivated by the above question we introduce a notion of weak tail domination of random vectors. We show that if the dominating vector has a regular distribution (including Gaussian case), weak tail domination yields strong tail domination (Theorem \(\Box\)). In particular Oleszkiewicz question has positive answer provided that the so-called Bernoulli Conjecture holds true. Finally we show that in general weak tail domination does not yield comparison of means or medians of norms even if the distribution of dominated vector is Gaussian.

*Research partially supported by MEiN Grant 1 PO3A 012 29.
Definition 1. Let $X$ and $Y$ be random vectors with values in some Banach space $F$. We say that the tails of $Y$ are weakly dominated by tails of $X$ and denote it by $Y \prec_\omega X$ if

$$P(\|x^*(Y)\| \geq t) \leq P(\|x^*(X)\| \geq t) \text{ for all } x^* \in F^*, t > 0.$$ 

The following regularity property of random vectors will give us a tool to pass from weak to strong comparison.

Definition 2. We say that a random vector $X$ with values in $F$ is $K$-regular for some $K < \infty$ if there exists a sequence $(x^*_n) \subset F^*$ such that

$$\|x_n^*(X)\| \log(n+2) = (E|x_n^*(X)|^{\log(n+2)})^{1/\log(n+2)} \leq K E\|X\| \text{ for } n = 1, 2, \ldots.$$ 

and

$$B_{F^*} = \{x^* \in F^*: \|x^*\| \leq 1\} \subset \text{cl}_X(\text{conv}\{\pm x_n^*: n \geq 1\}),$$

where for $A \subset F^*$, $\text{cl}_X(A)$ denotes the closure of $A$ with respect to the $L^2$ distance $d_x(x^*, y^*) := \sqrt{E|\|x^*(X) - y^*(X)\|_2|^2}$.

Proposition 1. If $X$ is $K$-regular and $Y \prec_\omega X$, then $E\|Y\| \leq 20 K E\|X\|$.

Proof. Let $x_n^*$ be as in Definition 2. We have for any $t > 0$,

$$P\left(\sup_{n \geq 1} |x_n^*(Y)| \geq t\right) \leq \sum_{n \geq 1} P(\|x_n^*(Y)\| \geq t) \leq \sum_{n \geq 1} t^{-\log(n+2)} E|x_n^*(Y)|^{\log(n+2)} \leq \sum_{n \geq 1} \left(\frac{K E\|X\|}{t}\right)^{\log(n+2)}.$$

Notice that $d_Y(x^*, y^*) \leq d_X(x^*, y^*)$, hence $B_{F^*}^Y$ is contained also in the closure of absolute convex of $\pm x_n^*$ with respect to $d_Y$ metric and thus

$$E\|Y\| \leq E \sup_{n \geq 1} |x_n^*(Y)| \leq K E\|X\| \left(e^2 + \int_{e^2}^{\infty} \left(t K E\|X\|\right) dt\right) \leq K E\|X\| \left(e^2 + \sum_{n=1}^{\infty} \int_{e^2}^{\infty} t^{-\log(n+2)} dt\right) \leq 20 K E\|X\|. \qedhere$$

2
Theorem 1. Let $X_1, X_2, \ldots$ be independent copies of symmetric random vector $X$. Suppose that there exist constants $K < \infty$ and $\alpha, \beta > 0$ such that for all $n = 1, 2, \ldots$

i) random vector $(X_1, \ldots, X_n)$ with values in $l_{\infty}^n(F)$ is $K$-regular,

ii) $P(\max_{i \leq n} \|X_i\| \geq \alpha E \max_{i \leq n} \|X_i\|) \geq \beta$.

Then for any random vector $Y$ such that $Y \prec \omega X$ we have

$$P(\|Y\| \geq t) \leq \frac{2}{\beta} P\left(\|X\| \geq \frac{\alpha t}{80K}\right).$$

The main idea how to derive comparison of tails from comparison of means is not new - it goes back at least to the paper of de la Peña, Montgomery-Smith and Szulga [2].

Proof. We may obviously assume that $Y$ is symmetric, by $Y_1, Y_2, \ldots$ we will denote independent copies of $Y$. Let $n \geq 2$ be such that

$$\frac{2}{n} \geq P(\|Y\| \geq t) \geq \frac{1}{n}.$$

Then $P(\max_{i \leq n} \|Y_i\| \geq t) \geq 1 - (1 - 1/n)^n \geq 1/2$, hence $E \max_{i \leq n} \|Y_i\| \geq t/2$. Let $\eta$ be r.v. independent of $(Y_i)$ such that $P(\eta = 1) = P(\eta = 0) = 1/2$, then by Theorem 3.2.1 of [3], $\eta(Y_1, \ldots, Y_n) \prec \omega (X_1, \ldots, X_n)$, where both variables are considered as random vectors in $l_{\infty}^n(F)$. By Proposition 1,

$$\frac{t}{4} \leq E \max_{i \leq n} \|\eta Y_i\| = E \|\eta(Y_1, \ldots, Y_n)\|_{l_{\infty}^n(F)} \leq 20KE \|(X_1, \ldots, X_n)\|_{l_{\infty}^n(F)} = 20KE \max_{i \leq n} \|X_i\|.$$

Property ii) yields

$$\beta \leq P\left(\max_{i \leq n} \|X_i\| \geq \frac{\alpha t}{80K}\right) \leq nP\left(\|X\| \geq \frac{\alpha t}{80K}\right),$$

so $P(\|X\| \geq \alpha t/(80K)) \geq \beta/n \geq \beta P(\|Y\| \geq t)/2$. \hfill $\square$

Remark 1. By the Paley-Zygmund inequality (cf. [3], Lemma 0.2.1), the comparison of first and second moments of maxima,

$$E \max_{i \leq n} \|X_i\|^2 \leq C(E \max_{i \leq n} \|X_i\|)^2 \quad (1)$$

implies property ii) of previous theorem with $\alpha = 1/2$ and $\beta = 1/(4C)$. 

3
Remark 2. Both Proposition 1 and Theorem 1 hold (with constants depending on \(C_1\) and \(C_2\)) if we replace the condition \(Y \prec_\omega X\) by the condition

\[
P(|x^*(Y)| \geq t) \leq C_1 P(|x^*(X)| \geq t/C_2) \text{ for all } x^* \in F^*, t > 0. \tag{2}
\]

Indeed, if \(\eta\) is a random variable independent of \(Y\) with \(P(\eta = 1) = 1 - P(\eta = 0) = 1/C_1\), then condition (2) implies \(\eta Y/C_2 \prec_\omega X\).

Let us give few examples of random vectors satisfying the assumptions of Theorem 1.

1. Any centered Gaussian vector on a separable Banach space is \(L\)-regular with universal \(L\). This is a consequence of majorizing measure theorem (cf.\([5]\) and \([6]\), Theorem 2.1.8). Since a product of Gaussian measures is again Gaussian, property i) holds with \(K = L\). Moments of Gaussian vectors are comparable so by Remark 1 also property ii) holds with \(\alpha = 1/2\) and universal \(\beta\).

2. Let \((\eta_i)\) be a sequence of independent symmetric real r.v.'s with logarithmically concave tails satisfying \(\Delta_2\) condition and \(v_i \in F\) be such that \(X = \sum v_i \eta_i\) is a.s. convergent. Then \(X\) is \(K\)-regular with constant \(K\) depending only on \(\Delta_2\) constant (\([1]\), Theorem 3). Random variable \((X_1, \ldots, X_n)\) has an analogous series representation in \(l_\infty^n(F)\), so property i) holds. It can be also checked that (1) is satisfied with universal \(C\).

3. Positive answer to Bernoulli Conjecture (\([6]\), Chapter 4) would imply the \(L\)-regularity of Rademacher series. Since (1) holds for \(X\) being a Rademacher sum with vector coefficients, Theorem 1 would give positive answer to Oleszkiewicz question.

We conclude with an example showing that weak tail domination does not yield any comparison of strong parameters even if the dominated vector has Gaussian distribution.

Example. Let \(F = l_\infty^n\), \(Y = \sum_{i=1}^n g_i e_i\) and \(X = 9(|g_1| + 1) \sum_{i=1}^n \eta_i e_i\), where \(g_i\) are i.i.d. \(\mathcal{N}(0, 1)\) and \(\eta_i\) are i.i.d. r.v.'s with uniform distribution on \([-1, 1]\), independent of \(g_1\).

To show that tails of \(Y\) are weakly dominated by tails of \(X\) it is enough to check that

\[
P(|\langle u, Y \rangle| \geq t) \leq P(|\langle u, X \rangle| \geq t) \text{ for } u \in S^{n-1}, t \geq 0. \tag{3}
\]
Let us fix \( u \in S^{n-1} \). For any \( t > 0 \) we have

\[
P(|\langle u, Y \rangle| \geq t) = P(|g_1| \geq t).
\]

By the Paley-Zygmund inequality,

\[
P\left( \left| \sum_{i=1}^{n} u_i \eta_i \right| \geq \frac{1}{3} \right) = P\left( \left| \sum_{i=1}^{n} u_i \eta_i \right|^2 \geq \frac{1}{3} E \left| \sum_{i=1}^{n} u_i \eta_i \right|^2 \right)
\]

\[
\geq \left(1 - \frac{1}{3}\right)^2 \frac{E \left| \sum_{i=1}^{n} u_i \eta_i \right|^2}{E \left| \sum_{i=1}^{n} u_i \eta_i \right|^4} \geq \frac{4}{27},
\]

thus

\[
P(|\langle u, X \rangle| \geq t) \geq \frac{4}{27} P(3(|g_1| + 1) \geq t) \geq \frac{4}{27} P\left(|g_1| \geq \frac{t}{3}\right).
\]

Using the simple estimate \( 2t \exp\left(-\frac{(2t)^2}{2}\right)/\sqrt{2\pi} \leq P(|g| \geq t) \leq \exp\left(-\frac{t^2}{2}\right) \),

we immediately get (3) for \( t \geq 3 \). For \( 0 \leq t \leq 3 \) we have

\[
P(|\langle u, X \rangle| \leq t) \leq P\left(9 \left| \sum_{i=1}^{n} u_i \eta_i \right| \leq t \right) \leq \frac{\sqrt{2t}}{9} \leq t \frac{P(|g_1| \leq 3)}{3}
\]

\[
\leq P(|g_1| \leq t) = P(|\langle u, Y \rangle| \leq t),
\]

where to get the second inequality we used Ball’s upper bound on cube sections [1]. Hence (3) holds also for \( t \in [0,3] \).

Thus \( Y \preceq_{\omega} Y \). However \( E\|Y\| = E \max_{i \leq n} |g_i| \geq \sqrt{\log n/L} \) and \( E\|X\| \leq 9E(|g_1| + 1) \leq 18 \).

Acknowledgments. The author would like to thank prof. S. Kwapień for suggesting the method used in the proof of Theorem 1.

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