PARABOLIC THEORY AS A HIGH-DIMENSIONAL LIMIT OF ELLIPTIC THEORY

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ABSTRACT. The aim of this article is to show how certain parabolic theorems follow from their elliptic counterparts. This technique is demonstrated through new proofs of five important theorems in parabolic unique continuation and the regularity theory of parabolic equations and geometric flows. Specifically, we give new proofs of an $L^2$ Carleman estimate for the heat operator, and the monotonicity formulas for the frequency function associated to the heat operator, the two-phase free boundary problem, the flow of harmonic maps, and the mean curvature flow. The proofs rely only on the underlying elliptic theorems and limiting procedures belonging essentially to probability theory. In particular, each parabolic theorem is proved by taking a high-dimensional limit of the related elliptic result.

1. INTRODUCTION

Experts have long realized the parallels between elliptic and parabolic theory of partial differential equations. It is well-known that elliptic theory may be considered a static, or steady-state, version of parabolic theory. And in particular, if a parabolic estimate holds, then by eliminating the time parameter, one immediately arrives at the underlying elliptic statement. Producing a parabolic statement from an elliptic statement is not as straightforward. In this article, we demonstrate a method for producing parabolic theorems from their elliptic analogues. Specifically, we show that certain parabolic estimates may be obtained by taking high-dimensional limits of the corresponding elliptic result.

The idea to consider parabolic theory as a high-dimensional limit of elliptic theory was used by Perelman as a motivation for introducing what is now known as the Perelman reduced volume, [14] Section 6. The methods of the proof, as well as the general philosophy that parabolic theory is a high-dimensional limit of elliptic theory, are discussed in the blog of Tao, [20]. Our set-up will be simpler than that of the Ricci flow, and we will be able to use a form of classical probabilistic formulae, essentially going back to Wiener [22], with a slight modification used in [19].

The method of obtaining parabolic theorems by taking high-dimensional limits is demonstrated through five new proofs. The first is a proof of an $L^2$ Carleman estimate for the operator $\Delta + \partial_t$. This Carleman estimate was proved by Escauriaza in [8] and Tataru in [21], with further analysis by Koch and Tataru in [13]. The second new proof, originally proved by Poon in [15], shows that the frequency function associated to the heat equation is monotonically non-decreasing. These two theorems, motivated by their elliptic counterparts, allowed the authors of [8], [15] and [21] to use the established techniques for elliptic theory to prove that strong unique continuation also holds for solutions to the heat equation. This was a major step forward in the theory of unique continuation for parabolic equations. The third new proof is of a monotonicity formula for two-phase free boundary parabolic problems. This formula was proved in [4] by Caffarelli and Kenig in [5] to prove regularity of solutions to parabolic equations and their singular perturbations. The fourth new proof in this article is of a monotonicity formula for the flow of harmonic maps. The original proof is due to Struwe, [18] (and many other proofs since). And the fifth new proof is of a monotonicity formula for mean curvature flow, which was proved by Huisken in [12]. These two theorems were crucial in the development of regularity theory for geometric flows. The parabolic theorems mentioned here were discovered independently, but we show that they in fact follow from their elliptic counterparts in a common way. The starting point of each new proof is a classical formula used in probability together with a related calculation from [19].
The author hopes that the techniques presented in this article may find other applications. In particular, if a certain elliptic result is known to hold in every dimension, then it may be possible to prove the corresponding parabolic result using the ideas presented here.

The article is organized as follows. In Section 2, we develop the connection between the elliptic and parabolic theory by presenting Wiener's calculation from [22] and its variant in [19]. Section 3 contains a collection of statements that will be referred to throughout the article. The $L^2$ parabolic Carleman estimate is proved in Section 4. The frequency function theorem for the heat operator is presented in Section 5. Section 6 contains the monotonicity formulae for two-phase free boundary problems. In Section 7, harmonic maps are introduced and the monotonicity formula is stated and proved. The results for minimal surfaces and mean curvature flow are given in Section 8.

2. Measure Theoretic Details

Within this section, we establish the two main tools of this article, Lemmas 1 and 2. In all subsequent sections, these lemmas allow us to pass from a known elliptic notion to the corresponding parabolic result.

We start with some classical ideas concerning random walks, going back to Wiener [22]. An explanation of these standard ideas is also available in Sverak's notes [19]. Consider $d$ particles, each one moving randomly in one spatial dimension. Let $x_1, x_2, \ldots, x_d$ denote the coordinates of these particles. Rather than imposing a condition on the step size, we instead impose the more universal condition that if each $x_j$ makes $n$ random steps, denoted $y_{j,1}, y_{j,2}, \ldots, y_{j,n}$, then for some fixed $t > 0$

$$|y|^2 = \sum_{j=1}^{d} [y_{j,1}^2 + \ldots + y_{j,n}^2] = 2dt. \quad (1)$$

Assuming that each $x_j$ starts at the origin, after these $n$ steps, the new positions will be

$$x_j = y_{j,1} + y_{j,2} + \ldots + y_{j,n}. \quad (2)$$

To understand the probability law for the events $(y_{1,1}, \ldots, y_{1,n}, \ldots, y_{d,1}, \ldots, y_{d,n})$, assume that the vectors $(y_{1,1}, \ldots, y_{1,n}, \ldots, y_{d,1}, \ldots, y_{d,n})$ are distributed over the $n \cdot d - 1$ dimensional sphere of radius $\sqrt{2dt}$ uniformly with respect to the canonical surface measure. If the surface measure is normalized to have total measure equal to 1, then this surface measure, $\mu_{n,d}^t$, is given by

$$\mu_{n,d}^t = \frac{1}{|S^{n \cdot d - 1}| (2dt)^{\frac{n \cdot d - 1}{2}}} \sigma_{n \cdot d - 1},$$

where $\sigma_{n \cdot d - 1}$ denotes the canonical surface measure of the sphere.

Let $Y_{n,d} = \{(y_{1,1}, \ldots, y_{1,n}, \ldots, y_{d,1}, \ldots, y_{d,n})\} = \mathbb{R}^{n \cdot d}$, $X_{d} = \{(x_1, x_2, \ldots, x_d)\} = \mathbb{R}^d$, $T = \{t\} = \mathbb{R}$. Define a function

$$f_{n,d} : Y_{n,d} \to X_{d}$$

by

$$f_{n,d} (y_{1,1}, \ldots, y_{1,n}, \ldots, y_{d,1}, \ldots, y_{d,n}) = (y_{1,1} + \ldots + y_{1,n}, \ldots, y_{d,1} + \ldots + y_{d,n}). \quad (3)$$

In other words, for each $j = 1, \ldots, d$, equation (2) holds.

We wish to compute the push-forward of $\mu_{n,d}^t$ by $f_{n,d}$. Let $\nu_{n,d}^t = f_{n,d}^\# \mu_{n,d}^t$. For simplicity, we may replace $f_{n,d}$ above with

$$f_{n,d} (y_{1,1}, y_{1,2}, \ldots, y_{1,n}, \ldots, y_{d,1}, y_{d,2}, \ldots, y_{d,n}) = (\sqrt{n} y_{1,1}, \sqrt{n} y_{2,1}, \ldots, \sqrt{n} y_{d,1}),$$

since the two maps are related by an orthogonal transformation that leaves the measure unchanged. Therefore, we write $x_j = \sqrt{n} y_{j,1}$ in what follows. The push-forward is computed in two steps. First, we push-forward the measure $\mu_{n,d}^t$ by the projection

$$f_{n,d} (y_{1,1}, y_{1,2}, \ldots, y_{1,n}, \ldots, y_{d,1}, y_{d,2}, \ldots, y_{d,n}) \mapsto (y_{1,1}, y_{2,1}, \ldots, y_{d,1}),$$
then we dilate by a factor of $\sqrt{n}$. A computation shows that the projection gives

$$\frac{1}{|S^{n-1} (2dt)^\frac{d-1}{2}|} |S^{n-1-d} (2dt)^\frac{d-1}{2}| \left( 1 - \frac{y^2_{1,1} + \ldots + y^2_{d,1}}{2dt} \right)^{\frac{n-d-2}{2}} dy_{1,1} \ldots dy_{d,1}.$$

Using $x_j = \sqrt{n} y_{j,1}$, we see that

$$V_{n,d}' = \frac{|S^{n-1-d} (2nt)|}{|S^{n-1} (2nt)^\frac{d-1}{2}|} \left( 1 - \frac{x^2_1 + \ldots + x^2_d}{2nt} \right)^{\frac{n-d-2}{2}} dx_1 \ldots dx_d = \frac{|S^{n-1-d} (2nt)|}{|S^{n-1} (2nt)^\frac{d-1}{2}|} \left( 1 - \frac{|x|^2}{2nt} \right)^{\frac{n-d-2}{2}} dx,$$

where $dx = dx_1 dx_2 \ldots dx_d$.

**Remark.** At this point, we notice that the above expression is not necessarily well-defined when the argument is negative, or when $2nt < |x|^2$. But notice that by (2), standard inequalities, and (1),

$$|x|^2 = \sum_{j=1}^d x_j^2 = \sum_{j=1}^d (y_{j,1} + \ldots + y_{j,n})^2 \leq n \sum_{j=1}^d (y_{j,1}^2 + \ldots + y_{j,n}^2) = n \cdot 2nt.$$

Thus, the argument is always non-negative and the expression is well-defined for all $n \in \mathbb{N}$.

Using the definition of push-forward and some simplifications, we arrive at the following classical result.

**Lemma 1.** If $\phi: X_n \to \mathbb{R}$ is a continuous, compactly supported function, then

$$\frac{1}{|S^{n-1}|} \int_{\{|y| = \sqrt{2nt}\}} \phi (f_{n,d} (y)) (2dt)^\frac{d-1}{2} \sigma_{n-1} = \frac{|S^{n-1-d} (2nt)|}{|S^{n-1} (2nt)^\frac{d-1}{2}|} \int_{X_d} \phi (x) (2nt)^\frac{d-1}{2} \left( 1 - \frac{|x|^2}{2nt} \right)^{\frac{n-d-2}{2}} dx.$$

(4)

Following Sverak in [19], we now broaden this viewpoint so that $t$ is a parameter, instead of a fixed constant. Define a function

$$F_{n,d} : Y_{n,d} \to X_d \times T$$

by

$$F_{n,d} (y_{1,1}, \ldots , y_{1,n}, \ldots , y_{d,1}, \ldots , y_{d,n}) = \left( y_{1,1} + \ldots + y_{1,n}, \ldots , y_{d,1} + \ldots + y_{d,n}, |y|^2 / 2d \right)$$

(5)

In other words, (1) and (2) both hold. We now seek a measure $\mu_{n,d}$ on $Y_{n,d}$ with the property that $F_{n,d}$ projects the slices $\{ y : |y|^2 = 2dt \}$ onto the measures $V_{n,d}'$. That is,

$$F_{n,d#} (\mu_{n,d}) = \int_0^\infty V_{n,d}' dt$$

From here we have

$$\mu_{n,d} = \frac{1}{d |S^{n-1}| |y|^{n-d-2}} dy$$

Simplifications lead to the following lemma.

**Lemma 2.** If $\phi : X_n \times T \to \mathbb{R}$ is a continuous, compactly supported function, then

$$\frac{1}{d |S^{n-1}|} \int_{Y_{n,d}} \phi (F_{n,d} (y)) \frac{1}{|y|^{n-d-2}} dy = \frac{|S^{n-1-d} (2nt)|}{|S^{n-1} (2nt)^\frac{d-1}{2}|} \int_0^\infty \int_{X_d} \phi (x,t) (2nt)^\frac{d-1}{2} \left( 1 - \frac{|x|^2}{2nt} \right)^{\frac{n-d-2}{2}} dx dt$$

(6)
3. Preliminaries

Here we will collect facts that will be used throughout the article. To start, we state a lemma that relates the derivatives of \( u \) and \( v \), whenever \( u \) and \( v \) satisfy the relation \( v(y) = u(F_{n,d}(y)) \). This lemma will be referred to throughout the article. The proof of each statement follows from the chain rule.

**Lemma 3.** Let \( Z \) be a space and let \( u : X_d \times T \to Z \). Suppose \( v : Y_{n,d} \to Z \) is such that \( v(y) = u(F_{n,d}(y)) \). Then the following hold:

\[
\frac{\partial v}{\partial y_{i,j}} = \frac{\partial u}{\partial x_i} \cdot \frac{y_{i,j}}{d} + \frac{\partial u}{\partial t} \quad (7)
\]

\[
\frac{\partial^2 v}{\partial y_{i,j} \partial y_{k,l}} = \frac{\partial^2 u}{\partial x_i \partial x_k} \cdot \frac{y_{i,j}}{d} + \frac{\partial^2 u}{\partial x_i \partial t} \cdot \frac{y_{i,j} y_{k,l}}{d^2} + \frac{\partial^2 u}{\partial x_k \partial t} \cdot \frac{y_{i,j}}{d^2} + \frac{\partial^2 u}{\partial t^2} \cdot \frac{\partial u}{\partial t} \quad (8)
\]

\[
\Delta v = n \left( \Delta u + \frac{\partial u}{\partial t} \right) + \frac{2}{d} (x,t) \cdot \nabla_{(x,t)} \left( \frac{\partial u}{\partial t} \right) \quad (9)
\]

\[
y \cdot \nabla v = x \cdot \nabla u + 2t \frac{\partial u}{\partial t} \quad (10)
\]

\[
|\nabla v|^2 = n |\nabla u|^2 + \frac{2}{d} \left( (x,t) \cdot \nabla_{(x,t)} u \right) \frac{\partial u}{\partial t} \quad (11)
\]

The following limits will be used repeatedly. By Stirling’s formula and limit laws, it can be shown that

\[
\lim_{n \to \infty} (2n)^{-d} \left| \frac{S_n^d - 1 - d}{S_n^d - 1} \right| = \left( \frac{1}{4\pi} \right)^{d/2}, \quad (12)
\]

\[
\lim_{n \to \infty} \left( 1 - \frac{|x|^2}{2ndt} \right)^{-d} = \exp \left( - \frac{|x|^2}{4t} \right). \quad (13)
\]

4. Carleman Estimates

Within this section, we use an elliptic Carleman estimate to prove its parabolic analogue. The main tool used in this proof is Lemma 2.

The following elliptic Carleman estimate is the \( L^2 \) case of Theorem 1 from \cite{3}. The original theorem was used to establish unique continuation properties of functions that satisfy \( |\Delta u| \leq |V| |u| \), for \( u \in H^2_{\text{loc}}(\Omega), V \in L^w_{\text{loc}}(\Omega) \), where \( w > \frac{N}{2} \), and \( \Omega \subseteq \mathbb{R}^N \) is open and connected.

**Theorem 1** \cite{3}, Theorem 1). For any \( \tau \in \mathbb{R} \) and all \( v \in H^{2,2}_{\text{c}}(\mathbb{R}^N \setminus \{0\}) \), the following inequality holds

\[
\left\| \left| y \right|^{-\tau + 2} \Delta v \right\|_{L^2(\mathbb{R}^N)} \geq c(\tau,N) \left\| \left| y \right|^{-\tau} v \right\|_{L^2(\mathbb{R}^N)}, \quad (14)
\]

where

\[
c(\tau,N) = \inf_{\ell \in \mathbb{Z}_{\geq 20}} \left( \frac{N}{2} + \ell + \tau - 2 \right) \left( \frac{N}{2} + \ell - \tau \right).
\]

**Remark.** In order for this theorem to be meaningful, we must ensure that \( c(\tau,N) \geq 0 \). We see that this holds if \( \tau \notin \pm \left( \frac{N}{2} - 2 \right) \). If we are in this situation, then \( c(\tau,N) \geq C\delta N \), where \( \delta = \text{dist} \left( \tau, \pm \left( \frac{N}{2} - 2 \right) \right) \).

The following parabolic Carleman estimate is the \( L^2 \) version of Theorem 1 from \cite{8}. The original theorem was used to prove strong unique continuation of solutions to the heat equation.
Theorem 2 ([8], Theorem 1). Let \( d \geq 1 \). Let \( \alpha \in \mathbb{R} \) be such that \( \beta = 2\alpha - \frac{d}{2} - 1 > 0 \) is not an integer. Then there is a constant \( C \) depending only on \( d \) and \( \varepsilon = \text{dist} (\beta, \mathbb{Z}_{\geq 0}) \) such that the inequality

\[
\int_0^\infty \int_{\mathbb{R}^d} t^{-2\alpha} e^{-\frac{\alpha}{2t}} |u|^2 \, dx \, dt \leq C (d, \varepsilon) \int_0^\infty \int_{\mathbb{R}^d} t^{-2\alpha+2} e^{-\frac{\alpha}{2t}} |\Delta u + \partial_t u|^2 \, dx \, dt, \tag{15}
\]

holds for every \( u \in C_0^\infty (\mathbb{R}^{d+1}_+ \setminus \{(0,0)\}) \).

We now show that Theorem 2 follows from the elliptic result, Theorem 1 and the results of Section 3.

Proof. Let \( u \in C_0^\infty (\mathbb{R}^{d+1}_+ \setminus \{(0,0)\}) \). Then, \( u : X_d \times T \to \mathbb{R} \). For every \( n \in \mathbb{N} \), let \( v_n : Y_{n,d} \to \mathbb{R} \) satisfy

\[
v_n (y) = u (F_{n,d} (y)).
\]

Since \( u \in C_0^\infty \), then \( v_n \) and \( \Delta v_n \) satisfy the hypotheses of Lemma 2 so

\[
\int_{Y_{n,d}} |v_n (y)|^2 |y|^{-2\varepsilon} \, dy = \int_{Y_{n,d}} |u (F_{n,d} (y))|^2 |y|^{n-d-2-2\varepsilon} \, dy
\]

\[
= d |S^{n-d-1-d}| \int_0^\infty \int_{X_d} |u (x,t)|^2 (2dt)^{\frac{d-1-2\varepsilon}{d} - 2\varepsilon} x \left( 1 - \frac{|x|^2}{2ndt} \right)^{\frac{nd-2d-2}{2}} \, dx \, dt
\]

\[
= \frac{1}{2} (2d)^{\frac{d-2}{2} - \varepsilon} n^{\frac{d}{2} + 2} |S^{n-d-1-d}| \int_0^\infty \int_{X_d} |u (x,t)|^2 \left( t^{\frac{d}{2} + 1 - 2\varepsilon} (2nt)^{\frac{d-2d-2}{2}} \right) \left( 1 - \frac{|x|^2}{2ndt} \right)^{\frac{nd-2d-2}{2}} \, dx \, dt,
\]

and

\[
\int_{Y_{n,d}} |\Delta v_n (y)|^2 |y|^{4-2\varepsilon} \, dy
\]

\[
= n^2 d |S^{n-d-1-d}| \int_0^\infty \int_{X_d} |\Delta u + \partial_t u + \frac{2}{n-d} \cdot \nabla (x,t) \partial_t u|^2 (2dt)^{\frac{d+1}{2} - 2\varepsilon} \left( 1 - \frac{|x|^2}{2ndt} \right)^{\frac{nd-2d-2}{2}} \, dx \, dt
\]

\[
= \frac{1}{2} (2d)^{\frac{d-2}{2} + 2 - \varepsilon} n^{\frac{d}{2} + 2} |S^{n-d-1-d}| \int_0^\infty \int_{X_d} |\Delta u + \partial_t u + \frac{2}{n-d} \cdot \nabla (x,t) \partial_t u|^2 \left( t^{\frac{d}{2} + \frac{d-2d-2}{2}} \right) \left( 1 - \frac{|x|^2}{2ndt} \right)^{\frac{nd-2d-2}{2}} \, dx \, dt,
\]

where the second line is due to (2) from Lemma 3.

By Theorem 1 and the remark that follows,

\[
\int_{Y_{n,d}} |\Delta v_n (y)|^2 |y|^{4-2\varepsilon} \, dy \geq C \delta_n^2 n^2 \int_{Y_{n,d}} |v_n (y)|^2 |y|^{-2\varepsilon} \, dy,
\]

where \( \delta_n = \text{dist} (\tau_n, \pm (\mathbb{N} + \frac{d-2}{2})) \).

Combining (16), (17), and (18), we see that

\[
\frac{1}{2} (2d)^{\frac{d-2}{2} + 2 - \varepsilon} n^{\frac{d}{2} + 2} \int_0^\infty \int_{X_d} |\Delta u + \partial_t u + \frac{2}{n-d} \cdot \nabla (x,t) \partial_t u|^2 \left( t^{\frac{d}{2} + \frac{d-2d-2}{2}} \right) \left( 1 - \frac{|x|^2}{2ndt} \right)^{\frac{nd-2d-2}{2}} \, dx \, dt
\]

\[
\geq C \delta_n^2 n^{\frac{1}{2}} (2d)^{\frac{d-2d-2}{2}} n^{\frac{d}{2} + 2} \int_0^\infty \int_{X_d} |u (x,t)|^2 \left( t^{\frac{d}{2} + \frac{d-2d-2}{2}} \right) \left( 1 - \frac{|x|^2}{2ndt} \right)^{\frac{nd-2d-2}{2}} \, dx \, dt.
\]
Setting \( 2\alpha = \tau_n - \frac{n \cdot d - 2}{2} \) and simplifying gives

\[
C \delta_0^2 \int_0^\infty \int_{X_d} |u(x, t)|^2 t^{-2\alpha} \left( 1 - \frac{|x|^2}{2ndt} \right) \frac{d^d x}{d^d x} dt \leq \int_0^\infty \int_{X_d} |\Delta u + \partial_t u|^2 t^{-2\alpha + 2} \left( 1 - \frac{|x|^2}{2ndt} \right) \frac{d^d x}{d^d x} dt
\]

(19)

\[
+ \frac{4}{n^2 d^2} \int_0^\infty \int_{X_d} [(x, t) \cdot \nabla_{(x, t)} u] \frac{d^d x}{d^d x} dt
\]

If \( \alpha \in \mathbb{R} \) is such that \( \beta = 2\alpha - d - 1 > 0 \) is not an integer, then \( \tau_n > \frac{n \cdot d}{2} \). Therefore,

\[
\delta_n \geq \text{dist} \left( \tau_n, \pm \left( \mathbb{N} + \frac{n \cdot d - 2}{2} \right) \right) = \text{dist} \left( \tau_n - \frac{n \cdot d}{2}, \mathbb{Z}_+ \right) = \text{dist} (\beta, \mathbb{Z}_+) = \varepsilon.
\]

(20)

We now take the limit as \( n \to \infty \) in inequality (19). Using the fact that \( u \) is smooth and compactly supported, along with observations (20), (12), and (13), we see that

\[
\frac{Ce^2}{8d^2} \int_0^\infty \int_{X_d} |u(x, t)|^2 t^{-2\alpha} e^{-\frac{|x|^2}{2n^2 d^2}} \frac{d^d x}{d^d x} dt \leq \int_0^\infty \int_{X_d} |\Delta u + \partial_t u|^2 t^{-2\alpha + 2} e^{-\frac{|x|^2}{n^2 d^2}} \frac{d^d x}{d^d x} dt,
\]

as required.

\[
5. \ \text{Frequency Functions}
\]

In this section, we explore the non-trivial connection between frequency functions for solutions to elliptic and parabolic equations. In particular, we use the monotonicity result for harmonic functions in conjunction with Lemma 1 to prove the corresponding monotonicity result for solutions to the heat equation.

In [10] and [11], the authors study the properties of frequency functions and use their results to prove a strong unique continuation theorem for solutions to elliptic partial differential equations. To do this, they generalize the following result due to Almgren from [1] for frequency functions associated to harmonic function.

**Theorem 3.** Let \( v : \mathbb{R}^N \to \mathbb{R} \). Set

\[
H(r; v) = \int_{\partial B_r} |v(y)|^2 dS(y) \quad \text{and} \quad D(r; v) = \int_{B_r} |\nabla v(y)|^2 dy
\]

\[
N(r; v) = \frac{r D(r)}{H(r)}.
\]

If \( \Delta v = 0 \), then \( N(r; v) \) is monotonically non-decreasing in \( r \).

We use this elliptic result above to reprove the parabolic version from [15], restated using the notation from [8]. This result was a crucial tool in the proof of strong unique continuation of the heat equation.

**Theorem 4 ([15]).** Let \( u : \mathbb{R}^d \times (0, T) \to \mathbb{R} \). Set \( G_t(x) = t^{-d/2} \exp \left( -\frac{|x|^2}{4t} \right) \) and define

\[
\mathcal{H} (t; u) = \int_{\mathbb{R}^d} |u(x, t)|^2 G_t(x) dx
\]

\[
\mathcal{D} (t; u) = \int_{\mathbb{R}^d} |\nabla u(x, t)|^2 G_t(x) dx
\]

\[
\mathcal{N} (t; u) = \frac{t \mathcal{D} (t)}{\mathcal{H} (t)}.
\]

If \( \Delta u + \frac{\partial u}{\partial t} = 0 \), then \( \mathcal{N} (t; u) \) is monotonically non-decreasing in \( t \).

**Proof.** Let \( u \) be a solution to \( \Delta u + \frac{\partial u}{\partial t} = 0 \) in \( \mathbb{R}^d \times (0, T) \). Then, \( u : X_d \times (0, T) \to \mathbb{R} \). For every \( n \in \mathbb{N} \), let \( u_n \) be a solution to

\[
\Delta u_n + \frac{\partial u_n}{\partial t} = -\frac{2}{n \cdot d} (x, t) \cdot \nabla x \cdot \frac{\partial u_n}{\partial t}.
\]

(21)
The sequence $\{u_n\}_{n \in \mathbb{N}}$ is constructed so that for each $t \in (0, T)$, $\lim_{n \to \infty} u_n(\cdot, t) = u(\cdot, t)$ in $H^1(\mathbb{R}^d, G_t(x) \, dx)$. We need to be able to apply Lemma 1 and we need the integrals involving $u_n$ to approach those for $u$.

Set $\tilde{Y}_{n,d} = F_{n,d}^{-1}(X_d \times (0, T)) \subset Y_{n,d}$. Let $v_n : \tilde{Y}_{n,d} \to \mathbb{R}$ be given by

$$v_n(y) = u_n(F_{n,d}(y)).$$

Then by (9) from Lemma 3 and (26),

$$\Delta v_n = n \left( \Delta u_n + \frac{\partial u_n}{\partial t} \right) + \frac{2}{d} \langle x, t \rangle \cdot \nabla (\langle x, t \rangle) \left( \frac{\partial u_n}{\partial t} \right) = 0.$$ 

Therefore, each $v_n$ is a harmonic function, so we may apply Theorem 3. Furthermore, each $v_n$ inherits the smoothness properties of $u_n$, so we may apply Lemma 1.

By Lemma 1

$$H \left( \sqrt{2dt}; v_n \right) = \int_{|y| = \sqrt{2dt}} |v_n(y)|^2 \sigma_{n,d-1}$$

$$= (2dt)^{n-d-1} \int_{|y| = \sqrt{2dt}} |u_n(f_{n,d}(y))|^2 (2dt)^{-\frac{n-d-1}{2}} \sigma_{n,d-1}$$

$$= (2dt)^{\frac{n-d}{2}} \left| S^{n-d-1} \right| \int_{X_d} |u_n(x,t)|^2 (2ndt)^{-\frac{n}{2}} \left( 1 - \frac{|x|^2}{2ndt} \right)^{\frac{n-d-1}{2}} dx. \quad (22)$$

By the divergence theorem and Lemma 1

$$D \left( \sqrt{2dt}; v_n \right) = \int_{|y| \leq \sqrt{2dt}} |\nabla v_n|^2 dY$$

$$= \frac{1}{2\sqrt{2dt}} \int_{|y| = \sqrt{2dt}} v_n(y) \cdot \nabla v_n(y) \sigma_{n,d-1}$$

$$= \frac{(2dt)^{\frac{n}{2}}}{2d} \left| S^{n-1-d} \right| \int_{X_d} u_n(x,t) \left( \frac{x}{2t} \cdot \nabla u_n + \partial_t u_n \right) (2ndt)^{-\frac{n}{2}} \left( 1 - \frac{|x|^2}{2ndt} \right)^{-\frac{n-d-1}{2}} dx,$$

where the last equality follows from (10) in Lemma 3.

Therefore, by Theorem 3 with $r = \sqrt{2dt}$, we may simplify to see that

$$N \left( \sqrt{2dt}; v_n \right) = \frac{\sqrt{2dt}D \left( \sqrt{2dt}; v_n \right)}{H \left( \sqrt{2dt}; v_n \right)}$$

$$= \frac{t \int_{X_d} u_n(x,t) \left( \frac{x}{2t} \cdot \nabla u_n + \partial_t u_n \right) t^{-\frac{d}{2}} \left( 1 - \frac{|x|^2}{2ndt} \right)^{-\frac{n-d-1}{2}} dx}{\int_{X_d} |u_n(x,t)|^2 t^{-\frac{d}{2}} \left( 1 - \frac{|x|^2}{2ndt} \right)^{-\frac{n-d-1}{2}} dx} \quad (23)$$

is non-decreasing in $t$.

We now take the limit as $n \to \infty$ in (23). Using the properties of $\{u_n\}$ and (13), we see that

$$\lim_{n \to \infty} \int_{X_d} |u_n(x,t)|^2 t^{-\frac{d}{2}} \left( 1 - \frac{|x|^2}{2ndt} \right)^{-\frac{n-d-1}{2}} dx$$

$$= \int_{X_d} |u(x,t)|^2 G_t(x) \, dx,$$
\[
\lim_{n \to \infty} \int_{X_d} u_n(x,t) \left( \frac{x}{2t} \cdot \nabla u_n + \partial_t u_n \right) t^{-\frac{d}{2}} \left( 1 - \frac{|x|^2}{2nt} \right) dx
= \int_{X_d} u(x,t) \left( \frac{x}{2t} \cdot \nabla u + \partial_t u \right) t^{-\frac{d}{2}} \exp \left( -\frac{|x|^2}{4t} \right) dx
= \int_{X_d} |\nabla u(x,t)|^2 G_t(x) dx,
\]
where we used integration by parts and that \( \nabla G_t(x) = -\frac{x}{2t} G_t(x) \) to reach the final equality. Therefore,
\[
\lim_{n \to \infty} N \left( \sqrt{2d/t}; v_n \right) = \frac{t \int_{X_d} |\nabla u(x,t)|^2 G_t(x) dx}{\int_{X_d} |u(x,t)|^2 G_t(x) dx} = \mathcal{N}(t; u).
\]
Since each \( N \) was non-decreasing in \( t \), the conclusion follows. \( \square \)

6. FREE BOUNDARY PROBLEMS

In [2], the authors study two-phase free boundary elliptic problems. The monotonicity formula presented below is a key tool in their work. This formula is used to establish Lipschitz continuity of minimizers, to identify blow-up limits, and to prove differentiability of the free boundary when \( N = 2 \).

**Theorem 5** ([2], Lemma 5.1). Let \( v_1, v_2 \) be two non-negative continuous harmonic functions defined in \( B_R \), the ball of radius \( R \) in \( \mathbb{R}^N \). Assume that \( v_1 v_2 \equiv 0 \) and \( v_1(0) = v_2(0) = 0 \). Then for all \( r < R \),
\[
\phi(r; v) = \frac{1}{r^d} \left( \int_{B_r} |y|^{2-N} |\nabla v_1(y)|^2 dy \right) \left( \int_{B_r} |y|^{2-N} |\nabla v_2(y)|^2 dy \right),
\]
is monotonically non-decreasing in \( r \).

Motivated by its application to the regularity theory of two-phase free boundary elliptic problems, the parabolic analogue of this formula was proved by Caffarelli in [4]. This two-phase monotonicity formula was extended in [5] and used to prove uniform Lipschitz estimates for solutions to singular perturbations of variable coefficient parabolic free boundary problems, where the linear parabolic operators are second-order divergence form with Dini top order coefficients.

**Theorem 6** ([4], Theorem 1). Let \( u_1, u_2 \) be two disjoint non-negative functions defined in \( \mathbb{R}^d \times [0,T] \). Assume that \( \Delta u_i + \partial_t u_i = 0 \) for \( i = 1,2 \), \( u_1 u_2 \equiv 0 \) and \( u_1(0,0) = u_2(0,0) = 0 \). Assume also that \( u_1 \) and \( u_2 \) have moderate growth at infinity. Let \( G_t(x) = t^{-\frac{d}{2}} \exp \left( -\frac{|x|^2}{4t} \right) \). Then for all \( \tau < T \),
\[
\Phi(\tau; u) = \frac{1}{\tau^d} \left( \int_0^\tau \int_{\mathbb{R}^d} |\nabla u_1|^2 G_t(x) dx dt \right) \left( \int_0^\tau \int_{\mathbb{R}^d} |\nabla u_2|^2 G_t(x) dx dt \right),
\]
is monotonically non-decreasing in \( \tau \).

We will reprove this theorem using only Theorem 5 and the tools developed in the early sections of this article.

**Proof:** Let \( u_1, u_2 \) be as in the statement of the theorem. For every \( n \in \mathbb{N} \), let \( u_{i,n} \geq 0 \) be a solution to
\[
\Delta u_{i,n} + \frac{\partial u_{i,n}}{\partial t} = -\frac{2}{n} \frac{d}{8} (x,t) \cdot \nabla x \frac{\partial u_{i,n}}{\partial t},
\]
The sequence \( \{u_{i,n}\}_{n \in \mathbb{N}} \) is constructed so that for each \( t \in (0, T) \), \( \lim_{n \to \infty} u_{i,n}(\cdot, t) = u_i(\cdot, t) \) in \( H^1(\mathbb{R}^d, G_t(x) \, dx) \).

We also impose the conditions that \( u_{1,n}u_{2,n} \equiv 0 \) and \( u_{1,n}(0, 0) = u_{2,n}(0, 0) = 0 \) for each \( n \in \mathbb{N} \).

As in previous proofs, if we let \( v_n : F_{n,d} \to \mathbb{R} \) be given by

\[
v_{i,n}(y) = u_{i,n}(F_{n,d}(y)),
\]

then each \( v_{i,n} \) is a harmonic function. Furthermore, \( v_{1,n} \geq 0, v_{2,n} \geq 0, v_{1,n}v_{2,n} \equiv 0 \) and \( v_{1,n}(0) = v_{2,n}(0) = 0 \) for each \( n \in \mathbb{N} \). Since each pair \( v_{i,n}, v_{2,n} \) satisfies the hypotheses of Theorem 5 then

\[
\varphi(\tau; v_n) := \frac{2^{d+2n-2d}d}{|S^{m-1}|} \Phi \left( \sqrt{2d \tau}, v_n \right)
\]
is monotonically increasing in \( \tau \) for each \( n \).

By a corollary to Lemma 2 we have

\[
\frac{1}{d} \int_{B_{\varphi \sqrt{\tau}}} \varphi(F_{n,d}(y)) \frac{1}{|y|^{n-d-2}} \, dy = |S^{m-1}| \int_0^\tau \int_{X_d} \varphi(x, t) (2nt)^{-\frac{d}{2}} \left( 1 - \frac{|x|^2}{2nt} \right)^{\frac{n-d-2}{2}} \, dx \, dt.
\]

Using (27) and (11), we see that

\[
\varphi(\tau; v_n) = \frac{1}{\tau^2} \frac{(2nt)^d}{(n \cdot d |S^{n-1}|)} \left( \int_{B_{\varphi \sqrt{\tau}}} |\nabla v_{1,n}(y)|^2 \frac{1}{|y|^{n-d-2}} \, dy \right) \left( \int_{B_{\varphi \sqrt{\tau}}} |\nabla v_{2,n}(y)|^2 \frac{1}{|y|^{n-d-2}} \, dy \right)
\]

\[
= \frac{1}{\tau^2} \prod_{i=1}^2 \left( \int_0^\tau \int_{X_d} |\nabla u_{i,n}|^2 + \frac{2}{n \cdot d} \left[ (x, t) \cdot \nabla_{(x, t)} u_{i,n} \right] \frac{\partial u_{i,n}}{\partial t} \right) t^{-\frac{d}{2}} \left( 1 - \frac{|x|^2}{2nt} \right)^{\frac{n-d-2}{2}} \, dx \, dt.
\]

Taking the limit, we see that

\[
\lim_{n \to \infty} \varphi(\tau; v_n) = \Phi(\tau; u)
\]
is also monotonically increasing in \( \tau \).

\[
\square
\]

7. Harmonic Maps into Spheres

In this section, we use a monotonicity result for harmonic maps to derive the proof of the parabolic analogue. Before stating the results, we must introduce some notation. We borrow the notation and introductory statements from [9].

Let \( m, N \geq 2 \). Let \( U \subset \mathbb{R}^N \) be smooth, and \( S^{m-1} \) denote the unit sphere in \( \mathbb{R}^m \). A function \( v = (v^1, \ldots, v^m) \) in the Sobolev space \( H^1(U; \mathbb{R}^m) \) belongs to the space \( H^1(U; S^{m-1}) \) if \( |v| = 1 \) almost everywhere.

**Definition 1.** A function \( v \in H^1(U; S^{m-1}) \) is a weakly harmonic mapping of \( U \) into \( S^{m-1} \) provided

\[
-\Delta v = |Dv|^2 v
\]

holds weakly in \( U \). That is, for every test function \( w = (w^1, \ldots, w^m) \in H^1_0(U; \mathbb{R}^m) \cap L^\infty(U; \mathbb{R}^m) \), we have

\[
\int_U Dv : Dw \, dx = \int_U |Dv|^2 v \cdot w \, dx,
\]

where we use the notation

\[
Dv = \left( \left. \frac{\partial v^j}{\partial x_k} \right|_{1 \leq i \leq m, 1 \leq k \leq N} \right)
\]

\[
Dv : Dw = \frac{\partial v^j}{\partial x_k} \frac{\partial w^j}{\partial x_k} \quad |Dv|^2 = Dv : Dv.
\]
Now let $g : \partial U \to S^{m-1}$ be a given smooth function and set
\[ \mathcal{A} = \{ w \in H^1(U; S^{m-1}) : w = g \text{ on } \partial U \text{ in the trace sense} \}. \]

Then (28) is the Euler-Lagrange equation for the variational problem of minimizing the Dirichlet energy
\[ I[w] = \int_U |Dw|^2 \, dx \]
over all $w \in \mathcal{A}$. If $v$ is a minimizer of $I[\cdot]$ over $\mathcal{A}$, then $v$ satisfies (29) and the following equality
\[ \int_U |Dv|^2 (\text{div } \zeta) - \frac{\partial v}{\partial x^i} \frac{\partial v}{\partial x^j} \frac{\partial \zeta^k}{\partial x^i} \, dx = 0 \]
for every vector field $\zeta = (\zeta^1, \ldots, \zeta^d) \in C^1_0(U; \mathbb{R}^d)$.

**Definition 2.** A function $v \in H^1(U; S^{m-1})$ is said to be a weakly stationary harmonic map from $U$ into the sphere $S^{m-1}$ if $u$ satisfies (29) and (30) for all test functions $w$ and $\zeta$ as above.

One way to understand this definition is that (29) states that $v$ is stationary with respect to the variations of the target $S^{m-1}$, while (30) states that $v$ is stationary with respect to variations of the domain $U$. Note that if $v$ is smooth, then (30) is an immediate consequence of (29) by taking $w = Du \cdot \zeta$.

The following is the monotonicity property for weakly stationary harmonic maps. For generalizations and important applications to regularity theory, see [16],[17]. The presentation here is from [9].

**Theorem 7 ([9]).** Suppose $x \in U \subset \mathbb{R}^N$ and $R > 0$ is such that $B(x, R) \subset U$. For all $r \in (0, R)$, if $v$ is a weakly stationary harmonic map from $U$ into $S^{m-1}$, then the quantity
\[ \phi(r; v) = \frac{1}{r^{N-2}} \int_{B(x, r)} |Dv(y)|^2 \, dy \]
is monotonically non-decreasing in $r$.

To understand that parabolic analogue, we now introduce the evolution of harmonic maps. The presentation is based on Struwe’s paper [18]; however, it is simplified since we only focus on targets that are spheres.

Let $u : \mathbb{R}^d \times \mathbb{R} \to S^{m-1}$ be a solution to
\[ \partial_t u + \Delta u + |Du|^2 u = 0. \]
(32)

We say that a map $u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^m$ is regular if $u$ and $Du$ are uniformly bounded, and $\frac{\partial u}{\partial t}$, $D^2 u$ belong to $L^p_{loc}$ for all $p < \infty$. We now have enough notation to state a version of the parabolic analogue of Theorem 7. The statement has been reformulated for our purposes. This result was applied to regularity theory of geometric flows.

**Theorem 8 ([18], Lemma 3.2).** Let $u : \mathbb{R}^d \times [0, T] \to S^{m-1}$ be a regular solution to (32) with $|Du(x, t)| \leq c < \infty$ uniformly. Let $G_t(x) = \frac{1}{t^{d/2}} \exp \left( -\frac{|x|^2}{4t} \right)$. Then the function
\[ \Phi(t; u) = t \int_{\mathbb{R}^d} |Du(x)|^2 G_t(x) \, dx \]
is monotonically non-decreasing in $t$.

**Proof.** By assumption, $u : X_d \times [0, T] \to S^{m-1}$. For every $n \in \mathbb{N}$, let $u_n : X_d \times [0, T] \to S^{m-1}$ be a smooth solution to
\[ \Delta u_n + \partial_t u_n + |Du_n|^2 u_n = -\frac{2}{n \cdot d} \left[ (x, t) \cdot \nabla_{(x,t)} \left( \frac{\partial u_n}{\partial t} \right) + \left( (x, t) \cdot \nabla_{(x,t)} u_n \right) \cdot \frac{\partial u_n}{\partial t} \right] u_n \]
(33)
The sequence \( \{u_n\}_{n \in \mathbb{N}} \) is constructed so that for each \( t \in [0, T] \), \( \lim_{n \to \infty} u_n(\cdot, t) = u(\cdot, t) \) in \( H^1(\mathbb{R}^d, G_t(x) \, dx) \).

Let \( v_n : \bar{Y}_{n,d} \rightarrow S^{m-1} \) be given by

\[
v_n(\gamma) = u_n(f_{n,d}(\gamma)),
\]

where \( \bar{Y}_{n,d} = F_{n,d}^{-1}(\mathcal{X}_d \times [0, T]) \subset Y_{n,d} \). Then by (12) and (11) from Lemma 3 along with (33), we see that

\[
\Delta v_n + |Dv_n|^2 v_n = n \left( \Delta u_n + \frac{\partial u_n}{\partial t} + |Du_n|^2 u_n \right) + \frac{2}{d} \left( (x, t) \cdot \nabla_{(x,t)} \left( \frac{\partial u_n}{\partial t} \right) + ((x, t) \cdot \nabla_{(x,t)} u_n) \cdot \frac{\partial u_n}{\partial t} u_n \right)
\]

\[
= 0.
\]

Thus, each \( v_n \) is a solution to (28). Since each \( u_n \) is smooth, then \( v_n \) is a weakly stationary harmonic map.

Now let \( \zeta_h(y) = \mu_h(|y|) y \), where

\[
\mu_h(s) = \begin{cases} 
1 & s \leq r \\
1 + \frac{r-s}{h} & r \leq s \leq r+h \\
0 & r \geq r+h
\end{cases}
\]

Plugging \( \zeta_h \) into (30) with \( v_n \) and letting \( h \to 0^+ \), we see that

\[
(n \cdot d - 2) \int_{B(0,r)} |Dv_n|^2 \, dy = r \int_{\partial B(r,0)} |Dv_n|^2 \sigma_{n,d-1} - \frac{2}{r} \int_{\partial B(0,r)} |y \cdot \nabla v_n|^2 \sigma_{n,d-1}.
\]

For each \( n \in \mathbb{N} \), let

\[
\psi_n(t) = \frac{(2d)^{d/2}}{2n^2 |S^{d-1}|} \phi \left( \frac{\sqrt{2dt} \cdot v_n}{\sqrt{2n}} \right).
\]

Since each \( v_n \) is a weakly stationary harmonic map, then by Theorem 7, it follows that \( \psi_n \) is monotonically non-decreasing in \( t \) for every \( n \).

Using (34), we have

\[
\phi \left( \frac{\sqrt{2dt} \cdot v_n}{\sqrt{2n}} \right) = \frac{1}{(2dt)^{d/2}} \int_{B(0,\sqrt{2dt})} |Dv_n|^2 \, dy
\]

\[
= \frac{2dt}{(n \cdot d - 2)} \int_{\{|y| = \sqrt{2dt}\}} |Dv_n|^2 \left( 2dt \right)^{-\frac{d-1}{2}} \sigma_{n,d-1} - \frac{2}{(n \cdot d - 2)} \int_{\{|y| = \sqrt{2dt}\}} |y \cdot \nabla v_n|^2 \left( 2dt \right)^{-\frac{d-1}{2}} \sigma_{n,d-1}.
\]

Since \( v_n(\gamma) \{|y| = \sqrt{2dt}\} = u_n(f_{n,d}(\gamma)), \) then we may apply Lemma 1 to both terms above. By (11) from Lemma 3

\[
\int_{\{|y| = \sqrt{2dt}\}} |Dv_n|^2 \left( 2dt \right)^{-\frac{d-1}{2}} \sigma_{n,d-1}
\]

\[
= |S^{d-1}| \int_{\mathcal{X}_d} \left[ n |Du_n|^2 + \frac{2}{d} ((x, t) \cdot \nabla_{(x,t)} u_n) \cdot \frac{\partial u_n}{\partial t} \right] (2ndt)^{-\frac{d}{2}} \left( 1 - \frac{|x|^2}{2ndt} \right)^{\frac{n-d-2}{2}} \, dx.
\]

And using (10) from Lemma 3

\[
\int_{\{|y| = \sqrt{2dt}\}} |y \cdot Dv_n|^2 \left( 2dt \right)^{-\frac{d-1}{2}} \sigma_{n,d-1}
\]

\[
= |S^{d-1}| \int_{\mathcal{X}_d} \left| x \cdot \nabla u_n + 2t \frac{\partial u_n}{\partial t} \right|^2 (2ndt)^{-\frac{d}{2}} \left( 1 - \frac{|x|^2}{2ndt} \right)^{\frac{n-d-2}{2}} \, dx.
\]
It follows that
\[ \psi_n(t) = \frac{t}{(1 - \frac{2}{n^d})} \int_{X_d} |Du_n|^2 t^{-\frac{d}{2}} \left( 1 - \frac{|x|^2}{2ndt} \right)^{\frac{nd-d-2}{2}} dx \]
\[ - \frac{1}{(n \cdot d - 2)} \int_{X_d} \left[ |x \cdot \nabla u_n|^2 + 2t (x \cdot \nabla u_n) \frac{\partial u_n}{\partial t} + 2t^2 \left[ \frac{\partial u_n}{\partial t} \right]^2 \right] t^{-\frac{d}{2}} \left( 1 - \frac{|x|^2}{2ndt} \right)^{\frac{nd-d-2}{2}} dx. \]

Mimicking the arguments from the proofs above, we see that
\[ \lim_{n \to \infty} \psi_n(t) = t \int_{X_d} |Du|^2 G_t(x) dx =: \Phi(t; u) \]
is also monotonically non-decreasing in \( t \), proving the theorem. □

8. **Minimal Surfaces and Mean Curvature Flow**

The theory of minimal surfaces is vast. We use the following proposition from [6] as our definition of a minimal surface. There are a number of alternate ways to define minimal surfaces, as described in [7], for example.

**Proposition 1.** \( \Sigma^N \subset \mathbb{R}^k \) is a minimal surface iff the restrictions of the coordinate functions of \( \mathbb{R}^k \) to \( \Sigma \) are harmonic functions.

The next theorem is a monotonicity result for minimal surfaces. The statement appears in [6], Proposition 4.1 and Lemma 4.2. This formula is useful in the regularity theory of minimal surfaces.

**Theorem 9.** Suppose that \( \Sigma^N \subset \mathbb{R}^k \) is a minimal surface and let \( w_0 \in \mathbb{R}^k \). Then the function
\[ \Theta_{w_0}(s; \Sigma) = \frac{\text{Vol}(B_s(w_0) \cap \Sigma)}{\text{Vol}(B_s \subset \mathbb{R}^N)} \]
is monotonically non-decreasing in \( s \). Furthermore,
\[ \frac{d}{ds} \Theta_{w_0}(s; \Sigma) = \frac{1}{\omega_{NyN+1}} \int_{\partial B_s \cap \Sigma} \frac{|(w - w_0)\perp|^2}{|(w - w_0)\top|^2}. \]

The parabolic analogue of the minimal surface equation is mean curvature flow. Let \( \{M_t\} \subset \mathbb{R}^{d+1} \) be a 1-parameter family of smooth hypersurfaces. Then \( \{M_t\} \) flows by mean curvature if
\[ z_t = H(z) = \Delta_M z, \]
where \( z \) are the coordinates on \( \mathbb{R}^{d+1} \) and \( H = -H^\nu \) denotes the mean curvature vector. The following is the monotonicity formula due to Huisken [12]. For convenience, we reverse the time direction and present a reformulation of Huisken’s original statement.

**Theorem 10 ([12], Theorem 3.1).** If a smooth 1-parameter family of hypersurfaces \( M_t \), satisfies \( z_t + \Delta_M z = 0 \) in \( \mathbb{R}^{d+1} \times [0, T] \), then the density ratio
\[ \vartheta(t; M_t) = \int_{M_t} t^{-d/2} \exp \left( -\frac{|z|^2}{4t} \right) \]
is monotonically non-decreasing in \( t \). Furthermore,
\[ \frac{d}{dt} \vartheta(t; M_t) = \int_{M_t} \left| H + \frac{z\perp}{2t} \right|^2 t^{-d/2} \exp \left( -\frac{|z|^2}{4t} \right). \]
Proof. Let $M_t$ be a smooth 1-parameter family of $d$-dimensional hypersurfaces that flows by backwards mean curvature. Assume that each $M_t$ is given by a graph. Then there exists a function $u : X_d \times [0, T] \to \mathbb{R}$ such that $M_t$ is given in $\mathbb{R}^{d+1}$ by the coordinates
\[
(x_1(t), \ldots, x_d(t), u(x_1(t), \ldots, x_d(t), t))
\]
Thus, the coordinates of $M_t$ are $z = (x, u(x, t))$. The unit outward normal is given by
\[
v = \frac{(-\nabla u, 1)}{\sqrt{1 + |\nabla u|^2}},
\]
and the first and second fundamental forms are
\[
g_{ij} = \delta_{ij} + \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}, \quad g^{ij} = \delta_{ij} - \frac{1}{1 + |\nabla u|^2} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}, \quad |g| = 1 + |\nabla u|^2
\]
\[
h_{ij} = \frac{-1}{\sqrt{1 + |\nabla u|^2}} \frac{\partial^2 u}{\partial x_i \partial x_j}.
\]
It follows that
\[
H = g^{ij} h_{ij} = \frac{-\Delta u}{\sqrt{1 + |\nabla u|^2}} + \frac{1}{1 + |\nabla u|^2} \left( 1 \right)^{3/2} \sum_{i,j=1}^{d} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}.
\]
By the assumption that $M_t$ flows by backwards mean curvature, we see that
\[
(x_1'(t), \ldots, x_d'(t), \nabla u(x_1(t), \ldots, x_d(t), t) \cdot (x_1'(t), \ldots, x_d'(t)) + \partial_t u)
\]
\[
= \left( \frac{\Delta u}{\sqrt{1 + |\nabla u|^2}} - \frac{1}{1 + |\nabla u|^2} \sum_{i,j=1}^{d} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} \right) \frac{(\nabla u, -1)}{\sqrt{1 + |\nabla u|^2}}
\]
A simplification shows
\[
\left( 1 + |\nabla u|^2 \right) \left( \Delta u + \partial_t u \right) - \sum_{i,j=1}^{d} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0,
\]
\[
\frac{\partial u}{\partial t} = H \sqrt{1 + |\nabla u|^2}.
\]
For each $n \in \mathbb{N}$, let $M^{(n)}_t$ be a smooth 1-parameter family of $d$-dimensional hypersurfaces that are given locally by coordinates
\[
(x_1(t), \ldots, x_d(t), u_n(x_1(t), \ldots, x_d(t), t)),
\]
where $u_n : X_d \times [0, T] \to \mathbb{R}$ satisfies
\[
\left( 1 + |\nabla u_n|^2 \right) \left( \Delta u_n + \partial_t u_n \right) - \sum_{i,j=1}^{d} \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} \frac{\partial^2 u_n}{\partial x_i \partial x_j} = \frac{1}{n} \Gamma(u_n) + \frac{1}{n^2} \Lambda(u_n),
\]
with
\[
\Gamma(u_n) = \frac{1}{d} \left[ \frac{\partial u_n}{\partial t} \left[ |\nabla u_n|^2 - 2 \left( \Delta u_n + \frac{\partial u_n}{\partial t} \right) \nabla (x,t) u_n \right] - 2 \left( 1 + |\nabla u_n|^2 \right) (x,t) \cdot \nabla (x,t) \left( \frac{\partial u_n}{\partial t} \right) \right]
\]
\[ \Lambda(u_n) = \frac{1}{d} \left[ \sum_{i,k=1}^{d} \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_k} \left( x_k \frac{\partial^2 u_n}{\partial x_i \partial t} + x_i \frac{\partial^2 u_n}{\partial x_k \partial t} \right) + \frac{\partial u_n}{\partial t} \sum_{i,k=1}^{d} \frac{\partial^2 u_n}{\partial x_i \partial x_k} \sum_{j,l=1}^{n} \left( y_{i,j} \frac{\partial u_n}{\partial x_i} + y_{i,j} \frac{\partial u_n}{\partial x_j} \right) \right] \\
+ \frac{1}{d^2} \left[ 2 \left( \frac{\partial u_n}{\partial t} \right)^2 (x,t) \cdot \nabla (x,t) u_n + \frac{\partial^2 u_n}{\partial t^2} \sum_{i,k=1}^{d} \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_k} + \left( \frac{\partial u_n}{\partial t} \right) \sum_{i,k=1}^{d} \frac{\partial^2 u_n}{\partial x_i \partial x_k} \sum_{j,l=1}^{n} y_{i,j} y_{j,l} \right] \\
+ \frac{1}{d^3} \left[ \frac{\partial u_n}{\partial t} \frac{\partial^2 u_n}{\partial t^2} \sum_{i,k=1}^{d} \sum_{j,l=1}^{n} \left( y_{i,j} y_{j,l} \frac{\partial u_n}{\partial x_i} + y_{i,j} y_{j,l} \frac{\partial u_n}{\partial x_k} \right) + \left( \frac{\partial u_n}{\partial t} \right) \sum_{i,k=1}^{d} \sum_{j,l=1}^{n} \left( y_{i,j} y_{j,l} \frac{\partial^2 u_n}{\partial x_i \partial t} + y_{i,j} y_{j,l} \frac{\partial^2 u_n}{\partial x_k \partial t} \right) \right] \\
+ \frac{1}{d^4} \left( \frac{\partial u_n}{\partial t} \frac{\partial^2 u_n}{\partial t^2} \sum_{i,k=1}^{d} \sum_{j,l=1}^{n} y_{i,j}^2 y_{j,l} - \frac{4}{d^2} [(x,t) \cdot \nabla (x,t) u] \left( (x,t) \cdot \nabla (x,t) \left( \frac{\partial u_n}{\partial t} \right) \right) \left( \frac{\partial u_n}{\partial t} \right) \right] (40) \]

We construct the sequence of surfaces \( \{ M_t^{(n)} \} \) so that \( \lim_{n \to \infty} M_t^{(n)} = M_t \). In other words, we require that

\[ \lim_{n \to \infty} \vartheta \left( t; M_t^{(n)} \right) = \vartheta \left( t; M_t \right). \]

Now for each \( n \in \mathbb{N} \), let \( \Sigma_n \) be an \( n \times d \)-dimensional hypersurface that is given by the following coordinates in \( \mathbb{R}^{n+d+1} \)

\[ \left( y_{1,1}(t), \ldots, y_{d,n}(t), \frac{1}{\sqrt{n}} v_n \left( y_{1,1}(t), \ldots, y_{d,n}(t) \right) \right), \]

where \( v_n : \mathcal{P}_{n,d} \to \mathbb{R} \), \( \mathcal{P}_{n,d} = F_{n,d}^{-1} (X_d \times [0,T]) \subset Y_{n,d} \), is given by

\[ v_n (y) = u_n \left( F_{n,d} (y) \right). \]

By analogy with the computations above, the unit outward normal is

\[ \vec{\nu} = \left( \begin{array}{c} -\nabla v_n \\ \sqrt{n + |\nabla v_n|^2} \end{array} \right), \]

and the first and second fundamental forms are

\[ g_{i,j,k,l} = \delta_{i,k} \delta_{j,l} + \frac{1}{n} \frac{\partial v_n}{\partial y_{i,j}} \frac{\partial v_n}{\partial y_{k,l}} \quad g^{i,j} = \delta_{i,k} \delta_{j,l} - \frac{1}{n + |\nabla v_n|^2} \frac{\partial v_n}{\partial y_{i,j}} \frac{\partial v_n}{\partial y_{k,l}} \quad |\vec{g}| = 1 + \frac{1}{n} |\nabla v_n|^2 \]

\[ h_{i,j} = \frac{-1}{\sqrt{n + |\nabla v_n|^2}} \frac{\partial^2 v_n}{\partial y_{i,j} \partial y_{k,l}}. \]

It follows that

\[ \tilde{H} = g^{i,j} h_{i,j} = -\frac{\Delta v_n}{\sqrt{n + |\nabla v_n|^2}} + \frac{1}{(n + |\nabla v_n|^2)^{3/2}} \sum_{i,k=1}^{d} \sum_{j,l=1}^{n} \frac{\partial v_n}{\partial y_{i,j}} \frac{\partial v_n}{\partial y_{k,l}} \frac{\partial^2 v_n}{\partial y_{i,j} \partial y_{k,l}}. \]

Therefore, the mean curvature of \( \Sigma_n \) vanishes iff

\[ (n + |\nabla v_n|^2) \Delta v_n - \sum_{i,k=1}^{d} \sum_{j,l=1}^{n} \frac{\partial v_n}{\partial y_{i,j}} \frac{\partial v_n}{\partial y_{k,l}} \frac{\partial^2 v_n}{\partial y_{i,j} \partial y_{k,l}} = 0. \]
Therefore, is monotonically non-decreasing in $\frac{\partial u}{\partial t}$.

And by (7) and (8),

$$
\sum_{i,k=1}^{d} \frac{\partial v_n}{\partial y_{i,j}} \frac{\partial v_n}{\partial y_{k,l}} \frac{\partial v_n}{\partial y_{i,j}} \frac{\partial v_n}{\partial y_{k,l}} = n^2 \sum_{i,k=1}^{d} \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_k} \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_k} + n \frac{\partial u_n}{\partial t} |\nabla u_n|^2 + \Lambda (u_n) + \frac{4}{d^2} [(x,t) \cdot \nabla (\cdot (x,t)) u_n] \left( \frac{\partial u_n}{\partial t} \right)
$$

Therefore,

$$
\left( n + |\nabla v_n|^2 \right) \Delta v_n - \sum_{i,k=1}^{d} \sum_{j=1}^{n} \frac{\partial v_n}{\partial y_{i,j}} \frac{\partial v_n}{\partial y_{k,l}} \frac{\partial v_n}{\partial y_{i,j}} \frac{\partial v_n}{\partial y_{k,l}} = n^2 \left( 1 + |\nabla u_n|^2 \right) \left( \Delta u_n + \frac{\partial u_n}{\partial t} \right) - \sum_{i,k=1}^{d} \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_k} \frac{\partial^2 u_n}{\partial x_i \partial x_k} - n \Gamma (u_n) - \Lambda (u_n)
$$

is 0,

by (38), so each $\Sigma_n$ is a minimal surface and we may apply Theorem 9.

Let $w$ denote the coordinates of $\Sigma_n$, that is, $w = \left( y, \frac{1}{\sqrt{n}} v_n (y) \right)$. Then, since $\Delta_{\Sigma_n} |w|^2 = 2nd$, it follows that

$$
\text{Vol} \left( B_{\sqrt{2dt}} (0) \cap \Sigma_n \right) = \frac{1}{2nd} \int_{\Sigma_n \cap \{ |w|^2 \le 2dt \}} \Delta_{\Sigma_n} |w|^2 \\
= \frac{1}{nd} \int_{\Sigma_n \cap \{ |w|^2 = 2dt \}} \left| w \right|^2,
$$

where the last line follows from Stokes’ theorem. Now define

$$
\Upsilon_n (t) = (2nd)^{\frac{d}{2}} \left| \frac{S^{n-d-1}}{S^{n-d-1-d}} \right| \Theta_0 \left( \sqrt{2dt}; \Sigma_n \right).
$$

By Theorem 9

$$
\Upsilon_n (t) = \frac{(2nd)^{\frac{d}{2}}}{\left| S^{n-d-1-d} \right|} (2dt)^{-nd/2} \int_{\Sigma_n \cap \{ |w|^2 = 2dt \}} |w|^2
$$

is monotonically non-decreasing in $t$ for each $n \in \mathbb{N}$. 

As $n \to \infty$, the set \( \left\{ y = 2dt - \frac{|v_0(y)|^2}{n} \right\} \) will approach a sphere of radius $r_n$ in the $y$-coordinates. Note also that \( \lim_{n \to \infty} r_n = \sqrt{2dt} \). Therefore,

\[
Y_n(t) = \frac{(2nd)^{d/2}}{\sqrt{\Omega_n}} \int_\Omega \left( 1 - \frac{|v_n|^2}{2ndt} \right)^{\frac{d-1}{2}} \left( 1 - \frac{(y \cdot \nabla v_n - v_n)^2}{2ndt} \right)^{\frac{d-1}{2}} \frac{r_n^{-(n-d-1)} + \varepsilon(n)}{n} \]

where $\varepsilon$ is some error function for which \( \lim_{n \to \infty} \varepsilon(n) = 0 \). An application of Lemmas \( \Pi \) and \( \Sigma \) shows that

\[
Y_n(t) = (2d)^{d/2} \int_t \left( 1 - \frac{|u_n|^2}{2ndt} \right)^{\frac{d-1}{2}} r_n^{-d} \left( 1 - \frac{|x|^2}{2ndt} \right)^{\frac{d-1}{2}} \times \left[ 1 + |\nabla u_n|^2 + \frac{1}{nd} \left( 2 \frac{\partial u_n}{\partial t} x \cdot \nabla u_n + \frac{r_n^2}{d} \left( \frac{\partial u_n}{\partial t} \right)^2 - \frac{(x \cdot \nabla u_n - u_n + \frac{r_n^2}{d} \frac{\partial u_n}{\partial t})^2}{2t} \right) \right] dx + \varepsilon(n)
\]

is also monotonically non-decreasing in $t$ for each $n \in \mathbb{N}$.

As usual, we now take $n \to \infty$ to conclude that

\[
\lim_{n \to \infty} Y_n(t) = (2d)^{d/2} \int_t (2dt)^{-d/2} \exp \left( -\frac{|x|^2 + |u|^2}{4t} \right) \sqrt{1 + |\nabla u|^2} \] dx

is also monotonically non-decreasing in $t$, proving the first statement in the theorem.

By Theorem 9 and some simplifying computations,

\[
\frac{d}{dt} Y_n(t) = \sqrt{\frac{d}{2t} \Omega_n (2nd)^{d/2}} \int_\Omega \left( 1 - \frac{|v_n|^2}{2ndt} \right)^{\frac{d-1}{2}} \left| \frac{w}{w^T} \right|^2 \frac{|w|^2}{r_n^{-(n-d-1)}} \left( 1 + \frac{|\nabla v_n|^2}{n} \right) + \tilde{\varepsilon}(n),
\]

where $\tilde{\varepsilon}(n)$ denotes another error function. By Lemma 3

\[
\frac{1}{4t^2} \sqrt{\frac{d}{2t} \Omega_n} \left| \frac{w}{w^T} \right|^2 \sqrt{1 + \frac{|\nabla v_n|^2}{n}}
\]

\[
= \frac{1}{4t^2} \sqrt{1 + |\nabla u_n|^2 + \frac{1}{nd} \left( 2x \cdot \nabla u_n + \frac{r_n^2}{d} \frac{\partial u_n}{\partial t} \right) \frac{\partial u_n}{\partial t} - \frac{1}{2ndt} \left( x \cdot \nabla u_n + \frac{r_n^2}{d} \frac{\partial u_n}{\partial t} - u_n \right)^2}
\]

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Thus, by Lemma 1 and the computation above,
\[
\frac{d}{dt} \mathcal{Y}_n(t) = (2d)^{\frac{d}{2}} \int_{X_d} \frac{1}{4rt^2} \left( x \cdot \nabla u_n + \frac{r^2}{d} \frac{\partial u_n}{\partial t} - u_n \right)^2 \times \left( 1 - \frac{u_n^2}{2ndt} \right) \frac{u_n^{d-1}}{r_n^d} \left( 1 - \frac{|z|^2}{2ndt} \right) ^{\frac{d-d-2}{2}} \ dx + \tilde{e}(n)
\]

We now take \( n \to \infty \) to see that
\[
\lim_{n \to \infty} \frac{d}{dt} \mathcal{Y}_n(t) = \int_{X_d} \left[ \frac{x \cdot \nabla u - u + \frac{\partial u}{\partial t}}{1 + |\nabla u|^2} \right]^2 t^{-\frac{d}{2}} \exp \left( -\frac{|z|^2 + u^2}{4t} \right) \sqrt{1 + |\nabla u|^2} \ dx
\]
\[
= \int_{M_t} \left| \mathbf{H} + \frac{z^+}{2t} \right|^2 t^{-\frac{d}{2}} \exp \left( -\frac{|z|^2}{4t} \right),
\]

since \( \mathbf{H} = -H \mathbf{v} \) and \( z^+ = (z \cdot \mathbf{v}) \mathbf{v} \), where we have used (35) and (37). This completes the proof. \( \square \)

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