Supersymmetric derivation of the hard core deuteron’s bound state

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Abstract

A supersymmetric construction of potentials describing the hard core interaction of the neutron-proton system for low energies is proposed. It considers only the binding energy case and uses the approximation of the Yukawa potential given by Hulthén. Recent experimental data for the binding energy of the deuteron are used to give the involved orders of magnitude.

Key-Words: Supersymmetry, Deuteron binding energy

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The neutron-proton (n-p) system is the simplest of the composite nuclei. It can exist in a stable bound state, the deuteron. This is a nuclide for which no excited bound states are known, i.e., it exists only in its ground state. For a n-p system in which there are no electrostatic forces (i.e., the force due to the two magnetic moments can be considered as small as an irrelevant correction to the nuclear forces) the interaction is depicted by a short range central force. Forms of the related potentials commonly used are the square well, gaussian and Yukawa potentials (see for example [1].) Among them, the gaussian and the square well potentials have deserved special attention because their mathematical simplicity. On the other hand, the Yukawa potential has a deeper theoretical significance because its close connection with the ‘exchange forces’ responsible of the binding between nucleons. Historically, this potential is often quoted by the prediction of the existence of the $\pi$ meson.

Hulthén noticed that the Yukawa potential can be conveniently approximated by the expression

$$V(r) = -\frac{V_0}{e^{r/\alpha} - 1}, \quad (1)$$

where $r$ is the distance between the nucleons, $\alpha > 0$ is a fixed length (the range of the potential) and $V_0 > 0$ is a fixed energy (the strength of the potential).

For low energies, the Schrödinger equation for the Yukawa potential is not solvable analytically and the potential (1) is used as the stand in for it (see [3] and references quoted therein). In this range of energies the problem of describing the n-p interactions can be characterized in two forms: that concerning to bound states or to scattering states. In the following, we shall only focus on the former case. On the other hand, empirically is known that the deuteron has a total spin 1. The obvious interpretation is that the spins of the neutron and the proton are parallel in the ground state, which would thus be described by a $^3S_1$-state. Therefore, we will restrict ourselves to the study of the nuclear binding energy $S$-states.

Let us stress now that the nucleons cannot approach each other closer than a certain distance, otherwise, the nucleus would not show an almost constant nuclear density. Then we must resort to phenomenological arguments [4]. The usual approach is to modify the nuclear potential at small distances to be consistent with the experimental data. A simpler approach considers the so-called hard core model which is characterized by a short range infinite repulsion inside the attractive nuclear interaction [5]. In other words, a realistic potential must contain more than a radial term of the form (1): it must also include an infinitely high barrier term.

The present paper investigates the supersymmetric nature of the hard core deuteron’s binding energy by doing calculations on the Hulthén’s potential. We shall show that the new potential so derived presents a repulsive barrier term and can be either isospectral (e.g. it shares the same spectrum) or almost isospectral (the same spectrum except the ground state) to the Hulthén’s potential. As a particular case, the new potential is chosen to be a representation of the hard core nuclear force describing the n-p low energy binding interactions. The involved orders of
magnitude are given by using the experimental data of $-2.22456614(41)$ MeV for
the deuteron binding energy $E_d$ recently reported in [3].

As usual for potentials depending on $r$, the Schrödinger equation for the Hulthén’s
potential reduces to an eigenvalue equation for a particle in a one dimensional
effective potential $V(x) = \ell(\ell + 1)/x^2 - V(x)$, where $\ell$ is the azimuthal quantum
number and $x = r/\alpha$ is a dimensionless radial coordinate. We are looking for the $S$
states of binding energy (i.e., negative energies and $\ell = 0$), therefore the following
equation holds

$$\frac{d^2}{dx^2} + \frac{V_0}{e^x - 1} - k^2 \psi(x) = 0,$$

(2)

where $\psi(x) \equiv xR(x)$, with $R(x)$ the standard radial wavefunction, and $E = -k^2 = (2\mu\alpha^2/\hbar^2)\mathcal{E}$, $V_0 = (2\mu\alpha^2/\hbar^2)V_0$, with $\mu$ the reduced mass.

The standard procedure of solution carries out the eigenfunctions

$$\psi_n(x) = C_n e^{-kx}(1 - e^{-x})_2F_1(2k + 1 + n, 1 - n, 2k + 1; e^{-x}), \quad n = 1, 2, \ldots,$$

(3)

with $C_n$ a normalization constant

$$C_n \equiv \alpha^{-3/2} \frac{\Gamma(n + 2k)}{\Gamma(n + 1)\Gamma(2k + 1)} [2k(n + k)(n + 2k)]^{1/2}.$$

The corresponding eigenvalues are given by

$$E_n = -k_n^2 = -\left(\frac{V_0 - n^2}{2n}\right)^2, \quad V_0 > n^2, \quad n = 1, 2, \ldots$$

(4)

Remark that the problem involves two mutually dependent parameters, $V_0$ and $k$.
In practice, the eigenvalue of the energy may be given by experiment while the
strength $V_0$ of the potential is to be determined. Therefore, equation (4) can be
used to evaluate $V_0$ in terms of $E_n$.

The dotted curve on Figure 1 represents the potential allowing only one
bound state, the deuteron’s ground state, labeled by a subscript $H$. In Figure 2 we
have plotted the well known corresponding probability density $|\psi_H(x)|^2$.

As regards the supersymmetric scheme, we have in the first place to look for a
superpotential $w(x)$ solving the Riccati equation

$$-w'(x) + w^2(x) = V(x) - \epsilon,$$

(5)

where the prime denotes derivative with respect to $x$ and the factorization energy
$\epsilon = -\kappa^2$ is, in principle, any real number. By a simple calculation we get for the
particular solution

$$w(x) = \kappa - \frac{1}{e^x - 1}, \quad \kappa > 0,$$

(6)
which is useful in the cases when the strength can be rewritten as \( V_0 = 1 + 2\kappa \). As usual, the susy partner \( \tilde{V} \) of \( V \) is given by the shape invariance condition \([8]\)

\[
\tilde{V}(x) = V(x) + 2w'(x),
\]

leading to

\[
\tilde{V}(x) = -\frac{1 + 2\kappa}{e^x - 1} + \frac{1}{2\sinh^2(x/2)}.
\]

Potential \([8]\) is a well known result in susy quantum mechanics (see \([9]\)). It has been used to study susy phase-equivalent potentials \([10]\) and to establish some interesting connections between the susy and the variational method \([11]\). Observe now the appearance of the r.h.s. term in \([8]\). This term presents a singularity of order \( 2/x^2 \) at origin and behaves just as a repulsive centrifugal term with \( \ell = 1 \) in the neighborhood of \( x = 0 \).

\[
\tilde{V}(x) \sim -\frac{1 + 2\kappa}{x} + \frac{2}{x^2}, \quad x<<1.
\]

As the value \( x = 1 \) implies \( r = \alpha \), the approximation \([9]\) holds in the range of the initial potential \( V(x) \). In the region \( x > 1 \), the potential \( \tilde{V}(x) \) rapidly becomes negligible (see Figure 1.) Here, the strength \( V_0 = 1 + 2\kappa \) plays the role of a coupling constant.

The above results can be used to determine the eigenfunctions and eigenvalues connected with the new potential \( \tilde{V}(x) \). The procedure consists now in factorizing the related Hamiltonians by

\[
H = A^\dagger A + \epsilon, \quad \tilde{H} = AA^\dagger + \epsilon,
\]

with

\[
A \equiv \frac{d}{dx} + w(x).
\]

It is straightforward to check that equations \([5], [7], [11]\) automatically lead to \([10]\). Then, an intertwining relationship holds:

\[
\tilde{H}A = AH.
\]

If \( \psi(x) \) is an eigenfunction of \( H \) with eigenvalue \( E \), equation \([12]\) gives

\[
\tilde{H}(A\psi) = E(A\psi), \quad A\psi \neq 0.
\]

Therefore, if \( \psi \in L^2(\mathbb{R}) \), we get the normalized eigenstate of \( \tilde{H} \)

\[
\tilde{\psi}(x) = (E - \epsilon)^{-1/2}A\psi(x).
\]

Now, let us stress on the information displayed by equations \([10], [13]\). First, in the case when \( \epsilon = E_n \), for any \( n = 1, 2, \ldots \), the l.h.s. equation in \([10]\) applies on \( \psi_n(x) \)
as \( H\psi_n(x) = E_n\psi_n(x) \), and consequently \( A^\dagger A\psi_n(x) = 0 \). It is easy to check that

\[ A\psi_n(x) = 0 \]

is a sufficient condition to get square integrable functions. Therefore, equation (13) means that \( \psi_n(x) \) has not a susy partner \( \tilde{\psi}_n(x) \) and the couple of Hamiltonians \( \Pi \) corresponds to a case of unbroken supersymmetry \( \mathcal{F} \).

On the other hand, when \( \epsilon \neq E_n \) for every \( n = 1, 2, \ldots \), there is no square integrable eigenfunction of \( H \) annihilated by \( A \), and equation (13) means that every \( \psi_n(x) \) will have a susy partner \( \tilde{\psi}_n(x) \). Now, from the r.h.s. of (10), it is clear that a function \( \tilde{\psi}_n(x) \), obeying \( A^\dagger\tilde{\psi}_n = 0 \), leads to \( \tilde{H}\tilde{\psi}_n(x) = \epsilon\tilde{\psi}_n(x) \). Hence, if \( \tilde{\psi}_n \in L^2(\mathbb{R}) \), it must be added to the new set \( \{\tilde{\psi}\} \). When \( \tilde{\psi}_n(x) \) is a square integrable function the Hamiltonians (10) present unbroken supersymmetry, otherwise they correspond to a case of broken supersymmetry \( \mathcal{F} \).

For the superpotential we are dealing with, one gets \( \tilde{\psi}_n(x) \propto e^{\kappa x}(1-e^{-x})^{-1} \), which is obviously not square integrable in \([0, \infty)\) for \( \kappa > 0 \). Then, the supersymmetric behaviour of the Hamiltonians (10) lies in the selection of \( \epsilon \), i.e., if \( \epsilon \) is chosen to be either an eigenvalue of \( H \) or not.

In the following we shall consider the case when the initial potential \( V(x) \) allows the binding of only two states. The purpose of this convention will be apparent in the sequel. To get a system with only two energy levels, the strength of the Hulthen’s potential has to be in the domain \( 4 < V \) in the sequel. To get a system with only two energy levels, the strength of the Hulthen’s potential has to be in the domain \( 4 < V \) in the sequel.

In order to get an idea of the orders of magnitude involved, we note that, although there is no a priori reason why \( \alpha \) should not be different for different sorts of the stationary systems described by \( V(x) \), we can take its numerical value as \( \alpha = 3\bar{f} \). Such assertion is justified by the fact that the mean distance between nucleons (i.e., the size of the nucleus) is in the range of 2 or 4 \( \bar{f} \). Therefore, we get \( (\hbar^2/\alpha^2m_p) \approx 4.6113 \text{ MeV} \). Here, we have assumed that the neutron and proton masses are equal to \( m_p \), hence \( 2\mu = m_p \). In this way, the experimental value of the deuteron binding energy \( E_d \) becomes in a dimensionless value \( E_d \approx -0.4825 \) (\( k_d \approx 0.6946 \)). Remark that, for the above selected domain of \( V_0 \), the deuteron energy lies in the domain of the exited state \( E_2 \) and not in the domain of the ground state \( E_1 \).

Let us consider now the factorization energy fixed as \( \epsilon = E_1 = -k_1^2 \). In this case, as discussed above, we will have a couple (14) with unbroken supersymmetry. Hence \( A\psi_1(x) = 0 \), and the potential \( \tilde{V}(x) \), with \( V_0 \equiv 1 + 2\kappa = 1 + 2k_1 \), misses the ground state of \( V(x) \) and admits only one bound state (see equation (13)):

\[ \tilde{\psi}(x) \equiv (E_2 - E_1)^{-1/2} \left[ \frac{d}{dx}\ln \psi_2(x) + w(x) \right] \psi_2(x), \]  

(14)

with eigenvalue

\[ E_2 = -\left( \frac{V_0 - 4}{4} \right)^2 = -\left( \frac{2k_1 - 3}{4} \right)^2. \]  

(15)
We go a steep further and impose that the numerical value of $E_2$ be determined by experiment and let it be equal to $E_d$, the dimensionless value of the deuteron binding energy, therefore

$$k_1 = \frac{4k_d + 3}{2} \simeq 2.8892, \quad V_0 = 4k_d + 4 \simeq 6.7784$$

which agrees with the previously stated domains for $V_0$, $E_1$ and $E_2$. Then, an initial potential (1), with range $\alpha = 3\tilde{f}$ and strength $V_0 \simeq 31.2572$ MeV, has a susy partner (8) allowing the binding of a single state $\tilde{\psi}(x)$ with an energy exactly equal to $E_d$.

The main characteristic of this new potential is its centrifugal term, which reduces the range of approaching between the nucleons. Therefore, we have constructed a radial potential describing the hard core binding interaction between the nucleons.

The probability density connected with the ground state (and single!) eigenfunction (14) of the hard core Hamiltonian $\tilde{H}$, for the numerical data mentioned above, has been plotted on Figure 2. Its features can be contrasted with those of the no core Hamiltonian $H$, labeled by $|\psi_H(x)|^2$, in the same figure. Observe the displacement to the right of $\tilde{\psi}$ with respect to $\psi_H$. An easy calculation shows that

$$\tilde{\psi}(x) \sim (1 - e^{-x}) \psi_H(x),$$

hence, near the origin, $\tilde{\psi}$ goes to zero as $x^2$ whereas $\psi_H$ goes just as $x$, and the probability to find the state $\tilde{\psi}(x)$ near to zero is minor than the probability to find $\psi_H(x)$ at the same place.

We shall now discuss some of the various implications of our results. First, the susy procedure accomplishes the derivation of a short range potential $\tilde{V}(x)$ allowing only one bound state. This potential exhibits many of the qualitative features concerned with a hard core potential. It behaves predominantly as an effective radial potential $V_\ell(x)$, with $\ell = 1$, near the origin (see equation (9)) and goes negligible with the increasing of $x$ in the region $x > 1$.

Second, from Figure 2, it is clear that there is a considerable probability of finding the two nucleons at distances larger than $\alpha$. Therefore, the nuclear force connected with $\tilde{V}(x)$ plays a relevant role only in a weak way because the neutron and the proton are outside each other’s range so much of the time. That is, of course, a well known feature in the behaviour of the deuteron’s ground state [1, 12]. It is interesting to remark on the smoothest of $\tilde{\psi}(x)$, this is a nodeless function, just as one might expect for the eigenfunction of a single stable bound state. Therefore, the wavefunction (14) corresponds to the hard core ground state eigenfunction of the deuteron.

In summary, the method developed in the paper relied on the assumption that it is possible to describe the interacting system of a proton plus a neutron by a Schrödinger equation. The method allowed the construction of a hard core potential for the deuteron and the estimation of the potential energy necessary to give the observed deuteron’s binding energy. It is remarkable that even very refined experiments at low energies do not suffice to determine more than the range and strength of the involved potential, leading the detailed shape completely indeterminate.
Observe that the hard core hypothesis makes the strength (14), of the potential (1), increased by $V_0 \simeq 3V_0H$, where $V_0H = 2k_d + 1$. In general, the nuclear forces are quite complicated and the assumption of a pure $^3S$-state does not suffice to explain the deuteron quadrupole moment, this makes necessary to introduce a tensor force (14). However, it is well known that the Hulthén’s potential gives a good approximation for the binding energy in the terms discussed at the very beginning of the paper. On the other hand, as mentioned above, there are different ways to modify the nuclear potential in order to get a hard core model, the potential derived here could be tested by the nucleon-nucleon scattering approaches, where the hard core hypothesis seems to be compatible with the empirical data (13).

As regards the unbroken susy results, we have to say that recent susy treatments, involving general solutions to the Riccati equation (3), have shown the way to introduce susy partner potentials sharing exactly the same spectra (13). That is, we could obtain a new Hamiltonian $\tilde{H}$ which, joined in a susy couple with $H$, present unbroken supersymmetry by extending the superpotential (3) to a general solution of (3). Although that method has been successfully applied in the study of some interesting potentials (see e.g. (14, 13)), it is out of the present scope and will be given elsewhere.

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Figure 1: The susy partner potentials (1) and (8) with $\alpha = 3\hat{f}$ and $V_0 \simeq 31.2572\text{ MeV}$. The deuteron’s binding energy $E_d \simeq -2.2245\text{ MeV}$ has been ticked on the frame. The potential $V_H$ has a strength $V_H \simeq 11.0173\text{ MeV}$ and the same value of $\alpha$. 
Figure 2: The deuteron’s ground state probability density for the cases without core $|\tilde{\psi}(x)|^2$, and hard core $|\tilde{\psi}(x)|^2$, with $\tilde{\psi}(x)$ given in (14) and the values displayed in Figure 1. Observe the displacement to the right of $|\tilde{\psi}(x)|^2$ with respect to $|\psi_H(x)|^2$. 