A REMARK ON DISK PACKINGS AND NUMERICAL INTEGRATION OF HARMONIC FUNCTIONS

STEFAN STEINERBERGER

ABSTRACT. We are interested in the following problem: given an open, bounded domain \( \Omega \subset \mathbb{R}^2 \), what is the largest constant \( \alpha = \alpha(\Omega) > 0 \) such that there exist an infinite sequence of disks \( B_1, B_2, \ldots, B_N, \ldots \subset \mathbb{R}^2 \) and a sequence \( (n_i) \) with \( n_i \in \{1, 2\} \) such that

\[
\sup_{N \in \mathbb{N}} N^{\alpha} \left\| \chi_{\Omega} - \sum_{i=1}^{N} (-1)^{n_i} \chi_{B_i} \right\|_{L^1(\mathbb{R}^2)} < \infty,
\]

where \( \chi \) denotes the characteristic function. We prove that certain (somewhat peculiar) domains \( \Omega \subset \mathbb{R}^2 \) satisfy the property with \( \alpha = 0 \). For these domains there exists a sequence of points \( (x_i)_{i=1}^{\infty} \in \Omega \) with weights \( (a_i)_{i=1}^{\infty} \) such that for all harmonic functions \( u : \Omega \to \mathbb{R} \)

\[
\left| \int_{\Omega} u(x) dx - \sum_{i=1}^{N} a_i u(x_i) \right| \leq C_\Omega \frac{\|u\|_{L^\infty(\Omega)}}{N^{0.53}},
\]

where \( C_\Omega \) depends only on \( \Omega \). This gives a Quasi-Monte Carlo method for harmonic functions which improves on the probabilistic Monte-Carlo bound \( \|u\|_{L^2(\Omega)} / N^{0.5} \) without introducing a dependence on the total variation. We do not know which decay rates are optimal.

1. Introduction

1.1. Harmonic functions. This paper aims to describe some progress in a problem that arose at the Oberwolfach Workshop 1340 ‘Uniform Distribution Theory and Applications’, where it was motivated by a talk of the author on a related problem [7]. We describe our question in its simplest possible setting: let \( \Omega \subset \mathbb{R}^2 \) be some bounded domain and let \( u : \Omega \to \mathbb{R} \) be a harmonic function, i.e. assume it satisfies

\[
\Delta u = 0, \quad \text{where} \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]

is the Laplacian. Is there a Quasi Monte Carlo method able to exploit this information effectively to compute an approximation (including an error estimate) of

\[
\int_{\Omega} u(x) dx?
\]

The key ingredient suggesting that this might indeed be the case is the mean-value property: let \( B(x, r) \) denote the disk with radius \( r \) centered at \( x \in \mathbb{R}^2 \). If \( u \) is harmonic in a neighbourhood of \( B(x, r) \), then

\[
u(x) = \frac{1}{r^2} \int_{B(x,r)} u(z) dz.
\]

This means that exact integration over disks can be done with one function evaluation. In particular, if one had a sequence of disks \( B_i \) such that

\[
\sup_{N \in \mathbb{N}} N^{\alpha} \left\| \chi_{\Omega} - \sum_{i=1}^{N} (-1)^{n_i} \chi_{B_i} \right\|_{L^1(\mathbb{R}^2)} \leq C_\Omega
\]

for some \( \alpha > 0 \), then this gives a Quasi Monte Carlo method for harmonic functions

\[
\left| \int_{\Omega} u(x) dx - \sum_{i=1}^{N} a_i u(x_i) \right| \leq C_\Omega \frac{\|u\|_{L^\infty(\Omega)}}{N^{\alpha}},
\]
where $x_i$ is the center of $B_i$ and $a_i = (-1)^{n_i}|B_i|$. Conversely, since the constant function 1 is harmonic any such Quasi Monte Carlo method gives a sequence of disks $B_i$ centered at $x_i$ with radius $r$ given via $r^2\pi = a_i$ and $n_i = 1 - (\text{sgn}(a_i) + 1)/2$ such that

$$\sup_{N \in \mathbb{N}} N^a \left\| \chi_\Omega - \sum_{i=1}^{N} (-1)^{n_i} \chi_{B_i} \right\|_{L^1(\mathbb{R}^2)} \leq C_\Omega.$$  

1.2. Main Result. We will prove a result for the following (quite restricted but nontrivial) type of domains: we say $\Omega \subset \mathbb{R}^2$ is finitely disk-covered if there exists a finite number of closed disks $B_1, B_2, \ldots, B_k$ such that any two disks meet at most in a single point that 'span' $\Omega$ in the following way: every point in $x \in \Omega$ is either contained in one of the disks or lies in a region surrounded by three disks such that any two out of these three disks touch in a point.

![Figure 1. A simple example of a finitely disk-covered domain and the underlying disks.](image1)

These sets are quite peculiar, however, at least any bounded, simply connected domain with a smooth boundary can be approximated in the Gromov-Hausdorff metric by a sequence of finitely disk-covered sets: it suffices to consider disks in a lattice arrangement (either hexagonal or rectangular in which case one has to add a final disk in the middle of every area) and approximate the desired domain using this lattice (rescaled to the desired level of accuracy of the approximation).

![Figure 2. A simple approximation of a rectangle: circles of equal radius in a rectangular grid with smaller circles filling up the holes.](image2)

Our consideration of this particular class of sets is twofold: it effectively cuts off the possibility of harmonic functions with large growth at the boundary of the domain since numerical integration close to the boundary of a finitely disk-covered domain can be done with finitely many function evaluations; secondly, the arising structure allows us to exploit recent advances in the study of Apollonian packings.

**Theorem.** Let $\Omega$ be finitely disk-covered. Then there exists a sequence of disks $B_i$ such that

$$\sup_{N \in \mathbb{N}} N^{0.53} \left\| \chi_\Omega - \sum_{i=1}^{N} \chi_{B_i} \right\|_{L^1(\mathbb{R}^2)} \leq C_\Omega.$$  

where $C_\Omega$ depends only on $\Omega$.

We believe that the statement is not optimal and that one should be able to construct sequences with a larger exponent in $N$ if one were to exploit the fact that some of the disks may have negative coefficients (something that is not used in the statement here). It should certainly be possible to prove some bounds for, say, the class of convex domains (numerical experiments suggest that the randomized greedy algorithm – picking a random point and, if it is not contained in one the existing disks, add the largest disk possible without introducing intersections – corresponds to $\alpha \sim 0.2$ for
convex domains). Since we never actually use the possibility of subtracting characteristic functions (i.e. \( n_i = 1 \)), we can immediately deduce that for any \( p \geq 1 \)
\[
\sup_{N \in \mathbb{N}} N^{\frac{dp}{d}} \left\| \chi_{\Omega} - \sum_{i=1}^{N} \chi_{B_i} \right\|_{L^p(\mathbb{R}^2)} \leq C_{\Omega}
\]
but there is no reason to assume that this should be in any way optimal.

1.3. Quasi-Monte Carlo. As outlined above, the statement immediately implies a Quasi Monte Carlo method. The same considerations as above suggest that there is no reason to assume this might be optimal.

**Corollary.** Let \( \Omega \) be finitely disk-covered. Then there exists a universal sequence \((x_i)_{i=1}^{\infty}\) of points in \( \Omega \) and a sequence \((a_i)_{i=1}^{\infty}\) of nonnegative reals with the following property: if \( \Delta u = 0 \) in a neighbourhood of \( \Omega \), then
\[
\left| \int_{\Omega} u(x)\,dx - \sum_{i=1}^{N} a_i u(x_i) \right| \leq C_{\Omega} \frac{\|u\|_{L^\infty(\Omega)}}{N^{0.53}},
\]
where \( C_{\Omega} \) depends only on \( \Omega \).

Let us emphasize the difference to classical QMC methods: a Quasi-Monte-Carlo method is based on the simple approximation
\[
\int_{[0,1]^2} u(x)\,dx \sim \frac{1}{N} \sum_{i=1}^{N} u(x_i)
\]
for a set of points \((x_i)_{i=1}^{N}\). The well-known Koksma-Hlawka inequality gives
\[
\left| \int_{[0,1]^2} u(x)\,dx - \frac{1}{N} \sum_{i=1}^{N} u(x_i) \right| \leq D_N(x_i) V(f),
\]
where \( D_N \) denotes the discrepancy of the point set and \( V(f) \) the total variation in the sense of Hardy-Krause. We refer to the classical monographs of Kuipers & Niederreiter [3], Drmota & Tichy [4] and Dick & Pillichshammer [2] for further information. We emphasize that there exist point sets such that \( D_N(x_i) \sim N^{-1} \log N \). However, and this is crucial, our bound is independent of the total variation of the function. Indeed, for the harmonic function (given in polar coordinates)
\[
u_m(r, \theta) = r^m \cos (m\theta),
\]
on some domain, we easily see that \( V(u_m) \sim m \), which can be made arbitrarily large; in contrast, our bound is independent of \( m \).

1.4. Possible extensions. If we were to modify the approximation scheme using a suitably rescaling, then for suitable points and weights the approximation
\[
\int_{\Omega} u(x)\,dx \sim \left( \frac{|\Omega|}{\sum_{i=1}^{N} a_i} \right) \sum_{i=1}^{N} a_i u(x_i)
\]
should yield even better results: what decay properties can be proven? Another natural conjecture is that, at least for finitely disk-covered domains, even
\[
\left| \int_{\Omega} u(x)\,dx - \sum_{i=1}^{N} a_i u(x_i) \right| \leq C_{\Omega} \frac{\|u\|_{L^1(\Omega)}}{N^{0.53}}
\]
might be true.
2. The Proof

Proof of the Theorem. The proof is constructive: since $\Omega$ is finitely disk-covered, we are initially given a finite set of disks $D_1, D_2, D_3, \ldots, D_k$ associated to $\Omega$ with centers $x_1, \ldots, x_k$. The mean-value theorem implies that for any harmonic $u$

$$\sum_{i=1}^{k} |D_i| u(x_i) = \int_{\bigcup_{i=1}^{k} D_i} u(x) dx.$$  

This is already precise on some part of the domain. The idea is to cover the rest of the domain with smaller and smaller disks (on each of which exact integration can again be performed). Let us consider a connected component of $\Omega \setminus \bigcup_{i=1}^{k} D_i$.

By assumption, it is bounded by three disks any two of which mutually touch in a point. Then there exists precisely one circle contained within the connected domain that is tangent to all three boundary circles: the statement dates back to Apollonius. Such a configuration of 4 circles is known as a Descartes configuration: given a Descartes configuration, it is possible to construct three additional circles within the three gaps. Iterating this process yields an Apollonian packing. For each connected component of $\Omega \setminus \bigcup_{i=1}^{k} D_i$ (of which there are only finitely many) we construct

Figure 3. Left: a Descartes configuration. Right: adding three additional circles

the associated Apollonian packing and then define an infinite sequence of disks $E_1, E_2, \ldots$ by ordering the union of the disks created by the Apollonian packings and the finitely many disks $D_1, D_2, \ldots, D_k$ by size. Let $x_i$ denote the center of $E_i$. Using the mean-value property, we get that

$$\left| \sum_{i=1}^{N} |E_i| u(x_i) - \int_{\Omega} u(x) dx \right| \leq \int_{\Omega \setminus \bigcup_{i=1}^{N} E_i} |u(x) dx| \leq \left| \Omega \setminus \bigcup_{i=1}^{N} E_i \right| \|u\|_{L^\infty(\Omega)}.$$  

It remains to control the speed with which the disks exhaust the set. Here we use a recent result of Kontorovich & Oh [4]: generalizing an earlier result of Boyd [1], they show the cardinality of disks with curvature $\kappa$ bounded from above by $T$ behaves as

$$c_1 \cdot T^\alpha \leq \# \{ i \in \mathbb{N} : \kappa(E_i) \leq T \} \leq c_2 \cdot T^\alpha$$  

for a universal constant $\alpha \sim 1.30568 \ldots$ (this approximation is due to McMullen [6]) and constants $c_1, c_2$ depending on the particular Apollonian packing. We consider merely finite number of Apollonian packings at the same time and may thus fix the constants $c_1, c_2$ in what follows. This implies that

$$c_2 T^\alpha \leq \# \left\{ i \in \mathbb{N} : T \leq \kappa(E_i) \leq \left( \frac{2c_2}{c_1} \right)^{\frac{1}{\alpha}} \right\} \leq \left( \frac{2c_2}{c_1} - c_1 \right) T^\alpha \quad (\ast).$$
This estimate controls the number of disks with curvature in a certain interval and shows that for
on average there are \( \sim T^{\alpha-1} \) disks with curvature \( T \leq \kappa \leq T + 1 \). We have that
\[
\left| \Omega \setminus \bigcup_{i=1}^{\infty} E_i \right| = 0
\]
and therefore
\[
\left| \Omega \setminus \bigcup_{\kappa(E_i) \leq T} E_i \right| = \left| \bigcup_{\kappa(E_i) \geq T} E_i \right|.
\]
A disk with curvature \( \kappa \) has measure \( \pi/\kappa^2 \) and therefore using \((\diamond)\)
\[
\left| \Omega \setminus \bigcup_{\kappa(E_i) \leq T} E_i \right| \leq \sum_{n=0}^{\infty} \frac{\pi}{(2c_2/c_1)^n T^2} \# \left\{ i \in \mathbb{N} : \left( \frac{2c_2}{c_1} \right)^n T \leq \kappa(E_i) \leq \left( \frac{2c_2}{c_1} \right)^{n+1} T \right\}
\]
\[
\leq \sum_{n=0}^{\infty} \frac{\pi}{(2c_2/c_1)^n T^2} \left( \frac{2c_2}{c_1} - c_1 \right) \left( \frac{2c_2}{c_1} \right)^n T^\alpha
\]
\[
\leq c \cdot T^{\alpha-2},
\]
for some constant \( c \). If we define \( N \) to be the number of circles with curvature bounded from above
by \( T \), then
\[
N \sim T^\alpha \quad \text{and thus} \quad T^{\alpha-2} \sim N^{2-\alpha} \sim \frac{1}{N^{2/\alpha}}.
\]
Since \( \alpha \sim 1.30568 \ldots \), we have that
\[
\frac{2-\alpha}{\alpha} = 0.536 \ldots
\]
and this yields the result. \( \square \)

3. Open problems

3.1. Optimal decay rates. The natural question is which decay rates are optimal. Our proof
may be regarded as a greedy algorithm: the big open question is the following: given a domain
\( \Omega \subset \mathbb{R}^2 \), is it true that the best approximation of \( \chi_\Omega \) is always given by
\[
\chi_\Omega \sim \chi_{B_1} + \chi_{B_2} + \cdots + \chi_{B_n}
\]
for a sequence of balls \((B_i)\) or whether there exist more interesting configurations for which
\[
\chi_\Omega \sim \pm \chi_{B_1} \pm \chi_{B_2} \pm \cdots \pm \chi_{B_n}
\]
yields a better result for a suitable choice of signs.

3.2. Harmonic functions on fractal sets. We conjecture that on finitely disk-covered domains
for the sequence of disks constructed in the argument and an arbitrary harmonic function \( u \)
actually the following stronger inequality should be true
\[
\left| \int_{\Omega} u(x) dx - \sum_{i=1}^{N} a_i u(x_i) \right| \leq C_N \frac{\|u\|_{L^1(\Omega)}}{N^{0.53}}.
\]
We emphasize that this is not a geometric statement about the constructed packing of disks and
that the statement is trivially false for arbitrary functions \( u \). Our reasoning behind conjecturing
such an inequality is the fact that the set
\[
\Omega \setminus \bigcup_{i=1}^{N} B_i
\]
has a very fractal structure
while harmonic functions have strong rigidity properties. It seems extremely natural to assume
that harmonic functions cannot differ too much on fractal sets from their average behavior.
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Stefan Steinerberger, Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany
E-mail address: steinerb@math.uni-bonn.de