A RAY-KNIGHT THEOREM FOR SPECTRALLY POSITIVE STABLE PROCESSES

BY WEI XU

1Department of Mathematics, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany. xuwei@math.hu-berlin.de

We generalize a classical second Ray-Knight theorem to spectrally positive stable processes. It is shown that the local time processes are solutions of certain stochastic Volterra equations driven by Poisson random measure and they belong to a class of fully novel non-Markov branching processes, named as rough continuous-state branching processes. Also, we prove the weak uniqueness of solutions to the stochastic Volterra equations by providing explicit exponential representations of the characteristic functionals in terms of the unique solutions to some associated nonlinear Volterra equations.

1. Introduction. This paper is preoccupied with generalizing the classical second Ray-Knight theorem to spectrally positive stable processes. The classical second Ray-Knight theorem was originally proved by Ray [35] and Knight [26] independently to understand the law of the Brownian local time processes at the first time the amount of local time accumulated at some level exceeds a given value. It is shown that for a constant ζ > 0 the Brownian local time process \( \{L_B(t, \tau^B_B(\zeta)): t \geq 0\} \) with \( \tau^B_B(\zeta) := \inf\{s \geq 0: L_B(0, s) \geq \zeta\} \) turns to be a \( \zeta \)-dimensional Bessel process. It is also a critical Feller branching diffusion and solves the stochastic differential equation

\[
Y_\zeta(t) = \zeta + \int_0^t \int_0^t Y_\zeta(s) 2W_B(ds, dz),
\]

where \( W_B(ds, dz) \) is a Gaussian white noise on \( \mathbb{R}_+^2 \) with density \( dsdz \); see Theorem 3.1 in [9].

For a general Markov process, its local time process has the Markov property if and only if it has continuous paths; see Theorem 1.1 in [11]. Therefore, it is usually a challenge to generalize the classical Ray-Knight theorems to Markov processes with jumps and limited work has been done in understanding the law of their local time processes. In a considerable important work, Eisenbaum et al. [12] provided a generalization of the classical second Ray-Knight theorem to any strongly symmetric recurrent Markov process \( \{S(t) : t \geq 0\} \) with state space \( \mathbb{V} \) being a locally compact separable metric space, i.e., there exists a mean-zero Gaussian process \( \{G_x : x \in \mathbb{V}\} \) such that for any \( b \in \mathbb{R} \) and \( \zeta > 0 \),

\[
\{L_S(x, \tau^S_S(\zeta)) + (G_x + b)^2/2 : x \in \mathbb{V}\} = \{(G_x + \sqrt{2\zeta + b^2})^2/2 : x \in \mathbb{V}\}
\]

in distribution. It is recently proved again in [36] using a martingale related to the reversed vertex-reinforced jump process. When \( S \) is a real-valued symmetric stable process with index \( 1 + \alpha \in (1, 2] \), the mean-zero Gaussian process turns to be a fractional Brownian motion with Hurst index \( \alpha/2 \). Specially, it is a standard Brownian motion when \( \alpha = 2 \) and the equivalence (1.2) is an alternate formulation of the second Ray-Knight theorem for Brownian motion.

Our main interest is in a one-dimensional spectrally positive stable process \( \{\xi(t) : t \geq 0\} \) with index \( 1 + \alpha \in (1, 2] \). It is a nonsymmetric, discontinuous Markov process defined on a complete probability space \( (\Omega, \mathcal{F}, P) \) endowed with filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfying the usual hypotheses. Its distribution is usually characterized by the Laplace transform \( \mathbb{E}[\exp\{-\lambda \xi(t)\}] = \exp\{t \Phi(\lambda)\} \) for \( \lambda \geq 0 \) and the Laplace exponent \( \Phi(\lambda) \) is of the form

\[
\Phi(\lambda) = b\lambda + c\lambda^{1+\alpha} = b\lambda + \int_0^\infty (e^{-\lambda y} - 1 + \lambda y) \nu_\alpha(dy),
\]

MSC2020 subject classifications: Primary 60G51, 60J55; secondary 60G22, 60J80, 60F05.

Keywords and phrases: Ray-Knight theorem, spectrally positive stable process, stochastic Volterra equation, rough continuous-state branching process, marked Hawkes point process.
where $b \geq 0$, $c > 0$ and the Lévy measure $\nu_\alpha(dy)$ is given by

$$
\nu_\alpha(dy) := \frac{c\alpha(\alpha + 1)}{\Gamma(1 - \alpha)}y^{-\alpha - 2}dy, \quad y > 0.
$$

Let $\{W(x) : x \in \mathbb{R}\}$ denote the scale function of $\xi$, which is identically zero on $x \in (-\infty, 0)$ and characterized on $[0, \infty)$ as a strictly increasing function whose Laplace transform is given by

$$
\int_0^\infty e^{-\lambda x}W(x)dx = \frac{1}{\Phi(\lambda)}, \quad \lambda > 0.
$$

The scale function $W$ is infinitely differentiable on $(0, \infty)$ with derivative denoted as $W'$.

Let $\{L_\xi(y) : y \in \mathbb{R}\}$ be the local time process of $\xi$, which is a joint continuous two-parameter process satisfying the well-known occupation density formula

$$
\int_0^t f(\xi(s))ds = \int f(y)L_\xi(y,t)dy.
$$

for every nonnegative measurable function $f$ on $\mathbb{R}$; readers may refer to Chapter V in [4] for details. Specially, the process $\{L_\xi(0,t) : t \geq 0\}$ is non-decreasing and $L_\xi(0,\infty) = \infty$ a.s. if and only if $b = 0$. This allows us to define its right inverse $\{\tau_\xi^L(\zeta) : \zeta \geq 0\}$ by $\tau_\xi^L(\zeta) = \infty$ if $\zeta \geq L_\xi(0,\infty)$ and

$$
\tau_\xi^L(\zeta) := \inf\{t \geq 0 : L_\xi(0,t) \geq \zeta\}, \quad \text{if } \zeta \in [0, L_\xi(0,\infty)).
$$

It is usual to interpret $\tau_\xi^L(\zeta)$ as the first time that the amount of local time accumulated at level $0$ exceeds $\zeta$. Our first main result in this paper is the following generalization of the classical second Ray-Knight theorem (1.1) to $\xi$.

**THEOREM 1.1.** For any $\zeta > 0$, conditioned on $\tau_\xi^L(\zeta) < \infty$ the local time process $\{L_\xi(t, \tau_\xi^L(\zeta)) : t \geq 0\}$ is the unique weak solution to the stochastic Volterra equation

$$
X_\zeta(t) = \zeta(1 - bW(t)) + \int_0^t \int_0^\infty \int_0^\infty (W(t - s) - W(t - s - y))\tilde{\nu}_a(ds,dy,dz),
$$

where $\tilde{\nu}_a(ds,dy,dz)$ is a compensated Poisson random measure on $(0, \infty) \times \mathbb{R}_+^2$ with intensity $dsdu_a(dy)dz$. Moreover, it is Hölder-continuous of any order strictly less than $\alpha/2$ on any bounded interval and the Hölder coefficient has finite moments of all orders.

The stochastic Volterra integral in (1.6) is well defined as an Itô integral. Indeed, there exists a constant $C > 0$ such that $|W(t) - W(t - y)| \leq Ct^{\alpha - 1} \cdot (t \wedge y)$ for any $t, y \geq 0$. A simple calculation shows that for any $t \geq 0$,

$$
\int_0^t ds \int_0^\infty |W(t - s) - W(t - s - y)|^2\nu_\alpha(dy) < Ct^{\alpha}
$$

and hence the stochastic Volterra integral has finite quadratic variation. The Hölder regularity and the exact modulus of continuity of local times processes have already provided in [7] and [3] respectively. For the case $b = 0$, the finiteness of moments of the Hölder coefficient has been proved in [14]. Different to methods developed in the previous literature, we prove the Hölder regularity with the help of (1.6) and the Kolmogorov continuity theorem. The finiteness of moments of the Hölder coefficient is proved by using the Garsia-Rodemich-Rumsey inequality.

Conditioned on the event $\tau_\xi^L(\zeta) < \infty$, the local time process $L_\xi(\cdot, \tau_\xi^L(\zeta))$ will fall into the trap 0 in finite time, i.e. $\tau_0 := \inf\{t \geq 0 : L_\xi(t, \tau_\xi^L(\zeta)) = 0\} < \infty$ a.s. and $L_\xi(\tau_0 + t, \tau_\xi^L(\zeta)) = 0$ for any $t \geq 0$. However, this asymptotic result is difficult to be obtained from the solution $X_\zeta$ to (1.6) because the Markov property fails to hold for $X_\zeta$. Indeed, by interpreting the stochastic Volterra integral as a
convolution in intuition, we observe that (1.6) is path-dependent and its solutions may move continually into the negative half line after hitting 0. Fortunately, thanks to the weak uniqueness, we can assert that 0 is an absorbing state for any solution to (1.6).

When \( b = 0 \), the \( Lévy \) process \( ξ \) is recurrent with \( L_ξ(0, \infty) = \infty \) a.s. and hence the conditional law in Theorem 1.1 reduces to an unconditional law. When \( b > 0 \), the process \( ξ \) is transient with \( L_ξ(0, \infty) < \infty \) a.s., then with positive probability \( τ^L_ξ(ζ) < \infty \) for any \( ζ \geq 0 \). We now generalize the classical second Ray-Knight theorem with unconditional law to the transient spectrally positive stable process \( ξ \) with \( b > 0 \). For \( λ \geq 0 \), let \( \{U^λ(x, dy) : x \in \mathbb{R}\} \) be the potential measures of \( ξ \), also known as the resolvent kernel, with

\[
U^λ(x, A) := \int_0^∞ e^{-λt} \mathbb{P}\{ξ(t) \in A\} dt
\]

for any set \( A \) in the Borel \( σ \)-algebra \( \mathcal{B}(\mathbb{R}) \). Specially, the measure \( U^λ(0, dy) \) is absolutely continuous with respect to the Lebesgue measure and its density, denoted as \( \{u^λ(y) : y \in \mathbb{R}\} \), is bounded, positive and continuous; see Theorem 16 and 19 in [4, p.61-65]. The process \( \{τ^L_ξ(ζ) : ζ \geq 0\} \) is a subordinator killed at an independent exponential time with mean \( u^0(0) \) and its Laplace transform is of the form

\[
\mathbb{E}\{\exp\{-λτ^L_ξ(ζ)\}\} = \exp\{-ζ/u^0(0)\}, \quad λ \geq 0.
\]

This yields that \( L_ξ(0, \infty) \) is an exponential random variable with mean \( u^0(0) \). In the next theorem we define \( g \) to be an exponential random variable with mean \( u^0(0) \), which is independent of the Poisson random measure \( N_λ(ds, dy, dz) \).

**THEOREM 1.2.** When \( b > 0 \), the local process \( \{L_ξ(t, τ^L_ξ(L_ξ(0, \infty))) : t \geq 0\} \) is the unique weak solution to (1.6) with \( ζ = g \).

In the second part of this paper we understand the law of the solution \( X_ξ \) by providing explicit representations for its characteristic functionals in terms of solutions to some associated nonlinear Volterra equation. In addition, we should indicate that the weak uniqueness of solutions to (1.6) is proved by verifying the uniqueness of solutions to the associated nonlinear Volterra equations. As we mentioned before, because of the stochastic Volterra integral, solutions to (1.6) are neither Markov nor semi-martingale, which makes it not possible to prove the pathwise uniqueness using the method developed in [9, 15]. In the next theorem we define

\[
K(t) := \frac{c \cdot t^{-α}}{Γ(1-α)}, \quad t > 0
\]

and \( \mathcal{V}_α \) to be an operator acting on an \( \mathbb{C} \)-valued function \( f \) by

\[
\mathcal{V}_α f(t) := \int_0^∞ \left( \exp\left\{ \int_{t-(t-y)^+}^t f(r) dr \right\} - 1 - \int_{t-(t-y)^+}^t f(r) dr \right) ν_α(dy), \quad t \geq 0.
\]

Let \( \mathbb{C}_- \) be the space of all complex numbers with non-positive real part. Let \( B(\mathbb{R}^+; \mathbb{C}_-) \) be the space of all bounded \( \mathbb{C}_- \)-valued functions on \( \mathbb{R}^+ \). Denote by \( f * g \) the convolution of two functions \( f \) and \( g \) on \( \mathbb{R}^+ \), i.e., \( f * g(t) = \int_0^t f(t-s)g(s)ds \) for any \( t \geq 0 \).

**THEOREM 1.3.** For any \( λ \in \mathbb{C}_- \) and \( g \in B(\mathbb{R}^+; \mathbb{C}_-) \), we have

\[
\mathbb{E}\{\exp\{λX_ξ(T) + g * X_ξ(T)\}\} = \exp\{ζ \cdot K * v_λ^g(T)\}, \quad T \geq 0,
\]

where \( \{v_λ^g(t) : t \geq 0\} \) is the unique global solution to the nonlinear Volterra equation

\[
v_λ^g(t) = λW'(t) + (g + \mathcal{V}_α v_λ^g) * W'(t).
\]
Noting that the derivative function $W'$ is singular at 0 and so is the solution $v^0_{\lambda}(t)$, we should consider $K * v^0_{\lambda}(0)$ as the right limit of $K * v^0_{\lambda}$ at 0. Indeed, we show that $K * W'(t) \sim 1$ as $t \to 0^+$ and $V_\alpha v^0_{\lambda}(t)$ can be uniformly bounded by $C t^{\alpha - 1}$ in the neighbor of 0. Thus as $t \to 0^+$,

$$K * v^0_{\lambda}(0) \sim K * v^0_{\lambda}(t) = \lambda + o(t^\alpha)$$

and hence the equality (1.9) still holds true for $T = 0$. Actually, the representation (1.9)-(1.10) indicates that the solution $X_\xi$ is a fully novel non-Markov branching system. We refer it as a rough continuous-state branching process because of the following two properties:

- It has the branching property and 0 is an absorbing state. Indeed, for any $\zeta_1, \zeta_2 > 0$, let $X_{\zeta_1}$ and $X_{\zeta_2}$ be two independent unique weak solutions to (1.6) with initial state $\zeta_1$ and $\zeta_2$ respectively. Then we have $X_{\zeta_1} + X_{\zeta_2} = X_{\zeta_1 + \zeta_2}$ in distribution, where $X_{\zeta_1 + \zeta_2}$ is the unique weak solution to (1.6) with initial state $\zeta_1 + \zeta_2$;
- Compared to the Feller branching diffusion that is locally Hölder-continuous of any order strictly less than 1/2, the process $X_\xi$ has a lesser degree of Hölder regularity and hence its simple paths are much rougher.

As the last result in this paper, we provide in the next theorem alternate representations for the stochastic Volterra equation (1.6) and the nonlinear Volterra equation (1.10) in terms of fractional integration and differential equation. Let $I^\rho_t$ and $D^\rho_t$ be the Riemann-Liouville fractional integral and derivative operator of order $\rho \in (0, 1]$ modified by a constant $a > 0$. They act on a measurable function $f$ on $\mathbb{R}_+$ according to

$$I^\rho_t f(t) := \frac{1}{a \Gamma(\rho)} \int_0^t (t-s)^{\rho-1} f(s) ds \quad \text{and} \quad D^\rho_t f(t) := \frac{a}{\Gamma(1-\rho)} \frac{d}{dt} \int_0^t (t-s)^{-\rho} f(s) ds.$$

For simplicity, we write $I^\rho_t = I^\rho_1$ and $D^\rho_t = D^\rho_1$.

THEOREM 1.4. The stochastic Volterra equation (1.6) is equivalent to

$$X_\xi(t) = \zeta - b \int_0^t \frac{(t-s)^{\alpha-1}}{\alpha \Gamma(\alpha)} X_\xi(s) ds + \int_0^t \int_0^\infty \int_0^{t-s} \int_{(t-s-y)-r}^{t-s} r^{\alpha-1} \frac{d}{dr} N_\alpha(ds, dy, dz)$$

and the nonlinear Volterra equation (1.10) is equivalent to

$$D^\alpha_t v^0_{\lambda}(t) = -b v^0_{\lambda}(t) + g(t) + V_\alpha v^0_{\lambda}(t), \quad I_{c}^{1-\alpha} v^0_{\lambda}(0) = \lambda.$$

Our main idea of the generalization of the second classic Ray-Knight theorem to spectrally positive stable processes is characterizing the the cluster points of local times processes of a sequence of nearly recurrent compound Poisson processes with negative drift and Pareto-distributed jump distribution. An important tool used in the characterizations is a new representation, named as Hawkes representation, for the local time processes of compound Poisson processes in terms of stochastic Volterra-Fredholm integral equation driven by a Poisson random measure. It is established based on the following two properties:

- The local time process of a compound Poisson process with drift $-1$ stopped at hitting 0 is equal in distribution to a homogeneous, binary Crump-Mode-Jagers process (CMJ-process) starting from one ancestor; see [29].
- A homogeneous, binary CMJ-process can be reconstructed as the density process of a marked Hawkes point measure with arrivals and marks of events representing the birth times and life-lengths of offsprings; see [21, 37].
The main difficulty we encountered is the limit characterization of the stochastic Volterra integrals in the new representation. Indeed, because of the absence of martingale property, the instruments provided by modern probability theory, e.g., the martingale representation theorem and martingale problems, are out of work. To overcome this difficulty, we first find a good approximation for the stochastic Volterra integral by exploring the genealogy and evolution of the CMJ-process. The tightness of the approximation processes is proved by using Kolmogorov tightness criterion. For the convergence in finite-dimensional distributions, we first construct a martingale that is equal to the approximation process in the finite-dimensional distributions at the given time points. Then we show that the martingales converge to a limit process whose finite-dimensional distributions at the given time points equal to those of the stochastic Volterra integral in (1.6).

Different types of limit results have been established for self-exciting system in many literature, but none of them contains our case. A functional central limit theorem has been established for multivariate Hawkes processes in [2] and for Marked Hawkes point measures in [21] respectively. For the nearly unstable Hawkes process with light-tailed kernel, Jaisson and Rosenbaum [24] proved the weak convergence of the rescaled intensity to a Feller diffusion, which was generalized to multivariate marked Hawkes point processes and their shot noise processes in [37]. For the heavy-tailed case, they also proved that the rescaled point process converges weakly to the integral of a rough fractional diffusion; see [13, 25]. However, they left the weak convergence of the rescaled intensity as an open problem.

Our exponential representations (1.9) of the characteristic functionals are established by extending the method developed in the proof of Theorem 4.3 in [1]. The key step in the proof is to write the Doob martingale related to \( \lambda X_\zeta(T) + (g + \nu_\alpha v_\zeta^2) \ast X_\zeta(T) \) as a stochastic integral with respect to a compensated Poisson random measure. Furthermore, because of the impact of the nonlinear operator \( \mathcal{V} \) defined by (1.8), the existence and uniqueness of solution to (1.10) turn to be a challenge. Indeed, different to the nonlinear Volterra equations that have been widely studied in many literature, e.g., [1, 18], we shall see that the operator \( \mathcal{V} \) not only fails to be Lipschitz continuous but also turns (1.10) into path-dependent. To overcome this difficulty, we first give a crucial prior-estimate about the asymptotics of solutions to (1.10). It not only reveals the convex sets in which solutions are but also guarantees that any local solution can be successfully extended on the whole half real line. We then improve the standard proofs well developed in the previous literature and apply them together with the prior-estimate to solve (1.10).

Organization of the paper. In Section 2, we provide some properties of the scale function and then introduce a sequence of compound Poisson processes whose local time processes converge weakly to that of a spectrally positive stable process. In Section 3 we give the Hawkes representations for the local times processes of compound Poisson processes. The local time processes of spectrally positive stable processes is characterized in Section 4 and the representation (1.9) of their characteristic functionals is proved in Section 5. The existence and uniqueness of solution to (1.10) are proved in Section 6. Finally, we prove Theorem 1.4 in Section 7.

Notation. For the background and notation of Lévy processes we refer to [4]. For notational convenience, we write \( \mathbb{R}_+ := [0, \infty) \), \( \mathbb{R}_- := (-\infty, 0] \), \( \mathbb{C}_- := \{ z \in \mathbb{C} : \Re z \in \mathbb{R}_- \} \) and \( i\mathbb{R} := \{ z \in \mathbb{C} : \Re z = 0 \} \). For any \( x \in \mathbb{R} \), let \( x^+ = \max\{x, 0\} \), \( x^- = \min\{x, 0\} \) and \( \lfloor x \rfloor \) be the floor of \( x \). Given a topology space \( \mathbb{E} \), let \( \mathcal{D}((0, \infty), \mathbb{E}) \) be the space of càdlàg \( \mathbb{E} \)-valued functions endowed with Skorokhod topology.

Let \( \Delta_h \) and \( \nabla_h \) be the forward and backward difference operator with step size \( h > 0 \) respectively, i.e., \( \Delta_h f(x) := f(x + h) - f(x) \) and \( \nabla_h f(x) := f(x) - f(x - h) \). Throughout this paper, we make the conventions
\[
\int_x^y = \int_y^x = \int_{(x, y]} = \int_{(y, x]} = \int_{[x, y]} = \int_{[y, x]} = \int_{(0, \infty)} = \int_{(\infty, 0)}
\]
for \( x, y \in \mathbb{R} \) satisfying that \( y \geq x \). For any \( 0 \leq a \leq b \leq \infty \) and \( p \in (0, \infty] \), let \( L^p(U; \mathbb{C}) \) be the space of \( \mathbb{C} \)-valued all \( p \)-integrable functions, i.e.,
\[
L^p((a, b]; \mathbb{C}) := \left\{ f : U \mapsto \mathbb{C} : \|f\|_{L^p_{(a, b)}} := \int_a^b |f(x)|^p \, dx \right\}.
\]
For simplicity, we write \( \|f\|_{L^p_b} = \|f\|_{L^p([0,b])} \).

We use \( C \) to denote a positive constant whose value might change from line to line.

2. Preliminaries.

2.1. The scale function. As shown by SDE-representation of local time processes, the scale function \( W \) plays a crucial role in our following proofs and analysis. We here provide some elements about the regularity and asymptotic behavior of this function. Define the Mittag-Leffler function \( E_{\alpha,\alpha} \) on \( \mathbb{R}_+ \) by

\[
E_{\alpha,\alpha}(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha(n+1))}.
\]

It is locally Hölder continuous with index \( \alpha \); see [19] for a precise definition of it and a survey of some of its properties, e.g. for any \( a \geq 0 \) we have the well-known Laplace transform

\[
(2.1) \quad \int_0^\infty e^{-\lambda x} ax^{a-1} E_{\alpha,\alpha}(-a \cdot y^\alpha) dx = \frac{a}{a + \lambda^\alpha}, \quad \lambda \geq 0.
\]

Applying integration by parts to (2.1), we have

\[
\int_0^\infty e^{-\lambda x} W'(x) dx = \int_0^\infty \lambda e^{-\lambda x} W(x) dx = \frac{1}{b + c\lambda^\alpha}, \quad \lambda > 0.
\]

From this and (2.1), we see that the derivative function \( W' \) is of the form:

\[
W'(x) = c^{-1} x^{a-1} \cdot E_{\alpha,\alpha}(-b/c \cdot x^\alpha), \quad x \geq 0.
\]

Specially, when \( b = 0 \) we have \( E_{\alpha,\alpha}(0) = 1/\Gamma(\alpha) \) and

\[
W'(x) = \frac{x^{a-1}}{c\Gamma(\alpha)}, \quad W(x) = \frac{x^\alpha}{c\Gamma(1 + \alpha)}.
\]

When \( b > 0 \), according to (2.1) we see that \( bW' \) is the density function on \( \mathbb{R}_+ \) called Mittag-Leffler density function and hence \( W(x) \to 1/b \) as \( x \to \infty \). Moreover, the following asymptotic properties of \( W \) and \( W' \) are direct consequences of the properties of Mittag-Leffler density function; see [19, 34, 33]. When \( b > 0 \), we have as \( x \to 0^+ \),

\[
W'(x) \sim \frac{x^{a-1}}{c\Gamma(\alpha)}, \quad W(x) \sim \frac{x^\alpha}{c\Gamma(1 + \alpha)}
\]

and as \( x \to \infty \),

\[
W'(x) \sim \frac{c\alpha \cdot x^{-a-1}}{\Gamma(1 - \alpha)}, \quad W(x) \sim \frac{1}{b} - \frac{c x^{-a}}{\Gamma(1 - \alpha)}.
\]

In conclusion, for \( b \geq 0 \) there exists a constant \( C > 0 \) such that for any \( t > 0 \),

\[
(2.3) \quad W'(t) \leq Ct^{\alpha-1} \quad \text{and} \quad W(t) \leq Ct^\alpha.
\]

Recall the function \( K \) defined by (1.7). For \( t > 0 \), let

\[
L_K(t) := t^{\alpha-1} \frac{1}{c\Gamma(\alpha)}.
\]

It is easy to show that the two functions \( L_K \) and \( K \) satisfy the Sonine equation, i.e. \( K \ast L_K \equiv 1 \), while \((L_K, K)\) is known as Sonine pair. In addition, when \( b > 0 \) the function \( bW' \) is the resolvent of \( bL_K \), which is usually introduced by means of the resolvent equation

\[
(2.4) \quad bW' = bL_K - (bL_K) \ast (bW').
\]

Convolving both sides of this resolvent equation by \( K \) and then dividing them by \( b \), we have

\[
(2.5) \quad K \ast W' = 1 - bW.
\]
2.2. Compound Poisson Processes. Let $\Lambda$ be a Pareto II distribution on $\mathbb{R}_+$ with location $0$ and shape $\alpha + 1$, i.e.,

$$\Lambda(dx) = (x + 1)^{-\alpha - 2}dx.$$ 

For $n \geq 1$, let $\{\xi^{(n)}(t) : t \geq 0\}$ be a compound Poisson process with drift $-1$, arrival rate $\gamma_n > 0$ and jump size distribution $\Lambda$. It is a spectrally positive Lévy process with Laplace exponent

$$\phi^{(n)}(\lambda) := \lambda + \int_0^\infty (e^{-\lambda x} - 1)\gamma_n\Lambda(dx), \quad \lambda \geq 0.$$ 

We are interested in the case $E[\xi^{(n)}(1)] = \gamma_n/\alpha - 1 \leq 0$ in which the function $\phi^{(n)}$ increases strictly to infinity. For any probability measure $\nu$ on $\mathbb{R}$, we denote by $P_\nu$ and $E_\nu$ the law and expectation of a Lévy process started from $\nu$, respectively. When $\nu = \delta_x$ is a Dirac measure at point $x$, we write $P_x = P_{\delta_x}$ and $E_x = E_{\delta_x}$. For simplicity, we also write $P = P_0$ and $E = E_0$. For any $A \in \mathcal{F}$, we denote by $P(A)$ the conditional law given the event $A$.

Let $\{L_{\xi^{(n)}}(y,t) : y \in \mathbb{R}, t \geq 0\}$ be the local time process of $\xi^{(n)}$ with

$$L_{\xi^{(n)}}(y,t) := \#\{s \in (0,t] : \xi^{(n)}(s) = y\}.$$ 

We have $L_{\xi^{(n)}}(y,\infty) = \infty$ a.s. if and only if $E[\xi^{(n)}(1)] = 0$; equivalently, if and only if $\gamma_n = \alpha$. The inverse local time $\{\tau_{\xi^{(n)}}^L(k) : k = 0, 1, \ldots\}$ at level $0$ is given by $\tau_{\xi^{(n)}}^L(k) = \infty$ if $k > L_{\xi^{(n)}}(y,\infty)$ and

$$\tau_{\xi^{(n)}}^L(k) := \min\{t > 0 : L_{\xi^{(n)}}(0,t) = k\}, \quad k = 0, 1, \ldots, L_{\xi^{(n)}}(y,\infty).$$

We now consider the structure of the local time process $\{L_{\xi^{(n)}}(t, \tau_{\xi^{(n)}}^L(k)) : t \geq 0\}$. Denote by $\tau_{\xi^{(n)}}^+$ the first passage time of $\xi^{(n)}$ in $[0, \infty)$, i.e., $\tau_{\xi^{(n)}}^+ := \inf\{t > 0 : \xi^{(n)}(t) \geq 0\}$. Note that $\xi^{(n)}$ will move from negative half line into positive half line by jumping, i.e., if $\xi^{(n)}(\tau_{\xi^{(n)}}^+ - ) < 0$ we have $\xi^{(n)}(\tau_{\xi^{(n)}}^+) > 0$ a.s. By Theorem 17(ii) in [4, p.204], we have $E[\xi^{(n)}(\tau_{\xi^{(n)}}^+/)] \leq 0$ under $P(\cdot | \tau_{\xi^{(n)}}^+ < \infty)$ is equal to distribution to the size-biased distribution of $\Lambda$ defined by

$$\Lambda^*(dx) := \alpha\overline{\Lambda}(x)dx = \alpha(1 + x)^{-1-\alpha}dx.$$ 

Here $\overline{\Lambda}(x) := \Lambda(x, \infty)$ is the tail distribution of $\Lambda$. Since the simple paths before the first passage time $\tau_{\xi^{(n)}}^+$ do not contribute to the local time process $\{L_{\xi^{(n)}}(t, \tau_{\xi^{(n)}}^L(k)) : t \geq 0\}$, by the strong Markov property of $\xi^{(n)}$ we can get the following proposition immediately.

**Proposition 2.1.** For any $n, k \geq 1$, the local time process $\{L_{\xi^{(n)}}(t, \tau_{\xi^{(n)}}^L(k)) : t \geq 0\}$ has the same law under either $P(\cdot | \tau_{\xi^{(n)}}^L(k) < \infty)$ or $P_{\Lambda^*}(\cdot | \tau_{\xi^{(n)}}^L(k) < \infty)$. Moreover, it is equal in distribution to the sum of $k$ i.i.d. copies of $\{L_{\xi^{(n)}}(t, \tau_{\xi^{(n)}}^L(1)) : t \geq 0\}$ under $P_{\Lambda^*}$.

We now consider under the following condition, the asymptotic behavior of the rescaled process $\{\xi_0^{(n)}(t) : t \geq 0\}$ with $\xi_0^{(n)}(t) := n^{-1}\xi^{(n)}(n^{1+\alpha}t)$ and its local time process $\{L_{\xi_0^{(n)}}(y,t) : y,t \geq 0\}$.

**Condition 2.2.** Assume that $n^{\alpha}(1 - \gamma_n/\alpha) \to \beta \geq 0$ as $n \to \infty$.

It is easy to see that the Lévy process $\xi_0^{(n)}$ has Laplace exponent $n^{1+\alpha}\phi^{(n)}(\lambda/n)$ for $\lambda \geq 0$. Under Condition 2.2, a routine computation shows that $n^{1+\alpha}\phi^{(n)}(\lambda/n) \to \phi_0(\lambda) := \beta\lambda + \Gamma(1 - \alpha)\lambda^\alpha$ as $n \to \infty$. By Corollary 4.3 in [23, p.440], we have $\xi_0^{(n)} \to \xi_0$ weakly in $\mathcal{D}([0,\infty), \mathbb{R})$, where $\xi_0$ is a spectrally positive $\alpha$-stable process with Lévy exponent $\phi_0$. For the process $\xi_0$, we can define the local time process $\{L_{\xi_0}(y,t) : y \in \mathbb{R}, t \geq 0\}$, inverse local time $\{\tau_{\xi_0}^L(\zeta) : \zeta \geq 0\}$, scale function $W_0$ and
its derivative $W_t'$ in the same way as for $\xi$. Applying the occupation density formula, we immediately have $\{L_{\xi(t)}(y, t) : y \in \mathbb{R}, t \geq 0\}$ is equal almost surely to $\{n^{-\alpha}L_{\xi(t)}(ny, n^{1+\alpha}t) : y \in \mathbb{R}, t \geq 0\}$, i.e. for every nonnegative measurable function $f$ on $\mathbb{R}$,

$$\int_{\mathbb{R}} f(y)L_{\xi(t)}(y, t)dy = \int_{0}^{t} f(n^{-1}\xi(n)^{(1+\alpha)}/s))ds = \int_{\mathbb{R}} f(y) \cdot n^{-\alpha}L_{\xi(t)}(ny, n^{1+\alpha}t)dy.$$  

(2.7) For any $\zeta > 0$, we also have $\tau_{\xi(t)}(\zeta) = n^{-1-\alpha}\tau_{\xi(t)}(\zeta n^\alpha)$ a.s. and hence

$$\{L_{\xi(t)}(t, \tau_{\xi(t)}(\zeta)) : t \geq 0\} = \{n^{-\alpha}L_{\xi(t)}(nt, \tau_{\xi(t)}(\zeta n^\alpha)) : t \geq 0\}, \quad \text{a.s.}$$  

(2.8) The following convergence result for the local time processes $\{L_{\xi(t)}(\cdot, \tau_{\xi(t)}(\zeta))\}_{n \geq 1}$ comes from Theorem 2.4 in [30].

**Lemma 2.3.** For $\zeta > 0$, the local time process $\{L_{\xi(t)}(t, \tau_{\xi(t)}(\zeta)) : t \geq 0\}$ under $\mathbb{P}\{\cdot|\tau_{\xi(t)}(\zeta) < \infty\}$ converges weakly to $\{L_{\xi(t)}(t, \tau_{\xi(t)}(\zeta)) : t \geq 0\}$ under $\mathbb{P}\{\cdot|\tau_{\xi(t)}(\zeta) < \infty\}$ in $\mathcal{D}([0, \infty); \mathbb{R}^+)$ as $n \to \infty$.

3. Hawkes representation. In this section we provide the new representations for the local time processes of compound Poisson processes in terms of marked Hawkes point processes and then rewrite them as solutions to a class of stochastic Volterra integral equations. These new representations will play the key role in characterizing the local time processes of spectrally positive stable processes.

3.1. Marked Hawkes point processes. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with filtration $\{\mathcal{F}_t : t \geq 0\}$ that satisfies the usual hypotheses and $\mathbb{U}$ be a Lusin topological space endowed with the Borel $\sigma$-algebra $\mathscr{B}$. Let $\{\eta_k : k = 1, 2, \cdots\}$ be a sequence of increasing, $(\mathcal{F}_t)$-adapted random times and $\{\eta_k = 1, 2, \cdots\}$ be a sequence of i.i.d. $\mathbb{U}$-valued random variables with distribution $\nu_H(du)$. We assume that $\eta_k$ is independent of $\{\eta_j : j = 1, 2, k\}$ for any $k \geq 0$. In terms of these two sequences we define the $(\mathcal{F}_t)$-random point measure

$$N_H(ds, du) := \sum_{k=1}^{\infty} 1_{\{\eta_k \in ds, \eta_k \in du\}}$$  

(3.1) on $(0, \infty) \times \mathbb{U}$. We say $N_H(ds, du)$ is a marked Hawkes point measure on $\mathbb{U}$ if the embedded point process $\{N_H(t) : t \geq 0\}$ defined by $N_H(t) := N_H((0, t], \mathbb{U})$ has an $(\mathcal{F}_t)$-intensity $\{\gamma \cdot Z(t) : t \geq 0\}$ with $\gamma > 0$ and

$$Z(t) = \mu(t) + \sum_{k=1}^{\infty} \phi(t - \sigma_k, \eta_k), \quad t \geq 0,$$

for some kernel $\phi : \mathbb{R}_+ \times \mathbb{U} \to [0, \infty)$ and some $\mathcal{F}_0$-measurable, nonnegative functional-valued random variable $\{\mu(t) : t \geq 0\}$. We usually interpret $\phi(\cdot, u)$ and $\mu$ as the impacts of an event with mark $u$ and all events prior to time $t$ on the arrival of future events respectively.

By the independence between $\eta_k$ and $\{\eta_i : i = 1, \cdots, k\}$ for any $k \geq 1$, we see that the random point measure $N_H(ds, du)$ defined by (3.1) has the intensity $\gamma Z(s-)dsH(du)$. Following the argument in [22, p.93], on an extension of the original probability space we can define a time-homogeneous Poisson random measure $N(ds, du, dz)$ on $(0, \infty) \times \mathbb{U} \times \mathbb{R}_+$ with intensity $\gamma dsH(du)dz$ such that

$$N_H(ds, du) = \int_{0}^{\infty} N(ds, du, dz)$$

and hence the intensity $Z(t)$ at time $t$ can be rewritten into

$$Z(t) = \mu(t) + \int_{0}^{t} \int_{\mathbb{U}} \int_{0}^{\infty} \phi(t - s, u)N(ds, du, dz).$$
Before proceeding to establish another stochastic Volterra integral representation for the intensity process, we need to introduce several quantities associated to the kernel. Denote by \( \{ \phi_H(t) : t \geq 0 \} \) the mean impacts of an event on the arrival of future events with

\[
\phi_H(t) := \gamma \cdot \int_U \phi(t, u) \nu_H (du).
\]

We always assume \( \phi_H \) is locally integrable. Let \( \{ R_H(t) : t \geq 0 \} \) be the resolvent of \( \phi_H \) defined as the unique solution to

\[
R_H(t) = \phi_H(t) + \phi_H * R_H(t).
\]

It is usual to interpret \( R_H \) as the mean impacts of an event and its triggered events on the arrivals of future events. In addition, we introduce a two-parameter function on \( \mathbb{R}_+ \times U \)

\[
R(t, u) = \phi(t, u) + R_H * \phi(t, u)
\]

to describe the mean impacts of an event with mark \( u \) on the arrivals of future events. An argument similar to the one used in Section 2 in [?] induces the following proposition immediately.

**Proposition 3.1.** The intensity process \( Z \) satisfies the stochastic Volterra integral equation

\[
Z(t) = \mu(t) + \int_0^t R_H(t - s) \mu(s) ds + \int_0^t \int_U \int_0^{Z(s -)} R(t - s, u) \tilde{N}(ds, du, dz),
\]

where \( \tilde{N}(ds, du, dz) := N(ds, du, dz) - \gamma dv_H (du) dz \).

**3.2. Hawkes representation.** We now give a Hawkes representation for local time processes of the compound Poisson processes defined in Section 2.2 based on their connection with a class of homogeneous, binary CMJ-processes; see the next lemma. This connection was first established by Lambert in [29]; see also Theorem 3.2 in [31], based on the observation that the jumping contour process of a homogeneous, binary CMJ-tree starting from one ancestor is a compound Poisson process with drift \(-1\); conversely, the local time process of a compound Poisson process with drift \(-1\) stopped at hitting 0 is equal in distribution to a homogeneous, binary CMJ-process starting from one ancestor.

**Lemma 3.2.** For \( n, k \geq 1 \), the local time process \( \{ L_{\xi^{(n)}}(t, \tau_{\xi^{(n)}}^{(L)}(k)) : t \geq 0 \} \) under \( P(\cdot | \tau_{\xi^{(n)}}^{(L)}(k) < \infty) \) is equal in distribution to a homogeneous, binary CMJ-process defined by the following properties:

(P1) There are \( k \) ancestors with residual life distributed as \( \Lambda^* \);

(P2) Offsprings have a common life-length distribution \( \Lambda \);

(P3) Each individual gives birth to its children following a Poisson process with rate \( \gamma_n \).

To establish the Hawkes representation for the local time process, we first need to clarify the genealogy of the associated CMJ-process defined properties (P1)-(P3). In the \( n \)-th population, denote by \( \ell_{0,i} \) the residual life of the \( i \)-th ancestor at time 0. We sort all offsprings by their birth times. For the \( i \)-th offspring, we endow it with the pair \((\varsigma_i, \ell_i)\) to represent its birth time and life-length respectively. The population size at time \( t \), denoted as \( Z^{(n)}(t) \), can be written as

\[
Z^{(n)}(t) = \sum_{j=1}^k 1_{\{ \ell_{0,j} > t \}} + \sum_{i \leq t} 1_{\{ \ell_i > t - \varsigma_i \}}.
\]

We define a random point measure \( N_H^{(n)}(dt, dy) \) associated to the sequence \( \{(\varsigma_i, \ell_i) : i = 1, 2, \ldots \} \) by

\[
N_H^{(n)}(dt, dy) = \sum_{i=1}^{\infty} 1_{\{ \varsigma_i \in dt, \ell_i \in dy \}}.
\]
By the branching property, at time $t$ a new offspring would be born at the rate $\gamma_n Z^{(n)}(t-)$. Thus the random point measure $N_H^{(n)}(ds, dy)$ has an $(\Theta_t)$-intensity $\gamma_n Z^{(n)}(t-)dt\Lambda(dy)$. Following the argument in the last section, we can rewrite the population process $Z^{(n)}$ into

\begin{equation}
Z^{(n)}(t) = \sum_{j=1}^{k} 1_{\{\ell_j > t\}} + \int_{0}^{t} \int_{0}^{\infty} \int_{0}^{\infty} 1_{\{y > t-s\}} N_0^{(n)}(ds, dy, dz),
\end{equation}

where $N_0^{(n)}(ds, dy, dz)$ is a Poisson random measure on $(0, \infty) \times \mathbb{R}^2$ with intensity $\gamma_n ds\Lambda(dy)dz$. Thus $N_H^{(n)}(ds, dy)$ is a marked Hawkes point measure on $\mathbb{R}$. Applying Proposition 3.1 to (3.2), we can immediately get the following Hawkes representation for the population process $Z^{(n)}$ and hence for the local time process $L_{\xi^{(n)}}(\cdot, \tau_{\xi^{(n)}}^{L}(k))$.

**Lemma 3.3.** For $n, k \geq 1$, the local time process $\{L_{\xi^{(n)}}(t, \tau_{\xi^{(n)}}^{L}(k)) : t \geq 0\}$ under $\mathbf{P}(\cdot | \tau^{L}_{\xi^{(n)}}(k) < \infty)$ is equal in distribution to the unique solution of the stochastic Volterra integral equation

\begin{equation}
Z^{(n)}(t) = \sum_{j=1}^{k} 1_{\{\ell_j > t\}} + \int_{0}^{t} R^{(n)}_{H}(t-s) \sum_{j=1}^{n} 1_{\{\ell_j > s\}} ds
+ \int_{0}^{t} \int_{0}^{\infty} \int_{0}^{\infty} Z^{(n)}(s-) R^{(n)}(t-s, y) N_0^{(n)}(ds, dy, dz),
\end{equation}

where $\tilde{N}_0^{(n)}(ds, dy, dz) := N_0^{(n)}(ds, dy, dz) - \gamma_n ds\Lambda(dy)dz$, $R^{(n)}_{H}$ and $R^{(n)}$ are defined by

\begin{align}
R^{(n)}_{H}(t) &= \gamma_n \Lambda(t) + \gamma_n \Lambda * R^{(n)}_{H}(t), \\
R^{(n)}(t, y) &= 1_{\{y > t\}} + \int_{0}^{t} R^{(n)}_{H}(t-s) \cdot 1_{\{y > s\}} ds.
\end{align}

**4. Characterizations of local time processes.** In this section we first provide a characterization for the local time process $L_{\xi^{(n)}}(\cdot, \tau_{\xi^{(n)}}^{L}(\zeta))$ as the cluster point of the sequence $\{L_{\xi^{(n)}}(\cdot, \tau_{\xi^{(n)}}^{L}(\zeta))\}_{n \geq 1}$ and then generalize it to $L_{\zeta^{(n)}}(\cdot, \tau_{\zeta^{(n)}}^{L}(\zeta))$. In the sequel of this section, we always assume Condition 2.2 holds.

For $n \geq 1$, let $Z^{(n)}$ be the unique solution to (3.3) with $k = [\zeta n^\alpha]$. Apparently the equivalence (2.8) along with Lemma 3.2 implies that the process $\{L_{\xi^{(n)}}(t, \tau_{\xi^{(n)}}^{L}(\zeta)) : t \geq 0\}$ under $\mathbf{P}(\cdot | \tau_{\xi^{(n)}}^{L}(\zeta) < \infty)$ is equal in distribution to the process $\{X_{\zeta}^{(n)}(t) : t \geq 0\}$ with $X_{\zeta}^{(n)}(t) := n^{-\alpha} Z^{(n)}(nt)$. From Lemma 2.3, we have $X_{\zeta}^{(n)}$ converges weakly to a limit process, denoted as $X_{0, \zeta}$, in $\mathbf{D}([0, \infty), \mathbb{R})$ as $n \to \infty$ and the process $X_{0, \zeta}$ is equal in distribution to $\{L_{\xi^{(n)}}(t, \tau_{\xi^{(n)}}^{L}(\zeta)) : t \geq 0\}$ under $\mathbf{P}(\cdot | \tau_{\xi^{(n)}}^{L}(\zeta) < \infty)$. In addition, by Lemma 3.3 it is not difficult to verify that

\begin{equation}
X_{\zeta}^{(n)}(t) = n^{-\alpha} \sum_{i=1}^{[\zeta n^\alpha]} 1_{\{\ell_i > nt\}} + \int_{0}^{nt} \int_{0}^{\infty} \int_{0}^{\infty} X_{\zeta}^{(n)}(s-) n^{-\alpha} R^{(n)}(nt-s) n^{-\alpha} \sum_{i=1}^{[\zeta n^\alpha]} 1_{\{\ell_i > s\}} ds
+ \int_{0}^{t} \int_{0}^{\infty} \int_{0}^{\infty} X_{\zeta}^{(n)}(s-) n^{-\alpha} R^{(n)}(nt-s, ny) \tilde{N}_0^{(n)}(nds, ndy, n^\alpha dz),
\end{equation}

We now describe the intuitions about characterizing the cluster points of $\{X_{\zeta}^{(n)}\}_{n \geq 1}$, which is achieved by analyzing the asymptotics of each terms in (4.1). The detailed proofs will be provided later. By the Glivenko-Cantelli theorem, we may approximate the first two terms on the right side of (4.1) respectively with

\[
X_{\zeta}^{(n)}(0) \overline{\Lambda}(nt) \quad \text{and} \quad X_{\zeta}^{(n)}(0) \cdot R_{H}^{(n)} * \overline{\Lambda}(nt),
\]
where $\overline{\Lambda}(x) := \overline{\Lambda}(x, \infty)$. The approximation errors are denoted as $\varepsilon_{0}^{(n)}(t)$ and $\varepsilon_{1}^{(n)}(t)$ respectively. Integrating both sides of (3.4) on $(0, nt]$ and then changing the order of integration, we have

$$\frac{\alpha}{\gamma_n} || P^{(n)}_H ||_{L^1} = \alpha || \overline{\Lambda} ||_{L^1} + \alpha : || \overline{\Lambda} * P^{(n)}_H ||_{L^1} = 1 - \overline{\Lambda}(nt) + (1 - \overline{\Lambda}) * R^{(n)}_H(nt).$$

Here (2.6) is also used in the second equality above. Moving 1 and $R^{(n)}_H(nt)$ to the left side of the first equality, it turns into

$$\overline{\Lambda}(nt) + \overline{\Lambda} * R^{(n)}_H(nt) = 1 - \left(1 - \frac{\gamma_n}{\alpha}\right) \frac{\alpha}{\gamma_n} || R^{(n)}_H ||_{L^1}$$

$$= 1 - n^\alpha \left(1 - \frac{\gamma_n}{\alpha}\right) \frac{\alpha}{\gamma_n} \int_0^t n^{1-\alpha} R^{(n)}_H(ns)ds$$

and hence the sum of the first two terms on the right side of (4.1) can be well approximated by

$$\bar{X}_\zeta^{(n)}(t) := X^{(n)}_\zeta(0) - X^{(n)}_\zeta(0) \cdot n^\alpha \left(1 - \frac{\gamma_n}{\alpha}\right) \frac{\alpha}{\gamma_n} \int_0^t n^{1-\alpha} R^{(n)}_H(ns)ds.$$

It is observed from this result and (4.1) that the asymptotic behavior of $\{n^{1-\alpha}R^{(n)}_H(nt) : t \geq 0\}$ plays a crucial role in the weak convergence of $\{X^{(n)}_\zeta\}_{n \geq 1}$.

**Lemma 4.1.** For any $T > 0$, as $n \to \infty$ we have uniformly in $t \in [0, T]$,

$$\int_0^t n^{1-\alpha} R^{(n)}_H(ns)ds \to W_0(t) \quad \text{and} \quad \bar{X}^{(n)}_\zeta(t) \to \zeta(1 - \beta W_0(t)).$$

**Proof.** Apparently the second uniform convergence follows immediately by applying the first one and Condition 2.2 to (4.2). We now prove the first uniform convergence. For $\lambda \geq 0$, denote by $L_{\overline{\Lambda}}(\lambda)$ and $L_{R^{(n)}_H}(\lambda)$ the Laplace transform of $\overline{\Lambda}$ and $R^{(n)}_H$ respectively. Taking the Laplace transform of both sides of (3.4), we have $L_{R^{(n)}_H}(\lambda) = \gamma_n L_{\overline{\Lambda}}(\lambda) \left[1 + L_{R^{(n)}_H}(\lambda)\right]$ and hence $L_{R^{(n)}_H}(\lambda) = \gamma_n L_{\overline{\Lambda}}(\lambda) \cdot (1 - \gamma_n L_{\overline{\Lambda}}(\lambda))^{-1}$. Thus

$$\int_0^\infty e^{-\lambda t} n^{1-\alpha} R^{(n)}_H(nt)dt = n^\alpha L_{R^{(n)}_H}(\lambda/n) = \frac{\gamma_n L_{\overline{\Lambda}}(\lambda/n)}{n^\alpha (1 - \gamma_n L_{\overline{\Lambda}}(\lambda/n))} = \frac{\gamma_n L_{\overline{\Lambda}}(\lambda/n)}{n^\alpha (1 - \frac{\gamma_n}{\alpha}) + \frac{\gamma_n}{\alpha} \cdot n^\alpha (1 - L_{\overline{\Lambda}}(\lambda/n))},$$

where $L_{\Lambda^*}$ denotes the Laplace transform of the measure $\Lambda^*$. A simple calculation along with Condition 2.2 shows that $\gamma_n L_{\overline{\Lambda}}(\lambda/n) \to 1$ as $n \to \infty$. Using the fact that $\overline{\Lambda}(t) = (1 + t)^{\alpha}$ together with Karamata-Tauberian theorem; see, e.g., Theorem 8.1.6 in [6, p.333], we have $n^\alpha (1 - L_{\overline{\Lambda}}(\lambda/n)) \to \Gamma(1 - \alpha)\lambda^\alpha$ as $n \to \infty$ and hence

$$\int_0^\infty e^{-\lambda t} n^{1-\alpha} R^{(n)}_H(nt)dt \to \frac{1}{\beta + \Gamma(1 - \alpha)\lambda^\alpha}.$$

By (2.1) the function whose Laplace transform is equal to the last quantity is $W_0'$ and hence the desired result follows.

We now start to analyze the asymptotics of the stochastic Volterra integral on the right side of (4.1). By (3.5) we first notice that for $t, y \geq 0$,

$$n^{-\alpha} R^{(n)}(nt, ny) = n^{-\alpha} 1_{\{y > t\}} + \int_0^t n^{1-\alpha} R^{(n)}_H(n(t-s)) 1_{\{y > s\}} ds.$$
Apparently the first term on the right side of this equality vanishes uniformly as \( n \to \infty \). Additionally, Lemma 4.1 suggests us to approximate the second term with
\[
\int_{(t-y)^+}^t n^{1-\alpha} R_H^{(n)}(ns) ds \sim \nabla_y W_0(t).
\]
Thus the stochastic Volterra integral in (4.1) can be well approximated by
\[
M_0^{(n)}(t) := \int_0^t \int_0^\infty \int_0^\infty \nabla_y W_0(t-s) \tilde{N}_0^{(n)}(nds, ndy, n^\alpha dz)
\]
with the error denoted as \( \varepsilon_2^{(n)}(t) \). In conclusion, based on all estimates and notation above we have
\[
(4.3) \quad X_\xi^{(n)}(t) = \sum_{k=0}^{2} \varepsilon_k^{(n)}(t) + \hat{X}_\xi^{(n)}(t) + M_0^{(n)}(t).
\]
Lemma 4.1 shows that \( \hat{X}_\xi^{(n)} \) converges locally uniformly to \( \zeta(1 - \beta W_0) \) as \( n \to \infty \). Also, it will be shown that the random measure \( \tilde{N}_0^{(n)}(nds, ndy, n^\alpha dz) \) can be approximated by a compensated Poisson random measure \( \tilde{N}_0(ds, dy, dz) \) on \( (0, \infty) \times \mathbb{R}^2_+ \) with intensity \( ds v_0, \alpha(dy) dz \), which yields that \( M_0^{(n)}(t) \) is asymptotically equivalent to
\[
M_0(t) := \int_0^t \int_0^\infty \int_0^\infty \nabla_y W_0(t-s) \tilde{N}_0(ds, dy, dz).
\]
Finally, passing to the limit in (4.3), we can obtain that
\[
X_{0,\xi}(t) \sim \zeta(1 - \beta W_0(t)) + M_0(t).
\]
In order to turn this approximation into equality, it is necessary to verify that \( (\varepsilon_0^{(n)}, \varepsilon_1^{(n)}, \varepsilon_2^{(n)}, M_0^{(n)}) \) converges weakly to \( (0, 0, 0, M_0) \) in \( D([0, \infty),\mathbb{R}^4) \) as \( n \to \infty \).

4.1. The resolvent. As a preparation, we provide some upper bound estimates for the resolvent \( R_H^{(n)} \) and \( R^{(n)} \). They will play an important role in the following proofs and analysis.

**Proposition 4.2.** Let \( R_\lambda \) be the resolvent of \( \alpha \tilde{\Lambda} \), i.e., \( R_\lambda = \alpha \tilde{\Lambda} + \alpha \tilde{\Lambda} * R_\lambda \). We have \( R_\lambda \) is bounded, completely monotone and regularly varying at \( \infty \) with index \( \alpha - 1 \). Moreover, there exists a constant \( C > 0 \) such that for any \( t \geq 0 \) and \( n \geq 1 \),
\[
(4.4) \quad R_H^{(n)}(t) \leq R_\lambda(t) \leq C (t^{\alpha-1} \wedge 1).
\]

**Proof.** It is easy to verify that \( R_H^{(n)} \) and \( R_\lambda \) have the Neumann series \( \sum_{k=1}^\infty (\gamma_n \tilde{\Lambda})^{(k)} \) and \( \sum_{k=1}^\infty (\alpha \tilde{\Lambda})^{(k)} \) respectively. Here \( f^{(k)} \) is the \( k \)-th convolution of a function \( f \) on \( \mathbb{R}_+ \). Thus the second result follows directly from the first one and the fact that \( \gamma_n \leq \alpha \). We now prove the first result. For any \( k \geq 1 \), we have
\[
(-1)^k \frac{d^k}{dt^k} \tilde{\Lambda}(t) = (\alpha + 1) \cdots (\alpha + k) \cdot (1 + t)^{-\alpha-1-k} \geq 0,
\]
which induces that \( \tilde{\Lambda} \) is completely monotone, and so is \( R_\lambda \); see Theorem 3 in [17]. Notice that \( R_\lambda(0) = \alpha \), let \( \tilde{R}_\lambda(t) := \alpha - R_\lambda(t) \) for any \( t \geq 0 \), which is continuous and increasing. Using integration by parts, we have
\[
(4.5) \quad \int_0^\infty e^{-\lambda t} d\tilde{R}_\lambda(t) = \lambda \int_0^\infty e^{-\lambda t}[\alpha - R_\lambda(t)] dt = \alpha - \lambda \mathcal{L}_{R_\lambda}(\lambda).
\]
where \( \mathcal{L}_{R_\Lambda} \) denotes the Laplace transform of \( R_\Lambda \). Taking Laplace transform of both sides of \( R_\Lambda = \alpha \bar{\Lambda} + \alpha \bar{\Lambda} \ast R_\Lambda \), we have \( \mathcal{L}_{R_\Lambda}(\lambda) = \alpha \mathcal{L}_{\bar{\Lambda}}(\lambda) \left[ 1 + \mathcal{L}_{R_\Lambda}(\lambda) \right] \) for any \( \lambda > 0 \) and hence
\[
\mathcal{L}_{R_\Lambda}(\lambda) = \frac{\alpha \mathcal{L}_{\bar{\Lambda}}(\lambda)}{1 - \alpha \mathcal{L}_{\bar{\Lambda}}(\lambda)}.
\]

It is obvious that the numerator goes to 1 as \( \lambda \to 0^+ \). On the other hand, from the fact that
\[
\int_0^\infty \alpha \bar{\Lambda}(s)ds = (1 + t)^{-\alpha}
\]
and Theorem 8.1.6 in [6, p.333] we also have as \( \lambda \to 0^+ \),
\[
1 - \alpha \mathcal{L}_{\bar{\Lambda}}(\lambda) \sim \Gamma(1 - \alpha) \lambda^\alpha
\]
and hence \( \mathcal{L}_{R_\Lambda}(\lambda) \sim \lambda^{-\alpha}/\Gamma(1 - \alpha) \).

Taking this back into (4.5), we immediately have
\[
\int_0^\infty \alpha^{-1} e^{-\lambda t}d\tilde{R}_H(t) \sim 1 - \frac{\lambda^{1-\alpha}}{\alpha \Gamma(1 - \alpha)}.
\]

By using Theorem 8.1.6 in [6, p.333] again, we also have as \( t \to \infty \),
\[
1 - \frac{\tilde{R}_H(t)}{\alpha} \sim \frac{t^{\alpha-1}}{\alpha \Gamma(\alpha) \Gamma(1 - \alpha)} \text{ and } \quad R_H(t) \sim \frac{t^{\alpha-1}}{\Gamma(\alpha) \Gamma(1 - \alpha)}.
\]

\[\Box\]

**Proposition 4.3.** For any \( p > 1 \), there exists a constant \( C > 0 \) such that for any \( t \geq 0 \) and \( n \geq 1 \),
\[
\int_0^t ds \int_0^\infty \left| n^{-\alpha} R_H^{(n)}(ns, ny) \right|^p n^{\alpha+1} \Lambda(n dy) \leq C(1 + t)^{\alpha(p-1)}.
\]

**Proof.** From (3.5) and the Cauchy-Schwarz inequality we have the term on the left side of (4.6) can be bounded by the sum of the following four terms:

\[
J_1^{(n)}(t) := \int_0^t ds \int_0^\infty \left| n^{-\alpha} \mathbf{1}_{\{y > s\}} \right|^p n^{\alpha+1} \Lambda(n dy),
\]

\[
J_2^{(n)}(t) := \int_0^t ds \int_0^s n^{1-\alpha} R_H^{(n)}(nr)dr \left| n^{\alpha+1} \Lambda(ns) \right|,
\]

\[
J_3^{(n)}(t) := \int_0^t ds \int_s^{s/2} \int_{s-y}^s n^{1-\alpha} R_H^{(n)}(nr)dr \left| n^{\alpha+1} \Lambda(ny) \right|,
\]

\[
J_4^{(n)}(t) := \int_0^t ds \int_{s/2}^s \int_{s-y}^s n^{1-\alpha} R_H^{(n)}(nr)dr \left| n^{\alpha+1} \Lambda(ny) \right|.
\]

Here the constant \( C \) depends only on \( p \). We first have
\[
J_1^{(n)}(t) = C n^{(1-p)\alpha+1} \int_0^t \bar{\Lambda}(ns) ds \leq C n^{(1-p)\alpha} \int_0^\infty \bar{\Lambda}(s) ds \leq C n^{(1-p)\alpha} \leq C.
\]

From (4.4) we have \( \int_0^s R_H^{(n)}(r)dr \leq C s^\alpha \) and hence
\[
J_2^{(n)}(t) \leq C n^{(1-p)\alpha} \int_0^{nt} s^{(p-1)\alpha-1} ds \leq C \cdot t^{(p-1)\alpha}.
\]

Moreover, (4.4) also indicates that \( n^{1-\alpha} R_H^{(n)}(nr) \leq C r^{\alpha-1} \) for any \( r > 0 \). This immediately induces that \( \int_{s-y}^s n^{1-\alpha} R_H^{(n)}(nr)dr \leq C |s - y|^{\alpha-1} \cdot y \) and
\[
J_3^{(n)}(t) \leq C \int_0^t ds \int_0^{s/2} |s - y|^{p(\alpha-1)\cdot y} \left| n^{\alpha-2} dy \right| \leq C \int_0^t s^{p(\alpha-1)} ds \int_0^{s/2} y^{p-\alpha-2} dy \leq C t^{(k-1)\alpha}.
\]

Thus
\[
J_1^{(n)}(t) + J_2^{(n)}(t) + J_3^{(n)}(t) + J_4^{(n)}(t) \leq C t^{(k-1)\alpha}.
\]
For \( J^{(4)}(t) \), by Hölder’s inequality and (4.4) we have
\[
\left| \int_{s-y}^{s} n^{1-\alpha} R_H^{(n)}(nr)dr \right|^p \leq y^{p-1} \int_{s-y}^{s} |n^{1-\alpha} R_H^{(n)}(nr)|^p dr \leq C y^{p-1} \int_{s}^{t} r^{p(\alpha-1)}dr \leq C y^{p-1} s^{p(\alpha-1)+1}
\]
and
\[
J^{(4)}_t(t) \leq C \int_{0}^{t} s^{p(\alpha-1)+1} ds \int_{s/2}^{s} y^{p-\alpha-3} dy \leq C \int_{0}^{t} s^{p(\alpha-1)-1} ds \leq Ct^{p-1}\alpha.
\]
Putting all estimates together, we can immediately get the desired result. \( \square \)

4.2. Moment estimates. Before providing moment estimates of any order for \( \{X^{(n)}_\zeta(t)\}_{n \geq 1} \), we first give some high-order moment estimates for the stochastic integral driven by a compensated Poisson random measure \( \tilde{N}_1(ds,dy,dz) \) on \((0,\infty) \times \mathbb{R}^2_+\) with intensity \( ds\nu_1(dy)dz \), where \( \nu_1(dy) \) is an \( \sigma\)-finite measure on \( \mathbb{R}_+ \). For some \( k \geq 1 \) and \( T > 0 \), let \( f(t,s,y) \) be a measurable function on \( \mathbb{R}^3_+ \) satisfying
\[
\sup_{t,s \in [0,T]} \int_{0}^{\infty} |f(t,s,y)|^2 \nu_1(dy) < \infty
\]
and \( \{\omega(t) : t \geq 0\} \) be an \((\mathcal{F}_t)\)-progressive nonnegative process with \( \sup_{t \in [0,T]} \mathbb{E}[|\omega(t)|^{2k-1}] < \infty \). Using the Burkholder-Davis-Gundy inequality and Hölder’s inequality several times, we have for any \( t \in [0,T] \),
\[
\mathbb{E}\left[ \left( \int_{0}^{t} R_H^{(n)}(t-s) \cdot 1_{\{\ell_0,1<s\}}ds \right)^{2k-1} \right] \leq C \sum_{i=1}^{k} \left( \int_{0}^{t} ds \int_{0}^{\infty} |f(t,s,y)|^2 \nu(dy) \right)^{2k-1}.
\]

**Proposition 4.4.** For \( p \geq 0 \), there exists a constant \( C > 0 \) such that for any \( t \geq 0 \) and \( n \geq 1 \),
\[
\mathbb{E}\left[ \left( \int_{0}^{t} R_H^{(n)}(t-s) \cdot 1_{\{\ell_0,1<s\}}ds \right)^{p} \right] \leq C \cdot t^{(p-1)\alpha}.
\]

**Proof.** For \( p \in [0,1] \), by Jensen’s inequality and Proposition 4.2,
\[
\mathbb{E}\left[ \left( \int_{0}^{t} R_H^{(n)}(t-s) \cdot 1_{\{\ell_0,1<s\}}ds \right)^{p} \right] = \left( \mathbb{E}\left[ \int_{0}^{t} R_H^{(n)}(t-s) \cdot 1_{\{\ell_0,1<s\}}ds \right] \right)^{p} \leq \left( \mathbb{E}\left[ \int_{0}^{t} R_H^{(n)}(t-s) \cdot 1_{\{\ell_0,1<s\}}ds \right] \right)^{p} \leq C \left( \int_{0}^{t} (t-s)^{\alpha-1}s^{-\alpha}ds \right)^{p} < C.
\]
Here the constant \( C \) is independent of \( n, p \) and \( t \). For \( p > 1 \), by Hölder’s inequality and the previous result,
\[
\mathbb{E}\left[ \left( \int_{0}^{t} R_H^{(n)}(t-s) \cdot 1_{\{\ell_0,1<s\}}ds \right)^{p} \right] \leq \|R_H^{(n)}\|_{L_1}^{p-1} \cdot \mathbb{E}\left[ \int_{0}^{t} R_H^{(n)}(t-s) \cdot 1_{\{\ell_0,1<s\}}ds \right] \leq C \|R_H^{(n)}\|_{L_1}^{p-1}.
\]
From (4.4), we have \( \|R_H^{(n)}\|_{L_1} \leq \|R_H\|_{L_1} \leq C \cdot t^{\alpha} \) and the desired result follows. \( \square \)

**Lemma 4.5.** We have \( \sup_{n \geq 1} \sup_{t \in [0,T]} \mathbb{E}[|X^{(n)}_\zeta(t)|^{p}] < \infty \) for any \( p \geq 0 \) and \( T > 0 \).

**Proof.** It suffices to prove this result with \( p = 2^k \) for \( k \geq 0 \). Taking expectations on both sides of (4.1), we have
\[
\mathbb{E}[X^{(n)}_\zeta(t)] \leq \zeta + n^{-\alpha} \sum_{k=1}^{\lfloor n^\alpha \rfloor} \mathbb{E}\left[ \int_{0}^{nt} R_H^{(n)}(nt-s)1_{\{\ell_0,1<s\}}ds \right].
\]
From Proposition 4.4, there exists a constant $C > 0$ such that $\mathbb{E}[X_0^{(n)}(t)] \leq C$ for any $t \geq 0$ and $n \geq 1$. By mathematical induction, it suffices to prove that for some $k \geq 1$ this result holds for $p = 2^k$ under the assumption that it holds with $p = 2^{k-1}$. By the Cauchy-Schwarz inequality,

$$
\mathbb{E}[|X_0^{(n)}(t)|^{2^k}] \leq C + C\mathbb{E}\left[\left| \int_0^{nt} R_H^{(n)}(nt-s) n^{-\alpha} \sum_{k=1}^{[n^{\alpha}]} 1_{\{\ell_n, s > s\}} ds\right|^{2^k}\right]
$$

(4.8)

$$
+ \mathbb{E}\left[\left| \int_0^{nt} \int_0^{\infty} \int_0^{\infty} X_0^{(n)}(s-) n^{-\alpha} R^{(n)}(nt-s, ny) \tilde{N}\left(\left| n(s-d, ndy, n^\alpha dz\right|^{2^{k-1}}\right)\right|\right].
$$

Applying (4.7) together with Proposition 4.3 to the last expectation above, we have it is smaller than

$$
C \sum_{i=1}^k \int_0^t ds \int_0^{\infty} \left| n^{-\alpha} R^{(n)}(ns, ny) \right|^2 n^{\alpha + 1} \mathbb{E}(ndy) \leq Ck(1 + t)^{\alpha(2^k - 1)}.
$$

We now consider the first expectation on the right side of (4.8). We first notice that

$$
\left| \sum_{k=1}^{[n^{\alpha}]} \int_0^{nt} R_H^{(n)}(nt-s) 1_{\{\ell_n, s > s\}} ds\right|^{2^k} = \sum_{|k^{(n)}| = 2^k} \prod_{i=1}^k \left| \int_0^{nt} R_H^{(n)}(nt-s) 1_{\{\ell_n, s > s\}} ds\right|^{2^k}.
$$

Here the sum above is over all $k^{(n)} := (k_1, \cdots, k_{X_0^{(n)}(0)}) \in \mathbb{N}^{[n^{\alpha}]}$ with $|k^{(n)}| := \sum_{i=1}^{[n^{\alpha}]} k_i = 2^k$. From Proposition 4.4, we have

$$
\mathbb{E}\left[\left| \int_0^{nt} R_H^{(n)}(nt-s) n^{-\alpha} \sum_{k=1}^{[n^{\alpha}]} 1_{\{\ell_n, s > s\}} ds\right|^{2^k}\right] \leq C n^{-2^k \alpha} \sum_{n^{\alpha} = 2^k} (nt)^{\alpha(k_i - 1)^+}.
$$

By using multinomial distribution and then combination formula, we have the last term above equals to

$$
C n^{-2^k \alpha} \sum_{j=1}^{[n^{\alpha}]} \binom{[\frac{n^{\alpha}}{2^k}]}{j} (nt)^{\alpha(2^k - j)} \leq C \sum_{j=1}^{[n^{\alpha}]} \frac{(2^k - j)^{\alpha(2^k - j)}}{j!} \leq C(1 + t)^{\alpha(2^k - 1)}.
$$

Putting all estimates above together, we can immediately get the desired result. \(\square\)

### 4.3. Weak convergence of \(\{M_0^{(n)}\}_{n \geq 1}\)

Here we just prove the weak convergence on the time interval \([0, 1]\) and the general case can be proved in the same way. To get the tightness we need the following helpful properties of the scale function $W_0$. By (2.3) and the mean value theorem, there exists a constant $C > 0$ such that for any $x, h \in (0, 1)$,

$$
\Delta_h W_0(x) = \nabla_h W_0(x + h) \leq C [(x + h)^{\alpha} \wedge (x^{\alpha - 1} - 1)].
$$

(4.9)

**Proposition 4.6.** For $p > 1 + \alpha$, there exists a constant $C > 0$ such that for any $h \in [0, 1]$,

$$
\int_0^{1} |\Delta_h W_0(s)|^p (s + h)^{-\alpha - 1} ds + \int_0^{h} ds \int_0^{s} |\nabla_y W_0(s)|^{p + \alpha - 2} dy \leq C h^{(p - 1)\alpha}.
$$

**Proof.** From (4.9), we have $\Delta_h W_0(s) \leq C [h^{\alpha} \wedge (s^{\alpha - 1} - 1)]$ and hence

$$
\int_0^{1} |\Delta_h W_0(s)|^p (s + h)^{-\alpha - 1} ds \leq Ch^{p\alpha} \int_0^{h} (s + h)^{-\alpha - 1} ds + C \cdot h^p \int_h^{1} s^{p(\alpha - 1)} (s + h)^{-\alpha - 1} ds,
$$
which can be bounded by \( C \cdot h^{(p-1)\alpha} \). Similarly, by (4.9) we also have \( \nabla_y W_0(s) \leq C[(s^{\alpha-1}y) \wedge s^\alpha] \). This induces that
\[
\int_0^h ds \int_0^s |\nabla_y W_0(s)|^p y^{-\alpha-2} dy \leq C \int_0^h s^{\rho(\alpha)-1} ds \int_0^{s/2} y^{\rho-\alpha-2} dy + C \int_0^h s^{\rho\alpha} ds \int_0^s y^{-\alpha-2} dy.
\]
A simple calculation yields that it also can be bounded by \( Ch^{(p-1)\alpha} \).

PROPOSITION 4.7. For any \( p \geq 2 \), there exists a constant \( C > 0 \) such that for any \( h \in [0, 1] \),
\[
(4.10) \quad \int_0^1 ds \int_0^s |\nabla_y \Delta_h W_0(s)|^p y^{-\alpha-2} dy \leq Ch^{(p-1)\alpha}.
\]

PROOF. An argument similar to the one used in the proof of Proposition 4.6 along with the Cauchy-Schwarz inequality shows that the term on the left side of (4.10) can be bounded by
\[
\sup_{h \in [0, 1]} \int_0^1 ds \int_0^s (|\nabla_y W_0(s + h)|^p + |\nabla_y W_0(s)|^p) y^{-\alpha-2} dy < \infty
\]
and hence the inequality (4.10) holds for \( h > 1/8 \). We now consider the case with \( h \in [0, 1/8] \). We first split the term on the left side of (4.10) into the following four parts:

\[
\begin{align*}
J_1(h) &:= \int_0^{4h} ds \int_0^s |\nabla_y \Delta_h W_0(s)|^p y^{-\alpha-2} dy, \\
J_2(h) &:= \int_0^1 ds \int_0^s |\nabla_y \Delta_h W_0(s)|^p y^{-\alpha-2} dy, \\
J_3(h) &:= \int_0^{4h} ds \int_{s-h}^s |\nabla_y \Delta_h W_0(s)|^p y^{-\alpha-2} dy, \\
J_4(h) &:= \int_0^1 ds \int_{s/2}^s |\nabla_y \Delta_h W_0(s)|^p y^{-\alpha-2} dy.
\end{align*}
\]

By the Cauchy-Schwarz inequality, there exists a constant \( C > 0 \) such that
\[
(4.11) \quad J_1(h) \leq C \int_0^{4h} ds \int_0^s |\nabla_y W_0(s + h)|^p y^{-\alpha-2} dy + C \int_0^{4h} ds \int_0^s |\nabla_y W_0(s)|^p y^{-\alpha-2} dy.
\]

Applying Proposition 4.6 to the second term on the right side of the above inequality, we have it can be bounded by \( Ch^{(p-1)\alpha} \). In addition, from (4.9) we have \( \nabla_y W_0(s + h) \leq C(s + h - y)^{\alpha-1} y \) and hence
\[
\int_0^{4h} ds \int_0^s |\nabla_y W_0(s + h)|^p y^{-\alpha-2} dy \leq C \int_0^{4h} ds \int_0^s (s + h - y)^{\rho(\alpha)-1} y^{\rho-\alpha-2} dy
\]
\[
\leq Ch^{\rho(\alpha)-1} \int_0^{4h} ds \int_0^s y^{\rho-\alpha-2} dy \leq Ch^{(p-1)\alpha}.
\]

Taking these two estimates into (4.11), we have \( J_1(h) \leq Ch^{(p-1)\alpha} \). Similarly, from (4.9) we have \( \Delta_h W_0(y) \leq Ch^\alpha \) for any \( y \in (0, h] \) and hence
\[
J_2(h) \leq \int_0^1 ds \int_{s-h}^s |\Delta_h W_0(s)|^p y^{-\alpha-2} dy + \int_0^1 ds \int_0^h |\Delta_h W_0(y)|^p (s - y)^{-\alpha-2} dy
\]
\[
\leq Ch^p \int_0^1 s^{\rho(\alpha)-1} ds \int_{s-h}^s y^{-\alpha-2} dy + Ch^{\rho\alpha+1} \int_0^1 (s - h)^{-\alpha-2} ds \leq Ch^{(p-1)\alpha}.
\]
We now analyze $J_3(h)$ and $J_4(h)$. From (4.12), there exists a constant $C > 0$ such that for any $h > 0$ and $s \geq y > 0$,

$$|\nabla_y \Delta_h W_0(s)| \leq \int_0^y d\tilde{y} \int_0^\tilde{h} |W''_0(s + \tilde{h} - \tilde{y})|d\tilde{h}$$

$$\leq C \int_0^y d\tilde{y} \int_0^\tilde{h} |s + \tilde{h} - \tilde{y}|^{\alpha - 2}d\tilde{h} \leq Ch(s - y)^{\alpha - 2} \cdot (|s - y| \wedge y).$$

Plugging this into $J_3(h)$ and $J_4(h)$ yields

$$J_3(h) \leq Ch^p \int_{4h}^1 ds \int_{s/2}^{s-h} (s - y)^p (\alpha - 1) y^{-\alpha - 2}dy \leq Ch^{p\alpha} \int_{4h}^1 s^{-\alpha - 1}ds \leq Ch^{(p - 1)\alpha}$$

and

$$J_4(h) \leq Ch^p \int_{4h}^1 |s/2|^{p(\alpha - 2)} ds \int_{0}^{s/2} y^{p - \alpha - 2}dy \leq Ch^{(p - 1)\alpha}.$$

Putting all estimates above together, we can immediately get the desired result. 

PROPOSITION 4.8. For any $p \geq 2$ and $h \in [0, 1]$, there exists a constant $C > 0$ such that

$$\int_0^1 ds \int_s^{s+h} |\nabla_y W_0(s + h) - W_0(s)|^p y^{-\alpha - 2}dy \leq Ch^{(p - 1)\alpha}.$$ 

PROOF. We split the term on the left side of the above inequality into the following three parts:

$$J_1(h) := \int_0^h ds \int_s^{s+h} |\nabla_y W_0(s + h) - W_0(s)|^p y^{-\alpha - 2}dy,$$

$$J_2(h) := \int_0^h ds \int_s^{s+h} |\nabla_y W_0(s + h) - W_0(s)|^p y^{-\alpha - 2}dy,$$

$$J_3(h) := \int_0^1 ds \int_s^{s+h} |\nabla_y W_0(s + h) - W_0(s)|^p y^{-\alpha - 2}dy.$$

By the Cauchy-Schwarz inequality,

$$J_1(h) \leq C \int_0^h ds \int_s^{s+h} |\nabla_y W_0(s + h)|^p y^{-\alpha - 2}dy + C \int_0^h |W_0(s)|^p ds \int_s^{s+h} y^{-\alpha - 2}dy.$$

Since $W_0(s) \leq Cs^\alpha$, the second term on the right side of the above inequality can be bounded by $C \int_0^h s^{-\alpha - 1}|W_0(s)|^p ds \leq Ch^{(p - 1)\alpha}$. For the first term, choosing a constant $\theta$ satisfying that $1 + \alpha < p\theta < (1 - \alpha)^{-1}$, by (4.9) we have $\nabla_y W_0(s + h) \leq C(s + h)^{(1 - \theta)\alpha}(s + h - y)^{\theta(\alpha - 1)}y^{\theta}$. Thus

$$\int_0^h ds \int_s^{s+h} |\nabla_y W_0(s + h)|^p y^{-\alpha - 2}dy \leq \int_0^h ds \int_s^{s+h} (s + h)^{(1 - \theta)p\alpha}(s + h - y)^{p\theta(\alpha - 1)}y^{p\theta - \alpha - 2}dy$$

$$\leq Ch^{(1 - \theta)p\alpha} \int_0^h ds \int_s^{s+h} (s + h - y)^{\theta(\alpha - 1)}y^{p\theta - \alpha - 2}dy$$

$$\leq Ch^{(1 - \theta)p\alpha} \int_0^h s^{\theta - \alpha - 2}ds \int_0^s (y + h - s)^{\theta(\alpha - 1)}dy$$

$$\leq Ch^{(1 - \theta)p\alpha + p\theta(\alpha - 1) + 1} \int_0^h s^{\theta - \alpha - 2}ds \leq Ch^{(p - 1)\alpha}.$$
and hence $J_1(h) \leq C h^{(p-1)\alpha}$. For $J_2(h)$, we notice that $\nabla_y W_0(s + h) - W_0(s)$ for any $y \in [h, s + h]$ and by the Cauchy-Schwarz inequality,

$$J_2(h) \leq \int_0^h \int_h^{s+h} |\Delta h W_0(s)|^p y^{-\alpha - 2} dy ds + \int_0^h \int_h^{s+h} |W_0(s + h - y)|^p y^{-\alpha - 2} dy ds.$$ 

Since $\Delta h W_0(s) \leq C(s + h)^\alpha$ and $W_0(s + h - y) \leq C(s + h - y)^\alpha$, we have

$$J_2(h) \leq C \int_0^h (s + h)^\alpha ds \int_h^{s+h} y^{-\alpha - 2} dy + C \int_0^h ds \int_h^{s+h} (s + h - y)^\alpha y^{-\alpha - 2} dy.$$ 

The first term on the right side of the above inequality can be bounded by $C h^{-\alpha - 1} \int_0^h (s + h)^\alpha ds \leq C h^{(p-1)\alpha}$. Changing the order of integration in the second term, we have

$$\int_0^h ds \int_h^{s+h} (s + h - y)^\alpha y^{-\alpha - 2} dy = \int_0^h ds \int_0^{s} (s - y)^\alpha (y - h)^{-\alpha - 2} dy \leq \int_0^h (s + h)^{-\alpha - 2} ds \int_s^h (y - s)^\alpha dy \leq C h^{(p-1)\alpha}$$

and hence $J_2(h) \leq C h^{(p-1)\alpha}$. Similarly, for $J_3(h)$ we also have

$$J_3(h) \leq \int_1^h ds \int_1^{s+h} |\Delta h W_0(s)|^p y^{-\alpha - 2} dy ds + \int_1^h \int_s^{s+h} |W_0(s + h - y)|^p y^{-\alpha - 2} dy ds \leq C h^p \int_1^h s^{\alpha-1} ds \int_s^{s+h} y^{-\alpha - 2} dy + C \int_1^h \int_s^{s+h} (s + h - y)^\alpha y^{-\alpha - 2} dy ds.$$ 

The first term on the right side of the last inequality can be bounded by $C h^p \int_1^h s^{\alpha-1} - \alpha - 1 ds \leq C h^{(p-1)\alpha}$ and the second term equals to

$$C \int_1^h ds \int_0^h (h - y)^\alpha (y + s)^{-\alpha - 2} dy \leq C \int_1^h s^{-\alpha - 2} ds \int_0^h (h - y)^\alpha dy \leq C h^{(p-1)\alpha}.$$ 

\[ \square \]

**Lemma 4.9.** For any $p \geq 1$, there exist constants $C > 0$ such that for any $n \geq 1$ and $t, h \in [0, 1]$,

$$E[|\Delta h M_0^{(n)}(t)|^{2p}] \leq C h^{p\alpha}$$

and hence the sequence $\{M_0^{(n)}\}_{n \geq 1}$ is tight.

**Proof.** The second result follows directly from the first one. Indeed, choosing $p > (2/\alpha)$ we have $\sup_{n \geq 1} E[|\Delta h M_0^{(n)}(t)|^{2p}] \leq C h^2$ and the tightness follows from Kolmogorov tightness criterion; see Theorem 13.5 in [5]. We now prove the first result with $p = 2k$ and $k \geq 1$. Other cases can be proved in the same way. We first split $\Delta h M_0^{(n)}(t)$ into the following five parts:

$$M_{0,1}(t, h) := \int_t^{t+h} \int_0^{t+h-s} \int_0^{X^{(n)}(s-)} \nabla_y W_0(t + h - s) \tilde{N}_0^{(n)}(nds, ndy, n^\alpha dz),$$

$$M_{0,2}(t, h) := \int_t^{t+h} \int_0^{\infty} \int_0^{X^{(n)}(s-)} W_0(t + h - s) \tilde{N}_0^{(n)}(nds, ndy, n^\alpha dz),$$

$$M_{0,3}(t, h) := \int_0^{t} \int_0^{t-s} \int_0^{X^{(n)}(s-)} \nabla_y \Delta h W_0(t - s) \tilde{N}_0^{(n)}(nds, ndy, n^\alpha dz),$$

$$M_{0,4}(t, h) := \int_0^{t} \int_0^{t-s} \int_0^{X^{(n)}(s-)} \nabla_y W_0(t - s) \tilde{N}_0^{(n)}(nds, ndy, n^\alpha dz),$$

$$M_{0,5}(t, h) := \int_0^{t} \int_0^{t-s} \int_0^{X^{(n)}(s-)} \Delta h W_0(t - s) \tilde{N}_0^{(n)}(nds, ndy, n^\alpha dz).$$
Applying (4.7) to $\mathbb{E}[|M_0^{(n)}(t, h)|^{2k}]$ together with Lemma 4.5 and Proposition 4.6, we have

$$
\mathbb{E}[|M_0^{(n)}(t, h)|^{2k}] \leq C \sum_{i=1}^{k} \left( \int_{0}^{t} ds \int_{0}^{t+h-s} \left| \nabla_y W_0(t+h-s) \right|^{2^i} y^{-\alpha-2} dy \right)^{2^{k-i}} 
$$

\[ \leq C \sum_{i=1}^{k} \left( \int_{0}^{h} ds \int_{0}^{s} \left| \nabla_y W_0(s) \right|^{2^i} y^{-\alpha-2} dy \right)^{2^{k-i}} \]

\[ \leq C \sum_{i=1}^{k} h^{2^i(1-2^{-i})\alpha} \leq C h^{2^{k-1}}. \]

Similar results for other terms also can be gotten with the help of Proposition 4.7 and 4.8.

**Lemma 4.10.** As $n \to \infty$, we have $M_0^{(n)} \to M_0$ in the sense of finite-dimensional distributions.

**Proof.** It suffices to prove that $(M_0^{(n)}(t_1), \ldots, M_0^{(n)}(t_d)) \to (M_0(t_1), \ldots, M_0(t_d))$ in distribution for any $d \geq 1$ and $0 \leq t_1 < \cdots < t_d \leq 1$. For any $i \in \{1, \ldots, d\}$, define

$$
M_i^{(n)}(t) := \int_{0}^{t} \int_{0}^{t} \int_{0}^{t} X_{\zeta}^{(n)}(s-) \nabla_y W_0(t_i - s) \tilde{N}_0^{(n)}(nds, ndy, n^\alpha dz),
$$

$$
M_i(t) := \int_{0}^{t} \int_{0}^{t} \int_{0}^{t} X_{\zeta}^{(n)}(s) \nabla_y W_0(t_i - s) \tilde{N}(ds, dy, dz). \]

It is apparent that $M_i^{(n)}(t_i) = M_i^{(n)}(t_i)$ and $M_i(t_i) = M_0(t_i)$ a.s. Thus it suffices to prove

(4.14) \( (M_1^{(n)}, \ldots, M_d^{(n)}) \to (M_1, \ldots, M_k) \)

weakly in $\mathcal{D}([0,1], \mathbb{R}^d)$ as $n \to \infty$. By Skorokhod’s theorem, we may assume that $X_{\zeta}^{(n)} \to X_{\zeta}$ a.s. as $n \to \infty$ in the topology of $\mathcal{D}([0,\infty), \mathbb{R}^d)$. Let $\tilde{\nu}(dy, dz) := \nu_{0,\alpha}(dy)dz$ be a $\sigma$-finite measure on $\mathbb{R}^d_+$ and $L^2(\tilde{\nu})$ the collection of all functions on $\mathbb{R}^d_+$ that is square-integrable with respect to $\tilde{\nu}$. For any $n \geq 1$, let $\tilde{N}_0^{(n)}(t) := \tilde{N}_0^{(n)}([0,nt], ndy, n^\alpha dz)$ and $\tilde{N}_0(t) := \tilde{N}_0([0,t], dy, dz)$, which are two $L^2(\tilde{\nu})$-martingales; see Definition 3.5 in [28]. We can rewrite $M_i^{(n)}(t)$ and $M_i(t)$ into

$$
M_i^{(n)}(t) = \int_{0}^{t} F(s, X_{\zeta}^{(n)}(s-)) \cdot \tilde{N}_0^{(n)}(dt) \quad \text{and} \quad M_i(t) = \int_{0}^{t} F(s, X_{\zeta}(s-)) \cdot \tilde{N}(dt),
$$

where the function $F : \mathbb{R}^d_+ \to (L^2(\tilde{\nu}))^d$ is defined by

$$
F(s, x) := (\nabla W_0(t_1-s), \ldots, \nabla W_0(t_d-s)) \cdot 1_{\{s \leq x\}}.
$$

From Condition 2.2 and Theorem 2.7 in [28], as $n \to \infty$ we have $\gamma_n n^{\alpha+1} \Lambda(\nu dy) \to \alpha(\alpha+1) y^{-\alpha-2} dy$ and $\tilde{N}_0^{(n)} \to \tilde{N}_0$ in the sense that

$$
(\tilde{N}_0^{(n)}(f_1, \ldots, f_k)) \to (\tilde{N}_0(f_1, \ldots, f_k)).
$$
weakly in $D([0, \infty), \mathbb{R}^k)$ for any $f_1, \cdots, f_k \in L^2(\nu)$. Since $\{\tilde{N}_0^{(n)}(nds, ndy, n^\alpha dz)\}_{n \geq 1}$ is a sequence of Poisson random measures, the uniform tightness of $\{N_0^{(n)}\}_{n \geq 1}$ can be proved like the proof of Lemma 4.9 in [20]. Since $X_\xi^{(n)} \to X_{0, \xi}$ a.s., we have $F(\cdot, X_\xi^{(n)}(\cdot)) \to F(\cdot, X_{0, \xi}(\cdot))$ a.s. in $D([0, \infty), (L^2(\nu))^d)$. Using Theorem 5.5 in [28], we immediately get the weak convergence (4.14) and the proof is completed. □

4.4. Error processes. It remains to prove the error sequence $\{\sum_{i=0}^2 \varepsilon_i^{(n)}\}_{n \geq 1}$ converges weakly to 0. From (4.3), we have $\sum_{i=0}^2 \varepsilon_i^{(n)} = X_\xi^{(n)} - \tilde{X}_\xi^{(n)} - M_0^{(n)}$. Our previous results show that the three sequences $\{X_\xi^{(n)}\}_{n \geq 1}$, $\{\tilde{X}_\xi^{(n)}\}_{n \geq 1}$ and $\{M_0^{(n)}\}_{n \geq 1}$ are all $C$-tight. From Corollary 3.33 in [23, p.353], we have the error sequence $\{\sum_{i=0}^2 \varepsilon_i^{(n)}\}_{n \geq 1}$ is also $C$-tight, and hence it suffices to prove $\varepsilon_i^{(n)} \to 0$ in the sense of infinite dimensional distributions for $i \in \{0, 1, 2\}$.

**Lemma 4.11.** For any $T > 0$, we have $\|\varepsilon_0^{(n)}\|_{L^\infty} + \|\varepsilon_1^{(n)}\|_{L_T^\infty} \to 0$ in probability as $n \to \infty$.

**Proof.** Apparently the convergence $\|\varepsilon_0^{(n)}\|_{L^\infty} \to 0$ in probability follows directly from the Glivenko-Cantelli theorem. We now prove $\|\varepsilon_1^{(n)}\|_{L_T^\infty} \to 0$ in probability with $T \geq 1$. Firstly,

$$
\|\varepsilon_1^{(n)}\|_{L_T^\infty} \leq \Lambda \cdot \sup_{t \in [0,1/\alpha]} \left| \int_0^t R_H^{(n)}(nt - s) \frac{1}{[n^\alpha \zeta]} \sum_{k=1}^{[n^\alpha \zeta]} (1_{\{t_k > s\}} - \overline{\Lambda}(s)) ds \right|
$$

$$
(4.15) + \Lambda \cdot \sup_{t \in [1/\alpha, T]} \int_1^T R_H^{(n)}(nt - s) \frac{1}{[n^\alpha \zeta]} \sum_{k=1}^{[n^\alpha \zeta]} 1_{\{t_k > s\}} - \overline{\Lambda}(s) ds.
$$

From Lemma 4.2, the first term on the right side of the inequality above can be bounded by

$$
C \sup_{s \in [0,1]} \left| \frac{1}{[n^\alpha \zeta]} \sum_{k=1}^{[n^\alpha \zeta]} 1_{\{t_k > s\}} - \overline{\Lambda}(s) \right|
$$

which vanishes as $n \to \infty$ because of the Glivenko-Cantelli theorem. By Hölder’s inequality, for any $p, q > 1$ with $q > 2/\alpha$ and $1/p + 1/q = 1$ we see that the second term on the right side of (4.15) can be bounded by

$$
(4.16) C\|R_H^{(n)}\|_{L_{nT}^p} \cdot \int_1^\infty \left| \frac{1}{[n^\alpha \zeta]} \sum_{k=1}^{[n^\alpha \zeta]} 1_{\{t_k > s\}} - \overline{\Lambda}(s) \right|^q ds \frac{1}{1/q}.
$$

Since $p > 1/(1 - \alpha)$ and $R_H^{(n)}(t) \leq C(1 + t)^{\alpha - 1}$; see Lemma 4.2, we have

$$
(4.17) \|R_H^{(n)}\|_{L_{nT}^p} < C(1 + nT)^{\alpha - 1 + 1/p} = C(1 + nT)^{\alpha - 1/q}.
$$

For any $k \geq 1$, let $Y_k(s) := 1_{\{t_k > s\}} - \overline{\Lambda}(s)$ for any $s \geq 0$. Then the second term in (4.16) equals to

$$
\|Y_k\|_{L_{1, \infty}^q} = \left( \int_1^\infty |1_{\{t_k \leq s\}} - \overline{\Lambda}(s)|^q ds \right)^{1/q}
$$

$$
= \left( \int_1^\infty |1 - \overline{\Lambda}(s)|^q \cdot 1_{\{t_k \leq s\}} + |\overline{\Lambda}(s)|^q \cdot 1_{\{t_k > s\}} ds \right)^{1/q}
$$

$$
\leq \left( \int_1^{\overline{\Lambda}} |\overline{\Lambda}(s)|^q ds \right)^{1/q} + \left( \int_1^\infty |\overline{\Lambda}(s)|^q ds \right)^{1/q}
$$

By the Minkowski inequality and the fact that $\overline{\Lambda}(s) \leq C(1 + s)^{-\alpha}$,
we have, we have the first term on the right side of the second inequality above vanishes as $\theta \to \infty$ and Theorem 10.5 in [32, p.295-302] we have $\{Y_k\}_{k \geq 1}$ satisfies the central limit theorem and for any $\theta \in (2 - 2/(\alpha q), 2)$ there exists a constant $C > 0$ such that for any $n \geq 1$,

$$E \left[ \left\| \frac{1}{n^{\alpha q}} \sum_{k=1}^{n^{\alpha q}} Y_k \right\|_{L^2(\Omega, \mathcal{F}, \mathbb{P})}^\theta \right] \leq C n^{-\theta \alpha /2}.$$

Taking this and (4.17) back into (4.16), we have the second term on the right side of (4.15) can be bounded by $CT^{\alpha - 1/q} n^{\alpha - 1/q - \theta /2}$, which vanishes as $n \to \infty$ because $\alpha - 1/q - \theta /2 < 0$. \hfill \Box

**Lemma 4.12.** For any $T > 0$, we have $\sup_{t \in [0, T]} E[|\varepsilon_2^{(n)}(t)|^2] \to 0$ as $n \to \infty$.

**Proof.** By the definition of $\varepsilon_2^{(n)}(t)$, we can split it into the following two parts:

$$\varepsilon_2^{(n)}(t) := \int_0^t \int_{t-s}^{t} \frac{X_{s-}^{(n)}}{n^{\alpha}} N_0^{(n)}(ndy, ndx, n^{\alpha} dz),$$

$$\varepsilon_2^{(n)}(t) := \int_0^t \int_{t-s}^{t} \int_0^{\infty} \frac{X_{s-}^{(n)}}{n^{\alpha}} N_0^{(n)}(ndx, ndy, n^{\alpha} dz).$$

By the Burkholder-Davis-Gundy inequality and Lemma 4.5 we have,

$$\sup_{t \geq 0} E[|\varepsilon_2^{(n)}(t)|^2] \leq C \sup_{t \geq 0} E \left[ \int_0^t \int_{t-s}^{t} \frac{X_{s-}^{(n)}}{n^{\alpha}} N_0^{(n)}(ndx, ndy, n^{\alpha} dz) \right]$$

$$= C \sup_{t \geq 0} \int_0^t \int_{t-s}^{t} \frac{X_{s-}^{(n)}}{n^{\alpha}} \Lambda(n(s-t)) ds \leq C n^{-\alpha} \int_0^t \Lambda(s) ds,$$

which goes to 0 as $n \to \infty$. Similarly, for $\theta \in ((1 - \frac{\alpha}{1+\alpha})^+, 1 - \alpha)$ we also have

$$E[|\varepsilon_2^{(n)}(t)|^2] \leq C \int_0^t ds \int_0^{\infty} \left| \int_{s-y}^{s} n^{1-\alpha} R_H^{(n)}(nr) dr - \nabla_y W_0(s) \right|^2 n^{\alpha+1} \Lambda(n dy).$$

From Lemma 4.1 we have the first term on the right side of the second inequality above vanishes as $n \to \infty$ and hence the lemma will be proved by showing that the second term is finite. Notice that it can be bounded by

$$C \int_0^t |W_0(s)|^{2-\theta} s^{-\alpha-1} ds + C \int_0^t ds \int_0^{s} |\nabla_y W_0(s)|^{2-\theta} y^{-\alpha-2} dy$$

$$+ C \int_0^t ds \int_0^{\infty} \left| \int_{(s-y)^+}^{s} n^{1-\alpha} R_H^{(n)}(nr) dr \right|^{2-\theta} n^{\alpha+1} \Lambda(n dy).$$

Since $W_0(s) \leq Cs^\alpha$, we have the first integral above can be bounded by $C t^{\alpha(1-\theta)}$. By Proposition 4.6 and 4.3, the last two integrals can be bounded by $C t^{\alpha(1-\theta)}$ and $C (1 + t)^{\alpha(1-\theta)}$ respectively. \hfill \Box
Characterization of local time processes. We first generalize the classical second Ray-Knight theorem to \( \xi_0 \) by characterizing the limit process \( X_{0,\xi} \). From all lemmas in the previous sections, we have \((X_{1}^{(n)}, \sum_{i=0}^{n} \xi_i^{(n)}, \hat{X}_{1}^{(n)}, M^{(n)}) \to (X_{0,\xi}, 0, (1 - \beta W_0), M_0) \) weakly in \( D([0, \infty), \mathbb{R}^4) \) as \( n \to \infty \). An argument similar to the one used in Proposition 4.6 in [27]; see also the proof of Theorem 6.9(a) in [23, p.582], shows that

\[
X_{0,\xi}(t) = \zeta(1 - \beta W_0(t)) + \int_0^t \int_0^\infty \int_0^\infty \nabla_y W_0(t - s) \tilde{N}_0(ds, dy, dz).
\]

For the stable process \( \xi \) defined by (1.3), let \( c_0 := (c/\Gamma(1 - \alpha))^{1-\alpha} \) and \( \beta = b/c_0 \). We have \( \xi = c_0 \xi_0 \) in distribution. Like the argument in (2.7), we have the local time process \( \{L_\xi(t, \tau^\xi_t(\zeta)) : t \geq 0\} \) under \( P(\cdot|\tau^\xi_0(\zeta) < \infty) \) is equal in distribution to \( \{c_0^{-1} L_{\xi_0}(t/c_0, \tau^\xi_{\xi_0}(c_0 \zeta)) : t \geq 0\} \) under \( P(\cdot|\tau^\xi_0(c_0 \zeta) < \infty) \), which is equal in distribution to \( \{X_\zeta(t) := X_{0,\xi}(t/c_0)/c_0 : t \geq 0\} \). From (4.18) and the fact that \( W(t) = W_0(t/c_0)/c_0 \), we see that \( X_\zeta \) is a weak solution to

\[
X_\zeta(t) = X_{0,\xi}(t/c_0)/c_0 = \zeta(1 - b W(t)) + \int_0^t \int_0^\infty \int_0^\infty \nabla_y W(t - s) \tilde{N}_\alpha(ds, dy, dz),
\]

where \( \tilde{N}_\alpha(ds, dy, dz) := \tilde{N}_0(e_0^{-1}ds, e_0^{-1}dy, c_0 dz) \) is compensated Poisson process with intensity \( d\nu_\alpha(c_0^{-1}dy, c_0^{-1}dy, c_0 dz) \). It is easy to see that \( \nu_\alpha(c_0^{-1}dy) = \alpha(\alpha + 1)c_0^{\alpha + 1} y^{-\alpha - 2}dy = \nu_\alpha(dy) \). Consequently, conditioned on \( \tau^\xi_0(\zeta) < \infty \) the local time process \( \{L_\xi(t, \tau^\xi_t(\zeta)) : t \geq 0\} \) is a weak solution to (1.6).

We now consider the regularity of \( X_\zeta \). Apparently, the function \( \zeta(1 - bW) \) is Hölder continuous with exponent \( \alpha \). Repeating the proof of Lemma 4.5, we also can prove that \( \sup_{t \in [0,1]} E[|X_\zeta(t)|^p] < \infty \) for any \( p \geq 0 \). Denote by \( M(t) \) the stochastic integral in (1.6). Like the proof of Lemma 4.9, we also have for any \( p \geq 1 \) there exists a constant \( C > 0 \) such that \( E[|\Delta_h M(t)|^{2p}] \leq C h^{p\alpha} \) for \( t, h \in [0, 1] \). By the Kolmogorov continuity theorem, we have \( M \) is locally Hölder continuous of any order strictly less than \( \alpha/2 \) and so is \( X_\zeta \). For any \( \theta \in (0, \alpha) \), let \( \|M\|_{\theta,1} \) be the Hölder coefficient of \( W \) on \([0, 1]\), i.e.,

\[
\|M\|_{\theta,1} := \sup_{0 \leq s \leq t \leq 1} \frac{|M(t) - M(s)|}{|s - t|^\theta}.
\]

By the Garsia-Rodemich-Rumsey inequality; see Lemma 1.1 in [16] with \( \psi(u) = |u|^p \) and \( p(u) = |u|^{q+1}/p \) with \( p > 1 \) and \( q > 1/p \), there exists a constant \( C_{p,q} \) such that for any \( s, t \in [0, 1] \),

\[
|M(t) - M(s)| \leq C_{p,q} |t - s|^{pq - 1} \int_s^t dv \int_s^v \frac{|M(u) - M(v)|^p}{|u - v|^{pq + 1}} du.
\]

In particular, choosing \( p > 2(1 + \theta)/\alpha \) and \( q = 1/p + \theta \) we have

\[
\|M\|_{\theta,1}^p \leq C_{p,q} \int_0^1 dv \int_0^1 \frac{|M(u) - M(v)|^p}{|u - v|^{pq + 1}} du
\]

and

\[
E[\|M\|_{\theta,1}^p] \leq C_{p,q} \int_0^1 dv \int_0^1 \frac{E[|M(u) - M(v)|^p]}{|u - v|^{pq + 1}} du
\leq C_{p,q} \int_0^1 dv \int_0^1 |u - v|^{p\alpha/2 - \theta - 2} du < \infty.
\]

Up to now we have proved Theorem 1.1 expect the weak uniqueness of solutions to (1.6), which will be proved in Section 6. Theorem 1.2 follows directly from Theorem 1.1 and the law of total probability.
5. Characteristic functionals. In this section we prove the exponential representation (1.9) of the characteristic functionals of solutions to (1.6) with \( \lambda \in i \mathbb{R} \) and \( g \in B(\mathbb{R}^+, i \mathbb{R}) \). For the general case, it will be proved in Section 6.

For a pair \((v^\theta_{\lambda}, T^\theta_{\lambda})\) with \( T^\theta_{\lambda} \in (0, \infty) \) and a continuous \( \mathbb{C}\)-valued function \( v^\theta_{\lambda} \) on \([0, T^\theta_{\lambda})\), we say it is a noncontinuous solution to (1.10) if \( v^\theta_{\lambda} \) satisfies (1.10) on \([0, T^\theta_{\lambda})\) and \(|v^\theta_{\lambda}(t)| \to \infty \) as \( t \to T^\theta_{\lambda}^- \). If \( T^\theta_{\lambda} = \infty \), the function \( v^\theta_{\lambda} \) turns to a global solution to (1.10). In terms of \( v^\theta_{\lambda} \) and \( X_\lambda \) we first establish a martingale with the help of the following moment results. For any \( t, r \geq 0 \), taking expectations and \( \mathcal{F}_r \)-conditional expectations on both sides of (1.6), we have \( \mathbb{E}[X_\lambda(t)] = \zeta (1 - bW(t)) \) and

\[
(5.1) \quad \mathbb{E}[X_\lambda(t)| \mathcal{F}_r] = \zeta [1 - bW(t)] + \int_0^t \int_0^\infty \int_0^\infty X_\lambda(s) \nabla_y W(t-s) \tilde{N}_\alpha(ds,dy,dz).
\]

**Proposition 5.1.** For \( \lambda \in i \mathbb{R} \) and \( g \in B(\mathbb{R}^+, i \mathbb{R}) \), assume \((v^\theta_{\lambda}, T^\theta_{\lambda})\) is a noncontinuous solution to (1.10). Then the random variable \( Y_T := \lambda X_\lambda(T) + (g + \nabla_\lambda v^\theta_{\lambda}) \ast X_\lambda(T) \) is integrable for any \( T \in (0, T^\theta_{\lambda}) \) and the Doob martingale \( \{Y_T(t) := \mathbb{E}[Y_T|\mathcal{F}_t]: t \in [0, T]\} \) has the representation

\[
Y_T(t) = \zeta \cdot K * v^\theta_{\lambda}(T) + \int_0^t \int_0^\infty X_\lambda(s) \int_{(T-s-y)^+}^T v^\theta_{\lambda}(r) dr \tilde{N}_\alpha(ds,dy,dz).
\]

**Proof.** Noting that \( \mathbb{E}[X_\lambda(t)] = \zeta (1 - bW(t)) \leq \zeta \) for any \( t \geq 0 \), we immediately have \( \mathbb{E}[|Y_T|] \leq \zeta (|\lambda| + \|g\|_{L_1} + \|\nabla_\lambda v^\theta_{\lambda}\|_{L_1}) < \infty \) and

\[
Y_T(0) = \mathbb{E}[Y_T] = \zeta (1 - bW(T)) = \zeta \cdot (g + \nabla_\lambda v^\theta_{\lambda}) \ast (1 - bW)(T).
\]

Convolving both sides of (1.10) by \( K \) and then using (2.5), we have

\[
K * v^\theta_{\lambda} = \lambda \cdot K * W' + (g + \nabla_\lambda v^\theta_{\lambda}) \ast K * W' = \lambda (1 - bW) + (g + \nabla_\lambda v^\theta_{\lambda}) \ast (1 - bW)
\]

and hence \( Y_T(0) = \zeta \cdot K * v^\theta_{\lambda}(T) \). For \( t \in (0, T] \), from (5.1) we have

\[
Y_T(t) = \zeta \cdot K * v^\theta_{\lambda}(T) + \int_0^t \int_0^\infty X_\lambda(s) \lambda \nabla_y W(T-s) \tilde{N}_\alpha(ds,dy,dz)
\]

\[
+ \int_0^T (g + \nabla_\lambda v^\theta_{\lambda})(T-r) dr \int_0^\infty \int_0^\infty \int_0^\infty \nabla_y W(r-s) \tilde{N}_\alpha(ds,dy,dz).
\]

Changing the order of integration in the last term on the right side of (5.2), we have it equals to

\[
\int_0^t \int_0^\infty \int_0^\infty X_\lambda(r) r W'(s-r) ds \tilde{N}_\alpha(ds,dr,dy,dz)
\]

\[
= \int_0^t \int_0^\infty \int_0^\infty X_\lambda(r) r W'(s-r) ds \tilde{N}_\alpha(ds,dr,dy,dz)
\]

\[
= \int_0^t \int_0^\infty \int_0^\infty X_\lambda(r) r W'(s-r) ds \tilde{N}_\alpha(ds,dr,dy,dz)
\]

\[
= \int_0^t \int_0^\infty \int_0^\infty X_\lambda(r) r W'(s-r) ds \tilde{N}_\alpha(ds,dr,dy,dz)
\]

\[
= \int_0^t \int_0^\infty \int_0^\infty X_\lambda(r) r W'(s-r) ds \tilde{N}_\alpha(ds,dr,dy,dz)
\]

Plugging this into (5.2) and using the fact that

\[
\lambda \nabla_y W(s) + (g + \nabla_\lambda v^\theta_{\lambda}) \ast \nabla_y W(s) = \int_0^s \left( \lambda W'(r) + (g + \nabla_\lambda v^\theta_{\lambda}) \ast W'(r) \right) dr
\]

\[
= \int_0^s v^\theta_{\lambda}(r) dr,
\]
we see that the sum of the two stochastic integrals on the right side of (5.2) equals to
\[
\int_0^t \int_0^\infty X_\xi(s) \int_{T-s-y}^{T-s} v_\alpha^g(r) dr \tilde{N}_\alpha(ds, dy, dz).
\]
Here we have finished the proof. \qed

**Lemma 5.2.** Under the conditions of Proposition 5.1, we have (1.9) holds for any \( t \in (0, T_\lambda^g) \).

**Proof.** For any \( T > 0 \), we define a stochastic process \( \{Z_T(t) : t \in [0, T]\} \) with
\[
Z_T(t) := E[\lambda Z_\xi(T) + g * Z_\xi(T)|\mathcal{F}_t] + \int_t^T V_\alpha v_\lambda^g(T-s) E[Z_\xi(s)|\mathcal{F}_s] ds.
\]
From Proposition 5.1, we have
\[
Z_T(t) = Y_T(t) - \int_0^t V_\alpha v_\lambda^g(T-s) X_\xi(s) ds
\]
\[
= \zeta \cdot K * v_\lambda^g(T) - \int_0^t V_\alpha v_\lambda^g(T-s) X_\xi(s) ds
\]
\[
+ \int_t^T \int_0^\infty \int_{T-s-y}^{T-s} v_\alpha^g(r) dr \tilde{N}_\alpha(ds, dy, dz),
\]
which is a semi-martingale. Applying Itô’s formula to \( \exp\{Z_T(t)\} \) along with (1.10), we have for \( t \in [0, T] \),
\[
\exp\{Z_T(t)\} = \exp\{\zeta \cdot K * v_\lambda^g(T)\} + \text{local martingale}.
\]
Taking expectations on both sides of the equation above, we can get (1.9) by using the standard stopping time argument together with the fact that \( |\exp\{Z_T(t)\}| \leq 1 \). \qed

**6. Nonlinear Volterra equations.** In this section we first prove the existence and uniqueness of solutions to the nonlinear Volterra equation (1.10) with \( \lambda \in i\mathbb{R} \) and \( g \in B(\mathbb{R}_+; i\mathbb{R}) \). Together with Lemma 5.2, it yields the weak uniqueness and nonnegativity of solutions to (1.6). These finally allow us to come back to prove the existence and uniqueness of solutions to the nonlinear Volterra equation (1.10) for any \( \lambda \in \mathbb{C}_- \) and \( g \in B(\mathbb{R}_+; \mathbb{C}_-) \). We first in the following lemma provide a prior estimation for solutions to (1.10). It not only provides a convex set in which we search solutions, but also helps to extend any local solution on the whole half real line \( \mathbb{R}_+ \) successfully.

**Proposition 6.1.** Suppose the conditions of Proposition 5.1 holds. If \( T_\lambda^g < \infty \), we have
(6.1)
\[
\int_0^{T_\lambda^g} |\text{Re } v_\lambda^g(t)| dt < \infty.
\]

**Proof.** From Lemma 5.2, there exists a weak solution \( X_\xi \) to (1.6) associated to \((v_\lambda^g, T_\lambda^g)\) such that (1.9) holds for \( T \in (0, T_\lambda^g) \). Since \( X_\xi \) is continuous, we have \( \sup_{t \in [0, T_\lambda^g]} |X_\xi(t)| < \infty \) a.s. Thus
\[
0 < \inf_{t \in [0, T_\lambda^g]} |\exp\{\zeta \cdot K * v_\lambda^g(t)\}| \leq \sup_{t \in [0, T_\lambda^g]} |\exp\{\zeta \cdot K * v_\lambda^g(t)\}| \leq 1.
\]
and \( \sup_{t \in [0, T_\lambda^g]} K * |\text{Re } v_\lambda^g|(t) < \infty \). Noting that \( \phi * K \equiv 1 \) and \( \phi \) is locally integrable, we have
\[
\int_0^{T_\lambda^g} |\text{Re } v_\lambda^g(t)| dt = \phi * K * |\text{Re } v_\lambda^g|(T_\lambda^g) < \infty.
\]
\qed
Lemma 6.2. Under the conditions of Proposition 5.1, we have $T_\alpha^g = \infty$. Moreover, for any $T > 0$ there exists a constant $C > 0$ such that $|v_\alpha^g(t)| \leq C t^{\alpha - 1}$ for any $t \in [0, T]$.

Proof. Assume for contradiction that $T_\alpha^g < \infty$. From Proposition 6.1 and the fact that $|e^z - 1 - z| \leq |z|^2 e^{\|z\|}$ for any $z \in \mathbb{C}$, there exists a constant $C > 0$ such that for any $t \in (0, T_\alpha^g]$ and $x \geq 0$,

$$\int_{(t-x)^+}^t |v_\alpha^g(s)| ds \leq C \quad \text{and} \quad |\nabla_x v_\alpha^g(t)| \leq C \int_0^\infty \left( \int_{(t-x)^+}^t v_\alpha^g(s) ds \right)^2 \nu(x).$$

For any $t, x \geq 0$, integrating both sides of (1.10) over $(t-x)^+$, we have

$$\int_{(t-x)^+}^t v_\alpha^g(s) ds = \lambda \nabla_x W(t) + (g + \nabla_x v_\alpha^g) \ast \nabla_x W(t).$$

Similarly, we also have

$$|V_\alpha v_\alpha^g(t)| \leq C [J_1(t) + J_2(t) + J_3(t)] \text{ for any } t \in (0, T_\alpha^g],$$

where

$$\begin{align*}
J_1(t) &:= \int_0^\infty |\nabla_x W(t)|^2 \nu(x) dt, \quad J_2(t) := \int_0^\infty \|\nabla_x W\|_{L_t^1}^2 \nu(x) dt, \\
J_3(t) &:= \int_0^\infty |(V_\alpha v_\alpha^g) \ast \nabla_x W(t)|^2 \nu(x) dt.
\end{align*}$$

By (4.9) we first have

$$J_1(t) \leq C \int_{t/2}^{t/2} (t-x)^{(2\alpha-1)} x^{-\alpha} dx + C \int_{t/2}^{\infty} t^{2\alpha} x^{-\alpha-2} dx \leq C t^{\alpha - 1}.$$

In addition, the monotonicity of $W$ yields that $\|\nabla_x W\|_{L_t^1} = \|W\|_{L_t^1} \leq \|W\|_{L^1_{t-x}} \leq C t^{\alpha} \mu(x \wedge t)$ and hence

$$J_2(t) \leq C t^{2(\alpha+1)} \cdot t^{-\alpha-1} + C \int_0^t t^{2\alpha} x^{-\alpha} dx \leq C t^{\alpha+1}.$$

We now turn to analyze $J_3(t)$. It is apparent from the Cauchy-Schwarz inequality that $J_3(t) \leq J_{31}(t) + J_{32}(t) + J_{33}(t)$ for any $t \geq 0$ with $J_{31}(t) := |(V_\alpha v_\alpha^g) \ast W(t)|^2 \cdot \nu(x, t)$ and

$$\begin{align*}
J_{32}(t) &:= 2 \int_0^t \left( \int_{(t-x)^+}^t \nu_\alpha v_\alpha^g(s) W(t-s) ds \right)^2 \nu(x) dt, \\
J_{33}(t) &:= 2 \int_0^t \left( \int_{(t-x)^+}^t \nu_\alpha v_\alpha^g(s) \nabla_x W(t-s) ds \right)^2 \nu(x) dt.
\end{align*}$$

For $\theta \in (1, \frac{1}{\alpha})$ and $\eta \in (\frac{1}{\theta} - \frac{1+\alpha^2}{2}, \frac{1}{2\theta})$, define

$$H(t) := \int_0^t s^{-\theta \eta} \cdot |\nu_\alpha \circ v_\alpha^g(t-s)|^\theta ds, \quad t > 0.$$

Using Hölder’s inequality together with the fact that $W(s) \leq C s^\alpha$, we have

$$|(V_\alpha v_\alpha^g) \ast W(t)| = \int_0^t s^{-\eta} |\nu_\alpha v_\alpha^g(t-s)| \cdot s^\eta W(s) ds \leq \|H(t)\|^{1/\theta} \cdot \left( \int_0^t s^\eta W(s)^{\theta/(\theta-1)} ds \right)^{1-1/\theta} \leq C \|H(t)\|^{1/\theta} \cdot t^{\alpha+\eta+1-1/\theta}$$

and hence $J_{31}(t) \leq \|H(t)\|^{2/\theta} \cdot t^{\alpha+2\eta+1-2/\theta}$. Similarly, we also have

$$\int_{(t-x)^+}^t |\nabla_x v_\alpha^g(s)| W(t-s) ds = \int_0^x s^{-\eta} |\nu_\alpha v_\alpha^g(t-s)| \cdot s^\eta W(s) ds$$
\[
\left( \int_0^x s^{-\eta \theta} |V_{\alpha} v^\alpha(s) - t(s)|^{\theta} ds \right)^{1/\theta} \cdot \left( \int_0^x |s^\beta W(s)|^{\theta/(\theta-1)} ds \right)^{1-1/\theta},
\]
which can be bounded by \(C|H(t)|^{1/\theta} \cdot x^{\alpha+\eta+1-1/\theta}\) and hence \(J_{32}(t) \leq C|H(t)|^{1/\theta} \cdot t^{\alpha+2\eta+1-1/\theta}\). For \(J_{33}(t)\), we firstly analyze its inner integral like the argument before. By Hölder’s inequality,

\[
\int_0^t |V_{\alpha} v^\alpha(s) \nabla_x W(t-s) ds = \int_x^t (s-x)^{-\eta} |V_{\alpha} v^\alpha(t-s)| (s-x)^\eta \nabla_x W(s) ds
\]

\[
\leq \left( \int_x^t (s-x)^{-\eta} |V_{\alpha} v^\alpha(t-s)|^{\theta} ds \right)^{1/\theta} \cdot \left( \int_x^t (s-x)^\eta \nabla_x W(s) \right)^{\theta/(\theta-1)}
\]

Changing the variable of the integration in the first term on the right side of the inequality above, we have it equals to \(|H(t-x)|^{1/\theta}\). For \(t \in (1+\alpha, \eta+1-1/\theta, \wedge 1)\) we have \(\nabla_x W(s) \leq C s^{\alpha-1} \cdot (s-x)^{\ell(\alpha-1)} \cdot x^\ell\) and the second term on the right side of the inequality above can be bounded by

\[
C \left( t^{\ell(1-\alpha)} \int_x^t (s-x)^{(\eta+\ell)(\alpha-1)} \cdot x^\ell ds \cdot x\eta \right)^{\theta-1} \leq C t^{\ell+\eta-\ell+1-\frac{1}{\theta}} \cdot x^\ell.
\]

Taking these two estimates back into \(J_{33}(t)\), we have

\[
J_{33}(t) \leq C t^{2(\alpha+\eta-\ell+1-1/\theta)} \cdot \int_0^t |H(t-x)|^{\frac{\theta}{2}} \cdot x^{2\ell-\alpha-2} dx.
\]

Putting all estimate above together, we have

\[
(6.2)V_{\alpha} v^\alpha(t) \leq Ct^{\alpha-1} + C|H(t)|^{\frac{\theta}{2}} \cdot t^{\alpha+2\eta+1-\frac{\theta}{2}} + C t^{2(\alpha+\eta-\ell+1-1/\theta)} \int_0^t |H(t-x)|^{\frac{\theta}{2}} \cdot x^{2\ell-\alpha-2} dx.
\]

Raising both sides of the above inequality to the \(\theta\) power and then convolving them by power function \(t^{-\theta\eta}\), we have for any \(t > 0\),

\[
H(t) \leq C \int_0^t (t-s)^{-\eta} \cdot s^{\theta(\alpha-1)} ds + C \int_0^t (t-s)^{-\eta} \cdot |H(s)|^2 \cdot s^{\theta(\alpha+2\eta+1-2)} ds
\]

\[
+ C \int_0^t (t-s)^{-\eta} \cdot s^{2\theta(\alpha-\ell+1) -2} \left( \int_0^s |H(s-x)|^{\frac{\theta}{2}} \cdot x^{2\ell-\alpha-2} dx \right)^{\theta} ds.
\]

Since \(\eta \theta < 1/2\) and \(\theta(\alpha-1) < -1\), a simple calculation shows that the first term on the right side of the above inequality can be bounded by \(C t^{\eta \theta}\) for any \(t \in (0, T]\). Using Hölder’s inequality together with the fact that \(2\ell - \alpha - 2 < -1\), we have

\[
\left( \int_0^s |H(s-x)|^{\frac{\theta}{2}} \cdot x^{2\ell-\alpha-2} dx \right)^\theta = \left( \int_0^s |H(s-x)|^{\frac{\theta}{2}} \cdot x^{2\ell-\alpha-2} \cdot x^{(1-\frac{\theta}{2})} dx \right)^\theta
\]

\[
\leq \left( \int_0^s x^{2\ell-\alpha-2} dx \right)^{\theta-1} \cdot \left( \int_0^s |H(s-x)|^{\frac{\theta}{2}} \cdot x^{2\ell-\alpha-2} dx \right)^\theta
\]

\[
\leq C s^{(\theta-1)(2\ell-\alpha-1)} \cdot \int_0^s |H(s-x)|^{\frac{\theta}{2}} \cdot x^{2\ell-\alpha-2} dx ds.
\]

From this and the fact that \(2\theta(\alpha + \eta - \ell + 1) - 2 > 0\), we have the last term on the right side of (6.3) can be bounded by

\[
C t^{2\theta(\alpha+\eta-\ell+1) -2 + (\theta-1)(2\ell-\alpha-1)} \cdot \int_0^t (t-s)^{-\eta} ds \cdot \int_0^s |H(s-x)|^{\frac{\theta}{2}} \cdot x^{2\ell-\alpha-2} dx ds
\]

\[
= C t^{\theta(2\alpha+2\eta+1)-2\ell-\alpha+\alpha} \cdot \int_0^t |H(s)|^2 \int_0^{t-s} (t-s-x)^{-\eta} \cdot x^{2\ell-\alpha-2} dx ds.
\]
Here the equality above comes from changing the order of integration. Noting that $2\ell - \alpha - 2 > -1$ and $\eta \theta < 1/2$, the inner integral in the double integral on the right side of the equality above can be bounded by $C(t-s)^{-\eta \theta + 2\ell - \alpha - 1}$. Thus the last term on the right side of (6.3) can be bounded by
\[
CT^{\theta(\alpha+2\eta+1-2/\theta)} \cdot \int_0^t (t-s)^{-\eta \theta} |H(s)|^2 ds.
\]
Taking this back into (6.3) and noting that $\alpha + 2\eta + 1 - 2/\theta > 0$, we have for any $t \in (0, T^\theta_{\lambda})$,
\[
H(t) \leq CT^{-\eta \theta} + CT^{\theta(\alpha+2\eta+1-2/\theta)} \cdot \int_0^t (t-s)^{-\eta \theta} \cdot |H(s)|^2 dt.
\]
Using the fractional integral inequality; see Theorem 2.1 in [10], the function $H$ can be uniformly bounded on $(0, T^\theta_{\lambda})$ by the continuous solution to the fractional Riccati equation
\[
(6.4) \quad \psi(t) = Ct^{-\eta \theta} + C \int_0^t (t-s)^{-\eta \theta} \cdot |\psi(t)|^2 dt
\]
for some large constant $C > 0$. The existence and uniqueness of global solution to (6.4) has been proved in [1, 8]. Specially, by using Theorem 2a in [8] together with the fact that $1 - \eta \theta \in (1/2, 1)$, we have there exists some constant $C_\psi, r_\psi > 0$ such that $\psi(t) \leq C_\psi t^{-\eta \theta}$ for any $t \in (0, r_\psi)$. From this and the continuity of $\psi$, there exists a constant $C > 0$ such that $H(t) \leq \psi(t) \leq Ct^{-\eta \theta}$ for any $t \in (0, T^\theta_{\lambda})$.

Taking this back into (6.2), since $\eta < \frac{1}{2\theta} < 1/2$ we have for any $t \in (0, T)$,
\[
|V^-_\alpha(t)| \leq Ct^\alpha - 1 + C t^{\alpha + 1 - 2/\theta} + C t^{2\alpha + 2\eta - 2\ell + 2 - 2/\theta} \cdot \int_0^t (t-s)^{-2\eta} s^{2\ell - \alpha - 2} ds \leq Ct^{\alpha - 1}.
\]
Taking this back into (1.10) together with a simple calculation, we immediately get the second desired result with $T = T^\theta_{\lambda}$. This leads to a contradiction to our assumption that $T^\theta_{\lambda} < \infty$ and $\nu_\lambda(T^\theta_{\lambda}) = \infty$. \(\square\)

For $T > 0$, it is apparent from the last lemma that we should search solutions to (1.10) on $[0, T]$ in the set
\[
A_{T,K} := \{f : f(t) \in C \text{ and } |f(t)| \leq K t^{\alpha - 1} \text{ for any } t \in (0, T)\}
\]
for some $K > 0$. Noting that the last term on the right side of equation (1.10) is the main challenge in the proof of existence and uniqueness of the solution. The next two propositions provide some properties of the map $(V_\alpha f) * W'$ acting on elements $f \in A_{T,K}$.

**Proposition 6.3.** There exists a constant $C > 0$ such that $|(V_\alpha f) * W'(t)| \leq C K^2 e^{\frac{K}{\alpha} s^{\alpha}} t^{2\alpha - 1}$ for any $K, T > 0$, $f \in A_{T,K}$ and $t \in (0, T]$.

**Proof.** Noting that $\|f\|_{L^1} \leq \frac{K}{\alpha} s^{\alpha}$ for any $f \in A_{T,K}$ and $s \in [0, T]$. By the fact that $|e^x - 1 - x| \leq |x|^2 e^{\frac{1}{2}|x|}$ for any $x \in \mathbb{C}$, we have for any $s \in [0, T]$,
\[
(6.5) \quad V_\alpha f(s) \leq e^{\frac{K}{\alpha} s^{\alpha}} \int_0^\infty \left| \int_{(s-y)}^\infty f(r) dr \right|^2 \nu_\alpha(dy) \\
\leq e^{\frac{K}{\alpha} s^{\alpha}} \int_0^s f(r) dr \left| \nu_\alpha(s, \infty) + e^{\frac{K}{\alpha} s^{\alpha}} \int_s^{s/2} f(r) dr \right|^2 \nu_\alpha(dy) \\
+ e^{\frac{K}{\alpha} s^{\alpha}} \int_s^{s/2} f(r) dr \left| \nu_\alpha(s, \infty) \right|^2 \nu_\alpha(dy).
\]
It is easy to see that the first term on the right side of the second inequality in (6.5) can be bounded by $C K^2 e^{\frac{K}{\alpha} s^{\alpha}} t^{\alpha - 1}$ and the second term can be bounded by $C K^2 e^{\frac{K}{\alpha} s^{\alpha}} t^{\alpha - 1}$.


Here the constant $C$ is independent of $K$ and $s$. Moreover, the fact that $f(r) \leq K(s/2)^{\alpha - 1}$ for any $r \in [s/2, s]$ implies that

$$
\int_0^{s/2} \left| \int_{s-y}^s f(r) dr \right|^2 \nu_\alpha(dy) \leq CK^2 s^{2(\alpha - 1)} \int_0^{s/2} y^{-\alpha} dy \leq CK^2 s^{\alpha - 1}.
$$

Putting all estimates above together, we have $\mathcal{V}_\alpha f(t) \leq CK^2 e^{\frac{K}{\alpha} s^\alpha} s^{\alpha - 1}$ and hence

$$
(\mathcal{V}_\alpha f) * W'(t) \leq CK^2 e^{\frac{K}{\alpha} s^\alpha} \int_0^t s^{\alpha - 1}(t-s)^{\alpha - 1} ds \leq CK^2 e^{\frac{K}{\alpha} s^\alpha} t^{2\alpha - 1}.
$$

\[\square\]

**Proposition 6.4.** For $\theta \in (1, (1 - \alpha)^{-1})$, there exists a constant $C > 0$ such that for any $K, T > 0$ and $f_1, f_2 \in \mathcal{A}_{T, K}$,

$$
\left\| (\mathcal{V}_\alpha f_1 - \mathcal{V}_\alpha f_2) * W' \right\|_{L^p_T} \leq CK \cdot T^\alpha e^{\frac{K}{\alpha} T^\alpha} \left\| f_1 - f_2 \right\|_{L^p_T}.
$$

**Proof.** Let $\bar{f} := f_1 - f_2$. For any $x, z \in \mathbb{C}$, it is easy to see that

$$
| (e^{x^2} - 1 - x) - (e^{z^2} - 1 - z) | \leq (e^{|x|} + e^{|z|})(|x| + |z|)|x - z|.
$$

Noting that $\|f_1\|_{L^1_T} \vee \|f_2\|_{L^1_T} \leq \frac{K}{\alpha}$, we have

$$
\left| (\mathcal{V}_\alpha f_1(t) - \mathcal{V}_\alpha f_2(t)) \right| \leq 2e^{\frac{K}{\alpha} t^\alpha} \int_0^\infty \left[ \|f_1\|_{L^1_{(t-y), t}} + \|f_2\|_{L^1_{(t-y), t}} \right] \cdot \|\bar{f}\|_{L^1_{(t-y), t}} \nu_\alpha(dy).
$$

By Minkowski’s inequality, we have $\| (\mathcal{V}_\alpha f_1 - \mathcal{V}_\alpha f_2) * W' \|_{L^p_T} \leq C e^{\frac{K}{\alpha} T^\alpha} (\|J_1\|_{L^p_T} + \|J_2\|_{L^p_T} + \|J_3\|_{L^p_T})$, where

$$
J_1(t) := \int_0^t W'(t-s) \nu_\alpha(s, \infty) \cdot \left( \|f_1\|_{L^1_{s-y, s}} + \|f_2\|_{L^1_{s-y, s}} \right) \cdot \|\bar{f}\|_{L^1_{s-y, s}} \nu_\alpha(dy),
$$

$$
J_2(t) := \int_0^s W'(t-s) ds \int_{s/2}^s \left( \|f_1\|_{L^1_{s-y, s}} + \|f_2\|_{L^1_{s-y, s}} \right) \cdot \|\bar{f}\|_{L^1_{s-y, s}} \nu_\alpha(dy),
$$

$$
J_3(t) := \int_{s/2}^t W'(t-s) ds \int_0^{s/2} \left( \|f_1\|_{L^1_{s-y, s}} + \|f_2\|_{L^1_{s-y, s}} \right) \cdot \|\bar{f}\|_{L^1_{s-y, s}} \nu_\alpha(dy).
$$

By Hölder’s inequality, we have $\|\bar{f}\|_{L^1_{s-y, s}} \leq \|\bar{f}\|_{L^p_{s-y, s}} \cdot s^{1-1/\theta}$. Moreover, since $\|f_1\|_{L^1_T} + \|f_2\|_{L^1_T} \leq 2K/\alpha \cdot s^\alpha$ and $\nu_\alpha(s, \infty) \leq Cs^{-\alpha - 1}$, we have

$$
J_1(t) \leq CK \int_0^t (t-s)^{\alpha - 1} s^{-1/\theta} \|\bar{f}\|_{L^p_{s-y, s}} ds \leq C t^{\alpha - 1/\theta} \|\bar{f}\|_{L^p_T} \quad \text{and} \quad \|J_1\|_{L^p_T} \leq CK \cdot T^\alpha \cdot \|\bar{f}\|_{L^p_T}.
$$

For any $y \in [0, s]$ we also have $\|f_1\|_{L^1_{s-y, s}} + \|f_2\|_{L^1_{s-y, s}} \leq 2K/\alpha \cdot s^\alpha$ and $\|\bar{f}\|_{L^1_{s-y, s}} \leq \|\bar{f}\|_{L^p_{s-y, s}} \cdot s^{1-1/\theta}$. Thus

$$
J_2(t) \leq CK \int_0^t s^{\alpha + 1 - 1/\theta} \nu_\alpha(s/2, s) \cdot (t-s)^{\alpha - 1} \|\bar{f}\|_{L^p_{s-y, s}} ds \leq CK t^{\alpha - 1/\theta} \|\bar{f}\|_{L^p_T}
$$

and hence $\|J_2\|_{L^p_T} \leq CK \cdot T^\alpha \|\bar{f}\|_{L^p_T}$. We now turn to $J_3$. Noting that $\|f_1\|_{L^1_{s-y, s}} + \|f_2\|_{L^1_{s-y, s}} \leq K(s-y)^{\alpha - 1} \cdot y$ for $y \in (0, s)$, the inner integral in $J_3$ can be bounded by

$$
CK \int_0^{s/2} (s-y)^{\alpha - 1} y^{-\alpha - 1} \int_{s-y}^s \left| \bar{f}(r) \right| dr \, dy = CK \int_0^{s/2} \left| \bar{f}(s-y) \right| dy \int_y^{s/2} (s-r)^{\alpha - 1} r^{-\alpha - 1} dr.
$$
The equality above comes from changing the order of integration. Taking this back into $J_3(t)$, we have for $\eta \in (0, \frac{1}{\theta(1-\alpha)})$,

$$J_3(t) \leq CK \int_0^t (t-s)^{\alpha-1} \int_0^s (s-y)^{\alpha-1} |\tilde{f}(y)|dyds$$

$$= CK \int_0^t |\tilde{f}(s)|ds \int_s^t (t-y)^{\alpha-1} y^{\alpha-1} (y-s)^{-\alpha} dy$$

$$= CK \int_0^t |\tilde{f}(s)|ds \int_0^{t-s} (t-s-y)^{\alpha-1} (y+s)^{\alpha-1} y^{-\alpha} dy$$

$$\leq CK \int_0^t s^{(1-\eta)(\alpha-1)} |\tilde{f}(s)|ds \int_0^{t-s} (t-s-y)^{\alpha-1} y^{\eta(\alpha-1)-\alpha} dy$$

$$\leq CK \int_0^t (t-s)^{\eta(\alpha-1)} s^{(1-\eta)(\alpha-1)} |\tilde{f}(s)|ds.$$

Applying Young’s and then Hölder’s inequality, we have

$$\|J_3\|_{L^q} \leq CK \left( \int_0^T t^{\theta(\alpha-1)} dt \right)^{1/\theta} \cdot \int_0^T s^{(1-\eta)(\alpha-1)} |\tilde{f}(s)|ds$$

$$\leq CKT^{\eta(\alpha-1)+1/\theta} \cdot \left( \int_0^T s^{(1-\eta)(\alpha-1)-1} ds \right)^{\theta/\theta-1} \|\tilde{f}\|_{L^q} \leq CK \cdot T^\alpha \|\tilde{f}\|_{L^q}.$$

Putting all estimates above together we can get the desired result immediately. \hfill $\Box$

**Lemma 6.5.** For any $\lambda \in i\mathbb{R}$ and $g \in B(\mathbb{R}_+; i\mathbb{R})$, the nonlinear Volterra equation (1.10) has a unique continuous global solution.

**Proof.** Our proof is based on Banach’s fixed point theorem. We first prove the existence of the solutions for small times. For some constants $K > 1$ and $\delta \in (0, 1)$ that will be specified later, we consider the map $\mathcal{R}_0$ that acts on functions $f \in \mathcal{A}_{\delta,K}$ according to

$$\mathcal{R}_0 f := \lambda W' + (g + V_{\alpha} f) * W'.$$

From (2.3) and Proposition 6.3, we have for any $f \in \mathcal{A}_{\delta,K}$ and $t \in (0, \delta]$,

$$\|\mathcal{R}_0 f(t)\| \leq C (\lambda + \|g\|_{\infty} + K^2 \delta^\alpha e^{\frac{K}{2C} \delta^\alpha}) \cdot t^\alpha - 1.$$

On the other hand, for $\theta \in (1, (1-\alpha)^{-1})$, by Proposition 6.4 we also have for any $f_1, f_2 \in \mathcal{A}_{\delta,K}$,

$$\|\mathcal{R}_0 f_1 - \mathcal{R}_0 f_2\|_{L^q} = \|V_{\alpha} f_1 - V_{\alpha} f_2\|_{L^q} \leq C K \cdot \delta^\alpha e^{\frac{K}{2C} \delta^\alpha} \|f_1 - f_2\|_{L^q}.$$

We choose $K > 2C(\lambda + \|g\|_{\infty})$ and $\delta^\alpha < \frac{e^{\frac{K}{2C} \delta^\alpha}}{2CK} \wedge 1$. This yields $\mathcal{R}_0 f(t) \leq K t^\alpha - 1$ and $\|\mathcal{R}_0 f_1 - \mathcal{R}_0 f_2\|_{L^q} \leq \|f_1 - f_2\|_{L^q}$. Thus $\mathcal{R}_0$ is a contractive map from $\mathcal{A}_{\delta,K}$ to itself. It can easily shown that $\mathcal{A}_{\delta,K}$ is a closed, bounded and convex subset in $L^\theta(\mathbb{R}_+)$. By Banach’s fixed point theorem there exists a unique fixed point $v_0 \in \mathcal{A}_{\delta,K}$, which is a solution to (1.10) in the sense that $v_0 = \mathcal{R}_0 v_0$ in $L^\theta((0, \delta])$. Let $v_0^\theta := \mathcal{R}_0 v_0$. By the properties of the convolution, the function $v_0^\theta$ is continuous and equivalent to $v_0$ in $L^\theta((0, \delta])$. It is not difficult to verify that $v_0^\theta$ is a solution to (1.10) on $(0, \delta]$. 

\[
\leq CK s^{\alpha-1} \int_0^{s/2} y^{-\alpha} |\tilde{f}(s-y)|dy \\
\leq CK s^{\alpha-1} \int_0^s (s-y)^{-\alpha} |\tilde{f}(y)|dy.
\]
We now extend this to a unique solution on the whole half line $\mathbb{R}_+$. Denote by $T$ the collection of all $t > 0$ such that (1.10) has a continuous solution on $(0, t]$. We assert that $T$ is an open interval containing $(0, \delta]$. Indeed, for any $t_0 \in T$, assume $\nu^\alpha_{t_0}$ is a solution to (1.10) on $(0, t_0]$. For $t \geq 0$, define

$$h(t) := \lambda W'(t_0 + t) + W' * g(t_0 + t) + \int_0^t \nu^\alpha_{t_0}(s)W'(t_0 + t - s)ds.$$  

From Lemma 6.2, there exists a constant $C > 0$ such that $\nu^\alpha_{t_0}(t) \leq Ct^{\alpha - 1}$ for any $t \in (0, t_0]$ and hence

$$h(t) \leq C(t_0^{\alpha - 1} + (t_0 + t)^\alpha + \int_0^{t_0} s^{\alpha - 1}(t_0 - s)^{\alpha - 1}ds \leq C[t_0^{\alpha - 1} + (t_0 + t)^\alpha].$$

Define a map $\mathcal{R}_1$ acting on functions $f \in B(\mathbb{R}_+; \mathbb{C})$ by

$$\mathcal{R}_1 f := h + W' * (\nu f).$$

For $T, K > 0$, define a closed ball $B_{T,K} := \{f \in B(\mathbb{R}_+; \mathbb{C}) : \sup_{t \in [0,T]}|f(t)| \leq K\}$ in $L^1([0,T])$. A calculation similar to (6.5) induces that there exists a constant $C > 0$ such that for any $K, T > 0$, $f \in B_{T,K}$ and $t \in [0,T]$,

$$|\nu f(t)| \leq CK^2 e^{Kt} \int_0^\infty (x^2 + t^2)x^{-\alpha - 2}dx \leq CK^2 e^{Kt} \cdot t^{1 - \alpha}$$

and

$$|\mathcal{R}_1 f(t)| \leq C[t^{\alpha - 1} + (t_0 + t)^\alpha + K^2 e^{Kt} \cdot t].$$

In addition, for any $f_1, f_2 \in B_{T,K}$, let $\bar{f} := f_1 - f_2$. Like the calculation in (6.6), we also have for any $t \in [0,T]$,

$$|\nu f_1(t) - \nu f_2(t)| \leq CK^2 \int_0^\infty x^{-\alpha - 1}dx \int_0^t (t-x)^{\alpha} |\bar{f}(r)|dr$$

and

$$|\nu f_1(t) - \nu f_2(t)| \leq CK^2 \cdot \left[\|\bar{f}\|_{L^1} \cdot t^{\alpha - \alpha} + \int_0^t x^{-\alpha - 1}dx \int_t^\alpha |\bar{f}(r)|dr\right]$$

and

$$|\mathcal{R}_1 f_1(t) - \mathcal{R}_1 f_2(t)| \leq CK^2 \cdot \left[\|\bar{f}\|_{L^1} \cdot \int_0^t (t-s)^{\alpha - 1}s^{\alpha - \alpha}dsight.$$  

$$+ \int_0^t (t-s)^{\alpha - 1}ds \int_0^t (s-x)^{-\alpha}|\bar{f}(x)|dx\right] \leq CK^2 e^{Kt} \cdot \|\bar{f}\|_{L^1}.$$  

Here the constant $C$ is independent of $K$ and $t$. Consequently, there exists a constant $C > 0$ such that for any $T, K > 0$ and $f_1, f_2 \in B_{T,K}$,

$$\|\mathcal{R}_1 f_1 - \mathcal{R}_1 f_2\|_{L^1} \leq CKe^{KT} T \cdot \|\bar{f}\|_{L^1}. $$

Choosing $K > 2C\left[t^{\alpha - 1} + (1 + t_0)^\alpha\right]$ and $T \leq \frac{e^{-2K}}{2CK} \leq 1$, we have $\mathcal{R}_1$ is a contractive map from $B_{T,K}$ to itself. Applying Banach’s fixed point theorem again, there exists a unique fixed point $v_1 \in B_{T,K}$ satisfying $v_1 = \mathcal{R}_1(v_1)$ in $L^1([0,T])$. Let $\tilde{v}_1 = \mathcal{R}_1(v_1)$, which is continuous on $[0,T]$ and is equivalent to $v_1$ in $L^1([0,T])$. Defining $\nu^\alpha_{t_0}(t_0 + t) = \tilde{v}_1(t)$ for any $t \in [0,T]$, one can verify that $\nu^\alpha_{t_0}$ solves (1.10) on $(0, t_0 + T]$. To finish extending the solution on $\mathbb{R}_+$, it suffices to prove $T_{\text{max}} := \sup\{T\} = \infty$, which follows directly from Lemma 6.2. Indeed, if $T_{\text{max}} < \infty$ we will have $\nu^\alpha_{t_0}(t) \to \infty$ as $t$ increases to $T_{\text{max}}$, which is a contradiction to Lemma 6.2.

It remains to prove the uniqueness. Suppose $\nu^\alpha_{t_0}$ and $\tilde{\nu}^\alpha_{t_0}$ are two solutions to (1.10). By Lemma 6.2, we have both of them belongs to $A_{t_0,K}$ for some $K > 0$ and any $t \in (0, 1)$. Like the first paragraph of this proof, for some $\delta_1 \in (0, 1)$ we have $\nu^\alpha_{t_0} = \tilde{\nu}^\alpha_{t_0}$ on $(0, \delta_1]$. If $t_1 := \inf\{t \geq \delta_1 : \nu^\alpha_{t_0}(t) \neq \tilde{\nu}^\alpha_{t_0}(t)\} < \infty$, the continuity of the two solutions induces that they do not equal to each other in a neighbor of $t_1$. On
the other hand, by Lemma 6.2 there exists a constant $K > 0$ such that $\sup_{s \in [0,1]} v_\delta^g(t_1 + s) \leq K$ and $\sup_{s \in [0,1]} \tilde{v}_\delta^g(t_1 + s) \leq K$. An argument similar to the second part of this proof shows that $v_\delta^g(t_1 + s) = \tilde{v}_\delta^g(t_1 + s)$ for any $s \in [0, \varepsilon]$ and some $\varepsilon \in (0, 1]$, which is a contradiction to our previous assumption and hence the uniqueness follows.

**Proof for Theorem 1.1.** As we have mentioned in the end of Section 4, it remains to prove the uniqueness in law for the solution $X_\zeta$ to (1.6). For any $T > 0$, we consider $\{X_\zeta(t) : t \in [0, T]\}$ as an $C([0, T]; \mathbb{R})$-valued random variable, whose law is uniquely determined by the characteristic functionals $E[\exp\{\int_0^T f(t)X_\zeta(t)dt\}]$ with $f \in B([0, T]; i\mathbb{R})$. Define $g(t) = f(T - t)$ for $t \in [0, T]$. We have $E[\exp\{\int_0^T f(t)X_\zeta(t)dt\}] = E[\exp\{g \cdot X_\zeta(T)\}] = \exp\{\zeta \cdot K_\alpha v_\delta^g(T)\}$. Thus the uniqueness for solution $v_\delta^g$ to (1.10) immediately induces that the weak uniqueness of solutions holds for (1.6).

**Proof for Theorem 1.3.** From Theorem 1.1, we have solutions to (1.6) are nonnegative and hence the characteristic functional $E[\exp\{\lambda X_\zeta(t) + g \cdot X_\zeta(t)\}]$ is well defined for any $\lambda \in \mathbb{C}_-$ and $g \in B(\mathbb{R}_+; \mathbb{C}_-)$. We can prove (1.9) immediately by modifying the proofs of Proposition 5.1 and Lemma 5.2. Moreover, the $L^1$-upper bound (6.1) still holds for any noncontinuous solution to (1.10) and hence the results of Lemma 6.2 also hold. Repeating the proof of Lemma 6.5, we can prove the existence and uniqueness of global solution to (1.10) for any $\lambda \in \mathbb{C}_-$ and $g \in B(\mathbb{R}_+; \mathbb{C}_-)$. \qed

7. Fractional integral representations. We in this section prove the two equivalence relations in Theorem 1.4. When $b = 0$, they follow directly from (2.2). Firstly, the equivalence between (1.10) and (1.12) follows directly from the equation (2.4), i.e. applying the fractional integral $I_c^\alpha$ on both sides of (1.12), we have $v_\delta^g$ solves (1.12) if and only if it satisfies $v_\delta^g = \lambda L_K - b L_K * v_\delta^g + L_K * (g + \mathcal{V}_\alpha v_\delta^g)$. By using Theorem 4.6 in [18, p.48] together with (2.4), we have

$$v_\delta^g = \lambda(L_K - b W' * L_K) + (L_K - b W' * L_K) * (g + \mathcal{V}_\alpha v_\delta^g) = \lambda W' + W' * (g + \mathcal{V}_\alpha v_\delta^g).$$

We now turn to prove the equivalence between (1.6) and (1.11) with the help of the stochastic Fubini lemma and the property

$$I_c^{1-\alpha}W'(t) = \int_0^t (t - s)^{-\alpha} \frac{W'(s)}{\Gamma(1-\alpha)} ds = c^{-1}(1 - bW(t))I_{t \geq 0}. \quad (7.1)$$

We first apply $\lambda(1-\alpha)$-th order fractional integration to both sides of (1.6). By Fubini’s lemma we have the $\lambda(1-\alpha)$-th order fractional integral of the first term on the right side of (1.6) equals to

$$\frac{T^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{b}{\Gamma(1-\alpha)} \int_0^T (t - s)^{-\alpha} W(t) dt = \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{b}{\Gamma(1-\alpha)} \int_0^T dt \int_0^t s^{-\alpha} W'(t - s) ds \quad (7.2)$$

For the $\lambda(1-\alpha)$-th order fractional integral of the stochastic integral in (1.6), by the stochastic Fubini lemma we have it equals to

$$\int_0^T \int_0^\infty \int_0^x \frac{X_\zeta(t)}{\Gamma(1-\alpha)} \nabla_y W(s - t) ds \tilde{N}_\alpha(dt, dy, dz). \quad (7.3)$$

Applying integration by substitution and then Fubini’s lemma to the integrant above, from (7.1) we have it equals to

$$\int_0^{T-t} \frac{(T - t - s)^{-\alpha}}{\Gamma(1-\alpha)} \nabla_y W(s) ds = \int_0^{T-t} \frac{s^{-\alpha} ds}{\Gamma(1-\alpha)} \int_{T-t-s}^{T-t-s} W'(r) dr.$$
\[\begin{align*}
&= \int_0^{T-t} \frac{s^{-\alpha} ds}{\Gamma(1-\alpha)} \left[ \int_s^{T-t} W'(r-s)dr - \int_{s+y}^{T-t} W'(r-s-y)dr \right] \\
&= \int_0^{T-t} ds \left[ \int_0^s \frac{W'(s-r)}{\Gamma(1-\alpha)} r^{-\alpha} dr - \int_0^{s-y} \frac{W'(s-r-y)}{\Gamma(1-\alpha)} r^{-\alpha} dr \right] \\
&= \int_0^T \left[ (1-bW(s)) - (1-bW(s-y)) 1_{\{s \geq y\}} \right] \frac{ds}{c} \\
&= \int_0^T \left[ (1-bW(s-t)) - 1_{\{s-t \geq y\}} (1-bW(s-t-y)) \right] \frac{ds}{c}.
\end{align*}\]

Taking this back into the stochastic integral (7.3) and then using the stochastic Fubini lemma, we have it equals to

\[\begin{align*}
\int_0^T \int_0^t \int_0^\infty X_\zeta(s) \frac{1}{c} \left[ (1-bW(t-s)) - 1_{\{t-s \geq y\}} (1-bW(t-s-y)) \right] \tilde{N}_\alpha(ds, dy, dz) \\
&= \int_0^T \int_0^t \int_0^\infty \frac{X_\zeta(s)}{c} 1_{\{t-s \geq y\}} \tilde{N}_\alpha(ds, dy, dz).
\end{align*}\]

Combining this together with (7.2), we have

\[\begin{align*}
\int_0^T dt \int_0^t \int_0^\infty X_\zeta(s) \frac{1}{c} \left[ (1-bW(t-s)) - 1_{\{t-s \geq y\}} (1-bW(t-s-y)) \right] \tilde{N}_\alpha(ds, dy, dz) \\
&= \frac{\zeta T^{1-\alpha}}{\Gamma(2-\alpha) } \int_0^T dt \int_0^t \int_0^\infty \frac{X_\zeta(s)}{c} 1_{\{t-s \geq y\}} \tilde{N}_\alpha(ds, dy, dz) \\
&\quad - \int_0^T b \frac{dt}{c} \left[ \zeta (1-bW(t)) + \int_0^t \int_0^\infty \frac{X_\zeta(s)}{c} \nabla_y W(t-s) \tilde{N}_\alpha(ds, dy, dz) \right] \\
&= \frac{\zeta T^{1-\alpha}}{\Gamma(2-\alpha) } - \int_0^T b \frac{X_\zeta(t)}{c} dt + \int_0^T dt \int_0^t \int_0^\infty \frac{X_\zeta(s)}{c} 1_{\{t-s \geq y\}} \tilde{N}_\alpha(ds, dy, dz).
\end{align*}\]

(7.4)

Taking \((1-\alpha)\)-th order derivatives on both sides of above equation, we have the term on the left side of the first equality above turns to be \(X_\zeta(T)\) and the two terms on the right side of the second equality above respectively equal to

\[\begin{align*}
\zeta \frac{d}{dT} \int_0^T \frac{(T-t)^{\alpha-1} t^{1-\alpha}}{\Gamma(2-\alpha)\Gamma(\alpha)} dt = \zeta
\end{align*}\]

and

\[\begin{align*}
\frac{1}{\Gamma(\alpha)} \frac{d}{dT} \int_0^T (T-t)^{\alpha-1} dt \int_0^t \frac{b}{c} X_\zeta(s) ds &= \frac{1}{\Gamma(\alpha)} \frac{d}{dT} \int_0^T t^{\alpha-1} dt \int_0^T \frac{b}{c} X_\zeta(s) ds \\
&= \frac{1}{\Gamma(\alpha)} \frac{d}{dT} \int_0^T \frac{b}{c} \int_0^T s^{\alpha-1} X_\zeta(t-s) ds \\
&= \frac{b}{c} \int_0^T \frac{(T-t)^{\alpha-1}}{\Gamma(\alpha)} X_\zeta(t) dt.
\end{align*}\]

Applying integration by substitution and then the stochastic Fubini lemma to the \((1-\alpha)\)-th order fractional derivative of the last stochastic integral in (7.4), we have it equals to

\[\begin{align*}
\frac{d}{dT} \int_0^T t^{\alpha-1} dt \int_0^T dr \int_0^{r-t} \int_0^\infty X_\zeta(s) 1_{\{r-t-s \geq y\}} \frac{ds}{c\Gamma(\alpha)} \tilde{N}_\alpha(ds, dy, dz) \\
&= \int_0^T r^{\alpha-1} dr \int_0^{T-r} \int_0^\infty X_\zeta(s) 1_{\{T-r-s \geq y\}} \frac{\tilde{N}_\alpha(ds, dy, dz)}{c\Gamma(\alpha)} \\
&= \int_0^T (T-r)^{\alpha-1} dr \int_0^{T-r} \int_0^\infty X_\zeta(s) 1_{\{s \geq y\}} \frac{\tilde{N}_\alpha(ds, dy, dz)}{c\Gamma(\alpha)} \\
&= \int_0^T \int_0^\infty X_\zeta(r) \int_r^T (T-s)^{\alpha-1} 1_{\{s-r \geq y\}} \frac{ds}{c\Gamma(\alpha)} \tilde{N}_\alpha(dr, dy, dz)
\end{align*}\]
Taking these three results above back into (7.4), we can get (1.11) immediately. □

Acknowledgements. The authors would like to thank Prof. Matthias Winkel for the helpful comments and discussion about the Hölder regularity of local time processes.

REFERENCES

[1] E. Abi-Jaber, M. Larsson, and S. Pulido, Affine Volterra processes, Ann. Appl. Probab., 29 (2019), pp. 3155–3200.
[2] E. Bacry, S. Delattre, M. Hoffmann, and J. F. Muzy, Some limit theorems for Hawkes processes and application to financial statistics, Stoch. Process. Appl., 123 (2013), pp. 2475–2499.
[3] M. T. Barlow, Necessary and sufficient conditions for the continuity of local time of Lévy processes, Ann. Probab., 16 (1988), pp. 1389–1427.
[4] J. Bertoin, Lévy Processes, Cambridge University Press, 1996.
[5] P. Billingsley, Convergence of Probability Measures, Wiley, New York, NY, 2 ed., 1999.
[6] N. H. Bingham, C. M. Goldie, and J. L. Teugels, Regular Variation, vol. 27, Cambridge University Press, 1987.
[7] E. S. Boylan, Local times for a class of Markov processes, Illinois J. Math., 8 (1964), pp. 19–39.
[8] G. Callegaro, M. Grasselli, and G. Pagès, Fast hybrid schemes for fractional Riccati equations (rough is not so tough), Math. Oper. Res., 46 (2021), pp. 221–254.
[9] D. A. Dawson and Z. Li, Stochastic equations, flows and measure-valued processes, Ann. Probab., 40 (2012), pp. 813–857.
[10] Z. Denton and A. S. Vatsala, Fractional integral inequalities and applications, Comput. Math. Appl., 59 (2010), pp. 1087–1094.
[11] N. Eisenbaum and H. Kaspi, A necessary and sufficient condition for the Markov property of the local time process, Ann. Probab., 21 (1993), pp. 1591–1598.
[12] N. Eisenbaum, H. Kaspi, M. B. Marcus, J. Rosen, and Z. Shi, A Ray-Knight theorem for symmetric Markov processes, Ann. Probab., 28 (2000), pp. 1781–1796.
[13] O. El Euch, M. Fukasawa, and M. Rosenbaum, The microstructural foundations of leverage effect and rough volatility, Finance Stoch., 22 (2018), pp. 241–280.
[14] N. Forman, S. Pal, D. Rizzolo, and M. Winkel, Uniform control of local times of spectrally positive stable processes, Ann. Appl. Probab., 28 (2018), pp. 2592–2634.
[15] Z. Fu and Z. Li, Stochastic equations of non-negative processes with jumps, Stoch. Process. Appl., 120 (2010), pp. 306–330.
[16] A. M. Garsia, E. Rodemich, H. Rumsey, and M. Rosenblatt, A real variable lemma and the continuity of paths of some Gaussian processes, Indiana Univ. Math. J., 20 (1970), pp. 565–578.
[17] G. Gripenberg, On positive, nonincreasing resolvents of Volterra equations, J. Differ. Equ., 30 (1978), pp. 380–390.
[18] G. Gripenberg, S.-O. Londen, and O. Staffans, Volterra integral and functional equations, Cambridge University Press, 1990.
[19] M. A. M. Haubold, H. J. and R. K. Saxena, Mittag-Leffler functions and their applications, J. Appl. Math., vol. 2011 (2011), p. Article ID 298628.
[20] U. Horst and W. Xu, The microstructure of stochastic volatility models with self-exciting jump dynamics, arXiv: 1911.12969, (2019).
[21] ———, Functional limit theorems for marked hawkes point measures, Stoch. Process. Appl., 134 (2021), pp. 94–131.
[22] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland/Kodansha, Amsterdam/Tokyo, 1989.
[23] J. Jacod and A. N. Shiryaev, Limit Theorems for Stochastic Processes, Springer, Berlin, 2003.
[24] T. Jaisson and M. Rosenbaum, Limit theorems for nearly unstable Hawkes processes, Ann. Appl. Probab., 25 (2015), pp. 600–631.
[25] ———, Rough fractional diffusions as scaling limits of nearly unstable heavy tailed Hawkes processes, Ann. Appl. Probab., 26 (2016), pp. 2860–2882.
[26] F. B. Knight, Random walks and a sojourn density process of brownian motion, Trans. Am. Math. Soc., 109 (1963), pp. 56–86.
[27] T. G. Kurtz and P. E. Protter, Weak limit theorems for stochastic integrals and stochastic differential equations, Ann. Probab., 19 (1991), pp. 1035–1070.
[28] W. XU, Weak convergence of stochastic integrals and differential equations. II. Infinite-dimensional case, In Probabilistic Models for Nonlinear Partial Differential Equations, 1627 (1996), pp. 197–285.

[29] A. LAMBERT, The contour of splitting trees is a Lévy process, Ann. Probab., 38 (2010), pp. 348–395.

[30] A. LAMBERT AND F. SIMATOS, Asymptotic behavior of local times of compound Poisson processes with drift in the infinite variance case, J. Theoret. Probab., 28 (2015), pp. 41–91.

[31] A. LAMBERT, F. SIMATOS, AND B. ZWART, Scaling limits via excursion theory: interplay between Crump-Mode-Jagers branching processes and processor-sharing queues, Ann. Appl. Probab., 23 (2013), pp. 2357–2381.

[32] M. LEDOUX AND M. TALAGRAND, Probability in Banach Spaces: isoperimetry and processes, North-Holland/Kodansha, Amsterdam/Tokyo, 1991.

[33] F. MAINARDI, On some properties of the Mittag-Leffler function $e_{\alpha}(-t^\alpha)$, completely monotone for $t > 0$ with $0 < \alpha < 1$, Discrete & Continuous Dynamical Systems - Series B, 19 (2014), pp. 2267–2278.

[34] M. MATHAI, A. M., AND H. J. HAUBOLD, Special Functions for Applied Scientists, Springer, New York., 2008.

[35] D. RAY, Sojourn times of diffusion processes, Illinois J. Math., 7 (1963), pp. 615–630.

[36] C. SABOT AND P. TARRES, Inverting ray-knight identity, Probab. Theory Related Fields, 165 (2016), pp. 559–580.

[37] W. XU, Diffusion approximations for marked self-exciting systems with applications to general branching processes., arXiv:2101.01288, (2021).