UNIFORM SHAPIRO-LOPATINSKI CONDITIONS AND
BOUNDARY VALUE PROBLEMS ON MANIFOLDS WITH
BOUNDED GEOMETRY

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ABSTRACT. We study the regularity of the solutions of second order boundary value problems on manifolds with boundary and bounded geometry. We first show that the regularity property of a given boundary value problem \((P, C)\) is equivalent to the uniform regularity of the natural family \((P_x, C_x)\) of associated boundary value problems in local coordinates. We verify that this property is satisfied for the Dirichlet boundary conditions and strongly elliptic operators via a compactness argument. We then introduce a uniform Shapiro-Lopatinski regularity condition, which is a modification of the classical one, and we prove that it characterizes the boundary value problems that satisfy the usual regularity property. We also show that the natural Robin boundary conditions always satisfy the uniform Shapiro-Lopatinski regularity condition, provided that our operator satisfies the strong Legendre condition. This is achieved by proving that “well-posedness implies regularity” via a modification of the classical “Nirenberg trick”. When combining our regularity results with the Poincaré inequality of (Ammann-Grosse-Nistor, preprint 2015), one obtains the usual well-posedness results for the classical boundary value problems in the usual scale of Sobolev spaces, thus extending these important, well-known theorems from smooth, bounded domains, to manifolds with boundary and bounded geometry. As we show in several examples, these results do not hold true anymore if one drops the bounded geometry assumption. We also introduce a uniform Agmon condition and show that it is equivalent to the coerciveness. Consequently, we prove a well-posedness result for parabolic equations whose elliptic generator satisfies the uniform Agmon condition.

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1. Introduction

We study the regularity of the solutions of general second order boundary value problems on manifolds with boundary and bounded geometry. We obtain several sufficient and several necessary conditions for the regularity of the solution in the usual scale of Sobolev spaces. To state these results in more detail, we need to introduce some notation.

One of the reasons for our interest in manifolds with bounded geometry is due to the fact that they provide a very convenient setting to study evolution equations in a geometric setting [16, 14, 29, 30, 23, 24, 59, 60, 20]. For instance, maximal regularity results on manifolds with bounded geometry were obtained by H. Amann [7, 8] and A. Mazzucato and V. Nistor [44]. Applications to Quantum Field Theory were obtained C. Bär and A. Strohmaier [15], by C. Gérard [28], and by W. Junker and E. Schrohe [38], to mention just a few of the many papers on the subject. In this paper, however, we establish some basic results for elliptic equations, which are useful in the study of evolution equation and for many other problems.

Let $M$ be a manifold with boundary and bounded geometry, see Definition 2.5, and consider a boundary value problem of the form

\[
\begin{align*}
Pu &= f \quad \text{in } M \\
Cu &= h \quad \text{on } \partial M.
\end{align*}
\]

More precisely, we are interested in regularity results, which generally say that if $u$ has a minimal regularity, but $f$ and $h$ have a good regularity, then $u$ also has a good regularity. See Definition 4.11 for a precise definition.

We provide three main methods to obtain a good regularity for $u$ based on:

1. the uniform regularity of the local problems;
2. the uniform Shapiro-Lopatinski conditions; and
3. the well-posedness in energy spaces.

The first two methods provide a characterization of the boundary value problems that satisfy regularity, but are not always easy to use in practice. Most of the
examples of boundary value problems that we know that are satisfying regularity are obtained from a well-posedness result in spaces of minimal regularity (energy spaces). In turn, these well-posedness results are obtained from coercivity.

Let us discuss now each of the three methods and the specific results and applications contained in this paper.

**Uniform regularity of the local problems.** This method is discussed in Section 5. In that section, we associate to each point \( x \) of \( M \) a local operator \( P_x \). This operator is associated to a boundary value problem \((P_x, C_x)\) if \( x \) is on the boundary. These operators are defined on small–but uniform size–coordinate patches around \( x \) (some care is needed close to the boundary). We introduce then uniform regularity conditions for these local operators. This amounts to regularity conditions for each of the local operators in such a way that the resulting constants are independent of \( x \) (see Definition 5.1). We then show that the initial operator satisfies regularity if, and only if, the associated family of local operators satisfies a uniform regularity condition. This is Theorem 5.6. We further show that if the family of local operators is compact and satisfies regularity, then it satisfies a uniform regularity condition. The compactness condition is satisfied, for instance, by strongly elliptic operators endowed with Dirichlet boundary conditions. This then yields right away a regularity result for these operators (Theorem 5.12).

**Uniform Shapiro-Lopatinski conditions.** This method is discussed in Section 6. In analogy with the classical case, we introduce a uniform Shapiro-Lopatinski regularity condition and we show that it characterizes the boundary value problems that satisfy regularity (Theorem 6.9). The usual Shapiro-Lopatinski conditions are not expressed in a quantitative way that will make them amenable to generalize right away to uniform conditions. For this reason, we go back to a more basic formulation of the Shapiro-Lopatinski condition in terms of the left invertibility of suitable model problems \((P_x^{(0)}, C_x^{(0)})\), one for each point of the boundary, and we require the norms of inverses of these problems to be bounded. If the problems \((P_x, C_x)\) are the local versions of the initial problem \((P, C)\), the problems \((P_x^{(0)}, C_x^{(0)})\) are the micro-local versions of \((P, C)\).

**Well-posedness in energy spaces.** This method is discussed in Section 7. While intellectually satisfying (especially in view of the classical results), the uniform Shapiro-Lopatinski is not very easy to use in practice. For this reason, we extend a result of Nirenberg [51] to prove that the problems that are well-posed in energy spaces satisfy regularity (Theorem 7.2). We then combine this method with the other two to check that the Robin boundary conditions \( e_1 \partial^\nu_a + Q \) (for suitable \( Q \), see Remark 4.4) satisfy, first, the classical (i.e. not necessarily uniform) Shapiro-Lopatinski boundary conditions, then that they satisfy the compactness condition, and hence, the uniform regularity for the local problems. This allows us to conclude a regularity result for mixed Dirichlet/Robin boundary conditions (with suitable \( Q \)). Let us explain this result in more detail.

For the rest of this introduction, \((M,g)\) will be a Riemannian manifold with boundary and bounded geometry and \( E \to M \) will be a vector bundle with bounded geometry. We assume that we are given a decomposition \( E|_{\partial_M} = F_0 \oplus F_1 \) as an orthogonal sum of vector bundles with bounded geometry and we let \( e_j \) denote the orthogonal projection \( E \to F_j \). Let \( a \) be a sesquilinear form on \( T^*M \otimes E \) and let
Q be a first order differential operator acting on the sections of \( F_1 \rightarrow \partial M \). To this data we associate the operator \( P_{(a,Q)} \) defined by

\[
(\tilde{P}_{(a,Q)}u,v) = \int_M a(\nabla u, \nabla v)dvol_g + \int_{\partial M} (Qu,v)dvol_{\partial g}.
\]  

We shall be interested in the second order operator \( \tilde{P} := \tilde{P}_{(a,Q)} + Q_1 \), where \( Q_1 \) is a first order differential operator. An operator of this form will be called a \textit{second order differential operator in divergence form}. Let \( \nu \) be the outer normal vector field to the boundary and \( r \) be the distance to the boundary, then \( dr(\nu) = 1 \) and \( \partial^\nu u := a(dr, \nabla u) \) is the associated conormal derivative, see Remarks 4.3, 4.4 and Example 4.8. We let \( P \) be obtained from \( \tilde{P} \) using the restriction to \( H^1_0 \). See Section 4.1 for more details. Our result is a regularity result for the problem

\[
\left\{ \begin{array}{ll}
Pu = f & \text{in } M \\
e_0 u = h_0 & \text{on } \partial M \\
e_1 \partial^\nu u + Qu = h_1 & \text{on } \partial M.
\end{array} \right.
\]  

Recall that a sesquilinear form \( a \) on a hermitian vector bundle \( V \rightarrow X \) is called \textit{strongly coercive} (or \textit{strictly positive}) if there is some \( c > 0 \) such that \( \Re a(\xi, \xi) \geq c|\xi|^2 \) for all \( x \in X \) and \( \xi \in V_x \). If the sesquilinear form \( a \) on \( T^*M \otimes E \) used to define \( P \) is strongly coercive, then \( P \) is said to satisfy the \textit{strong Legendre condition}. See also Definition 6.11.

**Theorem 1.1.** Let \( M \) be a manifold with boundary and bounded geometry, \( E \rightarrow M \) be a vector bundle with bounded geometry, and \( \tilde{P} := \tilde{P}_{(a,Q)} + Q_1 \) be second order differential operator in divergence form acting on \( (M,E) \), as above. Assume that \( \tilde{P} \) has coefficients in \( W^{\ell+1,\infty} \), that \( a \) is strongly coercive, and that \( Q + Q^* \) is a zero order operator. Then there is \( C > 0 \) such that, if \( u \in H^1(M; E) \), \( f := Pu \in H^{\ell-1}(M; E) \), \( h_0 := e_0 u|_{\partial M} \in H^{\ell+1/2}(\partial M; F_0) \), and \( h_1 := e_1 \partial^\nu u + Qu \in H^{\ell-1/2}(\partial M; F_1) \), then \( u \in H^{\ell+1}(M; E) \) and

\[
\|u\|_{H^{\ell+1}} \leq C (\|f\|_{H^{\ell-1}} + \|h_0\|_{H^{\ell+1/2}} + \|h_1\|_{H^{\ell-1/2}} + \|u\|_{H^1}).
\]

In Subsection 7.4, we will see why, in general, the bounded geometry assumption is useful. Many of our results extend to higher order equations, however, including this would have greatly extended the length of the paper. See nevertheless Remarks 4.13 and 7.3.

**Contents of the paper.** The results in this paper are a natural continuation of our joint paper with Bernd Ammann [10]. In that paper, we established some general geometric results on manifolds with boundary and bounded geometry and dealt almost exclusively with Dirichlet boundary conditions. Also, in [10] we restricted ourselves to the case of the Laplace operator. The main contribution of this paper is the fact that we consider general uniformly elliptic operators and we work under general boundary conditions. This requires some new, specific ideas and results. Further results are included in [9].

Here are the contents of the paper. Section 2 contains some basic definitions and some background results. Several of these results are from [10]. For instance, in this section we review the needed facts on manifolds with bounded geometry and we recall the definition of Sobolev spaces using partitions of unity. Section 3 contains some preliminary results on Sobolev spaces and differential operators. In particular,
we provide a description of Sobolev on manifolds with bounded geometry spaces
using vector fields. The first two sections can be skipped at a first reading by
the seasoned researcher. Section 4 introduces our differential operators and boundary
value problems, both of which are given in a variational (i.e. weak) form. We
show that, under some mild conditions, all non-degenerate boundary conditions are
equivalent to variational ones, which justifies us to concentrate on the later. Also in
this section we formulate the regularity condition. The last three sections describe
the three methods explained above and their applications. In the last section,
in addition to discussing the third method mentioned above (“Well-posedness in
energy spaces”), we also introduce a uniform Agmon condition, which, we show, is
equivalent to the coercivity of the operator. This then leads to an application to the
well-posedness of some parabolic equations on manifolds with bounded geometry.

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2. Background material and notation
We begin with some background material, for the benefit of the reader. More
precisely, we introduce here the needed function spaces and we recall the needed
results from [10] as well as from other sources. We also use this opportunity to fix
the notation, which is standard, so this section can be skipped at a first reading.
This section contains no new results. Throughout this paper, $M$ will be a (usually
non-compact) connected smooth manifold, possibly with boundary.

2.1. General notations and definitions. We begin with the most standard con-
cepts and some notation.

2.1.1. Continuous operators. Let $X$ and $Y$ be Banach spaces and $A: X \to Y$ be
a linear map. Recall that $A$ is continuous (or bounded) if, and only if,
$\|A\|_{X,Y} := \inf_{x \neq 0} \frac{\|Ax\|}{\|x\|} < \infty$. When the spaces on which
$A$ acts are clear, we shall drop them from the notation of the norm, thus write
$\|A\| = \|A\|_{X,Y}$. We say that $A$ is an isomorphism if it is a continuous bijection (in which case the inverse will also be
continuous, by the open mapping theorem).

2.1.2. The conjugate dual spaces. For complex vector spaces $V$ and $W$, a sesquilinear
map $V \times W \to \mathbb{C}$ will always be (complex) linear in $V$ and anti-linear in $W$.
A Hermitian form on $V$ is a positive definite sesquilinear map $(\cdot, \cdot): V \times V \to \mathbb{C}$,
which implies $(w, v) = (v, w)$. In order to deal with the complex version of the
Lax–Milgram lemma, we introduce the following notation and conventions.

Notations 2.1. Let $V$ be a complex vector space, usually a Hilbert space. We
shall denote by $\overline{V}$ the complex conjugate vector space to $V$. If $V$ is endowed with
a topology, we denote by $V'$ the (topological) dual of $V$. It will be convenient
to denote $V^* := (V')' \simeq \overline{V}'$. We can thus regard a continuous sesquilinear form
$B: V \times W \to \mathbb{C}$ as a continuous bilinear form on $V \times \overline{W}$, or, moreover, as a map $V \to W^*$. Let us assume that $V$ is a Hilbert space with inner product $(\cdot, \cdot): V \times V \to \mathbb{C}$. If
$T: V \to W$ is a continuous map of Hilbert spaces, then we denote by $T^*: V^* \to W^*$
the adjoint of $T$, as usual.
2.1.3. Vector bundles. Let $E \to M$ be a smooth real or complex vector bundle endowed with metric $(\cdot, \cdot)_E$. We denote by $\Gamma(M; E)$ the set of smooth sections of $E$. Let $E$ be endowed with a connection

$$\nabla^E : \Gamma(M; E) \to \Gamma(M; E \otimes T^*M).$$

We assume that $\nabla^E$ is metric preserving, which means that

$$X(\xi, \eta)_E = (\nabla_X \xi, \eta)_E + (\xi, \nabla_X \eta)_E.$$

If $F \to M$ is another vector bundle with connection $\nabla^F$, then we endow $E \otimes F$ with the induced product connection $\nabla^{E \otimes F}(\xi \otimes \eta) := (\nabla^E \xi) \otimes \eta + \xi \otimes \nabla^F \eta$.

We endow the tangent bundle $TM \to M$ with the Levi-Civita connection $\nabla^M$, which is the unique torsion free, metric preserving connection on $TM$. We endow $T^*M$ and all the tensor product bundles $E \otimes T^{*k}M := E \otimes (T^*M)^{\otimes k}$ with the induced tensor product connections.

2.2. Manifolds with boundary and bounded geometry. We now recall some basic material on manifolds with boundary and bounded geometry, mostly from [10], to which we refer for more details and references.

Definition 2.2. A vector bundle $E \to M$ with given connection is said to have totally bounded curvature if its curvature and all its covariant derivatives are bounded (that is, $\|\nabla^k R^E\|_\infty < \infty$ for all $k$). If $TM$ has totally bounded curvature, we shall then say that $M$ has totally bounded curvature.

Let $\exp_p^M : T_p M \to M$ be the exponential map at $p$ associated to the metric and

$$r_{\text{inj}}(p) := \sup \{ r \mid \exp_p^M : B^M_r(0) \to M \text{ is a diffeomorphism onto its image} \}$$

$$r_{\text{inj}}(M) := \inf_{p \in M} r_{\text{inj}}(p).$$

The following concept is classical and fundamental.

Definition 2.3. A Riemannian manifold without boundary $(M, g)$ is said to be of bounded geometry if $r_{\text{inj}}(M) > 0$ and if $M$ has totally bounded curvature.

If $M$ has boundary, clearly $r_{\text{inj}}(M) = 0$, so a manifold with non-empty boundary will never have bounded geometry in the sense of the above definition. However, Schick has found a way around this difficulty [58]. Let us recall a definition equivalent to his following [10]. The main point is to describe the boundary as a suitable submanifold of a manifold without boundary and with bounded geometry. Let us consider then a hypersurface $N \subset M$, i.e. a submanifold with dim $H = \dim M - 1$. Assume that $H$ carries a globally defined unit normal vector field $\nu$ and let $\exp^M : T^*_H M(0) \to M$ be the exponential in the direction of the chosen unit normal vector. By $H^N$ we denote the second fundamental form of $N$ (in $M$: $H^N(X, Y)\nu := \nabla_X Y - \nabla^N_X Y$).

Definition 2.4. Let $(M^m, g)$ be a Riemannian manifold of bounded geometry with a hypersurface $H = H^{m-1} \subset M$ and a unit normal field $\nu$ on $H$. We say that $H$ is a bounded geometry hypersurface in $M$ if the following conditions are fulfilled:

(i) $H$ is a closed subset of $M$;
(ii) $\| (\nabla^H)^k \|_{L^\infty} < \infty$ for all $k \geq 0$;
(iii) $\exp^H : H \times (-\delta, \delta) \to M$ is a diffeomorphism onto its image for some $\delta > 0$.

We shall denote by $r_H$ the largest value of $\delta$ satisfying this definition.
We have shown in [10] that \((H, g|_H)\) is then a manifold of bounded geometry in its own right. See also [23, 24] for a larger class of submanifolds of manifolds with bounded geometry.

**Definition 2.5.** A Riemannian manifold \(M\) with (smooth) boundary has bounded geometry if there is a Riemannian manifold \(\hat{M}\) with bounded geometry satisfying

(i) \(M\) is contained in \(\hat{M}\);

(ii) \(\partial M\) is a bounded geometry hypersurface in \(\hat{M}\).

**Example 2.6.** Lie manifolds have bounded geometry [11, 12]. It follows that, Lie manifolds with boundary are manifolds with boundary and bounded geometry.

For \(x, y \in M\), we let \(\text{dist}(x, y)\) denote the distance between \(x\) and \(y\) with respect to the metric \(g\).

**2.3. Coverings, partitions of unity, and Sobolev spaces.** We now introduce some basic notation and constructions for manifolds with boundary and bounded geometry. The boundary \(\partial M\) of a manifold with boundary \(M\) will always be a subset of \(M\): i.e. \(\partial M \subset M\). For the rest of the paper, we will always assume that \(E \rightarrow M\) has totally bounded curvature, and, for the rest of this section, we shall assume that \(M\) is a manifold with boundary and bounded geometry.

Let \(\nu\) be the inner unit normal vector field of \(\partial M\) and let \(r_\partial\) as in the bounded geometry condition for \(\partial M\) in \(M\), Definition 2.4. Moreover, for a metric space \(X\), we shall denote by \(B_N^X(p)\) the open ball of radius \(r\) centered at \(p\) and set \(B_r^X(p) := B_r^X(p)\). We shall identify \(T_pM\) with \(\mathbb{R}^{m+1}\) and, respectively, \(T_p\partial M\) with \(\mathbb{R}^m\), using an orthonormal basis, thus obtaining a diffeomorphism \(\exp^M_p : B_{r+1}^M(0) \rightarrow B_r^M(p)\).

For \(r < \frac{1}{2}\min\{r_{inj}(\partial M), r_{inj}(M), r_\partial\}\) we define the maps

\[
\begin{aligned}
\kappa_p : B_{2r}^M(0) \times [0, 2r) &\rightarrow M, \quad \kappa_p(x, t) := \exp^M_\partial (tv_q), \quad \text{if } p \in \partial M, \quad q := \exp^M_\partial (x) \\
\kappa_p : B_{r+1}^M(0) &\rightarrow M, \quad \kappa_p(v) := \exp^M_p (v), \quad \text{if } \text{dist}(p, \partial M) \geq r,
\end{aligned}
\]

with range

\[
U_p(r) := \begin{cases} 
\kappa_p(B_{2r}^M(0) \times [0, 2r)) \subset M & \text{if } p \in \partial M \\
\kappa_p(B_{r+1}^M(0)) = \exp^M_p (B_{r+1}^M(0)) \subset M & \text{otherwise.}
\end{cases}
\]

**Definition 2.7.** Let \(r_{FC} := \min\left\{ \frac{1}{8} r_{inj}(\partial M), \frac{1}{4} r_{inj}(M), \frac{1}{8} r_\partial \right\}\) and \(0 < r \leq r_{FC}\). Then \(\kappa_p : B_r^M(0) \times [0, r) \rightarrow U_p(r)\) is called a Fermi coordinate chart at \(p \in \partial M\). The charts \(\kappa_p\) for \(\text{dist}(p, \partial M) \geq r\) are called geodesic normal coordinates.

To define our Sobolev spaces, we need suitable coverings of our manifold. For the sets in the covering that are away from the boundary, we will use geodesic normal coordinates, whereas for the sets that intersect the boundary, we will use the Fermi coordinates introduced in Definition 2.7.

**Definition 2.8.** Let \(M\) be a manifold with boundary and bounded geometry and let \(0 < r \leq r_{FC} := \min\left\{ \frac{1}{8} r_{inj}(\partial M), \frac{1}{4} r_{inj}(M), \frac{1}{8} r_\partial \right\}\), as in Definition 2.7. A subset \(\{p_\gamma\}_{\gamma \in \mathbb{N}}\) is called an \(r\)-covering subset of \(M\) if the following conditions are satisfied:

(i) For each \(R > 0\), there exists \(N_R \in \mathbb{N}\) such that, for each \(p \in M\), the set \(\{\gamma \in \mathbb{N} | \text{dist}(p_\gamma, p) < R\}\) has at most \(N_R\) elements.
(ii) For each \( \gamma \in \mathbb{N} \), we have either \( p_\gamma \in \partial M \) or \( \text{dist}(p_\gamma, \partial M) \geq r \), so that \( U_\gamma := U_{p_\gamma}(r) \) is defined, compare to (4).

(iii) \( M \subset \bigcup_{\gamma=1}^\infty U_\gamma \).

**Remark 2.9.** If \( 0 < r < r_{FC} \), then we can always find an \( r \)-covering subset of \( M \), since \( M \) is a manifold with boundary and bounded geometry [35, Remark 4.6]. Moreover, it then follows from (i) of Definition 2.8 that the coverings \( \{ U_\gamma \} \) of \( M \) and \( \{ U_\gamma \cap \partial M \} \) of \( \partial M \) are uniformly locally finite.

We shall need the following class of partitions of unity defined using \( r \)-covering sets. Recall the definition of the sets \( U_\gamma := U_{p_\gamma}(r) \) from Equation (4).

**Definition 2.10.** A partition of unity \( \{ \phi_\gamma \}_{\gamma \in \mathbb{N}} \) of \( M \) is called an \( r \)-uniform partition of unity associated to the \( r \)-covering set \( \{ p_\gamma \} \subset M \), see Definition 2.8, if the support of each \( \phi_\gamma \) is contained in \( U_\gamma \) and \( \sup_\gamma \| \phi_\gamma \| W^{\ell,\infty}(M) < \infty \) for each fixed \( \ell \in \mathbb{Z}_+ \).

In order to deal with boundary value problems with values in a vector bundle (that is, with boundary value problems for systems), we will also need the concept of synchronous trivializations, which we briefly recall here:

**Definition 2.11.** Let \( M \) be a Riemannian manifold with boundary and bounded geometry, and let \( E \to M \) be a Hermitian vector bundle with metric connection. Let \( (U_\gamma, \kappa_\gamma, \phi_\gamma) \) be Fermi and geodesic normal coordinates on \( M \) together with an associated \( r \)-uniform partition of unity as in the definitions above. If \( p_\gamma \in M \setminus U_\gamma(\partial M) \), then \( E|_{U_\gamma} \) is trivialized by parallel transport along radial geodesics emanating from \( p_\gamma \). If \( p_\gamma \in \partial M \), then we trivialize \( E|_{U_\gamma} \) as follows: First we trivialize \( E|_{U_\gamma \cap \partial M} \) along the underlying geodesic normal coordinates on \( \partial M \). Then, we trivialize by parallel transport along geodesics emanating from \( \partial M \) and being normal to \( \partial M \). The resulting trivializations are called synchronous trivializations along Fermi coordinates and are maps

\[
\xi_\gamma : \kappa_\gamma^{-1}(U_\gamma) \times \mathbb{C}^t \to E|_{U_\gamma}
\]

where \( t \) is the rank of \( E \).

We shall need a definition of Sobolev spaces using partitions of unity and “Fermi coordinates” [35] and a few standard results. In the scalar case, these results were stated in [10]. Here we stress the vector valued case. First, we have the following proposition that is a direct consequence of Theorems 14 and 26 in [35].

**Proposition 2.12.** Let \( M \) be a Riemannian manifold with boundary and bounded geometry. Let \( \{ \phi_\gamma \} \) be an \( r \)-uniform partition of unity associated to an \( r \)-covering set \( \{ p_\gamma \} \subset M \) and let \( \kappa_\gamma = \kappa_{p_\gamma} \) be as in Definition 2.7. Let \( E \to M \) be a vector bundle with totally bounded curvature with trivializations \( \xi_\gamma \) as in Definition 2.11. Then

\[
|||u|||^p := \sum_\gamma \| \xi_\gamma^*(\phi_\gamma u) \|_{W^{\ell,p}}^p
\]

defines a norm equivalent to the standard norm on \( W^{s,p}(M;E) \), \( s \in \mathbb{R} \), \( 1 < p < \infty \).

As in the scalar case [10], the space \( \Gamma_c(M;E) \) of smooth, compactly supported sections of \( E \) is dense in \( W^{s,p}(M;E) \), for \( s \in \mathbb{R} \) and \( 1 < p < \infty \). This is obtained by truncating the sum. As usual, we shall let \( H^s(M;E) := W^{\ell,2}(M;E) \), \( s \in \mathbb{R} \). Similarly, we shall need the following extension of the trace theorem to the case of...
manifolds with boundary and bounded geometry, see Theorem 27 in [35] (see [10] for more references).

**Theorem 2.13** (Trace theorem). Let $M$ be a manifold with boundary and bounded geometry and let $E \to M$ have totally bounded curvature. Then, for every $s > 1/2$, the restriction to the Dirichlet part of the boundary $\text{res} : C^\infty_c(M) \to C^\infty_c(\partial_D M)$ extends to a continuous, surjective map

$$\text{res} : H^s(M; E) \to H^{s-\frac{1}{2}}(\partial_D M; E).$$

Let $\partial_n$ be the normal derivative at the boundary. We then denote by $H^s_0(M; E)$ the kernel of the restrictions maps $\text{res} \circ \partial_n^i$, $0 \leq j \leq m - 1$. It is known that $H^{-m}(M; E^*)$ identifies with $H^s_0(M; E^*)$.

See also [6, 11, 39, 40, 60, 63, 64] for related results, in particular, for the use of the partitions of unity. See [25] and its outgrowth [26] for an introduction to manifolds of bounded geometry. See [19, 37, 41, 61] for the general results on Sobolev spaces not proved above.

### 3. Preliminary results

We now include some preliminary results. This section consists mostly new results, but advanced reader can nevertheless skip this section at a first reading. *In this section we assume that $M$ is a manifold with boundary and bounded geometry and that $E \to M$ is a vector bundle with totally bounded curvature (and, hence, with bounded geometry).*

#### 3.1. Alternative characterizations of Sobolev spaces

We now provide an alternative description of Sobolev spaces using vector fields. If $A : V_1 \to V_2$ is a linear map of normed spaces, we define its *minimum reduced modulus* $\gamma(A)$ by

$$\gamma(A) := \inf_{\xi \in V_1/\ker(A)} \sup_{\eta \in \ker(A)} \frac{\|A\xi\|}{\|\xi + \eta\|},$$

i.e. $\gamma(A)$ is the largest number $\gamma$ satisfying $\|A\xi\| \geq \gamma \|\xi + \ker(A)\|_{V_1/\ker(A)}$ for all $\xi \in V_1$ where $\|\cdot\|_{V_1/\ker(A)}$ is the quotient norm on $V_1/\ker(A)$. As it is well known, if $V_1$ is complete, a standard application of the Open Mapping Theorem gives that the image of $A$ is closed if, and only if, $\gamma(A) > 0$. If $V_1$ and $V_2$ are Hilbert spaces and $A$ is surjective, we have $\gamma(A)^{-2} = \|(AA^*)^{-1}\|$. We have the following alternative characterization of Sobolev spaces in terms of vector fields. This alternative definition is more intuitive and easier to use in analysis. It is based on a the choice of a suitable finite family of vector fields, whose existence is assured by the following lemma.

**Lemma 3.1.** Let $(M, g)$ be a manifold with boundary and of bounded geometry. Then there exist vector fields $X_1, X_2, \ldots, X_N \in W^{\infty,\infty}(M; TM)$ such that for any $x \in M$ the map $\Phi_x : \mathbb{R}^N \ni (\lambda_1, \ldots, \lambda_N) \mapsto \sum \lambda_i X_i(x) \in T_x M$ is onto at any $x$ and $\inf_x \gamma(\Phi_x) > 0$. Then we have that $[X_i, X_j] = \sum_{k=1}^N C_{ij}^k X_k$ and $\nabla X_i X_j = \sum_{k=1}^N G_{ij}^k X_k$ for some functions $G_{ij}^k, C_{ij}^k \in W^{\infty,\infty}$ and any $1 \leq i, j \leq N$. Moreover, we can choose $X_1$ of length one and normal to the boundary of $M$ and $X_j$ tangent to the boundary for $j > 1$.

*Proof.* We use a covering by Fermi coordinates resp. geodesic normal coordinates $(U_\beta, \kappa_\beta)$ as in Definition 2.8 together with an associated partition of unity $\phi_\beta$ as
in Definition 2.10. We recall that all transition function $\kappa_\alpha \circ \kappa_\beta^{-1}$, the $\phi_\beta$ and all their derivatives are uniformly bounded, see [35, 60]. Let $x_\beta^i$ be the coordinates corresponding to the chart $\kappa_\beta$. Note that the Christoffel symbols in each chart and their derivatives are also uniformly bounded due to the bounded geometry, [35, Lemma 3.10].

Let those coordinates be ordered such that $x_\beta^1 = r$, the distance to the boundary, for $U_\beta \cap \partial M \neq \emptyset$. Let $X_1 := \sum_{U_\beta \cap \partial M \neq \emptyset} \phi_\beta \partial_r$. Then, $X_1$ is normal to $\partial M$.

To define $X_j$ for $j > 1$ we divide $\{U_\beta\}_\beta$ into finitely many disjoint subsets $V_i$, $i = 1, \ldots, d$, such that the $U_\beta$’s in any $V_i$ are pairwise disjoint. This is always possible since the covering $\{U_\beta\}_\beta$ is uniformly locally finite. We set

$$X_{i,j} := \sum_{U_\beta \in V_i} \phi_\beta \partial_{x_\beta^j}$$

for $j \in \{2, \ldots, m\}$

$$X_{i,1} := \sum_{U_\beta \in V_i, U_\beta \cap \partial M = \emptyset} \phi_\beta \partial_{x_\beta^1}$$

Then, $X_1, X_{i,j} \in W^{\infty, \infty}(M, TM)$. Moreover, all $X_{i,j}$ are tangent to the boundary by construction. By renaming the vector fields we obtain $X_1, \ldots, X_N$. Then $\Phi_x$ is onto by construction for all $x \in M$.

Next we show that the reduced minimum modulus is uniformly bounded from below: Let $x \in M$. Since the covering is uniformly locally finite, there is a chart $\kappa_\alpha$ around $x$ with $\phi_\alpha(x) > c > 0$, $c$ independent of $x$. Since $\partial_{x_\alpha}^\beta$, span $T_x M$ for all $(\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N$ there is a $(\mu_1, \ldots, \mu_N) \in \mathbb{R}^N$ with $\mu_j = 0$ if $X_j$ was not built from $\kappa_\beta$. Thus,

$$\gamma(\Phi_x) \geq \inf_{(\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N} \frac{\sum_i \mu_i X_i(x)}{\sum_i |\mu_i|^2} \geq \inf_{(\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N} \frac{c \sum_i \mu_i^2}{\sum_i \mu_i^2} = c.$$ 

We have $\partial_{x_\beta^j} = \sum_i a_{ij}^\alpha \partial_{x_\alpha}$ with $a_{ij}^\alpha \in W^{\infty, \infty}$ with bounds independent of $\alpha$ and $\beta$ since this is true for the transition functions $\kappa_\alpha \circ \kappa_\beta^{-1}$. Thus, together with the uniform bounds on the corresponding Christoffel symbols we get $\nabla \partial_{x_\alpha} \partial_{x_\beta^j} = \sum_k b_{ij}^k \partial_{x_\alpha} \partial_{x_\beta^k}$ with $b_{ij}^k \in W^{\infty, \infty}$. Altogether, we have

$$\nabla X_i X_j = \sum_{U_\beta \in V_i, U_\alpha \in V_j} \phi_\beta \nabla \partial_{x_\beta^j} (\phi_\alpha \partial_{x_\alpha}) = \sum_{U_\beta \in V_i, U_\alpha \in V_j, k} \phi_\beta (b_{ij}^k \partial_{x_\beta^j} \phi_\alpha + \phi_\alpha b_{ij}^k) \partial_{x_\beta^k}$$

$$= \sum_k G_{ij}^k X_k \quad \text{with} \quad G_{ij}^k = \sum_{U_\beta \in V_i, U_\alpha \in V_j} (\delta_{ij} \partial_{x_\beta^j} \phi_\alpha + \phi_\alpha b_{ij}^k) \in W^{\infty, \infty}$$

where we used in the last step that the supports of any two $U_\beta$ in $V_i$ are disjoint.

By, $[X_i, X_j] = \nabla X_i X_j - \nabla X_j X_i$ the remaining claim follows. \qed

**Proposition 3.2.** Let $X_1, X_2, \ldots, X_N \in W^{\infty, \infty}(M; TM)$ be such that for any $x \in M$ the map $\Phi_x : \mathbb{R}^N \ni (\lambda_1, \ldots, \lambda_N) \mapsto \sum \lambda_i X_i(x) \in T_x M$ is onto at any $x$ and $\inf_x \gamma(\Phi_x) > 0$. Then

$$W^{\ell, p}(M; E) = \{ u \mid \nabla^E X_{k_1} \nabla^E X_{k_2} \cdots \nabla^E X_{k_j} u \in L^p(M; E), j \leq \ell, k_1, \ldots, k_j \leq N \},$$

$$1 \leq p \leq \infty.$$ If $E$ has furthermore totally bounded curvature, we have also

$$W^{\ell, p}(M; E) = \{ u \mid \nabla^E X_{k_1} \nabla^E X_{k_2} \cdots \nabla^E X_{k_j} u \in L^p(M; E) \},$$

where, this time, $1 \leq k_1 \leq k_2 \leq \ldots \leq k_j \leq N$, $j \leq \ell$. 

We notice that the condition that $\Phi_x: \mathbb{R}^N \to T_xM$ be surjective implies that $\dim \ker \Phi_x = N - \dim M$ is constant. For compact $M$ then $\inf_{x \in M} \gamma(\Phi_x) > 0$ is automatically satisfied.

**Proof.** Let
\[
\mathcal{X} := \{ u \mid \nabla^E_{X_{k_1}} \nabla^E_{X_{k_2}} \ldots \nabla^E_{X_{k_j}} u \in L^p(M; E), \ j \leq \ell, \ k_1, \ldots, k_j \leq N \}.
\]
The fact that the vector fields $X_k$ are bounded with bounded covariant derivatives gives that $\nabla_{X_k}: W^{j+1,p}(M; E) \to W^{j,p}(M; E)$ is bounded for any $j \geq 0$ and any $p$. This gives $W^{\ell,p}(M; E) \subset \mathcal{X}$.

To prove the converse, we proceed by induction as follows. First of all, the assumption that $\inf_{x} \gamma(\Phi_x) > 0$ together with the fact that $\Phi_x$ is surjective for all $x$ gives that $\Phi^*_x(\Phi_x \Phi^*_x)^{-1}$ is a (pointwise bounded) right inverse to $\Phi_x$, regarded as a map $(\mathbb{R}^N)^{\otimes \ell} \to TM^{\otimes \ell}$. (Here $\mathbb{R}^N$ is the trivial vector bundle of rank $N$.)

Let us denote by $\Psi$ the extension of this map to sections of the corresponding vector bundles, and, by abuse of notation, also by
\[
\Psi: L^p(M; T^*M^{\otimes \ell} \otimes E) \to L^p(M; (\mathbb{R}^N)^{\otimes \ell} \otimes E)
\]
the induced maps. Then $\Psi$ is continuous and a right inverse of (the map defined by) $\Phi$. Therefore $\Psi$ is a homeomorphism onto its image. That gives that $\Psi(\xi) \in L^p(M; (\mathbb{R}^N)^{\otimes \ell} \otimes E)$ if, and only if, $\xi \in L^p(M; T^*M^{\otimes \ell} \otimes E)$. By taking the $(k_1, k_2, \ldots, k_\ell)$ component of $\Psi(\xi)$, we obtain that $(\nabla^E)^{\ell} u \in L^p(M; T^*M^{\otimes \ell} \otimes E)$ if, and only if, $\langle (\nabla^E)^{\ell} u, X_{k_1} \otimes X_{k_2} \otimes \ldots \otimes X_{k_\ell} \rangle \in L^p(M; E)$ for all $k_1, k_2, \ldots, k_\ell \in \{1, 2, \ldots, N\}$. To use induction, we notice that $\langle (\nabla^E)^{\ell} u, X_{k_1} \otimes X_{k_2} \otimes \ldots \otimes X_{k_\ell} \rangle = \nabla^E_{X_{k_1}} \nabla^E_{X_{k_2}} \ldots \nabla^E_{X_{k_\ell}} u$ is given by lower order terms of the same kind.

For the last part, we also notice that we can commute $\nabla^E_{X_{k_1}}$ with $\nabla^E_{X_{k_2}}$ up to totally bounded terms using Lemma 3.1. \hfill \Box

### 3.2. Sobolev spaces without using connections

Recall that we assume that $E \to M$ has totally bounded curvature. One way to think of such vector bundles is given by the following lemma:

**Lemma 3.3.** Let us assume that $M$ is a manifold with bounded geometry (possibly with boundary) and that $E \to M$ is a Hermitian vector bundle of totally bounded curvature. Then there exists a fibrewise isometric embedding $E \subset M \times C^N$ into the trivial $N$-dimensional vector bundle with the standard metric such that, if $e$ denotes the orthogonal projection onto $E$, then $e \in M_N(W^{\infty,\infty}(M))$ and the connection of $E$ is equivalent to the Grassmann (projection) connection of the embedding (i.e. the difference of both connections is in $W^{\infty,\infty}$). Conversely, if $e \in M_N(W^{\infty,\infty}(M))$ and $E := e(M \times C^N)$, then $E$ with the Grassmann connection has totally bounded curvature.

**Proof.** Let us consider for each open subset $U_\gamma$ as above the synchronous trivialization $\xi_\gamma: E|_{U_\gamma} \to \kappa^{-1}_\gamma(U_\gamma) \times C^t$ from Equation (5), with $C^t$ the typical fiber above $p_\gamma$. Then $\phi_\gamma^{1/2} \xi_\gamma$ extends to a vector bundle map $E \to M \times C^t$ that is in $W^{\infty,\infty}$ since $M$ has bounded geometry and the connection on $E$ has totally bounded curvature. Let $N = N_5$, with $N_5$ as in Definition 2.8. By the construction of the sets $U_\gamma$, we can divide the set of all $\gamma$’s into $N + 1$ disjoint subsets $\Gamma_k$, such that, for each fixed $k$ and any $\gamma, \gamma' \in \Gamma_k$, the sets $U_\gamma$ and $U_{\gamma'}$ are disjoint, by the construction of the sets $U_\gamma$. Let $\Psi_k := \sum_{\gamma \in \Gamma_k} \phi_\gamma^{1/2} \xi_\gamma$ and $\Psi := (\Psi_1, \Psi_2, \ldots, \Psi_{N+1}): E \to M \times C^t(N+1)$.
be the resulting bundle morphism. Then $\Psi$ is isometric, it is in $W^{\infty,\infty}$, and hence $e := \Psi^*\Psi$ is the desired projection. The equivalence of the connections follows from the uniform boundedness of the Christoffel symbols and their derivatives associated to the synchronous trivialization [35, Remark 5.3].

This lemma may be used to reduce differential operators acting on vector bundles to matrices of scalar differential operators. It also gives the following characterization of Sobolev spaces of sections of a vector bundle.

**Proposition 3.4.** We use the notation of the last lemma. Then

$$W^{m,p}(M; E) = eW^{m,p}(M)^N, \quad 1 \leq p \leq \infty.$$ 

We can now use this proposition to derive a description of Sobolev spaces on manifolds with bounded geometry that is completely independent of the use of connections.

**Remark 3.5.** The standard definition of the norm on Sobolev spaces is using powers of $\nabla$ [10, 35]. For instance $W^{k,\infty}(M) := \{u \mid \nabla^j u \in L^\infty(M), 0 \leq j \leq k\}$ (alternatively, it is the space of functions with uniformly bounded derivatives of order $\leq k$ in any normal geodesic coordinate chart on $B^r_n$, for any fixed $r$ less than the injectivity radius $r_{\text{inj}}(M)$ of $M$). We can define $W^{k,\infty}(M; TM)$ similarly. Let then $X_1, X_2, \ldots, X_N$ as in Proposition 3.2 and Lemma 3.1, $1 \leq p < \infty$. Then

$$W^{\ell,p}(M) = \{u \mid X_{k_1}X_{k_2} \ldots X_{k_\ell}u \in L^p(M), \ j \leq \ell, \ 1 \leq k_1 \leq \ldots \leq k_\ell \leq N\}.$$ 

Together with Proposition 3.4, this gives a description of Sobolev spaces without using connections.

### 3.3. Differential operators and partitions of unity.

A differential operator on $E$ is an expression of the form $Pu := \sum_{j=0}^k a_j \nabla^j u$, with $a_j$ a section of $\text{End}(E) \otimes TM^\otimes j$. It can thus simply be regarded as a formal collection of coefficients. In particular, we do not identify the differential operator with the maps that it induces (since it induces many). A differential operator $Pu = \sum_{j=0}^k a_j \nabla^j u$ will be said to have coefficients in $W^{\ell,\infty}$ if $a_j \in W^{\ell,\infty}(M; \text{End}(E) \otimes TM^\otimes j)$. If $\ell = 0$, we shall say that $P$ has bounded coefficients. If $\ell = \infty$, we shall say that $P$ has totally bounded coefficients. The continuity of the contraction map

$$W^{\ell,\infty}(M; \text{End}(E) \otimes TM^\otimes j) \otimes W^{\ell,p}(M; T^*M^\otimes j \otimes E) \rightarrow W^{\ell,p}(M; E),$$

gives that a differential operator $P = \sum_{j=0}^k a_j \nabla^j$ with coefficients in $W^{\ell,\infty}$ defines a continuous map

$$P = \sum_{j=0}^k a_j \nabla^j : W^{\ell+k,p}(M; E) \rightarrow W^{\ell,p}(M; E), \quad \ell \geq 0.$$ 

**Lemma 3.6.** Let $k \geq 0$ and let $P$ be an order $\ell$ differential operator with coefficients $a_j \in W^{k+1,\infty}(M; \text{End}(E) \otimes TM^\otimes j)$. Let $\phi \in W^{k+\ell+1,\infty}(M)$. Then the commutator $[P, \phi]$ defines a continuous linear map $H^{k+\ell}(M; E) \rightarrow H^{k+1}(M; E)$. Moreover, if $\{\phi_\gamma\}_\gamma$ is a bounded family in $W^{k+\ell+1,\infty}(M)$, then operator norms of $[P, \phi_\gamma] : H^{k+\ell}(M; E) \rightarrow H^{k+1}(M; E)$ are bounded uniformly in $\gamma$. 

Proof. We have $[P, \phi]u = \sum_{j=1}^{k+1} a_j \sum_{s=0}^{j-1} \nabla^{j-s} \phi \nabla^s u$. Thus,

$$\| [P, \phi]u \|_{H^{k+1}} \leq C \sum_{r=0}^{k+1} \sum_{j=0}^{\ell} \| \nabla^r (a_j \sum_{s=0}^{j-1} \nabla^{j-s} \phi \nabla^s u) \|_{L^2}$$

and the claim follows by the regularity assumptions on $a_j$ and $\phi$. □

4. Variational boundary conditions and regularity

We now introduce “differential operators in divergence form” from a global point of view. The natural boundary value problem associated to differential operators in divergence will be called \textit{variational boundary value problems}. In this subsection, we introduce and take a first look at these variational boundary value problems. We will see that, under some mild assumptions on our differential operator $P$, any non-degenerate boundary value problem is equivalent to a variational one. This allows to reduce the study of the former to that of the latter, for which several general regularity results will be obtained in the following sections. On the other hand, the degenerate boundary value problems are known to behave in a significantly different way than the non-degenerate ones (see, for instance, [49] and the references therein).

The possibility of reducing non-degenerate boundary value problems to variational ones seems not to have been explored too much in the literature.

We will continue to assume that $M$ is a smooth manifold with smooth boundary. Moreover, we will assume that $TM$ has totally bounded curvature, but \textit{we will not assume that $M$ has bounded geometry}, since we want to allow $M$ to be a domain in a Euclidean space. Recall that all differential operators are assumed to have bounded coefficients and $E \to M$ has totally bounded curvature.

4.1. Sesquilinear forms and operators in divergence form. It is important in applications to consider operators “in divergence form,” which we will define below shortly. They provide a slightly different class of differential operators than the operators with coefficients in $L^\infty$ considered above and will be useful in order to treat the Dirichlet and Robin problems on the same footing. To introduce second order differential operators in divergence form, we shall need the following data and assumptions:

\begin{assumption}
Let $M$ be a smooth manifold with smooth boundary, $E \to M$ a vector bundle, $a$ a sesquilinear form on $T^*M \otimes E$, and first order differential operators $Q$ and $Q_1$ satisfying

(A1) $TM$ and $E$ have totally bounded curvature.

(A2) $E|_{\partial M} = F_0 \oplus F_1$, with $F_0$ and $F_1$ with totally bounded curvature and $F_0$ is the orthogonal complement of $F_1$.

(A3) $a = (a_x)_{x \in M}$ is a measurable, bounded family of hermitian sesquilinear forms

$$a = (a_x)_{x \in M}, \quad a_x : T_x^* M \otimes E_x \times T_x^* M \otimes E_x \to \mathbb{C}.$$

(A5) $Q$ has $W^{1, \infty}$ coefficients and acts on $(\partial M, F_1)$.

(A4) $Q_1$ has $L^\infty$ coefficients and acts on $(M, E)$.

(A6) $V$ is a closed subspace $H^1_0(M; E) \subset V \subset H^1(M; E)$.

\end{assumption}
The family \((a_x)\) defines a section \(a\) of the bundle \(((T^*M \otimes E) \otimes (T^*M \otimes \overline{E}))'\). In general, we say that a section \(a = (a_x)_{x \in M}\) is a bounded sesquilinear form on \(T^*M \otimes E\) if it is an \(L^\infty\)-section of \(((T^*M \otimes E) \otimes (T^*M \otimes \overline{E}))'\).

4.1.1. The Dirichlet (sesquilinear) form. Using our assumptions 4.1, we first define

\[
B_a(u,v) := \int_M a(\nabla u, \nabla v) \, dvol_g,
\]

which is the Dirichlet form associated to \(a = (a_x)_{x \in M}\). Then the Dirichlet form \(B: V \times V \to \mathbb{C}\) associated to \(a\), \(Q\), and \(Q_1\) is

\[
B(u,v) := B_a(u,v) + (Q_1 u,v)_{L^2(M,E)} + (Qu|_{\partial M},v|_{\partial M})_{L^2(\partial M;F_1)},
\]

where, initially, \(u, v \in V \cap H^\infty(M;E)\), and then we then extend \(B\) to a sesquilinear linear form \(B: V \times V \to \mathbb{C}\) by continuity. In the future, we shall usually write \((Qu,v)_{L^2(\partial M)}\) instead of \((Qu|_{\partial M},v|_{\partial M})_{L^2(\partial M;F_1)}\).

4.1.2. The induced operator \(\hat{P}\). The continuous, sesquilinear form \(B: V \times V \to \mathbb{C}\) defines a linear map (or operator)

\[
\hat{P}: V \to V^*, \quad \langle \hat{P}(v), w \rangle := B(v,w), \quad v, w \in V.
\]

If \(B = B_a\), we shall denote by \(\hat{P}_a\) the associated operator. We note that \(B\) and \(\hat{P}\) depend on the choice of the metric \(g\), although this will typically not be shown in the notation, since the metric will be fixed.

**Definition 4.2.** We shall say that a differential operator \(\hat{P}: V \to V^*\) obtained as in Equation (8) is a second order differential operator in divergence form.

Recall that \(H^{-1}(M;E^*) \simeq H^1_0(M;E)^*\). Using the metric on \(E\), we shall identify \(H^{-1}(M;E^*) \simeq H^{-1}(M;E)\). Since \(H^1_0(M;E) \subset V\), we obtain the natural map \(V^* \to H^1_0(M;E)^* \simeq H^{-1}(M;E^*)\), and hence \(\hat{P}\) gives rise to a map \(P: V \to H^{-1}(M;E^*)\). Clearly, \(P\) does not depend on \(Q\), whereas this is in general not the case for \(\hat{P}\). In fact, \(Q\) only enters in the boundary conditions, see Example 4.8. Similarly, \(\hat{P}\) extends to a map \(\hat{P}: H^1(M;E) \to H^1(M;E)^*\). We notice that \(P\) (and hence also \(\hat{P}\)) determines the form \(a\), which is the principal symbol of \(P\). We shall say that \(a\) is the sesquilinear form associated to \(P\). More precisely, let us identify \(((T^*M \otimes E) \otimes (T^*M \otimes \overline{E}))' \simeq (T^*M \otimes T^*M)^* \otimes \text{End}(E)\) using the metric on \(E\). Then the quadratic function

\[
T^*M \ni \xi \to a(\xi, \xi) \in \text{End}(E)
\]

is the principal symbol of \(P\). In fact, it would be actually more natural to start with \(a \in TM \otimes TM \otimes \text{End}(E)\). However, as we always consider a metric on \(E\), this makes not difference for us.

In a certain way, \(P\) is the “true” differential operator associated to \(B\), whereas \(\hat{P}\) includes, in addition to \(P\), also boundary terms. To better understand this statement as well as the difference between \(P\) and \(\hat{P}\), a calculation in local coordinates is contained in Example 4.8 below, see also [41, 21].
4.2. Variational boundary value problems. We now examine the relation between the operators in divergence form (i.e. of the form \( P \)) and boundary value problems. In particular, we discuss the weak formulation of the Robin problem. See also [4, 41, 46, 52, 53] for the weak formulation of boundary value problems. We assume that we are given a direct sum decomposition

\[ E|_{\partial M} = F_0 \oplus F_1 \]

of the restriction of \( E \) to the boundary into a direct sum of two vector bundles with totally bounded curvature. We consider boundary differentials operators \( C_j : H^\infty(M; E) \to L^2(\partial M; F_j) \),

where \( C_0 \) is of order zero and \( C_1 = C_{10} + C_{11} \partial_\nu \), with \( C_{11} \) and \( C_{00} \) of order zero, \( \partial_\nu \) the covariant normal derivative at the boundary in the direction of the outer unit normal vector \( \nu \), and \( C_{10} \) only including derivatives tangential to \( \partial M \). Each of the operators \( C_0, C_{10}, \) and \( C_{11} \) factors through a map \( H^\infty(\partial M; E|_{\partial M}) \to L^2(\partial M; F_j) \), which will be denoted with the same symbol and we will assume to be differential operators with bounded coefficients. If \( C_0^{-1} \in L^\infty(\partial M; \text{End}(F_0)) \) and \( C_{11}^{-1} \in L^\infty(\partial M; \text{End}(F_1)) \) we shall say that the boundary conditions \( C = (C_0, C_1) \) are non-degenerate.

We are interested in boundary value problems of the form (1) where \( h = (h_0, h_1) \), \( h_j \in \Gamma(\partial M; F_j) \), and the relation \( Cu = h \) means \( C_0u = h_0 \) and \( C_1u = h_1 \). We shall regard the boundary conditions \( C_0, C_1 \) and \( (C'_0, C'_1) \) as equivalent if \( (C'_0, C'_1) = (D_0 C_0, D_1 C_1) \), where \( D_j \), for each \( j \), is a bounded automorphism of \( F_j \) with bounded inverse. The two sets of solutions of two equivalent boundary value problems are in a natural (continuous) bijection, which justifies looking at equivalence classes of boundary value problems. More precisely, let \( C \) and \( C' \) be equivalent boundary value problems and \( (D_0, D_1) \) be the automorphisms implementing the equivalence. Assume \( C \) and \( C' \) have totally bounded coefficients, for simplicity, and let \( h_0 \in \mathcal{H}^{k+1/2}(\partial M; F_0) \), \( h_1 \in \mathcal{H}^{k-1/2}(\partial M; F_1) \), \( h := (h_0, h_1) \) and \( h' := (D_0 h_0, D_1 h_1) \). Then the boundary value problems \( Pu = f \), \( Cu = h \) and \( Pu = f \), \( C'u = h' \) have the same solutions \( u \in \mathcal{H}^{k+1}(M; E) \).

4.2.1. Definition of variational boundary conditions. Let \( e_j \) be the orthogonal projection onto \( F_j \) at the boundary. Also, let \( B \) be the basic sesquilinear form defined in Equation (7) of the previous subsection. We shall consider for the rest of this paper

\[ V := \{ u \in \mathcal{H}^1(M; E) \mid e_0 u = 0 \text{ on } \partial M \}. \]

Spaces of this form with \( e_0 \) non-trivial (i.e. different from the restriction to some components of the boundary) arises in the study of the Hodge-Laplacian and in the study of the Ricci flow [56]. Let \( k \geq 1 \). For any \( h \in \mathcal{H}^{k-1/2}(\partial M; F_1) \) and \( f \in \mathcal{H}^{k-1}(M; E) \), we let

\[ F(v) := \int_M (f, v) \, d\text{vol}_g + \int_{\partial M} (h, v) \, d\text{vol}_{\partial g} , \]

where \( d\text{vol}_{\partial g} \) is the induced volume form on \( \partial M \). Then \( F \) defines a linear functional on \( V \) and hence an element of \( V^* \). We denote \( j_{k-1}(f, h) := F \) the induced map

\[ j_{k-1} : \mathcal{H}^{k-1}(M; E) \oplus \mathcal{H}^{k-1/2}(\partial M; F_1) \to V^*. \]
The formula for $F$ makes sense also for $k = 0$ and $f = 0$, in which case it is just the dual map to the restriction at the boundary. Recall that all our differential operators, including $\hat{P}$, have bounded coefficients.

**Remark 4.3.** We identify $(T^*M \otimes E \otimes T^*M \otimes E)'$ with $(T^*M \otimes T^*M)' \otimes \text{End}(E)$ using the metrics on $TM$ and $E$, respectively, as above. Then $a(dr, dr)$ can be regarded as a section of $\text{End}(E)|_{\partial M}$ and we have on $V$ that $a(dr, \nabla_\nu) = a(dr, dr)\partial_\nu + Q'$, where $Q'$ is a differential operator on $\Gamma(\partial M; E|_{\partial M})$ (that is, it does not involve normal derivatives at the boundary; it involves only tangential derivatives).

Note that we are not assuming the bundles $F_i$ to be orthogonal.

**Remark 4.4.** Let us assume, in the definition of $B(u, v)$, Equation (7), that $u \in H^2(M; E) \cap V$. Then we obtain

$$B(u, v) := (Pu, v) + \int_{\partial M} (\partial^\nu \nu u, v) \text{dvol}_g,$$

(14)

where $\text{dvol}_g$ is the volume form on $\partial M$, as before. We have $\partial^\nu \nu u = e_1 a(dr, dr \otimes \nabla_\nu u) + Q$, a first order differential operator. We shall also denote $\partial^\nu \nu u := a(dr, dr \otimes \nabla_\nu u)$, with $u$ a section of $F_1$ over $\partial M$. For reasons of symmetry (to have a class of operators stable under adjoints), one may want to consider in the minimal regularity case the following bilinear form

$$B(u, v) := B_0(u, v) + (Q_1 u, v)_{L^2(M)} + (u, Q_2 v)_{L^2(M)} + (Q u, v)_{L^2(\partial M)}.$$ 

In that case, the boundary operator $\partial^\nu \nu u$ will depend also on $Q_2$. However, if our operators have coefficients in $W^{1, \infty}$, which is the case when dealing with regularity estimates, as in this paper, then we can absorb $Q_2$ into $Q_1$ by taking adjoints, with the price of obtaining an additional boundary term, which, nevertheless, can then be absorbed into $Q$. See also Example 4.8, where the term $Q_2$ was kept in the formula.

**Lemma 4.5.** Assume $u \in H^2(M; E)$ and let $\hat{P}$ be as in Equation (8). The equation

$$\hat{P}(u) = j_{k-1}(f, h)$$

(15)

is then equivalent to the mixed Dirichlet/Robin boundary value problem

$$\begin{cases}
Pu = f & \text{in } M \\
e_0 u = 0 & \text{on } \partial M \\
e_1 \partial^\nu \nu u + Qu = h & \text{on } \partial M,
\end{cases}$$

(16)

where $\partial^\nu \nu u := a(dr, \nabla u)$ as above.

**Proof.** Indeed, $e_0 u = 0$ on $\partial M$ since $u$ is in $V$, which is, by definition the domain of $\hat{P}$. Let $F := j_{k-1}(f, h)$. The rest follows from Equation (7), which gives

$$\langle \hat{P}(u), w \rangle - F(w) = B(u, w) - \int_M (f, w) \text{dvol}_g - \int_{\partial M} (h, w) \text{dvol}_g = (Pu - f, w)_{M} - \int_{\partial M} (\partial^\nu \nu u + Qu - h, w) \text{dvol}_g.$$

The boundary conditions of this lemma will be the main object of study for us. As we will see below in Proposition 4.7, these boundary conditions turn out to be, in fact, quite general. Recall the following standard terminology.
Definition 4.6. We keep the notation of the Lemma 4.5. We shall say, as usual, that \( e_0 u = h_0 \) are the Dirichlet boundary conditions and \( e_1 \partial_\nu u + Qu = h_1 \) are the natural (or Robin) boundary conditions. Also, we shall say that \( P \) and the boundary conditions \( (e_0, e_1 \partial_\nu + Q) \) are equivalent. In particular, \( C \) and continuously on \( F \).

Non-degenerate boundary conditions are equivalent to variational ones for suitable \( a \), as we will see below.

Proposition 4.7. Let \( \hat{P} \) be a second order differential operator in divergence form associated to \( a \), regarded as a bounded, measurable section of \( T^* M \otimes T^* M \otimes \text{End}(E) \), such that \( e_1 a(dr, dr)e_1 \) is invertible in \( L^\infty(\partial M; \text{End}(F_1)) \). Let \( C = (C_0, C_1) \) be non-degenerate boundary conditions. Then there is a first order differential operator \( \hat{Q} \) with bounded coefficients such that the boundary conditions \( C \) and \( (e_0, e_1 \partial_\nu + Q) \) are equivalent. In particular, \( C \) is equivalent to some variational boundary conditions associated to \( \hat{P} \).

This proposition justifies our choice to consider only boundary value problems of the form (16) instead of the general form (1). Indeed, let \( \hat{P} \) be the differential operator in divergence form with boundary conditions \( (e_0, e_1 \partial_\nu + Q) \). Then the solutions of the equation \( \hat{P}(u) = F \) are in a natural bijections to the solutions of the boundary value problem \( Pu = f \) and \( Cu = h \), where \( f \) and \( h \) depend linearly and continuously on \( F \).

Proof. Let \( C_1 = C_{10} + C_{11} \partial_\nu \) as explained after Equation (11). We have \( C_0 e_0 = C_0 \) with \( C_0 \) invertible. Then, \( e_1 \partial_\nu + Q = e_1 a(dr, dr)e_1 \partial_\nu + Q' \) and \( e_1 a(dr, dr)e_1 C_{11}^{-1} C_1 = e_1 a(dr, dr)e_1 \partial_\nu + Q'' \), where \( Q' \) and \( Q'' \) are first order differential operators on \( F_1 \) with bounded coefficients. Since \( Q \) (and hence also \( Q' \)) can be chosen arbitrarily with these properties, we can certainly arrange that \( Q' = Q'' \) within the class of operators considered. \( \square \)

There is no good reason to chose \( C_0 \) other than \( e_0 \). On the other hand, there is no reason to expect, in general, that \( C_{11} \) be invertible. However, if \( C_{11} \) is not invertible, the behavior of the problem becomes completely different and, to the best of our knowledge, it is not fully understood at this time (see, however, [49] and the references therein).

Example 4.8. Recall the operator \( \hat{P} \) from Equation (8) and the related form \( B \) from Equation (7). Let us assume that \( M = U \subset \mathbb{R}^m \) is a submanifold with boundary of dimension \( m \). (Here \( \partial U \) denotes the boundary of \( U \) as a manifold with boundary, not as a subset of \( \mathbb{R}^m \)!)

Let \( E = \mathbb{C}^N \), \( F_0 = \mathbb{C}^{N_0} \), and \( F_1 = \mathbb{C}^{N_1} \) be trivial bundles with \( N = N_0 + N_1 \) all equipped with the standard metric. Finally, we assume that are matrix valued functions \( a_{ij}, b_i, b_j^*, c, d \in L^\infty(U; M_N) \). We will assume that the metric on \( U \) is the euclidean metric, since this will not really decrease the generality, but will simplify our notation. In this example, we choose

\[
V := \{ u \in H^1(U)^N | u_1 = u_2 = \ldots = u_{N_1} = 0 \text{ on } \partial U \}.
\]  

We let \( a \) denote the bilinear form on \( \mathbb{C}^{mN} \) associated to the matrix \( (a_{ij} \in (\mathbb{C}^N)^* \otimes \mathbb{C}^N)_{ij} \). We have \( \nabla_i = \partial_i \) since we are dealing with the trivial bundles with standard metric over the euclidean space. Let \( Q = (Q^{kl}_{ij})_{k,l=0}^{N_1+1} \) be a matrix first
order differential operator on \( \partial U \) and \((Pu)_k\) be the \( k \)th component of \( Pu \). This gives for all \( w \in V \)
\[
\langle \tilde{P}_{(a,Q)}u, w \rangle = \int_U a(\nabla u, \nabla w)dx + \int_{\partial U} (Qu, w)dS
\]
\[
= \sum_{i,j=1}^m \sum_{k,l=1}^N \int_U a_{ij}^{kl} \partial_j u_i \partial_j \nu l dx + \int_{\partial U} (Qu, w)dS,
\]
where \( dS \) is the induced volume form on \( \partial U \). In particular,
\[
(P_{(a,Q)}u)_k = (P_u)k = -\sum_{i,j=1}^m \sum_{l=1}^N \partial_i (a_{ij}^{kl} \partial_j u_l).
\]
Let \( Q_1 u := \sum_{j=1}^m b_j \partial_j u + cu \) and \( Q_2 u := \sum_{i=1}^m \sum_{j=1}^m b_i^j \partial_k u \), let \( u, v \in C^\infty(\overline{U})^N \cap V \), and let \( \nu \) be the outer, unit normal to \( \partial U \). The formula \( \int_U (\partial_j u) dx = \int_{\partial U} \nu_j u dS \) gives
\[
\langle \tilde{P}(u), w \rangle := \int_U a(\nabla u, \nabla w)dx + (Q_1 u, w)_U + (u, Q_2 w)_U
\]
\[
+ \sum_{k,l=N+1}^N \int_{\partial U} Q^{kl} u_k(x) \overline{w_l(x)} dS = \int_U Pu(x)w(x) dx + \int_{\partial U} \partial^P_u(x)w(x) dS.
\]
Let us extend \( Q \) to an \( N \times N \)-matrix by completing it with zeroes. In turn, this gives that
\[ Pu = P_u + \sum_{j=1}^m b_j \partial_j u + cu - \sum_{i=1}^m \partial_i (b_i^j u) \]
and
\[
(\partial^P_u)k := \begin{cases}
  \sum_{i=1}^N \left( \sum_{j=1}^m \nu_i a_{ij}^{kl} \partial_j u_l + \sum_i \nu_i b_i^{kl} u_l \right) + (Qu)_k & \text{if } k > N_1 \\
  0 & \text{otherwise}
\end{cases}
\]
If the coefficients \( b_i^j \) are differentiable and \( F_1 = 0 \), we can get rid of the operator \( Q_2 \) by absorbing it into \( Q_1 \). However, if these coefficients are not differentiable and we want a class of operators that is closed under adjoints, then we need to include the \( Q_2 \) term into the definition of \( P \) (or, rather, \( \tilde{P} \)). Moreover, if \( F_1 \neq 0 \), we see that \( \tilde{P}u \) contains an additional term compared to \( Pu \), meaning that \( \tilde{P}u - Pu \) is a distribution supported on \( \partial U \). Also, we see that \( \tilde{P} \) determines \( P \), but not the other way around.

4.2.2. Uniformly strongly elliptic operators. There are some classes of operators for which the conditions of Proposition 4.7 are almost automatically satisfied. Recall the following standard terminology.

**Definition 4.9.** We shall say that the operator \( \tilde{P}: V \to V^* \) defined using the sesquilinear form \( B \), see Equation (8), is **uniformly strongly elliptic** if there exists \( c_a > 0 \) such that
\[
\Re(a(\eta \otimes \xi, \eta \otimes \xi)) \geq c_a \|\eta\|^2 \|\xi\|^2
\]
for all \( x \) and all \( \xi \in E_x \) and \( \eta \in T^*_x M \). The associated operator \( P \) will be called **uniformly strongly elliptic**. A family of operators is called **uniformly strongly elliptic** if each operator is uniformly strongly elliptic and we can choose the same constant \( c_a \) for all operators in the family. We shall say that \( \tilde{P} \) is **uniformly elliptic** if there
exists $c_ε > 0$ such that, for every $ξ ∈ E_x$, there exists $ξ_1 ∈ E_x$, $∥ξ_1∥ = ∥ξ∥$, such that
\[|a(η ⊗ ξ, η ⊗ ξ_1)| ≥ c_ε∥η∥^2∥ξ∥^2\]
for all $x$ and $η ∈ T^*_x M$.

Note that in the above definition, we have taken advantage of the fact that our operator acts on sections of the same bundle. For operators acting between sections of different bundles, the definition will change in an obvious way (replacing the injectivity of the principal symbol with its invertibility). We see that if $P$ is uniformly strongly elliptic, then $e_1(a, η)ε_1$ is invertible in $L^∞(∂M; End(F_1))$. (If $e_1 = 1$, i.e. if $F = E|∂M$, it is enough to assume that $a$ is uniformly elliptic. We agree that if $F_1$ is the zero bundle on some component of $∂M$, then we consider every endomorphism of it to be invertible on that set.)

4.2.3. The scale of regularity for boundary value problems. Let $P$ be associated to $B$ as in Equation (8) and $V$ be as in Equation (12). We define
\[H^ℓ−1(M; E) := \begin{cases} H^{ℓ−1}(M; E) ⊕ H^{ℓ−1/2}(∂M; F_1) & \text{for } ℓ ≥ 1 \\ V^* & \text{for } ℓ = 0. \end{cases}\]

The natural exact sequence $0 → H^{ℓ−1/2}(∂M; F_1) → V^* → H^{−1}(M; E) → 0$, where the second map is induced by the trace map, see Theorem 2.13, shows that we have a natural scale of regularity spaces. In particular, $H^ℓ+1(M; E) ⊆ H^ℓ(M; E)$, $ℓ ≥ 0$. In general, the natural inclusion $H^ℓ(M; E) → H^ℓ−1(M; E) := V^*$ is given by the operators $j_ℓ$ defined in Equation (13). Let $\hat{P}_ℓ: H^ℓ+1(M; E) ∩ V → H^ℓ−1(M; E)$ be given by
\[\hat{P}_ℓ(u) := (Pu, e_1∂_ν u + Qu) \text{ for } ℓ ≥ 1. \tag{18}\]

Then the relation between $\hat{P}_ℓ$ and $\hat{P}$ is by (15) expressed in the commutativity of the diagram
\[H^ℓ+1(M; E) ∩ V \xrightarrow{\hat{P}_ℓ} H^ℓ−1(M; E) \xrightarrow{j_{ℓ−1}} H^ℓ(M; E) ∩ V \xrightarrow{\hat{P}} H^{ℓ−1}(M; E) \tag{19}\]

where the vertical arrows are the natural inclusions. Thus, although the definition of $\hat{P}_0 := \hat{P}: H^1(M; E) ∩ V → H^{−1}(M; E)$ is different from that of $\hat{P}_ℓ$ for $ℓ > 0$, it fits into a scale of regularity spaces.

4.3. Regularity conditions. The scale of regularity spaces provides a good setting to study regularity conditions

Notations 4.10. We shall denote by $D^{ℓ,j}(M; E)$ the set of pairs $(D, C)$, where $D$ is a second order differential operator defined on sections of $E → M$ and $C$ is an order $j$ boundary condition, with both $D$ and $C$ assumed to have coefficients in $W^{ℓ,∞}$. In case $M$ has no boundary (and thus there are no boundary conditions), we shall denote the resulting space $D^{ℓ,0}(M; E)$. If $E := \mathcal{L}$ (that is, if we are dealing with scalar boundary value problems), then we shall drop the vector bundle from the notation.
The assumption that \( C \) is an order \( j \) boundary condition implies that \( E|_{\partial M} = F_j \), and hence that \( F_{k-j} = 0 \). The general case just involves a more complicated notation. We endow the space \( D^{k,j}(M; E) \) with the norm defined by the maximum of the \( W^{\ell,\infty} \)-norms of the coefficients. Recall the following definition:

**Definition 4.11.** We say that \( (D, C) \in D^{k,j}(M; E) \), \( \ell \geq j + 1 \), satisfies an \( H^{\ell+1} \)-regularity estimate on \( M \) if there exists \( c_R > 0 \) with the following property: For any \( w \in H^k(M; E) \), \( \ell \geq k \geq j + 1 \), with compact support in \( M \) such that \( Dw \in H^{k-1}(M; E) \) and \( Cw \in H^{k-j+1/2}(\partial M; E) \), we have \( w \in H^{k+1}(M; E) \) and

\[
\|w\|_{H^{k+1}(M; E)} \leq c_R \left( \|Dw\|_{H^{k-1}(M; E)} + \|w\|_{H^k(M; E)} + \|Cw\|_{H^{k-j+1/2}(\partial M; E)} \right).
\]

If \( D \in D^{k,0}(M; E) \), we just drop the term \( \|Cw\|_{H^{k-j+1/2}(\partial M; E)} \). If \( j = 1 \) and if \( (D, C) \) are obtained from a variational formulation, then we allow also \( \ell = 1 \), with the relations \( Dw \in L^2(M; E) \) and \( Cw \in H^{1/2}(\partial M; E) \) being replaced with \( \tilde{D}w = j_0(f, h) \), with \( (f, h) \in L^2(M; E) \oplus H^{1/2}(\partial M; E) \) and the estimate is replaced with

\[
\|w\|_{H^2(M; E)} \leq c_R \left( \|f\|_{L^2(M; E)} + \|w\|_{H^1(M; E)} + \|h\|_{H^{1/2}(\partial M; E)} \right),
\]

in which case we also say that \( \tilde{D} \) satisfies an \( H^{\ell+1} \)-regularity estimate on \( M \).

The following comments on this definition are in order.

**Remark 4.12.** (i) The condition that \( w \) have compact support in \( M \) does not imply that it vanishes near the boundary \( \partial M \).

(ii) Typically, we shall consider boundary value problems coming from a variational formulation, but, at this time, it is not necessary to make this assumption.

(iii) The conditions that \( D \) and \( C \) have coefficients in \( W^{\ell,\infty} \) are needed so that \( Dw \in H^{k-1}(M; E) \) and \( Cw \in H^{k-j+1/2}(\partial M; E) \) for \( w \in H^{k+1}(M; E) \), \( k \leq \ell \).

(iv) Let \( (D', C') \in D^{k,j}(M; E) \) be a second boundary value problem such that \( D - D' \) has order \( < 2 \) and \( C - C' \) has order \( < j \). Then \( (D, C) \) satisfies an \( H^{\ell+1} \)-regularity estimate on \( M \) if, and only if, \( (D', C') \) does so.

**Remark 4.13.** The careful reader might have noticed already that, apart from notation, we have not really used the fact that our operator is second order, except maybe in Proposition 4.7. For instance, for an order \( 2m \) problem, one would consider

\[ a \in L^\infty(M; T^{\otimes m}M \otimes T^{\otimes m}M \otimes \text{End}(E)), \]

but the form \( B \) would involve several boundary terms relating the normal derivatives at the boundary. For \( V \) we would take an intermediate subspace \( H^m_0(M; E) \subset V \subset H^m(M; E) \). This could be given by a sub-bundle \( F_1 \subset E^{2m}_{\partial M} \) representing the possible values of \( (u, \partial_0 u, \ldots, \partial_{\nu}^{m-1}u) \) at the boundary:

\[ V := \{ u \in H^m(M; E) \mid (u, \partial_0 u, \ldots, \partial_{\nu}^{m-1}u) \in L^\infty(\partial M; F_1) \}. \]

Nevertheless, non-local conditions are also important [42]. One could then proceed to deal with higher order problems as in [41], for instance. Since Proposition 4.7 plays an important role in justifying our choice of essentially always using variational boundary value conditions and in view of the many complications that arise when dealing with higher order problems (see [2, 41, 55, 57]), we decided to restrict ourselves to the case of second order operators. See also Remark 7.3.
5. Uniform regularity estimates for families

We shall consider the same framework as in the previous section, in particular, 
\((M, g)\) will continue to be a Riemannian manifold with smooth boundary \(\partial M\) such that \(TM\) has totally bounded curvature. Also, we continue to assume that \(E \to M\) has totally bounded curvature.

5.1. Compact families of boundary value problems. We use the same notation as in the previous section. In particular, \(B, P, a, e_1 e_k + Q, C = (C_0, C_1), \) and \(F_0 \otimes F_1 = E|_{\partial M}\) are as in the previous section. In particular, \(\hat{P}\) will always represent a second order differential operator in divergence form associated to the sesquilinear form \(a;\) in particular, (the quadratic map associated to) \(a\) is the principal symbol of \(P.\)

We shall need a uniform version of Definition 4.11.

**Definition 5.1.** Let \(S \subset \mathcal{D}^{\ell,j}(M; E)\) and \(N \subset M\) a submanifold with boundary \(\partial N = \partial M \cap N\) of \(M.\) We shall say that \(S\) satisfies a uniform \(H^{\ell+1,j}\)-regularity estimate on \(N \subset M\) if each \((D, C) \in S\) satisfies an \(H^{j+1,1/2}\)-regularity estimate on \(N \subset M\) with a bound \(c_R\) (in Definition 4.11) independent of \((D, C) \in S.\) If \(S \subset \mathcal{D}^{\ell,j}(M; E),\) we just consider \(D \in S.\)

Note that \(S\) in the above definition is not assumed to be bounded. However, typically, we shall obtain the independence of the bound \(c_R\) by assuming also that \(S\) is compact (or, sometimes, just pre-compact).

**Proposition 5.2.** Assume that \(S \subset \mathcal{D}^{\ell,j}(M; E)\) is compact and that each \((D, C)\) satisfies an \(H^{\ell+1,j}\)-regularity estimate on \(M.\) Also, assume that the restriction \(H^j(M; E) \to H^{j-1/2}(\partial M; E)\) is continuous for all \(j \geq 1.\) Then \(S\) satisfies a uniform \(H^{\ell+1,j}\)-regularity estimate on \(M.\) The same result holds if \(S \subset \mathcal{D}^{\ell,j}(M; E).\)

The assumption that the restriction (trace map) \(H^j(M; E) \to H^{j-1/2}(\partial M; E)\) is continuous for all \(j \geq 1\) is, of course, satisfied if \(M\) is an open subset of a manifold with boundary and bounded geometry \(M_0\) [35] (see Theorem 2.13). In particular, this is the case if \(M\) is the domain of a Fermi coordinate chart. Also, note that here the boundary is in the sense of manifolds with boundary, and not in the sense of point-set topology.

**Proof.** Consider first the case \(S \subset \mathcal{D}^{\ell,j}(M; E).\) Let us assume the contrary and show that we obtain a contradiction. That is, let us assume that there exist sequences \((D_i, C_i) \in S\) and \(w_i \in H^{k+1}(M; E),\) with compact support in \(M,\) such that

\[
\|w_i\|_{H^{k+1}(M; E)} > 2^j \left( \|D_i w_i\|_{H^{k-1}(M; E)} + \|w_i\|_{H^k(M; E)} + \|C_i w_i\|_{H^{k-1/2}(\partial M; E)} \right).
\]

Since \(S\) forms a compact subset in \(\mathcal{D}^{\ell,j}(M; E),\) by replacing \((D_i, C_i)\) with a subsequence, if necessary, we can assume that \((D_i, C_i)\) converges. Let us denote the limit with \((D, C) \in S.\) Thus, there is a sequence \(\epsilon_i \to 0\) with

\[
\|D_i w\|_{H^{k-1}(M; E)} \geq \|D w\|_{H^{k-1}(M; E)} - \epsilon_i \|w\|_{H^{k+1}(M; E)},
\]

\[
\|C_i w\|_{H^{k-1/2}(\partial M; E)} \geq \|C w\|_{H^{k-1/2}(\partial M; E)} - \epsilon_i \|w\|_{H^{k+1/2}(\partial M; E)}.
\]

Together with the assumed continuity of the trace map, this implies that there exists \(c' > 0\) such that
\[ \|w_i\|_{H^{k+1}(M;E)} > 2^i \left( \|Dw_i\|_{H^{k-1}(M;E)} + \|w_i\|_{H^k(M;E)} + \|Cw_i\|_{H^{k-j+1/2}(\partial M;E)} - c'\epsilon_i \|w_i\|_{H^{k+1}(M;E)} \right), \]

for all \( i \). On the other hand, \((D, C)\) satisfies, by assumption, an \( H^{k+1}\) - regularity estimate. Consequently, there is a \( c > 0 \) such that

\[ \|Dw_i\|_{H^{k-1}(M;E)} + \|w_i\|_{H^k(M;E)} + \|Cw_i\|_{H^{k-j+1/2}(\partial M;E)} \geq c^{-1}\|w_i\|_{H^{k+1}(M;E)}, \]

and hence we obtain

\[ \|w_i\|_{H^{k+1}(M;E)} \geq 2^i \left( c^{-1} - c'\epsilon_i \right) \|w_i\|_{H^{k+1}(M;E)}. \]

For \( i \to \infty \), this gives the desired contradiction since \( 2^i(c^{-1} - c'\epsilon_i) \to \infty \), for \( i \to \infty \), and \( \|w_i\|_{H^{k+1}(M;E)} \neq 0 \). This completes the proof if \( S \subset \mathcal{D}^{k,0}(M;E) \).

If \( S \subset \mathcal{D}^{k,0}(M;E) \), the proof is obtained by simply dropping the terms that contain \( Cw \) from the above proof. (We can further replace \( 2\epsilon_i \) with \( \epsilon_i \), but that is not essential.)

Recall that a relatively compact subset is a subset whose closure is compact.

**Proposition 5.3.** Let \( N \subset M \) be a relatively compact open subset and let \( S \subset \mathcal{D}^{k+1,j}(M;E) \) be a bounded subset. Assume that every \( \mathcal{D}^{k,j}(N;E) \)-limit \((\hat{D}, \hat{C})\) of a sequence \((D_i, C_i) \in S\) satisfies an \( H^{k+1}\) - regularity estimate on \( N \). Then \( S \) satisfies a uniform \( H^{k+1}\) - regularity estimate on \( N \). The same result holds if \( S \subset \mathcal{D}^{k+1,0}(M;E) \).

We remark that in this proposition the compactness condition of Proposition 5.2 is replaced by a higher regularity assumption on the coefficients. This is needed in order to use the Arzela-Ascoli Theorem. Moreover, we note that by choosing a constant sequence, we see that the assumptions imply that each element in \( S \) satisfies an \( H^{k+1}\) - regularity estimate on \( N \). We also note that \((\hat{D}, \hat{C})\) in the statement automatically satisfies \((\hat{D}, \hat{C}) \in \mathcal{D}^{k,j}(N;E)\).

**Proof of Proposition 5.3.** We treat explicitly only the case \( S \subset \mathcal{D}^{k+1,j}(M;E) \), the case \( S \subset \mathcal{D}^{k+1,0}(M;E) \) being completely similar. Since the coefficients of all boundary value problems in \( S \) are bounded in \( W^{k+1,\infty}(M) \), the set of coefficients of the operators \((D, C) \in S\) is precompact in \( W^{k,\infty}(N) \), by the Arzela-Ascoli theorem. Let \( K \) be the closure of \( S \) in \( \mathcal{D}^{k,j}(N,E) \), which is therefore a compact set. Moreover, our assumptions imply that every element in \( K \) satisfies an \( H^{k+1}\) - regularity estimate. Proposition 5.2 then implies the result.

We shall need the following lemma.

**Lemma 5.4.** Let \( S \subset \mathcal{D}^{k,j}(M;E) \) satisfy a uniform \( H^{k+1}\) - regularity estimate on \( M \). If \( N \subset M \) is an open subset, then \( S \) satisfies a uniform \( H^{k+1}\) - regularity estimate on \( N \). The same result holds in the boundaryless case.

**Proof.** Let \( w \in H^k(N;E) \) with compact support. Then \( w \in H^k(M;E) \) and the uniform regularity estimate for \( w \) on \( M \) yields the desired result.

**Remark 5.5.** To deal with the general case when \( C \) is not a boundary condition of fixed order \( j \), but rather combines a boundary condition \( C_1 \) of order one and a boundary condition \( C_0 \) of order zero, we just replace \( \|Cw\|_{H^{k+1/2}(\partial M;E)} \) with \( \|C_0w\|_{H^{k+1/2}(\partial M;E)} \) in this case, we may have that both \( F_0 \neq 0 \) and \( F_1 \neq 0 \).
5.2. **Higher regularity and bounded geometry.** The relevance of uniform regularity conditions introduced in Section 5 is that it allows us to obtain higher regularity on manifolds with boundary and bounded geometry and suitable boundary conditions as follows.

Let \((M, g)\) be a Riemannian manifold with boundary and bounded geometry, as before. Let \((P, C)\) be a boundary value problem on \(M\). We shall assume that \((P, C)\) comes from a variational formulation, since most of the results will require this assumption. (This means that if \(C = (C_0, C_1)\), then \(P\) can be identified with \((P, C_1)\) and \(C_0\) is simply the projection onto \(F_0\) at the boundary.) We assume, for notational simplicity, that \(C\) has constant order \(j\) at the boundary. For the same reasons, we also assume \(P\) acts and takes values in the same vector bundle \(E\) (that is, \(E_1 = E\)), as before. The results of this subsection hold, however, in full generality (when the vector bundles \(E, E_1\), and \(F\) of Section 5 are distinct), but with some obvious changes. Let \(0 < r \leq r_{FC}\), as in Definition 2.7. Recall that \(U_p\) and \(\kappa_p\) from (4) and \(\xi_p\) from Definition 2.11 are such that either \(p \in \partial M\) or \(\operatorname{dist}(p, \partial M) \geq r\). We denote by \((P_p, C_p)\) the induced boundary value problems on \(\kappa_p^{-1}(U_p) = B_{2r}^m(0) \times [0, 2r)\), if \(p \in \partial M\). Then \(P_p = \xi_p^* \circ P \circ (\xi_p)_*\), and \(C_p = \xi_p^* \circ C \circ (\xi_p)_*\), with the obvious notation, meaning that the operators correspond through the diffeomorphisms \(\xi_p\). If \(\operatorname{dist}(p, \partial M) \geq r\), there is no \(C_p\) and we obtain a differential operator \(P_p\) on \(B_r^{m+1}(0)\). Let \(t\) denote the rank of \(E\) and let

\[
\mathcal{F}_b := \{(P_p, C_p) \mid P \in \partial M\} \subset \mathcal{D}^0; (B_{2r}^m(0) \times [0, 2r); \mathbb{C}^t)
\]

\[
\mathcal{F}_i := \{\{P_p\} \mid \operatorname{dist}(p, \partial M) \geq r\} \subset \mathcal{D}^0; (B_r^{m+1}(0); \mathbb{C}^t),
\]

be the induced boundary and interior families of operators. Note that we always equip \(B_{2r}^m(0) \times [0, 2r)\), (respectively, \(B_r^{m+1}(0)\)) with the euclidean metric.

**Theorem 5.6.** Let \((M, g)\) be a Riemannian manifold with boundary and bounded geometry and let \(E \rightarrow M\) be a Hermitian vector bundle with totally bounded curvature. Let \((P, C) \in \mathcal{D}^k; (M; E)\). If each of the families \(\mathcal{F}_b := \{(P_p, C_p) \mid p \in \partial M\}\) and \(\mathcal{F}_i := \{\{P_p\} \mid \operatorname{dist}(p, \partial M) \geq r\}\) (see Equation 20 and above) satisfy a uniform \(H^{k+1}\)-regularity estimate, then \((P, C)\) satisfies an \(H^{k+1}\)-regularity estimate. The converse is also true.

**Proof.** This follows from Definition 4.11 of uniform order \(k\) regularity estimates, from Proposition 2.12, and from Lemma 3.6. Let us choose the \(r\)-uniform partition of unity \(\phi_s\) used in those results such that \(\partial_s \phi_s\) vanishes at the boundary. This can be done first by choosing an \(r\)-uniform partition of unity on \(\partial M\).

\[
\|u\|_{H^{k+1}}^2 \leq \sum_{\gamma} \|\xi_{\gamma}^*(\phi_s u)\|_{H^{k+1}}^2 \leq \sum_{\gamma} \left(\|P\gamma \xi_{\gamma}^*(\phi_s u)\|_{H^{k-1}}^2 + \|C\gamma \xi_{\gamma}^*(\phi_s u)\|_{H^{k-1}}^2 \right)
\]

\[
\leq \sum_{\gamma} \left(\|\xi_{\gamma}^* P(\phi_s u)\|_{H^{k-1}}^2 + \|\xi_{\gamma}^* C(\phi_s u)\|_{H^{k-1}}^2 + \|\xi_{\gamma}^* (\phi_s u)\|_{H^k}^2 \right)
\]

\[
\leq (\|Pu\|_{H^{k-1}}^2 + \|Cu\|_{H^{k-1}}^2 + \|u\|_{H^k})^2 + \sum_{\gamma} \left(\|\xi_{\gamma}^* P(\phi_s u)\|_{H^{k-1}}^2 + \|\xi_{\gamma}^* C(\phi_s u)\|_{H^{k-1}}^2 \right)
\]

\[
+ \sum_{\gamma} \|C(\phi_s u)\|_{H^{k-1}}^2 \]
since the trivializations $\xi_r$ have uniformly bounded norms. Next we notice that
$$\sum_{\gamma} \| [P, \phi_{\gamma}] u \|_{H^{k-1}}^2 \lesssim \| u \|_{H^k}^2$$
since the family $\phi_{\gamma}$ is uniformly locally finite and by Lemma 3.6. The boundary term is treated similarly: We have
$$\sum_{\gamma} \| [C, \phi_{\gamma}] u \|_{H^{k-1}}^2 \lesssim \| u \|_{H^k}^2.$$ Here the last inequality is given by the trace theorem. For the first
inequality we note that for $j = 0$ the commutator is actually zero. For $j = 1$
we have $[C, \phi_{\gamma}] = [C_0, \phi_{\gamma}]$ since multiplication by $\phi_{\gamma}$ and $\partial_\nu$ commute, given
the product form of $\phi_{\gamma}$ near the boundary. Together with Lemma 3.6, this completes
the proof. □

Remark 5.7. The method of proof of Theorem 5.6 will yield similar global results in
other classes of spaces, as long as the local regularity results are available and
as long as a local description of these spaces using partitions of unity is available.
This is the case for the $L^p$-Sobolev spaces, $1 < p < \infty$, for which we have both the
description using partitions of unity (Proposition 2.12) and the local regularity results [32].

5.3. Regularity for Dirichlet boundary conditions. The results of the previous
sections were tailored to deal with the Dirichlet boundary conditions. It is
known [4, 3, 61] that strongly elliptic operators with Dirichlet boundary conditions
satisfy regularity conditions. We formulate this well-known result as a lemma for
further use. As before, $E \to M$ will be a vector bundle with bounded geometry.

Lemma 5.8. Let $(P, C) \in \mathcal{D}^{0,0}(B^m_r(0) \times [0, r); E)$, $0 < r \leq \infty$, be a uniformly
strongly elliptic boundary value problem with Dirichlet boundary conditions. Then
$P$ satisfies an $H^{r+1}$-regularity estimate on $M$. The same result holds for a uniformly
elliptic operator $P \in \mathcal{D}^{0,0}(B^m(0); E)$, $0 < r \leq \infty$.

From this we obtain

Corollary 5.9. Let $S \subset \mathcal{D}^{r+1,0}(B^m_r(0) \times [0, r); E)$ be a bounded, uniformly strongly
elliptic family of boundary value problems on $B^m_r(0) \times [0, r) \subset \mathbb{R}^{m+1}$ equipped
with the euclidean metric, $r \leq \infty$. We assume that the boundary conditions are
all Dirichlet. Then the family $S$ satisfies a uniform $H^{r+1}$-regularity estimate on
$B^m_r(0) \times [0, r')$, $r' < r$.

Proof. Let $(D_n, C_n) \in S$ converge to $(D, C) \in \mathcal{D}^{r,j}(B^m_r(0) \times [0, r'); E)$. Then $D$
is a uniformly strongly elliptic operator because the parameter $c_n$ stays away from
0 on $S$, in view of Definition 4.9. Lemma 5.8, then gives that $(D, C)$ satisfies an
$H^{r+1}$-regularity estimate, since the type of boundary conditions (Dirichlet or
Neumann) do not change by taking limits. This allows us to use Proposition 5.3
for the relatively compact subset $N := B^m_r(0) \times [0, r')$ of $M := B^m_r(0) \times [0, r)$
to obtain the result. □

Analogously, (in fact, even more directly, since we do not have to take boundary
conditions into account), we obtain

Corollary 5.10. Let $S \subset \mathcal{D}^{r+1,0}(B^{m+1}_r(0); E)$ be a bounded uniformly elliptic family
of differential operators on $B^{m+1}_r(0) \subset \mathbb{R}^{m+1}$, for some $0 < r \leq \infty$. Then the
family $S$ satisfies a uniform $H^{r+1}$-regularity estimate on $B^{m+1}_r(0) \subset \mathbb{R}^{m+1}$ for any
$r' < r$.

Remark 5.11. The regularity results of this section extend to the $L^p$-Sobolev spaces
$W^{k,p}$, $1 < p < \infty$, with essentially the same proofs by using also the results in [32].
Note that in Corollaries 5.9 and 5.10 we use a slightly stronger assumption on the coefficients of our operators than usually, namely we require them to have $W^{l+1,\infty}$-regularity (usually we require only $W^{l,\infty}$-regularity). This is required since we will use Proposition 5.3. Combining these result, we obtain the following.

**Theorem 5.12.** Let $P$ be a uniformly strongly elliptic second order differential operator with coefficients in $W^{l+1,\infty}$ acting on sections of $E \to M$. Then there exists $c > 0$ such that, if $u \in H^l(M;E)$, $Pu \in H^{l-1}(M;E)$, and $u|_{\partial M} \in H^{l+1/2}(\partial M;E)$, then $u \in H^{l+1}(M;E)$ and

$$\|u\|_{H^{l+1}(M;E)} \leq c \left( \|Pu\|_{H^{l-1}(M;E)} + \|u\|_{H^l(M;E)} + \|u\|_{H^{l+1/2}(\partial M;E)} \right).$$

*Proof.* Corollaries 5.9 and 5.10 show that the assumptions of Theorem 5.6 are satisfied. That theorem immediately gives our result. □

An alternative proof of this result is obtained using the methods of Section 7. The advantage of the method used in this section is that it applies right away to higher order equations.

### 6. A Uniform Shapiro-Lopatinski Regularity Condition

The case of Neumann boundary conditions seems to be different (at least for systems, see [3, 4, 61, 62]). This case, as well as that of Robin boundary conditions motivates, in part, the results of this section. In particular, we introduce a uniform version of the Shapiro-Lopatinski condition [2, 3, 4, 22, 36, 43, 50, 61], which turns out to characterize the operators satisfying regularity. Our approach has several points in common to the ones in [41, 55]. To deal with the concrete case of Robin (and Neumann) boundary conditions, we use positivity to check that the uniform Shapiro-Lopatinski regularity conditions are satisfied.

#### 6.1. Homogeneous Sobolev spaces and regularity conditions.

We need first the following homogeneous (with respect to dilations) versions of the usual Sobolev spaces. This setting was used for similar purposes in [55]. Let $\hat{u}$ be the Fourier transform of $u$, regarded as a tempered distribution (the normalizations are not important here). Consider the semi-norm

$$|u|^2_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |\xi|^2 |\hat{u}(\xi)|^2 d\xi,$$

and $\dot{H}^s(\mathbb{R}^n) := \{u \mid |u|_{H^s(\mathbb{R}^n)} < \infty \}$. Here $u$ is such that $|\xi|^s |\hat{u}(\xi)|$ is a function (but $|\hat{u}(\xi)|$ is not assumed to be a function, which allows us to include polynomials of low degree in $\dot{H}^s(\mathbb{R}^n)$). When $s \in \mathbb{Z}_+$, the seminorm $|u|_{H^s(\mathbb{R}^n)}$ is equivalent to the (usual) $\sum_{|\alpha| = s} \|\partial^\alpha u\|_{L^2(\mathbb{R}^n)}$, which allows us to define in this case also $|u|_{H^s(\mathbb{R}^n)}$ for a function $u$ defined only on a half-space of the form $\mathbb{R}^{n-1} \times [0, \infty)$. The reason for considering the semi-norms $|\cdot|_{H^s}$ and the spaces $\dot{H}^s(\mathbb{R}^n)$ is that they have good dilation properties (see the next lemma). These definitions extend right away to functions with values in $\mathbb{C}^N$ yielding the spaces $\dot{H}^s(\mathbb{R}^n; \mathbb{C}^N) = \dot{H}^{s}((\mathbb{R}^n)^N)$.

**Lemma 6.1.** Let $s > 0$.

(i) Let $\alpha_s(f)(s) = f(ts)$, then $|\alpha_s(f)|_{H^s} = t^{s-n/2} |f|_{H^s}$.

(ii) $\lim_{t \to \infty} t^{-s+n/2} \|\alpha_s(f)\|_{H^s} = |f|_{H^s}$. 


(iii) If $T$ is an order $k$, homogeneous, constant coefficient differential operator, then it defines continuous maps $T: \mathring{H}^s(\mathbb{R}^n) \to \mathring{H}^{s-k}(\mathbb{R}^n)$.

(iv) If $s \in \mathbb{N}$, the restrictions $\mathring{H}^s(\mathbb{R}^n) \to \mathring{H}^s(\mathbb{R}^n_+)$ and $\mathring{H}^s(\mathbb{R}^n) \to \mathring{H}^{s-1/2}(\mathbb{R}^{n-1})$ are continuous and surjective and, if also $T$ is as in (iii) and has order $k \leq s$, $T: \mathring{H}^s(\mathbb{R}^n_+) \to \mathring{H}^{s-k}(\mathbb{R}^n_+)$ is continuous.

Proof. The proof is standard [27, 37, 41, 61] and easy. We include a few details for the benefit of the reader.

(i), (ii), and (iii) are direct calculations based on the definition and the formulas for the norms in terms of the Fourier transform.

(iv) is slightly less trivial, but it is known in the classical case and our case can either be proved directly following the same method as in the case of the classical Sobolev spaces or it can be reduced to the classical Sobolev spaces using (ii). Indeed, for the continuity, this is immediate, as the restriction maps commute with $\alpha_1$. For the surjectivity, one has to argue also that there exist right inverses for the restriction that are $\alpha_1$-invariant. This is done by choosing a partition of unity that is invariant with respect to $\alpha_2$ and then choosing a right inverse to the restriction on one coordinate patch that is continuous for the classical norms. We replicate this right inverse for all patches using $\alpha_2$. This yields a continuous right inverse $H^{s-1/2}(\mathbb{R}^{n-1/2}) \to H^s(\mathbb{R}^n)$ for the restriction $H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1/2})$ that is, moreover, invariant for $\alpha_2$. We obtain a fully $\mathbb{R}^*_+$-invariant inverse by integrating over the compact group $\mathbb{R}^*_+/2\mathbb{Z}$. This right inverse will work also for the homogeneous spaces.

Let $D$ be an $N \times N$ matrix of second order differential operators on $B^m_r(0) \times [0, r)$, and let $C$ be a boundary differential operator of order $j$. Assume the coefficients have $W^{l, \infty}$ smoothness, that is, that $(D, C) \in \mathcal{D}^{l,j}(B^m_r(0) \times [0, r); \mathbb{C}^N)$. Here $r > 0$, and the case $r = \infty$ is not excluded.

Definition 6.2. Let $\ell \geq j + 1$. We shall say that $(D, C) \in \mathcal{D}^{l,j}(B^m_r(0) \times [0, r); \mathbb{C}^N)$ satisfies an $H^{\ell+1}$-regularity estimate on $B_r^n(0) \times [0, r)$ if there exists $c_{SL}$ such that for all $\ell \geq k \geq j + 1$, we have

$$|w|_{H^{k-1}(\mathbb{R}^m)} \leq c_{SL} \left( |Dw|_{H^{k-1}(\mathbb{R}^m)} + |Cw|_{H^{k-j+1/2}(\mathbb{R}^m)} \right),$$

for all $w$ smooth with compact support in $B^m_r(0) \times [0, r)$, where we have removed the vector bundle from the notation for the norms. If $(D, C)$ are obtained from a variational formulation, then we allow also $\ell = k = j + 1$ and in that case we assume that $Dw = j_0(f, h)$ and we replace the right hand side with $|f|_{L^2(\mathbb{R}^m)} + |h|_{H^{1/2}(\mathbb{R}^m)}$. This case is, in fact, crucial in applications, since it is the one obtained using coercivity to prove well-posedness.

This definition is very similar to Definition 4.11, except that we consider semi-norms instead of norms. Also, we only require $w$ to be smooth. However, an operator satisfying the conditions in Definition 6.2, will satisfy also those of Definition 4.11. We formulate this result as a lemma, for further use.

Lemma 6.3. Assume $(D, C) \in \mathcal{D}^{l,j}(B^m_r(0) \times [0, r); \mathbb{C}^N)$ satisfies an $H^{\ell+1}(B^m_r(0) \times [0, r)^N$-regularity estimate, then it satisfies an $H^{\ell+1}$-regularity estimate for all $\ell \geq$
Thus \( C = \text{matrix valued functions} \). Recall that given by \((D, C)\) coefficients frozen at \( j = 0 \), \( j + 1 \) if \( C \) is a variational boundary condition. This result extends to uniform conditions.

**Proof.** Denote \( M = B^n(0) \times [0, r) \) for the simplicity of the notation. Assume first that \( k \geq 2 \). Let \( w \in \Gamma(M; E) \) be smooth with compact support in \( M \) such that \( Dw \in H^{k-1}(M); E \) and \( Cw \in H^{k+j+1/2}(\partial M; E) \). We need to show that \( w \in H^{k+1}(M; E) \). We have

\[
\|w\|_{H^{k+1}(M; E)} \leq \|w\|_{H^{k+1}(M; E)} + \|w\|_{H^{k}(M; E)}
\]

\[
\leq c_{SL}(\|Dw\|_{H^{k-1}} + \|Cw\|_{H^{k-j+1/2}(\partial M; E)}) + \|w\|_{H^{k}(M; E)}
\]

\[
\leq c(\|Dw\|_{H^{k-1}(M; E)} + \|w\|_{H^{k}(M; E)} + \|Cw\|_{H^{k-j+1/2}(\partial M; E)}).
\]

This result is extended to \( w \in H^{k}(M; E) \) by a continuity and density argument using mollifying functions. The proof for \( k = j = 1 \) and variational boundary conditions is similar. \( \square \)

**Remark 6.4.** Let us assume that \( r = \infty \) and consider the map

\[
(D, C): \dot{H}^{k+1}(\mathbb{R}^{m+1}_+) \rightarrow \dot{H}^{k-1}(\mathbb{R}^{m+1}_+) \oplus \dot{H}^{k-j+1/2}(\mathbb{R}^{m})^N,
\]

(22) given by \((D, C)(u) = (Du, Cu)\). Then \((D, C)\) satisfies an \( \dot{H}^{\ell+1}(\mathbb{R}^{m+1}_+) \)-regularity estimate if, and only if, \((D, C)\) is injective with closed range for all \( j + 1 \leq k \leq \ell \). Equivalently, we have that \((D, C)\) is injective and its minimal reduced module \( \gamma(D, C) > 0 \), see Subsection 3.1. (In this case, the number \( \gamma(D, C) \) is the least \( c_{SL} \) in Definition 6.2.) In case \( D \) and \( C \) have constant coefficients, this is similar to the Shapiro-Lopatinski condition.

### 6.2. A global Shapiro-Lopatinski regularity condition

Let us assume first that we are on a Euclidean space and that \( D = \sum_{|\alpha| \leq 2} a_{\alpha} \partial^{\alpha} \), with \( a_{\alpha} \) smooth, matrix valued functions. Recall that \( j \) is the order of the boundary conditions and that we assume for simplicity, that only one of the vector bundles \( F_0 \) and \( F_1 \) (from \( E|_{\partial M} = F_0 \oplus F_1 \)) is non-zero. Thus, according to our notational convention, if \( j = 0 \), \( C = C_0 \) is a smooth, matrix valued function on \( B^n(0) \) and, if \( j = 1 \), \( C = C_1 = C_{10} + C_{11} \partial_n \), with \( C_{10} = \sum_{|\alpha| \leq 1} c_{\alpha} \partial^{\alpha} \) a first order differential operator on \( B^n(0) \) and \( C_{11} \) a smooth, matrix valued function on \( B^n(0) \). Here \( \partial_n = -\partial_n \) is the outward pointing normal derivative, where \( \partial_n \) is the partial derivative with respect to the last variable. We denote by \((D^{(0)}, C^{(0)})\) the principal part of \((D, C)\) with coefficients frozen at 0, that is,

\[
D^{(0)} := \sum_{|\alpha| = 2} a_{\alpha}(0) \partial^{\alpha}
\]

(23)

\[
C^{(0)}_0 := C_0(0), \quad \text{if } j = 0,
\]

(24)

\[
C^{(0)}_1 := \sum_{|\alpha| = 1} c_{\alpha}(0) \partial^{\alpha} + C_0(0) \partial_n, \quad \text{if } j = 1.
\]

(25)

Thus \((D^{(0)}, C^{(0)})\) is a homogeneous, constant coefficient boundary value problem on \( \mathbb{R}^{m+1}_+ \). In case both \( F_0 \) and \( F_1 \) are non-zero, we let \( C^{(0)}(0, C^{(0)}_1) \).

Motivated by Remark 6.4, we introduce the following definition.
**Definition 6.5.** We shall say that \((D, C) \in D^\ell (B^n_0(0) \times [0, r); \mathbb{C}^N)\) satisfies the \(H^{\ell+1}\)-Shapiro-Lopatinski regularity condition at 0 if \((D^{(0)}, C^{(0)})\) satisfies an \(H^{\ell+1}(\mathbb{R}_{+}^{n+1})^N\)-regularity estimate.

Let us turn now to the case of a manifold with boundary. The above definition generalizes for \((D, C) \in D^\ell (M; E)\) and any point \(x \in \partial M\).

**Notations 6.6.** Let \(T^+_x M\) be the half-space of \(T_x M\) corresponding to the inward pointing vectors at \(x \in \partial M\). Let \((D_x, C_x)\) be the induced operator (defined only a neighborhood of 0) on \(T^+_x M\). The principal part \((D^{(0)}_x, C^{(0)}_x)\) of \((D_x, C_x)\) with coefficients frozen at \(x\) will then be a matrix of constant coefficient differential operators on \(T^+_x M\).

**Definition 6.7.** We shall say that \((D, C) \in D^\ell (M; E)\) satisfies the \(H^{\ell+1}\)-Shapiro-Lopatinski regularity condition at \(x \in \partial M\) if \((D^{(0)}_x, C^{(0)}_x)\) satisfies the \(H^{\ell+1}\)-Shapiro-Lopatinski regularity condition at 0 on \(T^+_x M\).

The above condition is closely related to the condition of “regularity upon freezing the coefficients” introduced in [61, Equation (11.30)] and used, for instance, in [45]. We are ready now to globalize the Shapiro-Lopatinski regularity condition.

**Definition 6.8.** We shall say that \((D, C) \in D^\ell (M; E)\) satisfies a uniform \(H^{\ell+1}\)-Shapiro-Lopatinski regularity condition (at \(\partial M\)) if it satisfies the \(H^{\ell+1}\)-Shapiro-Lopatinski regularity condition at \(x\) for each \(x \in \partial M\) and the constant \(c_{SL}\) of Definition 6.2 can be chosen independently of \(x \in \partial M\).

We thus see that \((D, C) \in D^\ell (M; E)\) satisfies the uniform, \(H^{\ell+1}\)-Shapiro-Lopatinski regularity condition (at \(\partial M\)) if there exists \(c_{SL} > 0\) such that, for all \(x \in \partial M\) and all \(1 \leq k \leq \ell\), we have

\[
|w|_{H^{\ell+1}((T^+_x M)} \leq c_{SL} \left( |D^{(0)}_x w|_{H^{\ell+1-1/2}(T^+_x M)} + |C^{(0)}_x w|_{H^{\ell+1-1/2}(T^+_x M)} \right). \tag{26}
\]

We now apply these notions to a manifold \(M\) with boundary and bounded geometry. For any \(x \in M\), we denote by \((D_x, C_x)\) (or simply by \(D_x\), if \(x \notin \partial M\)) the operators (operator) on \(B^n_0(0) \times [0, r)\) (on \(B^n_0(0)\)) induced by \((D, C)\) in Fermi coordinates around \(x\). We let \(\mathcal{F}_x = \{(D_x, C_x)\} \times D^\ell (\partial M)\) and \(\mathcal{F}_x = \{D_x\} \times \partial M\) if \(x \in \partial M\).

**Theorem 6.9.** Assume that \(M\) is a manifold with boundary and bounded geometry and that \(E \rightarrow M\) is a vector bundle with bounded geometry. Let \((D, C) \in D^\ell (M; E)\). The following are equivalent.

(i) \((D, C)\) satisfies an \(H^{\ell+1}\)-regularity estimate on \(M\).

(ii) The family \(\mathcal{F}_x \cup \mathcal{F}_x = \{(D_x, C_x)\} \times \partial M\) and \(\{D_x\} \times \partial M\) satisfies a uniform \(H^{\ell+1}\)-regularity estimate.

(iii) \(D\) is uniformly elliptic on \(M\) and \((D, C)\) satisfies a uniform \(H^{\ell+1}\)-Shapiro-Lopatinski regularity condition (at \(\partial M\)).

(iv) \(D\) is uniformly elliptic and \((D, C)\) satisfies a uniform \(H^2\)-Shapiro-Lopatinski regularity condition (at \(\partial M\)).

If \((D, C)\) are obtained from a variational formulation, then the above conditions are equivalent also to

(v) \(D\) is uniformly elliptic and \((D, C)\) satisfies a uniform, \(H^1\)-Shapiro-Lopatinski regularity condition (at \(\partial M\)).
Note that usually we assume \((D, C) \in \mathcal{D}^{l,j}(M; E)\), whereas here we assume \((D, C) \in \mathcal{D}^{l+1,j}(M; E)\). This is needed for the proof of \((iii) \Rightarrow (i)\).

**Proof.** The implication \((i) \Rightarrow (ii)\) follows from the definitions by localization. (This is the converse of Theorem 5.6.) To obtain \((ii) \Rightarrow (iii)\) for each \(x \in \partial M\), we consider \(\xi\) with compact support on \(T^+_x M\) and \(\xi(v) = \xi(e^{-1}v)\). Using a chart around \(x\) with Fermi coordinates as in Subsection 2.3 and using the corresponding bounds of the transition function, the uniform \(H^{l+1}\)-regularity estimate for each \(\epsilon > 0\) from Definition 4.11 implies
\[
\epsilon^{k+1-n/2} \|\xi\|_{H^{k+1}(T^+_x M; E)} \leq \epsilon^{k+1-n/2} \bar{c}_R \left( \|\xi\|_{H^k(T^+_x M; E)} + \|D_x \xi\|_{H^{k+1}(T^+_x M; E)} + \|C_x \xi\|_{H^{k+1+1/2}(\partial T^+_x M; E)} \right)
\]
for some \(\bar{c}_R\) independent of \(x\). Passing to the limit as \(\epsilon \to 0\) and using Lemma 6.1(ii) we obtain
\[
|\xi|_{H^{k+1}(T^+_x M; E)} \leq \bar{c}_R \left( \|D_x^{(0)} \xi\|_{H^{k-1}(T^+_x M; E)} + \|C_x^{(0)} \xi\|_{H^{k-1+1/2}(\partial T^+_x M; E)} \right).
\]
This gives right away that \((D, C)\) satisfies a uniform, \(H^{l+1}\)-Shapiro-Lopatinski regularity condition at \(\partial M\). The same argument combined with a Fourier transform at an arbitrary interior point (and \(0 < r < r_{FC}\) arbitrary) and a perturbation argument \([36, 61]\) gives that \(D\) is uniformly elliptic on \(M \setminus \partial M\). Hence \(D\) is uniformly elliptic on \(M\).

The implications \((iii) \Rightarrow (iv) \Rightarrow (v)\) are trivial. We have that, in fact, they are equivalent. This is seen in the same way in which one proves higher regularity for boundary value problems using divided differences. See any textbook, in particular \([27, 37, 41, 61]\).

To complete the proof, it is enough then to show that \((iii) \Rightarrow (i)\). We want to estimate \(\|u\|_{H^{l+1}}\) in terms of \(\|Du\|_{H^{k-1}}, \|Cu\|_{H^{k-1+1/2}}\) and \(\|u\|_{H^k}\). This is done using an \(r\)-partition of unity \(\phi_r\) as in Definition 2.10 and following almost word for word the proof of Theorem 5.6, but choosing \(r > 0\) small enough. Let us use the notation of the proof of that theorem. Then the only difference in the estimate of the proof is that we need to replace \(P_\gamma\) and \(C_\gamma\) with their principal parts \(P^{(0)}_\gamma\) and \(C^{(0)}_\gamma\). We need to choose \(r > 0\) and with coefficients frozen at \(p_\gamma\). We note that the family \((P^{(0)}_\gamma, C^{(0)}_\gamma)\) satisfies a uniform \(H^{l+1}\)-regularity estimate, in view of Lemma 6.3. The lower order terms can be absorbed into the weaker norm \(\|u\|_{H^k}\). We then use the fact that \(\|\sqrt{P} u\|_{H^{k-1}} \leq C(r) \|u\|_{H^{l+1}}\), for \(u\) with support in the ball of radius \(r\) centered at \(\gamma\) and with \(C(r)\) independent of \(\gamma\) and \(C(r) \to 0\) as \(r \to 0\). We have \(C(r) \to 0\) when \(r \to 0\) since \(P\) has coefficients in \(W^{l+1, \infty}\). To obtain regularity estimates in the interior (away from the boundary), we use the uniform ellipticity of the operator. For more details, one can consult also Proposition 11.2 of \([61]\), which is a similar result with a similar proof.

In particular, the sequence \((ii) \Rightarrow (iii) \Rightarrow (i)\) gives a new proof of Theorem 5.6. We obtain the following consequence.

**Corollary 6.10.** Let \((D, C) \in \mathcal{D}^{l,j}(M; E)\). If \((D, C)\) satisfies an order \(H^2\)-regularity estimate on \(M\), then \((D, C)\) satisfies an \(H^{l+1}\)-regularity estimate on \(M\).

It would be interesting to investigate the relation between the results of this paper and those of Karsten Bohlen \([17, 18]\).
6.3. A uniform Agmon condition. In view of the results on the uniform Shapiro-Lopatinski conditions and of the usefulness of positivity apparent in the next section, we now consider the coercivity of our operators, in the same spirit as the uniform Shapiro-Lopatinski conditions. The notation and the approach are very similar. We keep the notation of Section 4. In particular, $\hat{P}$ will be a second order differential operator in divergence form with associated boundary conditions $C = \{C_0, C_1\}$, as in Subsection 5.2. From now on, $M$ will be a manifold with boundary and bounded geometry.

Recall the following standard terminology.

Definition 6.11. A sesquilinear form $a$ on a hermitian vector bundle $V \rightarrow X$ is called strongly coercive (or strictly positive) if there is some $c > 0$ such that $\Re a(\xi, \xi) \geq c|\xi|^2$ for all $x \in X$ and $\xi \in V_x$. If the sesquilinear form $a$ on $T^*M \otimes E$ used to define $P$ is strongly coercive, then $P$ is said to satisfy the strong Legendre condition.

Let $V \subset H$ be a continuous inclusion of Hilbert spaces (with non-closed image, in general). Let $V^*$ be the complex conjugate of $V$ (with pairing $V \times V^* \rightarrow \mathbb{C}$ restricting to the scalar product of $H$ on $V \times H$. Recall that an operator $T: V \rightarrow V^*$ is coercive on $V$ if it satisfies the Gårding inequality, that is, if there exist $\gamma > 0$ and $R \in \mathbb{R}$ such that, for all $u \in V$,
\[ \Re \langle Tu, u \rangle \geq \gamma \|u\|^2_V - R\|u\|^2_H. \tag{27} \]

Then $T + \lambda$ is strongly coercive for $\Re(\lambda) > R$ (see Definition 6.11), and hence it satisfies the conditions of the Lax-Milgram lemma. Therefore, it satisfies regularity in view of the results of the next section. Coercive operators on bounded domains were characterized by Agmon in [1] as strongly elliptic operators satisfying suitable conditions at the boundary (which we shall call the “Agmon condition.”). We shall need a uniform version of this condition, to account for the non-compactness of the boundary.

Let now $\hat{P}^* \rightarrow V^*$ be as in Equation (8) (so it is associated to the sesquilinear form $B$ and has principal symbol $a$). Let $C$ be the boundary conditions associated to $\hat{P}$. That is, $C = (e_0x, e_1x_0, e_1x_0^a + Q)$. We let $e_0x$ and $e_1x$ be the values at $x$ of the endomorphisms $e_0$ and $e_1$. Similarly, let $Q_x^{(0)}$ be the principal part of $Q$ with coefficients frozen at 0. Let $P_x^{(0)}$ be the principal part of the operator $P$ and $C_x^{(0)} = (e_0x, e_1x_0^a + Q_x^{(0)})$ be the principal part of the boundary conditions $C$ with coefficients frozen at some $x \in \partial M$, as in Notation 6.6. Let $B_x^{(0)}$ be the associated Dirichlet bilinear form to $P_x^{(0)}$ equipped with the above boundary conditions (again with coefficients frozen at $x$), that is
\[ B_x^{(0)}(u, v) := \int_{T_x^+} a_x^{(0)}(du, dv) + \int_{T_x, \partial M} (Q_x^{(0)}u, v). \tag{28} \]

This defines a continuous sesquilinear form on
\[ V_x := \{u \in H^1(T_x^+M; E_x) \mid u|_{T_x, \partial M} \in (F_1)x\}. \]

The associated operator in divergence form $\hat{P}_x^{(0)}$ will be called the principal part of $\hat{P}$ with coefficients frozen at $x$. 

Definition 6.12. We say that \( \tilde{P} \) satisfies the uniform Agmon condition (on \( \partial M \)) if it is uniformly strongly elliptic and if there exists \( C > 0 \) such
\[
\langle \tilde{P}^{(0)}(u), u \rangle = B_x^{(0)}(u, u) \geq C|u|^2_{H^1},
\]
for all \( x \in \partial M \) and all \( u \in C^\infty_c(T_x^+M) \) with \( \epsilon_{0x}u = 0 \) on the boundary of \( T_x^+M \).

We have then the following result that is proved, \textit{mutatis mutandis}, as the regularity result of Theorem 6.9.

Theorem 6.13. Let \( M \) be a manifold with boundary and bounded geometry, \( E \to M \) be a vector bundle with bounded geometry, and \( \tilde{P} \) be a second order differential operator in divergence form, as above. We assume \( \tilde{P} \) has coefficients in \( W^{1,\infty} \). We have that \( \tilde{P} \) is coercive on \( V := \{ u \in H^1(M; E) \mid \epsilon_{0x}u = 0 \text{ on } \partial M \} \) if, and only if, \( \tilde{P} \) is uniformly strongly elliptic and it satisfies the uniform Agmon condition on \( \partial M \).

The proof is essentially the same as that of Theorem 6.9, more precisely, of the equivalence (i) \( \Leftrightarrow \) (iii). We need at least \( W^{1,\infty} \) to make the partition of unity argument work.

Remark 6.14. As is well-known, coercivity estimates lead to solutions of evolution equations \([7, 8, 41, 54]\). Let \( H^1_0(M; E) \subset V \subset H^1(M; E) \) be the space defining our variational boundary value problem, see 4.1, Equation (12), and Remark 4.13. Let \( V^* \) be the complex conjugate dual of \( V \), as before, \( \mathcal{W} := L^2(0,T; V), \ T > 0, \) so that \( \mathcal{W}^* = L^2(0,T; V^*) \). Assume \( P : V \to V^* \) is coercive (i.e. it satisfies Equation (27)). Then Theorem 4.1 of [41, Section 3.4.4] states that, for any \( f \in \mathcal{W}^* \), there exists a unique \( w \in \mathcal{W} \cap C([0,T], L^2(M; E)) \) such that \( \partial_t w(t) - Pw(t) = f(t) \) and \( w(0) = 0 \). Moreover, we also have \( w \in H^1(0,T; V^*) \).

This leads to the following result.

Theorem 6.15. Let us assume that \( \tilde{P} \) is as in Remark 6.14, whose notation we continue to use, and that \( \tilde{P} \) satisfies the uniform Agmon condition. Then, for any \( f \in \mathcal{W}^* \), there is a unique \( w \in \mathcal{W} \cap C([0,T], L^2(M; E)) \) such that \( \partial_t w(t) - Pw(t) = f(t) \) and \( w(0) = 0 \). Moreover, we also have \( w \in H^1(0,T; V^*) \).

For manifolds with bounded geometry (no boundary), this result was proved in [44]. The result in [44] was generalized to higher order equations in [8]. For the particular case mixed (Dirichlet/Neumann) boundary conditions and scalar equations, this result was proved in [7].

7. Coercivity, Legendre condition, and regularity

In this section we use coercivity (or positivity) to obtain regularity results. As an application, we study mixed Dirichlet/Robin boundary conditions for operators satisfying the strong Legendre condition. We continue to assume that \( M \) is a manifold with boundary and bounded geometry and that \( E \to M \) is a vector bundle with bounded geometry.

7.1. Well-posedness in energy spaces implies regularity. Often the regularity conditions (including the Shapiro-Lopatinski ones discussed below) are obtained from the invertibility of the given operator. This is the case also in the bounded geometry setting, owing to Proposition 3.2. The results of this subsection will be used to deal with the Neumann and Robin boundary conditions. The approach
below is based on the so called “Nirenberg trick” (see also [45] and the references therein).

**Lemma 7.1.** Let \( X_j \) be as in Lemma 3.1 and \( Y = X_j \), for some \( j > 1 \) fixed.

(i) There exists a one parameter group of diffeomorphisms \( \phi_t : M \to M \), \( t \in \mathbb{R} \), that integrates \( Y \), that is \( \frac{d}{dt} f(\phi_t(x))|_{t=0} = (Y f)(x) \), for any \( x \in M \) and any smooth function \( f : M \to \mathbb{C} \).

(ii) Let \( \pi : E \to M \) be a vector bundle. Then there exists a one parameter group of diffeomorphisms \( \psi_t : E \to E \), \( t \in \mathbb{R} \), with \( \pi \circ \psi_t = \phi_t \circ \pi \), that integrates \( \nabla_Y \), that is \( \frac{d}{dt}(\psi_t \circ \xi)|_{t=0} = \nabla_Y \xi \), for any smooth section \( \xi : M \to E \).

**Proof.** (i) Let \( \hat{M} \) be as in Definition (2.5). Let \( \hat{Y} \in W^{\infty,\infty}(\hat{M}, TM) \) be an extension of \( Y \). Then \( \hat{Y} \) is a bounded vector field on a complete manifold. Hence, by [5, Sec. 3.9] it admits a global flow, i.e. a smooth solution \( \phi : \mathbb{R} \times \hat{M} \to \hat{M} \) of

\[
\frac{d}{dt} \phi(t, p) = \hat{Y}(p), \quad \phi(0, p) = p
\]

such that \( \phi_t := \phi(t, .) \) is a one-parameter family of diffeomorphisms of \( \hat{M} \). Since \( \hat{Y} \) is tangent to \( \partial \hat{M} \subset \hat{M} \), \( \phi(t, \partial \hat{M}) = \partial \hat{M} \). Since \( \partial \hat{M} \) divides \( \hat{M} \) into two parts, \( \phi_t \) restricts to diffeomorphisms of \( \hat{M} \).

(ii) We choose a connection \( \nabla^E \) on \( E \). Let \( \psi_t : E \to E \) be defined by \( e \mapsto e(t) \) where \( e(t) \) is the solution of \( \nabla^E \nabla^E_{\psi_t}(p) = \phi_t(p) e(t) = 0 \) with \( e(0) = e \). By the standard properties of parallel transport, respectively of the underlying linear ordinary differential equation, we have the global existence and uniqueness of the solution and the claimed properties. \( \square \)

We then obtain the following abstract regularity result. Let \( X_j \) be as in Lemmas 3.1 and 7.1. We can assume that \( X_1 \) is a unit vector field normal at the boundary. Recall that \( V := H^1(M; E) \cap \{ u|_{\partial M} \in \Gamma(\partial M; F_1) \} \).

**Theorem 7.2.** Let \( \hat{P} \) be a second order differential operator in divergence form with associated form \( a \) and \( W^{\infty,\infty} \)-coefficients. Let \( \zeta := a(X_1, X_1) \) and let us assume that \( \zeta \) is invertible and \( \zeta^{-1} \) bounded. Also, let us assume that \( \hat{P} : V \to V^* \) is a continuous bijection (i.e., an isomorphism). Then \( P \) satisfies an \( H^{\ell+1} \)-regularity estimate on \( M \).

**Proof.** The proof is classic, except maybe the fact that we have a slightly weaker assumption on the coefficients; typically one requires \( C^{k+1} \)-coefficients in textbooks. We include, nevertheless, a very brief outline of the proof. Recall that the proof is done by induction, with the general step the same as the first step (going from well-posedness to \( H^2 \)-regularity). Let us assume then that \( \ell = 1 \).

Let \( F \in j_0(L^2(M; E) \oplus H^{1/2}(M; F_1)) \) and let \( \hat{P}u = F \), with \( u \in V \). For simplicity, let us assume \( E \) is one-dimensional. In general, we just replace \( X_j \) with \( \nabla_{X_j} \). We want to show that \( u \in H^2(M) \), with continuous dependence on \( F \). To this end, in view of Proposition 3.2, it is enough to check that \( X_j X_j u \in L^2(M) \), with continuous dependence on \( F \), since we already know that \( u \in V \subset H^1(M) \), with continuous dependence on \( F \).

In particular, \( X_j u \in L^2(M) \). Nirenberg’s trick is to give conditions on \( X_j \) such that we can formally apply \( X_j \) to the equation \( \hat{P}u = F \) to obtain that \( X_j u \in V \) is the unique solution of

\[
\hat{P}(X_j u) = [\hat{P}, X_j](u) + X_j(F) \in V^*.
\]
This is possible whenever \([\hat{P}, X_j] : V \to V^*\) continuously and \(X_j\) generates a continuous parameter semi-group on \(V\) (see [13] for a general approach and more details). These conditions are satisfied since \(\hat{P}\) has coefficients in \(W^{1,\infty}\) and if \(j > 1\), since then \(X_j\) is tangent to the boundary and we can invoke Lemma 7.1. This argument gives that \(X_1X_ju \in L^2(M)\) if at least one of the \(i\) and \(j\) is \(> 1\).

It remains to prove that \(X_1^2u \in L^2(M)\). This is proved using the equation

\[
\zeta X_1^2 u = Pu - \sum_{i+j>2} c_{ij} X_i X_j u \in L^2(M),
\]

since we can choose \(c_{ij} \in L^\infty(M; End(E))\) and \(Pu \in L^2(M; E)\), by assumption. (Recall that we assumed that \(X_1\) is a unit vector everywhere on the boundary.)

**Remark 7.3.** Except the above theorem, it is very likely that most of the results obtained so far extend to higher order equations, but we have not checked all the details. The above theorem will require, however, some additional ideas in order to extend it to the setting of Remark 4.13.

### 7.2. Robin vs Shapiro-Lopatinski

Let us discuss, as an example, Robin (and hence also Neumann) boundary conditions from the perspective of the Shapiro-Lopatinski conditions. We do that now in the case of a model problem on the half-space \(\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times [0, \infty)\). The sesquilinear form \(a\) of Section 4 (see Assumption 4.1) is now simply a sesquilinear form on \(C^\infty SET = \mathbb{R}^n \otimes \mathbb{C}^N\). We allow now \(F_0\) and \(F_1\) to be both non-trivial, which amounts to a decomposition \(N = N_0 + N_1\), \(n_j \geq 0\). We shall need also the operator \(Q\), which is now a \(N_1 \times N_1\) matrix of constant coefficients differential operators on \(\mathbb{R}^{n-1}\) acting on the last \(N_1\) components of \(\mathbb{C}^N\). The bilinear form that we consider is then

\[
B(u, v) := \int_{\mathbb{R}^n_+} \left[ a(du, dv) + c(u, v) \right] dx + \int_{\mathbb{R}^{n-1}} (Qu, v) dx',
\]

where \(c\) is a scalar and \(x = (x', x_n)\). This is a particular case of the form considered in Equation (7). Recall the definition of a strongly coercive form, Definition 6.11. Note that strong coercivity implies uniform strong ellipticity as in Definition 4.9.

**Lemma 7.4.** Let us assume that \(Q + Q^*\) is of order zero and that \(a\) is strongly coercive. Then for \(c\) (of Equation 30) large enough, there is a \(\gamma > 0\) such that

\[
B(u, u) \geq \gamma ||u||_{H^1(\mathbb{R}^n_+)}^2
\]

for all \(u \in H^1(\mathbb{R}^n_+)\).

**Proof.** We identify \(\mathbb{R}^{n-1}\) with the boundary of the half-space \(\mathbb{R}^n_+\), as usual. Let \(c_Q > 0\) be a bound for the norm of the matrix \(\frac{1}{2}(Q + Q^*)\). We have, using first the definition

\[
B(u, u) := \int_{\mathbb{R}^n_+} \left[ a(du, du) + c(u, u) \right] dx + \int_{\mathbb{R}^{n-1}} (Qu, u) dx'
\]

\[
\geq c_a ||u||_{H^1(\mathbb{R}^n_+)}^2 + c ||u||_{L^2(\mathbb{R}^n_+)}^2 - c_Q ||u||_{L^2(\mathbb{R}^{n-1})}
\]

\[
\geq c_a \frac{1}{2} ||u||_{H^1(\mathbb{R}^n_+)}^2
\]

for \(c\) large. The last statement is proved in the same way one proves the trace inequality \(||u||_{L^2(\mathbb{R}^{n-1})} \leq cr ||u||_{H^1(\mathbb{R}^n_+)}\), but using also Lebesque’s dominated convergence theorem. \(\square\)
We obtain the following consequence.

**Corollary 7.5.** Let $a$ and $Q$ be as in Lemma 7.4 and let $\tilde{P}$ be the differential operator in divergence form associated to the form $B$ of Equation (30). Then $\tilde{P}$ (or, $(P, e_0, e_1 \partial_\nu + Q)$) satisfies an $H^{\ell+1}$-regularity estimate on $\mathbb{R}_+^n$ for all $\ell$. In particular, $(P(0), e_0, e_1 \partial_\nu + Q(0))$ satisfies the $H^{\ell+1}$-Shapiro-Lopatinski regularity condition at 0.

**Proof.** Let $V := \{u \in H^1(\mathbb{R}_+^n) | e_0 u = 0 \text{ on } \mathbb{R}_+^{n-1}\}$. We then have that the restriction of $B$ to $V \times V$ satisfies the assumptions of the Lax-Milgram lemma [32], and hence $\tilde{P} : V \rightarrow V^*$ is an isomorphism. Theorem 7.2 then gives the first part of the result. The last part follows from Theorem 6.9.

### 7.3. Mixed Dirichlet/Robin boundary conditions

Let us turn now back to the study of mixed Dirichlet/Robin boundary value problems on $M$. Let $M$ be a manifold with bounded geometry with a decomposition $E|_{\partial M} = F_0 \oplus F_1$ as the direct sum of two vector bundles with bounded geometry. We consider the same bilinear form $B$ as in Section 4, see Equation (7), with the data defining it as in Assumptions (4.1). In particular, $B$ is obtained from a form $a$ that can be interpreted as the principal symbol of the associated operators $P$ and $\tilde{P}$. If $a$ is strongly coercive, we say that $P$ (or $\tilde{P}$) satisfies the strong Legendre condition.

The associated boundary conditions are then

$$C_0 u = e_0 u|_{\partial M} \quad \text{and} \quad C_1 u = e_1 \partial_\nu u + Q u|_{\partial M} \quad (32)$$

for some first order differential operator $Q$ acting on sections of $F_1$. We let

$$||Cv||_k := ||C_0 v||_{H^{k+1/2}(\partial M; F_0)} + ||C_1 v||_{H^{k-1/2}(\partial M; F_1)}.$$  

We say that the boundary conditions $C$ are **mixed Dirichlet/Robin boundary conditions.** We shall need the following analog of Corollary 5.9.

**Corollary 7.6.** Let $S \subset D^{\ell+1,0}(B_r^m(0) \times [0, r); E)$ be a bounded family of boundary value problems on $B_r^m(0) \times [0, r) \subset \mathbb{R}^{m+1}$ equipped with the euclidean metric, $r \leq \infty$. We assume that the family $S$ satisfies a uniform strong Legendre condition and all the boundary conditions are $\text{are Robin boundary conditions of the form}$ $(e_0, e_1 \partial_\nu + Q,)$, with $Q + Q^*$ of order zero and bounded on $S$. Then the family $S$ satisfies a uniform $H^{\ell+1}$-regularity estimate on $B_r^m(0) \times [0, r')$, $r' < r$.

**Proof.** Let $(D_n, C_n) \in S$ converge to $(D, C) \in D^{\ell,0}(B_r^m(0) \times [0, r'); E)$, with $C_n = (e_0, e_1 \partial_\nu + Q_n)$, with $Q_n + Q^*_n$ of order zero. Then $D$ satisfies the strong Legendre condition because the parameter $c_n$ in the definition of the strong Legendre condition (the fact that $a$ is strongly coercive) is assumed to stay away from 0 on $S$. The limit of Robin boundary conditions is again a Robin boundary condition and the condition that $Q + Q^*$ be scalar is also preserved under limits. Corollary 7.5 then gives that $(D, C)$ satisfies an $H^{\ell+1}$-regularity estimate. This allows us again to use Proposition 5.3 for the relatively compact subset $N := B_{r'}^m(0) \times [0, r')$ of $M := B_r^m(0) \times [0, r)$ to obtain the result. □

We are ready now to prove the result stated in the Introduction, Theorem 1.1.

**Proof.** The proof of Theorem 1.1 is the same as that of Theorem 5.12. Indeed Corollaries 7.6 and 5.10 show that the assumptions of Theorem 5.6 are satisfied. That theorem immediately gives our result. □
Notice that in the statement of the theorem, we have dropped the condition that $B$ be of constant order (it is, nevertheless, of locally constant order).

Let us assume now that we have a partition of the boundary $\partial M = \partial_D M \cup \partial_R M$ as a disjoint union of two open and closed subsets and that $F_0 = E|_{\partial_D M}$ and $F_1 = E|_{\partial_R M}$. By combining Theorem 7.2 with the Poincaré inequality [9], we can prove following well-posedness result for the mixed Dirichlet/Robin boundary value problem. Let $A \subset \partial M$ (see [9] for details). Recall that we say that $(M, A)$ has finite width [10] if the distance to $A$ is bounded uniformly on $M$ and $A$ intersects all connected components of $M$.

**Theorem 7.7.** We use the same notation as in Theorem 1.1, in particular, $M$ is a manifold with boundary and bounded geometry and $P = (P, \partial^0_D u + Q)$ has coefficients in $W^{k, \infty}$ and satisfies the strong Legendre condition. Assume that $F_0 = E|_{\partial_D M}$, that $F_1 = E|_{\partial_R M}$, that $Q + Q^* \geq 0$, that there exists $\epsilon > 0$ and an open and closed subset $\partial PR M \subset \partial R M$ such that $Q + Q^* \geq \epsilon$ on $\partial PR M$, and, finally, that $(M, \partial D M \cup \partial PR M)$ has finite width. Then the boundary value problem

$$
\begin{cases}
P u = f \in H^{k-1}(M; E) & \text{in } M \\
u = h_D \in H^{k+1/2}(\partial_D M; E) & \text{on } \partial D M \\
\partial^0_D u + Q u = h_R \in H^{k-1/2}(\partial_R M; E) & \text{on } \partial R M,
\end{cases}
$$

has a unique solution $u \in H^{k+1}(M; E)$, $k \geq 0$, and this solution depends continuously on the data, $0 \leq k \leq \ell$.

Note that the more general form of the Robin boundary conditions considered in this paper (i.e. corresponding to a splitting of $E$ at the boundary, is useful for treating the Hodge-Laplacian. See also [47, 62]. See [61] also for results on the Robin problem on smooth domains. See [21, 31] for some results on the Robin problem on non-smooth domains.

**7.4. The bounded geometry of the boundary is needed.** Let us provide now an example of a manifold $\Omega$ with smooth metric and smooth boundary $\partial \Omega$ that satisfies the Poincaré inequality, and hence such that $\Delta: H^1_0(\Omega) \to H^{-1}(\Omega)$ is an isomorphism, but such that $\Delta: H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$ is not onto. In other words, the operator $D = \Delta$ with Dirichlet boundary condition does not satisfy an $H^2$-regularity estimate on $\Omega$, in spite of the fact that it has coefficients in $W^{N, \infty}$. Our example is based on the loss of regularity for problems on concave polygonal domains.

Let $G \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. It will be convenient to consider only closed domains. Let $m \in \mathbb{N}$ (we shall need only the case $m = 1$). We denote by $H^m(G)$ the space of functions with $m$ derivatives in $L^2$. It is the set of restrictions of functions from $H^m(\mathbb{R}^n)$ to $G$. We let $H^m_G(\mathbb{R}^n)$ be the set of distributions in $H^m(\mathbb{R}^n)$ with support in $G = \overline{G}$ and $H^m_0(G)$ be the closure of the set of test functions with support in $G$ in $H^m(\mathbb{R}^n)$, as usual. It is known that $H^1_G(\mathbb{R}^n) = H^1_0(G) = \ker(H^1(G) \to L^2(G))$. See [61] for the case $G$ smooth. The Lipschitz case is completely similar and follows from $H^1_0(G) = \ker(H^1(G) \to L^2(G))$, see, for example [48].

Our construction of the manifold with boundary $\Omega$ is based on the following lemma.
Lemma 7.8. Let $G$ and $\omega_1 \supset \omega_2 \supset \ldots \supset \omega_n \supset \ldots$ be closed, bounded domains in $\mathbb{R}^n$ with Lipschitz boundary such that $G := \cap_1^\infty \omega_n$. Let $f \in L^2(G)$, $u_n \in H^1_0(\omega_n)$, and $w \in H^2_0(G)$ satisfy $\Delta u_n = f$ on $\omega_n$ and $\Delta w = f$ on $G$. If the domains $\omega_n$ have smooth boundary and $w \notin H^2(G)$, then $\|u_n\|_{H^2(\omega_n)} \to \infty$.

Proof. We have that

$$H^1_0(\omega_1) \supset H^1_0(\omega_2) \supset \ldots \supset H^1_0(\omega_n) \ldots \supset H^1_0(G).$$

Let $\xi \in \cap_1^\infty H^1_0(\omega_n)$. Then $\xi$ has support in $G = \cap_1^\infty \omega_n$, and hence $\xi$ has support in $G$, which gives $\xi \in H^1_0(\mathbb{R}^n) = H^1_0(G)$, by the discussion preceding this lemma. This shows that $H^1(G) = \cap_1^\infty H^1_0(\omega_n)$. Let $B(u,v) := \int_{\mathbb{R}^n} (\nabla u, \nabla v) \, d\text{vol}$. Then $B$ induces an inner product on $H^1_0(\omega_1)$ equivalent to the initial inner product. Moreover, the relations

$$B(u_n, v) = (f, v) = B(u_n+1, v), \quad v \in H^1_0(\omega_{n+1})$$

show that $u_{n+1}$ is the $B$-orthogonal projection of $u_n$ onto $H^1_0(\omega_{n+1})$. Similarly, $w$ is the $B$-orthogonal projection of $u_n$ onto $H^1_0(G)$. Since $H^1(G) = \cap_1^\infty H^1_0(\omega_n)$, we obtain that $u_n \to w$ in $H^1_0(\omega_1)$.

To prove that $\|u_n\|_{H^2(\omega_n)} \to \infty$ if $w \notin H^2(G)$, we shall proceed by contradiction. Let us assume then that this is not the case. Then, by passing to a subsequence, we may assume that $\|u_n\|_{H^2(\omega_n)}$ is bounded. Then $\|u_n|_G\|_{H^2(G)} \leq \|u_n\|_{H^2(\omega_n)}$ also forms a bounded sequence. By passing to a subsequence again, we may assume then that $u_n|_G$ converges weakly in $H^2(G)$ to some $\tilde{w}$, by the Alaoglu-Bourbaki theorem. Hence $u_n|_G \to \tilde{w}$ weakly in $H^1(G)$ (even in norm, since $H^2(G) \to H^1(G)$ is compact). We have $\|u_n|_G - \tilde{w}\|_{H^1(G)} \leq \|u_n - w\|_{H^1(\omega_1)} \to 0$. Hence $u_n|_G \to w$ in $H^1(G)$. Consequently, $\tilde{w} = w$, which is a contradiction, since we have assumed that $w \notin H^2(G)$.

We are ready now to construct our manifold $\Omega$. Let $G$ be a bounded domain whose boundary is smooth, except at one point, where we have an angle $> \pi$ (so $G$ is not convex). It is known then that there exists $u \in H^1_0(G)$, $u \notin H^2(G)$ such that $f := \Delta u \in L^2(M)$. See [22, 34, 33, 50]. In fact, the space of functions $\phi \in L^2(W)$ such that $\Delta^{-1}\phi \in H^2(G) \cap H^1_0(G)$ is of codimension one in $L^2(G)$. Let $\ldots \subset \omega_{n+1} \subset \omega_n \subset \ldots \subset \omega_1$ be a sequence of closed, smooth, bounded domains whose intersection is $G$. Then we can take for $\Omega$ the (disjoint) union of all the domains $\omega_n \times \{n\}$, $n \in \mathbb{N}$.

Let us check that $\Omega$ has the desired properties. We construct $\phi \in L^2(\Omega)$ by taking $\phi := c_n f$ on $\omega_n$, $c_n > 0$. Let $u_n \in H^1_0(\omega_n)$ be the unique solution of $\Delta u_n = f$. We can choose $c_n$ such that $\sum_n c_n^2 < \infty$, and hence $\phi \in L^2(\Omega)$, but $\sum_n c_n^2 \|u_n\|_{H^2(\omega_n)}^2 = \infty$, since the sequence $\|u_n\|_{H^2(\omega_n)}$ is unbounded, by the last lemma. We have that $\Omega \subset \omega_1 \times \mathbb{N}$, and hence it satisfies the Poincaré inequality for all functions in $H^1_0(\Omega)$. Let $u \in H^1_0(\Omega)$ be the unique solution of the equation $\Delta u = \phi$ [10]. Then $u = c_n u_n$ on $\omega_n$, by the uniqueness of $u_n$, and hence $\|u\|_{H^2(\Omega)}^2 = \sum_1^\infty c_n^2 \|u_n\|_{H^2(\omega_n)}^2 = \infty$. That is, $u \notin H^2(\Omega)$. Note that $\Omega$ is not of bounded geometry: indeed, the second fundamental form of $\omega_n$ cannot be uniformly bounded in $n$, since the “limit” $G$ of the domains $\omega_n$ has a corner.

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