Thermodynamics of Non-Relativistic Scattering Theory

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Abstract

We use thermodynamic Bethe ansatz to study nonrelativistic scattering theory of low energy excitations of 1D Hubbard model, using the $S$-matrices proposed by Essler and Korepin. This model can be described by two types of excitation states, holons and spinons, as asymptotic states. In the attractive regime, the spinon is massive while the holon is massless. The situation is reversed with a repulsive coupling. We derive that the central charge of the Hubbard model in the IR limit is $c = 1$ while it vanishes in the UV limit. The contribution is due to the massless degree of freedom, i.e. the holon for the attractive regime, and the massive mode decouples completely. This result is consistent with various known results based on lattice Bethe ansatz computations. Our results make it possible to use the $S$-matrices of the excitations to compute more interesting quantities like correlation functions.

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There has been considerable interest in the interplay of integrable quantum field theory and statistical mechanics\cite{1}. In particular, a lot of progress in this relationship has been made in two-dimensional models. One of the most useful methods in these models is the factorizable $S$-matrix theory\cite{2}. In $1+1$-dimensional integrable field theories where infinite number of conservation laws exist, the $S$-matrices are factorizable into two-body elastic $S$-matrices, which satisfy the Yang-Baxter equation. With known particle spectrum and additional symmetries, one can determine the $S$-matrices exactly\cite{2} by solving the Yang-Baxter equation. Another method is to diagonalize the Hamiltonian using Bethe ansatz and find physical particle states and their $S$-matrices\cite{3}. In addition to their importance as physical amplitudes of scattering between asymptotic particle states, exact $S$-matrices can give other interesting quantities like the central charges of underlying conformal field theories (CFT’s) from the finite size effects, conformal dimensions of the operators, and even the correlation functions.

Our motivation is to establish the $S$-matrix approach to study the non-relativistic lattice models like the 1D Hubbard model\cite{5}. Although the Bethe ansatz method is quite useful in finding eigenvalue spectra, excitation states, and their thermodynamic properties, it is not so useful in finding other important quantities, in particular, correlation functions. Recently, Eßler and Korepin derived the $S$-matrices for low lying excitations of the 1D Hubbard model\cite{8}. What concerns us here, as a first step towards the complete $S$-matrix bootstrap of the low lying excitations of the Hubbard model, is how to confirm the validity of these matrices. For this purpose, we employ thermodynamic Bethe ansatz (TBA) for 2D QFT which is now a standard tool\cite{9} to check the $S$-matrices. In the original formulation of factorizable scattering theory, one is interested in the scattering of relativistic excitations. However, in interacting 1-D quantum systems, such as the Hubbard model, there are in general several low-energy excitations, with complicated dispersion relations and with different Fermi velocities. Also, the motions of low energy excitations are far from relativistic. Thus, what we should do is to extend this TBA method to non-relativistic theories to compute the central charges in
the IR and UV limits and compare with results of finite size effects[4].

Among the lattice models of the strongly correlated electron systems in low dimensions, it is believed that the two-dimensional Hubbard model provides some of the properties of high-$T_c$ superconductivity[4]. Furthermore, a strong quantum fluctuation in low dimensions suggests common features in 2D and 1D Hubbard models. Fortunately 1D Hubbard model can be exactly diagonalized via the Bethe-ansatz technique[5]. Its thermodynamic properties, such as the susceptibility, the magnetization, and the low-temperature specific heat, for the repulsive ($U > 0$) and the attractive ($U < 0$) on-site interactions have been studied in the literature[6]. It is noticeable that the excitation spectrum is described by the spin and charge excitations, i.e. spinons and holons, and the spin (charge) excitation possesses a gap in the attractive (repulsive) Hubbard model as long as the on-site interaction $U$ exists. The low-energy charge (spin) excitations for the attractive (repulsive) case are proportional to those for the antiferromagnetic Heisenberg chain irrespective of the strength of the interaction and of the electron filling. Woynarovich and Eckle[7] have analyzed the finite-size corrections in the Hubbard model for the repulsive and half-filled case, and their analysis yields the central charge of the Virasoro algebra $c = 1$ which is contributed by the spin excitations, while the contribution of the charge excitations is negligible only if the on-site repulsion $U$ is not so weak.

Let us now examine the dispersion relations of the excitations in the attractive case. Note that the results for the repulsive case can be obtained from the attractive case by interchanging the roles of holons and spinons. The holon energy in terms of rapidity $\lambda$ is given by

$$ E_c(\lambda) = 2 \int_0^\infty \frac{d\omega}{\omega} J_1(\omega) \cos(\omega \lambda) \cosh(\omega U), \quad (1) $$

whereas, momentum is given by

$$ P_c(\lambda) = -\int_0^\infty \frac{d\omega}{\omega} J_0(\omega) \sin(\omega \lambda) \cosh(\omega U), \quad -\frac{\pi}{2} \leq P_c(\lambda) \leq \frac{\pi}{2}. \quad (2) $$

In the above $U$ is the coupling constant of the on-site interaction of the 1D Hubbard
model. Notice that the holon is massless and \( E_c \sim v_c P_c \), as \( \lambda \to \infty \), where \( v_c \), the Fermi velocity, is given by \( v_c = E'_c(\infty)/P'_c(\infty) \).

Spinon has the following dispersion relations, for energy,

\[
E_s(k) = 2|U| - 2\cos k + 2\int_0^\infty \frac{d\omega}{\omega} \frac{J_1(\omega) \cos(\omega \sin k)}{\cosh(\omega U)} \exp(-|\omega U|),
\]

and momentum,

\[
P_s(k) = k - \int_0^\infty \frac{d\omega}{\omega} \frac{J_0(\omega) \sin(\omega \sin k)}{\cosh(\omega U)} \exp(-|\omega U|), \quad -\frac{\pi}{2} \leq P_s(k) \leq \frac{\pi}{2}.
\]

We see that the spinon stays massive for finite \( U \) for all values of the rapidity.

Recently, using these dispersion relations, Êßler and Korepin derived the scattering matrices of Hubbard model from the Bethe ansatz solution, by generalizing a method of extracting \( S \)-matrix from the asymptotics of the wave functions of the scattering state[8]. These \( S \)-matrices of the excitations on the lattice are well-defined as far as the wave packets of two excitations are well-separated. The complete scattering matrix is \( 16 \times 16 \) dimensional and is in a block diagonal form consisting of 4 blocks. Each of the blocks describe scatterings of holon-holon, spinon-holon, holon-spinon, and spinon-spinon, respectively, as follows:

\[
S = \begin{pmatrix}
S_{cc}(u) & 0 & 0 & 0 \\
0 & S_{sc}(w) & 0 & 0 \\
0 & 0 & S_{cs}(w) & 0 \\
0 & 0 & 0 & S_{ss}(v)
\end{pmatrix},
\]

where the holon-holon scattering amplitude is,

\[
S_{cc}(u) = -\frac{\Gamma \left( \frac{1+iu}{2} \right) \Gamma \left( \frac{1-iu}{2} \right)}{\Gamma \left( \frac{1-iu}{2} \right) \Gamma \left( \frac{1+iu}{2} \right)} \left( \frac{u}{u+i} I + \frac{i}{u+i} P \right), \quad u = \frac{\lambda - \lambda'}{2|U|},
\]
with $\Gamma$ being the gamma function. In the above $\textbf{I}$ and $\textbf{P}$ are the $4 \times 4$ identity and permutation matrices respectively. This $S$-matrix has the same form as that of the spin $\frac{1}{2}$ Heisenberg antiferromagnet and of the $SU(2)_1$ WZNW model\cite{[11]}. The spinon-spinon scattering amplitude is

$$S_{ss}(v) = \frac{\Gamma \left( \frac{1-iv}{2} \right) \Gamma \left( \frac{1+iv}{2} \right) \Gamma \left( \frac{1-iv}{2} \right) \Gamma \left( \frac{1+iv}{2} \right)}{\frac{v}{v-i} \textbf{I} - i \frac{v}{v-i} \textbf{P}}, \quad v = \frac{\sin k - \sin k'}{2|U|}, \quad (7)$$

and can be obtained from $S_{cc}$ by setting $u \rightarrow -v$. The spinon-holon scattering amplitude is,

$$S_{sc}(w) = -i \frac{1 + i \exp(\pi w)}{1 - i \exp(\pi w)} \textbf{I}, \quad w = \frac{\lambda - \sin k}{2|U|}. \quad (8)$$

We have the same form for holon-spinon scattering amplitude $S_{cs}$. Notice that $S_{sc}$ and $S_{cs}$ approach constant values as $w \rightarrow \infty$.

Let us now formulate the TBA for non-relativistic scattering. The TBA computes the Casimir energy of a theory on a circle of length $R$ with $S$-matrices and particle spectrum as input data\cite{[9]}. With a temperature $T = 1/R$ the configuration of minimizing free energy gives the ground state energy of the system, which is again related to the central charges of the underlying UV CFT by

$$E_{\text{ground}}(R) \sim -\sum_i \frac{\pi c_i}{6v_i R}, \quad (9)$$

where $v_i$ are Fermi velocities for excitations in the system and $c_i$ are the corresponding effective central charges.

Consider $N$ particles in a box of length $L$ with periodic boundary condition(PBC). Moving a $k$-th particle of type $a$ with energy $E_a(\theta_k)$ and momentum $P_a(\theta_k)$ all the way by exchanging with other particles and coming back to the original configuration using PBC, we get

$$e^{-iLP_a(\theta_k)} \prod_{i=1, i \neq k}^N S_{aa_i}(\theta_k - \theta_i) = 1, \quad (10)$$

where the index $a_i$ specifies species of the $i$-th particle. In general the product of $S$-matrices is a large size matrix called transfer matrix and one should diagonalize this
by some techniques. However, we consider a diagonal scattering theory first because the non-diagonal case can be understood as a slight modification. Taking logarithms on both sides of Eq.(10) gives

\[-LP_a(\theta_k) + \sum_{i=1}^{N} \frac{1}{i} \ln S_{aa_i}(\theta_k - \theta_i) = 2\pi n_k,\]

with an arbitrary integer \(n_k\). In the thermodynamic limit, \(N, L \to \infty\), one can express Eq.(11) in terms of density of states like

\[2\pi \rho_a(\theta) = -LP_a'(\theta) + \sum_b \int d\theta' \rho_b^1(\theta')\phi_{ab}(\theta - \theta'),\]

where \(\rho_a(\theta)\) and \(\rho_b^1(\theta)\) are the densities of allowed and occupied states, respectively, and \(\phi_{ab}\) is the logarithmic derivatives of \(S\)-matrices \(S_{ab}\). In terms of ‘pseudo-energies’ \(\epsilon_a\) defined by \(e^{-\epsilon_a} = \rho_a^1/(\rho_a - \rho_a^1)\), one can express the ground state energy by

\[E_{\text{ground}}(R) = -\frac{1}{R} \sum_a \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} RP_a'(\theta)L_{\epsilon_a}(\theta),\]

where \(L_\epsilon(\theta) = \ln[1 + e^{-\epsilon(\theta)}]\). \(\epsilon_a\) is determined by the minimizing condition of the free energy which is the following set of nonlinear equations:

\[\epsilon_a(\theta) = RE_a(\theta) - \sum_b \phi_{ab} * L_{\epsilon_b}(\theta),\]

where * denotes rapidity convolution, \(f * g(\theta) = \int_{-\infty}^{\infty} d\theta' f(\theta - \theta')g(\theta')\). As we mentioned above, the sum in Eq.(13) for non-diagonal theories has to been taken with care. Diagonalization of transfer matrix brings in additional ‘massless’ particles, which do not contribute to the central charge directly in the sum, due to masslessness, but which nevertheless affect the massive particle distributions. Additional care for non-relativistic scattering is that the participating particles can have different Fermi velocities, unlike in the relativistic case where all have the light speed as Fermi velocity.

For the non-diagonal theories, the product of \(S\)-matrices in Eq.(10) is replaced by the eigenvalues of the transfer matrix. For 1D Hubbard model the nondiagonal matrices
$S_{cc}$ and $S_{ss}$ are six vertex model type:

$$S_{\alpha} = \begin{pmatrix}
    a_{\alpha} & 0 & 0 & 0 \\
    0 & b_{\alpha} & c_{\alpha} & 0 \\
    0 & c_{\alpha} & b_{\alpha} & 0 \\
    0 & 0 & 0 & a_{\alpha}
\end{pmatrix}, \quad \alpha = c, s. \quad (15)$$

The eigenvalues of the transfer matrices and the associated constraint equations can be derived by algebraic Bethe Ansatz method to be

$$\Lambda_{\alpha}(\theta) = \prod_{i=1}^{N} a_{\alpha}(\theta - \theta_i) \prod_{r=1}^{M} a_{\alpha}(y_r - \theta) + \prod_{i=1}^{N} b_{\alpha}(\theta - \theta_i) \prod_{r=1}^{M} a_{\alpha}(\theta - y_r), \quad (16)$$

and

$$\prod_{i=1}^{N} b_{\alpha}(y_k - \theta_i) \prod_{r=1}^{M} b_{\alpha}(y_r - y_k) a_{\alpha}(y_k - y_r) = -1. \quad (17)$$

From the $S$-matrices Eqs.(6) and (7), one can read off the corresponding matrix elements to evaluate the explicit eigenvalues. The holon and the spinon sectors are coupled by diagonal scattering matrix $S_{sc}$.

Hence we have two kinds of the periodic conditions for the Bethe wave functions of holons and spinons in terms of $\Lambda_{c}$, $\Lambda_{s}$, and $S_{cs}$ and two sets of the constraint equations. For simplicity, we concentrate on holon sector first and will generalize the argument to spinons. From Eqs.(6) and (17) the constraint equation for holon sector becomes

$$\prod_{i=1}^{N} \beta_k - \theta_i - i|U| \prod_{r=1}^{M} \beta_k - \beta_r + 2i|U| = -1, \quad (18)$$

where we have introduced the shifted rapidities $\beta_i = y_i - i|U|$ to have the unitary form. It is well known, from the analogy in the antiferromagnetic Heisenberg chain, that in the thermodynamic limit, $N \to \infty$, the general solutions of these equations are the strings consisting of $n$-pseudoparticles of roots $\beta_{r,j} = \beta_r^0 + i|U|(n + 1 - 2j)$, where $\beta_r^0$ is real, $j = 1, \cdots, n$, and $n = 1, \cdots, \infty$. Such a string is a bound state of $n$-pseudoparticles.
and can be interpreted as a fictitious massless particle of real rapidity $\beta^0_r$. Since the length of the strings could be infinitely long, there are infinite number of such massless particles. Similarly the constraint equations for spinons can be understood in the context of another kind of pseudoparticles designated by the rapidities $\alpha$’s which also form the string-solutions in the thermodynamic limit. Therefore what we have is a diagonal scattering theory of holons, spinons and infinite number of massless particles associated with them. The scattering amplitudes of $n$-th massless particle with holon is

$$S_n(\theta) = \frac{\theta - i n|U|}{\theta + i n|U|},$$

and the scattering amplitudes between the massless particles are

$$S_{nm}(\theta) = \frac{\theta + i(n-m)|U|}{\theta - i(n-m)|U|} \times \frac{\theta + i(n+m)|U|}{\theta - i(n+m)|U|} \cdots \frac{\theta + i(n-2)|U|}{\theta - i(n+2)|U|} \cdots \frac{\theta + i(n+2)|U|}{\theta - i(n-2)|U|} \left[\frac{\theta + i(n+2)|U|}{\theta - i(n+2)|U|}\right]^2.$$  

For the spinons, the corresponding scattering amplitudes are obtained by the replacement $\theta \rightarrow -\theta$.

The minimizing condition of the free energy using the above mentioned $S$-matrices leads to an infinite set of non-linearly coupled equations. It is a standard procedure to use Fourier transformation on the TBA equations and to simplify the equations in terms of a unified kernel $\varphi = \left(4|U| \cosh \frac{\pi \theta}{2|U|}\right)^{-1}$:

$$RE_c(\theta) = \epsilon_0(\theta) + \varphi \ast (L_{\epsilon_1} + L_{\epsilon_0})(\theta)$$

$$0 = \epsilon_n(\theta) + \varphi \ast (L_{\epsilon_{n-1}} + L_{\epsilon_n})(\theta), \quad n \geq 1$$

$$RE_s(\theta) = \eta_0(\theta) + \varphi \ast (L_{\eta_1} + L_{\eta_0})(\theta)$$

$$0 = \eta_n(\theta) + \varphi \ast (L_{\eta_{n-1}} + L_{\eta_n})(\theta), \quad n \geq 1.$$  

These TBA equations have the incidence structure of an infinite chain, where a pair of semi-infinite chains of SU(2) invariant factorized scatterings are joined together as shown in the following picture:
While the TBA equations are non-linearly coupled equations and are hard to solve explicitly, it is rather easy for the cases of UV ($R \to 0$) and IR ($R \to \infty$) limits. However, the analysis of these two limits depends on the energy-momentum dispersion relations.

(i) Contrary to the relativistic case where the central charge can still get nontrivial contributions in the UV limit because $E(\theta)$ can be large enough to overcome vanishing $R$ in Eq.(13) at large rapidity, the central charge becomes zero for the Hubbard model as $R \to 0$ since $P'_c(\theta)$ and $P'_s(\theta)$ are bound over all values of rapidities.

(ii) The situation changes in the IR limit. When $U$ is finite, the spinons become massive and do not contribute to the central charge because the pseudo-energy $\eta_0(\theta) \sim RE_s(\theta) \to \infty$ for all values of $\theta$. After decoupling spinon sector, we have only semi-infinite chain of holon sector. For a finite value of $\theta$, $E_c(\theta)$ is non-zero and from Eq.(21), $\epsilon_0(\theta)$ becomes infinite. Therefore, only non-vanishing contribution to the central charge comes from $\theta \to \infty$ limit. In this limit, taking a derivative on Eq.(14) and substituting $P'_c$ with $E'_c/v_c$ into Eq.(13), we can now invoke the standard TBA analysis to evaluate the central charge in terms of the pseudo-energies at $\theta = 0$ and $\infty$. Note that the Fermi velocity in the denominator of Eq.(9) is canceled by the $v_c$ in the above substitution. The final result for the central charges is

$$c_{\text{eff}} = \frac{6}{\pi^2} \sum_{n=0}^{\infty} [\mathcal{L}\left(\frac{x_n^{\infty}}{1+x_n^{\infty}}\right) - \mathcal{L}\left(\frac{x_n^{0}}{1+x_n^{0}}\right)],$$

(22)

where $\mathcal{L}(x)$ is the Rogers dilogarithmic function;

$$\mathcal{L}(x) = -\frac{1}{2} \int_{\alpha}^{x} dt \left[ \frac{\ln(1-t)}{t} + \frac{\ln t}{(1-t)} \right],$$

(23)

and we have defined $x_n^{\infty} = \exp[-\epsilon_n(\infty)]$ and $x_n^{0} = \exp[-\epsilon_n(0)]$. The TBA equations,
Eq. (21), can be rewritten as a set of algebraic equations for \( x_n \)'s:

\[
x_0^\infty = (1 + x_1^\infty)^{\frac{1}{2}}, \quad x_n^\infty = (1 + x_{n-1}^\infty)^{\frac{1}{2}}(1 + x_{n+1}^\infty)^{\frac{1}{2}}, \quad n \geq 1, \quad (24)
\]

\[
x_0^0 = 0, \quad x_n^0 = (1 + x_{n-1}^0)^{\frac{1}{2}}(1 + x_{n+1}^0)^{\frac{1}{2}}, \quad n \geq 1.
\]

These have solutions \( x_n^\infty = (n + 2)^2 - 1 \) and \( x_n^0 = (n + 1)^2 - 1 \). Since \( x_n^0 = x_{n-1}^\infty \), only \( x_\infty^\infty \) survives to give \( c_{\text{eff}} = 1 \) after using \( \mathcal{L}(1) = \frac{\pi^2}{6} \).

As we claimed in the beginning, the central charge we computed comes from the holon sector while the spinon sector decouples. This is what happens for finite \( U \) but seems valid even vanishing \( U \) as far as \( U > 1/R \). In the literature, a discontinuity between \( U \to 0 \) and \( U = 0 \) has been predicted such that if \( U = 0 \) the central charge will be 2 because the model is nothing but a theory with four free fermions. In our analysis, we use the holon and spinon \( S \)-matrices which are valid for non-vanishing \( U \). Within this validity, our result is what has been noticed in the literature from different computations. With this confirmation for the Eßler-Korepin \( S \)-matrices, we established the \( S \)-matrix program for the non-relativistic models. The most noticeable difference of the non-relativistic theories from the relativistic ones is that the IR limit of the former models corresponds to the UV limit of the latter. This is the limit where massless degree of freedom survives and gives correct central charges. We hope our result can be a starting point for the application of the \( S \)-matrices to various lattice problems.

What we are particularly interested in is to compute correlation functions using the form-factor approach\[14\]. In this scenario, correlation functions of any local operator are expressed in terms of the form-factors which can be computed from the exact \( S \)-matrices. Although one needs to sum an infinite terms, this can be realized as the sum converges very fast. We hope to report this result elsewhere.
Acknowledgement

We thank Choonkyu Lee and Sungkil Yang for very helpful remarks. The work of C.A. and K.J.B.L. has been supported in part by KOSEF 941-0200-003-2. The work of S.N. is supported in part by non-directed research fund, Korea Research Foundation, 1993, and by Ministry of Education (RIBS). We also acknowledge a partial support from CTP/SNU.

References

[1] See for example, V.E. Korepin, N.M. Bogoliubov, and A.G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge Univ. Press, New York, 1993), and references therein.

[2] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. (N.Y.) **120** (1979) 253.

[3] See for example, L.D. Faddeev, in *Recent Advances in Field Theory and Statistical Mechanics*, (North Holland, Amsterdam, 1984).

[4] P.W. Anderson, Science **235** (1987) 1196.

[5] E.H. Lieb and F.Y. Wu, Phys. Rev. Lett **20** (1968) 1445.

[6] M. Takahashi, Prog. Theo. Phys. **47** (1972) 69; **52** (1974) 103; H. Shiba, Phys. Rev. B**6** (1972) 930; T.B. Bahder and F. Woynarovich, Phys. Rev. B**33** (1986) 2114; K. Lee and P. Schlottmann, Phys. Rev. B**40** (1989) 9104.

[7] F. Woynarovich and H-P. Eckle, J. Phys. A. **20** (1987) L443.

[8] F. Essler and V. E. Korepin, “$SU(2) \times SU(2)$-Invariant Scattering Matrix of the Hubbard Model”, ITP-SB-93-45.

[9] Al.B. Zamolodchikov, Nucl. Phys. B**342** (1990) 695.

[10] A.B. Zamolodchikov and Al.B. Zamolodchikov, Nucl. Phys. B**379** (1992) 602.

[11] P.B. Wiegmann, Phys. Lett. B**141B** (1984) 217.
[12] F. Woynarovich, J. Phys. A 22 (1989) 4243.

[13] See the appendix of A.B. Zamolodchikov and A.I. Zamolodchikov, Nucl. Phys. B379 (1992) 602.

[14] See for example F.A. Smirnov, in Introduction to Quantum Group and Integrable Massive Models of Quantum Field Theory, (World Scientific, Singapore, 1990) and references therein.