SOME OBSERVABLE RESULTS OF THE
RETARDED BOHM’S THEORY

ALI SHOJAI* & MEHDI GOLSHANI**

Department of Physics, Sharif University of Technology
P.O.Box 11365-9161 Tehran, IRAN

and

Institute for Studies in Theoretical Physics and Mathematics,
P.O.Box 19395-5531, Tehran, IRAN

*Email: SHOJAI@PHYSICS.IPM.AC.IR

**Fax: 98-21-8036317
SOME OBSERVABLE RESULTS OF THE RETARDED
BOHM’S THEORY

A. Shojai & M. Golshani

ABSTRACT

It is shown that the retarded Bohm’s theory has at least four novel properties. (1) The center of mass of an isolated two-body system is accelerated. (2) Hydrogen-like atoms are unstable. (3) The distribution function differs from the standard one. (4) The definition of energy needs some care.

1 INTRODUCTION

Nonrelativistic Bohm’s theory (NBT) can be formulated in terms of the following three postulates[1]:

(I)— Any system of particles is always accompanied by an objectively real field \( \psi(\vec{x}_1, \ldots, \vec{x}_N; t) \), which satisfies the Schrödinger equation:

\[
\frac{i\hbar}{\partial t} \psi = \sum_{i=1}^{N} \frac{\hbar^2}{2m_i} \nabla_i^2 \psi + V(\vec{x}_1, \ldots, \vec{x}_N; t) \psi
\]

(1)
(II)— Particles move according to:

\[
\frac{d\vec{r}_i(t)}{dt} = -i \frac{\hbar}{m_i} \left[ \nabla_i \left\{ \ln \left( \frac{\psi}{\sqrt{\psi^* \psi}} \right) \right\} \right]_{\{\vec{x}_j\} = \{\vec{r}_j(t)\}}
\]  

(3)

(III)— The distribution function of an ensemble of such system is given by:

\[
\rho(\vec{x}_1 \cdots \vec{x}_N; t) = \psi^* \psi
\]

It can be shown that the second and third postulates are compatible. That is, the motion predicted by (II) preserves the distribution function given by (III)[1,2].

A simple-minded extension of this formalism to the relativistic domain (i.e. simply writing the Lorentz covariant analogues of (1)–(3)) fails[2,3]. This is mainly because NBT is highly non-local.

Recently a local relativistic Bohm’s theory, called retarded Bohm’s theory (RBT)[4], is introduced. It is basically founded on the assumption that for the calculation of the position of some particle, the position of others should be evaluated at the retarded times. This means that instead of (2) we must use:

\[
\frac{d\vec{r}_i(t)}{dt} = -i \frac{\hbar}{m_i} \left[ \nabla_i \left\{ \ln \left( \frac{\psi}{\sqrt{\psi^* \psi}} \right) \right\} \right]_{\{\vec{x}_j\} = \{\vec{r}_j(t_{ij})\}}
\]

where \( t_{ij} \) is the retarded time of the \( j \)th particle with respect to the \( i \)th particle, defined by:

\[
t_{ij} = t - \frac{|\vec{r}_i(t) - \vec{r}_j(t_{ij})|}{c}
\]
This formalism is manifestly relativistic, in the sense that actions are propagated by light’s velocity.

There are at least three questions about this theory. The first one concerns the problem of whether it has any prediction beyond the standard quantum mechanics which can be checked experimentally? The inventor of RBT and others have presented some samples of such experiments[4,5]. In this work we shall present two other ones. These involve the self-acceleration of the center of mass of an isolated two-body system, and some sort of unstability in Hydrogen-like atoms.

The second question is about the consistensy between (3) and (4). Finally the third question is related to the problem of defining the energy of a system in RBT. These questions are also investigated in this paper.

2 Self-acceleration Effect

In NBT as in the classical mechanics the center of mass of an isolated system is not accelerated. For an isolated two-body system we have:

\[ V(\vec{x}_1, \vec{x}_2; t) = V(|\vec{x}_1 - \vec{x}_2|) \]  

(6)

It can be easily shown that the solution of Schrödinger equation is:

\[ \psi(\vec{x}_1, \vec{x}_2; t) = \Phi(\vec{x}_1 - \vec{x}_2)e^{i\vec{K} \cdot \vec{x}_{c.m}} e^{-iE t/\hbar} \]  

(7)
where:

\[ \vec{X}_{\text{c.m.}} = \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1 + m_2} \]  
(8)

The center of mass velocity is:

\[ \vec{v}_{\text{c.m.}} = \frac{\hbar \vec{K}}{m_1 + m_2} = \text{constant} \]  
(9)

where we have used the fact that:

\[ \vec{\nabla}_1 \Phi = -\vec{\nabla}_2 \Phi \]  
(10)

On the other hand, in RBT the right-hand-side of (4) is not calculated at the same time for the two particles, and since there is in general an asymmetry between \( t_{12} \) and \( t_{21} \),

as can be seen from the Fig. (1), the center of mass velocity would not be a constant.

In order to clarify this point, let us study a Hydrogen-like atom with \( \vec{K} = 0 \) and:

\[ \Phi = R_{nl}(r)P_{l1}(\theta)e^{i\phi} \]  
(11)

where \( r, \theta \) and \( \phi \) are the spherical coordinates of the relative distance \( \vec{x} = \vec{x}_1 - \vec{x}_2 \).

It can be shown easily that:

\[ t_{12} = t - \frac{a}{c} \]  
(12)

\[ t_{21} = t - \frac{\tilde{a}}{c} \]  
(13)
where:
\[
a = r + \frac{1}{c}(x\dot{x} + y\dot{y}) + \mathcal{O}(v^2/c^2) \tag{14}
\]
\[
\tilde{a} = r - \frac{1}{c}(x\dot{x} + y\dot{y}) + \mathcal{O}(v^2/c^2) \tag{15}
\]

The equations of motion (4) lead to the following relations:
\[
m_1\dot{x}_1 = -\frac{\hbar}{r^2} \left[ y - \frac{2y}{rc}(x\dot{x} + y\dot{y}) + \frac{r}{c}\dot{y}_2 \right] + \mathcal{O}(v^2/c^2) \tag{16}
\]
\[
m_2\dot{x}_2 = \frac{\hbar}{r^2} \left[ y + \frac{2y}{rc}(x\dot{x} + y\dot{y}) - \frac{r}{c}\dot{y}_1 \right] + \mathcal{O}(v^2/c^2) \tag{17}
\]
\[
m_1\dot{y}_1 = \frac{\hbar}{r^2} \left[ x - \frac{2x}{rc}(x\dot{x} + y\dot{y}) + \frac{r}{c}\dot{x}_2 \right] + \mathcal{O}(v^2/c^2) \tag{18}
\]
\[
m_2\dot{y}_2 = -\frac{\hbar}{r^2} \left[ x + \frac{2x}{rc}(x\dot{x} + y\dot{y}) - \frac{r}{c}\dot{x}_1 \right] + \mathcal{O}(v^2/c^2) \tag{19}
\]

So that the components of the velocity of the center of mass is given by:
\[
(m_1 + m_2)\dot{X}_{c.m.} = \frac{2\hbar y}{c r^3} [x(\dot{x}_1 + \dot{x}_2) + y(\dot{y}_1 + \dot{y}_2)] - \frac{\hbar}{cr}(\dot{y}_1 + \dot{y}_2) + \mathcal{O}(v^2/c^2) \tag{20}
\]
\[
(m_1 + m_2)\dot{Y}_{c.m.} = -\frac{2\hbar x}{c r^3} [x(\dot{x}_1 + \dot{x}_2) + y(\dot{y}_1 + \dot{y}_2)] + \frac{\hbar}{cr}(\dot{x}_1 + \dot{x}_2) + \mathcal{O}(v^2/c^2) \tag{21}
\]
which shows that the center of mass velocity is not zero, as is predicted by NBT. The center of mass is self-accelerated.

An estimation of the center of mass velocity is simple. From the above equations we have:
\[
v_1 \text{ or } v_2 \sim \frac{\hbar}{m r}
\]
\[
v_{c.m.} \sim \frac{\hbar^2}{m^2 c r^2} \sim \frac{10^{-68}}{10^{-60} \times 10^8} \frac{1}{10^{-20}} \sim 10^4 \text{ m/sec} \sim 10^{-4}c
\]
which must be an observable effect. If one considers a gas of such atoms and assumes that this self-acceleration is converted to heat via collisions, one has:

\[
\frac{1}{2}mv_{\text{c.m.}}^2 \sim \frac{3}{2}kT = \Rightarrow T \sim 10^6 K
\]

Thus, this effect can be observed as the self-heating of a gas of atoms having a temperature of about ten degrees of Kelvin!!

3 Unstability of Hydrogen-like Atoms

Now, we want to investigate to what extent the motion of a Hydrogen-like atom differs from the standard one, i.e. the one predicted by NBT. For simplicity we assume that \( m_1 = m_2 = m \). The equations (16)–(19) can be written in the spherical coordinates in the following form (we assume that \( \theta = \pi/2 \)):

\[
\dot{X}_{\text{c.m.}} = \dot{Y}_{\text{c.m.}} = 0 \tag{22}
\]

\[
\dot{r} = -\frac{2c}{1 - r^2/\alpha^2} \tag{23}
\]

\[
\dot{\phi} = \frac{(2\alpha c/r) \cos \phi - \dot{r}(\sin \phi - (\alpha/r) \cos \phi)}{r \cos \phi - \alpha \sin \phi} \tag{24}
\]

where:

\[
\alpha = \frac{\hbar}{mc} \tag{25}
\]
The solutions of (23) and (24) are:

\[
\frac{r^3}{3\alpha^2} - r = \frac{r_0^3}{3\alpha^2} - r_0 + 2ct 
\]
(26)

\[
r = r_0 + \alpha \phi 
\]
(27)

where \( r_0 = r(t = 0) \). Note that as \( c \to \infty \) these are the same as the results of NBT. A glance at these relations leads to the strange result that this sort of atom is not stable, in the sense that \( r \) is an increasing function of time. In fact \( r \) increases from \( 1 \times 10^{-10} \) meter to 1 meter in the time interval:

\[
t \sim \frac{m^2c}{6\hbar^2} \sim 10^{16} \text{ sec} \sim 10^9 \text{ year} \sim 10^{-1} \text{ times of the age of the universe.}
\]

4 The Distribution Function

Since we calculate the motion of particles via (4), rather than (2), there is no necessity for (3) to be consistent. In fact, in order to have a consistent theory, one must calculate the distribution function via the conservation law of particles, not using (3). The conservation law can be written as:

\[
\frac{\partial \rho}{\partial t} + \vec{V} \cdot (\rho \vec{v}) = 0 
\]
(28)

or equivalently as

\[
\frac{\partial \ln \rho}{\partial t} + \vec{v} \cdot \nabla (\ln \rho) + \nabla \cdot \vec{v} = 0 
\]
(29)
To solve this equation for the system presented in the previous section, perturbatively, we write:

\[
\ln \rho = \ln \rho_0 (r, \theta) + \alpha \eta (r, \theta, \phi; t) + \cdots
\]  

(30)

where \( \rho_0 \) is the result of NBT. Equation (29) leads to the following result, for first order terms:

\[
\frac{\partial \eta}{\partial t} + \frac{2 \alpha c}{r^2} \frac{\partial (\ln \rho_0)}{\partial r} + \frac{2 \alpha c}{r^2} \frac{\partial \eta}{\partial \phi} = 0
\]  

(31)

with the solution:

\[
\eta (r, \theta; t) = -\frac{2 \alpha c}{r^2} \frac{\partial (\ln \rho_0)}{\partial r} t
\]  

(32)

So that:

\[
\rho = \rho_0 e^{-\frac{2 \alpha c}{r^2} \frac{\partial (\ln \rho_0)}{\partial r} \alpha t + \mathcal{O}(\alpha^2)}
\]  

(33)

which shows the unstability of Hydrogen-like atoms clearly. Note that we have not normalised \( \rho \), because this must be done when all orders are calculated.

This is again an observable effect, because change in the distribution function would be reflected in physical quantities like electric dipole moment, magnetic quadrupole moment, etc.
5 The Problem of Energy

As it is stated by Squires [4], RBT is ambiguous when the \( \psi \)-field is an explicit function of time. This is because we do not know what time to use in (4).

One way out of this problem is to postulate that such explicit time dependencies should be evaluated at time \( t_{ii} = t \).

In NBT, the energy of the system is:

\[
E = i\hbar \left[ \frac{\partial}{\partial t} \left\{ \ln \left( \frac{\psi}{\sqrt{\psi^* \psi}} \right) \right\}_{\{\vec{x}_j\} = \{\vec{r}_j(t)\}} \right] \tag{34}
\]

which can be shown to be equivalent to:

\[
E = \frac{1}{2} \sum_{i=1}^{N} m_i |\dot{\vec{r}}_i(t)|^2 + [V(\vec{x}_1, \cdots, \vec{x}_N; t) + Q(\vec{x}_1, \cdots, \vec{x}_N; t)]_{\{\vec{x}_j\} = \{\vec{r}_j(t)\}} \tag{35}
\]

where the quantum potential is given by:

\[
Q(\vec{x}_1, \cdots, \vec{x}_N; t) = \sum_{i=1}^{N} \frac{\hbar^2}{2m_i} \frac{\nabla_i^2 \sqrt{\psi^* \psi}}{\sqrt{\psi^* \psi}} \tag{36}
\]

The extension of (34) and (35) to RBT is ambiguous. But if \( \psi \) has a time dependence like \( e^{-iEt/\hbar} \) then (34) works and leads to \( E = \mathcal{E} \).

It seems to us that there is a natural solution to this problem. First, we note that the classical potential \( V \) in (35) is in fact a retarded potential. That is, for the system in section 3, it is equal to:

\[
V(\vec{x}_1, \vec{x}_2; t) = q_1 \varphi(\vec{x}_1; t) + q_2 \varphi(\vec{x}_2; t) - q_1 \vec{A}(\vec{x}_1; t) \cdot \vec{r}_1 - q_2 \vec{A}(\vec{x}_2; t) \cdot \vec{r}_2 \tag{37}
\]
where $\varphi$ and $\vec{A}$ are the retarded electromagnetic potentials and $q_1$ and $q_2$ are the charges of two particles. Clearly in the calculation of $\mathcal{E}$ we must use $V(\vec{r}_1(t), \vec{r}_2(t); t)$.

On the other hand, the $i$th term in the sum in (36) represents the $i$th particle contribution to the quantum potential. So, we define:

$$\tilde{Q} = \sum_{i=1}^{N} \left[ -\frac{\hbar^2}{2m_i} \frac{\nabla^2_{\vec{x}_j}}{\sqrt{\psi^* \psi}} \right] \{\vec{x}_j\} = \{\vec{r}_j(t_i)\}$$ (38)

and assume that the correct energy of the system is given by:

$$\mathcal{E} = \frac{1}{2} \sum_{i=1}^{N} m_i [\dot{\vec{r}}_i(t)]^2 + V(\vec{r}_1(t), \cdots, \vec{r}_N(t); t) + \tilde{Q}$$ (39)

It is obvious that this is very different from $E$ of the example in section 3.

6 References

[1]– D. Bohm, Phys. Rev. 85 (1952) 166, 180.

[2]– P.R. Holland, The quantum theory of motion, (1993) Cambridge University Press, London/New York.

[3]– D. Bohm, B.J. Hiley, and P.V. Kaloyerou, Phys. Rep. 144, 6 (1987) 321, 349.

[4]– E.J. Squires, Phys. Lett. A 178 (1993) 22.

[5]– S. Mackman and E.J. Squires, Found. Phys. 25 (1995) 391.
