SEPARATION OF VARIABLES FOR TYPE $D_n$ HITCHIN SYSTEMS ON HYPERELLIPTIC CURVES

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Abstract. We use separation of variables to find explicit formulas for the Darboux coordinates for the Hitchin systems on a hyprelliptic curve for simple Lie algebras of type $D_n$.

Introduction

The action-angle coordinates for the Hitchin systems were known only in case of Lie algebra $\mathfrak{sl}(2)$ and a curve of genus 2 ([9]), until recently. A description of the class of spectral curves for the Hitchin systems on hyperelliptic curves of any rank was given in [3]. For Lie algebras of type $A_n, B_n, C_n$ the Darboux coordinates were found there in the explicit form.

The main goal of this paper is to find the Darboux coordinates for the Hitchin systems on a hyperelliptic curve for simple Lie algebras of type $D_n$, according to explicit description of the spectral curve, given in [3]. Nevertheless, we were able to explicitly obtain the Darboux coordinates only for the simplest case of Lie algebra $\mathfrak{so}(4)$. Note that the isomorphism $\mathfrak{so}(4) \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ doesn’t give any result, because this isomorphism is an outer isomorphism and it doesn’t preserve the spectral curve.

Consider a Hitchin system on a hyprelliptic curve $\Sigma_g$ of genus $g$, given by $y^2 = P_{2g+1}(x)$, for a classical Lie algebra $\mathfrak{g}$. This system is defined by the Lax operator $L$ ([2], [3], [5]). The spectral curve is given by the equation $\det(\lambda - L) = 0$. In terms of $x, y$ it looks like:

\begin{equation}
R(\lambda, x, y, H) = \lambda^n + \sum_{i=1}^{n} \lambda^{n-i}r_i(x, y, H) = 0,
\end{equation}

where $n$ is the dimension of the standard representation of the Lie algebra $\mathfrak{g}$, $r_i(x, y, H)$ are meromorphic functions on $\Sigma_g$ such that $r_i(x, y) = \chi_i(L(x, y))$, $\chi_i (i = 1, \ldots, n)$ are invariant polynomials of the Lie algebra $\mathfrak{g}$, $H$ are Hamiltonians of the system (which arise as coefficients of the expansion of those invariant polynomials over a basis of meromorphic functions).
1. Hamiltonians

Consider the Hitchin system for a simple Lie algebra \( \mathfrak{g} \) of type \( D_n \). This algebra \( \mathfrak{g} \) has \( n \) fundamental invariants of orders \( \{2, 4, \ldots, 2(n-1), n\} \) and the spectral curve is given by the following equation

\[
R(\lambda, x, y, H) = \lambda^{2n} + \sum_{i=1}^{n-1} \lambda^{2n-2i} r_i(x, y, H) + (r_0(x, y, H))^2 = 0,
\]

where \( r_i(x, y, H) \) is a fundamental invariant of degree \( 2i \), \( (i = 1, \ldots, (n-1)) \), \( r_0(x, y, H) \) is the invariant of degree \( n \) (Pfaffian).

According to [3], a fundamental invariant \( r_i \) of degree \( d_i \) can be expanded over the basis formed by two series of functions: \( \{1, x, \ldots, x^{d_i(g-1)}\} \) and \( \{y, yx, \ldots, yx^{(d_i-1)(g-1)-2}\} \) as follows:

\[
r_i(x, y, H) = \sum_{k=0}^{d_i(g-1)} H_{ik} x^k + \sum_{s=0}^{(d_i-1)(g-1)-2} H_{is} y^s.
\]

Consider the Hitchin system for the Lie algebra \( \mathfrak{so}(4) \) on a hyperelliptic curve of genus 2. The dimension of the space of spectral invariants is \( N = \dim \mathfrak{g} \cdot (g-1) = 6 \). Lie algebra \( \mathfrak{so}(4) \) has two fundamental invariants of degree 2, which are expanded over the basis \( \{1, x, x^2\} \), according to (1.1). Hence, the explicit equation for the spectral curve is

\[
\lambda^4 + (H_4 + xH_5 + x^2H_6)\lambda^2 + (H_1 + xH_2 + x^2H_3)^2 = 0.
\]

We can find the Hamiltonians using the fact that the spectral curve passes through points \( (\lambda_i, x_i) \). In other words, our problem is reduced to solving a system of 6 equations in variables \( H = \{H_j| j = 1, \ldots, 6\} \). This leads us to the following proposition.

**Proposition 1.1.** System for \( \mathfrak{g} = \mathfrak{so}(4) \) can be reduced to an equation of degree 4 in one variable and, as a consequence, is solvable in radicals.

**Proof.** Performing Gaussian elimination, one can eliminate variables \( H_4, H_5, H_6 \). Thus, the system is reduced to a system of three homogeneous equations with a nonzero right part:

\[
a_{11}H_1^2 + a_{12}H_2^2 + a_{13}H_3^2 + b_{11}H_1H_2 + b_{12}H_1H_3 + b_{13}H_2H_3 = \tilde{\lambda}_i, \; i = 1, 2, 3.
\]

Consider a matrix \( \{a_{ij}\}_{i,j=1,2,3} \) of the coefficients of quadratic terms. Performing Gaussian elimination again we obtain a diagonal matrix. Dividing the second two equations by \( \frac{b_{13}}{a_{33}} \) and \( \frac{b_{23}}{a_{33}} \) respectively, we obtain the following system.

\[
\begin{cases}
\tilde{a}_{11}H_1^2 + \tilde{b}_{11}H_1H_2 + \tilde{b}_{12}H_1H_3 + \tilde{b}_{13}H_2H_3 = \tilde{\lambda}_1 \\
\tilde{a}_{22}H_2^2 + \tilde{b}_{21}H_1H_2 + \tilde{b}_{22}H_1H_3 + \tilde{b}_{23}H_2H_3 = \tilde{\lambda}_1 \\
\tilde{a}_{33}H_3^2 + \tilde{b}_{31}H_1H_2 + \tilde{b}_{32}H_1H_3 + \tilde{b}_{33}H_2H_3 = \tilde{\lambda}_1
\end{cases}
\]

On the next step, subtracting the first equation from the second two, we get a system of two equations with a nonzero right part. Next, we divide both sides by \( H_1 \) to eliminate one variable. After all the before-mentioned, we get following system:

\[
\tilde{a}_{11}H_2^2 + \tilde{a}_{12}H_3^2 + c_{11}\tilde{H}_2\tilde{H}_3 + c_{12}\tilde{H}_2 + c_{13}\tilde{H}_3 = \tilde{\lambda}_i, \; i = 1, 2, \; \tilde{a}_{11} = 0, \; \tilde{a}_{22} = 0.
\]
Finally, we can express $\tilde{H}_2$ via $\tilde{H}_3$ from the first equation to get the equation of degree 4 in one variable $\tilde{H}_3$. This equation is solvable in radicals. Thus we obtain the explicit expression for the action coordinates for the Hitchin system for $D_2$. \hfill \square

In the general case of $D_n$, $n > 2$, according to the before-mentioned method, we can reduce original system to a system of homogeneous equations with nonzero right part in $(2n-1)(g-1)$ variables. In general, such systems are not solvable in radicals \cite{6}. In this case, the Hamiltonians of the Hitchin system are given by implicit formulas $R(\lambda_i, x_i, y_i, H) = 0$, $i = 1, \ldots, N$.

2. Angle coordinates

Recall from \cite{3} that the symplectic form on a phase space of the Hitchin system of any rank on a hyperelliptic curve is given by the formula:

\begin{equation}
\sigma = \sum_i d\lambda_i \wedge \frac{dx_i}{y_i},
\end{equation}

where $\frac{dx}{y}$ is a holomorphic differential on the hyperelliptic curve. For further calculations we can bring this form to the canonical one via the appropriate change of coordinates

$$(\lambda_i, x_i, y_i) \mapsto (\lambda_i, \tilde{x}_i, y_i), \quad \tilde{x}_i = \int \frac{dx}{y}.$$  

Furthermore, the following formula for coordinates \{\varphi_j\}, which are conjugate with coordinates \{\hat{H}_j\}, can be obtained from \cite{8} (1.7):

\begin{equation}
\varphi_j = -\sum_{i=1}^{N} \int \frac{R'_{\hat{H}_j}(\lambda, x, y, H)}{yR_{\lambda}(\lambda, x, y, H)} dx.
\end{equation}

(the $y$ in denominator appears due to the fact that our coordinates are not in a canonical form first and we have to do a change of variables first). Similar, but more specific formulas were given in \cite{8, 10, 11}.

To obtain formula (2.2) one can use the method of separation of variables. We have a system of $N = \dim \mathfrak{g} \cdot (g-1)$ equations $R_i(\lambda_i, x_i, y_i, H) = 0$ (the equations of the spectral curve passing through $N$ points $(\lambda_i, x_i, y_i)$). Suppose that $\frac{\partial S}{\partial x_i} = r(\tilde{x}_i, H)$, hence:

$$S = \sum_{i=1}^{N} \int r(\tilde{x}, H) d\tilde{x} = \sum_{i=1}^{N} \int r(x, H) \frac{dx}{y}.$$  

So we can put $\frac{\partial S}{\partial x_i}$ in the spectral curve equation instead of $\lambda_i$ and find the coordinates $\varphi_j = \frac{\partial S}{\partial \hat{H}_j}$ from the expression $\frac{d}{\partial \hat{H}_j}(R(r(x, H), x, y, H)) = 0$. In particular,

$$\frac{\partial R}{\partial \lambda} \frac{\partial r}{\partial \hat{H}_j} + \frac{\partial R}{\partial \hat{H}_j} = 0.$$
And find the coordinates \( \{ \varphi_j \} \) from the equation:

\begin{equation}
\varphi_j = - \sum_{i=1}^{N} \left( x_i, y_i \right) \frac{\partial r}{\partial H_j} \frac{dx}{y} = - \sum_{i=1}^{N} \int \frac{R'_{H_j}(\lambda, x, y, H) \, dx}{R'_{\lambda}(\lambda, x, y, H) \, y}.
\end{equation}

According to the formula (2.2) and to the explicit formulas for the spectral curve (0.1) and (1.1) we can obtain the coordinates \( \{ \varphi_j \} \) for the Hitchin systems for any classical system of roots on a hyperelliptic curve of any genus (see [3] for the root systems \( A_n, B_n, C_n \)). In particular,

**Corollary 2.1.** For Lie algebra \( \mathfrak{so}(4) \) the coordinates \( \{ \varphi_j \} \) are of the form (2.3), where:

- \( R'_{H_j}(\lambda, x, y, H) = x^{j-1}(H_1 + xH_2 + x^2H_3) \), \( j = 1, 2, 3 \),
- \( R'_{H_j}(\lambda, x, y, H) = \lambda^2 x^{j-4} \), \( j = 4, 5, 6 \),
- \( R'_{\lambda}(\lambda, x, y, H) = 4\lambda^3 + 2\lambda(H_4 + xH_5 + x^2H_6) \).

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