On the moduli spaces of left invariant metrics on cotangent bundle of Heisenberg group

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September 2, 2021

Abstract

The main focus of the paper is the investigation of moduli space of left invariant pseudo-Riemannian metrics on the cotangent bundle of Heisenberg group. Consideration of orbits of the automorphism group naturally acting on the space of the left invariant metrics allows us to use the algebraic approach. However, the geometrical tools, such as classification of hyperbolic plane conics, will often be required.

For metrics that we obtain in the classification, we investigate geometrical properties: curvature, Ricci tensor, sectional curvature, holonomy and parallel vector fields. The classification of algebraic Ricci solitons is also presented, as well as classification of pseudo-Kähler and pp-wave metrics. We get the description of parallel symmetric tensors for each metric and show that they are derived from parallel vector fields. Finally, we investigate the totally geodesic subalgebras by showing that for any subalgebra of the observed algebra there exists a metric that makes it totally geodesic.

Key words: cotangent bundle of Heisenberg group, moduli space, pseudo-Riemannian metrics, Ricci solitons, pp-waves, pseudo-Kähler metrics, parallel symmetric tensors, totally geodesic subalgebras.

MSC 2020: 22E25, 22E60, 53B30, 53B35

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The research of the first two authors was supported by the Serbian Ministry of Education, Science and Technological Development through Faculty of Mathematics, University of Belgrade.
1 Introduction

Left invariant (pseudo-)Riemannian metrics on Lie groups provide a lot of interesting examples of distinguished metrics, in particular Einstein and Ricci soliton metrics. Hence, it is a natural question whether a given Lie group $G$ admits some special left invariant (pseudo-)Riemannian metrics or not. The main difficulty in metric classification lies in the fact that the space of all left invariant metrics on a specific Lie group can be quite large.

That space of metrics is called the moduli space and is defined as the orbit space of the action of $\mathbb{R}^\times \text{Aut}(g)$ on the space $\mathfrak{M}(G)$ of left invariant metrics on $G$. Here, $\text{Aut}(g)$ denotes the automorphism group of the corresponding Lie algebra and $\mathbb{R}^\times$ is the scalar group. There are two, in some way dual, approaches to the classification problem, both based on the moduli space of left invariant (pseudo-)Riemannian metrics on the Lie group. The first one is to fix a Lie algebra basis in a way to make the commutator relations as simple as possible and then to adapt the inner product to it by action of the automorphism group. This approach was first introduced by Milnor [33] who used it to classify all left invariant Riemannian metrics on three-dimensional unimodular Lie groups. The second one is to start from the basis that makes the inner product take the most basic form, while allowing the Lie brackets to be arbitrary, but satisfying the Jacobi identity and in this way defining a hypersurface of feasible Lie brackets. Note that in both cases the orbits of $\text{Aut}(g)$ induce the isometry classes, while $\mathbb{R}^\times$ gives rise to the scaling. For the more detailed outline of each approach, we refer to [22, 28].

Interestingly, while the Riemannian case is well studied and understood, the pseudo-Riemannian case appears more challenging and still has a variety of open questions. The Milnor’s classification of 3-dimensional Lie groups with left invariant positive definite metric [33] has become a classic reference, while the corresponding Lorentz classification [13] followed twenty years later. In dimension four, only partial results are known. The classification of 4-dimensional Riemannian Lie groups is due to Bérard-Bérgery [4]. Jensen [24] has studied Einstein homogeneous spaces with Riemannian (positive definite) metric, while Karki and Thompson [26] studied Einstein manifolds that arise from right invariant Riemannian metrics on a 4-dimensional Lie group. Calvaruso and Zaeim [8] have classified Lorentz left invariant metrics on the Lie groups that are Einstein or Ricci-parallel,
using the previously mentioned second approach. The classification in the case of nilpotent Lie groups in small dimensions was extensively studied in both the Riemannian [29] and the pseudo-Riemannian setting [5, 23, 41]. Recent results include the classification of pseudo-Riemannian metrics for 4-dimensional solvable Lie groups [42] and in positive definite case, the moduli space for 6-dimensional nilpotent Lie groups admitting complex structure with the first Betti number equal to 4 has been determined [36]. In arbitrary dimension, one must mention the Lorentz classification of left invariant metrics on Heisenberg group $H_{2n+1}$ [43] and classification of Ricci solitons on nilmanifolds [30].

The cotangent bundles play significant role in standard description of physical systems, both for particles and for fields (see e.g. [2]). In particular, they appear as the configuration space of some mechanical systems and are frequently endowed with rich algebraic and geometric structures (see e.g. [14,15,17]). In this paper we are interested in the cotangent bundle of the Heisenberg group $H_3$, mainly because this group is a constant topic of research due to its properties and various areas of application. For example, Herman Weyl was led to an explicit realization of the Heisenberg group while trying to answer a question of physical equivalence of the Schrödinger’s and Heisenberg’s picture.

The paper is organized as follows.

First, in Section 2, we review some basic facts about the algebra $T^*H_3$ and its automorphism group.

In Section 3 we classify all non isometric left invariant pseudo-Riemannian metrics on $T^*H_3$. For the classification we use the second approach described above: we fix the commutators and act with automorphisms of the algebra $T^*H_3$ to find representatives of the metrics. The restriction of the metric on the derived subalgebra $T^*H_3'$ plays very important role in the analysis. Every induced signature of $T^*H_3'$ is discussed in separate subsection and in each case we have to apply different geometrical and algebraic methods for the classification. For example, in the case when induced metric is Lorentzian we must include some classical results from the projective geometry, while the degenerate case requires more subtle analysis that heavily depends on the signature of the degenerate subspace and often includes the use of euclidean and hyperbolic rotations. The results are summarized in Theorem 3.1.

Section 4 is devoted to investigation of the geometrical properties of the obtained metrics. First, we investigate the curvature properties (Proposition 4.1) and scalar curvature (Propositon 4.2). We show that all parallel vector fields are null in Proposition 4.3. The holonomy of metrics is quite diverse and described in Proposition 4.4. However, we leave some deeper understanding of the holonomy for further research.

In Subsection 4.2 we classify metrics which are algebraic Ricci solitons. In the Riemannian case (see [30]) such metric would be unique up to homotety, but since we work in pseudo-Riemannian settings, we have several non isometric metrics that are shrinking, expanding or steady solitons.

In Subsection 4.3 we consider the invariant complex structure obtained by Salamon [38] in his classification of complex structures on nilpotent Lie algebras, split to subsets according the value of its first and second Betti numbers of $M = \Gamma \backslash G$, where $\Gamma$ is a discrete subgroup of $G$. It is known that the corresponding symplectic structure is 5-dimensional and that the non-flat, Ricci-flat, pseudo-Kähler metrics are admissible (see [12]). In this section we classify pseudo-Kähler metrics and show that they all belong to the same family of metrics (Proposition 4.7).

It is known that every two left invariant metrics with same geodesics are affinely equivalent (see [6]) and that the difference of two such metric is invariant parallel symmetric tensor. In Subsection 4.4 in Proposition 4.8 we show that all such tensors can be obtained using parallel
vectors and therefore, from [25] it follows that metrics admitting such tensors are Riemannian extensions of Euclidean space.

There are lots of known facts about the totally geodesic subalgebras of a nilpotent Lie algebra (see e.g. [7]). Hence, the Subsection 4.5 is devoted to their investigation. Interestingly, for every subalgebra \( h \) of \( T^*_h3 \) there exists a metric that makes it totally geodesic, as shown in Proposition 4.9.

2 Preliminaries

Let us briefly recall the construction of cotangent Lie algebra.

Cotangent algebra \( T^*g \) of Lie algebra \( g \) is semidirect product of \( g \) and its cotangent space \( g^* \)

\[
T^*g := g \ltimes \text{ad}^* g^*,
\]

i.e. the commutators are defined by

\[
[(x, \phi), (y, \psi)] := ([x, y], \text{ad}^* (x)(\psi) - \text{ad}^* (y)(\phi)), \quad x, y \in g, \quad \phi, \psi \in g^*.
\] (1)

By \( \text{ad}^* : g \to \text{gl}(g^*) \) we denote the coadjoint representation

\[
(\text{ad}^* (x)(\phi))(y) := -\phi(\text{ad}(x)(y)) = -\phi([x, y]).
\]

The Heisenberg Lie algebra \( h_3 \) is 3-dimensional nilpotent Lie algebra defined by non-zero commutators

\[
[x_1, x_2] = x_3.
\] (2)

The cotangent algebra \( T^*h_3 \) of \( h_3 \) is 6-dimensional irreducible, 2-step nilpotent algebra with maximal abelian ideal of rank 4 and 3-dimensional center (see [34, Type 3] or [44, Type III3]).

For the convenience, we will fix the basis \( e = (e_1, e_2, e_3, e_4, e_5, e_6) \) such that the Lie algebra \( T^*h_3 \) is defined by non-zero commutators:

\[
[e_1, e_2] = e_6, \quad [e_1, e_3] = -e_5, \quad [e_2, e_3] = e_4.
\] (3)

Note that this relations can be written in the form

\[
[e_i, e_j] = \varepsilon_{ijk} e_{3+k}
\] (4)

where \( \varepsilon_{ijk} \) is totally antisymmetric Levi-Civita symbol and \( i, j, k \in \{1, 2, 3\} \). The commutator subalgebra \( T^*h_3' = [T^*h_3, T^*h_3] \) and the central subalgebra \( Z(T^*h_3) \) coincide

\[
T^*h_3' = \mathbb{R}\langle e_4, e_5, e_6 \rangle = Z(T^*h_3).
\]

Lemma 2.1. The group of automorphisms of Lie algebra \( T^*h_3 \) in basis \( e \) with commutators (3) is given in block-matrix form

\[
\text{Aut}(T^*h_3) = \left\{ \begin{pmatrix} A & 0 \\ B & A^* \end{pmatrix} \mid \det A \neq 0 \right\}
\] (5)

where \( A^* := (\det A)A^{-T} \) and \( A, B \) are \( 3 \times 3 \) matrices, or equivalently as

\[
\text{Aut}(T^*h_3) = \left\{ \begin{pmatrix} \pm (\sqrt{\det C}) C^{-T} & 0 \\ B & C \end{pmatrix} \mid \det C > 0 \right\}
\] (6)
Proof. By the definition, automorphism $F: T^*\mathfrak{h}_3 \to T^*\mathfrak{h}_3$ is linear bijective map satisfying

$$F([u,v]) = [F(u), F(v)], \quad u, v \in T^*\mathfrak{h}_3.$$ 

Automorphism $F$ maps vectors $e_1, e_2, e_3$ to arbitrary vectors

$$F(e_j) = \sum_{i=1}^{3} a_{ij} e_i + \sum_{i=1}^{3} b_{ij} e_{3+i} = a_{ij} e_i + b_{ij} e_{3+i}. \quad (7)$$

where $3 \times 3$ matrix $B = (b_{ij})$ is arbitrary and $3 \times 3$ matrix $A = (a_{ij})$ must be regular. In the last relation we dropped the summation sign assuming summation over repeated indices, as we will do in the sequel. The automorphism $F$ must preserve the commutator subalgebra. This can be written as

$$F(e_{3+j}) = c_{ij} e_{3+i}, \quad j = 1, 2, 3 \quad (8)$$

where $C = (c_{ij})$ is some $3 \times 3$ matrix. This explains the zero block in the matrix (5). Now we find relation between matrices $A$ and $C$.

Using (4) and (8) we get

$$F([e_i, e_j]) = \varepsilon_{ijk} c_{kp} e_{3+p}, \quad (9)$$

$$[F(e_i), F(e_j)] = [a_{ki} e_k + b_{ki} e_{3+k}, a_{mj} e_m + b_{mj} e_{3+m}] = [a_{ki} e_k, a_{mj} e_m]$$

$$= a_{ki} a_{mj} \varepsilon_{kmp} e_{3+p}. \quad (10)$$

Comparing relations (9) and (10) we get

$$\varepsilon_{ijk} c_{pk} = \varepsilon_{kmp} a_{ki} a_{mj},$$

or equivalently,

$$c_{pk} = \varepsilon_{kij} \varepsilon_{kmp} a_{ki} a_{mj} = A_{pk}$$

where $A_{pk}$ is cofactor of element $a_{pk}$ of matrix $A$. Therefore, $A^* = (\det A)(A^{-1})^T = C$ as claimed.

To obtain the second representation take determinant of the relation $C = A^* = (\det A)A^{-T}$ to obtain $\det C = (\det A)^2 > 0$.

3 Classification of metrics

In this section we classify non-isometric left invariant metrics of any signature on $T^*\mathfrak{h}_3$.

If $\mathfrak{g}$ is a Lie algebra and $\langle \cdot, \cdot \rangle$ inner product on $\mathfrak{g}$ the pair $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is called a metric Lie algebra. The structure of metric Lie algebra uniquely defines left invariant pseudo-Riemannian metric on the corresponding simple connected Lie group $G$ and vice versa.

Metric algebras are said to be isometric if there exists an isomorphism of Euclidean spaces preserving the curvature tensor and its covariant derivatives. This translates to the condition that metric algebras are isometric if and only if they are isometric as pseudo-Riemannian spaces (see [1, Proposition 2.2]). Although two isomorphic metric algebras are also isometric, the converse is not true. In general, two metric algebras may be isometric even if the corresponding Lie algebras are non-isomorphic. The test to determine whether any two given solvable metric algebras (i.e. solvmanifolds) are isometric was developed by Gordon and Wilson in [20]. However, by the results of Alekseevskii [1, Proposition 2.3], in the completely solvable case, isometric means isomorphic.
Since Lie algebra $T^*h_3$ is nilpotent and therefore completely solvable, non-isometric metrics on $T^*h_3$ are the non-isomorphic ones.

The isomorphic classes of different left invariant metrics on $T^*h_3$ can be seen as orbits of the automorphism group $\text{Aut}(T^*h_3)$ naturally acting on a space of left invariant metrics. This allows us to use the algebraic approach, although often more geometrical tools are required.

In basis $e$ of $T^*h_3$ metric $\langle \cdot, \cdot \rangle$ is represented by a symmetric $6 \times 6$ matrix $S_e = ((e_i, e_j))$, that we refer as metric matrix. The problem of classification of metrics on $T^*h_3$ is reduced to finding conjugacy classes of symmetric matrices under the action of group $\text{Aut}(T^*h_3)$:

$$S_f = F^T S_e F, \quad F \in \text{Aut}(T^*h_3).$$

(11)

In simple terms we want to find new basis $f = (f_1, f_2, f_3, f_4, f_5, f_6)$ of $T^*h_3$ with brackets of form (3) such that the metric matrix $S_f$ in that basis is as simple as possible. Since commutator algebra $T^*h_3' = \mathbb{R}(e_4, e_5, e_6)$ is invariant under $\text{Aut}(T^*h_3)$ we cannot change its metrical character, i.e. its signature.

Therefore, given symmetric metric matrix $S_e$ in basis $e$, we find its canonical form depending on restriction of metric $\langle \cdot, \cdot \rangle$ on $T^*h_3'$.

Let $S'_e$ be the symmetric $3 \times 3$ matrix representing the restriction. The restriction of the action (11) on $S'$ by automorphism $F \in \text{Aut}(T^*h_3)$ of the form (6) is $C^T S'_e C$. Since $C$ is an arbitrary matrix of positive determinant this action brings $S'_e$ into canonical form given by matrix $\text{diag}(\mu_1, \mu_2, \mu_3)$, $\mu_i \in \{1, -1, 0\}$. To establish the notation let

$$E_{30} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I, \quad E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E_{20} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_{10} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_{00} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(12)

$$E_{03} = -E_{30}, \quad E_{12} = -E_{21}, \quad E_{02} = -E_{20}, \quad E_{01} = -E_{10}.$$

The indexes of $E_{pq}$ denote the signature $(p, q)$, i.e. the number of, respectively, positive and negative vectors in the canonical form of $S'_e$.

Therefore, by choosing matrix $C$ in automorphism $F$ such that the restriction of metric on $T^*h_3'$ is in the canonical form $E_{pq}$, the matrix of metric $\langle \cdot, \cdot \rangle$ in new basis becomes

$$S_{pq} = F^T S_e F = \begin{pmatrix} S & M \\ M^T & E_{pq} \end{pmatrix},$$

(13)

where $M = (m_{ij})$ is arbitrary and $S = (s_{ij})$ is symmetric $3 \times 3$ matrix.

To simplify $S_{pq}$ further, we wish to choose an automorphism from the subgroup that preserves $E_{pq}$ part of matrix $S_{pq}$

$$\text{Aut}(E_{pq}) = \left\{ F \in \text{Aut}(T^*h_3) \mid C^T E_{pq} C = E_{pq} \right\}.$$

(14)

Groups $\text{Aut}(E_{pq})$ and $\text{Aut}(E_{qp})$ are isomorphic. In other cases these groups are fundamentally different and therefore we have to discuss each case of $S_{pq}$ given by (13) separately.
3.1 $T^*\mathfrak{h}_3'$ is definite (case $S_{30}$ and $S_{03}$)

In this case

$$\text{Aut}(E_{30}) = \left\{ \begin{pmatrix} \pm A & 0 \\ B & A \end{pmatrix} \mid A^T A = I, \det A > 0 \right\}$$

(15)

i.e. $A \in SO(3)$ is orthogonal and $B$ arbitrary $3 \times 3$ matrix.

Suppose that in basis $e$ the metric $\langle \cdot, \cdot \rangle$ is represented by matrix $S_{30}$ or $S_{03}$ given by (13). Find a new basis $f$ corresponding to automorphisms $F \in \text{Aut}(E_{30})$ of form (7) for $a_{ij} = \delta_{ij}$, i.e. matrix is identity matrix $A = I$. From the form of $F$ given by (15), we also have $F(e_{3+i}) = e_{3+i}$. Then

$$\langle F(e_j), F(e_{3+k}) \rangle = \langle e_j + b_{ij} e_{3+i}, e_{3+k} \rangle = \langle e_j, e_{3+k} \rangle + b_{ij} e_{3+i}, e_{3+k} \rangle = m_{jk} + b_{ij} \delta_{ik}.$$  

(16)

Therefore, for $b_{jk} = -m_{kj}$ i.e. for $B = -M^T$ we obtain

$$\langle F(e_j), F(e_{3+k}) \rangle = 0, \ j, k \in \{1, 2, 3\}.$$  

Therefore, $F$ brings matrix $S_{30}$ to the form

$$\begin{pmatrix} S & 0 \\ 0 & E_{30} \end{pmatrix} \text{ or } \begin{pmatrix} S & 0 \\ 0 & E_{03} \end{pmatrix}$$

where $S = S^T$ has changed, but we denote it by the same letter to simplify notation. Since, symmetric matrix $S$ can be diagonalized by orthogonal matrix $A$, by using automorphism $F$ of the form (15) we finally get canonical form for definite $T^*\mathfrak{h}_3'$

$$S_{30} = \begin{pmatrix} \Lambda & 0 \\ 0 & E_{30} \end{pmatrix} \text{ or } S_{03} = \begin{pmatrix} \Lambda & 0 \\ 0 & E_{03} \end{pmatrix},$$

(17)

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ and $\lambda_1 \geq \lambda_2 \geq \lambda_3$ are different from zero and of arbitrary sign.

3.2 $T^*\mathfrak{h}_3'$ is Lorentzian (case $S_{21}$ and $S_{12}$)

The admissible automorphisms are

$$\text{Aut}(E_{21}) = \text{Aut}(E_{12}) = \left\{ \begin{pmatrix} \pm A & 0 \\ B & A \end{pmatrix} \mid A^T E_{21} A = E_{21}, \det A > 0 \right\}$$

(18)

i.e. $A \in SO(2,1)$ and $\pm A \in O(2,1)$ and $B$ arbitrary $3 \times 3$ matrix.

Suppose that in basis $e$ the metric $\langle \cdot, \cdot \rangle$ is represented by matrix $S_{21}$ or $S_{12}$ given by (13).

By similar calculations as in (16), one can choose matrices $A = I$ and $B = -E_{21} M^T$ of automorphism $F \in \text{Aut}(E_{21})$ such that in new basis $\langle \cdot, \cdot \rangle$ has the form

$$\begin{pmatrix} S & 0 \\ 0 & E_{21} \end{pmatrix} \text{ or } \begin{pmatrix} S & 0 \\ 0 & E_{12} \end{pmatrix}.$$  

(19)

Now, $3 \times 3$ symmetric matrix $S$ can be of definite or Lorenzian signature. In order to preserve the form (19), we can act by automorphism $F \in \text{Aut}(E_{21})$ having $B = 0$. This reduces to finding equivalence classes of action of group $SO(2,1)$ on Riemannian and on Lorenzian symmetric matrix $S$.  

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3.2.1 $T^*\mathfrak{h}_3'$ is Lorentzian, $T^*\mathfrak{h}_3'\perp$ is Riemannian

The case when $T^*\mathfrak{h}_3'\perp$ is Riemannian is much simpler of the two cases.

Lemma 3.1. Let $S$ be a symmetric matrix with positive eigenvalues. Then there exists matrix $A \in SO(2,1)$ such that $A^T S A$ is diagonal.

Proof. There exists orthogonal matrix $T \in SO(3)$ such that

$$T^{-1}ST = D = \text{diag}(d_1, d_2, d_3), \quad d_i > 0.$$  

Then $S = TDT^{-1}$ and we denote symmetric matrix $\sqrt{S} = T\sqrt{D}T^{-1}$, where $\sqrt{D} = \text{diag}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})$. The matrix $\sqrt{S}^{-1}E_{21}\sqrt{S}^{-1} = (\sqrt{S}^{-1})^T E_{21} \sqrt{S}^{-1}$ is also symmetric (and has the same signature as $E_{21}$). Therefore it can be diagonalized by orthogonal matrix $R \in SO(3)$

$$R^T ((\sqrt{S}^{-1})^T E_{21} \sqrt{S}^{-1}) R = \text{diag}(\frac{1}{\delta_1^2}, \frac{1}{\delta_2^2}, -\frac{1}{\delta_3^2}) = \Delta^{-1} E_{21} \Delta^{-1}$$

where $\Delta = \text{diag}(\delta_1, \delta_2, \delta_3)$. If we denote $A = \sqrt{S}^{-1} R \Delta$, then $\det A > 0$ and

$$A^T E_{21} A = E_{21}, \quad A^T S A = \Delta^2 = \text{diag}(\delta_1^2, \delta_2^2, \delta_3^2),$$

which completes the proof. □

Suppose that metric $\langle \cdot, \cdot \rangle$ is represented by matrix $S_{21}$ or $S_{12}$ given by (19) and matrix $S$ is Lorentzian i.e. of signature (2,1) (or signature (1,2)). Finding canonical form of $S_{21}$ using automorphism $F \in \text{Aut}(E_{21})$ given by (18) with $B = 0$ brings metric to the canonical form

$$S_{21} = \begin{pmatrix} \pm \Delta & 0 \\ 0 & E_{21} \end{pmatrix} \quad \text{or} \quad S_{12} = \begin{pmatrix} \pm \Delta & 0 \\ 0 & E_{12} \end{pmatrix},$$

where $\Delta = \text{diag}(\delta_1^2, \delta_2^2, \delta_3^2)$.

3.2.2 $T^*\mathfrak{h}_3'$ is Lorentzian, $T^*\mathfrak{h}_3'\perp$ is Lorentzian

Suppose that metric $\langle \cdot, \cdot \rangle$ is represented by matrix $S_{21}$ (or $S_{12}$) given by (19) and matrix $S$ is Lorentzian i.e. of signature (2,1) (or signature (1,2)).

Finding canonical form of $S_{21}$ using automorphism $F \in \text{Aut}(E_{21})$ given by (18) reduces to:

**Problem 1.** Find equivalence classes of symmetric matrices $S$ of Lorentzian signature under the action of group $O(2,1)$.

It is useful to consider group $O(2,1)$ as group of isometries of hyperbolic plane. It is best seen in Klein projective model of hyperbolic plane [11].

Any symmetric non-degenerate matrix $H$ can be regarded as projective conic $\Gamma(H)$ with equation

$$\Gamma(H): \quad 0 = x^T H x,$$

where $x = (x_1 \ x_2 \ x_3)^T$ denotes column vector of homogenous coordinates $(x_1 : x_2 : x_3)$. For instance, the Absolute of Klein model $0 = x_1^2 + x_2^2 - x_3^2$ is conic $\Gamma(E_{21})$. We restrict our attention
only to conics represented by symmetric matrix of signature $\binom{2}{1}$ since the case of signature $\binom{3}{0}$ (and $\binom{0}{3}$) we covered in the previous subsection. Also, signature $\binom{3}{0}$ matrix $S$ represents “empty set” conic in real projective geometry.

Projective map $x \rightarrow Cx$, represented by non-degenerate $3 \times 3$ matrix $C$ maps conic $\Gamma(H)$ to conic $\Gamma(C^T HC)$. Hence, condition $C \in O(2,1)$ for the matrix of projective map is equivalent to preserving the Absolute $\Gamma(E_{21})$, i.e. $C$ is hyperbolic isometry.

Moreover, if $H = S$, the matrix of the metric we want to simplify, then we can regard metric $S$ as “conic” $\Gamma(S)$. Therefore, the Problem 1 of classification of metrics is equivalent to the problem of classification of hyperbolic conics:

**Problem 1*. Find canonical forms of projective conics under group of hyperbolic isometries.**

Note, that conic $\Gamma(S)$ must not belong to interior of Absolute (i.e. hyperbolic plane, or de Sitter space), since group $O(2,1)$ also acts on the its exterior (anti de Sitter space).

The classification of hyperbolic conics is classical and well known result [31,40]. In the original paper [40] there are nine types of conics in the classification, but in more recent literature [27,31,37] 12 types appear. However, all those classifications are mostly given by pictures only. In the paper [19] there are equations, but the classification is too complicated and in our case we don’t need to distinguish between all 12 types. We obtain only 4 types, since we consider conics in projective plane as a whole, rather than conics in hyperbolic plane which is intersection of projective plane and interior of the Absolute. Our classification that follows uses concept explained in [37].

We recall some basic facts about hyperbolic isometries in projective Klein model (see e.g. [11]). Let $\Gamma = \Gamma(H), H^T = H$ be a non-degenerate conic. Point $P(\xi_1: \xi_2: \xi_3)$ is said to be a pole and line $p : p_1 x_1 + p_2 x_2 + p_3 x_3 = 0$, i.e. $p(p_1 : p_2 : p_3)$ its polar with respect to $\Gamma$ if $\lambda p = HP$, where $\lambda \neq 0$ is used to emphasize the homogenous nature of coordinates. Observe that $P \in p$ if and only if $P \in \Gamma(H)$. It is well known that projective maps (or changes of coordinates) $x \rightarrow Cx$ preserve the pol-polar relation.

The group of hyperbolic isometries is generated by homologies $\phi_P$ (Klein reflection) with center $P \notin \Gamma(E_{21})$ and its polar $p$ with respect to the Absolute. The Klein reflection $\phi_P(M)$ of point $M$ is defined as point $M'$ such that points $M, M', P, PM$ are harmonic, where $PM$ is intersection of $PM$ and $p$.

In the sequel, we are interested in two conics: for $H = E_{21}$ conic $\Gamma(E_{21})$ representing the Absolute and defining the group of admissible transformations $O(2,1)$, and for $H = S$, conic $\Gamma(S)$ representing the metric we want to simplify.

The conic $\Gamma(S)$ is invariant with respect to Klein reflection $\phi_P$ if $P$ and $p$ are also common pol and polar for both conics $\Gamma(E_{21})$ and $\Gamma(S)$. In that case point $P$ is referred as center of symmetry and $p$ as line of symmetry of $\Gamma(S)$. The main idea is that equation of conic will simplify if the coordinates of its center of symmetry are “nice”.

The condition that $P$ and $p$ are common pol and polar for both $\Gamma(E_{21})$ and $\Gamma(S)$ is $\lambda_1 P = E_{21}P, \lambda_2 p = SP$, or equivalently

$$SP = \lambda E_{21}P \iff (E_{21}S)P = \lambda P, \lambda \neq 0.$$  \hspace{1cm} (22)

Nontrivial solution $P \neq (0 : 0 : 0)$ of that equation exists if and only if

$$\chi_S(\lambda) := \det(S - \lambda E_{21}) = 0.$$ \hspace{1cm} (23)
Note that \( \chi_S(\lambda) \) is not characteristic polynomial of matrix \( S \). Moreover, from (22) is clear that solution of (23) is eigenvalue, and \( P \) is eigenvector of non-square matrix \( E_{21}S \).

By multiplying (22) by \( PT \) from the left we obtain that for common pole \( P \)

\[
|P|^2 = P^TSP = \lambda PT E_{21}P = \lambda |P|^2, \tag{24}
\]

where we have denoted by \( |P|^2 \) the norm of \( P \) with respect to metric \( S \) (i.e. \( \langle \cdot, \cdot \rangle \)) and by \( |P| \) the norm with respect to “hyperbolic” metric defined by \( E_{21} \).

Since \( \chi_S(\lambda) \) is of degree 3 there is at least one real eigenvalue \( \lambda_1 \neq 0 \) corresponding to common pole \( P_1 \).

Case 1. \( |P_1| > 0 \) (equivalently \( P_1 \) is in the exterior of the Absolute)

We can choose new pseudo-orthogonal basis \( f = (f_1, f_2, f_3) = C \in O(2,1) \) of \( T^*b_3 \) such that \( f_1 = \frac{P_1}{|P_1|}, \) and \( f_2, f_3 \) are arbitrary. In new coordinates \( P_1(1 : 0 : 0), \) the matrix of the Absolute is unchanged and

\[
\lambda p_1 = E_{21}P_1 = (1 \ 0 \ 0)^T.
\]

The matrix \( S \) of metric \( \langle \cdot, \cdot \rangle \) has changed to \( \hat{S} = C^TSC = (\hat{s}_{ij}) \), which we want to determine. But, regardless of the change of coordinates \( P_1 \) and \( p_1 \) are pol and polar with respect to the same conic \( \Gamma(\hat{S}) \):

\[
(\lambda \ 0 \ 0)^T = \lambda p_1 = \hat{S}P_1 = (\hat{s}_{11}, \hat{s}_{12}, \hat{s}_{13})^T,
\]

and we have \( \hat{s}_{12} = \hat{s}_{13} = 0 \). Moreover

\[
\hat{s}_{11} = \langle f_1, f_1 \rangle = |f_1|^2 = \lambda_1 |f_1|^2 = \lambda_1.
\]

Therefore, in the case \( |P_1|^2 > 0 \), we may assume that the matrix \( S \) of the metric is of the form

\[
S = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & s_{22} & s_{23} \\
0 & s_{23} & s_{33}
\end{pmatrix} \tag{25}
\]

Now we discuss the possible Jordan forms of the matrix \( E_{21}S \).

Case 1a) \( E_{21}S \) is diagonalizable: \( E_{21}S \sim \text{diag}(\lambda_1, \lambda_2, \lambda_3) \)

If \( P_1, P_2, P_3 \) are corresponding eigenvectors, then the triangle \( P_1P_2P_3 \) is autopolar with respect to both conics \( \Gamma(E_{21}) \) and \( \Gamma(S) \). This means that \( P_i \) is pol of the line \( P_jP_k \) for all distinct \( i, j, k \).

Since \( |P_1| > 0 \), i.e. \( P_1 \) is in the exterior of the Absolute, it is easy to prove that exactly one of \( P_2 \) and \( P_3 \) has to be in the interior—let it be \( P_3 \). Therefore: \( |P_1|^2, |P_2|^2 > 0, |P_3|^2 < 0 \). As in Case 1., and more, we choose

\[
f_1 = \frac{P_1}{|P_1|}, \quad f_2 = \frac{P_2}{|P_2|}, \quad f_3 = \frac{P_3}{|P_3|}.
\]

The fact that \( P_1P_2P_3 \) is autopolar ensures that \( f = (f_1, f_2, f_3) = C \in O(2,1) \). We already know that in the new basis (because of the choice of \( f_1 \)) the matrix of metric conic is of the form (25). The new coordinates of the points are \( P_1(1 : 0 : 0), P_2(0 : 1 : 0), P_3(0 : 0 : 1) \). Using the fact that
$p_2(0:1:0)$ is polar of the pole $P_2$ with respect to the metric conic $\Gamma(S)$ we obtain $s_{23} = 0$. It is easy to check that canonical form is

$$S = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

(26)

where two of $\lambda_i$ are positive and one is negative.

**Case 1b**) $E_{21}S$ has Jordan form $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_3 \end{pmatrix}$.

One calculates that $E_{21}S$ has double eigenvalues if and only if

$$(s_{22} + s_{33})^2 - 4s_{23}^2 = 0 \iff s_{23} = \pm \frac{s_{22} + s_{33}}{2}.$$  

It is easy to check that the automorphism $C = \text{diag}(1,1,-1) \in O(2,1)$ changes $s_{23}$ to $-s_{23}$, so we can suppose that $s_{23} = \frac{s_{22} + s_{33}}{2}$. We obtain canonical form

$$S = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & s_{22} & \frac{s_{22} + s_{33}}{2} \\ 0 & \frac{s_{22} + s_{33}}{2} & s_{33} \end{pmatrix}, \quad \lambda_1 > 0, \quad s_{22} \neq s_{33}.$$  

(27)

The condition on the coefficients ensure that signature of $S$ is $(2,1)$.

**Case 1c**) $E_{21}S$ has Jordan form $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & \bar{z} \end{pmatrix}$, $z \in \mathbb{C}$.

One obtains that $E_{21}S$ has complex conjugate eigenvalues if and only if $(s_{22} + s_{33})^2 - 4s_{23}^2 < 0$. Suppose that $\lambda_1 < 0$. Then both eigenvalues of matrix $S' = \begin{pmatrix} s_{22} & s_{23} \\ s_{23} & s_{33} \end{pmatrix}$ must be positive, i.e. $s_{22}s_{33} - s_{23}^2 > 0$. From the previous two inequalities, we obtain $(s_{22} - s_{33})^2 < 0$, a contradiction. Therefore, case $\lambda_1 < 0$ is impossible. For $\lambda_1 > 0$ we have $s_{22}s_{33} - s_{23}^2 < 0$. Matrix $S'$ represents restriction of the metric $S$ to the plane spanned by $e_2$ and $e_3$ which is of signature $(1,1)$. Null vectors in that plane are

$$v_\pm = \pm s_{33}e_2 + (\mp s_{23} + \sqrt{-\det S'})e_3.$$  

Product of their squared norms (with respect to inner product $E_{21}$)

$$|v_-|^2 |v_+|^2 = s_{33}((s_{22} + s_{33})^2 - 4s_{23}^2)$$

is negative and therefore we may choose $v_+$ to be positive and $v_-$ negative. By hyperbolic rotation

$$f_2 = \cosh \phi e_2 + \sinh \phi e_3, \quad f_3 = \sinh \phi e_2 + \cosh \phi e_3$$

for some $\phi$ we can achieve that $f_3 = v_-$ (it wouldn’t be possible if $v_\pm$ are null or positive). In the new basis $f_1 = e_1$, $f_2$, $f_3 = v_-$ we have $s_{33} = \langle f_3, f_3 \rangle = |v_-|^2 = 0$ (since $f_3$ is chosen to be null vector).

Hence, we obtain the canonical form

$$S = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & s_{22} & s_{23} \\ 0 & s_{23} & 0 \end{pmatrix}, \quad \lambda_1 > 0, \quad s_{23} \neq 0.$$  

(28)
Case 2. \( |P_1| = 0 \) (equivalency \( P_1 \) on the Absolute)

In this case the Jordan form of \( E_{12}S \) is \( \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix} \). From the relation (24) we obtain that \( |P_1|^2 = 0 \) and therefore, 
\[ P_1 \in \Gamma(E_{21}) \cap \Gamma(S), \]
i.e. the \( P_1 \) belong to the intersection of the conics. After rotation, we can assume that \( P_1 \) is any point on the Absolute, for example \( P_1(0 : -1 : 1) \). The polar \( p_1 \) with respect to the Absolute is \( p_1(0 : 1 : 1) \). But \( p_1 \) is also the polar of \( P_1 \) with respect to \( \Gamma(S) \):
\[ \lambda p_1 = sP_1 \iff s_{12} = s_{13}. \quad (29) \]

From the condition that \( P_1 \) belongs to \( \Gamma(S) \) we obtain
\[ s_{33} = -s_{22} + 2s_{23}. \quad (30) \]

Condition that \( \lambda_1 \) is triple root of \( \chi_S \) is equivalent to
\[ s_{22} = s_{11} + s_{23}. \quad (31) \]

Taking into account relations (29), (30) and (31) we obtain that another intersection point of \( \Gamma(S) \) and the Absolute is
\[ M(-4s_{13}s_{23} : 4s_{13}^2 - s_{23}^2 : 4s_{13}^2 + s_{23}^2) \]

We would like to map point \( M \) to \( M_0(1 : 0 : 1) \) by transformation \( C \in O(2,1) \) while fixing point \( P_1 \). The required transformation is homology with center \( \{P\} = MM_0 \cap P_1 \) and axis being its polar \( p = E_{21}P \). One can show that the matrix of that homology is
\[ C = \begin{pmatrix} 8s_{13}^2 & 4s_{13}(s_{23} - 2s_{13}) & 4s_{13}(s_{23} - 2s_{13}) \\ 4s_{13}(s_{23} - 2s_{13}) & -s_{23}^2 - 4s_{23}s_{13} + s_{23}^2 & (s_{23} - 2s_{13})^2 \\ 4s_{13}^2 & -(s_{23} - 2s_{13})^2 & -12s_{13}^2 + 4s_{23}s_{13} - s_{23}^2 \end{pmatrix}. \]

Therefore, we suppose that \( M_0(1 : 0 : 1) \in \Gamma(S) \) or equivalently \( s_{23} = 2s_{13} \) to obtain canonical form
\[ S = \begin{pmatrix} s_{11} & s_{13} & s_{13} \\ s_{13} & s_{11} - 2s_{13} & -s_{13} \\ s_{13} & -s_{13} & -s_{11} - 2s_{13} \end{pmatrix}, \quad s_{11} \neq 0. \quad (32) \]

The signature of this matrix is always Lorentzian. Note that if \( s_{13} = 0 \), we get the previously considered diagonal form (26).

Case 3. \( |P_1| < 0 \) (equivalency \( P_1 \) is in the interior of the Absolute)

Since \( |P_1| < 0 \) we can choose basis \( (f_1, f_2, f_3) \in O(2,1) \) such that \( f_1 = \frac{P}{|P_1|} \). Similarly to Case 1., one obtains that the metric in that basis has matrix
\[ S = \begin{pmatrix} s_{11} & s_{12} & 0 \\ s_{12} & s_{22} & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}. \]
Zeroes of the characteristic polynomial (23) are:
\[-\lambda_1, \lambda_{2/3} = \frac{s_{11} + s_{22} \pm \sqrt{4s_{12}^2 + (s_{22} - s_{12})^2}}{2}.\]

We see that the polynomial can have no multiple roots, nor complex conjugated roots, and we obtain only the Case 1a), i.e. canonical form of metric is (26).

### 3.3 \( T^*\mathfrak{h}_3' \) is degenerate of rank 2 (metrics \( S_{20}, S_{02}, S_{11} \))

#### 3.3.1 Case \( S_{20}, S_{02} \)

Suppose that in basis \( e \) the metric \( \langle \cdot, \cdot \rangle \) is represented by matrix \( S_{20} \) or \( S_{02} \) given by (13). Therefore, we look for canonical form
\[
\begin{pmatrix}
S & M \\
M^T & \pm E_{20}
\end{pmatrix}
\]
(33)
with \( S \) and \( M \) as simple as possible. We first describe the group of isometries of degenerate inner product \( E_{20} \).

**Lemma 3.2.** The subgroup of \( \text{Gl}_3(\mathbb{R}) \) that preserves degenerate quadratic form represented by matrix \( E_{20} \) is
\[
O_3(2,0) = \left\{ \begin{pmatrix}
\lambda & a & b \\
0 & \cos \phi & \mp \sin \phi \\
0 & \sin \phi & \pm \cos \phi
\end{pmatrix} \mid a, b, \phi, \lambda \in \mathbb{R}, \lambda \neq 0 \right\}. \quad (34)
\]

From this lemma and Lemma 2.1 we derive the subgroup of \( \text{Aut}(T^*\mathfrak{h}_3) \) preserving form of matrix \( S_{20} \) or \( S_{02} \)
\[
\text{Aut}(E_{20}) = \text{Aut}(E_{02}) = \left\{ \begin{pmatrix}
\pm A & 0 \\
B & A^*
\end{pmatrix} \right\}, \quad (35)
\]

\[
A = \begin{pmatrix}
\lambda & 0 & 0 \\
a & \cos \phi & \frac{\cos \phi}{\lambda} \\
b & \frac{\sin \phi}{\lambda} & \frac{\cos \phi}{\lambda}
\end{pmatrix}, \quad A^* = \begin{pmatrix}
\frac{\lambda}{\lambda} & -a \cos \phi + b \sin \phi & -b \cos \phi - a \sin \phi \\
0 & \cos \phi & \mp \sin \phi \\
0 & \sin \phi & \pm \cos \phi
\end{pmatrix}.
\]

We denote automorphism \( F \in \text{Aut}(E_{20}) \) of the form (35) by \( F(\lambda, a, b, \phi, B, \lambda \neq 0) \).

Subalgebra \( T^*\mathfrak{h}_3' = \mathbb{R}(e_4, e_5, e_6) \) is degenerated, and from (12) and (13) we see that \( e_4 \in T^*\mathfrak{h}_3 \). Moreover \( T^*\mathfrak{h}_3' \cap T^*\mathfrak{h}_3 \) has codimension one in \( T^*\mathfrak{h}_3 \).

Finding canonical form of the metric \( S = S_{20} \) consists of several steps where we apply automorphisms in very certain order. To simplify the notation we will always denote the resulting matrix by \( S \) and keep the same notation for its entries, although the entries change.

The automorphism are quite restrictive in the plane \( \mathbb{R}(e_2, e_3) \) where we basically can choose new basis only by rotation. We have three cases that correspond to the following geometrical situations:

- **Case 1.** \( \mathbb{R}(e_2, e_3) \cap T^*\mathfrak{h}_3 \) is non-null.
- **Case 2.** \( \mathbb{R}(e_2, e_3) \subset T^*\mathfrak{h}_3 \).
- **Case 3.** \( \mathbb{R}(e_2, e_3) \cap T^*\mathfrak{h}_3 \) is null vector.
The first step is common for all three cases.

**Step 1.** On matrix \( S = S_{20} \) we first apply automorphism \( F(1, 0, 0, \phi, B) \) where

\[
\begin{align*}
\cos \phi &= \frac{m_{31}}{\sqrt{m_{21}^2 + m_{31}^2}}, & \sin \phi &= \frac{m_{21}}{\sqrt{m_{21}^2 + m_{31}^2}}, \\
 b_{22} &= -m_{22}m_{31} + m_{21}m_{32}, & b_{32} &= -m_{23}m_{31} + m_{21}m_{33}.
\end{align*}
\]

This results with the matrix \( F^T SF \) with \( m_{12} = m_{22} = m_{32} = 0. \)

**Case 1.** \( s_{22} \neq 0, m_{31} \neq 0 \)

**Step 2.** We apply automorphism \( F(1, a, b, 0, B) \) where

\[
\begin{align*}
 a &= \frac{m_{11}s_{23} - m_{31}s_{12}}{m_{31}s_{22}}, & b &= -m_{11} + \sqrt{m_{31}}, & b_{12} &= -\frac{s_{23}}{m_{31}},
\end{align*}
\]

\( b_{13} \) is complicated, so it is omitted and the remaining \( b_{ij} \) are zero. After this action we get \( s_{12} = 0 = s_{13}, \quad m_{11} = \sqrt{m_{13}}. \)

**Step 3.** We apply automorphism \( F(1, 0, 0, 0, B) \) where

\[
\begin{align*}
 b_{21} &= -m_{12}, & b_{23} &= -m_{32}, & b_{31} &= -m_{13}, & b_{33} &= -m_{33}, & b_{11} &= \frac{m_{11}^2 + m_{13}^2 - m_{13}}{2\sqrt{m_{13}}},
\end{align*}
\]

Here we have to use complicated parameter \( b_{13} \) again and the remaining \( b_{ij} \) are zero. The resulting matrix of metric has \( s_{11} = m_{12} = m_{13} = m_{32} = m_{33} = 0. \)

**Step 4.** The automorphism \( F(\lambda, 0, 0, 0, B) \), with \( \lambda = \sqrt[3]{m_{13}}, B = 0 \) simultaneously sets \( m_{11} = m_{13} = 1, \) and we obtain canonical form (33) of the metric, with:

\[
S = \begin{pmatrix}
0 & 0 & 0 \\
0 & s_{22} & 0 \\
0 & 0 & s_{33}
\end{pmatrix}, \quad s_{22} \neq 0, s_{33} \neq 0, \quad M = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]  \( \text{(36)} \)

**Case 2.** \( m_{31} = 0 \)

**Step 2.** By automorphism \( F(1, 0, 0, 0, B), b_{23} = -m_{32}, b_{33} = -m_{33} \) we get matrix \( F^T SF \) with \( m_{32} = 0 = m_{33}. \)

**Step 3.** Now we achieve \( s_{11} = s_{12} = s_{13} = 0 = m_{12} = m_{13} \) with appropriate choice of parameters \( a, b, b_{12}, b_{13}, b_{11}. \)

**Step 4.** The automorphism \( F(\lambda, 0, 0, \phi, B), \lambda = m_{11}, B = 0, \) where \( \phi \) is chosen in such way that corresponding rotation diagonals in \( \mathbb{R}^2, e_3 \) yield the canonical form (33), with:

\[
S = \begin{pmatrix}
0 & 0 & 0 \\
0 & s_{22} & 0 \\
0 & 0 & s_{33}
\end{pmatrix}, \quad s_{22}, s_{33} \neq 0, \quad M = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]  \( \text{(37)} \)

**Case 3.** \( s_{22} = 0 \)

In similar way, but without use of rotation, we obtain canonical form (33):

\[
S = \begin{pmatrix}
0 & s_{12} & 0 \\
s_{12} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad s_{12} \neq 0, \quad M = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]  \( \text{(38)} \)
3.3.2 Case $S_{11}$

Suppose that the metric $\langle \cdot, \cdot \rangle$ is represented by matrix $S_{11}$ given by (13).

When $T^*h_3'$ has the signature $(0, +, -)$, we have the following automorphisms:

$$\text{Aut}(E_{11}) = \left\{ \begin{pmatrix} \pm A & 0 \\ B & A^* \end{pmatrix} \right\},$$

(39)

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ a & \cosh \phi & \sinh \phi \\ b & \sinh \phi & \cosh \phi \end{pmatrix}, \quad A^* = \begin{pmatrix} \lambda^2 & -a \cosh \phi + b \sinh \phi & -b \cosh \phi + a \sinh \phi \\ 0 & \cosh \phi & -\sinh \phi \\ 0 & 0 & \cosh \phi \end{pmatrix}. $$

In the plane $\mathbb{R}\langle e_2, e_3 \rangle$ the automorphisms act as hyperbolic rotation which doesn’t necessarily diagonalize metric in that plane. To precisely describe that action we need the following lemma.

**Lemma 3.3.** Equivalence classes of symmetric matrix $S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ under the action $F^T SF$ where $F \in SO(1,1)$ are

$$\begin{pmatrix} a' & 0 \\ 0 & c' \end{pmatrix} \quad \text{if} \quad 4b^2 \neq (a+c)^2, \quad (40)$$

$$\begin{pmatrix} 0 & \frac{c-a}{2} \\ \frac{c-a}{2} & c-a \end{pmatrix} \quad \text{if} \quad 4b^2 = (a+c)^2, |c| > |a|, \quad (41)$$

$$\begin{pmatrix} a-c & \frac{a-c}{2} \\ \frac{a-c}{2} & 0 \end{pmatrix} \quad \text{if} \quad 4b^2 = (a+c)^2, |c| < |a|. \quad (42)$$

Under the $F$ which is anti-isometry, i.e. $F^T \text{diag}(1,-1)F = \text{diag}(-1,1)$, canonical forms (41) and (42) are equivalent.

**Proof.** Denote $E_{11} = \text{diag}(1,-1)$. The group $SO(1,1)$ consists of hyperbolic rotations and their negatives

$$SO(1,1) = \{ F \in GL_2(\mathbb{R}) \mid F^T E_{11} F = E_{11}, \det F = 1 \} = \left\{ \pm \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix} \mid \phi \in \mathbb{R} \right\}.$$  

**Case 1.** $(a+c)^2 - (2b)^2 > 0$ :

It is straightforward to check that hyperbolic rotation by “angle” $\phi$ such that

$$\cosh 2\phi = \lambda |a + c|, \quad \sinh 2\phi = -2\lambda \text{sgn}(a+c)b$$

where $\lambda = (|a+c|^2 - (2b)^2)^{-\frac{1}{4}}$ is determined from the condition $\cosh^2 2\phi - \sinh^2 2\phi = 1$, diagonalizes matrix $S$ and we obtain the form (40).

**Case 2.** $(a+c)^2 - (2b)^2 < 0$ :

In this case we diagonalize $S$ with hyperbolic rotation such that

$$\cosh 2\phi = \lambda |2b|, \quad \sinh 2\phi = -2\lambda \text{sgn}(b)(a+c)$$

and $\lambda = ((2b)^2 - (a+c)^2)^{-\frac{1}{4}}$. 

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**Case 3.** \((a + c)^2 - (2b)^2 = 0:\)

Suppose that \(b = \frac{a+c}{2}\). In this case null directions of metric \(S\) are \((1, -1)\) and \((c, -a), a > 0\). We will apply hyperbolic rotation such that one of basis vectors is null. The case \(|a| = |c|\) is either diagonal or impossible. If \(|a| > |c|\) we take hyperbolic rotation such that

\[
\cosh \phi = \frac{a}{\sqrt{a^2 - c^2}}, \quad \sinh \phi = \frac{-c}{\sqrt{a^2 - c^2}},
\]

to obtain canonical form (42). If \(|a| < |c|\) we take hyperbolic rotation

\[
\cosh \phi = \frac{c}{\sqrt{c^2 - a^2}}, \quad \sinh \phi = \frac{-a}{\sqrt{c^2 - a^2}},
\]

to obtain canonical form (41). Note that these two cases are not equivalent under the action of \(O(1,1)\) since the null direction of metric \(S\) belongs to either set of time-like or set of space-like vectors of metric \(\text{diag}(1, -1)\) that are preserved under the action of \(SO(1,1)\) (and \(O(1,1)\) as well).

However, if we admit anti-isometries we can change time-like and space-like vectors, and these two cases are equivalent.

The case \(b = -\frac{a+c}{2}\) is similar.

The classification of metrics of type \(S_{11}\) is similar the case of type \(S_{20}\) with possible difference only when a rotation is used. Hence, in the sequel, we follow the steps from Subsection 3.3.1. We look for the canonical form

\[
\begin{pmatrix}
S \\ M \\ M^T \\ \pm E_{11}
\end{pmatrix}
\]

(43)

with \(S\) and \(M\) as simple as possible.

Hyperbolic rotation is not transitive on the vectors of the plane. In “regular” cases the hyperbolic rotation can be used instead of Euclidean rotation and we obtain metrics (43) with \(S, M\) given by (36), (37) or (38).

Now we discuss “singular” cases. Already in the **Step 1.** the hyperbolic rotation is not possible if \(m_{21} = \pm m_{31} \neq 0\). In fact, those two cases are equivalent by an anti isometric automorphism, so we consider case \(m_{21} = m_{31}\). After long and detailed analysis we obtain two nonequivalent metrics

\[
\begin{pmatrix}
S \\ M \\ M^T \\ E_{11}
\end{pmatrix}, \quad S = \begin{pmatrix}
s_{11} & 0 & 0 \\ 0 & s_{22} & 0 \\ 0 & 0 & 0
\end{pmatrix}, \quad s_{11}, s_{22} \neq 0, \quad M = \begin{pmatrix}
0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0
\end{pmatrix}, \quad (44)
\]

\[
\begin{pmatrix}
S \\ M \\ M^T \\ E_{11}
\end{pmatrix}, \quad S = \begin{pmatrix}
0 & s_{12} & 0 \\ s_{12} & 0 & 0 \\ 0 & 0 & 0
\end{pmatrix}, \quad s_{12} > 0, \quad M = \begin{pmatrix}
0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0
\end{pmatrix}. \quad (45)
\]

If \(m_{21} \neq \pm m_{31}\) then in **Step 1.** we can achieve \(m_{21} = 0\) using hyperbolic rotation and proceed with the remaining steps.

In **Case 1.** rotation is not used, and no additional canonical forms are obtained.
In **Case 2.** rotation is used in **Step 4.** to diagonalize metric in $\mathbb{R}(e_2, e_3)$ plane. According to Lemma 3.3 this is not always possible with hyperbolic rotation, so we obtain additional metrics

\[
\begin{pmatrix}
S & M \\
M^T & E_{11}
\end{pmatrix}, \quad S = \begin{pmatrix}
0 & 0 & 0 \\
0 & s_{22} & \frac{1}{2}|s_{22}| \\
0 & \frac{1}{2}|s_{22}| & 0
\end{pmatrix}, \quad s_{22} \neq 0,
\]

\[
M = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad (46)
\]

\[
\begin{pmatrix}
S & M \\
M^T & E_{11}
\end{pmatrix}, \quad S = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \frac{1}{2}|s_{33}| \\
0 & \frac{1}{2}|s_{33}| & s_{33}
\end{pmatrix}, \quad s_{33} \neq 0,
\]

\[
M = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Finally, in **Case 3.** rotation is not used so the classification of metrics with center of signature $(0, +, -)$ is complete.

### 3.4 $T^*h_3$ is degenerate of rank 1 (case $S_{10}$, $S_{01}$)

Suppose that in basis $e$ the metric $\langle \cdot, \cdot \rangle$ is represented by matrix $S_{10}$ or $S_{01}$ given by (13). Therefore, we look for canonical form

\[
\begin{pmatrix}
S & M \\
M^T & \pm E_{10}
\end{pmatrix}
\]

with $S$ and $M$ as simple as possible.

When $T^*h_3$ has the signature $(0, 0, +)$ or $(0, 0, -)$ we have the following group of automorphisms preserving its canonical form

\[
\text{Aut}(E_{10}) = \text{Aut}(E_{01}) = \left\{ \begin{pmatrix}
\pm A & 0 \\
B & A^*
\end{pmatrix} \right\},
\]

\[
A = \begin{pmatrix}
a_{11} & a_{12} & 0 \\
a_{21} & a_{22} & 0 \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}, \quad A^* = \begin{pmatrix}
a_{22}a_{33} & -a_{21}a_{33} & a_{21}a_{32} - a_{22}a_{31} \\
a_{12}a_{33} & a_{11}a_{33} & a_{12}a_{31} - a_{11}a_{32} \\
0 & 0 & a_{11}a_{22} - a_{12}a_{21}
\end{pmatrix},
\]

with the condition $(a_{11}a_{22} - a_{12}a_{21})^2 = 1$. Notice that the automorphism of the form:

\[
F = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
\]

switches places of elements $m_{31}$ and $m_{32}$ in matrix $M$. Hence, we distinguish between two cases: if $m_{31} \neq 0$ and if $m_{31} = m_{32} = 0$. It is worth noting that this is not a simple algebraic distinction. These two cases will generate completely different geometric properties (see Proposition 4.4 (iii) below).

**Case 1.** $m_{31} \neq 0$ In this case, we can obtain the following form of $M$:

\[
\begin{pmatrix}
0 & m_{12} & 0 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

by taking the next steps.
**Step 1.** The appropriate choice of elements $a_{21}$ and $a_{31}$ in (49) gives us $m_{11} = m_{32} = 0$, while we can set $a_{22}$ and $a_{32}$ in a way that $m_{21} = t$, $m_{13} = t^2$, where $t \neq 0$ is an arbitrary parameter that will be normalized later. Finally, by setting $a_{21}$, $m_{22} = 0$ is obtained.

**Step 2.** Now, by selecting the last row of matrix $B$, we get $m_{13} = m_{23} = m_{33} = 0$.

**Step 3.** Choosing the remaining elements of matrix $B$, the matrix $S$ in (48) is reduced to $\pm \lambda E_{01}$, $\lambda \neq 0$.

**Step 4.** In the last step, we normalize both $\lambda$ and $t$ and get the metric (48) with $S = \pm E_{01}$ and $M$ taking the form (50).

**Case 2.** $m_{31} = m_{32} = 0$ Since the elements $a_{31}$ and $a_{32}$ do not act on the matrix $M$, the problem reduces to the action $A^T MA$ of matrix:

$$A = \begin{pmatrix} \bar{A} & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{A} \in SL(2).$$

In the first step, depending on the nature of eigenvalues of matrix $M$, we can choose matrix $\bar{A}$ such that upper-left $2 \times 2$ submatrix of $M$ takes one of the following three forms:

$$\begin{pmatrix} m_{11} & 0 \\ 0 & m_{22} \end{pmatrix}, \quad \begin{pmatrix} m_{11} & m_{12} \\ -m_{12} & m_{11} \end{pmatrix}, \quad \begin{pmatrix} m_{11} & 0 \\ 1 & m_{11} \end{pmatrix}. $$

Next, we can repeat **Step 2.** and **Step 3.** from the above. Finally, in the last step we again make the basis vector $e_3$ to be unit. Therefore, our metric $S_{10} = (\pm E_{01}, M, E_{01})$, with $M$ taking one of the three forms:

$$\begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} m_{11} & m_{12} & 0 \\ -m_{12} & m_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} m_{11} & 0 & 0 \\ 1 & m_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (51)$$

Note that the case of metric $S_{01}$ can be considered completely analogously.

### 3.5 $T^*h_3'$ is degenerate of rank 0 (case $S_{00}$)

The last case of totally degenerate center is the only case that can be considered using purely algebraic approach. Suppose that in basis $e$ the metric $\langle \cdot, \cdot \rangle$ is represented by matrix $S_{00}$ given by (13).

The group of admissible automorphisms is

$$\text{Aut}(E_{00}) = \left\{ \begin{pmatrix} \pm A & 0 \\ B & A^* \end{pmatrix} \right\}, \quad (52)$$

where $A$, $\text{det} A \neq 0$ and $B$ are arbitrary $3 \times 3$ matrices.

If we take the automorphism $F$ of the form (52) and act on the matrix $S_{00}$, we get:

$$\begin{pmatrix} A^T & B^T \\ 0 & (A^*)^T \end{pmatrix} \begin{pmatrix} S & M \\ M^T & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ B & A^* \end{pmatrix} = \begin{pmatrix} A^T SA + A^T MB + (A^T MB)^T \\ (A^T MA^*)^T \end{pmatrix}. \quad (53)$$
The non-degeneracy of metric matrix $S_{00}$ gives us that the matrix $M$ must also be regular. Hence, by setting $B = -\frac{1}{2}M^{-1}SA$, the matrix (53) takes the form:

$$
\begin{pmatrix}
0 & A^TMA^* \\
(A^TMA^*)^T & 0
\end{pmatrix}
$$

and the only thing left to do is to choose a regular matrix $A$ such that $A^TMA^*$ has the simplest form. However, one must have in mind that the matrix $M$ is not symmetric, therefore it is not necessarily diagonalizable. At least one eigenvalue of $M$ must be real and the remaining two can either be real (with some multiplicity) or complex conjugate. Therefore, the possible canonical Jordan forms of $M$ are:

$$
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix},
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 1 \\
0 & 0 & \lambda_3
\end{pmatrix},
\begin{pmatrix}
\lambda_1 & 1 & 0 \\
0 & \lambda_1 & 1 \\
0 & 0 & \lambda_1
\end{pmatrix},
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & -\lambda_3 \\
0 & \lambda_3 & \lambda_2
\end{pmatrix}.
$$

We can make one more step to further simplify these forms: by setting the automorphism matrix to be diagonal, in (54) we can obtain $\lambda_1 = 1$.

Note that all these metrics have neutral signature.

### 3.6 Main result

The previous extensive analysis proves the following theorem.

**Theorem 3.1.** The non-isometric left invariant metrics on $T^*h_3$ in basis $e$ with commutators (3) are represented by matrices $S_{pq} = (S,M,E_{pq})$ of the form (13):

(i) if $T^*h_3'$ is non-degenerate:

- $S_{30} = (S,0,\pm E_{30}),$ where $S$ takes the form (17);
- $S_{21} = (S,0,\pm E_{21}),$ where $S$ takes one of the forms (21), (26), (27), (28) or (32);

(ii) if $T^*h_3'$ is degenerate of rank 2:

- $S_{20} = (S,M,\pm E_{20}),$ where $S$ and $M$ take one of the forms (36), (37) or (38);
- $S_{11} = (S,M,\pm E_{11}),$ where $S$ and $M$ take one of the forms (36) or (38);
- $S_{11} = (S,M,E_{11}),$ where $S$ and $M$ take one of the forms (37), (44), (45), (46) or (47);

(iii) if $T^*h_3'$ is degenerate of rank 1:

- $S_{10} = (\pm E_{10},M,\pm E_{10}),$ where $M$ takes one of the forms (50) or (51) (here all four combinations of $\pm$ can occur);

(iv) if $T^*h_3'$ is degenerate of rank 0:

- $S_{00} = (0,M,0),$ where $M$ takes one of the forms (54) with $\lambda_1 = 1.$

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4 Geometrical properties of left invariant metrics

In this section the metrics obtained in Theorem 3.1 are further investigated. First, their curvature properties are of interest and then we briefly consider the holonomy algebras for each metric. Also, we get the description of parallel symmetric tensors for each metric and show that they are derived from parallel vector fields. Special types of metrics, such as pp-waves or Ricci solitons, are also examined. Since $T^*h_3$ is even dimensional, it is natural to investigate the invariant complex and symplectic structures. Consequently, the classification of pseudo-Kähler metrics is obtained. In the end, the well known facts about the totally geodesic subalgebras of a nilpotent Lie algebra are summarized and it is shown that for every subalgebra of $T^*h_3$ there exists at least one metric that makes it totally geodesic.

4.1 Curvature and holonomy of the metrics

If $S$ is matrix corresponding to $\langle \cdot, \cdot \rangle$ metric in basis $(e_1, \ldots, e_6)$, the algebra of its isometries $so(S) \cong so(p, q)$, $p + q = 6$ is spanned by endomorphisms $e_i \wedge e_j$, $1 \leq i < j \leq 6$ defined by $(e_i \wedge e_j)(x) := \langle e_j, x \rangle e_i - \langle e_i, x \rangle e_j, \quad x \in T^*h_3$. (55)

For the left invariant vector fields $x, y, z \in T^*h_3$ Koszul’s formula reduces to

$$2\langle \nabla_x y, z \rangle = \langle [x, y], z \rangle - \langle [y, z], x \rangle + \langle [z, x], y \rangle,$$

which allows us to calculate Levi-Civita connection $\nabla$ of metric $\langle \cdot, \cdot \rangle$. The curvature $R$ and Ricci tensor $\rho$ are given by:

$$R(x, y)z = \nabla_x (\nabla_y z) - \nabla_y (\nabla_x z) - \nabla_{[x, y]} z, \quad \rho(x, y) = Tr(z \mapsto R(z, x)y).$$

The scalar curvature is defined as trace of Ricci operator.

Metric $\langle \cdot, \cdot \rangle$ is said to be flat if the corresponding curvature tensor is zero everywhere, i.e. $R = 0$, and it is locally symmetric if $\nabla R = 0$. Similarly, metric is Ricci-flat if $\rho = 0$ and Ricci-parallel if $\nabla \rho = 0$.

We can further simplify the above definitions having in mind that we investigate curvature operators on the nilpotent Lie group. Let us define operators $ad^*_x, j_x$ and $\varphi_x$:

$$\langle ad_x y, z \rangle = \langle y, ad^*_x z \rangle, \quad j_x y := ad^*_y x; \quad \varphi_x := ad_x + ad^*_x.$$

Then the following lemma holds.

Lemma 4.1 ([1]). In case of nilpotent Lie algebra $\mathfrak{g}$ the curvature and Ricci tensors are given by:

$$R(x, y) = \frac{1}{2}(j_{[x, y]} + [j_x, ad^*_y] + [ad^*_x, j_y]) - \frac{1}{4}([\varphi_x, \varphi_y] + [j_x, \varphi_y] + [\varphi_x, j_y] - [j_x, j_y]),$$

$$\rho(x, y) = -\frac{1}{4}tr(j_x \circ j_y) - \frac{1}{2}tr(ad_x \circ ad^*_y),$$

for all left invariant vector fields $x, y \in \mathfrak{g}$.

In the following statement we describe curvature and Ricci curvature of metrics on $T^*h_3$ depending on signature of induced metric of $T^*h_3$. 

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Proposition 4.1. (i) If $T^*h_3'$ is nondegenerate the metric can’t be flat or Ricci flat.

(ii) If $T^*h_3'$ is degenerate of rank 2, the metrics $S_{20} = (S, M, \pm E_{20})$, where $S$ and $M$ take the form (37), and $S_{11} = (S, M, E_{11})$, where $S$ and $M$ take one of the forms (37), (46) or (47), are Ricci-parallel. Specially, Ricci-flat are the metrics:

\[
S_{20} = (S, M, \pm E_{20}), \quad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_{22} & 0 \\ 0 & 0 & \pm 1 - s_{22} \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
S_{11} = (S, M, E_{11}), \quad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_{22} & 0 \\ 0 & 0 & -1 + s_{22} \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
S_{11} = (S, M, E_{11}), \quad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 + \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

(iii) If $T^*h_3'$ is degenerate of rank 1, the corresponding metrics $S_{10} = (\pm E_{10}, M, E_{10})$, where $M$ takes one of the forms (51), are locally symmetric and Ricci-flat. Specially, metrics:

\[
S_{10} = (E_{10}, M, E_{10}), \quad M = \begin{pmatrix} \lambda \pm \sqrt{3} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
S_{10} = (-E_{10}, M, E_{10}), \quad M = \begin{pmatrix} \lambda & \pm \sqrt{3} & 0 \\ -\sqrt{3} & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

are flat. Also, the Ricci-parallel metric occurs and it has the form:

\[
S_{10} = (\pm E_{10}, M, \pm E_{10}), \quad M = \begin{pmatrix} 0 & \lambda & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\]

(iv) If $T^*h_3'$ is degenerate of rank 0, the corresponding metrics are flat.

(v) The only examples of Einstein metrics (i.e. metrics with proportional Ricci curvature and metric tensors) are the trivial, Ricci-flat ones.

Proof. The proof is straightforward, but long, since it is a case by case calculation for every canonical form. We give some details for metric $S_{10} = (E_{10}, M, E_{10})$ with $T^*h_3'$ of rank 1, where is $M$ given by (50).

Using (56) we obtain Levi-Civita connection of the metric in terms of non zero derivations.
(having in mind the relation $[x, y] = \nabla_x y - \nabla_y x$)

$$\nabla e_1 e_1 = m_{12}(-e_2 + e_3), \quad \nabla e_1 e_2 = \frac{1}{2} e_6, \quad \nabla e_1 e_3 = -e_5$$

$$2\nabla e_1 e_6 = -e_2 + e_3 - e_4 = \nabla e_2 e_3 = \nabla e_3 e_3,$$

$$\nabla e_2 e_2 = e_2 - e_3, \quad \nabla e_2 e_6 = \frac{1}{2m_{12}} e_5.$$  \hfill (57)

Note, that vectors $e_4, e_5$ are parallel. In Proposition 4.3 was proven that all parallel vectors are given by their linear combination.

Using Lemma 4.1, or directly from the definition of curvature, we obtain that non zero curvature operators are given by:

$$R(e_1, e_2) = 3 \frac{4m_{12}}{m_{12}} (-e_2 \wedge e_5 + e_3 \wedge e_5) + \frac{4m_{12} - 3}{m_{12}} e_4 \wedge e_5 + \frac{1}{2} e_4 \wedge e_6,$$

$$R(e_1, e_3) = e_4 \wedge e_5 + \frac{1}{2} e_4 \wedge e_6, \quad R(e_1, e_6) = -\frac{1}{4m_{12}} e_5 \wedge e_6,$$

$$R(e_2, e_6) = R(e_3, e_6) = -\frac{1}{2m_{12}} e_4 \wedge e_5.$$

By using Lemma 4.1 again we obtain that the only nonzero component of Ricci tensor is

$$\rho(e_1, e_1) = -\frac{1}{2}.$$  \hfill (58)

One can easily check that this metric is Ricci parallel, $\nabla \rho \equiv 0$. Also, we check (see the proof of Proposition 4.4) that $\nabla R \not\equiv 0$, and therefore the metric is not locally symmetric.

In [33] Milnor proved that in the Riemannian case if the Lie group $G$ is solvable, then every left invariant metric on $G$ is either flat, or has strictly negative scalar curvature. Notice that in the pseudo-Riemannian setting this does not hold. In the case of non-degenerate $T^*h_3'$, depending on a signature, scalar curvatures can be positive, negative or zero while in case of degenerate $T^*h_3'$, all but two metrics have zero scalar curvature. More precisely, we have the following statement which can be obtained from Proposition 4.1 by direct computation.

**Proposition 4.2.** (i) Scalar curvature of metrics $(S, 0, E_{pq})$, $p+q = 3$ on $T^*h_3$ with nondegenerate center $T^*h_3'$ is given by

$$\tau = -\frac{\text{trace}(SE_{pq})}{2 \det S}.$$  

(ii) Metrics $(S, M, E_{pq})$, $p + q = 2$ with $S$ and $M$ given by (36) and (38) have non-zero scalar curvature given, respectively, by $\tau = \mp \frac{\epsilon}{2 e_2 e_3}$ and $\tau = \pm \frac{\epsilon}{2 e_1 e_2}$, where $\epsilon$ is element in position $(3, 3)$ of $E_{pq}$.

(iii) All other metrics on $T^*h_3$ with degenerate center $T^*h_3'$ have zero scalar curvature $\tau = 0$.

**Example 1.** In [9, Example 5.1] the authors considered canonical metric defined by:

$$\langle (x, \alpha), (x', \alpha') \rangle = \alpha'(x) + \alpha(x'), \quad \forall x, x' \in h_3, \quad \alpha, \alpha' \in h_3^*.$$
This metric is neutral signature and ad-invariant, meaning that \(\langle [x, y], z \rangle = -\langle y, [x, z] \rangle\), for all \(x, y, z \in T^*\mathfrak{h}_3\). Notice that this is a special case of our metric \(S_{00} = (0, M, 0)\), when \(M\) is the identity matrix. This is the only ad-invariant metric on \(T^*\mathfrak{h}_3\) which confirms the result recently obtained in [10].

In the sequel we find all parallel vector fields of metrics on \(T^*\mathfrak{h}_3\). Their presence has important consequences on holonomy group of the metrics as well as on the existence of parallel symmetric tensors (see Section 4.4). They are characterized by the following lemma.

**Lemma 4.2.** The left invariant vector field \(x \in \mathfrak{g}\) on metric Lie algebra \((\mathfrak{g}, \langle \cdot, \cdot \rangle)\) is parallel (that is \(\nabla_y x = 0\) for all \(y \in \mathfrak{g}\)) if and only if \(x \perp \mathfrak{g}'\) and \(\text{ad}^*_x = -\text{ad}_x\).

**Proof.** Note that the Koszul’s formula (56) can be re-written in the following form:

\[
\nabla_y x = \frac{1}{2} (\text{ad}_y x - \text{ad}^*_y x - \text{ad}_x^* y), \quad x, y \in \mathfrak{g}.
\]

The statement of the lemma follows from this directly.\qed

**Proposition 4.3.** Let metric on \(T^*\mathfrak{h}_3\) be given by the matrix \(S\) from Theorem 3.1. In all cases parallel vector fields are null. Moreover:

(i) If \(T^*\mathfrak{h}_3'\) is non-degenerate, than there are no parallel vector fields;

(ii) If \(T^*\mathfrak{h}_3'\) is degenerate of rank 2, the only parallel vector fields are \(x \in \mathbb{R}(e_4)\);

(iii) If \(T^*\mathfrak{h}_3'\) is degenerate of rank 1, parallel vector fields are \(x \in \mathbb{R}(e_4, e_5)\);

(iv) If \(T^*\mathfrak{h}_3'\) is totally degenerate, all vectors from \(T^*\mathfrak{h}_3' = \mathbb{R}(e_4, e_5, e_6)\) are parallel.

Note that, since nilpotent group is simply connected, the restricted holonomy group coincides with full holonomy group. According to Ambrose-Singer theorem the holonomy algebra is generated by curvature operators \(R(x, y)\) and their covariant derivatives of any order. We know that holonomy algebra is subalgebra of isometry algebra, i.e. \(\text{so}(p, q)\), where \((p, q)\) denotes the signature of the metric.

The results are summarized in the following proposition.

**Proposition 4.4.** Let the non-flat metric on \(T^*\mathfrak{h}_3\) be given by the matrix \(S\) from Theorem 3.1. In all cases parallel vector fields are null. Moreover:

(i) If \(T^*\mathfrak{h}_3'\) is non-degenerate, the corresponding metrics have full holonomy algebra, \(\text{hol}(S) = \text{so}(p, q)\), \(p + q = 6\).

(ii) If \(T^*\mathfrak{h}_3'\) is degenerate of rank 2, then the following cases can occur:

\(\circ\) the holonomy algebra is 10-dimensional \(\text{so}(p, q)\), \(p + q = 5\), if the corresponding metric is \(S_{20} = (S, M, \pm E_{20})\) or \(S_{11} = (S, M, \pm E_{11})\), where matrices \(S\) and \(M\) take one of the forms (36) or (38), and \(S_{11} = (S, M, E_{11})\), with \(S\) and \(M\) taking the form (44);

\(\circ\) the holonomy algebra is 9-dimensional isomorphic to \(\text{sl}_2(\mathbb{R}) \ltimes \mathfrak{g}_{6,54}\), where \(\mathfrak{g}_{6,54}\) is six dimensional solvable algebra with five dimensional nilradical, (see [35]), in case of the metric \(S_{11} = (S, M, E_{11})\), with \(S\) and \(M\) taking the form (45);

\(\circ\) the holonomy algebra is 4-dimensional and isomorphic to \(\mathbb{R}^4\), if the corresponding metric is \(S_{20} = (S, M, \pm E_{20})\) or \(S_{11} = (S, M, \pm E_{11})\), where matrices \(S\) and \(M\) take the form (37), \(S_{11} = (S, M, E_{11})\), with \(S\) and \(M\) taking one of the forms (46) or (47).
(iii) If $T^\ast b_4$ is degenerate of rank 1, the non-flat metrics $S_{10} = (\pm E_{10}, M, \pm E_{10})$, where $M$ takes one of the forms (51), have holonomy algebra isomorphic to $\mathbb{R}$, while if $M$ take the form (50), the holonomy algebra is 5-dimensional and isomorphic to the 2-step nilpotent algebra given by the commutators $[h_1, h_3] = [h_2, h_4] = h_5$.

Proof. The proof is case by case for all types of metrics. We illustrate it for case (iii), i.e. for metric $S_{10} = (E_{10}, M, E_{10})$, where $M$ is given by (50), the same that we discussed in the proof of Proposition 4.1. From there we know that curvature operators

$$ r_1 := R(e_1, e_2), \quad r_2 := R(e_1, e_3), \quad r_3 := R(e_1, e_6), \quad r_4 := R(e_2, e_3), $$

are linearly independent and generate the space $\mathbb{R} \langle \{ R(e_i, e_j) \mid i, j = 1, \ldots, 6 \} \rangle$. Using connection formulas (57) we calculate their derivatives and see that

$$ r_5 := \nabla_{e_1} R(e_1, e_3) = \frac{1}{4} (e_2 \wedge e_4 - e_3 \wedge e_4) $$

is the only operator not belonging to $\mathbb{R} \langle r_1, r_2, r_3, r_4 \rangle$. Now we calculate covariant derivatives of $r_1, \ldots, r_5$ and see that they all belong to $\mathbb{R} \langle r_1, r_2, r_3, r_4, r_5 \rangle$. Therefore the holonomy algebra is spanned by curvature operators and their first covariant derivatives and

$$ \text{hol}(S) = \mathbb{R} \langle r_1, r_2, r_3, r_4, r_5 \rangle \subset o(4,2), $$

(59)

since signature of $S$ is $(4,2)$ for all $m_{12} \neq 0$. Now we obtain nonzero commutators

$$ [r_1, r_3] = \frac{1}{3} r_4, \quad [r_1, r_5] = -\frac{3}{8} r_4, \quad [r_2, r_3] = \frac{1}{4} r_4, $$

which after setting

$$ h_1 = r_2, \quad h_2 = -3r_1 + 4r_2, \quad h_3 = r_3, \quad h_4 = \frac{2}{9} r_5, \quad h_5 = \frac{1}{4} r_4, $$

yields the form formulated in the statement.

Now, let us discuss in more detail the case (ii2). Similar to the previous consideration, we get that the holonomy algebra is given by the following non-zero commutators:

$$ [h_1, h_2] = 2h_2, \quad [h_2, h_3] = h_3, \quad [h_3, h_7] = h_4, \quad [h_5, h_7] = h_6, \quad [h_7, h_8] = h_9, $$

$$ [h_1, h_3] = h_3, \quad [h_2, h_6] = h_4, \quad [h_3, h_9] = -h_3, \quad [h_5, h_9] = -h_5, \quad [h_7, h_9] = 2h_7, $$

$$ [h_1, h_4] = h_4, \quad [h_4, h_8] = -h_3, \quad [h_6, h_8] = -h_5, \quad [h_8, h_9] = -2h_8, \quad (60) $$

$$ [h_1, h_5] = -h_5, \quad [h_4, h_9] = h_4, \quad [h_6, h_9] = h_6, $$

$$ [h_1, h_6] = -h_6. $$

By Levi decomposition, we know that the algebra $\text{hol}(S)$ is a semidirect product of its maximal solvable ideal and a semisimple Lie algebra. Note that $\mathbb{R} \langle h_7, h_8, h_9 \rangle \cong sl_2(\mathbb{R})$ and that $\mathbb{R} \langle h_1, \ldots, h_6 \rangle$ is isomorphic to the 6-dimensional solvable Lie algebra denoted by $\mathfrak{g}_{6,54}$ (with $\lambda = 1, \gamma = 2$) in the classification of Mubarakzyanov [35, Table 4]. Hence, $\text{hol}(S) \cong sl_2(\mathbb{R}) \ltimes \mathfrak{g}_{6,54}$, where the form of
\[ \pi : \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{g}_{6,54} \] is retrieved from the relations (60):

\[
\pi(x) = \rho(x_7h_7 + x_8h_8 + x_9h_9) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x_9 & -x_7 & 0 & 0 \\
0 & 0 & x_8 & -x_9 & 0 & 0 \\
0 & 0 & 0 & 0 & x_9 & -x_7 \\
0 & 0 & 0 & 0 & x_8 & -x_9
\end{pmatrix}.
\]

Recall that a metric \( g \) is called \( pp \)-wave metric if there exists a parallel null vector field \( v \) such that \( R(u, w) = 0 \) for all \( u, w \in v^\perp \).

**Proposition 4.5.** Left invariant \( pp \)-wave metrics are \( S_{20} = (S^1, M, \pm E_{20}) \), \( S_{11} = (S^k, M, \pm E_{11}) \), \( k = 1, 2 \), where

\[
M = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad S^1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & s_{22} & 0 \\
0 & 0 & s_{33}
\end{pmatrix}, \quad S^2 = \begin{pmatrix}
0 & 0 & 1/2 |s_{22}| \\
0 & s_{22} & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad s_{22}, s_{33} \neq 0,
\]

and \( S_{10} = (\pm E_{10}, M, \pm E_{10}) \), where \( M \) takes one of the forms (51).

**Proof.** We have already noticed that the basis vector \( e_4 \) is parallel null vector field for all metrics from the proposition. The space orthogonal to \( e_4 \) is spanned by vectors \( e_2, \ldots, e_6 \) in every case, except for metric \( S_{10} \) with

\[
M = \begin{pmatrix}
m_{11} & m_{12} & 0 \\
-m_{12} & m_{11} & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

where it is spanned by vectors \( e_3, \ldots, e_6 \). However, it is a straightforward calculation to show that \( R(e_i, e_j) = 0 \), \( i, j = 2, \ldots, 6 \), in all cases. Hence, the metrics are \( pp \)-waves. \( \square \)

**Corollary 4.1.** All left invariant metrics on \( T^*\mathfrak{h}_3 \) with abelian holonomy algebra \( \mathbb{R}^k \), \( k = 1, 4 \), are homogenous \( pp \)-wave metrics.

### 4.2 Algebraic Ricci solitons on \( T^*\mathfrak{h}_3 \)

Since we have already seen that the only Einstein metrics are the trivial ones, the next step is to consider a weaker condition - Ricci soliton metrics, i.e. nilsolitons. Since every homogenous Ricci soliton is algebraic it suffices to consider algebraic Ricci solitons. The left invariant metric on a Lie group is called \textit{algebraic Ricci soliton} if it satisfies: \( \text{Ric} = \gamma I + D \), where \( \gamma \) is an arbitrary constant, \( \text{Ric} \) is Ricci operator and \( D \) denotes the derivation of a Lie algebra. A Ricci soliton is said to be shrinking, steady or expanding according to \( \gamma > 0, \gamma = 0 \) or \( \gamma < 0 \), respectively.

**Proposition 4.6.** Nilsolitons on \( T^*\mathfrak{h}_3 \) are:

(i) expanding \( (\gamma = -5/2\pi) \), in case of the metric \( S_{30} = (S, 0, E_{30}) \), with \( S = \text{diag}(\lambda, \lambda, \lambda) \);

(ii) shrinking \( (\gamma = 5/2\pi) \), in case of the metric \( S_{21} = (S, 0, E_{21}) \), with \( S = \text{diag}(\lambda, \lambda, -\lambda) \);
(iii) steady \((\gamma = 0)\), in case of metrics \(S_{20} = (S^1, M^2, \pm E_{20})\), \(S_{11} = (S^k, M^2, E_{11})\) or \(S_{10} = (\pm E_{10}, M^1, \pm E_{10})\), where matrices \(S^k\) and \(M^j\), \((k=1,2,3, j=1,2)\), take the forms:

\[
S^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_{22} & 0 \\ 0 & 0 & s_{33} \end{pmatrix}, \quad S^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_{22} & \frac{1}{2} |s_{22}| \\ 0 & \frac{1}{2} |s_{33}| & s_{33} \end{pmatrix}, \quad S^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} |s_{33}| \\ 0 & \frac{1}{2} |s_{33}| & s_{33} \end{pmatrix},
\]

\[
M^1 = \begin{pmatrix} 0 & \lambda & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad M^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

**Proof.** The proof requires analysis for each metric from the classification. Let us prove positive result for case (i), the other cases can be analyzed in a similar manner.

For metric \(S_{30} = (S, 0, E_{30})\), with \(S = \text{diag}(\lambda_1, \lambda_2, \lambda_3)\) the Ricci operator is diagonal, hence \(D = \text{Ric} - \gamma I\) also has the diagonal form:

\[
D = \text{diag}(-\frac{\lambda_2 + \lambda_3}{2\lambda_1 \lambda_2 \lambda_3} - \gamma, \frac{\lambda_1 + \lambda_3}{2\lambda_1 \lambda_2 \lambda_3} - \gamma, \frac{\lambda_1 + \lambda_2}{2\lambda_1 \lambda_2 \lambda_3} - \gamma, 1, 1, 1 - \gamma, 1, 1, 1 - \gamma).
\]

Since \(D\) is derivation, it must satisfy the condition \(D[x, y] = [x, Dy] + [Dx, y]\), for all \(x, y \in T^*_3\). By solving this system of equations, we get that \(\lambda_1 = \lambda_2 = \lambda_3\) and \(\gamma = \frac{2}{2\lambda_1}\).

By this, we showed that not all metrics \(S_{30} = (S, 0, E_{30})\), where \(S\) takes the form \((17)\), admit nilsolitons. They exist only in positive definite and neutral signature case, i.e. only if \(S = \lambda E_{30}\), \(\lambda \neq 0\).

It was proven in \([30]\) that Riemannian left homogenous Ricci soliton (equivalently, algebraic Ricci soliton) metric on nilpotent Lie group is unique up to isometry and scaling. Proposition 4.6 confirms that result for the metric Lie algebra \(T^*_3\). However it also shows that the result doesn’t hold in pseudo-Riemannian setting since some of Ricci soliton metrics (i)-(iii) have the same signature, but are not homotetic.

### 4.3 Pseudo-Kähler metrics on \(T^*_3\)

Now, let us classify the pseudo-Kähler metrics on \(T^*_3\).

**Almost complex structure** on a Lie algebra \(\mathfrak{g}\) is an endomorphism \(J : \mathfrak{g} \rightarrow \mathfrak{g}\) satisfying \(J^2 = -\text{id}\).

If \(J\) is integrable in the sense that the Nijenhuis tensor

\[
N_J(x, y) = [x, y] - [Jx, Jy] + J[Jx, y] + J[x, Jy]
\]

of \(J\) vanishes, i.e. if it satisfies the condition \(N_J(x, y) = 0\), for all \(x, y \in \mathfrak{g}\), then it is called complex structure on \(\mathfrak{g}\).

The center of \(T^*_3\) is 3-dimensional, hence it cannot admit an abelian complex structure, i.e. complex structure satisfying \([x, y] = [Jx, Jy]\), meaning that the center of the algebra must be \(J\)-invariant (consequently, even dimensional). However, every complex structure on \(T^*_3\) is 3-step nilpotent (see \([9, Proposition 4.1i]\) or \([12]\)) and they are all equivalent to the following structure (see \([12,32,38]\)):

\[
Je_1 = e_2, \quad Je_3 = -e_6, \quad Je_4 = e_5. \tag{61}
\]

Complex structure \(J\) is said to be Hermitian if it preserves the metric: \(\langle Jx, Jy \rangle = \langle x, y \rangle\), for all \(x, y \in \mathfrak{g}\).
Example 2. Let us fix the basis where the complex structure $J$ has the form (61). One can check that $J$ is Hermitian, if the corresponding metric is the positive definite metric $S_{30} = (S, 0, E_{30})$, with $S = \text{diag}(\lambda, \lambda, 1)$, $\lambda > 0$.

A symplectic structure on a Lie algebra $g$ is a closed 2-form $\Omega \in \bigwedge^2 g^*$ of maximal rank. A pair $(J, \Omega)$, where $J$ is complex and $\Omega$ symplectic, is called a pseudo-Kähler structure if $\Omega(Jx, Jy) = \Omega(x, y)$ holds for all $x, y \in g$.

We already know from [12, Proposition 3.9.i)] that the algebra $T^*h_3$ have a complex structure admitting a five-dimensional set of compatible symplectic forms. Denote by $\{e^1, \ldots, e^6\}$ the dual basis of $\{e_1, \ldots, e_6\}$. The Maurer-Cartan equations on $T^*h_3$ are given by:

$$de^1 = de^2 = de^3 = 0, \quad de^4 = e^2 \wedge e^3, \quad de^5 = -e^1 \wedge e^3, \quad de^6 = e^1 \wedge e^2.$$ 

The symplectic structure $\Omega = \sum_{i < j} a_{ij}e^i \wedge e^j$, $a_{ij} \in \mathbb{R}$, has to be closed $d\Omega = 0$ and compatible with complex structure $J$ given by (61). Hence, it takes the form:

$$\Omega = a_{12}e^1 \wedge e^2 + a_{13}(e^1 \wedge e^3 - e^2 \wedge e^6) + a_{14}(e^1 \wedge e^4 + e^2 \wedge e^5 - 2e^3 \wedge e^6) + a_{15}(e^1 \wedge e^5 - e^2 \wedge e^4) + a_{16}(e^1 \wedge e^6 + e^2 \wedge e^3). \quad (62)$$

The pseudo-Kähler pair $(J, \Omega)$ originates an Hermitian structure on a Lie algebra $g$ by means of defining a metric $\langle \cdot, \cdot \rangle$ as

$$\langle x, y \rangle = \Omega(Jx, y), \quad (63)$$

for all $x, y \in g$. For this Hermitian structure the condition of parallelism of $J$ with respect to the Levi-Civita connection for $\langle \cdot, \cdot \rangle$ is satisfied. In this case, a pair $(J, \langle \cdot, \cdot \rangle)$ is called a pseudo-Kähler metric on $g$.

We know from [12, Corollary 3.2] that the algebra $T^*h_3$ has compatible pairs $(J, \Omega)$ since it admits both symplectic and nilpotent complex structures. It was proven in [3, Theorem A] that the metric associated to any compatible pair $(J, \Omega)$ cannot be positive definite, since $T^*h_3$ is not abelian. Therefore, the metric from Example 2 is not pseudo-Kähler. However, from [18] follows that any pseudo-Kähler metric on $T^*h_3$ is Ricci-flat. Here, we give their classification and explicit form.

Proposition 4.7. The Lie algebra $T^*h_3$ admits Ricci-flat pseudo-Kähler metrics that are not flat. Every pseudo-Kähler metric on $T^*h_3$ is equivalent to $S_{10} = (E_{10}, M, E_{10})$ where $M$ has form of the second matrix in (51).

Proof. Let us fix the basis where complex structure $J$ is given by (61) and symplectic form $\Omega$ by (62). The compatibility condition (63) for $(J, \Omega)$ gives us that the restriction of the metric on $T^*h_3'$ must be degenerate of rank 1. One calculates that the metric itself is represented by a 5-parameter symmetric matrix:

$$S = \begin{pmatrix}
-a_{12} & 0 & a_{16} & -a_{15} & a_{14} & -a_{13} \\
0 & -a_{12} & -a_{13} & -a_{14} & -a_{15} & -a_{16} \\
a_{16} & -a_{13} & -2a_{14} & 0 & 0 & 0 \\
-a_{15} & -a_{14} & 0 & 0 & 0 & 0 \\
a_{14} & -a_{15} & 0 & 0 & 0 & 0 \\
-a_{13} & -a_{16} & 0 & 0 & 0 & -2a_{14}
\end{pmatrix}, \quad a_{14} \neq 0.$$
By examining curvature properties, we conclude that this metric is locally symmetric and Ricci-flat, but not flat. From Proposition 4.1 (iii), we know that this metric must be equivalent to a metric from one of the families $S_{10} = (\pm E_{10}, M, \pm E_{10})$, where $M$ takes one of the forms (51). We can do even more, we can find a specific form of the automorphism matrix $F$ such that the matrix $F^T SF$ has one of two following forms:

$$S_{10} = (E_{10}, M, E_{10}), \quad M = \begin{pmatrix} \lambda & \frac{1}{2} & 0 \\ -\frac{1}{2} & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{where } \lambda = -\frac{a_{15}}{2a_{14}}.$$

In this case, the complex structure is $J' = FJF^{-1}$, while the explicit formula for the corresponding symplectic forms can be retrieved from (63).

**Remark 4.1.** In [39], the author considered three symplectic structures that are special cases of the symplectic structure given by (62). For each of those structures a corresponding metric was obtained. However, the author did not notice that all of these metrics are equivalent.

**Remark 4.2.** The previous proposition also shows that differences between metrics in the classification (Theorem 3.1) are very geometrical, rather then only algebraic.

### 4.4 Geodesically equivalent metrics

We say that a metric $\langle \cdot, \cdot \rangle$ on a connected manifold $M^n$ is geodesically equivalent to $\langle \cdot, \cdot \rangle$, if every geodesic of $\langle \cdot, \cdot \rangle$ is a reparameterized geodesic of $\langle \cdot, \cdot \rangle$. We say that they are affinely equivalent, if their Levi-Civita connections coincide. We call a metric $\langle \cdot, \cdot \rangle$ geodesically rigid, if every metric $\langle \cdot, \cdot \rangle$, geodesically equivalent to $\langle \cdot, \cdot \rangle$, is proportional to $\langle \cdot, \cdot \rangle$ (by the result of H. Weyl the coefficient of proportionality is a constant). In Riemannian case if metric is not decomposable (not a product of two metrics) it is geodesically rigid. Therefore, it makes sense to look for geodesically equivalent metrics only in pseudo-Riemannian case.

As it was proven in [6], any two geodesically equivalent invariant metrics on a homogenous space are affinely equivalent. This is particularly true for left invariant metrics on Lie groups. If invariant metric doesn’t admit nonproportional affinely equivalent invariant metric we call it invariantly rigid.

Non-proportional, affinely equivalent metric $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle$ are both parallel with respect to the mutual Levi-Civita connection and therefore their difference is parallel symmetric tensor. Such tensors are closely related to description of holonomy groups [21]. Metrics admitting such tensors are fully described on general pseudo-Riemannian manifold in [25] as either Riemannian extensions or using certain complex metrics. In Proposition 4.8 we show that such (not invariantly rigid) left invariant metrics on $T^*h_3$ are Riemannian extensions. Moreover all such parallel tensors on $T^*h_3$ are “made of” parallel vector fields in the following way.

Suppose that $v_1, \ldots, v_r$ are parallel vector fields with respect to metric $\langle \cdot, \cdot \rangle$ and $v_1^*, \ldots, v_r^*$ 1-forms metrically dual to those vectors. It is easy to check that for any constants $C_{mn} = C_{nm}, n, m = 1, \ldots r$, metric

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle + C_{nm}v_n^* \otimes v_m^*$$

(64)

is affinely equivalent to $\langle \cdot, \cdot \rangle$, or equivalently symmetric tensor $C_{nm}v_n^* \otimes v_m^*$ is parallel.

In [25] it was shown that such metric $\langle \cdot, \cdot \rangle$ is Riemannian extension of Euclidean space.

To classify non invariantly rigid metrics on $T^*h_3$ we follow the algorithm proposed in [6]. To simplify the notation, the matrix $S$ will be used to denote the metric $\langle \cdot, \cdot \rangle$. 


Proposition 4.8. If $\mathfrak{h}_3^*$ is non-degenerate, the corresponding left invariant metrics are geodesically rigid. If $\mathfrak{h}_3^*$ is degenerate, non trivial affinely equivalent metrics exist and they are obtained exactly metrics obtained using parallel null vector fields by (64).

Proof. It is clear that if original metric $\langle \cdot, \cdot \rangle$ has parallel vector fields that the metric (64) is affinely equivalent to it.

To prove the converse we do case by case analysis for each metric from our classification.

Let $S$ be a symmetric matrix representing a left invariant metric $\langle \cdot, \cdot \rangle$ in basis $\{e_1, \ldots, e_6\}$ and $\omega$ its Levi-Civita connection matrix of 1-forms. As proven in [6, Proposition 3.1], left invariant metric $\overline{S}$ is geodesically equivalent to $S$ if and only if its matrix $\overline{S}$ in basis $\{e_1, \ldots, e_6\}$ belongs to the subspace

$$\text{aff}(S) := \{ \overline{S} \mid \overline{S}\omega + \omega^T \overline{S} = 0 \}. \quad (65)$$

Since $\omega$ is a matrix of 1-forms, therefore the given relations are six matrix equations.

If $S$ is such that $\mathfrak{h}_3^*$ is non-degenerate, we directly check that $\text{aff}(S)$ is one-dimensional, hence $S$ is geodesically rigid. This also follows (without calculation) from the fact that such metrics have full holonomy algebra (Proposition 4.4). Namely, if metric is not geometrically rigid it can’t have full holonomy (see [6]).

We illustrate the proof for metric $S = S_{10} = (E_{10}, M, E_{10})$ with $\mathfrak{h}_3^*$ of rank 1, where is $M$ given by (50). The connection matrix $\omega$ can be calculated from relations (57) and we get that $\text{aff}(S)$ is space of matrices

$$\lambda S + \begin{pmatrix} c^{11} & c^{12} & c^{12} & 0 & 0 & 0 \\ c^{12} & c^{22} & c^{22} & 0 & 0 & 0 \\ c^{12} & c^{22} & c^{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda, c^{11}, c^{12}, c^{22} \in \mathbb{R}. \quad (66)$$

In fact, $\text{aff}(S)$ is set of all parallel symmetric (left invariant) tensors for metric $S$ and we see that it is 4-dimensional. Now, we will prove that it is made of parallel vectors using formula (64). Parallel vectors for metric $S$ are $v_1 = e_4$ and $v_2 = \frac{1}{m_{12}} e_5$ (Proposition 4.3). Their metrically dual forms are

$$v_1^* = e^2 + e^3, \quad v_2^* = e^1,$$

where $(e^1, \ldots, e^6)$ is basis of one forms dual to vectors $(e_1, \ldots, e_6)$ in sense that $e^i(e_j) = \delta^i_j$. Now we see that

$$c^{11} (v_1^* \otimes v_1^*) + c^{12} (v_1^* \otimes v_2^* + v_2^* \otimes v_1^*) + c^{22} (v_2^* \otimes v_2^*) \quad (67)$$

are exactly parallel symmetric tensors in (66) not proportional to $S$. They are obtained from parallel vector fields using (64).

Remark 4.3. One can check that non-proportional, affinely equivalent metrics are related by an automorphism of the group. This means that the corresponding Lie groups equipped with these metrics posses a family of automorphisms that are not isometries, but preserve geodesics.

Remark 4.4. Note that if the metric is Ricci-parallel, i.e. $\nabla \rho = 0$, then $\rho \in \text{aff}(S)$. Obviously, the converse is not true: not all non invariantly rigid metrics are Ricci-parallel.
4.5 Totally geodesic subalgebras of $T^*\mathfrak{h}_3$

A subalgebra $\mathfrak{h}$ of a metric algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is said to be \textit{totally geodesic} if $\nabla_y z \in \mathfrak{h}$ for all $y, z \in \mathfrak{h}$. If \( \mathfrak{h}^\perp \) denotes the orthogonal complement of $\mathfrak{h}$ in $T^*\mathfrak{h}_3$, then, as a direct consequence of Koszul’s formula, we get that $\mathfrak{h}$ is totally geodesic subalgebra of $T^*\mathfrak{h}_3$ if and only if

$$
\langle [x, y], z \rangle + \langle [x, z], y \rangle = 0, \quad \text{for all } x \in \mathfrak{h}^\perp, y, z \in \mathfrak{h}.
$$

(68)

It is said that $\mathfrak{h}^\perp$ is \textit{\( \mathfrak{h} \)-invariant} if $[x, y] \in \mathfrak{h}^\perp$, for all $x \in \mathfrak{h}^\perp$, $y \in \mathfrak{h}$. A nonzero element $y \in T^*\mathfrak{h}_3$ is called \textit{geodesic} if it spans a totally geodesic subalgebra $\mathfrak{h}$ and it can be characterized by the condition that $\mathfrak{h}^\perp$ is $\mathfrak{h}$-invariant. For nilpotent Lie groups, there exists an inner product for which a nonzero element $y$ is geodesic (see e.g. [7]).

The algebra $T^*\mathfrak{h}_3$ is 2-step nilpotent with the derived algebra coinciding with the algebra center and it is nonsingular in a sense of Eberlein [16, Definition 1.4], meaning that for each non-central element $x \in T^*\mathfrak{h}_3$ the adjoint map $\text{ad}(x)$ is surjective onto $Z(T^*\mathfrak{h}_3)$. The nonsingularity condition is equivalent to the following statement: for each inner product on $T^*\mathfrak{h}_3$, the only geodesics are the vectors contained in the centre $Z(T^*\mathfrak{h}_3)$ of $T^*\mathfrak{h}_3$ or orthogonal to it (see [7, Proposition 1.11]). Also, every vector subspace of $Z(T^*\mathfrak{h}_3)$ and every subalgebra that is orthogonal to $T^*\mathfrak{h}_3'$ are totally geodesic subalgebras of $T^*\mathfrak{h}_3$ (see [7, Proposition 1.5]).

A classification of totally geodesic subalgebras of Lie algebras of Heisenberg type was given by Eberlein [16], while in [7], the authors considered an example of 6-dimensional nilpotent Lie algebra with 2-dimensional center.

\textbf{Proposition 4.9.} For every subalgebra $\mathfrak{h}$ of $T^*\mathfrak{h}_3$ there exists a metric that makes it totally geodesic.

\textit{Proof.} As previously stated, geodesics are the vectors contained in the centre of $T^*\mathfrak{h}_3$ or orthogonal to it. Hence, for every metric algebra corresponding to the non-degenerate center, all basis vectors are geodesic. Let us examine more closely the metric $S_{21} = (S, 0, \pm E_{21})$ where $S$ takes the following form:

$$
S = \begin{pmatrix}
  s_{11} & s_{13} & s_{13} \\
  s_{13} & s_{11} - 2s_{13} & -2s_{13} \\
  s_{13} & -2s_{13} & s_{11} - 2s_{13}
\end{pmatrix}, \quad s_{11} \neq \pm 2s_{13}, \ s_{11}, s_{13} \neq 0.
$$

The other cases can be analyzed analogously.

Let $\mathfrak{h}$ be $n$-dimensional ($n < 5$) subalgebra of metric algebra $(T^*\mathfrak{h}_3, S_{21})$. Then $\mathfrak{h}$ is one of the following algebras:

(i) $2$-dimensional abelian algebra $\mathbb{R}^2 \cong \mathbb{R}(x, y)$, $x \in T^*\mathfrak{h}_3$, $y \in T^*\mathfrak{h}_3'$;

(ii) $3$-dimensional abelian algebra $\mathbb{R}^3 \cong \mathbb{R}(x, y, z)$, $x \in T^*\mathfrak{h}_3$, $y, z \in T^*\mathfrak{h}_3'$;

(iii) $3$-dimensional Heisenberg algebra $\mathfrak{h}_3$;

(iv) $4$-dimensional 2-step nilpotent algebra $\mathfrak{h}_3 \oplus \mathbb{R}$.

First, let us consider the abelian case. Since every subalgebra of $T^*\mathfrak{h}_3'$ is totally geodesic by [7, Proposition 1.5], we can consider only the case when $x \in T^*\mathfrak{h}_3'$ has a 2-dimensional center. Hence, if $\mathfrak{h} = \mathbb{R}^k$, $k = 2, 3$, then $\mathfrak{h}^\perp$ is spanned by the vectors from $T^*\mathfrak{h}_3'$ that are not already in $\mathfrak{h}$. Hence (68) is trivially satisfied.

The nilpotent case is very similar. There are precisely three Heisenberg subalgebras of $T^*\mathfrak{h}_3$: $\mathfrak{h}^1 = \mathbb{R}(e_2, e_3, e_4)$, $\mathfrak{h}^2 = \mathbb{R}(e_1, e_3, e_5)$ and $\mathfrak{h}^3 = \mathbb{R}(e_1, e_2, e_6)$. The 4-dimensional algebras have the
form $\mathfrak{h}^k \oplus e_j$, $k = 1, 2, 3$, $j = 4, 5, 6$, $j \neq k + 3$. In all these cases the orthogonal complement is contained in the algebra center and (68) is again satisfied. Interestingly, the subalgebras $\mathfrak{h}^1$ and $\mathfrak{h}^3$ are totally geodesic subalgebras for every metric corresponding to the non-degenerate center.

Now, the only thing left is to find an example in dimension five. It is a straightforward check that the metrics $S_{10} = (\pm E_{10}, M, \pm E_{10})$, with $M = \text{diag}(\lambda, \lambda, 0)$, admit all three 5-dimensional totally geodesic subalgebras isomorphic to $\mathfrak{h}^3 \oplus \mathbb{R}^2$.

A subspace $\mathfrak{h} \subseteq (\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is called isotropic if $\langle x, y \rangle = 0$, for all $x, y \in \mathfrak{h}$, i.e. $\mathfrak{h} \subseteq \mathfrak{h}^\perp$. Additionally, $\mathfrak{h}$ is called totally isotropic if $\mathfrak{h} = \mathfrak{h}^\perp$.

**Example 3.** It was mentioned in [9, Example 5.2] that on $T^*\mathfrak{h}_3$ equipped with the canonical metric from Example 1 both spaces $T^*\mathfrak{h}_3'$ and $(T^*\mathfrak{h}_3')^*$ are totally isotropic. Here, we can see that both of these spaces are totally isotropic if the metric corresponds to the degenerate center $T^*\mathfrak{h}_3'$ of rank 0. For the same four families of metrics, the totally geodesic subalgebra $\mathfrak{h}_3 \cong \mathbb{R}\langle e_2, e_3, e_4 \rangle$ is also totally isotropic.

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