Lattice analogues of W-algebras and Classical Integrable Equations

Abstract

We propose a regular way to construct lattice versions of W-algebras, both for quantum and classical cases. In the classical case we write the algebra explicitly and derive the lattice analogue of Boussinesq equation from the Hamiltonian equations of motion. Connection between the lattice Faddeev-Takhtadjian-Volkov algebra [1] and q-deformed Virasoro is also discussed.

1 Introduction

Recently there has been a great interest in the lattice analogues of 2D field theoretical models possessing conformal invariance [1-4]. For the first time this interest arised in connection with Liouville model [1-3]. As is well known, its quantization in continuous limit is rather difficult in the range of central charges 1 < c < 25, where usual perturbation theory is inapplicable. 'Latticization' of the world sheet can be viewed as some alternative to usual point-splitting method of regularization. That’s why lattice conformal theories give another way of quantization, more convenient in some cases in comparison with the that of based on normal ordering procedure.
An additional motivation for studying the lattice analogue of conformal invariance comes from the traditional understanding of Integrable Massive models as appropriate deformations of Conformal ones [5-7]. It seems highly desirable that this important concept be formulated purely on purely lattice language. Some hypothetical application of lattice theories can be connected with the future non-perturbative string theory. It is a common folklore [8,9] that the total String Phase Space contains conformal theories as fixed points, integrable models as interpolating trajectories between them, and all the others (lattice, q-deformed, etc...) filling in the remnant holes. In connection with this, an interesting question of relationship between the various types of q-deformations and ‘latticization’ naturally arised last time [4,10-12].

Up to now, only the lattice analogue of Virasoro algebra was studied [1-3]. The aim of present paper is to demonstrate the general method of obtaining lattice W-algebras (LW) analogous to classical series WA, WB, WD. Further on lattice WG algebra will be denoted as LWG. The method is effective on both the classical and quantum levels, although the explicit calculations are presented only for the classical limit of LW3 ≡ LWA2. We also discuss the relationship between Faddeev-Takhtadjan-Volkov (FTV) lattice Virasoro algebra and Lie algebraic q-deformation of Virasoro algebra, obtained by the authors recently [12].

For the reminder we bring here some basic definitions and formulas concerning the theory of classical series of W-algebras. As our main interest is in studying minimal models, we discuss in this paper only those W-algebras possessing maximally degenerated representations [13]. So we consider some semisimple algebra G and associated with it WG-algebra generated by the set of currents \{A^p(z)\}, where p spin.

Denote the system of simple roots as \(\Phi_s = \{\alpha_r\}_{r=1}^l\) and consider the theories with energy-momentum tensors (EMT) of the form:

\[
T(z) = -t_{ij}(\partial\phi_i \partial\phi_j)(z) + 2i\alpha_0 \rho \cdot \partial^2 \phi(z)
\]

where \(t_{ij}\) is a restriction of Killing form to the Cartan subalgebra and \(\rho\) is a halfsum of positive roots. Round brackets as usual mean normal ordering. Minimal model \(\mathcal{M}_{p,q}\) is described by the EMT with central charge

\[
c_{p,q} = l \left(1 - \frac{\tilde{h}(\tilde{h} + 1)(p - q)^2}{pq}\right)
\]

The main property of these models is that all the correlation functions of the primary fields can be calculated in the Coulomb gas representation after introducing of the so-called screening charges (SC). They are defined as follows:

\[
Q_r^\pm = \oint \frac{dz}{2\pi i} (e^{i\alpha_\pm \alpha_r \cdot \phi})(z)
\]
where $\alpha_\pm$ is determined from the condition that the conformal dimension of the integrand be equal to one:

$$\alpha_\pm^2 - 2\alpha_0\alpha_\pm - 1 = 0$$

The crucial facts about SC’s we use are [13-14]: (i) they form the root part of Cartan-Weyl basis of quantum algebra $U_q(G)$ with $q = \exp(i\pi\alpha_\pm^2)$; (ii) they commute with all the generators of $W G$-algebra. In order to define the right lattice analogue of $W G$-algebra, we should consider the lattice versions of these statements. This will be done in next section.

## 2 General prescriptions

In this section we give the basic formulas and definitions concerning the lattice minimal models. The naturality of the method first proposed by Feigin [15] is guaranteed by its full parallelism with continuous case. Consider for example, the relation between the two screened vertex operators (VO) corresponding to simple roots:

$$V_{\alpha_i}^\pm(x) V_{\alpha_j}^\pm(z) = (x - z)^{\alpha_\pm^2 A_{ij}} (V_{\alpha_i}^\pm(x) V_{\alpha_j}^\pm(z))$$  \hspace{1cm} (2)

where $A_{ij}$ is Cartan matrix. In order to obtain its lattice analog introduce first the natural order in $\Phi_s$. For the case of $A_l$ we have $\alpha_i = e_i - e_{i+1}$. Then we put lattice VO on $l$ slightly shifted adjacent lattices such that $a_i(n)$ be left with respect to $a_j(n)$ if $i < j$. Thus lattice version of the eq.(2) is

$$a_i(m)a_j(n) = q^{A_{ij}} a_j(n)a_i(m), \text{ when } n > m \hspace{1cm} (2')$$

and

$$a_i(n)a_{i+1}(n) = q^{-1} a_{i+1}(n)a_i(n)$$

$$a_i(n)a_j(n) = a_j(n)a_i(n), \text{ when } |i - j| \geq 2$$

Note that $q$ now can be treated as free parameter contrary to the continuous case where it was rigidly determined by the formula in the end of Introduction.

Now we introduce a lattice analogue of the SC as

$$Q_i^\pm = \sum_n (a_i(n))^{\pm 1}$$  \hspace{1cm} (3)
and define $LWG$-algebra as zero-graded part of kernel of the adjoint action of lattice SO’s, where gradation is defined via the rules ($\bar{a} \equiv a^{-1}$): 

$$deg(a_r(n)) = 1 \quad deg(\bar{a}_r(n)) = -1$$

The most common anzats satisfying the condition stated is given by the following series ($LW_3$ - case)

$$L = \sum_{k,p,r,s} f_{k,p,r,s} b_n c_{n+1} c_{n+2} b_n$$

$$W = \sum_{k,p,r,s,t,u} g_{k,p,r,s,t,u} b_n c_{n+1} c_{n+2} b_n$$

where basic commutation relations $(2')$ here take the form

$$b_n b_{n+\Delta} = q^2 b_{n+\Delta} b_n \quad c_n c_{n+\Delta} = q^2 c_{n+\Delta} c_n$$

$$b_n c_{n+\Delta} = q^{-1} c_{n+\Delta} b_n \quad b_n c_n = q^{-1} c_n b_n$$

Commuting these two operators with the SC $Q_b$ one obtains the following Quantum Master Equations (QME)

$$-q^{-(k+1)}[k+1] f_{k+1,p,r,s} + q^{k+r} f_{k,p,r,s} - q^{-(r+1)}[s-r-1] f_{k,p,r+1,s} = 0$$

$$q^{-(k+1)}[k+1] g_{k+1,p,r,s,t,u} + q^{k-r}[k+p+r] g_{k,p,r,s,t,u} +
q^{r-t-1}[s+r+t-1] g_{k,p,r-1,s,t,u} + q^t [t+u-1] g_{k,p,r-1,s,t-1,u} = 0$$

Commuting with $Q_c$ gives two conjugated equations differing from the above by the replacement $(k,r) \leftrightarrow (s,p)$ and $(k,r,t) \leftrightarrow (s,p,u)$.

QME for lattice generators $A^p(n)$ forming higher $LW$-algebras can be obtained from the quite analogous considerations. We do not bring here the corresponding cumbersome expressions for generators and their QME’s, saying only that the single rule being used to obtain them is that each monomial entering the formula like (4) for spin-$p$ generator should be a zero-graded $p$-local expression, i.e. including the operators from $a_i(n)$ to $a_i(n+p)$ for all $i$. We give here the solutions of QME’s (6) for the reader could imagine what the matter is. Functions $f$ and $g$ as well as their higher analogues differ from zero in some half-infinite region, however it turns out that QME
cannot be solved through the characteristic method, as it gives only zero-part of the solution. The intuition for solving QME is provided by their classical limit, which will be discussed in details in Sect.3. In the formula below \( \binom{x}{y} \) stands for quantum binomial coefficient \( \frac{|x|!}{|y|!(x-y)!} \).

\[
 f_{k,p,r,s} = \left( 1 + \frac{(k+p)r}{[p][s]} \right) (-1)^{k+p+r+s} q^{(k-r)^2+(s-p)(s-p+1)pr} \cdot \binom{k+p-1}{k} \cdot \binom{r+s-1}{r}
\]

when \( k, p, r, s > 0 \) and analogous rather cumbersome expression for \( g_{k,p,r,s,t,u} \). Formally looking at eq.(7), one sees that it is not invariant under the replacement of the indices mentioned above. In fact, this can be removed by the appropriate redefinition of \( f \) on the boundary of the region \( k, p, r, s > 0 \).

3 LW3-algebra in the classical limit

We begin this section from defining the classical limit of the basic VO’s \( b_n \) and \( c_n \) and introduce more convenient variables

\[ p_n \equiv \bar{b}_n b_{n+1} \quad \text{and} \quad d_n \equiv c_n \bar{c}_{n+1} \]

Basic commutators in the classical limit become

\[
\{b_n, b_m\} = \{c_n, c_m\} = \epsilon(n-m) \\
\{b_n, c_m\} = \theta(n-m)
\]

where \( \epsilon(n) \) and \( \theta(n) \) are standardly defined as

\[
\epsilon(n) = \begin{cases} 
1, & \text{if } n > 0 \\
0, & \text{if } n = 0 \\
-1, & \text{if } n < 0
\end{cases} \\
\theta(n) = \begin{cases} 
1, & \text{if } n \geq 0 \\
0, & \text{otherwise}
\end{cases}
\]

Searching for the generators of LW3 in the form (5) from the condition of their commutativity with SC one obtains Classical Master Equations by simply omitting factors of the type \( q^{something} \) and replacing quantum numbers by usual ones in QME’s (6). Solving these simplified equations, one readily obtains the classical versions of the series like (7). Because of absence of unpleasant \( q \)-factors these series can be converted into finite expressions. This allows one to calculate the algebra explicitly. Operators \( L_n \) and \( W_n \) of ”spin” (=locality) 2 and 3 respectively are given by the following expressions:

\[
L_n = (p_{n+1}d_n + p_{n+1} + d_n)(1 + p_n + d_n)^{-1}(1 + p_{n+1} + d_{n+1})^{-1} \\
W_n = d_n p_{n+2}(1 + p_n + d_n)^{-1}(1 + p_{n+1} + d_{n+1})^{-1}(1 + p_{n+2} + d_{n+2})^{-1}
\]

After a straightforward but rather tedious calculation one can find that these operators form the following closed algebra:
\{L_n, L_{n+1}\} = (L_n L_{n+1} - W_n)(1 - L_n - L_{n+1}) \quad \{L_n, L_{n+2}\} = -L_n L_{n+1} L_{n+2} + W_n L_{n+2} + W_{n+1} L_n

\{L_n, W_n\} = \{W_n, L_{n+1}\} = -W_n L_n L_{n+1} + W_n^2

\{W_n, L_{n+3}\} = -W_n L_{n+2} L_n + W_n W_{n+2} \quad \{W_n, L_{n+2}\} = W_n L_{n+2}(1 - L_{n+1} - L_{n+2}) + W_n W_{n+1}

\{L_n, W_{n+1}\} = L_n W_{n+1}(1 - L_n - L_{n+1}) + W_n W_{n+1} \quad \{L_n, W_{n+2}\} = -L_n L_{n+1} W_{n+2} + W_n W_{n+2}

\{W_n, W_{n+1}\} = W_n W_{n+1}(1 - L_n - L_{n+2}) \quad \{W_n, W_{n+2}\} = W_n W_{n+2}(1 - L_{n+1} - L_{n+2})

\{W_n, W_{n+3}\} = -W_n W_{n+3} L_{n+2}

(10)

Strange as it might seemed that operators \(L_n\) do not form a subalgebra in \(LW_3\). However, looking attentively at the first line of commutation relations (10), one can see that setting \(W_n = 0\) one obtains FTV algebra [1].

4 Lattice Boussinesq Hierarchy

In this section we consider the direct lattice analogue of Boussinesq hierarchy, related to classical \(W_3\) algebra in the continuous limit. First we define the notion of lattice hamiltonian. Fundamental requirement for an operator to be called hamiltonian is that it commute with some auxiliary \(sl(3)\)-algebra. However, in this paper we will elaborate with more indirect, but convenient criteria: hamiltomians which will be defined below are in involution with each other and give the set of equations of motion (EM) becoming the well-known Boussinesq hierarchy in the continuous limit.

Define the zeroth hamiltonian as

\[ \mathcal{H} \equiv \mathcal{H}^{(0)} = \frac{1}{3} \sum_n \ln W_n \] (11)

The pair \((\mathcal{H}, Poisson Bracket (10))\) generates the following EM:

\[ \dot{L}_n = \{\mathcal{H}, L_n\} = (W_{n-1} - W_n) + L_n(L_{n+1} - L_{n-1}) \]

\[ \dot{W}_n = \{\mathcal{H}, W_n\} = W_n(L_{n+2} - L_{n-1}) \] (12)

It can be shown, that the system (12) is completely integrable, however in this paper we explicitly demonstrate some weaker property that it is bi-hamiltonian and there exists a recursion operator, allowing one to built the infinite set of integrals of motion (IM). Below the several first IM are presented:

\[ \mathcal{H}^{(1)} = \sum_n L_n \]
\[ \mathcal{H}^{(2)} = \sum_n \left( \frac{L_n^2}{2} + L_n L_{n+1} - W_n - L_n \right) \]  

(13)

The interesting points, related to correct continuous limit will be discussed in more details in next section.

5 Bi-hamiltonian structure of Lattice Boussinesq Hierarchy (LBH)

In this section we demonstrate the existence of two pairs (Hamiltonian, Symplectic Structure), consistent with each other in the following sense:

\[ \dot{\Psi} = \{\mathcal{H}^{(0)}, \Psi\}_2 = \{\mathcal{H}^{(1)}, \Psi\}_1 \]

where \( \Psi \) is any operator from the set \( \{L_n, W_n\} \). Then the recursion operator \( \mathcal{R} : \mathcal{H}^{(n)} \rightarrow \mathcal{H}^{(n+1)} \) has the form

\[ \mathcal{R} = \Omega_1^{-1}\Omega_2 \]

where \( \Omega_i \) is formal denotion of \( i \)-th symplectic structure. We recall the standard rules of building the first structure. Consider the Miura map \( \mathcal{M} : \{\lambda_n, \omega_n\} \ni \psi \rightarrow \Psi \in \{L_n, W_n\} \), where \( \{\lambda_n, \omega_n\} \) is some set of variables with given Poisson bracket \( \Omega \). Clearly, \( \Omega_2 \) is uniquely determined by \( \Omega \) via the Miura map as

\[ \Omega_2[\Psi_1, \Psi_2] := \Omega[\mathcal{M}(\psi_1), \mathcal{M}(\psi_2)] \]

Quite analogous formula can be written for the definition of the first structure:

\[ \Omega_1[\Psi_1, \Psi_2] := \mathcal{M}(\Omega[\mathcal{M}^{-1}(\Psi_1), \mathcal{M}^{-1}(\Psi_2)]) \]  

(14)

It can be shown by direct calculation that the following definition of the basic structure \( \Omega \)

\[
\begin{align*}
\{\lambda_n, \lambda_{n+1}\} &= \lambda_n \lambda_{n+1} - \omega_n & \{\omega_n, \omega_{n+1}\} &= \omega_n \omega_{n+1} \\
\{-\lambda_{n+2}, \omega_n\} &= \lambda_{n+2} \omega_n & \{\lambda_n, \omega_{n+1}\} &= \lambda_n \omega_{n+1} \\
\{\omega_n, \omega_{n+2}\} &= \omega_n \omega_{n+2}
\end{align*}
\]

(15)

with the explicit realization

\[ \lambda_n = p_n + d_n \quad \text{and} \quad \omega_n = p_n d_{n+1} \]

provides the bi-hamiltonian structure of the hierarchy (12-13).

Taking \( \mathcal{H}^{(0)} \) and \( \mathcal{H}^{(1)} \) from the previous section, \( \Omega_2 \) from eq.(10) and \( \Omega_1 \) from (14) one obtains the bi-hamiltonian form of LBH.
Here we would like to drop reader’s attention to the following interesting property of the first structure \( \Omega_1 \) related to the explicit form of \( LW_3 \)-algebra (10). Namely, \( \Omega_1 \) can be viewed as a contraction of \( \Omega_2 \) if we introduce the following natural gradation (\( \deg(\text{operator})=\text{spin}-1 \))

\[
\deg(L_n) = 1 \quad \deg(W_n) = 2 \quad \deg(\Omega_2) = 0
\]

and rewrite the algebra (10) in terms of graded operators \( \epsilon L_n \) and \( \epsilon^2 W_n \). One can easily check that

\[
\text{Eqs.}(15) = \lim_{\epsilon \to 0} \text{Eqs.}(10)
\]

It is noteworthy, that the same property is valid for the bi-hamiltonian structure of Volterra model. The details on this point will be given elsewhere [18]. We conclude this section by remark concerning the continuous limit. Note, firstly, that the operator \( W_n \) as defined by (9) consists in fact of two parts of locality 3 and 2. This means that the true spin-3-field is some appropriate combination of \( W_n \) and \( L_n \). Secondly, in the continuous limit the operators (9) can be expanded in the series on \( \Delta \), which begin from the constant. This means that the correspondent Hamiltonians should be appropriately regularized before taking the limit. Straightforward calculation gives the following expansions \( (x \equiv n \Delta) \)

\[
W_n \rightarrow \frac{1}{27}(1 - \Delta^2 u(x) - \frac{\Delta^3}{2}w(x))
\]

\[
L_n \rightarrow \frac{1}{3}(1 - \frac{\Delta^2}{3}u(x))
\]

where \( u(x) \) and \( w(x) \) are the fields forming usual classical \( W_3 \) algebra. Thus if we define, for example

\[
\tilde{\mathcal{H}}^{(0)} \equiv \frac{6}{\Delta^3} \sum_n (L_n - \frac{1}{3}lnW_n - \frac{1}{3} - ln3)
\]

\[
\tilde{\mathcal{H}}^{(1)} \equiv -\frac{9}{\Delta^2} \sum_n (L_n - \frac{1}{3})
\]

then \( \tilde{\mathcal{H}}^{(0,1)} \) become usual Boussinesq integrals

\[
\tilde{\mathcal{H}}^{(0)} \rightarrow \int dx \ w(x)
\]

\[
\tilde{\mathcal{H}}^{(1)} \rightarrow \int dx \ u(x)
\]

and \( \tilde{\mathcal{H}}^{(0)} \) in continuous limit generates well-known Boussinesq equations [17]

\[
\dot{u} = -u_{xx} + 2w_x \quad , \quad \dot{w} = w_{xx} - \frac{2}{3}u_{xxx} - \frac{2}{3}uu_x
\]
6 Q-deformation as "latticization"

In this section we are going to discuss the question, lying partly aside, but immediately related to the problem under consideration. In their recent paper on Lie-algebraic $q$-deformations of Virasoro algebra [12] authors demonstrated that this operation, leading to splitting of multiple poles in quantum operator product expansion (OPE), considered in classical limit means automatic transition from continuum to the lattice with step $\Delta$, if the deformation parameter is given by $q = \exp(2\pi i \Delta)$. Below we give a short list of interesting formulas and try to understand the relation between the algebras discussed in previous sections and in [12].

Lie-algebraic $q$-deformations of Virasoro and Superconformal algebras were found by Chaichian and Presnajder [16] and independently by the present authors [12]. These algebras naturally originate from the free fields models. We discuss here only the $q$-deformation of $Vir$. Consider free bosonic field $a_p$ with $q$-deformed commutation relations

$$[a_p, a_s] = [p]\delta_{p+s,0}$$

The following family of operators can be built from $a_n$:

$$l_\alpha^p = \sum_s \{\alpha(2s+p)\} a_{-s}a_{p+s}$$

where curl brackets denote the natural "fermionic" analog of usual quantum number $\{x\} \equiv \frac{q^x+q^{-x}}{q^x-q^{-x}}$. Direct calculation shows that $l_\alpha^p$ form two-loop Lie algebra

$$[l_\alpha^p, l_\beta^s] = [((\beta+1)p-(\alpha+1)s)l_\alpha^{p+s}+1 - ((\beta-1)p-(\alpha-1)s)l_\alpha^{p+s-1} + \ldots$$

$$+[(\beta+1)p+(\alpha-1)s]l_\alpha^{p+s-1} - ((\beta-1)p+(\alpha+1)s)l_\alpha^{p+s+1} + \text{central terms}$$

We have omitted central extension as it is irrelevant in the given context. We note only that in the limit $q \to 1$ the central term tends to right expression $p(p^2-1)$ (see [12] for detailed discussion on this point) Quantum OPE has the form $(T^\alpha(x) \equiv \sum_p l_\alpha^p x^{-p-2})$:

$$T^\alpha(z) T^\beta(w) = \frac{T^{\alpha+\beta+1}(wq^{\alpha+1})}{z-wq^{\alpha+\beta+2}} - \frac{T^{\alpha+\beta+1}(wq^{-\alpha-1})}{z-wq^{-\alpha-\beta-2}} + \ldots$$

When going to the classical limit it becomes

$$\{T^\alpha(x), T^\beta(y)\}_{P.B.} = T^{\alpha+\beta+1}(y + \Delta(\alpha+1))\delta(x-y - \Delta(\alpha + \beta+2)) - \ldots$$

Thus one observes that new parameter $\Delta$ naturally appears in the model as a shift on coordinate space and should obviously be interpreted as a lattice constant. Indeed, it is convenient to introduce a lattice with step $\Delta$ and to rewrite the previous equation as follows

$$\{\Lambda_n^\alpha, \Lambda_m^\beta\} = \Lambda^{\alpha+\beta+1}_{m+n+\alpha+1} \delta_{n,m+\alpha+\beta+2} - \Lambda^{\alpha+\beta+1}_{m-n-\alpha-1} \delta_{n,m-\alpha-\beta-2} + \ldots$$
where
\[ \Lambda_n^\alpha \equiv \int_{n\Delta}^{(n+1)\Delta} dx \, T^\alpha(x) \]

Superconstruction bases on the fact that the same algebra (17) is formed by the following bilinear operators
\[ M_p^\alpha = \sum_s [\alpha(2s + p)] \psi_{-s}\psi_{p+s} \]  \hspace{1cm} (19)
where \( \psi_p \) is a fermion field with deformed anticommutator
\[ \{\psi_p, \psi_s\} = \{p\} \delta_{p+s,0} \]

However, if one considers the same object (19) built from usual fermions then one obtains another two-loop algebra, non-isomorphic to (17). In somewhat analogous property takes place in the zoo of lattice algebras. Analogues of Vir-operators entering the LW-algebras even do not form closed subalgebras. However, as it follows from the remark in the end of Sect.3, one can hope, that LW\(_k\)-algebras with \( k < n \) can be extracted from LW\(_n\)-algebra by setting to zero generators with spins > \( k \). In any case, it seems that the lattice analogue of the notion of conformal invariance, if existing, cannot be formulated in purely algebraic terms. From the other side, the remarkable property of FTV-algebra in classical limit found in [12] is that it can be rewritten as a Lie algebra isomorphic to that of (18).

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