Vibration Analysis of Circular and Annular Plate under different Boundary Conditions

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Abstract

Objectives: Circular and circinate plates are unit wide utilized in machinery as basic elements and studied torsional vibration behaviour. Methods/Analysis: A degree analytical model is developed, and also the natural frequencies and mode shapes unit calculated for the torsional vibration underneath totally different boundary conditions. Findings: A physical insight is gained into the torsional vibration behaviour of circular and circinate plates. Results show that the torsional vibration behaviour of circular and circinate plates is totally different compared therewith of circular shafts. Force moment in circinate plates is transmitted via ring layers from the inner ring to the outer ring and contrariwise. Novelty/Improvement: additionally, the torsional vibration magnitude in terms of angular displacement decreases with increase in radius. This will be explained by the very fact that, once vibration energy is transmitted from the inner ring to the outer ring, the circle becomes larger and also the energy density becomes lower. The associate degree analytical results area unit verified exploitation an FEA model. Sensible agreement is shown between the analytical and FEA models.

Keywords: Annular Plate, Circular Plate, Mode Shape, Natural Frequency, Torsional Vibration

1. Introduction

Circular and circinate plates of machine members with moving parts like gears, turbine wheels, circular saws, and disk brakes, hard disks and optical disks such as CD-ROM and DVD’s generates mechanical forces during normal operation. The torsional vibration study is required for dynamics of rotational machinery, where power and torque moment are transmitted from engines or motors to the operating machinery via the gear and shaft systems. The classical one-dimensional model for the torsional vibration of uniform shafts or rods has been used for generations.1 The torsional vibration has conjointly been studied on rotor, gear, and shaft systems. The Finite Component Methodology (FEM) was accustomed analyse the torsional vibration of gear-branched systems.2 In this paper, the governing equation for the torsional vibration of skinny, circular and annulate plates comes. Then the torsional natural frequencies and mode shapes square measure calculated for four sets of boundary conditions. The torsional vibration behaviour of skinny, circular and circinate plates is analysed to get the physical insight.

2. Equation of Motion

Stresses in circular rings and disks, curved bars of narrow rectangular cross section with a circular axis, etc of
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The torsion of thin, circular and annular plates can be modelled as a plane stress problem.

\[
\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0
\]

(1a)

\[
\frac{1}{r} \left[ \frac{\partial \sigma_\theta}{\partial r} + \frac{2\tau_{r\theta}}{r} + S = 0 \right.
\]

(1b)

Where \(\sigma_r\) and \(\sigma_\theta\) are the normal stress in the radial and circumferential directions, respectively. \(\tau_{r\theta}\) is the shear stress. \(R\) and \(S\) are the body force in the radial and circumferential directions, respectively. The shear, normal stress and strain are axisymmetric and vary only in the radial direction. Equilibrium equations (1a) and (1b) can therefore be simplified

\[
\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0
\]

(2a)

\[
\frac{1}{r} \left[ \frac{\partial \sigma_\theta}{\partial r} + \frac{2\tau_{r\theta}}{r} + S = 0 \right.
\]

(2b)

Equations (2a) and (2b) now become uncoupled from each other.

\[\langle T \rangle + \langle T + \frac{\partial T}{\partial r} dr \rangle\]

A ring layer in the annular plate is schematically shown in Figure 1(a), where \(a\) and \(b\) are the inner and outer radii of the annular plate, respectively. Considering the isolated ring from the annular plate at radius \(r\), the torque moments applied to the inner and outer surfaces of the ring are denoted by

\[\langle T \rangle\]

respectively, as shown in Figure 1(b). Torque moment \(T\) can be calculated according to the shear stress on the inner ring surface as

\[T = 2\pi r \delta \tau_{r\theta}\]

(3)

According to the Newton’s Second Law, the equation of motion for torsional vibration of the ring layer can be written as

\[\rho A \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial T}{\partial r} dr\]

(4)

Where \(\psi\) is angular displacement of the ring, \((\partial^2 \psi/\partial t^2)\) is angular acceleration, \(\rho\) is the density of the plate, \(I_p\) is the rotational inertia moment of the ring about the plate centre \(O\). The rotational inertia moment of the ring about the plate centre \(O\) can be calculated by
The relationship between the shear stress and strain is given by Hook’s Law, where \( G \) is the shear modulus.

\[
\tau_{r\theta} = G \gamma \theta \quad (6)
\]

The relationship between the shear strain \( \gamma_{r\theta} \) and angular displacement \( \Psi \) of the ring can be derived from the geometrical relationship in the polar coordinates \((r, \theta)\) between the shear strain and translational displacement.

\[
\gamma_{r\theta} = \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \quad (7)
\]

Where \( u \) denotes the radial displacement and \( v \) denotes the circumferential displacement and is given by

\[
v = r \Psi \quad (8)
\]

As the shear strain \( \gamma_{r\theta} \) is axisymmetric the equation (7) can be given in the form

\[
\gamma_{r\theta} = r \frac{\partial \psi}{\partial r} \quad (9)
\]

Substituting equation (8) in equation (9) results in the relationship between the shear strain and the angular displacement is

\[
\gamma_{r\theta} = Gr \frac{\partial \psi}{\partial r} \quad (10)
\]

Substituting equation (10) in equation (6) results the shear stress is

\[
\tau_{r\theta} = Gr \frac{\partial \psi}{\partial r} \quad (11)
\]

The equation of motion for torsional vibration of the ring layer can be obtained by differentiating the above torque \( (T) \) w.r.t to \( r \) then the equation (11) becomes

\[
\frac{\partial T}{\partial r} = 2\pi r^3 G \frac{\partial \psi}{\partial r} \quad (12)
\]

Substituting the equations (12) and (5) in Motion equation (4) then

\[
\frac{\partial^2 \psi}{\partial t^2} = G \rho \left\{ \frac{\partial \psi}{\partial r} \right\} \frac{3}{r} + \frac{\partial^2 \psi}{\partial r^2} \quad (13)
\]

This is the final Motion equation for torsional vibration of the ring layer. When the radius is close to zero, \( 3/r \) reaches infinite and Equation (13) becomes trivial. This is for the solution of circular plate.

### 3. Calculation of Natural Frequencies and Mode Shapes

Solution to above equation (13) may take the form

\[
\psi = \phi(r) e^{i\omega t} \quad (14)
\]

Where \( \omega \) is the angular frequency and \( \Phi(r) \) represents a function of \( r \) that defines the natural mode shapes of the torsional vibration of circular and annular plate. Differentiate the above equation (14) w.r.t to \( r \) and \( t \) and substitute the above \( \Psi \) value in equation of motion (13), it becomes

\[
\left[ \frac{\partial^2 \Phi}{\partial r^2} + 3 \frac{x}{r} \frac{\partial \Phi}{\partial r} \right] + k^2 \Phi = 0 \quad (15)
\]

\[
c = \sqrt{\frac{G}{\rho}} \quad (16)
\]

is the torsional wave propagating speed in the radial direction, which is same as the torsional wave speed in a uniform rod propagating in the axial direction. Introducing non-dimensional variable

\[
x = \frac{r}{b} \quad (17)
\]

and substituting \( x \) for \( r \) in equation (15) results in

\[
\left[ \frac{\partial^2 \Phi}{\partial x^2} + 3 \frac{\partial \Phi}{\partial x} \right] + k^2 \Phi = 0 \quad (18)
\]

\[
k = \frac{\rho \omega}{c} \quad (19)
\]

is the total wave number within radius \( b \). Equation (18) is to be solved using numerical method under the following four set of boundary conditions.

1. Inner ring free and outer ring free (free-free)

\[
\left\{ \begin{array}{l}
X = \frac{a}{b}, \frac{d \phi}{dx} = 0
\end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{l}
X = 1, \frac{d \phi}{dx} = 0
\end{array} \right\}
\]

2. Inner ring free and outer ring clamped (free-clamped)

\[
\left\{ \begin{array}{l}
X = \frac{a}{b}, \frac{d \phi}{dx} = 0
\end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{l}
X = 1, \phi = 0
\end{array} \right\}
\]
3 Inner ring clamped and outer ring free (clamped-free)

\[
\begin{align*}
X = \frac{a}{b}, \phi = 0 \quad \text{and} \quad X = 1, \frac{d\phi}{dx} = 0
\end{align*}
\]

4 Inner ring clamped and outer ring clamped (clamped-clamped)

\[
\begin{align*}
X = \frac{a}{b}, \phi = 0 \quad \text{and} \quad X = 1, \phi = 0
\end{align*}
\]

The Equation (18) is solved by using fourth order Runge–Kutta method under the four sets of boundary conditions.

4. Results

The Natural Frequencies and Mode Shapes of the torsional vibration of circular and annular plates are calculated under the four sets of boundary conditions. The results for the natural frequencies are listed in Table 1 in terms of wave number \( k \) for the first three vibrations modes (except the rigid body mode). The ratio of inner to the outer radii of the annular plate in the calculations is chosen to be 0.01, 0.25, 0.50, and 0.80. The annular plate with ratio 0.01 approximately represents a circular plate. The outer radius of the annular plate is chosen to be \( b = 100 \) and the plate thickness is unity. The Young’s modulus of the plate material is \( 2.1 \times 10^{11} \) N/m\(^2\), the poison’s ratio is 0.3 and the material density is 7800 Kg/m\(^3\). These parameters are used for the FE model calculations to verify the analytical model.

The natural frequencies (Table 1) increases with the ratio of the inner to the outer radius for all the boundary conditions considered because the torsional stiffness of an annular plate increase with radius. For the plate 0.01, which is regarded as circular plate, the natural frequencies are almost the same under free-free and clamped-free boundary conditions and under the free-clamped and clamped-clamped boundary condition as well because the torsional stiffness at inner radius is very small, if the inner radius is small.

The torsional vibration mode shapes are calculated and plotted as shown in Figure 2.0 to Figure 6.0 for the free-free and clamped-free boundary conditions and clamped-free and clamped-clamped boundary conditions. All the mode shapes show a decaying behavior with increasing \( x \) in terms of the angular displacements magnitude. The decay rate becomes lower with increasing ratio \( a/b \) and it can be explained by the governing equation (18) where the damping co-efficient \( 3/x \) varies inversely-proportionally with \( x \). Thus a smaller ratio \( a/b \) causes high damping effects, where a large ratio \( a/b \) results in smaller damping effects.

![Graph Between "X" and Phi](image1)

**Figure 2.** Torsional vibration mode shapes of annular plates under free-free boundary. \( a/b=0.01, \ a/b=0.25 \ a/b=0.5 \ a/b=0.08. \)
**Figure 3.** Torsional vibration mode shapes of annular plates under clamped boundary. \(a/b=0.01, \ a/b=0.25, \ a/b=0.5, \ a/b=0.08.\)

**Figure 4.** Torsional vibration mode shapes of annular plates under clamped-free boundary. \(b=0.01, \ a/b=0.25, \ a/b=0.5, \ a/b=0.08.\)

**Table 1.** Natural Frequencies of torsional vibrations of annular plate

| Boundary conditions | Inner to Outer Radius = \(a/b\) | \(\omega_n = k_n / b \sqrt{G/\rho}, \ n=1,2,3,\ldots\) |
|---------------------|-----------------|------------------|
|                     |                 | K1 | K2 | K3  |
| Free-Free           | 0.01            | 5.14 | 11.62 | 14.80 |
|                     | 0.25            | 5.32 | 9.14  | 13.12 |
|                     | 0.50            | 6.81 | 12.86 | 19.05 |
|                     | 0.80            | 15.86 | 31.49 | 47.17 |
| Free-clamped        | 0.01            | 3.83 | 7.02  | 10.17 |
|                     | 0.25            | 3.90 | 7.45  | 11.29 |
|                     | 0.50            | 4.55 | 10.09 | 16.13 |
|                     | 0.80            | 8.95 | 23.97 | 39.52 |
| Clamped-Free        | 0.01            | 5.14 | 11.65 | 14.84 |
|                     | 0.25            | 6.17 | 10.41 | 14.62 |
|                     | 0.50            | 9.18 | 15.56 | 21.89 |
|                     | 0.80            | 23.26 | 39.09 | 54.85 |
| Clamped-Clamped     | 0.01            | 3.83 | 7.03  | 10.20 |
|                     | 0.25            | 4.45 | 8.54  | 12.68 |
|                     | 0.50            | 6.39 | 12.62 | 18.89 |
|                     | 0.80            | 15.74 | 31.43 | 47.13 |
5. Conclusion

The torsional vibration of circular and ringed plates is studied. Results shows that torsional vibration behavior of circular and ringed plate is totally different compared thereupon of slender rods. Torsion moment in circular and ringed plate is transmitted via ring layers from inner ring to the outer ring or from the outer ring to the inner ring, wherever as in slender rods, it’s transmitted from one finish to a different through the cross-sections. As torsional stiffness of ringed plates will increase with radius, the vibration magnitude in terms of the angular displacement decreases with increasing radius. The comparison of natural frequencies from analytical and finite element analysis models as shown in Table 2. Calculations of the torsional vibration modes are simple and straightforward using the analytical approach, where as in FEM the torsional modes have to be extracted from the hundreds of vibration modes of annular plates. Moreover it is difficult to gain the physical insight into the torsional vibration behavior using FEM and this can easily be achieved using the analytical approach.

| Table 2. Comparison of Natural Frequencies from analytical and FEA models |
|---------------------------------------------------------------|
| Boundary Conditions | a/b | Natural Frequencies (Hz) | 1st Order | 2nd Order | 3rd Order |
|                    |     |                          | Analytical | FEA | Analytical | FEA | Analytical | FEA |
| FREE-FREE          | 0.01 | 26.337                    | 26.302     | 59.541 | 59.515 | 75.836 | 75.791 |
|                    | 0.25 | 27.260                    | 27.246     | 46.834 | 46.834 | 67.227 | 67.212 |
|                    | 0.50 | 34.894                    | 34.897     | 65.588 | 65.851 | 97.613 | 97.626 |
|                    | 0.80 | 81.267                    | 81.284     | 161.357 | 160.110 | 241.702 | 241.490 |
| FREE-CLAMPED       | 0.01 | 19.625                    | 19.624     | 35.971 | 35.931 | 52.111 | 52.106 |
|                    | 0.25 | 19.983                    | 19.958     | 38.174 | 38.180 | 57.850 | 57.809 |
|                    | 0.50 | 23.314                    | 23.286     | 51.701 | 51.669 | 82.651 | 82.625 |
|                    | 0.80 | 45.860                    | 45.837     | 122.824 | 122.810 | 202.503 | 202.120 |
| CLAMPED-FREE       | 0.01 | 26.337                    | 26.313     | 59.695 | 59.569 | 76.041 | 75.877 |
|                    | 0.25 | 31.615                    | 31.582     | 53.341 | 53.341 | 74.914 | 74.908 |
|                    | 0.50 | 47.039                    | 47.004     | 79.730 | 79.738 | 112.166 | 112.250 |
|                    | 0.80 | 119.186                   | 119.186    | 200.300 | 200.300 | 281.055 | 281.055 |
| CLAMPED-CLAMPED    | 0.01 | 19.625                    | 19.630     | 36.022 | 35.951 | 52.265 | 52.150 |
|                    | 0.25 | 22.802                    | 22.778     | 43.759 | 43.725 | 64.956 | 64.956 |
|                    | 0.50 | 32.742                    | 32.742     | 64.665 | 64.658 | 96.793 | 96.793 |
|                    | 0.80 | 80.653                    | 80.600     | 161.049 | 161.360 | 241.497 | 241.460 |
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