de Sitter group and Einstein-Hilbert Lagrangian

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Abstract

Axial vector torsion in the Einstein-Cartan space $U_4$ is considered here. By picking a particular term from the $SO(4,1)$ Pontryagin density and then modifying it in a $SO(3,1)$ invariant way, we get a Lagrangian density with Lagrange multipliers. Then considering torsion and torsion-less connection as independent fields, it has been found that $\kappa$ and $\lambda$ of Einstein-Hilbert Lagrangian, appear as integration constants in such a way that $\kappa$ has been found to be linked with the topological Nieh-Yan density of $U_4$ space.

PACS numbers: 04.20.Cv, 04.20.Fy

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1. INTRODUCTION

In Riemannian manifold if metric and affine connection are treated as two independent entities then the local geometry is endowed with two independent tensors - curvature and torsion. Einstein considered only curvature to be enough for the most economical and successful theory of space-time. Cartan on the other hand considered Einstein’s view as a special case\(^1\). In the recent background of abstract geometrical framework for a consistent quantum theory of gravity the concept of torsion can be easily welcomed.

It is a natural belief that constants of nature have topological origin. In the well-known Einstein-Hilbert Lagrangian of gravity there are two constants - the gravitational constant and the cosmological constant. These constants are ad hoc in nature to justify classical gravitation. Moreover the constancy of the fundamental constants and, in particular, that of the gravitational constant has been questioned for a long time\(^2\) and it led the early attempts to unify gravity with electromagnetism\(^3, 4\). Modern theories, like the string theory, link fundamental constants with some extra dimensions and also predict variableness of these constants\(^5\). Recent cosmological data\(^6\) and its analysis\(^7\) indicates a possible variation of the gravitational constant as it has been found that the peak luminosities of distant supernovae appear to be fainter than predicted by standard big bang cosmology. So it demands that in a generalized theory of gravity, like Scalar-Tensor theory, one should treat Newton’s constant \(G\) in a different manner.

It is well known\(^8\) that torsion and curvature of any manifold are related to translation and rotation respectively. In particular torsion is more precisely related to broken translation gauge fields within the framework of nonlinear realization of the local space time group\(^9\). So to exploit both the symmetries of translation and rotation any gravitational Lagrangian must contain torsion together with curvature.

Today, gauge theory provides the theoretical base of all modern unification attempts in particle physics. In field theory gauge potentials become a standard tool for describing interactions with very different symmetries. And apparently the single gap in the modern gauge picture still remains : gauging the external or space-time symmetries of field and particles, that includes the gauge gravity also.

Kibble\(^10\) and Sciama\(^11\) pointed out that the Poincaré group, which is the semi-direct product of translation and Lorentz rotation, is the underlying gauge group of gravity and
found the so-called Einstein-Cartan theory where mass-energy of matter is related to the curvature and spin of matter is related to the torsion of space-time.

Apart from being the most studied group\[12\], the Poincaré group has some remarkable properties\[13\]: (1) it is the basic local group of physics; (2) it classifies the elementary particles, giving them both spin and momentum; (3) it has a very clear relationship with space-time. One major drawback of Poincaré group is that it is a non-semi simple group which implies that there is no Lagrangian yielding its Yang-Mills equations\[14\]. There exists a general procedure\[15\] to check whether or not a set of field equations leads to a coherent theory, i.e. a theory that can be quantized. If we apply it to Yang-Mills equations for non-semi simple groups, we find that they are never consistent. Here we see that though the Poincaré group is the classical group for relativistic kinematics it cannot be given a quantum version. Now by minimal addition of extra terms this inconsistent theory can be transformed to a good theory and we find a Lagrangian of a gauge theory for a semi-simple group, the de Sitter group, where vierbeins have been connected to the de Sitter boost parameters\[16\]. In this way, the de Sitter gauge theory comes up as the corrected Poincaré gauge theory. Alternatively, there are other approaches where de Sitter group based Yang-Mills theories are shown to be producing either Ashtekar formulation of gravity\[17\] or Einstein-Cartan version of general relativity\[18\]. One can also derive standard Einstein-Hilbert gravity from de Sitter gauge theory by choosing a particular Euler type four form as Lagrangian density\[19\].

From geometrical point of view there is an important connection between the de Sitter group and the Poincaré group. It is a well known fact that the Poincaré group can be obtained from the de Sitter group by an appropriate Inönü-Wigner contraction\[20,21\]. Here the contraction is achieved when the de Sitter pseudo-radius $R \to \infty$. By this contraction the curvature of the de Sitter space $R \to 0$ where $R \propto R^{-2}$. We know that the de Sitter space is a solution of Einstein’s equation for an empty space with cosmological constant $\Lambda = R/4$. Hence by the Inönü-Wigner contraction, the de Sitter space reduces to Minkowski space which is a sourceless solution of Einstein’s equation with a vanishing cosmological constant. Instead of taking $R \to \infty$ if we take $R \to 0$ then we find that the de Sitter group reduces to the “conformal” Poincaré group where the Poincaré translation is replaced by the special conformal transformation and the de Sitter space reduces to a cone-space whose geometry is gravitationally related to an infinite cosmological constant\[22\].

Recently a gravitational Lagrangian has been been proposed\[23\] where a Lorentz invariant
part of the de Sitter Pontryagin density has been treated as the Einstein-Hilbert Lagrangian. In this paper we shall try to establish the constancy of this gravitational constant under the background where the cosmological constant is the only source of gravitation.

2. TOPOLOGICAL DENSITIES AND GRAVITY LAGRANGIAN

Cartan’s structural equations for a Riemann-Cartan space-time $U_4$ are given by

$$T^a = de^a + \omega^a_b \wedge e^b$$

(1)

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b,$$

(2)

here $\omega^a_b$ and $e^a$ represent the spin connection and the local frames respectively.

In $U_4$ there exists two invariant closed four forms. One is the well known Pontryagin density $P$ and the other is the less known Nieh-Yan(NY) density $N$ given by

$$P = R^a_b \wedge R^b_a,$$

(3)

$$N = d(e^a \wedge T^a) = T^a \wedge T_a - R^a_b \wedge e^a \wedge e^b.$$ (4)

The minimal Lagrangian density of a spin-$\frac{1}{2}$ field $\psi$ with an external gravitational field with torsion is given by

$$L_D = \frac{i}{2} \{ \bar{\psi}^* \gamma \wedge D\psi + \bar{D\psi} \wedge *\gamma\psi \} + *m\bar{\psi}\psi$$

$$-\frac{1}{4} A \wedge \bar{\psi} \gamma_5 *\gamma\psi,$$

(5)

where $\gamma = \gamma^a e_a$, $D$ is the torsion-free exterior derivative, $A$ is the axial vector part of the torsion one form and $*$ is the Hodge duality operator. Therefore, considering the source in the matter Lagrangian, we can simply assume that the torsion is given by an axial vector only.

In presence of axial vector torsion, one naturally gets the Nieh-Yan density from

$$N = -R^a_b \wedge e^a \wedge e^b = -*N \eta,$$

(6)

where $\eta := \frac{1}{24} \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d$ (7)

is the invariant volume element. It follows that $*N$, the Hodge dual of $N$, is a scalar density of dimension $(\text{length})^{-2}$. 


It has been shown in an earlier paper\cite{28}, when a ‘direction vector’ (vortex line) is attached with a space-time point $x_\mu$, this eventually leads to the $SL(2, C)$ gauge theory of gravitation including torsion. The space time manifold now corresponds to the de Sitter space $M^{4,1}$. In this framework $SO(4,1)$ Pontryagin density can be written as \cite{23,29}

$$R_A^B \wedge R_B^A = R_a^b \wedge R_b^a + 2 \frac{1}{l^2} N$$

$$= \tilde{R}_a^b \wedge \tilde{R}_b^a - \frac{1}{3} R N + 2 \frac{1}{l^2} N, \quad (8)$$

where $\tilde{R}_{ab} := R_{ab} - \frac{1}{12}^* N \eta_{ab}$, $\eta_{ab} := \frac{1}{21} \epsilon_{abc} \epsilon^c \wedge e^d$, \hspace{1cm} (9)

such that $R = \tilde{R} = \frac{1}{2}^* (\tilde{R}_{ab} \wedge \eta_{ab})$, \hspace{1cm} (11)

and $\tilde{R}_{ab} \wedge e_a \wedge e_b = (R_{ab} - \frac{1}{12}^* N \eta_{ab}) \wedge e_a \wedge e_b$

$$= 0, \quad (12)$$

$l$ is a fundamental length constant and $A, B = 0,1,..,4$. Usually, when Lorentz group is embedded into the de Sitter group, $l$ is called the radius of the universe and is related to the cosmological constant \cite{29}. The above decomposition \cite{8,9} is always possible in $U_4$ provided, the divergence four form of the axial-vector torsion, $N \neq 0$. This serves no restriction on the gauge field part of the $SO(3,1)$ connection one form $\omega^{ab}$ in the background of a broken $SO(4,1)$ gauge theory. In other word the above breakup guarantees that, in the tangent space, only $SO(3,1)$ symmetry is preserved and $SO(4,1)$ symmetry is broken. The two parts $\tilde{R}_{ab}$ and $\frac{1}{12}^* N \eta_{ab}$ of $R_{ab}$ are two independent covariant two forms under $SO(3,1)$ rotation in the tangent space. From (11) and (12) we see that $\tilde{R}_{ab}$ and $\frac{1}{12}^* N \eta_{ab}$ are respectively connected to $R$ and $N$. Hence by the above decomposition we are separating two irreducible parts of the curvature w. r. t. the one form $e^a$ and consequently neither any Bianchi identity is violated nor it leads to any constraint on $N$.

Recently, it has been shown that \cite{23}, in $U_4$ space one can locally consider a particular term from this $SO(4,1)$ Pontryagin density as the gravitational Lagrangian \cite{30}, given by

$$\mathcal{L}_0 = -^* NR \eta. \quad (13)$$

This Lagrangian looks like the Einstein-Hilbert Lagrangian,

$$\mathcal{L}_{EH} = \frac{1}{\kappa} R \eta, \quad (14)$$

5
provided \( *N = \frac{1}{\kappa} \), where \( \kappa \) is Einstein’s gravitational constant, and the torsional contribution disappears from the scalar curvature \( R \). It is a well-known result[31] in \( U_4 \) space, in the case of axial vector torsion, that

\[
R = R_E + \frac{1}{4} * (T \wedge *T),
\tag{15}
\]

where \( R_E \) represents scalar curvature when the connection is without torsion and \( T \) is the torsion three form representing axial vector torsion \( A \). Now if we assume the axial vector \( A \) to be a null vector then (15) reduces to

\[
R = R_E.
\tag{16}
\]

This can be guaranteed by introducing a \( SO(3,1) \) invariant Lagrangian density \( L_1 \), given by

\[
L_1 = \zeta * (T \wedge *T)(T \wedge *T),
\tag{17}
\]

where \( \zeta \) is a dimensionless Lagrange multiplier. So far \( SO(3,1) \) invariance is concerned, torsion can be separated from the connection as torsional part of the \( SO(3,1) \) connection transforms like a tensor i.e. when the local frame also transforms like a \( SO(3,1) \) valued one form (not connection one form) in a broken \( SO(4,1) \) gauge theory. In this direction it is important to define a torsion-free exterior differentiation through a field equation involving the connection and the local frame only. So we introduce another Lagrangian density \( L_2 \), given by,

\[
L_2 = *(b_a \wedge \nabla e^a)(b_a \wedge \nabla e^a),
\tag{18}
\]

where \( \nabla \) represents exterior differentiation with respect to a \( SO(3,1) \) connection one form \( \tilde{\omega}^{ab} \) and \( b_a \) is a two form with one internal index and of dimension \((\text{length})^{-1}\). If we treat \( b_a \) as Lagrange multiplier then it ensures that \( \nabla \) represents torsion-free exterior differentiation. By this way torsion has become decoupled from the connection part of the theory. It has become independent of the one form \( e^a \), in particular, owing to its fundamental existence as a metric independent tensor in the affine connection in \( U_4 \), we treat here the three form \( T = e^a \wedge T_a \) as more fundamental than the one form \( T^{ab} = \omega^{ab} - \tilde{\omega}^{ab} \). Without any ambiguity and for future consistency, we can consider a connection independent cosmological density, given by

\[
L_3 = -\Lambda \eta ,
\tag{19}
\]
where $\Lambda$ is a constant whose value is to be ascertained later on. Now we are in a position to define the total gravitational Lagrangian density in empty space as

$$\mathcal{L}_G = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3,$$

$$= -^* NR\eta + \zeta^*(T \wedge ^*T)(T \wedge ^*T) + ^*(b_a \wedge \nabla e^a)(b_b \wedge \nabla e^b) - \Lambda \eta,$$

where $N = dT$ and $R\eta = -\frac{1}{2}(d\bar{\omega}^{ab} + \bar{\omega}^a \wedge \bar{\omega}^b) \wedge \eta_{ab}$. To start with this Lagrangian we have altogether 69 independent components of the field variables $e^a, T, \bar{\omega}^{ab}, b^a$ and $\zeta$.

### 3. EULER-LAGRANGE EQUATIONS AND GRAVITATIONAL CONSTANT

The Lagrangian $\mathcal{L}_G$, which is defined in the previous section, is only Lorentz invariant under rotation in the tangent space where de Sitter boosts are not permitted. As a consequence $T$ can be treated independently of $e^a$ and $\bar{\omega}^{ab}$. Then following reference [32], we independently vary $e^a, \nabla e^a, T, dT, R^{ab}, b^a$ and $\zeta$ and find

$$\delta \mathcal{L}_G = \delta e^a \wedge \frac{\partial \mathcal{L}_G}{\partial e^a} + \delta \nabla e^a \wedge \frac{\partial \mathcal{L}_G}{\partial \nabla e^a} + \delta T \wedge \frac{\partial \mathcal{L}_G}{\partial T} + \delta dT \frac{\partial \mathcal{L}_G}{\partial dT} + \delta R^{ab} \wedge \frac{\partial \mathcal{L}_G}{\partial R^{ab}} + \delta b^a \wedge \frac{\partial \mathcal{L}_G}{\partial b^a} + \delta \zeta \frac{\partial \mathcal{L}_G}{\partial \zeta},$$

$$\delta \omega^{ab} \wedge \frac{\partial \mathcal{L}_G}{\partial \omega^{ab}}.$$
\[
\frac{\partial L_G}{\partial T} = 4\zeta^* (T \wedge \ast T)^* T \\
\frac{\partial L_G}{\partial (dT)} = R \\
\frac{\partial L_G}{\partial R^{ab}} = \frac{1}{4} \ast N\epsilon_{abcd} e^c \wedge e^d = \frac{1}{2} \ast N\eta_{ab} \\
\frac{\partial L_G}{\partial b^a} = 2^* (b_b \wedge \nabla e^b) \nabla e_a \\
\frac{\partial L_G}{\partial \zeta} = \ast (T \wedge \ast T)(T \wedge \ast T)
\]

Where

\[
R_a := \frac{1}{2} \frac{\partial (R\eta)}{\partial e^a} = \frac{1}{4} \epsilon_{abcd} R^{bc} \wedge e^d = -G^b_a \eta_b \\
G^b_a := R^b_a - \frac{1}{2} R^b \delta^a_a \\
\eta_a := \frac{\partial \eta}{\partial e^a} = \frac{1}{3!} \epsilon_{abcd} e^b \wedge e^c \wedge e^d \\
\tau^a_b := T^{a\mu\nu} T_{b\mu\nu} \quad \text{and} \quad \tau := \tau^a_a
\]

From above, Euler-Lagrange equations for \(\zeta\) and \(b_a\) give us

\[
T \wedge \ast T = 0 \\
\nabla e_a = 0
\]

i.e. \(T\) corresponds to a null axial vector and \(\nabla\) is torsion free. Using this result in (23), (24) and (25) we get

\[
\frac{\partial L_G}{\partial e^a} = -\ast N(2R_a - R\eta_a) - \Lambda \eta_a \\
\frac{\partial L_G}{\partial (\nabla e^a)} = 0 \\
\frac{\partial L_G}{\partial T} = 0
\]

Using these results in equations from (22) to (27), we get Euler-Lagrange equations of \(e^a\), \(T\) and \(\bar{\omega}^{ab}\), given by

\[
\ast N(2R_a - R\eta_a) + \Lambda \eta_a = 0 \\
dR = 0 \\
\nabla(\ast N \eta_{ab}) = 0
\]
Using (35), the last equation yield
\[ d^* N = 0 \] \hspace{2cm} (42)

From equations (40) and (42) we can write
\[ *N = \frac{1}{\kappa} \quad \text{and} \quad R = 2\lambda \quad \text{(say)}. \] \hspace{2cm} (43)

Using (43) in (39) and then using (30) we get
\[ G^b_a = -\frac{1}{2}\lambda\delta^b_a, \] \hspace{2cm} (44)

where, for consistency, \( \lambda = \kappa\Lambda \). This last equation is the Einstein’s equation of gravity in the presence of the cosmological constant \( \lambda \) corresponding to the Einstein-Hilbert Lagrangian
\[ \mathcal{L}_{EH}^{CC} = \frac{1}{\kappa}(R - \lambda)\eta. \] \hspace{2cm} (45)

We know that, though torsion one form \( T^{ab} = \omega^{ab} - \bar{\omega}^{ab} \) is a part of the \( SO(3,1) \) connection, it does not transform like a connection form under \( SO(3,1) \) rotation in the tangent space and thus it imparts no constraint on the gauge degree of freedom of the Lagrangian. By this way the role of torsion in the underlying manifold has become multiplicative rather than additive one and the quadratic first part of the Lagrangian \( \mathcal{L}_G \) looks like \textit{torsion} \( \otimes \) \textit{curvature}[33]. In other words - the additive torsion is decoupled from the theory but not the multiplicative one. This indicates that torsion is uniformly nonzero everywhere. In the geometrical sense, this implies that micro local space-time is such that at every point there is a direction vector (vortex line) attached to it. This effectively corresponds to the non commutative geometry having the manifold \( M_4 \times Z_2 \) where the discrete space \( Z_2 \) is just not the two point space[34] but appears as an attached direction vector. This has direct relevance in the quantization of a fermion where the discrete space appears as the internal space of a particle[28]. This becomes relevant if we consider that fermions are the basic building units of matter. The existence of a globally defined null vector field \( *T \) with non-zero divergence then corresponds to the axial vector current leading to chiral anomaly. Now, in the background of the minimal action of a spinor field given in (5) and even in massless case, there is a divergent contribution of torsion to chiral anomaly given by[29]
\[ d\left\langle j_5 \right\rangle = A(x) \] \hspace{2cm} (46)
where
\[ j_5 = \bar{\psi}^* \gamma_5 \psi, \quad \gamma = \gamma^a e_a \]
and
\[ A(x) = 2 \sum_n \eta \psi_n^\dagger \gamma_5 \psi_n. \]

Then under standard regularization by the square of the Dirac operator in Einstein-Cartan space
\[ A = \lim_{M \to \infty} \frac{1}{8\pi^2} \left[ R^{ab} \wedge R_{ab} + 2M^2 (T_a \wedge T^a) - R_{ab} \wedge e^a \wedge e^b + O(M^{-2}) \right] \] (47)

As chiral anomaly appears as the quantum mechanical symmetry breaking, torsion in this sense represents the quantum effect. Mielke et al have questioned the contribution of the NY term to the chiral anomaly as well as its non triviality after renormalization. Contribution of the NY term to chiral anomaly has been confirmed by Obukhov et al and in an independent analysis, computing the index of the Dirac operator on a four dimensional compact manifold, it has been shown that the integral of the NY term is necessarily an integer, it is the difference of two Chern classes $SO(5)$ and $SO(4)$ and therefore being topological $N$ is non-renormalizable. In our present analysis this bears an important implication as we see from (43) that topological $N$ globally defines the gravitational constant, at least in the case where the cosmological constant is the only source of gravity, by the equation
\[ N = -\frac{1}{\kappa} \eta. \] (48)

It is important to note that, in some other approach, some multiplicative torsion $\otimes$ curvature terms also appear in 3D gravity but they seem to be devoid of any topological interpretation. Such terms do not appear in 4D.

Hence (44) implies that our starting Lagrangian $\mathcal{L}_G$ is equivalent to the Einstein-Hilbert Lagrangian in vacua in presence of a cosmological constant where the two constants $\kappa$ and $\lambda$ are constants of integration but also, by (43), $\lambda$ is half of the $SO(3,1)$ scalar curvature and the topological Nieh-Yan density is $(-\frac{1}{\kappa})$ times the invariant volume element. Also the form of the starting Lagrangian $\mathcal{L}_G$ implies that constancy of the gravitational constant depends upon the fact that the source term in $\mathcal{L}_G$ is independent of the $SO(3,1)$ gauge
connection. This suggests that in a gravity theory with $SO(3, 1)$ connection, torsion, local frame and matter, if we want gravitational constant to be an on-shell constant, then the material source term should be independent of the $SO(3, 1)$ connection.

4. DISCUSSION

Recent cosmological evidence\cite{6, 42} suggests that cosmological constant seems to be present evermore in the cosmological data. Theoretically, cosmological constant appears when one considers a four dimensional manifold that is due to compactification\cite{43} of a five dimensional manifold with the signature of a (anti)de Sitter space time\cite{44}. This implies that in the local tangent space the gauge group structure is either $SO(4, 1)$ or $SO(3, 2)$. To keep Lorentz invariance intact (anti)de Sitter boost is forbidden in the tangent space. So it is justified to consider the Lagrangian as a particular $SO(3, 1)$ invariant part of the full $SO(4, 1)$ Pontryagin density.

It is important to note that, in our present formalism, the only assumption is that the torsion is represented by a null axial vector and the corresponding Lagrangian is a particular term of the $SO(4, 1)$ Pontryagin density in such a way that the $SO(3, 1)$ invariance of the theory is maintained. The presence of the null axial vector at each space time point suggests that the space time manifold is characterized by the presence of a ‘direction vector’ (vortex line) attached to each point which is the source of torsion. It may be remarked that the degrees of freedom of this theory is minimally extended from that of Einstein-Hilbert theory with torsion contributing to the additional degree. As a result $\kappa$ and $\lambda$ have got their definite geometrical meaning in $U_4$ space in comparison to their standard meaning of being simply constants such that, in empty space, $\lambda$ is half of the $SO(3, 1)$ scalar curvature and $(-\frac{1}{\kappa})$ is the proportionality constant between the topological Nieh-Yan density and the invariant volume four form $\eta$. Moreover, being constants of integration, $\kappa$ and $\lambda$ might have got their fixed values in the Early Universe when the bulk matter was created.

In a recent paper\cite{45} it has been shown that, in the gravity without metric formalism of gravity, when one performs a particular canonical transformation of the field variables, CP-violating $\theta$-term appears in the Lagrangian together with the cosmological term. This supports the finding of this note when we consider that the torsion, being an axial vector, has a certain role to play in CP-violation. Indeed, the topological $\theta$-term of ‘gravity without
metric formalism’ is linked with the topological Nieh-Yan density of $U_4$ geometry. CP-violation or non-violation by topological terms has been discussed also in reference [27]. Link of torsion with CP-violation can be found also in $SL(2, C)$ gauge approach of gravity [46]. Thus arrow of time may play a significant role in the geometrical origin of torsion and hence of the gravitational constant.

ACKNOWLEDGEMENT

I wish to thank Prof. Pratul Bandyopadhyay, Indian Statistical Institute, Kolkata, for his valuable remarks and fruitful suggestions on this problem.

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