Tensor products of Bands and ideals

Mohamed Amine BEN AMOR *
Ömer GOK†
Damla YAMAN‡

* Research Laboratory of Algebra, Topology, Arithmetic, and Order
Department of Mathematics, Faculty of Mathematical, Physical and Natural Sciences of Tunis
Tunis-El Manar University, 2092-El Manar, Tunisia
† ‡ Department of Mathematics
Yildiz Technical University, Istanbul

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Abstract

In this study we give an answer to the open question about the Riesz tensor product of ideals and bands of a Riesz space. We prove among other results that the Dedekind completion of the Riesz tensor product of two ideals is again an ideal.

1 Introduction

Tensor product of different vector lattices and structures has been a significant field of interest and has been studied and improved in time by authors like H.H.Schaefer, J.J. Grobler, C.C.A. Labuschagne and G. Buskes since D. H. Fremlin first introduced the Riesz tensor product. Mainly we refer to [7], [5] and [12] on the subject. In [8], Fremlin proved the existence and the uniqueness of the tensor product in the following theorem. (For other constructions see [11] or [15]).
Theorem 1.1. Let $E$ and $F$ be Archimedean Riesz spaces. Then there is an Archimedean Riesz space $G$ and a Riesz bimorphism $\varphi : E \times F \to G$ such that

(i) whenever $H$ is an Archimedean Riesz space and $\psi : E \times F \to H$ is a Riesz bimorphism, there is a unique Riesz homomorphism $T : G \to H$ such that $T \varphi = \psi$;

(ii) $\varphi$ induces an embedding $\hat{\varphi} : E \otimes F \to G$;

(iii) $(ru-D) \hat{\varphi}[E \otimes F]$ is dense in $G$ in the sense that for every $w \in G$, there exist $x_0 \in E$ and $y_0 \in F$ such that for every $\epsilon > 0$, there is an element $v \in \hat{\varphi}[E \otimes F]$ such that $|w - v| \leq \epsilon \hat{\varphi}(x_0 \otimes y_0)$;

(iv) if $w > 0$ in $G$, then there exist $x \in E^+$ and $y \in F^+$ such that $0 < \hat{\varphi}(x \otimes y) \leq w$.

All such Archimedean Riesz spaces $G$ are isomorphic as Riesz spaces and this unique Archimedean Riesz space $G$ is called the Fremlin tensor product of $E$ and $F$ and is denoted by $E\overline{\otimes}F$.

The Fremlin tensor product has the following additional properties:

$(ru-D)_+$ (Positive relative uniform density property). For $w \in G_+$, there exists an element $(x, y) \in E_+ \times F_+$ such that for every $\epsilon > 0$, there exists an element $v \in (E_x)_+ \otimes (F_y)_+$ with $|w - v| \leq \epsilon \hat{\varphi}(x_0 \otimes y_0)$

(PUM) (Positive universal mapping property). Let $G$ be a relatively uniformly complete Archimedean Riesz space and $\psi : E \times F \to G$ be a positive linear mapping. Then there exists a unique positive linear mapping $\tau : E\overline{\otimes}F \to G$ such that $\tau \circ \varphi = \psi$.

(B) If $h \in E\overline{\otimes}F$, there exists an element $(x, y) \in E_+ \times F_+$ such that $|h| \leq x \otimes y$.

In 2018, an open question regarding the tensor products of order ideals and bands was presented by Azouzi, Ben Amor and Jaber in [4]. They asked whether or not the tensor product of two ideals or two bands in a Riesz space is again an ideal or a band. In this present work we give an affirmative answer to this question not for the Riesz tensor product but for the Dedekind complete tensor product. Indeed, Grobler in [10] introduced the Dedekind complete tensor product as the Dedekind completion of the Riesz tensor product, that is, $E\overline{\boxtimes}_\delta F = (E\overline{\otimes}F)\delta$. He stated and proved the following theorem [10, Theorem 5.2] which will be useful in the sequel.

Theorem 1.2. Let $E, F$ be two Archimedean Riesz spaces. The tensor product $E\overline{\otimes}_\delta F$ is a Dedekind complete vector lattice with the following properties:
(D3) $E \otimes F$ is a vector subspace of $E \odot F$ and $E \odot F$ is a Riesz subspace of $E \odot \delta F$.

(D5) If $G$ is a Dedekind complete vector lattice and $\psi : E \times F \to G$ is an order continuous Riesz bimorphism, then there exists an order continuous Riesz homomorphism $\tau : E \odot \delta F \to G$ such that $\tau \circ \sigma = \psi$.

(D4) The Dedekind complete Riesz subspace generated in $E \odot \delta F$ by $E \otimes F$ is equal to $E \otimes \delta F$.

(OD) For every $0 < h \in E \odot \delta F$ there exists an element $(x, y) \in E_+ \times F_+$ such that $0 < x \otimes y \leq h$.

(B) If $h \in E \odot \delta F$, there exists an element $(x, y) \in E_+ \times F_+$ such that $|h| \leq x \otimes y$.

(PUM) Let $G$ be a Dedekind complete Archimedean Riesz space and let $\psi : E \times F \to G$ be an order continuous positive bilinear mapping. Then there exists a unique order continuous positive linear mapping $\tau : E \odot \delta F \to G$ such that $\tau \circ \sigma = \psi$.

Most recently, G. Buskes and P. Thorn in preprint [6] present a counter example for tensor product of two ideals to be an ideal. In the light of these recent works, we try to present an overall look at this subject including tensor product of principal bands.

2 Preliminaries

Some fundamental notions and tools are given in this section. The reader can find further information in [3] or in [13] or in [14].

A real vector space $E$ equipped with an order relation $\geq$ compatible with the algebraic structure of $E$ which has the following properties is called an ordered vector space.

i. If $u \geq v$ for $u, v \in E$ then $u + w \geq v + w$ for all $w \in E$.

ii. If $u \geq v$ for $u, v \in E$ then $\lambda u \geq \lambda v$ for each $\lambda \geq 0$.

An ordered vector space $E$ is called a Riesz space if the supremum (or infimum) of every nonempty finite subset of $E$ exists in $E$.

A Riesz space $E$ is called Archimedean if $x, y \in E^+$ and $nx \leq y$ for each $n \in \mathbb{N}$ imply $x = 0$. Throughout this study all Riesz spaces will be assumed to be Archimedean.
A Riesz space is called Dedekind complete whenever every nonempty subset which is bounded above has a supremum (or every nonempty subset which is bounded below has an infimum). A Dedekind complete Riesz space $L$ is said to be a Dedekind completion of the Riesz space $E$ whenever $E$ is Riesz isomorphic to a majorizing order dense Riesz subspace of $L$. Every Archimedean Riesz space has a unique (up to a lattice isomorphism) Dedekind completion [3, Theorem 2.24].

Recall that a Riesz subspace of a Riesz space $E$ is a subspace of $E$ that is closed under the lattice operations in $E$. A subset $A$ of a Riesz space is called solid whenever $|x| \leq |y|$ and $y \in A$ implies $x \in A$. A solid vector subspace of a Riesz space is called an ideal. It is easy to see that every ideal is a Riesz subspace. The ideal generated by a vector $x$ is defined as

$$E_x = \{ y \in E : \exists \lambda > 0 \text{ and } |y| \leq \lambda |x| \}$$

and is called principal ideal.

A vector subspace $A$ of an ordered vector space $E$ is said to be majorizing $E$ whenever for each $x \in E$ there exists some $y \in A$ such that $x \leq y$.

A Riesz subspace $A$ of a Riesz space $E$ is called order dense in $E$ whenever for every $0 < x \in E$ there exists some $y \in A$ with $0 < y \leq x$.

A net $\{x_\alpha\}$ in a Riesz space is said to be order convergent to a vector $x$ whenever there exists another net $\{y_\alpha\}$ which satisfies $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for all $\alpha$. A subset $A$ of a Riesz space is called order closed whenever a net $\{x_\alpha\}$ in $A$ order converges to a vector $x$ and $x \in A$ holds. An order closed ideal is called a band. The band generated by a vector $x$ in a Riesz space $E$ is called principal band and given by

$$B_x = \{ y \in E : |y| \wedge n|x| \uparrow |y| \}$$

A band $B$ in a Riesz space $E$ that satisfies $E = B \oplus B^d$ is called a projection band.

The disjoint complement $A^d$ of a nonempty subset $A$ of a Riesz space $E$ is defined by

$$A^d = \{ u \in E : u \perp v \text{ for all } v \in A \}.$$ 

$A^d$ is an ideal and a band in $E$. Note that $A \subseteq A^{dd}$ where $A^{dd} = (A^d)^d$. $A^{dd}$ is the band generated by $A$. If $A$ is a band then $A = A^{dd}$.
A vector \( e > 0 \) in a Riesz space \( E \) is called a weak order unit whenever the band generated by \( e \) is equal to \( E \), i.e., \( B_e = E \). Every positive vector of a Riesz space is a weak order unit in the band it generates.

Let \( e \) and \( x \) be positive vectors of a Riesz space \( E \). \( x \) is called a component of \( e \) whenever \( x \land (e - x) = 0 \).

A Riesz subspace \( A \) of a Riesz space \( E \) is called regular if the embedding of \( A \) into \( E \) preserves the arbitrary suprema and infima.

**Theorem 2.1.** [2, Theorem 1.23] Every order dense Riesz subspace of a Riesz space is a regular Riesz subspace.

**Theorem 2.2.** [2, Theorem 1.24] Every ideal \( A \) of a Riesz space \( E \) is order dense in \( A^{dd} \). In particular, an ideal \( A \) is order dense in \( E \) if and only if \( A^d = \{0\} \).

The following lemma may seem to be out of interest of this work. However it serves an important function in the further material.

**Lemma 2.3.** Let \( u \) be a positive element in the Riesz space \( E \). Then the principal band \( B_u \) generated by \( u \) in \( E \) is order dense and majorizing in \( B^\delta_u \), the principal band generated by \( u \) in the Dedekind completion \( E^\delta \) of \( E \).

**Proof.** Since \( E \) is order dense in \( E^\delta \), it follows that \( E \) is regular in \( E^\delta \). This latter fact yields to

\[
B_u \subset B^\delta_u.
\]

Moreover \( B_u \) is order dense in \( B^\delta_u \). Indeed, if \( x \) is a strictly positive element in \( B^\delta_u \), then \( 0 < x \land u \). \( E \) is order dense in \( E^\delta \), so there is some \( y \) in \( E \) such that

\[
0 < y \leq x \land u \leq u.
\]

It follows that \( y \) is in \( B_u \) and \( 0 < y \leq x \). Consider now a positive element \( x \) in \( B^\delta_u \). Then

\[
x = \sup_n \{x \land nu\}
\]

Since \( E \) is majorizing in \( E^\delta \), there is some \( y \) in \( E \) such that

\[
x = \sup_n \{x \land nu\} \leq \sup_n \{y \land nu\}
\]
Put \( y_n = y \wedge nu \) for every \( n \). As \((y_n)\) order converges in \( E^\delta \), \((y_n)\) is a Cauchy sequence in \( E^\delta \). It follows that there is \((z_{n,m})\) a decreasing sequence to zero in \( E^\delta \) such that

\[ |y_n - y_m| \leq z_{n,m} \]

For every \( n, m \) put \( z'_{n,m} = z_{n,m} \) if \( z_{n,m} \) is in \( E \) otherwise, \( z_{n,m} - |y_n - y_m| > 0 \) which yields to the existence of some \( z''_{n,m} \) in \( E \) such that \( 0 < z''_{n,m} \leq z_{n,m} - |y_n - y_m| \). So we can take \( z'_{n,m} = z''_{n,m} + |y_n - y_m| \). It follows that

\[ |y_n - y_m| \leq z''_{n,m} \leq z_{n,m}. \]

The sequence \((z''_{n,m})\) order converges in \( E \) to zero and \((y_n)\) is a Cauchy sequence in \( E \). From the classical Birkhoff inequality (see Theorem 1.9 in [3]) we have

\[ |y_n - y_m| \leq (n - m)u \]

It follows that

\[ |y_n - y_m| \leq (n - m)u \wedge z''_{n,m} \leq z''_{n,m} \]

That is \((y_n)\) is a Cauchy sequence in \( B_u \). This latter fact yields to the existence of \( y^* \) in \( B_u^\delta \), the Dedekind completion of \( B_u \) such that \((y_n)\) order converges to \( y^* \). It follows that \( x \leq y^* \). As \( B_u \) is majorizing in \( B_u^\delta \), it follows that there is some \( t \) in \( B_u \) such that \( 0 \leq x \leq t \)

and this completes the proof.

\[ \square \]

### 3 The Riesz tensor product of Riesz subspaces

Throughout this section \( E \) and \( F \) will be Archimedean Riesz subspaces.

**Lemma 3.1.** Let \( E \) and \( F \) be Riesz spaces, \( x, x' \) be strictly positive elements in \( E \) and \( y, y' \) be strictly positive elements in \( F \). If \( x \otimes y \leq x' \otimes y' \), then \( x \) belongs in the principal ideal generated by \( x' \), and \( y \) belongs in the principal ideal generated by \( y' \).

**Proof.** Let \( E_{x \vee x'} \) be the principal ideal generated by \( x \vee x' \) and \( F_{y \vee y'} \) be the principal ideal generated by \( y \vee y' \). As in the construction of the Fremlin tensor product made by Schaefer in [15], the tensor product \( E_{x \vee x'} \otimes F_{y \vee y'} \) becomes a vector subspace of \( C(K_{x \vee x'}) \otimes C(K_{y \vee y'}) \) which in turn can be viewed as a subspace of \( C(K_{x \vee x'} \times K_{y \vee y'}) \) namely the subspace of all functions \( h(s, t) = \sum_i f_i(s)g_i(t) \) where \( f_i \) are in \( C(K_{x \vee x'}) \)
and \( g_i \) are in \( C(K_{y \lor y'}) \). \( K_{x \lor x'} \) and \( K_{y \lor y'} \) are compact topological spaces. \( x \) and \( x' \) can be viewed as \( \hat{x} \) and \( \hat{x}' \) in \( C(K_{x \lor x'}) \) and \( y \) and \( y' \) can be viewed as \( \hat{y} \) and \( \hat{y}' \) in \( C(K_{y \lor y'}) \). Since \( x \odot y \leq x' \odot y' \), it follows that

\[
\hat{x}(s)\hat{y}(t) \leq \hat{x}'(s)\hat{y}'(t)
\]

for every \( s \) and \( t \) in \( K_{x \lor x'} \) and \( K_{y \lor y'} \) respectively. As \( y \) is strictly positive then there is some \( t_0 \) in \( K_{y \lor y'} \) such that \( \hat{y}(t_0) > 0 \). It follows that

\[
\hat{x}(s) \leq \frac{\hat{y}'(t_0)}{\hat{y}(t_0)} \hat{x}'(s)
\]

for every \( s \) in \( K_{x \lor x'} \). This latter fact yields to \( x \) belongs in the principal ideal generated by \( x' \). We proceed by the same way to prove that \( y \) belongs in the principal ideal generated by \( y' \) and we are done. \( \square \)

**Proposition 3.2.** Let \( E \) and \( F \) be Riesz spaces. Let \( e \) be a positive element in \( E \), and \( f \) be a positive element in \( F \). Let \( E_0 \) be the principal ideal generated by \( e \) and \( E_f \) be the principal ideal generated by \( f \), then \( E_0 \odot F_f \) is an order dense Riesz subspace of \( (E \otimes F)_{e \odot f} \), where \( (E \otimes F)_{e \odot f} \) is the principal ideal in \( (E \otimes F) \) generated by \( e \otimes f \).

*Proof.* By [10, Proposition 3.1] and the boundedness property (B), it follows that \( E_0 \odot F_f \) is a Riesz subspace of \( (E \otimes F)_{e \odot f} \). Let \( 0 < u \in (E \otimes F)_{e \odot f} \). Then there is a positive real number \( \alpha \) such that \( 0 < u \leq \alpha e \otimes f \). By the (OD) Property, there are two positive elements \( x \in E \) and \( y \in F \) such that \( 0 < x \odot y \leq u \). It follows that \( 0 < x \odot y \leq \alpha e \otimes f \). Lemma 3.1 yields to \( 0 \leq x \in E_0 \) and \( 0 \leq y \in F_f \). That is \( x \odot y \in E_0 \odot F_f \). This completes the proof. \( \square \)

**Theorem 3.3.** Let \( E \) and \( F \) be Riesz spaces. Let \( e \) be a positive element in \( E \), and \( f \) be a positive element in \( F \). Let \( B_e \) be the principal band generated by \( e \) and \( B_f \) be the principal band generated by \( f \), then \( B_e \odot B_f \) is an order dense Riesz subspace in the principal band in \( E \otimes F \) generated by \( e \otimes f \).

*Proof.* First, we need to prove that \( B_e \odot B_f \subset \{ e \odot f \}^{dd} \). To this end, pick a positive element \( u \) in \( E \otimes F \) such that \( u \land e \odot f = 0 \), and a positive element \( v \) in \( B_e \odot B_f \). By the (B) property, there are two positive elements \( x \in B_e \) and \( y \in B_f \) such that

\[
0 \leq v \leq x \odot y.
\]
As $B_e$ and $B_f$ are principal bands, it follows that
\[
x = \bigvee_n x \wedge ne
\]
\[
y = \bigvee_m y \wedge mf
\]

From the order continuity of the Riesz tensor product (see [10, Corollary 3.4]), it follows that
\[
0 \leq u \wedge v \leq u \wedge x \otimes y
\]
\[
0 \leq u \wedge v \leq u \bigvee_n \bigvee_m (x \wedge ne) \otimes (y \wedge mf)
\]
\[
0 \leq u \wedge v \leq \bigvee_n \bigvee_m (u \wedge (x \wedge ne) \otimes (y \wedge mf))
\]
\[
0 \leq u \wedge v \leq \bigvee_n \bigvee_m (u \wedge ne \otimes mf)
\]
\[
0 \leq u \wedge v \leq \bigvee_n \bigvee_m (n + m)(u \wedge e \otimes f) = 0
\]

It follows that $B_e \overline{\otimes} B_f \subset \{e \otimes f\}^{dd}$. Now, we have that
\[
E_e \overline{\otimes} F_f \subset B_e \overline{\otimes} B_f \subset \{e \otimes f\}^{dd}
\]
and
\[
E_e \overline{\otimes} F_f \subset (E \overline{\otimes} F)_{e \otimes f} \subset \{e \otimes f\}^{dd}
\]

Theorem 1.24 in [2] yields to $(E \overline{\otimes} F)_{e \otimes f}$ is order dense in $\{e \otimes f\}^{dd}$. From Proposition 3.2, it follows that $E_e \overline{\otimes} F_f$ is order dense in $(E \overline{\otimes} F)_{e \otimes f}$. This makes an end to our proof.

In order to prove the theorem we will show that $\{e \otimes f\}^{dd}$ is included in $(B_e \overline{\otimes} B_f)^{dd}$. For this aim, we consider $u$ a positive element in $E \overline{\otimes} F$ such that $u \wedge e \otimes f = 0$, and let $v$ be a positive element in $B_e \overline{\otimes} B_f$. By the (B) property in $B_e \overline{\otimes} B_f$, it follows that there is some $x_1$ in $B_e$ and $y_1$ in $B_f$ such that $0 \leq v \leq x_1 \otimes y_1$. The (OD) property yields to the fact that there is some $x$ in $E$ and $y$ in $F$ such that $0 \leq x \otimes y \leq u \wedge v \leq x_1 \otimes y_1$. This latter fact together with Lemma 3.1 imply that $x$ is in $B_e$ and $y$ is $B_f$. It follows that
\[
0 \leq x \otimes y \wedge e \otimes f \leq u \wedge e \otimes f = 0.
\]

It follows then $B_e \overline{\otimes} B_f \subset \{e \otimes f\}^{dd}$. \qed
The next proposition is a generalization of [10, Porposition 7.1].

**Proposition 3.4.** Let $e$ and $f$ be weak order units of the Riesz spaces $E$ and $F$ respectively. Then $e \otimes f$ is a weak unit in $E \hat{\otimes} F$.

**Proof.** Let $0 \leq u$ in $E \hat{\otimes} F$ such that $u \wedge e \otimes f = 0$, we have to show that $u = 0$.

- First assume that $u = x \otimes y$ in $E_+ \otimes F_+$ then from $x \otimes y \wedge e \otimes f = 0$ follows that $(x \wedge ne) \otimes (y \wedge f) = 0 \ \forall n \in \mathbb{N}$

  The fact that $\{x \wedge ne\} \uparrow x$, together with the order continuity of the Riesz tensor product (see [10]), yield to

  $$x \otimes (y \wedge f) = 0$$

  We proceed again in the same way to obtain that $u = x \otimes y = 0$

- Now, if $0 < u \in E \hat{\otimes} F$ and $u \wedge e \otimes f = 0$, there exists by (O.D.) property $x \in E_+$ and $y \in F_+$ such that

  $$0 < x \otimes y \leq u$$

  but then $x \otimes y \wedge e \otimes f = 0$ and by the first step we obtain that $x \otimes y = 0$ which is impossible therefore $u$ has to be null.

The last point makes an end to our proof and we obtain that $e \otimes f$ is a weak order unit in $E \hat{\otimes} F$. \qed

4 The Dedekind complete Riesz tensor product of Riesz subspaces

**Lemma 4.1.** Let $E$ and $F$ be Riesz spaces. Let $e$ be a positive element in $E$, and $f$ be a positive element in $F$. Let $E_e$ be the principal ideal generated by $e$ and $E_f$ be the principal ideal generated by $f$, then the principal ideal $(E \hat{\otimes} F)_{e \otimes f}$ is equal to $(E \hat{\otimes} F)_{e \otimes f}^{\delta}$, where $(E \hat{\otimes} F)_{e \otimes f}^{\delta}$ is the Dedekind completion of $(E \hat{\otimes} F)_{e \otimes f}$.

**Proof.** First observe that $(E \hat{\otimes} F)_{e \otimes f}$ is a Dedekind complete ideal containing $(E \hat{\otimes} F)_{e \otimes f}$ (see for example Theorem 12.4 in [16]). It follows that $(E \hat{\otimes} F)_{e \otimes f}^{\delta}$ is included in
Lemma 2.5 in [D]. The set \( u \) is in \((E \otimes \delta F)_{e \otimes f} \). This latter fact yields to \( E \) Dedekind complete Riesz space \( \alpha \). The fact that \( u \) is in \((E \otimes \delta F)_{e \otimes f} \) is equivalent to (see for example [1])

\[
\begin{align*}
    u &= \sup \{ v \in E \otimes F \text{ such that } 0 \leq v \leq u \} \\
    &= \sup \{ v \in E \otimes F \text{ such that } 0 \leq v \leq u \leq \alpha(e \otimes f) \} \\
    &= \sup \{ v \in (E \otimes F)_{e \otimes f} \text{ such that } 0 \leq v \leq u \}
\end{align*}
\]

This latter fact yields to \( u \) is in \((E \otimes F)_{e \otimes f} \) which makes an end to our proof.

\[ \square \]

**Theorem 4.2.** Let \( E \) and \( F \) be Riesz spaces. Let \( e \) be a positive element in \( E \), and \( f \) be a positive element in \( F \). Let \( E_e \) be the principal ideal generated by \( e \) and \( E_f \) be the principal ideal generated by \( f \), then \( E_e \otimes \delta F_f \) is equal to the principal order ideal \((E \otimes \delta F)_{e \otimes f}\).

**Proof.** From lemma 3.2, it follows that \( E_e \otimes \delta F_f \) is an order dense Riesz subspace of \((E \otimes F)_{e \otimes f}\). Since \((E \otimes F)_{e \otimes f}\) is full in \((E \otimes \delta F)_{e \otimes f}\) (see [1] for example), it follows that \( E_e \otimes \delta F_f \) is an order dense Riesz subspace of \((E \otimes \delta F)_{e \otimes f}\). Now Pick \( u \) a positive element in \((E \otimes \delta F)_{e \otimes f}\), there exists some positive real number \( \alpha \) such that \( 0 \leq u \leq \alpha e \otimes f \). This, together with lemma 3.2, yield to

\[
    u = \sup_{(E \otimes \delta F)} \{ v \in E_e \otimes \delta F_f \text{ such that } 0 \leq v \leq u \leq \alpha e \otimes f \}
\]

The set \( \mathcal{D} = \{ v \in E_e \otimes \delta F_f \text{ such that } 0 \leq v \leq u \leq \alpha e \otimes f \} \) is bounded above in the Dedekind complete Riesz space \( E_e \otimes \delta F_f \), then there is \( u^* \in E_e \otimes \delta F_f \) such that

\[
    u^* = \sup_{E_e \otimes \delta F_f} \{ v \in E_e \otimes \delta F_f \text{ such that } 0 \leq v \leq u \leq \alpha e \otimes f \}
\]

Lemma 2.5 in [9], theorem 1.23 in [2] together with the order denseness of \( E_e \otimes \delta F_f \) in \((E \otimes \delta F)_{e \otimes f}\), yield to

\[
    u = u^*.
\]

That is \( E_e \otimes \delta F_f = (E \otimes \delta F)_{e \otimes f} \), which makes an end to the proof.

\[ \square \]

**Corollary 4.3.** Let \( E \) and \( F \) two Archimedean Riesz spaces. Let \( e \) be a positive element in \( E \), and \( f \) be a positive element in \( F \). Let \( E_e \) be the principal ideal generated by \( e \) and \( F_f \) be the principal ideal generated by \( f \). If \( E \otimes F \) is Dedekind complete then \( E_e \otimes \delta F_f \) is an ideal in \( E \otimes \delta F \).

**Theorem 4.4.** Let \( E \) and \( F \) two Archimedean Riesz spaces and \( A \) and \( B \) be two ideals of \( E \) and \( F \) respectively. Then \( A \otimes \delta B \) is an ideal in \( E \otimes \delta F \).
Proof. Pick a positive element $u$ in $A\widehat{\otimes}_\delta B$ and a positive element $v$ in $E\widehat{\otimes}_\delta F$, such that $0 \leq v \leq u$.

From the (B) property, there are $(x, y)$ in $A_+ \times B_+$ such that $0 \leq v \leq u \leq x \otimes v$.

The principal ideals generated by $x$ in $A$ and $y$ in $B$ are principal ideals in $E$ and $F$ respectively, it follows from Theorem 4.2 that $u$ is in the principal ideal $E_x \widehat{\otimes}_\delta F_y$. This yields to

$$v \in E_x \widehat{\otimes}_\delta F_y \subset A\widehat{\otimes}_\delta B.$$ 

As an immediate corollary, one can derive the next corollary

**Corollary 4.5.** Let $E$ and $F$ two Archimedean Riesz spaces and $A$ and $B$ be two ideals of $E$ and $F$ respectively. If $E \widehat{\otimes} F$ is Dedekind complete then $A\widehat{\otimes}_\delta B$ is an ideal in $E \widehat{\otimes} F$.

**Lemma 4.6.** Let $E$ and $F$ be Riesz spaces. Let $e$ be a positive element in $E$, and $f$ be a positive element in $F$. Let $B^\delta_e$ be the principal band generated by $e$ and $B^\delta_f$ be the principal band generated by $f$. Then for every positive element $v$ in $(e \otimes f)^{dd}$, there exists a positive element $u$ in $B^\delta_e \otimes_\delta B^\delta_f$ such that $0 \leq v \leq u$.

That is $B^\delta_e \otimes_\delta B^\delta_f$ is majorizing in $(e \otimes f)^{dd}$.

**Proof.** Let $v$ be a positive element in $(e \otimes f)^{dd}$. From the (B) property, there are two positive elements $x$ in $E_+$ and $y$ in $F_+$ such that $0 < v \leq x \otimes y$.

Let $B^\delta_e$ be the principal band generated by $e$ in $E^\delta$ and $B^\delta_f$ be the principal band generated by $f$ in $F^\delta$. Then, there are positive elements $x_1$ in $B^\delta_e$, $x_2$ in $(B^\delta_e)^d$, $y_1$ in $B^\delta_f$ and $y_2$ in $(B^\delta_f)^d$ such that

$$x = x_1 + x_2 \quad \text{and} \quad y = y_1 + y_2$$
Then, the inequalities
\begin{align*}
0 < v \wedge ne \otimes f &\leq (x_1 + x_2) \otimes (y_1 + y_2) \wedge ne \otimes f \\
&\leq x_1 \otimes y_1 \wedge ne \otimes f + x_2 \otimes y_1 \wedge ne \otimes f \\
& + x_1 \otimes y_2 \wedge ne \otimes f + x_2 \otimes y_2 \wedge ne \otimes f \\
&\leq x_1 \otimes y_1 \wedge ne \otimes f + (x_2 \wedge e) \otimes (nf \vee y_1) \\
& + (ne \vee x_1) \otimes (f \wedge y_2) + (x_2 \wedge e) \otimes (nf \vee y_2) \\
&\leq x_1 \otimes y_1 \wedge ne \otimes f \\
\end{align*}
hold for every positive \(n\), which yields to
\[0 < v \leq x_1 \otimes y_1. \tag{4.1}\]

It follows from lemma 2.3 that there exist \(x^*\) in \(B_e\) and \(y^*\) in \(B_f\) such that \(x_1 \leq x^*\) and \(y_1 \leq y^*\). This together with inequality 4.1, yield to
\[0 < v \leq x^* \otimes y^*\]
and we are done. \(\square\)

**Theorem 4.7.** Let \(E\) and \(F\) be Riesz spaces. Let \(e\) be a positive element in \(E\), and \(f\) be a positive element in \(F\). Let \(B_e\) be the principal band generated by \(e\) and \(B_f\) be the principal band generated by \(f\) then \(B_e \otimes_\delta B_f\) is equal to the principal band in \(E \otimes_\delta F\) generated by \((e \otimes f)\).

**Proof.** From Theorem 4.4, it follows that \(B_e \otimes_\delta B_f\) is an ideal in \(E \otimes_\delta F\). This together with Theorem 3.3, yield to the fact that \(B_e \otimes_\delta B_f\) is an order dense ideal in \((e \otimes f)^{dd}\). In order to prove that \(B_e \otimes_\delta B_f\) is a band, we have to prove that if \(u_\alpha\) is a net in \((B_e \otimes_\delta B_f)^+\) such that \(u_\alpha \uparrow u\) for some \(u\) in \((e \otimes f)^{dd}\), then \(u\) is in \((B_e \otimes_\delta B_f)\). Indeed, by lemma 4.6,
\[0 \leq u_\alpha \leq u \leq v\]
for some \(v\) in \(B_e \otimes_\delta B_f\). As, \(B_e \otimes_\delta B_f\) Dedekind complete, it follows that there is some \(u^*\) in \(B_e \otimes_\delta B_f\) such that \(u_\alpha \uparrow u^*\). The fact that \(B_e \otimes_\delta B_f\) is order dense and then regular in \((e \otimes f)^{dd}\) yields to
\[u = u^*\]
and then
\[B_e \otimes_\delta B_f = (e \otimes f)^{dd}.\]
This makes an end to our proof. \(\square\)
Corollary 4.8. Let $E$ and $F$ be two Riesz spaces with weak units $e$ and $f$ respectively. Then if $A_1$ and $A_2$ are projection bands in $E$ and $F$ respectively then $A_1 \circledast \delta A_2$ is a projection band in $E \circledast \delta F$.

Proof. If $A_1$ is a projection band in $E$, then $A_1 = B_{P_{A_1}(e)}$, the principal band in $E$ generated by $P_{A_1}(e)$. $A_2$ is then the principal band in $F$ generated by $P_{A_2}(f)$. It follows that

$$A_1 \circledast \delta A_2 = B_{P_{A_1}(e)} \circledast \delta B_{P_{A_2}(f)}$$

This together with Theorem 4.7 make an end to our proof. \qed

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