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A detailed investigation into near degenerate exponential random graphs

Received: date / Accepted: date

Abstract The exponential family of random graphs has been a topic of con-
tinued research interest. Despite the relative simplicity, these models capture
a variety of interesting features displayed by large-scale networks and allow
us to better understand how phases transition between one another as tuning
parameters vary. As the parameters cross certain lines, the model asymptot-
ically transitions from a very sparse graph to a very dense graph, completely
skipping all intermediate structures. We delve deeper into this near degener-
ate tendency and give an explicit characterization of the asymptotic graph
structure as a function of the parameters.

Keywords Exponential random graphs · Phase transitions · Near degener-
acy

Mathematics Subject Classification (2000) 05C80 · 82B26

1 Introduction

Exponential random graphs represent an important and challenging class
of models, displaying both diverse and novel phase transition phenomena.
These rather general models are exponential families of probability distribu-
tions over graphs, in which dependence between the random edges is defined
through certain finite subgraphs, in imitation of the use of potential energy to
provide dependence between particle states in a grand canonical ensemble of
statistical physics. They are particularly useful when one wants to construct
models that resemble observed networks as closely as possible, but without

Mei Yin’s research was partially supported by NSF grant DMS-1308333.

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specifying an explicit network formation mechanism. Consider the set $G_n$ of all simple graphs $G_n$ on $n$ vertices ("simple" means undirected, with no loops or multiple edges). By a $k$-parameter family of exponential random graphs we mean a family of probability measures $P^n_{β}$ on $G_n$ defined by, for $G_n \in G_n$,

$$P^n_{β}(G_n) = \exp \left\{ n^2 \left( \beta_1 t(H_1, G_n) + \cdots + \beta_k t(H_k, G_n) - \psi^β_n \right) \right\},$$

where $β = (β_1, \ldots, β_k)$ are $k$ real parameters, $H_1, \ldots, H_k$ are pre-chosen finite simple graphs (and we take $H_1$ to be a single edge), $t(H_i, G_n)$ is the density of graph homomorphisms (the probability that a random vertex map $V(H_i) \to V(G_n)$ is edge-preserving),

$$t(H_i, G_n) = \frac{|\text{hom}(H_i, G_n)|}{|V(G_n)||V(H_i)|},$$

and $ψ^β_n$ is the normalization constant,

$$ψ^β_n = \frac{1}{n^2} \log \sum_{G_n \in G_n} \exp \left\{ n^2 \left( \beta_1 t(H_1, G_n) + \cdots + \beta_k t(H_k, G_n) \right) \right\}. \quad (3)$$

Intuitively, we can think of the $k$ parameters $β_1, \ldots, β_k$ as tuning parameters that allow one to adjust the influence of different subgraphs $H_1, \ldots, H_k$ of $G_n$ on the probability distribution and analyze the extent to which specific values of these subgraph densities "interfere" with one another. Since the real-world networks the exponential models depict are often very large in size, our main interest lies in exploring the structure of a typical graph drawn from the model when $n$ is large.

This subject has attracted enormous attention in mathematics, as well as in various applied disciplines. Many of the investigations employ the elegant theory of graph limits as developed by Lovász and coauthors (V.T. Sós, B. Szegedy, C. Borges, J. Chayes, K. Vesztergombi, . . .) [8] [9] [10] [17] [18]. Building on earlier work of Aldous [1] and Hoover [15], the graph limit theory creates a new set of tools for representing and studying the asymptotic behavior of graphs by connecting sequences of graphs to a unified graphon space equipped with a cut metric. Though the theory itself is tailored to dense graphs, serious attempts have been made at formulating parallel results for sparse graphs [2] [4] [6] [7] [12] [19]. Applying the graph limit theory to $k$-parameter exponential random graphs and utilizing a large deviations result for Erdős-Rényi graphs established in Chatterjee and Varadhan [13], Chatterjee and Diaconis [11] showed that when $n$ is large and $β_2, \ldots, β_k$ are non-negative, a typical graph drawn from the “attractive” exponential model

$$\psi^β_∞ = \lim_{n \to ∞} ψ^β_n;$$

$$ψ^β_∞ = \sup_{0 ≤ u ≤ 1} \left( β_1 u^e(H_1) + \cdots + β_k u^e(H_k) - \frac{1}{2} u \log u - \frac{1}{2} (1 - u) \log(1 - u) \right), \quad (4)$$
where $e(H_i)$ is the number of edges in $H_i$. They also noted that in the edge-
(multiple)-star model where $H_j$ is a $j$-star for $j = 1, \ldots, k$, due to the unique
structure of stars, maximizers of the variational problem for $\psi^\beta_\infty$ for all pa-
rameter values satisfy (4) and a typical graph drawn from the model is al-
ways Erdős-Rényi. Since the limiting normalization constant is the generating
function for the limiting expectations of other random variables on the graph
space such as expectations and correlations of homomorphism densities, these
crucial observations connect the occurrence of an asymptotic phase transition
in (1) with an abrupt change in the solution of (4) and have led to further
exploration into exponential random graph models and their variations.

Being exponential families with finite support, one might expect exponen-
tial random graph models to enjoy a rather basic asymptotic form, though
in fact, virtually all these models are highly nonstandard as the size of the
network increases. The 2-parameter exponential random graph models have
therefore generated continued research interest. These prototype models are
simpler than their $k$-parameter extensions but nevertheless exhibit a wealth
of non-trivial attributes and capture a variety of interesting features dis-
played by large networks. The relative simplicity furthermore helps us better
understand how phases transition between one another as tuning parameters
vary and provides insight into the expressive power of the exponential con-
struction. In the statistical physics literature, phase transition is often asso-
ciated with loss of analyticity in the normalization constant, which gives rise
to discontinuities in the observed statistics. For exponential random graph
models, phase transition is characterized as a sharp, unambiguous partition
of parameter ranges separating those values in which changes in parameters
lead to smooth changes in the homomorphism densities, from those special
parameter values where the response in the densities is singular.

For the “attractive” 2-parameter edge-triangle model obtained by taking
$H_1 = K_2$ (an edge), $H_2 = K_3$ (a triangle), and $\beta_2 \geq 0$, Chatterjee and Di-
aconis [11] gave the first rigorous proof of asymptotic singular behavior and
identified a curve $\beta_2 = q(\beta_1)$ across which the model transitions from very
sparse to very dense, completely skipping all intermediate structures. In fur-
ther works (see for example, Radin and Yin [24], Aristoff and Zhu [3]), this
singular behavior was discovered universally in generic 2-parameter models
where $H_1$ is an edge and $H_2$ is any finite simple graph, and the transition
curve $\beta_2 = q(\beta_1)$ asymptotically approaches the straight line $\beta_2 = -\beta_1$ as
the parameters diverge. The double asymptotic framework of [11] was later
extended in [26], and the scenario in which the parameters diverge along
general straight lines $\beta_1 = a\beta_2$, where $a$ is a constant and $\beta_2 \to \infty$, was
considered. Consistent with the near degeneracy predictions in [3] [11] [24],
asymptotically for $a \leq -1$, a typical graph sampled from the “attractive”
2-parameter exponential model is sparse, while for $a > -1$, a typical graph
is nearly complete. Although much effort has been focused on 2-parameter
models, $k$-parameter models have also been examined. As shown in [25], near
degeneracy and universality are expected not only in generic 2-parameter
models but also in generic $k$-parameter models. Asymptotically, a typical
graph drawn from the “attractive” $k$-parameter exponential model where
$\beta_2, \ldots, \beta_k \geq 0$ is sparse below the hyperplane $\sum_{i=1}^k \beta_i = 0$ and nearly com-
complete above it. For the edge-(multiple)-star model, the desirable star feature relates to network expansiveness and has made predictions of similar asymptotic phenomena possible in broader parameter regions. Related results may be found in Häggström and Jonasson [14], Park and Newman [21], Bianconi [5], Lubetzky and Zhao [20], Radin and Sadun [22], [23], and Kenyon et al. [15].

These theoretical findings have advanced our understanding of phase transitions in exponential random graph models, yet some important questions remain unanswered. Previous investigations identified near degenerate parameter regions in which a typical sampled graph looks like an Erdős-Rényi graph $G(n, u^*)$, where the edge presence probability $u^* \to 0$ or 1, but the speed of $u^*$ towards these two degenerate states is not at all clear. When a typical graph is sparse ($u^* \to 0$), how sparse is it? When a typical graph is nearly complete ($u^* \to 1$), how dense is it? Can we give an explicit characterization of the near degenerate graph structure as a function of the parameters?

The rest of this paper is dedicated towards these goals. Some of the ideas for the sparse case were partially implemented in [27]. Theorem 1 considers generic “attractive” 2-parameter exponential random graph models and Theorem 2 derives parallel results for “repulsive” edge-(single)-star models. The asymptotic characterization of $u^*$ obtained in these theorems then make possible a deeper exploration into the asymptotics of the limiting normalization constant of the exponential model in Theorem 3, which indicates that though a typical graph displays Erdős-Rényi feature, the simplified Erdős-Rényi graph and the real exponential graph are not exact asymptotic analogs in the usual statistical physics sense. In the sparse region, the Erdős-Rényi graph does seem to reflect the asymptotic tendency of the exponential graph more accurately, as the two interpretations of the limiting normalization constant coincide when the parameters diverge. Lastly, Theorems 4 and 5 further extend the near degenerate analysis from 2-parameter exponential random graph models to $k$-parameter exponential random graph models.

2 Investigating near degeneracy

This section explores the exact asymptotics of generic 2-parameter exponential random graph models where $\beta_2 \geq 0$ near degeneracy. The analysis is then extended to $\beta_2 < 0$ for the edge-(single)-star model. By including only two subgraph densities in the exponent,

$$P_n^\beta(G_n) = \exp \left( n^2 \left( \beta_1 t(H_1, G_n) + \beta_2 t(H_2, G_n) - \psi_n^\beta \right) \right),$$

where $H_1$ is an edge and $H_2$ is a different finite simple graph, the 2-parameter models are arguably simpler than their $k$-parameter generalizations. As illustrated in Chatterjee and Diaconis [11], when $n$ is large and $\beta_2$ is non-negative, a typical graph drawn from the “attractive” 2-parameter exponential model [5] behaves like an Erdős-Rényi graph $G(n, u^*)$, where the edge presence probability $u^*(\beta_1, \beta_2)$ is picked randomly from the set of maximizers of a
As explained in the previous paragraph, in the large graph $G$ the origin, a typical graph drawn from the model looks like an Erdős-Rényi model (5) where $\beta$. For $u$-totic edge presence probability $e^{2\beta_1}$, is quite evident. Theorem 1 Consider an “attractive” 2-parameter exponential random graph model \cite{2} where $\beta_2 \geq 0$. For large $n$ and $(\beta_1, \beta_2)$ sufficiently far away from the origin, a typical graph drawn from the model looks like an Erdős-Rényi graph $G(n, u^*)$, where the edge presence probability $u^*(\beta_1, \beta_2)$ satisfies:

- $u^* \approx e^{2\beta_1}$ if $\beta_1 \leq -\beta_2$,
- $u^* \approx 1 - e^{-2(\beta_1 + p\beta_2)}$ if $\beta_1 > -\beta_2$.

Proof As explained in the previous paragraph, in the large $n$ limit, the asymptotic edge presence probability $u^*(\beta_1, \beta_2)$ of a typical sampled graph is prescribed according to the maximization problem (7). For $(\beta_1, \beta_2)$ whose magnitude is sufficiently big, $u^* \to 0$ when $\beta_1 \leq -\beta_2$ and $u^* \to 1$ when $\beta_1 > -\beta_2$.

For $\beta_1 \leq -\beta_2$, we rewrite (7) in the following way:

$$\frac{u^*}{e^{2\beta_1 + 2\beta_2 p(u^*)^{p-1}}} = 1 - u^*. \quad (8)$$

| $\beta_1$ | $\beta_2$ | $u^*(\beta_1, \beta_2)$ | $e^{2\beta_1}$ | $1 - e^{-2(\beta_1 + 3\beta_2)}$ |
|---|---|---|---|---|
| $-2$ | 1 | 0.01802 | 0.01832 | |
| $-2$ | 2 | 0.01806 | 0.01832 | |
| 0 | 1 | 0.99745 | 0.99752 | |
| 2 | 1 | 0.99995 | 0.99995 | |

Table 1 Asymptotic comparison for “attractive” edge-triangle model near degeneracy.
Take $0 < \epsilon < 1/p$, since $u^* \to 0$, for $(\beta_1, \beta_2)$ sufficiently far away from the origin, $(u^*)^{p-1} < \epsilon$. Using $1 - u^* \leq 1$, we then have

$$u^* \leq e^{2\beta_1 + 2\beta_2 p \epsilon} \leq e^{2\beta_2 (-1 + p \epsilon)}.$$  \hspace{1cm} (9)

This implies that

$$2\beta_2 p (u^*)^{p-1} \leq 2\beta_2 p e^{2\beta_2 (p-1)(-1 + p \epsilon)} \to 0$$  \hspace{1cm} (10)

as $\beta_2$ gets sufficiently large. Using $1 - u^* \to 1$ again, this further shows that $u^*$ asymptotically behaves like $e^{2\beta_1}$.

For $\beta_1 > -\beta_2$, we rewrite (7) in the following way:

$$v^* e^{-2\beta_1 - 2\beta_2 p (1 - v^*)^{p-1}} = 1 - v^*,$$  \hspace{1cm} (11)

where $v^* = 1 - u^* \to 0$. Going one step further, we separate $1 - (p - 1) v^*$ from $(1 - v^*)^{p-1}$:

$$v^* e^{-2\beta_1 - 2\beta_2 p + 2\beta_2 p (p - 1) v^*} = (1 - v^*) e^{-2\beta_2 p \sum_{s=2}^{p-1} (v^*)^{s-1}} \leq 1 - v^*,$$  \hspace{1cm} (12)

as the dominating term in the exponent $-2\beta_2 p (p-1)(-v^*)^2$ carries a negative sign. Take $0 < \epsilon < 1/p$, since $v^* \to 0$, for $(\beta_1, \beta_2)$ sufficiently far away from the origin, $v^* < \epsilon$. Using $1 - v^* \leq 1$, we then have

$$v^* \leq e^{-2\beta_1 - 2\beta_2 p + 2\beta_2 p (p - 1) \epsilon} \leq e^{2\beta_2 (1 - p + p (p - 1) \epsilon)} = e^{2\beta_2 (p-1)(-1 + p \epsilon)}.$$  \hspace{1cm} (13)
This implies that

$$2\beta_2(p-1)v^* \leq 2\beta_2 p(p-1)e^{2\beta_2(p-1)(-1+p\epsilon)} \rightarrow 0$$

(14)
as $\beta_2$ gets sufficiently large, and since $v^* \rightarrow 0$, also implies that the sum of all the terms in the exponent $-2\beta_2 p\sum_{s=2}^{p-1}(p-1)(-v^*)^s \rightarrow 0$. Using $1 - v^* \rightarrow 1$ again, this further shows that $v^*$ asymptotically behaves like $e^{-2(\beta_1 + p\beta_2)}$, or equivalently, $u^*$ asymptotically behaves like $1 - e^{-2(\beta_1 + p\beta_2)}$.

In the edge-(single)-star model where $H_2$ is a star with $p$ edges, due to the unique structure of stars, maximizers of the variational problem for the limiting normalization constant $\psi_\infty^\beta$ when the parameter $\beta_2 < 0$ again satisfies (7), and the near degeneracy predictions in Theorem 1 may be extended from the upper half-plane to the lower half-plane. It was shown in [27] that for large $n$ and $(\beta_1, \beta_2)$ sufficiently far away from the origin, a typical graph drawn from the “repulsive” edge-(single)-star model where $\beta_2 < 0$ is indistinguishable from an Erdős-Rényi graph $G(n, u^*)$, where the edge presence probability $u^* \rightarrow 0$ when $\beta_1 < 0$ and $u^* \rightarrow 1$ when $\beta_1 > -p\beta_2$. As regards the speed of $u^*$ towards these two degenerate states, simulation results suggest that just as in the “attractive” situation, $u^*$ is asymptotically $e^{2\beta_1}$ in the sparse case and is asymptotically $1 - e^{-2(\beta_1 + p\beta_2)}$ in the nearly complete case. See Table 2. Even for $\beta$ with small magnitude, the asymptotic tendency of $u^*$ is quite evident.

**Theorem 2** Consider a “repulsive” edge-(single)-star model obtained by taking $H_2$ a star with $p$ edges and $\beta_2 < 0$ in (5). For large $n$ and $(\beta_1, \beta_2)$

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**Fig. 2** Asymptotic tendency in “attractive” edge-triangle model.
Table 2  Asymptotic comparison for edge-2-star model near degeneracy.

| $\beta_1$ | $\beta_2$ | $u^*(\beta_1, \beta_2)$ | $e^{2\beta_1}$ | $1 - e^{-2(\beta_1 + 2\beta_2)}$ |
|---|---|---|---|---|
| $-2$ | $-4$ | $0.01435$ | $0.01832$ | |
| $-2$ | $-2$ | $0.01588$ | $0.01832$ | |
| $-3$ | $1$ | $0.00250$ | $0.00248$ | |
| $-3$ | $3$ | $0.00255$ | $0.00248$ | |
| $1$ | $1$ | $0.99750$ | $0.99752$ | |
| $4$ | $-1$ | $0.98317$ | $0.98168$ | |

sufficiently far away from the origin, a typical graph drawn from the model looks like an Erdős-Rényi graph $G(n, u^*)$, where the edge presence probability $u^*(\beta_1, \beta_2)$ satisfies:

- $u^* \approx e^{2\beta_1}$ if $\beta_1 < 0$,
- $u^* \approx 1 - e^{-2(\beta_1 + p\beta_2)}$ if $\beta_1 > -p\beta_2$.

Proof  For $(\beta_1, \beta_2)$ whose magnitude is sufficiently big, we examine the maximization problem (7) separately when $\beta_1 < 0$ and when $\beta_1 > -p\beta_2$.

First for $\beta_1 < 0$. Assume that $\beta_1 \leq \delta_1\beta_2$ for some fixed but arbitrary $\delta_1 > 0$. We rewrite (7) in the following way:

$$ u^* e^{2\beta_1} = (1 - u^*) e^{2\beta_2 p(u^*)^{p-1}} \leq 1 - u^* $$

(15)

Using $1 - u^* \leq 1$, we then have

$$ u^* \leq e^{2\beta_1} \leq e^{2\delta_1\beta_2} $$

(16)

This implies that

$$ 2|\beta_2| p(u^*)^{p-1} \leq 2|\beta_2| p e^{2\delta_1\beta_2(p-1)} \to 0 $$

(17)

as $\beta_2$ gets sufficiently negative. Using $1 - u^* \to 1$ again, this further shows that $u^*$ asymptotically behaves like $e^{2\beta_1}$.

Next for $\beta_1 > -p\beta_2$. Assume that $\beta_1 \geq -(p + \delta_2)\beta_2$ for some fixed but arbitrary $\delta_2 > 0$. We rewrite (7) in the following way:

$$ v^* \leq e^{-2\beta_1 - 2\beta_2 p(1 - u^*)^{p-1}} = 1 - v^* $$

(18)

where $v^* = 1 - u^* \to 0$. Going one step further, we separate 1 from $(1 - v^*)^{p-1}$:

$$ \frac{v^*}{e^{-2\beta_1 - 2\beta_2 p}} = (1 - v^*) e^{-2\beta_2 p \sum_{s=1}^{p-1} (\frac{v^*}{1-v^*})^s} \leq 1 - v^* $$

(19)

as the dominating term in the exponent $2\beta_2 p (p-1)v^*$ carries a negative sign. Using $1 - v^* \leq 1$, we then have

$$ v^* \leq e^{-2\beta_1 - 2\beta_2 p} \leq e^{2\delta_2\beta_2} $$

(20)
This implies that

$$2|\beta_2|p(p - 1)v^* \leq 2|\beta_2|p(p - 1)e^{2\beta_2} \to 0$$  \hspace{1cm} (21)$$
as $\beta_2$ gets sufficiently negative, and since $v^* \to 0$, also implies that the sum of all the terms in the exponent $-2\beta_2p \sum_{s=1}^{p-1} (p^{-1})(-v^*)^s \to 0$. Using $1 - v^* \to 1$ again, this further shows that $v^*$ asymptotically behaves like $e^{-2(\beta_1 + p\beta_2)}$, or equivalently, $u^*$ asymptotically behaves like $1 - e^{-2(\beta_1 + p\beta_2)}$.

Though the 2-parameter exponential random graph $G_n$ looks like an Erdős-Rényi random graph $G(n, u^*)$ in the large $n$ limit, we also note some marked dissimilarities. The limiting normalization constant $\psi_\infty$ for the 2-parameter exponential model (5) is given by (6), while the “equivalent” Erdős-Rényi model yields that $\psi_\infty$ is $-\log(1 - u^*)/2$. Since $u^*$ is nonzero for finite $(\beta_1, \beta_2)$, the two different interpretations of the limiting normalization constant indicate that the simplified Erdős-Rényi graph and the real exponential model are not exact asymptotic analogs in the usual statistical physics sense. When the relevant Erdős-Rényi graph is near degenerate, Theorems 1 and 2 give the asymptotic speed of $u^*$ as a function of $\beta_1$ and $\beta_2$, allowing a deeper analysis of the asymptotics of $\psi_\infty$ in the following Theorem 3. The theorem is formulated in the context of the edge-(single)-star model, since the asymptotics of $u^*$ are known in broader parameter regions for this model, but the statement for the “attractive” situation ($\beta_2 \geq 0$) applies without modification to generic 2-parameter models. See Figures 2 and 3. We also note that, in the sparse region, the Erdős-Rényi graph seems to reflect the asymptotic tendency of the exponential random graph more accurately, as the two interpretations of the limiting normalization constant do coincide when the parameters diverge.

**Theorem 3** Consider an edge-(single)-star model obtained by taking $H_2$ a star with $p$ edges in (3). For $(\beta_1, \beta_2)$ sufficiently far away from the origin, the limiting normalization constant $\psi_\infty$ satisfies:

- $\psi_\infty \asymp e^{2\beta_1}/2$ if $\beta_1 \leq -\beta_2$ and $\beta_1 < 0$.
- $\psi_\infty \asymp \beta_1 + \beta_2$ if $\beta_1 > -\beta_2$ and $\beta_1 > -p\beta_2$.

**Proof** For $(\beta_1, \beta_2)$ whose magnitude is sufficiently big, we examine the limiting normalization constant $\psi_\infty$ separately in the sparse region and in the nearly complete region.

In the sparse region ($\beta_1 \leq -\beta_2$ and $\beta_1 < 0$),

$$\psi_\infty = \beta_1 u^* + \beta_2 (u^*)^p - \frac{1}{2} u^* \log u^* - \frac{1}{2} (1 - u^*) \log(1 - u^*)$$ \hspace{1cm} (22)$$

From Theorems 1 and 2, $\beta_2(u^*)^{p-1} \to 0$ and $-\log(1 - u^*) \asymp u^* \asymp e^{2\beta_1}$. This shows that $\psi_\infty$ asymptotically behaves like $e^{2\beta_1}/2$. 

Fig. 3 Asymptotic tendency in edge-2-star model.

In the nearly complete region ($\beta_1 > -\beta_2$ and $\beta_1 > -p\beta_2$),

$$\psi_\beta^\infty = \beta_1(1 - v^*) + \beta_2(1 - v^*)^p - \frac{1}{2} v^* \log v^* - \frac{1}{2} (1 - v^*) \log(1 - v^*)$$

$$= \beta_2(1 - v^*)^p (1 - p) - \frac{1}{2} \log v^*. \quad (23)$$

From Theorems 1 and 2, $\beta_2 v^* \to 0$ and $v^* \approx e^{-2(\beta_1 + p\beta_2)}$. This shows that $\psi_\beta^\infty$ asymptotically behaves like $\beta_1 + \beta_2$.

We may also analyze the asymptotics of $\psi_\beta^\infty$ along the boundaries of the near degenerate region. The boundary of the sparse region is given by $\beta_1 = 0$ and $\beta_2 < 0$, and $u^*$ satisfies

$$\beta_2 p(u^*)^{p-1} = \frac{1}{2} \log \left( \frac{u^*}{1 - u^*} \right). \quad (24)$$

Though $u^* \to 0$ depends on $\beta_2$ in a complicated way, the asymptotic behavior of $\psi_\beta^\infty$ can be characterized:

$$\psi_\beta^\infty = \frac{1 - p}{2p} u^* \log \frac{u^*}{1 - u^*} - \frac{1}{2} \log(1 - u^*). \quad (25)$$

Using $\log(1 - u^*) \approx -u^*$, this shows that $\psi_\beta^\infty$ asymptotically behaves like $(1 - p) u^* \log u^*/(2p)$. We recognize that the asymptotic behaviors of $\psi_\beta^\infty$ on the boundary of and inside the sparse region are different: Inside, $\psi_\beta^\infty$ is
asymptotically \( u^*/2 \) and converges to 0, while on the boundary, \( \psi^\beta_\infty \) though also converges to 0 is at a much slower rate.

The boundary of the nearly complete region is given by \( \beta_1 = -p\beta_2 \) and \( \beta_2 < 0 \), and \( u^* \) satisfies

\[
- \beta_2 p + \beta_2 p (1 - v^*)^{p-1} = -\frac{1}{2} \log \left( \frac{v^*}{1-v^*} \right) .
\] (26)

Though \( v^* = 1 - u^* \) converges to 0 depends on \( \beta_2 \) in a complicated way, the asymptotic behavior of \( \psi^\beta_\infty \) can be characterized:

\[
\psi^\beta_\infty = \beta_2 (1 - p) (1 - v^*) \geq \frac{1 - p}{2p} (1 - v^*) \log \left( \frac{v^*}{1-v^*} \right) - \frac{1}{2} \log v^* .
\] (27)

Since the dominating term on the left of (26) is \( -\beta_2 p (p - 1)v^* \), using \( \log(1 - v^*) \to 0 \), we then have \( \beta_2 p (p - 1)v^* \approx \log v^*/2 \), which shows that \( |\beta_2| \) is asymptotically larger compared with \( |\log v^*| \) and further shows that \( \psi^\beta_\infty \) asymptotically behaves like \( \beta_2 (1 - p) \). We recognize that the asymptotic behaviors of \( \psi^\beta_\infty \) on the boundary of and inside the nearly complete region coincide.

3 Further discussion

This section extends the investigation into near degeneracy from generic 2-parameter exponential random graph models to generic \( k \)-parameter exponential random graph models. For “attractive” models where \( \beta_2, \ldots, \beta_k \geq 0 \), we derive parallel results concerning the asymptotic graph structure and the limiting normalization constant. Using similar methods, more results can be deduced for the “repulsive” edge-(multiple)-star model where \( \beta_2, \ldots, \beta_k < 0 \).

As illustrated in Chatterjee and Diaconis [11], when \( n \) is large and \( \beta_2, \ldots, \beta_k \) are non-negative, a typical graph drawn from the \( k \)-parameter exponential model behaves like an Erdős-Rényi graph \( G(n, u^*) \), where the edge presence probability \( u^*(\beta_1, \ldots, \beta_k) \) is picked randomly from the set of maximizers of (4), and thus satisfies

\[
\beta_1 + \beta_2 e(H_2)(u^*)^{e(H_2)-1} + \cdots + \beta_k e(H_k)(u^*)^{e(H_k)-1} = \frac{1}{2} \log \left( \frac{u^*}{1-u^*} \right) ,
\] (28)

where \( e(H_i) \) is the number of edges in \( H_i \). We take \( H_1 \) to be an edge and assume without loss of generality that \( 1 = e(H_1) \leq \cdots \leq e(H_k) \).

**Theorem 4** Consider an “attractive” \( k \)-parameter exponential random graph model (4) where \( \beta_2, \ldots, \beta_k \geq 0 \). For large \( n \) and \( (\beta_1, \ldots, \beta_k) \) sufficiently far away from the origin, a typical graph drawn from the model looks like an Erdős-Rényi graph \( G(n, u^*) \), where the edge presence probability \( u^*(\beta_1, \ldots, \beta_k) \) satisfies:

- \( u^* \approx e^{2\beta_1} \) if \( \sum_{i=1}^k \beta_i \leq 0 \),
- \( u^* \approx 1 - e^{\sum_{i=1}^k -2\beta_i e(H_i)} \) if \( \sum_{i=1}^k \beta_i > 0 \).
Proof The proof follows a similar line of reasoning as in the proof of Theorem 12. Expectedly though, the argument is more involved since we are working with $k$-parameter families rather than 2-parameter families. For $\sum_{i=1}^k \beta_i \leq 0$, we rewrite (28) in the following way:

$$\frac{u^*}{e^{2\beta_1 + \sum_{i=2}^k 2\beta_i \epsilon(H_i)(u^*)^{e(H_i)-1}}} = 1 - u^*. \quad (29)$$

Take $0 < \epsilon < 1/e(H_k)$, since $u^* \to 0$, for $(\beta_1, \ldots, \beta_k)$ sufficiently far away from the origin, $(u^*)^{e(H_i)-1} < \epsilon$ for $2 \leq i \leq k$. Using $1 - u^* \leq 1$, we then have

$$u^* \leq e^{2\beta_1 + \sum_{i=2}^k 2\beta_i \epsilon(H_i)\epsilon} \leq e^{\sum_{i=1}^k 2\beta_i (-1+e(H_i)\epsilon)}. \quad (30)$$

This implies that

$$2\beta_j \epsilon(H_j) (u^*)^{e(H_j)-1} \leq 2\beta_j \epsilon(H_j) e^{(e(H_j)-1) \sum_{i=2}^k 2\beta_i (-1+e(H_i)\epsilon)} \to 0 \quad (31)$$

for all $2 \leq j \leq k$ as $\beta_2, \ldots, \beta_k$ get sufficiently large. Using $1 - u^* \to 0$ again, this further shows that $u^*$ asymptotically behaves like $e^{2\beta_1}$.

For $\sum_{i=1}^k \beta_i > 0$, we rewrite (28) in the following way:

$$\frac{v^*}{e^{-2\beta_1 - \sum_{i=2}^k 2\beta_i \epsilon(H_i)\epsilon (1-v^*)^{e(H_i)-1}}} = 1 - v^*, \quad (32)$$

where $v^* = 1 - u^* \to 0$. Going one step further, for $2 \leq i \leq k$, we separate $1 - (e(H_i) - 1)v^*$ from $(1-v^*)^{e(H_i)-1}$:

$$\frac{v^*}{e^{\sum_{i=1}^k -2\beta_i \epsilon(H_i)\epsilon + \sum_{i=2}^k 2\beta_i \epsilon(H_i)\epsilon (e(H_i)-1)v^*}} = (1-v^*)e^{-\sum_{i=2}^k 2\beta_i \epsilon(H_i)\epsilon e(H_i) - 1} \leq 1 - v^*, \quad (33)$$

as the dominating term in the exponent $-\sum_{i=2}^k 2\beta_i \epsilon(H_i)\epsilon (e(H_i)-1)(-v^*)^2$ carries a negative sign. Take $0 < \epsilon < 1/e(H_k)$, since $v^* \to 0$, for $(\beta_1, \ldots, \beta_k)$ sufficiently far away from the origin, $v^* < \epsilon$. Using $1 - v^* \leq 1$, we then have

$$v^* \leq e^{\sum_{i=1}^k -2\beta_i \epsilon(H_i)\epsilon + \sum_{i=2}^k 2\beta_i \epsilon(H_i)\epsilon (e(H_i)-1)\epsilon} \leq e^{\sum_{i=2}^k 2\beta_i (1-e(H_i))\epsilon (e(H_i)-1)\epsilon} = e^{\sum_{i=2}^k 2\beta_i (1-e(H_i))\epsilon (e(H_i)-1)\epsilon}. \quad (34)$$

This implies that

$$2\beta_j \epsilon(H_j) (e(H_j)-1)v^* \leq 2\beta_j \epsilon(H_j) (e(H_j)-1)e^{\sum_{i=2}^k 2\beta_i (1-e(H_i))\epsilon (e(H_i)-1)\epsilon} \to 0 \quad (35)$$

for all $2 \leq j \leq k$ as $\beta_2, \ldots, \beta_k$ get sufficiently large, and since $v^* \to 0$, also implies that the sum of all the terms in the exponent $-\sum_{i=2}^k 2\beta_i \epsilon(H_i)\epsilon (e(H_i)-1)(-v^*)^2 \to 0$. Using $1 - v^* \to 1$ again, this further shows that $v^*$ asymptotically behaves like $e^{\sum_{i=1}^k -2\beta_i \epsilon(H_i)}$, or equivalently, $u^*$ asymptotically behaves like $1 - e^{\sum_{i=1}^k -2\beta_i \epsilon(H_i)}$. 
Theorem 5 Consider an “attractive” $k$-parameter exponential random graph model \[ \mathcal{G} \] where $\beta_2, \ldots, \beta_k \geq 0$. For $(\beta_1, \ldots, \beta_k)$ sufficiently far away from the origin, the limiting normalization constant $\psi^x_{\infty}$ satisfies:

- $\psi^x_{\infty} \simeq e^{2\beta_1}/2$ if $\sum_{i=1}^k \beta_i \leq 0$,
- $\psi^x_{\infty} \simeq \sum_{i=1}^k \beta_i$ if $\sum_{i=1}^k \beta_i > 0$.

Proof For $(\beta_1, \ldots, \beta_k)$ whose magnitude is sufficiently big, we examine the limiting normalization constant $\psi^x_{\infty}$ separately in the sparse region and in the nearly complete region.

In the sparse region ($\sum_{i=1}^k \beta_i \leq 0$),

$$
\psi^x_{\infty} = \sum_{i=1}^k \beta_i (u^*)^{\epsilon(H_i)} - \frac{1}{2} u^* \log u^* - \frac{1}{2} (1 - u^*) \log (1 - u^*) \tag{36}
$$

From Theorem 4 $\beta_i (u^*)^{\epsilon(H_i)} \to 0$ for all $2 \leq i \leq k$ and $-\log(1 - u^*) \simeq u^* \simeq e^{2\beta_1}$. This shows that $\psi^x_{\infty}$ asymptotically behaves like $e^{2\beta_1}/2$.

In the nearly complete region ($\sum_{i=1}^k \beta_i > 0$),

$$
\psi^x_{\infty} = \sum_{i=2}^k \beta_i (1 - v^*)^{\epsilon(H_i)} - \frac{1}{2} v^* \log v^* - \frac{1}{2} (1 - v^*) \log (1 - v^*) \tag{37}
$$

From Theorem 4 $\beta_i v^* \to 0$ for all $2 \leq i \leq k$ and $v^* \simeq e^{\sum_{i=1}^k -2\beta_i \epsilon(H_i)}$. This shows that $\psi^x_{\infty}$ asymptotically behaves like $\sum_{i=1}^k \beta_i$.

In the edge-(multiple)-star model, due to the unique structure of stars, maximizers of the variational problem for the limiting normalization constant $\psi^x_{\infty}$ satisfies (28) for any $\beta_1, \ldots, \beta_k$, and the near degeneracy predictions may be extended to the “repulsive” region. Using similar techniques as in 27, it may be shown that for $(\beta_1, \ldots, \beta_k)$ sufficiently far away from the origin and $\beta_2, \ldots, \beta_k$ all negative, $u^* \to 0$ when $\beta_1 < 0$ and $u^* \to 1$ when $\sum_{i=1}^k \beta_i \epsilon(H_i) > 0$. Then analogous conclusions as in Theorems 4 and 5 may be drawn:

- $u^* \simeq e^{2\beta_1}$ and $\psi^x_{\infty} \simeq e^{2\beta_1}/2$ if $\sum_{i=1}^k \beta_i \leq 0$ and $\beta_1 < 0$,
- $u^* \simeq 1 - e^{\sum_{i=1}^k -2\beta_i \epsilon(H_i)}$ and $\psi^x_{\infty} \simeq \sum_{i=1}^k \beta_i$ if $\sum_{i=1}^k \beta_i > 0$ and $\sum_{i=1}^k \beta_i \epsilon(H_i) > 0$.

We omit the proof details.
Acknowledgements

The author is very grateful to the anonymous referees for the invaluable suggestions that greatly improved the quality of this paper. She appreciated the opportunity to talk about this work in the Special Session on Topics in Probability at the 2016 AMS Western Spring Sectional Meeting, organized by Tom Alberts and Arjun Krishnan.

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