SIMULATIONS OF ELECTRON MAGNETOHYDRODYNAMIC TURBULENCE

Jungyeon Cho\textsuperscript{1,2} and A. Lazarian\textsuperscript{2}

\textsuperscript{1} Department of Astronomy and Space Science, Chungnam National University, Daejeon, Korea; cho@canopus.cnu.ac.kr
\textsuperscript{2} Department of Astronomy, University of Wisconsin, Madison, WI 53706, USA; lazarian@astro.wisc.edu

Received 2009 April 3; accepted 2009 June 11; published 2009 July 21

ABSTRACT

We present numerical simulations of electron magnetohydrodynamic (EMHD) and electron reduced MHD (ERMHD) turbulence. Comparing scaling relations, we find that both EMHD and ERMHD turbulence show similar spectra and anisotropy. We develop new techniques to study anisotropy of EMHD turbulence. Our detailed study of anisotropy of EMHD turbulence supports our earlier result of \( k_1 \propto k_{1/3} \) scaling, where \( k_1 \) and \( k_{1/3} \) are wavenumbers parallel and perpendicular to local direction of magnetic field, respectively. We find that the high-order statistics show a scaling that is similar to the She–Leveque scaling for incompressible hydrodynamic turbulence and different from that of incompressible MHD turbulence. We observe that the bispectra, which characterize the interaction of different scales within the turbulence cascade, are very different for EMHD and MHD turbulence. We show that both decaying and driven EMHD turbulence have the same statistical properties. We calculate the probability distribution functions (PDFs) of MHD and EMHD turbulence and compare them with those of interplanetary turbulence. We find that, as in the case of the solar wind, the PDFs of the increments of magnetic field strength in MHD and EMHD turbulence are well described by the Tsallis distribution. We discuss implications of our results for astrophysical situations, including the advection-dominated accretion flows and magnetic reconnection.

Key words: MHD – solar wind – turbulence

1. INTRODUCTION

Turbulence at scales below the proton gyroradius is of great importance in many astrophysical applications. Such turbulence involving motions of electrons is essential for understanding the small-scale magnetic field dynamics of plasmas. It is also important for understanding of magnetic fields in the crust of a neutron star (Goldreich & Reisenegger 1992). This turbulence has been measured at by solar wind probes (Leamon et al. 1999). The origin of the small-scale turbulence in a magnetized plasma is easy to understand if we think of what is happening with turbulent motions at small scales. When turbulence is driven on large scales, turbulence energy cascades down to smaller scales. The nature of magnetized turbulence from the outer scale to the proton gyroradius scale is relatively well known. Magnetized turbulence above the proton gyroradius can be described by the standard magnetohydrodynamics (MHD). As the name implies, the standard MHD treats the plasma as a single fluid. MHD turbulence can be decomposed into cascades of Alfvén, fast and slow modes (Goldreich & Sridhar 1995; Lithwick & Goldreich 2001; Cho & Lazarian 2002, 2003). While fast and slow modes get damped at larger scales, Alfvénic modes can cascade down to the proton gyroradius scale. Near and below the proton gyroradius scale, single-fluid description fails and we should take into account kinetic effects. Then what will happen to the Alfvén modes that reach the proton gyroradius scale?

Recent years, the nature of small-scale MHD turbulence in the solar wind has drawn interest from researchers (Howes et al. 2008a, 2008b; Matthaeus et al. 2008; Saito et al. 2008; Gary et al. 2008; Schekochihin et al. 2009; Dmitruk & Matthaeus 2006). In situ measurements of the solar wind show magnetic fluctuations over a broad range of frequencies. In the rest frame of the spacecrafts, the magnetic fluctuations show a broken power-law spectrum. For example, Leamon et al. (1999) reported that, at \( \sim 0.2 \) Hz, the spectrum breaks from a \( \nu^{-1.67} \) power law to a \( \nu^{-2.91} \) power law. In general, the spectral breakpoint lies in the range \( 0.2 \) Hz \( \lesssim \nu \lesssim 0.5 \) Hz (see Saito et al. 2008). The \( \nu^{-1.67} \) power law at \( \nu \lesssim 0.2 \) Hz seems to be relatively robust and represents inertial range of Alfvénic MHD turbulence. However, the power index for \( \nu \gtrsim 0.5 \) Hz seems to vary between \( -2 \) and \( -4 \) (Leamon et al. 1998; Smith et al. 2006). This range, characterized by a steeper power-law index, is termed “dispersion range” (Stawicki et al. 2001), which is different from the dissipation range. The true dissipation scale of turbulence may lie at the end of the dispersion range. When we convert frequency to length scale, the broken power law implies that the magnetic energy spectrum changes from a \( k^{-1.67} \) inertial range spectrum to a steeper dispersion range spectrum, as we move from large scales to small scales. The transition from the inertial range to the dispersion range occurs near the proton gyroscale \( \rho_p \). (However, it is also possible that it occurs at the ion inertial scale \( d_i = \rho_i/\sqrt{\beta_i} \), where \( \beta_i \) is the ion plasma \( \beta \). See discussion in Schekochihin et al. 2009.) The identity of the dispersion range turbulence is still under debate.

Small-scale magnetized turbulence also plays important roles in other astrophysical objects. Among them, the crust of neutron stars gives us useful insights on the electron MHD (EMHD) model of small-scale magnetized turbulence. In the crust of neutron stars, ions are virtually immobile and, thus, the ion gyroradius scale can be regarded as infinite. Therefore, turbulence in the crust should be similar to small-scale turbulence. Since ions are immobile, they provide a smooth charge background and electrons carry all the current, so that

\[
\nu_c = -\frac{\mathbf{J}}{n_e e} = -\frac{c}{4\pi n_e e} \nabla \times \mathbf{B} \tag{1}
\]

where \( \nu_c \) is the electron velocity, \( \mathbf{J} \) is the electric current density, \( \mathbf{B} \) is the magnetic field, \( c \) is the speed of light, \( n_e \) is the electron number density, and \( e \) is the absolute value of the electric charge.
Inserting this relation into the magnetic induction equation, we can obtain the EMHD equation

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{c}{4\pi n_i e} \nabla \times [\nabla \times \mathbf{B}] + \eta \nabla^2 \mathbf{B},$$

(2)

where $\eta$ is the magnetic diffusivity (see Kingsep et al. 1990 for details about EMHD). Goldreich & Reisenegger (1992) first showed that magnetized turbulence in the crust of neutron stars can be described by the EMHD equation. They discussed the properties of EMHD turbulence in neutron stars and argued that EMHD turbulence can enhance ohmic dissipation of magnetic field in isolated neutron stars (see also Cumming et al. 2004).

In this paper, we will focus on spectrum and anisotropy of EMHD turbulence. Earlier researchers convincingly showed that energy spectrum of EMHD turbulence is steeper than Kolmogorov’s $k^{-5/3}$ spectrum (Biskamp et al. 1996, 1999; Ng et al. 2003). They found that the energy spectrum follows

$$E(k) \propto k^{-7/3}.$$  

(3)

The steep energy spectrum can be explained by the following Kolmogorov-type argument (Biskamp et al. 1996). Suppose that the eddy interaction time for eddies of size $l$ is the usual eddy turnover time $t_{\text{can,l}} \sim l/v_l$. Since $v \propto \nabla \times \mathbf{B}$ (Equation (1)), this becomes $t_{\text{can,l}} \propto l^2/b_l$. Combining this with the constancy of spectral energy cascade rate ($b_l^2/t_{\text{can,l}} = \text{constant}$), one obtains $E(k) \propto k^{-7/3}$. Note that $E(k)$ and $b_l$ are related by $kE(k) \sim b_l^2$. Since earlier researchers convincingly obtained the $k^{-7/3}$ spectrum, more focus is given to anisotropy of EMHD turbulence.

Cho & Lazarian (2004; hereafter CL04) derived the expression for anisotropy

$$k_{||} \propto k^{1/3}_\perp,$$  

(4)

where $k_{||}$ and $k_\perp$ should be understood as wavenumbers parallel and perpendicular to the local magnetic field, the same way as parallel and perpendicular wavenumbers are understood in Goldreich & Sridhar (1995) model of MHD turbulence.\(^5\)

The expression (4), as we discuss in Section 3, follows from the application of the critical balance notion to the EMHD cascade. In this paper, we only consider strong EMHD turbulence. Discussions on weak EMHD turbulence can be found in Galtier & Bhattacharjee (2003) and Galtier (2006). In Section 2, we compare EMHD and recently proposed electron reduced MHD (ERMHD; Schekochihin et al. 2009) formalisms. In Section 3, we discuss expected scaling relations of EMHD and ERMHD turbulence. In Section 4, we describe our numerical methods. In Section 5, we compare EMHD and ERMHD turbulence. In this section, we perform simulations in an elongated numerical box ($786 \times 256^2$). In Section 6, we present detailed study of anisotropy from an EMHD simulation with 512$^3$ resolution. In Section 7, we present high-order statistics and bispectra of EMHD turbulence. In Section 8, we calculate the probability distribution functions (PDFs) and briefly compare them with the solar wind data. We give discussions in Section 9 and summary in Section 10.

\(^5\) A wavelet description would be more appropriate, as usual wavenumbers are defined in the global magnetic field frame. We keep this in mind, while using the traditional wavenumber notations.

2. EMHD AND ERMHD

Alfvén modes are incompressible and thus less prone to collisionless damping (Barnes 1966; Kulsrud & Pearce 1969). Therefore, in the solar wind, it is generally accepted that Alfvénic turbulence provides a suitable description of fluid motions on scales larger than the proton gyroscale. The physics of strong Alfvén turbulence is relatively well understood (see, e.g., Goldreich & Sridhar 1995).

However, turbulence on scales smaller than the proton gyroscale is not well understood. Nevertheless, there are several models for this small-scale turbulence. Here, we consider only fluid-like models of the small-scale turbulence. In particular, we consider EMHD model and ERMHD model. The former has been studied since 1990s (Kingsep et al. 1990; Biskamp et al. 1996) and the latter has been proposed only recently (Schekochihin et al. 2009). Both models can be derived from the generalized Ohm’s law.

2.1. EMHD

EMHD can be viewed Hall MHD in the limit of $k \rho_i \gg 1$, where $\rho_i$ is the ion gyroradius. Hall MHD equations are very similar to the standard MHD ones. But, there are also differences. The most important difference is that Hall MHD is based on the generalized Ohm’s law

$$\mathbf{E} = -\frac{\mathbf{v}}{c} \times \mathbf{B} + \frac{\mathbf{J} \times \mathbf{B}}{n_i e c} + \frac{\mathbf{J}}{\sigma},$$  

(5)

while the standard MHD uses the “standard” Ohm’s law

$$\mathbf{E} = -\frac{\mathbf{v}}{c} \times \mathbf{B} + \frac{\mathbf{J}}{\sigma}.$$  

(6)

Here, $\mathbf{v}$ is the fluid velocity, $\mathbf{E}$ is the electric field, and $\sigma$ is the conductivity. Compared with that of the standard MHD, the induction equation of Hall MHD has an extra term (the $\mathbf{J} \times \mathbf{B}$ term on the right-hand side):

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E},$$

$$= \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times \frac{\mathbf{J} \times \mathbf{B}}{n_i e} + \frac{c^2}{4\pi \sigma} \nabla^2 \mathbf{B}$$

$$= \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times \frac{c(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi n_i e} + \eta \nabla^2 \mathbf{B},$$

(7)

where $\eta = c^2/(4\pi \sigma)$ and we use $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{J}$.

When we normalize magnetic field to a velocity (i.e., $\mathbf{B}/\sqrt{4\pi \rho}$) and we use $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{J}$, Equation (7) becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - d_i \nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}] + \eta \nabla^2 \mathbf{B},$$  

(Hall MHD)

(8)

where

$$d_i = \frac{c}{4\pi n_i m_i} \frac{\sqrt{4\pi n_i e m_i}}{4\pi n_i e} = \frac{c}{\sqrt{4\pi n_i e Z(\rho_i c^3)^2}} = \frac{c}{\omega_p} = \rho_i \sqrt{\beta_i}.$$  

(9)

Here, $\rho_i$ is the ion gyroradius, $Z = n_e/\rho_i = \eta_i / e$, and $\beta_i$ is the ion plasma beta:

$$\beta_i = \frac{n_i k_B T}{B^2/8\pi},$$  

(10)
where $k_B$ is the Boltzmann constant. The order of magnitude values of the first and the second terms on the right-hand side are

$$\nabla \times (\mathbf{v} \times \mathbf{B}) \sim b^2/l, \quad (11)$$

$$d_i \nabla \times ((\nabla \times \mathbf{B}) \times \mathbf{B}) \sim d_i b^2/l^2, \quad (12)$$

where $l$ is the scale of interest and we assume $v \sim b$. Equation (8) reduces to the standard MHD induction equation for $l \gg d_i$, while it reduces to the EMHD equation for $l \ll d_i$:

$$\frac{\partial \mathbf{B}}{\partial t} = -d_i \nabla \times ((\nabla \times \mathbf{B}) \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \quad \text{(EMHD)} \quad (13)$$

In usual collisionless plasmas, transition from standard MHD to EMHD occurs at the ion inertial scale $d_i$. When ions are immobile and provide a homogeneous background, as in the crust of a neutron star, EMHD can be applicable even at the outer scale of turbulence, which can be larger than $d_i$ defined by the first two expressions in Equation (9). Note that the anisotropy of turbulence (i.e., $k_\parallel \gg k_\perp$) at $d_i$ (in usual plasmas) or on the energy injection scale (in case ions are immobile) is not a necessary condition for EMHD.

### 2.2. ERMHD

Schekochihin et al. (2009) first derived ERMHD from kinetic RMHD equations. They also gave derivation of ERMHD equations from the generalized Ohm's law. Therefore, the starting point of ERMHD may also be the generalized Ohm's law and derivation of ERMHD is identical to the EMHD case up to Equation (7) (see previous subsection). Note that the velocity $v$ in Equation (7) denotes ion velocity $v_i$ and that RMHD, hence ERMHD, assumes anisotropy of turbulence (i.e., $k_\perp \gg k_\parallel$).

However, ERMHD assumes that the term

$$\nabla \times (\mathbf{v} \times \mathbf{B}) \approx -\mathbf{B} \nabla \cdot \mathbf{v} \quad (14)$$

in Equation (7) may not be negligible in the limit of $k_B v_i \gg 1$. That is, ERMHD assumes $v_i \approx 0$ in this limit, but $\nabla \cdot \mathbf{v} = \nabla \cdot v_i \neq 0$ (see Schekochihin et al. 2009). Taking this into account, one can rewrite the magnetic induction equation as

$$\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{B} \nabla \cdot \mathbf{v} - \frac{c}{4 \pi e n_e} \nabla \times [\mathbf{B} \cdot \nabla \mathbf{B}], \quad (15)$$

which becomes

$$\frac{\partial \mathbf{B}_\perp}{\partial t} = -\frac{c}{4 \pi e n_e} \nabla \times [\mathbf{B} \cdot \nabla \mathbf{B}_\perp], \quad (16)$$

$$\frac{\partial \mathbf{B}_\parallel}{\partial t} = -\mathbf{B}_0 \nabla \cdot \mathbf{v} - \frac{c}{4 \pi e n_e} \nabla \times [\mathbf{B} \cdot \nabla \mathbf{B}_\parallel], \quad (17)$$

where $\mathbf{B}_\parallel$ and $\mathbf{B}_\perp$ denote the components of magnetic field parallel and perpendicular to the mean field $\mathbf{B}_0$, respectively (see Appendix A of this paper and Appendix C of Schekochihin et al. 2009). Here, we drop the dissipation term for simplicity.

Finally, from the continuity equation and the assumption of pressure balance, we can rewrite the above equations as

$$\frac{\partial \mathbf{B}_\perp}{\partial t} = -\frac{c}{4 \pi e n_e} \nabla \times [\mathbf{B} \cdot \nabla \mathbf{B}_\parallel], \quad (18)$$

$$\frac{\partial \mathbf{B}_0}{\partial t} = -\frac{\beta_i (1 + Z/\tau)}{2 + \beta_i (1 + Z/\tau)} \frac{c}{4 \pi e n_e} \nabla \times [\mathbf{B} \cdot \nabla \mathbf{B}_\perp] \quad (19)$$

(see Equations (A8) and (A14)), which become

$$\frac{\partial \mathbf{b}}{\partial t} = -\frac{\rho_i}{\sqrt{\beta_i}} \sqrt{\alpha} \nabla \times \left( \frac{\mathbf{B}}{4 \pi n_i m_i} \cdot \nabla \mathbf{b} \right), \quad (20)$$

where

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \quad (21)$$

$$\mathbf{b} = \mathbf{b}_\perp + \beta_i \hat{\mathbf{z}}, \quad (22)$$

$$\mathbf{b}_0 = \mathbf{b}_\perp + \sqrt{\frac{1}{\alpha}} \mathbf{b}_\parallel \hat{\mathbf{z}}, \quad (23)$$

$$\alpha \equiv \frac{\beta_i (1 + Z/\tau)}{2 + \beta_i (1 + Z/\tau)} \cdot \frac{1}{\beta_i}, \quad (24)$$

Here, $\mathbf{B}_0 (= \mathbf{B}_0 \hat{\mathbf{z}})$ is the mean magnetic field, $\hat{\mathbf{z}}$ is a unit vector along the direction of $\mathbf{B}_0$, $Z = q_i / e$ ($e = |q_i|$) is the ion-to-electron charge ratio, $\tau = T_i / T_e$ is the temperature ratio, $n_i$ is the average electron number density. The vector $\mathbf{b}_\perp$ (in Fourier space) is parallel to $\hat{\mathbf{z}} \times \mathbf{k}_\perp$, where $\mathbf{k}_\perp = \mathbf{k}_\perp / k_\perp$. Therefore, in Fourier space $\mathbf{b}_\parallel$ lies in the plane spanned by $\hat{\mathbf{z}} \times \mathbf{k}_\perp$ and $\hat{\mathbf{z}}$, which means $\mathbf{b}_\parallel$ is perpendicular to $\mathbf{k}_\perp$.

### 3. EXPECTED SCALING RELATIONS

In the presence of a strong mean magnetic field, turbulence energy tends to cascade in the direction perpendicular to the mean field. As a result, in Fourier space, modes with $k_\perp \gg k_\parallel$ are predominantly excited. In real space, characteristic scales parallel to the mean field ($l_0$) tend to be larger than those perpendicular to it ($l_\perp$). This is referred to as anisotropy of turbulence.

In the case of standard MHD turbulence, this global anisotropy has been studied since early 1980s (Shebalin et al. 1983). On the other hand, Goldreich & Sridhar (1995) showed that there exists a regime of turbulence in which a critical balance is maintained between wave motions (with timescale of $t_w \sim l_\perp / V_A$) and hydrodynamic motions (with timescale of $l_\perp / v_i$). This is the so-called strong turbulence regime and they found a certain relation between $k_\parallel$ and $k_\perp$, $k_\parallel \propto k_\perp^{2/3}$ in the regime. This scale-dependent anisotropy was numerically confirmed by Cho & Vishniac (2000) and Maron & Goldreich (2001). Cho & Vishniac (2000) showed that this scale-dependent anisotropy can be measured only in a local coordinate frame which is aligned with the locally averaged magnetic field direction. The necessity of using a local frame is due to the fact that eddies are aligned along the local mean magnetic field, rather than the global mean field $\mathbf{B}_0$. We call this kind of anisotropy as local anisotropy.

In the case of EMHD turbulence, anisotropy has been studied only recently. Dastgeer et al. (2000) and Dastgeer & Zank (2003) numerically studied global anisotropy of two-dimensional EMHD turbulence. On the other hand, Ng et al.
Anisotropy of ERMHD is expected to be similar to that of EMHD (Schechkinin et al. 2009). This is understandable from Equation (20). Note that $b = \mathbf{b}_\perp + b_\parallel \mathbf{z}$ and $\mathbf{B} = \mathbf{b}_\perp + \sqrt{\alpha} b_\parallel \mathbf{z}$ in the equation. The factor $\alpha$ is less than or equal to 1. If we assume $b_\parallel \sim \sqrt{\alpha} b_\perp$, then the parallel (or $z$) component of true magnetic field $b_\parallel$ is equal to $\sim \sqrt{\alpha} b_\perp \leq b_\perp$. If $\alpha \sim 1$ and, hence, $\mathbf{b} \sim \mathbf{b}_\perp$, then Equation (20) becomes the usual EMHD equation. Therefore, it is trivial to show that anisotropy of ERMHD is similar to that of EMHD. On the other hand, if $\alpha \ll 1$ and, hence, $\mathbf{b} = \mathbf{b}_\perp + \sqrt{\alpha} b_\parallel \sim \mathbf{b}_\perp$, then the ERMHD equation becomes different from the EMHD equation. However, in general, the parallel component of $\mathbf{b}$ does not contribute much to the term $\mathbf{B} \cdot \nabla \mathbf{b}$, because $b_\parallel \mathbf{z} \cdot \nabla \sim b_\parallel k_\parallel \ll b_\perp \cdot \nabla \sim b_\perp k_\parallel$ in the presence of anisotropy, $k_\parallel \ll k_\perp$. This means that what matters for energy cascade is the perpendicular component of $\mathbf{b}$, which is not directly affected by the value of $\alpha$. Therefore, even in case of $\alpha \ll 1$, we expect that anisotropy of ERMHD is similar to that of EMHD. However, this conjecture needs to be tested by numerical calculations.

4. NUMERICAL METHODS

As described earlier, EMHD and ERMHD equations involve with time evolution of magnetic field only. Since magnetic field is divergence-free, we can use incompressible numerical schemes to solve the equations.

4.1. The EMHD Code

We adopt a pseudospectral code to solve the normalized EMHD equations in a periodic box of size $2\pi$:

$$
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}] + \eta' \nabla^2 \mathbf{B},
$$

(25)

where magnetic field, time, and length are normalized by a mean field $B_0$, the whistler time $t_w = L^2(\omega_{pe}/c)^2/\Omega_e$ ($\Omega_e$ is the electron gyro frequency), and a characteristic length scale $L$ (see, for example, Galtier & Bhattacharjee 2003). The resistivity $\eta'$ in Equation (25) is dimensionless. The dispersion relation of a whistler waves in this normalized units is $\omega = k k_\perp B_0$. The magnetic field consists of the uniform background field and a fluctuating field: $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$. The strength of the uniform background field, $B_0$, is set to 1. We use up to 512$^3$ collocation points. At $t = 0$, the random magnetic field is restricted to the range $2 \leq k < 5$ in wavevector space. The amplitudes of the random magnetic field at $t = 0$ is $\sim 1.2$. In order to have a more extended inertial range, we use hyperdiffusivity for the diffusion terms.4 The power of hyperdiffusivity is set to 3 for all simulations, so that the dissipation term in the above equation is replaced with $\eta'(\nabla^2)^3 \mathbf{B}$. We perform both driven and decaying turbulence simulations.

4.2. The ERMHD Code

The ERMHD equation is only slightly different from the EMHD equation. Therefore, we also adopt a pseudospectral code to solve the normalized ERMHD equation in a rectangular periodic box of size $2\pi \times 2\pi \times 6\pi$:

$$
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times [\mathbf{B} \cdot \nabla \mathbf{B}] + \eta' \nabla^2 \mathbf{B},
$$

(26)

where

$$
\mathbf{B} \equiv B_0 + \mathbf{b},
$$

(27)

$$
\mathbf{b} = b_\perp + \sqrt{\alpha} b_\parallel \mathbf{z},
$$

(28)

$$
\dot{\mathbf{t}} \equiv \sqrt{\alpha} \dot{t},
$$

(29)

where $\eta'$ is the dimensionless resistivity, $b_\perp$ and $b_\parallel \mathbf{z}$ are the perpendicular and parallel components of magnetic field, respectively, and definition of $\alpha$ is given in Equation (24). Note that the ERMHD equation is very similar to the EMHD equation. Numerical setup is therefore similar to that of the EMHD case, except the size of the simulation box.

We use an elongated numerical box for ERMHD to satisfy the condition $k_\parallel \gg k_\perp$. The numerical box is three times longer in the direction of the mean magnetic field. The wavenumbers along the mean field direction have fractional values

$$
k_\parallel = 1/3, 2/3, 1, 4/3, \cdots,
$$

(30)

while those of perpendicular direction have integer values.5 We perform only decaying turbulence simulations. At $t = 0$, Fourier modes with $1/3 \leq k_\perp \leq 1$ and $4.5\sqrt{2} \leq k_\parallel \leq 4.5\sqrt{2}$ are excited. The strength of the mean magnetic field ($B_0$) is 1 and, the rms value of the random magnetic field (actually $b$) at $t = 0$ is $\sim 0.071$. Therefore, the critical balance, $b k_\perp / B_0 k_\parallel = 1$, is roughly satisfied at $t = 0$. For comparison, we perform a similar EMHD simulation (Run E256D-EL).

5. RESULTS: EMHD AND ERMHD

We compare EMHD and ERMHD turbulence in Figure 1. The left panel of Figure 1 shows time evolution and energy spectra of decaying EMHD and ERMHD runs (see Runs E256D-EL, ER256D1-EL, and ER256D8-EL in Table 1). The inset shows time evolution of magnetic energy density. The solid curve is for ERMHD with $\sqrt{\alpha} = 1$, the dotted curve for ERMHD with $\sqrt{\alpha} = 1/8$, and the dashed curve for EMHD. The horizontal axis represents time multiplied by $\sqrt{\alpha}$ and the vertical axis twice the magnetic energy density. In case of ERMHD with $\sqrt{\alpha} = 1/8$ (i.e., Run ER256D8-EL), the vertical axis is not twice the magnetic energy density. In case of ERMHD with $\sqrt{\alpha} = 1/8$, the dotted curve for ERMHD with $\sqrt{\alpha} = 1/8$, and the dashed curve for EMHD. The horizontal axis represents time multiplied by $\sqrt{\alpha}$ and the vertical axis twice the magnetic energy density. In case of ERMHD with $\sqrt{\alpha} = 1/8$ (i.e., Run ER256D8-EL), the vertical axis is not twice the magnetic energy density. In case of ERMHD with $\sqrt{\alpha} = 1/8$, the dotted curve for ERMHD with $\sqrt{\alpha} = 1/8$, and the dashed curve for EMHD. The horizontal axis represents time multiplied by $\sqrt{\alpha}$ and the vertical axis twice the magnetic energy density. In case of ERMHD with $\sqrt{\alpha} = 1/8$ (i.e., Run ER256D8-EL), the vertical axis is not twice the magnetic energy density. In case of ERMHD with $\sqrt{\alpha} = 1/8$, the dotted curve for ERMHD with $\sqrt{\alpha} = 1/8$, and the dashed curve for EMHD. The horizontal axis represents time multiplied by $\sqrt{\alpha}$ and the vertical axis twice the magnetic energy density. In case of ERMHD with $\sqrt{\alpha} = 1/8$ (i.e., Run ER256D8-EL), the vertical axis is not twice the magnetic energy density. In case of ERMHD with $\sqrt{\alpha} = 1/8$, the dotted curve for ERMHD with $\sqrt{\alpha} = 1/8$, and the dashed curve for EMHD. The horizontal axis represents time multiplied by $\sqrt{\alpha}$ and the vertical axis twice the magnetic energy density. In case of ERMHD with $\sqrt{\alpha} = 1/8$ (i.e., Run ER256D8-EL), the vertical axis is not twice the magnetic energy density. In case of ERMHD with $\sqrt{\alpha} = 1/8
We compare anisotropy of EMHD and ERMHD in the right panel of Figure 1. We obtain the plot at $\sqrt{\alpha}t = 0.6$. All three curves show similar scaling relations, $k_1 \propto k_1^{1/3}$. Note however that, although we do not show it in this paper, the anisotropy tends to get stronger as turbulence decays further. We will describe the method of obtaining anisotropy and its limitations in Section 6.1.

To summarize, we observe that both EMHD and ERMHD turbulence exhibit similar spectra and anisotropy. Spectra and time evolution of magnetic energy density of ERMHD turbulence seem to be invariant under the following transformation,

$$B \rightarrow \tilde{B}, \quad t \rightarrow \sqrt{\alpha}t. \quad (31)$$

6. RESULTS: SCALING OF EMHD

In the previous section, we have studied turbulence in a rectangular box elongated along the mean field direction. Note that the numerical resolution in the perpendicular direction is $256 \times 256$, which limits the inertial range of turbulence. To increase the inertial range we need to increase numerical resolution in the perpendicular direction. Studying ERMHD is more difficult than studying EMHD because we need to use an elongated numerical box for ERMHD to satisfy the condition $k_\perp \gg k_1$. For EMHD, there is no necessity of using such an elongated numerical box. Therefore, EMHD is more suitable for high-resolution simulations. Since EMHD turbulence and ERMHD turbulence exhibit similar scaling relations, we only consider a high-resolution EMHD simulation in this section.

We perform a decaying EMHD simulation with $512^3$ grid points (Run E512D in Table 1). At $t = 0$, Fourier modes with $2 \leq k < 5$ are excited. Here, $k = \sqrt{k_\parallel^2 + k_\perp^2}$ and, thus, the initial perturbation is isotropic (i.e., $k_1 \sim k_\perp$). The strength of the mean magnetic field ($B_0$) is 1 and the rms value of the random magnetic field ($b$) is $\sim 1.21$. Therefore, the critical balance, $bk_\perp/B_0k_1 = 1$, is roughly satisfied at $t = 0$.

In this section, we focus on anisotropy of EMHD turbulence. CL04 showed that the anisotropy of EMHD turbulence is consistent with $k_1 \propto k_1^{1/3}$. However, since it is difficult to measure anisotropy of EMHD turbulence, more careful assessments of anisotropy are necessary. In this subsection, we develop and test new techniques for anisotropy. We analyze anisotropy of turbulence after turbulence has developed a full inertial range.

The energy spectra in the left panel of Figure 2 show how the inertial range develops. At $t = 0$ only large-scale (i.e., small $k$) Fourier modes are excited. The dashed curve in Figure 2 shows the initial spectrum. As the turbulence decays, the initial energy cascades down to small scales and, as a result, small-scale (i.e., large $k$) modes are excited. When the energy reaches the dissipation scale at $k \gtrsim 100$, the energy spectrum goes down without changing its slope (the dotted and the solid curves). The slope at this stage is very close to that of the predicted spectrum:

$$E(k) \propto k^{-7/3}. \quad (32)$$

Notes.

1. In this case, the value of $\tilde{\delta}$ at $t = 0$ is listed.

2. In cases of driven turbulence, the average driving-scale wavenumber is listed.

### Table 1

| Run        | Resolution | $B_0$ | $b$ at $t = 0$ | $k_\perp$ at $t = 0$ | $k_\parallel$ at $t = 0$ | Comments          |
|------------|------------|-------|----------------|----------------------|----------------------|-------------------|
| E512D      | 512$^3$    | 1     | 1.21           | $2 \leq k < 5$      | $2 \leq k < 5$       | Decaying          |
| E256D      | 256$^3$    | 1     | 1.21           | $2 \leq k < 5$      | $2 \leq k < 5$       | Decaying          |
| E256F      | 256$^3$    | 1     | 1.21           | $k \sim 2.5$        | $k \sim 2.5$         | Forced            |
| E256D-EL   | 768 $\times$ 256$^2$ | 1 | 0.071 | $4.5/\sqrt{2} \leq k_\perp \leq 4.5/\sqrt{2}$ | $1/3 \leq k_\perp \leq 1$ | Decaying |
| ER256D1-EL | 768 $\times$ 256$^2$ | 1 | 0.071 | $4.5/\sqrt{2} \leq k_\perp \leq 4.5/\sqrt{2}$ | $1/3 \leq k_\perp \leq 1$ | Decaying, $\sqrt{\alpha} = 1$ |
| ER256D8-EL | 768 $\times$ 256$^2$ | 1 | 0.071 | $4.5/\sqrt{2} \leq k_\perp \leq 4.5/\sqrt{2}$ | $1/3 \leq k_\perp \leq 1$ | Decaying, $\sqrt{\alpha} = 1/8$ |
| MHD512F    | 512$^3$    | 0.8   | 0              | $k \sim 2.5$        | $k \sim 2.5$         | Forced            |
| MHD256F    | 256$^3$    | 1     | 0              | $k \sim 2.5$        | $k \sim 2.5$         | Forced            |
For the study of anisotropy, we use the data cube at $t \approx 0.46$. The solid curve in the left panel of Figure 2 represents the spectrum at that time.

The right panel of Figure 2 shows spectra of electric field. Both the total electric field (solid curve) and its $x$-component (dashed curve) show spectra compatible with $k^{-1/3}$. (Note that we assume $B_0 = B_l \hat{z}$.) This result is consistent with the ones obtained by gyrokinetic simulations (Howes et al. 2008b) and Hall MHD simulations (see Dmitruk & Matthaeus 2006; Matthaeus et al. 2008). Bale et al. (2005) reported a similar electric fluctuation spectrum in the solar wind.

6.1. Anisotropy: Method 1

In CL04, we noted that the quantity $B_L \cdot \nabla b_l$ is proportional to $B_L k |b_l|$, where $B_L$ is the local mean field, $b_l$ is the fluctuating field at scale $l$, and $k_l$ is the wavenumber parallel to the local mean magnetic field. We obtained $B_L$ by eliminating Fourier modes whose perpendicular wavenumber is greater than $k/2$. We obtained the fluctuating field $b_l$ by eliminating Fourier modes whose perpendicular wavenumber is less than $k/2$. In CL04, we calculated anisotropy in Fourier space:

$$k_1(k_\perp) \approx \left( \frac{\sum_{k \leq |k|} |B_L \cdot \nabla b_l|^2}{B_L^2 \sum_{k \leq |k|} |b_l|^2} \right)^{1/2}.$$  \hfill (33)

The reason why we did the calculation in Fourier space is that there might be unknown contamination from scales other than $l$. We plot anisotropy obtained this way in the left panel of Figure 3.

We note that, when we obtain the fluctuating field $b_l$ by filtering out Fourier modes whose perpendicular wavenumber is less than $k/2$ or greater than $2k$, we may calculate anisotropy directly in real space:

$$B_L \cdot \nabla b_l \approx B_L k |b_l| \rightarrow k_1 \approx \left( \frac{|\langle B_L \cdot \nabla b_l \rangle|^2}{B_L^2 b_l^2} \right)^{1/2}.$$  \hfill (34)

We plot anisotropy obtained this way in the right panel of Figure 3. The result is compatible with the expected anisotropy, $k_1 \propto k_\perp^{1/3}$. The limitation of this method is that the choice of the filtering wavenumbers, $k/2$ and $2k$, is an arbitrary one. When we use filtering wavenumbers of $k/\sqrt{2}$ and $\sqrt{2}k$, we get a slightly weaker anisotropy.

6.2. Anisotropy: Method 2

In the previous subsection, we assumed that $B_L \cdot \nabla b_l \propto B_L k |b_l|$. Is this really correct? If small-scale eddies are aligned with local mean field, the above assumption should be true. However, if the major axes of small-scale eddies are substantially misaligned with local large-scale mean field, the quantity $B_L \cdot \nabla b_l$ may sample perpendicular wavenumber of the eddies.

In this subsection, we investigate alignment of small-scale eddies with respect to local mean magnetic field.

We first visualize the alignment effect. The left panel of Figure 4 shows a snapshot of magnetic field structure in a plane parallel to the global mean magnetic field. The global mean magnetic field is parallel to the horizontal axis in the Figure. The white lines in Figure represent local mean magnetic field. We obtain the local mean field by filtering out Fourier modes with $k > 15$. The contours represent magnetic energy density of small-scale field. We obtain the small-scale field by filtering out Fourier modes with $k \leq 15$ or $k \geq 60$. Since the alignment effect, if any, should be three-dimensional effect in nature, we may not be able to visualize the effect correctly in a two-dimensional plane. Nevertheless, we can observe that small-scale eddies are well aligned with the local mean field directions.

If small-scale eddies are well aligned with the local mean field, the correlation length along the local mean field should be larger than that of perpendicular directions. Furthermore, when the alignment effect is present, the parallel correlation length should follow $l_\parallel \propto l^{1/3}$ (or, $k_\parallel^{-1/3}$) power law. We investigate the behavior of parallel and perpendicular correlation lengths by changing the scale $l$. We define the correlation lengths as the distances at which the two-point correlation function $\langle b_l(r_1) \cdot b_l(r_2) \rangle$ drops by $50\%$. The parallel correlation length is measured along the directions of local mean field, $B_L$, and the perpendicular one for perpendicular directions. In the right panel of Figure 4, we plot the parallel and perpendicular correlation lengths as functions of small-scale eddy size $l \propto 1/k$. The parallel correlation lengths (upper curve) are of course longer than the perpendicular ones (lower curve). On average, the parallel correlation lengths seem to follow the expected $l^{1/3}$ scaling. However, the slope is shallower than $1/3$ for small-size eddies and steeper than $1/3$ for large-size eddies. Roughly speaking, the perpendicular correlation lengths follow $l^1$ scaling, which is reasonable.

6 We use a similar filtering method as in the previous subsection. We obtain the local mean field, $B_L$, by eliminating Fourier modes whose perpendicular wavenumber is greater than $k/2$. We obtain the fluctuating field, $b_l$, by eliminating Fourier modes whose perpendicular wavenumber is less than $k/2$ or greater than $2k$. Note that $l = 2\pi/k$, or in terms of grid units $l = N_{side}/k$, where $N_{side} = 512$ in Run E512D.
Figure 3. Anisotropy of EMHD turbulence (Run E512D). Left: the old method (Equation (33)) is used. Right: a new method (Equation (34)) is used.

Figure 4. Anisotropy based on local correlation lengths (Run E512D). Left: visualization. White lines denote local mean magnetic field $\mathbf{B}_L$ obtained by filtering out large-$k$ Fourier modes. Contours show structure of small-scale eddies obtained by retaining Fourier modes near the scale of interest and filtering out all other scale Fourier modes. The global mean field $\mathbf{B}_0$ is parallel to the horizontal axis. Right: a new method based on correlation lengths in local frame. The parallel correlation length $l_\parallel$, for example, is the average of the correlation length measured along the local mean magnetic field direction at each point. (Note that the local mean magnetic field directions are marked by white lines in the left panel.) In Run E512D, $N_{side} = 512$.

The anisotropy of EMHD turbulence revealed by the filtering process is consistent with the $k_\parallel \propto k_\perp^{1/3}$ scaling.

6.3. Anisotropy: Multi-Point Second-Order Structure Functions

We may visualize scale-dependent anisotropy using the second-order structure function calculated in the local frame, which is aligned with local mean magnetic field $\mathbf{B}_L$:

$$SF_2(\mathbf{r}_\parallel, \mathbf{r}_\perp) = \langle |\mathbf{B}(\mathbf{x} + \mathbf{r}) - \mathbf{B}(\mathbf{x})|^2 \rangle_{\text{avg over } \mathbf{x}}, \quad (35)$$

where $\mathbf{r} = \mathbf{r}_\parallel \hat{\mathbf{r}}_\parallel + \mathbf{r}_\perp \hat{\mathbf{r}}_\perp$. The vectors $\hat{\mathbf{r}}_\parallel$ and $\hat{\mathbf{r}}_\perp$ are unit vectors parallel and perpendicular to the local mean field $\mathbf{B}_L$, respectively. See Cho & Vishniac (2000) for the detailed discussion of the local frame. The left panel of Figure 5 is an example of such visualization. The horizontal axis corresponds to the local mean field directions. The shapes of contours illustrate scale-dependent anisotropy: the large eddies are roughly isotropic and smaller eddies are more elongated. However, we should be careful when we derive a quantitative scaling relation for anisotropy from the contour diagram.

The two-point second-order structure function of a variable $A$

$$SF_2(r) = (\delta A_r)^2 = \langle |A(\mathbf{x} + \mathbf{r}) - A(\mathbf{x})|^2 \rangle \quad (36)$$

is in general related to the energy spectrum of the variable, $E_A(k)$, through

$$SF_2(r) = (\delta A_r)^2 \propto k E_A(k), \quad (37)$$

where $k \propto 1/r$ is the wavenumber. Therefore, when $E_A(k) \propto k^{-m}$, we have

$$SF_2(r) \propto r^{m-1}. \quad (38)$$

For example, in fully developed Kolmogorov turbulence ($E(k) \propto k^{-5/3}$), the scaling relation of the second-order longitudinal structure function is given by

$$SF_2(r) = (\delta v_r)^2 \propto kk^{-5/3} \propto k^{-2/3} \propto r^{2/3}. \quad (39)$$

However, when the slope of the turbulence spectrum is steeper than $k^{-3}$, the relation in Equation (37) becomes invalid and $SF_2(r) \propto r^2$ regardless of the slope of the energy spectrum (see Appendix B). In EMHD, the energy spectrum expressed in terms of $k_\perp$ scales as $E(k_\perp) \propto k_\perp^{-7/3}$ and we expect that the second-order structure function scales as

$$SF_2(0, r_\perp) \propto r_\perp^{4/3}. \quad (40)$$

On the other hand, the energy spectrum expressed in terms of $k_\parallel$ is expected to be $E(k_\parallel) \propto k_\parallel^{-3}$ when anisotropy scales as $k_\parallel \propto k_\perp^{1/3}$ (see CL04). Therefore, the one-to-one correspondence between the second-order structure function and the energy spectrum is not valid for the parallel direction. This means we cannot use the second-order structure function to reveal true anisotropy.

We have shown that the two-point second-order structure function is not suitable for quantitative study of anisotropy in EMHD turbulence. However, it is possible to construct multi-point second-order structure functions that can be used for variables with steep energy spectra (see Falcon et al. 2007; Lazarian & Pogosyan 2008; see also Appendix C). The three-point $SF_2$ was used by Falcon et al. (2007) and Lazarian & Pogosyan (2008). The three-point $SF_2$ will work for energy spectrum as steep as $\sim k^{-5}$. In Appendix C, we discuss how to construct four-point and five-point structure functions. The five-point $SF_2$ works for energy spectrum as steep as $\sim k^{-9}$.
The left panel of Figure 5 is obtained with this five-point structure function. The middle panel of the Figure shows the parallel structure functions, SF2(∥, 0), and the perpendicular structure functions, SF2(⊥, 0). We plot the results of three-point (dotted lines) and five-point (solid lines) structure functions. In the perpendicular direction, both structure functions reasonably follow the expected scaling of $r^{4/3}$. In the parallel directions, however, we observe that, when $r_\parallel \gtrsim 10$, the structure functions are much shallower than the expected scaling of $r^2$, which is from $\text{SF}_2 \propto k_\perp E(k_\perp) \propto r_\perp^4$. In the parallel directions, the five-point structure function is steeper than the three-point one.

One should note that it is very difficult to obtain a well-defined power-law scaling for the parallel direction. This is because the inertial range in the parallel direction, if any, is extremely short. In the perpendicular direction, the inertial range spans from $k_\perp \sim 3$ to the dissipation scale ($k_\perp \sim 100$). Suppose that $k_\parallel \propto k_\perp^{1/3}$ is the true anisotropy. Then, in the parallel direction, the inertial range spans from $k_\parallel \sim 3$ to $k_\parallel \sim 3 \times (100/3)^{1/3} \sim 10$. Therefore, it is virtually hopeless to reveal true anisotropy from contour diagram. Indeed, the right panel of Figure 5 shows that the anisotropy derived from the relation between semimajor axis and semiminor axis of contours is not really consistent with the $k_\parallel \propto k_\perp^{1/3}$ relation. Both the solid curve (five-point) and the dotted curve (three-point) show a similar scaling. The average slope is approximately $-0.5$. Therefore, we can conclude that true anisotropy is similar to or stronger than this. Note that the dashed curve (two-point) shows a milder anisotropy (or, a steeper slope). This means that multi-point structure functions are indeed better for steep spectrum. Although structure functions may not be suitable to reveal true anisotropy for EMHD, it is promising that multi-point structure functions seem to resolve steeper spectrum better.

**6.4. Decaying Versus Forced EMHD Turbulence**

It is generally true that both decaying and driven turbulence show a similar scaling. However, in the presence of strong mean magnetic field, it is not clear whether or not they really show a similar scaling. Consider a decaying strongly magnetized EMHD turbulence. Let us assume that the critical balance is roughly satisfied at $t = 0$. The strength of random magnetic field drops as time goes on. But, the strength of the mean magnetic field remains same. Therefore, critical balance will be soon destroyed. Of course, there is a narrow window of time during which critical balance is satisfied and we can study critically balanced EMHD turbulence during the time interval. Nevertheless, it is worth comparing driven turbulence and decaying turbulence. In this section, we show that forced EMHD turbulence exhibits similar scaling relations as decaying EMHD turbulence.

Numerical setup for driven EMHD turbulence is similar to that of decaying EMHD turbulence. Forcing is done in Fourier space. The forcing term consists of 21 Fourier components with $2 \leq k \leq \sqrt{12}$. The peak of energy injection is at $k \approx 2.5$. The forcing is statistically isotropic. The strength of the mean magnetic field $B_0$ is 1. We adjusted the amplitudes of the forcing components so that $b \approx 1$. Therefore, the driven EMHD turbulence satisfies the condition for critical balance. This run is designated as Run E256F in Table 1.

Inset in Figure 6 shows time evolution of magnetic energy density (in fact $B^2 = B_0^2 + b^2$). The magnetic energy reaches a stationary state after $t \approx 1$. The main plot of Figure 6 shows magnetic energy spectrum at three different time points marked by the arrows in the inset. Line styles of the spectra are the same as those of the arrows. The slopes of all three energy spectra are consistent with $-7/3$.

We plot anisotropy in the right panel of Figure 6. We use the technique described in Section 6.1. Anisotropy of driven EMHD turbulence is also consistent with the $k_\parallel \propto k_\perp^{1/3}$ scaling. All in all, driven MHD turbulence shows similar spectrum and anisotropy as decaying EMHD turbulence.

**6.5. Inverse Cascade of EMHD Turbulence**

In Wareing & Hollerbach (2009), the inverse cascade was reported as a part of decaying two-dimensional EMHD turbulence. The energy spectrum in Shaikh & Zank (2005) also shows a clear evidence of inverse cascade in driven two-dimensional EMHD turbulence. However, the difference of three-dimensional and two-dimensional turbulence makes one wonder whether the inverse cascade is present in decaying three-dimensional EMHD turbulence.

We perform another decaying EMHD turbulence simulation. The numerical setup is almost identical to the one described in Section 4.1. However, we use a different initial condition. At $t = 0$, Fourier modes with $16 \leq k \leq 32$ are excited (see the solid curve in the left panel of Figure 7). The numerical resolution is $256^3$. This run is not listed in Table 1.

The left panel of Figure 7 shows spectral behavior of the decaying EMHD turbulence. The dotted line shows the spectrum shortly after the simulation. The dotted line clearly shows that most of the energy cascades down to small scales. Of course, the energy cascading down to small scales will dissipate away below $k \gtrsim 80$. It also shows that some of the energy goes to larger scales, the amount of which is smaller than that in
the two-dimensional EMHD case (see Wareing & Hollerbach 2009). At later times, the peak to the energy spectrum moves to larger scales, which probably means that we see a self-similar decaying of energy that leads to the increase of the integral scale (see, for example, Biskamp 2003), rather than a signature of the inverse energy cascade that is observed in two-dimensional hydrodynamic turbulence. Nevertheless, we clearly observe that EMHD turbulence can generate more coherent magnetic field from a smaller scale, hence less coherent, magnetic field. Although this phenomenon is not the inverse cascade per se, we can still call it inverse cascade because it does show that small amount of energy goes to larger scales. However, in order to avoid confusions, we will use the term "small amount of inverse cascade" whenever possible.

We compare our results with the spectral behavior of the decaying three-dimensional MHD turbulence in the right panel of Figure 7. At \( t = 0 \), velocity field has a flat spectrum between \( k = 16 \) and \( k = 32 \) (solid curve). The magnetic field at \( t = 0 \) has only the uniform component, the strength of which is set to 1. The numerical resolution is also 256\(^3\) and this run is not listed in Table 1. At later times, the velocity field generates fluctuating magnetic field mainly between \( k = 16 \) and \( k = 32 \). We see that MHD turbulence also shows small amount of inverse cascade. Overall spectral behavior of decaying MHD turbulence is similar to that of decaying EMHD turbulence. However, there are also differences. For example, the gradual shift of the peak of the energy spectrum to larger scale is less pronounced in the MHD case. The slopes of the energy spectra on large scales are also different.

7. HIGH-ORDER STATISTICS AND BISPECTRUM OF EMHD

In this section, we present other statistical properties of EMHD turbulence. We use a data cube from Run E512D, which is the same data cube as we considered in Section 6.

7.1. High-Order Structure Functions

High-order structure functions are used for the study of intermittency, which refers to the non-uniform distribution of structures. The structure functions of order \( p \) for magnetic field is defined by

\[
SF_p(r) = \langle |B(x) - B(x + r)|^p \rangle_{\text{avg over } x}. \tag{41}
\]

Traditionally, researchers use high-order structure functions of velocity to probe dissipation structures of turbulence. In fully developed hydrodynamic turbulence, the (longitudinal) velocity structure functions \( SF_p = \langle (v(x + r) - v(x)) \cdot \hat{r} \rangle \) are expected to scale as \( r^{\zeta_p} \). One of the key issues in this field is the functional form of the scaling exponents \( \zeta_p \). There are several models for \( \zeta_p \). Roughly speaking, the dimensionality of the dissipation structures plays an important role.

Assuming one-dimensional worm-like dissipation structures, She & Leveque (1994) proposed the scaling relation

\[
\zeta_p^{\text{SL}} = p/9 + 2[1 - (2/3)^{p/3}] \tag{42}
\]

for incompressible hydrodynamic turbulence. On the other hand, assuming two-dimensional sheet-like dissipation structures,
Müller & Biskamp (2000) proposed the relation

\[ \xi_p^{MB} = \frac{p}{9} + 1 - (1/3)^{p/3} \]  

for incompressible MHD turbulence. There are other models for intermittency. But, in this paper, we only consider above mentioned two models because they are relevant to incompressible variables.

In this paper, we only consider structure functions in the perpendicular directions. That is, in our calculations, the vector \( r \) (see Equation (41)) is perpendicular to the direction of the local mean field, \( \mathbf{B}_\parallel \). The reason why we do not consider parallel direction is that structure functions are not suitable for revealing true scaling relation in the parallel directions. In Figure 8, we plot the relative scaling exponent \( \xi_p/\xi_3 \) for EMHD turbulence.\(^8\)

It is interesting that the two-point structure functions show a different scaling exponent compared with other multi-point structure functions. This result is not surprising because we already observed that the two-point second-order structure function shows a different behavior compared with other multi-point structure functions in the right panel of Figure 5. The scaling exponents based on the three-point, the four-point, and the five-point structure functions are consistent with the She-Leveque model.

7.2. Bispectrum

In astronomy, the bispectrum is a tool widely used in cosmology and gravitational wave studies (Fry 1998; Scoccimarro 2000; Liguori et al. 2006). Here, we briefly describe the definition of the bispectrum (see Burkhart et al. 2009). The bispectrum is closely related to the power spectrum. The Fourier transform of the second-order cumulant, i.e., the autocorrelation function, is the power spectrum while the Fourier transform of the third-order cumulant is known as the bispectrum. In a discrete system, the bispectrum is defined as:

\[ \text{BS}(\vec{k}_1, \vec{k}_2) = \sum_{\vec{k}_1=\text{const}} \sum_{\vec{k}_2=\text{const}} \hat{A}(\vec{k}_1) \cdot \hat{A}(\vec{k}_2) \cdot \hat{A}^*(\vec{k}_1 - \vec{k}_2), \]  

where \( k_1 \) and \( k_2 \) are the wavenumbers of two interacting waves and \( \hat{A}(\vec{k}) \) is the original discrete data with finite number of elements with \( \hat{A}^*(\vec{k}) \) representing the complex conjugate of \( \hat{A}(\vec{k}) \). As shown in Equation (44), the bispectrum is a complex quantity which will measure both phase and magnitude information between different wave modes.

Original formalism for bispectrum is suitable for scalar variables. Since we deal with magnetic field, which is a vector quantity, we simply take z-component of magnetic field. Note that \( B_\parallel = B_0 \hat{z} \). In Figure 9, we plot bispectrum of driven EMHD (left panel), decaying EMHD (middle panel), and driven MHD (right panel). For the driven ERMHD and MHD turbulence cases, we take the data after turbulence has reached statistically stationary state (\( t \sim 3 \) and \( t \sim 45 \), respectively). The decaying EMHD case was taken at \( t \sim 3 \) (see the inset of Figure 1). Both driven EMHD (left panel) and decaying EMHD (middle panel) cases show similar bispectrum. However, driven MHD (right panel) shows a substantially different bispectrum. All cases show that amplitudes of bispectra are highest when \( k_1 = k_2 \), which means a high correlation of modes at \( k_1 = k_2 \). The amplitudes of bispectra, hence strengths of nonlinear interactions, drop as we move away from the diagonal lines (i.e., the lines of \( k_1 = k_2 \)). The shape of isocountours depends on the nature of nonlinear interactions. The case of the standard MHD (right panel) shows wider isocountours than EMHD cases, which means nonlinear interactions in standard MHD turbulence drop more slowly as we move away from the \( k_1 = k_2 \) line. This demonstrates that nonlinear interactions in ERMHD and MHD cases are very different.

8. COMPARISON WITH THE SOLAR WIND TURBULENCE

In this subsection, we compare statistics of MHD/EMHD turbulence and turbulence in the solar wind. We consider quantities related to the PDFs of fluctuations in increments of the magnetic field strength \( B \)

\[ dB \equiv \langle (B(x + r) - B(x)) \rangle_x / \langle B(x) \rangle_x, \]  

where \( B = |B| \) is the strength of magnetic field and the angled brackets \( \langle \rangle_x \) denote average taken over \( x \). In this section, we are only concerned with characteristics of turbulence directly measured in space plasmas.

8.1. PDFs for \( dB \)

In the solar wind, the PDFs for increments of the magnetic field strength, \( dB \), is measured by

\[ dB(\tau) \equiv \langle (B(t + \tau) - B(t)) \rangle_t / \langle |B(t)| \rangle_t, \]  

where \( B(t) \) is the strength (or the averaged strength) of magnetic field at time \( t \) and \( \langle \rangle_t \) denotes average taken over time. The observed PDFs for \( dB \) on scales from 1 hr to 128 days are well described by the Tsallis distribution (Burlaga et al. 2007; Burlaga & Vinas 2005), which is given by

\[ y(x) = A \left[ 1 + (q - 1) \frac{x^2}{\mu^2} \right]^{1/(q-1)}, \]  

in hydrodynamic turbulence, the relative scaling exponents are expressed relative to \( \xi_3 \) since Kolmogorov’s four-fifth law is valid for the third-order longitudinal structure function: \( SF_3 \propto r \), where the constant of proportionality is \(-4/5\) times the energy injection rate. Therefore, it is natural to use \( \xi_3 \) for the relative scaling exponents. In MHD, there also exists a similar exact relation for a third-order structure function expressed in terms of Elsasser variables (Politano & Pouquet 1998). However, according to the exact correlation law obtained by Galtier (2008), there is no simple relation between \( SF_3 \) and \( r \) in EMHD and, therefore, the use of \( \xi_3 \) for the relative scaling exponents is not justified. Nevertheless, for the sake of comparison with existing models, we use \( \xi_3 \) for the Figure.
where $A$, $q$, and $w$ are constants. The function is proportional to a Gaussian function for small $x$,
\[ y(x) \approx A \left(1 - \frac{x^2}{w^2}\right) \approx A \exp\left(-\frac{x^2}{w^2}\right), \quad \text{as } x \to 0, \quad (48) \]
and a power law for large $x$,
\[ y(x) \propto x^{-2/(q-1)}, \quad \text{as } x \to \infty. \quad (49) \]

The shape of the function is determined by the values of $q$ and $w$. In the limit of $q \to 1$, the function reduces to a Gaussian distribution. Therefore, the value of $q$ is a measure of non-Gaussianity. The value of $w$ is related to the width of the distribution. Since $\tau > 1$ hr, the measured PDFs in Burlaga et al. (2007) and Burlaga & Vinas (2005) reflect fluctuations of MHD turbulence.

Figure 10 shows PDFs of MHD (top panel) and EMHD (middle and bottom panels) turbulence. The solid lines are PDFs of $dB$ in our numerical data and the dotted curves are fits of the PDFs to the Tsallis distribution. We use the Levenberg–Marquardt algorithm (Levenberg 1944; Marquardt 1963; see Press et al. 1992) for fitting. Each curve in the panels represents the PDF for a particular separation. As we move from the lowest curve in each panel, the separations in grid units are 2, 6, 10, 17, 29, 50, 88, respectively. Each curve is displaced vertically from the one below by a factor of 10. We set the value of each PDF at $dB = 0$ to 1 before displacement, for the sake of clarity.

The PDFs may depend on the direction of $r$ in Equation (45). That is, the PDFs for the parallel direction can be different from those for the perpendicular direction. Therefore, we calculate PDFs for the parallel and perpendicular directions separately. We try both local and global frames to define parallel and perpendicular directions. Overall, we consider four cases: the parallel direction in global frame ($r || B_0$), the parallel direction in local frame ($r || B_L$), the perpendicular direction in global frame ($r \perp B_0$), and the perpendicular direction in local frame ($r \perp B_L$).

Our calculations show that there is no big difference in the PDFs of the above mentioned four cases. General trend is that the PDF is close to a Gaussian function when separation is large (upper curves in each panel) and it deviates from a Gaussian function when the separation is small (lower curves in each panel). We note however that, for a given separation (for example, see the lowest curve in each panel, which corresponds to $r = 2$ grid units), the widths of the PDFs vary, depending on the direction of $r$.

We can confirm quantitatively the trend that the PDF is close to a Gaussian distribution when separation is large and it deviates from a Gaussian distribution when the separation is small. In Figure 11, we plot the values of $q$. The left panel is for EMHD turbulence (Run E512D) and the right panel for MHD turbulence (Run MHD512F). As we mentioned, the value of $q$ is close to 1 when the PDF is close to a Gaussian distribution. Indeed, when the separation is large, $q$ is very close to 1. The value of $q$ deviates from 1 as the separation gets smaller, which means that the PDF deviates from a Gaussian distribution. The overall trend is consistent with the observations of the MHD-scale fluctuations in the solar wind (see Burlaga & Vinas 2005; Burlaga et al. 2007). But, this should be taken as a very approximate statement, because the behavior of $q$ in the solar wind is very complicated.

In each panel of Figure 11, we plot the $q$ values of four different directions mentioned above. In general, all four cases show a similar trend, although the $q$ values for the parallel cases are slightly larger than those for the perpendicular cases.
In Figure 12, we plot the values of \( w \). The left panel is for EMHD turbulence (Run E512D) and the right panel for MHD turbulence (Run MHD512F). It is interesting that the values of \( w \) in EMHD cases show steeper dependence on separation than those of MHD cases. One should also note that the behavior of \( q \) as well as \( w \) for the parallel directions in EMHD cases may be dominated by large-scale fluctuations, because the energy spectrum \( E(k) \) is too steep (see Section 6.3 and Appendix B).

Figure 12 shows that the values of \( w \) for the perpendicular direction (dotted lines) are systematically larger than those for the parallel direction (solid lines). Figure 13 shows that \( w \) parameter indeed depends on the direction of \( r \). The curves in the left panel are the PDFs for different values of \( \theta \), which is the angle between the local mean magnetic field and the separation vector \( r \). The curves in top or bottom panel correspond to \( \theta = 0^\circ \) (lowermost curve), \( 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ \), and \( 90^\circ \) (uppermost curve), respectively. We can clearly see that the width of the PDF gets larger as the angle increases. Note that the \( w \) parameter of Tsallis distribution is related to the width of the PDF. The right panel quantitatively shows dependence of \( w \) on \( \theta \). The \( w \) parameter is smallest when \( \theta = 0^\circ \) (= parallel direction) and it is largest when \( \theta = 90^\circ \) (perpendicular direction).

8.2. PDFs for Wavelet Transform Coefficients

The wavelet transform is sometimes used for the study of the magnetic field fluctuation in the solar wind. For example, Alexandrova et al. (2008) used the Morlet wavelet transform

\[
W_i(\tau, t) = \sum_{j=0}^{N-1} B_i(t_j)\psi[(t_j - t)/\tau],
\]

where \( B_i(t_j) \) is the \( i \)th component of the magnetic field at \( t_j = t_0 + j \Delta t \), and \( \psi(u) \propto \exp(-u^2/2) \) is the Morlet wavelet. In this subsection, we briefly consider PDFs of the Morlet wavelet transform coefficient \( W_i \).

We perform the Morlet wavelet transform using our numerical data. We only consider the parallel (or \( z \) component of magnetic field in global frame. Note that, in our numerical calculations, \( \tau \) in Equation (50) is proportional to separation \( r \) and the global mean magnetic field \( \mathbf{B}_0 \) is parallel to \( z \)-direction.

Figure 14 shows that the Tsallis distributions roughly fit the PDFs of EMHD turbulence (Run E512D). The horizontal axis is \( W_i(\tau)/\langle W_i(\tau)^2 \rangle^{1/2} \). In general the PDFs for large separations (upper curves in each panel) are close to Gaussian distributions and those for small separations (lower curves in each panel) show departure from Gaussian ones. The PDFs for the parallel direction (lower panel) show stronger departure from Gaussian distributions, which is confirmed by the large values of \( q \) for small separations (not shown in this paper). In the right panel of the Figure, we show the flatness (or kurtosis) as a function of the scale, which is defined by

\[
\langle W_i^4 \rangle/\langle W_i^2 \rangle^2.
\]

We show both MHD and EMHD cases in the same plot. In the case of EMHD, we show the flatness of both \( dB_z \) and \( \mathbf{B}_0 \). In the case of MHD, we show the flatness of \( dB_z \) only. When the separation is large, the flatness is very close to the Gaussian value of 3. In general, it increases as the separation decreases, which is consistent with observations (Carbone et al. 2004; Alexandrova et al. 2008; see also Burlaga & Vinas 2005). Note that the flatness of \( W_i \) of EMHD turbulence in the parallel direction shows a strong dependence on the separation.
Figure 13. Angular dependence of the \( w \) parameter of Tsallis distribution. Left: the PDFs for \( dB \) and Tsallis fit. The curves in each panel correspond to \( \theta = 0^\circ \) (lowermost curve), \( 15^\circ \), \( 30^\circ \), \( 45^\circ \), \( 60^\circ \), \( 75^\circ \), and \( 90^\circ \) (uppermost curve), respectively. \( \theta \) is the angle between local mean magnetic field and the direction of \( r \). Right: dependence of \( w \) on \( \theta \). The horizontal axis is \( \theta \) in degrees. Runs E512D (EMHD) and MHD512F (MHD) are used. All calculations are done in local frame.

Figure 14. PDF and flatness (kurtosis) of the Morlet wavelet transformation coefficient \( W_z \), where \( z \) is parallel to the global mean field \( B_0 \). Left: the Tsallis distribution approximately fits the PDFs. In each panel, curves correspond to \( r = 2 \) (lowermost curve), 6, 10, 17, 29, 50, 88 (uppermost curve), respectively. Right: flatness is larger for smaller separation. When the separations are large, the measured values of the flatness are very close to 3, which is the same as the flatness of the normal distribution.

9. DISCUSSIONS

9.1. Anisotropy of EMHD

Anisotropy of EMHD is impossible to measure using the traditional second-order structure function. In CL04, we proposed to a different technique which allowed us to actually get \( k_{||} \sim k_{\perp}^{1/3} \), which confirmed our theoretical prediction.

However, as the problem of measuring anisotropy is important, in the present paper we provided not only higher resolution simulations, but also a few new techniques of measuring anisotropy. All these techniques provided results consistent with our earlier finding.

Unlike CL04, in this paper we did not limit ourselves by the simulations of the decaying turbulence, but also studied driven turbulence. We did not observe differences in the slope or spectrum between the two cases.

9.2. EMHD and Kinetic Alfven Wave Turbulence

As we have shown in Section 2, there are two approaches to describing the turbulence over scales less than the proton gyroscale. The traditional approach assumes that the term proportional to \( \nabla \cdot \mathbf{v} \) (see Equation (14)) is negligible, while the approach in Schekochihin et al. (2009) insists on keeping this term. We performed numerical simulations with and without the term and obtained virtually identical results for the turbulence spectrum and anisotropy. This result shows importance of EMHD to analyze collisionless plasmas.

CL04 derived the anisotropy of the EMHD turbulence and confirmed the obtained scaling numerically. Recently, Howes et al. (2008b) used a gyrokinetic code and obtained the expected \( k^{-7/3} \) energy spectrum for magnetic fluctuations below the proton gyroscale (see also discussions in Matthaeus et al. 2008 and Howes et al. 2008a). Anisotropy has not been studied with a gyrokinetic code yet.

9.3. MHD and EMHD Turbulence

Within the paper, we have employed a set of different statistical measures to characterize EMHD turbulence. We compared the results with those obtained for MHD turbulence. We found both similarities and differences between the two types of turbulence.

MHD turbulence and EMHD turbulence have different energy spectra and anisotropy in spite of the fact that their scaling relations are derived from the same principles: constancy of energy cascade and critical balance. Our confirmation of the anisotropy scaling predicted in CL04 testifies that the importance of the critical balance goes beyond MHD.

Surprisingly, the high-order statistics shows a similar scaling: scaling exponents of both MHD turbulence (see Cho et al. 2002) and EMHD turbulence are compatible with those of the She–Leveque scaling.\(^9\) Does it mean that the She–Leveque scaling is so universal that it fails to distinguish different

\(^9\) Note however that we considered turbulence in strongly magnetized medium and that the SFs we measured are for directions perpendicular to the local mean field \( B_L \).
types of turbulence? What is the underlying reason for this universality? These are still open questions to be addressed by the future research. Interestingly enough, the PDFs for increments of magnetic field are well described by the Tsallis function. The correspondence between the space measurements and Tsallis description can be found, for example, in Burlaga & Vinas (2005), Leubner & Vörös (2005), and Burlaga et al. (2007). Now we confirmed the correspondence with direct numerical simulations. Another statistics, namely, the bispectrum is very different for MHD and EMHD. The bispectrum measures nonlinear interactions. The difference observed confirms the difference in cascading of the two types of turbulence.

9.4. Inverse Cascade in EMHD

Our simulations in Section 6.5 show the existence of small amount of inverse cascade in decaying EMHD turbulence. Although we have not performed simulations with driven turbulence, we expect that small amount of inverse cascade is an intrinsic part of the EMHD cascade. That is, we expect that generation of small amount of larger scale, hence more coherent, magnetic field from small-scale turbulence is an intrinsic part of the EMHD cascade. This kind of inverse cascade has important implications in the three-dimensional MHD case. When there exists turbulent velocity cascade, this kind of inverse cascade enables generation of coherent field on the outer scale of turbulence from a small-velocity cascade, this kind of inverse cascade enables generation of coherent field on the outer scale of turbulence from a small-velocity cascade, this kind of inverse cascade enables generation of coherent field on the outer scale of turbulence from a small-velocity cascade, this kind of inverse cascade enables generation of coherent field on the outer scale of turbulence from a small-velocity cascade, this kind of inverse cascade enables generation of coherent field on the outer scale of turbulence from a small-velocity cascade, this kind of inverse cascade enables generation of coherent field on the outer scale of turbulence from a small-velocity cascade, this kind of inverse cascade enables generation of coherent field on the outer scale of turbulence from a small-velocity cascade, this kind of inverse cascade enables generation of coherent field on the outer scale of turbulence from a small-velocity cascade, this kind of inverse cascade enables generation of coherent field on the outer scale of turbulence from a small-velocity cascade, this kind of inverse cascade enables generation of coherent field on the outer scale of turbulence from a small-velocity cascade, this kind of inverse cascade enables generation of coherent field on the outer scale of turbulence from a small-velocity cascade, this kind of inverse cascade enables generation of coherent field on the outer scale of turbulence from a small-velocity cascade, this kind of inverse cascade enables generation of coherent field on the outer scale of turbulence from a small-velocity cascade. This kind of inverse cascade has important implications in the three-dimensional MHD case. When there exists turbulent velocity cascade, this kind of inverse cascade enables generation of coherent field on the outer scale of turbulence from a small-scale seed magnetic field (see Cho et al. 2009). Note, however, that the existence of turbulent velocity cascade that can amplify the magnetic field is assumed here. In EMHD, on the contrary, the fluctuations of magnetic field and velocity are connected with each other from the very beginning. Nevertheless, one can speculate that the small amount of inverse cascade in EMHD turbulence can help to create large-scale magnetic structures when the driving is at small scales. An interesting implication of this kind of inverse EMHD cascade could be its contribution to the creation of the dipole field, when magnetic field is generated by the thermomagnetic instability in the crust of a neutron star.

9.5. Comparison with Observations

Space plasma measurements allow for a direct comparison with the results of turbulence simulations. This comparison is essential, as numerically turbulence can be studied only for relatively small Reynolds and Lundquist numbers. As a result, validation of the obtained scaling in realistic astrophysical environments is absolutely essential (see an extended discussion of the issue in Lazarian et al. 2009).

It is encouraging that the spectra and the PDFs of the increments of magnetic field strength obtained numerically show correspondence with observations. We believe that more detailed comparisons are necessary.

9.6. EMHD Turbulence Anisotropy and the ADAF Model

The issue of why accretion disks around black holes are not as luminous as one would expect is a burning question, addressing which is required for explaining the low luminosity of black hole environments in the centers of galaxies. One of the ideas proposed was that of advection-dominated accretion flows (ADAFs; Narayan & Yi 1995; Narayan et al. 1998). According to this idea the low luminosity of the accreting material is due to the low rate of the transfer of energy of the turbulent accreting flow to electrons. It is postulated in the model that protons carry the lion’s share of the flow energy and thus the emission is suppressed. Is it so?

Quataert & Gruzinov (1999) discussed transition from standard Alfvénic MHD turbulence to EMHD turbulence in ADAFs. As Alfvénic turbulence reaches the proton gyroradius scale, some of the turbulence energy goes to protons through collisionless damping. The major heating mechanism for protons right above the proton gyroscale is the transit time damping (TTD) caused by non-zero parallel magnetic field fluctuations. Heating of protons is sensitive to $\beta_i \approx (v_i/v_A)^2$, where $v_A$ is the Alfvén speed. When $\beta_i \gg 1$, more protons are available for efficient interaction with Alfvén waves. Therefore, most of the turbulence energy goes to protons before it reaches the proton gyroscale. However, when $\beta_i \sim 1$, heating of protons is marginal and most of the turbulence energy will cascade down further, crossing the proton gyroscale (see Quataert 1999; Quataert & Gruzinov 1999).

The Alfvén waves will be converted into EMHD waves (whistlers) below the proton gyroradius scale. When EMHD turbulence is anisotropic (i.e., $k_\parallel \ll k_\perp$), heating of protons by EMHD turbulence will be marginal because “protons sample a rapidly varying electromagnetic field in the course of a Larmor orbit (Quataert & Gruzinov 1999).” In this paper, we confirmed that anisotropy of EMHD turbulence ($k_\parallel \propto k_\perp^{1/2}$) is strong. Therefore, in this case, the remaining energy (that is, the energy that has survived the collisionless damping before the proton gyroradius scale) cannot heat protons. Instead, it will heat electrons when it reaches the electron gyroradius scale. The bottom line is that the strong anisotropy of EMHD turbulence will make heating of protons by EMHD turbulence rather difficult in collisionless astrophysical plasmas with $\beta_i \lesssim 1$ (see Quataert & Gruzinov 1999).

9.7. Other Implications

Other applications of the EMHD model include collisionless magnetic reconnection in laboratory and space plasmas (Bulanov et al. 1999; Biskamp et al. 1995; Avinash et al. 1998). Turbulence on microscale may be a source of anomalous resistivity, which can stabilize X-point reconnection. How important this type of reconnection is is a subject of debates. For instance, a model of magnetic reconnection in Lazarian & Vishniac (1999) appeals to the magnetic field weak stochasticity, rather than microphysical plasma effects to explain fast reconnection. Simulations by Kowal et al. (2009) successfully tested the Lazarian & Vishniac (1999) model, which may mean that in many astrophysically important cases the reconnection is fast irrespective of the plasma properties. The anomalous effects, however, may be important for the initiation of the reconnection when the original level of turbulence in the system is low. In addition, in a partially ionized gas where the field wandering, which is the key element of Lazarian & Vishniac (1999) model, is partially suppressed, the anomalous plasma effects can be important for reconnection (Lazarian et al. 2004). Incidentally, in the latter paper it is predicted that MHD turbulence is not killed by neutral friction in the partially ionized gas, but it gets resurrected at the scales at which neutrals and ions decouple. The resurrected cascade involves only ions and electrons, not neutrals, and may proceed as EMHD cascade below the proton gyroscale.

In addition, properties of EMHD turbulence are important for understanding of physics of neutron star crusts (Cumming et al. 2004; Harding & Lai 2006) and acceleration of particles in solar flares (Liu et al. 2006). While in most cases, we view the EMHD cascade as the continuation of the MHD cascade below
the proton gyroradius, for the crust of a neutral star the EMHD turbulence can be present on much larger scales. The main assumption of the EMHD is that one can ignore the motions of protons. This is definitely the case of the neutron star’s solid crust.

10. SUMMARY

We have found the following results.

1. EMHD and ERMHD show identical scaling relations (both spectra and anisotropies).
2. High-resolution EMHD simulation confirms $k^{-7/3}$ spectrum obtained by earlier studies (Biskamp et al. 1996, 1999; Ng et al. 2003; CL04). The spectrum of electric field is consistent with $k^{-1/3}$ spectrum obtained by earlier studies (see Schekochihin et al. 2009; Howes et al. 2008b; Dmitruk & Matthaeus 2006).
3. Our detailed study of anisotropy using different techniques supports $k_1 \propto k_2^{1/3}$ the EMHD scaling obtained by CL04.
4. Decaying EMHD turbulence and driven EMHD turbulence show the same scaling.
5. When we use three or larger number of points to define the structure functions, the scaling exponents of high-order structure functions follow a scaling similar to that of incompressible hydrodynamic turbulence.
6. Bispectrum of ERMHD turbulence, reflecting the coupling of different scales in the cascade, looks very different from that of standard MHD one.
7. The PDFs of the increment of the magnetic field strength in EMHD and MHD cases are well described by the Tsallis distribution. The general trend of the PDFs is consistent with observations of the solar wind.

We thank the anonymous referee for useful suggestions/comments. J.C.’s work was supported by the Korea Research Foundation grant funded by the Korean Government (KRF-2006-331-C00136) and by KICOS through the grant K20702020016-07E0200-01610 provided by MOST. A.L. acknowledges the support by the NSF grants ATM 0648699 and AST 0808118. Both authors are supported by the NSF Center for Magnetic Self-Organization in Laboratory and Astrophysical Plasmas.

APPENDIX A

DERIVATION OF ERMHD EQUATIONS

Schekochihin et al. (2009) first derived ERMHD from kinetic RMHD equations. They also gave derivation of ERMHD equations from the generalized Ohm’s law. Therefore, the starting point of ERMHD may also be the generalized Ohm’s law. Derivation of ERMHD is almost identical to the EMHD case.

Here, we briefly summarize the derivation of the ERMHD equation from the generalized Ohm’s law. Detailed derivation can be found in Appendix of Schekochihin et al. (2009). The magnetic induction equation reads

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v}_e \times \mathbf{B}),$$  \hspace{1cm} (A1)

where

$$\mathbf{v}_e = \mathbf{v}_i - \frac{1}{e n_e} = \mathbf{v}_i - \frac{c}{4\pi e n_e} \nabla \times \mathbf{B},$$  \hspace{1cm} (A2)

(see Appendix C of Schekochihin et al. 2009). Here, we ignored magnetic dissipation. Note that the usual EMHD equations are also based on these equations. Substituting Equation (A2) into Equation (A1), we get

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times [\mathbf{v}_i \times \mathbf{B} - \frac{c}{4\pi e n_e} (\nabla \times \mathbf{B}) \times \mathbf{B}],$$  \hspace{1cm} (A3)

$$= -\mathbf{B} \nabla \cdot \mathbf{v}_i - \frac{c}{4\pi e n_e} \nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}],$$  \hspace{1cm} (A4)

$$\approx -\mathbf{B} \nabla \cdot \mathbf{v}_i - \frac{c}{4\pi e n_e} \nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}],$$  \hspace{1cm} (A5)

which is the same as the EMHD equation except the $\mathbf{B} \nabla \cdot \mathbf{v}_i$ term on the right. Here, we assume $\mathbf{v}_i \approx 0$, but $\mathbf{v} \cdot \mathbf{v}_i = \mathbf{v} \cdot \mathbf{v} \neq 0$. We ignore the last term of Equation (A4) because it is $\sim O(\epsilon^2)$. According to the RMHD ordering (see Section 2 of Schekochihin et al. 2009), we have

$$\frac{\delta n_e}{n_e} \sim \epsilon \ll 1,$$  \hspace{1cm} (A7)

where $\delta n_e$ is the fluctuating density.

For $b_\parallel$ and $b_\perp$, we have

$$\frac{\partial b_\parallel}{\partial t} = -\frac{c}{4\pi e n_e} \nabla \cdot [\mathbf{B} \cdot \nabla b_\parallel],$$  \hspace{1cm} (A8)

$$\frac{\partial b_\perp}{\partial t} = -\mathbf{B}_0 \nabla \cdot \mathbf{v}_i - \frac{c}{4\pi e n_e} \nabla \cdot [\mathbf{B} \cdot \nabla b_\perp].$$  \hspace{1cm} (A9)

where we use $\nabla \times \approx \nabla \perp$ and $\mathbf{B} \nabla \cdot \mathbf{v}_i = \mathbf{B}_0 \nabla \cdot \mathbf{v}_i + O(\epsilon^2)$. Here, $\epsilon \sim k_1/k_\perp$.

From the continuity equation, we can rewrite the $\mathbf{B}_0 \nabla \cdot \mathbf{v}_i$ as

$$\mathbf{B}_0 \nabla \cdot \mathbf{v}_i = -\mathbf{B}_0 \left( \frac{\partial}{\partial t} + \mathbf{v}_\perp \cdot \nabla_\perp \right) \frac{2}{\beta_i (1 + Z/\tau)} \frac{b_\parallel}{B_0},$$  \hspace{1cm} (A10)

$$= \mathbf{B}_0 \left( \frac{\partial}{\partial t} + \mathbf{v}_\perp \cdot \nabla_\perp \right) \frac{2}{\beta_i (1 + Z/\tau)} \frac{b_\parallel}{B_0},$$  \hspace{1cm} (A11)

where we use $\mathbf{B}_0 = b_\parallel \hat{z}, \mathbf{v}_\perp \times b_\parallel \hat{z} = O(\epsilon^2) \approx 0$, and $b_\parallel/B_0 = -(\beta_i/2)(1 + Z/\tau)(\delta n_e/n_e)$, which is equivalent to

$$\frac{\beta_i}{\beta_i (1 + Z/\tau)} \frac{2}{\beta_i (1 + Z/\tau)} \frac{\partial b_\parallel}{\partial t} = \frac{2}{\beta_i (1 + Z/\tau)} \frac{\partial b_\parallel}{\partial t},$$  \hspace{1cm} (A12)
the pressure balance. Here, \( Z = q_i/e \) \((e = |q_e|)\) is the ion-to-electron charge ratio, \( \tau = T_i/T_e \) is the temperature ratio, \( n_e \) is the mean electron number density, \( \delta n_e \) is the fluctuating electron number density, \( \beta_i = 8\pi n_e k_B T_i/B_0^2 \) is the ion plasma beta. Therefore, Equation (A9) becomes

\[
\frac{\partial \mathbf{B}_i}{\partial t} = -\beta_i (1 + Z/\tau) \frac{e}{2 + \beta_i (1 + Z/\tau)} 4\pi \mathbf{V}_\perp \times [\mathbf{B} \cdot \nabla \mathbf{B}_i]. \tag{A14}
\]

In this paper, we use notations different from those in Schekochihin et al. (2009). The notations in this paper and in Schekochihin et al. (2009) are related by

\[
b \leftrightarrow \delta \mathbf{B}, \tag{A15}
\]

\[
b_1 \leftrightarrow \delta B_1, \tag{A16}
\]

\[
b_\perp \leftrightarrow \delta \mathbf{B}_\perp \tag{A17}
\]

\[
n_e \leftrightarrow n_{0e}. \tag{A18}
\]

APPENDIX B
TWO-POINT SECOND-ORDER STRUCTURE FUNCTION AND SPECTRUM

We can obtain the two-point second-order structure function of a variable \( A \) from \( E_A(k) \):

\[
\text{SF}_2(r) = \langle \| A(\mathbf{x} + \mathbf{r}) - A(\mathbf{x}) \|^2 \rangle, \tag{B1}
\]

\[
= \int d^3 \mathbf{x} \int d^3 \mathbf{k} \frac{\mathbf{k}}{e^{i \mathbf{k} \cdot \mathbf{r}} - 1} \times \int d^3 \mathbf{k} \frac{\mathbf{k}}{e^{i \mathbf{k} \cdot \mathbf{r}} - 1}, \tag{B2}
\]

\[
= \int d^3 \mathbf{k} \frac{\mathbf{k}}{e^{i \mathbf{k} \cdot \mathbf{r}} - e^{-i \mathbf{k} \cdot \mathbf{r}}}, \tag{B3}
\]

\[
= 2\pi \int dk \frac{k}{2} \left[ 1 - \sin kr \right], \tag{B4}
\]

\[
\propto 8\pi r^{m-1} \int_{kr}^{\infty} d(kr)(kr)^{-m} \left[ 1 - \sin \frac{kr}{kr} \right], \tag{B5}
\]

where we assume

\[
k^2 \frac{\mathbf{k}}{e^{i \mathbf{k} \cdot \mathbf{r}} - e^{-i \mathbf{k} \cdot \mathbf{r}}} \propto k^{-m} \tag{B7}
\]

for \( k_0 < k < \infty \). In the limit of \( kr \ll 1 \), the integrand is proportional to \( \sim (kr)^{2-m} \). Therefore, if \( m < 3 \), we can rewrite Equation (B6) as

\[
\text{SF}_2(r) \propto 8\pi r^{m-1} \int_{kr}^{\infty} d(kr)(kr)^{-m} \left[ 1 - \sin \frac{kr}{kr} \right] \propto r^{m-1}. \tag{B8}
\]

If \( m > 3 \), however, the integral in Equation (B6) is roughly proportional to \( (kr)^{-m} = (kr)^{-m} \) and we have

\[
\text{SF}_2(r) \propto r^{m-1} (kr)^{m} \propto r^2. \tag{B9}
\]

In summary, we have

\[
\text{SF}_2(r) \propto \begin{cases} r^{m-1} & \text{if } m < 3 \\ r^2 & \text{if } m > 3. \end{cases} \tag{B10}
\]

In case of Kolmogorov, \( m = 5/3 < 3 \) and we have \( \text{SF}_2(r) \propto r^{m-1} = r^{2/3} \). In case of EMHD, \( m \) for the perpendicular direction \( (m = 7/3) \) is still smaller than \( 3 \) and we have \( \text{SF}_2(r) \propto r^{m-1} = r^{4/3} \). However, that for the parallel direction is expected to be larger than \( 3 \) if turbulence is anisotropic. Therefore, the two-point second-order structure function for the parallel direction is not suitable for revealing the true scaling exponent and we will have \( \text{SF}_2(r) \propto r^2 \) for the parallel direction.

APPENDIX C
MULTI-POINT SECOND-ORDER STRUCTURE FUNCTION AND SPECTRUM

Falcon et al. (2007) and Lazarian & Pogosyan (2008) used the three-point second-order structure function that can work with a steeper \( E(k) \):

\[
\text{SF}_2(r) = \langle \| A(\mathbf{x} + \mathbf{r}) - 2A(\mathbf{x}) + A(\mathbf{x} - \mathbf{r}) \|^2 \rangle, \tag{C1}
\]

\[
= \int d^3 \mathbf{x} \int d^3 \mathbf{k} \mathbf{u}(k)e^{i\mathbf{k} \cdot \mathbf{x}}(e^{i\mathbf{k} \cdot \mathbf{r}} - 2 + e^{-i\mathbf{k} \cdot \mathbf{r}}) \times \int d^3 \mathbf{k} \mathbf{u}(k)e^{i\mathbf{k} \cdot \mathbf{x}}(e^{i\mathbf{k} \cdot \mathbf{r}} - 2 + e^{-i\mathbf{k} \cdot \mathbf{r}}), \tag{C2}
\]

\[
= \int d^3 \mathbf{k} \mathbf{u}(k)e^{i\mathbf{k} \cdot \mathbf{r}}(e^{i\mathbf{k} \cdot \mathbf{r}} - 2 + e^{-i\mathbf{k} \cdot \mathbf{r}}) \times \left[ e^{i\mathbf{k} \cdot \mathbf{r}} - 2 + e^{-i\mathbf{k} \cdot \mathbf{r}} \right], \tag{C3}
\]

\[
= 2\pi \int d^2 \mathbf{k} \frac{\mathbf{k}}{e^{i \mathbf{k} \cdot \mathbf{r}} - e^{-i \mathbf{k} \cdot \mathbf{r}}} \int d\theta \sin \theta \left[ e^{2i \mathbf{k} \cdot \mathbf{r} \cos \theta} + e^{-2i \mathbf{k} \cdot \mathbf{r} \cos \theta} + 6 - 4e^{i \mathbf{k} \cdot \mathbf{r} \cos \theta} - 4e^{-i \mathbf{k} \cdot \mathbf{r} \cos \theta} \right], \tag{C4}
\]

\[
= 8\pi \int d^2 \mathbf{k} \frac{\mathbf{k}}{e^{i \mathbf{k} \cdot \mathbf{r}} - e^{-i \mathbf{k} \cdot \mathbf{r}}} \left[ \frac{\sin 2kr}{2kr} + 3 - 4 \frac{\sin kr}{kr} \right], \tag{C5}
\]

\[
\propto 8\pi r^{m-1} \int_{kr}^{\infty} d(kr)(kr)^{-m} \left[ 4 - 4 \frac{\sin kr}{kr} - 1 - \sin \frac{2kr}{2kr} \right], \tag{C6}
\]

where we assume \( k^2 \frac{\mathbf{k}}{e^{i \mathbf{k} \cdot \mathbf{r}} - e^{-i \mathbf{k} \cdot \mathbf{r}}} \propto k^{-m} \) for \( k_0 < k < \infty \). In the limit of \( kr \ll 1 \), the integrand is proportional to \( \sim (kr)^{4-m} \). Therefore, the integral is roughly a constant (i.e., independent of \( r \)), if \( m < 5 \). If \( m > 5 \), however, the integral is roughly proportional to \( (kr)^{4-m} = (kr)^{4-m} \). Therefore, we have

\[
\text{SF}_2(r) \propto \begin{cases} r^{m-1} & \text{if } m < 5 \\ r^{m-1} (kr)^{5-m} \propto r^4 & \text{if } m > 5. \end{cases} \tag{C7}
\]
Similarly, we can construct a four-point second-order structure function:

\[ SF_2(r) = (|A(x + 3r) - 3A(x + r) + 3A(x - r) - 3A(x - 3r)|^2), \quad (C8) \]

which scales as

\[ SF_2(r) \propto \begin{cases} 
  r^{m-1} & \text{if } m < 7, \\
  r^{m-1}(k_0r)^{7-m} & \text{if } m > 7.
\end{cases} \quad (C10) \]

We can also construct a five-point second-order structure function:

\[ SF_2(r) = (|A(x+2r) - 4A(x+r) + 6A(x) - 4A(x-r) + A(x-2r)|^2), \quad (C11) \]

which scales as

\[ SF_2(r) \propto \begin{cases} 
  r^{m-1} & \text{if } m < 9, \\
  r^{m-1}(k_0r)^{9-m} & \text{if } m > 9.
\end{cases} \quad (C12) \]

REFERENCES

Alexandrova, O., Carbone, V., Veltri, P., & Sorriso-Valvo, L. 2008, ApJ, 674, 1153
Avinash, K., Bulanov, S. V., Esirkepov, T., Kow, P., Pegoraro, F., Sasorov, P. S., & Sen, A. 1998, Phys. Plasmas, 5, 2849
Bale, S. D., Kellogg, P. J., Mozer, F. S., Horbury, T. S., & Rene, H. 2005, Phys. Rev. Lett., 94, 215002
Barnes, A. 1966, Phys. Fluids, 9, 1483
Biskamp, D. 2003, Magnetohydrodynamic Turbulence (Cambridge: Cambridge Univ. Press)
Biskamp, D., Schwartz, E., & Celani, A. 1998, Phys. Rev. Lett., 81, 4855
Biskamp, D., Schwartz, E., & Drake, J. F. 1995, Phys. Rev. Lett., 75, 3850
Biskamp, D., Schwartz, E., & Drake, J. F. 1996, Phys. Rev. Lett., 76, 1264
Biskamp, D., Schwartz, E., Zeiler, A., Celani, A., & Drake, J. F. 1999, Phys. Plasmas, 6, 751
Bulanov, S. V., Pegoraro, F., & Sakharov, A. S. 1992, Phys. Fluids B, 4, 2499
Burkhart, B., Falceta-Gonçalves, D., Kowal, G., & Lazarian, A. 2009, ApJ, 693, 250
Burlaga, L. F., & Vilas, A.-F. 2005, J. Geophys. Res. (Space Phys.), 110, 7110
Burlaga, L. F., Vilas, A.-F., & Wang, C. 2007, J. Geophys. Res. (Space Phys.), 112, 7206
Carbone, V., Bruno, R., Sorriso-Valvo, L., & Lepreti, F. 2004, Planet. Space Sci., 52, 953
Cho, J., & Lazarian, A. 2002, Phys. Rev. Lett., 88, 245001
Cho, J., & Lazarian, A. 2003, MNRAS, 345, 325
Cho, J., & Lazarian, A. 2004, ApJ, 615, L41 (CL04)
Cho, J., Lazarian, A., & Vishniac, E. T. 2002, ApJ, 564, 291
Cho, J., & Vishniac, E. 2000, ApJ, 539, 273
Cho, J., Vishniac, E. T., Beresnyak, A., Lazarian, A., & Ryu, D. 2009, ApJ, 693, 1449
Cumming, A., Arras, P., & Zweibel, E. 2004, ApJ, 609, 999
Galtier, S., & Bhattacharjee, A., Germschewski, K., & Galtier, S., 2003, Phys. Plasmas, 10, 1954
Goldreich, P., & Sridhar, S. 1995, ApJ, 438, 761
Goldreich, P., & Reisenegger, A. 1992, ApJ, 395, 250
Goldreich, P., & Sridhar, S. 1995, ApJ, 438, 763
Harding, A. K., & Lai, D. 2000, Rep. Prog. Phys., 69, 2631
Howes, G. W., Cowley, S. C., Dorland, W., Hammett, G. W., Quataert, E., Schekochihin, A. A., & Tasutono, T. 2008a, Phys. Rev. Lett., 101, 145002
Howes, G. W., Dorland, W., Cowley, S. C., Hammett, G. W., & Quataert, E., Schekochihin, A. A., & Tasutono, T. 2008b, Phys. Rev. Lett., 101, 065004
Kong, A. S., Chukbar, K. V., & Yan’kov, V. V. 1990, in Reviews of Plasma Physics, Vol. 16, ed. B. B. Kadomtsev (New York: Consultants Bureau), 243
Kowal, G., Lazarian, A., Vishniac, E. T., & Otmanowska-Mazur, K. 2009, in press (arXiv:0903.2052)
Kulsrud, R., & Pearce, W. P. 1969, ApJ, 156, 445
Lazarian, A., Beresnyak, A., Yan, H., Opfer, M., & Liu, Y. 2009, Space Science Rev., 143, 387
Lazarian, A., & Pogosyan, D. 2008, ApJ, 686, 350
Lazarian, A., & Vishniac, E. T. 1999, ApJ, 517, 700
Lazarian, A., Vishniac, E. T., & Cho, J. 2004, ApJ, 603, 180
Leamon, R. J., Smith, C. W., Ness, N. F., Matthaeus, W. H., & Wong, H. K. 1998, J. Geophys. Res., 103, 4775
Leamon, R. J., Smith, C. W., Ness, N. F., & Wong, H. K. 1999, J. Geophys. Res., 104, 22331
Leubner, M. P., & Vörös, Z. 2005, ApJ, 618, 547
Levengiai, K. 1944, Q. Appl. Math., 2, 164
Liguori, M., Hansen, F. K., Komatsu, E., Marasce, S., & Riotta, A. 2006, Phys. Rev. D, 73, 043505
Lithwick, Y., & Goldreich, P. 2001, ApJ, 562, 279
Liu, S., Petrovian, V., & Mason, G. M. 2006, ApJ, 636, 462
Maron, J., & Goldreich, P. 2001, ApJ, 554, 1175
Marquardt, D. 1963, SIAM J. Appl. Math., 11, 431
Matthaeus, W. H., Servidio, S., & Dimitrakis, P. 2008, Phys. Rev. Lett., 101, 149501
Müller, W.-C., & Biskamp, D. 2000, Phys. Rev. Lett., 84, 475
Narayan, R., Mahadevan, R., Grindlay, J. E., Popham, R. G., & Gammie, C. 1998, ApJ, 492, 554
Narayan, R., & Yi, I. 1995, ApJ, 452, 710
Ng, C. S., Bhattacharjee, A., Germschewski, K., & Galtier, S., 2003, Phys. Plasmas, 10, 1954
Pappalardo, H., & Pouquet, A. 1998, Phys. Rev. E, 57, 21
Press, W. H., Treuold, S. A., Vetterling, W. T., & Flannery, B. P. 1997, Numerical Recipes in Fortran (2nd ed.; Cambridge: Cambridge Univ. Press)
Quataert, E., 1999, in ASP Conf. Ser. 161, High Energy Processes in Accreting Black Holes, ed. J. Poutanen & R. Svensson (San Francisco, CA: ASP), 404
Quataert, E., & Grosselin, A. 1999, ApJ, 520, 248
Saito, S., Gary, S. P., Li, H., & Narita, Y. 2008, Phys. Plasmas, 15, 102305
Schechkind, A. A., Cowley, S. C., Dorland, W., Hammett, G. W., Howes, G. G., Quataert, E., & Tasutono, T. 2009, ApJ, 182, 310
Scoccimarro, R. 2000, ApJ, 544, 597
Shakura, N., & Zank, G. P. 2005, Phys. Plasmas, 12, 122310
She, Z.-S., & Leveque, E. 1994, Phys. Rev. Lett., 72, 336
Shebalin, J. V., Matthaeus, W. H., & Montgomery, D. 1983, J. Plasma Phys., 29, 525
Smith, C. W., Hamilton, K., Vasquez, B. J., & Leamon, R. J. 2006, ApJ, 645, L85
Stawicki, O., Gary, S. P., & Li, H. 2001, J. Geophys. Res., 106, 8273
Wareing, C. J., & Hollerbach, R. 2009, Phys. Plasmas, 16, 042307