Scaling solutions to 6D gauged chiral supergravity

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Abstract. We construct explicitly time-dependent exact solutions to the field equations of six-dimensional (6D) gauged chiral supergravity, compactified to 4D in the presence of up to two three-branes situated within the extra dimensions. The solutions we find are scaling solutions, and are plausibly attractors which represent the late-time evolution of a broad class of initial conditions. By matching their near-brane boundary conditions to physical brane properties, we argue that these solutions (together with the known maximally symmetric solutions and a new class of non-Lorentz-invariant static solutions, which we also present here) describe the bulk geometry between a pair of three-branes with non-trivial on-brane equations of state.
1. Introduction

The discovery of the existence of D-branes [1] within string theory has led to a fundamental
rethinking of the kinds of effective theories which can describe the low-energy limit of a
fundamental theory. The broadening of the collective mind to which this rethinking has led
has permitted new progress to be made on a number of the ‘naturalness’ issues which seem
to plague our understanding of the low-energy theories—i.e. the Standard Model plus General
Relativity—which describe the world we see around us.

This progress has come about largely because the possibility of trapping low-energy
particles on surfaces within higher dimensional spacetime changes how one must think about
naturalness issues. For instance, an important property of four dimensions is the equivalence of
a vacuum energy with a cosmological constant, and so also with nonzero four-dimensional (4D)
curvature. This connection underlies the cosmological constant problem [2, 3], which amounts
to the difficulty in understanding why the observed universe can be so flat given that quantum
fluctuations generically make the vacuum energy enormously large.

This connection can be broken in higher dimensional brane configurations, inasmuch as
known higher dimensional solutions show that large 4D energy sources can co-exist with zero 4D
curvature, as has been pointed out for pure gravity with co-dimension one [4, 5] and co-dimension
two [6]–[8] geometries, as well as for co-dimension two geometries within supergravity [9]–[12]
(see also the appendix of [12]). They can co-exist in this way because within the extra-dimensional
context the 4D energy density turns out to source the curvature of the extra dimensions rather
than the curvature of the four dimensions within which the energy density exists.
Whether or not this observation can lead to a solution of the cosmological constant problem depends on identifying the assumptions which are required in order to ensure that the 4D curvature is sufficiently small, and on whether or not these choices are natural or if they must be fine tuned. The hope is that the reformulation of the cosmological constant problem in this way can usefully recast the issues into a form which might allow progress that was not possible in the 4D context. 6D supergravity provides a particularly attractive framework within which to test these tuning issues in detail, since in six dimensions it is possible to have internal geometries which are large enough to allow the Casimir energy in these dimensions to be of order the size of the observed cosmological constant, allowing a variety of attractive phenomenological possibilities [13]. In this paper, we use 6D chiral gauged supergravity to explore one of the fine-tuning issues which arise within this extra-dimensional context.

There are two different kinds of fine-tunings against which one must be vigilant within these extra-dimensional models. The first of these is the requirement to tune against quantum fluctuations. That is, if parameters are chosen to ensure that four dimensions are flat, do these dimensions remain flat once the scale of the theory is lowered by integrating out particles having successively lower masses? This is the traditional cosmological constant problem, and this paper has nothing to add to the ongoing investigations as to whether the higher dimensional models require tuning in this way [14]. Our focus here is instead on the second fine-tuning issue, which arises in any formulation (such as the higher dimensional scenarios of interest) for which the dark energy density is time-dependent. This second question asks whether having an acceptable present-day cosmology requires an inordinately finely tuned adjustment of initial conditions before the advent of the present epoch’s Big Bang cosmology.

In order to do this, we shall construct several nontrivial exact solutions of the Nishino–Sezgin 6D chiral gauged supergravity [15, 16]. These solutions will correspond, in general, to the introduction of non-trivial matter sources on the brane, but in specific cases they describe the response of the bulk geometry to pure tension branes whose tensions are not otherwise fine-tuned. In this sense, we are probing the cosmological constant problem in these 6D compactifications, by inferring the brane geometries induced, via the bulk geometry, from arbitrary brane tensions. From a cosmological point of view, it is also interesting to have explicit solutions for branes with arbitrary equations of state, and couplings to the bulk fields such as the dilaton. We take a pragmatic approach of first looking for bulk solutions, and then inferring the properties of the branes for which these would be consistent solutions.

The remainder of the paper is organized as follows: we close this section with a brief summary of the relevant field equations. Section 2 then describes the properties of a three-parameter class of static solutions having two compact dimensions. They include the two-parameter family of previously known solutions which are maximally symmetric in the four noncompact dimensions, as well as a new set of static solutions which break the 4D Lorentz symmetry. These solutions generically contain singularities which we can associate with the positions of two Source branes. Section 3 derives a broad class of time-dependent solutions describing geometries for which the sizes of the various dimensions scale with a power of time. A related class of solutions describing one brane within two noncompact dimensions is then described in section 4. In section 5, we argue that these solutions are rich enough to describe the late-time behaviour of a large class of Source branes. We provide the matching conditions

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5 For a survey of many of the higher-dimensional supergravities see, for example, [17].

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which allow the asymptotic behaviour of the bulk fields to be related to physical properties of these branes like energy density, pressure and dilaton coupling. We close in section 6 with a brief summary and conclusions.

1.1. Field equations

The bosonic part of the Lagrangian density for 6D chiral gauged supergravity [15]–[17] is given by

\[ \mathcal{L} = -\frac{1}{2\kappa^2} g^{MN} [R_{MN} + \partial_M \phi \partial_N \phi] - \frac{1}{4} e^{-\phi} F_{MN} F^{MN} - \frac{2g^2}{\kappa^4} e^\phi, \]  

(1.1)

where \( \phi \) is the 6D scalar dilaton and \( F = dA \) is the appropriate field strength for the gauge potential, \( A_M \), which gauges a specific Abelian \( R \)-symmetry, whose gauge coupling, \( g \), has dimensions of inverse mass. We keep the 6D Planck scale, \( \kappa^2 = M_6^{-4} \), explicit for ease of comparison with the various conventions which are used in the literature. These expressions set to zero some of the bosonic fields of 6D supergravity, including possible matter hyperscalars and gauge potentials, \( \Phi^a = A_M^a = 0 \), and Kalb–Ramond fields, \( G^{MNP} = 0 \)—a choice which is consistent with the corresponding field equations (for solutions with nonzero hyperscalars see, however, [20]).

This action leads to the following field equations:

\[ D_M (e^{-\phi} F^{MN}) = 0 \]  

(Maxwell),

\[ \square \phi + \frac{\kappa^2}{4} e^{-\phi} F_{MN} F^{MN} - \frac{2g^2}{\kappa^2} e^\phi = 0 \]  

(dilaton),

\[ R_{MN} + \partial_M \phi \partial_N \phi + \kappa^2 e^{-\phi} F_{MP} F^P_N + \frac{1}{2} (\square \phi) g_{MN} = 0 \]  

(Einstein),

(1.2)

to which the bulk of the remainder of this paper is dedicated to solving, in order to find explicit static and time-dependent compactifications to four dimensions. To this end, we divide the six coordinates \( x^M, M = 0, \ldots, 5 \), into 4D coordinates \( x^\mu, \mu = 0, \ldots, 3 \), and 2D coordinates \( x^m, m = 4, 5 \). When required, the three spatial coordinates of the noncompact four dimensions are denoted \( x^i, i = 1, 2, 3 \).

2. Static solutions

We start by describing a broad class of static compactifications. For some of these solutions the noncompact four dimensions are maximally symmetric, and these solutions have been described in the literature. We supplement these with new static solutions which break the 4D Lorentz symmetry.

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6 The curvature conventions used here are those of Weinberg’s book [18], and differ from those of Misner et al [19] only by an overall sign in the Riemann tensor.
2.1. Ansätze

We start by writing out the field equations for configurations which are (i) time-independent; (ii) translation and rotation invariant in the three noncompact spatial dimensions; and (iii) are axially symmetric in the extra dimensions. That is, we take

\[ ds^2 = -e^{2\hat{\sigma}(\eta)} \, dt^2 + e^{2\hat{\sigma}(\eta)} \, dx_i \, dx^i + e^{2\hat{\varphi}(\eta)} \, d\theta^2, \]
\[ A_\theta = \hat{a}_\theta(\eta) \quad \text{and} \quad \varphi = \hat{\varphi}(\eta), \]

leading to the following system of coupled ordinary differential equations:

\[
\begin{align*}
\hat{a}''_	heta + (\hat{w}' + 3\hat{a}' - \hat{v}' - \hat{b}' - \hat{\varphi}')\hat{a}_\theta &= 0 \quad \text{(Maxwell)} \\
\hat{\varphi}'' + (\hat{w}' + 3\hat{a}' - \hat{v}' + \hat{b}')\hat{\varphi}' + \frac{k^2}{2} e^{-2\hat{b} - \hat{\varphi}} (\hat{a}_\theta')^2 - \frac{2g^2}{k^2} e^{2\hat{\varphi}+\hat{\varphi}} &= 0 \quad \text{(dilaton)} \\
\hat{w}'' + (\hat{w}' + 3\hat{a}' - \hat{v}' + \hat{b}')\hat{w}' - \frac{k^2}{4} e^{-2\hat{b} - \hat{\varphi}} (\hat{a}_\theta')^2 + \frac{g^2}{k^2} e^{2\hat{\varphi}+\hat{\varphi}} &= 0 \quad \text{((i) Einstein)} \\
\hat{a}'' + (\hat{w}' + 3\hat{a}' - \hat{v}' + \hat{b}')\hat{a}' - \frac{k^2}{4} e^{-2\hat{b} - \hat{\varphi}} (\hat{a}_\theta')^2 + \frac{g^2}{k^2} e^{2\hat{\varphi}+\hat{\varphi}} &= 0 \quad \text{((ii) Einstein)} \\
\hat{b}'' + (\hat{w}' + 3\hat{a}' - \hat{v}' + \hat{b}')\hat{b}' + \frac{3k^2}{4} e^{-2\hat{b} - \hat{\varphi}} (\hat{a}_\theta')^2 + \frac{g^2}{k^2} e^{2\hat{\varphi}+\hat{\varphi}} &= 0 \quad \text{((θθ) Einstein)} \\
\hat{w}'' + 3\hat{a}'' + \hat{b}'' + (\hat{w}')^2 + 3(\hat{a}')^2 + (\hat{b}')^2 + (\hat{\varphi}')^2 - (\hat{w}' + 3\hat{a}' + \hat{b}')\hat{\varphi}' + \frac{3k^2}{4} e^{-2\hat{b} - \hat{\varphi}} (\hat{a}_\theta')^2 + \frac{g^2}{k^2} e^{2\hat{\varphi}+\hat{\varphi}} &= 0.
\end{align*}
\]

Here ' denotes a derivative with respect to \( \eta \). These equations must be supplemented with boundary conditions at the locations of the two branes.

Although this appears to provide six equations for the six unknown functions \( \hat{w}, \hat{v}, \hat{b}, \hat{\varphi} \) and \( \hat{a}_\theta \), this is deceptive because we can ensure that one of these functions (say \( \hat{\varphi} \)) takes any particular form simply by appropriately changing the coordinate \( \eta \). However, one combination of these five equations, found by taking the combination \((\eta\eta)-(tt)-(ii)-(θθ)\), can be thought of as a ‘constraint’ on the evolution of the fields into the \( \eta \) direction because all second derivatives, \((d/d\eta)^2\), drop out:\(^7\)

\[
\kappa^2 e^{-2\hat{b} - \hat{\varphi}} (\hat{a}_\theta')^2 = 6(\hat{a}')^2 + 6\hat{a}' \hat{w}' + 6\hat{a}' \hat{b}' + 2\hat{b}' \hat{w}' - (\hat{\varphi}')^2 + \frac{4g^2}{k^2} e^{2\hat{\varphi}+\hat{\varphi}}. \quad (2.3)
\]

As a consequence of the Bianchi identities, one of the remaining five equations is then not independent and can be derived from the derivative of the constraint and the other four equations.

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\(^7\) This equation follows directly from writing the \((\eta\eta)\) Einstein equation in terms of the Einstein tensor, \(G_{\eta\eta} + \kappa^2 T_{\eta\eta} = 0\), and it is the Bianchi identity which ensures that it holds for all \( \eta \) once it is imposed on ‘initial conditions’ at \( \eta = \eta_0 \). This is the Hamiltonian constraint associated with the lapse function exp \( \hat{\varphi} \) in the spacelike ADM formalism of gravity.
A considerable simplification to this system of equations can be obtained by performing the redefinitions

\[
\hat{w} = 3\xi + \frac{(y - x)}{4} \\
\hat{a} = -\xi + \frac{(y - x)}{4} \\
\hat{v} = \ln N + \frac{(5y - x + 2z)}{4} \\
\hat{b} = \frac{(3x + y + 2z)}{4} \\
\hat{\phi} = \frac{(x - y - 2z)}{2}.
\]

Then one can demonstrate that the system of equations (2.2) and the constraint (2.3) follow from the Euler–Lagrange equations associated with the action

\[
S = \int d\eta \left[ N^{-1}[\eta^2 - (y')^2 + (z')^2 + 12(\xi')^2 + \kappa^2 e^{-2x}(\hat{\alpha})^2] + N \left( \frac{4g^2}{\kappa^2 e^{2y}} \right) \right],
\]

here \( N \) plays the role of a Lagrange multiplier and can be set to unity after variation. This system can be solved exactly, giving (with \( N = 1 \)):

\[
\hat{a}_0 = q \int d\eta e^{2x} \\
e^{-x} = \frac{kq}{\lambda_1} \cosh[\lambda_1(\eta - \eta_1)] \\
e^{-y} = \frac{2g}{\kappa \lambda_2} \cosh[\lambda_2(\eta - \eta_2)] \\
\hat{z} = z_0 + \lambda_3 \eta \\
\hat{\xi} = \xi_0 + \lambda_4 \eta
\]

and where the constraint equation amounts to the condition \( \lambda_3^2 = \lambda_1^2 + \lambda_2^2 + 12 \lambda_4^2 \). We can set two of the parameters \( \eta_1, \eta_2, z_0, \xi_0 \) to zero by coordinate rescalings without loss of generality.

2.2. Lorentz-invariant solutions

A special case of these solutions is \( \lambda_4 = 0 \), for which \( \xi' = 0 \) and so \( \hat{a}' = \hat{w}' \). The resulting solutions are therefore Lorentz invariant, and reduce to those that have been found earlier in the literature [10, 12, 21, 22]. In this case, the general solutions can be written as

\[
ds^2 = \mathcal{W}^2(\eta) \eta_{\mu\nu} dx^\mu dx^\nu + A^2(\eta) [\mathcal{W}^8(\eta) d\eta^2 + d\theta^2] \\
F_{\eta\theta} = \left( \frac{qA^2}{\mathcal{W}^2} \right) e^{-\lambda_3 \eta} \text{ and } e^{-\phi} = \mathcal{W}^2 e^{\lambda_3 \eta},
\]

8 The conventions of [21] may be obtained from ours by taking \( R_{\mu\nu} \rightarrow -R_{\mu\nu}, \phi \rightarrow -\phi/2 \) and \( \kappa^2 = 1/2 \), while those of [10] differ from those here only by the choice \( \kappa^2 = 1 \).
where

\[ \mathcal{W}^4 = \left( \frac{k^2 q^2}{2g \lambda_1} \right) \frac{\cosh[2 \lambda_1 (\eta - \eta_1)]}{\cosh[\lambda_2 (\eta - \eta_2)]} \]

\[ A^{-4} = \left( \frac{2k^2 q^2 g \lambda_1}{\lambda_2} \right) e^{-2 \lambda_3 \eta} \cosh^3[2 \lambda_1 (\eta - \eta_1)] \cosh[\lambda_2 (\eta - \eta_2)] \] (2.8)

along with the constraint \( \lambda_3^2 = \lambda_1^2 + \lambda_2^2 \).

A further special case is obtained for \( \lambda_3 = \lambda_4 = 0 \), and \( \eta_1 = \eta_2 \) for which \( \hat{\varphi} = \hat{\varphi}_0 \) and \( \hat{w} = \hat{a} = \hat{w}_0 \) are constants. Changing variables to proper distance, \( d\rho = e^{\hat{\varphi}_0} d\eta \), the form field becomes

\[ F_{\rho \theta} = \hat{a}' = \pm \frac{2g}{k^2} e^{\hat{\varphi}_0} \hat{B}(\rho), \] (2.9)

where primes now indicate differentiation with respect to \( \rho \) and \( \hat{B}(\rho) = e^{\hat{b}(\rho)} \) satisfies

\[ \frac{\hat{B}''}{\hat{B}} = \hat{b}'' + \left( \frac{\hat{b}'}{k^2} \right)^2 = -\frac{4g^2}{k^2} e^{\hat{\varphi}}, \] (2.10)

with solution \( \hat{B}(\rho) = B_0 \sin(2g e^{\hat{\varphi}_0/2}\rho/k) \). These are the rugby-ball generalizations [9] of the older Salam–Sezgin solution [15] to gauged chiral 6D supergravity.

2.3. Asymptotic forms

These solutions describe geometries which become singular at \( \eta = \pm \infty \), which are interpreted as being the positions of the three branes which source this configuration. Since it is ultimately the internal structure of the brane (if any) which is responsible for resolving these singularities, one expects this structure to be related to the asymptotic limit of the above solutions in the near-brane limit. We therefore pause here to outline what this asymptotic near-brane behaviour is.

To this end it is useful to adopt Gaussian-normal (GN) coordinates near the brane for which \( ds^2 = \hat{g}_{ab} dx^a dx^b + d\rho^2 \), where \( x^a \) denotes the five other coordinates, \( \{ x^a \} = \{ x^\mu, \theta \} = \{ t, x^i, \theta \} \), with the brane position being described by \( \rho = 0 \). We take the asymptotic form of the bulk fields in the near-brane limit (\( \rho \gg \ell \)) to be generically given by a power law [22]

\[ ds^2 \sim -[c_w(H_1 \rho)^\omega]^2 dt^2 + [c_\alpha(H_1 \rho)^\alpha]^2 \delta_{ij} dx^i dx^j + d\rho^2 + [c_\phi(H_1 \rho)^{\beta-1}]^2 \rho^2 d\theta^2 \]

\[ e^\phi \sim c_\phi(H_1 \rho)^{\rho} \] and \( F^{\rho\theta} \sim c_f(H_1 \rho)^{\gamma} \),

(2.11)

where \( \omega, \alpha, \beta, \gamma, c_w, c_\alpha, c_\phi, c_\theta, c_\phi \) and \( c_f \) are constants, and \( H_1 \) is an arbitrary scale. With these choices the extrinsic curvature, \( K_{ab} = \frac{1}{2} \partial_a \hat{g}_{ab} \), of the constant-\( \rho \) surfaces becomes

\[ K_i^j \approx \frac{\omega}{\rho}, \quad K_i^i \approx \frac{\alpha}{\rho} \delta_i^j \] and \( K_\phi^\rho \approx \frac{\beta}{\rho} \),

(2.12)

up to contributions that are subleading for small \( \rho \).
Only two of the five powers $\alpha$, $\beta$, $\omega$, $\gamma$ and $p$ defined above are independent, since the bulk field equations impose the following two conditions among them
\[
\omega^2 + 3\alpha^2 + \beta^2 + p^2 = \omega + 3\alpha + \beta = 1 \quad \text{and} \quad \gamma = p - 1. \tag{2.13}
\]
For example, explicit calculation with the general static solutions given above gives
\[
\alpha_\pm = \frac{\lambda_2 - \lambda_1 \pm 4\lambda_4}{5\lambda_2 - \lambda_1 \mp 2\lambda_3}, \quad \omega_\pm = \frac{\lambda_2 - \lambda_1 \mp 12\lambda_4}{5\lambda_2 - \lambda_1 \mp 2\lambda_3}, \tag{2.14}
\]
\[
\beta_\pm = \frac{\lambda_2 + 3\lambda_1 \mp 2\lambda_3}{5\lambda_2 - \lambda_1 \mp 2\lambda_3}, \quad p_\pm = -\frac{2(\lambda_2 - \lambda_1 \mp 2\lambda_3)}{5\lambda_2 - \lambda_1 \mp 2\lambda_3}, \tag{2.15}
\]
at the brane positions $\eta \to \pm \infty$. As is easily verified, these satisfy the expressions (2.13) above.
In later sections, we relate these powers to the physical properties of the branes which source these geometries, following arguments which generalize those of [8, 12, 22].

### 3. Scaling solutions

We now generalize the previous discussion to a new class of time-dependent scaling configurations which provide exact solutions to the same 6D field equations. The idea behind the construction is to assume a metric of the general form
\[
ds^2 = -t^c N^2 \, dt^2 + q_{ij}(t^{c/2} \beta^i \, dt + t^{c/2 + 1} \, dx^i)(t^{c/2} \beta^j \, dt + t^{c/2 + 1} \, dx^j),
\tag{3.1}
\]
for arbitrary real $c$, where $N$ and $\beta^i$ represent the lapse and shift functions just as in the usual ADM decomposition and $q_{ij}$ is the metric in the $n$-dimensional hypersurface with normal vector $\beta^i$. These quantities are assumed to be independent of time: $N = N(x^i)$, $\beta^i = \beta^i(x^i)$ and $q_{ij} = q_{ij}(x^i)$. This is the most general metric, up to coordinate transformations, preserving a single time-like homothetic Killing vector that acts as $t \to \lambda t$, $g_{MN} \to \lambda^{2+c} g_{MN}$.

The key point to recognize is that with this choice the components of the Ricci tensor also scale as a simple power of $t$: $R_{tt} \propto t^{-2}$, $R_{ti} \propto t^{-1}$ and $R_{ij} \propto t^0$. Because of this, and of the scale-invariance of the supergravity equations, it is also possible to scale the other fields in the problem with $t$ in such a way as to ensure that the field equations are also proportional to specific powers of $t$. Once this is done, then the problem of solving the field equations becomes an exercise in determining the profiles of the metric functions in one lower dimension.

The most general form for the dilaton and the gauge field necessary for equation (3.1) to be a consistent solution of the equations of motion is determined by the scaling symmetry to be
\[
\phi = \tilde{\phi}(x^i) - (2 + c) \log t \tag{3.2}
\]
\[
A_\mu = \begin{pmatrix}
    t^{-1} A_{0}(x^i) \\
    A_{i}(x^i)
\end{pmatrix}.
\tag{3.3}
\]

On substitution of these forms into the 6D equations of motion, we find that the time dependence completely decouples as expected from the scaling symmetry, and we are left with an effective
5D theory for the profiles $q_{ij}, \beta_i, N, \tilde{\phi}, A_0, A_i$. In practice the resulting 5D system is formidable and so in what follows, we shall consider special cases of these scaling solutions. In particular, we concentrate on the case where the variables only depend on one of the spatial dimensions (namely the radial direction $\eta$). Furthermore, we will work in the special case where only one component of the gauge field does not vanish, in particular $A_0 = 0$. All these assumptions greatly simplify this effective theory.

3.1. Warped scaling solutions

From now on we shall only consider scaling solutions which are warped on a single extra dimension. We consider the special case where the metric is

$$
\text{d}s^2 = (H_0 t)^c [-e^{2w(\eta)} \text{d}t^2 + e^{2a(\eta)} \delta_{ij} \text{d}x^i \text{d}x^j] + (H_0 t)^{2+c}[e^{2v(\eta)} \text{d}\eta^2 + e^{2b(\eta)} \text{d}\theta^2],
$$

while

$$
A_\theta = a_\theta(\eta) \quad \text{and} \quad e^\phi = \frac{e^{w(\eta)}}{(H_0 t)^{2+c}}.
$$

Although this metric is not precisely of the form of equation (3.1), it can easily be shown to be equivalent by redefining $x^i = \tilde{t} \tilde{x}^i$. From now on, we shall work with the form (3.4) for convenience. Here, $H_0$ is a constant of dimension inverse time and the functions $w, a, v, b, a_\theta$ and $\phi$ are to be determined by solving the field equations. In some cases, it will be natural to take $H_0 < 0$ so that the direction of increasing time corresponds to $t \rightarrow 0^-$. By virtue of the way the Ricci tensor scales with $t$, with the above ansatz all of the field equations reduce to the following set of coupled ordinary differential equations which govern the $\eta$-dependence of the various undetermined functions. The Maxwell equation is

$$
a''_0 + (w' + 3a' - b' - v' - \psi) a'_0 = 0,
$$

while the dilaton equation similarly becomes

$$
\phi'' + (w' + 3a' - v' + b') \phi' + (2 + c)(2c + 1)e^{2(v-w)} H_0^2 + \frac{k^2}{2} e^{-2b-\psi} (a'_0)^2 - \frac{2k^2}{2} e^{2a+2v} = 0.
$$

The Ricci tensor for this class of metrics is easily computed and leads to the following components for the Einstein equations. The $(t\eta)$ component is

$$
(2c + 1)w' + 3a' + (2 + c) \phi' = 0,
$$

while the $(tt)$ equation is

$$
w'' + (w' + 3a' - v' + b') w' + \frac{k^2}{4} e^{-2b-\psi} (a'_0)^2 + \frac{k^2}{2} e^{2a+2v} - (c^2 + \frac{c}{2} + 2c + 4) e^{-2w+2v} H_0^2 = 0.
$$

The $(\theta\theta)$ equation is

$$
b'' + (w' + 3a' - v' + b') b' + \frac{3k^2}{4} e^{-2b-\psi} (a'_0)^2 + \frac{k^2}{2} e^{2a+2v} - \frac{1}{2} (c + 2)(2c + 1) e^{-2w+2v} H_0^2 = 0,
$$

(3.10)
The logic for solving these equations is to use equation (3.8) to solve for \((t\eta)\) and finally the \((\eta\eta)\) equation is

\[
\begin{align*}
\frac{3\kappa^2}{4} e^{-2b-\varphi}(a_\theta')^2 + \frac{g^2}{\kappa^2} e^{2v+\psi} - \frac{1}{2}(c+2)(2c+1) e^{-2w+2v} H_0^2 &= 0,
\end{align*}
\]

where \(\prime = d/d\eta\). Equation counting proceeds much as for the static solutions, although with an important difference. The difference is the inclusion of time dependence, which makes two of the components of the Bianchi identity nontrivial rather than one. This implies that in this case two of the field equations are not independent of the others, rather than just one. Related to this is the existence in this case of two constraint equations which do not involve \(d^2\) / \(d\eta^2\), which the Bianchi identities ensure are preserved when evolved using the field equations in the \(\eta\) direction. These constraints can be taken to be the \((t\eta)\) Einstein equation, equation (3.8), and the combination \((\eta\eta)-(tt)-(3(ii)-(\theta\theta)):\n
\[
(\varphi')^2 - 6(w' + a' + b')a' - 2b'w' + \kappa^2 e^{-2b-\varphi}(a_\theta')^2 - \frac{4g^2}{\kappa^2} e^{2v+\psi} + 4(c^2 + c + 1) e^{-2w+2v} H_0^2 = 0.
\]

The logic for solving these equations is to use equation (3.8) to solve for \(a\) and equation (3.13) to solve for \(a_\theta\). Once this is done, we may ignore the Maxwell and \((ii)\) Einstein equations as they are redundant. We are then left with three independent equations—i.e. the dilaton and the \((tt)\) and \((\theta\theta)\) Einstein equations—for evolving the remaining three independent functions—\(w\), \(b\) and \(\varphi\)—into the \(\eta\) direction.

As for the static case, the system of equations is simplified by making the choice

\[
\begin{align*}
w &= 3\xi + (y - x)/4 \\
a &= -\xi + (y - x)/4 \\
v &= \ln N + (5y - x + 2z)/4 \\
b &= (3x + y + 2z)/4 \\
\varphi &= (x - y - 2z)/2.
\end{align*}
\]

The \((t\eta)\) equation now amounts to the condition

\[
\xi(\eta) \equiv \frac{1}{4}(w(\eta) - a(\eta)) = \frac{(2 + c)}{6c} z(\eta),
\]

up to an irrelevant integration constant. The remaining equations of motion and \((\eta\eta)-(tt)-(3(ii)-(\theta\theta))\) follow from the Euler–Lagrange equations of the action

\[
S = \int d\eta \left\{ N^{-1} \left[ (x')^2 - (y')^2 + \frac{4(1 + c + c^2)}{3c} (z')^2 + \kappa^2 e^{-2z}(a_\theta')^2 \right] + N \left[ \frac{4g^2}{\kappa^2} e^{2y} - 4H_0^2 (1 + c + c^2) e^{2y-2z/c} \right] \right\},
\]

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where $N$ plays the role of a Lagrange multiplier and can be set to unity after variation. As before $a_0$ and $x$ decouple and can be explicitly integrated:

$$a_0 = q \int d\eta e^{2x}$$

$$e^{-x} = \frac{kq}{\lambda_1} \cosh[\lambda_1(\eta - \eta_1)],$$

but $y$ and $z$ do not. Their equations follow from the reduced action

$$S = \int d\eta \left\{ N^{-1} \left[ -(y')^2 + \frac{4(1 + c + c^2)}{3c^2} (z')^2 \right] + N \left[ -\frac{\lambda_1^2}{k^2} e^{2y} - 4H_0^2 (1 + c + c^2) e^{2y - 2z/c} \right] \right\}.$$ (3.18)

With the choice $N = 1$, the resulting field equations are

$$y'' + \frac{4g^2}{k^2} e^{2y} - 4H_0^2 (1 + c + c^2) e^{2y - 2z/c} = 0.$$ (3.19)

$$z'' - 3cH_0^2 e^{2y - 2z/c} = 0.$$ (3.20)

The asymptotic form of these solutions as $\eta \to \pm \infty$ is $y \to \lambda_2^\pm \eta$ and $z \to \lambda_3^\pm \eta$ and the constraint implies $(\lambda_2^\pm)^2 = \lambda_1^2 + 4(1 + c + c^2)(\lambda_3^\pm)^2/(3c^2)$. Although we have not found closed-form analytic solutions of these equations, they are straightforward to integrate numerically, with the result having qualitatively the same form as the de Sitter solutions considered in [22]. (This relation with the de Sitter solutions is not surprising since these solutions may be obtained as the special case $c = -2$ of the above scaling solutions.)

3.2. Useful special cases

There are several special cases of the previous solutions which are of particular interest.

3.2.1. Connection to 4D scaling solutions. If $c = -1$ then the metric and dilaton have the time dependence

$$ds^2 = \frac{1}{t} \hat{g}_{\mu\nu}(y) dx^\mu dx^\nu + t \hat{g}_{mn}(y) dy^m dy^n$$ and $e^\phi = \frac{e^\phi}{t},$ (3.21)

which implies in particular that $\sqrt{-\hat{g}} g^{\mu\nu}$ is independent of $t$.

Such a scaling solution has a simple interpretation in the limit where the extra dimensions are large enough to justify a description in terms of an appropriate low-energy 4D effective theory [23]. In the classical limit, this theory contains two massless modes, corresponding to the 4D metric and one combination of the dilaton, $\phi$, and radius, $r$, of the extra dimensions (for which $r^2 e^\phi$ is fixed) which parameterizes a flat direction of the 4D scalar potential. The above scaling solution describes a time-dependent scaling along this flat direction with a fixed metric in the 4D Einstein frame. (More general choices for $c$ also rescale the Einstein-frame 4D metric.)
Explicit scaling solutions to the field equations of the effective 4D theory describing these modes are known, many of which are attractor solutions to which a broad class of initial conditions are drawn [24]. The solutions found here show how to extend those of the effective 4D theory to see the profiles of the other nonzero KK modes. Because the 4D solutions are attractors for the 4D field equations, we might also expect that the same may be true for the higher dimensional solutions found here.

3.2.2. Pure tension branes. It is known that even for the special case of pure tension branes, maximally symmetric solutions to the 6D field equations only exist when the tensions of the two branes are adjusted relative to one another [10, 12, 21]. One might hope that the above scaling solutions might describe the late time behaviour of the solutions in the case that the brane tensions are not adjusted in the appropriate way.

We now show that a subset of the solutions found above can indeed describe this situation. In order to do so, we must identify when the asymptotic form of the solutions near the branes have the pure tension form where \( a \sim w \) i.e. \( \xi \to 0 \). Since the \((\eta t)\) implies \( \xi' = (2 + c)z'/\langle 6c \rangle \) there are only two circumstances for which \( \xi' \to 0 \) near a brane:

(i) \( z' \to 0 \), or (ii) \( c = -2 \).

(3.22)

3.2.3. Special case \( c = \infty \)

Case (i) of (3.22) corresponds to the special case where the geometry near the brane has a conical singularity, since this always requires \( z' \to 0 \) in the near-brane limit. While this is always possible for one brane, in general it is not possible for both branes since the equation of motion for \( z \) is

\[
 z'' - 3cH_0^2 e^{2y - 2z/c} = 0,
\]

and so \( |z''| > 0 \). There is however, one special case for which this can be achieved. Redefining \( H_0 = |c|^{-1} \hat{H}_0 \) and taking the limit \( c \to \infty \), we recognize that the originally coupled system for \( y, z \) decouples,

\[
y'' + \frac{4g^2}{k^2} e^{2y} - 4\hat{H}_0^2 e^{2y} = z'' = 0.
\]

(3.24)

For which the conical solution is

\[
e^{-y} = \lambda_1^{-1} \sqrt{\left(\frac{4g^2}{k^2} - 4\hat{H}_0^2\right)} \cosh[\lambda_1 (\eta - \eta_2)], \quad z = z_0.
\]

(3.25)

As before, we can choose \( z_0 = 0 \) without loss of generality. Although the metric, equation (3.4), appears to be singular in this limit, this is only a consequence of an inconvenient choice for the time coordinate. If we instead convert to ‘proper’ time, \( \tau \), defined by \( d\tau = t^{1/2} dt \), then \( \tau \propto t^{1 + c/2} \) and the metric of equation (3.4) has a smooth large-\( c \) limit:

\[
ds^2 = -e^{2w(\eta)} d\tau^2 + \tau^2 [e^{2u(\eta)} \delta_{ij} dx^i dx^j + e^{2v(\eta)} d\eta^2 + e^{2h(\eta)} d\theta^2].
\]

(3.26)
We now convert back to ‘conformal’ time, $t$, using $d\tau \propto \tau \, dt$, then $\tau \sim e^{\tilde{H}_0 t/2}$, and

$$d\tau = e^{2w(\eta)} \, dr^2 + e^{2a(\eta)} \, dx^i \, dx^j + e^{2\epsilon(\eta)} \, d\eta^2 + e^{2b(\eta)} \, d\theta^2,$$

(3.27)

As $\eta \to \pm \infty$, $e^{b(\eta) - v(\eta)} \sim (kq)^{-1} \sqrt{4g^2/\kappa^2 - 4\tilde{H}_0^2 e^{\pm\lambda_1(\eta_1 - \eta_2)}}$. Converting to proper radius $d\rho = e^{v(\eta)} \, d\eta$ and identifying with the conical deficit form $d\rho^2 + (1 - \delta)^2 \rho^2 \, d\theta^2$, we infer the deficit angles

$$\delta_{\pm} = 1 - \lambda_1 \sqrt{4g^2/\kappa^2 - 4\tilde{H}_0^2} \, e^{\pm\lambda_1(\eta_1 - \eta_2)}.$$  

(3.28)

It is clear that by choosing any two of $\lambda_1, \eta_1, \eta_2$ and $\tilde{H}_0$ appropriately, we can match this solution on to two conical branes of arbitrary tensions.

Since in conformal time, all the ‘scale factors’ are exponential, this remains true in the 4D Einstein frame. Such a solution arises when the effective 4D equation of state is $w = -1/3$, precisely at the transition from acceleration to deceleration. By itself this behaviour cannot be responsible for late time dark energy in the form of quintessence.

3.2.4. Special case $c = -2$. Option (ii) of (3.22) makes the choice $c = -2$, since in this case the $(t\eta)$ Einstein equation implies the strong statement that $\xi' = \frac{1}{4}(w - a) = 0$ everywhere throughout the bulk. For this special case, the 6D geometry is everywhere maximally symmetric in the noncompact four dimensions, taking the form

$$ds^2 = e^{2w(\eta)} \, ds_{dS^4}^2 + e^{2\epsilon(\eta)} \, d\eta^2 + e^{2b(\eta)} \, d\theta^2,$$

(3.29)

where $ds_{dS^4}$ is the 4D de Sitter metric. These are just the de Sitter geometries considered in [22]. Note that there are no de Sitter solutions where both branes are conical.

3.3. Asymptotic forms

We now re-examine the near-brane behaviour of these scaling solutions in order to connect their properties to those of the source branes. If we repeat the analysis of asymptotic forms given earlier for static solutions for the scaling solutions, with near-brane asymptotic form assumed to be given by equation (3.4), we find two changes relative to the static case. The simplest change is that the assumed time dependence implies that the coefficients $c_w, c_a$ etc., now depend explicitly on $t$, with

$$c_w^2(t) \propto c_a^2(t) \propto t^c, \quad c_\phi^2(t) \propto t^{2+e} \quad \text{and} \quad c_\theta^2(t) \propto t^{-2-e}.$$  

(3.30)

The second change is to do with the relationship among the powers $\alpha, \beta, \gamma, \omega$ and $p$ which is dictated by the bulk equations. For instance, in the static case these equations required the powers to be related to one another by the Kasner-like conditions $\omega + 3\alpha + \beta = \omega^2 + 3\alpha^2 + \beta^2 + p^2 = 1$. These conditions also apply for the scaling solutions, unchanged by $c$ and $\tilde{H}_0$ because the relevant terms in the field equations are subdominant in powers of $\rho$. However, in the scaling case there is also a new constraint, equation (3.8), coming from the $(t\eta)$ Einstein equation, which implies the following for the asymptotic powers:

$$(2c + 1)\omega + 3\alpha + (2 + c)p = 0.$$  

(3.31)
3.4. Generalized scaling solutions

The previous scaling solutions admit a straightforward generalization by allowing for the two compact dimensions to scale differently with time. The new ansatz is

$$ds^2 = (H_0 t)^4 \left[ -e^{2\nu(x)} dt^2 + e^{2\nu(x)} \delta_{ij} dx^i dx^j \right] + (H_0 t)^{2+2\epsilon} \left[ e^{2\nu(x)} d\eta^2 + (H_0 t)^{2s} e^{2h(x)} d\phi^2 \right],$$

with

$$A_\theta = (H_0 t)^4 a_\theta(\eta) \quad \text{and} \quad e^\phi = \frac{e^\nu(x)}{(H_0 t)^{2+s}}.$$

The technique for demonstrating that this is a solution follows as before. First, we can show that all the powers of $t$ drop out and the equations reduce to a system for the radial profiles alone. As before it is helpful to perform the redefinition (3.14) to the $x, y, z, \xi$ variables (we shall choose $N = 1$). In this case the $(\eta t)$ constraint implies

$$\xi' = \frac{1}{6(2c + s)} (2e^{-2s} \kappa^2 s a_\theta a'_\theta + 2sx' + (4 + 2c + s)z').$$

(3.34)

This can be solved for $\xi$ at least formally. In this case, we do not find a simple action for the system, but the remaining equations of motion can be expressed as

$$a''_\theta - 2a'_\theta x' - (1 + 2c)sH_0^2 e^{2y + 6z - 6s} a_\theta = 0,$$

$$x'' - e^{-6\epsilon - 2s} [e^{2x + 2y + z} H_0^2 s(1 + 2c + s) + e^{2s} H_0^2 \kappa^2 s^2 a_\theta^2 - e^{6s} \kappa^2 (a'_\theta)^2] = 0$$

(3.35)

$$y'' - \frac{1}{k^2} e^{-6\epsilon - 2x + 2y} \left[ -2e^{2x} [2e^{6s} g^2 - e^x H_0^2 \kappa^2 (2 + 2c^2 + s + s^2 + 2c(1 + s))] + e^x H_0^2 \kappa^4 s^2 a_\theta^2 \right] = 0,$$

$$z'' - \frac{1}{k^2} e^{-6\epsilon - 2x + 2y} H_0^2 [e^{2x} [6c - s(s - 3)] - k^2 s^2 a_\theta^2] = 0,$$

along with the second constraint

$$3(2c + s)^2 \left\{ (x')^2 + (z')^2 - (y')^2 + e^{-2s} \kappa^2 (a'_\theta)^2 - \frac{4g^2}{k^2} e^{2y} + e^{2y + 6z - 6s} H_0^2 [2(2 + 2c^2 + (2c + s)(1 + s))] + e^{-2s} \kappa^2 s^2 a_\theta^2 \right\} + [(4 + 2c + s)z' + 2sx' + e^{-2s} \kappa (a'_\theta)^2]^2 = 0.$$

(3.36)

In this case, it is necessary to resort to numerics to make further progress; nevertheless we have demonstrated that such a class of solutions exists.

3.4.1. Scaling symmetry. True to its name, the generalized scaling solution admits a scaling symmetry such that under $t \to \lambda t, \tilde{x} \to \lambda \tilde{x}$ and $\theta \to \lambda^{-2} \theta$, the metric transforms as $g_{MN} \to \lambda^{2+\epsilon} g_{MN}$. Infinitesimally this corresponds to the existence of a vector $V$

$$V = \frac{\partial}{\partial t} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - s\theta \frac{\partial}{\partial \theta}.$$

(3.37)
which satisfies the homothetic Killing vector condition \[25\]

\[ \mathcal{L}_V g_{MN} = -(D_M V_N + D_N V_M) = -a g_{MN}, \] (3.38)

for constant $a = 2 + c$. Spacetimes admitting such a vector are also known as self-similar. This vector is only globally well defined for $s = 0$ since for nonzero $s$ the interval $\theta \in [0, 2\pi]$ is not mapped on to itself. The dilaton and gauge field transform as eigenfunctions of this Killing vector

\[ \mathcal{L}_V e^\phi = -(2 + c) e^\phi, \quad \mathcal{L}_V A = -(2 + c) A. \] (3.39)

4. Noncompact conical solutions

In this section, we shall present some special time-dependent solutions that describe conical branes. Unlike the previous solutions, they describe a single brane in an uncompactified space. Nevertheless it is reasonable to expect that they could describe the near-brane geometry of a more general time-dependent two-brane compact solution which is not described by the metrics already given.

Our ansatz for the metric is motivated by the form of the scaling solutions

\[ ds^2 = -\tilde{a}^2(t) \tilde{W}^8(t) \tilde{N}^2(t) \, dt^2 + \tilde{W}^2(t) (e^{\tilde{\xi}(t)} \, d\tilde{x}^2 + e^{-3\tilde{\xi}(t)} \, dr^2) + r^2 \tilde{a}^2(t) \, d\theta^2, \] (4.1)

along with $\phi = \tilde{\varphi}(t)$ and gauge potential $A_\theta = r^2 Q(t)$. Satisfying the $(r, t)$ Einstein equation requires $Q(t) = \tilde{Q} = \text{constant}$ and $\tilde{\xi} = (3\tilde{y} - 2\tilde{z})/12 + \tilde{\xi}_0$. This is equivalent to saying $\tilde{W}^2 e^{-3\tilde{\xi}} = \tilde{a}^2 e^{-3\tilde{\xi}_0}$, and so we see that assuming we have normalized $\theta$ to lie on the interval $[0, 2\pi]$, then the metric describes a conical brane with deficit angle $2\pi(1 - e^{3\tilde{\xi}_0})$. It is useful to make the change of variables

\[ \ln \tilde{a} = (\tilde{x} - \tilde{y} + \tilde{z})/4, \] (4.2)

\[ \ln \tilde{W} = (2\tilde{x} + \tilde{y})/8, \] (4.3)

\[ \tilde{\varphi} = -(\tilde{x} + \tilde{y} + \tilde{z})/2. \] (4.4)

As usual the remaining equations of motion and constraint can be derived from the action

\[ S = \int dt \left[ \tilde{N}^{-1} (6\tilde{x}^2 - 3\tilde{y}^2 - 2\tilde{z}^2) + \tilde{N} \left( 24e^{2\tilde{\xi}} \frac{\tilde{y}^2}{\kappa^2} + 24e^{2\tilde{\xi} + 2\tilde{\varphi}} \kappa^2 q^2 \right) \right]. \] (4.5)

This action looks qualitatively similar to the earlier actions, except for the signs of the kinetic terms.

5. Brane and bulk dynamics

In this section, we relate the asymptotic form of the above solutions to brane properties, and use the result to argue that the solutions capture the late-time evolution of a pair of brane sources with nontrivial equations of state.
5.1. Matching to brane properties

It is possible to make a general statement of how brane properties dictate the asymptotic forms of the bulk fields in the near-brane limit, at least for branes for which gravity contributes negligibly to the total brane stress energy. This section makes this statement explicit for the 6D case, following arguments presented in [8, 12].

5.1.1. A charged aside. Before doing so, it is worth briefly pausing to develop some intuition from the analogous problem in electromagnetism. Consider for this purpose the electrostatic potential, $\varphi(r)$, generated by a collection of point charges, $Q_i$, situated at various positions, $r_i(t)$, within three spatial dimensions. In this case, we know that some features of the resulting field are governed purely by the properties of individual charges, while others depend on the overall configuration of all of the charges.

Typically the field very near the source charges depends purely on the properties of the nearby source, with the asymptotic behaviour having Coulomb form, $\varphi(r) \to Q_i / |r - r_i|$, as $|r - r_i| \to 0$. This form is ultimately dictated by Gauss’s law, which constrains the local electric field, $E = -\nabla \varphi$, in terms of the local charge distribution. On the other hand, whether or not a given charge configuration is static—i.e. whether $\dot{r}_i = 0$—is a type of question which cannot be purely determined using only near-charge properties, since it requires knowledge of the global positioning of all of the charges.

This same kind of distinction arises also in the case of interest here: the gravitational fields sourced by a collection of branes. Again some features of the bulk geometry near the brane are completely dictated by the physical properties of the brane, while others—most notably the time-dependence of the geometry—depend on the complete configuration of branes which are present. The next few sections identify which of the properties of the bulk fields are which.

5.1.2. Thick branes and effective currents. In this section, we use the bulk field equations to show that local brane properties determine the near-brane form for radial derivatives (in the GN coordinates of subsection 2.3) of the dilaton and gauge potential, $\partial_\rho \phi$ and $F_{\rho a}$, as well as various combinations of the metric $g_{ab}$ and its extrinsic curvature, $K_{ab} = \frac{1}{2} \partial_\rho R_{ab}$. Other quantities depend on the properties of all of the branes which source the solution, and so cannot be inferred purely from the local properties of nearby branes.

To establish these points in the appendix, we review the arguments of [8] (see also [12, 26]) and imagine regarding the brane source to be a ‘thick’ brane, which physically extends over a small proper distance, $0 < \rho < \ell$. Within this region, we understand that the microphysical brane structure modifies the bulk equations, equations (1.2), to include new sources which are present only for $\rho < \ell$ and whose presence acts to smooth out the interior geometry at $\rho = 0$. Use of the field equations allows one to relate some of the near-brane properties of the external bulk fields to particular averages of these sources over the thick branes.

If in particular these new sources depend only on the radial coordinate, involve only weak gravitational fields and the external bulk fields satisfy the asymptotic near-brane power-law behaviour of equations (2.11), then the constants in this asymptotic form are related to the brane’s energy density, $\varepsilon = -\ell'$, pressures in the three noncompact and $\theta$ directions, $p_i = \ell_i'$ and...
\[ p_\theta = t_\theta^0, \text{ dilaton 'charge',} \sigma, \text{ and Maxwell current, } j^\theta, \text{ by simple expressions:} \]

\[ \kappa^2 \varepsilon \approx 2\pi [1 - c_\theta (3\alpha + \beta)(H_1 \ell)^{\beta - 1}] = 2\pi [1 - c_\theta (1 - \omega)(H_1 \ell)^{\beta - 1}] \]

\[ \kappa^2 p_i \approx 2\pi [c_\theta (\omega + 2\alpha + \beta)(H_1 \ell)^{\beta - 1} - 1] \delta_i^j = 2\pi [c_\theta (1 - \alpha)(H_1 \ell)^{\beta - 1} - 1] \delta_i^j \]

\[ \kappa^2 t_\theta^0 \approx 2\pi c_\theta (\omega + 3\alpha)(H_1 \ell)^{\beta - 1} = 2\pi c_\theta (1 - \beta)(H_1 \ell)^{\beta - 1} \]

\[ \kappa^2 \sigma = 2\pi c_\theta p(H_1 \ell)^{\beta - 1} \text{ and } j^\theta = 2\pi c_\theta c_f (H_1 \ell)^{\beta + \beta - 2}. \]

These expressions identify which features of the near-brane bulk solutions are governed purely by the local properties of the brane, reproducing standard results in the case of a conical singularity with defect angle \(2\pi \delta\) in the special case \(\alpha = \omega = p = 0, \beta = 1\) and \(c_\theta = 1 - \delta\). All of the remaining near-brane bulk-field properties—including in particular the time-dependence of the 4D metric—cannot be similarly determined purely from local information, and so depend in detail on the properties of both of the source branes.

Notice that the above relations impose relations among the stress-energy components \(t_\theta^i\) once they are combined with the bulk field equations. In particular, the relations \(\omega + 3\alpha + \beta = 1\) and \(\omega^2 + 3\alpha^2 + \beta^2 + p^2 = 1\) imply the components of the stress energy tensor must satisfy

\[ t_a^i = t_i^j + \sum_j t_j^i + t_\theta^0 = \frac{8\pi}{\kappa^2} [C_\theta - 1], \]

\[ (t_i^j)^2 + \sum_i (t_i^j)^2 + (t_\theta^0)^2 + \sigma^2 = \frac{8\pi^2}{\kappa^4} [2(C_\theta - 1)^2 + C_\theta (1 - \beta)], \]

where \(C_\theta = c_\theta (H \ell)^{\beta - 1}\). For the simplest case of time-dependent scaling solutions discussed in subsection 3.1, these must be supplemented by the additional relation following from equation (3.31)

\[ (2c + 1)t_i^j + \sum_i t_i^j - (2 + c)\sigma = \frac{4\pi}{\kappa^2} (C_\theta - 1)(c + 2). \]

This equation is equivalent to the conservation of stress-energy on the brane. In general, in the presence of a dilaton coupling and current, the brane stress energy is not independently conserved but can couple with the bulk fields.

5.1.3. Stress energies for the scaling solutions. We are now in a position to calculate the brane stress energies that give rise to a specific scaling solution. For simplicity, we shall consider the solutions of subsection 3.1. Since the near-brane limit corresponds to \(\eta \to \pm \infty\), it is sufficient to consider the asymptotic form of the scaling solutions, for which the metric components behave as exponentials. After a simple coordinate transformation, we can put the metric in the Kasner form

\[ ds^2 = (H_0 t)^{2c_\pm} [dt^2 + f^\pm_a (H_1 r)^{2\alpha_\pm} \delta_{ij} dx^i dx^j] + (H_0 t)^{2c_\pm} [dr^2 + f^\pm_\theta (H_1 r)^{2\beta_\pm} d\theta^2]. \]
We note that it is important to transform to GN coordinates in order to make contact with the Kasner exponents are given by formulae similar to the static solutions of the brane at $\kappa$ and the dilaton and form-field are

$$
\beta_\pm = \frac{\pm \lambda_2^+ + 3\lambda_1 \mp 2\lambda_3^+}{\pm 5\lambda_2^+ - \lambda_1 \mp 2\lambda_3^+}, \quad p_\pm = -\frac{2(\pm \lambda_2^+ - \lambda_1 \mp 2\lambda_3^+)}{\pm 5\lambda_2^+ - \lambda_1 \mp 2\lambda_3^+},
$$

where the $\lambda$'s are defined in subsection 3.1. To directly compare with the boundary conditions, we must put the metric in GN coordinates. This can be achieved approximately with the transformation $r = (H_0 t)^{-1-c/2} \rho$, at the cost of introducing new $O(\rho^2)$ contributions to the $dr^2$ and $dt \ d\rho$ components of the metric. These new contributions are negligible compared to those listed in equation (5.4) in the $\rho \to 0$ limit provided only that $\omega < 1$, which is in practice not a restrictive assumption. The result is a metric which takes the form

$$
ds^2 = -f^\pm_\rho(H_0 t)^{-\omega_\pm(2+c)}(H_1 \rho)^{2\omega_\pm} \, dt^2 + f_\rho^\pm(H_0 t)^{-\omega_\pm(2+c)}(H_1 \rho)^{2\omega_\pm} \, d\rho^2 + d\delta_i^j \, dx^i \, dx^j,
$$

and the dilaton and form-field are

$$
e^\phi = f^\pm_\phi(H_0 t)^{-(2+c)(1+p_{\pm}/2)}(H_1 \rho)^{p_{\pm}},
$$

$$
F^{\rho \theta} = f^\pm_\rho(H_0 t)^{-(2+c)(1+p_{\pm}/2)}(H_1 \rho)^{p_{\pm}-1}.
$$

We note that it is important to transform to GN coordinates in order to make contact with the time-dependence of the brane geometry. In particular, while the coordinate position of the edge of the brane at $\rho = \ell$ is constant in GN coordinates, the $t$-dependence of the change of variables between $r$ and $\rho$ implies that it is instead located along a curve $C(r, t) = \ell$ in another coordinate system.

Finally, we can read of the time-dependent brane stress-energies, dilaton coupling and current,

$$
\kappa^2 \varepsilon^\pm = 2\pi[1 - (1 - \omega_{\pm})f_\rho^\pm(H_0 t)^{(1-\beta_{\pm})(1+c/2)}(H_1 \ell)^{\beta_{\pm}-1}],
$$

$$
\kappa^2 p_i^\pm = -2\pi[1 - (1 - \alpha_{\pm})f_\rho^\pm(H_0 t)^{(1-\beta_{\pm})(1+c/2)}(H_1 \ell)^{\beta_{\pm}-1}],
$$

$$
\kappa^2 p_{\rho}^\pm = 2\pi(1 - \beta_{\pm})f_\rho^\pm(H_0 t)^{(1-\beta_{\pm})(1+c/2)}(H_1 \ell)^{\beta_{\pm}-1},
$$

$$
\kappa^2 \sigma_{\pm} = 2\pi\rho_{\pm}f_\rho^\pm(H_0 t)^{(1-\beta_{\pm})(1+c/2)}(H_1 \ell)^{\beta_{\pm}-1},
$$

$$
\frac{df_\rho^\pm}{f_\rho^\pm} = 2\pi f_\phi^\pm f_\rho^\pm(H_0 t)^{-\omega_{\pm}(3+p_{\pm} - \beta_{\pm})(1+c/2)}(H_1 \ell)^{\beta_{\pm}+p_{\pm}}.
$$

In the special case of a conical singularity we have $\beta = 1$ and so $\alpha = \omega = p = 0$, giving:

$$
\kappa^2 \varepsilon = -\kappa^2 p_i = 2\pi[1 - f_\rho], \quad \kappa^2 p_{\rho} = \kappa^2 \sigma = 0, \quad \frac{df_\phi}{f_\phi} = 2\pi f_\phi f_\rho(H_0 t)^{-\omega_{\pm}(2+c)}(H_1 \ell).
$$
The validity of the expressions (5.10) rests on the weak-field assumption used in deriving the boundary conditions in the appendix. Taking them literally for all time, we would infer that the brane energy becomes negative at some point in the past or the future, something which is clearly unphysical. This arises because the effective deficit angle becomes negative. It seems likely that this is an artefact of our prescription for the boundary conditions, or that the bulk solutions are no longer valid in this regime. To resolve this it will be necessary to build smooth, defect-like models of the branes and analyse the relationship between the boundary conditions and the bulk solutions. We feel that, nevertheless, the scaling solutions we have found here will at least represent a consistent description of the bulk in one or more asymptotic regimes.

5.1.4. Delta-function sources and conical singularities. It is a common practice to represent the brane sources in terms of a point-like delta-function in the two extra dimensions, with an action of the form

\[ S_b = -\int d^4x \sqrt{-\gamma} f(\phi) = -\int d^6x \sqrt{-\gamma} f(\phi) \delta^2(x), \]

where \( \gamma_{\mu\nu} = g_{MN} \partial_{\mu} x^M \partial_{\nu} x^N \) denotes the induced metric on the brane. In this case, direct variation of the brane action with respect to bulk fields would lead to

\[ T_{MN}^{(b)} = t_{\mu\nu}^{(b)} \delta^M_{\mu} \delta^N_{\nu} \frac{\delta^2(x)}{\sqrt{g_2}}, \quad J_N^{(b)} = j_{\nu}^{(b)} \delta^N_{\nu} \frac{\delta^2(x)}{\sqrt{g_2}} \quad \text{and} \quad P^{(b)} = \sigma^{(b)} \frac{\delta^2(x)}{\sqrt{g_2}}, \]

with

\[ t_{\mu\nu}^{(b)} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_b}{\delta \gamma_{\mu\nu}} = -f(\phi) \gamma^{\mu\nu}, \quad \sigma^{(b)} = -\frac{1}{\sqrt{-\gamma}} \frac{\delta S_b}{\delta \phi} = f'(\phi), \]

and \( j_{\nu}^{(b)} = -\delta S_b / \delta A_\nu = 0. \)

However, it is important to realize that such an assumption requires that the bulk fields remain regular at the brane position, \( \rho \to 0 \), so that it makes sense to evaluate the bulk fields there. The above expressions show that this is not generic for branes having co-dimension \( \geq 2 \), since it requires the powers \( \alpha, \beta, \omega \) and \( p \) all to be non-negative. If the brane couples to the Maxwell field then the condition on \( p \) strengthens to \( p \geq 1 \), which is consistent with the bulk field equation

\[ \alpha^2 + \beta^2 + \omega^2 + p^2 = 1 \]

only if \( p = 1 \) and \( \alpha = \beta = \omega = 0 \). Otherwise the representation of the brane in terms of a \( \delta \)-function source can be overly restrictive. More generally one can instead describe the low-energy bulk dynamics by excising the brane positions from the bulk geometry, and describing the brane sources in terms of boundary dynamics on the resulting co-dimension one boundary of the excised region [27].

5.2. Parameter counting

Section 2 provides an eight-parameter family of static solutions, with independent parameters \( \lambda_2, \lambda_3, \lambda_4, q, \eta_1, \eta_2, \zeta_0 \) and \( \xi_0 \). \( \lambda_1 \) can be fixed by means of the constraint equation

\[ \lambda_1^2 = \lambda_2^2 + \lambda_3^2 + 12 \lambda_4^2. \]

Of these, two (say \( \eta_2 \) and \( \xi_0 \)) can be removed by appropriately choosing units in the noncompact four dimensions, since this can be used to set \( a \) and \( w \) to any convenient value at any one position. (Note that \( b(\eta_0) \) cannot be similarly set to vanish in this way without changing the
periodicity condition $\theta \simeq \theta + 2\pi$. We also do not use the classical scale invariance of the bulk field equations to remove $z_0$ by allowing $\phi$ to be set to any desired value at any one point, $\eta = \eta_0$. We do not do so because this scale invariance may be broken by the brane-bulk couplings, and so may not be consistent with the boundary conditions given in the previous section.) This leaves a total of $8 - 2 = 6$ physical parameters in the static solutions presented above.

A similar counting applies to the generalized scaling solutions, whose time dependence introduces three more parameters: $c$, $s$ and $H_0$. Integrating the remaining equations to obtain the field profiles as functions of $\eta$ may be expected to introduce only five more constants (rather than six, due to the additional Bianchi-identity related constraint relative to the static case) into the general scaling solution as well, leading us to expect there to be a total of $5 + 3 = 8$ parameters in these solutions.

We may now count the number of parameters, which should be expected of a general bulk configuration which is sourced by two branes. For the symmetries of interest, each brane is characterized by the five quantities $\epsilon, p_i, p_\theta, j_\theta$ and $\sigma$. With two source branes there are ten such parameters in total. To this should also be added the integer which measures the total magnetic flux through the extra dimensions, so long as we also include the topological constraint [10] which relates this integer to the magnetic currents on each brane. This leaves us with a total of ten independent parameters describing the physics which sources the bulk fields, one of which is quantized to be an integer. This represents two more parameters than our general time-dependent solutions have available to accommodate.

Scaling solutions are often attractor solutions towards which general time-dependent configurations tend after the passage of any initial transients. If this is also so for the 6D supergravity field equations, we would be led to the following attractive picture. It has long been known [10, 12, 21] that static (and maximally symmetric, but curved [22]) solutions can only exist if the properties of the two source branes are appropriately adjusted relative to one another. But it has been unknown what happens to the bulk geometry in the generic case where such adjustments are not made, although it has been suspected that these would produce time-dependent bulk configurations. Based on the above considerations, in the generic case, we expect that the bulk is indeed time-dependent, and in particular this time-dependence approaches one of the scaling solutions given here (once transients pass) at late times.

6. Conclusions

6D supergravity provides a fascinating laboratory for investigating the issues which underly the cosmological constant problem, largely because six dimensions is both simple enough to allow the construction of explicit solutions, yet rich enough to exhibit an interesting variety of properties. In particular, it provides the simplest setting within which a collection of positive-tension branes can combine to produce vanishing 4D curvature. This makes it a very fruitful arena in which to explore how natural are the choices which must be made in order to ensure acceptably flat 4D worlds.

Our main result in this paper is to provide a new class of static and time-dependent solutions to the full field equations of gauged chiral 6D supergravity. For both classes of solutions, there are four warped, noncompact dimensions (which need not be maximally symmetric), and two curved

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9 We note that $H_1$ is not an independent parameter.
compact dimensions which are taken to be axially symmetric. The time-dependent solutions are of the self-similar scaling type, and as such are plausibly attractors which robustly describe the late-time limit of the time-dependent evolution generated by fairly generic initial conditions.

The solutions we find diverge at (at most) two positions within the extra dimensions, which we interpret to be the positions of two space-filling three-branes. This interpretation is supported by examining the asymptotic behaviour near the branes, which has a relatively simple interpretation in terms of brane stress-energy and dilaton coupling, at least in the limit that the gravitational fields involved in the brane structure are very weak.

The picture which emerges of brane-bulk dynamics is this. Much like for a collection of static electric charges within a compact space, the fields produced by a generic set of sources is time dependent. It is nonetheless possible to achieve static solutions provided the properties of the two branes involved are appropriately adjusted relative to one another. In the supersymmetric case the resulting static solutions are also marginally stable (classically) (this conclusion can have qualifications due to subtleties concerning the boundary conditions [28]), with the marginal perturbations being along a flat direction whose presence is guaranteed by a classical scale invariance of the 6D supergravity equations. We believe our scaling solutions to be the natural endpoint for motions which start out along these flat directions.

What do these results imply for the 6D self-tuning mechanism for the cosmological constant? It must be emphasized that they do not at all address the central issue of technical naturalness—i.e. the question of whether the choices required for flat (or slowly varying) cosmologies are stable against renormalization. What they do address is the ancillary issue of initial conditions, which arises in any theory for which the dark energy density is evolving in time. They do so by identifying how special the solutions are which can give acceptable 4D cosmologies.

Although such acceptable solutions do exist [24], their existence requires both an acceptably shallow potential and logarithmic corrections to this potential (such as can be generated by Casimir energies in the two extra dimensions [14]). However, the slow roll due to potentials like these coming from a quantum origin can only be relevant if they are not dominated by the fast motion driven by larger classical forces, and the motion described by the generic solutions obtained here would be much too fast to provide an acceptable 4D cosmology. So any description of the dark energy in terms of 6D dynamics along these lines must presuppose initial conditions which exclude the generic motion found here, and so rely on the universe being prepared with branes whose properties are sufficiently well adjusted relative to one another to give close to static classical dynamics. One might hope that such special initial conditions might have their explanation in the same way as did the initial condition problems of the standard Hot Big Bang cosmology: that is, in terms of the (possibly inflationary) dynamics of still earlier epochs of the universe.

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Appendix. Brane bulk matching

In this appendix, we summarize the connection between the near-brane asymptotic behaviour of the bulk solutions and some of the physical properties of these branes, following the arguments made in [8, 21]. To this end, we choose GN coordinates as described in subsection 2.3.

Our goal is to use the bulk field equations to show that local brane properties determine the near-brane form of the radial derivatives of the bulk fields: $\partial_\rho \phi$, $F_{\rho a}$ and $K_{ab} = \frac{1}{2} \partial_\rho g_{ab}$. Other quantities, such as the time dependence of the near-brane metric, depend on the properties of global properties involving all of the branes which source the solution, and so cannot be inferred purely from the properties of the brane at $\rho = 0$.

To establish these points, we follow [8, 12, 26] and imagine regarding the brane source to be a ‘thick’ brane, which physically extends over a small proper distance, $0 < \rho < \ell$. Within this region we understand that the microphysical brane structure modifies the bulk equations, equations (1.2), to include new sources in such a way as to ensure that all quantities remain smooth as $\rho \to 0$. We quantify this smoothness by assuming that the resulting bulk fields obey the conditions $K_{\mu \nu} = O(\rho^2)$, $K_{\theta \theta} = 2\rho$, $F_{a \rho} = O(\rho)$ and $\partial_\rho \phi = O(\rho)$ as $\rho \to 0$. We define the new sources which are required in this way from the equations

$$G_{MN} + \kappa^2 T_{MN} = D_M F^{MN} - J^N = \Box \phi - \kappa^2 P = 0, \quad (A.1)$$

where $G_{MN}$ is the Einstein tensor. The source currents $J_N = J_{N}^{(B)} + J_{N}^{(b)}$ and $P = P^{(B)} + P^{(b)}$ include both the contributions of the bulk fields, ‘(B)’, as inferred from equations (1.2) plus new brane contributions, ‘(b)’, which vanish for $\rho > \ell$. The stress energy receives similar kinds of contributions, $T_{MN} = T_{MN}^{(B)} + T_{MN}^{(b)}$.

Now comes the main argument. Consider for simplicity a purely radial source profile within the thick brane, with

$$d^2 s = g_{\mu \nu}(\rho) \, dx^\mu \, dx^\nu + g_{mn}(\rho) \, dx^m \, dx^n = g_{\mu \nu}(\rho) \, dx^\mu \, dx^\nu + e^{2b(\rho)} \, d\theta^2 + d\rho^2, \quad (A.2)$$

and so on for the other bulk fields. The point of the definitions given above is that the field equations allow the sources to be written as total derivatives, allowing their averages over the thick-brane volume\(^{10}\) to give expressions depending only on the boundary values at $\rho = 0$ and $\rho = \ell$. This is simplest to see for the dilaton, for which

$$\kappa^2 \sigma \equiv \int_{\rho<\ell} \, d^2 x \sqrt{g_2} \kappa^2 P = \int_{\rho<\ell} \, d^2 x \sqrt{g_2} \Box \phi = \int_0^{2\pi} \, d\theta [\sqrt{g_2} s^M \partial_M \phi]_{\rho=\ell}^{\rho=0} = 2\pi \{e^\phi \partial_\rho \phi\}_{\rho=\ell}, \quad (A.3)$$

where $s^M$ is the outward-pointing unit normal to surfaces of fixed $\rho$ and the last equality uses the above mentioned requirement that $\partial_\rho \phi$ must vanish at $\rho = 0$. Equation (A.3) gives an explicit relation between $\partial_\rho \phi|_{\rho=\ell}$ and purely local brane properties.

\(^{10}\)Multipole generalizations can be similarly defined by appropriately weighting the integrands by functions of $\theta$. 

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A similar argument may be made for the current \( J^M \). Averaging over the thick brane volume gives

\[
j^a = \int_{\rho<\ell} \frac{d^2x}{\sqrt{g}} J^a = \int_{\rho<\ell} \frac{d^2x}{\sqrt{g}} D_M F^{Ma} = \int_0^{2\pi} d\theta [\sqrt{g} s M F^{Ma}]_{\rho=\ell}^{\rho=0} = 2\pi [e^b F^{i\alpha}]_{\rho=\ell},
\]

(A.4)

with the contribution from \( \rho = 0 \) again vanishing due to the condition that \( F^{\alpha \beta} \) must vanish there. We see from this how the near-brane limit of \( F^{\alpha \beta} \) relates to local brane properties.

A similar result for the energy density is more difficult to define in general, however progress is possible for purely radial profiles if the gravitational binding energy of the thick brane is negligible. In this case, averaged stress energy components can be defined by

\[
\kappa^2 t^a_b = \int_{\rho<\ell} \frac{d^2x}{\sqrt{g}} 2 T^a_b = - \int_{\rho<\ell} \frac{d^2x}{\sqrt{g}} G^a_b.
\]

(A.5)

It is useful at this point to specialize to the particular metric, \( ds^2 = -e^{2w(\rho)} \, dt^2 + e^{2a(\rho)} \, \delta_{ij} \, dx^i \, dx^j + e^{2b(\rho)} \, d\rho^2 + d\rho^2 \), for which \( \sqrt{g} = e^b \) and the nonvanishing components of the Einstein tensor are

\[
-G_i^i = 3a'' + b'' + 6(a')^2 + 3a'b' + (b')^2 \\
-G_j^j = [w'' + 2a'' + b'' + 2w'a' + 3(a')^2 + 2a'b' + (w')^2 + w'b' + (b')^2] \delta_j^j \\
-G_\theta^\theta = w'' + 3a'' + 3w'a' + (w')^2 + 6(a')^2,
\]

(A.6)

where \( ' = d/d\rho \). The main point now is this: although the above integrals depend on the detailed metric profiles, the result simplifies in the limit of weak gravitational fields for which the expression may be linearized in the derivatives \( w', a' \) and \( b' \), where \( e^{2b} = \rho^2 e^{2\beta} \). In this case, the integrals may be performed explicitly, leading to the expressions

\[
k^2 t_i^i \approx 2\pi [e^b (3a' + b')]_{\rho=0}^\ell \\
k^2 t_j^j \approx 2\pi [e^b (w' + 2a' + b')]_{\rho=0}^\ell \delta_j^j \\
k^2 t_\theta^\theta \approx 2\pi [e^b (w' + 3a')]_{\rho=0}^\ell.
\]

(A.7)

We now specialize the above expressions by evaluating them using the smooth limit at \( \rho = 0 \) and the asymptotic form of the bulk metrics near \( \rho = \ell \):

\[
\frac{ds^2}{\sqrt{g}} \approx -[c_w(H_1 \rho)^\omega] \, dt^2 + [c_a(H_1 \rho)^\alpha] \delta_{ij} \, dx^i \, dx^j + d\rho^2 + [c_\phi(H_1 \rho)^{\phi - 1}] \rho^2 \, d\theta^2 + \rho^2 \, F^{\alpha \beta} \approx c_f(H_1 \rho)^\gamma \quad \text{and} \quad e^\phi \approx c_\phi(H_1 \rho)^\phi.
\]

(A.8)

Here, \( H_1 \) is an arbitrary dimensionful scale, while \( \omega, \alpha, \beta, p, c_w, c_a, c_\phi, c_f \), and \( c_\phi \) are constants, for which the bulk field equations imply the conditions \( \omega + 3\alpha + \beta = \omega^2 + 3\alpha^2 + \beta^2 + p^2 = 1 \) and \( \gamma = p - 1 \). For this metric, the relevant derivatives are \( w'(\ell) = \omega/\ell, a'(\ell) = \alpha/\ell, b'(\ell) = \beta/\ell \) and \( \phi'(\ell) = p/\ell \), while the 2D volume element is \( \sqrt{g} = c_\phi(H_1 \rho)^\phi \).

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The relevant quantities at $\rho = 0$ are $e^b w'_0 = e^b a'_0 = 0$ and $e^b b'_0 = 1$, and so up to contributions that are sub-leading for small $\ell$, we find

$$\kappa^2 t_i^j \approx 2\pi [c_\theta (3\alpha + \beta)(H_1 \ell)^{\beta-1} - 1] = 2\pi [c_\theta (1 - \omega)(H_1 \ell)^{\beta-1} - 1]$$

$$\kappa^2 t_i^j \approx 2\pi [c_\theta (\omega + 2\alpha + \beta)(H_1 \ell)^{\beta-1} - 1]$$

$$\kappa^2 t_i^0 \approx 2\pi c_\theta (\omega + 3\alpha)(H_1 \ell)^{\beta-1} = 2\pi c_\theta (1 - \beta)(H_1 \ell)^{\beta-1}$$

$$\kappa^2 \sigma = 2\pi c_\theta p(H_1 \ell)^{\beta-1}$$

and $j^0 = 2\pi c_\theta c_\theta (H_1 \ell)^{\alpha+\beta-1}$.

These are the expressions which are used in the main body of the text. Notice that in the case of a purely conical singularity having defect angle $0 < \delta < 1$ in the bulk geometry, we have $\alpha = \omega = p = 0$ and $\beta = 1$ and $c_\theta = 1 - \delta$, in which case the above expressions simplify to the following results for the energy density, $\varepsilon$, 3D pressure, $p_i$, and off-brane pressure, $p_0$: 

$$\kappa^2 \varepsilon = -\kappa^2 t_i^j \approx 2\pi \delta, \quad \kappa^2 p_i = \kappa^2 t_i^j \approx -2\pi \delta$$

$$\kappa^2 \sigma = 2\pi c_\theta p(H_1 \ell)^{\beta-1}$$

and $j^0 = 2\pi c_\theta c_\theta (H_1 \ell)^{\alpha+\beta-1}$. ($A.9$)

$$\kappa^2 \varepsilon = -\kappa^2 t_i^j \approx 2\pi \delta, \quad \kappa^2 p_i = \kappa^2 t_i^j \approx -2\pi \delta \quad \text{and} \quad \kappa^2 p_0 = \kappa^2 t_i^0 \approx 0. \quad (A.10)$$

References

[1] Polchinski J 1996 TASI Lectures on D-branes Preprint hep-th/9611050
[2] Weinberg S 1989 Rev. Mod. Phys. 61 1
[3] Polchinski J 2006 The cosmological constant and the string landscape Preprint hep-th/0603249
[4] Arkani-Hamed N, Dimopoulos S, Kaloper N and Sundrum R 2000 A small cosmological constant from a large extra dimension Phys. Lett. B 480 193 (Preprint hep-th/0001197)
Kachru S, Schulz M B and Silverstein E 2000 Self-tuning flat domain walls in 5d gravity and string theory Phys. Rev. D 62 045021 (Preprint hep-th/0001206)
[5] Forste S, Lalak Z, Lavignac S and Nilles H P 2000 A comment on self-tuning and vanishing cosmological constant in the brane world Phys. Lett. B 481 360 (Preprint hep-th/0002164)
Forste S, Lalak Z, Lavignac S and Nilles H P 2000 J. High Energy Phys. JHEP09(2000)034 (Preprint hep-th/0006139)
[6] Chen J-W, Luty M A and Pontón E 2000 J. High Energy Phys. JHEP09(2000)012 (Preprint hep-th/0003067)
Carroll S M and Guica M M 2003 Sidestepping the cosmological constant with football-shaped extra dimensions Preprint hep-th/0302067
Navarro I 2003 Co-dimension two compactifications and the cosmological constant problem J. Cosmol. Astropart. Phys. JCAP09(2003)004 (Preprint hep-th/0302129)
[7] Navarro I 2003 Spheres, deficit angles and the cosmological constant Class. Quantum Grav. 20 3603 (Preprint hep-th/0305014)
Nilles H P, Papazoglou A and Tasinato G 2004 Selftuning and its footprints Nucl. Phys. B 677 405 (Preprint hep-th/0309042)
Bostock P, Gregory R, Navarro I and Santiago J 2004 Einstein gravity on the Co-dimension 2 brane? Phys. Rev. Lett. 92 221601 (Preprint hep-th/0311074)
Vinet J and Cline J M 2004 Can Co-dimension-two branes solve the cosmological constant problem? Phys. Rev. D 70 083514 (Preprint hep-th/0406141)

Graesser M L, Kile J E and Wang P 2004 Gravitational perturbations of a six dimensional self-tuning model Phys. Rev. D 70 024008 (Preprint hep-th/0403074)

Kofinas G 2005 On braneworld cosmologies from six dimensions, and absence thereof Preprint hep-th/0506035

Mukohyama S, Sendouda Y, Yoshiguchi H and Kinoshita S 2005 Warped flux compactification and brane geometry J. Cosmol. Astropart. Phys. JCAP07(2005)013 (Preprint hep-th/0506050)

[8] Navarro I and Santiago J 2005 Gravity on Co-dimension 2 brane worlds J. High Energy Phys. JHEP02(2005)007 (Preprint hep-th/0411250)

[9] Aghababaie Y, Burgess C P, Parameswaran S and Quevedo F 2004 Towards a naturally small cosmological constant from branes in 6D supergravity Nucl. Phys. B 680 389–414 (Preprint hep-th/0304256)

Burgess C P 2005 Towards a natural theory of dark energy: supersymmetric large extra dimensions AIP Conf. Proc. 743 417 (Preprint hep-th/0411140)

[10] Aghababaie Y, Burgess C P, Cline J M, Firouzjahi H, Parameswaran S, Quevedo F, Tasinato G and Zavala I 2003 Warped brane worlds in six dimensional supergravity J. High Energy Phys. JHEP09(2003)037 (Preprint hep-th/0308064)

[11] Burgess C P 2004 Supersymmetric large extra dimensions and the cosmological constant: an update Ann. Phys. 313 283–401 (Preprint hep-th/0402200)

Garriga J and Porrati M 2004 Football shaped extra dimensions and the absence of self-tuning J. High Energy Phys. JHEP08(2004)028 (Preprint hep-th/0406158)

[12] Burgess C P, Quevedo F, Tasinato G and Zavala I 2004 General axisymmetric solutions and self-tuning in 6D chiral gauged supergravity J. High Energy Phys. JHEP11(2004)069 (Preprint hep-th/0408109)

[13] Azuelos G, Beauchemin P H and Burgess C P 2005 Phenomenological constraints on extra-dimensional scalars J. Phys. G: Nucl. Part. Phys. 31 1 (Preprint hep-ph/0401125)

Burgess C P, Matias J and Quevedo F 2005 MSLED: a minimal supersymmetric large extra dimensions scenario Nucl. Phys. B 706 71 (Preprint hep-ph/0404135)

Beauchemin P H, Azuelos G and Burgess C P 2004 Dimensionless coupling of bulk scalars at the LHC J. Phys. G: Nucl. Part. Phys. 30 N17 (Preprint hep-ph/0407196)

Matias J and Burgess C P 2005 MSLED, neutrino oscillations and the cosmological constant J. High Energy Phys. JHEP09(2005)052 (Preprint hep-ph/0508156)

Callin P and Burgess C P 2005 Deviations from Newton’s law in supersymmetric large extra dimensions Preprint hep-ph/0511216

[14] Burgess C P and Hoover D 2005 UV sensitivity in supersymmetric large extra dimensions: the Ricci-flat case Preprint hep-th/0504004

Ghilencea D M, Hoover D, Burgess C P and Quevedo F 2005 Casimir energies for 6D supergravities compactified on T(2)/Z (N) with Wilson lines J. High Energy Phys. JHEP09(2005)050 (Preprint hep-th/0506164)

Hoover D and Burgess C P 2006 Ultraviolet sensitivity in higher dimensions J. High Energy Phys. JHEP01(2006)058 (Preprint hep-th/0507293)

Burgess C P, de Rham C, Hoover D and Tolley A J Brane loops and naturalness in 6D supergravity, in preparation

[15] Nishino H and Sezgin E 1984 Phys. Lett. B 144 187

Nishino H and Sezgin E 1986 The complete N = 2, D = 6 supergravity with matter and Yang-Mills couplings Nucl. Phys. B 278 353

Randjbar-Daemi S, Salam A, Sezgin E and Strathdee J 1985 Phys. Lett. B 151 351

Salam A and Sezgin E 1984 Chiral compactification on Minkowski ×S^2 of N = 2 Einstein-Maxwell supergravity in six-dimensions Phys. Lett. B 147 47

[16] Marcus N and Schwarz J H 1982 Phys. Lett. B 115 111

D’Auria R, Fre P and Regge R 1983 Consistent supergravity in six-dimensions without action invariance Phys. Lett. B 128 44

New Journal of Physics 8 (2006) 324 (http://www.njp.org/)
Tanii Y 1984 N = 8 supergravity in six-dimensions Phys. Lett. B 145 197
Romans L J 1986 The F(4) gauged supergravity in six-dimensions Nucl. Phys. B 269 691–711
[17] Salam A and Sezgin E (ed) 1989 Supergravities in Diverse Dimensions Vol I and II (Singapore: World Scientific)
[18] Weinberg S 1972 Gravitation and Cosmology (New York: Wiley)
[19] Misner C W, Thorne K P and Wheeler J A 1970 Gravitation (San Francisco: Freeman)
[20] Randjbar-Daemi S and Sezgin E 2004 Scalar potential and dyonic strings in 6d gauged supergravity Nucl. Phys. B 692 346 (Preprint hep-th/0402217)
Kehagias A 2004 A conical tear drop as a vacuum-energy drain for the solution of the cosmological constant problem Phys. Lett. B 600 133 (Preprint hep-th/0406025)
Randjbar-Daemi S and Rubakov V A 2004 4d-flat compactifications with brane vorticities J. High Energy Phys. JHEP10(2004)054 (Preprint hep-th/0407176)
Lee H M and Papazoglou A 2005 Brane solutions of a spherical sigma model in six dimensions Nucl. Phys. B 705 152 (Preprint hep-th/0407208)
Nair V P and Randjbar-Daemi S 2005 Nonsingular 4d-flat branes in six-dimensional supergravities J. High Energy Phys. JHEP03(2005)049 (Preprint hep-th/0408063)
Parameswaran S L, Tasinato G and Zavala I 2005 The 6D SuperSwirl Preprint hep-th/0509061
Lee H M and Ludeling C 2005 The general warped solution with conical branes in six-dimensional supergravity Preprint hep-th/0510026
Carter B M N, Nielsen A B and Wiltshire D L 2006 Hybrid brane worlds in the Salam-Sezgin model Preprint hep-th/0602086
[21] Gibbons G W, Guven R and Pope C N 2004 3-branes and uniqueness of the Salam-Sezgin vacuum Phys. Lett. B 595 498 (Preprint hep-th/0307238)
[22] Tolley A J, Burgess C P, Hoover D and Aghababaie Y 2005 Bulk singularities and the effective cosmological constant for higher co-dimension branes J. High Energy Phys. JHEP03(2006)091 (Preprint hep-th/0512218)
[23] Aghababaie Y, Burgess C P, Parameswaran S and Quevedo F 2003 Supersymmetry breaking and moduli stabilization from fluxes and six-dimensional supergravity J. High Energy Phys. JHEP03(2003)032 (Preprint hep-th/0212091)
[24] Albrecht A, Burgess C P, Ravndal F and Skordis C 2001 Natural quintessence and large extra dimensions Phys. Rev. D 65 123507 (Preprint astro-ph/0107573)
[25] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 Exact Solutions of Einstein’s Field Equations 2nd edn (Cambridge: Cambridge University Press)
[26] de Rham C and Tolley A J 2006 Gravitational waves in a codimension two braneworld J. Cosmol. Astropart. Phys. JCAP02(2006)003 (Preprint hep-th/0511138)
[27] Burgess C P, de Rham C, Hoover D and Tolley A J 2006 Effective boundary actions, in preparation
[28] Burgess C P, de Rham C, Hoover D and Tolley A J 2006 Kicking the rugby ball: perturbations of 6D gauged chiral supergravity, in preparation