Using the conservation laws for charge, energy, momentum, and angular momentum, we derive hydrodynamic equations for the charge density, local temperature, and fluid velocity, as well as for the polarization tensor, starting from local equilibrium distribution functions for particles and antiparticles with spin 1/2. The resulting set of differential equations extend the standard picture of perfect-fluid hydrodynamics with a conserved entropy current in a minimal way. This framework can be used in space-time analyses of the evolution of spin and polarization in various physical systems including high-energy nuclear collisions. We demonstrate that a stationary vortex, which exhibits vorticity-spin alignment, corresponds to a special solution of the spin-hydrodynamical equations.

1. Introduction: Non-central heavy-ion collisions create fireballs with large global angular momenta which may generate a spin polarization of the hot and dense matter in a way resembling the equilibrium magnetomechanical effects of Einstein and de Haas [1] and Barnett [2]. Consequently, much effort has recently been invested in studies of polarization and spin dynamics of particles produced in high-energy nuclear collisions, both from the experimental and theoretical point of view (for a recent review see [3]). In this context, theoretical studies have explored the role of the spin-orbit coupling [4–8] and global equilibrium of rotating bodies [9–13]. See also the kinetic models of spin dynamics [14–17] and the works on anomalous hydrodynamics [18, 19]. Recent work, based on the Lagrangian formulation of hydrodynamics, is reported in Refs. [20, 21]. Moreover, it has been suggested that the global angular momentum should be reflected in the polarization of observed hadrons, e.g., in the case of Λ hyperons and vector mesons [5, 6, 22, 23].

Surprisingly, no dynamical fluid-like framework has been developed so far which allows for space-time evolution of polarization effects, despite the fact that the studies of fluids with spin have a long history initiated already in 1930’s [24, 25]. Recent works have contributed to our understanding of global equilibrium/stationary states, which exhibit interesting features of vorticity-spin alignment [9, 12]. Moreover, polarization effects present during the final freeze-out stage of collisions, where particles cease to interact, have also been studied [11, 16, 26]. On the other hand, little is known about the changes of polarization during the collision process, especially, if the latter is described with the help of fluid dynamics.

In this work, we develop a general perfect-fluid framework for charged particles with spin 1/2, which allows for space and time dependent studies of polarized fluids. We show how the conservation laws for the charge current, energy-momentum, and angular momentum can be used consistently to obtain the dynamics of fluids with spin degrees of freedom. In addition to the standard equations, which determine the time dependence of the charge density $n$, temperature $T$ and fluid four-velocity $u^\mu$, our equations determine the dynamics of the polarization tensor $\omega^{\mu\nu}$. The evolution equation for polarization presented here provides a consistent theoretical framework for studying the effects of vorticity and polarization in hydrodynamic simulations of heavy-ion collisions. Furthermore, our approach is applicable also to other systems exhibiting collision dominated, collective behavior connected with non-trivial polarization effects.

We note at this point that in ideal hydrodynamics, the system is assumed to be in local thermodynamic equilibrium. Thus, the polarization of each fluid cell equals the local equilibrium value, specified by the thermodynamic variables. Consequently, the ideal fluid-dynamic framework presented here, does not account for the relaxation of spin degrees of freedom, e.g., through the spin-orbit interaction [27].

Below we use the following conventions and notation for the metric tensor, Levi-Civita’s tensor, and the scalar product: $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$, $\epsilon^{0123} = -\epsilon_{0123} = 1$, $a \cdot b = g_{\mu\nu} a^\mu b^\nu$. Throughout the text we use natural units with $c = \hbar = k_B = 1$.

2. Local distribution functions: Our starting point is the phase-space distribution functions for spin-1/2 particles and antiparticles in local equilibrium, as introduced in [11]. In order to incorporate the spin degrees of freedom, they are generalized from scalar functions to two by two spin density matrices for each value of the space-time position $x$ and momentum $p$,

$$f^+_{rs}(x, p) = \frac{1}{2m} \bar{u}_r(p) X^+ u_s(p),$$

$$f^-_{rs}(x, p) = -\frac{1}{2m} \bar{v}_r(p) X^- v_s(p).$$

The resulting set of differential equations extend the standard picture of perfect-fluid hydrodynamics with a conserved entropy current in a minimal way, allowing for space-time analyses of the evolution of spin and polarization in various physical systems including high-energy nuclear collisions. We demonstrate that a stationary vortex, which exhibits vorticity-spin alignment, corresponds to a special solution of the spin-hydrodynamical equations.
Here $m$ is the particle mass and $u_r(p)$ and $v_r(p)$ are bispinors (with the spin indices $r$ and $s$ running from 1 to 2), with the normalization $\bar{u}_r(p)u_s(p) = 2m \delta_{rs}$ and $\bar{v}_r(p)v_s(p) = -2m \delta_{rs}$. Note the minus sign and different ordering of spin indices in (2) compared to (1).

Following the notation used in [11], we introduce the matrices

$$X^\pm = \exp[\pm \xi(x) - \beta_\mu(x) p^\mu] M^\pm, \quad (3)$$

where

$$M^\pm = \exp \left[ \pm \frac{1}{2} \omega_{\mu\nu}(x) \Sigma^{\mu\nu} \right]. \quad (4)$$

In these equations, we use the notation $\beta_\mu = u^\mu / T$ and $\xi = \mu / T$, with the temperature $T$, chemical potential $\mu$ and four velocity $u^\mu$. The latter is normalized to $u^2 = 1$. Moreover, $\omega_{\mu\nu}$ is the polarization tensor, while $\Sigma^{\mu\nu}$ is the spin operator expressed in terms of the Dirac gamma matrices, $\Sigma^{\mu\nu} = (i/4) [\gamma^\mu, \gamma^\nu]$. For the sake of simplicity, we restrict ourselves to classical Boltzmann statistics in this work. However, given the closed form expression for $M^\pm$ obtained below, it is a straightforward exercise to generalize our discussion to Fermi-Dirac statistics.

The antisymmetric polarization tensor $\omega_{\mu\nu}$ can be represented by the following tensor decomposition

$$\omega_{\mu\nu} \equiv k_\mu u_{\nu} - k_\nu u_{\mu} + \epsilon_{\mu\nu\beta\gamma} u^\beta \omega^{\gamma}. \quad (5)$$

We note that any part of $k_\mu$ or $\omega_{\mu}$ that is parallel to $u^\mu$, is cancelled in (5). Hence, we can assume that both $k_\mu$ and $\omega_{\mu}$ are orthogonal to $u^\mu$, i.e., $k \cdot u = \omega \cdot u = 0$, and express the four-vectors $k_\mu$ and $\omega_{\mu}$ in terms of $\omega_{\mu\nu}$ using

$$k_\mu = \omega_{\mu\nu} u^{\nu}, \quad \omega_{\mu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \omega^{\nu\alpha} u^{\beta}. \quad (6)$$

This means that $k_\mu$ and $\omega_{\mu}$ are space-like four-vectors with only three independent components.

It is convenient to introduce the dual polarization tensor

$$\tilde{\omega}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \omega^{\alpha\beta} = \omega_{\mu} u_{\nu} - \omega_{\nu} u_{\mu} + \epsilon_{\mu\nu\alpha\beta} k^{\alpha} u^{\beta}. \quad (7)$$

Using Eqs. (5) and (7) one finds that the scalar contraction of the polarization tensor with itself gives $\frac{1}{2} \omega_{\mu\nu} \omega^{\mu\nu} = k \cdot k - \omega \cdot \omega$, whereas the contraction of the polarization tensor with its dual yields $\frac{1}{2} \tilde{\omega}_{\mu\nu} \omega^{\mu\nu} = 2k \cdot \omega$.

The exponential function in Eq. (4) is defined in terms of a power series, which can be resummed (most easily in the chiral representation of the $\gamma$ matrices, where $\Sigma^{\mu\nu}$ is block diagonal). Using the constraint

$$k \cdot \omega = 0 \quad (8)$$

we find the compact form

$$M^\pm = \cosh(\zeta) \pm \frac{\sinh(\zeta)}{2\zeta} \omega_{\mu\nu} \Sigma^{\mu\nu}, \quad (9)$$

where

$$\zeta \equiv \frac{1}{2} \sqrt{k \cdot k - \omega \cdot \omega}. \quad (10)$$

We now assume that $k \cdot k - \omega \cdot \omega \geq 0$, which in conjunction with Eq. (10) implies that $\zeta$ is real. We motivate these choices below, following Eqs. (16) and (20).

3. Basic observables: Using the distribution functions (1) and (2), we obtain the basic hydrodynamic quantities. The charge current is given by [28]

$$N^\mu = \int \frac{d^3 p}{(2\pi)^3 E_p} p^\mu \left[ \text{tr}(X^+) - \text{tr}(X^-) \right] = nu^\mu, \quad (11)$$

where “tr” denotes the trace over spinor indices and

$$n = 4 \cosh(\zeta) \sinh(\zeta) n_{0(0)} \quad (12)$$

is the charge density. Here $n_{0(0)}(T) = \langle (u \cdot p) \rangle_0$ is the number density of spin 0, neutral Boltzmann particles, obtained using the thermal average

$$\langle \cdots \rangle_0 \equiv \int \frac{d^3 p}{(2\pi)^3 E_p} (\cdots) e^{-\beta p}, \quad (13)$$

where $E_p = \sqrt{m^2 + p^2}$.

The energy-momentum tensor for a perfect fluid then has the form [28]

$$T^{\mu\nu} = \int \frac{d^3 p}{2(2\pi)^3 E_p} p^\mu p^\nu \left[ \text{tr}(X^+) + \text{tr}(X^-) \right] = \langle \varepsilon + P \rangle u^\mu u^\nu - Pg^{\mu\nu}, \quad (14)$$

where the energy density and pressure are given by

$$\varepsilon = 4 \cosh(\zeta) \cosh(\zeta) \xi(0) \quad (15)$$

and

$$P = 4 \cosh(\zeta) \cosh(\zeta) P(0), \quad (16)$$

respectively. In analogy to the density $n_{0(0)}(T)$, we define the auxiliary quantities $\xi(0)(T) = \langle (u \cdot p)^2 \rangle_0$ and $P(0)(T) = -(1/3) \langle [p \cdot p - (u \cdot p)^2] \rangle_0$. At this point we note that in the case where $\zeta$ is not real, one can find a generalized form of $M^\pm$ and consequently of all thermodynamic quantities, involving trigonometric functions. As this potentially leads to negative values of the pressure, we exclude such cases from the present investigation. We also note that the energy-momentum tensor (14) is symmetric, owing to the fact that we deal with classical particles that have a well defined relation between energy, momentum and velocity, $p = E_p v$ [29].

The entropy current is given by an obvious generalization of the Boltzmann expression

$$S^\mu = -\int \frac{d^3 p}{2(2\pi)^3 E_p} p^\mu \left( \text{tr} X^+(\ln X^+ - 1) \right. \left. + \text{tr} X^-(\ln X^- - 1) \right). \quad (17)$$
This leads to the following entropy density
\[ s = u_\mu S^\mu = \frac{\varepsilon + P - \mu n - \Omega w}{T}, \] (18)
where \( \Omega \) is defined through the relation \( \zeta = \Omega/T \) and
\[ w = 4 \sinh(\zeta) \cosh(\zeta) n_{(0)} \]. (19)
Equation (18) suggests that \( \Omega \) should be used as a thermodynamic variable of the grand canonical potential, in addition to \( T \) and \( \mu \). Taking the pressure \( P \) to be a function of \( T, \mu \) and \( \Omega \), we find
\[ s = \frac{\partial P}{\partial T}_{\mu, \Omega}, \quad n = \frac{\partial P}{\partial \mu}_{T, \Omega}, \quad w = \frac{\partial P}{\partial \Omega}_{T, \mu}. \] (20)
The modified thermodynamic relation (18) is analogous to the one obtained in Ref. [9]. We note that the thermodynamic relations (20) also suggest that \( \zeta \) should be real.

Moreover, we observe that the thermodynamic variable \( \Omega \) controls the polarization of the system. Hence, in the present framework, \( \Omega \) acts as a proxy for the spin-vorticity coupling, which provides a spin-dependent shift in the single-particle energies [30]. In global equilibrium, \( \Omega \) is a unique function of the thermal vorticity \( \zeta \) [11]. However, in local equilibrium and, in particular, in nonequilibrium systems, this relation may be relaxed.

In this paper we explore the dynamics of systems, where the local polarization and thermal vorticity are initialized as independent variables and evolve according to ideal hydrodynamics. Thus, we allow for an incomplete relaxation of the spin degrees of freedom during the prehydrodynamic stage, while in the hydrodynamic phase dissipative processes, in particular also spin relaxation, are neglected. We stress that this idealized framework allows for non-trivial spin dynamics, and hence provides the possibility to perform key studies of polarization phenomena in a hydrodynamic setting.

4. Basic conservation laws: Before we turn to the discussion of the spin observables let us analyze the basic conservation laws. The conservation of energy and momentum requires that
\[ \partial_\mu T^{\mu\nu} = 0. \] (21)
This equation can be split into two parts, one longitudinal and the other transverse with respect to \( u^\mu \):
\[ \partial_\mu [(\varepsilon + P) u^\mu] = u^\mu \partial_\mu P = dP/d\tau, \]
\[ (\varepsilon + P) \frac{du^\mu}{d\tau} = (g^{\alpha\alpha} - u^\alpha u^\alpha) \partial_\alpha P. \] (22)
Evaluating the derivative on the left-hand side of the first equation in (22) and using (20) we find
\[ T \partial_\mu (su^\mu) + \mu \partial_\mu (nu^\mu) + \Omega \partial_\mu (wu^\mu) = 0. \] (23)
The middle term in Eq. (23) vanishes due to charge conservation,
\[ \partial_\mu (nu^\mu) = 0. \] (24)
Thus, in order to have entropy conserved in our system (for the perfect-fluid description we are aiming at), we demand that
\[ \partial_\mu (wu^\mu) = 0. \] (25)
Consequently, using Eq. (24) and Eq. (25) we self-consistently arrive at the equation for conservation of entropy, \( \partial_\mu (su^\mu) = 0 \).

Note that in the absence of a net spin polarization, i.e., for \( \zeta = 0 \), Eq. (12) reduces to the standard expression for the net charge density \( n = 4 \sinh(\xi) n_{(0)} \). On the other hand, one may consider two linear combinations of Eqs. (24) and (25) leading to conservation equations of the form \( \partial_\mu [(n \pm w) u^\mu] = 0 \). Using Eqs. (12) and (19), we find \( n \pm w = 4 \sinh(\mu \pm \Omega)/T n_{(0)} \), which indicates that thermodynamic quantities corresponding to charge and spin of the particles couple. In fact, \( \Omega \) can be interpreted as a chemical potential related with spin. Interestingly, from a thermodynamic point of view, a system of particles with spin 1/2 can be seen as a two component mixture of scalar particles with chemical potentials \( \mu \pm \Omega \).

The resulting scheme, i.e., Eqs. (21), (24) and (25), can be regarded as a minimal extension of the standard perfect-fluid hydrodynamics of charged particles, where all dynamic equations follow from the conservation laws. We note that (21), (24) and (25) form a closed system of equations, which facilitates the study of spin dynamics. We may first solve these equations and subsequently use this solution as the dynamic background for the spin dynamics. Because of this property, we dub them the equations for hydrodynamic (spin) background.

5. Spin dynamics: Our approach is based on the conservation of angular momentum in the form \( \partial_\nu J^{\lambda, \mu, \nu} = 0 \), where \( J^{\lambda, \mu, \nu} = L^{\lambda, \mu, \nu} + S^{\lambda, \mu, \nu} \) with \( L^{\lambda, \mu, \nu} = x^{\mu} T^{\nu \lambda} - x^{\nu} T^{\lambda \mu} \) and \( S^{\lambda, \mu, \nu} \) being the spin tensor. Since the energy-momentum tensor \( T^{\mu \nu} \) is symmetric (see Eq. (14)), the spin tensor \( S^{\lambda, \mu, \nu} \) satisfies the conservation law [29],
\[ \partial_\lambda S^{\lambda, \mu, \nu} = 0. \] (26)
For \( S^{\lambda, \mu, \nu} \) we use the form [9]
\[ S^{\lambda, \mu, \nu} = \frac{1}{2} \int \frac{d^3 p}{2(2\pi)^3 E_p} p^\lambda \text{tr} [(X^+ - X^-) \Sigma^{\mu \nu}] = \frac{wu^\lambda}{4\zeta} \omega^{\mu \nu}. \] (27)
We note that Eq. (27) differs from that derived in [11]. We find that the additional terms given in Eq. (42) of [11] are inconsistent with both the vortex solution discussed below and the conservation law (26). Hence, we employ (27), which leads to a self-consistent framework.
Using Eq. (25) and introducing the rescaled polarization tensor \( \tilde{\omega}^{\mu \nu} = \omega^{\mu \nu} / (2\zeta) \), we obtain
\[
u^A \partial_A \tilde{\omega}^{\mu \nu} \equiv \frac{d\tilde{\omega}^{\mu \nu}}{dt} = 0,
\] (28)
with the normalization condition \( \tilde{\omega}^{\mu \nu} \tilde{\omega}_{\mu \nu} = 2 \). The tensor \( \tilde{\omega}^{\mu \nu} \) can be decomposed in a way analogous to Eq. (5), with the two rescaled four-vectors \( \tilde{k}_\mu = k_\mu / (2\zeta) \) and \( \tilde{\omega}_\mu = \omega_\mu / (2\zeta) \), satisfying the constraints
\[
\tilde{k} \cdot u = 0, \quad \tilde{\omega} \cdot u = 0, \quad \tilde{k} \cdot \tilde{\omega} = 0, \quad \tilde{k} \cdot \tilde{\omega} = 1,
\] (29)
which leave only four independent components in \( \tilde{k}_\mu \) and \( \tilde{\omega}_\mu \). This is expected, since the condition (8) removes one degree of freedom and another is eliminated by the rescaling with \( \zeta \). The latter is anyway determined by the hydrodynamic background equations.

The last condition in (29) is fulfilled by employing the parameterization
\[
\tilde{k}_\mu = m_\mu \sinh(\psi), \quad \tilde{\omega}_\mu = n_\mu \cosh(\psi).
\] (30)
The four-vectors \( m_\mu \) and \( n_\mu \) are space-like and normalized to \(-1\),
\[
m_\mu m^\mu = -1, \quad n_\mu n^\mu = -1.
\] (31)
Using (30) in (28) we then find two coupled equations
\[
\frac{dm_\mu}{d\tau} \sinh(\psi) + m_\mu \cosh(\psi) \frac{d\psi}{d\tau} + m_\nu a^\nu \sinh(\psi) u_\mu + \epsilon_{\mu \nu \beta \gamma} a^\nu a^\beta n^\gamma \cosh(\psi) = 0,
\]
\[
\frac{dn_\mu}{d\tau} \cosh(\psi) + n_\mu \sinh(\psi) \frac{d\psi}{d\tau} + n_\nu a^\nu \cosh(\psi) u_\mu + \epsilon_{\mu \nu \alpha \beta} a^\nu a^\alpha m^\gamma \sinh(\psi) = 0,
\] (32)
where \( a^\mu = du^\mu / d\tau \).

Equations (32) should preserve the normalization conditions (31) as well as the orthogonality constraints \( m \cdot u = n \cdot u = m \cdot n = 0 \). It is straightforward to convince oneself that these conditions are satisfied during the evolution of the system, provided they are satisfied on the initial hypersurface and the following equation is fulfilled by the variable \( \psi \),
\[
\frac{d\psi}{d\tau} = \epsilon_{\mu \nu \beta \gamma} m^\mu u^\nu a^\beta n^\gamma.
\] (33)

6. Vortex solution: In order to demonstrate how our framework works in practice, we consider a rigid rotation of the fluid around the \( z \)-axis. The hydrodynamic flow is defined by the four-vector \( u^\mu \) with the components
\[
u^0 = \gamma, \quad u^1 = -\gamma \tilde{\Omega} y, \quad u^2 = \gamma \tilde{\Omega} x, \quad u^3 = 0,
\] (34)
where \( \tilde{\Omega} \) is a positive constant, \( \gamma = 1 / \sqrt{1 - \tilde{\Omega}^2 r^2} \), and \( r \) denotes the distance from the center of the vortex in the transverse plane, \( r^2 = x^2 + y^2 \). Due to the limiting light speed, the flow profile (34) may be realized only within a cylinder with the radius \( R < 1/\tilde{\Omega} \). The total time (convective) derivative takes the form
\[
\frac{d}{d\tau} = u^\mu \partial_\mu = -\gamma \tilde{\Omega} \left( \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right).
\] (35)
Equation (35) can be used to find the fluid acceleration
\[
a^\mu = \frac{du^\mu}{d\tau} = -\gamma^2 \tilde{\Omega}^2 (0, x, y, 0).
\] (36)
As expected the spatial part of (36) points towards the centre of the vortex, as it describes the centripetal acceleration. It is easy to see that the equations for the hydrodynamic background are satisfied if \( T, \mu \) and \( \tilde{\Omega} \) are \( r \)-dependent and proportional to the local Lorentz-\( \gamma \) factor, namely
\[
T = T_0 \gamma, \quad \mu = \mu_0 \gamma, \quad \tilde{\Omega} = \Omega_0 \gamma,
\] (37)
with \( T_0, \mu_0 \) and \( \Omega_0 \) being constants. One possibility is that the vortex represents an unpolarized fluid with \( \omega^{\mu \nu} = 0 \) and thus, with \( \Omega_0 = 0 \). Another possibility is that the particles in the fluid are polarized and \( \Omega_0 \neq 0 \). In the latter case we expect the polarization tensor to have the structure
\[
\omega^{\mu \nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{\Omega}/T_0 & 0 \\ -\tilde{\Omega}/T_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\] (38)
where the parameter \( T_0 \) has been introduced to keep \( \omega^{\mu \nu} \) dimensionless. This form, when used in Eqs. (6), yields \( \tilde{k}_\mu = \tilde{\Omega}^2 (\gamma/T_0) (0, x, y, 0) \) and \( \omega_\mu = \tilde{\Omega} (\gamma/T_0) (0, 0, 0, 1) \). As a consequence, we find \( \zeta = \tilde{\Omega} / (2T_0) \), which, for consistency with the hydrodynamic background equations, implies
\[
\tilde{\Omega} = 2 \Omega_0.
\] (39)
The factor 2 is a consequence of the fact that we are dealing with spin-\( \frac{1}{2} \) particles. It follows that \( \tilde{k}_\mu = \gamma \tilde{\Omega} r (0, x/r, y/r, 0) \) and \( \omega_\mu = \gamma (0, 0, 0, 1) \), leading to \( m_\mu = (0, x/r, y/r, 0) \), \( n_\mu = (0, 0, 0, 1) \), \( \cosh(\psi) = \gamma \), and \( \sinh(\psi) = \gamma \tilde{\Omega} r \). With all these quantities determined, it is rather straightforward to show that Eqs. (32) are fulfilled. We observe that \( d\psi / d\tau = 0 \), since the four-vectors \( m^\mu \) and \( a^\mu \) are parallel. We also note that the polarization tensor given by Eq. (38) agrees with the thermal vorticity, namely
\[
\omega^{\nu \mu} = -\frac{1}{2} (\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)
\] (40)
as emphasized in [9, 11].

1 We stress again that (26) and (28) do not imply that the spin dynamics described by these equations is trivial. Numerical solutions of the hydrodynamic equations with spin presented here will be studied in a future publication.
At this juncture, one may ask which vortex solution, polarized or unpolarized, is realized in Nature. Within the present framework the answer is that both of them can be realized and this depends on the boundary and initial conditions imposed on the hydrodynamic evolution. Spin relaxation effects, not included in the present framework, drive the system to the state of maximum entropy.

7. Boost-invariant, polarized fluid: Our system of equations allows also for boost-invariant solutions describing polarized fluids. Assuming a one-dimensional, boost-invariant flow $u^\mu = (t/\tau, 0, 0, z/\tau)$, where $\tau = \sqrt{t^2 - z^2}$ is the longitudinal proper time, we find that the hydrodynamic background equations for $m \ll T$ are satisfied if $\mu/T$ and $\Omega/T$ are constant, while $T$ is given by the Bjorken solution $T = T_0(\tau_0/\tau)^{-1/3}$, with $T_0$ and $\tau_0$ being the initial temperature and proper time, respectively. One of the forms of the polarization tensor that satisfies Eq. (28) in this case is

$$\bar{\omega}_{\mu\nu} = \epsilon_{\mu\nu\beta\gamma} u^\beta v^\gamma,$$

where $v^\gamma = (z/\tau, 0, 0, t/\tau)$ [31]. Since $u^\lambda \partial_\lambda u^\mu = 0$ and $u^\lambda \partial_\lambda v^\mu = 0$, Eq. (28) holds also in the massive case where, however, the equations of the hydrodynamic background must be solved numerically.

8. Summary and discussion: In this work we have introduced a hydrodynamic framework, which includes the evolution of the spin density in a consistent fashion. Equations that determine the dynamics of the system follow solely from conservation laws. Thus, they can be regarded as a minimal extension of the well established perfect-fluid picture.

Our framework can be used to determine the space-time dynamics of fluid variables, now including also the polarization tensor, from initial conditions defined on an initial space-like hypersurface. This property makes them useful for practical applications in studies of polarization evolution in high-energy nuclear collisions and also in other physics systems exhibiting fluid-like, collective dynamics connected with non-trivial polarization phenomena. In particular, the possibility to study the dynamics of systems in local thermodynamic equilibrium represents an important advance compared to studies, where global equilibrium was assumed.

Straightforward generalizations of our approach are possible to multi-component fluids, to systems obeying Fermi-Dirac statistics and to systems with magnetic fields. It is also of central interest to extend our scheme by including dissipative effects. Of particular importance in the present context is clearly the relaxation of spin degrees of freedom, e.g., by means of a spin-orbit coupling, that would drive the system towards equilibrium polarization \footnote{See a closely related discussion of this issue, which follows Eq. (23) in Ref. [21].}, which in global equilibrium is defined by $\omega_{\mu\nu} = \omega_{\mu\nu}$ [11]. The presence of a polarization tensor introduces a preferred directions in space, which suggests that concepts of anisotropic hydrodynamics may be useful for further developments of our formalism.

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