A GENERAL RELATIVISTIC MAGNETOHYDRODYNAMIC MODEL OF HIGH FREQUENCY QUASI-PERIODIC OSCILLATIONS IN BLACK HOLE LOW-MASS X-RAY BINARIES

CHANG-SHENG SHI\textsuperscript{1,2,3} AND XIANG-DONG LI\textsuperscript{1,2}

\textsuperscript{1} Department of Astronomy, Nanjing University, Nanjing 210093, China; scs1217@gmail.com, lixd@nju.edu.cn
\textsuperscript{2} Key Laboratory of Modern Astronomy and Astrophysics (Nanjing University), Ministry of Education, Nanjing 210093, China
\textsuperscript{3} College of Material Science and Chemical Engineering, Hainan University, Hainan 570228, China

Received 2010 January 23; accepted 2010 March 25; published 2010 April 16

ABSTRACT

We suggest a possible explanation for the high frequency quasi-periodic oscillations (QPOs) in black hole (BH) low-mass X-ray binaries. By solving the perturbation general relativistic magnetohydrodynamic equations, we find two stable modes of the Alfvén wave in the accretion disks with toroidal magnetic fields. We suggest that these two modes may lead to the double high frequency QPOs if they are produced in the transition region between the inner advection-dominated accretion flow and the outer thin disk. This model naturally accounts for the 3:2 relation for the upper and lower frequencies of the QPOs, and the relation between the BH mass and QPO frequency.

Key words: accretion, accretion disks – black hole physics – magnetohydrodynamics (MHD) – X-rays: binaries

Online-only material: color figure

1. INTRODUCTION

Low-mass X-ray binaries (LMXBs) are binary systems consisting of a neutron star (NS) or black hole (BH) accreting from a low-mass ($\lesssim 1 M_\odot$) companion star. X-ray emission of LMXBs often shows fast X-ray variability in the form of high frequency quasi-periodic oscillations (HFQPOs), which frequently appear in pairs in certain state simultaneously (van der Klis 2006). Abramowicz & Kluzniak (2001) pointed out which frequently appear in pairs in certain state simultaneously form of high frequency quasi-periodic oscillations (HFQPOs), which frequently appear in pairs in certain state simultaneously (van der Klis 2006). Abramowicz & Kluzniak (2001) pointed out that the frequency ratio of the twin-peak HFQPOs in the BH source GRO J1655–40 equals 3/2, and that this commensurability of frequencies may be a signature of a non-linear resonance. Later, Abramowicz et al. (2003) found a signature of the same commensurable ratio in the twin-peak HFQPOs observed in an NS-LMXB, Sco X-1. Based on this observational evidence, Kluzniak and Abramowicz argued in several papers (Klužniak & Abramowicz 2001; Kluzniak et al. 2004; Kluzniak et al. 2004) that the twin-peak HFQPOs in both BH and NS sources are due to the same physical mechanism—a non-linear parametric resonance in accretion disk global oscillations.

However, while for the BH sources the commensurable ratio 3/2 was quickly confirmed and generally accepted (e.g., Remillard & McClintock 2006), the presence of the same commensurability in the NS sources is denied by several experts (e.g., Boutelier et al. 2009). There is no consensus whether the nature of the twin-peak HFQPOs in the two types of LMXBs is the same. We have proposed a mechanism for the twin kilohertz QPOs in NS-LMXBs using the magnetohydrodynamic (MHD) Alfvén wave oscillations, and the results seem to fit the observation well (Li & Zhang 2005; Shi & Li 2009). In this paper, we focus on an MHD explanation of the HFQPOs in BH-LMXBs.

Barret et al. (2005) measured the quality factor $Q$ for the HFQPOs measured in the NS-LMXB 4U 1608–52, and found that $Q \sim 200$. They argued that such high coherency is impossible to achieve from kinematic effects in orbital motion of hot spots, clumps, or other similar features located at the accretion disk surface, because these features are too quickly sheared out by the differential rotation. Although orbital motion cannot explain the HFQPOs in LMXBs, the frequencies of several fluid oscillatory modes are expressed by the three characteristic orbital frequencies: the “Keplerian” frequency, the “radial” epicyclic frequency, and the “vertical” epicyclic frequency. In the Kerr metric, these three orbital frequencies and the Lense–Thirring “frame-dragging” frequency have been listed (e.g., Perez et al. 1997). Several HFQPOs models use their ratios (in various combinations) to explain the observed 3/2 commensurability. Cui et al. (1998) suggested the Lense–Thirring nodal precession frequency near the inner stable circular orbit (ISCO) radius as the lower HFQPO frequency, such as the 300 Hz QPOs in GRO J1655–40. The relativistic precession model of Stella et al. (1999) applies to both BH and NS sources; the parastron precession frequency and the Keplerian frequency were taken as the lower and upper frequencies of the twin HFQPOs, respectively, whereas the QPOs at much lower frequencies were interpreted in terms of the Lense–Thirring nodal precession frequency. Wang et al. (2003, 2005) suggested that a non-axisymmetric magnetic coupling of a rotating BH with its surrounding accretion disk coexists with the Blandford–Znajek process. The two frequencies were supposed as the Keplerian frequencies of two hot spots, one near the inner edge of the disk and the other somewhere outside, respectively.

Wagoner et al. (2001) considered the modes of the disk-seismic wave, such as $g$-modes, $p$-modes, and $c$-modes as the explanation of the HFQPOs. They estimated the masses and angular momenta of some BHs with the measured frequencies of the HFQPOs when the $g$-modes or $c$-modes were selected. Rezzolla et al. (2003) discussed the inertial-acoustic modes in a small-size torus very close to the horizon of the BH while the centrifugal and pressure gradients were selected as the only restoring forces. In this model, the BH spin had to be very close to the maximal value to produce the 3:2 ratio. Tassev & Bertschinger (2008) investigated the kinematic density waves in the accretion disks when nothing but the gravity was considered as the restoring force, and several discrete radii were adopted. Several modes in pairs close the ratio (3:2) could be got, but the correct frequencies could not be reproduced. Similar to the parametric resonance models, it is difficult to explain why other modes such as the fundamental frequencies were not observed except the two modes in pairs.
In modeling HFQPOs in BH-LMXBs two points need to be mentioned. First, in most models the 3:2 ratio was often overemphasized and substituted into those models directly. In fact, the 3:2 ratio of the twin HFQPOs in BH-LMXBs is not rigorous but approximate. Second, the HFQPO frequencies were often considered invariable so that these frequencies (168, 113 Hz and 67, 41 Hz in GRS 1915+105, Remillard 2004; 450, 300 Hz in GRO J1655–40, Strohmayer 2001a; 276, 184 Hz in XTE J1550–564, Miller et al. 2001; 240, 165 Hz in H1743–322, Homan et al. 2005) were used to estimate the parameters of the BHs such as the masses and the spins. In fact, the frequencies are stable, i.e., there are small variations in them, rather than invariable (Miller et al. 2001; Remillard et al. 1999; Morgan et al. 1997; Strohmayer 2001b).

This paper is organized as follows. In Section 2, we suggest the basic model and get two stable modes of the Alfvén wave by solving the general relativistic magnetohydrodynamic (GRMHD) equations of the perturbed plasma in BH accretion disks. In Section 3, we compare the results with the observations and discuss their possible implications.

2. THE GRMHD MODEL OF THE HFQPOS IN BH-LMXBs

We consider that the HFQPOs of BH-LMXBs result from GRMHD waves caused by perturbations in the disk. According to Section 2.1 we can find that only one type of GRMHD wave, the steady Alfvén wave, can spread along the magnetic field lines to the energy release region. The simulation by Koide et al. (2002) also shows that a toroidal Alfvén wave can be generated by the rotational dragging of space. We assume that the puny disturbance does not change the metric of the spacetime.

2.1. The Modes of the GRMHD Wave

The oscillation modes of plasma in accretion disks with or without magnetic field, such as g-modes, p-modes, and c-modes, have ever been investigated (Wagoner et al. 2001; Fu & Lai 2009; Lai & Tsang 2009). Here, we discuss the GRMHD wave modes in ideal adiabatic magnetofluid. In the fiducial observer (FIDO) frame, which is a locally inertial frame, the line element can be written as $(ds)^2 = -c^2dt^2 + \sum_{i=1}^{3} (dx^i)^2$, where $c$ is the speed of light in vacuum, the Roman indices $(i)$ run from 1 to 3, and $(ct, x^1, x^2, x^3)$ are the coordinates of the FIDO frame. The metric of the Kerr spacetime in the Boyer–Lindquist coordinates $(ct, r, \theta, \phi)$ is introduced and the line element for the observer at infinity (i.e., in the laboratory frame) can be written as

$$(ds)^2 = -h_0^2(cdt')^2 + \sum_{i=1}^{3} \left[ h_i^2(dx')^2 - 2h_i^2\omega_i dt' dx' \right],$$

where

$$h_0^2 = 1 - \frac{2r_g}{\Sigma}, \quad h_1 = \frac{\Sigma}{\Delta}, \quad h_2 = \Sigma, \quad h_3 = \frac{2carr}{A}.$$ (2)

Here, $(ct', x'^1, x'^2, x'^3)$ are the coordinates of the laboratory frame, $h_0$, $h_1$, and $\omega_0$ are the metrics of the Kerr spacetime, $r_g = G M/c^2$, $a = J/c GM^2$ ($M$ and $J$ are the mass and the angular momentum of the BH, respectively), $G$ is gravitational constant, $\Sigma = r^2 + a^2 r_g^2 \cos^2 \theta$, $\Delta = r^2 - 2r_g + a^2 r_g^2$, and $A = (r^2 + a^2 r_g^2)^2 - a^2 r_g^2 \sin^2 \theta \Delta$. Since our discussion is limited to the accretion disk we take $\theta = \pi/2$ and get $\Sigma = r^2$ and $A = (r^2 + a^2 r_g^2)^2 - a^2 r_g^2 \Delta$, where $r$ is the distance of the plasma from the BH. Besides that the lapse function and the shift velocity can be expressed as

$$\alpha = \sqrt{h_0^2 + \sum_{i=1}^{3} \left( \frac{h_i\omega_i}{c} \right)^2} = \sqrt{\frac{r^4 - 2r^3 r_g + a^2 r_g^2 r^2}{r^4 + a^2 r_g^2 r^2 + 2a^2 r_g^2 r_g^2}},$$ (3)

and

$$\beta^i = \frac{h_i\omega_i}{ca} \text{ or } \beta = (\beta_1, \beta_2, \beta_3) = \left(0, 0, \frac{2ar_g^2}{r^2 - 2r_g + a^2 r_g^2} \right),$$ (4)

where $\beta$ is a vector parallel to the toroidal velocity of the plasma. We begin with the form of 3+1 split of the GRMHD equations as in Koide (2003). Previous investigations (Ruzmaikina et al. 1979; Tout & Pringle 1992; Ruediger et al. 1995; Hawley 2000; Moss & Shukurov 2004; Hirose et al. 2004) have shown that in the accretion disk around a BH the toroidal component of the magnetic field may be much stronger than the poloidal component, i.e., $B_t \ll B_p$ and $B_\theta \ll B_r$. In our analysis we also assume $v_r \ll v_\phi$ and $v_\theta \ll v_\phi$. The equations in the FIDO frame are then written as follows:

$$\frac{\partial (\rho \gamma \rho v + c^2 E)}{\partial t} = -\nabla \cdot \left[ \alpha (v \rho v + c^2 B) \right],$$ (5)

$$\frac{\partial P}{\partial t} = -\nabla \cdot [\alpha (\tilde{T} + c \beta P)] - (\varepsilon + \gamma \rho c^2) \nabla \rho + \alpha f_{\text{surf}} - P \cdot \tilde{\sigma},$$ (6)

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot [\alpha (c^2 P - \gamma \rho c^2 v + \epsilon c \beta)] - (\nabla \alpha) \cdot c^2 P - \tilde{T} : \tilde{\sigma},$$ (7)

$$\frac{\partial \tilde{\sigma}}{\partial t} = -\nabla \cdot [\alpha (E - c \beta \times B)],$$ (8)

$$\nabla \cdot B = 0,$$ (9)

$$E + v \times B = 0,$$ (10)

$$pp^{-1} = \text{constant},$$ (11)

where $v$ is the velocity of the plasma, $\rho$ the plasma density, $p$ the barometric pressure, $\Gamma$ the adiabatic index, $\gamma$ the Lorentz factor, $E = E / \sqrt{\mu_0}$, $B = B / \sqrt{\mu_0}$ (here $E$ is the electric field, $B$ the magnetic field, and $\mu_0$ the magnetic permeability in the vacuum), respectively. The bold characters denote vectors, and the superscript ~ corresponds to tensors. The energy–momentum tensor is

$$\tilde{T} = \left( \rho + \frac{B^2}{2} + \frac{E^2}{2c^2} \right) \tilde{I} + \frac{\psi}{c^2} \gamma^2 \nabla V - BB - \frac{1}{c^2} EE,$$ (12)

where $\psi = \rho c^2 + \frac{\Gamma \rho}{2}$ is the relativistic enthalpy density. The equivalent momentum density and energy density are

$$P = \frac{\psi}{c^2} \gamma^2 V + \frac{1}{c^2} E \times B,$$ (13)
We can simplify the above two tensors in the accretion disk in the Kerr spacetime as

\[
\sigma_{ij} = \begin{cases} 
0 & 0 \\
\frac{r_s}{\sqrt{r^2 - 2gr_s + a^2r_s^2}} & 0 \\
\frac{r_s}{\sqrt{r^2 + 2gr_s + a^2r_s^2}} & 0 \\
0 & 0 \\
\frac{r_s}{\sqrt{(r^2 - 2gr_s + a^2r_s^2)(r^2 + 2gr_s + a^2r_s^2)}} & 0 \end{cases}
\]

and

\[
G_{ij} = - \begin{cases} 
\frac{r_g}{\sqrt{r^2 - 2gr_g + a^2r_g^2}} & 0 \\
\frac{r_g}{\sqrt{r^2 + 2gr_g + a^2r_g^2}} & 0 \\
0 & 0 \\
\frac{r_g}{\sqrt{(r^2 - 2gr_g + a^2r_g^2)(r^2 + 2gr_g + a^2r_g^2)}} & 0 \end{cases}
\]

where we define the vector \( \mathbf{N} \equiv [(h_3/\dot{h}_1)(\partial \omega_3/\partial x^1), 0, 0] \). The magnetized accretion torus with a toroidal magnetic field around a Kerr BH can exist stably (Komissarov 2006), and the GRMHD equations in the steady state can be expressed as

\[
\frac{\partial (\rho v)}{\partial t} = -\nabla \cdot [\alpha \rho (\mathbf{v} + \mathbf{c} \beta)],
\]

\[
\frac{\partial \mathbf{P}}{\partial t} = -\nabla \cdot [\alpha (\mathbf{T} + c \beta \mathbf{P})] - (\xi_0 + \gamma \rho c^2)\nabla \alpha + \alpha f_{\text{curv, 0}} - \mathbf{P}_0 \cdot \mathbf{\tilde{v}},
\]

\[
\frac{\partial \xi_0}{\partial t} = -\nabla \cdot [\alpha (c^2 \mathbf{P} - \gamma c^2 \rho_0 \mathbf{v} - \gamma c^2 \rho_0 \mathbf{v} + \mathbf{c} \beta)] - (\nabla \alpha) \cdot c^2 \mathbf{P}_0 - \mathbf{\tilde{T}}_0 \cdot \mathbf{\tilde{v}},
\]

\[
\frac{\partial \mathbf{B}_0}{\partial t} = -\nabla \times \{\alpha (\mathbf{E}_0 - \mathbf{c} \beta \times \mathbf{B}_0)\},
\]

\[
\nabla \cdot \mathbf{B}_0 = 0,
\]

\[
\mathbf{E}_0 + \mathbf{v}_0 \times \mathbf{B}_0 = 0,
\]

\[
\rho_0 \rho_0 T = \text{constant},
\]

where the subscript 0 denotes the variables in steady state. Next we consider the GRMHD equations after the plasma is perturbed slightly:

\[
\frac{\partial (\rho v)}{\partial t} = -\nabla \cdot [\alpha \gamma \rho (\mathbf{v} + \mathbf{c} \beta)],
\]

\[
\frac{\partial \mathbf{P}}{\partial t} = -\nabla \cdot [\alpha (\mathbf{T} + c \beta \mathbf{P})] - (\xi_0 + \gamma \rho c^2)\nabla \alpha + \alpha f_{\text{curv, 0}} - \mathbf{P}_0 \cdot \mathbf{\tilde{v}},
\]

\[
\frac{\partial \xi_0}{\partial t} = -\nabla \cdot [\alpha (c^2 \mathbf{P} - \gamma c^2 \rho_0 \mathbf{v} - \gamma c^2 \rho_0 \mathbf{v} + \mathbf{c} \beta)] - (\nabla \alpha) \cdot c^2 \mathbf{P}_0 - \mathbf{\tilde{T}}_0 \cdot \mathbf{\tilde{v}},
\]

\[
\frac{\partial \mathbf{B}_0}{\partial t} = -\nabla \times \{\alpha (\mathbf{E}_0 - \mathbf{c} \beta \times \mathbf{B}_0)\},
\]

\[
\nabla \cdot \mathbf{B}_0 = 0,
\]

\[
\mathbf{E}_0 + \mathbf{v}_0 \times \mathbf{B}_0 = 0,
\]

\[
\rho_0 \rho_0 T = \text{constant},
\]

where the subscript 0 denotes the variables in steady state. Next we consider the GRMHD equations after the plasma is perturbed slightly:

\[
\frac{\partial (\rho v)}{\partial t} = -\nabla \cdot [\alpha \gamma \rho (\mathbf{v} + \mathbf{c} \beta)],
\]

\[
\frac{\partial \mathbf{P}}{\partial t} = -\nabla \cdot [\alpha (\mathbf{T} + c \beta \mathbf{P})] - (\xi_0 + \gamma \rho c^2)\nabla \alpha + \alpha f_{\text{curv, 0}} - \mathbf{P}_0 \cdot \mathbf{\tilde{v}},
\]

\[
\frac{\partial \xi_0}{\partial t} = -\nabla \cdot [\alpha (c^2 \mathbf{P} - \gamma c^2 \rho_0 \mathbf{v} - \gamma c^2 \rho_0 \mathbf{v} + \mathbf{c} \beta)] - (\nabla \alpha) \cdot c^2 \mathbf{P}_0 - \mathbf{\tilde{T}}_0 \cdot \mathbf{\tilde{v}},
\]

\[
\frac{\partial \mathbf{B}_0}{\partial t} = -\nabla \times \{\alpha (\mathbf{E}_0 - \mathbf{c} \beta \times \mathbf{B}_0)\},
\]

\[
\nabla \cdot \mathbf{B}_0 = 0,
\]

\[
\mathbf{E}_0 + \mathbf{v}_0 \times \mathbf{B}_0 = 0,
\]

\[
\rho_0 \rho_0 T = \text{constant},
\]
\[ P_s = \frac{\gamma^2}{c^2} (\psi_0 v_x + \psi_x v_0) + \frac{1}{c^2} E_x \times B_0 + \frac{1}{c^2} E_0 \times B_x, \] (40)

\[ \varepsilon = \psi_x \gamma^2 - p_x - \gamma p_x c^2 + B_0 \cdot B_x + \frac{E_0 \cdot E_x}{c^2}. \] (41)

After carrying out Fourier transformation \( e^{ik \cdot r - iw t} \) for Equations (32)–(36) and substituting Equations (37) and (38) into them, we get the following equations when \( v_0 \parallel B_0 \parallel \beta \) is considered:

\[ [\omega - \alpha (k \cdot v_0) - \alpha c (k \cdot \beta)] \rho_s = \alpha \rho_0 (k \cdot v_s), \] (42)

\[ \omega P_s - \alpha c (k \cdot \beta) P_s = \alpha k \cdot \tilde{T}_s + \alpha (\varepsilon_s + \gamma p_s c^2) k + i \alpha f_{\text{curv},s} - i P_s \cdot \tilde{\sigma}, \] (43)

\[ \omega \varepsilon_s - \alpha c (k \cdot \beta) \varepsilon_s = 2 \alpha c^2 (k \cdot P_s) - \alpha \gamma c^2 \rho_0 (k \cdot v_s) - \alpha \gamma c^2 (k \cdot v_0) \rho_s - i \tilde{T}_s \cdot \tilde{\sigma}, \] (44)

\[ [\omega - \alpha (k \cdot v_0) - \alpha c (k \cdot \beta)] B_s = \alpha (k \cdot v_s) B_0 - \alpha (k \cdot B_0) v_s, \] (45)

\[ k \cdot B_s = 0, \] (46)

where \( k \) is the wave vector and \( \omega \) is the oscillation frequency. When Equations (37) and (38) are substituted into Equations (39)–(41) and \( v_0 \parallel B_0 \parallel \beta \) is considered, Equations (39)–(41) can be converted to be

\[ \tilde{T}_s = \left( \frac{\Gamma_0}{\rho_0} \rho_s + B_0 \cdot B_s \right) \tilde{T} + \frac{\psi_0}{c^2} \gamma^2 V_0 v_0 \]

\[ + \frac{\psi_0}{c^2} \gamma^2 (V_0 v_x + V_x v_0) - (B_0 B_s + B_s B_0), \] (47)

\[ P_s = \frac{\gamma^2}{c^2} (\psi_0 v_x + \psi_x v_0) - \frac{1}{c^2} (V_0 \cdot B_0) B_s \]

\[ + \frac{1}{c^2} (B_x \cdot B_0) V_s - \frac{1}{c^2} (V_s \cdot B_0) B_0 + \frac{E_0^2}{c^2} v_x, \] (48)

\[ \varepsilon = \psi_x \gamma^2 - \frac{\Gamma_0}{\rho_0} \rho_s - \gamma p_x c^2 + B_0 \cdot B_s. \] (49)

If \( \omega = \alpha k \cdot v_0 + \alpha c k \cdot \beta \), we get an unphysical solution \( (\omega = 0) \) from Equations (42)–(49), because the conditions \( k \cdot v_s = 0 \) and \( k \cdot V_0 = 0 \) are derived from Equations (42)–(44). Now we consider \( \omega \neq \alpha k \cdot v_0 + \alpha c k \cdot \beta \) and discuss all the three types of MHD waves: the Alfvén wave, the ion-acoustic wave, and the magnetosonic wave.

### 2.1.1. The Alfvén Wave

The transportation direction of Alfvén waves is along the magnetic field line, so \( k \parallel v_0 \parallel B_0 \parallel \beta \), where \( \parallel \) denotes parallel and \( \perp \) the same direction. Since the Alfvén wave is a transverse wave, i.e., \( k \perp B_0, k \perp v_s \) or the \( \varphi \) component of the perturbed velocity, \( v_{s,\varphi} = 0 \).

From Equations (15)–(17), (42), (47), and (48) and the above results we obtain

\[ f_{\text{curv},s} = (0, 0, 0). \]

And

\[ P_s \cdot \tilde{\sigma} = \frac{\gamma^2}{c^2} (\psi_0 v_{\perp,0} + \psi_s v_0) \mathbf{N} = 0. \]

and

\[ \tilde{T} : \tilde{\sigma} = \sigma \left[ \frac{\gamma^2}{c^2} \psi_0 v_0 + \frac{\alpha k \cdot B^2_0}{\omega - \alpha (k \cdot v_0) - \alpha c (k \cdot \beta)} \right] v_{s,r}, \]

where \( v_{s,r} \) denotes the \( r \) component of the perturbed velocity. Substituting the above equations and Equations (42), (45)–(49) into Equations (43) and (44) we can get \( v_{s,r} = 0 \) and the dispersion equation

\[ [\omega - \alpha c (k \cdot \beta)]^2 \gamma^2 (\psi_0 v_0 + B^2_0) - (2 \alpha \gamma^2 \psi_0 v_0 k) [\omega - \alpha c (k \cdot \beta)] + \alpha^2 \gamma^2 \psi_0 (k^2 v_0^2) - \alpha^2 c^2 k^2 B^2_0 = 0. \] (50)

The frequencies of the Alfvén waves are solved as

\[ \omega = k \alpha \left[ \beta c + \frac{\gamma^2 \psi_0 v_0 \pm B_0 \sqrt{B_0^2 c^2 + (c^2 - v_0^2) \gamma^2 \psi_0}}{B_0^2 + \gamma^2 \psi_0} \right]. \] (51)

The group velocities of the Alfvén waves are in the same form of the phase velocities of the Alfvén waves and are

\[ \omega = \frac{d \omega}{dk} = k \alpha \left[ \beta c + \frac{\gamma^2 \psi_0 v_0 \pm B_0 \sqrt{B_0^2 c^2 + (c^2 - v_0^2) \gamma^2 \psi_0}}{B_0^2 + \gamma^2 \psi_0} \right]. \] (52)

These velocities can be simplified in the special relativity, i.e., \( r \to \infty, \beta_3 \to 0, \) and \( \alpha \to 1, \) as

\[ v_A = \frac{\gamma^2 v_0 \pm \sqrt{\frac{B_0^2}{\psi_0} + \frac{(c^2 - v_0^2)^2}{\gamma^2 c^2}}}{\eta^2 + 1} = \frac{v_0 \pm \eta \sqrt{1 + \eta^2 c}}{\eta^2 + 1}, \] (53)

where \( \gamma^2 \eta^2 = B_0^2 / \psi_0. \) Equation (53) is in the same form as the expression of De Villiers & Hawley (2003).

#### 2.1.2. The Ion-acoustic Wave and the Magnetosonic Wave

The ion-acoustic wave is a longitudinal wave without electromagnetic polarization so \( k \parallel v_s \) and \( B_s = 0. \) According to Equation (45) we have \( B_0 \parallel V_s \) owing to \( B_s = 0. \) This leads to the conclusion that \( k \parallel v_s \parallel B_0 \parallel \beta \), and that no ion-acoustic wave solution fits Equations (42)–(49).

The magnetosonic wave is another type of longitudinal wave with transverse electromagnetic polarization, i.e., \( k \parallel v_s, k \perp B_s \). The wave vector \( k \) is not parallel with \( B_0 \) under the condition \( B_0 \parallel V_0 \parallel \beta \), otherwise the wave is degenerated into the ion-acoustic wave and does not exist. Now we discuss the magnetosonic wave under two different conditions, respectively.

First, when \( k \perp V_0 \) we get \( k \parallel v_s, k \perp B_s, B_0 \parallel \beta \perp k, \) and Equations (42)–(49) can be simplified to be

\[ (\alpha \gamma^2 \psi_0 + \alpha \Gamma_0 - \alpha c^2 \beta_0 + \alpha \beta_0^2) (k \cdot v_s) = i \tilde{T} \cdot \tilde{\sigma}, \] (54)

\[ \omega^2 (\gamma^2 \psi_0 + B_0^2) v_s + \gamma^2 c^2 \beta_0 - \alpha \gamma^2 \psi_0 B_0 (k \cdot v_s) v_0 - \alpha \gamma^2 c^2 \left( 2 B_0^2 + \gamma^2 \beta_0 c^2 + \gamma^2 c^2 \beta_0^2 \right) v_s - \alpha \gamma^2 \psi_0 \left( 2 B_0^2 + \gamma^2 \beta_0 c^2 + \gamma^2 c^2 \beta_0^2 \right) v_s = 0. \] (55)
\[ \tilde{T}_s : \tilde{\sigma} = \frac{\gamma^2}{c^2} \frac{\beta \rho_3}{B_1} \frac{d \omega}{dx^2} \psi_0 v_0 v_{r,r}, \quad (56) \]

\[ \mathbf{P}_r : \tilde{\sigma} = \frac{\gamma^2}{c^2} \left( \rho_0 c^2 + \frac{\Gamma^2}{\Gamma - 1} \rho_0 \right) \frac{v_0}{\omega} (k \cdot v_s) \mathbf{N}, \quad (57) \]

where

\[ f_{\text{curv},s} = \left( f_{x,r}, 0, G_{31} \frac{\gamma^2}{c^2} \psi_0 v_0 v_{r,r} \right), \quad (58) \]

with

\[ f_{x,r} = - G_{21} \alpha (k \cdot v_s) \Gamma p_0 + B_0^2 \frac{\Gamma p_0 - B_0^2}{\omega} - G_{31} \left[ \frac{\alpha \Gamma p_0 - B_0^2}{\omega} \right] + \frac{\gamma^2}{c^2} \left( \rho_0 c^2 + \frac{\Gamma^2}{\Gamma - 1} \rho_0 \right) \frac{v_0^2}{\omega} (k \cdot v_s). \]

From Equations (54)–(58) we can obtain an unstable solution (an increasing wave or an attenuation wave) if

\[ \tilde{T}_s : \tilde{\sigma} = \frac{\alpha \gamma^2 \psi_0 + \alpha \Gamma \rho_0 - \alpha \gamma^2 \Gamma \rho_0 + \alpha B_0^2}{\Gamma \gamma^2 \rho_0 v_0} = - i, \]

or no solution when \( k \perp v_0 \). The possible unstable solution is

\[ \omega = A + \sqrt{A^2 + 4 \left( \gamma^2 \psi_0 + B_0^2 \right) \left( 2 B_0^2 + \gamma^2 \rho_0 c^2 + \frac{\gamma^2}{\Gamma - 1} \rho_0 \alpha^2 \gamma^2 c^2 k^2 \right)} \]

\[ 2 \gamma^2 \psi_0 + 2 B_0^2, \quad (59) \]

where

\[ A = \frac{c^2 f_{s,s} - c^2 (P_s : \tilde{\sigma})}{\Gamma \gamma^2 \rho_0 v_0 v_t}, \]

and \((P_s : \tilde{\sigma})\), represents the \( r \)-component of the vector \( P_s : \tilde{\sigma} \).

Second, when \( k \) is neither parallel with nor vertical to \( v_0 \) we get the solution from Equations (42)–(49) as follows:

\[ \omega = a c (k \cdot \beta) - \frac{i a (k \cdot v_0) (\tilde{T}_s : \tilde{\sigma}) + 2 a^2 \gamma^2 \gamma^2 \Gamma \rho_0 (k \cdot v_s) (k \cdot v_0)}{(\alpha \gamma^2 \psi_0 + \alpha \Gamma \rho_0 - \alpha \gamma^2 \Gamma \rho_0 + \alpha B_0^2) (k \cdot v_s) - (\alpha (B_0 - B_0^2) (k \cdot v_s) - i T_\gamma : \tilde{\sigma})}, \quad (60) \]

which is also unstable.

We summarize the results in Sections 2.1.1 and 2.1.2: (1) there are two stable Alfven wave modes, (2) the ion-accoustic wave does not exist, and (3) a few unstable modes of magnetoacoustic wave may emerge in the GRMHD in the BH accretion disks.

2.2. The Relation Between the Magnetic Energy Density and the Relativistic Enthalpy Density

It is widely believed that the toroidal magnetic field in accretion disks is generated by dynamo mechanism (Ruzmaikin et al. 1979; Tout & Pringle 1992; Ruediger et al. 1995; Hawley 2000; Moss & Shukurov 2004), and that accretion is driven by the magnetic stress (e.g., Matsumoto & Tajima 1995; Brandenburg et al. 1995; Stone et al. 1996). Accordingly the angular momentum conservation gives (Torkelsson 1998)

\[ M \rho v_\psi = 2 \pi r H r \frac{B'_r}{\mu_0}, \quad (61) \]

where \( H \) is the thickness of the accretion disk, \( B'_r \) and \( B'_\psi \) are the \( \varphi \)- and \( r \)-components of the magnetic field \( B' \), respectively, \( M(= 2 \pi H r v_\psi \rho) \) is the accretion rate, \( v_\psi \) and \( v_r \) are the \( \varphi \)- and \( r \)-components of the velocity of the accreting plasma, respectively. Suppose \( B'_\varphi = \gamma_\text{dyn} B'_r \) and \( v_r = \nu v_\psi \), the above equation can be simplified to be

\[ B'_\varphi^2 = \nu v_\psi^2 \rho \mu_0 \gamma_\text{dyn}. \quad (62) \]

The velocity \( v_\psi \) can be approximatively expressed as the velocity of the circular orbit relative to Bardeen observers, which can be got from Equations (8.354)–(8.359) of Camenzind (2007). The velocity of the plasma in the prograde orbit is

\[ v_\psi = \left[ (i^2 + a_*^2 + \frac{\gamma}{i} a_*^2) \times \frac{i^2 + a_*^2 - 2i}{i^3 \sqrt{i - 2i^2 \sqrt{i + a_*^2 i}} + a_*^2 i - a_*^2 + 2a_* i - 2a_* i} \right] \]

\[ \frac{\gamma}{c} \times \frac{i^2 - 2i + a_*^2}{i^2 + a_*^2}, \quad (63) \]

and in the retrograde orbit

\[ v_\psi = \left[ (i^2 + a_*^2 + \frac{\gamma}{i} a_*^2) \times \frac{4a_* \sqrt{i} - (i^2 + a_*^2 - 2i)}{i^3 \sqrt{i - 2i^2 \sqrt{i + a_*^2 i}} - a_*^2 i + a_*^2 + 2a_* i - 2a_* i} \right] \]

\[ \frac{\gamma}{c} \times \frac{i^2 - 2i + a_*^2}{i^2 + a_*^2}, \quad (64) \]

where \( a_* = |a|, i = r/r_c \). The two velocities return to the Keplerian velocity \( c/\sqrt{i} \) when \( i \to \infty \).

Since \( r_c \rho_0 \ll r_c \rho^2 \) in the accretion disks, we can express \( \frac{\nu}{\mu_0} = \gamma^2 \eta^2 (\rho_0 c^2 + \frac{\Gamma p_0}{\rho_0}) \) as \( \frac{\nu}{\mu_0} = B^2 \gamma^2 \eta^2 \rho_0 c^2 \) and \( \gamma^2 \eta^2 = \gamma_\text{dyn} \nu^2 c^4 / c^2 = \nu^2 c^2 \) from Equation (62), where \( \nu^2 c^2 \) is the Alfvén waves, i.e., Equation (51), as

\[ \omega = k \alpha \left[ \beta_2 c + \frac{\gamma^2 \pm \sqrt{1 + \frac{\nu^2}{c^2} \nu}}{\gamma^2 + \frac{\nu^2}{c^2}} v_\psi \right]. \quad (65) \]

When we consider the characteristic wavelength \( \lambda \sim r \), i.e., the wave number \( k \sim 2 \pi / r \), Equation (65) is reduced to be

\[ \omega \approx \frac{2 \pi}{r} c \xi = \frac{2 \pi}{r \xi} c \xi. \quad (66) \]

where \( \xi = \alpha \left[ \beta_3 + \frac{\gamma^2 \pm \sqrt{1 + \frac{\nu^2}{c^2} \nu}}{\gamma^2 + \frac{\nu^2}{c^2}} v_\psi \right]. \quad (67) \]

2.3. Estimate of the Parameter \( l' \)

Begelman & Pringle (2007) have investigated the structure of accretion disks with strong toroidal magnetic fields, and found that the thickness of the disks is higher than standard thin disks, but in line with observations (Robinson et al. 1999; Shafier & Misselt 2006)

\[ H = 0.48 a_*^{1/17} r_{10}^{9/68} M_{34}^{1/34} \left( \frac{M}{M_\odot} \right)^{-15/68}, \quad (67) \]
where $\alpha_s$ is the viscosity prescription, $M_{18} = M/10^{18}$ g s$^{-1}$, and $r_{10} = r/10^{10}$ cm. The corresponding radial velocity is

$$v_r \simeq \frac{3}{2} \alpha_s \left( \frac{H}{r} \right)^2 v_k,$$

(68)

where $v_k$ is the Keplerian velocity. Here, we adopt that $v_k \simeq v_k$ to estimate the value of $l'$. From Equations (67) and (68), the parameter $l$ is

$$l = \frac{v_r}{v_p} \simeq 0.3456 \alpha_{15/17} r_{10}^{5/34} M_{18}^{3/17} \left( \frac{M}{M_\odot} \right)^{-15/34}.$$  

(69)

Vishniac et al. (1990) have discussed the dynamo action by internal waves in accretion disks and suggested $B_\parallel/B_\phi \sim \alpha_s$, so we have

$$l' = l'_{\text{dyn}} \simeq 0.3456 \alpha_{15/17}^{-2/17} r_{10}^{9/34} M_{18}^{3/17} \left( \frac{M}{M_\odot} \right)^{-15/34}.$$  

(70)

The relatively small values of the power indices in Equation (70) indicate that $l'$ is not sensitively dependent on $\alpha_s$, $r$, $M$, and $M$. HFQPOs in BH-LMXBs were generally observed in the steep power-law (SPL) state, i.e., very high state (VHS), and it is very likely that the QPOs are associated with the region that produces hard X-ray emission. The accretion model for the VHS is a long-debated subject. It might contain an inner advection-dominated accretion flow (ADAF) surrounded by an outer thin disk (e.g., Yuan 2001). By analyzing the observational data of GRO J1655−40 and XTE J1550−564, McClintock & Remillard (2006) showed that the disks with blackbody radiation appear to truncate at a radius (∼100$r_g$) in the low/hard state, and that the truncated radius decreases when the power-law component becomes stronger and steeper. Theoretical investigations (Abramowicz et al. 1995; Honma 1996; Esin et al. 1997; Liu et al. 1999; Manmoto & Kato 2000; Rožańska & Czerny 2000; Narayan & McClintock 2008) also suggest that the transition radius ranges from ∼100$r_g$ to 10,000$r_g$. Accordingly, the transition radius in VHS is likely to be smaller than in the low/hard state, and for illustration here we adopt its value to be 50$r_g$. The X-ray luminosity in the VHS is often more than $0.2L_{\text{Edd}}$ where $L_{\text{Edd}}$ is the Eddington luminosity (McClintock & Remillard 2006), so we adopt a typical accretion rate $10^{18}$ g s$^{-1}$. We also take $\alpha_s \sim 0.1$ (King et al. 2007). From the above values we can estimate the parameter $l'$ as

$$l' \sim 0.048 \left( \frac{\alpha_{15/17}}{0.1} \right)^{-2/17} \left( \frac{r}{50r_g} \right)^{9/34} \left( \frac{M}{10^{18} \text{ g s}^{-1}} \right)^{3/17} \left( \frac{M}{7M_\odot} \right)^{-15/34}.  

(71)

Combining Equations (3), (4), (66), and (71) we get the two frequencies of the HFQPOs as

$$\nu = 1.2756 \times 10^7 \xi \left( \frac{M}{M_\odot} \right)^{-1},$$

(72)

where

$$\xi = \sqrt{\frac{i^4 - 2i^3 + a^2i^2}{i^4 + a^2i^2 + 2a^2i}} + \frac{2a}{\sqrt{i^4 - 2i^3 + a^2}} \left[ \frac{\nu_p}{\gamma_2} + \frac{\nu_p^2}{\gamma_2 + 0.04765v_p^2/c^2} \right].$$

3. RESULTS AND DISCUSSION

Equations (63), (64), and (72) indicate the existence of Alfvén waves with two frequencies in the accretion disk. The ratio of the upper and lower frequencies is close to 3:2, suggesting that these waves may account for the HFQPO pairs. Given the mass and spin of a BH, one can determine the radius where the QPOs are produced. In Table 1, we present the inferred radius by comparing Equation (72) with either the upper or the lower centroid QPO frequency of several BH-LMXBs. We adopt the averaged masses for GRO J1655−40 (6.0−6.6 $M_\odot$, McClintock & Remillard 2006), GRS 1915+105 (10−18 $M_\odot$, Greiner et al. 2001), and XTE J1550−564 (8.4−10.8 $M_\odot$, McClintock & Remillard 2006) and averaged dimensionless spins of GRO J1655−40 (0.65−0.75, Shafee et al. 2006) and GRS 1915+105 (0.98−1.0, McClintock et al. 2006). Because the spin of XTE J1550−564 has not been measured, we take it to be 1, 0.5, and 0. It is interesting to see that in most cases the radii are ∼70$r_g$, consistent with the transition radii between an ADAF and a thin disk discussed above. The frequencies of the HFQPOs mainly depends on the transition radius ($r_{tr}$) and the ratio of $l' = (v_r/v_p)/(B_\parallel/B_\phi)$. According to the discussion in Section 2.3 the parameter $l'$ changes little with the $\alpha_s$, $r$, $M$, and $M$; this may explain why the HFQPO frequencies are relatively stable during the VHS.

| Sources | $m$ | $a$ | $\nu_u$ | $\nu_{\nu}/\nu_1$ | $\nu_{\nu}/\nu_1$ |
|--------|----|----|--------|----------------|----------------|
| GRO J1655−40 | 6.3 | 0.7 | 450 | 66.7528 | 1.5504 |
| GRS 1915+105 | 14 | 0.99 | 168 | 75.6114 | 1.5514 |
| GRS 1915+105 | 14 | 0.99 | 67 | 139.7920 | 1.5570 |
| XTE J1550−564 | 9.6 | 1 | 276 | 69.8141 | 1.5503 |
| XTE J1550−564 | 9.6 | 0.5 | 276 | 69.8639 | 1.5515 |
| XTE J1550−564 | 9.6 | 0 | 276 | 69.9159 | 1.5526 |

Figure 1. Relation between the lower HFQPO frequencies and BH masses for GRO J1655−40, XTE J1550−564, and GRS 1915+105.

(A color version of this figure is available in the online journal.)

Our calculations show that for BH-LMXBs with measured HFQPOs the ratio of the two frequencies is generally around 1.5 (see Table 1).
Considering the similarities in BH and NS accretion disks, it is interesting to ask why the 1.5 frequency ratio is not evident in the HFQPOs in NS-LMXBs. This correlation can be naturally reproduced in our model. In Figure 1, we plot the predicted relation between $v_1$ and $m$ when $r_g$ changes from 66$r_g$ to 76$r_g$, which fit reasonably with the measured data.

We are grateful to the anonymous referee for helpful comments. This work was supported by the Natural Science Foundation of China (under grant 10873008) and the National Basic Research Program of China (973 Program 2009CB824800).

REFERENCES

Abramowicz, M. A., Bulik, T., Bursa, M., & Kluzniak, W. 2003, A&A, 404, L21

Abramowicz, M. A., Chen, X. M., Kato, S., Lasota, J. P., & Regev, O. 1995, ApJ, 438, 37

Abramowicz, M. A., & Kluzniak, W. 2001, A&A, 374, L19

Aly, J. J. 1985, A&A, 143, 19

Barret, D., Kluzniak, W., Olive, J. F., Paltani, S., & Skinner, G. K. 2005, MNRAS, 357, 1288

Begelman, M. C., & Pringle, J. E. 2007, MNRAS, 375, 1070

Boutelier, M., Barret, D., & Miller, M. C. 2009, MNRAS, 339, 1901

Brandenburg, A., Nordlund, A., Stein, R. F., & Torkelsson, U. 1995, ApJ, 446, 741

Cackett, E. M., et al. 2009, ApJ, submitted (arXiv:0908.1098)

Camenzind, M. 2007, Compact Objects in Astrophysics: White Dwarfs, Neutron Stars and Black Holes (Astronomy and Astrophysics Library; Berlin: Springer)

Cui, W., Zhang, S. N., & Chen, W. 1998, ApJ, 492, 53

De Villiers, J. P., & Hawley, J. F. 2003, ApJ, 589, 458

Esin, A., McClintock, J. E., & Narayan, R. 1997, ApJ, 489, 865

Fu, W., & Lai, D. 2009, ApJ, 690, 1386

Greiner, J., Cuby, J. G., & McCaughrean, M. J. 2001, Nature, 414, 522

Hawley, J. F. 2000, ApJ, 528, 462

Hirose, S., Krolik, J. H., De Villiers, J. P., & Hawley, J. F. 2004, ApJ, 606, 1083

Homan, J., et al. 2005, ApJ, 623, 383

Honnala, F. 1996, PASJ, 48, 77

King, A. R., Pringle, J. E., & Livio, M. 2007, MNRAS, 376, 1740

Kluzniak, W., & Abramowicz, M. A. 2001, Acta Phys. Pol. B, 32, 3605

Kluzniak, W., Abramowicz, M. A., Kato, S., Lee, W. H., & Stergioulas, N. 2004, ApJ, 603, L89

Kluzniak, W., Abramowicz, M. A., & Lee, W. H. 2004, in AIP Conf. Proc. 714, X-ray Timing 2003: Rossi and Beyond, ed. Ph. Kaaret, J. H. Swank, & F. K. Lamb (Melville, NY: AIP), 379

Koide, S. 2003, Phys. Rev. D, 67, 104010

Koide, S., Shibata, K., Kodoh, T., & Meier, D. L. 2002, Science, 295, 1688

Komissarov, S. S. 2006, MNRAS, 368, 993

Lai, D., & Tsang, D. 2009, MNRAS, 393, 979

Li, X. D., & Zhang, C. M. 2005, ApJ, 635, L57

Liu, B. F., Yuan, W., Meyer, F., Meyer-Hofmeister, E., & Xie, G. Z. 1999, ApJ, 527, 17

Mammono, T., & Kato, S. 2000, ApJ, 538, 295

Matsumoto, R., & Tajima, T. 1998, ApJ, 445, 767

McClintock, J. E., & Remillard, R. A. 2006, in Compact Stellar X-ray Sources, ed. W. Lewin & M. van der Klis (Cambridge Astrophys. Ser. 39; Cambridge: Cambridge Univ. Press), 157

McClintock, J. E., et al. 2006, ApJ, 652, 518

Miller, J. M., et al. 2001, ApJ, 563, 928

Morgan, E. H., Remillard, R. A., & Greiner, J. 1997, ApJ, 482, 993

Moss, D., & Shukurov, A. 2004, A&A, 413, 403

Narayan, R., & McClintock, J. E. 2008, New Astron. Rev., 51, 733

Perez, C. A., Silbergleit, A. S., Wagoner, R. V., & Lehr, D. E. 1997, ApJ, 476, 589

Remillard, R. A. 2004, AIP Conf. Proc., 714, 13

Remillard, R. A., & McClintock, J. E. 2006, ARA&A, 44, 49

Remillard, R. A., et al. 1999, ApJ, 517, L127

Rezzolla, L., Yoshida, S., Maccarone, T. J., & Zanotti, O. 2003, MNRAS, 344, L37

Robinson, E. L., Wood, J. H., & Wade, R. A. 1999, ApJ, 514, 952

Różańska, A., & Czerny, B. 2000, A&A, 360, 1170

Ruediger, G., Elstner, D., & Stepinski, F. T. 1995, A&A, 298, 934

Ruzmaikin, A. A., Turchaninov, V. I., Zeldovich, L. B., & Sokoloff, D. D. 1979, ApSS, 66, 369

Shafer, R. A., et al. 2006, ApJ, 636, 113

Shafer, A. W., & Misselt, K. A. 2006, ApJ, 644, 1104

Shi, C. S., & Li, X. D. 2009, MNRAS, 392, 264

Stella, L., Vietri, M., & Morsink, S. M. 1999, ApJ, 524, L63

Stone, J. M., Hawley, J. F., Gammie, C. F., & Balbus, S. A. 1996, ApJ, 463, 656

Strohmayer, T. E. 2001a, ApJ, 552, L49

Strohmayer, T. E. 2001b, ApJ, 554, 169

Tassev, S. V., & Bertschinger, E. 2008, ApJ, 686, 423

Torkelsson, U. 1998, MNRAS, 298, 55

Tout, C. A., & Pringle, J. E. 1992, MNRAS, 259, 604

van der Klis, M. 2006, in Compact Stellar X-ray Sources, ed. W. Lewin & M. van der Klis (Cambridge Astrophys. Ser. 39; Cambridge: Cambridge Univ. Press), 39

Vishniac, E. T., Jin, L., & Diamond, P. 1990, ApJ, 365, 648

Wagoner, R. V., Silbergleit, A. S., & Ortega-Rodriguez, M. 2001, ApJ, 559, L25

Wang, D., Ma, R., Lei, W., & Yao, G. 2003, MNRAS, 344, 473

Wang, D., Ye, Y., Yao, G., & Ma, R. 2005, MNRAS, 359, 36

Yuan, F. 2001, MNRAS, 324, 119