The Poincaré series of divisorial valuations in the plane defines the topology of the set of divisors

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In [9], it was proved that the Alexander polynomial in several variables of a (reducible) plane curve singularity (i.e., of the corresponding link) defines the topology of the curve singularity and therefore its minimal embedded resolution. To a plane curve singularity one associates a multi-index filtration on the ring \( \mathcal{O}_{\mathbb{C}^2, 0} \) of germs of functions of two variables defined by the orders of a function on irreducible components of the curve. In [3], there was computed the Poincaré series of this filtration which turned out to coincide with the Alexander polynomial of the curve. For a finite set of divisorial valuations on the ring \( \mathcal{O}_{\mathbb{C}^2, 0} \) corresponding to some components of the exceptional divisor of a modification of \((\mathbb{C}^2, 0)\), in [6], there was obtained a formula for the Poincaré series of the corresponding multi-index filtration similar to the one from [3]. Here we show that the Poincaré series of a set of divisorial valuations on the ring \( \mathcal{O}_{\mathbb{C}^2, 0} \) defines “the topology of the set of the divisors” in the sense that it defines the minimal resolution of this set up to combinatorial equivalence. In [8], there was defined a notion of the zeta-function of an ideal. This notion can be adapted to finite sets of ideals giving a notion of the “Alexander polynomial” (in several variables) of a set of ideals (a mixture of notions introduced in [7] and [8]). To a divisorial valuation on the ring \( \mathcal{O}_{\mathbb{C}^2, 0} \) there corresponds a natural ideal in the ring \( \mathcal{O}_{\mathbb{C}^2, 0} \). This ideal is generated by equations of irreducible curves whose strict transforms on the space of the modification

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intersect the corresponding divisor. One can show that the Poincaré series of a set of divisorial valuations on the ring \( O_{\mathbb{C}^2,0} \) coincides with the Alexander polynomial of the corresponding set of ideals. In these terms, one can say that the Alexander polynomial of a set of divisorial valuations on the ring \( O_{\mathbb{C}^2,0} \) defines the topology of the set of divisors. If one takes all the components of the exceptional divisor of a modification then the Poincaré series defines the topology of the modification also for normal surface singularities: [4]. Notice that in this case the Poincaré series is not, generally speaking, a topological invariant.

We also give a proof of the statement for curves somewhat simpler than the one in [9].

Let \( \pi : (X, D) \to (\mathbb{C}^2, 0) \) be a modification of the complex plane \( \mathbb{C}^2 \), i.e., a proper analytic map of a nonsingular surface \( X \) which is an isomorphism outside of the origin in \( \mathbb{C}^2 \) and such that \( D = \pi^{-1}(0) \) is a normal crossing divisor on \( X \). The modification \( \pi \) is obtained by a sequence of point blowing-ups. The exceptional divisor \( D \) is the union of irreducible components \( E_\sigma \) (\( \sigma \in \Gamma \)), each of them is isomorphic to the complex projective line \( \mathbb{CP}^1 \). The dual graph of the modification \( \pi : (X, D) \to (\mathbb{C}^2, 0) \) is the graph whose vertices correspond to irreducible components \( E_\sigma \) of the exceptional divisor \( D \) (i.e., to elements of the set \( \Gamma \)), two vertices are connected by an edge iff the corresponding components intersect (at a point). The dual graph of the modification \( \pi \) is a tree. The set \( \Gamma \) of vertices of the dual graph inherits a partial order defined by representation of the modification as a sequence of blowing-ups: a component \( E_{\sigma'} \) is “greater” than another component \( E_\sigma \) if the exceptional divisor of the minimal modification which contains \( E_{\sigma'} \) also contains \( E_\sigma \) (\( \sigma' > \sigma \)). Two modifications of the plane are combinatorially equivalent if their dual graphs together with the partial orders of vertices are isomorphic.

Let \( E_\sigma, \sigma \in \Gamma \), be a component of the exceptional divisor \( D \). For a function germ \( f \) from the ring \( O_{\mathbb{C}^2,0} \) of germs of functions of two variables, let \( v_\sigma(f) \) be the multiplicity of the lifting \( f \circ \pi \) of the function \( f \) to the space \( X \) of the modification along the component \( E_\sigma \) (\( v_\sigma(0) := \infty \)). The order function \( v_\sigma : O_{\mathbb{C}^2,0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\} \) defines a valuation on the field of quotients of the ring \( O_{\mathbb{C}^2,0} \): the divisorial valuation defined by the component \( E_\sigma \). A function \( v : O_{\mathbb{C}^2,0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\} \) is called an order function if \( v(\lambda f) = v(f) \) for \( \lambda \neq 0 \) and \( v(f_1 + f_2) \geq \min\{v(f_1), v(f_2)\} \). Let \( \dot{E}_\sigma \) be the smooth part of the component \( E_\sigma \) in the exceptional divisor \( D \), i.e. \( E_\sigma \) itself without intersection points with other components of \( D \). Let \( \dot{L}_\sigma \) be a germ of a smooth curve on the space \( X \) of the modification transversal to the component \( E_\sigma \) at a smooth point of \( D \), i.e., at a point of \( \dot{E}_\sigma \). The image \( L_\sigma = \pi(\dot{L}_\sigma) \subset (\mathbb{C}^2, 0) \) is called a curvette.
corresponding to the component \( E_\sigma \).

Let us fix \( r \) different components \( E_1, \ldots, E_r \) of the exceptional divisor \( \mathcal{D} \) (\( \{1, \ldots, r\} \subset \Gamma \)), let \( \underline{v} := (v_1, \ldots, v_r) \in \mathbb{Z}^r \) and \( \underline{v}(f) := (v_1(f), \ldots, v_r(f)) \) (\( f \in \mathcal{O}_{\mathbb{C}^2,0} \)). The map \( \underline{v} : \mathcal{O}_{\mathbb{C}^2,0} \to (\mathbb{Z} \cup \{\infty\})^r \) defines a multi-index filtration (by ideals) on the ring \( \mathcal{O}_{\mathbb{C}^2,0} \): for \( \underline{v} \in \mathbb{Z}^r \), the corresponding ideal is \( J(\underline{v}) = \{ f \in \mathcal{O}_{\mathbb{C}^2,0} : \underline{v}(f) \geq \underline{v} \} \). (Here \( (v_1, \ldots, v_r) \geq (v'_1, \ldots, v'_r) \) if and only if \( v_i \geq v'_i \) for all \( i = 1, \ldots, r \).) It is sufficient to define the ideal \( J(\underline{v}) \) only for nonnegative \( \underline{v} \), i.e. for \( \underline{v} \in \mathbb{Z}_{\geq 0}^r \). However, for the definition below, it is convenient to assume that \( \underline{v} \in \mathbb{Z}^r \).

Let \( L(t_1, \ldots, t_r) := \sum_{\underline{v} \in \mathbb{Z}^r} \dim (J(\underline{v})/J(\underline{v} + \underline{1})) \cdot t^{\underline{v}} \) be a Laurent series in the variables \( t_1, \ldots, t_r \) (generally speaking, infinite in all directions; here \( \underline{1} = (1, \ldots, 1) \)). One can see that along each line in the lattice \( \mathbb{Z}^r \) parallel to a coordinate one the coefficient at \( t^{\underline{v}} \) is the same for \( \underline{v} \) from the nonpositive part of the line. This implies that

\[
P'(t_1, \ldots, t_r) = L(t_1, \ldots, t_r) \cdot \prod_{i=1}^r (t_i - 1)
\]

is a power series in the variables \( t_1, \ldots, t_r \), i.e. an element of \( \mathbb{Z}[[t_1, \ldots, t_r]] \).

The series

\[
P_{\{v_i\}}(t_1, \ldots, t_r) = \frac{P'(t_1, \ldots, t_r)}{t_1 \cdots t_r - 1}
\]

is called the Poincaré series of the collection \( \{v_i\} \) of divisorial valuations.

The minimal resolution of a set of divisorial valuations is the minimal modification in which all the divisors appear.

**Theorem 1.** The Poincaré series of a set of divisorial valuations on the ring \( \mathcal{O}_{\mathbb{C}^2,0} \) defines the minimal resolution of the set of divisors up to combinatorial equivalence.

**Proof.** Let \( (m_{\sigma \delta}) \) be the inverse of minus the intersection matrix \((E_\sigma \circ E_\delta)\) of components of the exceptional divisor \( \mathcal{D} \). For \( \sigma \neq \delta \) the intersection number \((E_\sigma \circ E_\delta)\) is equal to 1 if the components \( E_\sigma \) and \( E_\delta \) intersect (at a point) and is equal to 0 otherwise. The self-intersection number \((E_\sigma \circ E_\sigma)\) is a negative integer. The numbers \( m_{\sigma \delta} \) are positive integers and \( \det(m_{\sigma \delta}) = 1 \). The number \( m_{\sigma \delta} \) is also equal to \( v_\delta(h_\sigma) = v_\sigma(h_\delta) \), where \( h_\sigma = 0 \) is an equation of a curvette \( L_\sigma \) corresponding to the component \( E_\sigma \) and is equal to the intersection number \((L_\sigma \circ L_\delta)\) of curvettes corresponding to the components \( E_\sigma \) and \( E_\delta \). For \( \sigma \in \Gamma \), let \( m_\sigma := (m_{\sigma 1}, \ldots, m_{\sigma r}) \in \mathbb{Z}_{\geq 0}^r \). Let \( \chi(Z) \) be the Euler characteristic of the space \( Z \). According to \([3]\) one has:

\[
P_{\{v_i\}}(t_1, \ldots, t_r) = \prod_{\sigma \in \Gamma} (1 - t^{m_\sigma})^{-\chi(E_\sigma)}.
\]
The Poincaré series $P_{\{v_i\}}(t_1, \ldots, t_r)$ as a power series in $t_1, \ldots, t_r$ itself defines the factorization of the form $\prod_{m \in \mathbb{Z}_{r+1} \setminus \{0\}} (1 - t^m)^{k_m}$. Moreover the equation (1) implies the projection formula: if $\{i_1, \ldots, i_\ell\} \subset \{1, \ldots, r\}$, then the Poincaré series of the $\ell$-index filtration corresponding to the divisorial valuations $v_{i_1}, \ldots, v_{i_\ell}$ is obtained from the Poincaré series $P_{\{v_i\}}(t_1, \ldots, t_r)$ by substituting the variables $t_i$ with $i \notin \{i_1, \ldots, i_\ell\}$ by 1.

**Remark.** The last property does not hold for the Poincaré series of the filtration defined by orders of a function germ on irreducible components of a plane curve singularity (see [3]). This makes the proof of the corresponding statement for curves (Theorem 2 below) somewhat different.

![Figure 1: The dual graph of a divisorial valuation.](image)

The dual graph of the minimal resolution of one divisorial valuation on the ring $\mathcal{O}_{\mathbb{C}^2, 0}$ has the form shown on Figure 1. Here the corresponding component of the exceptional divisor is marked by the circle $\circ$, $g$ is the number of Puiseux pairs of a curvette corresponding to the divisor, and the length $c$ of the “last tail” may be equal to zero $(\sigma_{g+1} = \tau_g)$. The Poincaré series of this divisorial filtration is equal to

$$P(t) = \frac{\prod_{i=1}^{\ell-1} (1 - t^{m_{\tau_i}})}{\prod_{i=0}^{\ell} (1 - t^{m_{\sigma_i}})}$$

(2)

where $m_\sigma := m_{\sigma_{g+1}}(= v(h_\sigma))$, $m_{\sigma_0} < m_{\sigma_1} < \ldots < m_{\sigma_\ell}$, $m_{\tau_1} < m_{\tau_2} < \ldots < m_{\tau_{\ell-1}}$, and either $\ell = g + 1$ (if the length $c$ of the last tail is positive) or $\ell = g$ (otherwise). Moreover, there is no cancellation in the right hand side of the equation (2) (since $m_{\tau_i} \neq m_{\sigma_j}$ except possible coincidence of $\tau_g$ and $\sigma_{g+1}$), i.e. all factors participate in (2) explicitly. One can see that the second situation ($\ell = g$) takes place if and only if $m_{\sigma_\ell}$ does not belong to the semigroup generated by $m_{\sigma_0}, \ldots, m_{\sigma_{\ell-1}}$. In this case $m_{\sigma_0}, \ldots, m_{\sigma_\ell}$ is the minimal set of generators of the semigroup of values of the divisorial valuation (see e.g. [2]). This is equivalent to the condition $e_{\ell-1} := \gcd(m_{\sigma_0}, \ldots, m_{\sigma_{\ell-1}}) > 1$. If $\ell = g + 1$, then the minimal set of generators of the semigroup is $m_{\sigma_0}, \ldots, m_{\sigma_{\ell-1}}$.  

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Moreover, \( c = m_{\sigma_i} - m_{\tau_{i-1}} \). Due to the projection formula, this means that the Poincaré series of the set of divisorial valuations defines the minimal resolution of each valuation.

Let us describe a divisorial valuation (or the corresponding minimal resolution) by the numbers \( m_0, \ldots, m_{g+1} \), where for \( i = 0, 1, \ldots, g+1 \), \( m_i := m_{\sigma_i} \), \( m_{g+1} = m_{\tau_g} \) if \( c = 0 \). In the last case the number \( m_{g+1} \) is determined by the Poincaré series of the filtration by the equation \( m_{g+1} = e_{g-1} m_{\sigma_g} \), where \( e_{g-1} = \gcd(m_0, \ldots, m_{g-1}) \) (in this case the factor \((1 - t^{m_{\tau_g}})\) itself does not appear in the equation \([2]\) for the Poincaré series of the valuation).

The topological type, or equivalently the dual graph of the minimal resolution, of a curve singularity is defined by the topological type of each branch plus the intersection multiplicities of pairs of branches (see [10], [1]). This implies that the dual graph of the minimal resolution of a set of divisorial valuations is determined by the dual graph of the minimal resolution for each divisor plus the intersection multiplicities of curvettes corresponding to pairs of divisors.

Therefore it remains to show that the Poincaré series \( P_{\{v_i\}}(t_1, \ldots, t_r) \) determines these intersection multiplicities. The projection formula permits to prove this for two valuations. Moreover, the discussion above shows that we can assume the dual graph of the minimal resolution of each divisor known. Let these two divisors be described by the numbers \( m_0, \ldots, m_{g+1} \) and \( m'_0, \ldots, m'_{g'+1} \) respectively (\( m_{g+1} = m_{g+1}' \sigma_{g+1}' \), \( m'_{g'+1} = m_{g'1}' \sigma_{g'1}' \)) where the vertices \( \sigma_{g+1} \) and \( \sigma_{g'1}' \) of the dual graph correspond to the divisorial valuations under consideration; see the explanation above). Just as in the case of one divisorial valuation, there is no cancellation in the equation \([1]\) for two valuations, i.e. all factors with \( \chi(E_\sigma) \neq 0 \) are present and for all of them the exponents \( m_\sigma \) are different. This can be shown, e.g., in the following way. Let \( s \in \Gamma \) be the maximal vertex which is \( \leq \) than both \( \sigma_{g+1} \) and \( \sigma'_{g'1} \). The statement about absence of cancellation follows from the following facts:

1) The exponents \( m_\sigma \) are strictly increasing on \( \Gamma \) with respect to the partial order on it.

2) The ratio \( m_{\sigma,1}/m_{\sigma,2} \) of the coordinates of the exponent \( m_\sigma \) as a function on \( \sigma \) is constant (say, equal to \( q \)) on the set of vertices \( \sigma \) such that \( \sigma \leq s \). This ratio is (strictly) greater than \( q \) on the set of vertices \( \sigma \) such that \( s < \sigma \leq \sigma_{g+1} \) and is (strictly) smaller than \( q \) on the set of vertices \( \sigma \) such that \( s < \sigma \leq \sigma'_{g'1} \).

(These properties were used in [2]. For a more precise description of the behaviour of the ratio \( m_{\sigma,1}/m_{\sigma,2} \) see in the proof of Theorem [2].)

Let \( m_\sigma \) be a maximal exponent in the right hand side of the equation \([1]\) among the factors with \( \chi(E_\sigma) = 1 \). For the coordinates of \( m_\sigma = (m_{\sigma,1}, m_{\sigma,2}) \) (=
In the total transform \( \pi \) of the curve singularity \( C \), the order of the function germ \( g \) is defined as above. For \( i \in \{1, \ldots, r\} \), let \( \varphi_i : (C, 0) \to (\mathbb{C}^2, 0) \) be a parametrization (uniformization) of the curve \( C \), i.e. \( \text{Im} \varphi_i = C_i \) and \( \varphi_i \) is an isomorphism between \( (C, 0) \) and \( (C_i, 0) \) outside of the origin. For \( g \in \mathcal{O}_{C^2, 0} \), let \( w_i(g) \) be the order of the function germ \( g \) on the component \( C_i \), i.e. the exponent of the leading term in the power series decomposition of the function germ \( g \circ \varphi_i \): \( g \circ \varphi_i(\tau) = a\tau^{w_i(g)} + \text{terms of higher degree} \), where \( a \neq 0 \) (if \( g \circ \varphi_i \equiv 0 \), \( w_i(g) = \infty \)). The order functions \( w_1, \ldots, w_r \) define a multi-index filtration on the ring \( \mathcal{O}_{C^2, 0} \). The Poincaré series of this filtration is called the Poincaré series of the plane curve singularity \( (C, 0) \). In [3] it was shown that the Poincaré series \( P_C(t_1, \ldots, t_r) \) coincides with the Alexander polynomial in several variables of the link \( C \cap S^3_\varepsilon \subset S^3_\varepsilon \), where \( S^3_\varepsilon \) is the sphere of small radius \( \varepsilon \) centred at the origin in \( \mathbb{C}^2 \).

Let \( \pi : (X, D) \to (\mathbb{C}^2, 0) \) be an embedded resolution of the plane curve singularity \( (C, 0) \), whose exceptional divisor \( D \) is the union of irreducible components \( E_\sigma, \sigma \in \Gamma \). Let \( m_{\sigma \delta} \) be defined as above. For \( i \in \{1, \ldots, r\} \), let \( \alpha_i \in \Gamma \) be the index of the component \( E_{\alpha_i} \) of the exceptional divisor \( D \) intersecting the strict transform of the component \( C_i \) of the curve, let \( m_{\sigma, i} := m_{\sigma \alpha_i} \), and let \( m_\sigma := (m_{\sigma, 1}, \ldots, m_{\sigma, r}) \in \mathbb{Z}_{\geq 0}^r \). Let \( \tilde{E}_\sigma \) be the smooth part of the component \( E_\sigma \) in the total transform \( \pi^{-1}(\bar{C}) \) of the curve \( C \), i.e. \( E_\sigma \) itself without intersection with other components of \( \pi^{-1}(C) \). According to [3] one has:

\[
P_C(t_1, \ldots, t_r) = \prod_{\sigma \in \Gamma} (1 - t^{m_\sigma})^{-\chi(\tilde{E}_\sigma)},
\]

(3)

Let \( (m_{\sigma \alpha_{i+1}}, m_{\sigma \alpha'_{i+1}}) \) one can distinguish three different situations (up to permutation of the divisors).

a) \( m_{\sigma_1} = m_{g+1} \). This takes place if, for the valuation \( v_1 \), the length \( c \) of the last tail is positive. Then the first valuation \( v_1 \) is equal to \( v_\sigma \) and the intersection multiplicity between the corresponding curvettes is equal to \( m_{\sigma_2} \).

b) \( m_{\sigma_1} = m_g \) and \( m_{\sigma_2} \neq m'_{g'} \). This takes place if, for the valuation \( v_1 \), the length of the last tail is equal to zero and the last dead end \( \sigma_g \) for this valuation does not participate in the minimal resolution of the valuation \( v_2 \). Then the intersection multiplicity between the curvettes is equal to \( e_{g-1} \cdot m_{\sigma_2} \).

c) \( m_{\sigma_1} = m_g \) and \( m_{\sigma_2} = m'_{g'} \). This takes place if the lengths of the last tails for both valuations are equal to zero and the last dead ends for them coincide. Then the intersection multiplicity between the curvettes is equal to the minimum between \( e_{g-1} \cdot m_{\sigma_1} \) and \( e_{g-1} \cdot m_{\sigma_2} \). \( \square \)
Theorem 2. The Poincaré series of a plane curve singularity defines the minimal resolution of the curve up to combinatorial equivalence.

Proof. The equation (3) gives the following “projection formula”: for \( i \in \{1, \ldots, r\} \) one has

\[
P_{C \setminus C_i}(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_r) = \left[ \frac{P_C(t_1, \ldots, t_r)}{1 - t^{m_{\alpha_i}}} \right]_{t_i = 1}.
\]

The minimal resolution of an irreducible plane curve singularity (a branch) has the form shown on Figure 1 with \( c = 0 \) and with an arrow at the vertex \( \tau_g \) corresponding to the strict transform of the curve. The number \( g \) is equal to the number of Puiseux pairs of the curve and the minimal resolution could be described by the set \( m_0, \ldots, m_g \), where, for \( i = 0, \ldots, g \), \( m_i = m_{\sigma_i} \) (this set coincides with the minimal set of generators of the semigroup of values of the curve). The Poincaré series of the branch is equal to

\[
P(t) = \frac{\prod_{i=1}^{g} (1 - t^{m_{\tau_i}})}{\prod_{i=0}^{g} (1 - t^{m_{\sigma_i}})}.
\] (4)

It is clear that the Poincaré series determines the topological type of the branch.

This implies that it suffices to show that from the Poincaré series of the plane curve singularity \( C \) one can recover the exponent \( m_{\sigma_{i_0}} \) and the semigroup of values \( S_{C_{i_0}} \) corresponding to some index \( i_0 \in \{1, \ldots, r\} \), i.e. to an irreducible component \( C_{i_0} \) of \( C \). Notice that, for \( j \neq i_0 \), the intersection multiplicity between the irreducible components \( C_{i_0} \) and \( C_j \) is equal to \( m_{\alpha_{i_0}, \alpha_j} \).

For \( \sigma, \sigma' \in \Gamma \), let \( s(\sigma, \sigma') \) be the index such that \([1, \sigma] \cap [1, \sigma'] = [1, s(\sigma, \sigma')] \) (here \([1, \sigma]\) is the geodesic in the dual graph joining the first (minimal) vertex \( 1 \) with the vertex \( \sigma \)). Let us fix \( j, k \in \{1, \ldots, r\} \). The ratio \( m_{\sigma,j}/m_{\sigma,k} \) as a function on \( \sigma \) is constant on \([1, s(\alpha_j, \alpha_k)]\) and it is strictly increasing on the geodesic \([s(\alpha_j, \alpha_k), \alpha_j]\) joining the vertex \( s(\alpha_j, \alpha_k) \) with \( \alpha_j \). For \( \sigma \notin [1, \alpha_j] \cup [1, \alpha_k] \), the ratio \( m_{\sigma,j}/m_{\sigma,k} \) is equal to \( m_{\sigma',j}/m_{\sigma',k} \), where \( \sigma' \) is the unique vertex such that \([1, \sigma'] = ([1, \alpha_j] \cup [1, \alpha_k]) \cap [1, \sigma] \). Just as in the proof of Theorem 1, one can see that there is no cancellation in the equation (3) (for the minimal resolution).

Let \( \sigma \in \Gamma \) be such that the exponent \( m_{\sigma} \) is a maximal one among the set of exponents \( m_{\sigma} \) with \( \chi(\mathcal{E}_\sigma) \neq 0 \). Notice that, in contrast to the case of divisorial valuations, a maximal exponent is reached among components with \( \chi(\mathcal{E}_\sigma) < 0 \).
If there exists \( \tau \in \Gamma \) such that \( \tau > \sigma \) and \( E_\tau \cap E_\sigma \neq \emptyset \) then \( m_\tau > m_\sigma \). Thus there exists an index \( j \in \{1, \ldots, r\} \) such that \( \sigma = \alpha_j \) (otherwise the exponent \( m_\sigma \) is not a maximal one). From the comments above, it follows that the indices \( j \) such that \( \sigma = \alpha_j \) are some of elements of the nonempty set \( A \subset \{1, \ldots, r\} \) consisting of the elements \( j \) such that

\[
\frac{m_{\sigma,j}}{m_{\sigma,k}} \geq \frac{m_{\tau,j}}{m_{\tau,k}} \quad \text{for } \forall k \in \{1, \ldots, r\} \text{ and } \forall \tau \in \Gamma \text{ such that } \chi(\hat{E}_\tau) \neq 0,
\]

i.e. such that the binomial \((1-t^{m_\tau})\) is present in the equation (3). Let \( \ell \in A \) be such that \( \alpha_\ell \neq \sigma \). Such \( \ell \) exists only if the strict transform of the component \( C_\ell \) of the curve \( C \) intersects transversally the last dead end corresponding to a branch \( C_j \) with \( \alpha_j = \sigma \). In this case \( \alpha_\ell < \alpha_j \) and therefore \( m_{\sigma,\ell} < m_{\sigma,j} \) for any \( j \) such that \( \alpha_j = \sigma \). Notice that, if such \( \ell \) exists, it is unique. This means that for \( i_0 \in A \) such that \( m_{\sigma,i_0} \geq m_{\sigma,j} \) for all \( j \in A \) one has \( \alpha_{i_0} = \sigma \). Notice that such an index \( i_0 \) is, in general, not unique.

Now the proof is a consequence of the following statement:

**Lemma 1.** The semigroup of values of the irreducible curve \( C_{i_0} \) is generated by the elements of the set:

\[
\{m_{\tau,i_0} : \text{ for } \tau \text{ such that } \chi(\hat{E}_\tau) = 1\} \cup \{m_{\sigma,j} : j \neq i_0\}.
\]

**Proof.** Obviously all elements of the described set belong to the semigroup of values of the curve \( C_{i_0} \). The minimal set of generators of the semigroup of values of \( C_{i_0} \) is the set \( m_0 < m_1 < \ldots < m_g \) where \( m_i := m_{\sigma,i} \) \((i = 0, 1, \ldots, g)\) (see Figure 1). If the exponent \( m_q \) corresponding to a dead end \( \sigma_q \) of the dual graph of \( C_{i_0} \) does not appear in the Poincaré series of \( C \), i.e. \( \chi(\hat{E}_{\sigma_q}) = 0 \), then there exists \( k \in \{1, \ldots, r\} \) such that the strict transform \( C'_k \) of the component \( C_k \) in the minimal resolution of \( C_{i_0} \) intersects \( E_{\sigma_q} \). If \( C'_k \) is not resolved yet, there must exist a component \( E_\tau \) with \( \chi(\hat{E}_\tau) = 1 \), \( \tau > \sigma_q \) (produced by blowing-ups at points corresponding to \( C'_k \)) such that \( m_{\tau,i_0} = m_{\sigma,i_0} = m_q \). Thus, the exponent \( m_q \) does not appear among the ones in the first set of the statement if and only if the strict transform \( C'_k \) of \( C_k \) in the minimal resolution of \( C_{i_0} \) is smooth and transversal to \( E_{\sigma_q} \). But in this case \( m_q \) coincides with the intersection multiplicity between \( C_{i_0} \) and \( C_k \) and so \( m_q = m_{\sigma,k} \).

This proves the Theorem.

**Remark.** One may consider the multi-index filtration defined by a set of valuations corresponding to several irreducible plane curve singularities and several divisorial valuations. One has a formula for the Poincaré series of
such filtration similar to (1) and (3). One could ask whether this
Poincaré series defines the dual graph of the minimal resolution of
the set of curves and divisors. The following example shows
that this is, generally speaking, not the case. Let us consider
the divisorial valuation and the curve defined by the
minimal resolution shown on Figure 2. The component $E_{p+2}$
of the divisor

$$
\begin{array}{c}
1 \\
2 \\
3 \\
\vdots \\
p \\
p+2 \\
p+1
\end{array}
$$

Figure 2: Example.

corresponds to a curvette of type $A_{2p}$, i.e. $\{y^2 + x^{2p+1} = 0\}$, the curve under consideration is $\{y = 0\}$. One can easily see that, for any $p$, the Poincaré series of the corresponding filtration is

$$
P(u, t) = (1 - tu^2)^{-1},
$$

where $u$ (respectively $t$) is the variable corresponding to the divisorial valuation (to the curve respectively).

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