Dirichlet series under standard convolutions: variations on Ramanujan’s identity for odd zeta values

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Abstract
Inspired by a famous identity of Ramanujan, we propose a general formula linearizing the convolution of Dirichlet series as the sum of Dirichlet series with modified weights; its specialization produces new identities and recovers several identities derived earlier in the literature, such as the convolution of squares of Bernoulli numbers by Dixit et al., or the Fourier expansion of the convolution of Bernoulli–Barnes polynomials by Komori et al.

Keywords Riemann zeta function · Ramanujan’s identity for odd zeta values · Dirichlet series · Bernoulli polynomials · Euler polynomials · Hurwitz zeta function · Bessel zeta function

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1 A brief history and introduction

Ramanujan made many beautiful and elegant discoveries in his short life of 32 years. One of the famous identities given by Ramanujan which has attracted the attention of several mathematicians over the years is the following:

**Theorem 1.1** (Ramanujan’s formula for \( \zeta(2n+1) \)) If \( \alpha \) and \( \beta \) are positive real numbers such that \( \alpha \beta = \pi^2 \) and if \( n \in \mathbb{Z} \setminus \{0\} \), then we have

\[
\alpha^{-n} \left\{ \frac{1}{2} \zeta(2n+1) + \sum_{m=1}^{\infty} \frac{m^{-2n-1}}{e^{2am} - 1} \right\} - (-\beta)^{-n} \left\{ \frac{1}{2} \zeta(2n+1) + \sum_{m=1}^{\infty} \frac{m^{-2n-1}}{e^{2\beta m} - 1} \right\} = 2^{2n} \sum_{k=0}^{n+1} \frac{(-1)^{k-1} B_{2k} B_{2n-2k+2}}{(2k)! (2n - 2k + 2)!} \alpha^{n-k+1} \beta^k,
\]

(1.1)

where \( B_n \) denotes the \( n \)-th Bernoulli number and \( \zeta(s) \) represents the Riemann zeta function.

This theorem appears as Entry 21 in Chapter 14 of Ramanujan’s second notebook [3, 173]. It also appears in a formerly unpublished manuscript of Ramanujan which was published in its original handwritten form with his Lost Notebook [1, formula (28), page. 318–322]. For a fascinating account of the history and an elementary proof of Ramanujan’s identity for odd zeta values (henceforth simply Ramanujan’s identity), we refer the reader to [3, 5, 9].

The first published proof of Theorem 1.1 is due to Malurkar [25], although he was not aware that this formula can be found in Ramanujan’s Notebooks. Grosswald too rediscovered this formula and studied it more generally in [18, 19]. Berndt [6, Theorem 2.2] derived a general formula from which both Euler’s formula for \( \zeta(2n) \) (2.1), and Ramanujan’s identity for odd zeta values (1.1) follow as special cases, thus showing that Euler’s and Ramanujan’s formulas are natural companions of each other.

Ramanujan’s identity also has a number of applications. For instance, a contemporary interpretation of the above identity, as given for instance in [20], is that it encodes fundamental transformation properties of Eisenstein series on the full modular group and their Eichler integrals. This observation is extended in [7, Section 5] to weight \( 2k+1 \) Eisenstein series of level 2 through secant Dirichlet series. Moreover, Ramanujan’s identity also has applications in theoretical computer science [21] in the analysis of special data structures and algorithms.

Berndt and Straub [5] show that, for \( n \) a positive integer, equation (1.1) can be rewritten in terms of hyperbolic cotangent sums and is equivalent to

\[
\alpha^{-n} \sum_{m=1}^{\infty} \frac{\coth(\alpha m)}{m^{2n+1}} - (-\beta)^{-n} \sum_{m=1}^{\infty} \frac{\coth(\beta m)}{m^{2n+1}} = 2^{2n} \sum_{k=0}^{n+1} (-1)^k B_{2k} B_{2n-2k+2} \alpha^{n-k+1} \beta^k,
\]
where $\alpha, \beta$ are positive real numbers such that $\alpha\beta = \pi^2$.

To the best of our knowledge, the first recorded proof of the above identity was given by Nanjundiah [26] in 1951. Later Kongsiriwong [23] derived several beautiful generalizations and analogues of this identity.

Several interesting corollaries can be derived from formula (1.1) of Ramanujan. For instance, if we substitute $\alpha = \beta = \pi$ in Eq. (1.1), we deduce, for odd $n > 0$,

$$\zeta(2n + 1) = \pi (2\pi)^{2n} \sum_{k=0}^{n+1} \frac{(-1)^{k-1} B_{2k} B_{2n-2k+2}}{(2k)! (2n-2k+2)!} - 2 \sum_{m=1}^{\infty} \frac{m^{-2n-1}}{e^{2\pi m} - 1}, \quad (1.2)$$

a formula apparently due to Lerch [24]. Identity (1.2) is quite remarkable since it tells us that $\zeta(2n + 1)$ is equal to a rational multiple of $\pi^{2n+1}$ plus a rapidly convergent series.

Over the years, many generalizations and analogues of Theorem 1.1 of different kinds were studied. We provide a new and a very general viewpoint on this identity: our main result is Theorem 2.2, which provides a formula for the $n$-fold convolution of an arbitrary Dirichlet series as a sum over $n$ Dirichlet series with modified weights. We then explore some special cases and rederive some other generalizations of Ramanujan’s formula. Our transformation is tailored toward producing quasimodular functions with a desired convolution as the corresponding error term. For instance, we can interpret Ramanujan’s identity as a quasimodular transformation, by which we mean a transformation of the form $f(z) = f(-1/z) + \varepsilon(z)$, where $\varepsilon$ is some error function given by a convolution of Bernoulli numbers.

Therefore, we can interpret Ramanujan’s identity for odd zeta values as a quasimodular transformation for Eisenstein series, with the error given as a convolution of Bernoulli numbers. The first key reinterpretation is that the Bernoulli numbers in Ramanujan’s identity are in fact zeta functions evaluated at integers. Then instead of considering Ramanujan’s identity as a quasimodular transformation, we solely focus on the error term involving Bernoulli numbers, now interpreted as zeta functions. We show that Ramanujan’s identity can be written, in fact, in terms of special values of the Riemann zeta function evaluated at even integers. This new perspective gives a completely elementary method to prove quasimodular transformations. Under different specializations, this may allow us to discover new quasimodular forms with desired error term under the transformation $z \mapsto -\frac{1}{z}$.

Our main result shows that Ramanujan’s formula is the special case of a general result for the convolution of arbitrary Dirichlet series, parametrized by a set of zeros $x_n$ and weights $a_n$. This follows our general program [29, 30] to show that various identities for zeta and multiple zeta functions are special cases of polynomial identities, which then reduce to zeta functions under appropriate specialization. We proposed the definition of structural multiple zeta identities, which hold for the quasisymmetric zeta function $\sum_{n=1}^{\infty} \frac{1}{x_n}$ rather than the standard Riemann zeta function $\sum_{n=1}^{\infty} \frac{1}{n^s}$. This is the analog of a zeta function in the ring of quasisymmetric functions. Given such a structural identity, we obtain many interesting special cases for free, such as identities for the zeta function built from the zeros of an entire function, finite polynomial identities, and linear combinations of zeta functions.
Here we continue this program, but we have to consider Dirichlet series with weights, since the weights also transform when we consider the convolution of Dirichlet series. The main advantage to our master theorem is that this reduces the complexity of the calculations to solely computing the zeta generating function

\[ \sum_{n=1}^{\infty} \zeta(n) z^n = \sum_{n=1}^{\infty} a_n \frac{z}{x_n - z}, \]  

(1.3)
as discussed in subsection (2.4).

2 Main results and notations

2.1 Introduction

The starting point of the present study is the realization that Ramanujan’s identity (1.1) can be rephrased as a convolution identity for the Riemann zeta function. More precisely, after replacing the Bernoulli numbers by their zeta counterparts given by Euler’s formula for \( \zeta(2n) \), namely

\[ \frac{B_{2n}}{(2n)!} = \frac{(-1)^{n-1} \zeta(2n)}{2^{2n-1} \pi^{2n}} \quad (n \in \mathbb{N}). \]  

(2.1)

Ramanujan’s identity expresses the convolution (in the sense defined in the Proposition below) of Riemann zeta functions as an extension of a zeta function, namely a Dirichlet series, as follows.

**Proposition 2.1** Let \( \alpha, \beta \) be positive numbers such that \( \alpha \beta = \pi^2 \) and let \( n \) be a non-negative integer. An equivalent form of Ramanujan’s identity (1.1) is

\[ \alpha^{-n} \left\{ \frac{1}{2} \zeta(2n+1) + \sum_{m=1}^{\infty} \frac{m^{-2n-1}}{e^{2\alpha m} - 1} - \frac{1}{2\alpha} \zeta(2n+2) \right\} \]

\[ - (-\beta)^{-n} \left\{ \frac{1}{2} \zeta(2n+1) + \sum_{m=1}^{\infty} \frac{m^{-2n-1}}{e^{2\beta m} - 1} - \frac{1}{2\beta} \zeta(2n+2) \right\} \]

\[ = \frac{(-1)^n}{\pi^{2n+2}} \sum_{k=1}^{n} \alpha^{n-k+1} (-\beta)^k \zeta(2k) \zeta(2n-2k+2). \]  

(2.2)

It is thus natural to ask whether the convolution of an arbitrary Dirichlet series is still a Dirichlet series. And surprisingly, the answer turns out to be positive, and our main result is about how Dirichlet series transform by convolution. When Dirichlet series are multiplied, we obtain the coefficients of the product in terms of a Dirichlet
convolution, as

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} \sum_{m=1}^{\infty} \frac{b_m}{m^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} a_d b_{n/d}.$$  

The key here is that we consider a standard convolution of the form \( \sum_{k=0}^{n} a_k b_{n-k} \) instead, though classically Dirichlet series do not support a standard convolution structure.

### 2.2 Notations

For a sequence of non-zero complex numbers \( \{x_n\} \) which we will call zeros (see Sect. 3.2) and a sequence of associated complex weights \( \{a_n\} \), we define the Dirichlet series by

$$\zeta_{x,a} (N) = \sum_{n=1}^{\infty} \frac{a_n}{x_n^N}.$$  \hspace{1cm} (2.3)

Note that we assume that the Dirichlet series is convergent for \( N \geq 1 \). If the series diverges at \( N = 1 \), but has a finite abscissa of convergence, we can obtain analogous results. Next, we call the function

$$\psi_{x,a} (z) = \sum_{N=1}^{\infty} \zeta_{x,a} (N) z^N,$$  \hspace{1cm} (2.4)

associated with \( \zeta_{x,a} \) the zeta generating function. Using a geometric series, we can check that the generating function \( \psi_{x,a} (z) \) can be expressed in terms of the weights \( \{a_n\} \) and zeros \( \{x_n\} \) as follows

$$\psi_{x,a} (z) = \sum_{n=1}^{\infty} a_n \frac{z}{x_n - z}.$$  \hspace{1cm} (2.5)

When there is no ambiguity, the associated sequence of weights \( \{a_n\} \) will be omitted in the notations, so that for two Dirichlet series \( \zeta_{x,a} \) and \( \zeta_{y,b} \), we will write simply

$$\psi_{x,a} (z) = \psi_{x} (z), \quad \zeta_{x,a} (N) = \zeta_{x} (N) \quad \text{and} \quad \psi_{y,b} (z) = \psi_{y} (z), \quad \zeta_{y,b} (N) = \zeta_{y} (N).$$

Next, we introduce the following notation: the modified sequence of weights \( \{a \cdot \psi_{y,b}\} \) is

$$\left( a \cdot \psi_{y,b} \right)_n = a_n \psi_{y,b} (x_n).$$
so that the corresponding Dirichlet series is

\[
\zeta_{x,a,\psi_{y,b}}(N) = \sum_{n=1}^{\infty} \frac{a_n \psi_{y,b}(x_n) x_n^N}{N^n}.
\]

Similarly, we will denote by \((a,\psi_{y,b},\psi_{z,c})\) the sequence defined by

\[
(a,\psi_{y,b},\psi_{z,c})_n = a_n \psi_{y,b}(x_n) \psi_{z,c}(x_n)
\]

with the associated Dirichlet series

\[
\zeta_{x,a,\psi_{y,b},\psi_{z,c}}(N) = \sum_{n=1}^{\infty} \frac{a_n \psi_{y,b}(x_n) \psi_{z,c}(x_n) x_n^N}{N^n}.
\]

Finally the convolution of two Dirichlet series is defined as

\[
(\zeta_{y,b} \ast \zeta_{x,a})(N + 1) = \sum_{k=1}^{N} \zeta_{y,b}(k) \zeta_{x,a}(N + 1 - k),
\]

and the \(n\)-fold convolution of \(n\) Dirichlet series \(\zeta_{x,(1),a^{(1)}}, \ldots, \zeta_{x,(n),a^{(n)}}\) as

\[
\left( \prod_{i=1}^{n} \zeta_{x,(i),a^{(i)}} \right)(N + 1) = \sum \zeta_{x,(1),a^{(1)}}(k_1) \ldots \zeta_{x,(n),a^{(n)}}(k_n),
\]

where the sum is over the set of indices

\[
\left\{ (k_1, k_2, \ldots, k_n) : 1 \leq k_i \leq N, \sum_{i=1}^{n} k_i = N + 1 \right\}.
\]

2.3 Main results

Our main result is a formula which expresses the \(n\)-fold convolution of a set of Dirichlet series as a sum of the same \(n\) Dirichlet series with modified weights.

**Theorem 2.2** Let \(n \geq 2\). For a set of \(n\) Dirichlet series \(\{\zeta_{x,(i),a^{(i)}}\}_{1 \leq i \leq n}\), we have

\[
\left( \prod_{i=1}^{n} \zeta_{x,(i),a^{(i)}} \right)(N + 1) = \sum \zeta_{x,(1),a^{(1)}}(k_1) \ldots \zeta_{x,(n),a^{(n)}}(k_n),
\]

where the argument \(N + 1\) is removed for clarity. The special case \(n = 2\) reads

\[
\zeta_{y,b} \ast \zeta_{x,a} = \zeta_{x,a,\psi_{y}} + \zeta_{y,b,\psi_{z}},
\]
while the case \( n = 3 \) is

\[
\zeta_{z,c} \ast \zeta_{y,b} \ast \zeta_{x,a} = \zeta_{x,a} \ast \psi_y \ast \psi_z + \zeta_{y,b} \ast \psi_x \ast \psi_z + \zeta_{z,c} \ast \psi_x \ast \psi_y. \tag{2.8}
\]

For the sake of clarity, let us rephrase identities (2.7) and (2.8) in a more explicit way:

\[
\left( \zeta_{y,b} \ast \zeta_{x,a} \right) (N + 1) = \sum_{n=1}^{\infty} \left\{ \frac{a_n \psi_y (x_n)}{x_n^{N+1}} + \frac{b_n \psi_x (y_n)}{y_n^{N+1}} \right\}, \tag{2.9}
\]

and

\[
\left( \zeta_{z,c} \ast \zeta_{y,b} \ast \zeta_{x,a} \right) (N + 1) = \sum_{n=1}^{\infty} \left\{ \frac{a_n \psi_y (x_n) \psi_z (x_n)}{x_n^{N+1}} + \frac{b_n \psi_x (y_n) \psi_z (y_n)}{y_n^{N+1}} + \frac{c_n \psi_z (z_n) \psi_y (z_n)}{z_n^{N+1}} \right\}. \tag{2.10}
\]

The proof of this Theorem 2.2 is provided in Sect. 4. It also reveals a natural algebra underlying the convolution of Dirichlet series which we now make explicit.

Identity (2.7) is a simple consequence of a geometric sum. Trying to deduce the three-fold convolution (2.8) from its two-fold counterpart (2.7), we use the associativity of convolution to obtain

\[
\zeta_{z,c} \ast \zeta_{y,b} \ast \zeta_{x,a} = \zeta_{z,c} \ast \left( \zeta_{y,b} \ast \zeta_{x,a} \right),
\]

and now its distributivity with respect to the addition to produce

\[
\zeta_{z,c} \ast \zeta_{y,b} \ast \zeta_{x,a} = \zeta_{z,c} \ast \zeta_{x,a} \ast \psi_y + \zeta_{z,c} \ast \zeta_{y,b} \ast \psi_x.
\]

Expanding both terms according to rule (2.7) produces

\[
\zeta_{z,c} \ast \zeta_{x,a} \ast \psi_y = \zeta_{z,c} \ast \psi_{x,a} \psi_y + \zeta_{x,a} \ast \psi_y \ast \psi_{z,c},
\]

and

\[
\zeta_{z,c} \ast \zeta_{y,b} \ast \psi_x = \zeta_{z,c} \ast \psi_{y,b} \psi_x + \zeta_{y,b} \ast \psi_x \ast \psi_{z,c}.
\]

Both Dirichlet series in \( x \) and \( y \) can be expressed in the more simple way

\[
\zeta_{x,a} \ast \psi_y \ast \psi_{z,c} = \zeta_{x,a} \ast \psi_y \ast \psi_z \quad \text{and} \quad \zeta_{y,b} \ast \psi_x \ast \psi_{z,c} = \zeta_{y,b} \ast \psi_x \ast \psi_z.
\]

The fact that sum of the two Dirichlet series in \( z \) simplifies to

\[
\zeta_{z,c} \ast \psi_{x,a} \psi_y + \zeta_{z,c} \ast \psi_{y,b} \psi_x = \zeta_{z,c} \ast \psi_x \ast \psi_y.
\]
is deduced from Lemma 4.1 provided and proved in Sect. 4, which we restate here

$$
\psi_{x,a}\psi_y (z) + \psi_{y,b}\psi_x (z) = \psi_{x,a} (z) \psi_{y,b} (z),
$$

(2.11)

finally producing

$$
\zeta, c * \zeta, b * \zeta, a = \zeta, x, a, \psi, y, \psi, z + \zeta, y, b, \psi, y, \psi, z + \zeta, z, c, \psi, x, \psi, y,
$$
as expected. This completes the proof of Eq. (2.8). Let us notice that the sum to product identity for generating functions (2.11) which underlies this convolution algebra for Dirichlet series is based on the innocent looking sum to product identity

$$
\left( \frac{z}{y - z} \right) \left( \frac{y}{x - y} \right) + \left( \frac{z}{x - z} \right) \left( \frac{x}{y - x} \right) = \left( \frac{z}{y - z} \right) \left( \frac{z}{x - z} \right).
$$

3 Particular cases and extensions

This section shows how our main result allows to recover different versions of convolution identities for Dirichlet series, but also how it can generate new ones. As an example, we derive new convolution identities for the Bessel zeta and Hurwitz zeta functions.

3.1 Generalized Ramanujan identity

3.1.1 A first generalization

Let us first consider the case $n = 2$ in the Ramanujan setup.

**Theorem 3.1** With arbitrary $\alpha, \beta > 0$, the choice

$$
x_n = \frac{n^2 \pi^2}{\beta}, \quad y_n = -\frac{n^2 \pi^2}{\alpha},
$$
in Eq. (2.7) produces

$$
\sum_{k=1}^{N} \zeta(2k) \zeta(2N + 2 - 2k) \alpha^{N-k+1} (-\beta)^k
= \frac{1}{2} \left[ \alpha^{N+1} + (-\beta)^{N+1} \right] \zeta(2N + 2) - \frac{\pi}{2} \alpha^{N+1} \sum_{n=1}^{\infty} \frac{1}{n^{2N+1}} \sqrt{\frac{\beta}{\alpha}} \coth \left( \pi n \sqrt{\frac{\beta}{\alpha}} \right) - \frac{\pi}{2} (-\beta)^{N+1} \sum_{n=1}^{\infty} \frac{1}{n^{2N+1}} \sqrt{\frac{\alpha}{\beta}} \coth \left( \pi n \sqrt{\frac{\alpha}{\beta}} \right).
$$

(3.1)
Although it seems that we do not need the constraint $\alpha \beta = \pi^2$ as in Eq. (1.1), the fact that identity (3.1), now expressed in terms of the ratio $\mu = \beta/\alpha$,

$$
\sum_{k=1}^{N} (-1)^{k-1} \xi(2k) \xi(2N + 2 - 2k) \mu^k = -\frac{1}{2} \left[ \mu^{N+1} + (-1)^{N+1} \right] \xi(2N + 2)
$$

$$
+ \frac{\pi}{2} \sqrt{\mu} \sum_{n=1}^{\infty} \frac{\coth (\pi n \sqrt{\mu})}{n^{2N+1}} + \frac{(-1)^{N+1} \pi}{2} \sum_{n=1}^{\infty} \frac{\mu^{N+1}}{n^{2N+1}} \coth \left( \frac{\pi n}{\sqrt{\mu}} \right),
$$

(3.2)

depends only on parameter $\mu$, shows that it is a one-parameter identity equivalent to (1.1).

### 3.1.2 Bernoulli numbers

We now make the following choice in Eq. (2.7):

$$
a_m = e^{2i\pi m y_1}, \quad x_m = \frac{2i\pi m}{\omega_1}, \quad b_m = e^{2i\pi m y_2}, \quad y_m = \frac{2i\pi m}{\omega_2},
$$

with $m \in \mathbb{Z} \setminus \{0\}$, so that we have

$$
\xi_x (n) = -\omega_1^n \frac{B_n (y_1)}{n!} \quad \text{and} \quad \xi_y (n) = -\omega_2^n \frac{B_n (y_2)}{n!}, \quad (n \in \mathbb{N}).
$$

This produces, assuming $0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1$, and $\Im (\omega_1/\omega_2) \neq 0$, the following identity

$$
\sum_{k=0}^{N+1} \omega_1^k \frac{B_k (y_1)}{k!} \omega_2^{N-k+1} \frac{B_{N+1-k} (y_2)}{(N+1-k)!}
$$

$$
= -\omega_1 \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{e^{2i\pi m y_2}}{\left( \frac{2i\pi m}{\omega_2} \right)^N} e^{2i\pi m \frac{\omega_1}{\omega_2} y_1} - 1
$$

$$
- \omega_2 \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{e^{2i\pi m y_1}}{\left( \frac{2i\pi m}{\omega_1} \right)^N} e^{2i\pi m \frac{\omega_2}{\omega_1} y_2} - 1.
$$

(3.3)

The proof of this identity is given in Sect. 4 and reveals the following generalization, that can be considered as the Fourier expansion of the Bernoulli–Barnes polynomial.
Theorem 3.2 If the coefficients \( \{\omega_i\}_{1 \leq i \leq n} \) are such that \( \Im(\omega_i/\omega_j) \neq 0, \ i \neq j \), and if \( 0 \leq y_i \leq 1 \) for \( 1 \leq i \leq n \), then we have

\[
\sum_{k_1,\ldots,k_n} \prod_{i=1}^n \omega_i^{k_i-1} \frac{B_{k_i} (y_i)}{k_i!} = - \sum_{i=1}^n \frac{1}{\omega_i} \sum_{m \neq 0} \frac{e^{2\imath \pi m y_i}}{\left( 2\imath \pi m / \omega_i \right)^N} \prod_{j \neq i} \frac{e^{2\imath \pi m \omega_j / \omega_i} y_j}{e^{2\imath \pi m \omega_j / \omega_i} - 1}, \tag{3.4}
\]

where the sum on the left-hand side is over the set of indices

\[
\left\{ (k_1, k_2, \ldots, k_n) : 0 \leq k_i \leq N + 1, \sum_{i=1}^n k_i = N + 1 \right\}.
\]

This identity is provided in [22] in the special case \( y = y_1 = \cdots = y_n \) and is derived using properties of the Barnes zeta function. Our proof relies solely on our main convolution result for Dirichlet zeta functions and on the classical Fourier series expansions

\[
\frac{1}{z^2} + 2 \sum_{m=1}^{\infty} \frac{1}{z^2 + \frac{4\pi^2 m^2}{\omega_i^2}} \cos(2\pi m y_1) = \frac{\omega_1}{2z} \cosh\left( (\omega_1 z) \left( \frac{1}{2} - y_1 \right) \right),
\]

and

\[
\sum_{m=1}^{\infty} \frac{m}{z^2 + \frac{4\pi^2 m^2}{\omega_i^2}} \sin(2\pi m y_1) = \frac{\omega_1^2}{8\pi} \sinh\left( (\omega_1 z) \left( \frac{1}{2} - y_1 \right) \right),
\]

which can be found, for example, as [27, equations (1.51) and (1.53)].

A version of Theorem 3.2 for Euler polynomials \( E_n(x) \), defined by generating function

\[
\sum_{n=0}^{\infty} \frac{E_n(x)}{n!} z^n = \frac{2}{e^z + 1} e^{z x},
\]

is deduced next.

Corollary 3.3 If \( \{\omega_i\}_{1 \leq i \leq n} \) are such that \( \Im(\omega_i/\omega_j) \neq 0, \ i \neq j \), and if \( 0 \leq x_i \leq \frac{1}{2} \) for \( 1 \leq i \leq n \), then we have

\[
\sum_{k_1,k_2,\ldots,k_n} \prod_{i=1}^n \omega_i^{k_i-1} \frac{E_{k_i} (2x_i)}{k_i!} = 2^{2N+n+3} \omega_1 \ldots \omega_n \sum_{i=1}^{n} \sum_{m \text{ odd}} \frac{e^{2\imath \pi m x_i}}{2\imath \pi m / \omega_i} \prod_{j \neq i} \frac{e^{2\imath \pi m \omega_j / \omega_i} x_j}{e^{2\imath \pi m \omega_j / \omega_i} - 1}, \tag{3.5}
\]
where as before the sum on the left-hand side is over the set of indices

\[
\left\{ (k_1, k_2, \ldots, k_n) : 0 \leq k_i \leq N + 1, \sum_{i=1}^{n} k_i = N + 1 \right\}.
\]

### 3.1.3 Squares of Bernoulli numbers

Our general Theorem 2.2 deals with single sums. However, a quick look at its proof shows that they can be replaced by multiple sums. We produce here the case of double sums. Starting from a sequence of weights \( \{a_{m,n}\} \) and a sequence of roots \( \{x_{m,n}\} \), let us consider the double Dirichlet sum

\[
\zeta_{x,a}(N) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{m,n}}{x_{m,n}^{N+1}},
\]

and its associated generating function

\[
\psi_{x,a}(z) = \sum_{N=1}^{\infty} \zeta_{x,a}(N) z^N = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} \frac{z}{x_{m,n} - z}.
\]

**Theorem 3.4** For two double sequences of weights \( \{a_{m,n}\} \) and \( \{b_{m,n}\} \) and two sequences of zeros \( \{x_{m,n}\} \) and \( \{y_{m,n}\} \), the convolution of the Dirichlet series \( \zeta_{x,a} \) and \( \zeta_{y,b} \) satisfies

\[
\left( \zeta_{y,b} * \zeta_{x,a} \right)(N + 1) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ a_{m,n} \psi_{y}(x_{m,n}) \frac{x_{m,n}^{N+1}}{y_{m,n}^{N+1}} + b_{m,n} \psi_{x}(y_{m,n}) \frac{y_{m,n}^{N+1}}{x_{m,n}^{N+1}} \right].
\]

The proof of this Theorem is omitted here as it only requires replacing simple sums by double sums in the proof of Eq. (2.7).

As a consequence of this result, we consider the simple choice

\[
a_{p,q} = 1, \quad b_{r,s} = 1, \quad x_{p,q} = -\frac{p^2 q^2}{\alpha^2}, \quad y_{r,s} = \frac{r^2 s^2}{\beta^2}.
\]

This produces the Dirichlet series

\[
\zeta_{x,a}(s) = \left(-\alpha^2\right)^s \zeta^2(2s), \quad \zeta_{y,b}(s) = \beta^2 \zeta^2(2s),
\]

and the zeta generating function

\[
\psi_{x}(z) = -\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{\alpha^2 z}{p^2 q^2 + \alpha^2 z} = -\sum_{n=1}^{\infty} \tau_0(n) \frac{\alpha^2 z}{n^2 + \alpha^2 z},
\]
with \( \tau_0(n) \) as the number of divisors of \( n \), and where we have applied the general formula

\[
\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} F(pq) = \sum_{n=1}^{\infty} \sum_{q|n} F(n) = \sum_{n=1}^{\infty} \tau_0(n) F(n).
\] (3.6)

Similarly, we find that

\[
\psi_y(z) = \sum_{m=1}^{\infty} \tau_0(m) \frac{\beta^2 z}{m^2 - \beta^2 z}.
\]

Applying (2.7) produces the following important identity.

**Corollary 3.5** For arbitrary positive real numbers \( \alpha \) and \( \beta \), we have

\[
\sum_{k=1}^{N} (-1)^k \alpha^{2k} \beta^{2N+2-2k} \zeta^2(2k) \zeta^2(2N + 2 - 2k) \\
= -\beta^{2N+2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \tau_0(n) \tau_0(m) \frac{\alpha^2}{m^{2N} + \alpha^2 m^2} \\
- \left( -\alpha^2 \right)^{N+1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \tau_0(n) \tau_0(m) \frac{\beta^2}{n^{2N} + \beta^2 n^2}.
\] (3.7)

This result provides an alternate but equivalent identity compared to the identity by Dixit and Gupta [11, Thm. 2.1]: assuming \( \alpha, \beta \in \mathbb{R}^+ \) and \( \alpha \beta = \pi^2 \),

\[
\left( -\beta^2 \right)^{-N} \left\{ \zeta^2(2N + 1) \left( \gamma + \log \left( \frac{\beta}{\pi} \right) - \frac{\zeta'(2N + 1)}{\zeta(2N + 1)} \right) + \sum_{n=1}^{\infty} \frac{\tau_0(n)}{n^{2N+1}} \Omega \left( \frac{\beta^2 n}{\pi^2} \right) \right\} \\
- \left( \alpha^2 \right)^{-N} \left\{ \zeta^2(2N + 1) \left( \gamma + \log \left( \frac{\alpha}{\pi} \right) - \frac{\zeta'(2N + 1)}{\zeta(2N + 1)} \right) + \sum_{n=1}^{\infty} \frac{\tau_0(n)}{n^{2N+1}} \Omega \left( \frac{\alpha^2 n}{\pi^2} \right) \right\} \\
= 2^{4N} \pi^{N+1} \sum_{j=0}^{N+1} \frac{(-1)^j B_{2j}^2 B_{2N+2-2j}^2}{(2j)!^2 ((2N + 2 - 2j)!^2 \left( \alpha^2 \right)^j \left( \beta^2 \right)^{N-j+1}}.
\] (3.8)

We have restated their transformation in terms of the Koshliakov kernel \( \Omega(x) \), which has equivalent expressions [11]

\[
\Omega(x) = -\gamma - \frac{1}{2} \log x - \frac{1}{4 \pi^2 x} + \frac{x}{\pi} \sum_{j=1}^{\infty} \frac{\tau_0(j)}{x^2 + j^2}.
\] (3.9)
\[ = 2 \sum_{j=1}^{\infty} \tau_0(j) \left( K_0 \left( 4\pi \exp(i\pi/4)\sqrt{jx} \right) + K_0 \left( 4\pi \exp(-i\pi/4)\sqrt{jx} \right) \right). \]

(3.10)

where \( K_0 \) represents the modified Bessel function of the second kind.

We note that although the definition of \( \Omega \) is rather unmotivated, it naturally arises in the theory of the divisor function. Ramanujan’s identity deals with the summation kernel \( (\exp(2\pi x) - 1)^{-1} \), which has a pole with residue 1 at \( x = 0 \), and poles at \( x = \pm in, n \geq 1 \), with residues \( \frac{\tau_0(n)}{2\pi} \). The Koshliakov kernel also has a pole with residue \( -\frac{1}{4\pi} \) at \( x = 0 \), and poles at \( x = \pm in \) for \( n \geq 1 \), with residues \( \frac{\tau_0(n)}{2\pi} \) instead. This allows us to construct generalizations of Ramanujan’s zeta identities involving divisor sums and Bessel functions instead.

To show the equivalence of these two transformations, the only tool we need is the partial fraction decomposition (3.9). We write

\[
\zeta^2(2N+1) \left( \gamma + \log \left( \frac{\alpha}{\pi} \right) - \frac{\zeta'(2N+1)}{\zeta(2N+1)} \right) + \sum_{n=1}^{\infty} \frac{\tau_0(n)}{n^{2N+1}} \Omega \left( \frac{\alpha^2 n}{\pi^2} \right) \\
= \sum_{n=1}^{\infty} \frac{\tau_0(n)}{n^{2N+1}} \left( \gamma + \log \left( \frac{\alpha}{\pi} \right) + \frac{1}{2} \log n \right) + \sum_{n=1}^{\infty} \frac{\tau_0(n)}{n^{2N+1}} \Omega \left( \frac{\alpha^2 n}{\pi^2} \right) \\
= \sum_{n=1}^{\infty} \frac{\tau_0(n)}{n^{2N+1}} \left( \gamma + \log \left( \frac{\alpha}{\pi} \right) + \frac{1}{2} \log n \right) \\
+ \sum_{n=1}^{\infty} \frac{\tau_0(n)}{n^{2N+1}} \left( -\gamma - \log \left( \frac{\alpha \sqrt{n}}{\pi} \right) - \frac{1}{4\pi} \frac{\pi^2}{\alpha^2 n} + \frac{\alpha^2 n}{\pi^2} \sum_{j=1}^{\infty} \frac{\tau_0(j)}{j^2 + \alpha^4 n^2} \right) \\
= \sum_{n=1}^{\infty} \frac{\tau_0(n)}{n^{2N+1}} \left( -\frac{\pi}{4\alpha^2 n} + \alpha^2 n \pi \sum_{j=1}^{\infty} \frac{\tau_0(j)}{\pi^4 j^2 + \alpha^4 n^2} \right) \\
= -\frac{\pi}{4\alpha^2} \zeta^2(2N+2) + \alpha^2 \pi \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{\tau_0(n)}{n^{2N}} \frac{\tau_0(j)}{\pi^4 j^2 + \alpha^4 n^2}. 
\]

We used the Dirichlet series identities

\[ \sum_{n=1}^{\infty} \frac{\tau_0(n)}{n^s} = \zeta^2(s), \]

and

\[ \sum_{m=1}^{\infty} \frac{\tau_0(m) \log m}{m^s} = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{\log pq}{p^s q^s} = 2 \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{\log p}{p^s q^s} = -2 \zeta(s) \zeta'(s). \]
Now repeat this argument with $\beta$, then add and use the fact that $\alpha\beta = \pi^2$. Finally, rewriting the zeta functions at even integers as Bernoulli numbers completes the proof.

With $c \in \mathbb{N}$, a natural extension of identity (3.6) reads

$$
\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} p^c F(pq) = \sum_{n=1}^{\infty} \tau_c(n) F(n),
$$

where $T_c(n)$ is the sum of $c$ powers of positive divisors function so that, with the choice

$$a_{p,q} = q^c, \quad b_{r,s} = s^d, \quad x_{p,q} = -\frac{p^2 q^2}{\alpha^2}, \quad y_{r,s} = \frac{r^2 s^2}{\beta^2},$$

we obtain the Dirichlet functions

$$\zeta_{x,a}(k) = (-1)^k \alpha^{2k} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{q^c}{p^{2k} q^{2k}} = (-1)^k \alpha^{2k} \zeta(2k) \zeta(2k-c),$$

and

$$\zeta_{y,b}(k) = \beta^{2k} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{s^d}{r^{2k} s^{2k}} = \beta^{2k} \zeta(2k) \zeta(2k-d).$$

This produces the following formula

**Corollary 3.6** For arbitrary negative integers $c$ and $d$, and arbitrary positive real numbers $\alpha$ and $\beta$, the following identity holds

$$
\sum_{k=1}^{N} (-1)^k \zeta(2k) \zeta(2k-c) \zeta(2N + 2 - 2k) \zeta(2N + 2 - 2k - d) \alpha^{2k} \beta^{2N+2-2k} \alpha^2 \beta^2 m^2 - \alpha^2 \beta^2 n^2.
$$

An open problem is to describe the right generalization of the Koshliakov kernel $\Omega(x)$, so that the method used to prove [11, Thm. 2.1] generates a transformation with free parameters $c$ and $d$. In [11], the authors used the integral representation for the Koshliakov kernel in terms of a Mellin transform, together with a shift of the line of integration.
3.2 The Bessel zeta case

When the numbers \( \{x_n\} \) are the roots of an analytic function of order 1, so that the function possesses, under appropriate conditions on convergence, a Weierstrass infinite product representation

\[
f(z) = f(0) \prod_{k=1}^{\infty} \left(1 - \frac{x}{x_k}\right),
\]

the generating function associated with the zeros \( \{x_n\} \) and with the coefficients \( a_n = 1 \) is easily computed as

\[
-\frac{f'(z)}{f(z)} = -z \frac{d}{dz} \log f(z) = -z \frac{d}{dz} \sum_{k=1}^{\infty} \log \left(1 - \frac{z}{x_k}\right) = z \sum_{k=1}^{\infty} \frac{1/x_k}{1 - z/x_k} = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \left(\frac{z}{x_k}\right)^{\ell} = \sum_{\ell=1}^{\infty} \xi_{x,1}(\ell) z^{\ell}.
\]  

(3.12)

Bessel functions and their zeros produce an opportunity to test our general formula in this case. One advantage of this parameterized family of functions is that a special case for the value of the parameter, namely \( \nu = \frac{1}{2} \), recovers the previous Riemann zeta setup.

The normalized Bessel function of the first kind \( j_{\nu}(z) \), defined by

\[
j_{\nu}(z) = 2^{\nu} \Gamma(\nu + 1) \frac{J_{\nu}(z)}{z^{\nu}},
\]  

(3.13)

is an entire function such that \( j_{\nu}(0) = 1 \). It has the infinite product expansion

\[
j_{\nu}(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\sqrt{j_{\nu,n}}^2}\right),
\]

which reveals the real numbers \( \{j_{\nu,n}\}_{n \geq 1} \) as the zeros of the Bessel function \( J_{\nu} \). The Bessel zeta function is defined in terms of these zeros as

\[
\zeta_{B,\nu}(s) = \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^{2s}},
\]  

(3.14)

and the corresponding zeta generating function reads

\[
\sum_{N=1}^{\infty} \zeta_{B,\nu}(N) z^N = -z \frac{d}{dz} \log j_{\nu}(\sqrt{z}) = \frac{z}{4(\nu + 1)} \frac{j_{\nu+1}(\sqrt{z})}{j_{\nu}(\sqrt{z})}.
\]  

(3.15)

Therefore, with \( I_{\nu} \) the modified Bessel function of the first kind, we deduce the following.
Theorem 3.7 The Bessel zeta function satisfies the convolution identity

\[
\sum_{k=1}^{N} (-1)^k \zeta_{B,v}(k) \zeta_{B,v}(N + 1 - k) \alpha^{N-k+1} \beta^k
= v \left( \alpha^{N+1} + (-\beta)^{N+1} \right) \zeta_{B,v}(N + 1)
- \frac{\alpha^{N+1}}{2} \sqrt{\frac{\beta}{\alpha}} \sum_{q=1}^{\infty} \frac{1}{j_{\nu+1,q}} \frac{I_{v-1}\left(\sqrt{\frac{\beta}{\alpha}} j_{\nu,q}\right)}{I_{\nu}\left(\sqrt{\frac{\beta}{\alpha}} j_{\nu,q}\right)}
- \frac{(-\beta)^{N+1}}{2} \sqrt{\frac{\alpha}{\beta}} \sum_{q=1}^{\infty} \frac{1}{j_{\nu+1,q}} \frac{I_{v-1}\left(\sqrt{\frac{\alpha}{\beta}} j_{\nu,q}\right)}{I_{\nu}\left(\sqrt{\frac{\alpha}{\beta}} j_{\nu,q}\right)}.
\]

Corollary 3.8 The special case \( \nu = \frac{1}{2} \) produces the zeros \( j_{\frac{1}{2},n} = \pm n \pi, \) where \( n \neq 0 \) so that

\[
j_{\frac{1}{2}}(z) = \frac{\sin(z)}{z} = \text{sinc}(z),
\]

and

\[
\frac{I_{-\frac{1}{2}}\left(\sqrt{\frac{\beta}{\alpha}} q\pi\right)}{I_{\frac{1}{2}}\left(\sqrt{\frac{\beta}{\alpha}} q\pi\right)} = \coth\left(\sqrt{\frac{\beta}{\alpha}} q\pi\right),
\]

which recovers the Riemann zeta version of Ramanujan’s identity for odd zeta values.

The special case \( \nu = \frac{3}{2} \) is interesting since the ratio \( \frac{I_{\frac{1}{2}}(z)}{I_{\frac{3}{2}}(z)} \) can be explicitly computed as

\[
\frac{I_{\frac{1}{2}}(z)}{I_{\frac{3}{2}}(z)} = \frac{z}{z \coth z - 1},
\]

while the numbers \( j_{\frac{3}{2},n} \) are the positive roots of the equation

\[ \tan(x) = x. \]

Hence, we have the following corollary.
Corollary 3.9 Define \( \{z_n\}_{n \geq 1} \) as the strictly positive roots of the equation

\[
\tan (x) = x,
\]

and the Bessel zeta function

\[
\zeta_{B, \frac{3}{2}} (s) = \sum_{n=1}^{\infty} \frac{1}{z_n^{2s}},
\]

then the following identity holds

\[
\sum_{k=1}^{N} (-1)^{k-1} \zeta_{B, \frac{3}{2}} (k) \zeta_{B, \frac{3}{2}} (N + 1 - k) \alpha^{N-k+1} \beta^k
\]

\[
= -\frac{3}{2} \zeta_{B, \frac{3}{2}} (N + 1) \left[ \alpha^{N+1} + (-\beta)^{N+1} \right]
\]

\[
+ \sum_{n=1}^{\infty} \frac{\alpha^N \beta}{\sqrt{\beta} \alpha z_n^2} \frac{1}{\coth \left( \frac{\beta}{\alpha} z_n \right) - 1}
\]

\[
+ \sum_{n=1}^{\infty} \frac{\beta^N \alpha}{\sqrt{\alpha} \beta z_n^2} \frac{(-1)^{N+1}}{\coth \left( \frac{\alpha}{\beta} z_n \right) - 1}.
\]

Corollary 3.10 The case \( \nu = -\frac{1}{2} \) for which

\[
J_{-\frac{1}{2}} (z) = \sqrt{\frac{2}{\pi}} \frac{\cos z}{\sqrt{z}} \quad \text{and} \quad j_{-\frac{1}{2}, n} = \frac{\pi}{2} (2n - 1),
\]

produces a zeta function which is the dissection of the Riemann zeta function,

\[
\zeta_{B, -\frac{1}{2}} (n) = \frac{2^{2n} - 1}{\pi^{2n}} \zeta(2n).
\]

Since

\[
\frac{I_{-\frac{1}{2}} (z)}{I_{-\frac{1}{2}} (z)} = \tanh z - \frac{1}{z},
\]

we obtain the following identity after simplification

\[
\sum_{k=1}^{N} (-1)^k \left( 2^{2k} - 1 \right) \zeta(2k) \left( 2^{2N+2-2k} - 1 \right) \zeta(2N + 2 - 2k) \alpha^{N-k+1} \beta^k
\]

\[
= -\alpha^{N+1} \frac{\sqrt{\beta}}{2 \pi \alpha} \sum_{q=1}^{\infty} \frac{\tanh \left( \frac{\pi}{2} \sqrt{\beta} (2q - 1) \right)}{(2q - 1)^{2N+1}}
\]

\[
- (-\beta)^{N+1} \frac{\sqrt{\alpha}}{2 \pi \alpha}.
\]
$$\times \sqrt{\frac{\alpha}{\beta}} \sum_{q=1}^{\infty} \tanh \left( \frac{\pi}{2} \sqrt{\frac{\alpha}{\beta}} (2q - 1) \right) \frac{1}{(2q - 1)^{2N+1}}.$$  

This identity has appeared several times earlier in the literature, for example, in the articles by Berndt [6], Malurkar [25], Chourasiya et al. [10, Remark 2] and by Dixit and Gupta [12, Corollary 4.2].

### 3.3 The Hurwitz zeta case

We look now at the Hurwitz zeta function defined by

$$\zeta_H (s; x) := \sum_{p=0}^{\infty} \frac{1}{(p + x)^s}, \quad (3.16)$$

and choose

$$x_n = \frac{(n + x)^2}{\beta}, \quad y_n = -\frac{(n + y)^2}{\alpha},$$

in Eq. (2.7), which requires the generating function

$$\sum_{n=0}^{\infty} \frac{z}{(n + x)^2 - z} = \frac{\sqrt{z}}{2} \left[ \psi (x + \sqrt{z}) - \psi (x - \sqrt{z}) \right],$$

where, in this whole subsection, $\psi (x)$ denotes the digamma function (not to be confused with the zeta generating function (2.4)). After simplification, the resulting equivalent identity is as follows.

**Theorem 3.11** The equivalent Ramanujan’s identity for the Hurwitz zeta function (3.16), with $\psi$ denoting the digamma function, is

$$\sum_{k=1}^{N} \zeta_H (2k; x) \zeta_H (2N + 2 - 2k; y) \beta^k (-\alpha)^{N+1-k}$$

$$= \frac{i}{2} \beta^{N+1} \sqrt{\frac{\alpha}{\beta}} \sum_{n=0}^{\infty} \frac{1}{(n + x)^{2N+1}}$$

$$\times \left[ \psi \left( y + i \sqrt{\frac{\alpha}{\beta}} (x + n) \right) - \psi \left( y - i \sqrt{\frac{\alpha}{\beta}} (x + n) \right) \right]$$

$$+ \frac{i}{2} (-\alpha)^{N+1} \sqrt{\frac{\beta}{\alpha}} \sum_{n=0}^{\infty} \frac{1}{(y + n)^{2N+1}}$$

$$\times \left[ \psi \left( x + i \sqrt{\frac{\beta}{\alpha}} (y + n) \right) - \psi \left( x - i \sqrt{\frac{\beta}{\alpha}} (y + n) \right) \right]. \quad (3.17)$$
3.4 A multisection case

In an attempt to specialize our main result (2.7), we look for an identity which involves an usual odd $\zeta(2N+1)$ term together with a $\zeta(2mN+1)$ term for a positive odd integer $m$ (this requirement will be explained in the forthcoming remark 3.13). This can be achieved using the following choice.

**Theorem 3.12** For $m$ an odd integer and $\alpha, \beta > 0$ arbitrary real numbers, the choices

\[ a_n = 1, \quad x_n = \frac{n^2}{\alpha}, \quad b_n = n^{m-1}, \quad y_n = -\frac{n^{2m}}{\beta}, \quad (3.18) \]

and the use of multisection identity

\[ \sum_{n=1}^{\infty} \frac{z^{2m}n^{m-1}}{z^{2m} - n^{2m}} = \frac{\pi z^m}{2m} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} (-1)^j \cot \left( \pi z e^{i \frac{j\pi}{m}} \right), \quad (3.19) \]

produces the following identity

\[
\sum_{k=1}^{N} \zeta(2k) \zeta(2m(N-k) + m + 1) \alpha^k (-\beta)^{N+1-k} = \alpha^{N+1} \frac{\pi \beta}{2m} \sqrt{\frac{\alpha}{\beta}} \sum_{n=1}^{\infty} \frac{1}{n^{2N+1}} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} (-1)^j \coth \left( \pi n \frac{1}{m} \left( \sqrt{\frac{\beta}{\alpha}} \right)^{\frac{1}{m}} e^{i \frac{j\pi}{m}} \right)
\]

\[
+ \frac{1}{2} (-\beta)^{N+1} \zeta(2mN + m + 1)
\]

\[- (-\beta)^{N+1} \frac{\pi \alpha}{2} \sqrt{\frac{\beta}{\alpha}} \sum_{n=1}^{\infty} \frac{1}{n^{2mN+1}} \coth \left( \pi \sqrt{\frac{\alpha}{\beta}} n^{m} \right). \quad (3.20) \]

**Remark 3.13** The requirement for the integer $m$ to be odd is due to the fact that identity (3.19) holds in this case only. We are unaware of an equivalent identity in the case where $m$ is even. In [17], Dixit and Maji derived the following identity: assuming $\alpha \beta^{N} = \pi^{N+1}$,

\[
\left( \alpha^{\frac{2m}{m+1}} \right)^{-N} \left( \frac{1}{2} \zeta(2mN+1) + \sum_{n=1}^{\infty} \frac{1}{n^{2mN+1}} \frac{e^{(2n)^{m} \alpha}}{\alpha - 1} \right)
\]

\[
= \left( -\beta^{\frac{2m}{m+1}} \right)^{-N} \frac{2N(m-1)}{m}
\]

\[
\times \left( \frac{1}{2} \zeta(2N+1) + \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} \sum_{n=1}^{\infty} \frac{(-1)^{\frac{m+2j+3}{2}}}{n^{2N+1}} \frac{e^{(2n)^{\frac{1}{m}} \beta \exp\left(\frac{i\pi j}{m}\right) \exp\left(\frac{1}{m}\right)}}{\exp\left(\frac{1}{m}\right) - 1} \right)
\]
\[ + (-1)^{N+\frac{m+1}{2}} 2^{mN} \sum_{j=0}^{N+\frac{m+1}{2m}} (-1)^j B_j B_{2m(N-j)+m+1} \frac{2j}{m+1} \beta^{m+2m^2(N-j)} \alpha^{m-1} \beta^{m+2m^2(N-j)} \frac{1}{(2j)! (2m(N-j)+m+1)!}. \]

(3.21)

This identity is a special case of identity (3.20) with the additional constraint \( \alpha \beta^{N} = \pi^{N+1} \).

### 3.5 Higher Herglotz function analogue of Ramanujan’s identity

As a special case, Theorem 2.2 gives the following transformation for a combination of the Vlasenko–Zagier higher Herglotz function \( F_k(x) \) which is analogous to Ramanujan’s identity (1.1) and is derived by Dixit et al. in [13, Corollary 2.4].

**Proposition 3.14** Let \( \alpha \) and \( \beta \) be two complex numbers such that \( \Re(\alpha) > 0, \Re(\beta) > 0 \) and \( \alpha \beta = 4\pi^2 \). Let \( \psi \) denote the digamma function. Then for \( m \in \mathbb{N} \), we have

\[
(-\beta)^{-m} \left\{ 2\gamma \xi(2m+1) + \sum_{n=1}^{\infty} \frac{1}{n^{2m+1}} \left( \psi \left( \frac{in\beta}{2\pi} \right) + \psi \left( -\frac{in\beta}{2\pi} \right) \right) \right\} \\
+ \alpha^{-m} \left\{ 2\gamma \xi(2m+1) + \sum_{n=1}^{\infty} \frac{1}{n^{2m+1}} \left( \psi \left( \frac{in\alpha}{2\pi} \right) + \psi \left( -\frac{in\alpha}{2\pi} \right) \right) \right\} \\
= -2 \sum_{k=1}^{m-1} (-1)^k \xi(2k+1) \xi(2m-2k-1) \alpha^{k-m} \beta^{-k}. \tag{3.22}
\]

Letting \( m = 1 \) in Proposition 3.14 gives the following modular relation that appears as Corollary 2.5 in [13].

**Corollary 3.15** Let \( \alpha, \beta \in \mathbb{C} \) such that \( \Re(\alpha) > 0, \Re(\beta) > 0 \) and \( \alpha \beta = 4\pi^2 \). Let \( \psi \) denote the digamma function. Then, the following identity holds:

\[
\frac{1}{\alpha} \left\{ 2\gamma \xi(3) + \sum_{n=1}^{\infty} \frac{1}{n^3} \left( \psi \left( \frac{in\alpha}{2\pi} \right) + \psi \left( -\frac{in\alpha}{2\pi} \right) \right) \right\} \\
= \frac{1}{\beta} \left\{ 2\gamma \xi(3) + \sum_{n=1}^{\infty} \frac{1}{n^3} \left( \psi \left( \frac{in\beta}{2\pi} \right) + \psi \left( -\frac{in\beta}{2\pi} \right) \right) \right\}. \tag{3.23}
\]
4 Proofs

4.1 Proof of Proposition 2.1

Ramanujan’s formula (1.1) for $\zeta(2n + 1)$ states that

$$\alpha^{-n} \left\{ \frac{1}{2} \zeta(2n + 1) + \sum_{m=1}^{\infty} \frac{m^{-2n-1}}{e^{2\pi m} - 1} \right\} - (-\beta)^{-n} \left\{ \frac{1}{2} \zeta(2n + 1) + \sum_{m=1}^{\infty} \frac{m^{-2n-1}}{e^{2\pi m} - 1} \right\} = 2^{2n} \sum_{k=0}^{n+1} \frac{(-1)^{k+1} B_{2k} B_{2n-2k+2}}{(2k)! (2n-2k+2)!} \alpha^{n-k+1} \beta^k.$$

We start rewriting this identity by extracting the boundary terms in the right-hand side sum

$$\alpha^{-n} \left\{ \frac{1}{2} \zeta(2n + 1) + \sum_{m=1}^{\infty} \frac{m^{-2n+1}}{e^{2\pi m} - 1} \right\} - (-\beta)^{-n} \left\{ \frac{1}{2} \zeta(2n + 1) + \sum_{m=1}^{\infty} \frac{m^{-2n-1}}{e^{2\pi m} - 1} \right\} = 2^{2n} \sum_{k=0}^{n} \frac{(-1)^{k-1} B_{2k} B_{2n-2k+2}}{(2k)! (2n-2k+2)!} \alpha^{n-k+1} \beta^k - \frac{(-1)^n 22^n B_{2n+2}}{(2n+2)!} \left( \alpha^{n+1} - \beta^{n+1} \right).$$

Euler’s formula for $\zeta(2n)$, namely

$$\frac{B_{2n}}{(2n)!} = \frac{(-1)^{n-1} \zeta(2n)}{2^{2n-1} \pi^{2n}},$$

can then be applied to all terms in this sum, producing

$$\alpha^{-n} \left\{ \frac{1}{2} \zeta(2n + 1) + \sum_{m=1}^{\infty} \frac{m^{-2n+1}}{e^{2\pi m} - 1} \right\} - (-\beta)^{-n} \left\{ \frac{1}{2} \zeta(2n + 1) + \sum_{m=1}^{\infty} \frac{m^{-2n-1}}{e^{2\pi m} - 1} \right\} = \frac{(-1)^n 22^n + 2}{(2\pi)^{2n+2}} \sum_{k=1}^{n} \zeta(2n) \zeta(2n-2k+2) \alpha^{n-k+1} \beta^k$$

$$- \frac{(-1)^n 22^n + 1}{(2\pi)^{2n+2}} \sum_{k=1}^{n} \zeta(2n) \zeta(2n-2k+2) \alpha^{n-k+1} \beta^k$$

$$= \frac{(-1)^n 22^n + 2}{(2\pi)^{2n+2}} \sum_{k=1}^{n} \zeta(2n) \zeta(2n-2k+2) \alpha^{n-k+1} \beta^k$$

$$- \frac{1}{2} \frac{\zeta(2n+2)}{\pi^{2n+2}} \left( \alpha^{n+1} - \beta^{n+1} \right).$$
\[-\frac{1}{2} \xi(2n+2) \left(\alpha^{n+1} - \beta^{n+1}\right)\]

\[= \frac{(-1)^n}{\pi^{2n+2}} \sum_{k=1}^{n} \xi(2n) \xi(2n-2k+2) \alpha^{n-k+1} \beta^k\]

\[+ \frac{1}{2} \xi(2n+2) \left(\frac{1}{\alpha^{n+1}} - \frac{1}{(-\beta)^{n+1}}\right)\]

\[= \frac{(-1)^n}{\pi^{2n+2}} \sum_{k=1}^{n} \xi(2n) \xi(2n-2k+2) \alpha^{n-k+1} \beta^k\]

\[+ \alpha^{-n} \left(\frac{1}{2\alpha} \xi(2n+2)\right) - (-\beta)^{-n} \left(\frac{1}{2\beta} \xi(2n+2)\right)\]

as desired. This completes the proof of Riemann zeta version of Ramanujan’s identity 2.1.

### 4.2 Proof of Theorem 2.2

The case with two terms \((n = 2)\) is a simple consequence of a geometric sum:

\[\sum_{k=1}^{N} \xi_{x,a}(k) \xi_{y,b}(N + 1 - k) = \sum_{m=1}^{\infty} a_n b_m \sum_{n=1}^{\infty} x_n^{N+1-k} y_m^{-k}\]

\[= \sum_{m=1}^{\infty} a_n b_m \sum_{n=1}^{\infty} \sum_{k=1}^{N} \left(\frac{y_m}{x_n}\right)^k\]

\[= \sum_{m=1}^{\infty} a_n b_m \sum_{n=1}^{\infty} \frac{1}{y_m^{N+1-k}} \sum_{k=1}^{N} \left(\frac{y_m}{x_n}\right)^k\]

\[= \sum_{m=1}^{\infty} a_n b_m \left\{\frac{1}{y_m} \frac{1}{x_n} - \frac{1}{y_m} \frac{1}{x_n} - \frac{1}{y_m} \frac{1}{x_n} \frac{1}{y_m}\right\}\]

\[= \sum_{m=1}^{\infty} \frac{b_m}{y_m^{N+1}} \psi_{x,a}(y_m) + \sum_{n=1}^{\infty} \frac{a_n}{x_n^{N+1}} \psi_{y,b}(x_n),\]

as desired. The case with \(n = 3\) terms is proved right after its statement in Theorem 2.2, and requires the following additional result.

**Lemma 4.1** The zeta generating functions \(\psi_{x,a}\) and \(\psi_{y,b}\) of two arbitrary Dirichlet series \(\xi_{x,a}\) and \(\xi_{y,b}\) satisfy the composition rule

\[\psi_{x,a} \psi_y(z) + \psi_{y,b} \psi_x(z) = \psi_{x,a}(z) \psi_{y,b}(z).\]

**Proof** We have

\[\psi_{x,a} \psi_y(z) + \psi_{y,b} \psi_x(z)\]
\[
= \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} a_p b_q \left[ \left( \frac{z}{x_p - z} \right) \left( \frac{x_p}{y_q - x_p} \right) + \left( \frac{z}{y_q - z} \right) \left( \frac{y_q}{x_p - y_q} \right) \right],
\]
and since
\[
\left( \frac{z}{x_p - z} \right) \left( \frac{x_p}{y_q - x_p} \right) + \left( \frac{z}{y_q - z} \right) \left( \frac{y_q}{x_p - y_q} \right) = \left( \frac{z}{y_q - z} \right) \left( \frac{z}{x_p - z} \right),
\]
we deduce that
\[
\psi_{x,a}(z) + \psi_{y,b}(z) = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} a_p b_q \left( \frac{z}{y_q - z} \right) \left( \frac{z}{x_p - z} \right)
= \sum_{p=1}^{\infty} a_p \left( \frac{z}{x_p - z} \right) \sum_{q=1}^{\infty} b_q \left( \frac{z}{y_q - z} \right)
= \psi_{x,a}(z) \psi_{y,b}(z)
\]
as desired. \qed

Remark 4.2  Note that in the above proof, to expand \( \psi_{x,a}(z) \) in terms of zeta functions at integer powers we would require \( |x_n| < |z| \). Then expanding \( (a, \psi_{y,b})_n = a_n \psi_{y,b}(x_n) \) as a geometric series requires \( |x_n| < |y_m| \) for all \( n, m \). However, after adding the sum of two modified Dirichlet series, we are left with two independent geometric series only involving \( x_n, z \) and \( y_n, z \). Then we can choose \( z \) large enough so that there are no convergence issues.

The case with \( n \) terms
\[
\sum_{i=1}^{n} \zeta_{x,i}, a(i) = \sum_{i=1}^{n} \zeta_{x,i}, a(i) \prod_{1 \leq k \neq i \leq n} \psi_{x,k}
\]
is proved by induction on \( n \). Here we use the notations
\[
\zeta_{x,i}, a(i) (N) = \sum_{p=1}^{\infty} \frac{a_p(i)}{(x_p(i))^N}.
\]
Assume this is true for \( n - 1 \) and consider
\[
\sum_{i=1}^{n} \zeta_{x,i}, a(i) = \zeta_{x,n}, a(n) \sum_{i=1}^{n} \zeta_{x,i}, a(i).
\]
Notice that by the induction hypothesis, this is (always computed at \( N + 1 \) so this argument is omitted for simplicity)
\[
\sum_{i=1}^{n} \zeta_{x,i}, a(i) = \zeta_{x,n}, a(n) \sum_{i=1}^{n-1} \zeta_{x,i}, a(i) \prod_{1 \leq k \neq i \leq n-1} \psi_{x,k}
\]
\[
\sum_{i=1}^{n-1} \xi^{(n)}, a^{(n)} \ast \xi^{(i)}, a^{(i)} \prod_{k \neq i} \psi^{(k)} = \sum_{i=1}^{n-1} \xi^{(n)}, a^{(n)} \ast \xi^{(i)}, a^{(i)} \prod_{k \neq i} \psi^{(k)}.
\]

Using (2.7), each convolution term in the sum is evaluated as

\[
\xi^{(n)}, a^{(n)} \ast \xi^{(i)}, a^{(i)} \prod_{k \neq i} \psi^{(k)} = \xi^{(n)}, a^{(n)} \xi^{(i)}, a^{(i)} \prod_{1 \leq k \leq n-1} \psi^{(k)}.
\]

Next, we show that

\[
\sum_{i=1}^{n-1} \xi^{(n)}, a^{(n)} \prod_{1 \leq k \leq n-1} \psi^{(k)} = \xi^{(n)}, a^{(n)} \prod_{k \neq i} \psi^{(k)},
\]

(4.1)

by showing equivalently that

\[
\sum_{i=1}^{n-1} \psi^{(i)}, a^{(i)} \prod_{1 \leq k \leq n-1} \psi^{(k)} (z) = \prod_{i=1}^{n-1} \psi^{(i)}, a^{(i)} (z).
\]

(4.2)

Starting from the elementary partial fraction decomposition

\[
\sum_{i=1}^{n-1} \frac{z}{x_i - z} \prod_{k \neq i} \frac{x_i}{x_k - x_i} = \prod_{i=1}^{n-1} \frac{z}{x_i - z},
\]

substituting each \(x_i\) with \(x_j^{(i)}\), multiplying by \(a_{n_1}^{(1)} \ldots a_{n_{n-1}}^{(n-1)}\) and summing over the indices \(n_1, \ldots, n_{n-1}\) produces

\[
\sum_{n_1, \ldots, n_{n-1}} a_{n_1}^{(1)} \ldots a_{n_{n-1}}^{(n-1)} \sum_{i=1}^{n-1} \frac{z}{x_i^{(i)} - z} \prod_{k \neq i} \frac{x_i^{(i)}}{x_k^{(i)} - x_i^{(i)}} = \sum_{n_1, \ldots, n_{n-1}} a_{n_1}^{(1)} \ldots a_{n_{n-1}}^{(n-1)} \prod_{l=1}^{n-1} \frac{z}{x_l^{(i)} - z}
\]

\[
= \sum_{n_1, \ldots, n_{n-1}} \psi^{(i)}, a^{(i)} \prod_{k \neq i} \psi^{(k)} (z).
\]

But the multiple sum on the left-hand side is recognized as

\[
\sum_{n_1, \ldots, n_{n-1}} a_{n_1}^{(1)} \ldots a_{n_{n-1}}^{(n-1)} \sum_{i=1}^{n-1} \frac{z}{x_i^{(n_1)} - z} \prod_{k \neq i} \frac{x_i^{(n_1)}}{x_i^{(k)} - x_i^{(i)}} = \sum_{i=1}^{n-1} \psi^{(i)}, a^{(i)} (z).
\]
which proves identity (4.2) and consequently identity (4.1). Therefore, we deduce that

\[
\zeta_{x}(n), a(n) \ast \sum_{i=1}^{n-1} \zeta_{x}(i), a(i) \prod_{k \neq i} \psi_{a}(k) = \sum_{i=1}^{n-1} \zeta_{x}(n), a(n) \ast \zeta_{x}(i), a(i) \prod_{k \neq i} \psi_{a}(k)
\]

\[
= \sum_{i=1}^{n-1} \zeta_{x}(i), a(i) \prod_{1 \leq k \neq i \leq n} \psi_{a}(k) + \sum_{i=1}^{n-1} \zeta_{x}(n), a(n) \psi_{a}(i) \prod_{1 \leq k \neq i \leq n-1} \psi_{a}(k)
\]

\[
= \sum_{i=1}^{n} \zeta_{x}(i), a(i) \prod_{1 \leq k \neq i \leq n} \psi_{a}(k) + \zeta_{x}(n), a(n) \prod_{1 \leq k \neq i \leq n} \psi_{a}(k)
\]

\[
= \sum_{i=1}^{n} \zeta_{x}(i), a(i) \prod_{1 \leq k \neq i \leq n} \psi_{a}(k)
\]

This completes the proof of Theorem 2.2.

\[\square\]

4.3 Proof of Theorem 3.1

With the choice

\[x_{n} = \frac{n^{2} \pi^{2}}{\beta}, \quad y_{n} = -\frac{n^{2} \pi^{2}}{\alpha},\]

the associated generating functions are

\[\psi_{x}(z) = \sum_{n=1}^{\infty} \frac{\beta z}{\pi^{2} n^{2} - \beta z} = \frac{1 - \sqrt{\beta z} \cot \sqrt{\beta z}}{2},\]

and

\[\psi_{y}(z) = -\sum_{n=1}^{\infty} \frac{\alpha z}{\pi^{2} n^{2} + \alpha z} = \frac{1 - \sqrt{\alpha z} \coth \sqrt{\alpha z}}{2},\]

so that

\[\psi_{x}(y_{n}) = \frac{1}{2} \left[1 - n \pi \sqrt{\frac{\beta}{\alpha}} \cot \left(n \pi \sqrt{\frac{\beta}{\alpha}} \right)\right] = \frac{1}{2} \left[1 - n \pi \sqrt{\frac{\beta}{\alpha}} \coth \left(n \pi \sqrt{\frac{\beta}{\alpha}} \right)\right].\]

Similarly, we find that

\[\psi_{y}(x_{n}) = \frac{1}{2} \left[1 - n \pi \sqrt{\frac{\alpha}{\beta}} \coth \left(n \pi \sqrt{\frac{\alpha}{\beta}} \right)\right],\]
and obtain
\[
\frac{1}{\pi^{2N+2}} \sum_{k=1}^{N} \zeta(2k) \zeta(2N + 2 - 2k) \beta^k (-\alpha)^{N+1-k} \\
= \beta^{N+1} \sum_{n=1}^{\infty} \frac{1}{\pi^{2N+2} n^{2N+2}} \left[ \frac{1}{2} \left( 1 - n\pi \sqrt{\frac{\alpha}{\beta}} \coth \left( n\pi \sqrt{\frac{\alpha}{\beta}} \right) \right) \right] \\
+ (-\alpha)^{N+1} \sum_{n=1}^{\infty} \frac{1}{\pi^{2N+2} n^{2N+2}} \left[ \frac{1}{2} \left( 1 - n\pi \sqrt{\frac{\beta}{\alpha}} \coth \left( n\pi \sqrt{\frac{\beta}{\alpha}} \right) \right) \right].
\]

After simplification, and exchanging $\alpha \mapsto -\alpha$ and $\beta \mapsto -\beta$, we find that
\[
\sum_{k=1}^{N} \zeta(2k) \zeta(2N + 2 - 2k) \alpha^{N+1-k} (-\beta)^k \\
= \frac{1}{2} \zeta(2N + 2) \left[ \alpha^{N+1} + (-\beta)^{N+1} \right] \\
- \frac{\pi}{2} \alpha^{N+1} \sum_{n=1}^{\infty} \frac{1}{n^{2N+1}} \sqrt{\frac{\beta}{\alpha}} \coth \left( \pi n \sqrt{\frac{\beta}{\alpha}} \right) \\
- \frac{\pi}{2} (-\beta)^{N+1} \sum_{n=1}^{\infty} \frac{1}{n^{2N+1}} \sqrt{\frac{\alpha}{\beta}} \coth \left( \pi n \sqrt{\frac{\alpha}{\beta}} \right).
\]

Denoting $\mu = \beta/\alpha$, we can rewrite the above identity as follows
\[
\sum_{k=1}^{N} (-1)^k \zeta(2k) \zeta(2N + 2 - 2k) \mu^k = \frac{1}{2} \zeta(2N + 2) \left[ (-1)^{N+1} + \mu^{N+1} \right] \\
- \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\sqrt{\mu}}{n^{2N+1}} \coth \left( \pi n \sqrt{\mu} \right) - \frac{\pi}{2} (-\mu)^{N+1} \sum_{n=1}^{\infty} \frac{1}{n^{2N+1}} \sqrt{\frac{1}{\mu}} \coth \left( \frac{\pi n \sqrt{\mu}}{\mu} \right).
\]

Simplifying further we finally obtain
\[
\sum_{k=1}^{N} (-1)^k \zeta(2k) \zeta(2N + 2 - 2k) \mu^k = \frac{1}{2} \zeta(2N + 2) \left[ \mu^{N+1} + (-1)^{N+1} \right] \\
- \frac{\pi}{2} \sqrt{\mu} \sum_{n=1}^{\infty} \frac{\coth \left( \frac{\pi n \sqrt{\mu}}{\mu} \right)}{n^{2N+1}} - (-1)^{N+1} \mu^{N+1} \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^{2N+1}} \coth \left( \frac{\pi n \sqrt{\mu}}{\mu} \right),
\]

which is the desired result. \[\square\]
4.4 Proof of Theorem 3.2

We start with identity (3.3), by computing, with \( \Re \) denoting the real part of \( z \),

\[
\psi_x(z) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{ze^{2i\pi ny_1}}{2i\pi n/\omega_1 - z} = 2\Re \sum_{n=1}^{\infty} \frac{ze^{2i\pi ny_1}}{2i\pi n/\omega_1 - z} = 2\Re \sum_{n=1}^{\infty} \left( \frac{ze^{2i\pi ny_1}}{2i\pi n/\omega_1} \right)^2 + z^2 \left( -\frac{2i\pi n}{\omega_1} - z \right)
\]

\[
= -2 \sum_{n=1}^{\infty} \frac{z^2 \cos(2\pi ny_1)}{\left( \frac{2i\pi n}{\omega_1} \right)^2 + z^2} - z \left( \frac{2\pi n/\omega_1}{2i\pi n/\omega_1} \right) \sin(2\pi ny_1).
\]

Both sums are identified as Fourier series expansions: using [27, (1.51) and (1.53)] under the conditions \( 0 \leq y_1 \leq 1 \) and \( 0 \leq y_2 \leq 1 \), we get

\[
\frac{1}{z^2} + 2 \sum_{n=1}^{\infty} \frac{\cos(2\pi ny_1)}{z^2 + \frac{4\pi^2}{\omega_1^2} n^2} = \frac{\omega_1}{2z} \frac{\cosh(\omega_1 z \left( \frac{1}{2} - y_1 \right))}{\sinh\left( \frac{\omega_1 z}{2} \right)},
\]

and

\[
\sum_{n=1}^{\infty} \frac{n}{z^2 + \frac{4\pi^2}{\omega_1^2} n^2} \sin(2\pi ny_1) = \frac{\omega_1^2}{8\pi} \frac{\sinh(\omega_1 z \left( \frac{1}{2} - y_1 \right))}{\sinh\left( \frac{\omega_1 z}{2} \right)},
\]

so that we have

\[
\psi_x(z) = 1 - \frac{\omega_1}{2} \frac{\cosh(\omega_1 z \left( \frac{1}{2} - y_1 \right))}{\sinh\left( \frac{\omega_1 z}{2} \right)} + z \frac{\omega_1}{2} \frac{\sinh(\omega_1 z \left( \frac{1}{2} - y_1 \right))}{\sinh\left( \frac{\omega_1 z}{2} \right)}
\]

\[
= 1 - \frac{\omega_1}{2} \frac{ze^{\omega_1 z \left( y_1 - \frac{1}{2} \right)}}{\sinh\left( \frac{\omega_1 z}{2} \right)} = (1 - \omega_1) \left( \frac{ze^{\omega_1 z}}{e^{\frac{\omega_1 z}{2}} - e^{-\frac{\omega_1 z}{2}}} \right) e^{\omega_1 z y_1}
\]

\[
= (1 - \omega_1) \left( \frac{ze^{\omega_1 z y_1}}{e^{\omega_1 z} - 1} \right).
\]

Therefore, we deduce that

\[
\psi_x(y_n) = \frac{\omega_1}{\omega_2} \left( 1 - 2\pi n \right) \left( \frac{e^{2\pi n/\omega_1 y_1}}{e^{2\pi n/\omega_1 y_2} - 1} \right),
\]

\[
\psi_y(x_n) = \frac{\omega_2}{\omega_1} \left( 1 - 2\pi n \right) \left( \frac{e^{2\pi n/\omega_2 y_2}}{e^{2\pi n/\omega_1 y_2} - 1} \right),
\]
\[
\sum_{n \in \mathbb{Z} \setminus \{0\}} \left( \frac{b_n}{y_n^{N+1}} \psi_x(y_n) + \frac{a_n}{x_n^{N+1}} \psi_y(x_n) \right)
= \sum_{n \in \mathbb{Z} \setminus \{0\}} \left( \frac{b_n}{y_n^{N+1}} \psi_x(y_n) \right) + \sum_{n \in \mathbb{Z} \setminus \{0\}} \left( \frac{a_n}{x_n^{N+1}} \psi_y(x_n) \right)
= \sum_{n \in \mathbb{Z} \setminus \{0\}} \left( \frac{e^{2\pi ny_2}}{(2\pi n)^{N+1}} \right) \left( 1 - 2\pi n \frac{\omega_2}{\omega_1} e^{2\pi n \frac{\omega_1}{\omega_2} y_2} \right)
\]

Therefore, we deduce that
\[
\sum_{k=1}^{N} \frac{\omega_1^k}{k!} \frac{B_k(y_1)}{\omega_2^{N+1-k}} \frac{B_{N+1-k}(y_2)}{(N + 1 - k)!} = -\omega_1^N \frac{B_{N+1}(y_2)}{(N + 1)!} - \omega_2^N \frac{B_{N+1}(y_1)}{(N + 1)!}
\]

and
\[
\sum_{k=0}^{N+1} \frac{\omega_1^k}{k!} \frac{B_k(y_1)}{\omega_2^{N+1-k}} \frac{B_{N+1-k}(y_2)}{(N + 1 - k)!} = -\omega_1 \sum_{n \in \mathbb{Z} \setminus \{0\}} \left( \frac{e^{2\pi ny_2}}{(2\pi n)^N} \right) \left( \frac{e^{2\pi n \frac{\omega_1}{\omega_2} y_1}}{e^{2\pi n \frac{\omega_1}{\omega_2} y_2}} - 1 \right) - \omega_2 \sum_{n \in \mathbb{Z} \setminus \{0\}} \left( \frac{2\pi n}{\omega_1} \right)^N \frac{e^{2\pi n \frac{\omega_1}{\omega_2} y_1}}{e^{2\pi n \frac{\omega_1}{\omega_2} y_2}} - 1
\]

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The extension
\[
\sum_{k_1, k_2, \ldots, k_n} \prod_{i=1}^{n} \omega_i^{k_i-1} \frac{B_{k_i}(y_i)}{k_i!} = -\sum_{i=1}^{n} \frac{1}{\omega_i} \sum_{\omega_i \in \mathbb{Z} \setminus \{0\}} e^{2\pi i ny_i} \left( \frac{2\pi n}{\omega_i} \right)^N \prod_{j \neq i} e^{2\pi n \omega_i^{-1} \omega_j y_j} - 1,
\]

based on the identity in Theorem 5 is straightforward and left as an exercise to the reader. \(\square\)

4.5 Proof of Corollary 3.3

Here, we use the representation of Euler polynomials as integrals of Bernoulli polynomials, that is
\[
\int_{x}^{x+\frac{1}{2}} B_n(z) \, dz = \frac{E_n(2x)}{2^n+1}.
\]

Starting from identity (3.4)
\[
\sum_{k_1, k_2, \ldots, k_n} \prod_{i=1}^{n} \omega_i^{k_i-1} \frac{B_{k_i}(y_i)}{k_i!} = -\sum_{i=1}^{n} \frac{1}{\omega_i} \sum_{m \in \mathbb{Z} \setminus \{0\}} e^{2\pi i my_i} \left( \frac{2\pi m}{\omega_i} \right)^N \prod_{j \neq i} e^{2\pi m \omega_i^{-1} \omega_j y_j} - 1,
\]

let us integrate over each variable \(y_i\) from \(x_i\) to \(x_i + \frac{1}{2}\). Thus the left-hand side is
\[
\sum_{k_1, k_2, \ldots, k_n} \prod_{i=1}^{n} \omega_i^{k_i-1} \frac{E_{k_i}(2x_i)}{2^{k_i+1} k_i!} = \frac{1}{2^{2N+2+n}} \sum_{k_1, k_2, \ldots, k_n} \prod_{i=1}^{n} \omega_i^{k_i-1} \frac{E_{k_i}(2x_i)}{k_i!}.
\]

The right-hand side can now be computed using
\[
\int_{x_j}^{x_j+\frac{1}{2}} e^{2\pi i m \omega_i^{-1} x_j} \, dy_j = \frac{\omega_i}{\omega_j} \left( e^{2\pi i m \omega_i^{-1} x_j} \right) \left( e^{i\pi m \omega_i^{-1} x_j} - 1 \right)
\]
and
\[
\int_{x_i}^{x_i+\frac{1}{2}} e^{2\pi i m y_i} \, dy_i = \left( e^{2\pi i m x_i} \right) \left( (-1)^m - 1 \right).
\]
We deduce the right-hand side after integration as

\[
\sum_{i=1}^{n} \frac{1}{\omega_i} \sum_{m \in \mathbb{Z} \setminus \{0\}} \left( \frac{e^{2\pi \omega_i x_i}}{2\pi m} \right)^N \prod_{j \neq i} \frac{\omega_j}{\omega_i} \left( \frac{e^{2\pi \omega_j \omega_i x_j}}{2\pi m} \right) \left( e^{i\pi m \omega_i x_i} - 1 \right)
\]

\[
= - \frac{2}{\omega_1 \omega_2 \ldots \omega_n} \sum_{i=1}^{n} \sum_{m \text{ odd}} \left( \frac{e^{2\pi \omega_i x_i}}{2\pi m} \right)^{N+n-1} \prod_{j \neq i} \frac{e^{i\pi m \omega_j \omega_i x_j}}{e^{i\pi m \omega_i x_i} + 1}
\]

and finally after simplification we obtain

\[
\sum_{k_1, k_2, \ldots, k_n} \prod_{i=1}^{n} \frac{E_i(2x_i)}{\omega_i^{k_i-1}} = 2^{2N+n+3} \sum_{i=1}^{n} \sum_{m \text{ odd}} \left( \frac{e^{2\pi \omega_i x_i}}{2\pi m} \right)^{N+n-1} \prod_{j \neq i} \frac{e^{i\pi m \omega_j \omega_i x_j}}{e^{i\pi m \omega_i x_i} + 1}
\]

which is the desired result. \qed

4.6 Proof of Corollary 3.5

The choice

\[
ap, q = 1, \quad b, r, s = 1, \quad x_{p, q} = -\frac{p^2 q^2}{\alpha^2}, \quad y_{r, s} = \frac{r^2 s^2}{\beta^2},
\]

produces the Dirichlet series

\[
\zeta_{x, a}(s) = \left(-\alpha^2\right)^s \zeta^2(2s), \quad \zeta_{y, b}(s) = \beta^2 s^2 \zeta^2(2s)
\]

and the zeta generating functions

\[
\psi_x(z) = -\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{\alpha^2 z^{p^2 q^2 + \alpha^2 z^2}}{p^2 q^2 + \alpha^2 z^2} = -\sum_{n=1}^{\infty} \tau_0(n) \frac{\alpha^2 z^n}{n^2 + \alpha^2 z^n}.
\]

Here, \(\tau_0(n)\) represents the number of divisors of \(n\), and we have applied the general formula

\[
\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} F(pq) = \sum_{n=1}^{\infty} \sum_{\substack{q \geq 1 \\text{q divides } n}} F(n) = \sum_{n=1}^{\infty} \tau_0(n) \cdot F(n).
\]
Similarly, we find that

\[ \psi_y(z) = \sum_{m=1}^{\infty} \tau_0(m) \frac{\beta^2 z}{m^2 - \beta^2 z}. \]

Applying (2.7) produces

\[
\sum_{k=1}^{N} (-1)^k \zeta^2(2k) \zeta^2(2N + 2 - 2k) \alpha^{2k} \beta^{2N+2-2k}
\]

\[
= -\beta^{2N+2} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{(rs)^{2N+2}} \sum_{n=1}^{\infty} \tau_0(n) \frac{\alpha^2 r^2 s^2}{\beta^2 n^2 + \alpha^2 r^2 s^2}
\]

\[
- ( -\alpha^2 )^{N+1} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{1}{(pq)^{2N+2}} \sum_{m=1}^{\infty} \tau_0(m) \frac{\beta^2 p^2 q^2}{\alpha^2 m^2 + \beta^2 p^2 q^2}
\]

\[
- ( \beta^2 )^{N+1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\tau_0(n) \tau_0(m)}{m^{2N}} \frac{\alpha^2}{\beta^2 n^2 + \alpha^2 m^2}
\]

\[
- ( -\alpha^2 )^{N+1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\tau_0(n) \tau_0(m)}{n^{2N}} \frac{\beta^2}{\alpha^2 m^2 + \beta^2 n^2}.
\]

as desired. \( \square \)

### 4.7 Proof of Corollary 3.6

With the choice

\[ a_{p,q} = q^c, \quad b_{r,s} = s^d \quad \text{and} \quad x_{p,q} = -\frac{p^2 q^2}{\alpha^2}, \quad y_{r,s} = \frac{r^2 s^2}{\beta^2}, \]

we obtain the generating functions

\[ \psi_x(z) = -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha^2 q^c z}{p^2 q^2 + z \alpha^2} = -\sum_{n=1}^{\infty} \tau_c(n) \frac{\alpha^2 z}{n^2 + \alpha^2 z}, \]

and

\[ \psi_y(z) = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{\beta^2 s^d z}{r^2 s^2 - z \beta^2} = \sum_{m=1}^{\infty} \tau_d(m) \frac{\beta^2 z}{m^2 - z \beta^2}, \]
associated with the Dirichlet functions

\[ \xi_{x,a}(k) = (-1)^k \alpha^{2k} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{q^c}{p^{2k}q^{2k}} = (-1)^k \alpha^{2k} \zeta(2k) \zeta(2k - c), \]

and

\[ \xi_{y,b}(k) = \beta^{2k} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{s^d}{r^{2k}s^{2k}} = \beta^{2k} \zeta(2k) \zeta(2k - d). \]

This produces

\[
\sum_{k=1}^{N} (-1)^k \zeta(2k) \zeta(2k - c) \zeta(2N + 2 - 2k) \zeta(2N + 2 - 2k - d) \alpha^{2k} \beta^{2N+2-2k}
\]

\[
= -\beta^{2N+2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\tau_c(n) \tau_d(m)}{m^{2N}n^{2N}} \frac{\alpha^2}{\beta^2 m^2 + \alpha^2 n^2}
\]

\[
- (-\alpha^2)^{N+1} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\tau_c(n) \tau_d(m)}{n^{2N}m^{2N}} \frac{\beta^2}{\alpha^2 m^2 + \beta^2 n^2}.
\]

as desired.

4.8 Proof of Theorem 3.7

We first compute the Bessel zeta generating function as follows:

\[
\sum_{N=1}^{\infty} \zeta_{B,v}(N) z^N = -z \frac{d}{dz} \log \frac{\sqrt{z} J_{v+1}(\sqrt{z})}{2 J_v(\sqrt{z})} = v - \frac{\sqrt{z}}{2} J_{v-1}(\sqrt{z}),
\]

where we used the linear recurrence

\[ J_{v+1}(\sqrt{z}) + J_{v-1}(\sqrt{z}) = \frac{2v}{\sqrt{z}} J_v(\sqrt{z}). \]

Now we make the choice

\[ x_n = \frac{j_{n,v}^2}{\beta}, y_n = -\frac{j_{n,v}^2}{\alpha}, \]

which produces

\[ \psi_x(z) = v - \frac{\sqrt{\beta z}}{2} J_{v-1}(\sqrt{\beta z}) \]
where the factor of $\sqrt{\beta}$ occurs since we consider $\zeta_{\alpha, \beta}$ instead of $\zeta_{B, \nu}$. Next, we use the fact that $J_{\nu}(iz) = i^\nu I_{\nu}(z)$ to obtain

$$\psi_x(x_n) = -\frac{1}{2} i \sqrt{\frac{\beta}{\alpha}} j_{n, \nu} J_{\nu-1}\left(\frac{i \sqrt{\beta}}{\alpha} j_{n, \nu}\right) = -\frac{1}{2} i \sqrt{\frac{\beta}{\alpha}} j_{n, \nu} I_{\nu-1}\left(\frac{\sqrt{\beta}}{\alpha} j_{n, \nu}\right),$$

while

$$\psi_y(y_n) = -\frac{1}{2} i \sqrt{\frac{\alpha}{\beta}} j_{n, \nu} J_{\nu-1}\left(\frac{i \sqrt{\alpha}}{\beta} j_{n, \nu}\right) = -\frac{1}{2} i \sqrt{\frac{\alpha}{\beta}} j_{n, \nu} I_{\nu-1}\left(\frac{\sqrt{\alpha}}{\beta} j_{n, \nu}\right).$$

We deduce

$$\sum_{k=1}^{N} \zeta_{B, \nu}(k) \zeta_{B, \nu}(N + 1 - k) \alpha^{N-k+1} (-\beta)^k$$

$$= \alpha^{N+1} \sum_{q=1}^{\infty} \frac{1}{j_{v, q}} \left( v - \sqrt{\frac{\beta}{\alpha}} j_{v, q} \right) I_{\nu-1}\left(\frac{\sqrt{\beta}}{\alpha} j_{v, q}\right) + (-\beta)^{N+1} \sum_{q=1}^{\infty} \frac{1}{j_{v, q}} \left( v - \sqrt{\frac{\alpha}{\beta}} j_{v, q} \right) I_{\nu-1}\left(\frac{\sqrt{\alpha}}{\beta} j_{v, q}\right).$$

Finally, after simplification, we obtain

$$\sum_{k=1}^{N} \zeta_{B, \nu}(k) \zeta_{B, \nu}(N + 1 - k) \alpha^{N-k+1} (-\beta)^k$$

$$= v \left(\alpha^{N+1} + (-\beta)^{N+1}\right) \zeta_{B, \nu}(N + 1)$$

$$- \frac{1}{2} \alpha^{N+1} \sqrt{\frac{\beta}{\alpha}} \sum_{q=1}^{\infty} \frac{1}{j_{v, q}} j_{v, q} I_{\nu-1}\left(\frac{\sqrt{\beta}}{\alpha} j_{v, q}\right) I_{\nu}\left(\frac{\sqrt{\beta}}{\alpha} j_{v, q}\right).$$
\[- \frac{1}{2} (-\beta)^{N+1} \sqrt{\frac{\alpha}{\beta}} \sum_{q=1}^{\infty} \frac{1}{J_{v, q}^{2N+1}} \frac{I_{v-1} \left( \sqrt{\frac{\alpha}{\beta}} j_{v, q} \right)}{I_{v} \left( \sqrt{\frac{\alpha}{\beta}} j_{v, q} \right)}, \]

which is the desired result.

\[\square\]

### 4.9 Proof of Theorem 3.11

Since we have the general identity
\[
\sum_{n=0}^{\infty} \frac{z}{(x+n)^2 - z} = \frac{\sqrt{z}}{2} \left[ \psi(x + \sqrt{z}) - \psi(x - \sqrt{z}) \right],
\]
with \(\psi\) the digamma function, choosing
\[
x_n = \frac{(x+n)^2}{\beta}, \quad y_n = -\frac{(y+n)^2}{\alpha},
\]
produces the zeta generating functions
\[
\psi_x(y_n) = \frac{1}{2} i \sqrt{\frac{\beta}{\alpha}} (y + n) \left[ \psi \left( x + i \sqrt{\frac{\beta}{\alpha}} (y + n) \right) - \psi \left( x - i \sqrt{\frac{\beta}{\alpha}} (y + n) \right) \right],
\]
and
\[
\psi_y(x_n) = \frac{1}{2} i \sqrt{\frac{\alpha}{\beta}} (x + n) \left[ \psi \left( y + i \sqrt{\frac{\alpha}{\beta}} (x + n) \right) - \psi \left( y - i \sqrt{\frac{\alpha}{\beta}} (x + n) \right) \right].
\]

Applying (2.7) and subtracting 1 to both \(x\) and \(y\) (as the summation index in the Hurwitz zeta function starts at 0) produces the following
\[
\sum_{k=1}^{N} \xi_H(x; 2k) \xi_H(y; 2N + 2 - 2k) \beta^k (-\alpha)^{N+1-k}
\]
\[
= \frac{1}{2} \beta^{N+1} \sqrt{\frac{\alpha}{\beta}} \sum_{n=0}^{\infty} \frac{1}{(x+n)^{2N+1}}
\]
\[
\times \left[ \psi \left( y + i \sqrt{\frac{\alpha}{\beta}} (x + n) \right) - \psi \left( y - i \sqrt{\frac{\alpha}{\beta}} (x + n) \right) \right]
\]
\[
+ \frac{1}{2} (-\alpha)^{N+1} \sqrt{\frac{\beta}{\alpha}} \sum_{n=0}^{\infty} \frac{1}{(y+n)^{2N+1}}
\]
\[
\times \left[ \psi \left( x + i \sqrt{\frac{\beta}{\alpha}} (y + n) \right) - \psi \left( x - i \sqrt{\frac{\beta}{\alpha}} (y + n) \right) \right].
\]
as desired. ☐

4.10 Proof of Theorem 3.12

Let us prove first the following multisection identity:

$$\sum_{p=1}^{\infty} \frac{z^{2m} p^{m-1}}{z^{2m} - p^{2m}} = \pi \frac{z^m}{2m} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} (-1)^j \cot \left( \pi \frac{e^{i\theta}}{m} \right). \quad (4.3)$$

Using the Mittag–Leffler expansion of the cotangent function,

$$\sum_{p=1}^{\infty} \frac{2z^2}{z^2 - p^2} = \pi z \cot (\pi z) - 1,$$

we deduce, with $m = 2n + 1$,

$$\pi z \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} (-1)^j \cot \left( \pi \frac{e^{i\theta}}{m} \right) = \sum_{j=-n}^{n} (-1)^j \pi z \cot \left( \pi \frac{e^{i\theta}}{m} \right)$$

$$= \sum_{j=-n}^{n} (-1)^j e^{-i\frac{\pi}{m}} \left( e^{i\frac{\pi}{m}} \pi z \right) \cot \left( \pi \frac{e^{i\theta}}{m} \right)$$

$$= \sum_{j=-n}^{n} (-1)^j e^{-i\frac{\pi}{m}} \left[ 1 + \sum_{p=1}^{\infty} \frac{2 \left( \frac{e^{i\theta}}{m} \right)^2}{ \left( \frac{e^{i\theta}}{m} \right)^2 - p^2 } \right]$$

$$= \sum_{j=-n}^{n} (-1)^j e^{-i\frac{\pi}{m}} + \sum_{p=1}^{\infty} \sum_{j=-n}^{n} (-1)^j \frac{2 \left( \frac{e^{i\theta}}{m} \right)^2 e^{-i\frac{\pi}{m}}}{ \left( \frac{e^{i\theta}}{m} \right)^2 - p^2 }.$$

The first sum

$$\sum_{j=-n}^{n} (-1)^j e^{-i\frac{\pi}{m}} = \sum_{j=-n}^{n} e^{i\frac{j\pi}{2n+1}} = \sum_{j=-n}^{n} e^{i\pi \frac{2n}{2n+1}} = 0,$$

since the summation range is over a residue system. Using a partial fraction decomposition, the inner sum in the second term is

$$2z^2 \sum_{j=-n}^{n} (-1)^j \frac{e^{i\frac{j\pi}{2n+1}}}{z^2 e^{i\frac{2j\pi}{2n+1}} - p^2} = 2z^2 m \left( \frac{p^{m-1} z^{m-1}}{z^{2m} - p^{2m}} \right),$$
so that we finally have

\[ \frac{\pi z^m}{2m} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} (-1)^j \cot\left(\pi z e^{j\pi n/m}\right) = \sum_{p=1}^{\infty} \frac{p^{m-1} z^{2m}}{z^{2m} - p^{2m}}. \]

Now let us check the main formula (3.20): the generating functions associated with the choices (3.18) are

\[ \psi_x(z) = \sum_{n=1}^{\infty} \frac{z}{(n^2/\alpha) - z} = \frac{1}{2} \left(1 - \pi \sqrt{\alpha z} \cot(\pi \sqrt{\alpha z})\right) \]

and, by identity (3.19),

\[ \psi_y(z^{2m}) = \sum_{n=1}^{\infty} \frac{z^{2m}}{(n^{2m}/\beta) - z^{2m}} = -\sum_{n=1}^{\infty} \frac{\beta z^{2m}}{z^{2m} + \beta z^{2m}} = -\sum_{n=1}^{\infty} \frac{z^{2m}}{n^{2m} + \beta \bar{z}^{2m}}, \]

with \( \bar{z} = z^{1/2m} \). Next, we have

\[ \psi_y(z^{2m}) = \frac{\pi z^{2m} \sqrt{\beta}}{2m} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} (-1)^j \coth\left(\pi z^{1/2m} e^{j\pi n/m}\right), \]

and

\[ \psi_y(z) = \frac{\pi \sqrt{\beta} \bar{z}}{2m} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} (-1)^j \coth\left(\pi (z^{1/2m} e^{j\pi n/m} \right). \]

Therefore, we deduce that

\[ \psi_x(y_n) = \frac{1}{2} \left(1 - \pi \sqrt{\frac{\alpha}{\beta}} n^{2m} \coth\left(\pi \sqrt{\frac{\alpha}{\beta}} n^m\right)\right), \]

and

\[ \psi_y(x_n) = \frac{\pi n \sqrt{\beta}}{2m \sqrt{\alpha}} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} (-1)^j \coth\left(\pi n^{1/m} \sqrt{\frac{\beta}{\alpha}} e^{j\pi n/m}\right). \]

The associated Dirichlet series are

\[ \zeta_x(k) = a^k \zeta(2k), \quad \zeta_y(k) = (-\beta)^k \sum_{n=1}^{\infty} \frac{n^{m-1}}{n^{2mk}} = (-\beta)^k \zeta(2mk - m + 1). \]
Thus, we obtain the convolution formula

\[
\sum_{k=1}^{N} \zeta(2k) \zeta(2m(N + 1 - k) - m + 1) \alpha^k (-\beta)^{N+1-k}
\]

\[
= (-\beta)^{N+1} \sum_{n=1}^{\infty} \frac{n^{m-1}}{n^{2m(N+1)}} \frac{1}{2} \left[ 1 - \pi \sqrt{\frac{\alpha}{\beta}} n^m \coth \left( \frac{\pi \sqrt{\frac{\alpha}{\beta}} n^m}{m} \right) \right]
\]

\[
+ \alpha^{N+1} \sum_{n=1}^{\infty} \frac{1}{n^{2N+2}} \frac{\pi n}{2m} \sqrt{\frac{\beta}{\alpha}} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} (-1)^j \coth \left( \frac{\pi n^{1/m}}{\sqrt[4]{\frac{\beta}{\alpha}}} e^{i \pi/m} \right),
\]

or, after simplification,

\[
\sum_{k=1}^{N} \zeta(2k) \zeta(2m(N - k) + m + 1) \alpha^k (-\beta)^{N+1-k}
\]

\[
= \alpha^{N+1} \frac{\pi}{2m} \sqrt{\frac{\beta}{\alpha}} \sum_{n=1}^{\infty} \frac{1}{n^{2N+1}} \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}} (-1)^j \coth \left( \frac{\pi n^{1/m}}{\sqrt[4]{\frac{\beta}{\alpha}}} e^{i \pi/m} \right)
\]

\[
+ \frac{1}{2} (-\beta)^{N+1} \zeta(2mN + m + 1)
\]

\[
- (-\beta)^{N+1} \frac{\pi}{2} \sqrt{\frac{\alpha}{\beta}} \sum_{n=1}^{\infty} \frac{1}{n^{2mN+1}} \coth \left( \frac{\pi \sqrt{\frac{\alpha}{\beta}} n^m}{m} \right),
\]

which is the desired result.

\[\square\]

4.11 Proof of Proposition 3.14

The choices

\[
x_n = -\frac{n^2}{\beta}, \quad y_n = \frac{n^2}{\alpha}, \quad a_n = b_n = \frac{1}{n},
\]

in our main setup produce

\[
\zeta_{x,a}(k) = (-\beta)^k \zeta(2k + 1) \quad \text{and} \quad \zeta_{y,b}(k) = \alpha^k \zeta(2k + 1).
\]

With \(\psi(z)\) the digamma function and \(\gamma\) the Euler-Mascheroni constant, the associated generating functions are

\[
\psi_{x,a}(z) = -\sum_{n=1}^{\infty} \frac{\beta z}{n(n^2 + \beta z)} = -\frac{1}{2} \left[ \psi \left( i\sqrt{\beta z} + 1 \right) - \psi \left( -i\sqrt{\beta z} + 1 \right) + 2\gamma \right]
\]
\[= -\frac{1}{2} \left[ \psi \left( i \sqrt{\beta z} \right) - \psi \left( -i \sqrt{\beta z} \right) + 2\gamma \right], \]

and

\[
\psi_{y,b}(z) = \sum_{n=1}^{\infty} \frac{\alpha z}{n(n^2 - \alpha z)} = -\frac{1}{2} \left[ \psi \left( i \sqrt{\alpha z} + 1 \right) - \psi \left( -i \sqrt{\alpha z} + 1 \right) + 2\gamma \right]
\]

\[
= -\frac{1}{2} \left[ \psi \left( i \sqrt{\alpha z} \right) - \psi \left( -i \sqrt{\alpha z} \right) + 2\gamma \right].
\]

Therefore we deduce

\[
\psi_{x,a} \left( y_n \right) = -\frac{1}{2} \left[ \psi \left( i n \sqrt{\frac{\beta}{\alpha}} \right) - \psi \left( -i n \sqrt{\frac{\beta}{\alpha}} \right) + 2\gamma \right],
\]

and

\[
\psi_{y,b} \left( x_n \right) = -\frac{1}{2} \left[ \psi \left( i n \sqrt{\frac{\alpha}{\beta}} \right) - \psi \left( -i n \sqrt{\frac{\alpha}{\beta}} \right) + 2\gamma \right].
\]

We deduce, for all \( \alpha, \beta \in \mathbb{R}^+ \)

\[
-\frac{1}{2} \left( -\beta \right)^{N+1} \sum_{n=1}^{\infty} \frac{1}{n^{N+2}} \left[ \psi \left( i n \sqrt{\frac{\beta}{\alpha}} \right) - \psi \left( -i n \sqrt{\frac{\beta}{\alpha}} \right) + 2\gamma \right]
\]

\[
-\frac{1}{2} \alpha^{N+1} \sum_{n=1}^{\infty} \frac{1}{n^{N+2}} \left[ \psi \left( i n \sqrt{\frac{\alpha}{\beta}} \right) - \psi \left( -i n \sqrt{\frac{\alpha}{\beta}} \right) + 2\gamma \right]
\]

\[
= \sum_{k=1}^{N} \xi(2k + 1) \xi(2N - 2k + 3) \alpha^{N-k+1} (-\beta)^k.
\]

Simplifying the left-hand side using \( \alpha\beta = 4\pi^2 \) produces the desired result. \( \square \)

5 Conclusion and further research

The main result of this paper, Theorem 2.2, follows from the direct application of the geometric sum formula: it simply expresses the \( n \)-fold convolution of Dirichlet series as the sum of \( n \) Dirichlet series with modified weights. One of its main advantages is that it explains why the \( \alpha\beta = \pi^2 \) condition is needed in these identities, which are essentially one-parameter identities. It also reduces the proof of similar identities to the computation of the zeta generating function (2.4).

Several paths have not been explored yet and those will be the subject of future work. One of the difficulties associated with our result is the determination of a closed form version for the generating function (2.4) associated with a choice of Dirichlet
series. A more thorough exploration of the known values of such generating functions would provide a better view on this family of equivalent identities.

Due to their multivariate nature, these results can be extended to a choice of more exotic Dirichlet series such as Multiple Zeta Values (MZVs) or Witten zeta functions. We explored a variant for double sums in Theorem 3.4, although we barely scratched the surface.

There are other relatively simple specializations of our main theorem, though for brevity we have not pursued these. For instance, let $j_{\nu}(z)$ denote a modified Bessel function of the first kind (as in (3.13)), $j_{\nu,k}$ the $k$-th zero of $j_{\nu}(z)$ ordered by absolute magnitude, and

$$\tilde{\zeta}_\nu(s) = \sum_{k=1}^{\infty} \frac{1}{j_{\nu+1}(j_{\nu,k}) j_{\nu,k}^{s+\frac{1}{2}}} ,$$

the alternate Bessel zeta function. Though the definition is somewhat unmotivated, it naturally appears when lifting identities for multiple zeta functions to identities for the zeros of Bessel zeta functions, with the Riemann zeta function being replaced by the alternate Bessel zeta function [29]. Let

$$P_n(z) \equiv \sum_{m=0}^{n-1} \frac{d^2}{dz^{2n}} \left( \frac{1}{j_\nu(z)} \right) \bigg|_{z=0} \frac{z^{2m}}{(2m)!} ,$$

denote the degree $2n - 2$ polynomial obtained from the truncated Taylor series expansion of $1/j_{\nu}(z)$ around 0. Using the arguments of [29, Theorem 10] shows that Krein’s expansion is equivalent to

$$\frac{1}{j_\nu(z)} = P_n(z) + 4 (\nu + 1) \sum_{k=0}^{\infty} \frac{z^{2n+2k}}{(2n+2k)!} \tilde{\zeta}_\nu(2n+2k) .$$

Then, we can recognize Krein’s expansion as a closed form for the zeta generating function (2.4) attached to the alternate Bessel zeta function. This specialization of our main theorem produces an identity relating the zeros $j_{\nu,k}$ of the Bessel $j_\nu$ function with $j_{\nu+1}(j_{\nu,k})$. Variants of Krein’s expansion also hold for other special function [28], and will likely give analogous results. Note that the zeros of many families of orthogonal polynomials, including Bessel functions, satisfy highly nontrivial sum relations [8]. Incorporating these known identities for Bessel zeros will likely simplify our results. We leave as an open problem the specialization of our identities to the zeros of other families of orthogonal polynomials.

There has been other work on generalizing Ramanujan-type reciprocity to various arbitrary Dirichlet series. Under suitable convergence conditions which we omit, let

$$F(s) \equiv \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} , \quad G(s) \equiv \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^s} ,$$
denote two arbitrary Dirichlet series which satisfy the zeta-type functional equation

$$\chi (s) := (2\pi)^s \Gamma (s) F (s) = (2\pi)^{s-\delta} \Gamma (\delta - s) G (\delta - s).$$

Then we have the reciprocity relation related to Lambert series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} = \left(\frac{2\pi}{z}\right)^{\delta} \sum_{n=1}^{\infty} b_n e^{-4\pi^2 \mu_n / z} + \frac{1}{2i\pi} \int_{C} \left(\frac{2\pi}{z}\right)^{t} \chi (t) \, dt,$$

where $C$ is any curve enclosing the poles of $F(s)$ and $G(s)$, along with other generalizations [4]. We wonder whether our main theorem can be applied to generate interesting quasimodular relations involving the $e^{-\lambda_n z}$ kernel, and whether other reciprocity relations for the Riemann zeta function lift to reciprocity relations for arbitrary Dirichlet series.

Dixit et al. have studied many generalizations and analogues of Ramanujan’s reciprocity formula [11, 14–17]. We have rederived some of their results in this paper. We invite the reader to use our main result to rigorously rederive some of their other reciprocity results, as this method may extend the parameter domains under which their identities hold. Our method may also extend to other quasimodular type transformations. For example, Dixit et al. study the higher Herglotz functions [13] and obtain several reciprocity relations for them. Another example is Ramanujan reciprocity over imaginary quadratic number fields [2, Theorem 1.3], though this would require the development of a number field analog of the Koshliakov kernel (3.9).

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