Consistency of quasi-maximum likelihood for processes with asymmetric laplacian innovation

Y.BOULAROUK 1  K.DJABALLAH 2

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Abstract: Strong consistency of the quasi-maximum likelihood estimator are given for a general class of multidimensional causal processes based on asymmetric laplacian innovation.

Keywords: Quasi-maximum likelihood estimator, strong consistency, multidimensional causal processes.
1 Introduction

Since 1970 the statistical modeling changed direction, it just that the statistician community no longer uses structural models of Keynesian inspiration, rather stochastic models which have subsequently found a wide application in different stages of disciplines. But despite the usefulness of these processes do not cover all the phenomena and they are all (ARMA 1970, VAR 1980, ARCH 1982,...) built under the hypothesis of normality is not definitely checked and which directly affects the Likelihood function used for parameter estimation. We give in this paper, for the first time, asymptotic properties, namely strong consistency and asymptotic normality (respectively, SC and AN for short), of the QMLE for many multivariate models with Asymmetric Laplace errors. To establish results in a unified way, we consider almost everywhere (a.e.) solutions $X = (X_t, t \in Z)$ of equations of the type

$$X_t = M_{\theta_0}(X_{t-1}, X_{t-2}, ...)\zeta_t + f_{\theta_0}(X_{t-1}, X_{t-2}, ...) t \in Z. \quad (1.1)$$

Here, $\theta_0$ is the parameter of interest, $M_{\theta_0}(X_{t-1}, X_{t-2}, ...)$ is a $(s \times p)$ random matrix having a.e. full rank $s$, $f_{\theta_0}(X_{t-1}, X_{t-2}, ..121.)$ is a $R^p$ random vector, the $R^p$ random vectors $\zeta = (\zeta^{(k)})_{1 \leq k \leq p}$ are independent and identically distributed satisfying standard assumptions $E[\zeta_0^{(k)}\zeta_0^{(k')}]=0$ for $kk'$ and $E[\zeta_0^{(k)^2}]=\text{Var}(\zeta_0^{(k)})=1$.

In this study we suppose that $\zeta = (\zeta^{(k)})_{1 \leq k \leq p}$ are are distributed according to an Asymmetric Laplacian law. Hence, it has the density function $g$ given by $g(\zeta) = \frac{2e^{\zeta^{'-1}m}}{(2\pi)^{p/2}|\Sigma|^{1/2}} \left(2+\Sigma^{-1}m\right)^{v/2} K_v \left(\sqrt{(2+\Sigma^{-1}m)(\zeta^{'-1}\zeta)}\right)$ with $m = 0$ and the standard conditions defined above are checked, this function becomes:

$$g(\zeta) = \frac{2}{(2\pi)^{p/2}} \left(\frac{\zeta^{'-1}m}{2}\right)^{v/2} K_v \left(\sqrt{2\zeta^{'-1}}\zeta\right) \quad (1.2)$$

where $v = 1 - \frac{p}{2}$ and $K_v(u)$ is the modified Bessel function of the third kind given by $K_v(u) = \frac{1}{\Gamma(\frac{v}{2})}\int_0^\infty e^{-us} s^{-v/2}ds$, $u > 0$.

Since Our sample is made up of the first $n$ terms of an IID sequence of Laplacian random variables. The probability density function of the vector $\zeta = (\zeta_t)_{1 \leq t \leq n}$ is :

$$f(\zeta_1, \zeta_2, ..., \zeta_n) = \frac{2^n}{(2\pi)^{np/2}} \prod_{t=1}^n \left[\left(\frac{\zeta_t^{'-1}m}{2}\right)^{v/2} K_v \left(\sqrt{2}\zeta_t^{'-1}\zeta_t\right)\right]$$

Through a change of variable $X_t = M\zeta_t + f_t$, we find the probability density function of $X$ given by $g(\zeta) = \frac{2K_v(\sqrt{2(X_t-f_t_{0})}^{(H')^{-1}}(X_t-f_t_{0})))}{(2\pi)^{np/2}\text{det}(H_t^{1/2})} \left(\frac{2}{(2\pi)^{np/2}} \left(\frac{\zeta^{'-1}m}{2}\right)^{v/2} K_v \left(\sqrt{2\zeta^{'-1}}\zeta\right)\right)$
and the log likelihood function is

$$L_n(\Theta) = \sum_{t=1}^{n} q_t(\Theta)$$  \hspace{1cm} (1.3)$$

With

$$q_t(\Theta) = \log \left( K\sqrt{2(X_t - f_\Theta(t))(X_t - f_\Theta(t))'} + \frac{V}{2} \log \left( (X_t - f_\Theta(t))(H_t^{-1}(X_t - f_\Theta(t))) \right) \right)$$
$$- \frac{1}{2} \log \left( \det(H_t) \right)$$  \hspace{1cm} (1.4)$$

$$\hat{\theta}_n := \arg\max_{\theta \in \Theta} \hat{L}_n(\theta).$$  \hspace{1cm} (1.5)$$

1.1 Definition of the parameter sets $\Theta(r)$ and $\bar{\Theta}(r)$

In proposition \[below we provide the existence of a stationary solution of the general model (1.1). Two conditions of different types are used: the first one is a Lipschitz condition on the functions $f$ and $M$ in (1.1), the second one is a restriction on the set of the parameters.

Let us assume that for any $\theta \in \mathbb{R}^d$, $x \mapsto f_\theta(x)$ and $x \mapsto M_\theta(x)$ are Borel functions on $(\mathbb{R}^m)^\infty$ and that $\text{Rank } M_\theta(x) = m$ for all $x \in (\mathbb{R}^m)^\infty$. Assume that there exist two sequences $(\alpha_j(f, \theta))_{j=1}^{\infty}$ and $(\alpha_j(M, \theta))_{j=1}^{\infty}$ satisfying, for all $x, y$ in $(\mathbb{R}^m)^\infty$,

$$\|f_\theta(x) - f_\theta(y)\| \leq \sum_{j=1}^{\infty} \alpha_j(f, \theta) \|x_j - y_j\|,
\|M_\theta(x) - M_\theta(y)\| \leq \sum_{j=1}^{\infty} \alpha_j(M, \theta) \|x_j - y_j\|.$$

We can define the set

$$\Theta(r) = \left\{ \theta \in \mathbb{R}^d / \sum_{j=1}^{\infty} \alpha_j(f, \theta) + (\mathbb{E}\|\xi_0\|^r)^{1/r} \sum_{j=1}^{\infty} \alpha_j(M, \theta) < 1 \right\}. \hspace{1cm} (1.6)$$

This set depends on the distribution of $\xi_0$ via the moments $\mathbb{E}\|\xi_0\|^r$. But thanks to the fact that $\mathbb{E}\left[\xi_0^{(k)}\xi_0^{(k')}\right] = 0$ for $k \neq k'$ and $\mathbb{E}[\xi_0^{(k)^2}] = \text{Var } (\xi_0^{(k)}) = 1$ the set $\Theta(2)$ simplifies:

$$\Theta(2) = \left\{ \theta \in \mathbb{R}^d / \sum_{j=1}^{\infty} \alpha_j(f, \theta) + \sqrt{p} \sum_{j=1}^{\infty} \alpha_j(M, \theta) < 1 \right\}. \hspace{1cm}$$

Proposition 1 If $\theta_0 \in \Theta(r)$ for some $r \geq 1$ there exists a unique causal ($X_t$ is independent of $(\xi_t)_{t \geq 1}$ for $t \in \mathbb{Z}$) solution $X$ to the equation (1.1) which is stationary and ergodic and satisfies $\mathbb{E}\|X_0\|^r < \infty$.  

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1.2 **Uniform assumptions on** \( \Theta \)**

Fix some compact subset \( \Theta \) of \( \mathbb{R}^d \). For any sequences \( x, y \) of \( (\mathbb{R}^m)_{\infty} \), the functions \( \theta \mapsto f_\theta(x) \) and \( \theta \mapsto M_\theta(x) \) are assumed to be continuous on \( \Theta \). Assume that \( \|f_\theta(0)\|_\Theta < \infty \) and \( \|M_\theta(0)\|_\Theta < \infty \). To settle the assumptions in a short way, let us introduce the generic symbol \( \Psi \) for any of the functions \( f, M \) or \( H \).

**\( (A1(\Psi)) \)** Let \( \alpha_j(\Psi) = \sup_{\theta \in \Theta} \alpha_j(\Psi, \theta) \) be such that \( \sum_{j \geq 1} \alpha_j(\Psi) < \infty \).

**\( (A2) \)** There exists \( H > 0 \) such that \( \inf_{\theta \in \Theta} \det(H_\theta(x)) \geq H \) for all \( x \in (\mathbb{R}^m)_{\infty} \).

**\( (A3(\Psi)) \)** The function \( \theta \in \Theta \mapsto \Psi_\theta(x) \) is 2 times continuously differentiable for all \( x \in (\mathbb{R}^m)_{\infty} \) and

\[
\left\| \frac{\partial \Psi_\theta(0)}{\partial \theta} \right\|_\Theta + \left\| \frac{\partial^2 \Psi_\theta(0)}{\partial \theta \partial \theta^\prime} \right\|_\Theta < \infty.
\]

Moreover assume that there exist two integrable sequences \( \left( \alpha_j^{(i)}(\Psi) \right)_{j \geq 1}, i = 1, 2 \), such that for all \( x, y \in (\mathbb{R}^m)_{\infty} \)

\[
\left\| \frac{\partial \Psi_\theta(x)}{\partial \theta} - \frac{\partial \Psi_\theta(y)}{\partial \theta} \right\|_\Theta \leq \sum_{j=1}^{\infty} \alpha_j^{(1)}(\Psi) \|x_j - y_j\|,
\]

\[
\left\| \frac{\partial^2 \Psi_\theta(x)}{\partial \theta \partial \theta^\prime} - \frac{\partial^2 \Psi_\theta(y)}{\partial \theta \partial \theta^\prime} \right\|_\Theta \leq \sum_{j=1}^{\infty} \alpha_j^{(2)}(\Psi) \|x_j - y_j\|.
\]

If \( \Psi = H \), \( \|x_j - y_j\| \) in the RHS terms is replaced with \( \|x_j x'_j - y_j y'_j\| \).

The last assumption on the derivatives is just needed for the asymptotic normality of the QMLE.

1.3 **Identifiability and variance conditions**

We assume the same identifiability condition as in Jeantheau \[7\]:

**\( (Id) \)** For all \( \theta \in \Theta \), \( (f_\theta^t = f_{\theta_0}^t \text{ and } H_\theta^t = H_{\theta_0}^t \text{ a.s.}) \Rightarrow \theta = \theta_0 \).

**\( (Var) \)** One of the families \( \left( \partial f_{\theta_0}^t / \partial \theta_i \right)_{1 \leq i \leq d} \) or \( \left( \partial H_{\theta_0}^t / \partial \theta_i \right)_{1 \leq i \leq d} \) is a.e. linearly independent, where:

\[
\frac{\partial f_\theta^t}{\partial \theta} := \frac{\partial f_\theta}{\partial \theta}(X_{t-1}, \ldots) \text{ and } \frac{\partial H_\theta^t}{\partial \theta} := \frac{\partial H_\theta}{\partial \theta}(X_{t-1}, \ldots).
\]
The condition (Var) is needed for ensuring finiteness of the asymptotic variance in the result on asymptotic normality.

**Proposition 2** Let be \( k_v(u) \) the Bessel function of third kind. For all real \( x,y \) there exist \( A,B \) constants which satisfies:

- (ii) \( \sup_{\theta \in \Theta} |k_v(x)| < Au^v \)
- (ii) \( \sup_{\theta \in \Theta} |k_v(y) - k_v(x)| < B|y - x| \), \( \forall v \leq 0 \)
- (iii) \( \sup_{\theta \in \Theta} |\log[k_\frac{1}{2}](y)] - \log[k_\frac{1}{2}](x)| | < B|y - x| \)

**Proof**:

(i) We have:

\[
k_v(u) = \frac{1}{2} \left( \frac{u}{2} \right)^v \int_0^\infty t^{-v-1} e^{-t - \frac{u^2}{4t}} dt
\]

\[
|k_v(u)| = \frac{1}{2} \left| \frac{u}{2} \right|^v \int_0^\infty t^{-v-1} e^{-t - \frac{u^2}{4t}} dt
\]

\[
= \frac{1}{2^{v+1}} |u|^v \int_0^\infty t^{-v-1} e^{-t - \frac{u^2}{4t}} dt
\]

\[
= \frac{1}{2^{v+1}} |u|^v \int_0^\infty t^{-v-1} e^{-t - \frac{u^2}{4t}} dt
\]

\[
\leq \frac{1}{2^{v+1}} |u|^v \int_0^\infty t^{-v-1} e^{-t} dt \leq Au^v \ \forall v < 0
\]

(ii)

\[
k_v(u) = \frac{1}{2} \left( \frac{u}{2} \right)^v \int_0^\infty t^{-v-1} e^{-t - \frac{u^2}{4t}} dt
\]

\[
= \frac{1}{2^{v+1}} \int_0^\infty t^{-v-1} e^{-t - \frac{u^2}{4t}} dt
\]

Let put \( g(u) = u^v e^{-\frac{u^2}{4}} \) \( (g'(u) = u^{v-1} e^{-\frac{u^2}{4}} (v - \frac{u^2}{2}) \lim_{x \to 0} g'(u) = a < 0 \) and \( \lim_{u \to \infty} g'(u) = 0 \)

\( g''(u) = u^v e^{-\frac{u^2}{4}} (v - \frac{u^2}{2} + 1) \) \( g''(u) = 0 \Rightarrow u = \pm \sqrt{-2t(v + 1)} \)

* a simple study of \( g' \) shows that it is bounded and so \( g \) is lipschitzian which implies that \( |g(y) - g(x)| \leq c|y - x| \)
\[ |k_v(y) - k_v(x)| \leq \frac{1}{2^{v+1}} \int_0^\infty t^{-v-1} e^{-t} |y^v e^{-\frac{y^2}{4}} - x^v e^{-\frac{x^2}{4}}| dt + \frac{1}{2^{v+1}} \int_0^\infty t^{-v-1} e^{-t} c |y - x| dt \]

\[ = \frac{c}{2^{v+1}} |y - x| \int_0^\infty t^{-v-1} e^{-t} dt \]

or:

\[ \int_0^\infty t^{-v-1} e^{-t} dt = \int_0^1 t^{-v-1} e^{-t} dt + \int_1^\infty t^{-v-1} e^{-t} dt < \infty \quad \text{since } v < 0 \]

\[ |k_0(y) - k_0(x)| \leq \int_0^1 \frac{|\cos(yt) - \cos(xt)|}{\sqrt{t^2 + 1}} dt + \frac{1}{2^{v+1}} \int_1^\infty t^{-v-1} e^{-t} |y^v e^{-\frac{y^2}{4}} - x^v e^{-\frac{x^2}{4}}| dt \]

\[ \leq \int_0^1 c |y - x| \frac{t}{\sqrt{t^2 + 1}} dt + \frac{1}{2^{v+1}} \int_1^\infty t^{-v-1} e^{-t} c |y - x| dt \]

\[ = c |y - x| \left[ \int_0^1 \frac{t}{\sqrt{t^2 + 1}} dt + \frac{1}{2^{v+1}} \int_1^\infty t^{-v-1} e^{-t} dt \right] \]

\[ = c |y - x| \left[ \sqrt{2} + \frac{1}{2^{v+1}} \int_1^\infty t^{-v-1} e^{-t} dt \right] < \infty \]

so

\[ \sup_{\theta \in \Theta} |k_v(y) - k_v(x)| \leq B |y - x| \quad \forall v \leq 0 \]

(iii) by definition we have \( k_v(u) = \frac{1}{2} (\frac{u}{2})^v \int_0^\infty t^{-v-1} e^{-t - \frac{u^2}{4}} dt, u > 0 \) in particular

\[ k_\frac{1}{2}(u) = \sqrt{\frac{\pi}{2u}} e^{-u} \]

\[ \log[k_\frac{1}{2}(u)] = \frac{1}{2} [\log\pi - \log(2u)] - u \]

else

\[ |\log[k_\frac{1}{2}(y)] - \log[k_\frac{1}{2}(x)]| = |(\frac{1}{2} [\log\pi - \log(2y)] - y) - (\frac{1}{2} [\log\pi - \log(2x)] - x)| \]

\[ = \frac{1}{2} [\log(2x) - \log(2y) + 2(x - y)] \leq c |y - x| \]

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Remark 1.1 This last proposition implies that $|\log[k_v(y)] - \log[k_v(x)]| \leq c|y - x|$ for any possible values of $v$.

Theorem 1 Assume that $\theta_0 \in \Theta(2) \cap \Theta$ and let $X$ be the stationary solution of (1.1). If $\theta_0 \in \Theta$, a compact set of $R^d$ such that assumptions $(D(\Theta)), (Id(\Theta)), (A0(f, \Theta))$ and $(A0(M, \Theta))$ hold with:

$$\alpha_j^f(f, \Theta) + \alpha_j^M(M, \Theta) + \alpha_j^H(H, \Theta) = O(j^{-l}) \text{ for some } l > 2$$

(1.7)

then the QMLE $\hat{\theta}_n$ defined in (1.3) is SC; that is, $\hat{\theta}_n \rightarrow \theta_0$ a.s.

Lemma 1 Assume that $\theta_0 \in \Theta(r)$ for $r \geq 2$ and $X$ is the stationary solution of (1). Let $\Theta$ be a compact set of $R^d$:

1. If $(A_0(f, \Theta))$ holds, then $\forall \theta \in \Theta, f_0^j \in L^r(C(\Theta, R^m))$ and

$$E[\|\hat{f}_n^j - f_0^j\|_\Theta^r] \leq E[\|X_0\|_r^r (\|\alpha_j(f)\|_\Theta) : \text{ for all } t \in N^*;$$

2. If $(A_0(M, \Theta))$ holds, then $\forall \theta \in \Theta, H_0^j \in L^{r/2}(C(\Theta, M_m))$ and there exists $C > 0$ not depending on $t$ such that

$$E[\|\hat{H}_n^j - H_0^j\|_\Theta^{r/2}] \leq C (\|\alpha_j^0(M, \Theta)\|_\Theta)^{r/2} \text{ for all } t \in N^*;$$

3. If $(A_0(H, \Theta))$ holds, then $\forall \theta \in \Theta, H_0^j \in L^{r/2}(C(\Theta, M_m))$ and

$$E[\|\hat{H}_n^j - H_0^j\|_\Theta^{r/2}] \leq E[\|X_0\|_r^r (\|\alpha_j^0(H, \Theta)\|_\Theta)^{r/2} \text{ for all } t \in N^*;$$

Moreover, under any of the two last conditions and with $(D(\Theta)), H_0^j$ is an invertible matrix and $\|((\hat{H}_n^j)^{-1})\|_\Theta \leq H^{-1/m}$.

Proof of Lemma 1. See Lemma 1 of Bardet and Weitenberger (1).

Proof of Theorem 1. The proof of the theorem is divided into two parts. In (i), a uniform (in $\theta$) law of large numbers on $(\hat{q}_t)_{t \in N^*}$ [defined in (1.3)] is established. In (ii), it is proved that $L(\theta) := E(q_0(\theta))$ has a unique maximum in $\theta_0$. Those two conditions lead to the consistency of $\hat{\theta}_n$. (i) Using Proposition 1, with $q_t = G(X_t, X_{t-1}, ...)$, one deduces that $(q_t)_{t \in Z}$ [defined in (3)] is a stationary ergodic sequence. From Straumann and Mikosch [8], we know that, if $(v_t)_{t \in Z}$ is a stationary ergodic sequence of random elements with values in $C(\Theta, R^m)$, then the uniform (in $\theta \in \Theta$) law of large numbers
is implied by $E||v_0||_\Theta \leq \infty$. As a consequence, $X = (q_t, t \in Z)$ satisfies a uniform ($in\theta \in \Theta$) strong law of large numbers as soon as $E[\sup_\theta |q_t(\theta)|] < \infty$. But, from the inequality $\log(x) \leq x$, for all $x \in [0, \infty[$ and Lemma 1, for all $t \in Z$,

$$|q_t(\theta)| \leq \log \left[ K_v \left( \sqrt{2(X_t - f_\Theta'(H)')^{-1}(X_t - f_\Theta')} \right) \right] + \frac{v}{2} (X_t - f_\Theta')' (H')^{-1} (X_t - f_\Theta') - \frac{1}{2} \log (\det(H_\Theta')$$

Note: $S(\Theta) = \frac{v}{2} (X_t - f_\Theta')' (H')^{-1} (X_t - f_\Theta') - \frac{1}{2} \log (\det(H_\Theta'))$

$$|S(\Theta)| \leq \frac{v}{2} \frac{||X_t - f_t'(\theta)||^2}{(H')^{-1}} + m \left[ \frac{1}{m} \log(H) + \frac{||H_\Theta'||}{(M')^{-1}} - 1 \right]$$

$$\Rightarrow \sup_{\theta \in \Theta} |S(\Theta)| \leq \frac{v}{2} \frac{||X_t - f_t'(\theta)||^2}{(H')^{-1}} + \frac{1}{2} |\log H| + \frac{m}{2} \frac{||H_\Theta'||}{(M')^{-1}} \quad (1.8)$$

But, $\forall t \in Z, E||X_t|| < \infty$ (see Proposition 1) and $E||f_t'(\theta)||^2_\Theta + E||H_\Theta'||^2_\Theta < \infty$ (see Lemma 1). As a consequence, the right-hand side of (1.8) has a finite first moment. Therefore, to proof that $q_t$ have a finite first order moment, we have to proof that $E \left( \sup_{\theta \in \Theta} \log \left[ K_v \sqrt{2(X_t - f_\Theta'(H)')^{-1}(X_t - f_\Theta')} \right] \right) < \infty$.

By the result of Proposition 2 we have $\sup_{\theta \in \Theta} |k_v(u)| < Au^v$

$$\log \left[ K_v \sqrt{2(X_t - f_\Theta'(H)')^{-1}(X_t - f_\Theta')} \right] \leq \log \left[ K_v (2(X_t - f_\Theta'(H)')^{-1}(X_t - f_\Theta')) \right] \leq A' \left[ (2(X_t - f_\Theta'(H)')^{-1}(X_t - f_\Theta')) \right]^\frac{v}{2} \leq A' \frac{|X_t - f_t'(\theta)||_\Theta}{H^{\frac{m}{2}}}

\Rightarrow \sup_{\theta \in \Theta} \left| \log \left[ K_v \left( \sqrt{2(X_t - f_\Theta'(H)')^{-1}(X_t - f_\Theta')} \right) \right] \right| \leq A' \frac{|X_t - f_t'(\theta)||_\Theta}{H^{\frac{m}{2}}}

whence:

$$E \sup_{\theta \in \Theta} \left| \log \left[ K_v \sqrt{2(X_t - f_\Theta'(H)')^{-1}(X_t - f_\Theta')} \right] \right| < \infty$$
and, therefore, 
\[ E[\sup_{\theta \in \Theta} |q_t(\theta)|] < \infty \]

The uniform strong law of large numbers for \((q_t(\theta))\) follows; hence,
\[ \left| \frac{L_n(\theta)}{n} - L(\theta) \right|_\Theta \to 0 \text{ a.s} \quad \text{with} \quad L(\theta) := E[q_0(\theta)] \]

Now, one shows that \(\frac{1}{n}\left| L_n - L_\theta \right|_\Theta \to 0 \text{ a.s}\). Indeed, for all \(\theta \in \Theta\) and \(t \in N^*\),

Let put

\[ A_t(\theta) = \frac{1}{2} v(X_t - f_0^t)'(H_t')^{-1}(X_t - f_0^t) + \det(H_0^t) \]

\[ B_t(\theta) = \log \left(K_v \sqrt{2(X_t - f_0^t)'(H_t')^{-1}(X_t - f_0^t)} \right) \]

\[ |\hat{q}_t(\theta) - q_t(\theta)| \leq |\hat{A}_t(\theta) - A_t(\theta)| + |\hat{B}_t(\theta) - B_t(\theta)| \quad (1.9) \]

\[ |\hat{A}_t(\theta) - A_t(\theta)| \leq \frac{v}{2} \left| (X_t - \hat{f}_0^t)'(\hat{H}_t')^{-1}(X_t - \hat{f}_0^t) - (X_t - f_0^t)'(H_t')^{-1}(X_t - f_0^t) \right| 
\]

\[ + \frac{1}{2} \left| \det(\hat{H}_0^t) - \det(H_0^t) \right| \]

\[ \leq \frac{1}{2C} \left| \det(\hat{H}_0^t) - \det(H_0^t) \right| + \frac{v}{2} \left| (X_t - \hat{f}_0^t)'[(\hat{H}_t')^{-1} - (H_t')^{-1}](X_t - \hat{f}_0^t) \right| 
\]

\[ + \frac{v}{2}(2X_t - \hat{f}_0^t - f_0^t)'(H_t')^{-1}(f_0^t - \hat{f}_0^t) \]

\[ \leq \frac{1}{2} H^{-1} \left| \det(\hat{H}_0^t) - \det(H_0^t) \right|_\Theta + \frac{v}{2} \left( \left| X_t \right| + \left| \hat{f}_0^t \right|_\Theta \right) \left| (\hat{H}_t')^{-1} - (H_t')^{-1} \right|_\Theta \]

\[ + \frac{v}{2} \left( \left| X_t \right| + \left| \hat{f}_0^t \right|_\Theta + \left| f_0^t \right|_\Theta \right) \left| H_t' \right|_\Theta \left| \hat{f}_0^t - f_0^t \right|_\Theta \quad (1.10) \]

on the one hand we have,
\[ \left| (\hat{H}_0^t)^{-1} - (H_0^t)^{-1} \right|_\Theta \leq \left| (\hat{H}_0^t) - H_0^t \right| \left| (H_0^t)^{-1} \right|_\Theta \]

on the other hand, for invertible matrix \(A \in M_m(R)\), and \(H \in M_m(R)\),

\[ \det(H_0^t) = \det(\hat{H}_0^t) + \det(\hat{H}_0^t).Tr \left( (\hat{H}_0^t)^{-1}'((\hat{H}_0^t)^{-1} - (H_0^t)^{-1}) \right) + o((\hat{H}_0^t)^{-1} - (H_0^t)^{-1}), \]

where \(\left| Tr \left( ((\hat{H}_0^t)^{-1})'((\hat{H}_0^t)^{-1} - (H_0^t)^{-1}) \right) \right| \leq \left| (\hat{H}_0^t)^{-1} \right|_\Theta \left| \hat{H}_0^t - H_0^t \right|_\Theta \). Using

the relation \(\left| (H_0^t)^{-1} \right| \geq H^{-m}\) for all \(t \in Z\), there exists \(C > 0\) not depending

on \(t\), such that inequality (8) becomes

\[ \sup_{\theta \in \Theta} |\hat{A}_t(\theta) - A_t(\theta)| \leq C \left( \left| X_t \right| + \left| \hat{f}_0^t \right|_\Theta + \left| f_0^t \right|_\Theta \right) \left( \left| \hat{H}_0^t - H_0^t \right|_\Theta + \left| \hat{f}_0^t - f_0^t \right|_\Theta \right) \]
\[ \left| \tilde{B}_t(\theta) - B_t(\theta) \right| \leq K_n \sqrt{2(X_t - f^{\ell}_0)'(\tilde{H}^t)^{-1}(X_t - f^{\ell}_0)} - K_n \sqrt{2(X_t - f^{\ell}_0)'(H^t)^{-1}(X_t - f^{\ell}_0)} \]

\[ \leq A \sqrt{2(X_t - f^{\ell}_0)'(\tilde{H}^t)^{-1}(X_t - f^{\ell}_0) - (X_t - f^{\ell}_0)'(H^t)^{-1}(X_t - f^{\ell}_0)} \]

\[ \leq \sqrt{2A} \left( (X_t - \tilde{f}^{\ell}_0)'(\tilde{H}^t)^{-1} - (H^t)^{-1} \right) (X_t - \tilde{f}^{\ell}_0) \]

Following the same approach for \( A \) found :

\[ \sup_{\theta \in \Theta} \left| \tilde{B}_t(\theta) - B_t(\theta) \right| \leq C' \left( \| X_t \| + \left\| \tilde{f}^{\ell}_0 \right\|_{\Theta} + \left\| f^{\ell}_0 \right\|_{\Theta} \right)^{\frac{1}{2}} \left( \left\| \tilde{H}_0 - H_0 \right\|_{\Theta} + \left\| \tilde{f}^{\ell}_0 - f^{\ell}_0 \right\|_{\Theta} \right)^{\frac{1}{2}} \]

From the Holder and Minkowski inequalities and by virtue of \( 3/2 = 1 + 1/2 \),

\[ E[ \sup_{\theta \in \Theta} \left| \tilde{A}_t(\theta) - A_t(\theta) \right|] \leq C \left( E[\| X_t \|] + \left\| \tilde{f}^{\ell}_0 \right\|_{\Theta} + \left\| f^{\ell}_0 \right\|_{\Theta} \right)^{\frac{1}{2}} \times \left( E[\| \tilde{H}_0 - H_0 \|_{\Theta}] + E[\| \tilde{f}^{\ell}_0 - f^{\ell}_0 \|_{\Theta}] \right)^{\frac{1}{2}} \]

\[ \leq C \left( E[\| \tilde{H}_0 - H_0 \|_{\Theta}] + E[\| \tilde{f}^{\ell}_0 - f^{\ell}_0 \|_{\Theta}] \right)^{\frac{1}{2}} \]  

\[ E[ \sup_{\theta \in \Theta} \left| \tilde{B}_t(\theta) - B_t(\theta) \right|^{1/2}] \leq C' \left( E[\| X_t \|] + \left\| \tilde{f}^{\ell}_0 \right\|_{\Theta} + \left\| f^{\ell}_0 \right\|_{\Theta} \right)^{\frac{1}{2}} \times \left( E[\| \tilde{H}_0 - H_0 \|_{\Theta}] + E[\| \tilde{f}^{\ell}_0 - f^{\ell}_0 \|_{\Theta}] \right)^{\frac{1}{2}} \]

\[ \leq C' \left( E[\| \tilde{H}_0 - H_0 \|_{\Theta}] + E[\| \tilde{f}^{\ell}_0 - f^{\ell}_0 \|_{\Theta}] \right)^{\frac{1}{2}} \]

with \( C > 0, C' > 0 \) not depending on \( \theta \) and \( t \). Now, consider, for \( n \in N \),

\[ S_n = \sum_{i=1}^{n} \frac{1}{i} \sup_{\theta \in \Theta} \left| \tilde{q}_i(\theta) - q_i(\theta) \right| \]

Applying the Kronecker lemma (see Feller [2], page 238), if \( \lim_n \rightarrow \infty S_n < \infty \text{a.s.} \), then \( \frac{1}{n} \| \tilde{L}_n - L_n \| \rightarrow 0 \text{a.s.} \). Following Feller’s arguments, it remains to show that, for all \( \varepsilon > 0 \),

\[ P(\forall n \in N, \exists m \text{ such that } |S_m - S_n| > \varepsilon) = P(A) = 0. \]
Let $\varepsilon > 0$, and denote

$$A_{m,n} := \{|S_m - S_n| > \varepsilon\}$$

for $m > n$. Notice that $A = \bigcap_{n \in \mathbb{N}} \bigcup_{m > n} A_{m,n}$. For $n \in \mathbb{N}^*$, the sequence of sets $(A_{m,n})_{m > n}$ is obviously increasing, and, if $A_n = \bigcup_{m > n} A_{m,n}$, then $\lim_{m \to \infty} P(A_{m,n}) = P(A_n)$. Observe that $(A_n)_{n \in \mathbb{N}}$ is a decreasing sequence of sets and, thus,

$$\lim_{n \to \infty} \lim_{m \to \infty} P(A_{m,n}) = \lim_{n \to \infty} P(A_n) = P(A).$$

It remains to bound $P(A_{m,n})$. From the Bienyamé–Chebyshev inequality,

$$P(A_{m,n}) = P \left( \sum_{t=n+1}^{m} \frac{1}{l} \sup_{\theta \in \Theta} |\hat{q}_t(\theta) - q_t(\theta)| > \varepsilon \right) \leq \frac{1}{\varepsilon^2} \mathbb{E} \left[ \left( \sum_{t=1}^{n} \frac{1}{l} \sup_{\theta \in \Theta} |\hat{q}_t(\theta) - q_t(\theta)| \right)^{\frac{2}{3}} \right] \leq \frac{1}{\varepsilon^{\frac{2}{3}}} \sum_{t=n+1}^{m} \frac{1}{l^{\frac{2}{3}}} \mathbb{E} \left( \sup_{\theta \in \Theta} |\hat{q}_t(\theta) - q_t(\theta)|^{\frac{2}{3}} \right).$$

Using (9) and condition (5), since $l > 3/2$, there exists $C > 0$ such that

$$\left( \sum_{j=1}^{\infty} \alpha_j^0(f,\Theta) + \alpha_j^0(M,\Theta) + \alpha_j^0(H,\Theta) \right)^{\frac{2}{3}} \leq \frac{C}{l^{2(l-1)/3}}$$

Thus,

$$t^{-\frac{2}{3}} \mathbb{E} \left( \sup_{\theta \in \Theta} |\hat{q}_t(\theta) - q_t(\theta)|^{\frac{2}{3}} \right) < C(l^{-2/3})$$

for some $C > 0$, and

$$\sum_{t=1}^{m} \frac{1}{l^{\frac{2}{3}}} \mathbb{E} \left( \sup_{\theta \in \Theta} |\hat{q}_t(\theta) - q_t(\theta)|^{\frac{2}{3}} \right) < \infty \quad a.s \quad l > 3/2$$

Thus,

$$\lim_{n \to \infty} \lim_{m \to \infty} P(A_{m,n}) \to 0 \text{ and } \frac{1}{n} \|\hat{L}_n - L_n\| \to 0 \text{ a.s}$$

(ii) En cours........

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