UNRAMIFIED CASE OF THE GROTHENDIECK–SERRE CONJECTURE
ABOUT TORSORS

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Abstract. We prove the unramified case of the Grothendieck–Serre conjecture: let $R$ be an unramified regular local ring (that is a local ring such that all its $\mathbb{Z}$-fibers are regular) and $G$ be a reductive $R$-group scheme. Then a $G$-torsor over $R$ is trivial, provided that it is trivial over the fraction field of $R$. In fact, we prove a slight generalization of that: $R$ is allowed to be a semilocal ring geometrically regular over a Dedekind domain. We prove some additional results concerning torsors that are of independent interest.

1. Introduction and Main results

Let $R$ be a regular local ring; let $G$ be a reductive group scheme over $R$. A conjecture of Grothendieck and Serre (see [Ser, Remarque, p.31], [Gro1, Remarque 3, p.26-27], and [Gro2, Remarque 1.11.a]) predicts that a $G$-torsor over $R$ is trivial, if it is trivial over the fraction field of $R$. Recently this has been proved in the case when $R$ contains an infinite field in [FP], it was extended to the case of finite fields in [Pan3]. In the mixed characteristic case the conjecture was previously known if $R$ is unramified (that is, the fibers of the projection $\text{Spec} R \to \text{Spec} \mathbb{Z}$ are regular) and $G$ is quasi-split, see [ˇCes].

The goal of this note is to extend this to the case of arbitrary reductive group scheme $G$. Precisely, we prove the following theorem, which is a generalization of [ˇCes, Thm. 9.1] to arbitrary reductive group schemes.

Theorem 1. Let $R$ be a Noetherian semilocal flat $\mathcal{O}$-algebra, where $\mathcal{O}$ is a semilocal Dedekind domain. Assume that all fibers of $\text{Spec} R \to \text{Spec} \mathcal{O}$ are geometrically regular. Let $G$ be a reductive $R$-group scheme and let $E$ be a $G$-torsor over $R$. If $E$ is trivial over the total ring of fractions of $R$, then $E$ is trivial.

This theorem will be proved in Section 7. Note that the conjecture is currently known in very few ramified cases (see the references in [ˇCes]).

Remark 1.1. In the situation of Theorem 1, $R$ is automatically regular. Indeed, Popescu’s Theorem [SP, Tag07GC] shows that $R$ is a filtered colimit of smooth finitely generated $\mathcal{O}$-algebras, which are regular by [SP, Tag07NF].

Corollary 1. Let $R$ be a semilocal flat $\mathcal{O}$-algebra, where $\mathcal{O}$ is a semilocal Dedekind domain. Assume that all fibers of $\text{Spec} R \to \text{Spec} \mathcal{O}$ are geometrically regular. Let $G$ be a reductive $R$-group scheme, then two $G$-torsors over $R$ that become isomorphic over the fraction field of $R$, are isomorphic. In other words, the restriction map

$$H^1_{\text{et}}(R, G) \to H^1_{\text{et}}(K(R), G)$$
is injective.

Proof. See the proof of [FP] Cor. 1. □

Let \((R, m)\) be a regular local ring. Recall that \(R\) is unramified if \(R/pR\) is also regular, where \(p\) is the characteristic of the residue field \(R/m\). It follows from Theorem 1 that the Grothendieck–Serre conjecture holds for unramified regular local rings:

Corollary 2. Let \(R\) be an unramified regular local ring and \(G\) be a reductive \(R\)-group scheme. Then a \(G\)-torsor over \(R\) is trivial, provided that it is trivial over the fraction field of \(R\).

Proof. If \(R\) contains a field, then this follows from [FP] Pan3. Otherwise, there is a unique integer prime number \(p\) such that \(p\) is neither zero, nor invertible in \(R\). Thus, \(R\) is a flat \(O\)-algebra, where \(O = \mathbb{Z}_p\). The special fiber of the projection \(\text{Spec} \ R \to \text{Spec} \ O\) is equal to \(R/pR\); by definition of unramifiedness it is regular, thus it is geometrically regular over \(\mathbb{Z}/p\mathbb{Z}\) because \(\mathbb{Z}/p\mathbb{Z}\) is perfect. The generic fiber of this projection is also regular because \(R\) is regular (see [SP Tag07C0]). Hence we can apply Theorem 1. □

1.1. History and overview of the proof. The Grothendieck–Serre conjecture can be generalized from local to semilocal case. Most of the papers cited deal with the semilocal case.

(a) The conjecture was proved in the case when the semilocal ring contains an infinite field in 2012 by Fedorov and Panin who completed the work of many people; see [FP] and the historical remarks therein. The conjecture was proved when the semilocal ring contains a finite field by Panin in 2014; see [Pan3].

(b) We briefly explain the strategy of the proof in the equicharacteristic case. The first step is to use Popescu's Theorem to reduce to the case when \(R\) is the semilocal ring of a finite set \(x\) of closed points on an integral affine scheme \(X\) smooth and of finite type over a field \(k\). By spreading out we may assume that the group scheme \(G\) and the \(G\)-torsor \(E\) are defined over \(X\). We may also assume that \(E\) is trivial away from a proper closed subscheme \(Y \subset X\).

(c) The second step is to fiber an open neighborhood \(X' \subset X\) of \(x\) over an open subset \(S \subset \mathbb{A}^{\dim X - 1}_k\) in such a way that the fibers are smooth curves and \(Y \cap X'\) is finite over the base. In fact, any generic projection does the job. Formally, this is accomplished by compactifying \(X\) and using Bertini's Theorem. As we will see momentarily, this is where the situation becomes drastically different in the mixed characteristic case. We note that the finite field case became approachable after finite field Bertini’s Theorem became available (see [Poo]).

(d) The third step is to replace \(X' \to S\) with \(C := X' \times_S \text{Spec} \ R \to \text{Spec} \ R\) and \(E\) with \(E' := p^*_1E\). The original torsor \(E\) is recovered as \(\Delta^*E\), where \(\Delta: \text{Spec} \ R \to C\) is the diagonal embedding. The data \((C \to \text{Spec} \ R, \Delta, E')\) can be improved until \(E'\) can be descended onto \(\mathbb{A}^1_R\).

(e) The last step is to show that if \(E'\) is a torsor over \(\mathbb{A}^1_R\) trivial away from an \(R\)-finite subscheme, then \(E'\) is trivial along any section of \(\mathbb{A}^1_R \to \text{Spec} \ R\). We note that originally this statement was only available in the case when \(G\) is simple and simply-connected, so one had to reduce to this case. One of the main results of [Fed3] shows that the statement is true for any reductive \(G\).

(f) Very little was known about the mixed characteristic case until 2015 (see [Fed2] and the historical remarks in loc. cit.) Here is the main ideas of loc. cit. Assume that \(R\) is unramified. Then the main
difficulty is that the argument in (c) fails. For example, if \( \dim X = 2 \), then the fibers of the projection \( X \to \text{Spec} \mathbb{Z}_{p^2} \) are already one-dimensional, so there is nothing to fiber. On the other hand, \( Y \) may not be finite over \( \text{Spec} \mathbb{Z}_{p^2} \), which is crucial for the following. In general, we “loose” one dimension because the projection \( X \to \text{Spec} \mathbb{Z}_{p^2} \) cannot be deformed.

Here is the main idea of [Fed2]: let \( G \) be quasi-split with a Borel subgroup scheme \( B \). The generic trivialization of \( E \) induces a generic reduction to \( B \). Since \( G/B \) is projective, this reduction can be extended to a complement of a subscheme of codimension two; call it \( Z \). Now we recover the lost dimension: there is a smooth morphism \( X' \to S \) similar to the above such that \( Z \cap X' \) is \( S \)-finite (for example, if \( \dim X = 2 \), then \( Z \) is a finite scheme, so it is automatically finite over \( S \)). One shows that this can be performed in such a way that there is \( Y \subset X' \), also \( S \)-finite, such that \( Y \supset Z \cap X' \) and \( X' - Y \) is affine. One then shows that \( E \) can be reduced to the unipotent radical \( R_u(B) \) of \( B \) on \( X' - Y \), which shows that \( E_{X' - Y} \) is trivial (because \( X' - Y \) is affine). The rest of the proof is very similar to the equal characteristic case.

To make the above ideas work, it is required in [Fed2] that \( X \) has a projective compactification satisfying some technical conditions. It is also required that \( G \) is split and only the local (rather than semilocal) case is considered.

(g) In 2020 Česnavičius (see [Ces]) was able to get rid of the assumptions on the compactification, as well as to generalize to quasi-split group schemes and to the semilocal rings. Also, the proof was streamlined in loc. cit., so we will generally follow it in this paper. The first main idea is to use Cohen–Macaulay compactifications. The second idea is that one need not choose \( Y \) as above at all. Thus, Česnavičius descends \( E' \) to \( \mathbb{A}^1_R \) by showing that \( R_u(B) \)-torsors can be descended along certain affine morphisms. The author was not able to generalize this to our situation, so a different path was followed (see Proposition 4.5 and Proposition 5.2).

(h) We also mention the paper [Pan1], where the conjecture is proved for \( \text{SL}_1(D) \), where \( D \) is an Azumaya algebra over an unramified regular local ring.

(i) The first main idea of this paper is to work on a cover \( \bar{U} \to U := \text{Spec} \mathbb{R} \) such that \( G_{\bar{U}} \) contains a Borel subgroup scheme \( \bar{B} \). We show that \( E_{\bar{U}} \) can be reduced to \( R_u(B) \) away from a subscheme \( Z \) of codimension two, and, moreover, this reduction generically comes from a generic trivialization of \( E \) (see Section 2.2). This is a strong property: we show in Proposition 5.1 that in this situation \( E \) itself is trivial over any affine subscheme of \( U \) disjoint from \( Z \). More precisely, we have the following theorem, which is more general, than what is needed for the proof of Theorem 1. Recall that strictly local rings of a scheme are the strict henselizations of its local rings. A scheme is \emph{geometrically factorial} if its strictly local rings are factorial. Note that regular schemes are geometrically factorial (combine [EGA4] Cor. 18.8.13] and [EGA4] Thm. 21.11.1])

\textbf{Theorem 2.} Let \( U \) be a Noetherian geometrically factorial affine semilocal scheme and \( G \) be a reductive \( U \)-group scheme. Let \( E \) be a generically trivial \( G \)-torsor over an open subscheme \( U_0 \subset U \). Then there is a closed subscheme \( Z \subset U \) of codimension at least two such that \( E \) is trivial over any open affine subscheme of \( U_0 \) that is disjoint from \( Z \).

The theorem will be proved in Section 5. As a corollary we prove the dimension one case of the Grothendieck–Serre conjecture (see [Nis] [PS] [Guo]).
Corollary 3. Let \( R \) be a Dedekind domain. Then the Grothendieck–Serre conjecture holds for \( R \).

Proof. Apply Theorem 2 to \( U_0 = U := \text{Spec} R \). Since \( \dim U = 1 \), \( Z \) must be empty. \( \square \)

(j) For the next step, following essentially [Fed2] and [Čes], one finds a smooth curve \( C \to \text{Spec} R \) with a section \( \Delta \), a torsor \( \mathcal{E}' \) over \( C \) such that \( \Delta^* \mathcal{E}' = \mathcal{E} \), and an \( R \)-finite closed subset \( Z \subset C \) such that \( \mathcal{E}' \) is reduced to a unipotent radical of a Borel subgroup scheme on a finite étale cover of \( C \) (Proposition 4.1). As before, after improving these data (Proposition 4.5), one gets an étale morphism \( C \to \mathbb{A}^1_R \) such that \( Z \) maps isomorphically onto a closed subscheme \( Z' \subset \mathbb{A}^1_R \) such that the preimage of \( Z' \) in \( C \) is equal to \( Z \).

The second main idea is that after shrinking \( C \) we can find a closed subscheme \( Y \subset C \) such that \( Y \) contains \( Z \), \( C - Y \) is affine, and the composition \( Y \to C \to \mathbb{A}^1_R \) is a locally closed embedding; denote its image by \( Y' \). The difference with the previous works is that \( Y \) is not necessarily finite over \( R \). Nevertheless, \( \mathcal{E}' \) is trivial over \( C - Y \) by affineness (Proposition 3.1). Thus, we can descend \( \mathcal{E}' \) to a torsor \( \mathcal{E}'' \) over an open neighborhood \( W \subset \mathbb{A}^1_R \) of \( Z' \). We show that it can be also reduced to the unipotent radical of a Borel subgroup scheme over a cover of \( W - Z' \). Note that \( Z' \) is \( R \)-finite, so we can enlarge \( Z' \) in such a way that it is still \( R \)-finite and that \( W - Z' \) is affine. Then \( \mathcal{E}'' \) is trivial over \( W - Z' \) and we can extend it to \( \mathbb{A}^1_R \) by gluing with a trivial torsor.

(k) The last step is to generalize Step (j) to the mixed characteristic case; see Theorem 3 in Section 6. This is accomplished by the following theorem of independent interest.

Theorem 3. Let \( U \) be an affine semilocal scheme. Let \( G \) be a reductive group scheme over \( U \). Assume that \( Z \) is a closed subscheme of \( \mathbb{A}^1_U \) finite over \( U \). Let \( \mathcal{E} \) be a \( G \)-torsor over \( \mathbb{A}^1_U \) trivial over \( \mathbb{A}^1_U - Z \). Then for every section \( \Delta : U \to \mathbb{A}^1_U \) of the projection \( \mathbb{A}^1_U \to U \) the \( G \)-torsor \( \Delta^* \mathcal{E} \) is trivial.

This theorem will be proved in Section 6. If \( U \) is a scheme over a field, then this is a slight generalization of [Fed3, Thm. 4]. The general case needs only minor modifications because most of the work is happening over the closed points of \( U \) anyways.

1.2. Notations. For a group scheme \( G \) we denote by \([G, G]\) its derived subgroup scheme, by \( G^\text{ad} \) its adjoint group scheme, and by \( R_u(G) \) its unipotent radical.

In this paper we work with right torsors; we only consider torsor for flat and finitely presented group schemes. If \( G \) is a \( T \)-group scheme and \( T' \to T \) is a morphism, we say “a \( G \)-torsor over \( T'' \) to mean a \( G \cdot T'' \)-torsor. If \( T'' \to T' \) is another morphism and \( \mathcal{E} \) is a \( G \)-torsor over \( T' \), then \( \mathcal{E} \times_{T'} T'' \) is naturally a torsor over \( T'' \) called “the pullback of \( \mathcal{E} \)”. Note that every section of a torsor gives a trivialization of this torsor. We say that a \( G \)-torsor over \( T' \) is generically trivial if its restriction to a dense open subscheme of \( T' \) is trivial. If \( T' \) is reduced, this is equivalent to the torsor being trivial over each generic point of \( T' \).

Notations as above, let \( H \) be a subgroup scheme of \( G \) and \( \mathcal{E} \) be a \( G \)-torsor over \( T' \). Consider the sheaf \( \mathcal{E}/H \) on the big fppf site \( \text{Sch}/T' \) given by the sheafification of the presheaf \( T'' \to \mathcal{E}(T'')/H(T'') \). A section \( s : T' \to \mathcal{E}/H \) is called a reduction of \( \mathcal{E} \) to \( H \). If \( s \) is such a reduction, then the \( s \)-pullback \( H \) of the \( H \)-torsor \( \mathcal{E} \to \mathcal{E}/H \) is an \( H \)-torsor over \( T' \). In this case, we have a canonical isomorphism \( H \times^H G = \mathcal{E} \) (indeed, this statement is enough to check fppf locally, so we may assume that \( \mathcal{E} \) is trivial).
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2. Reducing to a unipotent subgroup scheme

The main goal of this section is to prove Proposition 2.2. However, we start with a statement of independent interest. Recall from Section 1.1(i) that a scheme is geometrically factorial if its strictly local rings are factorial.

Proposition 2.1. Let $U$ be a Noetherian geometrically factorial affine semilocal scheme and $T$ be a $U$-torus. Then every generically trivial $T$-torsor over an open subscheme of $U$ is trivial.

Proof. We will derive the statement from [CTS] Thm. 4.1(iii)]. Let $U_0$ be an open subscheme of $U$. Since $U$ is geometrically factorial, it is locally factorial by [EGA4	extsuperscript{1}] Prop. 21.13.12]. Thus it is normal, so its connected components are integral. Hence, we can reduce to the case when $U$ is integral. Since $U$ is normal, by [SGA3] Exp. X, Thm. 5.16 $T$ is isotrivial. Hence, we can find a finite étale Galois cover $U' \to U$ that splits $T$. Let $U''_0 \to U_0$ be any finite étale subcover of $U' \times_U U_0 \to U_0$. Then $U''_0$ extends to a finite étale cover $U'' \to U$. Indeed, the statement is enough to prove after replacing $U$ by an étale cover, so we may assume that $U'$ is a disjoint union of schemes that project isomorphically onto $U$.

Since $U''$ is affine and semilocal, we have $\text{Pic} U'' = 0$ (see [SP] Tag02M9]. Since $U''$ is locally factorial (use again [EGA4	extsuperscript{1}] Prop. 21.13.12]), every line bundle over $U''_0$ extends to $U''$, so $\text{Pic} U''_0 = 0$. Thus, replacing $U'$ with any of its connected components and applying [CTS] Thm. 4.1(iii)], we see that every $T$-torsor over $U_0$ is trivial. □

Proposition 2.2. Let $U$ be an integral Noetherian geometrically factorial affine semilocal scheme and $G$ be a reductive $U$-group scheme with a maximal torus $T$. Let $U \to U$ be a finite étale cover. Let $\tilde{B} \subset G_{\tilde{U}}$ be a Borel subgroup scheme containing $T_{\tilde{U}}$. Let $E$ be a generically trivial $G$-torsor over an open subscheme $U_0 \subset U$. Denote by $\xi$ the generic point of $U$. Then there are

(a) A closed subset $Z$ of codimension at least 2 in $U$.

(b) A generic trivialization $s: \xi \to E$ such that the composition

\begin{equation}
\xi \times_U \tilde{U} \xrightarrow{s \times \text{id}} \mathcal{E}_{U_0} \times_U \tilde{U} \to \mathcal{E}_{U_0} \times_U \tilde{U}/R(u)(\tilde{B})
\end{equation}

extends to $(U_0 - Z) \times_U \tilde{U}$.

Proof. Let us abbreviate $\tilde{G} := G_{\tilde{U}}$, $\tilde{T} := T_{\tilde{U}}$, $\tilde{E} := \mathcal{E}_{U_0} \times_U \tilde{U}$. Let $s: \xi \to E$ be any generic trivialization of $E$. Consider the composition

$s'_B: \xi \times_U \tilde{U} \xrightarrow{s' \times \text{id}} \tilde{E} \to \tilde{E}/\tilde{B}$.

By [SGA3] Exp. XXVI, Cor. 3.6, Lm. 3.20 $\tilde{E}/\tilde{B}$ is projective over $U_0 \times_U \tilde{U}$. Since $U$ is geometrically factorial, $\tilde{U}$ is locally factorial by [EGA4] Prop. 21.13.12] so it is normal. Thus by the valuative criterion of properness we can extend $s'_B$ to a morphism $U'' \to \tilde{E}/\tilde{B}$, where $U''$ is an open subscheme
of $U_0 \times_U \bar{U}$ whose complement has codimension at least 2. Let $Z$ be the image of the closure of $U_0 \times_U \bar{U} - U'$ in $\bar{U}$ under the finite projection $\bar{U} \to U$. Then $Z$ is of codimension at least 2 in $U$ and $s'_B$ extends to $U' := (U_0 - Z) \times_U \bar{U}$. By a slight abuse of notation we denote the extension by the same letter so that now we have $s'_B : U' \to \tilde{E}/B$.

We will show how to modify $s'$ so that the morphism (11) (with $s$ replaced by $s'$) extends to $U'$ as well. Consider the $T$-torsor $\tilde{E}/R_u(B) \to \tilde{E}/B$. Let $\tilde{T} \to U'$ be its pullback via $s'_B$. Note that the composition

$$s'_u : \xi \times_U \bar{U} \xrightarrow{s' \times \text{id}} \tilde{E} \to \tilde{E}/R_u(B)$$

gives a section $\xi \times_U \bar{U} \to \tilde{T}$ because $s'_u$ lifts $s'_B$. This induces an isomorphism of $T$-torsors $\tilde{i} : \tilde{T}_\xi \to \tilde{T}_\xi$, where $\tilde{T}_\xi := T \times_U (\xi \times_U \bar{U})$, $\tilde{T}_\xi := T \times_U \tilde{T}$ $(\xi \times_U \bar{U})$. This isomorphism equips $\tilde{T}_\xi$ with the descent datum for the morphism $\xi \times_U \bar{U} \to \xi$.

**Lemma 2.3.** The above descent datum for the morphism $\xi \times_U \bar{U} \to \xi$ extends to the descent datum for the morphism $U' \to U_0 - Z$. In more detail, let $p_1, p_2 : U' \times_U U' \to U'$ be the two projections and let $p_{i, \xi}$ be their restrictions to $p_i^{-1}(\xi \times_U \bar{U})$, then the descent isomorphism

$$p_{1, \xi}(\tilde{T}_\xi) \to p_{2, \xi}(\tilde{T}_\xi)$$

induced by $\tilde{i}$ extends to a descent isomorphism $p_1^*\tilde{T} \to p_2^*\tilde{T}$. (Note that the cocycle condition extends by continuity.)

**Proof.** The statement is enough to prove after an étale surjective base change $V \to U$ with integral $V$. By [SGA3, Exp. XXIV, Cor. 4.1.6] there is a finite étale cover such $V \to U$ such that $G_V$ is split. We may assume that $V$ is connected, thus, by [SP, Tag0BQL], $V$ is integral.

Thus, we may assume that $G$ is split. We also reduce to the case when $\bar{U}$ is integral. Then every Borel subgroup scheme of $G$ containing $T_{\bar{U}}$ is the pullback of a Borel subgroup scheme $G$ containing $T$ (use [SGA3, Exp. XXII, Cor. 5.5.5(iii)]). Thus, we may assume that $B = B_{\bar{U}}$ for a Borel subgroup scheme $B \subset G$ containing $T$. Then the section $s'_B : U' \to \tilde{E}/B$ descends to a section $U_0 - Z \to \tilde{E}/B$, since it descends generically. Thus, $\tilde{T}$ is the pullback via $U' \to U_0 - Z$ of $T := \tilde{E}/R_u(B) \times_{\tilde{E}/B} (U_0 - Z)$. Now the statement is clear. \qed

We continue with the proof of Proposition 2.2. By the above lemma $\tilde{T}$ descends to a $T$-torsor $T$ over $U_0 - Z$. By construction, $\tilde{i}$ descends to a trivialization $\iota : T_\xi \to \tilde{T}_\xi$. By Proposition 2.1 $T$ is trivial. Let $t : U_0 - Z \to T$ be a trivialization. Under the isomorphism $\iota$, the restriction of $t$ to $\xi$ corresponds to a section $t' : \xi \to T$.

We can view $t'$ as a $\xi$-point of $G$. Let $s := s't' \in \xi(\xi)$ be the composition of $s'$ with the action of $G$ by $t'$. Consider the corresponding section $s_u : \xi \times_U \bar{U} \to \tilde{E}/R_u(B)$. We claim that $s_u$ extends to $U'$. First of all, the composition $\xi \times_U \bar{U} \xrightarrow{s_u} \tilde{E}/R_u(B) \to \tilde{E}/B$ coincides with $s'_B$. Thus, $s_u$ gives a generic section of $\tilde{T} = U' \times_{\tilde{E}/B} \tilde{E}/R_u(B)$. By construction, this section is equal to $t_\xi \times \text{id} : \xi \times_U U' \to T \times_U U' = \tilde{T}$. Thus, this section extends to a section

$$t \times \text{id} : U' \to \tilde{T} = \tilde{E}/R_u(B) \times_{\tilde{E}/B} U'.$$

Composing this section with the projection to $\tilde{E}/R_u(B)$, we get the required extension of $s_u$. \qed
The goal of this section is to prove the following proposition.

**Proposition 3.1.** Let $V$ be a reduced connected affine Noetherian scheme. Let $\Gamma$ be an abstract finite group and let $\tilde{V} \to V$ be a finite étale $\Gamma$-cover with a connected $\tilde{V}$.

Let $G$ be a reductive $V$-group scheme and let $T \subset [G, G]$ be a maximal torus. Assume that the torus $T_{\tilde{V}}$ is split. Assume that $\tilde{B} \subset [G, G]_{\tilde{V}}$ is a Borel subgroup scheme containing $T_{\tilde{V}}$.

Let $E$ be a $G$-torsor over $V$. Let $r: \tilde{V} \to E_{\tilde{V}}/R_u(\tilde{B})$ be a reduction of $E_{\tilde{V}}$ to $R_u(\tilde{B})$. Assume that generically $r$ comes from a generic trivialization of $E_{\tilde{V}}$. Then $r$ comes from a global trivialization of $E$, so, in particular, $E$ is trivial.

**Remark 3.2.** The assumption that $V$ and $\tilde{V}$ are connected is not needed to conclude that $E$ is trivial. Indeed, we immediately reduce to the case when $V$ is connected. Next, $\Gamma$ acts on the connected components of $\tilde{V}$. Fix a connected component $V' \subset \tilde{V}$ and let $\Gamma' \subset \Gamma$ be its isotropy subgroup. Then $V' \to V$ is a finite étale $\Gamma'$-cover, so we may apply the proposition to conclude that $E$ is trivial.

This proposition will be proved in Section 3.2 after some preliminaries. The phrase “generically comes from a trivialization” means that there is a trivialization of $E$ over a dense open subset of $V$ (equivalently, over the generic points of $V$) such that the induced $R_u(\tilde{B})$-reduction of $E_{\tilde{V}}$ coincides with $r$ over this dense open subset.

### 3.1. Some preliminaries about reductive group schemes.

We start with a general statement. Recall that two Borel subgroup schemes of a reductive group scheme are in general position if their intersection is a torus, see [SGA3, Exp. XXII, Def. 5.9.1].

**Lemma 3.3.** Let $H \to S$ be a reductive group scheme. Let $U_{\pm}$ be the unipotent radicals of two Borel subgroup schemes in general position. Then the multiplication morphism $U_- \times U_+ \to H$ is a closed embedding.

**Proof.** Since the statement is étale local over $S$, we may assume that $H$ is split. Now $H$ admits a closed embedding into $GL_{n,S}$ for some $n$ and we may assume that the image of $U_+$ (resp. $U_-$) is contained in the subgroup scheme of upper (resp. lower) triangular matrices. Then it is enough to check the statement when $H = GL_{n,S}$, and $U_{\pm}$ are the subgroup schemes of lower and upper triangular unipotent matrices. This is checked by an explicit calculation. \(\square\)

We fix some notation through the end of Section 3. We claim that there is a root system $\Delta$ of $\tilde{G}$ with respect to $T$ (see [SGA3, Exp. XIX, Def. 3.6]). Indeed, this is true Zariski locally by [SGA3, Exp. XX, Prop. 2.1] (recall that $\tilde{T}$ is a split torus), and the statement follows from uniqueness of the root system and connectedness of $\tilde{V}$.

For $\alpha \in \Delta$ we have a root subgroup scheme $U_\alpha \subset \tilde{G}$ ([SGA3, Exp. XXII, Notations 1.3]). It follows from [SGA3, Exp. XXII, Prop. 5.5.5(iii)] in view of connectedness of $\tilde{V}$ that there is a unique system
of positive roots $\Delta_+ \subset \Delta$ (see [SGA3] Exp. XXI, Sect. 3.2]) such that
\[ \tilde{B} = \tilde{T} \cdot \prod_{\alpha \in \Delta_+} U_\alpha. \]

Then $\Delta = \Delta_- \cup \Delta_+$, where $\Delta_- := -\Delta_+$. Let $\Delta_0 \subset \Delta_+$ be a subset closed under addition (that is, $\gamma_1, \gamma_2 \in \Delta_0, \gamma_1 + \gamma_2 \in \Delta$ imply that $\gamma_1 + \gamma_2 \in \Delta_0$). Then, according to [SGA3] Exp. XXII, Cor. 5.6.5,
\[ U_0 := \prod_{\alpha \in \Delta_0} U_\alpha \]
is a unipotent subgroup scheme of $\tilde{G}$. Moreover, the product does not depend on the ordering of $\Delta_0$.

Such unipotent subgroup schemes of $\tilde{G}$ will be called standard. In particular, $R_u(\tilde{B})$ is standard.

**Remark 3.4.** Let $\Delta_0 \subset \Delta_+$ be a subset closed under addition and let $\alpha_1, \ldots, \alpha_n \in \Delta_0$. If $\alpha_1 + \alpha_2 + \ldots + \alpha_n \in \Delta_+$, then $\alpha_1 + \alpha_2 + \ldots + \alpha_n \in \Delta_0$. Indeed, set $\alpha := \alpha_1 + \alpha_2 + \ldots + \alpha_n$. We induct on $n$, the base case $n = 1$ being obvious. If $n > 1$, then by [SGA3] Exp. XXI, Lm. 3.1.1] there is $i$ such that $\alpha - \alpha_i \in \Delta_+$. By induction hypothesis, $\alpha - \alpha_i \in \Delta_0$. Thus, $\alpha = \alpha_i + (\alpha - \alpha_i) \in \Delta_0$ by definition of root subsets closed under addition.

Note that $\Gamma$ acts on $\Delta$ for it acts on $\tilde{G}$ preserving $\tilde{T}$.

**Lemma 3.5.** Let $U_0 = \prod_{\alpha \in \Delta_0} U_\alpha$ be a standard unipotent subgroup scheme of $\tilde{G}$. Then either $\Delta_0$ is $\Gamma$-invariant, or there are $\gamma_0 \in \Gamma$ and $\alpha \in \Delta_0 - (\Delta_0 + \Delta_0)$ such that if we set
\[ U_1 := \prod_{\beta \in \Delta_0 \setminus \{\alpha\}} U_\beta, \]
then
(a) $U_1$ is a standard subgroup scheme of $\tilde{G}$.
(b) The image of the multiplication morphism $\gamma_0(U_0) \times U_1 \to \tilde{G}$ is a reduced closed subscheme $Z \subset \tilde{G}$.
(c) The multiplication morphism $\iota: Z \times U_\alpha \to \tilde{G}$ is a closed embedding.

**Proof.** Note that
\[ \tilde{B}_- := \tilde{T} \cdot \prod_{\alpha \in \Delta_-} U_\alpha \]
is a Borel subgroup scheme in general position with $B$. We have
\[ U_+ := R_u(\tilde{B}) = \prod_{\alpha \in \Delta_+} U_\alpha \text{ and } U_- := R_u(\tilde{B}_-) = \prod_{\alpha \in \Delta_-} U_\alpha. \]

Set $\Delta'_0 := \Delta_+ \cap (\Delta_0 + \Delta_0)$. It follows by induction on the root length that $\Delta_0 - \Delta'_0$ generates $\Delta_0$. Note that for any $\alpha \in \Delta_0 - \Delta'_0$ the set $\Delta_0 - \{\alpha\}$ is closed under addition so part (a) is automatic.

Assume that $\Delta_0$ is not $\Gamma$-invariant. The there is $\gamma_0 \in \Gamma$ such that $\gamma_0(\Delta_0) \not\supset \Delta_0$. Then, since $\Delta_0 - \Delta'_0$ generates $\Delta_0$, there is $\alpha \in \Delta_0 - \Delta'_0$ such that $\alpha \not\in \gamma_0(\Delta_0)$.

Consider the composition of multiplication morphisms
\[ \prod_{\beta \in \gamma_0(\Delta_0) \cap \Delta_-} U_\beta \times \prod_{\beta \in (\gamma_0(\Delta_0) \cap \Delta_+) \cup (\Delta_0 - \{\alpha\})} U_\beta \to U_- \times U_+ \to \tilde{G}. \]
It follows from Lemma 3.3 that this morphism is a closed embedding. It is clear that its image $Z$ coincides with the image of $\gamma_0(U_0) \times U_1$. This proves part (ii).

To prove part (iii), note that we have a closed embedding defined as the composition
\[
\prod_{\beta \in \gamma_0(\Delta_0) \cap \Delta} U_{\beta} \times \prod_{\beta \in (\gamma_0(\Delta_0) \cap \Delta_+ \cup (\Delta_0 - \{\alpha\}))} U_{\beta} \times U_{\alpha} \rightarrow U_{-} \times U_{+} \rightarrow \tilde{G}.
\]

Part (iii) follows from that. □

Let $\Delta_0 \subset \Delta_+$ be a $\Gamma$-invariant subset closed under addition and let $U_0 := \prod_{\alpha \in \Delta_0} U_{\alpha}$ be the corresponding standard unipotent subgroup scheme of $\tilde{G}$. Then $U_0$ is $\Gamma$-invariant so there is a unique subgroup scheme $U \subset [G, G]$ such that $U_{\tilde{V}} = U_0$. The following lemma is similar to [SGA3, Exp. XXVI, Cor. 2.2].

**Lemma 3.6.** Every $U$-torsor over $V$ is trivial.

**Proof.** Recall that for a locally free sheaf $L$ on $V$ of finite rank, we have an abelian $V$-group scheme $\mathcal{W}(L)$ whose underlying space is the total space of $L$. We claim that there is a filtration $U \supset U_1 \supset \ldots \supset U_n = 0$ such that all quotients $U_i/U_{i+1}$ are of the form $\mathcal{W}(L_i)$. We induct on $|\Delta_0|$, the base case being trivial. Set $\Delta'_0 := (\Delta_0 + \Delta_0) \cap \Delta$, then $\Delta'_0$ is a closed subset proper subset of $\Delta_0$. Let $U' := \prod_{\alpha \in \Delta'_0} U_{\alpha}$. Note that $\Delta'_0$ and thus $\Delta_0 - \Delta'_0$ are $\Gamma$-invariant. Thus there is a unique locally free sheaf $L$ on $V$ such that $L_{\tilde{V}} = \text{Lie}(U_0/U')$. On the other hand, there is a unique subgroup scheme $U_1 \subset G$ such that $(U_1)_{\tilde{V}} = U'$ and $U/U_1 = \mathcal{W}(L)$. The claim follows by induction.

Now the statement follows from the long exact sequence of non-abelian cohomology and affineness of $V$; see the proof of [SGA3, Exp. XXVI, Cor. 2.2] for details. □

3.2. **Proof of Proposition 3.1** We need another simple lemma.

**Lemma 3.7.** Let $T$ be a scheme, $H$ be a flat and finitely presented $T$-group scheme, and $\mathcal{H}$ be an $H$-torsor over a scheme $T$. Let $\Gamma$ be an abstract finite group and let $\tilde{T} \rightarrow T$ be a finite étale $\Gamma$-cover. Let $U \subset H_{\tilde{T}}$ be a $\tilde{T}$-subgroup scheme. Let $r : \tilde{T} \rightarrow \mathcal{H}_{\tilde{T}}/U$ be a reduction of $\mathcal{H}_{\tilde{T}}$ to $U$. Let $s$ be a trivialization of $\mathcal{H}_{\tilde{T}}$ lifting $r$. Then $s$ comes from a trivialization of $\mathcal{H}$ if and only if there is $u \in U(\tilde{T})$ such that $su^{-1}$ is $\Gamma$-invariant, that is, for all $\gamma \in \Gamma$ we have $\gamma(s) = su^{-1}\gamma(u)$.

Note that $U(\tilde{T}) \subset H(\tilde{T})$ and $\Gamma$ acts on $H(\tilde{T})$ through its action on $\tilde{T}$. Also, by $\mathcal{H}_{\tilde{T}}/U$ we mean the sheafification of the presheaf $T' \mapsto \mathcal{H}(T')/U(T')$ on the big fppf site of $\tilde{T}$-schemes.

**Proof.** Assume that $r$ comes from a trivialization $s_1$ of $\mathcal{H}$. Then $s = (s_1|_{\tilde{T}})u$ for some $u \in U(\tilde{T})$. Since $s_1|_{\tilde{T}}$ is $\Gamma$-invariant, we see that for all $\gamma \in \Gamma$ we have $\gamma(s) = su^{-1}\gamma(u)$. The converse statement is obtained by reversing the argument. □

We return to the proof of Proposition 3.1 Set $\tilde{E} := \mathcal{E}_{\tilde{V}}$. We construct inductively a sequence of standard unipotent subgroup schemes $R_{\alpha}(B) = U_0 \supset U_1 \supset \ldots \supset U_n$ such that $U_n = \prod_{\alpha \in \Delta_n} U_{\alpha}$ for a $\Gamma$-invariant closed under addition subset $\Delta_n \subset \Delta$, and reductions
\[
r_i : \tilde{V} \rightarrow \tilde{E}/U_i
\]
such that $r_i$ generically comes from a trivialization of $\mathcal{E}$ and the induced reduction to $U_0$ (that is, the composition $\tilde{V} \rightarrow \tilde{E}/U_i \rightarrow \tilde{E}/U_0$) coincides with $r$. We can take $r_0 = r$. Assume that $r_i$ has already been constructed. We may assume that $U_i$ is not $\Gamma$-invariant, else we take $U_n = U_i$.

Applying Lemma 3.5 to $U_i$, we get $\gamma_0 \in \Gamma$, $\alpha \in \Delta_+$, and a standard subgroup scheme $U_{i+1} \subset U_i$ such that $U_i = U_\alpha U_{i+1}$ (use again [SGA3], Exp. XXII, Cor. 5.6.5]). Since $U_i$ is standard and $\tilde{V}$ is affine, the $U_i$-torsor over $\tilde{V}$ corresponding to $r_i$ is trivial (argue as in the proof of [SGA3], Exp. XXVI, Cor. 2.2]). Thus, we can lift $r_i$ to $s \in \tilde{E}(\tilde{V})$. By definition of right torsors for any $\gamma \in \Gamma$ there is a unique $g_\gamma \in G(\tilde{V})$ such that $s = \gamma(s)g_\gamma$.

By assumption, $r_i$ comes from a trivialization of $\mathcal{E}$ on a dense open subset of $V$. Thus, according to Lemma 3.7 there is a dense open $\Gamma$-invariant subset $V' \subset \tilde{V}$ and $u \in U_i(V')$ such that for all $\gamma \in \Gamma$ we have $g_\gamma|_{V'} = \gamma(u)^{-1}u$. Let us write $u = u_1 u_\alpha$, where $u_1 \in U_{i+1}(V')$, $u_\alpha \in U_\alpha(V')$. We have for all $\gamma \in \Gamma$

$$g_\gamma|_{V'} = (\gamma(u^{-1})u_1) u_\alpha.$$  

We see that $g_{\gamma_0}|_{V'}$ factors uniquely through the closed embedding $\iota$, where $\iota$ is from Lemma 3.5[\(\alpha\)]. Since $\tilde{V}$ is reduced, $g_{\gamma_0}$ also factors uniquely through $\iota$:

$$g_{\gamma_0} = \iota \circ (\bullet, \tilde{u}_\alpha),$$

where $\tilde{u}_\alpha \in U_\alpha(\tilde{V})$. We see that $\tilde{u}_\alpha|_{V'} = u_\alpha$. Consider $s' := s\tilde{u}_\alpha^{-1} \in \tilde{E}(\tilde{V})$. We see from (2) that for all $\gamma \in \Gamma$

$$\gamma(s')|_{V'} = (s'|_{V'})u_1^{-1}\gamma(u_1).$$

Applying Lemma 3.7 again, we see that the composition

$$r_{i+1} : \tilde{V} \to \tilde{E} \to \tilde{E}/U_{i+1}$$

comes generically from a trivialization of $\mathcal{E}$. Also, since $\tilde{u}_\alpha \in U_0(\tilde{V})$, the $U_0$-reduction induced by $r_{i+1}$ coincides with $r$.

We have proved by induction that there is a reduction of $\mathcal{E}_{\tilde{V}}$ to a standard $\Gamma$-invariant subgroup scheme $U_n$, inducing $r$. Moreover, this reduction comes from a generic trivialization of $\mathcal{E}$. Since $U_n$ is $\Gamma$-invariant, there is a unique unipotent subgroup scheme $U \subset G$ such that $U_n = U_{\tilde{V}}$. Then $\Gamma$ acts on reductions $\tilde{V} \to \mathcal{E}_{\tilde{V}}/U_n$. A reduction coming generically from a trivialization of $\mathcal{E}$ must be $\Gamma$-invariant, so it comes from a reduction of $\mathcal{E}$ to $U$; denote the corresponding $U$-torsor by $\mathcal{U}$.

By Lemma 3.6 $\mathcal{U}$ is trivial. A trivialization of $\mathcal{U}$ gives a trivialization of $\mathcal{E}$. By construction, this trivialization induces $r$. \hfill \Box

4. Lifting to a relative curve and preparation for descent

4.1. Fibering into curves. We fix some notations through the end of Section 5. Let $G$ be a reductive group scheme over a scheme $S$ and let $T \subset G$ be a maximal torus. Let $I(G, T)$ be the scheme of Borel subgroup schemes of $G$ containing $T$. Then $I(G, T) \to S$ is a $W$-torsor (see [SGA3], Exp. XXII, Cor. 5.5.5(ii)]), where $W := N(T)/T$ is the Weyl group scheme of $G$ with respect to $T$. Thus, $I(G, T)$ is finite and étale over $C$ (see [SGA3], Exp. XXII, Sect. 3.1]). Let $B(G, T) \subset G_{I(G, T)}$ be the tautological Borel subgroup scheme. Let $U(G, T) := R_u(B(G, T))$ be the unipotent radical.
The following proposition is an analogue of [ˇČes, Prop. 4.2] (cf. also [Fed2, Prop. 4.4]). The main difference is that the torsor $\mathcal{E}'$ is not reduced to a unipotent subgroup scheme on a subscheme whose complement is finite over the base but only on a canonical finite cover of such a subscheme.

**Proposition 4.1.** Let $X$ be an integral affine scheme smooth, of finite type, and of positive dimension over a semilocal Dedekind domain. Let $x \subset X$ be a finite subset of closed points, let $G$ be a reductive $R$-group scheme, where $R := \mathcal{O}_{X,x}$, and let $T \subset G$ be a maximal torus. Let $\mathcal{E}$ be a generically trivial $G$-torsor, then there are

(a) a smooth integral affine $R$-scheme $C$ of pure relative dimension 1 with the generic point $\xi$;

(b) a section $\Delta \in C(R)$;

(c) an $R$-finite closed subscheme $Z \subset C$ of codimension at least two in $C$;

(d) a reductive $C$-group scheme $G'$ with a maximal torus $T'$ such that the $\Delta$-pullback of $T' \subset G'$ is $T \subset G$;

(e) a $G'$-torsor $\mathcal{E}'$ with a generic trivialization $s' : \xi \to \mathcal{E}'$ such that $\Delta^* \mathcal{E}' \approx \mathcal{E}$ and the composition

$$\xi \times_C I(G', T') \xrightarrow{s' \times \text{id}} \mathcal{E}'_{I(G', T')/U(G', T')}$$

extends to $(C - Z) \times_C I(G', T')$.

We recall that $I(G', T')$ is a finite étale cover of $C$ and $U(G', T')$ is a unipotent subgroup scheme of $G'_{I(G', T')}$.

**Proof.** Similar to [ˇČes, Prop. 4.2] in view of our Proposition 2.2. In more detail, denote the semilocal Dedekind domain by $\mathcal{O}$. Set $U := \text{Spec} R$. Since $U$ is regular, it is geometrically factorial (combine [EGA4i, Cor. 18.8.13] and [EGA4i, Thm. 21.11.1]). Let $Z' \subset U$ and $s$ be the data provided by Proposition 2.2 applied to $U_0 = U$, $I(G, T) \to U$, $B(G, T)$, and $\mathcal{E}$. By spreading out and replacing $X$ by an open neighborhood of $x$, we may assume that $G$ and $T$ are defined over $X$ (in which case $I(G, T)$ becomes a finite étale cover of $X$). Further, we may assume that $Z' = Z'' \times_X U$, where $Z''$ is a closed subscheme of $X$ of codimension at least two and that the composition

$$\eta \times_X I(G, T) \xrightarrow{s \times \text{id}} \mathcal{E}_{I(G, T)} \to \mathcal{E}_{I(G, T)/U(G, T)},$$

where $\eta$ is the generic point of $X$, extends to $(X - Z'') \times_X I(G, T)$.

Applying Proposition 2.2, we obtain an affine open subscheme $X' \subset X$ containing $x$, an affine open subscheme $S \subset \mathcal{A}_\mathcal{O}^{\dim X - 1}$, and a smooth morphism $X' \to S$ of pure relative dimension one such that $Z'' \cap X'$ is finite over $S$.

We now let $C := X' \times_S U$ and $Z := (Z'' \cap X') \times_S U$. We let $\Delta : U \to C$ be the diagonal embedding. Now we replace $C$ with its connected component containing $\Delta(U)$ and restrict $Z$ to this component. Let $G'$, $T'$, and $\mathcal{E}'$ be the pullbacks of $G$, $T$, and $\mathcal{E}$ under the projection $C \to X'$.

The generic trivialization $s$ of $\mathcal{E}$ gives rise to a generic trivialization $s'$ of $\mathcal{E}'$. Since $I(G', T') = I(G, T) \times_S U$, the extension property for $\mathcal{E}$ implies the extension property in $\mathcal{E}'$. All the conditions of the proposition are now satisfied by construction. \qed
4.2. Equating the group schemes. We will later show that in Proposition 4.1 we may assume that the data $G' \supset T'$ is the pullback of $G \supset T$ under the projection $C \to \text{Spec} R$. To that end, we will use Proposition 4.2 below. We start with some preliminaries.

**Lemma 4.2.** Let $R$ be an integral domain and let $T \subset G^n_{m,R}$ be a subtorus. Let $P^n_R$ be the standard compactification of $G^n_{m,R} \subset \mathbb{A}^n_R$. Let $\tilde{T}$ be the closure of $T$ in $P^n_R$. Then $\tilde{T}$ is fiberwise dense in $T$.

**Proof.** Consider the standard affine cover $P^n_R = \bigcup_{i=0}^n U_i$, where $U_i \approx \mathbb{A}^n_R$. It is enough to show that for all $i$ the intersection $T \cup U_i$ is fiberwise dense in its closure in $U_i$. Denote the closure of $T$ in $U_i$ by $\mathcal{T}_i$.

We identify the character lattice of $G^n_{m,R}$ with $\mathbb{Z}^n$ in such a way that $U_i = \text{Spec} R[\mathbb{Z}_{\geq 0}^n]$. Let $\Lambda := \text{Hom}_R(T, G_{m,R})$ be the character lattice of $T$ (this is a free abelian group). The embedding $T \hookrightarrow G^n_{m,R}$ corresponds to a surjective homomorphism $\pi: \mathbb{Z}^n \to \Lambda$. Let $X = \text{Spec} R[\pi(\mathbb{Z}_{\geq 0}^n)]$. Then $X$ is a closed integral subscheme of $U_i$ containing $T$. Since $\pi(\mathbb{Z}_{\geq 0}^n)$ generates $\Lambda$, we see that $T$ is open in $X$, so that $X = \mathcal{T}_i$.

Similarly, for any point $u \in \text{Spec} A$, we see that $X_u = (\mathcal{T}_i)_u$ is the closure of $T_u$ in $\mathbb{A}^n_u$ so that $T_u$ is dense in $(\mathcal{T}_i)_u$. □

The following proposition about tori over normal semilocal schemes may be of independent interest.

**Proposition 4.3.** Let $W$ be a semilocal normal Noetherian affine scheme and let $T$ be a $W$-torus. Let $\mathcal{T}$ be $T$-torsor. Then

(a) $T$ has a fiberwise compactification. That is, there is projective $W$-scheme $\mathcal{T}$ such that $T$ is a fiberwise dense open subscheme of $\mathcal{T}$;

(b) similarly, $\mathcal{T}$ has a fiberwise compactification. That is, there is projective $W$-scheme $\mathcal{T}$ such that $\mathcal{T}$ is a dense open subscheme of $\mathcal{T}$.

(c) If $U$ is a closed subscheme of $W$ and $\delta: U \to \mathcal{T}$ is a section of $\mathcal{T} \to W$, then there is a closed subscheme $\tilde{W} \subset \mathcal{T}$ finite and étale over $W$ and containing $\delta(U)$.

**Proof.** (3) Since $W$ is Noetherian and normal, by [Tho Cor. 3.2(3)] we have an embedding $T \to GL_{n,W}$ for some $n > 0$. Put $N := n^2$, and let $P^N_W = \mathbb{P}(\mathfrak{gl}_{n,W} \oplus \mathcal{O}_W)$ be the standard compactification of $GL_{n,W}$. Let $\mathcal{T}$ be the closure of $T$ in $P^N_W$. We need to check that $T$ is fiberwise dense in $\mathcal{T}$. This is enough to check on an étale cover, so we may assume that $T$ is a diagonalizable (=split) torus. Thus we may assume that $T$ is contained in the torus $G^n_{m,W}$ of diagonal matrices and the statement follows from Lemma 4.2 (we note that the connected components of the étale cover are integral by [SP Tag0BQL]).

(0) We take $\mathcal{T} := T \times^T \mathcal{T}$.

(2) Let $w \in W$ be a closed point. Using Bertini’s Theorem (see [Poo and SGA43, Exp. XI, Thm. 2.1(ii)]), we see that for large $d$ there is a degree $d$ hypersurface $H_{1,w}$ in $P^N_W$ intersecting $\mathcal{T}_w$ transversally and such that $\dim(H_{1,w} \cap (\mathcal{T}_w - \mathcal{T}_w)) < \dim(H_{1,w} \cap \mathcal{T}_w)$. If $w \in U$, we may arrange it so that $\delta(w) \in H_{1,w}$. Since $W$ is semilocal, we can lift the hypersurfaces $H_{1,w}$, where $w$ ranges over the closed points of $W$, to a hypersurface $H_1 \subset P^N_W$ containing $\delta(U)$. Thus, $H_1 \cap \mathcal{T}$ is smooth over $W$. Next, we find a hypersurface $H_2 \subset P^N_W$ intersecting $H_1 \cap \mathcal{T}$ transversally, containing $\delta(U)$, and such that for all closed points $w$ in $W$ the dimension of $H_{2,w} \cap H_{1,w} \cap (\mathcal{T}_w - \mathcal{T}_w)$ is smaller than dimension
of $H_{2,W} \cap H_{1,W} \cap T_w$. Repeating this procedure, we find a closed subscheme $\widetilde{W} \subset \mathcal{T}$ finite over $W$, such that $\widetilde{W} \cap \mathcal{T}$ is étale over $W$, and $\widetilde{W} \supset \delta(U)$. Moreover, the dimensional inequalities show that $\widetilde{W}$ does not intersect the infinity divisor $\mathcal{T} - \mathcal{T}$, so $\widetilde{W} \subset \mathcal{T}$. (Cf. [FP] Prop. 4.1] and [Pan2 Lm. 4.3].) 

The following lemma is analogous to [PSV] Prop. 5.1], [Pan2 Thm. 4.1], and [Ces Lm. 5.1].

**Proposition 4.4.** Let $W$ be a semilocal normal Noetherian affine scheme. Assume that $G_1$ and $G_2$ are reductive $W$-group schemes of the same type. Assume that $T_i$ is a maximal torus of $G_i$ ($i = 1, 2$). Assume that $U \subset W$ is a closed subscheme and $\iota: (G_1)_U \to (G_2)_U$ is an isomorphism sending $(T_1)_U$ isomorphically onto $(T_2)_U$.

Then there is a connected finite étale cover $\pi: \widetilde{W} \to W$ with a section $\delta: U \to \widetilde{W}$ and an isomorphism $\iota: (G_1)_\widetilde{W} \to (G_2)_\widetilde{W}$ sending $(T_1)_\widetilde{W}$ isomorphically onto $(T_2)_\widetilde{W}$ and such that $\delta^*\iota = \iota$.

**Proof.** Let $G_1^{ad}$ be the adjoint group scheme of $G_1$ and let $T_1^{ad}$ be the image of $T_1$ in $G_1^{ad}$. Let $A$ be the group scheme of automorphisms of $G_1$ preserving $T_1$. Then by [SGA3 Exp. XXIV, Prop. 2.1] and [SGA3 Exp. XXIV, Thm. 1.3(ii)] $A$ is an extension of the étale locally constant group scheme of outer automorphisms of $G_1$ by the normalizer $N(T_1^{ad})$ of $T_1^{ad}$ in $G_1^{ad}$.

Let $\tilde{I}$ be the scheme of isomorphisms $G_1 \to G_2$ taking $T_1$ isomorphically onto $T_2$. Then $\tilde{I}$ is an $A$-torsor over $W$ ([SGA3 Exp. XXIV, Cor. 2.2(i)])]. Thus $I := \tilde{I}/T_1^{ad}$ is étale locally constant over $W$. Note that $\iota$ gives a section $\delta: U \to \tilde{I}$. Let $\delta': U \to I$ be the composition of $\delta$ with the projection to $I$.

By [SGA3 Exp. X, Cor. 5.14] the connected components of $I$ are finite over $W$. They are also étale over $W$. Note that $U' := \delta'(U)$ intersects only finitely many components, denote their union by $I'$, so that $I'$ is finite and étale over $U$ and $U' \subset I'$. Then $\delta$ decomposes as $\delta'' \circ \delta'$, where $\delta'': U' \to \tilde{I}$. Applying Proposition 4.3(c) to the $T_1^{ad}$-torsor $I' \times_\tilde{I} \tilde{I} \to I'$ and its section $\delta''$, we get a subscheme $\tilde{W} \subset I' \times_\tilde{I} \tilde{I}$ finite and étale over $W$ and containing $\delta''(U') = \delta(U)$. Since $\tilde{W}$ is a subscheme of $\tilde{I}$, we get an isomorphism $\iota: (G_1)_\tilde{W} \to (G_2)_\tilde{W}$ sending $(T_1)_\tilde{W}$ isomorphically onto $(T_2)_\tilde{W}$. By construction, $\delta^*\iota = \iota$. 

4.3. **Preparation to the descent.** We now show that in Proposition 4.4 we may assume that the data $G' \supset T'$ is the pullback of $G \supset T$ under the projection $C \to \text{Spec} R$. We also change our relative curve $C$ in such a way that the torsor can be descended to $\mathbb{A}_R^1$.

The following is an analogue of [Ces Prop. 6.5]. One difference is that we have an additional closed subscheme $Y \subset C$ with affine complement but we do not require $Y$ to be finite over $\text{Spec} R$.

**Proposition 4.5.** Let $X$ be an integral affine scheme smooth, of finite type, and of positive dimension over a semilocal Dedekind domain. Let $x \subset X$ be a finite subset of closed points, let $G$ be a reductive
Consider the data $T$ and $Z$. Applying Proposition 4.4 to the semilocalization of $I$ in item (d), note that we have an isomorphism $E \cong \text{onto a closed subscheme} (\text{open subscheme} C)$.

Let $\xi \times_R I(G, T) \frac{s' \times \text{id}}{\underline{\mathcal{E}}} \times_R I(G, T) \rightarrow (\mathcal{E} \times_R I(G, T))/U(G, T)$ extends to $(C - Z) \times_R I(G, T)$.

(e) a closed subscheme $Y \subset C$ such that $Z \cup \Delta(Spec R) \subset Y$ and $C - Y$ is affine;

(f) an étale $R$-morphism $C \rightarrow W$, where $W$ is open in $\mathbb{A}^1_R$ and affine, that maps $Y$ isomorphically onto a closed subscheme $Y' \subset W$ with $Y \cong Y' \times_{\mathbb{A}^1_R} C$.

Proof. Consider the data $Z \subset C \rightarrow \text{Spec } R$, $\Delta$, $T' \subset G'$, $\mathcal{E}'$, and $s'$, provided by Proposition 4.1. Applying Proposition 4.4 to the semilocalization of $Z \cup \Delta(Spec R)$ in $C$ and spreading out, we find an open subscheme $C' \subset C$ containing $Z \cup \Delta(Spec R)$, a finite étale morphism $C'' \rightarrow C'$ with integral $C''$, a section $\Delta': \text{Spec } R \rightarrow C''$ lifting $\Delta$, and an isomorphism $i: G_{C''} \rightarrow G_{C''}$ taking $T_{C''}$ isomorphically onto $T'_{C''}$ and such that $(\Delta'; i)$ is the isomorphism of item (d) of Proposition 4.1.

Put $\mathcal{E}'' := \mathcal{E}''$, we view $\mathcal{E}$ as a $G$-torsor using the isomorphism $i$. The generic trivialization $s'$ of $\mathcal{E}'$ induces a generic trivialization $s''$ of $\mathcal{E}''$. Now renaming $C'' \rightarrow C$, $\Delta' \rightarrow \Delta$, $Z \times_C C'' \rightarrow Z$, $\mathcal{E}'' \rightarrow \mathcal{E}'$, and $s'' \rightarrow s'$, we obtain items (ii)–(iv) of the proposition (to see that we have the extension property in item (ii), note that we have an isomorphism $I(G', T') = I(G, T) \times_R C$).

Set $T := Z \cup \Delta(Spec R)$. Arguing exactly as in the proof of [Ces, Prop. 6.5], we see that we may change the data so that there is an étale morphism $C \rightarrow \mathbb{A}^1_R$ mapping $T$ isomorphically onto a closed subscheme $T' \subset \mathbb{A}^1_R$ such that $T = T' \times \mathbb{A}^1_R C$. Let us give more details. Applying [Ces, Lm. 6.1] to the closed subscheme $T \subset C$, we see that we can change the data so that for all $x \in x$ we have

$$\# \{z \in T_x : [k(z) : k(x)] = d\} < \# \{z \in \mathbb{A}^1_R : [k(z) : k(x)] = d\} \text{ for every } d \geq 1.$$

Now, applying [Ces, Lm. 6.3] with $Y = \emptyset$ and replacing $C$ with an appropriate affine Zariski neighborhood of $T$, we get a flat morphism $C \rightarrow \mathbb{A}^1_R$ such that $T$ maps isomorphically onto a closed subscheme $T' \subset \mathbb{A}^1_R$ and such that $T = T' \times_{\mathbb{A}^1_R} C$. This morphism is étale in a neighborhood of $T$, so, shrinking $C$, we obtain the desired morphism (see [Fed1, Lm. 2.11]).

It remains to construct $Y$ and $W$. We need a general lemma.

**Lemma 4.6.** Let $R$ be a semilocal integral Nagata ring, let $\varphi: C \rightarrow \mathbb{A}^1_R$ be an étale morphism of affine schemes. Let $T \subset C$ be a closed subscheme finite over $R$. Assume that $\varphi$ maps $T$ isomorphically onto a closed subscheme $T' \subset \mathbb{A}^1_R$ and that we have scheme theoretically $\varphi^{-1}(T') = T$.

Then there is an open affine subscheme $C' \subset C$ and a closed subscheme $Y \subset C'$ containing $T$ such that $C' - Y$ is affine, $\varphi$ maps $Y$ isomorphically onto a locally closed subscheme $Y' \subset \mathbb{A}^1_R$, and we have scheme theoretically $\varphi^{-1}(Y') \cap C' = Y$. 
Proof. We let $\overline{C}$ be the normalization of $\mathbb{P}^1_R$ in the function field of $C$. Since $R$ is a Nagata ring, normalization gives a finite morphism $\overline{\varphi}: \overline{C} \to \mathbb{P}^1_R$. Since $C$ is normal, it can be identified with an open subscheme of $\overline{C}$, and we have $\overline{\varphi}|_C = \varphi$.

Since $\varphi$ is finite, $C$ is $R$-projective. Let $u \in \text{Spec } R$ be a point. Since $\varphi$ is finite, the fiber $\overline{C}_u$ is 1-dimensional. Next, $T = \varphi^{-1}(T') \cap C$ is open and closed (being $R$-finite) in $\varphi^{-1}(T')$, so we can write

$$\varphi^{-1}(T') = T \cup T_2,$$

where $T_2$ is $R$-finite. We claim that there is a closed subscheme $\overline{Y} \subset \overline{C}$ such that $\overline{Y}$ is $R$-finite, $\overline{Y} \supset T$, $\overline{Y} \cap T_2 = \emptyset$, and $\overline{C} - \overline{Y}$ is affine. Indeed, if $R$ is local this follows from [Fed2] Lm. 3.5(ii) applied to $T_1 = T$ and $T_2$; the generalization to the semilocal case is immediate.

Let $i$ be the restriction of $\overline{\varphi}$ to $\overline{Y}$; this is a finite morphism. Note that $i^{-1}(T') = T$. Thus the induced morphism of coherent sheaves $\mathcal{O}_{\mathbb{P}^1_R} \to i_* \mathcal{O}_{\overline{Y}}$ is an isomorphism on $T'$, so there is an open neighborhood $W'$ of $T'$ in $A^1_R$ such that this morphism is surjective over $W'$. Then the restriction of $\overline{\varphi}$ to $\overline{\varphi}^{-1}(W') \cap \overline{Y}$ is a closed embedding. Hence, the restriction of $\varphi$ to $Y := C \cap \overline{\varphi}^{-1}(W') \cap \overline{Y}$ is an immersion. Thus there is an open subscheme $W'' \subset W'$ such that the above restriction factors through a closed embedding $Y \to W''$; denote its image by $Y'$.

Note that $Y$ contains $T$ and $Y$ is a closed subscheme of $C \cap \overline{\varphi}^{-1}(W''')$. Moreover, the étale morphism $\varphi|_{C \cap \overline{\varphi}^{-1}(W''')}$ has a section over $Y'$ whose image is $Y$. Thus (see [Fed1] Lm. 2.11), there is an affine open neighborhood $C'$ of $Y$ in $C \cap \overline{\varphi}^{-1}(W''')$ such that $\varphi^{-1}(Y') \cap C' = Y$. \hfill $\square$

Now we finish the proof of Proposition 4.5. Note that $R$ is excellent and therefore Nagata ring. Let $Y' \subset C' \subset C$ and $Y'' \subset A^1_R$ be obtained by applying the above lemma. Let $W'$ be an open subscheme of $A^1_R$ such that $Y'$ is closed in $W'$. Let $W$ be an affine neighborhood of $T'$ in $W'$. Replace $Y'$ with $Y' \cap W$, $C'$ with $C' \times_{A^1_R} W$, and $Y$ with $Y \cap C'$. Note that $C'$ and $C' - Y$ remain affine as intersections of two open affine subschemes. It remains to rename $C' \mapsto C$. \hfill $\square$

5. Descending to $A^1$

The goal of this section is to descend the data of Proposition 4.5 to $A^1_R$; see Proposition 5.2 below. We will also prove Theorem 2. We start with a general statement. We use the notations from Section 4.

Lemma 5.1. Let $U$ be an affine connected semilocal scheme and let $G$ be a reductive $U$-group scheme with a maximal torus $T$. Then there are

(a) a finite étale $\Gamma$-cover $\widetilde{U} \to U$ with connected $\widetilde{U}$, where $\Gamma$ is an abstract group such that the torus $(T \cap [G, G])_{\widetilde{U}}$ is split;

(b) a $U$-morphism $\pi: \widetilde{U} \to I(G, T)$.

(c) A Borel subgroup scheme $B \subset [G, G]_{\widetilde{U}}$ such that $(T \cap [G, G])_{\widetilde{U}}$ is its maximal torus and $R_u(B) = \pi^* U(G, T)$.

Proof. Let $G_{\text{spl}}$ be a split reductive group scheme of the same type as $[G, G]$. Let $T_{\text{spl}} \subset G_{\text{spl}}$ be a split maximal torus. Let $A$ be the group scheme of automorphisms of $G_{\text{spl}}$ preserving $T_{\text{spl}}$. Then by [SGA3_3] Exp. XXIV, Prop. 2.1 $A$ is an extension of the finite constant group scheme of outer
automorphisms of $G_{\text{spl}}$ by the normalizer $N(T_{\text{spl}})$ of $T_{\text{spl}}$ in $G_{\text{spl}}$ (in fact, this extension is non-

The quotient $A/T_{\text{spl}}$ is the finite constant group scheme; denote the corresponding 

abstract finite group by $\Gamma$, so that $\Gamma$ is isomorphic to the product of the Weyl group of $G_{\text{spl}}$ with the 
group of its outer automorphisms.

Let $\hat{U}$ be the scheme of isomorphisms $G_{\text{spl}} \to [G, G]$ sending $T_{\text{spl}}$ isomorphically onto $T \cap [G, G]$. 

Then $\hat{U}$ is an $A$-torsor over $U$. Indeed, the statement is enough to check étale locally over $U$, in which 
case we may assume that $(T \subset [G, G]) = (T_{\text{spl}} \subset G_{\text{spl}})$. Set $(\hat{U} := \hat{U}/T_{\text{spl}})$, then $\hat{U}$ is a finite étale 

$\Gamma$-cover of $U$.

We claim that there is an isomorphism $(G_{\text{spl}}|_{\hat{U}}) \to [G, G]|_{\hat{U}}$ sending $(T_{\text{spl}}|_{\hat{U}})$ isomorphically onto 

$(T \cap [G, G])|_{\hat{U}}$. Indeed, such isomorphisms $\rho$ are in bijections with sections of the $T_{\text{spl}}$-torsor $\hat{U} \to \hat{U}$.

By \cite{SGA3} Exp. IX, Prop. 2.11(i) $T_{\text{spl}}$ is split, so we need only to check that $\text{Pic}(\hat{U}) = 0$, which 
follows because $U$ is affine and semilocal (see \cite{SP} Tag02M9). By construction, $(T \cap [G, G])|_{\hat{U}}$ is split.

Let $B_{\text{spl}}$ be a Borel subgroup scheme of $G_{\text{spl}}$ containing $T_{\text{spl}}$. Let $\hat{B} \subset [G, G]|_{\hat{U}}$ be the Borel 

subgroup scheme corresponding to $B_{\text{spl}}$ under $\rho$. By \cite{SGA3} Exp. XXII, Prop. 6.2.8(ii) there is a unique 

Borel subgroup scheme $B' \subset G_{\hat{U}}$ such that $B' \cap [G, G]|_{\hat{U}} = \hat{B}$. By definition of $I(G, T)$, $B'$ gives a 

$U$-morphism $\pi: \hat{U} \to I(G, T)$ such that $\pi^* B(G, T) = B'$. Then 

$$R_u(\hat{B}) = R_u(B') = \pi^* U(G, T).$$

It remains to replace $\hat{U}$ with any its connected component, replace $\Gamma$ with the isotropy subgroup of 

this component, and restrict $\rho$ and $B$ to this component. (Cf. Remark 3.2) \hfill $\square$

Proof of Theorem 2. As explained in the proof of Proposition 2.1 the connected components of $U$ 

are integral, so we may assume that $U$ is integral from the very beginning. By \cite{SGA3} Exp. XIV, 
Cor. 3.20 (which generalizes immediately to the semi-local case) we can choose a maximal torus 
$T \subset G$. Applying Proposition 2.2 to $I(G, T) \to U$ and $B(G, T)$, we get a closed subset $Z \subset U$ 
of codimension at least two and a generic trivialization of $\mathcal{E}$ inducing a generic $R_u(\hat{B})$-reduction of 
$\mathcal{E}_{U_0 \times_U I(G, T)}$ that extends to a reduction $r: (U_0 - Z) \times_U I(G, T) \to \mathcal{E}_{U_0 \times_U I(G, T)}/U(G, T)$.

Let a finite $\Gamma$-cover $\hat{U} \to U$, a morphism $\hat{U} \to I(G, T)$, and a Borel subgroup scheme $\hat{B} \subset [G, G]|_{\hat{U}}$ be those 
provided by Lemma 5.1. Pulling back the reduction $r$, we get a reduction 

$$r': (U_0 - Z) \times_U \hat{U} \to \mathcal{E}_{U_0 \times_U \hat{U}}/U(G, T).$$

Let $W$ be any affine open subscheme of $U_0$ disjoint from $Z$. Then Proposition 5.2 applied to the $\Gamma$- 
cover $W \times_U \hat{U} \to W$, the maximal torus $T_W \cap [G_W, G_W]$ of $G_W$, the Borel subgroup scheme $\hat{B}_{W \times_U \hat{U}}$, and the restriction of $r'$ to $W \times_U \hat{U}$, shows that $\mathcal{E}_W$ is trivial. \hfill $\square$

Proposition 5.2. Let $X$ be an integral affine scheme smooth, of finite type, and of positive dimension 
over a semilocal Dedekind domain. Let $x \subset X$ be a finite subset of closed points, let $G$ be a reductive 
$R$-group scheme, where $R := \text{Spec} \mathcal{O}_{X,x}$. For a generically trivial $G$-torsor $\mathcal{E}$ there are 

(a) a closed $R$-finite subscheme $Y \subset \mathbb{A}^1_R$;

(b) a $G$-torsor $\mathcal{E}'$ over $\mathbb{A}^1_R$ trivial away from $Y$;

(c) a section $\Delta \in \mathbb{A}^1_R(R)$ such that $\Delta^* \mathcal{E}' \simeq \mathcal{E}$. 

Proof. Set $U := \text{Spec } R$. By [SGA3², Exp. XIV, Cor. 3.20] (which generalizes immediately to the semi-local case) we can choose a maximal torus $\mathbf{T} \subset \mathbf{G}$. Consider the data $Z \subset Y \subset C \to W, \Delta, \mathcal{E}', s'$ provided by Proposition 4.3.

Let a finite $\Gamma$-cover $\bar{U} \to U$, a morphism $\bar{U} \to I(\mathbf{G}, \mathbf{T})$, and a Borel subgroup scheme $\bar{\mathbf{B}} \subset [\mathbf{G}, \mathbf{B}]_{\bar{U}}$ be those provided by Lemma 5.1. It follows from item (ii) of Proposition 4.5 upon restricting to $\bar{U}$, that we have a reduction

$$
r : (C - Z) \times_R \bar{U} \to (\mathcal{E}' \times_R \bar{U}) / R_u(\bar{\mathbf{B}})
$$

that generically comes from the trivialization $s'$ of $\mathcal{E}'$.

We claim that $(C - Y) \times_R \bar{U}$ is connected. Indeed, the image of the $\Gamma$-invariant morphism $\Delta \times \text{id}_G : \bar{U} \to C \times_R \bar{U}$ is connected. Since $\Gamma$ acts transitively on the connected components of $C \times_R \bar{U}$, this implies that $C \times_R \bar{U}$ is connected. Thus, its open subscheme $(C - Y) \times_R \bar{U}$ is also connected. Since $C - Y$ is affine, we can apply Proposition 4.3 to get a trivialization $s$ of $\mathcal{E}'$ over $C - Y$ such that the induced $R_u(\bar{\mathbf{B}})$-reduction of $\mathcal{E}'_{(C - Y) \times_R \bar{U}}$ is equal to the restriction of $r$ to $(C - Y) \times_R \bar{U}$.

Let $\Delta'$ be the composition of $\Delta$ and $C \to W$. Let $Z'$ be the image of $Z$ under the isomorphism $Y \to Y'$. We have the following elementary distinguished square (see [MV, Ch. 3, Def. 1.3] and [Fed1, Def. 3])

$$
\begin{array}{c}
C - Y \longrightarrow C \\
\downarrow \\
W - Y' \longrightarrow W.
\end{array}
$$

(4)

We use this square and $s$ to glue $\mathcal{E}'$ with the trivial $\mathbf{G}$-torsor over $W - Y'$ (see [CTO, Prop. 2.6]). We obtain a $\mathbf{G}$-torsor $\mathcal{E}''$ over $W$ with a distinguished trivialization $s''$ away from $Y'$.

The elementary distinguished square (4) gives by base change an elementary distinguished square

$$
\begin{array}{c}
(C - Y) \times_R \bar{U} \longrightarrow (C - Z) \times_R \bar{U} \\
\downarrow \\
(W - Y') \times_R \bar{U} \longrightarrow (W - Z') \times_R \bar{U}.
\end{array}
$$

Since sections of schemes satisfy the Nisnevich descent, we can glue the reduction $r$ with the reduction of $\mathcal{E}''$ to $R_u(\bar{\mathbf{B}})$ over $(W - Y') \times_R \bar{U}$ induced by $s''$ to obtain a reduction of $\mathcal{E}''$ to $R_u(\bar{\mathbf{B}})$ over $(W - Z') \times_R \bar{U}$.

We claim that we can find an $R$-finite closed subscheme $Y'' \subset W$ such that $Z' \cup \Delta'([\text{Spec } R]) \subset Y''$ and $W - Y''$ is affine. If $R$ is local, this follows from [Fed2, Lemma 3.5(ii)] applied to $\mathbb{P}^1_R \to \text{Spec } R$, $T_1 = Z' \cup \Delta'([\text{Spec } R])$, and $T_2 = \mathbb{P}^1_R - W$. The generalization to the semilocal case is immediate. By construction, we have a reduction of $\mathcal{E}''$ to $R_u(\bar{\mathbf{B}})$ over $(W - Y'') \times_R \bar{U}$. Applying again Proposition 4.3, we see that $\mathcal{E}''_{W - Y''}$ is trivial. Note that $Y''$ is closed in $\mathbb{A}^1_R$ (being finite over $R$). Gluing $\mathcal{E}''$ with the trivial $\mathbf{G}$-torsor over $\mathbb{A}^1_R - Y''$, we obtain a $\mathbf{G}$-torsor $\mathcal{E}'''$ over $\mathbb{A}^1_R$ trivial away from $Y''$. It remains to rename $Y'' \mapsto Y$, $\mathcal{E}''' \mapsto \mathcal{E}'$, and $\Delta' \mapsto \Delta$.

\[ \square \]
6. Torsors over $\mathbb{A}^1$

The goal of this section is to prove Theorem 3. Recall the notion of a topologically trivial torsor \cite[Def. 2.1]{Fed3}. Recall that a semisimple group scheme of adjoint type can be written as the product of Weil restrictions of simple group schemes along finite connected étale covers (see \cite[Exp. XXIV, Prop. 5.10]{SGA3}). We start with the following theorem.

**Theorem 4.** Let $U$ be a connected affine semilocal scheme. Let $G$ be a reductive group scheme over $U$ with center $Z$; write

$$G^{\text{ad}} := G/Z \simeq \prod_{i=1}^r G^i,$$

where $G^i$ is the Weil restriction of a simple $U_i$-group scheme $\overline{G}^i$ via a finite étale morphism $U_i \to U$.

Let $Z \subset \mathbb{A}^1_U$ be a closed subscheme finite over $U$. Let $G$ be a $G$-torsor over $\mathbb{P}^1_U$ such that its restriction to $\mathbb{P}^1_U - Z$ is trivial and such that for all closed points $u \in U$ the $G_u$-torsor $(G_{\mathbb{P}^1_U})/Z_u$ is topologically trivial.

Let $Y \subset \mathbb{A}^1_U$ be a closed subscheme finite and étale over $U$. Assume that $Y \cap Z = \emptyset$. Assume further that for each $i = 1, \ldots, r$ there is an open and closed subscheme $Y^i \subset Y \times_U U_i$ satisfying two properties: (i) the pullback of $\overline{G}^i$ to $Y^i$ is isotropic and (ii) for every closed point $v \in U_i$ such that $\overline{G}^i_v$ is isotropic we have $\text{Pic}(\mathbb{P}^1_v - Y^i_v) = 0$. Finally, assume that the relative line bundle $\mathcal{O}_{\mathbb{P}^1_U}(1)$ trivializes on $\mathbb{P}^1_U - Y$.

Then the restriction of $G$ to $\mathbb{P}^1_U - Y$ is also trivial.

**Proof.** If $U$ is a scheme over a field, then this is \cite[Thm. 6]{Fed3}. By inspection, the proof goes through in the general case except that the reference to \cite[Prop. 2.12]{Fed3} should be replaced with the reference to \cite[Lm. 8.3]{Ces}. \hfill $\square$

Next, we need a lemma.

**Lemma 6.1.** Let $U_i$, $U$, and $\overline{G}^i$ be as in the formulation of the theorem. Let $T \subset \mathbb{A}^1_U$ be a closed subscheme finite over $U$. Then there is a scheme $Y$ finite and étale over $U_i$ such that $\overline{G}^i_Y$ is isotropic and the following condition is satisfied:

(*) If $v$ is a closed point of $U_i$ such that $\overline{G}^i_v$ is isotropic, then the $k(v)$-degrees of the points of $Y_v$ are coprime.

Moreover, this $Y$ can be chosen so that there is a closed $U$-embedding $Y \to \mathbb{A}^1_U - T$, where $Y$ is viewed as a $U$-scheme via the composition $Y \to U_i \to U$.

**Proof.** Let $\mathcal{P}$ be the $U_i$-scheme classifying the non-trivial parabolic subgroup schemes of $\overline{G}^i$. By \cite[Exp. XXVI, Cor. 3.6]{SGA3} $\mathcal{P}$ is smooth and projective over $U_i$. Thus, for some $N > 0$ we have a $U_i$-embedding $\mathcal{P} \to \mathbb{P}^N_U$. Let $v$ be a closed point of $U_i$. If $\overline{G}^i_v$ is isotropic, then we have a $k(v)$-rational point $\overline{v}$ on the fiber $\mathcal{P}_v$.

Next, we argue as in the proof of Proposition \cite[Lemma 2.8]{LS83}. Using Bertini’s Theorem (see \cite[Exp. XI, Thm. 2.1(ii)]{Poo} and \cite[Exp. XI, Thm. 2.1(ii)]{SGA4}), we see that for large $d$ there is a degree $d$ hypersurface $H_{1,v}$ in $\mathbb{P}^N_v$ intersecting $\mathcal{P}_v$ transversally and containing the point $\overline{v}$. We lift the hypersurfaces $H_{1,v}$, where $v$ ranges over the closed points of $U_i$, to a hypersurface $H_1 \subset \mathbb{P}^N_{U_i}$. Thus, $H_1$ intersects $\mathcal{P}$ transversally and contains all
the points \( \tilde{v} \). Next, we find a hypersurface \( H_2 \subset \mathbb{P}^N_{U_i} \) intersecting \( H_1 \cap \mathcal{P} \) transversally and containing all the points \( \tilde{v} \). Repeating this procedure, we find a closed subscheme \( Y' \subset \mathcal{P} \) finite and étale over \( U_i \) such that for all closed points \( v \) such that \( \overline{G}_v \) is isotropic, \( Y'_v \) has a \( k(v) \)-rational point. Since \( Y' \subset \mathcal{P} \), \( \overline{G}_{Y'} \) is isotropic. (Cf. [FP, Prop. 4.1] and [Pan2, Lm. 4.3].)

Now we choose a prime number \( p_1 \) large enough that

(i) If \( u' \in T \) is a point lying over a closed point \( u \in U \), then \( p_1 > [k(u') : k(u)] \).

(ii) for \( n > p_1 \) and a closed point \( u \in U \) such that \( k(u) \) is finite, the number of degree \( n \) points in \( A^1_{k(u)} \) is larger than the number of points in the \( u \)-fiber of \( Y' \).

Choose a prime number \( p_2 > p_1 \). For every closed point \( w \in Y' \) choose a monic polynomial \( h_w \in k(w)[t] \) of degree \( p_1 + p_2 \) such that: (i) if \( k(w) \) is finite, then \( h_w \) is the product of two irreducible polynomials of degrees \( p_i \); if \( k(w) \) is infinite, then \( h_w \) is a separable polynomial having a root in \( k(w) \).

Note that \( Y' \) is affine and semilocal, since it is finite over \( U \). Write \( Y' := \text{Spec} \, A \). Then we can find a monic polynomial \( h \in A[t] \) that reduces to \( h_w \) at each closed point \( w \). Since \( h \) is monic, the scheme \( Y := \text{Spec} \, A[t]/(h) \) is finite over \( Y' \). Since \( Y \) has a morphism to \( Y' \), we see that \( \overline{G}_Y \) is isotropic.

Condition (*) is satisfied by the choice of \( h_w \).

It remains to check that there is a closed embedding \( Y' \to A^1_U - T \). View \( Y' \) as a \( U \)-scheme. Let \( u \in U \). If \( k(u) \) is finite, there is a closed \( k(u) \)-embedding \( Y_u \to A^1_u - T_u \) because of our conditions (i) and (ii). If \( k(u) \) is infinite, there is a closed \( k(u) \)-embedding \( Y_u \to A^1_u - T_u \) because \( Y_u \) is separable over \( k(u) \). By the Chinese Remainder Theorem these embeddings can be extended to an embedding \( Y \to A^1_U - T \), since \( U \) is affine and semilocal.

\( \square \)

**Proof of Theorem** 3. We use the notations from the formulation of the theorem. We may assume that \( U \) is connected. Applying an affine transformation to \( A^1_U \), we may assume that \( \Delta \) is the horizontal section \( \Delta(U) = U \times 1 \). We can extend the \( G \)-torsor \( E \) to a \( G \)-torsor \( \overline{E} \) over \( P^1_U \) by gluing it with the trivial \( G \)-torsor over \( P^1_U - \mathbb{Z} \). Let \( d \) be the degree of the simply-connected central cover of \( G^{ad} \) (see [Con, Exercise 6.5.2]). Consider the morphism \( P^1_Z \to P^1_{\mathbb{Z}} : z \mapsto z^d \); let \( \psi : P^1_U \to P^1_U \) be the base change of this morphism. Consider the \( G \)-torsor \( \psi^* \overline{E} \) over \( P^1_U \). For a closed point \( u \in U \) write \( \overline{E}_u := \overline{E}_{P^1_u} \). Then by [Gil, Thm. 3.8(a)] the \( G^{ad} \)-torsor \( \overline{E}_u/Z_u \) is Zariski locally trivial. By [Fed3, Prop. 2.3] the \( G^{ad} \)-torsor \( \psi^* \overline{E}_u/Z_u \) is topologically trivial. Since the morphism \( \psi \) has a section over \( U \times 1 \), it is enough to show that \( \psi^* \overline{E}_{U \times 1} \) is trivial. Note that \( \psi^* \overline{E} \) is trivial over \( P^1_U - \psi^{-1}(Z) \).

Now using Lemma 6.1 we construct inductively \( U_i \)-schemes \( Y_i \) satisfying the conditions of the lemma with closed embeddings \( \iota_i : Y_i \to A^1_U \) such that the subschemes \( \iota_i(Y_i) \) are disjoint from each other and from \( \psi^{-1}(Z) \cup (U \times 1) \).

Take \( Y = (U \times 1) \cup \bigsqcup_{i=1}^r Y^i \). Note that \( Y^i \) is an open and closed subscheme of \( Y \times_U U_i \) and by construction \( \overline{G}_i \) is isotropic over \( Y^i \). Let \( v \) be a closed point of \( U_i \) such that \( \overline{G}_v \) is isotropic. Then condition (*) of Lemma 6.1 shows that \( \text{Pic}(A^1_v - Y^i_v) = 0 \). It remains to apply Theorem 4 to \( \psi^{-1}(Z) \subset A^1_U, \psi^* \overline{E}, \) and \( Y \).  \( \square \)
Let $R$, $G$, and $E$ be as in Theorem 1. Since $R$ is regular, Spec $R$ is the disjoint union of its irreducible components, so we may assume that $R$ is integral. By Popescu’s Theorem ([SP, Tago07GC]) we may assume that $R$ is a semilocal ring of a finite set $x$ of closed points on an integral affine scheme $X$ smooth over a semilocal Dedekind domain. By Corollary 3 we may assume that $X$ is of positive dimension over this Dedekind domain. Thus, we are in the situation of Proposition 5.2. It remains to use Theorem 3.

This completes the proof of Theorem 1. □

References

[Čes] Kestutis Česnavičius. Grothendieck–Serre in the quasi-split unramified case. In Forum of Mathematics, Pi, volume 10. Cambridge University Press, 2022.

[Con] Brian Conrad. Reductive group schemes. http://math.stanford.edu/~conrad/papers/luminysga3smf.pdf, 2014.

[CTO] Jean-Louis Colliot-Thélène and Manuel Ojanguren. Espaces principaux homogènes localement triviaux. Inst. Hautes Études Sci. Publ. Math., (75):97–122, 1992.

[CTS] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc. Principal homogeneous spaces under flasque tori: applications. J. Algebra, 106(1):148–205, 1987.

[EGA4] Alexander Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. Inst. Hautes Études Sci. Publ. Math., (32):361, 1967.

[Fed1] Roman Fedorov. On the purity conjecture of Nisnevich for torsors under reductive group schemes. ArXiv e-prints, 2109.10332.

[Fed2] Roman Fedorov. On the Grothendieck–Serre conjecture on principal bundles in mixed characteristic. Transactions of the AMS, 375(1):559–586, 2022.

[Fed3] Roman Fedorov. On the grothendieck-serre conjecture about principal bundles and its generalizations. Algebra & Number Theory, 16(2):447–465, 2022.

[FP] Roman Fedorov and Ivan Panin. A proof of the Grothendieck–Serre conjecture on principal bundles over regular local rings containing infinite fields. Publications mathématiques de l'IHÉS, 122(1):169–193, 2015.

[Gil] Philippe Gille. Torsseurs sur la droite affine. Transform. Groups, 7(3):231–245, 2002.

[Gro1] Alexander Grothendieck. Torsion homologique et sections rationnelles. In Anneaux de Chou et applications, Séminaire Claude Chevalley, number 3. Paris, 1958.

[Guo] Ning Guo. The Grothendieck–Serre conjecture over semilocal Dedekind rings. Transformation Groups, pages 67–87. North-Holland, Amsterdam, 1968.

[PS] Ivan Alexandrovich Panin and Anastasia Konstantinovna Stavrova. On the Grothendieck–Serre conjecture concerning principal G-bundles over semilocal Dedekind domains. Journal of Mathematical Sciences, 222(4):453–462, 2017.
[PSV] Ivan Panin, Anastasia Stavrova, and Nikolai Vavilov. On Grothendieck-Serre’s conjecture concerning principal $G$-bundles over reductive group schemes: I. *Compos. Math.*, 151(3):535–567, 2015.

[Ser] Jean-Pierre Serre. Espaces fibrés algébrique. In *Anneaux de Chow et applications, Séminaire Claude Chevalley*, number 3. Paris, 1958.

[SGA3.2] Michel Demazure and Alexander Grothendieck. *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 152. Springer-Verlag, Berlin, 1970.

[SGA3.3] Michel Demazure and Alexander Grothendieck. *Schémas en groupes. III: Structure des schémas en groupes réductifs*. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 153. Springer-Verlag, Berlin, 1970.

[SGA4.3] Théorie des topos et cohomologie étale des schémas. *Tome 3*. Lecture Notes in Mathematics, Vol. 305. Springer-Verlag, Berlin, 1973. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck and J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat.

[SP] Aise Johan De Jong et al. The stacks project.

[Tho] Robert W. Thomason. Equivariant resolution, linearization, and Hilbert’s fourteenth problem over arbitrary base schemes. *Adv. in Math.*, 65(1):16–34, 1987.

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