Dynamical response of dissipative helical edge states

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Quantum spin Hall insulators are characterized by topologically protected counter-propagating edge states. Here we study the dynamical response of these helical edge states under a time-dependent flux biasing, in presence of a heat bath. It is shown that the relaxation time of the edge carriers can be determined from a measurement of the dissipative response of topological insulator (TI) disks. The effects of various perturbations, including Zeeman coupling and disorder, are also discussed.

Introduction.—The hallmark of two-dimensional (2D) quantum spin Hall (QSH) TIs consists in the existence of dissipationless conducting edge states in the absence of any time-reversal breaking perturbations [4]. Due to spin-orbit coupling and a particular bulk band structure, the spin of the edge carriers is tied to their direction of motion [2,3]. These helical edge states have been reported experimentally in HgTe/CdTe [4] and InAs/GaSb [5] quantum wells. So far, most of the studies have covered the equilibrium or ground-state physics of helical edge states, while less is known about their dynamics and the associated relaxation mechanisms. Only recently, the problem of dissipation has gained attention in the context of topological insulators (TI) [6] and topological superconductors [7].

Recently, it has been proposed that Floquet type of TIs can be engineered by applying a proper external drive on a trivial band insulator or on a semimetal [8]. Floquet bands have already been reported in time-resolved photoemission experiments on three dimensional TIs [9], and their topological nature is under active debate. Relaxation phenomena are crucial to establish such nonequilibrium steady states of matter, and insure the balance between the energy injected by the drive and the energy dissipated towards microscopic degrees of freedom of the environment.

In the meantime, experimental progresses have been achieved in extracting typical relaxation times of carriers in coherent conductors such as normal-superconducting (NS) rings [10][11]. The idea is to couple a small coherent system, characterized by a flux-dependent spectrum, to a multimode superconducting resonator. The dissipative and nondissipative magnetic susceptibility of unconnected samples is obtained by measuring the energy shifts and nondissipative magnetic susceptibility of unconnected quantum wells. So far, most of the studies have covered the equilibrium or ground-state physics of helical edge states, while less is known about their dynamics and the associated relaxation mechanisms. Only recently, the problem of dissipation has gained attention in the context of topological insulators (TI) [6] and topological superconductors [7].

In view of these experimental advances, this Letter addresses the dynamical response of the generic helical edge state of a 2D TI coupled to a thermal bath and threaded by a time-dependent flux $\Phi(t)$, which is the superposition of a dc flux $\phi$ and a small alternating flux at a single frequency $\omega$ (see experiments [10][11]). It is obtained that the dissipative response of the helical edge state exhibits a characteristic phase-dependent signature: a single peak is located either at $\phi = 0$ or at $\phi = \phi_0/2$, depending on the electronic filling. This peak has a maximal amplitude when the frequency is equal to the relaxation rate of the edge carriers. In contrast to standard metallic rings [12][13] or NS rings [10][11], the extraction of the carrier lifetime is simplified by a selection rule which forbids interband transitions between left and right spin-polarized movers. This is a dynamical manifestation of the edge states’ helical structure. These analytical results are validated in a comparison with lattice simulations of the Bernevig-Hughes-Zhang (BHZ) model for HgCd/CdTe quantum wells [3].

Model and formalism.—Let us consider a disk of a 2D TI under a perpendicular time-dependent uniform magnetic field $B(t)$ (Fig. 1). The focus stays here on the response of the helical edge liquid which encloses a flux $\Phi(t) = \phi + \delta\phi(t)$. The dc flux $\phi$ is varied arbitrarily, while the time-dependent oscillatory flux $\delta\phi(t) = \phi_{ac}\cos\omega t$ has a small amplitude with respect to the flux quantum. Since the edge carriers are exchanging energy with a heat bath, the dynamical susceptibility $\chi(\omega)$ gains a dissipative component at finite frequency.

FIG. 1. (Color online). A QSH TI disk under a time-dependent perpendicular magnetic field $B(t)$, in the meanders of a superconducting resonator, which serves as a measuring device. The Hamiltonian [1] models the counterpropagating edge states (red and blue) which enclose a flux $\Phi(t) = \phi + \delta\phi(t)$. The dc flux $\phi$ is varied arbitrarily, while the time-dependent oscillatory flux $\delta\phi(t) = \phi_{ac}\cos\omega t$ has a small amplitude with respect to the flux quantum. Since the edge carriers are exchanging energy with a heat bath, the dynamical susceptibility $\chi(\omega)$ gains a dissipative component at finite frequency.

The Hamiltonian for a two-dimensional (2D) TI is given by $H = H_0 + H'(t)$, where $H_0$ describes the equilibrium state of the TI disk and $H'(t)$ represents the dissipative part.

\[
H_0 = \hbar v_F L \left( -\frac{\partial}{\partial \theta} + \frac{\phi}{\phi_0} \right) \sigma_3, \quad H'(t) = \frac{e v_F}{L} \delta\phi(t) \sigma_3. \tag{1}
\]
The Fermi velocity of the carriers is \( v_F \), the length of the edge state, \( L \), and the angular coordinate, \( \theta \). The \( \sigma_3 \) matrix is the standard diagonal spin Pauli matrix. The Hamiltonian in Eq. (1) only includes the orbital effect of the magnetic field. In principle, the magnetic field also couples to the electronic spin, but it turns out that this Zeeman coupling, \( H_z = -\mu_B B(t) \sigma_3 \), has the same structure as the orbital one. Moreover this Zeeman coupling is much smaller than the orbital coupling by a factor equal to ratio of the orbital magnetic moment of the helical loop to the spin magnetic moment of the electron \( \mu_B \).

In the absence of a time-dependent drive \( (\delta \phi(t) = 0) \), the helical liquid is described by the low-energy effective Hamiltonian \( H_0 \), and it supports a robust persistent current \( I_{\text{per}}(\phi) \), characterized by a maximal amplitude \( I_0 = ev_F/L \) at zero temperature \[13\]. The flux-dependent energy levels

\[
\epsilon_{n\sigma}(\phi) = \epsilon_n(\phi) = \sigma \hbar \omega_0 \left(n + \frac{\phi}{\phi_0}\right),
\]

are discrete and identified by an angular momentum \( n \) and a spin \( \sigma \) quantum numbers, which are gathered in the notation \( n = (n, \sigma) \). The corresponding energy eigenstates solve the Schrödinger equation \( H_0|\Psi_n\rangle = \epsilon_n(\phi)|\Psi_n\rangle \), where \( |\Psi_n\rangle \) are the eigenspinors of \( \sigma_3 \) times \( e^{in\theta} \). The energy spacing between adjacent levels of a given spin and flux is denoted by \( \hbar \omega_0 = \hbar v_F/L \). Each energy level carries a flux-independent current \( i_n = -\sigma I_0 = -\sigma e v_F/L \).

Let us consider that the quantum edge states are coupled to a thermal bath containing many degrees of freedom. These degrees of freedom could have various distinct microscopic origins: electromagnetic modes of the external circuit, phonons, bulk states of the disk, etc. Then, in response to the finite driving term \( \delta \phi(t) = \phi_{ac} \cos \omega t \), the edge can support both nondissipative \( (I_{\text{ac}} \cos \omega t) \) and dissipative \( (I_{\text{ac}} \sin \omega t) \) ac steady currents. This response is captured by a complex frequency-dependent susceptibility \( \chi(\omega) = \chi(\omega) + i\chi'(\omega) \). In the present setup, the TI disk is unconnected and therefore it only exchanges energy with the environment, while the number of particles remains fixed. This precludes the possibility of gapping the edge states due to contamination of an edge channel with electrons of opposite spin originating from the environment.

Here we will not investigate the microscopic mechanisms leading to dissipation, but rather provide a generic and phenomenological model to describe it in the case of a weak coupling to the environment. To this aim, we consider the evolution the system under the following kinetic equation for the reduced density operator \( \rho(t) \) (obtained after tracing out the environment degrees of freedom \[12\] \[13\] \[13\]):

\[
\frac{\partial \rho(t)}{\partial t} + \frac{i}{\hbar} [H(t), \rho(t)] = -\gamma [\rho(t) - \rho_{eq}(t)].
\]

The matrix \( \gamma \) phenomenologically represents the relaxation rates for populations and coherences in the density matrix operator. The master equation indicates that starting from an out-of-equilibrium situation, the density operator will evolve towards the quasi-equilibrium density matrix \( \rho_{eq}(t) = \exp[\beta(H(t) - \mu) + 1]^{-1} \). Because the system exchanges only heat with the environment, the number of particles is fixed. Consequently, the chemical potential \( \mu \) is not constant and depends generally on flux, number of particles, temperature, and time. Nevertheless, here \( \mu \) can be taken constant in flux, due to the particular flux dependence of the last occupied energy level for a given parity of electron number. Moreover, the time dependence of \( \mu \) brings only a negligible contribution to the dissipative response in comparison with other competing terms (see Supplemental Material (SM)).

In the linear response approximation \( (\phi_{ac} \ll \phi_0) \), the master Eq. (3) is solvable, and the complex linear susceptibility in the basis of the time-independent Hamiltonian \( H_0 \) can be decomposed into three parts \[12\] \[13\].

\[
\chi(\omega) = \chi_{\text{per}} + \chi_D(\omega) + \chi_{ND}(\omega).
\]

The static part of the susceptibility \( \chi_{\text{per}} \) is purely real and it is due to the persistent current in the system. The second and third terms are called diagonal and nondiagonal with reference to the \( H_0 \) eigenstate basis. The diagonal susceptibility \( \chi_D \) describes only the intraband response of the system, while the nondiagonal susceptibility \( \chi_{ND} \) is related to interband transitions. Note that all the terms in Eq. (1) depend on the static flux \( \phi \) and temperature.

**Helical edge states’ susceptibility.** — Let us first discuss the adiabatic part of the susceptibility. It is directly related to the persistent current physics. In the absence of the time-dependent perturbation, the susceptibility reads \( \chi_{\text{per}} = \frac{\partial}{\partial \phi} \langle \sum_n i_n f_n \rangle \), which is the derivative of the persistent current with respect to the dc flux \( \phi \). In this case, the sum runs over the angular momentum and spin quantum numbers. The functions \( f_n \) represent henceforth the Fermi-Dirac distribution function for static Hamiltonian \( H_0 \), \( f_n = f(\epsilon_n(\phi)) \).

The perturbation \( H'(t) \) commutes with \( H_0 \) and it cannot induce spin flips or changes in the angular momentum of the electrons. Because the system does not exchange electrons with the environment the spin and angular quantum numbers remain conserved. This selection rule forbids interband transitions and it implies that the nondiagonal susceptibility \( \chi_{ND}(\omega, T, \phi) \) vanishes. Therefore, dissipation can occur only through intraband relaxation processes.

This is a remarkable simplification with respect to the complicated situation of multilevels systems encountered in experiments for normal (and Josephson) rings, where separating the three contributions in Eq. (4) is a difficult and subtle task \[11\] \[13\] \[16\]. Therefore, the linear susceptibility of the helical edge contains only two terms: \( \chi(\omega) = \chi_{\text{per}} + \chi_D(\omega) \), the first being given the derivative...
The eventual contribution of bulk states to the insulating term are equidistant and have the same absolute value of and constant in flux. It is reasonable to assume identical approximating the diagonal rates simulations. The problem is further simplified when applied flux. This model is a useful lattice regularization of the effective 4-band Dirac model describing the topological transition in HgTe/CdTe quantum wells [3]. The different terms are tensor products of the Pauli matrices $\sigma$ and $\tau$ describing internal (orbital and spin) degrees of freedom. Since we are primarily concerned with the edges, we use the hollow cylinder geometry, with base circumference $L_x$ and height $L_y$. The system is taken in a topological insulating phase (bulk gap $\simeq 2A$) and at half filling $N_{1/2} = 2L_x L_y$. There are two helical edge states located around the bottom ($x = 1$) and top ($x = L_y$) bases of the cylinder. Provided $L_y$ is large enough, these two edge states do not overlap and exhibit equal and independent orbital responses $\chi(\omega)$.

In Fig. 2(a) and (b) the diagonal susceptibility shows a peak at zero flux which is maximal when the frequency is exactly equal to the relaxation rate $\gamma_D$. The magnetic signal from the helical model (Eq. (1)) is scaled by a factor of two in order to account for the two helical edge liquids in the lattice BHZ model (top and bottom of the cylinder). The match between the helical and the BHZ models holds at different driving frequencies, temperatures, or diagonal potential $\mu$ reads as

$$\frac{\chi''(\omega)}{\chi_0} = \frac{4\omega \gamma_D}{\omega^2 + \gamma_D^2} \left[ \frac{1}{2} + \sum_{m=1}^{\infty} \frac{m T/T^*}{\sinh(m T/T^*)} \right] \cos(2\pi m \frac{\phi}{\phi_0}) \cos(2\pi m \mu \hbar \omega_0) \tag{6}$$

The susceptibility is measured in units of $\chi_0 = \frac{\hbar}{\gamma_D} = \frac{e^2 v_F}{4\pi\hbar^2 n}$. The characteristic temperature $T^*$ is proportional to the level spacing, $T^* = \hbar v_F/(\pi k_B L)$. It immediately follows that there is a peak in the dissipative susceptibility of the current at zero flux. If the fermionic parity is changed by adding or subtracting a single particle, the chemical potential changes by $\hbar \omega_0/2$ and the peak moves to half-integer flux $\phi/\phi_0 = \pm 0.5$ (SM).

Since the current matrix is diagonal, there is no damping rate for coherences $\rho_{nn}$. As such the dissipation is entirely captured by the evolution for the populations $\rho_{nn}$. In this case, the result would be the same, if one used the relaxation-time approximation, usually employed to describe the onset of resistance in metallic rings [17] (SM).

Comparison with the BHZ model.— We now present numerical simulations supporting the analytical results above. We use the Bernevig-Hughes-Zhang (BHZ) model on a square lattice, described by the Hamiltonian [3]:

$$H = \sum_{x=1}^{L_x} \sum_{y=1}^{L_y} c_{xy}^\dagger \left[ \frac{i}{2\hbar} \sigma_1 \tau_3 + B \sigma_3 \tau_0 \right] c_{x+1y} + \text{H.c.} \tag{7}$$

$$+ \left( \frac{A}{2\hbar} \sigma_2 + B \sigma_3 \right) \tau_0 c_{xy} + \left( \frac{M}{2} - 2B \right) \sigma_3 \tau_0 c_{xy}$$

...
rates $\gamma_D$. Indeed the diagonal susceptibility depends crucially on the states near the chemical potential and thus at half-filling it is well approximated by that of the edge states inhabiting the gap, while the bulk contribution is negligible. The two pairs of edge states must be well separated otherwise hybridization of edge states leads to a vanishing zero-flux susceptibility. Furthermore the diagonal susceptibility in zero flux decreases with temperature, but it maintains a maximum at $\omega = \gamma_D$ (Fig. 2(c)). If a pair of particles is added, the susceptibility-flux characteristic is shifted by half-integer flux quantum, such that the susceptibility peak moves to $\phi/\phi_0 = 0.5$. At odd number of particles the peaks are smaller and appear at both $\phi/\phi_0 = 0$ and 0.5 (see SM).

In contrast with the 1D helical model Eq. (1), the 2D lattice model allows transitions between the bulk states. These transitions induce a large contribution only to the nondiagonal susceptibility $\langle \chi''_D \rangle / \chi_0$, which scales with the number of electrons in the system, while the diagonal contribution only scales with the edge length. $\chi''_D (\omega) \propto L_x$. Nevertheless, this large bulk-states contribution is almost flux independent for large systems, thereby allowing an easy extraction of the flux-dependent edge contribution (see SM) and determination of the lifetime $\gamma_D^{-1}$ of the edge states. Note that the nondiagonal dissipative response $\langle \chi''_N \rangle$ has been evaluated under the assumption that all damping rates for coherences (in Eq. (3)) are equal and constant in flux or temperature, $\gamma_{nn} = \gamma_{ND}$.

Disorder effects.— The addition of scalar disorder does not destroy the edge states. The signature peak in the dissipative diagonal susceptibility slowly decreases, however it does not vanishes, if the disorder strength is smaller than the bulk gap ($\simeq 2 A$ for the parameters in simulations of Fig. 3(d)). As disorder strength increases and becomes larger than the bulk gap, dips can develop in the diagonal susceptibility in random samples. On average, the susceptibility at large disorder becomes more and more flat and flux independent (Fig. 3(d)).

It is important to remark that this situation is different from the case of a regular system with nonrelativistic fermions. Indeed, rings with nonrelativistic fermions present energy level crossings in the ballistic limit which are not protected against disorder: infinitesimal scalar disorder removes the degeneracies and yields a vanishing zero-flux diagonal susceptibility (instead of the peak predicted in the topologically protected edge state).

Effect of an in-plane field.— So far we have derived the dynamical response of the helical edge due to a purely orbital coupling arising from a perpendicular magnetic field. An additionally static in-plane field induces a Zeeman coupling between spin up and spin down. In the helical model, we consider a constant term proportional to a spin-mixing matrix $\sigma_1$, $H = H_0 + H'(t) + V \sigma_1$. In this case the edge states are gapped out in zero flux, leading to a vanishing dissipative diagonal susceptibility. Moreover, the in-gap states now bring a nondiagonal susceptibility $\langle \chi''_N \rangle$ (see SM). The response will depend on the nondiagonal damping rates $\gamma_{nn}$, which renders the analysis more difficult.

A constant Zeeman field is implemented also in the BHZ model (see SM). The agreement between the helical and BHZ models still holds (Fig. 3(a) and (b)). The bulk is largely unaffected by the flux, and its contribution to susceptibility remains almost constant in flux. The features in the nondiagonal susceptibility can be accounted by the edge state contribution, shifted with a large constant dissipative bulk contribution (Fig. 3(c)). Note that very small spin-mixing still opens a gap at time-reversal invariant fluxes. Then the vanishing level current leads to a dip in diagonal susceptibility at zero flux. Only energy states close to these flux values are affected for very small Zeeman fields. These leads to troughs in the diagonal susceptibility of small width in comparison to the overall
width of the signal.

In experiments, the presence of spin mixing due to Zeeman fields can prove problematic, as the contribution from the nonдиagonal susceptibility may completely mask the features of the diagonal susceptibility. As the magnetic field increases the diagonal susceptibility diminishes and the contribution from the nonдиagonal signal dominates the diagonal response. Depending on the damping rates for coherences, size of the system, and the driving frequency, the single peak in signal can split in many components. Finally, if the nonдиagonal susceptibility washes out completely the diagonal response, one recovers again a large single peak indicating that the largest dissipation is present in zero magnetic flux.

Using the material parameters \([4]\) of the HgTe/CdTe quantum wells, we estimate the relevant quantities. Using the Fermi velocity for HgCdTe quantum wells of thickness \(d \approx 7\text{nm}\) is approximately \(v_F \approx 5.5 \times 10^7 \text{m/s}\). Therefore the characteristic temperature for a ring of size \(L = 0.5\mu\text{m}\) is \(T^* \approx 2.67\text{K}\). The distance between levels at the Fermi surface is \(\hbar \omega_0\), which for our given wire sets the characteristic frequency \(\omega_0 \approx 6.9 \times 10^{12} \text{s}^{-1}\). The characteristic current for the same ring length \(I_0 \approx 1.76 \times 10^{-7} \text{A}\). Therefore the characteristic dimensional susceptibility reads \(\chi_0 \approx 4.26 \times 10^7 \text{A}^2/\text{J}\). In order to explore the physics of the edge states, the temperature was taken smaller than the gap, which implies the constraint \(T < \frac{2\pi L_a}{\hbar} \frac{T^*}{a}\) for \((A, B, M) = (1, 0.6, 1)\) (where the gap \(\approx 2\)). In simulations, the constant magnetic field that gaps the edge states yields a Zeeman energy \(V\) as large as \(0.1\hbar \omega_0\) and, therefore, a transversal magnetic field \(B_T \leq 0.1 \frac{\hbar v_F}{\mu_B T}\) (or \(B_T \leq 7.8T\) for the given parameters).

Conclusions.—In this Letter, we have studied the dissipative response of a 2D QSH insulator under the effect of a small time-dependent driving in flux. Using a helical model for the edge states and exact diagonalization of a tight-binding BHZ insulator, we have proven that the contribution of the edge states and the bulk can be differentiated. Crucially, the lifetime of the edge states can be identified by searching the frequency where the dissipative response is maximal. While the bulk can indeed bring a large contribution to susceptibility, it can be eliminated by observing, that it is almost constant in flux.

Moreover, the measurement of the diagonal dissipative susceptibility is sensitive to the gapping of the edge states (either due to hybridization between pairs of edge states brought in spatial proximity or due to a Zeeman fields at zero flux). The peak in the diagonal susceptibility can be splitted in more than two peaks in these cases.

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RINCUSCRIPTIBILITY FOR A SMALL TIME-DEPENDENT PERTURBATION

This section reviews the linear response theory yielding the total susceptibility for rings threaded by a time-dependent flux, following Ref. [12].

Let us consider a static Hamiltonian $H_0$ with a time perturbation $H'(t)$ due to an oscillating flux $\Phi(t)$. The unperturbed Hamiltonian $H_0$ represents a lattice system with a discrete number of states. It has a set of energy eigenvalues $\epsilon_n$ and an orthonormal set of eigenvectors $\{|n\rangle\}$. The following static operators are used throughout the Supplement: the current $J = -\partial H_0/\partial \phi$ and susceptibility $X = \partial J/\partial \phi$ operators. They are generally not diagonal in the basis of $H_0$.

The flux decomposes into a static flux $\phi$ and a time-dependent component $\delta \phi(t)$. It is sufficient to consider a single Fourier component for the oscillating frequency $\omega$, such that the flux reads $\Phi = \phi + \phi_{ac} e^{i\omega t}$. The amplitude of the time-dependent component is very small $\phi_{ac} \ll \phi_0$, such that all flux-dependent quantities are expanded to first order in it.

The system is connected to a bath and it is described by a reduced density matrix $\rho$. Under the effect of the flux, a current $I$ is induced in the ring

$$I = \text{Tr}[\rho(t)J(t)],$$

where $J$ is the time-dependent current operator. We are looking for the response $\chi$ of the system defined as the change in the induced current due to the variation of the flux, defined through the functional relation

$$\chi = \frac{\delta I}{\delta \phi}. \quad (9)$$

The density matrix evolves under the master equation [12] [13]:

$$\frac{\partial \rho(t)}{\partial t} + i\hbar [H(t), \rho(t)] = -\gamma [\rho(t) - \rho_0(t)]. \quad (10)$$

The change in the current in the approximation of linear response is

$$\delta I = \sum_n \langle n|\delta \rho J + \rho_0 \delta J|n \rangle, \quad (11)$$

where the sum is over the eigenstates of the unperturbed Hamiltonian. The density matrix for the unperturbed system is $\rho_0 = \sum_n f_n |n\rangle \langle n|$ with the Fermi-Dirac function depending on the energies of the unperturbed system $f_n \equiv f(\epsilon_n)$.

The master equation is evaluated in the eigenstate basis of $H_0$, $\{|n\rangle\}$. The chemical potential is expanded near the static value $\mu = \mu_0 + \delta \mu(t)$. Similarly the density operator is expanded near the unperturbed density operator $\rho = \rho_0 + \delta \rho$. In a basis of $H_0$ eigenstates, the master equation is represented as a set of differential equation. To linear order in $\delta \phi$, the equations for the matrix elements of the density operator are decoupled,

$$\delta \rho_{nn}(\omega) = -\frac{\delta f_n}{\partial \epsilon_n} i\gamma_{nn} \left( J_{nn} \delta \phi + \delta \mu \right),$$

$$\delta \rho_{mn}(\omega) = -\frac{f_m - f_n}{\hbar \omega_{mn}} \left( \omega_{mn} - i\gamma_{mn} \right) J_{mn} \delta \phi, \quad (12)$$

where in the last equation $m \neq n$. The condition that the number of particles is fixed reads as $\text{Tr}[\delta \rho] = 0$. This determines the change in the chemical potential with the flux:

$$\frac{\delta \mu}{\delta \phi} = -\sum_n \frac{\delta f_n}{\partial \epsilon_n} \frac{J_{nn}}{n}. \quad (13)$$

We will examine at the end of the section conditions for neglecting this term.

To first order in $\delta \phi$, the change in the current operator reads $\delta J = X \delta \phi$. The induced current in linear response is obtained using the equations for the density matrix [12] with the current variation $\delta J$ in Eq. (11). Finally, Eq. (12) yields a susceptibility that has diagonal and nondiagonal elements in the state basis of $H_0$,

$$\chi(\omega) = \sum_n X_{nn} f_n - J_{nn} (J_{nn} + \frac{\delta \mu}{\delta \phi} \frac{\delta f_n}{\partial \epsilon_n} i\gamma_{nn} + \omega - \sum_{m,n} |J_{mn}|^2 f_m - f_n \frac{\omega_{mn} - i\gamma_{mn}}{\hbar \omega_{mn}} - \omega - i\gamma_{mn}). \quad (14)$$

The primed sum denote in the following that $m \neq n$. This formula is especially useful in the numerical determination of the susceptibility as it does not depend on the flux discretization.

The above equation is simplified using a sum rule from equating the second order perturbation theory for the
eigenvalues $\epsilon_n(\phi)$ of the Hamiltonian $H(t)$ and the Taylor expansion for the energy $\epsilon_n(\phi + \delta \phi)$. Consequently, the static current and susceptibility operators are expressed as

$$J_{nn} = -\frac{\partial \rho_n}{\partial \phi} = i_n, \quad X_{nn} = \frac{\partial \rho_n}{\partial \phi} - 2 \sum_{m \neq n} |J_{mn}|^2 \frac{\chi_{n|n}}{\hbar \omega_{mn}}.$$  \hfill (15)

The explicit formula for the tripartite susceptibility, $\chi = \chi_{\text{per}} + \chi_D + \chi_{\text{ND}}$, follows using the sum rule in Eq. (14):

$$\chi(\omega) = \sum_n \frac{\partial (im_n f_n)}{\partial \phi} - J_{nn}(J_{nn} + \frac{\delta \mu}{\delta \phi}) \frac{\partial f_n}{\partial \epsilon_n} \gamma_{nn} - i \omega$$

$$- \sum_{m,n} |J_{mn}|^2 \frac{f_m - f_n}{\hbar \omega_{mn}} \gamma_{mn} - i \omega + \gamma_{mn}.$$  \hfill (16)

The first term represents the persistent current contribution $\chi_{\text{per}}$, while the second and the third terms stand, respectively, for the complex diagonal and nondiagonal susceptibilities.

The dissipative response for the system follows readily,

$$\chi'(\omega) = -\sum_n J_{nn}(J_{nn} + \frac{\delta \mu}{\delta \phi}) \frac{\partial f_n}{\partial \epsilon_n} \omega \gamma_D + \gamma_D^2$$

$$- \sum_{m,n} |J_{mn}|^2 \frac{f_m - f_n}{\hbar \omega_{mn}} \omega \gamma_{mn} + \gamma_{mn}.$$ \hfill (17)

Note that if there is no damping rate for coherences ($\gamma_{nn} = 0$) or vanishing nondiagonal current components ($J_{mn} = 0$), for a finite number of discrete levels, and a negligible variation in the chemical potential, then the susceptibility reads as

$$\chi = \sum_n \frac{\partial \rho_n}{\partial \phi} f_n - \frac{1}{\tau_n} (\rho_{nn} - f_n),$$ \hfill (18)

which is the same expression which can be obtained using the simpler time-relaxation approximation

$$\frac{\partial \rho_{nn}}{\partial t} = -\frac{1}{\tau_n} (\rho_{nn} - f_n),$$ \hfill (19)

with $\tau_n = \gamma_{nn}^{-1}$.

Lastly, let us return to the issue of the chemical potential variation $\delta \mu$. The condition that the number of particles is fixed for the time-independent problem $\sum_n f_n = N$, yields a constraint on the static chemical potential $\mu_0$. It follows from Eqs. (13) and (15) that in the linear response theory $\delta \mu(t)/\delta \phi = \delta \mu_0/\delta \phi$. Therefore a constant chemical potential with respect to the static flux, will have no time variations. This will prove important in the next section in the case of the helical model.

The chemical potential variation will equally prove negligible in the half-filling BHZ from a different point of view. Its contribution to the diagonal susceptibility is small in comparison to the other terms in Eq. (14). For example, after factoring out the dynamical dependence, the susceptibility $\chi_{\delta \mu}$ due to the variation of $\mu$ reads as

$$\chi_{\delta \mu} \propto \frac{\left( \sum_m \frac{\partial f_n}{\partial \epsilon_n} J_{nn} \right)^2}{\sum_m \frac{1}{\hbar \omega_n} \frac{\delta f_n}{\partial \epsilon_n}} \ll \frac{1}{\hbar \omega_n} J_{nn}^2.$$

This can is readily understood in the topological insulator case with helical edge states in the gap. The terms in the above sum contain mainly the contribution from the edge states, which are close to the Fermi energy. But the helical states have a linear energy-flux dispersion, and any current $J_{nn}$ has a partner with the same magnitude, but different sign, at a given energy. Then the numerators on the left hand side of the inequality give a vanishing contribution. For a finite temperature the denominator is finite and hence the response $\chi_{\delta \mu}$ is negligible. On the right hand side all the currents are squared, leading to a large contribution as observed in the body of the Letter.

**HELICAL MODELS**

### Dirac ring

The Dirac ring is described by the Hamiltonian $H = H_0 + H'(t)$

$$H_0 = \frac{\hbar v_F}{L} \left( -i \frac{\partial}{\partial \theta} + \frac{\phi}{\phi_0} \right) \sigma_3, \quad H'(t) = \frac{ev_F}{L} \sigma_3 \delta \phi(t).$$ \hfill (21)

In this case there is an infinite number of discrete eigenvalues with a linear dispersion, $\epsilon_{n\sigma} = \sigma (n + \phi/\phi_0)$, where $\sigma$ indicates the spin degree of freedom $\sigma = \pm$ for $\uparrow$, respectively $\downarrow$. The spin component of the wave function are the eigenstates of $\sigma_3$ operator. The wave functions read as

$$|n \uparrow\rangle = e^{i n \theta - i \epsilon_{n \uparrow} t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |n \downarrow\rangle = e^{i n \theta - i \epsilon_{n \downarrow} t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ \hfill (22)

Then the current operator matrix reads as

$$i_{n\sigma} = \langle n\sigma | J | n'\sigma' \rangle = -\sigma_0 \delta_{n_n', \sigma' \sigma}, \quad I_0 = \frac{ev_F}{L}.$$ \hfill (23)

The model contains an infinite number of occupied states, which contribute to the persistent current. Thus the persistent current may not be a convergent sum, and subsequently $\chi_{\text{per}} = \partial I_{\text{per}}/\partial \phi$ may not be defined. The authors have explicitly obtained in Ref. [13] the persistent current in the helical model using a regularization of the sum over the infinite number of states. The final result is finite and matches lattice results,

$$\frac{I_{\text{per}}}{I_0} = \sum_{m=1}^{\infty} \frac{2T/T^*}{\pi \sinh(mT/T^*)} \sin(2\pi m \frac{\phi}{\phi_0}) \cos(2\pi m \frac{\mu}{\hbar \omega_0}).$$ \hfill (24)
This regularization renders the persistent susceptibility well defined. In contrast, the diagonal part of the susceptibility $\chi_D$ is always well defined as it contains predominantly the contribution from states near the Fermi surface due to the term $\partial f_{n \sigma}/\partial \epsilon_{n \sigma}$.

Note that this current expression was obtained in the grand canonical ensemble for constant $\mu$. Nevertheless, it can be connected with the case where the number of particles in the system is fixed. When $\mu = n_0 \omega_0/2$ with $n_0$ integer, $\mu$ does not depend on the static flux and stands for a fixed number of particles. Because of the symmetry of the energy states, the $\mu = 0$ case represents the half-filling case in the lattice models. Changing the number of particles by $n$ equivalent to a change in the chemical potential $\Delta \mu = n \hbar \omega_0$. Consequently, the change in the fermion parity leads to a shift by $n \phi_0/2$ in the current-flux characteristic. Due to gauge invariance, all physical quantities are periodic in $\phi_0$. Then adding an even number of particles is equivalent to the starting situation.

Because the chemical potential is constant in flux, the diagonal susceptibility reads

$$
\chi_D = -\sum_{n \sigma} \gamma_{n \sigma}^2 \frac{\partial f_{n \sigma}}{\partial \epsilon_{n \sigma}} \frac{i \omega}{\gamma_{nn} - i \omega},
$$

with $\gamma$ spin independent.

The diagonal susceptibility $\chi_D$ can be expressed entirely in terms of the persistent current susceptibility $\chi_{\text{per}}$. After summing the spin degrees and algebraic manipulation of the sums in $\chi_D$, it follows that the diagonal susceptibility reads as

$$
\frac{\chi_D}{\chi_0} = \frac{i \omega}{\gamma_{nn} - i \omega} \left(2 + \frac{\chi_{\text{per}}}{\chi_0}\right).
$$

The equation is then used to obtain the total susceptibility $\chi = \chi_{\text{per}} + \chi_D$. The dissipative susceptibility showed in the body of the Letter follows in the approximation of identical $\gamma_{nn} = \gamma_D$ by taking the complex part of Eq. (26):

$$
\chi''(\omega) = \frac{4 \omega \gamma_D}{\omega^2 + \gamma_D^2} \left[2 + \sum_{m=1}^{\infty} \frac{m T/T^*}{\sinh(m T/T^*)} \right. \\
\times \cos(2 \pi m \frac{\phi}{\phi_0}) \cos(2 \pi m \frac{\mu}{\hbar \omega_0}) \left. \right].
$$

Gapped Dirac ring

The helical edge states are gapped in zero flux by adding a constant term which mixes the spin. This produces a vanishing diagonal susceptibility in zero flux.

The static Hamiltonian reads as

$$
H_0 = \hbar \omega_0 \left(-i \frac{\partial}{\partial \phi} + \frac{\phi}{\phi_0}\right) \sigma_3 + V \sigma_1,
$$

with the energy $\pm \epsilon_n$,

$$
\epsilon_n = \left[\hbar^2 \omega_0^2 (n + \phi/\phi_0)^2 + V^2\right]^{1/2}.
$$

Let us consider again the same perturbation $H'(t)$, containing the time-oscillating flux. There are no current operator matrix elements between states with different angular momentum. Nevertheless, there are matrix elements between different spins.

The current diagonal and nondiagonal matrix elements are

$$
|J_{nn}| = I_0 \frac{\epsilon_n (m = 0)}{\epsilon_n}, \quad |J_{n \bar{n}}| = I_0 \frac{V}{\epsilon_n},
$$

obeying the conservation law $J_{nn}^2 + J_{n \bar{n}}^2 = I_0^2$. We have denoted here $\langle n | J | n \rangle = J_{nn}$.

Without loss of generality, the chemical potential is taken at zero, $\mu = 0$. The diagonal and nondiagonal susceptibility follow readily; the diagonal part reads as

$$
\chi_D = \frac{i \omega}{\gamma_D - i \omega \frac{\pi^2 T^*}{T} \sum_n \frac{i^2}{\hbar \omega_0} \cosh^{-2} \left(\frac{\pi^2 T^*}{T} \frac{\epsilon_n}{\hbar \omega_0}\right)}.
$$

The sum runs over all angular momenta $n$. Due to the fast decaying hyperbolic cosine at large $n$ the sum is quickly converging.

The nondiagonal susceptibility after summing over the spin degree of freedom reads as

$$
\chi_{\text{ND}} = -\sum_n \frac{J_{n \bar{n}}^4}{\epsilon_n} \tanh \left(\frac{\pi^2 T^*}{T} \frac{\epsilon_n}{\hbar \omega_0}\right) \frac{\omega (\omega + i \gamma_{nn})}{(\omega + i \gamma_{nn})^2 - 4 \epsilon_n^2}
$$

The dissipative susceptibilities are obtained by taking the imaginary part in the above equations. These results allow direct comparison with dissipative susceptibilities in the BHZ model at half filling.

APPLICATION TO AN IDEAL BHZ MODEL

To test the pertinence of using the helical model to deduce properties for edge states in a topological insulator, we consider Bernevig-Hughes-Zhang (BHZ) model on a square lattice. In numerical simulations, we determine the linear response of the system to the time-dependent flux $\Phi(t)$ and show that it reproduces quite well the analytical results from the helical models.

The susceptibility in linear response is entirely determined by the lattice geometry, static Hamiltonian, static flux $\phi$, driving frequency $\omega$, damping rates $\gamma_{nn}$, temperature $T$, and number of particles $N$. The susceptibility is determined in simulations using the linear response Eq. (14). The following subsection will discuss the model, its ingredients, and the various numerical tests used to extract the dissipative susceptibility.
Model

The BHZ model is implemented on a square lattice. The Hamiltonian for the infinite system reads as

$$H = \sum_{\mathbf{k}} c_{\mathbf{k}}^\dagger \mathcal{H}(\mathbf{k}) c_{\mathbf{k}}, \quad (33)$$

where the spin indices for the creation and annihilation operators are implied from the structure of the first-quantized Hamiltonian,

$$\mathcal{H}(\mathbf{k}) = \begin{pmatrix} h(\mathbf{k}) & 0 \\ 0 & h^*(\mathbf{-k}) \end{pmatrix} \quad (34)$$

with

$$h(\mathbf{k}) = A[\sin(k_x)\sigma_1 + \sin(k_y)\sigma_2] + [M - 2B(2 - \cos k_x - \cos k_y)]\sigma_3. \quad (35)$$

The coefficients $A$, $B$, and $M$ are material dependent parameters, which are taken in the simulation without reference to their exact values for the HgTe model. Nonetheless, the parameters must obey a set of constraints in order for the system to be in a topological phase, in which edge states are localized near the two bases of the two bases of the BHZ cylinder: $A \neq 0$ and $M/B \in (0, 8)$. In numerical simulations, the common choice was: $A = M = 1$ and $B = 0.6$.

A finite square patch is cut out along the primitive lattice vectors from the infinite system and it is fashioned into a hollow cylinder. In the cylinder geometry, the coordinate $x$ counts the sites along the base of the cylinder, while $y$ counts the sites along the height of the cylinder (Fig. 4). Due to translational invariance in $x$ direction, the momentum $k$ parallel to the base is a good quantum number. Therefore, in a mixed representation, states can be described by momentum $k$, and real space, height index $y$.

There is a static flux $\phi$ threading the hollow cylinder. This is implemented in the lattice model through the Peierls substitution:

$$k \rightarrow k + \frac{2\pi a}{L_x} \frac{\phi}{\phi_0}. \quad (36)$$

Therefore the current operator for the system threaded by the flux reads

$$J = -i_0 \sum_{k_y} c_{k_y}^\dagger \left[ \cos \left( k + \frac{2\pi \phi}{L_x \phi_0} \right) \sigma_1 \tau_3 - 2\frac{B}{A} \sin \left( k + \frac{2\pi \phi}{L_x \phi_0} \right) \sigma_3 \tau_0 \right] c_{k_y}. \quad (37)$$

Here, we have introduced another spin Pauli matrix $\tau$ relating the two blocks in the Hamiltonian (34).

The static Hamiltonian is diagonalized and one has access to its $4L_xL_y$ eigenstates $\epsilon_{\mathbf{q}}$ and eigenvectors $\{\mathbf{q}\}$.

It is apparent that the BHZ model has a chiral symmetry reflecting the property that each positive energy state has a partner at negative energy. Moreover, any state is at least twofold spin degenerate in zero flux.
For the given parameters, $A = 1$, $B = 0.6$, and $M = 1$, the model is in a topological insulating phase with a bulk gap $\approx 2A$. The edge states connect the bulk bands and traverse the gap. They are states living in the energy bulk gap and having a linear dispersion relation in momentum $k$ and in static flux $\phi$. The flux removes their spin degeneracy except at a set of flux values where the system recovers time-reversal invariance, $\phi = n\phi_0/2$, with $n$ any integer. Additionally, there is a degeneracy due to the fact that there are two edges each accommodating a pair of edge states. Because the absolute slope of the states is constant, the assumption that the diagonal rates are identical $\gamma_{nn} = \gamma_D$ is in effect. In contrast, the bulk states do not vary with the flux $\phi$, and therefore quantities that depend on them will be almost constant in flux.

In the following, we work at (or close to) half filling, deep in the bulk gap. The dissipative susceptibilities in the model at half filling can be directly computed. The current operator matrix elements are available. In contrast to the helical model, there are nondiagonal components, $J_{mn} \neq 0$ ($m \neq n$). At half filling, the chemical potential does not depend on the flux $\mu = 0$ and allows us to introduce temperature in the model only through the equilibrium Fermi-Dirac functions $f_n = f(\epsilon_n(\phi))$. The dependence of the chemical potential on the flux at different fillings will be discussed below.

Subsequently, the susceptibility in the model is computed using Eq. (14). The bulk states contribute little to the diagonal susceptibility since $\chi_D$ depends on states near the Fermi surface. At half filling, only the edge states are energetically close to $\mu = 0$, and they yield the characteristic peak the dissipative susceptibility (see Letter). The edge states do not contribute to the nondiagonal susceptibility $\chi_{ND}$ since the driving frequency cannot induce spin flips or changes in angular momentum. In contrast, since the current has off-diagonal components between the bulk edge states, they yield a large paramagnetic contribution to the nondiagonal dissipative susceptibility. However for large systems, the bulk states and the nondiagonal susceptibility depend little on the flux (see Fig. 5a). Therefore the characteristic peak in the dissipative susceptibility contains information only from the edge states. The peak is maximal when the driving frequency is equal to the diagonal dissipation rate $\gamma_D$. This allows in turn to determine the lifetime of the edge states $\gamma_D^{-1}$.

Parity effects in the lattice model

The total number of available states in the lattice model is $N = 4L_xL_y$. In the body of the Letter, we have worked at half filling $N_{1/2} = 2L_xL_y$, where the number of particles is even. In the present section, we discuss effects due to changes from half filling, while still remaining in the bulk gap.

The particle number enters into the equation through the chemical potential, in the Fermi-Dirac function

$$f_n = \frac{1}{e^{\beta(\epsilon_n-\mu)} + 1}.$$ (38)

The condition that the number of particles is fixed imposes constraints on the chemical potential. In particular, $\mu$ is determined from the normalization condition $N = \text{Tr}[f_n]$, with the trace over all the eigenstates. Thus the chemical potential is generally a function of the flux and the number of particles $N$. Nevertheless for certain constant values, $\mu$ does not vary with the number of particles. When $\mu = n\hbar\omega_0/2$, with $n$ any integer, there is always an even number of particles $N$ in the model. Indeed, since all the energy states are at least twofold degenerate, fixing the chemical potential at $\mu = n\hbar\omega_0/2$ allows one to scan the ground state in an entire period in the energy-flux dispersion, while conserving the particle number. In contrast, a constant $\mu$ cannot capture the cases with odd number of particles in the lattice.

In case of even $N$, when adding or subtracting $2n$ particles at half filling, the current-flux and susceptibility-flux characteristics will shift by half flux quantum $\phi_0/2$ for $n$ odd, and due to gauge invariance, they will be identical for $n$ even.

In case of odd $N$, the situation is more complicated, with a flux-dependent chemical potential. In contrast with the helical model or the BHZ even filling, there is an additional term in the diagonal susceptibility, ensuring the conservation of particle number,

$$\chi_D = -\sum_n J_{nn}(J_{nn} + \frac{\partial \mu}{\partial \phi}) \frac{\partial f_n}{\partial \epsilon_n} \frac{i\omega}{\gamma_{nn} - i\omega}. \quad (39)$$

The chemical potential is obtained by inverting numerically the relation $\text{Tr}[f_n] = N$. All odd particle cases are distinguished by peaks in the susceptibility both at $\phi = 0$ and $\phi = \phi_0/2$. These peaks are smaller in amplitude in comparison with the even cases.

The upshot of the section is that there is a dependence modulo 4 on the number of particles in the model. For even number of particles there will be a shifts from $\phi/\phi_0 = 0$ to $0.5$ of the susceptibility signal. For odd number of particles, the signal is split between peaks at both time-invariant flux values.

The helical model can be used to provide more understanding to these parity effects. It can account and explain the particular features in the response.

Comparison with the helical model

Before studying the dynamical response of the system, let us compare the helical and the BHZ models at zero temperature in the absence of driving. The interesting physics in this case is that of equilibrium persistent currents.
In the BHZ model there are one pair of helical states at both bases of the cylinder. Hence one has to employ two helical models to account for the lattice model. Moreover, in order to compare the models, it is necessary to scale the physical quantities according to the appropriate level spacing. Energies in the helical model are scaled with \( \hbar \omega_0 \), while in the BHZ model, they are scaled with the level spacing at the Fermi surface, \( \frac{2\pi}{a} \), with \( a \) the lattice constant and \( A \) a BHZ model parameter. Similarly all the other characteristic quantities, \( I_0 \) and \( \chi_0 \), are related between the two models. Finally, we work in units where the lattice spacing is dimensionless \( a = 1 \) and \( \hbar = 1 \).

In the numerical simulation for the BHZ model, we obtain a persistent current which close to half filling \( N_{1/2} = 2L_x L_y \) depends modulo 4 on the number of particles. This is represented for relevant cases in Fig. 5(c). \( N_{1/2}, N_{1/2} \pm 1 \) and \( N_{1/2} + 2 \). This can be understood by tracking the particles near zero energy. At half filling \( N_{1/2} \) there are two filled states and two empty states at zero energy and zero flux. The \( 4I_0 \) discontinuity in the current at zero flux indicates that the ground state at negative and positive flux is quite different. At small negative flux there is an imbalance, two filled right-moving states and two empty left-moving states, at positive flux it is the reverse. If all the states are filled in zero-flux, one encounters the same difference in the ground-state moved at half-integer flux \( \phi / \phi_0 = \pm 1/2 \). In other words, at \( N_{1/2} + 2 \), the current-flux characteristic has shifted by \( \phi_0 / 2 \). At odd number of particles \( N_{1/2} \pm 1 \), discontinuities appear at integer and half-integer flux, but the amplitude was halved. This is because the current carried by the almost fourfold degenerate states near \( \phi = 0 \) and \( \phi = \phi_0 / 2 \) is always only \( \pm I_0 \).

The helical model can account perfectly for the persistent currents in the BHZ model. Let us denote the lattice persistent current as \( I_{\text{per}}^{bhz} \). Two helical models are required to mimic the two pair of edge states in the BHZ model. Let us denote by \( I_{\text{per}}(\mu) \) the current for one helical model [24]. As noted before, adding a particle is equivalent to varying the chemical potential in the helical model by half energy spacing \( \Delta \mu = \hbar \omega_0 / 2 \). Then the currents in the BHZ model are obtained by adding or subtracting particles in the two helical models. For the cases with an even number of particles, represented in the Fig. 5(c).

\[
\frac{I_{\text{per}}^{bhz}(N_{1/2})}{I_0^{bhz}} = 2 \frac{I_{\text{per}}(0)}{I_0}, \quad \frac{I_{\text{per}}^{bhz}(N_{1/2} + 2)}{I_0^{bhz}} = 2 \frac{I_{\text{per}}(\hbar \omega_0 / 2)}{I_0}.
\]

Similarly, for the odd particle cases, one extra particle is added or extracted in one of the helical models

\[
\frac{I_{\text{per}}^{bhz}(N_{1/2} \pm 1)}{I_0^{bhz}} = \frac{I_{\text{per}}(0) + I_{\text{per}}(\pm \hbar \omega_0 / 2)}{I_0}.
\]

Thus the helical models explain the modulo 4 pattern in the persistent current simulations.

The above arguments hold qualitatively also at higher temperature in the presence of driving. For even particle cases, the signal in the susceptibility is correctly given by doubling the signal in the helical cases. For the odd particle numbers, the BHZ signal is not exactly given by the sum of two helical models shifted by \( \Delta \mu = \hbar \omega_0 / 2 \). The direct sum of helical models would predict a signal split at both \( \phi = 0 \) and \( \phi = \phi_0 / 2 \), and half the size of the signal in the even case. In the simulation we see that indeed the signal is split, but its amplitude is 50% higher than the predicted signal under the above simple argument (Fig. 5(d)).

Throughout the body of the Letter, we have worked at half filling with the number of particles \( N_{1/2} = 2L_x L_y \). In this case the chemical potential \( \mu = 0 \) is constant as a function of the static flux.

**Scalar disorder**

To test the robustness of the signal to addition of disorder, the initial BHZ model under flux is enriched with scalar disorder on-site disorder

\[
H' = w_j \sum_j c_j^\dagger \sigma_0 \tau_0 c_j.
\]

The on-site disorder \( w \) is a random variable, uniformly distributed in the interval \([-W/2, W/2]\), where \( W \) is the disorder amplitude. In simulations, the disorder is taken in units of model parameter \( A \).

As a proof of principle, we consider ideally thin cylinders \( L_y = 8a \). This allows exploring long cylinder lengths averaged over many disorder realizations and obtaining readily the diagonal dissipative response. The system shows sensitivity to disorder, and the value of the disorder average of the susceptibility decreases continuously with disorder. Nevertheless, the dissipative susceptibility never
Constant Zeeman field

Finally, the BHZ model is subjected to a constant transversal Zeeman field which gaps the edge states in zero flux. The field mixes the spin states and leads to the observed decrease in the diagonal susceptibility at zero flux.

The perturbation added to the Hamiltonian is constant for all the sites in the lattice

$$H''' = V \sum_j c_j^\dagger \sigma_0 \tau_1 c_j,$$

where $j$ runs over all the sites in the cylinder and $V$ is a constant Zeeman energy. The Zeeman term anticommutes with the BHZ Hamiltonian (34) and mixes the spin states. To compare the response in the helical and BHZ models, the field is scaled with the respective energy level spacing near the Fermi energy, $\hbar \omega_0$. This yields again agreement between the diagonal response in the two systems (Fig. 3 in the Letter). Very small fields with respect to the bulk gap can still create infinitesimal gaps in the helical edge states at zero flux. This affects the states infinitesimally close to the zero flux by creating a vanishing diagonal susceptibility. This is reflected as a dip of very small width in the diagonal signal.

In the presence of a magnetic field the effect of large coherence damping rates $\gamma_{ND}$ can wash out the diagonal susceptibility signal. The peak at zero flux in the non-diagonal dissipative susceptibility, due to spin mixing of the edge channels, dominates the diagonal susceptibility dip predicted from the vanishing of the level current at the time-reversal invariant fluxes. Additionally, interband transitions between the bulk states is enhanced at larger damping rates $\gamma_{ND}$. The bulk states contribution remains almost constant in flux and it adds to uniformly increase the overall dissipative susceptibility (Fig. 7).

vanishes as in the case of scalar disorder in a nonrelativistic fermion systems.