Limiting temperature from a parton gas with power-law tailed distribution

T. S. Biró∗ and A. Peshier

Institute for Theoretical Physics, University of Giessen, D 35392 Heinrich-Heine-Ring 16, Giessen, Germany

(Dated: June 21, 2005)

We combine Tsallis distributed massless partons to an effective thermal prehadron spectrum by folding. A limiting temperature and a mass spectrum combined of three exponentials emerge by this procedure. Meson and baryon resonance spectra have different polynomial prefactors.

PACS numbers: 02.70.Ns, 05.90.+m

The idea to treat the bulk of mesonic and baryonic resonances as a statistical system stems from Rolf Hagedorn[1]. The mass spectrum is exponential, multiplied with an originally negative power of the mass, which fact gives rise to a maximal, limiting temperature, \( T_H \), when heating a hadron gas. The exponential factor with the main part of the hadron energy, i.e. the rest mass, is the most essential feature in this assumption, power factors and depending on them the quantitative fit to the Hagedorn temperature, \( T_H \), vary. A recent compilation of hadron resonances by Broniowski, Florkowski and Glozman shows that this idea works well beyond the data used for original fits, although baryons and mesons seem to follow a slightly different line[2]. Explanations for the origin of this exponential mass spectrum date back to the MIT bag model, where this behavior was demonstrated by Kapusta[3]. Bugaev and others point out that such a system is a perfect thermostat forcing the same temperature to any finite system in thermal contact[4]. The picture of Hagedorn resonances also fit well to lattice QCD results on the equation of state[5] and is under consideration in recent microscopic models of quark matter dehadrinization in heavy-ion collisions[6].

In this paper we demonstrate how another approach may support the occurrence of an essentially exponential mass spectrum of hadrons. We consider massless thermal partons, but with a generalized equilibrium distribution with power-law tail asymptotics. This distribution will then be folded to mesonic or baryonic one-particle distributions of energy, or - at zero rapidity - transverse mass and transverse momenta, respectively. Such stationary distributions are conform with the basic laws of statistical physics and may be considered as a description of the intermediate \( (p_T \approx 1-4 \text{ GeV}) \) part of the observed hadron spectra[7,8]. This approach delivers qualitatively interesting results, in particular a characteristic difference between the mesonic and baryonic mass spectrum.

There are numerous occurrences of power-law tailed statistical distributions in nature. In particular hadron transverse spectra stemming from elementary particle or heavy ion collisions can be well fitted at mid rapidity by a formula reflecting \( m_T \)-scaling[2,10,11,12,13,14,15]:

\[
    f(p_T) \sim (1 + m_T/E_c)^{-v} .
\]

Interpreting these spectra in terms of the single particle energy, one considers \( E = m_T = \sqrt{p_T^2 + m^2} \) for a relativistic particle with mass \( m \). This formula describes a Tsallis distribution, which was conjectured earlier by using axiomatic thermodynamical arguments[16]. This differs from the traditional interpretation of such spectral tails in particle physics, when these are treated as non-equilibrium phenomena. The very high-\( p_T \) part, expected to stem from jet fragmentation, may still allow for a statistical interpretation below \( p_T = 6-8 \text{ GeV} \).

Distributions with a power-law like tail are encountered in several different statistical models[5,17,18,19,20,21,22,23,24]. They are investigated as generic distributions in the non-extensive thermodynamics[25,26,27], based on a generalization of the Boltzmann-entropy, encountered first in informatics problems[16,28]. Without being able to exclude the non-equilibrium interpretation of the power-law tail in particle physics, we explore here some consequences of a stationary state with Tsallis distributed extreme relativistic particles (massless partons).

We consider a massless parton gas with binary collisions obeying a general, non-extensive energy composition rule[21,22], \( E_{12} = h(E_1, E_2) \). Whenever this rule is associative, the one-particle energy can be mapped onto an additive quasi-energy \( X(E_{12}) = X(E_1) + X(E_2) \), with the help of a strict monotonic function, obeying \( X(0) = 0 \). For the Tsallis stationary distribution one uses

\[
    X(E) = \frac{1}{a} \ln (1 + aE) ,
\]

with an energy scale \( a = 1/E_c \), related to the microscopic dynamics. This gives rise to the following non-extensive composition rule:

\[
    h(x,y) = x + y + axy .
\]

The stationary one-particle distribution under these conditions becomes

\[
    f(E_i) = \frac{1}{Z_1(\beta)} \exp(-X(E_i)/T) ,
\]

with a temperature \( T \) determined by the conserved total (quasi) energy \( X(E_{tot}) \) and particle number. The

∗On leave from KFKI RMKI Budapest, Hungary
the non-extensive thermodynamics, in certain kinetic models leading to a stationary state of the sum of the quasi-energies, systems.

\[ F_N(E) = \Delta EX'(E) \prod_{i=1}^{N} d\Gamma_i \delta(X(E)) - \sum_{i=1}^{N} X(E_i) f(E_i). \]

Here \( X'(E) \) stands for the derivative of the strict mono-tomic mapping function \( X(E) \), \( \Delta E \) is the width of the \( N \)-particle energy shell and the \( d\Gamma_i \) integration measures refer to the one-particle phase space factors. The factORIZATION is usually a good approximation for values of \( N \) being still negligible besides the total number of particles. A check of this formula for \( N = 1 \) expresses the one-particle energy distribution as being proportional to the one-particle phase space distribution:

\[ F_1(E) = \frac{V}{2\pi^2} \beta^2 \Delta E f(E). \]

The \( N \)-particle quasi-energy distribution we are seeking for is then given in a form normalized to one in an energy shell of width \( \Delta E \) as

\[ g_N(X) = \Delta E \int_{-\infty}^{+\infty} ds \frac{e^{isX}}{2\pi} \prod_{j=1}^{N} \left[ \int d\Gamma_j f(E_j) e^{-isX(E_j)} \right]. \]

Utilizing now the equilibrium one-particle quasi-energy distribution, we obtain

\[ \int d\Gamma_j f(E_j) e^{-isX(E_j)} = \frac{Z_1(\beta + is)}{Z_1(\beta)}. \]

The \( N \)-particle quasi-energy distribution we are seeking for is then given in a form normalized to one in an energy shell of width \( \Delta E \) as

\[ g_N(X) = \Delta E \int_{-\infty}^{+\infty} ds \frac{e^{isX}}{2\pi} \left( \frac{Z_1(\beta + is)}{Z_1(\beta)} \right)^N. \]

Such integrals may be evaluated in a saddle point approxi-mation, which is good for large \( N \) as long as no singular-ity has been encountered in the expansion of \( \ln Z_1(\beta + is) \). The result is a Gaussian distribution in \( X(E) \). Irrespective to this approximation, as long as the factorization assumption is valid, the exact expectation value is given by

\[ \langle X(E) \rangle = -N \frac{\partial}{\partial \beta} \ln Z_1(\beta), \]

and the square width by

\[ \delta X(E)^2 = N \frac{\partial^2}{\partial \beta^2} \ln Z_1(\beta). \]

In case of the Tsallis distribution using \( \Gamma \) for massless particles one obtains

\[ Z_1(\beta) = \frac{V_d}{(2\pi)^d} (d - 1)! \prod_{k=1}^{d} B_k^{-1} \]

with spatial volume \( V_d \) in \( d \) dimensions and with the factors \( B_k = \beta - ka \). The expectation value of the quasi-energy per particle becomes

\[ \epsilon_1 = \frac{\langle X(E) \rangle}{N} = \sum_{k=1}^{d} B_k^{-1} \]

while the unit square width contribution is given by

\[ \delta_1 = \frac{\delta X^2}{N} = \sum_{k=1}^{d} B_k^{-2}. \]

All these expressions lose their conventional interpretability for \( \beta \leq da \) in \( d \) dimensions. The value, \( T_H = 1/(da) \) is a limiting temperature for positive values of the parameter \( a \), at which the physically relevant quasi-energy per particle diverges, so there is no use of further heating at this temperature. More and more energy given to the system would raise the temperature less and less. For \( a > 0 \), i.e. for attractive correction, the energy per particle is limited by \( (1 + 1/2 + 1/3)E_c \) from above, but any temperature may occur (cf. Fig. 1).

The \( N \)-particle energy distribution can be obtained also exactly in this case. The Fourier-integral \( \Gamma \) has \( N \)-fold poles at the values \( s_k = -iB_k \) for each directional degree of freedom \( k = 1, \ldots, d \). The evaluation of such integrals is somewhat involved in a general number of dimensions, so we restrict our further analysis to the cases \( d = 1 \) and \( d = 3 \). For a one-dimensional Tsallis distribution we obtain

\[ g_{N+1}(X) = \Delta E B_1 \left( \frac{B_1 X}{N!} \right)^N \exp (X/E_c - X/T). \]

In this case a Hagedorn exponential emerges with the limiting temperature \( T_H = E_c \). Considering on the other hand \( F_{N+1}(E) = g_{N+1}(X) X'(E) \), i.e. the \( N + 1 \)-particle distribution of the (not conserved) naive energy expression only, the exponentially growing factor does not occur. It is due to \( X'(E) = e^{-aX} \) for the Tsallis distribution:

\[ F_{N+1}(E) = \Delta E B_1^{N+1} \left( \frac{\ln(1 + aE)}{aN} \right)^N \frac{1}{(1 + aE)^{-1/aT}}. \]
However, still $\beta > a$ or $T < T_H$ must be satisfied, and for positive $a$ values increasing quasi-energy per particle does not raise the temperature above $T_H$.

The corresponding expression in $d = 3$ dimensions is more involved. There occur three different exponential factors with lowest limiting temperature $T_H = E_c/3$. The general dependence on $\beta$ can be factorized out by shifting the integration variable $s$ to $s + i\beta$:

$$g_{N+1}(X) = \frac{V^{N+1} \Delta E}{(\pi^2 Z^1(\beta))^{N+1}} e^{-\beta X} \Phi_N(X),$$

with $\Phi_N(X)$ depending on the dynamical input parameters only, but not on the temperature $T = 1/\beta$:

$$\Phi_N(X) = \int \frac{ds}{2\pi} e^{isX} \left( \prod_{k=1}^3 B_k^{-1}(is) \right)^{N+1}.$$  

Factorizing out the ideal thermal factor $e^{-\beta X}/Z_N$, the rest can be regarded as the mass spectrum of the $N$-parton system at $X = m$ and $\Delta E = \Delta m$:

$$\rho_N(m) = \left( \frac{V}{\pi^2} \right)^N \Phi_{N-1}(m).$$

The functions $\Phi_N(X)$ obey the recursion rule

$$\Phi_N(X) = \int_0^X dt \Phi_{N-1}(t) \Phi_0(X - t)$$

with the starting point of the recursion,

$$\Phi_0(X) = \frac{1}{2a^2} \left( e^{3aX} - 2e^{2aX} + e^{aX} \right).$$

Particular important cases are $N = 0$ (partons), $N = 1$ (mesons or diquarks) and $N = 2$ (direct baryons). Besides the already known $\Phi_0(X)$ (eq. 21) we obtain

$$\Phi_1(X) = \frac{(aX - 3)e^{3aX} + 4aX e^{2aX} + (aX + 3) e^{aX}}{4a^5}$$

and

$$\Phi_2(X) = \frac{1}{16a^8} \left[ ((aX)^2 - 9aX + 24) e^{3aX} - 8 ((aX)^2 + 6) e^{2aX} + ((aX)^2 + 9aX + 24) e^{aX} \right].$$

For any $N$ the result contains three exponentials giving rise to a lowest limiting temperature of $T_H = E_c/3 = 1/3a$ for positive values of the parameter $a$.

Fig. 2 shows the non-degenerate, integrated mass spectra. Values published on the Particle Data Group homepage are summed up in mass histograms. The respective numbers are fitted as $N_M = 1 + A f_1((m - m_M)/3T_H)$ and $N_B = 1 + A^2 f_2((m - m_B)/3T_H)$ with $A = V/ \pi^2 a^3$ and the integral functions $f_n(x) = \int_0^x \Phi_n(t) dt$. The fits to the data are most sensitive to the value of $T_H$, which however may be compensated by changing the assumed volume, $V$. Keeping $m_M = 0.14$ GeV and $m_B = 0.94$ GeV, only $A$ and $T_H$ are varied. Our best fit gives a relatively high value, $T_H = 0.35$ GeV (meaning $E_c = 1.05$ GeV) and a volume of $V = 261 f m^3$ (a sphere with a radius of 4 fm, or a box with a length of 6.4 fm). Above the masses where the data seem to deviate from the fast growing part, the fit cannot be followed any more. According to ref. 2, newest data raise the experimental curve higher. Our idea, different from both the string and bag model consideration, seem to agree with the difference between the meson and baryon mass spectra, as well as with a polynomial upcurving of the baryon spectrum.

In conclusion we pointed out that Tsallis distributed massless partons can be combined to an effective mesonic
and baryonic mass spectrum by considering the conserved quasi-energy as the hadron energy. Besides an ideal thermal factor, $e^{-\beta X/Z_N}$, a further energy dependent factor results from the folding of parton distributions. It can be regarded as a thermal (pre)hadron mass spectrum emerging from a statistical hadronization picture. The prediction of this folding, while having two parameters (a volume and the Hagedorn temperature), gives an acceptable qualitative agreement with the known hadronic mass spectrum. In this picture a natural difference emerges between mesonic and baryonic resonances due to their different foldness by parton coalescence. The characteristic temperature, $T_H = 1/3a \approx 350$ MeV is a limiting temperature: one cannot increase the temperature above this value, not even with an infinite amount of energy. The parameter $E_c = 1/a = 3T_H \approx 1.05$ GeV provides at the same time the scale where the power-law tail of individual pr-t-spectra starts to dominate the exponential part, and it is intimately related to the typical pair-interaction energy, due to $h(E_1,E_2) - E_1 - E_2 = E_1E_2/E_c$.

Acknowledgment

This work has been supported by BMBF, by the Hungarian National Science Fund OTKA (T49466) and by DFG due to a Mercator Professorship for T.S.B.

[1] R. Hagedorn: Nucl.Phys.B 24, 93, 1970; C. J. Hamer, S. C. Frautschi: Phys.Rev.D 24, 2125, 1971; R. Hagedorn, J. Ranft: Nucl.Phys.B 48, 157, 1972; R. Hagedorn: Riv. Nuovo Cimento 6N 10, 1, 1984; R. Hagedorn, K. Redlich: Z.Phys.C 27, 541, 1985; G. J. Burgers, C. Fugelsang, R. Hagedorn, V. Kuvshinov: Z.Phys.C 46, 465, 1990; J. Letessier, J. R. Rafelski, A. Tounsi: Phys.Lett.B 328, 499, 1994.

[2] W. Broniowski, W. Florkowski, L. Ya. Glozman: Phys.Rev.D 117503, 2004.

[3] A. Chodos et.al., Phys.Rev.D 9, 3471, 1974; J. I. Kapusta, Phys.Rev.D 23, 2444, 1981; Nucl.Phys.B 196, 1, 1982.

[4] D. B. Blaschke, K. A. Bugaev: Prog.Part.Nucl.Phys. 53, 197, 2004; L. G. Moretto, K. A. Bugaev, J. B. Elliott, L. Phair: nucl-th/0504010, K. A. Bugaev, J. B. Elliott, L. G. Moretto, L. Phair: hep-ph/0504011.

[5] A. Tawfik, Phys.Rev.D 71, 054502, 2005; K. Redlich, F. Karsch, A. Tawfik: J.Phys.G 30, s1271, 2004; F. Karsch, K. Redlich, A. Tawfik: Eur.Phys.J.C 29, 549, 2003; F. Karsch, K. Redlich, A. Tawfik: Phys.Lett.B 571, 67, 2003.

[6] C. Greiner, P. Knoch-Steinheimer, F. M. Liu, I. A. Shovkovy, H. Stocker: J.Phys.G 31, s725, 2005; hep-ph/0412005.

[7] T. S. Biro, G. Purcsel, M. Muller, Acta Phys.Hung. A 21, 85, 2004.

[8] T. S. Biro, G. Györgyi, A. Jakovác, G. Purcsel: J.Phys.G 31, s759, 2005; hep-ph/0409157.

[9] PHENIX collaboration: J. M. Heuer et.al. Acta Phys. Hung. Heavy Ion Phys. 15, 291, 2003; K. Adcox et.al. Phys.Rev.C 69, 024904, 2004; S. S. Adler et.al. Phys.Rev.C 69, 034909, 034910, 2004; K. Reygers et.al. Nucl.Phys.A 734, 74, 2004.

[10] STAR collaboration: C. Adler et.al. Phys.Rev.Lett. 87, 112303, 2001; J. Adams et.al. Phys.Rev.Lett. 91, 172302, 2003; A. A. P. Suaide et.al. Braz.J.Phys. 34, 300, 2004; R. Witt, nucl-ex/0403021.

[11] ZEUS collaboration: Eur.Phys.J. C11, 251, 1999.

[12] I. Bediaga, E. M. F. Curado, J. M. de Miranda, Z.Phys.C 22, 307, 1984; Z.Phys.C 73, 229, 1997.

[13] M. Gazdzicki, M. Gorenstein, Phys.Lett.B 517, 250, 2001.

[14] C. Beck, Physica A 286, 164, 2000, cond-mat/0301354.

[15] J. Schaffner-Bielich, D. Kharzeev, L. McLerran, R. Venugopalan, Nucl.Phys.A 705, 494, 2002.

[16] C. Tsallis, J.Stat.Phys. 52, 50, 1988; Physica A 221, 277, 1995; Braz.J.Phys. 29, 1, 1999; F. Prato, C. Tsallis, Phys.Rev.E 60, 2389, 1999; V. Latora, A. Rapisarda, C. Tsallis, Phys.Rev.E 64, 056134, 2001; Physica A 305, 129, 2002.

[17] C. Anteneodo, C. Tsallis, Physica A 324, 89, 2003; T. S. Biró, A. Jakovác, Phys.Rev.Lett.94:132302, 2005.

[18] G. Wilk, Z. Włodarczyk, Physica A 305, 227, 2002; Chaos, Solitons and Fractals 13, 581, 2002; Phys.Rev.Lett. 84, 2770, 2000.

[19] A. Bialas, Phys.Lett.B 466, 301, 1999.

[20] W. Florkowski, Acta Pol. B 35, 799, 2004.

[21] T. Kodama, H.-T. Elze, C. E. Augier, T. Koide, Europhysics Letters 70, 439, 2005.

[22] T. J. Sherman, J. Rafelski, Lecture Notes in Physics 633, 377, 2003.

[23] D. Walton, J. Rafelski, Phys.Rev.Lett. 84, 31, 2000.

[24] S. H. Hansen, Cluster temperatures and non-extensive thermo-statistics, astro-ph/0501393.

[25] C. Tsallis, E. P. Borges, cond-mat/0305121; C. Tsallis, E. Bragati, cond-mat/0306506; C. Tsallis, Braz.J.Phys. 29, 1, 1999; A. Plastino, A. R. Plastino, Braz.J.Phys. 29, 50, 1999.

[26] Q. A. Wang, A. Le Mêhauté, J.Math.Phys. 43, 5079, 2002; Q. A. Wang, Chaos, Solitons and Fractals 13, 581, 2002; Eur.Phys.J.B 26, 357, 2002.

[27] L. Borland, Phys.Rev.E 57, 6634, 1998; D. H. Zanette, Chaos, Solitons and Fractals 13, 581, 2002; Chaos, Solitons and Fractals 13, 581, 2002; Eur.Phys.J.B 26, 357, 2002.

[28] A. Plastino, A. R. Plastino, Braz.J.Phys. 29, 1, 1999; P. Prato, C. Tsallis, Braz.J.Phys. 29, 1, 1999; A. Plastino, A. R. Plastino, Braz.J.Phys. 29, 50, 1999.

[29] A. A. P. Suaide et.al. Braz.J.Phys. 34, 300, 2004; hep-ph/0409157.

[30] ZEUS collaboration: Eur.Phys.J. C11, 251, 1999.

[31] L. G. Moretto, K. A. Bugaev, J. B. Elliott, L. Phair: nucl-th/0504010.

[32] A. Tawfik, Phys.Rev.D 71, 054302, 2005; K. Redlich, F. Karsch, A. Tawfik: J.Phys.G 30, s1271, 2004; F. Karsch, K. Redlich, A. Tawfik: Eur.Phys.J.C 29, 549, 2003; F. Karsch, K. Redlich, A. Tawfik: Phys.Lett.B 571, 67, 2003.

[33] C. Greiner, P. Knoch-Steinheimer, F. M. Liu, I. A. Shovkovy, H. Stocker: J.Phys.G 31, s725, 2005; hep-ph/0412005.

[34] T. S. Biró, G. Purcsel, B. Müller, Acta Phys.Hung. A 21, 85, 2004.