GINIBRE INTERACTING BROWNIAN MOTION IN INFINITE DIMENSIONS IS SUB-DIFFUSIVE

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Abstract. We prove that the tagged particles of infinitely many Brownian particles in \( \mathbb{R}^2 \) interacting via a logarithmic (two-dimensional Coulomb) potential with inverse temperature \( \beta = 2 \) are sub-diffusive. The associated delabeled diffusion is reversible with respect to the Ginibre random point field, and the dynamics are thus referred to as the Ginibre interacting Brownian motion. If the interacting Brownian particles have interaction potential \( \Psi \) of Ruelle class and the total system starts in a translation-invariant equilibrium state, then the tagged particles are always diffusive if the dimension \( d \) of the space \( \mathbb{R}^d \) is greater than or equal to two. That is, the tagged particles are always non-degenerate under diffusive scaling. Our result is, therefore, contrary to known results. The Ginibre random point field has various levels of geometric rigidity. Our results reveal that the geometric property of infinite particle systems affects the dynamical property of the associated stochastic dynamics.

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1. Introduction

We consider a system \( \mathbf{X} = (X^i)_{i \in \mathbb{N}} \) of infinitely many Brownian particles moving in \( \mathbb{R}^d \) and interacting through a translation-invariant, two-body potential \( \Psi(x) \). \( \mathbf{X} \) is described by the infinite-dimensional stochastic differential equation (ISDE)

\[
X^i_t - X^i_0 = B^i_t - \frac{\beta}{2} \int_0^t \sum_{j \neq i} \nabla \Psi(X^i_u - X^j_u) du,
\]

where \( B^i \) (\( i \in \mathbb{N} \)) denotes independent \( d \)-dimensional Brownian motions and \( \beta \geq 0 \) is the inverse temperature, which is taken as a constant. The solution \( \mathbf{X} \) provides a description of the interacting Brownian motion \([12, 13, 28, 4, 33]\).

The delabeled process \( \mathbf{X} = \{X_t\}_{t \in [0, \infty)} \) associated with \( \mathbf{X} \) is given by

\[
X_t = \sum_{i \in \mathbb{N}} \delta_{X^i_t},
\]

where \( \delta_a \) is the delta measure at \( a \in \mathbb{R}^d \) and \( \mathbf{X} \) is a configuration-valued process by definition.

We suppose that the delabeled process \( \mathbf{X} \) is reversible with respect to a translation-invariant equilibrium state \( \mu^{\Psi, \beta} \). In many cases, we expect the existence of such an equilibrium state. For example, if \( \Psi \) is a Ruelle-class potential (i.e., it is super-stable and regular in the sense of Ruelle), then the associated translation-invariant canonical Gibbs measures exist. Here, super-stability is a condition that prevents
infinitely many particles agglomerating in a bounded domain, and regularity means the integrability of interactions at infinity and therefore provides an interpretation of the Dobrushin–Lanford–Ruelle equation [27].

We investigate the tagged particles $X^i = \{X^i_t\}_{t \in [0, \infty)}$ in the system. Although the total delabeled system $X$ is a $\mu^{\Psi, \beta}$-reversible Markov process, each tagged particle $X^i$ is a non-Markov process because the total system affects it in a complicated fashion. Applying the Kipnis–Varadhan theory, it can nevertheless be seen that the motion of each tagged particle always reverts to Brownian motion under diffusive scaling [7, 3, 24, 17, 32]. That is,

$$\lim_{\epsilon \to 0} \epsilon X_{t/\epsilon}^i = \sigma B_t.$$  

The constant may depend on the initial configuration, and we therefore introduce $\alpha$, defined as the average of $(1/2)\sigma^2$. The constant matrix $\alpha$ is called the self-diffusion matrix.

Once such convergence of motion is established under this fairly general situation, it is natural and important to inquire about the positivity of the self-diffusion matrix $\alpha$.

Historically, there was a conjecture that $\alpha = 0$ for hard-core potentials and sufficiently large activities in multi-dimensional grand canonical Gibbs measures; cf. [1]. This conjecture seemed to be plausible because the presence of a hard core should suppress the motion of tagged particles. However the contrary was proved in [18]. Indeed, $\alpha$ is always positive definite if $d \geq 2$ and $\Psi$ is a Ruelle-class potential corresponding to a hard core.

The discrete similarity is a tagged particle problem of exclusion processes in $\mathbb{Z}^d$. As for the simple exclusion processes, Kipnis–Varadhan [11] proved that $\alpha$ is always positive definite except for the nearest neighborhood jump in one space dimension. Spohn [29] proved that $\alpha$ is always positive definite for general exclusion processes with Ruelle-class potentials when $d \geq 2$.

We note that the set of the Gibbs measures with Ruelle-class potentials is the standard class of random point fields in both continuous and discrete spaces. We hence consider, on good grounds, that it is reasonable to believe that self-diffusion matrices are always positive definite for $d \geq 2$. Nevertheless, we present the antithesis in the present paper.

The Ginibre interacting Brownian motion $X = (X^i)_{i \in \mathbb{N}}$ is a system of infinite-many Brownian particles moving in $\mathbb{R}^2$ and interacting via the two-dimensional Coulomb potential $\Psi(x) = -\log |x|$ with inverse temperature $\beta = 2$. The stochastic dynamics $X = \{X_i\}$ are then described by the ISDE

$$X^i_t - X^i_0 = B^i_t + \int_0^t \lim_{R \to \infty} \sum_{|X^i_u - X^j_u| < R, j \neq i} \frac{X^j_u - X^j_i}{|X^i_u - X^j_u|^2} \, du.$$  

The associated delabeled process $X$ is reversible with respect to the Ginibre random point field $\mu_{\text{Gin}}$. By definition, $\mu_{\text{Gin}}$ is a random point field on $\mathbb{R}^2$ for which the $n$-point correlation function $\rho_{\text{Gin}}^n$ with respect to the Lebesgue measure is given by

$$\rho_{\text{Gin}}^n(x_1, \ldots, x_n) = \det([K_{\text{Gin}}(x_i, x_j)])_{1 \leq i, j \leq n} \quad \text{for each } n \in \mathbb{N},$$

where $K_{\text{Gin}} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}$ is the exponential kernel defined by

$$K_{\text{Gin}}(x, y) = \pi^{-1} e^{-\frac{|x|^2}{2} - \frac{|y|^2}{2}} e^{\bar{x} \bar{y}}.$$
Here, we identify $\mathbb{R}^2$ as $\mathbb{C}$ by the correspondence $\mathbb{R}^2 \ni x = (x_1, x_2) \mapsto x_1 + \sqrt{-1}x_2 \in \mathbb{C}$, and $\bar{y} = y_1 - \sqrt{-1}y_2$ gives the complex conjugate of $y$ under this identification.

It is known that $\mu_{\text{Gin}}$ is translation and rotation invariant. Furthermore, $\mu_{\text{Gin}}$ is tail trivial \cite{23}. Let $S$ be the configuration space over $\mathbb{R}^2$ defined by \cite{13}. A measurable map $l: S \to (\mathbb{R}^2)^N$ defined for $\mu_{\text{Gin}}$-a.s. $s \in S$ called a labeling map if $l(s) = (l_i(s))_{i \in \mathbb{N}}$ satisfies $\sum_i \delta_{l_i(s)} = s$. A typical example of $l$ is $|l_i(s)| < |l_{i+1}(s)|$ for all $i \in \mathbb{N}$. This label is well defined for $\mu_{\text{Gin}}$-a.s. $s$.

For $\mu_{\text{Gin}} \circ \Gamma^{-1}$-a.s. $s = (s_i)_{i \in \mathbb{N}}$, \cite{11} has a solution $X = (X^i)_{i \in \mathbb{N}}$, of which the delabeled process $X$ is $\mu_{\text{Gin}}$-reversible \cite{20}. The ISDE has a unique strong solution starting at $\mu_{\text{Gin}} \circ \Gamma^{-1}$-a.s. $s$ under a reasonable constraint. We refer to Section 7 in \cite{26} and references therein for the existence and uniqueness of solutions of \cite{11}.

Let $P_s$ be the distribution of the solution $X$ of \cite{11} staring at $s \in (\mathbb{R}^2)^N$.

**Theorem 1.1.** For each $i \in \mathbb{N},$

$$\lim_{\epsilon \to 0} \epsilon X^i_{/\epsilon^2} = 0 \text{ weakly in } C([0, \infty); \mathbb{R}^2)$$

under $P_s$ in $\mu_{\text{Gin}} \circ \Gamma^{-1}$-probability.

**Remark 1.1.** The claim in Theorem \cite{11} means that for any bounded continuous function $F$ on $C([0, \infty); \mathbb{R}^2)$ and $\kappa > 0$, it holds that for each $i \in \mathbb{N},$

$$\lim_{\epsilon \to 0} \mu_{\text{Gin}} \circ \Gamma^{-1} \left( \left\{ s \in (\mathbb{R}^2)^N; \left| \int F(\epsilon X^i_{/\epsilon^2}) dP_s - F(0) \right| > \kappa \right\} \right) = 0.$$ 

Here $0 = \{0_t\}$ of $F(0)$ denotes the constant path with value 0.

Recently, it has become clear that the Ginibre random point field has various geometric rigidities, specifically a small variance property according to Shirai \cite{31}, the number rigidity according to Ghosh and Peres \cite{6}, and the dichotomy of reduced Palm measures \cite{25}. These properties are different from those of the Poisson random point field and the Gibbs measure with a Ruelle-class potential, which have been extensively studied as the standard class of random point fields appearing in statistical physics.

These geometric properties affect the dynamical properties. Indeed, from these rigidities, our theorem demonstrates that geometric rigidities yield dynamical rigidity in the sense of the sub-diffusivity of each tagged particle of the natural infinite-particle system given by \cite{11}.

We present results of simulations of the Poisson random point field and the Ginibre random point field in [figure].

\[\text{[figure]}\]
\( \alpha[\mu_{\text{Gin}}] = O \), we use the geometric theorems of the Ginibre random point field mentioned above.

We now explain the critical geometric rigidities that we use to prove Theorem 1.1.

Let \( \mu_{\text{Gin},x} \) be the reduced Palm measure conditioned at \( x = \sum_{i=1}^{m} \delta_{x_i} \). By definition, \( \mu_{\text{Gin},x} \) is the regular conditional probability of \( \mu_{\text{Gin}} \) conditioned at \( \{x_1, \ldots, x_m\} \) such that

\[
\mu_{\text{Gin},x}(ds) = \mu_{\text{Gin}}(\cdot - \sum_{i=1}^{m} \delta_{x_i} | s(\{x_i\}) \geq 1 \text{ for } i = 1, \ldots, m).
\]

In [25], we proved the following dichotomy of the reduced Palm measures of \( \mu_{\text{Gin}} \).

**Lemma 1.1** ([25, Theorem 1.1]). Assume that \( x(\mathbb{R}^2) = m \) and \( y(\mathbb{R}^2) = n \) for \( m, n \in \{0\} \cup \mathbb{N} \), where we take \( \mu_{\text{Gin},x} = \mu_{\text{Gin}} \) if \( m = 0 \). The following then holds.

1. If \( m \neq n \), then \( \mu_{\text{Gin},x} \) and \( \mu_{\text{Gin},y} \) are singular relative to each other.
2. If \( m = n \), then \( \mu_{\text{Gin},x} \) and \( \mu_{\text{Gin},y} \) are mutually absolutely continuous.

We find that \( \mu_{\text{Gin},x} \) is continuous in \( x = (x_1, \ldots, x_m) \in (\mathbb{R}^2)^m \), where \( x = \sum_{i=1}^{m} \delta_{x_i} \). This follows from the explicit formula of the Radon–Nikodym density \( d\mu_{\text{Gin},x}/d\mu_{\text{Gin}} \) in Lemma 1.2.

**Lemma 1.2** ([25, Theorems 1.2]). Let \( x, y \in S_m \) for \( m \in \mathbb{N} \). Then

\[
\frac{d\mu_{\text{Gin},x}}{d\mu_{\text{Gin},y}} = \frac{1}{Z_{x,y}} \lim_{r \to \infty} \prod_{|s_j| < r} \frac{|x - s_j|^2}{|y - s_j|^2},
\]

where \( |x - s_j| = \prod_{i=1}^{m} |x_i - s_j| \). The normalization constant \( Z_{x,y} \) is given by

\[
Z_{x,y} = \frac{\det[K_{\text{Gin}}(x_i, x_j)]_{i,j=1}^{m}}{\det[K_{\text{Gin}}(y_i, y_j)]_{i,j=1}^{m}} |\Delta(y)|^2 \left| \Delta(x) \right|.
\]

Here, \( Z_{x,y} \) is a unique smooth function on \( (\mathbb{R}^2)^m \times (\mathbb{R}^2)^m \) defined by continuity when the denominator has vanished. Furthermore, \( \Delta \) denotes the difference product for \( m \geq 2 \) and \( \Delta(x) = 1 \) for \( m = 1 \).

Intuitively, the dichotomy in Lemma 1.1 indicates the following phenomena. Suppose that we remove a finite unknown number \( m \) of particles \( \{s_i_1, \ldots, s_i_m\} \) from a sample point \( s = \sum \delta_{s_i} \) of \( \mu_{\text{Gin}} \). We then deduce the number \( m \) from information of \( s_\phi := \sum_{i \in \mathbb{N} \setminus \{i_1, \ldots, i_m\}} \delta_{s_i} \). Such a structure is the same as periodic random point fields. Although a sample point \( s \) of \( \mu_{\text{Gin}} \) has enough randomness as seen from the simulation in Fig 1, we can infer the number of the removed particles exactly for \( \mu_{\text{Gin}} \)-a.s. \( s \).

For \( d = 1 \), we proved that the non-collision of particles always implies sub-diffusivity [22]. (See also [8, 30].) Using the variational formula of the self-diffusion constant, the proof in [22] relies on the construction of a sequence of functions that reduces this constant to zero. This crucially uses the total order structure of non-collision particle systems in \( \mathbb{R} \), which is specific in one-dimension.

A key point of the proof of Theorem 1.1 is to construct such a sequence of functions without using the total order structure. Indeed, we shall use the above-mentioned geometric rigidity of the Ginibre random point field to accomplish this procedure.

We shall present general theorems concerning the sub-diffusivity of the interacting Brownian motions for \( d \geq 2 \) and prove Theorem 1.1 as a specific example of the general theorems (Theorem 1.2 and Theorem 1.3).
We set \( S_R = \{ x \in \mathbb{R}^d : |x| < R \} \). Let \( S \) be the set consisting of the configurations of \( \mathbb{R}^d \). By definition, \( S \) is given by

\[
S = \{ s = \sum_i \delta_{s_i} : s(S_R) < \infty \text{ for all } R \in \mathbb{N} \}.
\]

We endow \( S \) with the vague topology, under which \( S \) is a Polish space. \( S \) is called the configuration space over \( \mathbb{R}^d \).

Let \( \{ \vartheta_x \}_{x \in \mathbb{R}^d} \) be the translation operator on \( S \) such that for \( s = \sum_i \delta_{s_i} \),

\[
\vartheta_x(s) = \sum_i \delta_{s_i - x}.
\]

Then, \( \vartheta_x : S \to S \) is a homeomorphism for each \( x \in \mathbb{R}^d \), and \( s \mapsto \vartheta_x(s) \) is continuous on \( x \in \mathbb{R}^d \) for each \( s \in S \). Furthermore, \( (x, s) \mapsto \vartheta_x(s) \) is continuous.

A probability measure \( \mu \) on \( (S, \mathcal{B}(S)) \) is called a random point field. Let \( \mu \) be a random point field on \( \mathbb{R}^d \). We say \( \mu \) is translation (shift) invariant if

\[
\mu = \mu \circ \vartheta_x^{-1} \quad \text{for all } x \in \mathbb{R}^d.
\]

We assume the following.

(A1) \( \mu \) is translation invariant and \( \mu(\{s(\mathbb{R}^d) = \infty\}) = 1 \).

The translation invariance of \( \mu \) implies \( \mu(\{s(\mathbb{R}^d) = \infty\}) = 1 \) if \( \mu \) is not a zero measure. Thus, the second assumption in (A1) yields no restriction in practice.

A symmetric and locally integrable function \( \rho^n : (\mathbb{R}^d)^n \to [0, \infty) \) is called the \( n \)-point correlation function of \( \mu \) with respect to the Lebesgue measure if \( \rho^n \) satisfies

\[
\int_{A_1 \times \cdots \times A_m} \rho^n(x_1, \ldots, x_n) dx_1 \cdots dx_n = \int \prod_{i=1}^m \frac{s(A_i)!}{(s(A_i) - k_i)!} \mu(ds)
\]

for any sequence of disjoint bounded measurable sets \( A_1, \ldots, A_m \in \mathcal{B}(S) \) and a sequence of natural numbers \( k_1, \ldots, k_m \) satisfying \( k_1 + \cdots + k_m = n \). When \( s(A_i) - k_i < 0 \), according to our interpretation, \( s(A_i)!/(s(A_i) - k_i)! = 0 \) by convention. We make an assumption.

(A2) \( \mu \) has a locally bounded \( k \)-point correlation function for each \( k \in \mathbb{N} \).

We set the projections \( \pi_R, \pi_R^\circ : S \to S \) such that

\[
\pi_R(s) = s(\cdot \cap S_R), \quad \pi_R^\circ(s) = s(\cdot \cap S_R^\circ).
\]

Let \( \Psi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\} \) be a measurable function satisfying \( \Psi(x) = \Psi(-x) \). We take \( \Psi \) as an interaction potential of \( \mu \). Let \( S_R^m = \{ s \in S : s(S_R) = m \} \). We set \( \Lambda_R^m = \Lambda(\cdot \cap S_R^m) \), where \( \Lambda \) is the Poisson random point field whose intensity is the Lebesgue measure.

Definition 1.1 (21). We say a random point field \( \mu \) on \( \mathbb{R}^d \) is a \( \Psi \)-quasi-Gibbs measure with inverse temperature \( \beta \) if its regular conditional probabilities

\[
\mu_{R,x}^{m} = \mu(\pi_R(x) \in \cdot | \pi_R^\circ(s) = \pi_R^\circ(s), x(S_R) = m)
\]

satisfy, for all \( R, m \in \mathbb{N} \) and \( \mu \)-a.s. \( s \),

\[
\frac{1}{\Gamma(1 + \beta \mathbb{N}_R^\circ(s))} \Lambda_R^m(dx) \leq \mu_{R,x}^{m}(dx) \leq \frac{1}{\Gamma(1 + \beta \mathbb{N}^\circ(s))} \Lambda_R^m(dx).
\]
Here, \( c_{1,1}(r, m, s) \) is a positive constant depending on \( m \) and \( \pi_{t}^{\nu}(s) \). For two measures \( \mu, \nu \) on a \( \sigma \)-field \( \mathcal{F} \), we write \( \mu \preceq \nu \) if \( \mu(A) \leq \nu(A) \) for all \( A \in \mathcal{F} \). Furthermore, \( \mathcal{H}_{R}^{\mu} \) is the Hamiltonian on \( S_{R} \) defined by

\[
\mathcal{H}_{R}^{\mu}(x) = \sum_{x_{j}, x_{k} \in \delta_{S}, 1 \leq j < k \leq m} \Psi(x_{j} - x_{k}) \quad \text{for} \quad x = \sum_{i} \delta_{x_{i}}.
\]

We assume that \( \mu \) has a well-behaved, local density with respect to \( \Lambda \) as follows.

\[(A3) \] \( \mu \) is a \( \Psi \)-quasi-Gibbs measure with inverse temperature \( \beta \). Furthermore, \( \Psi \) is upper semi-continuous.

Let \( S_{s,i} \) be the set consisting of an infinite number of single point measures such that

\[
S_{s,i} = \{ s \in S ; s(\{ x \}) \in \{0, 1\} \text{ for all } x \in \mathbb{R}^{d}, s(\mathbb{R}^{d}) = \infty \}.
\]

Let \( \mathcal{M} \) be the set of all measures on \( \mathbb{R}^{d} \). Let \( \pi : (\mathbb{R}^{d})^{\mathbb{N}} \rightarrow \mathcal{M} \) be such that \( \pi((s_{i})_{i \in \mathbb{N}}) = \sum_{i \in \mathbb{N}} \delta_{s_{i}}. \) A function \( \iota_{s,i} : (\mathbb{R}^{d})^{\mathbb{N}} \rightarrow (\mathbb{R}^{d})^{\mathbb{N}} \) is called a label if for all \( s \in S_{s,i} \)

\[
\pi(\iota_{s}(s)) = s.
\]

We write \( \iota(s) = (\iota_{s}(s))_{i \in \mathbb{N}} \in (\mathbb{R}^{d})^{\mathbb{N}}. \) Then we have \( s = \sum_{i \in \mathbb{N}} \delta_{\iota_{s}(s)}. \)

Recall that \( S_{s,i} \) is endowed with the vague topology. Let \( W(S_{s,i}) \) be the set consisting of \( S_{s,i} \)-valued continuous path \( w = w_{t} \) on \([0, \infty)\). We call a function \( \{ t_{i} \}_{i \in [0, \infty)} \) defined on \( W(S_{s,i}) \) a label-valued path if \( \iota_{t}(w) = \iota_{t}(w) \in (\mathbb{R}^{d})^{\mathbb{N}}, t_{t}(w) \) is \( \sigma[w_{t}; 0 \leq t \leq t] \)-measurable for each \( t \in [0, \infty) \), and

\[
\pi(\iota_{t}(w)) = w_{t}.
\]

We call \( \iota_{t} \) the \( i \)-th label-valued path of \( \iota_{t} \). We say the \( i \)-th label-valued path \( \iota_{t} \) is continuous if \( \iota_{t}(w) \) is continuous in \( t \) under the vague topology. We say a label-valued path \( \iota_{t} = (\iota_{t})_{i \in \mathbb{N}} \in W(S_{s,i}) \) is continuous if the \( i \)-th label-valued path \( \iota_{t} \) is continuous for each \( i \in \mathbb{N} \).

We note that as an \( \mathbb{R}^{d} \)-valued path, \( \iota_{t}(w) \) is not necessary continuous even if \( \delta_{\iota_{t}(w)} \) is continuous under the vague topology. This fact comes from the difference between the vague topology and that derived by the Euclidean metric. Indeed, if \( \lim_{t \rightarrow T} |\iota_{t}(w)| = \infty \), then \( \delta_{\iota_{t}(w)} \) converges to zero measure under the vague topology.

We say an \( S_{s,i} \)-valued continuous path \( w = \{ w_{t} \} \in W(S_{s,i}) \) has neither explosion nor entering if, for any continuous label-valued path \( \iota_{t} \),

\[
\sup_{0 \leq t \leq T} |\iota_{t}(w)| < \infty \quad \text{for each } T, i \in \mathbb{N}.
\]

For an \( S_{s,i} \)-valued continuous path \( w = \{ w_{t} \} \in W(S_{s,i}) \) and a continuous label-valued path \( \iota_{t} \), we set the \( (\mathbb{R}^{d})^{\mathbb{N}} \)-valued path as

\[
(1.6) \quad w_{t} = \iota_{t}(w).
\]

We endow \( (\mathbb{R}^{d})^{\mathbb{N}} \) with the product topology. For an \( S_{s,i} \)-valued continuous path \( w \in W(S_{s,i}) \) with neither explosion nor entering and a label \( \iota_{t} \), we find a unique continuous label-valued path \( \iota_{t} \) such that the labeled path \( w = \{ w_{t} \} \) is continuous in \( t \) and \( w \) can be written as \( w_{t} = \sum_{i \in \mathbb{N}} \delta_{\iota_{t}(w)} \) for all \( t \) (see Lemma 8.1).
From (A2) and (A3), there exists a \(\mu\)-reversible diffusion \((P_s, X_t)\). (See [21].) Let
\[
P_\mu(\cdot) = \int P_s(\cdot)\mu(ds).
\]
Let \(W_{NE}(S_{a,i})\) be the set consisting of \(S_{a,i}\)-valued continuous paths on \([0, \infty)\) with neither explosion nor entering. From (A1)-(A3) and \(d \geq 2\), we find that the \(\mu\)-reversible diffusion \((P_s, X_t)\) has the non-explosion and non-collision properties (see Lemma 10.2 in [26]). Hence,
\[
(1.7)
P_\mu(X \in W_{NE}(S_{a,i})) = 1.
\]
From \((1.7)\) and Lemma 8.1, we have a continuous labeled process \(X = (X^i)_{i \in \mathbb{N}}\) such that \(X_t = \sum_{i \in \mathbb{N}} \delta_{X^i_t}\). The process \(X\) is an \((\mathbb{R}^d)^{\mathbb{N}}\)-valued diffusion [19, 20, 26].

The diffusion \((P_s, X_t)\) is associated with the Dirichlet form \((\mathcal{E}^\mu, \mathcal{D}^\mu)\) on \(L^2(\mu)\) given in Section 2.1. Let \((\mathcal{E}^\mu, \mathcal{D}^\mu)\) be the Dirichlet form defined by (2.4). It then holds that \(\mathcal{D}^\mu \subset \mathcal{D}^\mu\) and that \((\mathcal{E}^\mu, \mathcal{D}^\mu)\) restricted on \(\mathcal{D}^\mu\) coincides with \((\mathcal{E}^\mu, \mathcal{D}^\mu)\). Thus, \((\mathcal{E}^\mu, \mathcal{D}^\mu)\) is an extension of \((\mathcal{E}^\mu, \mathcal{D}^\mu)\). In particular, we see
\[
(1.8)
(\mathcal{E}^\mu, \mathcal{D}^\mu) \leq (\mathcal{E}^\mu, \mathcal{D}^\mu).
\]
The Dirichlet forms \((\mathcal{E}^\mu, \mathcal{D}^\mu)\) and \((\mathcal{E}^\mu, \mathcal{D}^\mu)\) are called the lower and upper Dirichlet forms, respectively [9]. For a given \(\mu\) satisfying (A2) and (A3), such Dirichlet forms exist [21].

To prove sub-diffusivity, we consider a Dirichlet form being an extension of \((\mathcal{E}^\mu, \mathcal{D}^\mu)\). Let \((\mathcal{E}^\perp, \mathcal{D}^\perp)\) be as in (2.29). Then as we see in (2.30), we have
\[
(1.9)
(\mathcal{E}^\perp, \mathcal{D}^\perp) \leq (\mathcal{E}^\mu, \mathcal{D}^\mu).
\]
Taking (1.8) and (1.9) into account, we make an assumption.
\[
(1.10)
(\mathcal{E}^\perp, \mathcal{D}^\perp) = (\mathcal{E}^\mu, \mathcal{D}^\mu).
\]
We refer to Theorem 3.2 in [9] for a sufficient condition of the identity \((\mathcal{E}^\mu, \mathcal{D}^\mu) = (\mathcal{E}^\mu, \mathcal{D}^\mu)\). With a slight modification, the result in Theorem 3.2 in [9] can be applied to the identity \((\mathcal{E}^\perp, \mathcal{D}^\perp) = (\mathcal{E}^\mu, \mathcal{D}^\mu)\) in (1.10).

Let \(\text{Tail}(S) = \bigcap_{R \in \mathbb{N}} \sigma(\pi^R)\) be the tail \(\sigma\)-field. We assume the following.
\[
(1.11)\mu \text{ is tail trivial. That is, } \mu(A) \in \{0, 1\} \text{ for all } A \in \text{Tail}(S).
\]
Assumption (1.11) causes no restriction. A quasi-Gibbs measure \(\mu\) has a decomposition with tail trivial components concerning the tail \(\sigma\)-field. (See Section 3.3 and Lemma 14.2 in [26].) Each component of the decomposition satisfies almost the same assumptions as \(\mu\) and has well-behaved properties as does \(\mu\). (See Theorem 3.2 in [26].)

For \(y, s \in S\), we write \(y \prec s\) if
\[
y(\{x\}) \leq s(\{x\}) \text{ for all } x \in \mathbb{R}^d.
\]
Note that for \(s, y \in S\) such that \(y \prec s\), the difference \(s - y\) belongs to \(S\). Let \(\mu_y\) be the regular conditional probability such that
\[
(1.12)\mu_y = \mu(\cdot | y \prec y).
\]
The random point field \(\mu_y\) is called the reduced Palm measure conditioned at \(y\). If \(y(\mathbb{R}^d) < \infty\), then the existence of reduced Palm measures is well known. The concept of reduced Palm measures in (1.10) is a generalization of that in (1.2).
In Lemma [5.1] we shall construct the reduced Palm measure $\mu_y$ for $y$ such that $\gamma(\mathbb{R}^d) = \infty$ if $\mu$ is decomposable in the sense of Definition [1.2].

Let $S_m = \{ s \in S : s(\mathbb{R}^d) = m \}$ for $m \in \{ 0 \} \cup \mathbb{N}$. By definition, $S_0$ consists of the zero measure. We assume that the reduced Palm measure $\mu_y$ exists for each $y \in S_m$, where $m \in \{ 0 \} \cup \mathbb{N}$. Here, by convention, $\mu_y = \mu$ if $y \in S_0$. For a set $S^0_\infty$, we take $S^0_\infty$ in such a way that

$$S^0_\infty = \bigcup_{s \in S^0_\infty} \{ t : t \prec s, (s - t)(\mathbb{R}^d) = m \}.$$

By definition, $S^0_\infty$ is the collection of configurations $t$ obtained by removing $m$-particles from a configuration $s \in S^0_\infty$. For subsets $A, B \subset S$, we set $A + B = \{ s + t : s \in A, t \in B \}$. Taking the dichotomy of the Ginibre RPF in Lemma 1.1 into account, we introduce the following concept.

Definition 1.2. We call $\mu$ $k$-decomposable with $\{S^0_\infty\}_{m=0}^k$ if $\mu$ satisfies

(1.11) $S^0_\infty \cap S^0_n = \emptyset$ for $m \neq n \in \{ 0, \ldots, k \}$,

(1.12) $S^0_\infty = S_m + S^0_\infty$ for all $m \in \{ 0, \ldots, k \}$,

(1.13) $\mu_y(S^0_\infty) = 1$ for all $y \in S_m, m \in \{ 0, \ldots, k \}$.

We shall use the concept of being one-decomposable in Section 6 which is a key point of the proof of the main result. Hence, we assume the following.

(A6) $\mu$ is one-decomposable with $\{S^0_\infty, S^1_\infty\}$.

We now state our main theorem.

Theorem 1.2. Assume $d \geq 2$. Assume (A1)–(A6). Let $X = (X^i)_{i \in \mathbb{N}}$ be the labeled process defined after (1.7) with $X_0 = \{ s \}$, where $\{ s \}$ is a label. The solution $X = (X^i)_{i \in \mathbb{N}}$ then satisfies, for each $i \in \mathbb{N}$,

$$\lim_{\epsilon \to 0} \epsilon X^i_{t/\epsilon} = 0 \text{ weakly in } C([0, \infty); \mathbb{R}^d)$$

under $P_\epsilon$ in $\mu$-probability.

We write $\mu_x = \mu(-\delta_x | \{ x \}) \geq 1$ for $x \in \mathbb{R}^d$. We make an assumption.

(A7) (1) $\mu_x$ and $\mu_y$ are singular relative to each other.

(2) $\mu_x$ and $\mu_y$ are mutually absolutely continuous for each $x, y \in \mathbb{R}^d$.

Theorem 1.3. Assume $d \geq 2$. Assume (A1)–(A5) and (A7). We then obtain the same result as in Theorem 1.2.

The Ginibre random point field satisfies (A7) by Lemma [1.1]. In Lemma 5.3 we deduce (A6) from (A7). Note that (A7) does not follow from (A6). (See Example 5.2)

We prepare a set of notations for a function on $S$ following [16]. Let $S^m_R = \{ s \in S : s(S_R) = m \}$ for $R \in \mathbb{N}$ and $m \in \{ 0 \} \cup \mathbb{N}$. Then $S$ is decomposed as

(1.14) $S = \bigcup_{m=0}^{\infty} S^m_R$.

Let $S'^m_R = S_R \times \cdots \times S_R$ be the $m$-product of $S_R$. For $s \in S^m_R$, $x^m_R(s) = (x^i_R(s))_{i=1}^m \in S'^m_R$ is called an $S^m_R$-coordinate of $s$ if $\pi_R(s) = \sum_{i=1}^{m} \delta_{x^i_R(s)}$. 
For a function $f : S \to \mathbb{R}$ and $R, m \in \mathbb{N}$, let $f_{R,m}^m(x)$ be the function satisfying

\[(1.15) \quad f_{R,m}^m(*) : S \times S_R^m \to \mathbb{R}\]

and the following conditions.

\[(1.16) \quad f_{R,s}^m(x)\text{ is a permutation invariant function in } x \text{ on } S_R^m \text{ for each } s \in S,\]

\[(1.17) \quad f_{R,s(1)}^m(x) = f_{R,s(2)}^m(x) \text{ if } \pi_R^c(s(1)) = \pi_R^c(s(2)) \text{ for } s(1), s(2) \in S_R^m,\]

\[(1.18) \quad f_{R,s}^m(x_R^m(s)) = f(s) \text{ for } s \in S_R^m,\]

\[(1.19) \quad f_{R,s}^m(x) = 0 \text{ for } s \notin S_R^m.\]

Note that $f_{R,s}^m$ is unique and $f(s) = \sum_{m=0}^{\infty} f_{R,s}^m(x_R^m(s))$ for each $R \in \mathbb{N}$ and $s \in S$. Here by convention, $x_R^0(s) = 0$ for $s \in S_R^0$. $f_{R,s}^m(\emptyset) = f(\pi_R(s))$ for $s \in S_R^0$, and $f_{R,s}^m = 0$ for $s \notin S_R^0$. Note that $\pi_R(s)$ is the zero measure for $s \in S_R^0$. The function $f_{R,s}^m$ is thus constant on $S_R^m$. Although the $S_R^m$-coordinate $x_R^m(s)$ of $s$ is not unique, $f_{R,s}^m$ is well defined by (1.16).

For a bounded set $A$, we set $x_A^m(s)$ and $f_{A,s}^m(x)$ similarly as above by replacing $S_R$ by $A$.

Let

\[(1.20) \quad D_\mu = \{ f : S \to \mathbb{R} ; f \text{ is } \mathcal{B}(S)\text{-measurable,} \]

\[f_{R,s}^m(x)\text{ is smooth in } x \text{ on } S_R^m \text{ for all } R, m \in \mathbb{N}, s \in S \} \]

We say a function $f : S \to \mathbb{R}$ is local if $f$ is $\sigma[\pi_R]$-measurable for some $R \in \mathbb{N}$. We set

\[(1.21) \quad D_\circ = \{ f \in D_\mu ; f \text{ is local }, \quad D_{ob} = \{ g \in D_\circ ; g \text{ is bounded}, \}
\]

\[C_0^\infty(\mathbb{R}^d) \otimes D_{ob} = \{ \sum_{i=1}^{m} f_i(x)g_i(y) ; f_i \in C_0^\infty(\mathbb{R}^d), \quad g_i \in D_{ob}, \quad m \in \mathbb{N} \}.\]

We next specify $X$ in terms of the ISDE. For this, we recall the concept of logarithmic derivative of $\mu$. Let $\mu^{[1]}$ be the reduced one-Campbell measure of $\mu$ such that

\[(1.22) \quad \mu^{[1]}(dxds) = \rho^1(x)\mu_x(ds)dx,\]

where $\rho^1$ is the one-point correlation function of $\mu$ with respect to the Lebesgue measure and $\mu_x$ is the reduced Palm measure conditioned at $x \in \mathbb{R}^d$. By definition, $\mu_x$ is the regular conditional probability defined as

\[\mu_x(ds) = \mu(\cdot - \delta_x | s(\{x\}) \geq 1).\]

Note that the one-point correlation function $\rho^1$ of $\mu$ exists by (A2) and is constant by (A1).

**Definition 1.3 (218).** An $\mathbb{R}^d$-valued function $d^\mu \in L^1_{\text{loc}}(\mu^{[1]})^d$ is called the logarithmic derivative of $\mu$ if, for all $\varphi \in C_0^\infty(\mathbb{R}^d) \otimes D_{ob}$,

\[\int_{\mathbb{R}^d \times S} d^\mu(x,y)\varphi(x,y)\mu^{[1]}(dxdy) = -\int_{\mathbb{R}^d \times S} \nabla_x \varphi(x,y)\mu^{[1]}(dxdy).\]

Here $\mu^{[1]}_{R} = \mu^{[1]}(\cdot \cap \{S_R \times S\})$ and $L^1_{\text{loc}}(\mu^{[1]}) = \bigcap_{R=1}^{\infty} L^1(\mu^{[1]}_R)$. 


Once $d\mu$ is calculated, we obtain the ISDE describing the labeled process $X = (X^i)_{i \in \mathbb{N}}$. Let $X_i^0 = \sum_{j \neq i} \delta_{X_j}$. We consider the ISDE

$$X_i^t - X_i^0 = B^i_t + \frac{1}{2} \int_0^t d\mu(X^i_u, X^i_u) du,$$

where $d\mu$ is reversible diffusion. Let $X_i^\wedge t = \sum_{j \neq i} \delta_{X_j}^t$. We consider the ISDE

$$X_i^t - X_i^0 = B^i_t + \frac{1}{2} \int_0^t \sum_{u} \delta_{X^i_u} du,$$

(1.23)

$$X_0^0 = l(s).$$

(1.24)

Then, under (A2) and (A3), (1.23)–(1.24) have a weak solution starting at $l(s)$ for $\mu$-a.s. $s$ such that the associated delabeled process $X$ is a $\mu$- reversible diffusion associated with the Dirichlet form $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ on $L^2(\mu)$ [20]. Furthermore, under mild constraints, the ISDE (1.23) has a unique strong solution. In particular, a weak solution is unique in law for $\mu$-a.s. $s$ [26, 10].

The logarithmic derivative $d\mu_{\text{Gin}}$ of $\mu_{\text{Gin}}$ is given by

$$d\mu_{\text{Gin}}(x, s) = \lim_{r \to \infty} \sum_{|x - s_i| < r} \frac{2(x - s_i)}{|x - s_i|^2} \text{ in } L^2_{\text{loc}}(\mu_{\text{Gin}}).$$

(1.25)

Hence, taking $\mu = \mu_{\text{Gin}}$ in (1.23), we obtain the ISDE (1.1) (see [20, 26]).

We explain the idea of the proof of the main theorems (Theorem 1.1, Theorem 1.2, and Theorem 1.3). It is known (cf. [24, 17]) that the self-diffusion matrix $\alpha = (\alpha_{p,q})_{p,q=1}^{d}$ satisfies the variational formula

$$\alpha_{p,p} = \inf \left\{ \int \frac{1}{2} \sum_{q=1}^{d} \left| D^{\text{trn}}_q f - \delta_{p,q} \right|^2 + \mathbb{D}[f, f] \, d\mu_0 : f \in \mathcal{D}_* \right\}.$$

Here, $D^{\text{trn}}_q = (D^{\text{trn}}_q)_{q=1}^{d}$ is the generator of the translation on $S$ defined by (2.38) and $\mathbb{D}$ is the carré du champ defined by (2.2). Furthermore, $\mathcal{D}_*$ is the subset of $\mathcal{D}_*$ defined by (2.50).

We shall construct a sequence of functions $\chi_{L,p,q}$ such that

$$\lim_{L \to \infty} D^{\text{trn}}_q \chi_{L,p}(s) = \delta_{p,q},$$

(1.26)

$$\lim_{L \to \infty} \mathbb{D}[\chi_{L,p}, \chi_{L,q}](s) = 0.$$

(1.27)

At first glance, it is difficult to construct such a sequence of functions satisfying these two conditions. This is because the second condition suggests that the limit function is a constant, while the first condition states that it is not. To resolve this issue, we focus on the tail $\sigma$-field Tail($S$).

We note that, from (2.1)–(2.2), all tail measurable functions $f$ satisfy

$$\mathbb{D}[f, f] = 0.$$

We also remark that a tail measurable function $f \in \mathcal{D}_*$ is not necessarily continuous under the vague topology. Indeed, it happens that, in general,

$$\lim_{R \to \infty} f(\pi_R(s)) \neq f(s)$$

even if $f_m^R(x)$ in (1.15) for $f$ is constant for each $R, m \in \mathbb{N}$ and $s \in S$.

Let $S_\infty = \{ s \in S : s(\mathbb{R}^d) = \infty \}$. If $f$ is tail measurable, then $f$ is constant on $S \setminus S_\infty$. In contrast, $f$ is not necessarily constant on $S_\infty$, as we see in (1.28), even if $f$ is tail measurable. It should be noted that $\mu(S_\infty) = 1$ by (A1). Thus, it may possible to construct a sequence of tail measurable functions $f$ satisfying (1.26) and (1.27).
Unfortunately, as a consequence of (A5), a tail measurable function becomes a constant for $\mu$-a.s. and thus does not satisfy (1.26). Hence, we shall introduce the $\sigma$-field $G_\infty$ in (6.4). This $\sigma$-field is larger than $\text{Tail}(S)$ and we can construct a sequence of $G_\infty$-measurable functions satisfying both (1.26) and (1.27).

From (A6), we shall construct the function $\chi_{L,p}(s)$ in (7.3). Using (A1), we deduce that $\{\chi_{L,p}\}$ satisfies (1.26). Furthermore, the function $\chi_{L,p}(s)$ is $G_\infty$-measurable, and thus satisfies (1.27).

The remainder of the paper is organized as follows. In Section 2, we introduce various diffusion processes and the associated Dirichlet forms related to the tagged particle problem. In Section 3, we present a sufficient condition such that the limit self-diffusion matrix vanishes. In Section 4, we recall the Kipnis–Varadhan theory and prove an invariance principle of the additive functional of reversible diffusion processes. In Section 5, we introduce the concept of reduced Palm measures conditioned at infinitely many particles on decomposable configuration spaces. This concept is one of the main tools of our analysis. In Section 6, we introduce the mean-rigid $\sigma$-field $G_\infty$, which yields the mean-rigid conditioning of random point fields. This $\sigma$-field is also a key point of the proof of the main theorems. In Section 7, we complete the proof of the main theorems (Theorem 1.1, Theorem 1.2, and Theorem 1.3). In Appendix 1 (Section 8), we prove Lemma 8.1. In Appendix 2 (Section 9), we prove $(2.17)$.

2. Preliminaries: Delabeled processes, one-labeled processes, tagged particle processes, and environment processes

In this section, we present four stochastic dynamics related to the tagged particle problem, and their Dirichlet forms.

2.1. Delabeled processes. Let $D_\bullet$ be as in (1.20). For $f, g \in D_\bullet$ and $s = \sum_i \delta_{s_i}$, we set

$$(2.1) \quad D^m_R[f, g](s) = \frac{1}{2} \sum_{s_i \in S_R, \sum_i \delta_{s_i}} (s_i f^m_{R,s_i} s_i, g^m_{R,s_i} s_i)^{R_d}(x^m_R(s)).$$

Here, $f^m_{R,s_i}$ is as in (1.15) for $f \in D_\bullet$, $\nabla s_i = (\partial s_{i1}, \ldots, \partial s_{id})$, and $x^m_R(s)$ is an $S'_R$-coordinate of $s$ introduced after (1.14). Note that $D^m_R[f, g](s)$ is independent of the choice of the $S'_R$-coordinate $x^m_R(s)$ and is well defined. We set

$$(2.2) \quad D_R = \sum_{m=1}^{\infty} D^m_R.$$ 

We see that $D_R[f, f](s)$ is non-decreasing in $R$ for each $f \in D_\bullet$ and $s \in S$. Hence, we set

$$(2.3) \quad D[f, f](s) = \lim_{R \to \infty} D_R[f, f](s) \leq \infty.$$

Thus, we set the carré du champs $D$ by polarization.

Let $(E_R^\mu, D^\mu_R)$ be the bilinear form such that

$$(2.4) \quad E_R^\mu(f, g) = \int_S D_R[f, g] d\mu,$$

$$(2.5) \quad D^\mu_R = \{ f \in D_\bullet : E_R^\mu(f, f) < \infty, f \in L^2(\mu) \}.$$
Using the method in [15][21][9], we deduce from (A3) that \((\mathcal{E}_R^\mu, \mathcal{D}_R^\mu, \cdot)\) is closable on \(L^2(\mu)\). Hence, we denote by \((\mathcal{E}_R^\mu, \mathcal{D}_R^\mu,\cdot)\) its closure. Clearly, the sequence of the closed forms \((\mathcal{E}_R^\mu, \mathcal{D}_R^\mu)\) is increasing in the sense that

\[
\mathcal{E}_R^\mu(f, f) \leq \mathcal{E}_{R+1}^\mu(f, f) \quad \text{for all } f \in \mathcal{D}_{R+1}^\mu, \quad \mathcal{D}_R^\mu \supset \mathcal{D}_{R+1}^\mu.
\]

Let \((\mathcal{E}^\mu, \mathcal{D}^\mu)\) be the limit closed form on \(L^2(\mu)\) given by

\[
(2.4) \quad \mathcal{E}^\mu(f, f) = \lim_{R \to \infty} \mathcal{E}_R^\mu(f, f), \quad \mathcal{D}^\mu = \{ f \in \bigcap_{R=1}^\infty \mathcal{D}_R^\mu; \lim_{R \to \infty} \mathcal{E}_R^\mu(f, f) < \infty, \}. \]

Let \(\mathcal{D}_0\) be as in [12][21]. Clearly, \(\mathcal{D}_0 \subset \mathcal{D}_\cdot\). We set

\[
\mathcal{D}_{R,0}^\mu = \{ f \in \mathcal{D}_0; \mathcal{E}_R^\mu(f, f) < \infty, f \in L^2(\mu), f \circ \sigma_{[R]}\text{-measurable}\}.
\]

Because \(\mathcal{D}_{R,0}^\mu \subset \mathcal{D}_{R,\cdot}^\mu\) and \((\mathcal{E}_R^\mu, \mathcal{D}_R^\mu, \cdot)\) is closable on \(L^2(\mu)\), \((\mathcal{E}_R^\mu, \mathcal{D}_R^\mu, \cdot)\) is closable on \(L^2(\mu)\). We then denote the closure on \(L^2(\mu)\) as \((\mathcal{E}_R^\mu, \mathcal{D}_R^\mu)\). We easily see that \((\mathcal{E}_R^\mu, \mathcal{D}_R^\mu)\) is decreasing in \(R\). Indeed, we find that

\[
\mathcal{E}_R^\mu(f, f) = \mathcal{E}_{R+1}^\mu(f, f) \quad \text{for all } f \in \mathcal{D}_R^\mu, \quad \mathcal{D}_R^\mu \subset \mathcal{D}_{R+1}^\mu.
\]

The limit \((\mathcal{E}_R^\mu \cup \bigcup_{R \geq 0} \mathcal{D}_R^\mu)\) is closable on \(L^2(\mu)\). This follows from \(\bigcup_{R \geq 0} \mathcal{D}_R^\mu \subset \mathcal{D}^\mu\). We denote the closure of \((\mathcal{E}_R^\mu, \bigcup_{R \geq 0} \mathcal{D}_R^\mu)\) on \(L^2(\mu)\) as \((\mathcal{E}^\mu, \mathcal{D}^\mu)\).

By construction, it is clear that

\[
(2.5) \quad (\mathcal{E}^\mu, \mathcal{D}^\mu) \leq (\mathcal{E}^\mu, \mathcal{D}^\mu).
\]

We call \((\mathcal{E}^\mu, \mathcal{D}^\mu)\) the lower Dirichlet form and \((\mathcal{E}^\mu, \mathcal{D}^\mu)\) the upper Dirichlet form. The names come from the relation (2.5). Under mild constraints, these two Dirichlet forms coincide. (See Theorem 3.2 in [9] and Theorem 3.1 in [25].)

Loosely speaking, the identity \((\mathcal{E}^\mu, \mathcal{D}^\mu) = (\mathcal{E}^\mu, \mathcal{D}^\mu)\) holds if the weak solution of ISDE (1.23) and (1.24) is unique in law for \(\mu\)-a.s. and the one-point correlation function of \(\mu\) has at most exponential growth at infinity [9]. The last condition is obvious in the current situation because \(\mu\) is translation invariant.

We next introduce carré du champs perpendicular to the diagonal line \(s_1 = s_2 = \cdots\), or equivalently to the translation operator \(D_{\text{trn}} = (D_{\text{trn}}^p)_{p=1}^d\) defined by (2.38).

For this purpose, we prepare a set of notations.

For a set \(A \subset \mathbb{R}^d\) and \(s = \sum_i s_i \in \mathcal{S}\), we set

\[
\frac{\partial}{\partial \Gamma(A)} = \sum_{s_i \in A} \frac{\partial}{\partial s_i}.
\]

We consider a partial derivative perpendicular to \(\partial/\partial \Gamma(A)\) in \(A\). The orthogonal projection of \(\partial/\partial s_i\) onto the subspace perpendicular to \(\partial/\partial \Gamma(A)\) is then given by

\[
\frac{\partial}{\partial s_i} - \frac{1}{s(A) \partial \Gamma(A)} \frac{\partial}{\partial \Gamma(A)}.
\]

We note that \(s(A)\) becomes the number of particles in \(A\) for \(s = \sum_i s_i\).

The orthogonality mentioned above is with respect to the inner product such that

\[
\left( \frac{\partial}{\partial s_{i,p}}, \frac{\partial}{\partial s_{j,q}} \right) = \delta_{i,j} \delta_{p,q}.
\]
where \( s_i = (s_{i,p})_{p=1}^d, s_j = (s_{j,q})_{q=1}^d \in \mathbb{R}^d \). For each \( s_i \in A \), we have

\[
(2.6) \quad \left( \frac{\partial}{\partial s_i} - \frac{1}{s(A)} \frac{\partial}{\partial \Gamma(A)}, \frac{1}{s(A)} \frac{\partial}{\partial \Gamma(A)} \right) = 0.
\]

Let \( A \subset B \). Then

\[
(2.7) \quad \left( \frac{1}{\sqrt{s(A)} \partial \Gamma(A)} \frac{\partial}{\sqrt{s(B)} \partial \Gamma(B)}, \frac{1}{\sqrt{s(B)} \partial \Gamma(B)} \right) = 0.
\]

Let \( T_1 = S_1 \) and \( T_R = S_R \setminus S_{R-1} \) for \( R \geq 2 \). Note that \( \{T_R\}_{R \in \mathbb{N}} \) is a partition of \( \mathbb{R}^d \). For each \( R \in \mathbb{N} \), let \( \{T_R^m\}_{m \in \{0\} \cup \mathbb{N}} \) be the partition of \( S \) such that

\[
T_R^m = \{ s \in S : s(T_R) = m \}.
\]

We set the carré du champ such that, for \( f \in \mathcal{D}_S \) and \( s = \sum_i \delta_{s_i}, \)

\[
(2.8) \quad T_R^m[f, f](s) = \frac{1}{2} \sum_{m=1}^{\infty} 1_{T_R^m}(s) \sum_{s_i \in T_R} \left| \left( \frac{\partial}{\partial s_i} - \frac{1}{s(T_R)} \frac{\partial}{\partial \Gamma(T_R)} \right) f^m_{T_R, s} \right|^2.
\]

Here for \( f \), the functions \( f^m_{T_R, s} \) are given by \((1.15) - (1.19)\). Let

\[
(2.9) \quad T_R^Q[f, f](s) = \frac{1}{2} \sum_{m=1}^{\infty} 1_{T_R^m}(s) \left| \frac{1}{\sqrt{s(T_R)} \partial \Gamma(T_R)} f^m_{T_R, s} \right|^2 = \frac{1}{2} \sum_{m=1}^{\infty} 1_{T_R^m}(s) \frac{1}{s(T_R)} \left| \frac{\partial}{\partial \Gamma(T_R)} f^m_{T_R, s} \right|^2.
\]

**Lemma 2.1.** For each \( f \in \mathcal{D}_S \) and \( R \in \mathbb{N}, \)

\[
(2.10) \quad \mathbb{D}_R[f, f] = \sum_{Q=1}^{R} T_R^1[f, f] + \sum_{Q=1}^{R} T_R^Q[f, f].
\]

**Proof.** For \( s \in T_Q^m \), we see

\[
(2.11) \quad \sum_{s_i \in T_Q} \left| \frac{\partial}{\partial s_i} f^m_{T_Q, s} \right|^2
\]

\[
= \sum_{s_i \in T_Q} \left| \left( \frac{\partial}{\partial s_i} - \frac{1}{s(T_Q)} \frac{\partial}{\partial \Gamma(T_Q)} \right) f^m_{T_Q, s} \right|^2 + \left( \frac{1}{s(T_Q)} \frac{\partial}{\partial \Gamma(T_Q)} \right) f^m_{T_Q, s} \right|^2
\]

\[
= \sum_{s_i \in T_Q} \left| \left( \frac{\partial}{\partial s_i} - \frac{1}{s(T_Q)} \frac{\partial}{\partial \Gamma(T_Q)} \right) f^m_{T_Q, s} \right|^2 + \frac{1}{s(T_Q)} \left| \frac{\partial}{\partial \Gamma(T_Q)} f^m_{T_Q, s} \right|^2.
\]

Hence from \((2.8), (2.9), \) and \((2.11), \) we deduce

\[
\frac{1}{2} \sum_{m=1}^{\infty} 1_{T_Q^m}(s) \sum_{s_i \in T_Q} \left| \frac{\partial}{\partial s_i} f^m_{T_Q, s} \right|^2
\]

\[
= \frac{1}{2} \sum_{m=1}^{\infty} 1_{T_Q^m}(s) \left\{ \sum_{s_i \in T_Q} \left| \left( \frac{\partial}{\partial s_i} - \frac{1}{s(T_Q)} \frac{\partial}{\partial \Gamma(T_Q)} \right) f^m_{T_Q, s} \right|^2 + \frac{1}{s(T_Q)} \left| \frac{\partial}{\partial \Gamma(T_Q)} f^m_{T_Q, s} \right|^2 \right\}
\]

\[
= T_Q^1[f, f](s) + T_Q^Q[f, f](s).
\]

Summing both sides over \( Q = 1, \ldots, R \), we obtain \((2.10). \)
Let $S_R^m = \{ s \in S : s(S_R) = m \}$. Note that $S_{R-1} \cup T_R = S_R$ and $S_{R-1} \cap T_R = \emptyset$ for $R \in \mathbb{N}$. For $R \in \mathbb{N}$ such that $2 \leq R$, we set

(2.12) \[ \mathbb{U}^+_R(f,f)(s) = \frac{1}{2} \sum_{m=1}^{\infty} 1_{S_R^m}(s) \left( \frac{1}{\sqrt{s(S_{R-1})}} \frac{\partial}{\partial \Gamma(S_{R-1})} - \frac{\sqrt{s(S_R)}}{s(S_R)} \frac{\partial}{\partial \Gamma(S_R)} \right) f_{R,s}^m \]

\[ + \frac{1}{2} \sum_{m=1}^{\infty} 1_{S_R^m}(s) \left( \frac{1}{\sqrt{s(T_R)}} \frac{\partial}{\partial \Gamma(T_R)} - \frac{\sqrt{s(S_R)}}{s(S_R)} \frac{\partial}{\partial \Gamma(S_R)} \right) f_{R,s}^m. \]

Let $D_1^+ = T_1^+$ and for $R \geq 2$

(2.13) \[ D_1^+[f,f](s) = \sum_{Q=1}^{R} T^+_Q[f,f](s) + \sum_{Q=2}^{R} U^+_Q[f,f](s). \]

For $R \in \mathbb{N}$, we set

(2.14) \[ \mathbb{U}^+_R(f,f)(s) = \frac{1}{2} \sum_{m=1}^{\infty} 1_{S_R^m}(s) \left( \frac{1}{\sqrt{s(S_R)}} \frac{\partial}{\partial \Gamma(S_R)} f_{R,s}^m \right)^2 \]

\[ = \frac{1}{2} \sum_{m=1}^{\infty} 1_{S_R^m}(s) \left( \frac{\partial}{\partial \Gamma(S_R)} f_{R,s}^m \right)^2. \]

Not that $T_1 = S_1$ by definition. Hence from (2.9) and (2.14), we have

(2.15) \[ T_1^+[f,f](s) = \mathbb{U}^+_1[f,f](s). \]

**Lemma 2.2.** For each $f \in D_\bullet$ and $R \in \mathbb{N}$,

(2.16) \[ D_R[f,f](s) = D_1^+[f,f](s) + \mathbb{U}^+_R[f,f](s). \]

**Proof.** From (2.13), (2.12), and (2.14), we have

(2.17) \[ \mathbb{U}^+_{R-1}[f,f](s) + T^+_R[f,f](s) = \mathbb{U}^+_R[f,f](s) + \mathbb{U}^+_R[f,f](s) \quad \text{for } R \geq 2. \]

We give a detail calculation of (2.17) in Appendix 2 (Section 9). Using (2.17), we have

(2.18) \[ \mathbb{U}^+_1[f,f](s) + \sum_{Q=2}^{R} T^+_Q[f,f](s) = \sum_{Q=2}^{R} \mathbb{U}^+_Q[f,f](s) + \mathbb{U}^+_R[f,f](s). \]

Hence from (2.15) and (2.18), we deduce

(2.19) \[ \sum_{Q=1}^{R} T^+_Q[f,f](s) = \sum_{Q=2}^{R} \mathbb{U}^+_Q[f,f](s) + \mathbb{U}^+_R[f,f](s). \]

Using (2.10), (2.13), and (2.19), we obtain (2.16). \qed

We set

(2.20) \[ D^-[f,f](s) = \sum_{Q=1}^{\infty} T^+_Q[f,f](s) + \sum_{Q=2}^{\infty} \mathbb{U}^+_Q[f,f](s). \]

Then from (2.13), $D^+_R[f,f](s)$ is increasing in $R$ and satisfies

(2.21) \[ \lim_{R \to \infty} D^+_R[f,f](s) = D^-[f,f](s). \]
Lemma 2.3. For each \( f \in \mathcal{D}_0 \),

\[
\mathbb{D}^1[f,f](s) \leq \mathbb{D}[f,f](s). 
\]

Furthermore, for any \( f \in \mathcal{D}_0 \) and all \( s \in S \) such that

\[
\lim_{R \to \infty} \mathcal{U}^*_R[f,f](s) = 0,
\]

we have

\[
\mathbb{D}^1[f,f](s) = \mathbb{D}[f,f](s).
\]

In particular, for \( f \in \mathcal{D}_0 \) such that \( \mathbb{D}[f,f](s) < \infty \), \( 2.24 \) holds and \( \mathbb{D}^1[f,f](s) < \infty \).

Proof. Note that both \( \mathbb{D}^1[f,f](s) \) and \( \mathbb{D}[f,f](s) \) are increasing in \( R \) and thus have the limits \( \mathbb{D}^1[f,f](s) \) and \( \mathbb{D}[f,f](s) \). Then, we obtain \( 2.22 \) from \( 2.16 \). Furthermore, from \( 2.16 \), we see that, if \( 2.23 \) holds, then the both limits coincide. Hence, we obtain \( 2.24 \).

For each \( f \in \mathcal{D}_0 \), we find \( Q \in \mathbb{N} \) such that \( f \) is \( \sigma(Q) \)-measurable. Because \( s(S_Q) < \infty \) and \( f \in \mathcal{D}_0 \) such that \( \mathbb{D}[f,f](s) < \infty \), we see that \( 2.23 \) holds. This yields \( 2.24 \) from \( 2.16 \). The claim \( \mathbb{D}^1[f,f](s) < \infty \) follows from \( \mathbb{D}[f,f](s) < \infty \) and \( 2.24 \). \( \square \)

We set for \( s = \sum_i \delta_{s_i} \),

\[
\Upsilon_{1}^{R,m} = \left\{ v = \frac{\partial}{\partial s_i} - \frac{1}{s(T_R)} \frac{\partial}{\partial \Gamma(T_R)} ; s_i \in T_R, \ s \in \mathbb{T}_R^m \right\}, \ R \in \mathbb{N},
\]

\[
\Upsilon_{2}^{R,m,n} = \left\{ v = \frac{1}{s(S_{R-1})} \frac{\partial}{\partial \Gamma(S_{R-1})} - \frac{\sqrt{s(S_{R-1})}}{s(S_R)} \frac{\partial}{\partial \Gamma(S_R)} ; s \in \mathbb{S}_{R-1}^m \cap \mathbb{T}_R^m \right\}, \ R \geq 2,
\]

\[
\Upsilon_{3}^{R,m,n} = \left\{ v = \frac{1}{s(T_R)} \frac{\partial}{\partial \Gamma(T_R)} - \frac{\sqrt{s(T_R)}}{s(S_R)} \frac{\partial}{\partial \Gamma(S_R)} ; s \in \mathbb{T}_R^m \cap \mathbb{S}_R^n \right\}, \ R \geq 2.
\]

Let

\[
\Upsilon_1 = \bigcup_{m \in \mathbb{N}} \Upsilon_{1}^{R,m}, \ \Upsilon_2 = \bigcup_{m,n \in \mathbb{N}} \Upsilon_{2}^{R,m,n}, \ \Upsilon_3 = \bigcup_{m,n \in \mathbb{N}} \Upsilon_{3}^{R,m,n}.
\]

Let \( \Upsilon_1 = \bigcup_{R=1}^{\infty} \Upsilon_1^R \) and \( \Upsilon_k = \bigcup_{R=2}^{\infty} \Upsilon_k^R \) for \( k = 2,3 \). We set

\[
\Upsilon = \Upsilon_1 \cup \Upsilon_2 \cup \Upsilon_3.
\]

If \( v \in \Upsilon_1^R \), then \( v \) is a partial derivative on \( T_R \). To be precise, \( v \in \Upsilon_1^{R,m} \) is a partial derivative on \( T_R^m \). We disregard \( m \) and simply call a partial derivative on \( T_R \) for convenience. Because \( v \) is a local operator, we can regard \( v \) as a partial derivative on any domain \( A \) including \( T_R \). For a function \( f \) on \( S \) and \( v \in \Upsilon_1^{R,m} \), we denote \( f \in \text{Dom}(v) \) if \( f_{T_R,s}^m(x) \) is in the domain of \( v \), where \( f_{T_R,s}^m \) is a representation of \( f \) defined by \( 1.15 \)-\( 1.19 \). We set \( v_f(s) = 0 \) for \( s \notin T_R^m \) and

\[
v_f(s) := v_f_{T_R,s}^m(x) \quad \text{for} \ s \in T_R^m \text{ such that } \pi_{T_R}(s) = \pi(x).
\]

For a function \( f \) on \( S \) and \( v \in \Upsilon_k^{R,m,n} \), \( k = 2,3 \), we define \( f \in \text{Dom}(v) \) in the same fashion. Let \( f \) be a function on \( S \). For \( v \in \Upsilon_2^{R,m,n} \), we set \( v_f(s) = 0 \) for \( s \notin \mathbb{S}_{R-1}^m \cap \mathbb{T}_R^m \) and

\[
v_f(s) = v_f_{S_{R-1}^m}^m(x) \quad \text{for} \ s \in \mathbb{S}_{R-1}^m \cap \mathbb{T}_R^m \text{ such that } \pi_{S_R}(s) = \pi(x).
\]
For \( v \in Y_{R,m,n}^{\mathbb{R}}, \) we set \( \nu f(s) = 0 \) for \( s \notin \mathbb{T}_R^m \cap S_R \) and
\[
\nu f(s) := \nu f_{\mathbb{R},\mathbb{R}}(x) \quad \text{for } s \in \mathbb{T}_R^m \cap S_R \text{ such that } \pi_{\mathbb{R},\mathbb{R}}(s) = \pi(x).
\]

Let \( \mathbb{D}[f,g] = \frac{1}{2}(\nu f, \nu g)_{\mathbb{R}} \) be the carré du champ generated by \( \nu \). Then, \( \mathbb{Y} \) is the collection of partial derivatives constituting the carré du champ \( \mathbb{D}^\perp \) defined by (2.20). That is,
\[
(2.27) \quad \mathbb{D}[f,g] = \sum_{\nu \in \mathbb{Y}} \mathbb{D}[f,g].
\]
We set
\[
(2.28) \quad \mathbb{E}[f,f] = \int_{\mathbb{T}} \mathbb{D}[f,f](s) d\mu.
\]

Let \( \mathbb{D}_R^\perp \) be as in (2.13) and let \( (\mathbb{E}_R^\perp, \mathbb{F}_R^\perp) \) be the bilinear form such that
\[
\mathbb{E}_R^\perp(f,g) = \int_{\mathbb{T}} \mathbb{D}^\perp_R(f,g) d\mu,
\]
\[
\mathbb{F}_R^\perp = \{ f:S \to \mathbb{R} : f \in \bigcap_{\nu \in \mathbb{Y}} \text{Dom}(f), \mathbb{E}_R^\perp(f,f) < \infty, \nu^k f \text{ is bounded for each } \nu \in \mathbb{Y}, k \in \mathbb{N} \}.
\]

**Lemma 2.4.** Assume (A2) and (A3). Then we obtain the following.

1. \( (\mathbb{E}_R^\perp, \mathbb{F}_R^\perp) \) is closable on \( L^2(\mu) \).
2. \( (\mathbb{E}^\perp, \mathbb{F}^\perp) \) is closable on \( L^2(\mu) \).

**Proof.** For \( v \in \mathbb{Y} \), let \( \mathbb{E}^v(f,g) = \int_{\mathbb{T}} \mathbb{D}[f,g] d\mu \). Then, \( (\mathbb{E}^v, \mathbb{F}^\perp) \) is closable on \( L^2(\mu) \) by (A3). Hence from (2.13), \( (\mathbb{E}^\perp, \mathbb{F}^\perp) \) is a countable sum of closable forms. This yields (1).

From (2.13), (2.20), (2.21), and (2.28), we see that \( (\mathbb{E}^\perp, \mathbb{F}^\perp) \) is the increasing limit of \( (\mathbb{E}_R^\perp, \mathbb{F}_R^\perp) \). Hence from (1), we deduce (2). \( \square \)

Let \( (\mathbb{E}_R^\perp, \mathbb{F}_R^\perp) \) be the closure of \( (\mathbb{E}_R^\perp, \mathbb{F}_R^\perp) \) on \( L^2(\mu) \). By (2.13), \( (\mathbb{E}_R^\perp, \mathbb{F}_R^\perp) \) is increasing in \( R \). Let \( (\mathbb{E}^\perp, \mathbb{F}^\perp) \) be the closed form given by the increasing limit of \( \{ (\mathbb{E}_R^\perp, \mathbb{F}_R^\perp) \} \). Then
\[
(2.29) \quad \lim_{R \to \infty} \mathbb{E}_R^\perp(f,f) = \mathbb{E}^\perp(f,f), \quad \mathbb{F}^\perp = \{ f \in \bigcap_{R=1}^\infty \mathbb{D}_R^\perp : \mathbb{E}_R^\perp(f,f) < \infty \}.
\]
Furthermore, \( (\mathbb{E}^\perp, \mathbb{F}^\perp) \) coincides with the closure of \( (\mathbb{E}^\perp, \mathbb{F}^\perp) \) on \( L^2(\mu) \).

From (2.22) and \( \mathbb{D}^\perp \supset \mathbb{D}^\mu \), we deduce
\[
(2.30) \quad (\mathbb{E}^\perp, \mathbb{F}^\perp) \subseteq (\mathbb{E}^\mu, \mathbb{D}^\mu), \quad \mathbb{D}^\perp \supset \mathbb{D}^\mu.
\]

**Lemma 2.5.** Assume (A2)-(A4). Then
\[
(2.31) \quad (\mathbb{E}^\perp, \mathbb{F}^\perp) = (\mathbb{E}^\mu, \mathbb{D}^\mu) = (\mathbb{E}^\mu, \mathbb{D}^\mu).
\]

**Proof.** (2.31) follows from (A4), (1.8), and (2.30) immediately. \( \square \)

In [20, 21, 26], we take \( (\mathbb{E}^\mu, \mathbb{D}^\mu) \) as the Dirichlet form describing the delabeled diffusion. The identity (2.31) implies that the domain \( (\mathbb{E}^\mu, \mathbb{D}^\mu) \) coincides with the (superficially) larger domains \( (\mathbb{E}^\perp, \mathbb{F}^\perp) \) and \( (\mathbb{E}^\mu, \mathbb{D}^\mu) \).
Let $T^n_t$ be the Markovian semi-group on $L^2(\mu)$ associated with $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ on $L^2(\mu)$.
It is known that there exists a delabeled diffusion $(P_s, X_t)$ associated with the Dirichlet form $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ on $L^2(\mu)$ \cite{10}. By construction, $(P_s, X_t)$ is $\mu$-reversible and $T^n_t f(s) = E_s[f(X_t)]$ for each $f \in L^2(\mu)$.

From Lemma 2.5 the diffusion $(P_s, X_t)$ is also associated with $(\mathcal{E}^\perp, \mathcal{D}^\perp)$ and $(\mathcal{E}^\mu, \mathcal{D}^\mu)$.

2.2. One-labeled processes. Let $X = \sum_{i \in \mathbb{N}} \delta_{X_i}$ be the delabeled diffusion in Section 2.1. Then, using (1.7), we construct the labeled process $X = (X^i)_{i \in \mathbb{N}}$. From the labeled process $X = (X^i)_{i \in \mathbb{N}}$, we construct the natural one-labeled process $(X^1, X^{1\circ})$ such that $X^{1\circ} = \{X^{1\circ}_t\}$ is the delabeled process given by $X^{1\circ}_t = \sum_{i=2}^{\infty} \delta_{X^i_t}$.

We shall present the Dirichlet form associated with the one-labeled process $(X^1, X^{1\circ})$. We define $\nabla f(x) = \frac{1}{2}(\nabla f, \nabla g)_{\mathbb{R}^d}(x)$.

We naturally regard $\nabla$ and $\mathbb{D}$ as the carré du champs on $C_r^{\infty}(\mathbb{R}^d) \otimes \mathcal{D}_\bullet$ in such a way that, for $f = f_1 \otimes f_2$ and $g = g_1 \otimes g_2$,

$$\nabla f, g) = \nabla f, g_1) f_2 g_2, \quad \mathbb{D}[f, g] = f_1 g_1 \mathbb{D}[f_2, g_2].$$

Let $\mu^{[1]}$ be the reduced one-Cambell measure of $\mu$ given by (1.22). We set

$$\mathcal{E}^{[1]}(f, g) = \int_{\mathbb{R}^d \times S} \{\nabla f, g\} + \mathbb{D}[f, g]\} d\mu^{[1]},$$

$$\mathcal{D}^{[1]} = \{f \in C_r^{\infty}(\mathbb{R}^d) \otimes \mathcal{D}_\bullet : \mathcal{E}^{[1]}(f, f) < \infty, f \in L^2(\mu^{[1]}))\}.$$

We define $\mathcal{E}^{[1], \perp}$ by (2.32) through replacing $\mathbb{D}$ with $\mathbb{D}^{\perp}$. We define $\mathcal{D}^{[1], \perp}$ by (2.32) through replacing $\mathbb{D}$ and $\mathcal{E}^{[1]}$ with $\mathcal{D}^{\perp}$ and $\mathcal{E}^{[1], \perp}$, respectively. Furthermore, we define $\mathcal{D}^{[1]}_0$ by (2.32) through replacing $\mathbb{D}$ with $\mathbb{D}_0$.

Lemma 2.6. Assume (A1)–(A4). Then the following hold.
(1) $(\mathcal{E}^{[1], \perp}, \mathcal{D}^{[1], \perp})$ and $(\mathcal{E}^{[1]}, \mathcal{D}^{[1], \perp})$ are closable on $L^2(\mu^{[1]}))$.

(2) $(\mathcal{E}^{[1]}, \mathcal{D}^{[1], \perp})$ is closable on $L^2(\mu^{[1]}))$.

(3) Let $(\mathcal{E}^{[1], \perp}, \mathcal{D}^{[1], \perp})$, $(\mathcal{E}^{[1]}, \mathcal{D}^{[1], \perp})$, and $(\mathcal{E}^{[1]}, \mathcal{D}^{[1], \perp})$ be the closures of $(\mathcal{E}^{[1], \perp}, \mathcal{D}^{[1], \perp})$, $(\mathcal{E}^{[1]}, \mathcal{D}^{[1], \perp})$, and $(\mathcal{E}^{[1]}, \mathcal{D}^{[1], \perp})$ on $L^2(\mu^{[1]}))$, respectively. Then,

$$(\mathcal{E}^{[1], \perp}, \mathcal{D}^{[1], \perp}) = (\mathcal{E}^{[1], \perp}, \mathcal{D}^{[1], \perp}) = (\mathcal{E}^{[1]}, \mathcal{D}^{[1], \perp}).$$

(4) The one-labeled process $(X^1, X^{1\circ})$ is the diffusion associated with $(\mathcal{E}^{[1], \perp}, \mathcal{D}^{[1], \perp})$, $(\mathcal{E}^{[1]}, \mathcal{D}^{[1], \perp})$, and $(\mathcal{E}^{[1]}, \mathcal{D}^{[1], \perp})$ on $L^2(\mu^{[1]}))$.

Proof. Similar to the case for Theorem 2.3 in \cite{21}, we can prove that $\mu^{[1]}$ is a quasi-Gibbs measure with upper semi-continuous potential $\Psi$ in the sense of \cite{21} (with an obvious generalization of the concept). Hence, from Lemma 2.1 in \cite{21}, we obtain (1). The claim (2) follows from (1) because $\mathcal{D}^{[1]}_0 \subset \mathcal{D}^{[1], \perp} \subset \mathcal{D}^{[1], \perp} \subset \mathcal{D}^{[1], \perp}$.

From $\mathcal{D}^{[1]}_\perp = \mathcal{D}^{\perp} = \mathcal{D}^{\perp}$, and (A4), we deduce that

$$C_r^{\infty}(\mathbb{R}^d) \otimes \mathcal{D}^{\perp} \subset C_r^{\infty}(\mathbb{R}^d) \otimes \mathcal{D}^{\perp} \subset C_r^{\infty}(\mathbb{R}^d) \otimes \mathcal{D}^{\perp} = C_r^{\infty}(\mathbb{R}^d) \otimes \mathcal{D}^{\perp}.$$
It is easy to see that
\begin{equation}
C_0^\infty(\mathbb{R}^d) \otimes D_0 \subset C_0^\infty(\mathbb{R}^d) \otimes D_0.
\end{equation}
From (2.34) and (2.35), we obtain \( (\mathcal{E}[1], D[1]) \geq (\mathcal{E}[1], D[1]) \). The inverse inequality is obvious. We thus conclude (3).

Using Lemma 2.3 and Theorem 2.4 in [19], we see that \((X^1, X^{1^0})\) is the diffusion associated with the Dirichlet form \( (\mathcal{E}[1], D[1]) \) on \( L^2(\mu[1]) \). Hence, using the identity in (3), we find that the diffusion \((X^1, X^{1^0})\) is also associated with \((\mathcal{E}[1], D[1])\) and \((\mathcal{E}[1], D[1])\) on \( L^2(\mu[1]) \).

2.3. Tagged particle processes. The tagged particle problem of interacting Brownian motions is to prove the diffusive scaling limit of each particle in the system \([7, 17, 18]\). The standard device used for this problem is to introduce the process \([7, 3, 17, 18]\). The standard device used for this problem is to introduce the process \([7, 3, 17, 18]\). The standard device used for this problem is to introduce the process \([7, 3, 17, 18]\). The standard device used for this problem is to introduce the process \([7, 3, 17, 18]\). The standard device used for this problem is to introduce the process \([7, 3, 17, 18]\). The standard device used for this problem is to introduce the process \([7, 3, 17, 18]\). The standard device used for this problem is to introduce the process \([7, 3, 17, 18]\). The standard device used for this problem is to introduce the process \([7, 3, 17, 18]\).

Example 2.1. We present the ISDE of \((X, Y)\) for the Ginibre interacting Brownian motion. Using (1.1) and (2.30), we obtain
\begin{align*}
    dX_t &= dB_t^1 - \lim_{R \to \infty} \left( \sum_{|Y^i_t| < R, j \in \mathbb{N}} \frac{Y^j_t}{|Y^j_t|^2} \right) dt, \\
    dY^i_t &= \sqrt{2} dB_t^i + \frac{Y^i_t}{|Y^i_t|^2} dt + \lim_{R \to \infty} \left( \sum_{|Y^j_t| < R, j \in \mathbb{N}} \frac{Y^j_t}{|Y^j_t|^2} \right) dt + \lim_{R \to \infty} \left( \sum_{|Y^j_t - Y^i_t - |Y^i_t|^j| < R, j \in \mathbb{N}, |Y^i_t|} \frac{Y^j_t - Y^i_t}{|Y^j_t - Y^i_t|^2} \right) dt.
\end{align*}

Here, \( \{\hat{B}^i\}_{i \in \mathbb{N}} \) is the collection of \( d \)-dimensional Brownian motions given by
\begin{equation}
    \hat{B}_t^i = \frac{1}{\sqrt{2}} (B_t^{i+1} - B_t^1).
\end{equation}
We note that \( \{\hat{B}^i\}_{i \in \mathbb{N}} \) are not independent and each \( \hat{B}^i \) is (equivalent in law to) the \( d \)-dimensional standard Brownian motion. We note that the second equation above is self-contained as an equation of \( Y = (Y^i)_{i \in \mathbb{N}} \). We shall see that the delabeled process \( Y = \sum \delta_{Y^i} \) of \( Y \) is a diffusion process with invariant probability measure \( \mu_{\text{Gin}, 0} \) in Section 2.3. This property is critical when we apply the Kipnis–Varadhan theory to the tagged particle problem [3, 11, 17].

We note that, although \((X, Y)\) is also a diffusion with state space \( \mathbb{R}^d \times (\mathbb{R}^d)^\mathbb{N} \), there exists no associated Dirichlet space. Indeed, suitable invariant measures are lacking for \((X, Y)\). In contrast, \((X, Y)\) is a diffusion with an invariant measure. As a result, it has the associated Dirichlet space. This fact is important in the analysis of the Dirichlet form version of the Kipnis–Varadhan theory in [17]. We shall specify the Dirichlet form associated with \((X, Y)\).
Let \( \vartheta_x \) be as in (1.4). Let \( D_{\text{trn}} = (D_{\text{trn}}^d)_{p=1} \) be such that
\[
D_{\text{trn}}^p f(s) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \{ f(\vartheta_{\epsilon s}(s)) - f(s) \}.
\]
Here \( \epsilon_p \) is the unit vector in the \( p \)-direction. We set
\[
(\nabla - D_{\text{trn}})[f, g] = \frac{1}{2}((\nabla - D_{\text{trn}})^{1}f, (\nabla - D_{\text{trn}})^{2}g)_{\mathbb{R}^d}. 
\]
We introduce the Dirichlet form such that
\[
E^{XY}(f, g) = \int_{\mathbb{R}^d \times S} (\nabla - D_{\text{trn}})[f, g] + \mathbb{D}[f, g] dx \times \mu_0, 
\]
\[
D^{XY}_{\ominus} = \{ f \in C_0^\infty(\mathbb{R}^d) \otimes (D_{\text{trn}} \cap D) ; E^{XY}(f, f) < \infty, f \in L^2(dx \times \mu_0) \}.
\]
Here, \( D_{\text{trn}} \) is the domain of \( D_{\text{trn}} \): i.e., \( D_{\text{trn}} \) is the set of functions for which the limit in (2.38) exists for all \( s \in S \) and \( p = 1, \ldots, d \).

We define \( E^{XY, \perp} \) through (2.40) by replacing \( \mathbb{D} \) with \( \mathbb{D}^{\perp} \). We define \( D^{XY, \perp}_{\ominus} \) through (2.41) by replacing \( D \) and \( E^{XY} \) with \( D^{\perp} \) and \( E^{XY, \perp} \), respectively. We define \( D^{XY} \) through (2.41) by replacing \( D \) with \( \mathbb{D}^{\ominus} \).

**Lemma 2.7.** Make the same assumptions as Lemma 2.6. Then the following hold.
1. \( (E^{XY, \perp}, D^{XY, \perp}_{\ominus}) \) and \( (E^{XY}, D^{XY}_{\ominus}) \) are closable on \( L^2(dx \times \mu_0) \).
2. \( (E^{XY}, D^{XY}_{\ominus}) \) is closable on \( L^2(dx \times \mu_0) \).
3. Let \( (E^{XY, \perp}, D^{XY, \perp}_{\ominus}), (E^{XY, \perp}, D^{XY}_{\ominus}) \) and \( (E^{XY}, D^{XY}_{\ominus}) \) be the closures of \( (E^{XY, \perp}, D^{XY, \perp}_{\ominus}) \), \( (E^{XY, \perp}, D^{XY}_{\ominus}) \), and \( (E^{XY}, D^{XY}_{\ominus}) \) on \( L^2(dx \times \mu_0) \), respectively. Then,
\[
(E^{XY, \perp}, D^{XY, \perp}) = (E^{XY, \perp}, D^{XY}_{\ominus}) = (E^{XY}, D^{XY}_{\perp}) \text{ on } L^2(dx \times \mu_0).
\]
4. The diffusion \( (X, Y) \) is associated with \( (E^{XY, \perp}, D^{XY, \perp}) = (E^{XY}, D^{XY}) \) on \( L^2(dx \times \mu_0) \).

**Proof.** Let \( \{ \vartheta_x \}_{x \in \mathbb{R}^d} \) be the translation operator on \( S \) given by (1.4). Let \( \iota \) be the transformation on \( \mathbb{R}^d \times S \) defined by
\[
\iota(x, s) = (x, \vartheta_x(s)).
\]
Using (A1) and (2.42), we deduce that
\[
\mu^{[1]} \circ \iota^{-1} = dx \times \mu_0.
\]
Hence, we have the unitary transformation between \( L^2(\mu^{[1]}) \) and \( L^2(dx \times \mu_0) \) such that
\[
(f \circ \iota, g \circ \iota)_{L^2(\mu^{[1]})} = (f, g)_{L^2(dx \times \mu_0)}.
\]
From (2.39) and (2.42), we deduce that
\[
(\nabla [f \circ \iota, g \circ \iota] + \mathbb{D} \perp [f \circ \iota, g \circ \iota]) \circ \iota^{-1} = (\nabla - D_{\text{trn}})[f, g] + \mathbb{D} \perp [f, g],
\]
\[
(\nabla [f \circ \iota, g \circ \iota] + \mathbb{D} [f \circ \iota, g \circ \iota]) \circ \iota^{-1} = (\nabla - D_{\text{trn}})[f, g] + \mathbb{D} [f, g].
\]
Using (2.40), (2.43), and (2.45), we find the isometry between the bilinear forms such that
\[
E^{[1], \perp}(f \circ \iota, g \circ \iota) = E^{XY, \perp}(f, g),
\]
\[
E^{[1]}(f \circ \iota, g \circ \iota) = E^{XY}(f, g).
\]
Using (2.44) and (2.46) together with Lemma 2.6, we obtain (1).
Because \( \mathcal{D}_{\infty}^{XY} \subset \mathcal{D}_{\infty}^{XY} \), we obtain (2) from (1). Recall that \((\mathcal{E}^{[1]}, \mathcal{D}^{[1], \perp}) = (\mathcal{E}^{[1]}, \mathcal{D}^{[1], \perp}) = (\mathcal{E}^{[1]}, \mathcal{D}^{[1], \perp}) \) by Lemma 2.6 (3). Then, (3) follows from this and (2.40).

We regard \( \iota \) as the transformation of \( \mathcal{C}([0, \infty); \mathbb{R}^d \times \mathcal{S}) \), denoted by the same symbol \( \iota \), such that \( \iota(X, X^{1\circ}) = \{(X_t, X^{1\circ}_t)\}_{t \in [0, \infty)} \). Then

\[
\iota(X, X^{1\circ}) = (X, Y).
\]

Using Lemma 2.6 (4) and (2.41), (2.45), and (2.47), we obtain (4). \( \square \)

2.4. Environment processes. Let \( Y \) be the environment process given by (2.37). Note that \( Y \) itself is a diffusion. Hence, we specify the Dirichlet form associated with \( Y \). We set

\[
\mathcal{D}^{trn}[f, g] = \frac{1}{2}(\mathcal{D}^{trn} f, \mathcal{D}^{trn} g)_{\mathbb{R}^d}.
\]

Let \( \mathcal{D}^Y \) and \( \mathcal{D}^{Y, \perp} \) be the carré du champ defined by

\[
\begin{align*}
\mathcal{D}^Y[f, g] &= \mathcal{D}^{trn}[f, g] + \mathcal{D}[f, g], \\
\mathcal{D}^{Y, \perp}[f, g] &= \mathcal{D}^{trn}[f, g] + \mathcal{D}^\perp[f, g].
\end{align*}
\]

Let \((\mathcal{E}^Y, \mathcal{D}^Y)\) be the bilinear form defined by

\[
\begin{align*}
\mathcal{E}^Y(f, g) &= \int_S \mathcal{D}^Y[f, g]d\mu_0, \\
\mathcal{D}^Y &= \{ f \in \mathcal{D}^{trn} \cap \mathcal{D}^\bullet; \mathcal{E}^Y(f, f) < \infty, f \in L^2(\mu_0) \}.
\end{align*}
\]

We define \( \mathcal{E}^{Y, \perp} \) by (2.41) through replacing \( \mathcal{D}^Y \) with \( \mathcal{D}^{Y, \perp} \). We define \( \mathcal{D}^{Y, \perp} \) by (2.41) through replacing \( \mathcal{D}^\bullet \) and \( \mathcal{E}^Y \) with \( \mathcal{D}^\perp \) and \( \mathcal{E}^{Y, \perp} \), respectively. That is,

\[
\begin{align*}
\mathcal{E}^{Y, \perp}(f, g) &= \int_S \mathcal{D}^{Y, \perp}[f, g]d\mu_0, \\
\mathcal{D}^{Y, \perp} &= \{ f \in \mathcal{D}^{trn} \cap \mathcal{D}^\perp; \mathcal{E}^{Y, \perp}(f, f) < \infty, f \in L^2(\mu_0) \}.
\end{align*}
\]

We define \( \mathcal{D}^Y \) by (2.41) through replacing \( \mathcal{D}^\bullet \) with \( \mathcal{D}_0 \). In [19], it was proved that \((\mathcal{E}^Y, \mathcal{D}^Y)\) is closable on \( L^2(\mu_0) \). Let \((\mathcal{E}^Y, \mathcal{D}^Y)\) be the closure of \((\mathcal{E}^Y, \mathcal{D}^Y)\) on \( L^2(\mu_0) \).

For a random variable \( Z \) and a probability measure \( \nu \), we write \( Z \sim \nu \) if the distribution (law) of \( Z \) coincides with \( \nu \).

**Lemma 2.8.** Make the same assumptions as Lemma 2.6. Then the following hold.

(1) \((\mathcal{E}^{Y, \perp}, \mathcal{D}^{Y, \perp})\) and \((\mathcal{E}^Y, \mathcal{D}^Y)\) are closable on \( L^2(\mu_0) \).

(2) Let \((\mathcal{E}^{Y, \perp}, \mathcal{D}^{Y, \perp})\) and \((\mathcal{E}^Y, \mathcal{D}^Y)\) be the closures of \((\mathcal{E}^{Y, \perp}, \mathcal{D}^{Y, \perp})\) and \((\mathcal{E}^Y, \mathcal{D}^Y)\) on \( L^2(\mu_0) \), respectively. Then,

\[
(\mathcal{E}^{Y, \perp}, \mathcal{D}^{Y, \perp}) = (\mathcal{E}^Y, \mathcal{D}^Y) = (\mathcal{E}^Y, \mathcal{D}^Y).
\]

(3) The diffusion \( Y \) is associated with \((\mathcal{E}^{Y, \perp}, \mathcal{D}^{Y, \perp})\) and \((\mathcal{E}^Y, \mathcal{D}^Y)\) on \( L^2(\mu_0) \).

(4) Let \((X, Y)\) be the diffusion in Lemma 2.7(4). Suppose that \((X_0, Y_0) \sim \mu_0 \times \mu_0 \), where \( m \) is a probability measure on \( \mathbb{R}^d \). The distribution of \( Y \) of \((X, Y)\) is then the same as that of the diffusion \( Y \) associated with \((\mathcal{E}^{Y, \perp}, \mathcal{D}^{Y, \perp})\) and \((\mathcal{E}^Y, \mathcal{D}^Y)\) such that \( Y_0 \sim \mu_0 \).
Proof. We can prove (1) similarly as for Lemma 5.3 and Lemma 5.4 in [19]. In [19], the claim was proved for \((E^Y, D^Y_\mu)\). The proof in [19] is still valid for \((E^{Y,1}, D^{Y,1}_\mu)\) and \((E^{Y,2}, D^{Y,2}_\mu)\). Indeed, in [19], the claim follows from [19, Lemma 5.1], which is a counterpart of Lemma 2.7. Hence, Lemma 2.8 follows from Lemma 2.7.

Claim (2) follows from Lemma 2.7 (3). In [19], we proved that \(Y\) is associated with \((E^Y, \bar{D}^Y)\) on \(L^2(\mu_0)\). Hence, (3) follows from this and (2).

Claim (4) for the upper form \((E^Y, \bar{D}^Y)\) was proved in Lemma 5.4 (1) in [19]. Hence, combining this with the identity in (2.34), we obtain (4). 

3. A SUFFICIENT CONDITION FOR THE VANISHING OF THE SELF-DIFFUSION MATRIX

In this section, \(\mu\) is a random point field on \(\mathbb{R}^d\) satisfying the conditions (A1)–(A4).

Let \(\mathbb{D}\) be as in (2.2). Let \(\mathbb{D}^Y\) be the carré du champ on \(\mathcal{D}_{\mu}\) defined by (2.48). Let \((E^Y, D^Y_\mu)\) be the bilinear form defined by (2.49) and (2.50). Taking the decomposition \(D^Y = D^{\text{trn}} + \mathbb{D}\) in (2.48) into account, we set

\[
E^{Y,1}(f, g) = \int_S D^{\text{trn}} [f, g] d\mu_0, \quad E^{Y,2}(f, g) = \int_S \mathbb{D} [f, g] d\mu_0.
\]

From this, we have a decomposition of the Dirichlet form such that

\[
E^Y(f, g) = E^{Y,1}(f, g) + E^{Y,2}(f, g) \quad \text{for } f, g \in D^Y_\mu.
\]

Using (3.1) and the obvious inequalities \(E^{Y,i}(f, f) \leq E^Y(f, f)\), we extend the domain of \(E^{Y,i}\) from \(D^Y_\mu\) to \(\mathbb{D}^Y\), where \(i = 1, 2\). Hence, (3.2) yields

\[
E^Y(f, g) = E^{Y,1}(f, g) + E^{Y,2}(f, g) \quad \text{for } f, g \in \mathbb{D}^Y.
\]

Because of Lemma 2.8, \((E^Y, D^Y_\mu)\) is closable on \(L^2(\mu_0)\). Meanwhile, each of \((E^{Y,1}, D^{Y,1}_\mu)\) and \((E^{Y,2}, D^{Y,2}_\mu)\) is not necessarily closable on \(L^2(\mu_0)\). Still, (3.3) makes sense under the inner product \(E^Y\).

We see that \(\mathbb{D}^Y\) is a Hilbert space with inner product \(\langle \cdot, \cdot \rangle_{L^2(\mu_0)}\) and \(\mathbb{D}^Y\) is a pre-Hilbert space with inner product \(E^Y\). Let \(\sim\) be the equivalence relation on \(\mathbb{D}^Y\) such that \(f \sim g\) if and only if \(E^Y(f - g, f - g) = 0\). The quotient space \(\mathbb{D}^Y / \sim\) is a pre-Hilbert space with inner product \(\bar{E}^Y\) such that

\[
\bar{E}^Y(\bar{f}, \bar{g}) = E^Y(f, g) \quad \text{for } f, g \in \mathbb{D}^Y,
\]

where \(\bar{f} = f / \sim\) and \(\bar{g} = g / \sim\). The completion \(\bar{D}^Y\) of \(\mathbb{D}^Y / \sim\) is then a Hilbert space with inner product \(\bar{E}^Y\).

Let \(D^{\text{trn}}_p\) be as defined in (2.38). For \(p = 1, \ldots, d\) and \(g \in \mathbb{D}^Y_\mu\), we set

\[
F_p(g) = \int_S \frac{1}{2} D^{\text{trn}}_p g d\mu_0.
\]

By the Schwartz inequality, (3.2), and (3.4), we obtain for any \(g \in \mathbb{D}^Y_\mu\)

\[
|F_p(g)|^2 \leq \frac{1}{2} E^{Y,1}(g, g) \leq \frac{1}{2} E^Y(g, g) = \frac{1}{2} \bar{E}^Y(\bar{g}, \bar{g}).
\]

From (3.6), we regard \(F_p\) as a bounded linear functional on \(\mathbb{D}^Y\) and \(\bar{D}^Y\), and we denote it by the same symbol \(F_p\).
Lemma 3.1. For \( p = 1, \ldots, d \), there exists a unique solution \( \psi_p \in \mathcal{D}^Y \) of the equation

\[
\mathcal{E}^Y(\psi_p, g) = F_p(g) \quad \text{for all } g \in \mathcal{D}^Y.
\]

Proof. Because we regard \( F_p \) as a bounded linear functional of the Hilbert space \( \mathcal{D}^Y \) with inner products \( \mathcal{E}^Y \), Lemma 3.1 is obvious from the Riesz theorem. \( \Box \)

We consider a resolvent equation. For each \( \lambda > 0 \) and \( p = 1, \ldots, d \), let \( \psi_{\lambda,p} \in \mathcal{D}^Y \) be the unique solution of the equation such that for any \( g \in \mathcal{D}^Y \)

\[
(3.7) \quad \mathcal{E}^Y(\psi_{\lambda,p}, g) + \lambda(\psi_{\lambda,p}, g)_{L^2(\mu_0)} = F_p(g).
\]

Lemma 3.2. For each \( p = 1, \ldots, d \), the following hold.

1. \( \{\psi_{\lambda,p}\}_{\lambda > 0} \) is an \( \mathcal{E}^Y \)-Cauchy sequence in \( \mathcal{D}^Y \) satisfying

\[
(3.8) \quad \lim_{\lambda,\lambda' \to 0} \mathcal{E}^Y(\psi_{\lambda,p} - \psi_{\lambda',p}, \psi_{\lambda,p} - \psi_{\lambda',p}) = 0,
\]

\[
(3.9) \quad \lim_{\lambda \to 0} \lambda \|\psi_{\lambda,p}\|_{L^2(\mu_0)}^2 = 0.
\]

2. \( \{\tilde{\psi}_{\lambda,p}\}_{\lambda > 0} \) is an \( \mathcal{E}^Y \)-Cauchy sequence in \( \tilde{\mathcal{D}}^Y \) satisfying

\[
(3.10) \quad \lim_{\lambda \to 0} \mathcal{E}^Y(\tilde{\psi}_{\lambda,p} - \psi_p, \tilde{\psi}_{\lambda,p} - \psi_p) = 0.
\]

Here, \( \psi_p \in \tilde{\mathcal{D}}^Y \) is the limit of \( \{\tilde{\psi}_{\lambda,p}\}_{\lambda > 0} \) in \( \mathcal{E}^Y \).

Proof. Lemma 3.2 follows from the standard argument; see [11] or Proposition 2.2 in [24] for detail. \( \Box \)

We make an assumption.

\( \textbf{(A8)} \) For \( p = 1, \ldots, d \), there exists an \( \mathcal{E}^Y \)-Cauchy sequence \( \{h^L_p\} \) in \( \mathcal{D}_Y^\bullet \) such that

\[
(3.11) \quad \lim_{L \to \infty} \left\{ \int_\Sigma \frac{1}{2} \sum_{q=1}^d \left| D^{\text{trn}}_{l_{q}} h^L_p - \delta_{p,q} \right|^2 d\mu_0 + \mathcal{E}^Y(\psi_{\lambda,p}, \psi_{\lambda,p}) \right\} = 0.
\]

Here, we set \( D^{\text{trn}} = (D^{\text{trn}})^d_{q=1} \) and \( \delta_{p,q} \) is the Kronecker delta.

Lemma 3.3. Assume \( \textbf{(A8)} \). Then, for \( p = 1, \ldots, d \),

\[
(3.12) \quad \lim_{\lambda \to 0} \left\{ \int_\Sigma \frac{1}{2} \sum_{q=1}^d \left| D^{\text{trn}}_{l_{q}} \psi_{\lambda,p} - \delta_{p,q} \right|^2 d\mu_0(dy) + \mathcal{E}^Y(\psi_{\lambda,p}, \psi_{\lambda,p}) \right\} = 0.
\]

Proof. From (3.11), we easily deduce that

\[
(3.13) \quad \lim_{L \to \infty} \mathcal{E}^Y(h^L_p, g) = \int_\Sigma \frac{1}{2} D^{\text{trn}} g d\mu_0 \quad \text{for all } g \in \mathcal{D}_Y^\bullet.
\]

Because \( \{h^L_p\} \) is an \( \mathcal{E}^Y \)-Cauchy sequence, \( \{\tilde{h}^L_p\} \) is an \( \mathcal{E}^Y \)-Cauchy sequence in \( \tilde{\mathcal{D}}^Y \) by (3.4). By (3.4) and (3.13), \( \{\tilde{h}^L_p\} \) is a weak convergent sequence in the Hilbert space \( \tilde{\mathcal{D}}^Y \). By (3.5) and Lemma 3.1, \( \psi_p \) is the limit of the Cauchy sequence \( \{\tilde{h}^L_p\} \). Hence, we obtain

\[
(3.14) \quad \lim_{L \to \infty} \mathcal{E}^Y(\psi_p - \tilde{h}^L_p, \psi_p - \tilde{h}^L_p) = 0.
\]
We write $\tilde{E}^{Y,1}(f) = \tilde{E}^{Y,1}(f, f)$. Then,
\[
\int \frac{1}{2} \sum_{q=1}^{d} \left| D_{q}^\text{trn} \psi_{\lambda,p} - \delta_{p,q} \right|^2 \mu_0(dy) 
\leq 2 \int \left\{ \frac{1}{2} \sum_{q=1}^{d} \left| D_{q}^\text{trn} \psi_{\lambda,p} - D_{q}^\text{trn} h_L \right|^2 + \frac{1}{2} \sum_{q=1}^{d} \left| D_{q}^\text{trn} h_L - \delta_{p,q} \right|^2 \right\} \mu_0(dy) 
= 2\tilde{E}^{Y,1}(\psi_{\lambda,p} - \psi_p + \psi_p - \tilde{h}_p^L) + 2 \int \frac{1}{2} \sum_{q=1}^{d} \left| D_{q}^\text{trn} h_L - \delta_{p,q} \right|^2 \mu_0(dy) 
\leq 4\tilde{E}^{Y,1}(\psi_{\lambda,p} - \psi_p) + 4\tilde{E}^{Y,1}(\psi_p - \tilde{h}_p^L) + 2 \int \frac{1}{2} \sum_{q=1}^{d} \left| D_{q}^\text{trn} h_L - \delta_{p,q} \right|^2 \mu_0(dy).
\]
Taking $L \to \infty$ and then $\lambda \to 0$ in the last line, each term vanishes by (3.10), (3.14), and (3.11). Hence, we find that the first term in (3.12) converges to zero.

We next calculate the second term in (3.12). We write $\tilde{E}^{Y,2}(f) = \tilde{E}^{Y,2}(f, f)$. Then,
\[
\begin{align*}
\tilde{E}^{Y,2}(\psi_{\lambda,p}) &= \tilde{E}^{Y,2}((\psi_{\lambda,p} - \psi_p) + (\psi_p - \tilde{h}_p^L)) \\
&\leq 3\{\tilde{E}^{Y,2}(\psi_{\lambda,p} - \psi_p) + \tilde{E}^{Y,2}(\psi_p - \tilde{h}_p^L) + \tilde{E}^{Y,2}(\tilde{h}_p^L)\} \\
&= 3\{\tilde{E}^{Y,2}(\psi_{\lambda,p} - \psi_p) + \tilde{E}^{Y,2}(\psi_p - \tilde{h}_p^L) + \tilde{E}^{Y,2}(\tilde{h}_p^L)\}.
\end{align*}
\]
Using (3.10), (3.14), and (3.11), we find that the last line converges to zero as $L \to \infty$ and then $\lambda \to 0$. Hence, we find that the second term in (3.12) converges to zero.

Collecting these, we obtain (3.12). \qed

4. INVARIANCE PRINCIPLE WITH THE VANISHING OF THE SELF-DIFFUSION MATRIX

Let $(X, Y)$ be as in Lemma 2.7. From Lemma 2.7 (3) and (4), we deduce that $(X, Y)$ is the diffusion associated with $(E^{XY}, D^{XY})$ on $L^2(dx \times \mu_0)$. We take $(X_0, Y_0) \sim_{\text{law}} m \times \mu_0$. The distribution of the second component $Y$ of $(X, Y)$ is independent of $m$ from Lemma 2.8 (4).

Let $Y$ be as in Lemma 2.8 (3), we see that $Y$ is the diffusion associated with $(E^Y, D^Y)$ on $L^2(\mu_0)$. We take $Y_0 \sim_{\text{law}} \mu_0$. The distribution of $Y$ then coincides with that of the second component $Y$ in $(X, Y)$ from Lemma 2.8 (4). Furthermore, $Y$ is a $\mu_0$-stationary Markov process.

The Dirichlet forms $(E^{XY}, D^{XY})$ on $L^2(dx \times \mu_0)$ and $(E^Y, D^Y)$ on $L^2(\mu_0)$ are quasi-regular. (See [15] for the definition of quasi-regularity). Indeed, quasi-regularity of the upper Dirichlet forms $(E^{XY}, D^{XY})$ on $L^2(dx \times \mu_0)$ and $(E^Y, D^Y)$ on $L^2(\mu_0)$ were proved in [19]. Then, using the identities of the upper and lower Dirichlet forms in Lemma 2.7 and Lemma 2.8, we obtain the quasi-regularity of the lower Dirichlet forms $(E^{XY}, D^{XY})$ on $L^2(dx \times \mu_0)$ and $(E^Y, D^Y)$ on $L^2(\mu_0)$. Once we have verified quasi-regularity, we can apply the Dirichlet form theory developed in [5] to the associated diffusion processes $(X, Y)$ and $Y$.

Let $\psi_\lambda = (\psi_{\lambda,p})_{p=1}^{d}$ be the function given by (3.7). Note that
\[
(1 \otimes \psi_\lambda)(X_t, Y_t) = \psi_\lambda(Y_t).
\]
From the\textsuperscript{(4.7)} similar to using the Schwartz inequality, the obtained result is
more,\textsuperscript{(4.8)} $M$
\begin{equation}
\text{(4.6)}
\end{equation}
\begin{equation}
\text{Proof.}\text{ Assume } (A1)-(A4) \text{ and } (A8). \text{ Then, } \{M_{\lambda}\}_{\lambda>0} \text{ are continuous } L^2\text{-martingales with stationary increments. Furthermore, } \{M_{\lambda}\}_{\lambda>0} \text{ satisfies}\n\end{equation}
\begin{equation}
\text{(4.2)}
\end{equation}
\end{equation}
\begin{equation}
\text{Proof.}\text{ Note that } x_p \otimes 1 + 1 \otimes \psi_{\lambda,p} \text{ is locally in } D^{XY}\text{ and, for } \varphi \in D_0^{XY},
\end{equation}
\begin{equation}
\text{(4.3)}
\end{equation}
\end{equation}
\begin{equation}
\text{Using } \text{(4.1)} \text{ and } \text{(4.3), we find that } M_{\lambda} \text{ is a continuous } L^2\text{-martingale. (See Theorem 5.2.2 and Theorem 5.2.4 in \cite{5}). Recall that } (X_0,Y_0) \sim_{\text{law}} m \times \mu_0 \text{ and } Y \text{ is a } \mu_0\text{-stationary diffusion process. Using these, } \text{(2.40)}, \text{ and Lemma 2.8 (4), we have}\n\end{equation}
\begin{equation}
\text{(4.4)}
\end{equation}
\text{Applying Lemma 3.3 to the right-hand side of } \text{(4.4)}, \text{ we obtain } \text{(4.2).}
\end{equation}
\begin{equation}
\text{Theorem 4.1. Assume } (A1)-(A4) \text{ and } (A8). \text{ Then,}\n\end{equation}
\begin{equation}
\text{(4.5)}
\end{equation}
\end{equation}
\begin{equation}
\text{Proof.}\text{ Using } \text{(4.1)}, \text{ we see for each } t \text{ that}\n\end{equation}
\begin{equation}
\text{(4.6)}
\end{equation}
\text{From Lemma 4.1, } M_{\epsilon}\text{ is an } L^2\text{-martingale with stationary increments. Furthermore, } M_{\epsilon}\text{ satisfies } \text{(4.2). Hence, we find that}\n\end{equation}
\begin{equation}
\text{(4.7)}
\end{equation}
\text{From the } \mu_0\text{-stationarity of } Y \text{ and } \text{(3.9), we deduce that}\n\end{equation}
\begin{equation}
\text{(4.8)}
\end{equation}
\text{Similarly, using the Schwartz inequality, the } \mu_0\text{-stationarity of } Y, \text{ and } \text{(3.9), we obtain}\n\end{equation}
\begin{equation}
\text{(4.9)}
\end{equation}
\text{With the reason mentioned above, the distribution of } \psi_{\lambda}(Y_t) \text{ is independent of } m \text{ and coincides with the distribution of } Y \text{ with } Y_0 \sim_{\text{law}} \mu_0. \text{ We note that } \{\psi_{\lambda}(Y_t)\} \text{ has a continuous modification because } \psi_{\lambda} \text{ is in the domain of the quasi-regular Dirichlet form } (E^Y, D^Y). \text{ Hence, we always take a continuous version of } \psi_{\lambda}(Y_t) \text{ and denote it by the same symbol.}\n\end{equation}
\text{Let } M_{\lambda} = (M_{\lambda,p})_{p=1}^d, \lambda > 0, \text{ be a continuous process such that}\n\begin{equation}
\text{(4.1)}
\end{equation}
Clearly, \( \lim_{\epsilon \to 0} E[|\epsilon X_0|^2] = 0 \). Hence, putting (4.17) into (4.19), we obtain
\[
(4.10) \quad \lim_{\epsilon \to 0} E[|\epsilon X_{t/\epsilon^2}|^2] = 0.
\]
From (4.10), we deduce for any \( 0 < t_1 < \cdots < t_k \) that
\[
(\epsilon X_0, \epsilon X_{t_1/\epsilon^2}, \ldots, \epsilon X_{t_k/\epsilon^2}) = (0, 0, \ldots, 0) \quad \text{in law as } \epsilon \to 0.
\]
This implies
\[
(4.11) \quad \lim_{\epsilon \to 0} \epsilon X_{t/\epsilon^2} = 0 \quad \text{in f.d.d. in } \mu_0\text{-measure}.
\]
Applying the Lyons–Zheng decomposition to \( X \), we deduce that \( \{\epsilon X_{t/\epsilon^2}\} \) is tight in \( C([0, \infty); \mathbb{R}^d) \). Hence, combining this with (4.11), we obtain (4.5). \( \square \)

5. Decomposition of configuration spaces and duality

In this section, we introduce the decomposition of the configuration space and duality. In the next lemma, we prove the dual relation to (1.12). This relation is a consequence of (A6) or the dichotomy in Lemma 1.1. Recall that \( S_m = \{s \in S; s(\mathbb{R}^d) = m\} \) for \( m \in \mathbb{N} \). Let \( \mu_y \) be as in (1.10).

**Lemma 5.1.** Assume (A5). Assume that \( \mu \) is \( k \)-decomposable with \( \{S_m\}^k_{m=0} \). Then, for each \( y \in S_m \) and \( m \in \{0, \ldots, k\} \), \( \mu_y \) exists and satisfies
\[
(5.1) \quad \mu_y(S_m) = 1.
\]

**Proof.** Let \( y_R = \pi_R(y) \) and \( A_R(y) = \{s; y_R \prec s\} \). Let \( \sigma[A_R(y)] = \{A_R(y), A_R(y)^c, S, \emptyset\} \) and
\[
(5.2) \quad \mathcal{A}_R(y) = \sigma[A_R(y)] \vee \sigma[\pi^c_R].
\]
By construction,
\[
(5.3) \quad \mathcal{A}_R(y) \supset \mathcal{A}_{R'}(y) \quad (R < R').
\]
Let \( A(y) = \{s; y \prec s\} \). Then \( \sigma[A(y)] = \{A(y), A(y)^c, S, \emptyset\} \). From (5.2) and (5.3),
\[
(5.4) \quad \bigcap_{R=1}^{\infty} \mathcal{A}_R(y) = \sigma[A(y)] \vee \text{Tail}(S).
\]
Because \( y_R(\mathbb{R}^d) < \infty \), the (not reduced) Palm measure \( \mu(\cdot | y_R \prec s) \) exists as
\[
(5.5) \quad \mu(\cdot | y_R \prec s) = \mu(\cdot | A_R(y)).
\]
Using the martingale convergence theorem and (5.3)–(5.5), we find that
\[
(5.6) \quad \lim_{R \to \infty} \mu(A | \mathcal{A}_R(y)) = \mu(A | \sigma[A(y)] \vee \text{Tail}(S))
= \mu(A | \sigma[A(y)]) \quad \text{by (A5)}.
\]
Note that \( \sigma[A(y)] = \{A(y), A(y)^c, S, \emptyset\} \), and \( A(y) = \{y \prec s\} \). Because the measurable space \( (S, \mathcal{B}(S)) \) has a countably determining class, we deduce
\[
(5.7) \quad \mu(A(y) | y \prec s)(t) = 1 \quad \text{for } t \in A(y) = \{s; y \prec s\}.
\]
Thus, the Palm measure \( \mu(\cdot | y \prec s) \) exists. Using (5.7), we find that the reduced Palm measure \( \mu_y = \mu(\cdot | y \prec s) \) exists. From (4.11) and (4.12) with \( \mu(S_0^c) = 1 \), we deduce that \( \mu_y \) satisfies (5.1). \( \square \)
Definition 5.1. Let $\mu$ be $k$-decomposable with $\{S^0_m\}_{m=0}^k$. We say $\mu$ is $k$-translation invariant if, for each $0 \leq m \leq k$,

\begin{align}
(5.8) \quad & \vartheta_a(S^0_m) = S^0_m \quad \text{for all } a \in \mathbb{R}^d, \\
(5.9) \quad & \mu_y = \mu_{\vartheta_a(y)} \circ \vartheta_a^{-1} \quad \text{for all } y \in S^0_m, \ a \in \mathbb{R}^d.
\end{align}

If $m = 0$, then we interpret $\mu_y$ as the translation invariance of $\mu$ as in (1.4).

Lemma 5.2. Assume that $\mu$ is $k$-decomposable with $\{S^0_m\}_{m=0}^k$ and translation invariant. Assume that the set $S^0_0$ is translation invariant. Then, $\mu$ is $k$-translation invariant.

Proof. Using Lemma 5.1, we find that the reduced Palm measure $\mu_y$ exists for $y \in S^0_0$. Clearly, $S^0_m$ is translation invariant. By assumption, the set $S^0_0$ is translation invariant. Then, (5.8) follows from (1.12) and the translation invariance of $S^0_m$ and $S^0_0$. The equation (5.9) follows from (5.2), (5.6), and the definition of reduced Palm measures.

Lemma 5.3. Assume (A7). There then exists $\{S^0_0, S^0_1\}$ such that (A6) holds with $\{S^0_0, S^0_1\}$ and that $S^0_0$ is translation invariant.

Proof. From (A7) (1), there exists $\{S^0_0, S^0_1\}$ satisfying $\mu(S^0_0) = 1$, $\mu_0(S^0_1) = 1$, and $S^0_0 \cap S^0_1 = \emptyset$. Using (A7) (2), we find that $\mu_x(S^0_1) = 1$ for all $x \in \mathbb{R}^d$. Hence, $\mu$ is one-decomposable.

We next construct a translation invariant version $U$ of $S^0_0$. Let $K_\epsilon$ be a compact set such that $K_\epsilon \subset S^0_0$ and that $\mu(K_\epsilon) > 1 - \epsilon$. We take $K_\epsilon$ to be increasing in $\epsilon$. Let

$$U_{R,\epsilon} = \bigcup_{|x| \leq R} \vartheta_x(K_\epsilon), \quad U_R = \bigcup_{\epsilon > 0} U_{R,\epsilon}, \quad U = \bigcup_{R \in \mathbb{N}} U_R.$$ 

Note that the translation operator $\vartheta_x$ on $S$ is a homeomorphism for each $x \in \mathbb{R}^d$ and that $s \mapsto \vartheta_x(s)$ is continuous in $x \in \mathbb{R}^d$ for each $s$. Hence, $(x, s) \mapsto \vartheta_x(s)$ is continuous. Because $\{|x| \leq R\} \times K_\epsilon$ is compact and the map $(x, s) \mapsto \vartheta_x(s)$ is continuous, we deduce that $U_{R,\epsilon}$ is a compact set in $S$. Hence, $U$ is a Borel set in $S$ because $U$ is the increasing limit of compact sets.

Because $\mu(U_{\epsilon > 0}K_\epsilon) = 1$ and $\bigcup_{\epsilon > 0}K_\epsilon \subset U$, we obtain $\mu(U) = 1$. By construction, $U$ is translation invariant. Taking $S^0_0$ as $U$, we complete the proof of Lemma 5.3.

Example 5.1. From Lemma 1.1, the Ginibre random point field satisfies (A7). Hence, the Ginibre random point field is one-translation invariant and one-decomposable from Lemma 5.2 and Lemma 5.3. We easily see that the Ginibre random point field is $k$-translation invariant and $k$-decomposable for each $k \in \mathbb{N}$.

Example 5.2. Let $\{v_1, \ldots, v_d\}$ be a set of independent vectors in $\mathbb{R}^d$. Let $L$ be the lattice given by $\{v_1, \ldots, v_d\}$ and let $T$ be the associated torus; i.e.,

$$L = \{ \sum_{p=1}^{d} n_p v_p : n_p \in \mathbb{Z}, \ 1 \leq p \leq d \}, \quad T = \{ \sum_{p=1}^{d} t_p v_p \subset \mathbb{R}^d : t_p \in [0, 1), \ 1 \leq p \leq d \}.$$ 

We set $\tau : T \to S$ by $\tau(x) = \sum_{v \in L} \delta_{x + v}$. Let $m$ be the uniform distribution on $T$. We call $\mu = m \circ \tau^{-1}$ a periodic random point field. By construction, $\mu$ is $k$-decomposable and $k$-translation invariant for any $k \in \mathbb{N}$ but does not satisfy (A7).
6. Mean-rigid conditioning

In this section, we introduce the concept of a mean-rigid $\sigma$-field $\mathcal{G}_\infty$. We define the functions $N_R$ and $M_R$ on $S$ such that

\begin{equation}
N_R(s) = s(S_R), \quad M_R(s) = \sum_{s_i \in S_R} s_i.
\end{equation}

Replacing $S_R$ by $T_R$ in (6.1), we define $N_{T_R}$ and $M_{T_R}$. For a function $f$, a $\sigma$-field $\mathcal{F}$, and a random point field $\mu$, we say $f$ is $\mathcal{F}$-measurable for $\mu$-a.s. if a $\mu$-version of $f$ is $\mathcal{F}$-measurable.

**Definition 6.1.** (1) A random point field $\mu$ on $\mathbb{R}^d$ is said to be number-rigid if, for each $R \in \mathbb{N}$, the function $N_R(s)$ (resp. $N_{T_R}$) is $\sigma[\pi_R^c]$-measurable (resp. $\sigma[\pi_{T_R}^c]$-measurable) for $\mu$-a.s.

(2) A random point field $\mu$ on $\mathbb{R}^d$ is said to be mean-rigid if $\mu$ is number-rigid and, for each $R \in \mathbb{N}$, the function $M_R(s)$ (resp. $M_{T_R}(s)$) is $\sigma[\pi_R^c]$-measurable (resp. $\sigma[\pi_{T_R}^c]$-measurable) for $\mu$-a.s.

If $\mu$ is mean-rigid, then by definition it holds for each $R \in \mathbb{N}$ and for $\mu$-a.s. $s$, there exist $a = a(\pi^c_R(s))$, $b = b(\pi^c_R(s))$, $a' = a'(\pi^c_{T_R}(s))$, and $b' = b'(\pi^c_{T_R}(s))$ such that

$$
\mu(\{s \in S : M_R(s) = a(\pi^c_R(s)), N_R(s) = b(\pi^c_R(s))\}) = 1,
$$

$$
\mu(\{s \in S : M_{T_R}(s) = a'(\pi^c_{T_R}(s)), N_{T_R}(s) = b'(\pi^c_{T_R}(s))\}) = 1.
$$

Our concept of number rigidity and mean rigidity is a slightly weaker property than that of the concept in [7, 2]. In [7, 2], rigidity is posed for all Borel sets with Lebesgue-negligible boundary. In our concept, rigidity is posed only for $S_R$ and $T_R$. This reduction is enough for our purpose.

Mean rigidity is a critical property for sub-diffusivity. Our proof of sub-diffusivity uses this property. The Ginibre random point field is number-rigid but not mean-rigid. Hence, we introduce an algorithm to transform a function on a probability space with a number-rigid random point field to a function that is measurable with respect to a mean-rigid $\sigma$-field.

For $R \in \mathbb{N}$, let $\mathcal{G}_R$ and $\mathcal{H}_R$ be the sub $\sigma$-fields of $\mathcal{B}(\mathcal{S}_T^c)$ given by

\begin{equation}
\mathcal{G}_R = \sigma[N_R, M_R, \pi_R^c],
\end{equation}

\begin{equation}
\mathcal{H}_R = \sigma[N_{T_R}, M_{T_R}, \pi_{T_R}^c].
\end{equation}

We set the mean-rigid $\sigma$-field $\mathcal{G}_\infty$ as follows.

\begin{equation}
\mathcal{G}_\infty = \bigcap_{R=1}^{\infty} \mathcal{G}_R.
\end{equation}

**Lemma 6.1.** (1) For each $R \in \mathbb{N}$,

\begin{equation}
\mathcal{G}_R \supset \mathcal{G}_{R+1},
\end{equation}

\begin{equation}
\mathcal{H}_R \supset \mathcal{G}_R.
\end{equation}

(2) Let $\vartheta_a$ be the translation on $S$ as in [1, 4]. Then, for each $a \in \mathbb{R}^d$,

\begin{equation}
\vartheta_a(\mathcal{G}_\infty) = \mathcal{G}_\infty.
\end{equation}
Proof. Recall that $T_{R+1} = S_{R+1} \setminus S_R$. Then we see $\pi^c_R = \pi_{T_{R+1}} + \pi^c_{R+1}$. Hence, 

\[
\sigma[N_R, M_R, \pi^c_R] = \sigma[N_R, M_R, \pi_{T_{R+1}}, \pi^c_{R+1}] \supset \sigma[N_{R+1}, M_{R+1}, \pi^c_{R+1}].
\]

This together with (6.3) yields (6.5).

From $T_R = S_R \setminus S_{R-1}$, we find $\pi^c_T = \pi_{R-1} + \pi^c_R$. Hence, we see 

\[
\sigma[N_{T_R}, M_{T_R}, \pi^c_{T_R}] = \sigma[N_{T_R}, M_{T_R}, \pi_{R-1}, \pi^c_R] \supset \sigma[N_R, M_R, \pi^c_R].
\]

This together with (6.3) yields (6.6).

Without loss of generality, we set $|a| < 1$. Let $S_R(a) = \vartheta_a(S_R)$. Then, 

(6.8) 

\[S_{R-|a|} \subset S_R(a) \subset S_{R+|a|}.\]

Replacing $S_R$ with $S_R(a)$ in (6.1), we define $N_{R,a}$ and $M_{R,a}$. Let $\pi^c_{R,a} = \pi_{S_R(a)}$.

Then from (6.8), we find that 

(6.9) 

\[G_{R-|a|} \supset \sigma[N_{R,a}, M_{R,a}, \pi^c_{R,a}] \supset G_{R+|a|}.
\]

From (6.2), we find that $\vartheta_a(G_R) = \sigma[N_{R,a}, M_{R,a}, \pi^c_{R,a}]$. Hence, from this and (6.9), we deduce that 

(6.10) 

\[\bigcap_{R=1}^{\infty} G_{R-|a|} \supset \bigcap_{R=1}^{\infty} \vartheta_a(G_R) \supset \bigcap_{R=1}^{\infty} G_{R+|a|}.
\]

Using (6.2) and (6.10) and noting 

\[\bigcap_{R=1}^{\infty} \vartheta_a(G_R) = \vartheta_a\left(\bigcap_{R=1}^{\infty} G_R\right),\]

we obtain $G_\infty \supset \vartheta_a(G_\infty) \supset G_\infty$, which yields (6.7). \hfill $\Box$

Let $\mu_0(\cdot | G_R), R \in \mathbb{N} \cup \{\infty\}$, be the regular conditional probabilities.

Lemma 6.2. $\mu_0(\cdot | G_\infty)$ is one-translation invariant. That is, for each $a \in \mathbb{R}^d$, 

\[\mu_{0-a}(\cdot | G_\infty) = \mu_0(\cdot | G_\infty) \circ \vartheta_a^{-1}.
\]

Proof. Let $A \in \mathcal{B}(\mathbb{S})$. Then from the martingale convergence theorem, (6.4), and (6.5) 

\[\mu_0(A | G_\infty)(s) = \lim_{R \to \infty} \mu_0(A | G_R)(s) \quad \text{in } L^1(\mu_0) \text{ and } \mu_0\text{-a.s.s.}
\]

Hence, the claim follows from the translation invariance of $\mu$ and (6.7). \hfill $\Box$

Lemma 6.3. Assume that $f$ is $G_\infty$-measurable. Then $f \in \mathcal{D}_+$ and 

(6.11) 

\[
\mathbb{D}_+[f, f] = 0.
\]

Proof. We write $s = \sum_i \delta_{s_i}$ as before. Suppose $v \in \Upsilon^{R,m}_1$. Then for $s \in \Upsilon^{m}_R$, 

\[v = \frac{\partial}{\partial s_t} - \frac{1}{m} \frac{\partial}{\partial T(R)}, \quad s_t \in T_R.
\]

We note $s(T_R) = N_{T_R}(s)$ and $\sum_{s_i \in T_R} s_i = M_{T_R}(s)$. Note that $f$ is $G_\infty$-measurable by assumption. Then, $f$ is measurable with respect to $\mathcal{H}_R = \sigma[N_{T_R}, M_{T_R}, \pi^c_{T_R}]$ from (6.3) and (6.6). Hence, we see that $f$ is a function of $N_{T_R}(s), M_{T_R}(s)$, and
From (2.7) and (6.14), we see that
\[ H \]
Hence using (2.27), we conclude
\[ \text{(6.18)} \]
(6.18),
From (2.7) and (6.17), we see that
\[ f_{\mathcal{R},\pi_{\mathcal{R}}}(s_1, \ldots, s_m) = f_{\mathcal{R},\pi_{\mathcal{R}}}(s_1, \ldots, s_m) = f_{\mathcal{R},\pi_{\mathcal{R}}}(s) = f_{\mathcal{R},\pi_{\mathcal{R}}}(s) (s_1 + \cdots + s_m). \]
The vector \( v \) is perpendicular to \( \partial/\partial \Gamma(T_R) \) as we see in (6.10). Hence from (6.12), we obtain
\[ \text{(6.13)} \]
\( v f_{\mathcal{R},\pi_{\mathcal{R}}}(s_1, \ldots, s_m) = v f_{\mathcal{R},\pi_{\mathcal{R}}}(s_1 + \cdots + s_m) = 0. \)
Suppose \( v \in \mathcal{T}_{\mathcal{R},m,n}^m \). Then \( s(S_{R-1}) = m \) and \( s(S_R) = m + n \). We find
\[ \text{(6.14)} \]
\[ v = \frac{1}{\sqrt{m}} \frac{\partial}{\partial \Gamma(S_{R-1})} - \frac{\sqrt{m}}{m + n} \frac{\partial}{\partial \Gamma(S_R)}. \]
From (6.4), \( f \) is \( \mathcal{G}_R \)-measurable. Hence, \( f \) is a function of \( N_R(s), M_R(s), \) and \( \pi_{\mathcal{R}}(s) \). Note that \( N_{R-1}(s) = m \) and \( N_R(s) = m + n \) on \( S_{R-1} \cap \mathcal{T}_R \) and \( f_{\mathcal{R},\pi_{\mathcal{R}}}(s) = f_{\mathcal{R},\pi_{\mathcal{R}}}(s) \cdot \). By definition, \( M_R(s) = s_1 + \cdots + s_{m+n} \). Hence, there exists a function \( f_{\mathcal{R},\pi_{\mathcal{R}}}(s) \) on \( \mathbb{R}^d \) such that
\[ \text{(6.15)} \]
\[ f_{\mathcal{R},\pi_{\mathcal{R}}}(s_1, \ldots, s_{m+n}) = f_{\mathcal{R},\pi_{\mathcal{R}}}(s_1 + \cdots + s_{m+n}). \]
Suppose \( v \in \mathcal{T}_{\mathcal{R},m,n}^m \). Then \( s(T_R) = m \) and \( s(S_R) = n \). We find
\[ \text{(6.16)} \]
\[ v f_{\mathcal{R},\pi_{\mathcal{R}}}(s_1, \ldots, s_{m+n}) = 0. \]
From (6.4), \( f \) is \( \mathcal{G}_R \)-measurable. Hence, \( f \) is a function of \( N_R(s), M_R(s), \) and \( \pi_{\mathcal{R}}(s) \). Note that \( N_{R}(s) = m \) and \( N_R(s) = n \) on \( \mathcal{T}_R \cap S_R \) and that \( f_{\mathcal{R},\pi_{\mathcal{R}}}(s) = f_{\mathcal{R},\pi_{\mathcal{R}}}(s) \cdot \). By definition, \( M_R(s) = s_1 + \cdots + s_n \). Thus, there exists a function \( f_{\mathcal{R},\pi_{\mathcal{R}}}(s) \) on \( \mathbb{R}^d \) such that
\[ \text{(6.17)} \]
\[ f_{\mathcal{R},\pi_{\mathcal{R}}}(s_1, \ldots, s_n) = f_{\mathcal{R},\pi_{\mathcal{R}}}(s_1 + \cdots + s_n). \]
Suppose \( v \in \mathcal{T}_{\mathcal{R},m,n}^m \). Then \( s(T_R) = m \) and \( s(S_R) = n \). We find
\[ \text{(6.18)} \]
\[ v f_{\mathcal{R},\pi_{\mathcal{R}}}(s_1, \ldots, s_n) = 0. \]
From (2.7) and (6.17), we see that \( v \) is perpendicular to \( \partial/\partial \Gamma(S_R) \). Hence from (6.18),
\[ \text{(6.19)} \]
\[ v f_{\mathcal{R},\pi_{\mathcal{R}}}(s_1, \ldots, s_n) = 0. \]
Putting (6.13), (6.16), and (6.19) together and recalling (2.25) and (2.26), we obtain
\[ \mathbb{D}_v[f,f] = 0 \quad \text{for all } v \in \mathcal{T}. \]
Hence using (2.27), we conclude \( f \in \mathcal{D}^\perp \) and (6.11). \( \square \)
7. Proof of Theorem 1.1, Theorem 1.2 and Theorem 1.3

In this section, we construct a sequence of cut-off $G_\infty$-measurable functions. Using this, we shall verify (A8) in Theorem 7.1. The goal of this section is to complete the proof of Theorem 1.1, Theorem 1.2, and Theorem 1.3.

We assume that $\mu$ is one-decomposable with $\{S_0, S_1^\circ\}$ and translation invariant. Let $y \in S_1^\circ$ and let $\mu_y$ be as in (5.1). Applying Lemma 5.1, we obtain

$$
\mu_y(S_1) = 1.
$$

Using (7.1), we set the probability measure $\sigma_y$ on $\mathbb{R}^d$ by

$$
\sigma_y(A) = \mu_y(\{x(A) = 1\}) \quad \text{for } A \in B(\mathbb{R}^d).
$$

Let $\mu_0 = \mu(\cdot - \delta_0)\mathbb{1}(\{0\}) \geq 1$ be the reduced Palm measure of $\mu$ conditioned at the origin 0 as before. Let $\mu_0(\cdot | G_\infty)(s)$ be the regular conditional probability of $\mu_0$ as in Lemma 6.2.

For $L \in \mathbb{N}$, let $\xi_L: \mathbb{R} \to \mathbb{R}$ be a smooth non-decreasing function such that

$$
0 \leq \xi'_L(t) \leq 2, \quad \xi_L(t) = \begin{cases} L + 1 & L + 2 \leq t \\ t & |t| \leq L \\ -L - 1 & t \leq -L - 2. \end{cases}
$$

We write $x = (x_p)_{p=1}^d \in \mathbb{R}^d$. For each $s \in S$, we set $\chi_L = (\chi_{L,p})_{p=1}^d$ by

$$
\chi_{L,p}(s) = \begin{cases} \int_{S_1^\circ} \chi_L(x_p) \sigma_y(dx) \mu_0(dy | G_\infty)(s) & s \in S_1^\circ \\ 0 & s \notin S_1^\circ.
\end{cases}
$$

We note that $\chi_{L,p}$ is $G_\infty$-measurable by construction and that $\chi_{L,p}$ is neither a continuous nor local function on $S$. Let $D_q^{\text{trn}}$ be as in (2.38). Recall that

$$
D_q^{\text{trn}}f(s) = \lim_{h \to 0^+} \frac{1}{h} \{ f(\partial_h e_q(s)) - f(s) \}.
$$

Let $D^\gamma$ be such that

$$
D^\gamma = \bigcap_{q=1}^d \left\{ f : D_q^{\text{trn}}f(s) \text{ exists for all } s \in S_1^\circ \right\}.
$$

Lemma 7.1. (1) $\chi_L \in D^\gamma$ and $|D_q^{\text{trn}}\chi_{L,p}(s)| \leq 2$, $q = 1, \ldots, d$.

(2) $\{\chi_L\}$ is an $E^{Y,1}$-Cauchy sequence as $L \to \infty$ satisfying

$$
\lim_{L \to \infty} D^{\text{trn}}\chi_L(s) = E \quad \text{for all } s \in S_1^\circ \text{ and in } L^2(\mu_0).
$$

Here $E = (\delta_{p,q})_{p,q=1}^d$ is the unit matrix of order $d$.

Proof. From Lemma 6.2 we deduce

$$
\mu_0(dy | G_\infty)(s) = \mu_{0-a}(\partial_a(\cdot)) \in dy | G_\infty)(\partial_a(s)).
$$

From Lemma 5.2 we have for all $s \in S_1^\circ$

$$
\sigma_y(dx) = \sigma_{\partial_a(y)}(d(x-a)).
$$
Hence using (7.3), (7.5), and (7.6), we obtain for all $s \in S$:

\[
\frac{1}{n} \left( \chi_{L,p}(\partial h_e)(s) - \chi_{L,p}(s) \right)
\]

(7.7)

\[
= \frac{1}{n} \int_{S^2} \left\{ \int_{\mathbb{R}^d} \xi_L(x_p)\sigma_{h_e}(y) \mu_0(dy|G)(s) - \int_{\mathbb{R}^d} \xi_L(x_p)\sigma_y(dx) \right\} \mu_0(dy|G)(s)
\]

(7.8)

\[
= \frac{1}{n} \int_{S^2} \left\{ \int_{\mathbb{R}^d} \xi_L(x_p + h_e)(y) \sigma_y(dx) - \int_{\mathbb{R}^d} \xi_L(x_p)\sigma_y(dx) \right\} \mu_0(dy|G)(s)
\]

(7.9)

\[
= \int_{S^2} \int_{\mathbb{R}^d} \frac{1}{n} \left\{ \xi_L(x_p + h_e) - \xi_L(x_p) \right\} \sigma_y(dx) \mu_0(dy|G)(s).
\]

(7.10)

Using (7.2), (7.3), and (7.7), we find $\chi \in D$ and that for all $s \in S$:

\[
D_{tr}^{\chi} \chi_{L,p}(s) = \delta_{p,q} \int_{S^2} \int_{\mathbb{R}^d} \xi_L(x_p)\sigma_y(dx) \mu_0(dy|G)(s).
\]

(7.11)

Hence, $|D_{tr}^{\chi} \chi_{L,p}(s)| \leq 2$ from (7.2) and (7.5). We have thus obtained (1).

From (7.2), (7.8), and the Lebesgue convergence theorem, we obtain (2).

The main result of this section is as follows.

**Theorem 7.1.** Assume (A1)–(A6). Then, (A8) holds.

**Proof.** By construction, $\chi_{L,p}$ is $G$-measurable. Hence from Lemma 6.3, we see

(7.9)

\[
\chi_{L,p} \in D_{\sigma}^{\chi}, \quad \mathbb{D}^{\chi}[\chi_{L,p}, \chi_{L,p}] = 0.
\]

From Lemma 7.4, we see $\chi_{L,p} \in D$. Combining these, we obtain $\chi_{L,p} \in D_{\sigma}^{\chi}$, where $D_{\sigma}^{\chi}$ is defined by (7.24). From $\chi_{L,p} \in D_{\sigma}^{\chi}$ and Lemma 2.8 (2), we deduce

(7.10)

\[
\chi_{L,p} \in D_{\sigma}^{\chi}.
\]

From Lemma 7.4 (2) and (7.24), we see that $\{\chi_{L,p}\}$ is a $\mathcal{E}_{\sigma}$-Cauchy sequence as $L \to \infty$ and satisfies

\[
\lim_{L \to \infty} \frac{1}{S} \sum_{q=1}^{d} \left| D_{tr}^{\chi} \chi_{L,p} - \delta_{p,q} \right|^2 \mu_0 = 0,
\]

(7.11)

From $\chi_{L,p} \in D_{\sigma}^{\chi}$, we obtain (A8).

We now complete the proof of Theorem 1.2 and Theorem 1.3.

**Proof of Theorem 1.2.** By assumption, (A1)–(A6) hold. Hence from Theorem 7.1, (A8) holds. All the assumptions of Theorem 1.2 are thus satisfied. We therefore obtain Theorem 1.2.

**Proof of Theorem 1.3.** Theorem 1.3 follows from Theorem 1.2 and Lemma 5.3.

To prove Theorem 1.1, we prepare a sequence of lemmas.
Lemma 7.2. For $s = \sum_i \delta_{s_i}$, we set $s_i^\circ = \sum_{j \neq i} \delta_{s_j}$. Then

\begin{equation}
\lim_{R \to \infty} \int_S \frac{1}{|S_R|^2} \sum_{s_i \in S_R} \frac{1}{2} d^\text{Rin}(s_i, s_i^\circ) |\mu_{\text{Gin}}(ds) = 0.
\end{equation}

Proof. Let $\mathbb{I} = [0, 1]^2$ and
\[ F(s) = \sum_{s_i \in \mathbb{I}} \frac{1}{2} d^\text{Rin}(s_i, s_i^\circ). \]

Recall that $d^\text{Rin} \in L^2_{\text{loc}}(\mu_{\text{Gin}}^{[1]})$ by (1.25) and that $\mu_{\text{Gin}}$ is reflection and translation invariant. Hence, we see
\begin{equation}
\int_S |F(s)|^2 \mu_{\text{Gin}}(ds) < \infty, \quad \int_S F(s) \mu_{\text{Gin}}(ds) = 0.
\end{equation}

We note that $\partial_a(\mathbb{I}) \cap \partial_b(\mathbb{I}) = \emptyset$ for $a, b \in \mathbb{Z}^2$ with $a \neq b$. We set
\[ I_R = \{ z \in \mathbb{Z}^2; \partial_z(\mathbb{I}) \subset \overline{S_R} \}, \]
\[ J_R = \{ z \in \mathbb{Z}^2; \partial_z(\mathbb{I}) \notin \overline{S_R}, \partial_z(\mathbb{I}) \cap \partial S_R \neq \emptyset \}. \]

We then have
\begin{equation}
\sum_{s_i \in S_R} \frac{1}{2} d^\text{Rin}(s_i, s_i^\circ) = \sum_{z \in I_R} F(\partial_z(s)) + \sum_{z \in J_R} \sum_{s_i \in \partial_z(\mathbb{I}) \cap \overline{S_R}} \frac{1}{2} d^\text{Rin}(s_i, s_i^\circ).
\end{equation}

For a set $A$, we denote the volume by $|A|$. Note that
\begin{equation}
\lim_{R \to \infty} \frac{\sum_{z \in J_R} \partial_z(\mathbb{I})}{|S_R|} = 1, \quad \lim_{R \to \infty} \frac{\sum_{z \in J_R} |\partial_z(\mathbb{I})|}{|S_R|} = 0.
\end{equation}

Recall that $\mu_{\text{Gin}}$ is tail trivial. Hence, $\mu_{\text{Gin}}$ is ergodic under the translation $\{\partial_x\}_{x \in \mathbb{R}^2}$. Hence using ergodic theorem, we deduce
\begin{equation}
\lim_{R \to \infty} \frac{\overline{|S_R|}}{|S_R|} = \frac{1}{\rho_{\text{Gin}}^2} = \pi \quad \text{in } L^2(\mu_{\text{Gin}}).
\end{equation}

Using ergodic theorem again, we obtain from (7.15)–(7.17)
\begin{equation}
\lim_{R \to \infty} \frac{1}{|S_R|} \sum_{s_i \in S_R} \frac{1}{2} d^\text{Rin}(s_i, s_i^\circ) = F(s) \quad \text{in } L^1(\mu_{\text{Gin}}).
\end{equation}

Combining this with (7.14), we obtain (7.13). \qed

Let $(\mathcal{E}_R^\mu, \mathcal{D}_R^\mu)$ be the Dirichlet form defined after (2.3). Let $(\mathcal{E}_R^\gamma, \mathcal{D}_R^\gamma)$ be the Dirichlet form defined before (2.29). Here we take $\mu = \mu_{\text{Gin}}$. The carré du champs of these Dirichlet forms are then given by $\mathcal{D}_R$ and $\mathcal{D}_R^\perp$. From (2.10), we have
\begin{equation}
\mathcal{D}_R^\perp[f, f](s) = \mathcal{D}_R[f, f](s) - \mathcal{U}_R^\perp[f, f](s).
\end{equation}

We consider the stochastic differential equations describing the labeled dynamics given by $(\mathcal{E}_R^\mu, \mathcal{D}_R^\mu)$ and $(\mathcal{E}_R^\gamma, \mathcal{D}_R^\gamma)$ on $L^2(\mu)$. We take a label $l = (l^i)_{i \in \mathbb{N}}$ on $S_{x, i}$ such that $|l^i(s)| \leq |l^{i+1}(s)|$ for all $i \in \mathbb{N}$. Let $m = s(S_R)$. The stochastic differential
equation $\mathbf{X}^R = (X^{R,i})$ for the Dirichlet form $(\mathcal{E}^R, D^R)$ satisfying $\mathbf{X}^R_0 = \mathbf{i}(s)$ is then given by

$$(7.19) \quad dX^{R,i}_t = dB^i_t + \frac{1}{2}d\mu_{\text{Gin}}(X^{R,i}_t, \sum_{j \neq i}^\infty \delta_{X^{R,j}_t})dt + \frac{1}{2}n^R(X^{R,i}_t)dL^R_i, \quad 1 \leq i \leq m,$$

$$dL^R_i = d\partial S_R(X^{R,i}_t)dL^R_i, \quad 1 \leq i \leq m,$$

$$X^{R,i}_t = X^{0,i}_t = \mathbf{i}(s), \quad m < i < \infty.$$

Here $d\mu_{\text{Gin}}$ is the logarithmic derivative of $\mu_{\text{Gin}}$ given by (1.25), $n^R$ is the inner normal vector on $\partial S_R$, and $L^R_i$ are non-negative increasing processes. From (7.18) and (7.19), we see that the stochastic differential equation for the Dirichlet form $(\mathcal{E}^R_0, D^R_0)$ on $L^2(\mu)$ is given by

$$(7.20) \quad dX^{R,i}_t = dB^i_t + \frac{1}{2}d\mu_{\text{Gin}}(X^{R,i}_t, \sum_{j \neq i}^\infty \delta_{X^{R,j}_t})dt + \frac{1}{2}n^R(X^{R,i}_t)dL^R_i,$$

$$- \frac{1}{m} \sum_{k=1}^m dB^i_t - \frac{1}{m} \sum_{k=1}^m \frac{1}{2}d\mu_{\text{Gin}}(X^{R,i}_t, \sum_{j \neq k}^\infty \delta_{X^{R,j}_t})dt$$

$$- \frac{1}{m} \sum_{k=1}^m \frac{1}{2}n^R(X^{R,i}_t)dL^R_i,$$

$$1 \leq i \leq m,$$

$$dL^R_i = d\partial S_R(X^{R,i}_t)dL^R_i, \quad 1 \leq i \leq m,$$

$$X^{R,i}_t = X^{0,i}_t = \mathbf{i}(s), \quad m < i < \infty.$$

To simplify the first equation in (7.20), we set $B^R, K^R$, and $L^R$ as

$$(7.21) \quad B^R_t := - \frac{1}{s(S_R)} \sum_{k=1}^{s(S_R)} B^k_t,$$

$$(7.22) \quad K^R_t := - \int_0^t \frac{1}{s(S_R)} \sum_{k=1}^{s(S_R)} \frac{1}{2}d\mu_{\text{Gin}}(X^{R,R,i}_u, \sum_{j \neq k}^\infty \delta_{X^{R,j}_u})du,$$

$$(7.23) \quad L^R_t := - \int_0^t \frac{1}{s(S_R)} \sum_{k=1}^{s(S_R)} \frac{1}{2}n^R(X^{R,R,i}_u)dL^R_{u,i}.$$

Using these notations and $m = s(S_R)$, we write (7.20) as

$$(7.24) \quad X^{R,i}_t - X^{0,i}_t = B^i_t + \int_0^t \frac{1}{2}d\mu_{\text{Gin}}(X^{R,R,i}_u, \sum_{j \neq i}^\infty \delta_{X^{R,j}_u})du$$

$$+ \int_0^t \frac{1}{2}n^R(X^{R,R,i}_u)dL^R_{u,i} + B^R_t + K^R_t + L^R_t.$$

These three terms come from $-\mathcal{U}_R^T$ in (7.18). We shall estimate $B^R + K^R + L^R$.

Lemma 7.3. For each $T \in \mathbb{N}$ and $\epsilon > 0$,

$$(7.25) \quad \lim_{R \to \infty} P \left( \max_{0 \leq t \leq T} |B^R_t| \geq \epsilon \right) = 0,$$

$$(7.26) \quad \lim_{R \to \infty} P \left( \max_{0 \leq t \leq T} |K^R_t| \geq \epsilon \right) = 0.$$
Proof. From ergodic theorem, we have $\lim_{R \to \infty} \frac{s(S_R)}{|S_R|} = \rho_{\text{Gin}}^R = 1/\pi$ for $\mu_{\text{Gin}}$-a.s. and in $L^2(\mu_{\text{Gin}})$. Recall that $B^k$, $k \in \mathbb{N}$, in (7.21) are independent Brownian motions. Hence, we see that for each $T > 0$ and each $\epsilon > 0$

$$\lim_{R \to \infty} P \left( \max_{0 \leq t \leq T} |B^R_t| \geq \epsilon \right) = \lim_{R \to \infty} P \left( \max_{0 \leq t \leq T} \frac{1}{s(S_R)} \sum_{k=1}^{\infty} B^k_t \geq \epsilon \right) = 0.$$

This implies (7.25). For each $T \in \mathbb{N}$ and $\epsilon > 0$, we see

$$P \left( \max_{0 \leq t \leq T} |K^R_t| \geq \epsilon \right) = P \left( \max_{0 \leq t \leq T} \left| \int_0^t \frac{1}{s(S_R)} \sum_{k=1}^{\infty} \frac{1}{2} d\mu_{\text{Gin}}(X^{R,i,k}_u, \sum_{j \neq k} \delta_{X^{R,i,j}_u}) du \right| \geq \epsilon \right) \leq P \left( \int_0^T \left| \frac{1}{s(S_R)} \sum_{k=1}^{\infty} \frac{1}{2} d\mu_{\text{Gin}}(X^{R,i,k}_u, \sum_{j \neq k} \delta_{X^{R,i,j}_u}) \right| du \geq \epsilon \right) \leq \frac{1}{\epsilon} E \left[ \int_0^T \left| \frac{1}{s(S_R)} \sum_{k=1}^{\infty} \frac{1}{2} d\mu_{\text{Gin}}(X^{R,i,k}_u, \sum_{j \neq k} \delta_{X^{R,i,j}_u}) \right| du \right] = \frac{T}{\epsilon} E \left[ \int_S \left| \frac{1}{s(S_R)} \sum_{i \in S_R} \frac{1}{2} d\mu_{\text{Gin}}(S_i, S^R_t) \right| \right] \mu_{\text{Gin}}(ds) = \frac{T}{\epsilon} \int_S \left| \frac{1}{s(S_R)} \sum_{i \in S_R} \frac{1}{2} d\mu_{\text{Gin}}(S_i, S^R_t) \right| \mu_{\text{Gin}}(ds).$$

Hence applying Lemma 7.2 to the last line, we obtain (7.26). \qed 

Using Lyons-Zheng decomposition, we see that $X^{R,i}_t$, $1 \leq i \leq s(S_R)$, can be written as a sum of martingale additive functionals $M^{R,i}$ and $M^{R,i*}$ such that

$$X^{R,i}_t = X_0^{R,i} - X^{R,i}_0 = \frac{1}{2} \left( M^{R,i} + M^{R,i*} \right),$$

where $r_T(X)_t = X_{T-t}$ is the time reversal on $[0, T]$ and $M^{R,i}$ and $M^{R,i*}$ are defined by

$$M^{R,i} = B^i_t - \frac{1}{s(S_R)} \sum_{k=1}^{\infty} B^k_t,$$

$$M^{R,i*} = M^{R,i}(r_T) - M^{R,i}(r_T).$$

Hence using the martingale inequality, we deduce from (7.27) that $X^{R,i} = (X^{R,i}_t)$ is tight in $C([0, T]; \mathbb{R}^2)^{\mathbb{N}}$. Then, we denote an arbitrary convergent subsequence by the same symbol $X^{R,i} = (X^{R,i}_t)$ and its limit by $X^i = (X^i_t)$. That is,

$$\lim_{R \to \infty} X^{R,i} = X^i \quad \text{weakly in } C([0, T]; \mathbb{R}^2)^{\mathbb{N}}.$$
Lemma 7.4. For each $T \in \mathbb{N}$ and $\epsilon > 0$,

\begin{equation}
\lim_{R \to \infty} \sup_{0 \leq t \leq T} \frac{1}{s} \sum_{k=1}^{s} \left| X_{t}^{R,i,k} - X_{0}^{R,i,k} - B_{t}^{k} + B_{t}^{R} \right| \geq \epsilon
\end{equation}

\begin{equation}
\leq P \left( \max_{0 \leq t \leq T} \frac{1}{s} \sum_{k=1}^{s} \left| X_{t}^{i,k} - X_{0}^{i,k} - B_{t}^{k} \right| \geq \epsilon \right) \quad \text{for each } Q \in \mathbb{N},
\end{equation}

(7.31)

Proof. Let $B = (B^{i})_{i \in \mathbb{N}}$ be an $(\mathbb{R}^{2})^{\mathbb{N}}$-valued Brownian motion. From (7.20) and (7.29), we see that $(X_{t}^{R,i,k}, B^{k})_{k \in \mathbb{N}}$ converge weakly to $(X_{t}^{i,k}, B^{k})$ in $C([0, T] ; (\mathbb{R}^{2})^{\mathbb{N}} \times (\mathbb{R}^{2})^{\mathbb{N}} \times \mathbb{R}^{2})$ as $R$ goes to $\infty$. Hence, we obtain (7.30) because $A$ is a closed set. Here, we set

\[ A = \{ (u, v, w) \in C([0, T] ; \mathbb{R}^{2})^{\mathbb{N}} \times (\mathbb{R}^{2})^{\mathbb{N}} \times \mathbb{R}^{2}) : \max_{0 \leq t \leq T} \left| u_{t} - u_{0} - v_{t} - w_{t} \right| \geq \epsilon \}, \]

where $u = (u^{i})$ and $v = (v^{i})$. From (7.20), (7.21), and (7.28), we have for each $k \in \mathbb{N}$

\begin{equation}
X_{t}^{i,k} - X_{0}^{i,k} = \frac{1}{2} \left( B_{t}^{k} + B_{T-t}^{k}(r_{T}) - B_{T}^{k}(r_{T}) \right).
\end{equation}

Applying the martingale inequality to the right-hand side of (7.32), we obtain (7.31). \hfill \square

Let $b = b(x, y)$ be the drift coefficient of ISDE (1.1). Then $b = (1/2)d^{\mu_{\text{Gin}}}$ and

\begin{equation}
b(x, y) = \lim_{s \to \infty} \sum_{|x-y| < s} \frac{x-y}{|x-y|^{2}} \quad \text{in } L_{\text{loc}}^{2}(\mathbb{R}^{2} \times S, \mu_{\text{Gin}}^{[1]}).
\end{equation}

We quote a result from [9]. We note that the choice of parameter set $(r, s, p)$ below depends on the limit ISDE, and the number of parameters does not have any special meaning.

Lemma 7.5 ([9]). There exists $b_{r,s,p} = b_{r,s,p}(x, y)$ such that

\begin{equation}
b_{r,s,p} \in C_{b}(\mathbb{R}^{2} \times S),
\end{equation}

\begin{equation}
\lim_{r \to \infty} \lim_{s \to \infty} \sup_{p \to \infty} \| b_{r,s,p} - b \|_{L_{\text{loc}}^{1}(\mathbb{R}^{2} \times S, \mu_{\text{Gin}}^{[1]})} = 0 \quad \text{for } \mu_{\text{Gin}} - \text{a.s.}.
\end{equation}

Here, $\mu_{\text{Gin}}^{[1]}$ is the regular conditional probability such that $\mu_{\text{Gin}}^{[1]}(\cdot) = \mu_{\text{Gin}}(\cdot | \sigma[\pi_{R}])$ and $\mu_{\text{Gin}}^{[1]}$ is the one-Campbell measure of $\mu_{\text{Gin}}^{[1]}$.

Proof. Lemma 7.5 follows from Lemma 6.1 in [9]. Here, we note that assumption (6.4) of Lemma 6.1 in [9] follows from Lemma 8.2 in [26]. \hfill \square
Lemma 7.6. For each \( T \in \mathbb{N} \) and \( \epsilon > 0 \),

\[
(7.36) \quad \limsup_{R \to \infty} P \left( \sup_{0 \leq t \leq T} \left| \frac{1}{s(S_Q)} \sum_{k=1}^{s(S_Q)} \int_0^t \frac{1}{2} d\mu_{\text{Gin}} \left( X_u^{R,+,k}, \sum_{j \neq k} \delta_{X_u^{+,j}} \right) du \right| \geq \epsilon \right) \\
\leq P \left( \sup_{0 \leq t \leq T} \left| \frac{1}{s(S_Q)} \sum_{k=1}^{s(S_Q)} \int_0^t \frac{1}{2} d\mu_{\text{Gin}} \left( X_u^{+,k}, \sum_{j \neq k} \delta_{X_u^{+,j}} \right) du \right| \geq \epsilon \right), \quad Q \in \mathbb{N},
\]

\[
(7.37) \quad \lim_{Q \to \infty} P \left( \sup_{0 \leq t \leq T} \left| \frac{1}{s(S_Q)} \sum_{k=1}^{s(S_Q)} \int_0^t \frac{1}{2} d\mu_{\text{Gin}} \left( X_u^{+,k}, \sum_{j \neq k} \delta_{X_u^{+,j}} \right) du \right| \geq \epsilon \right) = 0.
\]

Proof. Similarly as Lemma 4.2 in \([9]\), we have from (7.29) and (7.28)–(7.30),

\[
(7.38) \quad \sup_{R \in \mathbb{N}} \sum_{k=1}^{m} E[|X_t^{R,+,k} - X_u^{R,+,k}|^4] \leq \left| \frac{2}{1} \right| m |t - u|^2 \quad \text{for } 0 \leq t, u \leq T,
\]

\[
(7.39) \quad \lim_{a \to \infty} \inf_{R \to \infty} P \left( \max_{1 \leq k \leq m} \sup_{0 \leq t \leq T} |X_t^{R,+,k}| \leq a \right) = 1 \quad \text{for each } m \in \mathbb{N},
\]

\[
(7.40) \quad \lim_{l \to \infty} \inf_{R \in \mathbb{N}} P \left( I_{r,T}(X_{r,T}^{R,+,l}) \leq l \right) = 1 \quad \text{for each } r \in \mathbb{N},
\]

where \( c_{7.1} > 0 \) is a constant and

\[
I_{r,T}(X^{R,+,l}) = \sup \{ k \in \mathbb{N} \mid |X_t^{R,+,k}| \leq r \text{ for some } 0 \leq t \leq T \}.
\]

From (7.29) and (7.38)–(7.40), we deduce for each \( k \in \mathbb{N} \)

\[
(7.41) \quad \lim_{R \to \infty} \left( X_u^{R,+,k}, \sum_{j \neq k} \delta_{X_u^{+,j}} \right) = \left( X_u^{+,k}, \sum_{j \neq k} \delta_{X_u^{+,j}} \right) \quad \text{weakly in } C([0,T]; \mathbb{R}^2 \times S).
\]

Recall that \((1/2)d\mu_{\text{Gin}} = b\) and set

\[
Z_t^R := \frac{1}{s(S_Q)} \sum_{k=1}^{s(S_Q)} \int_0^t b_{r,s,p}(X_u^{R,+,k}, \sum_{j \neq k} \delta_{X_u^{+,j}}) du,
\]

\[
Z_t := \frac{1}{s(S_Q)} \sum_{k=1}^{s(S_Q)} \int_0^t b_{r,s,p}(X_u^{+,k}, \sum_{j \neq k} \delta_{X_u^{+,j}}) du.
\]

Then from \( b_{r,s,p} \in C_b(\mathbb{R}^2 \times S) \) and (7.41), we have

\[
(7.42) \quad \lim_{R \to \infty} Z^R = Z \quad \text{weakly in } C([0,T]; \mathbb{R}^2).
\]

Hence using (7.32), we deduce

\[
(7.43) \quad \limsup_{R \to \infty} P \left( \sup_{0 \leq t \leq T} |Z_t^R| \geq \epsilon \right) \leq P \left( \sup_{0 \leq t \leq T} |Z_t| \geq \epsilon \right).
\]
Because the delabeled process $X^{R,\perp} = \sum_{k=1}^{\infty} \delta_{X^{R,\perp,k}}$ is $\mu_{\text{Gin}}$-reversible, we have

\begin{equation}
E\left[\sup_{0 \leq t \leq T} \left| \int_0^t \frac{1}{s(S_Q)} \sum_{k=1}^{s(S_Q)} \{b - b_{r,s,p}\}(X^{R,\perp,k}_u, \sum_{j \neq k}^{\infty} \delta_{X^{R,\perp,j}})du \right| \right]
\leq T \int_{S} \frac{1}{s(S_Q)} \sum_{s_i \in S_Q} \{b - b_{r,s,p}\}(s_i, s_i^c) \mu_{\text{Gin}}(ds).
\end{equation}

Similarly, we have

\begin{equation}
E\left[\sup_{0 \leq t \leq T} \left| \int_0^t \frac{1}{s(S_Q)} \sum_{k=1}^{s(S_Q)} \{b - b_{r,s,p}\}(X^{\perp,k}_u, \sum_{j \neq k}^{\infty} \delta_{X^{\perp,j}})du \right| \right]
\leq T \int_{S} \frac{1}{s(S_Q)} \sum_{s_i \in S_Q} \{b - b_{r,s,p}\}(s_i, s_i^c) \mu_{\text{Gin}}(ds).
\end{equation}

Hence from (7.43), (7.44) and (7.45), we obtain (7.36).

Let $X^{R,\perp}$ and $X^{\perp}$ be the delabeled processes with initial distribution $\mu_{\text{Gin}}$ associated with $X^{R,\perp} = (X^{R,\perp,k})_{k \in \mathbb{N}}$ and $X^{\perp} = (X^{\perp,k})_{k \in \mathbb{N}}$, respectively. Then by definition,

$$X^{R,\perp}_t = \sum_{k=1}^{\infty} \delta_{X^{R,\perp,k}}, \quad X^{\perp}_t = \sum_{k=1}^{\infty} \delta_{X^{\perp,k}}.$$  

We note that $X^{R,\perp}$ is associated with the Dirichlet form $(E^{\perp}, D^{\perp}_R)$ on $L^2(\mu_{\text{Gin}})$. In particular, $X^{R,\perp}$ is a $\mu_{\text{Gin}}$-reversible Markov process. As we see (7.24), $(E^{\perp}, D^{\perp})$ on $L^2(\mu_{\text{Gin}})$ is the increasing limit of $(E^{\perp}_R, D^{\perp}_R)$ on $L^2(\mu_{\text{Gin}})$. From (7.24), $X^{R,\perp}$ converges weakly to $X^{\perp}$ in $C([0, T]; \mathbb{R}^N)$. Hence, the finite-dimensional distributions of $X^{R,\perp}$ converge to those of $X^{\perp}$. In particular, $X^{\perp}$ is a $\mu_{\text{Gin}}$-reversible Markov process associated with $(E^{\perp}, D^{\perp})$ on $L^2(\mu_{\text{Gin}})$. Hence, we deduce (7.46) with a similar calculation as (7.44), where

\begin{equation}
E\left[\sup_{0 \leq t \leq T} \left| \int_0^t \frac{1}{s(S_Q)} \sum_{k=1}^{s(S_Q)} \frac{1}{2} d\mu_{\text{Gin}}(X^{\perp,k}_u, \sum_{j \neq k}^{\infty} \delta_{X^{\perp,j}})du \right| \right]
\leq T \int_{S} \frac{1}{s(S_Q)} \sum_{s_i \in S_Q} \frac{1}{2} d\mu_{\text{Gin}}(s_i, s_i^c) \mu_{\text{Gin}}(ds).
\end{equation}

Using Chebyshev’s inequality, we deduce (7.37) from (7.46) and Lemma 7.2 □
Lemma 7.7. For each $Q, T \in \mathbb{N}$ and $\epsilon > 0$,

\begin{equation}
\lim_{R \to \infty} P \left( \max_{0 \leq t \leq T} \frac{1}{s(\mathcal{S}_Q)} \sum_{k=1}^{s(\mathcal{S}_Q)} \int_0^t \frac{1}{2} \mathbf{n}^R(X_t^{R, \perp, k}) dL^R_t \right) \geq \epsilon = 0.
\end{equation}

Proof. Using (7.47) and the martingale inequality, we find positive constants $c_{7.2}$ and $c_{7.3}$ independent of $R, k \in \mathbb{N}$ such that

\begin{equation}
P \left( \max_{0 \leq t \leq T} |X_t^{R, \perp, k} - X_0^{R, \perp, k}| \geq h \right)
\leq P \left( \max_{0 \leq t \leq T} |\mathcal{M}_t^{R, k}| \geq h \right) + P \left( \max_{0 \leq t \leq T} |\mathcal{M}_t^{R, k} - 1| \geq h \right)
= 2P \left( \max_{0 \leq t \leq T} |\mathcal{M}_t^{R, k}| \geq h \right)
\leq \int_0^\infty e^{-t^2/\epsilon^2} dt \quad \text{for all } h > 0.
\end{equation}

Note that $L_t^{R, \perp, k}$ increases only on $\partial S_R$. Hence from (7.48), we have (7.47).

Lemma 7.8. For each $\epsilon > 0$,

\begin{equation}
\lim_{R \to \infty} P \left( \max_{0 \leq t \leq T} |\mathcal{L}_t^R| \geq 4\epsilon \right) = 0.
\end{equation}

Proof. Using (7.20) and (7.23), we deduce for each $1 \leq k \leq s(\mathcal{S}_R)$

\begin{equation}
\mathcal{L}_t^R = X_t^{R, \perp, k} - X_0^{R, \perp, k} - B_t^k - \int_0^t \frac{1}{2} d\mu^{\text{Gin}}(X_u^{R, \perp, k}, \sum_{j \neq k} \delta_{X_u^{R, \perp, j}}) du
- \int_0^t \frac{1}{2} \mathbf{n}^R(X_u^{R, \perp, k}) dL^R_u - B_t^R - K_t^R.
\end{equation}

Note that the left-hand side of (7.50) is independent of $1 \leq k \leq s(\mathcal{S}_R)$. We fix $Q$ and sum over $k = 1, \ldots, s(Q)$. Then, for any $Q \leq R$, we have

\begin{equation}
\mathcal{L}_t^R = \frac{1}{s(\mathcal{S}_Q)} \left\{ \sum_{k=1}^{s(\mathcal{S}_Q)} X_t^{R, \perp, k} - X_0^{R, \perp, k} - B_t^k - \int_0^t \frac{1}{2} d\mu^{\text{Gin}}(X_u^{R, \perp, k}, \sum_{j \neq k} \delta_{X_u^{R, \perp, j}}) du
- \int_0^t \frac{1}{2} \mathbf{n}^R(X_u^{R, \perp, k}) dL^R_u - B_t^R - K_t^R \right\}.
\end{equation}

Taking $R \to \infty$ in (7.51) and using (7.30), (7.36), (7.47), and Lemma 7.3, we deduce for each $Q \in \mathbb{N}$

\begin{equation}
\lim_{R \to \infty} P \left( \max_{0 \leq t \leq T} |\mathcal{L}_t^R| \geq 4\epsilon \right) \leq P \left( \max_{0 \leq t \leq T} \left| \frac{1}{s(\mathcal{S}_Q)} \sum_{k=1}^{s(\mathcal{S}_Q)} \left( X_t^{\perp, k} - X_0^{\perp, k} - B_t^k \right) \right| \geq \epsilon \right)
+ P \left( \max_{0 \leq t \leq T} \left| \frac{1}{s(\mathcal{S}_Q)} \sum_{k=1}^{s(\mathcal{S}_Q)} \left( \int_0^t \frac{1}{2} d\mu^{\text{Gin}}(X_u^{\perp, k}, \sum_{j \neq k} \delta_{X_u^{\perp, j}}) du \right) \right| \geq \epsilon \right).
\end{equation}

Taking $Q \to \infty$ and applying (7.31) and (7.37) to the right-hand side, we obtain (7.49).
Proof of Theorem 1.1. It is well known that $\mu_{\text{Gin}}$ satisfies (A1) and (A2). Assumption (A3) was proved in Theorem 2.3 in [21].

We use Theorems 3.1 and 3.2 in [9] to prove that the Ginibre random point field satisfies (A4). From Section 7 in [26], Section 7.4 in [9], and Example 7.3 in [10], we find that the Ginibre random point field satisfies the assumptions of Theorems 3.1 and 3.2 in [9]. Hence, we obtain the convergence of the solution $X^R = (X^R,i)$ of (7.19) to the solution $X = (X^i)$ of (143) and the identity $(\mathcal{E}^{\mu_{\text{Gin}}}, \mathcal{D}^{\mu_{\text{Gin}}}) = (\mathcal{E}^{\mu_{\text{Gin}}}, \mathcal{D}^{\mu_{\text{Gin}}})$ from Theorems 3.1 and 3.2 in [9].

To use Theorem 3.1 in [9], we need (A6) in [9]. This assumption requires a cut-off function $b_{r,s,p}$ satisfying (7.34) and (7.35) for the coefficient of (7.19). Recall that the finite-volume stochastic differential equation for $(\mathcal{E}_R^i, D_R^i)$ is given by (7.20).

From Lemma 7.3 and Lemma 7.8, we see that the difference of the coefficient in stochastic differential equation (7.20) from (7.19) is negligible as $R \to \infty$ in the sense that

$$\lim_{R \to \infty} P \left( \max_{0 \leq t \leq T} |B_t^R + K_t^R + L_t^R| \geq \epsilon k \right) = 0 \quad \text{for each } \epsilon > 0.$$

Hence, we can prove that the solution $X^R,\perp = (X^{R,\perp,i})$ of (7.20) converges in law in $C([0,T]; (\mathbb{R}^2)^N)$ to that of ISDE (1.1) in a similar fashion as Theorem 3.1 in [9]. Thus, $X^\perp$ is a weak solution of (1.1) from (7.29).

Because both $X^\perp$ and $X$ are weak solutions of (1.1), we deduce $X^\perp = X$ in law from the uniqueness of weak solutions of ISDE (1.1). We refer to Corollary 3.2 and Theorem 7.1 in [20] for the uniqueness of weak solutions of (1.1).

From (7.20), $(\mathcal{E}^\perp, D^\perp)$ on $L^2(\mu_{\text{Gin}})$ is the increasing limit of $(\mathcal{E}_R^i, D_R^i)$ on $L^2(\mu_{\text{Gin}})$. Hence, the delabeled process given by $X^\perp$ is associated with $(\mathcal{E}^\perp, D^\perp)$ on $L^2(\mu_{\text{Gin}})$. Recall that the delabeled process given by $X$ is associated with $(\mathcal{E}^{\mu_{\text{Gin}}}, \mathcal{D}^{\mu_{\text{Gin}}})$ on $L^2(\mu_{\text{Gin}})$. Hence, we obtain $(\mathcal{E}^\perp, D^\perp) = (\mathcal{E}^{\mu_{\text{Gin}}}, \mathcal{D}^{\mu_{\text{Gin}}})$ from $X^\perp = X$ in law. We refer to Theorem 3.2 in [9]. We thus obtain (A4).

Assumption (A5) was proved in [23], [14]. Assumption (A7) follows from Lemma 1.3. Hence, using Theorem 1.3 we obtain Theorem 1.1. 

8. Appendix 1

We endow $(\mathbb{R}^d)^N$ with the product topology with metric $\varrho$ such that

$$\varrho(x, y) = \sum_{i \in N} \frac{1}{2^i} \max\{1, |x_i - y_i|\},$$

where $x = (x_i) \in (\mathbb{R}^d)^N$. We note that an $(\mathbb{R}^d)^N$-valued path $w = (w_i^t)_{i \in N}$ continuous if and only if each component $w_i^t$ is continuous.

Lemma 8.1. Let $w \in W_{NE}(S_{s,i})$ be an $S_{s,i}$-valued continuous path with neither explosion nor entering. Let $l$ be a label. Then, there exist a unique $(\mathbb{R}^d)^N$-valued continuous path $w_t$ and a unique label-valued path $l_t$ such that $w_t = l_t(w)$ for all $t$ and that $l_0 = l$.

Proof. Let $k \in \mathbb{N}$ be fixed. We shall prove that there exists a label-valued path $l_t = (l_t^i)_{i \in N}$ such that $l_t^i(w) \in \mathbb{R}^d$ is continuous for all $t \in [0, \infty)$ and that $l_0 = l$.

Let $l_t = (l_t^i)_{i \in N}$ be a label-valued path such that $l_0(w) = l(w_0)$. We first prove that there exists $l_t$ such that $\delta_{l_t^i(w)}$ is continuous in $t$ under the vague topology.
For \( \epsilon > 0 \), let \( S_{\epsilon, t} = \{ x \in \mathbb{R}^d : |x - l^k_t(w)| < \epsilon \} \) be the \( \epsilon \)-neighborhood of \( l^k_t(w) \).

Because \( w_0\{l^k_t(w)\} = 1 \), \( w_t\{\{x\}\} = 0 \) or \( 1 \) for all \( x \in \mathbb{R}^d \) and \( t \in [0, \infty) \), and \( w_t \) is continuous under the vague topology, there exist an \( \epsilon_0\)-neighborhood \( S_{\epsilon_0, 0} \) of \( l^k_0(w) \) and \( \eta_0 > 0 \) such that \( w_t(S_{\epsilon_0, 0}) = 1 \) for all \( 0 \leq t \leq \eta_0 \). Hence, we find a label-valued path \( l^k_t \) such that \( \delta_{l^k_t}(w) \) is continuous in \( 0 \leq t < \eta_0 \) under the vague topology and that such a \( k \)-label \( l^k_t \) is unique in \( [0, \eta_0) \).

We set

\[
(8.1) \quad \tau^k = \inf\{t > 0 : \inf_{u \in [0, t]} \inf_{j \neq k} |l^u_t(w) - l^j_t(w)| = 0\}.
\]

Then, we find that \( \tau^k \geq \eta_0 > 0 \).

For each \( t < \tau^k \), there exists an \( \epsilon_t\)-neighborhood \( S_{\epsilon_t, t} \) of \( l^k_t(w) \) such that \( w_t(S_{\epsilon_t, t}) = 1 \). Hence, there exists an \( \eta_t > 0 \) such that

\[
(8.2) \quad w_u(S_{\epsilon_t/2, t}) = 1 \quad \text{for all } u \in [t - \eta_t, t + \eta_t] \cap [0, \infty).
\]

From (8.2), we can extend the \( k \)-th label \( l^k_t \) such that \( \delta_{l^k_t}(w) \) is continuous in \( u \) under the vague topology in \( (t - \eta_t, t + \eta_t) \cap [0, \infty) \) and such an extension is unique. Hence, we can extend the \( k \)-th label \( l^k_t \) to \( l^k_0 \) such that \( \delta_{l^k_t}(w) \) is continuous in \( [0, \tau^k) \) under the vague topology. Such a label \( l^k_t \) is unique in \( 0 \leq t < \tau^k \) because of (8.2).

Suppose \( \tau^k < \infty \). Because \( w \) has no explosion, we see that \( \limsup_{t \to \tau^k} |l^k_t(w)| < \infty \). Hence, we can choose a subsequence \( t_\ell < \tau^k \) such that \( \lim_{\ell \to \infty} t_\ell = \tau^k \) and \( \lim_{\ell \to \infty} l^k_{t_\ell}(w) = a \) for some \( a \in \mathbb{R}^d \).

Let \( U_\epsilon(a) = \{ |a - x| \leq \epsilon \} \). Note that \( w_t = \sum_\ell \delta_{l^k_t(w)} \) is continuous under the vague topology and that \( U_\epsilon \) is a closed set. Hence using (8.1), we see that for each \( \epsilon > 0 \)

\[
w_{\tau^k}(U_\epsilon(a)) \geq \limsup_{t \to \tau^k} w_t(U_\epsilon(a)) \geq 2.
\]

Hence, \( w_{\tau^k}(\{a\}) \geq 2 \), which yields contradiction. We thus see that \( \tau^k = \infty \) and \( l^k_t(w) \) is continuous in \( 0 \leq t < \infty \) as \( \mathbb{R}^d \)-valued process.

Let \( t > 0 \). With a similar argument as above and using the assumption that \( w \) has no entering, we can extend the \( k \)-th label \( l^k_t \) of any label \( l = (l^i)_{i \in \mathbb{N}} \) at time \( t \) toward opposite direction such that \( l^k_t(w) \) is continuous and unique in \( [0, t] \).

We have thus seen that the \( k \)-th label \( l^k_t \) of any label \( l = (l^i)_{i \in \mathbb{N}} \) at time \( t = 0 \) can be extended to \( [0, \infty) \) uniquely in such a way that \( l^k_t(w) \) is continuous as an \( \mathbb{R}^d \)-valued process.

This argument holds for each \( k \in \mathbb{N} \). Hence, any label \( l = (l^i)_{i \in \mathbb{N}} \) can be extended to a label-valued path \( l_t \), \( t \in [0, \infty) \), such that \( l_t(w) \) is a \( (\mathbb{R}^d)^N \)-valued continuous process uniquely. We have thus obtained the claim. \( \square \)
9. APPENDIX 2: PROOF OF (2.17)

From (2.12) and (2.14), we have

\[
U_{R}^[\gamma][f, f](s) + U_{R}^[\gamma][f, f](s) = \frac{1}{2} \sum_{m=1}^{\infty} s_{R}^{m} \left( \frac{1}{\sqrt{s_{R}-1}} \frac{\partial f_{R,x}^{m}}{\partial \Gamma(S_{R})} - \frac{1}{s_{R}} \frac{\partial f_{R,x}^{m}}{\partial \Gamma(S_{R})} \right)^{2}
\]

\[
\frac{1}{2} \sum_{m=1}^{\infty} s_{R}^{m} \left( \frac{1}{\sqrt{s_{R}} s_{(T)}} \frac{\partial f_{R,x}^{m}}{\partial \Gamma(S_{R})} - \frac{1}{s_{R}} \frac{\partial f_{R,x}^{m}}{\partial \Gamma(S_{R})} \right)^{2}
\]

\[
\frac{1}{2} \sum_{m=1}^{\infty} s_{R}^{m} \left( \frac{1}{\sqrt{s_{R} s_{(T)}}} \frac{\partial f_{R,x}^{m}}{\partial \Gamma(S_{R})} - \frac{1}{s_{R}} \frac{\partial f_{R,x}^{m}}{\partial \Gamma(S_{R})} \right)^{2}
\]

Using \(s_{R-1} + s_{(T)} = s_{R}\) and \(\frac{\partial}{\partial \Gamma(S_{R-1})} + \frac{\partial}{\partial \Gamma(S_{R})} = \frac{\partial}{\partial \Gamma(S)}\), we see

\[
\left( \frac{1}{\sqrt{s_{R-1}}} \frac{\partial f_{R,x}^{m}}{\partial \Gamma(S_{R-1})} - \frac{1}{s_{R}} \frac{\partial f_{R,x}^{m}}{\partial \Gamma(S_{R})} \right)^{2}
\]

\[
\left( \frac{1}{\sqrt{s_{R}}} \frac{\partial f_{R,x}^{m}}{\partial \Gamma(S_{R})} - \frac{1}{s_{R}} \frac{\partial f_{R,x}^{m}}{\partial \Gamma(S_{R})} \right)^{2}
\]

Putting (9.2) into (9.1) and using (2.9) and (2.14), we have

\[
U_{R}^[\gamma][f, f](s) + U_{R}^[\gamma][f, f](s) = \frac{1}{2} \sum_{m=1}^{\infty} s_{R}^{m} \left( \frac{1}{\sqrt{s_{R}-1}} \frac{\partial f_{R,x}^{m}}{\partial \Gamma(S_{R})} - \frac{1}{s_{R}} \frac{\partial f_{R,x}^{m}}{\partial \Gamma(S_{R})} \right)^{2}
\]

\[
\frac{1}{2} \sum_{m=1}^{\infty} s_{R}^{m} \left( \frac{1}{\sqrt{s_{R}} s_{(T)}} \frac{\partial f_{R,x}^{m}}{\partial \Gamma(S_{R})} - \frac{1}{s_{R}} \frac{\partial f_{R,x}^{m}}{\partial \Gamma(S_{R})} \right)^{2}
\]

\[
\frac{1}{2} \sum_{m=1}^{\infty} s_{R}^{m} \left( \frac{1}{\sqrt{s_{R} s_{(T)}}} \frac{\partial f_{R,x}^{m}}{\partial \Gamma(S_{R})} - \frac{1}{s_{R}} \frac{\partial f_{R,x}^{m}}{\partial \Gamma(S_{R})} \right)^{2}
\]

We have thus completed the proof of (2.17). 

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