The Geometry of Reduction: Compound Reduction and Overlapping State Space Domains

Joshua Rosaler

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Abstract
The relationship whereby one physical theory encompasses the domain of empirical validity of another is widely known as “reduction.” Elsewhere I have argued that one influential methodology for showing that one physical theory reduces to another, associated with the so-called “Bronstein cube” of theories, rests on an oversimplified and excessively vague characterization of the mathematical relationship between theories that typically underpins reduction. I offer what I claim is a more precise characterization of this relationship, which here is based on a more basic notion of reduction between distinct models (one from each theory) of a single physical system. Reduction between two such models, I claim, rests on a particular type of approximation relationship between group actions over the models’ state spaces, characterized by a particular function between the model state spaces and a particular subset of the more encompassing model’s state space. Within this approach, I show formally in what sense and under what conditions reduction is transitive, so that reduction of a model 1 to another model 2 and reduction of model 2 to a third model 3 entails direct reduction of model 1 to model 3. Building on this analysis, I consider cases in which reduction of a model 1 to a model 3 may be effected via distinct intermediate models 2a and 2b, and motivate a set of formal consistency requirements between distinct “reduction paths” having the same models as their “end points”. These constraints are explicitly shown to hold in the reduction of a model of non-relativistic classical mechanics (model 1) to a model of relativistic quantum mechanics (model 3), which may be effected by a composite reduction that proceeds either via a model of non-relativistic quantum mechanics (model 2a) or a model of relativistic classical mechanics (model 2b). I offer some brief speculations as to whether and how this sort of consistency requirement might serve to constrain the reductions relating other theories and models, including the relationship that the Standard Model and general relativity must bear to any viable unification of these frameworks.
1 Introduction

A certain widely accepted view dictates that theories in physics progress toward increasing generality and unification. Galileo’s theory of terrestrial gravitation gave way to Newton’s more encompassing Universal Theory of Gravitation, which in turn gave way to Einstein’s still more encompassing theory of general relativity, which, it is expected, will in turn give way to some even more encompassing quantum theory of gravitation. Similarly, the theories of electrostatics and magnetostatics both gave way to Maxwellian electrodynamics, which gave way to quantum electrodynamics, which gave way to the Standard Model, which in turn, it is expected, will give way to a theory that also encompasses gravitational phenomena. The relationship between any pair of successive theories in such a sequence, whereby the later theory encompasses the domain of empirical validity of the earlier theory, is known as “reduction.” Reduction is the link between successive theories on which claims of increasing unification in physics are ultimately rooted. Indeed, the requirement that general relativity reduce to a theory of quantum gravity is part of what it means to be a theory of quantum gravity. Likewise, the requirement that the Standard Model reduce to beyond the Standard Model (BSM) theories is a necessary requirement on BSM theories, and the need to simultaneously reduce both general relativity and the Standard Model is a definitional requirement on any “theory of everything.”

In many cases, the conventional wisdom of steadily increasing unification in physics implicitly takes for granted reduction between certain established theories without explicit proof, on the basis of a certain philosophically motivated faith in the mathematical unity of nature. Nevertheless, the detailed mathematical relationships between theories that underpin reduction in many cases are far from trivial, as in the case of the reduction of classical to quantum theories, where a disparate range of concerns about quantum measurement, decoherence, the $\hbar \to 0$ limit, Ehrenfest’s Theorem, and other results all play a role. Likewise, concerns about renormalization make the reduction of quantum mechanics to quantum field theory, which is also widely taken for granted, more subtle than is typically suggested in most of discussions of the matter.

It is reasonable to expect that lessons gleaned from more careful study of relationships between established theories may be useful in the study of relationships between speculative models of new physics and current theories. For example, it is reasonable to expect that general foundational issues concerning decoherence, quantum measurement, and renormalization in the context of known quantum theories will

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1 There exist conflicting conventions as to the precise use of the term “reduction.” On one usage, it is the more encompassing theory that “reduces to” the less encompassing theory; such uses reflect the general connotation of reduction that suggests simplification, as in “6/4 reduces to 3/2.” On another set of usages, it is rather the less encompassing theory that “reduces to” the more encompassing theory; such uses reflect the alternative connotation of reduction as subsumption into a more general framework, as in the claim that “chemistry reduces to physics” or “thermodynamics reduces to statistical mechanics.” We will adopt the latter connotation here, so that the less encompassing description is understood to reduce to the more encompassing description. For further discussion of the distinction between these uses of the term “reduce,” see Nickles’ [21].

2 This point is also emphasized in Crowther [5].

3 Instead of “A reduces to B” one sometimes says that “B reduces A.”
have some role to play in understanding the connection between already established theories and whatever the correct theory of quantum gravity turns out to be.

The present discussion seeks to further develop the methodology of reduction elaborated in [25, 27], and [24]. Section 2 rehearses arguments to the effect that one popular approach to reduction, associated with the “Bronstein cube” of physical theories and based on the notion that reduction consists simply in taking limits of constants of nature, is flawed in its characterization of the technical requirements for reduction. Section 3 reviews the core elements of an alternative approach, based on the notion that reduction between theories fundamentally concerns a more basic notion of reduction between distinct models of the same physical system, and requires the identification of quantities within the reducing (more encompassing) model that approximately satisfy the same mathematical relations as physically salient structures of the reduced model in cases where the reduced model is empirically successful. Where relations between many models of fundamental physics are concerned, this instantiation can be shown to hold by virtue of a certain geometrical relationship between specific group actions over the state spaces of the two models. Section 4 advances the central claims of this article, which concern the manner in which distinct inter-model reductions may be composed to yield a direct reduction between theoretically remote descriptions of the same physical system, and advances a set of consistency requirements in cases where a single reduction may be effected via multiple distinct compound reductions differing in their intermediate layer of description. This consistency requirement is illustrated in the special case of the reduction of non-relativistic classical mechanics to relativistic quantum mechanics, which may be effected via an intermediate model either of non-relativistic quantum mechanics or of classical relativistic mechanics. On the view that reduction is simply a matter of taking some limit, critiqued in Sect. 2, it is sometimes suggested that the classical limit \( \hbar \to 0 \) should commute with the non-relativistic limit \( c \to \infty \), although presentations of the Bronstein cube typically do not specify precisely in what sense this might actually turn out to be the case, or which specific quantities one should be taking the limit of. The consistency condition described here serves to illustrate more explicitly one important sense in which the quantum-to-classical and relativistic-to-non-relativistic transitions commute with each other (see Fig. 1). Some brief speculative remarks are then offered as to how these consistency conditions might serve to constrain the relationship between potential models of new physics and models drawn from established theories. Section 5 is the Conclusion.

2 The Role of Limits in Reduction

Limiting relations, in which the mathematical structures of the reduced theory are recovered as some parameter in the reducing theory is varied, play a crucial role in

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4 Theories of “fundamental physics” here are understood to include non-statistical mechanical and non-thermodynamic theories, such as classical mechanics, quantum mechanics, relativistic quantum mechanics, quantum field theory, and quantum gravity. While many of the claims about reduction defended here also apply in the context of thermodynamics and statistical mechanics, these theories also introduce novel probabilistic aspects that demand special treatment.
Fig. 1 Two distinct strategies for mapping the reductions among physical theories. On the left is the so-called Bronstein cube of physical theories (a larger version of which is reproduced in Fig. 2), described in Sect. 2, which can be interpreted either as a vague heuristic or as a more stringent prescription for reduction, which holds that the mathematical correspondences underpinning reduction can be characterized simply as a matter of taking a limit of some constant of nature (e.g., $\hbar$, $c$, $G$). On the right is an alternative picture based on a certain geometrical relationship between state space models, described in Sects. 3 and 4. The image on the right (a larger version of which is reproduced in Fig. 8), which concerns the relationship between the classical and non-relativistic domains of relativistic quantum theory, concerns a set of theories associated with the bottom face of the Bronstein cube. This latter approach rests on the identification of a special subset of each reducing model’s state space, within which a certain function of the reducing model’s state approximately mimics the transformation properties of the high-level model’s state.

many reductions: the laws of Newtonian mechanics are recovered as an approximation to those of special relativity for speeds much less than that of light; effective field theories describing some species of light particle are recovered as an approximation to more encompassing effective field theories in the limit where other particle species are much more massive; thermodynamic regularities are recovered as an approximation to those of statistical mechanics when the number of degrees of freedom becomes very large; the list continues. The prevalence of limits across so many cases has given rise to a manner of speaking in which reduction is designated simply as the requirement that one theory be a limit or limiting case of another. That is, reduction is sometimes taken to require, seemingly as a matter of definition, that one theory be a mathematical limit of another. Yet, given that the ultimate aim of reduction is to show that all physical systems that can be modeled in the reduced theory can be modeled more precisely in the reducing theory (which also should incorporate phenomena outside the scope of the reduced theory) it is important to keep in mind that limiting relations are merely a means to this end, rather than the ultimate goal of reduction. For this reason, it has been argued that one approach to reduction, based on the notion that reduction is essentially about showing one theory to be a limit of another, and exemplified by the so-called “Bronstein cube” of physical theories, rests on a flawed understanding of the mathematical relationship between theories according to which one theory encompasses the domain of empirical validity of another.

As shown in Fig. 2, the Bronstein cube places the theories of modern physics—including some as yet undiscovered theory of quantum gravity, which is understood to encompass the empirical domains of validity of both general relativity and quantum field theory—at the corners of a cube, where passage between theories at opposite ends of a given edge is effected through a mathematical limit in which some constant of
nature is varied. According to the figure, transition from quantum to classical theories, represented by passage along the four edges parallel to the $\hbar$ axis, is effected by taking the limit $\hbar \to 0$ of vanishing Planck’s constant. Transition from relativistic to non-relativistic theories, represented by passage along the four horizontal edges, is effected by taking the limit $c \to \infty$ in which the speed of light becomes infinite. Transition from gravitational to non-gravitational theories, represented by passage along the four vertical edges, is effected by taking the limit $G \to 0$ of vanishing Newton’s constant. The Bronstein cube is thought to originate in a paper by Gamow, Ivanenko, and Landau, [14], and is further discussed in [4,8,30]. Recently, Oriti has proposed an extension of the cube, which he calls the “Bronstein hypercube,” to include a fourth axis corresponding to the number $N$ of degrees of freedom in a system [22].

The difficulties with this way of thinking about the relationships among the theories of modern physics have been discussed extensively elsewhere, so we review them only briefly here. Oriti himself, a leading authority on the Bronstein cube, writes that it provides only “an extremely rough sketch of theoretical physics.” As he explains, “It does not account even remotely for the complexity of phenomena that are actually described by the mentioned frameworks. And it does not say anything about the very many subtleties involved in actually moving from one framework to the other, and back from there” [22]. Moreover, the picture of inter-theory reduction provided by the cube is extremely vague in its requirements. Apart from the demand that some limit be involved in the transition between theoretical frameworks, this approach does not
offer any clear recipe for determining whether reduction holds in a given case. Given a pair of theories, how, precisely does one determine whether one is a limit of the other, or whether one reduces to the other? Does the existence of any limiting relationship between any two quantities of the theories suffice to establish one theory as a limit of another? Or is it necessary that these limits involve specific types of quantities within the theories? If the latter, specifically what sorts of quantities in the reduced theory must be recovered as limits of quantities in the reduced theory? Does every quantity in the reducing theory need to go over in the limit to some quantity in the reduced theory? If we interpret the limits of the cube naively and literally, it is clear that the limiting relations suggested by the cube do not hold; for example, naively taking the $\hbar \to 0$ limit of Schrödinger’s equation gives nonsense, not classical mechanics. More needs to be said about how one decides, given a pair of theories, whether one is a limit of another, and also how this serves to ensure subsumption of one theory’s domain by another. If the sole contents of the Bronstein cube is merely that the limits associated with edges of the cube are somehow related to the task of showing that one theory encompasses the domain of empirical validity of another, then it must be regarded as more of a vague heuristic than a well-formulated account of the requirements for reduction. In attempting to understand how one theory encompasses the domain of empirical validity of another, much more needs to be said in the context of concrete systems about how the reducing theory serves to represent those physical degrees of freedom that are well-described by the reduced theory.

A second worry, apart from vagueness, concerns whether the limits in the diagram commute, as the diagram indicates that they should [3,15]. For example, the diagram indicates that it should be possible to pass, say, from (relativistic) quantum field theory to Newtonian mechanics, by first taking the limit $\hbar \to 0$ and then the limit $c \to \infty$, or alternatively by first taking the limit $c \to \infty$ and then the limit $\hbar \to 0$. On the other hand, it means little to say that these limits should commute until one specifies the particular set of $\hbar$- and $c$-dependent quantities that one is taking these limits of. Many presentations of the Bronstein cube approach neglect to specify what these quantities are or how they should be determined for a given pair of theories. In the next sections, we will see how the notion that different reductions commute can be made more precise.

A third concern is that the limits are based on varying constants of nature, which are fixed for real systems; thus, it is sometimes argued that the relevance of such limiting relations for real systems is obscure [3,15]. A recent effort to address this objection in the context of the limit $\hbar \to 0$ has been made by Feintzeig, who attempts to interpret these limits not as counterfactual changes in the constants of nature, but as changes of the units in which the numerical values of these constants are given [10]. A similar effort in the context of the limit $c \to \infty$ has recently been made by Fletcher [11].

A fourth objection, expounded at length in [26], is that the approach to reduction based on the Bronstein cube treats reduction as a purely formal mathematical relationship, in the sense that it should be possible to determine whether one theory reduces to another given knowledge only of their mathematical frameworks. This view arises naturally from the tendency to see limiting relations as the end goal of reduction. However, recalling that reduction requires one theory to subsume the domain of empirical validity of another, it is clear that knowledge of the mathematical frameworks of the theories alone is not sufficient to determine whether one reduces to the other. It is also
necessary to have some empirical knowledge of the domain of empirical success of
the reduced theory, and of the precision with which it describes systems in its domain,
so that one knows just how closely and in what specific contexts the mathematical
frameworks of the theories need to dovetail in order to ensure that one subsumes the
domain of the other.

The Bronstein cube, and the notion that reduction is simply a matter of taking
limits of constants of nature, is useful perhaps as a rough, partial heuristic. Here I wish
to warn against the temptation to take the picture of reduction suggested by the
cube too seriously, and to conclude merely by virtue of the existence of some formal
limiting relation between theories that reduction has been shown to hold. Particularly
in the context of quantum-classical relations, it seems clear that the limit $\hbar \to 0$ alone
does not suffice for reduction since complex mechanisms associated with decoherence
and quantum measurement also must play a role in reduction in these cases.

3 Reduction Between State Space Models

As we have noted, one way of understanding what it means for one theory to reduce to
another is through the concept of domain subsumption: i.e., theory $A$ reduces to theory
$B$ if the domain of empirical validity of theory $A$ is strictly contained in the domain
of empirical validity of theory $B$. In other words, every physical system well-modeled
by theory $A$ is modeled at least as accurately in theory $B$, and theory $B$ also models
systems that are not well-modeled in theory $A$. For a given system $K$ in theory $A$’s
domain of applicability - which must also be in theory $B$’s domain of applicability if
$A$ reduces to $B$ - what is the nature of the mathematical relationship between theory
$A$’s and theory $B$’s models of system $K$ such that both consistently describe the same
physical behavior?

3.1 Reduction and Instantiation

An answer to the preceding question can be formulated in terms of the general concept
of instantiation. Broadly, what it is for $X$ to instantiate $Y$ another is for $X$ to fill the
role played by $Y$. Both a bob on a spring and a pendulum oscillating at small angles
approximately instantiate the model of a classical harmonic oscillator; the operating
systems on a Mac and PC both provide distinct instantiations of Microsoft Word; more
prosaically, Abraham Lincoln and Barack Obama instantiate the role of president of
the United States. In the context of reduction, a more encompassing theory $B$ is able to
represent those features of system $K$ that are well described by theory $A$ by virtue of
the fact that theory $B$ approximately instantiates certain physically salient features of
theory $A$. Expanding on ideas of Dennett, Wallace characterizes reduction as follows:
“This instantiation relation (I claim) is the right way of understanding the relationship
between different scientific theories—the sense in which one theory may be said to
‘reduce’ to another. Crucially: this “reduction”, on the instantiation model, is a local
affair: it is not that one theory is a limiting case of another per se, but that, in a particular
situation, the ‘reducing’ theory instantiates the ‘reduced’ one” ’ [31], Chap. 2.
Building on this characterization, reduction of theory $T_h$ to theory $T_l$ requires that any circumstance under which the behavior of a real system $K$, understood as some set of physical degrees of freedom, is accurately modeled in $T_h$ also be a circumstance under which it can be modeled more accurately and universally in $T_l$. (The subscripts $h$ and $l$ designate the “high-level” or less encompassing description and the “low-level” or more encompassing description, respectively.) Thus, reduction between theories $T_h$ and $T_l$ is grounded in reductions between specific models $M_h$ (drawn from $T_h$) and $M_l$ (drawn from $T_l$) of systems $K$ in $T_h$’s domain of empirical validity. Model $M_h$ of system $K$ reduces to model $M_l$ of system $K$ by virtue of the fact that certain quantities defined within $M_l$ approximately instantiate the physically relevant properties and relations described by $M_h$ in all cases where $M_h$ furnishes an approximate description of $K$’s behavior.

The purpose of the discussion below is to mathematically formalize the notion of reduction in terms of instantiation in order to answer the question, “What sort of mathematical relationship must hold between two models of the same physical system in order that one model reduce to another—i.e., in order that one model provide a strictly more detailed, accurate, and broadly applicable description of the system?” Here, we understand a model $M$ of physical system $K$ to be specified by a state space $S$, which in the cases of interest to us here possesses the structure of a differentiable manifold (and often also the structure of a vector space), a notion of distance between states generated by a metric or norm defined over the manifold, and some further structure defined over the state space manifold that prescribes the dynamics of the model (e.g., Lagrangian, Hamiltonian, equations of motion).

What does it mean for one model $M_h$ of some set of physical degrees of freedom $K$ to reduce to another model $M_l$? On the understanding of reduction advocated here, it means that every circumstance in which the physical degrees of freedom associated with system $K$ are well modeled by $M_h$ is a circumstance in which these same degrees of freedom are modeled more accurately by $M_l$. However, the state spaces of $M_l$ and $M_h$ may represent different degrees of freedom (dof’s), where the dof’s represented by the latter “supervene on,” or are uniquely determined by, those represented by the former. This dependence or supervenience relation is typically represented by a fixed function $B : S_l \to S_h$ from the low-level state space $S_l$ to the high-level state space $S_h$ that establishes a mathematical bridge between the models and the physical degrees of freedom that they describe. The quantity $B(x_l)$, whose behavior is determined entirely by the transformation properties of the low-level state $x_l$ specified by $M_l$, specifies $M_l$’s representation of the degrees of freedom that are represented by the high-level model’s state $x_h$. For example, the center of mass of a classical composite object (such as a baseball or planet) may be modeled either by a classical Hamiltonian model whose state space directly represents the possible positions and momenta of the object’s center of mass or by a more detailed classical model describing the behavior of the object’s microscopic constituents, where the physical degrees of freedom described by the high-level macroscopic model are represented instead by a particular function $B$ of the microscopic state. That is, a particular function of the

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5 Property A (e.g., a macrostate) supervenes on property B (e.g., a microstate) if and only if there can be no difference in A without a difference in A. The value of property B uniquely determines the value of property A while the reverse is typically not the case.
positions and momenta of the microscopic constituents approximately instantiates the macroscopic Hamiltonian dynamics of the phase space point used to describe the motion of the ball’s center of mass. However, as we will see explicitly below, in cases of reduction $B(x_l)$ typically only mimics the behavior of the high-level state $x_h$ prescribed by $M_h$ when $x_l$ resides in some restricted subset $d \subset S_l$. For reduction to occur, the induced trajectory $B(x_l(t))$ must approximate the high-level trajectory $x_h(t)$ roughly within the empirically determined margin of error $\delta_{emp}$ within which the high-level $x_h(\tau)$ is known to track the real physical behavior of the degrees of freedom $K$. Moreover, it must do so over the empirical timescale $T_{emp}$ over which the high-level trajectory $x_h(t)$ tracks $K$ within $\delta_{emp}$.

In more general contexts, the relevant parameter characterizing dynamical flows in the state space may not necessarily be time, but some more general variable (or set of variables) $\tau$ that parametrizes physical solutions of the model. Generally, reduction requires that for every physically realistic\(^6\) solution $x_h(\tau)$ of $M_h$, there exist some physically realistic solution $x_l(\tau)$ of $M_l$ such that $|B(x_l(\tau)) - x_h(\tau)|_h < 2\delta_{emp}$ over ranges of $\tau$ for which $x_h(\tau)$ approximates the behavior of $K$ within $\delta_{emp}$.\(^7\) That is, the high-level trajectory $B(x_l(\tau))$ induced by the low-level solution $x_l(\tau)$ via $B$ approximately instantiates the high-level solution $x_h(\tau)$ over the required range of $\tau$.

In the next subsection, we discuss how these requirements can be formalized in terms of flows associated with group actions over the models’ respective state spaces.

3.2 Formal Requirements: Relating State Space Group Actions Over a Restricted State Space Domain

This section explains how the state-space-based approach to reduction described in general terms in the previous subsection may be formalized as a relationship between group actions over the state spaces of the reduced and reducing models. The mathematical style of the discussion will be informal so as not to distract from the central concepts. However, I explain in broad terms how some of the more detailed technicalities can be filled in more rigorously.

Here, we focus on the large set of cases where both high- and low-level models can be formulated in terms of first-order, deterministic dynamical equations of motion over some state space possessing the structure of a differentiable manifold and a metric that defines a notion of distance between states\(^8\):

\(^6\) “Physically realistic” here indicates that the trajectory approximates the behavior of the real physical degrees of freedom described by the model to within some specified error tolerance.

\(^7\) The difference between $x_h(\tau)$ and $B(x_l(\tau))$ is required to be less than $2\delta_{emp}$ rather than $\delta_{emp}$ because if the $x_h(\tau)$ and $B(x_l(\tau))$ both approximate $K$’s behavior within error bound $\delta_{emp}$, then they may differ from each other by at most $2\delta_{emp}$.

\(^8\) As suggested above, many features of the approach to reduction described here can be extended straightforwardly to reductions involving models with stochastic dynamics, with certain important modifications to accommodate the probabilistic nature of the models. For example, approximate equality of induced and high-level state space trajectories is replaced by approximate equality with high likelihood. The understanding of reduction in terms of instantiation may also be extended to non-dynamical models such as the Ideal Gas model although we do not explore this here.
\[
\frac{dx^\mu_h}{d\tau} = V^\mu_h \big|_{x_h},
\]
\[
\frac{dx^\mu_l}{d\tau} = V^\mu_l \big|_{x_l},
\]
where \( \tau \) is a flow parameter (usually time), the high-level dynamics are generated by vector field \( V^\mu_h \big|_{x_h} = \frac{dx^\mu_h}{d\tau} \in T_{x_h} S_h \) and the low-level dynamics by \( V^\mu_l \big|_{x_l} = \frac{dx^\mu_l}{d\tau} \in T_{x_l} S_l \). The \( M_h \)-prescribed evolution of an arbitrary initial high-level state \( x^0_h \in S_h \) is then given by \( x_h(\tau) = \left[ e^{\tau V_h} x_h \right]_{x_h=x^0_h} \). Likewise, the \( M_l \)-prescribed evolution of an arbitrary initial low-level state is given by \( x_l(\tau) = \left[ e^{\tau V_l} x_l \right]_{x_l=x^0_l} \).

Since the degrees of freedom described by \( M_h \) are presumed to supervene on the degrees of freedom described by \( M_l \), there will be some function \( B : S_l \to S_h \) that serves to characterize the mathematical dependence of the former on the latter and thereby to identify \( M_l \)'s representation, \( B(x_l) \), of the degrees of freedom described by \( M_h \). Informally, reduction requires that whenever \( x_h(\tau) \) accurately describes the behavior of the system \( K \), there exist a physically possible low-level trajectory \( x_l(\tau) \) such that \( x_h(\tau) \approx B(x_l(\tau)) \), or more explicitly,

\[
\left[ e^{\tau V_h} x_h \right]_{x_h=B(x^0_l)} \approx B \left( \left[ e^{\tau V_l} x_l \right]_{x_l=x^0_l} \right). \tag{1}
\]

In other words, mapping up to the high-level state space with \( B \) and then applying the high-level dynamics should yield approximately the same result as applying the low-level dynamics and then applying \( B \); that is, dynamical evolution should approximately commute with the function \( B \).

Let us make this more precise. Consider first the notion of approximate equality implied by the symbol \( \approx \) in (1). When can the high-level trajectory \( x_h(t) \) and the induced trajectory \( B(x_l(t)) \) be considered approximately equal? Since reduction requires that \( M_l \)'s description of the physical degrees of freedom \( K \) be more accurate than \( M_h \)'s, it is sensible to require that \( x_h(t) \) and \( B(x_l(t)) \) agree within the empirical margin of error \( \delta_{emp} \) within which \( B(x_l(t)) \) is known to track the behavior of \( K \). Moreover, this margin should be respected over ranges \( T_{emp} \) of the flow parameter \( \tau \) for which the trajectory \( x_h(t) \) tracks \( K \) within \( \delta_{emp} \). Thus, relation (1) can be stated more precisely as the requirement that

\[
\left| \left[ e^{\tau V_h} x_h \right]_{x_h=B(x^0_l)} - B \left( \left[ e^{\tau V_l} x_l \right]_{x_l=x^0_l} \right) \right|_h < 2\delta_{emp} \tag{2}
\]
for \( 0 \leq \tau < T_{emp} \) (see Fig. 3). In classical phase space, for example, the norm \( \| \cdot \|_h \) can be taken as the Euclidean norm; in Hilbert space, it can be taken as the norm associated

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9 For example, the deterministic equations of classical mechanics are only known to describe the trajectory of Saturn’s moon Hyperion over certain limited timescales, beyond which classical predictability is lost due to quantum and classically chaotic effects. See for example [32] and [33] for further details. Quantum models should only be required to recover classical trajectories over timescales for which these classical trajectories furnish an accurate representation of the system in question.
In cases where a single system $K$ can be described both by a high-level model $M_h$ and a low-level model $M_l$, which may describe more fundamental degrees of freedom, reduction of $M_h$ to $M_l$ requires that the quantity $B(x_l(\tau))$ (dotted curve in $S_h$) approximate $x_h(\tau)$ (solid curve in $S_h$) within empirical margin $2\delta_{\text{emp}}$ (indicated by light grey lines in $S_h$) with the Hilbert space inner product. It is particularly important to emphasize that the relation (2) will generally hold only for as long as the low-level state $x_l(\tau)$ remains in some restricted subset $d \subset S_l$ of the low-level state space; when it leaves this subset, the difference between $x_h(\tau)$ and $B(x_l(\tau))$ will exceed this margin. When these two representations of $K$ diverge, reduction requires the low-level model’s representation $B(x_l(\tau))$ to be the more accurate of the two.

It is useful to note that the condition (1) will hold if the quantity $B(x_l(\tau))$ approximately satisfies the high-level equations of motion satisfied by $x_h(\tau)$:

$$\frac{dB^\mu(x_l(\tau))}{d\tau} \approx V^\mu_h |_{B(x_l(\tau))}.$$  \hspace{1cm} (3)

For (2) to hold, it suffices that the approximate equality (3) hold to within a margin equal to $\delta_{\text{emp}}/T_{\text{emp}}$. In general, (3) will hold only for $x_l$ in some restricted subset $d \subset S_l$. Once the flow $x_l(\tau)$ carries the evolution out of $d$, the approximation in (3), characterized by $\delta_{\text{emp}}$ and $T_{\text{emp}}$, will cease to hold. \hspace{1cm} (11) Applying the Chain Rule to (3) and substituting

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\[ \text{One can see that (1) follows from (3) by integrating both sides of (3) with respect to the parameter } \tau. \]

\[ \text{It is worth emphasizing here that relation (3) bears a close resemblance in certain respects to the requirements for reduction proposed by Ernest Nagel and Kenneth Schaffner, for whom reduction required that it be possible to logically deduce approximate versions of the laws of the reduced theory from those of the reducing theory via the use of “bridge laws” [20,29]. The approach here is in some ways similar in spirit to Nagel/Schaffner approach in showing that, by virtue, of the low-level model’s equations of motion, } B(x_l) \text{ approximately satisfies the high-level model’s equations of motion. However, unlike their approach, the} \]
If the high-level dynamical vector field \( V_h \) evaluated at \( B(x_l) \) is approximately equal to the push-forward of the low-level vector field \( V_l \) (dotted arrows in upper state space) evaluated at \( x_l \), for all \( x_l \in d \), then the trajectories induced on \( S_h \) through \( B \) by the integral curves of \( V_l \) in \( d \) will approximate the integral curves of \( V_h \) in the image domain \( B(d) \subset S_h \).



\[
V_l^{\mu}(x_l) \quad \text{for} \quad \frac{dx_l^\mu}{d\tau} = V_l(x_l), \quad \text{one can easily check that (3) and (1) will be satisfied if the push forward of } V_l \text{ under } B, \text{ evaluated at any } x_l \in d, \text{ is approximately equal to } V_h \text{ evaluated at } B(x_l) \text{ (see Fig. 4):}
\]

\[
\begin{align*}
\frac{\partial B^\mu}{\partial x_l^\alpha} \bigg|_{x_l} V_l^{\alpha}(x_l) & \approx V_h^{\mu}(B(x_l)) \quad \text{for } x_l \in d. 
\end{align*}
\]  

In fact, one may take the condition (4) for given \( V_l \) and \( V_h \), where \( \approx \) holds within \( \delta_{emp}/T_{emp} \), to define the domain \( d \subset S_l \). That is, one may define \( d \) as the set of states for which (4) holds within the required empirically established margin of approximation for the given dynamical vector fields \( V_l \) and \( V_h \). Note that all reference to the flow parameter \( \tau \) is removed from this formulation of the matching condition between the dynamical flows prescribed by the different models, which concerns the relationship between the dynamical vector fields of the two models rather than between their integral curves

Footnote 11 continued
requirements for reduction here are formalized mathematically within the specific context of group actions over state space manifolds. Moreover, reduction here concerns relations between two specific models of a single fixed system, rather than between entire theories as in the Nagel/Schaffner approach. For recent discussion of Nagel and Schaffer’s approach to reduction, see [1,7,29].

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If the high-level dynamical vector field $V_h$ evaluated at $B(x_l)$ is approximately equal to the push-forward of the low-level vector field $V_l$ (dotted arrows in upper state space) evaluated at $x_l$, for all $x_l \in d$, then the trajectory induced on $S_h$ through $B$ by the integral curves of $V_l$ in $d$ will approximate the integral curves of $V_h$ in the image domain $B(d) \subset S_h$ as in Eq. (1). Relation (4) can be seen as a “differential” requirement for reduction, while the relation (2) is the corresponding “integral” requirement.

In fact, it sometimes happens that the relationship (4) extends beyond the particular vector fields $V_h$ and $V_l$ that generate the models’ dynamical evolution to include the full algebra of physical symmetry generators 12 of these models. In such cases, if \{\(V_h, U^1_h, \ldots, U^n_h\)\} is a basis for the algebra of vector field generators over $S_h$ of $M_h$’s physical symmetries, then there exist corresponding \{\(V_l, U^1_l, \ldots, U^n_l\)\} in the algebra of vector field generators over $S_l$ of $M_l$’s physical symmetries such that

$$\partial B^\mu / \partial x^\alpha \bigg|_{xl} [V_l, U^i_l]^\alpha \bigg|_{xl} \approx [V_h, U^i_h]^\mu \bigg|_{B(x_l)} \quad \text{for } x_l \in d \subset S_l \quad \text{for all } 1 \leq i \leq n$$

$$\partial B^\mu / \partial x^\alpha \bigg|_{xl} [U^i_l, U^j_l]^\alpha \bigg|_{xl} \approx [U^i_h, U^j_h]^\mu \bigg|_{B(x_l)} \quad \text{for } x_l \in d \subset S_l \quad \text{for all } 1 \leq i, j \leq n;$$

see Fig. 5. The margins of error in the approximate equalities are set to ensure that the generated flows agree within the required empirically determined margin of error. The domain $d$ can be defined as the set of low-level states, if any, for which the approximate relations (5) hold within the necessary empirically determined margin of approximation.

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12 Physical symmetries are understood here as those that map between physically distinct states when the symmetries are interpreted as active transformations, while gauge symmetries only map between redundant representations of a single physical state and never between physically distinct states.
For explicit demonstration of this claim with regard to the relationship between unitary and canonical group actions over quantum and classical state spaces (respectively), see [27]. Like the well-known analysis of Inonu and Wigner concerning group contractions, this relationship reveals a particular type of mathematical connection between the symmetry groups of different theories [17]. However, it is distinct from the analysis of Wigner and Inonu in that on the current approach, the dovetailing between group actions is restricted to low-level group actions operating within a certain subset $d$ of the reducing model’s state space; on Inonu and Wigner’s approach, there is no such state space restriction. Moreover, Wigner and Inonu’s approach does not rely essentially, as the present approach does, on the mapping $B : S_l \rightarrow S_h$ between state spaces that identifies the low-level model’s proxy for the high-level state $x_h$.

### 3.3 Examples

Let us now examine several examples of the general relationship between state space models just described. These examples are inter-related, and specially chosen to illustrate the main claims of the next two sections, which concern the manner in which distinct inter-model reductions may be composed to form new reductions, and the consistency requirements that must be satisfied when a single reduction may be effected via multiple composite reductions differing in their intermediate layer of description.\(^{13}\) I present each example first by specifying state spaces and equations of motion of the models involved and then detailing the form of the function $B$, the relevant state space domain $d \subset S_l$, and the specific form of (3), which in each case can be shown to hold using the low-level equations of motion. For concreteness, we consider all models as alternative descriptions of the same physical system, consisting of a massive charged particle (e.g., proton) in a background electromagnetic potential - however these methods should apply to any system whose behavior is well-described by multiple models within the general class considered here. While these examples for the most part formulate known results within the general framework just presented, their primary purpose is to show how reductions can be formally composed to yield new reductions, and to illustrate the formal consistency conditions that should be satisfied when a single reduction may be effected as the composition of two component reductions in more than one way. In the following sections, we will begin to see the usefulness of this framework both in clarifying the nature of the relationship between the quantum-to-classical and relativistic-to-non-relativistic transitions, as well as the relationship between speculative theories of new physics and currently established theories. They will furthermore reveal a particular mathematical sense in which different sorts of inter-theoretic transition (e.g., classical-to-quantum and non-relativistic-to-relativistic) may commute with each other.

\(^{13}\) Further examples of this relationship may be found in [25, 27], and [24].
3.3.1 Newtonian Mechanics (NM)/ Quantum Mechanics (QM)

Both Newtonian (i.e., non-relativistic classical) and quantum mechanical models can be used within a certain regime to accurately describe the evolution of a charged particle such as a proton in a background electric field. The state space of the classical model, $S_c = \Gamma$, is the classical one-particle phase space $\Gamma$; its equations of motion are Hamilton’s equations parametrized by the proton mass $m$, charge $q$, electrostatic potential $V(x)$ and static magnetic vector potential $A(x)$. The state space of the quantum model, $S_q = H$ is the one-particle Hilbert space (let us focus on the position degree of freedom, ignoring spin); its equation of motion is Schrodinger’s equation, parametrized by the same $m$, $q$, $V(x)$, $A(x)$. The bridge function $B : H \rightarrow \Gamma$ and domain $d \subset H$ in this case are given by:

$$B(|\psi\rangle) = \left( \langle \psi | \hat{x} | \psi \rangle, \langle \psi | \hat{p} | \psi \rangle \right)$$

$$d = \{ \text{“narrow wave packet states”} \}$$

where the wave packets in $d$ are narrow in the specific sense that their position space widths are small by comparison with the characteristic length scale on which the potentials $V(x)$ and $A(x)$ vary. By the generalization of Ehrenfest’s Theorem for a charge in a static background electromagnetic field with potentials $V$ and $A$, it follows from Schrodinger’s equation that $|\psi\rangle \in d$,

$$\frac{d}{dt} \langle \hat{p} \rangle \approx qE(\langle \hat{x} \rangle) + q \frac{d}{dt} \langle \hat{x} \rangle \times B(\langle \hat{x} \rangle)$$

$$\frac{d}{dt} \langle \hat{x} \rangle \approx \frac{1}{m} \left( \langle \hat{p} \rangle - qA(\langle \hat{x} \rangle) \right);$$

where $E(x) = -\nabla V(x) - \frac{1}{c} \frac{\partial A(x)}{\partial t} = -\nabla V(x)$ and $B(x) = \nabla \times A(x)$. The terms on the right-hand side can be understood as components of the Hamiltonian vector field that generates dynamical flows over classical phase space. Thus, $B(|\psi\rangle)$ approximately satisfies the classical Hamilton equations, and so satisfies (3). This will only hold over the timescales over which $|\psi\rangle$ remains in $d$ - i.e. the timescales over which the wave packet remains sufficiently narrowly peaked in position. This timescale will generally be longer for larger values of $m$, which reduce the rate of wave packet spreading under the Schrodinger evolution. The neglected errors in the approximate equalities $\approx$ are proportional to the position and momentum widths of the quantum state $|\psi\rangle$.

14 Note that despite the subatomic nature of the charged particles that are collided in accelerators like the LHC, it is the classical Lorentz Force Law, rather than a quantum equation, that is used to guide their motions.
15 Assuming that effects of decoherence can be ignored—which is realistic for such small systems—the dynamics of the charged particle can be modeled by a purely unitary dynamics to a good approximation.
16 See, e.g., [12] for proof and discussion of this result.
17 See, e.g., [24], Chap. 2 for further discussion of this well-known result.
3.3.2 Quantum Mechanics (QM)/ Relativistic Quantum Mechanics (RQM)

Models of both non-relativistic and relativistic quantum mechanics can be used to accurately describe the behavior of a low-momentum (<< $mc$), but not necessarily localized, charge such as a proton. The state space of the non-relativistic quantum model, $S_h = \mathcal{H}_{Pauli}$, now explicitly including spin, is the Pauli Hilbert space of 2-spinors; the equation of motion of such a model is the Pauli equation for non-relativistic spin-1/2 particles, with parameter values $m$ and $q$, and background potentials $V(x)$, $A(x)$. The state space of the RQM model, $S_l = \mathcal{H}_{Dirac}$, is the Dirac Hilbert space of 4-spinors; the model’s equation of motion is the Dirac equation with parameters $q$, $m$, $V$, and $A$ as above. The bridge function $B : \mathcal{H}_{Dirac} \rightarrow \mathcal{H}_{Pauli}$ and domain $d \subset \mathcal{H}_{Dirac}$ are given by

$$
B^a[\psi^a(x, t)] = e^{imt} P^a_d \psi^a(x, t)
$$

$$
d = \{ \text{“low-momentum, positive energy 4-spinors”} \}
$$

where $\psi^a(x, t)$ is the 4-spinor wave function, and the operator $P^a_d$ projects 4-spinors onto their upper two components. It is well known that for low-momentum four-spinors in the Dirac representation—i.e., for $\psi^a(x) \in d$ - the upper two components approximately satisfy the Pauli equation:

$$
i \hbar \frac{\partial}{\partial t} \left( e^{imt} P^a_d \psi^a(x, t) \right) \approx \left\{ \frac{1}{2m} \left[ \sigma \cdot (-i\hbar \nabla - qA(x)) \right]^2 + qV(x) \right\}^{\alpha\beta} \left( e^{imt} P^\beta_d \psi^a(x, t) \right).
$$

(7)

This shows that (3) is satisfied in this case; see, e.g., [6] for proof of this claim. This approximate equality will only hold for as long as the function $\psi^a(x, t)$ remains in the subset $d$ of low-momentum states. The neglected error in the approximation $\approx$ is proportional to $\frac{\mu^4}{m^4}$, where $\mu$ is the upper limit of modes in the momentum-space expansion of $\psi^a(x, t)$.

3.3.3 Special Relativity (SR)/ Relativistic Quantum Mechanics (RQM)

Both relativistic classical and relativistic quantum models may be used to describe a charge with any kinetic energy, as long as its quantum mechanical wave packet is not too spread out. The state space of the classical model is the relativistic classical phase space $S_h = \Gamma_{rel}$; the equations of motion are the relativistic Hamilton equations with parameters $m$, $q$, $V(x)$, and $A(x)$ as above. The state space of the quantum model is the Dirac Hilbert space $S_l = \mathcal{H}_{Dirac}$; the equations of motion are as in the previous example. The bridge function $B : \mathcal{H}_{Dirac} \rightarrow \Gamma_{rel}$ and domain $d \subset \mathcal{H}_{Dirac}$ are given by

$$
B[\psi^a(x)] = \left( \int d^3x \ x \ \psi^a(\hat{x}) \psi^a(x), \int d^3x \ \psi^a(\hat{x})(-i\hbar \nabla)\psi^a(x) \right) = \left( \langle \hat{x} \rangle, \langle \hat{p} \rangle \right)
$$

$$
d = \{ \text{“narrow 4-spinor wave packets”} \}.
$$
$B$ maps a 4-spinor into the corresponding expectation values of 3-position and 3-momentum, and $d$ consists of Dirac spinor fields that are narrowly peaked in position and momentum (although the degree to which they may be simultaneously peaked in both is of course constrained by the uncertainty principle). In this case, one can prove a relativistic analogue of Ehrenfest’s Theorem, from which it follows that expectation values approximately satisfy the relativistic Lorentz Force Law:

\[ \frac{d}{dt} \langle \hat{\mathbf{p}} \rangle \approx qE(\langle \hat{\mathbf{x}} \rangle) + q \frac{d}{dt} \langle \hat{\mathbf{x}} \rangle \times B(\langle \hat{\mathbf{x}} \rangle) \]
\[ \frac{d}{dt} \langle \hat{\mathbf{x}} \rangle \approx \frac{1}{m\gamma(\langle \hat{\mathbf{p}} \rangle)} \langle \hat{\mathbf{p}} \rangle \]

(8)

for $\psi^a(x) \in d$; see [24], Chap. 4 for proof of this claim. Thus, (3) holds in this case as well, but only for as long the wave packet remains sufficiently narrow in both position and momentum (within the constraints of the uncertainty principle). The neglected errors in the approximation $\approx$ are proportional to the position and momentum space widths of the Dirac wave packet.

### 3.3.4 Newtonian Mechanics (NM)/ Special Relativity (SR)

A slow-moving charge may be described either by a model of Newtonian mechanics or of classical relativistic mechanics. As we have seen, the state space of the classical model is the classical one-particle phase space $S_h = \Gamma$; its equations of motion are Hamilton’s equations parametrized by the mass $m$ and charge $q$ and potentials $V(x)$, $A(x)$. The state space of the classical model is the relativistic classical phase space $S_h = \Gamma_{rel}$; the equations of motion are the relativistic Hamilton equations with parameters $m$, $q$, $V(x)$, $A(x)$ as above. The bridge function $B : \Gamma_{rel} \to \Gamma$ and domain $d \subset \Gamma_{rel}$ are given simply by

\[ B(x, p) \equiv (x, p), \]
\[ d = \{(x, p) \in \Gamma_{rel} | |v| << c\} \]

where $v = \frac{cp-qA(x)}{\sqrt{(p-\frac{q}{c}A(x))^2 + m^2c^2}}$. The relation (3) in this case takes the form,

\[ \frac{dp}{dt} = qE(x) + q \frac{dx}{dt} \times B(x) \]
\[ \frac{dx}{dt} \approx \frac{1}{m} (p - qA(x)) \]

(9)

and follows straightforwardly from expansion in powers of $\frac{v}{c}$. As always, this approximation holds only for as long as the low-level state remains in the subset $d$ - that is, as long as $|v| << c$. The neglected terms in the approximate equality $\approx$ are proportional to $\frac{v^3}{c^3}$.
3.4 Extension to Other Cases

The class of models considered here, in which the state space has the structure of a differentiable manifold with metric and the state dynamics can be specified as the integral curves of some vector field over the state space, includes many examples in addition to the ones treated above. Like models of QM, the effective field theories (EFTs) of elementary particle physics can be formulated in terms of a unitary evolution on some Hilbert space, so that reduction between two EFTs—e.g., reduction of QED or Fermi weak theory to electroweak theory, or of electroweak theory to the Standard Model—can be treated using the methods employed here. Likewise, Newtonian and Einsteinian models of gravity, as well as canonical models of quantum gravity, all belong to this class of models, so that reductions connecting these models can also be approached in using the methodology of bridge functions and state space domains.

3.5 A Note on Gauge Theories

The framework for reduction described above assumes that the reduced and reducing models both can be formulated in terms of deterministic flows generated by some dynamical vector fields over their respective state spaces. While these conditions are met by many pairs of models between which reduction holds, care must be taken when considering reductions involving gauge theories, since it is not the dynamics of the gauge fields themselves that is deterministic (prior to fixing a gauge), but rather the dynamics of gauge invariant quantities constructed from the gauge fields. While the dynamics of the gauge fields can be made deterministic by fixing a gauge, the choice of gauge is purely a matter of convention and has no physical import. In recovering the empirical successes of a model associated with some gauge theory, it is only necessary to approximate the gauge invariant features of the state evolution by some induced evolution $B(x_l(\tau))$. In the case of classical electrodynamics, this would consist only of the transverse components of the electromagnetic 4-potential $A^\mu(x)$. In quantized

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18 The reduction of Newtonian gravity to general relativity has been examined extensively in the work of Ehlers and Rohrlich [9,23]. I leave it to future work to assess the compatibility of Ehlers’ and Rohrlich’s treatments with the methodology described here. However, it is possible to offer a preliminary sketch of how an analysis of this reduction within the present framework would proceed. The high-level state space would be the classical phase space of N particles and the high-level model’s dynamics would be specified by a Hamiltonian with Newtonian gravitational potential. The low-level model could be formulated in terms of a 3 + 1 splitting of spacetime (assuming global hyperbolicity), so that its state space consists of configurations of the 3-metric and corresponding conjugate momenta, together with the configurations and conjugate momenta of any matter degrees of freedom. The dynamics of such a model would be given in terms of a Hamiltonian formulation of Einstein’s field equations coupled with some evolution equations for the matter degrees of freedom. The state space domain of such a reduction would consist of 3-metric configurations and momenta consistent with a weak-field approximation (i.e., close to the flat Minkowski metric), and of states of the matter degrees of freedom in which the kinetic energy is much smaller than the rest energy. The bridge function would map the low-level state into the high-level state space by mapping the coordinates and momenta of the matter degrees of freedom into a locally inertial coordinate system. While the gravitational degrees of freedom are dynamical in the low-level model, in the high-level Newtonian model they are not, so that there are no gravitational degrees of freedom represented in the high-level Newtonian model’s state space. Instead, the influence of the gravitational degrees of freedom described by the low-level model manifest their influence within the high-level model by serving to determine the form Hamiltonian of the high-level model.
gauge theories, one likewise should only demand to recover gauge-invariant features of the state evolution from a reducing model, although the precise manner of doing so depends on the specific method of quantization that is used to define the quantized gauge theory.

4 Compound Reduction and Overlapping State Space Domains

Intuitively, it is natural to expect that reduction should be transitive: that is, if, relative to some physical system $K$, model $M_1$ reduces to model $M_2$ and model $M_2$ reduces to model $M_3$, then model $M_1$ should reduce directly to model $M_3$. Here, we will see how this intuition can be formalized straightforwardly within the general framework described in the previous section, which takes a reduction between two models $M_h$ and $M_l$ to be specified by a subset $d \subset S_l$ of the low-level model’s state space and “bridge” function $B : S_l \to S_h$ from the low- to the high-level state space.

Denote the bridge function and state space domain for the reduction of $M_1$ to $M_2$ (abbreviated $1 \to 2$) respectively as $B_1^2$ and $d_2^1$, and the bridge function and state space domain for the reduction of $M_2$ to $M_3$ (abbreviated $2 \to 3$) respectively as $B_2^3$ and $d_3^2$. Then the bridge function and state space domain of the direct reduction of $M_1$ to $M_3$ (abbreviated $1 \to 3$) are given respectively by

$$B_1^3(x_3) = B_2^2(B_3^2(x_3)),$$

$$d_3^1 = d_3^2 \cap B_3^{2,-1}(d_2^1),$$

(10)

where $B_3^{2,-1}(d_2^1)$ is the inverse image of $d_2^1$ under $B_3^2$. Reduction of $M_1$ to $M_3$ occurs only if the set $d_3^1$ is non-empty—that is, only if the image $B_3^2(d_3^2)$ of the domain $d_3^2$ of the low-level reduction $2 \to 3$ overlaps with the state space domain $d_2^1$ of the high-level reduction $1 \to 2$. The set $d_3^1 = d_3^2 \cap B_3^{2,-1}(d_2^1)$ is then just the subset of the state space domain $d_3^2$ of the $2 \to 3$ reduction whose image under $B_3^2$ lies in the state space domain $d_2^1$ of the $1 \to 2$ reduction. See Fig. 6 for a diagrammatic illustration.

Reduction of $M_1$ to $M_3$ requires that the quantity $B_1^3(x_3(\tau))$, whose behavior is governed by the dynamics of $M_3$, approximately satisfies the equations of motion of $M_1$ when $x_3(\tau) \in d_3^1 = d_3^2 \cap B_3^{2,-1}(d_2^1)$:

$$\frac{d}{d\tau} B_2^1(B_3^2(x_3(\tau))) \approx V_1^\mu \big|_{B_2^2(B_3^2(x_3(\tau)))} \quad \text{for} \quad x_3 \in d_3^2 \cap B_3^{2,-1}(d_2^1),$$

(11)

where on the left-hand-side we have made use of the substitution $B_3^2(x_3) = B_1^2(B_2^2(x_3))$. The approximate equality $\approx$ is characterized by the parameters $\delta_{emp}^1$ and $T_{emp}^1$, where $\delta_{emp}^1$ is an empirically determined upper bound on the error of $M_1$ in describing the relevant physical degrees of freedom in question, and $T_{emp}^1$ is a lower bound on the range of the parameter $\tau$ over which this approximation holds.

The reduction $1 \to 3$ follows deductively from the reductions $1 \to 2$ and $2 \to 3$. Although (11) is not difficult (just tedious) to prove rigorously using exact inequalities
A direct reduction of model $M_1$ to model $M_3$ can be achieved by composing the reduction of model $M_1$ to model $M_2$, specified by the function $B_1^1 : S_2 \rightarrow S_1$ and state space domain $d_1^2 \subset S_2$, with the reduction of model $M_2$ to model $M_3$, specified by the function $B_2^2 : S_3 \rightarrow S_2$ and state space domain $d_2^3 \subset S_3$. The composite reduction of $M_1$ to model $M_3$ is given by the function $B_1^3(x_3) = B_1^1(B_2^2(x_3))$ and state space domain $d_3^2 \cap B_3^{2,-1}(d_1^2)$ of the form (2), we can see intuitively how (11) follows from the reductions $1 \rightarrow 2$ and $2 \rightarrow 3$ using approximate equalities. In the reduction $1 \rightarrow 2$, the fact that the $M_2$-prescribed trajectory $x_2(\tau) \in S_2$ exactly satisfies $M_2$’s equations of motion implies that the quantity $B_1^2(x_2(\tau)) \in S_1$ approximately satisfies the $M_1$ equations of motion if $x_2(\tau) \in d_1^1$. By contrast with $x_2(\tau)$, whose evolution is prescribed by the model $M_2$, the quantity $x_3(\tau) \equiv B_2^3(x_3(\tau))$, whose evolution is prescribed by $M_3$, only approximately satisfies the $M_2$ equations when $x_3(\tau) \in d_2^2$. Assuming continuity of all functions $B$, the fact that $B_1^2(x_2(\tau))$ approximately satisfies the $M_1$ equations of motion should not be significantly altered when the exact solution $x_2(\tau)$ to the $M_2$ equations of motion is replaced by the approximate solution $x_3'(\tau)$. Thus $B_2^3(x_3'(\tau))$ also approximately satisfies the $M_1$ equations of motion, albeit to a coarser degree of approximation.

With the substitution $x_2'(\tau) = B_3^3(x_3(\tau))$, this entails that $B_2^1(B_3^2(x_3(\tau)))$, whose dynamics are prescribed by $M_3$, approximately satisfies the $M_1$ equations of motion when $x_3(\tau) \in d_3^2 \cap B_3^{2,-1}(d_1^2)$—as we wished to show. 19

Using the Chain Rule, one can remove all reference to the flow parameter $\tau$ in (11) via a relation of the type (4):

$$\frac{\partial B_2^{1,\mu}}{\partial x_2^\alpha} \bigg|_{B_1^2(x_3)} \frac{\partial B_3^{2,\alpha}}{\partial x_3^\beta} \bigg|_{x_3} V_3^\beta \bigg|_{x_f} \approx V_1^\mu \bigg|_{B_2^1(B_3^2(x_3))} \quad \text{for} \quad x_3 \in d_3^2 \cap B_3^{2,-1}(d_1^2), \quad (12)$$

where $\frac{\partial B_1^{1,\alpha}}{\partial x_3^\alpha} \bigg|_{x_3} = \frac{\partial B_2^{1,\alpha}}{\partial x_2^\alpha} \bigg|_{B_1^2(x_3)} \frac{\partial B_3^{2,\mu}}{\partial x_3^\beta} \bigg|_{x_3}$. The requirements for reduction are then formulated exclusively in terms of the vector fields that generate the dynamical evolution over the models’ state spaces, rather than in terms of these flows themselves. One can further extend this push–forward relationship to draw a direct connection between the

19 See [24], Chap. 1 for a more formal proof of this claim.
different algebras of physical symmetries over the various state spaces involved in a composite reduction, via relations of the form (5).

### 4.1 Consistency Requirements Between Alternative “Reduction Paths”

In certain cases, a single direct reduction $1 \rightarrow 3$ of model $M_1$ to $M_3$ can be effected via either of two intermediate models $M_{2a}$ or $M_{2b}$. That is, the reduction $1 \rightarrow 3$ can be effected through both the composition of the reductions $1 \rightarrow 2a$ and $2a \rightarrow 3$, which we abbreviate $1 \rightarrow 2a \rightarrow 3$, and through the composition of the reductions $1 \rightarrow 2b$ and $2b \rightarrow 3$, which we abbreviate $1 \rightarrow 2b \rightarrow 3$. We will see an example of this in the following subsection, where the reduction of a non-relativistic classical model to a model of relativistic quantum mechanics may be effected either via a model of non-relativistic quantum mechanics or a model of relativistic classical mechanics.

In examining such cases, I aim to formalize the notion that certain types of inter-theoretic transition, such as between quantum and classical and between relativistic and non-relativistic theories, commute. In particular, I aim to do this with greater mathematical precision and greater clarity of physical interpretation than the vague suggestion that limits along a single face of the Bronstein cube, such as the limits $\hbar \rightarrow 0$ and $\frac{1}{c} \rightarrow 0$, should somehow commute. In particular, the Bronstein cube methodology (to the extent that it constitutes a well-defined methodology at all) makes no mention of the need for bridge functions or restricted state space domains in showing how the structures of the reducing model approximately instantiate the physically salient mathematical relations of the reduced model.

In cases where it is possible to effect the reduction $1 \rightarrow 3$ either via the path $1 \rightarrow 2a \rightarrow 3$ or $1 \rightarrow 2b \rightarrow 3$, we are free to consider the direct reduction of model $M_1$ to $M_3$ directly, without regard to intermediary models $M_{2a}$ or $M_{2b}$. Since it is the same quantity in $M_3$ that instantiates the degrees of freedom described by $M_1$, irrespective of whether we choose to view the reduction $1 \rightarrow 3$ as a composite $1 \rightarrow 2a \rightarrow 3$ or $1 \rightarrow 2b \rightarrow 3$, the bridge function $B^1_3(x_3)$ and state space domain $d^3_1$ should be approximately independent of which of these “reduction paths” is used:

$$ B^1_3(x_3) \approx B^1_{2a}(B^2_a(x_3)) \approx B^1_{2b}(B^2_b(x_3)) \text{ for } x_3 \in d^3_1 \quad (13) $$

where

$$ d^1_3 = d^2_3 \cap B^{2a,-1}_3(d^1_{2a}) = B^{2b,-1}_3(d^1_{2b}) \cap d^2_b \quad (14) $$

(see Fig. 7). The approximate equivalence $\approx$ indicates that there is a small “halo” of non-uniqueness to the bridge function $B^1_3(x_3)$, corresponding to a small neighborhood of functions of $x_3$ all of which approximately instantiate the same high-level (i.e., $M_1$) behavior within the empirical margin of approximation $\delta^1_{empr}$. This margin reflects the fact that for the reducing model to encompass the empirical success of the reduced model, the former only needs to recover the evolutions prescribed by the reduced model to the extent that the latter actually succeed at providing an accurate representation.
In many cases, the direct reduction $M_1 \rightarrow M_3$ of a model $M_1$ (with state space $S_1$) to an empirically more encompassing model $M_3$ (with state space $S_3$) may be effected via two (or more) compound reductions $M_1 \rightarrow M_2a \rightarrow M_3$ or $M_1 \rightarrow M_2b \rightarrow M_3$, where $M_2a$ and $M_2b$ have state spaces $S_{2a}$ and $S_{2b}$, respectively. The bridge function $B_{13}$ and state space domain $d_{13}$ characterizing the direct reduction $1 \rightarrow 3$ are approximately independent of which of these two paths, $1 \rightarrow 2a \rightarrow 3$ or $1 \rightarrow 2b \rightarrow 3$, one chooses. This is illustrated below for the relationship between non-relativistic classical models and relativistic quantum models. Consistency between the paths is ensured if

$$B_{2a3}(B_{12a}(d_{13})) \approx B_{2b3}(B_{12b}(d_{13})) \approx B_{13}(d_{13}),$$

of the system itself. The requirement (14) that the domains recovered by the distinct reduction paths be equal will hold if

$$B_{32a}(d_{3}) = d_{2a},$$

$$B_{32b}(d_{3}) = d_{2b}.$$  \hspace{1cm} (15)

In particular, both paths then yield

$$d_{3} = d_{3}^{2a} \cap d_{3}^{2b}.$$  \hspace{1cm} (16)

Concrete illustration of these relations in the reduction of a model of Newtonian mechanics to a model of relativistic quantum mechanics is provided in Sect. 4.2. It is also worth emphasizing that these consistency requirements extend easily to cases where more than two “reduction paths” are available between the reduced and reducing models, and in which the individual reduction paths chain together more than three distinct models.
4.2 Commutation of Quantum-to-Classical and Relativistic-to-Non-relativistic Transitions

Let us check the conditions (13) and (14) explicitly for the case of the reduction NM \rightarrow RQM in the description of a slow-moving heavy charge, which may be effected via either of the composite reductions NM \rightarrow QM \rightarrow RQM or NM \rightarrow SR \rightarrow RQM.

Considering first the path NM \rightarrow QM \rightarrow RQM, let us determine the bridge function and state space domain specified by the relations (10), recalling the bridge functions for the NM \rightarrow QM and QM \rightarrow RQM reductions in Section 3, and the fact that \( x_{RQM} \) is specified by the 4-spinor wave function \( \psi^a(x) \). Composing these functions yields,

\[
B_{RQM}^{NM}(x_{RQM}) = B_{QM}^{NM}(B_{RQM}^{QM}(x_{RQM})) = \left( \sum_{a=1}^{2} \int d^3 x \, \psi^a(x) \psi^a(x), \sum_{a=1}^{2} \int d^3 x \, \psi^a(x)(-i \nabla) \psi^a(x) \right). \tag{17}
\]

Note that the sum in the expectation values is only over the upper two 4-spinor components. Likewise, recalling the state space domains of the reductions NM \rightarrow QM and QM \rightarrow RQM from Sect. 3, we can identify the state space domain \( d_{RQM}^{NM} \) for the composite reduction NM \rightarrow QM \rightarrow RQM using the relation (10):

\[
d_{RQM}^{NM} = d_{RQM}^{QM} \cap B_{RQM}^{QM, \sim 1}(d_{QM}^{NM}) = d_{RQM}^{QM} \cap B_{RQM}^{QM, \sim 1}(\{ \text{“narrow wave packets”} \}) = \{ \text{“low-momentum, positive energy 4-spinors”} \} \cap \{ \text{“positive energy 4-spinors narrowly peaked in upper two components”} \} = \{ \text{“low-momentum, narrowly peaked, positive-energy 4-spinors”} \} \tag{18}
\]

In the third line, we have made use of the fact that the inverse image of \( d_{QM}^{NM} \), the set of narrowly peaked non-relativistic wave functions, under \( B_{RQM}^{QM} \), which projects 4-spinors onto their upper two components, is the set of 4-spinor wave packets narrowly peaked in their upper two components. In the fourth equality, the fact that wave packets in the overlap domain are low-momentum entails that their lower two components are negligible in size. There is also some tension between the requirement that wave packets be restricted to low-momentum Fourier components and that they be narrowly peaked in position, since narrowly peaked wave packets require high-momentum modes. However, the intersection between the set of narrow wave packets and the set of low-momentum wave packets may still be sizable (depending on the precise margins one uses to define “narrow” and “low-momentum”), and assuming that the external potential fields do not vary too sharply, supports a robust approximation of the Newtonian evolution by the evolution induced by the RQM model. That is, wave
packets can be simultaneously sufficiently narrow and sufficiently non-relativistic to support both the classical and non-relativistic approximations together.

Now consider the form of (10) for the alternative reduction path $NM \rightarrow SR \rightarrow RQM$:

$$B_{RQM}^{NM}(x_{RQM}) = B_{SR}^{NM}(B_{RQM}^{SR}(x_{RQM}))$$

$$= \left( \sum_{a=1}^{4} \int d^3x \, \psi_{a}^{\dagger}(x)\psi_{a}(x), \right.$$ \n
$$\left. \sum_{a=1}^{4} \int d^3x \, \psi_{a}^{\dagger}(x)(-i\nabla)\psi_{a}(x) \right). \tag{19}$$

Note that the sum is now over all 4-spinor components and not just the upper two. Informally, we can identify the state space domain for the composite reduction $NM \rightarrow SR \rightarrow RQM$ as

$$d_{RQM}^{NM} = d_{RQM}^{SR} \cap B^{-1}_{RQM}(d_{SR}^{NM}),$$

$$= d_{RQM}^{SR} \cap B^{-1}_{RQM} (\text{“low-momentum phase space points”})$$

$$\cap \{ \text{“narrowly peaked, positive energy 4-spinors”} \}$$

$$\cap \{ \text{“low-momentum, positive energy 4-spinors”} \}$$

$$\cap \{ \text{“low-momentum, narrowly peaked, positive energy 4-spinors”} \}. \tag{20}$$

In the second line, we have made use of the fact that the inverse image of $d_{SR}^{NM}$, the set of low-velocity phase space points, under $B_{RQM}^{SR}$, which maps 4-spinor wave functions into expectation values of position and momentum, is the set of low-momentum, positive-energy 4-spinor wave packets. Thus, we see that the state space domains for the two composite reductions are the same along the two reduction paths. Both consist of narrow, low-momentum, positive energy 4-spinors.

Note also that $d_{SR}^{NM} = d_{RQM}^{QM} \cap d_{RQM}^{SR}$, where $d_{RQM}^{QM}$ is the set of low-momentum 4-spinors and $d_{RQM}^{SR}$ is the set of 4-spinors narrowly peaked in their upper two components. That is, the state space domain of the reduction $NM \rightarrow RQM$ is simply the overlap between the state space domains $d_{RQM}^{QM}$ and $d_{RQM}^{SR}$ of the two intermediate models. On the other hand, the composite bridge functions for the two reduction paths, $B_{RQM}^{QM}(B_{RQM}^{QM}(x_{RQM}))$ and $B_{SR}^{NM}(B_{RQM}^{SR}(x_{RQM}))$, are not strictly speaking equal. The first composite bridge function only involves a summation over the upper two 4-spinor components while the second involves a summation over all four components. However, over $d_{RQM}^{NM}$, the composite bridge functions associated with the two reduction paths approximately agree, since the contribution of the 3, 4 components to the sum in (19) is very small over this subset of the RQM state space and so can be neglected.
Fig. 8 The direct reduction of the Newtonian model $M_{NM}$ of a localized, slow-moving (i.e., non-relativistic) charge to the RQM model $M_{RQM}$ of the same charge may be effected via either of the compound reductions $NM \rightarrow QM \rightarrow RQM$ or $NM \rightarrow SR \rightarrow RQM$. The bridge function $B_{RQM}^{NM}$ and state space domain $d_{RQM}^{NM}$ characterizing the direct reduction $NM \rightarrow RQM$ are approximately independent of which of these two paths one chooses. This illustrates one concrete sense in which the quantum-to-classical and relativistic-to-non-relativistic transitions commute.

Thus, we see from these two composite reductions that

$$B_{QM}^{NM}(B_{RQM}^{QM}(x_{RQM})) \approx B_{SR}^{NM}(B_{RQM}^{SR}(x_{RQM}))$$

(21)

for $x_{RQM} \in d_{RQM}^{NM}$, where

$$d_{RQM}^{NM} = d_{RQM}^{QM} \cap B_{RQM}^{QM,-1}(d_{RQM}^{NM}) = d_{RQM}^{SR} \cap B_{RQM}^{SR,-1}(d_{RQM}^{NM})$$

(22)

and therefore that the requirements (13) and (14) are satisfied in this case (see Fig. 8). To be rigorous, one should show in that errors characterizing the approximate equalities (21) are smaller than the empirical error bound $\delta_{emp}$ within which $M_{NM}$ is known to successfully describe the physical degrees of freedom under consideration.

The approach to reduction based on the specification of bridge functions $B: S_l \rightarrow S_h$ and state space domains $d \subseteq S_l$ thus illustrates explicitly one important sense in which the quantum-to-classical and relativistic-to-nonrelativistic transitions commute. From the perspective of the state space domains, the state space domain $d_{RQM}^{NM}$ of the reduction $NM \rightarrow RQM$ is the same irrespective of whether one first applies the restriction to non-relativistic states $d_{RQM}^{QM}$ and then to classical states $d_{RQM}^{SR}$, or instead applies these restrictions in the reverse order; this simply reflects the fact that $A \cap B = B \cap A$ for arbitrary sets $A$ and $B$. From the perspective of bridge functions, the specific quantity identified as the RQM model’s proxy for an NM model’s phase space point is independent of whether one first applies the map from to the non-relativistic state space and then from there to the classical non-relativistic state space, or instead first applies the map to the relativistic classical phase space and
then to the non-relativistic classical phase space. Within the margin of approximation and timescales for which the high-level Newtonian model successfully describes the position and momentum of the physical charge in question, the bridge function $B_{RQM}^{NM} : \mathcal{H}_{RQM} \rightarrow \Gamma_{NM}$ and state space domain $d_{RQM}^{NM} \subset \mathcal{H}_{RQM}$ characterizing the direct reduction of $NM \rightarrow RQM$ are independent of whether the reduction is performed via the path $NM \rightarrow QM \rightarrow RQM$ or the path $NM \rightarrow SR \rightarrow RQM$.

In either case, the quantity $B_{RQM}^{NM} [\psi^a ((x))]$, understood as a composition of bridge functions associated with the component reductions along either path, can be shown to approximately satisfy the high-level non-relativistic classical Hamilton equations when $[\psi^a (x)] \in d_{RQM}^{NM}$:

\[
\frac{d}{dt} \left( \sum_{a=1}^{4(\text{or } 2)} \int d^3 x \, \psi^{a\dagger}(x, t)(-i\hbar \nabla - qA(x))\psi^a(x, t) \right) \approx E \left( \sum_{a=1}^{4(\text{or } 2)} \int d^3 x \, \psi^{a\dagger}(x, t)\psi^a(x, t) \right) + \frac{1}{m} \left( \sum_{a=1}^{4(\text{or } 2)} \int d^3 x \, \psi^{a\dagger}(x, t)(-i\hbar \nabla - qA(x))\psi^a(x, t) \right) \times B \left( \sum_{a=1}^{4(\text{or } 2)} \int d^3 x \, \psi^{a\dagger}(x, t)\psi^a(x, t) \right) 
\]

(23)

\[
\frac{d}{dt} \left( \sum_{a=1}^{4(\text{or } 2)} \int d^3 x \, \psi^{a\dagger}(x, t)\psi^a(x, t) \right) \approx \frac{1}{m} \left( \sum_{a=1}^{4(\text{or } 2)} \int d^3 x \, \psi^{a\dagger}(x, t)(-i\hbar \nabla - qA(x))\psi^a(x, t) \right) \times \left( \sum_{a=1}^{4(\text{or } 2)} \int d^3 x \, \psi^{a\dagger}(x, t)\psi^a(x, t) \right) 
\]

(24)

where the electric and magnetic fields are defined in terms of the potentials appearing in the Dirac equation by the relations $E(\langle x \rangle) \equiv -\nabla V(\langle x \rangle)$ and $B(\langle x \rangle) \equiv \nabla \times A(\langle x \rangle)$, with $\langle x \rangle = \int d^3 x \, \psi^{a\dagger}(x, t)\psi^a(x, t)$. The upper index on the sums in the above expression may be taken as either 2 or 4 since the lower two components are negligible over $d_{RQM}^{NM}$. Thus, the relation (11) is satisfied simultaneously for both paths $NM \rightarrow QM \rightarrow RQM$ and $NM \rightarrow SR \rightarrow RQM$.

Here, we see that a quantity constructed within the remote and theoretically abstract realm of four-spinors and Dirac matrices instantiates the much more intuitive and familiar behavior of Newtonian mechanics. In principle, reduction of the RQM model to even more fundamental descriptions such as a model of QED could be used to effect a direct reduction of $M_{NM}$ to $M_{QED}$, which in turn would enable us to replace the expression for the Newtonian state in terms of 4-spinor wave functions with an expression in terms of, say, the QED state. Through the use of bridge functions and state space domain restrictions, it is possible to describe familiar, intuitive phenomena
more direct to our experience in the abstract theoretical terms needed to characterize phenomena and degrees of freedom far more remote from our experience. In principle, it should be possible via these methods (or a suitably generalized version of them within the instantiation-based way of conceptualizing reduction) to embed the phenomena of everyday experience into the more universal, but far more abstract, theoretical framework furnished by some particular model of quantum gravity. Thus, the use of bridge functions and state space domains facilitates more explicit articulation of the reductionist ideal according to which the theories and models of physics become progressively more universal, which has driven theoretical progress since Galileo and Kepler and currently motivates the search for a theory of quantum gravity and a “theory of everything.”

4.3 Making Approximate Equalities Exact in the Limit

The preceding analysis of the classical and non-relativistic domains of an RQM model kept the constants $\hbar$ and $c$ fixed at their physical values. There it was typically the case that many of the equalities between quantities in a high-level model $M_h$ and their instantiations by some low-level model $M_l$ were approximate in nature. In this section, we explain how these approximate equalities, which hold in a certain sense “on the way” to the limit, can be made exact by counterfactually varying the values of the constants $\hbar$ and $c$.

Consider first the non-relativistic approximation to relativistic theories. Before considering the counterfactual limit, it is worth inquiring to what extent the approximate equalities in relations of the form (1) and (3) can be made exact without changing the value of $c$. In the reduction $NM \rightarrow SR$, the approximate equality becomes exact only when $\gamma = 1$—that is, when the velocity is exactly zero. For any non-zero value of $v$ with $c$ fixed, $\gamma > 1$, so that the equality cannot be exact. Thus, under these assumptions, for fixed $c$ the non-relativistic approximation cannot hold exactly except when $v = 0$. Likewise, in the reduction $QM \rightarrow RQM$, the upper two components of the Dirac 4-spinor satisfy the Pauli equation exactly only in the case of 4-spinors that do not possess Fourier modes for which $|k| > 0$—i.e., 4-spinors that are spatially constant. Thus, in both reductions, the approximate equalities become exact only in the highly restricted case of physical states in which the system is at rest or lacks any component with finite momentum. If, on the other hand, one takes the formal, counterfactual limit $c \rightarrow \infty$, then the approximate equalities (7) and (9) should become exact for all states in $\mathcal{H}_{Dirac}$ and $\Gamma_{rel}$, so that state space domain restrictions are no longer needed to recover approximate (or exact) validity of the high-level equations.

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20 Within the recent philosophical literature, the metaphysical basis for and implications of such a reductionist view, including careful analysis of what it means for one theory to be more “fundamental” than another, is examined extensively in the work of Ladyman, French, Saatsi, McKenzie, and many others [13,18,19,28].

21 Butterfield employs this distinction, between behavior in the limit, and on the way to the limit, in order to propose a reconciliation between ostensibly clashing concepts of reduction and emergence in physics [2].

22 For recent extensive discussion of non-relativistic limits, including in the context of gravitational theories, see [11].
of motion as applied to $B(x_t)$. However, it is precisely for this reason that attempts to effect these reductions on a Bronstein-style approach overlook the necessity of restricting to a particular domain $d$ of the physical state space when identifying those quantities in the reducing model that approximately instantiate the regularities of the reduced model, and when circumscribing the circumstances under which they do so.

In the case of quantum-to-classical transitions, we can likewise ask whether the approximate equalities in relations of the form (1) and (3) can be made exact without changing the value of $\hbar$. The requirement that expectation values approximately satisfy classical equations of motion, reflected in (6) and (8), depends on the possibility of wave packets that are simultaneously narrowly peaked both in position and momentum: if they are too narrowly peaked in momentum, they will have a wide spatial spread with respect the characteristic length scale of the background potential, so that Ehrenfest’s Theorem no longer implies approximately classical evolutions for expectation values (not to mention the fact that states with wide spatial spread are inherently non-localized and therefore non-classical); on the other hand, if the quantum state is too narrowly peaked in position (such as in the case of a delta function), it will contain Fourier modes of arbitrarily high momentum and therefore evolve classically only for the very briefest of instants, so that classicality fails to persist for any extended period of time. Thus, a compromise between localization in position and momentum is required to have states that evolve approximately classically for an extended period of time. But the uncertainty principle $\Delta x \Delta p \geq \frac{\hbar}{2}$ famously limits the degree to which one can have both at the same time. For fixed $\hbar$, quantum states therefore must have some spread, which in turn implies that the approximate validity of classical equations for expectation values in generic background potentials cannot be made exact no matter what the state, and will always have some error associated with the finite widths of the state (except in the very special case of the harmonic oscillator, where expectation values evolve exactly classically for all times no matter what the state). While the absolute error in (6) and (8) can never be made to vanish for finite $\hbar$ by a particular choice of state because of the uncertainty principle, there is perhaps a sense in which the relative error, such as the ratio of the absolute error to the magnitude of the classical terms in (6) and (8), can be made to vanish by focusing on progressively larger or more energetic systems.

Feintzeig has recently proposed a “factual” interpretation of the limit $\hbar \to 0$ in which changes to the numerical value of $\hbar$ are not interpreted counterfactually, but rather are induced by a change of units associated with descriptions of the system at increasingly large length scales. Feintzeig’s focus is specifically on the relationship between classical and quantum algebras of observables reflected in the axioms of deformation quantization. By contrast, for the distinct type of formal correspondence considered here, in which quantum expectation values evolve approximately classically over a certain subset of the quantum state space, a mere change of units does not suffice in (6) and (8) to reduce the size of the ignored quantum correction terms relative to the classical terms that are kept, since both the neglected correction terms and classical terms have the same units, and therefore scale by the same factor under a change of units.

On the other hand, if we permit ourselves to vary $\hbar$ counterfactually, then in the limit $\hbar \to 0$ it is possible to have wave packets that are arbitrarily narrowly peaked in
both position and momentum. In this case, there is no tradeoff between the quality of
the approximation in (6) and the length of time for which these approximations hold.
In the limit \( h \to 0 \), it is possible for expectation values to satisfy classical equations of
motion exactly by choosing states that are infinitely narrowly peaked in both position
and momentum. However, the limit \( h \to 0 \) by itself does not imply that expectation
values satisfy classical equations of motion exactly, since a quantum wave packet may
have finite width in both position and momentum even as \( h \to 0 \). Since the uncertainty
principle is an inequality, nothing prevents \( \Delta x \Delta p \) from being arbitrarily larger than
\( \frac{\hbar}{2} \), no matter how small \( h \) is. In such a case, expectation values will not generally
evolve classically in the limit \( h \to 0 \), since there will still be some error arising from
the finite widths of the wave packet that may remain in this limit (see [27] for more
detailed discussion of this point). For exact agreement between classical trajectories
and trajectories of quantum expectation values, one must take the limits in which the
position and momentum widths of the quantum state vanish, in addition to the limit
\( h \to 0 \).

4.4 Speculations on New Physics: Relating the Standard Model \( \to \) TOE and
General Relativity \( \to \) TOE Reductions

In the introduction, we speculated that a careful examination of the requirements for
reduction involving familiar theories might offer insight into the precise nature of
the relationship that theories/models such as the Standard Model (SM) and general
relativity (GR) might bear to any candidate “theory of everything” (TOE). How, if at
all, might the formal methodological considerations discussed above clarify the nature
of the relationship between current theories and any viable theory of quantum gravity,
and more specifically, any “theory of everything”? Any theory of quantum gravity, understood broadly as a quantum theory of space-
time, must by definition reduce general relativity in the sense of recovering its empirical
successes. By contrast, any “theory of everything” (TOE) (as string theory aspires to be)
must reduce both general relativity and the Standard Model, and so is necessarily
also a theory of quantum gravity. Whatever the correct unification of the Standard
Model and general relativity turns out to be, one common vision of such a unification
is a single mathematical structure specified by a single state space and set of dynamical
equations - i.e., a model - that incorporates all phenomena captured by these theories.
While some models apply only narrowly and locally, others, such as the Standard
Model, are extremely broad in scope; a TOE would naturally lie at one extreme end
of this spectrum, insofar as it is produces a single mathematical model - a “model
of everything,” (or MOE) - incorporating all physical phenomena. In keeping with
common usage, I will use the acronym TOE to designate such a model.

Applying the approach to reduction described above, it is reasonable to expect
that there exist functions \( B_{TOE}^{SM} \) and \( B_{TOE}^{GR/CE} \), and subsets \( d_{TOE}^{SM} \subset S_{TOE} \) and
\( d_{TOE}^{GR/CE} \subset S_{TOE} \), such that \( B_{TOE}^{SM}(x_{TOE}) \) approximately instantiates the equations
Fig. 9 Classical electrodynamics (CED) lies in the overlap of the domains of empirical validity of the Standard Model (SM) and general relativity with an electromagnetic stress energy tensor (GR/CED). It should therefore be possible to effect the reduction of classical electrodynamics (CED) to the correct theory of everything (TOE) - whatever that may be - either via the composite reduction $CED \rightarrow SM \rightarrow TOE$ or via the composite reduction $CED \rightarrow GR/CED \rightarrow TOE$. The bridge function $B_{CED}^{TOE}$ and state space domain $d_{TOE}^{CED}$ should then be approximately independent of the path that one uses to perform the reduction of motion of $SM^{23}$ for $x_{TOE} \in d_{TOE}^{SM}$ and such that $B_{TOE}^{GR/CED}(x_{TOE})$ approximately instantiates the equations of motion of the $GR/CED$ model for $x_{TOE} \in d_{TOE}^{GR}$. In string theory, for example, $d_{TOE}^{GR/CED}$ is sometimes thought to include coherent states of the gravitational field, which are the quantum gravitational analogue to narrow wave packets considered in the context of ordinary quantum mechanics. $^{24}$ Moreover, classical electrodynamics (CED) lies in the intersection of the domains of empirical validity of $SM$ and of $GR/CED$ (a model of GR with an electromagnetic term in its stress-energy tensor), so it should be possible as in the $NM \rightarrow RQM$ reduction to effect the reduction $CED \rightarrow TOE$ via distinct composite reductions, $CED \rightarrow SM \rightarrow TOE$ and $CED \rightarrow GR/CED \rightarrow TOE$. $^{25}$ Then, by analogy with the above discussion of the classical and non-relativistic domains of RQM, we should demand that

$$B_{TOE}^{CED}(x_{TOE}) \approx B_{TOE}^{SM}(B_{TOE}^{SM}(x_{TOE})) \approx B_{TOE}^{CED}(B_{TOE}^{GR/CED}(x_{TOE}))$$ (25)

$^{23}$ This presumably would be some quantum field theoretic Schrödinger equation for the Standard Model quantum state.

$^{24}$ See, Huggett and Vistarini’s [16], and sources therein, for discussion of the role of coherent states in the relationship between string theory and classical general relativity.

$^{25}$ The relationship between GR and SM cannot be characterized as a case of reduction since it is not true that the domain of either is contained in that of the other. However, as we discuss here, their domains overlap. On this overlap domain, which includes the domain of CED, we expect a weaker relationship - what Crowther has called “correspondence” - to hold. Correspondence requires distinct theories to approximately agree in cases where their domains of validity overlap. As Crowther notes, reduction is a special case of correspondence in which the overlap of the theories’ domains is the entire domain of one of the theories [5].
for $x_{TOE} \in d^{SM}_{TOE} \cap d^{GR/CED}_{TOE} \subset S_{TOE}$. This tells us that whatever $M_{TOE}$ turns out to be, and whatever the state space domains of $d^{SM}_{TOE}$ and $d^{GR/CED}_{TOE}$ turn out to be, $d^{SM}_{TOE}$ and $d^{GR/CED}_{TOE}$ must overlap, and within this overlap the composite bridge functions $B^{CED}_{GR/CED}(B^{GR/CED}_{TOE}(x_{TOE}))$ and $B^{CED}_{SM}(B^{SM}_{TOE}(x_{TOE}))$, constructed respectively via the compound reductions $CED \rightarrow SM \rightarrow TOE$ and $CED \rightarrow GR/CED \rightarrow TOE$, should approximately satisfy the coupled Maxwell equations and Lorentz Force Law equations. This is illustrated in Fig. 9.

5 Conclusion

Questions about the precise requirements for reduction bear importantly on efforts to construct new theories in physics, since these theories are often required to recover the empirical successes of established theories. Perhaps the most important examples from the perspective of modern physics are attempts to construct a quantum theory of gravity, and relatedly, attempts to construct a theory that recovers the empirical successes of both the Standard Model and general relativity. I have critiqued one common approach to reduction, associated with the Bronstein cube of physical theories and the notion that reduction is simply a matter of taking limits, as being either too vague or simplistic in its characterization of these requirements. I’ve defended an alternative point of view based on the notion that reduction concerns the relationship between distinct models of the same physical system and requires identification of quantities in the reducing model that approximately instantiate the physically salient structures of the reduced model. By employing the strategy of identifying bridge functions and state space domains, it is possible to formalize the notion that reduction is transitive—i.e., that reductions can be formally composed or chained together to directly relate mathematical models with widely differing mathematical and conceptual frameworks (e.g., Newtonian mechanics and relativistic quantum mechanics). Unlike presentations of the Bronstein cube approach to reduction, the approach described enables a clear and explicit articulation of the sense in which quantum-to-classical and relativistic-to-non-relativistic transitions commute, as well as suggesting new ways in which current theories should serve to constrain speculative models of new physics.

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