A Smoothed Impossibility Theorem on Condorcet Criterion and Participation

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Abstract. In 1988, Moulin [29] proved an insightful and surprising impossibility theorem that reveals a fundamental incompatibility between two commonly-studied axioms of voting: no resolute voting rule (which outputs a single winner) satisfies Condorcet Criterion and Participation simultaneously when the number of alternatives $m$ is at least four. In this paper, we prove an extension of this impossibility theorem using smoothed analysis: for any fixed $m \geq 4$ and any voting rule $r$, under mild conditions, the smoothed likelihood for both Condorcet Criterion and Participation to be satisfied is at most $1 - \Omega(n^{-3})$, where $n$ is the number of voters that is sufficiently large. Our theorem immediately implies a quantitative version of the theorem for i.i.d. uniform distributions, known as the Impartial Culture in social choice theory.

Keywords: Smoothed Analysis · Voting · Condorcet Criterion · Participation

1 Introduction

Social choice theory studies the design and analysis of mechanisms that aggregate agents’ preferences over multiple alternatives to make a group decision. Desirable normative properties of the mechanism, called axioms, were proposed as the basis for comparing mechanisms. For example, the Condorcet Criterion (CC for short) requires that the mechanism $r$ must choose the Condorcet winner whenever it exists. The Condorcet winner is the alternative who beats all other alternatives in head to head competitions. As another example, Participation (Par for short) requires that no agent has incentive to abstain from voting.

In 1988, Moulin [29] proved an insightful impossibility theorem that reveals a fundamental incompatibility between CC and Par (C&P impossibility for short): no resolute voting rule (which outputs a single winner) satisfies CC and Par simultaneously when the number of alternatives $m$ is at least four. The theorem is surprising because each axiom is believed to be a natural and mild requirement for voting, and indeed, each is satisfied by many commonly-studied voting rules. Moreover, when $m = 3$, both are satisfied by the maximin rule. The proof is based on worst-case analysis—a profile that violates CC or Par is constructed in different cases.

In addition to the worst-case analysis, there is a large body of literature on the quantitative versions of impossibility theorems, trying to understand their practical relevance. For example, quantitative versions of Arrow’s impossibility theorem [18, 19, 25, 26], Gibbard-Satterthwaite theorem [5, 13, 17, 27, 43], and the ANR impossibility theorem [40] were
obtained. These theorems reveal average-case limitations of voting rules w.r.t. satisfaction of combinations of axioms under certain probability models on agents’ preferences.

We are not aware of a quantitative version of the C&P impossibility, despite the large body of literature on the likelihood of CC and likelihood of PAR. Moreover, many previous works adopted the i.i.d. uniform distribution, known as the Impartial Culture (IC), which has been widely criticized of being unrealistic [33]. This limits the practical significance of the quantitative impossibility theorems under IC.

1.1 Our Contributions

In this paper, we prove an impossibility theorem on the simultaneous satisfaction of CC and PAR under the smoothed social choice framework [40], which is inspired by the smoothed complexity analysis [37] and is more general than IC or i.i.d. distributions in general. We are given a set of \( m \geq 4 \) alternatives, denoted by \( \mathcal{A} \), and a set of distributions \( \Pi \) over linear orders over \( \mathcal{A} \), denoted by \( \mathcal{L}(\mathcal{A}) \). There are \( n \geq 1 \) agents. Each agent’s preferences are modeled by a random variable whose distribution is in \( \Pi \). For any function \( X \) that maps each voting rule \( r \) and each profile \( P \) (the collection of agents’ preferences) to \( \{0, 1\} \), the smoothed likelihood of \( X \), denoted by \( \tilde{X} \), is defined to be the following worst average-case value [40, 42].

\[
\tilde{X}_{\Pi}^{\min}(r, n) \triangleq \inf_{\bar{\Pi} \in \Pi^n} \Pr_{P \sim \bar{\Pi}}(X(r, P) = 1)
\]  

That is, \( \tilde{X}_{\Pi}^{\min}(r, n) \) is the lower bound on the probability for \( X \) to be 1 over the profile that is generated from any combination of distributions in \( \Pi^n \). In other words, imagine that an adversary tries to minimize the value of \( X \), but can only choose the distribution of preferences for each agent, instead of their (deterministic) preferences as in the classical worst-case analysis. Then, \( \tilde{X} \) measures the minimum (expected) value the adversary can achieve. When \( X \) represents a desirable property, a low \( \tilde{X} \) value is negative news, because it states that the adversary can choose agents’ “ground truth” preferences, so that \( X \) holds with small probability.

It is not hard to see that smoothed analysis generalizes the worst-case analysis (where \( \Pi \) contains deterministic distributions for all linear orders) and the i.i.d. case (where \( |\Pi| = 1 \)), though its practical relevance largely depends on the choice of \( \Pi \). In this paper, we prove the following smoothed impossibility theorem for the simultaneous satisfaction of CC and PAR, i.e., \( X = \text{C&P} \), under mild assumptions on \( \Pi \).

**Theorem 1 (Smoothed Impossibility of C&P, Informally Put).** For any fixed \( m \geq 4 \), any resolute voting rule \( r \), and any \( \Pi \) over \( \mathcal{L}(\mathcal{A}) \) that satisfies mild conditions, there exist a constant \( C > 0 \) and a constant \( N \in \mathbb{N} \) such that for every \( n \geq N \),

\[
\text{C&P}_{\Pi}^{\min}(r, n) < 1 - C \cdot n^{-3}
\]

The theorem states that the simultaneous satisfaction of CC and PAR for any voting rule cannot be higher than \( 1 - \Omega(n^{-3}) \), or in other words, the adversary can make the likelihood of violation of CC or PAR to be at least \( \Omega(n^{-3}) \). A straightforward corollary of
the theorem leads to a quantitative impossibility theorem on C&P under IC (Corollary 1 in Section 3).

We believe that Theorem 1 is a mild impossibility theorem, because the simultaneous satisfaction of CC and PAR can still converge to 1 in large elections regardless of the adversary’s choice. In fact, for many commonly-studied voting rules that satisfies CC, the smoothed satisfaction of C&P (which is the same as the smoothed satisfaction of PAR) converges to 1 at an $O(n^{-0.5})$ rate [42]. How to close the gap between $-3$ and $-0.5$ is an open question for future work.

**Proof Techniques.** Moulin [29]'s proof is constructive. First, a profile $\hat{P}$ of 25 rankings is constructed. Then, it was shown that a violation of C&P exists in a profile that is obtained from $\hat{P}$ by appending no more than 10 votes.

At a high level, Theorem 1 is proved by calculating the smoothed likelihood of profiles that are closely related to the ones constructed by Moulin [29]. However, this is more challenging than it sounds due to the following two technical challenges.

- First, in Moulin’s proof, the size of the profile where C&P is violated varies from 25 to 35, whereas in Theorem 1 the number of votes is fixed to be $n$.
- Second, bounding the (smoothed) likelihood of profiles under which Moulin’s argument about $\hat{P}$ holds appears highly challenging, because the main technique tool used in previous work for smoothed analysis in social choice [40, 42], i.e., representing the properties of interest as unions of finitely many polyhedra, cannot be directly applied.

We overcome the two challenges in two steps.

- In Step 1, we address the first challenge by noticing that adding a ranking is effectively equivalent to subtracting its reverse, and then define series of five $n$-profiles, called *trails*, as the counterparts to the process of adding votes in Moulin’s proof.
- In Step 2, we address the second challenge by lower-bounding the smoothed likelihood of violations of C&P via weighted counting of a class of profiles that are closely related to $\hat{P}$ on the trails defined in Step 1, and notice that such profiles can be represented by a polyhedron. Then, we invoke the PMV-in-Polynomial theorem [41, Theorem 1] to bound the smoothed likelihood.

### 1.2 Related Work and Discussions

**Condorcet criterion (CC).** The axiom was proposed by Condorcet in 1785 [4] and has “nearly universal acceptance” [35, p. 46]. Many commonly-studied voting rules satisfy CC, except positional scoring rules [8] and multi-round-score-based elimination rules, such as STV. There is a large body of literature on theoretical characterizations of the *Condorcet efficiency*, which is the probability for the Condorcet winner to win conditioned on its existence [7, 9, 14, 31, 34]. Computer simulations were used to study the likelihood of CC beyond positional scoring rules, see, e.g., [11, 12, 24, 32]. The smoothed satisfaction of CC for commonly-studied voting rules was investigated recently [42].
The participation axiom (Par). The axiom was introduced to describe voting rules where the no-show paradox [10] does not occur. Par is satisfied by all positional scoring rules. As mentioned earlier in the introduction, when there are three alternatives, the maximin rule satisfies CC and Par, but for every \( m \geq 4 \), no voting rule satisfies both CC and Par [29]. The likelihood of Par under commonly studied voting rules w.r.t. IC was explicitly posed as an open question by Berg and Lepelley [3] in 1994, and has been investigated in a series of works including [20, 21, 39], see [15, Chapter 4.2.2]. The smoothed satisfaction of Par for commonly-studied voting rules was investigated recently [42], showing that for many commonly studied voting rules that satisfy CC, including maximin, ranked pairs, Schulze, and Copeland, the smoothed satisfaction of Par is \( 1 - O(n^{-0.5}) \).

Quantitative and Smoothed Impossibility Theorems. There is a large body of literature on quantitative versions of impossibility theorems in social choice under IC. For example, quantitative versions of Arrow’s impossibility theorem [1] were proved [18, 19, 25, 26]. A quantitative Gibbard-Satterthwaite theorem [16, 36] was proved for \( m = 3 \) by Friedgut et al. [13], and the theorem was subsequently developed in [5, 17, 43], and the general case was resolved by Mossel and Racz [27]. In judgement aggregation, Nehama [30] and Filmus et al. [6] developed quantitative characterizations of AND-homomorphism as oligarchy, whose worst-case version was due to List and Pettit [22, 23]. Xia [40] proved a smoothed version of the ANR impossibility theorem on anonymity and neutrality, whose worst-case version was due to Moulin [28].

Smoothed Analysis in General. Smoothed analysis has been applied to a wide range of computational problems, including mathematical programming, machine learning, equilibrium analysis, see the survey by Spielman and Teng [38]. Recently, Baumeister et al. [2] and Xia [40] independently proposed to conduct smoothed analysis in social choice. In this paper, we follow the latter work to study the smoothed satisfaction of C&P as defined in Equation (1), and our proof uses the PMV-in-Polyhedron theorem in [40, 41]. We do not see an easy way of applying techniques in previous work to prove Theorem 1 as commented earlier.

2 Preliminaries

For any \( q \in \mathbb{N} \), we let \( [q] = \{1, \ldots, q\} \). Let \( \mathcal{A} = [m] \) denote the set of \( m \geq 3 \) alternatives. Let \( \mathcal{L}(\mathcal{A}) \) denote the set of all linear orders over \( \mathcal{A} \). Let \( n \in \mathbb{N} \) denote the number of voters (agents). Each voter uses a linear order \( R \in \mathcal{L}(\mathcal{A}) \) to represent his or her preferences, called a vote, where \( a \succ_R b \) means that the agent prefers alternative \( a \) to alternative \( b \). The vector of \( n \) voters’ votes, denoted by \( P \), is called a (preference) profile, sometimes called an \( n \)-profile. The set of \( n \)-profiles for all \( n \in \mathbb{N} \) is denoted by \( \mathcal{L}(\mathcal{A})^* = \bigcup_{n=1}^{\infty} \mathcal{L}(\mathcal{A})^n \).

For any profile \( P \), let \( \text{Hist}(P) \in \mathbb{R}_{\geq 0}^{\mathcal{L}(\mathcal{A})^*} \) denote the anonymized profile of \( P \), also called the histogram of \( P \), which contains the total number of each linear order in \( \mathcal{L}(\mathcal{A}) \) according to \( P \). A voting rule maps any profile to a set of alternatives, called the winners. A resolute voting rule \( r \) maps any profile to a set that consists of a single winner.
(Un)weighted majority graphs and the Condorcet winner. For any profile $P$ and any pair of alternatives $a, b$, let $P[a \succ b]$ denote the number of votes in $P$ where $a$ is preferred to $b$. Let $\text{WMG}(P)$ denote the weighted majority graph of $P$, whose vertices are $A$ and whose weight on edge $a \rightarrow b$ is $w_P(a, b) = P[a \succ b] - P[b \succ a]$. Let $\text{UMG}(P)$ denote the unweighted majority graph, which is the unweighted directed graph that is obtained from $\text{WMG}(P)$ by keeping the edges with strictly positive weights.

The Condorcet winner of a profile $P$ is the alternative that only has outgoing edges in $\text{UMG}(P)$. Let $\text{CW}(P)$ denote the set of Condorcet winners in $P$. Notice that $|\text{CW}(P)| \leq 1$ and the domain of $\text{CW}()$ can be naturally extended to all weighted or unweighted directed graphs.

Axioms of voting. We focus on per-profile axioms [40] in this paper. A per-profile axiom is a function $X$ that maps a voting rule $r$ and a profile $P$ to $\{0, 1\}$, where 0 (respectively 1) means that $r$ violates (respectively, satisfies) the axiom at $P$. Then, the classical (worst-case) satisfaction of the axiom under $r$ becomes $\min_{P \in \mathcal{L}(A)^n} X(r, P)$. For example, the CONDORCET CRITERION (CC) is characterized by a function $\text{CC}$ such that $\text{CC}(r, P) = 1$ if and only if either (1) there is no Condorcet winner under $P$, or (2) the Condorcet winner is the winner of $P$ under $r$. PARTICIPATION (PAR) is characterized by a function $\text{PAR}$ such that $\text{PAR}(r, P) = 1$ if and only if no voter has incentive to abstain from voting. Formally, let $P = (R_1, \ldots, R_n)$, then

$$[\text{PAR}(r, P) = 1] \iff [\forall j \leq n, r(P) \succeq_{R_j} r(P - R_j)],$$

where $P - R_j$ is the $(n - 1)$-profile that is obtained from $P$ by removing the $j$-th vote $R_j$. That is, $P - R_j = (R_1, \ldots, R_{j-1}, R_{j+1}, \ldots, R_n)$. For any pair of alternatives $a$ and $b$, we write $\{a\} \succeq_{R_j} \{b\}$ if and only if $a \succeq_{R_j} b$, i.e., either $a = b$ or voter $j$ prefers $a$ to $b$.

Let $\text{C&P}$ denote the simultaneous satisfaction of $\text{CC}$ and $\text{PAR}$, i.e., $\text{C&P}(r, P) = \text{CC}(r, P) \times \text{PAR}(r, P)$. Equivalently, $\text{C&P}(r, P) = 1$ if and only if $\text{CC}(r, P) = \text{PAR}(r, P) = 1$.

Smoothed satisfaction of axioms. Given a per-profile axiom $X$, a set $\Pi$ of distributions over $\mathcal{L}(A)$, a voting rule $r$, and $n \in \mathbb{N}$, the smoothed satisfaction of $X$ under $r$ with $n$ agents, denoted by $X^\min_{\Pi}(r, n)$, is defined in Equation (1) in the Introduction. We note that the “min” in the superscript means that the adversary aims at minimizing the satisfaction of $X$.

Next, we formally define the assumptions on $\Pi$ in this paper.

Definition 1 (Strictly positive and closed $\Pi$) We say that a distribution $\pi$ over $[q]$ is strictly positive (by $\epsilon$ for some $\epsilon > 0$), if for every $j \in [q]$, $\pi(j) \geq \epsilon$. We say that a set $\Pi$ of distributions over $[q]$ is strictly positive (by $\epsilon$), if all distributions in $\Pi$ are strictly positive by $\epsilon$. $\Pi$ is said to be closed, if it is a closed subset of the probability simplex in $\mathbb{R}^q$.

For example, let $\Pi_{\text{IC}} = \{\pi_{\text{uni}}\}$, where $\pi_{\text{uni}}$ is the uniform distribution over $\mathcal{L}(A)$. Then, $\Pi_{\text{IC}}$ is strictly positive and closed. Moreover, $\widetilde{\text{C&P}}^\min_{\Pi_{\text{IC}}}(r, n)$ becomes the likelihood of the simultaneous satisfaction of $\text{CC}$ and $\text{PAR}$ under $\text{IC}$. Let us look at another more informative example that illustrates smoothed analysis beyond i.i.d. distributions.
Example 1 Let \( m = 3 \) and \( \Pi = \{\pi^1, \pi^2\} \), where \( \pi^1 \) and \( \pi^2 \) represents the following two distributions over \( \mathcal{L}(A) \).

| \( \pi^1 \) | \( 1 \) | \( 2 \) | \( 3 \) |
|---|---|---|---|
| \( \pi^1 \) | \( 1/4 \) | \( 1/12 \) | \( 1/6 \) |
| \( \pi^2 \) | \( 1/12 \) | \( 1/4 \) | \( 1/6 \) |

It follows that \( \Pi \) is strictly positive (by 1/12) and closed. Let \( n = 2 \), we have

\[
\widehat{\text{C&P}}_{\Pi}^\min (r, 2) = \min \left\{ \Pr_{P \sim (\pi^1, \pi^1)}(\text{C&P}(r, P) = 1), \Pr_{P \sim (\pi^1, \pi^2)}(\text{C&P}(r, P) = 1), \Pr_{P \sim (\pi^2, \pi^1)}(\text{C&P}(r, P) = 1), \Pr_{P \sim (\pi^2, \pi^2)}(\text{C&P}(r, P) = 1) \right\}
\]

3 A Smoothed Impossibility Theorem on CC and Par

The main result of this paper is the following smoothed impossibility theorem on the simultaneous satisfaction of CC and Par. Let \( \text{CH}(\Pi) \) denote the convex hull of \( \Pi \).

Theorem 1 (Smoothed Impossibility of CC and Par) For any fixed \( m \geq 4 \), any (resolve) rule \( r \) over \( \mathcal{L}^*(A) \), and any strictly positive and closed \( \Pi \) over \( \mathcal{L}(A) \) with \( \pi_{\text{uni}} \in \text{CH}(\Pi) \), there exists a constant \( C > 0 \) and \( N \in \mathbb{N} \) such that for every \( n \geq N \),

\[
\widehat{\text{C&P}}_{\Pi}^\min (r, n) < 1 - C \cdot n^{-3}
\]

We believe that the theorem is quite general because it holds for any fixed \( m \geq 4 \), any strictly positive and close \( \Pi \) with \( \pi_{\text{uni}} \in \text{CH}(\Pi) \) (and \( \pi_{\text{uni}} \) may not be in \( \Pi \)), and any sufficiently large \( n \). The inequality in the theorem may not hold for small \( n \), because it is possible that \( \text{C&P}(r, P) = 1 \) for all \( n \)-profiles \( P \), which means that \( \widehat{\text{C&P}}_{\Pi}^\min (r, n) = 1 \). A straightforward application of Theorem 1 to \( \Pi_{\text{IC}} = \{\pi_{\text{uni}}\} \) leads to the following corollary.

Corollary 1 (Quantitative Impossibility of CC and Par under IC) For any fixed \( m \geq 4 \) and any (resolve) rule \( r \) over \( \mathcal{L}^*(A) \), there exists a constant \( C > 0 \) and \( N \in \mathbb{N} \) such that for every \( n \geq N \),

\[
\Pr_{P \sim \text{IC}}(\text{C&P}(r, P) = 1) < 1 - C \cdot n^{-3},
\]

where \( P \) is an \( n \)-profile generated from IC.

As another example, let \( \Pi \) denote the set of distributions in Example 1. Even though \( \pi_{\text{uni}} \notin \Pi \), we have \( \pi_{\text{uni}} = \frac{1}{2} \cdot \pi^1 + \frac{1}{2} \cdot \pi^2 \in \text{CH}(\Pi) \). Therefore, Theorem 1 can be applied.

Proof of Theorem 1. Recall from “Proof Techniques” in Section 1.1 that the theorem is proved by calculating the smoothed likelihood of profiles that are closely related to the ones constructed by Moulin [29]. We first prove the theorem for the 2 \( \| \) n case, then comment on how to modify the proof for the 2 \( \| \) n case.

The proof proceeds in the following steps. In Step 0, we briefly recall Moulin [29]’s proof and define notation that will be used later. In Step 1, we define trials of \( n \)-profiles...
as counterparts to the sequences of profiles in Moulin [29]'s proof. Step 2 lower bounds the smoothed likelihood of violations of CC or PAR by the smoothed likelihood of a class of profiles that are closely related to the profile used by Moulin [29], and this class of profiles can be represented by a polyhedron. Then we apply the PMV-in-Polyhedron theorem [41, Theorem 1] to the polyhedron to characterize the smoothed likelihood.

**Step 0 for 2 \n: Recalling Moulin [29]'s proof.** Moulin [29]'s proof starts with constructing a 25-profile whose WMG is in a set \( \mathcal{G}_M \) defined as follows.

**Definition 2** Let \( \mathcal{G}_M \) denote the set of weighted graphs \( G \) with weights \( w_G \) over \( A \), such that

- its restriction on \( \{1, 2, 3, 4\} \) is the graph in Figure 1 (a), and

\[
\begin{align*}
\text{(a) WMG}(\hat{P}) & & \text{(b) WMG}(P^+) \\
1 & \leftarrow & 3 & \leftarrow & 2 \\
5 & \rightarrow & 7 & \rightarrow & 9 \\
4 & \rightarrow & 7 & \rightarrow & 3 \\
 & & & & \\
1 & \leftarrow & 7 & \leftarrow & 2 \\
9 & \rightarrow & 3 & \rightarrow & 5 \\
4 & \rightarrow & 3 & \rightarrow & 3 \\n\end{align*}
\]

Fig. 1. WMGs restricted on \( \{1, 2, 3, 4\} \).

- for any \( a \in \{1, 2, 3, 4\} \) and \( b \in \{5, \ldots, m\} \), we have \( w_G(a, b) \geq 5 \).

Then, Moulin [29] proved that for any profile \( \hat{P} \) whose WMG is in \( \mathcal{G}_M \) and any voting rule, there exists a profile \( P' \) with \( |P'| \leq 10 \) such that \( \hat{P} \cup P' \) violates CC or PAR.

To present the proof, let us first define some notation. For any profile \( P \), any ranking \( R \in L(A) \), and any \( j \in \mathbb{N} \), let \( P + j \times R \) denote the profile obtained from \( P \) by appending \( j \) copies of \( R \) at the end of \( P \). Let \( MS_P(a) = \min_{b \neq a} w_P(b, a) \) denote the min score of alternative \( b \) in \( P \), which is used to define the maximin rule and will also be used in following lemma, which is the key tool in Moulin [29]’s proof.

**Lemma 1 (Moulin [29]’s tool)** For any resolute voting rule \( r \) over \( L^*(A) \), any profile \( P \), and any pair of different alternatives \( a \) and \( b \), if \( r(P) = \{a\} \) and \( w_P(b, a) + MS_P(b) > 0 \), then

\[
\exists 0 \leq j \leq |MS_P(b)| + 1 \text{ such that } C\&P(r, P + j \times [a \succ b \succ \text{others}]) = 0,
\]

where alternatives in “others” are ranked alphabetically.
Proof: We briefly recall the proof by Moulin [29] for completeness. To see why Lemma 1 is true, suppose \( r(P) = \{a\} \) and \( w_P(b, a) + \text{MS}_P(b) > 0 \). If \( \text{MS}_P(b) > 0 \) then Lemma 1 immediately holds for \( j = 0 \) because \( b \) is the Condorcet winner while \( r(P) = \{a\} \neq \{b\} \). Suppose \( \text{MS}_P(b) \leq 0 \). Let us consider the procedure of adding up to \( |\text{MS}_P(b)| + 1 \) copies of \([a \succ b \succ \text{others}]\) to \( P \) one by one, and let \( P^* \) denote the last profile, i.e.,

\[
P^* = P + (|\text{MS}_P(b)| + 1) \times [a \succ b \succ \text{others}]
\]

According to PAR, \( r(P^*) = \{a\} \), but \( b \) is the Condorcet winner under \( P^* \), which means that CC must be violated during the process.

Moulin [29] then apply Lemma 1 in the following cases depending on \( r(\hat{P}) \), where we recall that \( \hat{P} \) is a profile such that \( \text{WMG}(\hat{P}) \in G_M \). For the sake of contradiction, suppose \( r \) satisfies CC and PAR. When recalling Moulin’s proof below, we will define rankings \( R_{1,1}^*, R_{1,2}^*, R_{1}^*, \ldots, R_m^* \) that will be used later.

- **Case 1.** If \( r(\hat{P}) = \{1\} \), then let \( R_1^* = [2 \succ 1 \succ 3 \succ 4 \succ \text{others}] \), where alternatives in “others” are ranked alphabetically. Let

\[
P^+ = \hat{P} + 4 \times R_1^*
\]

If the winner under \( P^+ \) is not 1 or 2, then for some \( 1 \leq j \leq 4 \), PAR is violated at \( \hat{P} + j \times R_1^* \), which is a contradiction. Therefore, we must have \( r(P^+) \subset \{1, 2\} \). The WMG of \( P^+ \) is shown in Figure 1 (b). We now apply Moulin [29]’s tool (Lemma 1) in the following two subcases.

- **Case 1.1.** If \( r(P^+) = \{1\} \), then notice that \( w_P(2, 1) = 7 > 5 = |\text{MS}_P(2)| \), which means that, let \( R_{1,1}^* = [1 \succ 2 \succ \text{others}] \), by Lemma 1, we have

\[
\exists 1 \leq j \leq 6 \text{ s.t. } \text{C&P}(r, P^+ + j \times R_{1,1}^*) = 0
\]

- **Case 1.2.** If \( r(P^+) = \{2\} \), then notice that \( w_P(3, 2) = 5 > 3 = |\text{MS}_P(3)| \), which means that, let \( R_{1,2}^* = [2 \succ 3 \succ \text{others}] \), by Lemma 1, we have

\[
\exists 1 \leq j \leq 4 \text{ s.t. } \text{C&P}(r, P^+ + j \times R_{1,2}^*) = 0
\]

- **Case 2.** If \( r(\hat{P}) = \{2\} \), then notice that \( w_{\hat{P}}(3, 2) = 9 > 7 = |\text{MS}_{\hat{P}}(3)| \), which means that, let \( R_2^* = [2 \succ 3 \succ \text{others}] \) (which is the same as \( R_{1,2}^* \)), by Lemma 1, we have

\[
\exists 1 \leq j \leq 8 \text{ s.t. } \text{C&P}(r, \hat{P} + j \times R_2^*) = 0
\]

- **Case 3.** If \( r(\hat{P}) = \{3\} \), then notice that \( w_{\hat{P}}(4, 3) = 7 > 5 = |\text{MS}_{\hat{P}}(4)| \), which means that, let \( R_3^* = [3 \succ 4 \succ \text{others}] \), by Lemma 1, we have

\[
\exists 1 \leq j \leq 6 \text{ s.t. } \text{C&P}(r, \hat{P} + j \times R_3^*) = 0
\]

- **Case i for each \( 4 \leq i \leq m \).** If \( r(\hat{P}) = \{i\} \) for some \( 4 \leq i \leq m \), \( w_P(1, i) \geq 5 > 3 = |\text{MS}_{\hat{P}}(1)| \), which means that, let \( R_i^* = [i \succ 1 \succ \text{others}] \), by Lemma 1, we have

\[
\exists 1 \leq j \leq 4 \text{ s.t. } \text{C&P}(r, \hat{P} + j \times R_i^*) = 0
\]

In each case, at least one of CC and PAR is violated at a profile that is obtained from \( \hat{P} \) by adding no more than 10 votes, which is a contradiction and concludes Moulin [29]’s proof.
Step 1 for 2 \( \uparrow n \): Trails and their properties. To analyze the likelihood of C&P for fixed \( n \), notice that adding two copies of \( R \) has the same effect of changing a \( \text{Rev}(R) \) vote to \( R \), where \( \text{Rev}(R) \) represents the reverse order of \( R \). We will call such an operation a flip. This motivates us to define trails that correspond to the sequences of profiles in Moulin [29]'s proof.

Definition 3 (Trails) For any \( n \)-profile \( P \), any \( 0 \leq j^* \leq n \), any \( 0 \leq j \leq j^* \), and any \( \ell = (\ell_1, \ldots, \ell_{j^*}) \in [n]^{j^*} \) that consists of \( j^* \) different numbers, we let \( \text{Flip}_\ell(P,j) \) denote the \( n \)-profile obtained from \( P \) by flipping the preferences of voters \( \{\ell_1, \ldots, \ell_j\} \). The length-\((j^*+1)\) trail represented by \( \ell \) is the following sequence of \( n \)-profiles:

\[
P \xrightarrow{\text{flip } \ell_1 \text{-th vote}} \text{Flip}_\ell(P,1) \xrightarrow{\text{flip } \ell_2 \text{-th vote}} \text{Flip}_\ell(P,2) \rightarrow \cdots \rightarrow \text{Flip}_\ell(P,j^*)
\]

Example 2 (A Trail) A length-3 trail represented by \( \ell = (3,1) \) is shown below.

\[
\begin{align*}
P = \text{Flip}_\ell(P,0) & \xrightarrow{\text{flip 3rd vote}} \text{Flip}_\ell(P,1) & \xrightarrow{\text{flip 1st vote}} \text{Flip}_\ell(P,2) \\
\{1 > 2 > 3 > 4, \} & \xrightarrow{\text{flip 3rd vote}} \{1 > 2 > 3 > 4, \} & \xrightarrow{\text{flip 1st vote}} \{4 > 3 > 2 > 1, \}
\end{align*}
\]

We further define the following \( m+1 \) types of trails that correspond to the \( m+1 \) sequences of profiles used in Moulin [29]'s proof presented above, i.e., in cases 1.1., 1.2, 2, \ldots, \( m \), where linear orders \( R_{1,1}^*, R_{1,2}^*, R_1^*, \ldots, R_m^* \) are defined.

Definition 4 (Moulin Trails) For any \( n \)-profile \( P = (R_1, \ldots, R_n) \) that contains \( 5 \times \mathcal{L}(\mathcal{A}) \) and any \( 2 \leq i \leq m \), define the set of type-\( i \) trails (starting at \( P \)), denoted by \( T_{P,i} \), as follows:

\[
T_{P,i} = \{(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5) \in [n]^5 : \forall j \leq 5, R_{\ell_j} = \text{Rev}(R_{i}^*) \}
\]

Furthermore, we define type-1.1 trail, denoted by \( T_{1,1} \), and type-1.2 trail, denoted by \( T_{1,2} \), as follows.

\[
T_{P,1,1} = \{(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5) \in [n]^5 : R_{\ell_2} = R_{\ell_5} = \text{Rev}(R_1^*), R_{\ell_1} = R_{\ell_3} = R_{\ell_4} = \text{Rev}(R_{1,1}^*) \}
\]

\[
T_{P,1,2} = \{(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5) \in [n]^5 : R_{\ell_2} = R_{\ell_5} = \text{Rev}(R_1^*), R_{\ell_1} = R_{\ell_4} = R_{\ell_3} = \text{Rev}(R_{1,1}^*) \}
\]

Let

\[
T_P = T_{P,1,1} \cup T_{P,1,2} \cup \bigcup_{i=2}^{m} T_{P,i}
\]

That is, a trail starting at \( P \) is a type-\( i \) trail (for \( i \in \{1.1, 1.2, 2, \ldots, m\} \)) if and only if for every \( 1 \leq j \leq 5 \), the \( j \)-th profile in the trail is obtained from its predecessor \((j - 1)-\)th profile in the trial) by flipping the preferences of an agent whose preferences are the same
as the $j$-th vote in the sequence of case $i$ in Moulin [29]'s proof presented in Step 0 above. Notice that the length of each trail in $T_P$ defined in Definition 4 is fixed to be 6.

Next, we prove a claim (Claim 1) that states that for any profile $\hat{P}$ whose WMG is in $G_M$ and that contains sufficient number of each ranking (formally defined as $(n, \gamma)$-profiles in Definition 5 below), there are $\Theta(n^5)$ trails in $T_{\hat{P}}$, each of which contains an $n$-profile that violates CC or PAR (or both).

**Definition 5 ($(n, \gamma)$-profiles)*** For any $n \in \mathbb{N}$ and $\gamma > 1$, an $n$-profile $P$ is said to be an $(n, \gamma)$-profile, if for every $R \in L(A)$, the number of $R$ votes in $P$ is at least $\gamma n + 5$.

**Claim 1** For any resolute voting rule $r$ over $L^*(A)$, any $0 < \gamma$, and any $(n, \gamma)$-profile $\hat{P}$ whose WMG is in $G_M$, there exist $\Theta(n^5)$ trails $\vec{\ell} \in T_{\hat{P}}$, such that for each of which there exists $1 \leq j \leq 5$ such that

$$C&P\left(r, \text{Flip}_{\vec{\ell}}(\hat{P}, j)\right) = 0$$

**Proof:** We modify Moulin [29]'s proof presented in Step 0 above by subtracting the reverse of rankings to keep the total number of rankings to be $n - 1$ or $n$. Notice that because $r$ may not satisfy anonymity, there are multiple ways to insert $R$ or remove $\text{Rev}(R)$. Formally, we define $\uparrow_{\vec{\ell}}$ to replace the procedure of appending votes in Moulin [29]'s proof presented above.

**Definition 6** For any $(n, \gamma)$-profile $P = (R_1, \ldots, R_n)$, any $j^* \in \mathbb{N}$, any $\vec{\ell} = (\ell_1, \ldots, \ell_{j^*}) \in [n]^{j^*}$ whose components (1) are all different, and (2) correspond to voters with the same vote in $P$, and any $1 \leq j \leq j^*$, define

$$\uparrow_{\vec{\ell}}(P, j) = \begin{cases} \text{Flip}_{\vec{\ell}}(P, \frac{j}{2}) & \text{if } 2 \mid j \\ \text{Flip}_{\vec{\ell}}(P, \frac{j - 1}{2}) - R_{\ell_{\frac{j+1}{2}}} & \text{if } 2 \nmid j \end{cases},$$

where $\text{Flip}_{\vec{\ell}}(P, \frac{j - 1}{2}) - R_{\ell_{\frac{j+1}{2}}}$ is the $(n - 1)$-profile obtained from $\text{Flip}_{\vec{\ell}}(P, \frac{j - 1}{2})$ by removing the $\ell_{\frac{j+1}{2}}$-th vote.

For example, $\uparrow_{\vec{\ell}}(P, 1)$ is the $(n - 1)$-profile that is obtained from $P$ by removing the $\ell_1$-th vote. $\uparrow_{\vec{\ell}}(P, 2)$ is the $n$-profile that is obtained from $\uparrow_{\vec{\ell}}(P, 1)$ by inserting $\text{Rev}(R_{\ell_1})$, so that it becomes the $(\ell_1)$-th vote; or equivalently, $\uparrow_{\vec{\ell}}(P, 2)$ can be obtained from $P$ by flipping the $(\ell_1)$-th vote; etc. A concrete instance of $\uparrow_{\vec{\ell}}$ is illustrated in the following example.

**Example 3** Continuing the setting of Example 2, $\uparrow_{\vec{\ell}}(\hat{P}, j)$ for $0 \leq j \leq 4$ is shown as follows.
Let CC that under certain conditions, the trail contains at least one violation of CC.

\[ \varepsilon : 2 \prec 3 \prec 4, \quad 2 \prec 3 \prec 4 \prec 1, \quad 3 \prec 4 \prec 1 \prec 2 \]

\[ P = \uparrow_{\ell} (P, 0) \rightarrow \uparrow_{\ell} (P, 1) \rightarrow \uparrow_{\ell} (P, 2) \]

The next lemma extends Moulin’s tool (Lemma 1) to trails (Definition 3), which states that under certain conditions, the trail contains at least one violation of CC or PAR.

**Lemma 2 (Moulin [29]'s tool on trails)** Let \( r \) be a resolute voting rule over \( L^*(A) \), and let the \( n \)-profile \( P \) and \( \ell \in [n] \) satisfy the premises in Definition 6. Let \( R = \text{Rev}(a \succ b \succ \text{others}) \) denote the vote by voters in \( \ell \). If \( r(P) = \{a\} \) and \( w_P(b, a) > |MSP(b)| \), then

\[ \exists 1 \leq j \leq |MSP(b)| + 1 \text{ such that } 2 \mid j \text{ and } C & P(r, \uparrow_{\ell} (P, j)) = 0 \]

**Proof:** Let \( j^* = |MSP(b)| \). The proof is done by analyzing the following sequence of \( j^* + 1 \) profiles:

\[ P = \uparrow_{\ell} (P, 0) \rightarrow \uparrow_{\ell} (P, 1) \rightarrow \uparrow_{\ell} (P, 2) \rightarrow \cdots \rightarrow \uparrow_{\ell} (P, j^*) \]

If \( r(\uparrow_{\ell} (P, j^*)) \neq \{b\} \), then CC is violated at \( \uparrow_{\ell} (P, j^*) \), which is an \( n \)-profile because \( |MSP(b)| \) has the same parity as \( n \), which is an odd number. This proves the lemma. Now, suppose \( r(\uparrow_{\ell} (P, j^*)) = \{b\} \). Because \( r(P) = \{a\} \), there exists \( j \geq 1 \) such that \( r(\uparrow_{\ell} (P, j - 1)) = \{a\} \neq r(\uparrow_{\ell} (P, j)) \). Let \( j' \) denote the smallest such \( j \). We prove that Lemma 2 holds by discussing the following two cases.

- If \( 2 \not| j' \), then \( |\uparrow_{\ell} (P, j')| = n - 1 \) and due to the minimality of \( j' \), we have

\[ \uparrow_{\ell} (P, j') = \uparrow_{\ell} (P, j' - 1) - R_{\ell} \]

If \( a \) is not the winner, then

\[ \text{Recall that } a \text{ is ranked at the bottom of } R_{\ell} = \text{Rev}(a \succ b \succ \text{others}). \text{ Therefore, } \]

PAR is violated at \( \uparrow_{\ell} (P, j' - 1) \), where the winner is \( a \) but voter \( \frac{j' + 1}{2} \) has incentive to abstain from voting, so that the winner is improved from \( a \) to any other alternative.

- If \( 2 \mid j' \), then \( |\uparrow_{\ell} (P, j')| = n \) and

\[ \uparrow_{\ell} (P, j' - 1) = \uparrow_{\ell} (P, j') - \text{Rev} \left( R_{\ell} \right) \]

If \( a \) is the winner, then

\[ \text{Recall that } a \text{ is ranked at the bottom of } R_{\ell} = \text{Rev}(a \succ b \succ \text{others}). \text{ Therefore, } \]
Recall that \( a \) is ranked at the top of
\[
\text{Rev}
\left(\frac{R_R'}{\tau}\right) = \text{Rev}(R) = [a > b > \text{others}]
\]

Therefore, \( \text{PAR} \) is violated at \( \uparrow_{\ell} (\hat{P}, j') \), where the winner is not \( a \), but agent \( \frac{j'}{2} \) has incentive to abstain from voting, so that the winner is improved to \( a \).

This proves Lemma 2.

Notice that when \( 2 \mid j \), we have \( \uparrow_{\ell} (\hat{P}, j) = \text{Flip}_P(\hat{P}, \frac{j}{2}) \). The rest of the proof is done by applying Lemma 2 to the \( m + 1 \) cases in Moulin [29]'s proof following trails in \( T_P \). More precisely, we discuss the following \( m + 1 \) cases.

- **Case 1.** Suppose \( r(\hat{P}) = \{1\} \). For every \( \ell = (\ell_1, \ldots, \ell_5) \in T_{\hat{P}, 1.1} \cup T_{\hat{P}, 1.2} \), if \( r(\text{Flip}_P(\hat{P}, 2)) \not\in \{1, 2\} \), then \( \text{PAR} \) is violated at \( \text{Flip}_P(\hat{P}, 1) \) or \( \text{Flip}_P(\hat{P}, 2) \).
  - **Case 1.1.** If \( r(\text{Flip}_P(\hat{P}, 2)) = \{1\} \), then for every \( \tilde{\ell}^* = (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5) \in T_{\hat{P}, 1.1} \), we apply Lemma 2 to \( \text{Flip}_P(\hat{P}, 2) \), which means that CC and/or \( \text{PAR} \) is violated at \( \text{Flip}_P(\hat{P}, j) \) for some 3 \( \leq j \leq 5 \).
  - **Case 1.2.** If \( r(\text{Flip}_P(\hat{P}, 2)) = \{2\} \), then similarly, for every \( \tilde{\ell}^* = (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5) \in T_{\hat{P}, 1.2} \), we apply Lemma 2 to \( \text{Flip}_P(\hat{P}, 2) \), which means that CC and/or \( \text{PAR} \) is violated at \( \text{Flip}_P(\hat{P}, j) \) for some 3 \( \leq j \leq 5 \).

Therefore, in Case 1 the number of trails in \( T_{\hat{P}, 1.1} \cup T_{\hat{P}, 1.2} \) that contain a violation of CC or \( \text{PAR} \) is at least \( \min(|T_{\hat{P}, 1.1}|, |T_{\hat{P}, 1.2}|) = \Theta(n^5) \), because \( \hat{P} \) is an \((n, \gamma)\)-profile.

- **Case 2** for \( 2 \leq i \leq m \). Suppose \( r(\hat{P}) = \{i\} \). For every \( \tilde{\ell} = (\ell_1, \ldots, \ell_5) \in T_{\hat{P}, i} \), following the same reasoning as in Moulin [29]'s proof presented in Step 1 above, CC and/or \( \text{PAR} \) is violated at \( \text{Flip}_P(\hat{P}, j) \) for some 1 \( \leq j \leq 5 \). Notice that the total number of trails in \( T_{\hat{P}, i} \) is \( \Theta(n^5) \).

This proves Claim 1.

**Step 2 for** \( 2 \nmid n \): Lower-bounding the smoothed likelihood of violations. Let \( P_{n, \gamma} \) denote the set of all \((n, \gamma)\)-profiles whose WMGs are in \( \mathcal{G}_M \). We will bound the smoothed probability for CC or \( \text{PAR} \) to be violated by bounding \( S_{n, \gamma, \bar{\pi}} \) defined as follows. For any \( \bar{\pi} \in \Pi^n \), let
\[
S_{n, \gamma, \bar{\pi}} = \sum_{P \in P_{n, \gamma}} \sum_{\tilde{\ell} \in T_P} \sum_{j=1}^5 \Pr_{P \sim \bar{\pi}} \left( P = \text{Flip}_P(\hat{P}, j) \right) \times \left( 1 - \text{C&P} \left( r, \text{Flip}_P(\hat{P}, j) \right) \right)
\]
In words, \( S_{n, \gamma, \bar{\pi}} \) is the sum of probabilities of profiles \( P \) that satisfy the following two conditions:

1. **(1)** \( P \) is on a trail starting at a profile in \( P_{n, \gamma} \), and
2. **(2)** CC or \( \text{PAR} \) is violated at \( P \).
Notice that some profiles are intentionally counted multiple times in $S_{n,\gamma,\vec{\pi}}$. Also notice that for any $\hat{P} \in \mathcal{P}_{n,\gamma}$ and any profile $P^*$ on any trail in $\mathcal{T}_P$, because the length of the trail is 6, we have

$$\Pr_{P \sim \vec{\pi}} (P = P^*) \geq \left( \frac{1 - \epsilon}{\epsilon} \right)^5 \Pr_{P \sim \vec{\pi}} (P = \hat{P}) = \Omega \left( \Pr_{P \sim \vec{\pi}} (P = \hat{P}) \right) \quad (2)$$

By Claim 1, there are $\Omega(n^5)$ trails starting at $\hat{P}$, each of which contains a violation of CC or PAR (different trails may contain the same profile). Therefore, each trail contributes $\Omega(\Pr_{P \sim \vec{\pi}} (P = \hat{P}))$ to $S_{n,\gamma,\vec{\pi}}$. By (2), we have

$$S_{n,\gamma,\vec{\pi}} = \Omega \left( n^5 \cdot \Pr_{P \sim \vec{\pi}} (P \in \mathcal{P}_{n,\gamma}) \right) \quad (3)$$

Next, we prove

$$S_{n,\gamma,\vec{\pi}} = O \left( n^5 \cdot \Pr_{P \sim \vec{\pi}} (C\&P(r, P) = 0) \right) \quad (4)$$

In fact, (4) follows after noticing that, in general, any $n$-profile $P^*$ is on $O(n^5)$ trails (regardless of whether $C\&P(r, P^*) = 0$ and/or whether the trails start at profiles in $\mathcal{P}_{n,\gamma}$). To see this, notice that for any trail type $i \in \{1,1,1,2,3,\ldots,m\}$, any position $0 \leq j \leq 5$ on the trail, and any $\vec{\ell} = (\ell_1, \ldots, \ell_5) \in [n]^5$, if $P^*$ is the $j$-th profile on the trail, then the trail is uniquely determined. Therefore, the total number of trails that $P^*$ can possibly be on is no more than

$$\left( m + 1 \right) \times 6 \times n^5 = O(n^5)$$

Combining Inequalities (3) and (4), for constant $\gamma > 0$, we have

$$\Pr_{P \sim \vec{\pi}} (C\&P(r, P) = 0) = \frac{1}{O(n^5)} S_{n,\gamma,\vec{\pi}} \quad \text{according to } (4)$$

$$= \frac{\Theta(n^5)}{O(n^5)} \Pr_{P \sim \vec{\pi}} (P \in \mathcal{P}_{n,\gamma}) \quad \text{according to } (3)$$

$$= \Theta \left( \Pr_{P \sim \vec{\pi}} (P \in \mathcal{P}_{n,\gamma}) \right) \quad (5)$$

The next claim characterizes the smoothed probability of $\Pr_{P \sim \vec{\pi}} (P \in \mathcal{P}_{n,\gamma})$ in (5), where $\gamma = \frac{1}{4m!}$.

**Claim 2** There exists $N \in \mathbb{N}$ such that for every $n > N$ with $2 \nmid n$,

$$\sup_{\vec{\pi} \in \Pi^n} \Pr_{P \sim \vec{\pi}} (P \in \mathcal{P}_{n,\frac{1}{4m!}}) = \Theta(n^{-3})$$

**Proof:** Let $\gamma = \frac{1}{4m!}$. The claim is proved by first modeling the histograms of profiles in $\mathcal{P}_{n,\frac{1}{4m!}}$ as a polyhedron $\mathcal{H}^{\vec{\pi}}$, then applying [41, Theorem 1]. $\mathcal{H}^{\vec{\pi}}$ is defined by three classes of constraints. The first class $A_1^M \cdot (\vec{x})^\top \leq \left( \vec{b}_1^M \right)^\top$ guarantees that the restriction of $\text{WMG}(\vec{x})$ on $\{1,2,3,4\}$ is the same as Figure 1 (a). The second class $A_2^M \cdot (\vec{x})^\top \leq \left( \vec{b}_2^M \right)^\top$
guarantees that the restriction of WMG($\vec{x}$) on edges between $\{1, 2, 3, 4\}$ and $\{5, \ldots, m\}$ is as specified in the definition of $G_M$ (Definition 2). The third class $A_3^M \cdot (\vec{x})^\top \leq (b_3^M)^\top$ guarantees that each ranking appears at least $2\gamma n = \frac{n}{2m}$ times, which is larger than $\gamma n + 5$ for any sufficiently large $n$.

Definition 7 ($H^{G_M}$) Let $A^{GM} = \begin{bmatrix} A_1^M \\ A_2^M \\ A_3^M \end{bmatrix}$ and $b^{GM} = [b_1^M, b_2^M, b_3^M]$, where

- For every edge $a \rightarrow b$ between $\{1, 2, 3, 4\}$ with positive weight $w_G(a, b)$ in Figure 1 (a), $A_1^M$ has two rows Pair$_{a,b}$ and Pair$_{b,a}$, and the corresponding $b_1^M$ value is $w_G(a, b)$ and $-w_G(a, b)$, respectively.
- For every $a \in \{1, 2, 3, 4\}$ and every $b \in \{5, \ldots, m\}$, $A_2^M$ has a row Pair$_{b,a}$ with corresponding $b_2^M$ value $-5$.
- For every $R \in L(A)$, $A_3^M$ has a row whose $x_R$ coefficient is $1 - 2m!$ and other coefficients are 1, and the corresponding $b_3^M$ value is 0.

Let

$$H^{G_M} = \left\{ \vec{x} \in \mathbb{R}^m : A^{GM} \times (\vec{x})^\top \leq (b^{GM})^\top \right\}$$

It is not hard to see that $H^{G_M}$ characterizes $P_{n, \gamma}$, in the sense that for any profile $P$, $WMG(P) \in P_{n, \gamma}$ if and only if $\text{Hist}(P) \in H^{G_M}$. Therefore, we have

$$\sup_{\vec{\pi} \in P_n} \Pr_{P \sim \vec{\pi}} (P \in P_{n, \frac{1}{4m}}) = \sup_{\vec{\pi} \in P_n} \Pr_{P \sim \vec{\pi}} (\text{Hist}(P) \in H^{G_M})$$

Next, we recall [41, Theorem 1] to prove that the right hand side of this equation is $\Theta(n^{-3})$.

**Theorem 1 in [41]** (Smoothed Likelihood of PMV-in-polyhedron, sup part). Given any $q \in \mathbb{N}$, any closed and strictly positive $\Pi$ over $[q]$, and any polyhedron $H = \left\{ \vec{x} \in \mathbb{R}^q : A \times (\vec{x})^\top \leq (b)^\top \right\}$, where $A$ is an integer matrix, for any $n \in \mathbb{N}$, we have

$$\sup_{\vec{\pi} \in P_n} \Pr (X_{\vec{\pi}} \in H) = \begin{cases} 0 & \text{if } H \text{ is inactive} \\
\exp(-\Theta(n)) & \text{if } H \text{ is active and } H_{\leq 0} \cap \text{CH}(\Pi) = \emptyset \\
\Theta(n^{(\dim(H_{\leq 0}) - q)/2}) & \text{otherwise} \end{cases}$$

In this theorem, $X_{\vec{\pi}}$ is the Poisson Multinomial Variable (PMV) that is the histogram of $n$ independent random variables over $[q]$ distributed as $\vec{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$, respectively. $H_{\leq 0}$ is the characteristic cone of $H$. That is,

$$H_{\leq 0} = \left\{ \vec{x} \in \mathbb{R}^q : A \times (\vec{x})^\top \leq (0)^\top \right\}$$
\(\mathcal{H}\) is active (at \(n\)) if there exists \(\bar{x} \in \mathbb{Z}_{\geq 0}^n \cap \mathcal{H}\) such that \(\bar{x} \cdot \mathbf{1} = n\). \(\dim(\mathcal{H}_{\leq 0})\) is the dimension of \(\mathcal{H}_{\leq 0}\), which is the dimension of the smallest linear space that contains \(\mathcal{H}_{\leq 0}\) and is equivalent to \(m! - \text{Rank}(\mathbf{A})\).

To apply [41, Theorem 1], we let \(q = m!\) and \(\mathcal{H} = \mathcal{H}^{G_M}\). It follows that \(\tilde{X}_{\bar{\pi}} = \text{Hist}(P)\), where \(P\) is generated from \(\tilde{\pi}\). To prove the claim, it suffices to prove that (1) the polynomial case of [41, Theorem 1] holds, and (2) \(\dim(\mathcal{H}_{\leq 0}) = m! - 6\). These are proved in the following two steps, respectively.

- **Step 1:** The polynomial case of [41, Theorem 1] holds. We first prove that \(\mathcal{H}^{G_M}\) is active for any sufficiently large \(n\), which implies that the 0 case does not hold. Our proof is by construction. Let \(\tilde{P}\) denote the 25-profile constructed by Moulin [29]. Let \(n' = n - 25 - \lceil \frac{n - 25}{m!} \rceil \times m!\), which is an even number because we assumed that \(n\) is odd. Let

\[
P = \tilde{P} + \left\lfloor \frac{n - 25}{m!} \right\rfloor \times \mathcal{L}(\mathbf{A}) + \frac{n'}{2} \left( [1 \succ 2 \succ \cdots \succ m], [m \succ m - 1 \succ \cdots \succ 1] \right)
\]

Then, for any \(n > 4m! + 25\), it is not hard to verify that \(\text{Hist}(P) \in \mathcal{H}^{G_M}\), which means that \(\mathcal{H}^{G_M}\) is active.

Next, recall that \(\pi_{\text{uni}} \in \text{CH}(\Pi)\) and \(\mathcal{H}^{G_M}_{\leq 0}\) is the characteristic cone of \(\mathcal{H}^{G_M}\), that is,

\[
\mathcal{H}^{G_M}_{\leq 0} = \left\{ \bar{x} \in \mathbb{R}^{m!} : \mathbf{A}^{G_M} \times (\bar{x})^\top \leq \begin{pmatrix} 0 \end{pmatrix}^\top \right\}
\]

It is not hard to verify that \(\pi_{\text{uni}} \in \mathcal{H}^{G_M}_{\leq 0}\). Therefore, \(\text{CH}(\Pi) \cap \mathcal{H}^{G_M}_{\leq 0} \neq \emptyset\), which means that the exponential case does not hold. The only remaining possibility is that the polynomial case of [41, Theorem 1] holds.

- **Step 2:** \(\dim(\mathcal{H}^{G_M}_{\leq 0}) = m! - 6\). We first note that \(\dim(\mathcal{H}^{G_M}_{\leq 0}) \leq m! - 6\), because \(\mathbf{A}_1^{G_M} \times (\bar{x})^\top \leq \begin{pmatrix} 0 \end{pmatrix}^\top\) implies that \(\mathbf{A}_1^{G_M} \times (\bar{x})^\top = \begin{pmatrix} 0 \end{pmatrix}^\top\), and \(\text{Rank}(\mathbf{A}_1^{G_M}) = 6\), where 6 represents the six edges among \(\{1, \ldots, 6\}\). Next, we prove \(\dim(\mathcal{H}^{G_M}_{\leq 0}) \geq m! - 6\). Let \(P'\) denote an arbitrary profile whose UMG has \(\binom{m}{2} - 6\) edges: all edges from \(\{1, 2, 3, 4\}\) to \(\{5, \ldots, m\}\) and edges between alternatives in \(\{5, \ldots, m\}\) (in arbitrary directions). In other words, only the six edges between \(\{1, 2, 3, 4\}\) are missing, and other edges are compatible with the definition of \(\mathcal{G}_M\). It follows that \(\text{Hist}(P' + |P'| \times \mathcal{L}(\mathbf{A}))\) is an interior point of \(\mathcal{H}^{G_M}_{\leq 0}\), such that the essential equalities of \(\mathbf{A}^{G_M}\) are \(\mathbf{A}_1^{G_M}\). This proves that \(\dim(\mathcal{H}^{G_M}_{\leq 0}) = m! - 6\).

This concludes the proof of Claim 2.

Therefore, when \(2 \nmid n\), Theorem 1 follows after inequality (5) and Claim 2.

**When** \(2 \nmid n\), the proof is similar to the \(2 \nmid n\) case with the following modifications. First, the weights on edges in Figure 1 are doubled, because now they must be even numbers. Second, the lengths of trails (Definition 4) are also doubled, i.e. the length becomes 10 for \(2 \nmid n\) case. Third, \(\mathbf{B}^{G_M}\) (Definition 7) is also doubled. Finally, in (3) and (4), \(n^5\) becomes \(n^{10}\).
4 Summary and Future Work

We prove a smoothed impossibility theorem on Condorcet Criterion and Participation. An immediate open question is to close the gap between the $n^{-3}$ rate proved in the theorem and the best known rate of $n^{-0.5}$ of existing voting rules. The problem appears challenging even under IC. Another open question is to characterize the the smoothed likelihood of C&P impossibility for the max-adversary, especially for II’s such that $\pi_{uni} \notin II$. More generally, we believe that developing smoothed versions of other impossibility theorems, especially those where quantitative versions were obtained under IC, such as Arrow’s, Gibbard-Satterthwait, and various impossibility theorems in judgement aggregation, is a promising direction for future work.

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