Edge local complementation for logical cluster states

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Abstract. A method is presented for the implementation of edge local complementation (ELC) in graph states, based on the application of two Hadamard operations and a single controlled-phase (CZ) gate. As an application, we demonstrate an efficient scheme for constructing a one-dimensional logical cluster state based on the five-qubit quantum error-correcting code, using a sequence of ELCs. A single physical CZ operation, together with local operations, is sufficient to create a logical CZ operation between two logical qubits. This approach in concatenation may allow one to create a hierarchical quantum network for quantum information tasks.

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1. Introduction

Multipartite entangled states are fundamental resources for quantum computation, with many mysteries yet to be understood [1]. A particularly useful and interesting set of multipartite entangled states is the so-called graph states [2]. These are quantum states associated with mathematical graphs, where vertices represent qubits in superposition states and edges represent the maximally entangling controlled-phase (CZ) gates between them. Building complex graph states is a difficult task in practice (i.e. in experiments), because it requires the application of CZ gates between arbitrary qubits; that said, considerable strides towards this have been made in recent years [3]. It is, nevertheless, useful to consider the circumstances under which specific multipartite graph states can be constructed efficiently.

A class of particularly useful graph states is the quantum error-correcting codes (QECCs). These are used to prevent quantum information leakage, since quantum information is generically fragile against interactions with the environment [4]. Standard QECCs can protect quantum information against an arbitrary error on a single qubit. Several schemes for measurement-based quantum computation with embedded quantum error correction have been proposed in recent years, but the structure of logical cluster states is very complex [5–7]. Very recently, a concatenation scheme for a single logical qubit encoded in the five-qubit QECC (5QECC) was studied in the graph-state context [8]. While topological approaches to fault tolerance in graph-state quantum computation yield higher error thresholds [9], directly encoding the quantum information in QECC graphs might turn out to be more practical experimentally if efficient methods for constructing these states can be found.

We propose that multipartite graph states, which are useful in constructing logical cluster states with 5QECC, can be efficiently built by local Hadamard operations from simpler graph states. In this paper, we prove that the mathematical operation called edge local complementation (ELC) [10], which is defined by a series of local complementation (LC) operations on a graph [2, 11], is efficiently realizable in specific graph states because it is equivalent to the action of local Hadamard operations. From the mathematical point of view, LC transforms a given graph into another, with a different adjacency matrix; in practice, LC of a given vertex complements the subgraph corresponding to its neighborhood. From the quantum information point of view, LC corresponds to a set of local operations on a given graph state that therefore preserves any entanglement measure, yet describes a different graph state. Yet the cost of generating the new graph from a completely unentangled state would be significantly higher if the total number of edges is larger than in the original graph state. Our results indicate that the apparently complex nature of multipartite 5QECC states should not in itself be an impediment to their experimental generation, because they are in fact generically simple graphs under ELC.

This paper is organized as follows. We introduce the graph state notation in section 2. The definition of ELC and its equivalence to Hadamard operations in graph states are discussed in section 3. In section 4, we present the step-wise method of building one-dimensional (1D) logical cluster states. Finally, we summarize our results and present our future research interests.

2. Background

Let us begin with the definition of graphs and graph operations. In graph theory, a graph $G = (V, E)$ is given by $N$ vertices $V = \{a_1, \ldots, a_N\}$ and edges $E$ corresponding to a linked line between two adjacent (neighboring) vertices. We only consider simple graphs with no self-loops.
and no multiple edges. If a vertex \( c \in V \) is chosen in a graph, the other vertices are represented by its \( n \) neighboring vertices \( \mathcal{N}(c) = \{b_1, \ldots, b_n\} \in V \) and outer vertices \( V \setminus \{c, b_1, \ldots, b_n\} = \{o_1, \ldots, o_{N-n-1}\} \in V \). The neighborhood of all of the vertices is defined by the adjacency matrix \( A \), an \( N \times N \) symmetric matrix with elements \( A_{ij} = 1 \) iff \( \{a_i, a_j\} \in E \).

All simple graphs correspond to a class of quantum states called graph states [2], in which each vertex is represented by a qubit in a superposition state and an edge corresponds to the application of a maximally entangling gate. Specifically, an \( N \)-qubit graph state \( |G\rangle \) is defined as

\[
|G\rangle = \bigotimes_{i>j} (CZ_{i,j})^{A_{ij}} |+\rangle^{\otimes N},
\]

where \( |\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2} \) are the \( \pm 1 \)-eigenstates of \( X \) (here \( \{X, Y, Z\} \) are the \( 2 \times 2 \) Pauli operators) and \( CZ = \text{diag}(1, 1, 1, -1) \) is the controlled-phase (controlled-\( Z \)) gate acting between two qubits. Graph states can be defined in at least two equivalent ways, both of which will prove useful for our purposes. Because the CZ operations can be written as

\[
CZ_{i,j} = \frac{1}{2}(I_i I_j + I_i Z_j + Z_i I_j - Z_i Z_j)
\]

\[
= \frac{1}{2}(1 + (-1)^{x_i + x_j} + (-1)^{y_i} - (-1)^{y_i + y_j}) = (-1)^{y_i + y_j},
\]

where \( I_i \) is the identity matrix applied at site \( i \), the graph state is given by a quadratic form of a Boolean function \( p(x) \)

\[
|G\rangle = \frac{1}{\sqrt{2^N}} (-1)^{p(x)} |x_1 \cdots x_N\rangle,
\]

where \( x_j \in \{0, 1\} \) and \( p(x) = \sum_{i<j} A_{ij} x_i x_j \) [12]. Obviously, the value \( x_i x_j = 1 \) iff \( x_i = x_j = 1 \) (otherwise the value is 0), so \( p(x) \) is a quadratic polynomial representing the graph adjacency matrix. Alternatively, the state \( |G\rangle \) is the fixed eigenvector, with unit eigenvalue, of \( N \) independent commuting operators

\[
S(a) = X_a \bigotimes_{b=\neq a} Z_b,
\]

i.e. \( S(a)|G\rangle = |G\rangle \) for all \( a \in V \). Because the \( \{S(a), a \in V\} \) generate a set of \( 2^N \) stabilizer operators \( S \), these \( N \) generators uniquely define \( |G\rangle \).

Figures 1(b) and (c) show two simple and important examples of graph states that are not equivalent under LU transformations: the star \( S_N \) and cycle \( C_N \) graphs. The star graphs correspond to GHZ states:

\[
|S_n\rangle = \frac{1}{\sqrt{2^{n+1}}} (-1)^q(x) |x_{c_1} x_{a_1} \cdots x_{a_n}\rangle,
\]

with \( q(x) = \sum_{i=1}^n x_{c_i} x_{a_i} \); these are LC-equivalent to the complete graphs \( K_{n+1} \). Figure 1(b) depicts \( |S_4\rangle \equiv |g\rangle \). The cycle graph state is equal to

\[
|C_N\rangle = \frac{1}{\sqrt{2^N}} (-1)^r(x) |x_{a_1} \cdots x_{a_N}\rangle,
\]

with \( r(x) = x_{a_1} x_{a_N} + \sum_{i=2}^{N-1} x_{a_i} x_{a_{i+1}} \). Figure 1(c) shows \( |C_5\rangle \equiv |\varnothing\rangle \). The former is related to classically encoded graph states and the latter to 5QEC [13].

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Figure 1. A simple example of ELC on a graph state is shown in (a). The graph consists of four qubits (blue circles) and initially has three edges (red lines), forming a linear graph. ELC is given by three sequential (vertex) LCs at $c_1$, $c_2$, and $c_1$, respectively. A five-qubit star graph state $|S_5\rangle \equiv |g\rangle$ is shown in (b); panel (c) depicts a pentagon cycle graph state $|C_5\rangle \equiv |pentagon\rangle$.

LC and ELC are two operations used to classify locally equivalent graphs that are generally inequivalent under isomorphism (vertex permutation). The action of local complementation LC$(a)$ at the vertex $a$ transforms the graph $G$ by replacing the subgraph associated with the neighboring vertices $N(a)$ with its complement [11]. The new graph generated by LC$(a)$ on $G$ is locally equivalent to the original graph. It is important to note that the LC$(a)$ operation does not affect the edges of outer vertices in the graph $G$; only the neighborhood of vertex $a$ is affected. The action of ELC$(a, b)$ on the edge $\{a, b\} \in E$ is defined by three LCs: ELC$(a, b) = LC(a)\cdot LC(b)\cdot LC(a)\cdot LC(b)\cdot LC(a)\cdot LC(b)$. The action of ELC on the edge $\{a, b\}$ can be understood as follows. Consider any pair of vertices $\{c, d\} \in E$, where $c$ is a neighbor of $a$ but not $b$, and $d$ is a neighbor of $b$ but not $a$ (or vice versa); alternatively, $c$ and $d$ can both be neighbors of $a$ and $b$. ELC then corresponds to complementing the edge between $c$ and $d$; that is, if $\{c, d\} \in E$, then delete the edge, and add it if $\{c, d\} \notin E$. In addition, the neighborhoods of $a$ and $b$ are replaced by one another. ELC has been investigated in order to recognize the edge-local equivalence of two graphs [14] and to understand the relationship between classical codes and graphs [15].

In the context of graph states, local equivalence implies that one graph state can be transformed into another by the action of single-qubit (i.e. local) operations. It is well known that two graph states that are equivalent under stochastic local operations and classical communication (SLOCC) must also be equivalent under the local unitary (LU) operations [16]. A long-standing conjecture held that LU equivalence also implied equivalence under the action of Clifford-group elements (operations that map the Pauli group to itself), although this was recently proved to be false in general [17].

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Nevertheless, the transformations LC (and therefore ELC) on graph states can be expressed solely in terms of local Clifford operations [2]:

\[
LC(a) = \sqrt{-iX_a} \bigotimes_{b \in \mathcal{N}(a)} \sqrt{iZ_b} \propto \sqrt{S(a)},
\]

where \( t \equiv \sqrt{-1} \). Suppose that \(|G\rangle\) possesses the qubit \( c_1 \) (called a core qubit) connected to \( n \) neighboring qubits. The action of \( LC(c_1) \) corresponds to the application of \( \frac{n(n-1)}{2} \) CZ operations on the graph state, creating edges between \( \mathcal{N}(c_1) \) if there were none and removing them otherwise \((CZ^2 = I)\). Although entanglement between the two graph states is the same due to the invariance of entanglement under LU operations, the number of effective CZ operations (i.e. the number of edges) differs. ELC on the edge \( \{a, b\} \) would then correspond to the operation

\[
ELC(a, b) = \sqrt{-iX_a} \bigotimes_{c \in \mathcal{N}(a)} \sqrt{iZ_c} \sqrt{-iX_b} \bigotimes_{d \in \mathcal{N}(b)} \sqrt{iZ_d} \sqrt{-iX_a} \bigotimes_{f \in \mathcal{N}(a)} \sqrt{iZ_f},
\]

where \( \mathcal{N}' \) and \( \mathcal{N}'' \) are reminders that the neighborhoods themselves change under the LC operations. Recognizing that \( a \) and \( b \) remain neighbors, this can be rewritten as

\[
ELC(a, b) = (-i)H_a \otimes H_b \bigotimes_{c \in \mathcal{N}(a), b} \sqrt{iZ_c} \bigotimes_{d \in \mathcal{N}(b), a} \sqrt{iZ_d} \bigotimes_{f \in \mathcal{N}(a), b} \sqrt{iZ_f},
\]

where \( H_a = (X_a + Z_a)/\sqrt{2} = \sqrt{-iX_a}\sqrt{iZ_a} \sqrt{-iX_a} \) is the Hadamard operator on qubit \( a \). One of the goals of this paper is to show that the result of this operation on graph states can be expressed in the simpler form \( ELC(a, b)|G\rangle = H_a \otimes H_b |G\rangle \), requiring the application of far fewer local operations.

Simple examples of LC and ELC are shown in figure 1(a). The initial graph state \(|G\rangle\) consists of four qubits and three edges. After the first \( LC(c_1) \), because no edge exists between two neighboring qubits of \( c_1 \) in state \(|G\rangle\), an edge is drawn between them. After \( LC(c_2) \), the edge on qubits \( a_1 \) and \( c_1 \) is deleted by a rule of the LC because two sequential CZ operations become the identity between \( a_1 \) and \( c_1 \). Finally, after the last \( LC(c_1) \), the number of edges is four on the final graph state, which is represented by \( ELC(c_1, c_2)|G\rangle \), although all four graph states are locally equivalent.

3. Edge local complementation (ELC) via Hadamard gates

Consider two disconnected graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) and their respective graph states \(|G_1\rangle\) and \(|G_2\rangle\); each possesses a core vertex (qubit), \( c_1 \in V_1 \) and \( c_2 \in V_2 \), respectively. A CZ operation is then applied to the two core qubits, linking the two graph states into a single connected graph, \(|G_a\rangle = CZ_{c_1c_2}|G_1\rangle \otimes |G_2\rangle\). If a Hadamard operation is then applied to each core qubit, the graph \(|G_a\rangle\) is transformed into another locally equivalent graph state, \(|G_H\rangle = H_{c_1} \otimes H_{c_2} |G_a\rangle\). Below, we show that the state \(|G_H\rangle\) is the edge local complement of \(|G_a\rangle\), i.e. that \(|G_H\rangle = H_{c_1} \otimes H_{c_2} |G_a\rangle = ELC(c_1, c_2)|G_a\rangle\). It is important to note that the equivalence of edge complementation on \( \{c_1, c_2\} \in E \) with the application of Hadamard operations on \( c_1 \) and \( c_2 \) is only valid if \( \mathcal{N}(c_1) \cap \mathcal{N}(c_2) \neq \emptyset \), i.e. that prior to the application of \( CZ_{c_1c_2} \), the neighborhoods of \( c_1 \) and \( c_2 \) were completely disjoint. Our results do not apply to graphs where \( c_1 \) and \( c_2 \) share a neighborhood (other than themselves).
The main theorem of this paper is the following.

**Theorem 3.1.** Consider two graph states, \( |G_1 \rangle = \frac{1}{\sqrt{2^n}}(-1)^{\sum_{i>j} A_{ij}^{(1)} x_i x_j} |x_1 \cdots x_n \rangle \) and \( |G_2 \rangle = \frac{1}{\sqrt{2^n}}(-1)^{\sum_{i>j} A_{ij}^{(2)} x_i x_j} |x_1 \cdots x_n \rangle \), defined by adjacency matrices \( A^{(1)} \) and \( A^{(2)} \) on independent vertex sets \( V_1 \in \{ a_1^{(1)}, \ldots, a_N^{(1)} \} \) and \( V_2 \in \{ a_1^{(2)}, \ldots, a_N^{(2)} \} \), respectively. If core qubits, \( c_1 \) and \( c_2 \), are chosen at random from each of these vertex sets, and are entangled with one another by means of a CZ gate, then

\[
H_{c_1} H_{c_2} C Z_{c_1, c_2} |G_1 \rangle |G_2 \rangle = E L C (c_1, c_2) C Z_{c_1, c_2} |G_1 \rangle |G_2 \rangle,
\]

where the EL C operator on the edge \( \{ c_1, c_2 \} \) is \( E L C (c_1, c_2) = L C (c_1) L C (c_2) L C (c_1) \), and the (vertex) LC operator at qubit a complements the edge set of its neighborhood \( N(a) \).

**Proof.** The core qubits \( c_1 \) and \( c_2 \) have neighborhood \( N(c_1) = \{ b_1^{(1)}, \ldots, b_n^{(1)} \} \) and \( N(c_2) = \{ b_1^{(2)}, \ldots, b_m^{(2)} \} \), respectively. The remaining vertices of the graphs \( G_1 \) and \( G_2 \) are \( V_1 \setminus \{ c_1, b_1^{(1)}, \ldots, b_n^{(1)} \} = \{ o_1^{(1)}, \ldots, o_{N_1-1}^{(1)} \} \in V_1 \) and \( V_2 \setminus \{ c_2, b_1^{(2)}, \ldots, b_m^{(2)} \} = \{ o_1^{(2)}, \ldots, o_{N_2-1}^{(2)} \} \in V_2 \), respectively. Performing a CZ operation between these core qubits, the graph state \( |G_u \rangle \) is

\[
|G_u \rangle = C Z_{c_1, c_2} |G_1 \rangle |G_2 \rangle = (-1)^{x_1 x_2} |G_1 \rangle |G_2 \rangle,
\]

\[
= \frac{1}{\sqrt{2^{N_1+N_2}}}(-1)^{q_1(x)+q_2(x)} |x_1^{(1)} \cdots x_n^{(1)} \rangle |x_1^{(2)} \cdots x_m^{(2)} \rangle,
\]

\[
q_1(x) = \sum_{i>j} A_{ij}^{(1)} x_i^{(1)} x_j^{(1)} = x_{c_1} \sum_{i=1}^n x_{b_i^{(1)}} + \sum_{i>j} A_{ij}^{(1)} x_i^{(1)} x_j^{(1)},
\]

\[
q_2(x) = \sum_{i>j} A_{ij}^{(2)} x_i^{(2)} x_j^{(2)} = x_{c_2} \sum_{i=1}^m x_{b_i^{(2)}} + \sum_{i>j} A_{ij}^{(2)} x_i^{(2)} x_j^{(2)}.
\]

**3.1. Two Hadamards applied to core qubits**

Consider

\[
|G_H \rangle = H_{c_1} H_{c_2} |G_u \rangle = \frac{1}{2} (X_{c_1} X_{c_2} + X_{c_1} Z_{c_2} + Z_{c_1} X_{c_2} + Z_{c_1} Z_{c_2}) |G_u \rangle.
\]

This can be simplified by noting that for \( b_j \in N(c_1) \)

\[
X_{c_1} (-1)^{x_1 x_{b_j}} = (-1)^{(x_1+1) x_{b_j}} X_{c_1},
\]

\[
Z_{c_1} (-1)^{x_1 x_{b_j}} = (-1)^{x_1} (-1)^{x_1 x_{b_j}} = (-1)^{x_1 (x_{b_j}+1)}.
\]

One then obtains

\[
H_{c_1} H_{c_2} (-1)^{x_1 x_2} = \frac{1}{2} (-1)^{x_1 x_2} \left[ -(-1)^{(x_1+x_2)} X_{c_1} X_{c_2} + X_{c_1} + X_{c_2} + (-1)^{(x_1+x_2)} \right].
\]
Applying this to the remaining operators in equation (11) yields
\[
|G_H\rangle = \frac{(-1)^{\nu_1+\nu_2}}{\sqrt{2^{N_1+N_2}}} \frac{1}{2} \prod_{k,k'} \left[ (-1)^{x_k^{(1)} + x_{k'}^{(2)}} \right] \langle x_{d_1^{(1)}} \cdots x_{d_{N_1}^{(1)}} | x_{d_1^{(2)}} \cdots x_{d_{N_2}^{(2)}} \rangle
\]
\[
+ (-1)^{x_k^{(1)} + x_{k'}^{(2)}} \frac{1}{2} \prod_{k,k'} \left[ (-1)^{x_k^{(1)} + x_{k'}^{(2)}} \right] \langle x_{d_1^{(1)}} \cdots x_{d_{N_1}^{(1)}} | x_{d_1^{(2)}} \cdots x_{d_{N_2}^{(2)}} \rangle
\]
\[
= \frac{(-1)^{\nu_1+\nu_2}}{\sqrt{2^{N_1+N_2}}} (-1)^{(q_1(x)+q_2(x)+q_3(x))} \langle x_{d_1^{(1)}} \cdots x_{d_{N_1}^{(1)}} | x_{d_1^{(2)}} \cdots x_{d_{N_2}^{(2)}} \rangle,
\]
where
\[
q_3(x) = (x_1 + x_2) \left( \sum_{k} x_{b_k^{(1)}} + \sum_{k'} x_{b_{k'}^{(2)}} \right) + \sum_{k,k'} x_{b_k^{(1)}} x_{b_{k'}^{(2)}}.
\]
Finally, one can combine all the terms to obtain
\[
|G_H\rangle = H_{c_1} H_{c_2} CZ_{c_1,c_2} |G_1\rangle |G_2\rangle = \frac{(-1)^{p(x)}}{\sqrt{2^{N_1+N_2}}} \langle x_{d_1^{(1)}} \cdots x_{d_{N_1}^{(1)}} | x_{d_1^{(2)}} \cdots x_{d_{N_2}^{(2)}} \rangle,
\]
where
\[
p(x) = x_1 x_2 + x_{c_1} \sum_{k=1}^{m} x_{b_k^{(1)}} + x_{c_2} \sum_{k=1}^{n} x_{b_k^{(2)}} + \sum_{k=1}^{m} \sum_{k'=1}^{m} x_{b_k^{(1)}} x_{b_{k'}^{(2)}}
\]
\[
+ \sum_{i > j} A_{ij}^{(1)} x_{d_i^{(1)}} x_{d_j^{(1)}} + \sum_{i > j} A_{ij}^{(2)} x_{d_i^{(2)}} x_{d_j^{(2)}}.
\]

3.2. ELC on core qubits

Recall that edge local complementation ELC\((c_1, c_2)\) on the edge \{\(c_1, c_2\)\} is described by the three local complementations LC\((c_1)\)LC\((c_2)\)LC\((c_1)\) = LC\((c_2)\)LC\((c_1)\)LC\((c_2)\). Suppose that the first LC is performed on \(|G_u\rangle\) at qubit \(c_1\). The result is that all neighboring qubits of \(c_1\) are explicitly connected to each other (adding an edge to an existing edge annihilates it). The additional edges are given by the quadratic form
\[
r_1(x) = \sum_{i > j} x_{d_i^{(1)}} x_{d_j^{(1)}} + x_{c_2} \sum_{i} x_{b_i^{(1)}},
\]
Next one complements the neighborhood of qubit \(c_2\), which is given by the quadratic form \(x_{c_2} \left( x_{c_1} + \sum_i x_{b_i^{(1)}} + \sum_i x_{b_i^{(2)}} \right)\); the result is \(x_{c_1} \left( \sum_i x_{b_i^{(1)}} + \sum_i x_{b_i^{(2)}} \right) + \sum_{i > j} x_{d_i^{(1)}} x_{d_j^{(1)}} + \sum_{i > j} x_{d_i^{(2)}} x_{d_j^{(2)}} + \sum_{i,j} x_{x_i^{(1)}} x_{b_j^{(2)}}\). The total additional edges are then given by the quadratic form
\[
r_2(x) = x_{c_2} \sum_i x_{b_i^{(1)}} + x_{c_1} \left( \sum_i x_{b_i^{(1)}} + \sum_i x_{b_i^{(2)}} \right) + \sum_{i > j} x_{d_i^{(2)}} x_{d_j^{(2)}} + \sum_{i,j} x_{b_i^{(1)}} x_{b_j^{(2)}}.
\]
Lastly, one complements the neighborhood of qubit \( c_1 \), which is given by the quadratic form
\[
x_{c_1} \left( x_{c_2} + \sum_i x_{b_i^{(1)}} + \sum_i x_{b_i^{(2)}} \right) = x_{c_1} \left( x_{c_2} + \sum_i x_{b_i^{(2)}} \right); \text{the result is simply } x_{c_2} \sum_i x_{b_i^{(2)}} + \sum_{i \neq j} x_{b_i^{(2)}} x_{b_j^{(2)}}.
\]
The quadratic form for the additional edges after this final operation is
\[
r_3(x) = x_{c_2} \left( \sum_i x_{b_i^{(1)}} + \sum_i x_{b_i^{(2)}} \right) + x_{c_1} \left( \sum_i x_{b_i^{(1)}} + \sum_i x_{b_i^{(2)}} + \sum_{i \neq j} x_{b_i^{(1)}} x_{b_j^{(2)}} \right).
\]
(23)

Combining this result with the remaining terms in the quadratic form (11), the graph resulting from the ELC becomes
\[
\text{ELC}(c_1, c_2) |G_u\rangle = \frac{(-1)^{p(x)}}{\sqrt{2^{N_1+N_2}}} |a_1^{(1)} \cdots a_{N_1}^{(1)} |b_1^{(2)} \cdots b_{N_2}^{(2)}\rangle,
\]
(24)
where
\[
p(x) = x_{c_1} x_{c_2} + x_{c_1} \sum_i x_{b_i^{(2)}} + x_{c_2} \sum_i x_{b_i^{(2)}} x_{b_j^{(2)}} + \sum_{i \neq j} A_{ij}^{(2)} x_{a_i^{(1)}} x_{a_j^{(1)}} + \sum_{i \neq j} A_{ij}^{(2)} x_{a_i^{(2)}} x_{a_j^{(2)}}.
\]
(25)
which is identical to the quadratic form (20).

Equation (20) shows that when Hadamard gates are applied to both (core) qubits of a single edge between two graphs, the result is a new graph state corresponding to the effective application of \( 2(n+m)+nm \) CZ operations. These operations have the effect of replacing the original neighborhood of each core qubit with the neighborhood of the other core qubit (and vice versa), while simultaneously adding the neighborhood of a given core qubit to the neighborhood of the other. That is, from the edge set one deletes the combinations \( \{c_1, b_k^{(1)}\} \) and \( \{c_2, b_i^{(2)}\} \) and adds the combinations \( \{c_1, b_i^{(2)}\} \), \( \{c_2, b_k^{(1)}\} \) and \( \{b_k^{(1)}, b_i^{(2)}\} \). In other words, the Hadamard operations have complemented the neighborhood of the edge \( \{c_1, c_2\} \) or performed ELC. Of particular interest is the special case when both of the original graphs \( |G_1\rangle \) and \( |G_2\rangle \) are star graphs with the core qubit corresponding to the maximum-degree vertex, i.e. where \( \{o_1^{(1)}\} \notin V_1 \) and \( \{o_2^{(2)}\} \notin V_2 \). Then the resulting graph would be completely bipartite, with every vertex of the first group \( \{c_1, b_1^{(1)}, \ldots, b_{n_1}^{(1)}\} \) connected to every vertex of the second group \( \{c_2, b_1^{(2)}, \ldots, b_{n_2}^{(2)}\} \) [18].

### 3.3. Vertex local complementation

The above analysis proves that the application of Hadamard operations to the core qubits \( c_1 \) and \( c_2 \) is equivalent to ELC on the edge \( \{c_1, c_2\} \). It is not obvious that ELC based on the formal definition of LC given in equation (7), \( \text{LC}(c_1) = \sqrt{-iX} X \prod_{p=0}^{N(c_1)} \sqrt{Z_{b_p}}, \) reproduces the same result. Although graph transformations effected by this expression have already been discussed in [2] in the context of vertex LC, ELC using this operator was not explicitly explored in that work. In fact, as shown below, the application of these unitary gates in order to effect ELC requires local operations in addition to the two Hadamard gates.

It is convenient to write
\[
\sqrt{-iX} = [-I+iX]/\sqrt{2}, \quad \sqrt{iZ} = [iI+Z]/\sqrt{2}.
\]
(26)
The action of these on quadratic forms is
\[
-\sqrt{-1} X_{c_1} (-1)^{x_{c_1} b_j} = (-1)^{x_{c_1} b_j} \left[ -1 + i (-1)^{b_j} X_{c_1} \right] / \sqrt{2},
\]
\[
\sqrt{i} Z_{b_j} (-1)^{x_{c_1} b_j} = (-1)^{x_{c_1} b_j} \left[ i + (-1)^{b_j} \right] / \sqrt{2}.
\]

Suppose that one has an arbitrary graph state \( |G \rangle \) defined by the quadratic form \( p(x) \) whose neighborhood of the qubit \( c_1 \) is \( N(c_1) = \{ b_1, \ldots, b_n \} \), i.e. where \( p(x) \) includes the term \( c_1 \sum_j b_j \). LC on the vertex \( c_1 \) then yields
\[
\text{LC}(c_1) = \sqrt{-1} X_{c_1} \prod_{j=1}^n \sqrt{i} Z_{b_j} (-1)^{x_{c_1} b_j} = \frac{1}{2^{n/2}} \prod_{j=1}^n (-1)^{x_{c_1} b_j} \left[ i + (-1)^{b_j} \right] \left[ -1 + i (-1)^{b_j} X_{c_1} \right].
\]

When this LC operator is applied to the graph state \( |G \rangle \), the \( X_{c_1} \) operator will act only on its eigenstates and will effectively disappear. The effect of the various terms above is then equivalent to the new quadratic form
\[
p(x)' = p(x) + \sum_{j > k} b_j b_k - \sum_i b_i.
\]

In other words, LC\((c_1)\) has complemented the neighborhood of the qubit \( c_1 \), by effectively applying CZ entangling operations to all of its neighbors. In addition, it has applied Z gates to all the neighbors. These are local operations that commute with the CZs and are therefore unimportant. That said, complete equivalence (rather than simply unitary equivalence) under ELC would then require the application of additional unitary gates beyond the two Hadamard gates.

4. Application: the efficient generation of one-dimensional logical cluster states

We now discuss a novel and useful application of the theory of ELC to quantum information processing. In a previous work [6], we showed that logical cluster states corresponding to 5QECC can be made with logical CZ operations consisting of many CZ operations among the physical qubits. A linear \( N \)-qubit logical cluster state is given by
\[
|CS_L^L \rangle = \prod_{i=1}^{N-1} C_{i+1} |+^L \rangle_1 \otimes \cdots \otimes |+^L \rangle_N,
\]
where \( C^L \) is a logical CZ operation between two logical qubits and \( |\pm^L \rangle = (|0^L \rangle \pm |1^L \rangle)/\sqrt{2} \). For \( |CS_L^L \rangle \) with 5QECC, 25 physical CZ operations are required in order to construct a logical CZ operation from \( |+\rangle^{10} \) (see figure 3 in [6]). The construction of many-qubit logical cluster states requires so many entangling operations to build logical CZ gates as to be impractical for realistic quantum information processing. In this context, the ELC provides an efficient solution.
to this conundrum: a single physical CZ operation and two Hadamard operations are sufficient to build a logical CZ operation between two logical qubits.

First we will review how to encode a physical qubit into a logical qubit with 5QECC. One begins begin with a qubit in state $|0\rangle_{a_1}$ and four auxiliary qubits in $|++-+\rangle_{a_2-a_5}$. After a Hadamard operation on qubit $a_1$ and four CZ operations between $a_1$ and the others, one obtains the five-qubit GHZ-type graph state $|g\rangle_A$ (see figure 1(b) but with $c_1$ replaced by $a_1$ and $a_{1,2,3,4}$ replaced by $d_{2,3,4,5}$). After an additional Hadamard operation on qubit $a_1$ in $|g\rangle_A$, the state is equal to a five-qubit GHZ state

$$|\text{CS}_2\rangle_{AB} = CZ_{AB}^L E_A^L E_B^L |+\rangle_{a_2-a_5}^\otimes 10 = CZ_{AB}^L |+\rangle_A |+\rangle_B. \tag{34}$$

Because $E_A^L E_B^L |+\rangle_{a_2-a_5}^\otimes 10 = C_5 \prod_{i=2}^5 CZ_{a_i,a_1}^L |+\rangle_{a_2-a_5}^\otimes 4$ in this case, 35 physical CZ operations in total are required in order to build $|\text{CS}_2^L\rangle_{a_1 b_1}$ from $|0\rangle_{a_1} |+\rangle_{b_1}^\otimes 10$. In the second method, one creates classically encoded graph states by means of ELCs; the quantum encoding is then applied to the classically encoded states in order to obtain logical cluster states. Initially, the core qubit $a_1$ of one classical state is entangled with its counterpart $b_1$ in the other state, yielding a two-qubit cluster state $|\text{CS}_2\rangle_{a_1 b_1} = ((|0\rangle_{a_1} |+\rangle_{b_1} + |1\rangle_{a_1} |-)_{b_1})/\sqrt{2}$. Note that the first Hadamard operations $(H_{a_1} \otimes H_{b_1})$ in $E_A^L E_B^L$ leave the state $|\text{CS}_2^L\rangle_{a_1 b_1}$ invariant. After the GHZ-type CZ operations are performed between $a_1$ ($b_1$) and $a_i$ ($b_i$) ($i = 2, 3, 4, 5$), through the operation $\prod_i \prod_j CZ_{a_i,a_j}^L CZ_{b_i,b_j}^L$, a connected graph state $|G_u\rangle$ is obtained (see figure 2). When two Hadamard operations are subsequently applied to $a_1$ and $b_1$ in $|G_u\rangle$, the...
Figure 2. The ELC consisting of three sequential LCs makes the same transformation as the operation of two Hadamard operations on core qubits. Green lines indicate new edges created by each LC, whereas red lines indicate the pre-existing edges. The graph state \(|G_H\rangle\) is completely bipartite.

The resulting state is transformed into another graph state \(|G_H\rangle\), given by

\[
|G_H\rangle_{AB} = \frac{1}{2} \left[ (+)^{\otimes 5}_{a_1-a_5} \left( (+)^{\otimes 5}_{b_1-b_5} + (-)^{\otimes 5}_{b_1-b_5} \right) + (-)^{\otimes 5}_{a_1-a_5} \left( (+)^{\otimes 5}_{b_1-b_5} - (-)^{\otimes 5}_{b_1-b_5} \right) \right].
\] (35)

This state is a classically encoded two-qubit cluster state.

In figure 2, it is shown that the action of three LCs on the core vertices \(a_1\) and \(b_1\) provides the desired CZ operations among the physical qubits, reproducing the state (35). The resulting graph is known as a complete bipartite graph state [18]: each of the vertices in one neighborhood (corresponding to logical register A or B) is connected with all the vertices of the other neighborhood, and vice versa. While it is possible to construct \(|G_H\rangle_{AB}\) directly by applying 25 CZ operations starting with \((+)^{10}\), it can be efficiently built using only nine CZ operations plus two local operations. For the quantum encoding scheme, the final state is given by

\[
|CS_L^2\rangle_{AB} = C_{a_1-a_5}^{\ominus} C_{b_1-b_5}^{\ominus} |G_H\rangle_{AB} = E_A^L E_B^L CZ_{a_1b_1} |+\rangle^{10}.
\] (36)

Therefore, the state \(|CS_L^2\rangle\) can be efficiently built by 19 CZ operations with the help of two Hadamard operations, instead of 35 CZ operations, and the logical CZ operation expressed by

\[
CZ_{AB} E_A^L E_B^L = E_A^L E_B^L CZ_{a_1b_1}
\] (37)

shows that a single physical CZ operation is sufficient to create a logical CZ operation between logical qubits.

While the encoding procedure for graph states is straightforward to implement, its interpretation in terms of ELC is not obvious in general. For example, any encoding of a cluster...
state with an odd number of qubits is difficult to express in terms of ELCs, each requiring an even number of Hadamard operations. The interpretation of encoding linear 2N-qubit cluster states through ELC is straightforward, however.

Consider, for example, the linear four-qubit logical cluster state. First one assigns five qubits each to registers A, B, C and D. After assigning a core qubit from each, designated \(a_1, b_1, c_1\) and \(d_1\), respectively, one prepares the linear four-qubit cluster state \(|\text{CS}_4\rangle = CZ_{a_1,b_1} CZ_{b_1,c_1} CZ_{c_1,d_1} |+\rangle^4_{a_1-d_1}\). The encoding consists of acting on each register with \(\prod_J E^L_J |\text{CS}_4\rangle |+\rangle^{16}\) for \(J = A, B, C, D\), where \(E^L_J\) is given in equation (33). The first step is to perform four Hadamard operations on \(|\text{CS}_4\rangle\). Applying two Hadamard operations on qubits \(a_1\) and \(b_1\), the intermediate graph state is equal to

\[
|\Psi_{\text{inter}}\rangle_{a_1-d_1} = \text{ELC}(a_1, b_1)|\text{CS}_4\rangle = CZ_{a_1,b_1} CZ_{a_1,c_1} CZ_{c_1,d_1} |+\rangle^4_{a_1-d_1},
\]

using the results of ELC. Because \(c_1\) and \(d_1\) share an edge but their neighborhoods are disjoint, it is reasonable to associate the subsequent Hadamard operations on qubits \(c_1\) and \(d_1\) with another ELC on the edge \(\{c_1, d_1\}\). The resulting state is equal to another linear four-qubit cluster state, with but with the vertex labels permuted:

\[
|\text{CS}_4\rangle = \text{ELC}(c_1, d_1)|\Psi_{\text{inter}}\rangle_{a_1-d_1} = CZ_{a_1,b_1} CZ_{a_1,d_1} CZ_{c_1,d_1} |+\rangle^4_{a_1-d_1}.
\]

After four GHZ-type operations and the second set of four Hadamard operations on \(|\text{CS}_4\rangle_{a_1-d_1} |+\rangle^{16}\), again corresponding to two ELCs, the outcome is a linear four-qubit cluster state with classical encoding. The Hadamard operations not only effect the ELC; they also reverse the permutation of the vertex labels above. Finally, the quantum encoding scheme on all the qubits yields a logical four-qubit cluster state \(|\text{CS}_4^L\rangle\), which is sufficient for universal quantum computation with 5QECC [6]. This procedure can be trivially extended to any even-length chain, by applying Hadamard gates in pairs on nearest-neighbor edges in order to implement ELCs from the left boundary of the chain to the right.

5. Summary and remarks

The main result presented in this paper is a proof that the action of ELC on a graph state can be effected solely through the use of two Hadamard operations applied to the edge qubits. A crucial assumption in this proof is that the neighborhoods of the edge qubits were disjoint, i.e. that the neighbors \(N(a)\) of the first edge qubit \(a\) were different from the neighbors \(N(b)\) of the second qubit \(b\). Under this restriction, ELC interchanges the respective neighborhoods, i.e. \(N(a) \leftrightarrow N(b)\), while simultaneously making neighbors of all the neighbors. In principle, this transformation would require a large number of either LU operations on the graph-state qubits or entangling gates between various qubits. A distinct advantage of the present scheme is the large savings in the number of (local) operations required.

As an example of the utility of this insight, we show how ELC can be used to efficiently create classically encoded cluster states and 1D logical cluster states based on the five-qubit error-correcting code, for an even number of logical qubits. In this scheme, a physical CZ operation, together with local operations, is sufficient to create a logical CZ operation between two logical qubits.

Arbitrary encoded graph states can be obtained by a straightforward extension of the procedure described above. The operations encoding a logical qubit, equation (33), are local to the physical qubits comprising the logical qubit and therefore commute with one another.
It therefore suffices to first construct the desired graph state with the core qubits, associate four ancillae to each core qubit and operate independently with equation (33) on each five-qubit register.

Multipartite entangled states that fundamentally include fault tolerance might be desirable for practical measurement-based quantum computing and multipartite quantum communication [8, 19]. For a generalized scheme of level-$l$ logical graph states based on our proposal, the same encoding procedure can be used repeatedly. Since the level-1 logical graph state is made by our protocol, a level-$l$ concatenated logical graph state becomes an initial state to create a level-$(l+1)$ concatenated one ($|\pm_i^l \rangle \rightarrow |\pm_i^{l+1} \rangle$) with the help of the classical and quantum coding schemes described above. This concatenated method may also be useful in building multi-party quantum networks similar to classical complex networks in hierarchical organization [20].

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