Quantum Electrodynamical Bloch Theory with Homogeneous Magnetic Fields

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We propose a solution to the problem of Bloch electrons in a homogeneous magnetic field by including the quantum fluctuations of the photon field. A generalized quantum electrodynamical (QED) Bloch theory from first principles is presented. In the limit of vanishing quantum fluctuations we recover the standard results of solid-state physics, for instance, the fractal spectrum of the Hofstadter butterfly. As a further application we show how the well known Landau physics is modified by the photon field and that Landau polaritons emerge. This shows that our QED-Bloch theory does not only allow to capture the physics of solid-state systems in homogeneous magnetic fields, but also novel features that appear at the interface of condensed matter physics and quantum optics.

Cavity QED materials is a growing research field bridging quantum optics [11, 2], polaritonic chemistry [3–7], and materials science, such as light-induced new states of matter achieved with classical laser fields [8, 9]. Photomatter interactions have recently been suggested to modify electronic properties of solids, such as superconductivity and electron-phonon coupling [10–14]. On the other hand, materials in classical magnetic fields are known to give rise to a variety of novel phenomena such as the Landau levels [15], the integer [16, 17] and the fractional quantum Hall effect [18], and the quantum fractal of the Hofstadter butterfly [19] which can be now accessed experimentally with high resolution [20–22]. One of the open questions in this field is whether Bloch theory is applicable for solids in the presence of a homogeneous magnetic field. The homogeneous magnetic field breaks explicitly translational symmetry. This issue was solved to some extent by introducing the magnetic translation group. However, the magnetic translation group puts fundamental limitations on the possible directions and values of the strength of the magnetic field [17, 23, 24].

In this Letter, by combining QED with solid-state physics, we provide a consistent and comprehensive theory for solids interacting with homogeneous electromagnetic fields, both classical and quantum. Our main findings are as follows: (i) The quantum fluctuations of the electromagnetic field allow us to restore translational symmetry that is broken due to an external homogeneous magnetic field (see Fig. 1). (ii) We generalize Bloch theory and provide a Bloch central equation for electrons in a solid in the presence of a homogeneous magnetic field and its quantum fluctuations. (iii) Applying our framework to the case of a 2D solid in a perpendicular homogeneous magnetic field, in the limit of no quantum fluctuations, we recover the fractal spectrum of the Hofstadter butterfly (see Fig. 2). (iv) In the case of a 2D electron gas in a cavity and under the influence of a perpendicular homogeneous magnetic field we find Landau polariton states [25, 27]. The spectrum of the Landau polaritons (in atomic units) is

$$E_{j,k_{w}} = k_{w}^2/2M + \Omega (j + 1/2).$$

(1)

The frequency of the upper polariton is $$\Omega = \sqrt{\omega_c^2 + \omega_p^2}$$ and depends on the cyclotron frequency $$\omega_c$$ and an effective plasma frequency $$\omega_p$$. The kinetic energy $$k_{w}^2/2M$$
corresponds to the lower polariton (see Fig. 3) and will be explained in detail in what follows.

Translational Symmetry with Homogeneous Magnetic Fields.—Non-relativistic QED describes electrons minimally coupled to the electromagnetic field, both classical and quantum. For the description of the photon field we follow the standard procedure of assuming a finite box of length $L$ and volume $V = L^3$, in which we model the electromagnetic field [1 2 28]. In the usual case of a solid, the volume $V$ does not constitute a physical quantity. In this case the local electron density $n_e = N/V$ is the quantity to work with, since the volume $V$ and the number of electrons $N$ tend to infinity in such a way that the local electron density $n_e$ is constant. On the other hand, if we consider a solid confined in a cavity, the mode volume determines the coupling of the cavity modes to the electrons [29 30] and the volume becomes a physical quantity. Our starting point in both cases is the Pauli-Fierz Hamiltonian [1 2 28]

$$\hat{H} = \frac{1}{2m_e} \sum_{j=1}^{N} \left[ \left( i\hbar \nabla_j + \frac{e}{c} \hat{A}(r_j) + \frac{e}{c} \hat{A}_{\text{ext}}(r_j) \right) \right]^2 + v_{\text{ext}}(r_j)$$

$$+ \frac{1}{4\pi\epsilon_0} \sum_{j<k}^{N} \frac{e^2}{|r_j - r_k|} + \hbar \omega_n \left( \hat{a}_n^\dagger \hat{a}_n + \frac{1}{2} \right),$$

(2)

where we neglected the Pauli (Stern-Gerlach) term. Here $\hat{A}(r)$ is the quantized vector potential of the electromagnetic field in Coulomb gauge, $\nabla \cdot \hat{A}(r) = 0$, given by

$$\hat{A}(r) = \left( \frac{\hbar c^2}{\epsilon_0 V} \right)^{\frac{1}{2}} \sum_{n,\lambda} \frac{\epsilon_{n,\lambda}}{\sqrt{2\omega_n}} \left[ \hat{a}_{n,\lambda} \delta_{n,\lambda} r + \hat{a}_{n,\lambda}^\dagger \delta_{n,\lambda} e^{-i\kappa_n \cdot r} \right].$$

(3)

Further, $\kappa_n = 2\pi n/L$ are wave vectors with $n = (n_x, n_y, n_z) \in \mathbb{Z}^3$, $\omega_n = c|\kappa_n|$ are the allowed frequencies, $\epsilon_0$ the vacuum permittivity, and $\epsilon_{n,\lambda}$ are the transversal polarization vectors of each photon mode [1 2]. The operators $\hat{a}_{n,\lambda}$ and $\hat{a}_{n,\lambda}^\dagger$ are annihilation and creation operators, respectively, and obey canonical commutation relations $[\hat{a}_{n,\lambda}, \hat{a}_{m,\mu}^\dagger] = \delta_{nm} \delta_{\lambda\mu}$. By introducing the displacement coordinates $q_{n,\lambda}$ and their conjugate momenta $\partial/\partial q_{n,\lambda}$ we can define $\hat{a}_{n,\lambda} = [q_{n,\lambda} + \partial/\partial q_{n,\lambda}] / \sqrt{2}$ and $\hat{a}_{n,\lambda}^\dagger = [q_{n,\lambda} - \partial/\partial q_{n,\lambda}] / \sqrt{2}$. The quantized field in our theory captures the real back-reaction of matter to the electromagnetic field. Such back-reactions are essential in solid-state physics as, e.g., captured by the semi-classical microscopic-macroscopic connection that determines the induced fields inside a material [29 31]. In the setting of cavity QED these back-reactions can be enhanced by cavity confinement, and in this case the quantized field models the influence coming from the cavity modes.

Moreover, $\hat{A}_{\text{ext}}(r)$ is a general external vector potential. Here we are interested in the case of a homogeneous magnetic field, and for that purpose we choose the external vector potential in Landau gauge [15] $\hat{A}_{\text{ext}}(r) = -e_x B y$ which gives rise to a constant magnetic field in the $z$-direction, $B_{\text{ext}} = \nabla \times \hat{A}_{\text{ext}}(r) = e_z B$.

Within the framework of Bloch theory [32], the external potential is assumed periodic, $v_{\text{ext}}(r) = v_{\text{ext}}(r + R_n)$, where $R_n$ is a Bravais lattice vector. In order to analyze conveniently the external vector potential, which depends on the electronic coordinates only in $y$-direction and is polarized along the $x$-direction, we choose the lattice vectors as follows

$$R_n = x_n e_x + y_n e_y + z_n e_z = n a_x e_x + m a_y e_y + l a_z e_z.$$  

(4)

Having a periodic external potential and a uniform magnetic field, one would expect a periodic solution using Bloch theory. Yet, it is obvious that the external vector potential $A_{\text{ext}}(r)$ breaks translational symmetry since it is linear in $y$. The quantized vector potential $\hat{A} = e_y q \sqrt{\hbar^2/\epsilon_0 V}$. The Pauli-Fierz Hamiltonian, after expanding the covariant kinetic energy, takes the form

$$\hat{H} = \frac{\hbar^2}{2m_e} \sum_{j=1}^{N} \left[ -\frac{\hbar^2}{2m_e c^2} \nabla_j^2 + \frac{ie\hbar}{m_e c} (\hat{A} + \hat{A}_{\text{ext}}(r_j)) \cdot \nabla_j + v_{\text{ext}}(r_j) \right]$$

$$+ \frac{1}{4\pi\epsilon_0} \sum_{j<k}^{N} \frac{e^2}{|r_j - r_k|} + \frac{\epsilon^2}{2m_e c^2} \sum_{j=1}^{N} \left( \hat{A} + \hat{A}_{\text{ext}}(r_j) \right)^2$$

$$+ \hbar \omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right).$$

(5)

But what exactly does the optical limit mean for a solid? The optical limit is valid in cases where the wavelength of the electromagnetic field is much larger than the size of the electronic system. But solids compared to the size of an atom are infinitely large systems. This holds especially within the context of Bloch theory where full periodicity is assumed. This implies that in the optical limit the wavelength of the field should be infinite and the frequency should tend to zero. Naively, taking $\omega \to 0$ in $\hat{A}$ seems to lead to divergencies in Eq. (5). However, if the limit is performed consistently, which means that we take the back-reaction of matter due to the square of the vector potential into account, no divergencies arise.

To that end, we isolate the purely photonic part of $\hat{H}$ which includes only one bare photon mode of frequency $\omega$ plus the square of the vector potential $\hat{H}_p = \hbar \omega \left( \hat{a}^\dagger \hat{a} + 1/2 \right) + \hat{A}^2 N e^2 / 2m_e c^2$. In terms of the photon coordinate $q$ and the conjugate momentum $\partial_q = \partial/\partial q$ it is $\hat{H}_p = \hbar \omega / 2 \left( -\partial_q^2 + q^2 \right) + q^2 N e^2 h / 2m_e \omega_0 V$. By introducing the dressed frequency parameter given by
The quantized vector potential in the optical limit is 

\[ \hat{H}_p = \hbar \omega / 2 (-\partial_n^2 + u^2) \]

where the frequency \( \omega_p \) is the well-known plasma frequency which depends on the electron density \( n_e \) and is given by \( \omega_p = \sqrt{n_e e^2 / m_e \epsilon_0} \). The vector potential as a function of the new coordinate \( \omega / \omega_e \) is periodic in the electronic coordinates because substituting \( \hat{A} \) in the back into Eq. (5) we obtain the Hamiltonian in the optical limit

\[
\hat{H}_{opt} = -\frac{\hbar^2}{2m_e} \sum_{j=1}^{N} \nabla_j^2 + \frac{i e}{m_e c} \sum_{j<k}^{N} \left( \hat{A} + \hat{A}_{ext}(r_j) \right) \cdot \nabla_j \\
+ \frac{e^2}{2mc^2} \sum_{j=1}^{N} \left( \hat{A} + \hat{A}_{ext}(r_j) \right)^2 - \frac{\hbar \omega_p}{2} \partial_u^2.
\]

The quantized vector potential in the optical limit is \( \hat{A} = e_x u \sqrt{\hbar c^2 / \epsilon_0 \omega_p} \). For a periodic potential \( \hat{H}_{opt} \) is still not periodic in the electronic coordinates because the external vector potential is still linear in \( y \). But the optical Hamiltonian \( \hat{H}_{opt} \) is periodic under the generalization translation

\[
(r_j, u) \rightarrow (r_j + \mathbf{R}_n, u + B y_m \sqrt{\hbar c^2 / \epsilon_0 \omega_p}).
\]

This proves our claim that in the optical limit the broken translational symmetry, caused by the homogeneous magnetic field, gets restored (see Fig. 1). We further note, that if we include a time-dependent homogeneous external vector potential as well \[ 23-33 \], we can treat a solid subject to a homogeneous electric as well as magnetic field.

**QED-Bloch Theory with Homogeneous Magnetic Fields.**—Having restored the broken translational symmetry we will move a step further and derive a Bloch central equation for periodic solids in homogeneous magnetic fields. Instead of expressing the unfeasible many-electron interacting problem of Eq. (6), we will employ the independent electron approximation which resembles the usual approach of density-functional theory (DFT). Such an approach is perfectly consistent with Bloch theory, which is not a theory of one electron in a periodic potential, but of many non-interacting electrons in a periodic potential. Thus, to account for the collective coupling of the electrons to the photon field, we use an effective plasma frequency that captures the back-reaction. Any further exchange and correlation effects would need the inclusion of effective fields as introduced in quantum-electrodynamical DFT (QEDFT) \[ 6, 36 \]. Introducing the cyclotron frequency \( \omega_c = eB / m_e c \), the optical Hamiltonian of Eq. (6) in the independent electron approximation is

\[
\hat{H}_{opt} = -\frac{\hbar^2}{2me} \nabla^2 + i e \left( u \sqrt{\hbar \omega_p / m_e - y \omega_c} \right) \cdot \nabla \\
+ \frac{e^2}{2mc^2} \left( u \sqrt{\hbar \omega_p / m_e - y \omega_c} \right)^2 - \frac{\hbar \omega_p}{2} \partial_u^2.
\]

For convenience we work in units where \( \hbar = m_e = e = 1 \). The Hamiltonian \( \hat{H}_{opt} \) is invariant under the translation given by Eq. (7) that acts on both, the electronic and photonic configuration space. In order to describe properly this symmetry we switch to a new set of polaritonic coordinates given by

\[
v = \frac{\sqrt{\omega p u - \omega_c y}}{\sqrt{2}}, w = \frac{m_p \sqrt{\omega p u + m e \omega_c y}}{\sqrt{2M}},
\]

where the mass parameters are \( m_p = 1 / \omega_p^2, m_e = 1 / \omega^2 \), and \( M = (m_p + m_e) / 2 \). In this coordinate system the Hamiltonian \( \hat{H}_{opt} \) becomes

\[
\hat{H}_{opt} = -\left( \partial_v^2 + \partial_w^2 + \partial_u^2 / M \right) / 2 + i \sqrt{2} \omega_c \partial_x \\
+ \hat{v}_{ext}(r) - \Omega^2 \partial_u^2 / 4 + v^2
\]

with \( \Omega^2 = 1 / \omega_p^2 + 1 / m_p + \omega^2 + \omega_c^2 \) and \( r = (x, w / \sqrt{2} \omega_c - m_p v / \sqrt{2M} \omega_c, z) \). The coordinates \( v \) and \( w \) are independent since the respective momenta and positions commute. The Hamiltonian \( \hat{H}_{opt} \) includes a harmonic oscillator \( \hat{H}_v = \Omega^2 \partial_u^2 / 4 + v^2 \) which has the Hermite functions \( \phi_j(v) \) as eigen-states and its spectrum is \( \epsilon_j = \Omega(j + 1/2) \) with \( j \in \mathbb{N}_0 \). \( \hat{H}_v \) can be written equivalently in terms of annihilation and creation operators \( \hat{H}_v = \Omega(b \hat{b} + 1/2) \), \( \hat{b} = v / \sqrt{\Omega} \), which includes the Bloch’s theorem in those coordinates. Thus, the eigen-functions of the Hamiltonian \( \hat{H}_{opt} \) can be written with the ansatz

\[
\Psi_k(r_w, v) = e^{i k \cdot r_w} U_k(r_w, v)
\]

where \( r_w = (x, w, z) \). Here \( U_k(r_w, v) \) is periodic along \( r_w = (x, w, z) \) with periodicities \( a_x, \sqrt{2} \omega_c a_y, \) and \( a_z \), respectively. One important aspect of our version of the Bloch ansatz above is that it is a polaritonic Bloch ansatz, in the sense that both coordinates \( v \) and \( w \) are combined coordinates. The crystal momentum \( k = (k_x, k_w, k_z) \), corresponds to \( r_w \), and \( k_x \) is a polaritonic quantum number. Moreover, the polaritonic unit cell in \( w \)-direction scales linearly with the strength of the magnetic field (see Fig. 1). The same feature appears also in the case of the so-called magnetic unit cell, but the magnetic unit cell allows only field strengths which generate a rational magnetic flux through a unit cell \[ 17 \]. This is a consequence of invariance under the magnetic translation group introduced by Zak \[ 23, 24 \]. On the contrary, the polaritonic unit cell puts no restrictions on the allowed magnetic strengths.
Since the function $U^k(r_w, v)$ is periodic in $r_w$, we can expand it in a Fourier series along $r_w$. For the $v$ coordinate of the polaritonic Bloch ansatz we use the eigen-functions of the harmonic oscillator $\hat{H}_v$. Thus,

$$\Psi_k(r_w, v) = e^{i k \cdot r_w} \sum_{n, j} U_{n, j} e^{i G_n r_w} \phi_j(v),$$

(12)

where $G_n = (G_n^x, G_n^y, G_n^z)$ = $2\pi (n/a_z, m/\sqrt{2}\omega_c a_y, l/a_z)$ is the reciprocal lattice vector. The external potential is expanded in a Fourier series as well.

$$v_{\text{ext}}(r) = \sum_n V_n e^{i G_n r} e^{-i G_n^m m}_v v/M,$$

(13)

Substituting Eqs. (12) and (13) into the Hamiltonian $\hat{H}_{\text{opt}}$ of Eq. (10), and then acting from the left with $\langle \phi_i |$ and eliminating the plane waves we obtain

$$\left[ \frac{(k_x + G_n^x)^2}{2} + \frac{(k_w + G_n^y)^2}{2M} + \frac{(k_z + G_n^z)^2}{2} + \mathcal{E}_i - E_k \right] U_{n, i}^k - \sqrt{2}(k_x + G_n^x) \sum_j \langle \phi_i | v \phi_j \rangle U_{n, j}^k \langle \phi_i | e^{-i G_n^m m}_v v/M | \phi_j \rangle = 0.$$

(14)

Using the Hermite recursion relations we find for the matrix $\langle \phi_i | v \phi_j \rangle = \sqrt{\Omega \sqrt{\Omega}} \delta_{i+1, j-1} + \sqrt{\Omega + 1} \delta_{i, j+1}$). The exponential in the last term of Eq. (14) can be written as a displacement operator using $\hat{b}$ and $\hat{b}^\dagger$,

$$e^{-i G_n^m m}_v v/M = e^{\alpha_{m, n} \hat{b} - \alpha_{m, n}^* \hat{b}^\dagger} = \hat{D}(\alpha_{m, n})$$

(15)

where $\alpha_{m, n} = -i G_n^m m v \sqrt{\Omega}/2M$. The matrix representation of this displacement operator in the basis $\{\phi_i(v)\}$ is given by

$$\langle \phi_i | \hat{D}(\alpha_{m, n}) | \phi_j \rangle = \sqrt{\frac{j!}{\Omega^n}} \frac{-\nu_{m, n}^2}{2} L_{ij}^{(-\nu_{m, n}^2/2)} (|\alpha_{m, n}^2|),$$

(16)

where $i \geq j$ and $L_{ij}^{(-\nu_{m, n}^2/2)} (|\alpha_{m, n}^2|)$ are the associated Laguerre polynomials. Using Eq. (16) and the expression for the matrix $\langle \phi_i | v \phi_j \rangle$ we obtain the generalized Bloch central equation

$$\left[ \frac{(k_x + G_n^x)^2}{2} + \frac{(k_w + G_n^y)^2}{2M} + \frac{(k_z + G_n^z)^2}{2} + \mathcal{E}_i - E_k \right] U_{n, i}^k - \sqrt{ \Omega } \left[ \sqrt{1 + U_{n, i+1}^k} + \sqrt{1 - U_{n, i-1}^k} \right] \frac{(k_x + G_n^x) \sum_j \langle \phi_i | v \phi_j \rangle U_{n, j}^k \langle \phi_i | e^{-i G_n^m m}_v v/M | \phi_j \rangle = 0. $$

(17)

Equation (17) gives the spectrum and the eigen-functions for electrons inside a solid under the influence of a constant magnetic field, in the case where quantum fluctuations of the field due to the electron density are also taken into account. We would like to mention that Eq. (17) also holds in the limit where the plasma frequency goes to zero, i.e., $\omega_p \to 0$. In this limit all the parameters involved in Eq. (17) depend only on the strength of the external magnetic field, since they take the values, $M \to \infty$, $\Omega \to \omega_c$, and $\alpha_{m, n} \to -i \pi \sqrt{2 (m - m^') \sqrt{\omega_e a_y}}$. This means that the physics of periodic structures in homogeneous magnetic fields (17) is recovered in our theory. More concretely we, for instance, recover the fractal pattern of the Hofstadter butterfly, depicted in Fig. 2, with the choice of a cosine lattice potential in the lowest Landau level. Detailed explanations on how the Hofstadter butterfly was obtained and how the quantized fields can modify the fractal spectrum will be presented in a forthcoming publication.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Energy spectrum of a 2D solid in a perpendicular homogeneous magnetic field as a function of the inverse relative flux.}
\end{figure}

**Landau Polaritons.**—In the following we show how the photon field modifies well-known results of condensed matter physics in the special case of no external potential, $v_{\text{ext}}(r) = 0$. In this case $\hat{H}_{\text{opt}}$ of Eq. (10) can be diagonalized analytically, which is a consequence of the generalized translational symmetry. For the part of the Hamiltonian depending on the coordinates $r_w = (x, y, z)$ the eigen-functions are plane waves given by $e^{i k \cdot r}$ and by applying the Hamiltonian $\hat{H}_{\text{opt}}$ on these plane waves we obtain

$$\hat{H}_{\text{opt}}[k] = k_x^2/2 + k_y^2/2M - \Omega^2 \delta_{i, j}^2/4 + (v - k_z/\sqrt{2})^2.$$

The eigen-functions of the shifted harmonic oscillator are the Hermite functions $\phi_j(v - k_z/\sqrt{2})$ with spectrum $\mathcal{E}_j = \Omega (j + 1/2)$. The eigen-functions of $\hat{H}_{\text{opt}}$ are

$$\Psi_{k, j}(r_w, v) = e^{i k \cdot r_w} \phi_j \left( v - k_z/\sqrt{2} \right).$$

(18)
Restricting our model to the case of a 2D electron gas \((k_z = 0)\) confined in a cavity, under the influence of a perpendicular homogeneous magnetic field, we find the energy spectrum to be given by Eq. \((1)\). This spectrum is similar to the one derived by Landau \[15\], but there is a major difference. The eigen-functions of Eq. \((18)\) are functions of the combined polaritonic coordinates \(v\) and \(w\). Thus, these states should be interpreted as Landau polaritons. Such Landau polaritons have been recently observed experimentally \[25, 27\]. More specifically in \[27\], Landau polaritons were observed in a strained Germanium 2D hole gas with 2D density \(n^{2D} = 1.3 \times 10^{12} \text{ cm}^{-2}\) and effective mass \(m^* = 0.0675m_e\), confined in cavity with frequency \(\omega = 2\pi \times 208 \text{ GHz}\).

In this setting we can define the effective plasma frequency \[10\] in terms of the 2D density and the cavity frequency \(\omega_p^2 = e^2n^{2D}/m^*\epsilon_0L = e^2n^{2D}\omega/2\pi cm^*\epsilon_0\). Here \(L\) is the cavity length and \(\omega = 2\pi c/L\) is the fundamental cavity frequency. Using the parameters reported in \[27\] we find the effective plasma frequency \(\omega_p = 0.653 \text{ THz}\). Fig. \[3\] shows the polaritonic dispersions as a function of the cyclotron frequency. The lower polariton branch in fact is a continuum (shaded area in Fig. \[3\]) for which we assume the polaritonic kinetic term \(k_w^2\) to be restricted in \(k_w^2 \in [0, 1/2] \text{ THz}^{-1}\) \[? \]. This reproduces qualitatively and quantitatively the data reported in \[27\]. Importantly, the fact that the lower polariton branch is a continuum is a direct consequence of the translational symmetry in the generalized polaritonic coordinate \(w\) (see Fig. \[1\]). As discussed in \[27\] generic models, such as the Hopfield model, fail to account for this behavior. Finally, we would like to point out that in the case of no cavity confinement, the plasma frequency goes to zero and we obtain the original Landau levels since \(\Omega \rightarrow \omega_c\) and \(M \rightarrow \infty\).

Conclusions.—In this Letter we demonstrated how the translational symmetry can be restored for Bloch electrons in the presence of a homogeneous magnetic field by taking the fluctuations of the field into account. We derived a Bloch central equation \[17\] which gives the spectrum of electrons in solids with a homogeneous magnetic field in the presence but also in the absence of the field fluctuations. The solutions of this equation in the limit of zero fluctuations reproduce the already known results of Bloch electrons in magnetic fields, such as the quantum Hall effect \[16, 17\] and the spectrum of the Hofstadter butterfly \[19\]. The derived central equation puts no limitations on the strength of the magnetic field and thus allows to scan through the whole continuum of field strengths for the first time. Moreover, in the case of a 2D electron gas in the presence of a homogeneous magnetic field and confined in a cavity we find Landau polaritons. Landau-polaritonic states have recently been observed \[25, 27\] and can be used to control magneto-transport properties of 2D gases \[26\]. These Landau polaritons have direct implications on related phenomena such as the quantum Hall effects and the Hofstadter butterfly. Our theory provides a first principles framework for this new emerging field where condensed matter physics meets cavity QED. We propose that cavity QED confinement of 2D materials will allow for the observation of such polaritonic effects.

![FIG. 3. Upper (red line) and lower (blue line and shaded area) polariton branches of Eq. (1) as a function of the cyclotron frequency \(\omega_c\) = \(eB/m^*c\). The lowest energy of the upper polariton (UP), \(\Omega/2\), asymptotically reaches the dispersion of the lowest Landau level \(\omega_c/2\) (orange dashed line). The lower polariton (LP) branch consists of the whole shaded area around the blue line.](image)

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