On a Pseudo-Formal Linearization Method via the Orthogonal Polynomial Approximation

Hitoshi Takata\textsuperscript{1} and Kazuo Komatsu\textsuperscript{2}

\textsuperscript{1}Kagoshima University, 1-21-40 Korimoto, Kagoshima 890-0065, Japan
\textsuperscript{2}National Institute of Technology, Kumamoto College, 2659-2 Suya, Koshi, Kumamoto 861-1102, Japan
E-mail: kaz@kumamoto-nct.ac.jp

Abstract This paper presents a method of pseudo-formal linearization based on the orthogonal polynomial approximation for nonlinear systems. The given nonlinear system is piecewise linearized by the formal linearization approach using the Legendre, Chebyshev, Hermite, or Laguerre polynomial functions. The resulting formal linear systems are smoothly united into a single linear one by an automatic choosing function. A nonlinear observer is synthesized as an application of this method. Numerical examples show that the accuracy of approximation by this method is greatly improved when appropriate orthogonal polynomial functions are chosen.

Keywords: nonlinear system, pseudo-formal linearization, orthogonal polynomial expansion, nonlinear observer, linearization function

1. Introduction

The problems of linearization for nonlinear systems have been studied for many years [1]-[8]. One of the earliest reports was made by Sternberg using a theorem of Poincaré [1]. For nonlinear estimation and control systems, the Taylor expansion method truncated at the first order is the most popular and practical linearization [2]. This linearization is effective only for small oscillations or almost linear systems. The linearization problem has also been investigated by geometric methods [3]-[5]. Although many interesting results have been achieved, they are generally not easily applicable to practical systems. As another easier applicable approach, a formal linearization method has been studied [6],[7]. Moreover, it has been expanded to a pseudo-formal linearization method via the Taylor polynomial approximation [8].

In this paper, we consider the pseudo-formal linearization method via the orthogonal polynomial approximation such as Legendre, Chebyshev, Hermite, or Laguerre polynomials [9] as follows. A region of the given nonlinear system is divided into some subdomains considering the nonlinearity. Next, a formal linearization function, which consists of polynomials, is defined. Then on each subdomain, a given dynamic nonlinear system is approximated by the orthogonal polynomial expansion and is linearized with respect to the formal linearization function. Finally, the resulting linearized systems on subdomains are smoothly united by an automatic choosing function to make a single linear system. The error bounds of this proposed method indicate that the accuracy of the approximation is improved when the domain is reasonably divided and the order of the polynomials increases. A nonlinear observer is also synthesized to illustrate how to apply this method. Numerical experiments show that the presented pseudo-formal linearization can effectively improve the accuracy.

2. Statement of Problem

A nonlinear dynamic equation is described by

\[ \Sigma_1 : \dot{x}(t) = f(x(t)), \quad x(0) = x_0 \in D \subset \mathbb{R}^n \]  

(1)

where \( t \) denotes time, \( \cdot = d/dt, x = [x_1, \cdots, x_n]^T \) is an \( n \)-dimensional state vector, and \( f \) is a sufficiently smooth nonlinear vector-valued function. \( D \) is a domain denoted by the Cartesian product

\[ D = \prod_{i=1}^{n} [m_i - p_i, m_i + p_i] \quad (m_i \in \mathbb{R}, p_i > 0) \]

The problem is to translate these nonlinear systems into pseudo-formal linear systems by using the orthogonal polynomial approximation.
3. Pseudo-Formal Linearization

First, we introduce a vector-valued separable function in order to linearize the given nonlinear state differential equation in Eq. (1) as

\[ C : D \rightarrow R^L \tag{2} \]

In this paper, let \( C = [I : 0] \) (\( I : L \times L \) unit matrix) for simplicity. Considering the nonlinearity of the given nonlinear dynamic system and letting \( D \) be a domain of \( C^{-1} \), the domain \( D \) is divided into \((M + 1)\) subdomains

\[ D = \bigcup_{k=0}^{M} D_k \tag{3} \]

where

\[ D_M = D - \bigcup_{k=0}^{M-1} D_k \]

and \( C^{-1}(D_0) \geq 0 \). \( D_k(0 \leq k \leq M - 1) \) endowed with a lexicographic order is the Cartesian product

\[ D_k = \prod_{j=1}^{L} [a_{kj}, b_{kj}) \] \( (a_{kj} < b_{kj}) \]

We here introduce an automatic choosing function of the sigmoid type [8],

\[ I_k(\zeta) = \prod_{j=1}^{L} \left( 1 - \frac{1}{1 + \exp (2\mu (\zeta_j - a_{kj}))} \right) \]

\[ - \frac{1}{1 + \exp (-2\mu (\zeta_j - b_{kj}))}, \quad (0 \leq k \leq M - 1) \tag{4} \]

\[ I_M(\zeta) = 1 - \sum_{k=0}^{M-1} I_k(\zeta) \]

so that

\[ \sum_{k=0}^{M} I_k(\zeta) = 1 \tag{5} \]

where

\[ \zeta = [\zeta_1, \cdots, \zeta_L]^T = C(x) \]

and \( \mu \) is a positive real value. \( I_k(\zeta) \) is analytic and almost unity on \( D_k \); otherwise, it is almost zero (see Fig. 1).

Secondly, the state vector \( x \) is changed into \( y \) so that we can use the chosen orthogonal polynomial approximation at each \( k \) \((0 \leq k \leq M)\). Let \( y \) be

\[ y = \mathcal{P}^{(k)}_0(x - \mathcal{M}^{(k)}) \in \mathcal{D}_0 \tag{7} \]

where

\[ y = [y_1, \cdots, y_L, y_{L+1}, \cdots, y_n]^T \]

\[ \mathcal{M}^{(k)} = [m_1^{(k)}, \cdots, m_L^{(k)}, m_{L+1}, \cdots, m_n]^T \]

From Eq. (1),

\[ \dot{y}(t) = \mathcal{P}^{(k)}_0^{-1} f(\mathcal{P}^{(k)} y(t) + \mathcal{M}^{(k)}) \tag{10} \]

Thirdly, we define an \( N \)-th order formal linearization function that consists of polynomials defined by

\[ \phi(x) \triangleq [x_1, x_2, \cdots, x_n, x_1^2, x_2^2, \frac{x_1 x_2}{2!}, \frac{x_1 x_3}{3!}, \cdots, \frac{x_n^2}{2!}, \cdots, x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}, \frac{x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}}{r_1! r_2! \cdots r_n!} \cdots \frac{N! N! \cdots N!}{N! ! N! ! \cdots N! !}]^T \]

\[ = [\phi(10), \cdots, \phi(r_1 \cdots r_n)(x), \cdots, \phi(N \cdots N)(x)]^T \tag{11} \]

Deriving the derivative of each element of \( \phi \) yields

\[ \dot{\phi}(r_1 \cdots r_n)(x) = \frac{\partial}{\partial x} \phi(r_1 \cdots r_n)(x) \cdot \dot{x} \]

\[ = \frac{\partial}{\partial y} \phi(r_1 \cdots r_n)(x) \cdot f(x) = \frac{\partial}{\partial y} \mathcal{P}^{(k)}_0^{-1} \]

\[ \times \phi(r_1 \cdots r_n)(\mathcal{P}^{(k)} y + \mathcal{M}^{(k)}) \cdot f(\mathcal{P}^{(k)} y + \mathcal{M}^{(k)}) \]

\[ \triangleq G^{(k)}_{(r_1 \cdots r_n)}(y) \tag{12} \]

Fourthly, we apply the orthogonal polynomial approximation approach in Sect. 4 to \( G^{(k)}_{(r_1 \cdots r_n)}(y) \) in Eq. (12). Then, on each subdomain \( D_k \), \( G^{(k)}_{(r_1 \cdots r_n)}(y) \) is approximated by

\[ G^{(k)}_{(r_1 \cdots r_n)}(y) \approx \sum_{q_1=0}^{N} \cdots \sum_{q_n=0}^{N} C^{(k)}_{(r_1 \cdots r_n)}(q_1 \cdots q_n) n \prod_{i=1}^{n} T_{q_i}(y_i) \]
where \( C_{(q_1 \cdots q_n)} \) is the coefficient obtained by the least-squares method. Therefore, 
\[
\dot{\phi}_{(r_1 \cdots r_n)}(x) = \sum_{j_1=0}^{N} \cdots \sum_{j_n=0}^{N} A^{(k)}_{(r_1 \cdots r_n)} \phi_{(j_1 \cdots j_n)}(x) \\
+ R_{N+1}^{(k)}(x)
\]
where 
\[
A^{(k)}_{(r_1 \cdots r_n)} = \frac{\partial (j_1 + \cdots + j_n)}{\partial x_r} \times G_{(r_1 \cdots r_n)} \left( r^{(k)(-1)}(x - M^{(k)}) \right) \bigg|_{x=0}
\]
\[R_{N+1}^{(k)}(x)\]
is the approximation error. Therefore, 
\[
\dot{\phi}_{(r_1 \cdots r_n)}(x) = \left[ A^{(k)(0)}_{(r_1 \cdots r_n)}, A^{(k)(0)(-1)}_{(r_1 \cdots r_n)}, \cdots, A^{(k)(0 \cdots -N)}_{(r_1 \cdots r_n)}, \cdots, A^{(k)(0 \cdots N)}_{(r_1 \cdots r_n)} \right] \phi(x) + A^{(k)(0 \cdots -0)}_{(r_1 \cdots r_n)} + R_{N+1}^{(k)}(r_1 \cdots r_n)(x)
\]
Thus, it follows that on a subdomain \( D_k \), 
\[
\dot{\phi}(x) = A^{(k)} \phi(x) + b^{(k)} + R_{N+1}^{(k)}(x)
\]
where 
\[
A^{(k)} = \left[ A^{(k)(j_1 \cdots j_n)}_{(r_1 \cdots r_n)} \right], b^{(k)} = \left[ A^{(k)(0 \cdots -0)}_{(r_1 \cdots r_n)} \right]
\]
\[R_{N+1}^{(k)}(x) = \left[ R_{N+1}^{(k)}(r_1 \cdots r_n)(x) \right]
\]
We unite \((M+1)\) linearized systems (Eq. (14)) on the subdomains into a single linear system on the whole domain by using Eq. (5) as
\[
\dot{\phi}(x) = \sum_{k=0}^{M} \phi(x) I_k(\zeta)
\]
\[
= \sum_{k=0}^{M} \left( A^{(k)} \phi(x) + b^{(k)} + R_{N+1}^{(k)}(x) \right) I_k(\zeta)
\]
\[
= \tilde{A}(\zeta) \phi(x) + \tilde{b}(\zeta) + \tilde{R}_{N+1}(x, \zeta)
\]
where
\[
\tilde{A}(\zeta) = \sum_{k=0}^{M} A^{(k)} I_k(\zeta), \quad \tilde{b}(\zeta) = \sum_{k=0}^{M} b^{(k)} I_k(\zeta)
\]
\[
\tilde{R}_{N+1}(x, \zeta) = \sum_{k=0}^{M} R_{N+1}^{(k)}(x) I_k(\zeta)
\]
Finally, a pseudo-formal linearization system is defined as
\[
\Sigma_2 : \dot{z}(t) = \tilde{A}(\zeta) z(t) + \tilde{b}(\zeta), \quad z(0) = \phi(x(0))
\]
The resulting system (Eq. (16)) is a generalization of the standard formal linearization [7] because \( \tilde{A}(\zeta) \) and \( \tilde{b}(\zeta) \) are functions of \( \zeta \) in Eq. (4). From Eq. (11), its inversion is carried out by evaluating
\[
\tilde{x}(t) = [I, 0, \cdots, 0] z(t)
\]
as the approximated value of \( x(t) \), where \( I \) is the \( n \times n \) unit matrix.

For this approach, we have the following error bounds. Let \( \| \cdot \| \) be the Euclidean norm.

**Theorem 1**
An error bound when a nonlinear system is approximated by the pseudo-formal linearization is
\[
\varepsilon_{N+1} = \max_k \left\{ \sup_x \left\{ \| R_{N+1}^{(k)}(x) \| : x \in D_k \right\} : 0 \leq k \leq M \right\}
\]
where
\[
\| R_{N+1}^{(k)}(x) \| = \left[ \sum_{r_1 + \cdots + r_n = N} \left( \sum_{r_1 + \cdots + r_n = 1} R_{N+1}^{(k)}(r_1 \cdots r_n)(x) \right)^2 \right]^{1/2}
\]

**Theorem 2**
An error bound of the pseudo-formal linearization for a nonlinear dynamic system is
\[
\| x(t) - \tilde{x}(t) \| \leq \| \phi(x(0)) - z(0) \| + \frac{\varepsilon_{N+1}(\| A \|_{\text{max}}^t - 1)}{\| A \|_{\text{max}}}
\]
where
\[
\| A \|_{\text{max}} = \max \{ \| A^{(k)} \| : 0 \leq k \leq M \}
\]
(Proof. See Ref.[8])

4. **Orthogonal Polynomial Approximation**

The orthogonal polynomial approximation is one of the most popular and important approaches for nonlinear functions [9]. From the property of the Fourier series for orthogonal functions, even if truncated at any finite order \( N \), the resulting equation is always optimal in the sense of minimizing
\[
\int_{D_0} G_{(q_1 \cdots q_n)}^{(k)}(y) \left[ \sum_{q_1=0}^{N} \cdots \sum_{q_n=0}^{N} C_{(q_1 \cdots q_n)}^{(k)(r_1 \cdots r_n)} \prod_{i=1}^{n} T_{q_i}(y_i) \right]^2 W(y) dy
\]
where
\[
\| X \|^2 = X^T X, \quad D_0 = D_0 \times \cdots \times D_0
\]
\[ W(y) = \prod_{i=1}^{n} W(y_i), \quad dy = dy_1 \cdots dy_n \]

\( D_0 \) is the basic domain, and \( W(y_i) \) is the weighting function. Namely, \( C_{(q_1, \ldots, q_n)}^{(k)(r_1, \ldots, r_n)} \) is the coefficient obtained by the least-squares method.

In this paper, as examples, we use the following simple orthogonal polynomials in Eq. (13), which can be expanded into more complicated ones.

(A) Legendre polynomial
For \( D_0 = [-1, 1] \ni y_i \) and \( W(y_i) = 1 \) for all \( 1 \leq i \leq n \), it follows that
\[
T_0(y_i) = 1, T_1(y_i) = y_i, T_2(y_i) = (3y_i^2 - 1)/2
\]
\[
T_3(y_i) = (5y_i^3 - 3y_i)/2, T_4(y_i) = (35y_i^4 - 30y_i^2 + 3)/8
\]
\[
T_5(y_i) = (63y_i^5 - 70y_i^3 + 15y_i)/8, \ldots
\]
\[
C_{(q_1, \ldots, q_n)}^{(k)(r_1, \ldots, r_n)} = \prod_{i=1}^{n} (q_i + 1/2) \int_{D_0} G_{(r_1, \ldots, r_n)}^{(k)}(y) T_{q_i}(y_i)dy
\]
(23)

(B) Chebyshev polynomial
For \( D_0 = [-1, 1] \ni y_i \) and \( W(y_i) = 1/\sqrt{1 - y_i^2} \) for all \( 1 \leq i \leq n \), it follows that
\[
T_0(y_i) = 1, T_1(y_i) = y_i, T_2(y_i) = 2y_i^2 - 1
\]
\[
T_3(y_i) = 4y_i^3 - 3y_i, T_4(y_i) = 8y_i^4 - 8y_i^2 + 1
\]
\[
T_5(y_i) = 16y_i^5 - 20y_i^3 + 5y_i, \ldots
\]
\[
C_{(q_1, \ldots, q_n)}^{(k)(r_1, \ldots, r_n)} = \frac{2n-\gamma}{\pi^n} \int_{D_0} G_{(r_1, \ldots, r_n)}^{(k)}(y) \prod_{i=1}^{n} T_{q_i}(y_i)dy
\]
\[
\times \prod_{i=1}^{n} \frac{T_{q_i}(y_i)}{\sqrt{1 - y_i^2}} dy
\]
(24)

where
\[
\gamma = \{ \text{the number of } q_i = 0 : 1 \leq i \leq n \}
\]

(C) Hermite polynomial
For \( D_0 = (-\infty, \infty) \ni y_i \) and \( W(y_i) = e^{-y_i^2} \) for all \( 1 \leq i \leq n \), it follows that
\[
T_0(y_i) = 1, T_1(y_i) = 2y_i, T_2(y_i) = 4y_i^2 - 2
\]
\[
T_3(y_i) = 8y_i^3 - 12y_i, T_4(y_i) = 16y_i^4 - 48y_i^2 + 12
\]
\[
T_5(y_i) = 32y_i^5 - 160y_i^3 + 120y_i, \ldots
\]
\[
C_{(q_1, \ldots, q_n)}^{(k)(r_1, \ldots, r_n)} = \prod_{i=1}^{n} \frac{1}{2^r q_i! \sqrt{\pi}} \int_{D_0} G_{(r_1, \ldots, r_n)}^{(k)}(y) \prod_{i=1}^{n} T_{q_i}(y_i)e^{-y_i^2}dy
\]
\[
\times \prod_{i=1}^{n} (T_{q_i}(y_i)e^{-y_i^2})dy
\]
(25)

Hermite polynomials can be directly used in Eqs. (7)–(13) even though the region is \([-1, 1] \subset D_0 \).

(D) Laguerre polynomial
For \( D_0 = [0, \infty) \ni y_i \) and \( W(y_i) = e^{-y_i} \) for all \( 1 \leq i \leq n \), it follows that
\[
T_0(y_i) = 1, T_1(y_i) = y_i + 1, T_2(y_i) = y_i^2 - 4y_i + 2
\]
\[
T_3(y_i) = -y_i^3 + 9y_i^2 - 18y_i + 6
\]
\[
T_4(y_i) = y_i^4 - 16y_i^3 + 72y_i^2 - 96y_i + 24
\]
\[
T_5(y_i) = -y_i^5 + 25y_i^4 - 200y_i^3 + 600y_i^2 - 600y_i + 120, \ldots
\]
\[
C_{(q_1, \ldots, q_n)}^{(k)(r_1, \ldots, r_n)} = \prod_{i=1}^{n} \frac{1}{(q_i)!} \int_{D_0} G_{(r_1, \ldots, r_n)}^{(k)}(y) \prod_{i=1}^{n} (T_{q_i}(y_i)e^{-y_i})dy
\]
(26)

In this polynomial approximation, Eqs. (8) and (9) should be changed to
\[
m_j^{(k)} = a_{kj}, p_j^{(k)} = \frac{b_{kj} - a_{kj}}{\beta_j}
\]
(27)
\[
D_0 = \prod_{i=1}^{n} [0, \beta_i) \quad (\beta_i > 0)
\]
(28)
so that \( D_0 \) can approach \( \prod_{i=1}^{n} [0, \infty) \) as \( \beta_i \to \infty \).

5. Nonlinear Observer

We here synthesize a nonlinear observer as an application of the above pseudo-formal linearization. We assume that the nonlinear dynamic system is the same as Eq. (1),
\[
\dot{x}(t) = f(x(t))
\]
(29)
and the measurement equation is
\[
y(t) = h(x(t))
\]
(30)
where \( y \in \mathbb{R}^m \) is a measurement vector and \( h = [h_1, \ldots, h_m]^T \) is a sufficiently smooth nonlinear vector-valued function.

The nonlinear dynamic system in Eq. (29) is transformed into the linear system in Eq. (16) by the pseudo-formal linearization in Sect. 3. In order to linearize the measurement equation with respect to the linearization function \( \phi \), we apply the orthogonal polynomial expansion shown in Eqs. (12) to (14) as follows.
\[
h_r(x) = h_r(\mathcal{P}^{(k)}y(t) + \mathcal{M}^{(k)})
\]
\[
\approx \sum_{q_1=0}^{N_1} \cdots \sum_{q_n=0}^{N_n} C_{(q_1, \ldots, q_n)}^{(r)(k)} \prod_{i=1}^{n} T_{q_i}(y_i) \triangleq \tilde{G}_r^{(k)}(y)
\]
(31)
where \( C^{(k)}(r) \) is the coefficient obtained by the least-squares method.

\[
h_r(x) \approx \sum_{j_1=0}^{N} \cdots \sum_{j_n=0}^{N} H^{(k)}(r_{j_1,\ldots,j_n}) \phi_{j_1,\ldots,j_n}(x)
\]

where

\[
H^{(k)}(r_{j_1,\ldots,j_n}) = \frac{\partial^{j_1+\ldots+j_n}}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}} \times \hat{G}^{(k)}_r \left( \mathcal{P}^{(k)}_r (x - M^{(k)}) \right) \bigg|_{x=0}
\]

Thus, \( h_r(x) \) on a subdomain \( D_k \) is approximated by the formal linearization function as

\[
h_r(x) \approx [H^{(k)}(0^{10...0}), H^{(k)}(0^{11...0}), \ldots, H^{(k)}(1^{11...0}), \ldots, H^{(k)}(N^{10...0})] \phi(x) + H^{(k)}(0^{10...0})
\]

and the linear measurement equation with respect to \( \phi \) is obtained as

\[
y \approx \left[ H^{(k)}(j_1,\ldots,j_n) \right] \phi(x) + \left[ H^{(k)}(0^{10...0}) \right]
\]

\[
\triangleq H^{(k)} \phi(x) + d^{(k)}
\]

Applying Eq. (5) to Eq. (32) yields

\[
y \approx \sum_{k=0}^{M} \left( H^{(k)} \phi(x) + d^{(k)} \right) I_k(\zeta)
\]

\[
= \sum_{k=0}^{M} H^{(k)} I_k(\zeta) \phi(x) + \sum_{k=0}^{M} d^{(k)} I_k(\zeta)
\]

Thus, a pseudo-formal linearization system for the measurement equation is approximately derived as

\[
y(t) = \tilde{H}(\zeta) z(t) + \tilde{d}(\zeta)
\]

where

\[
\tilde{H}(\zeta) = \sum_{k=0}^{M} H^{(k)} I_k(\zeta), \quad \tilde{d}(\zeta) = \sum_{k=0}^{M} d^{(k)} I_k(\zeta)
\]

The linear observer theory is applied to the linearized system in Eqs. (16) and (34) and an identity observer is obtained as

\[
\hat{z}(t) = \hat{A}(\zeta) \hat{z}(t) + \hat{b}(\zeta) + K(y - \tilde{H}(\zeta) \hat{z}(t) - \tilde{d}(\zeta))
\]

\[
= \sum_{k=0}^{M} \left\{ (A^{(k)} \hat{z}(t) + b^{(k)})
\right. \\
\left. + K^{(k)} (y(t) - H^{(k)} \hat{z}(t) - d^{(k)}) \right\} I_k(\zeta)
\]

where \( \zeta = C(\hat{z}) \) and \( K^{(k)} \) is the observer gain on a subdomain \( D_k \) given by

\[
K^{(k)} = S^{(k)} \mathcal{P}^{(k)}_r H^{(k)}
\]

\( P^{(k)} \) satisfies the matrix Riccati equation

\[
A^{(k)} P^{(k)} + P^{(k)} A^{(k)T} + Q^{(k)} - P^{(k)} H^{(k)T} S^{(k)}^{-1} H^{(k)} P^{(k)} = 0
\]

\( Q^{(k)} \) and \( S^{(k)} \) are arbitrary real symmetric positive definite matrices.

From Eq. (17), the estimate \( \hat{z}(t) \) of a nonlinear observer becomes

\[
\hat{z}(t) = [I, 0, \ldots, 0] \hat{z}(t)
\]

6. Numerical Experiments

To show the effectiveness of the approach, numerical experiments on the presented pseudo-formal linearization and the nonlinear observer are illustrated.

6.1 Pseudo-formal linearization

Consider the simple nonlinear dynamic system

\[
\dot{x} = x - x^2, \quad D = [-\frac{1}{4}, \frac{5}{4}] \subset R
\]

The parameters are set as \( M = 2, \mu = 20, \zeta = x \) in Eqs. (3) and (4). \( D \) is divided into \( D = \bigcup \limits_{k=0}^{2} D_k \) where

\[
D_0 = [-\frac{1}{4}, \frac{1}{4}], D_1 = [\frac{1}{4}, \frac{3}{4}], D_2 = [\frac{3}{4}, \frac{5}{4}]
\]

The system parameters are set as

\[
M^{(0)} = 0, M^{(1)} = 0.5, M^{(2)} = 1
\]

\[
\mathcal{P}^{(k)} = 0.25 \quad (k = 0, 1, 2)
\]

for the Legendre, Chebyshev, and Hermite expansions, and

\[
M^{(0)} = -0.25, M^{(1)} = 0.25, M^{(2)} = 0.75
\]

\[
\mathcal{P}^{(k)} = \frac{1}{2 \beta} \quad (k = 0, 1, 2)
\]

for the Laguerre expansion. Taylor expansion points are set at

\[
\tilde{x}_0 = 0, \tilde{x}_1 = 0.5, \tilde{x}_2 = 1
\]

The pseudo-formal linearizations of Eq. (16) when \( N = 1 \) become

\[
\begin{align*}
(A) \quad \dot{z} & = (z - 1/48) I_0 + (11/48) I_1 + (-z + 47/48) I_2 \\
(B) \quad \dot{z} & = (z - 1/32) I_0 + (7/32) I_1 + (-z + 31/32) I_2 \\
(C) \quad \dot{z} & = (z - 1/32) I_0 + (7/32) I_1 + (-z + 31/32) I_2 \\
(D) \quad \dot{z} & = (3/2 \cdot z + 1/16) I_0 + (1/2 \cdot z + 1/16) I_1 \\
& \quad + (-1/2 \cdot z + 9/16) I_2 \\
(T) \quad \dot{z} & = (z) I_0 + (1/4) I_1 + (-z + 1) I_2
\end{align*}
\]
Consider the nonlinear dynamic system

$$x(1) = 0 \quad \text{and} \quad x(0) = 0.1 \quad \text{for} \quad y = \sin(x_1(t)) + x_2(t) \equiv h(x(t))$$

Assume the measurement equation

$$y = \sin(x_1(t)) + x_2(t) \equiv h(x(t))$$

In order to apply the linearization, we set $C(x) = x_1$, because of its highest nonlinearity $\sin(x_1)$ and consider the domain of $x$ as

$$D = [-\frac{3}{4}\pi, \frac{3}{4}\pi] \times [-2.6, 0.4]$$

We divide this whole domain into three subdomains $(M = 2)$ (see Fig. 1) as

$$D_0 = [-\frac{3}{4}\pi, -\frac{1}{4}\pi], D_1 = [-\frac{1}{4}\pi, \frac{1}{4}\pi], D_2 = [\frac{1}{4}\pi, \frac{3}{4}\pi]$$

Table 1 Error bounds of $J_N(t)$

| $J_N(t)$ | $J_N(t)$ | $J_N(t)$ | $J_N(t)$ |
|-----------|-----------|-----------|-----------|
| Legendre  | Chebyshev | Hermite   | Laguerre  |
| N = 1     | N = 2     | N = 3     | N = 4     |
| (A)       | (B)       | (C)       | (D)       |
| 0.041666  | 0.006250  | 0.002958  | 0.000200  |
| 0.031250  | 0.0063906 | 0.000251  | 0.000246  |
| 0.250000  | 0.125000  | 0.031250  | 0.015625  |
| 0.062500  | 0.015625  | 0.001953  | 0.002446  |

These results indicate that the approximation of the pseudo-formal linearization can be improved when the orthogonal polynomial approximation is used appropriately.

### 6.2 Nonlinear observer

We simulate nonlinear observers using the presented method in Eq. (36) and the Taylor expansion (T) [8]. Consider the nonlinear dynamic system

$$\dot{\delta} + a\dot{\delta} + b\sin\delta = c, \quad \delta \in R$$

When $x_1 = \delta$, $x_2 = \dot{\delta}$, $a = b = 1$, and $c = 0$, the system yields

$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ -\sin(x_1(t)) - x_2(t) \end{bmatrix} = f(x(t))$$

Assume the measurement equation

$$y = \sin(x_1(t)) + x_2(t) \equiv h(x(t))$$

In order to apply the linearization, we set $C(x) = x_1$, because of its highest nonlinearity $\sin(x_1)$ and consider the domain of $x$ as

$$D = [-\frac{3}{4}\pi, \frac{3}{4}\pi] \times [-2.6, 0.4]$$

We divide this whole domain into three subdomains $(M = 2)$ (see Fig. 1) as

$$D_0 = [-\frac{3}{4}\pi, -\frac{1}{4}\pi], D_1 = [-\frac{1}{4}\pi, \frac{1}{4}\pi], D_2 = [\frac{1}{4}\pi, \frac{3}{4}\pi]$$

The system parameters are set as

$$M^{(0)} = \begin{bmatrix} \pi \\ -1.1 \end{bmatrix}, M^{(1)} = \begin{bmatrix} 0 \\ -1.1 \end{bmatrix}, M^{(2)} = \begin{bmatrix} \pi \\ -1.1 \end{bmatrix}$$

$$P^{(k)} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, (k = 0, 1, 2)$$

for the Legendre, Chebyshev, and Hermite expansions, and

$$M^{(0)} = \begin{bmatrix} -1 \\ -2.6 \end{bmatrix}, M^{(1)} = \begin{bmatrix} 0 \\ -2.6 \end{bmatrix}, M^{(2)} = \begin{bmatrix} 1 \\ -2.6 \end{bmatrix}$$

$$P^{(k)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, (\beta_1 = 100, \beta_2 = 8, k = 0, 1, 2)$$

for the Laguerre expansion. The Taylor expansion points are set at

$$\{\hat{x}_{10}, \hat{x}_{11}, \hat{x}_{12}\} = \{-\frac{1}{2}\pi, 0, \frac{1}{2}\pi\}$$

The parameters for the nonlinear observer are set as

$$x(0) = [1.8, -2.5]^T, \mu = 10, \hat{x}(0) = [0.8, -1]^T$$

$$Q^{(k)} = \frac{1}{2}I, S^{(k)} = 1 (k = 0, 1, 2)$$

Figure 3 shows the true value $x(t)$ and the estimates $\hat{x}(t)$ in Eq. (36) in which $\hat{x}(t) = \hat{x}(\text{Legendre}), \hat{x}(\text{Chebyshev}), \hat{x}(\text{Hermite}), \hat{x}(\text{Laguerre})$ and $\hat{x}(\text{Taylor})$ when $N = 2$. Figure 4 shows the logarithmic integral square errors of the estimation

$$J(t) = \int_0^t \|x(\tau) - \hat{x}(\tau)\|^2 d\tau$$

The Hermite approach improves the value of $\log_{10} J(t)$ by about 4% compared with the Taylor approach in Fig. 4. For this pseudo-formal linearization method, the above examples indicate the following.
(1) The Taylor expansion approach is part of the basis of this method.

(2) The Chebyshev approach has one of the best error bounds, as shown in Table 1, and its error decreases as the order \( N \) increases.

(3) Since each subdomain in Eq. (3) is finitely limited, the Laguerre polynomial approximation might not be suitable for the examples in this section, because its basic domain is \( D_0 = [0, \infty) \ni y \) and its largest weight is set at \( y = 0 \).

(4) The Hermite and Chebyshev approaches give good results, as shown in Figs. 2–4.

We must consider the problem of selecting the best parameters, because these results greatly depend on the parameters. When this method is applied to practical systems, one might consider the approximation errors and its computation times.

7. Conclusions

We have studied a pseudo-formal linearization method for nonlinear systems based on the orthogonal polynomial expansion considering terms up to a higher order to improve the accuracy of the linearization. From the results of the previous sections, we consider that the best selection of the orthogonal polynomial functions, the order, and the division of the domain will enable us to make better utilization of the proposed method. The application of this method to practical systems such as electric power systems is left for future studies.

References

[1] S. Sternberg: Local contractions and a theorem of Poincaré, Am. J. Math., Vol. 79, No.4, pp. 809-824, 1957.
[2] Y. N. Yu, K. Vongsuriya and L. N. Wedman: Application of an optimal control theory to a power system, IEEE Trans. Power Appl. Syst., Vol. PAS-89, No.1, pp. 55-62, 1970.
[3] R. W. Brockett: Feedback invariants for nonlinear systems, Proc. IFAC Congress, pp. 1115-1120, 1978.
[4] R. Su: On the linear equivalents of nonlinear systems, Syst. Control Lett., Vol. 2, No.1, pp. 48-52, 1982.
[5] A. J. Krener: Approximate linearization by state feedback and coordinate change, Syst. Control Lett., Vol. 5, pp. 181-185, 1984.
[6] K. Komatsu and H. Takata: A formal linearization by the Chebyshev interpolation and its applications, Proc. IEEE CDC, Vol. 1, pp. 70-75, 1996.
[7] K. Komatsu and H. Takata: Design of nonlinear observer by using augmented linear system based on formal linearization of polynomial type, Int. J. Comput. Electr. Autom. Control Inf. Eng., Vol. 3, No.11, pp. 2523-2526, 2009.
[8] H. Takata and K. Komatsu: A pseudo-formal linearization of polynomial type for nonlinear systems and its applications, J. Signal Process., Vol. 22, No.1, pp. 9-16, 2018.
[9] T. Akasaka: Numerical Computation, Corona Pub., 1974. (in Japanese)

Hitoshi Takata received his B.S. degree in electrical engineering from Kyushu Institute of Technology in 1968 and his M.S. and Dr. Eng. degrees in electrical engineering from Kyushu University in 1970 and 1974, respectively. He is currently a Professor Emeritus at Kagoshima University. His research interests include the control, linearization and identification of nonlinear systems.
Kazuo Komatsu received his B.S. degree in computer science and Dr. Eng. degree in electrical engineering from Kyushu Institute of Technology in 1985 and 1995, respectively. He is currently a Professor at the Department of Human-Oriented Information Systems Engineering in National Institute of Technology, Kumamoto College. His research interests include formal linearization for nonlinear systems and its applications. He is a member of the RISP.

(Received February 4, 2019; revised April 23, 2019)