Four-wave mixing in spin-orbit coupled Bose-Einstein condensates

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We describe possibilities of spontaneous, degenerate four-wave mixing (FWM) processes in spin-orbit coupled Bose-Einstein condensates. Phase matching conditions (i.e., energy and momentum conservation laws) in such systems allow one to identify four different configurations characterized by involvement of distinct spinor states in which such a process can take place. We derived these conditions from first principles and then illustrated dynamics with direct numerical simulations. We found, among others, the unique configuration, where both probe waves have smaller group velocity than pump wave and proved numerically that it can be observed experimentally under proper choice of the parameters. We also reported the case when two different FWM processes can occur simultaneously. The described resonant interactions of matter waves is expected to play important role in the experiments of BEC with artificial gauge fields.

I. INTRODUCTION

Traditionally four-wave mixing (FWM) process is associated with photon interactions via a non-linear polarization. It is a third-order parametric process in which two particles (from two so-called writing or pump beams - one particle from each beam) are annihilated when passing through a non-linear medium, and at the same time two new particles (constituting probe and signal beams) are generated. In optics FWM is commonly associated with the third order Kerr nonlinearity. The phenomenon is ubiquitous (see e.g., \cite{1,2}) and its applications are very widespread, including: fiber optic communication (very often not welcome), wavelength conversion, parametric amplification, optical regeneration, optic phase conjugation, correction of aberrations of images etc.

FWM can be observed also for massive particles as it was predicted and observed in cold atomic gasses two decades ago \cite{3,4} (see also \cite{5}). In such statement resonantly interacting particles are neutral atoms rather than photons. Flexibility of control of trapping potentials, as well as of nonlinear interactions, in atomic systems open interesting perspectives of managing both momentum conservation and energy conservation laws through the interplay of additional linear and nonlinear potentials. This issue was already explored in Refs. \cite{6,7}. In this paper we explore a similar idea of controlling FWM processes through artificially created gauge potentials.

In order to consider FWM in a specific medium, one has to identify the characteristic eigenmodes: the elementary solutions to the linearized equations of motion in a form of plane waves, identified by their wavevectors and frequencies which satisfy the so-called phase matching conditions, that are reminiscent of momentum and energy conservation laws. These are often quite demanding constraints depending on the particular form of dispersion relation characteristic for the system under investigation. For instance, in one dimension they cannot be satisfied for a system of cold atoms obeying parabolic dispersion relation and confined to (quasi-)one dimension. The situation can be improved by artificial change of dispersion law using linear optical lattices \cite{6} or by the manipulation of the wavenumbers of the matter waves involved in the process by means of nonlinear lattices \cite{7}. These modifications introduce the internal texture to the propagation medium, making it inherently inhomogeneous.

If a system has a spinor nature, i.e., consists of two subsystems, an alternative way to manipulate the linear properties of the medium, even preserving homogeneity, is to employ coherent coupling of the constituents. In optics, for example, one can satisfy the matching conditions for the FWM of the light propagating in homogeneous coupled waveguides with gain and losses \cite{8}. In atomic similar situation naturally occurs for spinor Bose-Einstein, where coupling between two atomic states by means of the spin-orbit coupling (SOC) allows one to manipulate the dispersion relation in the presence of external potential. This idea becomes attractive since recently, a spin-orbit-coupled Bose-Einstein condensate (SOC-BEC) of hyperfine states of \(^{85}\)Rb, was created experimentally \cite{9,10}.

The main goal of this paper is to show that with properly adjusted SOC, one can satisfy the phase matching conditions even for a homogeneous one-dimensional SOC-BECS. Importantly, the SOC properties in atomic systems are highly tunable \cite{13}, i.e. the matching conditions reported below are experimentally feasible.

Due to the spinor nature of the one-dimensional (1D) SOC-BEC, it is characterized by two branches of the dispersion relation. As a result, the matching conditions can be readily satisfied. Moreover, unlike in the case BECs without SOC, now one can find a diversity of distinct FWM processes, where the interacting waves, as well as waves generated may represent different spinor states at the same values of parameters of the system (similarly to the FWM with laser pulses reported in \cite{8}). It is a goal of the present study to identify the FWM processes
available in the FWM in SOC-BECs and show they can be efficient enough to be observed.

The organization of the paper is as follows: in Sec. II we discuss phase matching conditions in a quasi-1D SOC-BEC, and we focus on the degenerate case, where the central, pump wave serves as a source for stimulated enhancement of two probe waves. Here we borrow the terminology from optics but we make no distinction between signal and probe beams. Next we identify four possible configurations of FWM and in Sec. III we perform feasibility study using real time simulations for all predicted configurations. The outcomes are summarized in Conclusion.

II. PHASE MATCHING CONDITIONS

A. General relations

Let us consider a quasi-1D SOC-BEC which is described by a two-component order parameter \( \Psi(x,t) = (\Psi_1(x,t), \Psi_2(x,t))^T \) (hereafter T stands for transposition). The dynamics of the spinor \( \Psi(x,t) \) is governed by the coupled Gross-Pitaevskii equations (GPEs):

\[
 i\partial_t \Psi = H\Psi + \frac{1}{2}G(\Psi)\Psi, 
\]

where

\[
 H = \frac{1}{2} \left( -\partial_x^2 + \Omega \cdot \sigma - i\alpha \sigma_x \partial_x \right) 
\]

is the linear mean field Hamiltonian of two component BEC, \( \alpha \) is the SOC strength, \( \Omega \) is the vector of the Zeeman coupling (we admit external magnetic field), \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) is the vector of Pauli matrices \( \sigma_{x,y,z} \), the nonlinearity \( 2 \times 2 \) matrix is given by

\[
 G(\Psi) = \text{diag} \left( g_1|\Psi_1|^2 + g_2|\Psi_2|^2, g_1|\Psi_1|^2 + g_2|\Psi_2|^2 \right) 
\]

with intra- and inter-component interactions \( g_{1,2} \) and \( g \), respectively, and we use the dimensionless units where \( \hbar = m = 1 \).

To address the matching conditions for the FWM we start with the eigenmodes of the linear spectral problem, representing them in the form of the plane waves

\[
 \Psi_{\pm}(x,t) = e^{ikx - i\mu_{\pm}(k)t} \psi_{\pm}(k), 
\]

where \( \psi_{\pm}(k) = (\psi_{\pm}^{(1)}(k), \psi_{\pm}^{(2)}(k))^T \) is a constant (i.e. \( x \)- and \( t \)-independent) spinor, \( k \) is a mode wavenumber, \( \mu_{\pm}(k) \) is its frequency, and \( \pm \) indicate the upper (\("+"\)) and lower (\("-"\)) branches of the spectrum [i.e., \( \mu_{-}(k) \leq \mu_{+}(k) \)]. We will concentrate in the case of the Zeeman field in the \((x,z)\) plane, i.e. \( \Omega = (\Omega_x,0,\Omega_z) \) [Without loss of generality we fix \( \Omega_x,\Omega_z \geq 0 \), for which we compute the two branches of the dispersion relation

\[
 \mu_{\pm}(k) = \frac{k^2}{2} \pm \frac{\varepsilon(k)}{2}, 
\]

Here

\[
 \varepsilon(k) = \sqrt{\Omega_z^2 + \tilde{\Omega}^2(k)} 
\]

with \( \tilde{\Omega}(k) = \alpha k + \Omega_z \), is the gap between the spectral branches at a given \( k \). The lower (-) and the upper (+) branches of the respective eigenvectors are defined as:

\[
 \psi_{\pm}(k) = \frac{1}{\sqrt{\tilde{\Omega}^2 + (\varepsilon(k) \mp \Omega_z)^2}} \left( \pm \varepsilon(k) - \Omega_z \right). 
\]

To reduce the number of parameters, here we investigate the degenerate FWM process, where two input spinor states are identical. We label their wavevectors by \( k_1 \) and using the optical terminology, we call them pump waves. Two spinors that are created in the FWM process with central wavevectors \( k_2 \) and \( k_3 \), will be referred to as probe waves. Respectively, the conservation of wavenumbers and frequencies of the pump and probe waves, are expressed in a form of the phase-matching conditions

\[
 2k_1 = k_2 + k_3, 
\]

\[
 2\mu_{\nu_1}(k_1) = \mu_{\nu_2}(k_2) + \mu_{\nu_3}(k_3). 
\]

The indexes \( \nu_j \) \((j = 1,2,3)\) refer to either \("+"\) or \("-"\) branch of the spectrum.

Below we consider only the cases \( k_{2,3} \neq k_1 \), excluding the trivial case of self-phase modulation where all wavenumbers are equal. We note that the system obeys gauge, rather than Gallilean, invariance, at which the generalized momentum

\[
 \Pi = \int_{-\infty}^{\infty} \Psi^\dagger \left( -i\partial_x + \frac{\alpha}{2} \sigma_x \right) \Psi dx 
\]

is conserved: \( d\Pi/dt = 0 \). For our consideration this means that the input wavenumber \( k_1 \) cannot be set arbitrarily to zero without changing the spinor eigenstates. It also means that at zero Zeeman field \( \Omega = 0 \), the linear Hamiltonian \( H \) is gauge equivalent to the usual one-dimensional Schrödinger Hamiltonian \( H_0 = -\partial_x^2 \) which does not support the matching conditions \((\ref{eq:matching_condition})\) and \((\ref{eq:matching_condition_2})\).

In other words, while SOC controls the waves involved in resonant processes, the FWM itself requires nonzero Zeeman field.

For the following consideration it is convenient to rewrite \((\ref{eq:matching_condition})\) as

\[
 k_2 = k_1 + q, \quad k_3 = k_1 - q. 
\]

Now matching condition for frequencies \((\ref{eq:matching_condition_2})\) can be rewritten in the form

\[
 2q^2 = 2s_1 \varepsilon(k_1) - s_2 \varepsilon(k_1 + q) - s_3 \varepsilon(k_1 - q), 
\]

where \( s_j = \pm 1 \). Since each wave belongs to either upper or lower branch, these are eight different equations for given \( k_1 \) and \( q \). However only four of them have nontrivial solutions. To justify this we first notice that if
Table I. Possible configurations of degenerate FWM processes (positive and negative $q$ are included). The first column are numbers identifying configurations, which corresponds to the specific choice of the spectrum branches for the pump and probe waves, indicated in second, third, and forth columns, respectively. In the last column we show the maximal number, $N_{\text{max}}$, of $q$ values solving Eq. (12).

| Configuration | $s_1$ | $s_2$ | $s_3$ | $N_{\text{max}}$ |
|---------------|-------|-------|-------|-----------------|
| 1             | 1     | 1     | 1     | 2               |
| 2             | 1     | 1     | 1     | 2               |
| 3             | 1     | 1     | 1     | 2               |
| 4             | 1     | 1     | 1     | 4               |

$s_2 = s_3$, the condition (12) is symmetric under $q \leftrightarrow -q$ exchange. If however $s_2 \neq s_3$, then (12) is symmetric with respect to simultaneous change $(s_2, q) \leftrightarrow (s_3, -q)$. This allows one to restrict the analysis to the case $q > 0$. Next we use $\varepsilon(k) > 0$ property and conclude that the case $(s_1, s_2, s_3) = (-1, 1, 1)$ does not have solutions since the right hand side of Eq. (12) becomes negative. Let us now consider $k_1 \geq 0$ (the case $k_1 < 0$ is fully analogous). For non-negative values of $k_1$ we find the following inequalities

$$0 \leq q^2 \leq \varepsilon_+ (k_1 - q) \leq \varepsilon_+ (k_1 + q),$$

excluding the cases $(-1, 1, -1)$ and $(-1, -1, 1)$. Hence, the initial pulse from the lower brunch of the spectrum may originate degenerate FWM in processes involving modes from the lowest branch only (this is the configuration 4 in the Table I below). Finally, using inequalities (13) one can exclude also the case $(1, 1, 1)$, leaving only four possible configurations summarized in the Table I.

In the last column of the Table I we list the maximal number of solutions for particular configuration. In what follows we present analytical and graphic considerations that led us to these counts.

**B. Analysis of possible configurations**

By straightforward algebraic manipulations we can eliminate square-root terms in the equation (12) (simple sequence of transfers and squaring). As a result all four phase matched processes listed in the Table I are determined by the following cubic equations

$$Q^3 - \left(1 + 4s_1 \sqrt{\omega_1^2 + \omega_2^2}\right)Q^2 + \left(2s_1 \sqrt{\omega_2^2 + \omega_3^2} + 5\omega_2^2 + 5\omega_3^2\right)Q - \omega_3^2 - 2s_1 \sqrt{\omega_2^2 + \omega_3^2} (\omega_1^2 + \omega_2^2) = 0$$

where $\tilde{\omega} = \tilde{\Omega}/\alpha^2$, $\omega_2 = \Omega_2/\alpha^2$ and $Q = q^2/\alpha^2$. For obvious reasons we are interested only in positive roots of (14) and exclude the root $Q = 0$ (i.e. $q = 0$) which does not correspond to FWM but to the self-phase modulation.

A number of real roots of Eq. (14) is determined by the sign of the discriminant

$$\Delta_{s_1} = \tilde{\omega}^2 \left[15\tilde{\omega}^2 + 4s_1 \sqrt{\omega_2^2 + \omega_3^2} (\omega_1^2 + \omega_2^2 + 3) - 12\omega_2^2 - 4\right].$$

If $\Delta_{s_1} > 0$, there exists one real root. Three distinct real solutions exist if $\Delta_{s_1} < 0$. At $\Delta_{s_1} = 0$ all roots are real and at least one is multiple.

Now we inspect systematically configurations listed in Tab. I which manifest qualitatively different types of dynamics. Starting with the last one, we set $s_1 = -1$. Now the discriminant (15) depends on the two parameters $\{\tilde{\omega}, \omega_2\}$ for different values of which $\Delta_{s_1}$ can acquire any sign or be zero. Analyzing Vieta formulae one can exclude the possibility of all three real roots being positive. It means that in the configuration 4 (see the Table I) there may exist either one or two real positive roots of Eq. (14). Taking into account that roots appear in pairs, $\pm q$, this corresponds to at most four possible arrangements allowed by the phase matching condition (12).

Next we turn to the configurations 1, 2, and 3 in the
phase matching equations. Comparing the values of the LHS and RHS of Eq. (12) at $q = 0$ and at $q \to \infty$, we conclude that each of the configurations 1, 2 and 3 must have at least one root for $q > 0$.

The roots $\pm q_1, q_2, q_3$ of Eq. (12) are located at the crossing of the blue dashed curve and solid curves. In (a) and (b) the SOC strength is $\alpha = 2$ and $\alpha = 10$, respectively. The red, pink and black lines correspond to the configurations 1, 2 and 3 (see table (I)). Panel (c) exemplifies configuration 4; here light green line represents RHS of Eq. (12) for $\alpha = 5$, dark blue for $\alpha = 7.67$ and brown for $\alpha = 9$. In panel (d) we fix $\alpha = 9$ and vary $k_1$ to show regions of zero, one and two roots.

The fourth configuration ($s_1 = s_2 = s_3 = -1$) is illustrated in panel (c). Here we observe three different possibilities: no positive solutions of the phase matching condition (12) (the light green curve does not cross the dash blue curve at $\alpha = 5$); one positive root (the deep green and dash blue curves tangent to each other when $\alpha \approx 7.67$), and two positive solutions (the brown curve crosses the dash blue curve in two points, when $\alpha = 9$).

Finally, in the panel (d) we varied $k_1$ (while holding $\alpha = 9$) and shown the regions where the phase matching equation of the fourth configuration supports one positive (the black curves) and two positive solutions (the pink curves). In all panels (a)-(d) we fixed $\Omega_x = 2.5$ and $\Omega_z = 8$.

C. Matching of group velocities

While no matching conditions on the group velocities is imposed, for practical observation of different scenarios of FWM in numerical simulations, the issue of the group velocities (GVs)

$$v_\pm(k) = \frac{\partial \mu_\pm(k)}{\partial k} = k \pm \frac{\tilde{\Omega}}{2 \sqrt{\Omega_z^2 + \Omega_z^2}},$$

becomes relevant. On the one hand where all wavepackets move with respect to each other, it is important that the spinor involved in the process have similar values of GVs: otherwise fast separation of wavepackets in space may drastically reduce the conversion efficiency. On the other hand, GVs should have sufficient difference in order to observe spatial separation of the probe wavepackets. Thus in addition to solving the matching conditions we set a task of finding optimal conditions in the context of FWM numerical simulations (they are presented in the next section).

Let us shortly discuss optimal choices of GVs in different configurations. For each configuration listed in Table I at a given $k_1$ one can determine $q$, i.e., the wavenumbers $k_2$ and $k_3$, and consequently their GVs. The results are summarized in Fig. 2.

The results are summarized in Fig. 2. The GVs associated with different configurations of degenerated FWM presented in Table I versus momentum of the pump wave. Panel (a) shows the first configuration with $\Omega_x = 2.5$, $\Omega_z = 4$ and $\alpha = 3$. Panel (b) shows second and third configurations, with $\Omega_x = 3$, $\Omega_z = 8$ and $\alpha = 10$, by the solid and dashed black lines, respectively. Panel (c) illustrates the forth configuration with $\Omega_x = 6$, $\Omega_z = 4$ and $\alpha = 7$. In all panels, the group velocities of the pump (probe) wavepackets are shown by thick red (black) curves. The blue dots marked on the red and black curves indicate the points where we do numerical simulations. The dynamics of FWM at these particular points are shown in the Figures 3 - 6.

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First, we note, that the GVs of different branches cross each other at $k_0 = -\Omega_x/\alpha$, when $\Omega(k_0) = 0$. Away from the crossing point we have two parallel linear asymptotes: $v_\pm(k) \to k \pm 1/2$ at $k \to \infty$. In Fig. 2 (a) we observe that the GV of the pump wavepacket, $v_+(k_1)$, is close
to either $v_-(k_2)$ or $v_-(k_3)$, almost for all $k_1$ except the vicinity of $k_1 = 0$. This means that to obtain clear separation of the wavepackets, generated in the FWM at relatively short time intervals, $k_1$ should be chosen close to zero. Then, the separation between velocities grows rapidly enough allowing direct observation of separated pulse. On the other hand the time that pulses overlap is still long enough to generate substantial four wave mixing signal.

Fig. 3 (b) shows the dependence of GVs of phase matched wavepackets versus pump momentum for configurations 2 and 3 from the Table I. We again observe that in some regions GVs are close to each other or even coincide what does not allow observation of separation of the generated wavepackets from the initial one. However an interesting situation occurs in the vicinity of $k_1 = 0$. Here GVs of the second and third waves have bigger absolute values than $v_+(k_1)$ and the same sign. In this case both created waves move faster that the initial wavepacket. We should emphasize that this is not common in the usual realizations of FWM and this is solely due to the SOC coupling. Note that in these configurations it is also possible to initiate FWM process where one of the velocities of the probe waves is smaller and one bigger than the that of the pump. The situation is more complicated for the case of the fourth configuration in Table I as shown in Fig. 2 (c). The (thick) red curve represents GV of the pump wave of the negative branch $v_-(k_1)$ [see Eq. (16)].

The other (black) curves, that have forms of three loops, represent GVs of generated waves [$v_-(k_2)$ and $v_-(k_3)$] that correspond to other (non-trivial) solutions. Like in the previous cases, to reach significant separation of the pulses in the real space we choose $k_1$ in a region far from the crossing of the curves.

Fig. 3. Initial (blue) and final (red) states of FWM process of configuration 1 from Table I. The moduli of the first component $|\Psi_1(x,t)|$ and of its Fourier transform $|\Phi_1(k,t)|$ are shown in upper row while the corresponding quantities of the second component are presented in the lower row. Insets show the respective temporal evolutions. The parameters are: $\Omega_z = 3$, $\alpha = 0.5$, $\Omega_x = 3$, $k_1 = -0.45$, $k_2 = 3.704$, $k_3 = -4.204$, $g = 0.8$, $g_1 = 0.808$, $g_2 = 0.792$. Time of the evolution is equal to $t = 300$, the total norm $N \approx 78$, $A_1 = 1$, $A_2 = 0.2$, $A_3 = 0$ and $w = 60$. The correspondence between the picks and the spinor states is indicated inside each panel.

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III. NUMERICAL RESULTS

Equipped with the solutions of matching conditions and with the ideas of optimization the conversion efficiency in terms of the GVs we now turn to direct numerical simulations of the configurations of the FWM processes summarized in Table I. In order to find favorable conditions to observe clear evidence of specific FWM process, first one has to select proper momentum $k_1$. Note that phase matching will automatically determine all participating wavepackets GVs as explained in the previous section. Then, appropriate initial widths and amplitudes of the pump and probe waves need to be adjusted to ensure long enough and strong enough nonlinear interaction.

In all simulations we used the wavepackets having
equal widths and completely overlapping at \( t = 0 \), i.e.,
\[
\Psi(x, t = 0) = e^{-x^2/w^2} \sum_{j=1}^{3} A_j \psi_{s_j}(k_j) e^{i k_j x}. \tag{17}
\]
Here \( A_j \) are initial amplitudes the wave-packets with the central wavevectors \( k_j \) of the spinors defined in accordance with Eq. (7).

FIG. 5. Initial (blue) and final (red) states of FWM process of the third configuration from Table I. The moduli of the first spinor component \( |\Psi_t(x, t)| \) and of its Fourier transform \( |\Phi_t(k, t)| \) are shown in upper row while the corresponding quantities of the second spinor component are presented in the lower row. Insets show the respective temporal evolutions. Values of other parameters are: \( \Omega_s = 4, \alpha = 7, \Omega_s = 6, k_1 = -0.71, k_2 = -7.091, k_3 = 5.671, g = 0.3, q_1 = 0.303, q_2 = 0.297 \). Time of evolution \( t = 300 \) and the total norm \( N \approx 107 \). Initial amplitudes are \( A_1 = 1, A_2 = 0, A_3 = 0.2 \) and width \( w = 80 \). The correspondence between the picks and the spinor states is indicated inside each panel.

For the FWM process corresponding to configuration 1, the initial state is formed with \( A_1 = 1, A_2 = 0.2, A_3 = 0 \) and \( s_j \) are chosen according to Table I: \( s_1 = 1, s_2 = -1 \) and \( s_3 = -1 \). In Fig. 3 we show an example of the FWM for this configuration. Due to the FWM process, by the expense of the highly populated initial state \( A_1 \) we observe strong amplification of the seed state and growth of the third matter wave with phase matched momentum \( k_3 \). This process is depicted with snapshots at the beginning and end (i.e., at \( t = 0 \) and \( t = 300 \)) of the simulations in Fig. 3 where main panels (a), (c), [(b), (d)] refer to the first [second] spinor component. In particular blue contours in panels (a), (b) represent initially overlapping pump \( (k_1) \) and probe \( (k_2) \) waves in the configuration space. They are fully separated after evolution time \( (t = 300) \) due to the difference in GVs and new, clearly visible, wave of central momentum \( k_3 \) is generated. Panels (c), (d) show the corresponding features in the Fourier space. Here we distinct two waves as narrow blue peaks, at initial time and again three waves at the end of evolution. The most explicit feature is substantial broadening of all participating matter waves during the evolution. The inset in each panel shows full time evolution of modulus of the spinor components - (a) and (b) in the real space and (c) and (d) in the momentum space.

In the next two figures we illustrate the FWM process corresponding the second and third configuration from Table I with GV configuration shown in panel (b) of the Fig. 3. As mentioned above in these two configurations the exist two different roots \( (q_2 \neq q_3) \) of phase matching condition (12). As one can see directly in panels (a) and (b) of Fig. 3 these configurations are related by the transformation \( q_1 \to -q_2 \) and \( q_2 \to -q_1 \), i.e. the analysis of second and third configuration are analogous.

Interestingly, in Fig. 4 both probe and created waves are generated in the same side of the pump wave. The evolution of the probe wavepackets in Fig. 4 looks qualitatively similar to that one shown in Fig. 3. However, since each newborn wavepacket bears a quasi-spin, the emergent spinors (more precisely the left propagating waves) are different in these cases.

Turning to the fourth in the Table IV, we recall that in this case one can obtain up to five solutions from the phase matching condition (including the trivial case of
values of parameters were slightly different from those used for panel (c), but for the sake of clarifications we propose to look at panel (d) of the Fig. 7. Due to the strong spreading there is substantial overlap of wavepackets, but when we look at them in the momentum space all peaks can be clearly identified. Close inspection reveals oscillations on the pump wave due to the self-phase modulation mentioned above.

IV. CONCLUSIONS

In our study we analyzed the four-wave mixing process in Bose-Einstein condensates with spin-orbit coupling. We found all phase matched configurations for degenerate case where two identical initial states interact with two probe ones. We performed numerical simulations to illustrate the dynamics in which we seeded one of the probe and observed stimulated growth of the latter combined with resonant generation of extra waves. We found unique conditions where both probe waves have smaller group velocity than pump wave, and also reported the case when two FWM process can occur simultaneously. This kind of four-wave mixing can play important role in the experiments of BEC with artificial gauge fields.

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