TOPOLOGICAL ENTROPY ON CLOSED SETS IN $[0, 1]^2$

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Abstract. We generalize the definition of topological entropy due to Adler, Konheim, and McAndrew [AKM] to set-valued functions from a closed subset $A$ of the interval to closed subsets of the interval. We view these set-valued functions, via their graphs, as closed subsets of $[0, 1]^2$. We show that many of the topological entropy properties of continuous functions of a compact topological space to itself hold in our new setting, but not all. We also compute the topological entropy of some examples, relate the entropy to other dynamical and topological properties of the examples, and we give an example of a closed subset $G$ of $[0, 1]^2$ that has 0 entropy but $G \cup \{(p, q)\}$, where $(p, q) \in [0, 1]^2 \setminus G$, has infinite entropy.

1. Introduction

Generalized inverse limits, or inverse limits with set-valued functions, a subject studied only since 2003 with its introduction by Bill Mahavier, provides an entirely new way to study multi-valued functions, a way that does not lose information under iteration. But it is increasingly apparent that they also offer a rich source of new examples of dynamical systems and continua. In fact, they offer a sort of lab in which one can make mathematical experiments - and then have a real chance, with some effort, of understanding (via some sort of coding) deeply the resulting topology and dynamics of the example, and of how the topology and dynamics are interacting.

In this paper we generalize the idea of topological entropy to closed subsets of $[0, 1]^2$, and later to closed subsets of $[0, 1]^n$, for $n$ a positive integer greater than 1. We reduce the problem of computing topological entropy in our context to one of counting the “boxes” (elements of our grid covers) certain sets generated by our closed subset of $[0, 1]^n$ intersect. We also relate the topological entropy of the examples we give to the topology and dynamics of the examples.

James Kelly and Tim Tennant [KT] have recently studied topological entropy of set-valued functions using Bowen’s definition. Our focus is different, as we use the original definition using open covers due to Adler, Konheim and McAndrew [AKM]. There is some overlap of results, and we point these out as they occur. Our results do agree with theirs.

Suppose $X$ is a compact metric space. Recall that if $f : X \to X$ is a continuous function, the inverse limit space generated by $f$ is

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$\lim (X, f) := \{(x_0, x_1, \ldots) : x_i \in X \text{ for each } i \text{ and for each } i \geq 1, x_{i-1} = f(x_i)\}$,
which we can abbreviate as $\lim f$. The map $f$ on $X$ induces a natural homeomorphism $\sigma$ on $\lim f$, which is called the shift map, and is defined by
$$\sigma((x_0, x_1, \ldots)) = (x_1, x_2, \ldots)$$
for $(x_0, x_1, \ldots)$ in $\lim f$.

In [Bo2], R. E. Bowen showed that the topological entropy of the shift map $\sigma$ on $\lim f$ is equal to the topological entropy of $f$, where topological entropy of a continuous function on a compact metric space has the original definition due Adler, Konheim, and McAndrew [AKM] and later to Dinaburg [D] and Bowen [Bo1]. (While defined differently, these two definitions of topological entropy for a continuous map on a compact metric space coincide.)

Generalized inverse limits, or inverse limits with set-valued functions, are a generalization of (standard) inverse limits. Here, rather than beginning with a continuous function $f$ from a compact metric space $X$ to itself, we begin with an upper semicontinuous function $f$ from $X$ to the closed subsets of $X$. In other words, now our function $f$ is set-valued. The generalized inverse limit, or the inverse limit with set-valued mappings, associated with this mapping is the set

$$\lim (X, f) := \{(x_0, x_1, \ldots) : x_i \in X \text{ for each } i, \text{ and for each } i \geq 1, x_{i-1} \in f(x_i)\},$$
which is a closed subspace of $\Pi_{i \geq 0}X$ endowed with the product topology. (As is the case with standard inverse limits, these can be defined in much more general settings, but we do not need those here.) Here again, the shift map $\sigma$ defined above takes $\lim (X, f)$ onto itself, but it is no longer a homeomorphism: $\sigma : \lim (X, f) \rightarrow \lim (X, f)$ is a continuous surjection.

The topic of generalized inverse limits is currently an intensely studied area of continuum theory, with papers from many authors at this point. See [B], [BCM1], [BCM2], [BCM3], [BK], [CR] [GK1], [GK2], [I], [IM2], [I1], [I2], [I3], [I4], [I5], [L], [M], [N1], [N2], [N3], [N4], and [V], for example. (This list is far from exhaustive, with the number of papers on generalized inverse limits now over 60.) Tom Ingram and Bill Mahavier included a chapter on these spaces in their book [IM1], and since then Tom Ingram has written another book on the topic, [I6]. While most of the research has been on understanding the topology of these spaces, some researchers have recently turned to understanding the dynamical properties, since, for a set-valued map $f : X \rightarrow 2^X$, $(\lim f, \sigma)$ is a discrete dynamical system. (See [RT], [KT], [KN].)

2. Background and Notation

Sometimes it is convenient to index our factor spaces, sometimes not. Suppose for each integer $i \geq 0$, $I_i = [0, 1] = I$. The Hilbert cube is $I^\infty = [0, 1]^\infty = \Pi_0^\infty I_i$. 

$\lim (X, f) := \{(x_0, x_1, \ldots) : x_i \in X \text{ for each } i \text{ and for each } i \geq 1, x_{i-1} = f(x_i)\}$,
We often need to talk about various projections from a subset of $I^\infty$ into an interval or a product of intervals. Unless it leads to confusion, for a subset $X$ of $I^\infty$, and a point $x = (x_0, x_1, \ldots)$ in $X$, $\pi_i(x) = x_i$. (That is, we do not specify the momentary domain of $\pi_i$.) Likewise, if $N$ is a positive integer, $x = (x_0, x_1, \ldots, x_N)$, $x \in X \subset I^{N + 1}$, then $\pi_i(x) = x_i$ for $0 \leq i \leq N$. Also, we make following definitions.

- We use both $\mathbb{N}$ and $\mathbb{Z}^+$ to denote the positive integers.
- Let $m \geq 0$ be an integer less than the integer $n$. Then $\langle m, n \rangle = \{m, m + 1, \ldots, n\}$, and we call $\langle m, n \rangle$ the integer interval from $m$ to $n$. Then $\pi_{\langle m, n \rangle}(x) = (x_m, x_{m + 1}, \ldots, x_n)$. We define $\langle m, \infty \rangle$ to be the set $\{m, m + 1, \ldots\}$.
- Let $A = \{n_1, n_2, \ldots\}$ denote a subset of the nonnegative integers (not necessarily listed in order, and either finite or infinite). Then $\pi_A(x) = (x_{n_1}, x_{n_2}, \ldots)$.
- If $A$ is a subset of the space $X$, then $A^\circ$ denotes the interior of $A$ in $X$, and $\overline{A}$ denotes the closure of $A$ in $X$.
- Suppose $x = (x_0, x_1, \ldots, x_n)$ is a point in $I^{n + 1}$ and $y = (y_0, y_1, \ldots)$ is a point in $I^\infty$. Then we define $x \oplus y$ to be the point $(x_0, \ldots, x_n, y_0, y_1, \ldots)$ in $I^\infty$.
- The metric we use on $I^\infty$ is $d(x, y) = \sum_{i=0}^{\infty} \frac{|\pi_i(x) - \pi_i(y)|}{2}$, where $x$ and $y$ are points in $I^\infty$.
- The graph of the set-valued function $f : I \to 2^I$ is the set $\Gamma(f) = \{(x, y) : y \in f(x)\}$.
- The set-valued function $f : I \to 2^I$ is upper semicontinuous at the point $x$ in $I$ if for each open set $V$ in $I$ that contains $f(x)$, there is an open set $U$ in $I$ that contains $x$, and if $z \in U$, then the set $f(z) \subset V$. The function $f$ is upper semicontinuous if it is upper semicontinuous at each point $x$ in $I$. The function $f$ is upper semicontinuous if and only if $\Gamma(f)$ is closed in $I \times I$. (See [A] and [IM1].)
- The set-valued function $f : I \to 2^I$ is called surjective if for each $y \in I$, there is $x \in I$ such that $y \in f(x)$.
- The shift map $\sigma : I^\infty \to I^\infty$ is defined by $\sigma((x_0, x_1, x_2, \ldots)) = (x_1, x_2, \ldots)$. The shift map takes $I^\infty$ continuously onto itself. Also, if $f : I_1 \to 2^{I_{1-1}}$ for each $i > 0$, and $M = \liminf f$, then $\sigma(M) \subset M$. If $f$ is surjective, then $\sigma(M) = M$, and $M$ is invariant under the action of $\sigma$. However, unless $f : I \to I$ is a function, $\sigma$ is not one-to-one.
- For $A \subset I_0 \times I_1$, define $A^{-1} = \{(x, y) : (y, x) \in A\}$. More generally, if $N$ is a positive integer and $A \subset \Pi_{i=0}^N I_i$, then $A^{-1} = \{(x_N, x_{N-1}, \ldots, x_1, x_0) \in \Pi_{i=0}^N I_i : (x_0, x_1, \ldots, x_{N-1}, x_N) \in A\}$.
- Suppose $X, Y$ are topological spaces, and $\mathcal{U}$ is a collection of sets that covers $X$. Then $\mathcal{U} \times Y$ denotes the collection $\{u \times Y : u \in \mathcal{U}\}$, which covers $X \times Y$.
- Suppose $\alpha$ is a collection of (open) sets in the space $X$, and $H \subset X$. Then $\alpha \cap H := \{A \cap H : A \in \alpha\}$.

**Topological Entropy Using Open Covers.** For completeness, we review the traditional version of topological entropy (due to Adler, Konheim, and McAndrew [AKM]) and its properties here and follow to a large extent the discussion in Peter
Walters’ book [W]. We conclude the subsection with theorems on topological entropy due to Bowen [Bo2] that we use. In the next section we recycle and generalize this definition to our new setting.

Definitions.

- If $U$ is a finite collection of sets, define $N^*(U)$ to be the cardinality of the collection $U$. If $U$ is an open cover of the compact topological space $X$, let $N(U)$ denote the number of sets in a finite subcover of $U$ of smallest cardinality. Define the entropy $H(U)$ by $H(U) = \log N(U)$.
- If $U$ is a finite collection of open sets that covers the set $G$, then a subcover $U'$ of $G$ in $U$ is minimal if there does not exist a subcover of $G$ in $U$ of smaller cardinality.
- If $U$ and $V$ are are open covers of a space $X$, define the join $U \lor V$ to be the collection $U \lor V = \{ u \cap v : u \in U, v \in V \}$ of open sets. The join $U \lor V$ is also an open cover of the space $X$. We can likewise define, for a finite collection $\{U_i\}_{i=1}^n$ of open covers of $X$, the join $\lor_{i=1}^n U_i$.
- If $U$ and $V$ are are open covers of the compact topological space $X$, then $U$ is a refinement of $V$ if each $u \in U$ is contained in some $v \in V$. We will say that $V \prec U$ and also that $U \succ V$. Note that if $U$ is a subcover of $X$ in $V$, then $U$ is both a subcollection of $V$ and a refinement of $V$, and $V \prec U$.

Remarks. Suppose $\alpha$ and $\beta$ are open covers of the compact topological space $X$. Then

1. $H(\alpha) \geq 0$.
2. $H(\alpha) = 0$ if and only if $N(\alpha) = 1$ if and only if $X \in \alpha$.
3. If $\alpha \prec \beta$, then $H(\alpha) \leq H(\beta)$.
4. $H(\alpha \lor \beta) \leq H(\alpha) + H(\beta)$.
5. If $T : X \to X$ is a continuous map, then $H(T^{-1}(\alpha)) \leq H(\alpha)$. If $T$ is also surjective, then $H(T^{-1}(\alpha)) = H(\alpha)$.

(See [W] for proofs of (3), (4), and (5) above.)

We will need the following lemma, which is used in the proof of Theorem 1 and in our results.

Lemma 1. [W] If $\{a_n\}_{n \geq 1}$ is a sequence of nonnegative real numbers such that $a_{n+p} \leq a_n + a_p$ for each $n, p \in \mathbb{N}$, then $\lim_{n \to \infty} \frac{a_n}{n}$ exists and equals $\inf_{n \in \mathbb{N}} \frac{a_n}{n}$. 


Theorem 1. (See [W].) If $\alpha$ is an open cover of $X$ and $T : X \to X$ is continuous, then $\lim_{n \to \infty} \frac{H(\bigvee_{i=0}^{n-1} T^{-i}(\alpha))}{n}$ exists.

**Definition.** If $\alpha$ is an open cover of the compact topological space $X$, and $T : X \to X$ is continuous, then the entropy of $T$ relative to $\alpha$ is $h(T, \alpha)$ given by

$$h(T, \alpha) = \lim_{n \to \infty} \frac{H(\bigvee_{i=0}^{n-1} T^{-i}(\alpha))}{n}.$$ 

**Remarks.**

1. $h(T, \alpha) \geq 0$.
2. If $\alpha < \beta$, then $h(T, \alpha) \leq h(T, \beta)$.
3. $h(T, \alpha) \leq H(\alpha)$.

(See [W] for proofs of (1), (2), and (3) above.)

**Definition.** If $T : X \to X$ is continuous, the topological entropy $h(T)$ of $T$ is given by

$$h(T) = \sup_{\alpha} h(T, \alpha)$$

where $\alpha$ ranges over all open covers of $X$.

**Remarks.**

1. $\infty \geq h(T) \geq 0$.
2. In the definition of $h(T)$ one can take the supremum over finite open covers of $X$. This follows from the fact that if $\alpha < \beta$, then $h(T, \alpha) \leq h(T, \beta)$.
3. If $id_X$ denotes the identity map from $X$ to $X$, then $h(id_X) = 0$.
4. If $Y$ is a closed subset of $X$ and $T(Y) = Y$, then $h(T|Y) \leq h(T)$.

**Theorem 2.** (See [W].) If $X_1, X_2$ are compact spaces and $T_i : X_i \to X_i$ are continuous for $i = 1, 2$, and if $\phi : X_1 \to X_2$ is a continuous map with $\phi(X_1) = X_2$ and $\phi \circ T_1 = T_2 \circ \phi$, then $h(T_1) \geq h(T_2)$. If $\phi$ is a homeomorphism, then $h(T_1) = h(T_2)$.

**Theorem 3.** (See [W].) If $T : X \to X$ is a homeomorphism of a compact space $X$, then $h(T) = h(T^{-1})$.

**Theorem 4.** (See [W], Thm. 7.10.) If $T : X \to X$ is a continuous map of a compact metric space $X$, then $h(T^n) = nh(T)$.

**Theorem 5.** ([Bo2, Proposition 5.2]) Suppose $f : X \to X$ is a continuous surjective map on a compact Hausdorff space. If $\sigma$ denotes the induced shift homeomorphism on the inverse limit space $\lim(X,f)$, then $h(\sigma) = h(f)$.

Suppose $X$ is a compact metric space and $f : X \to X$ is continuous. A point $x$ in $X$ is called a wandering point if there is an open set $U$ containing $x$ such that $U \cap (\bigcup_{m \neq 0, m \in \mathbb{Z}} f^m(U)) = \emptyset$. (This is the definition Bowen gives in [Bo2].)

Today most authors use this definition: ([W]) A point $x$ is called wandering for $f$ if there is an open set $U$ containing $x$ such that the sets $f^{-n}(U)$, $n \geq 0$, contain no point of $U$.
are mutually disjoint. The proposition below shows that these two definitions are equivalent. (Surely this is known, but do not know where this is shown, so we give a proof.)

**Proposition 1.** Bowen’s and Walters’ definition of a wandering point are equivalent.

**Proof.** Suppose $X$ is a compact metric space and $f : X \to X$ is continuous.

Let us prove that $\Omega_f^B = \Omega_f^W$.

Suppose $x \in \Omega_f^W$. Then for every neighbourhood $U$ of $x$ exists $n \geq 1$ such that $f^{-n}(U) \cap U \neq \emptyset$. But then $\cup_{m \in \mathbb{Z}} (f^{-m}(U) \cap U) = U \cap (\cup_{m \neq 0, m \in \mathbb{Z}} f^m(U)) \neq \emptyset$. Hence, $x \in \Omega_f^B$ and so $\Omega_f^W \subseteq \Omega_f^B$.

Let us prove that $\Omega_f^B \subseteq \Omega_f^W$. Suppose $x \in \Omega_f^B$ and $U$ arbitrary open set containing $x$. Then $U \cap (\cup_{m \neq 0, m \in \mathbb{Z}} f^m(U)) \neq \emptyset$, hence $\cup_{m \neq 0, m \in \mathbb{Z}} (U \cap f^m(U)) \neq \emptyset$. Therefore, exists $m \in \mathbb{Z}$ such that $U \cap f^m(U) \neq \emptyset$. If $m < 0$ statement is true, so let us assume that $m > 0$. Then there is some $y \in U \cap f^m(U)$. Since $y \in f^m(U)$, there is $x' \in U$ such that $f^m(x') = y$. But then, since $f^m(x') = y \in U$, we have $x' \in (f^{-1})^{-m}(U) = f^{-m}(U)$ so $x' \in U \cap f^{-m}(U) \neq \emptyset$ and therefore $x \in \Omega_f^W$.

Hence, $\Omega_f^B = \Omega_f^W$.

We will call a nonempty open set $U$ with the property that $U \cap (\cup_{m \neq 0, m \in \mathbb{Z}} f^m(U)) = \emptyset$ a **simple wandering set**.

If a point in $X$ is not a wandering point, we call it a **nonwandering point**. The set of nonwandering points of $X$ under $f$ is denoted $\Omega_f$ or just $\Omega$ if no ambiguity results. Thus,

$$\Omega_f = \{ x \in X : \text{for each open set } U \text{ containing } x, U \cap (\cup_{m \neq 0, m \in \mathbb{Z}} f^m(U)) \neq \emptyset \}.$$

The nonwandering set $\Omega_f$ is closed and invariant under $f$, in the sense that $f(\Omega_f) \subseteq \Omega_f$ ([W], Theorem 5.6, p. 124). Hence, the set of wandering points $X \setminus \Omega_f$ is open in $X$.

**Theorem 6.** (See [Bo2]) Let $f : M \to M$ be a continuous map on a compact metric space. If $\Omega$ is the wandering set of $f$, then $h(f) = h(f)\Omega$.

**Mahavier Products.** The Mahavier product is a useful tool for studying subsets of a generalized inverse limit. (See [GK2] and [BK].) They have nice topological and algebraic properties. A closed subset of a standard inverse limit (where the bonding maps are single valued functions - so actually a function from the space to itself) is a subinverse limit. **Unfortunately, a closed subset of a generalized inverse limit need not be a “subgeneralized inverse limit”**. The Mahavier product allows one to consider closed subsets of the generalized inverse limit space, whether or not they are subgeneralized inverse limits. It also makes it easy to consider “finite" generalized inverse limits and their subsets. Suppose $n \geq 2$. Then $\{(x_0, x_1, \ldots, x_n) : x_{i-1} \in f(x_i) \text{ for } n \geq i > 0\}$ is what we mean by a **finite generalized inverse limit**. (For standard inverse limits over intervals, these sets are always arcs topologically, and therefore of limited interest. This is not the case for generalized inverse limits.)
Define the Mahavier product [GK2] as follows:
Suppose $X$, $Y$, and $Z$ are sets. Suppose $A \subset X \times Y$ and $B \subset Y \times Z$. Define
\[ A \star B = \{(x, y, z) : (x, y) \in X \times Y, \text{ and } (y, z) \in Y \times Z\} \]
to be the Mahavier product of $A$ and $B$. Thus, $A \star B \subset X \times Y \times Z$. If $A = \{(a, b)\}$ and $B = \{(b, c)\}$, then we write $(a, b) \star (b, c)$ to mean $A \star B = \{(a, b)\} \star \{(b, c)\}$. Note that $A \star B = (A \times Z) \cap (X \times B)$. In the definition, the factor space $Y$ is acting as the “link up” factor; that is, if $a \in A$, then there is a point $c \in A \star B$ such that the projection of $c$ to $X \times Y$ is $a$ only if some point $b \in B$ has its projection to $Y$ the same as the projection of $a$ to $Y$. In particular, if the projection of $A$ to $Y$ does not intersect the projection of $B$ to $Y$, then $A \star B = \emptyset$.

If, in addition, $W$ is a set and $C \subset Z \times W$, we can form the Mahavier products $(A \star B) \star C$ and $A \star (B \star C)$. One of the nice algebraic properties of Mahavier products is that they are associative. Hence, $(A \star B) \star C = A \star (B \star C)$, and we can just write $A \star B \star C$ without ambiguity. Also, if $A, B$ are open (closed), then $A \star B$ is open (closed).

Now suppose that for each $i \geq 0$, $X_i$ is a set, and $A_i \subset X_{i-1} \times X_i$. Then, using induction, we can form the Mahavier products
\[ A_1 \star A_2 \star \cdots \star A_n \subset X_0 \times \cdots \times X_n \]
for each positive integer $n$. We often write $\star_{i=1}^n A_i$ for $A_1 \star A_2 \star \cdots \star A_n$. If $A_i = A, 1 \leq i \leq n$ then
\[ A \star A \star \cdots \star A = \star_{i=1}^n A. \]
We also define $A_1 \star A_2 \star \cdots$ to be the set of points $x = (x_0, x_1, \ldots) \in \Pi_{i=0}^\infty X_i$ such that $(x_{i-2}, x_{i-1}, x_i) \in A_{i-1} \star A_i$ for each $i \geq 2$, and we write $\star_{i=1}^\infty A_i$ for $A_1 \star A_2 \star \cdots$. Note that if $A_i$ is closed in $X_{i-1} \times X_i$, then $\star_{i=1}^\infty A_i$ is closed in $\Pi_{i=0}^\infty X_i$.

Note that if $f : I \to 2^I$ is a bonding function with graph $\Gamma(f)$, and $G := (\Gamma(f))^{-1} = \{(y, x) : (x, y) \in \Gamma(f)\}$, then
\[ M = \lim_{\leftarrow \infty} f = G \star G \star G \star \cdots = \star_{i=1}^\infty G. \]

In the sequel, we consider Mahavier products of the form $G \star H$ when $G \subset I^N$ and $H \subset I^M$. There is some ambiguity about what the set $G \star H$ is in this situation. In this paper we always mean the following: Write $I^N$ as $I^{N-1} \times I$ for $G$ and $I^N$ as $I \times I^{N-1}$ for $H$. Then $G \subset I^{N-1} \times I$ and $H \subset I \times I^{N-1}$ and
\[ G \star H = \{(x, y, z) : (x, y) \in G \text{ and } (y, z) \in H, \text{ where } x \in I^{N-1}, y \in I, z \in I^{N-1}\}. \]

3. Preliminary Results

While we are mostly interested in closed sets that are the graphs of upper semi-continuous functions from $I$ to $2^I$, we do not need for our closed set to be such a graph in order to define its topological entropy. Also, often, the entropy of such a graph is determined by a closed subset of the graph - sometimes a finite subset of
the graph. Moreover, in order to discuss the entropy properties of closed subsets of $I^2$, we need to define the topological entropy of closed subsets of $I^n$ for $n$ a positive integer greater than 1.

Before we define topological entropy for closed subsets of $[0, 1]^n$, we need some background information on the closed sets and open covers we are using.

The following examples demonstrate that if $G$ is a closed subset of $[0, 1]^n$, then (1) it may be the case that $G := \star_{i=1}^\infty G = \emptyset$ (and that $G$ is of limited interest), and (2) even if $G := \star_{i=1}^\infty G \neq \emptyset$, it may be the case that $\sigma(G) \neq G$.

Example 1. Suppose $G$ is the closed subset $[\frac{3}{4}, 1] \times [0, \frac{1}{4}]$ of $I_0 \times I_1$. Then $G := \star_{i=1}^\infty G = \emptyset$. In fact, $G \ast G = \emptyset$. (It is equally easy to construct empty examples in higher dimensions.)

Example 2. Let $L_0 = I_0 \times \{p\}$ and $L_1 = I_0 \times \{q\}$, where $0 \leq p < q \leq 1$. Suppose $G$ is the closed subset $L_0 \cup L_1$ of $I_0 \times I_1$. Then $G := \star_{i=1}^\infty G$ is a Cantor set of arcs, and $\sigma(G)$ is a Cantor set and is a proper subset of $G$.

Proof. Let $C = \{s = (s_1, s_2, \ldots) : s_i \in \{p, q\} \text{ for each } i > 0\}$. Then $C$ is a Cantor set contained in $G$. Moreover, for each $s \in C$, $I_0 \times \{s\}$ is an arc contained in $G$, and $G = \cup \{L_0 \times \{s\} : s \in C\}$. Hence, $G$ is a Cantor set of arcs. Since $\sigma(G) = C$, $C$ is a proper subset of $G$.

Example 3. Suppose $0 \leq p < q \leq 1$, $L_p = I_0 \times I_1 \times \{p\}$, and $L_q = I_0 \times I_1 \times \{q\}$. Then if $s, t \in \{p, q\}$, $L_s \ast L_t = I^2 \times \{s\} \times I \times \{t\}$. If $s = s_1, s_2, \ldots$ is a sequence each member of which is either $p$ or $q$, let $G_s = L_{s_1} \ast L_{s_2} \ast \cdots = I^2 \times \{s_1\} \times I \times \{s_2\} \times I \times \{s_3\} \cdots$. If $G = L_p \cup L_q$, then $G$ is a closed subset of $I^3$, and $\star_{i=1}^\infty G = \cup \mathcal{M}$, where $\mathcal{M} = \{M_s : s = s_1, s_2, \ldots \text{ is a sequence each member of which is either } p \text{ or } q\}$. Each $M_s$ is homeomorphic to the Hilbert cube, so if $M = \star_{i=1}^\infty G$, then $M$ is topologically a Cantor set of Hilbert cubes (with $M_s \cap M_t = \emptyset$ when $s, t$ are different sequences of $p$’s and $q$’s). Note that because $G \subset I_0 \times I_1 \times I_2$, $\sigma(M)$ is not a subset of $M$, but $\sigma^2(M) \subset M$. Let $M' = \cup \{s_1 \times I \times s_2 \times I \times s_3 \times \cdots : s_1, s_2, \ldots \text{ is a sequence each member of which is either } p \text{ or } q\}$. Then $\sigma^2(M) = M' \subset M$, but $M' \neq M$. However, $\sigma^2(M') = M'$.

The following propositions give some natural conditions under which $\star_{i=0}^\infty G \neq \emptyset$ for $G$ a closed subset of $[0, 1]^n$ ($n > 1$).

**Proposition 2.** If $G$ is a nonempty closed subset of $I_0 \times I_1$, then $G = \star_{i=0}^\infty G \neq \emptyset$ if and only if for every integer $m \geq 2$, $\star_{i=1}^m G \neq \emptyset$. If $G$ is a nonempty closed subset of $I_0 \times I_1 \times \cdots \times I_n$, then $G = \star_{i=1}^\infty G \neq \emptyset$ if and only if for every integer $m \geq 2$, $\star_{i=1}^m G \neq \emptyset$.

**Proof.** If $G = \star_{i=1}^\infty G \neq \emptyset$ it follows from the definition that $\star_{i=1}^m G \neq \emptyset$, $\forall m \in \mathbb{N}$. Now, suppose $\star_{i=1}^m G \neq \emptyset$ for every integer $m \geq 2$. We will inductively define a point in $G = \star_{i=1}^\infty G$. First observe the following: If $(x_0, \ldots, x_{m-1}, x_m) \in \star_{i=1}^m G$ for some integer $m \geq 2$ then $(x_0, \ldots, x_{m-1}) \in \star_{i=1}^{m-1} G$. (**)}
For $m = 2$ we have $G \star G \neq \emptyset$ so from the above it follows that there is a point $(x, y) \in G$ and $z \in [0, 1]$ such that $(x, y, z) \in G \star G$.

Now, for given $m = k$ we have that $G \star G \neq \emptyset$, so there are points $(x_0, x_1, \ldots, x_k) \in \pi_k G$ and $x_{k+1} \in [0, 1]$ such that $(x_0, x_1, \ldots, x_k, x_{k+1}) \in \pi_{k+1} G$. This follows from the assumption and assumption $G \star G \neq \emptyset$.

Therefore we have constructed a sequence $x_0, x_1, \ldots, x_k, \ldots$, such that for each positive integer $m \geq 2, (x_0, x_1, \ldots, x_m) \in \pi_m G$ i.e. $(x_0, x_1, \ldots, x_k, \ldots) \in \pi_{\infty} G$. Therefore we have $G = \pi_{\infty} G \neq \emptyset$. The proof of the second statement is similar so we omit it.

**Proposition 3.** Let $G$ be a nonempty closed subset of $I_0 \times I_1$. If there is some point $(x, y) \in G$ such that $(y, x)$ is also in $G$, then $G = \pi_{\infty} G \neq \emptyset$. If $G$ is a nonempty closed subset of $\prod_{n=0}^{\infty} I_0$ and if there is some point $(x_0, \ldots, x_n) \in G$ such that $(x_n, \ldots, x_0)$ is also in $G$, then $G = \pi_{\infty} G \neq \emptyset$.

**Proof.** Let $(x, y) \in G$ such that $(y, x) \in G$. Then, for each integer $m \geq 2$ we have that $(x, y, x, \ldots, x) \in \pi_m G$ if $m$ is even or $(x, y, x, \ldots, y) \in \pi_m G$ if $m$ is odd. In both cases $\pi_{\infty} G \neq \emptyset$ and therefore, from the previous proposition it follows that $G = \pi_{\infty} G \neq \emptyset$.

The proof of the second statement is similar and we omit it.

**Corollary 1.** If $G = G^{-1} \neq \emptyset$, where $G^{-1} = \{(y, x) : (x, y) \in G\}$ is a nonempty closed subset of $I_0 \times I_1$, then $G = \pi_{\infty} G \neq \emptyset$.

**Proof.** Since $G \neq \emptyset$, there is some point $(x, y) \in G$. Since $G = G^{-1}$, the point $(y, x) \in G$. From previous proposition it follows that $G \neq \emptyset$.

**Proposition 4.** If $G$ is a nonempty closed subset of $\prod_{n=0}^{\infty} I_0$ and $\pi_n(G) \subset \pi_0(G)$, then $G = \pi_{\infty} G \neq \emptyset$.

**Proof.** The set $G \neq \emptyset$, so there is some $(x_0, \ldots, x_n) \in G$. Since $\pi_n(G) \subset \pi_0(G)$, there are points $x_{n+1}, x_{n+2}, \ldots, x_{2n}$ in $I$ such that $(x_n, x_{n+1}, \ldots, x_{2n}) \in G$. Thus $(x_0, \ldots, x_n, \ldots, x_{2n}) \in G \star G$. Then since $x_{2n} \in \pi_0(G)$, there is $(x_0, \ldots, x_{3n}) \in G$ such that $(x_0, \ldots, x_{3n}) \in G \star G \star G$, and we can continue this process indefinitely, obtaining a point in $G$.

**Proposition 5.** Suppose $n$ is a positive integer. If $G$ is a closed subset of $I_0 \times I_1$ that contains a finite set of points $\{(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_0)\}$, then $G = \pi_{\infty} G \neq \emptyset$. Furthermore, $G$ contains a point of period $n$ under the action of $\sigma$.

**Proof.** The point $(x_0, x_1, \ldots, x_n, x_0, \ldots, x_n) \in G$, so $G \neq \emptyset$. Let $y_0 = (x_0, \ldots, x_n)$ and $y_n = (x_n, x_0, \ldots, x_{n-1})$. For each $0 < i < n$, let $y_i = (x_i, \ldots, x_{n-1})$. For $0 \leq i \leq n$, let $z_i = y_i \oplus y_{i+1} \oplus y_{i} \oplus \ldots$. Then each $z_i \in G$, and $\sigma(z_i) = z_{i+1}$ for $0 \leq i < n$, and $\sigma(z_n) = z_0$. Hence, $\sigma^n(z_0) = z_0$.

**Proposition 6.** If $G$ is a nonempty closed subset of $I_0 \times I_1$ and $G = \pi_{\infty} G \neq \emptyset$, then $\sigma(G) \subset G$. If $G$ is a nonempty closed subset of $I_0 \times I_1 \times \cdots \times I_n$ and $G = \pi_{\infty} G \neq \emptyset$, then $\sigma^n(G) \subset G$.

**Proof.** We prove the first statement. The proof for the second statement is similar. Suppose $x = (x_0, x_1, \ldots) \in \sigma(G)$. Then there is $y = (y_0, y_1, \ldots) \in G$ such that
\( \sigma(y) = x \). Now \( \sigma(y) = (y_1, y_2, \ldots) = x \), so \( x_{i-1} = y_i \) for each \( i > 0 \). Since \( y \in G \), for each \( i > 0 \), \( (y_{i-1}, y_i) \in G \). Then for each \( i > 1 \), \( (y_{i-2}, y_{i-1}) \in G \). Then \( x \in G \).

Proposition 4 can be generalized to \( G \subset I^{n+1}, n > 1 \), too. First we give an example, then state the more general proposition.

**Example 4.** Suppose \( G \) is a nonempty and closed subset of \( I^{2+1} \) such that
\[
\{(x_0, x_1, x_2), (x_2, x_3, x_4), (x_4, x_5, x_6)\} \subset G \text{ and } x_6 = x_0.
\]
Then \( (x_0, x_1, x_2, x_3, x_4, x_5, x_6) = (x_0, x_1, x_2, x_3, x_4, x_5, x_0) \in G \star G \star G \) and the point \( y_0 = (x_0, x_1, x_2, x_3, x_4, x_5) \oplus (x_0, x_1, x_2, x_3, x_4, x_5) \oplus \cdots \in *_{i=1}^{\infty} G = G \) and \( G \neq \emptyset \). Now \( \sigma^2(G) \subset G \) and
\[
\begin{align*}
\sigma^2(y_0) &= (x_2, x_3, x_4, x_5, x_0, x_1, \ldots, x_5, x_0, \ldots) = y_1 \\
\sigma^2(y_1) &= (x_4, x_5, x_0, x_1, \ldots, x_5, x_0, \ldots) = y_2 \\
\sigma^2(y_2) &= (x_0, x_1, \ldots, x_5, x_0, \ldots) = y_0
\end{align*}
\]
Therefore, \( y_0 \) is a point in \( G \) and \( \sigma^3(y_0) = y_0 \). So \( y_0 \) is a point in \( G \) of period 3 under the action of \( \sigma^2 \).

**Proposition 7.** Suppose \( G \) is a closed subset of \( I^{n+1}, n \in \mathbb{N} \), and \( p \in \mathbb{N}, p > 1 \). If \( y_0 = (x_{i_0}, \ldots, x_{i_1}n) \) \( \in G \) for 0 \( \leq i \leq p-1 \), and \( x_{pn} = x_0 \), then \( z_0 = y_0 \oplus y_1 \oplus \cdots \oplus y_{p-1} \star y_0 \star y_1 \star \cdots \star y_{p-1} \star \cdots \in G = *_{i=1}^{\infty} G \neq \emptyset \), and \( z_0 \) is a period \( p \) point under the action of \( \sigma^n \) in \( G \).

**Proof.** The proof is straightforward and we omit it. \( \square \)

**Proposition 8.**
\begin{enumerate}
\item Let \( G \) be a nonempty closed subset of \( I_0 \times I_1 \) such that \( G \neq \emptyset \). Let \( \cap_{i=0}^{\infty} \sigma^i(G) = G^* \). Then \( G^* \neq \emptyset \) and \( G^* \subset G \). Furthermore, \( \sigma(G^*) = G^* \).
\item Let \( G \) be a nonempty closed subset of \( I_0 \times I_1 \times \cdots \times I_n \) such that \( G \neq \emptyset \). Let \( \cap_{i=0}^{\infty} \sigma^i(G) = G^* \). Then \( G^* \neq \emptyset \) and \( G^* \subset G \). Furthermore, \( \sigma^n(G^*) = G^* \).
\end{enumerate}

**Proof.** We give a proof of (1). The proof of (2) is similar. Since \( \sigma^n(G) \subset \sigma^{n-1}(G) \) for \( n > 0 \), \( G^* \neq \emptyset \) and \( G^* \subset G \). It is also easy to see that \( \sigma(G^*) = G^* \). \( \square \)

For \( G \) is a nonempty closed subset of \( I_0 \times I_1 \times \cdots \times I_n \) and \( G = *_{i=1}^{\infty} G \neq \emptyset \), we will call the set \( G^* = \cap_{i=0}^{\infty} \sigma^i(G) \) the kernel of \( G \).

**Grid covers.** Suppose \( K \) is a closed subset of \( I^\infty \). Let \( \tau = \{\tau_1, \ldots, \tau_n\} \) be a minimal open cover of \([0, 1]\) by open intervals. Let \( N \) be a positive integer. The grid generated by \( \tau \) for \( N \) is the collection \( T \) of basic open sets in \( I^\infty \)
\[
T = \{\tau_{i_0} \times \tau_{i_1} \times \cdots \times \tau_{i_N} \times I^\infty : i_j \in \{1, n\}\}
\]
Since \( T \) is an open cover of \( I^\infty \) by basic open sets, it is therefore also a cover of \( K \) by basic open sets. We will say that \( T \) is a grid cover of \( K \). Likewise,
\[
S = \{\tau_{i_0} \times \tau_{i_1} \times \cdots \times \tau_{i_N} : i_j \in \{1, n\}\}
\]
is a grid cover of \( I^{N+1} \) by basic open sets, and is also therefore a cover of any closed subset \( L \) of \( I^{N+1} \).

Surely the following propositions are known, but we include them just to make sure our grid covers “do the job” that any open cover of a compact subset of \( I^n \) or \( I^\infty \) would do.
Proposition 9. Suppose $K$ is a closed subset of $I^\infty$. If $\mathcal{U}$ is an open cover of $K$ by open sets in $I^\infty$, then there is a grid cover $T$ of $I^\infty$ such that $T' = \{ o \in T : o \cap K \neq \emptyset \}$ refines $\mathcal{U}$ and covers $K$. If $\mathcal{V}$ is an open cover of $K$ by open sets in the subspace $K$, then there is a grid cover $T$ of $I^\infty$ such that $T^* = \{ o \cap K : o \in T \}$ refines $\mathcal{V}$ and covers $K$.

Proof. Suppose $\mathcal{U}$ is an open cover of $K$ by open sets in $I^\infty$. Then there is a collection $\mathcal{V}$ of basic open sets in $I^\infty$ that refines $\mathcal{U}$ and covers $K$. We also assume that for each $v \in \mathcal{V}$, each projection $\pi_k(v)$ is an open interval (relative to $[0,1]$). Since $K$ is compact, there is a finite subcover $\mathcal{V}'$ of $\mathcal{V}$. Let $\mathcal{V}' = \{ v_1, \ldots, v_m \}$. There is a collection $\mathcal{W}$ of basic open sets such that $\mathcal{W} = \{ w_1, \ldots, w_m \}$, $w_i \subset v_i$ for $1 \leq i \leq m$, and $\mathcal{W}$ covers $K$. Again, we can choose the collection $\mathcal{W}$ so that for each $w \in \mathcal{W}$, each projection $\pi_k(w)$ is an open interval relative to $[0,1]$. Since each $w_i$ is a basic open set, there is some positive integer $N$ such that $\pi_j(w_i) = I$ for each $j > N$.

Let $\pi_k(w_i) = [a_{i,k}, b_{i,k}]$ for each $0 \leq k \leq N$, $1 \leq i \leq m$, and let

$$\mathcal{E} = \{ x : x \in [a_{i,k}, b_{i,k}], 0 \leq k \leq N, 1 \leq i \leq n \} \cup \{ 0,1 \}.$$ 

Since $\mathcal{E}$ is a finite subset of $I$, we can list the members of $\mathcal{E}$ in increasing order as $\mathcal{E} = \{ 0 = t_0, t_1, \ldots, t_\gamma = 1 \}$. Then each $\pi_k(w_i)$ is a unique union of consecutive intervals of the form $[t_{i-1}, t_i]$. If $x = (x_0, x_1, \ldots) \in K$, then there is some $w_i$ such that $x \in w_i$, which implies that for each $k \leq N$, $x_k \in \pi_k(w_i)$, and there is some $t_{j_k}$ such that $x_k \in [t_{j_k}, t_{j_k+1}]$. Thus, $x \in \Pi_k^N [t_{j_k}, t_{j_k+1}] \times I^\infty$.

Suppose $\epsilon > 0$. Let $\pi_k(w_i)^+ = (a_{i,k} - \epsilon, b_{i,k} + \epsilon) \cap [0,1]$ for each $0 \leq k \leq N$, $1 \leq i \leq m$. Let $w_i^+ = \Pi_k^N \pi_k(w_i)^+ \times I^\infty$. We can choose $\epsilon > 0$ so small that $(1) \epsilon < \min_{k=0}^{\gamma-1} \{ (b_{i,k} - a_{i,k}) \}$, and $(2) w_i^+ \subset w_i$. Then each $w_i^+$ is a union of members of $\Gamma = \{ \Pi_k^N ((t_{j_k} - \epsilon, t_{j_k+1} + \epsilon) \cap [0,1]) \times I^\infty : j_k \in [0,\gamma) \}$. Hence, if $\Gamma^* = \{ g \in \Gamma : g \subset w_i^+ \text{ for some } i \text{ and } g \cap K \neq \emptyset \}$, then $\Gamma^* > \mathcal{W} \lor \mathcal{V} > \mathcal{U}$ and $\Gamma^*$ covers $K$.

The proof of the last statement now follows easily, so we omit it.

\[ \square \]

Proposition 10. Suppose $M$ is a positive integer and $K$ is a closed subset of $I^{M+1}$. If $\mathcal{U}$ is an open cover of $K$ by open sets in $I^{M+1}$, then there is a grid cover $T$ of $I^{M+1}$ such that $T' = \{ o \in T : o \cap K \neq \emptyset \}$ refines $\mathcal{U}$ and covers $K$. If $\mathcal{V}$ is an open cover of $K$ by open sets in the subspace $K$, then there is a grid cover $T$ of $I^{M+1}$ such that $T^* = \{ o \cap K : o \in T \}$ refines $\mathcal{V}$ and covers $K$.

Proof. The proof is similar to the proof of Proposition 9 and we omit it.

\[ \square \]
4. Topological entropy of closed subsets of $[0,1]^2$

We index our intervals for bookkeeping purposes. For convenience, we also write $I^\infty$ for $\prod_{i=0}^{\infty} I_i$ (for $m$ a positive integer). Suppose $G$ is a closed subset of $I_0 \times I_1$. We can define the **topological entropy** of $G$ as follows:

1. First, let $\alpha = \{a_1, \ldots, a_n\}$ be a minimal open cover of $I_0$ by open intervals. Then $N^*(\alpha) = n$. For each positive integer $m > 0$, let 
   \[ \alpha^m = \{ \prod_{j=0}^{m-1} a_{k_j} : k_j \in \{1, n\}, 0 \leq j \leq m - 1 \}. \]

2. If $K$ is a closed subset of $\prod_{i=0}^{m-1} I_i$ ($m > 1$ a positive integer or $m = \infty$), and $U$ is a collection of open sets in $\prod_{i=0}^{m-1} I_i$ that covers $K$, let $N(K, U)$ denote the least cardinality of a subcover of $K$ in $U$.

3. Then $\alpha^2 = \{a_i \times a_j : 1 \leq i, j \leq n\}$ is a cover of $G$ by open subsets of $I_0 \times I_1$, and $N(G, \alpha^2) \leq n^2$.

4. Now 
   \[ \alpha^3 = \{a_{i_0} \times a_{i_1} \times a_{i_2} : i_k \in \{1, n\}, 0 \leq k \leq 2\}, \]
   is a cover of $G \star G$ by open sets in $\prod_{j=0}^{m-1} I_j$ and $N(G \star G, \alpha^3) \leq n^3$.

5. Note that $\alpha^3 \star \alpha^2$ contains more sets than does $\alpha^3$ since it contains sets of the form $(a_i \times a_j) \star (a_k \times a_l)$ for $i, j, k, l \leq n$, and $(a_i \times a_j) \star (a_k \times a_l) = a_i \times (a_j \cap a_k) \times a_l$, which is nonempty as long as $a_j \cap a_k \neq \emptyset$. However, a minimal subcover of $G \star G$ in $\alpha^3 \star \alpha^2$ has the same number of elements as a minimal subcover of $\alpha^3$, since each set $a_i \times (a_j \cap a_k) \times a_l$ is contained at least one member of $\alpha^3$.

6. We can continue this process for each $m \in \mathbb{N}$:
   \[ \alpha^{m+1} = \{ \prod_{j=0}^{m} a_{k_j} : k_j \in \{1, n\}, 0 \leq j \leq m \} \]
   is an open cover of $\star_{i=1}^m G$ and $N(\star_{i=1}^m G, \alpha^{m+1}) \leq n^{m+1}$. Again, a minimal subcover of $\star_{i=1}^m G$ by elements of $\star_{i=1}^m \alpha^2$ has the same number of elements as a minimal subcover of $\star_{i=1}^m G$ by elements of $\alpha^{m+1}$. Since using the cover $\star_{i=1}^m \alpha^2$ is sometimes more convenient, we continue to use both covers.

7. Suppose $G$ is a closed subset of $I_0 \times I_1$. Let $G = \star_{i=1}^{\infty} G$.
   - For each positive integer $m$, $0 \leq N(\star_{i=1}^m G, \alpha^{m+1}) \leq n^{m+1}$. If $G \neq \emptyset$, $0 < N(\star_{i=1}^m G, \alpha^{m+1})$.
   - If $G \neq \emptyset$, $1 = N(\star_{i=1}^m G, \alpha^{m+1})$ if and only if there is a sequence $\alpha_{j_0}, \alpha_{j_1}, \ldots, \alpha_{j_m}$ (with each $1 \leq j_i \leq n$) such that $G \subset (\alpha_{j_0} \times \cdots \times \alpha_{j_m}) \times I^\infty$.
   - If $\alpha, \beta$ are both minimal open covers of $I_0$ by open intervals and $\alpha < \beta$, then for each $m > 0$, $N(\star_{i=1}^m G, \alpha^{m+1}) \leq N(\star_{i=1}^m G, \beta^{m+1})$.

**Proof.** Let $k = N(\star_{i=1}^m G, \beta^{m+1})$. Let $\{B_1, B_2, \ldots, B_k\}$ be a subcover of $\star_{i=1}^m G$ in $\beta^{m+1}$ of minimal cardinality. For each $1 \leq i \leq k$, there is some $A_i \in \alpha^{m+1}$ such that $B_i \subset A_i$. Then $\{A_1, A_2, \ldots, A_k\}$ is a subcover of $\star_{i=1}^m G$ in $\alpha^{m+1}$ of cardinality $k$. Hence, $N(\star_{i=1}^m G, \alpha^{m+1}) \leq N(\star_{i=1}^m G, \beta^{m+1})$.

**□**

- If $\alpha$ is a minimal open cover of $I_0$ by open intervals, $k, l$ are positive integers, and $K \subset \star_{i=1}^l G$, $L \subset \star_{i=1}^k G$, $K, L$ are closed, then
\(\alpha^{k+1} \ast \alpha^l\) is a cover of \(\Pi^{k+1}_i I_i\) and of \(K \ast L\) by open sets in \(\Pi^{k+1}_i I_i\), as is \((\sum_{i=1}^{k+1} G) \ast (\sum_{i=1}^{l} G) = \sum_{i=1}^{k+1+l} G\). Furthermore, \(N(\sum_{i=1}^{k+1} G, \alpha^{k+1}) = N(\sum_{i=1}^{k} G, \alpha^{k+1}) N(\sum_{i=1}^{l} G, \alpha^l)\).

**Proof.** Showing that \(\alpha^{k+1} \ast \alpha^l\) and \((\sum_{i=1}^{k+1} G) \ast (\sum_{i=1}^{l} G) = \sum_{i=1}^{k+1+l} G\) are open covers of \(\Pi^{k+1}_i I_i\) and of \(K \ast L\) by open sets in \(\Pi^{k+1}_i I_i\) is straightforward and we omit it. Let \(\{A_1, A_2, \ldots, A_p\}\) be a subcover of \(\sum_{i=1}^{k+1} G\) in \(\alpha^{k+1}\) of minimal cardinality and let \(\{B_1, B_2, \ldots, B_q\}\) be a subcover of \(\sum_{i=1}^{l} G\) in \(\alpha^l\) of minimal cardinality. Then \(\{A_i \ast B_j : 1 \leq i \leq p, 1 \leq j \leq q\}\) is a subcover of \(\sum_{i=1}^{k+1} G\) in \(\alpha^{k+1} \ast \alpha^l\), and

\[
N(\sum_{i=1}^{k+1} G, \alpha^{k+1} \ast \alpha^l) = N(\sum_{i=1}^{k} G, \alpha^{k+1}) N(\sum_{i=1}^{l} G, \alpha^l) \\
\leq N(\sum_{i=1}^{k} G, \alpha^{k+1}) N(\sum_{i=1}^{l} G, \alpha^l) \\
= N(\sum_{i=1}^{k} G, \sum_{i=1}^{l} G) N(\sum_{i=1}^{k} G, \sum_{i=1}^{l} G).
\]

\[\square\]

- If \(\alpha, \beta\) are both minimal open covers of \(I_0\) by open intervals, then for each \(m > 0\), \(\alpha^{m+1} \cup \beta^{m+1} = (\alpha \cup \beta)^{m+1}\), and \(N(\sum_{i=1}^{m} G, \alpha^{m+1} \cup \beta^{m+1}) \leq N(\sum_{i=1}^{m} G, \alpha^{m+1}) N(\sum_{i=1}^{m} G, \beta^{m+1})\).

**Proof.** Showing that \((\alpha \cup \beta)^{m+1} = \alpha^{m+1} \cup \beta^{m+1}\) is straightforward and we omit it. Let \(\{A_1, A_2, \ldots, A_k\}\) be a subcover of \(\sum_{i=1}^{m} G\) in \(\alpha^{m+1}\) of minimal cardinality and let \(\{B_1, B_2, \ldots, B_l\}\) be a subcover of \(\sum_{i=1}^{m} G\) in \(\beta^{m+1}\) of minimal cardinality. Then \(\{A_i \cap B_j : 1 \leq i \leq k, 1 \leq j \leq l\}\) is a subcover of \(\sum_{i=1}^{m} G\) in \(\alpha^{m+1} \cup \beta^{m+1}\), and \(N(\sum_{i=1}^{m} G, \alpha^{m+1} \cup \beta^{m+1}) \leq N(\sum_{i=1}^{m} G, \alpha^{m+1}) N(\sum_{i=1}^{m} G, \beta^{m+1})\).

\[\square\]

- If \(K\) is a closed subset of \(G \subset I_0 \times I_1\), \(m\) is a positive integer, and \(\alpha = \{\alpha_1, \ldots, \alpha_n\}\) is a minimal open cover of \(I_0\) by open intervals, then \(N(\sum_{i=1}^{m} K, \alpha^{m+1}) \leq N(\sum_{i=1}^{m} G, \alpha^{m+1})\).

**Proof.** Suppose \(\{A_1, A_2, \ldots, A_k\}\) is an open subcover of minimum cardinality of \(\sum_{i=1}^{m} G\) in \(\alpha^{m+1}\). Since \(\sum_{i=1}^{m} K \subset \sum_{i=1}^{m} G\), \(\{A_1, A_2, \ldots, A_k\}\) is also an open subcover of \(K\). Hence, \(N(\sum_{i=1}^{m} K, \alpha^{m+1}) \leq N(\sum_{i=1}^{m} G, \alpha^{m+1})\).

\[\square\]

- Suppose \(l\) and \(m\) are positive integers. Then \(\alpha^{l+1}\) is a grid cover of \(\sum_{i=1}^{l} G, \alpha^{l+1}\) is a grid cover of \(\sum_{i=1}^{l+1} G\), and \(\alpha^{l+1} \times \Pi^{l+1}_i I_i\) is an open cover of \(\sum_{i=1}^{l+1} G\). Then \(N(\sum_{i=1}^{l} G, \alpha^{l+1}) \leq N(\sum_{i=1}^{l+1} G, \alpha^{l+1} \times \Pi^{l+1}_i I_i) \leq N(\sum_{i=1}^{l+1} G, \alpha^{l+1} \times \Pi^{l+1}_i I_i)\).

**Proof.** Suppose \(\{B_j \times \Pi^{l+1}_i I_i : j=1\}^{k}_1\) is a subcover of \(\sum_{i=1}^{l+1} G\) in \(\alpha^{l+1} \times \Pi^{l+1}_i I_i\), of least cardinality. Then \(\{B_j : j=1\}^{k}_1\) is a subcover of \(\sum_{i=1}^{l+1} G\) in \(\alpha^{l+1}\) of least cardinality. Hence, \(N(\sum_{i=1}^{l+1} G, \alpha^{l+1}) \leq N(\sum_{i=1}^{l+1} G, \alpha^{l+1} \times \Pi^{l+1}_i I_i)\). Since \(\alpha^{l+1} \times \Pi^{l+1}_i I_i\) refines \(\alpha^{l+1} \times \Pi^{l+1}_i I_i\), \(N(\sum_{i=1}^{l+1} G, \alpha^{l+1} \times \Pi^{l+1}_i I_i) \leq N(\sum_{i=1}^{l+1} G, \alpha^{l+1} \times \Pi^{l+1}_i I_i)\). The result follows.

\[\square\]
If \( \alpha = \{\alpha_1, \ldots, \alpha_n\} \) is a minimum open cover of \( I_0 \) by intervals, \( G \) is a closed subset of \( I_0 \times I_1 \) and \( G \neq \emptyset \), then 
\[
\lim_{m \to \infty} \frac{\log N(\star_{i=1}^m G, \alpha^{m+1})}{m} = \lim_{m \to \infty} \frac{\log N(\star_{i=1}^m G, \star_{i=1}^m \alpha)}{m}
\]
exists.

**Proof.** Let \( a_m = \log N(\star_{i=1}^m G, \alpha^{m+1}) = \log N(\star_{i=1}^m G, \star_{i=1}^m \alpha) \) for each \( m \in \mathbb{N} \). Then 
\[
1 \leq N(\star_{i=1}^m G, \alpha^{m+1}) \leq n^{m+1}, \quad \text{so}
\]
\[
0 \leq a_m = \log N(\star_{i=1}^m G, \alpha^{m+1}) \leq (m+1) \log n.
\]
By Lemma [1] it suffices to show that \( a_{m+k} \leq a_m + a_k \). We have
\[
a^{m+k+1} \subset (\star_{i=1}^m \alpha^2) * (\star_{i=m+1}^{m+k} \alpha^2),
\]
and
\[
N(\star_{i=1}^{m+k} G, \alpha^{m+k+1}) = N(\star_{i=1}^{m+k} G, (\star_{i=1}^{m+k} \alpha^2) * (\star_{i=m+1}^{m+k} \alpha^2)).
\]
Since \( N(\star_{i=1}^m G, \alpha^{m+1}) \) is the cardinality of a minimal subcover of \( \star_{i=1}^m G \)
in \( \star_{i=1}^m \alpha^2 \), and \( N(\star_{i=1}^m G, \alpha^{m+1}) \) is the cardinality of a minimal subcover of \( \star_{i=1}^m \alpha = \star_{i=1}^{m+k} G \) in \( \star_{i=m+1}^{m+k} \alpha^2 \), \( (\star_{i=1}^m \alpha^2) * (\star_{i=m+1}^{m+k} \alpha^2) \) is a cover of \( \star_{i=1}^{m+k} G \)
in \( \prod_{i=0}^{m+k} I_i \). Thus,
\[
N(\star_{i=1}^{m+k} G, \alpha^{m+k+1}) = N(\star_{i=1}^m G, (\star_{i=1}^{m+k} \alpha^2) * (\star_{i=m+1}^{m+k} \alpha^2)) =
\]
\[
N(\star_{i=1}^{m+k} G, (\star_{i=1}^m \alpha^2) * (\star_{i=m+1}^{m+k} \alpha^2)) \leq N(\star_{i=1}^m G, \star_{i=1}^m \alpha^2) N(\star_{i=1}^m G, \star_{i=m+1}^{m+k} \alpha^2),
\]
and we have
\[
a_{m+k} = \log N(\star_{i=1}^{m+k} G, \alpha^{m+k+1}) \leq \log N(\star_{i=1}^m G, \star_{i=1}^m \alpha^2) N(\star_{i=1}^m G, \star_{i=m+1}^{m+k} \alpha^2) = a_m + a_k. \quad \square
\]

(9) If \( G \neq \emptyset \), define \( \text{ent}(G, \alpha) \) to be
\[
\text{ent}(G, \alpha) = \lim_{m \to \infty} \frac{\log N(\star_{i=1}^m G, \alpha^{m+1})}{m}.
\]
If \( G = \emptyset \), define \( \text{ent}(G, \alpha) = 0 \).

(10) **Proposition 11.** If \( \alpha = \{\alpha_1, \ldots, \alpha_n\} \) is a minimum open cover of \( I_0 \) by intervals, \( G \) is a closed subset of \( I_0 \times I_1 \), then we have the following:

(a) \( \text{ent}(G, \alpha) \geq 0 \).

(b) If \( \alpha < \beta \), \( \alpha, \beta \) both minimal covers of \( I_0 \) by open intervals, then \( \text{ent}(G, \alpha) \leq \text{ent}(G, \beta) \).

(c) If \( K \) is a closed subset of \( G \subset I_0 \times I_1 \), \( m \) is a positive integer, and \( \alpha = \{\alpha_1, \ldots, \alpha_n\} \) is a minimal open cover of \( I_0 \) by open intervals, then \( \text{ent}(K, \alpha) \leq \text{ent}(G, \alpha) \).

**Proof.** (a) follows directly from the definition of \( \text{ent}(G, \alpha) \).

Let us prove (b): For each positive integer \( m \), \( \alpha^m < \beta^m \). If \( \{B_1, \ldots, B_k\} \) is a minimal subcover of \( \star_{i=1}^m G \) in \( \beta^{m+1} \), then for each \( 1 \leq i \leq k \), there is some \( A_i \in \alpha^{m+1} \) such that \( B_i \subset A_i \). Thus, \( \{A_1, \ldots, A_k\} \) is a subcover of \( \star_{i=1}^m G \) in \( \alpha^{m+1} \), and \( \text{ent}(G, \alpha) \leq \text{ent}(G, \beta) \).
(c) follows directly from the fact that $N(\star_{i=1}^\infty K, \alpha^{m+1}) \leq N(\star_{i=1}^m G, \alpha^{m+1})$ for each positive integer $m$. \hfill \Box

(11) Finally, we define $\text{ent}(G) = \sup\{\text{ent}(G, \alpha)\}$, where $\alpha$ ranges over all minimal covers of $I_0$ by open intervals (in $I_0$).

**Theorem 7.** Let $G$ be a closed subset of $I_0 \times I_1$ and $G^{-1} = \{(x, y) : (y, x) \in G\}$. Then $\text{ent}(G) = \text{ent}(G^{-1})$.

**Proof.** For a positive integer $m$, note that a point $(x_0, x_1, \ldots, x_m) \in \star_{i=1}^m G$ if and only if $(x_m, x_{m-1}, \ldots, x_0) \in \star_{i=1}^m G^{-1}$. Suppose $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ is a minimal cover of $I_0$ by open intervals. Suppose $m$ is a positive integer. Then $\alpha_0 \times \alpha_1 \times \cdots \times \alpha_m \in \alpha^{m+1}$ if and only if $\alpha_{i_m} = \alpha_{i_{m-1}} \times \cdots \times \alpha_0 \in \alpha^{m+1}$, and $\alpha_{i_0} \times \alpha_{i_1} \times \cdots \times \alpha_{i_m}) \cap \star_{j=1}^m G \neq \emptyset$ if and only if $(\alpha_{i_m} \times \alpha_{i_{m-1}} \times \cdots \times \alpha_0) \cap \star_{j=1}^m G^{-1} \neq \emptyset$. Then $N(\star_{i=1}^m G, \alpha^{m+1}) = N(\star_{i=1}^m G^{-1}, \alpha^{m+1})$ for each $m$. Hence, $\text{ent}(G, \alpha) = \text{ent}(G^{-1}, \alpha)$ for each cover $\alpha$, and the result follows. \hfill \Box

**Remark** Theorem 7 above overlaps with Corollary 3.6 of [KT].

**Proposition 12.** If $\mathcal{V}$ is an open cover (in $I^\infty$) of $\star_{i=1}^\infty G$, then $\sigma^{-1}(\mathcal{V}) := \{\sigma^{-1}(v) : v \in \mathcal{V}\} = \{I_0 \times v : v \in \mathcal{V}\}$ is also an open cover (in $I^\infty$) of $\star_{i=1}^\infty G$.

**Proof.** Suppose $x = (x_0, x_1, \ldots) \in \star_{i=1}^\infty G$. Then $\sigma(x) = (x_1, x_2, \ldots) \in \star_{i=1}^\infty G$, and there is some $v \in \mathcal{V}$ such that $\sigma(x) \in v$. Since $I_0 \times v = \sigma^{-1}(v)$, $x \in \sigma^{-1}(v) \in \sigma^{-1}(\mathcal{V})$. \hfill \Box

If $\mathcal{U}$ is an open cover of $I^\infty$, $G$ is a closed subset of $I_0 \times I_1$, and $G = \star_{i=1}^\infty G \neq \emptyset$, let $\mathcal{U}^* = \{u \cap G : u \in \mathcal{U}\}$ denote the corresponding open cover of $G$ by open sets in $G$.

**Theorem 8.** Suppose $G$ is a closed subset of $I_0 \times I_1$, $G = \star_{i=1}^\infty G \neq \emptyset$, and $\sigma(G) = G$. If $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ is a minimal open cover of $I_0$ by open intervals, then $\text{ent}(G, \alpha) = h(\sigma, (\alpha^{M+1} \times I^\infty)^*)$ for each positive integer $M$.

**Proof.** Let $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ be a minimal open cover of $I_0$ by intervals.

Fix the positive integer $M$. Let

$$\mathcal{V} = \left\{ \prod_{j=0}^M \alpha_{i_j} \times I^\infty : \alpha_{i_j} \in \alpha \text{ and } \left( \prod_{j=0}^M \alpha_{i_j} \times I^\infty \right) \cap G \neq \emptyset \right\},$$

and let

$$\mathcal{U} = \{ \alpha_{k_0} \times \alpha_{k_1} \times \cdots \times \alpha_{k_M} \times I^\infty : \{k_j\}_{j=0}^{M+1} \text{ is a sequence of members of } \{1, \ldots, n\} \text{ of length } M + 2 \}. $$
For \( v = \prod_{j=0}^{M} \alpha_{ij} \times I^\infty \in \mathcal{V} \), \( \sigma^{-1}(v) = I_0 \times \prod_{j=0}^{M} \alpha_{ij} \times I^\infty \). Then

\[
\sigma^{-1}(\mathcal{V}) \cup \mathcal{V} = \left\{ \sigma^{-1}(v) \cap w : v = \prod_{j=0}^{M} \alpha_{ij} \times I^\infty, w = \prod_{j=0}^{M} \alpha_{kj} \times I^\infty \in \mathcal{V} \right\}
\]

\[
= \left\{ \left( I_0 \times \prod_{j=0}^{M} \alpha_{ij} \times I^\infty \right) \cap \left( \prod_{j=0}^{M} \alpha_{kj} \times I^\infty \right) : \{i_j\}_{j=0}^{M} \text{ and } \{k_j\}_{j=0}^{M} \text{ are finite sequences of members of } \{1, \ldots, n\} \text{ of length } M + 1 \right\}.
\]

If \( v = \prod_{j=0}^{M} \alpha_{ij} \times I^\infty \in \mathcal{V} \) and \( w = \prod_{j=0}^{M} \alpha_{kj} \times I^\infty \in \mathcal{V} \), then

\[
\sigma^{-1}(v) \cap w = \left( I_0 \times \prod_{j=0}^{M} \alpha_{ij} \times I^\infty \right) \cap \left( \prod_{j=0}^{M} \alpha_{kj} \times I^\infty \right)
\]

\[
= \alpha_{k_0} \times (\alpha_{k_1} \cap \alpha_{i_0}) \times \ldots \times (\alpha_{k_M} \cap \alpha_{i_{M-1}}) \times \alpha_{i_M} \times I^\infty
\]

\[
\subset \alpha_{k_0} \times \alpha_{k_1} \times \ldots \times \alpha_{k_M} \times \alpha_{i_M} \times I^\infty.
\]

Hence, the collection \( \sigma^{-1}(\mathcal{V}) \cup \mathcal{V} \) refines the collection \( \mathcal{U} \). Then \((\sigma^{-1}(\mathcal{V}) \cup \mathcal{V})^* \) refines the collection \( \mathcal{U}^* \), so \( \mathcal{U}^* < (\sigma^{-1}(\mathcal{V}) \cup \mathcal{V})^* \), and \( N(\mathcal{G}, \mathcal{U}^*) \leq N(\mathcal{G}, (\sigma^{-1}(\mathcal{V}) \cup \mathcal{V})^*) \).

But \( \mathcal{U} \) also refines \( \sigma^{-1}(\mathcal{V}) \cup \mathcal{V} \), and so \( \mathcal{U}^* \) refines \( (\sigma^{-1}(\mathcal{V}) \cup \mathcal{V})^* \). Thus, \( N(\mathcal{G}, \mathcal{U}^*) \geq N(\mathcal{G}, (\sigma^{-1}(\mathcal{V}) \cup \mathcal{V})^*) \). Then \( N(\mathcal{G}, \mathcal{U}^*) = N(\mathcal{G}, (\sigma^{-1}(\mathcal{V}) \cup \mathcal{V})^*) \).

Note that \( N(\mathcal{G}, \mathcal{U}^*) = N(\sigma_{i=1}^{M+1} G, \alpha^{M+2}) \).

We can conclude: By similar arguments, for each positive integer \( l \),

\[
N(\mathcal{G}, (\nu_{i=0}^l \sigma^{-1} \mathcal{V})^*) = N(\mathcal{G}, \alpha^{M+l+1} \times I^\infty) = N(\sigma_{i=1}^{M+l+1} G, \alpha^{M+l+1}).
\]

Now \( \mathcal{V} = \alpha^{M+1} \times I^\infty \), and for \( l \) a positive integer, \( N(\mathcal{G}, (\nu_{i=0}^l \sigma^{-1} \mathcal{V})^*) = N(\sigma_{i=1}^{M+l} G, \alpha^{M+l+1}) \).

Then \( \log(N(\mathcal{G}, (\nu_{i=0}^l \sigma^{-1} \mathcal{V})^*)) = \log(N(\sigma_{i=1}^{M+l} G, \alpha^{M+l+1})) \). It follows that

\[
h(\sigma, \alpha^{M+1} \times I^\infty) = \lim_{l \to \infty} \frac{\log(N(\mathcal{G}, (\nu_{i=0}^l \sigma^{-1} \mathcal{V})^*))}{l} = \lim_{l \to \infty} \frac{\log(N(\sigma_{i=1}^{M+l} G, \alpha^{M+l+1}))}{l},
\]

while

\[
\text{ent}(G, \alpha) = \lim_{l \to \infty} \frac{\log(N(\sigma_{i=1}^{M+l} G, \alpha^{M+l+1}))}{l}.
\]

For each positive integer \( k \), let \( \log(N(\sigma_{i=1}^{M+k} G, \alpha^{M+k+1})) = a_k \).

Then \( \log(N(\sigma_{i=1}^{M+l} G, \alpha^{M+l+1})) = a_l \) and \( \log(N(\sigma_{i=1}^{M+l+1} G, \alpha^{M+l+2})) = a_{M+l} \). Furthermore, \( a_l \leq a_{M+l} \leq a_M + a_l \). (This is because \( N(\sigma_{i=1}^{M+l} G, \alpha^{M+l+1}) \leq N(\sigma_{i=1}^{M+l} G, \alpha^{M+l+1})N(\sigma_{i=1}^{M+1} G, \alpha^{M+1}) \). Thus \( \frac{a_l}{l} \leq \frac{a_{M+l}}{l} \leq \frac{a_M}{l} + \frac{a_l}{l} \). By Lemma 1, \( \lim_{l \to \infty} \frac{a_l}{l} \) exists, and

\[
\lim_{l \to \infty} \frac{a_l}{l} \leq \lim_{l \to \infty} \frac{a_{M+l}}{l} \leq \lim_{l \to \infty} \frac{a_{M+l}}{l} = \lim_{l \to \infty} \frac{a_{M+l}}{l} = \lim_{l \to \infty} \frac{a_l}{l}.
\]
It follows that
\[
\lim_{l \to \infty} \frac{\log(N(\ast_{l=1}^k G, \alpha^{l+1}))}{l} = \lim_{l \to \infty} \frac{\log(N(\ast_{l=1}^{M+l} G, \alpha^{M+l+1}))}{l},
\]
and thus, \( \text{ent}(G, \alpha) = h(\alpha, (\alpha^{M+1} \times I^\infty)^*) \) for each positive integer \( M \).

**Theorem 9.** Suppose \( G \) is a closed subset of \( I_0 \times I_1 \), \( G = \ast_{l=1}^\omega G \), and \( \sigma(G) = G \). If \( \alpha = \{\alpha_1, \ldots, \alpha_n\} \) is a minimal open cover of \( I_0 \) by open intervals, then \( \text{ent}(G) = h(\sigma) \).

**Proof.** Since each open cover of \( G \) is refined by the grid cover \( \alpha^{M+1} \times I^\infty \) for some \( M \) and minimal open cover \( \alpha \) by intervals of \( I_0 \), the result follows.

**Remark.** Theorem 9 overlaps with Theorem 3.1 of [KT].

**Theorem 10.** If \( f : I \to I \) is a continuous function and \( G \) is the graph of \( f^{-1} \), then \( h(f) = \text{ent}(G) \).

**Proof.** This follows from Theorem 8 and Bowen’s result that \( h(f) = h(\sigma) \) in [Bo2].

5. **Topological entropy of closed subsets of \([0, 1]^{N+1}\)**

If \( X \) is a compact metric space and \( f : X \to X \) is continuous, then for each positive integer \( k \), \( h(f^k) = kh(f) \) (See [W Theorem 7.10]). This well-known result for continuous mappings does not hold for an upper semicontinuous mappings \( F : X \to 2^X \).

Suppose \( F : X \to 2^X \) is upper semicontinuous. We can define \( F^2 : X \to 2^X \) by \( F^2(x) = \bigcup_{y \in F(x)} F(y) \). Then, inductively, for \( n > 2 \), \( F^n(x) = \bigcup_{y \in F^{n-1}(x)} F(y) \). In [KT], using Bowen’s ideas of \((n, \epsilon)\)-separated and \((n, \epsilon)\)-spanning, they define the topological entropy \( h(F) \). They show that, \( h(\sigma) = h(F) \) (where \( \sigma \) denotes the shift \( \sigma : \lim(X, f) \to \lim(X, f) \)). Hence if \( X = [0, 1] \), \( G \) denotes the graph of \( F \) (and thus \( G \) is a closed subset of \([0, 1]^{17}\)), \( h(F) = h(\sigma) = \text{ent}(G^{-1}) = \text{ent}(G) \). (See Theorem 7 and Theorem 9 of the previous section). In [KT], the following theorem (re-phrased a bit for our setting) is proved:

**Theorem 11.** [KT Theorem 5.4] Suppose \( X \) is a compact metric space, \( F : X \to 2^X \) is upper semicontinuous, and \( k \in \mathbb{N} \). Then
\[
h(F) \leq h(F^k) \leq kh(F).
\]

It is not the case \( h(F^k) = kh(F) \) always as the following example (from [KT]) shows.

**Example 5.** [KT Example 5.7] Define \( F : I \to 2^I \) by
\[
F(x) = \begin{cases} 
{x + \frac{1}{2} - x}, & x \leq \frac{1}{2} \\
{x - \frac{1}{2}}, & x \geq \frac{1}{2}
\end{cases}
\]
Then \( F^2 \neq F \), but \( h(F^2) = h(F) = \log 2 \).
However, an analogous result to
\[ h(f^k) = kh(f) \]
holds for Mahavier products. Note that if \( G = \Gamma(F^{-1}) \), where \( F : X \rightarrow 2^X \) is upper semicontinuous, then \( \Gamma(F^k) = \pi_{\{k,0\}}(k \ast_{i=1}^k G) \). Thus, only the beginning input and final output are kept, while the “intermediate” values are forgotten, when iterating a function in the usual way. This does not happen with the Mahavier product: all possibilities are retained.

The purpose of this section is to show that if \( G \) is a closed subset of \( I^2 \), and \( k \) is a positive integer, then
\[ \text{ent}(\star_{i=1}^k G) = k \text{ent}(G). \]

Before we can show this, we must define and explore \( \text{ent}(H) \) for \( H \) a closed subset of \( I^{N+1} \), for \( N \) a positive integer.

1. Let \( \alpha = \{\alpha_1, \ldots, \alpha_n\} \) be a minimal open cover of \( I_0 \) by intervals. Let
\[ \beta = \{\Pi_{j=0}^N \alpha_{k_j} : k_j \in (1,n), 0 \leq j \leq N\}. \]

Hence, \( \beta \) is the grid cover of \( \Pi_{i=0}^N I_i \) determined by \( \alpha \), and \( \beta \) therefore covers \( H \). Since \( N^\ast(\beta) = n(N+1) := n_\beta \), we can list the members of \( \beta = \{\beta_1, \beta_2, \ldots, \beta_{n_\beta}\} \). For each positive integer \( m > 1 \), let
\[ \beta^m = \{\Pi_{j=0}^{m-1} \beta_{k_j} : k_j \in (1,n_\beta), 0 \leq j \leq m - 1\}. \]

Then \( \beta \ast \beta = \{\beta_i \ast \beta_j : 1 \leq i, j \leq n_\beta\} \) is a cover of \( H \ast H \) by open subsets of \( \Pi_{i=0}^N I_i \), and \( N(H \ast H, \beta \ast \beta) \leq n_\beta^2 \). Note that \( \beta \ast \beta \) refines \( \alpha^{2N+1} \) and \( \beta \ast \beta \) is refined by \( \alpha^{2N+1} \), so \( N(H \ast H, \beta \ast \beta) = N(H \ast H, \alpha^{2N+1}) \).

2. We can continue this process for each \( m \in \mathbb{N} \):
\[ \star_{i=1}^m \beta = \{\star_{j=1}^m \beta_{k_j} : k_j \in (1,n_\beta), 1 \leq j \leq m\} \]
is an open cover of \( \star_{i=1}^m H \) and \( N(\star_{i=1}^m H, \star_{i=1}^m \beta) \leq n_\beta^m \). Again, a minimal subcover of \( \star_{i=1}^m H \) by elements of \( \star_{i=1}^m \beta \) has the same number of elements as a minimal subcover of \( \star_{i=1}^m H \) by elements of \( \alpha^{mN+1} \). Since using the cover \( \star_{i=1}^m \beta \) is sometimes more convenient, we continue to use both covers. Without loss of generality, we may assume that a minimal subcover (in both \( \alpha^{mN+1} \) and \( \star_{i=1}^m \beta \)) consists of sets of the form \( \Pi_{j=0}^N \alpha_{k_j} \), where each \( k_j \in (1,n) \).
(4) Suppose $H$ is a closed subset of $\prod_{i=0}^{N} I_i$. Let $\mathbf{H} = \star_{i=1}^{\infty} H$.

- For each positive integer $m$, $0 \leq N(\star_{i=1}^{m} H, \alpha^{mN+1}) = N(\star_{i=1}^{m} H, \star_{i=1}^{m} \beta) \leq m^{N+1}$, and $0 \leq N(\star_{i=1}^{m} H, \alpha^{mN+1}) = N(\star_{i=1}^{m} H, \star_{i=1}^{m} \beta) \leq m \beta$. If $\mathbf{H} \neq \emptyset$, $0 < N(\star_{i=1}^{m} H, \star_{i=1}^{m} \beta)$.

- If $\mathbf{H} \neq \emptyset$, $1 = N(\star_{i=1}^{m} H, \star_{i=1}^{m} \beta)$ if and only if there is a sequence $\alpha_{j_0}, \alpha_{j_1}, \ldots, \alpha_{j_m}$ (with each $1 \leq j_i \leq n$) such that $\mathbf{H} \subset (\alpha_{j_0} \times \ldots \times \alpha_{j_m}) \times I^\infty$.

- As before, if $\alpha, \gamma$ are both minimal open covers of $I_0$ by open intervals and $\alpha < \gamma$, then for each $m > 0$, $N(\star_{i=1}^{m} H, \alpha^{mN+1}) \leq N(\star_{i=1}^{m} H, \gamma^{mN+1})$.

- As before, if $\alpha, \gamma$ are both minimal open covers of $I_0$ by open intervals, then for each $m > 0$, $\alpha^{mN+1} \cup \gamma^{mN+1} = (\alpha \cup \gamma)^{mN+1}$, and $N(\star_{i=1}^{m} H, \alpha^{mN+1} \cup \gamma^{mN+1}) \leq N(\star_{i=1}^{m} H, \alpha^{mN+1})N(\star_{i=1}^{m} H, \gamma^{mN+1})$.

- If $K$ is a closed subset of $H \subset \prod_{i=0}^{N} I_i$, $m$ is a positive integer, and $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ is a minimal open cover of $I_0$ by open intervals, then $N(\star_{i=1}^{m} K, \alpha^{mN+1}) \leq N(\star_{i=1}^{m} H, \alpha^{mN+1})$.

- Suppose $l$ and $m$ are positive integers. Then $\alpha^{lN+1}$ is a grid cover of $\star_{i=1}^{l} H$, $\alpha^{lN+mN+1}$ is a grid cover of $\star_{i=1}^{m} H$, and $\alpha^{lN+1} \times \prod_{i=1}^{m} l N(I_i)$ is an open cover of $\star_{i=1}^{m} H$. Then $N(\star_{i=1}^{l} H, \alpha^{lN+1}) \leq N(\star_{i=1}^{m} H, \alpha^{mN+1} \times \prod_{i=1}^{l} N(I_i)) \leq N(\star_{i=1}^{l} H, \alpha^{lN+mN+1})$.

(5) If $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ is a minimal open cover of $I_0$ by intervals, $H$ is a closed subset of $\prod_{i=0}^{N} I_i$ and $\mathbf{H} \neq \emptyset$, then $\lim_{m \to \infty} \frac{\log N(\star_{i=1}^{m} H, \alpha^{mN+1})}{m}$ exists.

Proof. Let $a_m = \log N(\star_{i=1}^{m} H, \alpha^{mN+1}) = \log N(\star_{i=1}^{m} H, \star_{i=1}^{m} \beta)$ for each $m \in N$. Then $1 \leq N(\star_{i=1}^{m} H, \alpha^{mN+1}) \leq m^{N+1}$, so $0 \leq a_m = \log N(\star_{i=1}^{m} H, \alpha^{mN+1}) \leq (mN + 1) \log n$.

By Lemma[1], it suffices to show that $a_{m+k} \leq a_m + a_k$. We have $\alpha^{m+k+1} \subset \star_{i=1}^{m} \beta \star_{i=1}^{k} \beta = \star_{i=1}^{m+k} \beta$, so $\alpha^{m+k+1}$ refines $\star_{i=1}^{m+k} \beta$. Then $N(\star_{i=1}^{m+k} H, \alpha^{m+kN+1}) = N(\star_{i=1}^{m+k} H, \star_{i=1}^{m+k} \beta)$. Since $N(\star_{i=1}^{m+k} H, \alpha^{m+kN+1})$ is the cardinality of a minimal subcover of $\star_{i=1}^{m+k} H$ in $\star_{i=1}^{m+k} \beta$, and $N(\star_{i=1}^{m+k} H, \alpha^{kN+1})$ is the cardinality of a minimal subcover of $\star_{i=1}^{k} H$ in $\star_{i=1}^{k} \beta$, $\star_{i=1}^{m+k} \beta$ is a cover of $\star_{i=1}^{m+k} H$ in $\prod_{i=0}^{m+k} N(I_i)$.

Thus, $N(\star_{i=1}^{m+k} H, \alpha^{m+kN+1}) = N(\star_{i=1}^{m+k} H, \star_{i=1}^{m+k} \beta) = N(\star_{i=1}^{m+k} H, \star_{i=1}^{m+k} \beta) \leq N(\star_{i=1}^{m} H, \star_{i=1}^{m+k} \beta)N(\star_{i=1}^{k} H, \star_{i=1}^{k} \beta)$, and we have $a_{m+k} = \log(N(\star_{i=1}^{m+k} H, \alpha^{m+kN+1})) \leq \log(N(\star_{i=1}^{m} H, \star_{i=1}^{m+k} \beta)N(\star_{i=1}^{k} H, \star_{i=1}^{k} \beta)) = \log(N(\star_{i=1}^{m} H, \star_{i=1}^{m+k} \beta) + \log(N(\star_{i=1}^{k} H, \star_{i=1}^{k} \beta)) = a_m + a_k.$
Proof. The proof is similar to that of Theorem 12, so we omit it.

Theorem 12. Let $H$ be a closed subset of $\prod_{i=0}^N I_i$, and $H^{-1} = \{(x_0, x_1, \ldots, x_N) \in H \}$. Then $\text{ent}(H) = \text{ent}(H^{-1})$.

Proof. The proof is similar to that of Theorem 12, so we omit it.

Proposition 13. If $\mathcal{V}$ is an open cover (in $I^\infty$) of $\ast_{i=1}^\infty H$, with $H$ a closed subset of $\prod_{i=0}^N I_i$, then $\sigma^{-N}(\mathcal{V}) := \{\sigma^{-N}(v) : v \in \mathcal{V} \} = \{\prod_{i=0}^{N-1} I_i \times v : v \in \mathcal{V} \}$ is also an open cover (in $I^\infty$) of $\ast_{i=1}^\infty H$.

Proof. The proof is similar to that of Proposition 12, so we omit it.

Suppose $H$ is a closed subset of $\prod_{i=0}^N I_i$ and $H = \ast_{i=1}^\infty H$. If $\mathcal{U}$ is an open cover of $I^\infty$, let $\mathcal{U}^* = \{u \cap H : u \in \mathcal{U} \}$ denote the corresponding open cover of $H$ by open sets in $H$.

Theorem 13. Suppose $H$ is a closed subset of $\prod_{i=0}^N I_i$, $H = \ast_{i=1}^\infty H \neq \emptyset$, and $\sigma^N(H) = H$. Suppose $M$ is a positive integer. If $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ is a minimal open cover of $I_0$ by open intervals, then $\text{ent}(H, \alpha) = h(\sigma^N, (\alpha^{M+1} \times I^\infty)^*)$.

Proof. This proof is similar to the proof of Theorem 8 but a little more difficult technically. Let $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ be a minimal open cover of $I_0$ by intervals.

Fix the positive integer $M$. Let

$$\mathcal{V} = \left\{ \prod_{j=0}^{MN} \alpha_{ij} \times I^\infty : \alpha_{ij} \in \alpha \quad \text{and} \quad \left( \prod_{j=0}^{MN} \alpha_{ij} \times I^\infty \right) \cap H \neq \emptyset \right\},$$

and let

$$\mathcal{U} = \left\{ \alpha_{k_0} \times \alpha_{k_1} \times \ldots \times \alpha_{k_{(M+1)N}} \times I^\infty : \{k_j\}_{j=0}^{(M+1)N} \text{ is a sequence of members of } \{1, \ldots, n\} \text{ of length } (M+1)N + 1 \right\}.$$
For $v = \prod_{j=0}^{MN} \alpha_{i_j} \times I^\infty \in V$, $\sigma^{-N}(v) = \prod_{i=0}^{N-1} I_i \times \prod_{j=0}^{MN} \alpha_{i_j} \times I^\infty$. Then

$$\sigma^{-N}(V) \vee V = \left\{ \sigma^{-N}(v) \cap w : v = \prod_{j=0}^{MN} \alpha_{i_j} \times I^\infty, w = \prod_{j=0}^{MN} \alpha_{k_j} \times I^\infty \in V \right\}$$

$$= \left\{ \left( \prod_{i=0}^{N-1} I_i \times \prod_{j=0}^{MN} \alpha_{i_j} \times I^\infty \right) \cap \left( \prod_{j=0}^{MN} \alpha_{k_j} \times I^\infty \right) : \{i_j\}_{j=0}^{MN} \text{ and } \{k_j\}_{j=0}^{MN} \right\}$$

are finite sequences of members of $\{1, \ldots, n\}$ of length $(M+1)N + 1$.

If $v = \prod_{j=0}^{MN} \alpha_{i_j} \times I^\infty \in V$ and $w = \prod_{j=0}^{MN} \alpha_{k_j} \times I^\infty \in V$, then

$$\sigma^{-N}(v) \cap w = \left( \prod_{i=0}^{N-1} I_i \times \prod_{j=0}^{MN} \alpha_{i_j} \times I^\infty \right) \cap \left( \prod_{j=0}^{MN} \alpha_{k_j} \times I^\infty \right)$$

$$= \prod_{i=0}^{N-1} \alpha_{k_i} \times (\alpha_{k_0} \cap \alpha_{i_0}) \times \ldots \times (\alpha_{k_{MN}} \cap \alpha_{i_{(M-1)N}}) \times \prod_{l=0}^{N(M-1)+1} \alpha_{i_l} \times I^\infty$$

$$\subset \alpha_{k_0} \alpha_{k_1} \ldots \alpha_{k_{MN}} \alpha_{i_{(M-1)N+1}} \ldots \alpha_{i_{(M-1)N+1}} \times I^\infty.$$

Hence, the collection $\sigma^{-1}(V) \vee V$ refines the collection $\mathcal{U}$. Then $(\sigma^{-N}(V) \vee V)^*$ refines the collection $\mathcal{U}^*$, so $\mathcal{U}^* < (\sigma^{-N}(V) \vee V)^*$, and $N(\mathbf{H}, \mathcal{U}^*) \leq N(\mathbf{H}, (\sigma^{-N}(V) \vee V)^*)$.

But $\mathcal{U}$ also refines $\sigma^{-N}(V) \vee V$, and so $\mathcal{U}^*$ refines $(\sigma^{-N}(V) \vee V)^*$. Thus, $N(\mathbf{H}, \mathcal{U}^*) \geq N(\mathbf{H}, (\sigma^{-N}(V) \vee V)^*)$. Then $N(\mathbf{H}, \mathcal{U}^*) = N(\mathbf{H}, (\sigma^{-N}(V) \vee V)^*)$.

Note that $N(\mathbf{H}, \mathcal{U}^*) = N_{*_{i=1}^{(M+1)N}} \mathbf{H}, \alpha^{MN+1}$.

We can continue: By similar arguments, for each positive integer $l$, $N(\mathbf{H}, (\forall_{i=0}^{l} \sigma^{-iN}(V)^*)) = N_{*_{i=1}^{(M+l)N+1}} \mathbf{H}, \alpha^{MN+1} \times I^\infty = N_{*_{i=1}^{(M+l)N+1}} \mathbf{H}, \alpha^{MN+1}$. Now $V = \alpha^{MN+1} \times I^\infty$, and for $l$ a positive integer, $N(\mathbf{H}, (\forall_{i=0}^{l} \sigma^{-iN}(V)^*)) = N_{*_{i=1}^{(M+l+1)N+1}} \mathbf{H}, \alpha^{MN+1}$. Then $\log(N(\mathbf{H}, (\forall_{i=0}^{l} \sigma^{-iN}(V)^*)) = \log(N_{*_{i=1}^{(M+l)N+1}} \mathbf{H}, \alpha^{MN+1})$. It follows that

$$h(\sigma^N, \alpha^{MN+1} \times I^\infty) = \lim_{l \to \infty} \frac{\log(N(\mathbf{H}, (\forall_{i=0}^{l} \sigma^{-iN}(V)^*)))}{l}$$

$$= \lim_{l \to \infty} \frac{\log(N_{*_{i=1}^{(M+l)N+1}} \mathbf{H}, \alpha^{MN+1}))}{l},$$

while

$$\text{ent}(\mathbf{H}, \alpha) = \lim_{l \to \infty} \frac{\log(N_{*_{i=1}^{(M+l)N+1}} \mathbf{H}, \alpha^{MN+1}))}{l}.$$
Suppose Example 6. The dynamics of the shift map $\sigma$. By Lemma [1] $\lim_{l \to \infty} \frac{a_l}{l}$ exists, and

$$\lim_{l \to \infty} \frac{a_l}{l} \leq \lim_{l \to \infty} \frac{a_{l+M}}{l} \leq \lim_{l \to \infty} \left(\frac{a_l}{l} + \frac{a_{M}}{l}\right) = \lim_{l \to \infty} \frac{a_l}{l} + \lim_{l \to \infty} \frac{a_{M}}{l} = \lim_{l \to \infty} \frac{a_l}{l}.$$ 

It follows that

$$\lim_{l \to \infty} \frac{\log(N(\bigcup_{i=1}^{l} H, \alpha^{(N+1)}))}{l} = \lim_{l \to \infty} \frac{\log(N(\bigcup_{i=1}^{M+l} H, \alpha^{(M+l)N+1}))}{l},$$

and thus, $\text{ent}(G, \alpha) = h(\sigma^{N}, (\alpha^{MN+1} \times I^{\infty})^*)$. 

**Theorem 14.** Suppose $H$ is a closed subset of $\Pi_{i=0}^{N} I_i$, $H = \bigstar_{i=1}^{\infty} I_i$, and $\sigma^{N}(H) = H$. If $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ is a minimal open cover of $I_0$ by open intervals, then $\text{ent}(H) = h(\sigma^{N})$.

**Proof.** Since each open cover of $H$ is refined by the grid cover $\alpha^{MN+1} \times I^{\infty}$ for some $M$ and minimal open cover $\alpha$ by intervals of $I_0$, the result follows.

**Theorem 15.** Suppose $G$ is a closed subset of $I_0 \times I_1$, $G = \bigstar_{i=1}^{\infty} G \neq \emptyset$, and $\sigma(G) = G$. Then $\text{ent}(\bigstar_{i=1}^{k} G) = k \text{ent}(G)$.

**Proof.** Suppose $k$ is a positive integer. Let $G$ be a closed subset of $I_0 \times I_1$ such that $G = \bigstar_{i=1}^{\infty} G \neq \emptyset$, and $\sigma(G) = G$, and let $H = \bigstar_{i=1}^{k} G \subset \Pi_{i=0}^{k} I_i$. Let $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ be a minimal open cover of $I_0$ by open intervals. Then for each positive integer $m$, $\bigstar_{i=1}^{m} H = \bigstar_{i=1}^{m} (\bigstar_{i=1}^{k} G) = \bigstar_{i=1}^{mk} G$. Hence, $N(\bigstar_{i=1}^{mk} G, \alpha^{mk+1}) = N(\bigstar_{i=1}^{m} H, \alpha^{mk+1})$.

Then

$$\text{ent}(G, \alpha) = \lim_{m \to \infty} \frac{1}{m} \log(N(\bigstar_{i=1}^{m} G, \alpha^{m+1})) = \lim_{m \to \infty} \frac{1}{mk} \log(N(\bigstar_{i=1}^{mk} G, \alpha^{mk+1}))$$

$$= \frac{1}{k} \lim_{m \to \infty} \frac{1}{m} \log(N(\bigstar_{i=1}^{mk} G, \alpha^{mk+1})) = \frac{1}{k} \text{ent}(\bigstar_{i=1}^{mk} G, \alpha^{mk+1}).$$

Thus, for every minimal cover $\alpha$ by open intervals of $I_0$,

$$k \text{ent}(G, \alpha) = \text{ent}(\bigstar_{i=1}^{mk} G, \alpha^{mk+1}).$$

The result follows.

### 6. Computation and Application of Topological Entropy

In this section we compute the topological entropy for some closed subsets $G$ of $I^2$. While most authors prefer the Bowen approach (using $(n, \epsilon)$-spanning sets and $(n, \epsilon)$-separating sets), we find that “counting the boxes” (i.e., computing $N(\bigstar_{i=1}^{m} G, \alpha^{m+1})$) for $G$ a closed subset of $I^2$ and $\alpha$ a minimal open cover of $I$ by intervals) is often easy and natural. We also explore the relationship between $G$ and $G = \bigstar_{i=1}^{\infty} G$, and investigate the interaction of the topology of $G$ and the dynamics of the shift map $\sigma$.

**Example 6.** Suppose $G = I^2$. Then $\text{ent}(G) = \infty$.  

TOPOLOGICAL ENTROPY ON CLOSED SETS IN $[0, 1]^2$

Proof. Suppose $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ is a minimal open cover of $I_0$ by open intervals. Then for each positive integer $m$, $N(\star_{i=1}^m G, \alpha^{m+1}) = n + 1$. Thus,

$$\text{ent}(G, \alpha^2) = \lim_{m \to \infty} \frac{1}{m} \log N(\star_{i=1}^m G, \alpha^{m+1}) = \lim_{m \to \infty} \frac{1}{m} \log n + 1 = \lim_{m \to \infty} \frac{m + 1}{m} \log n = \log n.$$

Then

$$\sup_{\alpha} \text{ent}(G, \alpha^2) = \sup_{\alpha} \log n = \infty.$$  

□

Remark While we computed the entropy of Example 6 using our boxes, that $\text{ent}(G) = \infty$ follows from [KT, Theorem 7.1].

Example 7. Let $G$ denote the union of the diagonal from $(0, 0)$ to $(1, 1)$ and two points where $(x, y)$ is an arbitrary point in $I \times I$ such that $x \neq y$ and the second point is $(y, x)$. Then, $\text{ent}(G) = \log 2$.

Proof. The set $G$ is the union of the diagonal and two points, $G \star G$ is the union of the diagonal from $(0, 0, 0)$ to $(1, 1, 1)$ and six points, $\star_{i=1}^m G$ is the union of the diagonal and $2^{m+1} - 2$ different points that don’t lie on the diagonal. Namely, points in $\star_{i=1}^m G$ that are not on the diagonal have $m + 1$ coordinates and every coordinate is either $x$ or $y$, so there are $2^{m+1} - 2$ different points which do not lie on diagonal. So, for intervals in $\alpha$ sufficiently small we have

$N \left( G, \alpha^2 \right) = n + 2$, $N \left( G \star G, \alpha^3 \right) = n + 6$, …, $N \left( \star_{i=1}^m G, \alpha^{m+1} \right) = n + 2^{m+1} - 2$.

Hence,

$$\text{ent}(G, \alpha) = \lim_{m \to \infty} \frac{1}{m} \log (n + 2^{m+1} - 2) = \log 2.$$  

Therefore, $\text{ent}(G) = \log 2$. □

In the example above, note that the topological entropy being $\log 2$ is completely determined by the four point subset $G' = \{(x, y), (y, x), (x, x), (y, y)\}$, i.e., $\text{ent}(G') = \log 2$. In [KT], it was shown (Proposition 6.4) that it can never happen for a continuous function $f : X \to X$ on a compact metric space that the
entropy $h(f)$ is determined by a finite set. They also showed (Proposition 6.1) that if $a \neq b$ in $X$ compact metric, $F : X \to 2^X$ is upper semicontinuous, and $F(a) \supseteq \{a, b\}, F(b) \supseteq \{a, b\}$, then $h(F) \geq \log 2$.

In the next example we find, for each positive integer $n > 1$, finite sets $G_n$ such that $\text{ent } G_n = \log n$. Of course, it is well known that for any positive number $r$, there are continuous functions $f$ on the interval such that $h(f) = r$.

**Example 8.** For each $n \in \mathbb{N}$ there exists set $G_n \subseteq I \times I$ such that $\text{ent } (G_n) = \log n$.

**Proof.** Let $n \in \mathbb{N}$ be arbitrary. We define $G_n$ in following way:

$$G_n = \left\{ \left( \frac{k}{n-1}, \frac{l}{n-1} \right) : k, l \in \{0, 1, \ldots, n-1\} \right\}.$$  

Set $G_n$ is union of $n^2$ points, $G_n \ast G_n$ is union of $n^3$ points, $\star_{i=1}^{m} G_n$ is union of $n^{m+1}$ different points so for intervals in $\alpha$ sufficiently small we have:

$$N \left( G, \alpha^2 \right) = n^2, N \left( G \ast G, \alpha^3 \right) = n^3, \ldots, N \left( \star_{i=1}^{m} G, \alpha^{m+1} \right) = n^{m+1}.$$  

We have

$$\text{ent} \left( G, \alpha \right) = \lim_{m \to \infty} \frac{1}{m} \log \left( n^{m+1} \right) = \log n.$$  

Therefore, $\text{ent} \left( G \right) = \log n$. \hfill \(\square\)

The following example is discussed in [KT]. They used Bowen’s result (Theorem 6) to show that the entropy is 0. Since the proof of Bowen’s theorem is nontrivial, quite delicate, and uses notation that is not defined and probably out of date, we calculate the entropy directly. We explore this example in more depth later, as it turns out to be quite interesting, and “just barely” has entropy 0.

**Example 9.** (The Triangle Example) Let $G$ to be the set $\{(x, y) \in I \times I : x \geq y\}$. Then $\text{ent } G = 0$.

**Proof.** Let $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ be an open cover of $I$ where $n \in \mathbb{N}$ is arbitrary and let $m \in \mathbb{N}, m \geq 2$ be arbitrary. We have $\star_{i=1}^{m} G = \{(x_0, \ldots, x_m) : x_i \geq x_{i+1}, 0 \leq i \leq m-1\}$. The elements of cover of $\star_{i=1}^{m} G$ are of the form $\alpha_{i_1} \times \alpha_{i_2} \times \ldots \times \alpha_{i_{m+1}}$, where $i_j \in \{1, 2, \ldots, n\}$. From the definition of the set $\star_{i=1}^{m} G$ it follows that $i_j \geq i_{j'}$.  

![Figure 3](image-url)
when \( j' \geq j \). Therefore, calculating the number \( N(\star_{i=1}^m G, \alpha^{m+1}) \) is equivalent to calculating the number of \((m+1)\)-tuples \((i_1, i_2, \ldots, i_{m+1})\) such that \( i_j \geq i_{j'} \), whenever \( j' \geq j \), where \( i_j \in \{1, 2, \ldots, n\} \). That number is equal to the binomial coefficient \( \binom{m+1+n-1}{m+1} = \binom{m+n}{m+1} = \binom{m+n}{n-1} \). Therefore, \( N(\star_{i=1}^m G, \alpha^{m+1}) = \binom{m+n}{n-1} \).

Now we have:

\[
\text{ent}(G, \alpha) = \lim_{m \to \infty} \frac{1}{m} \log \left( \frac{n^m + n^{m+1}}{n^m} \right) = \lim_{m \to \infty} \frac{n-1}{m} \log(m + n) = 0.
\]

Therefore, \( \text{ent}(G, \alpha) = 0 \) and hence \( \text{ent}(G) = 0 \). \[\square\]

The two following examples are immediate consequences of the previous example. The next three examples were considered by Ingram in [16].

**Example 10.** [17, Example 2.14] Let \( G = \{(x, x) \mid x \in I\} \cup \{(1) \times I\} \). Then \( \text{ent}(G) = 0 \).

**Example 11.** [17, Example 2.2] Let \( G = \{(0) \times [0, 1]\} \cup ([0, 1] \times \{1\}) \). Then, \( \text{ent}(G) = 0 \).

**Example 12.** [17, Example 2.3] Let \( G = \{(0) \times [0, 1]\} \cup ([0, 1] \times \{1\}) \). Then \( \text{ent}(G) = \infty \).

**Proof.** Let \( L_1 = \{0\} \times [0, 1] \) and \( L_2 = [0, 1] \times \{0\} \). For arbitrary \( m \in \mathbb{N} \), \( \star_{i=1}^m G \) contains \( L_1 \ast L_2 \ast \ldots \ast L_1 \ast L_2 \) if \( m \) is even and \( L_1 \ast L_2 \ast \ldots \ast L_2 \ast L_1 \) if \( m \) is odd. Either way, \( N(\star_{i=1}^m G, \alpha^{m+1}) > n^{m+1} \). Therefore,

\[
\text{ent}(G, \alpha) \geq \lim_{m \to \infty} \frac{1}{m} \log \left( n^{m+1} \right) = \frac{1}{2} \log n.
\]

and it follows \( \text{ent}(G) = \infty \). \[\square\]

The Maribor Monster Example below has been studied by several researchers, as it has quite interesting topology. (See [BCMM1], [16], [KN], for example.) The continuum that forms the inverse limit is a \( \lambda \)-dendroid that is both \( \frac{1}{2} \)-indecomposable and hereditarily decomposable. The dynamical behavior of \( \sigma \) on the inverse limit is a bit easier to understand. There is one invariant arc \( K = \{(x, 1-x, x, 1-x, \ldots) \} \):
that repels all other points, and there is an attracting invariant Cantor set

Example 13. (Maribor Monster Example) Let \( G = (I \times \{0\}) \cup \{(x, 1-x) \mid x \in I\} \).

\[
\text{Then } \text{ent}(G) = \frac{1+\sqrt{5}}{2}, \text{ i.e., the so-called "golden ratio".}
\]

Proof. Let us denote with \( L_1 \) line from \((1,0)\) to \((0,0)\) and with \( L_2 \) line from \((1,0)\) to \((0,1)\). We have \( G = L_1 \cup L_2 \). In arbitrary product \(*_{i=1}^m L_{i_j} \), \(i_j \in \{1,2\}\), let \( i_0 \) be first index such that \( L_{i_0} = L_1 \). Next coordinate has to be 0 and after that we have only zeros and ones such that we cannot have two neighboring ones. If \( L_i = L_2, \forall i \) then the product is arc from \((0,1,0,\ldots)\) to \((1,0,1,\ldots)\). So we have

\[
N \left( *_{i=1}^m G, \alpha^m \right) + N \left( *_{i=1}^{m-1} G, \alpha^{m-1} \right) \leq N \left( *_{i=1}^{m+1} G, \alpha^{m+1} \right)
\]

and hence \( nF_{m+2} \leq N \left( *_{i=1}^m G, \alpha^m \right) \leq n(F_{m+3} - 1) \), where \( F_m \) is \( m \)-th Fibonacci number. Therefore, we have that

\[
\text{ent}(G, \alpha) = \lim_{m \to \infty} \frac{1}{m} \log N \left( *_{i=1}^m G, \alpha^m \right) = \lim_{m \to \infty} \frac{1}{m} \log n(F_{m+3} - 1) = \frac{1+\sqrt{5}}{2}.
\]

and \( \text{ent}(G) = \log \frac{1+\sqrt{5}}{2} \). \( \square \)

*Figure 5. Set \( G \) from Example 13*

Example 14. Let \( a \in \mathbb{N}, a > 1 \) be arbitrary and let \( G_a = \left\{ \left( \frac{k}{a-1}, 0 \right) \mid k \in \{0,\ldots,a-1\} \right\} \cup \left\{ (0, \frac{k}{a-1}) \mid k \in \{1,\ldots,a-1\} \right\} \subseteq I^2 \). Then \( \text{ent}(G_a) = \log \frac{1+\sqrt{1+4a}}{2} \).

Proof. Let us denote number of points in \(*_{i=1}^m G_a\) with \( N_m \).

We prove \( N_m = N_{m-1} + aN_{m-2} \).

Proof is combinatorial: let us observe arbitrary \((m+1)\) – tuple in \(*_{i=1}^m G_a\). If we have 0 on the first coordinate, second can be any number from \( \{0, \frac{1}{a-1}, \frac{2}{a-1}, \ldots, 1\} \) so we can get any \( m \) – tuple. If we have non-zero as first coordinate, second coordinate has to be zero, third can be anything (as above) so we can get any \( (m-1) \) – tuple.
Therefore, we get recurrence relation $N_m = N_{m-1} + a \, N_{m-2}$ with initial values $N_1 = 2a + 1$ and $N_2 = (a + 1)^2 + a$. Solving it using characteristic polynomial $x^2 - x - a = 0$ we get

$$N_m = a_0 \left( \frac{1 + \sqrt{1 + 4a}}{2} \right)^m + b_0 \left( \frac{1 - \sqrt{1 + 4a}}{2} \right)^m$$

where $a_0$ and $b_0$ are positive real numbers obtained from initial values. Therefore, for intervals in $\alpha$ sufficiently small we have

$$N \left( \star_{i=1}^m G, \alpha^{m+1} \right) = N_m$$

and

$$\lim_{m \to \infty} N \left( \star_{i=1}^m G, \alpha^{m+1} \right) = \lim_{m \to \infty} N_m.$$

Now,

$$\text{ent}(G, \alpha) = \lim_{m \to \infty} \frac{\log N_m}{m}.$$ 

By simple calculations we get

$$\text{ent}(G, \alpha) = 1 + \sqrt{1 + 4a}.$$

\[\Box\]

**Example 15.** Let $G = \{ (x_0, x_1) \in I_0 \times I_1 : x_1 \leq x_0^2 \}$ and let $bL = ([0, 1] \times \{0\}) \cup (\{1\} \times [0, 1])$. Then $\text{ent}(G) = \text{ent}(bL) = 0$.

**Proof.** The sets $G$ and $bL$ are subsets of the set $G$ in Triangle Example so it follows from that and Proposition 11 (c). \[\Box\]

Before stating the proposition we give a new notion. If $H$ is closed subset of $[0, 1]^{n+1}$, define $\pi_{(0,n)}$ to be the map from $H$ to $[0, 1] \times [0, 1]$ defined by $\pi_{(0,n)}(x) = (x_0, x_n)$ for $x = (x_0, x_1, \ldots, x_n) \in H$. For $G$ a closed subset of $[0, 1] \times [0, 1]$, and $n$ positive integer, let $G^{0,n} = \pi_{(0,n)}(\star_{i=1}^n G)$. Therefore, $G^{0,n}$ is a closed subset of $[0, 1] \times [0, 1]$. 

\[\Box\]
With \( \lim_{H_d} \) we denote limit with respect to the Hausdorff metric. Let us recall that Hausdorff metric:
\[
H_d(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}.
\]

**Proposition 14.** Let \( G \) be a connected and closed subset of \([0, 1]^2\) such that \( \lim_{H_d} G^{0,n} = bL \) where \( bL \) is from the previous example. Then \( \text{ent}(G) = 0 \).

**Proof.** We have \( \lim_{H_d} G^{0,n} = bL \) i.e. for each \( \epsilon > 0 \) there is positive integer \( n_0 \) such that for every positive integer \( n, n \geq n_0 \) it follows \( H_d(G^{0,n}, bL) < \epsilon \). We divide the proof in several steps:

(i) \( G \) does not contain any point on diagonal not equal to \((0, 0)\) and \((1, 1)\).

Assume the contrary, i.e. if it does contain point \((x, x), x \neq 0, 1\), then \( G^{0,n} \)
also contains that point, for all \( n \in \mathbb{N} \). Then, \( d((x, x), bL) = \left\{ \begin{array}{ll}
x, & x \leq \frac{1}{2} \\ 1 - x, & x > \frac{1}{2} \end{array} \right. \)
but either way it is greater than 0.

Therefore, we get \( H_d(G^{0,n}, bL) \geq d((x, x), bL) > 0, \forall n \in \mathbb{N} \), hence we cannot have \( \lim_{H_d} G^{0,n} = bL \).

(ii) \( \pi_0(G) = [0, 1] \) and \( \pi_1(G) = [0, 1] \).

Suppose \( \pi_0(G) = J_0 \) where \( J_0 \) is a closed and proper subset of \([0, 1]\). Then, \( G^{0,n} \subseteq J_0 \times [0, 1] \subset [0, 1] \times [0, 1] \) for all \( n \in \mathbb{N} \). Since \( bL \) contains \([0, 1] \times \{0\}\), there exists a point \((x, 0) \in bL \setminus (J_0 \times \{0\})\) such that \( d((x, 0), J_0) = d_0 > 0 \)
(because \( J_0 \) is closed). Now we have that \( 0 < d_0 = d((x, 0), J_0 \times [0, 1]) \leq d((x, 0), G^{0,n}) \) and therefore \( H_d(G^{0,n}, bL) \geq d_0 > 0 \). So we get \( \lim_{H_d} G^{0,n} \neq bL \) which is contradiction. Similarly, \( \pi_1(G) = [0, 1] \).

(iii) \( G \) doesn’t contain point \((x_0, y_0)\) above the diagonal.

Suppose that \( G \) contains a point \((x_0, y_0)\) above the diagonal, i.e. \( y_0 > x_0 \).

The set \( G \) is closed, connected and by (ii), \( \pi_0(G) = [0, 1] \) and \( \pi_1(G) = [0, 1] \),
therefore \( G \) intersects the diagonal in some point \((x, x), x \in [0, 1]\). By (i) it follows that \( x \) is equal to 0 or 1.

Suppose that \( x = 0 \). Since \( G \) is connected and closed, it contains a sub-continuum connecting points \((x_0, y_0)\) and \((0, 0)\) which we denote with \( K \).

Since \( K \) is a subcontinuum, for each positive integer \( n \) there exists finite sequence of points \( x_{n-1} < x_{n-2} < \ldots < x_0 < y_0 \) in \([0, 1]\) such that \( (x_{n-1}, x_{n-2}, \ldots, x_0, y_0) \in \mathcal{K}_{n=1} K \). Therefore, since \( K \subseteq G \), we have \( H_d(G^{0,n}, bL) \geq H_d(K^{0,n}, bL) \geq d((x_0, y_0), bL) > 0 \), hence we cannot have \( \lim_{H_d} G^{0,n} = bL \).

If we suppose that \( x = 1 \), we get the contradiction in the same way.

From (i) and (iii) it follows that set \( G \) is under the diagonal, except points \((0, 0)\) and \((1, 1)\) i.e.
\[
(\forall (x, y) \in G \setminus \{(0, 0), (1, 1)\}, \ y < x.) \ (*)
\]

Now we have that \( G \) is a subset of the set in the Triangle Example and therefore \( \text{ent}(G) = 0 \).

\[\square\]

Now we return to the Triangle Example and its properties. Let \( M \) denote the Mahavier product \( \times_{n=1} G \) produced by this example. It was noted in [KT] that \( M \) contains copies of the Hilbert cube. It may be homeomorphic to it.
The dynamics of σ on M are quite easy to describe. If \( x = (x_0, x_1, \ldots) \) is a point in \( M \), then \( x_0 \geq x_1 \geq \cdots \) and, as a consequence, if \( x^* = \inf\{x_i : i \geq 0\} \), then \( x, \sigma(x), \sigma^2(x), \ldots \) converges to the point \( x^* = (x^*, x^*, \ldots) \). And if you look backwards to where points “come from”, so, consider the double-sided inverse limit \( x = (\ldots, x_{-2}, x_{-1}, x_0, x_1, \ldots) \), then if \( x^* = \sup\{x_{-i} : i \geq 0\} \), then \( x, \sigma^{-1}(x), \sigma^{-2}(x), \ldots \) converges to the point \( x^* = (x^*, x^*, \ldots) \).

In the Triangle Example points can “slow down” and hang around as long as they wish before continuing to their destination. The propositions below make this precise. There are many different definitions of chaotic maps (having positive entropy is one of them), but sensitive dependence on initial conditions is a property that virtually all chaotic maps, or maps chaotic on some subset, share. The usual definition of sensitive dependence on initial conditions is given below. The Triangle Example doesn’t have that, but it has a sort of weak version of it, and also a definition of sensitive dependence on initial conditions is given below. The Triangle Example doesn’t have that, but it has a sort of weak version of the specification property. We define weak sensitivity and show that the Triangle Example has weak sensitivity on a subset.

Suppose \( X \) is a compact metric space and \( f : X \to X \) is a map. Then \( f \) has sensitive dependence on initial conditions (SDIC) if there is some \( \delta > 0 \) such that for each \( \epsilon > 0 \) and each point \( x \) in \( X \), there are a point \( y \) in \( X \) with \( d(x, y) < \epsilon \), and an integer \( n > 0 \) such that \( d(f^n(x), f^n(y)) > \delta \).

Suppose \( X \) is a compact metric space and \( f : X \to X \) is a map. Then \( f \) has weak sensitivity (WS) at the point \( x \) if there is some \( \delta > 0 \) such that for each \( \epsilon > 0 \) there are a point \( y \) in \( X \) with \( d(x, y) < \epsilon \) and an integer \( n > 0 \) such that \( d(f^n(x), f^n(y)) > \delta \). \( f \) has weak sensitivity (WS) on the set \( A \subset X \) if there is some \( \delta > 0 \) such that for each \( \epsilon > 0 \) and each \( x \in A \) there are a point \( y \) in \( X \) with \( d(x, y) < \epsilon \) and an integer \( n > 0 \) such that \( d(f^n(x), f^n(y)) > \delta \).

**Proposition 15.** Suppose \( 1 > \delta > 0 \), and \( M_\delta \) is the closed subset \( \{y = (y_0, y_1, \ldots) \in M : y_i \geq \delta \text{ for each } i\} \). Then \( \sigma \) has WS at each point of \( M_\delta \).

**Proof.** Let \( \epsilon > 0 \) and \( x = (x_0, x_1, \ldots) \in M_\delta \) be arbitrary. Suppose \( y = (x_0, x_1, \ldots, x_{i_0}, 0, \ldots) \) where \( i_0 \) is positive integer such that \( i_0 > \log_2 \frac{1}{\epsilon} \). We have \( d(x, y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i} = \sum_{i=i_0+1}^{\infty} \frac{\epsilon}{2^i} \geq \epsilon \) and \( \sigma^{i_0+1}(y) = 0 \) (where \( 0 = (0, 0, 0, \ldots) \)). Hence, \( d(\sigma^{i_0+1}(x), \sigma^{i_0+1}(y)) = d((x_{i_0+1}, x_{i_0+2}, \ldots), 0) = \sum_{i=0}^{\infty} \frac{|x_{i_0+1+i} \cdot \epsilon}{2^i} \geq \delta \sum_{i=0}^{\infty} \frac{1}{2^i} = 2\delta > \delta \) and we are done.

**Proposition 16.** Let \( M' = \bigcup_{i=1}^{\infty} M_{1/i} \). Then \( M' \) is dense and connected in \( M \) and is a union of closed sets each of which has WS. However, each \( M_{1/i} \) has empty interior and the \( M_{1/i} \)'s are nested, so \( M' \) is a first category \( F_\sigma \)-set. (Also, note that \( M' \) itself does not have WS.)

**Notation.** For \( 0 \leq x \leq 1 \), let \( \varphi = (x, x, x, \ldots) \), and note that \( \varphi \in M \). Let the diagonal, denoted \( \Delta \), be the following subset of \( M \):

\[ \Delta = \{ \varphi : 0 \leq x \leq 1 \} \]
Notation. Suppose \( z \in [0, 1] \). Let \( z^m \) denote the point \( z, z, z, \ldots, z \) in \( I^m \). If \( z_i \in [0, 1] \) for each positive integer \( i \), and \( m_i \) is a positive integer for each positive integer \( i \), let \( z_1^{m_1} \oplus z_2^{m_2} \oplus \cdots \) denote the concatenation \((z_1, z_1, \ldots, z_1, z_2, z_2, \ldots, z_2, \ldots)\).

The following shows that we have a weak form of a shadowing or specification property with the triangle example.

Proposition 17. Suppose \( 1 > z_1 > z_2 > \cdots > 0 \), and \( \{m_i\}_{i=1}^{\infty} \) is a sequence of natural numbers greater than 1. Then if \( x = x_1^{m_1} \oplus x_2^{m_2} \oplus \cdots \), \( d(x, s) \leq 2^{-m_{j+1}}, x \in M \), then \( d(\sigma(x), (s) \leq 2^{-m_{j+1}}, \sigma \in \sigma^{m_{j+1}}(x), s) \leq 2^{-m_{j+1}}, \) and so on. Suppose, in addition, \( \{n_i\}_{i=1}^{\infty} \) a sequence of natural numbers greater than 1, and \( y = z_1^{m_1+n_1} \oplus z_2^{m_2+n_2} \oplus \cdots \). Then \( d(\sigma^j(y), (s) \leq 2^{-m_{j+1}} \) for \( 0 \leq j \leq 1 \), \( d(\sigma^{m_1+n_1+j}(y), (s) \leq 2^{-m_{j+1}} \) for \( 0 \leq j \leq 2 \), and so on.

Proof. Let \( \pi_i(x) = x_i, i \geq 0 \). The proof is straightforward: Note that \( d(x, s) = \sum_{i=0}^{\infty} |x_i - s_i| = \sum_{i=0}^{\infty} |x_i - s_i| \leq 2^{-m_{j+1}}, \) and then that \( d(\sigma(x), (s) = \sum_{i=0}^{\infty} |\sigma(x_i - s_i)| = \sum_{i=0}^{\infty} |x_{n_1+j} - s_{n_1+j}| \leq 2^{-m_{j+1}} \). This pattern continues. The second part is just an extension of the first.

\[ \square \]

Proposition 18. In the Triangle Example \( M \), for each \( 0 < a < 1 \), the open set \( V_a = M \cap ((a,1) \times [0,a) \times I^\infty) \) is a simple wandering set for \( \sigma \). For each \( n \in \mathbb{N} \), the set \( M \cap (I^n \times V_a) \) is also a simple wandering set. The nonwandering set \( \Omega_a = \Delta = \{ x = (x, x, x, \ldots) : 0 \leq x \leq 1 \} \), and so the wandering set \( M \setminus \Omega_a = M \setminus \Delta = \{ x = (x_0, x_1, \ldots) \in M : \text{for some } i, x_i \neq x_{i+1} \} \).

Proof. Suppose \( V_a \cap \sigma^m(V_a) \neq \emptyset \) for some \( m \neq 0 \). If \( m > 0 \), then there is some \( x = (x_0, x_1, x_2, \ldots) \in V_a \) such that \( \sigma^m(x) \in V_a \). But then \( x_0 \in (a,1] \) and \( x_0 \in [0,a) \) and \( x_0 \geq x_1 \geq x_2 \geq \cdots \). Since \( \sigma^m(x) = (x_m, x_{m+1}, \ldots) \), \( x_m < a \), so \( x_m \notin [a,1] \). This is a contradiction.

Suppose \( m < 0 \). Then there is some \( x = (x_0, x_1, x_2, \ldots) \in V_a \cap \sigma^m(V_a) \). Then \( \sigma^{-m}(x) = (x_{-m}, x_{-m+1}, \ldots) \in V_a \). But \( x_0 \in (a,1] \), \( x_1 \in [0,a) \), and \( x_0 \geq x_1 \geq x_2 \geq \cdots \), and again we have a contradiction. Thus, \( V_a \) is a wandering set for each \( 0 < a < 1 \).

It is straightforward to show that \( M \setminus (I^n \times V_a) \) for each \( n > 0 \) is also wandering. If \( x = (x_0, x_1, \ldots) \notin \Delta \), then there is least \( i \geq 0 \) such that \( x_i > x_{i+1} \). There is some \( 0 < a < 1 \) such that \( x_i > a > x_{i+1} \). If \( i = 0 \), then \( x \notin V_a \). If \( i > 0 \), then \( x \in I^i \times V_a \). Hence \( M \setminus \Delta \) is a subset of the wandering set.

Clearly, \( \Delta \subset \Omega_a \). Then \( \Delta = \Omega_a \) and \( M \setminus \Delta = M \setminus \Omega_a \).

\[ \square \]

What happens if we add a point to the set \( G \) in \([0,1]^2\) above? The sequence of lemmas below discusses this.
Lemma 2. Let $H' = \{(x, 0) : 0 \leq x \leq 1\} \cup \{(1, y) : 0 \leq y \leq 1\}$ and $H = H' \cup \{(0, 1)\}$. Then $H' \subset G$ and $H$ has $\infty$ entropy.

Proof. That $H' \subset G$ is obvious. So let us prove that the entropy of $H$ is $\infty$. Suppose that $n$ is a positive integer. Let $L_n = \{(i/n, 0) : 0 \leq i < n\} \cup \{(1, i/n) : 0 \leq i \leq n\} \cup \{(0, 1)\}$. Then $L_n \subset H$ and $|L_n| = 2n + 2$, where $|L_n|$ denotes the cardinality of this finite set.

Then $L_n = \bigstar_{i=0}^n L_n \subset \bigstar_{i=0}^n H$ is a Cantor set. In fact, $L_n = \{(s_0, s_1, \ldots) : s_i \in \{j/n : 0 \leq j \leq n\}; \text{ if } s_i = k/n, \text{ where } 0 < k < n, \text{ then } s_{i+1} = 0; \text{ if } s_i = 1, \text{ then } s_{i+1} = k/n \text{ where } 0 \leq k \leq n, \text{ and if } s_i = 0, \text{ then } s_{i+1} = 0 \text{ or } s_{i+1} = 1\}$. Note that the point $(0, 1, i/n, 0) \in \bigstar_{j=1}^3 L_n$ for $0 \leq i \leq n$. Let $A_n = \{(0, 1, k/n, 0) : 0 \leq k \leq n\} \subset \bigstar_{j=1}^3 L_n$. Note that $N(A_n, \alpha^{3+1}) = n + 1$ (as long as $\alpha$ has sufficiently small intervals and is chosen so that no $\frac{1}{n}$ is in 2 intervals), and, in general, for $m > 0$, $N(\bigstar_{i=1}^n A_n, \alpha^{3m+1}) = (n + 1)^m \leq N(\bigstar_{j=1}^{3m} L_n, \alpha^{3m+1})$.

Then

$$\text{ent}(L_n, \alpha) = \lim_{m \to \infty} \frac{\log(N(\bigstar_{j=1}^m L_n, \alpha^{m+1}))}{m} \geq \lim_{3m \to \infty} \frac{\log(N(\bigstar_{i=1}^n A_n, \alpha^{3m+1}))}{3m} = \lim_{3m \to \infty} \frac{\log(n + 1)^m}{3m} \geq \frac{\log(n + 1)}{3}.$$

Then for each $\alpha$ with sufficiently small and carefully chosen intervals, $\text{ent}(L_n, \alpha) = \frac{\log(n + 1)}{3}$, so $\text{ent}(L_n) \geq \frac{\log(n + 1)}{3}$. Since $\text{ent}(H) \geq \text{ent}(L_n) \geq \frac{\log(n + 1)}{3}$ for each $n$, the result follows.

Lemma 3. Suppose $0 \leq p < q \leq 1$. Let $H^{(p,q)} = \{(x, p) : p \leq x \leq q\} \cup \{(q, y) : p \leq y \leq q\}$ and $H^{(p,q)} = H^{(p,q)} \cup \{(p, q)\}$. Then $H^{(p,q)} \subset G$ and $H^{(p,q)}$ has $\infty$ entropy.

Proof. This is similar to the last result. We let, for each $n$, $L_n^{(p,q)} = \{(p + i \Delta t, p) : 0 \leq i \leq n\} \cup \{(q, p + i \Delta t) : 0 \leq i \leq n\} \cup \{(p, q)\}$, where $\Delta t = \frac{2 \pi}{n}$. Then it follows, as above, that $\text{ent}(L_n^{(p,q)}) = \frac{\log(n + 1)}{3}$. Since $L_n^{(p,q)} \subset H^{(p,q)}$, the result follows.

Then we have the following result:

Theorem 16. Suppose $0 \leq p < q \leq 1$. Then $\text{ent}(G \cup \{(p, q)\}) = \infty$, while $\text{ent}(G) = 0$.

Suppose $X$ is a metric space, $f : X \to X$ is a map.

1. A point $x$ in $X$ has period $n$ if $f^n(x) = x$. A point $x$ in $X$ has prime period $n$ if $f^n(x) = x$, but $f^j(x) \neq x$ for $0 < j < n$. The point $x$ is periodic.
(2) $f$ is transitive if for each pair $U, V$ of open sets in $X$, there is $n$ such that $f^n(U) \cap V \neq \emptyset$.

**Theorem 17.** Let $G \cup \{(0,1)\} = G^+$. Then $G^+ = \bigoplus_{i=1}^{\infty} G^+$ under the action of $\sigma$ has (1) a dense set of periodic points, (2) has periodic points of all periods (nontrivially), and (3) $\sigma$ is transitive.

**Proof.**

(1) Suppose $x = (x_0, x_1, \ldots) \in G^+$, and $x \in U = U_0 \times U_1 \times \cdots \times U_n \times I^\infty$, a basic open set. Let $z = (x_0, x_1, \ldots, x_n)$. Then $w = z \oplus (0,1) \oplus z \oplus (0,1) \oplus \cdots \in U \cap G^+$, and $w$ is periodic. Hence the set of periodic points of $G^+$ is dense in $G^+$.

(2) $\Delta \subset G^+$, so $G^+$ has fixed points (period one points). The point $(0, 1, 0, 1, 0, \ldots) \in G^+$ has period 2. If $0 < x < 1$, then $(x, 0, 1, x, 0, 1, \ldots) \in G^+$ has prime period 3. If $0 < x < y < 1$, then the point $(y, x, 0, 1, y, x, 0, 1, \ldots) \in G^+$ has prime period 4. And, so on. Hence $G^+$ has prime period $n$ points for each positive integer $n$.

(3) Suppose $U, V$ are open sets in $G^+$. Without loss of generality, we can assume that $U, V$ are basic open sets. So, let $U = G^+ \cap (U_0 \times U_1 \times \cdots \times U_n \times I^\infty)$ and let $V = G^+ \cap (V_0 \times V_1 \times \cdots \times V_m \times I^\infty)$. Suppose $x = (x_0, x_1, \ldots) \in U$ and $y = (y_0, y_1, \ldots) \in V$. Then consider the point $z = (x_0, x_1, \ldots, x_n) \oplus (0, 1) \oplus y$. The point $z \in G^+$, $z \in U$, and $\sigma^{n+1}(z) = y$. Thus, $\sigma^{n+3}(U) \cap V \neq \emptyset$. Hence $\sigma$ is transitive.

□

**Corollary 2.** The map $\sigma$ on $G^+$ has sensitive dependence on initial conditions and is chaotic in the sense of Devaney.

**Proof.** That $\sigma$ has sensitive dependence on initial conditions follows from [BBCDS]. Devaney’s definition of chaos requires (1) transitivity, (2) dense set of periodic points, and (3) sensitive dependence on initial conditions. (The paper cited above shows that SDIC follows from transitivity and dense set of periodic points.)

□

**Notation.** Suppose $0 < p < q < 1$. Let $T^{(p,q)} = \{(x,y) : q \geq x \geq y \geq p\}$ and let $T^{(p,q)} = \bigoplus_{i=1}^{\infty} T^{(p,q)}$.

**Lemma 4.** Then $\sigma(T^{(p,q)}) = T^{(p,q)}$ and $\sigma$ on $T^{(p,q)}$ (1) admits a dense set of periodic points, (2) admits periodic points of all periods (nontrivially), and (3) is transitive. Hence, $\sigma$ on $T^{(p,q)}$ is chaotic in the sense of Devaney. Furthermore, $\sigma|T^{(p,q)}$ has entropy $\infty$.

**Proof.** Using the same (slightly adjusted) arguments as those for Theorem 1, Theorem 2, and Corollary 2 shows this.

□

Let $W^{(p,q)} := \bigoplus_{i=1}^{\infty} (G \cup \{(p,q)\})$, which contains $T^{(p,q)}$. Under $\sigma$, $T^{(p,q)}$ is invariant and closed in $W^{(p,q)}$. But do all points of $W^{(p,q)} \setminus (T^{(p,q)} \cup \Delta)$ wander?

(1) Suppose $x$ is in $W^{(p,q)}$, $x$ is eventually in $T^{(p,q)}$, but is not in $T^{(p,q)}$. Then there is least $N \geq 0$ and $z = (z_0, z_1, \ldots) \in T^{(p,q)}$ such that $x = (x_0, x_1, \ldots, x_N) \oplus z$, and $x_i > q$, $0 \leq i \leq N$. Suppose $x_N > a > q$. Then $x \in V_a = W^{(p,q)} \cap (I^N \times (a,1] \times [0,a) \times I^\infty)$. Hence, $x_N \in (a,1]$ and $z_0 = x_{N+1} \in [0,a)$. 

□
If \( y = (y_0, y_1, \ldots) \in \sigma^{-1}(V_a) = W^{(p,q)}(I^{N+1} \times (a,1] \times [0,a) \times I^\infty) \), then \( y \not\in V_a \), because \( y_i > q \) for \( 0 \leq i \leq N + 1 \), and \( y_{N+1} \notin [0,a) \). Likewise, \( \sigma^{-m}(V_a) \cap V_a = \emptyset \) for \( m > 0 \).

If \( y = (y_0, y_1, \ldots) \in \sigma(V_a) = W^{(p,q)}(I^{N-1} \times (a,1] \times [0,a) \times I^\infty) \) (provided \( N > 0 \)), then \( y \notin V_a \), because \( y_i \geq q \) for \( 0 \leq i \leq N - 1 \), and \( y_N \notin (a,1] \). Likewise, \( \sigma^m(V_a) \cap V_a = \emptyset \) for \( N + 1 > m > 0 \). If \( m \geq N + 1 \), then \( \sigma^m(V_a) \cap V_a = \emptyset \) since any point \( z = (z_0, z_1, \ldots) \) of \( \sigma^m(V_a) \) has each coordinate less than or equal to \( q < a \) and so cannot be in \( V_a \). Hence, each such \( x \) is wandering.

(2) Suppose \( x \) is eventually fixed but is not eventually in \( T^{(p,q)} \). Then either there is some \( z > q \) or some \( z < p \) such that for some \( m > 0 \), \( \sigma^m(x) = \emptyset \). If there is \( z > q \) such that for some \( m > 0 \), \( \sigma^m(x) = \emptyset \), then each coordinate of \( x \) is greater than \( q \), and \( x \) is wandering (for the same reason as it was wandering in \( G \)).

If there is \( z < p \) such that for some \( m > 0 \), \( \sigma^m(x) = \emptyset \), then there is some greatest \( N \) such that \( x_N \geq z \). Let \( x_N > a > z \). Then \( x \in U_a = W^{(p,q)}((a,1]^{N+1} \times [0,a) \times I^\infty) \). Again, for each \( m \), \( U_a \cap \sigma^m(U_a) = \emptyset \), so \( x \) is wandering.

(3) Suppose \( x \) is neither eventually fixed nor eventually in \( T^{(p,q)} \). Then either each \( x_i > q \), or eventually \( x_i < p \). If each \( x_i > q \), then there is least \( n \) such that \( x_n > x_{n+1} \) (since \( x \) is not eventually fixed). Then there is \( a \) such that \( x_n > a > x_{n+1} \). If \( U_a = W^{(p,q)}((a,1]^{n+1} \times [0,a) \times I^\infty) \), then \( U_a \) is wandering and contains \( x \), so again we have a wandering point.

Suppose that eventually \( x_i < p \). Then there is least \( n \) such that \( x_n < p \) and \( x_n > x_{n+1} \) (again, \( x \) is not eventually fixed). Then if \( x_n > a > x_{n+1} \), and \( U_a = W^{(p,q)}((a,1]^{n+1} \times [0,a) \times I^\infty) \), then \( U_a \) is wandering and contains \( x \), so such a point is wandering.

**Proposition 19.** Let \( W^{(p,q)} := \cap_{i=1}^\infty (G \cup \{(p,q)\}) \). Then \( W^{(p,q)} \) contains \( T^{(p,q)} \) and contains \( \Delta \). Under \( \sigma \), \( T^{(p,q)} \) is invariant and closed in \( W^{(p,q)} \), as is \( \Delta \). Furthermore, all points of \( W^{(p,q)} \setminus (T^{(p,q)} \cup \Delta) \) wander.

**Theorem 18.** Suppose \( D \) is a closed region in \([0,1] \times [0,1] \) with nonempty connected interior and such that if \( D^\circ \) is the interior of \( D \), then \( D = \overline{D^\circ} \). Then either \( \text{ent}(D) = 0 \) or \( \text{ent}(D) = \infty \).

**Proof.** Either \( D \subset G \), or \( D \subset \overline{[0,1]^2 \setminus G} \), or \( D^\circ \) intersects the diagonal \( \{(x,x) : 0 \leq x \leq 1\} \). If \( D \subset G \), or \( D \subset \overline{[0,1]^2 \setminus G} \), then \( \text{ent}(D) = 0 \). Otherwise, \( D^\circ \) intersects the diagonal, and there is some \( 0 \leq p < q \leq 1 \) such that \( T^{(p,q)} \subset D \), so \( \text{ent}(D) = \infty \). \( \square \)
Theorem 19. The entropy of $H$ is at least $\log(k)$, i.e., $\text{ent}(H) \geq \log(k)$.

Proof. Since $\pi_0(H_i) = [0,1]$, $N(H_i, \alpha^{N+1}) \geq n$ for each $i$, and $N(H, \alpha^{N+1}) \geq kn$ for $\alpha$ with sufficiently small intervals. For each $m \geq 1$, there are $k^m$ sets $H_{s_1} \ast H_{s_2} \ast \ldots \ast H_{s_m}$, with this collection being mutually disjoint. If $L_m = \{H_{s_1} \ast H_{s_2} \ast \ldots \ast H_{s_m} : s_i \in \{1,\ldots,k\} \text{ for each } 1 \leq i \leq m\}$, then $N(L_m, \alpha^{mN+1}) \geq k^m n$, and
\[
\lim_{m \to \infty} \frac{\log(N(L_m, \alpha^{mN+1}))}{m} \geq \lim_{m \to \infty} \frac{\log(k^m n)}{m} = \lim_{m \to \infty} \frac{m \log(k) + \log(n)}{m} = \log(k).
\]

Since
\[
\lim_{m \to \infty} \frac{\log(N(L_m, \alpha^{mN+1}))}{m} \leq \lim_{m \to \infty} \frac{\log(N(H, \alpha^{mN+1}))}{m},
\]
the result follows. \(\square\)

7. Entropy versus box counting dimension

Superficially at least, defining entropy with box covers appears to be similar to the way the box counting dimension of a set is defined. For a minimal open cover \( \alpha \) of \([0, 1] \), its corresponding grid cover \( \alpha_n \) of \( \prod_{i=0}^{n} I_i \), and a subset \( A \) of \( \prod_{i=0}^{n} I_i \), let \( N^*(A, \alpha^{n+1}) \) denote the number of members of \( \alpha^{n+1} \) that intersect \( A \). Let \( |\alpha| \) denote the length of the largest interval in \( \alpha \). The box counting dimension of \( A \) can be defined as
\[
\lim_{|\alpha| \to 0} \frac{\log(N^*(A, \alpha^{n+1}))}{\log(|\alpha|)},
\]
provided this limit exists. Unfortunately it does not always exist.

For us, the closed sets \( G \) in \( I_0 \times I_1 \) in which we have been interested so far, have topological dimension the same as their box counting dimension. And this is also true of \( \bigotimes_{i=1}^{n} G \), i.e., the box counting dimension of \( \bigotimes_{i=1}^{n} G \) is the same as the topological dimension of \( \bigotimes_{i=1}^{n} G \). Hence, relative to box counting dimension, our sets are rather simple.

Box counting dimension is intended to measure the dimension of a fractal set. There are many other ways to measure this dimension, such as Hausdorff dimension and correlation dimension. They are not equivalent in general. On the other hand, entropy is designed to measure how much points of a set interact with other points of the set (in a very particular way).

Consider the following example:

**Example 16.** Suppose \( G = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \). Then the box counting dimension of \( G \) is 0, as is the box counting dimension of \( \bigotimes_{i=1}^{n} G \) for each \( n \), since \( \bigotimes_{i=1}^{n} G \) is finite for each \( n \). However, the entropy of \( G \) is \( \log 2 \), since for each minimal open cover \( \alpha \) of \([0, 1] \) by open intervals,
\[
\frac{\log(N(\bigotimes_{i=1}^{n} G, \alpha^{n+1}))}{n} = \frac{\log(2^{n+1})}{n} = \frac{(n + 1) \log 2}{n},
\]
so
\[
\lim_{n \to \infty} \frac{\log(N(\bigotimes_{i=1}^{n} G, \alpha^{n+1}))}{n} = \lim_{n \to \infty} \frac{(n + 1) \log 2}{n} = \log 2.
\]
It follows that \( \sup_{\alpha} \text{ent}(G, \alpha) = \log 2 \).

Similar results hold for most, if not all, our examples. But we provide one last example that shows box counting dimension can also be greater than the topological entropy.
Example 17. Let \( G = \{(x,x) : 0 \leq x \leq 1\} \), i.e., \( G \) is the diagonal from \((0,0)\) to \((1,1)\) in \([0,1]^2\). Then \( \text{ent}(G) = 0 \) and \( \dim_{\text{box}}(\ast_{i=1}^{\infty} G) = 1 \) ([F], p. 48).

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