Improvement of the conformal map combined with the Sinc approximation for unilateral rapidly decreasing functions

Tomoaki Okayama\(^1\) and Tomoya Shiraishi\(^2\)

\(^1\)Graduate School of Information Sciences, Hiroshima City University, 3-4-1 Ozuka-higashi, Asaminami-ku, Hiroshima 731-3194, Japan
\(^2\)Kunisaki High School, 1974 Tsurugawa, Kunisaki-machi, Kunisaki, Oita 873-0503, Japan

Abstract

The Sinc approximation is a very efficient for bilateral rapidly decreasing functions. Even for non-bilateral rapidly decreasing functions, the Sinc approximation can work accurately if combined with an appropriate conformal map. Appropriate conformal maps for typical cases have been proposed in the literature. In the case of unilateral rapidly decreasing functions, however, it can be improved further. In this paper, we employ an improved conformal map for the case of unilateral rapidly decreasing functions. In addition, we present a computable error bound for the improved approximation formula.

Keywords Sinc approximation, conformal map, error bound, verified computation

Research Activity Group Quality of Computation

1. Introduction and summary

The Sinc approximation is a function approximation formula expressed as follows:

\[
F(x) \approx \sum_{k=-M}^{N} F(kh)S(k, h)(x), \quad x \in \mathbb{R},
\]

where \(h\) is the mesh size, \(M\) and \(N\) are truncation numbers, and \(S(k, h)(x)\) is the so-called Sinc function defined by

\[
S(k, h)(x) = \frac{\sin[\pi(x/h - k)]}{\pi(x/h - k)}.
\]

If the function to be approximated is analytic and decays exponentially as \(x \to \pm \infty\), i.e., \(F(x)\) is a rapidly decreasing function, the Sinc approximation shows quite fast convergence. Even if the given function \(f(t)\) is not a rapidly decreasing function, an appropriate conformal map \(t = g(x)\) will transform \(f(t)\) to a rapidly decreasing function \(f(g(x))\). Therefore, putting \(F(x) = f(g(x))\) in (1), we may obtain accurate approximation. For this reason, the Sinc approximation is often combined with a conformal map. Appropriate conformal maps for typical cases have been usefully summarized by Stenger [1].

In this article, we are concerned with functions \(f(t)\) that decay exponentially as \(t \to \infty\), but decay algebraically as \(t \to -\infty\). We call such functions unilateral rapidly decreasing functions. According to Stenger’s list, an appropriate conformal map for unilateral rapidly decreasing functions is

\[
t = \psi(x) = \sinh(\log(\text{arcsinh}(e^{x}))).
\]

In fact, this conformal map transforms unilateral rapidly decreasing functions to (bilateral) rapidly decreasing functions, which allows the Sinc approximation to converge rapidly in this case as well.

This study aims to improve the conformal map (2). In the area of numerical integration, the following conformal map

\[
t = \phi(x) = 2\sinh(\log(\log(1 + e^{x})))
\]

has recently been proposed [2] as a replacement for \(t = \psi(x)\), and performance improvement by the replacement has been reported. Our idea is based on the result; that is, instead of \(t = \psi(x)\) in (2), we propose to employ \(t = \phi(x)\) in (3) combined with the Sinc approximation (1). Further, as a subsidiary objective of this study, we give a computable error bound for the improved approximation formula. This error bound guarantees the accuracy of the approximation, which is useful for verified computation.

The rest of this article is organized as follows. The existing and new error bound theorems are stated in Section 2. Numerical examples are presented in Section 3. Finally, the proof for the newly presented theorem is provided in Section 4.

2. Existing and new theorems

First, some relevant notations are introduced. Let \(\mathcal{D}_{d}\) denote a strip complex domain defined by \(\mathcal{D}_{d} = \{\zeta \in \mathbb{C} : |\text{Im}\, \zeta| < d\}\) for \(d > 0\). In addition, let \(\mathcal{D}_{d}^{-} = \{\zeta \in \mathcal{D}_{d} : \text{Re}\, \zeta < 0\}\) and \(\mathcal{D}_{d}^{+} = \{\zeta \in \mathcal{D}_{d} : \text{Re}\, \zeta \geq 0\}\). Then, the error of the combination of the conformal map \(t = \psi(x)\) and the Sinc approximation (1) is analyzed as follows.

**Theorem 1** (Stenger [1, Theorem 1.5.13]) Assume that \(f\) is analytic in \(\psi(\mathcal{D}_{d})\) with \(0 < d \leq \pi/2\), and that there...
exist positive constants $K_+, K_-, \alpha$, and $B$ such that
\[ |f(z)| \leq K_-|z|^{-\alpha} \tag{4} \]
holds for all $z \in \psi(\mathcal{D}_R^+)$, and
\[ |f(z)| \leq K_+|e^{-z}|^B \]
holds for all $z \in \psi(\mathcal{D}_R^+)$, let $\beta = B/2$, let $\mu = \min\{\alpha, \beta\}$, let $M$ and $N$ be defined as
\[ \begin{align*}
M &= n, \quad N = \lceil\alpha/\beta\rceil \quad (\text{if } \mu = \alpha), \\
N &= n, \quad M = \lceil\beta n/\alpha\rceil \quad (\text{if } \mu = \beta),
\end{align*} \tag{5} \]
and let $h$ be defined as
\[ h = \sqrt{\frac{\pi d}{\mu n}} \tag{6} \]
Then, there exists a constant $C$ independent of $n$ such that
\[ \sup_{t \in \mathbb{R}} \left| f(t) - \sum_{k=-M}^{N} f(\psi(kh))S(k,h)(\psi^{-1}(t)) \right| \leq C\sqrt{n} e^{-\sqrt{\mu n}}. \]

In this article, we present the following theorem. The proof is given in Section 4.

**Theorem 2** Assume that $f$ is analytic in $\phi(\mathcal{D})$ with $0 < \theta < \pi$, and that there exist positive constants $K_+$, $K_-$, $\alpha$, and $\beta$ such that (4) holds for all $z \in \phi(\mathcal{D}_d^+)$, and
\[ |f(z)| \leq K_+|e^{-z}|^\beta \]
holds for all $z \in \phi(\mathcal{D}_R^+)$, let $\mu = \min\{\alpha, \beta\}$, let $M$ and $N$ be defined as (5), and let $h$ be defined as (6). Then, we have
\[ \sup_{t \in \mathbb{R}} \left| f(t) - \sum_{k=-M}^{N} f(\phi(kh))S(k,h)(\phi^{-1}(t)) \right| \leq 2C_d e^{-2\sqrt{\mu n}} + C_T \sqrt{n} e^{-\sqrt{\mu n}}, \]
where $C_d$ and $C_T$ are constants defined by
\[ C_d = K_+ \frac{e}{(1 - \log 2)(e-1)\cos(d/2)} \alpha \]
\[ + \frac{K_+}{\beta} \left(\frac{e^{1/2}}{\cos(d/2)}\right)^\beta, \tag{7} \]
\[ C_T = K_+ \frac{1}{\alpha} + \frac{K_+}{\beta} \left(\frac{e^{1/2}}{\cos(d/2)}\right)^\beta. \tag{8} \]

A big difference between Theorems 1 and 2 lies in the parameters $\mu$ and $d$, which affect the convergence rate: $O(\sqrt{n} e^{-\sqrt{\mu n}})$. The value of $\beta$ in Theorem 1 is less than that in Theorem 2 because the decay rate as $z \rightarrow \infty$ is halved ($\beta = B/2$) in Theorem 1, but it is not halved in Theorem 2. Therefore, $\mu = \min\{\alpha, \beta\}$ in Theorem 2 may be larger than that in Theorem 1. As for $d$, the range is limited as $d < \pi/2$ in Theorem 1, whereas $d < \pi/2$ in Theorem 2. This difference originates from the conformal maps $\psi$ and $\phi$ (see [2] for details). Therefore, $d$ in Theorem 2 may be larger than that in Theorem 1.

Another big difference lies in the constants. In Theorem 1, only existence of the constant $C$ is stated, and we cannot compute its value. In contrast, in Theorem 2, the constant prior to $\sqrt{n} e^{-\sqrt{\mu n}}$ is explicitly written in a computable form, which enables us to obtain the quantitative value of the error bound. This is useful for verified computation.

### 3. Numerical examples

In this section, we present numerical results for the following unilateral rapidly decreasing functions:
\[ f_1(t) = 1 - \frac{1}{2} \left( 1 - \frac{t}{4+t^2} \right) \frac{1}{1+e^{t^2/2}}, \]
\[ f_2(t) = \frac{1}{\sqrt{1 + (t/2)^2}} \frac{e^{-t/2}}{1 - (t/2)^2}. \]
The function $f_1$ satisfies the assumptions in Theorem 1 with $\alpha = 2$, $\beta = \pi/4$, $d = 3/2$, and for some $K_+$ and $K_-$. It also satisfies the assumptions in Theorem 2 with $\alpha = 2$, $\beta = \pi/2$, $d = 3/2$, $K_+ = 9/4$, and $K_- = e^2 + 1$. In addition, $f_2$ satisfies the assumptions in Theorem 1 with $\alpha = 1$, $\beta = 1/2$, $d = 3/2$, and for some $K_+$ and $K_-$. It also satisfies the assumptions in Theorem 2 with $\alpha = \beta = 1$, $d = 3$, $K_+ = 7/5$, and $K_- = 47/2$. All programs are written in C with double-precision floating-point arithmetic. The results are shown in Figs. 1 and 2. Both graphs show that the new method converges faster than Stenger’s method for both functions $f_1$ and $f_2$. Further, the error bound of Theorem 2 (dotted line) surely includes the observed error (solid line with filled circle).

### 4. Proofs

#### 4.1 Sketch of the proof

Applying $t = \phi(x)$ and putting $F(x) = f(\phi(x))$, we have
\[ \left| f(t) - \sum_{k=-M}^{N} f(\phi(kh))S(k,h)(\phi^{-1}(t)) \right| \leq 2C_d e^{-2\sqrt{\mu n}} + C_T \sqrt{n} e^{-\sqrt{\mu n}}. \]

#### 4.2 Proofs

**Lemma 3** Let the assumptions of Theorem 2 be satisfied. Then, putting $F(x) = f(\phi(x))$, we have
\[ \sup_{x \in \mathbb{R}} \left| F(x) - \sum_{k=-\infty}^{\infty} F(kh)S(k,h)(x) \right| \leq \frac{2C_d e^{-\pi d/h}}{\pi d(1 - e^{-2\pi d/h})}, \]
where $C_d$ is the constant defined in (7).

We bound the truncation error as follows. The proof is given in Section 4.3.
Lemma 4 Let the assumptions of Theorem 2 be satisfied. Then, putting $F(x) = f(\phi(x))$, we have
\[
\sup_{x \in \mathbb{R}} \left| \sum_{k=-\infty}^{M-1} F(kh) S(k, h)(x) + \sum_{k=N+1}^{\infty} F(kh) S(k, h)(x) \right| 
\leq \frac{C_T}{h} e^{-\mu h},
\]
where $C_T$ is the constant defined in (8).

Combining Lemmas 3 and 4, using (6), we establish Theorem 2. This completes the proof.

4.2 Estimate of the discretization error

The following function space is important to bound the discretization error.

Definition 5 Let $\mathcal{D}_d(\epsilon)$ be a rectangular domain defined for $0 < \epsilon < 1$ by
\[
\mathcal{D}_d(\epsilon) = \{ \zeta \in \mathbb{C} : |\text{Re}\, \zeta| < 1/\epsilon, \ |\text{Im}\, \zeta| < d(1 - \epsilon) \}.
\]
Then, $\mathcal{H}^1(\mathcal{D}_d)$ denotes the family of all functions $F$ that are analytic in $\mathcal{D}_d$ such that the norm $\mathcal{N}_1(F, d)$ is finite, where
\[
\mathcal{N}_1(F, d) = \lim_{\epsilon \to 0} \int_{\partial \mathcal{D}_d(\epsilon)} |F(\zeta)| d|\zeta|.
\]
The discretization error for a function belonging to this function space is analyzed as follows.

Theorem 6 (Stenger [3, Theorem 3.1.3]) Let $F \in \mathcal{H}^1(\mathcal{D}_d)$. Then, we have
\[
\sup_{x \in \mathbb{R}} \left| F(x) - \sum_{k=-\infty}^{\infty} F(kh) S(k, h)(x) \right| 
\leq \frac{\mathcal{N}_1(F, d) e^{-\eta d/h}}{\eta d (1 - e^{-2\eta d/h})}.
\]

Therefore, we should show $F \in \mathcal{H}^1(\mathcal{D}_d)$ and estimate $\mathcal{N}_1(F, d)$ under the assumptions of Theorem 2. The next result is crucial for the purpose.

Lemma 7 (Okayama et al. [2, Lemma 5.4]) Assume that $F$ is analytic in $\mathcal{D}_d$ with $0 < d < \pi$, and that there exist positive constants $K_+$, $K_-$, $\alpha$, and $\beta$ such that
\[
|F(\zeta)| \leq K_+ \frac{d^{1/2} \log(1 + e^\zeta)}{1 + e^\zeta} \beta
\]
holds for all $\zeta \in \mathcal{D}_d^+$, and
\[
|F(\zeta)| \leq K_- \frac{\log(1 + e^{\zeta})}{[\log(1 + e^{\zeta})]^\alpha [1 + \log(1 + e^{\zeta})]}\]
holds for all $\zeta \in \mathcal{D}_d^-$. Then, $F$ belongs to $\mathcal{H}^1(\mathcal{D}_d)$, and $\mathcal{N}_1(F, d)$ is bounded as
\[
\mathcal{N}_1(F, d) \leq 2 C_D,
\]
where $C_D$ is the constant defined in (7).

The assumptions of Lemma 7 are satisfied under the assumptions of Lemma 3. Therefore, combining Theorem 6 and Lemma 7, we obtain Lemma 3.

4.3 Estimate of the truncation error

The next result is crucial.

Lemma 8 (Okayama et al. [2, Lemma 5.5]) Let the same assumptions in Lemma 7 be satisfied. Let $\mu = \min\{\alpha, \beta\}$, and let $M$ and $N$ be defined as (5). Then, we have
\[
\sum_{k=-\infty}^{M-1} |F(kh)| + \sum_{k=N+1}^{\infty} |F(kh)| \leq \frac{C_T}{h} e^{-\mu h},
\]
where $C_T$ is the constant defined in (8).

The assumptions of Lemma 8 are satisfied under the assumptions of Lemma 4. Therefore, using this lemma and $|S(k, h)(x)| \leq 1$ for all $x \in \mathbb{R}$, we obtain Lemma 4.

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