NONCOMMUTATIVE HYPERBOLIC METRICS

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Abstract. We characterize certain noncommutative domains in terms of noncommutative holomorphic equivalence via a pseudometric that we define in purely algebraic terms. We prove some properties of this pseudometric and provide an application to free probability.

1. Introduction

Noncommutative functions are (countable) families of functions defined on matrices of increasing dimension over a base set, usually with some structure (vector space, operator space, $C^*$-algebra, von Neumann algebra etc) which satisfy certain compatibility conditions, to be described below. We exploit these conditions to describe metric/geometric properties of noncommutative domains in purely algebraic terms and to study properties of noncommutative maps of such domains. Our results seem to be relevant to the study of certain classical several complex variables maps, in the spirit of [3, 13].

2. Noncommutative domains, functions and kernels

2.1. Noncommutative functions. Noncommutative functions originate in Joseph L. Taylor’s work [29, 30] on spectral theory and functional calculus for $k$-tuples of noncommuting operators. We largely follow [24] in our presentation of noncommutative sets and functions. We refer to [24] for details on, and proofs of, the statements below.

Let us introduce the following notation: if $S$ is a nonempty set, we denote by $S_{m \times n}$ the set of all matrices with $m$ rows and $n$ columns having entries from $S$. If $S = \mathbb{F}$ is a field, then we use the standard notation $GL_n(\mathbb{F})$ for the group of matrices $X$ in $\mathbb{F}^{n \times n}$ which are invertible (that is, there exists $X^{-1} \in \mathbb{F}^{n \times n}$ such that $XX^{-1} = X^{-1}X = I_n$, where $I_n$ is the diagonal matrix having the multiplicative unit of $\mathbb{F}$ on the diagonal and zero elsewhere). We will work almost exclusively with subsets of operator spaces and operator systems (linear subspaces of the algebra $B(\mathcal{H})$ of bounded operators over a Hilbert space $\mathcal{H}$ – which we assume to be separable – which contain the unit $1$ of $B(\mathcal{H})$, are norm-closed and selfadjoint - see [24]). However some of our definitions hold in much broader generality. Given a complex vector space $\mathcal{V}$, a noncommutative set is a family $\Omega_{nc} := (\Omega_n)_{n \in \mathbb{N}}$ such that

(a) for each $n \in \mathbb{N}$, $\Omega_n \subseteq \mathcal{V}^{n \times n}$;
(b) for each $m, n \in \mathbb{N}$, we have $\Omega_m \oplus \Omega_n \subseteq \Omega_{m+n}$.

The noncommutative set $\Omega_{nc}$ is called right admissible if in addition the condition (c) below is satisfied:

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(c) for each \( m, n \in \mathbb{N} \) and \( a \in \Omega_m, b \in \Omega_n, w \in \mathcal{V}^{m \times n} \), there is an \( \epsilon > 0 \) such that
\[
\begin{bmatrix}
a & zw \\
0 & b
\end{bmatrix} \in \Omega_{m+n} \text{ for all } z \in \mathbb{C}, |z| < \epsilon.
\]

Left admissible sets are defined similarly, except that \( zw \) appears in the lower left corner of the matrix.

Given complex vector spaces \( \mathcal{V}, \mathcal{W} \) and a noncommutative set \( \Omega_{nc} \subseteq \prod_{n=1}^{\infty} \mathcal{V}^{n \times n} \), a noncommutative function is a family \( f := (f_n)_{n \in \mathbb{N}} \) such that \( f_n : \Omega_n \to \mathcal{W}^{n \times n} \) and
\begin{enumerate}[(1)]  
  \item \( f_m(a) + f_n(b) = f_{m+n}(a \oplus b) \) for all \( m, n \in \mathbb{N}, a \in \Omega_m, b \in \Omega_n \);  
  \item for all \( n \in \mathbb{N} \), \( f_n(T^{-1}aT) = T^{-1}f_n(a)T \) whenever \( a \in \Omega_n \) and \( T \in GL_n(\mathbb{C}) \) such that \( T^{-1}aT \) belongs to the domain of definition of \( f_n \).
\end{enumerate}

These two conditions are equivalent to the requirement that \( f \) respects intertwinings by scalar matrices:

\begin{enumerate}[(1)]  
  \item For all \( m, n \in \mathbb{N}, a \in \Omega_m, b \in \Omega_n, S \in \mathbb{C}^{m \times n} \), we have
\[
aS = Sb \implies f_m(a)S = Sf_n(b).
\]
\end{enumerate}

If \( \mathcal{V}, \mathcal{W} \) are operator spaces, it is shown in [24, Theorem 7.2] that, under very mild openness conditions on \( \Omega_{nc} \), local boundedness for \( f \) implies each \( f_n \) is analytic as a map between Banach spaces. More specifically, if \( \Omega_{nc} \) is finitely open (that is, for all \( n \in \mathbb{N} \), the intersection of \( \Omega_n \) with any finite dimensional complex subspace is open) and \( f \) is locally bounded on slices (that is, for every \( n \in \mathbb{N} \), for every \( a \in \Omega_n \) and \( b \in \mathcal{V}^{n \times n} \), there exists an \( \epsilon > 0 \) such that the set \( \{ f_n(a+zb) : z \in \mathbb{C}, |z| < \epsilon \} \) is bounded in \( \mathcal{W}^{n \times n} \)), then each \( f_n \) is Gâteaux complex differentiable on \( \Omega_n \) (see Section 2.3 below). Indeed, this is a consequence of the following essential property of noncommutative functions:

if \( \Omega_{nc} \) is admissible, \( a \in \Omega_n, c \in \Omega_m, b \in \mathcal{V}^{n \times m} \) such that \( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \Omega_{n+m} \), then there exists a linear map \( \Delta f_{n,m}(a, c) : \mathcal{V}^{n \times m} \to \mathcal{W}^{n \times m} \) such that
\[
\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} f_n(a) & \Delta f_{n,m}(a, c)(b) \\ 0 & f_m(c) \end{bmatrix}.
\]

This implies in particular that \( f_{n+m} \) extends to the set of all elements \( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \) such that \( a \in \Omega_n, c \in \Omega_m, b \in \mathcal{V}^{n \times m} \) (see [24, Section 2.2]). Two properties of this operator that are important for us are
\begin{enumerate}[(3)]  
  \item \( \Delta f_{n,n}(a, c)(a - c) = f(a) - f(c) = \Delta f_{n,n}(c, a)(a - c), \quad \Delta f_{n,n}(a, b) = f'_n(a)(b) \), the derivative of \( f_n \) in \( a \) applied to the element \( b \in \mathcal{V}^{n \times m} \). Moreover, \( \Delta f(a, c) \) as functions of \( a \) and \( c \), respectively, satisfy properties similar to the ones described in items (1), (2) above – see [24, Sections 2.3–2.5] for details (for convenience, from now on we shall suppress the indices denoting the level for noncommutative functions, as it will almost always be obvious from the context).
\end{enumerate}

**Example 2.1.** There are many examples of noncommutative functions. We provide here three.

1. The best known is provided by the classical theory of analytic functions of one complex variable: if \( D \) is a simply connected domain in \( \mathbb{C} \) and \( f : D \to \mathbb{C} \) is analytic, then \( f \) is the first level of an nc map \( f : \prod_{n=1}^{\infty} A \in \mathbb{C}^{n \times n} : \sigma(A) \subseteq D \to \prod_{n=1}^{\infty} \mathbb{C}^{n \times n} \) given by the classical analytic functional calculus: \( f_n(A) = (2\pi i)^{-1} \int_{\gamma} (A - \zeta I_n)^{-1} f(\zeta) \, d\zeta \), for some simple closed curve \( \gamma \) which surrounds once counterclockwise the spectrum \( \sigma(A) \) of \( A \).
(2) If \( P(X_1, \ldots, X_k) \) is a polynomial in \( k \) non-commuting indeterminates \( X_1, \ldots, X_k \) and \( \mathcal{A} \) is a \( C^* \)-algebra, then the evaluation \( P(a_1, \ldots, a_k) \), \( a_j \in \mathcal{A}^{m \times n} \), \( n \in \mathbb{N} \), is an nc function. More generally, this can be extended to power series \( P \) with (finite or infinite) radius of convergence (see, for instance, [28]).

(3) If \( \mathcal{A} \) is a unital \( C^* \)-algebra and \( B \subseteq \mathcal{A} \) is an inclusion of \( C^* \)-algebras which share the same unit, assume that \( E: \mathcal{A} \to B \) is a unit-preserving conditional expectation. If \( X = X^* \in \mathcal{A} \), then the map \( G_X \) defined by \( G_X(a) = (E \otimes \text{Id}_{\mathbb{C}^{m \times n}}) [(b - X \otimes I_n)^{-1}] \), \( b \in B^{m \times n} \), is an nc function (see [32, 33]). Its domain is the set of all \( b \) such that \( b - X \otimes I_n \) is invertible. The noncommutative upper half-plane \( \bigcup_{n=1}^{\infty} \{ b \in B^{m \times n} : (b - b^*)/2i > 0 \} \) is a natural nc subdomain on which \( G_X \) is defined.

2.2. Noncommutative kernels. This section follows mostly [11]. Let \( \Omega_{nc} \) be a noncommutative subset of the operator space \( \mathcal{V} \). Consider two other operator spaces \( \mathcal{V}_0 \) and \( \mathcal{V}_1 \). Denote by \( \mathcal{L}(\mathcal{V}_0, \mathcal{V}_1) \) the space of linear operators from \( \mathcal{V}_0 \) to \( \mathcal{V}_1 \). A global kernel on \( \Omega_{nc} \) is a function \( K: \Omega_{nc} \times \Omega_{nc} \to \mathcal{L}(\mathcal{V}_0, \mathcal{V}_1)_{nc} \) such that

\( a \in \Omega_{nc}, c \in \Omega_{nc} \implies K(a, c) \in \mathcal{L}(\mathcal{V}_0^{m \times n}, \mathcal{V}_1^{m \times n}) \)

\( K \left( \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}, \begin{bmatrix} c & 0 \\ 0 & \bar{c} \end{bmatrix} \right) \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{2,1} & P_{2,2} \end{bmatrix} = \begin{bmatrix} K(a, c)(P_{1,1}) & K(a, c)(P_{1,2}) \\ K(\bar{a}, \bar{c})(P_{2,1}) & K(\bar{a}, \bar{c})(P_{2,2}) \end{bmatrix} \)

for any \( m, m, n, n \in \mathbb{N} \), \( a \in \Omega_{nc}, \bar{a} \in \Omega_{nc}, c \in \Omega_{nc}, \bar{c} \in \Omega_{nc}, P_{1,1} \in \mathcal{V}_0^{m \times n}, P_{1,2} \in \mathcal{V}_0^{m \times n}, P_{2,1} \in \mathcal{V}_0^{n \times n}, P_{2,2} \in \mathcal{V}_0^{n \times n} \) (that is, \( \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{2,1} & P_{2,2} \end{bmatrix} \in \mathcal{V}_0^{(m+n) \times (n+m)} \)). Obviously, condition (5) can be extended to evaluations of \( K \) in diagonal matrices with arbitrarily many blocks on the diagonal. The kernel \( K \) is called an affine noncommutative kernel if in addition to condition (1), it respects intertwinings:

\( a \in \Omega_{nc}, \bar{a} \in \Omega_{nc}, S \in \mathcal{C}^{m \times m} \) are such that \( Sa = \bar{a}S \),
\( c \in \Omega_{nc}, \bar{c} \in \Omega_{nc}, T \in \mathcal{C}^{n \times n} \) are such that \( cT = T\bar{c} \),
\( P \in \mathcal{V}_0^{m \times n} \implies SK(a, c)(P)T = K(\bar{a}, \bar{c})(SPT) \).

Conditions (4) and (6) are equivalent to conditions (1), (5) and

\( a, \bar{a} \in \Omega_{nc}, S \in \text{GL}_m(\mathbb{C}) \) are such that \( SaS^{-1} = \bar{a} \),
\( c, \bar{c} \in \Omega_{nc}, T \in \text{GL}_n(\mathbb{C}) \) are such that \( T^{-1}cT = \bar{c} \),
\( P \in \mathcal{V}_0^{m \times n} \implies K(\bar{a}, \bar{c})(P) = SK(a, c)(S^{-1}PT^{-1})T \).

If \( f: \Omega_{nc} \to \mathcal{W}_{nc} \) is a noncommutative map, then \( \Omega_{nc} \times \Omega_{nc} \ni (a, c) \mapsto \Delta f(a, c) \in \mathcal{L}(\mathcal{V}, \mathcal{W})_{nc} \) satisfies the above conditions (see [24, Proposition 2.15]).

We call \( K \) a noncommutative (nc) kernel if \( K \) satisfies (1) and respects intertwinings in the following sense:

\( a \in \Omega_{nc}, \bar{a} \in \Omega_{nc}, S \in \mathcal{C}^{m \times m} \) are such that \( Sa = \bar{a}S \),
\( c \in \Omega_{nc}, \bar{c} \in \Omega_{nc}, T \in \mathcal{C}^{n \times n} \) are such that \( Tc = \bar{c}T \),
\( P \in \mathcal{V}_0^{m \times n} \implies SK(a, c)(P)T^* = K(\bar{a}, \bar{c})(SPT^*) \).

Conditions (4) and (8) are equivalent to conditions (1), (5) and

\( a, \bar{a} \in \Omega_{nc}, S \in \text{GL}_m(\mathbb{C}) \) are such that \( SaS^{-1} = \bar{a} \),
\( c, \bar{c} \in \Omega_{nc}, T \in \text{GL}_n(\mathbb{C}) \) are such that \( TcT^{-1} = \bar{c} \),
\( P \in \mathcal{V}_0^{m \times n} \implies K(\bar{a}, \bar{c})(P) = SK(a, c)(S^{-1}P(T^{-1})^*)T^* \).
Observe that if \( K \) is an affine nc kernel, then \( (a,c) \mapsto K(a,c^*) \) is an nc kernel.

We say that a noncommutative kernel \( K \) is a completely positive noncommutative (cp nc) kernel if in addition

\[ a \in \Omega_m, P \geq 0 \text{ in } \mathcal{V}_0^{m \times m} \implies K(a,a)(P) \geq 0 \text{ in } \mathcal{V}_1^{m \times m} \text{ for all } m \in \mathbb{N}, \]

If \( \mathcal{V}_0, \mathcal{V}_1 \) are \( C^* \)-algebras, then (10) is equivalent to requiring that for all \( N \in \mathbb{N}, m_1, m_2, \ldots, m_N \in \mathbb{N}, \)

\[ a^{(j)} \in \Omega_{m_j}, P_j \in \mathcal{V}_0^{N \times m_j}, 1 \leq j \leq N \implies \sum_{i,j=1}^N b_{ij}^* K(a^{(i)}, a^{(j)})(P_i^* P_j) b_{ij} \geq 0 \]

(see [11 Proposition 2.2]). If \( K(a,a) \) is completely positive, then it is also completely bounded and \( \|K(a,a)\| = \|K(a,a)\|_{cb} = \|K(a,a)(1)\| \).

**Example 2.2.** Let \( A \) be a \( C^* \)-algebra. The simplest non-constant nc kernel is \( \mathcal{A}_{nc} \times \mathcal{A}_{nc} \ni (a,c) \mapsto a \cdot c^* \in \mathcal{L}(A,A)_{nc} \). That is, for \( m,n \in \mathbb{N}, a \in \mathcal{A}_{nc}^{m \times m}, c \in \mathcal{A}_{nc}^{n \times n}, P \in \mathcal{A}_{nc}^{m \times n} \), we have \( (a,c) \mapsto (P \mapsto aPc^*) \). More generally, if \( G, H \) are nc functions from \( \Omega_{nc} \subseteq \mathcal{V}_{nc} \) to \( \mathcal{A}_{nc} \), then \( (a,c) \mapsto G(a) \cdot H(c)^* \) is an nc kernel. One can further pre-compose this kernel with a completely bounded map \( \Psi : A \rightarrow A \):

\[ \Omega_m \times \Omega_n \ni (a,c) \mapsto [A^{m \times n} \ni P \mapsto G(a)(\text{Id}_{C^{m \times n}} \otimes \Psi)(P)H(c)^*] \in A^{m \times n} \]

is an nc kernel. If \( G = H \) and \( \Psi \) is completely positive, then this is a cp nc kernel. In a certain sense, all nc kernels are of this form (we refer to [11 Theorem 3.1] for the precise statement). Note also that \( (a,c) \mapsto [P \mapsto G(a)(\text{Id}_{C^{m \times n}} \otimes \Psi)(P)H(c)^*] \) is an affine nc kernel.

**Example 2.3.** One of the main objectives of this paper is to analyze certain metric properties of noncommutative sets. An important class of such sets is given precisely by noncommutative kernels. Let \( \mathcal{A} \) be a \( C^* \)-algebra, \( \mathcal{V} \) be an operator space and \( \Omega_{nc} \subseteq \mathcal{V}_{nc} \) be an nc set. Assume that \( K : \Omega_{nc} \times \Omega_{nc} \rightarrow \mathcal{L}(A)_{nc} \) is a noncommutative kernel. We may define the set

\[ \mathcal{D}_K := \bigcap_{n=1}^{\infty} \{ a \in \Omega_n : K(a,a)(I_n) > 0 \}. \]

Observe that if \( K \) were assumed instead to be an affine nc kernel, then the above definition would change to \( \mathcal{D}_n = \{ a \in \Omega_n : K(a,a^*)(I_n) > 0 \} \). Clearly \( \mathcal{D}_K \) may be empty or equal to \( \Omega_{nc} \).

If \( a \in \Omega_m, \tilde{a} \in \Omega_{m\overline{m}}, \) then, by (4) and (5), \( K(a \oplus \tilde{a}, a \oplus \tilde{a})(I_{m \oplus \overline{m}}) = K(a,a)(I_m \oplus I_{\overline{m}}) \)

\[ = \begin{bmatrix} K(a,a)(I_m) & 0 \\ 0 & K(\tilde{a}, \tilde{a})(I_{\overline{m}}) \end{bmatrix} > 0. \]

Thus, under the weaker assumptions that \( K \) is a global kernel, we are guaranteed that \( \mathcal{D}_K \) is a noncommutative set. Under our assumption that \( K \) is a noncommutative kernel, we have in addition that for any \( S \in GL_m(\mathbb{C}), \)

\[ K(SaS^{-1}, (S^{-1}aS^*)(I_m) = SK(a,a)(S^{-1}I_mS)S^{-1} = SK(a,a)(I_m)S^{-1}. \]

Thus, if \( S \) is unitary (that is, \( S^* = S^{-1} \)), then \( K(SaS^*, SaS^*)(I_m) > 0 \) whenever \( K(a,a)(I_m) > 0 \). We conclude that if \( K \) is an nc kernel on \( \Omega_{nc} \), then \( \mathcal{D}_K \) is a noncommutative set which is invariant with respect to conjugation by scalar unitary matrices.
Some of the more famous examples of noncommutative sets are given by nc kernels:

(i) The noncommutative upper half-plane \( H^+(A) = \prod_{n=1}^\infty H^+(A^{n\times n}) \), where \( H^+(A^{n\times n}) = \{ a \in A^{n\times n} : \Im a > 0 \} \) (we remind the reader that \( \Im b = (b - b^*)/2i, \Re b = (b + b^*)/2 \), so that \( b = \Re b + i\Im b \)). The kernel in this case is \( K(a,c)(P) = (aP - (cP^*)^*)/2i, a \in A^{m\times m}, c \in A^{n\times n}, P \in A^{m\times n} \). It is easy to verify that this is a globally defined nc kernel. This set is important in free probability (see \[32, 33\]).

(ii) The unit ball \( B_1(A) = \prod_{n=1}^\infty B_1(A^{n\times n}) \), where \( B_1(A^{n\times n}) = \{ a \in A^{n\times n} : \|a\| < 1 \} \) (the norm considered being the C*-norm on \( A^{n\times n} \)). Here the kernel is even simpler: \( K(a,c)(P) = 1 - aPc^* \).

(iii) More generally, if \( G \) is a noncommutative function with values in \( A \), we could define \( H^+(A)_G \) by using the kernel \( K(a,c)(P) = (G(a)P - (G(c)P^*)^*)/2i \) and \( B_1(A)_G \) by using the kernel \( K(a,c)(P) = 1 - G(a)P(c)^* \).

However, some are not:

(iv) Consider \( \mathcal{N}(A) = \prod_{n=1}^\infty \{ a \in A^{n\times n} : a^n = 0 \} \). Clearly \( \mathcal{N}(A) \) is closed under direct sums, and, moreover, if \( S \in GL_n(\mathbb{C}) \) and \( a \in A^{n\times n} : a^n = 0 \), then \( (SaS^{-1})^n = Sa^nS^{-1} = 0 \). So this set is in fact invariant under conjugation by all of \( GL_n(\mathbb{C}) \), not just by the unitary group. This is because \( \mathcal{N}(A) \) is “thin,” in the sense that it has empty interior in all the natural topologies on nc sets (see below). Thus, one cannot expect that \( \mathcal{N}(A) \) is of the form \( D_K \) for an nc kernel \( K \).

### 2.3. Three topologies on noncommutative sets.

As already stated, operator spaces constitute the natural framework for noncommutative function theory. We recall that (see, for instance, \[20\]) if \( V \) is an operator space, then

\[
\|a + \tilde{a}\|_{m+m'} = \max\{\|a\|_m, \|\tilde{a}\|_{\tilde{m}}\}, \quad m, \tilde{m} \in \mathbb{N}, a, \tilde{a} \in V_1^{m\times m}, \tilde{a} \in V_1^{\tilde{m}\times \tilde{m}},
\]

and

\[
\|SaT\|_n \leq \|S\|\|a\|_m\|T\|, \quad m, n \in \mathbb{N}, a \in V_1^{m\times m}, S \in C_1^{n\times m}, T \in C_1^{m\times n}.
\]

A topology naturally compatible with these norm conditions is the uniformly-open topology. It has as basis balls defined the following way: if \( c \in V_1^{s\times s} \) and \( r \in (0, +\infty) \), then

\[
B_{nc}(c, r) = \prod_{n=1}^\infty \left\{ a \in V_1^{sn\times sn} : \|a - \oplus_{j=1}^n c\|_{sn} < r \right\}.
\]

This topology is not Hausdorff. A noncommutative function \( f \) defined on a noncommutative set \( \Omega_{nc} \subseteq V_{nc} \) with values in an operator space is said to be uniformly analytic if \( \Omega_{nc} \) is uniformly open, and \( f \) is uniformly locally bounded and complex differentiable at each level. It is shown in \[24\] Corollary 7.28 that \( f \) is analytic if and only if it is uniformly locally bounded (that is, the requirement of complex differentiability at each level is automatically satisfied by an nc function which is uniformly locally bounded on a uniformly open nc set).

The second important topology (already mentioned above) is the finitely open topology: a set \( \Omega_{nc} \subseteq V_{nc} \) is called finitely open if for any \( n \in \mathbb{N} \), the intersection of \( \Omega_n \) with any finite dimensional subspace \( X \) of \( V_1^{n\times n} \) is open in the Euclidean topology of \( X \). It is shown in \[24\] Theorem 7.2 that if \( f \) is a noncommutative function defined on \( \Omega_{nc} \) which is locally bounded on slices, then \( f \) is analytic on slices, in the sense that for any \( n \in \mathbb{N} \) and any finite dimensional subspace \( X \) of \( V_1^{n\times n} \), \( f|_X \) is analytic as a function of several complex variables.
Finally, one can also consider the topology in which a set $\Omega_{nc}$ is open in $\mathcal{V}_{nc}$ if and only if $\Omega_n$ is open in the topological vector space topology of $\mathcal{V}_{nc}^{n \times n}$ for all $n \in \mathbb{N}$. Observe that such a set is also finitely open. We refer to it as the level topology.

3. A (pseudo)distance on noncommutative sets

Let $\mathcal{V}$ be a complex topological vector space. As we progress through the paper, we put more and more structure on $\mathcal{V}$, but for our first definition, we need nothing more than the axioms of a complex topological vector space. For now we endow $\mathcal{V}_{nc}^{n \times m}$, $n, m \in \mathbb{N}$, with the usual (product) topology. Let $\mathcal{D}$ be a noncommutative subset of $\mathcal{V}_{nc}$ and consider the following properties:

1. For any $n \in \mathbb{N}$, $D_n$ is open in $\mathcal{V}_{nc}^{n \times n}$;
2. If $U$ is a unitary $n \times n$ complex matrix and $a \in D_n$, then $UaU^* \in D_n$;
3. If $a \in \mathcal{V}_{nc}^{n \times n}, c \in \mathcal{V}_{nc}^{m \times m}$ are such that $\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \in D_{n+m}$, then $a \in D_n, c \in D_m$. (Note that this is a sort of “converse” of part (b) of the definition of noncommutative sets.)

Let $S_{n,m} = \{ g : \mathcal{V}_{nc}^{n \times m} \to [0, +\infty) : g(tb) = tg(b) \forall t \geq 0 \}$ (with the convention $0 \times (+\infty) = +\infty$), and define $S = \bigcap_{n,m \in \mathbb{N}} S_{n,m}$. Define a function $\delta_D : \mathcal{D} \times \mathcal{D} \to S$ such that $\delta_D(a,c) \in S_{n,m}$ whenever $a \in D_n, c \in D_m$, by

$$
\delta_D(a,c)(b) = \left( \sup \left\{ t \in [0, +\infty) : \begin{bmatrix} a & sb \\ 0 & c \end{bmatrix} \in D_{n+m} \text{ for all } s \in [0,t] \right\} \right)^{-1},
$$

with the convention $1/0 = +\infty$. Observe first that $\delta_D(a,c)$ is indeed well-defined because noncommutative sets respect direct sums: $\begin{bmatrix} a & sb \\ 0 & c \end{bmatrix} \in D_{n+m}$ at least for $s = 0$. Second, $\delta_D(a,c)(b) = 0 \iff \begin{bmatrix} a & sb \\ 0 & c \end{bmatrix} \in D_{n+m}$ for all $s \in [0, +\infty)$. Third, if $s_0 \in (0, +\infty)$ is given, then, as indicated in the definition, $\delta_D(a,c)(s_0b) = s_0\delta_D(a,c)(b)$.

Indeed, if $\delta_D(a,c)(b) = 0$ or $+\infty$, then the statement is obvious. Else, if $\begin{bmatrix} a & sb \\ 0 & c \end{bmatrix} \in D_{n+m}$ for all $s \in [0,\delta_D(a,c)(b)^{-1})$, then $\begin{bmatrix} a & sb \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & \frac{r}{s_0} s_0b \\ 0 & c \end{bmatrix}$, so that $\begin{bmatrix} a & r(s_0b) \\ 0 & c \end{bmatrix} \in D_{n+m}$ for all $r \in [0, (s_0\delta_D(a,c)(b))^{-1})$, which shows that $s_0\delta_D(a,c)(b) = \delta_D(a,c)(s_0b)$.

Remark 3.1. A complex vector space $\mathcal{V}$ endowed with a topology for which the multiplication with positive scalars is continuous (a requirement automatically satisfied by a topological vector space), the quantity $\delta$ is upper semicontinuous in its three variables whenever it is defined on an nc set which satisfies property (1) above. Indeed, consider such an nc set $\Omega \subseteq \mathcal{V}_{nc}$. It is enough to prove the statement at level one. Thus, consider three nets $\{a_i\}_{i \in I}, \{c_i\}_{i \in I}$, and $\{b_i\}_{i \in I}$ converging to $a, c \in \Omega_1$ and $b \in \mathcal{V}$, respectively. Let $t \in (0, +\infty)$ be chosen so that $\begin{bmatrix} a_i & 0 \\ 0 & c_i \end{bmatrix} \in \Omega_2$ for all $s \leq t$. Since $\Omega_2$ is open in the topology of $\mathcal{V}_{nc}^{2 \times 2}$, there exists an $t_0 \in I$ such that $\begin{bmatrix} a_i & 0 \\ 0 & c_i \end{bmatrix} \subseteq \Omega_2$ for all $i \geq t_0$ (we have used here the compactness of $[0, t]$). Thus, $t^{-1} > \delta(a,c)(b)$
implies that \( t^{-1} > \delta(a_i, c_i)(b_i) \) for all \( i \) large enough. This implies that
\[
\limsup_{i \to t} \delta(a_i, c_i)(b_i) \leq \delta(a, c)(b), \quad a, c \in \Omega_1, b \in V.
\]
This shows that \( \delta \) is upper semicontinuous on nc sets that satisfy property (i) under very mild conditions on the topology of the underlying vector space. Remarkably, under the supplementary hypothesis that the intersection \( \partial \Omega_{2k} \cap \left[ \begin{array}{cc} a & b \\ 0 & c \end{array} \right] \) is discrete for all \( b \in V^{k \times k}, a, c \in \Omega_k \), the exact same argument applied to the complement of \( \Omega \) shows that \( \delta \) is lower semicontinuous, and thus continuous.

The following proposition is straightforward, but, unless some of the hypotheses (i) – (iii) from above are assumed, it may well be vacuous.

**Proposition 3.2.** Let \( V, W \) be two complex topological vector spaces and let \( D \) and \( E \) be two noncommutative subsets of \( V_{nc} \) and \( W_{nc} \), respectively. Assume that \( f : D \to E \) is a function such that

(a) for any \( a \in D_n \), we have \( f(a) \in E_n \);

(b) \( f \) respects direct sums;

(c) if \( a \in D_n, c \in D_m \) and \( b \in V^{n \times m} \) are such that \( \left[ \begin{array}{cc} a & b \\ 0 & c \end{array} \right] \in D_{n+m} \), then there exists a function of three variables denoted \( \Delta f(a, c)(b) \) such that \( \Delta f(a, c)(tb) = t\Delta f(a, c)(b) \) for all \( t \in [0, +\infty) \) with the property that \( tb \) is in the domain of \( \Delta f(a, c)(\cdot) \), and \( f \) satisfies

\[
f \left( \left[ \begin{array}{cc} a & b \\ 0 & c \end{array} \right] \right) = \left[ \begin{array}{cc} f(a) & \Delta f(a, c)(b) \\ 0 & f(c) \end{array} \right].
\]

Then

\[
\delta_D(a, c)(b) \geq \delta_E(f(a), f(c))(\Delta f(a, c)(b)), \quad a \in D_n, c \in D_m, b \in V^{n \times m}.
\]

Note that the hypothesis on the homogeneity of \( \Delta f(a, c)(b) \) in \( b \) is meaningful only if there exists some interval \( (t, r) \) such that \( \left[ \begin{array}{cc} a & sb \\ 0 & c \end{array} \right] \in D_{n+m} \) for all \( s \in (t, r) \). Otherwise, one can simply define \( \Delta f(a, c)(sb) \) as \( s\Delta f(a, c)(b) \).

**Proof.** The statement is tautological: consider \( a \in D_n, c \in D_m \) and \( b \in V^{n \times m} \) such that \( \left[ \begin{array}{cc} a & sb \\ 0 & c \end{array} \right] \in D_{n+m} \) for all \( s \in [0, t_0] \). If \( t_0 = +\infty \), then \( \delta_D(a, c)(b) = 0 \) and

\[
f \left( \left[ \begin{array}{cc} a & sb \\ 0 & c \end{array} \right] \right) = \left[ \begin{array}{cc} f(a) & \Delta f(a, c)(sb) \\ 0 & f(c) \end{array} \right] = \left[ \begin{array}{cc} f(a) & s\Delta f(a, c)(b) \\ 0 & f(c) \end{array} \right]
\]

for all \( s \in [0, +\infty) \), so that \( \delta_E(f(a), f(c))(\Delta f(a, c)(b)) = 0 \). If \( t_0 = 0 \) (i.e. \( \delta_D(a, c)(b) = +\infty \)), then the inequality \( \delta_D(a, c)(b) \geq \delta_E(f(a), f(c))(\Delta f(a, c)(b)) \) is obvious thanks to hypothesis (b). Finally, if \( t_0 = \delta_D(a, c)(b)^{-1} \in (0, +\infty) \), then

\[
f \left( \left[ \begin{array}{cc} a & sb \\ 0 & c \end{array} \right] \right) = \left[ \begin{array}{cc} f(a) & \Delta f(a, c)(sb) \\ 0 & f(c) \end{array} \right] = \left[ \begin{array}{cc} f(a) & s\Delta f(a, c)(b) \\ 0 & f(c) \end{array} \right] \in E_{n+m}
\]

for all \( s \in [0, t_0] \), which implies \( t_0 = \delta_D(a, c)(b)^{-1} \leq \delta_E(f(a), f(c))(\Delta f(a, c)(b))^{-1} \). This concludes the proof. \( \square \)
Remark 3.3. (1) If we assume hypotheses (1) for \( \mathcal{D} \), then for any \( a \in \mathcal{D}_n, b \in \mathcal{D}_m \), we are guaranteed that there exists a \( t_0 \in (0, +\infty) \) such that

\[
\begin{bmatrix}
a & sb \\
0 & c
\end{bmatrix} \in \mathcal{D}_{n+m} \text{ for all } s \in [0, t_0].
\]

Thus, under a very mild assumption of openness in a complex topological vector space, we are guaranteed that \( \delta_{\mathcal{D}}(a, c)(b) \) is finite (possibly zero).

(2) Assumption (2) on \( \mathcal{D} \) is sufficient (although not necessary) in order to guarantee that

\[
\begin{bmatrix}
a & z b \\
0 & c
\end{bmatrix} \in \mathcal{D}_{n+m} \text{ for all } z \in \mathbb{C}, \ |z| < \delta_{\mathcal{D}}(a, c)(b)^{-1}.\]

Indeed, one simply conjugates \( \begin{bmatrix} a & sb \\ 0 & c \end{bmatrix} \) with the unitary \( \begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix} \in \mathbb{C}^{(n+m)\times(n+m)}, \) where \( \theta \) is the argument of \( z \).

(3) If, in Proposition 3.2, the sets \( \mathcal{D} \) and \( \mathcal{E} \) are assumed to satisfy hypotheses (1) and (2), and in addition \( b \mapsto \Delta f(a, c)(b) \) satisfies \( \Delta f(a, c)(zb) = z\Delta f(a, c)(b) \), then we are guaranteed that the statement of the proposition is not vacuous. In particular,

Corollary 3.4. If \( f : \mathcal{D} \to \mathcal{E} \) is a locally bounded noncommutative function on a finitely open subset, then \( f \) satisfies \( \delta_{\mathcal{E}}(f(a), f(c))(\Delta f(a, c)(b)) \leq \delta_{\mathcal{D}}(a, c)(b), \ a \in \mathcal{D}_n, c \in \mathcal{D}_m, b \in \mathcal{V}_{n\times m}, n, m \in \mathbb{N} \).

Next, we study some of the properties of \( \delta_{\mathcal{D}} \) in more detail.

Lemma 3.5. Assume that the noncommutative subset \( \mathcal{D} \) of \( \mathcal{V}_{nc} \) satisfies properties (2) and (3). For any unitary matrices \( U \in \mathbb{C}^{n\times n}, V \in \mathbb{C}^{m\times m}, a_1, a_2 \in \mathcal{D}_n, c_1, c_2 \in \mathcal{D}_m, b_{11} \in \mathcal{V}_{n\times n}, b_{12} \in \mathcal{V}_{n\times m}, b_{21} \in \mathcal{V}_{m\times n}, b_{22} \in \mathcal{V}_{m\times m}, \) we have

\[
\delta_{\mathcal{D}}(Ua_1U^*, Vc_2V^*)(Ub_{12}V^*) = \delta_{\mathcal{D}}(a_1, c_2)(b_{12})
\]

(14)

\[
\delta_{\mathcal{D}} \left( \begin{bmatrix} a_1 & 0 \\ 0 & c_1 \end{bmatrix}, \begin{bmatrix} a_2 & 0 \\ 0 & c_2 \end{bmatrix} \right) \left[ \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix} \right] = \max \{ \delta_{\mathcal{D}}(a_1, c_1)(b_{11}), \delta_{\mathcal{D}}(a_2, c_2)(b_{22}) \},
\]

(15)

\[
\delta_{\mathcal{D}} \left( \begin{bmatrix} a_1 & 0 \\ 0 & c_1 \end{bmatrix}, \begin{bmatrix} a_2 & 0 \\ 0 & c_2 \end{bmatrix} \right) \left[ \begin{bmatrix} 0 & b_{12} \\ b_{21} & 0 \end{bmatrix} \right] = \max \{ \delta_{\mathcal{D}}(a_1, c_2)(b_{12}), \delta_{\mathcal{D}}(c_1, a_2)(b_{21}) \}.
\]

(16)

Proof. Relation (14) follows trivially from hypothesis (2): for \( s \geq 0 \), we have the chain of equivalences

\[
\begin{bmatrix}
a_1 & sb_{12} \\
0 & c_2
\end{bmatrix} \in \mathcal{D}_{n+m} \iff \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} a_1 & sb_{12} \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & V^* \end{bmatrix} \in \mathcal{D}_{n+m}
\]

\[
\iff \begin{bmatrix} Ua_1U^* & sUb_{12}V^* \\ 0 & Vc_2V^* \end{bmatrix} \in \mathcal{D}_{n+m}.
\]

A slight variation of this trick proves (15) and (16). Let

\[
U_0 = \begin{bmatrix}
0_{m\times n} & I_m & 0_{m\times n} & 0_{m\times m} \\
0_{n\times n} & 0_{n\times m} & I_n & 0_{n\times m} \\
I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times m} \\
0_{m\times n} & 0_{m\times m} & 0_{m\times n} & I_m
\end{bmatrix},
\]
a complex \((2n + 2m) \times (2n + 2m)\) unitary matrix. For any \(s \geq 0\), we have

\[
\begin{bmatrix}
a_1 & 0 & 0 & sb_{12} \\
0 & c_1 & sb_{21} & 0 \\
0 & 0 & a_2 & 0 \\
0 & 0 & 0 & c_2
\end{bmatrix} \in \mathcal{D}_{2(n+m)} \iff U_0 \begin{bmatrix}
a_1 & 0 & 0 & sb_{12} \\
0 & c_1 & sb_{21} & 0 \\
0 & 0 & a_2 & 0 \\
0 & 0 & 0 & c_2
\end{bmatrix} U_0^* \in \mathcal{D}_{2(n+m)} 
\]

\[
\iff \begin{bmatrix}
c_1 & sb_{21} & 0 & 0 \\
0 & a_2 & 0 & 0 \\
0 & 0 & a_1 & sb_{12} \\
0 & 0 & 0 & c_2
\end{bmatrix} \in \mathcal{D}_{m+n+n+m} 
\]

\[
\iff \begin{bmatrix}
c_1 & sb_{21} & 0 & 0 \\
0 & a_2 & 0 & 0 \\
0 & 0 & a_1 & sb_{12} \\
0 & 0 & 0 & c_2
\end{bmatrix}, \begin{bmatrix}
a_1 & sb_{12} \\
0 & c_2
\end{bmatrix} \in \mathcal{D}_{n+m},
\]

where we have used property (3) in the last equivalence and property (2) in the first. This proves (16). Relation (15) is proved the same way.

\(\square\)

The next lemma shows that, in a certain way, \(\delta_D\) is itself a sort of noncommutative function.

**Lemma 3.6.** Assume that \(\mathcal{D} \subseteq \mathcal{V}_{nc}\) satisfies properties (2) and (3). For any \(n, m \in \mathbb{N}, a \in \mathcal{D}_n, c \in \mathcal{D}_m, b \in \mathcal{V}^{n \times m}\), and any \(k \in \mathbb{N}\), we have

\[
\delta_D(I_k \otimes a, I_k \otimes c)(Z \otimes b) = \delta_D(a, c)(b)\|ZZ^*\|^{\frac{2}{k}}, \quad Z \in \mathbb{C}^{k \times k}.
\]

**Proof.** We shall prove this lemma in two steps. In the first step, we assume that \(a = c\) (and implicitly \(m = n\)). Consider unitary matrices \(U, V^* \in \mathbb{C}^{k \times k}\) which diagonalize \(Z\):

\[
UZV^* = \text{diag}(\lambda_1, \ldots, \lambda_k), \text{ where } 0 \leq \lambda_1 \leq \cdots \leq \lambda_k = \|Z^*Z\|^{\frac{1}{2}} \text{ are the singular values of } Z.
\]

Then

\[
\begin{bmatrix}
U \otimes 1 & 0 \\
0 & V \otimes 1
\end{bmatrix}
\begin{bmatrix}
I_k \otimes a & Z \otimes b \\
0 & I_k \otimes a
\end{bmatrix}
\begin{bmatrix}
U^* \otimes 1 & 0 \\
0 & V^* \otimes 1
\end{bmatrix}
\begin{bmatrix}
I_k \otimes a & UZV^* \otimes b \\
0 & I_k \otimes a
\end{bmatrix}.
\]

Thus, by property (2),

\[
\begin{bmatrix}
I_k \otimes a & sZ \otimes b \\
0 & I_k \otimes a
\end{bmatrix} \in \mathcal{D}_{2kn}
\text{ if and only if the matrix }
\begin{bmatrix}
a & 0 & \cdots & 0 & s\lambda_1 b & 0 & \cdots & 0 \\
0 & a & \cdots & 0 & s\lambda_2 b & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a & 0 & \cdots & s\lambda_k b \\
0 & 0 & \cdots & 0 & a & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & a
\end{bmatrix} \in \mathcal{D}_{2kn}.
\]
Successive permutations transform this into the condition
\[
\begin{bmatrix}
a & s\lambda_1 b & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & a & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & a & s\lambda_j b & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & a & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & a & s\lambda_k b \\
0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & a \\
\end{bmatrix} \in \mathcal{D}_{2kn},
\]
i.e.
\[
\text{diag}\left(\begin{array}{ccc}
a & s\lambda_1 b \\
0 & a \\
\vdots & \vdots \\
0 & 0 \\
\end{array}\right), \ldots, \begin{array}{ccc}
a & s\lambda_k b \\
0 & a \\
\vdots & \vdots \\
0 & 0 \\
\end{array}\right) \in \mathcal{D}_{2kn}.
\]
By property (3) we have that this happens if and only if each block \(\begin{bmatrix} a & s\lambda_j b \\ 0 & a \end{bmatrix}\) belongs to \(\mathcal{D}_{2n}\). Since the largest singular value \(\lambda_k\) of \(Z\) equals \(\|Z^*Z\|^{\frac{1}{2}}\), the first step is proved.

In order to prove the second step, we use equation (16) of Lemma 3.5, which guarantees that
\[
\delta_\mathcal{D}(I_k \otimes a, I_k \otimes (Z \otimes b)) = \delta_\mathcal{D}\left(\begin{bmatrix} I_k \otimes a & 0 \\ 0 & I_k \otimes c \end{bmatrix}, \begin{bmatrix} I_k \otimes a & 0 \\ 0 & I_k \otimes c \end{bmatrix}\right)\left(\begin{bmatrix} 0 & Z \otimes b \\ 0 & 0 \end{bmatrix}\right).
\]
By conjugating with a permutation matrix, it follows, again via Lemma 3.5 and the first step, that
\[
\delta_\mathcal{D}\left(\begin{bmatrix} I_k \otimes a & 0 \\ 0 & I_k \otimes c \end{bmatrix}, \begin{bmatrix} I_k \otimes a & 0 \\ 0 & I_k \otimes c \end{bmatrix}\right) = \delta_\mathcal{D}\left(I_k \otimes a, I_k \otimes c\right)\left(\begin{bmatrix} 0 & Z \otimes b \\ 0 & 0 \end{bmatrix}\right)
= \delta_\mathcal{D}\left(\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}\right)\left(\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right) \|Z^*Z\|^{\frac{1}{2}}
= \delta_\mathcal{D}(a, c)(b) \|Z^*Z\|^{\frac{1}{2}}.
\]

With these two lemmas, we can prove now the main result of this section. For simplicity, denote
\[
\tilde{\delta}_\mathcal{D}(a, c) := \delta_\mathcal{D}(a, c)(a - c).
\]

**Theorem 3.7.** Assume that \(\mathcal{D} \subseteq \mathcal{V}_{nc}\) satisfies properties (2) and (3). The following statements are equivalent for any \(m, n \in \mathbb{N}\):

(i) For any \(a \in \mathcal{D}_n, c \in \mathcal{D}_m, \delta_\mathcal{D}(a, c)(b) = 0 \implies b = 0;\)
(ii) For any \(a \in \mathcal{D}_n, \delta_\mathcal{D}(a, a)(b) = 0 \implies b = 0;\)
(iii) For any \(a, c \in \mathcal{D}_n, \tilde{\delta}_\mathcal{D}(a, c) = 0 \implies a = c;\)
(iv) For any \(k \in \mathbb{N}\), there exists no non-constant noncommutative function \(f : \mathbb{C}_{nc} \rightarrow (\mathcal{D}_k)_{nc}\).

By \((\mathcal{D}_k)_{nc}\) we denote all the levels of \(\mathcal{D}\) which are multiples of \(k\).
and the above definition. Kobayashi metrics. The following corollary is a direct consequence of Proposition 3.2 between two hyperbolic domains is a contraction with respect to the corresponding
for ˜
∈ D
a,c,v
all) the conditions of Theorem 3.7 are satisfied.
We shall call such a finite sequence
matrix and an application of Lemma 3.6, we conclude that
\[ \delta_D\left(\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}\right) \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \delta_D(a, c)(b). \]
Finally, to prove (iv) \(\implies\) (ii), assume that we found \(a_0 \in D_{n_0}\) and \(b_0 \in \mathbb{Z}^{n_0 \times n_0} \setminus \{0\}\) such that \(\delta_D(a_0, a_0)(b_0) = 0\). We build the linear noncommutative function \(f(Z) = \begin{bmatrix} a_0 & 0 \\ 0 & a_0 \end{bmatrix} + Z \otimes \begin{bmatrix} 0 & b_0 \\ 0 & 0 \end{bmatrix}, Z \in C^{p \times p}\). By a conjugation with a permutation matrix and an application of Lemma 3.6 we conclude that \(f\) takes values in \(D_{2p n_0}\), so that (iv) does not hold. This completes the proof.

The function \(\tilde{\delta}_D\) allows us to define a distance (possibly degenerate) on \(D\), by mimicking the definition of the Kobayashi distance, with \(\tilde{\delta}_D\) playing the role of Lempert function.

**Definition 3.8.** If \(D\) is a noncommutative set in an operator space satisfying assumptions 2 and 3, then for any \(n \in \mathbb{N}, a, c \in D_n\),
\[ \tilde{d}_D(a, c) = \inf \left( \left\{ \sum_{j=1}^{N} \tilde{\delta}_D(a_{j-1}, a_j) : a_j \in D_n, 0 \leq j \leq N, a_0 = a, a_N = c, N \in \mathbb{N} \right\} \right). \]
We shall call such a finite sequence \(a = a_0, a_1, \ldots, a_N = c\) a division of \(\tilde{d}_D(a, c)\).

The function \(\tilde{d}_D : D \times D \to [0, +\infty]\) fails to separate the points of \(D\) if one (and hence all) the conditions of Theorem 3.7 are satisfied.

It is quite easy to show that \(\tilde{d}\) is a distance. Indeed, since \(\tilde{\delta}_D(a, c) = \tilde{\delta}_D(c, a)\), it follows that \(\tilde{d}_D(a, c) = \tilde{d}_D(c, a)\). So only the triangle inequality remains to be proved. Let \(a, c, v \in D_n\). If \(a_0 = a, a_1, \ldots, a_N = c\) and \(a_N = c, \ldots, a_{N+p-1} = a, a_{N+p} = v\) are divisions for \(\tilde{d}_D(a, c)\) and \(\tilde{d}_D(c, v)\), respectively, then \(a_0 = a, a_1, \ldots, a_N, a_{N+1}, \ldots, a_{N+p} = v\) is a division for \(\tilde{d}_D(a, v)\). In particular,
\[ \sum_{j=1}^{N} \tilde{\delta}_D(a_{j-1}, a_j) + \sum_{j=N+1}^{N+p} \tilde{\delta}_D(a_{j-1}, a_j) \geq \inf \left\{ \sum_{j=1}^{M} \tilde{\delta}_D(d_{j-1}, d_j) : d_0, \ldots, d_j \text{ division of } \tilde{d}_D(a, v) \right\} \]
for all divisions \(a_0 = a, a_1, \ldots, a_N = c\) for \(\tilde{d}_D(a, c)\) and \(a_N = c, \ldots, a_{N+p-1}, a_{N+p} = v\) for \(\tilde{d}_D(c, v)\). Taking infimum separately after each division provides
\[ \tilde{d}_D(a, c) + \tilde{d}_D(c, v) \geq \tilde{d}_D(a, v). \]

The most general version of the Schwarz-Pick Lemma tells us that an analytic map between two hyperbolic domains is a contraction with respect to the corresponding Kobayashi metrics. The following corollary is a direct consequence of Proposition 3.2 and the above definition.
Corollary 3.9. Let \( \mathcal{D}, \mathcal{E} \) be two noncommutative sets satisfying assumptions (2) and (3). Let \( f: \mathcal{D} \to \mathcal{E} \) be a noncommutative function. Then \( f \) is a contraction with respect to the above-defined metric:

\[
\tilde{d}_\mathcal{E}(f(a), f(c)) \leq \tilde{d}_\mathcal{D}(a, c), \quad a, c \in \mathcal{D}_n, n \in \mathbb{N}.
\]

Note that assuming also hypothesis (1) in the above corollary guarantees that the two sides of the inequality above are both finite (possibly zero).

Until now we have made no assumptions on the openness of \( \mathcal{D} \). As seen in Remark 3.1, hypotheses (1)—(3) guarantee that \( \delta_\mathcal{D} \) is upper semicontinuous in its three variables, and in particular so is \( \delta_\mathcal{D} \). Thus, we may define an infinitesimal version of \( \tilde{d}_\mathcal{D} \).

**Definition 3.10.** If \( \mathcal{D} \) is a noncommutative set in an operator space satisfying assumptions (1)—(3), then for any \( n \in \mathbb{N}, a, c \in \mathcal{D}_n \),

\[
d_\mathcal{D}(a, c) = \inf \left\{ \int_{[0,1]} \delta(a(t), a(t))(a'(t)) \, dt : \right. \\
a: [0,1] \to \mathcal{D}_n \text{ continuously differentiable, } a(0) = a, a(1) = c \left. \right\}.
\]

Note that the openness of \( \mathcal{D}_n \) implies that \( \delta(a(t), a(t))(a'(t)) \) is finite for all \( t \in [0,1] \). Since upper semicontinuous functions attain their supremum, this shows that \{\( \delta(a(t), a(t))(a'(t)) : t \in [0,1] \)\} is a bounded set, and the integrals defining \( d_\mathcal{D} \) are necessarily finite, so that \( d_\mathcal{D} \) is well-defined and finite (possibly zero). The fact that \( d_\mathcal{D} \) is a (possibly degenerate) metric follows easily: as before, it is only the triangle inequality that needs to be verified. If \( a, v, c \in \mathcal{D}_n \), then the above infimum over all paths from \( a \) to \( c \) is necessarily no greater than the infimum over all paths from \( a \) to \( c \) which go through \( v \). Since \( \delta_\mathcal{D} \) is continuous and paths which are continuous and differentiable everywhere except at one point can be approximated arbitrarily well by paths which are differentiable everywhere, it follows immediately that \( d_\mathcal{D}(a, c) \leq d_\mathcal{D}(a, v) + d_\mathcal{D}(v, c) \).

Another application of Proposition 3.2 shows that noncommutative functions are contractions also with respect to \( d_\mathcal{D} \). We record this fact below.

Corollary 3.11. Let \( \mathcal{D}, \mathcal{E} \) be two noncommutative sets satisfying assumptions (1)—(3). Let \( f: \mathcal{D} \to \mathcal{E} \) be a noncommutative function. Then \( f \) is a contraction with respect to the above-defined metric:

\[
d_\mathcal{E}(f(a), f(c)) \leq d_\mathcal{D}(a, c), \quad a, c \in \mathcal{D}_n, n \in \mathbb{N}.
\]

We establish next the relation between \( \tilde{d}_\mathcal{D} \) and \( d_\mathcal{D} \) under the assumptions (1)—(3). As an immediate consequence of the upper semicontinuity of \( \delta \) (Remark 3.1), we obtain for any differentiable path \( a \) defined on \([0,1]\) and any \( t \in [0,1] \) the relation

\[
\limsup_{h \to 0} \delta_\mathcal{D}(a(t), a(t+h)) \left( \frac{a(t+h) - a(t)}{h} \right) \leq \delta_\mathcal{D}(a(t), a(t))(a'(t)).
\]

(When \( t = 0 \) or \( t = 1 \), the limit should of course be taken one-sided.) In particular given an arbitrary path \( a \), a division of \([0,1]\) translates into a division of \( \delta_\mathcal{D}(a,c) \). Given \( \varepsilon > 0 \), for any \( t \in [0,1] \) there exists \( \eta_{t, \varepsilon} > 0 \) such that \( \delta_\mathcal{D}(a(t), a(t+h)) \left( \frac{a(t+h) - a(t)}{h} \right) < \delta_\mathcal{D}(a(t), a(t))(a'(t)) + \varepsilon \) for any \(|h| < \eta_{t, \varepsilon} \). The family \( \{ t - \eta_{t, \varepsilon}, t + \eta_{t, \varepsilon} \}_{0 \leq t \leq 1} \) is an open cover of \([0,1]\), so that we may extract a finite subcover \( (t_0 - \eta_{t_0, \varepsilon}, t_0 + \eta_{t_0, \varepsilon}), (t_1 - \eta_{t_1, \varepsilon}, t_1 + \eta_{t_1, \varepsilon}), \ldots , (t_N - \eta_{t_N, \varepsilon}, t_N + \eta_{t_N, \varepsilon}) \), \( t_1 < \cdots < t_N \). Let \( t_0 = 0, t_{N+1} = 1 \). By choosing the smallest among \( \eta_{t_j, \varepsilon}, 1 \leq j \leq N \), and increasing the number of points \( t_j \) if necessary, we may assume
that \( \eta_{1,2} = \cdots = \eta_{N,2} = \eta > 0 \) and \( t_j \in (t_{j-1} - \eta, t_{j-1} + \eta) \cap (t_{j+1} - \eta, t_{j+1} + \eta) \). Then

\begin{align}
\hat{d}_D(a,c) &\leq \sum_{j=0}^{N} \hat{d}_D(a(t_j), a(t_{j+1})) \\
&= \sum_{j=0}^{N} (t_{j+1} - t_j) \delta_D(a(t_j), a(t_{j+1})) \left( \frac{a(t_{j+1}) - a(t_j)}{t_{j+1} - t_j} \right) \\
&< \sum_{j=0}^{N} (t_{j+1} - t_j) \delta_D(a(s_j), a(s_j))(a'(s_j)) + \varepsilon \quad (s_j \in [t_j, t_{j+1}])
\end{align}

(18)

\begin{align}
&\leq \sum_{j=0}^{N} (t_{j+1} - t_j) \int_{[t_j, t_{j+1}]} \delta_D(a(t), a(t))(a'(t)) \, dt + \varepsilon \\
&= \int_{[0,1]} \delta_D(a(t), a(t))(a'(t)) \, dt + \varepsilon.
\end{align}

We have used in (17) the definition of \( \hat{d}_D \), and in relation (18) the fact that we may choose \( s_j \) arbitrarily in \([t_j, t_{j+1}]\), and we decide to choose an \( s_j \) such that

\[ \delta_D(a(s_j), a(s_j))(a'(s_j)) \leq \frac{1}{t_{j+1} - t_j} \int_{[t_j, t_{j+1}]} \delta_D(a(t), a(t))(a'(t)) \, dt. \]

Since \( a \) has been arbitrarily chosen, it follows that \( \hat{d}_D(a,c) \leq d_D(a,c) \) for all \( a, c \) belonging to the same level of \( D \). Thus,

(19)

\[ \hat{d}_D \leq d_D \quad \text{for all } \mathcal{D} \subset \mathcal{V}_{nc} \text{ satisfying hypotheses (1)} - (3). \]

Since we have shown that \( \delta \) and \( \hat{\delta} \) generate distances, it is natural to ask what topology one may expect those distances to determine on the original space. In the most general case, we are able to make only the following statement:

**Proposition 3.12.** Assume that \( \mathcal{D} \) is a noncommutative subset of a topological vector space \( \mathcal{V} \) which satisfies assumptions (1) – (3). If \( n \in \mathbb{N} \) is given and a subset \( A \) of \( \mathcal{D}_n \) is open in the topology generated by \( d_D \), then it is open in the product topology induced by \( \mathcal{V} \) on \( \mathcal{V}^{n \times n} \).

**Proof.** Assume that \( a \in \mathcal{D}_n \) is given. For any net \( \{a_i\}_{i \in I} \subseteq \mathcal{D}_n \) which converges to \( a \) in the product topology of \( \mathcal{V}^{n \times n} \), we have by Remark (3.1) that

\[ 0 = \hat{\delta}_D(a, a) \geq \limsup_{i \in I} \hat{\delta}_D(a_i, a) \geq \limsup_{i \in I} \hat{d}_D(a_i, a) \geq 0. \]

Thus, \( \lim_{i \in I} \hat{d}_D(a_i, a) = 0 \) whenever \( \{a_i\}_{i \in I} \subseteq \mathcal{D}_n \) converges to \( a \) in the product topology of \( \mathcal{V}^{n \times n} \). This completes our proof. \( \square \)

4. **A Smooth (Pseudo)Metric**

We have seen above that simple properties of noncommutative sets allow us to define a distance which is often nondegenerate, and with respect to which analytic noncommutative functions are natural contractions. These results have a “metric space” flavour. In this section we consider the case when the distance defined has a “differential geometry” flavour.
4.1. Hypotheses. Let $\mathcal{V}$ be an operator space and $\mathcal{J}, \mathcal{K}$ be $C^*$-algebras. Let $\mathcal{O}_{nc} \subseteq \mathcal{V}_{nc}$ be a noncommutative set, and assume that $G: \mathcal{O}_{nc} \times \mathcal{O}_{nc} \to \mathcal{L}(\mathcal{J}, \mathcal{K})$ is an affine noncommutative kernel. Recall that if $(a, c) \mapsto G(a, c)$ is an affine $nc$ kernel, then $(a, c) \mapsto G(a, c^*)$ is an $nc$ kernel. We prefer to work with the affine kernel $G$ because we will often need to take its derivative (or, rather, difference-differential) on both the first and second coordinate (which we denote by $0\Delta G(a; d', c)$ and $\Delta G(a; c; c')$, respectively). Consider the following properties:

1. $\mathcal{O}_{nc}$ is uniformly open and $G$ is locally uniformly bounded. Thus, $G$ is uniformly analytic in each of its two variables.
2. $\mathcal{O}_{nc}$ is finitely open and $G$ is locally bounded on slices. Thus, $G$ is analytic on slices in each of its two variables.
3. $\mathcal{O}_{nc}$ is open in the level topology and $G$ is locally bounded on slices. Thus, $G$ is analytic on slices in each of its two variables.
4. For any $n \in \mathbb{N}$ and $a, c \in \mathcal{O}_n$ such that $a^*, c^* \in \mathcal{O}_n$, we have $G(a, c)(v)^* = G(c^*, a^*)(v)$ for $v = v^* \in \mathcal{J}^{n \times n}$;
5. $\{a \in \mathcal{O}_{nc}: G(a, a^*)(1) > 0\} \neq \emptyset$.
6. At each level at which the set $\{a \in \mathcal{O}_{nc}: G(a, a^*)(1) > 0\}$ is nonempty, we have $\|G(a, a^*)(1)^{-1}\| \to +\infty$ as $a$ tends to the norm-topology boundary of $\{a \in \mathcal{O}_{nc}: G(a, a^*)(1) > 0\}$.
7. Let $\Omega$ be a connected component of $\{a \in \mathcal{O}_{nc}: G(a, a^*)(1) > 0\}$. For any given $a \in \Omega_n, c \in \mathcal{T}_n$, we have $G(a, c^*)(1)$ invertible as an element in the $C^*$-algebra $K_n^{n \times n}$.
8. The function $G$ is analytic on a neighbourhood of $\Omega_n \times \Omega_n$ for each $n \in \mathbb{N}$.

In our results below, we will assume various subsets of the above hypotheses. We would like to emphasize at this moment already that they are not very restrictive, and important families of kernels satisfy all of them.

Pick a point $a_0 \in \{a \in \mathcal{O}_{nc}: G(a, a^*)(1) > 0\}$ at the first nonempty level. Let $\mathcal{D}_{G,nc}$ be the connected component of $a_0$ (i.e. at each multiple $k$ of the level in which $a_0$ occurs, we consider the connected component of $a_0 \otimes 1_k$). In all applications we are currently aware of, the set $\mathcal{O}_{nc}$ is considerably bigger than $\mathcal{D}_{G,nc}$. It seems in fact that at the present level of knowledge in this field, analyticity of $G$ on the boundary of $\mathcal{D}_{G,nc}$ is necessary in order to obtain powerful results about arbitrary functions defined on it. Given the case of single-variable analytic functions, that is probably not so surprising. However, for the purposes of the next section, this hypothesis is not needed.

We would like to emphasize that if $G(a, a^*)$ is completely positive, then the condition $G(a, a^*)(1) > 0$ can be replaced by the condition $G(a, a^*)(x) > 0$ for any $x > 0$. Indeed, one implication is obvious. Conversely, if $x > 0$, then it is invertible and $x \geq \|x^{-1}\|^{-1} > 0$, so that $0 < \|x^{-1}\|^{-1}G(a, a^*)(1) \leq G(a, a^*)(x)$. For our purposes, completely positive kernels are “bad”: they generate a degenerate pseudometric. However, in the following, to the extent possible, we shall perform our computations in such a way as to be able to draw conclusions for both the case $G(a, a^*)$ completely positive and $G(a, a^*)(1) > 0$ (without the assumption that $G(a, a^*)$ is positive).

4.2. The smooth pseudometric. The following proposition gives a noncommutative version of a hyperbolic pseudometric. This version is given in terms of the defining functions of the domains in question and its definition is purely algebraic. It is clear that noncommutative domains admit hyperbolic pseudometrics level-by-level. However, there would be an appropriation to think that they are related to the pseudometric
we define here. We will see later that in some cases our pseudometric indeed generates the Kobayashi metric, while in others it does not. As in Theorem 3.7 and as in the classical theory of several complex variables, for the pseudometric to be nondegenerate, it is necessary that the domains do not contain holomorphic images of complex lines (i.e. copies of \( \mathbb{C} \)) at any level.

Consider \( G \) satisfying properties [(1), (2) or (3)], (4), and (5), and define \( \mathcal{D}_{G,nc} \) as above. Without loss of generality, we assume that \( \mathcal{D}_{G,1} \neq \emptyset \). Recall that the spectrum of an operator \( V \) on a Hilbert space is denoted by \( \sigma(V) \). For \( a \in \mathcal{D}_{G,n}, c \in \mathcal{D}_{G,m}, b \in \mathcal{V}^{n \times m}, \) we have

\[
\delta_{\mathcal{D}_{G,nc}}(a, c)(b) = \max \left\{ 0, \sup \sigma \left( G(a, a^*)(1)^{-1/2} \left[ \frac{\partial G(a, c, c^*)(b, 1)}{G(c, c^*)(1)} \right]^{-1} \right) \right\},
\]

and, when \( m = n \),

\[
\delta_{\mathcal{D}_{G,nc}}(a, c) = \delta_{\mathcal{D}_{G,nc}}(a, c)(a - c).
\]

It will be seen below that

\[
\delta_{\mathcal{D}_{G,nc}}(a, c) = \max \left\{ 0, \sup \sigma \left( G(a, a^*)(1)^{-1/2} G(a, a^*)(1) G(c, c^*)(1)^{-1} \right. \right.
\]

\[
\times G(c, a^*)(1) G(a, a^*)(1)^{-1} \left. \frac{1}{2} - 1 \right) \right\}^{1/2}.
\]

It will be apparent that these two objects coincide with the ones defined in Section 3 for the particular case of domains defined via inequalities of the type described in hypothesis (5) above.

Consider another function \( H \) defined on some noncommutative subset of \( \mathcal{W}_{nc} \), which satisfies the same properties as \( G \). We define \( \mathcal{D}_{H,nc} \) the same way as \( \mathcal{D}_{G,nc} \).

**Proposition 4.1.** Let \( \mathcal{D}_{G,nc}, \mathcal{D}_{H,nc} \) be two domains defined as above. Let \( f : \mathcal{D}_{G,nc} \to \mathcal{D}_{H,nc} \) be a noncommutative map. For any \( n, m \in \mathbb{N}, a \in \mathcal{D}_{G,n}, c \in \mathcal{D}_{G,m}, b \in \mathcal{V}^{n \times m}, \) we have

\[
\delta_{\mathcal{D}_{H,nc}}(f(a), f(c))(\Delta f(a, c)(b)) \leq \delta_{\mathcal{D}_{G,nc}}(a, c)(b).
\]

If \( m = n \), then

\[
\delta_{\mathcal{D}_{H,nc}}(f(a), f(c)) \leq \delta_{\mathcal{D}_{G,nc}}(a, c).
\]

and

\[
\frac{H(f(a), f(a^*)^*(1)^{-1/2} H(f(a), f(c^*)^*(1)) H(f(c), f(c^*)^*(1)^{-1} H(f(c), f(a^*)^*(1)
\]

\[
\times H(f(a), f(a^*)^*(1)^{-1/2} \right)\]

\[
\leq G(a, a^*)(1)^{-1/2} G(a, c^*)(1) G(c, c^*)(1)^{-1} G(c, a^*)(1) G(a, a^*)(1)^{-1/2} - 1 \right\].

In addition,

\[
H(f(a), f(c^*)^*(1)) H(f(c), f(c^*)^*(1)^{-1} H(f(c), f(a^*)^*(1) - H(f(a), f(a^*)^*(1)
\]

\[
\leq H(f(a), f(a^*)^*(1) \times
\]

\[
G(a, a^*)(1)^{-1/2} G(a, c^*)(1) G(c, c^*)(1)^{-1} G(c, a^*)(1) G(a, a^*)(1)^{-1/2} - 1 \right\].
Remark 4.2. If in addition $H(u, v^*)(1)H(v, v^*)(1)^{-1}H(v, u^*)(1) - H(u, u^*)(1) \geq 0$ for all $u, v \in \mathcal{D}_{H, nc}$, then relation (24) is equivalent to
\[
\left\|H(f(a), f(a^*)^*(1)^{-\frac{1}{2}}H(f(a), f(c^*))^*(1)
\right\| \\
\leq \left\|G(a, a^*)^*(1)^{-\frac{1}{2}}G(a, c^*)^*(1)G(c, c^*)^*(1)^{-1}G(a, a^*)^*(1)G(a, a^*)^*(1)^{-\frac{1}{2}} - 1\right\|.
\]

Remark 4.3. The condition $H(u, v^*)(1)H(v, v^*)(1)^{-1}H(v, u^*)(1) - H(u, u^*)(1) \geq 0$ for all $u, v \in \mathcal{D}_{H, nc}$ is satisfied by a large and important class of noncommutative domains. In particular, it is satisfied by generalized noncommutative half-planes and generalized noncommutative balls. Indeed, consider the upper half-plane from Example 2.3(i), $H^+(\mathcal{A}^{n \times n}) = \{b \in \mathcal{A}^{n \times n} : \exists b > 0\}$. It is given by the affine kernel $H(a, c)(P) = (2i)^{-1}(aP - Pc)$. Then the inequality reduces to
\[
\frac{a - c^*}{2i} \left(\frac{c - c^*}{2i}\right)^{-1} - \frac{a - a^*}{2i} \geq 0.
\]
But
\[
\frac{a - c^*}{2i} \left(\frac{c - c^*}{2i}\right)^{-1} - \frac{a - a^*}{2i} = \frac{1}{4}(a - c) \left(\frac{c - c^*}{2i}\right)^{-1}(a - c)^* \geq 0
\]
whenever $a \neq c$ in the upper half-plane. One can generalize this to kernels of the form $H(a, c)(P) = (2i)^{-1}(h(a)P - Ph(c^*))$, for some noncommutative function $h$.

A better-known class of kernels is given by the formula $H(a, c)(P) = 1 - h(a)Ph(c^*)^*$ with $h$ a noncommutative function. Then $H(a, c^*)(1) = 1 - h(a)h(c)^*$, so from the point of view of the above inequality it is enough to consider the case when $h$ is the identity function. If $H(a, c)(P) = 1 - aPc$, this comes down to:
\[
(1 - ac^*)(1 - cc^*)^{-1}(1 - ca^*) - (1 - aa^*)
\]
\[
= 1 - ca^* + (c - a)c^*(1 - cc^*)^{-1}c(c - a)^* + (c - a)c^* - 1 + aa^*
\]
\[
= (c - a)c^*(1 - cc^*)^{-1}c(c - a)^* + cc^* + aa^* - ac^* - ca^*
\]
\[
= (c - a)c^*(1 - cc^*)^{-1}c(c - a)^* + (c - a)(c - a)^* \geq 0
\]
whenever $a \neq c$ satisfy $\|a\|, \|c\| < 1$.

Note that this also proves that
\[
\left\|H(a, a^*)^*(1)^{-\frac{1}{2}}H(a, c^*)(1)H(c, c^*)(1)^{-1}H(c, a^*)(1)^{-\frac{1}{2}} - 1\right\| = 0
\]
for $H(a, c)(P) = 1 - h(a)Ph(c^*)^*$ if and only if $h(a) = h(c)$. So the pseudodistance defined by this formula separates points if and only if $h$ is injective. The same fact holds for the generalized half-plane.

Proof of Proposition 4.4. The proof is based on showing that formula (20) for $\delta_{\mathcal{D}_{G, nc}}$ coincides with the definition (12) in the particular case of a domain defined by a noncommutative kernel as in assumption (5). An application of Proposition 3.2 will allow us to conclude. On the way to proving (22), we will obtain formulas allowing us to argue that (23) and (24) hold by applying the same principle as in the proof of Proposition 3.2. In some cases, for future reference, we will perform computations which are slightly more involved than absolutely necessary.
Thus, let us start by evaluating $G$ on elements $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $a \in \mathcal{D}_{G,n}, c \in \mathcal{D}_{G,m}$. We have $G\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & c'' \end{bmatrix}\right)$, where $(I_{n+m}) > 0$ whenever $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \mathcal{D}_{G,n+m}$. As $a \in \mathcal{D}_{G,n}, c \in \mathcal{D}_{G,m}$, we can use the properties of nc functions/kernels to write explicitly the entries of this matrix. For future reference, we consider the general case, with $P_{11} \in \mathcal{J}^{n \times n}, P_{12} \in \mathcal{J}^{n \times m}, P_{21} \in \mathcal{J}^{m \times n}, P_{22} \in \mathcal{J}^{m \times m}$, $a_1, a_2 \in \mathcal{D}_{G_{a_1}}, c_1, c_2 \in \mathcal{D}_{G_{c_1}}, b_1 \in \mathcal{V}^{n \times m}, b_2 \in \mathcal{V}^{m \times n}$. According to condition (7) in the definition of affine kernels,

$$
G\left(\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & c_2 \end{bmatrix}\right) = G\left(\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}, \begin{bmatrix} c_2 & b_2 \\ 0 & a_2 \end{bmatrix}\right) = G\left(\begin{bmatrix} c_1 & b_1 \\ 0 & c_2 \end{bmatrix}, \begin{bmatrix} c_2 & b_2 \\ 0 & a_2 \end{bmatrix}\right)
$$

On the other hand,

$$
G\left(\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}, \begin{bmatrix} c_2 & b_2 \\ 0 & a_2 \end{bmatrix}\right) \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = G\left(\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}, \begin{bmatrix} c_2 & b_2 \\ 0 & a_2 \end{bmatrix}\right) \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + 0 \Delta G\left(\begin{bmatrix} a_1 & c_1 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} c_2 & b_2 \\ 0 & a_2 \end{bmatrix}\right) \begin{bmatrix} P_{12} & P_{11} \\ P_{22} & P_{21} \end{bmatrix}
$$

We identify each of the two components of this column vector.

$$
G\left(\begin{bmatrix} a_1 & c_1 \\ 0 & a_2 \end{bmatrix}\right) \begin{bmatrix} P_{12} & P_{11} \\ P_{22} & P_{21} \end{bmatrix} = G\left(\begin{bmatrix} a_1 & c_1 \\ 0 & a_2 \end{bmatrix}\right) \begin{bmatrix} P_{12} & P_{11} \\ P_{22} & P_{21} \end{bmatrix}
$$

and

$$
G\left(\begin{bmatrix} c_1 & b_1 \\ 0 & c_2 \end{bmatrix}\right) \begin{bmatrix} P_{12} & P_{11} \\ P_{22} & P_{21} \end{bmatrix} = G\left(\begin{bmatrix} c_1 & b_1 \\ 0 & c_2 \end{bmatrix}\right) \begin{bmatrix} P_{12} & P_{11} \\ P_{22} & P_{21} \end{bmatrix}
$$

Finally,

$$
0 \Delta G\left(\begin{bmatrix} a_1 & c_1 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} c_2 & b_2 \\ 0 & a_2 \end{bmatrix}\right) \begin{bmatrix} b_1 & P_{22} \\ P_{21} & P_{21} \end{bmatrix} =
$$

$$
0 \Delta G\left(\begin{bmatrix} a_1 & c_1 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} c_2 & b_2 \\ 0 & a_2 \end{bmatrix}\right) \begin{bmatrix} b_1 & P_{22} \\ P_{21} & P_{21} \end{bmatrix} + 0 \Delta G\left(\begin{bmatrix} a_1 & c_1 \\ 0 & a_2 \end{bmatrix}, \begin{bmatrix} c_2 & b_2 \\ 0 & a_2 \end{bmatrix}\right) \begin{bmatrix} b_1 & P_{22} \\ P_{21} & P_{21} \end{bmatrix}
$$

a row vector with two components. To centralize all results, if

$$
G\left(\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}, \begin{bmatrix} c_2 & b_2 \\ 0 & a_2 \end{bmatrix}\right) \begin{bmatrix} P_{12} & P_{11} \\ P_{22} & P_{21} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix},
$$

then

$$
G_{11} = G(a_1, c_2)([P_{12}]) + 0 \Delta G(a_1, c_1, c_2) ([b_1, [P_{22}])
$$
(27) \[ G_{12} = 1 \Delta G(a_1, c_2; a_2)([P_{12}], b_2) \]
+ \[ G(a_1, a_2)([P_{11}]) + \frac{1}{2} \Delta \Delta G(a_1, c_2; a_2)(b_1, [P_{22}], b_2) \]
+ \[ \frac{1}{2} \Delta G(a_1, c_2; a_2)(b_1, [P_{22}]) \].

(28) \[ G_{21} = G(c_1, c_2)([P_{22}]), \]

(29) \[ G_{22} = \frac{1}{2} \Delta G(c_1, c_2; a_2)([P_{22}], b_2) + G(c_1, a_2)([P_{21}]). \]

For any C*-algebra \( A \), \[ \begin{pmatrix} a & u^* \\ v & w \end{pmatrix} \in \mathcal{A}^{(n+m) \times (n+m)} \) is strictly positive if and only if \( u > 0, w > 0 \) and \( v^*u^{-1}v < w \) (or, equivalently, \( u > vw^{-1}v^* \) - see [25] Chapter 3).

Thus, \( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathcal{D}_{G,n+m} \) if and only if

\[ a \in \mathcal{D}_{G,n}, c \in \mathcal{D}_{G,m} \]
and \( G(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \begin{pmatrix} \bar{a}^* \\ \bar{b}^* \end{pmatrix}) \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} > 0. \]

The requirement of positivity for \( G \) applied to a block-diagonal \( P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} \), means

\[ 0 < G(c, c^*)([P_{22}]), \]
\[ 0 < G(a, a^*)([P_{11}]) + \frac{1}{2} \Delta \Delta G(a,c; c^*) (b, [P_{22}], b^*), \]

and

\[ \frac{1}{2} \Delta G(a, c; c^*) (b, [P_{22}]) [G(c, c^*)([P_{22}])]^{-1} \frac{1}{2} \Delta G(c, c; a^*) ([P_{22}], b^*) \]
\[ < G(a, a^*)([P_{11}]) + \frac{1}{2} \Delta \Delta G(a,c; c^*) (b, [P_{22}], b^*). \]

(Note that if \( G_1(x, x^*) \) were cp, by letting \( P_{11} \) go to zero in the above, we’d conclude that the map \( P_{22} \mapsto \frac{1}{2} \Delta \Delta G_2(a, c, c^*) (b, [P_{22}], b^*) \) is necessarily a completely positive map whenever \( b \) so that \( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathcal{D}_{G,n+m} \).\)

Given \( a, c \) as above, by the openness of \( \mathcal{O}_{ac} \), which is a consequence of condition (5) and of the analyticity of \( G \), we know that there is an \( \epsilon > 0 \) depending on \( a, c \) so that

1 If the requirement in the definition of \( \mathcal{D}_{G,ac} \) were that

\[ G(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \begin{pmatrix} \bar{a}^* \\ \bar{b}^* \end{pmatrix}) \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} > 0 \]

for all \( P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} > 0 \) in \( \mathcal{J}^{(n+m) \times (n+m)} \) - so \( P_{12} = P_{21} \) for all \( n, m \), then, according to the above formula applied to \( a_1 = a, a_2 = a^*, c_1 = c, c_2 = c^*, b_1 = b, b_2 = b^* \), the requirement \( \begin{pmatrix} G_{12} & G_{11} \\ G_{22} & G_{21} \end{pmatrix} > 0 \), would become

\[ 0 < G(c, c^*)([P_{22}]), \]
\[ 0 < G(a, a^*)([P_{11}]) + \frac{1}{2} \Delta \Delta G(a,c; c^*) (b, [P_{22}], b^*) \]
\[ + \frac{1}{2} \Delta G(a,c; a^*) ([P_{12}], b^*) + \frac{1}{2} \Delta G(a,c; a^*) ([P_{22}], b^*) \]

and

\[ [G(a, c^*)([P_{12}]) + \frac{1}{2} \Delta G(a,c; c^*) (b, [P_{22}]) [G(c, c^*)([P_{22}])]^{-1} \times (G(c,c; a^*) ([P_{22}], b^*) + \frac{1}{2} \Delta G(a,c; a^*) ([P_{22}], b^*) \]
\[ < G(a, a^*)([P_{11}]) + \frac{1}{2} \Delta \Delta G(a,c; c^*) (b, [P_{22}], b^*) + \frac{1}{2} \Delta G(a,c; a^*) ([P_{12}], b^*) \]
\[ + \frac{1}{2} \Delta G(a,c; a^*) ([P_{22}], b^*). \]
We record for future reference the expressions for $K$, recalling the definition (12) for $\delta$,

$$\varepsilon_0 := \sup \left\{ t \in (0, +\infty) : \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in D_{G,n+m} \forall s < t \right\} \in (0, +\infty].$$

Observe that if

$$1\Delta_0 \Delta G(a; c, c^*; a^*)(b_0, 1, b_0^*) \geq 0 \Delta G(a; c, c^*)(b_0, 1)G(c, c^*)(1)^{-1} \Delta G(c, c^*; a^*)(1, b_0^*)$$

then $D_{G,n+m}$ contains a complex line. Indeed, one simply divides by $|z|^2$ in (31).

We argue that $\delta_{D_{G,n}}(a, c)(b)$ is indeed given in this case by formula (20). If $\varepsilon_0 < +\infty$, then it can be written as

$$\delta_{D_{G,n}}(a, c)(b)^2 = \varepsilon_0^{-2} = \sup \left\{ \varphi \left( G(a, a^*)(1)^{-1/2} \left[ \begin{array}{c} 0 \Delta G(a; c, c^*)(b_0, 1) [G(c, c^*)(1)^{-1} \Delta G(c, c^*; a^*)(1, b_0^*)] \\
- 1 \Delta_0 \Delta G(a; c, c^*; a^*)(b_0, 1, b_0^*) \end{array} \right] G(a, a^*)(1)^{-1/2} \right) : \varphi : K^{n \times n} \to \mathbb{C} \text{ state} \right\}.$$

Thus, formula (20) holds. By Proposition 5.2 we conclude relation (22). If $\varepsilon_0 = +\infty$, there is nothing to prove.

Consider now the case $b_0 = \epsilon(a - c)$ for some arbitrary $\epsilon \in \mathbb{C}$. We apply Equation (4) to write

- $\Delta_0 \Delta G(a; c, a^*)(\epsilon(a - c), [P_{21}]) = G(a, a^*)((\epsilon P_{21})) - G(c, a^*)((\epsilon P_{21}))$;
- $1\Delta_0 \Delta G(a, c^*; a^*)([P_{12}], \tau(a - c)^*) = G(a, a^*)((\tau P_{12})) - G(a, c^*)((\tau P_{12}))$;
- $0\Delta G(a; c, c^*)(\epsilon(a - c), [P_{22}]) = G(a, c^*)((\epsilon P_{22})) - G(a, c^*)((\epsilon P_{22}))$;
- $1\Delta_0 \Delta G(c, c^*; c^*)(\epsilon(a - c), [P_{22}], \tau(a - c)^*) = G(a, a^*)((\epsilon P_{22})) - G(a, c^*)((\epsilon P_{22}))$.

We record for future reference the expressions for $G_{ij}$ corresponding to $b_0 = \epsilon(a - c)$.

$$G_{11} = G(a, a^*)((P_{12}) + \epsilon G(a, c^*)((P_{22})) - G(c, c^*)((P_{22}))$$

$$G_{12} = G(a, a^*)((\epsilon P_{21}) + \tau(P_{12})) - G(a, c^*)((\tau P_{12})) - G(c, a^*)((\epsilon P_{21}))$$

$$G_{21} = G(c, c^*)((P_{22}))$$

$$G_{22} = G(c, c^*)((P_{21}) + \tau(G(a, a^*)((P_{22})) - G(c, c^*)((P_{22})))$$

For $\epsilon = 1$, we obtain that for any state $\psi$ on $K^{n \times n}$ and $\varepsilon > 0$, there is a state $\varphi$ on $K^{n \times n}$ depending on $\varepsilon$ such that

$$\psi \left( H(f(a), f(c^*))(1)^{-1/2} H(f(a), f(c^*)) H(f(c), f(c^*))^{-1} \times H(f(c), f(a^*)) H(f(a), f(a^*))^{-1/2} - 1 \right) \varepsilon$$

$$\leq \varphi \left( G(a, a^*) H^{-1} G(c, c^*)^{-1} G(c, c^*)^{-1} G(a, a^*) G(a, a^*)^{-1/2} \right).$$
Recall that \( \psi, \varphi \) are states, so that \( \varphi(1) = \psi(1) = 1 \), which implies that

\[
\psi \left( H(f(a), f(a^*)) (1)^{\frac{1}{2}} H(f(a), f(c^*)) (1)^{-\frac{1}{2}} \right) - \varepsilon \\
\times H(f(c), f(a^*)) (1)^{\frac{1}{2}} H(f(a), f(c^*)) (1)^{-\frac{1}{2}} - \varepsilon \\
\leq \varphi \left( G(a, a^*) (1)^{\frac{1}{2}} G(a, c^*) (1)^{-\frac{1}{2}} \left( G(c, c^*) (1)^{-1} G(c, a^*) (1) G(a, a^*) (1)^{-\frac{1}{2}} \right) \right).
\]

Clearly the elements under the states above are nonnegative, so this reduces to

\[
\left\| H(f(a), f(a^*)) (1)^{\frac{1}{2}} H(f(a), f(c^*)) (1)^{-\frac{1}{2}} \right\| \\
\times H(f(c), f(a^*)) (1)^{\frac{1}{2}} H(f(a), f(c^*)) (1)^{-\frac{1}{2}} \\
\leq \left\| G(a, a^*) (1)^{\frac{1}{2}} G(a, c^*) (1)^{-\frac{1}{2}} \left( G(c, c^*) (1)^{-1} G(c, a^*) (1) G(a, a^*) (1)^{-\frac{1}{2}} \right) \right\|.
\]

The last inequality of our proposition, (24), is a trivial consequence of the selfadjointness of the elements involved, together with the previous results. \qed

We are not automatically able to conclude the norm-inequality (24) only because the norm of the left-hand side might be achieved at the lower bound of the spectrum. However, assuming the hypothesis of Remark 4.2 guarantees this is not the case. It wouldn’t be unreasonable to suppose that this hypothesis is satisfied in most cases of interest. So we discuss next three things related to it.

**Remark 4.4.** First, not surprisingly, the inequality \( G(a, a^*) (1) - G(a, c^*) (1) G(c, c^*) (1)^{-1} G(c, a^*) (1) \geq 0 \), opposite to the one introduced in Remark 4.2, cannot hold under the assumption of no complex lines in \( \mathcal{D}_{G, nc} \). Indeed, if we put \( b = \epsilon (a - c) \) in formulas (20), (27), (28) and (29) for \( a_1 = a_2 = a, c_1 = c_2 = c \), we obtain, according to (32) with \( \epsilon > 0, P_{11} = P_{22} = 1, P_{12} = P_{21} = 0 \), the matrix inequality

\[
\begin{bmatrix}
G(a, a^*) (1 + \epsilon^2) - \epsilon^2 G(a, c^*) (1) - c^2 G(c, a^*) (1) + c^2 G(c, c^*) (1) & \epsilon G(a, c^*) (1) - G(c, a^*) (1) \\
\epsilon G(c, a^*) (1) - G(c, c^*) (1) & G(c, c^*) (1)
\end{bmatrix} > 0.
\]

Multiplying with

\[
\begin{bmatrix}
1 & \epsilon \\
0 & 1
\end{bmatrix}
\]

left and its adjoint right does not change the positivity of the matrix, so that

\[
\begin{bmatrix}
(1 + \epsilon^2) G(a, a^*) (1) & \epsilon G(a, c^*) (1) \\
\epsilon G(c, a^*) (1) & G(c, c^*) (1)
\end{bmatrix} > 0,
\]

for any \( \epsilon > 0 \) such that

\[
\begin{bmatrix}
a & \epsilon(a - c) \\
0 & c
\end{bmatrix} \in \mathcal{D}_{G, 2n}. \quad \text{Since } \mathcal{D}_{G, 2n} \text{ contains no complex line, it follows that there is an } \epsilon_0 (a, c) > 0 \text{ maximal beyond which the matrix inequality above fails. Thus, necessarily } G(a, a^*) (1) - G(a, c^*) (1) G(c, c^*) (1)^{-1} G(c, a^*) (1) \geq 0.
\]

**Remark 4.5.** Second, we observe that certain obvious transformations of \( G \) have similarly obvious effects on \( \mathcal{D}_{G, nc} \). For example, composing \( G \) with a completely positive unital map increases \( \mathcal{D}_{G, nc} \). Indeed, if \( \Phi \) is such a map, then \( G(a, a^*) (1) > \varepsilon 1 \implies \Phi(G(a, a^*) (1)) > \varepsilon \Phi(1) = \varepsilon 1 \), so that \( \mathcal{D}_{G, nc} \subseteq \mathcal{D}_{\Phi \circ G, nc} \). Subtracting a positive multiple of 1 from \( G \) decreases \( \mathcal{D}_{G, nc} \), adding increases it. However, if for any \( t \in \mathbb{R} \setminus \sigma(G(c, c^*) (1)) \) we let

\[
f(t) = |G(a, c^*) (1) - t1| |G(c, c^*) (1) - t1|^{-1} |G(c, a^*) (1) - t1| - |G(a, a^*) (1) - t1|,
\]

we get
It is quite easy to see that this set is noncommutative: if
\[ f(t) = (G(c, c^*)(1) - t1)^{-1}(G(c, a^*)(1) - t1) - 1 \]
\[ \times (G(c, a^*)(1) - t1)(G(c, c^*)(1) - t1) - 1 \geq 0, \]
(recall hypothesis (3) which states that \( G(a, c)(1)^* = G(c^*, a^*)(1) \), and
\[ f''(t) = 2 \left[ 1 - (G(a, c^*)(1) - t1)(G(c, c^*)(1) - t1)^{-1} \right] \]
\[ \times (G(c, c^*)(1) - t1)^{-1} \left[ 1 - (G(c, a^*)(1) - t1)^{-1}(G(c, a^*)(1) - t1) \right] \geq 0, \]
for all \( t \in \mathbb{R} \setminus \sigma(G(c, c^*)(1)) \). This means that for any state \( \varphi \), the map \( t \mapsto \varphi \circ f \)
is convex and increasing on each connected component of \( \mathbb{R} \setminus \sigma(G(c, c^*)(1)) \). Clearly,
\[ \lim_{t \to \pm \infty} \|f(t) - (G(a, c^*)(1) + G(c, c^*)(1) + G(c, a^*)(1) - G(a, a^*)(1))\| = 0, \]
so \( a \) is such that \( G(a, c^*)(1) + G(c, c^*)(1) + G(c, a^*)(1) - G(a, a^*)(1) \geq 0 \) (for example, \( c \in D_{G_n} \) and
\( a \) close to \( c \)), then \( f(t) \geq 0 \) for all real \( t \) in the connected component of \(-\infty\). Conversely,
if \( G(a, c^*)(1) + G(c, c^*)(1) + G(c, a^*)(1) - G(a, a^*)(1) \leq 0 \), then \( f(t) \leq 0 \) for all real \( t \)
in the connected component of \(+\infty\).

**Remark 4.6.** Finally, the function \( \delta \) has been defined in terms of the length of a “ray”
in a given direction. In this remark we look at the whole set of points \( b \) for which the
upper triangular matrix \[
\begin{pmatrix}
a & b \\
0 & c
\end{pmatrix}
\]
belongs to the chosen noncommutative set. Consider a
nc set \( D \) which satisfies property (2). Fix \( m, n \in \mathbb{N} \) and \( a \in D_n, c \in D_m \). Let
\[
\gamma(a, c)_{nc} = \bigcup_{k \in \mathbb{N}} \left\{ b \in \mathcal{D}^{m \times mk}: \begin{bmatrix} I_k \otimes a & b \\ 0 & I_k \otimes c \end{bmatrix} \in D_{km + kn} \right\}.
\]
It is quite easy to see that this set is noncommutative: if \( b = b_1 \otimes b_2 \) with \( b_j \in \gamma(a, c)_{k_j}, \)
\( j = 1, 2 \), then
\[
\begin{bmatrix}
I_{k_1 + k_2} \otimes a & b \\
0 & I_{k_1 + k_2} \otimes c
\end{bmatrix} = \begin{bmatrix}
I_{k_1} \otimes a & b_1 & 0 \\
0 & I_{k_2} \otimes a & b_2 \\
0 & 0 & I_{k_1} \otimes c & 0 \\
0 & 0 & 0 & I_{k_2} \otimes c
\end{bmatrix},
\]
By permuting rows 2 and 3 and columns 2 and 3 (which comes to the conjugation with
a scalar matrix), we obtain
\[
\begin{bmatrix}
I_{k_1} \otimes a & b_1 & 0 & 0 \\
0 & I_{k_1} \otimes c & 0 & 0 \\
0 & 0 & I_{k_2} \otimes a & b_2 \\
0 & 0 & 0 & I_{k_2} \otimes c
\end{bmatrix},
\]
which belongs to \( D_{(n+m)(k_1+k_2)} \) because \( D \) is a noncommutative set.

Similarly, \( \gamma(a, c)_{nc} \) is invariant by conjugation with scalar unitary matrices: if \( b \in \mathcal{D}^{m \times mk}, \)
for any unitary matrix \( U \in \mathbb{C}^{k \times k}, \)
\[
\begin{bmatrix}
I_k \otimes a & UbU^* \\
0 & I_k \otimes c
\end{bmatrix} = \begin{bmatrix} U \otimes I_n & 0 \\ 0 & U \otimes I_m \end{bmatrix} \begin{bmatrix} I_k \otimes a & b \\ 0 & I_k \otimes c \end{bmatrix} \begin{bmatrix} U^* \otimes I_n & 0 \\ 0 & U^* \otimes I_m \end{bmatrix},
\]
which belongs to \( D_{k(m+n)} \) by assumption (2) on the set \( D \).

Given the unitary invariance of the set \( \gamma(a, c)_{nc} \) and Lemma 4.6 one is justified in
asking whether \( \gamma(a, c)_{nc} \) is in fact matrix convex. That turns out to be false in general.

Let us recall Wittstock’s definition of a matrix convex set (see [19 Section 3]): a
matrix convex set is a noncommutative set \( K = \bigcup_n K_n \) such that for any \( S \in \mathbb{C}^{r \times n} \)
satisfying \( S^* S = I_n \), we have \( S^* K_r S \subseteq K_n \). Since \( \gamma(a, c)_{nc} \) is invariant by conjugation
with scalar unitary matrices, matrix convexity of $\mathfrak{T}(a,c)_{nc}$ is equivalent to the following statement: for any $k < k' \in \mathbb{N}$ and $b \in \mathfrak{T}(a,c)_{k'}$, we have $[I_k \ 0 \ b \ I_k] \in \mathfrak{T}(a,c)_k$, i.e. the upper right $k \times k$ corner of $b$ is an element of $\mathfrak{T}(a,c)_{nc}$ whenever $b$ is. There is a simple counterexample to this statement: consider the unit disk $\mathbb{D}$ in the complex plane, and the noncommutative set

$$\mathcal{D} = \prod_{k \in \mathbb{N}} \{ A \in \mathbb{C}^{k \times k} : \sigma(A) \subseteq \mathbb{D}, \| A \| < k \}.$$ 

This is clearly a noncommutative set (if $A_j \in \mathcal{D}_{k_j}$, then $\| A_1 \oplus A_2 \| = \max\{ \| A_1 \|, \| A_2 \| \} < \max\{ k_1, k_2 \} < k_1 + k_2$ which is unitarily invariant ($\| U^*AU \| = \| A \|$). However,

$$\begin{bmatrix}
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \in \mathcal{D}_4, \quad \text{while} \quad \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} \notin \mathcal{D}_2,$$

which means that $\begin{bmatrix} 3 \\ 0 \end{bmatrix} \in \mathfrak{T}(0,0)_2$, while $3 \notin \mathfrak{T}(0,0)_1$.

However, there are important classes of nc sets for which the set $\mathfrak{T}$ is matrix convex. One such example is the class of generalized half-planes (see Remark 4.3). Consider an injective nc map $h : \mathcal{V}_{nc} \to \mathcal{A}_{nc}$ for some unital $C^*$-algebra $\mathcal{A}$. Recall that a generalized half-plane is

$$H^+_k(\mathcal{V}) = \prod_{n=1}^{\infty} \{ a \in \mathcal{V}^{n \times n} : h(a) + h(a)^* > 0 \}.$$ 

Then elements $b \in \mathfrak{T}(a,c)_{nc}$ must satisfy

$$(\mathbb{R}h(a))^{-1/2} \Delta h(a,c)(b)(\mathbb{R}h(c))^{-1} \Delta h(a,c)(b)^*(\mathbb{R}h(a))^{-1/2} < 4 \cdot 1.$$ 

That is, for any $k' \in \mathbb{N}$,

$$(I_{k'} \otimes \mathbb{R}h(a))^{-1/2} \Delta h(I_{k'} \otimes a, I_{k'} \otimes c)(b)(I_{k'} \otimes \mathbb{R}h(c))^{-1}$$

$$\times \Delta h(I_{k'} \otimes a, I_{k'} \otimes c)(b)^*(I_{k'} \otimes \mathbb{R}h(a))^{-1/2}$$

$$= \left[ \sum_{l=1}^{k'} (\mathbb{R}h(a))^{-1/2} \Delta h(a,c)(b_{il})(\mathbb{R}h(c))^{-1} \Delta h(a,c)(b_{il})^*(\mathbb{R}h(a))^{-1/2} \right]_{1 \leq i,j \leq k'}$$

$$< 4I_{k'} \otimes 1.$$ 

If one fixes such a $k' > 1$ in $\mathbb{N}$ and $b \in \mathfrak{T}(a,c)_{k'}$, proving matrix convexity comes to proving that the upper right $k \times k$ corner of $b$ is in $\mathfrak{T}(a,c)_k$ for all $0 < k < k'$. That is,

$$\left[ \sum_{l=1}^{k} (\mathbb{R}h(a))^{-1/2} \Delta h(a,c)(b_{il})(\mathbb{R}h(c))^{-1} \Delta h(a,c)(b_{il})^*(\mathbb{R}h(a))^{-1/2} \right]_{1 \leq i,j \leq k}$$

$$< 4I_k \otimes 1.$$ 

Denoting $P_k$ the projection onto the first $k$ coordinates of $\mathbb{C}^{k' \times k'}$, the above relation is equivalent to

$$(P_k I_{k'} \otimes \mathbb{R}h(a))^{-1/2} \Delta h(I_{k'} \otimes a, I_{k'} \otimes c)(b)(P_k I_{k'} \otimes \mathbb{R}h(c))^{-1}$$

$$\times \Delta h(I_{k'} \otimes a, I_{k'} \otimes c)(b)^*(P_k I_{k'} \otimes \mathbb{R}h(a))^{-1/2} < 4P_k I_{k'} \otimes 1 = 4I_k \otimes 1.$$ 

This is implied by the general fact that $AA^* < 4I_{k'} \implies PAPA^*P \leq 4P = 4I_k$. Indeed, clearly $AA^* < 4I_{k'} \implies PAA^*P \leq 4P = 4I_k$ and $P \leq I_{k'} \implies (PA)(P^*A)^* \leq
Proposition 4.7. Let \( \mathcal{V} \) be an operator system and consider an injective noncommutative function \( h \) defined on \( \mathcal{V}_{\text{nc}} \) with values in a unital \( C^\ast \)-algebra \( A \). Define the kernel
\[
H(a,c) = 1 - h(a) \cdot h(c^\ast)
\]
and the set
\[
\mathcal{D}_{H,\text{nc}} = \prod_{n=1}^{\infty} \{ a \in \mathcal{V}^{n \times n} : H(a,a^\ast)(I_n) > 0 \}.
\]
Then \( d_{\mathcal{D},\text{nc}} \) coincides level-by-level with the Kobayashi distance on \( \mathcal{D}_{H,\text{nc}} \).

Proof. Let us start by noting that, according to relation (37), the infinitesimal Poincaré (or hyperbolic) metric on the unit disk \( \mathbb{D} \), \( \kappa_{\mathbb{D}}(z,v) \), coincides with \( \delta_{\mathbb{D}}(z,z)(v) \): they both equal \( \frac{|v|}{1 - |z|^2} \). Thus, the metric generated by \( \delta_{\mathbb{D}}(z,z)(v) \) coincides with the one generated by \( \kappa_{\mathbb{D}}(z,v) \), the Poincaré metric. We denote by \( k_A(\cdot,\cdot) \) the Kobayashi distance on the set \( A \). Recall that the Kobayashi metric is the largest metric (with the given normalization) that is decreasing under holomorphic mappings. For any \( n \in \mathbb{N} \) and \( f : \mathbb{D} \to \mathcal{D}_{H,n} \), we have
\[
k_{\mathcal{D}_{H,n}}(f(z),f(w)) \leq k_{\mathbb{D}}(z,w) = d_{\mathbb{D}}(z,w), \quad z,w \in \mathbb{D}.
\]
Thus, \( d_{\mathcal{D}_{H,n}} \leq k_{\mathcal{D}_{H,n}} \).

Let \( x \in \mathcal{D}_{H,n}, b \in \mathcal{V}^{n \times n} \). We claim that
\[
\delta_{\mathcal{D}_{H,n}}(x,x)(b) = \left[ \sup \left\{ t \geq 0 : \exists \mathcal{D}_{\text{nc}} \to (\mathcal{D}_{H,2n})_{\text{nc}}, f(0) = I_2 \otimes x, \Delta f(0,0)(1) = t \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \right\} \right]^{-1}.
\]
By \( (\mathcal{D}_{H,n})_{\text{nc}} \) we of course denote the subset of \( \mathcal{D}_{H,\text{nc}} \) formed of all levels which are multiples of \( n \). Indeed, inequality \( \leq \) follows easily: as shown in Proposition 3.2 and Lemma 3.5 if \( f \) is as in the right-hand side of the above relation, then
\[
t \delta_{\mathcal{D}_{H,n}}(x,x)(b) = \delta_{\mathcal{D}_{H,n}}(x,x)(tb)
\]
\[
= \delta_{\mathcal{D}_{H,2n}} \left( \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}, \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \right) \left( \begin{bmatrix} 0 & tb \\ 0 & 0 \end{bmatrix} \right).
\]
We show the reverse inequality by finding an “extremal” function. Let \( \iota = \frac{1}{\delta_{\mathcal{D}_{H,n}}(x,x)(b)} \).

Consider the nc function \( f = \{ f_p \}_{p \in \mathbb{N}}, f_p(Z) = \begin{bmatrix} I_p \otimes x & 0 \\ 0 & I_p \otimes x \end{bmatrix} + Z \otimes \begin{bmatrix} 0 & I_p \otimes b \\ 0 & 0 \end{bmatrix}, Z \in \mathbb{C}^{p \times p}, \|Z\| < 1, p \in \mathbb{N} \). According to Lemmas 3.6 and 3.5 we have
\[
\delta_{\mathcal{D}_{H,n}}(x,x)(b) > \delta_{\mathcal{D}_{H,n}}(x,x)(b)\|Z\|
\]
\[
= \delta_{\mathcal{D}_{H,n}} \left( \begin{bmatrix} I_p \otimes x & 0 \\ 0 & I_p \otimes x \end{bmatrix}, \begin{bmatrix} I_p \otimes x & 0 \\ 0 & I_p \otimes x \end{bmatrix} \right) \left( Z \otimes \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \right),
\]
whenever \( Z \) is a contraction. Thus, \( f \) takes values in \( (\mathcal{D}_{H,2n})_{\text{nc}} \) and for \( Z = 1 \in \mathbb{C} \), we actually reach the supremum in the above equality.
5. A Classification of Noncommutative Domains of Holomorphy

We turn now towards a classification of noncommutative domains (with respect to the level topology) which contain no complex lines at any level, up to noncommutative holomorphic equivalence, in terms of \( \delta \). In this section we assume that \( V \) is a Banach space, and when dealing with domains defined by kernels, we further assume that \( V \) is an operator space. Observe that if \( f: D \to E \) is a noncommutative automorphism (i.e. a map which is bijective at each level, with analytic inverse), then inequality stated in Corollary 5.4 must hold in both directions (for \( f \) and \( f^{-1} \)), so they must become equalities. That is,

\[
\delta_D(a, c) = \delta_E(f(a), f(c)), \quad a, c \in D, n \in \mathbb{N}.
\]

Conversely, assume that there is a function \( f \) as above such that equality \([34]\) holds for all \( a, c \in D, n \in \mathbb{N} \). Then it follows trivially that \( f \) is injective. Indeed, if not, there would be an \( n \in \mathbb{N} \) and points \( a \neq c \in D \) such that \( f(a) = f(c) \). Then we would have \( 0 = \delta_E(f(a), f(c)) = \delta_D(a, c) \), a contradiction, according to Theorem 3.7 to the hypothesis that \( D \) contains no complex lines.

Proving the surjectivity of \( f \) as a consequence of equality \([34]\) is not possible in full generality. We make the following assumption about our domains:

Given a noncommutative set \( D \) in the noncommutative extension \( V_{nc} \) of a Banach space \( V \), which is invariant under conjugation with scalar matrices,

For any \( n \in \mathbb{N} \) and \( a \in D_n, \) if \( \{ c_k \}_{k \in \mathbb{N}} \subset D_n \) satisfies

\[
\lim_{k \to \infty} \inf_{x \in D_n} \| x - c_k \| = 0,
\]

then

\[
\lim_{k \to \infty} \delta_D(a, c_k) = +\infty.
\]

This hypothesis does not exclude the possibility that \( \delta_D \equiv +\infty \).

**Theorem 5.1.** Consider two noncommutative domains \( D \) and \( E \) in a given space \( V_{nc} \) which are invariant under conjugation by unitary scalar matrices and contain no complex lines, and a noncommutative function \( f: D \to E \). Assume that both \( D \) and \( E \) satisfy hypothesis \([34]\). Then the following are equivalent:

1. \( f \) satisfies \( \tilde{\delta}_D(a, c) = \tilde{\delta}_E(f(a), f(c)), \) \( a, c \in D \).
2. \( f \) is a bijective noncommutative map, with noncommutative inverse.

The reader might worry about a trivial counterexample: the map from the nc disk to the nc bidisk sending \( z \) to \((z, 0)\). However, we excluded this possibility by the way we formulated our statement: in this case, the nc disk is equal to its boundary in the “environment” in which the bidisk lives, so according to \([35]\), its \( \delta \) would have to be constantly equal to infinity.
an obvious contradiction. Thus, \( \tilde{D} \leq \tilde{D} \mathcal{E} \) for all \( a, c \in D \), which means \( f^{(-1)}(a') = a, f^{(-1)}(c') = c \). By Proposition 3.2,
\[
\tilde{D}(f(a), f(c)) = \tilde{D}(a', c') \leq \tilde{D}(f^{(-1)}(a'), f^{(-1)}(c')) = \tilde{D}(a, c).
\]

(1) \( \implies \) (2): We have already seen that under condition (1), \( f \) is injective. Thus, we need to show that \( f \) is also surjective. Once we showed that, the noncommutativity of the correspondence \( a' \to f^{(-1)}(a') \) allows us to conclude. The essential part of the proof is in the following quite obvious lemma, which we nevertheless state separately, since it might be of independent interest.

**Lemma 5.2.** Consider a noncommutative domain \( D \) and a noncommutative subset \( D' \subset D \). Assume that both \( D \) and \( D' \) are invariant under conjugation by scalar unitary matrices and satisfy hypothesis (33). If \( \tilde{D}(a, c) = \tilde{D}(a, c) \) for all \( a, c \in D' \), then \( D = D' \).

**Proof.** The proof of this lemma is utterly trivial: assume towards contradiction that there exist points in \( D \setminus D' \). Pick a point \( x \in D \cap \partial D' \) (by \( \partial D' \) we understand the boundary of the set \( D' \) at the corresponding level \( n \) in the norm topology of the Banach space \( V^{n \times n} \)) and a point \( a \in D' \). By the definition of the boundary, there exists a sequence \( \{c_k\}_{k \in \mathbb{N}} \subset D' \) converging to \( x \) in norm. In particular, \( \{c_k\}_{k \in \mathbb{N}} \) satisfies the condition of hypothesis (33), so that \( \tilde{D}(a, c_k) \to +\infty \) as \( k \to \infty \). By Remark 3.1, we have
\[
\infty > \tilde{D}(a, x) \geq \limsup_{n \to \infty} \tilde{D}(a, c_k) = \limsup_{n \to \infty} \tilde{D}(a, c_k) = \infty,
\]
an obvious contradiction. Thus, \( D = D' \), as claimed. \( \square \)

Consider the set \( f(D) \subset \mathcal{E} \). For any \( x, y \in f(D) \), there exist unique \( a, c \in D \) such that \( f(a) = x, f(c) = y \). It follows from Proposition 3.2 that \( \tilde{D}(x, y) \leq \tilde{D}(a, c) \). Since \( f(D) \subset \mathcal{E} \), we necessarily have \( \tilde{D}(x, y) \geq \tilde{D}(x, y) \). Together with the hypothesis of (1), we obtain
\[
\tilde{D}(a, c) = \tilde{D}(f(a), f(c)) = \tilde{D}(x, y) \leq \tilde{D}(x, y) \leq \tilde{D}(a, c),
\]
so that \( \tilde{D}(x, y) = \tilde{D}(x, y) \) for all \( x, y \in f(D) \). By Lemma 5.2, we conclude that \( f(D) = \mathcal{E} \). \( \square \)

**Remark 5.3.** It turns out that in Theorem 5.1 we cannot dispense with the requirement that \( \delta \) blows up at the boundary. The following counterexample, similar to the one in Remark 4.6, shows what goes wrong if this requirement is dropped. Consider a domain \( D \subset \frac{1}{2} \mathbb{D} \subset \mathbb{C} \) and define the nc set
\[
D = \prod_{n=1}^{\infty} \{ A \in \mathbb{C}^{n \times n} : \sigma(A) \subset D, \|A\| < 1 \}.
\]

The proof from Remark 4.6 applies to show that \( D \) is a unitarily invariant noncommutative set which is open at each level. However, a direct computation shows that
\[
\left[ \begin{array}{cc}
a & b \\
0 & c
\end{array} \right] < 1 \text{ if and only if } aa^* + bb^* < 1, cc^* < 1, \text{ and } bc^*(1 - cc^*)^{-1}cb^* < 1 - aa^* - bb^*
\]
(this holds in an arbitrary \( C^* \)-algebra). Since the other restriction in the definition of \( D \) is on the spectrum of the matrix \( A \), it only affects \( a \) and \( c \); there is no other restriction...
Proposition 5.4. Under certain conditions, strict inclusion of domains leads to strict inequalities between $M \iff a$

However, for any choice of selfadjoints $a$ and $c$, we have that $\hat{\delta}_D(a,c) \leq \frac{4}{3}$. Thus, $\hat{\delta}$ stays bounded (by 4/3) on the intersection of the selfadjoints with $D$.

On the other hand, we have $D \subseteq B_1(\mathbb{C})$, the nc unit ball of $\mathbb{C}$, and $\delta_{B_1(\mathbb{C})}|D = \delta_D$.

In the context of Lemma 5.2 we record here the “opposite” case: we show that, under certain conditions, strict inclusion of domains leads to strict inequalities between the associated distances.

**Proposition 5.4.** Consider an operator space $V$. Let $D' \subset D$ be an inclusion of noncommutative domains in $V_{nc}$. Assume that

1. $M := \sup_{n \in \mathbb{N}} \sup_{x \in D'_n} \|x\| < +\infty$;
2. $m := \inf_{n \in \mathbb{N}} \inf \{\|x - w\| : x \in D'_n, w \in V^{n \times n} \setminus D_n\} > 0$.

Then there exists a constant $k \in [0, 1)$ such that $k\hat{\delta}_{D'} \geq \hat{\delta}_D$.

**Proof.** Let $n$ be a fixed level, and pick $a, c \in D'_n$. By definition,

$$\hat{\delta}_{D'}(a,c)^{-1} = \sup \left\{ t > 0 : \begin{bmatrix} a & s(c-a) \\ 0 & c \end{bmatrix} \in D'_n \text{ for all } s < t \right\},$$

$$\hat{\delta}_D(a,c)^{-1} = \sup \left\{ t > 0 : \begin{bmatrix} a & r(c-a) \\ 0 & c \end{bmatrix} \in D_n \text{ for all } r < t \right\}.$$

We know that the distance from $D'_n$ to $V^{2n \times 2n} \setminus D_n$ is at least $m$, so that

$$\|\hat{\delta}_D(a,c)^{-1} - s\|\{c-a\} = \left\| \begin{bmatrix} a & s(c-a) \\ 0 & c \end{bmatrix} - \begin{bmatrix} a & \hat{\delta}_D(a,c)^{-1}(c-a) \\ 0 & c \end{bmatrix} \right\| \geq m$$

whenever $\begin{bmatrix} a & s(c-a) \\ 0 & c \end{bmatrix} \in D'_n$, and thus, $\hat{\delta}_D(a,c)^{-1} - \hat{\delta}_{D'}(a,c)^{-1} \geq \frac{m}{\|c-a\|}$ for any $a, c \in D'_n, a \neq c$. It follows that

$$\frac{\hat{\delta}_{D'}(a,c)}{\hat{\delta}_D(a,c)} \geq 1 + \frac{m\hat{\delta}_{D'}(a,c)}{\|a-c\|} = 1 + m\hat{\delta}_{D'}(a,c) \left( \frac{a-c}{\|a-c\|} \right).$$

We bound from below $\delta_{D'}(a, c)(b)$ when $\|b\| = 1$ and $a, c \in D'_n$. We have $\delta_{D'}(a, c)(b) > \xi \iff \delta_{D'}(a, c)(b)^{-1} < \xi^{-1}$; but any element in $D'_n$ has norm bounded from above by $M$, so $\begin{bmatrix} a & sb \\ 0 & c \end{bmatrix} \in D'_n$ implies $|s| = \|sb\| \leq M$. Thus, $\delta_{D'}(a, c)(b) \geq M^{-1}$. We obtain

$$\frac{\delta_{D'}(a,c)}{\delta_D(a,c)} \geq 1 + \frac{m}{M^2},$$

for the constant $k = \frac{M}{m+M} < 1$.

For the distance $\hat{d}$, we have

**Corollary 5.5.** Under the assumptions, and with the notations, of Proposition 5.4 we have $k\hat{d}_{D'} \geq \hat{d}_D$. 


Proof. For any $n \in \mathbb{N}$, $a, c \in \mathcal{D}_n$, and division $a = a_0, a_1, \ldots, a_N = c \in \mathcal{D}_n$, we have

$$k \sum_{j=1}^{N} \tilde{d}_{\mathcal{D}'}(a_{j-1}, a_j) > \sum_{j=1}^{N} \tilde{d}_{\mathcal{D}}(a_{j-1}, a_j).$$

Taking infimum in the left side provides $k \tilde{d}_{\mathcal{D}'}(a, c)$. Increasing the number of divisions in the right hand side can only decrease the infimum, so that $k \tilde{d}_{\mathcal{D}'}(a, c) \geq \tilde{d}_{\mathcal{D}}(a, c)$. \hfill \Box

As a side benefit, we obtain from the proof of Proposition 5.4 that on bounded domains in operator spaces, $\tilde{d}$ and the norm are locally equivalent. We have already seen in Proposition 3.12 that if $\|a_k - a\| \to 0$, then $\tilde{d}_{\mathcal{D}}(a_k, a) \to 0$ and thus $\tilde{d}_{\mathcal{D}}(a_k, a) \to 0$. Now assume that in a bounded domain $\mathcal{D}$ we have a sequence $\{a_k\}_{k \in \mathbb{N}} \subset \mathcal{D}$ and a point $a \in \mathcal{D}$ so that $\tilde{d}_{\mathcal{D}}(a_k, a) \to 0$ as $k \to \infty$. We have seen in the proof of Proposition 5.4 that $\tilde{d}_{\mathcal{D}}(a, c)(b) \geq M^{-1}$ if $\mathcal{D}$ is included in a norm-ball of radius $M$, uniformly in $a, c \in \mathcal{D}_n, b \in \mathcal{V}^{n \times n}$, $\|b\| = 1$, $n \in \mathbb{N}$. Thus, $\tilde{d}_{\mathcal{D}}(a, c) \geq M^{-1}\|a - c\|$, so that for any division $a = a_0, a_1, \ldots, a_N = c$ of $\tilde{d}_{\mathcal{D}}(a, c)$, we have $\sum_{j=1}^{N} \tilde{d}_{\mathcal{D}}(a_{j-1}, a_j) \geq M^{-1}\sum_{j=1}^{N} \|a_j - a_{j-1}\| \geq M^{-1}\|a - c\|$. Thus, $\tilde{d}_{\mathcal{D}}(a, c) \geq M^{-1}\|a - c\|$. Applying this to $c = a_k$ yields $\lim_{k \to \infty} \|a - a_k\| = 0$. We have proved

**Proposition 5.6.** If $\mathcal{D}$ is a bounded nc domain in an operator space $\mathcal{V}$ and $n \in \mathbb{N}$, then on any subset $A \subset \mathcal{D}_n$ which is at a positive distance from $\mathcal{D}_n$, the topologies induced by $\tilde{d}_{\mathcal{D}}$ and the norm of $\mathcal{V}^{n \times n}$ coincide.

**Remark 5.7.** A very similar proof shows that the result stated in Proposition 5.6 holds also for bounded strict subsets of half-planes.

### 6. An Application to a Problem in Free Probability

In this section, we use some of the tools introduced before in order to study a problem in free probability. We consider a $C^*$-noncommutative probability space $(M, E, B)$, where $B \subseteq M$ is a unital inclusion of $C^*$-algebras and $E: M \to B$ is a unit-preserving conditional expectation. Elements in $M$ are called operator-valued (or, sometimes, $B$-valued) random variables. If $X = X^* \in M$, we define the distribution of $X$ with respect to $E$ to be the collection of multilinear maps

$$\mu_X = \{m_{n, X}, n \in \mathbb{N}\},$$

where $m_{0, X} = 1 \in B \subseteq M$, $m_{1, X} = E[X] \in B$, and

$$m_{n, X}: B \times \cdots \times B \to B, \quad m_{n, X}(b_1, \ldots, b_{n-1}) = E[Xb_1Xb_2 \cdots Xb_{n-1}X], n > 1.$$

Such distributions are encoded analytically by the noncommutative Cauchy-Stieltjes transform (see Example 2.1(3)):

$$G_{X, n}(b) = E \left[ (b - I_n \otimes X)^{-1} \right], \quad n \in \mathbb{N}, b \in B^{n \times n}, \Im b > 0.$$

This is a noncommutative function mapping the noncommutative upper half-plane of $B$ into the noncommutative lower half-plane (see, for instance, [22]). It has several good properties, including the fact that $\Im G_{X, n}(b) < 0$, so that $F_{X, n}(b) := G_{X, n}(b)^{-1}$ exists and maps elements of positive imaginary part into elements of positive imaginary part. Moreover, it has been shown in [13] that $\Im F_{X, n}(b) \geq \Im b$, so that $h_{X, n}(b) := F_{X, n}(b) - b, \Im b > 0$, takes values elements of nonnegative imaginary part.
It has been shown in [10] that for any given selfadjoint \( X \in M \) and completely positive map \( \rho: B \to B \) such that \( \rho - \text{Id}_B \) is still completely positive on \( B \), there exists a selfadjoint \( X_\rho \) in a possibly larger \( C^* \)-algebra containing \( M \) such that \( E \) extends to this possibly larger algebra and the following relations hold:

\[
G_{X_\rho,n}(b) = G_{X,n}(\omega_\rho(b)), \quad \omega_\rho(b) = b + (\rho - \text{Id}_B)h_{X,n}(\omega_\rho(b)), \quad \exists b > 0, n \in \mathbb{N}.
\]

In terms of the free probability significance of \( X_\rho \), we only mention that \( \mu_{X_\rho} = \overline{\mu}_X \), and refer the interested reader to [10] for details. We wish to mention, however, that, thanks to a trick due to Hari Bercovici, understanding free convolution powers indexed by completely positive maps suffices in order to understand free additive convolutions of operator-valued distributions, so, in a certain sense, \( \{\overline{\mu}_X: \rho \text{ and } \rho - \text{Id}_B \text{ completely positive}\} \) is the most general object to understand in the context of free convolutions of operator-valued distributions.

All of the above has been done for selfadjoint operators that belong to \( M \), that is, bounded selfadjoint operators. We will apply our results in order to show that, under certain hypotheses, this can be also done for unbounded operators \( X = X^* \) affiliated to \( M \), making a step in the direction of a full generalization of the results of [16]. Our hypotheses will be the following:

(H1) \( B \) and \( X \) generate an algebra of (unbounded) operators \( B(X) \), such that the spectral projections of any selfadjoint element of \( B(X) \) belong to \( M \). In particular, the distribution of any selfadjoint element from \( B(X) \) with respect to any continuous linear functional on \( M \) must be a probability measure;

(H2) \( E [\Im (b - X)^{-1}] < 0 \) whenever \( \Im b > 0 \) in \( B \).

Hypothesis (H1) is very natural, in the sense that otherwise there would hardly be a way to conceive a \( B \)-valued distribution of \( X \). It is clearly satisfied under the assumption that \( M \) is a finite factor. Hypothesis (H2) deserves a few more comments. It is natural in terms of allowing for the analytic functions tools (including the R-transform of Voiculescu - see [31, 34]) to be deployed. But it can be also viewed as a measure of nondegeneracy of \( E \): indeed, let \( b = u + iv, u = u^*, v > 0 \). Then

\[
E [\Im (b - X)^{-1}] = E [\Im ((u - X) + iv)^{-1}]
\]

\[
= -v^{-\frac{1}{2}}E \left[ \left( \left( v^{-\frac{1}{2}}(u - X)v^{-\frac{1}{2}} \right)^2 + 1 \right)^{-1} \right] v^{-\frac{1}{2}},
\]

so that \( E [\Im (b - X)^{-1}] < 0 \) if and only if \( E \left[ \left( \left( v^{-\frac{1}{2}}(u - X)v^{-\frac{1}{2}} \right)^2 + 1 \right)^{-1} \right] > 0 \). It is clear that, since \( X \) is unbounded, \( 0 = \min \sigma \left( \left( v^{-1/2}(u - X)v^{-1/2} \right)^2 + 1 \right)^{-1} \). Also,

\[
k := \left\| \left( \left( v^{-1/2}(u - X)v^{-1/2} \right)^2 + 1 \right)^{-1} \right\| \leq 1.
\]

Thus, non-invertibility of \( E [\Im (b - X)^{-1}] \) becomes equivalent to the equality

\[
E \left[ \left( \left( v^{-\frac{1}{2}}(u - X)v^{-\frac{1}{2}} \right)^2 + 1 \right)^{-1} - k \right] = \left\| \left( \left( v^{-\frac{1}{2}}(u - X)v^{-\frac{1}{2}} \right)^2 + 1 \right)^{-1} - k \right\|.
\]

That is, \( E \) is isometric on an element which is not in \( B \). Thinking in terms of the duals of \( M \) and \( B \), respectively, this tells us that there exists an element \( \varphi \) of norm one in the dual of \( B \) such that \( \left( \left( v^{-\frac{1}{2}}(u - X)v^{-\frac{1}{2}} \right)^2 + 1 \right)^{-1} - k \) reaches its norm on \( \varphi \circ E \). Thus,
hypothsis (H2) is implied by the requirement that elements in $M$ but not in $B$ do not reach their norms on $B^* \circ E$. It may be worth mentioning that in the case of a tracial $W^*$-probability space with normal faithful trace state $\tau$ which is left invariant by $E$, hypothesis (H2) comes to stating that \( \left( u^{-\frac{i}{2}}(u - X)u^{-\frac{i}{2}} \right)^2 + 1 \)^{-1} - k does not reach its norm on $L^2(B, \tau)$, and in the case when $B$ is finite dimensional, (H2) is equivalent to not allowing algebraic relations between $X$ and elements in $B$.

In this section, we shall show that the fixed point equation (38) has a nontrivial solution also when $X$ is unbounded, but still satisfies hypotheses (H1) and (H2) above. Unfortunately, that is not exactly sufficient in order to characterize the distribution of $X_\rho$ for all possible unbounded random variables $X$ as above, as shown in [35]. However, it does cover a significant number of special cases, including of many unbounded operators with no moments.

**Theorem 6.1.** Consider a noncommutative function $h: H^+(B) \to B$ such that $\exists h(b) \geq 0$ and $\lim_{y \to +\infty} \frac{3h(yb + y\delta b)}{y} = 0$ in the wo-topology for all $b \in H^+(B)$. For any given $b > 0$, the map $w \mapsto b + h(w)$ has a unique attracting fixed point in $H^+(B)$, to be denoted by $\omega(b)$, and the correspondence $b \mapsto \omega(b)$ is a noncommutative self-map of $H^+(B)$, hence, in particular, analytic.

**Proof.** It is clearly enough to prove the theorem at level 1. Thus, fix $\delta h_0 > \epsilon_0 1 > 0$. For any $n \geq 1$, the map $h_0: w \mapsto b_0 \otimes I_n + h(w)$ sends $H^+(B^{n \times n})$ into $H^+(B^{n \times n}) + i\delta \epsilon 1$, so that, as a noncommutative map, it sends $H^+(B)$ to $H^+(B) + i\delta \epsilon 01$.

We re-write the proof of Corollary 4.4 for this context: if $\exists \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in H^+(B^{2 \times 2})$, then $\exists h_0 \left( \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) \in H^+(B^{2 \times 2}) + i\delta \epsilon 01. That means $(\exists h_0(a) - \delta \epsilon 01)^{-1/2} \Delta h_0(a, c)(b)(\exists h_0(c) - \delta \epsilon 01)^{-1} \Delta h_0(a, c)(b)^*(\exists h_0(a) - \delta \epsilon 01)^{-1/2} \leq \|(3a)^{-1/2}b(3c)^{-1/2}\| \cdot 1$ for all $a, c \in H^+(B), b \in B$. We re-write this as

$$\Delta h_0(a, c)(b)(\exists h_0(c) - \delta \epsilon 01)^{-1} \Delta h_0(a, c)(b)^* \leq \|(3a)^{-1/2}b(3c)^{-1/2}\| \cdot 1.$$

Multiplying left and right by $(\exists h_0(a))^{-1/2}$, we obtain

$$\begin{align*}
(\exists h_0(a))^{-1/2} \Delta h_0(a, c)(b)(\exists h_0(c) - \delta \epsilon 01)^{-1} \Delta h_0(a, c)(b)^* (\exists h_0(a))^{-1/2} &\leq \|(3a)^{-1/2}b(3c)^{-1/2}\| \cdot 1 - \epsilon_0(\exists h_0(a))^{-1} \\
&\leq \|(3a)^{-1/2}b(3c)^{-1/2}\| \cdot 1 - \epsilon_0(\exists h_0(a))^{-1} \\
&= \|(3a)^{-1/2}b(3c)^{-1/2}\| \cdot 1 - \epsilon_0(\exists h_0(a))^{-1}.
\end{align*}$$

Since $xx^* \leq M \cdot 1 \iff x^*x \leq M \cdot 1$, we immediately obtain

$$(\exists h_0(a))^{-1/2} \Delta h_0(a, c)(b)(\exists h_0(c))^{-1} \Delta h_0(a, c)(b)^* (\exists h_0(a))^{-1/2} \leq \|(3a)^{-1/2}b(3c)^{-1/2}\| \cdot 1 - \epsilon_0(\exists h_0(a))^{-1} \cdot 1 - \epsilon_0(\exists h_0(c))^{-1}.$$ (39)

Applying this to $b = a - c$ yields

$$(\exists h_0(a))^{-1/2} (h_0(a) - h_0(c))(\exists h_0(c))^{-1} (h_0(a) - h_0(c))^* (\exists h_0(a))^{-1/2} \leq \|(3a)^{-1/2}(a - c)(3c)^{-1/2}\| \cdot 1 - \epsilon_0(\exists h_0(a))^{-1} \cdot 1 - \epsilon_0(\exists h_0(c))^{-1}.$$ (40)

It thus follows that if $\omega(h_0) \in H^+(B) + i\delta \epsilon 01$ is a fixed point for $h_0$, then it must be the unique and attracting fixed point of $h_0$. Indeed, for an arbitrary $a \in H^+(B)$, if we let
\[ r = \| (3a)^{-1/2} (a - \omega(b_0))(3\omega(b_0))^{-1/2} \|, \]

it follows that
\[ h_0(B(\omega(b_0), 2r)) \subset B(\omega(b_0), 2r), \]

where, as in [13, Proposition 3.2], we denote
\[ (41) \quad B(c, t) = \{ a \in B : \| (3a)^{-1/2} (a - c)(3c)^{-1/2} \| \leq t \}. \]

It has been shown in [13, Proposition 3.2] that \( B(\omega(b_0), 2r) \) is bounded in norm in the sense that
\[ \| d \| \leq \| \Re \omega(b_0) \| + \| 3\omega(b_0) \| \left( 2r^2 + 1 + 2r\sqrt{r^2 + 1} + 2r \sqrt{2r^2 + 1 + 2r\sqrt{r^2 + 1}} \right), \]

and that it is bounded away from the boundary of \( H^+(B) \) in the sense that
\[ \Im d \geq \frac{1}{2 + 4r^2} 3\omega(b_0), \quad d \in B(\omega(b_0), 2r). \]

Thus, for any \( N \in \mathbb{N} \), we have, by an iteration of (10),
\[
\left\| (3h_0^N(a))^{-\frac{1}{2}} (h_0^N(a) - \omega(b_0))(3\omega(b_0))^{-\frac{1}{2}} \right\|^2 \\
= \left\| (3h_0^N(a))^{-\frac{1}{2}} (h_0^N(a) - h_0^N(\omega(b_0)))(3h_0^N(\omega(b_0)))^{-\frac{1}{2}} \right\|^2 \\
\leq \| (3a)^{-\frac{1}{2}} (a - \omega(b_0))(3\omega(b_0))^{-\frac{1}{2}} \|^2 \\
\times \prod_{j=1}^{N} \| (1 - \epsilon_0)(3h_0^{\frac{j}{2}}(a))^{-1} \| (1 - \epsilon_0)(3h_0^{\frac{j}{2}}(\omega(b_0)))^{-1} \| \\
= \| (3a)^{-\frac{1}{2}} (a - \omega(b_0))(3\omega(b_0))^{-\frac{1}{2}} \|^2 \\
\times (1 - \epsilon_0)(\|3\omega(b_0))^{-1}\| N \prod_{j=1}^{N} (1 - \epsilon_0)(3h_0^{\frac{j}{2}}(a))^{-1} \|). \]

Letting \( N \) go to infinity sends \( (1 - \epsilon_0)(\|3\omega(b_0))^{-1}\| N \prod_{j=1}^{N} (1 - \epsilon_0)(3h_0^{\frac{j}{2}}(a))^{-1} \| \) to zero, so that
\[ \lim_{N \to \infty} \left\| (3h_0^N(a))^{-\frac{1}{2}} (h_0^N(a) - \omega(b_0))(3\omega(b_0))^{-\frac{1}{2}} \right\|^2 = 0. \]

Recall that \( x^{-1} \geq \frac{1}{\|x\|} \) for any positive operator \( x \). Since
\[ \frac{1}{2 + 4r^2} 3\omega(b_0) \leq 3h_0^N(a) \leq \| \Re \omega(b_0) \| + \| 3\omega(b_0) \| (4r + 1)^2, \]

we have
\[
\left\| (3h_0^N(a))^{-\frac{1}{2}} (h_0^N(a) - \omega(b_0))(3\omega(b_0))^{-\frac{1}{2}} \right\|^2 \\
\geq \frac{\| h_0^N(a) - \omega(b_0) \|^2}{\| 3\omega(b_0) \| \| 3h_0^N(a) \|} \\
\geq \frac{\| h_0^N(a) - \omega(b_0) \|^2}{\| 3\omega(b_0) \| \| \Re \omega(b_0) \| + \| 3\omega(b_0) \| (4r + 1)^2}, \]

which allows us to conclude that
\[ \lim_{N \to \infty} \left\| h_0^N(a) - \omega(b_0) \right\| = 0, \]

uniformly on bounded sets which are at strictly positive norm-distance from the complement of \( H^+(B) \).
Iterating in relation (39) for \( a = c = \omega(b_0) \) yields

\[
\frac{\|h'_0(\omega(b_0))^{\circ N}(b)\|}{\|3\omega(b_0)\|} \leq \frac{\|3\omega(b_0)\|^{-1/2}h'_0(\omega(b_0))^{\circ N}(b)(3\omega(b_0))^{-1/2}\|}{\|3\omega(b_0)\|^{-1/2}b(3\omega(b_0))^{-1/2}\|(1 - \varepsilon_0\|3\omega(b_0))^{-1}\|)^N} \leq \frac{\|3\omega(b_0)\|^{-1}\|(1 - \varepsilon_0\|3\omega(b_0))^{-1}\|)^N\|b\|}{\|3\omega(b_0)\|^{-1}\|(1 - \varepsilon_0\|3\omega(b_0))^{-1}\|)^N\|b\|},
\]

which implies that

\[
\frac{\|h'_0(\omega(b_0))^{\circ N}(b)\|}{\|3\omega(b_0)\|} \leq \|3\omega(b_0)\|\|3\omega(b_0)\|^{-1}\|(1 - \varepsilon_0\|3\omega(b_0))^{-1}\|)^N\|b\|
\]

for all \( b \in B \), so that

\[
\frac{\|h'_0(\omega(b_0))^{\circ N}(b)\|}{\|3\omega(b_0)\|} \leq \|3\omega(b_0)\|\|3\omega(b_0)\|^{-1}\|(1 - \varepsilon_0\|3\omega(b_0))^{-1}\|)^N\|b\|
\]

the norm of \( [h'_0(\omega(b_0))]^{\circ N} \) being the norm of a bounded linear map on the \( C^* \)-algebra \( B \). Thus, for \( N > \frac{\log(\|3\omega(b_0)\|\|3\omega(b_0)\|^{-1}\|)}{-\log(1 - \varepsilon_0\|3\omega(b_0))^{-1}\|)} \), we have \( \|h'_0(\omega(b_0))^{\circ N}\| < 1 \). In general, if a linear operator \( T \) on a Banach space \( B \) satisfies \( \|\|T\|\| < 1 \), we may write \( \sum_{j=0}^{k-1} T^j = (1 + T + \cdots + T^{N-1}) + T^N(1 + T + \cdots + T^{N-1}) + T^{2N}(1 + T + \cdots + T^{N-1}) + \cdots + T^{(k-1)N}(1 + T + \cdots + T^{N-1}) = (1 + T + \cdots + T^{N-1}) \sum_{j=0}^{k-1} (T^N)^j \), which tends to \( (1 + T + \cdots + T^{N-1})(1 - T)^{-1} \) as \( k \to \infty \). Since \( N \) is fixed, it follows easily that in fact so does \( \sum_{j=0}^{k} T^j \). A simple algebraic manipulation shows that \( (1 + T + \cdots + T^{N-1})(1 - T)^{-1} = (1 - T)^{-1} \). Thus, \( \text{Id}_B - h'_0(\omega(b_0)) \) is invertible as a linear self-map of the Banach space \( B \). By the implicit function theorem for analytic maps on Banach spaces, it follows that \( \omega \) depends analytically on \( b_0 \). This result, together with the properties of fixed points for noncommutative maps proved in [2], allow us to conclude that \( \omega \) is a noncommutative map on a noncommutative neighbourhood of \( b_0 \).

All of the above has been established under the assumption that a fixed point \( \omega(b_0) \) exists. We have not proved its existence, though. Relation (40) would allow us easily to prove such an existence along the lines of the above proof if we could somehow guarantee the boundedness of the iterates \( \{h_n^N(a)\}_{N \in \mathbb{N}} \) for some given \( a \in H^+(B) \). Unfortunately, this does not seem possible to do in a direct way. Thus, we show the existence of the fixed point \( \omega(b_0) \) by a perturbative argument, most of which is contained in the following proposition, which, we believe, might be of independent interest. Define

\[
k_0(a) = -h_0(-a^{-1})^{-1}, \quad a \in H^+(B).
\]

As \( \Im h(a) > \varepsilon_0, \) it follows that \( k_0(H^+(B)) \subseteq \{ w : \|w - i(2\varepsilon_0)^{-1}\| < (2\varepsilon_0)^{-1}\} \), the noncommutative ball centered at an imaginary multiple of the identity.

**Proposition 6.2.** For any \( a \in H^+(B) \cup \{0\} \), the fixed-point equation \( x = a + k_0(x) \) has a unique solution \( x(a) \) in \( H^+(B) \). \( x(a) \) is a noncommutative function of \( a \) whenever \( a \in H^+(B) \), and \( x(0_m \oplus 0_n) = x(0_m) \oplus x(0_n) \) for all \( m, n \in \mathbb{N} \).

**Proof.** Note that the set \( a + k_0(H^+(B)) \) is bounded and bounded away from the complement of \( H^+(B) \). Thus, the argument used above allows us to conclude the existence, uniqueness and analyticity of \( x \) on \( H^+(B) \). The existence of \( x(0) \) in \( H^+(B) \) is the only difficult part of the proof. For this, we shall use some results from [12], specifically Proposition 3.1, Remark 3.2(2), and Corollary 3.3, together with the definition of a noncommutative version of horodisks in the noncommutative upper half-plane (see [12] Relation (22))). These results have been formulated for functions of a slightly different nature, but it is very easy to see that all elements of the proofs involved adapt to bounded functions like \( k_0 \) which satisfy \( k_0(a^*) = k_0(a) \).
We claim that \( x(H^+(B)) = \{m + in: m = m^*, n > \Im k_0(m + in)\} \). Since \( x(a) = a + k_0(x(a)) \), the inclusion \( \subseteq \) is quite obvious. To prove \( \supseteq \), recall that the map \( B^{sa} \ni p \mapsto \Re x(p + iq) \in B^{sa} \) is a bijection for any given \( q > 0 \) (see \cite{12} Corollary 3.3). We also know that there exists a smooth function \( g_q: B^{sa} \to \{b \in B: \Im b > 0\} \) such that \( g_q(\Re x(p + iq)) = \Im x(p + iq) \). In particular, for any \( m \in B^{sa} \), there exists a unique \( n > 0 \) such that \( g_q(m) = n \): we have

\[
m + in = x(p + iq) = p + iq + k_0(x(p + iq)) = p + iq + k_0(m + in),
\]

so that \( p = m - \Re k_0(m + in), q = n - \Im k_0(m + in) \). This proves \( \supseteq \).

Since \( k_0(H^+(B)) \) is bounded, it follows that for any pair \( m = m^*, n > 0 \), we have \( yn > k_0(m + iyn) \) for all sufficiently large \( y \in (0, +\infty) \). Thus, we may define

\[
0 \leq t_{m,n} = \inf \{y > 0: sn > \Im k_0(m + isn) \text{ for all } s > y\}.
\]

We argue that for all \( s > t_{m,n} \), we have \( sn > \Im k_0(m + isn) \), and for all \( 0 \leq s < t_{m,n} \), we have \( sn \not\supset \Im k_0(m + isn) \). The argument is virtually identical to the one in \cite{14} Lemma 5.8 and is based on related works in the case of scalar, classical distributions by Biane \cite{17} and by Huang \cite{22}, so we will only sketch it. We consider the map \( \mathbb{C}^+ \ni z \mapsto \varphi(m + zn - k_0(m + zn)) \in \mathbb{C} \) for an arbitrary state \( \varphi \). If \( H(z) = \frac{\varphi(m) + z - \varphi(k_0(m + zn))}{\varphi(n)} \), then

\[
\lim_{y \to +\infty} \frac{H(0)}{i(y)} = 1 \quad \text{and} \quad \Im H(z) \leq \Im z.
\]

Then Huang’s version \cite{22} Section 3 of Biane’s results \cite{17} Lemmas 2 and 4] applies to \( H \) to guarantee that if \( \frac{\varphi(k_0(m + iyn))}{\varphi(n)} \geq y_0 \), then \( \frac{\varphi(k_0(m + iyn))}{\varphi(n)} \geq y \) for all \( y \in (0, y_0) \). Since this holds for any state \( \varphi \), our claim follows.

Obviously, there are two possibilities: either \( t_{m,n} > 0 \) or \( t_{m,n} = 0 \). Consider first the case when \( t_{m,n} = 0 \). Pick a state \( \varphi \) on \( B \) and \( n > 0 \). We have

\[
\left\| \frac{\varphi(k_0(m + iyn))}{\varphi(n)} - \varphi(n) \right\|^2 \geq \frac{|\varphi(k_0(m + iyn)) - \varphi(k_0(m + iyn'))|^2}{3\varphi(k_0(m + iyn))3\varphi(k_0(m + iyn'))},
\]

which in its own turn implies

\[
\left| n - \frac{\varphi(k_0(m + iyn))}{\varphi(n)} \right|^2 \geq \frac{3\varphi(k_0(m + iyn))}{3\varphi(k_0(m + iyn'))} - \frac{3\varphi(k_0(m + iyn'))}{3\varphi(k_0(m + iyn))} \right|^2.
\]

As \( t_{m,n} = 0 \), we have \( \frac{\varphi(k_0(m + iyn))}{\varphi(n)} < \varphi(n) \) for all \( y > 0 \), so that necessarily

\[
0 \leq \lim_{y \to 0} \frac{\Im \varphi(k_0(m + iyn))}{y} \leq \frac{2 + \left\| n - \frac{\varphi(k_0(m + iyn))}{\varphi(n)} \right\|^2 + \sqrt{2 + \left\| n - \frac{\varphi(k_0(m + iyn))}{\varphi(n)} \right\|^2)^2 - 4}}{2}.
\]

As this holds for any \( n > 0 \) and any state \( \varphi \) on \( B \), we conclude that \( k_0 \) satisfies the hypotheses of \cite{13} Theorem 2.3. Thus, if there exists a pair \( m = m^*, n > 0 \) such that \( t_{m,n} = 0 \), then for any \( n > 0 \),

\[
\lim_{y \to 0} k_0(m + iyn) = \alpha = \alpha^*
\]

exists in the norm topology. However, observe that since \( k_0(H^+(B)) \subseteq \{w: \|w - i(2\epsilon_0)^{-1}\| < (2\epsilon_0)^{-1}\} \), and the limit is in norm, we must have \( \alpha = 0 \).

Now consider the case when \( t_{m,n} > 0 \). As seen above, for any \( y > 0 \), there exist \( p_y = m - \Re k_0(m + iyn), q_y = yn - \Im k_0(m + iyn) \) such that \( x(p_y + iq_y) = p_y + \)
$t_m,n = 0$, and then $k_0$ has a Julia-Caratheodory derivative at $m$, and \( \lim_{y \to 0} k_0(m + i y n) = 0 \) in norm for all $n' > 0$, or $t_{m,n} > 0$, and then $x$ extends analytically around $p_{t_{m,n}} + i q_{t_{m,n}} = m - i(\Re k_0(m + i t_{m,n}) + i(t_{m,n}n - 3k_0(m + i t_{m,n}))).$

We apply this to $m = 0$. Assume towards contradiction that $t_{0,n} = 0$ for some $n > 0$. Recall from [12, Relation (22)] the definition of the pseudo-horodisks at zero in “direction” $n$ (with $n$ normalized so that $\|n\| = 1$):

\[
\mathcal{H}(0,n) = \{ w \in H^+(B) : (w - 0)^*(3w)^{-1}(w - 0) \leq n \} = \{ w \in H^+(B) : n^{-1/2}3w + n^{-1/2}\Re w(3w)^{-1}\Re wn^{-1/2} \leq 1 \},
\]

and

\[
\mathcal{H}(0,n) = \{ w \in H^+(B) : (w - 0)^*(3w)^{-1}(w - 0) < n \} = \{ w \in H^+(B) : n^{-1/2}3w + n^{-1/2}\Re w(3w)^{-1}\Re wn^{-1/2} < 1 \}.
\]

Note that the only selfadjoint element in $\mathcal{H}(0,n)$ is zero. Indeed, by definition, if $w \in \mathcal{H}(0,n)$, then $n \geq \Re w(3w)^{-1}\Re w + 3w$, so that if $\|3w\| \to 0$, then necessarily $\|\Re w\| \to 0$ (in fact one can easily obtain the estimate $3w > \Re wn^{-1}\Re w \geq \frac{(\Re w)^2}{\|w\|^2} = (\Re w)^2$). Consider $B(i yn, y^{-1/2}), y > 0$, with $B$ defined in relation (11). We have:

\[\mathcal{H}(0,n) \subseteq \bigcap_{0 < t < 1} \bigcup_{0 < y < t} B(i yn, y^{-1/2}) \subseteq \mathcal{H}(0,n).\]

This has been shown in [12], but we will provide a sketch of the proof below. Thus, assume towards contradiction that $a \in \mathcal{H}(0,n)$, but $a \not\in \bigcap_{0 < t < 1} \bigcup_{0 < y < t} B(i yn, y^{-1/2}).$

Then there exist a $t_0 \in (0,1)$ such that $a \not\in B(i yn, y^{-1/2})$ for any $y \in (0, t_0)$. That is, $(a - i y n)^*(3a)^{-1}(a - i y n) \not\leq n$ for all $y \in (0, t_0)$. At the same time, there exists an $\epsilon_{a,n} \in (0, +\infty)$ such that $a^*(3a)^{-1}a \leq n - \epsilon_{a,n}$. However, $(a - i y n)^*(3a)^{-1}(a - i y n) = a^*(3a)^{-1}a + y(\Re(3a)^{-1}a - i a^*(3a)^{-1}n + y n(3a)^{-1}n) \leq a^*(3a)^{-1}a + y(2n\|3a\|^{-1}\|3a\|^{-1}\|a\| + y\|n\|^2\|3a\|^{-1}) < a^*(3a)^{-1}a + \epsilon_{a,n} \leq n$ for all $y \in (0, \sqrt{\|a\|^2 + \epsilon_{a,n}\|3a\|^{-1}\|a\|})$ (recall that $\|n\| = 1$). This is a contradiction. Thus the first inclusion holds. The second inclusion is equally simple: $a \in B(i yn, y^{-1/2})$ for some sequence $y_j$ decreasing to zero is equivalent to $a^*(3a)^{-1}a + y_j(\Re(3a)^{-1}a - i a^*(3a)^{-1}n + y_j n(3a)^{-1}n) \leq n$ for all $j \in \mathbb{N}$, which implies $a^*(3a)^{-1}a \leq n$, that is, $a \in \mathcal{H}(0,n)$. We have

\[(44) \quad k_0 \left( \mathcal{H}(0,n) \right) \subseteq k_0 \left( \bigcap_{0 < t < 1} \bigcup_{0 < y < t} B(i yn, y^{-1/2}) \right) \subseteq \bigcap_{0 < t < 1} k_0 \left( \bigcup_{0 < y < t} B(i yn, y^{-1/2}) \right) = \bigcap_{0 < t < 1} \bigcup_{0 < y < t} k_0(B(i yn, y^{-1/2})).\]
Recall that \( iyn = x(p_y + iq_y) = p_y + iq_y + k_0(x(p_y + iq_y)) = p_y + iq_y + k_0(y_n) \), that is, \( iyn = x(p_y + iq_y) \) is a fixed point for \( w \mapsto p_y + iq_y + k_0(w) \). Thus,

\[
 k_0(B(y_n, y^{-1/2})) \subseteq B(iyn, y^{-1/2}) - (p_y + iq_y).
\]

We have seen that \( p_y = -\Re k_0(iyn) \) tends to zero in norm (in fact \( \|p_y/y\| \) is bounded as \( y \to 0 \)), and \( q_y = y_n - \Re k_0(iyn) \to 0 \) in norm as \( y \to 0 \) (in fact, \( \|q_y/y\| \) is uniformly bounded for \( y \in (0, 1) \)).

We claim that

\[
\bigcap_{0 < t < 1} \bigcup_{0 < y < t} (B(iyn, y^{-1/2}) - (p_y + iq_y)) \subseteq \mathcal{H}(0, n).
\]

Assume that is not the case. Then there exists

\[
a_0 \in \bigcap_{0 < t < 1} \bigcup_{0 < y < t} (B(iyn, y^{-1/2}) - (p_y + iq_y)) \setminus \mathcal{H}(0, n),
\]

that is, for all \( t \in (0, 1) \), there exists \( 0 < y < t \) such that \( a_0 \in B(iyn, y^{-1/2}) - (p_y + iq_y) \), and yet \( a_0^*(3a_0)^{-1}a_0 \not\in n \). So (representing \( B \) on a Hilbert space via the GNS construction), there exists a unit vector \( \xi \) and a number \( \eta > 0 \) such that

\[
(3a_0)^{-1}a_0 \xi, a_0 \xi > (n \xi, \xi) + \eta
\]

and \( a_0 = a_0 - p_y - iq_y \), where \( (a_0 - iyn)^*(3a_0)^{-1}(a_0 - iyn) \leq n \). Thus, we found a sequence \( \{y_j\}_{j \in \mathbb{N}} \) decreasing to zero such that

\[
(a_0 + p_y + iq_y - iy_j n)^*(3a_0 + q_y)^{-1}(a_0 + p_y + iq_y - iy_j n) \leq (n \xi, \xi).
\]

Expanding, we obtain

\[
\begin{align*}
&(3a_0 + q_y)^{-1}a_0 \xi, a_0 \xi + 2\Re \langle (3a_0 + q_y)^{-1}a_0 \xi, (p_y + iq_y - iy_j n) \xi \rangle \\
&+ \langle (3a_0 + q_y)^{-1}(p_y + iq_y - iy_j n) \xi, (p_y + iq_y - iy_j n) \xi \rangle \\
&\leq (n \xi, \xi).
\end{align*}
\]

From \((46)\) and \((47)\) together we obtain (by cancelling \( (n \xi, \xi) \))

\[
\langle (3a_0)^{-1}a_0 \xi, a_0 \xi \rangle - \eta
\]

\[
\begin{align*}
&> \langle (3a_0 + q_y)^{-1}a_0 \xi, a_0 \xi \rangle + 2\Re \langle (3a_0 + q_y)^{-1}a_0 \xi, (p_y + iq_y - iy_j n) \xi \rangle \\
&+ \langle (3a_0 + q_y)^{-1}(p_y + iq_y - iy_j n) \xi, (p_y + iq_y - iy_j n) \xi \rangle.
\end{align*}
\]

We re-arrange this relation to get

\[
\begin{align*}
&(3a_0)^{-1}a_0 \xi, a_0 \xi - \langle (3a_0 + q_y)^{-1}a_0 \xi, a_0 \xi \rangle - \eta \\
&= \langle (3a_0 + q_y)^{-1}q_y, (3a_0)^{-1}a_0 \xi, a_0 \xi \rangle - \eta \\
&> 2\Re \langle (3a_0 + q_y)^{-1}a \xi, (p_y + iq_y - iy_j n) \xi \rangle \\
&+ \langle (3a_0 + q_y)^{-1}(p_y + iq_y - iy_j n) \xi, (p_y + iq_y - iy_j n) \xi \rangle.
\end{align*}
\]

Since \( \lim_{j \to \infty} \|q_y\| = \lim_{j \to \infty} \|p_y\| = \lim_{j \to \infty} y_j = 0 \), when we take limit as \( j \to \infty \) in the above inequality, we obtain \( -\eta > 0 \), a contradiction. Thus,

\[
\bigcap_{0 < t < 1} \bigcup_{0 < y < t} (B(iyn, y^{-1/2}) - (p_y + iq_y)) \subseteq \mathcal{H}(0, n).
\]
Combining this with relations (44) and (45), we obtain
\[ k_0(H(0, n)) \subseteq H(0, n). \]

Quite trivially,
\[ a \in H(0, n) \iff n^{-1/2}a^*(3a)^{-1}a^{-1/2} \leq 1 \]
\[ \iff n^{-1/2}3a^{-1/2} + n^{-1/2}\Re(a)^{-1} \Re a^{-1/2} \leq 1 \]
\[ \iff (3a + \Re(a)^{-1}\Re a)^{-1} \geq n^{-1} \]
\[ \iff \Im(-a^{-1}) \geq n^{-1}. \]

Thus, \( H(0, n) \) is mapped bijectively (and as a noncommutative set) onto the set \( \{ a \in H^+(B) : 3a \geq n^{-1} \} \) by the idempotent correspondence \( a \mapsto -a^{-1} \). By the definition of \( k_0 \) (see (13)), it follows that \( h(\{ a \in H^+(B) : 3a \geq n^{-1} \}) \subseteq \{ a \in H^+(B) : 3a \geq n^{-1} \} \).

However, our hypothesis on \( h \) states that \( \lim_{y \to \infty} \langle (3\Re(a + iy3a)\xi, \xi) \rangle = 0 \) for any \( a \in H^+(B) \) and unit vector \( \xi \). This means that given \( u = u^*, v > 0 \), there exists an \( y \) depending on \( u, v \), and \( \xi \) such that \( 3h(u + iyv)\xi, \xi \rangle < y(\xi, \xi)/2 \). But \( 3h(u + iyv) \geq \Im(u + iyv) = yv \), a contradiction. This concludes the proof of our proposition. \( \square \)

Proposition 6.2 shows that the map \( w \mapsto b_0 + h(w) \) has an attracting fixed point in \( H^+(B) \). The results of [2] allow us to conclude the proof of Theorem 6.1. \( \square \)

In order to argue that Theorem 6.1 solves to a certain extent the problem of defining free convolution powers of unbounded selfadjoint random variables, let us show that if \( X = X^* \in M \), then \( h_X(b) = E[(b - X)^{-1}] - b \) satisfies the hypothesis of Theorem 6.1. Fix \( b = u + iv, u = u^*, v > 0 \). Then
\[
h_X(u + zv) = E[(u - X + zv)^{-1}] - u - zv
\]
\[
= v^{1/2}E \left[ \left( z + v^{-1/2}(u - X)v^{-1/2} \right)^{-1} \right] v^{1/2} - u - zv
\]
\[
= v^{1/2} \left\{ E \left[ \left( z + v^{-1/2}(u - X)v^{-1/2} \right)^{-1} \right] - z - v^{-1/2}uv^{-1/2} \right\} v^{1/2}.
\]

We argue that \( h_X \) satisfies the hypothesis of Theorem 6.1. This means (via a polarization argument) to show that \( \lim_{y \to \infty} \langle (3h_X(u + iyv)\xi, \xi) \rangle = 0 \). First, since \( \Im(m + in)^{-1} = -(mn^{-1}m + n)^{-1}, \Re(m + in)^{-1} = n^{-1}m(mn^{-1}m + n)^{-1} \), we have
\[
3E \left[ \frac{1}{z - Y} \right] = -E \left[ \frac{y}{y^2 + (x - Y)^2} \right] < 0, \quad \Re E \left[ \frac{1}{z - Y} \right] = E \left[ \frac{x - Y}{y^2 + (x - Y)^2} \right],
\]
where \( z = x + iy \). Thus,
\[
3E \left[ \frac{1}{z - Y} \right]^{-1} =
\]
\[
\left\{ E \left[ \frac{x - Y}{y^2 + (x - Y)^2} \right] \frac{y}{y^2 + (x - Y)^2} \right\}^{-1} + E \left[ \frac{y}{y^2 + (x - Y)^2} \right]^{-1}
\]
\[
\leq E \left[ \frac{y}{y^2 + (x - Y)^2} \right]^{-1},
\]
which makes
\[ \Im E \left[ (z - Y)^{-1} \right]^{-1} - y \leq y \left( E \left[ \frac{y^2}{y^2 + (x - Y)^2} \right]^{-1} - 1 \right). \]

Dividing by \( y \) provides us with the majorizing term \( E \left[ \frac{y^2}{y^2 + (x - Y)^2} \right]^{-1} - 1 \). This, as a function of \( y \), is decreasing, as it can be seen by taking the (classical) derivative with respect to \( y \):
\[
\partial_y E \left[ \frac{y^2}{y^2 + (x - Y)^2} \right]^{-1} = \frac{- E \left[ \frac{y^2}{y^2 + (x - Y)^2} \right]^{-1} E \left[ \frac{2y(x - Y)^2}{(y^2 + (x - Y)^2)^2} \right] E \left[ \frac{y^2}{y^2 + (x - Y)^2} \right]^{-1}} \leq 0.
\]

Thus, \( E \left[ \frac{y^2}{y^2 + (x - Y)^2} \right]^{-1} - 1 \) is a decreasing function of \( y \). If it does not decrease to zero, then there exists a positive operator \( 0 \neq c \geq 0 \) which belongs to the universal envelopping von Neumann algebra of \( B \) such that \( \lim_{y \to \infty} E \left[ \frac{y^2}{y^2 + (x - Y)^2} \right]^{-1} - 1 = c \) in the weak operator topology. Multiplying left and right by \((1 + c)^{-1/2}\) allows us to conclude that \((1 + c)^{1/2} E \left[ (z - Y)^{-1} \right] (1 + c)^{1/2}\) belongs to the norm-ball of center \(-i/(2y)\) and radius \(1/(2y)\). Taking the imaginary part and multiplying by \( y \) yields
\[
\lim_{y \to \infty} (1 + c)^{1/2} E \left[ \frac{y^2}{y^2 + (x - Y)^2} \right] (1 + c)^{1/2} = 1
\]
in the wo-topology. Thus\footnote{We use here that if \( 0 < b_j \) decreases to 1, then \( 0 < b_j \) increases to 1; this can be seen by evaluating \((1 - b_j)^{1/2} \xi, \xi)^2 = (b_j^{1/2} (b_j^{-1} - 1)^{1/2} \xi, \xi)^2 \leq (b_j \xi, \xi)/(b_j^{-1} - 1) \xi, \xi).\)

Composing this with any wo-continuous state \( \varphi \) on the universal envelopping algebra of \( B \) provides us with a state \( \varphi \circ E \) on \( M \) with respect to which the distribution of \( Y \) is not a probability, contradicting (H1).

**Corollary 6.3.** Under hypotheses (H1) and (H2), if the distribution of \( X \) is encoded by the restriction of \( E[(b - X)^{-1}] \) to \( H^+(B) \), then \( \tilde{\mu}_X^{\varphi} \) is well-defined for all cp maps \( \rho: B \to B \) such that \( \rho - \text{Id}_B \) is still cp.

**Proof.** Apply Theorem 6.1 to \( h(w) = (\rho - \text{Id}_B) h_{X,n}(w) \). \( \square \)

Let us briefly comment on the Nevanlinna representation of \( h_X \). If \( X \in M \), results of [27] guarantee the existence of an extension of \( B \) in which there exists a bounded selfadjoint element \( X \) and of a completely positive map \( \rho: B(X) \to B \) such that \( h_X(b) = -E[X] + \rho \left[ (X - b)^{-1} \right] \), \( b \in H^+(B) \). As in the case of the classical Nevanlinna representation, for unbounded operators \( X \), \( \rho \) is not anymore the appropriate cp map. We define \( \eta: B(X) \to B \) by \( \eta[a] = \rho \left[ (X - i)^{-1} a (X + i)^{-1} \right] \). The correspondence becomes now
\[
h_X(b) = \Re h_X(i) + \eta \left[ (X - b)^{-1} + b + b(X - b)^{-1} b \right], \quad \Im b > 0.
\]
Observe that indeed $\Im h(i) = \eta |i|$. Rewriting this map as
\[ h_X(b) = \Re h_X(i) + \eta \left[(X - b)^{-1} - X + X(X - b)^{-1}X\right] \]
makes it clear that it maps $H^+(B)$ in its closure.

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