Reconstruction and Repair Degree of Fractional Repetition Codes

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Abstract—Given a Fractional Repetition (FR) code, finding the reconstruction and repair degree in a Distributed Storage Systems (DSS) is an important problem. In this work, we present algorithms for computing the reconstruction and repair degree of FR Codes.

I. INTRODUCTION

Distributed Storage Systems (DSSs) use coding theory to provide reliability in the system. Recently a new class of regenerating codes known as "repair by transfer codes" were used to optimize disk I/O in the system [1]. In this work, we consider DSS that use Distributed Replication-based Simple Storage (DRESS) Codes consisting of an inner Fractional Repetition (FR) code and an outer Maximum Distance Separable (MDS) code to optimize various parameters of DSS [2], [3]. Codes which has rate at least the capacity of the system are known as universally good codes [2]. To find out the universally good codes one has to find the reconstruction degree $k$ (minimum number of nodes one has to contact to yield the entire data) and the repair degree $d$ (number of nodes needs to be contacted in case of failure of a node) in such a DSS. To the best of our knowledge there is no algorithm known for finding the reconstruction degree $k$ of a given FR code. It is easy to compute the repair degree $d$ for strong FR codes as it is the degree of any node, however for weak FR codes no algorithm is known for computing the repair degree. Motivated by this in this work, we present algorithms for computing the reconstruction and repair degree of FR codes.

This paper is organized as follows. Section 2 collects necessary background on FR codes. In Section 3 we present the algorithms for computing the reconstruction degree and in Section 4 we present an algorithm for computing repair degree. Section 5 concludes with general remarks.

II. BACKGROUND

In an $(n,k,d)$ DSS, data is stored on $n$ nodes in such a fashion such that user can get the data by connecting any $k$ nodes ($k \leq n$) in a smart way) on $n$ nodes such that each node $U_i, 1 \leq i \leq n$ has $\alpha_i$ packets [2], [3].

Definition 1. (Fractional Repetition Code): A Fractional Repetition (FR) code denoted by $C(n,\theta,\alpha,\rho)$ with replication factor $\rho$, for a DSS with parameter $(n, k, d)$, is a collection $\mathcal{C}$ of $n$ subsets $U_1, U_2, \ldots, U_n$ of a set $\Omega = \{1, 2, \ldots, \theta\}$, which satisfies the following conditions:

- Every member of $\Omega$ appears exactly $\rho$ times in the collection $\mathcal{C}$.
- $|U_i| = \alpha_i$ $(\forall i = 1, 2, \ldots, n)$

where $\alpha = \max \{\alpha_i\}_{i=1}^n$.

Clearly, FR codes satisfy the equation (1) [3].

$$n\alpha = \rho\theta + \delta,$$

(1)
where $\theta$ packets are replicated $\delta$ times among $n$ nodes (each having weakness $\delta_j$) and $\rho$ is total weight of FR codes [3]. Thus $\delta$ is given by $\delta = \sum_{i=1}^{n} \delta_i = \sum_{i=1}^{n} (\alpha - \alpha_i)$ [3].

**Remark 2.** For strong FR codes [2], $\delta = 0$ then equation (7) reduces to $n\alpha = \rho\theta$, also in this case $\alpha_i = \alpha = d, \forall 1 \leq i \leq n$.

**Example 3.** For FR code $C(7, 8, 4, 3)$ a possible node packet distribution is shown in Table I. Note that in this example $\delta = \sum_{i=1}^{n} \delta_i = 4$ and it satisfies the relation $n\alpha = \rho\theta + \delta$.

**Definition 4.** (Node-packet distribution incidence matrix): For FR code $C(n, \theta, \alpha, \rho)$ a node-packet distribution incidence matrix $M_{n \times \theta}$ is a matrix with its entries $m_{ij}$ given as

$$m_{ij} = \begin{cases} 
0 & \text{if } j \in \Omega \text{ s.t. } i \notin U_i \\
1 & \text{if } j \in \Omega \text{ s.t. } j \in U_i.
\end{cases}$$

The column support of each column $M_j$, $1 \leq j \leq \theta$ of $M$ is denoted by $H_j = \text{Supp}(M_j) = \{i | m_{ij} \neq 0\}$. We will need this in Section IV.

**Remark 5.** Clearly, for a given FR code $C(n, \theta, \alpha, \rho)$ its node-packet distribution incidence matrix $M_{n \times \theta}$ has the following properties.

1) Weight of $i$th row of $M$ is $\alpha_i$.
2) Weight of each column of $M$ is $\rho$.

**Example 6.** Node-packet distribution incidence matrix $M_{7 \times 8}$ for FR code $C(7, 8, 4, 3)$ of Example 3 is

$$M_{7 \times 8} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}.$$  

Given a $(n, k, d)$ DSS, one has to find a good FR code $C(n, \theta, \alpha, \rho)$ which matches with the parameters of DSS. Note that the parameter $k$ in DSS is known as the reconstruction degree of DSS. If one wants to get the entire file one has to contact any $k$ nodes in DSS. However if we look at the definition of FR code $C$ one finds that it is independent of $k$ (there is no direct formula for calculating the reconstruction degree). This motivates us to define the reconstruction degree of FR code $C$ as the the number $k_{FR}$ so that if one wants the entire file (total $(\theta - 1)$ packets as one remaining packet one can get using MDS codes) one has to contact smallest set of any $k_{FR}$ nodes in FR code. Clearly, $k \leq k_{FR}$. In order to find the value $k_{FR}$ of a FR code we also define another reconstruction degree $k^*$ of FR code as the smallest subset of nodes of $C$, that allows recovering the entire data (all $(\theta - 1)$ packets). Clearly, we also have $k^* \leq k_{FR}$. We present an algorithm [1] to compute $k^*$. This gives a lower bound on actual $k_{FR}$.

**Example 7.** Reconstruction degree $k^*$ of the code $C(7, 8, 4, 3)$ of Example 3 is 2 because using $U_2$ and $U_5$ one can get at least 7 packets and $k_{FR} = 4$ as contacting any 4 nodes will give us at least 7 packets. Another interesting example showing this difference is Figure 7 of [2], where FR code $C(6, 9, 3, 3)$ has $k^*_3 = 3$ ($v_1, v_2$ and $v_3$ will give at least 8 packets) and $k_{FR} = 4$ (any 4 nodes will give at least 8 packets).

In Section III we consider an algorithm for computing $k^*$, and hence a lower bound on $k_{FR}$. For a $(n, k, d)$ DSS one can define the rate of FR code $C$ as

$$R_C(k) = \min_{i \in S} |U_i|$$

where $S \subseteq \{1, 2, \ldots, n\}$ and $|S| = k$.

It is clear that $R_C(k)$ of FR code $C$ is the number of guaranteed distinct packets which an user will get when any $k$ nodes are contacted in $C$. Thus finding the reconstruction degree $k$ is very useful in finding the rate of FR code.

**Example 9.** Given a $(7, k, d)$ DSS, and a FR code $C(7, 8, 4, 3)$ of Example 3 the rate of the code is 4 for $k = 3$ and rate is 6 for $k = 4$.

### III. Algorithm for Computing Reconstruction Degree $k$

In order to find the reconstruction degree of a FR code, one can always delete one packet from all the nodes (W.L.O.G, usually delete last packet $\theta$) as we can recover it using the parity of MDS codes. Hence for constructing entire data it is sufficient to reconstruct only $(\theta - 1)$ packets. Thus WLOG we delete last packet $\theta$ in the algorithm [1].

We now consider an example to compute reconstruction degree $k^*$ using algorithm [1].

**Example 10.** Consider a FR code $(5, 9, 4, 2)$ as shown in Table II.

| Nodes | Packet distribution |
|-------|---------------------|
| $U_1$ | 1, 2, 3, 4          |
| $U_2$ | 1, 6, 9             |
| $U_3$ | 2, 7, 9             |
| $U_4$ | 3, 5, 6, 8          |
| $U_5$ | 4, 7, 8             |

- **Note that since $n = 5$, after removing any packet (say last packet 9) we get $V^5 = \{V_1, V_2, V_3, V_4, V_5\}$, where $V_1 = \{1, 2, 3, 4\}$, $V_2 = \{1, 6\}$, $V_3 = \{2, 5, 7\}$, $V_4 = \{3, 5, 6, 8\}$, $V_5 = \{4, 7, 8\}$ each having cardinality as $\{4, 3, 3, 4, 3\}$ respectively.**
- **Further since there is no set $V_p \ s.t. \ V_p \subseteq V_q \ (1 \leq p, q \leq 5) \ so \ step \ 1 \ yields \ V^m = V^5 = \{V_1, V_2, V_3, V_4, V_5\}.**
Algorithm 1 Algorithm to compute reconstruction degree $k^*$

**Require:** Node packet distribution of FR code after removing the last packet $\theta$ from all $n$ nodes of $V^n = \{V_1, V_2, ..., V_n\}$.

**Ensure:** $k^\text{app}_{UP} = \text{Reconstruction degree}$
1: For $1 \leq i, j, m \leq n$, if $\exists V_i$ and $V_j$ s.t. $V_j \subseteq V_i$ then delete all such $V_j$ for all possible nodes $V_i$ and list remaining collection of nodes as $V^{m,n} = \{V_{i_1}, V_{i_2}, ..., V_{i_m}\}$, $|V_{i_j}| = \alpha_{i_j}$ = number of packets in node $V_{i_j}$.
2: Let $V' = \{V_{i_j} \in V^{m,n} | 1 \leq j \leq m \}$ and $|V_{i_j}| = \max(\alpha_{i_j})$.

3: Pick an arbitrary set $V_{i_j} \in V'$, and call this set as $P$. Set the counter $k_3 = 1$, $1 \leq k_3 \leq m$ and $1 \leq \lambda \leq |V'| = l$.
4: If $\exists V_{i,\mu}, (1 \leq j' \leq m) \in V^{m,n}$ s.t. $V_{i,\mu} \cap P = \phi$ then go to step 5 otherwise jump to step 6.
5: Pick $V_{i,\mu}, (1 \leq j' \leq m) \in V^{m,n}$ which has maximum cardinality among all $V_{i,\mu}$ in $V^{m,n}$ with $V_{i,\mu} \cap P = \phi$. Update $P = P \cup V_{i,\mu}$, update counter $k_3 = (k_3 + 1)$ and go to step 4.
6: If $\exists V_{i,\mu}, (1 \leq r \leq m) \in V^{m,n}$ s.t. $V_{i,\mu} \not\subset P$ then go to step 7 otherwise go to step 8.
7: Pick $V_{i,\mu}, (1 \leq r' \leq m) \in V^{m,n}$ which has maximum $|V_{i,\mu}| \cap P(l)$ among all $V_{i,\mu}$ in $V^{m,n}$ having the condition $V_{i,\mu} \not\subset P$ then update $P = P \cup V_{i,\mu}$, update counter $k_3 = (k_3 + 1)$ and go to step 6.
8: If $1 \leq \lambda \leq l$, then store $k_3$ in $k_3'$ and set $k_3 = k_3 + 1$ and perform step 4 for $P = V_{i,\mu}, (1 \leq \lambda \leq m) \in V'$ s.t. $V_{i,\mu} \not\subset V'$, otherwise report $k^\text{app}_{UP} = \min(k_3')$.

**Remark 11.** Note that in general, algorithm [7] computes an upper bound on $k^*$. However, in Example [10] algorithm gives an exact value of $k^*$, i.e., $k^*_{UP} = k^* = 3$. Table [11] presents a case of FR code $\mathcal{G}: (5, 8, 4, 2)$ for which $k^* = 2$ and $k^*_{UP} = 3$. Further note that at the cost of complexity, one can modify the algorithm [7] at step 3, by taking $P$ on all possible nodes in $V_m$ to yield an exact reconstruction degree $k^*$. In particular, for strong FR code this algorithm will always give an exact value of $k^*$.

### Table III. Node-Packet Distribution for FR Code $\mathcal{G}: (5, 8, 4, 2)$

| Nodes | Packets distribution |
|-------|---------------------|
| $U_1$ | 1,2,3,4             |
| $U_2$ | 1,2,5,7             |
| $U_3$ | 3,4,6,8             |
| $U_4$ | 7,8                 |
| $U_5$ | 6                  |

Arguments similar to Algorithm [1] can be used to give an algorithm for computing the exact reconstruction degree $k_{FR}$ as shown in Algorithm [2].

Algorithm 2 Algorithm to compute reconstruction degree $k_{FR}$

**Require:** A set of packets $\Omega = \{1,2, ..., \theta\}$ and node packet distribution of FR code with $n$ nodes $U^n = \{U_1, U_2, ..., U_n\}$.

**Ensure:** Exact reconstruction degree $k_{FR}$.
1: For $1 \leq m \leq n$ set $U^m = \{U_1, U_2, ..., U_m\}$. Take $m = n$.
2: Pick the set $U_m \in U^m$ and call this set as $P$. Set the counter $k_3 = 1$, $1 \leq k_3 \leq m$ and $1 \leq \lambda \leq n$. If $\Omega \cap P = \phi$ or singleton set then go to step 6 otherwise go to step 3.
3: If $\exists U_j, (1 \leq j \leq m) \in U^m$ s.t. $U_j \cap P = \phi$ then go to step 4 otherwise jump to step 5.
4: Pick an arbitrary $U_j, (1 \leq j' \leq m) \in U^m$ which has maximum cardinality among all $U_j$ in $U^m$ with $U_j \cap P = \phi$. Update $P = P \cup U_j$, update counter $k_3 = (k_3 + 1)$. Again if $\Omega \cap P = \phi$ or singleton set then go to step 6 otherwise go to step 3.
5: Pick $U_j, (1 \leq r \leq m) \in U^m$ s.t. $U_j \not\subset P$ which has maximum $|U_j| \cap P(l)$ among all $U_j \in U^m$ having the condition $U_j \not\subset P$ then update $P = P \cup U_j$, update counter $k_3 = (k_3 + 1)$. Once again if $\Omega \cap P = \phi$ or singleton set then go to step 6 otherwise go to step 5.
6: Stor $k_3$ in $k_3'$ and set $k_3 = k_3 + 1$.
7: If $1 \leq \lambda \leq n$ then calculate $U^{m-1} = U^m \setminus \{U_m\}$ and perform step 2 for $P = U_j', (1 \leq j' \leq m) \in U^{m-1}$, otherwise report $k_{FR} = \max(k_3')$.

In Section IV we focus our attention to repair degree which is another important parameter of DSS.

### IV. ALGORITHM FOR COMPUTING REPAIR DEGREE

Given a $(n,k,d)$ DSS, in case of a node failure, it can be repaired by contacting any $d$ nodes [2], [4]. Thus $d$ is known as the repair degree of a node. In case of FR codes, the repair of a node is Table based, i.e., one has to contact specific set of nodes for repair. However, in case of strong FR code $\mathcal{G}(n, \theta, \alpha, \rho)$, we have $\alpha = d$ for every node so it is easy to calculate the repair degree. Moreover, in case of weak FR code, if repair degree of a node $U_i, 1 \leq i \leq n$ is denoted by $d_i$, then $d_i \leq \alpha_i = |U_i| \leq \alpha$ since in the worst case all $\alpha_i$ packets can be recovered by contacting some $\alpha_i$ nodes and $\alpha$ is maximum size of any node. As expected we also have...
$d_i \leq (n - 1)$. A list of repair degree for all 5 nodes for FR code $\mathcal{C}(7,8,4,3)$ of Example 3 is given in Table 1. Note that the repair degree is much less than the number of packets in a node for weak FR code as compared to strong FR code where it is equal to the size of each node. Thus computing the repair degree of weak FR codes is an interesting problem. Algorithm 3 computes the repair degree $d_i$ for any node $U_i$.

Algorithm 3 Algorithm to compute Repair Degree $d_i$

Require: Incidence matrix $M_{n \times b}$ of FR code.
Ensure: Repair degree $d_i$ of node $U_i$.

1: For each node $i$, $1 \leq i \leq n$ let $S_i^{(i)} = \{ H_j \setminus \{i\} \mid i \in H_j, 1 \leq j \leq \theta \}$

2: Compute $T \subseteq \{1,2,\ldots,\theta\}$ s.t. $|T|$ is maximum among all possible subsets and for $t \in T$, $H_t \setminus \{i\} \in S_i^{(i)}$, and

$\bigcap H_t \setminus \{i\} \neq \phi$. Set counter $l_0 (1 \leq q \leq n) = |T| - 1$

3: Update $S_i^{(i)} = S_i^{(i)} \cup (H_t \setminus \{i\}), \forall t \in T$. 

4: If $S_i^{(i)} = \phi$ then $d_i = \alpha_i - \sum_{i=1}^{q} \lambda_i$, where $\alpha_i = |V_i|$, otherwise set $q = q + 1$ and go to step 2.

Example 12. Consider the following node-packet distribution incidence matrix $M_{11 \times 8}$ for FR code $\mathcal{C} : (11,8,2,3)$. 

\[
M_{11 \times 8} = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

According to algorithm 3 the calculation of repair degree $d_i$ where $i \in \{1,2,\ldots,11\}$ for the node-packet distribution incidence matrix $M_{11 \times 8}$ is as follows.

- $H_1 = \{1,8,5\}$, $H_2 = \{2,5,9\}$, $H_3 = \{5,9,3\}$, $H_4 = \{8,1,5\}$, $H_5 = \{2,6,8\}$, $H_6 = \{9,4,7\}$, $H_7 = \{10,7,1\}$, $H_8 = \{2,6,11\}$.

- If we want to compute repair degree for $5^{th}$ node (i.e. $d_5$) then pick the all $H_j$ s.t. $5 \in H_j$ i.e. $H_1$, $H_2$, $H_3$ and $H_4$.

- Now $S_5^{(5)} = \{H_1 \setminus \{5\}, H_2 \setminus \{5\}, H_3 \setminus \{5\}, H_4 \setminus \{5\}\}$.

- But $\bigcap_{r \in \{1,2,3,4\}} H_r \setminus \{5\} = \phi$ and there is no any common element among any three sets chosen from the $S_5^{(5)}$.

- Now updated $S_5^{(5)}$ is $S_5^{(5)} = \{H_2 \setminus \{5\}, H_3 \setminus \{5\}\}$ then we have $H_2 \setminus \{5\} \cap H_3 \setminus \{5\} = \{9\} \neq \phi$ here $T = \{2,3\}$ so $l_2 = 2 - 1 = 1$.

- Now Repair degree ($d_5$) = $\alpha_5 - l_1 - l_2 = 2$ where $\alpha_5$ is weight of $5^{th}$ row in node-packet distribution incidence matrix $M_{11 \times 8}$.

V. CONCLUSION

In this paper, we presented algorithms for computing reconstruction degree of FR code $\mathcal{C}(n,\theta,\alpha,\rho)$. Given a FR code we define the reconstruction degree $k^*$ as the smallest subset of nodes when contacted will give the entire data and provided algorithm for computing it. This gives a lower bound on the actual reconstruction degree $k_{FR}$ of FR code, which is defined as the, smallest number of any $k_{FR}$ nodes when contacted will yield the entire data. At the cost of complexity, we also provided an algorithm for computing exact $k_{FR}$. Finally we show the significance of weak FR codes over strong FR codes using repair degree of FR codes. We also present an algorithm for computing repair degree for weak FR codes.

REFERENCES

[1] N. Shah, K. Rashmi, P. Vijay Kumar, and K. Ramchandran, “Distributed storage codes with repair-by-transfer and nonachievability of interior points on the storage-bandwidth tradeoff,” Information Theory, IEEE Transactions on, vol. 58, no. 3, pp. 1837–1852, 2012.

[2] S. El Rouayheb and K. Ramchandran, “Fractional repetition codes for repair in distributed storage systems,” in Communication, Control, and Computing (Allerton), 2010 48th Annual Allerton Conference on, Oct. 2010, pp. 1510 –1517.

[3] M. K. Gupta, A. Agrawal, and D. Yadav, “On weak dress codes for cloud storage,” CoRR, vol. abs/arXiv/1302.3681, 2013.

[4] A. Dimakis, K. Ramchandran, Y. Wu, and C. Suh, “A survey on network codes for distributed storage,” Proceedings of the IEEE, vol. 99, no. 3, pp. 476 –489, march 2011.

[5] A. G. Dimakis, B. Godfrey, Y. Wu, M. J. Wainwright, and K. Ramchandran, “Network coding for distributed storage systems,” CoRR, vol. abs/0803.0632, 2008.

[6] S. Anil, M. K. Gupta, and T. A. Gulliver, “Enumerating some fractional repetition codes,” CoRR, vol. abs/1303.6801, 2013.