Trace theorem for quasi-Fuchsian groups

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Abstract. We complete the proof of the Trace Theorem in the quantized calculus for quasi-Fuchsian groups which was stated and sketched, but not fully proved, on pp. 322–325 of the book Noncommutative geometry of the first author.

Bibliography: 34 titles.

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§1. Introduction

We first recall how quasi-Fuchsian groups were obtained by Bers [2] from a pair of cocompact Fuchsian groups $\Gamma_1, \Gamma_2$ and a given group isomorphism $\alpha: \Gamma_1 \to \Gamma_2$. All required notations and notions used below are explained in §2. The quasi-Fuchsian group $G = G(\Gamma_1, \Gamma_2, \alpha)$ is a discrete subgroup $G \subset \text{PSL}(2, \mathbb{C})$ which simultaneously uniformizes the compact Riemann surfaces $X_j = \mathbb{D}/\Gamma_j$, $j = 1, 2$, (where $\mathbb{D}$ is the unit disc in $\mathbb{C}$) in the following sense (see [5]).

(1) There is a Jordan curve $C \subset \mathbb{C} = S^2$ invariant under any $g \in G$ and such that the action of $G$ on $C$ is minimal (every orbit is dense).

(2) Let $\Sigma_{\text{int}}$ and $\Sigma_{\text{ext}}$ be the connected components of the complement of $C$. There are conformal diffeomorphisms $Z: \mathbb{D} \to \Sigma_{\text{int}}$, $Z': \mathbb{D} \to \Sigma_{\text{ext}}$ and group isomorphisms $\pi: G \to \Gamma_1$, $\pi': G \to \Gamma_2$ such that

$$g \circ Z = Z \circ \pi(g), \quad g \circ Z' = Z' \circ \pi'(g), \quad \pi'(g) = \alpha(\pi(g)) \quad \forall g \in G.$$ 

Furthermore, the group $G = G(\Gamma_1, \Gamma_2, \alpha)$ possesses the following properties:

(i) $G$ is finitely generated;

(ii) $G$ does not contain elliptic or parabolic elements.

The Jordan curve $C = \Lambda(G)$ is a quasi-circle whose Hausdorff dimension $p$ is strictly bigger than 1 except when $\Gamma_1$ and $\Gamma_2$ are conjugate Fuchsian groups (see [5], Theorem 2).

The main result of this paper is the following theorem which appears as Theorem 17 on p.324 of [13]. It gives a formula for the $p$-dimensional geometric probability measure on $C = \Lambda(G)$ in terms of the quantized differential $[F, Z]$ of the Riemann mapping $Z: \mathbb{D} \to \Sigma_{\text{int}}$ understood as a function on the circle $S^1 = \partial \mathbb{D}$

$\nu$ on $\overline{\mathbb{C}}$ is called $p$-dimensional geometric (relative to $G$) if $d(\nu \circ g)(z) = |g'|^p(z) d\nu(z)$ for every $g \in G$. Here $g'$ is the complex derivative. (See §2.4.)

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(to which it extends by continuity using the Carathéodory theorem; see [25]). Here $F$ is the Hilbert transform on the circle; equivalently, $F = 2P - 1$, where $P$ is the Riesz projection and the algebra $L_\infty(\partial \mathbb{D})$ is identified with its natural action on the Hilbert space $L_2(\partial \mathbb{D})$ by pointwise multiplication. The basic formula depends on the fact that, unlike for distributional derivatives, one can take the $p$th power $|[F, Z]|^p$ of the absolute value of the quantized differential $[F, Z]$. The nice geometric properties of the quasi-Fuchsian groups $G = G(\Gamma_1, \Gamma_2, \alpha)$ are used crucially in the proof and we formulate our result in slightly greater generality and in more intrinsic terms without reference to the joint uniformization.

**Theorem 1.1.** Let $G$ be a finitely generated quasi-Fuchsian group without parabolic elements. Let $p > 1$ be the Hausdorff dimension of $C = \Lambda(G)$, and let $\nu$ be the (unique) $p$-dimensional geometric probability measure on $\Lambda(G)$. Then

(a) $[F, Z] \in \mathcal{L}_{p, \infty}$;

(b) for every $f \in C(\Lambda(G))$ and for every bounded trace$^2$ $\varphi$ on $\mathcal{L}_{1, \infty}$, there exists a constant $c(G, \varphi) < \infty$ such that

$$
\varphi((f \circ Z) \cdot |[F, Z]|^p) = c(G, \varphi) \cdot \int_{\Lambda(G)} f(t) \, d\nu(t); \quad (1.1)
$$

(c) for any Dixmier trace $\text{Tr}_\omega$, where the extended limit $\omega$ is power invariant, one has $c(G, \text{Tr}_\omega) > 0$.

The statement (c) provides a large class of traces for which $c(G, \varphi) > 0$. The notion of power invariance for the limiting process $\omega$ is explained in §7.

Theorem 1.1 was stated in [13] and the proof$^3$ was sketched there after the statement of the Theorem and using a number of lemmas but the reference [538] in [13] was never published and the detailed proof is thus unpublished even if the various steps were described in [13]. It is thus very valuable to make them available while proving a more general result and introducing variants in the proposed proof in [13]. The variants concern the estimate of the growth of the Poincaré series which in [13] is attributed to Corollary 10 of [33] but the precise relation with the two forms of the absolute Poincaré series is assumed without a precise reference. This relation is due to the convex co-compactness of the action of the quasi-Fuchsian group inside hyperbolic 3-space, but in this paper the same estimate is obtained using a different method. The other important point not contained in [13] is the proof of Lemma 11, which is stated there without proof (see [13], Ch. IV, §3.3).

The authors dedicate this paper to Dennis Sullivan.

**§ 2. Preliminaries**

**2.1. General notation.** Fix throughout a separable infinite-dimensional Hilbert space $H$. We let $\mathcal{L}(H)$ denote the $*$-algebra of all bounded operators on $H$. It becomes a $C^*$-algebra when equipped with the uniform operator norm (denoted here by $\| \cdot \|_\infty$). For a compact operator $T$ on $H$, let $\lambda(k, T)$ and $\mu(k, T)$ denote

$^2$In particular, for every Dixmier trace.

$^3$This was joint work with D. Sullivan to whom the first author is indebted for his generosity in sharing his geometric insight.
its \( k \)th eigenvalue and \( k \)th largest singular value (these are the eigenvalues of \(|T|\) arranged in descending order). The sequence \( \mu(T) = \{\mu(k,T)\}_{k \geq 0} \) is referred to as the singular value sequence of the operator \( T \). The standard trace on \( \mathcal{L}(H) \) is denoted by \( \text{Tr} \). For an arbitrary operator \( 0 \leq T \in \mathcal{L}(H) \) we set
\[
n_T(t) := \text{Tr}(E_T(t,\infty)), \quad t > 0,
\]
where \( E_T(a,b) \) stands for the spectral projection of a self-adjoint operator \( T \) corresponding to the interval \( (a,b) \). Fix an orthonormal basis in \( H \) (the particular choice of the basis is inessential). We identify the algebra \( l_\infty \) of bounded sequences with the subalgebra of all diagonal operators with respect to the chosen basis. For a given sequence \( \alpha \in l_\infty \) we denote the corresponding diagonal operator by \( \text{diag}(\alpha) \).

2.2. Principal ideals \( \mathcal{L}_{p,\infty} \) and infinitesimals of order \( 1/p \). For a given \( p \), \( 0 < p < \infty \), we let \( \mathcal{L}_{p,\infty} \) denote the principal ideal in \( \mathcal{L}(H) \) generated by the operator \( \text{diag}(\{(k+1)^{-1/p}\}_{k \geq 0}) \). Equivalently,
\[
\mathcal{L}_{p,\infty} = \{T \in \mathcal{L}(H) : \mu(k,T) = O((k+1)^{-1/p})\}.
\]
These ideals, for different \( p \), all admit an equivalent description in terms of spectral projections, namely
\[
T \in \mathcal{L}_{p,\infty} \iff n_{|T|} \left( \frac{1}{n} \right) = O(n^p). \tag{2.1}
\]
We also have
\[
|T|^p \in \mathcal{L}_{1,\infty} \iff \mu^p(k,T) = O((k+1)^{-1}) \iff T \in \mathcal{L}_{p,\infty}. \tag{2.2}
\]
The ideal \( \mathcal{L}_{p,\infty} \), \( 0 < p < \infty \), is equipped with a natural quasi-norm\(^4\)
\[
\|T\|_{p,\infty} = \sup_{k \geq 0} (k+1)^{1/p} \mu(k,T), \quad T \in \mathcal{L}_{p,\infty}.
\]
However, for \( 1 < p < \infty \), it is technically convenient to use the equivalent norm
\[
\|T\|_{p,\infty} = \sup_{n \geq 0} (n+1)^{1/p-1} \sum_{k=0}^{n} \mu(k,T), \quad T \in \mathcal{L}_{p,\infty}.
\]
The following H"older property (see \cite{9}, Ch. 11, §6) is widely used throughout the paper:
\[
A_k \in \mathcal{L}_{p_m,\infty}, \quad 1 \leq m \leq n \quad \Rightarrow \quad \prod_{m=1}^{n} A_m \in \mathcal{L}_{p,\infty}, \quad \frac{1}{p} = \sum_{m=1}^{n} \frac{1}{p_m}. \tag{2.3}
\]
\(^4\)A quasi-norm satisfies the norm axioms, except that the triangle inequality is replaced by \( \|x+y\| \leq K(\|x\| + \|y\|) \) for some uniform constant \( K > 1 \).
Similarly, let \((X, \kappa)\) be a measure space (finite or infinite, atomless or atomic). We define the function space

\[ L_{p, \infty}(X, \kappa) = \{ \kappa\text{-measurable } x : \mu(t, x) = O(t^{-1/p}) \}. \]

In [13], a compact operator \(T \in \mathcal{L}(H)\) is called an infinitesimal. It is said to be of order \(\alpha > 0\) if it belongs to the ideal \(L_{1, \infty}^{\alpha}\). Equality (2.3) shows the fundamental fact that the order of the product of infinitesimals is the sum of their orders.

### 2.3. Traces on \(L_{1, \infty}\)

**Definition 2.1.** If \(\mathcal{I}\) is an ideal in \(\mathcal{L}(H)\), then a unitarily invariant linear functional \(\varphi : \mathcal{I} \to \mathbb{C}\) is said to be a trace.

Since \(U^{-1}TU - T = [U^{-1}, TU]\) for all \(T \in \mathcal{I}\) and all unitaries \(U \in \mathcal{L}(H)\), and since the unitaries span \(\mathcal{L}(H)\), it follows that the traces are precisely the linear functionals on \(\mathcal{I}\) satisfying the condition

\[ \varphi(TS) = \varphi(ST), \quad T \in \mathcal{I}, S \in \mathcal{L}(H). \]

The latter may be reinterpreted as the vanishing of the linear functional \(\varphi\) on the commutator subspace which is denoted \([\mathcal{I}, \mathcal{L}(H)]\) and defined to be the linear span of all commutators \([T, S]\), where \(T \in \mathcal{I}\) and \(S \in \mathcal{L}(H)\).

It is shown in [24], Lemma 5.2.2, that \(\varphi(T_1) = \varphi(T_2)\) whenever \(T_1\) and \(T_2\), \(0 \leq T_1, T_2 \in \mathcal{I}\), are such that the singular value sequences \(\mu(T_1)\) and \(\mu(T_2)\) coincide. For \(p > 1\) the ideal \(L_{p, \infty}\) does not admit a nonzero trace while for \(p = 1\) there exists a plethora of traces on \(L_{1, \infty}\) (see [17] or [24], for example). An example of a trace on \(L_{1, \infty}\) is the Dixmier trace introduced in [14] which we now explain.

**Definition 2.2.** The dilation semigroup on \(L_\infty(0, \infty)\) is defined by setting

\[ (\sigma_s x)(t) = x\left(\frac{t}{s}\right), \quad t, s > 0. \]

In this paper a dilation invariant extended limit means a state on the algebra \(L_\infty(0, \infty)\) invariant under \(\sigma_s\), \(s > 0\), which vanishes on every function with bounded support.

**Dixmier trace.** Let \(\omega\) be a dilation-invariant extended limit. Then the functional \(\text{Tr}_\omega : \mathcal{L}_{1, \infty}^+ \to \mathbb{C}\) defined by setting\(^5\)

\[ \text{Tr}_\omega(A) = \omega\left(t \to \frac{1}{\log(1 + t)} \int_0^t \mu(u, A) \, du\right), \quad 0 \leq A \in \mathcal{L}_{1, \infty}, \]

is additive and therefore extends to a trace on \(\mathcal{L}_{1, \infty}\). We call such traces Dixmier traces.

These traces clearly depend on the choice of the functional \(\omega\) on \(L_\infty(0, \infty)\). Using a slightly different definition, this notion of trace was applied in [13] in the setting of noncommutative geometry. We also remark that the assumption used by Dixmier

\(^5\)Here the singular value function is defined by \(\mu(A) = \sum_{k \geq 0} \mu(k, A) \chi(k, k+1).\)
of translation invariance for the functional $\omega$ is redundant (see [13], Ch. IV, §2.3 or [24], Theorem 6.3.6).

An extensive discussion of traces and more recent developments in the theory may be found in [24], including a discussion of the following facts.

(a) All Dixmier traces on $L_{1,\infty}$ are positive.
(b) All positive traces on $L_{1,\infty}$ are continuous in the quasi-norm topology.
(c) There exist positive traces on $L_{1,\infty}$ which are not Dixmier traces (see [32]).
(d) There exist traces on $L_{1,\infty}$ which fail to be continuous (see [17]).

2.4. Kleinian groups. A Fuchsian (Kleinian) group is Poincaré’s name for a discrete subgroup of $\text{PSL}(2, \mathbb{R})$ ($\text{PSL}(2, \mathbb{C})$, respectively). We are interested in Kleinian groups which are obtained by deforming certain Fuchsian groups. A nice deformation of a Fuchsian group uniformizing a compact Riemann surface is called by Bers a quasi-Fuchsian group (see [2]). The corresponding action on the complex sphere $\overline{\mathbb{C}}$ is topologically conjugate to the action of the Fuchsian group, and Poincaré noticed the deformation of the round circle of the Fuchsian group into a topological Jordan curve with remarkable properties. This ‘so-called curve’ in the words of Poincaré is now understood to have very nice conformally self-similar properties. We give below the formal Definitions 2.3 and 2.4 of Kleinian, Fuchsian and quasi-Fuchsian groups and work with intrinsic properties of Kleinian groups with no mention of the deformation.

We let $\text{SL}(2, \mathbb{C})$ be the group of all $2 \times 2$ complex matrices with determinant 1. We identify the group $\text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\{\pm 1\}$ and its action on the complex sphere $\overline{\mathbb{C}}$ (see [25]) by fractional linear transformations. The element $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in \text{SL}(2, \mathbb{C})$ represents the mapping $z \rightarrow \frac{g_{11}z + g_{12}}{g_{21}z + g_{22}}$, $z \in \mathbb{C}$.

The following definition of a Kleinian group is taken from [26], Ch. II, §A. We refer the reader to [26] for more advanced properties of Kleinian groups.

**Definition 2.3.** Let $G \subset \text{PSL}(2, \mathbb{C})$ be a discrete subgroup. We say that

(a) $G$ is freely discontinuous at the point $z \in \overline{\mathbb{C}}$ if there exists a neighbourhood $U \ni z$ such that $g(U) \cap U = \emptyset$ for every $g \in G$, $g \neq 1$;
(b) $G$ is Kleinian if it is freely discontinuous at some point $z \in \overline{\mathbb{C}}$.

The set of all points $z \in \overline{\mathbb{C}}$ at which $G$ is not freely discontinuous is called the limit set of $G$ and is denoted by $\Lambda(G)$. This set is either infinite or consists of 0, 1 or 2 points. The latter three cases correspond to the so-called elementary Kleinian groups, which are usually dropped from consideration.

The two parts of the definition below can be found in [26] on p. 103 and p. 192, respectively.

**Definition 2.4.** A Kleinian group $G$ is called:

(a) Fuchsian (of the first kind) if its limit set is a circle;
(b) quasi-Fuchsian if its limit set is a closed Jordan curve.

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6More precisely what we call ‘quasi-Fuchsian’ corresponds to ‘quasi-Fuchsian of the first kind’.
It is known that the limit set of a finitely-generated quasi-Fuchsian group (which is not Fuchsian) has Hausdorff dimension strictly greater than 1 (see Corollary 1.7 in [11]).

It is known that \((\mathbb{C}\backslash \Lambda(G))/G\) is a Riemann surface for an arbitrary Kleinian group \(G\). The following definition is taken from [11].

**Definition 2.5.** A Kleinian group \(G\) is called *analytically finite* if its Riemann surface \((\mathbb{C}\backslash \Lambda(G))/G\) is of finite type; that is, a finite union of compact surfaces with at most finitely many punctures and branch points.

We need the important notion of a \(p\)-dimensional geometric measure on \(\mathbb{C}\).

**Definition 2.6.** Let \(G\) be a Kleinian group. The measure \(\nu\) on \(\mathbb{C}\) is called \(p\)-dimensional geometric (relative to \(G\)) if \(d(\nu \circ g)(z) = |g'|^p(z) \, d\nu(z)\) for every \(g \in G\).

An important condition for existence and uniqueness of geometric measures can be found in [34] (see Theorem 1 there). Our proof of Theorem 1.1, (b) also delivers, via the Riesz Representation Theorem, the existence of a \(p\)-dimensional geometric measure concentrated on \(\Lambda(G)\) for the case when \(p\) is the Hausdorff dimension of \(\Lambda(G)\).

A subgroup in \(G\) is called parabolic if it fixes exactly one point in \(\mathbb{C}\).

The notion of a fundamental domain \(\mathbb{F} \subset \mathbb{C}\) of a Kleinian group \(G\) is defined in [26], Ch. II, § G. In particular, the sets \(\{g \mathbb{F}\}_{g \in G}\) are pairwise disjoint.

We also need the notion of the Hausdorff dimension of a set \(X \subset \mathbb{C}\) (applied to the set \(\Lambda(G)\) in this text).

**Definition 2.7.** We say that the Hausdorff dimension of a set \(X \subset \mathbb{C}\) does not exceed \(q\) if for each \(\varepsilon > 0\) there exist balls \(B(a_i,r_i)\) such that

\[
X \subset \bigcup_i B(a_i,r_i), \quad \sum_i r_i^q < \varepsilon.
\]

The infimum of all such \(q\) is called the *Hausdorff dimension of a set* \(X \subset \mathbb{C}\).

**Remark 2.8.** In what follows, we may assume without loss of generality that our group \(G\) does not contain elliptic elements (an element is called elliptic if it is conjugate to a rotation). By Selberg’s Lemma, there is a torsion-free subgroup \(G_0 \subset G\) which has finite index in \(G\). The limit set of \(G_0\) is the limit set of \(G\).

Since every finite index subgroup in a finitely generated group is itself finitely generated (see p. 55 in [30]), it follows that the conditions of Theorem 1.1 hold for the group \(G_0\). The proof of this theorem constructs a geometric measure for the subgroup of \(\operatorname{PSL}(2, \mathbb{C})\) of invariance of the limit set of \(G_0\) and hence for the group \(G\). Moreover, the uniqueness of the geometric measure for \(G_0\) implies uniqueness for \(G\). In addition to that, the group \(G_0\) does not contain elliptic elements. Indeed, an elliptic element is conjugate in \(\operatorname{PSL}(2, \mathbb{C})\) to a unitary element. Since \(G_0\) is discrete, it follows that every elliptic element has finite order; since \(G_0\) is torsion free, it follows that there are no elliptic elements.

This remark was written for the reason that some authors do not allow branches in the Riemann surfaces. It is sometimes hard to check whether a particular paper in the reference allows branches or not. The Riemann surface of a Kleinian group without elliptic elements does not have branches, which makes it easier for the reader.
2.5. Action of $\text{PSL}(2, \mathbb{C})$ on hyperbolic space. Let us briefly recall how the group $\text{PSL}(2, \mathbb{C})$ acts on the three-dimensional hyperbolic space. We refer the reader to §1.2 in [18] for details.

By definition, the unit ball model $\mathbb{B}$ of hyperbolic space is the open unit ball of $\mathbb{R}^3$ equipped with the following Riemannian metric:

$$ds^2 = \frac{4(u_0^2 + (du_1)^2 + (du_2)^2)}{(1 - u_0^2 - u_1^2 - u_2^2)^2}, \quad u = (u_0, u_1, u_2) \in \mathbb{B}.$$  

The Riemannian metric generates a distance in $\mathbb{B}$. We do not need the (complicated) distance formula, but only the fact that (see formula (2.5) of §1.2 in [18])

$$\text{dist}(u, 0) = \log \left(\frac{1 + |u|}{1 - |u|}\right), \quad u \in \mathbb{B}. \quad (2.4)$$

Here $u = (u_0, u_1, u_2)$ is identified with the quaternion $u_0 + u_1 i + u_2 j$ and $|u|$ denotes the norm of the quaternion (which coincides with the Euclidean norm of $u$).

For a matrix $g \in \text{SL}(2, \mathbb{C})$, consider the matrix $\pi(g)$ of quaternions defined as follows:

$$\pi(g) = \frac{1}{2} \begin{pmatrix} 1 & -j \\ -j & 1 \end{pmatrix} g \begin{pmatrix} 1 & j \\ j & 1 \end{pmatrix} = \begin{pmatrix} a & c' \\ c & a' \end{pmatrix}, \quad |a|^2 - |c|^2 = 1. \quad (2.5)$$

Here the quaternions $a$ and $c$ are given by the following formulæ:

$$a = \frac{1}{2} (g_{11} + \bar{g}_{22}) + \frac{1}{2} (g_{12} - \bar{g}_{21})j, \quad c = \frac{1}{2} (g_{21} + \bar{g}_{12}) + \frac{1}{2} (g_{22} - \bar{g}_{11})j. \quad (2.6)$$

Note that $|a|^2 - |c|^2 = 1$. The operation $a \rightarrow a'$ is the inner automorphism implemented by the quaternion $k$; it acts as follows:

$$(a_0 + a_1 i + a_2 j + a_3 k)' = a_0 - a_1 i - a_2 j + a_3 k \quad \forall a_j \in \mathbb{R}.$$  

The action of the group $\text{SL}(2, \mathbb{C})$ on $\mathbb{B}$ is given by

$$\pi(g): u \rightarrow (au + c')(cu + a')^{-1}, \quad u \in \mathbb{B}. \quad (2.7)$$

By Proposition 1.2.3 in [18] this action consists of isometries of $\mathbb{B}$. Formulae (2.4), (2.6) and (2.7) are crucially used in the proof of Lemma 3.1 below.

2.6. Bochner integration. The following definition of measurability can be found, for instance, in [21] (see Definition 3.5.4 there).

**Definition 2.9.** Let $X$ be a Banach space. A function $f: (\mathbb{R}, \infty) \rightarrow X$ is called:

(a) **strongly measurable** if there exists a sequence of $X$-valued simple functions converging to $f$ almost everywhere;

(b) **weakly measurable** if the mapping $s \rightarrow \langle f(s), y \rangle$ is measurable for every $y \in X^*$.

If the Banach space $X$ is separable, then the Pettis Measurability Theorem (see Theorem 3.5.3 in [21], for example) states the equivalence of the notions above.
A strongly measurable function \( f \) is Bochner integrable if
\[
\int_{-\infty}^{\infty} \|f(s)\|_X \, ds < \infty.
\]
(2.8)

Theorem 3.7.4 in [21] states that there exists a sequence \( \{f_n\}_{n \geq 0} \) of simple \( X \)-valued functions such that
\[
\int_{-\infty}^{\infty} \|(f_n - f)(s)\|_X \, ds \to 0, \quad n \to \infty.
\]
The Bochner integral is now defined as
\[
\int_{-\infty}^{\infty} f(s) \, ds \overset{\text{def}}{=} \lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(s) \, ds.
\]
Its key feature is that
\[
\left\| \int_{-\infty}^{\infty} f(s) \, ds \right\|_X \leq \int_{-\infty}^{\infty} \|f(s)\|_X \, ds.
\]

2.7. Weak integration in \( \mathcal{L}(H) \). The following definitions (and subsequent construction of a weak integral) are folklore. For example, one can look at p. 77 of [31] and put the topological space \( X \) there to be \( \mathcal{L}(H) \) equipped with the strong operator topology. Every functional on \( X \) can be written as a linear combination of \( x \to \langle x\xi, \eta \rangle \), \( \xi, \eta \in H \).

**Definition 2.10.** A function \( s \to f(s) \) with values in \( \mathcal{L}(H) \) is measurable in the weak operator topology if, for all vectors \( \xi, \eta \in H \), the function
\[
s \to \langle f(s)\xi, \eta \rangle, \quad s \in \mathbb{R},
\]
is measurable.

For such functions there is a notion of weak integral. Note that the scalar-valued mapping
\[
s \to \sup_{\|\xi\|, \|\eta\| \leq 1} \langle f(s)\xi, \eta \rangle = \|f(s)\|_\infty, \quad s \in \mathbb{R},
\]
is measurable.

Let the function \( f: \mathbb{R} \to \mathcal{L}(H) \) be measurable in the weak operator topology. We say that \( f \) is integrable in the weak operator topology if
\[
\int_{\mathbb{R}} \|f(s)\|_\infty \, ds < \infty.
\]
(2.9)

Define a sesquilinear form
\[
(\xi, \eta) \to \int_{\mathbb{R}} \langle f(s)\xi, \eta \rangle \, ds, \quad \xi, \eta \in H.
\]
It is immediate that
\[
|(\xi, \eta)| \leq \int_{\mathbb{R}} \|f(s)\|_\infty \, ds \cdot \|\xi\| \|\eta\|, \quad \xi, \eta \in H.
\]
That is, for a fixed \( \xi \in H \) the mapping \( \eta \rightarrow (\xi, \eta) \) defines a bounded anti-linear functional on \( H \). It follows from the Riesz Lemma (description of the dual of a Hilbert space) that there exists an element \( x_\xi \in H \) such that \( (\xi, \eta) = \langle x_\xi, \eta \rangle \). The mapping \( \xi \rightarrow x_\xi \) is linear and bounded. The operator which maps \( \xi \) to \( x_\xi \) is called the \textit{weak integral} of the mapping \( s \rightarrow f(s), s \in \mathbb{R} \).

The so-defined weak integral possesses the following properties.

(a) If the mapping \( s \rightarrow f(s) \) is integrable in the weak operator topology, then

\[
\left\| \int_{-\infty}^{\infty} f(s) \, ds \right\|_\infty \leq \int_{-\infty}^{\infty} \| f(s) \|_\infty \, ds.
\]

(b) If the mapping \( s \rightarrow f(s) \) is integrable in the weak operator topology and if \( A \in \mathcal{L}(H) \), then \( s \rightarrow A \cdot f(s) \) is also integrable in the weak operator topology and

\[
\int_{\mathbb{R}} A \cdot f(s) \, ds = A \cdot \int_{\mathbb{R}} f(s) \, ds.
\]

(c) If the mapping \( s \rightarrow f(s) \) is Bochner integrable in some Banach ideal in \( \mathcal{L}(H) \), then it is integrable in the weak operator topology. Its Bochner integral then equals the weak one.

\section*{2.8. Double operator integrals.}

Here we state the definition and basic properties of Double Operator Integrals which were developed by Birman and Solomyak in [6]–[8]. We refer the reader to [29] for the proofs and for more advanced properties.

Heuristically, the double operator integral \( T_{\phi}^{X,Y} \), where \( X \) and \( Y \) are self-adjoint operators and \( \phi \) is a bounded Borel measurable function on \( \text{Spec}(X) \times \text{Spec}(Y) \), is defined using the spectral decompositions:

\[
T_{\phi}^{X,Y}(A) = \iint \phi(\lambda, \mu) \, dE_X(\lambda)A \, dE_Y(\mu).
\]

This formula defines a bounded operator from \( \mathcal{L}_2 \) to \( \mathcal{L}_2 \). However, we want to consider it as a bounded operator on other ideals—and this leads to difficulty unless the function \( \phi \) is ‘good’ enough.

To specify the class of ‘good’ functions, we use the integral tensor product of [28], of \( L^\infty(\text{Spec}(X), \mu_X) \) by \( L^\infty(\text{Spec}(Y), \mu_Y) \) where the \( \mu \) denote the spectral measures. The integral projective tensor products were introduced in [28] where it was proved that the maximal class of functions for which the double operator integrals can be defined for arbitrary bounded linear operators coincides with the integral projective tensor product of \( L^\infty(\text{Spec}(X), \mu_X) \) by \( L^\infty(\text{Spec}(Y), \mu_Y) \). Thus, we consider only those functions \( \phi \) which admit a representation

\[
\phi(\lambda, \mu) = \int_{\Omega} a(\lambda, s)b(\mu, s) \, d\kappa(s), \quad (2.10)
\]

where \( (\Omega, \kappa) \) is a measure space and where

\[
\int_{\Omega} \sup_{\lambda \in \text{Spec}(X)} |a(\lambda, s)| \cdot \sup_{\mu \in \text{Spec}(Y)} |b(\mu, s)| \, d\kappa(s) < \infty. \quad (2.11)
\]
For those functions, we write
\[ T^{X,Y}_\phi (A) = \int_\Omega a(X,s)Ab(Y,s) \, d\kappa(s), \]  
(2.12)
where the latter integral is understood in the weak sense (the integrand is measurable in the weak operator topology and (2.9) holds thanks to (2.11)).

For the function \( \phi \) from the integral tensor product, we have (see Theorem 4 in [29]) that \( T^{X,Y}_\phi : L^1 \to L^1 \) and \( T^{X,Y}_\phi : L^\infty \to L^\infty \). In particular, \( T^{X,Y}_\phi : L^p,\infty \to L^p,\infty \) for \( p > 1 \).

One of the key properties of Double Operator Integrals is that they respect algebraic operations (see Proposition 2.8 in [27] or formula (1.6) in [10], for example). Namely,
\[ T^{X,Y}_{\phi_1 + \phi_2} = T^{X,Y}_{\phi_1} + T^{X,Y}_{\phi_2} \quad \text{and} \quad T^{X,Y}_{\phi_1 \cdot \phi_2} = T^{X,Y}_{\phi_1} \circ T^{X,Y}_{\phi_2}. \]  
(2.13)

2.9. Fredholm modules. The following is taken from [13].

Definition 2.11. Let \( \mathcal{A} \) be a \( \ast \)-algebra represented on the Hilbert space \( H \). Let \( F \in L(H) \) be a self-adjoint unitary operator. We call the triple \((F,H,\mathcal{A})\) a Fredholm module if \([F,a]\) is compact for every \( a \in \mathcal{A} \).

The infinitesimal \([F,a]\) is called the quantum derivative of the element \( a \) (see Ch. IV in [13] for studies of quantum derivatives).

A Fredholm module is called \((p,\infty)\)-summable if \([F,a] \in L_{p,\infty} \) for every \( a \in \mathcal{A} \).

Part (a) of Theorem 1.1 exactly states that the Fredholm module \((F,L^2(S^1),\mathcal{A})\) is \((p,\infty)\)-summable, where \( \mathcal{A} \) is the \( \ast \)-algebra generated by \( Z \).

§3. Proof of Theorem 1.1, part (a)

3.1. Growth of matrix coefficients in \( G \). Let \( G \) be a Kleinian group. As stated in Corollary II.B.7 in [26], the series \( \sum_{g \in G} |g'(z)|^2 \) converges for a.e. \( z \in \overline{C} \) (with respect to the Lebesgue measure). The critical exponent of \( G \) is defined\(^7\) as follows:
\[ p = \inf \left\{ q : \sum_{g \in G} |g'(z)|^q < \infty \text{ for a.e. } z \in \overline{C} \right\} \]
(see p. 323 of [13], for example).

Let \( \|g\|_\infty \) denote the uniform norm of the matrix \( g \in SL(2,\mathbb{C}) \) as an operator on the Hilbert space \( \mathbb{C}^2 \). Equip our countable group \( G \) with counting measure and define \( l_{p,\infty}(G) \) as in §2.2.

Lemma 3.1. Let \( G \subset PSL(2,\mathbb{C}) \) be a Kleinian group. If \( p \) is its critical exponent, then \( \{\|g\|_\infty^2\}_{g \in G} \subset l_{p,\infty}(G) \).

Proof. By Corollary 5 in [33] (see also the right-hand estimate in Corollary 10 in [33]),
\[ \text{Card}\{\{g \in G : \dist((\pi(g))(0),0) \leq r\}\} \leq C e^{pr}. \]

\(^7\)Sullivan uses a slightly different definition in [33], but they are equivalent.
Using the formula (2.4) and denoting $e^{-r}$ by $t$, we arrive at

\[
\text{Card}\left(\left\{ g \in G : \frac{1 - |(\pi(g))(0)|}{1 + |(\pi(g))(0)|} \geq t \right\}\right) \leq Ct^{-p}.
\]

Since $|(\pi(g))(0)| < 1$, it follows that

\[
\text{Card}\left(\left\{ g \in G : 1 - |(\pi(g))(0)|^2 \geq 4t \right\}\right) \leq Ct^{-p}.
\]

Since $|a'| = |a|$ and $|c'| = |c|$, it follows from (2.7) that

\[
(\pi(g))(0) = c'(a')^{-1} \quad \text{and, therefore,} \quad 1 - |(\pi(g))(0)|^2 = 1 - \frac{|c|^2}{|a|^2} = \frac{1}{|a|^2}.
\]

Thus,

\[
\text{Card}\left(\left\{ g \in G : \frac{1}{|a|^2} \geq 4t \right\}\right) \leq Ct^{-p}.
\]

It is immediate from (2.6) that $|a| \leq 2\|g\|_{\infty}$. Therefore,

\[
\text{Card}\left(\left\{ g \in G : \frac{1}{4\|g\|_{\infty}^2} \geq 4t \right\}\right) \leq Ct^{-p}.
\]

This concludes the proof.

By Theorem 5 in [26], Ch. II, §B, $g_{21} \neq 0$ for every $g \in G$, $g \neq 1$. This allows us to state a stronger version of Lemma 3.1.

**Lemma 3.2.** Let $G$ be a Kleinian group and let $p$ be the critical exponent of $G$. If $\infty$ is not in the limit set of $G$, then $\{ |g_{21}|^{-2} \}_{1 \neq g \in G} \in l_{p,\infty}(G)$.

**Proof.** By assumption $\infty \notin \Lambda(G)$. Hence $G$ is freely discontinuous at $\infty$ (see Definition 2.3). It follows that $\{ g(\infty) \}_{1 \neq g \in G}$ is a bounded set. Note that $g(\infty) = g_{11}/g_{21}$. Thus, $|g_{11}| = O(|g_{21}|)$.

Clearly,

\[
g^{-1} = \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}.
\]

Applying the preceding paragraph to the element $g^{-1}$ we conclude that $|g_{22}| = O(|g_{21}|)$.

By Theorem 5 in [26], Ch. II, §B, the sequence $\{ |g_{21}| \}_{1 \neq g \in G}$ is bounded from below. Thus,

\[
|g_{12}| = \left| \frac{g_{11}g_{22} - 1}{g_{21}} \right| \leq \frac{|g_{11}| \cdot |g_{22}|}{|g_{21}|} + \frac{1}{|g_{21}|} = O(|g_{21}|) + O(1) = O(|g_{21}|).
\]

Combining the estimates in the preceding paragraphs we conclude that $\|g\|_{\infty} = O(|g_{21}|)$. The assertion follows from Lemma 3.1. The lemma is proved.

The following lemma provides the converse to Lemma 3.2 (under additional assumptions on the group $G$).
Lemma 3.3. Let $G \subset \text{PSL}(2, \mathbb{C})$ be as in Theorem 1.1. There exists $C > 0$ such that

$$\left\{ \frac{1}{(k + 1)^{1/p}} \right\}_{k \geq 0} \leq C \mu \left( \left\{ |g_{21}|^{-2} \right\}_{1 \neq g \in G} \right).$$

Proof. By Theorem 4 of [3] the group $G$ is a quasiconformal deformation of a Fuchsian group of the first kind. In particular, its limit set $\Lambda(G)$ is a quasi-circle. By Theorem 12 in [19] the Hausdorff dimension of $\Lambda(G)$ is strictly less than 2. The group $G$ is finitely generated and thus by the Ahlfors Finiteness Theorem, $G$ is analytically finite. It follows now from Theorem 1.2 in [11] that $G$ is geometrically finite. Theorem 1 in [34] states that the critical exponent equals $p$. It is proved in [4] that a geometrically finite Kleinian group without parabolic elements is convex co-compact. Thus, the results of §3 in [33] are applicable.

By the left-hand estimate in Corollary 10 in [33],

$$\text{Card}(\{ g \in G : \text{dist}(\pi(g)(0), 0) \leq r \}) \geq Ce^{pr}.$$  

Using the formula (2.4) and denoting $e^{-r}$ by $t$, we arrive at

$$\text{Card} \left( \left\{ g \in G : \frac{1 - |(\pi(g))(0)|}{1 + |(\pi(g))(0)|} \geq t \right\} \right) \geq Ct^{-p}.$$  

Since $|(\pi(g))(0)| < 1$, it follows that

$$\text{Card}(\{ g \in G : 1 - |(\pi(g))(0)|^2 \geq t \}) \geq Ct^{-p}.$$  

Since $|a'| = |a|$ and $|c'| = |c|$, it follows that

$$(\pi(g))(0) \stackrel{(2.7)}{=} c'(a')^{-1} \quad \text{and therefore} \quad 1 - |(\pi(g))(0)|^2 = 1 - \frac{|c|^2}{|a|^2} \stackrel{(2.5)}{=} \frac{1}{|a|^2}.$$  

Thus,

$$\text{Card}(\{ g \in G : |a|^2 \leq t^{-1} \}) \geq Ct^{-p}.$$  

We infer from (2.6) that

$$4|a|^2 = |g_{11} + \bar{g}_{22}|^2 + |g_{12} - \bar{g}_{21}|^2 \quad \text{and} \quad 4|c|^2 = |g_{21} + \bar{g}_{12}|^2 + |g_{22} - \bar{g}_{11}|^2.$$  

By the parallelogram rule

$$8|a|^2 \geq 4|a|^2 + 4|c|^2 = 2|g_{11}|^2 + 2|g_{22}|^2 + 2|g_{12}|^2 + 2|g_{21}|^2 \geq 2|g_{21}|^2.$$  

It follows that

$$\text{Card} \left( \left\{ g \in G : \frac{1}{4}|g_{21}|^2 \leq t^{-1} \right\} \right) \geq Ct^{-p}.$$  

This concludes the proof.

3.2. When does the quantum derivative fall into $L_{p, \infty}$? In this subsection we find a sufficient condition for the quantum derivative to belong to the ideal $L_{p, \infty}$, $p > 1$. A similar result for the ideal $L_p$ is available as Theorem 4 and Proposition 5 on p. 316 of [13]. We get the required estimate by real interpolation.
Let $\alpha \neq -1$ and let $\nu_\alpha$ be the measure on $\mathbb{D}$ defined by

$$d\nu_\alpha(z) = |\alpha + 1|(1 - |z|^2)^\alpha \, dm(z),$$

where $m$ is the normalised Lebesgue measure on $\mathbb{D}$. For $\alpha > -1$ this is a finite measure space; for $\alpha < -1$, this is an infinite measure space. Let $\text{Hol}(\mathbb{D})$ be the space of all holomorphic functions on $\mathbb{D}$. The symbol $[\cdot, \cdot]_{\theta, \infty}$ denotes the functor of real interpolation (see, for example, Definition 2.g.12 in [23]).

**Lemma 3.4.** If $1 < p_0 < 2$, then

$$[L_{p_0}(\mathbb{D}, \nu_{p_0-2}) \cap \text{Hol}(\mathbb{D}), L_2(\mathbb{D}, \nu_0) \cap \text{Hol}(\mathbb{D})]_{\theta, \infty} = [L_{p_0}(\mathbb{D}, \nu_{p_0-2}), L_2(\mathbb{D}, \nu_0)]_{\theta, \infty} \cap \text{Hol}(\mathbb{D}).$$

**Proof.** Clearly, $L_2(\mathbb{D}, \nu_0) \cap \text{Hol}(\mathbb{D})$ is a closed subset in $L_2(\mathbb{D}, \nu_0)$. By Proposition 1.2 in [20], $L_{p_0}(\mathbb{D}, \nu_{p_0-2}) \cap \text{Hol}(\mathbb{D})$ is a closed subspace in $L_{p_0}(\mathbb{D}, \nu_{p_0-2})$, so the left-hand side is well defined.

The following map (see Proposition 1.4 in [20]) is called the Bergman projection:

$$(P_0 f)(z) = \int_{\mathbb{D}} \frac{f(w) \, d\nu_0(w)}{(1 - z \bar{w})^2}, \quad z \in \mathbb{C}.$$ 

By Theorem 1.10 in [20], we have that

$$P_0 : L_{p_0}(\mathbb{D}, \nu_{p_0-2}) \rightarrow L_{p_0}(\mathbb{D}, \nu_{p_0-2}) \cap \text{Hol}(\mathbb{D})$$

is a bounded mapping. Also, by the same theorem we have that

$$P_0 : L_2(\mathbb{D}, \nu_0) \rightarrow L_2(\mathbb{D}, \nu_0) \cap \text{Hol}(\mathbb{D})$$

is a bounded mapping.

Therefore, for the left-hand side (LHS) of the equality in the statement of Lemma 3.4, we have

$$\text{LHS} = [P_0(L_{p_0}(\mathbb{D}, \nu_{p_0-2})), P_0(L_2(\mathbb{D}, \nu_0))]_{\theta, \infty} = P_0([L_{p_0}(\mathbb{D}, \nu_{p_0-2}), L_2(\mathbb{D}, \nu_0)]_{\theta, \infty}) = [L_{p_0}(\mathbb{D}, \nu_{p_0-2}), L_2(\mathbb{D}, \nu_0)]_{\theta, \infty} \cap \text{Hol}(\mathbb{D}).$$

The proof is complete.

The following lemma describes the class of functions $f$ on the unit circle $\partial \mathbb{D}$ whose quantum derivative belongs to the weak ideal $\mathcal{L}_{p, \infty}$, $p > 1$. Here the function space $L_{p, \infty}(\mathbb{D}, \nu_{-2})$ is defined in §2.2.

**Lemma 3.5.** Suppose $f : \partial \mathbb{D} \rightarrow \mathbb{C}$ has an extension to an analytic function on $\mathbb{D}$. For $p > 1$, we have

$$\|[F, f]\|_{p, \infty} \leq c_p \|h\|_{L_{p, \infty}(\mathbb{D}, \nu_{-2})},$$

where $h(z) = (1 - |z|^2)|f'(z)|$, $z \in \mathbb{D}$. 
Proof. Let $C_p$ be the collection of all $f : \mathbb{D} \to \mathbb{C}$ such that the mapping $z \to (1 - |z|^2)f(z)$, $z \in \mathbb{D}$, belongs to the space $L_{p,\infty}(\mathbb{D}, \nu_{-2})$. If $1/p = (1 - \theta)/p_0 + \theta/2$, then

$$[L_{p_0}(\mathbb{D}, \nu_{p_0-2}), L_2(\mathbb{D}, \nu_0)]_{\theta, \infty} = C_p.$$  

Let

$$A_{1/p}^{1/p} = \{ f \in \text{Hol}(\mathbb{D}) : f' \in L_p(\mathbb{D}, \nu_{p-2}) \}$$

and let

$$D_p = \{ f \in \text{Hol}(\mathbb{D}) : f' \in C_p \}.$$  

It follows from Lemma 3.4 that

$$[A_{1/p_0}^{1/p_0}, A_{2}^{1/2}]_{\theta, \infty} = D_p.$$  

By Theorem 4 and Proposition 5 on p. 316 of [13],

$$[F, f] \in \mathcal{L}_p \iff f \in A_{1/p}^{1/p}, \quad 1 < p < \infty.$$  

Applying real interpolation methods to the Banach couples $(A_{p_0}^{1/p_0}, A_{2}^{1/2})$ and $(\mathcal{L}_{p_0}, \mathcal{L}_2)$, we infer

$$\| [F, f] \|_{p, \infty} = \| [F, f] \|_{[\mathcal{L}_{p_0}, \mathcal{L}_2]_{\theta, \infty}} \leq c_p \| f \|_{[A_{p_0}^{1/p_0}, A_{2}^{1/2}]_{\theta, \infty}} = \| f \|_{D_p}.$$  

Lemma 3.5 is proved.

3.3. Proof of Theorem 1.1, part (a). We are now ready to prove the first part of our main result.

As explained in the (first few lines of the) proof of Lemma 3.3, the group $G$ is geometrically finite. By Theorem 1 in [34], the critical exponent $\delta$ equals the Hausdorff dimension $p$ of $\Lambda(G)$. Note that $p > 1$ by Theorem 2 in [5].

Consider $G$ acting on $\Sigma_{\text{int}}$. Let $\pi$ be the action of $G$ on the unit disc by the formula

$$g \circ Z = Z \circ \pi(g). \quad (3.1)$$  

Since every $\pi(g)$ is a conformal automorphism of the unit disc, it is automatically fractional-linear (see [25]). Thus, $\pi(G)$ is a group of fractional-linear transformations preserving the unit circle, that is, a Fuchsian group, and its limit set is the unit circle $\partial \mathbb{D}$, thus it is Fuchsian of the first kind. As a group, $\pi(G)$ is isomorphic to $G$ and is, therefore, finitely generated.

We claim that the Fuchsian group $\pi(G)$ does not contain parabolic elements. Assume the contrary: let $g \in G$ be such that $\pi(g)$ is parabolic. Hence there exists a fixed point $w_0 \in \partial \mathbb{D}$ of $\pi(g)$ such that $(\pi(g))^n w \to w_0$ as $n \to \pm \infty$ for every $w \in \mathbb{D}$. Let $w = Z(z)$, $z \in \Sigma_{\text{int}}$, and let $w_0 = Z(z_0)$, $z_0 \in \Lambda(G)$. By (3.1), $g^n(z) \to z_0$ as $n \to \pm \infty$. Hence, $g \in G$ is parabolic\footnote{An element $g \in \text{PSL}(2, \mathbb{C})$ is either parabolic or diagonalizable. If $g$ is diagonalizable, then (after conjugating $g$ by a fractional-linear transform) we have that $g : z \to az$ for every $z \in \mathbb{C}$. If $|a| < 1$, then $g^n(z) \to 0$ as $n \to \infty$ and $g^n(z) \to \infty$ as $n \to -\infty$ for every $z \in \mathbb{C}$, $z \neq 0$. If $|a| > 1$, then $g^n(z) \to 0$ as $n \to -\infty$ and $g^n(z) \to \infty$ as $n \to \infty$ for every $z \in \mathbb{C}$, $z \neq 0$. If $|a| = 1$ and $a \neq 1$, then the sequence $\{g^n(z)\}_{n \in \mathbb{Z}}$ diverges as $n \to \infty$ and as $n \to -\infty$ for every $z \in \mathbb{C}$, $z \neq 0$.}, which is not the case.
Since \( \pi(G) \) is finitely generated and of the first kind, it follows from Theorem 10.4.3 in [1] that the Riemann surface \( \mathbb{D}/\pi(G) \) has finite area. Taking into account that \( \pi(G) \) does not have parabolic elements, we infer from Corollary 4.2.7 in [22] that the Riemann surface \( \mathbb{D}/\pi(G) \) is compact. By Corollary 4.2.3 and Theorem 3.2.2 in [22], \( \pi(G) \) admits a fundamental domain \( \mathcal{F} \) which is compactly supported in \( \mathbb{D} \).

**Step 1.** We claim that there exists a finite constant such that for every \( g \in G \),

\[
\sup_{z \in \pi(g)\mathcal{F}} (1 - |z|^2)|Z'(z)| \leq \frac{\text{const}}{|g_{21}|^2}.
\]

Indeed, we have \( z = \pi(g)w \), where \( w \in \mathcal{F} \). We have\(^9\)

\[
1 - |z|^2 = (1 - |w|^2)|\pi(g)(w)|.
\]

It follows from the chain rule that

\[
(1 - |z|^2)|Z'(z)| = (1 - |w|^2) \cdot |Z'(\pi(g)w)| \cdot |(\pi(g))'(w)| = (1 - |w|^2) \cdot |(Z \circ \pi(g))'(w)|.
\]

It follows from (3.1) and the chain rule that

\[
(1 - |z|^2)|Z'(z)| \overset{(3.1)}{=} (1 - |w|^2) \cdot |(g \circ Z)'(w)| = |g'(Z(w))| \cdot (1 - |w|^2)|Z'(w)|. \tag{3.2}
\]

Since \( g'(u) = (g_{21}u + g_{22})^{-2} \) and \( g^{-1}(\infty) = -g_{22}/g_{21} \), it follows that

\[
|g'(Z(w))| = \frac{1}{|g_{21}Z + g_{22}|^2} = \frac{1}{|g_{21}|^2} \cdot \frac{1}{|Z(w) - g^{-1}(\infty)|^2}. \tag{3.3}
\]

Thus, for \( z \in \pi(g)\mathcal{F} \), since \( g^{-1}(\infty) \) stays in the unbounded component of the complement of the limit set \( \Lambda(G) \) and thus \( |Z(w) - g^{-1}(\infty)| \geq \text{dist}(Z(\mathcal{F}), \Lambda(G)) \), we have

\[
(1 - |z|^2)|Z'(z)| \leq \frac{1}{|g_{21}|^2} \cdot \frac{1}{\text{dist}^2(Z(\mathcal{F}), \Lambda(G))} \cdot \sup_{w \in \mathcal{F}} (1 - |w|^2)|Z'(w)|.
\]

Since \( \mathcal{F} \) is compact and \( Z'|_{\mathcal{F}} \) is continuous, the claim follows.

**Step 2.** Let \( h(z) = (1 - |z|^2)|Z'(z)| \) (see also the statement of Lemma 3.5). It follows from Step 1 that

\[
\|h\|_{L^p,\infty}(\mathbb{D}, \nu_{-2}) \leq \|h\chi_{\mathcal{F}}\|_{L^p,\infty}(\mathbb{D}, \nu_{-2}) + \text{const} \cdot \left\| \sum_{1 \neq g \in G} \frac{1}{|g_{21}|^2} \chi_{\pi(g)\mathcal{F}} \right\|_{L^p,\infty}(\mathbb{D}, \nu_{-2}).
\]

Recall that \( \mathcal{F} \) is compactly supported in \( \mathbb{D} \) and therefore \( \nu_{-2}(\mathcal{F}) < \infty \). Let \( \nu_{-2}(\mathcal{F}) = a \). Elements of the group \( \pi(G) \) are conformal automorphisms of the unit disc, hence

\[
\frac{|dk(w)|}{1 - |k(w)|^2} = \frac{|\beta w + \bar{\alpha}|^{-2}}{1 - |\alpha w + \beta|^2} \frac{|dw|}{|\beta w + \bar{\alpha}|^2} = \frac{|dw|}{|\beta w + \bar{\alpha}|^2 - |\alpha w + \beta|^2} = \frac{|dw|}{1 - |w|^2}.
\]

\(^9\)This is a standard fact. Let \( k : w \rightarrow \frac{\alpha w + \beta}{\beta w + \bar{\alpha}} \), \( |\alpha|^2 - |\beta|^2 = 1 \), be an arbitrary conformal automorphism of the unit disc. We have
isometries of the hyperbolic plane $\mathbb{H}^2$. The measure $\nu_{-2}$ is a volume form of $\mathbb{H}^2$ and is therefore invariant with respect to its isometries. Hence, $\nu_{-2}$ is $\pi(G)$-invariant.\footnote{This fact can also be seen directly as follows. Let $k: z \to \frac{\alpha z + \beta}{\bar{\beta}z + \bar{\alpha}}$, $|\alpha|^2 - |\beta|^2 = 1$, be an arbitrary conformal automorphism of the unit disc. Its Jacobian is exactly $|k'(z)|^2$. Thus, $d(\nu_{-2} \circ k)(z) = \frac{d(m \circ k)(z)}{(1 - |k(z)|^2)^2} = \frac{|k'(z)|^2}{(1 - |k(z)|^2)^2} \, dm(z) = \frac{dm(z)}{(1 - |z|^2)^2} = d\nu_{-2}(z)$. This shows conformal invariance of the measure $\nu_{-2}$.} It follows that

$$\nu_{-2}(\pi(g)\mathcal{F}) = a \quad \text{for every } g \in G. \quad (3.4)$$

Thus,

$$\mu\left(\sum_{1 \neq g \in G} \frac{1}{|g|_2^2} \chi(\pi(g)\mathcal{F})\right) = \mu\left(\left\{\frac{1}{|g|_2^2}\right\}_{1 \neq g \in G} \otimes \chi(0,a)\right),$$

where $\mu$ on the left-hand side is computed in the measure space $(\mathbb{D}, \nu_{-2})$ and $\mu$ on the right-hand side is computed in the algebra $(G \times (0, \infty), \text{Card} \times m)$. Hence

$$\|h\|_{L_{p,\infty}(\mathbb{D}, \nu_{-2})} \leq \|h\|_\infty \|\chi(0,a)\|_{p,\infty} + \text{const} \cdot \left\|\left\{\frac{1}{|g|_2^2}\right\}_{1 \neq g \in G} \otimes \chi(0,a)\right\|_{p,\infty}.$$

It follows now from Lemma 3.2 that $h \in L_{p,\infty}(\mathbb{D}, \nu_{-2})$. The assertion follows by applying Lemma 3.5 to an analytic function $Z$. Part (a) of Theorem 1.1 is proved.

The next lemma is the core part of the proof of Theorem 1.1, part (c). Its proof is similar to that of Theorem 1.1, part (a).

**Lemma 3.6.** If $G \subset \text{PSL}(2, \mathbb{C})$ is as in Theorem 1.1, then

$$\liminf_{s \to 0} s\|F, Z\|_{p+s} > 0.$$

**Proof.** Let $h(z) = (1 - |z|^2)|Z'(z)|$, $z \in \mathbb{D}$. For every $g \in G$, $g \neq 1$, it follows from (3.2) and (3.3) (in the proof of Theorem 1.1, (a)) that

$$\inf_{z \in \pi(g)\mathcal{F}} (1 - |z|^2)|Z'(z)| \geq \frac{1}{|g|_2^2} \cdot \inf_{w \in \mathcal{F}} (1 - |w|^2)|Z'(w)| \cdot \inf_{w \in \mathcal{F}} \frac{1}{|Z(w) - g^{-1}(\infty)|^2}$$

$$\geq \frac{1}{|g|_2^2} \cdot \inf_{w \in \mathcal{F}} (1 - |w|^2)|Z'(w)| \cdot \frac{1}{(||Z||_\infty + |g^{-1}(\infty)|)^2} \geq \text{const} \cdot \frac{1}{|g|_2^2}.$$

We have $\nu_{-2}(\pi(g)\mathcal{F}) = a$ for every $g \in G$. Since the sets $\{\pi(g)\mathcal{F}\}_{g \in G}$ are pairwise disjoint, it follows that

$$\|h\|_{L_{p+s}(\mathbb{D}, \nu_{-2})} \geq \text{const} \cdot \left\|\{1 - |g|_2^2\}_{1 \neq g \in G}\right\|_{p+s}.$$

We infer from Lemma 3.3 that

$$\left\|\{1 - |g|_2^2\}_{1 \neq g \in G}\right\|_{p+s} \geq \text{const} \cdot \left\|\{(k + 1)^{-1/p}\}_{k \geq 0}\right\|_{p+s} \geq \frac{\text{const}}{s}, \quad s \downarrow 0.$$

By Proposition 5 on p. 316 of [13],

$$\|Z\|_{A_{p+s}^{1/(p+s)}} \geq c_p \|h\|_{L_{p+s}(\mathbb{D}, \nu_{-2})} \geq \frac{\text{const}}{s}, \quad s \downarrow 0.$$
Since $Z$ is an analytic function on $\mathbb{D}$, it follows from Theorem 4 on p.316 of [13] that
\[ \| [F, Z] \|_{p+s} \geq c_p^{-1} \| Z \|_{B_{p+s}^{1/(p+s)}} = c_p^{-1} \| Z \|_{A_{p+s}^{1/(p+s)}} \geq \frac{\text{const}}{s}, \quad s \downarrow 0. \]
This completes the proof.

§ 4. Integration in $(L_{p,\infty})_0, p > 1$

Lemma 4.1. Let $s \to Z(s)$ be a bounded function from $\mathbb{R}$ to $(L_{p,\infty})_0$. If it is measurable in the weak operator topology, then it is weakly measurable\(^{11}\) in $(L_{p,\infty})_0$.

**Proof.** Let $\gamma$ be a bounded linear functional on $(L_{p,\infty})_0$. By the noncommutative Yosida-Hewitt theorem (see [16]) $\gamma$ extends to a normal functional on $L_{p,\infty}$. Let $L_{q,1}$ be the Lorentz space which is the K"{o}the dual\(^{12}\) of $L_{p,\infty}$. There exists $x \in L_{q,1}$ such that $\gamma(y) = \text{Tr}(xy), \quad y \in L_{p,\infty}$.

Fix $n \in \mathbb{N}$ and choose a finite rank operator $x_n$ such that $\| x - x_n \|_{q,1} < 1/n$. By assumption the scalar-valued function $f_n : s \to \text{Tr}(x_n Z(s)), \quad s \in \mathbb{R}$, is measurable. On the other hand
\[ |f - f_n|(s) \leq \| Z(s) \|_{p,\infty} \| x - x_n \|_{q,1} \]
and therefore
\[ \| f - f_n \|_{\infty} \leq \frac{1}{n} \sup_{s \in \mathbb{R}} \| Z(s) \|_{p,\infty}. \]
Hence $f_n$ converges to $f$ uniformly. Since the limit of a sequence of measurable functions is measurable, the weak measurability of the mapping $s \to Z(s)$ follows.

The lemma is proved.

Lemma 4.2. Let $s \to Z(s)$ be a bounded function from $\mathbb{R}$ to $(L_{p,\infty})_0$ which is measurable in the weak operator topology. If
\[ \int_{\mathbb{R}} \| Z(s) \|_{p,\infty} \, ds < \infty, \quad (4.1) \]
then $s \to Z(s)$ is a Bochner integrable function from $\mathbb{R}$ to $(L_{p,\infty})_0$. We have
\[ \int_{\mathbb{R}} Z(s) \, ds \in (L_{p,\infty})_0. \quad (4.2) \]

**Proof.** By Lemma 4.1 the mapping $s \to Z(s)$ is weakly measurable from $\mathbb{R}$ to $(L_{p,\infty})_0$. Since $(L_{p,\infty})_0$ is separable, it follows from Theorem 3.5.3 in [21] that the mapping $s \to Z(s)$ is strongly measurable from $\mathbb{R}$ to $(L_{p,\infty})_0$ (in the sense of Definition 3.5.4 in [21]). Using Theorem 3.7.4 in [21] and (4.1) we obtain that the mapping $s \to Z(s)$ is Bochner integrable from $\mathbb{R}$ to $(L_{p,\infty})_0$. The inclusion (4.2) follows now from the definition of Bochner integral (see Definition 3.7.3 in [21]). The proof is complete.

\(^{11}\)See Definition 2.9.
\(^{12}\)See [16] for the definition and basic properties of K"{o}the duals.
In what follows, we use the notation $A^z$ for the complex power of a positive operator $A \in \mathcal{L}(H)$ defined as follows for all $z \in \mathbb{C}$ with positive real part $\text{Re}(z) \geq 0$. Let $f_z : [0, \infty) \to \mathbb{C}$ be the Borel function given by

$$f_z(x) = \begin{cases} e^{z \log(x)}, & x > 0, \\ 0, & x = 0. \end{cases}$$

We set $A^z = f_z(A)$, where the right-hand side is defined by means of the functional calculus. In particular this defines the imaginary power $A^{is} = f_{is}(A)$ for $s \in \mathbb{R}$. One has $A^{z+z'} = A^z A^{z'}$ for $z, z' \in \mathbb{C}$ with positive real parts $\text{Re}(z) \geq 0$ and $\text{Re}(z') \geq 0$.

One has $f_z(xy) = f_z(x)f_z(y)$ for $\text{Re}(z) \geq 0$ and $x, y \geq 0$. Thus using the convention $0^z = 0$ for $z \in \mathbb{C}$, $\text{Re}(z) \geq 0$, (in particular, $0^{is} = 0$) one has the formula

$$\lambda^{is} A^{is} = (\lambda A)^{is}, \quad s \in \mathbb{R}, \quad \lambda \geq 0, \quad 0 \leq A \in \mathcal{L}(H),$$

which is used repeatedly in Lemmas 5.1 and 5.2.

**Lemma 4.3.** Let $A_1, A_2, A_3 \in \mathcal{L}(H)$ be positive and let $X_1, X_2, X_3, X_4 \in \mathcal{L}(H)$. The mapping

$$s \mapsto X_1 A_1^{is} X_2 A_2^{is} X_3 A_3^{is} X_4, \quad s \in \mathbb{R},$$

is measurable in the weak operator topology.

**Proof.** For every bounded positive operator $A$, the mapping $s \mapsto A^{is}$ is strongly continuous. Indeed, let $\log_{\text{fin}}$ be a Borel function on $[0, \infty)$ defined by

$$\log_{\text{fin}}(x) = \begin{cases} \log(x), & x > 0, \\ 0, & x = 0. \end{cases}$$

We have that $\log_{\text{fin}}(A)$ is an unbounded self-adjoint operator. Thus the mapping

$$s \mapsto A^{is} = \exp(is \log_{\text{fin}}(A)) \cdot E_A(0, \infty)$$

is strongly continuous by Stone’s theorem.

Thus, for arbitrary vectors $\xi, \eta \in H$ the mapping

$$s \mapsto \langle X_1 A_1^{is} X_2 A_2^{is} X_3 A_3^{is} X_4 \xi, \eta \rangle, \quad s \in \mathbb{R},$$

is continuous. In particular, the latter scalar-valued mapping is measurable and our vector-valued mapping is measurable, in the weak operator topology. The lemma is proved.

### §5. Proof of the key ‘commutator’ estimate

This section contains a modification of Lemma 11 stated on p. 321 of [13]. The proofs here were obtained with the help of Denis Potapov.

In this section integrals are understood in the weak sense (see §2.7) unless explicitly specified otherwise.
Lemma 5.1. For every $p > 1$ there exists a Schwartz function $h$ such that, for every $0 \leq X, Y \in \mathcal{L}(H)$,

$$X^p - Y^p = V - \int_{\mathbb{R}} X^{is} V Y^{-is} h(s) \, ds.$$ 

Here $V = X^{p-1}(X - Y) + (X - Y)Y^{p-1}$.

Proof. Define a function $g$ by setting

$$g(t) = 1 - \frac{e^{pt/2} - e^{-pt/2}}{(e^{t/2} - e^{-t/2})(e^{(p-1)t/2} + e^{-(p-1)t/2})}, \quad t \in \mathbb{R}, \ t \neq 0, \ g(0) = \left(1 - \frac{p}{2}\right).$$

It is an even function of $t$, it is smooth at $t = 0$ with the Taylor expansion

$$g(t) = \left(1 - \frac{p}{2}\right) + \frac{1}{24}(p^3 - 3p^2 + 2p)t^2 + \cdots,$$

and

$$g(t) = \frac{e^{2t} - e^{pt}}{(e^{t} - 1)(e^{pt} + e^{t})},$$

so that $g = 0$ for $p = 2$, and $g(t)$ is equivalent to $e^{(1-p)t}$ as $t \to \infty$ for $p < 2$ and to $-e^{-t}$ for $p > 2$. Similarly, all derivatives of $g$ have exponential decay at $\infty$. Thus $g$ is a Schwartz function. Set $h$ to be the Fourier transform of $g$, so that $h$ is also a Schwartz function. Set

$$\phi_1(\lambda, \mu) = g \left( \log \left( \frac{\lambda}{\mu} \right) \right) \quad \forall \lambda, \mu > 0,$$

$$\phi_1(0, \mu) = 0 \ \forall \mu \geq 0 \quad \text{and} \quad \phi_1(\lambda, 0) = 0 \ \forall \lambda \geq 0,$$

so that our function $\phi_1$ is defined on $[0, \infty) \times [0, \infty)$. Note that it is not continuous at $(0,0)$. One has

$$\phi_1(\lambda, \mu) = 1 - \frac{\lambda^p - \mu^p}{(\lambda - \mu)(\lambda^{p-1} + \mu^{p-1})}, \quad \lambda, \mu > 0, \ \lambda \neq \mu. \quad (5.1)$$

We claim that

$$\phi_1(\lambda, \mu) = \int_{\mathbb{R}} h(s) \lambda^{is} \mu^{-is} \, ds, \quad \lambda, \mu \geq 0. \quad (5.2)$$

Indeed, we have

$$g(t) = \int_{\mathbb{R}} h(s) e^{i\lambda t} \, ds, \quad t \in \mathbb{R}.$$ 

For $\lambda, \mu > 0$ we set $t = \log(\lambda/\mu)$ and obtain

$$\phi_1(\lambda, \mu) = \int_{\mathbb{R}} h(s) e^{is \log(\lambda/\mu)} \, ds = \int_{\mathbb{R}} h(s) \lambda^{is} \mu^{-is} \, ds.$$ 

For $\lambda = 0$ or $\mu = 0$ the left-hand side of (5.2) vanishes by the definition of $\phi_1$, while the right-hand side vanishes due to the convention $0^is = 0$. Thus, (5.2) holds for all $\lambda, \mu \geq 0$. Set

$$\phi_2(\lambda, \mu) = (\lambda^{p-1} + \mu^{p-1})(\lambda - \mu), \quad \lambda, \mu \geq 0.$$
This function is bounded on $\text{Spec}(X) \times \text{Spec}(Y)$ and the same holds for
\[
\phi_3(\lambda, \mu) = (X^{p-1} + \mu^{p-1})(\lambda - \mu) - (\lambda^p - \mu^p) \quad \forall \lambda, \mu \geq 0.
\]
The equality $\phi_3 = \phi_1 \phi_2$ holds on $[0, \infty) \times [0, \infty)$. Indeed this follows from (5.1) for $\lambda, \mu > 0$, $\lambda \neq \mu$. For $\lambda = \mu > 0$ one has $\phi_1(\lambda, \lambda) = 1 - p/2$, $\phi_2(\lambda, \lambda) = 0$ and $\phi_3(\lambda, \lambda) = 0$. If $\lambda = 0$ or $\mu = 0$ one has $\phi_1(\lambda, \mu) = 0$ and $\phi_3(\lambda, \mu) = 0$.

It follows from the definition (2.12) of Double Operator Integrals and since $X, Y \geq 0$ that
\[
T_{\phi_1}^{X,Y}(A) = \int_\mathbb{R} h(s)X^{is}AY^{-is} \, ds. \quad (5.3)
\]
Indeed, since $h$ is a Schwartz function, the condition (2.11) holds and therefore (2.12) reads as (5.3). Here the integral on the right-hand side is understood in the weak sense. Measurability of the integrand is guaranteed by Lemma 4.3 and condition (2.9) follows from the inequality
\[
\|h(s)X^{is}AY^{-is}\|_\infty \leq |h(s)| \cdot \|A\|_\infty, \quad s \in \mathbb{R},
\]
and from the fact that $h$ is a Schwartz (and hence integrable) function. In particular, $T_{\phi_1}^{X,Y} : \mathcal{L}_\infty \to \mathcal{L}_\infty$.

Using formulae (2.10) and (2.12) we obtain that $T_{\phi_2}^{X,Y} : \mathcal{L}_\infty \to \mathcal{L}_\infty$ and
\[
T_{\phi_2}^{X,Y}(A) = X^pA - X^{p-1}AY + XAY^{p-1} - AY^p.
\]
The function $\phi_3$ bounded on $\text{Spec}(X) \times \text{Spec}(Y)$, $T_{\phi_3}^{X,Y} : \mathcal{L}_\infty \to \mathcal{L}_\infty$ and
\[
T_{\phi_3}^{X,Y}(A) = (X^pA - X^{p-1}AY + XAY^{p-1} - AY^p) - (X^pA - AY^p).
\]
We have $\phi_3 = \phi_1 \phi_2$ on $\text{Spec}(X) \times \text{Spec}(Y)$, and thus
\[
T_{\phi_1}^{X,Y}(V) \overset{(2.13)}{=} T_{\phi_1}^{X,Y}(T_{\phi_2}^{X,Y}(1)) = T_{\phi_3}^{X,Y}(1) = V - (X^p - Y^p).
\]
The assertion of the lemma follows now from (5.3).

Lemma 5.2 below can be proved without any compactness assumption on the operator $B$; however, the proof becomes much harder. We impose a compactness assumption due to the fact that $B$ is compact in Lemma 5.3 (the only place where we use Lemma 5.2).

**Lemma 5.2.** Let $0 \leq A, B \in \mathcal{L}(H)$. If $1 < p < \infty$ and $B$ is compact, then
\[
B^pA^p - (A^{1/2}BA^{1/2})^p = "T(0)" - \int_\mathbb{R} T(s)h(s) \, ds,
\]
where we denote, for brevity, $Y = A^{1/2}BA^{1/2}$ while
\[
T(s) = B^{p-1+is}[B, A^{p+is}]Y^{-is} + B^{p-1+is}A^{p-1/2+is}[A^{1/2}, B]Y^{-is} + B^{is}[B, A^{1+is}]Y^{p-1-is} + B^{is}A^{1/2+is}[A^{1/2}, B]Y^{p-1-is}
\]
and
\[
"T(0)" := B^{p-1}[B, A^p] + B^{p-1}A^{p-1/2}[A^{1/2}, B] + [B, A]Y^{p-1} + A^{1/2}[A^{1/2}, B]Y^{p-1}.
\]
Proof. By assumption $B$ is compact, and therefore one can write $B = \sum_j \lambda_j p_j$, where \( \{p_j\} \) is a family of mutually orthogonal projections such that $\sum_j p_j = 1$. We have

$$B^p A^p - Y^p = \sum_j p_j (B^p A^p - Y^p) = \sum_j p_j ((\lambda_j A)^p - Y^p).$$

Applying Lemma 5.1 to the expression in brackets we obtain\(^{13}\)

$$B^p A^p - Y^p = \sum_j p_j (V_j - \int_\mathbb{R} (\lambda_j A)^i s V_j Y^{-i s} h(s) ds), \quad (5.4)$$

where

$$V_j = (\lambda_j A)^{p-1}(\lambda_j A - Y) + (\lambda_j A - Y)Y^{p-1} = (\lambda_j A)^p - (\lambda_j A)^{p-1}Y + \lambda_j AY^{p-1} - Y^p.$$  

Therefore, we get $\sum_j p_j V_j = B^p A^p - B^{p-1}A^{p-1}Y + BAY^{p-1} - Y^p = \text{“}T(0)\text{”}$. Moreover, we have

$$\sum_j p_j (\lambda_j A)^i s V_j = \sum_j p_j ((\lambda_j A)^{p+i s} - (\lambda_j A)^{p-1+i s}Y + (\lambda_j A)^{1+i s}Y^{p-1} - (\lambda_j A)^{i s}Y^p)$$

$$= \sum_j p_j \lambda_j^{p+i s} A^{p+i s} - \sum_j p_j \lambda_j^{p-1+i s} A^{p-1+i s}Y$$

$$+ \sum_j p_j \lambda_j^{1+i s} A^{1+i s}Y^{p-1} - \sum_j p_j \lambda_j^{i s} A^{i s}Y^p.$$

By the functional calculus, we have

$$\sum_j p_j (\lambda_j A)^i s V_j = B^{p+i s} A^{p+i s} - B^{p-1+i s}A^{p-1+i s}Y + B^{1+i s} A^{1+i s}Y^{p-1} - B^{i s} A^{i s}Y^p$$

$$= B^{p-1+i s}(BA^{p+i s} - A^{p-1+i s}Y) + B^{i s}(BA^{1+i s} - A^{i s}Y)Y^{p-1}$$

$$= B^{p-1+i s} [B, A^{p+i s}] + B^{p-1+i s} A^{p-1+i s}(AB - Y)$$

$$+ B^{i s} [B, A^{1+i s}] Y^{p-1} + B^{i s} A^{i s}(AB - Y) Y^{p-1}$$

$$= B^{p-1+i s} [B, A^{p+i s}] + B^{p-1+i s} A^{p-1+i s} A^{1/2}[A^{1/2}, B]$$

$$+ B^{i s} [B, A^{1+i s}] Y^{p-1} + B^{i s} A^{i s} A^{1/2}[A^{1/2}, B] Y^{p-1}.$$

Substituting the last equality into (5.4) completes the proof.

The following lemma is the main result of this section. It provides the key estimate used in the proof of Theorem 1.1, part (b). In [13], Ch. IV, §3.β, the corresponding Lemma 11 is stated without proof.

Lemma 5.3. Let $0 \leq A \in L_\infty$ and let $0 \leq B \in L_{p,\infty}$, $1 < p < \infty$. If $[A^{1/2}, B] \in (L_{p,\infty})_0$, then

$$B^p A^p - (A^{1/2} BA^{1/2})^p \in (L_{1,\infty})_0.$$

---

\(^{13}\)In this and the subsequent formulae imaginary powers are defined as in § 4. The convention $0^{is} = 0$ is used.
Proof. Consider the formula for $B^p A^p - (A^{1/2} B A^{1/2})^p$ obtained in Lemma 5.2. We have

$$B^p A^p - (A^{1/2} B A^{1/2})^p = "T(0)" - B^{p-1} \cdot (I + II) - (III + IV) \cdot Y^{p-1},$$

where $Y$ is defined as in Lemma 5.2 and where

$$I = \int_{\mathbb{R}} B^{is} [B, A^{p+is}] Y^{-is} h(s) \, ds,$$

$$II = \int_{\mathbb{R}} B^{is} A^{p-1/2+is} [A^{1/2}, B] Y^{-is} h(s) \, ds,$$

III = \int_{\mathbb{R}} B^{is} [B, A^{1+is}] Y^{-is} h(s) \, ds,$$

IV = \int_{\mathbb{R}} B^{is} A^{1/2+is} [A^{1/2}, B] Y^{-is} h(s) \, ds.$$

Step 1. We show that $I \in (\mathcal{L}_{p, \infty})_0$.

Without loss of generality suppose $0 \leq A \leq 1$. For a fixed $s \in \mathbb{R}$ the function $x \mapsto x^{p+is}$ can be uniformly approximated by polynomials $f_m$ on the interval $[0,1]$. It is immediate that

$$[B, A^{p+is}] - [B, f_m(A)] = B(A^{p+is} - f_m(A)) - (A^{p+is} - f_m(A))B.$$

Thus,

$$\|[B, A^{p+is}] - [B, f_m(A)]\|_{p, \infty} \leq 2\|B\|_{p, \infty}\|A^{p+is} - f_m(A)\|_{\infty} \to 0, \quad m \to \infty.$$

Due to the assumption $[A^{1/2}, B] \in (\mathcal{L}_{p, \infty})_0$,

$$[B, A] = A^{1/2} [B, A^{1/2}] + [B, A^{1/2}] A^{1/2} \in (\mathcal{L}_{p, \infty})_0.$$

Thus,

$$[B, A^k] = \sum_{l=0}^{k-1} A^l [B, A] A^{k-1-l} \in (\mathcal{L}_{p, \infty})_0.$$

It follows that $[B, f_m(A)] \in (\mathcal{L}_{p, \infty})_0$. Thus,

$$[B, A^{p+is}] \in (\mathcal{L}_{p, \infty})_0. \quad (5.5)$$

By hypothesis, $B \in \mathcal{L}_{p, \infty}$. We infer from $0 \leq A \leq 1$ that $A^{p+is}$ is a contraction for every $s \in \mathbb{R}$. Hence

$$\|[B, A^{p+is}]\|_{p, \infty} \leq 2\|B\|_{p, \infty}\|A^{p+is}\|_{\infty} \leq 2\|B\|_{p, \infty}. \quad (5.6)$$

It follows from Lemma 4.3 that the mapping

$$s \to B^{is} [B, A^{p+is}] Y^{-is} h(s), \quad s \in \mathbb{R},$$

is measurable in the weak operator topology. Combining Lemma 4.2 and (5.6), we infer that $I \in (\mathcal{L}_{p, \infty})_0$.

Step 2. By Step 1 we have that $I \in (\mathcal{L}_{p, \infty})_0$. Repeating the argument in Step 1 for III and using $[A^{1/2}, B] \in (\mathcal{L}_{p, \infty})_0$ for II and IV we obtain that also $II, III, IV \in (\mathcal{L}_{p, \infty})_0$. 
The next assertion is similar to (2.3) and follows immediately from Corollary 2.3.16.b in [24]: if $X \in (L_p, \infty)_0$ and $0 \leq Z \in L_p$, then $XZ^{p-1} \in (L_1, \infty)_0$ and $Z^{p-1}X \in (L_1, \infty)_0$. Since $B, Y \in L_p$, it follows that
$$B^{p-1} \cdot (I + \Pi) \in (L_1, \infty)_0 \quad \text{and} \quad (III + IV) \cdot Y^{p-1} \in (L_1, \infty)_0.$$  

Also, we have by Lemma 5.2

$$T(0) = B^{p-1}[B, A^p] + B^{p-1}A^{p-1/2}[A^{1/2}, B] + [B, A]Y^{p-1} + A^{1/2}[A^{1/2}, B]Y^{p-1}.$$  

Setting $s = 0$ in (5.5) we obtain that $[B, A^p] \in (L_p, \infty)_0$. By the commutator assumption and Leibniz rule
$$[B, A] = [B, A^{1/2}]A^{1/2} + A^{1/2}[B, A^{1/2}] \in (L_p, \infty)_0.$$  

Since $B, Y \in L_p$, it follows that “$T(0)$” $\in (L_1, \infty)_0$.

Combining these results we complete the proof of Lemma 5.3.

§6. Proof of Theorem 1.1, part (b)

For a detailed study of commutator estimates for the absolute value function, we refer the reader to [15] or [12].

Lemma 6.1. Let $A, B \in \mathcal{L}(H)$. If $[A, B] \in (L_p, \infty)_0$ and $[A, B^*] \in (L_p, \infty)_0$ then $[A, |B|] \in (L_p, \infty)_0$.

Proof. For self-adjoint $B$ the assertion is proved in [15]. Let $B \in \mathcal{L}(H)$ be arbitrary and consider the operators $C$ and $D$ on the Hilbert space $H \otimes \mathbb{C}^2$ given by

$$C = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}.$$  

We have

$$[C, D] = \begin{pmatrix} 0 & [A, B] \\ [A, B^*] & 0 \end{pmatrix} \in (L_p, \infty)_0.$$  

Since $D$ is self-adjoint, it follows from Theorem 3.4 in [15] that $[C, |D|] \in (L_p, \infty)_0$. However,

$$|D| = \begin{pmatrix} |B^*| & 0 \\ 0 & |B| \end{pmatrix}.$$  

Thus,

$$[C, |D|] = \begin{pmatrix} [A, |B^*|] & 0 \\ 0 & [A, |B|] \end{pmatrix}.$$  

This concludes the proof.

The following lemma is Lemma 10, part (3), on p. 320 of [13].

Lemma 6.2. If $T, S \in \mathcal{L}_{p, \infty}$ are such that $T - S \in (L_{p, \infty})_0$, then $|T|^p - |S|^p \in (L_{1, \infty})_0$.

The following lemma crucially uses Lemma 5.3. Recall the lightened notation: the algebra $L\infty(\partial \mathbb{D})$ is identified with its natural action on the Hilbert space $L_2(\partial \mathbb{D})$ by pointwise multiplication.
Lemma 6.3. Let $f \in C(S^1)$ be such that $[F, f] \in \mathcal{L}_{p, \infty}$. Let $g \in \text{SL}(2, \mathbb{C})$ be such that the function $u = \frac{g_{11}f + g_{12}}{g_{21}f + g_{22}}$ is well defined and bounded. Then

$$
||[F, u]|^p \leq ||[F, f]|^p |g_{21}f + g_{22}|^{-2p} + (\mathcal{L}_{1, \infty})_0. \tag{6.1}
$$

Proof. Since $u$ is bounded, it follows that $f$ is separated from $-\frac{g_{22}}{g_{21}} \in \mathbb{C}$. Thus, $v = (g_{21}f + g_{22})^{-1} \in C(S^1)$. If $g_{21} = 0$, then the assertion is trivial. Further, we assume that $g_{21} \neq 0$. Clearly, $u = \frac{g_{11}}{g_{21}}\frac{1}{g_{21}f + g_{22}} v$. Thus,

$$
[F, u] = \frac{1}{g_{21}}[F, v] = \frac{1}{g_{21}} \cdot v[F, g_{21}f + g_{22}]v = v[F, f]v.
$$

Therefore,

$$
[F, u] = [F, f]v^2 + [v, [F, f]] \cdot v.
$$

Since $v \in C(S^1)$, it follows from Theorem 8, a) on p. 319 of [13] that

$$
[F, u] \in [F, f]v^2 + (\mathcal{L}_{p, \infty})_0.
$$

By Lemma 6.2, we have (everywhere in the proof below, LHS means the left-hand side of (6.1))

$$
\text{LHS} \in ||[F, f]|^2|^p + (\mathcal{L}_{1, \infty})_0.
$$

Equivalently,

$$
\text{LHS} \in ||[F, f]|^2|^p + (\mathcal{L}_{1, \infty})_0.
$$

Since $v^2 \in C(S^1)$, it follows from Theorem 8, a) on p. 319 of [13] that

$$
[[F, f], v^2] \in (\mathcal{L}_{p, \infty})_0.
$$

By Lemma 6.1, we have

$$
||[[F, f], v^2] \in (\mathcal{L}_{p, \infty})_0. \tag{6.2}
$$

It follows from Lemma 6.2 that

$$
\text{LHS} \in |v^2|[F, f]|^p + (\mathcal{L}_{1, \infty})_0.
$$

Equivalently,

$$
\text{LHS} \in |v^2|[F, f]|^p + (\mathcal{L}_{1, \infty})_0.
$$

Since $|v| \in C(S^1)$, it follows from Theorem 8, a) on p. 319 of [13] that

$$
[[F, f], |v|] \in (\mathcal{L}_{p, \infty})_0.
$$

By Lemma 6.1, we have

$$
||[F, f], |v|| \in (\mathcal{L}_{p, \infty})_0. \tag{6.3}
$$

We have

$$
|v^2|[F, f]| = |v| \cdot |[F, f]| \cdot |v| - |v| \cdot ||[F, f], |v||,
$$

Thus,

$$
|v^2|[F, f]| \in |v| \cdot |[F, f]| \cdot |v| + (\mathcal{L}_{p, \infty})_0.
$$
It follows from Lemma 6.2 that
\[ |v|^2 \cdot ||F,f||^p \in |v| \cdot ||F,f|| \cdot |v|^p + (\mathcal{L}_{1,\infty})_0. \]
Thus,
\[ \text{LHS} \in |v| \cdot ||F,f|| \cdot |v|^p + (\mathcal{L}_{1,\infty})_0. \]

Set \( A = |v|^2 \) and \( B = ||F,f|| \). We have
\[ \text{LHS} \in (A^{1/2}BA^{1/2})_p + (\mathcal{L}_{1,\infty})_0. \]

On the other hand (6.3) reads as follows: \( [B,A^{1/2}] \in (\mathcal{L}_{p,\infty})_0 \). It follows now from Lemma 5.3 that
\[ \text{LHS} \in B_pA_p + (\mathcal{L}_{1,\infty})_0. \]

This is exactly (6.1) and the proof of Lemma 6.3 is complete.

We also need the following auxiliary lemma. Page 314 of [13] mentions a corresponding assertion for the Dirac operator on the line and the action of \( \text{SL}(2,\mathbb{R}) \). Those settings (and results) are unitarily equivalent.

**Lemma 6.4.** The mapping \( h \to U_h, h \in \text{SU}(1,1) \), defined by the formula
\[ (U_h\xi)(z) = \xi\left(\frac{\alpha z + \beta}{\beta z + \alpha}\right) \frac{1}{\beta z + \alpha}, \quad \xi \in L_2(\partial \mathbb{D}), \quad |z| = 1, \]
where
\[ h = \begin{pmatrix} \alpha & \beta \\ \beta & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1, \]
is a unitary representation of the group \( \text{SU}(1,1) \) on the Hilbert space \( L_2(\partial \mathbb{D}) \) which commutes with \( F \).

**Proof.** The fact that \( h \to U_h \) is a homomorphism is simple, and we omit the proof.

First, we show this representation is unitary. Indeed, we have
\[ \langle U_h\xi, U_h\xi \rangle = \frac{1}{2\pi} \int_{\partial \mathbb{D}} |\xi \circ h|^2(e^{it}) \cdot \frac{1}{|\beta e^{it} + \bar{\alpha}|^2} \, dt. \]

On the circle \( \partial \mathbb{D} \) we have
\[ h: e^{it} \to e^{is} \overset{\text{def}}{=} \frac{\alpha e^{it} + \beta}{\beta e^{it} + \bar{\alpha}}. \]

Thus,
\[ \frac{ds}{dt} = \frac{1}{i} e^{-is} \cdot \frac{d(e^{is})}{dt} = \frac{\bar{\beta} e^{it} + \bar{\alpha}}{i(\alpha e^{it} + \beta)} \cdot \frac{1}{(\beta e^{it} + \bar{\alpha})^2} \cdot ie^{it} = \frac{1}{|\beta e^{it} + \bar{\alpha}|^2}. \]

Thus,
\[ \langle U_h\xi, U_h\xi \rangle = \frac{1}{2\pi} \int_{\partial \mathbb{D}} |\xi|^2(e^{is}) \, ds = \langle h, h \rangle, \]

Thus, \( U_h \) is indeed a unitary operator.
Thus, \( \kappa \) admits a representation of the form

\[
(U_h \kappa)(z) = \frac{(\alpha z + \beta)^n}{(\beta z + \alpha)^{n+1}} = (\alpha)^{-n-1}(\alpha z + \beta)^n \left( 1 + \frac{\beta}{\alpha} z \right)^{-n-1}
\]

\[
= (\alpha)^{-n-1}(\alpha z + \beta)^n \sum_{m=0}^{\infty} \left( \frac{n-1}{m} \right) \left( \frac{\beta}{\alpha} z \right)^m.
\]

The series converges uniformly on the unit circle \( S^1 \) because \( |\beta| < |\alpha| \). The series contains only positive powers of \( z \) and therefore \( P_+ U_h \kappa = U_h \kappa \).

It follows from the preceding paragraph that \( P_+ U_h P_+ = U_h P_+ \). Taking the adjoint we obtain \( P_+ U_h^{-1} P_+ = P_+ U_h^{-1} \). Replacing \( h \) with \( h^{-1} \) we obtain \( P_+ U_h P_+ = P_+ U_h \). Thus, \( P_+ U_h = U_h P_+ \). It follows that \( U_h \) commutes with \( F \).

We are now ready to prove our main result.

**Proof of Theorem 1.1, part (b).** Consider the linear functional on \( C(\Lambda(G)) \) defined by the formula

\[
f \to \varphi((f \circ Z) \cdot |[F, Z]|^p), \quad f \in C(\Lambda(G)),
\]

where \( \varphi \) is a continuous trace on \( \mathcal{L}_{1,\infty} \).

It follows from boundedness of \( \varphi \) and (2.2) that

\[
|\varphi((f \circ Z) \cdot |[F, Z]|^p)| \leq \|\varphi\|_{\mathcal{L}_{1,\infty}} \|f \circ Z\| \|[F, Z]\|_p^p, \quad f \in C(\Lambda(G)).
\]

Thus, our functional is bounded and, by the Riesz Representation Theorem, it admits a representation of the form

\[
\varphi((f \circ Z) \cdot |[F, Z]|^p) = \int_{\Lambda(G)} f(t) \, d\kappa(t), \quad f \in C(\Lambda(G)). \tag{6.4}
\]

Here \( \kappa \) is some Radon measure on \( \Lambda(G) \).

We claim that

\[
\int_{\Lambda(G)} (f \circ g^{-1})(t) \, d\kappa(t) = \int_{\Lambda(G)} f(t) |g'(t)|^p \, d\kappa(t), \quad f \in C(\Lambda(G)), \quad g \in G. \tag{6.5}
\]

To see this, let \( \pi(G) \subset SU(1, 1) \) be the Fuchsian group as in the proof of part (a). Let \( h \to U_h \) be its unitary representation given in Lemma 6.4. It is immediate that

\[
U_{\pi(g)}(\xi \cdot \eta) = (\xi \circ \pi(g)) \cdot U_{\pi(g)}(\eta), \quad \xi \in L_\infty(\partial \mathbb{D}), \quad \eta \in L_2(\partial \mathbb{D}).
\]

Thus,

\[
U_{\pi(g)} Z U_{\pi(g)}^{-1} = Z \circ \pi(g) = g \circ Z, \quad (f \circ g^{-1} \circ Z) = U_{\pi(g)}^{-1} (f \circ Z) U_{\pi(g)}.
\]

Since \( U_{\pi(g)} \) commutes with \( F \), it follows from the preceding formula that

\[
(f \circ g^{-1} \circ Z)[[F, Z]]^p = U_{\pi(g)}^{-1} (f \circ Z) U_{\pi(g)}[[F, Z]]^p
\]

\[
= U_{\pi(g)}^{-1} (f \circ Z) U_{\pi(g)}[[F, Z]]^p \cdot U_{\pi(g)} = U_{\pi(g)}^{-1} (f \circ Z) [[F, g \circ Z]]^p \cdot U_{\pi(g)}.
\]
It follows from the unitary invariance of the trace $\varphi$ that

$$\varphi((f \circ g^{-1} \circ Z) \cdot [F, Z] |^p) = \varphi((f \circ Z) \cdot [F, g \circ Z] |^p).$$

By Lemma 6.3 with $f = Z$,

$$|[F, g \circ Z] |^p \in [F, Z] |^p \cdot (|g'|^p \circ Z) + (\mathcal{L}_{1,\infty})_0.$$ 

Since $\varphi$ vanishes on $(\mathcal{L}_{1,\infty})_0$, it follows that

$$\varphi((f \circ g^{-1} \circ Z) \cdot [F, Z] |^p) = \varphi((f \circ Z) \cdot [F, Z] |^p \cdot (|g'|^p \circ Z)).$$

This proves (6.5). In other words, $\kappa$ is a geometric measure.

As explained in the (first few lines of the) proof of Lemma 3.3, the group $G$ is geometrically finite. Theorem 1 in [34] states that the geometric (probability) measure on $\Lambda(G)$ is unique. Setting $c(G, \varphi) = \kappa(\Lambda(G))$ completes the proof of Theorem 1.1, part (b).

§ 7. Proof of Theorem 1.1, part (c)

Let us introduce the power semigroup as follows.

$$(P_s x)(t) = x(t^s), \quad t, s > 0.$$ 

If $\omega$ is an extended limit which is invariant under $P_s$ (we say that it is power invariant), then $\omega \circ \log$ is a state on $L_\infty(-\infty, \infty)$ which is dilation invariant. This state vanishes on every function whose support is bounded from above and is therefore identified with a dilation invariant extended limit on $L_\infty(0, \infty)$.

In this section we consider those extended limits which are dilation and power invariant. The following assertion is available as Theorem 8.6.8 in [24]. For the convenience of the reader, we present a short proof here.

Lemma 7.1. If $\omega$ is a dilation and power-invariant extended limit, then

$$\text{Tr}_\omega(A) = (\omega \circ \log) \left( t \to \frac{1}{t} \text{Tr}(A^{1+1/t}) \right), \quad 0 \leq A \in \mathcal{L}_{1,\infty}.$$ 

Proof. We have\textsuperscript{14}

$$\text{RHS} = (\omega \circ \log) \left( t \to \frac{1}{t} \sum_{n \geq 0} (n + 1)^{-\frac{1}{t}} \cdot ((n + 1)\mu(n, A))^{1+1/t} \right).$$

We have

$$|(n+1)\mu(n, A) - ((n+1)\mu(n, A))^{1+1/t}| \leq \sup \{|x-x^{1+1/t}| : 0 \leq x \leq \|A\|_{1,\infty} \} = O\left( \frac{1}{t} \right).$$

\textsuperscript{14}Here and in what follows RHS denotes the right-hand side of the equality in the statement of the lemma. — The Editor's note.
as \( t \to \infty \). Therefore,

\[
\text{RHS} = (\omega \circ \log) \left( t \to \frac{1}{t} \sum_{n \geq 0} (n + 1)^{-1/t} \mu(n, A) \right).
\]

Now set

\[
\beta = \sum_{n \geq 0} \mu(n, A) \chi_{(\log(n+1), \infty)}.
\]

Clearly, \( \beta(u) = O(u) \) as \( u \uparrow \infty \). Using Theorem 8.6.7 in [24] we infer

\[
(\omega \circ \log) \left( t \to \frac{\beta(t)}{t} \right) = (\omega \circ \log) \left( t \to \frac{h(t)}{t} \right),
\]

where

\[
h(t) = \int_{0}^{\infty} e^{-u/t} d\beta(u) = \sum_{n \geq 0} (n + 1)^{-1/t} \mu(n, A).
\]

Thus,

\[
\text{RHS} = (\omega \circ \log) \left( t \to \frac{1}{t} \log(n+1) < t \sum_{n \geq 0} \mu(n, A) \right) \overset{\text{def}}{=} \omega \left( t \to \frac{1}{\log(t)} \sum_{n+1 < t} \mu(n, A) \right).
\]

Since \( A \in \mathcal{L}_{1,\infty} \), it follows that

\[
\int_{0}^{t} \mu(s, A) \, ds = \sum_{n+1 < t} \mu(n, A) + O(1).
\]

This completes the proof.

**Corollary 7.2.** If \( \omega \) is a dilation and power-invariant extended limit, then \( c(G, \text{Tr}_{\omega}) > 0 \).

**Proof.** Let \( T = ||[F, Z]||^{p} \). It follows from Lemma 3.6 that

\[
\liminf_{s \to 0} s \text{Tr}(T^{1+s}) > 0.
\]

Therefore,

\[
(\omega \circ \log) \left( t \to \frac{1}{t} \text{Tr}(T^{1+1/t}) \right) > 0.
\]

The assertion follows now from Lemma 7.1. The corollary is proved.

The proof of Theorem 1.1 is complete.

**Remark 7.3.** The existence of a Dixmier trace \( \varphi \) on \( \mathcal{L}_{1,\infty} \) such that \( \varphi(T) \neq 0 \) follows from the weaker estimate \( \limsup_{s \to 0} s \text{Tr}(T^{1+s}) > 0 \). Indeed, assume the contrary, that is \( \varphi(T) = 0 \) for every Dixmier trace \( \varphi \). It follows from Theorem 9.3.1 in [24] that

\[
\lim_{s \to 0} s \text{Tr}(T^{1+s}) = 0,
\]

which is not the case. Since \( \varphi(T) = c(G, \varphi) \), the assertion follows.
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