COUNTING HYPERBOLIC COMPONENTS
IN THE MAIN MOLECULE

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ABSTRACT. We count the number of hyperbolic components of period \( n \) that lie on the main molecule of the Mandelbrot set. We give a formula for how to compute the number of these hyperbolic components of period \( n \) in terms of the divisors of \( n \) and in the prime power case, an explicit formula is derived.

1. Introduction

Consider the map \( f_c(z) = z^2 + c \), where \( c \) is the parameter variable and \( z \) is the dynamic variable, both taking on complex values. Following [7], we denote by \( K(f_c) \) the filled Julia set, defined by taking the union of bounded orbits for \( f_c \). Let the Mandelbrot set, \( M \), be the compact subset of all parameter values \( c \) such that \( K(f_c) \) is connected. Recall that we may look at the period of the hyperbolic components of \( M \) as shown in Figure 1. The main cardioid is the hyperbolic component that consists of the parameter values such that \( f_c \) has an attracting fixed point in \( \mathbb{C} \) [10].

**Definition 1.1.** The main molecule is the union of all hyperbolic components attached to the main cardioid through a chain of finitely many components. Let \( M(n) \) denote the number of hyperbolic components of period \( n \) on the main molecule.

\[\text{Figure 1. Some periods of hyperbolic components of } M. \text{ The hyperbolic components of period 6 on the main molecule are labeled in black.}\]

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The closure of the main molecule is the locus of parameters for which the core entropy is zero. In this note, we are interested in computing \( M(n) \) for various \( n \) and we shall give formulas for \( M(n) \). Apart from the main molecule, Lutzky has given formulas for the number of hyperbolic components of period \( n \) [4], Kiwi and Rees have given formulas for certain hyperbolic components in the space of quadratic rational maps [3], and Milnor and Poirier have studied hyperbolic components for polynomials of general degree [8]. In our case, the formulas for \( M(n) \) turn out to be related to the Euler phi function and other combinatorial functions. For example, Figure 1 is an illustration that shows \( M(6) = 6 \).

2. Preliminary Results

From [1], there is a simple way to count how many hyperbolic components there are attached to specifically the main cardioid.

Lemma 2.1. For any \( n \in \mathbb{N} \), there are \( \varphi(n) \) hyperbolic components of period \( n \) attached to the main cardioid, where \( \varphi \) is the Euler phi function.

Proof. Let \( H \) be a hyperbolic component of period \( n \) attached to the main cardioid. If \( n = 1 \), then the main cardioid itself is the only such hyperbolic component so there is \( \varphi(1) = 1 \) component of period 1 attached to the main cardioid. Now suppose \( n > 1 \). Then, the root point of \( H \) is a parabolic point of ray period \( n \) [7]. For \( c \neq 1/4 \), let \( f_c(z) = z^2 + c \) be a quadratic polynomial with a parabolic fixed point at \( \alpha \in \mathbb{C} \) with multiplier, \( \lambda \), a primitive \( n \)th root of unity. This multiplier must have the form \( \lambda = e^{2\pi i \theta} \) where \( \theta \in \mathbb{Q}/\mathbb{Z} \) is the rotation number. Each hyperbolic component has an associated rotation number. Because the number of primitive \( n \)th roots of unity is given by \( \varphi(n) \), there are \( \varphi(n) \) possible rotation numbers for \( H \) hence \( \varphi(n) \) hyperbolic components of period \( n \). □

Corollary 2.2. Let \( p, n \in \mathbb{N} \). For any hyperbolic component \( H \) of period \( p \), there are \( \varphi(n) \) hyperbolic components of period \( pn \) attached to \( H \).

Proof. Douady and Hubbard [2] introduced a tuning map \( i_H : M \to M \) whose image is a baby Mandelbrot set (that is, a homeomorphic copy of \( M \)) [9]. From [10], the idea of the tuning map is to take \( c \in M \) and for each bounded component of the Fatou set corresponding to \( f_c \), replace each of these bounded components with a homeomorphic copy of the filled Julia set corresponding to some parameter in \( H \). This yields a set homeomorphic to the filled Julia set of \( f_{i_H(c)} \). The tuning map \( i_H \) also maps the main cardioid onto \( H \) and maps hyperbolic components of period \( n \) onto hyperbolic components of period \( pn \). By Lemma 2.1, there must then be \( \varphi(n) \) such hyperbolic components. □

Theorem 2.3. Let \( n \in \mathbb{N} \) and write \( n = d_1 \cdots d_k \), where \( d_1, \ldots, d_k \) are divisors of \( n \) with \( d_i > 1 \) for every \( i \in \{1, \ldots, k\} \). Then, we have

\[
M(n) = \sum_{n=d_1 \cdots d_k \atop d_i \geq 1} \varphi(d_1) \cdots \varphi(d_k).
\]

Example 2.4. Before presenting the proof, let us compute \( M(12) \), the number of hyperbolic components of period 12 attached to the main cardioid through a chain of finitely many components. We need to take into account all possible ordered
partitions of 12 into its divisors. For instance, $6 \cdot 2$ and $2 \cdot 6$ represent two different such partitions. With this in mind, we have

$$M(12) = \varphi(12) + \varphi(6)\varphi(2) + \varphi(2)\varphi(6) + \varphi(4)\varphi(3) + \varphi(3)\varphi(4) + \varphi(3)\varphi(2)\varphi(2) + \varphi(2)\varphi(3)\varphi(3) + \varphi(2)\varphi(2)\varphi(3)$$

$$= 4 + (2 \cdot 1) + (1 \cdot 2) + (2 \cdot 2) + (2 \cdot 2) + (2 \cdot 1 \cdot 1) + (1 \cdot 2 \cdot 1) + (1 \cdot 1 \cdot 2)$$

$$= 22.$$  

Figure 2 illustrates the period 12 hyperbolic components attached to the main cardioid through the period 2 and 3 hyperbolic components. In particular, (A) corresponds to the terms $\varphi(2)\varphi(6), \varphi(2)\varphi(3)\varphi(2),$ and $\varphi(2)\varphi(2)\varphi(3).$ (B) along with the other period 3 hyperbolic component corresponds to the terms $\varphi(3)\varphi(4)$ and $\varphi(3)\varphi(2)\varphi(2).$

With the intuition from this example, we now present the proof of Theorem 2.3.

Proof. We proceed by induction on the number of divisors. For $n = 1,$ $M(1) = 1.$ Suppose $n \in \mathbb{N}$ has only one divisor, $d_1 > 1.$ Then by Lemma 2.1,

$$M(n) = \varphi(n) = \sum_{n=d_1}^{d_1=1} \varphi(d_1)$$

so the base case holds. Now suppose that the formula in the statement of the lemma holds for $k$ divisors, where $k \in \mathbb{N}.$ We show that it also holds for $k + 1$ divisors. Let $n \in \mathbb{N}$ have $k + 1$ divisors. Then, using Lemma 2.1 we can count the number of hyperbolic components of period $n$ attached to the main cardioid and using Corollary 2.2 we can count the number of hyperbolic components of period $n$ attached to the components already attached to the main cardioid. Thus, we have

$$M(n) = \varphi(n) + \varphi(d_1)M\left(\frac{n}{d_1}\right) + \cdots + \varphi(d_{k+1})M\left(\frac{n}{d_{k+1}}\right).$$
Applying the inductive hypothesis to each $M\left(\frac{n}{d_i}\right)$ for $i \in \{1, ..., k + 1\}$, we may write the equation above as a sum over all the divisors of $n$. This is equivalent to:

$$M(n) = \sum_{\substack{n=d_1...d_{k+1} \atop d_i > 1}} \varphi(d_1) \cdot \ldots \cdot \varphi(d_{k+1}).$$

With this formulation of $M(n)$, we can define $M(n)$ in a recursive way depending on the Euler $\varphi$ function and the divisors of $n$.

**Corollary 2.5.** For any $n \in \mathbb{N}$,

$$M(n) = \sum_{d|n \atop d > 1} \varphi(d) M\left(\frac{n}{d}\right).$$

**Proof.** This follows from the inductive step of the proof of Theorem 2.3. □

### 3. Prime Powers

In two different ways, we will find an explicit formula for positive integers that are prime powers. First, we begin with an example for small prime powers.

**Example 3.1.** Let $n, p \in \mathbb{N}$ with $p$ prime.

(a) Suppose $n = p = p \cdot 1$. Then, $M(p) = \varphi(p) = p - 1$ as all components of period $p$ lie on the main cardioid.

(b) Now let $n = p^2 = p^2 \cdot 1 = p \cdot p$. Then, there are $\varphi(p^2)$ components of period $p^2$ on the main cardioid. There are more components to count here because of the components of period $p$ attached to the main cardioid. On these components, there are components of period $p$ that must be counted. There are $\varphi(p) \cdot \varphi(p)$ such components. In total,

$$M(p^2) = \varphi(p^2) + (\varphi(p))^2 = p(p - 1) + (p - 1)^2 = (p - 1)(2p - 1).$$

(c) Finally, suppose $n = p^3 = p^3 \cdot 1 = p^2 \cdot p = p \cdot p^2 = p \cdot p \cdot p$. Applying similar reasoning as before and computing, our total is now

$$M(p^3) = \varphi(p^3) + \varphi(p^2)\varphi(p) + \varphi(p)\varphi(p^2) + (\varphi(p))^3 = (p - 1)(2p - 1)^2.$$

This example sheds light on the general case of when $n$ is an arbitrary prime power, leading to the following result.

**Theorem 3.2.** For any $n, k \in \mathbb{N}$, if $n = p^k$, then $M(p^k) = (p - 1)(2p - 1)^{k-1}$. 
Proof. Let $p \in \mathbb{N}$ be a prime. Applying Corollary 2.4 for $n = p^k$, we obtain:

\[ M(p^k) = \sum_{1 \leq h < k} \varphi(p^h)M(p^{k-h}) \]

\[ = \sum_{1 \leq h < k} (p^{h-1}(p - 1)^2(2p - 1)^{k-h-1}) + p^{k-1}(p - 1) \]

\[ = \frac{(2p - 1)^{k-1}(p - 1)^2}{p} \left( \sum_{1 \leq h < k} \left( \frac{p}{2p-1} \right)^h \right) + p^{k-1}(p - 1) \]

\[ = \frac{(2p - 1)^{k-1}(p - 1)^2}{p} \left( 1 - \left( \frac{p}{2p-1} \right)^k \right) + p^{k-1}(p - 1) \]

\[ = (p - 1)(2p - 1)^{k-1}. \]

\[ \blacktriangleleft \]

4. Products of Distinct Primes

Consider now the case where the positive integer $n$ is of the form $n = p_1...p_m$, where $p_1, ..., p_m$ are distinct primes. Before considering the general case, let us again look at a simple example.

Example 4.1. Let $n = p_1p_2$, where $p_1$ and $p_2$ are distinct primes. As before, we need to consider the number of ways to write this product, namely $p_1p_2 \cdot 1$, $p_1 \cdot p_2$, and $p_2 \cdot p_1$. Recall to determine the number of hyperbolic components with period $p_1p_2$, we need to count the components of period $p_1p_2$ on the main cardioid, the components of period $p_2$ on the components of period $p_1$ on the main cardioid, and the components of period $p_1$ on the components of period $p_2$ on the main cardioid. Therefore, we must have

\[ M(p_1p_2) = \varphi(p_1p_2) + \varphi(p_1)\varphi(p_2) + \varphi(p_2)\varphi(p_1) = 3\varphi(p_1p_2) = 3(p_1 - 1)(p_2 - 1). \]

For the general case of $n = p_1...p_m$, we first need to determine how many ways we can write this product of primes in a similar manner as the example above. This can be done by a recursive process. Instead of thinking about how to write these products of primes, we can consider the equivalent problem of determining the number of ordered partitions of $\{1,...,m\}$. Let this number be represented by $N(m)$. Define $N(0) = 1$. It is clear that $N(1) = 1$. The ordered partitions of $\{1,2\}$ are $\{\{1\}, \{2\}\}, \{\{2\}, \{1\}\},$ and $\{\{1,2\}, \{\}\}$ so $N(2) = 3$. One can check by hand that in fact $N(3) = 13$ and $N(4) = 75$. The numbers $N(m)$ are known as the ordered Bell numbers or Fubini numbers and from [3], they satisfy

\[ N(m) \sim \frac{m!}{2(\log(2))^{m+1}}. \]

The following is a well-known lemma about the ordered Bell numbers.

Lemma 4.2. Let $n = p_1p_2 \cdots p_m$ be a product of distinct primes. Then,

\[ N(m) = \sum_{k=1}^{m} \binom{m}{k} N(m-k). \]

Proof. Let $1 \leq k \leq m$. Begin by choosing $k$ numbers from $\{1,...,m\}$. The number of ordered partitions of the remaining $m-k$ numbers is $N(m-k)$. Because there are
As \( n \) tends to zero as \( k \) partitions fixing \( k \) numbers and ordering the rest. As we are counting the total number of ordered partitions as \( k \) ranges between 1 and \( m \), we have

\[
N(m) = \sum_{k=1}^{m} \left( \begin{array}{c} m \\ k \end{array} \right) N(m-k).
\]

\( \square \)

In this case where \( n \) is a product of distinct primes, the following theorem shows that the number of hyperbolic components of period \( n \) on the main molecule is closely related to the ordered Bell numbers.

**Theorem 4.3.** Let \( n = p_1 p_2 \cdots p_m \), a product of distinct primes. Then, we have

\[
M(n) = M(p_1 \cdots p_m) = N(m)(p_1 - 1) \cdots (p_m - 1) = N(m)\varphi(n).
\]

**Proof.** By Theorem 2.3, if we write \( n = d_1 \cdots d_k \), where \( d_1, \ldots, d_k \) are divisors of \( n \) with \( d_i > 1 \) for every \( i \in \{1, ..., k\} \), then we have

\[
M(p_1 \cdots p_m) = \sum_{n=d_1 \cdots d_k \atop d_i > 1} \varphi(d_1) \cdots \varphi(d_k).
\]

For \( r_1 \in \{1, \ldots, m\} \), we have that \( d_1 = p_{i_1} \cdots p_{i_{r_1}} \) where \( p_{i_1}, \ldots, p_{i_{r_1}} \) are the primes \( p_1, \ldots, p_m \). Therefore,

\[
\varphi(d_1) = \varphi(p_{i_1} \cdots p_{i_{r_1}}) = \varphi(p_{i_1}) \cdots \varphi(p_{i_{r_1}}).
\]

For \( r_2 \in \{1, \ldots, m - r_1\} \), \( d_2 = p_{j_1} \cdots p_{j_{r_2}} \) where \( p_{j_1}, \ldots, p_{j_{r_2}} \) are the remaining \( m - r_1 \) primes. Similarly,

\[
\varphi(d_2) = \varphi(p_{j_1} \cdots p_{j_{r_2}}) = \varphi(p_{j_1}) \cdots \varphi(p_{j_{r_2}}).
\]

Continuing in this way, at the \( k \)-th step, we will have exhausted all of the \( m \) primes. We may then write \( n = d_1 \cdots d_k \) and note that

\[
\varphi(d_1) \cdots \varphi(d_k) = \varphi(p_1) \cdots \varphi(p_m).
\]

We are summing over all possible ways of writing \( n \) in terms of its divisors. Each term in the sum is of the form \( \varphi(p_1) \cdots \varphi(p_m) \) and there are \( N(m) \) possible ways to do so. Therefore,

\[
M(p_1 \cdots p_m) = N(m)\varphi(p_1) \cdots \varphi(p_m) = N(m)(p_1 - 1) \cdots (p_m - 1) = N(m)\varphi(n).
\]

\( \square \)

Now let \( p_m \) denote the \( m \)th prime number. As before, let \( n = p_1 \cdots p_m \). We then have

\[
\frac{M(n)}{N(m) \cdot n} = \frac{N(m)(p_1 - 1) \cdots (p_m - 1)}{N(m) \cdot p_1 \cdots p_m} = \frac{1}{p_1} \cdots \frac{1}{p_m} < 1
\]

As \( n \to \infty \) we must have \( m \to \infty \) and it is well-known that the product above tends to zero as \( m \to \infty \). Therefore, \( M(n) = o(N(m)n) \) as \( n \to \infty \).
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References

1. S. Bullett and P. Sentenac. *Ordered orbits of the shift, square roots, and the devil’s staircase*, Math. Proc. Camb. Phil. Soc., 115 (1994), 451-481.
2. A. Douady and J. Hubbard. *Étude Dynamique de Polynômes Complexes*, Publications Math. d’Orsay, Orsay, France: Université de Paris-Sud, Dép de Math. 1984.
3. J. Kiwi and M. Rees. *Counting hyperbolic components*, J. Lond. Math. Soc. (2), 88 (2013), 669-698.
4. M. Lutzky. *Counting hyperbolic components of the Mandelbrot set*, Phys. Lett. A, 177 (1993), 338-340.
5. O. A. Gross, *Preferential Arrangements*, Amer. Math. Monthly, 69 (1962), 4-8, DOI 10.2307/2312725.
6. J. Milnor. *Dynamics in One Complex Variable*, 3rd ed., Princeton Univ. Press, Oxfordshire, 2006.
7. J. Milnor. *Periodic Orbits, External Rays and the Mandelbrot Set: An Expository Account*, Géométrie complexe et systèmes dynamiques - Colloque en l’honneur d’Adrien Douady Orsay, 261 (2000), 57.
8. J. Milnor and A. Poirier. *Hyperbolic components in spaces of polynomial maps*. [arXiv:math/9202210] 1992.
9. G. Tiozzo. *Topological Entropy of Quadratic Polynomials and Dimension of Sections of the Mandelbrot Set*, Adv. Math., 273 (2015), 651-715.
10. S. Zakeri. *External Rays and the Real Slice of the Mandelbrot Set*, Ergodic Theory Dynam. Systems, 23 (2003), 637-660.

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