Studying Bounds on Lepton Flavor Violating (LFV) Decay Processes

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Abstract

This dissertation reviews the Standard Model formalism as well as the Lepton Flavour Violating (LFV) decay processes which cause its extension, known as the physics beyond the SM. Firstly, using the experimental bounds on three body LFV decays, the corresponding bounds on two body LFV decays are reviewed. The dynamical suppression of three body LFV decays due to momentum dependent couplings is also reviewed. Secondly, the role of the LFV decays to explain the LSND excess is discussed in detail, for which the experimental bounds on three body LFV decays, i.e. $\mu \rightarrow 3e$ are used to constraint the coupling $\tilde{g}_{Z\mu e}$, which is needed to calculate the anomalous muon decay $\mu \rightarrow e\nu\bar{\nu}$. Then comparing the effective coupling of anomalous muon decay to $r > 1.6 \times 10^{-3}[1]$, it is proved that LFV is not the correct hypothesis to explain the LSND excess. Finally, LFV decays at loop order are studied in Seesaw model of neutrino masses [19] where the smallness of the Seesaw neutrino mass may be naturally realized with $m_N$ (mass of right-handed singlet neutrinos) of order 1 TeV. It is shown that the Higgs mass of a new scalar doublet with lepton number $L = -1$ needed in the model has to be larger than 50 TeV to get the branching ratio of $\mu \rightarrow 3e$ to be consistent with the existing bound on $\mu \rightarrow 3e$. This defeats the original motivation of the model, namely that there is no physics beyond the TeV energy scale.
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Chapter 1

Introduction

The most fundamental element of physics is the reduction principle. The large variety of macroscopic forms of matter can be traced back according to this principle, to a few microscopic constituents which interact by a small number of forces. The reduction principle has provided a guide to unraveling of the structure of physics from the macroscopic world through atomic and nuclear physics to particle physics. The laws of nature are summarized in the Standard Model of the particle physics.

Standard Model is one of the successful model of the 20th century proposed by the Glashow, Weinberg and Salam. The weak and electromagnetic interactions are unified in the electroweak Standard Model. This model has provided the plenty of successful predictions with an impressive level of precision.

In quantum electrodynamics (QED) the interactions are specified by the gauge principle. The electroweak Standard Model is based on the gauge
symmetry group $SU(2) \times U(1)$. $SU(2)$ is non-Abelian electroweak-isospin group, to which three $W$ gauge fields are associated and $U(1)$ is the Abelian hypercharge group. The associated $B$ field and the neutral component of the $W$ triplet field mix to form the photon field $A$ and a new neutral electroweak field $Z$. This gauge group is spontaneously broken by the Higgs mechanism (see for example 2.3.1). In the Standard Model, all particles acquire their masses by interaction with another particle, the Higgs Boson. Using the knowledge of basic symmetry group, the gauge invariant Lagrangian has been written, giving not only the interactions of the various fields including fermions but also the mass relationships for the fermions, gauge bosons and Higgs boson (see for example 2.3.2).

Despite lot of success of the Standard Model, no body can say that it is the end of physics. There are some limitations (see for example 2.4) which necessitate the extension of the Standard Model.

When we advertize the Standard Model, we have said that it is a model whose foundation is symmetry. In this model only those reactions are allowed which conserve the individual as well as the total lepton-flavour. Thus the reactions violating the lepton flavour can not be contained in the Standard Model and cause its extension. This extension is commonly known as physics beyond the Standard Model.

The problem of the physics beyond the Standard Model has been studied for a considerable length of time. In the past few years substantial progress has been made to understand this physics. Lepton Flavor Violating (LFV) interactions are among the most promising candidates to understand this new physics.
Using the experimental bounds on the three body Lepton Flavor Violating decays we have found the bounds on the two body LFV decays. These bounds are suppressed by the form factors and was named as the dynamical suppression of the LFV bounds. The detailed study of these LFV decays is presented in chapter 3. The bounds thus found on the two body LFV decays are used for further studies of LFV decays in some interesting reactions.

Today there is evidence for a very important new property of the neutrinos i.e. they have mass and, as a result mix with each other to lead the phenomenon of neutrino oscillation. Evidences that the neutrinos are massive particles come from three anomalous effects, the LSND excess, the atmospheric anomaly and solar neutrino deficit. But the atmospheric and the solar results are the most convincing one. The LSND has the small probability compared to the atmospheric and solar anomalies. In order to incorporate neutrino mass we must violate the Standard Model and introduce the new concept of LFV. A Seesaw model of neutrino masses [19] which involve right handed singlet neutrinos of mass $m_N$ of the order TeV, is also discussed using the LFV physics at loop order. The main purpose of this model is that there is no new physics beyond 1 TeV. It is however, shown that even if one keeps $m_N$ at 1 TeV, the present bounds on $\mu \rightarrow 3e$ requires that new Higgs boson, necessary in this model, to have mass larger than 50 TeV. This necessitates a new physics beyond the TeV energy scale defeating the original motivation of the model. This is discussed in chapter 4, where it is also shown that the LFV anomalous muon decay $\mu \rightarrow e\nu\bar{\nu}$ cannot significantly contribute to the LSND DAR result, at least when we use $\tilde{g}_{Z\mu e}$ as constrained by $\mu \rightarrow 3e$ in a model independent way.
Chapter 2

The Electroweak Theory

2.1 History

The proposal of the symmetry group for the Electroweak Theory, $SU(2) \times U(1)$, was made by Glashow in 1961. His motivation was to unify weak and electromagnetic interactions into a symmetry group that contained $U(1)_{em}$. The prediction includes the existence of four physical vector boson eigenstates, $W^\pm$, $Z$, and $\gamma$, obtained from the rotations of the weak eigenstate. In particular, the rotation by the weak angle $\theta_w$ which defines the $Z$ weak boson was introduced already in this work. Furthermore, the correct structure of weak neutral current mediated by $Z$-boson was also obtained. The massive weak bosons $W^\pm$ and $Z$ were considered as mediators of weak interactions. This model has serious problem of giving masses to $W^\pm$ and $Z$, since the gauge symmetry would predict their masses to be zero. The vector boson masses $M_W$ and $M_Z$ were parameters introduced by hand and the in-
teraction Lagrangian was that of the IVB Theory. The mass term for vector bosons in the Lagrangian not only destroyed the gauge symmetry but also normalizability of the theory.

Another key ingredient for the building of the Electroweak Theory is provided by the Goldstone Theorem which was introduced by Nambu in 1960 and proved and studied with generality by Goldstone in 1961 and by Goldstone, Salam and Weinberg in 1962. This theorem states the existence of massless spinless particles as an implication of spontaneous symmetry breaking of global symmetries.

The spontaneous symmetry breaking of local (gauge) symmetries, needed for the breaking of the electroweak symmetry $SU(2) \times U(1)$, was studied by P.Higgs, F. Englert and R. Brout, Guralnik, Hagen and Kibble in 1964. The procedure for this spontaneous breakdown of gauge symmetries is referred to as the Higgs Mechanism.

The electroweak theory as it is now known was formulated by Weinberg and independently by Salam in 1968 who incorporated the gauge group $SU(2) \times U(1)$ introduced by the Glashow earlier. This theory, commonly called Glashow-Weinberg Salam Model or Standard Model (SM), was built with the help of the gauge principle and incorporated all the good phenomenological properties of the pregauged theories of the weak interactions, and in particular those of the IVB theory. It incorporated the idea of spontaneous breaking of the gauge symmetry by introducing Higgs doublet. In this way the weak vector bosons acquired their masses without destroying the normalizability of the gauge theory. The SM is indeed a gauge theory of the electroweak interactions based on the gauge symmetry $SU(2) \times U(1)$ and
the intermediate vector bosons, $\gamma, W^{\pm}$ and $Z$ are the four associated gauge bosons. The gauge boson masses, $M_W$ and $M_Z$, are generated by the Higgs Mechanism in the Electroweak Theory and, as a consequence, it respects unitarity at all energies and is renormalizable.

The important proof of renormalizability of gauge theories with and without spontaneous symmetry was provided by 't Hooft in 1971.

The first firm indication that the Standard Model was the correct theory of Electroweak interaction was the discovery of weak neutral Current in 1973 as predicted by the model. This also provided the first measurement of $\sin^2 \theta_w$. By using this experimental input for $\theta_w$ and the values of electromagnetic coupling and $G_F$, the SM provided the first estimates for $M_W$ and $M_Z$ which were discovered experimentally in 1983 at the predicted masses. Another important ingredients of the SM are: fermion family replication, quark mixing and CP violation.

The success of the SM was clearly the discovery of the gauge bosons $W^{\pm}$ and $Z$ at the SpS collider at CERN in 1983. Since then there have been plenty of tests of the SM even at quantum level [2].

2.2 Choice of the group $SU(2) \times U(1)$

In order to give an argument for the choice of $SU(2) \times U(1)$ gauge group for electroweak unification, it is sufficient to consider the $e^{-}\nu$ component of the charged weak current that we write now in the form,

$$J_\mu = \bar{\nu}\gamma_\mu \left(\frac{1-\gamma_5}{2}\right)e = \bar{\nu}_L\gamma_\mu e_L = \bar{\psi}_L\gamma_\mu \tau^+\psi_L$$
\[ J^\dagger_\mu = \bar{e}_\gamma^\mu \left( \frac{1 - \gamma^5}{2} \right) \nu = \bar{\nu}_L \gamma^\mu \nu_L = \bar{\psi}_L \gamma^\mu \tau^- \psi_L \]

and we have introduced the lepton doublet notation and the \( \tau_i \) \( (i = 1, 2, 3) \) are Pauli matrices,

\[
\psi_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \quad \bar{\psi}_L = \begin{pmatrix} \bar{\nu}_L \\ \bar{e}_L \end{pmatrix}, \quad \tau_\pm = \frac{1}{2} (\tau_1 \pm i\tau_2)
\]

\[
\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The three generators \( I_i \) of unitary, unimodular group \( SU(2) \) satisfy the commutation relation

\[
[I_i, I_j] = i\epsilon_{ijk} I_k
\]

ei the same commutation relation as satisfied by \( \tau_i \) viz,

\[
[\tau_i, \tau_j] = 2i\epsilon_{ijk} \tau_k,
\]

since for the fundamental representation of \( SU(2) \)

\[
I_i = \frac{\tau_i}{2}
\]

Note that in the charged currents there are just two generators \( I_1 \) and \( I_2 \). A third generator \( I_3 \) is needed in order to close the \( SU(2) \) algebra. This implies the third current that is relevant for the electroweak interactions,

\[
J^3_\mu = \bar{\psi}_L \gamma^\mu \frac{\tau_3}{2} \psi_L = \frac{1}{2} (\bar{\nu}_L \gamma^\mu \nu_L - \bar{e}_L \gamma^\mu e_L).
\]

Obviously this can not be identified with the \( J^\text{em}_\mu \) which is \(-\bar{e}_\gamma^\mu e\). This clearly indicate that \( SU(2) \) is not sufficient for electroweak unification and it
must be extended and the simple extension is to consider the group $SU(2) \times U(1)$.

The Gell-Mann Nishijima relation,

$$Q = I_3 + \frac{Y}{2}$$

where,

$Q =$ electric charge, $I_3 =$ weak isospin, $Y =$ weak hypercharge; implies

$$Y = \begin{cases} -1 & \text{for } \nu_L \\ -1 & \text{for } e_L \\ -2 & \text{for } e_R \end{cases}$$

The corresponding relation among the current is

$$J_{\mu}^{em} = J_{\mu}^3 + \frac{1}{2} J_{\mu}^Y$$

Thus

$$J_{\mu}^Y = 2 \left( J_{\mu}^{em} - J_{\mu}^3 \right)$$

Therefore, if the following are used as inputs

$$J_{\mu}^{em} = (-1) \bar{e}_L \gamma_\mu e_L + (-1) \bar{e}_R \gamma_\mu e_R$$

$$J_{\mu}^3 = \left( -\frac{1}{2} \right) \bar{e}_L \gamma_\mu e_L + \left( \frac{1}{2} \right) \bar{\nu}_L \gamma_\mu \nu_L$$

one can get $J_{\mu}^Y$ as output,

$$J_{\mu}^Y = 2 \left( J_{\mu}^{em} - J_{\mu}^3 \right) = (-1) \bar{e}_L \gamma_\mu e_L + (-2) \bar{e}_R \gamma_\mu e_R + (-1) \bar{\nu}_L \gamma_\mu \nu_L.$$
This clearly indicates that

\[ Y = -1 \quad \text{for} \quad \left( \begin{array}{c} e_L \\ \nu_L \end{array} \right) \]

\[ Y = -2 \quad \text{for} \quad e_R \]

Due to symmetry breaking the two neutral currents \( J_3^\mu \) and \( J_{em}^\mu \) will mix to give two physical currents of which one must be identified with electromagnetic current \( J_{em}^\mu \) and the second current will be new currents. These currents will be coupled to physical vector bosons \( A_\mu \) and \( Z_\mu \).

\[
g_2 J_3^\mu W_3\mu + \frac{1}{2} \hat{g} J_Y^\mu B_\mu = e J_{em}^\mu A_\mu + g_Z J_Z^\mu Z_\mu \quad (2.1)
\]

where,

\[
A_\mu = \cos \theta_w B_\mu + \sin \theta_w W_3\mu, \\
Z_\mu = \cos \theta_w W_3\mu - \sin \theta_w B_\mu; \quad \theta_w = \text{weak angle}.
\]

From here

\[
W_3\mu = \sin \theta_w A_\mu + \cos \theta_w Z_\mu, \\
B_\mu = \cos \theta_w A_\mu - \sin \theta_w Z_\mu.
\]

Thus, Eq. (2.1) becomes

\[
g_2 J_3^\mu [\sin \theta_w A_\mu + \cos \theta_w Z_\mu] + \frac{1}{2} \hat{g} J_Y^\mu [\cos \theta_w A_\mu - \sin \theta_w Z_\mu] = e J_{em}^\mu A_\mu + g_Z J_Z^\mu Z_\mu.
\]
So,

\[ eJ_{\mu}^m = g_2 J^3_\mu \sin \theta_w + \frac{1}{2} \dot{g} \cos \theta_w J^Y_\mu = -e [\bar{e}_L\gamma_\mu e_L + \bar{e}_R\gamma_\mu e_R] \]

\[ = \frac{1}{2} g_2 \sin \theta_w [-\bar{e}_L\gamma_\mu e_L + \bar{\nu}_L\gamma_\mu \nu_L] \]

\[ + \frac{1}{2} \dot{g} \cos \theta_w [(-1)\bar{e}_L\gamma_\mu e_L + (-2)\bar{\nu}_R\gamma_\mu e_R + (-1)\bar{\nu}_L\gamma_\mu \nu_L]. \]

This simply gives us

\[ g_2 \sin \theta_w = e \]

\[ \dot{g} \cos \theta_w = e \]

Hence

\[ \tan \theta_w = \frac{g_2}{\dot{g}}. \]

Similarly

\[ g_Z J^Z_\mu = g_2 J^3_\mu \cos \theta_w - \frac{1}{2} \dot{g} J^Y_\mu \sin \theta_w \]

\[ = \frac{g_2}{\cos \theta_w} \left[ -\bar{e}_L\gamma_\mu e_L \left( \frac{1}{2} - \sin^2 \theta_w \right) + \frac{1}{2} \bar{\nu}_L\gamma_\mu \nu_L + \sin^2 \theta_w \bar{\nu}_R\gamma_\mu e_R \right]. \]

This will give us the neutral current \( J^Z_\mu = J^{NC}_\mu \) and the corresponding coupling \( g_Z \). This is the main indication of the electroweak unification.

\section*{2.3 The Electroweak Standard Model}

The Electroweak Standard Model is the commonly accepted theory of the fundamental electroweak interactions [3]. When we want to advertise the virtue of the Standard Model, we say that it is a model whose foundation is
symmetry [4]. It is a gauge invariant Quantum Field Theory based on the symmetry group $SU(2) \times U(1)$, which is spontaneously broken by the Higgs mechanism.

The Electroweak Standard Model consists of three components.

1): The basic constituents of matter are leptons and quarks which are realized in three families of identical structure:

| Leptons | $\nu_e$ | $\nu_\mu$ | $\nu_\tau$ |
|---------|---------|-----------|-----------|
|         | $e^-$   | $\mu^-$   | $\tau^-$  |

| Quarks  | $u$ | $c$ | $t$ |
|---------|-----|----|----|
|         | $d$ | $s$ | $b$ |

We will concentrate on the leptonic sector only.

2): Four different forces act between leptons and quarks: The electromagnetic and weak forces are unified in the Standard Model. The fields associated with these forces are spin 1 fields, describing the photon $\gamma$ and the electroweak gauge bosons $W^\pm$ and $Z$.

3): The third component of the Standard Model is the Higgs mechanism.

Before going to the deep discussion of the Standard Model, we have to require some theoretical basis [5].

### 2.3.1 The Theoretical Base

The fundamental forces of the Electroweak Standard Model, the electromagnetic and the weak force, are mediated by gauge fields. The concept could consistently be extended to massive gauge field by introducing the Higgs
mechanism which generates masses without destroying the underlying gauge symmetries of the theory [6].

1): Gauge Sector

Gauge invariant theories are invariant under gauge transformations of fermion fields: \( \psi \rightarrow U\psi \). \( U \) is either a phase factor for Abelian transformations or Unitary matrix for non-Abelian transformation acting on the multiplets of the fermion field \( \psi \). Now if the theory guarantees the local transformation for which \( U \) depends on the space-time point \( x \), the usual space-time derivatives \( \partial_\mu \) must be extended to covariant derivatives \( D_\mu \) which includes a new vector field \( V_\mu \):

\[
i \partial_\mu \rightarrow iD_\mu = i\partial_\mu - gV_\mu
\]

\( g \) defines the universal gauge coupling of the system. Under local gauge transformations the gauge field \( V_\mu \) is transformed by a rotation plus a shift:

\[
V_\mu \rightarrow UV_\mu U^{-1} + ig^{-1} [\partial_\mu U] U^{-1}.
\]

But in contrast to this, the curl \( F \) of \( V_\mu \),

\[
F_{\mu\nu} = -ig^{-1} [D_\mu, D_\nu]
\]

is just rotated under gauge transformation.

The Lagrangian describing the system of spinor fermions and vectorial gauge bosons for massless particles can be written in the compact form as follows:

\[
L [\psi, V] = \bar{\psi}iD\psi - \frac{1}{2}TrF^2
\]
It incorporates the following interactions:

Fermion-gauge bosons

\[-g \bar{\psi} V \psi.\]

Three bosons couplings

\[ig \text{Tr} \left( \partial_\nu V_\mu - \partial_\mu V_\nu \right) \left[ V_\mu, V_\nu \right].\]

Four boson couplings

\[\frac{1}{2} g^2 \text{Tr} \left[ V_\mu, V_\nu \right]^2.\]

2): Higgs mechanism

What is called the Higgs mechanism is the extension of the spontaneous symmetry breaking to create massive vector bosons in a gauge invariant theory. As the SM is a gauge theory, the $SU(2) \times U(1)$ gauge invariance requires masses of the gauge bosons to be zero, since the presence of an explicit mass term for the gauge bosons in the Lagrangian violates gauge invariance. The Higgs mechanism circumvents this constraint by beginning with a gauge invariant theory having massless gauge bosons. The $W^\pm$ and $Z^0$ masses were generated by spontaneously breaking the local gauge symmetry from $SU(2) \times U(1) \rightarrow U(1)_{em}$, which was achieved by the introduction of a self-interacting complex scalar field, $\Phi$, transforming as an $SU(2)$ doublet. The doublet field $\Phi$ and its complex conjugate together comprise four independent fields. Spontaneous symmetry breaking was implemented by giving one of the neutral fields a nonzero vacuum expectation value,

\[\langle \phi \rangle \equiv \langle 0 | \phi | 0 \rangle = \frac{v}{\sqrt{2}} \neq 0.\]
Of the four fields in the Lagrangian before spontaneous symmetry breaking, three fields become the longitudinal degrees of freedom of the vector bosons $W^\pm$ and $Z^0$; the photon coupled to the remaining symmetry group $U(1)_{em}$-generators, remains massless.

One neutral scalar particle remains in the physical sector of the theory. This is the so-called Salam-Weinberg Higgs particle, which the $SU(2) \times U(1)$ model predicts to exist.

Since the same Higgs doublet is used to give masses to the bosons and fermions, which have Yukawa couplings with the scalar fields, the $SU(2) \times U(1)$ model predicts the couplings of the Higgs particles with all the known bosons and fermions but makes no prediction about its mass. This could be traced back to the fact that in Salam-Weinberg theory, the Higgs particle mass is function of the unknown quartic Higgs-boson coupling constant.

### 2.3.2 Formulation of the Electroweak Standard Model

**The Matter Sector**

The fundamental fermions, as families with left handed isospin doublets and right handed isospin singlets appear in the fundamental representation of the the group $SU(2) \times U(1)$. It is realized that the symmetry pattern in the first, second and third generation of the fermions is same,

\[
\begin{bmatrix}
\nu_e \\
e^- \\
\end{bmatrix}_L \quad \begin{bmatrix}
\nu_e_R \\
e^-_R \\
\end{bmatrix} \quad \begin{bmatrix}
\nu_\mu \\
\mu^- \\
\end{bmatrix}_L \quad \begin{bmatrix}
\nu_\mu_R \\
\mu^-_R \\
\end{bmatrix} \quad \begin{bmatrix}
\nu_\tau \\
\tau^- \\
\end{bmatrix}_L \quad \begin{bmatrix}
\nu_\tau_R \\
\tau^-_R \\
\end{bmatrix}
\]

The symmetry structure cannot be derived in the Standard Model. It is an experimental fact that in weak interactions the parity is not conserved. The
different isospin assigned to the left handed and right handed field allows for maximal parity violation in the weak interactions. So the experimental observation is incorporated in the natural way.

The relationship between the electric charge $Q$ and basic quantum numbers is described by Gell-Mann-Nishijima relation

$$Q = I_3 + \frac{Y}{2}.$$ 

**Interactions**

The interactions of the Standard Model are summarized by the three terms in the basic Lagrangian:

$$L = L_G + L_F + L_H \quad (2.2)$$

which are specified in the following way:

**Gauge fields**

$SU(2) \times U(1)$ is a non-Abelian group which is generated by the isospin operators $I_1, I_2, I_3$ and the hypercharge $Y$. Each of these generalized charges is associated with a vector field: a triplet of vector fields $W^{1,2,3}_\mu$ with $I_{1,2,3}$ and a singlet field $B_\mu$ with $Y$. The isotriplet $W^a_\mu, a = 1, 2, 3$ and isosinglet $B_\mu$ lead to the field strength tensors

$$W^a_{\mu \nu} = \partial_\mu W^a_\nu + \partial_\nu W^a_\mu + g_2 \varepsilon_{abc} W^b_\mu W^c_\nu$$

$$B_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \quad (2.3)$$

$g_2$ is defined as the coupling constant for non-Abelian gauge group $SU(2)$.
Using the equation (2.3) the pure gauge field Langragian can be written as follows

\[ L_G = -\frac{1}{4} W^a_{\mu\nu} W^{\mu\nu,a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \]  

(2.4)

It is invariant under the non-Abelian \( SU(2) \times U(1) \) transformation.

**Fermion fields and fermion-gauge interactions**

The left-handed fermion fields of each lepton family

\[ \psi^L_j = \begin{pmatrix} \psi^L_{j+} \\ \psi^L_{j-} \end{pmatrix} \]

with family index \( j \) are grouped into \( SU(2) \) doublets with component index \( \sigma = \pm \), and the right-handed fields into singlets

\[ \psi^R_j = \psi^{R}_{j\sigma}. \]

Each left and right-handed multiplet is an eigenstate of the weak hypercharge \( Y \) such that the relation (2.3) is fulfilled. The Covariant derivative

\[ D_\mu = \partial_\mu - ig_2 I_a W^a_\mu + ig_1 \frac{Y}{2} B_\mu \]  

(2.5)

induces the fermion-gauge field interaction via the minimal substitution rule

\[ L_F = \sum_j \bar{\psi}^L_j i\gamma^\mu D_\mu \psi^L_j + \sum_{j,\sigma} \bar{\psi}^{R}_{j\sigma} i\gamma^\mu D_\mu \psi^{R}_{j\sigma}. \]  

(2.6)

g_1 is the coupling constant for the Abelian \( U(1) \) gauge group.
Higgs field and Higgs interaction

For spontaneous breaking of the $SU(2) \times U(1)$ symmetry leaving the electromagnetic gauge group $U(1)$ unbroken, a single complex scalar doublet field with hypercharge $Y = 1$

$$\phi(x) = \begin{pmatrix} \phi^+ (x) \\ \phi^0 (x) \end{pmatrix} \quad \text{(2.7)}$$

is coupled to the gauge fields through

$$L_H = (D_\mu \phi)^+ (D^\mu \phi) - V(\phi) \quad \text{(2.8)}$$

with the covariant derivative

$$D_\mu = \partial_\mu - ig_2 I_a W^a_\mu + ig_1 B^\mu_2.$$ 

The Higgs field self interaction

$$V(\phi) = -\mu^2 \phi^+ \phi + \frac{\lambda}{4} (\phi^+ \phi)^2 \quad \text{(2.9)}$$

is constructed in such a way that $\phi$ has a non vanishing vacuum expectation value, i.e.

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

with

$$v = \frac{2\mu}{\sqrt{\lambda}} \quad \text{(2.10)}$$

The field (2.7) can be written in the following way,

$$\phi(x) = \begin{pmatrix} \phi^+ (x) \\ (v + H(x) + i\chi(x))/\sqrt{2} \end{pmatrix} \quad \text{(2.11)}$$
where the field components $\phi^+, H$, $\chi$ have vacuum expectation values zero.

Exploiting the invariance of the Langragian, the components $\phi^+, \chi$ can be gauged away; this means that they are unphysical (Higgs ghosts or would be Goldstone bosons). In this particular gauge, the unitarity gauge, the Higgs field has simple form

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}.$$

The real field $H(x)$ which describes small oscillations about the ground state defines the physical Higgs field.

The Higgs field components have triplet and quartic self couplings following from $V$ and couplings to the gauge fields via the kinetic term of Eq.(2.8).

In addition, Yukawa couplings of the fermions are introduced in order to make the fermion massive. The Lagrangian for the Yukawa term can be written as follows

$$L_{\text{Yukawa}} = g_l \left( \nu_L \phi^+ l_R + \bar{l}_R \phi^- \nu_L + \bar{l}_L \phi^0 l_R + \bar{l}_R \phi^{0*} l_L \right)$$

(2.12)

The fermion mass terms follow from the $v$- part of $\phi^0$ [3].

The Lagrangian $L$ summarizes the laws of physics for the electromagnetic and the weak interactions between the leptons, and it predicts the form of self-interaction between the gauge fields. Moreover, the specific form of the Higgs interaction generates the mass of the fundamental particles, the leptons, the gauge bosons and the Higgs boson itself, and it predicts the interactions of the Higgs particle [5].
Masses and mass eigenstates of particles

In the unitary gauge the mass terms are extracted by substituting

$$\phi \rightarrow 0, \frac{v}{\sqrt{2}}$$

in the basic Higgs Lagrangian (2.8). The apparent $SU(2)$ seems to be lost thereby, but only superficially so and remain present in the hidden form; the resulting Lagrangian preserves an apparent local $U(1)$ gauge symmetry which is identified with the electromagnetic gauge symmetry: $SU(2) \times U(1) \rightarrow U(1)_{em}$ [5].

Gauge Bosons

The mass matrix of the gauge boson in the basis $(\vec{W}, B)$ takes the form

$$M_V^2 = \frac{1}{4} v^2 \begin{pmatrix} g_W^2 & g_W^2 & g_W^2 & g_W^2 \\ g_W^2 & g_W \dot{g}_W & g_W \dot{g}_W & g_W \dot{g}_W \\ g_W \dot{g}_W & g_W \dot{g}_W & g_W \dot{g}_W & g_W \dot{g}_W \end{pmatrix}. \quad (2.13)$$

This gives the mass of the vector boson in non diagonal form. The mass of the charged weak bosons is obvious

$$M_{W^\pm}^2 = \frac{1}{4} g_W^2 v^2.$$ 

As eigenstates related to the two masses $M_{W^\pm}^2$ the charged $W^\pm$ boson state may be defined as

$$W_\mu^\pm = \frac{1}{\sqrt{2}} \left[ W_\mu^1 \mp W_\mu^2 \right]. \quad (2.14)$$
For the neutral bosons, the mass term gives the matrix

\[
M_{VN}^2 = \frac{1}{4} \begin{pmatrix} g_W^2 & gw \hat{g}_W \\ gw \hat{g}_W & g_W^2 \end{pmatrix} v^2 \tag{2.15}
\]

Since \( \det(M_{VN}^2) = 0 \), therefore one of the eigenvalue of \( M_{VN}^2 \) is zero. The above matrix is diagonalized by defining the fields \( A_\mu, Z_\mu \):

\[
A_\mu = \cos \theta_W B_\mu + \sin \theta_W W_3^\mu \tag{2.16}
\]

\[
Z_\mu = -\sin \theta_W B_\mu + \cos \theta_W W_3^\mu \tag{2.17}
\]

In matrix form the above equations can be written as follows;

\[
\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos \theta_W & \sin \theta_W \\ -\sin \theta_W & \cos \theta_W \end{pmatrix} \begin{pmatrix} B_\mu \\ W_3^\mu \end{pmatrix} \tag{2.18}
\]

Then we get

\[
M_A^2 = 0 \quad A_\mu: \text{photon} \tag{2.19}
\]

\[
M_Z^2 = \frac{1}{4} \left( g_W^2 + g_W^2 \right) v^2 = \frac{1}{4} g_W^2 v^2 \left( \frac{1}{\cos^2 \theta_W} \right) \tag{2.20}
\]

Where

\[
\tan \theta_W = \frac{\hat{g}_W}{g_W} \tag{2.21}
\]

i.e. the electroweak mixing angle \( \theta_W \) is defined by the ratio of the \( SU(2) \) and \( U(1) \) couplings.

Introduce a parameter

\[
\rho = \frac{M_W^2}{M_Z^2 \cos^2 \theta_W}
\]
Now using the value of $M_Z^2$, we get

$$\rho = 1$$

This is the consequence of the fact that Higgs field is a doublet under $SU(2)_L$ [7].

Experimentally the mixing angle turns out to be large, i.e. $\sin^2 \theta_W \simeq 0.23$. The fact that the experimental value for $\sin^2 \theta_W$ is far away from the limits 0, or 1, indicate a large mixing effect. This supports the interpretation that the electromagnetic and the weak interactions are indeed manifestations of a unified electroweak interaction even though the underlying symmetry group $SU(2) \times U(1)$ is not simple. It may therefore be concluded that the electromagnetic and the weak interactions are truly unified in the Glashow-Salam-Weinberg theory of the electroweak interactions.

The ground-state value of the Higgs field is related to the Fermi coupling constant. From the low-energy relation $G_F/\sqrt{2} = g_W^2/8M_W^2$ in $\beta$ decay and combine with the mass relation $M_{W^\pm}^2 = 1/4g_W^2v^2$, the value of $v$ can be derived:

$$v = \left[1/\sqrt{2}G_F\right]^{1/2}$$

$$\simeq 246 GeV$$

(2.22)

The typical range for electroweak phenomena, defined by the weak masses $M_W$ and $M_Z$, is of the order 100GeV.
Fermions

The leptons are endowed with mass by means of Yukawa interactions with the Higgs ground state:

$$M_f = g_f v \sqrt{2}. \quad (2.23)$$

Though the masses of chiral fermion fields can be introduced in a consistent way via the Higgs mechanism, the Standard Model does not provide predictions for the experimental values of the Yukawa couplings $g_f$ and, as consequence, of the masses. The theory of the masses is not available yet.

The Higgs Bosons

The real field $H(x)$ which describes small oscillations about the ground state tells us the mass of the physical neutral scalar particles with mass

$$M_H = \mu \sqrt{2} = \sqrt{\lambda} v. \quad (2.24)$$

It can not predicted in the Standard Model since the quartic coupling $\lambda$ is an unknown parameter.

We conclude this session with the following remarks:

1). A definite prediction of electroweak unification is the existence of weak neutral currents with the same effective couplings as charged currents. This current has been found experimentaly.

2). The existence of the vector bosons $W^\pm, Z$ with definite masses which have also been discovered.

3). The theory has one free parameter $\sin^2 \theta_W$. 

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2.4 Limitations of the Standard Model

There is no confirmed experimental evidence from accelerators against the Standard Model, and several possible extensions have been ruled out. Nevertheless, there is no thinking physicist could imagine that the Standard Model is the end of physics. Even if one accepts the strange set of group representations and hypercharge that it requires, the Standard Model contains at least 19 parameters. Moreover, many more parameters are required if one wishes to accommodate non-accelerator observations. For example, the neutrino masses and mixing introduce at least 7 parameters: 3 masses, 3 mixing angles and 1 CP violating phase [8].

2.5 Summary

In this chapter, the $SU(2)$ and $U(1)$ gauge group have been reviewed, which provides a background knowledge for the understanding of the Electroweak Standard Model. The concept of how vector bosons $W^\pm$, $Z$ acquire masses have also been discussed in this chapter. The neutrino masses and mixings which are the weaknesses of the Standard Model have been mentioned here and will be studied in detail in the last chapter.
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Chapter 3

Generic Feature of Lepton
Flavour Violation

3.1 Introduction

It is generally believed that the Standard Model of electroweak interactions is a low-energy approximation to a more fundamental theory. Yet there is no clear experimental evidence either to guide its extension to additional physical processes or to predict the model parameters. The Standard Model incorporates the lepton family-number conservation, which has been empirically verified to high precision but is not a consequence of a known gauge theory. Indeed many theoretical extensions to the Standard Model allow lepton family-number violation within a range that can be tested by experiment [1].

The predications of the rate for a given family-number nonconserving
process vary among these extensions, and thus provide a test of the model. Many possibilities have been explored, and the present experimental limits for a wide variety of the processes have been tabulated. Thus the Lepton Flavour Violation (LFV) processes that are strongly suppressed in the Standard Model by powers of (small) neutrino masses may provide signals for new physics. At present we have stringent bound for $\mu$ decays, e.g.

$$BR(\mu \rightarrow 3e) \leq 10^{-12}$$

and some what weaker $O(10^{-16})$ bounds on LFV $\tau$ decays [2].

The possibility of large samples of decaying vector bosons [$V = J/\psi, \Upsilon$, and $Z^0$] and the clear signature provided by $\mu^\pm e^\pm$ final states suggest searching for LFV two-body decays

$$V \rightarrow \mu^\pm e^\pm.$$ 

Here we show that rather simple considerations based on unitarity, provide rather strong constraints on two-body LFV processes. Hence, most three-body $\mu$ and $\tau$ LFV decays are likely to provide more sensitive tests of lepton flavour violation, rather than the two-body decays.

### 3.2 Basic consideration and calculation

Let us assume that a vector boson $V_i$ (here $V_i$ could be either a fundamental state, such as the $Z^0$ or the quark-antiquark bound state such as the $\phi, J/\psi$, or $\Upsilon$) couples to $\mu^\pm e^\pm$. If it couples also to $e^+ e^-$, as all the states above do, then by unitarity $V$ exchange contributes also to $\mu \rightarrow 3e$. Let us define
the effective couplings between vector boson \( V_i \) and \( \mu^\pm \mu^\mp \) as \( \tilde{g}_{V\mu e} \), and the corresponding effective Lagrangian can be written as

\[
\mathcal{L}_{eff} = \tilde{g}_{V\mu e} \bar{\mu} \gamma^\alpha e V^\alpha + H.c. \tag{3.1}
\]

From the Feynman diagram, the amplitude \( A(\mu \to 3e) \) can be written as

\[
A(\mu \to 3e) = \bar{u}_e(k_3)\tilde{g}_{V\mu e} \gamma^\alpha u_\mu(p) \frac{g_{\alpha\beta}}{M^2_V - s} \bar{u}_e(k_2)g_{Vee} \gamma^\beta v_e(k_1)
\]

\[
= \bar{u}_e(k_3)\gamma^\alpha u_\mu(p) \bar{u}_e(k_2)\gamma_\alpha v_e(k_1) \tilde{g}_{V\mu e}g_{Vee} \frac{1}{M^2_V - s}. \tag{3.2}
\]

Here \( g_{Vee} \) is the effective coupling of the vector boson \( V_i \) to \( e^+e^- \), while \( s \equiv (k_1 + k_2)^2 \leq m^2_\mu \).

(There are, of course, also potential axial vector couplings of \( V \) to \( e^+e^- \), which contribute to this process. These can be included in the above, but as we shall see they do not change qualitative discussion).

Since we are dealing with the low energy process, as a first approximation, it is sensible to neglect \( s \) in comparison with \( M^2_V \). Therefore the Eq. (3.2) takes the form,

\[
A(\mu \to 3e) = \bar{u}_e(k_3)\gamma^\alpha u_\mu(p) \bar{u}_e(k_2)\gamma_\alpha v_e(k_1) \tilde{g}_{V\mu e}g_{Vee} \frac{1}{M^2_V}. \tag{3.3}
\]

In order to calculate the decay width we have to calculate \(|A|^2\) which is given by

\[
|A|^2 = AA^\dagger = \bar{u}_e(k_3)\gamma^\alpha u_\mu(p) \bar{u}_e(k_2)\gamma_\alpha v_e(k_1) \tilde{g}_{V\mu e}g_{Vee} \frac{1}{M^2_V} \left( \frac{\tilde{g}_{V\mu e}g_{Vee}}{M^2_V} \right)^2. \tag{3.4}
\]
Taking the spin average for the initial state and summation over spins for final state particles, we get

$$\sum_{\text{spins}=\pm \frac{1}{2}} |A|^2 = \frac{1}{16} [Tr \left[ \gamma_\alpha \left( k_1 - m_e \right) \gamma_\mu \left( k_2 + m_e \right) \right]$$

$$\times Tr \left[ \gamma^\alpha \left( \not{p} + m_\nu \right) \gamma^\mu \left( k_3 + m_e \right) \right] \left( \frac{g_{\nu\mu} g_{\rho\nu}}{M_V^2} \right)^2. \quad (3.5)$$

Assuming that the external particles (electrons) are massless. Therefore, Eq. (3.5) takes the form

$$\sum_{\text{spins}=\pm \frac{1}{2}} |A|^2 = 2 [(k_1 \cdot p) (k_2 \cdot k_3) + (k_1 \cdot k_3) (k_2 \cdot p)]. \quad (3.6)$$

We next carry the phase space integrations, starting with the integrals over the electrons momenta, given by

$$I^{\mu\nu} (q) = \int d^3 k_1 d^3 k_2 \frac{k_1^\mu k_2^\nu}{E_1 E_2} \delta^4 (k_1 + k_2 - q) \quad (3.7)$$

where

$$q \equiv p - k_3. \quad (3.8)$$

It follows from the Lorentz covariance of the integral (3.7) that the most general form is

$$I^{\mu\nu} (q) = g^{\mu\nu} A \left( q^2 \right) + q^\mu q^\nu B \left( q^2 \right). \quad (3.9)$$

From this equation it follows that

$$g_{\mu\nu} I^{\mu\nu} (q) = 4A \left( q^2 \right) + q^2 B \left( q^2 \right)$$

$$q_\mu q_\nu I^{\mu\nu} (q) = q^2 A \left( q^2 \right) + \left( q^2 \right)^2 B \left( q^2 \right). \quad (3.10)$$

This can be obtained by using Eq. (3.9).
Since we have been taking the electron masses to be zero so that $k_1^2 = k_2^2 = k_3^2 = 0$ and, on account of the $\delta$-function in (3.7),

$$q^2 = (k_1 + k_2)^2 = 2(k_1 \cdot k_2)$$

$$\frac{q^2}{2} = k_1 \cdot k_2$$

(3.11)

In order to find $A(q^2)$ and $B(q^2)$, we calculate the expression on the left hand sides of (3.10). From Eq. (3.7) we obtain

$$g_{\mu\nu}I^{\nu\nu}(q) = \int d^3k_1 d^3k_2 \frac{g_{\mu\nu}k_1^\mu k_2^\nu}{E_1 E_2} \delta^4(k_1 + k_2 - q)$$

$$= \int d^3k_1 d^3k_2 \frac{(k_1 \cdot k_2)}{E_1 E_2} \delta^4(k_1 + k_2 - q).$$

(3.12)

Using Eq. (3.11), the above equation takes the form

$$g_{\mu\nu}I^{\nu\nu}(q) = \frac{q^2}{2} \int d^3k_1 d^3k_2 \frac{\delta^4(k_1 + k_2 - q)}{E_1 E_2}$$

$$\equiv \frac{1}{2} q^2 I(q^2).$$

(3.13)

We see from its definition that the integral $I(q^2)$ is an invariant, so that it can be evaluated in any coordinate system. For our convenience we shall choose the centre-of-mass system of two electrons. In this system

$$k_1 = -k_2$$

And for the massless electron

$$E_1 = \sqrt{|k_1|^2 + m_e^2}$$

$$= |k_1|$$

also

$$E_2 = |k_2|.$$
So from above two results, we can write

\[ E_1 = E_2 = E. \]  

(3.14)

Hence by removing the integration on \( k_2 \), we get

\[
I \left( q^2 \right) = \int \frac{d^3k_1}{E^2} \delta (2E - q_0) \\
= \int \frac{E^2 dE}{E^2} \delta (2E - q_0) d\Omega \\
= 2\pi
\]  

(3.15)

and from Eq. (3.13)

\[
g_{\mu\nu}I^{\mu\nu} (q) = \pi q^2.
\]  

(3.16)

Similarly, calculating \( q_\mu q_\nu I^{\mu\nu} (q) \), one gets

\[
q_\mu q_\nu I^{\mu\nu} (q) = \int d^3k_1 d^3k_2 \frac{q_\mu q_\nu k_1^\mu k_2^\nu}{E_1 E_2} \delta^4 (k_1 + k_2 - q) \\
= \int d^3k_1 d^3k_2 \frac{(q \cdot k_1) (q \cdot k_2)}{E_1 E_2} \delta^4 (k_1 + k_2 - q).
\]  

(3.17)

This is obtained by using Eq. (3.7). Also using Eq. (3.11), we got

\[
q_\mu q_\nu I^{\mu\nu} (q) = \frac{(q^2)^2}{4} I \left( q^2 \right) \\
= \frac{1}{2} \pi \left( q^2 \right)^2.
\]  

(3.18)

Substituting Eqs. (3.16) and (3.18) in Eq. (3.7), we get

\[
\pi \left( q^2 \right) = 4A \left( q^2 \right) + q^2 B \left( q^2 \right)
\]  

(3.19)

\[
\frac{1}{2} \pi \left( q^2 \right)^2 = q^2 A \left( q^2 \right) + \left( q^2 \right)^2 B \left( q^2 \right).
\]  

(3.20)
Solving (3.19) and (3.20) for the values of $A(q^2)$ and $B(q^2)$, we have

$$A(q^2) = \frac{\pi (q^2)}{6}$$
$$B(q^2) = \frac{2\pi}{6}.$$

Hence Eq. (3.9) leads to

$$I^{\mu\nu}(q) = \frac{\pi}{6} \left( g^{\mu\nu} q^2 + 2q^\mu q^\nu \right). \quad (3.21)$$

Then the partial muon decay rate can be written as

$$d\Gamma = \frac{1}{(2\pi)^3} \frac{1}{2E_\mu} \frac{d^3k_3}{E_3} \int \frac{d^3k_1}{E_1} \frac{d^3k_2}{E_2} \sum_{\text{spins}=\pm 1} |A|^2 (2\pi)^4 \delta^4(k_1 + k_2 - q)$$

$$= \left( \frac{\bar{\nu}_{e} g_{\nu e}}{M_V^2} \right)^2 \frac{1}{(2\pi)^3} \frac{1}{2E_\mu} \frac{d^3k_3}{E_3} \times \left[ p_\mu k_3^\nu \left( g^{\mu\nu} q^2 + 2q^\mu q^\nu \right) \right]$$

$$= \left( \frac{\bar{\nu}_{e} g_{\nu e}}{M_V^2} \right)^2 \frac{1}{(2\pi)^3} \frac{1}{2E_\mu} \frac{d^3k_3}{E_3} \times 4 \left[ (p \cdot k_3) q^2 + 2 (k_3 \cdot q) (p \cdot q) \right]. \quad (3.22)$$

Using (3.21), (3.22) becomes

$$d\Gamma = \left( \frac{\bar{\nu}_{e} g_{\nu e}}{M_V^2} \right)^2 \frac{1}{(2\pi)^3} \frac{1}{2E_\mu} \frac{d^3k_3}{E_3}$$

$$\times \left[ p_\mu k_3^\nu \left( g^{\mu\nu} q^2 + 2q^\mu q^\nu \right) \right]$$

$$= \left( \frac{\bar{\nu}_{e} g_{\nu e}}{M_V^2} \right)^2 \frac{1}{(2\pi)^3} \frac{1}{2E_\mu} \frac{d^3k_3}{E_3} \times 4 \left[ (p \cdot k_3) q^2 + 2 (k_3 \cdot q) (p \cdot q) \right]. \quad (3.23)$$

This is obtained by using Eq. (3.7).

Finally, we must integrate Eq. (3.23) over all momenta $k_3$ of the emitted electron. For a muon at rest, i.e. in the rest frame of muon, we have

$$p = (E_\mu, 0) = (m_\mu, 0)$$
$$k_3 = (E_3, k_3). \quad (3.24)$$
Also

\[ q_0 = m_\mu - E_3 \]
\[ q = -k_3 \quad (3.25) \]

Now, we will calculate the terms involving in Eq. (3.23), i.e.

\[ q^2 = (p + k_3)^2 = \left( m_\mu^2 - 2m_\mu E_3 \right), \]
\[ k_3^2 = 0 \]

\[ k_3.p = m_\mu E_3 \]
\[ k_3.q = m_\mu E_3 \]
\[ p.q = m_\mu (m_\mu - E_3) \quad (3.26) \]

By substuting these values Eq. (3.23) takes the form

\[ d\Gamma = \left( \frac{\bar{g}_{\nu_\mu}g_{\nu e}}{M_V^2} \right)^2 \frac{1}{(2\pi)^6} \frac{1}{2m_\mu} \frac{d^3k_3}{E_3} \frac{\pi}{6} 4 \]
\[ \times \left[ (m_\mu E_3) \left( m_\mu^2 - 2m_\mu E_3 \right) + 2 \left( m_\mu E_3 \right) m_\mu \left( m_\mu - E_3 \right) \right] \]
\[ = \left( \frac{\bar{g}_{\nu_\mu}g_{\nu e}}{M_V^2} \right)^2 \frac{1}{(2\pi)^5} \frac{2\pi}{6} E_3^2 dE_3 d\Omega \left[ 3m_\mu^2 - 4m_\mu E_3 \right]. \quad (3.27) \]

Integrating Eq. (3.27) over all directions \( \Omega \) of the emitted electron and over its complete range of energies \( 0 \leq E_3 \leq \frac{1}{2}m_\mu \), we obtain the total decay rated

\[ \Gamma = \left( \frac{\bar{g}_{\nu_\mu}g_{\nu e}}{M_V^2} \right)^2 \frac{4\pi}{(2\pi)^5} \frac{2\pi}{6} m_\mu \int_0^{\frac{1}{2}m_\mu} E_3^2 \left[ 3m_\mu^2 - 4m_\mu E_3 \right] dE_3 \]
\[ = \left( \frac{\bar{g}_{\nu_\mu}g_{\nu e}}{M_V^2} \right)^2 \frac{4\pi}{(2\pi)^5} \frac{2\pi}{6} m_\mu^5 \frac{5}{16} \]
\[ \Gamma (\mu \to 3e) = \frac{1}{2} \left( \frac{\bar{g}_{\nu_\mu}g_{\nu e}}{M_V^2} \right)^2 \frac{m_\mu^5}{192\pi^3} \quad (3.28) \]
Comparing the above contribution to the $\mu \to 3e$ process to that of the ordinary muon decay, $\mu \to e\nu\bar{\nu}$, which proceeds via $W$ exchange and (almost) identical kinematics, gives the relation

$$\frac{\Gamma [\mu \to 3e]_{V-\text{exch.}}}{\Gamma [\mu \to e\nu\bar{\nu}]} \approx \frac{g_{e\nu}^2 g_{e\mu}^2}{M_V^4} / \frac{g_W^4}{M_W^4}.$$ (3.29)

This ratio is defined as the branching ratio. Therefore

$$[B (\mu \to 3e)]_{V-\text{exch.}} \approx \frac{g_{e\nu}^2 g_{e\mu}^2}{M_V^4} / \frac{g_W^4}{M_W^4}.$$ (3.30)

Since

$$\Gamma (V \to e^+e^-) \sim g_{e\nu}^2 M_V,$$

and

$$\Gamma (V \to \mu^+e^-) \sim g_{e\mu}^2 M_V,$$

while

$$\Gamma (W \to e\nu) \sim g_W^2 M_W,$$

we can rewrite the above expression as

$$[B (\mu \to 3e)]_{V-\text{exch.}} \approx \frac{\Gamma (V \to e^+e^-) \Gamma (V \to \mu^+e^-)}{\Gamma (W \to e\nu)} \left( \frac{M_W}{M_V} \right)^6.$$ (3.31)

Using $[BR (\mu \to 3e)]_{V-\text{exch.}} \leq 10^{-12}$ and other data pertaining to the $e^+e^-$ widths of the various vector mesons $V_i$, we find a set of bounds for the two body LFV branching ratios of these vector mesons, i.e.

$$\Gamma (V \to \mu^+e^-) \approx \frac{[BR (\mu \to 3e)]_{V-\text{exch.}} \Gamma^2 (W \to e\nu)}{\Gamma (V \to e^+e^-)} \left( \frac{M_V}{M_W} \right)^6.$$ (3.32)
These bounds are calculated by dividing the above decay width by the full width of the vector mesons. Using the numerical values these bounds become

\[
\mathcal{B}(Z \rightarrow \mu e) \leq 5 \times 10^{-13} \quad (3.33)
\]

\[
\mathcal{B}(J/\psi \rightarrow \mu e) \leq 4 \times 10^{-13} \quad (3.34)
\]

\[
\mathcal{B}(\Upsilon \rightarrow \mu e) \leq 2 \times 10^{-9} \quad (3.35)
\]

\[
\mathcal{B}(\Phi \rightarrow \mu e) \leq 4 \times 10^{-17} \quad (3.36)
\]

All the vector (or pseudoscalars) used as intermediaries in deriving the bounds in Eqs. (3.33–3.36) are not on-shell. Thus we must entertain the possibility that their contribution to the three-body decays considered are reduced. This could at least weaken the various strong bounds obtained above. Now we will focus on the possible mechanism for the reduction of such bounds.

### 3.2.1 Dynamical suppression of the LFV bounds

The source of suppression is connected to possible “form factor” effect due to the dynamics which would, for example, reduce the contribution of various $V_i$ states to $\mu \rightarrow 3e$ compared to the naive expectations. However the effect of these form factors should be minimal or controllable if the LFV is induced by physics at scales much higher than the EW scale of the $Z$ mass. These effects of dynamics are beautifully discussed by Illuna, Jack and Riemann in their recent papers [3, 4]. Following these two papers, first of all we will discuss the dynamical suppression in the two body decay of $Z \rightarrow l_1^+ l_2^-$ and then move to $\mu \rightarrow 3e$. 

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The rare process $\mu \rightarrow e\gamma$ is the classic example of the lepton flavour violating process. The previous limit on its branching ratio is [5]

$$\mathcal{B}(\mu \rightarrow e\gamma) < 4.9 \times 10^{-11}.$$ 

This reaction has not been observed so far, and the best experimental upper limit of its branching fraction is [4]

$$\mathcal{B}(\mu \rightarrow e\gamma) < 1.2 \times 10^{-11}$$

i.e. it is improved by the factor of 4.1.

At the $Z$ factory LEP, searches for similar processes, through the $Z$ boson, became possible:

$$Z \rightarrow e\mu.$$ 

The best experimental limit on its branching ratio is (95% C.L.)

$$\mathcal{B}(Z \rightarrow e^+\mu^-) < 1.7 \times 10^{-6}.$$ 

Let’s discuss this reaction in detail.

The most general matrix element for the interaction of an on-shell vector boson with a fermionic current, as shown in fig. 3.2.1, may be described by four dimensionless form factors. At one loop level it is convenient to parameterize the amplitude as

$$A = -\frac{ig\alpha_W}{4\pi} \varepsilon^\rho \bar{u}_e (p_2) \Gamma_\rho v_\mu (p_1),$$

with $\alpha_W = \frac{g^2}{(4\pi)}$, $\varepsilon$ being the boson polarization vector and

$$\Gamma_\rho = \gamma_\rho (f_V - f_A\gamma_5) + \frac{q^\nu}{M_W} (i f_M + f_E\gamma_5) \sigma_{\rho\nu},$$

(3.38)
\[ \sigma_{\rho\nu} = \frac{i}{2} (\gamma_\rho \gamma_\nu - \gamma_\nu \gamma_\rho) \]

In Eq. (3.38), the form factors \( f_V \) and \( f_A \) stands for vector and axial-vector couplings and \( f_M \) and \( f_E \) for magnetic and electric dipole moments/transitions of equal/unlike final fermions. The form factors depend on the momentum transfer squared \( Q^2 = (p_2 + p_1)^2 \). In principle all the four form factors must be non-zero. In order to find the decay width we have to find \( |A|^2 \) which is

\[ |A|^2 = AA^\dagger. \quad (3.39) \]

So

\[
|A|^2 = \left( \frac{g_{\alpha W}}{4\pi} \right)^2 \left[ 8 (p_2 \cdot p_1) \left\{ |f_V|^2 + |f_A|^2 \right\} + \frac{\{|f_E|^2 + |f_M|^2\}}{M_W^2} \left\{ 16 (p_2 \cdot q) (p_1 \cdot q) - 4 (q \cdot q) (p_2 \cdot p_1) \right\} \right]. \quad (3.40)
\]

Using the spin averages, above result becomes

\[
\sum_{\text{spins}} |A|^2 = \left( \frac{g_{\alpha W}}{4\pi} \right)^2 \frac{1}{12} \left[ 8 (p_2 \cdot p_1) \left\{ |f_V|^2 + |f_A|^2 \right\} + \frac{\{|f_E|^2 + |f_M|^2\}}{M_W^2} \left\{ 16 (p_2 \cdot q) (p_1 \cdot q) - 4 (q \cdot q) (p_2 \cdot p_1) \right\} \right]. \quad (3.41)
\]

The decay width can be calculated as

\[
\Gamma = \frac{1}{2M_Z} \int \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \sum_{\text{spins}} |A|^2 (2\pi)^4 \delta^4 (q - p_1 - p_2) \\
= \frac{\alpha_{\alpha W}^3}{24\pi^2} M_Z \left[ |f_V|^2 + |f_A|^2 + \frac{1}{2c_W^2} (|f_E|^2 + |f_M|^2) \right]. \quad (3.42)
\]
The branching ratio can be obtained by dividing the decay width obtained in Eq. (3.42) by the total width of $Z$ boson.

$$
B[Z \to \mu e] = \frac{\alpha^3 \cdot M_Z}{24 \pi^2 \Gamma_Z} \left[ |f_V|^2 + |f_A|^2 + \frac{1}{2c_W^2} \left( |f_E|^2 + |f_M|^2 \right) \right]. \quad (3.43)
$$

where

$$
\Gamma_Z \approx \frac{\alpha \cdot W}{c_W^2} \cdot M_Z,
$$

These form factors are model dependent. It is easily seen that the branching fraction can be approximated as

$$
B(Z \to \mu e) \sim \left( \frac{\alpha W}{\pi} \right)^2 \sim 10^{-6} \quad (3.44)
$$

This is in agreement with its experimental value. Now we shall discuss these suppression in $\mu \to 3e$ decay.

**CASE-1:**

First discuss the vector-axial-vector form factor contribution at $V_{\mu e}$ vertex to $\mu \to 3e$, i.e. instead of the $\gamma^\alpha$ we have $\gamma^\alpha (f_V + f_A \gamma_5)$. Therefore the amplitude becomes

$$
A(\mu \to 3e) = \bar{u}_e(k_3) \gamma^\alpha (f_V + f_A \gamma_5) u_\mu(p) \bar{u}_e(k_2) \gamma_\alpha v_e(k_1) \frac{\bar{g}_{V_{\mu e}} g_{V_{ee}}}{M_V^2 - s} = \bar{u}_e(k_3) \gamma^\alpha (f_V + f_A \gamma_5) u_\mu(p) \bar{u}_e(k_2) \gamma_\alpha v_e(k_1) \frac{\bar{g}_{V_{\mu e}} g_{V_{ee}}}{M_V^2} \quad (3.45)
$$

Using the same procedure as for the previous cases $|A|^2$, we get

$$
|A|^2 = 32 \left[ |f_V|^2 + |f_A|^2 \right] \left[ (k_{1,p}) (k_2.k_3) + (k_1.k_3) (k_2.p) \right] \left( \frac{\bar{g}_{V_{\mu e}} g_{V_{ee}}}{M_V^2} \right)^2 \quad (3.46)
$$
Using the spin averages, the above expression becomes

\[
\sum_{\text{spins}=\pm \frac{1}{2}} |A|^2 = 2 \left[ |f_V|^2 + |f_A|^2 \right] \left[ (k_1.p) (k_2.k_3) + (k_1.k_3) (k_2.p) \right] \left( \frac{\tilde{g}_{\nu e} g_{V e}}{M_V^2} \right)^2.
\]

(3.47)

Solving the delta function using the technique described above, the decay width becomes

\[
\Gamma (\mu \rightarrow 3e) = \frac{1}{2} \left[ |f_V|^2 + |f_A|^2 \right] \left( \frac{\tilde{g}_{\nu e} g_{V e}}{M_V^2} \right)^2 \frac{m_\mu^5}{192\pi^3}.
\]

(3.48)

The corresponding branching ratio becomes

\[
\mathcal{B} (\mu \rightarrow 3e) \approx \left[ |f_V|^2 + |f_A|^2 \right] \left( \frac{\tilde{g}_{\nu e}^2 g_{V e}^2}{M_V^4} / \frac{g_W^4}{M_W^4} \right).
\]

(3.49)

CASE-2:

Now consider the electric and magnetic form factor contribution to the \( \mu \rightarrow 3e \) at \( V_i \mu e \) vertex. The amplitude can be written as

\[
A = \bar{u}_e(k_3) \frac{q^\nu}{M_W} (i f_M + f_E \gamma_5) \sigma_{\alpha\nu} u_\mu(p) \bar{u}_e(k_2) \gamma^\alpha v_e(k_1) \frac{\tilde{g}_{\nu e} g_{V e}}{M_V^2}.
\]

(3.50)

Using the same procedure as before the decay width for this reaction becomes

\[
\Gamma (\mu \rightarrow 3e) = \frac{1}{2} \frac{1}{80} \left( \frac{\tilde{g}_{\nu e}^2 g_{V e}^2}{M_V^4} \right)^2 \left[ |f_M|^2 + |f_E|^2 \right] \frac{m_\mu^2}{M_W^2} \frac{m_\mu^5}{192\pi^3}.
\]

(3.51)

The branching ratio for this case becomes

\[
\mathcal{B} (\mu \rightarrow 3e) \approx \frac{1}{80} \left[ |f_M|^2 + |f_E|^2 \right] \frac{m_\mu^2}{M_W^2} \left( \frac{\tilde{g}_{\nu e}^2 g_{V e}^2}{M_V^4} / \frac{g_W^4}{M_W^4} \right).
\]

(3.52)

Hence, the branching ratios of the three body decays are suppressed by a factor \( \frac{m_\mu^2}{M_W^2} \).
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Figure 3.1: $\mu \rightarrow 3e$

Figure 3.2: $Z \rightarrow \mu e$
Chapter 4

Lepton Flavour Violation in LSND and Seesaw model of neutrino masses

4.1 Introduction

The question of whether or not neutrinos have a nonzero mass has remained one of the most tantalizing issue in the present day physics. In the Standard Model of electroweak theory neutrinos are considered to be massless. But there is no compelling theoretical reason behind this assumption [1]. Hints that the neutrinos are massive particles comes from the observation of three anomalous effects,

1. the LSND (Liquid Scintillator Neutrino Detector) excess [2, 3, 4],

2. the atmospheric anomaly [5, 6, 7] and
3. the solar neutrino deficit [8, 9, 10, 11].

In particular, the atmospheric results are the most convincing ones. All the three effects can be naturally explained in terms of neutrino flavour oscillations, which will occur when neutrino propagate through space, if there masses are non-degenerate and the weak and mass eigenstates are mixed.

4.1.1 The neutrino flavour oscillation

However, in order to explain all three experimentally observed effects in terms of neutrino flavour oscillation, one is forced to invoke additional sterile neutrino states [12] to accommodate the very different frequencies of oscillations, given by three different mass squared $\Delta m^2$'s, indicated by three different effects. The existence of such neutrino is currently unresolved problem and clearly demonstrates that the neutrino sector is not fully understood. So due to the unappealing theoretical feature of a light sterile neutrino, it is interesting to look for alternatives that could explain the LSND excess with the known three light neutrinos.

From a phenomenological point of view, we recall that the neutrino flavour oscillation hypothesis predicts a well defined dependence of phenomena as a function of neutrino energy, characterized by the so called $L/E$ behaviour, where $L$ is distance between source and detector and $E$ is neutrino energy. So far, no experiment has conclusively demonstrated such a $L/E$ dependence of the anomalous effect, with may be expectation of SuperKamiokande data which favours the dependence $\alpha LE^n$ where $n = -1$ [13].
In such an unclear situation, it is possible to envisage “non-flavour oscillation” mechanism to explain part of neutrino data.

Aside from theoretical argument against sterile neutrino we argue that, from a phenomenological point of view, the LSND effect is peculiar: it has a small probability, measured to be $(2.5 \pm 0.6 \pm 0.4) \times 10^{-3}$ [13], in contrast to the solar and atmospheric neutrino anomalies, which are large. Hence, LSND is a natural candidate for an interpretation involving a different physics than in atmospheric and solar neutrino flavour oscillation.

### 4.2 LSND puzzle

We recall that the LSND effect was first reported as an excess of $\bar{\nu}_e$’s in the $\bar{\nu}_\mu$ flux from the $\mu^+$ decay at rest (DAR) process. The neutrino beam is obtained with 800 kinetic energy protons hitting a series of targets, producing secondary pions. Most of $\pi^+$ come to rest and decay through the sequence $\pi^+ \rightarrow \mu^+ \nu_\mu$; followed by $\mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu$, supplying the experiment with the $\bar{\nu}_\mu$’s with a maximum energy of 52.8 MeV. The intrinsic contamination of $\bar{\nu}_e$’s coming from the symmetrical decay chain starting with $\pi^-$ is estimated to be small since most negatively charged mesons are captured before they decay.

The excess of $\bar{\nu}_e$’s, explained in terms of neutrino flavour transitions of the type $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$, occur via the reactions:

\[
\begin{align*}
\mu^+ &\rightarrow e^+\nu_e\bar{\nu}_\mu \\
\bar{\nu}_\mu \overset{\text{vacuum}}{\longrightarrow} \bar{\nu}_e
\end{align*}
\]
\[ \bar{\nu}_e p \rightarrow e^+ n \]
i.e. anti-neutrinos are produced by the \( \mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu \) and detected by \( \bar{\nu}_e p \rightarrow e^+ n \).

There is another evidence in favour of the neutrino flavour oscillation which was reported in the Decay In Flight (DIF). But due to its lower statistical significance, we concentrate on the hint from stopped muon, and ignore the DIF result.

The LSND claim is contradicted by the latest KARMEN2 results [14], however the experimental sensitivity is marginal to conclusively exclude or confirm completely the LSND excess. A new experiment, MiniBOONE [15] would confront the flavour oscillation hypothesis with a very high statistical accuracy. A negative result from MiniBOONE experiment would indicate that the neutrino flavour oscillation is not the correct hypothesis to explain the excess seen in LSND. It would however not contradict other possible non-flavour-oscillation interpretation of the effect. In particular, LFV decays would play a role in the interpretation of LSND excess. A neutrino factory is an ideal machine to probe such anomalous decays of muon.

The aim of this work is to investigate another approach. We assume that the three neutrinos interact through Lepton Flavour Violation interactions, which are forbidden in the Standard Model. This is an attractive possibility, because various extensions of the Standard Model which predict the neutrino masses also give rise to such new interactions. These interactions can affect the LSND excess. The motivation for the LFV is that the branching probability for LSND is too small \( (2.5 \pm 0.6 \pm 0.4) \times 10^{-3} \) compared to
atmospheric and solar anomalies. We analyze the consequence of small Lepton Flavour Violating interactions to explain the LSND excess, and check whether this scenario will be feasible.

4.2.1 Basic consideration and calculation

Let us assume that a vector boson $V_i$ (here $V_i$ could be either a fundamental state, such as the $Z^0$ or the quark-antiquark bound state such as the $\phi, J/\psi$, or $\Upsilon$) couples to $\mu^\pm e^\pm$. If it couples also to $e^+e^-$, as all the states above do, then by unitarity its exchange contributes also to $\mu \rightarrow 3e$. Let us define the effective couplings between vector boson $V_i$ and $\mu^\pm e^\mp$ as $\tilde{g}_{V_{\mu e}}$, and the corresponding effective Lagrangian can be written as

$$L_{\text{eff}} = \tilde{g}_{V_{\mu e}} \bar{\mu} \gamma^\alpha e V^\alpha + H.c. \quad (4.1)$$

Here, the reaction under consideration is

$$\mu^+ \rightarrow e^+ \nu_l \bar{\nu}_l,$$

therefore among these vector bosons only the $Z^0$ will contribute. $l$ is any of the three known leptons, i.e. $e$, $\mu$, or $\tau$. The coupling at the LFV vertex remains the same, but at the neutrino vertex it will be changed and is obtained by the Standard Model. We also assume that the LFV decay proceeds through a similar diagram as the Standard Model muon decay, however with interchanged neutrino flavour as shown in fig. 4.3.1.

Using Feynman rules, the corresponding amplitude can be written as

$$A(\mu \rightarrow e\nu_l\bar{\nu}_l) = \frac{\tilde{g}_{Z_{\mu e}} g_{Z_{e\mu}} g_{\alpha\beta}}{M^2_Z - s} \left[ \bar{\nu}(p_1) \gamma^\alpha v(p_2) \bar{u}(p_3) \gamma^\beta (1 - \gamma_5) v(p_4) \right]$$
Here $g_{V \nu \bar{\nu}}$ is the effective coupling of the $Z$-boson to $\nu \bar{\nu}$. Also for low energy, we have to neglect $s$ in comparison to the $M_Z$. Therefore Eq. (4.2) takes the form

$$A(\mu \to e\nu_l \bar{\nu}_l) = \frac{\tilde{g}_{Z\mu\gamma} g_{Z\nu\bar{\nu}}}{M_Z^2} [\bar{v}(p_1) \gamma^\alpha v(p_2) \bar{u}(p_4) \gamma^\alpha (1 - \gamma_5) v(p_3)]$$

(4.3)

Now as usual we have to calculate $|A|^2$, which is

$$\sum_{\text{spins}=\pm \frac{1}{2}} |A|^2 = \frac{1}{16} \left( \frac{\tilde{g}_{Z\mu\gamma} g_{Z\nu\bar{\nu}}}{M_Z^2} \right)^2 \text{Tr} \{ \slashed{p}_1 \gamma^\alpha, \slashed{p}_2 \gamma^\beta \} \times \text{Tr} \{ \slashed{p}_4 \gamma^\alpha (1 - \gamma_5), \slashed{p}_3 \gamma^\beta (1 - \gamma_5) \}.$$  

(4.4)

Solving these traces, the above Eq. becomes

$$\sum_{\text{spins}=\pm \frac{1}{2}} |A|^2 = \left( \frac{\tilde{g}_{Z\mu\gamma} g_{Z\nu\bar{\nu}}}{M_Z^2} \right)^2 4 \left[ (p_1 p_4) (p_2 p_3) + (p_1 p_3) (p_2 p_4) \right].$$  

(4.5)

Then the partial decay rate for the muon can be written as

$$d\Gamma = \frac{1}{(2\pi)^9} \frac{1}{2E_\mu} \frac{d^3 k_3}{E_3} \int \frac{d^3 k_1 d^3 k_2}{E_1 E_2} \sum_{\text{spins}=\pm \frac{1}{2}} |A|^2 (2\pi)^4 \delta^4 (p_1 + p_2 - q),$$

(4.6)

where

$$q = p_3 + p_4.$$  

Using the same technique as we have used in the previous chapter, we got the total decay width as follows

$$\Gamma(\mu \to e\nu_l \bar{\nu}_l) = \left( \frac{\tilde{g}_{Z\mu\gamma} g_{Z\nu\bar{\nu}}}{M_Z^2} \right)^2 \frac{m_{\mu}^5}{192\pi^3}.$$  

(4.7)
Comparing the above contribution to the $\mu \to e\nu\bar\nu_l$ process to that of the ordinary muon decay, $\mu \to e\nu\bar\nu$, which proceeds via $W$ exchange and (almost) identical kinematics, gives the relation

$$\frac{\Gamma (\mu \to e\nu\bar\nu_l)_{Z-exch.}}{\Gamma (\mu \to e\nu\bar\nu)} = \left[ \frac{\Gamma (\mu \to e\nu\bar\nu_l)_{Z-exch.}}{\Gamma (\mu \to 3e)_{V-exch.}} \right] \times \left[ \frac{\Gamma (\mu \to 3e)_{V-exch.}}{\Gamma (\mu \to e\nu\bar\nu)} \right]$$

$$\mathcal{B} (\mu \to e\nu\bar\nu_l)_{Z-exch.} = \left[ \frac{\Gamma (\mu \to e\nu\bar\nu_l)_{Z-exch.}}{\Gamma (\mu \to 3e)_{V-exch.}} \right] \times \mathcal{B} (\mu \to 3e)_{V-exch.} \quad (4.8)$$

Using the value of the $\Gamma (\mu \to 3e)_{Z-exch.}$, Eq. (4.8) becomes

$$\mathcal{B} (\mu \to e\nu\bar\nu_l)_{Z-exch.} \approx \left( \frac{g_{Z\nu\bar\nu}}{g_{Zee}} \right)^2 \times \mathcal{B} (\mu \to 3e)_{V-exch.}. \quad (4.9)$$

We know that

$$[\mathcal{B} (\mu \to 3e)]_{V-exch.} \leq 10^{-12}.$$ 

Since

$$\Gamma (Z \to e^+e^-) \sim g_{Zee}^2 M_Z$$

and

$$\Gamma (Z \to \nu_l\bar\nu_l) \sim g_{Z\nu\bar\nu}^2 M_Z.$$ 

So Eq. (4.8) becomes

$$\mathcal{B} (\mu \to e\nu\bar\nu_l)_{Z-exch.} \leq \frac{\Gamma (Z \to \nu_l\bar\nu_l)}{\Gamma (Z \to e^+e^-)} \times 10^{-12} \leq 2 \times 10^{-12}. \quad (4.10)$$

As it is seen from the above equation the branching ratio for the this lepton flavour violating decay is the same as that of the $\mu \to 3e$. This is because of the fact that the reaction kinematic is same, and also in both the cases the external particles are assumed to be the massless. The factor of 2 is due
to the fact that the couplings between electron-positron is different then the two neutrinos.

It is already mentioned that, in order to explain the LSND result, the effective New Physics coupling should satisfy

$$r = \left| \frac{G_N^\nu}{G_F} \right|^2 = (2.5 \pm 0.6 \pm 0.4) \times 10^{-3}. \quad (4.11)$$

We define $G_N^\nu$ to be the effective coupling of the anomalous muon decay. Thus, at the 90% C.L. we need [4]

$$r > 1.6 \times 10^{-3}. \quad (4.12)$$

Now let’s calculate this for the reaction discussed here. As we know that

$$\frac{G_F}{\sqrt{2}} = \frac{g_{W}^{2}}{M_{W}^{2}}. \quad (4.13)$$

Thus following the Eq. (4.12), we can write the effective coupling as follows

$$\frac{G_{F}^{\text{eff}}}{\sqrt{2}} = \frac{\bar{g}_{Z \mu e} g_{Z \nu \bar{\nu}}}{M_{Z}^{2}}. \quad (4.14)$$

Now the ratio of the two couplings become

$$\frac{G_{F}^{\text{eff}}}{G_F} = \frac{\bar{g}_{Z \mu e} g_{Z \nu \bar{\nu}}}{M_{Z}^{2}} \frac{1}{G_F} \quad (4.15)$$

So $r$ becomes

$$r = \left| \frac{G_{F}^{\text{eff}}}{G_F} \right|^2$$

$$= \frac{\Gamma (\mu \rightarrow 3e)_{Z - \text{exch}}}{\Gamma (\mu \rightarrow e\nu \bar{\nu})} \frac{1}{2} \frac{g_{Z \nu \bar{\nu}}^{2}}{g_{Z e^+e^-}^{2}} \quad (4.16)$$

As and using the value of the $g_{Z \nu \bar{\nu}}^{2}$ and $g_{Z e^+e^-}^{2}$ from the Standard Model, we get

$$r \sim 10^{-12}. \quad (4.17)$$
Thus, comparing with Eq. (4.12) which requires \( r > 1.6 \times 10^{-3} \), we learn that the LFV anomalous muon decays \( \mu \rightarrow e\nu \bar{\nu} \) cannot significantly contribute to the LSND DAR result, at least when we use \( g_{\mu^+e^-} \) as constrained by \( \mu \rightarrow 3e \) in a model independent way.

Thus the excess found by the LSND is not due to these Lepton Flavour Violating decays. So we cannot say that whether this excess is due to neutrino flavour oscillations or due to LFV decays which might occur through some exotic mechanism leading to \( \mu^+ \rightarrow e^+ \bar{\nu}_e \bar{\nu}_\mu \). Hence, we can say that it is still unresolved puzzle of the neutrino physics.

### 4.3 Seesaw model of neutrino masses

In the minimal standard gauge model of quarks and leptons, each of three known neutrinos (\( \nu_e, \nu_\mu, \nu_\tau \)) appears only as a member of left-handed \( SU(2) \) lepton doublet

\[
\psi_i = (\nu_i, l_i)_L
\]

and the Higgs sector contains only one scalar doublet

\[
\Phi = (\phi^+, \phi^0).
\]

As a result, neutrinos are massless in this model. Experimentally there is now a host of evidence for neutrino oscillations, and that is most naturally explained if neutrinos are massive and mix with each other. Theoretically there is no compelling reason for massless neutrinos, and any extension beyond the minimal Standard Model often allows them to massive. There exists
already a vast literature on specific models of neutrino masses and mixing [16].

Here we will make the following simple observations. In the minimal Standard Model of particle interactions, the massless neutrinos acquire naturally small Majorana masses through the effective dimension-five operator

\[ \frac{1}{\Lambda} (\nu_i \phi^0 - l_i \phi^+) (\nu_j \phi^0 - l_j \phi^+) , \] (4.20)

where \( \Lambda \) is a large effective mass scale, and \( \Phi = (\phi^+, \phi^0) \) is the usual Higgs doublet with the non-zero expectation value, \( \langle \phi^0 \rangle = v \). All models of neutrino mass and mixing (which have the same light particle content as at the minimal Standard Model) can be explained by the operator

\[ \Lambda^{-1} \phi^0 \phi^0 \nu_i \nu_j . \] (4.21)

Different models are merely different reasons for this operator. The intermediate heavy particle in case of the operator defined in Eq. (4.20) is clearly a fermion singlet. Let’s call it \( N_i \) and let its mass be \( m_N \) and its coupling to \( \nu_i \) be \( f_i \). Also we can identify \( f_i v \) or simply \( f v \) as a Dirac mass \( m_D \) linking \( \nu_i \) to \( N_i \) and the neutrino mass matrix is introduced so that

\[ m_\nu = \frac{m_D^2}{m_N} , \] (4.22)

so that \( \Lambda = m_N/f^2 \) in Eq. (4.20). This is, of course, just the well known canonical seesaw mechanism, with \( N_i \) identified as the right-handed neutrino with a large Majorana mass. Given that \( m_\nu \) is at most of order \( 1eV \) and \( f \) should be too small, the usual thinking is that \( m_N \) has to be very large, i.e., \( m_N \gg v \). As such, this famous mechanism must be accepted on faith, because there cannot be any direct experimental test of its validity.
Consider now the possibility that there is no new physics beyond the $TeV$ energy scale. This is an intriguing idea proposed recently in theories of large extra dimensions [17]. Instead of using an ingredient supplied by the large extra dimensions, a model has recently been proposed [19] to show how Eq. (4.22) may be realized naturally with $m_N$ of order $1\, TeV$ in a simple extension of the Standard Model. This means that $m_D$ should be small i.e., $m_D \ll 10^2 GeV$. If it comes from zeroth component of $\Phi$ i.e., $\phi^0$ as in the Standard Model, that would not be natural; but as shown below, it will come instead from another with naturally small vacuum expectation value. This new realization of the seesaw mechanism will allow direct experimental test of its validity, as discussed below.

Consider the minimal Standard Model with three lepton families:

$$\begin{pmatrix} \nu_i \\ l_i \end{pmatrix}_L \sim (1, 2, -1/2), \quad l_{iR} \sim (1, 1, -1),$$

where their transformations under the standard $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group are indicated above. Now add three neutral fermion singlets

$$N_{iR} \sim (1, 1, 0),$$

but instead of assigning them the lepton number $L = 1$, so that they can pair up with the lepton doublet through the interaction $\bar{N}_R (\nu_L\phi^0 - l_L\phi^+)$, $L = 0$ is assigned to forbid this Yukawa term. To complete the model, a new scalar doublet, called leptoquark [18]

$$\begin{pmatrix} \eta^+ \\ \eta^0 \end{pmatrix} \sim (1, 2, 1/2)$$

is introduced with lepton number $L = -1$. Hence the terms

$$\frac{1}{2} M_i N_{iR}^2 + f_{ij} \bar{N}_{iR} (\nu_{jL}\eta^0 - l_{jL}\eta^+) + H.c.$$

(4.26)
appear in the Lagrangian. The effective operator of Eq. (4.20) for neutrino mass is then replaced by one with $\eta$ instead of $\phi$, and, if $\langle \eta^0 \rangle = u$ is naturally small, the corresponding scale $\Lambda$ will not have to be so large and $M_i$ of Eq. (4.26) may indeed be of order 1 TeV.

The Higgs potential of this model is given by

$$V = m_1^2 \Phi^\dagger \Phi + m_2^2 \eta^\dagger \eta + \frac{1}{2} \lambda_1 (\Phi^\dagger \Phi)^2 + \frac{1}{2} \lambda_2 (\eta^\dagger \eta)^2 + \lambda_3 (\Phi^\dagger \Phi) (\eta^\dagger \eta) + \lambda_4 (\Phi^\dagger \eta) (\eta^\dagger \Phi) + \mu_{12}^2 (\Phi^\dagger \eta + \eta^\dagger \Phi),$$

where the $\mu_{12}^2$ term breaks $L$ explicitly but softly. Note that, given the particle content of this model, the $\mu_{12}^2$ term is the only possible soft term which also breaks $L$.

The equations of constraints for $\langle \phi^0 \rangle = v$ and $\langle \eta^0 \rangle = u$ are

$$v \left[ m_1^2 + \lambda_1 v^2 + (\lambda_3 + \lambda_4) u^2 \right] + \mu_{12}^2 u = 0 \quad (4.28)$$

$$u \left[ m_2^2 + \lambda_2 u^2 + (\lambda_3 + \lambda_4) v^2 \right] + \mu_{12}^2 v = 0. \quad (4.29)$$

Consider the case

$$m_1^2 < 0, \quad m_2^2 > 0, \quad |\mu_{12}^2| \ll m_2^2, \quad (4.30)$$

then

$$v^2 \simeq - \frac{m_1^2}{\lambda_1}, \quad u \simeq - \frac{\mu_{12}^2 v}{m_2^2 + (\lambda_3 + \lambda_4) v^2}. \quad (4.31)$$

Hence it is very clear that $u$ may be very small compared to $v (= 174 \text{ GeV})$. For example, if $m_2 \sim 1 \text{ TeV}$, $|\mu_{12}^2| \sim 10 \text{ GeV}^2$, then $u \sim 1 \text{ MeV}$. The relative smallness of $|\mu_{12}^2|$ may be attributed to the fact that it corresponds to the explicit breaking of the lepton number in $V$ of Eq. (4.27). The usual
argument here is that, if $|\mu^2_{12}|$ were zero, then as seen from Eq. (4.31) the model’s symmetry is increased, i.e., the lepton number would not be broken. Hence the assumption that it is small compared to $|m^2_1|$ or $m^2_2$ is “natural.” One thing is very clear from here that if $|\mu^2_{12}|$ were much smaller, then neutrino masses would be too small to account for the present observation of the neutrino oscillations.

The $6 \times 6$ mass matrix spanning $[\nu_e, \nu_\mu, \nu_\tau, N_1, N_2, N_3]$ is now given by

$$
\mathcal{M}_\nu = \begin{pmatrix}
0 & 0 & 0 & f_{e1}u & f_{e2}u & f_{e3}u \\
0 & 0 & 0 & f_{\mu1}u & f_{\mu2}u & f_{\mu3}u \\
0 & 0 & 0 & f_{\tau1}u & f_{\tau2}u & f_{\tau3}u \\
f_{e1}u & f_{\mu1}u & f_{\tau1}u & M_1 & 0 & 0 \\
f_{e2}u & f_{\mu2}u & f_{\tau2}u & 0 & M_2 & 0 \\
f_{e3}u & f_{\mu3}u & f_{\tau3}u & 0 & 0 & M_3
\end{pmatrix}.
$$

(4.32)

The mixing between $\nu$ and $N$ is thus of the order $fu/M$, which will allow the physical $N$ to decay through its small component of $\nu$ to $l^\pm W^\mp$. The effective mass matrix spanning the light neutrino is then

$$
\mathcal{M}_{ij} = \sum_k \frac{f_{ik}f_{jk}u^2}{M_k}.
$$

(4.33)

and is of order 1 $eV$ if $f$ is of order unity.

There are five physical Higgs boson with masses given by

$$
m^2_{h^\pm} = m^2_2 + \lambda_3 v^2 + (\lambda_2 - \lambda_4) u^2 - \mu^2_{12} u/v,
$$

(4.34)

$$
m^2_A = m^2_2 + (\lambda_3 + \lambda_4) v^2 + \lambda_2 u^2 - \mu^2_{12} u/v,
$$

(4.35)

$$
m^2_{h_1} = 2\lambda_1 v^2 + \mathcal{O}(u^2),
$$

(4.36)

$$
m^2_{h_2} = m^2_2 + (\lambda_3 + \lambda_4) v^2 + \mathcal{O}(u^2).
$$

(4.37)
The \( m_{h_1^2} \) behaves very much like the ordinary Higgs boson. The new scalar particles of this model, i.e. \( h^\pm, A, \) and \( h_2^0 \) (all with mass\( \sim m_2 \)), as well as \( N_{iR} \), are now accessible to direct experimental discovery in future accelerators.

In summary, a new seesaw model of neutrino mass is proposed, where a second scalar doublet \( (\eta^+, \eta^0) \) with lepton number \( L = -1 \) is added to the minimal Standard Model together with three neutral right-handed fermion singlets \( N_i \) with lepton number \( L = 0 \). Thus \( N_i \) is allowed to have a Majorana mass \( m_N \) as well as interaction \( f_{ij} \bar{N}_{iR} (\nu_j L \eta^0 - l_j L \eta^+) \). Hence \( m_\nu \) is proportional to \( \langle \eta^0 \rangle^2 / m_i \) and, if \( \langle \eta^0 \rangle \ll \langle \phi^0 \rangle \), \( m_N \) may be of the order 1 TeV and be observable experimentally. This is accomplished with the Higgs potential of Eq. (4.27), where \( L \) is broken explicitly and uniquely with the soft term \( \Phi^\dagger \eta + \eta^\dagger \Phi \).

As the Lepton Flavor Violation is discussed in all models of neutrino mass. It is argued that in this model, there is no LFV at tree level for charged leptons. However, it does occur in one loop through \( \eta \) and \( N \) exchange [19].

The aim of this work is to test this model for the lepton flavor violating decays, i.e. \( \mu \rightarrow 3e \) and \( \mu^+ \rightarrow e^+ \nu\bar{\nu} \) in one loop process. We have to calculate the branching ratio of \( \mu \rightarrow 3e \) and compare it to its experimental bounds.

### 4.3.1 Basic consideration and calculation

First of all we will discuss the reaction \( \mu \rightarrow 3e \). The corresponding box diagrams can be shown in fig. 4.3.1. The couplings at each vertex can be taken from Eq. (4.32). Instead of \( \gamma \) matrices we have unity at each vertex,
because each Higgs boson is a scalar.

Using Feynman rules, the amplitude can be written as follows [20]:

\[
 iT (\mu \rightarrow 3e) = 2 \sum_i (f^*_{\mu i} f_{ei} f^*_{ei}) \int \frac{d^4k}{(2\pi)^4} \bar{v}(p_1) \left[ \frac{i}{k - m_i} \right] v(p_2) \\
\times \bar{u}(p_3) \left[ \frac{-i}{k^2 - m_i^2} \right] v(p_4) \\
= 2 \sum_i (f^*_{\mu i} f_{ei} f^*_{ei}) \int \frac{d^4k}{(2\pi)^4} \bar{v}(p_1) \left[ k + m_i \right] v(p_2) \\
\times \bar{u}(p_3) \left[ k + m_i \right] v(p_4) \\
= \frac{2}{(2\pi)^4} \sum_i (f^*_{\mu i} f_{ei} f^*_{ei}) \left\{ \bar{v}(p_1) \gamma^\alpha v(p_2) \bar{u}(p_3) \gamma^\beta v(p_4) I^\alpha\beta \right\} \\
+ m_i \left\{ \bar{v}(p_1) \gamma^\alpha v(p_2) \bar{u}(p_3) v(p_4) I^\alpha + \bar{v}(p_1) \gamma^\beta v(p_2) \bar{u}(p_3) \gamma^\beta v(p_4) \right\} I^\beta \\
+ m_i^2 \left\{ \bar{v}(p_1) v(p_2) \bar{u}(p_3) v(p_4) I \right\},
\]

where

\[
 I^\alpha = \int d^4k \frac{k^\alpha}{(k^2 - m_h^2)^2} (k^2 - m_i^2)^2 (4.39) \\
 I^\beta = \int d^4k \frac{k^\beta}{(k^2 - m_h^2)^2} (k^2 - m_i^2)^2 (4.40) \\
 I = \int d^4k \frac{1}{(k^2 - m_h^2)^2} (k^2 - m_i^2)^2 (4.41)
\]

It is assumed that the loop momenta is very high, i.e. $k \rightarrow \infty$. So there is no change in momenta at the vertices, and also the dominating integral is the first one. Before calculating the $|T|^2$, we have to do the loop integration. To do this we use the technique known as Feynman parameterization [21].

\[
 \frac{1}{a^2b^2} = 6 \int_0^1 dz \frac{z(1-z)}{[b + (a-b)z]^4}.
\]
For our above mentioned case \( a \) and \( b \) are

\[
\begin{align*}
a &= k^2 - m_i^2 \\
b &= k^2 - m_h^2.
\end{align*}
\tag{4.43}
\]

So

\[
[b + (a - b)z] = k^2 - m_h^2 + (m_h^2 - m_i^2)z \\
= k^2 + m_h^2 [(1 - x_i)z - 1] \\
= k^2 + s,
\tag{4.44}
\]

where \( x_i = \left(\frac{m_i^2}{m_h^2}\right) \) and \( s = m_h^2 [(1 - x_i)z - 1] \). Then, Eq. (4.39) becomes

\[
I^{\alpha\beta} = 6 \int_0^1 dz \times z (1 - z) \int d^4k \frac{k^\alpha k^\beta}{[k^2 + s]^4} \\
= 6 \int_0^1 dz \times z (1 - z) \times \left[ \frac{i\pi^2 \Gamma (1) \times g^{\alpha\beta}}{2\Gamma (4) \times s} \right] \\
= 6 \int_0^1 dz \times z (1 - z) \times \left[ \frac{i\pi^2 g^{\alpha\beta}}{2 \times 6m_h^2 [(1 - x_i)z - 1]} \right],
\tag{4.45}
\]

because

\[
\int d^4k \frac{k^\alpha k^\beta}{[k^2 + s + i\epsilon]^n} = \frac{i\pi^2 \Gamma (n - 3)}{2\Gamma (n)} \times \frac{g^{\alpha\beta}}{s^{n-3}}, \quad n \geq 3.
\]

Therefore, after solving the \( z \) integration Eq. (4.45) takes the form

\[
I^{\alpha\beta} = \frac{i\pi^2 g^{\alpha\beta}}{2m_h^2} \times \left[ \frac{1 - x_i^2 + x_i \ln (x_i^2)}{2 (x_i - 1)^3} \right].
\tag{4.46}
\]

Using the same technique for Eq. (4.40) becomes

\[
I^{\alpha} = \int d^4k \frac{k^\alpha}{(k^2 - m_h^2)^2 (k^2 - m_i^2)^2} \\
= \int d^4k \frac{k^\alpha}{(k^2 - s)^2} = 0,
\tag{4.47}
\]

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because
\[ \int d^4k \frac{k^\alpha}{[k^2 + s + i\epsilon]^n} = 0 \quad n \geq 3. \]

Also define
\[ f_{\mu i}^* f_{ei}^* f_{ei} f_{\mu i} = \xi_i. \]  
(4.48)

Hence finally Eq. (4.38) becomes
\[ T(\mu \rightarrow 3e) = \frac{\pi^2}{(2\pi)^4 m_H^2} \times \sum_i \xi_i \left\{ \bar{v}(p_1) \gamma^\alpha v(p_2) \bar{u}(p_3) \gamma_\alpha v(p_4) \right\} \]
\[ \times \left\{ 1 - x_i^2 + x_i \ln \left( x_i^2 \right) \right\} \]
\[ \times \left\{ 1 - x_j^2 + x_j \ln \left( x_j^2 \right) \right\} \]
\[ \times 2 \left[ (p_1 p_4) (p_2 p_3) + (p_1 p_3) (p_2 p_4) \right]. \]  
(4.49)

Now the square of the amplitude becomes
\[ \sum_{spins} |T|^2 = \left( \frac{\pi^2}{(2\pi)^4 m_H^2} \right)^2 \sum_{i,j} \xi_i \xi_j^* \left\{ \frac{1 - x_i^2 + x_i \ln \left( x_i^2 \right)}{2 (x_i - 1)^3} \right\} \]
\[ \times \left\{ 1 - x_j^2 + x_j \ln \left( x_j^2 \right) \right\} \]
\[ \times 2 \left[ (p_1 p_4) (p_2 p_3) + (p_1 p_3) (p_2 p_4) \right]. \]

Define
\[ A(x_i) = \left\{ \frac{1 - x_i^2 + x_i \ln \left( x_i^2 \right)}{2 (x_i - 1)^3} \right\}. \]

Then the above result takes the form
\[ \sum_{spins} |T|^2 = \left[ \frac{\pi^2}{(2\pi)^4 m_H^2} \right]^2 \sum_{i,j} \xi_i \xi_j^* A(x_i) A(x_j) \times \left[ (p_1 p_4) (p_2 p_3) + (p_1 p_3) (p_2 p_4) \right]. \]  
(4.50)

The corresponding decay width becomes
\[ \Gamma[\mu \rightarrow 3e] = \left[ \frac{\pi^2}{(2\pi)^4 m_H^2} \right]^2 \sum_{i,j} \xi_i \xi_j^* A(x_i) A(x_j) \times \frac{m_\mu^5}{192 \times \pi^3}, \]  
(4.51)
and the branching ratio becomes

\[
B[\mu \to 3e] = \frac{\Gamma[\mu \to 3e]}{\Gamma[\mu \to e\nu\bar{\nu}]} = \left[ \frac{1}{(2\pi)^42m_h^4} \sum_{i,j} \xi_i^* \xi_j A(x_i)A(x_j) \right] \frac{M_W^6}{\Gamma^2[W \to e\nu] \times 2^5}.
\]

(4.52)

Now let’s calculate the other reaction, i.e. \( \mu^+ \to e^+\nu_l\bar{\nu}_l \).

The corresponding box diagrams can be shown in fig. 4.3.1. Using Feynman rules, the amplitude can be written as follows

\[
iT(\mu^+ \to e^+\nu_l\bar{\nu}_l) = 2 \sum_i \left( f_{\mu l}^* f_{e l}^* f_{\nu l}^* f_{\bar{\nu} l} \right) \int \frac{d^4k}{(2\pi)^4} \bar{v}(p_1) \left[ \frac{i}{k - m_i} \right] \times v(p_2) \bar{u}(p_3) \left[ \frac{i}{k - m_i} \right] v(p_4) \left[ \frac{-i}{k^2 - m_{h^+}^2} \right] \left[ \frac{-i}{k^2 - m_{b^2}^2} \right] = 2 \sum_i \left( f_{\mu l}^* f_{e l}^* f_{\nu l}^* f_{\bar{\nu} l} \right) \int \frac{d^4k}{(2\pi)^4} \bar{v}(p_1) \left[ k + m_i \right] v(p_2) \times \bar{u}(p_3) \left[ k + m_i \right] v(p_4) \frac{1}{k^2 - m_{h^+}^2} \left( \frac{1}{k^2 - m_i^2} \right)^2 \frac{-i}{k^2 - m_{b^2}^2}.
\]

(4.53)

Using the same assumption here that the loop moment is very high, i.e. \( k \to \infty \). The above integral can be solved by using Feynman parameterization [22]

\[
\frac{1}{a^2bc} = 6 \int_0^1 dx \int_0^x dy \frac{(1 - x)}{[a + (b - a) x + (c - b) y]^4}
\]

Therefore

\[
\Gamma^{\alpha\beta} = \int d^4k \frac{k^\alpha k^\beta}{(k^2 - m_i^2)^2 (k^2 - m_{h^+}^2) (k^2 - m_{b^2}^2)} = \frac{-i\pi^2}{4m_i^2} A(x_{1i}, x_{2i}) g^{\alpha\beta},
\]

(4.54)
where
\[ x_{1i} = \frac{m_{h_{1i}}^2}{m_i^2}, \quad x_{2i} = \frac{m_{h_{2i}}^2}{m_i^2}, \]
and
\[ A(x_{1i}, x_{2i}) = \frac{J(x_{1i}) - J(x_{2i})}{x_{1i} - x_{2i}} \]
\[ J(x_{1i}) = \frac{1}{1 - x_{1i}} + \frac{x_{1i} \ln x_{1i}}{(1 - x_{1i})^2} \] (4.55)

Define
\[ f_{ii}^* f_{ii}^* f_{ii} = \xi_i. \] (4.56)

Using these results Eq. (4.53) takes the form
\[ iT \left( \mu^+ \rightarrow e^+ \nu \bar{\nu} \right) = \left[ \frac{-i\pi^2}{2m_i^2 (2\pi)^4} \right] \sum_i \xi_i A(x_{1i}, x_{2i}) \times \{ \bar{v}(p_1) \gamma^\alpha v(p_2) \bar{u}(p_3) \gamma^\alpha v(p_4) \}. \] (4.57)

Now, after calculating \( \sum_{spins} |T|^2 \), the decay width becomes
\[ \Gamma \left[ \mu^+ \rightarrow e^+ \nu \bar{\nu} \right] = \left[ \frac{\pi^2}{2m_i^2 (2\pi)^4} \right] \sum_{i,j} \xi_i \xi_j A(x_{1i}, x_{2i}) A(x_{1j}, x_{2j}) \frac{m_{\mu}^5}{192 \times \pi^3} \]
\[ = \left[ \frac{1}{32m_i^2 \pi^2} \right] \sum_{i,j} \xi_i \xi_j A(x_{1i}, x_{2i}) A(x_{1j}, x_{2j}) \frac{m_{\mu}^5}{192 \times \pi^3}. \] (4.58)

Then the branching ratio becomes
\[ B \left[ \mu^+ \rightarrow e^+ \nu \bar{\nu} \right] = \left[ \frac{1}{32m_i^2 \pi^2} \right] \sum_{i,j} (\xi_i \xi_j A(x_{1i}, x_{2i}) A(x_{1j}, x_{2j})) \frac{M_W^6}{2 \times \Gamma^2 [W \rightarrow e\nu]}. \] (4.59)

In order to calculate the numerical value of the branching ratio for \( \mu \rightarrow 3e \), we will fix the value of \( m_i = 1 \text{ TeV} \) and also assume that all the couplings...
are of the order unity. As $m_h^2 \sim m_2^2$, therefore if $m_2 \gg 100$ GeV its effect on
the radiative parameters
\[
\Delta S = \frac{1}{24\pi} \frac{\lambda_4 v^2}{m_2^2},
\]
\[
\Delta T = \frac{1}{96\pi} \frac{1}{s^2 c^2 M_Z^2 m_2^2} \frac{\lambda_1^2 v^4}{m_2^2},
\]
is negligible small and will not change the excellent experimental fit of the
minimal Standard Model.

To have a feeling about the branching ratios let us take $m_h = 900$ GeV,
the branching ratio for $\mu \rightarrow 3e$ becomes
\[
\mathcal{B} = 3.6 \times 10^{-6}
\]
which is much higher then the experimental bound on the branching ratio of
$\mu \rightarrow 3e$, i.e. $\mathcal{B}(\mu \rightarrow 3e)_{\text{V-exch.}} \leq 10^{-12}$. For the mass smaller than 900 GeV
the branching ratio grows higher and higher. Thus from here it is obvious
that $m_h > 1$ TeV. By varying the Higgs mass the corresponding branching
ratio becomes
\[
m_h = 10\text{TeV} \quad \mathcal{B} = 1.0 \times 10^{-8}
\]
\[
m_h = 30\text{TeV} \quad \mathcal{B} = 3.1 \times 10^{-11}.
\]
It can be easily seen that with the increase of Higgs mass, the branching ratio
decreases. As the Higgs mass reaches to 50 TeV, branching ratio becomes
\[
\mathcal{B} = 3.6 \times 10^{-12}
\]
which is comparable to the experimental value. Thus we have concluded
that the Higgs mass is greater than 50 TeV to explain LFV decays in Seesaw
model of neutrino masses [19].

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Conclusion

The aim of this work has been to understand the physics beyond the Standard Model by considering Lepton Flavour Violating (LFV) decay processes. Using the experimental bounds on the three body LFV decays, we have calculated the bounds on the corresponding two body decays. The Dynamical suppression of three body LFV decays due to momentum dependent couplings have also been discussed in detail.

The experimental bounds on the three body LFV decay, $\mu \rightarrow 3e$ constrained the coupling $\tilde{g}Z_{\mu e}$, which is helpful to calculate the anomalous muon decays. The anomalous muon decay $\mu \rightarrow e\nu\bar{\nu}$ could not significantly contribute to the LSND DAR result. This has been concluded by comparing the effective coupling of anomalous muon decay with $r > 1.6 \times 10^{-3}$ [4]. LFV decays at a loop order have also been discussed in Seesaw model of neutrino masses which involve right handed singlet neutrinos of mass $m_N$ of order TeV [19]. The main purpose of the model is that there is no new physics beyond 1 TeV. It has been shown that even if one keeps $m_N$ at 1 TeV the present bounds on $\mu \rightarrow 3e$ requires that new Higgs bosons, necessary in this model should have their mass larger than 50 TeV. Thus it is necessary to have a
new physics beyond the TeV energy scale defeating the original motivation of the model.
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Figure 4.1: Anomalous muon decay

Figure 4.2: Box diagram for $\mu \rightarrow 3e$

Figure 4.3: Box diagram for anomalous muon decay
Figure 4.3
