ON THE TAUTOLOGICAL RING OF $\mathcal{M}_g$

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Abstract. We prove that any product of tautological classes of $\mathcal{M}_g$ of degree $d$ (in the Chow ring of $\mathcal{M}_g$ with rational coefficients) vanishes for $d > g - 2$ and is proportional to the class of the hyperelliptic locus in degree $g - 2$.

1. Results

Fix an integer $g \geq 2$ and denote by $C^n_g$ the moduli space of tuples $(C, x_1, \ldots, x_n)$, where $C$ is a smooth connected projective curve of genus $g$ and $x_1, \ldots, x_n$ are (not necessarily distinct) points of $C$; we also write $\mathcal{M}_g$ for $C^0_g$. Forgetting the $i$th point defines a morphism $C^n_g \to C^{n-1}_g$ whose relatively dualizing sheaf is denoted by $\omega_i$ ($i = 1, \ldots, n$). We write $K_i$ for the first Chern class of $\omega_i$, considered as an element of the Chow group $A^1(C^n_g)$ (with rational coefficients); for $n = 1$ we also write $K$.

Our main result is:

(1.1) Theorem. Any monomial of degree $d$ in the classes $K_1, \ldots, K_n$ is a linear combination of the classes of the irreducible components of the locus parametrizing tuples $(C, x_1, \ldots, x_n)$ admitting a morphism $C \to \mathbb{P}^1$ of degree $\leq g + n$ such that the fiber over $0$ (resp. $\infty$) has at most $g + n - d - 1$ points (resp. is a singleton) and $\{x_1, \ldots, x_n\}$ is contained in one of these two fibers. (Hence such a class is zero when $d > g + n - 2$.) All monomials of degree $g + n - 2$ are proportional to the class of the locus parametrizing tuples $(C, x_1, \ldots, x_n)$ with $C$ hyperelliptic and $x_1 = \cdots = x_n$ a Weierstraß point.

The direct image of $K^{d+1}$ in $A^d(\mathcal{M}_g)$ is the Mumford–Morita–Miller tautological class $\kappa_d$. Mumford showed in his fundamental paper [4] that the subring of $A^\bullet(\mathcal{M}_g)$ generated by these classes (the tautological ring of $\mathcal{M}_g$) is already generated by $\kappa_1, \ldots, \kappa_{g-2}$. On the basis of many calculations Carel Faber has made the intriguing conjecture that this ring has the formal properties of the even-dimensional cohomology ring of a projective manifold of dimension $g - 2$, i.e., satisfies Poincaré duality and a Lefschetz decomposition. We offer the following support for this conjecture:

(1.2) Theorem. Any product of tautological classes that has degree $d$ is a linear combination of the classes of the irreducible components of the locus parametrizing curves $C$ admitting a morphism $C \to \mathbb{P}^1$ of degree $\leq 2g - 2$ totally ramified over $\infty$ and with at most $g - 1 - d$ points over $0$ (hence is zero when $d > g - 2$). All such classes of degree $g - 2$ are proportional to the class of the hyperelliptic locus.

A finer analysis of our proof may well yield that $\kappa_1^{g-2}$ is a nonzero multiple of the hyperelliptic class, but it is not known whether the latter is actually nonzero.
The proof of the theorems uses the flag of subvarieties of \( \mathcal{M}_g \) introduced by Arbarello [1], a variant of which was exploited by Diaz [2] to prove that \( \mathcal{M}_g \) has no complete subvarieties of dimension \( g - 2 \). Our simple key result (2.4) serves as a substitute for Diaz’s lemma 2 in [2] and can be used in that paper to eliminate the use of compactifications of Hurwitz schemes (see (2.8)). The proof of the second assertion of each theorem involves an application of the Fourier transform for abelian varieties, due to Mukai and Beauville.

In this paper we only consider Chow groups with respect to rational equivalence, tensorized with \( \mathbb{Q} \), and graded by codimension, notation: \( A^* \). If \( X \) is a variety that is smooth, or more generally, that admits a smooth Galois covering, then there is an intersection product \( A^k(X) \otimes A^l(X) \rightarrow A^{k+l}(X) \).

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2. Proofs

(2.1) Let \( C \) be a smooth projective curve of genus \( g \) and let \( D_0 \) and \( D_\infty \) be positive divisors on \( C \) that are linearly equivalent, but whose supports are disjoint. Then there is a finite morphism \( \pi : C \rightarrow \mathbb{P}^1 \) such that \( \pi^*(i) = D_i \) \( (i = 0, \infty) \). If \( p \in C \) occurs in \( D_i \) with multiplicity \( m_p > 0 \), then \( \pi \) determines an isomorphism of \( \mathbb{C} \cong T^*_i \mathbb{P}^1 \) onto \( T^*_p \mathbb{C}^\otimes m_p \). However, \( \pi \) is not unique for it is defined up to natural action of \( \mathbb{C}^\times \) on \( \mathbb{P}^1 \). That ambiguity can be eliminated as follows.

Let \( R \) denote the part of the ramification divisor of \( \pi \) that lies over \( \mathbb{P}^1 - \{0, \infty\} \). If \( c \) denotes the number of points of \( \text{supp}(D_0 + D_\infty) \), then the Riemann-Hurwitz formula implies that the degree \( r \) of \( R \) is equal to \( 2g - 2 + c \). If \( \pi_*(R) = \sum \pi_i(z_i) \), then \( \pi \) can be normalized in such a way that \( \prod \pi_i(z_i) = 1 \). This normalization is unique up to multiplication by an \( r \)th root of unity. So for \( p \) and \( m_p \) as above, and \( \pi \) normalized, the corresponding generator of \( T^*_p \mathbb{C}^\otimes m_p \) raised to the \( r \)th power gives a canonical generator of \( T^*_p \mathbb{C}^\otimes m_p \).

This argument works just as well in families and so we obtain:

(2.2) Proposition. Let \( f : C \rightarrow S \) be a projective family of smooth genus \( g \) curves with reduced base. Let \( D_0 \) and \( D_\infty \) be positive relative divisors on \( C \) whose supports are disjoint and are étale over \( S \). Suppose that their difference is linearly equivalent to the pull-back of a divisor on \( S \). Then for every section \( x : S \rightarrow C \) of \( f \) with image in the support of \( D_0 + D_\infty \), a suitable positive tensor power of \( x^*\omega_{C/S} \) is trivial.

We shall use the following simple fact:

(2.3) Lemma. Let \( L_1, \ldots, L_d \) be line bundles on a variety \( V \) and let \( V = V^0 \supset V^1 \supset \cdots \supset V^d \) be a chain of closed subvarieties such that \( L_k \) is trivial on \( V^{k-1} \rightarrow V^k \). Then \( c_1(L_1) \cdots c_1(L_d) \) has support in \( V^d \).

The key result we need is:

(2.4) Lemma. Let \( d \) be a positive integer and let \( \{(C_t, x_t, P_t)\}_{t \in \Delta} \) be an analytic family of triples consisting of a smooth connected projective curve \( C_t \), a point \( x_t \in C_t \), and a pencil \( P_t \) on \( C_t \) containing \( d(x_t) \). Assume that for \( t \neq 0 \), \( P_t \) has no base points and let \( R_t \) be the part of the ramification divisor on \( C_t - x_t \) of the associated morphism \( C_t \rightarrow P_t \). If \( R_0 \) is the limit of \( R_t \) for \( t \rightarrow 0 \), then the multiplicity of \( x_0 \) in \( R_0 \) is also the multiplicity of \( x_0 \) as a fixed point of \( P_0 \).
Proof. Represent the family by a smooth analytic morphism \( t : C \to \Delta \) with section \( x : \Delta \to C \). Extend \( t \) to a chart \((z, t)\) at \( x_0 \) such that \( z = 0 \) is the image of \( x \) at \( x_0 \). In terms of these coordinates generators of \( P_t \) can be represented by \( z^d \) and a holomorphic function \( A(z, t) = \sum_{i \neq d} a_i(t)z^i \) which is divisible neither by \( t \) nor by \( z \). The first index \( k \) for which \( a_k(0) \neq 0 \) is \( < d \) and is equal to the multiplicity of \( x_0 \) as fixed point of \( P_0 \). In the domain of the chart, the divisor \( R_t \) is given by locus where the \( z \)-derivatives of \( A \) and \( z^d \) are proportional, i.e., by the divisor of \( \sum_{i \neq d}(i - d)a_i(t)z^i \) \( (t \neq 0) \). This expression is not divisible by \( z \) or \( t \) so that \( R_0 \) is given by \( \sum_{i \neq d}(i - d) a_i(0)z^i \). So \( x_0 \) occurs with multiplicity \( k \) in \( R_0 \).

An immediate consequence is an amplification of a result due to Arbarello [1] and Diaz [2]:

**(2.5) Corollary.** Suppose that in the situation of (2.4) there exists an analytic section \( \{D_t \in P_t \}_{t \in \Delta} \) such that for \( t \neq 0 \), \( \text{supp}(D_t) \) is disjoint with \( x_t \) and has \( d - r \) points, whereas \( D_0 = d(x_0) \). Then \( P_0 \) can be written as \( r(x_0) + P' \).

(2.6) If \( d \) is positive integer, then we have moduli space \( P(d) \) of triples \((C, x, P)\) with \( C \) a smooth projective curve of genus \( g \), \( x \in C \) and \( P \) a pencil on \( C \) containing \((d)x\). The existence of this is clear if \( d > 2g - 2 \), for then this is just a bundle of projective spaces of dimension \( d - g - 1 \) over \( C_g \); the remaining cases \( d \leq 2g - 2 \) follows from this by simply viewing \( P(d) \) as the locus in \( P(2g - 1) \) parametrizing triples \((C, x, P)\) for which \( x \) is a fixed point in \( P \) of multiplicity \( 2g - 1 - d \). This implies that we also have defined a moduli space \( Z \) of tuples \((C, x_1, \ldots, x_n, x, D, P)\) with \( C \) a smooth projective curve of genus \( g \), \( x_1, \ldots, x_n, x \in C \), \( P \) a pencil on \( C \) containing \((n + g)x\), \( D \) a degenerate member of \( P \) and \( \{x_1, \ldots, x_n\} \subset \text{supp}(D) \). Notice that \( D \) and \( x \) determine \( P \) unless \( D = (n + g)(x) \). The forgetful morphism \( f : Z \to C_g^n \) is clearly proper.

The tuples for which \( \text{supp}(D) \) has at most \( g + n - 1 - k \) points outside \( x \) define a closed subvariety \( Z^k \) of \( Z \). It is clear that \( Z^{n+g-1} \) can be identified with the set of tuples \((C, x, \ldots, x, (n + g)x, P)\) with \( P \) a pencil through \((n + g)(x)\).

(2.7) Lemma. For \( k < g + n - 1 \), \( Z^k - Z^{k+1} \) is Zariski-open in an affine variety of pure dimension \( 3g - 3 + n - k \) and \( f^*K_x|Z^k - Z^{k+1} = 0 \) \((i = 1, \ldots, n)\).

Proof. Let \( k < g + n - 1 \) and let \( W \) be a connected component of \( Z^k - Z^{k+1} \). If \((C, x_1, \ldots, x_n, x, D, P)\) represents an element of \( W \), then write \( D = m(x) + D' \) with \( x \notin \text{supp}(D') \) so that \( \text{supp}(D') \) has exactly \( n + g - k \) points. There is a finite morphism \( \pi : C \to \mathbb{P}^1 \) with \( \pi^*(0) = D' \) and \( \pi^*(\infty) = (g + n - m)(x) \). The part of the ramification divisor \( R \) of \( \pi \) over \( \mathbb{P}^1 \) \(-\{0, \infty\}\) has by Riemann-Hurwitz degree \( 2g - 2 + (g + n - k) = 3g - 2 + n - k \).

The multiplicity \( m \), the multiplicity of \( x_i \) in \( D \), and the stratum of the diagonal stratification of \( C_g^{n+1} \) containing \((C, x_1, \ldots, x_n, x)\) only depend on \( W \). So assigning to \((C, x_1, \ldots, x_n, x, P)\) the \( C^* \)-orbit of \( \pi_R \) defines a flat, quasi-finite morphism from \( W \) to the quotient of a \((3g - 3 + n - k)\)-dimensional torus by an action of the symmetric group. So \( W \) is pure of dimension \( 3g - 3 + n - k \). Proposition (2.2) implies that \( f^*K_x|W \) is trivial.

Proof of the first clause of (1.1). Let \( X^k \) be the union of irreducible components of \( Z^k \) that are distinct from \( Z^{n+g-1} \). It can be shown that \( Z^{n+g-1} \) is actually an irreducible component of \( Z^k - Z^{k+1} \).

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irreducible component of $Z$ and so $X^0 \neq Z$.) The restriction $f : X^0 \to C^n_g$ is clearly proper. It is also surjective, because for given $(C, x_1, \ldots, x_n)$, the morphism

$$(y, y_1, \ldots, y_{g-1}) \in C^g \mapsto \left[ -(n+g)y + 2(x_1) + \sum_{i=2}^{n} (x_i) + \sum_{j=1}^{g-1} (y_j) \right] \in J(C)$$

is onto. Observe that $X^{n+g-1} = \emptyset$.

We claim that $f(X^k \cap Z^{n+g-1}) \subset f(X^{k+1})$. For if $(C, x_1, \ldots, x_n, (n+g)x, P)$ represents an element of $X^k \cap Z^{n+g-1}$, then by (2.5), $P$ will be of the form $(k+1)x + P'$ with $P'$ a pencil of degree $n + g - k - 1$. So $P$ has a member $\neq (n+g)(x)$ with at most $n + g - k - 2$ points.

(2.8) Since $f(X_k) - f(X_{k+1})$ admits a finite covering that is Zariski-open in an affine variety, it cannot contain a complete curve. From this we recover Diaz’s theorem which asserts that $C^n_g$ does not contain a complete subvariety of dimension $> g + n - 2$.

In order to complete the proof of (1.1) we need two more results, one algebraic, one topological.

(2.9) Lemma. Let $f : A \to S$ be a family of abelian varieties of dimension $g$ and let $d$ be a positive integer. Then the class of the locus $A(d)$ of points of order $d$ is a positive multiple of the class of the zero section in $A^g(A)$. (The coefficient is the number of elements in $(\mathbf{Z}/d)^{2g}$ of order $d$.)

Proof. We use the Fourier transform for abelian varieties introduced by Mukai, developed by Beauville and extended to abelian schemes by Deninger–Murre [2]. Mukai’s transform gives an (in general inhomogeneous) isomorphism $F : A(A) \to \tilde{A}(\tilde{A})$, where $\tilde{A} \to S$ is the dual family. We shall compare the images of the two classes in $A(\tilde{A})$ under $F$.

Let $k$ be an positive integer relative prime to $d$. Multiplication by $k$ in $A$ maps $A(d)$ isomorphically onto itself. So the class of $A(d)$ in $A^g(A)$ is fixed under $k_*$. Lemma (2.18) of [2] implies that then $F([A(d)]) \in A^g(\tilde{A})$. Since the projection induces an isomorphism $A^0(\tilde{A}) \to A^0(\tilde{A})$, the lemma follows.

(2.10) Lemma. Let $\pi : C \to \mathbb{P}^1$ be a covering of degree $d$ by a smooth connected curve that is totally ramified over $0$ and $\infty$ such that the part $D$ of the discriminant in $\mathbb{P}^1 - \{0, \infty\}$ is reduced. Then there exists a disk neighborhood $B$ of supp$(D)$ in $\mathbb{P}^1 - \{0, \infty\}$ such that for $p \in \partial B$, the monodromy group of $\pi$ over $(B - \text{supp}(D), p)$ is a single transposition $(a', a'')$. Moreover, if $\sigma$ is the monodromy of a simple loop in $\mathbb{P}^1 - \text{int}(B)$ around $0$ based at $p$, then $a'' = \sigma^r(a')$ for some divisor $r$ of $d$ and $\pi$ factors through the covering $z \in \mathbb{P}^1 \to z' \in \mathbb{P}^1$.

Proof. We choose a base point $p \in \mathbb{P}^1$ outside the discriminant and we put $F := \pi^{-1}(p)$. By a simple arc we shall mean an embedded interval connecting $p$ with a point of the discriminant that does not meet the discriminant along the way. A
simple arc $\alpha$ determines up to isotopy (relative $p$ and the discriminant) a simple loop based at $p$ around a point of the discriminant and hence a monodromy transformation $\tau_\alpha \in \text{Aut}(F)$. A collection of simple arcs that do not meet outside $p$ shall be called an arc system. Notice that the directions of departure of the members of such a collection determine a cyclic ordering (our preference is clockwise) of these.

We begin by fixing a simple arc $\omega$ connecting $p$ with $0$. We write $\sigma$ for $\tau_\omega$; this is a $d$-cycle in $\text{Aut}(F)$. Any transposition $\tau$ in $\text{Aut}(F)$ can be written $(a, \sigma^l(a))$ for some $l \in \{0, 1, \ldots, \frac{1}{2}d\}$; this means that $\sigma \tau$ is the product of two disjoint cycles of length $l$ and $d-l$. Let us call $l$ the mesh of $\tau$.

Let $\alpha_1$ be an simple arc to a point of supp$(D)$ that forms with $\omega$ an arc system and is such that $\tau := \tau_{\alpha_1}$ has minimal mesh $r$. Write $\sigma \tau = \sigma' \sigma''$ with $\sigma'$ and $\sigma''$ disjoint cycles of length $r$ resp. $d-r$ and denote by $F'$ and $F''$ the corresponding parts of $F$. Notice that $\tau_{\alpha_1}$ interchanges some $a' \in F'$ with some $a'' \in F''$.

Let $\beta$ be another simple arc to a point of supp$(D)$ such that $(\omega, \alpha_1, \beta)$ is a clockwise oriented arc system. Then $\tau_\beta$ cannot commute with $\sigma''$; if it did, then it would interchange two points of $F'$ and would therefore have a mesh $< r$. It may happen that $\tau_\beta$ commutes with $\sigma'$. But not every choice for $\beta$ can be like this, for then $\sigma'$ would commute with the monodromy around $\infty$ and this is impossible as the latter is a $d$-cycle.

So for some $\beta$, $\tau_\beta$ interchanges some $b' \in F'$ with some $b'' \in F''$. If we modify $\beta$ by letting it first wind $k$ times around the union of $\omega$ and $\alpha_1$, then its monodromy gets conjugated by $(\sigma' \sigma'')^\pm k$. In this way we can arrange that $b'' = a''$. If $b' \neq a'$, then a straightforward verification shows that $\tau_\beta$ would have a smaller mesh than $r$. So $b' = a'$ and hence $\tau_\beta = \tau$. This argument proves more: the fact that for every integer $k$ the mesh of the $(\sigma' \sigma'')^k$-conjugate of $\tau_\beta$ is $\geq r$ implies that $r$ divides $d$. We put $\alpha_2 := \beta$.

We now prove with induction on $l$ that for $l \leq \deg(D)$ there is an arc system $(\alpha_1, \alpha_2, \ldots, \alpha_l)$ in clockwise cyclic order such that $\tau_{\alpha_i} = \tau$ for $i = 1, \ldots, l$. The lemma then follows: we already showed that $r$ divides $d$, and it is easy to see that the asserted factorization exists. So suppose we found such an arc system $(\alpha_1, \alpha_2, \ldots, \alpha_l)$ for some $l \geq 2$.

First assume $l$ even. Then the monodromy around the union of these arcs is equal to $\sigma$ and so the above argument yields simple arcs $\beta_1, \beta_2$ such that $\tau_{\beta_1} = \tau_{\beta_2}$ and $(\omega, \alpha_1, \ldots, \alpha_l, \beta_1, \beta_2)$ is an arc system in clockwise order. Since $\tau_{\beta_1}$ does not commute with $\sigma''$, we can modify $\beta_1$ and $\beta_2$ by letting both go round the union of $(\omega, \alpha_1, \ldots, \alpha_l)$ the same number of times first, to ensure that $\tau_{\beta_1} = \tau_{\beta_2}$ moves $a''$.

If $\tau_{\beta_i}$ does not commute with $\sigma'$, then the argument above shows that in fact $\tau_{\beta_i} = \tau$ and so we managed to take two induction steps.

If $\tau_{\beta_i}$ does commute with $\sigma'$, then let $\beta_i'$ be obtained from $\beta_i$ by going round $\alpha_l$ first. Then $(\omega, \alpha_1, \ldots, \alpha_{l-1}, \beta'_i, \beta'_2, \alpha_l)$ is in clockwise order and $\tau_{\beta_i'} = \tau_{\beta_2'}$ interchanges an element of $F'$ with an element of $F''$ Next modify the $\beta'_1$ and $\beta'_2$ by letting them first encircle $(\omega, \alpha_1, \ldots, \alpha_{l-1})$ the same number of times as to arrange that $\tau_{\beta_i'}$ moves $a''$ (this might cause them to meet $\alpha_l$ in a point $\neq p$). Then $\tau_{\beta_i'} = \tau$ and hence we have made the induction step.

It remains to do the induction step for $l$ odd. That is handled in the same way as the case $l = 1$.

Proof of the second clause of (1.1). Notice that $X^{n+g-2}$ parametrizes the triples $(C, x, y)$, where $C$ is smooth of genus $g$, $x, y \in C$ are distinct and $d(x) = d(y)$ for all $x, y$. For each such triple, let $\alpha_{x, y}$ be the unique simple arc of length $d(x)$ from $x$ to $y$. Then $\alpha_{x, y}$ commutes with $\sigma$ and $\sigma'$. The idea is to construct a sequence of arcs $\alpha_{x_1, y_1}, \alpha_{x_2, y_2}, \ldots, \alpha_{x_n, y_n}$ such that each $\alpha_{x_i, y_i}$ commutes with $\tau$ and $\tau_{\alpha_{x_i, y_i}} = \tau_{\alpha_{x_{i+1}, y_{i+1}}}$.

This can be done by induction on $n$. When $n = 1$, the construction is trivial. For $n > 1$, choose $\alpha_{x_n, y_n}$ such that $\tau_{\alpha_{x_n, y_n}} = \tau_{\alpha_{x_{n-1}, y_{n-1}}}$ and $\tau_{\alpha_{x_1, y_1}, \ldots, x_{n-1}, y_{n-1}} = \tau_{\alpha_{x_n, y_n}}$. Then $\tau_{\alpha_{x_1, y_1}, \ldots, x_{n-1}, y_{n-1}, x_n, y_n}$ is a $d$-cycle in $\text{Aut}(F)$ and hence has the desired property.

Thus, for any triple $(C, x, y)$, we have constructed a sequence of arcs $\alpha_{x_1, y_1}, \alpha_{x_2, y_2}, \ldots, \alpha_{x_n, y_n}$ such that $\tau_{\alpha_{x_1, y_1}, \ldots, x_n, y_n} = \tau_{\alpha_{x_1, y_1}, \ldots, x_{n-1}, y_{n-1}}$ for all $n$. This completes the proof of the second clause of (1.1).
some \( d \in \{2, \ldots, n+g\} \). By our previous discussion this defines a closed subvariety \( Y \) of \( \mathcal{C}_g^2 \) of pure codimension \( g \). The assertion that is to be proved will follow if we show that the classes of the irreducible components of \( Y \) are proportional in \( A^g(\mathcal{C}_g^2) \). Our first business is therefore to describe these irreducible components.

For \( d \geq 2 \), let \( Y_d \subset \mathcal{C}_g^2 \) be the locus parametrizing triples \((C, x, y)\) for which \((x) - (y)\) has order \( d \) in \( J(C) \). For such \((C, x, y)\) we have a morphism \( \pi : C \to \mathbb{P}^1 \) of degree \( d \) such that \( \pi^*(0) = d(x) \), \( \pi^*(\infty) = d(y) \) and \( \pi \) does not factor through a cover \( z \in \mathbb{P}^1 \mapsto z^r \in \mathbb{P}^1 \) for some \( r > 1 \). The previous lemma shows that all such covers are of the same topological type. This implies that \( Y_d \) is irreducible. So every irreducible component of \( Y \) is equal to some \( Y_d \).

Let \( \mathcal{J}_g \to \mathcal{M}_g \) be the universal Jacobian and let \( q : \mathcal{C}_g^2 \to \mathcal{J}_g \) be the Abel-Jacobi map \((C, x, y) \mapsto (x) - (y) \in J(C)\). Then \( Y_d = q^{-1}\mathcal{J}_g(d) \). Since \( Y_d \) has the correct codimension \( g \) in \( \mathcal{C}_g^2 \), it follows that \([Y_d]\) is a positive multiple of \( q^*[\mathcal{J}_g(d)] \). According to (2.9), \( [\mathcal{J}_g(d)] \) is a positive multiple of the class of the zero section in \( A^g(\mathcal{J}_g) \) and so the proof is complete.

**Proof of (1.2).** First observe that the direct image of \( K_1^{1+d_1} \cdots K_n^{1+d_n} \) under the forgetful morphism \( \mathcal{C}_g^n \to \mathcal{M}_g \) equals \( \kappa_{d_1} \cdots \kappa_{d_n} \). Now the direct image of the class of an irreducible component of \( X^k \) of codimension \( k \) under \( \mathcal{C}_g^n \to \mathcal{M}_g \) is zero unless the image has the correct codimension \( k-n \). In particular, a nonzero image requires \( k \geq n \). It follows that any product in the tautological classes of degree \( d \) can be represented by a linear combination of the irreducible components of the locus in \( \mathcal{M}_g \) that parametrizes the curves \( C \) that admit a covering \( \pi : C \to \mathbb{P}^1 \) of degree \( \leq 2g-2 \) totally ramified over \( \infty \) and with at most \( g-1-d \) points over 0. The rest follows immediately from (1.1).

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