ON WEIGHTED MEANS AND $MN$-CONVEX FUNCTIONS

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ABSTRACT. In this paper, we give more general definitions of weighted means and $MN$-convex functions. Using these definitions, we also obtain some generalized results related to properties of $MN$-convex functions. The importance of this study is that the results of this paper can be reduced to different convexity classes by considering the special cases of $M$ and $N$.

1. INTRODUCTION

The notions of convexity and concavity of a real-valued function of a real variable are well known [16]. The generalized condition of convexity, i.e. $MN$-convexity with respect to arbitrary means $M$ and $N$, was proposed in 1933 by Aumann [2]. Recently many authors have dealt with these generalizations. In particular, Niculescu [15] compared $MN$-convexity with relative convexity. Andersen et al. [3] examined inequalities implied by $MN$-convexity. In [3], Anderson et al. studied certain generalizations of these notions for a positive-valued function of a positive variable as follows:

**Definition 1.** A function $M : (0, \infty) \times (0, \infty) \to (0, \infty)$ is called a Mean function if

(M1) $M(u, v) = M(v, u)$,
(M2) $M(u, u) = u$,
(M3) $u < M(u, v) < v$ whenever $u < v$,
(M4) $M(\lambda u, \lambda v) = \lambda M(u, v)$ for all $\lambda > 0$.

**Example 1.** For $u, v \in (0, \infty)$

$$M(u, v) = A(u, v) = A = \frac{u + v}{2}$$

is the Arithmetic Mean,

$$M(u, v) = G(u, v) = G = \sqrt{uv}$$

is the Geometric Mean,

$$M(u, v) = H(u, v) = H = A^{-1}(u^{-1}, v^{-1}) = \frac{2uv}{u + v}$$

is the Harmonic Mean,

$$M(u, v) = L(u, v) = L = \begin{cases} \frac{u-v}{\ln u-\ln v} & u \neq v \\ \frac{u}{u} & u = v \end{cases}$$
is the Logarithmic Mean,
\[ M(u, v) = I(u, v) = I = \begin{cases} \frac{1}{e} \left( \frac{u^e}{v} \right)^{\frac{1}{e-1}} & u \neq v \\ u & u = v \end{cases} \]
is the Identric Mean,
\[ M(u, v) = M_p(u, v) = M_p = \begin{cases} A^{1/p}(u^p, v^p) = \left( \frac{u^p + v^p}{2} \right)^{1/p} & p \in \mathbb{R} \setminus \{0\} \\ p = 0 \end{cases} \]
is the \( p \)-Power Mean. In particular, we have the following inequality
\[ M_{-1} = H \leq M_0 = G \leq L \leq I \leq A = M_1. \]

Anderson et al. in \cite{1} developed a systematic study to the classical theory of continuous and midconvex functions, by replacing a given mean instead of the arithmetic mean.

**Definition 2.** Let \( M \) and \( N \) be two means defined on the intervals \( I \subset (0, \infty) \) and \( J \subset (0, \infty) \) respectively, a function \( f : I \rightarrow J \) is called \( MN \)-midpoint convex if it satisfies
\[ f(M(u, v)) \leq N(f(u), f(v)) \]
for all \( u, v \in I \).

The concept of \( MN \)-convexity has been studied extensively in the literature from various points of view (see e.g. \cite{1, 2, 12, 15}),

Let \( A(u, v, \lambda) = \lambda u + (1 - \lambda)v \), \( G(u, v, \lambda) = u^\lambda v^{1-\lambda} \), \( H(u, v, \lambda) = uv/(\lambda u + (1 - \lambda)v) \) and \( M_p(u, v, \lambda) = (\lambda u^p + (1 - \lambda)v^p)^{1/p} \) be the weighted arithmetic, geometric, harmonic, power of order \( p \) means of two positive real numbers \( u \) and \( v \) with \( u \neq v \) for \( \lambda \in [0, 1] \), respectively. \( M_p(u, v, \lambda) \) is continuous and strictly increasing with respect to \( \lambda \in \mathbb{R} \) for fixed \( p \in \mathbb{R} \setminus \{0\} \) and \( a, b > 0 \) with \( a > b \). See \cite{9, 12, 13, 14, 15} for some kinds of convexity obtained by using weighted means.

The aims of this paper, a general definition of weighted means and a general definition of \( MN \)-convex functions via the weighted means is to give. In recent years, many studies have been done by considering the special cases of \( M \) and \( N \).

The importance of this study is that some properties of \( MN \)-convex functions and some related inequalities have been proven in general terms.

2. **Main Results**

**Definition 3.** A function \( M : (0, \infty) \times (0, \infty) \times [0, 1] \rightarrow (0, \infty) \) is called a weighted mean function if

(WM1) \( M(u, v, \lambda) = M(v, u, 1 - \lambda) \),
(WM2) \( M(u, v, \lambda) = u \),
(WM3) \( u < M(u, v, \lambda) < v \) whenever \( u < v \) and \( \lambda \in [0, 1] \),
(WM4) \( M(\alpha u, \alpha v, \lambda) = \alpha M(u, v, \lambda) \) for all \( \alpha > 0 \),
(WM5) \( \lambda \in [0, 1] \) be fixed. Then \( M(u, v, \lambda) \leq M(w, v, \lambda) \) whenever \( u \leq w \) and \( M(u, v, \lambda) \leq M(u, \omega, \lambda) \) whenever \( v \leq \omega \).
(WM6) \( u, v \in (0, \infty) \) be fixed and \( u \neq v \). Then \( M(u, v, \cdot) \) is a strictly monotone and continuous function on \([0, 1] \).
then the following identities hold: property (WM2), then we obtained the identity (2.1). By using similar method, if we take \( M(u, v, s), M(v, w, s), \lambda \) for all \( u, v, w \in (0, \infty) \) and \( s, \lambda \in [0, 1] \).

**Remark 1.** According to the above definition every weighted mean function is a mean function with \( \lambda = 1/2 \). Also, By (WM6) we can say that for each \( x \in [u, v] \subseteq (0, \infty) \) there exists a \( \lambda \in [0, 1] \) such that \( x = M(u, v, \lambda) \). Moreover;

i.) If \( M(u, v, .) \) is a strictly increasing, then \( M(u, v, 0) = u \) and \( M(u, v, 1) = v \) whenever \( u < v \) (i.e. \( M(u, v, \lambda) \) is in the positive direction)

ii.) If \( M(u, v, .) \) is a strictly decrasing, then \( M(u, v, 0) = v \) and \( M(u, v, 1) = x \) whenever \( u < v \) (i.e. \( M(u, v, \lambda) \) is in the negative direction) and \( M(u, v, .)([0, 1]) = \{ \min \{ u, v \}, \max \{ u, v \} \} \).

**Remark 2.** Throughout this paper, we will assume that different weighted means have the same direction unless otherwise stated.

**Example 2.**

\[
M(u, v, \lambda) = A(u, v, \lambda) = A_\lambda = (1 - \lambda)u + \lambda v
\]

is the Weighted Arithmetic Mean,

\[
M(u, v, \lambda) = G(u, v, \lambda) = G_\lambda = u^{1/\lambda}v^{1/\lambda}
\]

is the Weighted Geometric Mean,

\[
M(a, b, s, \lambda) = M_p(u, v, \lambda) = M_p,\lambda = \left\{ \begin{array}{ll}
A^{1/p}(u^p, v^p, \lambda) = \{(1 - \lambda)x^p + \lambda y^p)^{1/p} & p \in \mathbb{R}\setminus\{0\} \\
G(u, v, \lambda) = u^{1-\lambda}v^\lambda & p = 0
\end{array} \right.
\]

is the \( p \)-Power Mean. In particular, we have the following inequality

\[
M_{1,\lambda} = H_a \leq M_{0,\lambda} = G_\lambda \leq M_{1,\lambda} = A_\lambda \leq M_{p,\lambda}
\]

for all \( x, y \in (0, \infty), t \in [0, 1] \) and \( p \geq 1 \).

**Proposition 1.** If \( M : (0, \infty) \times (0, \infty) \times [0, 1] \to (0, \infty) \) is a weighted mean function, then the following identities hold:

\[
M(M(a, M(a, b, s), \lambda), M(b, M(a, b, s), \lambda), s) = M(a, b, s)
\]

(2.1)

\[
M(M(a, b, \lambda), M(b, a, \lambda), 1/2) = M(a, b, 1/2).
\]

(2.2)

**Proof.** If we take \( v = w = M(a, b, s) \), \( u = a \) and \( z = b \) in (WM7) and we use the property (WM2), then we obtained the identity (2.1). By using similar method, if we take \( u = w = a, v = z = b \) and \( s = 1/2 \) in (WM7) and we use the properties (WM1) and (WM2), then we obtained the identity (2.2).

**Definition 4.** Let \( M \) and \( N \) be two weighted means defined on the intervals \( I \subseteq (0, \infty) \) and \( J \subseteq (0, \infty) \) respectively, a function \( f : I \to J \) is called \( MN \)-convex (concave) if it satisfies

\[
f(M(u, v, \lambda)) \leq (\geq) N(f(u), f(v), \lambda)
\]

for all \( u, v \in I \) and \( \lambda \in [0, 1] \).
The condition (WM8) in Definition 3 shows us that the function $M(u,v,.)$ is both $MM$-convex and $MM$-concave on $[0,1]$ for fixed $u,v \in (0,\infty)$. It is easily seen that weighted means mentioned in the Example 2 hold the condition (WM8).

We note that by considering the special cases of $M$ and $N$, we obtain several different convexity classes as $AA$-convexity (classical convexity), $AG$-convexity (log-convexity), $GA$-convexity, $GG$-convexity (geometrically convexity), $HA$-convexity (harmonically convexity), $M_p$-$A$-convexity ($p$-convexity), etc. For some convexity types, see ([9, 6, 14, 15]).

**Definition 5.** Let $M$ and $N$ be two weighted means defined on the intervals $[u,v] \subseteq (0,\infty)$ and $J \subseteq (0,\infty)$ respectively and $f:[u,v] \rightarrow J$ be a function. We say that $f$ is symmetric with respect to $M(u,v,1/2)$, if it satisfies

$$f(M(u,v,\lambda)) = f(M(u,v,1-\lambda))$$

for all $\lambda \in [0,1]$.

**Theorem 1.** Let $M$ and $N$ be two weighted means defined on the intervals $[u,v] \subseteq (0,\infty)$ and $J \subseteq (0,\infty)$ respectively. If function $f:[u,v] \rightarrow J$ is $MN$-convex, then the function $f$ is bounded.

**Proof.** Let $K = \max \{f(u), f(v)\}$. For any $z = M(u,v,\lambda)$ in the interval $[u,v]$, By using $MN$-convexity of $f$ and (WM3) we have

$$f(z) \leq N(f(u),f(v),\lambda) \leq K.$$ 

the function $f$ is also bounded from below. For any $z \in (u,v)$, there exists a $\lambda_0 \in (0,1)$ such that $z = M(u,v,\lambda_0)$, then by using $MN$-convexity of $f$ and (2.2) we have

$$f(M(u,v,1/2)) = f(M(z,M(v,u,\lambda_0),1/2)) \leq N(f(z),f(M(v,u,\lambda_0)),1/2).$$

On the other hand, if $f(z) = f(M(v,u,\lambda_0))$, then $N(f(z),f(M(v,u,\lambda_0)),1/2) = f(z)$ and thus the function $f$ is also bounded from below.

If $f(z) \neq f(M(v,u,\lambda_0))$, then there exists $\mu_0 \in (0,1)$ such that

$$N(f(z),f(M(v,u,\lambda_0)),1/2) = \mu_0 f(z) + (1-\mu_0) f(M(v,u,\lambda_0)).$$

By the inequality (2.3) and using $K$ as the upper bound, we have

$$f(z) \geq \frac{1}{\lambda_0} [f(M(u,v,1/2)) - (1-\lambda_0) f(M(v,u,\lambda_0))]$$

$$\geq \frac{1}{\lambda_0} [f(M(u,v,1/2)) - (1-\lambda_0)K] = k.$$ 

Thus, we obtain $f(z) \geq \max \{k, f(u)\}$ for any $z \in [u,v]$. This completes the proof. \qed

**Theorem 2.** Let $M$ and $N$ be two weighted means defined on the intervals $I \subseteq (0,\infty)$ and $J \subseteq (0,\infty)$ respectively. If the functions $f,g:I \rightarrow J$ are $MN$-convex, then $N(f(\cdot),g(\cdot),1/2)$ is a $MN$-convex function.

**Proof.** Since $f$ and $g$ are $MN$-convex functions, we have

$$f(M(u,v,\lambda)) \leq N(f(u),f(v),\lambda)$$

and

$$g(M(u,v,\lambda)) \leq N(g(u),g(v),\lambda)$$.
for all \( x, y \in I \) and \( t \in [0, 1] \). Then by (WM5) and (WM7) we have
\[
N(f(.), g(.), 1/2)(M(u, v, \lambda)) = N(f(M(u, v, \lambda)), g(M(u, v, \lambda)), 1/2) \leq N(N(f(u), f(v), t), N(g(u), g(v), \lambda), 1/2) = N(N(f(.), g(.), 1/2)(u), N(f(.), g(.), 1/2)(v), \lambda).
\]
This completes the proof. \( \square \)

We can give the following results for different convexity classes by considering the special cases of \( M \) and \( N \).

**Corollary 1.** Let \( I, J \subseteq (0, \infty) \) and \( f, g : I \rightarrow J \).
\[ i.) \] If \( f \) and \( g \) are convex functions, then \( A(f(.), g(.), 1/2) = (f + g)/2 \) is also convex function.
\[ ii.) \] If \( f \) and \( g \) are \( GA \)-convex functions, then \( A(f(.), g(.), 1/2) = (f + g)/2 \) is also \( GA \)-convex function.
\[ iii.) \] If \( f \) and \( g \) are harmonically convex functions, then \( A(f(.), g(.), 1/2) = (f + g)/2 \) is also harmonically convex function.
\[ iv.) \] If \( f \) and \( g \) are \( p \)-convex functions, then \( A(f(.), g(.), 1/2) = (f + g)/2 \) is also \( p \)-convex function.
\[ v.) \] If \( f \) and \( g \) are log-convex functions, then \( G(f(.), g(.), 1/2) = \sqrt{fg} \) is also log-convex function.
\[ vi.) \] If \( f \) and \( g \) are \( GG \)-convex functions, then \( G(f(.), g(.), 1/2) = \sqrt{fg} \) is also \( GG \)-convex function.
\[ vii.) \] If \( f \) and \( g \) are \( HG \)-convex functions, then \( G(f(.), g(.), 1/2) = \sqrt{fg} \) is also \( HG \)-convex function.
\[ viii.) \] If \( f \) and \( g \) are \( AH \)-convex functions, then \( H(f(.), g(.), 1/2) = 2fg/(f + g) \) is also \( AH \)-convex function.

**Remark 3.** In Corollary 1 we gave results only for some convexity types. It is possible to increase the results by considering another special cases of \( M \) and \( N \).

**Theorem 3.** Let \( M \) and \( N \) be two weighted means defined on the intervals \( I \subseteq (0, \infty) \) and \( J \subseteq (0, \infty) \) respectively. If \( f : I \rightarrow J \) is a \( MN \)-convex function and \( \alpha > 0 \), then \( \alpha f \) is a \( MN \)-convex function.

**Proof.** By using \( MN \)-convexity of \( f \) and (WM4), we have
\[
\alpha f(M(u, v, \lambda)) \leq \alpha N(f(u), f(v), \lambda) \leq N(\alpha f(u), \alpha f(v), \lambda).
\]
This completes the proof. \( \square \)

**Theorem 4.** Let \( M, N \) and \( K \) be three weighted means defined on the intervals \( I \subseteq (0, \infty), J \subseteq (0, \infty) \) and \( L \subseteq (0, \infty) \) respectively. If \( f : I \rightarrow J \) is a \( MN \)-convex function and \( g : J \subseteq (0, \infty) \rightarrow L \) is nondecreasing and \( NK \)-convex function, then \( g \circ f \) is a \( MK \)-convex function.

**Proof.** By using \( MN \)-convexity of \( f \), we have
\[
f(M(u, v, \lambda)) \leq N(f(u), f(v), \lambda).
\]
Since \( g \) is \( NK \)-convex and nondecreasing function
\[
g(f(M(u, v, \lambda))) \leq g(N(f(u), f(v), t)) \leq K(g(f(u)), g(f(v)), \lambda).
\]
This completes the proof. \( \square \)
Theorem 5. Let \( M \) and \( N \) be two weighted means defined on the intervals \( I \subseteq (0, \infty) \) and \( J \subseteq (0, \infty) \) respectively. If the function \( f : I \to J \) is \( MN \)-convex, \( M \leq A \) and \( N \leq A \) (\( A \) is the weighted arithmetic mean), then \( f \) satisfies Lipschitz condition on any closed interval \([a, b] \) contained in the interior \( I^o \) of \( I \). Consequently, \( f \) is absolutely continuous on \([a, b] \) and continuous on \( I^o \).

Proof. Choose \( \varepsilon > 0 \) so that \( a - \varepsilon \) and \( b + \varepsilon \) belong to \( I \), and let \( m_1 \) and \( m_2 \) be the lower and upper bounds for \( f \) on \([a - \varepsilon, b + \varepsilon] \). If \( u \) and \( v \) are distinct points of \([a, b] \) and we choose a point \( z \) such that

\[
y = M(u, z, t), \quad t = \frac{|v - u|}{\varepsilon + |v - u|},
\]

then

\[
f(v) \leq N(f(u), f(z), \lambda) \leq A(f(u), f(z), \lambda) = f(u) + \lambda|f(z) - f(u)|
\]

\[
f(v) - f(u) \leq \lambda|f(z) - f(u)| \leq \lambda(m_2 - m_1) \leq K|v - u|
\]

where \( K = (m_2 - m_1)/\varepsilon \). Since this is true for any \( u, v \in [a, b] \), we conclude that \( |f(v) - f(u)| \leq K|v - u| \) as desired.

Next we recall that \( f \) is absolutely continuous on \([a, b] \) if, corresponding to any \( \varepsilon > 0 \), we can produce a \( \delta > 0 \) such that for any collection \( \{(a_i, b_i)\}_{i=1}^n \) of disjoint open subintervals of \([a, b] \) with \( \sum_{i=1}^n (b_i - a_i) < \delta \), \( \sum_{i=1}^n |f(b_i) - f(a_i)| < \varepsilon \). Clearly the choice \( \delta = \varepsilon/K \) meets this requirement.

Finally the continuity of \( f \) on \( I^o \) is a consequence of the arbitrariness. \( \Box \)

Theorem 6. Let \( M \) and \( N \) be two weighted means defined on the intervals \( I \subseteq (0, \infty) \) and \( J \subseteq (0, \infty) \) respectively. If function \( f_\lambda : I \to J \) be an arbitrary family of \( MN \)-convex function \( s \) and let \( f(u) = \sup_\lambda f_\lambda(u) \). If \( K = \{x \in I : f(x) < \infty\} \) is nonempty, then \( K \) is an interval and \( f \) is \( MN \)-convex function on \( K \).

Proof. Let \( t \in [0, 1] \) and \( u, v \in K \) be arbitrary. Then

\[
f(M(u, v, \lambda)) = \sup_\alpha f_\alpha(M(u, v, \lambda)) \leq \sup_\alpha (N(f_\alpha(u), f_\alpha(v), \lambda)) \leq N\left(\sup_\alpha f_\alpha(u), \sup_\alpha f_\alpha(v), \lambda\right) = N(f(u), f(v), \lambda) < \infty.
\]

This shows simultaneously that \( K \) is an interval, since it contains every point between any two of its points, and that \( f \) is \( MN \)-convex function on \( K \). This completes the proof of theorem. \( \Box \)

Theorem 7 (Hermite-Hadamard’s inequalities for \( MN \)-convex functions). Let \( M \) and \( N \) be two weighted means defined on the intervals \( I \subseteq (0, \infty) \) and \( J \subseteq (0, \infty) \) respectively. If function \( f : I \to J \) is \( MN \)-convex, then we have

\[
f(M(u, v, 1/2)) \leq \int_0^1 N(f(M(u, v, \lambda)), f(M(u, v, 1 - \lambda)), 1/2) \, d\lambda \leq N(f(u), f(v), 1/2)
\]

for all \( u, v \in I \) with \( u < v \).
Proof. Since \( f : I \to \mathbb{R} \) is a \( MN \)-convex function, by using (2.2) we have
\[
f(M(u, v, 1/2)) = f(M(M(u, v, \lambda), M(u, v, 1 - \lambda), 1/2)) \\
\leq N(f(M(u, v, \lambda)), f(M(u, v, 1 - \lambda)), 1/2)
\]
for all \( u, v \in I \) and \( \lambda \in [0, 1] \). Further, integrating for \( \lambda \in [0, 1] \), we have
\[
(2.5) \quad f(M(u, v, 1/2)) \leq \int_{0}^{1} N(f(M(u, v, \lambda)), f(M(u, v, 1 - \lambda)), 1/2) \, d\lambda.
\]
Thus, we obtain the left-hand side of the inequality (2.4) from (2.5).

Secondly, By using \( MN \)-convexity of \( f \) and (WM5) with (2.2), we get
\[
N(f(M(u, v, \lambda)), f(M(u, v, 1 - \lambda)), 1/2) \\
\leq N(N(f(u), f(v), \lambda), N(f(u), f(v), 1 - \lambda), 1/2) \\
= N(f(u), f(v), 1/2).
\]
Integrating this inequality with respect to \( \lambda \) over \([0, 1]\), we obtain the right-hand side of the inequality (2.4). This completes the proof. □

We can give the following some results for different convexity classes by considering the special cases of \( M \) and \( N \). It is possible to increase the results by considering another special cases of \( M \) and \( N \).

Corollary 2. Let \( I, J \subseteq (0, \infty) \) and \( f : I \to J \).

i.) If \( f \) is convex function, then we have the following well-known celebrated Hermite-Hadamard’s inequalities for convex functions
\[
f(A(u, v, 1/2)) = f\left(\frac{u + v}{2}\right) \\
\leq \int_{0}^{1} A(f(A(u, v, \lambda)), f(A(u, v, 1 - \lambda)), 1/2) \, d\lambda \\
= \frac{1}{2(v - u)} \int_{u}^{v} f(x) + f(u + v - x) \, dx \\
= \frac{1}{v - u} \int_{u}^{v} f(x) \, dx \\
\leq A(f(u), f(v), 1/2) = \frac{f(u) + f(v)}{2}.
\]

ii.) If \( f \) is GA-convex function, then we have the following Hermite-Hadamard’s inequalities for GA-convex functions (see [7] Theorem 3.1. with \( s = 1 \))
\[
f(G(u, v, 1/2)) = f(\sqrt{uv}) \\
\leq \int_{0}^{1} A(f(G(u, v, \lambda)), f(G(u, v, 1 - \lambda)), 1/2) \, d\lambda \\
= \frac{1}{2(ln v - ln u)} \int_{u}^{v} f(x) + f\left(\frac{uv}{x}\right) \, dx \\
= \frac{1}{ln v - ln u} \int_{u}^{v} \frac{f(x)}{x} \, dx \\
\leq A(f(u), f(v), 1/2) = \frac{f(u) + f(v)}{2}.
\]
iii.) If $f$ is harmonically convex function, then we have the following Hermite-Hadamard’s inequalities for harmonically-convex functions (see [6, 2.4. Theorem])

\[ f(H(u,v,1/2)) = f \left( \frac{2uv}{u+v} \right) \]

\[ \leq \int_0^1 A(f(H(u,v,\lambda)), f(H(u,v,1-\lambda)), 1/2) \, d\lambda \]

\[ = \frac{uv}{2(v-u)} \int_u^v f(x) + f \left( \left[u^{-1} + v^{-1} - x^{-1}\right]^{-1} \right) \frac{dx}{x^2} \]

\[ = \frac{uv}{v-u} \int_u^v \frac{f(x)}{x^2} \, dx \]

\[ \leq A(f(u), f(v), 1/2) = \frac{f(u) + f(v)}{2}. \]

iv.) If $f$ is $p$-convex function ($p \neq 0$), then we have the following Hermite-Hadamard’s inequalities for $p$-convex functions (see [9, Theorem 2])

\[ f(M_p(u,v,1/2)) = f \left( \left[ \frac{u^p + v^p}{2} \right]^{1/p} \right) \]

\[ \leq \int_0^1 A(f(M_p(u,v,\lambda)), f(M_p(u,v,1-\lambda)), 1/2) \, d\lambda \]

\[ = \frac{p}{2(v^p - u^p)} \int_u^v f(x) + f \left( \left[u^p + v^p - x^p\right]^{1/p} \right) \frac{dx}{x^{1-p}} \]

\[ = \frac{p}{v^p - u^p} \int_u^v \frac{f(x)}{x^{1-p}} \, dx \]

\[ \leq A(f(u), f(v), 1/2) = \frac{f(u) + f(v)}{2}. \]

v.) If $f$ is log-convex function, then we have the following Hermite-Hadamard’s inequalities for log-convex functions (see [5, Theorem 2.1])

\[ f(A(u,v,1/2)) = f \left( \frac{u + v}{2} \right) \]

\[ \leq \int_0^1 G(f(A(u,v,\lambda)), f(A(u,v,1-\lambda)), 1/2) \, d\lambda \]

\[ = \frac{1}{v-u} \int_u^v \sqrt{f(x)f(u+v-x)} dx \]

\[ \leq G(f(u), f(v), 1/2) = \sqrt{f(u)f(v)}. \]

vi.) If $f$ is $GG$-convex function, then we have the following Hermite-Hadamard’s inequalities for $GG$-convex functions (see [8, the inequality (7)])

\[ f(G(u,v,1/2)) = f \left( \sqrt{uv} \right) \]

\[ \leq \int_0^1 G(f(G(u,v,\lambda)), f(G(u,v,1-\lambda)), 1/2) \, d\lambda \]

\[ = \frac{1}{\ln(v) - \ln(u)} \int_u^v \sqrt{f(x)f\left(\frac{uv}{x}\right)} \frac{dx}{x} \]

\[ \leq G(f(u), f(v), 1/2) = \sqrt{f(u)f(v)}. \]
Corollary 3. Let \( I, J \subseteq (0, \infty) \) and \( f : I \to J \).

\[
\text{vii.) If } f \text{ is HG-convex function, then we have }
\]
\[
f (H(u, v, 1/2)) = f \left( \frac{2uv}{u+v} \right) 
\leq \int_{0}^{1} G \left( f (H(u, v, \lambda)), f (H(u, v, 1-\lambda)), 1/2 \right) d\lambda 
= \frac{uv}{v-u} \int_{u}^{v} f(x)f \left( \frac{[u^{-1} + v^{-1} - x^{-1}]^{-1}}{x^2} \right) dx 
\leq G( f(u), f(v), 1/2) = \sqrt{f(u)f(v)}.
\]

\[
\text{viii.) If } f \text{ is AH-convex function, then we have }
\]
\[
f (A(u, v, 1/2)) = f \left( \frac{u+v}{2} \right) 
\leq \int_{0}^{1} H \left( f (A(u, v, \lambda)), f (A(u, v, 1-\lambda)), 1/2 \right) d\lambda 
= \frac{2}{v-u} \int_{u}^{v} \frac{f(x)f(u+v-x)}{f(x)+f(u+v-x)} dx 
\leq A( f(u), f(v), 1/2) = \frac{f(u)+f(v)}{2}.
\]

**Theorem 8.** Let \( M \) and \( N \) be two weighted means defined on the intervals \( [u, v] \subseteq (0, \infty) \) and \( J \subseteq (0, \infty) \) respectively. If function \( f : [u, v] \to J \) is MN-convex and symmetric with respect to \( M(u, v, 1/2) \), then we have

\[
f (M(u, v, 1/2)) \leq f(x) \leq N (f(u), f(v), 1/2)
\]

for all \( x \in I \).

**Proof.** Let \( x \in [u, v] \) be arbitrary point. Then there exists a \( \lambda \in [0, 1] \) such that \( x = M(u, v, \lambda) \). Since \( f : [u, v] \to J \) is a MN-convex function and symmetric with respect to \( M(u, v, 1/2) \), by using (2.2) we have

\[
f (M(u, v, 1/2)) = f (M (M(u, v, \lambda), M(u, v, 1-\lambda), 1/2)) 
\leq N (f (M(u, v, \lambda)), f (M(u, v, 1-\lambda)), 1/2) 
= f(x).
\]

Thus, we obtain the left-hand side of the inequality (2.6). Secondly, By using MN-convexity of \( f \) and (WM5) with (2.2), we get

\[
f(x) = N (f (M(u, v, \lambda)), f (M(u, v, 1-\lambda)), 1/2) 
\leq N (N(f(u), f(v), \lambda), N(f(u), f(v), 1-\lambda), 1/2) 
= N (f(u), f(v), 1/2).
\]

This completes the proof. \( \square \)

We can give the following some results for different convexity classes by considering the special cases of \( M \) and \( N \). It is possible to increase the results by considering another special cases of \( M \) and \( N \).

**Corollary 3.** Let \( I, J \subseteq (0, \infty) \) and \( f : I \to J \).
i.) If \( f \) is a convex function and symmetric with respect to \((u + v)/2\), then we have the following inequalities for convex functions (see [4, Theorem 2]):

\[
\frac{f(u + v)}{2} \leq f(x) \leq \frac{f(u) + f(v)}{2}.
\]

ii.) If \( f \) is a GA-convex function and symmetric with respect to \(\sqrt{uv} \), then we have the following inequalities for convex functions (see [10, Theorem 2.9]):

\[
f(\sqrt{uv}) \leq f(x) \leq \frac{f(u) + f(v)}{2}.
\]

iii.) If \( f \) is a \( p \)-convex function and symmetric with respect to \(\left(\frac{u^p + v^p}{2}\right)^{1/p} \), then we have the following inequalities for convex functions (see [11, Theorem 2.2]):

\[
f\left(\frac{u^p + v^p}{2}\right)^{1/p} \leq f(x) \leq \frac{f(u) + f(v)}{2}.
\]

3. Conclusion

The aim of this article is to determine that a mean is called the weighted mean when it meets what conditions, and also is to give a general definition of MN-convex functions. The importance of this study is that some properties of MN-convex functions and some related inequalities have been proven in general terms via this general definition of MN-convex functions.

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