Solutions of the Modified Chiral Model in (2+1) Dimensions.

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Abstract. This paper deals with classical solutions of the modified chiral model on $\mathbb{R}^{2+1}$. Such solutions are shown to correspond to products of various factors which we call time-dependent unitons. Then the problem of solving the system of second-order partial differential equations for the chiral field is reduced to solving a sequence of systems of first-order partial differential equations for the unitons.

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I. INTRODUCTION

As is well known [1, 2], some physical phenomena (relativity, particle physics, etc) lead, in their mathematical description to nonlinear partial differential equations. Some of these equations are elliptic (when we deal with time independent problems), others are hyperbolic or parabolic. However, with few exceptions, nonlinear partial differential equations are hard to solve (especially when they involve more than one spatial variable). Although in some cases static solution can be found the generalization to the time dependent problems often presents a formidable task.

A simple model which exemplifies many of the points mentioned above is the so-called relativistic chiral model in (2+1) dimensions. This model involves a chiral field \( J(x, y, t) \) which takes its values in \( SU(N) \). The equation of motion for this field is given by

\[
\partial_\mu (J^{-1} \partial^\mu J) = 0, \quad J \in SU(N),
\]

and is, in fact, the Euler-Lagrange equation corresponding to the Lagrangian density

\[
\mathcal{L} = \frac{1}{2} \text{tr} [(J^{-1} \partial_\mu J)(J^{-1} \partial^\mu J)].
\]

Here the indices \( \mu, \nu, \alpha \) range over the values 0, 1, 2, with \( x^\mu = (t, x, y) \) and \( \partial_\mu \) denotes partial differentiation with respect to \( x^\mu \). Thus \( J \) describes maps of \( \mathbb{R}^{2+1} \) into \( SU(N) \) and, when we restrict our attention to static fields, we have maps from \( \mathbb{R}^2 \) into \( SU(N) \). If we restrict our attention further and consider only static fields whose energy is finite (see below) then the problem reduces to finding harmonic maps of \( S^2 \rightarrow SU(N) \).

Such harmonic maps were studied by Uhlenbeck [3] who, in a seminal paper, showed how to construct these maps in an explicit form. Her construction involved the introduction of the concept of a uniton. She then showed how starting from a given solution one can find a new one by “the addition of a uniton”. This observation allowed her to demonstrate that all solutions can be obtained from the trivial ones \( ie \ J=\text{const} \) by the addition of a certain number (in fact \( \leq (N - 1) \)) of unitons.

This beautiful result relied heavily on the integrability of the chiral model in (2+0) dimensions (for more details see [4]). When we consider the same model in (2+1) dimensions, \( ie \) when we want to solve [4] we immediately run into a problem as it is not integrable and so Uhlenbeck’s method does not apply.
Some time ago Ward [5] presented a model in (2+1) dimensions which is an integrable generalization of the static chiral model in (2+0) dimensions. The model involves an $SU(N)$ valued chiral field whose equation of motion now takes the form

$$(\eta^{\mu\nu} + \epsilon^{\mu\nu\alpha} V_\alpha) \partial_\mu (J^{-1} \partial_\nu J) = 0.$$  \hspace{1cm} (3)

Here the tensor $\eta^{\mu\nu}$ is the inverse Minkowski metric, given by $\eta^{\mu\nu} = \text{diag}(-1, 1, 1)$; $\epsilon^{\mu\nu\alpha}$ is the fully antisymmetric tensor with $\epsilon^{012} = 1$; and $V_\alpha$ is the constant unit vector pointing in the $x$-direction, $V_\alpha = (0, 1, 0)$.

Notice that although (3) is not relativistically covariant (as $V_\alpha$ is nonzero) the solutions of Uhlenbeck are static solutions of (3). As the model is integrable many solutions of (3) can be easily found. The first solutions of this model [5] exhibited many properties seen in other integrable models (like wave-like behaviour with a phase shift, soliton-like structures passing through each other, etc). Recently, Ward [6], Ioannidou [7] and even more recently Anand [8] have found further interesting solutions of (3). These solutions described the evolution of soliton-like extended structures whose properties were quite similar to properties of solitonic objects in the $O(3)$ sigma model (ie they exhibit the $90^\circ$ scattering).

The approach of Ward and Ioannidou involved treating fields of the model as the limiting cases of the Riemann problem with zeros. In this paper we return to Uhlenbeck’s construction of static solutions and show that her construction can be generalized to give, in general nonstatic, solutions of the modified chiral model of Ward (ie of (3)). Our approach reproduces the solutions reported in [6] and [7] and presents an unusual but interesting application of Uhlenbeck’s approach.

**II. THE MODIFIED MODEL**

As shown by Ward [5] the modified chiral model (3) has the same conserved energy density as the original chiral model (1) and this energy is given by

$$E = -\frac{1}{2} \text{tr} \left[ (J^{-1} J_t)^2 + (J^{-1} J_x)^2 + (J^{-1} J_y)^2 \right].$$  \hspace{1cm} (4)

To specify the problem of finding solutions of (3) completely we have to state the boundary conditions satisfied by $J$. Thus in order to ensure the finiteness of energy of
the solutions, we require

\[ J = J_c + J_\theta(\theta) r^{-1} + O(r^{-2}), \quad r \to \infty, \]  

(5)

where \( z = r e^{i\theta} \) (with \( z = x + iy \)), \( J_c \) is a constant matrix, and \( J_\theta \) is independent of \( t \).

Equation (5) is completely integrable, in the sense that can be written as the compatibility condition for a pair of linear equations involving a spectral parameter \( \lambda \). Define

\[ A_z = \frac{1}{2} J^{-1}(\partial_z - \frac{1}{2} i\partial_t)J, \quad A_{\bar{z}} = \frac{1}{2} J^{-1}(\partial_{\bar{z}} + \frac{1}{2} i\partial_t)J, \]  

(6)

where \( A_z, A_{\bar{z}} \) are \( N \times N \) matrices depending on \( x, y, t \) but not on \( \lambda \).

Then the pair of linear operators,

\[ \partial_z + (1 - \lambda) A_{\bar{z}}, \quad \partial_{\bar{z}} + (1 - \lambda^{-1}) A_z, \]  

(7)

with \( \partial_z = \partial_z + \frac{1}{2} i\lambda^{-1} \partial_t, \ \partial_{\bar{z}} = \partial_{\bar{z}} - \frac{1}{2} i\lambda \partial_t \), commute for all \( \lambda \in \mathbb{C}^* = \mathbb{C} - \{0\} \), as a consequence of (5). Therefore one can find a solution of \( E_\lambda \) of the equations

\[ \partial_z E_\lambda = (1 - \lambda^{-1}) E_\lambda A_z, \quad \partial_{\bar{z}} E_\lambda = (1 - \lambda) E_\lambda A_{\bar{z}}. \]  

(8)

**Theorem 1** If \( J : \mathbb{R}^{2+1} \to SU(N) \) is harmonic, then there exist a \( GL(N, \mathbb{C}) \)-valued function \( E_\lambda \) of (5) which is analytic, holomorphic in \( \lambda \in \mathbb{C}^* \) and unitary for \( |\lambda| = 1 \) or \( \bar{\lambda} = \lambda^{-1} \) such that \( E_{-1} = J \) and

\[ E_\lambda = (E_{-1}^{-1})^\dagger, \]  

(9)

where \( \dagger \) denotes complex conjugate transpose. Conversely, if \( E_\lambda \) is holomorphic and analytic; and \( A_z, A_{\bar{z}} \) defined by (6) are independent of \( \lambda \), then \( J = E_{-1} \) is harmonic.

This follows from (5) and \( E_\lambda \) is called an extended solution corresponding to \( J \). System (8) is overdetermined and in order for a solution \( E_\lambda \) to exist, \( A_z \) and \( A_{\bar{z}} \) have to satisfy integrability conditions,

\[ (\partial_z + \frac{1}{2} i\partial_t)A_{\bar{z}} + (\partial_{\bar{z}} - \frac{1}{2} i\partial_t)A_z = 0, \]  

\[ (\partial_z - \frac{1}{2} i\partial_t)A_{\bar{z}} - (\partial_{\bar{z}} + \frac{1}{2} i\partial_t)A_z - 2[A_z, A_{\bar{z}}] = 0. \]  

(10)

Notice that, the first equation of (10) is equivalent to (5).
III. UHLENBECK’S CONSTRUCTION

It has been known for some time that all static solutions of grassmannian models are also solutions of the static chiral model \[9\]; although, not much has been known about other solutions. However, Uhlenbeck \[3\] proved that any harmonic map from the Riemann sphere $S^2$ to the unitarity group $U(N)$ can be factorized into a product of a finite number of factors (so-called unitons) involving maps into Grassmannians $G_k(C^N)$ each of which satisfies a system of first-order partial differential equations. Thus the problem of solving a system of second-order partial differential equations for a static harmonic map has been reduced to having to solve a sequence of systems of first-order partial differential equations for the unitons. This observation can be generalized to the non-static case (3).

We will seek solutions of (8) of the form

$$E_\lambda = \prod_{k=0}^{n} (1 + (\lambda - 1)R_k),$$

for some $n$, where $R_k$ are Hermitian projectors of the form $R_k = (q_k^\dagger \otimes q_k)/|q_k|^2$ ($R_0$ is a constant matrix, ie 0-uniton), which satisfy some first order differential equations; and $q_k$ are $N$-dimensional vectors which, in general, depend on $z, \bar{z}, t$ and $|q_k|^2 = q_k \cdot q_k^\dagger$. Note that a constant $E_\lambda$ is a 0-uniton. Then, following Uhlenbeck, we consider

$$\tilde{E}_\lambda = E_\lambda (1 + (\lambda - 1)R_n),$$

and find

**Theorem 2** Let $E_\lambda : C^* \times R^{2+1} \rightarrow GL(N,C)$ be an extended harmonic map $E_\lambda : R^{2+1} \rightarrow SU(N)$ for $|\lambda| = 1$. Then $\tilde{E}_\lambda = E_\lambda (1 + (\lambda - 1)R_n)$ is an extended map for a Hermitian projection $(R_n)$ if and only if

$$(1 - R_n)(\partial_t R_n - 2iA_\bar{z} R_n) = 0,$$

$$R_n(\partial_\bar{z} R_n - A_\bar{z}(1 - R_n)) = 0,$$

where $A_\bar{z} = -(A_\bar{z})^\dagger$ given by (8).

From the definition of the extended solution, $\tilde{A}_\bar{z} = (1 - \lambda)^{-1}\tilde{E}_\lambda^{-1} \partial_\bar{z}\tilde{E}_\lambda$ and so

$$\tilde{A}_\bar{z} = (1 + (\lambda^{-1} - 1)R_n) \left( A_\bar{z}(1 + (\lambda - 1)R_n) - (\partial_\bar{z} - \frac{1}{2} i\lambda \partial_t) R_n \right),$$

(14)
using the fact that $\tilde{E}^{-1}_{\lambda} = (1 + (\lambda^{-1} - 1)R_n)E^{-1}_\lambda$ and $A_z = (1 - \lambda)^{-1}E_\lambda \partial_z E_\lambda^{-1}$. But, from Theorem 1 we know that $\tilde{E}_\lambda$ is an extended solution if the $\lambda$ and $\lambda^{-1}$ terms vanish, which leads to the system (13).

Next we observe that

**Theorem 3** Suppose that $E_\lambda$ is an extended solution and so is $\tilde{E}_\lambda = E_\lambda(1 + (\lambda - 1)R_n)$. Then if $A_{\bar{z}} = \frac{1}{2}E^{-1}_\lambda \partial_z E_\lambda$ and $\tilde{A}_{\bar{z}} = \frac{1}{2}\tilde{E}^{-1}_\lambda \partial_z \tilde{E}_\lambda$, they are related by

$$\tilde{A}_{\bar{z}} = A_{\bar{z}} - \partial_{\bar{z}} R_n + \frac{1}{2} i \partial_t R_n. \quad (15)$$

This follows automatically from the terms in (14) which do not contain $\lambda$. In general, by induction, one can show that

$$A_{\bar{z}} \equiv (\partial_{\bar{z}} + \frac{1}{2} i \partial_t) \sum_{k=0}^{n} R_k, \quad (16)$$

and so, that the corresponding $R_k$ satisfy the following equations

$$R_n \left( \partial_{\bar{z}} R_n - \sum_{k=0}^{n-1} (-\partial_{\bar{z}} + \frac{1}{2} i \partial_t) R_k (1 - R_n) \right) = 0. \quad (17)$$

Observe that for $n = 1$, system (17) gives $\partial_t R_1 R_1 = 0$ and $R_1 \partial_{\bar{z}} R_1 = 0$; i.e. the equations for the holonomic static solutions (instantons) of the grassmannian models. This is the case as

$$\partial_t R_1 R_1 \equiv q_1 q_1^\dagger \partial_t q_1 \frac{q_1 \otimes q_1}{|q_1|^4} - \partial_t q_1 q_1 \frac{q_1 \otimes q_1}{|q_1|^4} = 0, \quad (18)$$

which is satisfied, if and only if, $q_1$ is independent of $t$. Using the same argument for $R_1 \partial_{\bar{z}} R_1 = 0$ we deduce that $q_1 = q_1(z)$. For $n \neq 1$, we have more general and, in general, $t$-dependent solutions.

Now, let us concentrate on (3) and discuss the construction of its solutions. If we set, $\tilde{E}_{-1} = J$ and $E_{-1} = J_0$ in (12), then the chiral field $J$ (due to Theorem 2) is of the form

$$J = J_0(1 - 2R_n), \quad (19)$$

and satisfies (3) if $J_0$ does and if $R_n$ satisfies (13). This describes a time-dependent multi-soliton solution of (3) since it satisfies (10), i.e.

$$(\partial_{\bar{z}} + \frac{i}{2} \partial_t) \tilde{A}_{\bar{z}} + (\partial_{\bar{z}} - \frac{i}{2} \partial_t) \tilde{A}_{\bar{z}} \equiv \left( \frac{1}{2} \partial_{\bar{z}} + \frac{i}{2} \partial_t \right) \left( - (1 - 2R_n)(\partial_{\bar{z}} + \frac{i}{2} \partial_t)R_n + (1 - 2R_n)A_{\bar{z}}(1 - 2R_n) \right)$$
\[
+ \left( \partial_z - \frac{i}{2} \partial_t \right) \left( (1 - 2R_n)(\partial_z - \frac{i}{2} \partial_t)R_n + (1 - 2R_n)A_z(1 - 2R_n) \right) \\
= \left( \partial_z + \frac{i}{2} \partial_t \right) \left( A_z - (\partial_z - \frac{i}{2} \partial_t)R_n \right) \\
+ \left( \partial_z - \frac{i}{2} \partial_t \right) \left( A_z + (\partial_z + \frac{i}{2} \partial_t)R_n \right) \\
= 0,
\]
\tag{20}
\]

using equations (13), (19) and (19).

So, to construct time-dependent solutions of the $SU(N)$ model (3), one can start from a constant solution (zero uniton) and add to it one uniton. This solution will be nonconstant - but it is static. Then, to this one uniton, one can add a second uniton, then a third one, and so on. As we are going to see, the number of soliton-like structures in a $n$-uniton configuration is not related to $n$ (i.e. the procedure of the “addition of a uniton” can both increase or decrease the number of unitons). In the static chiral model (4) Uhlenbeck’s theorem shows that for $U(N)$ the largest uniton number is less than $N$. In our case, the $t$-dependence of the system (17) invalidates this argument; the number of unitons can be arbitrary. In the static $SU(2)$ case considered by Uhlenbeck the only solutions are those described by constant matrices (0-unitons) and factors constructed from holomorphic functions (1-unitons). In the modified chiral model the 0- and 1-uniton solutions are the same but then, in addition, we have further solutions corresponding to two and more unitons. These additional solutions are nonstatic. We do not know, at this stage, whether there is any bound on the uniton number so that all solutions correspond to field configurations of up this number.

IV. THE $SU(2)$ AND $SU(3)$ CASES.

In this section we shall construct and discuss, as an example, a 2-uniton solution of the $SU(3)$ model (3). First of all, observe that all fields of the $SU(2)$ model can be embedded into the $SU(3)$ model so that the $SU(2)$ solutions are automatically also solutions of the $SU(3)$ model. In what follows we will recover some of them as special cases of the derived expressions.

Let us observe that the following expression describes a non-static 2-uniton solution:

\[
J = K \left( 1 - 2 \frac{q_1^\dagger \otimes q_1}{|q_1|^2} \right) \left( 1 - 2 \frac{q_2^\dagger \otimes q_2}{|q_2|^2} \right),
\tag{21}
\]
where \( q_i \) for \( i = 1, 2 \) are the three-dimensional vectors given by

\[
q_1 = (1, f, g), \\
q_2 = (1 + |f|^2 + |g|^2) (1, f, g) - 2i (t f' + h) (f, -(1 + |g|^2), g f) - 2i (t g' + h_1) (g, g f, -(1 + |f|^2)),
\]

and \( K \) is a constant matrix. Here \( f \) and \( g \) are rational meromorphic functions of \( z \). The field \( J \) takes values in \( SU(3) \), is smooth everywhere in \( \mathbb{R}^{2+1} \) (as the two vectors \( q_i \) are nowhere zero). It satisfies the boundary condition (9) and the equation of motion (3).

Now let indicate how the solution (21) was constructed and how further solutions can be obtained. One way of proceeding is to take the \( q_i \) vectors of the form \( q_i = (1, u_i, v_i) \). Then, for the 1-uniton solution corresponding to (17), the \( q_1 \) vector \( q_1 = (1, u_1, v_1) \) (as we have shown) is given in terms of the complex variable \( z \). Thus we write \( q_1 = (1, f, g) \) where \( f \) and \( g \) are functions of \( z \) only. Then the system (17) for \( n = 2 \) reduces to the following first order differential equations for the function \( u_2 \) and \( v_2 \):

\[
\partial_z u_2 = \frac{G}{(1 + |f|^2 + |g|^2)} (-u_2 + f), \quad \partial_z v_2 = \frac{G}{(1 + |f|^2 + |g|^2)} (-v_2 + f),
\]

where \( G = \bar{f}'(-u_2 + f) + \bar{g}'(-v_2 + g) + (u_2 g - v_2 f)(\bar{f} g' - \bar{f}' g) \) and

\[
\partial_t u_2 = 2i \left( \frac{1 + u_2 \bar{f} + v_2 \bar{g}}{1 + |f|^2 + |g|^2} \right) (u_2(\bar{f} f' + \bar{g} g') + \bar{g}(f' g - g' f) + f'), \\
\partial_t v_2 = 2i \left( \frac{1 + u_2 \bar{f} + v_2 \bar{g}}{1 + |f|^2 + |g|^2} \right) (v_2(\bar{f} f' + \bar{g} g') - \bar{f}(f' g - g' f) + g'),
\]

and \( ' \) denotes the derivative with respect to \( z \). A few lines of algebra allow us to show that \( q_2 \) of (22) is the general solution of the equations of the above system. Notice that when \( v_1 = 0 \) (ie \( g = 0 \), (21) corresponds to the field \( J \) of the \( SU(2) \) model (3).

Calculating the energy density for (21) we find,

\[
\mathcal{E} = \mathcal{E}_0 + 4\mathrm{tr} \left( (\partial_z - \frac{i}{2} \partial_t) R_2 (\partial_z + \frac{i}{2} \partial_t) R_2 \right) - 4\mathrm{tr} \left( (\partial_z - \frac{i}{2} \partial_t) R_2 \partial_z R_2 \right) - 4\mathrm{tr} \left( (\partial_z + \frac{i}{2} \partial_t) R_2 R_2 \partial_z R_2 \right),
\]

where \( \mathcal{E}_0 \) is the energy density of the 1-uniton (ie static) field. As is well known \( \mathcal{E}_0 \) is a total derivative and so, probably, also is \( \mathcal{E} \) but, so far, we have not managed to prove this. However, we have looked at some special cases and have found that the total energy \( E \), obtained by integrating \( \mathcal{E} \), is quantized in units of \( 8\pi \), ie is given by \( E = 8N\pi \). \( N \) appears to be related to the number of topological structures.
Let us discuss some special cases:

(a) The embedding of $SU(2)$ solitons, where we set

$$g = f, \quad h = h_1,$$

(26)

Such soliton solutions are localized along the direction of motion; they are not, however, of constant size. Their height (maximum of $\mathcal{E}$), is time dependent. In fact, for $(\deg f \geq \deg h)$ no scattering occurs, i.e. the solitons are located at the centre-of-mass forming a totally symmetric ring configuration. Otherwise, there are $(\deg f - 1)$ static solitons at the centre-of-mass of the system accompanied by $N = \deg h - \deg f + 1$ solitons (located at $t + z^N = 0$; since $J$ departs from its asymptotic value $J_c$ when $tf' + h = 0$) accelerating towards the solitons in the centre, scattering at an angle of $\pi/N$, and then decelerating as they separate. This phenomenon has been observed, firstly, in the $SU(2)$ model (3) (cf. [6, 7]) and is a feature of systems (with nontrivial topology) admitting topological solitons.

As an example, let us take $f(z) = z$ and $h(z) = z^2$. The corresponding field represents a configuration of two solitons that undergo a $90^0$ scattering. The energy density of this solution is

$$\mathcal{E} = 32 \frac{1 + 12r^2 + 12r^4 + 32r^4t^2 + 8t^2 - 16(x^2 - y^2)t}{[1 + 12r^4 + 4r^2 + 16(x^2 - y^2)t + 1 + 8t^2]^2},$$

(27)

and is symmetric under the interchange $t \to -t$, $x \to y$, $y \to x$. Figure 1 illustrates what happens close to $t = 0$.

(b) In a more general case, i.e. for arbitrary values of our parameters, the configuration represents a multi-soliton solution that is localized close to the centre of the mass and deforms as the time passes. As an example, let us present a typical case by taking $f(z) = z$, $g(z) = 2z$, $h(z) = z^2$ and $h_1(z) = 3$. In this time-dependent solution, which corresponds to a configuration of three solitons, the energy density of the field, for large (negative) $t$, is in the shape of a ring which, as time passes, deforms to three peaks and then expands again to a ring. Figure 2 presents pictures of the corresponding energy densities at some representative values of time.

Returning to the scattering of our soliton-like structures we have seen that an embedding of the $SU(2)$ model provides an example of such a scattering. A further example is
provided by a genuine $SU(3)$ case when we take

$$f = \rho^2(z), \quad g = \sqrt{2} \rho(z),$$

where $\rho(z)$ is an arbitrary function.

One example is presented in Figure 3, where we have chosen $\rho = z$, $h = z^3$ and $h_1 = 0$. This configuration can be seen to consist of two static solitons at the centre-of-mass accompanied by two solitons accelerating towards the centre, scattering at right angles, and then decelerating as they separate.

V. CONCLUSIONS AND FURTHER COMMENTS

In this paper we have shown how to adapt Uhlenbeck’s construction of uniton solutions of the chiral model in (2+0) dimensions to the, time-dependent, uniton-like solutions of the modified chiral model in (2+1) dimensions. It seems likely that there are many more interesting solutions still to be found. One could, for example, investigate the existence of time-dependent solutions based on other static ones constructed by Uhlenbeck.

As for static fields both models are the same, all Uhlenbeck’s solutions are also solutions of the modified chiral model. However, at the level of two unitons and beyond, our construction has given us genuine time-dependent solutions of the modified chiral model. Looking at some examples we have seen that, in some cases, their field configurations can be thought of as representing scatterings of some soliton-like objects, sometimes accompanied also by static structures. The total energy appears to be quantized - thus showing that during their scattering the soliton-like structures must change their size.

We have not succeeded yet in gaining the full understanding of various properties of our solutions. These problems are currently being investigated.

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References
[1] P. D. B. Collins, A. D. Martin abd E. J. Squires, Particle Physics and Cosmology, (J. Wiley, 1989).

[2] M. J. Ablowitz and P. A. Clarkson, Soliton, Nonlinear Evolution Equations and Inverse Scattering (CUP, 1991).

[3] K. Uhlenbeck, J. Diff. Geom. 30, 1 (1989).

[4] W. J. Zakrzewski, Low dimensional sigma models (IOP, 1989).

[5] R. S. Ward, J. Math. Phys. 29, 386 (1988).

[6] R. S. Ward, Phys. Lett. A 208, 203 (1995).

[7] T. Ioannidou, J. Math. Phys. 37, 3422 (1996).

[8] C. Anand, Ward’s Solitons II: Exact Solutions, preprint.

[9] H. Eichenherr and M. Forger, Nucl. Phys. B 164, 528 (1980).

Figure Captions.

**Figure 1:** Energy density of the embedding of $SU(2)$ solitons, at increasing times.

**Figure 2:** Energy density at increasing times, for a three ring-shaped solitons.

**Figure 3:** Energy density at increasing times showing a $90^\circ$ scattering between pure $SU(3)$ solitons.
$t = -2$

$t = 0$

$t = 2$
