Dynamics of the beam laying on the elastic basis of the Pasternak model carrying the moving constant load

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Abstract. A system consisting of a guide along which a heavy mass moves is considered. A guide is considered to be a homogeneous beam lying on the elastic base of the Pasternak model. The features of bending wave generation by a moving load are studied. Based on the general solution of the problem, an expression is obtained for the pressure of the waves (the forces of resistance to motion).

The dynamic behavior of elastic systems with moving loads has been given enough attention in the literature, which is associated with a wide practical application [1-15]. Of particular interest are the problems of oscillations of beams on a Winkler basement, which are used as models describing the dynamics of the railroad track [10, 14]. In connection with the development of high-speed highways, the requirements for the model and accuracy of calculations are increasing. Using the P. L. Pasternak model [16], which is characterized by two bed coefficients (compression coefficient and shear coefficient), on the one hand, it is possible to preserve the simplicity of the mathematical apparatus of the inherent Winkler model [17], and on the other, to obtain more reliable results.

Consider a system consisting of a guide along which a heavy mass moves according to an unknown law \( x = l(t) \) (Fig. 1).

![Figure 1. Guide with a moving load.](image)

We consider the guide lying on the elastic basement of the Pasternak model as a homogeneous beam, the density of the Lagrange function of which has the form:

\[
\lambda = \frac{1}{2} \left( \rho Fu_x^2 - JEu_{xx}^2 - h_1 u^2 - h_2 u_{xx}^2 \right),
\]
where \( u(x, t) \) – transversal displacement of the midline of the beam; \( \rho F \) – linear density; \( J \) – moment of inertia, \( E \) – Young modulus, \( h_1 \) and \( h_2 \) – the coefficient of the "bed" for compression and the coefficient of "shift" of the base of the beam, respectively.

The Lagrange function of a moving load (heavy mass) will have the form

\[
L = m\left(\ddot{u}_0^2 + \dot{l}^2\right)/2 - mg\dot{u}_0,
\]

where \( u_0(t) \) – transversal mass displacement \( m \).

The mutual dynamic behavior of the beam and the load moving along it is described by a system of equations

\[
\begin{align*}
\dddot{u} + \alpha^2 u_{xxxx} - c_{ll}^2 u_{xx} + \omega_0^2 u &= 0, \quad (1) \\
u(x = l(t) + 0, t) &= u(x = l(t) - 0, t) = u(l(t), t) = u_0(t), \quad (2) \\
u_x(x = l(t) + 0, t) &= u_x(x = l(t) - 0, t), \quad (3) \\
u_{xx}(x = l(t) + 0, t) &= u_{xx}(x = l(t) - 0, t), \quad (4) \\
m\ddot{u}_0(t) &= -\rho F[a^2 u_{xxxx} - c_{ll}^2 u_x - lu_t] + mg, \quad (5)
\end{align*}
\]

Here, square brackets mean the difference between the limiting values of the quantities standing in them to the right and left of the moving boundary \( x = l(t) \); \( \alpha = \sqrt{J/E/\rho F}; \quad c_{ll} = \sqrt{h_2/\rho F}; \quad \omega_0 = \sqrt{h_1/\rho F} \) – lowest frequency of waves excited in a beam; \( F_{pr} \) – wave pressure; \( Q \) – external force. To complete the statement of the problem, one should set the initial conditions and the need for the absence of deflections at infinity \( u(x, t) \to 0 \) при \( x \to \pm \infty \).

Assuming the motion to be uniform \( (l(t) = Vt, \quad V = \text{const}) \), and using the developed approach to the study of this kind of problems [8], we will look for an established (stationary) solution on the left \( (x < Vt) \) and on the right \( (x > Vt) \) of the moving load on the form

\[
A \exp[i(\omega t - kx)],
\]

where \( A, \omega, k \) – constant complex values.

Then the problem of wave kinematics (determination of frequencies \( \omega \), wave numbers \( k \) and critical velocities, when passing through which the picture of wave formation changes qualitatively) is reduced to solving the dispersion equation

\[
-\omega^2 + \alpha^2 k^4 + c_{ll}^2 k^2 + \omega_0^2 = 0,
\]

In conjunction with kinematic invariant [8]:

\[
\omega - Vk = 0,
\]

which expresses the fact that the phases of the emitted waves are equal to zero at the point where the moving constant source of disturbances is located.

From the condition for the degeneracy of the roots of system (7) - (8) of equations, we find the expression for the critical velocity \( V_c = \sqrt{2\alpha \omega_0 + c_{ll}^2} \), which, apparently, was first obtained in [18]. This speed coincides with the minimum phase velocity of wave propagation for this model (Fig. 2) and exceeds that for the beam of the Euler – Bernoulli model on a Winkler base [8].

In Fig. 2 are presented in dimensionless form \( (\hat{\varphi}_{ph} = v_{ph}/\sqrt{\alpha \omega_0}, \hat{\varphi}_{gr} = v_{gr}/\sqrt{\alpha \omega_0} \) \) dependences of phase \( (v_{ph} = \omega/k) \) and group velocities \( (v_{gr} = d\omega/dk) \) on the wavenumber \( k \) for the various parameter \( \beta \) values \( (\beta = 0.1; 2; 10) \). When the wavelength is \( 2\pi \sqrt{\frac{\alpha}{\omega_0}} \).
the phase velocity is the same as the group velocity and reaches its minimum value equal to $\sqrt{2\alpha \omega_0 + c_{II}^2}$.

![Diagram](a) ![Diagram](b) ![Diagram](c)

**Figure 2.** Dimensionless dependencies of phase and group velocities on the wavenumber.

It can be seen (Fig. 2) that in the range from 0 to 1 for dimensionless wave numbers (which vary from 0 to $\frac{\omega_0}{\alpha}$ – in dimension variables), the values of the phase velocities of the waves exceed the values of their group velocities, therefore, the dispersion is normal. Anomalous dispersion is observed when this range is exceeded ($\tilde{k} > 1$). If $c_{II} = \sqrt{2\alpha \omega_0}$, then the dependence of the group velocity on the wave number is linear (Fig. 2b) as for the beam of the Euler – Bernoulli model without taking into account the elastic base.

Given that at infinity, the beam deflections are limited, and traveling waves divert energy from the object, i.e.

$$
\begin{cases}
\text{Im} k > 0 & \text{when } x < Vt, \\
V_{gr} < V & \text{when } x > Vt,
\end{cases}
$$

($V_{gr} = d\omega/dk$ - group velocity of the waves) from (7) and (8) it follows that on the left to the load ($x < Vt$)

$$
k_{1,2} = \left( \pm \sqrt{2\alpha \omega_0 - c_{II}^2 + V^2} + i \sqrt{2\alpha \omega_0 + c_{II}^2 - V^2} \right) (2\alpha)^{-1},
$$

while on the right to the load when $x > Vt$
\[ k_{3,4} = \left( \pm \sqrt{2\alpha \omega_0 - c_{II}^2 + V^2} - i \sqrt{2\alpha \omega_0 + c_{II}^2 - V^2} \right) (2\alpha)^{-1}, \]
\[ \omega_{1-4} = k_{1-4}V \]

Therefore, for a stationary load or moving at a speed less than critical, the field of transversal displacements is localized near the source and is a superposition of oscillations decaying exponentially.

Zero-frequency source moving at a speed \( V > V_c \), it does not create its own field, but it radiates four waves, two of which run ahead of the moving load, and the other two follow it, diverting energy from it. Wave numbers and wave frequencies are determined by the formulas

\[ k_{1,2} = \pm \left( V^2 - c_{II}^2 - \sqrt{(V^2 - c_{II}^2)^2 - 4\alpha^2 \omega_0^2} \right)^{1/2} (2\alpha^2)^{-1/2}, \]
\[ k_{3,4} = \pm \left( V^2 - c_{II}^2 + \sqrt{(V^2 - c_{II}^2)^2 - 4\alpha^2 \omega_0^2} \right)^{1/2} (2\alpha^2)^{-1/2}, \]
\[ \omega_{1-4} = k_{1-4}V \]

Thus, the solution to problem (1) - (6), which describes the beam vibrations under the action of a uniformly moving load, can be represented in the form

\[ u(x,t) = \begin{cases} 
A_1 e^{(\omega_1 t - k_1 x)} + A_2 e^{(\omega_2 t - k_2 x)} & \text{when } x \leq Vt \\
A_3 e^{(\omega_3 t - k_3 x)} + A_4 e^{(\omega_4 t - k_4 x)} & \text{when } x \geq Vt'
\end{cases} \]

where the amplitudes \( A_i \) (\( i = 1,4 \)) have the following form

\[ A_1 = \frac{\text{imag}}{\text{imag}} e^{(k_1 - k_2)(k_1 - k_3)(k_1 - k_4)}, \quad A_2 = \frac{\text{imag}}{\text{imag}} e^{(k_2 - k_1)(k_2 - k_3)(k_2 - k_4)}, \]
\[ A_3 = \frac{\text{imag}}{\text{imag}} e^{(k_3 - k_1)(k_3 - k_2)(k_3 - k_4)}, \quad A_4 = \frac{\text{imag}}{\text{imag}} e^{(k_4 - k_1)(k_4 - k_2)(k_4 - k_3)} \]

The wave amplitudes increase indefinitely at a critical load speed \( V = V_c \). In Fig. 3 shows the profile of the deflection of the beam under load dependence on its speed is less than critical for various values of parameter \( \beta \) (\( \beta = 0; 0.1; 2; 10 \)).

**Figure 3.** Profile of the deflection of the beam under load dependence on its speed.

Based on the general solution of the problem, we obtain the following expression for the pressure of the waves (forces of resistance to motion [19]).
\[ F_{pr} = \begin{cases} 0, & V < V_*, \\ \frac{(mg)^2}{2\rho F\sqrt{(V^2 - c_i^2)^2 - 4\alpha^2\omega_0^2}}, & V > V_* \end{cases} \]

Since the own field does not exert pressure on the load, then for \( V < V_* \) we have \( F_{pr} \equiv 0 \) (Fig. 4). The case \( V = V_* \) is a "resonant" value of speed, accompanied by unlimited growth of \( F_{pr} \). In Fig. 4, the first curve is constructed for a beam on a Winkler base, the second is based on a Pasternak model. Starting at speed \( V = V_* \) (as mentioned above), the system emits waves of two waves to the left and to the right of the load, traveling in \(+x\) direction, similar to the Vavilov – Cherenkov effect [20], exerting pressure on the load. It is seen that with \( V > V_* \) wave pressure force \( F_{pr} \) is always directed against to the motion (inhibitory).

![Figure 4. The wave pressure force dependence on velocity.](image)

Note that the considered problem complements the cycle of studies of the problems of wave dynamics and the stability of the movement of high-speed objects along the rail guides of the rocket track [21-24]. The obtained results can serve as a methodological and computational support when setting up experiments on high-speed acceleration of a payload on a rocket track.

**Acknowledgements**
The work was supported by Russian Science Foundation (project 20-19-00613).

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