RANDOM ATTRACTOR OF STOCHASTIC BRUSSELATOR SYSTEM WITH MULTIPLICATIVE NOISE

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Abstract. Asymptotic dynamics of stochastic Brusselator system with multiplicative noise is investigated in this work. The existence of random attractor is proved via the exponential transformation of Ornstein-Uhlenbeck process and some challenging estimates. The proof of pullback asymptotic compactness here is more rigorous through the bootstrap pullback estimations than a non-dynamical substitution of Brownian motion by its backward translation. It is also shown that the random attractor has the attracting regularity to be an \((L^2 \times L^2, H^1 \times H^1)\) random attractor.

1. Introduction. In this work, we shall prove the existence of the random attractor for the following stochastic Brusselator system with multiplicative noise,

\[
du = (d_1 \Delta u + a - (b + 1)u + u^2 v) \, dt + \rho u \circ dW(t),
\]

\[
dv = (d_2 \Delta v + bu - u^2 v) \, dt + \rho v \circ dW(t),
\]

for \((t, x) \in \mathbb{R} \times \Gamma\), where \(\Gamma \subset \mathbb{R}^n (n \leq 3)\) is a bounded domain with a locally Lipschitz continuous boundary, given the homogeneous Dirichlet boundary condition

\[
u(t,x) = u(t,x) = 0, \quad t > t_0, \quad x \in \partial \Gamma,
\]

and an initial condition

\[
u(t_0, x) = u_0(x), \quad v(t_0, x) = v_0(x).
\]

All the coefficients \(d_1, d_2, a, b\) and \(\rho\) are arbitrarily given positive constants. \(\{W(t)\}_{t \in \mathbb{R}}\) is a one-dimensional, two-sided standard Wiener process (Brownian motion) on a probability space which will be specified later. The terms \(\rho u \circ dW(t)\) and \(\rho v \circ dW(t)\) indicate that the stochastic PDEs (1.1)-(1.2) are interpreted as the corresponding stochastic integral equations in the Stratonovich sense.

The original Brusselator equations were proposed in [11] as a system of ODEs and the diffusive Brusselator equations have been used as a typical mathematical model for morphogenesis and trimolecular autocatalytic reactions in physical chemistry and mathematical biology, cf. [9] and the references in [16].

The concept of random attractor for random dynamical system was first introduced in [7, 8] in the study of the asymptotic dynamics of Navier-Stokes equations.

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and other PDEs with multiplicative and additive white noise. The fundamental results on random dynamical systems and related topics have been summarized in [1].

When dealing with random dynamics and the existence of random attractor for stochastic partial differential equations with multiplicative noise, we usually transform the stochastic PDEs into deterministic ones with random parameters and random initial data through the exponential transformation of Brownian motion. In this work, however, we take the approach of the exponential transformation of the Ornstein-Uhlenbeck process. This transformation does change the structure of the original equations and produces the non-autonomous terms, cf. (2.10) and (2.11). It demands more challenging and sophisticated pullback a priori estimates, other than the non-dynamical substitution of $\omega$ by $\theta_t \omega$ as in some other publications.

Another notable aspect is that the Brusselator reaction-diffusion system does not satisfy the usual dissipative condition, cf. [16], and the bootstrap method used in the paper is also different from the decomposition method for the deterministic global attractors in [16]. The result in this work shows that the approach of the Ornstein-Uhlenbeck transformation unifies the treatment of the stochastic PDEs with multiplicative noise and with additive noise in regard to pullback asymptotic dynamics, especially for the reaction-diffusion systems with some sort of weak and hidden dissipativity.

In this paper the initial data or solutions are not restricted to be nonnegative and there are no restrictions on any of the positive parameters in the equations (1.1)–(1.2).

The rest of the paper is organized as follows. In Section 2 we present preliminary concepts on random dynamical system and random attractors. In Section 3 we prove the pullback absorbing property of the Brusselator random dynamical system. In Section 4 we show the pullback asymptotic compactness. In Sections 5 we reach the main results on the existence of a random attractor and its $L^2 \times L^2$ to $H^1 \times H^1$ attracting regularity.

2. Preliminaries and formulation. In this section, we recall the concepts of random dynamical system and random attractor. We refer to [1, 3, 6, 7] for more details. Let $(X, \|\cdot\|_X)$ be a real separable Banach space with Borel $\sigma$-algebra $\mathcal{B}(X)$ and let $(\Omega, \mathcal{F}, P)$ be a probability space. $\mathbb{R}^+ = [0, \infty)$.

**Definition 2.1.** $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ is called a metric dynamical system (MDS) if $\theta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, $\theta_0$ is the identity on $\Omega$, $\theta_{t+s} = \theta_t \circ \theta_s$ for all $s, t \in \mathbb{R}$ and $\theta_t P = P$ for all $t \in \mathbb{R}$ on $\Omega$.

**Definition 2.2.** A continuous random dynamical system (RDS) on $X$ over a metric dynamical system $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ is a mapping

$$\varphi : \mathbb{R}^+ \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x),$$

which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable such that for every $\omega \in \Omega$, the following conditions hold:

(i) $\varphi(0, \omega, \cdot)$ is the identity on $X$;

(ii) Cocycle property: $\varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s \omega, \varphi(s, \omega, \cdot))$ for all $t, s \in \mathbb{R}^+$;

(iii) $\varphi(\cdot, \omega, \cdot) : \mathbb{R}^+ \times X \to X$ is strongly continuous.

**Definition 2.3.** A continuous stochastic flow on a Banach space $X$ over a metric dynamical system $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ is a family of mappings $S(t, \tau, \omega) : X \to X$ for $t \geq \tau \in \mathbb{R}$ and $\omega \in \Omega$ with the following conditions:
Definition 2.4. A random set in \( X \) is a set-valued function \( B(\omega) : \Omega \to 2^X \) whose graph \( \{ (\omega, x) : x \in B(\omega) \} \subset \Omega \times X \) is an element of the product \( \sigma \)-algebra \( \mathcal{F} \times \mathcal{B}(X) \). A bounded random set \( B(\omega) \subset X \) means that there is a random variable \( r(\omega) \in \mathbb{R}^+ \) such that \( ||B(\omega)|| := \sup_{x \in B(\omega)} ||x|| \leq r(\omega) \) for all \( \omega \in \Omega \). A random set \( B(\omega) \) is called compact (respectively precompact) if for each \( \omega \in \Omega \) the set \( B(\omega) \) is compact (respectively precompact) in \( X \). A bounded random set is called tempered with respect to \( (\theta_t)_{t \in \mathbb{R}} \) on \( (\Omega, \mathcal{F}, P) \), if for each \( \omega \) and for any constant \( c > 0 \),

\[
\lim_{t \to \infty} e^{-ct} ||B(\theta^{-t}\omega)|| = 0.
\]

A random variable \( R : (\Omega, \mathcal{F}, P) \to (0, \infty) \) is called tempered with respect to \( \{\theta_t\}_{t \in \mathbb{R}} \) if for each \( \omega \),

\[
\lim_{t \to \pm\infty} \frac{1}{t} \log R(\theta^{-t}\omega) = 0.
\]

A collection \( \mathcal{D} \) of random subsets of \( X \) is called inclusion-closed, if \( D = \{ D(\omega) \}_{\omega \in \Omega} \in \mathcal{D} \) and \( \mathcal{D} = \{ D(\omega) \subset D(\omega) : \omega \in \Omega \} \) imply that \( \hat{D} \in \mathcal{D} \). In this case, the collection \( \mathcal{D} \) is called a universe. In the paper, we define \( \mathcal{D} \) to be the universe of all the tempered random sets in a specified phase space \( X \). Note that all bounded non-random sets are included in \( \mathcal{D} \).

Definition 2.5. Let \( \mathcal{D} \) be a collection of random subsets of \( X \). A random set \( K \in \mathcal{D} \) is called a \( \mathcal{D} \)-pullback absorbing set with respect to an RDS \( \varphi \) over the MDS \( (\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}}) \), if for any \( \omega \in \Omega \) and any bounded set \( B(\omega) \in \mathcal{D} \) there exists a finite time \( t_B(\omega) > 0 \) such that

\[
\varphi(t, \theta^{-t}\omega, B(\theta^{-t}\omega)) \subset K(\omega) \quad \text{for all } t \geq t_B(\omega).
\]

Definition 2.6. Let \( \mathcal{D} \) be a collection of random subsets of \( X \). Then an RDS \( \varphi \) is \( \mathcal{D} \)-pullback asymptotically compact in \( X \) if for each \( \omega \in \Omega \), \( \{ \varphi(t_n, \theta^{-t_n}\omega, x_n) \}_{n=1}^{\infty} \) has a convergent subsequence in \( X \) whenever \( t_n \to \infty \), and \( x_n \in B(\theta^{-t_n}\omega) \) for any given \( B \in \mathcal{D} \).

Definition 2.7. Let a universe \( \mathcal{D} \) of tempered random sets in a Banach space \( X \) be given. A random set \( A \in \mathcal{D} \) is called a random attractor in \( \mathcal{D} \) for a given RDS \( \varphi \) on \( X \) over the MDS \( (\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}}) \), if the following conditions are satisfied for each \( \omega \in \Omega \):

(i) \( A \) is a compact random set.

(ii) \( A \) is invariant in the sense that

\[
\varphi(t, \omega, A(\omega)) = A(\theta_t\omega), \quad \forall \ t \geq 0;
\]

(iii) \( A \) attracts every set \( B \in \mathcal{D} \) in the pullback sense that for every \( \omega \in \Omega \) one has

\[
\lim_{t \to \infty} \text{dist}_X(\varphi(t, \theta^{-t}\omega, B(\theta^{-t}\omega)), A(\omega)) = 0,
\]

where the Hausdorff semi-distance is given by \( \text{dist}_X(Y, Z) = \sup_{y \in Y} \inf_{z \in Z} ||y - z||_X \) for subsets \( Y \) and \( Z \) in \( X \).

We have the following proposition on the existence of random attractor due to Crauel and Flandoli [7, Theorem 3.11].
Proposition 2.8. Given a Banach space $X$ and a collection $\mathcal{D}$ of random sets of $X$, let $\varphi$ be a continuous RDS on $X$ over an MDS $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$. Suppose that there exists a closed pullback absorbing set $\{K(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ and $\varphi$ is pullback asymptotically compact with respect to $\mathcal{D}$, then the RDS $\varphi$ has a unique random attractor $A = \{A(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ whose basin is $\mathcal{D}$ and given by

$$A(\omega) = \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} \varphi(t, \theta^{-\tau}_t \omega, K(\theta^{-\tau}_t \omega)).$$

Define the product Hilbert spaces

$$H = [L^2(\Gamma)]^2, \quad E = [H^1_0(\Gamma)]^2, \quad \Pi = [H^1_0(\Gamma) \cap H^2(\Gamma)]^2.$$ 

The norm and inner-product of $H$ or $L^2(\Gamma)$ will be denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$, respectively. The norm of $L^p(\Gamma)$ or the product space $L^p(\Gamma) = [L^p(\Gamma)]^2$ will be denoted by $\| \cdot \|_{L^p}$ if $p \neq 2$. By the Poincaré inequality and the homogeneous Dirichlet boundary condition (1.3), there is a constant $\lambda > 0$ such that

$$\| \nabla \xi \|^2 \geq \lambda \| \xi \|^2, \quad \text{for } \xi \in H^1_0(\Gamma) \text{ or } E,$$

and we take $\| \nabla \|$ to be the equivalent norm of the space $E$ or the space $H^1_0(\Gamma)$. We shall use $| \cdot |$ to denote the Lebesgue measure or a vector norm in a Euclidean space.

The linear sectorial operator

$$A = \begin{pmatrix} d_1 \Delta & 0 \\ 0 & d_2 \Delta \end{pmatrix} : D(A)(= \Pi) \rightarrow H$$

is the generator of an analytic $C_0$-semigroup on the Hilbert space $H$, cf. [14]. By Sobolev embedding theorem, $H^1_0(\Gamma) \hookrightarrow L^n(\Gamma)$ is a continuous embedding for $n \leq 3$. Thus there is a positive constant $\zeta > 0$ associated with the Sobolev imbedding inequality

$$\| \varphi \|_{L^n(\Gamma)} \leq \zeta \| \varphi \|_E = \zeta \| \nabla \varphi \|, \quad \text{for any } \varphi \in E.$$  \hfill (2.3)

By Hölder inequality we have

$$\| u^2 \|^2 \leq \| u \|_{L^6}^2 \| v \|_{L^6}, \quad \text{for } u, v \in L^6(\Gamma),$$

and the nonlinear mapping

$$f(u, v) = \begin{pmatrix} a - (b + 1)u + u^2v \\ bu - u^2v \end{pmatrix} : E \rightarrow H,$$

is a locally Lipschitz continuous mapping defined on $E$.

Let $\{W(t)\}_{t \in \mathbb{R}}$ be the standard one-dimensional two-sided Wiener process in the probability space $(\Omega, \mathcal{F}, P)$, where

$$\Omega = \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \},$$

the $\sigma$-algebra $\mathcal{F}$ is generated by the compact-open topology on $\Omega$, and $P$ is the corresponding Wiener measure on $\mathcal{F}$. The shift mapping $\theta_t$ is defined by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}.$$ 

Then $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ is the canonical MDS and the stochastic process $\{W(t, \omega) = \omega(t) : t \in \mathbb{R}, \omega \in \Omega\}$ is the canonical Wiener process (Brownian motion).

Consider the Ornstein-Uhlenbeck process

$$z(\theta_t \omega) = - \int_{-\infty}^{0} e^s(\theta_t \omega)(s) ds = - \int_{-\infty}^{0} e^s \omega(t + s) ds + \omega(t),$$

\hfill (2.5)
which solves the linear stochastic differential equation
\[ dz + zdt = dW(t). \]  

(2.6)

The following proposition is quoted from [3].

**Proposition 2.9.** Let the metric dynamical system \((Ω, F, P, θ_t)\) and the Ornstein-Uhlenbeck process \(\{z(θ_tω)\}_{t ∈ R}\) be defined as above. Then there is a \(θ_t\)-invariant set \(Ω ∈ Ω\) of full \(P\)-measure such that for every \(ω ∈ Ω\), the following statements hold.

1. The Ornstein-Uhlenbeck process \(\{z(θ_tω)\}_{t ∈ R}\) has the asymptotically sublinear growth property, i.e.
\[ \lim_{t → ±∞} \frac{|z(θ_tω)|}{|t|} = 0, \]
(2.7)

2. \(z(θ_tω)\) is continuous in \(t\) and, for any fixed \(t_0 ∈ R\),
\[ \lim_{t → ±∞} \frac{1}{t - t_0} \int_{t_0}^{t} z(θ_sω) ds = 0, \]
(2.8)

In the sequel we consider \(ω ∈ Ω\) only and will always write \(Ω\) for \(Ω\).

As the main approach to investigating the random dynamics of stochastic PDEs, we convert the stochastic Brusselator system (1.1)-(1.2) to a system of pathwise PDEs with the random parameter \(ω(t)\) and random initial data. Make the transformation
\[ U = e^{-ρz(θ_tω)} u, \quad V = e^{-ρz(θ_tω)} v, \]
(2.9)

where \(z(θ_tω)\) is the Ornstein-Uhlenbeck process in (2.5). Then
\[ dU = -ρe^{-ρz(θ_tω)} u ◦ dz + e^{-ρz(θ_tω)} du, \]
\[ dV = -ρe^{-ρz(θ_tω)} v ◦ dz + e^{-ρz(θ_tω)} dv. \]

In view of the equation (2.6), \(dz + zdt = dW(t)\), the system (1.1)-(1.2) is transformed by (2.9) to the following random PDE problem:
\[ \frac{dU}{dt} = d_1 ∆U + ae^{-ρz(θ_tω)} (b + 1)U + e^{2ρz(θ_tω)} U^2 V + ρz(θ_tω)U, \]
(2.10)
\[ \frac{dV}{dt} = d_2 ∆V + bU - e^{2ρz(θ_tω)} U^2 V + ρz(θ_tω) V, \]
(2.11)

for \(ω ∈ Ω, x ∈ Γ\) and \(t > t_0\), with the homogeneous Dirichlet boundary condition
\[ U(t, ω, x) = V(t, ω, x) = 0, \quad t > t_0 ∈ R,\ x ∈ ∂Γ, \ ω ∈ Ω, \]
(2.12)

and the initial condition at \(t = t_0 ∈ R\),
\[ U(t_0, ω, x) = U_0(ω, x) = e^{-ρz(θ_{t_0ω})} u_0(x), \quad V(t_0, ω, x) = V_0(ω, x) = e^{-ρz(θ_{t_0ω})} v_0(x). \]
(2.13)

For every \(ω ∈ Ω\), the problem (2.10)-(2.13) of the pathwise nonautonomous partial differential equations can be written as
\[ \frac{dg}{dt} = Ag + F(g, θ_tω), \]
(2.14)
\[ g(t_0, ω, x) = g_0 = (U_0(ω, x), V_0(ω, x))^T, \]
where \( g(t, \omega; t_0, g_0) = (U(t, \omega; t_0, U_0), V(t, \omega; t_0, V_0))^T \) and

\[
F(g, \theta t_\omega) = \left( \begin{array}{c}
 ae^{-\rho z(\theta t_\omega)} - (b + 1)U + e^{2\rho z(\theta t_\omega)}U^2V + \rho z(\theta t_\omega)U \\
 bU - e^{2\rho z(\theta t_\omega)}U^2V + \rho z(\theta t_\omega)V
\end{array} \right)
\]

for any \( t \geq t_0 \) with initial data

\[ g_0(\omega) = (U_0(\omega, \cdot), V_0(\omega, \cdot))^T = (e^{-\rho z(\theta t_0 \omega)}u_0(\cdot), e^{-\rho z(\theta t_0 \omega)}v_0(\cdot))^T. \]

By conducting \textit{a priori} estimates on the Galerkin approximations of the initial value problem (2.14) and the compactness argument, c.f. [5], but with the extra care on the non-autonomous terms from the random noise, we can prove the local existence and uniqueness of the weak solution \( g(t, \omega; t_0, g_0), t \in [t_0, T(\omega, g_0)] \) for some \( T(\omega, g_0) > t_0 \), which depends continuously on the initial data.

By the parabolic regularity [14, Theorem 48.5], every weak solution turns out to be a strong solution for \( t > t_0 \) in the existence interval. Similar to Lemma 1.2 in [16], every weak solution \( g(t, \omega; t_0, g_0) \) of (2.14) on the maximal interval of existence has the property

\[ g(t, \omega; t_0, g_0) \in C([t_0, T_{\text{max}}); H) \cap C^1([t_0, T_{\text{max}}); H) \cap L^2([t_0, T_{\text{max}}); E). \]

Below we shall study the global existence and the asymptotic dynamics of the weak solutions of the problem (2.14).

3. Pullback absorbing property. For brevity, we write \( U(t, \omega; t_0, U_0), V(t, \omega; t_0, V_0) \) as \( U(t, \omega), V(t, \omega) \) or simply \( U, V \), similarly we write weak solution \( g(t, \omega; t_0, g_0) \) as \( g(t, \omega) \) or \( g \).

\textbf{Lemma 3.1.} \textit{For any given tempered random variable} \( R(\omega) > 0 \) \textit{and any initial data} \( h_0 = (u_0, v_0) \in H \) \textit{with} \( \|(u_0, v_0)\| \leq R(\omega) \), \textit{there exists a time} \( -\infty < T(R, \omega) \leq -1 \) \textit{sufficiently large such that the weak solution} \( g(t, \omega) = (U(t, \omega; t_0, U_0), V(t, \omega; t_0, V_0)) \) \textit{of the problem of the random Brusselator system (2.10)-(2.13) exists on} \( [t_0, 0] \) \textit{for any initial time} \( t_0 \leq T(R, \omega) \).

\textit{Moreover, for terminal time} \( t \in [-4, 0] \) \textit{when} \( t_0 \leq \min\{T(R, \omega), -4\} \), \textit{there exists a random variable} \( M(t, \omega) \) \textit{independent of initial data such that the weak solution satisfies}

\[ \left\| g \left(t, \omega; t_0, e^{-\rho z(\theta t_0 \omega)}h_0\right) \right\|^2 \leq M(t, \omega), \quad t \geq t_0, \quad \omega \in \Omega. \]  

\textbf{Proof.} \textit{Taking the inner product of (2.11) with} \( V(t, \omega) \), \textit{we get}

\[ \frac{1}{2} \frac{d}{dt} \|V\|^2 + d_2 \|\nabla V\|^2 = -\epsilon e^{2\rho z(\theta t_\omega)} \int_{\Gamma} \left( UV - \frac{1}{2} be^{-2\rho z(\theta t_\omega)} \right)^2 dx + \frac{b^2}{4} \|\Gamma\| e^{-2\rho z(\theta t_\omega)} + \rho z(\theta t_\omega) \|V\|^2. \]

\textit{It follows that, in the maximal interval of existence} \( [t_0, T_{\text{max}}) \),

\[ \frac{d}{dt} \|V\|^2 + 2\lambda d_2 \|V\|^2 \leq \frac{d}{dt} \|V\|^2 + 2d_2 \|\nabla V\|^2 \leq 2\rho z(\theta t_\omega) \|V\|^2 + \frac{b^2}{4} \|\Gamma\| e^{-2\rho z(\theta t_\omega)}. \]
Multiplying the above inequality by \(e^{\int_{t_0}^t (2\rho z(\theta, \omega) - 2\lambda d_2)ds}\) and then integrating it over \([t_0, t]\) where \(t_0 < -4 \leq t \leq 0\), we obtain

\[
\|V(t, \omega; t_0, g_0)\|^2 \leq \|V_0\|^2 e^{\int_{t_0}^t (2\rho z(\theta, \omega) - 2\lambda d_2(t - t_0))} + \frac{1}{2} b^2 |\Gamma| \int_{t_0}^t e^{\int_{t_0}^\tau (2\rho z(\theta, \omega) - 2\lambda d_2)ds - 2\rho z(\theta, \omega)d\tau} d\tau. \tag{3.4}
\]

The next step is key to the pullback estimates. We will get rid of the dependence on the initial time and data by the asymptotic decay of the Ornstein-Uhlenbeck process. The arguments go as follows.

From (2.7) and (2.8), for every random variable \(R(\omega) > 0\), there exists a time \(T_1(R, \omega) < -4\) such that for any \(t_0 \leq T_1(R, \omega)\), and \(t \in [-4, 0]\), we have

\[
\frac{1}{(t - t_0)} \int_{t_0}^t 2\rho z(\theta, \omega) ds - 2\lambda d_2 \leq -\lambda d_2, \quad e^{-\lambda d_2(t - t_0)} e^{-\rho z(\theta, \omega) R^2(\omega)} \leq 1. \tag{3.5}
\]

The improper integral associated with the integral on the right-hand side of (3.4),

\[
\int_{-\infty}^t \exp \left\{ \int_{t}^\tau (2\rho z(\theta, \omega) - 2\lambda d_2)ds - 2\rho z(\theta, \omega) \right\} d\tau, \tag{3.6}
\]

is convergent for the following reason. By (2.7) and (2.8) there exists \(T_2(\omega) < -4\) such that for any \(\tau \leq T_2(\omega)\), we have

\[
\exp \left[ \int_\tau^t (2\rho z(\theta, \omega) - 2\lambda d_2)ds - 2\rho z(\theta, \omega) \right] = \exp \left[ (t - \tau) \left( \frac{\int_\tau^t 2\rho z(\theta, \omega) ds}{t - \tau} - 2\lambda d_2 - \frac{2\rho z(\theta, \omega)}{t - \tau} \right) \right] \leq e^{-\lambda d_2(t - \tau)} \tag{3.7}
\]

and

\[
\int_{-\infty}^\tau e^{-\lambda d_2(t - \tau)} d\mu \leq \int_{-\infty}^{T_2} e^{-\lambda d_2(t - \mu)} d\mu = \frac{1}{\lambda d_2} e^{\lambda d_2(T_2 - t)}. \tag{3.8}
\]

Therefore, we have the following estimates that for \(t_0 \leq T_1(R, \omega)\) and \(t \in [-4, 0]\),

\[
\|V(t, \omega; t_0, g_0)\|^2 \leq \|V_0\|^2 e^{\int_{t_0}^t (2\rho z(\theta, \omega)ds - 2\lambda d_2(t - t_0))} + \frac{1}{2} b^2 |\Gamma| \int_{t_0}^t e^{\int_{t_0}^\tau (2\rho z(\theta, \omega) - 2\lambda d_2)ds - 2\rho z(\theta, \omega)d\tau} d\tau. \tag{3.9}
\]

Next deal with the \(U\) component through the transformation

\[
y(t, x, \omega) = U(t, x, \omega) + V(t, x, \omega). \tag{3.10}
\]

Adding up (2.10) and (2.11), we get

\[
\frac{dy}{dt} = d_1 \Delta y + (d_1 - d_2) \Delta V - y + V + ae^{-\rho z(\theta, \omega)} + \rho z(\theta, \omega)y. \tag{3.11}
\]
Take the inner product of (3.11) with $y(t)$ to obtain
\[
\frac{1}{2} \frac{d}{dt} \|y\|^2 + d_1 \|\nabla y\|^2
\]
\[
= (d_2 - d_1) \int t \left\| \nabla V \cdot \nabla y \right\| dt + (\rho_2(\theta_d) - 1) \|y\|^2 + \int t \left( V y dt + ae^{-\rho_2(\theta_d)} \int t y dt \right)
\]
\[
\leq \frac{|d_2 - d_1|^2}{2d_1} \|\nabla V\|^2 + \frac{d_1}{2} \|\nabla y\|^2 + (\rho_2(\theta_d) - 1) \|y\|^2 + \|V\|^2 + \frac{1}{2} \|y\|^2 + a^2|\Gamma|e^{-2\rho_2(\theta_d)}.
\]
(3.12)

It follows that
\[
\frac{d}{dt} \|y\|^2 + d_1 \|\nabla y\|^2 \leq \frac{|d_2 - d_1|^2}{2d_1} \|\nabla V\|^2
\]
\[
+ (2\rho_2(\theta_d) - 1) \|y\|^2 + 2\|V\|^2 + 2a^2|\Gamma|e^{-2\rho_2(\theta_d)}.
\]
(3.13)

Multiplying the above inequality by $e^{\int_{t_0}^t (2\rho_2(\theta_d) - 1) ds}$ and then integrating it over $[t_0, t]$, where $t_0 < -4 < t \leq 0$. Then there exists a time $T_3(R, \omega) < -4$ such that for any $t_0 \leq T_3(R, \omega)$, $t \in [-4, 0]$, we have
\[
\|y(t, \omega; t_0, g_0)\|^2 \leq \|y_0\|^2 e^{\int_{t_0}^t 2\rho_2(\theta_d) ds - (t - t_0)}
\]
\[
+ \int_{t_0}^t e^{\int_{t_0}^\tau (2\rho_2(\theta_d) - 1) ds} \left[ \frac{|d_2 - d_1|^2}{d_1} \|\nabla V(\tau)\|^2 + 2\|V(\tau)\|^2 + 2a^2|\Gamma|e^{-2\rho_2(\theta_d)} \right] d\tau
\]
\[
\leq 1 + \int_{t_0}^t e^{\int_{t_0}^\tau (2\rho_2(\theta_d) - 1) ds} \left[ \frac{|d_2 - d_1|^2}{d_1} \|\nabla V(\tau)\|^2 + 2\lambda \|\nabla V(\tau)\|^2 + 2a^2|\Gamma|e^{-2\rho_2(\theta_d)} \right] d\tau.
\]
(3.14)

Now treat the last integral in (3.14). Multiply (3.3) by $e^{\int_{t_0}^t (2\rho_2(\theta_d) - 1) ds}$ and then integrate it on $[t_0, t]$, for $t_0 < -4 < t \leq 0$. We find that there exists $T_4(R, \omega) < -4$ such that the following inequality holds,
\[
2d_2 \int_{t_0}^t e^{\int_{t_0}^\tau (2\rho_2(\theta_d) - 1) ds} \|\nabla V(\tau, \omega; t_0, g_0)\|^2 d\tau
\]
\[
\leq \int_{t_0}^t e^{\int_{t_0}^\tau (2\rho_2(\theta_d) - 1) ds} (2\rho_2(\theta_d) \|V(\tau)\|^2 + \frac{1}{2} b^2|\Gamma| e^{-2\rho_2(\theta_d)}) d\tau
\]
\[
+ e^{\int_{t_0}^t (2\rho_2(\theta_d) - 1) ds} \|V(t_0)\|^2 + \int_{t_0}^t \|V(\tau)\|(-2\rho_2(\theta_d) + 1) e^{\int_{t_0}^\tau (2\rho_2(\theta_d) - 1) ds} d\tau
\]
\[
\leq \frac{1}{2} b^2|\Gamma| \int_{t_0}^t e^{\int_{t_0}^\tau (2\rho_2(\theta_d) - 1) ds} e^{-2\rho_2(\theta_d)} d\tau + 1 + \int_{t_0}^t \|V(\tau)\| e^{\int_{t_0}^\tau (2\rho_2(\theta_d) - 1) ds} d\tau,
\]
(3.15)

when $t_0 \leq T_4(R, \omega)$. As for the term $\int_{t_0}^t \|V(\tau)\| e^{\int_{t_0}^\tau (2\rho_2(\theta_d) - 1) ds} d\tau$ in (3.15), from (3.4), there exists $T_5(R, \omega) \leq T_3(R, \omega)$ such that for $t_0 \leq T_5(R, \omega)$ we have
\[
\int_{t_0}^t e^{\int_{t_0}^\tau (2\rho_2(\theta_d) - 1) ds} \|V(\tau, \omega; t_0, V_0)\|^2 d\tau
\]
\[
\leq \int_{t_0}^t e^{\int_{t_0}^\tau (2\rho_2(\theta_d) - 1) ds} \|V_0\|^2 e^{\int_{t_0}^\tau (2\rho_2(\theta_d) ds - 2\lambda d_2(\tau - t_0))} d\tau
\]
\[
+ \frac{b^2|\Gamma|}{2} \int_{t_0}^t e^{\int_{t_0}^\tau (2\rho_2(\theta_d) - 2\lambda d_2) ds - 2\rho_2(\theta_d)} d\xi d\tau.
\[ \leq \int_{t_0}^{t} e^{f_t(2\rho z(\theta, \omega) + \max\{-1, -2\lambda_d\}) ds} \| V_0 \|^2 d\tau \]
\[ + \frac{b^2|\Gamma|}{2} \int_{t_0}^{t} \int_{t}^{\xi} e^{f_t(2\rho z(\theta, \omega) - 1) ds} e^{f_t(2\rho z(\theta, \omega) - 2\lambda_d) ds - 2\rho z(\theta, \omega) d\tau} d\xi \]
\[ \leq (t - t_0) \| V_0 \|^2 e^{f_t(2\rho z(\theta, \omega) + \max\{-1, -2\lambda_d\}) ds} \]
\[ + \frac{b^2|\Gamma|}{2} \int_{t_0}^{t} \int_{t}^{\xi} e^{f_t(2\rho z(\theta, \omega) + f_t(2\rho z(\theta, \omega) - 2\lambda_d) ds) - 2\rho z(\theta, \omega) d\tau d\xi} \]
\[ \leq 1 + \frac{b^2|\Gamma|}{2} \int_{t}^{\xi} (t - \xi) e^{f_t(2\rho z(\theta, \omega) ds + \max\{-1, -2\lambda_d\}(t - \xi) - 2\rho z(\theta, \omega) d\xi}, \]

Note that the last improper integral above is convergent by the similar calculation as in (3.7) and (3.8). The stochastic process given by
\[ C_1(t, \omega) = \frac{b^2|\Gamma|}{4d_2} \int_{-\infty}^{t} e^{f_t(2\rho z(\theta, \omega) - 1) ds - 2\rho z(\theta, \omega) d\tau} d\tau + \frac{1}{2d_2} \]
\[ + 1 + \frac{b^2|\Gamma|}{2} \int_{t}^{\xi} (t - \xi) e^{f_t(2\rho z(\theta, \omega) ds + \max\{-1, -2\lambda_d\}(t - \xi) - 2\rho z(\theta, \omega) d\xi} \]
is tempered due to (2.7) and (2.8).

By (3.14), (3.15) and (3.16), for \( t_0 \leq \min\{T_3(R, \omega), T_4(R, \omega), T_5(R, \omega)\} \), we have
\[ \| y(t, \omega; t_0, g_0) \|^2 \leq C_2(t, \omega), \quad \text{for } -4 \leq t \leq 0, \]

where
\[ C_2(t, \omega) = 1 + \left( \frac{|d_2 - d_1|^2}{d_1} + 2\lambda \right) C_1(t, \omega) + 2a^2 \int_{-\infty}^{t} e^{f_t(2\rho z(\theta, \omega) - 1) ds - 2\rho z(\theta, \omega) d\tau}. \]

Set \( T(R, \omega) = \min\{T_1(R, \omega), T_2(R, \omega), T_3(R, \omega), T_4(R, \omega), T_5(R, \omega)\} \). Then we have, for \( t_0 \leq T(R, \omega) \) and \( t \in [-4, 0], \)
\[ \| g(t, \omega; t_0, g_0) \|^2 = \| U(t, \omega; t_0, g_0) \|^2 + \| V(t, \omega; t_0, g_0) \|^2 \]
\[ = \| y(t, \omega; t_0, g_0) - V(t, \omega; t_0, g_0) \|^2 + \| V(t, \omega; t_0, g_0) \|^2 \]
\[ \leq 2\| y(t, \omega; t_0, g_0) \|^2 + 3\| V(t, \omega; t_0, g_0) \|^2 \leq M(t, \omega), \]
where
\[ M(t, \omega) = 3C_2(t, \omega) + 3 \left( 1 + \frac{b^2|\Gamma|}{2} \int_{-\infty}^{t} e^{f_t(2\rho z(\theta, \omega) - 2\lambda_d) ds - 2\rho z(\theta, \omega) d\tau} d\tau \right). \]
The proof is completed. \( \square \)

If \( g(t, \omega; \tau, g_0) \) is the weak solution to problem (2.10)-(2.13), then
\[ h(t, \omega; \tau, h_0) = S(\tau, \omega) h_0 = e^{\rho z(\theta, \omega)} g(t, \omega; \tau, g_0), \quad t \geq \tau, \]

where
\[ h_0 = (u_0, v_0), \quad g_0 = e^{-\rho z(\theta, \omega)} h_0, \]
is the solution to the original stochastic Brusselator problem (1.1)-(1.4).
By the uniqueness of weak solution of $g(t, \omega; \tau, g_0)$ and the stationary increment property of Brownian Motion, we can verify that $S(t, \tau, \omega)$ is a stochastic flow on $H$, namely,

$$S(t, s, \omega)S(s, \tau, \omega) = S(t, \tau, \omega), \quad \text{for } \tau \leq s \leq t,$$

$$S(t, s, \omega) = S(t - s, 0, \theta_s \omega), \quad \text{for } s \leq t.$$ 

The second equality means that

$$e^{\rho z(\theta \omega)} g(t, \omega; s, e^{-\rho z(\theta \omega)} h_0) = e^{\rho z(\theta_1 - \omega)} g(t-s, \theta_s \omega; 0, e^{-\rho z(\omega)} h_0) \quad \text{for all } s \leq t. \tag{3.21}$$

Define the Brusselator random dynamical system $\varphi : \mathbb{R}^+ \times \Omega \times H \to H$ over the MDS $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ by

$$\varphi(t - \tau, \theta_\tau \omega, h_0) = S(t, \tau, \omega)h_0 = e^{\rho z(\theta \omega)} g(t, \omega; \tau, g_0), \tag{3.22}$$

where $t \geq \tau \in \mathbb{R}, \omega \in \Omega, h_0 \in H$. The cocycle property of the mapping $\varphi$ can be checked by $\varphi(3.22)$, $\varphi(3.21)$ and the properties of stochastic flow in Definition 2.3.

From (3.22), we have the pullback relation

$$\varphi(t, \theta_{-\tau} \omega, (u_0, v_0)) = e^{\rho z(\omega)} g(0, \omega; -t, e^{\rho z(\theta_{-\tau} \omega)} h_0), \tag{3.23}$$

in which $g(0, \omega; -t, g_0), t \geq 0$, can be called the pullback quasi-trajectory from $g_0$, which is not a trajectory but the terminal values at time $t = 0$ of the bunch of weak solutions $g(0, \omega; -t, g_0)$ starting from $g_0$ more and more backward at time $-t$. We shall deal with the pullback quasi-trajectories to investigate the pullback asymptotic behavior of the Brusselator random dynamical system $\varphi$.

**Lemma 3.2.** For the Brusselator random dynamical system $\varphi$ on $H$ over the MDS $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$, there exists a $\mathcal{D}$-pullback absorbing set $B_0(\omega)$, which is the random ball centered at the origin with the radius $M_0(\omega)$ given by

$$M_0(\omega) = e^{\rho z(\omega)} \left[ 3C_2(0, \omega) + 3 \left( 1 + \frac{b^2 |\Gamma|}{2} \int_{-\infty}^{0} e^{\int_{\theta^*}^{s}(2\rho z(\theta, \omega) - 3\lambda d\theta - 2\rho z(\theta, \omega)) d\tau} \right) \right].$$

**Proof:** This is a direct consequence of Lemma 3.1 and the characterization of the pullback quasi-trajectories of this Brusselator RDS $\varphi$. Note that $M_0(\omega)$ is a tempered random variable and $B_0 \in \mathcal{D}$. 

Furthermore we show the pullback absorbing property of the $V$-component of the random Brusselator system $(2.10)$-$\text{(2.13)}$ in the Banach space $L^6(\Gamma)$. This is a key step to pave the way toward the proof of the pullback asymptotic compactness in the next section.

**Lemma 3.3.** For any given initial data $(u_0, v_0) \in E$ and terminal time $t \in [-4, 0]$, there exists a random time $T_0(\|(u_0, v_0)\|_{L^6}, \omega) \leq -4$ and a positive random variable $P(t, \omega)$ such that for any initial time $t_0 \leq T_0(\|(u_0, v_0)\|_{L^6}, \omega)$ we have

$$\|V(t, \omega; t_0, g_0)\|_{L^6} \leq P(t, \omega), \quad -4 \leq t \leq 0. \tag{3.24}$$

**Proof:** Taking the inner product of $(2.11)$ with $V^3$, we obtain

$$\frac{1}{4} \frac{d}{dt} \int_{\Gamma} V^4(t, x) dx + 3d_2 \|V(t) \nabla V(t)\|^2$$
Next we use the bootstrap method to take the inner product of (2.11) with $V^5$ and obtain

$$
\frac{1}{6} \frac{d}{dt} \int_{\Gamma} V^6(t, x) dx + 5d^2 \left\| V^2(t) \nabla V(t) \right\|^2
= \int_{\Gamma} \left( bUV^5 - e^{2\rho z(\theta, t)}U^2V^6 + \rho z(\theta, t)V^6 \right) dx
$$
Then there is a random variable $T_0(\|y_0\|_{L^6}, \omega) \leq -4$ such that for every $\omega \in \Omega$, $t_0 \leq T_0(\|y_0\|_{L^6}, \omega)$ and $t \in [-4, 0]$, we have

$$\frac{d}{dt} \|V(t)\|_{L^6}^6 + 10\lambda d_2 \|V(t)\|_{L^6}^6 \leq \frac{d}{dt} \|V(t)\|_{L^6}^6 + 10d_2 \|V^3(t)\|^2$$

$$\leq 2\rho z(\theta, \omega) \|V(t)\|_{L^6}^6 + \frac{1}{2} b^2 e^{-2\rho z(\theta, \omega)} \|V(t)\|_{L^4}^4.$$  

(3.29)

It follows that

$$\|V(t)\|_{L^6}^6 \leq \left[ \int_0^t \frac{1}{2} b^2 e^{-2\rho z(\theta, \omega)} V^4 + \frac{1}{2} e^{2\rho z(\theta, \omega)} U^2 V^6 - e^{2\rho z(\theta, \omega)} U^2 V^6 + \rho z(\theta, \omega) V^6 \right] dx.$$

(3.28)

Then there is a random variable $T_0(\|y_0\|_{L^6}, \omega) \leq -4$ such that for every $\omega \in \Omega$, $t_0 \leq T_0(\|y_0\|_{L^6}, \omega)$ and $t \in [-4, 0]$, we have

$$\|V(t, \omega; t_0, y_0)\|_{L^6}^6 \leq 2\rho z(\theta, \omega) \|V(t)\|_{L^6}^6 + \frac{1}{2} b^2 e^{-2\rho z(\theta, \omega)} \|V(t)\|_{L^4}^4.$$  

(3.30)
where
\[
P(t, \omega) = 3 + \frac{b^6 |\Gamma|}{8} \int_{-\infty}^{t} e^{-2\rho z(\theta, \omega)} d\tau \int_{-\infty}^{t} e^{-2\lambda d_2(t-\eta)-2\rho z(\theta, \omega)} d\eta \\
\cdot \int_{-\infty}^{t} e^{I_1^t(2\rho z(\theta, \omega)-2\lambda d_2) d\tau} e^{-2\rho z(\theta, \omega)} d\xi.
\]

Note that the three improper integrals above are convergent by the similar calculations as shown in (3.7) and (3.8). The proof is completed.

The next lemma is instrumental to the proof of pullback asymptotic compactness in the next section.

**Lemma 3.4.** Let \((t_0, t_1)\) satisfy \(-4 \leq t_0 < t_1 < 0\) and \((u_0, v_0) \in H\) with \(\|(u_0, v_0)\| \leq R(\omega)\), where \(R(\omega) > 0\) is any given random variable as in Lemma 3.1. If the weak solution \(g(t, \omega; t_0, g_0)\) satisfies \(\|g(t, \omega; t_0, g_0)\| \in E\) with
\[
\|g(t_1, \omega; t_0, g_0)\|_E \leq G(\omega),
\]
where \(G(\omega) > 0\) is any given random variable, then there exists a random variable \(D(t, G, \omega) > 0\) such that
\[
\|V(t, \omega; t_0, g_0)\|_{L^4} \leq D(t, G, \omega), \quad \text{for any } t \in [t_1, 0], t_0 \leq \min\{T(R, \omega), -4\},
\]
where \(T(R, \omega)\) is the same as in Lemma 3.1.

**Proof.** Fix an initial time \(t_0 \leq \min\{T(R, \omega), -4\}\). Integrate (3.26) over \([t_1, t]\) to get
\[
\|V(t, \omega; t_0, g_0)\|_{L^4} \leq \|V(t_1, \omega; t_0, g_0)\|_{L^4}^4 e^{I_1^t(2\rho z(\theta, \omega)-6\lambda d_2) d\tau}
\]
\[
+ \frac{1}{2} b^2 \int_{t_1}^{t} e^{I_1^\tau(2\rho z(\theta, \omega)-6\lambda d_2) d\tau} e^{-2\rho z(\theta, \omega)} \|V(\tau, \omega; t_0, g_0)\|_{E}^2 d\tau
\]
\[
\leq \delta^4 G^4(\omega) e^{I_1^t(2\rho z(\theta, \omega)-6\lambda d_2) d\tau}
\]
\[
+ \frac{1}{2} b^2 \int_{t_1}^{t} e^{I_1^\tau(2\rho z(\theta, \omega)-6\lambda d_2) d\tau} e^{-2\rho z(\theta, \omega)} \|V(\tau, \omega; t_0, g_0)\|_{E}^2 d\tau
\]
\[
\leq \delta^4 G^4(\omega) e^{I_1^t(2\rho z(\theta, \omega)-6\lambda d_2) d\tau} + \frac{b^2}{2} \int_{t_1}^{t} e^{I_1^\tau(2\rho z(\theta, \omega)-6\lambda d_2) d\tau} e^{-2\rho z(\theta, \omega)} d\tau
\]
\[
+ \frac{1}{4} b^4 |\Gamma| \int_{t_1}^{t} e^{-2\rho z(\theta, \omega)} e^{I_1^\tau(2\rho z(\theta, \omega)-6\lambda d_2) d\tau} \int_{-\infty}^{\tau} e^{I_1^\xi(2\rho z(\theta, \omega)-2\lambda d_2) d\xi} e^{-2\rho z(\theta, \omega)} d\xi d\tau,
\]
(3.32)
in which the last inequality follows from the use of (3.4) and \(\delta\) is the constant of the Sobolev embedding \(H^1_0(\Gamma) \hookrightarrow L^4(\Gamma)\),
\[
\|\varphi\|_{L^4(\Gamma)} \leq \delta \|\varphi\|_E, \quad \text{for any } \varphi \in E.
\]

Put
\[
\Pi(t, \omega) = \delta^4 \tilde{P}^4(\omega) e^{I_1^t(2\rho z(\theta, \omega)-6\lambda d_2) d\tau} + \frac{1}{2} b^2 \int_{t_1}^{t} e^{I_1^\tau(2\rho z(\theta, \omega)-6\lambda d_2) d\tau} e^{-2\rho z(\theta, \omega)} d\tau
\]
\[
+ \frac{1}{4} b^4 |\Gamma| \int_{t_1}^{t} e^{-2\rho z(\theta, \omega)} e^{I_1^\tau(2\rho z(\theta, \omega)-6\lambda d_2) d\tau} \int_{-\infty}^{\tau} e^{I_1^\xi(2\rho z(\theta, \omega)-2\lambda d_2) d\xi d\tau} e^{-2\rho z(\theta, \omega)} d\xi d\tau,
\]
(3.33)
Fix any initial time \( t_0 \leq \min\{T(R, \omega), -4\} \). By integrating (3.29) over \([t_1, t]\) and using (2.3), we get

\[
\|V(t, \omega; t_0, g_0)\|_{L^6}^6 \\
\leq \|V(t_1, \omega; t_0, g_0)\|_{L^6}^6 \cdot e^{\int_{t_1}^t (2\rho_2(\theta, \omega) + 10\lambda_d)(t - t_0)} \\
+ \frac{1}{2} b^2 \int_{t_1}^t e^{\int_{\tau}^t (2\rho_2(\theta, \omega) + 10\lambda_d)(s - t_0)} \|V(\tau)\|_{L^6}^6 d\tau \\
\leq \zeta^6 G^6(\omega)e^{\int_{t_1}^t (2\rho_2(\theta, \omega) - 6\lambda_d)(s - t_0)} + \frac{1}{2} b^2 \int_{t_1}^t e^{\int_{\tau}^t (2\rho_2(\theta, \omega) - 6\lambda_d)(s - t_0)} \Pi(\tau, \omega) d\tau.
\] (3.34)

Then (3.31) is valid with

\[
D(t, G, \omega) = \zeta^6 G^6(\omega)e^{\int_{t_1}^t (2\rho_2(\theta, \omega) - 6\lambda_d)(s - t_0)} + \frac{1}{2} b^2 \int_{t_1}^t e^{\int_{\tau}^t (2\rho_2(\theta, \omega) - 6\lambda_d)(s - t_0)} \Pi(\tau, \omega) d\tau.
\]

The proof is completed.

4. Pullback asymptotic compactness. In this section, we show that the Brus selator random dynamical system \( \varphi \) is pullback asymptotically compact in \( H \) by the following uniform Gronwall inequality, cf. [14].

**Proposition 4.1.** Given a natural number \( n > 1 \), let \( \beta, \zeta, \text{ and } h \) be nonnegative functions in \( L^1([-n, 0]; \mathbb{R}^+) \). Assume that \( \beta \) is absolutely continuous on \([-n, 0]\) and the following differential inequality is satisfied,

\[
\frac{d\beta}{dt} \leq \zeta \beta + h, \quad \text{for } t \in [-n, 0].
\]

If

\[
\int_t^{t+1} \zeta(\tau) d\tau \leq A, \quad \int_t^{t+1} \beta(\tau) d\tau \leq B, \quad \int_t^{t+1} h(\tau) d\tau \leq C,
\]

for any \( t \in [-n, -1] \), where \( A, B, \text{ and } C \) are some positive constants, then

\[
\beta(t) \leq (B + C)e^A \quad \text{for } t \in [-n + 1, 0].
\]

**Lemma 4.2.** For any given random variable \( R(\omega) > 0 \) and any initial data \((u_0, v_0) \in H \) with \( \|(u_0, v_0)\| \leq R(\omega) \), there exists a tempered random variable \( K(\omega) > 0 \), and a finite time \( T(R, \omega) < 0 \) such that if the initial time \( t_0 \leq T(R, \omega) \), then the weak solution \( g(t, \omega; t_0, g_0) \), where \( g_0 = e^{-\rho_2(\theta, \omega)}(u_0, v_0) \), of the problem of the random Brusselator reaction-diffusion system (2.10)-(2.13) satisfies \( g(0, \omega; t_0, g_0) \in E \) and

\[
\|g(0, \omega; t_0, g_0)\|_E^2 \leq K(\omega), \quad t_0 \leq T(R, \omega).
\] (4.1)

**Proof.** The proof is divided into three bootstrap steps. First we conduct estimates of the time average of the \( H_0^1(\Gamma) \)-norm for both the \( U \)-component and \( V \)-component of solutions on the time interval \([-4, -1] \). Second we apply the uniform Gronwall inequality (Proposition 4.1) to get the pointwise estimate of the \( U \)-component in
the time interval $[-2, 0]$. Third we use the results of previous two steps to get the pointwise estimate of the $V$-component in the time interval $[-1, 0]$.

**STEP 1.** In this step, we establish the time-average estimates of the of $E$-norm for the weak solutions $(U, V)$. Note that the estimate of $L^6(\Gamma)$-norm of the $V$-component of the weak solution has been obtained in Lemma 3.4. Since $z(\theta, \omega)$ is continuous in $t$, we see that $Z(\omega) = \max_{-4 \leq t \leq -1} |z(\theta, \omega)|$ is a positive constant for every given $\omega \in \Omega$. Fix the initial time $t_0 \leq \min\{T(R, \omega), -4\}$, here $T(R, \omega)$ comes from Lemma 3.1, integrate the second inequality of (3.3) over $[t, t+1]$, where $-4 \leq t \leq -1$, and by (3.9) we have

$$
\begin{align*}
&\int_t^{t+1} 2d_2\|\nabla V(\tau, \omega; t_0, g_0)\|^2 d\tau \\
\leq &\int_t^{t+1} 2\rho z(\theta, \omega) \left(1 + \frac{b^2|\Gamma|}{2} \int_{-\infty}^{\tau} e^{\int_0^\tau (2\rho z(\theta, \omega) - 2\lambda dgs) ds - 2\rho z(\theta, \omega)} d\xi \right) d\tau \\
&+ \frac{b^2|\Gamma|}{2} \int_t^{t+1} e^{-2\rho z(\theta, \omega)} d\tau + \|V(t)\|^2 \\
\leq &\int_{-4}^{0} 2c|z(\theta, \omega)| \left(1 + \frac{b^2|\Gamma|}{2} \int_{-\infty}^{\tau} e^{\int_0^\tau (2\rho z(\theta, \omega) - 2\lambda dgs) ds - 2\rho z(\theta, \omega)} d\xi \right) d\tau \\
&+ \frac{b^2|\Gamma|}{2} \int_{-4}^{0} e^{-2\rho z(\theta, \omega)} d\tau + 1 \\
&+ \frac{b^2|\Gamma|}{2} \max_{-4 \leq t \leq -1} \int_{-\infty}^{t} e^{\int_0^t (2\rho z(\theta, \omega) - 2\lambda dgs) ds - 2\rho z(\theta, \omega)} d\tau.
\end{align*}
$$

(4.2)

Then for $t_0 \leq \min\{T(R, \omega), -4\}$ and $-4 \leq t \leq -1$, we have

$$
\int_t^{t+1} \|\nabla V(\tau, \omega; t_0, g_0)\|^2 d\tau \leq \frac{K_1(\omega)}{2d_2},
$$

(4.3)

where

$$
K_1(\omega) = \int_{-4}^{0} 2\rho|z(\theta, \omega)| \left(1 + \frac{b^2|\Gamma|}{2} \int_{-\infty}^{\tau} e^{\int_0^\tau (2\rho z(\theta, \omega) - 2\lambda dgs) ds - 2\rho z(\theta, \omega)} d\xi \right) d\tau \\
+ \frac{b^2|\Gamma|}{2} \int_{-4}^{0} e^{-2\rho z(\theta, \omega)} d\tau + 1 \\
+ \frac{b^2|\Gamma|}{2} \max_{-4 \leq t \leq -1} \int_{-\infty}^{t} e^{\int_0^t (2\rho z(\theta, \omega) - 2\lambda dgs) ds - 2\rho z(\theta, \omega)} d\tau.
$$

In particular, let $t = -4$ and we have

$$
\int_{-4}^{-3} \|\nabla V(\tau, \omega; t_0, g_0)\|^2 d\tau \leq \frac{K_1(\omega)}{2d_2}.
$$

(4.4)

By the Mean Value Theorem, there is a time $t_1 \in [-4, -3)$ such that

$$
\|V(t_1, \omega; t_0, g_0)\|_E \leq \frac{K_1(\omega)}{2d_2}.
$$

(4.5)

Then by Lemma 3.4, there is a random variable $D(t, K_1/(2d_2), \omega) > 0$ such that

$$
\|V(t, \omega; t_0, g_0)\|_E^6 \leq D(t, K_1/(2d_2), \omega), \quad \text{for any } t \in [t_1, 0], t_0 \leq T(R, \omega).
$$

(4.6)
Fix any initial time $t_0 \leq \min\{T(R, \omega), -4\}$. Integrating the inequality of (3.13) over $[t, t+1]$, where $-4 \leq t \leq -1$, in view of (3.17) and (4.5) we have

\[
\int_t^{t+1} d_t \| \nabla y(\tau, \omega; t_0, g_0) \|^2 d\tau 
\]

\[
\leq \left( \frac{|d_2 - d_1|^2}{d_1} + 2\lambda \right) \int_t^{t+1} \| \nabla V \|^2 d\tau + \int_t^{t+1} (2\rho z(\theta, \omega) - 1) \| y(\tau) \|^2 d\tau + \| y(t) \|^2 
\]

\[
+ 2a^2 \Gamma \int_t^{t+1} e^{-2\rho z(\theta, \omega)} d\tau 
\]

\[
\leq \left( \frac{|d_2 - d_1|^2}{d_1} + 2\lambda \right) \frac{K_1(\omega)}{2d_2} + \max_{-4 \leq t \leq 0} C_2(t, \omega) \int_{-4}^0 |2\rho z(\theta, \omega) - 1| d\tau 
\]

\[
+ \max_{-4 \leq t \leq -1} C_2(t, \omega) + 2a^2 \Gamma \int_{-4}^0 e^{-2\rho z(\theta, \omega)} d\tau. 
\]

Consequently, for $t_0 \leq \min\{T(R, \omega), -4\}$ and $-4 \leq t \leq -1$, it holds that

\[
\int_t^{t+1} \| \nabla U(\tau, \omega; t_0, g_0) \|^2 d\tau = \int_t^{t+1} \| \nabla y(\tau, \omega; t_0, g_0) - \nabla V(\tau, \omega; t_0, g_0) \|^2 d\tau 
\]

\[
\leq \int_t^{t+1} \| \nabla V(\tau, \omega; t_0, g_0) \|^2 + \| \nabla V(\tau, t_0, \omega, V_0) \|^2 d\tau 
\]

\[
\leq \frac{K_1(\omega)}{d_2} + \frac{2K_2(\omega)}{d_1}, 
\]

where

\[
K_2(\omega) = \left( \frac{|d_2 - d_1|^2}{d_1} + 2\lambda \right) \frac{K_1(\omega)}{2d_2} + \max_{-4 \leq t \leq 0} C_2(t, \omega) \int_{-4}^0 |2\rho z(\theta, \omega) - 1| d\tau 
\]

\[
+ \max_{-4 \leq t \leq -1} C_2(t, \omega) + 2a^2 \Gamma \int_{-4}^0 e^{-2\rho z(\theta, \omega)} d\tau. 
\]

STEP 2. Now we conduct the estimates of $H^1_\omega(\Gamma)$-norm for the $U$-component of the weak solutions. Taking the inner product of (2.10) with $-\Delta U(t)$, we get

\[
\frac{1}{2} \frac{d}{dt} \| \nabla U \|^2 + d_1 \| \Delta U \|^2 + (b+1) \| \nabla U \|^2 
\]

\[
= \int_\Gamma \left( -ae^{-\rho z(\theta, \omega)} \Delta U - e^{2\rho z(\theta, \omega)} V \Delta U \right) dx + \rho z(\theta, \omega) \| \nabla U \|^2 
\]

\[
\leq \left( \frac{d_1}{4} + \frac{d_1}{4} \right) \| \Delta U \|^2 + \left( \frac{1}{4} \| \Delta u \| \right) e^{-2\rho z(\theta, \omega)} + \frac{1}{d_1} e^{4\rho z(\theta, \omega)} \int_\Gamma U^4 V^2 dx + \rho z(\theta, \omega) \| \nabla U \|^2. 
\]

It follows that

\[
\frac{d}{dt} \| \nabla U \|^2 + d_1 \| \Delta U \|^2 + 2(b+1) \| \nabla U \|^2 
\]

\[
\leq 2a^2 \Gamma \| e^{-2\rho z(\theta, \omega)} + \frac{2}{d_1} e^{4\rho z(\theta, \omega)} \| U \|_2 \| V \|_2 \| \nabla U \|^2 + 2\rho z(\theta, \omega) \| \nabla U \|^2 
\]

\[
\leq 2a^2 \Gamma \| e^{-2\rho z(\theta, \omega)} + \frac{2}{d_1} e^{4\rho z(\theta, \omega)} \| \nabla U \|^2 \| V \|_2 \| \nabla U \|^2 
\]

\[
\leq 2a^2 \Gamma \| e^{-2\rho z(\theta, \omega)} + \left( \frac{2}{d_1} e^{4\rho z(\theta, \omega)} \| \nabla U \|^2 \| V \|_2 \| \nabla U \|^2. 
\]
After dropping the terms $d_1\|\Delta U\|^2$ and $2(b + 1)\|\nabla U\|^2$ from the left-hand side of the inequality (4.9), it can be written as

\[
\frac{d\beta}{dt} \leq \alpha(t)\beta(t) + \gamma(t), \quad t \in [-3, 0],
\]

where

\[
\beta(t) = \|\nabla U\|^2, \\
\alpha(t) = \frac{2c^4}{d_1}e^{4\rho z(\theta, \omega)}\|\nabla U\|^2\|V\|_{L^2}^2 + 2\rho z(\theta, \omega), \text{ and} \\
\gamma(t) = \frac{2a^2|\Gamma|}{d_1}e^{-2\rho z(\theta, \omega)}.
\]

For $t_0 \leq T(R, \omega)$ and $-3 \leq t \leq -1$ and, we obtain the following estimates: By (4.8),

\[
\int_t^{t+1} \beta(\tau) d\tau = \int_t^{t+1} \|\nabla U\|^2 d\tau \leq \frac{K_1(\omega)}{d_2} + \frac{2K_2(\omega)}{d_1}.
\]

By (4.8) and (4.6)

\[
\int_t^{t+1} \alpha(\tau) d\tau = \int_t^{t+1} \frac{2c^4}{d_1}e^{4\rho z(\theta, \omega)}\|\nabla U\|^2\|V\|_{L^2}^2 d\tau + \int_t^{t+1} 2\rho z(\theta, \omega) d\tau \\
\leq \frac{2c^4}{d_1} \max_{-3 \leq \tau \leq 0} \left[ D^{1/3} \left( \tau, \frac{K_1}{2d_2}, \omega \right) e^{4\rho z(\theta, \omega)} \right] \left[ \frac{K_1(\omega)}{d_2} + \frac{2K_2(\omega)}{d_1} \right] + 2 \int_{-3}^0 g(z(\theta, \omega)) d\tau,
\]

and

\[
\int_t^{t+1} \gamma(\tau) d\tau = \frac{2a^2|\Gamma|}{d_1} \int_t^{t+1} e^{-2\rho z(\theta, \omega)} d\tau \leq \frac{2a^2|\Gamma|}{d_1} \int_{-3}^0 e^{-2\rho z(\theta, \omega)} d\tau.
\]

Apply the Uniform Gronwall Inequality (Proposition 4.1) with the above three estimates to get

\[
\|U(t, \omega; t_0, g_0)\|_E \leq K_3(\omega), \quad t \in [-2, 0], \ t_0 \leq T(R, \omega),
\]

where

\[
K_3(\omega) = \left( \frac{K_1(\omega)}{d_2} + \frac{2K_2(\omega)}{d_1} + \int_{-3}^0 \frac{2a^2|\Gamma|}{d_1} e^{-2\rho z(\theta, \omega)} d\tau \right) \exp \left\{ \frac{2c^4}{d_1} \max_{-3 \leq \tau \leq 0} \left[ D^{1/3} \left( \tau, \frac{K_1}{2d_2}, \omega \right) e^{4\rho z(\theta, \omega)} \right] \left[ \frac{K_1(\omega)}{d_2} + \frac{2K_2(\omega)}{d_1} \right] + 2 \int_{-3}^0 g(z(\theta, \omega)) d\tau \right\}.
\]

STEP 3. We wrap up the proof by conducting the estimates of $H_0^1(\Gamma)$-norm for the $V$-component of the weak solutions based on the results of Step 1 and Step 2. Taking the inner product of (2.11) with $-\Delta V(t)$, we get

\[
\frac{1}{2} \frac{d}{dt} \|\nabla V\|^2 + d_2\|\Delta V\|^2 \\
= \int_{\Gamma} \left( -bU \Delta V + e^{2\rho z(\theta, \omega)} U^2 V \Delta V \right) dx + \rho z(\theta, \omega)\|\nabla V\|^2 \\
\leq \left( \frac{d_2}{4} + \frac{d_2}{4} \right) \|\Delta V\|^2 + \frac{b^2}{d_2} \|U\|^2 + \frac{1}{d_2} e^{4\rho z(\theta, \omega)} \int_{\Gamma} U^4 V^2 dx + \rho z(\theta, \omega)\|\nabla V\|^2
\]
It follows that
\[
\frac{d}{dt} \|\nabla V\|^2 + d_2 \|\Delta V\|^2 \\
\leq 2b_2^2 \|U\|^2 + \frac{2}{d_2} e^{4\rho z(\theta_t \omega)} \|U\|_4^4 \|V\|^2_{L^6} + 2\rho z(\theta_t \omega) \|\nabla V\|^2 
\tag{4.12}
\]
\[
\leq 2b_2^2 \|U\|^2 + \frac{2\zeta_6}{d_2} e^{4\rho z(\theta_t \omega)} \|\nabla U\|^4 \|\nabla V\|^2 + 2\rho z(\theta_t \omega) \|\nabla V\|^2.
\]
After dropping the terms \(d_2 \|\Delta V\|^2\) from the left-hand side, the inequality (4.12) can be written as
\[
\frac{d}{dt} \tilde{\beta}(t) \leq \alpha(t) \tilde{\beta}(t) + \gamma(t), \quad t \in [-2, 0],
\tag{4.13}
\]
where
\[
\tilde{\beta}(t) = \|\nabla V\|^2,
\]
\[
\alpha(t) = \frac{2\zeta_6}{d_2} e^{4\rho z(\theta_t \omega)} \|\nabla U\|^4 + 2\rho z(\theta_t \omega),
\]
and
\[
\gamma(t) = \frac{2b_2^2}{d_2} \|U\|^2.
\]
For \(-2 \leq t \leq -1\) and \(t_0 \leq \min\{T(R, \omega), -4\}\), we have the following estimates: It follows from (4.3) that
\[
\int_t^{t+1} \tilde{\beta}(\tau) d\tau = \int_t^{t+1} \|\nabla V\|^2 d\tau \leq \frac{K_1(\omega)}{2d_2}.
\]
By (4.11),
\[
\int_t^{t+1} \alpha(\tau) d\tau = \frac{2\zeta_6}{d_2} \int_t^{t+1} e^{4\rho z(\theta_\tau \omega)} \|\nabla U\|^4 d\tau + 2 \int_t^{t+1} \rho z(\theta_\tau \omega) d\tau
\]
\[
\leq \frac{2\zeta_6}{d_2} \max_{-2 \leq \tau \leq 0} (e^{4\rho z(\theta_\tau \omega)}) K_3^2(\omega) + 2 \int_{-2}^0 c|z(\theta_\tau \omega)| d\tau,
\]
and by (4.8)
\[
\int_t^{t+1} \gamma(\tau) d\tau = \frac{2b_2^2}{d_2} \int_t^{t+1} \|U\|^2 d\tau \leq \frac{2b_2^2 \lambda^2}{d_2} \int_t^{t+1} \|\nabla U\|^2
\]
\[
\leq \frac{2b_2^2 \lambda^2}{d_2} \left( \frac{K_1(\omega)}{d_2} + \frac{2K_2(\omega)}{d_1} \right).
\]
Apply the Uniform Gronwall Inequality again with the above three estimates to obtain
\[
\|V(t, \omega; t_0, g_0)\|_{L^\infty}^2 \leq K_4(\omega), \quad t \in [-1, 0], \ t_0 \leq \min\{T(R, \omega), -4\},
\tag{4.14}
\]
where
\[
K_4(\omega) = \left( \frac{K_1(\omega)}{2d_2} + \frac{2b_2^2 \lambda^2}{d_2} \left( \frac{K_1(\omega)}{d_2} + \frac{2K_2(\omega)}{d_1} \right) \right) \cdot \exp \left[ \frac{2\zeta_6}{d_2} \max_{-2 \leq \tau \leq -1} (e^{4\rho z(\theta_\tau \omega)}) K_3^2(\omega) + 2 \int_{-2}^0 c|z(\theta_\tau \omega)| d\tau \right].
\]
Finally, set \(t = 0\) in (4.11) and (4.14). Thus (4.1) holds with \(K(\omega) = K_3(\omega) + K_4(\omega)\). The proof is completed. \(\square\)
5. Main results on the random attractor. In this section, we finally prove the existence of a random attractor for the Brusselator random dynamical system \( \varphi \) in the phase space \( H \). Moreover, we show that it has the \( H \) to \( E \) attracting regularity.

**Theorem 5.1.** For any positive parameters \( d_1, d_2, a, b \) and \( \rho \), there exists a unique random attractor \( \mathcal{A} = \{ \mathcal{A}(\omega) \}_{\omega \in \Omega} \) in the phase space \( H \) for the Brusselator random dynamical system \( \varphi \) over the MDS \( (\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}}) \).

**Proof.** By Lemma 3.2, the RDS \( \varphi \) has a pullback absorbing set with respect to the universe \( \mathcal{D} \), which is the closed random ball \( B_0(\bar{\omega}) \) centered at the origin with radius \( M_0(\omega) \) in \( H \).

Lemma 4.2 and the compact imbedding \( E \hookrightarrow H \) imply that the RDS \( \varphi \) is pullback asymptotically compact in \( H \) with respect to \( \mathcal{D} \).

According to Proposition 2.8, there exists a unique random attractor \( \mathcal{A} = \{ \mathcal{A}(\omega) \}_{\omega \in \Omega} \) in \( H \) for this RDS \( \varphi \), which is given by

\[
\mathcal{A}(\omega) = \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} \varphi(t, \theta_{-\tau} \omega, B_0(\theta_{-t} \omega)), \quad \omega \in \Omega
\]

Therefore, we reach the conclusion. \( \square \)

Now we show that the random attractor \( \mathcal{A}(\omega) \) is an \( (H, E) \) random attractor. This concept is a generalization of \( (H, E) \) global attractor introduced in [2].

**Definition 5.2.** Let \( \{ \Sigma(t, \omega) \}_{t \geq 0} \) be a random dynamical system on a Banach space \( X \) over a given metric dynamical system and let \( Y \) be a compactly imbedded subspace of \( X \). Let a universe \( \mathcal{D} \) of tempered random sets in a Banach space \( X \) be given. A subset \( \mathcal{A} \in \mathcal{D} \) is called an \( (X, Y) \) random attractor for this RDS, if \( \mathcal{A}(\omega) \) has the following properties:

(i) \( \mathcal{A} \) is a nonempty, compact, and invariant random set in \( Y \).

(ii) \( \mathcal{A} \) attracts any set \( B \in \mathcal{D} \) with respect to the \( Y \)-norm. Namely, there exists \( \tau = \tau_B > 0 \) such that \( \Sigma(t, \theta_{-\tau} \omega, B(\theta_{-t} \omega)) \subset Y \) for \( t > \tau \) and

\[
\operatorname{dist}_Y(\Sigma(t, \theta_{-\tau} \omega, B(\theta_{-t} \omega)), \mathcal{A}(\omega)) \to 0, \quad \text{as} \ t \to \infty.
\]

**Definition 5.3.** Let \( \mathcal{D} \) be a universe of sets in a Banach space \( X \). A random dynamical system \( (\varphi, \theta) \) on \( X \) is said to be \( \mathcal{D} \)-flattening if for every \( B \in \mathcal{D} \), \( \epsilon > 0 \) and \( \omega \in \Omega \), there exists a \( T_0(B, \omega, \epsilon) > 0 \) and a finite dimensional subspace of \( X_1(\epsilon) \) (which may depend on \( \epsilon \)) of \( X \), such that the following two conditions are satisfied.

(i) \( \bigcup_{t \geq T_0} Q \varphi(t, \theta_{-t} \omega, B(\theta_{-t} \omega)) \) is bounded in \( X \),

(ii) \( \|(I - Q)(\varphi(t, \theta_{-t} \omega, B(\theta_{-t} \omega)))\|_X < \epsilon \),

where \( Q : X \to X_1(\epsilon) \) is a bounded projection.

Let \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \to \infty \) as \( j \to \infty \) be the complete set of eigenvalues, each repeated to its multiplicity, of the differential operator \( -A : [H^2(\Gamma) \cap H_0^1(\Gamma)]^2 \to [L^2(\Gamma)]^2 \) defined by (2.2) and \( \{e_j\}_{j=1}^\infty \) be the corresponding eigenvectors. Let \( Q_n : H \to \operatorname{Span}\{e_1, \cdots, e_n\} \) be the orthogonal projection from \( H \) onto the subspace spanned by the first \( n \) eigenvectors. Then every \( u \in H \) has a unique orthogonal decomposition \( u = u_1 + u_2 \), where \( u_1 = Q_n u \) and \( u_2 = (I - Q_n) u \) are called low modes and high modes, respectively.

The following proposition is proved in [10].

**Proposition 5.4.** Given a uniformly convex Banach space \( X \) and a universe of random sets \( \mathcal{D} \) in \( X \), let \( \varphi \) be a continuous RDS on \( X \) over an MDS \( (\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}}) \).
Suppose that there exists a closed pullback absorbing set \( \{ K(\omega) \}_{\omega \in \Omega} \in \mathcal{D} \) and the RDS \( \varphi \) is \( \mathcal{D} \)-flattening, then the RDS \( \varphi \) has a unique random attractor \( A = \{ A(\omega) \}_{\omega \in \Omega} \in \mathcal{D} \) in \( X \), which is given by

\[
A(\omega) = \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} \varphi(t, \theta_{-\tau} \omega, K(\theta_{-\tau} \omega)).
\]

Next we show that the Brusselator random dynamical system \( \varphi \) possesses the flattening property in the more regular space \( E \). The first condition of the flattening property is clearly satisfied due to Lemma 4.2. To prove the second condition, we decompose the weak solution \( g(t, \omega; t_0, g_0) \) of the random Brusselator reaction-diffusion system (2.10)-(2.13) as the high modes and the low modes,

\[
g(t, \omega) = g_1(t, \omega) + g_2(t, \omega), \quad \text{where} \quad g_1 = Q_n g, \quad g_2 = (I - Q_n) g.
\]

**Lemma 5.5.** For any given \( \epsilon > 0 \) and any initial data \((u_0, v_0) \in H\) with \( \|(u_0, v_0)\| \leq R(\omega) \), where \( R(\omega) \) is an arbitrarily given tempered positive random variable, there exists a finite \( T_E(R, \omega) = \min \{ T(R, \omega), -4 \} < 0 \), where \( T(R, \omega) \) is given in Lemma 3.1, and a positive integer \( N(\epsilon, \omega) \) such that the high modes \( g_2(t, \omega; t_0, g_0) \) satisfy

\[
\| e^{\rho_z(\omega)} g_2(0, \omega; t_0, g_0) \|_E < \epsilon, \quad \text{for all} \quad t_0 \leq T_E(R, \omega), \quad n > N(\epsilon, \omega).
\]

**Proof.** Taking the inner product of (2.10) with \( -\Delta U_2(t) \), we get

\[
\frac{1}{2} \frac{d}{dt} \| \nabla U_2 \|^2 + d_1 \| \Delta U_2 \|^2 + (b + 1) \| \nabla U_2 \|^2
\]

\[
= \int \left( -ae^{-\rho_z(\theta, \omega)} \Delta U_2 - e^{2\rho_z(\theta, \omega)} U_2 \Delta U_2 \right) dx + \rho_z(\theta, \omega) \| \nabla U_2 \|^2
\]

\[
\leq \left( \frac{d_1}{4} + \frac{d_1}{4} \right) \| \Delta U_2 \|^2 + \frac{\alpha^2 |\Gamma|}{d_1} e^{-2\rho_z(\theta, \omega)} + \frac{1}{d_1} e^{4\rho_z(\theta, \omega)} \int \Gamma U^4 V^2 dx + \rho_z(\theta, \omega) \| \nabla U_2 \|^2.
\]

It follows that for any \( t_0 \leq T_E(R, \omega) \) and \( t \in [-1, 0] \),

\[
\frac{d}{dt} \| \nabla U_2 \|^2 + d_1 \lambda_{n+1} \| \nabla U_2 \|^2
\]

\[
\leq \frac{2a^2 |\Gamma|}{d_1} e^{-2\rho_z(\theta, \omega)} + \frac{2}{d_1} e^{4\rho_z(\theta, \omega)} \| U \|^4_{L^6} \| V \|^2_{L^6} + 2\rho_z(\theta, \omega) \| \nabla U_2 \|^2
\]

\[
\leq \frac{2a^2 |\Gamma|}{d_1} e^{-2\rho_z(\theta, \omega)} + \frac{2}{d_1} e^{4\rho_z(\theta, \omega)} \zeta^6 \| \nabla U \|^4 \| \nabla V \|^2 + 2\rho_z(\theta, \omega) \| \nabla U_2 \|^2
\]

\[
\leq \frac{2a^2 |\Gamma|}{d_1} e^{-2\rho_z(\theta, \omega)} + \frac{2}{d_1} e^{4\rho_z(\theta, \omega)} \zeta^6 K_3^2(\omega) K_4(\omega) + 2\rho_z(\theta, \omega) \| \nabla U_2 \|^2.
\]

Multiply (5.2) by \( e^{\int_{\sigma}^t (2\rho_z(\theta, \omega) - d_1 \lambda_{n+1}) ds} \) and integrate the resulting inequality over \([\sigma, t], \) for \( -1 \leq \sigma < t \leq 0 \) to obtain

\[
\| \nabla U_2(t, \omega; t_0, g_0) \|^2 \leq \| \nabla U_2(\sigma, \omega; t_0, g_0) \|^2 e^{\int_{\sigma}^t (2\rho_z(\theta, \omega) - d_1 \lambda_{n+1}) ds} + \frac{2a^2 |\Gamma|}{d_1} \int_{\sigma}^{t} e^{\int_{\tau}^t (2\rho_z(\theta, \omega) - d_1 \lambda_{n+1}) ds - 2\rho_z(\theta, \omega) d\tau} \]

\[
+ \frac{2}{d_1} e^{4\rho_z(\theta, \omega)} \zeta^6 K_3^2(\omega) K_4(\omega) + 2\rho_z(\theta, \omega) \| \nabla U_2 \|^2.
\]
Consequently, there exists an integer $n_1(\epsilon, \omega) \geq 1$ such that for every $\omega \in \Omega$ and $t \in [-1, 0]$,

$$
\|e^{\rho z(\theta \omega)} \nabla U_2(t, \omega; t_0, g_0)\|^2 \leq \max_{t \in [-1, 0]} e^{2\rho z(\theta \omega)} \|\nabla U_2(t, \omega; t_0, g_0)\|^2 < \frac{\epsilon^2}{2},
$$

whenever $t_0 \leq T_E(R, \omega)$, $n > n_1(\epsilon, \omega)$.

On the other hand, taking the inner product of (2.11) with $-\Delta V_2(t)$, we get

$$
\int \left(-b U \Delta V_2 + e^{2\rho z(\theta \omega)} U^2 V \Delta V_2 \right) dx + \rho z(\theta \omega) \|\nabla V_2\|^2 \\
= \left(\frac{d_2}{4} + \frac{d_2}{4}\right) \|\Delta V_2\|^2 + \frac{k_2}{d_2} \|U\|^2 + \frac{1}{d_2} e^{4\rho z(\theta \omega)} \int U^4 V^2 dx + \rho z(\theta \omega) \|\nabla V_2\|^2.
$$

It follows that

$$
\frac{d}{dt} \|\nabla V_2\|^2 + d_2 \|\Delta V_2\|^2 \\
\leq \frac{2b_2}{d_2} \|U\|^2 + \frac{2}{d_2} e^{4\rho z(\theta \omega)} \|U\|^2 \|V\|^2 + 2\rho z(\theta \omega) \|\nabla V_2\|^2 \\
\leq \frac{2b_2}{d_2} C_2(t, \omega) + \frac{2k_2}{d_2} e^{4\rho z(\theta \omega)} K_3(\omega) K_4(\omega) + 2\rho z(\theta \omega) \|\nabla V_2\|^2,
$$

for $t \in [-1, 0]$ and $t_0 \leq T_E(R, \omega)$. Multiply (5.5) by $e^{\int_{\sigma}^{t} (2\rho z(\theta \omega) - d_2 \lambda_{n+1}) ds}$ and integrate the resulting inequality over $[\sigma, t]$ where $-1 \leq \sigma < t \leq 0$ to get

$$
\|\nabla V_2(t, \omega; t_0, g_0)\|^2 \leq \|\nabla V_2(\sigma, \omega; t_0, g_0)\|^2 e^{\int_{\sigma}^{t} (2\rho z(\theta \omega) - d_2 \lambda_{n+1}) ds}
$$
Similar to (5.3) and (5.4), there exists an integer $N \geq 1$ such that for every $\omega \in \Omega$ and $t \in [-1,0]$,

$$
\|e^{\psi(t, \omega)} \nabla V_2(t, \omega; t_0, g_0)\| \leq \max_{t \in [-1,0]} e^{2\psi(t, \omega)} \| \nabla V_2(t, \omega; t_0, g_0)\| < \frac{\epsilon^2}{2},
$$

(5.7)

whenever $t_0 \leq T_E(R, \omega)$, $n > N(\epsilon, \omega)$. Adding up (5.4) and (5.7), we reach (5.1). The proof is completed.

**Theorem 5.6.** The random attractor $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega \in \Omega}$ for the Brusselator random dynamical system $\varphi$ shown in Theorem 5.1 is indeed an $(H, E)$ random attractor.

**Proof.** Since Lemma 5.5 along with (3.23) shows that the flattening property with respect to $E$ is satisfied by the RDS $\varphi$ and Lemma 4.2 together with (3.23) confirms that there exists a pullback absorbing ball centered at the origin with the radius $e^{\psi(\omega)} K(\omega)$ in the space $E$. Therefore, by Proposition 5.4 with $X = E$, there exists a unique random attractor $\mathcal{A}_E$ for the Brusselator random dynamical system $\varphi$ in $E$.

By the mutual attraction and the invariance of both random attractors $\mathcal{A}$ in $H$ from Theorem 5.1 and $\mathcal{A}_E$ in $E$, we see that $\mathcal{A}_E = \mathcal{A}$ and $\mathcal{A}$ is an $(H, E)$ random attractor. The proof is completed.

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