ON LESIEUR’S MEASURED QUANTUM GROUPOIDS

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ABSTRACT. In his thesis ([L1]), which is published in an expended and revised version ([L2]), Franck Lesieur had introduced a notion of measured quantum groupoid, in the setting of von Neumann algebras, using intensively the notion of pseudo-multiplicative unitary, which had been introduced in a previous article of the author, in collaboration with Jean-Michel Vallin [EV]. In [L2], the axioms given are very complicated and are here simplified.
1. Introduction

1.1. In two articles ([Val1], [Val2]), J.-M. Vallin has introduced two notions (pseudo-multiplicative unitary, Hopf-bimodule), in order to generalize, up to the groupoid case, the classical notions of multiplicative unitary [BS] and of Hopf-von Neumann algebras [ES] which were introduced to describe and explain duality of groups, and led to appropriate notions of quantum groups ([ES], [W1], [W2], [BS], [MN], [W3], [KV1], [KV2], [MNW]).

In another article [EV], J.-M. Vallin and the author have constructed, from a depth 2 inclusion of von Neumann algebras $M_0 \subset M_1$, with an operator-valued weight $T_1$ verifying a regularity condition, a pseudo-multiplicative unitary, which led to two structures of Hopf bimodules, dual to each other. Moreover, we have then constructed an action of one of these structures on the algebra $M_1$ such that $M_0$ is the fixed point subalgebra, the algebra $M_2$ given by the basic construction being then isomorphic to the crossed-product. We construct on $M_2$ an action of the other structure, which can be considered as the dual action. If the inclusion $M_0 \subset M_1$ is irreducible, we recovered quantum groups, as proved and studied in former papers ([EN], [E1]).

Therefore, this construction leads to a notion of "quantum groupoid", and a construction of a duality within "quantum groupoids".

1.2. In a finite-dimensional setting, this construction can be mostly simplified, and is studied in [NV1], [BSz1], [BSz2], [Sz], [Val3], [Val4], and examples are described. In [NV2], the link between these "finite quantum groupoids" and depth 2 inclusions of $II_1$ factors is given.

1.3. Franck Lesieur introduced in his thesis [L1] a notion of "measured quantum groupoids", in which a modular hypothesis on the basis is required. Mimicking in a wider setting the technics of Kustermans and Vaes [KV], he obtained then a pseudo-multiplicative unitary, which, as in the quantum group case, "contains" all the information of the object (the von Neuman algebra, the coproduct, the antipod, the co-inverse). Unfortunately, the axioms chosen then by Lesieur don’t fit perfectly with the duality (namely, the dual object does not fit the modular condition on the basis chosen in [L1]), and, for this purpose, Lesieur gave the name of "measured quantum groupoids" to a wider class [L2], whose axioms could be described as the analog of [MNW], in which a duality is defined and studied, the initial objects considered in [L1] being denoted now "adapted measured quantum groupoids". In [E3] had been shown that, with suitable conditions, the objects constructed in [EV] from depth 2 inclusions, are "measured quantum groupoids" in this new setting.
1.4. Unfortunately, the axioms given in ([L2], 4) are very complicated, and there was a serious need for simplification. This is the goal of this article.

1.5. This article is organized as follows:
In chapter 2 are recalled all the definitions and constructions needed for that theory, namely Connes-Sauvageot’s relative tensor product of Hilbert spaces, fiber product of von Neumann algebras, and Vaes’ Radon-Nikodym theorem.

The chapter 3 is a résumé of Lesieur’s basic result ([L2], 3), namely the construction of a pseudo-multiplicative unitary associated to a Hopf-bimodule, when exist a left-invariant operator-valued weight, and a right-invariant valued weight.

The chapter 4 is mostly inspired from Lesieur’s ”adapted measured quantum groupoids” ([L2], 9), with a wider hypothesis, namely, that there exists a weight on the basis such that the modular automorphism groups of two lifted weights (via the two operator-valued weights) commute. This hypothesis allows us to use Vaes’ theorem, and is a nice generalization of the existence of a relatively invariant measure on the basis of a groupoid. With that hypothesis, mimicking ([L2], 9), we construct a co-inverse and a scaling group.

In chapter 5, we go on with the same hypothesis. It allows us to construct two automorphism groups on the basis, which appear to be invariant under the relatively invariant weight introduced in chapter 4.

It is then straightforward to get that we are now in presence of Lesieur’s ”measured quantum groupoids” ([L2], 4) and chapter 6 is devoted to main properties of these.

1.6. The author is indebted to Frank Lesieur, Stefaan Vaes, Leonid Vainerman, and especially Jean-Michel Vallin, for many fruitful conversations.

2. Preliminaries

In this chapter are mainly recalled definitions and notations about Connes’ spatial theory (2.1, 2.3) and the fiber product construction (2.4, 2.5) which are the main technical tools of the theory of measured quantum theory.

2.1. Spatial theory [C1], [S2], [T]. Let $N$ be a von Neumann algebra, and let $\psi$ be a faithful semi-finite normal weight on $N$; let $\mathfrak{N}_\psi$, $\mathfrak{M}_\psi$, $H_\psi$, $\pi_\psi$, $\Lambda_\psi$, $J_\psi$, $\Delta_\psi$,... be the canonical objects of the Tomita-Takesaki construction associated to the weight $\psi$. Let $\alpha$ be a non-degenerate normal representation of $N$ on a Hilbert space $\mathcal{H}$. We may as well consider $\mathcal{H}$ as a left $N$-module, and write it then $\alpha\mathcal{H}$. Following ([C1], definition 1), we define the set of $\psi$-bounded elements of $\alpha\mathcal{H}$ as:

$$D(\alpha\mathcal{H}, \psi) = \{\xi \in \mathcal{H}; \exists C < \infty, \|\alpha(y)\xi\| \leq C\|\Lambda_\psi(y)\|, \forall y \in \mathfrak{N}_\psi\}$$
Then, for any $\xi$ in $D(\alpha H, \psi)$, there exists a bounded operator $R^{\alpha, \psi}(\xi)$ from $H_\psi$ to $\mathcal{H}$, defined, for all $y$ in $\mathcal{M}_\psi$ by:

$$R^{\alpha, \psi}(\xi)\Lambda_\psi(y) = \alpha(y)\xi$$

This operator belongs to $\text{Hom}_N(H_\psi, \mathcal{H})$; therefore, for any $\xi, \eta$ in $D(\alpha H, \psi)$, the operator:

$$\theta^{\alpha, \psi}(\xi, \eta) = R^{\alpha, \psi}(\xi)R^{\alpha, \psi}(\eta)^*$$

belongs to $\alpha(N)^\prime$; moreover, $D(\alpha H, \psi)$ is dense ([C1], lemma 2), stable under $\alpha(N)^\prime$, and the linear span generated by the operators $\theta^{\alpha, \psi}(\xi, \eta)$ is a weakly dense ideal in $\alpha(N)^\prime$.

With the same hypothesis, the operator:

$$\langle \xi, \eta \rangle^{\alpha, \psi} = R^{\alpha, \psi}(\eta)^*R^{\alpha, \psi}(\xi)$$

belongs to $\pi_\psi(N)^\prime$. Using Tomita-Takesaki’s theory, this last algebra is equal to $J_\psi\pi_\psi(N)J_\psi$, and therefore anti-isomorphic to $N$ (or isomorphic to the opposite von Neumann algebra $N^\alpha$). We shall consider now $\langle \xi, \eta \rangle^{\alpha, \psi}$ as an element of $N^\alpha$, and the linear span generated by these operators is a dense algebra in $N^\alpha$. More precisely ([C], lemma 4, and [S2], lemme 1.5b), we get that $\langle \xi, \eta \rangle^{\alpha, \psi}$ belongs to $\mathcal{M}_\psi$, and that:

$$\Lambda_\psi(\langle \xi, \eta \rangle^{\alpha, \psi}) = J_\psi R^{\alpha, \psi}(\xi)^*\eta$$

If $y$ in $N$ is analytical with respect to $\psi$, and if $\xi \in D(\alpha H, \psi)$, then we get that $\alpha(y)\xi$ belongs to $D(\alpha H, \psi)$ and that:

$$R^{\alpha, \psi}(\alpha(y)\xi) = R^{\alpha, \psi}(\xi)J_\psi\sigma_\psi^{-1/2}(y^*)J_\psi$$

So, if $\eta$ is another $\psi$-bounded element of $\alpha H$, we get:

$$\langle \alpha(y)\xi, \eta \rangle^{\alpha, \psi} = \sigma_\psi^{1/2}(y)\langle \xi, \eta \rangle^{\alpha, \psi}$$

There exists ([C], prop.3) a family $(e_i)_{i \in I}$ of $\psi$-bounded elements of $\alpha H$, such that:

$$\sum_i \theta^{\alpha, \psi}(e_i, e_i) = 1$$

Such a family will be called an $(\alpha, \psi)$-basis of $\mathcal{H}$.

It is possible ([EN] 2.2) to construct an $(\alpha, \psi)$-basis of $\mathcal{H}$, $(e_i)_{i \in I}$, such that the operators $R^{\alpha, \psi}(e_i)$ are partial isometries with final supports $\theta^{\alpha, \psi}(e_i, e_i)$ 2 by 2 orthogonal, and such that, if $i \neq j$, then $\langle e_i, e_j \rangle^{\alpha, \psi} = 0$. Such a family will be called an $(\alpha, \psi)$-orthogonal basis of $\mathcal{H}$.

We have, then:

$$R^{\alpha, \psi}(\xi) = \sum_i \theta^{\alpha, \psi}(e_i, e_i)R^{\alpha, \psi}(\xi) = \sum_i R^{\alpha, \psi}(e_i) \langle \xi, e_i \rangle^{\alpha, \psi}$$

$$\langle \xi, \eta \rangle^{\alpha, \psi} = \sum_i \langle \eta, e_i \rangle^{\alpha, \psi} \langle \xi, e_i \rangle^{\alpha, \psi}$$
the sums being weakly convergent. Moreover, we get that, for all \( n \in N \), \( \theta^{\alpha,\psi}(e_i)  \alpha(n)  e_i = \alpha(n)  e_i \), and \( \theta^{\alpha,\psi}(e_i, e_i) \) is the orthogonal projection on the closure of the subspace \( \{ \alpha(n)  e_i, n \in N \} \).

If \( \theta \in Aut N \), then it is straightforward to get that \( D(\alpha \circ \theta, \psi) = D(\alpha, \psi) \), and then, we get that, for any \( \xi, \eta \in D(\alpha, \psi) \):

\[
< \xi, \eta >_{\alpha \circ \theta, \psi} = \theta^{-1} ( < \xi, \eta >_{\alpha, \psi} )
\]

Let \( \beta \) be a normal non-degenerate anti-representation of \( N \) on \( \mathcal{H} \). We may then as well consider \( \mathcal{H} \) as a right \( N \)-module, and write it \( \mathcal{H}_\beta \), or consider \( \beta \) as a normal non-degenerate representation of the opposite von Neumann algebra \( N^\circ \), and consider \( \mathcal{H} \) as a left \( N^\circ \)-module.

We can then define on \( N^\circ \) the opposite faithful semi-finite normal weight \( \psi^\circ \); we have \( \mathcal{M}_{\psi^\circ} = \mathcal{M}_\psi^* \), and the Hilbert space \( H_{\psi^\circ} \) will be, as usual, identified with \( H_\psi \), by the identification, for all \( x \in \mathcal{M}_\psi \), of \( \Lambda_\psi(x^*) \) with \( J_\psi \Lambda_\psi(x) \).

From these remarks, we infer that the set of \( \psi^\circ \)-bounded elements of \( \mathcal{H}_\beta \) is:

\[
D(\mathcal{H}_\beta, \psi^\circ) = \{ \xi \in \mathcal{H}; \exists C < \infty, \| \beta(y^*) \xi \| \leq C \| \Lambda_\psi(y) \|, \forall y \in \mathcal{M}_\psi \}
\]

and, for any \( \xi \in D(\mathcal{H}_\beta, \psi^\circ) \) and \( y \in \mathcal{M}_\psi \), the bounded operator \( R^{\beta,\psi^\circ}(\xi) \) is given by the formula:

\[
R^{\beta,\psi^\circ}(\xi)J_\psi \Lambda_\psi(y) = \beta(y^*)\xi
\]

This operator belongs to \( Hom_{N^\circ}(H_\psi, \mathcal{H}) \). Moreover, \( D(\mathcal{H}_\beta, \psi^\circ) \) is dense, stable under \( \beta(N)' = P \), and, for all \( y \in P \), we have:

\[
R^{\beta,\psi^\circ}(y\xi) = yR^{\beta,\psi^\circ}(\xi)
\]

Then, for any \( \xi, \eta \in D(\mathcal{H}_\beta, \psi^\circ) \), the operator

\[
\theta^{\beta,\psi^\circ}(\xi, \eta) = R^{\beta,\psi^\circ}(\xi)R^{\beta,\psi^\circ}(\eta)^*
\]

belongs to \( P \), and the linear span generated by these operators is a dense ideal in \( P \); moreover, the operator-valued product \( < \xi, \eta >_{\beta,\psi^\circ} = R^{\beta,\psi^\circ}(\eta)^*R^{\beta,\psi^\circ}(\xi) \) belongs to \( \pi_\psi(N) \); we shall consider now, for simplification, that \( < \xi, \eta >_{\beta,\psi^\circ} \) belongs to \( N \), and the linear span generated by these operators is a dense algebra in \( N \), stable under multiplication by analytic elements with respect to \( \psi \). More precisely, \( < \xi, \eta >_{\beta,\psi^\circ} \) belongs to \( \mathcal{M}_\psi \) ([C], lemma 4) and we have ([S1], lemme 1.5)

\[
\Lambda_\psi(< \xi, \eta >_{\beta,\psi^\circ}) = R^{\beta,\psi^\circ}(\eta)^*\xi
\]

A \((\beta,\psi^\circ)\)-basis of \( \mathcal{H} \) is a family \((e_i)_{i \in I} \) of \( \psi^\circ \)-bounded elements of \( \mathcal{H}_\beta \), such that

\[
\sum_i \theta^{\beta,\psi^\circ}(e_i, e_i) = 1
\]
We have then, for all $\xi$ in $D(H_\beta)$:

$$\xi = \sum_i R^\beta,\psi^\alpha(e_i)\Lambda_\psi(\langle \xi, e_i \rangle)$$

It is possible to choose the $(e_i)_{i \in I}$ such that the $R^\beta,\psi^\alpha(e_i)$ are partial isometries, with final supports $\theta^\beta,\psi^\alpha(e_i, e_i)$ of 2 by 2 orthogonal, and such that $\langle e_i, e_j \rangle = 0$ if $i \neq j$; such a family will be then called a $(\beta, \psi^\alpha)$-orthogonal basis of $H$. We have then

$$R^\beta,\psi^\alpha(e_i) = \theta^\beta,\psi^\alpha(e_i, e_i)R^\beta,\psi^\alpha(e_i) = R^\beta,\psi^\alpha(e_i)$$

Moreover, we get that, for all $n$ in $N$, and for all $i$, we have:

$$\theta^\beta,\psi^\alpha(e_i, e_i)\beta(n)e_i = \beta(n)e_i$$

and that $\theta^\beta,\psi^\alpha(e_i, e_i)$ is the orthogonal projection on the closure of the subspace $\{\beta(n)e_i, n \in N\}$.

### 2.2. Jones’ basic construction and operator-valued weights.

Let $M_0 \subset M_1$ be an inclusion of von Neumann algebras (for simplification, these algebras will be supposed to be $\sigma$-finite), equipped with a normal faithful semi-finite operator-valued weight $T_1$ from $M_1$ to $M_0$ (to be more precise, from $M_1^+$ to the extended positive elements of $M_0$ (cf. [T] IX.4.12)). Let $\psi_0$ be a normal faithful semi-finite weight on $M_0$, and $\psi_1 = \psi_0 \circ T_1$; for $i = 0, 1$, let $H_i = H_{\psi_i}$, $J_i = J_{\psi_i}$, $\Delta_i = \Delta_{\psi_i}$ be the usual objects constructed by the Tomita-Takesaki theory associated to these weights. Following ([J], 3.1.5(i)), the von Neumann algebra $M_2 = J_1M'_0J_1$ defined on the Hilbert space $H_1$ will be called the basic construction made from the inclusion $M_0 \subset M_1$. We have $M_1 \subset M_2$, and we shall say that the inclusion $M_0 \subset M_1 \subset M_2$ is standard.

Following ([EN] 10.6), for $x$ in $\mathfrak{M}_{T_1}$, we shall define $\Lambda_{T_1}(x)$ by the following formula, for all $z$ in $\mathfrak{M}_{\psi_0}$:

$$\Lambda_{T_1}(x)\Lambda_{\psi_0}(z) = \Lambda_{\psi_1}(xz)$$

Then, $\Lambda_{T_1}(x)$ belongs to $\text{Hom}_{M_0^2^*}(H_0, H_1)$; if $x, y$ belong to $\mathfrak{M}_{T_1}$, then $\Lambda_{T_1}(x)\Lambda_{T_1}(y) = T_1(x^*y)$, and $\Lambda_{T_1}(x)\Lambda_{T_1}(y)^*$ belongs to $M_2$.

Using then Haagerup’s construction ([T], IX.4.24), it is possible to construct a normal semi-finite faithful operator-valued weight $T_2$ from $M_2$ to $M_1$ ([EN], 10.7), which will be called the basic construction made from $T_1$. If $x, y$ belong to $\mathfrak{M}_{T_1}$, then $\Lambda_{T_1}(x)\Lambda_{T_1}(y)^*$ belongs to $\mathfrak{M}_{T_2}$, and $T_2(\Lambda_{T_1}(x)\Lambda_{T_1}(y)^*) = xy^*$.

By Tomita-Takesaki theory, the Hilbert space $H_1$ bears a natural structure of $M_1 - M_0^0$-bimodule, and, therefore, by restriction, of $M_0 - M_0^*$-bimodule. Let us write $r$ for the canonical representation of $M_0$ on $H_1$, and $s$ for the canonical antirepresentation given, for all $x$ in $M_0$, by $s(x) = J_1r(x)^*J_1$. Let us have now a closer look to the subspaces.
D(H_{1s}, \psi_0^s) and D(\gamma, H_1, \psi_0). If x belongs to \mathfrak{M}_{T_1} \cap \mathfrak{M}_{\psi_1}, we easily get that J_1 \Lambda_{\psi_1}(x) belongs to D(\gamma, H_1, \psi_0), with:

\[ R_{s,\psi_0}(J_1 \Lambda_{\psi_1}(x)) = J_1 \Lambda_{T_1}(x)J_0 \]

and \Lambda_{\psi_1}(x) belongs to D(H_{1s}, \psi_0), with:

\[ R_{s,\psi_0}^{\ast}(\Lambda_{\psi_1}(x)) = \Lambda_{T_1}(x) \]

In ([E3], 2.3) was proved that the subspace $D(H_{1s}, \psi_0^s) \cap D(\gamma, H_1, \psi_0)$ is dense in $H_1$; let us write down and precise this result :

2.2.1. **Proposition.** Let us keep on the notations of this paragraph; let $T_{\psi_1, T_1}$ be the algebra made of elements $x$ in $\mathfrak{M}_{\psi_1} \cap \mathfrak{M}_{T_1} \cap \mathfrak{M}_{\psi_1}^* \cap \mathfrak{M}_{T_1}^*$, analytical with respect to $\psi_1$, and such that, for all $z \in \mathbb{C}$, $\sigma_{\psi_1}^z(x)$ belongs to $\mathfrak{M}_{\psi_1} \cap \mathfrak{M}_{T_1} \cap \mathfrak{M}_{\psi_1}^* \cap \mathfrak{M}_{T_1}^*$. Then:

(i) the algebra $T_{\psi_1, T_1}$ is weakly dense in $M_1$; it will be called Tomita’s algebra with respect to $\psi_1$ and $T_1$;

(ii) for any $x$ in $T_{\psi_1, T_1}$, $\Lambda_{\psi_1}(x)$ belongs to $D(H_{1s}, \psi_0^s) \cap D(\gamma, H_1, \psi_0)$;

(iii) for any $\xi$ in $D(H_{1s}, \psi_0^s)$, there exists a sequence $x_n$ in $T_{\psi_1, T_1}$ such that $\Lambda_{T_1}(x_n) = R_{s,\psi_0}^\ast(\Lambda_{\psi_1}(x))$ is weakly converging to $R_{s,\psi_0}^{\ast}(\xi)$ and $\Lambda_{\psi_1}(x_n)$ is converging to $\xi$.

**Proof.** The result (i) is taken from ([EN], 10.12); we get in ([E3], 2.3) an increasing sequence of projections $p_n$ in $M_1$, converging to 1, and elements $x_n$ in $T_{\psi_1, T_1}$ such that $\Lambda_{\psi_1}(x_n) = p_n \xi$. So, (i) and (ii) were obtained in ([E3], 2.3) from this construction. More precisely, we get that:

\[ T_1(x_n^* x_n) = \langle R_{s,\psi_0}^{\ast}(\Lambda_{\psi_1}(x_n)), R_{s,\psi_0}^{\ast}(\Lambda_{\psi_1}(x_n)) \rangle_{s,\psi_0} \]

\[ = \langle p_n \xi, p_n \xi \rangle_{s,\psi_0} \]

\[ = R_{s,\psi_0}^{\ast}(\xi)^* p_n R_{s,\psi_0}^{\ast}(\xi) \]

which is increasing and weakly converging to $\langle \xi, \xi \rangle_{s,\psi_0}$.

We finish by writing a proof of this useful lemma, we were not able to find in litterature :

2.2.2. **Lemma.** Let $M_0 \subset M_1$ be an inclusion of von neumann algebras, equipped with a normal faithful semi-finite operator-valued weight $T$ from $M_1$ to $M_0$. Let $\psi_0$ be a normal semi-finite faithful weight on $M_0$, and $\psi_1 = \psi_0 \circ T$; if $x$ is in $\mathfrak{M}_T$, and if $y$ is in $M_0' \cap M_1$, analytical with respect to $\psi_1$, then $xy$ belongs to $\mathfrak{M}_T$.

**Proof.** Let $a$ be in $\mathfrak{M}_{\psi_0}$; then $xa$ belongs to $\mathfrak{M}_{\psi_1}$, and $xya = xay$ belongs to $\mathfrak{M}_{\psi_1}$; moreover, let us consider the element $T(y^* x^* xy)$ of the positive extended part of $M_0^+$; we have:

\[ < T(y^* x^* xy), \omega_{\Lambda_{\psi_0}}(a) > = \psi_1(a^* y^* x^* xy) = \| \Lambda_{\psi_1}(xay) \|^2 = \]

\[ = \| J_{\psi_1} \sigma_{s,\psi_0}^1(y^*) J_{\psi_1} \Lambda_{\psi_1}(xa) \|^2 = \| J_{\psi_1} \sigma_{s,\psi_0}^1(y^*) J_{\psi_1} \Lambda_T(x) \Lambda_{\psi_0}(a) \|^2 \]
from which we get that $T(y^*x^*xy)$ is bounded and
\[
T(y^*x^*xy) \leq \|\sigma_{-i/2}(y^*)\|^2T(x^*x)
\]

\[\square\]

2.3. Relative tensor product \([C1], [S2], [T]\). Using the notations of 2.1, let now $\mathcal{K}$ be another Hilbert space on which there exists a non-degenerate representation $\gamma$ of $N$. Following J.-L. Sauvageot ([S2], 2.1), we define the relative tensor product $\mathcal{H}_\beta \otimes_\gamma \mathcal{K}$ as the Hilbert space obtained from the algebraic tensor product $D(\mathcal{H}_\beta, \psi_o) \otimes \mathcal{K}$ equipped with the scalar product defined, for $\xi_1, \xi_2$ in $D(\mathcal{H}_\beta, \psi_o)$, $\eta_1, \eta_2$ in $\mathcal{K}$, by
\[
(\xi_1 \otimes \eta_1 | \xi_2 \otimes \eta_2) = (\gamma(\xi_1, \xi_2), \eta_1 | \eta_2)
\]
where we have identified $N$ with $\pi_\psi(N)$ to simplify the notations.

The image of $\xi \otimes \eta$ in $\mathcal{H}_\beta \otimes_\gamma \mathcal{K}$ will be denoted by $\xi \otimes_\gamma \eta$. We shall use intensively this construction; one should bear in mind that, if we start from another faithful semi-finite normal weight $\psi'$, we get another Hilbert space $\mathcal{H}_\beta \otimes_\gamma \mathcal{K}$; there exists an isomorphism $U_{\psi, \psi'}$ from $\mathcal{H}_\beta \otimes_\gamma \mathcal{K}$ to $\mathcal{H}_{\psi', \gamma} \mathcal{K}$, which is unique up to some functorial property ([S2], 2.6) (but this isomorphism does not send $\xi \otimes_\gamma \eta$ on $\xi \otimes_\gamma \eta'$).

When no confusion is possible about the representation and the anti-representation, we shall write $\mathcal{H} \otimes_\psi \mathcal{K}$ instead of $\mathcal{H}_\beta \otimes_\gamma \mathcal{K}$, and $\xi \otimes_\psi \eta$ instead of $\xi \otimes_\gamma \eta$.

If $\theta \in \text{Aut}N$, then, using a remark made in 2.1, we get that the application which sends $\xi \otimes_\gamma \eta$ onto $\xi \otimes_{\psi \theta} \eta$ leads to a unitary from $\mathcal{H}_\beta \otimes_\gamma \mathcal{K}$ onto $\mathcal{H}_{\psi \theta \otimes \gamma} \mathcal{K}$.

For any $\xi$ in $D(\mathcal{H}_\beta, \psi_o)$, we define the bounded linear application $\lambda_\xi$ from $\mathcal{K}$ to $\mathcal{H}_\beta \otimes_\gamma \mathcal{K}$ by, for all $\eta$ in $\mathcal{K}$,
\[
\lambda_\xi(\eta) = \xi \otimes_\gamma \eta.
\]
We shall write $\lambda_\xi$ if no confusion is possible. We get ([EN], 3.10):
\[
\lambda_\xi^* = R_{\beta, \psi_o}(\xi) \otimes_\psi 1_{\mathcal{K}}
\]
where we recall the canonical identification (as left $N$-modules) of $L^2(N) \otimes_\psi \mathcal{K}$ with $\mathcal{K}$. We have:
\[
(\lambda_\xi^*)^* \lambda_\xi^* = \gamma(\xi, \xi)
\]

In ([S1] 2.1), the relative tensor product $\mathcal{H}_\beta \otimes_\gamma \mathcal{K}$ is defined also, if $\xi_1$,
\( \xi_2 \) are in \( \mathcal{H} \), \( \eta_1, \eta_2 \) are in \( D(\gamma \mathcal{K}, \psi) \), by the following formula:
\[
(\xi_1 \otimes \eta_1 | \xi_2 \otimes \eta_2) = (\beta(< \eta_1, \eta_2 >) \gamma, \psi) \xi_1 | \xi_2
\]
which leads to the definition of a relative flip \( \sigma_\psi \) which will be an isomorphism from \( \mathcal{H}_\beta \otimes_\gamma \mathcal{K} \) onto \( \mathcal{K}_\gamma \otimes_\beta \mathcal{H} \), defined, for any \( \xi \) in \( D(\mathcal{H}_\beta, \psi) \), \( \eta \) in \( D(\gamma \mathcal{K}, \psi) \), by:
\[
\sigma_\psi(\xi \otimes \eta) = \eta \otimes \psi_\omega \xi
\]
This allows us to define a relative flip \( \varsigma_\psi \) from \( \mathcal{L}(\mathcal{H}_\beta \otimes_\gamma \mathcal{K}) \) to \( \mathcal{L}(\mathcal{K}_\gamma \otimes_\beta \mathcal{H}) \) which sends \( X \) in \( \mathcal{L}(\mathcal{H}_\beta \otimes_\gamma \mathcal{K}) \) onto \( \varsigma_\psi(X) = \sigma_\psi X \sigma_\psi^* \). Starting from another faithful semi-finite normal weight \( \psi' \), we get a von Neumann algebra \( \mathcal{L}(\mathcal{H}_\beta \otimes_\gamma \mathcal{K}) \) which is isomorphic to \( \mathcal{L}(\mathcal{H}_\beta \otimes_\gamma \mathcal{K}) \), and a von Neumann algebra \( \mathcal{L}(\mathcal{K}_\gamma \otimes_\beta \mathcal{H}) \) which is isomorphic to \( \mathcal{L}(\mathcal{K}_\gamma \otimes_\beta \mathcal{H}) \); as we get that:
\[
\sigma_{\psi'} \circ U_{\beta,\gamma}^{\psi,\psi'} = U_{\gamma,\beta}^{\psi',\psi_\omega}
\]
we see that these isomorphisms exchange \( \varsigma_\psi \) and \( \varsigma_{\psi'} \). Therefore, the homomorphism \( \varsigma_\psi \) can be denoted \( \varsigma_N \) without any reference to a specific weight.

We may define, for any \( \eta \) in \( D(\gamma \mathcal{K}, \psi) \), an application \( \rho_\eta^{\beta,\gamma} \) from \( \mathcal{H} \) to \( \mathcal{H}_\beta \otimes_\gamma \mathcal{K} \) by \( \rho_\eta^{\beta,\gamma}(\xi) = \xi \otimes \beta \otimes_\gamma \eta \). We shall write \( \rho_\eta \) if no confusion is possible. We get that:
\[
(\rho_\eta^{\beta,\gamma})^* \rho_\eta^{\beta,\gamma} = \beta(< \eta, \eta >) \gamma, \psi
\]
We recall, following ([S2], 2.2b) that, for all \( \xi \) in \( \mathcal{H} \), \( \eta \) in \( D(\gamma \mathcal{K}, \psi) \), \( y \) in \( \mathcal{N} \), analytic with respect to \( \psi \), we have:
\[
\beta(y) \xi \otimes \psi_\omega \eta = \xi \otimes \psi_\omega \gamma(\sigma_{\psi,\omega_\eta}(y)) \eta
\]
Let \( x \) be an element of \( \mathcal{L}(\mathcal{H}) \), commuting with the right action of \( \mathcal{N} \) on \( \mathcal{H}_\beta \) (i.e. \( x \in \beta(N)' \)). It is possible to define an operator \( x \beta \otimes_\gamma 1_\mathcal{K} \) on \( \mathcal{H}_\beta \otimes_\gamma \mathcal{K} \). We can easily evaluate \( \| x \beta \otimes_\gamma 1_\mathcal{K} \| \), for any finite \( J \subset I \), for any \( \eta_i \) in \( \mathcal{K} \), we have:
\[
((x^* x \beta \otimes_\gamma 1_\mathcal{K})(\Sigma_{i \in J} e_i \beta \otimes_\gamma \eta_i))(\Sigma_{i \in J} e_i \beta \otimes_\gamma \eta_i)) = \sum_{i \in J} (\gamma(< x e_i, x e_i > \beta, \psi_\omega) \eta_i | \eta_i) \leq \| x \|^2 \sum_{i \in J} (\gamma(< e_i, e_i > \beta, \psi_\omega) \eta_i | \eta_i) = \| x \|^2 \| \Sigma_{i \in J} e_i \beta \otimes_\gamma \eta_i \|
\]
from which we get \( \| x \beta \otimes_\gamma 1_\mathcal{K} \| \leq \| x \| \).

By the same way, if \( y \) commutes with the left action of \( \mathcal{N} \) on \( \gamma \mathcal{K} \) (i.e. \( y \)
is in $\gamma(N')$, it is possible to define $1_{\mathcal{H}_\beta \otimes \gamma} y$ on $\mathcal{H}_\beta \otimes \gamma \mathcal{K}$, and by composition, it is possible to define then $x_{\beta \otimes \gamma} y$. If we start from another faithful semi-finite normal weight $\nu'$, the canonical isomorphism $U^{\psi, \psi'}_{\beta, \gamma}$ from $\mathcal{H}_\beta \otimes \gamma \mathcal{K}$ to $\mathcal{H}_\beta \otimes \gamma \mathcal{K}$ sends $x_{\beta \otimes \gamma} y$ on $x_{\beta \otimes \gamma} y$ ([S2], 2.3 and 2.6); therefore, this operator can be denoted $x_{\beta \otimes \gamma} y$ without any reference to a specific weight, and we get $\|x_{\beta \otimes \gamma} y\| \leq \|x\| \|y\|$. If $\theta \in \text{Aut} \mathcal{N}$, the unitary from $\mathcal{H}_\beta \otimes \gamma \mathcal{K}$ onto $\mathcal{H}_{\beta \otimes \gamma} \mathcal{K}$ sends $x_{\beta \otimes \gamma} y$ on $x_{\beta \otimes \gamma} y$.

With the notations of [2.1] let $(e_i)_{i \in I}$ a $(\beta, \psi')$-orthogonal basis of $\mathcal{H}$; let us remark that, for all $\eta$ in $\mathcal{K}$, we have:

$$e_i \beta \otimes \gamma \eta = e_i \beta \otimes \gamma (\langle e_i, e_i >_{\beta, \psi'} \rangle) \eta$$

On the other hand, $\theta_{\beta, \psi'}(e_i, e_i)$ is an orthogonal projection, and so is $\theta_{\beta, \psi'}(e_i, e_i) \beta \otimes \gamma 1$; this last operator is the projection on the subspace $e_i \beta \otimes \gamma (\langle e_i, e_i >_{\beta, \psi'} \rangle) \mathcal{K}$ ([E2], 2.3) and, therefore, we get that $\mathcal{H}_\beta \otimes \gamma \mathcal{K}$ is the orthogonal sum of the subspaces $e_i \beta \otimes \gamma (\langle e_i, e_i >_{\beta, \psi'} \rangle) \mathcal{K}$; for any $\Xi$ in $\mathcal{H}_\beta \otimes \gamma \mathcal{K}$, there exist $\xi_i$ in $\mathcal{K}$, such that $\gamma(\langle e_i, e_i >_{\beta, \psi'} \rangle) \xi_i = \xi_i$ and $\Xi = \sum_i e_i \beta \otimes \gamma \xi_i$, from which we get that $\sum_i \|\xi_i\|^2 = \|\Xi\|^2$.

Let us suppose now that $\mathcal{K}$ is a $\mathcal{N} - \mathcal{P}$ bimodule; that means that there exists a von Neumann algebra $\mathcal{P}$, and a non-degenerate normal anti-representation $\epsilon$ of $\mathcal{P}$ on $\mathcal{K}$, such that $\epsilon(\mathcal{P}) \subset \gamma(N)'$. We shall write then $\mathcal{K}_\epsilon$. If $y$ is in $\mathcal{P}$, we have seen that it is possible to define then the operator $1_{\mathcal{H}_\beta \otimes \gamma} \epsilon(y)$ on $\mathcal{H}_\beta \otimes \gamma \mathcal{K}$, and we define this way a non-degenerate normal antirepresentation of $\mathcal{P}$ on $\mathcal{H}_\beta \otimes \gamma \mathcal{K}$, we shall call again $\epsilon$ for simplification. If $\mathcal{H}$ is a $\mathcal{Q} - \mathcal{N}$ bimodule, then $\mathcal{H}_\beta \otimes \gamma \mathcal{K}$ becomes a $\mathcal{Q} - \mathcal{P}$ bimodule (Connes’ fusion of bimodules).

Taking a faithful semi-finite normal weight $\nu$ on $\mathcal{P}$, and a left $\mathcal{P}$-module $\zeta \mathcal{L}$ (i.e. a Hilbert space $\mathcal{L}$ and a normal non-degenerate representation $\zeta$ of $\mathcal{P}$ on $\mathcal{L}$), it is possible then to define $(\mathcal{H}_\beta \otimes \gamma \mathcal{K}) \epsilon \otimes \zeta \mathcal{L}$. Of course, it is possible also to consider the Hilbert space $\mathcal{H}_\beta \otimes \gamma (\mathcal{K}_\epsilon \otimes \zeta \mathcal{L})$. It can be shown that these two Hilbert spaces are isomorphics as $\beta(N)' - \zeta(\mathcal{P})'_{\nu}$-bimodules. (In ([V1] 2.1.3), the proof, given for $\mathcal{N} = \mathcal{P}$ abelian can be used, without modification, in that wider hypothesis). We shall write
then \( \mathcal{H}_\beta \otimes_\psi \mathcal{K}_\epsilon \otimes_\nu \mathcal{L} \) without parenthesis, to emphasize this coassociativity property of the relative tensor product.

Dealing now with that Hilbert space \( \mathcal{H}_\beta \otimes_\psi \mathcal{K}_\epsilon \otimes_\nu \mathcal{L} \), there exist different flips, and it is necessary to be careful with notations. For instance, \( 1_\beta \otimes \sigma_\nu \) is the flip from this Hilbert space onto \( \mathcal{H}_\beta \otimes_\nu (\mathcal{L}_\epsilon \otimes_\nu \mathcal{K}) \), where \( \gamma \) is here acting on the second leg of \( \mathcal{L}_\epsilon \otimes_\nu \mathcal{K} \) (and should therefore be written \( 1_\zeta \otimes_\nu \gamma \), but this will not be done for obvious reasons). Here, the parenthesis remains, because there is no associativity rule, and to remind that \( \gamma \) is not acting on \( \mathcal{L} \). The adjoint of \( 1_\beta \otimes \sigma_\nu \) is \( 1_\beta \otimes \sigma_\nu^o \).

The same way, we can consider \( \sigma_\psi 1_\epsilon \otimes_\nu \) from \( \mathcal{H}_\beta \otimes_\psi \mathcal{K}_\epsilon \otimes_\nu \mathcal{L} \) onto \( \mathcal{K}(\gamma \otimes_\beta \mathcal{H}) \otimes_\nu \mathcal{L} \).

Another kind of flip sends \( \mathcal{H}_\beta \otimes_\psi (\mathcal{L}_\epsilon \otimes_\nu \mathcal{K}) \) onto \( \mathcal{L}_\epsilon \otimes_\nu (\mathcal{H}_\beta \otimes_\gamma \mathcal{K}) \).

We shall denote this application \( \sigma_{\epsilon,\gamma}^{1,2} \) (and its adjoint \( \sigma_{\epsilon,\gamma}^{1,2} \)), in order to emphasize that we are exchanging the first and the second leg, and the representation \( \gamma \) and \( \epsilon \) on the third leg.

If \( \pi \) denotes the canonical left representation of \( N \) on the Hilbert space \( L^2(N) \), then it is straightforward to verify that the application which sends, for all \( \xi \in \mathcal{H}_\chi \), \( \chi \) normal faithful semi-finite weight on \( N \), and \( x \in \mathfrak{M}_\chi \), the vector \( \xi_\beta \otimes_{\chi} \pi J_\chi \Lambda_\chi (x) \) on \( \beta(x^\ast) \xi \), gives an isomorphism of \( \mathcal{H}^\beta \otimes_\chi \pi L^2(N) \) on \( \mathcal{H}_\chi \), which will send the antirepresentation of \( N \) given by \( n \mapsto 1_{\mathcal{H}^\beta} \otimes_\chi \pi J_\chi n^\ast J_\chi \) on \( \beta \).

If \( \mathcal{K} \) is a Hilbert space on which there exists a non-degenerate representation \( \gamma \) of \( N \), then \( \mathcal{K} \) is a \( N - \gamma(N) \) bimodule, and the conjugate Hilbert space \( \overline{\mathcal{K}} \) is a \( \gamma(N)^\ast - N \) bimodule, and, \( (S2) \), for any normal faithful semi-finite weight \( \phi \) on \( \gamma(N)^\ast \), the fusion \( \gamma \mathcal{K} \otimes \overline{\mathcal{K}}_{\phi} \) is isomorphic to the standard space \( L^2(N) \), equipped with its standard left and right representation.

Using that remark, one gets for any \( x \in \beta(N)^\ast : \)

\[
\| x_\beta \otimes_\gamma 1_{\mathcal{K}} \| \leq \| x_\beta \otimes_\gamma 1_{\mathcal{K}} \otimes_\nu 1_{\mathcal{K}} \| = \| x_\beta \otimes_\nu 1_{L^2(N)} \| = \| x \|
\]

from which we have \( \| x_\beta \otimes_\gamma 1_{\mathcal{K}} \| = \| x \| \).

2.4. Fiber product \([V1]\), \([EV]\). Let us follow the notations of \([2,3]\) and let now \( M_1 \) be a von Neumann algebra on \( \mathcal{H} \), such that \( \beta(N) \subset M_1 \), and \( M_2 \) be a von Neumann algebra on \( \mathcal{K} \), such that \( \gamma(N) \subset M_2 \). The von Neumann algebra generated by all elements \( x_\beta \otimes_\gamma y_\gamma \), where \( x \) belongs
to $M'_1$, and $y$ belongs $M'_2$ will be denoted $M'_1 \otimes^\gamma_{\gamma(N)} M'_2$ (or $M'_1 \otimes^N N M'_2$ if no confusion if possible), and will be called the relative tensor product of $M'_1$ and $M'_2$ over $N$. The commutant of this algebra will be denoted $M_1 \otimes^\gamma_{\gamma(N)} M_2$ (or $M_1 \otimes^N N M_2$ if no confusion is possible) and called the fiber product of $M_1$ and $M_2$, over $N$. If $\theta \in \text{Aut} N$, using a remark made in [2.3], we get that the von Neumann algebras $M_1 \otimes^\gamma_{\gamma(N)} M_2$ and $M_1 \otimes^\gamma_{\gamma(N)} M_2$ are spatially isomorphic, and we shall identify them.

It is straightforward to verify that, if $P_1$ and $P_2$ are two other von Neumann algebras satisfying the same relations with $N$, we have

$$M_1 \otimes^N N M_2 \cap P_1 \otimes^N N P_2 = (M_1 \cap P_1) \otimes^N N (M_2 \cap P_2).$$

Moreover, we get that $\zeta_N(M_1 \otimes^\gamma_{\gamma(N)} M_2) = M_2 \otimes^\gamma_{\gamma(N)} M_1$.

In particular, we have:

$$(M_1 \cap \beta(N)) \otimes^\gamma_{\gamma(N)} (M_2 \cap \gamma(N)) \subset M_1 \otimes^\gamma_{\gamma(N)} M_2$$

and:

$$M_1 \otimes^\gamma_{\gamma(N)} = (M_1 \cap \beta(N)) \otimes^\gamma_{\gamma(N)} 1.$$

More generally, if $\beta$ is a non-degenerate normal involutive antihomomorphism from $N$ into a von Neumann algebra $M_1$, and $\gamma$ a non-degenerate normal involutive homomorphism from $N$ into a von Neumann algebra $M_2$, it is possible to define, without any reference to a specific Hilbert space, a von Neumann algebra $M_1 \otimes^\gamma_{\gamma(N)} M_2$.

Moreover, if now $\beta'$ is a non-degenerate normal involutive antihomomorphism from $N$ into another von Neumann algebra $P_1$, $\gamma'$ a non-degenerate normal involutive homomorphism from $N$ into another von Neumann algebra $P_2$, $\Phi$ a normal involutive homomorphism from $M_1$ into $P_1$ such that $\Phi \circ \beta = \beta'$, and $\Psi$ a normal involutive homomorphism from $M_2$ into $P_2$ such that $\Psi \circ \gamma = \gamma'$, it is possible then to define a normal involutive homomorphism (the proof given in ([S1] 1.2.4) in the case when $N$ is abelian can be extended without modification in the general case):

$$\Phi_{\beta,\gamma} \Psi : M_1 \otimes^\gamma_{\gamma(N)} M_2 \mapsto P_1 \otimes^\gamma_{\gamma'} P_2.$$

Let $\Phi$ be in $\text{Aut} M_1$, $\Psi$ in $\text{Aut} M_2$, and let $\theta \in \text{Aut} N$ be such that $\Phi \circ \beta = \beta \circ \theta$ and $\Psi \circ \gamma = \gamma \circ \theta$, then, using the identification between $M_1 \otimes^\gamma_{\gamma(N)} M_2$ and $M_1 \otimes^\gamma_{\gamma\circ \theta} M_2$, we get the existence of an automorphism $\Phi_{\beta,\gamma} \Psi$ of $M_1 \otimes^\gamma_{\gamma(N)} M_2$.

In the case when $\gamma N$ is a $N - P^o$ bimodule as explained in [2.3], and $\zeta \mathcal{L}$ a $P$-module, if $\gamma(N) \subset M_2$ and $\epsilon(P) \subset M_2$, and if $\zeta(P) \subset M_3$, where $M_3$ is a von Neumann algebra on $\mathcal{L}$, it is possible to consider
then \((M_1 \beta \ast_\gamma M_2) \ast_\xi M_3\) and \(M_1 \beta \ast_\gamma (M_2 \epsilon \ast_\xi M_3)\). The coassociativity property for relative tensor products leads then to the isomorphism of these von Neumann algebras we shall write now \(M_1 \beta \ast_\gamma M_2 \epsilon \ast_\xi M_3\) without parenthesis.

2.5. Slice maps [E3]. Let \(A\) be in \(M_1 \beta \ast_\gamma M_2\), \(\psi\) a normal faithful semi-finite weight on \(N\), \(H\) an Hilbert space on which \(M_1\) is acting, \(K\) an Hilbert space on which \(M_2\) is acting, and let \(\xi_1, \xi_2\) be in \(D(H_\beta, \psi^0)\);

let us define:

\[
(\omega_{\xi_1, \xi_2} \beta \ast_\gamma id)(A) = (\lambda_{\xi_2}^\beta)^* A \lambda_{\xi_1}^\beta
\]

We define this way \((\omega_{\xi_1, \xi_2} \beta \ast_\gamma id)(A)\) as a bounded operator on \(K\), which belongs to \(M_2\), such that:

\[
(\omega_{\xi_1, \xi_2} \beta \ast_\gamma id)(A)\eta_1|\eta_2) = (A(\xi_1 \beta \otimes_\gamma \eta_1)|\xi_2 \beta \otimes_\gamma \eta_2)
\]

One should note that \((\omega_{\xi_1, \xi_2} \beta \ast_\gamma id)(1) = \gamma(\xi_1, \xi_2 > \beta, \psi^0)\).

Let us define the same way, for any \(\eta_1, \eta_2\) in \(D(\gamma K, \psi)\):

\[
(id \beta \ast_\gamma \omega_{\eta_1, \eta_2})(A) = (\rho_{\eta_2}^\beta)^* A \rho_{\eta_1}^\beta
\]

which belongs to \(M_1\).

We therefore have a Fubini formula for these slice maps: for any \(\xi_1, \xi_2\) in \(D(H_\beta, \psi^0), \eta_1, \eta_2\) in \(D(\gamma K, \psi)\), we have:

\[
< (\omega_{\xi_1, \xi_2} \beta \ast_\gamma id)(A), \omega_{\eta_1, \eta_2} > = < (id \beta \ast_\gamma \omega_{\eta_1, \eta_2})(A), \omega_{\xi_1, \xi_2} >
\]

Let \(\phi_1\) be a normal semi-finite weight on \(M_1^+\), and \(A\) be a positive element of the fiber product \(M_1 \beta \ast_\gamma M_2\), then we may define an element of the extended positive part of \(M_2\), denoted \((\phi_1 \beta \ast_\gamma id)(A)\), such that, for all \(\eta\) in \(D(\gamma L^2(M_2), \psi)\), we have:

\[
\|(\phi_1 \beta \ast_\gamma id)(A)\|^2 = \phi_1(id \beta \ast_\gamma \omega_{\eta})(A)
\]

Moreover, then, if \(\phi_2\) is a normal semi-finite weight on \(M_2^+\), we have:

\[
\phi_2(\phi_1 \beta \ast_\gamma id)(A) = \phi_1(id \beta \ast_\gamma \phi_2)(A)
\]

and if \(\omega_i\) are in \(M_1\), such that \(\phi_1 = sup_i \omega_i\), we have \((\phi_1 \beta \ast_\gamma id)(A) = sup_i(\omega_i \beta \ast_\gamma id)(A)\).

Let now \(P_1\) be a von Neuman algebra such that:

\[
\beta(N) \subset P_1 \subset M_1
\]
and let $\Phi_i$ ($i = 1, 2$) be a normal faithful semi-finite operator valued weight from $M_i$ to $P_i$; for any positive operator $A$ in the fiber product $M_1 \beta \ast \gamma M_2$, there exists an element $(\Phi_1 \beta \ast \gamma \psi id)(A)$ of the extended positive part of $P_1 \beta \ast \gamma M_2$, such that ([E3], 3.5), for all $\eta$ in $D(\gamma L^2(M_2), \psi)$, and $\xi$ in $D(L^2(P_1 \beta, \psi^o)$, we have :

$$\|\left(\Phi_1 \beta \ast \gamma \psi id\right)(A)\|^2 = \|\Phi_1(id \beta \ast \gamma \omega \eta)(A)\|^2$$

If $\phi$ is a normal semi-finite weight on $P$, we have :

$$(\phi \circ \Phi_1 \beta \ast \gamma \psi id)(A) = (\phi \beta \ast \gamma \psi)(\Phi_1 \beta \ast \gamma \psi id)(A)$$

We define the same way an element $(id \beta \ast \gamma \Psi_2)(A)$ of the extended positive part of $M_1 \gamma \ast \beta P_2$, and we have :

$$(id \beta \ast \gamma \Psi_2)((\Phi_1 \beta \ast \gamma \psi id)(A)) = (\Phi_1 \beta \ast \gamma \psi id)((id \beta \ast \gamma \Psi_2)(A))$$

Considering now an element $x$ of $M_1 \beta \ast \pi \pi(N)$, which can be identified (2.4) to $M_1 \cap \beta(N)'$, we get that, for $e$ in $\mathfrak{M}_\psi$, we have :

$$(id \beta \ast \pi \psi \omega J_\psi \Lambda \psi(e))(x) = \beta(e^* x)$$

Therefore, by increasing limits, we get that $(id \beta \ast \pi \psi)$ is the injection of $M_1 \cap \beta(N)'$ into $M_1$. More precisely, if $x$ belongs to $M_1 \cap \beta(N)'$, we have :

$$(id \beta \ast \pi \psi)(x \beta \otimes \pi 1) = x$$

Therefore, if $\Phi_2$ is a normal faithful semi-finite operator-valued weight from $M_2$ onto $\gamma(N)$, we get that, for all $A$ positive in $M_1 \beta \ast \gamma M_2$, we have :

$$(id \beta \ast \gamma \psi \circ \Phi_2)(A) \beta \otimes \gamma 1 = (id \beta \ast \gamma \Phi_2)(A)$$

With the notations of 2.11 let $(e_i)_{i \in I}$ be a $(\beta, \psi^o)$-orthogonal basis of $\mathcal{H}$; using the fact (2.3) that, for all $\eta$ in $\mathcal{K}$, we have :

$$e_i \beta \otimes \gamma \psi \psi \eta = e_i \beta \otimes \gamma \psi \psi (\langle e_i, e_i \rangle \beta, \psi^o) \eta$$

we get that, for all $X$ in $M_1 \beta \ast \gamma M_2$, $\xi$ in $D(\mathcal{H}_\beta, \psi^o)$, we have :

$$(\omega \xi, e_i, \beta \ast \gamma \psi id)(X) = \gamma(\langle e_i, e_i \rangle \beta, \psi^o)(\omega \xi, e_i, \beta \ast \gamma \psi id)(X)$$
2.6. **Vaes’ Radon-Nikodym theorem.** In [V] is proved a very nice Radon-Nikodym theorem for two normal faithful semi-finite weights on a von Neumann algebra $M$. If $\Phi$ and $\Psi$ are such weights, then are equivalent:
- the two modular automorphism groups $\sigma^\Phi$ and $\sigma^\Psi$ commute;
- the Connes’ derivative $[D\Psi : D\Phi]$, is of the form:

$$[D\Psi : D\Phi]_t = \lambda^{it^2/2} \delta^t$$

where $\lambda$ is a non-singular positive operator affiliated to $Z(M)$, and $\delta$ is a non-singular positive operator affiliated to $M$.

It is then easy to verify that $\sigma^\Psi_t(\delta^is) = \lambda^{ist} \delta^is$, and that

$$[D\Phi \circ \sigma^\Psi_t : D\Phi]_s = \lambda^{ist}$$

$$[D\Psi \circ \sigma^\Phi_t : D\Psi]_s = \lambda^{-ist}$$

Moreover, we have also, for any $x \in M^+$:

$$\Psi(x) = \lim_n \Phi((\delta^{1/2} e_n)x(\delta^{1/2} e_n))$$

where the $e_n$ are self-adjoint elements of $M$ given by the formula:

$$e_n = a_n \int_{\mathbb{R}^2} e^{-n^2x^2 - n^4y^4} e^{ix\delta^y} dxdy$$

where we put $a_n = 2n^2 \Gamma(1/2)^{-1} \Gamma(1/4)^{-1}$. The operators $e_n$ are analytic with respect to $\sigma^\Phi$ and such that, for any $z \in \mathbb{C}$, the sequence $\sigma^\Phi_z(e_n)$ is bounded and strongly converges to $1$.

In that situation, we shall write $\Psi = \Phi_{\delta}$ and call $\delta$ the modulus of $\Psi$ with respect to $\Phi$; $\lambda$ will be called the scaling operator of $\Psi$ with respect to $\Phi$.

Moreover, if $a \in M$ is such that $a\delta^{1/2}$ is bounded and its closure $\overline{a\delta^{1/2}}$ belongs to $\mathfrak{M}_\Phi$, then $a$ belongs to $\mathfrak{M}_\Psi$. We may then identify $\Lambda_{\Psi}(a)$ with $\Lambda_{\Phi}(a\delta^{1/2})$, $J_{\Psi}$ with $\lambda^{1/4} J_{\Phi}$, $J_{\Psi} \Delta_{\Psi}$ with $J_{\Phi} \Delta_{\Psi}^{-1}$.

### 3. Hopf-bimodules and Pseudo-multiplicative unitary

In this chapter, we recall the definition of Hopf-bimodules [3.1], the definition of a pseudo-multiplicative unitary [3.2], give the fundamental example given by groupoids [3.4], and construct the algebras and the Hopf-bimodules "generated by the left (resp. right) leg" of a pseudo-multiplicative unitary [3.3]. We recall the definition of left- (resp. right-) invariant operator-valued weights on a Hopf-bimodule; if we have both operator-valued weights, we then recall Lesieur’s construction of a pseudo-multiplicative unitary.
3.1. **Definition.** A quintuplet \((N, M, \alpha, \beta, \Gamma)\) will be called a Hopf-bimodule, following ([Val1], [EV] 6.5), if \(N, M\) are von Neumann algebras, \(\alpha\) a faithful non-degenerate representation of \(N\) into \(M\), \(\beta\) a faithful non-degenerate anti-representation of \(N\) into \(M\), with commuting ranges, and \(\Gamma\) an injective involutive homomorphism from \(M\) into \(M_{\beta \star \alpha} M\) such that, for all \(X\) in \(N\):

(i) \(\Gamma(\beta(X)) = 1_{\beta \otimes_{\alpha} \beta}(X)\)

(ii) \(\Gamma(\alpha(X)) = \alpha(X)_{\beta \otimes_{\alpha} \beta} 1\)

(iii) \(\Gamma\) satisfies the co-associativity relation:

\[
(\Gamma_{\beta \star_{\alpha} \beta} id)\Gamma = (id_{\beta \star_{\alpha} \beta} \Gamma)\Gamma
\]

This last formula makes sense, thanks to the two preceding ones and 2.4.

If \((N, M, \alpha, \beta, \Gamma)\) is a Hopf-bimodule, it is clear that \((N^o, M, \beta, \alpha, \varsigma_N \circ \Gamma)\) is another Hopf-bimodule, we shall call the symmetrized of the first one. (Recall that \(\varsigma_N \circ \Gamma\) is a homomorphism from \(M\) to \(M_{\beta \star_{\alpha} \beta} M\)).

If \(N\) is abelian, \(\alpha = \beta, \Gamma = \varsigma_N \circ \Gamma\), then the quadruplet \((N, M, \alpha, \alpha, \Gamma)\) is equal to its symmetrized Hopf-bimodule, and we shall say that it is a symmetric Hopf-bimodule.

Let \(\mathcal{G}\) be a groupoid, with \(\mathcal{G}^{(0)}\) as its set of units, and let us denote by \(r\) and \(s\) the range and source applications from \(\mathcal{G}\) to \(\mathcal{G}^{(0)}\), given by \(xx^{-1} = r(x)\) and \(x^{-1}x = s(x)\). As usual, we shall denote by \(\mathcal{G}^{(2)}\) (or \(\mathcal{G}_{s,r}^{(2)}\)) the set of composable elements, i.e.

\[
\mathcal{G}^{(2)} = \{(x, y) \in \mathcal{G}^{2}; s(x) = r(y)\}
\]

In [Y] and [Val1] were associated to a measured groupoid \(\mathcal{G}\) equipped with a Haar system \((\lambda^u)_{u \in \mathcal{G}^{(0)}}\) and a quasi-invariant measure \(\mu\) on \(\mathcal{G}^{(0)}\) (see [R1], [R2], [C2] II.5 and [AR] for more details, precise definitions and examples of groupoids) two Hopf-bimodules:

The first one is \((L^\infty(\mathcal{G}^{(0)}), \mu), L^\infty(\mathcal{G}, \nu), r_{\mathcal{G}}, s_{\mathcal{G}}, \Gamma_{\mathcal{G}}\), where \(\nu\) is the measure constructed on \(\mathcal{G}\) using \(\mu\) and the Haar system \((\lambda^u)_{u \in \mathcal{G}^{(0)}}\), where we define \(r_{\mathcal{G}}\) and \(s_{\mathcal{G}}\) by writing, for \(g\) in \(L^\infty(\mathcal{G}^{(0)})\):

\[
r_{\mathcal{G}}(g) = g \circ r
\]

\[
s_{\mathcal{G}}(g) = g \circ s
\]

and where \(\Gamma_{\mathcal{G}}(f)\), for \(f\) in \(L^\infty(\mathcal{G})\), is the function defined on \(\mathcal{G}^{(2)}\) by \((s, t) \mapsto f(st)\); \(\Gamma_{\mathcal{G}}\) is then an involutive homomorphism from \(L^\infty(\mathcal{G})\) into \(L^\infty(\mathcal{G}_{s,r}^{(2)})\) (which can be identified to \(L^\infty(\mathcal{G})_{s,r} L^\infty(\mathcal{G})\)).

The second one is symmetric; it is \((L^\infty(\mathcal{G}^{(0)}), L(\mathcal{G}), r_{\mathcal{G}}, r_{\mathcal{G}}, \hat{\Gamma}_{\mathcal{G}})\), where
Let $\mathcal{L}(\mathcal{G})$ be the von Neumann algebra generated by the convolution algebra associated to the groupoid $\mathcal{G}$, and $\hat{\Gamma}_G$ has been defined in [Y] and [Val1].

3.2. Definition. Let $N$ be a von Neumann algebra; let $H$ be a Hilbert space on which $N$ has a non-degenerate normal representation $\alpha$ and two non-degenerate normal anti-representations $\hat{\beta}$ and $\beta$. These 3 applications are supposed to be injective, and to commute two by two. Let $\nu$ be a normal semi-finite faithful weight on $N$; we can therefore construct the Hilbert spaces $H_{\beta} \otimes_{\alpha} N_{\nu}$ and $H_{\alpha} \otimes_{\beta} N_{\nu}$. A unitary $W$ from $H_{\beta} \otimes_{\alpha} N_{\nu}$ onto $H_{\alpha} \otimes_{\beta} N_{\nu}$ will be called a pseudo-multiplicative unitary over the basis $N$, with respect to the representation $\alpha$, and the anti-representations $\hat{\beta}$ and $\beta$ (we shall say it is an $(\alpha, \hat{\beta}, \beta)$-pseudo-multiplicative unitary), if :

(i) $W$ intertwines $\alpha$, $\hat{\beta}$, $\beta$ in the following way :

$$ W(\alpha(X) \beta \otimes_{\alpha} 1) = (1 \otimes_{\beta} \alpha(X))W $$
$$ W(1 \beta \otimes_{\alpha} \beta(X)) = (1 \otimes_{\beta} \beta(X))W $$
$$ W(\hat{\beta}(X) \beta \otimes_{\alpha} 1) = (\hat{\beta}(X) \otimes_{\beta} 1)W $$
$$ W(1 \beta \otimes_{\alpha} \hat{\beta}(X)) = (\beta(X) \otimes_{\beta} 1)W $$

(ii) The operator $W$ satisfies :

$$ (1_{\beta} \otimes_{\beta} W)(W \beta \otimes_{\alpha} 1_{\beta}) = $$
$$ = (W \otimes_{\beta} 1_{\beta})\sigma_{\alpha, \beta}^{2,3}(W \otimes_{\alpha} 1)(1_{\beta} \beta \otimes_{\alpha} \sigma_{\nu, \nu}) (1_{\beta} \beta \otimes_{\alpha} W) $$

Here, $\sigma_{\alpha, \beta}^{2,3}$ goes from $(H_{\alpha} \otimes_{\beta} H) \beta \otimes_{\alpha} H$ to $(H_{\beta} \otimes_{\alpha} H) \beta \otimes_{\alpha} H$, and $1_{\beta} \beta \otimes_{\alpha} \sigma_{\nu, \nu}$ goes from $H_{\beta} \otimes_{\alpha} (H_{\alpha} \otimes_{\beta} H)$ to $H_{\beta} \otimes_{\alpha} H_{\beta} \otimes_{\alpha} H$.

All the properties supposed in (i) allow us to write such a formula, which will be called the "pentagonal relation". One should note that this definition is different from the definition introduced in [EV] (and repeated afterwards). It is in fact the same formula, the new writing

$$ \sigma_{\alpha, \beta}^{2,3}(W \otimes_{\alpha} 1)(1_{\beta} \beta \otimes_{\alpha} \sigma_{\nu, \nu}) $$

is here replacing the rather awkward writing

$$ (\sigma_{\nu, \nu} \otimes_{\beta} 1_{\beta})(1_{\beta} \alpha \otimes_{\beta} W)\sigma_{\alpha, \beta}^{2,3}(1_{\beta} \beta \otimes_{\alpha} \sigma_{\nu, \nu}) $$

but denotes the same operator, and we suggest the reader to convince himself of this easy fact.
All the properties supposed in (i) allow us to write such a formula, which will be called the "pentagonal relation".
If we start from another normal semi-finite faithful weight \( \nu' \) on \( N \), we may define, using (2.3), another unitary \( W_{\nu'} = U_{\alpha,\beta}^{\nu',\nu} \) from \( \mathcal{H}_\beta \otimes \alpha \mathcal{F} \) onto \( \mathcal{H}_\beta \otimes \beta \mathcal{F} \). The formulae which link these isomorphims between relative product Hilbert spaces and the relative flips allow us to check that this operator \( W_{\nu'} \) is also pseudo-multiplicative; which can be resumed in saying that a pseudo-multiplicative unitary does not depend on the choice of the weight on \( N \).
If \( W \) is an \((\alpha, \beta, \beta)\)-pseudo-multiplicative unitary, then the unitary \( \sigma_\nu W^* \sigma_\nu \) from \( \mathcal{H}_\beta \otimes \alpha \mathcal{F} \) to \( \mathcal{H}_\alpha \otimes \beta \mathcal{F} \) is an \((\alpha, \beta, \beta)\)-pseudo-multiplicative unitary, called the dual of \( W \).

3.3. Algebras and Hopf-bimodules associated to a pseudo-multiplicative unitary. For \( \xi_1 \) in \( D(\alpha, \beta, \nu) \), \( \eta_1 \) in \( D(\beta, \beta, \nu^\nu) \), the operator \((\rho_{\eta_2}^{\alpha,\beta})^* W_{\nu,\nu}^{\beta,\alpha} \) will be written \((id * \omega_{\xi_2, \eta_2})(W)\); we have, therefore, for all \( \xi_1, \eta_1 \) in \( \mathcal{F} \):

\[
(id * \omega_{\xi_2, \eta_2})(W)\xi_1 | \eta_1 = (W(\xi_1 \beta \otimes \alpha \xi_2) | \eta_1 \alpha \otimes \beta \eta_2)
\]

and, using the intertwining property of \( W \) with \( \hat{\beta} \), we easily get that \((id * \omega_{\xi_2, \eta_2})(W)\) belongs to \( \hat{\beta}(N)^\prime \).
If \( x \) belongs to \( N \), we have \((id * \omega_{\xi_2, \eta_2})(W)\alpha(x) = (id * \omega_{\xi_2, \alpha(x)\eta_2})(W)\), and \( \beta(x)(id * \omega_{\xi_2, \eta_2})(W) = (id * \omega_{\beta(x)\xi_2, \eta_2})(W)\).
If \( \xi \) belongs to \( D(\alpha, \beta, \nu) \), then \( \eta_1 \) in \( D(\beta, \beta, \nu) \), we shall write \((id * \omega_{\xi})(W)\) instead of \((id * \omega_{\xi})(W)\).
We shall then write \( A_w(W) \) the weak closure of the linear span of these operators, which are right \( \alpha(N) \)-modules and left \( \beta(N) \)-modules. Applying ([E2] 3.6), we get that \( A_w(W)^* \) and \( A_w(W) \) are non-degenerate algebras (one should note that the notations of ([E2]) had been changed in order to fit with Lesieur’s notations). We shall write \( \mathcal{A}(W) \) the von Neumann algebra generated by \( A_w(W) \). We then have \( \mathcal{A}(W) \subset \beta(N)^\prime \).
For \( \xi_1 \) in \( D(\beta, \beta, \nu) \), \( \eta_1 \) in \( D(\alpha, \beta, \nu) \), the operator \((\lambda_{\eta_2}^{\alpha,\beta})^* W_{\nu,\nu}^{\beta,\alpha} \) will be written \((\omega_{\xi_2, \eta_1} * id)(W)\); we have, therefore, for all \( \xi_2, \eta_2 \) in \( \mathcal{F} \):

\[
((\omega_{\xi_2, \eta_1} * id)(W)\xi_2 | \eta_2 = (W(\xi_1 \beta \otimes \alpha \xi_2) | \eta_1 \alpha \otimes \beta \eta_2)
\]

and, using the intertwining property of \( W \) with \( \beta \), we easily get that \((\omega_{\xi_2, \eta_1} * id)(W)\) belongs to \( \beta(N)^\prime \). If \( \xi \) belongs to \( D(\beta, \beta, \nu) \cap D(\beta, \beta, \nu) \), we shall write \((\omega_{\xi} * id)(W)\) instead of \((\omega_{\xi} * id)(W)\).
We shall then write \( A_w(W) \) the weak closure of the linear span of these operators. It is clear that this weakly closed subspace is a non degenerate algebra; following ([EV] 6.1 and 6.5), we shall write \( \mathcal{A}(W) \) the von Neumann algebra generated by \( A_w(W) \). We then have \( \mathcal{A}(W) \subset \beta(N)^\prime \).
In ([EV] 6.3 and 6.5), using the pentagonal equation, we got that 
\( (N, \mathcal{A}(W), \alpha, \beta, \Gamma) \) and \( (N^0, \mathcal{A}(W), \hat{\beta}, \alpha, \hat{\Gamma}) \) are Hopf-bimodules, where \( \Gamma \) and \( \hat{\Gamma} \) are defined, for any \( x \) in \( \mathcal{A}(W) \) and \( y \) in \( \mathcal{A}(W) \), by:

\[
\Gamma(x) = W^*(1_{\alpha \otimes \hat{\beta}}x)W_{N^0}
\]

\[
\hat{\Gamma}(y) = W(y_{\beta \otimes \alpha}1)W^*
\]

In ([EV] 6.1(iv)), we had obtained that \( x \) in \( \mathcal{L}(\mathcal{F}) \) belongs to \( \mathcal{A}(W)' \) if and only if \( x \) belongs to \( \alpha(N)' \cap \beta(N)' \) and verifies

\[
(x_{\alpha \otimes \hat{\beta}}1)_N = W(x_{\beta \otimes \alpha}1)
\]

We obtain the same way that \( y \) in \( \mathcal{L}(\mathcal{F}) \) belongs to \( \mathcal{A}(W)' \) if and only if \( y \) belongs to \( \alpha(N)' \cap \hat{\beta}(N)' \) and verify \( (1_{\alpha \otimes \hat{\beta}}y)_N = W(1_{\beta \otimes \alpha}y) \).

Moreover, we get that \( \alpha(N) \subset \mathcal{A} \cap \hat{\mathcal{A}} \), \( \beta(N) \subset \mathcal{A} \), \( \hat{\beta}(N) \subset \hat{\mathcal{A}} \), and, for all \( x \) in \( N \):

\[
\Gamma(\alpha(x)) = \alpha(x)_{\beta \otimes \alpha}1_N
\]

\[
\Gamma(\beta(x)) = 1_{\beta \otimes \alpha}\beta(x)_N
\]

\[
\hat{\Gamma}(\alpha(x)) = 1_{\alpha \otimes \hat{\beta}}\alpha(x)_{N^0}
\]

\[
\hat{\Gamma}(\hat{\beta}(x)) = \hat{\beta}(x)_{\alpha \otimes \hat{\beta}}1_{N^0}
\]

Following ([E2], 3.7) If \( \eta_1, \xi_2 \) are in \( D(\alpha, \nu) \), let us write \((id * \omega_{\xi_2, \eta_1})(\sigma_{\nu, W})\) for \((\lambda_{\nu \nu}^\alpha)^*\rho_{\xi_2, \eta_1}^\alpha\); we have, therefore, for all \( \xi_1 \) and \( \eta_2 \) in \( \mathcal{F} \):

\[
(id * \omega_{\xi_2, \eta_1})(\sigma_{\nu, W})\xi_1|\eta_2 = (W(\xi_1\beta \otimes_{\alpha} \xi_2)|\eta_1\alpha \otimes_{\nu} \eta_2)
\]

Using the intertwining property of \( W \) with \( \alpha \), we get that it belongs to \( \alpha(N)' \); we write \( C_w(W) \) for the weak closure of the linear span of these operators, and we have \( C_w(W) \subset \alpha(N)' \). It had been proved in ([E2], 3.10) that \( C_w(W) \) is a non degenerate algebra; following ([E2] 4.1), we shall say that \( W \) is weakly regular if \( \overline{C_w(W)} = \alpha(N)' \). If \( W \) is weakly regular, then \( A_w(W) = \mathcal{A}(W) \) and \( A_w(W) = \mathcal{A}(\overline{W}) \) ([E2], 3.12).

3.4. **Fundamental example.** Let \( \mathcal{G} \) be a measured groupoid, with \( \mathcal{G}^{(0)} \) as space of units, and \( r \) and \( s \) the range and source functions from \( \mathcal{G} \) to \( \mathcal{G}^{(0)} \), with a Haar system \( (\lambda_u)_{u \in \mathcal{G}^{(0)}} \) and a quasi-invariant measure \( \mu \) on \( \mathcal{G}^{(0)} \).

Let us write \( \nu \) the associated measure on \( \mathcal{G} \). Let us note:

\[
\mathcal{G}^2_{r,s} = \{(x,y) \in \mathcal{G}^2, r(x) = r(y)\}
\]

Then, it has been shown [Val1] that the formula \( W_\mathcal{G}f(x,y) = f(x, x^{-1}y) \), where \( x, y \) are in \( \mathcal{G} \), such that \( r(y) = r(x) \), and \( f \) belongs to \( L^2(\mathcal{G}^2) \) (with respect to an appropriate measure, constructed from \( \lambda^u \) and \( \mu \)),

\( \mu \) being a quasi-invariant measure on \( \mathcal{G}^2 \).
is a unitary from $L^2(G^{(2)})$ to $L^2(G_{r,r}^2)$ (with respect also to another appropriate measure, constructed from $\lambda^n$ and $\mu$).

Let us define $r_G$ and $s_G$ from $L^\infty(G^{(0)})$ to $L^\infty(G)$ (and then considered as representations on $L(L^2(G))$, for any $f$ in $L^\infty(G^{(0)})$, by $r_G(f) = f \circ r$ and $s_G(f) = f \circ s$.

We shall identify ([Y], 3.2.2) the Hilbert space $L^2(G^{(2)})$ with the relative Hilbert tensor product $L^2(G,\nu) \otimes_{r_G} L^2(G,\nu)$, and the Hilbert space $L^2(G_{r,r}^2)$ with $L^2(G,\nu) \otimes_{r_G} L^2(G,\nu)$. Moreover, the unitary $W_G$ can be then interpreted [Val2] as a pseudo-multiplicative unitary over the basis $L^\infty(G^{(0)})$, with respect to the representation $r_G$, and anti-representations $s_G$ and $r_G$ (as here the basis is abelian, the notions of representation and anti-representations are the same, and the commutation property is fulfilled). So, we get that $W_G$ is a $(r_G, s_G, r_G)$ pseudo-multiplicative unitary.

Let us take the notations of 3.3; the von Neumann algebra $A(W_G)$ is equal to the von Neumann algebra $L^\infty(G,\nu)$ ([Val2], 3.2.6 and 3.2.7); using ([Val2] 3.1.1), we get that the Hopf-bimodule homomorphism $\Gamma$ defined on $L^\infty(G,\nu)$ by $W_G$ is equal to the usual Hopf-bimodule homomorphism $\Gamma_G$ studied in [Val1], and recalled in 3.1. Moreover, the von Neumann algebra $\widehat{A(W_G)}$ is equal to the von Neumann algebra $L(G)$ ([Val2], 3.2.6 and 3.2.7); using ([Val2] 3.1.1), we get that the Hopf-bimodule homomorphism $\Gamma$ defined on $L(G)$ by $W_G$ is the usual Hopf-bimodule homomorphism $\Gamma_G$ studied in [Y] and [Val1].

Let us suppose now that the groupoid $G$ is locally compact in the sense of [R1]; it has been proved in ([E2], 4.8) that $W_G$ is then weakly regular (in fact was proved a much stronger condition, namely the norm regularity).

3.5. Definitions ([L1], [L2]). Let $(N, M, \alpha, \beta, \Gamma)$ be a Hopf-bimodule, as defined in 3.1 a normal, semi-finite, faithful operator valued weight $T$ from $M$ to $\alpha(N)$ is said to be left-invariant if, for all $x \in M^+_\Gamma$, we have :

$$(id_{\beta^*_\alpha} T)_N \Gamma(x) = T(x)_{\beta^*_\alpha} 1_N$$

or, equivalently (2.5), if we choose a normal, semi-finite, faithful weight $\nu$ on $N$, and write $\Phi = \nu \circ \alpha^{-1} \circ T$, which is a normal, semi-finite, faithful weight on $M$ :

$$(id_{\beta^*_\alpha} \Phi)_N \Gamma(x) = T(x)$$

A normal, semi-finite, faithful operator-valued weight $T'$ from $M$ to $\beta(N)$ will be said to be right-invariant if it is left-invariant with respect to the symmetrized Hopf-bimodule, i.e., if, for all $x \in M^+_T$, we have :

$$(T'_{\beta^*_\alpha} id)_N \Gamma(x) = 1_{\beta^*_\alpha} T'(x)$$
or, equivalently, if we write \( \Psi = \nu \circ \beta^{-1} \circ T' \):

\[
\left( \Psi_{\beta^*\alpha} \right)_{N} \Gamma(x) = T'(x)
\]

In the case of a Hopf-bimodule, with a left-invariant normal, semi-finite, faithful operator valued weight \( T \) from \( M \) to \( \alpha(N) \), Lesieur had constructed an isometry \( U \) in the following way: let us choose a normal, semi-finite, faithful weight \( \nu \) on \( N \), and let us write \( \Phi = \nu \circ \alpha^{-1} \circ T \), which is a normal, semi-finite, faithful weight on \( M \); let us write \( H_\Phi, J_\Phi, \Delta_\Phi \) for the canonical objects of the Tomita-Takesaki theory associated to the weight \( \Phi \), and let us define, for \( x \) in \( N \), \( \hat{\beta}(x) = J_\Phi \alpha(x^*)J_\Phi \).

Let \( H_\Phi \) be a Hilbert space on which \( M \) is acting; then ([L2], theorem 3.14), there exists an unique isometry \( U_{H_\Phi} \) from \( H_{\alpha} \otimes^\nu \nu^o \Lambda_\Phi(a) \) to \( H_{\beta} \otimes^\nu \nu^o \Lambda_\Phi(a) \), such that, for any \( (\beta, \nu^o) \)-orthogonal basis \( (\xi_i)_{i \in I} \) of \( (H_{\Phi})_{\beta} \), for any \( a \) in \( \mathfrak{N}_T \cap \mathfrak{N}_\Phi \) and for any \( v \) in \( D((H_{\Phi})_{\beta}, \nu^o) \), we have

\[
U_{H_\Phi}(v_{\alpha} \otimes^\nu \nu^o \Lambda_\Phi(a)) = \sum_{i \in I} \xi_i \otimes^\nu \nu^o \Lambda_\Phi((\omega_{\nu, \xi_i} \beta^*\alpha \nu^o \nu)(\Gamma(a)))
\]

Then, Lesieur proved ([L2], theorem 3.37) that, if there exists a right-invariant normal, semi-finite, faithful operator valued weight \( T' \) from \( M \) to \( \beta(N) \), then the isometry \( U_{H_\Phi} \) is a unitary, and that \( W = U_{H_\Phi}^* \) is an \((\alpha, \hat{\beta}, \beta)\)-pseudo-multiplicative unitary from \( H_{\Phi} \otimes^\nu \nu^o H_{\Phi} \) to \( H_{\Phi} \otimes^\nu \nu^o H_{\Phi} \).

**Proposition** Let \((N, M, \alpha, \beta, \Gamma)\) be a Hopf-bimodule, as defined in 3.1; let us suppose that there exist a normal, semi-finite, faithful left-invariant operator valued weight \( T \) from \( M \) to \( \alpha(N) \) and a right-invariant normal, semi-finite, faithful operator valued weight \( T' \) from \( M \) to \( \beta(N) \); let us write \( \Phi = \nu \circ \alpha^{-1} \circ T \), and let us define, for \( n \) in \( N \):

\[
\hat{\beta}(n) = J_\Phi \alpha(n^*)J_\Phi
\]

Then the \((\alpha, \hat{\beta}, \beta)\)-pseudo-multiplicative unitary from \( H_{\Phi} \otimes^\nu \nu^o H_{\Phi} \) to \( H_{\Phi} \otimes^\nu \nu^o H_{\Phi} \) verifies, for any \( x, y, y_1, y_2 \) in \( \mathfrak{N}_T \cap \mathfrak{N}_\Phi \):

\[
(i \ast \omega_{J_\Phi \alpha(y_2), \beta^*\alpha \nu^o \nu} J_\Phi(x))(W) = (id_{\nu^o \nu} \omega_{J_\Phi \alpha(y_2), J_\Phi \alpha(y_1)}) \Gamma(x^*)
\]

**Proof.** This is just ([L2], 3.19).

**Remark** Clearly, the pseudo-multiplicative unitary \( W \) does not depend upon the choice of the right-invariant operator-valued weight \( T' \).
4. Coinverse and scaling group

In this chapter, we are dealing with a Hopf-bimodule \((N, \alpha, \beta, M, \Gamma)\), equipped with a left-invariant operator-valued weight \(T_L\), and a right-invariant operator-valued weight \(T_R\). If \(\nu\) denotes a normal semi-finite faithful weight on the basis, let \(\Phi\) (resp. \(\Psi\)) be the lifted normal faithful semi-finite weight on \(M\) by \(T_L\) (resp. \(T_R\)). Then, with the additional hypothesis that the two modular automorphism groups associated to the two weight \(\Phi\) and \(\Psi\) commute, we can construct a co-inverse, a scaling group and an antipod, using slight generalizations of the constructions made in ([L2],9) for "adapted measured quantum groupoids".

4.1. Definition. Let \((N, \alpha, \beta, M, \Gamma)\) be a Hopf-bimodule, equipped with a left-invariant operator-valued weight \(T_L\), and a right-invariant valued weight \(T_R\); let \(\nu\) be a normal semi-finite faithful weight on \(N\); we shall denote \(\Phi = \nu \circ \alpha^{-1} \circ T_L\) and \(\Psi = \nu \circ \beta^{-1} \circ T_R\) the two lifted normal semi-finite weights on \(M\). We shall say that the weight \(\nu\) is relatively invariant with respect to \(T_L\) and \(T_R\) if the two modular automorphisms groups \(\sigma^\Phi\) and \(\sigma^\Psi\) commute.

4.2. Lemma. Let \((N, \alpha, \beta, M, \Gamma)\) be a Hopf-bimodule, equipped with a left-invariant operator-valued weight \(T_L\), and a right-invariant valued weight \(T_R\); let \(\nu\) be a normal semi-finite faithful weight on \(N\), relatively invariant with respect to \(T_L\) and \(T_R\); we shall denote \(\Phi = \nu \circ \alpha^{-1} \circ T_L\) and \(\Psi = \nu \circ \beta^{-1} \circ T_R\) the two lifted normal semi-finite weights on \(M\). Let us suppose that the two modular automorphisms groups \(\sigma^\Phi\) and \(\sigma^\Psi\) commute, and let us denote \(\delta\) the modulus of \(\Psi\) with respect to \(\Phi\) and \(\lambda\) the scaling operator \((2.6)\). We shall use the notations of \([2.2.1]\)

Then :

(i) let \(x \in T_{\Phi,T_R}\) and \(n \in \mathbb{N}\) and \(y = e_n x\), with the notations of \([2.6]\); then \(y\) belongs to \(\mathcal{R}_\Psi \cap \mathcal{R}_{T_R}\), is analytical with respect to \(\Psi\), and the operator \(\sigma^\Psi_{-i/2}(y^*) \delta^{1/2}\) is bounded, and its closure \(\sigma^\Psi_{-i/2}(y^*) \delta^{1/2}\) belongs to \(\mathcal{R}_\Phi\); moreover, with the identifications made in \([2.2.1]\) we have :

\[
\Lambda_\Phi(\sigma^\Psi_{-i/2}(y^*) \delta^{1/2}) = J_\Psi \Lambda_\Psi(y)
\]

(ii) let \(E\) be the linear space generated by all such elements of the form \(\sigma^\Psi_{i/2}(y^*) \delta^{1/2}\), for all \(x \in T_{\Phi,T_R}\) and \(n \in \mathbb{N}\); then \(E\) is a weakly dense subspace of \(\mathcal{R}_\Phi\), and, for all \(z \in E\), \(\Lambda_\Phi(z) \in D((H_\Phi)_{\beta, \nu^o})\);

(iii) the linear set of all products \(\Lambda_\Phi(z), \Lambda_\Phi(z') >_{\beta, \nu^o}\) (for \(z, z'\) in \(E\)) is a dense subspace of \(N\).

Proof. As \(e_n\) is analytical with respect to \(\Psi\), \(y\) belongs to \(\mathcal{R}_\Psi \cap \mathcal{R}_{T_R}\), is analytical with respect to \(\Psi\), and \(\sigma^\Psi_{-i/2}(y^*) \delta^{1/2}\) is bounded ([V], 1.2); as \(\delta^{-1}\) is the modulus of \(\Phi\) with respect to \(\Psi\), we get that \(\sigma^\Psi_{i/2}(y^*) \delta^{1/2}\) belongs to \(\mathcal{R}_\Phi\); we identify \(\Lambda_\Phi(\sigma^\Psi_{-i/2}(y^*) \delta^{1/2})\) with \(\Lambda_\Psi(\sigma^\Psi_{-i/2}(y^*)) = J_\Psi \Lambda_\Psi(y)\), which is (i).
The subspace $E$ contains all elements of the form $\sigma_{i/2}^{\Psi}(x^*)\delta^{1/2}\sigma_{i/2}^{\Psi}(e_n)$ $(x \in T_\Psi T_R)$, and, by density of $T_\Psi T_R$ in $M$, we get that the closure of $E$ contains all elements of the form $a e_n \sigma_{i/2}^{\Psi}(e_n) = ae_n \sigma_{i/2}^{\Psi}(e_n)$, for all $a \in M$; now, as $e_n \sigma_{i/2}^{\Psi}(e_n)$ is converging to 1, we finally get that $E$ is dense in $M$; as $\Lambda_\Phi(E) \subset J_\Psi \Lambda_\Phi(\mathfrak{M}_\Psi \cap \mathfrak{M}_T R)$, we get, by 2.2, that, for all $z$ in $E$, $\Lambda_\Phi(z)$ belongs to $D(\Lambda_\Phi)$; more precisely, we have:

$$R^{\beta,\nu}(\Lambda_\Phi(e_n x)) = R^{\beta,\nu}(J_\Psi \Lambda_\Phi(e_n x)) = \Lambda_T R(e_n x)$$

Therefore, the set of elements of the form $\Lambda_\Phi(z), \Lambda_\Phi(z') >_{\beta,\nu}$ contains all elements of the form $\beta^{-1} \circ T_R(x^* e_n e_n x)$, for all $x$ in $T_\Psi T_R$ and $n \in \mathbb{N}$; as $T_R(x^* e_n e_n x) = \Lambda_T R(e_n x)^* \Lambda_T R(e_n x) = \Lambda_T R(e_n x)^* e_n e_n \Lambda_T R(x)$; so, its closure contains all elements of the form $\beta^{-1} \circ T_R(x^* x)$, and, therefore, it contains $\beta^{-1} \circ T_R(\mathfrak{M}_T R)$, which finishes the proof. □

4.3. Definition. As in ([L2], 9.2), we can define, for all $\lambda \in \mathbb{C}$, a closed operator $\Delta^{N^0}_{\beta,\alpha,\beta} \Delta^N_{\beta}$, with natural values on elementary tensor products; it is possible also to define a unitary antilinear operator $J^{N^0}_{\beta,\alpha,\beta} J^{N^0}_{\beta}$ from $H_{\Phi} \otimes^{\beta,\alpha,\beta} H_{\Phi}$ onto $H_{\Phi} \otimes^{\beta,\alpha,\beta} H_{\Phi}$ (whose inverse will be $J^{N^0}_{\beta,\alpha,\beta} J^{N^0}_{\beta}$); by composition, we define then a closed antilinear operator $S^{N^0}_{\Phi,\alpha,\beta,\alpha,\beta} S^{N^0}_{\Phi}$, with natural values on elementary tensor products, whose adjoint will be $F^{N^0}_{\beta,\alpha,\beta,\alpha,\beta} F^{N^0}_{\beta}$.

4.4. Proposition. For all $a, c$ in $(\mathfrak{M}_\Psi \cap \mathfrak{M}_T L)^*(\mathfrak{M}_\Psi \cap \mathfrak{M}_T R)$, $b, d$ in $T_\Psi T_R$ and $g, h$ in $E$, the following vector:

$$U^{*}_{H_{\Phi}} \Gamma(g^*)[\Lambda_\Phi(h)_{\beta,\alpha,\nu} (\lambda^{\beta,\alpha,\nu}_{\gamma}(\sigma_{\nu}(a^*)))^* U_{H_{\Phi}}(\Lambda_\Phi(a)_{\alpha,\beta,\nu}(\nu^*))]$$

belongs to $D(S^{N^0}_{\Phi,\alpha,\beta,\alpha,\beta} S^{N^0}_{\Phi})$, and the value of $\sigma_{\nu}(S^{N^0}_{\Phi,\alpha,\beta,\alpha,\beta} S^{N^0}_{\Phi})$ on this vector is equal to:

$$U^{*}_{H_{\Phi}} \Gamma(h^*)[\Lambda_\Phi(g)_{\beta,\alpha,\nu} (\lambda^{\beta,\alpha,\nu}_{\gamma}(\sigma_{\nu}(a^*)))^* U_{H_{\Phi}}(\Lambda_\Phi(c)_{\alpha,\beta,\nu}(\nu^*))]$$

Proof. The proof is identical to ([L2],9.9), thanks to 1.2(ii). □

4.5. Proposition. There exists a closed densely defined anti-linear operator $G$ on $H_{\Phi}$ such that the linear span of:

$$(\lambda^{\beta,\alpha}_{\gamma}(\sigma_{\nu}(a^*)))^* U_{H_{\Phi}}(\Lambda_\Phi(a)_{\alpha,\beta,\nu}(\nu^*))$$
with $a$, $c$ in $(\mathcal{M}_\Phi \cap \mathcal{N}_{T_L})^*(\mathcal{M}_\Phi \cap \mathcal{N}_{T_R})$, $b$, $d$ in $T_{\Phi,T_R}$, is a core of $G$, and we have:

$$G[(\lambda^\beta_\alpha(\sigma_\Phi^0(b^*)))^*U_{\Lambda_\Psi}(\Lambda_\Psi(a)_{\alpha,\beta}\Lambda_\Phi((cd)^*))] =$$

$$(\lambda^\beta_\alpha(\sigma_\Phi^0(d^*)))^*U_{\Lambda_\Psi}(\Lambda_\Psi(c)_{\alpha,\beta}\Lambda_\Phi((ab)^*))$$

**Proof.** The proof is identical to ([L2],9.10), thanks to 4.2(iii). □

4.6. **Theorem.** Let $(N,\alpha,\beta,M,\Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight $T_L$, and a right-invariant valued weight $T_R$; let $\nu$ be a normal semi-finite faithful weight on $N$, relatively invariant with respect to $T_L$ and $T_R$; we shall denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and $\Psi = \nu \circ \beta^{-1} \circ T_R$ the two lifted normal semi-finite weights on $M$. Let $G$ be the closed densely defined antilinear operator defined in 4.5, and let $G = IQ_1/2$ its polar decomposition. Then, the operator $D$ is positive self-adjoint and non singular; there exists a one-parameter automorphism group $\tau_t$ on $M$ defined, for $x \in M$, by:

$$\tau_t(x) = D^{-it}x D^{it}$$

We have, for all $n \in N$ and $t \in \mathbb{R}$:

$$\tau_t(\alpha(n)) = \alpha(\sigma^\nu_t(n))$$

$$\tau_t(\beta(n)) = \beta(\sigma^\nu_t(n))$$

which allows us to define $\tau_t\beta\alpha, \tau_t, \tau_t\beta\alpha, \sigma^\Phi_t$ and $\sigma^\Psi_t\beta\alpha, \tau_t\alpha, \tau_t\beta\alpha, \tau_t\alpha, \tau_t\beta\alpha, \tau_t\alpha, \tau_t\beta\alpha, \tau_t\alpha$. Moreover, we have:

$$\Gamma \circ \tau_t = (\tau_t)_{\beta\alpha} \tau_t \Gamma$$

$$\Gamma \circ \sigma^\Phi_t = (\tau_t)_{\beta\alpha} \sigma^\Phi_t \Gamma$$

$$\Gamma \circ \sigma^\Psi_t = (\tau_t)_{\beta\alpha} \tau_t \alpha, \tau_t\beta\alpha, \tau_t\alpha, \tau_t\beta\alpha, \tau_t\beta\alpha, \tau_t\alpha, \tau_t\beta\alpha, \tau_t\beta\alpha, \tau_t\alpha$$

**Proof.** The proof is identical to [L2], 9.12 to 9.28. □

4.7. **Theorem.** Let $(N,\alpha,\beta,M,\Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight $T_L$, and a right-invariant valued weight $T_R$; let $\nu$ be a normal semi-finite faithful weight on $N$, relatively invariant with respect to $T_L$ and $T_R$; we shall denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and $\Psi = \nu \circ \beta^{-1} \circ T_R$ the two lifted normal semi-finite weights on $M$. Let $G$ be the closed densely defined antilinear operator defined in 4.5, and let $G = IQ_1/2$ its polar decomposition. Then, the operator $I$ is antilinear, isometric, surjective, and we have $I = I^* = I^2$; there exists a *-antiautomorphism $R$ on $M$ defined, for $x \in M$, by:

$$R(x) = Ix^* I$$
such that, for all $t \in \mathbb{R}$, we get $R \circ \tau_t = \tau_t \circ R$ and $R^2 = \text{id}$.

For any $a, b$ in $\mathcal{M}_R \cap \mathcal{M}_T$, we have:

$$R((\omega J_{\Phi}(a) \beta^{*N} id)\Gamma(b*b)) = (\omega J_{\Phi}(b) \beta^{*N} id)\Gamma(a*a)$$

and for any $c, d$ in $\mathcal{M}_\Phi \cap \mathcal{M}_T$, we have:

$$R((id \beta^{*N} \omega J_{\Phi}(c))\Gamma(d*d)) = (id \beta^{*N} \omega J_{\Phi}(d))\Gamma(c*c))$$

For all $n \in N$, we have $R(\alpha(n)) = \beta(n)$, which allows us to define $R_{\beta^{*N}}R$ from $M_{\beta^{*N}}M$ onto $M_{\alpha^{*\beta}M^o}$ (whose inverse will be $R_{\alpha^{*\beta}R}$), and we have:

$$\Gamma \circ R = \varsigma_{N^o}(R_{\beta^{*N}}R)\Gamma$$

Proof. The proof is identical to [L2], 9.38 to 9.42. \hfill \Box

4.8. Theorem. Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight $T_L$, and a right-invariant valued weight $T_R$; let $\nu$ be a normal semi-finite faithful weight on $N$, relatively invariant with respect to $T_L$ and $T_R$; we shall denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$; then:

(i) $M$ is the weak closure of the linear span of all elements of the form $(\omega_{\beta^{*N}} id)\Gamma(x)$, for all $x \in M$ and $\omega \in M^*$ such that there exists $k > 0$ such that $\omega \circ \beta \leq kv$.

(ii) $M$ is the weak closure of the linear span of all elements of the form $(id \beta^{*N} \omega)\Gamma(x)$, for all $x \in M$ and $\omega \in M^*$ such that there exists $k > 0$ such that $\omega \circ \alpha \leq kv$.

(iii) $M$ is the weak closure of the linear span of all elements of the form $(id * \omega_{v,w})(W)$, where $v$ belongs to $D_{\alpha H_{\Phi}, \nu}$ and $w$ belongs to $D((H_{\Phi})_{\beta}, \nu^o)$.

Proof. The proof is identical to [L2], 9.25. \hfill \Box

4.9. Definition. Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight $T_L$, and a right-invariant valued weight $T_R$; let $\nu$ be a normal semi-finite faithful weight on $N$, relatively invariant with respect to $T_L$ and $T_R$; we shall denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and $\Psi = \nu \circ \beta^{-1} \circ T_R$ the two lifted normal semi-finite weights on $M$; let $\tau_t$ the one-parameter automorphism group constructed in [L6] and let $R$ be the involutive $*$-anti-automorphism constructed in [L7]. We shall call $\tau_t$ the scaling group of $(N, \alpha, \beta, M, \Gamma, T_L, T_R, \nu)$ and $R$ the converse of $(N, \alpha, \beta, M, \Gamma, T_L, T_R, \nu)$. Thanks to [L7] and [L8] we see that, $T_L$ and $\nu$ being given, $R$ does not depend on the choice of the right-invariant operator-valued weight $T_R$, provided that there exists a right-invariant operator-valued weight $T_R$ such that $\nu$ is relatively invariant with respect to $T_L$ and $T_R$. 


Similarly, from [4.6] one gets that, for all $x$ in $M$, $\omega \in M_*$ such that there exists $k > 0$ with $\omega \circ \alpha \leq k\nu$, $\omega' \in M_*$ such that there exists $k > 0$ with $\omega \circ \beta \leq k\nu$, one has:

\[
\tau_t((id_{\beta^*_{\alpha_k}}N \omega)\Gamma(x)) = (id_{\beta^*_{\alpha_k}}N \omega \circ \sigma^\Phi_{\alpha_k})\Gamma\sigma^\Phi_t(x)
\]

\[
\tau_t((\omega'_t \circ \alpha_k id)\Gamma(x)) = (\omega' \circ \sigma^{\Phi}_{\alpha_k}N id)\Gamma\sigma^{\Phi}_{-t}(x)
\]

So, $T_L$ and $\nu$ being given, $\tau_t$ does not depend on the choice of the right-invariant operator-valued weight $T_R$; provided that there exists a right-invariant operator-valued weight $T_R$ such that $\nu$ is relatively invariant with respect to $T_L$ and $T_R$.

4.10. Theorem. Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight $T_L$, and a right-invariant valued weight $T_R$; let $\nu$ be a normal semi-finite faithful weight on $N$, relatively invariant with respect to $T_L$ and $T_R$; we shall denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$; then, for any $\xi, \eta$ in $\mathcal{D}(\alpha H_\Phi, \nu) \cap \mathcal{D}(H_\Phi, \nu^\circ \alpha^{-1})$, $(id * \omega_{\xi,\eta})(W)$ belongs to $\mathcal{D}(\tau_{1/2})$, and, if we define $S = R\tau_{1/2}$, we have:

\[
S((id * \omega_{\xi,\eta})(W)) = (id * \omega_{\eta,\xi})(W)^*
\]

More generally, for any $x$ in $D(S) = D(\tau_{1/2})$, we get that $S(x)^*$ belongs to $D(S)$ and $S(S(x)^*)^* = x$; $S$ will be called the antipod of the measured quantum groupoid, and, therefore, the co-inverse and the scaling group, given by polar decomposition of the antipod, rely only upon the pseudo-multiplicative $W$.

Proof. It is proved similarly to [L2] 9.35 and 9.36.

4.11. Proposition. Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight $T_L$, and a right-invariant valued weight $T_R$; let $\nu$ be a normal semi-finite faithful weight on $N$, relatively invariant with respect to $T_L$ and $T_R$; let $\tau_t$ be the scaling group of $(N, \alpha, \beta, M, \Gamma, T_L, T_R, \nu)$ and $R$ the coinverse of $(N, \alpha, \beta, M, \Gamma, T_L, T_R, \nu)$; then:

(i) the operator-valued weight $RT_R R$ is left-invariant, the operator valued-weight $RT_R R$ is right-invariant, and $\nu$ is relatively invariant with respect to $RT_R R$ and $RT_L R$.

(ii) $\tau_t$ is the scaling group of $(N, \alpha, \beta, M, \Gamma, RT_R R, RT_L R, \nu)$

Proof. Let $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and $\Psi = \nu \circ \beta^{-1} \circ T_R$ the two lifted normal semi-finite weights on $M$ by $T_L$ and $T_R$; the lifted weight by $RT_R R$ (resp. $RT_L R$) is then $\Psi \circ R$ (resp. $\Phi \circ R$). As $\sigma^\Psi_0 R = R \circ \sigma^\Psi_0$ and $\sigma^\Phi_0 R = R \circ \sigma^\Phi_0$, we get that $\sigma^\Psi_0$ and $\sigma^\Phi_0$ commute, which is (i).
From 4.6 and 4.7, we get that:

\[ \Gamma \circ \sigma_{t}^{\Psi_{\circ}R} = \Gamma \circ R \circ \sigma_{-t}^{\Psi_{\circ}R} = \zeta_{N^{\circ}}(R_{\beta^{*}_{\alpha} R}) \Gamma \circ \sigma_{-t}^{\Psi_{\circ}R} \]

\[ = \zeta_{N^{\circ}}(R \circ \sigma_{-t}^{\Psi_{\circ}R} R_{\alpha^{*}_{\beta} R \circ \tau_{t} \circ R}) \zeta_{N} \Gamma = (\tau_{t}^{\beta^{*}_{\alpha} \sigma_{t}^{\Psi_{\circ}R}}) \Gamma \]

from which we get that, for all \( x \in M \) and \( \omega \in M^{*} \) such that there exists \( k > 0 \) such that \( \omega \circ \alpha < k \nu \), we have:

\[ \tau_{t}((id_{\beta^{*}_{\alpha} \omega}) \Gamma(x)) = (id_{\beta^{*}_{\alpha} \omega} \circ \sigma_{-t}^{\Psi_{\circ}R}) \Gamma(\sigma_{t}^{\Psi_{\circ}R}(x)) \]

from which we get, by 4.8, that \( \tau_{t} \) is the scaling group associated to \( RT_{R} \), \( RT_{L} \) and \( \nu \). 

\[ \square \]

5. Automorphism groups on the basis

In this section, with the same hypothesis as in chapter 4, we construct two one-parameter automorphism groups on the basis \( N \) (5.2), and we prove (5.7) that these automorphisms leave invariant the quasi-invariant weight \( \nu \). We prove also in (5.7) that the weight \( \nu \) is also quasi-invariant with respect to \( T_{L} \) and \( RT_{L}R \).

5.1. Lemma. Let \( (N, \alpha, \beta, M, \Gamma) \) be a Hopf-bimodule, equipped with a left-invariant operator-valued weight \( T_{L} \), and a right-invariant valued weight \( T_{R} \); let \( \nu \) be a normal semi-finite faithful weight on \( N \), relatively invariant with respect to \( T_{L} \) and \( T_{R} \). Let \( x \in M \cap \alpha(N)^{'} \) and \( y \in M \cap \beta(N)^{'} \). Then:

(i) \( x \) belongs to \( \beta(N) \) if and only if we have:

\[ \Gamma(x) = 1_{\beta^{\otimes_{\alpha} x}} \]

(ii) \( y \) belongs to \( \alpha(N) \) if and only if we have:

\[ \Gamma(y) = y_{\beta^{\otimes_{\alpha} 1}} \]

More generally, if \( x_{1}, x_{2} \) are in \( M \cap \alpha(N)^{'} \) and such that \( \Gamma(x_{1}) = 1_{\beta^{\otimes_{\alpha} x_{2}}} \), then \( x_{1} = x_{2} \in \beta(N) \).

Proof. The proof is given in [L2], 4.4.

5.2. Proposition. Let \( (N, \alpha, \beta, M, \Gamma) \) be a Hopf-bimodule, equipped with a left-invariant operator-valued weight \( T_{L} \), and a right-invariant valued weight \( T_{R} \); let \( \nu \) be a normal semi-finite faithful weight on \( N \), relatively invariant with respect to \( T_{L} \) and \( T_{R} \). Then, there exists a unique one-parameter group of automorphisms \( \gamma_{t}^{L} \) of \( N \) such that, for all \( t \in \mathbb{R} \) and \( n \in N \), we have:

\[ \sigma_{t}^{TL}(\beta(n)) = \beta(\gamma_{t}^{L}(n)) \]

\[ \sigma_{t}^{RTL}(\alpha(n)) = \alpha(\gamma_{t}^{L}(n)) \]
Moreover, the automorphism groups $\gamma^L$ and $\sigma^\nu$ commute, and there exists a positive self-adjoint non-singular operator $h_L \eta Z(N) \cap N^\gamma^L$ such that, for any $x \in N^+$ and $t \in \mathbb{R}$, we have:

$$\nu \circ \gamma^L_t(x) = \nu(h_L^t x)$$

Starting from the operator-valued weights $RT_R R$ and $RT_L R$, we obtain another one-parameter group of automorphisms $\gamma^R_t$ of $N$, such that we have:

$$\sigma^{RT_R R}_t(\beta(n)) = \beta(\gamma^R_t(n))$$
$$\sigma^{TR}_t(\alpha(n)) = \alpha(\gamma^R_t^{-1}(n))$$

and a positive self-adjoint non-singular operator $h_R \eta Z(N) \cap N^\gamma^R$ such that we have:

$$\nu \circ \gamma^R_t(x) = \nu(h_R^t x)$$

Proof. The existence of $\gamma^L_t$ is given by [L2], 4.5; moreover, from the formula $\sigma^\Psi_t \circ \sigma^\Psi_s(\beta(n)) = \sigma^\Psi_s \circ \sigma^\Psi_t(\beta(n))$, we obtain:

$$\beta(\gamma^L_t \circ \sigma^\nu_{-s}(n)) = \beta(\sigma^\nu_{-s} \circ \gamma^L_t(n))$$

which gives the commutation of $\gamma^L_t$ and $\sigma^\nu_{-s}$. The existence of $h_L$ is then straightforward. The construction of $\gamma^R$ and $h_R$ is just the application of the preceding results to $RT_R R$, $RT_L R$ and $\nu$.

5.3. Proposition. Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight $T_L$, and a right-invariant valued weight $T_R$: let $\nu$ be a normal semi-finite faithful weight on $N$, relatively invariant with respect to $T_L$ and $T_R$. Let $T'_L$ (resp. $T'_R$) be another left (resp. right)-invariant operator-valued weight; we shall denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$, $\Phi' = \nu \circ \alpha^{-1} \circ T'_L$, $\Psi = \nu \circ \beta^{-1} \circ T_R$ and $\Psi' = \nu \circ \beta^{-1} \circ T'_R$ the lifted normal semi-finite weights on $M$; then, we have:

$$\beta(h_{Lst}^{ist}) = (D\Psi' \circ \sigma^\Phi_t : D\Psi' \circ \tau_s)_s$$
$$\alpha(h_{Rst}^{ist}) = (D\Psi' \circ \sigma^\Psi_t : D\Psi' \circ \tau_s)_s$$

where $\tau_s$ is the scaling group constructed from $T_L$, $T_R$ and $\nu$ as well from $RT_R R$, $RT_L R$ and $\nu$ [4.6 and 4.17].

Proof. From [4.6] we get, for all $t \in \mathbb{R}$, $\Gamma \circ \sigma^\Phi_t \tau_{-t} = (id \circ_{N} N \gamma^N \tau_t) \Gamma$, and, therefore, by the right-invariance of $T'_R$, we get, for all $x \in M^+_{T'_R}$, that $\tau_t \sigma^\Phi_{T'_R} \sigma^\Phi_{T'_R} \tau_{-t}(x) = T'_R(x)$; let now $x \in M^+_{T'_R}$; $T'_R(x)$ is an element of the positive extended part of $N$ which can be written:

$$\int_0^\infty \lambda d\varepsilon_\lambda + (1 - p)\infty$$
where \( p \) is a projection in \( \beta(N) \), and \( e_\lambda \) is a resolution of \( p \). As \( x \) belongs to \( \mathcal{M}^+_{p^\prime} \), it is well known that \( p = 1 \), and \( T'_R(x) = \int_0^\infty \lambda d\epsilon_\lambda \). There exists also a projection \( q \) and a resolution of \( q \) such that:

\[
\tau_i \sigma^\Phi_{-t} T'_R \sigma^\Phi_t \tau_{-t}(x) = \int_0^\infty \lambda d\epsilon_\lambda + (1 - q) \infty
\]

and, for all \( \mu \in \mathbb{R}^+ \), we have, because \( e_\mu x e_\mu \) belongs to \( \mathcal{M}^+_{T'_R} \):

\[
e_\mu \left( \int_0^\infty \lambda d\epsilon_\lambda \right) e_\mu + e_\mu (1 - q) e_\mu \infty = e_\mu \tau_i \sigma^\Phi_{-t} T'_R \sigma^\Phi_t \tau_{-t}(x) e_\mu = \tau_i \sigma^\Phi_{-t} T'_R \sigma^\Phi_t \tau_{-t}(e_\mu x e_\mu) = T'_R(e_\mu x e_\mu) = \int_0^\mu \lambda d\epsilon_\lambda
\]

from which we infer that \((1 - q) e_\mu = 0\), and, therefore, that \( q = 1 \); then, we get that \( e_\mu \tau_i \sigma^\Phi_{-t} T'_R \sigma^\Phi_t \tau_{-t}(x) e_\mu \) is increasing with \( \mu \) towards \( T'_R(x) \). Therefore, we get that:

\[
\tau_i \sigma^\Phi_{-t} T'_R \sigma^\Phi_t \tau_{-t}(x) \subset T'_R(x)
\]

and, finally, the equality, for all \( x \in \mathcal{M}^+_{p^\prime} \):

\[
\tau_i \sigma^\Phi_{-t} T'_R \sigma^\Phi_t \tau_{-t}(x) = T'_R(x)
\]

Moreover, as we have, for all \( n \in N \)

\[
\tau_i \sigma^\Phi_{-t}(\beta(n)) = \beta(\sigma^\Psi_{-t}(n))
\]

we get, using \( 5.2 \) that, for all \( x \in \mathcal{M}^+_{p^\prime} \):

\[
\Psi'(\beta(h_{L}^{-t/2}) \sigma^\Phi_t \tau_{-t}(x) \beta(h_{L}^{-t/2})) = \Psi'(x)
\]

and, therefore, that, for all \( x \in M^+ \):

\[
\Psi'(\beta(h_{L}^{-t/2}) \sigma^\Phi_t \tau_{-t}(x) \beta(h_{L}^{-t/2})) \leq \Psi'(x)
\]

A similar calculation (with \( \tau_i \sigma^\Phi_{-t} \) instead of \( \sigma^\Phi_{-t} \)) leads to:

\[
\Psi'(\beta(h_{L}^{t/2}) \tau_i \sigma^\Phi_{-t}(x) \beta(h_{L}^{t/2})) \leq \Psi'(x)
\]

which leads to the equality, from which we get the first result. Applying this result to \( RT'_R R, R T'_R R \) and \( \nu \), we get, using again \( 4.11 \):

\[
\beta(h_{R}^{\emp}) = (D\Psi' \circ R \circ \sigma^\Psi_{-t} \circ D\Psi' \circ R \circ \tau_i)_s
\]

\[
= (D\Psi' \circ \sigma^\Psi_{-t} \circ R \circ D\Psi' \circ \tau_i \circ R)_s
\]

\[
= R[\gamma((D\Psi' \circ \sigma^\Psi_{-t} : D\Psi' \circ \tau_i \circ R)_s]]
\]

which leads to the result. \( \Box \)
5.4. Corollary. Let \((N, \alpha, \beta, M, \Gamma)\) be a Hopf-bimodule, equipped with a left-invariant operator-valued weight \(T_L\), and a right-invariant valued weight \(T_R\); let \(\nu\) be a normal semi-finite faithful weight on \(N\), relatively invariant with respect to \(T_L\) and \(T_R\). We shall denote \(\Phi = \nu \circ \alpha^{-1} \circ T_L\) and \(\Psi = \nu \circ \beta^{-1} \circ T_R\) the two lifted normal semi-finite weights on \(M\), \(R\) the converse and \(\tau_t\) the scaling group constructed in \([4,7]\) and \([4,0]\) we shall denote \(\lambda\) the scaling operator of \(\Psi\) with respect to \(\Phi\) \([2,0]\). Then, for all \(s, t\) in \(\mathbb{R}\):

(i) \((D\Psi : D\Phi \circ \tau_t)_s = \lambda^{ist}(h^{ist}_L)\)

(ii) \((D\Phi : D\Phi \circ \tau_t)_s = \lambda^{ist}(h^{ist}_R)\)

(iii) \((D\Phi : D\Phi \circ \sigma^R_{-t})_s = \lambda^{ist}(h^{ist}_R)\alpha(h^{-ist}_L)\)

(iv) \((D\Phi : D\Phi \circ \sigma^R_{-t})_s = \lambda^{ist}(h^{ist}_R)\beta(h^{-ist}_L)\).

Proof. Applying 5.3 with \(T_R = T_R\), as \((D\Psi : D\Phi)_s = \lambda^{ist}\) \([2,0]\), we obtain (i). Applying 5.3 with \(T_R' = T_L\), as \((D\Phi : D\Phi \circ \sigma^R_{-t})_s = \lambda^{ist}\), we obtain (ii). Applying 5.3 with \(T_R' = RT_L R\), we obtain :

\[
\beta(h^{ist}_L) = (D\Phi : D\Phi \circ \sigma^R_t : D\Phi \circ R \circ \tau_t)_s
\]

\[
= (D\Phi : D\Phi \circ \sigma^R_{-t} \circ R : D\Phi \circ \tau_t \circ R)_s
\]

\[
= R((D\Phi : D\Phi \circ \sigma^R_{-t} : D\Phi \circ \tau_t)_{-s})
\]

and, therefore \(\alpha(h^{ist}_L) = (D\Phi : D\Phi \circ \sigma^R_{-t} : D\Phi \circ \tau_t)_{-s}\) from which one gets :

\[
\alpha(h^{ist}_L) = (D\Phi : D\Phi \circ \sigma^R_{-t} : D\Phi \circ \tau_t)_s
\]

Using (ii), we get :

\[
(D\Phi : D\Phi \circ \sigma^R_{-t})_s = \lambda^{ist}(h^{ist}_R)\alpha(h^{-ist}_L)
\]

which is (iii). And applying 5.3 with \(T_R' = RT_L R\), we obtain (iv). \(\square\)

5.5. Lemma. Let \(M\) be a von Neumann algebra, \(\Phi\) a normal semi-finite faithful weight on \(M\), \(\theta_t\) a one parameter group of automorphisms of \(M\). Let us suppose that there exists a positive non singular operator \(\mu\) affiliated to \(M^\Phi\) such that, for all \(s, t\) in \(\mathbb{R}\), we have

\[
(D\Phi \circ \theta_t : D\Phi)_s = \mu^{ist}
\]

We have then, for all \(t \in \mathbb{R}\), \(\theta_t(\mu) = \mu\). Let us write \(\mu = \int_0^\infty \lambda d\nu\lambda\) the spectral decomposition of \(\mu\), and let us define \(f_n = \int_{1/n}^n d\nu\lambda\). We have then, for all \(a\) in \(M^\Phi\), \(t\) in \(\mathbb{R}\), \(n\) in \(\mathbb{N}\):

\[
\omega_{J_\Phi \Lambda_\Phi(af_n)} \circ \theta_t = \omega_{J_\Phi \Lambda_\Phi(\theta_{-t}(a)f_n \mu^{t/2})}
\]

Proof. Let us remark first that \(\theta_t(\mu) = \mu\), and, therefore, \(\theta_t(f_n) = f_n\).

On the other hand, for any \(a\) in \(M\), we have :

\[
\theta_{-t} \sigma^\Phi_s \theta_t(x) = \sigma^\Phi_{s \theta_t}(x) = \mu^{ist} \sigma^\Phi_s(x) \mu^{-ist}
\]

and then :

\[
\theta_{-t} \sigma^\Phi_s(x) = \mu^{ist} \sigma^\Phi_s \theta_t(x) \mu^{-ist}
\]
If now $x$ is analytic with respect to $\Phi$, we get that $\theta_{-t}(f_nxf_m)$ is analytic with respect to $\Phi$ and that:

$$f_n\theta_{-t}\sigma_{i/2}(x)f_m = \mu^{-t/2}f_n\sigma_{i/2}(\theta_{-t}(x))f_m\mu^{t/2}$$

Let us take now $a$ in $\mathfrak{N}_\Phi$, analytic with respect to $\Phi$; we have, for any $y$ in $M$:

$$\omega_{J_\Phi}(f_\nu|a)f_\mu = (\theta_t(y)J_\Phi\Lambda_\Phi(f_\nu|a)f_\mu|J_\Phi\Lambda_\Phi(f_\nu|a)f_\mu)$$

which, using the preceding remarks, is equal to:

$$\Phi(\sigma_{i/2}(\theta_{-t}(a)))f_m\mu^{t/2}y\mu^{t/2}f_m\sigma_{i/2}(\theta_{-t}(a))f_m\mu^{t/2}$$

and, making now $f$ increasing to $1$, we get that $\omega_{J_\Phi}(a|f_\nu|a)$ is equal to:

$$\Phi(\sigma_{i/2}(\theta_{-t}(a)))f_m\mu^{t/2}y\mu^{t/2}f_m\sigma_{i/2}(\theta_{-t}(a))$$

from which we get the result. 

5.6. **Lemma.** Let $(N, \alpha, \beta, M, \Gamma)$ be a Hopf-bimodule, equipped with a left-invariant operator-valued weight $T_L$, and a right-invariant valued weight $T_R$; let $\nu$ be a normal semi-finite faithful weight on $N$, relatively invariant with respect to $T_L$ and $T_R$. We shall denote $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and $\Psi = \nu \circ \beta^{-1} \circ T_R$, the two lifted normal semi-finite weights on $M$, $R$ the coinverse and $\tau_t$ the scaling group constructed in 4.7 and 4.6.

Then, we have:

(i) there exists a positive non singular operator $\mu_1$ affiliated to $M^\Phi$ and invariant under $\tau_t$, such that $(D\Phi \circ \tau_t : D\Phi)_s = \mu_1^{st}$; let us write $\mu_1 = \int_0^\infty \lambda d\lambda$ and $f_n = \int_{1/n}^n d\lambda$; we have then, for all $a$ in $\mathfrak{N}_\Phi$, $t$ in $\mathbb{R}$, $n$ in $\mathbb{N}$ and $x$ in $M^+$:

$$\omega_{J_\Phi}(\tau_t(a)f_n) = \omega_{J_\Phi}(a|f_n\mu_1^{t/2}) \circ \tau_t$$

$$T \circ \tau_t(x) = \alpha \circ \sigma^L \circ \alpha^{-1}(T(\mu_1^{t/2}x\mu_1^{-t/2}))$$

(ii) there exists a positive non singular operator $\mu_2$ affiliated to $M^\Phi$ and invariant under $\sigma_t^{\Phi_R}$, such that $(D\Phi \circ \sigma_t^{\Phi_R} : D\Phi)_s = \mu_2^{st}$; let us write $\mu_2 = \int_0^\infty \lambda d\lambda$ and $f'_n = \int_{1/n}^n d\lambda$; we have then, for all $b$ in $\mathfrak{N}_\Phi$, $t$ in $\mathbb{R}$ and $n$ in $\mathbb{N}$:

$$\omega_{J_\Phi}(bf'_n) \circ \sigma^{\Phi_R}_t = \omega_{J_\Phi}(\sigma^{\Phi_R}_t(b)|f'_n\mu_2^{t/2})$$

$$T(\sigma^{\Phi_R}_t(\mu_1^{t/2}x\mu_1^{-t/2})) = \alpha \circ \sigma^{L}_t \circ \alpha^{-1}(T(x))$$
Moreover, we have \( \mu_1^{is} = \lambda^{-is} \alpha(h_R^{-is}) \), \( \mu_2^{is} = \mu_1^{is} \alpha(h_L^{is}) \), and \( \alpha(h_L^{is}) \) belong to \( \alpha(N) \cap M^\Phi \). The non-singular operators \( \mu_1, \mu_2 \) and \( \alpha(h_L) \) commute two by two.

**Proof.** By 5.4(ii), we get that \((D\Phi \circ \tau_t : D\Phi)_s = \lambda^{-ist} \alpha(h_R^{-ist})\), as \( \lambda \) is positive non singular, affiliated to the center \( Z(M) \), and \( h_R \) is positive non singular affiliated to the center of \( N \), we get there exists \( \mu_1 \) positive non singular, affiliated to \( M^\Phi \) such that :

\[
\mu_1^{ist} = \lambda^{-ist} \alpha(h_R^{-ist}) = (D\Phi \circ \tau_t : D\Phi)_s
\]

We can then apply 5.5 to \( \tau_t \) and \( \tau_t(a) f_n \) (which belongs to \( \mathfrak{N}_\Phi \)) to get the first formula of (i). On the other hand, we get that \( \alpha \circ \sigma_{-t}^{-1} \circ \tau \circ \tau_t \) is a normal semi-finite operator-valued weight which verify, for all \( x \in M^+ \)

\[
\alpha \circ \sigma_{-t}^{-1} \circ \tau \circ \tau_t(x) = T(\mu_1^{t/2} x \mu_1^{t/2})
\]

from which we get the second formula of (i).

By 5.4(iii), we get that \((D\Phi \circ \sigma \phi_t : D\Phi)_s = \lambda^{-ist} \alpha(h_R^{-ist}) \alpha(h_L^{ist})\); with the same arguments, we get that there exists \( \mu_2 \) positive non singular, affiliated to \( M^\Phi \) such that :

\[
\mu_2^{ist} = \lambda^{-ist} \alpha(h_R^{-ist}) \alpha(h_L^{ist}) = (D\Phi \circ \sigma \phi_t : D\Phi)_s
\]

and we get the first formula of (ii) by applying again 5.5 with \( \sigma \phi_t \).

On the other hand, using 5.2 we get that \( \alpha \circ \gamma_{-t}^L \circ \alpha^{-1} \circ T \circ \sigma \phi_t \) is an operator-valued weight which verify, for all \( x \in M^+ \) :

\[
\nu \circ \alpha \circ \gamma_{-t}^L \circ \alpha^{-1} \circ T \circ \sigma \phi_t(x) = \nu(h_L^{-t/2} \alpha^{-1}(T \sigma \phi_t(x)) h_L^{-t/2})
\]

\[
= \Phi(\alpha(h_L^{-t/2} \sigma \phi_t(x)) \alpha(h_L^{-t/2}))
\]

\[
= \Phi \circ \sigma \phi_t(\alpha(h_L^{-t/2} x \alpha(h_L^{-t/2})]
\]

\[
= \Phi(\mu_2^{t/2} \alpha(h_L^{-t/2} x \alpha(h_L^{-t/2}) \mu_2^{t/2})
\]

from which we get, because \( \mu_2^{t/2} \alpha(h_L^{-t/2}) \) commutes with \( \alpha(N) \) :

\[
\alpha \circ \gamma_{-t}^L \circ \alpha^{-1} \circ T \circ \sigma \phi_t(x) = T(\mu_1^{t/2} \alpha(h_L^{-t/2}) x \alpha(h_L^{-t/2}) \mu_2^{t/2})
\]

or :

\[
T(\sigma \phi_t(x)) = \alpha \circ \gamma_t^L \circ \alpha^{-1}(T(\mu_1^{t/2} x \mu_1^{t/2}))
\]

from which we finish the proof. \( \square \)

5.7. **Proposition.** Let \((N, \alpha, \beta, M, \Gamma)\) be a Hopf-bimodule, equipped with a left-invariant operator-valued weight \( T_L \), and a right-invariant valued weight \( T_R \); let \( \nu \) be a normal semi-finite faithful weight on \( N \), relatively invariant with respect to \( T_L \) and \( T_R \). We shall denote \( \Phi = \nu \circ \alpha^{-1} \circ T_L \) and \( \Psi = \nu \circ \beta^{-1} \circ T_R \) the two lifted normal semi-finite weights on \( M, R \) the co-inverse and \( \tau_t \) the scaling group constructed in 4.4 and 4.6, let \( \lambda \) be the scaling operator of \( \Psi \) with respect to \( \Phi \) (2.6), \( \gamma^L \) and \( \gamma^R \) the two one-parameter automorphism groups of \( N \) introduced in 5.2.
; then, we have:

(i) for all $t \in \mathbb{R}$:

$$\Gamma \circ \tau_t = (\sigma_t^\Phi \beta_*^\alpha \sigma_{-t}^R) \Gamma = (\sigma_t^\Psi \beta_*^\alpha \sigma_{-t}) \Gamma$$

(ii) $h_L = h_R = 1$, and:

$$\nu \circ \gamma^L = \nu \circ \gamma^R = \nu$$

(iii) for all $s, t$ in $\mathbb{R}$:

$$(D\Phi : D\Phi \circ \tau_t)_s = \lambda^{ist}$$

$$(D\Psi : D\Psi \circ \tau_t)_s = \lambda^{ist}$$

(iv) for all $s, t$ in $\mathbb{R}$:

$$(D\Phi \circ \sigma_{-t}^{\Phi R} : D\Phi)_s = \lambda^{ist}$$

Therefore, the modular automorphism groups $\sigma^\Phi$ and $\sigma^{\Phi R}$ commute, the weight $\nu$ is relatively invariant with respect to $\Phi$ and $\Phi \circ R$ and $\lambda$ is the scaling operator of $\Phi \circ R$ with respect to $\Phi$; and we have $\tau_t(\lambda) = \lambda$, $R(\lambda) = \lambda$.

(v) there exists a non singular positive operator $q$ affiliated to $Z(N)$ such that $\lambda = \alpha(q) = \beta(q)$.

**Proof.** As, for all $n \in \mathbb{N}$, we have:

$$\sigma_{-t}^{\Phi R}(\alpha(n)) = R\sigma_t^\Phi R(\alpha(n)) = \alpha(\gamma^L_t(n))$$

and, by definition, $\sigma_t^\Phi(\beta(n)) = \beta(\gamma^L_t(n))$, using a remark made in [2.3] we may consider the automorphism $\sigma_{-t}^\Phi \beta_*^\alpha \sigma_t^{\Phi R}$ on $M \beta_*^\alpha M$; let’s take $a$ and $b$ in $\mathcal{H}_\Phi \cap \mathcal{H}_{TL}$; let’s write $h_L = \int_0^\infty \lambda de^L_\lambda$ and let us write $h_p = \int_{1/p}^p de^L_\lambda$; moreover, let’s use the notations of [5.6]; we have:

$$(id_{\beta_*^\alpha \omega J_\Phi \Lambda_\Phi(\alpha(h_p)f_m)}) (\sigma_{-t}^\Phi \beta_*^\alpha \sigma_t^{\Phi R}) \Gamma \circ \tau_t(f_n a^* a f_n)$$

is equal to:

$$\sigma_{-t}^\Phi \Gamma (id_{\beta_*^\alpha \omega J_\Phi \Lambda_\Phi(\alpha(h_p)f_m) \circ \sigma_t^{\Phi R}) \circ \tau_t(f_n a^* a f_n)$$

which, thanks to [5.6](ii), can be written, because $\alpha(h_p)$ belongs to $\alpha(N)' \cap \mathcal{M}_\Phi$, and therefore $\alpha(h_p)$ belongs to $\mathcal{H}_\Phi$:

$$\sigma_{-t}^\Phi \Gamma (id_{\beta_*^\alpha \omega J_\Phi \Lambda_\Phi(\sigma_{-t}^{\Phi R}(\alpha(h_p)))f_m \mu_{-t/2}^z}) \circ \tau_t(f_n a^* a f_n)$$

or:

$$R \sigma_{-t}^{\Phi R} \Gamma (id_{\beta_*^\alpha \omega J_\Phi \Lambda_\Phi(\sigma_{-t}^{\Phi R}(\alpha(h_p)))f_m \mu_{-t/2}^z}) \circ \tau_t(f_n a^* a f_n)$$

By [5.6] and [2.2.2] we know that $a f_n \mu_{1/2}^t$ belongs to $\mathcal{H}_\Phi \cap \mathcal{H}_{TL}$; using now [5.6](i), we get that $\tau_t(a f_n) = \tau_t(a) f_n$ belongs to $\mathcal{H}_\Phi \cap \mathcal{H}_{TL}$. 
On the other hand, by \textbf{5.6} and \textbf{2.2.2} we know that $b\alpha(h_p) f'_{m}$ belongs to $\mathfrak{M}_\Phi \cap \mathfrak{M}_{T_L}$; using now \textbf{5.6}(ii), we get that:

$$\sigma_{-t}^{\Phi R}(b\alpha(h_p) f'_{m} \mu_{-t/2}^1) = \sigma_{-t}^{\Phi R}(b) f'_{m} \mu_{-t/2}^1 \alpha(h_p) \alpha(h_{-t/2}^1)$$

belongs to $\mathfrak{M}_\Phi \cap \mathfrak{M}_{T_L}$, and so, using again \textbf{2.2.2}

$$\sigma_{-t}^{\Phi R}(b) f'_{m} \mu_{-t/2}^1 \alpha(h_p) = \sigma_{-t}^{\Phi R}(b) f'_{m} \mu_{-t/2}^1 \alpha(h_p) \alpha(h_{-t/2}^1)$$

belongs also to $\mathfrak{M}_\Phi \cap \mathfrak{M}_{T_L}$; therefore, we can use \textbf{4.7} and we get it is equal to:

$$R\sigma_{t}^{\Phi R}(id_{N}^\ast \omega_{J_\Phi A_\Phi(\tau_1(a)f_n)})\Gamma(\mu_2 f'_{m} \alpha(h_p) \sigma_{-t}^{\Phi R}(b^*b) \alpha(h_p) f'_{m} \mu_{-t/2}^1)$$

which can be written, thanks to \textbf{5.6}(i):

$$R\sigma_{t}^{\Phi R}(id_{N}^\ast \omega_{J_\Phi A_\Phi(a^\ast f_n)}^{1/2} \circ \tau_{-t}) \Gamma(\mu_2 f'_{m} \alpha(h_p) \sigma_{-t}^{\Phi R}(b^*b) \alpha(h_p) f'_{m} \mu_{-t/2}^1)$$

or, $\alpha(h_p)$, as well as $\mu_{-t/2} f'_{m}$, being invariant under $\sigma_{t}^{\Phi R}$:

$$R(id_{N}^\ast \omega_{J_\Phi A_\Phi(a^\ast f_n)}^{1/2}) \Gamma(\sigma_{-t}^{\Phi R}(b^*b) \alpha(h_p) f'_{m} \mu_{-t/2}^1)$$

and using \textbf{4.6} and again \textbf{4.7} we get it is equal to:

$$R[(id_{N}^\ast \omega_{J_\Phi A_\Phi(a^\ast f_n)}^{1/2}) \Gamma(\mu_2 f'_{m} \alpha(h_p) b^*b \alpha(h_p) f'_{m} \mu_{-t/2}^1)] = (id_{N}^\ast \omega_{J_\Phi A_\Phi(ba(h_p)f_n)}) \Gamma(\mu_1 f_1 \alpha^a a f_n \mu_{1/2}^{1/2})$$

Finally, we have proved that, for all $a, b$ in $\mathfrak{M}_\Phi \cap \mathfrak{M}_{T_L}$, $m, n, p$ in $\mathbb{N}$, we have:

$$(id_{N}^\ast \omega_{J_\Phi A_\Phi(ba(h_p)f_n)}) \Gamma(\sigma_{-t}^{\Phi R}(b^*b) \alpha(h_p) f'_{m} \mu_{-t/2}^1) = (id_{N}^\ast \omega_{J_\Phi A_\Phi(ba(h_p)f_n)}) \Gamma(\mu_1 f_1 \alpha^a a f_n \mu_{1/2}^{1/2})$$

But, for all $x, y \in M$, we have:

$$\omega_{J_\Phi A_\Phi(ba(h_p)f_n)}(x) = \omega_{J_\Phi A_\Phi(b)}(\alpha(h_p) f'_{m} x f'_{m} \alpha(h_p))$$

$$\omega_{J_\Phi A_\Phi(ba(h_p)f_n)}(y) = \omega_{J_\Phi A_\Phi(b)}(\alpha(h_p) f'_{m} \mu_{-t/2}^1 y \mu_{-t/2}^1 f'_{m} \alpha(h_p))$$

and, therefore, we get that:

$$(id_{N}^\ast \omega_{J_\Phi A_\Phi(b)}) [(1 \otimes \alpha \alpha(h_p) f'_{m}) \Gamma(\sigma_{-t}^{\Phi R}(b^*b) \alpha(h_p) f'_{m} \mu_{-t/2}^1)]$$

is equal to:

$$(id_{N}^\ast \omega_{J_\Phi A_\Phi(b)}) [(1 \otimes \alpha \alpha(h_p) f'_{m} \mu_{-t/2}^1) \Gamma(\mu_1 \alpha^a a f_n \mu_{1/2}^{1/2}) f'_{m} \alpha(h_p))$$
and, by density, we get that:

\[(1_{\beta} \otimes_{\alpha} f_m')(\sigma_{-t}^{\Phi} \beta \ast_{\alpha} \sigma_t^{\Phi \circ R})_{N} \circ \tau_l(f_n a^* a f_n)(1_{\beta} \otimes_{\alpha} f_m)\]

is equal to:

\[(1_{\beta} \otimes_{\alpha} f_m')(\sigma_{-t}^{\Phi} \beta \ast_{\alpha} \sigma_t^{\Phi \circ R})_{N} \circ \tau_l(f_n a^* a f_n)(1_{\beta} \otimes_{\alpha} f_m')\]

and, after making \(p\) going to \(\infty\), we obtain that:

\[(1_{\beta} \otimes_{\alpha} f_m')(\sigma_{-t}^{\Phi} \beta \ast_{\alpha} \sigma_t^{\Phi \circ R})_{N} \circ \tau_l(f_n a^* a f_n)(1_{\beta} \otimes_{\alpha} f_m')\]

is equal to \((*)\):

\[(1_{\beta} \otimes_{\alpha} f_m')(\sigma_{-t}^{\Phi} \beta \ast_{\alpha} \sigma_t^{\Phi \circ R})_{N} \circ \tau_l(f_n a^* a f_n)(1_{\beta} \otimes_{\alpha} f_m')\]

Let’s now take a file \(a_i\) in \(\mathfrak{N}_\Phi \cap \mathfrak{N}_{T_L}\) weakly converging to 1; we get that \((1_{\beta} \otimes_{\alpha} f_m')(\sigma_{-t}^{\Phi} \beta \ast_{\alpha} \sigma_t^{\Phi \circ R})_{N} \circ \tau_l(f_n a^* a f_n)(1_{\beta} \otimes_{\alpha} f_m')\) is equal to:

\[(1_{\beta} \otimes_{\alpha} f_m')(\sigma_{-t}^{\Phi} \beta \ast_{\alpha} \sigma_t^{\Phi \circ R})_{N} \circ \tau_l(f_n a^* a f_n)(1_{\beta} \otimes_{\alpha} f_m')(\sigma_{-t}^{\Phi} \beta \ast_{\alpha} \sigma_t^{\Phi \circ R})_{N} \circ \tau_l(f_n a^* a f_n)(1_{\beta} \otimes_{\alpha} f_m')(\sigma_{-t}^{\Phi} \beta \ast_{\alpha} \sigma_t^{\Phi \circ R})_{N} \circ \tau_l(f_n a^* a f_n)(1_{\beta} \otimes_{\alpha} f_m')\]

When \(n\) goes to \(\infty\), then \(f_n\) is increasing to 1, the first is increasing to \(1_{\beta} \otimes_{\alpha} f_m\), and the second is increasing to:

\[(1_{\beta} \otimes_{\alpha} f_m')(\sigma_{-t}^{\Phi} \beta \ast_{\alpha} \sigma_t^{\Phi \circ R})_{N} \circ \tau_l(f_n a^* a f_n)(1_{\beta} \otimes_{\alpha} f_m')(\sigma_{-t}^{\Phi} \beta \ast_{\alpha} \sigma_t^{\Phi \circ R})_{N} \circ \tau_l(f_n a^* a f_n)(1_{\beta} \otimes_{\alpha} f_m')(\sigma_{-t}^{\Phi} \beta \ast_{\alpha} \sigma_t^{\Phi \circ R})_{N} \circ \tau_l(f_n a^* a f_n)(1_{\beta} \otimes_{\alpha} f_m')\]

which is therefore bounded.

Taking now \(m\) going to \(\infty\), we get that the two non-singular operators \(\Gamma(\mu_1)\) and \(1_{\beta} \otimes_{\alpha} \mu_2\) are equal. Using \(5.4\) we get then that \(\mu_1\) is equal to \(\mu_2\) (and is affiliated to \(\beta(N)\)), from which we get, using \(5.6\) that \(h_L = 1\). Applying all these calculations to \((N, \alpha, \beta, M, \Gamma, RT_{T_R} R, T_R, \nu)\), we get that \(h_R = 1\), which is (ii).

Let’s come back to the equality \((*)\) above; we obtain that:

\[(1_{\beta} \otimes_{\alpha} f_m')(\sigma_{-t}^{\Phi} \beta \ast_{\alpha} \sigma_t^{\Phi \circ R})_{N} \circ \tau_l(f_n a^* a f_n)(1_{\beta} \otimes_{\alpha} f_m')\]

is equal to:

\[(1_{\beta} \otimes_{\alpha} f_m')(\sigma_{-t}^{\Phi} \beta \ast_{\alpha} \sigma_t^{\Phi \circ R})_{N} \circ \tau_l(f_n a^* a f_n)(1_{\beta} \otimes_{\alpha} f_m')\]

So, when \(n\) and \(m\) go to \(\infty\), we obtain:

\[(\sigma_{-t}^{\Phi} \beta \ast_{\alpha} \sigma_t^{\Phi \circ R})_{N} \circ \tau_l(a^* a) = \Gamma(a^* a)\]

which, by density, gives the first formula of (i), the second being given then by \(4.11\).

From (ii) and \(5.4\) (i) and (ii), we get (iii).

From (ii) and \(5.4\) (iii), we get that \((D \Phi \circ \sigma^{\Phi \circ R} : D \Phi) = \lambda^{\text{st}}\); therefore, as \(\lambda\) is affiliated to \(Z(M)\), we get the commutation of the modular groups \(\sigma^\Phi\) and \(\sigma^{\Phi \circ R}\). Using \(2.6\), we get that there exists \(\lambda_R\) positive
non singular affiliated to $Z(M)$ and $\delta_R$ positive non singular affiliated to $M$ such that $(D\Phi \circ R : D\Phi)_t = \lambda^{ist/2}_R \delta_R^t$, and the properties of $R$ allows us to write that $R(\lambda_R) = \lambda_R$. But, on the other hand, the formula $(D\Phi \circ \sigma^\Phi \circ R : D\Phi)_s = \lambda^{ist}_R$ (2.6), gives that $\lambda_R = \lambda$ and, therefore, we get that $R(\lambda) = \lambda$. The formula $\tau_t(\lambda) = \lambda$ comes from (iii), which finishes the proof of (iv).

By (i), we have $\lambda = \mu_1 = \mu_2$, and, as we had proved that $\mu_1$ is affiliated to $\beta(N)$, we get that $\lambda$ is affiliated to $\beta(N)$; as $R(\lambda) = \lambda$ by (iv), we get (v).

□

6. Measured Quantum Groupoids

In this chapter, we give a new definition (6.1) of a measured quantum groupoid, and, using [L2], we get some other results, namely on the modulus (6.3), the antipod (6.4), and the manageability of the pseudo-multiplicative unitary (6.5), all results borrowed from Lesieur.

6.1. Definition. An octuplet $(N, M, \alpha, \beta, \Gamma, T_L, T_R, \nu)$ will be called a measured quantum groupoid if:

(i) $(N, M, \alpha, \beta, \Gamma)$ is a Hopf-bimodule

(ii) $T_L$ is a normal semi-finite faithful operator-valued weight from $M$ to $\alpha(N)$, which is left-invariant, i.e. such that, for any $x \in M^+_L$ :

$$(id \beta^* \alpha_T L) \Gamma(x) = T_L(x) \beta \otimes_\alpha 1_N$$

(iii) $T_R$ is a normal semi-finite faithful operator-valued weight from $M$ to $\beta(N)$, which is right-invariant, i.e. such that, for any $x \in M^+_R$ :

$$(T_R \beta \otimes_\alpha id)_N \Gamma(x) = 1 \beta \otimes_\alpha T_R(x)$$

(iv) $\nu$ is a normal semi-finite faithful weight on $N$, which is relatively invariant with respect to $T_L$ and $T_R$, i.e. such that the modular automorphism groups $\sigma^\Phi$ and $\sigma^\Psi$ commute, where $\Phi = \nu \circ \alpha^{-1} \circ T_L$ and $\Psi = \nu \circ \beta^{-1} \circ T_R$.

Let $R$ be the co-inverse constructed in 4.7 thanks to 5.7, we get that $(N, M, \alpha, \beta, \Gamma, T_L, T_R, R, \nu)$ is a measured quantum groupoid (as well as $(N, M, \alpha, \beta, \Gamma, T_R R R, T_R, \nu)$). Moreover, $R$ (resp. $\tau_t$) remains the co-inverse (resp. the scaling group) of this measured quantum groupoid.

6.2. Remark. Let $(N, M, \alpha, \beta, \Gamma, T_L, T_R, \nu)$ be a measured quantum groupoid in the sense of 6.1 and let us denote $R$ (resp. $\tau_t$) the co-inverse (resp. the scaling group) constructed in 4.7 (resp. 4.6). Then $(N, M, \alpha, \beta, \Gamma, T_L, R, \tau, \nu)$ is a measured quantum groupoid in the sense of [L2], 4.1.

Contrarily if $(N, M, \alpha, \beta, \Gamma, T, R, \tau, \nu)$ is a measured quantum groupoid in the sense of [L2], 4.1, then $(N, M, \alpha, \beta, \Gamma, T, R \tau R, \nu)$ is a measured quantum groupoid in the sense of 6.1.
6.3. **Theorem.** Let \((N,M,\alpha,\beta,\Gamma,T_L,T_R,\nu)\) be a measured quantum groupoid; let us denote \(\Phi = \nu \circ \alpha^{-1} \circ T_L\), and let \(R\) be the co-inverse and \(\tau\) the scaling group constructed in \(4.7\) and \(4.6\). Let \(\delta_R\) be the modulus of \(\Phi \circ R\) with respect to \(\Phi\). Then, we have:

(i) \(R(\delta_R) = \delta_R^{-1}, \tau_t(\delta_R) = \delta_R\), for all \(t \in \mathbb{R}\).

(ii) we can define a one-parameter group of unitaries \(\delta_R^{it} \otimes_{\alpha} \delta_R^{it}_{\nu} N\) which acts naturally on elementary tensor products, which verifies, for all \(t \in \mathbb{R}\):

\[
\Gamma(\delta_R^{it}) = \delta_R^{it} \otimes_{\alpha} \delta_R^{it}_{\nu}
\]

**Proof.** Thanks to \(6.2\) we can rely on Lesieur's work \([L2]\); (i) is \([L2]\), 5.6; (ii) is \([L2]\), 5.20.

\(\square\)

6.4. **Proposition.** Let \((N,M,\alpha,\beta,\Gamma,T_L,T_R,\nu)\) be a measured quantum groupoid; let us denote \(\Phi = \nu \circ \alpha^{-1} \circ T_L\), and let \(R\) be the co-inverse and \(\tau\) the scaling group constructed in \(4.7\) and \(4.6\). Then:

(i) the left ideal \(N_{TL} \cap N_{\Phi} \cap N_{RTL} \cap N_{\Phi,\nu}\) is dense in \(M\), and the subspace \(\Lambda_{\Phi}(N_{TL} \cap N_{\Phi} \cap N_{RTL} \cap N_{\Phi,\nu})\) is dense in \(H_{\Phi}\).

(ii) there exists a dense linear subspace \(E \subset N_{\Phi}\) such that \(\Lambda_{\Phi}(E)\) is dense in \(H_{\Phi}\) and \(J_{\nu} \Lambda_{\Phi}(E) \subset D(\alpha H_{\Phi,\nu}) \cap D((H_{\Phi})_{\beta,\nu})\).

**Proof.** Part (i) is given by \([L2]\) 6.5; part (ii) by \([L2]\) 6.7.

\(\square\)

6.5. **Theorem.** Let \((N,M,\alpha,\beta,\Gamma,T_L,T_R,\nu)\) be a measured quantum groupoid; let us denote \(\Phi = \nu \circ \alpha^{-1} \circ T_L\), and let \(R\) be the co-inverse and \(\tau\) the scaling group constructed in \(4.7\) and \(4.6\). Then:

(i) there exists a one-parameter group of unitaries \(P^{it}\) such that, for all \(t \in \mathbb{R}\) and \(x \in N_{\Phi}\):

\[
P^{it} \Lambda_{\Phi}(x) = \lambda^{it/2} \Lambda_{\Phi}(\tau_t(x))
\]

(ii) for any \(y \in M\), we get:

\[
\tau_t(y) = P^{it} y P^{-it}
\]

(iii) we have:

\[
W(P^{it} \otimes_{\alpha} P^{it}) = (P^{it} \otimes_{\beta} P^{it}) W
\]

(iv) for all \(v \in D(P^{-1/2}), w \in D(P^{1/2}), p, q \in D(\alpha H_{\Phi,\nu}) \cap D((H_{\Phi})_{\beta,\nu})\), we have:

\[
(W^*(v_{\alpha} \otimes_{\beta} q) w_{\beta} \otimes_{\alpha} p) = (W(P^{-1/2} v_{\beta} \otimes_{\alpha} J_{\Phi} p) P^{1/2} w_{\alpha} \otimes_{\beta} J_{\Phi} q)
\]

The pseudo-multiplicative unitary will be said to be "manageable", with "managing operator" \(P\).

(v) \(W\) is weakly regular in the sense of \([E2]\), 4.1.

**Proof.** The proof is given in \([L2]\), 7.3. and 7.5.

\(\square\)
6.6. **Theorem.** Let \((N, M, \alpha, \beta, \Gamma, T_L, T_R, \nu)\) be a measured quantum groupoid; let us denote \(\Phi = \nu \circ \alpha^{-1} \circ T_L\), and let \(R\) be the co-inverse and \(\tau\) the scaling group constructed in 4.7 and 4.8. Let \(T'\) be another left-invariant operator-valued weight; let us write \(\Phi' = \nu \circ \alpha^{-1} \circ T'\) and let us suppose that:

(i) \((N, M, \alpha, \beta, \Gamma, T', RT'R, \nu)\) is a measured quantum groupoid;
(ii) \(\tau_t\) is the scaling group of this new quantum groupoid;
(iii) for all \(t \in \mathbb{R}\), the automorphism group \(\gamma^L_t\) of \(N\) defined by \(\sigma^\Phi_t(\beta(n)) = \beta(\gamma^L_t(n))\) commutes with \(\gamma^L_t\);

Then, there exists a strictly positive operator \(h\) affiliated to \(Z(N)\) such that \((DT': DT)_t = \beta(h^it)\). Moreover, we have then \(\gamma'^L = \gamma^L\).

**Proof.** This is [L2] 5.21. Then, we get:

\[
\beta(\gamma'^L_t(n)) = \sigma^\Phi_t(\beta(n)) = \beta(h^{-it})\beta(\gamma^L_t(n))\beta(h^{it}) = \beta(\gamma^L_t(n))
\]

\(\square\)

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