Comments on initial conditions for the Abraham-Lorentz(-Dirac) equation

Ofek Birnholtz\textsuperscript{1} \textsuperscript{*}

\textsuperscript{1}Racah Institute of Physics, Hebrew University, Jerusalem 91904, Israel.

(Dated: October 23, 2014)

An accelerating electric charge coupled to its own electromagnetic (EM) field both emits radiation and experiences the radiation’s reaction as a (self-)force. Considering the system from an Effective Field Theory perspective, and using the physical initial conditions of no incoming radiation can help resolve many of the problems associated with the often considered “notorious” Abraham-Lorentz / Abraham-Lorentz-Dirac equations.

I. BACKGROUND & MOTIVATION

Viewed as a single-particle equation, the century old equation of Abraham-Lorentz (AL) \cite{1,3} for the self-force on an accelerating electric charge has been notorious for involving many physically and mathematically unwanted behaviours \cite{1,3}. The AL equation,

$$m\ddot{x} = \frac{2}{3} \frac{q^2}{c^3} \dot{x} + F_{\text{ext}}(x, \dot{x}),$$

as well as Dirac’s \cite{4} relativistic generalization (the Abraham-Lorentz-Dirac (ALD) equation\textsuperscript{1}), involve the time derivative of the acceleration (“jerk”), and are thus 3\textsuperscript{rd} order ODE, unusual in mechanics. The most troubling of the problems associated with them include

- the requirement of a 3\textsuperscript{rd} initial condition
- causality violations due to pre-acceleration\textsuperscript{2}
- runaway/unstable solutions\textsuperscript{3}
- disagreement with the Larmor formula for energy output of a constantly accelerating particle \cite{27}.

Some considerations of the origin of the ALD force in the microscopic structure of the so-called point-particle have even lead to problems such as the “3\textsuperscript{rd} problem” \cite{14,17}, negative mass \cite{18,19}, and even suggestions that the fundamentals of EM of point particles be altogether replaced \cite{10,22}. Most attempts for solving these problems have suggested modifying the equation itself \cite{23,35}, predominantly by reducing the equation for the particle’s trajectory to a more familiar 2\textsuperscript{nd} order ODE, although this goes against the breadth of rigorous derivations for the AL/ALD equations \cite{1,7,36,43}.

Retaining the original equation, Dirac \cite{4} suggested introducing an un-intuitive termination condition rather than the extra initial condition, requiring both the existence and value of the terminal acceleration\textsuperscript{4}. However, even he realized this was a very contrived way of posing a physical question (backwards)\textsuperscript{5} and an invitation for causality violation. This condition also limits the scope of problems which can be solved to those where the acceleration approaches a finite constant at some (infinite or known finite) future time, and thus for many systems cannot be used. For example, the classical system of an EM (Coulombic) two-body system loses energy to radiation while shrinking its orbit, causing the orbital frequency and the accelerations to grow boundlessly, rather than approach a finite limit. Analogous systems have been treated successfully for general relativistic (GR) gravitational systems \cite{44,45}, see also Sec. IV, and Dirac’s condition falls short of helping.

Behind an opposite approach \cite{10,43} lies the realization that the charge is not isolated and that the radiation does not arise from the particle alone; the complete system of course consists of both the charge and the EM field everywhere. Realizing the missing information lies not in the particle’s future but in its past and in its surroundings, an alternative suggestion for initial data has been to supplement the particle’s initial position and velocity by full Cauchy data for the entire field, everywhere. This poses the problem at the opposite extreme: while in this approach no data is missing or guessed a-posteriori, too much information is required. This is because the initial conditions for the particle and for the field are not independent, and are in fact related via the field singularity at the particle’s position - with the dependence itself involving also velocity and acceleration (for example via the Liénard-Wiechert potential \cite{5}) - making the problem over-constrained.

\textsuperscript{1} Ofek.Birnholtz@mail.huji.ac.il
\textsuperscript{2} Relativistically, $x$ becomes $\dot{x}$, time derivatives are taken w.r.t proper time, and the jerk $\ddot{x}$ is to be replaced by $\dddot{x} - \dot{x} \ddot{x} \ddot{\nu}/\ddot{\nu}$. Note all equations are given in cgs units.
\textsuperscript{3} For example Dirac’s own remarks, “it is possible for a signal to be transmitted faster than light through the interior of an electron. The finite size of the electron now reappears in a new sense, the interior of the electron being a region of failure, not of the field equations of electromagnetic theory, but of some of the elementary properties of space-time” \cite{4}, see also \cite{13}.
\textsuperscript{4} e.g. Medina’s “the Lorentz-Abraham-Dirac formula for the radiation reaction of a point charge predicts unphysical motions that run away or violate causality” \cite{32}.
\textsuperscript{5} In his words, “We now have a striking departure from the usual ideas of mechanics. We must obtain solutions of our equations of motion for which the initial position and velocity of the electron are prescribes, together with its final acceleration, instead of solutions with all the initial conditions prescribed” \cite{4}.
II. EFT WITH NO INCOMING RADIATION

From the perspective of Effective Field Theory (EFT), we wish to separate the charge degrees of freedom from those of the fields. First we use the linearity of Maxwell’s field equations to separate the charge and the field itself generates from the background field. The remaining system of the charge coupled to its own field is described with a simple Action formulation of a source-field radiating system; we thus employ the standard methods of dimensional reduction, field doubling, and then integrating out the field degrees of freedom (“Balayage” - for a full explanation of the method see [37], for concise derivations of the AL/ALD radiation-reaction force see [38][39], and for EFT background [47][50]). We remain with an effective action describing the charge and the radiation-reaction on it, from which the Euler-Lagrange equation finally gives the equations of AL/ALD, with its high order time derivatives.

The key point lies in integrating out the radiation field: to do that, we solve the 2nd order wave equation away from the charge, finding two solutions which describe incoming and outgoing radiation. Upon plugging the solution back into the action to receive an effective action for the particle, we choose the solution of outgoing radiation, dropping the solution of incoming radiation, which amounts to the physical constraint of no incoming radiation.

This constraint of “no incoming radiation” is precisely the desired supplementary initial condition. It complements the initial conditions on particle’s mechanical degrees of freedom (of which there are only position and velocity, as usual) with an initial condition on the radiation field at past infinity (often denoted $\mathcal{I}^−$ in GR terminology [51]). It guarantees the particle cannot “drain” energy or angular momentum in from infinitely past and far away, but rather only radiates them outwards towards the future, which is physically desirable.

III. BENEFITS

A. Comparisons with previous approaches

Comparing with the requirement for full Cauchy data, this condition is more lenient and does away with over-constraints, as it imposes a condition only on the radiation field on $\mathcal{I}^−$ rather than on the complete background+radiation field everywhere (for more on such separations, see [52]). Comparing with Dirac’s termination condition, we first see that there is a class of problems which we are now able to pose and solve, but could not using his condition. These include the aforementioned Coulombic 2-body inspiral and other systems which can not approach a finite terminal acceleration, but are governed by radiating away (without absorbing) energy and angular momentum. Regarding the problems which did adhere to Dirac’s condition of terminal acceleration, from our condition we learn that they may not draw energy from radiation at any time in their history (not only terminally so), and thus that any non-zero terminal acceleration must be the consequence of background fields. In particular, systems where the terminal acceleration is non-zero must have a background field with unbounded spatial support - a physically unreasonable set-up.

A common practice regarding high derivatives in EFT is to treat them as small perturbations, first solving the low-order differential equations and only then adding the perturbative effects of the high derivatives [53][55]. Similarly many authors have suggested replacing the ALD equation itself with 2nd order ODE’s, by effectively substituting for the high derivative term only the leading order [23][24] or a series expansion [28]. These derivative expansions serve well to get rid of runaway solutions (by definition, because they are perturative), but have no regard for radiation absorption vs emission. Thus on their own, they might still produce erroneous solutions; under the restriction of no incoming radiation, they may be a good practical method.

We thus see the (sometimes implicit) condition of no incoming radiation both expands the problem set we may look at, adds physical insight, and corrects erroneous behaviours.

We also note that the Balayage process of tracing the field outwards to infinity and then back to the particle, treated as a point from outside, avoids the questions of particle internals and finite-size effects, such as those of bare / negative mass.

B. Further Implications

Examining initial conditions can also help resolve the paradox regarding the energy emitted by a particle experiencing a constant non-zero acceleration $\mathbf{a}$. In this case the work done by the AL force is (using eq. (I.1))

$$P = \mathbf{v} \cdot \frac{2}{3} \frac{q^2}{c^3} \mathbf{a} = 0,$$

while the Larmor formula predicts an average energy output of

$$P = \frac{2}{3} \frac{q^2}{c^3} \mathbf{a}^2 > 0,$$

in apparent contradiction. However, as the latter gives an only average over time, and as energy absorbed or radiated by the particle should only be defined asymptotically, a true comparison must refer to the time integrals of both formulae over the entire trajectory. These two integrals are trivially related by integration by parts, and thus their difference is seen to reside entirely in the boundary terms, i.e. in the initial and terminal accelerations. For acceleration over a finite time, this entirely suffices, as the boundary terms vanish. This argument breaks for eternal acceleration - but in this situation, according to the Weak Equivalence Principle [56] between
acceleration and curvature, spacetime itself is no longer asymptotically flat (but rather Rindler space [57]), and thus the "no incoming radiation" condition must be applied to those asymptotics, altering the definition of the radiation field and of the energy as viewed from infinity.

Here again the surprise fades upon consideration of the incoming radiation from the infinite past. Fig. (1) shows the well-known fact that a charge moving with constant velocity in vacuum neither gives away or absorbs energy (the energy of the EM field rotates around the particle and follows its trajectory). Fig. (2) shows that the situation is different in the vicinity of an external electric field, where the motion of the particle creates an energy flow within the domain of the field, drawing energy from the domain boundaries inwards. The particle’s fields may also cause rearrangements of the charges sustaining the external field, who in turn affect the approaching particle. This begins even before the particle itself penetrates the field’s domain, because even before the particle penetrates, its own EM field already has; causality is preserved for the particle and the fields.

IV. CONCLUSIONS: EM, QM & GRAVITY

The problem of moving bodies emitting radiation, losing energy and angular momentum to it, and in turn experiencing a reaction force has also been studied extensively in the context of GR. At the limit of slow velocities, the energy loss and radiation reaction are described by the Quadrupole formula [58], which can be thought of as an analogue of the AL formula. While AL describes dipole radiation (and thus involves 3 time derivatives), the Quadrupole formula involves 5 time derivatives, and is non-linear in the coordinates.

Introducing the physical properties of an electron in eq. (1.1), we find the typical timescale to be of order \( \sim 10^{-24} \) s, which is the time it takes light to cross the classical electron’s radius \( \sim 10^{-15} \) m. The Quantum Mechanical (QM) nature of the electron becomes relevant at much longer space/time scales, i.e the Compton wavelength \( \sim 10^{-12} \) m. We thus expect classical EM radiation-reaction effects to be masked by QM behaviours (for detailed discussions see [35, 59]). The situation is reversed for gravitational interactions, where the gravitational lengthscale is always at and above the Schwarzschild radius (at least \( \sim 10^3 \) m for a compact object of stellar mass), many orders of magnitude above the scale for quantum gravity, expected at the Planck scale \( \sim 10^{-35} \) m. Hence the 2-body problem of an inspiraling orbit, for example, is a classical problem in GR and features prominently in searches for gravitational waves, while in EM it must be treated quantum mechanically, and has in fact historically led to the very development of QM [60]. Discussions of classical radiation reaction are therefore much more natural to gravitational contexts than to EM.

However, aside from the quantum barrier relevant to one and not the other, the two problems are quite similar, and there is much mutual insight to be gained. In particular, examinations of the asymptotics of spacetime, and the initial conditions at \( \mathcal{I}^- \), have shed light on a century-old problem in EM. Also, we hold that
much can be learned about gravitational systems from studying the simpler corresponding problems in classical EM, where the EM case has the added benefits of linearity and less time derivatives (the GR Quadrupole formula has 5 time derivatives rather than ALD’s 3). Disregarding QM effects in such studies is legitimate when conclusions are desired eventually for gravitational systems, where QM is less relevant.

ACKNOWLEDGMENTS

The author thanks B. Kol, A. Yaron, S. Hadar, B. Rezende and P. Beniamini for helpful comments and discussions. This research was supported by an ERC Advanced grant to Tsvi Piran and by the Israel Science Foundation grant no. 812/11. It was part of the Einstein Research Project "Gravitation and High Energy Physics", which is funded by the Einstein Foundation Berlin.
