Searching for electrovac solutions
to Einstein–Maxwell equations
with the help of computer algebra system
\textbf{GRG}_{EC}

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\textbf{Abstract}

An example of application of the specialized computer algebra system \textbf{GRG}_{EC} to the searching for solutions to the source-free Maxwell and Einstein–Maxwell equations is demonstrated. The solution involving five arbitrary functions of two variables is presented in explicit form (up to quadratures).

1 \textbf{Introduction}

The specialized computer algebra system \textbf{GRG}_{EC} \cite{1} is intended for applications to the classical theory of gravity as well as adjacent problems related to the classical theory of field and geometry. In particular, \textbf{GRG}_{EC} is currently ‘aware’ of the majority of main characteristics of the geometry of a curved space utilized in Einstein’ theory. These are, for example, the bases in foliations of exterior forms connected with metric, the connection (including its Newman-Penrose representation), the curvature, its irreducible constituents and algebraic invariants, the general equations connecting these objects (Cartan’ structural equations, Bianchi’ equations, various algebraic identities), the field equations of the gravity theory (Einstein equations for vacuum and various matter content). This list can be continued with the basic elements of the Rainich theory, the theory of Lanczos potential, the methods of the symmetry description (Killing equations). \textbf{GRG}_{EC} system operates with major characteristics of such classical fields as electromagnetic field, massless spinor field (Weyl field), massive spinor fields (Dirac fields) including the both cases of the inclusion

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of interaction with electromagnetic field ("charged" Dirac field) and the absence of electromagnetic interaction, massless scalar field, conformally invariant scalar field, massive scalar field, massive vector field (Proca field), pressure-free dust matter, massive and null, etc.¹

Concerning the above characteristic of GRGEC capabilities, it has to be noted that there is, of course, a number of programming packages of symbolic manipulations with a similar application field (an excellent survey of the latter topic is given in Ref. [2], see also [3]). At the same time a general overview of the current state of this field of computer applications led a majority of authors to the conclusion that, in spite of noteworthy achievements of the applied computer algebra, it would be still premature now to single out any of the existing programming systems of the class implied. None of them can be assessed as an universal tool which reveals unconditional advantages over all the potential competitors and best suits for majority of applications. Accordingly, in practice, answering what is the best system of symbolic calculations for application to the gravity theory, one has to preliminary make definite what a class of problems to be tackled is implied. This observation allows one to suppose that GRGEC can find an own position in the row of computer-based tools utilized for theoretical investigations in the field of the gravity theory.

It is worth mentioning that one may find a lot of examples giving evidence that the practical usefulness of programming instruments intended to 'non-local' applications is a fairly subtle matter affected by a plenty of factors, often rather vague and sometimes fairly unexpected. In particular, it may be stated that any partial characteristic (similar to the above list the physical-mathematical material implemented in GRGEC) or any verbal description is not able to ultimately establish a proper measure of the actual advantages — as well as drawbacks — of a software given.

Nevertheless, there is an evident straightforward mean to gain some insight into the heart of the problem. Specifically, an overview of a proper set of manifest examples of the practical use of a programming system can serve a probe exhibiting its real characteristics. Concerning GRGEC system, the present work realizes a step just in that direction.

In order to demonstrate in acting the basic features of GRGEC system (and to exhibit some of its capabilities) we would like to learn the solving of a concrete problem, namely, the integrating of the system of Maxwell and Einstein–Maxwell equations. The result we shall derive is of a certain independent interest. The solution of the electrovac field equations obtained below with the help of — or, if one prefer, in collaboration with — GRGEC system involves five arbitrary functions of two variables. Though the calculation we picked does not pretend to be related, with regard to its complexity, to the category of record ones, it nevertheless gives, in our opinion, a clear evidence of a usefulness of the programming tool presented for the handling of such sort problems.

It is worth noting that our choice of the subject of consideration was severely conditioned by fairly simple 'pragmatic' reasons. On the one hand it is evident that consideration of a toy problem would obviously decrease a potential interest of the demonstration. In any case, it tells a reader nothing definite in connection with the real efficiency of the

¹This list should not regarded as ultimate one.
computer tool applied. On the other hand, in the opposite extreme case of complicated problems lying on the edge of the practical capabilities of the method, there would appear an evident technical obstacle strongly restricting the scope of discussion. Indeed, solutions of complicated problems are usually too bulky to be exhibited in sufficient details within frames of an article of any reasonable length\(^2\). In the best case the only initial posing of the problem and the final result with minimal discussion can be presented, the explanations of the essence and, all the more, the features of the course of the solving procedure being inevitably omitted. (As an indirect evidence of such a state of things we might refer to a lot of publications of investigations resorting to assistance of the computer algebra where a remark ”... with the help of computer algebra we found that...”, or the like, is the only ‘discussion’ of the relevant programming issues.)

In our case, as usually, a contradiction of the choice among two opposite extremes is to be settled by means of a compromise. Specifically, although the problem we shall discuss below is in principle tractable by means of a ‘manual’ calculation (being rather error prone though), definitely, nobody would qualify it at whole as a trivial one. It seems thus to be able to provide a proper material for the displaying the fashion of the work with the computer tool considered.

Reverting to the general characterizing of GRG\(_{EC}\) system, it has to be mentioned that one of the major goals pursued during its development was an attempt to release a user from the duty to ‘develop a program’ — at least, in the meaning usually associated with the latter pursuit. Rather, dealing with GRG\(_{EC}\) system, one has to merely describe, employing mostly ‘habitual’ words arranged in a sequence of natural ‘phrases’, and operating with more or less standard mathematical notations, what initial data are given and indicate what a result has to be generated. To that end, GRG\(_{EC}\) maintains the input language which is, probably, as close as possible to the one used for the representation of the relevant notions and the relationships taking place in the application field itself. Of course, comparatively sophisticated mathematical manipulations require from a user an additional control over the calculations. In particular, only he or she is able to decide what a way should be most fruitful under the specific conditions depending on the details of the method of the problem treatment and the data given. However this, anyway, means that a researcher may focus on the essence of the problem considered, the technical programming-related issues requiring considerably less outlay than the other symbolic computational tools would necessitate.

Currently, GRG\(_{EC}\) system enables one to calculate or to subject to the other processing more than two hundreds of the so called \textit{data objects} modeling the basic notions and relationships (equations) originated from the field theory in the curved space-time and the geometry. However GRG\(_{EC}\) is not accommodated for the purpose of the abstract index manipulating. The handling of data objects endowed with extrinsic indices (tensors, spinors, \textit{etc.}) is carried out utilizing, essentially, the explicit sets of their components attributed to a definite gauge.

\(^2\)It may not be definitely stated that, after all, this characteristic of the present paper proves to be unconditionally satisfactory but there is seen no ways to amend it in that respect.
ture’ over the well known general purposes computer algebra system Reduce developed by A. Hearn (see Ref. [4]). Accordingly, GRG_{EC} potentially possesses the same degree of portability as Reduce does. The programming language utilized for the GRG_{EC} realization is not Rlisp, the language of the own Reduce coding, but the Lisp dialect named STANDARD LISP, see Ref. [5], which is supported in frames of the PSL (Portable Standard Lisp) package, the bottom level of the Reduce infrastructure hierarchy\(^3\). It is important to emphasize that a user need be familiar with neither STANDARD LISP, nor with Reduce (although a knowledge of the latter system would not be completely useless of course especially in the cases of more sophisticated calculations). Specifically, GRG_{EC} creates an ‘opaque casing’ which mostly conceals the lower level data structures utilized and the inner ways of their handling. Reduce is thus a ‘lower level’ structure from viewpoint of GRG_{EC}.

A language exploited for the communication of a user with GRG_{EC} system is a product of the independent working up. It is closely related to the application field and, at the same time, mimics the elements of the natural language, utilizing a sort of partial simulation of English. Accordingly, the essence of GRG_{EC} programs is, as a rule, understandable for any specialist in the application field — even a dilettante in the programming as such. In the worst case, a quite moderate number of elucidations of specific issues could prove to be necessary (one will see this on examples considered in the present work).

The source code of GRG_{EC} amounts now to approximately 1.5 Mb. The compiled (32-bit) binary code occupies about 1 Mb of a disk space. GRG_{EC} is currently intended for a free distributing.

\section{2 Preliminaries}

It has to be noted that in the case of four dimensional Lorentz geometry GRG_{EC} uses for calculations a version of the null tetrad method based on Cartan’ structural equations incorporated with 2-spinors theory.

Unfortunately, it is problematic to point out a single source which would survey the formalism coinciding in all substantial details with the one implemented in GRG_{EC} system\(^4\). At the same time the univocal representation of a mathematical matter underlaid is of a notable importance for anybody who deals with a concrete computer realization of a mathematical formalism.

Specifically, a well known detailed presentation (now, essentially, unofficial standard) of the methods of spinor calculus is given in the well known monograph by Rindler & Penrose [6]. However, at first, the methods based on Cartan’ structural equations and their incorporating with spinor approach are not considered there in fact. At second, apart from some notational discrepancies, there is a distinction in the conventions fixing the sign of the spinor contraction, \(i.e\). the definition what a rule, either \(i_A = \epsilon_{AB} i^B\) or \(i_A = i^B \epsilon_{BA}\), is to be used for spinor indices manipulations. The choices adopted in [6] and in GRG_{EC} are opposite. \(^{[The point is that the both above definitions were (and are, see, \(e.g\)., [7, [8], \(PSL\) is not the exclusive possible base suitable for the Reduce implementation.\]

\(^3\)It is worth noting that PSL is not the exclusive possible base suitable for the Reduce implementation.\]

\(^4\)This does not mean of course that the latter is unique in any respect.
Appendix A: A system of notations and basic conventions being elaborated a decade earlier than Ref. [6], adopting alternative choices, appears.) Further, the methods of the calculus of exterior forms in a version adapted to the gravity theory are outlined in Ref. [9]. However, the spinor based description of the geometry and field theory is given there after a fashion fairly incomplete and insufficient for our purposes. We might also refer to the work [10] but it treats a specific problem and cannot serve a review of the formalism applied; besides, there are still some disagreements in conventions utilized in Ref. [10] and GRGEC, unfortunately.

We have to give therefore here a summary of the main elements of the mathematical methods which will be ‘applied by GRGEC system’ for the treatment of the problem considered in the main body of the paper. Besides, we discuss here its mathematical posing necessary for the subsequent processing by the methods of computer algebra.

### 2.1 Elements of mathematical formalism

Let 4-dimensional Lorentz metric be represented in the form of expansion

$$g = \vartheta^A \vartheta_B \otimes \vartheta^\dot{B} \vartheta_{\dot{A}}.$$

Here and below undotted spinor indices (denoted by upper-case Latin $A, B, \text{etc.}$) run over the two element set $\{0, 1\}$ while the dotted ones (dotted upper case $\dot{A}, \dot{B}, \text{etc.}$) run over the (distinct) set $\{\dot{0}, \dot{1}\}$ of ‘dotted integers’ (subjected, of course, the standard arithmetic). We use throughout the standard spinor notations normalized in accordance with the following convention which determines the spinor index manipulation rules [8]:

$$\iota_A = \epsilon_{AB} \iota^B, \quad \iota_B = \epsilon_{AB} \iota^A, \quad \iota_{\dot{A}} = \epsilon_{\dot{A}\dot{B}} \iota^\dot{B}, \quad \iota_{\dot{B}} = \epsilon_{\dot{A}\dot{B}} \iota^\dot{A}.$$

Here 2-dimensional Levi–Civita symbols $\epsilon_{**}, \epsilon_{**}$ may be defined as follows:

$$\epsilon_{AB} = -\epsilon_{BA}, \quad \epsilon_{01} = 1; \quad \epsilon^{AB} = -\epsilon^{BA}, \quad \epsilon^{01} = 1; \quad \epsilon_{\dot{A}\dot{B}} = -\epsilon_{\dot{B}\dot{A}}, \quad \epsilon_{\dot{0}\dot{1}} = 1; \quad \epsilon^{\dot{A}\dot{B}} = -\epsilon^{\dot{B}\dot{A}}, \quad \epsilon^{\dot{0}\dot{1}} = 1.$$

Further, $\vartheta^A$ (involved in Eq. (1) with distinct positions of indices) is the tetrad of the, generally speaking, complex-valued covectors (1–forms) which are null with regard to the metric $g$. In the case of the metric signature $(- + + +)$ assumed throughout they always can be picked on to satisfy the Hermitean-like symmetry relation

$$\vartheta_{0\dot{0}} = \vartheta_{0\dot{0}}, \quad \vartheta_{1\dot{1}} = \vartheta_{1\dot{1}}, \quad \vartheta_{0\dot{1}} = \vartheta_{1\dot{0}}.$$

5In connection with these remarks it should be mentioned that the predecessor of GRGEC is dated to 1981-2.
In such a case, the two tetrad elements $\vartheta_{00}, \vartheta_{11}$ are real while $\vartheta_{01}, \vartheta_{10}$ are necessarily complex. (The meaning of distinction in the format of numerical indexing which is observed in Eqs. (2) and (3) will be explained below.)

We shall refer to Eq. (3) as the real gauge condition. The imposing of this gauge does not completely fixes the tetrad: it may be still undergone gauge transformations whose set includes, in particular, the proper orthochronous Lorentz group.

Further, it is sometimes convenient to utilize another indexing of the tetrad. Let us consider the following re-definition:

$$\vartheta^0 = \vartheta_{01}, \quad \vartheta^1 = \vartheta_{10}, \quad \vartheta^2 = \vartheta_{00}, \quad \vartheta^3 = -\vartheta_{11}. \quad (4)$$

In terms of such an indexing Eq. (1) takes the form

$$g = 2\vartheta^0 \otimes \vartheta^1 + 2\vartheta^2 \otimes \vartheta^3, \quad (5)$$

where $\alpha \otimes \beta \equiv \frac{1}{2}\alpha \otimes \beta + \frac{1}{2}\beta \otimes \alpha$ denotes the symmetrized tensor product. We shall say that $\vartheta_{AB}$ is the tetrad endowed with spinor indices while Eq. (4) defines the tensorial (or tensor) tetrad indexing.

Important algebraic objects which characterize the property of space-time metricity are the two families of the so called $S$-forms constituting 2-index spinors. In particular, undotted $S$-forms are defined as follows:

$$S_{AB} = \frac{1}{2} \epsilon^{KL} \vartheta_{AK} \wedge \vartheta_{BL} = \frac{1}{2} \vartheta_{A}^{\hat{L}} \wedge \vartheta_{BL}. \quad (6)$$

Next, each metric induces a unique torsion-free metric compatible connection. Within framework assumed, it is described by the object endowed with a pair of symmetric undotted indices (which do not constitute a true spinor however) whose components are 1-forms. It is named undotted connection and is denoted $\Gamma_{AB}$. Due to indicial symmetry there are maximally three distinct components of undotted connection. They are denoted $\Gamma_{00}, \Gamma_{01}, \Gamma_{11}$. Notice that we used above so called summed spinor indexing which applies to any object symmetric with respect to all spinor indices of the same class (undotted and/or dotted ones) including the case of connection. Hereinafter, dealing with numeric values of spinor indices in explicit (expanded) formulae, the summed indices are used throughout (see Eqs. (3) and all the subsequent ones; Eqs. (2) are an exception). In particular, in the case of the undotted connection the complete collection of its components comprises $\Gamma_{00} \equiv \Gamma_0, \Gamma_{01} \equiv \Gamma_{10} \equiv \Gamma_0, \Gamma_{11} \equiv \Gamma_2$ (here at right hand side the summed indexing is used while at left the corresponding ‘numerical values’ of ‘ordinary’ spinor indices are substituted, cf. Eqs. (2)).

Given null tetrad, the connection forms can be computed from the so called first structural equations which are also named the first Cartan equations. There exist two their

\[6\]Within framework of the matrix language this point assumes introduction of Pauli matrices. We do not utilize them in an explicit form.

\[7\]As far as we know the simple but very useful notion of summed spinor indices was never explicitly introduced and used on a regular ground.
representations. The first version of the first structural equations involves the differentials of the tetrad elements and, in the case of the fulfillment of the real gauge condition, reads

$$d\theta_{AB} + \Gamma^C_A \wedge \theta_{CB} + \Gamma^C_B \wedge \theta_{AC} = 0.$$ (7)

Here the complex conjugation varies the class of connection indices, i.e. $\Gamma^C_B$ enters the equation, effectively, with two dotted indices, $\dot{B}$ and $\dot{C}$, which are connected with undotted $B$ and $C$, respectively, by means of the map sending $0$ to $\dot{0}$, $i$ to $\dot{i}$ and vice versa. [More precisely, the so called dotted connection forms $\Gamma_{\dot{B}}^C$, which are in general case algebraically independent on undotted ones, should be situated in Eq. (7) instead of $\Gamma^C_B$. However, under the real gauge condition, $\Gamma^C_{\dot{B}}$ and $\Gamma^C_B$ are mutually ‘Hermitean conjugated’ and we prefer to evade introduction of an additional object which is, essentially, superfluous for our current purposes.]

The second version of the first structural equations (first Cartan equations) involves differentials of $S$-forms instead of ones of tetrad elements and reads:

$$dS_{AB} + 2\Gamma^C_{(A} \wedge S_{B)C} = 0.$$ (8)

The both versions of the first structural equations (which are, essentially, equivalent) always can be algebraically resolved with respect to the connection forms.

The next important geometric relationship utilizes the so called undotted curvature spinor $\Omega_{AB} = \Omega_{BA}$ whose each component is a 2-form. The link of undotted curvature and undotted connection is yielded by the second structural equation\(^8\). It reads

$$\Omega^B_A = d\Gamma^B_A + \Gamma^K_A \wedge \Gamma^K_B.$$ (9)

Space-time is locally flat if and only if $\Omega^B_A$ (and its dotted counterpart) vanishes.

The specific relationships lying in foundation of the general relativity (Einstein equations for the general matter content and their particular case, Einstein–Maxwell equations, which we shall deal with in the present work alone) are formulated making use of the so called irreducible spinor curvatures. The latter involve in particular, the undotted Weyl spinor $\Psi_{ABCD} = \Psi_{(ABCD)}$ representing Weyl curvature tensor and its Hodge dual, Ricci spinor $\Phi_{AB\dot{C}\dot{D}} = \Phi_{(AB)(\dot{C}\dot{D})}$ equivalent to the traceless part of the Ricci curvature tensor, and the scalar curvature $R$. All these objects can be determined from the known undotted curvature (and $S$-forms) with the help of the equations displayed below:

$$i\Psi_{ABCD} \cdot \text{Vol} = \Omega_{(AB} \wedge S_{CD)}; \quad -i\Phi_{AB\dot{C}\dot{D}} \cdot \text{Vol} = \Omega_{AB} \wedge S_{\dot{C}\dot{D}}; \quad iR \cdot \text{Vol} = 4\Omega_{AB} \wedge S^{AB}.$$ (10)

Here the dotted $S$-forms $S_{\dot{C}\dot{D}}$ are defined similarly to undotted ones by the equation

$$S_{\dot{A}\dot{B}} = \frac{1}{2} \epsilon^{KL} \partial_{\dot{K}A} \wedge \partial_{\dot{L}B}$$

\(^8\)undotted one, there is also dotted second structural equation involving dotted curvature and dotted connection.
(cf. Eq. (6)) and
\[ \text{Vol} = \frac{i}{12} \theta^B_A \wedge \theta^C_B \wedge \theta^D_C \wedge \theta^A_D \]
is the volume element (nonzero 4-form) induced by the corresponding metric.

Now let us outline the form of the basic equations of electromagnetic theory within framework of the formalism under consideration. They are extremely transparent. Electromagnetic field is associated with the complex-valued 2-form \( \omega \) which is named \textit{undotted electromagnetic 2-form}. It has to be spanned by the undotted \( S \)-forms, \textit{i.e.}, admits the expansion
\[ \omega = \phi^{AB} S_{AB}. \] (11)
The above coefficients \( \phi_{AB} = \phi_{BA} \) constitute the so called \textit{undotted spinor of electromagnetic field}. The charge- and current-free Maxwell equations reduce to the closeness condition imposed on the undotted electromagnetic 2-form:
\[ d\omega = 0. \] (12)
We shall refer to it as \textit{undotted Maxwell equations}. In the case of the imposing of the real gauge condition (3) the complex conjugation of the undotted electromagnetic 2-form yields the \textit{dotted electromagnetic 2-form} \( \hat{\omega} = \bar{\omega} \) which admits a similar expansion
\[ \hat{\omega} = \phi^{\hat{A}\hat{B}} S_{\hat{A}\hat{B}}, \]
where the coefficients \( \phi_{\hat{A}\hat{B}} \) constitute the \textit{dotted spinor of electromagnetic field}. Finally, under a suitable choice of the physical units, Einstein–Maxwell equations are represented as follows:
\[ 2\Phi_{AB\hat{C}\hat{D}} = \phi_{AB} \phi_{\hat{C}\hat{D}}, \] (13)
\[ R = 0. \] (14)
Eq. (13) is called the \textit{spinor part of Einstein–Maxwell equations} while (14) is their \textit{scalar part}.

Now we may proceed with the description of the problem chosen for the demonstration of the application of the formalism outlined with the help of GRG_{EC}.

### 2.2 Radiative electrovacs

In the present work, we shall consider solutions of the source-free Maxwell and Einstein–Maxwell equations (electrovacs) which, under a proper gauge, satisfy the equation
\[ \Gamma_\theta = 0. \] (15)
Characterizing its geometric, the first important consequence follows from \( \Gamma_{\theta\theta} \)–component of Eq. (7) which now reads
\[ d\theta^2 + 2\theta^2 \wedge \text{Re} \Gamma_I = 0. \] (16)
Specifically, by virtue of Frobenius’ theorem, \( \theta^2 \) determines an integrable distribution. Said another way, there exists, at least locally, a scalar function \( x \) such that

\[
\theta^2 = \Xi_1 \, dx
\]

for some non-zero factor \( \Xi_1 \), the hypersurfaces \( x = \text{constant} \) being null.

Further, it has to be noted that the coefficients \( \rho, \sigma, \ldots \pi \) of the general expansion of undotted connection with respect to null tetrad

\[
\begin{align*}
\Gamma_0 &= \rho \theta^0 + \sigma \theta^1 + \tau \theta^2 - k \theta^3, \\
\Gamma_1 &= \alpha \theta^0 + \beta \theta^1 - \gamma \theta^2 + \varepsilon \theta^3, \\
\Gamma_2 &= \lambda \theta^0 + \mu \theta^1 - \nu \theta^2 + \pi \theta^3.
\end{align*}
\]

are nothing else but the Newman-Penrose (NP) spin coefficients, see Ref. [11], [9]. Some of them (namely those involved in expansion of \( \Gamma_0 \) and \( \Gamma_1 \)) describe the invariant properties of the null congruence determined by the covector \( \theta^2 \), i.e., generating hypersurfaces \( x = \text{constant} \). In particular, if \( k = 0 \) (that follows from (15)) this congruence is geodetic and possesses the complex expansion equal to \( \rho \) and shear \( |\sigma| \). Thus we immediately see that all the optical scalars vanish in the case under consideration.

Of course the ansatz (15) imposes severe restrictions on the space-time curvature as well. Specifically, the ‘\( \theta^0 \)’-component of Eq. (9) reads

\[
\Omega_0 = d\Gamma_0 + 2\Gamma_0 \wedge \Gamma_1
\]

and, thus, Eq. (15) immediately yields \( \Omega_0 = 0 \). This, in turn, implies in particular the following constraints on the components of the Weyl spinor and the scalar curvature (see Eq. (10)):

\[
0 = \Psi_0 = \Psi_1 = \Psi_2 + \frac{1}{12} R.
\]

Since in electrovac space-times the scalar curvature \( R \) vanishes (Eq. (14)), \( \Psi_2 \) vanishes as well and only \( \Psi_3, \Psi_4 \) may be non-zero.

The above restriction on the components of Weyl spinor implies that the space-times in question belong to Petrov–Penrose types \( \text{III} \) or \( \text{N} \) (or, perhaps, are conformally flat, the type \( \text{O} \)). The corresponding multiple principle null direction coincides with the one determined by the congruence of null geodesics generated by null hypersurfaces \( x = \text{constant} \). We have seen that all its optical scalars vanish.

It is known that gravitational fields with the above algebraic structure of the conformal curvature are usually associated with gravitational radiation spread in the 4-dimensional light-like direction, multiple principal direction. In particular, the parallelly propagated plane waves (\( pp \)-waves, see Ref. [9], section 21.5) satisfy all the above conditions. Due to these reasons and for convenience of further references, we shall name the solutions of the source-free Maxwell and Einstein–Maxwell equations satisfying the restriction (15) **radiative electrovacs**

\footnote{It should be mentioned however that there exist configurations describing radiation which do not belong to the class singled out. A well-known example is, in particular, Robinson-Trautman’ class of metrics, see Ref. [9]. On the other hand, there are ‘radiative’ spacetimes (in our terminology) which are unlikely to be associated with any radiation process, e.g., the Hauser’ solution [12].}.
In the present work, we find the ‘most general’ metric belonging to the class of radiative electrovacs, reducing all the field equations to several subsequent quadratures. Our solution does not actually includes all the radiative electrovacs but the metrics missed here (e.g. $pp$-waves) are either generated from it by means of appropriate limiting processes or arise following a particular ‘degenerated’ side-branches of the ‘generic’ integration procedure.

Now we are able to carry out a complete posing of the problem.

### 2.3 Posing of the problem

First of all we have to determine what a tetrad fits the restriction (15). To that end, we shall utilize some further implications inferred by Eq. (7). Specifically, it has been mentioned that its $\theta^{00}$-component implies Eq. (17). Absorbing the factor $\Xi_1$ by the tetrad element $\theta^3$ (which is an admissible gauge transformation, cf. Eq. (5)), we obtain that, without loss of generality, one may assume

$$\theta^2 = dx/\sqrt{2}. \tag{21}$$

Further, eliciting the $\theta^i$-component of Eq. (8), one obtains

$$dS_0 + 2S_0 \wedge \Gamma_I = 0,$$

where $S_0 = \theta^2 \wedge \theta^0$ (see Eqs. (4), (6)). Again, the Frobenius’ theorem implies

$$S_0 = e^\alpha d\hat{x} \wedge d\phi$$

for some functions $\alpha, \hat{x}, \phi$. By virtue of (21), without loss of generality, one may get $\hat{x} = 2x$ (we a priori know that by virtue of definition $S_0 = \theta^2 \wedge \ldots = \ldots dx \wedge \ldots$) and, then, executing, if necessary, some gauge transformation, we obtain

$$\theta^0 = e^\alpha d\phi/\sqrt{2} \Rightarrow \theta^1 = \overline{\theta^0} = e^{\bar{\alpha}}d\bar{\phi}/\sqrt{2}. \tag{22}$$

It is immediately clear from Eq. (5) that the only value of $(\text{Re } \alpha)$ is significant while the choice of $(\text{Im } \alpha)$ is a matter of gauge. Indeed $(\text{Im } \alpha)$ may be arbitrary, leaving the metric unaffected. Additionally, analyzing the above explicit equations, it is easy to see that the local coordinate transformation $\phi \to \Xi_2(\phi, x)$ for arbitrary $\Xi_2$ holomorphic with respect to $\phi$ (and satisfying the natural restriction $\Xi_2, \phi \neq 0^{10}$) still remains allowable, provided the corresponding transformation of $\alpha$ accompanies it. We reserve this possibility for a future use.

Noting that $\theta^0 \wedge \theta^1 \wedge \theta^2 \neq 0$, the variables $x, \phi, \bar{\phi}$ are functionally independent (and thus $\phi$ is complex and ‘functionally independent’ of $\bar{\phi}$, i.e. $\phi$ and $\bar{\phi}$ are a form of representation of two real independent coordinates). Let us use them as three of four coordinates and denote the fourth independent real coordinate $y$.

Thus, up to now, the only non-specified tetrad element is $\theta^3$. Its generic holonomic expansion reads

$$\sqrt{2} \theta^3 = \Xi_3 dy + 2 \text{Re}(\chi d\phi) + h dx,$$

10Hereinafter, a subscript following comma denotes partial derivative.
where the coefficient $\Xi_3$ may not vanish, $\chi$ (complex) and $h$ (real) are some functions. Then, executing the admissible change of the coordinate $y$, $y \rightarrow \int \Xi_3 \, dy$, one always may reduce (the counterpart of) $\Xi_3$ in new coordinates to a constant. Thus we may assume

$$\sqrt{2} \, \theta^3 = dy + 2 \, \text{Re}(\chi \, d\phi) + h \, dx. \tag{23}$$

Let us notice further that the transformation

$$y \rightarrow y + \Xi_4 \tag{24}$$

for arbitrary real function $\Xi_4 = \Xi_4(x, \phi, \bar{\phi})$ which is accompanied with the replacing

$$\chi \rightarrow \chi - \Xi_4, \quad \bar{\chi} \rightarrow \bar{\chi} - \Xi_4$$

and some transformation of $h$ is also a change of a gauge. (It does not affect any of equations considered above including the expansion (23).) Notice also that then we have the following subsidiary transformation;

$$\chi, \phi + \bar{\chi}, \phi \rightarrow \left(\chi, \phi + \bar{\chi}, \phi\right) - 2 \Xi_4, \phi \bar{\phi}. \tag{26}$$

It will be useful below.

Ultimately, we can formulate the problem as follows: We shall search for electrovacs whose metric is described by the null tetrad (22), (23) and possesses connection which satisfies the constraint (15).

## 3 Handling field equations with the help of computer algebra

In this section subdivided into several subsections we present the application of GRG$_{EC}$ system for the reduction of the field equations specifying radiative electrovacs. In each step, a separate series of calculations is carried out yielding some new information describing the structure of solution. Whenever its ‘amount’ exceeds some reasonable threshold, the calculation stops and further proceeds with the next step from the beginning, utilizing the relationships obtained so far as a new and more detailed initial data.

Any knowledge of GRG$_{EC}$ system is not presupposed. Accordingly, relevant explanations on this issue are given whenever necessary$^{11}$, making the discussion, essentially, self-contained.

### 3.1 Step 1: Coding of the initial data and inference of first implications

Generally speaking, the data to be read and processed by GRG$_{EC}$ is to be arranged in a form of the so called problem specification which is a regular text usually stored in a

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$^{11}$excluding some general means of the formulae coding which are utilized in majority of programming languages such as Fortran, Pascal, etc.
disk file. (One finds in Appendix A a sample of the workable problem specification.) The structure of problem specification is fairly transparent. Apart from a title and conclusion, it consists of a series of sections comprised, in turn, of several paragraphs and must, in particular, contain the section of (initial) DATA. The other sections (for example, the section of INSTRUCTIONS, the section of SUBSTITUTIONS, and some others) are optional.

Specifying initial DATA, a user has to explicitly list all the non-implemented (‘user-defined’) objects which he or she is going to utilize. Here the particular symbols of COORDINATES, specific FUNCTIONS (‘unknown’ to the system), etc., are, in particular, meant.

In our case all what we need at the beginning is explicitly DECLARED and defined as follows:

1 input: Data:
   input: declare COORDINATES x,y,phi,phi~;
3 input: declare SCALARS al,chi,h(x,y,phi,phi~);
input: declare REAL x,y,h;
5 input: declare COMPLEX CONJUGATED phi & phi~,al & al~,chi & chi~;
6 input: TETRAD comprises
... input: component|0=E**al d phi/sqrt(2),
8 input: component|1=C.C.(component|0),
... input: component|2=d x/sqrt(2),
10 input: component|3=(d y +2 RE(chi d phi) +h d x)/sqrt(2);
11 input: end of data.

Here the symbol ‘al’ stands for α (see Eq. (22)), ‘C.C.’ is the symbol of the complex conjugating operator while the origin of the other symbols seems to be evident. Continuing with a matter of notations, it is also useful to mention that here and below the enumeration at left hand side in the copies of fragments of GRGEC input scripts and output listings, as well as the marks at left ‘input:’ (line of input code), ‘output:’ (line of output file), etc., are not a part of the texts displayed. They were introduced for convenience of further references alone. For example, we can refer to the lines 1,3,5,6–10,11 involved in the DATA section displayed above.)

It is seen that the above script encodes, almost verbatim, the content of Eqs. (21, 22, 23). Nevertheless a number of comments, concerning some specific points of GRGEC input language, should be made.

First, the tilde symbol ‘~’ attached at right to some identifiers imitates the overscoring used to denote complex conjugation in the standard system of mathematical notations. For example, ‘phi~’ may be regarded as the representation of \( \bar{\phi} \), etc. However, contrary to the habit of mathematical notations, the additional tilde mark does not impose itself any actual relation between the corresponding identifiers. For example, \( \phi \) and \( \phi\sim \) are a priori not connected in any way. The encoding of the necessary relationship such as \( C.C.(\phi) = \bar{\phi} \) is realized by means of a separate declaration. See line 5.

Next, the term ‘SCALARS’ denotes, roughly, a sort of functions each of whose is at the same time a single symbol (identifier). The dependence upon the (fixed) set of arguments pointed out in the declaration (line 3) is supported by a special implicit mechanism.

---

12 One may organize a library of such sort data and load them from there.
13 It has no relation to the own Reduce’s scalars.
14 which, to be more precise, is varied by the coordinate transformations, if any
We also mention that the symbol $df$ (or, equivalently, $DF$) denotes partial derivative (as well as ordinary derivative of a function of a single argument which is not formally distinguished from a partial derivative). In particular, $df(chi,phi)$ encodes $\partial \chi / \partial \phi$. The derivative $\partial^2 \chi / \partial \phi / \partial \bar{\phi}$ is represented by the script $df(chi,phi,phi^{-})$, for $\partial^2 \chi / \partial \phi^2$ one has to write down $df(chi,phi,2)$, etc. (These notations are identical to ones supported by *Reduce* system.)

Finally, the record ‘component$\{j\}$’ (the delimiter | is here optional and might be dropped out), where ($j$) is a digit, refers to a single component of a data object, in our case TETRAD. Thus the record ‘component$\{1=C.C.(component\{0\})$’ (see line 8) represents the equation $\theta^0 = \bar{\theta}^1$, cf. Eqs. (3), (4).

Now we are ready to proceed with calculations.

Let GRGEC package was started and has read the problem specification consisting of a single section of DATA. We may to immediately control the calculations issuing the instructions from a keyboard. (Alternatively, they — or a part of them — might be put into a section of INSTRUCTIONS, provided it were included in the problem specification. The mixed form of control is also available.)

Thus, at first, after the auxiliary request to

*input*: turn on displaying of negative powers\(^{15}\)

affecting the format of output, we issue the instruction to

*input*: find UNDOTTED CONNECTION

The system response to it comes to the following output:

*output*: => find UNDOTTED CONNECTION
*output*: UNDOTTED CONNECTION is not found in the DATA section...
*output*: ...UNDOTTED CONNECTION has been calculated $\approx 1.0$ s.
*output*: Total time spent amounts to 3.7 seconds.

(Remark: Hereinafter, the symbol ‘$\equiv \equiv$’ denotes some blank space cut off from the copy of a line of an input script or an output listing in order to fit it to the page width.)

However, we want to look at the result (UNDOTTED CONNECTION) in explicit form. To that end, we have to initiate the action ‘TYPE’ directed to the data object we are interested in. The shortened form of the corresponding instruction looks as follows:

*input*: UGAMMA0?

Here UGAMMA0 is the reference to the component $\Gamma_0$ of the UNDOTTED CONNECTION (‘UGAMMA’ is its kernel name while ‘0’ is the value of the summed undotted index distinguishing the component, see section 2.1). The return of the instruction reads:

*output*: => UGAMMA0?
*output*: --– TYPE UGAMMA0
*output*: UGAMMA = ( - ---*E^0 *DF(chi^-,y)) d x + ( 1 a^-
*output*: 0 2
*output*: - (---*E^0 )*(DF(al,y) + DF(al^-,y))) d phi

\(^{15}\)which may be shortened to ‘turn on DIV’
Thus $\Gamma_0$ does not automatically vanish. This simply means that the expression for the TETRAD assumed is more general than we actually need. At the same time the condition ensuring the vanishing of $\Gamma_0$ is very simple: $a_l+a_l^\sim$ (i.e. $\alpha + \bar{\alpha}$) and $\chi^\sim$ (i.e. $\bar{\chi}$) have to be independent on $y$ ($y$). Additionally, we have seen above that the value of $a_l-a_l^\sim$ is completely gauge dependent. In particular, it always may be assumed to be independent on $y$ and, then, both $a_l$ and $a_l^\sim$ turns out to be independent on $y$ as well. Besides, $\chi$ is also independent on $y$. We ‘inform’ the system of these facts by means of the following instruction:

**input:** Let $DF(a_l, y)=0, DF(a_l^\sim, y)=0, DF(\chi, y)=0, DF(\chi^\sim, y)=0$

The response reads

**output:** => Let $DF(\alpha, y) = 0$, $DF(\alpha^\sim, y) = 0$, $DF(\chi, y) = 0$, $DF(\bar{\chi}, y) = 0$

**output:** Section of SUBSTITUTIONS is not found.

**output:** It is added to the problem specification.

**output:** The following substitution rules

1. $DF(\alpha, y) \rightarrow 0$
2. $DF(\alpha^\sim, y) \rightarrow 0$
3. $DF(\chi, y) \rightarrow 0$
4. $DF(\bar{\chi}, y) \rightarrow 0$

are put to the section of SUBSTITUTIONS.

**output:** Substitutions 1, 2, 3, 4 are now active

Now we can inspect the resulting structure of UNDOTTED CONNECTION. The instruction

**input:** evaluate $U\Gamma_0$, $U\Gamma_1$/d $\phi$/$\phi^\sim$, $U\Gamma_2$/d $\phi$/$\phi^\sim$

yields in particular the output

**output:** The expression $U\Gamma_0$ vanishes. $\{\sim\sim\} 0.7$ s.

**output:** The expression $U\Gamma_1$/d $\phi$/$\phi^\sim$ vanishes. $\{\sim\sim\} 0.8$ s.

**output:** The expression $U\Gamma_2$/d $\phi^\sim$ vanishes. $\{\sim\sim\} 0.8$ s.

**output:** Total time spent amounts to 4.5 seconds.

Thus the condition distinguishing radiative electromacs (Eq. (15)) is now satisfied. Additionally, we see that the holonomic expansion of $U\Gamma_1$ does not involve $d$ $y$-term while the expansion of $U\Gamma_2$ consists of the $d$ $x$- and $d$ $\phi^\sim$-terms alone. We shall make use of these facts below. (We might display the explicit expressions of all the components of UNDOTTED CONNECTION but these would be, essentially, useless here).

Our next operation implies the inferring of some information from the field equations. Specifically, let us ask the system to

**input:** obtain and isolate $\cdots$ the SCALAR PART of EINSTEIN-MAXWELL EQUATIONS;

**input:** show EQUATIONS

The answer is the following:

**output:** => obtain and isolate the SCALAR PART of EINSTEIN - MAXWELL EQUATIONS

**output:** --> OBTAIN SCALAR PART of EINSTEIN - MAXWELL EQUATIONS
...SCALAR PART OF EINSTEIN - MAXWELL EQUATIONS ⊲ has been obtained 4.0 s.

Total time spent amounts to 8.6 seconds.

-- ISOLATE SCALAR PART of EINSTEIN - MAXWELL EQUATIONS

Warning: The denominator E *E has been removed assuming it to be nonzero...

EQ. number 1 is put to the list of EQUATIONS.

Total time spent amounts to 8.7 seconds.

=> show EQUATIONS

al + al~

Eq. (1) reads 0=4*(E *DF(h,y,2) - DF(al,phi,phi~) - DF(al~,phi,phi~))

Evidently, it makes sense to

resolve EQUATION (1) w.r.t. df(h,y,2);

The result is Eq. (2) (we shall display it explicitly below) while the preceding EQUATION may be removed (by means of the instruction ‘erase EQUATION (1)’).

The last calculation we carry out in frames of the current step is the deriving of a consequence of Eq. (2) (which determines df(h,y,2), see below) by means of its differentiating with respect to the COORDINATE y. Specifically, let us claim to

isolate EQUATION DR(LHS(EQ(2))=RHS(EQ(2)),y)=0;

show EQUATIONS;

Remark: Here the symbol ‘DR’ is so called crossderivative operator which is similar the to partial derivative but takes into account the mutual tree-like dependences of the SCALARS, if any, regarding them as functions, not mere variables. Mathematically, one may regard DR(...,y) in the above instruction as a partial derivative with respect to y, provided a proper interpretation of dependences of variables (SCALARS and COORDINATES) is implied.) This calculation yield a new EQUATION and we obtain in particular the following output:

Eq. (3) reads DF(h,y,3)=0

Eq. (2) reads DF(h,y,2)

=DF(al,phi,phi~) + DF(al~,phi,phi~))

The current point is just reasonable for the terminating of the present (the first) step of the problem solving. The result having been derived is the following: we determined the dependence of all the unknowns on the COORDINATE y (which represents the affine parameter y along the multiple null principal congruence). Specifically, the SCALARS al, chi are independent on it (as well as al~, chi~) while h is a second order polynomial in y, the higher order coefficient being found in explicit form.

Additionally, we have seen that a gauge transformation allows to add any function of x,φ,¯φ to χ,χ. See Eq. (26). Thus since both χ and χ are independent on y the above 2-term sum can be nullified by means of a proper gauge choice. Then, there is a real ‘potential’ k = k(x,φ,¯φ) such that χ = i k, χ = −i k. Ultimately, noting that the functions χ,χ enter the tetrad in the form of the expression ℜ(χ dφ) alone, we have the reduced representation

ℜ(χ dφ) = −ℑ(k dφ) (27)
which will be used below in ‘formulae’ (GRG\textsubscript{EC} scripts) determining TETRAD.

An additional remark is now in order. It has to be mentioned that in this section we exhibited, essentially, the complete session of the work with GRG\textsubscript{EC}. Indeed, the input code used in the first step of calculations consists, at first, of the section of DATA which was displayed in the beginning and, at second, a series of instructions (assumed here to be issued from a keyboard) which was listed (and commented) in the course of the discussion as well\textsuperscript{16}. We also reproduced all the substantial output which follow the instructions execution (the only omission is the initialization messages and the outcome of some preliminary analysis of the initial data by the system including the copy of the structured problem specification).

Unfortunately, in the sequel, in many cases the authentic copies of output would occupy too much space to be given here. For example, the output of the only next (the second) step of calculation considered below is more that 30 kB long, i.e., would require at least 10 pages even in a condensed format. Hence we shall carry out below a strict selecting and shall display only those output which is actually necessary. On the contrary, the input will be displayed without omissions.

3.2 Step 2: Introduction of electromagnetic field and further reduction of field equations

The DATA incorporated in the problem specification of the second step of calculations develop ones used in the preceding step (see above lines 1–11). In particular, the symbols of COORDINATES need not be varied, of course. Similarly, the TETRAD is described by the same paragraph (a fragment of a section bounded at right by the semicolon ‘;’) which comprises lines 6–10, where however the equation (27) is taken into account. Thus now instead of the line 10, one has the following one:

\texttt{input: component|3=(d y -2 IM(df(k,phi) d phi) +h d x)/sqrt(2)};

Furthermore, now the identifier h is not an abstract ‘SCALAR’. Instead, it is specified an ABBREVIATION by means of the statement

\texttt{input: ABBREVIATION follows: h=R+Q*y+P*y**2};

(\textbf{Remark}: An identifier of ABBREVIATION is immediately replaced by the corresponding expression associated with it whenever it is met in a mathematical formula). Thus we realize the form of the dependence of h upon y (second order polynomial) inferred in the preceding section from the field equations. The thee symbols R,P,Q are the new SCALARS instead of the single h but, as opposed to h, they are independent on y. We shall need even more SCALARS. Their total collection looks now as follows:

\texttt{input: declare SCALARS a1,k,R,P,Q(x,phi,phi~),
gamma,tau,zeta,\lambda,mu(x,y,phi,phi~),
psi,ems(x,y,phi,phi~);}

(cf. line 3). Besides, let us

\textsuperscript{16}To be completely precise, the file \texttt{slang} containing the additional section of CONFIDENTIAL SLANG which installs the extended collection of ‘keywords’ is read and included in the problem specification (cf. the list line of the script displayed in Appendix A). We omit discussion of this facility here however. See [1].
Further, we may specify the structure of UNDOTTED CONNECTION basing on its properties derived in the preceding subsection. Specifically, UNDOTTED CONNECTION comprises

\begin{align*}
\text{component}_1 &= \gamma \, d\phi + \tau \, d\phi^\sim + \zeta \, dx, \\
\text{component}_2 &= (\mu \, d\phi^\sim + \lambda \, dx) / E^\alpha
\end{align*}

(Remark: Dropping out specification of the component\(0\) of UNDOTTED CONNECTION (\(\Gamma_0\) in the mathematical notation) in this working binding of the data object value, it is automatically nullified.) This representation of connection components merely expresses the form of their expansions implied by the equations displayed in the lines 43–44, 46–47, see the comment following them. The denominator \(E^\alpha\) (i.e. \(e^\alpha\)) of the component\(2\) is introduced for the sake of the later convenience (it makes some relationships simpler). The SCALARS \(\gamma, \tau, \mu, \lambda\) are assumed to be subjected no \textit{a priori} constraints, being therefore defined just by the expansions displayed in lines 83–84.

Accomplishing the DATA section of the current version of the problem specification, we add the following description of the electromagnetic field:

\begin{align*}
\text{component}_1 &= \psi, \\
\text{component}_2 &= \text{ems}
\end{align*}

Since the SCALARS \(\psi, \text{ems}\) still correspond to arbitrary functions (see line 77), it, essentially, only states that the component\(0\) (encoding \(\phi_0 = \phi_{00}\)) vanishes, implying additionally (by virtue of the statement displayed line 88) that \(\phi_0\) vanishes as well. It is worth noting that the vanishing of \(\phi_0\) is neither a restriction nor an ansatz since it can be deduced from Eqs. (19), (15) and the spinor part of Einstein–Maxwell equations (13) in a way similar to one used for the derivation of Eqs. (20).

Proceeding now with calculations, let us first of all examine implications of the first structural equations. Since, given the tetrad, they have to uniquely determine the connection forms, it may be expected that, in the case under consideration, the explicit values of the SCALARS \(\gamma, \tau, \mu, \lambda\) have to follow. This is indeed the case. Namely, the instruction

\begin{align*}
\text{obtain and isolate} \\
\text{the SECOND VERSION of the UNDOTTED FIRST CARTAN EQUATIONS}
\end{align*}

entails just the five EQUATIONS (automatically enumerated with the numbers 1,...,5).

Then, requiring to

\begin{align*}
\text{resolve EQUATIONS (1)-(5) w.r.t. } &\gamma, \tau, \mu, \zeta, \lambda \\
\text{(after that one may 'erase EQUATIONS (1)-(5)') and then, issuing the instructions}
\end{align*}

\begin{align*}
\text{turn on the displaying of negative powers;} \\
\text{show new EQUATIONS factoring out } &E^\alpha, E^\alpha^\sim
\end{align*}

we obtain the following output:

\begin{align*}
\text{Eq. (10) reads } &\gamma = - \frac{1}{17} \text{DF}(\alpha^\sim, \phi)
\end{align*}
Eq. (9) reads \( \tau = \frac{1}{2} \alpha + \alpha^* \)

Eq. (8) reads \( \mu = - \left(-\frac{1}{2} \varepsilon \right) \left\{ \partial(\alpha, x) + \partial(\alpha^*, x) \right\} - \partial(k, \phi, \phi^*) I \)

Eq. (7) reads \( \zeta = \frac{1}{4} \alpha + \frac{1}{4} \alpha^* - \frac{1}{2} \partial(k, \phi, \phi^*) I \)

Eq. (6) reads \( \lambda = - \partial(k, x, \phi) I - 2 \partial(k, \phi) I P y \)

These relationships are just the content of the first structural equations. Essentially, they define the connection forms, see lines 83–84.

We may regard the expressions determining auxiliary SCALARS \( \gamma, \tau \) as the ultimate formulae and use them for the complete eliminating these SCALARS (which are involved in the expansions of UNDOTTED CONNECTION alone). To that end, we have preliminarily to add substitution rules \( \text{LHS(EQ(9))} \rightarrow \text{RHS(EQ(9))}, \text{LHS(EQ(10))} \rightarrow \text{RHS(EQ(10))} \)

to the section of SUBSTITUTIONS. (Although we have not included this section it in the problem specification, it is automatically introduced.) (Remark: The above record of substitution rules mean that, bringing them into action, the expression at left in EQ[(9)] displayed above is replaced by the expression situated at right (which is indicated by the symbol ‘\( \rightarrow \)’); the second formula involving EQ[(10)] possesses similar meaning.)

The substitutions introduced are automatically endowed with the numbers 1 and 2, respectively\(^{17} \). They are used for the referring to the rules. It particular, we invoke the substitutions which ensure the eliminating of \( \gamma \) and \( \tau \) in favor of their values by means of the instruction \( \text{match the rules (1), (2) with UNDOTTED CONNECTION} \)

(In order to verify that the necessary transformation has been indeed carried out one may issue the instruction ‘type UNDOTTED CONNECTION’.)

Next we consider a component of Einstein field equations which was also used in the first step of calculation. The instruction

\( \text{obtain and isolate the SCALAR PART of EINSTEIN-MAXWELL EQUATIONS} \)

entails the only new EQUATION endowed with the number 11. Requiring to ‘show new EQUATION’, we obtain in particular the output

Eq. (11) reads \( 0 = 2 \varepsilon E (\zeta,y) - \partial(\alpha, \phi, \phi^*) - \partial(\alpha^*, \phi, \phi^*) \)

\(^{17}\)One might introduce any other numerical labels instead.
Notice that this EQUATION involves the derivative of the SCALAR \( zeta \) whose value has been specified by the above Eq. (7). A natural consequence of these two EQUATIONS, implying the elimination of \( DF(zeta,y) \), may be derived by means of the instruction

\[
\text{input: isolate EQUATION LHS(EQ(11))=RHS(EQ(11))}
\]

\[
\text{input: } -\text{COEFFN(RHS(EQ(11)),df(zeta,y)}*\text{DR(LHS(EQ(7))-RHS(EQ(7)),y)}
\]

This is an example of operation frequently used for reductions of overdetermined systems of equations with partial derivatives. Removing the second line we would obtain the EQUATION identical to the initial one (number 11). The additional term in the lower line is proportional to the derivative of the discrepancy \( \text{LHS(EQ(7))-RHS(EQ(7))} \) of Eq. (7). Thus the resulting EQUATION (number 12) is equivalent to Eq. (11) modulo Eq. (7). The coefficient in front of the derivative of the discrepancy is chosen in such a way to automatically entail the desirable simplification. In our case we have to eliminate \( df(zeta,y) \); hence it equals \( \text{COEFFN(RHS(EQ(11)),df(zeta,y))} \), i.e. the coefficient in front of the derivative \( df(zeta,y) \) involved in r.h.s. expression of Eq. (11) which is considered here as polynomial in \( df(zeta,y) \) (a linear function in fact).

It turns out that the last EQUATION obtained determines the value of the SCALAR \( P \). The instructions

\[
\text{input: resolve Eq. (12) w.r.t. } P; \text{ show new EQUATIONS}
\]

yield in particular

\[
\text{output: Eq. (13) reads } P=--*E
\]

\[
\text{output: } 2
\]

\[
\text{output: } *(\text{DF(al,phi,phi}) + \text{DF(al,phi,phi})
\]

which will play an important role in what follows.

Now the optimal way of the further reduction of the field equation is to proceed with Maxwell equations. Thus the next instruction reads:

\[
\text{input: obtain and isolate UNDOTTED MAXWELL EQUATIONS}
\]

It yields four ‘plain’ EQUATIONS (number 14-17) which can be found to determine a number of SCALAR derivatives. Specifically, the instructions

\[
\text{input: resolve Eqs. (14)-(17) w.r.t.}
\]

\[
\text{input: show new EQUATIONS}
\]

yield the corresponding explicit expressions (Eqs. (18)-(21)). We obtain in particular that

\[
\text{output: Eq. (21) reads } \text{DF(psi,y)}=0
\]

and

\[
\text{output: Eq. (19) reads } \text{DF(psi,phi})=0
\]

(We do not display the values of \( \text{DF(ems,y)} \) (Eq. (20)) and \( \text{DF(psi,phi)} \) (Eq. (18)) since it is only essential currently to know that these are nonzero). We see therefore that the function \( \psi \) (represented by the SCALAR \( psi \)) which equals the component \( \phi_1 \) of the undotted spinor of electromagnetic field (see line 86) depends on the coordinates \( x \) and \( \phi \) alone (represented by the symbols \( x, \phi \)). Since \( \phi \) is a complex variable \( \psi \) has to be holomorphic with respect to it (and is assumed to be \( C^\infty \) with respect to \( x \)). Further, we have to inform the algebraic processor that the constraints 132,133 are to be taken
into account throughout all the subsequent calculations. This is realized by means of the instruction

input: let the last EQUATION hold true,
input: the last but 2 EQUATION hold true

A series of the further manipulations with the collection of EQUATIONS obtained so far follows a standard routine procedure which widely applies for reducing of overdetermined systems of quasilinear equations with partial derivatives. Specifically, some of the equations available are differentiated and, then, combined in such a way to cancel out the higher order derivatives arisen. Such an operation is repeated until it yields new tractable relationships. Thus, following this scheme, the concrete transformations, which lead to the desirable simplifications and entail new data (new EQUATIONS), are chosen by ourselves with the help of the analysis of the intermediate output. In all the cases we meet the latter is fairly straightforward but, nevertheless, too bulky to be displayed here. Hence we shall not exhibit all these speculations and shall outline only the most substantial points of the derivation whose code is displayed below.

Specifically, the following series of instructions is issued:

136 input: isolate EQUATION DR(LHS(EQ(18)),y)=DR(RHS(EQ(18)),y);
137 input: show new EQUATIONS;
138 input: resolve Eqs. (20),(23) w.r.t. DF(ems,y),DF(ems,y,phi~);
139 input: show new EQUATION;
140 input: let (10) the last EQUATION hold true;
141 input: isolate EQUATION DR(LHS(EQ(20)),y)=DR(RHS(EQ(20)),y);
142 input: resolve Eqs. (25),(20) w.r.t. DF(ems,y,2),ems;
143 input: abolish substitution (10);
144 input: show Eq. (27);
145 input: let the last EQUATION hold true;
146 input: resolve Eqs. (18),(26) w.r.t. DF(k,phi,phi~),df(ems,y);
147 input: show new EQUATIONS;
148 input: resolve Eqs. (26),(28) w.r.t. df(ems,y),ems;
149 input: show new EQUATION;

Eq. (18) (produced by the instruction displayed in lines 129-130) expresses DF(psi,phi). By virtue of Eq. (21) l.h.s. of the EQUATION described in the line 136 vanishes. Explicitly, the corresponding new

input: Eq. (23) reads 0=DF(al,phi~)*DF(ems,y) + DF(ems,y,phi~)

(a part of the output of line 137). In turn, Eq. (20) expresses DF(ems,y) and lines 138-139 yield separate representations for DF(ems,y,phi~) and DF(ems,y). The instruction yields a single new EQUATION (expressing DF(ems,y,phi~))19. Line 140 initiates subsequent replacing of the derivative DF(ems,y,phi~) by the corresponding value, the rule being endowed with number 10. (In line 143 this substitution is disabled.) Eq. (20) determines DF(ems,y). Thus in the line 141 the second order derivative DF(ems,y,2) is introduced.

The instruction in line 142 eliminates ems from r.h.s. of the new EQUATION (number 25). One of the resulting EQUATIONS is most important, namely,

151 output: Eq. (27) reads DF(ems,y,2)=0

---

18For the first reading it might be recommended to skip the fragment below up to the line 151.
19Representation of DF(ems,y) does not vary; its inclusion to the instruction 138-139 is intended for the eliminating of DF(ems,y). Such a trick will be often used below.
Thus \( \text{ems} \) is linear in \( y \). Line 145 ensures the nullifying of \( \text{DF} (\text{ems}, y, 2) \) in all the subsequent calculations.

It is straightforward now to determine the coefficient in front of \( y \) in \( \text{ems} \) expansion. The instruction in line 146 yields a pair of EQUATIONS. In particular

\[
\text{Eq. (28)} \quad \text{DF} (\text{ems}, y) = E * \text{DF} (\text{psi}, \phi) \\
(\text{psi} \text{ does not depends on } y)
\]

Finally, line 148 yields representation of \( \text{ems} \) through \( \text{DF} (\text{ems}, \phi^\sim) \) and the expressions independent on \( y \) (Eq. (30)) which will be utilized later on.

Next, we shall deduce some useful consequences of Einstein–Maxwell equations. To that end, let us, at first,

input: obtain the SPINOR PART of EINSTEIN–MAXWELL EQUATIONS;
input: isolate the component \(|1|1\sim\) of the ABOVE EQUATIONS;
input: show new EQUATIONS

This yields in particular the output:

\[
\text{Eq. (31)} \quad 0 = -2*E * \text{DF} (\zeta, y) - E * \psi * \psi^\sim 
\]

\[
\text{Eq. (31)} \quad \text{DF} (a_1, \phi, \phi^\sim) + \text{DF} (a_1^\sim, \phi, \phi^\sim)
\]

Eq. (31) involves the derivative \((\zeta, y)\). On the other hand we had the 'explicit' representation of \( \zeta \) (Eq. (7), see lines 104-109). The dependence of \( \zeta \) on \( y \) is characterized by the SCALAR \( \Pi \) which, in turn, is determined by Eq. (13) (lines 124-127). These relationships (determining \( \text{DF} (\zeta, y) \)) can be represented by means of the two substitution rules which are introduced as follows:

input: add substitution rules
input: (30) \( \Pi \rightarrow \Pi (\text{LHS} (\text{EQ}(13)) - \text{RHS} (\text{EQ}(13))) \),
input: (31) \( \text{DF} (\zeta, y) \rightarrow \text{DF} (\zeta, y) - \text{MATCHING} (\text{DR} (\text{LHS} (\text{EQ}(7)) - \text{RHS} (\text{EQ}(7)), y), 30) \)

Notice that, formally, the rule (30) is trivial modulo Eq. (13). However, the term involving it is chosen in such a way to eliminate \( \Pi \) at right. Similarly, the rule (31) is trivial modulo some expression vanishing if Eq. (7) is satisfied. (Remark: Here the macro MATCHING applies the substitution rule whose number is given as the second argument\(^{20}\) \((i.e., \text{the rule (30)} \text{to the first argument.)}) The combined effect is the eliminating of the both \( \text{DF} (\zeta, y) \) and \( \Pi \) at right in the expression to be substituted in accordance with the rule (31). Accordingly, applying it to Eq. (31) by means of the instruction

input: isolate EQUATION \( 0 = \text{MATCHING} (\text{LHS} (\text{EQ}(31)) - \text{RHS} (\text{EQ}(31)), 31) \);

(here ‘31’ in EQ(31) is the number of an EQUATION while the last ‘31’ is the number of a substitution rule; their coincidence is occasional), the request ‘show new EQUATION’ yields in particular the following output:

\[
\text{Eq. (32)} \quad 0 = E * \psi * \psi^\sim + 2 * \text{DF} (a_1, \phi, \phi^\sim) 
\]

This EQUATION is worth a separate attention. It involves the SCALARS \( a_1, a_1^\sim, \psi, \psi^\sim \) (representing \( \alpha, \bar{\alpha}, \psi, \bar{\psi} \)) which may be regarded as functions of some fixed sets of arguments. In accordance with declaration shown in lines 75–77, \( a_1 \) and \( a_1^\sim \) depend on \( x, \)

\(^{20}\)An arbitrary number the rules is allowed to be invoked by the ‘MATCHING’ operator.
phi, and phi~ while more strong restriction is imposed on psi (see lines 132, 133): it may depend on x and phi alone (i.e., is holomorphic with respect to phi). Then, since psi and psi~ are COMPLEX CONJUGATED, the SCALAR psi~ depends on x and phi~ (and represents the function of x and φ holomorphic with respect to the second argument). Taking these relationships into account, we see that Eq. (28) may be regarded as Liouville equation with respect to unknown al+al~ (2 Re α) for some given psi, psi~ (ψ and ψ̄), holomorphic and antiholomorphic, respectively (cf. [9], Eq. (27.50)). Besides, we have seen above that the value of al-al~ is a matter of a gauge, provided it does not depend on the COORDINATE y. Other relevant gauge freedom involves the transformation of the COORDINATE phi (and, simultaneously, its complex conjugated phi~) which, in mathematical notations, is described by the formula φ → Ξ(φ, x) for arbitrary Ξ holomorphic with respect to φ, assuming the restriction Ξ(φ, x)φ ̸= 0 to be fulfilled. The latter fact has to be taken into account when analyzing solutions of Liouville equation and their gauge transformations.

It should be noted now that GRGEC is not able (and was not intended) to solve differential equations. We have to perform this work ourselves. In our case it is straightforward to find and write down the general solution to Eq. (32) (cf. [9], Eq. (27.51)). It is convenient to represent the solution in the form of substitution rules. They are introduced by the following instruction:

```
input: add rules
168 input: (40) al -> -log(psi)-log(1+phi*phi~/4),
   ... input: (41) al~ -> -log(psi~)-log(1+phi*phi~/4),
   ... input: (42) df(al,phi,phi~) -> DR(-log(psi)-log(1+phi*phi~/4),phi,phi~),
171 input: (43) df(al~,phi,phi~) -> DR(-log(psi~)-log(1+phi*phi~/4),phi,phi~);
```

(here, evidently, the third and the fourth rules are the consequences of the first and second ones) which just describe the general local solution to Eq. (32)).

It is instructive to show that we indeed deal with a solution the Liouville equation (Eq. (32)). Preliminarily, we have to inform the system on the properties of the SCALAR psi~ (as opposed to psi, we have not dealt with it yet in fact). It is convenient to do this referring to the corresponding EQUATIONS concerning its complex conjugated counterpart, psi, i.e. Eq. (19) and Eq. (21). Taking their complex conjugation we obtain the appropriate characteristics of psi~. To that end, let us issue the instructions

```
input: evaluate aux=C.C.(LHS(EQ(19)));
input: let (50) aux->C.C.(RHS(EQ(19)));
input: evaluate aux=C.C.(LHS(EQ(21)));
input: let aux=C.C.(RHS(EQ(21)));
```

(Remark: Here and below the (‘user-defined’) identifier aux plays role of auxiliary variable used for the temporary storing the expressions situated on r.h.s. of assignments in instructions with the action EVALUATE.) These substitution rules are labeled with numerical labels 50 (specified by ourselves) and 51 (the next number generated automatically).

Then the instruction

```
176 input: evaluate aux=MATCHING(LHS(EQ(32))-RHS(EQ(32)),42,43),
177 input: MATCHING(aux,40,41);
```

---

21This point should not be considered as its inalienable feature. Moreover, it could be worth introducing an interface with an appropriate ODE or PDE package. However such a facility has not been realized so far.
reports zero value which just means the expectable satisfaction of the Eq. (32) (see lines 164–166) by virtue of the relationships displayed in lines 168–171. (Let us remind that lines 176,177 re-evaluate the ‘discrepancy’ of Eq. (32) executing, at first, substitutions number 42,43, and then, substitutions number 40,41 which were specified in lines 168–171).

An additional useful result immediately follows from Eq. (13) and Eq. (32). Their appropriate superposition is realized by means of instructions

```
input: isolate EQUATION
        LHS(EQ(9))=RHS(EQ(9))
        -COEFFN(NUMR(RHS(EQ(9))),DF(al,phi,phi~))
        /DENM(RHS(EQ(9)))/2
        *(LHS(EQ(280))-RHS(EQ(28)));

input: show equation (29);
```

In particular, the last line entails the the response

```
output: Eq. (33) reads P= - ---*psi*psi~
```

The last action we shall carry out in frames of the current step is the explicit separating of the dependence of the SCALAR ems on the COORDINATE y. As we have seen it is a linear function of y (see line 151). Moreover, the coefficient in front of y is specified by Eq. (28) (lines 152–153). Thus it retains to find the free term which can be explicitly represented as ems–y*df(ems,y). Instead we shall calculate the derivative of (ems–y*df(ems,y))*E**al with respect to phi~. The corresponding instructions are straightforward, executing several transformations which are identical ones modulo Eq. (28) and Eq. (30):

```
input: evaluate
        expr=(ems-y*df(ems,y))*E**al
        aux=DR(expr,phi~),
        aux=aux-COEFFN(NUMR(aux),df(ems,phi~,y))/DENM(aux)
        *DR(LHS(EQ(28))-RHS(EQ(28)),phi~),
        aux=aux-COEFFN(NUMR(aux),df(ems,y))/DENM(aux)
        *(LHS(EQ(28))-RHS(EQ(28))),
        aux=aux-COEFFN(NUMR(aux),ems)/DENM(aux)
        *(LHS(EQ(30))-RHS(EQ(30))),
        aux

input: factoring out E**al,E**al~;
```

Here the line 197 plays role of the definition of expr to be estimated, line 198 introduces its derivative denoting it aux which is further undergone transformations ‘identical’ modulo Eq. (28), Eq. (30). The last two lines of the code exhibits its resulting representation:

```
output: The expression
output: aux
output: amounts to:
```
The dependence derived can be partially integrated by means of the extracting from the (value of) auxiliary variable $expr$ a series of appropriate expressions which, after the subtracting their corresponding derivatives from the value of $aux$ (equal to $\text{DR}(expr,\phi^-)$ modulo Eq. (28) and Eq. (30)), lead to the cancelling out certain terms of the latter. Specifically, the following instructions are to be issued:

```plaintext
input: evaluate
215 input: expr=expr -df(k,phi)* COEFFN(NUMR(aux),df(k,phi,phi^-))/DENM(aux),
       ... input: aux=DR(expr,phi^-),
       ... input: aux=aux-COEFFN(NUMR(aux),df(ems,phi^-),y)/DENM(aux)
       ... input: aux=aux-COEFFN(NUMR(aux),df(ems,phi^-),y)/DENM(aux)
       ... input: aux=aux-COEFFN(NUMR(aux),df(ems,phi^-),y)/DENM(aux)
       ... input: aux=aux-COEFFN(NUMR(aux),df(ems,phi^-),y)/DENM(aux)
222 input: *(LHS(EQ(29))-RHS(EQ(29))),
       input: aux;
```

Here the first application of the action ‘EVALUATE’ (lines 215–222) ‘integrates out’ the term proportional to $\text{df}(k,\phi)$ while the second application (lines 225–232) makes the same thing with the term proportional to $k$. The output of the last instruction looks as follows:

```plaintext
output: The expression
output: expr
output: amounts to:
239 output: al
... output: E *( - DF(ems,y)*y + ems)
... output: -1
242 output: + I *( - 2*DF(k,phi)*psi - DF(psi,phi)*k) {{}} 0.2 s.
output: The expression
output: aux - DR(-E**(al+al^-)*psi,x)
output: vanishes. {{}} 0.2 s.
```

Thus the derivative of the expression displayed in lines 239–242 with respect to $\phi^-$ equals the derivative of $-E**(al+al^-)*psi$ with respect to $x$. Hence we obtain the free ($y$-independent) term of $ems$ by a quadrature.

Now the amount of the new information deduced suffices to renew the problem specification.
3.3 Step 3: Accomplishing of reduction of the set of field equations

We modify the preceding problem specification in such a way to implement all the relationships obtained so far. We need now to

\[ \text{input: Declare SCALARS } k, R, Q(x, \phi, \phi^\sim), \]
\[ \text{input: } \psi, \beta(x, \phi), \psi^\sim(x, \phi^\sim), \]
\[ \text{input: } x(x, \phi, \phi^\sim), \]
\[ \text{input: } \rho(\phi, \phi^\sim), \]
\[ \text{input: } \lambda(y, Q, \psi, \psi^\sim, x, \phi, \phi^\sim), \]
\[ \text{input: } \zeta(y, Q, \rho, \psi, \psi^\sim, x, \phi, \phi^\sim), \]
\[ \text{input: } \mu(\rho, \psi, \psi^\sim, x, \phi, \phi^\sim), \]
\[ \text{input: } \text{ems0}(\rho, \psi, k, x, \phi, \phi^\sim), \]
\[ \text{input: } \text{ems1}(\rho, \psi, x, \phi); \]

In addition to the \textbf{SCALARS} used before and displayed in lines 75–77 some new ones are introduced. The collections of their ‘arguments’ are determined by the \textbf{VALUES} which has to be bound with them. The latter realize the relationships (equations) derived above. Specifically,

\[ \text{input: SCALAR VALUES follow:} \]
\[ \text{input: } \rho = 1 + \phi \cdot \phi^\sim / 4, \]
\[ \text{input: } \lambda = \text{DR}(P, \phi) \cdot y \cdot y^2 + y \cdot (\text{df}(Q, \phi) - 2i \cdot P \cdot \text{df}(k, \phi)), \]
\[ \text{input: } -i \cdot \text{df}(k, x, \phi) - i \cdot Q \cdot \text{df}(k, \phi) + \text{df}(R, \phi), \]
\[ \text{input: } \zeta = P \cdot y + \text{DR}(\text{al-}\text{al}^\sim, x) / 4, \]
\[ \text{input: } + Q / 2 + \text{df}(k, \phi, \phi^\sim) \cdot i / 2 / E^*(\text{al+al}^\sim), \]
\[ \text{input: } \mu = -\text{DR}(E^*(\text{al+al}^\sim), x) / 2 - i \cdot \text{df}(k, \phi, \phi^\sim), \]
\[ \text{input: } \text{ems1} = E^*(\text{al}^\sim) \cdot \text{df}(\psi, \phi), \]
\[ \text{input: } \text{ems0} = E^*(\text{al}^\sim) \cdot (-2i \cdot \psi \cdot \text{df}(k, \phi) - i \cdot k \cdot \text{df}(\psi, \phi)), \]
\[ \text{input: } -x_i - \text{df}(\beta, \phi) ; \]

As opposed to the interpretation followed to in the preceding steps, the symbols \text{al} (together with \text{al}^\sim) and \text{P} are now subsidiary objects. They are defined in accordance with the following statement:

\[ \text{input: } \text{ABBREVIATIONS comprise} \]
\[ \text{... input: } \text{al} = -\log(\rho) - \log(\psi), \]
\[ \text{... input: } \text{P} = -\psi \cdot \psi^\sim / 4, \]
\[ \text{268 input: } h = R + Q \cdot y + P \cdot y^2 ; \]

\textit{Cf.} lines 168, 193–195, 256. (Here \text{h} definition exactly replicates the one used above, see line 74).

The declaration of \textbf{REAL} objects (see line 78) is now to be replenished with the \textbf{SCALAR} \text{rho} while all the other new \textbf{SCALARS} are defined as complex (by means of the declaration ‘\textbf{COMPLEX CONJUGATED}’), being endowed with complex conjugated counterparts whose identifiers are marked with the character ‘\text{~}’ attached at right.

The specification of the basic geometric geometric objects — \textbf{TETRAD} and \textbf{UNDOTTED CONNECTION} — consists of the same formulae displayed in lines 6–10 and 83–84. As to the \textbf{UNDOTTED EM SPINOR}, its \textbf{component|1} looks as above while the \textbf{component|2} is endowed with the expansion revealing its dependence on \text{y}:

\[ \text{input: } \text{component|2} = \text{ems0} + y \cdot \text{ems1} ; \]

This line replaces line 87. In particular, the former \textbf{SCALAR} \text{ems} (depending on \text{y}) is now replaced by the two new ones, \text{ems0} and \text{ems1}, which are independent on \text{y}. 

25
The last — and novel — element of the problem specification is the following section of
input: Substitutions:
input: (1) lam->VAL(lam);
input: (2) zeta->VAL(zeta);
input: (3) mu->VAL(mu);
274 input: (4) rho->VAL(rho);
input: (5) ems1->VAL(ems1);
input: (6) ems0->VAL(ems0);
277 input: (7) df(xi,phi~)->DR(e**(al+al~)*psi,x);
input: (8) ems1~->VAL(ems1~);
input: (9) ems0~->VAL(ems0~);
input: end of substitutions.

These list of substitution rules will be utilized whenever necessary. All they but one (displayed in line 277) mean the replacing of the identifier of a SCALAR by the corresponding SCALAR VALUE (‘extracted’ by the macros VAL). The specific substitution situated in line 277 characterizes \( xi \) as the integral of the expression at right hand side with respect to the variable (complex COORDINATE) \( \phi^- \).

Now let us proceed with calculations.

At first, it is instructive to demonstrate how one can check that the current initial data verifies the part of field equations considered during the preceding steps of our calculation. This is carried out by means of the following instructions:
input: obtain the SECOND VERSION of the UNDOTTED FIRST CARTAN EQUATIONS,
input: UNDOTTED MAXWELL EQUATIONS,
input: and the SCALAR PART of EINSTEIN-MAXWELL EQUATIONS;
284 input: match substitution rules (1)-(4),(5)-(7) with
285 input: the ABOVE EQUATIONS;
286 input: renew and type
input: the ABOVE EQUATIONS;

These equations are satisfied not automatically but as a consequence of specific SCALAR VALUES given in the problem specification. Accordingly, their substituting realized by the instruction displayed in lines 284-285 is necessary. The output of the above action TYPE situated in line 286 says us that all the equations listed are satisfied. Thus to obtain the ‘complete’ solution of the problem it retains to
input: obtain the SPINOR PART of EINSTEIN-MAXWELL EQUATIONS
and ensure its fulfillment.

In this case, similarly to the above procedure, we also have to take into account the SCALAR VALUES given (evaluating derivatives of SCALARS, they are automatically taken into account). To that end, the following instructions are to be issued:
289 input: match substitution rules (1)-(3),(5)-(6),(8)-(9) with
... input: the ABOVE EQUATIONS;
291 input: isolate the ABOVE EQUATIONS;

The result of the line 291 is four differential EQUATIONS (marked with the ordinal numbers 1,2,3,4). The first of them is in fact trivial, being immediately fulfilled, provided the SCALAR VALUE for \( \rho \) (see line 256) is taken into account. Indeed, the instruction
input: Evaluate MATCHING(LHS(EQ(1))-RHS(EQ(1)),4);
reports zero result. (The substitution number 4 which is referred to here causes the replacing of the SCALAR rho by the corresponding VALUE, see lines 274,256.) On the contrary, a single Eq. (4) entails two more EQUATIONS (which prove to be non-trivial) since it means the vanishing of a linear function of y. Accordingly, the instruction

\textit{input: isolate}

\textit{input: EQUATION 0=COEFFN(LHS(EQ(4))-RHS(EQ(4)),y,1),}

\textit{input: EQUATION 0=COEFFN(LHS(EQ(4))-RHS(EQ(4)),y,0);}

constructs the EQUATIONS (their ordinal numbers are 5 and 6) which just express the vanishing of the first (the first line, the last parameter equals 1) and zero (the second line, the last parameter equals 0) order coefficients. The resulting collection of four EQUATIONS, which all the field equations have been reduced to, can be resolved with respect to derivatives of unknown SCALARS and displayed by means of the instructions

\textit{input: resolve EQUATIONS (2),(3),(5),(6) w.r.t.}

\textit{input: df(Q,phi~),df(Q,phi),df(R,phi,phi~),df(Q,phi,phi~);}

\textit{input: show new EQUATIONS;}

The output looks as follows:

\textit{output: Eq. (10) reads DF(Q,phi~)=(- 2*DF(k,phi,phi~,2)*I*psi*psi~*rho -}

\textit{the continuation dropped out consists of 13 non-empty lines}}

\textit{output: Eq. (9) reads DF(Q,phi)=}

\textit{output: \{\ldots\} 2*DF(k,phi,phi~)*DF(psi,phi)*I*psi*psi~*rho}

\textit{the continuation dropped out consists of 13 non-empty lines}}

\textit{304 output: Eq. (8) reads DF(R,phi,phi~)=}

\textit{output: \{\ldots\} 4*DF(k,phi,phi~,2)*DF(k,phi)*psi*psi~*rho}

\textit{the continuation dropped out consists of 59 non-empty lines}}

\textit{output: Eq. (7) reads DF(Q,phi,phi~)=(- DF(psi,x)*psi~}

\textit{output: \{\ldots\} + DF(psi,phi)*DF(beta~,phi~)*psi*psi~*rho}

\textit{the continuation dropped out consists of 9 non-empty lines}}

These EQUATIONS express the content of all the field equations which have not been satisfied so far. Their further reduction is now in order.

Specifically, we have three equations determining derivatives of the SCALAR Q. Thus they have to satisfy some consistency conditions. Calculating \(2*RHS(EQ(7))-DR(RHS(EQ(9)),\downarrow phi^-)-DR(RHS(EQ(10)),phi)\) (by means of the action EVALUATE), one obtains identical zero. This means that Eq. (7) does not yields a new information (and thus may be ERASEd). On the contrary, the condition of mutual consistency of Eq. (9) and Eq. (10) is nontrivial. We shall see it reduces in fact to the equation determining the SCALAR k. The optimal rout of its derivation is the following.

Let us introduce the auxiliary variable ‘expr’ which initially is assigned with the value Q. Then we make a series of additive transformations of expr. Simultaneously we shall
calculate another auxiliary variable $aux$ by the formula $aux=DR(expr,\phi)$, replacing here the derivative of $Q$ by its expression provided by Eq. (9). The corresponding instructions read:

```
input: evaluate expr=Q,
input:          expr=expr - df(k,phi,phi~)
input:          *COEFFN(NUMR(RHS(EQ(9))),df(k,phi,2,phi~))
input:          /DENM(RHS(EQ(9))),
input:          aux=DR(expr,phi)-(LHS(EQ(9))-RHS(EQ(9))),
input:          aux;
input: evaluate expr=expr - k*COEFFN(NUMR(aux),df(k,phi))/DENM(aux),
input:          aux=DR(expr,phi)-(LHS(EQ(9))-RHS(EQ(9))),
input:          aux;
input: evaluate expr=expr
input:          - df(psi,x)*COEFFN(NUMR(aux),df(psi,x,phi))/DENM(aux)
input:          - beta*COEFFN(NUMR(aux),df(beta,phi))/DENM(aux),
322 input:      aux=DR(expr,phi)-(LHS(EQ(9))-RHS(EQ(9))),
input:      aux;
```

The output of the last line reads:

```
output: The expression
output: aux
output: amounts to:
output: psi~*xi
output: ------{~~~} 0.6 s.
output: 2
```

In accordance with line 322 $aux$ equals modulo Eq. (9) the derivative of $expr$ with respect to $\phi$. This dependence can be integrated. Let us notice also that $expr$ possesses a complex value. The instructions

```
input: turn on the displaying of negative powers,
input: the support of complex numbers; 22
input: evaluate (expr+C.C.(expr))/2,(expr-C.C.(expr))/2/i;
```

yield the output:

```
output: The expression
output: (expr + C.C. (expr))/ 2
output: amounts to:
output: 1 1 -1 -1 1
output: -***DF(psi,x)*psi + -***DF(psi~,x)*psi~ + q - ***psi*beta~
output: 4 4 4
output: 1
output: - ---*beta*psi~
output: 4
output: {~~~} 0.2 s.
```

```
output: The expression
output: (expr - C.C. (expr))/ 2/ i
output: amounts to:
output: - DF(k,phi,phi~)*psi*psi~*rho - -***DF(psi,x)*psi
output: 4
```

---22 this one may be shortened to ‘turn on DIV,COMPLEX’
Thus after the integrating mentioned above, the separation of the real and imaginary parts of the equation obtained obviously yields explicit representation of \( Q \) and \( \text{DF}(k, \phi, \phi^*) \), respectively.

This result accomplishes the current (third) step of our calculations.

### 3.4 Step 4: Determination of \( R \)

It was shown above that unknown \( Q \) and the derivative \( \text{DF}(k, \phi, \phi^*) \) can be explicitly expressed in the form involving the integral of the SCALAR function \( \xi \) (depending on \( x, \phi, \phi^* \)) with respect to \( \phi \). It is reasonable therefore to introduce another SCALAR (we denote it \( \text{int}_\xi \)) whose derivative with respect to \( \phi \) equals \( \xi \). Another SCALAR independent of \( \phi \) arises as a ‘constant of integration’. We denote it \( \gamma^* \). \( \gamma^* \) may depend on \( x \) and \( \phi^* \). (Thus it represents the function \( \bar{\gamma}(x, \bar{\phi}) \) holomorphic with respect to \( \bar{\phi} \).)

Accordingly, the lines 247 and 248 of the declaration specifying SCALAR dependences now has to look as follows:

```plaintext
input: psi, beta(x, phi), gam^*, psi^*(x, phi^*),
```

while its other lines are not changed. Besides, the items ‘\( \text{gam} \) & \( \text{gam}^* \), \( \text{int}_\xi \) & \( \text{int}_\xi^* \)’ are added to the description of ‘COMPLEX CONJUGATED’ objects which now reads:

```plaintext
input: Declare COMPLEX CONJUGATED phi & phi^*, al & al^*, psi & psi^*,
```

Further, the list of ABBREVIATIONS (see lines 265–268) is replenished with two new items:

```plaintext
361 input: dd_k=-k/2/rho**2
... input: +i/(4 rho**2*psi*psi^*)
```

which represent the values of \( \text{DF}(k, \phi, \phi^*) \) and \( Q \), respectively, derived in the preceding subsection.

We shall use below the following

```plaintext
input: Substitutions:
```

We shall use below the following
Their meaning is manifest. (Remark: Let us remind that the macro ‘VAL’ returns the SCALAR VALUE of its argument, a SCALAR identifier.)

The aim of the present step of calculations is to express the last unknown SCALAR $R$ through the other SCALARS which may be regarded in this context as known functions. More precisely, we shall determine $df(R,\phi,\phi^-)$ (essentially, the 2-dimensional Laplacian of $R$), cf. line 304. $R$ is then obtained by means of a straightforward integrating (the solving of Poisson equation).

First of all, we deduce the equation determining $R$ from the Einstein-Maxwell equations by means of the executing appropriate substitutions of SCALAR VALUES. Specifically, the instructions

```plaintext
input: obtain the SPINOR PART OF EINSTEIN-MAXWELL EQUATIONS;
input: match substitution rules (12),(17),(18) with the ABOVE EQUATIONS;
input: renew the ABOVE EQUATIONS;
input: match substitution rules (15),(16) with the ABOVE EQUATIONS;
input: renew the ABOVE EQUATIONS;
input: match substitution rules (7),(77) with the ABOVE EQUATIONS;
input: renew the ABOVE EQUATIONS;
input: match substitution rules (2),(3),(5),(6),(22),(33),(55),(66) with the ABOVE EQUATIONS;
input: renew the ABOVE EQUATIONS;
input: match substitution rules (9),(10),(13),(14) with the ABOVE EQUATIONS;
input: renew the ABOVE EQUATIONS;
input: match substitution rules (15),(16),(8),(11) with the ABOVE EQUATIONS;
input: renew the ABOVE EQUATIONS;
input: renew the ABOVE EQUATIONS;
input: isolate the ABOVE EQUATIONS;
```
EQUATIONS. The first of them, Eq. (1), is in fact trivial. It is satisfied, provided the meaning of $\rho$ is taken into account (cf. the comment following lines 289–291). Concerning another one, it may be considered just as the equation determining $R$. The instructions

```plaintext
input: resolve Eq. (2) w.r.t. df(R,phi,phi~);
input: show new EQUATION;
```

yield in particular the following the output:

```plaintext
output: Eq. (3) reads DF(R,phi,phi~)=
```

```
```

(R.H.S. here involves neither $R$ nor its derivatives). The system of field equations in the representation described by the current problem specification entails no other relationships which involve SCALAR $R$.

Although Eq. (3) seems to be fairly involved, it is nevertheless quite tractable. We apply the method of the ‘partial integrating’ which was, essentially, already used above. The corresponding instructions are the following:

```plaintext
input: evaluate
```

```plaintext
input: expr=R,
input: aux=LHS(EQ(3))-RHS(EQ(3)),
422 input: expr=expr +k**2*COEFFN(COEFFN(NUMR(aux)
... input: ,df(k,phi)),df(k,phi~))
424 input: /DENM(aux)/2,
425 input: aux=DR(expr,phi,phi~)-(LHS(EQ(3))-RHS(EQ(3))),
input: aux=MATCHING(aux,8),
427 input: expr=expr-k*DF(psi~,x)
... input: *COEFFN(COEFFN(NUMR(aux),df(k,phi))
... input: ,df(psi~,x,phi~))/DENM(aux)
... input: -k*DF(psi,x)
... input: *COEFFN(COEFFN(NUMR(aux),df(k,phi~))
432 input: ,df(psi,x,phi))/DENM(aux),
433 input: aux=DR(expr,phi,phi~)-(LHS(EQ(3))-RHS(EQ(3))),
input: aux=MATCHING(aux,8),
435 input: expr=expr-k*psi~
... input: *COEFFN(COEFFN(NUMR(aux),df(k,phi))
... input: ,df(psi~,phi~))/DENM(aux)
... input: -k*psi
... input: *COEFFN(COEFFN(NUMR(aux),df(k,phi~))
440 input: ,df(psi,phi))/DENM(aux),
441 input: aux=DR(expr,phi,phi~)-(LHS(EQ(3))-RHS(EQ(3))),
input: aux=MATCHING(aux,8),
input: aux=MATCHING(aux,17,18),
input: aux=MATCHING(aux,15,16,7,77),
445 input: expr=expr-beta*beta~
... input: *COEFFN(COEFFN(NUMR(aux),df(beta,phi))
447 input: ,df(beta~,phi~))/DENM(aux),
448 input: aux=DR(expr,phi,phi~)-(LHS(EQ(3))-RHS(EQ(3))),
input: aux=MATCHING(aux,8),
input: aux=MATCHING(aux,17,18),
input: aux=MATCHING(aux,15,16,7,77),
input: aux=MATCHING(aux,19,20),
```
Their meaning is quite straightforward. Initially, the working variable $expr$ is endowed with the value $R$ (line 420). Further it is modified by means of the adding appropriate terms (lines 422–424, 427–432, 435–440, 445–447, 453–458) proportional to $k^2$, $k \cdot \text{DF}(\psi, x)$, $k \cdot \text{DF}(\psi, \psi)$, etc. The necessary coefficients are determined ‘automatically’, utilizing the value of the another working variable $aux$. The latter is each time equal modulo Eq. (3) to the second order derivative $\text{DR}(expr, \phi, \phi)$ of the current value of $expr$, (lines 425, 441, 448, 459). During these transformations the properties (in fact, definitions) of various variables expressed in the form of substitution rules (see above the section of Substitutions) are taken into account. (These are the instructions which involve the macro ‘MATCHING’.) In particular, after the execution of the instruction displayed in line 462, $aux$ still equals the derivative $\text{DR}(expr, \phi, \phi)$ modulo Eq. (3). Moreover, it is easy to see that this relationship just expresses (in a modified but equivalent form) the essence of Eq. (3).

Ultimately, the instructions

- turn on the displaying of negative powers;
- evaluate $aux, expr$ factoring out $R, k, \text{DF}(\psi, x), \text{DF}(\psi, \psi)$;

entails the following output:

The expression $aux$ amounts to:

$$
2 -3 -1 -2
$$

- $\text{DF}(\psi, x) \cdot \psi \cdot \psi^* \cdot \rho$

$$
1 -2 -2 -2
$$

- $\text{DF}(\psi, x) \cdot \text{DF}(\psi, \psi) \cdot \psi \cdot \psi^* \cdot \rho$

$$
2
$$

+ $\text{DF}(\psi, x) \cdot \psi \cdot \rho \cdot (\psi + \psi^* + \phi + \phi^*)$

$$
4
$$

- $\text{DF}(\psi, \psi) \cdot \psi \cdot \psi^* \cdot \rho$

$$
1 -2 -2 -2
$$

+ $\text{DF}(\psi, \psi) \cdot \psi \cdot \rho \cdot (\psi + \psi^* + \phi + \phi^*)$

$$
4
$$

- $\text{DF}(\psi, 2) \cdot \psi \cdot \psi^* \cdot \rho$

$$
2 -1 -2 -2 1
$$

+ $\text{DF}(\psi, 2) \cdot \psi \cdot \rho \cdot (-\psi - \psi^* + \phi + \phi^*)$

$$
4
$$

Entails the following output:

$$
\approx 0.4 \text{ s.}
$$
The expression amounts to:
\[ R + \frac{1}{4} k \psi \psi - \frac{1}{4} k \Delta (\psi) \psi \]
\[ + \frac{1}{4} k \Delta (\psi) \psi + k I (\psi \beta + \psi \gamma) \]
\[ + \psi \int_\xi - \beta \psi - \gamma \psi \]
\[ - \psi \int \xi + (\beta \beta + \beta \gamma) \]
\[ \approx 0.2 \text{ s.} \]

In view of all said above the flat Laplacian of \( expr \) equals \( aux \) which may be considered as a known function. This relation is easily integrable.

It is evident that the result deduced accomplishes the solving of the problem considered.

### 3.5 The solution

Let us summarize the above results re-casting the formulary input data and computer output to standard mathematical notations. Namely, we found a generic solution of Einstein–Maxwell equations which complies with Eq. (15). Its metric is described by the expansion (5) where the tetrad of 1-forms \( \theta^a \) is defined as follows (cf. [9], Eq. (27.46)):

\[
\begin{align*}
\theta^0 &= \rho^{-1} \psi^{-1} d\phi / \sqrt{2}, \quad \theta^1 = \theta^0, \quad \theta^2 = dx / \sqrt{2}, \\
\theta^3 &= (dy - 2 \text{Im}(k, \phi) d\phi) + (R + Q y - \frac{1}{4} |\psi|^2 y^2) dx / \sqrt{2}
\end{align*}
\]

while the non-zero components of the undotted spinor of the electromagnetic field comprise

\[
\phi_1 = \psi \quad \text{and} \quad \phi_2 = \rho \psi (-S + \psi, \phi, y).
\]

Here \( x, y \) are the real coordinates, \( \phi \) is the complex one (whose complex conjugated counterpart \( \bar{\phi} \) is considered as independent variable), the other symbols denote their functions. Specifically,

\[
\begin{align*}
\rho &= 1 + \frac{1}{4} |\phi|^2, \\
S &= \xi + \beta, \psi + 2i \psi k, \phi + ik \psi, \\
R &= D - \frac{1}{4} (|\beta|^2 + k^2 |\psi|^2) \\
&+ 2 \text{Re} (\nu + ik (\psi^{-1} \psi, x - \psi (\bar{\Psi} + \gamma + \bar{\beta}) - \beta (\bar{\Psi} + \gamma))) , \\
Q &= \frac{1}{2} \text{Re} (\psi^{-1} \psi, x - \psi (\bar{\Psi} + \gamma + \bar{\beta})).
\end{align*}
\]
Here real $D = D(x, \phi, \bar{\phi})$ and $k = k(x, \phi, \bar{\phi})$ satisfy the equations

$$
D_{\phi \bar{\phi}} = -\frac{1}{2}\rho^{-2}|(\psi^{-1})_{,x}|^2 - \frac{1}{4}|\xi|^2
- \frac{1}{2}\rho^{-2} \text{Re} \left( 2 \bar{\psi}^{-1} (\psi^{-1})_{,xx} + (\psi^{-1})_{,x} (\Upsilon + \bar{\Upsilon}) \right),
$$

$$
k_{\phi \bar{\phi}} + \frac{1}{2}\rho^{-2}k = -\frac{1}{2}\rho^{-2}|\psi|^{-2} \text{Im} \left( \psi^{-1} \psi_{,x} + \psi \left( \bar{\Upsilon} + \gamma + \bar{\beta} \right) \right). \tag{34}
$$

Complex functions $\xi = \xi(x, \phi, \bar{\phi})$, $\Upsilon = \Upsilon(x, \phi, \bar{\phi})$ are defined according to the following relationships:

$$
\xi_{,\phi} = \rho^{-2} (\bar{\psi}^{-1})_{,x}, \quad \Upsilon_{,\phi} = \xi. \tag{35}
$$

The functions $\psi = \psi(x, \phi) \neq 0$, $\beta = \beta(x, \phi)$, $\gamma = \gamma(x, \phi)$, $\nu = \nu(x, \phi)$ are arbitrary. They are assumed to be holomorphic with respect to the second (complex) argument (and smooth with respect to the first one).

It is worth mentioning that although the description of the solution presented involves some unsolved equations, all they are solved by means of subsequent quadratures. Indeed, there are three kinds of equations which, dropping out irrelevant complicating details, look as follows:

$$
(*) \quad \frac{\partial f(z, \bar{z})}{\partial z} = g(z, \bar{z}), \quad (**) \quad \frac{\partial^2 f(z, \bar{z})}{\partial z \partial \bar{z}} = g(z, \bar{z}), \quad (***) \quad (1 + z\bar{z})^2 \frac{\partial^2 f(z, \bar{z})}{\partial z \partial \bar{z}} + 2f = g(z, \bar{z}).
$$

Here $g = g(z_1, z_2)$ denotes some known function holomorphic with respect to the both arguments, i.e. admitting a holomorphic extension to some neighborhood of the ‘shell’ $\{z_1 = \bar{z}_2\} \subset \mathbb{C}^2$. Local solutions $f$ of these equations share the latter property. They can be represented in explicit form in terms of quadratures as follows:

$$
(*) \quad f = z \int_0^1 dt \, g(tz, \bar{z}), \quad (**) \quad f = z\bar{z} \int_0^1 dt_1 \int_0^1 dt_2 \, g(t_1z, t_2\bar{z}),
$$

$$
(***) \quad f = z\bar{z} \int_0^1 dt_1 \int_0^1 dt_2 \, \left( 1 - \frac{2z\bar{z}(1-t_1)(1-t_2)}{(1+z\bar{z})(1+z\bar{z}t_1t_2)} \right) g(t_1z, t_2\bar{z})
$$

$$
+ \text{Re} \left( \frac{dj(z)}{dz} - \frac{2zj(z)}{1+z\bar{z}} \right),
$$

the last formula (local general solution of inhomogeneous Helmholtz equation on a sphere) being of certain independent interest. Here $j = j(z)$ is an arbitrary holomorphic function playing role of ‘integration constant’. It is worth noting that the second term in (***) represents itself the general local solution of the homogeneous version ($g = 0$) of Eq. (***) and thus (***) is its general local solution in non-homogeneous case. In the formulae (*), (**) the ‘integration constants’ similar to $j(z)$ are dropped out since they had been explicitly introduced in the above Eqs. (28–35) describing the solution of the field equations.
This is just the origin of the arising of the arbitrary functions \( \nu, \gamma, \beta \). On the contrary, the arbitrary function which arises when integrating Eq. (34) (equivalent to (\(*\ast\ast\ast\)) with respect to \( k \) was not explicitly introduced, being implicitly involved in (the value of) \( k \). It is the fifth arbitrary function (the fourth one is \( \psi \)).

Thus, in total, the electrovac solution obtained involves \textit{five arbitrary functions of two variables} holomorphic with respect to one of them.

Concerning the aspect of the physical interpretation of the field configuration described by equations (28–35), one should take into account that a huge ‘amount’ of degrees of freedom the family of solutions possesses makes fairly difficult, if not impossible, to reveal and formulate it in a full generality. Nevertheless one may gain some insight into the problem analyzing suitable particular cases (subfamilies) of the field configurations described by the above equations. That way, a useful simple example is provided by the following their representative (cf. [9], Eq. (27.54)):

\[
\begin{align*}
\mathbf{g} &= dx \otimes (dy + (\text{Re} \nu(x, \phi) - |\psi(\phi)|^2 y^2) \, dx) + \frac{d\phi \otimes d\bar{\phi}}{|\psi(\phi)|^2 (1 + |\phi|^2)^2}, \\
\omega &= -d(\psi y) + \frac{d\phi \wedge d\bar{\phi}}{\psi (1 + |\phi|^2)^2}.
\end{align*}
\]

(Here, with regard to Eqs. (28–35), a minor modification of the gauge and some obvious elementary re-definitions were carried out). The arbitrary functions \( \psi(\phi), \nu(x, \phi) \) are holomorphic with respect to \( \phi \).

As opposed to the general case, the meaning of the latter field configuration is immediately manifest: it represents the generalized plane wave spread against the Bertotti–Robinson space-time [13] describing itself non-null homogeneous static electromagnetic field. Indeed, the Bertotti-Robinson field configuration arises in the case \( \nu = 0, \psi = \text{constant} \neq 0 \). On the other hand an appropriate limiting procedure corresponding to the nullifying the static electric (magnetic) field yields the \( pp \)-wave metric.

4 Summary and discussion

Summarizing, the paper presented exhibits an example demonstrating the application of the specialized computer algebra system \textsc{grgec} to the problem of the searching for solutions to the coupled Maxwell and Einstein–Maxwell equations. We investigated their general class complying ansatz (15). The latter equation imposes certain restrictions on the irreducible curvature spinors which give evidence (basing first of all on important particular cases) for the associating the configurations under consideration with radiation processes.

Reassigning all the routine calculations to \textsc{grgec}, we found a generic solution of the problem which turns out to involve five arbitrary functions of two arguments (see subsection 3.5). In agreement, with \textit{a priori} expectations its particular case can be interpreted as \( pp \)-wave spread against a background Bertotti-Robinson space-time, see Refs. [13], [9].

35
circumstance partially justifies the working term ‘radiative electrovacs’ (perhaps too wide in the context considered) which was used above for the referring to the class of space-times investigated.

The complete problem specification (GRGEC input code) which encodes the formulae (28–35) from subsection 3.5 and represents the solution found is displayed in Appendix A. Appendix B displays the corresponding GRGEC output which proves that we actually deal with a solution of the relevant field equations.

As it has been mentioned, the prevailing aim pursued in the present work is the characterizing of GRGEC system from viewpoint of its suitability for a practical application. Additionally, we simultaneously attempted, mostly after indirect fashion, to exhibit possible styles of its usage and to demonstrate some of its capabilities. In particular, this is one of the purposes the inclusion of section 3 which discusses the processing of the problem in the complete form.

It is instructive to mention that the treatment of the problem considered in subsections 2.2, 2.3 began immediately with consideration of a workable GRGEC program (we called it problem specification, see subsection 3.1). In particular we did not need, essentially, to preliminarily learn GRGEC input language and data structures to a notable extent, restricting ourselves to several remarks. At the same time it should be mentioned that, in principle, GRGEC is a fairly complex system whose complete description occupies hundreds of pages. One can estimate therefore a prominent clarity of the organization of GRGEC language which enables one to comprehend the essence of GRGEC input without becoming absorbed into specific programming-related issues. The content of Appendix A, where an integrated problem specification is exhibited, provides a nice demonstration of the latter circumstance.

To be more specific, one may distinguish, in principle, the two kinds of GRGEC code. The first one mostly follows a traditional imperative style of well-known programming languages. Typical examples of such a sort scripts are displayed in lines 215–222, 361–366, etc. Generally speaking, the instructions contained in these fragments of GRGEC script realize, in a sense, ‘low level’ mathematical calculations handling symbolic formulae. Their evident characteristic property is, as usually, a high degree of a detailed petty control by a user depending on intermediate results required. It should be noted however that GRGEC provides a rather limited set of facilities intended for such a ‘lower level’ algebraic programming. For example, no loops, conditional and branching operators, subroutines, etc., are supported. The point is that, though it might seem surprising, they are superfluous here in fact.

A foundation (and, hopefully, a source of potential advantages) of GRGEC lives in

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23For the sake to provide a performance scale for the timing labels shown in the output protocol (as well as for incidental ones occurred above in the fragments of output listings reproduced), we note that the IBM/PC compatible computer with AMD386DX 40 MHz CPU and 8 Mb of RAM was used. A fiducial point characterizing its (fairly moderate) performance rate is provided by the time of the expanding of \((a + b + c)^{100}\) by Reduce (without output). The corresponding script reads `\texttt{show time$(A+B+C)^{100}\text{showtime;}$}'. It reports about 87 seconds of CPU labour.

24The developing of new programs requires more detailed knowledge of course.
a distinct area. Essentially, in GRGEC the control over majority of calculations is based mostly on actions of a maximally high level, approaching the one which might be attributed to the relationships and notions characteristic of the application field (the geometry and the field theory) itself. As examples of the corresponding programming (a code of the ‘second’ kind) one could get the instructions displayed in lines 49–51 or 91,92,93. The series of instructions displayed below (having no relation to the problem considered in the present work) also serves an illustration of such a ‘super-high level’ coding:

```
input: find UNDOTTED WEYL SPINOR and compare it with sample;
input: classify the ABOVE STUFF;
input: calculate factorizing denominators UNDOTTED WEYL INVARIANTS
input: and write them to a disk file;
```

Illuminating the grounds of the approach utilized, one of the cornerstones of a ‘knowledge’ implemented in GRGEC and concerning the mathematical and physical theoretical issues is the collection of so called data objects which model the basic notions originated from the geometry and the field theory. In the present work, we dealt with a fairly moderate part of them which included TETRAD, UNDOTTED CONNECTION, UNDOTTED EM SPINOR, and some others. A full family of data objects implemented so far is rather numerous (although still not exhaustive) and also includes such entities as, for example, the DATA SAVED (to a disk storage), ALL the TRACES OF ENERGY-MOMENTA (known in the current point of calculation), COVARIANT DIFFERENTIALS OF WEYL 2-FORMS, and even DUPLICATED CHARGED DIRAC DERIVATIVE OF UNDOTTED DIRAC PHI-SPINOR. These are export names of data objects which are used for the referring to them in instructions. (There is also possibilities of the access to separate components, if any, of a data object. The corresponding examples can be found in lines 83, 84, 86, 87 and others.) In order to determine the value of some data object one should apply one of the actions ‘ELICIT’ (from a DATA included in the problem specification), ‘CALCULATE’ (from the other data objects), ‘FIND’ (i.e. partially ‘ELICIT’ and partially ‘CALCULATE’), ‘OBTAIN’ (used for equations). Depending on the current state of environment, the realization of such actions can be rather sophisticated invoking, in particular, (the models of) various relationships originated from the physical theory and geometry. Further, given the value of a data object, one may use for its transformation various substitution rules applying them either globally (by means of the actions EXCITE or LET) or locally (action MATCH). The substitutions with the help of Reduce’ package COMPACT are realised by the action MATCH COMPACTIFYING. The Reduce’ polynomial factorizer is invoked by means of the action RENEW FACTORIZING’. Alternatively, one may ‘RENEW FACTORIZING NUMERATORS’ or ‘RENEW FACTORIZING DENOMINATORS’. The Reduce’ routine performing decomposition of rational functions into simple fractions is invoked by means of the instruction ‘FRACTIONATE ⟨data_object⟩ WITH RESPECT TO25 (kernel)26; (or ‘FRACTIONATE CONDENSING . . . ’ which additionally re-arranges simple denominators into factors-SCALARS), etc.

Generally speaking, we have listed major part of actions intended for the ‘active’ handling of data objects. One sees that, in principle, the controlling facilities supported by

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25 These three ‘keywords’ may be shortened to ‘w.r.t.’.
26 This term closely corresponds to the notion of a kernel introduced and supported in Reduce.
GRGEC are in no way intricate while they reveal no potential limitations on the complexity of the calculation processes underlaid. It may be stated that GRGEC successively follows a tendency to ensure a maximal clarity and naturalness of the applied code.

It is also worth noting that GRGEC usually does not advertise about the ways of the realizing of the instructions performed. The listing displayed in Appendix B is an instructive illustration of the latter circumstance\textsuperscript{27}. Indeed, one finds no formulae in the output issued by the instructions executed. Nevertheless the main result — the confirmation of the satisfaction of the relevant equations — is clearly exhibited. (If one wants to be informed in more details, an additional instruction to ‘\texttt{type ALL KNOWN}’ would give rise to a lot of formulae.)

Similarly, as it was mentioned, the basic mathematical formalism which is used for majority of calculations discussed in the present work is the calculus of exterior forms\textsuperscript{28}. However, this circumstance nowhere explicitly manifests itself (except perhaps of instruction 38 and lines 43, 46 of output). This may be interpreted as a particular manifestation of the intention to release a user from superfluous details.

At the same time the implementation of exterior calculus (together with the standard tensor methods) constitutes itself a foundation of the library of next level routines realizing geometrical and physical theoretical relationships. These two interconnected underlaid ‘strata’ of GRGEC system are relatively independent from their ‘manager’ — the interpreter. On the other hand the narrow application field of programming system is determined mostly by the library of applied routines (and, to a less extent, by the basic mathematical tools implemented). Thus, having substituted another applied library (which should support an appropriate collection of data objects), one obtains the system with another application field which is controlled, essentially, in the same way. Thus the approach to the integrating of the constituents of GRGEC outlined possesses a flexibility which makes GRGEC a valuable subject of a practical interest worthing a further working up.

Resuming, it may be stated that GRGEC system can be successfully applied for practical calculations in the field of gravitation theory, demonstrating a high efficiency and excellent convenience in an application. It can be also estimated as a promising base for the further development of the efficient tools for doing computer analysis of a wide scope of problems in the field of theoretical physics.

References

[1] S.I. Tertychniy, I.G. Obukhova, \textit{SIGSAM Bulletin} \textbf{31}, (1997) 6.

[2] M.A.H. MacCallum, in: \textit{Recent developments in gravitation and mathematical physics}, (World Sci. Publishing, River Edge, NJ, 1996) p. 3.

\textsuperscript{27} A copy of output directed to a monitor screen is there presented. GRGEC additionally stores the output listing which contains somewhat more detailed information to a disk file.

\textsuperscript{28} GRGEC uses its own implementation independent of the Reduce\textsuperscript{\texttt{\textregistered}} sub-package \texttt{Excalc}. 
Appendix A

Here an example of the problem specification (input code processed by GRGEC) is displayed. It checks the fulfillment of the relevant field equations by the solution described in section 3.5.

Problem Electrovac Metric.

Data:

```plaintext
declare COORDINATES x,y,phi,phi~;
declare SCALARS psi,beta,gam,nu(x,phi),
rho(phi,phi~),
k,xi,int_xi,int_del(x,phi,phi~);
declare REAL x,y,k,rho,del,int_del;
declare COMPLEX CONJUGATED phi & phi~,
    psi & psi~,beta & beta~,gam & gam~,
    nu & nu~,
    xi & xi~,
    int_xi & int_xi~;
SCALAR VALUE follows: rho=1+phi*phi~/4;
ABBREVIATIONS comprise
    emS= xi +df(beta,phi) +2i*psi*df(k,phi) +i*k*df(psi,phi),
    dd_k=-k/rho**2/2
```
\[ +\frac{1}{4} i \rho \psi^2 \]
\[ - \frac{1}{4} \text{C.C.}(\text{aux}), \text{WHERE} \]
\[ \text{aux} = \text{df}(\psi, x)/\psi + \psi(\beta^{\ast} + \text{int}_{\xi} + \gamma), \]
\[ Q = \frac{1}{4} \text{C.C.}(\text{aux}), \text{WHERE} \]
\[ \text{aux} = -\text{df}(\psi, x)/\psi + \psi(\beta^{\ast} + \text{int}_{\xi} + \gamma), \]
\[ R = \int_{\text{del}} - (\beta\beta^{\ast} + k^{2} \psi \psi^{\ast})/4 \]
\[ + \text{C.C.}(\text{aux}), \text{WHERE} \]
\[ \text{aux} = \nu + i k(\text{df}(\psi, x)/\psi - \psi(\beta^{\ast} + \text{int}_{\xi} + \gamma)) \]
\[ - \beta(\text{int}_{\xi} + \gamma)/4, \]
\[ \text{del} = -\text{DR}(1/\psi, x)\text{DR}(1/\psi^{\ast}, x)/\rho^{2}/2 - \xi^{2}/4 \]
\[ - \text{C.C.}(\text{aux}), \text{WHERE} \]
\[ \text{aux} = 2 \text{DR}(1/\psi, x, 2)/\psi^{\ast} \]
\[ + \text{DR}(1/\psi, x)(\text{int}_{\xi} + \gamma)/\rho^{2}/2; \]
\[ \text{TETRAD} \text{ comprises} \]
\[ \text{component}[0] = \text{df}(\psi, x)/\rho \sqrt{2}, \]
\[ \text{component}[1] = \text{C.C.}(\text{component}[0]), \]
\[ \text{component}[2] = \text{df}(\psi, x)/\rho \sqrt{2}, \]
\[ \text{component}[3] = (\text{df}(\psi, x)/\rho \sqrt{2}) d x)/\rho \sqrt{2}; \]
\[ \text{UNDOTTED EM SPINOR} \text{ comprises} \]
\[ \text{component}[1] = \psi, \]
\[ \text{component}[2] = \rho \psi(-\text{emS} + y \text{df}(\psi, \phi)); \]
\[ \text{DOTTED EM SPINOR} \text{ is HERMITEAN CONJUGATED to UNDOTTED EM SPINOR}; \]
end of data.

Substitutions:

(4) \( \rho \rightarrow \text{VAL}(\rho) \);
(7) \( \text{df}(\xi, \phi^{\ast}) \rightarrow \text{DR}(1/\psi^{\ast}, x)/\rho^{2}; \)
(77) \( \text{df}(\xi^{\ast}, \phi) \rightarrow \text{DR}(1/\psi, x)/\rho^{2}; \)
(8) \( \text{df}(k, \phi, \phi^{\ast}) \rightarrow \text{dd}_{k}; \)
(9) \( \text{df}(k, \phi, 2, \phi^{\ast}) \rightarrow \text{DR}(\text{dd}_{k}, \phi); \)
(10) \( \text{df}(k, \phi, \phi^{\ast}, 2) \rightarrow \text{DR}(\text{dd}_{k}, \phi^{\ast}); \)
(15) \( \text{df}(\text{int}_{\xi}, \phi) \rightarrow \text{x}_{i}; \)
(16) \( \text{df}(\text{int}_{\xi^{\ast}}, \phi^{\ast}) \rightarrow \text{x}_{i^{\ast}}; \)
(17) \( \text{df}(\text{int}_{\xi}, \phi, \phi^{\ast}) \rightarrow \text{df}(\xi, \phi^{\ast}); \)
(18) \( \text{df}(\text{int}_{\xi^{\ast}}, \phi^{\ast}, \phi) \rightarrow \text{df}(\xi^{\ast}, \phi); \)
(19) \( \text{df}(\text{int}_{\xi}, \phi, 2) \rightarrow \text{df}(\xi, \phi); \)
(100) \( \text{df}(\text{int}_{\text{del}}, \phi, \phi^{\ast}) \rightarrow \text{del}; \)
end of substitutions.

Instructions:

obtain UNDOTTED MAXWELL EQUATIONS;
match substitution rule (7) with the ABOVE EQUATIONS;
obtain EINSTEIN-MAXWELL EQUATIONS;
match substitution rule (4) with the SCALAR PART of EINSTEIN-MAXWELL EQUATIONS;
match substitution rules (9),(10),(17),(18) with the SPINOR PART of EINSTEIN-MAXWELL EQUATIONS;
renew the ABOVE EQUATIONS;
match substitution rules (7),(77),(15),(16) with the ABOVE EQUATIONS;
renew the ABOVE EQUATIONS;
match substitution rules (4),(100) with the ABOVE EQUATIONS;
renew and type ALL the EQUATIONS;
Appendix B

The output of the processing by GRG\textsubscript{EC} the code displayed in Appendix A is given. (Some initialization messages including the copy of the input script are dropped out.).

Total time spent amounts to 4.5 seconds.

----  Processing of the problem ‘Electrovac Metric’  ----

COORDINATES are listed below:
\[x, y, \phi, \phi^*\]

SCALARS dependences are shown below:
- \(\nu\): \((x, \phi, \phi^*)\) (is added as C.C. of the SCALAR \(\nu\))
- \(\gamma\): \((x, \phi, \phi^*)\) (is added as C.C. of the SCALAR \(\gamma\))
- \(\beta\): \((x, \phi, \phi^*)\) (is added as C.C. of the SCALAR \(\beta\))
- \(\psi\): \((x, \phi, \phi^*)\) (is added as C.C. of the SCALAR \(\psi\))
- \(\rho\): \((\phi, \phi^*)\) (is added as C.C. of the SCALAR \(\rho\))
- \(\int_{\phi} \phi^*\): \((x, \phi, \phi^*)\) (is added as C.C. of the SCALAR \(\int_{\phi} \phi^*\))
- \(\xi\): \((x, \phi, \phi^*)\) (is added as C.C. of the SCALAR \(\xi\))
- \(\kappa\): \((x, \phi, \phi^*)\) (is added as C.C. of the SCALAR \(\kappa\))

No unrealized abbreviations have been specified...

Abbreviations processed are listed below:
- \(\text{emS, dd}_k\)
- \(Q, R, \text{del}\)

SCALAR VALUE is shown below:
\[
\phi \phi^* + 4
\]
\[
\frac{\rho}{4}
\]

No unrealized form abbreviations have been specified...

No form abbreviations have been specified...

Total time spent amounts to 6.1 seconds.

** The instructions given will be executed now **

===> obtain UNDOTTED MAXWELL EQUATIONS
...UNDOTTED MAXWELL EQUATIONS have been obtained \(\langle\langle\langle 0.6 \text{ s.}\)

Total time spent amounts to 6.7 seconds.

===> match substitution rule (7) with the ABOVE EQUATIONS
UNDOTTED MAXWELL EQUATIONS have been processed \(\langle\langle\langle 0.1 \text{ s.}\)

===> obtain EINSTEIN - MAXWELL EQUATIONS
...SPINOR PART OF EINSTEIN - MAXWELL EQUATIONS has been obtained \(\langle\langle\langle 6.4 \text{ s.}\)

...SCALAR PART OF EINSTEIN - MAXWELL EQUATIONS has been obtained \(\langle\langle\langle 6.7 \text{ s.}\)
Total time spent amounts to 13.4 seconds.

==> match substitution rule (4) with the SCALAR PART of EINSTEIN - MAXWELL EQUATIONS
SCALAR PART OF EINSTEIN - MAXWELL EQUATIONS has been processed \( \sim 0.2 \) s.

==> match substitution rules (9), (10), (17), (18) with the SPINOR PART of EINSTEIN - MAXWELL EQUATIONS
SPINOR PART OF EINSTEIN - MAXWELL EQUATIONS has been processed \( \sim 2.6 \) s.

==> renew the ABOVE EQUATIONS
SPINOR PART OF EINSTEIN - MAXWELL EQUATIONS has been renewed \( \sim 3.3 \) s.

==> match substitution rules (7), (77), (8), (15), (16) with the ABOVE EQUATIONS
SPINOR PART OF EINSTEIN - MAXWELL EQUATIONS has been processed \( \sim 5.7 \) s.

==> renew the ABOVE EQUATIONS
SPINOR PART OF EINSTEIN - MAXWELL EQUATIONS has been renewed \( \sim 5.9 \) s.

==> match substitution rules (4), (100) with the ABOVE EQUATIONS
SPINOR PART OF EINSTEIN - MAXWELL EQUATIONS has been processed \( \sim 6.4 \) s.

==> renew and type ALL the EQUATIONS
--> RENEW ALL the EQUATIONS
SPINOR PART OF EINSTEIN - MAXWELL EQUATIONS has been renewed \( \sim 6.5 \) s.
SCALAR PART OF EINSTEIN - MAXWELL EQUATIONS has been renewed \( \sim 6.7 \) s.
UNDOTTED MAXWELL EQUATIONS have been renewed \( \sim 6.7 \) s.

--> TYPE ALL the EQUATIONS
SPINOR PART OF EINSTEIN - MAXWELL EQUATIONS is satisfied
SCALAR PART OF EINSTEIN - MAXWELL EQUATIONS is satisfied
UNDOTTED MAXWELL EQUATIONS are satisfied

==> quit
Total time spent amounts to 20.3 seconds, garbage collection consumed 137 ms.
The DISK file for the copying is being closed... ...done
Quitting