LOWER BOUNDS FOR BLOW-UP IN A PARABOLIC-PARABOLIC KELLER-SEGEL SYSTEM

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ABSTRACT. This paper deals with a parabolic-parabolic Keller-Segel system, modeling chemotaxis, with time dependent coefficients. We consider non-negative solutions of the system which blow up in finite time $t^*$ and an explicit lower bound for $t^*$ is derived under sufficient conditions on the coefficients and the spatial domain.

1. Introduction. In 1970 Keller and Segel (see [6]) derived a mathematical model of a chemotaxis process by using a system of two parabolic equations, whose solutions describe the movement of cells in response to a presence of a chemical signal substance, with a non-homogeneous space distribution. Let $u(x,t)$ represent the cell density and $v(x,t)$ the concentration of the chemoattractant, then we have the following system

\[
\begin{aligned}
    u_t &= \Delta u - \chi \text{div}(u \nabla v), \quad (x,t) \in \Omega \times (0,\infty), \\
    \tau v_t &= D \Delta v - a v + b u, \quad (x,t) \in \Omega \times (0,\infty), \\
    u_v &= v_v = 0, \quad (x,t) \in \partial \Omega \times (0,\infty), \\
    u &= u_0(x) \geq 0 \quad v \equiv v_0(x) \geq 0, \quad x \in \Omega,
\end{aligned}
\]

(1)

where $\Omega$ is a bounded domain with smooth boundary, $u_v$ (i.e. $v_v$) is the derivative of $u$ (i.e. $v$) with respect the unit normal $v$ to $\partial \Omega$ directed outward. The coefficients $\chi, \tau, D, a, b$ are positive constants.

One interesting question is the "chemotactic collapse": from mathematical point of view it corresponds to the blow-up in finite time to solutions of system (1). There has been a large interest Keller-Segel type system with bounded or unbounded solutions: we refer to [2], [3], [4],[5], [16], [17], [18] and [20] and the references therein.

Other interesting results concerning blowing up solutions for other general parabolic problems, both equations and systems, are available in [7], [8], [9], [10],[11],[13] and [19].

We point out that Payne and Song in [15] derive a lower bound for the blow up time to solutions of system (1) when $\Omega$ is a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. They achieve such estimate by introducing a suitable auxiliary function satisfying a first order differential inequality. Additionally, in [1] the authors provide a numerical method by means of which, approximations for the blow-up time of solutions to (1) can be obtained.

On the other hand, natural observations and practical experiences show how in specific circumstances the parameters modeling the chemotaxis phenomena can also change in time.

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It justifies our interest to study a different model, where the positive coefficients in (1) are herein replaced by positive time dependent coefficients.

Specifically we consider the following problem

\[
\begin{align*}
  u_t &= \Delta u - k_1(t) \text{div}(u \nabla v), \quad (x, t) \in \Omega \times (0, t^*), \\
  v_t &= k_2(t) \Delta v - k_3(t) v + k_4(t) u, \quad (x, t) \in \Omega \times (0, t^*), \\
  u_\nu &= v_\nu = 0, \quad (x, t) \in \partial \Omega \times (0, t^*), \\
  u &= u_0(x) \geq 0, \quad v = v_0(x) \geq 0, \quad x \in \Omega,
\end{align*}
\]

where \( t^* \) is the blow-up time \((0 < t^* < \infty)\), \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with smooth boundary \( \partial \Omega \), \( u_\nu \) is the normal derivative on \( \partial \Omega \). The coefficients \( k_i(t) \) \((i = 1, 2, 3, 4)\) are positive and regular functions of \( t \), \( u_0(x) \) and \( v_0(x) \) are assumed non-negative on \( \Omega \) satisfying the compatibility conditions on \( \partial \Omega \).

System (2) represents the following situation: the chemoattractant spreads diffusively and decays with rate \( k_3(t) \) and \( k_3(t) \), respectively; it is also produced by the bacteria with rate \( k_4(t) \). The bacteria diffuse with mobility 1 and also drift in the direction of the gradient of concentration of the chemoattractant with velocity \( k_1(t) \nabla v \); \( k_1(t) \) is called chemosensitivity. Moreover, the Neumann boundary conditions mean that no flux with the external boundary is permitted.

The object of this paper is to study solutions of (2) which blow up in finite time \( t^* \). It is well known that when blow-up occurs at \( t^* \), explicit estimates are of a great practical interest, since, mostly, it is not possible an exact (accurate) computation of \( t^* \). More precisely, we derive sufficient conditions on the data in order to obtain an explicit lower bound for \( t^* \).

2. Preliminaries and main results. As remarked in the introduction our aim is to derive a lower bound (possible explicit) for the blow-up time \( t^* \) to solutions of (2).

We will make use of the following Sobolev type inequality, here presented in this general form.

**Lemma 2.1.** (Sobolev type inequality) Let \( v \) be a non-negative \( C^1 \) function, defined in a bounded domain \( \Omega \in \mathbb{R}^3 \) with the origin inside, assumed to be star-shaped and convex in two orthogonal directions. Then

\[
\int_{\Omega} v^{\frac{4}{n}} \, dx \leq \left[ \frac{3}{2\rho_0} \int_{\Omega} v^n \, dx + \frac{n}{2} \left( 1 + \frac{d}{\rho_0} \right) \int_{\Omega} v^{n-1} |\nabla v| \, dx \right]^\frac{2}{n},
\]

valid for \( n \geq 1 \), with \( \rho_0 := \min_{\partial \Omega} (x \cdot \nu) \) and \( d := \max_{\Omega} |x| \).

For the proof see [12] and [14].

From Lemma 2.1 we derive a bound for \( \int_{\Omega} v^3 \, dx \), to be used in the proof of the main theorem.

**Lemma 2.2.** Under the hypotheses of Lemma 2.1

\[
\int_{\Omega} v^3 \, dx \leq \sqrt{2 \left[ m_1^2 \left( \int_{\Omega} v^2 \, dx \right)^{\frac{2}{3}} + \frac{m_2^2}{4\varepsilon^3} \left( \int_{\Omega} v^2 \, dx \right)^{\frac{3}{2}} + \frac{3}{4} m_2^2 \epsilon \int_{\Omega} |\nabla v|^2 \, dx \right]},
\]

with \( m_1 := \frac{3}{2\rho_0} \), \( m_2 := 1 + \frac{d}{\rho_0} \) and \( \epsilon \) an arbitrary positive function.

**Proof.** We point out that (4) can be derived by (3), but we sketch the proof for reader convenience.

We use (3) with \( n = 2 \) and by using Schwarz inequality in the second integral, we get

\[
\int_{\Omega} v^3 \, dx \leq \left[ m_1 \int_{\Omega} v^2 \, dx + m_2 \left( \int_{\Omega} v^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla v|^2 \right)^{\frac{1}{2}} \, dx \right]^2.
\]
By applying first the arithmetic inequality \((a + b)^{\frac{3}{2}} \leq \sqrt{2}(a^\frac{3}{2} + b^\frac{3}{2})\), valid with \(a, b > 0\), and the Hölder inequality we obtain
\[
\int_{\Omega} v^3 dx \leq \sqrt{2} \left( m_1^2 \left( \int_{\Omega} v^2 dx \right)^{\frac{3}{2}} + m_2^3 \left[ \left( \int_{\Omega} v^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla v|^2 \right)^{\frac{1}{2}} \right] \right)
\leq \sqrt{2} \left( m_1^2 \left( \int_{\Omega} v^2 dx \right)^{\frac{3}{2}} + m_2^3 \left( \int_{\Omega} v^2 dx \right)^{\frac{3}{2}} + \frac{3}{4} \int_{\Omega} |\nabla v|^2 dx \right),
\]
where in the last inequality we use
\[
a^r b^{1-r} \leq ra + (1-r)b, \quad a, b > 0, \quad 0 < r < 1,
\]
and \(\epsilon\) is any positive function; the lemma is so proved.

Let us present our main result.
First we introduce the following auxiliary function
\[
W(t) = \alpha(t) \int_{\Omega} u^2 dx + \beta(t) \int_{\Omega} (\Delta v)^2 dx,
\]
which value at \(t = 0\) is \(W(0) = \alpha(0) \int_{\Omega} u_0^2 dx + \beta(0) \int_{\Omega} (\Delta v_0)^2 dx\); \(\alpha(t)\) and \(\beta(t)\) in (6) are positive and derivable functions in \([0, t^*]\), to be determined.

Now we give the following definition

**Definition 2.3.** The solution \((u, v)\) to system (2) blows up in \(W\)-measure at time \(t^*\) if
\[
\lim_{t \to t^*} W(t) = \infty.
\]

Now we state our main result, i.e. we derive an explicit lower bound for \(t^*\). For brevity we write \(k_i := k_i(t), \quad i = 1, 2, 3, 4\).

**Theorem 2.4.** Let \((u, v)\) be a solution of (2). Assume \(\Omega\) a bounded domain in \(\mathbb{R}^3\), with the origin inside, star-shaped and convex in two orthogonal directions. Let \(W\) defined in (6) and \((u, v)\) becomes unbounded at some time \(t^*\) in \(W\)-measure (7). Moreover assume that the coefficients \(k_i\) (for \(i = 1, 2, 3, 4\)) satisfy the following relation
\[
\frac{2k_4'}{k_4} - \frac{k_1'}{k_2} + 2k_3 \leq 0,
\]
and let be
\[
\begin{cases}
\beta(t) = e^{2K_3(t)}, \quad \text{with } K_3(t) = \int_0^t k_3(s) ds, \\
\alpha(t) = \frac{k_2}{k_1} \beta.
\end{cases}
\]
Then
\[
t^* \geq H^{-1} \left( \frac{1}{2W(0)^2} \right),
\]
with \(H^{-1}\) the inverse function of \(H(t) := \int_0^t \omega(\tau) d\tau\), \(\omega(\tau)\) being a positive function depending only on the data.

3. **Proof of Theorem 2.4.**

**Proof.** We show that \(W(t)\) defined on solution of the system (2) satisfies an appropriate differential inequality of the first order. By integrating such inequality we get the lower bound of \(t^*\).

By differentiating \(W(t)\) we have
\[
W'(t) = \alpha' \int_{\Omega} u^2 dx + \beta' \int_{\Omega} (\Delta v)^2 dx + 2\alpha \int_{\Omega} uu_t dx + 2\beta \int_{\Omega} \Delta v \Delta v_t dx.
\]
Now we focus our attention to the last two integrals in (10). By using the first equation in (2) and the divergence theorem, the first term can be written as

$$\int_{\Omega} uu_t dx = - \int_{\Omega} |\nabla u|^2 dx + \frac{k_1}{2} \int_{\Omega} u^2 \Delta v dx. \quad (11)$$

Moreover, it can be checked that

$$\int_{\Omega} \Delta v \Delta v_t dx = - \int_{\Omega} \nabla (\Delta v) \cdot \nabla v_t dx$$

$$= - k_2 \int_{\Omega} |\nabla (\Delta v)|^2 dx + k_3 \int_{\Omega} \nabla (\Delta v) \cdot \nabla v dx + k_4 \int_{\Omega} \nabla (\Delta v) \cdot \nabla u dx, \quad (12)$$

where in the last equality we have used the second equation in (2). First of all, we observe that

$$\int_{\Omega} \nabla (\Delta v) \cdot \nabla v dx = - \int_{\Omega} (\Delta v)^2 dx. \quad (13)$$

Now by using Schwarz inequality and (5), we have

$$\int_{\Omega} |\nabla (\Delta v)| \nabla u |dx = \left( \epsilon_1 \int_{\Omega} |\nabla (\Delta v)|^2 dx \right)^{\frac{1}{2}} \left( \frac{1}{\epsilon_1} \int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}}$$

$$\leq \frac{\epsilon_1}{2} \int_{\Omega} |\nabla (\Delta v)|^2 dx + \frac{1}{2 \epsilon_1} \int_{\Omega} |\nabla u|^2 dx, \quad (14)$$

where $\epsilon_1$ is an arbitrary positive and time depending function to be determined. We now combine (13) and (14) in (12) to get

$$\int_{\Omega} \Delta v \Delta v_t dx \leq \left( \frac{\epsilon_1}{2} k_4 - k_2 \right) \int_{\Omega} |\nabla (\Delta v)|^2 dx - k_3 \int_{\Omega} (\Delta v)^2 dx$$

$$+ k_4 \left( \frac{1}{2 \epsilon_1} \right) \int_{\Omega} |\nabla u|^2 dx. \quad (15)$$

Plugging (11) and (15) in (10) we lead to

$$W'(t) \leq \alpha' \int_{\Omega} u^2 dx + \beta' \int_{\Omega} (\Delta v)^2 dx + \left( -2 \alpha + \frac{k_4}{\epsilon_1} \beta \right) \int_{\Omega} |\nabla u|^2 dx$$

$$+ \alpha k_1 \int_{\Omega} u^2 \Delta v dx - (2k_2 - \epsilon_1 k_4) \beta \int_{\Omega} |\nabla (\Delta v)|^2 dx - 2k_3 \beta \int_{\Omega} (\Delta v)^2 dx. \quad (16)$$

Regarding term $\int_{\Omega} u^2 \Delta v dx$, by means of Hölder inequality and (5) we obtain

$$\int_{\Omega} u^2 \Delta v dx \leq \frac{2}{3} \epsilon_2 \int_{\Omega} u^3 dx + \frac{1}{3 \epsilon_2} \int_{\Omega} |\Delta v|^3 dx, \quad (17)$$

with $\epsilon_2$ another positive and time depending function to be chosen.

We observe that we are now under the hypotheses of Lemma 2.2, which can be applied to both terms in (17). In fact we can write

$$\int_{\Omega} u^3 dx \leq \sqrt{2} \left[ m_1 \left( \int_{\Omega} u^2 dx \right)^{\frac{2}{3}} + m_2 \left( \int_{\Omega} u^2 dx \right)^{\frac{3}{2}} \right]$$

$$+ \frac{3}{4} \epsilon_3 \left( \int_{\Omega} |\nabla u|^2 dx \right), \quad (18)$$
and
\[
\int_{\Omega} |\Delta v|^3 dx \leq \sqrt{2} \left[ m_1^3 \left( \int_{\Omega} |\Delta v|^2 dx \right)^{\frac{3}{2}} + m_2^2 \left\{ \frac{1}{4\epsilon_3^4} \left( \int_{\Omega} |\Delta v|^2 dx \right)^3 \right\} + \frac{3}{4} \epsilon_4 \left( \int_{\Omega} |\nabla (\Delta v)|^2 dx \right) \right] \tag{19}
\]
where \( \epsilon_3 \) and \( \epsilon_4 \) are any positive and time depending functions to be determined. Hence, by employing (18) and (19) in (17)
\[
\int_{\Omega} u^2 \Delta v dx \leq \frac{2\sqrt{2}}{3} \epsilon_3 [ m_1^3 \left( \int_{\Omega} u^2 dx \right)^{\frac{3}{2}} + m_2^2 \left\{ \frac{1}{4\epsilon_3^3} \left( \int_{\Omega} u^2 dx \right)^3 + \frac{3}{4} \epsilon_3 \left( \int_{\Omega} |\nabla u|^2 dx \right) \right\}]
+ \frac{2\sqrt{2}}{3} \epsilon_3 [ m_1^3 \left( \int_{\Omega} |\Delta v|^2 dx \right)^{\frac{3}{2}} + m_2^2 \left\{ \frac{1}{4\epsilon_3^3} \left( \int_{\Omega} |\Delta v|^2 dx \right)^3 \right\}]
+ \frac{3}{4} \epsilon_4 \left( \int_{\Omega} |\nabla (\Delta v)|^2 dx \right)].
\tag{20}
\]
By using (20) in (16) we obtain
\[
W'(t) \leq \alpha' \int_{\Omega} u^2 dx + \left[ -2\alpha + \frac{1}{\sqrt{2}} m_2 k_1 \alpha e_2^2 \epsilon_3 + \frac{k_1^2 \beta}{\epsilon_1} \right] \int_{\Omega} |\nabla u|^2 dx
+ \left[ k_1 \epsilon_1 \beta - 2k_3 \beta + \sqrt{2} \alpha k_1 m_2^3 \frac{\epsilon_2}{4\epsilon_3^2} \right] \int_{\Omega} |\nabla \Delta v|^2 dx
+ \frac{2\sqrt{2}}{3} \alpha k_1 \epsilon_2 \left[ m_1^3 \left( \int_{\Omega} u^2 dx \right)^{\frac{3}{2}} + m_2^2 \left\{ \frac{1}{4\epsilon_3^3} \left( \int_{\Omega} u^2 dx \right)^3 \right\} \right]
+ \frac{\sqrt{2}}{3} \epsilon_3 [ m_1^3 \left( \int_{\Omega} |\Delta v|^2 dx \right)^{\frac{3}{2}} + m_2^2 \left\{ \frac{1}{4\epsilon_3^3} \left( \int_{\Omega} |\Delta v|^2 dx \right)^3 \right\}]
+ (\beta' - 2k_3 \beta) \int_{\Omega} (\Delta v)^2 dx.
\tag{21}
\]
Now we choose \( \alpha(t) \) and \( \beta(t) \) in (6) as
\[
\alpha(t) = \frac{k_2^2 \beta}{k_2}, \quad \beta(t) = e^{2K_3(s)} t_3(s), \quad \text{with} \quad K_3(t) = \int_0^t k_3(s) ds, \tag{22}
\]
and the arbitrary functions \( \epsilon_i \) (for \( i = 1, \ldots, 4 \)) as
\[
\epsilon_1(t) = \frac{k_2}{k_4}, \quad \epsilon_2(t) = 1, \quad \epsilon_3(t) = \frac{\sqrt{2}}{k_1 m_2^2}, \quad \epsilon_4(t) = \frac{2\sqrt{2} k_2^2}{k_1 k_2^2 m_2^2}.
\tag{23}
\]
By using the values in (22) and (23), the coefficients of \( \int_{\Omega} |\nabla u|^2 dx \) and \( \int_{\Omega} |\nabla \Delta v|^2 dx \) in (21) vanish.
Moreover
\[
\alpha' = \frac{\beta k_2^2}{k_2} \left( \frac{2k_4'}{k_4} - \frac{k_2}{k_2} + 2k_3 \right) \leq 0,
\]
from hypothesis (8). In these circumstances we neglect in (21) the non-positive terms and drop the terms whose coefficients are zero due to the choice of \( \epsilon_i(t) \) and \( \alpha(t) \) and \( \beta(t) \). By using the inequality \( a^\gamma + b^\gamma \leq (a + b)^\gamma \), valid for \( \gamma > 1 \) and \( a \) and \( b \) non-negative, at the end we obtain
\[
W'(t) \leq AW^{\frac{2}{3}}(t) + BW^{\frac{3}{2}}(t), \tag{24}
\]
where
\[
\begin{align*}
A &= A(t) = \frac{\sqrt{2} m_1^3 \alpha k_1}{3} \max_{t \in [0, t^*]} \left( \frac{2\epsilon_2}{\alpha^2} \frac{1}{\epsilon_3^2 \beta^2} \right), \\
B &= B(t) = \frac{\sqrt{2} m_1^2 \alpha k_1}{12} \max_{t \in [0, t^*]} \left( \frac{2}{\epsilon_3^2 \alpha^2} \frac{1}{\epsilon_4} \right),
\end{align*}
\]
\[ W^2(t) \leq W_0^{-2} W^3(t), \quad t \in [t_1, t^*], \]  

Inserting (25) in (24), we obtain the desired differential inequality,

\[ \omega(t) \geq \frac{W'(t)}{W^3(t)}, \]  

being

\[ \omega(t) = W_0^{-3/2} A + B. \]  

By integrating (26) between \( t_1 \) and \( t^* \), we obtain

\[ H(t^*) := \int_0^{t^*} \omega(\tau) d\tau \geq \int_{t_1}^{t^*} \omega(\tau) d\tau \geq \frac{1}{2W_0^{1/2}}. \]  

Inequality (28) provides a lower bound \( T \) for \( t^* \) with

\[ T := H^{-1}\left(\frac{1}{2W_0^{1/2}}\right), \]

\( H^{-1} \) being the inverse of \( H \); in this way (9) is proved. \( \square \)

**Remark 1.** Since \( k_i \) are strictly positive and continuous functions in \([0, t^*]\), also \( \omega(t) \) of Theorem 2.4 is positive; therefore, \( H \) is defined in \( t^* \) and, in particular, \( 0 < H(t^*) = \lim_{t \to t^*} \int_0^t \omega(\tau) d\tau < \infty. \)

4. **Miscellaneous.** In this section we again consider system (2) and discuss how obtaining in Theorem 2.4 another lower bound for \( t^* \) and, moreover, how generalizing the same theorem to the plane case.

4.1. **Another lower bound.** Hypothesis (8) on the time depending coefficients \( k_i \), is not strictly necessary to derive a lower bound of \( t^* \); in fact, from (22) and (23), (21) is also reduced to

\[ W'(t) \leq \frac{2\sqrt{2}}{3} \alpha k_1 \epsilon_2 \left[ m_1^2 \left( \int_\Omega u^2 dx \right)^{1/2} + m_2^2 \left( \int_\Omega u^2 dx \right)^{3/2} \right] 
+ \frac{\sqrt{2}}{3\epsilon_2} k_1 \left[ m_1^2 \left( \int_\Omega (\Delta u)^2 dx \right)^{1/2} + m_2^2 \left( \int_\Omega (\Delta u)^2 dx \right)^{3/2} \right] 
+ \alpha' \int_\Omega u^2 dx + \beta' \int_\Omega (\Delta u)^2 dx, \]

where we have neglected the negative term \(-2k_3\beta \int_\Omega (\Delta u)^2 dx\). This implies

\[ W'(t) \leq A W^{1/2}(t) + B W^3(t) + CW(t), \]

where \( A \) and \( B \) have been previously computed and \( C \) is

\[ C = C(t) = \max_{t \in [0, t^*]} \left( \frac{\alpha'}{\alpha} \frac{\beta'}{\beta} \right), \]

\( \alpha \) and \( \beta \) given by (22).
Therefore, following the same reasoning of Theorem 2.4, if this function
\[ \tilde{\omega}(t) = W_0^{-\frac{3}{2}}A + B + CW_0^{-2}, \]
is considered, this inequality
\[ \tilde{H}(t^*) := \int_0^{t^*} \tilde{\omega}(\tau) d\tau \geq \int_t^{t^*} \tilde{\omega}(\tau) d\tau \geq \frac{1}{2W_0}, \]
provides another lower bound for \( t^* \), given by
\[ \tilde{T} = \tilde{H}^{-1}\left(\frac{1}{2W_0}\right). \]

Of course, since \( \tilde{\omega}(t) \geq \omega(t) \), \( \tilde{T} \leq T \), so that not considering hypothesis (8) returns a less accurate estimate of \( t^* \) than that given by (9).

4.2. The case of a plane domain. As we previously mentioned, relations (18) and (19) hold in a three dimensional domain. If problem (2) is considered in a convex domain \( \Omega \subset \mathbb{R}^2 \), this lower bound for \( t^* \) can be obtained:
\[ t^* \geq H^{-1}\left(\frac{1}{W_0}\right), \tag{29} \]
where \( H^{-1} \) is the inverse of
\[ H(t^*) := \int_0^{t^*} \bar{\omega}(\tau) d\tau, \]
\( \bar{\omega}(\tau) \) being a positive function depending only on the data.

In fact, by means of a similar technique used in [15], (18) and (19) can be replaced by
\[ \int_{\Omega} u^3 dx \leq \frac{\sqrt{2}}{3} m_1 \left( \int_{\Omega} u^2 dx \right)^{\frac{3}{2}} + \frac{\sqrt{2}}{2} m_2 \int_{\Omega} u^2 dx \left( \int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}}, \]
and
\[ \int_{\Omega} |\Delta v|^2 dx \leq \frac{\sqrt{2}}{3} m_1 \left( \int_{\Omega} |\Delta v|^2 dx \right)^{\frac{3}{2}} + \frac{\sqrt{2}}{2} m_2 \int_{\Omega} |\Delta v|^2 dx \left( \int_{\Omega} |\nabla (\Delta v)|^2 dx \right)^{\frac{1}{2}}, \]
m_1 and m_2 defined in Lemma 2.2.

In this way, by arranging the proof of Theorem 2.4, estimate (29) can be checked.

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