A hyperholomorphic line bundle on certain hyperkähler manifolds not admitting an $S^1$-symmetry

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Abstract
Generalizing work of Haydys [7] and Hitchin [8], we prove the existence of a hyperholomorphic line bundle on certain hyperkähler manifolds that do not necessarily admit an $S^1$ action. As examples, we consider the moduli space of (non-strongly) parabolic Higgs bundles, the moduli space of solutions to Nahm’s equations, and Nakajima quiver varieties.

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1 Introduction

A hyperholomorphic line bundle over a hyperkähler manifold is a line bundle with connection whose curvature is of type (1,1) in each complex structure. Conversely, given an integral 2-form that is of type (1,1) in each complex structure we can find a line bundle with connection of that curvature. Work of Haydys [7] and Hitchin [8, 9] has shown the existence of a canonical hyperholomorphic line bundle on a hyperkähler manifold admitting an $S^1$ action that preserves the metric and one complex structure while rotating the other two. Specifically, they prove

**Theorem 1.** Suppose $(M, g, \omega_I, \omega_J, \omega_K)$ is a hyperkähler manifold with an isometric action of $S^1$ such that

$$\mathcal{L}_X \omega_I = 0, \quad \mathcal{L}_X \omega_J = -\omega_K, \quad \mathcal{L}_X \omega_K = \omega_J,$$

or, equivalently,

$$d\alpha = 0, \quad d(J\alpha) = \omega_J, \quad d(K\alpha) = \omega_K,$$

where $X$ is the Killing vector field that generates the $S^1$ action and $\alpha = i_X \omega_I$. Then

$$\omega_I + dd^c_I \mu$$

is of type (1,1) in every complex structure, where $\mu \in C^\infty(M)$ is the moment map for the $S^1$-action (in symplectic structure $\omega_I$). In particular, if $(M, \omega_I)$ is prequantizable then $M$ admits a hyperholomorphic line bundle with the above form as its curvature.
This line bundle is used in a correspondence between hyperkähler and quaternion Kähler manifolds [7, 8] and also appears in physics in the case that $M$ is the moduli space of Higgs bundles [17].

We will generalize theorem 1 to

**Theorem 2.** Let $X$ be any vector field (not necessarily Killing) on a hyperkähler manifold $M$ and let $\alpha = i_X \omega_I$. Suppose there are 2-forms $F_1, F_2$ of type $(1,1)$ in each complex structure such that the following equations,

\[ \mathcal{L}_X \omega_I = 0, \quad \mathcal{L}_X \omega_J = -\omega_K - F_2, \quad \mathcal{L}_X \omega_K = \omega_J + F_1, \tag{1} \]

which are equivalent to

\[ d\alpha = 0, \quad d(J\alpha) = \omega_J + F_1, \quad d(K\alpha) = \omega_K + F_2, \tag{2} \]

are satisfied. Then

\[ \omega_I - d(I\alpha) \]

is of type $(1,1)$ in each complex structure.

Note that if $X$ comes from an $S^1$-action with moment map $\mu$, then $d_i^c \mu = -I\alpha$. Just as the typical examples of hyperkähler manifolds with $S^1$ actions satisfying the conditions of theorem 1 are cotangent bundles, we will see that hyperkähler manifolds satisfying the conditions of 2 look like twisted cotangent bundles.

Despite the odd first impression of the equations (1) and (2), they arise quite naturally. An example of a manifold satisfying the conditions of theorem 1 is the moduli space of Higgs bundles, with the $S^1$-action given by scaling the Higgs field. If one instead looks at the moduli space of Higgs bundles over a Riemann surface where the Higgs fields are allowed to have simple poles along a fixed divisor, then the moduli space $\mathcal{P}$ is a holomorphic Poisson manifold [14]. The symplectic leaves $\mathcal{M}$ of $\mathcal{P}$ are given by fixing the eigenvalues for the residues of the Higgs fields and have a hyperkähler structure [11, 16]. The $S^1$-action on $\mathcal{P}$ clearly does not preserve this foliation but we show that there is a canonical projection map from $T\mathcal{P}$ to $T\mathcal{M}$ under which the vector field generating the $S^1$-action on $\mathcal{P}$ projects to a vector field that satisfies (1) on any symplectic leaf.

The conditions of theorem 2 also naturally arise as the result of hyperkähler reduction on a hyperkähler manifold $M$ with $G$-basic 1-form $\alpha$ satisfying the conditions of theorem 1 (e.g. if $M$ has an $S^1$-action). Then
the conditions of theorem 1 may not descend to the hyperkähler quotient and need to be replaced by the weaker conditions of 2. This general set-up is discussed in section 3.

Associated to a hyperkähler manifold $M^{4n}$ is its twistor space $Z$, which is a complex manifold of complex dimension $2n + 1$ and fibers over $\mathbb{C}P^1$. There is a one-to-one correspondence between hyperholomorphic line bundles on $M$ and holomorphic line bundles on $Z$ that are trivial on twistor lines. In the case that $M$ admits an $S^1$ action, Hitchin [8] gives a Cech description of this line bundle over $Z$. Further, he shows that this line bundle has a meromorphic connection with singularities on the fibers over the north and south poles of $\mathbb{C}P^1$. We generalize this Cech description in the case that $M$ satisfies the conditions of theorem 2. The main difference is that there is no longer a meromorphic connection, as the curvature (whose $(1,1)$ part represents the Atiyah class of the holomorphic line bundle) of the analogous connection has terms involving the $(1,1)$-forms $F_1$ and $F_2$.

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2 Proof of theorem 2

We now prove theorem 2. Our proof is different than the one given in [8] and rests on the vanishing of the Nijenhuis tensor in each complex structure. We first note that the equivalence of equations 1 and 2 follows from Cartan’s homotopy formula and the facts that $i_X \omega_J = -K\alpha$ and $i_X \omega_K = J\alpha$.

Thus suppose we have a 1-form $\alpha$ on a hyperkähler manifold $M$ satisfying (2). Since $\omega_I$ is of type $(1,1)$ in the $I$ complex structure, to show that $\omega_I - d(I\alpha)$ is also $(1,1)$ in the $I$ complex structure we must show that $d(I\alpha)$ is, i.e. that $d(I\alpha)(v, w) = d(I\alpha)(v, w)$ for all vector fields $v$ and $w$. We have

$$d(I\alpha)(v, w) = -Iv \cdot \alpha(w) + Iw \cdot \alpha(v) - \alpha(I[v, w]).$$

The vanishing of the Nijenhuis tensor gives us

$$I[v, w] = I[v, w] - [v, w] - [v, w],$$
which implies that
\[
d(I\alpha)(Iv, Iv) = -Iv \cdot \alpha(w) + \alpha([Iv, w]) + Iv \cdot \alpha(v) + \alpha([v, Iv]) - \alpha(I[v, w])
\]
\[
= d\alpha(w, Iv) - w \cdot \alpha(Iv) + d\alpha(Iv, v) + iv \cdot \alpha(Iv) - \alpha(I[v, w])
\]
\[
= d\alpha(w, Iv) + d\alpha(Iv, v) + d(I\alpha)(v, w)
\]
\[
= d(I\alpha)(v, w),
\]

since \(d\alpha = 0\).

For the complex structure \(J\), we compute
\[
d(I\alpha)(Jv, Jw) = Jv \cdot \alpha(Kw) - Jw \cdot \alpha(Kv) - \alpha(I[Jv, Jw])
\]
\[
= (Jv \cdot \alpha(Kw) - \alpha(KJv) - \alpha(Iv, w)) + (Jw \cdot \alpha(Kv) - \alpha(Kv, Jw) - \alpha(I[v, w])
\]
\[
= d(K\alpha)(Jv, w) + w \cdot \alpha(KJv) + d(K\alpha)(v, Jw) - v \cdot \alpha(KJv) - \alpha(I[v, w])
\]
\[
= d(K\alpha)(Jv, w) + (K\alpha)(v, Jw) + v \cdot \alpha(Iw) - w \cdot \alpha(Iv) - \alpha(I[v, w])
\]
\[
= d(K\alpha)(Jv, w) + (K\alpha)(v, Jw) + d(I\alpha)(v, w)
\]
\[
= \omega_K(Jv, w) + F_2(Jv, w) + \omega_K(v, Jw) + F_2(v, Jw) + d(I\alpha)(v, w)
\]
\[
= \omega_K(Jv, w) + \omega_K(v, Jw) + d(I\alpha)(v, w),
\]

where we have used the Nijenhuis identity
\[
[Jv, Jw] = [v, w] + J[v, w] + [v, Jw]
\]
in the second equality and the last equality follows from the fact that \(F_2\) is of type (1,1) in \(J\). Thus to show that \(\omega_I - d(I\alpha)\) is of type (1,1) in \(J\), we need to show that
\[
\omega_I(Jv, Jw) - \omega_K(Jv, w) - \omega_K(v, Jw) = \omega_I(v, w).
\]

But the left hand side is
\[
g(IJv, Jw) - g(KJv, w) - g(Kv, Jw)
\]
\[
= g(Kv, Jw) + g(Iv, w) - g(Kv, Jw)
\]
\[
= g(Iv, w)
\]
\[
= \omega_I(v, w),
\]
as desired.

Since \(\omega_I - d(I\alpha)\) is of type (1,1) in complex structures \(I\) and \(J\), it is also of type (1,1) in complex structure \(K\).
3 Hyperkähler reduction

We will now see how the hyperholomorphic 2-form interacts with hyperkähler reduction. Suppose a hyperkähler manifold $M$ has a 1-form $\alpha$ satisfying

$$d\alpha = 0, \quad d(J\alpha) = \omega_J, \quad d(K\alpha) = \omega_K$$

(3)
as well as a hamiltonian action of a Lie group $G$ that preserves the hyperkähler structure and $\alpha$ (such an example is the case of an $S^1$ action, as in theorem 1, that commutes with the action of $G$). Let

$$\mu_G = (\mu_I, \mu_J, \mu_K) : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$$

be the moment map and denote by $M//G$ the hyperkähler reduction at 0, i.e. as a manifold $M//G$ is $\mu_G^{-1}(0)/G$ and the hyperkähler structure is induced from that of $M$ (see e.g. [10] for more details). We want to understand when $\alpha$ descends to the hyperkähler quotient $M//G$ and satisfies (2) for some $F_1$ and $F_2$.

If $M$ has an $S^1$ action with Killing vector field $X$ then a natural compatibility between the $S^1$ and $G$ actions one may want is the equations

$$X \cdot \mu_I = 0, \quad X \cdot \mu_J = -\mu_K, \quad X \cdot \mu_K = \mu_J,$$

(4)

which say that the moment map is equivariant with respect to the $S^1$-action on $\mathbb{R}^3$ given by rotation about $(1, 0, 0)$.

If $\mu$ is a moment map for symplectic form $\omega$ and $Y \in \mathfrak{g}$, we let $\mu_Y$ denote the function $x \mapsto \mu(x)(Y)$ on $M$. Then $d\mu_Y = i_Y \omega$ where $Y^*$ is the action vector field on $M$ coming from $Y$.

**Proposition 1.** If $\mu_G$ satisfies (4), which is equivalent to

$$\alpha(Y^*) = 0, \quad (J\alpha)(Y^*) = -\mu_J^Y, \quad (K\alpha)(Y^*) = -\mu_K^Y$$

for all $Y \in \mathfrak{g}$,

(5)

then $\alpha$ descends to a 1-form $\hat{\alpha}$ on the hyperkähler quotient $M//G$, which continues to satisfy (3).

**Proof.** By the previous comments, we have

$$X \cdot \mu_I^Y = d\mu_I^Y(X) = \omega_I(Y^*, X) = -\alpha(Y^*),$$

$$X \cdot \mu_J^Y = d\mu_J^Y(X) = \omega_J(Y^*, X) = \omega_I(X, KY^*) = (K\alpha)(Y^*),$$

(6)
and, similarly,
\[ X \cdot \mu^Y_K = -(J\alpha)(Y^*). \] (7)
This establishes that (4) and (5) are equivalent. But from (5), we see that \( \alpha, J\alpha \) and \( K\alpha \) are all \( G \)-basic when restricted to \( \mu_G^{-1}(0) \). Therefore they descend to forms on \( M//G \) that continue to satisfy (2). \( \square \)

If we only impose that \( \alpha \) be \( G \)-invariant, then equations (4) only hold up to locally constant functions:

**Proposition 2.** The form \( \alpha \) being \( G \)-invariant is equivalent to any of the following holding for all \( Y \in g \).

1. \([X, Y^*] = 0\).
2. \(\mathcal{L}_{Y^*}\alpha = 0\).
3. \(d(X \cdot \mu^Y_J) = 0\).
4. \(d(X \cdot \mu^Y_J + \mu^Y_K) = 0\).
5. \(d(X \cdot \mu^Y_K - \mu^Y_J) = 0\).
6. \(d(\alpha(Y^*)) = 0\).
7. \(d(J\alpha(Y^*) + \mu^Y_J) = 0\).
8. \(d(K\alpha(Y^*) + \mu^Y_K) = 0\).

**Proof.** Since \( \omega_I \) is \( G \)-invariant the \( G \)-invariance of \( \alpha \) is equivalent to \( X \) being \( G \)-invariant, which is equivalent to 1. We have
\[
\mathcal{L}_{Y^*}\alpha = \mathcal{L}_{Y^*}i_X\omega_I = i_X\mathcal{L}_{Y^*}\omega_I = i_{|Y^*,X|}\omega_I = i_{|Y^*,X|}\omega_I,
\]
from which we get (1) \(\Leftrightarrow\) (2) by the non-degeneracy of \( \omega_I \). Also,
\[
d(X \cdot \mu^Y_J) = d(\omega_I(Y^*, X)) = -dY^*\alpha = -\mathcal{L}_{Y^*}\alpha,
\]
showing that (2) \(\Leftrightarrow\) (3).

For (4) we have
\[
d(X \cdot \mu^Y_J) = \mathcal{L}_X d\mu^Y_J \\
= \mathcal{L}_X i_{Y^*}\omega_J \\
= i_{Y^*}\mathcal{L}_X\omega_J + i_{[X,Y^*]}\omega_J \\
= -i_{Y^*}\omega_K + i_{[X,Y^*]}\omega_J \\
= -d\mu^Y_K + i_{[X,Y^*]}\omega_J.
\]
Thus
\[
d(X \cdot \mu^Y_J + \mu^Y_K) = i_{[X,Y^*]}\omega_J,
\]
which gives (1) ⇔ (4) by the non-degeneracy of $\omega_J$. A similar calculation shows (1) ⇔ (5).

Cartan’s homotopy formula and the fact that $\alpha$ is closed gives (2) ⇔ (6). Finally, equations (6) and (7) show that (4) ⇔ (8) and (5) ⇔ (7).

In the examples we will consider (moduli spaces of parabolic Higgs bundles and solutions to Nahm’s equations), only the first equation in (5) is satisfied (i.e. $\alpha$ is $G$-basic but not necessarily $J\alpha$ or $K\alpha$). From (2), the functions $J\alpha(Y^*)$ and $K\alpha(Y^*)$ are locally constant on $G^{-1}(0)$. Assuming these are actually constant, we get linear maps

$$(J\alpha)_g : \mathfrak{g} \to \mathbb{R}, \ Y \mapsto (J\alpha)(Y^*)$$

and similarly for $K\alpha$.

**Proposition 3.** $(J\alpha)_g$ and $(K\alpha)_g$ are Lie algebra homomorphisms, i.e. they vanish on $[\mathfrak{g}, \mathfrak{g}]$.

**Proof.** Since $\omega_J$ is basic and $(J\alpha)(Y^*)$ is constant, we have

$$0 = \omega_J(Y_1^*, Y_2^*) = d(J\alpha)(Y_1^*, Y_2^*) = -J\alpha([Y_1^*, Y_2^*]) = -J\alpha([Y_1, Y_2]^*)$$

and similarly for $K\alpha$. □

We will denote the hyperkähler structures on $M//G$ by $\hat{\cdot}$, e.g. $\hat{\omega}, \hat{\omega}_I$, etc.

Let $\Omega \in \mathfrak{g}^2(\mu^{-1}_G(0); \mathfrak{g})$ be the curvature of the principal $G$-connection on $\mu^{-1}_G(0) \to M//G$ induced by the metric on $M$, which is of type (1,1) in all complex structures [5]. Then $(J\alpha)_g \circ \Omega$ and $(K\alpha)_g \circ \Omega$ give characteristic classes that obstruct the equation $d(\hat{J}\hat{\alpha} + i\hat{K}\hat{\alpha}) = \omega_J + i\omega_K$:

**Proposition 4.** Suppose $\alpha$ is $G$-basic, satisfies (3), and the functions $J\alpha(Y^*), K\alpha(Y^*)$ are constant for all $Y \in \mathfrak{g}$ (when restricted to $\mu^{-1}_G(0)$). Then $\alpha$ naturally descends to a 1-form $\hat{\alpha}$ on $M//G$ satisfying the conditions of theorem 2, i.e.

$$d\hat{\alpha} = 0, \ d(\hat{J}\hat{\alpha}) = \hat{\omega}_J + F_1, \ d(\hat{K}\hat{\alpha}) = \hat{\omega}_K + F_2,$$

with $F_1$ and $F_2$ of type (1,1) in each complex structure. Specifically,

$$F_1 = (J\alpha)_g \circ \Omega, \ F_2 = (K\alpha)_g \circ \Omega.$$  \hspace{1cm} (8)

This proposition follows from the following general fact:
Lemma 1. Suppose $P \xrightarrow{\pi} X$ is a principal $G$ bundle with connection of curvature $\Omega$, $\beta \in \mathfrak{a}^1(P)$ is $G$-invariant and $d\beta = \pi^*\gamma$ is $G$-basic. Then for all $Y \in \mathfrak{g}$, $\beta(Y^*)$ is locally constant and, assuming this is actually constant, we have

$$d\hat{\beta} = \gamma + \beta_y \circ \Omega$$

where $\hat{\beta}$ is the 1-form on $X$ coming from the connection and $\beta_y : \mathfrak{g} \to \mathbb{R}$ is the map $Y \mapsto \beta(Y^*)$.

Thus it may happen (as we will see in the examples) that while $\alpha$ naturally descends to $\hat{\alpha}$ on the hyperkähler quotient, the dual vector field $X$, while $G$-invariant, is not tangent to $\mu^{-1}_G(0)$. If $\hat{X}$ is the vector field on $M//G$ dual to $\hat{\alpha}$, then its horizontal lift to $\mu^{-1}_G(0)$ is the orthogonal projection of $X$ onto the level set and the $\mathbb{R}$-action determined by $\hat{X}$ may not be an $S^1$-action.

Unlike in the case of an $S^1$-action, the Kähler forms $\omega_J$ and $\omega_K$ are no longer exact. It is thus natural to ask when they are pre-quantizable, i.e. when their cohomology classes live in $H^2(M; 2\pi\mathbb{Z})$. By the previous proposition, $\omega_J$ and $\omega_K$ are cohomologous to $F_1$ and $F_2$, respectively. However, if the representation $i(J\alpha)_g : \mathfrak{g} \to i\mathbb{R}$ lifts to $G \to U(1)$ then the unitary line bundle $\mu^{-1}_G(0) \times_G \mathbb{C}$ has a connection with curvature $i(J\alpha)_g \circ \Omega = F_1$ (and similarly for $K\alpha$ and $F_2$). Thus we see

Corollary 1. If the representation $i(J\alpha)_g : \mathfrak{g} \to i\mathbb{R}$, (resp. $i(K\alpha)_g : \mathfrak{g} \to i\mathbb{R}$), lifts to $G \to U(1)$ then $\omega_J$ (resp. $\omega_K$) is prequantizable.

3.1 Push-down

There is a natural way to pushdown the hyperholomorphic line bundle on $M$ to $M//G$. We first note that topologically this line bundle is the prequantum line bundle for $\omega_I$ and so has the infinitesimal Kostant action of $G$. The hyperholomorphic connection is $G$-invariant since it is $\nabla = \nabla^{pq} - I\alpha$ with $I\alpha$ $G$-invariant. Now we restrict it to the level set $\mu^{-1}_G(0) \xrightarrow{\iota} M$, which is a principal $G$-bundle over $M//G$ that has a canonical connection given by the metric on $M$. Then the push-downed line bundle with connection on $M//G$ is defined via

$$(\hat{\nabla}, (\iota^*L)/G), \quad \hat{\nabla}_v = \nabla_{v^H},$$

where $v^H$ is a horizontal lift of $v$. Since the pushdown of $\nabla^{pq}$ is the prequantum connection $\nabla^{M//G,pq}$ on $M//G$, we have $\hat{\nabla} = \nabla^{M//G,pq} - I\hat{\alpha}$. There-
fore the hyperholomorphic line bundle on $M//G$ is obtained via pushdown from the hyperholomorphic line bundle on $M$.

4 The line bundle on twistor space

4.1 Twistor space

Recall that associated to a hyperkähler manifold is its twistor space $Z \rightarrow \mathbb{C}P^1$. As a smooth manifold, $Z = M \times \mathbb{C}P^1$ but the complex structure at $(x, \zeta)$ is $I_{\zeta} \oplus I_{\mathbb{C}P^1}$, where $I_{\zeta} = aI + bJ + cK$ for $\zeta = (a, b, c) \in \mathbb{S}^2 \simeq \mathbb{C}P^1$. There is a one to one correspondence between hyperholomorphic line bundles on $M$ and holomorphic line bundles on $Z$ that are trivial when restricted to twistor lines.

Let $T_V = \ker \pi_*$ be the vertical vectors and $d_V$ be the vertical de Rham differential on $\mathcal{A}_V = \Gamma(Z; \Lambda^\bullet T^*_V)$. Let $\mathcal{A}_Z^{p,q}(2)$ (resp. $\mathcal{A}_V^{p,q}(2)$) denote the space of $(p, q)$ forms on $Z$ (resp. sections of $\Lambda^{p,q}T^*_V$) with simple singularities on the divisor $Z_0 + Z_{\infty} = \{ \zeta = 0 \} \cup \{ \zeta = \infty \}$. $Z$ comes equipped with the following:

- A real structure, i.e. an anti-holomorphic involution
  \[
  \tau : Z \rightarrow Z, \ (x, \zeta) \mapsto \left( x, -\frac{1}{\zeta} \right).
  \]

- A vertical meromorphic symplectic form
  \[
  \omega = \frac{1}{i\zeta}(\omega_J + i\omega_K) + 2\omega_I + \frac{\zeta}{i}(\omega_J - i\omega_K) \in \mathcal{A}_Z^{2,0}(2)
  \]

Our construction will use the map
\[
\overline{\tau} : \mathcal{A}_Z^\bullet \rightarrow \mathcal{A}_Z^\bullet, \ \gamma \mapsto \overline{\tau \gamma}, \tag{9}
\]
which preserves the type decomposition of differential forms and commutes with $d$. 
4.2 The Lie algebroid

We can generalize the construction in [8] of the holomorphic line bundle on \( Z \) corresponding to the hyperholomorphic line bundle on a simply-connected hyperkähler manifold \( M \) with 1-form \( \alpha \) satisfying (2). Actually, as in [8], we will construct a holomorphic Lie algebroid extension

\[
0 \to \mathcal{O}_Z \to E \to T^{1,0}Z \to 0,
\]

isomorphism classes of which correspond to the Čech cohomology group \( H^1(d\mathcal{O}_Z) \). Such a Lie algebroid is equivalent to a line bundle if the characteristic class in \( H^2(Z; \mathbb{C}) \), which comes from the short exact sequence of sheaves \( 0 \to \mathbb{C} \to \mathcal{O}_Z \to d\mathcal{O}_Z \to 0 \), is integral.

Relative to a cover \( U_j \) of \( U = \{ \zeta \neq \infty \} \) we will construct \( \phi_j \in \mathfrak{A}^{1,0}(U_j - \{ \zeta = 0 \}) \) that satisfy the following

1. \( d\phi_j = d\phi_k \).
2. \( (\zeta \phi_j)|_{\zeta = 0} = (\zeta \phi_k)|_{\zeta = 0} \).
3. \( \tau^*(d\phi_j) = -d\phi_j \)

From this we see that \( \{ \phi_k - \phi_j \} \) gives a 1-cocycle of closed holomorphic 1-forms and therefore defines a holomorphic Lie algebroid on \( U \). To extend this to all of \( Z \) we observe that, by point 3., the forms \( -\tau^*(\phi_j) \) give a singular connection on a Lie algebroid over \( \{ \zeta \neq 0 \} \) of the same curvature. Therefore, the collection \( \{ \phi_k - \phi_j, \tau^*(\phi_k) - \phi_j, \tau^*(\phi_j) - \tau^*(\phi_k) \} \) gives a 1-cocycle of closed 1-forms on all of \( Z \).

When there is an \( S^1 \)-action, the connection \( \{ \phi_j, -\tau^*(\phi_k) \} \) on \( Z \setminus (Z_0 \cup Z_\infty) \) is holomorphic, but now the curvature picks up the term \( \frac{1}{\zeta}(F_1 + iF_2) + \zeta(F_1 - iF_2) \), which is of type (1,1). Thus the Atiyah class of the line bundle on \( Z \setminus (Z_0 \cup Z_\infty) \) is

\[
\left[ \frac{1}{\zeta}(F_1 + iF_2) + \zeta(F_1 - iF_2) \right] \in H^{1,1}(Z \setminus (Z_0 \cup Z_\infty)) \simeq H^1(\Omega^1_{Z \setminus (Z_0 \cup Z_\infty)}),
\]

which obstructs the existence of a meromorphic connection on the line bundle.

Since our construction follows [8] very closely and we do not use it in the following examples, the details appear in the appendix.
5 Examples

We now focus on three examples: moduli spaces of parabolic Higgs bundles, Nakajima quiver varieties, and moduli spaces of solutions to Nahm’s equations.

5.1 Moduli space of parabolic Higgs bundles

Following the construction of the moduli spaces of parabolic Higgs bundles of Konno [11] and Nakajima [16], we fix the following data:

- A closed Riemann surface $\Sigma$ with a topological vector bundle $E \to \Sigma$ of rank $r$ with trivial determinant bundle.
- A divisor $D = p_1 + \cdots + p_n$.
- A flag of $E_{p_j}$ for each $j$ (which we assume to be complete for simplicity).
- Parabolic weights $\alpha_{k}^{(j)}, k = 1, \ldots, r, j = 1, \ldots, n$, that satisfy
  \[ 0 \leq \alpha_{1}^{(j)} < \ldots < \alpha_{r}^{(j)} < 1. \]
- Numbers $\lambda_{k}^{(j)} \in \mathbb{C}, k = 1, \ldots, r, j = 1, \ldots, n$ such that $\sum_k \lambda_{k}^{(j)} = 0$. These will be the eigenvalues of the residues of the Higgs fields at the punctures.
- A singular hermitian metric $h$ that at each puncture $p_j$ takes the form
  \[ h = \text{diag}(|z_j|^{2\alpha_{1}^{(j)}}, \ldots, |z|^{2\alpha_{r}^{(j)}}) \]
  with respect to the flag, where $z_j$ is a local holomorphic coordinate vanishing at $p_j$.

Definition 1. A parabolic Higgs bundle (with respect to the above data) is a pair $(\bar{\partial}_E, \theta)$ where $\bar{\partial}_E$ is a holomorphic structure on $E$ and $\theta \in \Omega^1(E; \text{Par sl}(E)(D))$ is a meromorphic 1-form such that $\text{Res}_{p_j} \theta \in \text{sl}(E_{p_j})$ preserves the parabolic structure and has eigenvalues $\lambda_{1}^{(j)}, \ldots, \lambda_{r}^{(j)}$. 
Note that in much of the literature parabolic Higgs bundles refers to the special case where the residues are nilpotent. We will call such parabolic Higgs bundles strongly parabolic.

As a hyperkähler manifold, the moduli space is obtained via hyperkähler reduction of the infinite dimensional affine space

$$C = \{\text{singular } \mathfrak{su}(E, h)\text{-connections}\} \times \mathscr{A}_\Sigma^{1,0}(\Sigma; \text{Par } \mathfrak{sl}(E)(D))$$

where \(\mathscr{A}_\Delta^{1,0}(\Sigma; \text{Par } \mathfrak{sl}(E)(D))\) consists of all \(\theta \in \mathscr{A}^{1,0}(\Sigma - D; \mathfrak{sl}(E))\) such that

- \(z_j \theta\) is smooth at each \(p_j\), where \(z_j\) is a local holomorphic coordinate centered at \(p_j\), and \(\text{Res}_{p_j} \theta\) preserves the parabolic structure at \(p_j\).
- \(\text{Res}_{p_j} \theta\) has eigenvalues \(\{\lambda_k^{(j)}\}\).

\(C\) is an affine space modeled on \(\mathscr{A}^{0,1}(\Sigma; \mathfrak{sl}(E)) \times \mathscr{A}_\Sigma^{1,0}(\Sigma; \text{SPar } \mathfrak{sl}(E)(D))\), where \(\text{SPar } \mathfrak{sl}(E)\) is the space of traceless endomorphisms that are nilpotent at the punctures. The hyperkähler structure is given by

$$g((a, b), (a, b)) = 2i \int_{\Sigma} \text{tr}(a^* \wedge a + b \wedge b^*),$$

$$I(a, b) = (ia, ib), \quad J(a, b) = (ib^*, -ia^*), \quad K(a, b) = (-b^*, a^*),$$

for \(a \in \mathscr{A}^{0,1}(\Sigma; \mathfrak{sl}(E)), b \in \mathscr{A}_\Sigma^{1,0}(\Sigma; \text{SPar } \mathfrak{sl}(E)(D))\).

The group \(\mathcal{G} = \mathscr{A}^0(\text{Par } SU(E))\) of parabolic special unitary gauge transformations acts on the affine space preserving the hyperkähler structure and Hitchin’s equations

$$F_A + [\theta, \theta^*] = 0$$
$$\bar{\partial}_A \theta = 0$$

arise as the zero level set of the moment map \(\mu_{\mathcal{G}}\). The hyperkähler quotient \(\mathcal{M}\) is the moduli space of parabolic Higgs bundles. We note that to rigorously define this space, one must use weighted Sobolev spaces as in [11] and [16], but we ignore this technical issue.

If all of the \(\lambda_k^{(j)}\) are zero (or we consider non-singular Higgs bundles), then there is an \(S^1\)-action given by multiplication on the Higgs field. The moment map of this action is \(C \ni (A, \theta) \mapsto -i \int_{\Sigma} \text{tr}(\theta \wedge \theta^*)\) and its exterior derivative is

$$\alpha_{(A, \theta)}(a, b) = -i \int_{\Sigma} \text{tr}(\theta \wedge b^* + b \wedge \theta^*)$$
In the general case with $\lambda_k^{(j)}$ not all zero, the integral defining the moment map diverges but the integral defining $\alpha$ converges. This is because near a puncture $p_j$

\[ \theta = \left( \begin{array}{ccccc}
\lambda_1^{(j)} & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots \\
\ast & \cdots & \cdots & \lambda_r^{(j)}
\end{array} \right) \frac{dz}{z} + \text{higher order terms} \]

\[ b^* = |z|^\gamma \left( \begin{array}{ccccc}
0 & \cdots & \cdots & \cdots \\
\ast & \cdots & \cdots & \cdots \\
\ast & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots 
\end{array} \right) \frac{d\bar{z}}{\bar{z}} + \text{higher order terms} \]

where $\gamma > 0$. Then $\text{tr}(\theta \wedge b^*)$ is of the order $|z|^\gamma \frac{dz \wedge d\bar{z}}{|z|^2}$ which in polar coordinates $z = |z|e^{i\theta}$ is $-2i|z|^{1-\gamma}d|z|d\theta$, which is integrable.

It is straightforward to check that $\alpha$ satisfies

\[ d\alpha = 0, \quad d(J\alpha) = \omega_J, \quad d(K\alpha) = \omega_K, \quad \alpha(Y^*) = 0 \]

for $Y \in \mathfrak{g}^0(\text{Par su}(E))$ (here the action field is $Y_{(A, \theta)}^*(Y) = (\partial_A Y, [\theta, Y])$). Thus to invoke proposition 4, we just need to check that on $\mu_g^{-1}(0)$, the (necessarily locally constant) functions

\[ (A, \theta) \mapsto (J\alpha)_{(A, \theta)}(Y^*), (K\alpha)_{(A, \theta)}(Y^*) \]

are indeed constant, which happens if and only if the function $(J\alpha + iK\alpha)(Y^*)$ is constant in $(A, \theta)$.

**Proposition 5.** We have

\[ (J\alpha + iK\alpha)(\partial_A Y, [\theta, Y]) = -2 \sum_{j=1}^n \text{tr}(\text{diag}(\lambda_1^{(j)}, \ldots, \lambda_r^{(j)}) Y_{p_j}), \]

where the linear map $\text{diag}(\lambda_1^{(j)}, \ldots, \lambda_r^{(j)})$ is represented via the flag at $p_j$.

**Proof.** A straightforward computation gives

\[ (J\alpha + iK\alpha)(a, b) = 2 \int_{\Sigma} \text{tr}(\theta \wedge a). \]
Thus
\[
(J\alpha + iK\alpha)(\bar{\partial}_A Y, [\theta, Y]) = 2 \int_\Sigma \text{tr}(\theta \wedge \bar{\partial}_A Y)
\]
\[
= -2 \int_\Sigma \text{tr}(\bar{\partial}_A (\theta a))
\]
\[
= -2 \int_\Sigma d \text{tr}(\theta Y)
\]
\[
= -2 \sum_j \text{tr}(\text{Res}_{\theta_p} Y_{p_j}).
\]

where the second equality comes from $\bar{\partial}_A \theta = 0$ and the last line is a consequence of the residue theorem.

Therefore by proposition 4 and theorem 2

**Theorem 3.** The moduli space $\mathcal{M}$ has a hyperholomorphic line bundle (or hyperholomorphic Lie algebroid extension if $\omega_I$ is not quantizable) of curvature $2i\omega_I - 2id(I\alpha)$.

**Remark.** In the case of $SU(2)$ Higgs bundles, Konno [12] shows that $\omega_I$ is prequantizable if the parabolic weights satisfy
\[
2\alpha^{(l)}, \ldots, 2\alpha^{(n)}_r, \sum_{j=1}^n \alpha^{(j)} \in \mathbb{Z}.
\]

where $\alpha^{(j)}$ is the parabolic weight at the $j$th puncture.

On an open dense set, the space $\mathcal{M}$ is a twisted cotangent bundle over the moduli space $\mathcal{N}$ of parabolic vector bundles. This subspace of $\mathcal{M}$ is simply-connected since $\mathcal{N}$ is [2] and so we can use the construction in section 4 to construct the holomorphic Lie algebroid on the twistor space of the twisted cotangent bundle.

Over a puncture, we have
\[
\text{Par}\, \mathfrak{su}(E_{p_j}) \cong \{(it_1^{(j)}, \ldots, it_r^{(j)}) \in \mathfrak{u}(1)^r \mid t_1^{(j)} + \cdots + t_r^{(j)} = 0\}.
\]

and the representation $(J\alpha + iK\alpha)_{\text{Lie} g}$ is the composition
\[
\mathcal{A}^0(\text{Par}\, \mathfrak{su}(E)) \to \prod_{j=1}^n \text{Par}\, \mathfrak{su}(E_{p_j}) \xrightarrow{\lambda} \mathbb{C},
\]
where the last map is the representation of \( \prod_{j=1}^{n} u(1)^r \) that has weights \(-2\lambda_1^{(j)}, \ldots, -2\lambda_r^{(j)}\) on the \( j \)th factor. If \( \lambda_k^{(j)} \in \frac{r}{2}\mathbb{Z} \) then this last representation lifts to the Lie group

\[
\prod_{j=1}^{n} \text{Par}SU(E_p)/\mathbb{Z}_r
\]

\[
\cong \prod_{j=1}^{n} \left\{ (e^{2\pi it_1^{(j)}}, \ldots, e^{2\pi it_r^{(j)}}) \in U(1)^r \mid t_1^{(j)} + \cdots + t_r^{(j)} = 0 \right\}/\mathbb{Z}_r,
\]

in which case we also get a lift of the representation of \((J\alpha + iK\alpha)_{\text{Lie}G}\) to \( \mathcal{G} \) (here \( \mathbb{Z}_r \) is the group of constant gauge transformations, given by the \( r \)th roots of unity). Thus from corollary 1 we see

**Proposition 6.** The symplectic form \( \omega_J \) (resp. \( \omega_K \)) is pre-quantizable if \( \text{Im} \lambda_k^{(j)} \) (resp. \( \text{Re} \lambda_k^{(j)} \)) \( \in \frac{r}{2}\mathbb{Z} \) for all \( j, k \).

**Remark.** From this perspective we can relate the form \( F_1 + iF_2 \) to hyperholomorphic structures that occur in wall-crossing in physics [4]. We have the exact sequence

\[
1 \to \mathcal{G}_p \to \mathcal{G} \to \prod_{j=1}^{n} \text{Par}SU(E_p)/\mathbb{Z}_r \to 1,
\]

where \( \mathcal{G}_p \) is the normal subgroup of gauge transformations that restrict to the identity at every puncture. Then the moduli space \( \mathcal{M} \) can be obtained by performing hyperkähler reduction in steps: we can first form the hyperkähler quotient of \( \mathcal{C} \) by the action of \( \mathcal{G}_p \). The resulting (finite-dimensional) hyperkähler manifold, \( \mathcal{M}' \), will have a hamiltonian action of \( \prod_{j=1}^{n} \text{Par}SU(E_p)/\mathbb{Z}_r \) and taking the hyperkähler quotient gives \( \mathcal{M} \). If we let \( \mu_p \) denote this last moment map, then \( \mu_p^{-1}(0) \to \mathcal{M} \) is a principal \( \prod_{j=1}^{n} \text{Par}SU(E_p)/\mathbb{Z}_r \)-bundle. The curvature of this principal bundle will be a 2-form with values in \( su(E_p) \) which is \((1,1)\) in each complex structure [5]. This hyperholomorphic projective bundle is considered in the case of (non-singular) Higgs bundle in [4]. Now because of the complete flag we have a decomposition of the fibers \( E_p \) into a direct sum of lines, which gives a decomposition of the curvature 2-form in terms of scalar 2-forms. Then the 2-form \( F_1 + iF_2 \) is the linear combination of these 2-forms weighted by \((-2\text{ times})\) the eigenvalues of the residues of the Higgs fields.

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From (8) the form $F_1 + iF_2$ is $(J\alpha + iK\alpha)_{\text{Lie}(G)} \circ \Omega$, where $\Omega$ is the curvature of the connection on the infinite rank principal bundle $G \to \mu_{\mathcal{G}}^{-1}(0) \to \mathcal{M}$. Using similar arguments as in [3, 6], one sees that

$$\Omega(A, \theta)((a_1, b_1), (a_2, b_2)) = -2G_{(A, \theta)} \text{ad}^*_\theta(a_1, b_1)(a_2, b_2),$$

where $G_{(A, \theta)}$ is the Green’s operator in degree 0 of the complex

$$\mathcal{A}^0(\text{Par } \mathfrak{su}(E)) \xrightarrow{\nabla^A + \theta} \mathcal{A}^1(\mathfrak{su}(E)(D)) \to \cdots.$$

We therefore see that $F_1 + iF_2$ vanishes in the abelian case of $U(1)$-Higgs bundles. Indeed, in the $U(1)$ case Hitchin’s equations decouple and the moduli space is just the product space of holomorphic line bundles with $H^0(K(D))$, the space of meromorphic 1-forms with simple poles at each $p_j$ of residue $\lambda^j$. $H^0(K(D))$ is affine on $H^0(K)$ and picking a fixed parabolic Higgs field $\theta_0 \in H^0(K(D))$ determines a diffeomorphism that preserves the hyperkähler structures. From our set-up, and the discussion to take place in the following section, we see that there is actually a canonical choice of $\theta_0$ given by the vanishing locus of $\alpha$, i.e. $\theta_0$ is determined by

$$\int_{\Sigma} \theta_0 \wedge \bar{b} = 0$$

for all $b \in H^0(K)$. Equivalently,

$$\int_{\Sigma} \theta_0 \wedge a = 0 \text{ for all harmonic (0, 1) forms } a.$$

### 5.1.1 Relationship to the $S^1$-action on the moduli space of all parabolic Higgs bundles

The moduli space of all parabolic Higgs bundles (where the eigenvalues of residues are not fixed) is a holomorphic Poisson manifold [14] whose symplectic leaves are the hyperkähler manifolds defined above. From our perspective, we let $\mathcal{P}$ be the moduli space of solutions to Hitchin’s equations (10) as above except we no longer fix the eigenvalues of the residues of the Higgs field. There is a natural $S^1$-action on $\mathcal{P}$ given by scaling the Higgs field and we let $\tilde{X}$ denote the vector field on $\mathcal{P}$ that generates it. We will show that there is a canonical projection map $p : T\mathcal{P} \to T\mathcal{M}$, where $T\mathcal{M} \subset T\mathcal{P}$ is the distribution underlying the foliation by Higgs fields whose residues
have fixed eigenvalues, and that $p(\tilde{X})$ is the vector field $X$ of theorem 2 (i.e. $\alpha = i_{p(\tilde{X})}\omega_I$).

By linearizing (10), at a point $(A, \theta) \in \mathcal{P}$ we have

$$T_{(A,\theta)}\mathcal{P} = \left\{ (a, b) \mid \partial_E a - \bar{\partial}_E a^* + [\theta, b^*] + [b, \theta^*] = 0, \quad [a, \theta] + \bar{\partial}_E b = 0 \right\} \subset \mathcal{A}^{0,1}(\mathfrak{sl}(E)) \times \mathcal{A}^{1,0}(\text{Par}\mathfrak{su}(E))$$

and similarly for $T_{(A,\theta)}\mathcal{M}$, except that $b$ must lie in $\mathcal{A}^{1,0}(\text{Par}\mathfrak{sl}(E)(D))$. It is then straightforward to check that

$$\tilde{\omega}_I : T\mathcal{P} \otimes T\mathcal{M} \to \mathbb{R}, \quad \tilde{\omega}_I((a_1, b_1), (a_2, b_2)) = \int_{\Sigma} \text{tr}(a_1^* \wedge a_2 - a_2^* \wedge a_1 - b_1 \wedge b_2^* + b_2 \wedge b_1^*)$$

is well-defined and restricts to the Kähler form $\omega_I$ on any symplectic leaf. This in turn defines a natural projection $p : T\mathcal{P} \to T\mathcal{M}$ via $\omega_I(p(v), w) = \tilde{\omega}_I(v, w)$ for all $w \in T\mathcal{M}$. The vector field $\tilde{X}$ on $\mathcal{P}$ is given by $\tilde{X}_{(A,\theta)} = (0, i\theta)$ and $X$ is defined by $i_X \omega_I = \alpha$. From (11) and the definition of $\tilde{\omega}_I$, we therefore see that $p(\tilde{X}) = X$.

### 5.2 Nakajima quiver varieties

We first recall the following general set-up. Suppose $V$ is a hermitian vector space. Then $T^* V \simeq V \times V^*$ has the structure of a hyperkähler manifold, where $I$ is given by multiplication by $i$ and

$$J(v', w') = (-w'^*, v'^*), \quad (v', w') \in V \oplus V^* \simeq T_{(v,w)}(T^* V),$$

where $*: V \leftrightarrow V^*$ is the conjugate linear isomorphism determined by the hermitian metric. The action of $S^1$ on $T^* V$ given by scalar multiplication on $V^*$ satisfies the hypotheses of theorem 1 and the moment map of this action is

$$\mu : T^* V \to \mathbb{R}, \quad (v, w) \mapsto -\frac{1}{2}||w||^2.$$ 

Letting $\alpha = d\mu$, we record for later use the following useful equations:

$$\alpha_{(v,w)}(v', w') = -g((0, w), (0, w')) \quad (J\alpha + iK\alpha)_{(v,w)}(v', w') = (\omega_J + i\omega_K)((0, w), (v', w')).$$

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We will now consider the Nakajima quiver varieties [15]. Let $Q$ be a quiver (i.e. directed graph) with $n$ vertices labeled $\{1, 2, \ldots, n\}$ and let $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in \mathbb{Z}^n$. Let

$$M_Q = \bigoplus_{i \to j \in Q} \text{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \oplus \bigoplus_{k=1}^n \text{Hom}(\mathbb{C}^{w_k}, \mathbb{C}^{v_k})$$

be the space of representations of the framed quiver of $Q$ (here we are identifying $Q$ with its set of oriented edges). Then $M_Q$ is a hermitian vector space (coming from the standard hermitian structure on $\mathbb{C}^m$) so that, by the preceding discussion, $T^*M_Q$ is a hyperkähler vector space with $S^1$-action satisfying the conditions of theorem 1. If $H$ denotes the union of the edges of $Q$ along with the edges with the opposite orientation then we have

$$T^*M_Q = \bigoplus_{i \to j \in H} \text{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \oplus \bigoplus_{k=1}^n \text{Hom}(\mathbb{C}^{w_k}, \mathbb{C}^{v_k}) \oplus \bigoplus_{k=1}^n \text{Hom}(\mathbb{C}^{v_k}, \mathbb{C}^{w_k}).$$

Using the notation of [15], we write $(B, i, j) = (B_h, i_k, j_k) h \in H, k \in \{1, \ldots, n\}$ for an element of $T^*M_Q$, where $h \in H$ is an arrow from vertex $s(h)$ to $t(h)$, $B_h \in \text{Hom}(\mathbb{C}^{v_{s(h)}}, \mathbb{C}^{v_{t(h)}}, i_k \in \text{Hom}(\mathbb{C}^{w_k}, \mathbb{C}^{v_k})$, and $j_k \in \text{Hom}(\mathbb{C}^{v_k}, \mathbb{C}^{w_k})$. Then we have

$$(\omega_J + i\omega_K)((B, i, j), (B', i', j')) = \sum_{h \in H} \text{tr}(\epsilon(h) B_h B_{h'})$$

$$+ \sum_{k=1}^n \text{tr}(i_k j'_k - i'_k j_k), \quad (14)$$

where we are identifying a fiber of $T(T^*M_Q)$ with $T^*M_Q$ and $\epsilon(h) = 1$ if $h \in Q$ and $-1$ otherwise.

The group

$$G_v = \prod_{j=1}^n U(\mathbb{C}^{v_j})$$

acts on $T^*M_Q$ preserving the hyperkähler structure. The action vector field corresponding to $Y = (Y_1, \ldots, Y_n) \in g_v$ is

$$Y_{(B, i, j)} = (Y_{t(h)} B_h - B_h Y_{s(h)}, Y_{i_k} i_k, -j_k Y_k) h \in H, k \in \{1, \ldots, n\}$$

Let $Z_v \simeq u(1)^n$ denote the center of the Lie algebra $g_v$ of $G_v$ and fix elements $\zeta_\mathbb{R} = ((\zeta_\mathbb{R})_1, \ldots, (\zeta_\mathbb{R})_n) \in Z_v$ and $\zeta_\mathbb{C} = ((\zeta_\mathbb{C})_1, \ldots, (\zeta_\mathbb{C})_n) \in Z_v \otimes \mathbb{C}$. Then,
identifying $g_v$ with its dual via the hermitian inner product, hyperkähler moment maps are

$$
\mu_I : T^*M_Q \to g_v = \bigoplus_{k=1}^{n} u(\mathbb{C}^{v_{k}})
$$

$$
\mu_J + i\mu_K : T^*M_Q \to g_v \otimes \mathbb{C} = \bigoplus_{k=1}^{n} gl(\mathbb{C}^{v_{k}})
$$

whose $k$th components are

$$(\mu_I(B, i, j))_k = \frac{1}{2} \left( \sum_{h \in H,t(h)=k} (B_h B_h^* - B_h^* B_h + i_k i_k^* - j_k j_k) \right) - (\zeta_{R})_k \in u(\mathbb{C}^{v_{k}})$$

$$(\mu_J + i\mu_K)(B, i, j)_k = \sum_{h \in H,t(h)=k} (\epsilon(h) B_h B_h^* + i_k j_k) - (\zeta_{C})_k \in gl(\mathbb{C}^{v_{k}}),$$

where $\bar{h}$ denotes the edge $h$ with the opposite orientation. We let $\mathcal{M}_{(\zeta_{R}, \zeta_{C})}$ be the hyperkähler quotient with respect to this moment map.

From the discussion at the beginning of this section, $\alpha$ satisfies

$$d\alpha = 0, \quad d(J\alpha) = \omega_J, \quad d(K\alpha) = \omega_K$$

and from equations (12) and (13) along with (14), it is straightforward to see that

$$\alpha(Y^*) = 0$$

$$(J\alpha + iK\alpha)(Y^*) = -2 \sum_{j=1}^{n} (\zeta_{C})_j \text{tr}(Y_j),$$

where the last equation is restricted to the 0 level set of the moment map $(\mu_I, \mu_J, \mu_K)$. Thus by proposition 4, $\mathcal{M}_{(\zeta_{R}, \zeta_{C})}$ has a hyperholomorphic Lie algebroid. The above equation along with corollary 1 gives

**Proposition 7.** The symplectic form $\omega_J$ on $\mathcal{M}_{(\zeta_{R}, \zeta_{C})}$ is pre-quantizable if $\text{Im}(\zeta_{C})_1, \ldots, \text{Im}(\zeta_{C})_n \in \frac{1}{2}\mathbb{Z}$. Similarly, $\omega_K$ is pre-quantizable if $\text{Re}(\zeta_{C})_1, \ldots, \text{Re}(\zeta_{C})_n \in \frac{1}{2}\mathbb{Z}$. 

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5.3 Nahm’s equations

Now we will show that the moduli space of solutions to Nahm’s equations [1, 13] has a canonical 1-form \( \alpha \) satisfying (2). Let \( G \) be a compact Lie group with an \( \text{Ad} \)-invariant inner product \( \langle \cdot, \cdot \rangle \) on its Lie algebra \( \mathfrak{g} \). Fix \( \tau_1, \tau_2, \tau_3 \in \mathfrak{g} \) such that the intersection of the centralizers is a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \). We define

\[
A_{\tau_1,\tau_2,\tau_3} = \{ T_0 + iT_1 + jT_2 + kT_3 : [0, \infty) \to \mathfrak{g} \otimes \mathbb{H} \mid T_0 \to 0, T_i \to \tau_i, i=1,\ldots,3 \},
\]

where the convergence is exponentially fast. Write \( T \) for \((T_0, T_1, T_2, T_3)\).

The space \( A_{\tau_1,\tau_2,\tau_3} \) is an affine space modeled on \( A_{0,0,0} \) and we have a hyperkähler structure defined as follows. The metric is given by

\[
|| (t_0, t_1, t_2, t_3) ||^2 = \int_0^\infty \sum_{j=0}^3 \langle t_j(s), t_j(s) \rangle ds,
\]

\((t_0, t_1, t_2, t_3) \in \mathcal{A}_{0,0,0} \simeq T_{\tau_1,\tau_2,\tau_3} A_{\tau_1,\tau_2,\tau_3}
\]

and the complex structures \( I, J, K \) are given by right multiplication by \(-i, -j, -k\), respectively.

Let

\[
\mathcal{G} = \{ g : [0, \infty) \to G \mid g(0) = e, \lim_{s \to \infty} g(s) \in \exp \mathfrak{h} \}
\]

where \( e \in G \) is the identity element. Then \( \mathcal{G} \) acts on \( \mathcal{A}_{\tau_1,\tau_2,\tau_3} \) via

\[
g \cdot (T_0, T_1, T_2, T_3) = (\text{Ad}_g T_0 - \dot{g} g^{-1}, \text{Ad}_g T_1, \text{Ad}_g T_2, \text{Ad}_g T_3),
\]

preserving the hyperkähler structure. The action is Hamiltonian and we have the moment maps

\[
\mu_I(T_0, T_1, T_2, T_3) = \dot{T}_1 + [T_0, T_1] - [T_2, T_3]
\]

\[
\mu_J(T_0, T_1, T_2, T_3) = \dot{T}_2 + [T_0, T_2] - [T_3, T_1]
\]

\[
\mu_K(T_0, T_1, T_2, T_3) = \dot{T}_3 + [T_0, T_3] - [T_1, T_2],
\]

which are called Nahm’s equations. Let \( \mathcal{M} = \mathcal{A}_{\tau_1,\tau_2,\tau_3}/\mathcal{G} \) be the hyperkähler reduction at 0.

Now suppose \( \tau_2 = \tau_3 = 0 \). Then we have an \( S^1 \) action on \( \mathcal{A}_{\tau_1,0,0} \) via

\[
e^{i\theta} \cdot (T_0, T_1, T_2 + iT_3) = (T_0, T_1, e^{i\theta}(T_2 + iT_3)),
\]

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which commutes with the \( G \)-action. The moment map is given by \( T \mapsto -\int_0^\infty \langle T_2(s), T_2(s) \rangle + \langle T_3(s), T_3(s) \rangle \) and its exterior derivative is the \( G \)-invariant 1-form

\[
\alpha_T(t) = -\int_0^\infty \left( \langle T_2(s), t_2(s) \rangle + \langle T_3(s), t_3(s) \rangle \right) ds.
\]

If \( \tau_2, \tau_3 \neq 0 \) then the integral defining the moment map diverges but \( \alpha \) is still well-defined since \( t_2 \) and \( t_3 \) converge to 0 exponentially fast and \( T_2 \) and \( T_3 \) are bounded.

The action vector field corresponding to \( Y \in \text{Lie}(G) = \{ Y : [0, \infty) \to g \mid Y(0) = 0, \lim_{s \to \infty} Y(s) \in \mathfrak{h} \} \) is given by

\[
Y^{*}(T_0, T_1, T_2, T_3) = ([Y, T_0] - \dot{Y}, [Y, T_1], [Y, T_2], [Y, T_3]).
\]  

(15)

It is straightforward to verify that on \( \mathcal{A}_{\tau_1, \tau_2, \tau_3} \), \( \alpha \) satisfies

\[
d\alpha = 0, \ d(J\alpha) = \omega_J, \ d(K\alpha) = \omega_K, \ \alpha(Y^{*}) = 0.
\]

To invoke proposition 4, we just need to show that the (necessarily locally constant) functions \( T \mapsto (J\alpha)(Y^{*}), (K\alpha)(Y^{*}) \) are indeed constant on the level set of the moment map.

**Proposition 8.** On the solution space to Nahm’s equations, we have

\[
(J\alpha)(Y^{*}) = -\langle \tau_2, Y(\infty) \rangle, \ \ (K\alpha)(Y^{*}) = -\langle \tau_3, Y(\infty) \rangle.
\]

**Proof.** We have

\[
(J\alpha)(T) = \alpha_T(t_2, t_3, -t_0, -t_1) = \int_0^\infty \langle T_2, t_0 \rangle + \langle T_3, t_1 \rangle
\]

so that using (15) we compute

\[
(J\alpha)(Y^{*}) = \int_0^\infty \left( \langle T_2, [Y, T_0] - \dot{Y} \rangle + \langle T_3, [X, T_1] \rangle \right)
\]

\[
= \int_0^\infty \left( \frac{d}{ds} \langle T_2, Y \rangle + \langle \dot{T}_2, Y \rangle + \langle T_2, [Y, T_0] \rangle + \langle T_3, [X, T_1] \rangle \right)
\]

\[
= -\langle \tau_2, Y(\infty) \rangle + \int_0^\infty \langle \dot{T}_2 + [T_0, T_2] + [T_1, T_3], X \rangle
\]

\[
= -\langle \tau_2, Y(\infty) \rangle.
\]

The proof for \( K\alpha \) is similar. \( \square \)
From corollary 1 we have

**Proposition 9.** The symplectic form $\omega_J$ (resp. $\omega_K$) is pre-quantizable if the element in $h^*$ dual to $\tau_2$ (resp. $\tau_3$) via the Killing form lies in the weight lattice.

The space $\mathcal{M}$ is diffeomorphic to a complex coadjoint orbit of the complexification of $G$ [13] and is therefore simply-connected if $G$ is. Thus in this case the construction of section 4 can be used to construct the holomorphic line bundle on its twistor space.

## A Construction of the line bundle/Lie algebroid on twistor space

### A.1 Definition of $\varphi_j$

Let $F = F_1 + iF_2$ and

$$\tilde{F} = \frac{1}{\zeta} F - \zeta \overline{F} \in \mathfrak{A}_{\nu}^{1,1}(2).$$

Let $Y = X + i\zeta \frac{\partial}{\partial \zeta}$ where $X$ is the vector field for the infinitesimal $S^1$-action and the sum is via the $C^\infty$ decomposition $TZ = TM \oplus T\mathbb{C}P^1$.

Using (1), it is straightforward to verify that

$$\mathcal{L}_Y \omega = \tilde{F}. \tag{16}$$

Let $\{U_j\}$ be an open cover of $U$ such that on each $U_j$ we have:

- A holomorphic lift $Z_j \in \Gamma(U_j; T_{\nu}^{1,0})$ of $\frac{\partial}{\partial \zeta}$ satisfying

  $$\mathcal{L}_{Z_j} i\zeta \omega = 0, \tag{17}$$

which is equivalent to

  $$\mathcal{L}_{Z_j} \omega = -\frac{1}{\zeta} \omega. \tag{18}$$

- A 1-form $A_j \in \Gamma(U_j; \Lambda^{1,0}_{\nu} T_{\nu}^*)$ such that $dA_j = \overline{F}|_{U_j}$.  

\[1\]Such an $A_j$ exists since we can find (at least locally) a line bundle and connection with curvature $\tilde{F}$. Then the $(0,1)$ part defines a holomorphic structure so we can take $A_j$ to be the connection forms in a holomorphic gauge.
Each $Z_j$ defines a local holomorphic splitting $TZ \simeq TM \oplus T\mathbb{C}P^1$ over $U_j$. If $\mu \in \mathcal{A}_v^\bullet$ then we let $\mu_j \in \mathcal{A}_j^\bullet$ denote the corresponding differential form with respect to this splitting.

Define the vertical vector fields

$$X_{jk} = Z_k - Z_j \in \Gamma(TV|_{U_j \cap U_k})$$

and

$$T_j = Z_j - \frac{\partial}{\partial \zeta} \in \Gamma(TV|_{U_j}).$$

We will make use of the following.

**Lemma 2.** For any $\mu \in \Gamma(Z; \Lambda^\bullet T^*_V)$, we have

$$\mu_k - \mu_j = -d\zeta \wedge i_{X_{jk}} \mu$$

$$d\mu_j = (d\nu \mu)_j + d\zeta \wedge (\mathcal{L}_Z \mu)_j$$

and if $v \in \Gamma(TZ)$ projects to a vector field on $\mathbb{C}P^1$, then

$$\mathcal{L}_v \mu_j = (\mathcal{L}_v \mu)_j + d\zeta \wedge (i_{[Z_j, v]} \mu)_j,$$

where $v_j$ is the vertical part of $v$ with respect to the splitting determined by $Z_j$.

**Proof.** All of the equations are easily seen to be true when pulled back to $\Lambda^\bullet T^*_V$, so it is sufficient to check that the equations are true when contracted with $Z_j$ and then pulled back to $\Lambda^\bullet T^*_V$. Let $\iota : TV \to TZ$ be the inclusion. We have

$$\iota^* i_{Z_j} (\mu_k - \mu_j) = \iota^* i_{Z_j} \mu_k = -\iota^* i_{X_{jk}} \mu_k = -i_{X_{jk}} (d\zeta \wedge i_{X_{jk}} \mu),$$

proving the first equation. The second equation follows from

$$\iota^* i_{Z_j} d\mu_j = \iota^* (-d i_{Z_j} \nu \mu_j + \mathcal{L}_Z \nu \mu_j) = \iota^* \mathcal{L}_Z \nu \mu_j = \mathcal{L}_Z \nu \mu$$

and the third from

$$\iota^* i_{Z_j} \mathcal{L}_v \mu_j = \iota^* (\mathcal{L} v i_{Z_j} \mu_j + [i_{Z_j}, \mathcal{L}_v] \mu_j) = \iota^* i_{[Z_j, v]} \mu_j = i_{[Z_j, v]} \mu.$$

$\square$
Proposition 10. The 1-form
\[ i_{Y,i\zeta Z_j}\omega + i_{\iota T_j} \bar{F} - 2i\zeta A_j \in \mathcal{A}_V^1(U_j) \]
is (vertically) closed.

Proof. Using (16) and (18), we have
\[
d_V i_{Y,i\zeta Z_j}\omega = \mathcal{L}_{i_{Y,i\zeta Z_j}}\omega = [\mathcal{L}_Y, \mathcal{L}_{i\zeta Z_j}]\omega = -i\bar{F} - \mathcal{L}_{i\zeta Z_j} \bar{F}
\]
\[
= -i\bar{F} - \mathcal{L}_{i\zeta \frac{\partial}{\partial \zeta}} \bar{F} - \mathcal{L}_{i\iota T_j} \bar{F} = -i\bar{F} + \frac{i}{\zeta} F + i\zeta \bar{F} - d_V i_{\iota T_j} \bar{F}
\]
\[
= 2i\zeta \bar{F} - d_V i_{\iota T_j} \bar{F} = d_V(2i\zeta A_j) - d_V i_{\iota T_j} \bar{F}.
\]

Therefore, shrinking $U_j$ if necessary, we can find $f_j \in C^\infty(U_j)$ such that
\[
d_V f_j = i_{Y,i\zeta Z_j}\omega + i_{\iota T_j} \bar{F} - 2i\zeta A_j \in \mathcal{A}_V^1(U_j)
\tag{19}
\]

Proposition 11. The function $i_X i_{X,j_k} i_\zeta \omega - f_k + f_j$ is constant on $Z_0$.

Proof. On $Z_0$ we have
\[
d(f_k - f_j) = i_{[X,i\zeta X_{jk}]\omega + i_{\iota X_{jk}} \bar{F}}
\]
\[
= i_{[X,X_{jk}]} i_\zeta \omega - i_{X_{jk}} i_\zeta \bar{F}
\]
\[
= i_{[X,X_{jk}]} i_\zeta \omega + i_{X_{jk}} (\omega_J + i\omega_K) + i_{\iota X_{jk}} \bar{F}
\]

and
\[
di_{X_{jk}} i_\zeta \omega = \mathcal{L}_X i_{X_{jk}} i_\zeta \omega - i_X di_{X_{jk}} i_\zeta \omega
\]
\[
= i_{X_{jk}} \mathcal{L}_X i_\zeta \omega + i_{[X,X_{jk}]} i_\zeta \omega - i_{X} X_{jk} \mathcal{L}_X i_\zeta \omega
\]
\[
= i_{X_{jk}} \mathcal{L}_X i_\zeta \omega + i_{[X,X_{jk}]} i_\zeta \omega
\]
\[
= i_{\iota X_{jk}} \mathcal{L}_{Y - i\zeta \frac{\partial}{\partial \zeta}} \omega + i_{[X,X_{jk}]} i_\zeta \omega
\]
\[
= i_{\iota X_{jk}} \bar{F} + \zeta^2 i_{X_{jk}} \mathcal{L}_{\frac{\partial}{\partial \zeta}} \left( \frac{1}{i_{\zeta}} (\omega_J + i\omega_K) \right) + i_{[X,X_{jk}]} i_\zeta \omega
\]
\[
= i_{\iota X_{jk}} \bar{F} + i_{X_{jk}} (\omega_J + i\omega_K) + i_{[X,X_{jk}]} i_\zeta \omega.
\]
Thus $i_X i_{X_{jk}} i_\zeta \omega - f_k + f_j$ is a Čech 1-cocycle with values in the constant sheaf $\mathbb{C}$. By our assumption that $\pi_1 M = 0$, we have

**Corollary 2.** We can choose the $f_j$ so that the function $i_X i_{X_{jk}} i_\zeta \omega - f_k + f_j$ vanishes on $Z_0$.

We are now able to define the 1-forms $\varphi_j$ that will define the Lie algebroid.

**Definition 2.** Let

$$\varphi_j = i_Y \omega_j - \frac{f_j}{i_\zeta} d\zeta + 2\zeta A_j \in \mathcal{A}_Z^{1,0}(U_j).$$

**Proposition 12.** The form $\varphi_k - \varphi_j$ is non-singular at $Z_0$ and we have

$$d\varphi_j = i_Y \left( d\zeta \wedge \frac{1}{\zeta} \omega \right) + \frac{1}{\zeta} F + \zeta \tilde{F}.$$

**Proof.** From lemma 2 and corollary 2, at $Z_0$ we have

$$\varphi_k - \varphi_j = - i_Y \left( d\zeta \wedge i_{X_{jk}} \omega \right) - i_X i_{X_{jk}} \omega d\zeta$$

$$= - i_\zeta i_{X_{jk}} \omega + d\zeta \wedge i_X i_{X_{jk}} \omega - i_X i_{X_{jk}} \omega d\zeta$$

$$= - i_\zeta i_{X_{jk}} \omega.$$

is non-singular.

We use lemma 2 and (18) to compute

$$d(i_Y \omega_j) = \mathcal{L}_Y \omega_j - i_Y d\omega_j$$

$$= (\mathcal{L}_Y \omega)_j + d\zeta \wedge i_{[Z_j, Y_j]} \omega - i_Y \left( d\zeta \wedge \mathcal{L}_Z \omega \right)$$

$$= (\tilde{F})_j + d\zeta \wedge i_{[Z_j, Y_j]} \omega + i_Y \left( d\zeta \wedge \frac{1}{\zeta} \omega \right)$$

while

$$d \left( \frac{- f_j}{i_\zeta} d\zeta + 2\zeta A_j \right) = - \left( \frac{1}{i_\zeta} i_{[Y_j, i_\zeta Z_j]} \omega + i_{T_j} \tilde{F} - 2i^* A_j \right) \wedge d\zeta + 2d\zeta \wedge A_j + 2\zeta \tilde{F}$$

$$= d\zeta \wedge i_{[Y_j, Z_j]} + d\zeta \wedge i_{T_j} \tilde{F} + 2\zeta \tilde{F}.$$

Since $\tilde{F} = (\tilde{F})_j + d\zeta \wedge i_{Z_j} \tilde{F}$, summing these gives

$$d\varphi_j = i_Y \left( d\zeta \wedge \frac{1}{\zeta} \omega \right) + \tilde{F} + 2\zeta \tilde{F} = i_Y \left( d\zeta \wedge \frac{1}{\zeta} \omega \right) + \frac{1}{\zeta} F + \zeta \tilde{F}$$

as desired.
Corollary 3. The forms \( \varphi_k - \varphi_j \) give a cocycle of holomorphic 1-forms on \( U \) and therefore defines a holomorphic Lie algebroid on \( U \).

It is straightforward to check that \( Y \) and \( \omega \) are invariant under \( \tau^* \) while \( d\zeta/\zeta \) and \( 1/\zeta F + \zeta F \) each pick up a factor of \(-1\). Thus from the above proposition, we have \( \tau^*(d\varphi_j) = -d\varphi_j \). Therefore we can extend this Lie algebroid to all of \( Z \) by using the cocycle \( \{ \varphi_k - \varphi_j, \tau^*(\varphi_k) - \varphi_j, \tau^*(\varphi_j) - \tau^*(\varphi_k) \} \) relative to the cover \( \{ U_j \} \cup \{ \tau(U_j) \} \) of \( Z \). Then \( \{ \varphi_j, -\tau^*(\varphi_k) \} \) gives a connection with singularities at \( \zeta = 0, \zeta = \infty \) of curvature \( i_Y \left( d\zeta \wedge \frac{1}{\zeta} \omega \right) + \frac{1}{\zeta} F + \zeta F \).

A.2 Computation of the Atiyah class

We now show that the Lie algebroid constructed in the previous section is the same one determined by the \( (1,1) \)-form \( 2i \omega_I - 2id(I\alpha) \). Any holomorphic Lie algebroid on \( Z \) has an Atiyah class in \( H^1(\Omega^1_Z) \to H^1(\Omega^1_Z) \) coming from the long exact sequence in cohomology associated to the short exact sequence of sheaves

\[
0 \to dO_Z \to \Omega^1_Z \to d\Omega^1_Z \to 0.
\]

Since \( H^0(d\Omega_Z) = 0 \) for twistor space [8], the map \( H^1(dO_Z) \to H^1(\Omega^1_Z) \simeq H^{1,1}(Z) \) is injective so that a holomorphic Lie algebroid is completely determined by its Atiyah class in \( H^{1,1}(Z) \). We will prove

**Theorem 4.** The Atiyah class of the holomorphic Lie algebroid defined in the previous section is

\[
2i \omega_I - 2id(I\alpha) \in H^{1,1}(Z).
\]

To compute this characteristic class, we find a \( (1,0) \) form \( A - \frac{\mu}{\zeta} d\zeta \) whose residues at \( \zeta = 0 \) and \( \zeta = \infty \) agree with those of \( \varphi_j \) and \( \varphi_j' \), respectively. It then follows that the Atiyah class of this Lie algebroid in \( H^{1,1}(Z) \) is given by

\[
\bar{\partial} \varphi_j - \bar{\partial} \left( A - \frac{\mu}{\zeta} d\zeta \right).
\]

It is straightforward to verify

\[
i_X \omega_J = -K\alpha, \quad i_X \omega_K = J\alpha,
\]

from which we see that the singular part of \( \varphi_j \) at \( \zeta = 0 \) is

\[
(\zeta \varphi_j)_{\zeta=0} = (J\alpha + iK\alpha)_j + if_j d\zeta.
\]
If \( \gamma \) is a vertical differential form, we will abuse notation and write \( \gamma \) for the corresponding differential form on \( Z \) obtained via the global \( C^\infty \) splitting \( TZ = TM \times TCP^1 \). Then for any \( \gamma \in \mathcal{A}^1(M) \) we have

\[
\gamma - \gamma_j = \gamma(T_j)d\zeta.
\]

Therefore in terms of the \( C^\infty \) splitting we have

\[
(\zeta \varphi_j)|_{\zeta=0} = J\alpha + iK\alpha + (if_j|_{Z_0} - (J\alpha + iK\alpha)(T_j))d\zeta.
\]  

(21)

**Proposition 13.** On \( Z_0 \simeq M \), the functions \( if_j - (J\alpha + iK\alpha)(T_j) \) piece together to a global function \( \mu \) satisfying

\[
d\mu = 2\alpha.
\]

**Proof.** From corollary 2, the difference of these functions is

\[
ii_Xi_{Xjk}i\zeta \omega - (J\alpha + iK\alpha)(X_{jk}).
\]

By (20), the above becomes

\[
ii_Xi_{Xjk}i\zeta \omega + ii_{Xjk}i_X(\omega_J + i\omega_K) = 0
\]
on \( Z_0 \).

Now, up to first order in \( \zeta \), \( [Y, i\zeta Z_j] = i\zeta [X, Z_j] - \zeta T_j \). So from (19) on \( Z_0 \) we have

\[
df_j = ii_{[X, Z_j]}(\omega_J + i\omega_K) + ii_{T_j}(\omega_J + i\omega_K) + ii_{T_j}F
\]

\[
= ii_{[X, T_j]}(\omega_J + i\omega_K) + ii_{T_j}(\omega_J + i\omega_K + F)
\]

\[
= ii_{[X, T_j]}(\omega_J + i\omega_K) + ii_{T_j}(\omega_J + iK\alpha)
\]

\[
= ii_{[X, T_j]}(\omega_J + i\omega_K) - idi_{T_j}d(J\alpha + iK\alpha) + i\mathcal{L}_{T_j}(J\alpha + iK\alpha)
\]

\[
= ii_{[X, T_j]}(\omega_J + i\omega_K) - idi_{T_j}d(J\alpha + iK\alpha) + \mathcal{L}_{T_j}i_X(\omega_J + i\omega_K)
\]

\[
= (ii_{[X, T_j]} + \mathcal{L}_{T_j}i_X)(\omega_J + i\omega_K) - idi_{T_j}d(J\alpha + iK\alpha)
\]

\[
= i_X\mathcal{L}_{T_j}(\omega_J + i\omega_K) - idi_{T_j}d(J\alpha + iK\alpha).
\]

From the degree zero part (in \( \zeta \)) of (17), one sees that \( \mathcal{L}_{Z_j}(\omega_J + i\omega_K) = -2i\omega_I \) so that

\[
d\mu = ii_X\mathcal{L}_{T_j}(\omega_J + i\omega_K) = ii_X\mathcal{L}_{Z_j}(\omega_J + i\omega_K) = 2i_X\omega_I = 2\alpha.
\]

\( \square \)
Now let
\[ A = \frac{1}{\zeta}(J\alpha + iK\alpha) + 2iI\alpha + \zeta(J\alpha - iK\alpha), \]
which is a 1-form on \(Z\) of type (1,0). Then from (21) we see that
\[ \left( \zeta \left( A + \frac{\mu}{\zeta}d\zeta \right) \right)_{\zeta=0} = (\zeta \varphi_j)|_{\zeta=0}. \]

so that \(\varphi_j - A - \frac{\mu}{\zeta}d\zeta\) is non-singular at \(\zeta = 0\).

Since \(\overline{\tau}(A + \frac{\mu}{\zeta}d\zeta) = -A - \frac{\mu}{\zeta}d\zeta\), we see that \(\{\varphi_j - A - \frac{\mu}{\zeta}d\zeta, -\overline{\tau}(\varphi_j) - A - \frac{\mu}{\zeta}d\zeta\}\) is a non-singular connection for the Lie algebroid constructed in the previous section. Thus the Atiyah class is represented by \(\overline{\partial}(\varphi_j - A - \frac{\mu}{\zeta}d\zeta)\).

From proposition 12 we have
\[ \overline{\partial}\varphi_j = \frac{1}{\zeta}F + \zeta\overline{F}. \]

Now we have
\[ dA = \frac{1}{\zeta}(\omega_J + i\omega_K + F) + 2i\overline{d}(I\alpha) + \zeta(\omega_J - i\omega_K + \overline{F}) \]
\[ -\frac{1}{\zeta}d\zeta \wedge \left( \frac{1}{\zeta}(J\alpha + iK\alpha) - \zeta(J\alpha - iK\alpha) \right). \]

Now, since \(\frac{1}{\zeta}(\omega_J + i\omega_K) + 2i\omega_I + \zeta(\omega_J - i\omega_K)\) is of type (2,0), we have
\[ \left( \frac{1}{\zeta}(\omega_J + i\omega_K) + 2i\omega_I + \zeta(\omega_J - i\omega_K) \right)_{1,1} = -(2i\omega_I)_{1,1}. \]

Therefore, since \(2i\overline{d}(I\alpha) - 2i\omega_I\) is of type (1,1) we see that
\[ \overline{\partial}(\varphi_j - A - \frac{\mu}{\zeta}d\zeta) = 2i\omega_I - 2i\overline{d}(I\alpha) + \frac{1}{\zeta}d\zeta \wedge \left( \frac{1}{\zeta}(J\alpha + iK\alpha) - \zeta(J\alpha - iK\alpha) + 2\alpha \right)_{0,1}. \]

But this last term vanishes since
\[ \frac{1}{\zeta}(J\alpha + iK\alpha) + 2\alpha - \zeta(J\alpha - iK\alpha) \]
\[ = i \left( \frac{1}{\zeta}(J(I\alpha) + iK(I\alpha)) + 2iI(I\alpha) + \zeta(J(I\alpha) - iK(I\alpha)) \right) \]
is of type (1,0). This proves theorem 4.
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