MULTIPLE AHARONOV–BOHM EIGENVALUES: THE CASE OF THE FIRST EIGENVALUE ON THE DISK

LAURA ABATANGELO

ABSTRACT. It is known that the first eigenvalue for Aharonov–Bohm operators with half-integer circulation in the unit disk is double if the potential's pole is located at the origin. We prove that in fact it is simple as the pole a ≠ 0.

1. INTRODUCTION

In the present paper we are interested in the spectral properties of Schrödinger operators with Aharonov–Bohm vector potential (see e.g. [8, 27, 7]), acting on functions \( u : \mathbb{R}^2 \to \mathbb{C} \), i.e.

\[
(i \nabla + A_0^a)^2 u := -\Delta u + 2i A_0^a \cdot \nabla u + |A_0^a|^2 u, \quad (1.1)
\]

where the vector potential is singular at the point \( a \) and takes the form

\[
A_0^a(x_1, x_2) = \alpha \left( -\frac{x_2 - a_2}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2} \right). \quad (1.2)
\]

We address here its eigenvalues in the unit disk in the special case when circulation \( \alpha = \frac{\pi}{4} \).

In order to pose the problem, we address here the general functional setting. If \( \Omega \subseteq \mathbb{R}^2 \) is open, bounded and simply connected, for \( a \in \Omega \), we define the functional space \( H^{1,\alpha}_0(\Omega, \mathbb{C}) \) as the completion of \( C_0^\infty(\Omega \setminus \{a\}, \mathbb{C}) \) with respect to the norm

\[
\|u\|_{H^{1,\alpha}(\Omega, \mathbb{C})} := \left( \|\nabla u\|^2_{L^2(\Omega, \mathbb{C})} + \|u\|^2_{L^2(\Omega, \mathbb{C})} + \left\| \frac{u}{|x - a|} \right\|^2_{L^2(\Omega, \mathbb{C})} \right)^{1/2}.
\]

When the circulation of the vector potential is not an integer, i.e. \( \alpha \in \mathbb{R} \setminus \mathbb{Z} \), the latter norm is equivalent to the norm

\[
\|u\|_{H^{1,\alpha}(\Omega, \mathbb{C})} = \left( \|i \nabla + A_0^a\|_{L^2(\Omega, \mathbb{C})}^2 + \|u\|_{L^2(\Omega, \mathbb{C})}^2 \right)^{1/2},
\]

by the Hardy type inequality proved in [25] (see also [8] and [16, Lemma 3.1 and Remark 3.2])

\[
\int_{D_r(a)} |(i \nabla + A_0^a)u|^2 \, dx \geq \left( \min_{j \in \mathbb{Z}} |j - \alpha| \right)^2 \int_{D_r(a)} \left\| \frac{u(x)}{|x - a|^2} \right\|^2 \, dx,
\]

which holds for all \( r > 0, a \in \mathbb{R}^2 \) and \( u \in H^{1,\alpha}(D_r(a), \mathbb{C}) \). Here we denote as \( D_r(a) \) the disk of center \( a \) and radius \( r \).

By a Poincaré type inequality, see e.g. [5, A.3], we can consider the equivalent norm on \( H^{1,\alpha}_0(\Omega, \mathbb{C}) \)

\[
\|u\|_{H^{1,\alpha}_0(\Omega, \mathbb{C})} := \left( \|i \nabla + A_0^a\|_{L^2(\Omega, \mathbb{C})}^2 \right)^{1/2}.
\]

We set the eigenvalue problem

\[
\begin{cases}
(i \nabla + A_0^a)^2 \varphi = \lambda \varphi & \text{in } \Omega \\
\varphi = 0 & \text{on } \partial \Omega,
\end{cases} \quad (1.3)
\]
in a weak sense, that is $\lambda \in \mathbb{C}$ is an eigenvalue of problem (1.3) if there exists $u \in H^1_{0}\alpha(\Omega, \mathbb{C}) \setminus \{0\}$ (called eigenfunction) such that
\[
\int_{\Omega} (i\nabla + A_{a}^{\alpha}) u \cdot (i\nabla + A_{a}^{\alpha}) v \, dx = \lambda \int_{\Omega} u v \, dx \quad \text{for all } v \in H^1_{0}\alpha(\Omega, \mathbb{C}).
\]
From classical spectral theory, for every $(a, \alpha) \in \Omega \times \mathbb{R},$ the eigenvalue problem (1.3) admits a diverging sequence of real and positive eigenvalues $\{\lambda_{k}(a, \alpha)\}_{k \geq 1}$ with finite multiplicity. These eigenvalues also have a variational characterization given by
\[
\lambda_{k}(a, \alpha) = \min \left\{ \sup_{u \in W_{k} \setminus \{0\}} \frac{\int_{\Omega} |(i\nabla + A_{a}^{\alpha}) u|^{2}}{\int_{\Omega} |u|^{2}} : W_{k} \text{ is a linear } k\text{-dim subspace of } H^1_{0}\alpha(\Omega, \mathbb{C}) \right\}. \tag{1.4}
\]

The paper [6] started the study of multiple eigenvalues of this operator with respect both to the position of the pole $a \in \Omega$ and the circulation $\alpha \in (0, 1).$ It shows that multiple eigenvalues in general occur, even if under suitable assumptions they are very rare locally with respect to the two parameters. Here we just mention that these assumptions rely on the local behavior of the corresponding eigenfunctions. Moreover, to the best of our knowledge, no result is available about this rareness globally with respect to the two parameters, yet (on this general theme the interested reader can see [31]).

As already mentioned, in this paper we consider the eigenvalue problem when $\Omega$ is the unit disk $D := \{(x_{1}, x_{2}) \in \mathbb{R}^{2} : x_{1}^{2} + x_{2}^{2} < 1\}$ and the circulation $\alpha = \frac{1}{2},$ i.e. the problem
\[
\begin{cases}
(i\nabla + A_{a})^{2} \varphi = \lambda \varphi & \text{in } D \\
\varphi = 0 & \text{on } \partial D.
\end{cases} \tag{1.5}
\]
Throughout the paper we will erase the index $\alpha,$ since it is fixed $\alpha = \frac{1}{2}.$ Because of this choice, in view of the correspondence between the magnetic problem and a real Laplacian problem on a double covering manifold (see [17, 30]), the operator (1.4) behaves as a real operator. As a consequence, the nodal set of the eigenfunctions of operator (1.1) (i.e. the set of points where they vanish) is made of curves and not of isolated points as we could expect for complex valued functions. More specifically, the magnetic eigenfunctions always have an odd number of nodal lines ending at the singular point $a,$ and therefore at least one.

In particular, we are going to focus our attention on the first eigenvalue to problem (1.5) and to study its multiplicity as the pole is moving from the origin around the disk. One can prove that this situation fulfills the assumptions of [6] Theorem 1.6, so that we know that the origin is locally the only point where the first eigenvalue is double. The main result of the paper is then the following

**Theorem 1.1.** Let $\lambda(a)$ be the first eigenvalue of Problem (1.5), i.e. $\lambda(a) := \lambda_{1}(a, \frac{1}{2}).$ It is simple if and only if $a \neq 0.$

We recall that the necessary condition is still known (see [11]). The new result is in fact the sufficient condition. The proof relies essentially in two steps. Firstly, we observe that eigenvalue functions are radial functions. Thanks to the local analytic regularity of eigenvalues with respect to analytic perturbations of the problem, the double eigenvalue for $a = 0$ immediately splits in two locally analytic branches, which a priori can be the same. We will show that in fact they are really different by means of their Taylor expansion’s first terms. The first derivatives of the two branches at the origin can be computed in terms of the corresponding eigenfunctions’ asymptotic expansions in the spirit of [6]. This is the content of Section 3.

From a technical point of view, the disk gives us chances to compute eigenfunctions explicitly. This can be done by reducing problem (1.5) to a suitable weighted Laplace eigenvalue problem on the double covering and thanks to a certain spectral equivalence between Problem (1.5) and suitable Laplace eigenvalue problems with mixed boundary conditions (see Section 2). This is enough to prove that the first derivatives of the two aforementioned analytic branches computed at the origin are different, in particular with opposite sign, thus concluding Section 3.

The proof is concluded in Section 4 thanks to the continuity and monotonicity of the two branches up to the boundary of the domain.
1.1. **Motivations.** The interest in Aharonov-Bohm operators with half-integer circulation $\alpha = \frac{1}{2}$ is motivated by the fact that nodal domains of their eigenfunctions are strongly related to spectral minimal partitions of the Dirichlet Laplacian. The investigation carried out in [2, 3, 4, 14, 26, 29] highlighted a strong connection between nodal properties of eigenfunctions and asymptotic expansion of the function which maps the position of the pole $a$ in the domain to eigenvalues of the operator $(i\nabla + A_a)^2$ (see also [11] Section 3 for a brief overview).

The interest in the case of disk comes from the seminal papers [19] and [10], where the so-called Mercedes Star Conjecture is introduced and discussed. Roughly speaking, the conjecture evokes that the spectral minimal 3-partition for the disk is in fact the Mercedes Star partition (see [11] Figure 1).

For what concerns us, the disk gives us the opportunity to begin to tackle the interesting question about how rare multiple eigenvalues are with respect to the position of the pole globally in the domain. This is a first contribution to carry on the analysis started in [6]. On the other hand, the present paper is not dealing directly with the aforementioned conjecture, but it presents arguments which may be useful towards it. Finally, Theorem 1.1 validates numerical simulations presented in [10, Figure 1] for the first eigenvalue.

2. **Explicit eigenfunctions and eigenvalues**

The aim of this section is exploiting the symmetry of the disk in order to deduce peculiar features of eigenvalues to Problem (1.5). Firstly, we recall that the map $a \mapsto \lambda_k(a)$ is a radial function for any $k \in \mathbb{N} \setminus \{0\}$.

2.1. **Eigenfunctions in the double covering.** In the papers [17, Lemma 3.3] and [30, Section 3] it is shown that in case of half-integer circulation the considered operator is equivalent to the standard Laplacian in the double covering. We then briefly recall some basic facts about Aharonov–Bohm operators. For any $a \in \mathbb{R}^2$, we define $\theta_a : \mathbb{R}^2 \setminus \{a\} \rightarrow [0, 2\pi)$ the polar angle centered at $a$ such that

$$\theta_a(a + r(\cos t, \sin t)) = t, \quad \text{for } t \in [0, 2\pi).$$

Thus, it results (see [16, 17, 2] for deeper explanations) that $2A_a$ is gauge equivalent to $0$, as $2A_a = -ie^{-i\theta_a}\nabla e^{i\theta_a} = \nabla \theta_a$. We introduce the following antilinear and antiunitary operator

$$K_a u = e^{i\theta_a} \tilde{\nabla} u,$$

which depends on the position of the pole $a \in \Omega$ through the angle $\theta_a$. It results that $(i\nabla + A_a)^2$ and $K_a$ commute. The restriction of the scalar product to $L^2_{K_a}(\Omega) := \{u \in L^2(\Omega, \mathbb{C}) : K_a u = u\}$ gives it the structure of a real Hilbert space and commutation implies that eigenspaces are stable under the action of $K_a$. Then we can find a basis of $L^2_{K_a}(\Omega)$ formed by $K_a$-real eigenfunctions of $(i\nabla + A_a)^2$. Being allowed to consider $K_a$-real eigenfunctions of $(i\nabla + A_a)^2$ allows to reduce the analysis to the real operator $(i\nabla + A_a)^2_{K_a}(\Omega)$ in the real space $L^2_{K_a}(\Omega)$.

**Definition 2.1.** (14, Lemma 2.3), [17, Lemma 3.3]) Let $\Omega \subset \mathbb{R}^2$ be an open simply connected and bounded set. Let $a \in \Omega$ be the pole of the operator. The double covering of $\Omega$ is the set

$$\tilde{\Omega} := \{y \in \mathbb{C} : y^2 + a \in \Omega\}.$$

**Lemma 2.2.** (14, Lemma 2.3]) Let $\theta$ denote the angle of the polar coordinates in $\mathbb{R}^2$. If $\varphi$ is a $K_0$-real eigenfunction of the problem (1.5) for $a = 0$, then the function

$$\psi(y) := e^{-i\theta(y)} \varphi(y^2)$$

defined in $\tilde{\Omega}$ is real valued and it is a solution to the problem

$$\begin{cases}
-\Delta \psi = 4\lambda |y|^2 \psi & \text{in } \tilde{\Omega}, \\
\psi = 0 & \text{on } \partial \tilde{\Omega}.
\end{cases}$$

(2.2)

The second basic special feature of the disk is stated in the following
Lemma 2.3. When $a = 0$, the double covering of the unit disk $D$ can be identified with the twofold unit disk $D$.

Proof. By Definition 2.1 the double covering of the unit disk $D$ is

$$
\Omega_0 := \{y \in \mathbb{C} : y^2 \in D\}.
$$

If we identify $\mathbb{C}$ with $\mathbb{R}^2$ in the standard way and consider the polar coordinates $(x_1, x_2) = \rho (\cos \theta, \sin \theta)$, we need that

$$(y_1, y_2) = \rho^2 (\cos 2\theta, \sin 2\theta) \in D.$$ 

Then, observing that $y_1 = x_1^2 - x_2^2$ and $y_2 = 2x_1x_2$, a simple computation shows that $y_1^2 + y_2^2 = (x_1^2 + x_2^2)^2 < 1$. \hfill $\square$

Thanks to Lemma 2.3, we are in position to have an explicit expression of eigenfunctions to Problem 1.5 by means of Bessel and trigonometric functions.

Lemma 2.4. If $\lambda_0$ is an eigenvalue of the problem (2.2), then it is double and its eigenfunctions take the form

$$
\psi(\rho \cos \theta, \rho \sin \theta) = AJ_{n/2}(\sqrt{\lambda_0} \rho^2) \cos(n\theta) + BJ_{n/2}(\sqrt{\lambda_0} \rho^2) \sin(n\theta), \quad y = (\rho \cos \theta, \rho \sin \theta) \in \tilde{D}
$$

with $A, B \in \mathbb{R}$ and for some $n \in \mathbb{N} \setminus \{0\}$.

Coming back to the original problem (1.5) on the original domain $D$, $\lambda_0$ is a double eigenvalue of the problem (2.3) and its eigenfunctions take the form

$$
\varphi(r \cos t, r \sin t) = e^{i\frac{t}{2}} J_{n/2}(\sqrt{\lambda_0} r) \left( A \cos \left( n \frac{t}{2} \right) + B \sin \left( n \frac{t}{2} \right) \right), \quad x = (r \cos t, r \sin t) \in D. \tag{2.3}
$$

Proof. Standard separation of variables $\psi(\rho \cos \theta, \rho \sin \theta) = u(\rho) v(\theta)$ leads to

$$v(\theta) = C \text{ or } v(\theta) = A \cos(n\theta) + B \sin(n\theta) \text{ for } n \in \mathbb{N}$$

being $A, B, C \in \mathbb{R}$. The radial part produces a Bessel-type equation which reads

$$
\rho^2 \frac{d^2 u}{d\rho^2} + \rho \frac{du}{d\rho} + (4\lambda_0 \rho^4 - n^2) u(\rho) = 0
$$

whose solutions are given by the so-called modified Bessel functions $J_{n/2}(\sqrt{\lambda_0} \rho^2)$ or $J_{-n/2}(\sqrt{\lambda_0} \rho^2)$ (for the modified Bessel functions, see the book by Watson [33]). From the results in [16, 17] we know that the eigenfunction is regular at the origin, so its radial part will be given in terms of the only $J_{n/2}$. Imposing the boundary conditions at $\rho = 1$, we find $J_{n/2}(\sqrt{\lambda_0}) = 0$, which means that

$$
\lambda_0 = \alpha_{n/2,k}^2 \quad \text{for some } k \in \mathbb{N},
$$

where $\{\alpha_{n/2,k}\}_{k \in \mathbb{N}}$ denote the sequence of zeros of the Bessel function $J_{n/2}$. This concludes the first part of the statement. By virtue of Lemma 2.2 the rest of the statement follows. \hfill $\square$

Note that the case of the disk is covered by the paper [11]; the fact that every eigenvalue is double was already provided by [11, Proposition 5.3] in a more general context. Nevertheless, this is not the main point we are interested in.

We recall that there is a connection between the zeros of the Bessel functions (to this aim we refer to [33, Chapter XV]): in particular, the positive zeros of the Bessel function $J_{a,n/2}$ are interlaced with those of the Bessel function $J_{a,n/2}$, and by Porter’s Theorem the positive zeros of $J_{a,n/2}$ are interlaced with those of the Bessel function $J_{a,n/2}$. Then, denoting $\alpha_{n/2,k}$ the $k$-th zero of the Bessel function $J_{a,n/2}$, we have

$$0 < \alpha_{1/2,1} < \alpha_{1/2,1} < \alpha_{2/2,1} < \alpha_{2/2,2} < \alpha_{2/2,1} < \ldots$$

Remark 2.5. The first case is then $(n, k) = (1, 1)$ and it corresponds to the double first eigenvalue for the Aharonov–Bohm operator with half-integer circulation and pole at the origin.

The second case is $n = 3$ and $k = 1$, which produces the double third eigenvalue.
2.2. Isospectrality and consequences on eigenvalues. We introduce two auxiliary problems. Let us denote $D^+ := \{(x_1, x_2) \in D : x_2 > 0\}$.

**Definition 2.6.** (26) The two problems
\begin{align*}
\begin{cases}
-\Delta u = \lambda u & \text{in } D^+ \\
u = 0 & \text{on } \partial D^+ \setminus \{0\} \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \{0\} \\
\end{cases}
\end{align*}

are called Dirichlet–Neumann and Neumann–Dirichlet eigenvalue problem for the Laplacian in the upper half-disk, respectively.

We recall the following result proved in [11] (see also [26, Proposition 5.3]).

**Lemma 2.7.** (11) Let $a = (t, 0)$ for $t \in [0, 1]$. The set of the eigenvalues of Problem (1.5) $\{\lambda_j(t)\}_{j \geq 1}$ is the union (counted with multiplicity) of the sequences $\{\lambda_j^{DN}(t)\}_{j \geq 1}$ and $\{\lambda_j^{ND}(t)\}_{j \geq 1}$, being $\{\lambda_j^{DN}(t)\}_{j \geq 1}$ and $\{\lambda_j^{ND}(t)\}_{j \geq 1}$ the set of the eigenvalues of the Dirichlet–Neumann and Neumann–Dirichlet problems (2.4) respectively.

By virtue of the latter Lemma 2.7 and the continuity result stated in [26] for Aharonov–Bohm eigenvalues (see also [15, Section 10]), the following result holds true.

**Lemma 2.8.** (26, 15) Fix $k \in \mathbb{N}\setminus\{0\}$ and denote $\lambda_k^{DN}(t)$ ($\lambda_k^{ND}(t)$) the $k$-th eigenvalue of the Dirichlet–Neumann problem in (2.4) (Neumann–Dirichlet problem, respectively). Then the maps
\[ t \mapsto \lambda_k^{DN}(t) \quad t \mapsto \lambda_k^{ND}(t) \]
are continuous in $(-1, 1)$.

We observe that in this case the standard Courant–Fisher characterization of eigenvalues establishes
\[ \lambda_k^{DN}(t) = \min_{E \subseteq \mathcal{H}_t, \dim E = k} \max_{u \in E \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2}, \]
where
\[ \mathcal{H}_t := \left\{ u \in H^1(\Omega) : u = 0 \text{ on } \partial \Omega \setminus \{0\}, \frac{\partial u}{\partial \nu} = 0 \text{ on } \{0\} \right\}, \]
analogously for $\lambda_k^{ND}(t)$.

**Remark 2.9.** By (2.5), if $-1 < t_1 \leq t_2 < 1$ then $\mathcal{H}_{t_2} \subseteq \mathcal{H}_{t_1}$ and then $\lambda_j^{DN}(t_2) \geq \lambda_j^{DN}(t_1)$ for any $j \geq 1$, i.e. the function $t \mapsto \lambda_j^{DN}(t)$ is monotone non-decreasing for any $j \geq 1$. As well, the function $t \mapsto \lambda_j^{ND}(t)$ is monotone non-increasing for any $j \geq 1$.

In the case of the disk, one can even see it by noting that $\lambda_j^{DN}(t) = \lambda_j^{ND}(t)$ because of the symmetry of the disk.

Another consequence of Lemma 2.7 is the following result.

**Lemma 2.10.** Let us consider the problems in (2.4). For $t = 1$ we have $\lambda_1^{DN}(1) = \lambda_1^{ND}(1)$.

We note the latter result can be proved by direct computation, in terms of Bessel-type functions, as in the proof of Lemma 2.3.

Now, if $a = (t, 0)$ let us denote $\lambda_j(t)$ the $j$-th eigenvalue of the problem (1.5). By Lemma 2.7, symmetry of the disk and Remark 2.9 (non-increasing monotonicity of the map $t \mapsto \lambda_j^{DN}(t)$), we have
\[ \lambda_1(t) = \min \{\lambda_1^{DN}(t), \lambda_1^{ND}(t)\} = \min \{\lambda_1^{ND}(-t), \lambda_1^{ND}(t)\} = \lambda_1^{ND}(t) \quad \text{for any } t \in [0, 1). \]

We have as well
\[ \lambda_2(t) = \min \{\lambda_1^{DN}(t), \lambda_2^{ND}(t)\} = \lambda_1^{DN}(t) \quad \text{for any } t \in [0, 1), \]
where the last equivalence follows from Lemma 2.7, Remark 2.9 and Lemma 2.10, recalling that $\lambda_2^{ND}(0) = \lambda_2^{DN}(0) > \lambda_1^{DN}(0) = \lambda_1^{ND}(0)$. Indeed, if by contradiction there exists $t \in (0, 1)$ such that $\lambda_2^{ND}(t) < \lambda_1^{DN}(t)$, then Remark 2.9 implies $\lambda_2^{ND}(1) \leq \lambda_2^{ND}(t) < \lambda_1^{DN}(t) \leq \lambda_1^{DN}(1)$ which denies Lemma 2.10.
3. IMMEDIATE SPLITTING OF THE EIGENVALUE

The aim of this section is to show that as the pole is moved, then the double eigenvalue split and produce two locally different analytic branches of eigenvalues. The first one is strictly monotone decreasing whereas the second one is strictly monotone increasing in a small neighborhood of the origin, with respect to the distance of the pole from the origin. In order to do this, we are going to exploit the results achieved in Section 2. In addition, by rotational symmetry, we will restrict ourselves to the case when the pole is moving along x1-axis.

3.1. Analytic perturbation with respect to the pole. As already pointed out in the Introduction (see also [6, Section 2], [26]), as the pole moves not only the operator changes, but also this produces different variational settings: functional spaces depend on the position of the pole. In order to study the moving pole’s effect on eigenvalues, first of all we need to define a family of diffeomorphisms which allow us to set the eigenvalue problem on a fixed domain, in the spirit of [6, 26].

We consider a particular case of the local perturbation introduced in [6, Subsection 5.1]. Let \( \xi \in C^\infty(\mathbb{R}^2) \) be a cut-off function such that
\[
0 \leq \xi \leq 1, \quad \xi \equiv 1 \text{ on } D_{1/4}(0), \quad \xi \equiv 0 \text{ on } \mathbb{R}^2 \setminus D_{1/2}(0), \quad |\nabla \xi| \leq 16 \text{ on } \mathbb{R}^2.
\]
For \( a \in D_{1/4}(0) \), we define the local transformation \( \Phi_a \in C^\infty(\mathbb{R}^2, \mathbb{R}^2) \) by
\[
\Phi_a(x) = x + a\xi(x).
\]
Notice that \( \Phi_a(0) = 0 \) and that \( \Phi_a' \) is a perturbation of the identity
\[
\Phi_a' = I + a \otimes \nabla \xi = \begin{pmatrix} 1 & a_1 \frac{\partial \xi}{\partial x_1} & a_2 \frac{\partial \xi}{\partial x_2} \\ a_1 \frac{\partial \xi}{\partial x_1} & 1 & a_2 \frac{\partial \xi}{\partial x_2} \\ a_2 \frac{\partial \xi}{\partial x_2} & a_2 \frac{\partial \xi}{\partial x_2} & 1 \end{pmatrix},
\]
so that
\[
J_a(x) := \det(\Phi_a') = 1 + a_1 \frac{\partial \xi}{\partial x_1} + a_2 \frac{\partial \xi}{\partial x_2} + 1 = 1 + a \cdot \nabla \xi.
\]
Let \( R = 1/128 \). Then, if \( a \in D_R(0) \), \( \Phi_a \) is invertible, its inverse \( \Phi_a^{-1} \) is also \( C^\infty(\mathbb{R}^2, \mathbb{R}^2) \), see e.g. [28, Lemma 1]. Then, as in [6, Section 7], we define \( \gamma_a : L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C}) \) by
\[
\gamma_a(u) = \sqrt{J_a(u \circ \Phi_a)}.
\]
where \( J_a \) is defined in [28]. Such a transformation \( \gamma_a \) defines an isomorphism preserving the scalar product in \( L^2(\Omega, \mathbb{C}) \). Moreover, since \( \Phi_a \) and \( \sqrt{J_a} \) are \( C^\infty \), \( \gamma_a \) defines an algebraic and topological isomorphism of \( H^1,^{1,a}(\Omega, \mathbb{C}) \) in \( H^1,^{0,0}(\Omega, \mathbb{C}) \) and inversely with \( \gamma_a^{-1} \), see [28, Lemma 2]. We notice that \( \gamma_a^{-1} \) writes
\[
\gamma_a^{-1}(u) = \left( \sqrt{J_a \circ \Phi_a^{-1}} \right)^{-1}(u \circ \Phi_a^{-1}).
\]

With a little abuse of notation we define the application \( \gamma_a : (H^1,^{1,a}(\Omega, \mathbb{C}))^* \to (H^1,^{0,0}(\Omega, \mathbb{C}))^* \) in such a way that
\[
(H^1,^{1,a}(\Omega, \mathbb{C})),(\gamma_a(f),v)_{H^1,^{1,a}(\Omega, \mathbb{C})} = (H^1,^{0,0}(\Omega, \mathbb{C})),(f,\gamma_a^{-1}(v))_{H^1,^{0,0}(\Omega, \mathbb{C})},
\]
for any \( f \in (H^1,^{1,a}(\Omega, \mathbb{C}))^* \), and inversely for \( \gamma_a^{-1} : (H^1,^{0,0}(\Omega, \mathbb{C}))^* \to (H^1,^{1,a}(\Omega, \mathbb{C}))^* \).

We define the new operator \( G_a : H^1,^{0,0}(\Omega, \mathbb{C}) \to (H^1,^{0,0}(\Omega, \mathbb{C}))^* \) by the following relation
\[
G_a \circ \gamma_a = \gamma_a \circ (i\nabla + A_a)^2,
\]
(3.6)
being \( \gamma_a \) defined in [28] and [38]. By [28, Lemma 3] the domain of definition of \( G_a \) is given by \( \gamma_a(H^1,^{1,a}(\Omega, \mathbb{C})) \), it coincides with \( H^1,^{1,a}(\Omega, \mathbb{C}) \). Moreover, \( G_a \) and \( (i\nabla + A_a)^2 \) are spectrally equivalent, in particular they have the same eigenvalues with the same multiplicity and the map \( a \mapsto G_a \) is \( C^\infty(D_R(0), BL(H^1,^{0,0}(\Omega, \mathbb{C})), (H^1,^{1,a}(\Omega, \mathbb{C}))^*) \).

Now, let us consider the special case \( a = (a_1,0) \), which means moving the pole just along the \( x_1 \)-axis. For simplicity, in the following we denote
\[
t := a_1 \quad \text{ and } \quad G_t := G_{(a_1,0)}.
\]
Then, following the same argument in [26] Section 4, the family $t \mapsto G_t$ is an analytic family of type (B) in the sense of Kato with respect to the variable $t$. In order to prove it, by definition (see [24] Chapter 7, Section 4) we need to show that the quadratic form $q_t$ associated to $G_t$, defined as

$$q_t(u) = \langle H_t^\gamma(\Omega)u, G_t(u, u)H_t^\gamma(\Omega) \rangle,$$

is an analytic family of type (a) in the sense of Kato, i.e. it fulfills the following two conditions:

(i) the form domain is independent of $t$;

(ii) the form $q_t(u)$ is analytic with respect to the parameter $t$ for any $u$ in the form domain.

The first assertion follows from [3.10] (see [6], Section 7.1), whereas the second one follows from Lemmas 5.1, 5.2, 7.1 possibly shrinking the interval $(-R, R)$ where the parameter $t$ is varying. The Kato-Rellich perturbation theory gives some information in the case when the considered eigenvalue is not simple. Let $\lambda_0$ be any double eigenvalue of $G_0$. Then there exist a family of 2 linearly independent $L^2(\Omega)$-normalized eigenfunctions $\{u_j(t)\}_{j=1,2}$ relative to the associated eigenvalue $\mu_j(t)$ for $j = 1, 2$ which depend analytically on the parameter $t$ and such that for $j = 1, 2$, $\mu_j(t) = \lambda_0$ and $\mu_j(t)$ is an eigenvalue of the operator $G_t$. We recall that $G_t$ has the same eigenvalues with the same multiplicity as operator $(i\nabla + A(t, 0))^2$. Note that the 2 functions $u_j(t)$ are not a priori necessarily distinct. The Feynman-Hellmann formula (see [24], Chapter VII, Section 3) then tells us that

$$\mu_j'(0) = \langle H_t^\gamma(\Omega)u_j(0), G_t(0)[u_j(0)]u_j(0) \rangle_{H_t^\gamma(\Omega, \mathcal{C})}.$$

(3.7)

3.2. Computing the derivative at 0 of the two branches. The aim of this subsection is showing that the two (a priori not necessarily different) analytic branches $t \mapsto \mu_j(t)$, $j = 1, 2$, have a different derivative at $t = 0$. In order to do this, we refer to the paper [6]. In particular, for $j = 1, 2$ [5.5, 7] together with Lemma 8.2, Lemma 8.6 yield

$$\mu_j'(0) = \langle H_t^\gamma(\Omega, \mathcal{C})u_j(0), G(t, 0)[u_j(0)]u_j(0) \rangle_{H_t^\gamma(\Omega, \mathcal{C})} = \frac{\pi}{2}(A_j^2 - B_j^2)$$

(3.8)

where $A_j, B_j \in \mathbb{R}$ are the coefficients in the expansion (2.3).

What is left is detecting $u_j(0)$ for $j = 1, 2$. To this aim, we are going to exploit the symmetry property of the domain with respect to the $x_1$-axis. We refer to [11] and define the antunitary antilinear operator $\Sigma : L^2(D) \to L^2(D)$

$$\Sigma u := \bar{u} \circ \sigma,$$

being $\sigma(x_1, x_2) = (x_1, -x_2)$. We have that $\Sigma$ and $(i\nabla + A_0)^2$ commute (see [11], Section 5), as well as $\Sigma$ and $K_0$. This means $L^2_{K_0}$ is stable under the action of $\Sigma$. Thus, if we write

$$L^2_{K, \Sigma}(\Omega) := L^2_K(\Omega) \cap \ker(\Sigma - \Id) \quad L^2_{K, a, \Sigma}(\Omega) := L^2_K(\Omega) \cap \ker(\Sigma + \Id),$$

then we have the orthogonal decomposition

$$L^2_{K}(\Omega) = L^2_{K, \Sigma}(\Omega) \oplus L^2_{K, a, \Sigma}(\Omega).$$

(3.9)

We can therefore define the operators $(i\nabla + A_0)^2_{\Sigma}$ and $(i\nabla + A_0)^2_{a, \Sigma}$, restrictions of $(i\nabla + A_0)^2$ to $L^2_{K, \Sigma}(\Omega)$ and $L^2_{K, a, \Sigma}(\Omega)$ respectively. The spectrum of $(i\nabla + A_0)^2$ is the union (counted with multiplicities) of the spectra of $(i\nabla + A_0)^2_{\Sigma}$ and $(i\nabla + A_0)^2_{a, \Sigma}$. Lemma 2.7 is then completed by the following result.

Lemma 3.1. ([11], Propositions 5.7 and 5.8) If $u$ is a $K_0$-real $\Sigma$-invariant eigenfunction of $(i\nabla + A_0)^2$ then the restriction to $D^+$ of $e^{-\frac{i}{2}b_0}u$ is a real eigenfunction of the Dirichlet–Neumann problem in (2.4).

If $u$ is a $K_0$-real $a\Sigma$-invariant eigenfunction of $(i\nabla + A_0)^2$ then the restriction to $D^+$ of $e^{-\frac{i}{2}b_0}u$ is a real eigenfunction of the Neumann–Dirichlet problem in (2.4). Conversely, if $v$ is an eigenfunction of the Dirichlet–Neumann problem in $D^+$, if $\tilde{v}$ is the even extension of $u$ in $D$, the function $e^{\frac{i}{2}b_0}\tilde{v}$ is a $(K_0$-real) $\Sigma$-invariant eigenfunction of $(i\nabla + A_0)^2$. If $v$ is an eigenfunction of the Neumann–Dirichlet problem in $D^+$, if $\tilde{v}$ is the odd extension of $u$ in $D$, the function $e^{\frac{i}{2}b_0}\tilde{v}$ is a $(K_0$-real) $a\Sigma$-invariant eigenfunction of $(i\nabla + A_0)^2$. 
In view of (3.4) and (3.6) we have that \( u_1(0) \) and \( u_2(0) \) are two \( K_0 \)-real linearly independent eigenfunctions of \((i\nabla + A_0)^2\). Therefore via (3.9), Lemma 3.1 and Lemma 2.4 \( u_1(0) \) is \( \Sigma \)-invariant whereas \( u_2(0) \) is \( \Sigma \)-invariant. From Lemma 2.4 Remark 2.5 and the asymptotic expansion of the Bessel functions (see e.g. [33] Chapter 3) there exist \( A, B \in \mathbb{R} \setminus \{0\} \) such that

\[
\begin{align*}
u_1(r \cos t, r \sin t) &= e^{i \frac{t}{2} r^{1/2}} B \sin \frac{t}{2} + O(r^{3/2}) \quad \text{as } r \to 0^+ \quad (3.10) \\
u_2(r \cos t, r \sin t) &= e^{i \frac{t}{2} r^{1/2}} A \cos \frac{t}{2} + O(r^{3/2}) \quad \text{as } r \to 0^+. \quad (3.11)
\end{align*}
\]

Equations (3.7) and (3.8) immediately give

\[
\begin{align*}
\mu_1'(0) &= -\frac{\pi}{2} B^2 < 0, \quad (3.12) \\
\mu_2'(0) &= \frac{\pi}{2} A^2 > 0, \quad (3.13)
\end{align*}
\]

thus concluding the first step towards our main result.

4. Conclusion

We are now in position to conclude the proof of our main result.

**Proof of Theorem 1.1.** Thanks to rotational invariance of eigenvalues, it is sufficient to prove that if \( a = (t, 0) \) and \( \lambda_1(t) \) is the first eigenvalue of the problem (1.5), which is double for \( t = 0 \), then \( \lambda_1(t) \) is simple for any \( t \in (0, 1) \).

By the results of Section 3 there exists \( \delta > 0 \) such that the two analytic eigenbranches \( \mu_1(t) \) and \( \mu_2(t) \) are different for \( t \in (-\delta, \delta) \), since

\[
\mu_1'(0) < 0 \quad \text{whereas} \quad \mu_2'(0) > 0. \quad (4.1)
\]

Moreover, we have that

\[
\lambda_1(t) = \begin{cases} 
\mu_2(t) & \text{for } t \in (-\delta, 0] \\
\mu_1(t) & \text{for } t \in [0, \delta),
\end{cases} \quad (4.2)
\]

since \( \mu_j(t) \) are eigenvalues of the operator \( G_t \) which is spectral equivalent to \((i\nabla + A_0)^2\) with \( a = (t, 0) \). In order to prove that it is simple for \( t \in (0, 1) \), it will be sufficient to prove that \( \lambda_1(t) < \lambda_2(t) \) for \( t \in (0, 1) \). This is guaranteed by (4.2), (4.1), (2.6), (2.7) and Remark 2.9. This concludes the proof of Theorem 1.1. \( \square \)

![Figure 1](image-url)  
**Figure 1.** The double first Aharonov–Bohm eigenvalue \( \lambda_1(0) \) splits in two different branches of simple eigenvalues up to the boundary.
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References

[1] L. Abatangelo. Sharp asymptotics for the eigenvalue function of Aharonov–Bohm operators with a moving pole. Rend. Sem. Mat. Univ. Politec. Torino Bruxelles-Torino Talks in PDE’s Turin, May 2–5, 2016, Vol. 74, 2: 19–29, 2016.
[2] L. Abatangelo and V. Felli. Sharp asymptotic estimates for eigenvalues of Aharonov–Bohm operators with varying poles. Calc. Var. Partial Differential Equations, 54(4):3677–3903, 2015.
[3] L. Abatangelo and V. Felli. On the leading term of the eigenvalue variation for Aharonov–Bohm operators with a moving pole. SIAM J. Math. Anal., 48(4):2843–2868, 2016.
[4] L. Abatangelo, V. Felli, and C. Léna. On Aharonov–Bohm operators with two colliding poles. Advanced Nonlin. Studies, 17:283–296, 2017.
[5] L. Abatangelo, V. Felli, B. Noris, and M. Nys. Sharp boundary behavior of eigenvalues for Aharonov–Bohm operators with varying poles. J. Funct. Anal., 273: 2428–2487, 2017.
[6] L. Abatangelo, M. Nys. On multiple eigenvalues for Aharonov–Bohm operators in planar domains. Nonlinear Analysis, 2017, https://doi.org/10.1016/j.na.2017.11.010.02.
[7] R. Adami and A. Teta. On the Aharonov–Bohm Hamiltonian.lett. Math. Phys., 43(1):43–53, 1998.
[8] Y. Aharonov and D. Bohm. Significance of electromagnetic potentials in the quantum theory. Phys. Rev. (2), 115:485–491, 1959.
[9] A. A. Balinsky. Hardy type inequalities for Aharonov-Bohm magnetic potentials with multiple singularities. Math. Res. Lett., 10, no. 2:369–176, 2003.
[10] V. Bonnaillie-Noël and B. Helffer. On spectral minimal partitions: the disk revisited. Ann. Univ. Buchar. Math. Ser., 4(LXIII)(1):321–342, 2013.
[11] V. Bonnaillie-Noël, B. Helffer, T. Hoffmann-Ostenhof, Aharonov-Bohm Hamiltonians, isospectrality and minimal partitions. J. Phys. A (18), 42:185–203, 2009.
[12] V. Bonnaillie-Noël, B. Helffer. Numerical analysis of nodal sets for eigenvalues of Aharonov-Bohm Hamiltonians on the square with application to minimal partitions. Exp. Math., 20 no. 3: 304–322, 2011.
[13] V. Bonnaillie-Noël, C. Léna. Spectral minimal partitions of a sector. Discrete Contin. Dyn. Syst. Ser. B, 19 no. 1: 27–53, 2014.
[14] V. Bonnaillie-Noël, B. Noris, M. Nys, and S. Terracini. On the eigenvalues of Aharonov–Bohm operators with varying poles. Anal. PDE, 7(6):1365–1395, 2014.
[15] M. Dauge, B. Helffer. Eigenvalues variation. II. Multidimensional problems. J. Diff. Eq. 104: 236–297, 1993
[16] V. Felli, A. Ferrero, and S. Terracini. Asymptotic behavior of solutions to Schrödinger equations near an isolated singularity of the electromagnetic potential. J. Eur. Math. Soc. (JEMS), 13(1):119–174, 2011.
[17] B. Helffer, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and M. P. Owen. Nodal sets for groundstates of Schrödinger operators with zero magnetic field in non-simply connected domains. Comm. Math. Phys., 202(3):629–649, 1999.
[18] B. Helffer. On spectral minimal partitions: a survey. Milan J. Math., 78, no. 2: 575–590, 2010.
[19] B. Helffer, T. Hoffmann-Ostenhof. On minimal partitions: new properties and applications to the disk. Spectrum and dynamics, CRM Proc. Lecture Notes, 52, Amer. Math. Soc., Providence, RI: 119–135, 2010.
[20] B. Helffer, T. Hoffmann-Ostenhof. On a magnetic characterization of spectral minimal partitions. J. Eur. Math. Soc. (JEMS), 15, no. 6, 2081–2092, 2013.
[21] B. Helffer, T. Hoffmann-Ostenhof, S. Terracini. Nodal domains and spectral minimal partitions. Ann. Inst. H. Poincaré Anal. Non Linéaire, 26:101–635, 2009.
[22] B. Helffer, T. Hoffmann-Ostenhof, S. Terracini. Nodal minimal partitions in dimension 3. Discrete Contin. Dyn. Syst., 28, no. 2: 617–635, 2010.
[23] B. Helffer, T. Hoffmann-Ostenhof, S. Terracini. On spectral minimal partitions: the case of the sphere. Around the research of Vladimir Maz’ya. III, Int. Math. Ser. (N. Y.), 13, Springer, New York, 153–178, 2010.
[24] T. Kato. Perturbation theory for linear operators. Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, xxii+619 pp., 1995.
[25] A. Laptev and T. Weidl. Hardy inequalities for magnetic Dirichlet forms. In Mathematical results in quantum mechanics (Prague, 1998), volume 108 of Oper. Theory Adv. Appl., pages 299–305. Birkhäuser, Basel, 1999.
[26] C. Léna. Eigenvalues variations for Aharonov–Bohm operators. J. Math. Phys., 56(1):011502, 18, 2015.
[27] M. Melgaard, E. M. Ouhabaz, and G. Rozenblum. Negative discrete spectrum of perturbed multivortex Aharonov–Bohm Hamiltonians. Ann. Henri Poincaré, 5(5):979–1012, 2004.
[28] A. M. Micheletti. Perturbazione dello spettro dell’operatore di Laplace, in relazione ad una variazione del campo. Ann. Scuola Norm. Sup. Pisa (3), 26:151–169, 1972.

THE FIRST AB EIGENVALUE ON THE DISK 9
[29] B. Noris, M. Nys, and S. Terracini. On the Aharonov–Bohm operators with varying poles: the boundary behavior of eigenvalues. *Comm. Math. Phys.*, 339(3): 1101–1146, 2015.

[30] B. Noris, S. Terracini. Nodal sets of magnetic Schrödinger operators of Aharonov-Bohm type and energy minimizing partitions. *Indiana University Mathematics Journal* (4), 59: 1361–1403, 2010.

[31] M. Teytel. How rare are multiple eigenvalues? *Communications on Pure and Applied Mathematics*, 52: 917–934, 1999.

[32] J. C. Saut, R. Temam. Generic properties of nonlinear boundary value problems. *Comm. Partial Differential Equations* (3), 4: 293–319, 1979.

[33] G.N. Watson. A treatise on the theory of the Bessel functions. *Cambridge at the University press*, 1944.

Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca,
Via Cozzi 55, 20125 Milano, Italy.
E-mail address: laura.abatangelo@unimib.it