Quantum noise reduction in singly resonant optical devices

C. Cabrillo, J. L. Roldán, P. García-Fernández
Instituto de Estructura de la Materia, CSIC, Serrano 123, 28006 Madrid, Spain.
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Quantum noise in a model of singly resonant frequency doubling including phase mismatch and driving in the harmonic mode is analyzed. The general formulae about the fixed points and their stability as well as the squeezing spectra calculated linearizing around such points are given. The use of a nonlinear normalization allows to disentangle in the spectra the dynamic response of the system from the contributions of the various noisy inputs. A general “reference” model for one-mode systems is developed in which the dynamic aspects of the problem are not contaminated by static contributions from the noisy inputs. The physical insight gained permits the elaboration of general criteria to optimize the noise suppression performance. With respect to the squeezing in the fundamental mode the optimum working point is located near the first turning point of the dispersive bistability induced by cascading of the second order nonlinear response. The nonlinearities induced by conventional crystals appear enough to reach it being the squeezing ultimately limited by the escape efficiency of the cavity. In the case of the harmonic mode both, finite phase mismatch and/or harmonic mode driving allow for an optimum dynamic response of the system something not possible in the standard phase matched Second Harmonic Generation. The squeezing is then limited by the losses in the harmonic mode, allowing for very high degrees of squeezing because of the non-resonant nature of the mode. This opens the possibility of very high performances using artificial materials with resonantly enhanced nonlinearities. It is also shown how it is possible to substantially increase the noise reduction and at the same time to more than double the output power for parameters corresponding to reported experiments.

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I. INTRODUCTION

Second Harmonic Generation (SHG) has nowadays quite a long tradition as a mean of squeezed light generation [1–7]. The preferred experimental setup has been the doubly resonant configuration as, at least in principle, permits arbitrarily large squeezing. However, such scheme has been hampered by the technical difficulties arising from keeping the resonance in both modes simultaneously. Thus, in spite of the development of very ingenious stabilizing procedures [3], for the moment it has been only possible to maintain the double resonance for a few seconds. Certainly, this kind of experimental delicacy can hardly surprise when dealing with the generation of non-classical states of light. In view of such difficulties, some experimental efforts have been recently redirected to singly resonant configurations [4–6]. Although, the maximum noise suppression is then limited to a 90% [4], the efforts resulted in very stable intense squeezed light sources with degrees of squeezing even surpassing those reported in the doubly resonant counterparts [6]. This evolution highlights the importance of reducing to a minimum the technical demanding of new proposals in a so experimentally challenging field.

At the same time, singly resonant Optical Parametric Oscillation (OPO), the most successful method to squeeze the vacuum [8], has been generalized to singly resonant Optical Parametric Amplification (OPA), i.e. a laser driving in the harmonic mode has been added, again showing an extraordinary stability at quite high noise suppression values in the fundamental mode [10]. Although the squeezed beams are in this case much less intense than the in SHG counterpart, this setup permits a control of the phase of the squeezed quadrature, something which allowed a spectacular demonstration using quantum tomography, of the different kinds of squeezed states [11].

In view of this experimental success it seems timely to extend the quantum mechanical model beyond the pure phase matched cases. More specifically, we address here quantum noise reduction in an extension of the conventional singly resonant SHG to include also a coherent input in the harmonic mode as well as phase mismatch between the interacting waves.

On the other hand, an increasingly number of papers is being devoted to study quantum noise in systems combining different kinds of nonlinearities (see, for instance, [12–14] for some recent contributions). In particular, the combination of $\chi^{(2)}$ with Kerr-like $\chi^{(3)}$ nonlinearities in c.w. cavity systems has been quite extensively studied [15,17,22]. Even exact full quantum results have been obtained showing, for instance, the emergence of tristability not present in the classical counterpart [21]. With respect to the squeezing performance the results appear as very promising at least in degenerate doubly resonant configurations [15,22]. The simplest system from the implementation point of view,
combining this two kinds of nonlinearities that the authors can think of is precisely a singly resonant second order nonlinear system with phase mismatched interacting waves as then, by virtue of the cascading effect, an effective Kerr-like third order nonlinearity appears.

In order to make the search of strong noise reduction through the parameter space affordable we are bounded to the standard linearization procedures, the only capable of yielding analytical results. Inside the linear approximation perfect squeezing is possible at dynamic instabilities. We make use of this fact to find optimum working points showing up maximum squeezing. They are, however, an artifact of the method as the linear approximation breaks down at the instabilities. What matters for the practical implementation of new squeezed light sources (our ultimate goal) are the optimum paths through the parameter space approaching such points. Let us explain a little more what we mean. Fixed a particular parameter there will be a set of values of the remaining parameters (including the frequency as such) tuned up to yield the maximum noise reduction. When the chosen parameter is varied an optimum path is defined by the set of parameter values maximizing the noise reduction at any stage of the variation. The real optimum working point will be somewhere along these paths before reaching an instability. Thus, these paths would guide the experimentalist towards the optimum working point in the real experimental setup. The essence of our approach to find such paths will be to isolate the dynamic aspect of the squeezing behavior from the static contributions to the noise coming from the different inputs. A simple adequate normalization will disentangle the two aspects of the quantum noise behavior. This simplifies the analysis sufficiently to allow a characterization of the optimum paths.

Another crucial issue when dealing with the squeezing performance of a system is precisely the election of the most relevant parameter to compare the different configurations with each other or with the reported related experiments. There is no universal criterion to determine the squeezing efficiency of a given device. An efficient setup regarding power consumption, i.e., when compared for fixed input power could well be deceptive when compared for the same output power and perhaps inadequate to some spectroscopic applications. However, within the state of the art of the present squeezed light generators, the main concern is to improve the squeezing figures themselves having other considerations such as the power consumption a relative importance. Under this perspective probably the parameter of utmost importance as far as c.w. resonant systems are concerned is the energy load inside the cavity. Indeed, the usual causes of squeezing degradation such as blue-light-induced red absorption come from an excessive mean photon number inside the cavity capable of significantly degrade the material optical response at the relevant frequencies. These considerations will lead us to define another normalization this time useful for the evaluation of the squeezing efficiency with respect to the intra-cavity photon number.

The sketch of the article is as follows. In section II the quantum mechanical model is presented. In section III the evolution equations are linearized, the fixed points of the system obtained and their stability studied. Section IV gives all the formulae regarding quantum noise spectra in the system. In section V a general approach to one-mode systems is developed which allows the definition of general criteria to characterize the optimum paths and applied to the specific case here addressed. Finally the limits of the model and possible implementations are thoroughly discussed in section VI, concluding the article with a summary of the most relevant results obtained.

II. QUANTUM MECHANICAL MODEL

The system we want to address consists in a second order nonlinear medium coupling two modes of frequency \( \omega \) (fundamental) and \( 2\omega \) (harmonic) respectively and placed inside a ring cavity resonant only with the fundamental mode. We will also assume just one input-output mirror of finite reflectivity. The effect of phase mismatch when only the fundamental mode is driven has been experimentally studied in [23] where bistability induced by cascading was demonstrated. The classical evolution equation of the fundamental mode, \( \alpha \), as given in [24], reads

\[
\frac{d\alpha}{dt} = -\left[\gamma + i\delta + \nu K(\Delta k)\alpha^2\right] \alpha + \sqrt{2\gamma_c} \alpha_{in} .
\]  

The nonlinear coupling depends on the wave vector mismatch \( \Delta k = k(2\omega) - k(\omega) \) as \( K(\Delta k) = 2 \int_0^{L_m} \int_0^{2\pi} u^*(\Delta k, z) u(\Delta k, z') dz' dz / L_m^2 \) being \( L_m \) the length of the nonlinear medium, \( u(k, z) \) the spatial dependence of the resonator mode and \( \nu \) is proportional to the second order nonlinear susceptibility (see below). Splitting \( K \) in its real and imaginary parts, Eq. (2.1) can be recast as

\[
\frac{d\alpha}{dt} = -\left[\gamma + i|\alpha|^2 + i(\delta + \Gamma|\alpha|^2)\right] \alpha + \sqrt{2\gamma_c} \alpha_{in} .
\]  

For a plane wave geometry
\[ \mu \equiv \nu K_r = \nu \left( \sin \frac{\Delta k L_m}{2} \right)^2 \]  
\[ \Gamma \equiv \nu K_i = \frac{2
u}{\Delta k L_m} \left[ \sin \frac{\Delta k L_m}{2} \cos \frac{\Delta k L_m}{2} - 1 \right], \]

where \( K_r \) and \( K_i \) denote the real and imaginary part of \( K(\Delta k) \). In this way the nonlinear dynamics is divided in a nonlinear absorption (the up conversion of photons) and a nonlinear dispersion (the cascading effect). The behavior of both parameters with \( \Delta k \) is plotted in Fig. 1. Notice that for any finite \( \Delta k \) the nonlinear dispersion is also finite, periodically completely dominating (null frequency doubling).

Quantization of Eq. (2.2) is then accomplished independently for each effect. Regarding the nonlinear absorption we use the two-photon model proposed in \([26]\), while nonlinear dispersion is accounted for by a fourth order Hamiltonian, \( H = (\hbar \Gamma / 2) a^\dagger a^2 \) as in the standard theory of optical Kerr effect. It represents a Hamiltonian modification of the two-photon absorption model so that the quantum mechanical equation reads

\[ \frac{da}{dt} = -\left[ \gamma + i \delta + (\mu + i \Gamma) a^\dagger a \right] a + 2\sqrt{\mu} a^\dagger b_{in} + \sqrt{2\gamma_c} a_{in} + \sqrt{2\gamma_s} w_{in}, \]  

where Latin characters denote the annihilation operators for the corresponding classical (Greek characters) modes. Two extra terms not present in the classical analog appear, namely, a white noise input, \( w_{in} \), accounting for the fluctuations induced by the scattering and the absorption in the crystal \( (\gamma_s = \gamma - \gamma_c) \) and a parametric “gain” term coming from the, classically empty, incoming harmonic mode, \( b_{in} \). Eq. (2.4) is complemented with the boundary conditions

\[ a_{out} = \sqrt{2\gamma_c} a - a_{in}, \]  
\[ b_{out} = \sqrt{\mu} a^2 - b_{in}, \]  

from which the output spectra can be computed. Input fields are assumed to be in coherent states. In particular, allowing a coherent state different from the vacuum for the incoming harmonic mode we generalize the system to the case of driving both modes. In the case \( \Gamma = \delta = 0 \), the squeezing properties as well as the applicability to quantum nondemoliton measurements of this system have been studied in detail in \([27]\).

The used definitions for the creation operators give the following relations with the usual experimental parameters (see appendix in \([27]\)): the input and output powers are \( P_{\omega, in/out} = \hbar \omega \langle a_{in/out}^\dagger a_{in/out} \rangle \) and \( P_{2\omega, in/out} = \hbar 2\omega \langle b_{in/out}^\dagger b_{in/out} \rangle \); the circulating power is \( \hbar \omega \langle a^\dagger a \rangle / \tau \) being \( \tau \) the round-trip time and the single-pass power-conversion efficiency (in \( W^{-1} \)) is \( 2\tau^2 \nu / \hbar \omega \).

### III. LINEARIZED EVOLUTION EQUATIONS AND LINEAR STABILITY ANALYSIS

Defining fluctuation operators as
a linearization of Eqs. (2.4) and (2.5) yields

\[
\frac{d\alpha}{dt} = -\left[\gamma + i\delta + 2(\mu + i\Gamma)|\alpha|^2\right] \alpha + \left[2\sqrt{\mu}\beta_{in} - (\mu + i\Gamma)\alpha^2\right] \delta a^t + 2\sqrt{\mu} \alpha^* \delta b_{in} + \sqrt{2\gamma_c} \delta a_{in} + \sqrt{2\gamma_s} w_{in} ,
\]

(3.2)

and

\[
\begin{align*}
\delta a_{out} &= \sqrt{2\gamma_c} \delta a - \delta a_{in} , \\
\delta b_{out} &= 2\alpha \sqrt{\mu} \delta a - \delta b_{in} ,
\end{align*}
\]

(3.3a, 3.3b)

being \(\alpha_{in, out}, \beta_{in, out}\) the mean values of the corresponding input and output modes and \(\alpha\) a stable fixed point of the classical counterpart of Eq. (2.4), i.e.

\[
\frac{d\alpha}{dt} = -\left[\gamma + i\delta + (\mu + i\Gamma)|\alpha|^2\right] \alpha + 2\sqrt{\mu} \alpha^* \beta_{in} + \sqrt{2\gamma_c} \alpha_{in} .
\]

(3.4)

Equating to zero the l.h.s. of Eq. (3.4) a “state equation” for the fixed points is obtained, namely,

\[
\alpha = \frac{\sqrt{2\gamma_c} \left\{ [\gamma + \mu - i(\delta + \Gamma)n] \alpha_{in} + 2\sqrt{\mu} \beta_{in} \alpha_{in}^* \right\}}{(\gamma + \mu n)^2 + (\delta + \Gamma n)^2 - 4\mu |\beta_{in}|^2} ,
\]

(3.5)

with \(n = |\alpha|^2\). Let be \(\theta, \phi\) and \(\varphi\) the phases of \(\alpha, \alpha_{in}\) and \(\beta_{in}\) respectively. Then, dividing both sides of Eq. (3.5) by \(e^{i\varphi/2}\)

\[
\begin{align*}
|\alpha|e^{i(\theta - \varphi/2)} [\gamma + \mu n + i(\delta + \Gamma n)] \alpha_{in} + 2\sqrt{\mu} \beta_{in} \alpha_{in}^* e^{i(\phi - \varphi/2)} + 2\sqrt{\mu} |\beta_{in}| e^{-i(\phi - \varphi/2)} \alpha_{in} \\
= |\alpha_{in}| \sqrt{2\gamma_c} \left\{ (\gamma + \mu n - i(\delta + \Gamma n)) e^{i(\phi - \varphi/2)} + 2\sqrt{\mu} |\beta_{in}| e^{-i(\phi - \varphi/2)} \right\} .
\end{align*}
\]

(3.6)

Taking the squared modulus in both sides a quintic equation for \(n\) is obtained

\[
0 = n \left\{ (\gamma + \mu n)^2 + (\delta + \Gamma n)^2 - 4\mu |\beta_{in}|^2 \right\} - 2\gamma_c |\alpha_{in}|^2 \left\{ (\gamma + \mu n)^2 + (\delta + \Gamma n)^2 + 4\mu |\beta_{in}|^2 + 4\sqrt{\mu} |\beta_{in}| \{(\gamma + \mu n) \cos(2\phi - \varphi) + (\delta + \Gamma n) \sin(2\phi - \varphi)\} \right\} .
\]

(3.7)

The real and the imaginary part of Eq. (3.4) determine the \(\sin(\theta - \varphi/2)\) and \(\cos(\theta - \varphi/2)\) as functions of the solutions of Eq. (3.7)

\[
\begin{align*}
\cos(\theta - \varphi/2) &= \frac{|\alpha_{in}|}{|\alpha|} \sqrt{2\gamma_c} \left\{ (\gamma + \mu n + 2\sqrt{\mu} |\beta_{in}|) \cos(\phi - \varphi/2) + (\delta + \Gamma n) \sin(\phi - \varphi/2) \right\} \left\{ (\gamma + \mu n)^2 + (\delta + \Gamma n)^2 - 4\mu |\beta_{in}|^2 \right\} , \\
\sin(\theta - \varphi/2) &= \frac{|\alpha_{in}|}{|\alpha|} \sqrt{2\gamma_c} \left\{ (\gamma + \mu n - 2\sqrt{\mu} |\beta_{in}|) \sin(\phi - \varphi/2) - (\delta + \Gamma n) \cos(\phi - \varphi/2) \right\} \left\{ (\gamma + \mu n)^2 + (\delta + \Gamma n)^2 - 4\mu |\beta_{in}|^2 \right\} .
\end{align*}
\]

(3.8a, 3.8b)

Eq. (3.7) allows for numerical calculation of the fixed points given the input fields. But it can be interpreted also as a linear equation for \(|\alpha_{in}|^2\), i.e.

\[
2\gamma_c |\alpha_{in}|^2 = \frac{n \left\{ (\gamma + \mu n)^2 + (\delta + \Gamma n)^2 - 4\mu |\beta_{in}|^2 \right\}^2}{\gamma + \mu n + 2\sqrt{\mu} |\beta_{in}| e^{i(\phi - \varphi/2)} + 4\sqrt{\mu} |\beta_{in}| (\delta + \Gamma n) \sin(2\phi - \varphi) .
\]

(3.9)

The positive character of the r.h.s. is not always guaranteed and therefore not for every value of the parameters a real positive \(n\) is possible. Notice, however, that in the cases in which this happens a simultaneous change of the sign of \(\delta\) and \(\Gamma\) yields a consistent set of parameter values. As we shall see, this fact will have useful consequences in regarding the analysis of the quantum noise behavior in the system.

The stability of the fixed points is governed by the real part of the eigenvalues of the drift matrix associated with the linearized evolution equation (3.2). Very simple algebra yields

\[a = \alpha + \delta a,\]
\[a_{in, out} = \alpha_{in, out} + \delta a_{in, out},\]
\[b_{in, out} = \beta_{in, out} + \delta b_{in, out},\]
\[ \lambda_{\pm} = -(\gamma + 2\mu n) \pm \sqrt{|(\mu + i\Gamma)\alpha^2 - 2\sqrt{\mu} \beta_{in}|^2 - (\delta + 2\Gamma n)^2}. \] (3.10)

Provided that the real part of both eigenvalues are negative the fixed point will be stable. With respect to the phase-matched SHG case (\(\Gamma = 0\) and \(\beta_{in} = 0\), always stable), although both \(\Gamma\) and \(\delta\) alone tend to stabilize the dynamics, in combination are able of destabilize the system. A finite \(\beta_{in}\), on the other hand, can promote instability depending on its relative phase with respect to \(\alpha\), the case \(\theta - \varphi/2 = \pm \pi/2\) maximizing the effect. All of these new instabilities, however, correspond to zero eigenvalues without a finite imaginary part. In other words, contrary to the double resonant SHG there is no Hopf bifurcation and, consequently, no self-pulsing solution.

IV. SQUEEZING SPECTRA

For a given quadrature of the electric field, \(X_{\theta_x}^{\text{out}}(t) \equiv a_{\text{out}}(t)e^{-i\theta_x} + a_{\text{out}}^\dagger(t)e^{i\theta_x}\), the squeezing spectrum is simply the noise spectrum of such a quantity, i.e.

\[
S(\omega) = C \int_{-\infty}^{\infty} \langle \delta X_{\theta_x}^{\text{out}}(t) \delta X_{\theta_x}^{\text{out}}(t + \tau) \rangle e^{-i\omega \tau} d\tau
\]

\[
= C \int_{-\infty}^{\infty} \langle \delta X_{\theta_x}^{\text{out}}(\omega) \delta X_{\theta_x}^{\text{out}}(-\omega) \rangle d\omega',
\]

(4.1)

being \(C\) some normalization constant and the averages are assumed stationary. As a function of the annihilation and creation operators Eq. (4.1) is rewritten as

\[
S(\omega) = C \left[ \langle \delta a_{\text{out}}^\dagger(\omega) \delta a_{\text{out}}(-\omega) \rangle + \text{Re} \{ \exp(-i2\theta_x) \langle \delta a_{\text{out}}(\omega) \delta a_{\text{out}}(-\omega) \rangle \} \right],
\]

(4.2)

where use has been made of the stationarity of the average and \(\text{Re}\) denotes real part. From this expression it is evident that the noise is minimized and therefore the squeezing effect maximized for a quadrature phase such as

\[
S(\omega) = C \left[ \langle \delta a_{\text{out}}^\dagger(\omega) \delta a_{\text{out}}(-\omega) \rangle - \langle \delta a_{\text{out}}(\omega) \delta a_{\text{out}}(-\omega) \rangle \right],
\]

(4.3)

corresponding to a phase

\[
\theta_m = \frac{\nu(\omega) - \pi}{2},
\]

(4.4)

where \(\nu(\omega)\) is the phase of \(\langle \delta a_{\text{out}}(\omega) \delta a_{\text{out}}(-\omega) \rangle\). The spectrum of the conjugate quadrature (i.e. with a phase \(\nu(\omega)/2\)) corresponds to a plus sign in Eq. (4.3) and by virtue of the Heisenberg principle shows an excess noise above the vacuum. Taking \(C = 1\) (corresponding to vacuum noise units) and splitting Eq. (4.3) into a vacuum noise component plus a normally ordered part we finally arrive to

\[
S_{-,+}(\omega) = 1 + \langle \delta a_{\text{out}}^\dagger(\omega) \delta a_{\text{out}}(-\omega) \rangle \mp \langle \delta a_{\text{out}}(\omega) \delta a_{\text{out}}(-\omega) \rangle,
\]

(4.5)

for both the squeezing and the “stretching” spectra. After tedious but simple algebra, the spectra of the fundamental and second harmonic modes can be written as

\[
S_{-,+}^a(\omega) = 1 + 4\gamma_c |B|^2 \frac{N_{-,+}}{D},
\]

(4.6a)

\[
S_{-,+}^b(\omega) = 1 + 8\mu n |B|^2 \frac{N_{-,+}}{D},
\]

(4.6b)

where \(B = 2\sqrt{\mu} \beta_{in} - (\mu + i\Gamma)\alpha^2\) and

\[
N_{-,+} = 2|B|(\gamma + 2\mu n) \mp \sqrt{[(\gamma + 2\mu n)^2 - (\delta + 2\Gamma n)^2 + |B|^2 + \omega^2]^2 + 4(\gamma + 2\mu n)^2(\delta + 2\Gamma n)^2},
\]

(4.7a)

\[
D = [(\gamma + 2\mu n)^2 + (\delta + 2\Gamma n)^2 - |B|^2 - \omega^2]^2 + 4(\gamma + 2\mu n)^2 \omega^2.
\]

(4.7b)

The correlations defining the squeezing phase \(\nu(\omega)\) are given by

\[
\langle \delta a_{\text{out}}(\omega) \delta a_{\text{out}}(-\omega) \rangle = 4\gamma_c B [\omega^2 + |B|^2 + (\gamma + 2\mu n)^2 - (\delta + 2\Gamma n)^2 + i2(\gamma + 2\mu n)(\delta + 2\Gamma n)] / D,
\]

(4.8a)

\[
\langle \delta b_{\text{out}}(\omega) \delta b_{\text{out}}(\omega) \rangle = 8\mu n \alpha^2 B [\omega^2 + |B|^2 + (\gamma + 2\mu n)^2 - (\delta + 2\Gamma n)^2 + i2(\gamma + 2\mu n)(\delta + 2\Gamma n)] / D.
\]

(4.8b)

The trigonometric equations for the corresponding phases are quite complicated and rather useless. However, an interesting consequence can directly be drawn from Eqs. (4.8), namely, for detuning-\(s\) such as \(\delta + 2\Gamma n = 0\) the phases are independent of \(\omega\) equaling those of \(B\) and \(\alpha^2 B\) respectively.
V. SQUEEZING PERFORMANCE

In the previous sections we have developed the general raw formulae regarding quantum noise in the system. This section is devoted to the analysis of the quantum noise behavior implied by them and in particular to proceed with our program of finding optimum quantum noise reduction. However, before any specific assessment of the quantum noise performance we will elaborate a little more on the formulae (mainly by adequate normalizations) in order to gain physical insight and ease our task. As an aftermath we shall obtain general results going well beyond the specifics of the system addressed here.

A. General results concerning one-mode systems

Let us begin defining a nonlinear and a total decay rate as \( \gamma_{nl} \equiv 2\mu n \) and \( \gamma_t \equiv \gamma + \gamma_{nl} \) respectively. We shall scale the evolution with this total decay rate defining a dimensionless time \( \tau \equiv \gamma_t t \). In the spectra (4.6) the only dependence on \( \theta \) is through \( B \) disappearing for \( \beta_{in} = 0 \). It is also possible to restrict this dependence to such a term directly in Eq. (3.2) and Eq. (3.3) by means of appropriate phase shifts of the modes. All together account for

\[
\frac{d \delta c}{d\tau} = -\left[1 + i\Delta\right]\delta c + \tilde{B} \delta c^\dagger + \sqrt{2\tilde{\gamma}_{nl}} \delta d_{in} + \sqrt{2\tilde{\gamma}_c} \delta c_{in} + \sqrt{2\tilde{\gamma}_s} s_{in} ,
\]

where the tilde represents divided by \( \gamma_t \), \( \Delta = \tilde{\delta} + 2\tilde{\Gamma}n \), \( \tilde{B} = 2\sqrt{\tilde{\mu} \delta} - (\tilde{\mu} + i\tilde{\Gamma})n \) and the modes are redefined as

\[
\begin{align*}
    c &\equiv a e^{-i\theta}, \\
    c_{in,out} &\equiv \frac{a_{in,out}}{\sqrt{\gamma_t}} e^{-i\theta}, \\
    d_{in,out} &\equiv \frac{b_{in,out}}{\sqrt{\gamma_t}} e^{-i2\theta}, \\
    s_{in} &\equiv \frac{w_{in}}{\sqrt{\gamma_t}} e^{-i\theta}.
\end{align*}
\]

In agreement with the previous notation \( \delta_{in} \) denotes the mean value of \( d_{in} \). The boundary conditions of the new modes are

\[
\begin{align*}
    \delta c_{out} &= \sqrt{2\tilde{\gamma}_{nl}} \delta c - \delta c_{in} , \\
    \delta d_{out} &= \sqrt{2\tilde{\gamma}_{nl}} \delta c - \delta d_{in} .
\end{align*}
\]

For coherent states, the correlations of the new input modes remain as white noise but in the scaled time \( \tau \). We shall refer to the previous formulae as the tilde normalization.

The evolution equation (5.1) encodes the dynamic response of the intracavity system to a series of noisy input channels \( (\delta d_{in}, \delta c_{in} \) and \( s_{in} \)). Quantum Mechanical consistency, i.e., conservation of equal-time commutators, imposes a fluctuation-dissipation relation which under this normalization reads

\[
\tilde{\gamma}_{nl} + \tilde{\gamma}_c + \tilde{\gamma}_s = 1 .
\]

The evolution equation (5.1) along with Eqs. (5.3) are now written in such a way that the input-output couplings are real-valued as in the standard input-output formalism \[28,29\]. This is a completely general result. Provided a well-defined linearized theory in the sense of preserving equal-time commutators we only need the adequate set of phase shifts of the input channels (a trivial unitary transformation preserving commutators) making the couplings real-valued to obtain a theory formally equal to the standard input-output formalism simply because this is the theory preserving the equal-time commutators when the couplings are real-valued. Thus, for any system with only one effective mode there is a formulation in which the intracavity field follows

\[
\frac{d \delta c}{d\tau} = -\left[1 + i\Delta\right]\delta c + \tilde{B} \delta c^\dagger + \sum_{n=1}^{N} \sqrt{2\tilde{\gamma}_{nl}} \delta c_{in}^n ,
\]

with
The frequency scale $\gamma_t$ defining the dimensionless time $\tau$ is just the real part of the factor multiplying $\delta c$ after the phase shifts. N-1 of the input channels will have a time-reversed counterpart corresponding to the outgoing channels fulfilling
\[
\delta c_{\text{out}}^n = \sqrt{2\gamma_n} \delta c - \delta c_{\text{in}}^n.
\]
The remaining input channel will account for the irreversible losses. The corresponding spectra are related with the intracavity spectra by
\[
S_{n,+}^{\text{ref}}(\tilde{\omega}) = 1 + 2\gamma_n : S_{n,+}(\tilde{\omega}) :,
\]
where $S_{n,+}(\tilde{\omega})$ denotes the intracavity spectra and we have made use of the proportionality of normally ordered intracavity and outgoing correlations. The spectra $S_{n,+}^{\text{ref}}(\tilde{\omega})$ coincide with the spectra of the original formulation as the new outgoing modes are just a phase sift of the originals.

Let us now define a sort of “reference” system with only one time-reversible input channel, i.e.,
\[
\frac{d\delta c}{d\tau} = -[1 + i\Delta] \delta c + \tilde{B} \delta c^\dagger + \sqrt{2} \delta c_{\text{in}}^{\text{ref}},
\]
and
\[
\delta c_{\text{out}}^{\text{ref}} = \sqrt{2} \delta c - \delta c_{\text{in}}^{\text{ref}}.
\]
Obviously $S_{n,+}^{\text{ref}}(\tilde{\omega}) : = 2 : S_{n,+}(\tilde{\omega}) :$ so that we get finally
\[
S_{n,+}^{\text{ref}}(\tilde{\omega}) = 1 + \gamma_n : S_{n,+}(\tilde{\omega}) :.
\]
This is the central result of this section. Let us elaborate a little about its interpretation. Squeezing in a given output channel means that for a certain range of phase shifts the corresponding quadratures show an intensity of their fluctuations below that of the associated incoming channel (assumed in a coherent state). In view of Eqs. (5.7), the amplitude of the outgoing fluctuations is a coherent superposition of the intracavity and the incoming fluctuations. Squeezing is possible if an adequate correlation between $\delta c$ and the relevant input channel is established. But the intracavity field is nothing else than the dynamic response of the intracavity system to the incoming channels. The input channels are uncorrelated and so the dynamic response of the intracavity system to them. A given input channel can consequently correlate only with the dynamic response to itself. The presence of any other input channel can only degrade the effect. The great advantage of the tilde normalization is that makes this fact explicit. Indeed, Eq. (5.11) express the output spectra as the dynamic response of the system to an isolated input channel, i.e., $S_{n,+}^{\text{ref}}(\tilde{\omega}) :$; scaled down by the “static” contribution to the noise owing to the presence of extra input channels. The scale factor $\gamma_n$ is just the ratio between the coupling constant of the chosen output channel and the sum of all of them.

Eq. (5.11) greatly simplifies our task of finding the optimum path to maximum noise reduction as we can center our efforts onto the simple reference system described by Eqs. (5.9) and (5.10). Even more interesting the results concerning the reference system will be of general applicability to any one-mode system including as such any multiply resonant system under adiabatic elimination of all the modes but one. The normally ordered spectra of the reference system are easily calculated as
\[
S_{n,+}^{\text{ref}}(\tilde{\omega}) : = 4|\tilde{B}| \frac{2|\tilde{B}| + \sqrt{(1 + \tilde{\omega}^2 + |\tilde{B}|^2 - \Delta^2)^2 + 4\Delta^2}}{(1 - \tilde{\omega}^2 - |\tilde{B}|^2 + \Delta^2)^2 + 4\tilde{\omega}^2}.
\]

Our first step is to determine if the dynamic response is capable of a total noise suppression. Perfect squeezing can only occur at a dynamic instability. Equaling to zero the l.h.s. of Eq. (5.10) (the only possible unstable eigenvalue) and after proper normalization an equation determining the instability can be written as
\[
1 + \Delta^2 = |\tilde{B}|^2.
\]
Written in this way an interesting parallelism with the standard OPO below threshold shows up, i.e. an instability appears when the modulus of the “losses” coefficient equals that of the “parametric” coefficient, a sort of natural
extension of the condition for the instability in the conventional OPO for which the coefficients are real. Inserting the instability condition (5.13) in \( S_{ref}^\ast (\tilde{\omega}) \) : results in

\[
S_{ref}^\ast (\tilde{\omega}) = 4|\tilde{B}| \frac{2|\tilde{B}| - \sqrt{4|\tilde{B}|^2 + \tilde{\omega}^2(\tilde{\omega}^2 + 4)}}{\tilde{\omega}^2(\tilde{\omega}^2 + 4)}. \tag{5.14}
\]

Applying L'Hopital’s rule with respect to \( \tilde{\omega}^2 \), \( S_{ref}^\ast (\tilde{\omega}) \) : equals -1 at \( \tilde{\omega} = 0 \), that is, perfect squeezing is obtained at the instability, again in parallel with OPO. In other words, the dynamic response of the system assuming that the condition Eq. (5.13) is reachable, is capable of a complete suppression of quantum noise.

Spectrum (5.12) is simple enough to permit analytical optimization. Taking partial derivative in \( S_{ref}^\ast (\tilde{\omega}) \) with respect to \( \tilde{\omega} \) and equaling to zero, \( \tilde{\omega} = 0 \) appears as the optimum point whatever the values of \( \Delta \) and \( |\tilde{B}| \). The same applies to \( \Delta = 0 \) when taking partial derivative with respect to \( \Delta \). Notice that this last condition implies also a squeezing phase independent of the frequency. The optimized noise obtained imposing these two conditions simplifies to

\[
S_{opt} := -\frac{4|\tilde{B}|}{(1 + |\tilde{B}|)^2}, \tag{5.15}
\]

with a minimum at the instability \( |\tilde{B}| = 1 \) approached monotonically. These conditions \( (\tilde{\omega} = 0, \Delta = 0 \text{ and } |\tilde{B}| = 1) \) will help us in finding optimum paths. In particular, moving \( |\tilde{B}| \) from zero to one while maintaining \( \Delta = \tilde{\omega} = 0 \) defines an optimum path reaching the instability for the “reference” model.

An optimum path is defined solely by the squeezing spectrum leaving aside the “stretching” one. It is important to study also the accompanying excess noise on the conjugate quadrature for it could invalidate in practice the optimum path if this excess noise is unbearable high. The minimal excess noise production imposed by the Heisenberg principle corresponds to \( S_{-\ast}(\omega)S_{\ast}(\omega) = 1 \). It is a perfect complementary relation between quadratures: the deamplification of fluctuations in a given quadrature must equal the amplification of fluctuations in the conjugate. In that case we are dealing with a Minimum Uncertainty State (MUS) for those quadratures, the text-book definition of a squeezed state. Adding 1 to Eq. (5.12) and after some minor algebra

\[
S_{-\ast,+}^{ref}(\tilde{\omega}) = \frac{(2|\tilde{B}| \pm \sqrt{(\tilde{\omega}^2 + |\tilde{B}|^2 + 1 - \Delta^2)^2 + 4\Delta^2})}{(1 - \tilde{\omega}^2 - |\tilde{B}|^2 + \Delta^2)^2 + 4\tilde{\omega}^2}. \tag{5.16}
\]

Straightforward algebra leads to \( S_{ref}^\ast (\tilde{\omega})S_{ref}^\ast (\tilde{\omega}) = 1 \). The excess noise is minimum again in parallel to the standard OPO system with real coefficients.

**B. Standard normalization**

As discussed in the introduction we are here principally interested in the squeezing behavior with respect to the photon number \( n \). Unfortunately the tilde normalization is inappropriate to such a task as the frequency scale depends on \( n \) itself. It is far more convenient to use \( \gamma^{-1} \) as the time scale instead of \( \gamma^{-1} \) and to normalize the photon number as \( m = \nu n / \gamma \). In complete parallelism to the tilde normalization we have then,

\[
\frac{d\delta c}{dt} = -\left[ 1 + 2mK_r + i(\tilde{\delta} + 2mK_i) \right] \delta c + \left[ \sqrt{K_r} \eta_{in} - (K_r + iK_i)m \right] \delta c^\dagger + 2\sqrt{mK_r} \delta d_{in} + \sqrt{2\gamma_c} \delta e_{in} + \sqrt{2\gamma_s} s_{in}, \tag{5.17}
\]

where the hat represents divided by \( \gamma \) and

\[
\eta_{in} = \frac{2\sqrt{\nu}}{\gamma} \beta_{in} e^{-i2\theta}, \tag{5.18}
\]

which represents the harmonic mode input amplitude normalized to the value at the standard OPO threshold. The spectra (5.14) become now
C. Squeezing at the fundamental mode

Applying Eq. (5.11) to the fundamental mode

\[ S^a_{\gamma,+}(\tilde{\omega}) = 1 + \gamma_c : S^c_{\gamma,+}(\tilde{\omega}) : \]  

(5.21)

It is clear that the best performance corresponds to \( \gamma_{nl} = 0 \), that is, either \( n = 0 \) or \( \mu = 0 \), as then \( \gamma_c \) maximizes to \( \eta = \gamma_c / (\gamma_c + \gamma_s) \) (the escape efficiency of the cavity). The case \( n = 0 \) corresponds to the very well known case of squeezed vacuum generation. For \( \mu = 0 \) and finite \( n \) the system is formally equivalent to a resonant optical Kerr effect system whose quantum noise behavior has been amply studied previously \[30\]. The condition (5.14) reduces for \( \mu = 0 \) to \( \delta = -2n\Gamma \pm \sqrt{n^2\Gamma^2 - \gamma_s^2} \), the well known turning points of optical dispersive bistability \[30\], but with the nonlinear dispersion induced by cascading. Indeed, such cascading induced bistability has been experimentally demonstrated in \[25\]. Rewriting it within the hat normalization the condition reads

\[ \delta_{\pm} = -2mK_i \pm \sqrt{m^2K_i^2 - 1}, \]  

(5.22)

where \( K_i = -1/\pi \) as it is evaluated at \( \Delta kL_m = 2\pi \). Once \( \gamma_c \) is independent of \( \Delta k \) and \( n \), the optimum path with respect to \( m \) (the only remaining free parameter) is determined solely by the reference system. It corresponds to increase \( m \) till \( m = \pi \) (where the condition (5.22) is reached) while maintaining \( \delta = 2m/\pi \) and \( \omega = 0 \). Fig. 2 displays the evolution of both the maximum squeezing and the maximum excess noise following such a path for three values of the escape efficiency, namely, 0.9, 0.99 and the ideal 1. The noise is expressed in dB’s with respect to the vacuum noise. A Heisenberg limited excess noise appears in such a case as a specular image of the squeezing. The instability is signaled by the divergence in the excess noise. Above it, the curves shown are not physical as they correspond to unstable fixed points. The case \( \eta = 0.99 \) in Fig. 2 shows an excellent behavior with an almost Heisenberg limited excess noise till near the instability.

Fig. 3 illustrates the idea of optimum path by comparing the \( \eta = 0.99 \) plot of Fig. 2 against various cases with fixed values of \( \delta \). Below \( m = \pi \), for a given \( m \) the maximum squeezing is obtained when \( \Delta = 0 \) as expected. Above \( m = \pi \) it is not possible to reach the minimum noise of 1 - \( \eta \) fulfilling \( \Delta = 0 \).

Being \( \mu = 0 \) the formulae simplify enough for allowing a simple expression for the squeezing phase. More specifically, from \( B = -\Gamma \alpha_{out}^2 \), \( \theta_m = \theta + \pi/2 \). On the other hand substituting Eq. (5.5) in Eq. (2.50), results in \( \sqrt{2}\gamma_c\alpha_{out} = \alpha(\gamma_c - \gamma_s + i\Gamma n) \), giving a squeezing phase relative to that of the output field of

\[ \frac{\pi}{2} - \arctan \left( \frac{\Gamma n}{\gamma_c - \gamma_s} \right). \]
FIG. 2. Noise spectra at zero frequency of the fundamental mode following and optimum path for three escape efficiencies of the cavity including the ideal case \( \eta = 1 \). The curves above the divergences are not physical.

FIG. 3. Comparison among noise spectra at zero frequency (fundamental mode) following various paths in the parameter space. The optimum one corresponds to \( \Delta = 0 \). Above the divergences the results are not physical.
FIG. 4. Squeezing in the fundamental mode at zero frequency as a function of the phase mismatch for a low intracavity photon number.

At the instability $\Gamma_n = \gamma$ and for low $\gamma_n$ it approaches $45^\circ$. There is a possibility of an extra control of the squeezing phase not present in the conventional Kerr effect system by making use of the harmonic mode. Taking a finite $\mu$ but low enough so that $\mu n \ll \gamma$ and at the same time a $\beta_m$ high enough to imply $2\sqrt{\mu}\beta_m \approx \gamma$, still we will have $\gamma_{nl} \approx \gamma$ while the squeezing phase relative to the output field will depend on both the modulus and the relative phase between the input fields. In practice, however, maintaining $\mu n$ very low could imply a exceedingly high $\beta_m$ in order to have a $2\sqrt{\mu}\beta_m$ intense enough for a significant influence on the final phases. Of course, the instability point would accordingly depend on $\beta_m$.

There is no hope of any behavior similar to the reported in [15] as a competition between second and third order nonlinearities needs $\mu \neq 0$, opening the fundamental mode to the fluctuations of the input harmonic mode with strong deleterious effects. At most, some remnants of the enhanced efficiency coming from the competition between nonlinearities can be observed for low $\gamma_{nl}$. Then, as shown in Fig. 4, the best working point is not necessarily located at $\mu = 0$, i.e. maximum squeezing is obtained with a finite mismatch.

D. Squeezing at the harmonic mode

For the harmonic mode Eq. (5.11) yields

$$S^h_{-} (\tilde{\omega}) = 1 + \tilde{\gamma}_{nl} : S^{{ref} f}_{-} (\tilde{\omega}) :.$$  \hspace{1cm} (5.23)

Now, the situation is the complete opposite: the performance is favored by a finite $\mu$ in order to have a non-zero $\gamma_{nl}$ and a large $n$ to approach the ratio $\hat{\gamma}_{nl}$ to one. In fact, under ideal conditions of perfect dynamic noise suppression and no absorption and scattering losses ($\gamma_n = 0$), the squeezing in both modes are complementary in the sense of

$$S^a_{+} + S^h_{-} = 2 - \frac{\gamma_{nl}}{\gamma_t} = 1,$$  \hspace{1cm} (5.24)

a direct consequence of the fluctuation-dissipation relation (5.8). This complementarity has been previously reported for the doubly resonant degenerate parametric oscillator [31]. The maximum squeezing available for the harmonic mode whatever the dynamic response of the system is easily obtained by setting $S^{{ref} f}_{-} (\tilde{\omega}) : to -1 in Eq. (5.23)$, that is,

$$S_M = 1 - \frac{2mK_r}{1 + 2mK_r} = \frac{1}{1 + 2mK_r}.$$  \hspace{1cm} (5.25)

This static contribution to the noise is now nonlinear in the sense that it depends on the phase mismatch and $m$. An immediate consequence of Eq. (5.25) is the possibility of an arbitrarily large quantum noise reduction for any finite value of $K_r$. The $1/9$ limit of the conventional phase-matched SHG is therefore due to a failure of the setup to maximize the dynamic response of the system. Let us center then, firstly in the SHG-like case with $\beta_m = 0$ as
it includes the above mentioned conventional setup (the experiments in [4] and [6]). The instability points are now given by (directly in the hat normalization)

\[ \hat{\delta}_\pm = -2mK_i \pm \sqrt{m^2(K_i^2 - 3K_r^2) - 4K_rm - 1}. \] (5.26)

Both kinds of nonlinearities (dispersive and absorptive) are in this case necessary as the factor \( K_i^2 - 3K_r^2 \) needs to be positive to allow \( \hat{\delta}_\pm \) to be real. The phase-matched case is therefore excluded.

Fig. 5 shows \( K_i^2 - 3K_r^2 \) as a function of the phase mismatch and indeed near above \( \pi \) it is positive. Optimum approaches to (5.26) are now more difficult to evaluate than in the fundamental mode as both the dynamic processing of the noise and the static contribution from the noise inputs (encoded in \( \tilde{\gamma}_{nl} \)) depend on \( m \) and \( \Delta_k \). With respect to \( m \) is clear that the static part is optimized at \( m \to \infty \). This limit can be approached letting \( \tilde{\omega} = 0 \) and \( \hat{\delta} = -2K_im \) (i.e. \( \Delta = 0 \)). \(|\tilde{B}|\) reduces in this case of \( \beta_m = 0 \) to \( m\sqrt{K_i^2 + K_r^2}/(1+2mK_r) \) showing a monotonic increasing behavior with respect to \( m \) from 0 to the maximum (at \( m \to \infty \))

\[ |\tilde{B}| = \frac{1}{4} \left[ 1 + \left( \frac{K_i}{K_r} \right)^2 \right]. \] (5.27)

Notice that for \( K_i^2 = 3K_r^2 \) it consistently equals 1. We have then, both \( \tilde{\gamma}_{nl} = 1 \) and the fastest approach to 1 of \( |\tilde{B}| \) when \( m \to \infty \). Therefore, the squeezing along an optimum path with respect to \( \Delta k \) is given by substituting Eq. (5.27) in the spectrum (5.13) and then the obtained \( S_{opt} \) in Eq. (5.23). Obviously, in a real experiment \( m \) can be large but always finite. Let us take as a “large” \( m \) one giving a \( S_M \) around 20 dB as in the case \( \eta = 0.99 \) of Fig. 2. This corresponds to \( m = 50 \). Figure 6 displays \( S_{b-+} \) as a function of the phase mismatch in such a case. To illustrate the modulation exerted by \( S_M \) we have take this time \( \hat{\delta} \) equal to the real part of Eq. (5.26) plus a very small number. In this way the plot remains valid for the whole range of the \( \Delta kL_m \). While Eq. (5.24) is complex the condition \( \Delta = 0 \) is almost fulfilled and above the instability the noise suppression reduction follows \( S_M \). Again the pernicious effect of the instability regarding the excess noise has a very short range. For comparison \( S_M \) is also depicted.

The optimum path with respect to \( m \) is much more complicated to find because the intricate dependence of \( K_i \) and \( K_r \) with respect to the phase mismatch. Figure 6 has been generated finding numerically the minima of \( S_{b-+} \) while scanning the range of \( m \). For comparison the phase-matched case is also depicted showing an asymptotic behavior towards \(-10\log 9\). For low values of \( m \) the effect of \( \tilde{\gamma}_{nl} \) overwhelms the dynamic response so that the best value corresponds to maximize \( K_r \). As soon as the two curves depart from each other the dynamic response dominates the behavior and the minimum noise is at the instability as in Fig. 4. At this stage the optimum path begins to follow the instability all the time. It should be taken then, as a mathematical limit. However, in view of Fig. 6, before reaching it, bearable values of the excess noise are accessible with a slight diminution of the squeezing.

It is worth to mention that a squeezing as large as 48% induced by cascading has been very recently reported [32]. The cascading was due, however, to a detuning of the pump mode in a triply resonant non-degenerate OPO with a much lower finesse for the pump mode rather than by phase mismatch. Under such conditions a cascaded \( \chi^{(3)} \) is also induced leading ideally to perfect squeezing in the pump mode.
FIG. 6. Noise spectra at zero frequency of the harmonic mode following a nearly optimum path with respect to the phase mismatch for the SHG like case.

FIG. 7. Squeezing in the harmonic mode along an optimum path with respect to the normalized intracavity photon number \((m)\) for the SHG like case compared with the phase-matched SHG case.
FIG. 8. Noise spectra at zero frequency (harmonic mode) following an optimum path with respect to the normalized input harmonic amplitude ($\eta_{in}$). The curves are not physical above the divergences.

Although a finite mismatch allows to reach $S_M$, the overall optimum working point corresponding to $\Delta k = 0$ is out of reach. The question arises then, of if it is possible to fulfill Eq. (5.13) at $\Delta k = 0$. A glance at the definition of $B$ suggests it should be possible adding a driving to the harmonic mode. In such a case $\tilde{\gamma}_{nl}$ simplifies to $2m/(1 + 2m)$ obviously independent of $\eta_{in}$. Constructing an optimum path with respect to the harmonic input reduces then, to set $\Delta = \tilde{\omega} = 0$. Phase matching along with $\Delta = 0$ implies $K_r = 1$, $K_i = 0$ and $\delta = 0$ so that the instability condition (5.13) simplifies to

$$1 + 2m = |\eta_{in} - m|,$$

a perfectly achievable condition. We can further optimize by choosing the phase of $\eta_{in}$ adequately to approach $\tilde{B}$ to one as much as possible. The extreme cases correspond to $\eta_{in} = -(1 + m)$ and $\eta_{in} = 1 + 3m$. From Eqs. (3.8) and (3.9) it is easy to check that they correspond respectively to $\phi - \varphi/2 = \pi$ and $\phi - \varphi/2 = 0$. The negative case maximizes $\tilde{B}$. It has been previously reported in [27]. Taking squared modulus of Eq. (2.5b) the negative case appears as promoting harmonic output power while the converse is valid for the positive. The squeezing phase is also easy to calculate in this case. In particular, given the correlation (4.8b), $\nu(\omega)$ is determined by the phase of $\alpha^2 B$ (independent of $\omega$ as $\Delta = 0$), something proportional to

$$\epsilon^{i4\theta}.$$ 

The corresponding squeezing phases are $\theta_m = 2\theta + \pi$ for the negative case while for the positive case it changes from $\theta_m = 2\theta + \pi$ to $\theta_m = 2\theta + \pi/2$ at $\eta_{in} = m$. On the other hand, the output harmonic amplitude is proportional to (see Eq. (2.5b))

$$b_{out} \propto (\eta_{in} - 2m)e^{i(2\theta + \pi)}.$$ 

Consequently, the relative squeezing phase for the negative $\eta_{in}$ is $\pi$, i.e., amplitude squeezing. The positive case is more complicated. It remains equal $\pi$ (amplitude squeezing) till $\eta = m$. Above this value it changes to $\pm \pi/2$ depending on the sign of $\eta_{in}/2 - m$ yielding in any case phase squeezing. At a first glance, it appears there is a sudden change from amplitude to phase squeezing when the input phases are fix to $\phi - \varphi/2 = 0$ and $|\eta_{in}|$ passes through $m$. It is not so however, as at this point $\tilde{B} = 0$ and the state collapses to a coherent state with no squeezing. The situation is clearly depicted in figure 8 where $S^b_{+,0}(0)$ are displayed as a function of $\eta_{in}$ assumed real. The r.h.s. of the plot corresponds to $\phi - \varphi/2 = 0$ while the l.h.s. to $\phi - \varphi/2 = \pi$ and for negative ordinates it should be considered as an optimum path with respect to $\eta_{in}$. The behavior is completely symmetric with respect to $\eta_{in} = m$ where both the squeezing and the excess noise equal that of the vacuum.

The optimum path with respect to $m$ is now given by $S_M$ at $K_r = 1$. Figure 8 is the equivalent to Fig. 7 for the new situation. It represents the maximum efficiency as far as quantum noise reduction is concerned the system can yield in any way with respect to $m$. The improvement with respect to the standard phase-matched SHG as well as to the optimized SHG is certainly high.
FIG. 9. Maximum squeezing (harmonic mode) as a function of the normalized intracavity photon number \((m)\) in nonlinear second order singly resonant device. For comparison the phase-matched and optimized SHG cases are also shown.

VI. DISCUSSION AND CONCLUSIONS

Two are the main purposes of the present work. On one side, to gain physical insight about the origins of quantum noise in singly resonant systems. On the other, to explore their potential as squeezed light sources. The results shown in the previous sections certainly reveal a high potential of the studied configurations. An evaluation of the limits of the model in reproducing the real physical situation as well as a discussion of possible implementations, seems, therefore, in order.

One obvious idealization of the model is to assume perfectly coherent inputs neglecting the excess noise of real lasers something expected to have deleterious effects at low frequencies. White and coworkers [33] have developed an analytical approach to this problem resulting in an impressive agreement with the experiments. As expected, the excess noise completely destroys the squeezing at low frequencies. In their experiments, however, the deleterious effect was restricted to only 7 MHz by adding a mode cleaner to the system, the spectrum coinciding with the ideal one out of this range. Even better, in [11] the laser noise was shot-noise-limited down to 1 MHz, again using an external mode cleaner.

Considering as sensible the assumption of coherent states for the input modes as well as a value of \(m\) around 3 (we will see below it looks like the case) our main concern about the fundamental mode results summarized in Fig. 2 refers to the feasibility of the chosen escape efficiencies. The ratio \(\gamma_c/(\gamma_c + \gamma_s)\) is difficult to maximize in a resonant mode because, by its own resonant nature, \(\gamma_c\) must be rather low. Thus, in [25] it was only of 0.52, while in [3] it was 0.36. Even in [3], a doubly resonant system specifically designed to squeeze the fundamental mode, the escape efficiency was around 0.9, limiting the maximum squeezing achievable to 90% (in practice, a 52% of noise reduction was reached). It appears, then, that nowadays the \(\eta = 0.99\) should be taken rather as an ideal illustrative case.

In contrast, the ultimate limit for the noise suppression in the harmonic mode (Eq. (5.25)) is pushed up by the fundamental mode photon number, opening a way to bypass the usual untouchable limit imposed by the escape efficiency of the cavity (as in the fundamental mode). Therefore, the squeezing in the harmonic mode can be arbitrarily large under the ideal assumption that the energy load inside the cavity can be also arbitrarily large. However, this is not totally true as the model does not take into account the losses in the harmonic mode which necessarily limit the degree of noise suppression. We can estimate this limitation assuming the absorption in one single pass through the nonlinear material equivalent to the effect of a beam splitter with the adequate reflectivity. Taking an absorption of 0.6%/cm as in [3] and a length of 1 cm, the equivalent reflectivity would be of 6 \(10^{-3}\). The spectrum after the beam splitter is given by \(S_{\text{out}} = 1 + T : S_{\text{in}}\). Setting : \(S_{\text{in}} := -1\) and \(T = 1 - R\), the ultimate squeezing achievable is precisely \(R = 6 \times 10^{-3}\), i.e. - 22 dB. In other words, the chosen value of \(m = 50\) in Figs. 6 and 8 represents more or less the maximum the model can stand without the inclusion of the harmonic mode losses.

Of course, we still cannot assume \(m = 50\) as a realistic limit for the state of the art devices as \(m\) depends not only on the intra-cavity photon number but on the ratio \(\nu/\gamma\) between the nonlinearity and losses. This ratio must be high enough in order to prevent a degradation of the nonlinear optical response in the system as commented in the introduction. Besides, this ratio scales down the power available in the external sources. In view of these complications, probably the most reliable way of setting the physical scale of \(m\) is to compare the results with the reported experiments. In [3] the quoted noise reduction was of -5.2 dB. Setting to zero \(\Delta k\) and \(\beta_{\text{in}}\) in Eq. (6.19)
corresponding to phase-matched SHG, a -5.2 dB squeezing results at \( m = 2.5 \), far from the \( m = 50 \) limit. Fortunately, the limit (5.28) grows up quite quickly for low \( m \)'s (see Fig. 9). Thus, 10 dB of noise suppression are reached at \( m = 5 \), no such unthinkable value. However, -15 dB of noise reduction requires \( m = 15 \), while a -20 dB figure is at the \( m = 50 \) limit, an order of magnitude higher. New nonlinear materials seem the only possibility for such high squeezing degrees. A promising via consists in the use of resonant nonlinearities in asymmetric quantum wells (AQW). Huge nonlinearities have been demonstrated in frequency doubling experiments and even a tuning of the nonlinearity with a d.c. field [34]. Obviously, also the absorption is enhanced by the resonance. This can be a problem as the ratio \( \nu/\gamma \) could at the end of the day not be increased. To asses this possibility requires quite a detailed analysis out of the scope of the present work. We can foresee, however, a promising advantage in the fact that the losses in the harmonic mode have little influence on the performance. By maintaining a strong two photon resonance but relaxing the one photon counterpart (tuning with a d.c. field or by an adequate energy level engineering), the nonlinearity would be certainly enhanced while the losses at the fundamental mode would not increase so strongly, thus enhancing \( \nu/\gamma \). With only one passage through the cavity of the harmonic mode and taking into account that a very thin layer of material is capable of SHG [34], the corresponding deleterious effect cannot be very large. Even a more exciting possibility comes from the recent experimental demonstrations of absorption inhibition in AQW induced by quantum interference [35–37]. The absorption transparency and the resonant enhancement can be combined using an adequate quantum well engineering leading to very efficient frequency doublers (see [38], where precisely a scheme only resonant at the harmonic mode is proposed).

These are certainly promising perspectives but we should not dismiss the improvements arising at the range of the present nonlinear crystals performances. Let us center then, around \( m = 2.5 \). As shown in the previous section the best strategy corresponds to drive both modes with relative phases \( \varphi - \phi/2 = \pi/2 \) (negative \( \eta_{in} \)) and \( \Delta = 0 \).

In Fig. 10 the noise behavior till \( m = 5 \) is displayed for various “distances” to the instability (5.28). Even at half the instability \( \eta_{in} \) value, the squeezing at \( m = 2.5 \) grows from -5.1 dB (69%) to -7.2 dB (80%). The excess noise, on the other hand, rapidly increases at low \( m \)'s but it also saturates quickly to bearable values. The improvement, although nothing spectacular is quite substantial. In [27] it was not reckoned so because the noise suppression was studied as function of the input power. Given its nonlinear relation with \( m \) the improvement is much slower with respect to this variable. Besides the squeezing, the output power is also enhanced. Taking a negative \( \eta_{in} \) in Eq. (5.29), the output power results in

\[
P_{out} \propto (2m + |\eta_{in}|)^2,
\]

and thus, the harmonic mode input contributes constructively to it. As shown in Fig. 11 at half a way of the instability the power is nearly doubled. Although from the theoretical point of view the injection of a coherent signal in the harmonic mode looks quite harmless, the experimental implementation is not trivial. However, the remarkable achievements in [10,11] with the OPA strongly support the feasibility of the idea.

Finally, a word of caution about the design of the device. It is important to avoid the setting of oscillations out of the fundamental mode (the so called subharmonic pumped OPO [38]), something capable of destroying the noise reduction [41]. At a first glance, finite values of \( \eta_{in} \) would favor the effect by promoting the down conversion. But it
FIG. 11. The harmonic output power corresponding to the cases of Fig. 10.

is not necessarily so as the down conversion is encouraged only for a given range of the relative phase between the two driving fields. Thus, for the negative $\eta_{in}$ case studied above, being the harmonic output power maximized, the down conversion is minimized.

To conclude, let us summarize the most relevant results. Firstly, for any system with only one effective mode we have given a systematic approach capable of isolating the processing of quantum noise by the dynamic response of the system. This dynamic processing is maximized at zero frequency, zero generalized nonlinear detuning ($\Delta$ as defined in section [V]) and at a dynamic instability. The static contributions to the noise coming from the different noisy inputs can in some cases, move the overall optimum working point away from that corresponding to maximum dynamic noise suppression. In spite of such, to have a rule to maximize the dynamic quantum noise suppression resulted very useful to characterize the squeezing behavior when applied to a specific optical system. In particular, for the case of a singly resonant second order nonlinear device, the squeezing at the fundamental mode is limited by the escape efficiency of the cavity, the best working point being within figures of merit of conventional nonlinear crystals. In the harmonic mode high squeezing requires new materials but it is only limited by the losses in the non-resonant harmonic mode opening the possibility of using multiple quantum wells with resonantly enhanced nonlinearities. However, with standard nonlinear crystals still is possible a substantial improvement with respect to the reported experiments by injecting a coherent driving in the harmonic mode. Besides, the output power is highly enhanced.

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* E-mail: ccabrilo@foton0.iem.csic.es
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