Asymptotic Supergraviton States in Matrix Theory

Jan Plefka and Andrew Waldron

NIKHEF
P.O. Box 41882, 1009 DB Amsterdam
The Netherlands

Abstract: We study the Matrix theory from a purely canonical viewpoint. In particular, we identify free particle asymptotic states of the model corresponding to the 11D supergraviton multiplet along with the split of the matrix model Hamiltonian into a free and an interacting part. Elementary quantum mechanical perturbation theory then yields an effective potential for these particles as an expansion in their inverse separation. We discuss how our scheme can be used to compute the Matrix theory result for the 11D supergraviton $S$ matrix and briefly comment on non-eikonal and longitudinal momentum exchange processes.

1 The model.

Matrix theory [1] is the conjectured description of $M$ theory in terms of a supersymmetric matrix model. At low energies and large distances $M$ theory, by definition, reduces to 11D supergravity. In this talk we explicitly construct asymptotic particle states in Matrix theory to be identified with the 11D supergraviton multiplet and study the scattering of these states.

The Hamiltonian of the Matrix theory is that of ten dimensional $U(N)$ super Yang-Mills dimensionally reduced to $0 + 1$ dimensions [2] and arises from two disparate viewpoints. On the one hand, it is the regulating theory of the eleven dimensional supermembrane in light cone gauge quantization [3] and on the other, it is the effective Hamiltonian describing the short distance properties of $D0$ branes [4, 5, 6]. Employing the conjecture of [6], the finite $N$ model is to be identified with the compactification of a null direction of $M$ theory (henceforth called the $-$ direction). The

---

1Talk presented at the "31st International Symposium Ahrenshoop on the Theory of Elementary Particles" Buckow, September 2-6, 1997.
quantized total momentum of the $U(N)$ system in this direction is then given by $P_- = N/R$, where $R$ denotes the compactification radius.

We shall be primarily interested in the $U(2)$ theory, studying the Hilbert space of two supergravitons with momentum $P_- = 1/R$ each. The coordinates and Majorana spinors of the transverse nine dimensional space then take values in the adjoint representation of $U(2)$, i.e.

\begin{align*}
X_\mu &= X_\mu^0 \mathbb{1} + X_\mu^A i\sigma^A & \mu = 1, \ldots, 9 \\
\theta_\alpha &= \theta_\alpha^0 \mathbb{1} + \theta_\alpha^A i\sigma^A & \alpha = 1, \ldots, 16 
\end{align*} (1.1) (1.2)

where $\sigma^A$ are the Pauli matrices. We shall often employ a vector notation for the $SU(2)$ part in which $\vec{X}_\mu = (X_1^\mu, X_2^\mu, X_3^\mu) \equiv (X_A^\mu)$ and similarly for $\vec{\theta}$.

The Hamiltonian is then given by

$$H = H_{CoM} + \frac{1}{2} \vec{P}_\mu \cdot \vec{F}_\mu + \frac{1}{4} (\vec{X}_\mu \times \vec{X}_\nu)^2 + \frac{i}{2} \vec{X}_\mu \cdot \bar{\theta} \gamma_\mu \times \bar{\theta}$$ (1.3)

where $H_{CoM} = \frac{1}{2} RP_\mu^{-1} P_\mu^0$ is the $U(1)$ centre of mass Hamiltonian. Note that we are using a real, symmetric representation of the $SO(9)$ Dirac matrices in which the nine dimensional charge conjugation matrix is equal to unity.

The Hamiltonian (1.3) is augmented by the Gauss law constraint

$$\vec{L} = \vec{X}_\mu \times \vec{P}_\mu - \frac{i}{2} \bar{\theta} \times \bar{\theta} , \quad [L^A, L^B] = i \epsilon^{ABC} L^C$$ (1.4)

whose action is required to vanish on physical states.

The task is now to identify the free asymptotic two-particle states of the Hamiltonian (1.3) which describe the on-shell supergraviton multiplet of eleven dimensional supergravity. This problem manifestly factorises into a $U(1)$ centre of mass state and an $SU(2)$ invariant state describing the relative dynamics of the particles.

### 2 The centre of mass theory.

The eigenstates of the free $U(1)$ centre of mass Hamiltonian $H_{CoM}$ are

$$|k_\mu; h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\tilde{\alpha}}\rangle_0 = e^{ik_\mu X_\mu^0} |h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\tilde{\alpha}}\rangle_0$$ (2.5)

and possess transverse $SO(9)$ momentum $k_\mu$ and on-shell supergraviton polarisations $h_{\mu\nu}, B_{\mu\nu\rho}$ and $h_{\mu\tilde{\alpha}}$ (graviton, antisymmetric tensor and gravitino, respectively). The state $|h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\tilde{\alpha}}\rangle_0$ is the $44 \oplus 84 \oplus 128$ representation of the centre of mass spinor $\theta^0$ degrees of freedom. The construction of this state is carried out in detail in [8] and allows the explicit calculation of the spin dependence of Matrix theory supergraviton amplitudes. In order to define the fermionic vacuum and creation and

\footnote{Note that the polarisation tensors $h_{\mu\nu}, B_{\mu\nu\rho}$ and $h_{\mu\tilde{\alpha}}$ correspond to physical polarisations. The Matrix theory does away with unphysical timelike and longitudinal polarisations at the price of manifest eleven dimensional Lorentz invariance.}
annihilation operators one performs a decomposition of the $SO(9)$ Lorentz algebra with respect to an $SO(7) \otimes U(1)$ subgroup \cite{3}. This is done as follows. Firstly split vector indices $\mu = (1, \ldots, 9)$ as $(m = 1, \ldots, 7; 8, 9)$ so that an $SO(9)$ vector $V_\mu$ may be rewritten as $(V_m, V, V^*)$ where $V = V_8 + iV_9$ and $V^* = V_8 - iV_9$. For an $SO(9)$ spinor the same decomposition is made by complexifying, in particular, for the canonical spinor variables we have

$$\lambda = \frac{\theta^0 + i\theta^0}{\sqrt{2}}, \quad \lambda^\dagger = \frac{\theta^0 - i\theta^0}{\sqrt{2}},$$

where the subscript $\pm$ denotes projection by $(1 \pm \gamma_9)/2$. The canonical anticommutation relations are now $\{\lambda_\alpha, \lambda^\dagger_\beta\} = \delta_{\alpha\beta}$ where $\alpha, \beta = 1, \ldots, 8$ and we define the fermionic vacuum $|--\rangle$ by $\lambda |--\rangle = 0$. We denote the completely filled state by $|+\rangle = \lambda^\dagger_1 \cdots \lambda^\dagger_8 |--\rangle$. One finds then the following expansion for the supergraviton polarisation state

$$|h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\alpha}\rangle_0 = h|--\rangle + \frac{1}{4} h_m|--\rangle_m + \frac{1}{16} h_{mn}|\pm\rangle_{mn} + \frac{1}{4} h^*_m|+\rangle_m + h^*|--\rangle,$$

$$-\frac{\sqrt{3} i}{8} \left( B_{mn} |--\rangle_{mn} + \frac{i}{6} B_m|--\rangle_m + \frac{1}{6} B_{mnp}|\pm\rangle_{mnp} - B^*_m|--\rangle_{mn} \right)$$

$$-\frac{i}{\sqrt{2}} \left( h_a |--\rangle_a - \frac{1}{2} h_{ma} |--\rangle_{ma} + \frac{1}{2} h^*_{ma} |--\rangle_{ma} - h^*_a |--\rangle_a \right).$$

The states in (2.7) transform covariantly with respect to $SO(7) \otimes U(1)$ and are defined in \cite{8}.

### 3 Asymptotic states.

Relative motions are described in the Matrix theory by the constrained $SU(2)$ quantum mechanical matrix theory defined above. However, spacetime is only an asymptotic concept in this theory. In particular diagonal matrix configurations, i.e., those corresponding to Cartan generators of $SU(N)$, span flat directions in the matrix model potential and describe spacetime configurations \cite{1}. Transverse directions are described by supersymmetric harmonic oscillator degrees of freedom, as we will see below.

Due to the gauge constraint (1.4) quantum mechanical wavefunctions must be invariant under $SU(2)$ rotations so that there is no preferred Cartan direction. To find asymptotic states corresponding to supergraviton (i.e., spacetime) excitations in a gauge invariant way we proceed as follows. Let us suppose we wish to study states describing particles widely separated in the (say) ninth spatial direction, then we may simply declare the $SU(2)$ vector $\vec{X}_9$ to be large. The limit $|\vec{X}_9| = \sqrt{\vec{X}_9 \cdot \vec{X}_9} \to \infty$ is $SU(2)$ rotation (and therefore gauge) invariant. We search for asymptotic particle-like solutions in this limit.
To this end it is convenient to employ the (partial) gauge choice \[9\] in which one chooses a frame where \( \vec{X}_9 \) lies along the \( z \)-axis,

\[ X_9^1 = 0 = X_9^2. \]  

(3.8)

Calling \( X_9 = (0, 0, x) \) and \( \vec{X}_a = (Y^1_a, Y^2_a, x_a) \) (with \( a = 1, \ldots, 8 \)) the Hamiltonian in this frame then is \( H = H_V + H_B + H_F + H_4 \) where

\[
H_V = -\frac{1}{2x} (\partial_x)^2 x - \frac{1}{2} (\partial_{x_a})^2
\]

(3.9)

\[
H_B = -\frac{1}{2} \left( \frac{\partial}{\partial Y^I_a} \right)^2 + \frac{1}{2} r^2 Y^I_a Y^I_a
\]

(3.10)

\[
H_F = r \tilde{\theta}^I \gamma_9 \tilde{\theta}
\]

(3.11)

\[
H_4 = \text{"rest".}
\]

(3.12)

The sum of the Hamiltonians \( H_B \) and \( H_F \) is that of a supersymmetric harmonic oscillator with frequency \( r \) and describes excitations transverse to the flat directions. Particle motions in the flat directions correspond to the Hamiltonian \( H_V \) whereby we interpret the Cartan variables \( x_\mu = (x_a, x) \) asymptotically as the \( SO(9) \) space coordinates.

The Hilbert space may be treated as a “product” of superoscillator degrees of freedom and Cartan wavefunctions depending on \( x_\mu \) and the third component of \( \vec{\theta} \) via the identity

\[
H = \sum_{m,n} |m\rangle \langle m| H |n\rangle \langle n|
\]

(3.13)

where \( \{ |n\rangle \} \) denote the complete set of eigenstates of \( H_B \) and \( H_F \). Since the frequency \( r \) of the superoscillators is coordinate dependent, operators \( \partial/\partial x_\mu \) do not commute with \( |n\rangle \) so that this “product” is not direct. This construction allows us to study an “effective” Hamiltonian \( H_{mn}(x_\mu, \partial x_\mu, x^3) = \langle m| H |n\rangle \) for the Cartan degrees of freedom pertaining to asymptotic spacetime. In particular the free Hamiltonian is given by the diagonal terms \( H_0 \)

\[
H_0 = \sum_n |n\rangle \langle n| \left( H_V + H_B + H_F - \frac{c_n}{r^2} \right) |n\rangle \langle n|
\]

(3.14)

and the interaction Hamiltonian then reads \( H_{\text{int}} = H - H_{\text{CoM}} - H_0 \). Since supersymmetric harmonic oscillator zero point energies vanish, eigenstates of \( (3.14) \) are

\[
|k_\mu; h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\dot{\alpha}}\rangle = \frac{1}{x} e^{i k_\mu x_\mu} |h_{\mu\nu}, B_{\mu\nu\rho}, h_{\mu\dot{\alpha}}\rangle \otimes |0_B, 0_F\rangle
\]

(3.15)

The spinors \( \tilde{\theta} \) are built from \( \theta^1 \) and \( \theta^2 \) by complexification and a spin(9) rotation (see \( \text{[8]} \)). Note that \( r^2 \equiv x_a x_a + x^2 \).

We subtract terms \( c_n/r^2 \) to ensure the correct asymptotic behaviour of the interaction Hamiltonian. A detailed explanation of this point may be found in \( \text{[8]} \).
where \(|0_B, 0_F\rangle\) is the supersymmetric harmonic oscillator vacuum. These states satisfy the correct free particle dispersion relation

\[
H_0|k_\mu; h_\mu, B_{\mu\nu}, h_{\mu\bar{\nu}}\rangle = \frac{1}{2} k_\mu k_\mu |k_\mu; h_\mu, B_{\mu\nu}, h_{\mu\bar{\nu}}\rangle.
\]  

(3.16)

Here, the supergraviton polarisation multiplet \(|h_\mu, B_{\mu\nu}, h_{\mu\bar{\nu}}\rangle\) is built from the \(44 \oplus 84 \oplus 128\) representation of \(\theta^4\) as in (2.7).

Therefore, upon taking the direct product of an asymptotic state \(|3.15\rangle\) with a centre of mass eigenstate \(|2.5\rangle\)

\[
|1, 2\rangle = e^{i k^{\text{tot}}_{\mu} x_{\mu}^0} \frac{1}{x} e^{i k^{\text{tot}}_{\mu} x_{\mu}} \otimes |0_B, 0_F\rangle \otimes |h^1_{\mu}, B^1_{\mu\nu}, h^1_{\mu\bar{\nu}}\rangle |h^2_{\mu}, B^2_{\mu\nu}, h^2_{\mu\bar{\nu}}\rangle |0_B, 0_F\rangle \otimes |\theta^3\rangle
\]

one obtains a state describing a pair of supergravitons widely separated in the ninth spatial direction. Its interactions, which die off as \(x \to \infty\), are governed by \(H_{\text{int}}\).

### 4 Scattering amplitudes.

The \(2 \to 2\) supergraviton scattering amplitude is then obtained by elementary quantum mechanical scattering theory as

\[
\lim_{T \to \infty} \langle Y', Y' | e^{-iH T} |1, 2\rangle = \delta(k^{\text{tot}}_{\mu} - k^{\text{tot}}_{\mu}) \int 4\pi x^2 d^9 x_{\mu}
\]

\[
e^{-i k^{\text{tot}}_{\mu} x_{\mu}} (H^1, H^2 | H_{\text{Eff}}(x_\mu, \partial_\mu, \theta^3_3) | H^1, H^2) \frac{e^{i k^{\text{tot}}_{\mu} x_{\mu}}}{x}
\]

(4.18)

where we have introduced \(|H^1, H^2\rangle = |h^1_{\mu}, B^1_{\mu\nu}, h^1_{\mu\bar{\nu}}\rangle |h^2_{\mu}, B^2_{\mu\nu}, h^2_{\mu\bar{\nu}}\rangle |0_B, 0_F\rangle \otimes |\theta^3\rangle\) and similarly for \(|H^{1'}, H^{2'}\rangle\). The leading (Born) approximation to the “effective” Cartan Hamiltonian \(H_{\text{Eff}}\) is given by

\[
H^{(1)}_{\text{Eff}}(x_\mu, \partial_\mu, \theta^3_3) = \langle 0_B, 0_F | H_{\text{int}} |0_B, 0_F\rangle \tag{4.19}
\]

and higher order contribution are obtained from the Lippman-Schwinger expansion

\[
H_{\text{Eff}}(x_\mu, \partial_\mu, \theta^3_3) = \langle 0_B, 0_F | H_{\text{int}} |0_B, 0_F\rangle + \langle 0_B, 0_F | H_{\text{int}} \frac{1}{E - H_0 + i\epsilon} H_{\text{int}} |0_B, 0_F\rangle + \ldots \tag{4.20}
\]

which due to the scaling behaviours \(H_{\text{int}} \sim \mathcal{O}(x^{-1/2})\) and \(H_0 \sim \mathcal{O}(x)\) turns out to be an expansion in \(1/x\) the inverse separation of the two supergravitons \(8\).

The leading term of \(H_{\text{Eff}}\) is on dimensional grounds of order \(1/r^2\) and receives contribution at first and second order perturbation theory in the sense of (4.21). An explicit computation \(8\) shows that these two contributions precisely cancel. This supersymmetric cancellation is in accordance with the two loop semiclassical background field path integral calculation of \(11\) and yields a strong test of our

\[5\text{Similar computations have been performed in a different context in}\ 10.\]
Higher order contributions should then capture the revered $v^4/r^7$ potential for $D0$ particles \[12\].

Let us stress at this point, however, that the amplitudes (4.13) are restricted to the eikonal kinematical regime (i.e. high energy, straight line), as the asymptotic in- and outgoing supergraviton pairs are widely separated in the same (in this case 9th) spatial direction. Scattering amplitudes at arbitrary angles $(\sigma_{\mu\nu})$ may be obtained by performing an $SO(9)$ rotation of the outgoing state, i.e.

$$\langle 1', 2' \vert \exp(i \sigma_{\mu\nu} L^{\mu\nu}_{SO(9)}) \exp(-iHT) \vert 1, 2 \rangle$$

(4.21)

where $L^{\mu\nu}_{SO(9)}$ denotes the generator of $SO(9)$.

The $2 \rightarrow 1$ supergraviton scattering channel of the $U(2)$ Matrix theory hinges on the knowledge of the zero-energy groundstate $\vert GS \rangle$ of the $SU(2)$ supersymmetric quantum mechanics (which exists, according to \[11, 13\]). The “1” supergraviton state with $P_- = 2/R$ is then given by the direct product of the $U(1)$ centre of mass state $\vert k^I_{\mu}, \mathcal{H}^I \rangle_0$ with $\vert GS \rangle$. Therefore the $2 \rightarrow 1$ amplitude reads

$$\lim_{T \rightarrow \infty} \langle 1' \vert e^{-iHT} \vert 1, 2 \rangle = \langle k^I_{\mu}, \mathcal{H}^I \rangle_0 \otimes \langle GS \vert \exp(-iHT) \vert k^1_{\mu}, k^2_{\mu}, \mathcal{H}^1, \mathcal{H}^2 \rangle$$

(4.22)

since $H \vert GS \rangle = 0$. Exact knowledge of the state $\vert GS \rangle$ would yield us the complete non-perturbative answer for this process involving longitudinal momentum exchange. Recently there has been some progress towards uncovering the structure of the ground state \[14, 15\].

Acknowledgements

J. Plefka thanks the organizers for a stimulating symposium.

References

[1] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, Phys. Rev. D55 (1997) 5112, [hep-th/9610043].

For recent reviews see:

T. Banks, “Matrix Theory”, [hep-th/9710231],

D. Bigatti and L. Susskind, “Review of Matrix Theory”, [hep-th/9712072].

[2] M. Claudson and M.B. Halpern, Nucl. Phys. B250 (1985) 689;
R. Flume, Ann. Phys. 164 (1985) 189;
M. Baake, P. Reinicke, and V. Rittenberg, J. Math. Phys. 26 (1985) 1070.

[3] B. de Wit, J. Hoppe and H. Nicolai, Nucl. Phys. B305 (1988) 545.

[4] J. Polchinski, Phys. Rev. Lett. 75 (1995) 4724, [hep-th/9510017].

[5] E. Witten, Nucl. Phys. B460 (1996) 335, [hep-th/9510133].
[6] U. Danielson, G. Ferretti and B. Sundborg, Int. J. Mod. Phys. A11 (1996) 5463, hep-th/9603081; D. Kabat and P. Pouliot, Phys. Rev. Lett. 77 (1996) 1004, hep-th/9603127.

[7] L. Susskind, “Another Conjecture about $M$-atrix Theory”, hep-th/9704080.

[8] J. Plefka and A. Waldron, “On the quantum mechanics of $M$-atrix theory.”, hep-th/9710104, to appear in Nucl. Phys. B.

[9] B. de Wit, M. Lüscher and H. Nicolai, Nucl. Phys. B320 (1989) 135.

[10] S. Sethi and M. Stern, “D-Brane Bound States Redux”, hep-th/9705046.

[11] K. Becker and M. Becker, Nucl. Phys. B506 (1997) 48, hep-th/9705091; K. Becker, M. Becker, J. Polchinski and A. Tseytlin, Phys. Rev. D56 (1997) 3174, hep-th/9706072.

[12] M.D. Douglas, D. Kabat, P. Pouliot and S. Shenker, Nucl. Phys. B485 (1997) 85, hep-th/9608024.

[13] M. Porrati and A. Rozenberg, “Bound States at Threshold in Supersymmetric Quantum Mechanics”, hep-th/9708119.

[14] J. Hoppe, “On the Construction of Zero Energy States in Supersymmetric Matrix Models”, hep-th/9709132; “On the Construction of Zero Energy States in Supersymmetric Matrix Models II”, hep-th/9709217.

[15] M.B. Halpern and C. Schwartz, “Asymptotic Search for Ground States of $SU(2)$ Matrix Theory”, hep-th/9712133.