On the Hilbert scheme of the moduli space of vector bundles over an algebraic curve

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Abstract. Let $M(n, \xi)$ be the moduli space of stable vector bundles of rank $n \geq 3$ and fixed determinant $\xi$ over a complex smooth projective algebraic curve $X$ of genus $g \geq 4$. We use the gonality of the curve and $r$-Hecke morphisms to describe a smooth open set of an irreducible component of the Hilbert scheme of $M(n, \xi)$, and to compute its dimension. We prove similar results for the scheme of morphisms $\text{Mor}_P(\mathbb{G}, M(n, \xi))$ and the moduli space of stable bundles over $X \times \mathbb{G}$, where $\mathbb{G}$ is the Grassmannian $\mathbb{G}(n-r, \mathbb{C}^n)$. Moreover, we give sufficient conditions for $\text{Mor}_2^{\text{ns}}(\mathbb{P}^1, M(n, \xi))$ to be non-empty, when $s \geq 1$.

1. Introduction

Studying the geometry of varieties frequently involves the understanding of their subvarieties. The Hilbert scheme $\text{Hilb}_Y$, the Chow scheme $C(Y)$ and the scheme of morphisms $\text{Mor}(-, Y)$ each provides a certain compactification of the space of irreducible subvarieties of a projective variety $Y$.

Let $M(n, \xi)$ be the moduli space of stable vector bundles of rank $n \geq 3$ and fixed determinant $\xi$ over a complex smooth projective algebraic curve $X$ of genus $g \geq 4$. In this paper, we are interested in studying the Hilbert scheme $\text{Hilb}_{\rho}^{\rho}(M(n, \xi))$ for a fixed Hilbert polynomial $\rho$ of degree $> 1$.

It is worth pointing out that for Hilbert polynomials $\rho$ of degree 1 it was proved in [20] that there exists a component of $\text{Hilb}_{\rho}^{\rho}(M(2, \mathcal{O}))$ that provides a non-singular model for $M(2, \mathcal{O})$. For $M(2, \mathcal{O}_X(-1))$, the Hilbert scheme $\text{Hilb}_{M(2, \mathcal{O}_X(-1))}^{M(2, \mathcal{O}_X(-1))}$ and the Chow scheme of degree 1 curves coincide with the moduli space of stable maps $\text{Mor}_{1}(\mathbb{P}^1, M(2, \mathcal{O}_X(-1)))$ (see [12, 18]). In [11], Kiem proved that the Hilbert scheme $\text{Hilb}_{M(2, \mathcal{O}_X(-1))}^{2n+1}$ of conics, the scheme of morphisms $\text{Mor}_2(\mathbb{P}^1, M(2, \mathcal{O}_X(-1)))$ and the Chow scheme of conics are related by contractions. To prove these results, they used the Hecke correspondence and the space of Hecke curves. The Hecke correspondence for vector bundles, was introduced by Narasimhan and Ramanan in [19] and has proved to be one of the

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most powerful tools in the study of \( M(n, \xi) \). It can be summed up in the following diagram (see [19,20])

\[
\begin{array}{ccc}
\mathbb{P}(U_x) & \xrightarrow{\pi} & M(n, \xi(x)) \\
& \searrow \swarrow q & \\
& M(n, \xi) & ,
\end{array}
\]

where \( x \in X \), \( \pi : \mathbb{P}(U_x) \to M(n, \xi(x)) \) is the restriction of the projective Poincaré bundle to \( \{x\} \times M(n, \xi(x)) \) and, at a generic point \( F \in M(n, \xi) \), \( q : \mathbb{P}(U_x) \to M(n, \xi) \) is a projective fibration with fibre \( \mathbb{P}(F_x) \). Narasimhan and Ramanan used \((0,1)\) and \((1,0)\)-stable bundles (see Sect. 2.3) to define subschemes of \( M(n, \xi) \), called good Hecke cycles. The main concern in [19] was the description of the deformations of the moduli space \( M(n, \xi) \); moreover, the lines in the good Hecke cycles and their properties were implicitly described there. Such lines were called later Hecke curves by Hwang (see [8,9]) and they turn out to be the minimal rational curves in \( M(n, \xi) \) (see [9,17,24]). The moduli space and the tangent spaces of the minimal curves in \( M(n, \xi) \) have been studied principally for rank 2 in [4,5,8,9,17,24]. The interest in studying \( \text{Mor}(\mathbb{P}^1, M(n, \xi)) \) also has its origin in attempts to compute the quantum cohomology of the moduli space \( M(n, \xi) \), which has recently become an important topic of research (see e.g. [2,3,11,18,26]).

For an integer \( 0 < r < n \), the problem that we address in this article is to provide a description of the set of maps from the Grassmannians \( G = G(n-r, \mathbb{C}^n) \) to \( M(n, \xi) \). Our purpose is to describe a smooth open set which is both an irreducible component of \( \text{Hilb}^P_M(n, \xi) \) and of \( \text{Mor}_P(\mathbb{G}, M(n, \xi)) \), where \( P \) is the Hilbert polynomial associated to the Grassmannian \( \mathbb{G} \).

In order to state our results we recall from [16] that using \((k, \ell)\)-stable bundles, one can generalise the idea of the good Hecke cycles to obtain the notion of Hecke Grassmannians (see Sect. 2) in \( M(n, \xi) \). It has recently been noted by O. Mata-Gutiérrez that through a very general point \( E \in M(n, \xi) \), there exists Hecke Grassmannians passing through \( E \). The Hecke Grassmannian is called an \( r \)-Hecke cycle, with \( 0 < r < n \), if it defines a closed subscheme. The good Hecke cycles are precisely those where \( r = 1 \).

The set of \( r \)-Hecke cycles form an irreducible family (see Sect. 3). Let \( \mathcal{HG} \) be the irreducible component of the Hilbert scheme \( \text{Hilb}^P_M(n, \xi) \) of \( M(n, \xi) \) containing \( r \)-Hecke cycles. The main idea behind the study of \( \mathcal{HG} \) is to apply the \((k, \ell)\)-stability (see Sect. 2.3). For integers \( k, \ell, 1 \leq r \leq n-1 \) chosen such that

\[
0 \leq k(n-1) + \ell + r < (n-1)(g-1)
\]

and

\[
0 \leq k + (\ell + r)(n-1) < (n-1)(g-1),
\]

we construct in Sect. 3 a fibration \( p : \mathcal{A} \to X \) where the fibre \( p^{-1} \) is identified with the set of \((k, \ell)\)-stable vector bundles of rank \( n \) and determinant \( \xi(rx) \), for all \( x \in X \). Now we state our main result which is a generalisation of those in [20]
**Theorem 1.1.** If \((n, d) = 1\) and \(r\) is less than the gonality of \(X\) then

1. there is an algebraic isomorphism \(\Upsilon\) from \(A\) to an open subscheme of \(H_G\).
2. The Hilbert scheme \(H_G\) is smooth at \(\Upsilon(z)\), for any \(z \in A\).
3. \(\dim H_G = (n^2 - 1)(g - 1) + 1\).
4. Locally, the deformations of \(r\)-Hecke cycles are \(r\)-Hecke cycles.

It would be interesting to prove, in the non-coprime case, that our component is actually smooth and provides a non-singular model for \(M(n, \xi)\), as was proved for \(n = 2\) and \(\xi\) the trivial bundle in [19].

The proof of Theorem 1.1 is similar in spirit to the proof of [20, Theorem 5.13] (see also [25]). While some of the results in [19, 20] apply in our case, the main results need extra hypotheses, such as a condition on the gonality of the curve. We use the gonality of the curve to prove that any \(z \in A\) defines a closed embedding \(\phi_z : G \rightarrow M(n, \xi)\) of a Grassmannian \(G\) to \(M(n, \xi)\), and for the injectivity of \(\Upsilon : A \rightarrow H_G\). In our case we construct a diagram (see diagram 3.9)

\[
\begin{array}{ccc}
G & \xrightarrow{\Phi} & M(n, \xi) \\
\downarrow{\pi_1} & & \\
A & & \\
\end{array}
\]

with \(\pi_1\) a Grassmannian fibrations and the fibre of \(\Phi\) at \(F\) the Grassmannian bundle \(p : G(r, F) \rightarrow X\), for each \(F\) in a suitable open set \(B \subset M(n, \xi)\). This construction generalises that of [19, 20, 25].

One of the advantages of using Grassmannians \(G\) lies in the fact that they have natural subvarieties, namely the Schubert varieties. This allows us to study flag Hilbert schemes parameterising \(t\)-tuples \((G, Y_1, \ldots, Y_t)\) such that \(Y_1 \subset \cdots \subset Y_t \subset G\) are subschemes of \(M(n, \xi)\). However, this topic exceeds the scope of this paper. We just consider nested pairs of subschemes \(\mathbb{P}^1 \leftarrow G \rightarrow M(n, \xi)\). In particular, our viewpoint sheds some new light on the study of rational curves on \(M(n, \xi)\) allowing us to obtain many more rational curves of higher degrees.

**Corollary 1.2.** If \(\text{Mor}_{r}(\mathbb{P}^1, G) \neq \emptyset\) then \(\text{Mor}_{2ns}(\mathbb{P}^1, M(n, \xi)) \neq \emptyset\). Moreover, \(\dim \text{Mor}_{2ns}(\mathbb{P}^1, M(n, \xi)) \geq (n^2 - 1)(g - 1) + 1\).

The fibration \(A\) in Theorem 1.1 parameterises closed subschemes, morphisms \(f : G \rightarrow M(n, \xi)\) and stable vector bundles over \(X \times G\). To see this, recall that the morphisms from \(G\) to \(M(n, \xi)\) are parameterised by a locally Noetherian scheme \(\text{Hom}(G, M(n, \xi))\) and the quotient

\[
\text{Mor}_p(G, M(n, \xi)) = \text{Hom}_p(G, M(n, \xi))/\text{Aut}(G)
\]

can be defined by means of the Chow variety (see [3,7,22]).

We say that a morphism \(\lambda : G \rightarrow M(n, \xi)\) is an \(r\)-Hecke morphism if \(\lambda(G)\) is an \(r\)-Hecke cycle. Such morphisms have minimal degree \(2n\) (see Proposition 4.6 and Corollary 5.2) and are in an irreducible family parameterised by \(A\). Let \(\text{Mor}_{H, r}^p(G, M(n, \xi))\) be the irreducible component of the scheme \(\text{Mor}_p(G, M(n, \xi))\) that contains the \(r\)-Hecke morphisms.

The next theorem is analogous to Theorem 1.1.
Theorem 1.3. Under the hypotheses of Theorem 1.1 there is an algebraic injective morphism

\[ \Sigma : \mathcal{A} \to \text{Mor}_{\mathcal{P}}^{\mathcal{H},r}(\mathbb{G}, M(n, \xi)). \]

Moreover, \( \text{Mor}_{\mathcal{P}}^{\mathcal{H},r}(\mathbb{G}, M(n, \xi)) \) is smooth at the \( r \)-Hecke morphisms and

\[ \dim \text{Mor}_{\mathcal{P}}^{\mathcal{H},r}(\mathbb{G}, M(n, \xi)) = (n^2 - 1)(g - 1) + 1. \]

In particular, \( \dim \text{Mor}_{\mathcal{P}}^{\mathcal{H},1}(\mathbb{P}^2, M(3, \xi)) = 8g - 7. \)

For any \( z \in \mathcal{A} \), the morphism \( \Sigma(z) = \phi_z : \mathbb{G} \to M(n, \xi) \) is defined by the existence of a vector bundle \( \mathcal{P}_z \) over \( X \times \mathbb{G} \), which we call an \( r \)-Hecke bundle. The stability of \( \mathcal{P}_z \), with respect to any polarisation \( L \), is established in Proposition 5.5. Let \( M^{L,\mathcal{H}}_{X \times \mathbb{G}}(\mathcal{P}_z) \) be the irreducible component of the moduli space of \( L \)-stable vector bundles over \( X \times \mathbb{G} \) that contain \( \mathcal{P}_z \). Our last result describes a smooth open set of \( M^{L,\mathcal{H}}_{X \times \mathbb{G}}(\mathcal{P}_z) \). Furthermore, we see that locally the deformations of \( \mathcal{P}_z \) come from those of the curve, of the Grassmannian as well as those induced by the elements of \( H^0(\mathbb{G}, \phi_z^*(TM)) \).

Theorem 1.4. Under the hypotheses of Theorem 1.1

1. there is an algebraic injective morphism \( \Gamma : \mathcal{A} \to M^{L,\mathcal{H}}_{X \times \mathbb{G}}(\mathcal{P}_z). \)
2. \( M^{L,\mathcal{H}}_{X \times \mathbb{G}}(\mathcal{P}_z) \) is smooth at \( \Gamma(z) \), for all \( z \in \mathcal{A} \).
3. \( \dim M^{L,\mathcal{H}}_{X \times \mathbb{G}}(\mathcal{P}_z) = n^2 g + 1. \)
4. \( \dim M^{L,\mathcal{H}}_{X \times \mathbb{G}}(\mathcal{P}_z)/(\text{Aut}(X) \times \text{Aut}(\mathbb{G})) = (n^2 - 1)(g - 1) + 1. \)

This article is organised as follows. In Sect. 2, we summarise the relevant material on Grassmannians, \( (k, \ell) \)-stability and elementary transformations. Some of the recent results are reviewed in a more general setting. Section 3 is devoted to the study of the morphisms \( \phi_z \) and \( \Upsilon \). The main result, Theorem 1.1, is proved in the fourth section. In Sect. 5, we touch only a few aspects of the theory of rational curves on \( M(n, \xi) \) and prove Theorem 1.3 and 1.4. We raise some questions in Sect. 6.

Notation: Given a vector bundle \( E \) over \( X \) we denote by \( d_E \) the degree, by \( n_E \) the rank and by \( \mu(E) := \frac{d_E}{n_E} \) the slope of \( E \). For abbreviation, we write \( H^i(E) \) instead of \( H^i(X, E) \), whenever it is convenient. The Grassmannian of \( s \)-planes of a vector space or of a vector bundle \( V \) will be denoted by \( \mathbb{G}(s, V) \). By \( p_i \), we mean the natural projection to the \( i \)-factor. The trivial bundle over \( Y \) with fibre the vector space \( W \) will be denoted by \( \mathcal{O}_Y \times W \).

2. Grassmannians, \( (k, \ell) \)-stability and elementary transformations

Let \( M(n, d) \) be the moduli space of stable vector bundles of rank \( n \geq 3 \) and degree \( d \) over \( X \). The fibre of the determinant morphism \( \text{det} : M(n, d) \to \text{Pic}^d(X) \) at \( \xi \in \text{Pic}^d(X) \) is \( M(n, \xi) \). It is well known that \( M(n, \xi) \) is a Fano variety with
Picard group $\mathbb{Z}\Theta$, where $\Theta$ is an ample divisor. If $K_M$ is the canonical bundle then $-K_M = 2(n,d)\Theta$ (see [6]). When $n$ and $d$ are coprime $M(n,\xi)$ is projective, smooth of dimension $(n^2-1)(g-1)$ and there is a Poincaré bundle $\mathcal{U}$ and a Grassmannian Poincaré bundle $\mathbb{G}(s,\mathcal{U})$. If $n$ and $d$ have a common divisor, there is no universal vector bundle (see [21,23]). In [19, Proposition 2.4] it was proved that in the non-coprime case there exist an étale cover of $M(n,\xi)$ and a family $\mathcal{V}$ of stable bundles of rank $n$ and determinant $\xi$ with universal properties. However, as in the projective case (see [1]), there always exists a Grassmannian Poincaré bundle

$$\mathbb{G}(s,\mathcal{U}) \to X \times M(n,\xi)$$

with the property that its restriction to $X \times \{E\}$ is isomorphic to the Grassmannian bundle $\mathbb{G}(s,E)$ over $X$ that parameterises all the $s$-planes in the fibres of the stable vector bundle $E$ on $X$.

Since Grassmannians, elementary transformations and $(k,\ell)$-stability will be the main tools that we will use, we briefly review the essentials, and fix the notation.

### 2.1. Grassmannians

We will give only the principal properties of Grassmannians of vector spaces and of vector bundles that we use. For a fuller treatment we refer the reader to, e.g., [25].

Let $E$ be a vector bundle of rank $n$ over $X$. Let $p_E : \mathbb{G}(n-r,E) \to X$ be the Grassmannian bundle whose fibre at $x \in X$ is the Grassmannian $\mathbb{G}(n-r,E_x)$ of $(n-r)$-planes of $E_x$. It is clear that $\mathbb{G}(n-r,E) = \mathbb{G}(r,E^\ast)$.

Let

$$\xi_E : 0 \to S_E \to p_E^*E \xrightarrow{\alpha_E} Q_E \to 0$$

(2.1)

be the tautological exact sequence over $\mathbb{G}(n-r,E)$, where $S_E$ and $Q_E$ are the tautological bundles of rank $n-r$ and $r$, respectively. The tangent bundle of $\mathbb{G}(n-r,E)$, denoted by $T\mathbb{G}(E)$, fits in the following extension

$$\xi : 0 \to T_{p_E} \to T\mathbb{G}(E) \to p_E^*TX \to 0,$$

(2.2)

where $T_{p_E}$ is the tangent bundle to the fibres and $T_{p_E} = S_E^\ast \otimes Q_E$.

For any $x \in X$, let

$$\xi_{E_x} : 0 \to S_{E_x} \to O_{\mathbb{G}} \times E_x \xrightarrow{\alpha_{E_x}} Q_{E_x} \to 0$$

(2.3)

be the restriction of the sequence (2.1) to $\mathbb{G}(n-r,E_x)$.

The Grassmannian varieties $\mathbb{G} = \mathbb{G}(n-r,E_x)$ are Fano varieties with Picard group $\mathbb{Z}O_{\mathbb{G}}(1)$, where $O_{\mathbb{G}}(1)$ is an ample divisor. The tangent bundle $T\mathbb{G}$ of $\mathbb{G}$ is the vector bundle $S_{E_x}^\ast \otimes Q_{E_x}$. Since $\det(Q_{E_x}) = O_{\mathbb{G}}(1)$, $T\mathbb{G}$ has degree $n$. Denote by $\Omega^1\mathbb{G}$ the cotangent bundle of $\mathbb{G}$.

The following remark summarises the main properties of the cohomology of $S_{E_x}$, $T\mathbb{G}$ and $\Omega^1\mathbb{G}$, which will be needed in Section 4 (see for instance [25]).
Remark 2.1. Let $\mathbb{G}$ be the Grassmannian $\mathbb{G}(n-r, E_x)$

(1) $H^i(\mathbb{G}, S_{E_x}) = 0$ for all $i \geq 0$;
(2) $\dim H^0(\mathbb{G}, T\mathbb{G}) = n^2 - 1$ and $H^i(\mathbb{G}, T\mathbb{G}) = 0$ for $i \geq 1$;
(3) $H^1(\mathbb{G}, \Omega^1_{\mathbb{G}}) = \mathbb{C}$ and $H^i(\mathbb{G}, \Omega^1_{\mathbb{G}}) = 0$, for $i \neq 1$.

Given a morphism $\lambda : \mathbb{G} \to M(n, \xi)$ we define the degree of $\lambda$ as

$$d(\lambda) := \text{degree}(\lambda^*(-K_M)).$$

From [7, 4(c)], the morphisms from $\mathbb{G}$ to $M(n, \xi)$ are parameterised by a locally Noetherian scheme $\text{Hom}(\mathbb{G}, M(n, \xi))$. We can see $\text{Hom}(\mathbb{G}, M(n, \xi))$ as a subscheme of $\text{Hilb}(\mathbb{G} \times M(n, \xi))$ when we identify a morphism $\lambda$ with its graph $\Gamma_\lambda$ in $\mathbb{G} \times M(n, \xi)$. As it is well known, $\text{Hom}(\mathbb{G}, M(n, \xi))$ is the disjoint union of the sub-schemes $\text{Hom}_P(\mathbb{G}, M(n, \xi))$, for all polynomials $P$, where $\text{Hom}_P(\mathbb{G}, M(n, \xi))$ is the subscheme that parameterises morphisms $\lambda : \mathbb{G} \to M(n, \xi)$ with fixed Hilbert polynomial $P$. Denote by $\text{Mor}_P(\mathbb{G}, M(n, \xi))$ the scheme

$$\text{Hom}_P(\mathbb{G}, M(n, \xi))/\text{Aut}(\mathbb{G}),$$

where $\lambda \sim \lambda'$ if there exists $\beta \in \text{Aut}(\mathbb{G})$ such that $\lambda' = \lambda \circ \beta$. The scheme $\text{Mor}_P(\mathbb{G}, M(n, \xi))$ can be defined by means of the Chow variety (see [3, 7, 22]).

Remark 2.2. From [7, 13] we have that

(1) the expected dimension of $\text{Hom}_P(\mathbb{G}, M(n, \xi))$ is $h^0(\mathbb{G}, \lambda^*TM(n, \xi))$.
(2) $\text{Hom}_P(\mathbb{G}, M(n, \xi))$ is smooth at $\lambda$ if $h^1(\mathbb{G}, \lambda^*TM(n, \xi)) = 0$.
(3) $H^0(\mathbb{G}, T\mathbb{G})$ is the tangent space at the identity to the group of automorphisms of $\mathbb{G}$, and the image of the canonical morphism

$$H^0(\mathbb{G}, T\mathbb{G}) \to H^0(\mathbb{G}, \lambda^*(TM))$$

corresponds to the deformations of $\lambda$ by reparameterisations.

(4) The expected dimension of $\text{Mor}_P(\mathbb{G}, M(n, \xi))$ is the dimension of

$$H^0(\mathbb{G}, \lambda^*(TM))/H^0(\mathbb{G}, T\mathbb{G}).$$

(5) $\text{Mor}_P(\mathbb{G}, M(n, \xi))$ is smooth at $\lambda$ if $h^1(\mathbb{G}, \lambda^*TM(n, \xi)) = 0$.

To define morphisms $\lambda : \mathbb{G} \to M(n, \xi)$ we use the $r$-elementary transformations and $(k, \ell)$-stability.

2.2. $r$-Elementary transformations

Let $E$ be a vector bundle over $X$ of rank $n$ and determinant $\eta$ of degree $e$. For any $\varphi = (x, W \subset E_x) \in \mathbb{G}(n-r, E)$ consider a new vector bundle $E^W$ defined by the exact sequence of sheaves

$$\xi_{x, W} : 0 \to E^W \xrightarrow{\iota} E \xrightarrow{\alpha_W} \mathcal{O}_x \times (E_x/W) \to 0,$$  \hspace{1cm} (2.4)

where $\mathcal{O}_x \times (E_x/W)$ is the skyscraper sheaf with support in $x$ and fibre $E_x/W$. That is, $E^W = \text{Ker}(\alpha_W)$. The vector bundle $E^W$ is called the $r$-elementary transformation of $E$ in $\varphi = (x, W \subset E_x)$. Note that $E^W$ has rank $n$, degree $e - r$ and determinant $\eta \otimes \mathcal{O}(-rx)$. 

Remark 2.3. Let $E$ be a vector bundle over $X$.

(1) Denote by $\text{Ker}(\iota_x)$ the kernel of the homomorphism $E^W_x \xrightarrow{\iota_x} E_x$ between the fibres at $x$, induced by the sheaf map $\iota$ in (2.4). It has dimension $r$ and its annihilator $\text{Ker}(\iota_x)^\perp$ is a $n - r$ dimensional subspace in $(E^W_x)^*$, which is canonically isomorphic to $W^*$.

(2) The restriction of (2.4) to $x$ gives the exact sequence

$$0 \to \text{Ker}(\iota_x) \to E^W_x \xrightarrow{\iota_x} E_x \xrightarrow{\alpha_W} (E_x / W) \to 0,$$

which splits as

$$0 \to \text{Ker}(\iota_x) \to E^W_x \xrightarrow{\iota_x} W \to 0 \text{ and } 0 \to W \to E_x \xrightarrow{\alpha_W} (E_x / W) \to 0.$$

(2.6)

(The extension (2.5) and its splitting (2.6) will be relevant in Sect. 4, (see Proposition 4.2)).

(3) A point $\wp = (x, W \subset E_x)$ in $\mathbb{G}(n - r, E)$ defines the element

$$\tilde{\wp} := (x, \text{Ker}(\iota_x) \subset E^W_x)$$

in $\mathbb{G}(r, E^W)$.

(4) The $r$-elementary transformation of $(E^W)\ast$ in $(x, W^\ast \subset (E^W)^\ast_x)$ is $E^\ast$. So,

$$(E^W)\ast W^\ast = E^\ast,$$

and we have the exact sequence

$$\xi_{x, W^\ast} : 0 \to E^\ast \xrightarrow{\iota} (E^W)\ast \to \mathcal{O}_x \times ((E^W)^\ast_x / W^\ast) \to 0.$$

(2.7)

We are interested in describing the set

$$\Omega := \{(x, E \xrightarrow{\alpha} \mathcal{O}_x) : E \text{ and Ker(} \alpha \text{) are stable }\}.$$

That is, in describing the set of $r$-elementary transformations of stable bundles that give stable bundles. To describe $\Omega$ we use the $(k, \ell)$-stability.

2.3. $(k, \ell)$-Stability

Let $k$ and $\ell$ be integers. A vector bundle $E$ over $X$ is $(k, \ell)$-stable (see [20]) if for any proper subbundle $F$ of $E$

$$\frac{k(n - n_F) + \ell n_F}{nn_F} < \mu(E) - \mu(F).$$

In particular, if $k$ and $\ell$ are non-negative integers, $(k, \ell)$-stability implies stability. However, for negative values of $k$ and $\ell$, a $(k, \ell)$-stable bundle does not need to be stable (see [16]).
Denote by $A_{k,\ell}(n, d)$ the set of $(k, \ell)$-stable vector bundles of rank $n$ and degree $d$ over $X$ and let $A_{k,\ell}(n, \xi) := A_{k,\ell}(n, d) \cap M(n, \xi)$. Proposition 5.3 in [20] shows that $(k, \ell)$-stability is an open condition. In particular, $(k, \ell)$-stable bundles are very general in $M(n, \xi)$. A point is very general if it is outside of a countable union of subvarieties of dimension strictly smaller than the dimension of $M(n, \xi)$.

There are natural filtrations (see [16])

$$A_{k,\ell}(n, d) \supset A_{k,\ell+1}(n, d) \supset A_{k,\ell+2}(n, d)\ldots$$

and

$$A_{k,\ell}(n, d) \supset A_{k+1,\ell}(n, d) \supset A_{k+2,\ell}(n, d)\ldots$$

with $A_{0,0}(n, d) = M(n, d)$ and $A_{k_0,\ell_0}(n, d) = \emptyset$ if

$$k_0(n - 1) + \ell_0 \geq (n - 1)g \quad \text{or} \quad k_0 + \ell_0(n - 1) \geq (n - 1)g.$$  

The non-emptiness of $A_{k,\ell}(n, \xi)$ is established in [16, Proposition 2.4]. The following proposition is a reformulation of [16, Proposition 2.4] and [20, Lemma 5.5] (see also [16]) in terms of $r$-elementary transformations.

**Proposition 2.4.** If $k, \ell, r, n, d$ are integers such that

$$0 < k(n - 1) + \ell + r < (n - 1)(g - 1) \quad \text{and} \quad 0 < k + (\ell + r)(n - 1) < (n - 1)(g - 1)$$

(2.8)

then $A_{k,\ell}(n, d) \neq \emptyset$, $A_{k,\ell}(n, d) \subset M(n, d)$ and $A_{k,\ell}(n, \xi)$ is non-empty. Moreover, any $r$-elementary transformation of a $(k, \ell)$-stable bundle in $A_{k,\ell}(n, \xi)$ is $(k, \ell - r)$-stable; in particular, it is stable and general.

The principal significance of the above proposition is that it allows us to define maps from Grassmannians to $M(n, d)$. Indeed, for integers $k, \ell, r, n, d$ as in Proposition 2.4, and $z = (x, E) \in X \times A_{k,\ell}(n, d)$, we have a map

$$\phi_z : \mathbb{G}(n - r, E_x) \rightarrow M(n, d - r),$$

defined as $W \mapsto E^W$. Our aim is to define morphisms from Grassmannians to $M(n, \xi)$, for a fix $\xi \in Pic^d(X)$.

**Remark 2.5.** The map $\phi_z : \mathbb{G}(n - r, E_x) \rightarrow M(n, d - r)$ can be defined by considering just $(0, r)$-stable bundles. However, for our purpose (see Sect. 5) it is convenient to use $(k, \ell + r)$-stable bundles that satisfy (2.8), since in this case the vector bundles in the image $\phi_E(\mathbb{G}(n - r, E))$ will be general.
3. The morphisms $\phi_z$ and $\Upsilon$

We assume throughout the rest of the article, unless otherwise stated, that $k$, $\ell$ and $r$ are integers satisfying the inequalities in Proposition 2.4. For the construction of the morphisms we also assume that $(n, d) = 1$, and hence $(n, d + r) = 1$. However, we can always work in an étale cover (see [19, Proposition 2.4]).

To construct algebraic morphisms from Grassmannians to $M(n, \xi)$ we first define a fibration $\mathcal{A} \to X$, which depends on $k$, $\ell$, $r$, $n$ and $\xi$, but to streamline the notation, we omit such indexes.

Fix $\xi \in Pic^d(X)$. Define
\[ \vartheta : X \times A_k,\ell(n, d + r) \to Pic^d(X) \quad \text{as} \quad (x, E) \mapsto \mathcal{O}_X(-rx) \otimes \det(E). \]

Let $\mathcal{A}$ be the inverse image $\vartheta^{-1}(\xi)$. The natural map $\pi : \mathcal{A} \to X$ is a fibration with fibre $A_k,\ell(n, \xi(rx))$ at $x \in X$. Note that the $r$-elementary transformation associated to the elements in $\mathcal{A}$ have determinant $\xi$. For any $z = (x, E) \in \mathcal{A}$ we denote $\mathbb{G}(n - r, E_x)$ as $\mathbb{G}(z)$.

To prove that given $z = (x, E) \in \mathcal{A}$ the map
\[ \phi_z : \mathbb{G}(z) \to M(n, \xi) \quad W \mapsto E^W \]
is algebraic, we construct a family of stable bundles parameterised by $\mathbb{G}(z)$. The construction of the family is similar to that in [19, 20, 25]. Since it is relevant for our work, we recall the main details.

The construction goes as follows. Given $z = (x, E)$, denote by $\alpha_z$ the surjective homomorphism $\alpha_z : p_1^*E \to p_1^*\mathcal{O}_x \otimes p_2^*\mathcal{Q}_{E_x}$ of sheaves over $X \times \mathbb{G}(z)$ associated to the surjective morphism $\alpha_{E_x} : \mathcal{O}_G \times E_x \to \mathcal{Q}_{E_x}$ under the isomorphism
\[ H^0(\mathbb{G}(z), Hom(\mathcal{O}_G \times E_x, \mathcal{Q}_{E_x})) \cong H^0(X \times \mathbb{G}(z), Hom(p_1^*E, p_1^*\mathcal{O}_x \otimes p_2^*\mathcal{Q}_{E_x})), \]

where $p_i$ is the projection, for $i = 1, 2$.

Since $p_1^*\mathcal{Q}_{E_x} \otimes p_2^*\mathcal{Q}_{E_x}$ has a locally free resolution of length 1, the kernel of $\alpha_z : p_1^*E \to p_1^*\mathcal{O}_x \otimes p_2^*\mathcal{Q}_{E_x}$, denoted by $\mathcal{P}_z$, is locally free and fits in the exact sequence
\[ \xi_{x,E} : 0 \to \mathcal{P}_z \to p_1^*E \xrightarrow{\alpha_z} p_1^*\mathcal{O}_x \otimes p_2^*\mathcal{Q}_{E_x} \to 0 \quad (3.1) \]
of sheaves over $X \times \mathbb{G}(z)$. The restriction of (3.1) to $X \times \{W\}$ for any $W \in \mathbb{G}(z)$ is precisely the extension $\mathcal{P}_z|_{X \times \{W\}} = E^W$. Therefore, if $z \in \mathcal{A}$,
\[ \mathcal{P}_z \to X \times \mathbb{G}(z) \quad (3.2) \]
is a family of stable bundles of rank $n$ and determinant $\xi$ parameterised by $\mathbb{G}(z)$ and hence we have a morphism
\[ \phi_z : \mathbb{G}(z) \to M(n, \xi) \]
with the following properties.

\[ z \]
**Proposition 3.1.** Under the hypotheses of Proposition 2.4, if \( r \) is less than the gonality of \( X \) then, for \( z \in \mathcal{A} \), the morphism \( \phi_z : \mathbb{G}(z) \to M(n, \xi) \) is a closed embedding for any \( z \in \mathcal{A} \). Moreover, \( \phi_z(\mathbb{G}(z)) \subset A_{k,1-r}(n, \xi) \).

**Proof.** First observe that since \( r \) is less than the gonality of \( X \), \( h^0(\mathcal{O}(rx)) \leq 1 \). Thus, any morphism \( \xi \to \xi \otimes \mathcal{O}(rx) \) vanishes only at \( x \).

Let us prove the injectivity of \( \phi_z \). Suppose, contrary to our claim, that there exist \( W, V \in \mathbb{G}(z) \) such that \( W \neq V \) but \( F := E^W = E^V \). Hence, \( \bigwedge^n F = \xi \) where \( \xi = \bigwedge^n E \otimes \mathcal{O}_X(-rx) \). From (2.4), we have two non-zero linearly independent homomorphisms \( f_1, f_2 : F \to E \), that depend on \( W \) and \( V \) respectively. Choose \( y \neq x \in X \) and a linear combination \( \lambda_1(x) \cdot y + \lambda_2(x) \cdot y \) that is not an isomorphism.

Then the homomorphism \( g := \lambda_1(f_1) + \lambda_2(f_2) : F \to E \) has no maximal rank at \( y \). Therefore, the induced map \( \bigwedge^n g : \bigwedge^n F \to \bigwedge^n E \) defines a non-zero section in \( H^0(\mathcal{O}(rx)) \) that vanishes on \( y \) and \( x \), contrary to the gonality of \( X \). Therefore, \( \phi_z \) is injective, which is our claim.

The proof above gives more, namely

\[
h^0(X, \text{Hom}(E^W, E)) = 1, \tag{3.3}\]

for any \( W \in \mathbb{G}(z) \).

To prove the injectivity of the differential map

\[
d\phi_z : T_{[W]}\mathbb{G}(z) \to T_{E^W}M(n, \xi) \]

recall that

\[
T_{[W]}\mathbb{G}(z) = W^* \otimes E_x/W \subset (E^W)^*_x \otimes E_x/W
\]

and 

\[
T_{E^W}M(n, \xi) = H^1(X, \text{ad}(E^W)).
\]

Tensor the exact sequence (2.4) with \( (E^W)^* \) and apply cohomology to get the exact sequence

\[
0 \to H^0(\text{End}(E^W)) \to H^0((E^W)^* \otimes E) \to (E^W)^*_x \otimes E_x/W \xrightarrow{\delta} H^1(\text{End}(E^W)) \to \cdots \tag{3.4}
\]

Since \( E^W \) is stable and \( H^0(X, \text{Hom}(E^W, E)) \cong \mathbb{C} \) (see (3.3)), (3.4) shows that the coboundary map \( (E^W)^*_x \otimes E_x/W \xrightarrow{\delta} H^1(X, \text{End}(E^W)) \) is injective.

Hence, the restriction of \( \delta \) (or \(-\delta\)) to \( W^* \otimes E_x/W \) is injective and is precisely the differential \( d\phi_z \) (see [20, Lemma 5.10]). Therefore \( \phi_z : \mathbb{G}(z) \to M(n, \xi) \) is a closed embedding, which proves the proposition.

Given a pair \( z \in \mathcal{A} \), the image \( \phi_z(\mathbb{G}(z)) \subset M(n, \xi) \) defines a closed subscheme in \( M(n, \xi) \), called \( r \)-Hecke cycle, and hence a point \([\phi_z(\mathbb{G}(z))]\) in the Hilbert scheme \( \text{Hilb}_{M(n, \xi)} \). We will identify \( \mathbb{G}(z) \) with its image \( \phi_z(\mathbb{G}(z)) \) in \( M(n, \xi) \) when no confusion can arise. The morphism \( \phi_z \) is called \( r \)-Hecke morphism and the vector bundle \( P_r \to X \times \mathbb{G}(z) \) an \( r \)-Hecke bundle. In Sect. 5 we will parametrise such morphisms and bundles.
The construction of the morphism $\phi_z$ can be done for families of $(k, \ell)$-stable bundles. Indeed, let $f : E \to X \times T$ be a family of $(k, \ell)$-stable bundles of rank $n$ and degree $d + r$ parameterised by a scheme $T$. Define $\vartheta : X \times T \to Pic^d(X)$ as $(x, t) \mapsto \mathcal{O}_X(-rx) \otimes det(\mathcal{E}_t)$ and let $\mathcal{A}(T)$ be the inverse image $\vartheta^{-1}(\xi)$. Let $G(E) \xrightarrow{\pi_1} \mathcal{A}(T)$ be the Grassmannian bundle of $(n - r)$-planes associated to the restriction of $E$ to $\mathcal{A}(T)$. Let $\gamma : G(E) \to X$ be the composition $G(E) \xrightarrow{\pi_1} \mathcal{A}(T) \xrightarrow{p_1} X$ and $\Gamma := \Gamma_\gamma \subset X \times G(E)$ the divisor associated to the graph of $\gamma$ (strictly speaking, $\Gamma_\gamma$ is in $G(E) \times X$). As before, over $X \times G(E)$ we have a surjection

$$p_2^* \pi_1^*(\mathcal{E}) \xrightarrow{\tilde{\alpha}} \mathcal{O}_T \otimes p_2^* \mathcal{Q}_E \to 0,$$

where $Q_E$ is the tautological quotient bundle and $p_2 : X \times G(E) \to G(E)$ the projection. The kernel $P_E$, which fits in the exact sequence

$$0 \to P_E \to p_2^* \pi_1^*(\mathcal{E}) \xrightarrow{\tilde{\alpha}} \mathcal{O}_T \otimes p_2^* \mathcal{Q}_E \to 0,$$

is locally free. Hence, $P_E$ is a family of stable bundles parameterised by $G(E)$. Therefore, we have a morphism

$$\Phi_E : G(E) \to M(n, \xi)$$

and a diagram

$$G(E) \xrightarrow{\Phi_E} \mathcal{A}(T) \xrightarrow{\pi_1} M(n, \xi).$$

In particular, applying the above construction to the family defined by the restriction of the Poincaré bundle $U$ to $X \times A_{k, \ell}(n, d + r)$, we obtain the exact sequence [see (3.5)]

$$0 \to P_U \to p_2^* \pi_1^*(U) \xrightarrow{\tilde{\alpha}} \mathcal{O}_T \otimes p_2^* \mathcal{Q}_U \to 0,$$

over $X \times G(U)$ with $\pi_1 : G(U) \to \mathcal{A}$; and the diagram

$$G(U) \xrightarrow{\Phi} \mathcal{A} \xrightarrow{\pi_1} M(n, \xi).$$

where $\Phi$ is $\Phi_U$.

The above construction is functorial. If we denote by $\mathcal{H}G$ the irreducible component of $\text{Hilb} M(n, \xi)$ that contains the $r$-Hecke cycles $[\phi_z(G(z))]$, we get an algebraic morphism

$$\Upsilon : \mathcal{A} \to \mathcal{H}G,$$

defined as $z \mapsto [\phi_z(G(z))]$. 
Proposition 3.2. If \((n, d) = 1\) and \(r\) is less than the gonality of \(X\) then \(\Upsilon : \mathcal{A} \to \mathcal{HG}\) is injective.

Proof. Suppose the proposition were false. Then we could find \((x, E) = z_1 \neq z_2 = (y, F)\) in \(\mathcal{A}\) such that \([\phi_{z_1}(G(z_1))] = [\phi_{z_2}(G(z_2))]\). Since \(\phi_{z_1}\) and \(\phi_{z_2}\) are embeddings, we get an isomorphism \(\beta : G(z_1) \to G(z_2)\) that induces the following commutative diagrams

![Diagram](image)

and

![Diagram](image)

i.e. \(\phi_{z_1} = \phi_{z_2} \circ \beta\) and \(\tilde{p}_1 \circ (id, \beta) = p_1\).

By the universal properties of \(M(n, \xi)\),

\[
P_{z_1} \cong (id, \beta)^*(P_{z_2}) \otimes p_2^*(L) \tag{3.11}
\]

where \(L\) is a line bundle on \(G(z_1)\). But \(L\) is trivial since \((P_{z_1})_{|t \times G(z_1)}\) is trivial, for any \(t \neq x\) (see (3.1)). We thus get \(P_{z_1} \cong (id, \beta)^*(P_{z_2})\) and \(p_1^*P_{z_1} = \tilde{p}_1^*P_{z_2}\). Let

\[
0 \to p_1^*P_{z_1} \to p_1^*(p_1^*E) \xrightarrow{\alpha_*} p_1^*(p_1^*\mathcal{O}_x \otimes p_2^*Q_{E_x}) \to \ldots
\]

be the direct image sequence of (3.1) by \(p_1\). It follows that \(p_1^*(P_{z_1}) \cong E(-x)\), because

1. \(p_1^*p_1^*(E) \cong E\),
2. \(p_1^*(p_1^*\mathcal{O}_x \otimes p_2^*Q_{E_x}) \cong \mathcal{O}_x \otimes E\),
3. the morphism \(\alpha_*\) is the morphism associated to \(\alpha_{E_x}\),
4. the kernel of \(E \xrightarrow{\alpha_*} \mathcal{O}_x \otimes E\) is \(E(-x)\).

Similarly, \(\tilde{p}_1^*(P_{z_2}) \cong F(-y)\). Therefore, since \(r\) is less than the gonality of \(X\), the isomorphisms

\[
E(-x) \cong p_1^*(P_{z_1}) \\
\cong p_1^*(id, \beta)^*(P_{z_2}) \\
\cong \tilde{p}_1^*(P_{z_2}) \\
\cong F(-y)
\]

imply that \(x = y\) and \(E = F\), which proves the proposition. \(\square\)
4. The Hilbert scheme

To compute the differential map $d \Upsilon$ at $z \in \mathcal{A}$ we denote by $N_{G/M}$ the normal bundle of $G(z)$ in $M(n, \xi)$ and by $TM$ the tangent bundle of $M(n, \xi)$. These bundles fit into the following exact sequence

$$0 \to T_G(z) \to \phi_z^* TM \to N_{G/M} \to 0 \quad (4.1)$$

of vector bundles over $G(z)$. Then Theorem 1.1 follows from the next proposition.

**Proposition 4.1.** For any $z \in \mathcal{A}$,

1. $H^0(G(z), N_{G/M}) = T_z \mathcal{A}$.
2. $H^i(G(z), N_{G/M}) = 0$ for $i \geq 1$.
3. $N_{G/M}$ is generated by global sections.

The proof of Proposition 4.1 is somewhat lengthy, so we will split it into several lemmas.

From diagram (3.9) we have that for any $z \in \mathcal{A}$, $\pi_1^{-1}(z) = G(z)$ and $\Phi|_{G(z)} = \phi_z$. Hence, also from (3.9) we have the following commutative diagram

$$
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
T_{\Phi|G(z)} & = T_{\Phi|G(z)} \\
\downarrow & \downarrow \\
0 \to T_G(z) \to T_G(U)|_{G(z)} \to N_{G/G(U)} \to 0 \\
\parallel & \parallel \\
0 \to T_G(z) \to \phi_z^* TM \to N_{G/M} \to 0, \\
\downarrow & \downarrow \\
0 & 0
\end{array}
$$

(4.2)

where $N_{G/G(U)}$ is the normal bundle of $G(z)$ in $G(U)$.

As $G(z)$ is a fibre of the Grassmannian bundle $\pi_1 : G(U) \to \mathcal{A}$, the normal bundle $N_{G/G(U)}$ is the trivial bundle $O_{G(z)} \times T_z \mathcal{A}$. Hence, we have the exact sequence

$$0 \to T_{\Phi|G(z)} \to O_{G(z)} \times T_z \mathcal{A} \to N_{G/M} \to 0 \quad (4.3)$$

and the cohomology sequence

$$0 \to H^0(T_{\Phi|G(z)}) \to H^0(O_{G(z)}) \times T_z \mathcal{A} \to H^0(N_{G/M}) \to H^1(T_{\Phi|G(z)}) \to \ldots$$

(4.4)

In order to compute the cohomology of $T_{\Phi|G(z)}$, we restrict the extension (3.8) to the divisor $\Gamma_\gamma = G(U)$. To streamline the notation, we use the same notation for the restrictions.

The next proposition is slightly different from [19, Lemma 2.1, Proposition 4.8 and 4.12], but the proof works verbatim (see also [25] and Remark 2.3).
Proposition 4.2. The restriction of (3.8) to the divisor $\Gamma_\gamma$ induces the exact sequence

$$0 \to K \to \mathcal{P}_U \to p_2^* \pi_1^* U \xrightarrow{\alpha} Q_U \to 0$$

(4.5)

of vector bundles over $\Gamma_\gamma = \mathbb{G}(U)$, where $K$ is a vector bundle of rank $r$ such that $K \cong Q_U$. Moreover, the sequence (4.5) splits as

$$0 \to K \to \mathcal{P}_U \to S_U \to 0 \text{ and } 0 \to S_U \to p_2^* \pi_1^* U \xrightarrow{\alpha} Q_U \to 0.$$  

(4.6)

Note that the restriction of the sequences (4.6) to $((x, E), W \subset E_x) \in \mathbb{G}(U)$ are the sequences (2.6) in Remark 2.3.

Lemma 4.3. The Grassmannian bundle $g : \mathbb{G}(r, \mathcal{P}_U) \to \mathbb{G}(U)$ has a section

$$\sigma_K : \mathbb{G}(U) \to \mathbb{G}(r, \mathcal{P}_U)$$

such that

1. $\sigma_K^*(T_g)|_{\mathbb{G}(z)} \cong \Omega^1_{\mathbb{G}(z)}$.
2. $\sigma_K^*(T_{\pi_2|_{\sigma_K(\mathbb{G}(z))}}) \cong (T_{\Phi})|_{\mathbb{G}(z)}$, where $\pi_2 : \mathbb{G}(r, \mathcal{P}_U) \to M(n, \xi)$ is defined as $((x, F), Z \subset F_x) \mapsto F$.
3. $H^i(\mathbb{G}(z), (T_{\Phi})|_{\mathbb{G}(z)}) \cong H^i(\mathbb{G}(z), \sigma_K^*(T_{\pi_2|_{\sigma_K(\mathbb{G}(z))}}))$ for $i \geq 0$.

Proof. The subbundle $K \subset \mathcal{P}_U$ has rank $r$, and defines the section $\sigma_K : \mathbb{G}(U) \to \mathbb{G}(r, \mathcal{P}_U)$ such that $\sigma_K^*(T_g) \cong K \otimes \mathcal{P}_U/K$. Thus, from Proposition 4.2 we have that $\sigma_K^*(T_g) \cong Q_U^* \otimes S_U \cong \Omega^1_{\mathcal{P}_U}$.

The rest of the lemma follows from the definition of $\sigma_K$, since it is clear that $\pi_2 \circ \sigma_K = \Phi$, and that $\sigma_K$ is isomorphic to its image. Therefore, $\sigma_K^*(T_{\pi_2|_{\sigma_K(\mathbb{G}(z))}}) \cong T_{\Phi}|_{\mathbb{G}(z)}$ as claimed.

Remark 4.4. Strictly speaking, the morphisms in the above diagram are defined in open sets. Let $\overline{U} := \Phi^{-1}(\Phi(\mathbb{G}(U)))$ and $A := \pi_1(\overline{U})$. The open sets, and mainly the codimension of the complements, are relevant to compute cohomology groups of $M(n, \xi)$ (see [19]). However, in this paper, we will not use the cohomology groups of $M(n, \xi)$ in any essential way, so we use $\mathbb{G}(U)$, $\mathbb{G}(r, \mathcal{P}_U)$, $M(n, \xi)$ as targets of our morphisms.
Lemma 4.5. \( H^i(\sigma_K(\mathbb{G}(z)), (T_{\pi_2})|_{\sigma_K(\mathbb{G}(z))}) = 0 \) for \( i \geq 0 \).

Proof. From Proposition 3.1 we have that \( \phi_\zeta \) is an embedding. Therefore, the image \( \sigma_K(\mathbb{G}(z)) \) is transversal to the fibres of \( \pi_2 \). Since the fibre of \( \pi_2 \) at \( E^W \in \phi_\zeta(\mathbb{G}(n-r, E_x)) \) is the Grassmannian bundle

\[
p : \mathbb{G}(r, E^W) \to X,
\]

the restriction of \( (T_{\pi_2}) \) to \( \sigma_K(\mathbb{G}(z)) \) fits into the following exact sequence

\[
\zeta : 0 \to (T_g)|_{\sigma_K(\mathbb{G}(z))} \to (T_{\pi_2})|_{\sigma_K(\mathbb{G}(z))} \to \mathcal{O}_{\sigma_K(\mathbb{G}(z))} \times T_x X \to 0. \tag{4.8}
\]

From the cohomology sequence of (4.8),

\[
H^i(\sigma_K(\mathbb{G}(z)), (T_{\pi_2})|_{\sigma_K(\mathbb{G}(z))}) \cong H^i(\sigma_K(\mathbb{G}(z)), (T_{\pi_2})|_{\sigma_K(\mathbb{G}(z))}) \quad \text{for} \quad i \geq 2
\]

since \( H^i(\mathbb{G}(z), \mathcal{O}_{\mathbb{G}(z)}) = 0 \) for \( i \geq 1 \). Moreover, for \( i \geq 2 \),

\[
H^i(\sigma_K(\mathbb{G}(z)), (T_{\pi_2})|_{\sigma_K(\mathbb{G}(z))}) = 0
\]

by Lemma 4.3,(1) and Remark 2.1,(3).

The proof is completed by showing that

\[
H^0(\sigma_K(\mathbb{G}(z)), (T_g)|_{\sigma_K(\mathbb{G}(z))}) = H^0(\sigma_K(\mathbb{G}(z)), (T_{\pi_2})|_{\sigma_K(\mathbb{G}(z))}) = 0. \tag{4.9}
\]

The equality (4.9) follows from the cohomology of the sequence (4.8) and by recalling that \( \sigma_K(\mathbb{G}(z)) \subset \mathbb{G}(r, \mathcal{P}_x) \), \( T_x X = \mathbb{C} \) and

\[
H^1(\sigma_K(\mathbb{G}(z)), (T_g)|_{\sigma_K(\mathbb{G}(z))}) \cong H^1(\mathbb{G}(z), \Omega^1_{\mathbb{G}(z)}) = \mathbb{C}.
\]

It follows that \( H^1(\sigma_K(\mathbb{G}(z)), (T_{\pi_2})|_{\sigma_K(\mathbb{G}(z))}) = 0 \), and the proof is complete. \( \square \)

Proof of Proposition 4.1. To compute the cohomology of \( N_{\mathbb{G}/M} \) we use the cohomology exact sequence (4.4).

Combining Lemma 4.3,(3) with Lemma 4.5 we deduce that \( H^i(\mathbb{G}(z), T_{\phi}|_{\mathbb{G}(z)}) = 0 \) for \( i \geq 0 \). We conclude from (4.4) that \( H^i(\mathbb{G}(z), N_{\mathbb{G}/M}) = 0 \) for \( i \geq 1 \), hence that \( H^0(\mathbb{G}(z), N_{\mathbb{G}/M}) = T_z \mathcal{A} \), and finally that \( N_{\mathbb{G}/M} \) is generated by global sections, which is the desired conclusion. \( \square \)

The next proposition will be used to fix the Hilbert polynomial.

Proposition 4.6. For any \( z \in \mathcal{A} \), \( \deg(\phi_\zeta) = 2n \).

Proof. It follows from the exact sequence (4.3) that

\[
\deg(\phi_\zeta^*(T\mathbb{G})) = \deg(N_{\mathbb{G}/M}) + \deg(T\mathbb{G}(z)) = \deg(N_{\mathbb{G}/M}) + n.
\]

We see at once that \( \deg N_{\mathbb{G}/M} = n \), which is clear from the exact sequences in diagram (4.2). Therefore, \( \deg(\phi_\zeta^*(T\mathbb{M})) = 2n \) and \( \deg(\phi_\zeta) = \deg(\phi_\zeta^*(-K_M)) = 2n \) as claimed. \( \square \)
Applying Proposition 4.6 we conclude that $HG$ is a component of the Hilbert scheme $Hil^P_{M(n,\xi)}$ where $P$ is the Hilbert polynomial

$$P(m) = \chi(G, \phi(z)^*(-K_M)) = \chi(G, m(O_G(2n))).$$

We can now prove our main result, Theorem 1.1

Proof of Theorem 1.1. From Proposition 3.2 the morphism $\Upsilon : A \to \mathcal{H}G$, defined as $z \mapsto [\phi_z(G(z))]$ is injective. Clearly, the isomorphism $T_zA \to H^0(G(z), N_{G/M})$ in Proposition 4.1 is the differential of $\Upsilon$ at $z$.

Since $H^i(G(z), N_{G/M}) = 0$ for $i \geq 1$, $\mathcal{H}G$ is smooth at $[\phi_z(G(z))]$ for all $z \in A$ and $\dim \mathcal{H}G = (n^2 - 1)(g - 1) + 1$ (see Remark 2.2). Moreover, from the exact sequence (4.3), $N_{G/M}$ is generated by global sections and, locally, the deformations of an $r$-Hecke cycle are $r$-Hecke cycles. □

5. The Mor($G, M(n, \xi)$) scheme

Let $G$ be the Grassmannian $G(n - r, \mathbb{C}^n)$ and $P$ the Hilbert polynomial

$$P(m) = \chi(G, m(O_G(2n))).$$

In this section we apply the previous results to describe an open set of a component of the scheme $Mor_P(G, M(n, \xi))$.

Let $Hom_P(G, M(n, \xi))$ be the scheme of morphisms from $G$ to $M(n, \xi)$. Recall that to remove the dependency on the choice of coordinates of $G$, we take the quotient by the action of $Aut(G)$. Therefore,

$$Mor_P(G, M(n, \xi)) = Hom_P(G, M(n, \xi))/Aut(G).$$

Proposition 5.1. Any embedding $\lambda : G \to M(n, \xi)$ passing through general points has degree at least $2n$, with respect to $-K_M$.

Proof. Suppose that $\lambda : G \to M(n, \xi)$ has degree $t$. Let $\kappa : \mathbb{P}^1 \to G$ be a morphism of degree 1. Since $\lambda : G \to M(n, \xi)$ is an embedding, the composition $\lambda \circ \kappa : \mathbb{P}^1 \to M(n, \xi)$ is a rational curve and, from [24, Theorem 1], it has degree $\deg(\lambda \circ \kappa) = t \geq 2n$, which is our claim. □

Corollary 5.2. Under the hypotheses of Proposition 2.4 the r-Hecke morphisms have minimal degree.

Proof. The conditions (2.8) imply that the vector bundles in the image $\phi_z(G(z))$ are $(k, \ell)$-stable and hence very general (see Remark 2.5). Propositions 5.1 and 4.6 now show that $\deg(\phi_z)$ is minimal, which is the desired conclusion. □

The next proposition follows directly from Proposition 3.2.

Proposition 5.3. The morphism $\Sigma : A \to Mor_P(G, M(n, \xi))$, defined as $z \mapsto [\phi_z]$ is algebraic and injective. Moreover, $\Sigma(A)$ is contained in an irreducible component of $Mor_P(G, M(n, \xi))$. 

Denote by $\text{Mor}^\mathcal{H}_r(P, \mathbb{G}, M(n, \xi))$ the irreducible component of $\text{Mor}_P(\mathbb{G}, M(n, \xi))$ that contains the $r$-Hecke morphisms. We can now prove Theorem 1.3.

**Proof of Theorem 1.3.** The theorem follows from [7]. By the exact sequence (4.1) and Proposition 4.2,

$$\dim H^0(\mathbb{G}(z), \phi_z^* TM) = \dim H^0(\mathbb{G}(z), T \mathbb{G}(z)) + \dim H^0(\mathbb{G}(z), N_{\mathbb{G}/M})$$

and $H^i(\mathbb{G}(z), \phi_z^*(TM)) = 0$ for all $i \geq 1$. Recall that the image of the inclusion $H^0(\mathbb{G}(z), T \mathbb{G}(z)) \rightarrow H^0(\mathbb{G}(z), \phi_z^*(TM))$ corresponds to the deformations of $\phi_z$ by reparameterisations. Hence, the dimension of $\text{Mor}^\mathcal{H}_r(\mathbb{G}, M(n, \xi))$ is $\dim H^0(\mathbb{G}, N_{\mathbb{G}/M}) = (n^2 - 1)(g - 1) + 1$. This completes the proof of the theorem. \hfill \Box

Let us mention two important consequences of the theorem.

**Corollary 5.4.** If $\text{Mor}^\mathcal{H}_{2n}(\mathbb{P}^1, M(n, \xi))$ is the irreducible component of the space of Hecke curves in $M(n, \xi)$ then $\mathbb{G}(2, \mathcal{U}) \subseteq \text{Mor}^\mathcal{H}_{2n}(\mathbb{P}^1, M(n, \xi))$, where $\mathbb{G}(2, \mathcal{U})$ is the restriction of the Grassmannian Poincaré bundle to $\mathcal{U}$.

**Proof.** The proof is based on the following observation. The Hecke curves are lines in the projective space $\mathbb{P}(E_x)$. Therefore, the corollary follows from the identification of $\mathbb{G}(1, \mathbb{P}(E_x))$ and $\mathbb{G}(2, E_x)$. \hfill \Box

Let $\text{Mor}_s(\mathbb{P}^1, \mathbb{G})$ be the moduli space of stable maps from $\mathbb{P}^1$ to the Grassmannian $\mathbb{G}$, of degree $s$.

**Proof of Corollary 1.2.** The proof is immediate from the natural morphism

$$\text{Mor}_s(\mathbb{P}^1, \mathbb{G}) \times \text{Mor}_P(\mathbb{G}, M(n, \xi)) \rightarrow \text{Mor}_{2ns}(\mathbb{P}^1, M(n, \xi)),$$

induced by the composition. \hfill \Box

The remainder of this section will be devoted to the proof of Theorem 1.4.

Let us recall that if $L$ is an ample divisor on a projective variety $Y$, the $L$-degree $\deg_L(E)$ of a torsion-free sheaf $E$ on $Y$ is defined to be the intersection number $[\mathbb{C}(E)] \cdot [L]^{\dim Y - 1}$. The torsion-free sheaf $E$ is said to be $L$-stable if, for any proper subsheaf $\mathcal{F}$ of $E$,

$$\frac{\deg_L \mathcal{F}}{\text{rk} \mathcal{F}} < \frac{\deg_L E}{\text{rk} E}.$$  

Denote by $\mathcal{M}^L_Y(\mathbb{E})$ the moduli space of $L$-stable sheaves over $Y$ with the same numerical invariants as $\mathbb{E}$.

For any $z \in \mathcal{A}$, the morphism $\phi_z$ is defined by the existence of a vector bundle $\mathcal{P}_z$ over $X \times \mathbb{G}$ called $r$-Hecke bundle. Since $\text{Pic}(\mathbb{G}) = \mathbb{Z}$, $\text{Pic}(X \times \mathbb{G}) = \text{Pic}(X) \oplus \text{Pic}(\mathbb{G})$. Thus, any polarization $L$ on $X \times \mathbb{G}$ can be expressed in the form $L = a\alpha + b\beta$ with $a, b > 0$, where $\alpha$ is ample on $X$ and $\beta$ ample on $\mathbb{G}$.

**Proposition 5.5.** For any $z \in \mathcal{A}$, the $r$-Hecke bundle $\mathcal{P}_z \rightarrow X \times \mathbb{G}$ is $L$-stable with respect to any polarization $L$.

**Proof.** The vector bundle $\mathcal{P}_z \rightarrow X \times \mathbb{G}$ is a family of stable bundles parameterised by $\mathbb{G}$. By construction, for any $y \neq x$, $(\mathcal{P}_z)_{(y) \times \mathbb{G}} \cong \mathcal{O}_Y^n$. Therefore, the $L$-stability of $\mathcal{P}_z$ follows from [1, Lemma 2.2]. \hfill \Box
Let $M_{L,H}^{X,G}(P_z)$ be the irreducible component of the moduli spaces of $L$-stable vector bundles over $X \times G$ containing the $r$-Hecke bundles $P_z$, $z \in A$.

**Proof of Theorem 1.4.** The tangent space of the moduli space at $P_z$ is

$$H^1(X \times G, End(P_z)) = H^1(X \times G, O_{X \times G}) \oplus H^1(X \times G, Ad(P_z)).$$

From the Leray spectral sequence and the cohomology of (4.1) we have that

$$H^1(X \times G, Ad(P_z)) = H^0(G, \phi_z^*(T M)) = H^0(G, T G(z)) \oplus H^0(G, N_{G/M}).$$

Since $H^1(X \times G, O_{X \times G}) = H^1(X, O_X)$,

$$H^1(X \times G, End(P_z)) = H^0(G, T G(z)) \oplus H^0(G, N_{G/M}) \oplus H^1(X, O_X).$$

Therefore, the theorem follows from the equality $h^0(G, N_{G/M}) = \dim T_z A = (n^2 - 1)(g - 1) + 1$. Note that locally the deformations of $P_z$ come from those of the curve, from those of the Grassmannian and from $H^0(G, \phi_z^*(T M))$, and this is precisely the assertion of the theorem. □

6. Open questions

There are a number of interesting questions about the $r$-Hecke cycles, the Hecke morphisms and the Hecke bundles. We mention some of them, which are most interesting from the viewpoint of moduli spaces.

We have defined three open sets, namely:

1. $\Upsilon(A)$ of the Hilbert scheme $\mathcal{H}G$,
2. $\Sigma(A)$ of the moduli scheme $\text{Mor}_P^A(G, M(n, \xi))$,
3. $\Gamma(A)$ of the moduli space $M_{L,H}^{X,G}(P_z)$.

Let $\widetilde{\Upsilon}$, $\widetilde{\Sigma}$ and $\widetilde{\Gamma}$ be the closures of $\Upsilon(A)$, $\Sigma(A)$ and $\Gamma(A)$ respectively.

The natural questions are:

**Question 6.1.** What are the elements that compactify these open sets? Do the boundary points have a natural geometric meaning?

**Question 6.2.** What are the relationships among $\widetilde{\Upsilon}$, $\widetilde{\Sigma}$ and $\widetilde{\Gamma}$?

**Question 6.3.** Do they give a new compactification of the moduli space $M(n, \xi)$?

For each $\phi_z : G \to M(n, \xi)$, the image of the differential $d\phi_z$ defines a subspace $d\phi_z T G$ of $T E^w M$. The dual of the quotient $T E^w M / d\phi_z T G$ defines a subspace $\mathcal{GH}$ in the cotangent space

$$T_{E^w}^* M = H^0(X, End E^w \otimes K_X),$$

and hence a subspace of the Higgs bundles and of the spectral curves. This construction generalises that of [10].

**Question 6.4.** Describe the Higgs bundles and the spectral curves defined by $\mathcal{GH}$.
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