We explore the possibility that scale symmetry is a quantum symmetry that is broken only spontaneously and apply this idea to the Standard Model (SM). We compute the quantum corrections to the potential of the higgs field ($\phi$) in the classically scale invariant version of the SM ($m_\phi = 0$ at tree level) extended by the dilaton ($\sigma$). The tree-level potential of $\phi$ and $\sigma$, dictated by scale invariance, may contain non-polynomial effective operators, e.g. $\phi^6/\sigma^2$, $\phi^8/\sigma^4$, $\phi^{10}/\sigma^6$, etc. The one-loop scalar potential is scale invariant, since the loop calculations manifestly preserve the scale symmetry, with the DR subtraction scale $\mu$ generated spontaneously by the dilaton vev $\mu \sim \langle \sigma \rangle$. The Callan-Symanzik equation of the potential is verified in the presence of the gauge, Yukawa and the non-polynomial operators. The couplings of the non-polynomial operators have non-zero beta functions that we can actually compute from the quantum potential. At the quantum level the higgs mass is protected by spontaneously broken scale symmetry, even though the theory is non-renormalizable. We compare the one-loop potential to its counterpart computed in the “traditional” DR scheme that breaks scale symmetry explicitly ($\mu =$constant) in the presence at the tree level of the non-polynomial operators.
1 Motivation

In this letter we explore the idea that scale symmetry is a quantum symmetry and study its implications for physics beyond SM. However, this symmetry is broken in the real world. We shall consider here only spontaneous breaking of this (quantum) symmetry. One motivation of our study is that scale symmetry plays a role in the ultraviolet (UV) behaviour of the models. In particular the SM with a classical higgs mass parameter $m_\phi=0$ has an increased symmetry: it is scale invariant at the tree level; this was invoked \[1\] to protect $m_\phi$ naturally \[2\] from large quantum corrections, but a full quantum study is needed.

Consider a classically scale invariant theory. One known issue when studying scale symmetry at the quantum level is that the regularization of the loop corrections introduces a dimensionful parameter (subtraction scale $\mu$) which breaks explicitly the scale symmetry, thus destroying the symmetry we want to investigate and affecting the UV properties of the quantum theory. To avoid this, the UV regularization must preserve this symmetry. This is done by using a subtraction function $\mu(\sigma)$ which generates (dynamically) a subtraction scale $\mu(\langle \sigma \rangle)$ when the field $\sigma$ acquires a vev $\langle \sigma \rangle$ after spontaneous scale symmetry breaking. For details on this see \[3\] and recent examples at one-loop \[11\]-\[16\] and higher loops \[17\]-\[20\]. Here $\sigma$ is the Goldstone mode (dilaton) of the spontaneously broken scale symmetry \[3\].

The model we consider is a scale-invariant SM, defined as SM with classical $m_\phi=0$ and extended by the dilaton. The goal is to use this scale invariant regularization to compute quantum corrections to the scalar potential. The quantum result is scale invariant, so it can only have spontaneous scale symmetry breaking, with a flat direction for the dilaton $\langle \sigma \rangle$. For clarity, this result is then compared to that in the "usual" dimensional regularization (DR) of $\mu=$constant scale, which breaks explicitly the scale symmetry at the quantum level.

Let us consider first a simplified scale invariant (classical) theory (e.g. \[11\]-\[27\]) of two real scalar fields $\phi$ (higgs-like) and $\sigma$. The potential $V$ is an homogeneous function, having no dimensionful couplings, so

$$V(\phi, \sigma) = \sigma^4 W(\phi/\sigma), \quad \text{where} \quad W(\phi/\sigma) = V(\phi/\sigma, 1)$$

(1)

We assume that $V(\phi, \sigma)$ has spontaneous scale symmetry breaking i.e. that $\sigma$ acquires a non-zero vacuum expectation value $\langle \sigma \rangle \neq 0$. We thus search for such a solution and for the necessary condition for this spontaneous breaking to happen. With $\langle \sigma \rangle \neq 0$ it is then easy to see that the minimum conditions $V_\phi = V_\sigma = 0$ ($V_\alpha = \partial V/\partial \alpha$) are equivalent to

$$W(\rho) = W'(\rho) = 0, \quad \rho \equiv \phi/\sigma.$$  \hspace{1cm} (2)

These equations can have a common solution $\rho_0 \equiv \langle \phi \rangle/\langle \sigma \rangle$, if the couplings satisfy a partic-

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1By quantum scale symmetry we mean that the full 1PI quantum action is scale invariant.
2One could use a regularization that does not keep manifest scale symmetry and attempt to restore it “by hand” at the end, but this misses scale-invariant operators if the theory is non-renormalizable (see later).
3To be exact, the mass eigenstates may actually contain a small mixing of original $\phi, \sigma$. 

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ular condition (constraint), see below. Then a flat direction exists in the plane \((\phi, \sigma)\) with \(\phi = \rho_0 \sigma\). Indeed, if \((\langle \phi \rangle, \langle \sigma \rangle)\) is a ground state with \(V = 0\) then so is \((t\langle \phi \rangle, t\langle \sigma \rangle)\), \(t\) real. Also the second derivatives matrix \(V_{\alpha\beta}\) wrt \(\alpha, \beta = \phi, \sigma\) has \(\det(V_{\alpha\beta}) \propto (4W W'' - 3W'2) = 0\) on the ground state, so a massless state is indeed present corresponding to the flat direction. Finally, since \(\rho_0\) is a root of both \(W\) and of its derivative \(W'\), then \(W(\phi/\sigma) \propto (\phi/\sigma - \rho_0)^2\), while if \(V\) depends only on even powers of the scalar fields (our model below), then the general structure is

\[
W(\phi/\sigma) \propto (\phi^2/\sigma^2 - \rho_0^2)^2. \tag{3}
\]

Note that the vanishing vacuum energy \(V(\langle \phi \rangle, \langle \sigma \rangle) = 0\) follows from the (spontaneously broken) scale symmetry, see eq.(2). A scale invariant regularization of this theory leads to a scale invariant quantum potential, which thus remains of the form in eq.(1). Hence the above discussion around eqs.(1), (2), (3) remains true at the quantum level, including the possibility of spontaneous-only breaking of the scale symmetry.

One of the two minimum conditions in (2) fixes the ratio \(\rho_0 = \langle \phi \rangle/\langle \sigma \rangle\) in terms of the (dimensionless) couplings of the theory. Thus all vev’s of such theory, including \(\langle \phi \rangle\) are proportional to \(\langle \sigma \rangle \neq 0\) which is a (unknown) parameter of the theory. The second minimum condition, after eliminating \(\rho_0\) between the two equations in (2), gives a relation among the couplings of the theory in the order of perturbation in which \(V\) is computed. This means that one coupling, say \(\lambda_\sigma\) (the dilaton self-coupling) is defined in terms of the rest \(\lambda_\sigma = f(\lambda_{j\neq\sigma})\). This relation follows from demanding that \(V\) have a flat direction\(^4\) which is a consequence of our requiring that quantum scale symmetry be broken spontaneously. Such relation can be “arranged” by one initial classical tuning, with subsequent (quantum) tunings bringing “acceptable” form \(\mathcal{O}(\lambda_j)\) corrections to this relation, relative to the previous perturbation order\(^5\); this tuning ensures a vanishing vacuum energy \(V(\langle \phi \rangle, \langle \sigma \rangle) \sim W(\rho_0) = 0\), see conditions (2).

We stress that the above picture, that builds on previous studies\(^3\), is very different from that obtained in the “traditional” DR scheme (\(\mu =\)constant scale) that is often used in classically scale invariant models e.g.[18]-[27]; in such models scale symmetry is broken explicitly by the (regularization of) quantum effects and then conditions (1), (2) are not true anymore at quantum level and the flat direction is lifted by quantum corrections\(^6\).

What about the hierarchy problem? In the absence of gravity (not included here), the Standard Model has no hierarchy problem. However, this situation is no longer true under the reasonable assumption that there is some “new physics” beyond SM, e.g. a large vev of a new scalar field that couples to Higgs, etc. In the model we consider, defined by the scale invariant version of the SM extended by the dilaton, we have “new physics” beyond

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\(^4\)This agrees with the classical result of [26] that spontaneous breaking of conformal symmetry to Poincaré symmetry for a single scalar field happens if the dilaton potential \(g\chi^{2d/(d-2)}\) has \(g = 0\) (flat direction); (note our model is a non-renormalizable scale invariant quantum theory (see later) rather than conformal).

\(^5\)Perturbativity \(\lambda_\sigma < 1\) is maintained for a weak coupling \(\lambda_m\) between the visible (\(\phi\)) and hidden (\(\sigma\)) sectors, see later, eq.(11a) or eq.(11) which fixes \(\lambda_\sigma < 1\) in terms of the other couplings, for a small enough \(\lambda_m\).

\(^6\)The dilaton is then a pseudo-Goldstone which is light, so it is regarded as the SM Higgs in those models.
the SM, represented by the vev $\langle \sigma \rangle$ that breaks spontaneously the scale symmetry. $\langle \sigma \rangle$ can be very large compared to $\langle \phi \rangle$ where the latter fixes the electroweak scale; $\langle \sigma \rangle$ can then be regarded as a physical cut-off of the theory. We simply enforce such hierarchy by choosing a very weak coupling of the visible to the hidden sector of the dilaton. Such hierarchy is however stable under quantum corrections, so $m_\phi \sim \langle \phi \rangle \ll \langle \sigma \rangle$ without tuning at the quantum level and we verify this in our model at one-loop. This is expected to remain true to all orders in perturbation theory since scale symmetry is preserved by the regularization and is broken only spontaneously. We thus have an example of a quantum stable hierarchy, with a vanishing vacuum energy at the loop level, that follow from the demand of spontaneously broken quantum scale symmetry.

In the following we apply these ideas to the scale invariant version of the SM (with classical higgs mass $m_\phi = 0$) extended by the dilaton. The higgs and the dilaton have a potential dictated solely by the classical scale symmetry, so it can contain higher dimensional non-polynomial operators such as $\phi^6/\sigma^2$, $\phi^8/\sigma^4$, etc. We then compute the one-loop potential with a scale invariant regularization, so a flat direction is maintained at the quantum level. Even if the tree-level potential does not include the non-polynomial operators (by tuning their couplings to 0), they are generated at one-loop with finite coefficient or as two-loop or higher counterterms - this means the scale invariant quantum theory is non-renormalizable. Further, the quantum consistency of the theory is shown by verifying the Callan-Symanzik equation of the potential in the presence of the non-polynomial effective operators, gauge and Yukawa interactions. We also compare the scale-invariant one-loop potential to its counterpart computed in the “usual” DR scheme that breaks scale symmetry explicitly ($\mu=$constant), in the presence at tree level of these effective operators.

If scale symmetry is preserved by one-loop $V$, there is no dilatation anomaly which is a result of explicit scale symmetry breaking by quantum calculations with $\mu=$constant. Contrary to common lore, the couplings still run with momentum since the vanishing of the beta functions is not a necessary condition for scale invariance. Their one-loop running is identical to that in the theory with explicit scale symmetry breaking ($\mu=$constant), but at two-loop they start to differ in theories with spontaneous versus explicit breaking.

This analysis in flat space-time should be extended to include the effects of gravity which we ignored. Since Einstein gravity breaks scale symmetry, a natural setup to include such effects is to consider the Brans-Dicke-Jordan theory of gravity, see examples in. In such setup it may still be possible to perform a scale-invariant regularization and then examine such scale invariant theory at quantum level.

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7In the Brans-Dicke-Jordan theory of gravity, not considered here, one expects $\langle \sigma \rangle \sim M_{\text{Planck}}$.
8Such hierarchy can be generated dynamically or as in.
9One takes $|\lambda_m| \ll \lambda_\phi$; $\lambda_m$: coupling of hidden $(\sigma)$ to visible sector $(\phi)$; $\lambda_\phi$: higgs self-coupling, see later.
10Scale symmetry may also be broken at some high scale due to Landau poles of some of the couplings of the theory or due to other non-perturbative effects. We do not consider these effects here since they involve physics above Planck scale in which case the present flat space-time picture is not appropriate - one needs to upgrade this formalism to include Brans-Dicke-Jordan gravity, see e.g.
2 SM with a scale invariant one-loop potential

2.1 The tree-level scale invariant potential

Consider the SM Lagrangian with tree-level higgs mass \( m_\phi = 0 \), so it is scale invariant. The higgs sector is weakly coupled to the “hidden” sector of the dilaton \( \sigma \) with

\[
\mathcal{L} = |D_\mu H|^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - V_0
\]

where

\[
H = \left( \frac{G^+}{\sqrt{2}}(\phi + iG^0) \right)
\]

and

\[
V_0 = \frac{\lambda_\phi}{3!} (H^\dagger H)^2 + \frac{\lambda_m}{2} (H^\dagger H)\sigma^2 + \frac{\lambda_\sigma}{4!} \sigma^4 + \frac{4\lambda_6}{3} \frac{(H^\dagger H)^3}{\sigma^2} + \cdots
\]

where the dots stand for higher powers of \( H^\dagger H \). The neutral higgs (\( \phi \)) and dilaton part is

\[
V(\phi, \sigma) = \frac{1}{4!} \lambda_\phi \phi^4 + \frac{1}{4} \lambda_m \phi^2 \sigma^2 + \frac{1}{4!} \lambda_\sigma \sigma^4 + \frac{\lambda_6}{6} \frac{\phi^6}{\sigma^2} + \cdots
\]

We take \( \lambda_\phi, \lambda_\sigma > 0 \) and \( \lambda_m < 0 \) and that the two sectors of \( \phi \) and \( \sigma \) are weakly coupled, with \( |\lambda_m| < \lambda_\phi \). Regarding the terms suppressed by powers of \( \sigma \), they respect the (classical) scale symmetry of the action, so they can be present in the theory. They are well-defined since \( \sigma \) acquires spontaneously a vev \( \langle \sigma \rangle \neq 0 \) under conditions that we identify shortly (see (a) in eqs.\((9), (11) \) below). One can expand such terms about the ground state, into an infinite sum of familiar polynomial (effective) operators:

\[
\lambda_6 \frac{\phi^6}{\sigma^2} = \lambda_6 \frac{\phi^6}{\langle \sigma \rangle^2} \left( 1 - 2 \frac{\sigma'}{\langle \sigma \rangle} + 3 \frac{\sigma'^2}{\langle \sigma \rangle^2} + \cdots \right), \quad \sigma = \langle \sigma \rangle + \sigma', \quad \sigma' : \text{fluctuation}. \]

However, we prefer to use the form in eq.\((7) \) since it keeps manifest the scale symmetry of \( \mathcal{L} \). Finally, we keep \( \lambda_6 \neq 0 \) but set to 0 the coefficients of \( (H^\dagger H)^4/\sigma^4 \) and higher terms.

Consider first \( \lambda_6 = 0 \). We demanded spontaneous breaking of scale symmetry, so we seek the condition for which \( \langle \sigma \rangle \neq 0 \). The minimum of \( V \) exists if derivatives \( V_\phi = V_\sigma = 0 \), giving

\[
(a) : \quad \lambda_\sigma = \frac{9\lambda_m^2}{\lambda_\phi} \left[ 1 + \text{loops} \right] \quad \text{and} \quad (b) : \quad \frac{\langle \phi \rangle^2}{\langle \sigma \rangle^2} = \frac{-3\lambda_m}{\lambda_\phi} \left[ 1 + \text{loops} \right],
\]

so also \( \langle \phi \rangle \neq 0 \); here “loops” stands for loop corrections.

\[\text{Even if we set } \lambda_6, \lambda_8, \ldots = 0 \text{ at EW scale, such terms are generated in a quantum scale invariant theory at one-loop (with a finite coefficient) \[8\] or as two-loop counterterms \[10\], so their presence is inevitable. If instead } \mu = \text{constant (explicit breaking) and } \lambda_6, \lambda_8, \ldots = 0, \text{ such terms are never generated at quantum level.} \]
Let us then assume that $\lambda_\sigma$ is indeed that of (a) up to “loop” effects that one can identify order by order in perturbation theory and that we ignore for the classical discussion here. If (a) is true, we have spontaneous breaking of scale symmetry and

$$V = \frac{1}{4!} \lambda_\sigma \sigma^4 \left( \frac{\phi^2}{\sigma^2} + \frac{3\lambda_m}{\lambda_\sigma} \right)^2 \quad (10)$$

with $V = 0$ at the minimum. A flat direction, corresponding to the Goldstone of scale symmetry (dilaton) exists in the plane $(\phi, \sigma)$. The neutral higgs acquires a mass $m_\phi^2 = (\lambda_\phi/3)(1 - 3\lambda_m/\lambda_\phi)\langle \phi \rangle^2$, while the EW Goldstone bosons are massless. Thus, spontaneous scale symmetry breaking triggers EW symmetry breaking, with a vacuum energy $V = 0$.

Consider now $\lambda_6 \neq 0$, with $\lambda_6 > 0$ for a well-defined $V$ at large $\phi$. Then eqs. (10) become

(a): $\lambda_\sigma = \rho_0^2 \left[ 2\lambda_6 6^0 - 3 \lambda_m \right] + \text{loops}$, where

(b): $\rho_0^2 = \frac{\langle \phi \rangle^2}{\langle \sigma \rangle^2} = \frac{1}{12\lambda_6} \left[ - \lambda_\phi + (\lambda_\phi^2 - 72\lambda_6 \lambda_m)^{1/2} \right] + \text{loops} \quad (11)$

We assume from now on that $\lambda_\sigma$ is indeed given by relation (a), up to small quantum corrections (ignored here), to ensure spontaneous scale symmetry breaking; this relation is “protected” by scale symmetry. The potential is then

$$V = \frac{\lambda_6}{6} \sigma^4 \left( \frac{\phi^2}{\sigma^2} - \frac{\rho_0^2}{\sigma^2} \right)^2 \left( \phi^2 + \xi_0^2 \right) \quad (12)$$

in agreement with (3). Here $\xi_0^2 = (\lambda_\phi + 2(\lambda_\phi^2 - 72\lambda_6 \lambda_m)^{1/2})/(12\lambda_6) > 0$. If $\lambda_6 \to 0$ one recovers eq. (10). The neutral higgs mass can again be computed and recovers the above value for small $\lambda_6$; the dilaton is again massless, with the flat direction mildly changed by $\lambda_6$. To conclude, spontaneous scale symmetry breaking triggers EW symmetry breaking and ensures $V = 0$ on the ground state. We would like to know if this can remain true at quantum level.

The scale $\langle \sigma \rangle$ of “new physics” beyond SM should be larger than $\langle \phi \rangle \sim O(100)$ GeV. In Brans-Dicke-Jordan theory of gravity (not considered here) that can generalise this study, one actually expects $\langle \sigma \rangle \sim M_{\text{Planck}}$. So a hierarchy $\langle \phi \rangle \ll \langle \sigma \rangle$ may be generated dynamically \[28, 29\]. Here we take a common view of a very weak coupling of the hidden ($\sigma$) to visible ($\phi$) sector: $|\lambda_m| \ll \lambda_\phi$ \[38\]; then from eq. [11] $\lambda_\sigma \ll |\lambda_m|$. This classical “tuning” can ensure

12It is actually the generalization of (a) for $\lambda_6 \neq 0$ that we shall assume to be true, see later.

13Eqs. [11] for small $\lambda_6$ become $\lambda_\sigma = (0\lambda_m^2/\lambda_\phi) [1 + O(\lambda_6) + \text{loops}]$ and $\rho_0^2 = (-3\lambda_m/\lambda_\phi) [1 + O(\lambda_6) + \text{loops}]$.

14With $y = \phi^2/\sigma^2$, $V = (\lambda_6 \sigma^4/4!) (y + 3\lambda_m/\lambda_\phi)[y + 3\lambda_m/\lambda_\phi + (4\lambda_6/\lambda_\phi)(y^2 - 3y\lambda_m/\lambda_\phi + 9\lambda_m^2/\lambda_\phi^2)] + O(\lambda_6)$.

15One has

$$m_\phi^2 = \left[ -2 + \frac{\lambda_6}{6\lambda_\phi} \lambda_m \langle \sigma \rangle^2 + \rho_0^2 \left[ \frac{\lambda_\phi}{3} \left( \frac{\lambda_\phi}{6\lambda_\phi} - 1 \right) - 2\lambda_m \right] \right] (\sigma)^2 = -\lambda_m \left( 1 - \frac{3\lambda_m}{\lambda_\phi} \right) (\sigma)^2 + O(\lambda_6) \quad (13)$$

16This hierarchy is stable under renormalization group \[39\] due to a shift symmetry, $\sigma \to \sigma + \text{constant}$.

17Note this is not a tuning in the sense of cancellation of mass scales, seen in the mass hierarchy problem.
a hierarchy of scales $\langle \phi \rangle \ll \langle \sigma \rangle$ ($\lambda_6$ only brings sub-leading corrections, since the hierarchy is controlled by $\lambda_m$, the main coupling of the two sectors).

This concludes the picture of the classical potential with scale symmetry. At the quantum level, one question is whether the (quantum) scale symmetry, when spontaneously broken, maintains the hierarchy $m^2_\phi \sim \langle \phi \rangle^2 \ll \langle \sigma \rangle^2$ without additional tuning of the couplings. If quantum corrections $\lambda^2_\phi \langle \sigma \rangle^2$ are generated, a tuning of the higgs self-coupling $\lambda_\phi$ would be needed and this would re-introduce the hierarchy problem.

### 2.2 The one-loop scale invariant potential

Let us compute the one-loop potential by preserving scale symmetry at quantum level and thus avoid its explicit breaking by the UV regularization. The method is described in [4, 6, 7, 8, 9, 10]. To do this note we already have a vev $\langle \sigma \rangle$ that can act as subtraction scale. The starting point is in $d = 4 - 2\epsilon$ dimensions where the tree level potential is modified into

$$\tilde{V} = \mu(\sigma)^{2\epsilon} V, \quad \mu(\sigma) = z \sigma^{1/(1-\epsilon)}, \quad (14)$$

$\tilde{V}$ is thus scale invariant in $d = 4 - 2\epsilon$. The function $\mu(\sigma)$ generates a subtraction scale $\mu(\langle \sigma \rangle)$ when $\sigma$ acquires a vev spontaneously. The definition of $\mu(\sigma)$ follows on dimensional grounds, with $z$ an arbitrary dimensionless subtraction parameter [7]. If we set $\mu(\sigma)=$constant, one immediately recovers the “traditional” DR scheme that breaks explicitly the scale symmetry in $d = 4 - 2\epsilon$. We thus have two possible analytical continuations to $d = 4 - 2\epsilon$ of the classical scale invariant theory in $d = 4$: one is scale invariant (eq.(14)), the other is not ($\mu=$constant), and they lead to distinct quantum theories (of different symmetry) [8, 10], as discussed below. The one-loop potential in $d = 4 - 2\epsilon$ is then [3, 10] [8]

$$V_1 = \tilde{V} - \frac{i}{2} \int \frac{d^d p}{(2\pi)^d} \text{Tr} \ln \left[ p^2 - \tilde{V}_{ij} + i\epsilon \right]. \quad (15)$$

This is computed in the Landau gauge. The field dependent squared masses are eigenvalues of the matrix of second derivatives denoted $\tilde{V}_{ij}$ where subscripts $i, j$ stand for: the EW Goldstone scalars $G^0$, Re $(G^+)$, Im $(G^+)$, neutral higgs $\phi$ and dilaton $\sigma$. Unlike the EW Goldstone modes or fermions and gauge bosons, the field-dependent masses of $\phi$ and $\sigma$
acquire a correction $\propto \epsilon$ relative to their values induced by $V$ alone, from derivatives of $\mu(\sigma)$

$$m^2_1 = \frac{\mu(\sigma)^{2\epsilon}}{2} h_1^2 \phi^2, \quad m^2_W = \frac{\mu(\sigma)^{2\epsilon}}{4} g_2^2 \phi^2, \quad m^2_Z = \frac{\mu(\sigma)^{2\epsilon}}{4} (g_1^2 + g_2^2) \phi^2,$$

$$m^2_G = \frac{\mu(\sigma)^{2\epsilon}}{6} \left[ 3 \lambda_\phi \phi^2 + 3 \lambda_m \sigma^2 + 6 \lambda_\gamma \phi^4 \right],$$

$$M^2_k = \mu(\sigma)^{2\epsilon} \left[ m^2_k + \epsilon \delta_k \right], \quad k = \phi, \sigma.$$ (16)

where $m_t \ (h_t)$ is the field-dependent top mass (Yukawa coupling), $m_{W,Z}$ denote the gauge boson masses and $m_G$ denote the three EW Goldstone field-dependent masses. Finally $M^2_k$ are eigenvalues of $V_{\alpha\beta}$, while $m^2_k$ are eigenvalues of the $2 \times 2$ sub-matrix $V_{\alpha\beta}$ of $V_{ij}$ with

$$V_{\alpha\beta} = \partial^2 V / \partial \alpha \partial \beta, \quad \alpha, \beta = \phi, \sigma.$$

Then, one finds at one-loop ($\kappa = (4\pi)^2$)

$$V_1 = \mu(\sigma)^{2\epsilon} \left\{ V - \frac{1}{4\kappa} \sum_{j=\phi,\sigma,G,W,Z,t} n_j m^4_j(\phi, \sigma) \left( \frac{1}{\epsilon} - \ln \frac{m^2_j(\phi, \sigma)}{c_j \mu^2(\sigma)} \right) + \frac{4 (V_{\alpha\beta} N_{\beta\alpha})}{\mu^2(\sigma)} \right\}. \quad (17)$$

with summation over $\alpha, \beta = \phi, \sigma$ and $N_{\alpha\beta} = \mu(\mu_{\alpha} V_{\beta} + \mu_{\beta} V_{\alpha} - \mu_{\alpha} \mu_{\beta}) V$ and $\mu_{\alpha} = \partial \mu / \partial \alpha$. Also $n_j = \{3, 1, 6, 3, -12\}$ for $j = \{G, S, W, Z, t\}$, with $S = \phi, \sigma$; $c_j = 4\pi \epsilon^{3/2-\gamma_\kappa}$ if $j = \phi, \sigma, t, G$ and $c_j = 4\pi \epsilon^{5/6-\gamma_\kappa}$ if $j = W, Z$. The one-loop term $(V_{\alpha\beta} N_{\beta\alpha})$ is a new correction, absent in the case of $\mu =$constant (i.e. explicit scale symmetry breaking by the regularization).

The poles in the one-loop Lagrangian are cancelled by the counterterm $\delta L$.\[^{21}\]

$$\delta L_1 \equiv \frac{1}{2} \left( Z_\phi - 1 \right) \left( \partial_\mu \phi \right)^2 + \frac{1}{2} \left( Z_\sigma - 1 \right) \left( \partial_\mu \phi \right)^2$$

$$- \mu(\sigma)^{2\epsilon} \left\{ \frac{1}{4!} (Z_{\lambda_\phi} - 1) \lambda_\phi \phi^4 + \frac{1}{4} (Z_{\lambda_m} - 1) \lambda_m \phi^2 \sigma^2 + \frac{1}{4!} (Z_{\lambda_\gamma} - 1) \lambda_\gamma \phi^4 \right\}$$

$$+ \sum_{j=3,4,5,6} \frac{1}{2} (Z_{\lambda_{2j}} - 1) \lambda_{2j} \frac{\phi^{2j}}{\phi^{2j-4}} \right\}.$$ (18)

Introducing the notation:

$$Z_\xi = 1 + \frac{1}{\epsilon} \frac{\gamma_\xi}{\kappa}, \quad \xi = \lambda_\phi, \lambda_\sigma, \text{ etc.} \quad (19)$$

one identifies:

\[^{20}\text{In general, in terms of derivatives of tree level } V: m_k^2 = \frac{1}{4} \left[ \text{Tr} (V_{\alpha\beta}) \pm \left[ (\text{Tr} V_{\alpha\beta})^2 - 4 \det V_{\alpha\beta} \right]^{1/2} \right] \text{ and also } \delta_k = \mu(\sigma)^{-2} \left\{ \text{Tr} (N_{\alpha\beta}) \pm \left[ (\text{Tr} V_{\alpha\beta}) \text{Tr} N_{\alpha\beta} - 2 \rho \right] / \left[ (\text{Tr} V_{\alpha\beta})^2 - 4 \det V_{\alpha\beta} \right]^{1/2} \right\}. \text{ The expression of } \rho \text{ is } \rho = V_{\phi\phi} N_{\sigma\sigma} + V_{\phi\sigma} N_{\phi\sigma} - 2 V_{\phi\phi} N_{\phi\sigma}, \text{ where } N_{\phi\phi} = 0, N_{\sigma\sigma} = z^2 (2 \sigma V_{\phi} - V), N_{\phi\sigma} = z^2 \sigma V_{\phi}. \text{ }\]

\[^{21}\text{One can use } \sum_{k=\phi,\sigma} m_k^2 = V_{\phi\phi}^2 + V_{\phi\sigma}^2 + 2 V_{\phi\sigma}^2. \text{ }\]
\[ \gamma_{\lambda_{\phi}} = \frac{3}{2\lambda_{\phi}} \left( \frac{3}{2} g_1^2 + \frac{3}{4} (g_1^2 + g_2^2)^2 - 12 \lambda_{\phi}^4 + \frac{4}{3} \lambda_{\phi}^2 + \lambda_{m}^2 + 32 \lambda_{m} \lambda_{6} \right), \]

\[ \gamma_{\lambda_{m}} = \frac{1}{2} (2\lambda_{\phi} + \lambda_{\sigma} + 4\lambda_{m}), \]

\[ \gamma_{\lambda_{\sigma}} = \frac{3}{2} (\lambda_{\sigma} + 4\lambda_{m}/\lambda_{\sigma}). \] (20)

Notice that \( \lambda_{6} \) contributes to \( \gamma_{\lambda_{\phi}} \) and to the beta function of \( \lambda_{\phi} \) (see later). Finally

\[ \gamma_{\lambda_{6}} = \frac{3}{2} (6\lambda_{\phi} - 8\lambda_{m} + \lambda_{\sigma}), \]

\[ \gamma_{\lambda_{8}} = \frac{2 \lambda_{6}}{\lambda_{8}} (28\lambda_{6} + \lambda_{m}), \]

\[ \gamma_{\lambda_{10}} = 20 \frac{\lambda_{6}^2}{\lambda_{10}}, \quad \gamma_{\lambda_{12}} = \frac{3 \lambda_{6}^2}{\lambda_{12}}. \] (21)

Therefore, the non-polynomial operator \( \lambda_{6} \phi^6/\sigma^2 \) in the tree-level \( V \) generated new non-polynomial counterterms up to and including \( \phi^{12}/\sigma^8 \), of couplings \( \propto \lambda_{6} \). This effect is independent of whether the quantum calculation respects or not the scale symmetry (i.e. \( \mu \sim \sigma \) or \( \mu=\text{constant} \)). The generalisation to more such operators at the tree level is immediate.

The SM one-loop potential \( U_{1} \) is then

\[ U_{1} = V + V^{(1)} + V^{(1,n)}, \] (22)

where

\[ V^{(1)} = \frac{1}{48 \kappa} \sum_{j=\phi, \sigma; G, T, W, Z} n_{j} m_{2j}^4(\phi, \sigma) \ln \frac{m_{2j}^2(\phi, \sigma)}{c_{j}(z \sigma)^2}, \]

\[ V^{(1,n)} = \frac{1}{48 \kappa} \left[ (-16 \lambda_{m} \lambda_{\phi} - 18 \lambda_{m}^2 + \lambda_{\phi} \lambda_{\sigma}) \phi^4 - \lambda_{m} (48 \lambda_{m} + 25 \lambda_{\sigma}) \sigma^2 - 7 \lambda_{\sigma}^2 \sigma^4 \right. \]

\[ + \left. (\lambda_{\phi} \lambda_{m} + 6 \lambda_{6} \lambda_{\sigma}) \phi^6 \sigma^2 + 8 \lambda_{6} (4 \lambda_{\phi} - 2 \lambda_{m}) \phi^4 \sigma^2 + \lambda_{6} (192 \lambda_{6} + 2 \lambda_{\phi}) \frac{\phi^{10}}{\sigma^6} + 40 \lambda_{6}^2 \phi^{12} \sigma^8 \right]. \] (24)

\( U_{1} \) is manifestly scale invariant. Firstly, the Coleman-Weinberg (CW) term is modified into a scale invariant form \( V^{(1)} \) where we finally replaced \( \mu(\sigma) = z \sigma \) (see (14) for \( \epsilon \to 0 \)). Note that \( V^{(1)} \) contains new terms of the form \( \phi^8/\sigma^4 \ln[...], \phi^6/\sigma^2 \ln[...] \) of coefficients \( \propto \lambda_{6} \), that originate from \( m_{2j}^4(\phi, \sigma) \). In the “usual” DR scheme \( V^{(1)} \) has the same form, with \( (z \sigma) \to \mu \).

There is also a finite one-loop contribution \( V^{(1,n)} \) due to “evanescent” corrections \( (\propto \epsilon) \) to the field-dependent masses of \( \phi \) and \( \sigma \) (eq. (16)), induced by derivatives of \( \mu \sim \sigma \). Therefore, \( V^{(1,n)} \) is not present in the other case of \( \mu=\text{constant} \) when the regularization breaks the scale symmetry; thus \( V^{(1,n)} \) can distinguish between these two cases at one-loop.\textsuperscript{22}

Further, in the

\textsuperscript{22}These two cases are different quantum theories (have different symmetry).
classical decoupling limit of the hidden sector from the SM, $\lambda_m \to 0$ and $\lambda_6 \to 0$, then $V^{(1,n)}$ vanishes. $V^{(1,n)}$ also contains terms non-polynomial in fields like $\lambda_m \lambda \phi^6 / \sigma^2$ that remains present even if we set $\lambda_6 = 0$. At two-loop such non-polynomial operators, including higher order $\phi^8 / \sigma^4$, etc, emerge as two-loop counterterms even if we set $\lambda_6 = 0$.

Although we do not show it, one can immediately Taylor expand both $V^{(1)}$ and $V^{(1,n)}$ about the non-zero vev of $\sigma$, with $\sigma = \langle \sigma \rangle + \sigma'$. One then obtains a representation that contains an infinite sum of polynomial operators in the field fluctuations $(\phi', \sigma')$ suppressed by powers of $\langle \sigma \rangle$. However, in this case manifest scale symmetry of the quantum result is lost.

### 2.3 One-loop beta functions and Callan-Symanzik equation

To check the quantum consistency of the scalar potential, we verify the Callan-Symanzik equation for it. This is to ensure that physics is independent of the subtraction scale $\mu(\langle \sigma \rangle) = z(\langle \sigma \rangle)$. To this purpose we need the one-loop beta functions of all couplings, including those of the non-polynomial operators. These are computed from the condition that the bare coupling is independent of the subtraction parameter $z$.

For example $d/(d \ln z) \lambda^B = 0$, where $\lambda^B = \mu(\langle \sigma \rangle) Z_\phi Z_\sigma \lambda^B$ and $\phi^2_2 = Z_\phi \phi^2$. Using these relations, the beta function that is $\beta_{\lambda^B} = d\lambda^B/d(\ln z)$ becomes

$$
\beta_{\lambda^B} = \frac{2 \lambda^B}{\kappa} \left( \gamma_{\lambda^B} - 2 \gamma_\phi \right),
$$

with summation over $j$ with $\alpha_j = g_1^2, g_2^2, h_1^2, \lambda_\phi, \lambda_m, \lambda_\sigma, \lambda_6, \lambda_8$, etc. Next, using notation, one has

$$
\gamma_\phi = \frac{1}{\kappa} \left( \frac{3}{4} g_1^2 + \frac{9}{4} g_2^2 - 3 h_1^2 \right), \quad \gamma_\sigma = 0
$$

which can easily be computed in a scale invariant way. Relations similar to eq.(25) exist for the other beta functions. We then find

$$
\beta_{\lambda^B} = \frac{2 \lambda^B}{\kappa} \left( \gamma_{\lambda^B} - 2 \gamma_\phi \right),
\beta_{\lambda_m} = \frac{2 \lambda_m}{\kappa} \left( \gamma_{\lambda_m} - \gamma_\phi \right),
\beta_{\lambda_\sigma} = \frac{2 \lambda_\sigma}{\kappa} \gamma_{\lambda_\sigma}.
$$

---

23 Assuming one set $\lambda_6 = 0$ at tree level, some other subtraction scheme could eventually remove finite $\phi^4$, $\phi^2 \sigma^2$ or $\sigma^4$ terms in $V^{(1,n)}$, but could not remove the remaining $\lambda_m \lambda \phi^6 / \sigma^2$ that does not vanish for $\lambda_6 = 0$.

24 The two-loop beta functions of such terms are non-zero even if $\lambda_6 = 0$, so setting these to zero (at some scale) will not remove them since they are again generated at a different scale.

25 The dimensionless parameter $z$ tracks the dependence on the subtraction scale $\mu(\langle \sigma \rangle) = z(\langle \sigma \rangle)$.

26 $\gamma_\phi$ and $\gamma_\sigma$ have the same expression as when $\mu =$constant scale.
\( \beta_{\lambda_6} \) includes a correction due to \( \lambda_6 \), which is the coupling of the non-polynomial term that we included in the classical potential eq. (11). These one-loop beta functions are identical to those of the similar theory with a regularization that breaks scale symmetry explicitly \( (\mu = \text{constant}) \). We find in a similar way

\[
\beta_{\lambda_6} = \frac{2\lambda_6}{\kappa} (\gamma_{\lambda_6} - 3\gamma_\phi),
\]
\[
\beta_{\lambda_8} = \frac{2\lambda_8}{\kappa} (\gamma_{\lambda_8} - 4\gamma_\phi),
\]
\[
\beta_{\lambda_{10}} = \frac{2\lambda_{10}}{\kappa} (\gamma_{\lambda_{10}} - 5\gamma_\phi),
\]
\[
\beta_{\lambda_{12}} = \frac{2\lambda_{12}}{\kappa} (\gamma_{\lambda_{12}} - 6\gamma_\phi).
\] (28)

These beta functions of the non-polynomial operators are difficult to obtain by other methods (diagrammatic, etc). This justifies keeping these operators in a scale symmetric form (eq. (27)), rather than expanding them about the ground state in series of polynomial operators (eq. (28)).

The Callan-Symanzik equation of the scalar potential states the independence of the potential of the subtraction scale. At one-loop this gives

\[
\frac{d}{d \ln z} U_1(\phi, \sigma, \alpha_k) = \left( 2 \frac{\partial}{\partial z} + \beta_{\alpha_k} \frac{\partial}{\partial \alpha_k} + \gamma_\phi \frac{\partial}{\partial \phi} \right) U_1(\phi, \sigma, \alpha_k) = \mathcal{O}(\alpha_j^3). \quad (29)
\]

Here \( \alpha_k \) denote the couplings \( \lambda_\phi, \lambda_\sigma, \lambda_m, g_1^2, g_2^2, h_1^2, \lambda_6, \lambda_8, \lambda_{10}, \lambda_{12} \) which were found to have nonzero beta functions. Further \( \gamma_\phi = \frac{\partial \ln \phi}{\partial \ln z} = -\frac{1}{2} \frac{\partial \ln Z_\phi}{\partial \ln z} \) was found in eqs. (26), (19), while \( \gamma_\sigma = 0 \). Finally \( U_1(\phi, \sigma, \alpha_k) \) denotes the potential found in eq. (22) with the observation that all couplings are now replaced by their “running” versions. In particular the tree level potential (part of \( U_1 \)) is supplemented with the following terms with running couplings \( \lambda_{8,10,12} \)

\[
V \to V + \frac{\lambda_8}{8} \phi^8 + \frac{\lambda_{10}}{10} \phi^{10} + \frac{\lambda_{12}}{12} \phi^{12} \quad (30)
\]

These terms are present since the couplings \( \lambda_{8,10,12} \) (which had boundary values set to 0 at the EW scale, unlike \( \lambda_6 \neq 0 \)), have non-zero beta functions.

The only explicit \( z \)-dependent part in \( U_1 \) comes through the Coleman-Weinberg part \( V^{(1)} \) of eq. (22), while the terms involving the beta functions and anomalous dimension act only on the tree level part of the potential, in our one-loop approximation.

\footnote{However, at two-loop order the beta functions start to differ \cite{10} in our case of spontaneous scale symmetry breaking from the case of explicit breaking (by the regularization with \( \mu = \text{constant} \)). In this case the evanescent corrections \( \propto \epsilon \) to scalar field-dependent masses (higgs, dilaton) of the potential “meet” the \( 1/\epsilon^2 \) usual two-loop poles, to bring new poles \( \epsilon \times 1/\epsilon^2 \sim 1/\epsilon \) that demand new counterterms, thus modifying the beta functions, see \cite{10} for details.}
With the above results, checking the Callan Symanzik equation is immediate. We stress that this is verified in the presence of the non-polynomial operators that actually correspond to infinitely many polynomial operators when expanded about the ground state.

In conclusion, the potential is indeed independent of the subtraction scale \( z \langle \sigma \rangle \), so one can take any value for it. It is customary to set the subtraction scale equal to \( \langle \phi \rangle \), to minimise the log terms in the potential. In our scale invariant approach \( \mu(\sigma) = z \sigma \), so after scale symmetry breaking \( \mu(\langle \sigma \rangle) = \langle \phi \rangle \) if we take \( z = \langle \phi \rangle / \langle \sigma \rangle \), and we do so below. This means the couplings and fields are evaluated at the scale \( \langle \phi \rangle \).

### 2.4 The one-loop higgs mass

The one-loop corrected potential is scale invariant and it has a flat direction\(^{28}\), the dilaton, which remains massless at the quantum level\(^{29}\). We can compute the higgs mass \( m_{\tilde{\phi}} \) at one-loop by using

\[
m_{\tilde{\phi}}^2 = (U_1)_{\sigma\sigma} + (U_1)_{\phi\phi} \Big|_{\text{min}}
\]

where the subscripts denote derivatives with respect to the fields shown. We calculate the new ground state and the correction \( \delta m_{\tilde{\phi}}^2 \) to classical \( m_{\tilde{\phi}}^2 \) in the limit of an ultraweak coupling of the visible to the hidden sector \( |\lambda_m| \ll \lambda_\phi \) (giving \( \lambda_\sigma \ll |\lambda_m| \)). This was motivated earlier in that it ensures a classical hierarchy \( \langle \phi \rangle \ll \langle \sigma \rangle \). The new ground state is modified to

\[
\langle \phi \rangle^2 \langle \sigma \rangle^2 = -\frac{3}{16} \lambda_\phi \left[ 1 + \zeta \right], \quad \zeta = -\frac{\lambda_\phi}{4\kappa} \left[ 4 \ln \left( \lambda_\phi / 2 \right) - 8 \right]
\]

With the notation \( g^2 = g_1^2 + g_2^2 \), the one-loop correction is

\[
\delta m_{\tilde{\phi}}^2 = \frac{-\lambda_m}{\lambda_\phi} \langle \sigma \rangle^2 \left\{ 27 g^4 \left( \ln \frac{g^2}{4} + \frac{1}{3} \right) + 2 g_2^4 \left( \ln \frac{g_2^2}{4} + \frac{1}{3} \right) - 16 h_1^4 \left( \ln \frac{h_1^2}{2} - \frac{1}{3} \right) \right\}
\]

\[
+ 4 \lambda_\phi^2 \left[ 5 \ln \frac{\lambda_\phi^2}{12} - 8 + \ln 27 \right] \quad (33)
\]

This quantum correction remains proportional to \( \lambda_m \langle \sigma \rangle^2 \sim \mathcal{O}(100\text{GeV}) \), just like the tree-level value. Thus the initial classical hierarchy \( m_{\tilde{\phi}} \sim \langle \phi \rangle \ll \langle \sigma \rangle \) is stable in the presence of quantum corrections, without any quantum tuning of the couplings \( \lambda_\phi, m, \sigma \), in agreement with previous results \(^{30}\). An additional correction from \( \lambda_6 \neq 0 \) does not change this result since it is sub-leading in the limit of ultraweak coupling considered here (being suppressed

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\(^{28}\)See previous discussion in the Introduction around eqs.\(^{22}\).

\(^{29}\)This is not the case in the “traditional” DR scheme where scale symmetry is broken explicitly by the quantum calculation with \( \mu = \text{constant} \) and the dilaton becomes a pseudo-Goldstone mode.

\(^{30}\)In particular there is no term \( \lambda_6 \langle \sigma \rangle^2 \) that would require tuning the higgs self-coupling \( \lambda_\phi \), etc.

\(^{31}\)For the physical higgs mass there is also the usual correction of running from \( p^2 = 0 \) to \( p^2 = m_h^2 \),
by the large $\langle \sigma \rangle$. Finally, the spontaneous breaking of scale symmetry used here avoids the constraint of \cite{39} (derived using explicit breaking by the “usual” DR scheme) that demands new physics at the TeV scale.

### 2.5 What about the dilatation anomaly?

Let us analyze the situation of the dilatation current $D^\mu$ and its divergence \cite{32,7}. For a set of fields $\phi_j$ ($\phi$, $\sigma$, etc)

$$D^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_j)} (x^\nu \partial_\nu \phi_j + d_\phi) - x^\mu \mathcal{L},$$

$$\partial_\mu D^\mu = (d_\phi + 1) (\partial_\mu \phi_j) \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_j)} + d_\phi \phi_j \frac{\partial \mathcal{L}}{\partial \phi_j} - d\mathcal{L}, \quad (34)$$

with $d_\phi$ the mass dimension of $\phi$, $d_\phi = (d - 2)/2$ for a scalar in $d$ dimensions. For standard kinetic terms and using the equations of motion, we find for a potential $V$ in $d$ dimensions

$$\partial_\mu D^\mu = d\mathcal{V} - \frac{d - 2}{2} \phi_j \frac{\partial \mathcal{V}}{\partial \phi_j}. \quad (35)$$

Consider now that $\mathcal{V}$ is scale invariant at both classical and quantum level as in our case \cite{14} (also eq.(22)). Therefore, for a dimensionless parameter $\rho$, $\mathcal{V}$ has the property $\mathcal{V}(\rho \phi_j) = \rho^{2d/(d-2)} \mathcal{V}(\phi_j)$ in $d = 4 - 2\epsilon$ dimensions (homogeneous function). Differentiating this equation with respect to $\rho$ and then taking $\rho \to 1$ gives $2d/(d-2) \mathcal{V} = \phi_j \partial \mathcal{V}/\partial \phi_j$ so the rhs of eq.(34) then vanishes. Therefore $\partial_\mu D^\mu = 0$ at both classical and quantum level, so there is no anomalous breaking of the quantum scale symmetry. Nevertheless the couplings still “run” and have non-zero beta functions (eq.(27)) with their corresponding poles in $\rho$.

To understand this better, let us also see what happens if $\mathcal{V}$ is not scale invariant in $d = 4 - 2\epsilon$. This happens when $\mathcal{V} = \mu^{2\epsilon} \mathcal{V}(\phi_j)$ which is the case of the “traditional” DR scheme with explicit scale symmetry breaking, with $\mu$ a fixed scale (not a function of the fields) and $\mathcal{V}$ the potential, scale invariant in $d = 4$ (assuming no mass terms). Then $\mathcal{V}(\rho \phi_j) = \rho^{4} \mathcal{V}(\phi_j)$, but $\mathcal{V}$ is no longer scale invariant in $d = 4 - 2\epsilon$. From eq.(35)

$$\partial_\mu D^\mu = d \mu^{2\epsilon} \mathcal{V} - 2(d - 2) \mu^{2\epsilon} \mathcal{V}(\phi_j) = 2\epsilon \mu^{2\epsilon} \mathcal{V} = 2\epsilon \mu^{2\epsilon} \lambda_j \frac{\partial \mathcal{V}}{\partial \lambda_j}. \quad (36)$$

While at the classical level the rhs vanishes when $\epsilon \to 0$, at the quantum level the quartic couplings $\lambda_j$ in $\mathcal{V}$ acquire a pole $\beta_{\lambda_j}/\epsilon$ which cancels the $\epsilon$ in front, to give a finite non-zero rhs $\partial_\mu D^\mu \propto \beta_{\lambda_j}(\partial \mathcal{L}/\partial \lambda_j)$. This is the familiar scale anomaly breaking of the conservation of

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\footnote{We have $\mathcal{V} = \mu(\sigma)^{2\epsilon} \mathcal{V}$ in eqs.(14), while in eq.(22) $\mathcal{V} = \mu(\sigma)^{2\epsilon} U_1$ before $\epsilon \to 0.$}

\footnote{This property is shown using that: $\mathcal{V}(\phi_j) = \phi_j^k \mathcal{V}(\phi_j/\phi_k)$, $k = \text{fixed}$; since $[\mathcal{V}(\phi_j)] = d$, $\mathcal{V}(\phi_j/\phi_k) = 0$ and $[\phi_j] = (d - 2)/2$ then $\xi = 2d/(d - 2)$. Then $\mathcal{V}(\rho \phi_j) = (\rho \phi_k)^{\xi} \mathcal{V}(\phi_j/\phi_k) = \rho^{2\epsilon} \mathcal{V}(\phi_j)$ with $\xi$ as above.}
In conclusion, it is scale invariance of the action in $d=4-2\epsilon$ that ensures that no scale anomaly is present. This invariance in $d=4-2\epsilon$ is lost in the “usual” DR regularization with explicit breaking ($\mu=$constant). Thus, the vanishing of the beta function is not a necessary condition for the theory to be scale invariant; one must also specify how the quantum theory was regularized, with or without respecting its scale symmetry. In other words the non-vanishing of the beta function does not mean the theory cannot be scale invariant.

2.6 Further remarks

As mentioned, the vacuum energy vanishes in models with scale symmetry or with spontaneous breaking of it, see discussion after eq.(2). This protection remains in place at the quantum level provided this symmetry is respected by the quantum calculation itself. The initial classical tuning of the boundary values of the couplings, relation (a) in eq.(9):

$$\lambda_\sigma = 9\lambda_\phi^2/\lambda_\phi^2(1 + O(\lambda_\phi^2)),$$

assumed to be true in the paper (for spontaneous breaking to exist), receives loop corrections of order $O(\lambda_j)$. As a result the tuning of the couplings, that enforces $V_{\text{min}} = 0$ at the loop level (demanded by scale symmetry), is $O(\lambda_j)$ relative to its tree level case. More generally, in order $n$ the tuning is $O(\lambda_j)$ relative to that in order $n-1$, i.e. at the level of the precision of the perturbation theory calculation in that order.

The consistency of the boundary values for the running couplings with some high scale physics that must fix the value of $\langle \sigma \rangle$ should be investigated. This discussion requires one extend this quantum calculation to the case of curved space-time while respecting this symmetry. The appropriate setup is in the context of Brans-Dicke-Jordan theory of gravity. As discussed in [30], in such frame with non-minimal couplings, the dilaton (with derivative couplings) decouples and avoids “fifth force experiments”. For investigations along this direction see [28, 29, 31, 32, 33, 34, 35, 36, 37].

3 Conclusion

We explored the possibility that scale symmetry is a quantum symmetry of the SM that is broken (only) spontaneously. Following previous developments on this idea, we considered the case of the classically scale invariant version of the SM which has vanishing tree-level mass for the higgs ($\phi$) and is extended by the dilaton $\sigma$ (the Goldstone mode of scale symmetry). The vev $\langle \sigma \rangle \neq 0$ breaks the scale symmetry spontaneously and generates dynamically a subtraction scale $\mu \sim \langle \sigma \rangle$ that is necessary for quantum calculations.

The classical scalar potential is dictated by the scale symmetry only and may contain non-polynomial effective operators such as $\lambda_6\phi^6/\sigma^2$, $\lambda_8\phi^8/\sigma^4$, $\lambda_{10}\phi^{10}/\sigma^6$, $\lambda_{12}\phi^{12}/\sigma^8$, etc; even if at classical level it was conserved.

$^3$If $V$ contains mass terms, $\partial_\mu D^\mu$ also contains a “classical” breaking of scale symmetry term, $m^2\phi^2$.

$^3$Another possibility is to consider Einstein gravity which breaks the scale symmetry discussed here. Then scale symmetry is only an approximate symmetry and the dilaton is a pseudo-Goldstone mode that acquires a small mass and the vacuum energy is then non-zero.

$^3$
these may always be Taylor-expanded into a sum of infinitely-many polynomial operators in
fields fluctuations suppressed by powers of \( \langle \sigma \rangle \) (which can be regarded as a physical cutoff
of the theory); however, in such case the manifest scale symmetry of the theory is lost.

The one-loop computation of the potential respected the scale symmetry of the classical
Lagrangian. As a result, a scale invariant one-loop potential for the higgs and dilaton is
obtained. The quantum potential has corrections from gauge and Yukawa interactions and
also from the higher dimensional, non-polynomial operators. The latter were included in
the classical Lagrangian and their couplings \( (\lambda_6, \lambda_8, \lambda_{10}, \lambda_{12}, \text{etc}) \) are one-loop renormalized
with beta functions that we computed from the quantum potential. These beta functions are
difficult to compute by other means and are an important result of this work. Tuning these
couplings to zero at the tree-level will not avoid the presence of their corresponding operators
at the quantum level; these operators re-emerge at the quantum level with a finite one-loop
coefficient and as two-loop (scale-invariant) counterterms, due to the non-renormalizability of
theories with quantum scale invariance. The role of these (scale invariant) effective operators
which capture the effects of an infinite series of polynomial operators deserves further study.

The quantum consistency of the calculation was verified by showing that the Callan-
Symanzik equation of the quantum potential is respected in the presence of the non-polynomial
operators. We also showed the differences between the scale-invariant one-loop potential and
its counterpart computed in the “usual” DR scheme \( (\mu =\text{constant}) \) that breaks scale sym-
metry explicitly, in the presence at the tree level of non-polynomial operators.

In quantum scale invariant models all mass scales are generated by vacuum expectation
values of the fields, after spontaneous scale symmetry breaking; therefore, any mass hierarchy
is not primary or fundamental, but can be generated by a hierarchy of the (dimensionless)
couplings of the theory. The vacuum energy is vanishing at the loop level in the case of
spontaneously broken quantum scale symmetry provided one coupling is a function of the
rest; this ensures the flat direction exists. This can be arranged by one initial classical tuning,
with subsequent, quantum tunings of \( \mathcal{O}(\lambda_j) \) relative to previous order. This picture is in
contrast to the case when the regularization breaks explicitly the classical scale symmetry
of the action, leading to a different quantum theory (where the minimum of the potential is
non-zero).

It is possible to arrange a hierarchy \( m_\phi^2 \sim \langle \phi \rangle^2 \ll \langle \sigma \rangle^2 \) by choosing at the classical level
an ultraweak coupling \( \lambda_m \) between the SM and the hidden sector of the dilaton \( (|\lambda_m| \ll \lambda_\phi) \)
or by more elegant means (dynamics, etc). This hierarchy is stable at the one-loop level,
without additional tuning of the couplings and despite the presence of the non-renormalizable
operators mentioned. This UV behaviour should survive to higher orders due to the spont-
aneous (i.e. soft) scale symmetry breaking.
Appendix

For convenience, we present the expressions of the beta functions found in the text

$$\beta_{\lambda_6} = \frac{1}{\kappa} \left[ \lambda_6 \left( \frac{3}{4}g_1^2 + \frac{3}{4}g_1^4 + \frac{3}{2}g_1^2g_2^2 - 12h_t^4 \right) \right]$$

$$\beta_{\lambda_m} = \frac{2\lambda_m}{\kappa} \left[ \lambda_m + 2\lambda_m + \frac{1}{2}\lambda_\sigma - \left( \frac{3}{4}g_1^2 + \frac{9}{4}g_2^2 - 3h_t^2 \right) \right]$$

$$\beta_{\lambda_{10}} = \frac{10}{\kappa} \left[ 4\lambda_6 - \frac{10}{\kappa} \left( \frac{3}{4}g_1^2 + \frac{9}{4}g_2^2 - 3h_t^2 \right) \right]$$

$$\beta_{\lambda_{12}} = \frac{2}{\kappa} \left[ 3\lambda_6 - 6\lambda_{12} \left( \frac{3}{4}g_1^2 + \frac{9}{4}g_2^2 - 3h_t^2 \right) \right].$$

(A-1)

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