Coherent quantum oscillations in coupled traps with ultracold atoms

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The dynamics of two traps with ultracold atoms and connected by Josephson type coupling, is shown to exhibit a transition from dispersive dynamics to localized coherent oscillations. This transition is controlled by coupling strength and energy offset between the traps. The dynamics is also shown to be exactly that of a Heisenberg chain with a linear magnetic field. Possible applications for “quantum-state engineering” are discussed.

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Trapping and cooling of atomic gases has produced an interesting laboratory for analyzing the transition from classical to quantum behavior. The fascinating new phenomena include the creation of Bose-Einstein condensates in magneto-optical traps, and the realization of coupled quantum systems with BEC’s in ultracold, e.g., optical lattices. One can now analyze in some detail the properties of Josephson currents and Bloch oscillations that appear in the coupled systems, and entanglement of quantum states in individual traps. Possible use of trapped ultracold atoms in realizing a quantum computer has also been considered (for “quantum-state engineering” with conventional Josephson junctions see, e.g., the recent review [8]).

It is evident that quantum dynamics of these systems has become an important topic. This is particularly so if one tries to manipulate the dynamics as is needed, e.g., in the case of quantum computer. Also, rapid decoherence of oscillatory dynamics even without external perturbations would make controlled operations very difficult to perform. In coupled atom-field systems, such as, e.g., the one of two-level atoms interacting with a high-Q cavity, escaping photons form the main loss mechanism for quantum coherence independent of the internal dynamics of the system. It may thus be useful to consider systems in which controllable quantum dynamics can be realized without active involvement of a quantized electromagnetic field.

We consider in this Letter a coupled system of two traps with ultracold atoms, possibly in a condensed state so that they can each be described with one boson field \( b_{1,2}, b_{1,2}^\dagger \), with subindices denoting the trap. Experiments are now approaching the limit in which a fully quantum mechanical treatment of the problem is necessary, and it is actually this limit which is interesting from the standpoint explained above. First attempts have already been made to analyze this system quantum mechanically, and we shall concentrate here on the quantum dynamics of the system, which so far has remained mostly unexplored. The two traps are assumed to be connected by a Josephson type of coupling, and atoms are assumed to interact via delta function potentials so that their interactions are of the form \( cb_j^\dagger b_j^\dagger b_j b_j; \ j = 1,2 \). Here \( c \) is the interaction strength which for analytical simplicity is taken to be of opposite sign for the two traps (this is now experimentally possible, and results are expected to be similar for general \( c \) albeit much more difficult to achieve analytically). We have previously shown that for large Josephson coupling a natural quantum Hamiltonian for systems of this kind can be expressed in terms of the number operators \( \hat{N}_j = b_j^\dagger b_j \) and the exponential phase operators \( \phi_j = (\hat{N}_j + 1)^{-1/2} \), \( \phi_j^\dagger = (\hat{N}_j + 1)^{-1/2} \). Notice that these phase operators are one-sided unitary: \( \phi_j \phi_j^\dagger = 1 \), \( \phi_j^\dagger \phi_j = 1 - |0\rangle \langle 0| \), with \( |0\rangle \) appropriate vacua. In this case of two coupled traps the Hamiltonian thus becomes

\[
\hat{H}_{tr} = c\hat{N}_1^2 - c\hat{N}_2^2 + (\omega + \Delta)\hat{N}_1 + \omega \hat{N}_2 - \frac{g}{2} \left( \phi_1 \phi_2^\dagger + \phi_1^\dagger \phi_2 \right),
\]

in which \( \omega \) and \( \omega + \Delta \) are the single particle energies (frequencies) in the two traps. We include an offset \( \Delta \) in the energies (“detuning”) because this is an important parameter in the problem as will become evident below. \( g \) is the strength of Josephson coupling. The total number of particles is conserved, \( \{\hat{H}_{tr}, \hat{N}_1 + \hat{N}_2\} = 0 \), so we can consider without loss of generality the Hamiltonian \( \hat{H}_{tr} = -\omega(\hat{N}_1 + \hat{N}_2) + c(\hat{N}_1 + \hat{N}_2)^2 \),

\[
\hat{H} \equiv \Delta \hat{N}_1 - \frac{g}{2} \left( \phi_1 \phi_2^\dagger + \phi_1^\dagger \phi_2 \right),
\]

where \( \Delta = \Delta + 2c(\hat{N}_1 + \hat{N}_2) \). The ordinary (orthonormal) Fock states of the system can be formed from the vacuum states by operating with the phase operators,

\[
|N_1, N_2\rangle = (\phi_1^\dagger)^{N_1}(\phi_2^\dagger)^{N_2}|0\rangle_1|0\rangle_2
\]

such that \( (\hat{N}_1 + \hat{N}_2)|N_1, N_2\rangle = (N_1 + N_2)|N_1, N_2\rangle \equiv N|N_1, N_2\rangle \). In problems like entanglement of quantum states or quantum computing one is interested in the dynamics of a state which is not an eigenstate of the Hamiltonian. The Fock states are not eigenstates of Hamiltonian Eq. (2), and in view of possible applications of the system, we consider next the dynamics of Fock states. We find it instructive to consider the dynamics in terms of the correlator

\[
\Phi_{ii}(t) = \langle j, N - j | e^{-i\hat{H}t} | l, N - l \rangle,
\]
which describes the time evolution of the occupation of particles in the two traps, and has an obvious connection to probability of observing a given state.

This correlator can be solved exactly for the Hamiltonian \( H \). The most convenient way to find the solution is to first construct an “equation of motion” for the correlator, which for fixed subindex \( j \) takes the form

\[
-i \frac{2}{g} \frac{d}{dt} \Phi_{jl}^N(t) = \Phi_{jl+1}^N(t) + \Phi_{jl-1}^N(t) - \frac{2}{g} \Delta \Phi_{jl}^N(t),
\]

and a similar equation is found for fixed \( l \). The initial condition for the correlator is \( \Phi_{jl}^N(0) = \delta_{jl} \), and the boundary conditions are \( \Phi_{jl}^N(t) = 0 \) for \( j, l = -1, N+1 \). Notice that this differential equation is a generalization of the equation of motion for a lattice particle in a linear (electric) field \([16]\).

Consider first the case of vanishing “field” or “effective detuning”, i.e. \( \Delta = 0 \). In this case Eq. (5) can easily be solved and we find

\[
\Phi_{jl}^N(t) = \sum_{k=1}^{N+1} 2 e^{-itE_k} \frac{\sin \left[ \frac{\pi k(j+1)}{N+2} \right]}{N+2} \frac{\sin \left[ \frac{\pi k(l+1)}{N+2} \right]}{N+2},
\]

in which the (Bloch) spectrum \( E_k \) is given by \( E_k = -g \cos(\pi k/(N+2)) \). The time evolution of any initial state can now be followed. We show in Fig. 1 the probability \( |\Phi_{jl}^N(t)|^2 \) for the initial state with equal number of particles in both traps, \( j = l = N/2 \).

![Fig. 1](image_url)

**Fig. 1.** Probability \( |\Phi_{jl}^{(j)}(t)|^2 \) for \( j = 10 \) as a surface diagram in the \( tl \) plane for \( \Delta = 0 \), and \( \Phi_{jl}^N(0) = \delta_{10.10} \). Height is indicated with a grey scale such that black means zero height.

It is evident from this figure that there is no coherent oscillation in the occupation numbers, the probability is rapidly dispersed into the available Fock space. The behavior is analogous to the one found in \([17]\) for a “spin wave” in an optical lattice, but there the dynamics was not followed very far (see also discussion below for connection to the Heisenberg chain). Notice that the time evolution is not translationally invariant at intermediate time scales, and thus exhibits \([18]\) “aging”. There is however an almost complete recovery of the initial probability distribution at a late time (which is repeated periodically). The small wiggles in the probability that are clearly visible at very early times are the “Rabi oscillations” of \([13]\).

For non-zero effective detuning \( \Delta \neq 0 \), Eq. (5) can also be solved exactly. For the same initial condition as above,

\[
\Phi_{jl}^N(t) = \sum_{k=1}^{N+1} e^{-itE_k} B(\nu_k, \lambda; j+1) B(\nu_k, \lambda; l+1) \frac{\sum_{n=1}^{N+1} B^2(\nu_k, \lambda; n)}{\sum_{n=1}^{N+1} B^2(\nu_k, \lambda; n)},
\]

in which \( \lambda = g/\Delta \), and \( B(\nu, \lambda; k) \) are Lommel polynomials. In terms of the Bessel functions of the first and second kind \( B(\nu, \lambda; k) = \frac{1}{2\pi} [J_\nu(\lambda)Y_{\nu+k}(\lambda) - J_{\nu+k}(\lambda)Y_\nu(\lambda)] \). The boundary conditions given below Eq. (6) mean that \( B(\nu, \lambda; N+2) = 0 \), and the \( N+1 \) roots of this equation are the \( \nu_k \). The spectrum \( E_k \) is now \( E_k = -\Delta(\nu_k + 1) \). For small \( N/\lambda \) this spectrum is similar to the Bloch spectrum found above for zero effective detuning. There is a crossover around \( N \approx \lambda \) into a linear spectrum: \( E_k \approx \Delta k \) for \( N/\lambda \gg 1 \). This latter kind of spectrum is also called the Wannier-Stark ladder.

For \( N/\lambda \ll 1 \) we thus find the probability \( |\Phi_{jl}^N(t)|^2 \) behaves very similarly to that for zero effective detuning shown in Fig. 1. Around \( \lambda \approx N \) there is a transition to a completely different behavior. An excitation composed of a superposition of states that are localized in the Fock space is formed in the system such that coherent oscillation appears in the occupation probabilities of the two traps.
of Bose condensates). If we express this Hamiltonian Eq. (8) in the Fock space, we find that

$$\hat{H}_\infty = \Delta \sum n\langle n| - \frac{g}{2} \sum \langle n| (n+1) + |n+1\rangle\langle n|.$$  \hspace{1cm} (9)

This form of the Hamiltonian appears in many problems related to trapped gases of ultracold atoms, e.g., in the tight binding limit of a ring-shaped trap with a moving defect [14].

The correlator $\Phi_{jl}^N(t)$ can again be solved exactly with the result

$$\Phi_{jl}^N(t) = \sum_{k=1}^{\infty} e^{-itE_k} \frac{J_{j+1+\nu_k}(\lambda)J_{l+1+\nu_k}(\lambda)}{\sum_{n=1}^{\infty} J_n^{2+\nu_k}(\lambda)},$$  \hspace{1cm} (10)

where summation is over the $\nu_k$ determined by the solution of the “index equation” $J_{\nu_k}(\lambda) = 0$. The spectrum is now $E_k + \Delta = -\Delta|\nu_k|$, which is very similar to the Wannier-Stark spectrum for small $k$. The solution is also in this case a coherent oscillation in the occupation number of the finite trap similar to the solution shown in Fig. 3. The same “quantum features” of the solution can thus be realized with only one trap with a small number of weakly interacting particles.

For zero effective detuning coherent oscillations disappear as expected, and we find that asymptotically $|\Phi_{jl}^N(t)|$ decays as $t^{-3/2}$. The dynamical exponent $z = 3/2$ also appears in nonequilibrium growth processes with one-dimensional interfaces [15].

We shall finally make a connection to quantum computing even though a detailed analysis of this aspect of the problem cannot be given here. We show here that the correlators $\Phi_{jl}^N(t)$ considered above can be mapped exactly to the corresponding correlators of an isotropic Heisenberg spin 1/2 chain in a linear field, for which all logical operations of quantum computing have already been [21] constructed.

Consider the one spin-flip correlator (related to the “NOT” operation)

$$F_{jl}^{N+1} \equiv \langle 0 | \sigma_j^+ e^{-it\hat{H}_{xxx}} \sigma_l^- | 0 \rangle,$$  \hspace{1cm} (11)

where the ground state $|0\rangle$ is that with all spins pointing up, and the Heisenberg Hamiltonian $H_{xxx}$ is given by

$$\hat{H}_{xxx} = -\frac{g}{2} \sum_{n=1}^N \sigma_n^+ \sigma_{n+1}^- + \sigma_{n+1}^- \sigma_n^+ + \frac{1}{2} (\sigma_n^z \sigma_{n+1}^z - 1) + \frac{1}{2} \sigma_n^z.$$  \hspace{1cm} (12)

Consider the corresponding limit in the Bogolubov theory [19] of Bose condensates. If we express this Hamiltonian Eq. (8) in the Fock space, we find that

$$\hat{H}_\infty = \Delta \sum n\langle n| - \frac{g}{2} \sum \langle n| (n+1) + |n+1\rangle\langle n|.$$  \hspace{1cm} (9)

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Here $\sigma_n^\pm, \sigma_n^z$ are Pauli spin matrices, and $h_n$ is an external magnetic field. The constant factors in the Hamiltonian [13] are added to ensure that $H_{xxx}|0\rangle = 0$. As before we can derive an equation of motion for correlator [13], and find for fixed index $j$ that

FIG. 2. Probability $|\Phi_{jl}^N(t)|^2$ for $N/\lambda = 4$ in Fig. 2, where these features of the solution are clearly visible. For increasing $N/\lambda$ the number of localized states in the superposition decreases, and the probability becomes more and more localized around the center line (i.e. the initial value). This is evident from Fig. 3, which shows $|\Phi_{jl}^N(t)|^2$ for $N/\lambda = 10$.

FIG. 3. Probability $|\Phi_{jl}^N(t)|^2$ for $j = 10$ as a surface diagram in the $tl$ plane for $\lambda = 2$, and $\Phi_{jl}^N(0) = \delta_{10,10}$. Height is indicated with a grey scale such that black means zero height.

We could call the localized oscillatory solution as a “trapped quantum soliton”. We believe that this solution should be experimentally observable in coupled traps with ultracold atoms, in which the parameter $\lambda$ can fairly easily be controlled by controlling either $\Delta$ or $g$ or both. Notice also that the states that form the superposition are entangled. This system would thus be suitable for analyzing entanglement, and perhaps even quantum computing as “engineering” of states is possible.

It is also instructive to consider the limit in which one of the coupled traps becomes macroscopic (classical): $N \to \infty$ (i.e. either $N_1$ or $N_2$). For simplicity we consider this case only in the limit of weakly interacting particles and set $c = 0$. The correlator $\Phi_{jl}^\infty(t) = \langle j| \exp(-it\hat{H}_\infty)|l\rangle$ satisfies the same equation Eq. (3) as for finite $N$ but now the boundary conditions are $\Phi_{jl}^\infty(t) = 0$ for $j, l = -1$. The Hamiltonian becomes in this limit

$$\hat{H}_\infty = \Delta \hat{\tilde{N}} - \frac{g}{2} (\phi + \phi^\dagger).$$  \hspace{1cm} (8)

Notice that the (new) number operator $\hat{\tilde{N}}$ does not any more commute with the Hamiltonian (this is similar to the corresponding limit in the Bogolubov theory [19].

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Notice that the (new) number operator $\hat{\tilde{N}}$ does not any more commute with the Hamiltonian (this is similar to the corresponding limit in the Bogolubov theory [19].
\[- \frac{2}{g} \frac{d}{dt} F_{jl}^{N+1}(t) = F_{jl+1}^{N+1}(t) + F_{jl-1}^{N+1}(t) \]
\[- 2 \left( 1 + \frac{h_l}{g} \right) F_{jl}^{N+1}(t) \] (13)

with a similar equation for fixed \( l \). The initial condition to be applied here is \( F_{jl}^{N+1}(0) = \delta_{jl} \), and the boundary conditions appropriate for the lattice of \( N + 1 \) sites are \( F_{jl}^{N+1}(t) = 0; j, l = 0, N + 2 \). As the constant factor in the last term on the right side of Eq. (13) only gives a constant global phase in the solution, it is evident that the absolute value of a solution of this equation for a constant magnetic field is identical with the corresponding solution of Eq. (5) with \( \Delta = 0 \), and with that of Eq. (5) with \( \Delta \neq 0 \) for a linear magnetic field \( h_n = n h \). There is obviously a one-to-one mapping from the Heisenberg dynamics into that of coupled traps with ultracold atoms. The number of lattice sites in the Heisenberg chain plays the role of total particle number in the traps. Thus, e.g., the limit \( N \to \infty \) in the coupled traps, with Hamiltonian (\[8\]), corresponds to the limit of semi-infinite chain in the Heisenberg problem. Notice that for a non-zero linear magnetic field, the localized coherent oscillations of Figs. 2 and 3 appear also in the Heisenberg chain.

In conclusion, we have considered a model for two traps with ultracold atoms and with a strong Josephson coupling between them. We find that for non-zero effective detuning in the trap energies there is a transition to localized coherent oscillation in the occupation probabilities of the traps. Below the transition, i.e. for \( N/\lambda \ll 1 \), the dynamics is dispersive as in the case of vanishing offset. We also found that there is a one-to-one mapping between these dynamics and those of an isotropic Heisenberg chain in an appropriate magnetic field. This opens up the possibility in principle of realizing quantum computation, or simulation, in coupled-trap systems with only the detuning and the coupling of the traps as the relevant parameters.

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