DRINFE LD DOUBLES VIA DERIVED HALL ALGEBRAS 
AND BRIDGELAND HALL ALGEBRAS

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Abstract. Let \(A\) be a finitary hereditary abelian category. We give a Hall algebra presentation of Kashaev’s theorem on the relation between Drinfeld double and Heisenberg double. As applications, we obtain realizations of the Drinfeld double Hall algebra of \(A\) via its derived Hall algebra and Bridgeland Hall algebra of \(m\)-cyclic complexes.

1. Introduction

The Hall algebra \(\mathcal{H}(A)\) of a finite dimensional algebra \(A\) over a finite field was introduced by Ringel [9] in 1990. Ringel [8] proved that if \(A\) is a hereditary algebra of finite type, the twisted Hall algebra \(\mathcal{H}_v(A)\), called the Ringel–Hall algebra, is isomorphic to the positive part of the corresponding quantized enveloping algebra. In 1995, Green [3] generalized Ringel’s work to any hereditary algebra \(A\) and showed that the composition subalgebra of \(\mathcal{H}_v(A)\) generated by simple \(A\)-modules gives a realization of the positive part of the quantized enveloping algebra associated with \(A\). Moreover, he introduced a bialgebra structure on \(\mathcal{H}_v(A)\) via a significant formula called Green’s formula. In 1997, Xiao [13] provided the antipode on \(\mathcal{H}_v(A)\), and proved that the extended Ringel–Hall algebra is a Hopf algebra. Furthermore, he considered the Drinfeld double of the extended Ringel–Hall algebras, and obtained a realization of the full quantized enveloping algebra.

In order to give an intrinsic realization of the entire quantized enveloping algebra via Hall algebra approach, one tried to define the Hall algebra of a triangulated category (for example, [5], [12], [14]). Kapranov [5] considered the Heisenberg double of the extended Ringel–Hall algebras, and defined an associative algebra, called the lattice algebra, for the bounded derived category of a hereditary algebra \(A\). By using the fibre products of model categories, Toën [12] defined an associative algebra, called the derived Hall algebra, for a DG-enhanced triangulated category. Later on, Xiao and Xu [14] generalized the definition of the derived Hall algebra to any triangulated category with some homological finiteness conditions. In particular, the derived Hall algebra \(\mathcal{DH}(A)\) of the bounded derived category
of a hereditary algebra $A$ can be defined, and it is proved in [12] that there exist certain Heisenberg double structures in $\mathcal{DH}(A)$.

Recently, for each hereditary algebra $A$, Bridgeland [11] defined an associative algebra, called the Bridgeland Hall algebra, which is the Ringel–Hall algebra of 2-cyclic complexes over projective $A$-modules with some localization and reduction. He proved that the quantized enveloping algebra associated to $A$ can be embedded into its Bridgeland Hall algebra. This provides a beautiful realization of the full quantized enveloping algebra by Hall algebras. Afterwards, Yanagida [15] (see also [16]) showed that the Bridgeland Hall algebra of 2-cyclic complexes of a hereditary algebra is isomorphic to the Drinfeld double of its extended Ringel–Hall algebras. Inspired by the work of Bridgeland, Chen and Deng [2] introduced the Bridgeland Hall algebra $\mathcal{DH}_m(A)$ of $m$-cyclic complexes of a hereditary algebra $A$ for each nonnegative integer $m \neq 1$. If $m = 0$ or $m > 2$, the algebra structure of $\mathcal{DH}_m(A)$ has a characterization in [17], in particular, it is proved that there exist Heisenberg double structures in $\mathcal{DH}_m(A)$.

Kashaev [6] established a relation between the Drinfeld double and Heisenberg double of a Hopf algebra. Explicitly, he showed that the Drinfeld double is representable as a subalgebra in the tensor square of the Heisenberg double.

In this paper, let $\mathcal{A}$ be a finitary hereditary abelian category. We first give a Hall algebra presentation of Kashaev’s Theorem on the relation between Drinfeld double and Heisenberg double. Then we apply this presentation to the Bridgeland Hall algebra and derived Hall algebra of $\mathcal{A}$.

Throughout the paper, all tensor products are assumed to be over the complex number field $\mathbb{C}$. Let $k$ be a fixed finite field with $q$ elements and set $v = \sqrt{q} \in \mathbb{C}$. Let $\mathcal{A}$ be a finitary hereditary abelian $k$-category. We denote by $\text{Iso}(\mathcal{A})$ and $K(\mathcal{A})$ the set of isoclasses of objects in $\mathcal{A}$ and the Grothendieck group of $\mathcal{A}$, respectively. For each object $M$ in $\mathcal{A}$, the class of $M$ in $K(\mathcal{A})$ is denoted by $\hat{M}$, and the automorphism group of $M$ is denoted by $\text{Aut}(M)$. For a finite set $S$, we denote by $|S|$ its cardinality, and we also write $a_M$ for $|\text{Aut}(M)|$. For a positive integer $m$, we denote the quotient ring $\mathbb{Z}/m\mathbb{Z}$ by $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$. By convention, $\mathbb{Z}_0 = \mathbb{Z}$.

2. Preliminaries

In this section, we recall the definitions of Ringel–Hall algebra, Heisenberg double, and Drinfeld double (cf. [10, 13, 5]).

2.1. Hall algebras. For objects $M, N_1, \ldots, N_t \in \mathcal{A}$, let $g_{N_1, \ldots, N_t}^M$ be the number of the filtrations

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{t-1} \supseteq M_t = 0$$
such that $M_{i-1}/M_i \cong N_i$ for all $1 \leq i \leq t$. In particular, if $t = 2$, $g^M_{N_1N_2}$ is the number of subobjects $X$ of $M$ such that $X \cong N_2$ and $M/X \cong N_1$. One defines the Hall algebra $\mathcal{H}(\mathcal{A})$ to be the vector space over $\mathbb{C}$ with basis $[M] \in \text{Iso} (\mathcal{A})$ and with the multiplication defined by

$$[M] \diamond [N] = \sum_{[L]} g^M_{MN}[L].$$

By definition, it is easy to see that for each $1 < i < t$,

$$g^M_{N_1 \cdots N_i} = \sum_{[X]} g^M_{N_1 \cdots N_{i-1}X} g^X_{N_i} = \sum_{[Y]} g^M_{N_1 \cdots N_{i-1}Y} g^M_{YN_{i+1} \cdots N_t}.$$

For any $M, N \in \mathcal{A}$, define

$$\langle M, N \rangle := \dim_k \text{Hom}_\mathcal{A}(M, N) - \dim_k \text{Ext}^1_\mathcal{A}(M, N).$$

It induces a bilinear form

$$\langle \cdot, \cdot \rangle : K(\mathcal{A}) \times K(\mathcal{A}) \rightarrow \mathbb{Z},$$

known as the Euler form. We also consider the symmetric Euler form

$$(\cdot, \cdot) : K(\mathcal{A}) \times K(\mathcal{A}) \rightarrow \mathbb{Z},$$

defined by $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ for all $\alpha, \beta \in K(\mathcal{A})$.

The twisted Hall algebra $\mathcal{H}_v(\mathcal{A})$, called the Ringel–Hall algebra, is the same vector space as $\mathcal{H}(\mathcal{A})$ but with the twisted multiplication defined by

$$[M][N] = v^{\langle M, N \rangle} \cdot [M] \diamond [N].$$

We can form the extended Ringel–Hall algebra $\mathcal{H}_v^e(\mathcal{A})$ by adjoining symbols $K_\alpha$ for all $\alpha \in K(\mathcal{A})$ and imposing relations

$$K_\alpha K_\beta = K_{\alpha+\beta}, \ K_\alpha[M] = v^{(\alpha, \hat{M})} \cdot [M] K_\alpha.$$

Green introduced a (topological) bialgebra structure on $\mathcal{H}_v^e(\mathcal{A})$ by defining the co-multiplication as follows:

$$\Delta([L]K_\alpha) = \sum_{[M],[N]} v^{\langle M, N \rangle} \frac{a_{M1}a_{N1}}{a_L} g^L_{MN}[M] K_{N+\alpha} \otimes [N] K_\alpha, \text{ for any } L \in \mathcal{A}, \alpha \in K(\mathcal{A}).$$

That $\Delta$ is a homomorphism of algebras amounts to the following crucial formula.

**Theorem 2.1. (Green’s formula)** Given $M, N, M', N' \in \mathcal{A}$, we have the following formula

$$a_{M1}a_{N1}a_{M'1}a_{N'1} \sum_{[L]} g^L_{MN} g^L_{M'N'} \frac{1}{a_L} = \sum_{[A],[A'],[B],[B']} \frac{|\text{Ext}^1_\mathcal{A}(A, B')|}{|\text{Hom}_\mathcal{A}(A, B')|} g^M_{AA} g^N_{BB} g^M_{AB} g^N_{A'B'} a_A a_{A'} a_B a_{B'}.$$
2.2. **Heisenberg doubles.** Let $A$ and $B$ be Hopf algebras, and let $\varphi : A \times B \to \mathbb{C}$ be a Hopf pairing. The **Heisenberg double** $HD(A, B, \varphi)$ is defined to be the free product $A \ast B$ imposed by the following relations (with $a \in A$ and $b \in B$):

$$b \ast a = \sum \varphi(a_2, b_1)a_1 \ast b_2,$$

where and elsewhere we use Sweedler’s notation $\Delta(a) = \sum a_1 \otimes a_2$.

There exists a so-called Green’s pairing $\varphi_0 : \mathcal{H}_0(A) \times \mathcal{H}_0(A) \to \mathbb{C}$ defined by

$$\varphi_0([M]K_\alpha, [N]K_\beta) = \delta_{[M],[N]} \frac{v^{(\alpha, \beta)}}{a_M},$$

which is a Hopf pairing.

Now let us apply the construction of Heisenberg double to Ringel–Hall algebras. Let $H^+(A)$ (resp. $H^-(A)$) be the Ringel–Hall algebra $\mathcal{H}_0^+(A)$ with each $[M]K_\alpha$ rewritten as $\mu^+_{M}K^+_{\alpha}$ (resp. $K^-_{\alpha}K^-_{\alpha}$). Thus, considering $A = H^-(A)$, $B = H^+(A)$ and $\varphi = \varphi_0$, we obtain the **Heisenberg double Hall algebra**, denoted by $HD(A)$. By direct calculations, we give the characterization of $HD(A)$ via generators and generating relations (with $\alpha, \beta \in K(A)$ and $[M], [N] \in Iso(A)$) as follows (cf. [5]):

$$\mu^+_{M}\mu^+_{N} = \sum_{[L]} v^{(M,N)} g^{L}_{MN} \mu^+_{L}, \quad \mu^-_{M}\mu^-_{N} = \sum_{[L]} v^{(M,N)} g^{L}_{MN} \mu^-_{L};$$

$$K^+_{\alpha}\mu^+_{M} = v^{(\alpha, \hat{M})}\mu^+_{M}K^+_{\alpha}, \quad K^-_{\alpha}\mu^-_{M} = v^{(\alpha, \hat{M})}\mu^-_{M}K^-_{\alpha};$$

$$K^+_{\alpha}K^\pm_{\beta} = K^\pm_{\alpha+\beta}, \quad K^+_{\alpha}K^-_{\beta} = \mu^+ v^{(\alpha, \beta)}K^-_{\alpha};$$

$$K^\pm_{\alpha}\mu^-_{M} = \mu^-_{M}K^\pm_{\alpha}, \quad K^-_{\alpha}\mu^+_{M} = \mu^+ v^{-(\alpha, \hat{M})}\mu^+_{M}K^-_{\alpha};$$

$$\mu^+_{M}\mu^+_{N} = \sum_{[X],[Y]} v^{(N\hat{\gamma},\hat{X}\hat{\gamma})} g^{L}_{MN} K^+_{N\hat{\gamma}} \mu^+_{L};$$

where and elsewhere $\gamma^XY_{MN} = \frac{\alpha_{XY}}{a_{MN}} \sum_{[L]} a_L g^{M}_{LX} g^{N}_{LY}$.

Similarly, one defines the **dual Heisenberg double Hall algebra** $\hat{HD}(A)$, which is given by the generators and generating relations (with $\alpha, \beta \in K(A)$ and $[M], [N] \in Iso(A)$) as follows:

$$\nu^+_{M}\nu^+_{N} = \sum_{[L]} v^{(M,N)} g^{L}_{MN} \nu^+_{L}, \quad \nu^-_{M}\nu^-_{N} = \sum_{[L]} v^{(M,N)} g^{L}_{MN} \nu^-_{L};$$

$$K^+_{\alpha}\nu^+_{M} = v^{(\alpha, \hat{M})}\nu^+_{M}K^+_{\alpha}, \quad K^-_{\alpha}\nu^-_{M} = v^{(\alpha, \hat{M})}\nu^-_{M}K^-_{\alpha};$$

$$K^\pm_{\alpha}K^\pm_{\beta} = K^\pm_{\alpha+\beta}, \quad K^\pm_{\alpha}K^\pm_{\beta} = v^{(\alpha, \beta)}K^\pm_{\alpha};$$

$$K^\pm_{\alpha}\nu^+_{M} = \nu^+_{M}K^\pm_{\alpha}, \quad K^-_{\alpha}\nu^+_{M} = \nu^+ v^{-(\alpha, \hat{M})}\nu^-_{M}K^\pm_{\alpha};$$

$$\nu^+_{M}\nu^+_{N} = \sum_{[X],[Y]} \nu^{(N\hat{\gamma},\hat{X}\hat{\gamma})} g^{L}_{MN} K^+_{N\hat{\gamma}} \nu^+_{L}.$$
2.3. Drinfeld doubles. Let $A$ and $B$ be Hopf algebras, and let $\varphi : A \times B \to \mathbb{C}$ be a Hopf pairing. The Drinfeld double $D(A, B, \varphi)$ is defined to be the free product $A \ast B$ imposed by the following relations (with $a \in A$ and $b \in B$):

$$\sum \varphi(a_1, b_2)b_1 \ast a_2 = \sum \varphi(a_2, b_1) a_1 \ast b_2.$$  \hfill (2.13)

Applying the construction of Drinfeld double to the Ringel–Hall algebras $H^-(A)$ and $H^+(A)$, we obtain the Drinfeld double Hall algebra, denoted by $D(A)$, which is defined by the generators and generating relations (with $\alpha, \beta \in K(A)$, $[M], [N] \in \text{Iso}(A)$) as follows:

$$\omega^+_M \omega^+_N = \sum_{[L]} u^{(M,N)}_{M \to N} g^L_{MN} \omega^+_L; \quad \omega^-_M \omega^-_N = \sum_{[L]} u^{(M,N)}_{M \to N} g^L_{MN} \omega^-_L;$$ \hfill (2.14)

$$\mathcal{H}^+_{\alpha} \mathcal{H}^+_{\beta} = \mathcal{H}^+_{\alpha + \beta}, \quad \mathcal{H}^+_{\alpha} \mathcal{H}^-_{\beta} = \mathcal{H}^-_{\beta} \mathcal{H}^+_{\alpha};$$ \hfill (2.15)

$$\mathcal{H}^+_{\alpha} \mathcal{H}^+_{\beta} = \mathcal{H}^+_{\alpha + \beta}, \quad \mathcal{H}^-_{\alpha} \mathcal{H}^-_{\beta} = \mathcal{H}^-_{\beta} \mathcal{H}^+_{\alpha};$$ \hfill (2.16)

$$\sum_{[X],[Y]} u^{(M-\hat{X},\hat{N}-\hat{Y})}_{MN}\mathcal{H}^+_{\hat{X}} \omega^+_{\hat{X}} \omega^+_{\hat{Y}} = \sum_{[X],[Y]} u^{(M-\hat{X},\hat{N}-\hat{Y})}_{MN}\mathcal{H}^+_{\hat{X}} \omega^+_{\hat{X}} \omega^-_{\hat{Y}}.$$ \hfill (2.17)

3. Kashaev’s theorem: Hall algebra presentation

In this section, we prove Kashaev’s theorem \cite{Kashaev} Theorem 2] in the form of Ringel–Hall algebras. There are some similar constructions in \cite{Ringel}, but they are not so natural.

**Theorem 3.1.** There exists an embedding of algebras $I : D(A) \hookrightarrow HD(A) \otimes \check{HD}(A)$ defined on generators by

\begin{align*}
\mathcal{H}^+_{\alpha} &\mapsto K^+_{\alpha} \otimes K^+_{\alpha}, \quad \omega^+_M &\mapsto \sum_{[M_1],[M_2]} u^{(M_1,M_2)}_{[M_1, M_2]} \frac{a_{M_1} a_{M_2}}{a_M} g^M_{M_1 M_2} \mu^+_{M_1} \mu^+_{M_2} K^+_{M_2} \otimes \nu^+_{M_2}, \\
\mathcal{H}^-_{\alpha} &\mapsto K^-_{\alpha} \otimes K^-_{\alpha}, \quad \omega^-_M &\mapsto \sum_{[M_1],[M_2]} u^{(M_2,M_1)}_{[M_1, M_2]} \frac{a_{M_1} a_{M_2}}{a_M} g^M_{M_2 M_1} \mu^-_{M_1} \mu^-_{M_2} \check{K}^-_{M_1} \otimes \check{\nu}^+_{M_2},
\end{align*}

and

**Proof.** In order to prove that $I$ is a homomorphism of algebras, it suffices to show that the relations from (2.14) to (2.18) are preserved under $I$. We only prove the relations (2.14) and (2.18), since the other relations can be easily proved.

For the first relation in (2.14),

$$\sum_{[L]} u^{(M,N)}_{M \to N} g^L_{MN} I(\omega^+_L) = \sum_{[L],[L_1],[L_2]} u^{(M,N)+[L_1,L_2]}_{[L],[L_1],[L_2]} \frac{a_{L_1} a_{L_2}}{a_L} g^L_{MN} g^L_{L_1 L_2} \mu^+_{L_1} \mu^+_{L_2} K^+_{L_2} \otimes \nu^+_{L_2}.$$
\[ I(\omega_M^+)I(\omega_N^+) = \sum_{[M_1],[M_2],[N_1],[N_2]} \omega^{(M_1,M_2)+(N_1,N_2)} \frac{a_{M_1}a_{M_2}a_{N_1}a_{N_2}}{a_Ma_N} g^M_{M_1M_2}g^N_{N_1N_2}M_{M_1}^+M_{N_2}^+K_{M_2}^+K_{N_2}^+ \otimes \nu_{M_2}^+ \nu_{N_2}^+ \]

\[ = \sum_{[M_1],[M_2],[N_1],[N_2]} \omega^{x_0} \frac{a_{M_1}a_{M_2}a_{N_1}a_{N_2}}{a_Ma_N} g^M_{M_1M_2}g^N_{N_1N_2}M_{X}^+M_{Y}^+K_{M_2+N_2}^+ \otimes \nu_{M_2}^+ \nu_{N_2}^+ \]

\[ (x_0 = \langle M_1, M_2 \rangle + \langle N_1, N_2 \rangle + (M_2, N_1)) \]

\[ = \sum_{[M_1],[M_2],[N_1],[N_2],[L_1],[L_2]} \omega^{x_1} \frac{a_{M_1}a_{M_2}a_{N_1}a_{N_2}}{a_Ma_N} g^M_{M_1M_2}g^N_{N_1N_2}M_{X}^+M_{Y}^+L_{L_1}^+ \otimes \nu_{L_1}^+ \]

\[ (x_1 = \langle M_1, M_2 \rangle + \langle N_1, N_2 \rangle + (M_2, N_1) + \langle M_1, N_1 \rangle + \langle M_2, N_2 \rangle). \]

For each fixed \( L_1, L_2 \), noting that in (\( \star \)) \( \hat{M} = \hat{M}_1 + \hat{M}_2, \hat{N} = \hat{N}_1 + \hat{N}_2, \hat{L}_i = \hat{L}_1 + \hat{L}_2 \) for \( i = 1, 2 \), we obtain that \( x_1 = \langle M, N \rangle + \langle L_1, L_2 \rangle - 2\langle M_1, N_2 \rangle \). Thus, by Green’s formula, we conclude that

\[ \sum_{[M_1],[N_1],i=1,2} \omega^{x_1} \frac{a_{M_1}a_{M_2}a_{N_1}a_{N_2}}{a_Ma_N} g^M_{M_1M_2}g^N_{N_1N_2}M_{X}^+M_{Y}^+L_{L_1}^+ \otimes \nu_{L_1}^+ \]

and thus

\[ I(\omega_M^+)I(\omega_N^+) = \sum_{[L]} \omega^{(M,N)} g^L_{MN}I(\omega_L^+). \]

Similarly, we can prove that the second relation in (2.14) is also preserved under \( I \).

Now, we come to prove that the relation in (2.18) is preserved under \( I \). First of all, substituting \( \gamma_{MN}^{XY} = \frac{a_{ax}}{a_{ax}} \sum_{[L]} a_L g^M_{LX}g^N_{LY} \) into (2.18), we rewrite (2.18) as follows:

\[ \sum_{[X],[Y],[L]} \omega^{(L,M-N)} a_X a_Y a_L g^M_{LX} g^N_{LY} \mathcal{K}_L^{-} \omega_X^{-} \omega_Y^{-} = \sum_{[X],[Y],[L]} \omega^{(L,N-M)} a_X a_Y a_L g^M_{XL} g^N_{LY} \mathcal{K}_L^{+} \omega_X^{+} \omega_Y^{-} \]

On the one hand,

\[ \text{LHS} := \sum_{[X],[Y],[L]} \omega^{(L,M-N)} a_X a_Y a_L g^M_{LX} g^N_{LY} I(\mathcal{K}_L^{-})I(\omega_X^{-}) \]

\[ = \sum_{[X],[Y],[L],[X_1],[X_2],[Y_1],[Y_2],[N_1],[N_2]} \omega^{y_0} a_X a_{X_1}a_{X_2}a_{Y_1}a_{Y_2} a_L g^M_{LX} g^X_{X_1X_2} g^N_{Y_1Y_2} g^Y_{Y_1} g^Y_{Y_2} g^X_{X_1} g^X_{X_2} g^Y_{Y_1} g^Y_{Y_2} g^X_{X_1} g^X_{X_2} \]

\[ (y_0 = \langle \hat{L}, \hat{M} - \hat{N} \rangle + \langle X_1, X_2 \rangle + \langle Y_1, Y_2 \rangle) \]

\[ = \sum_{[L],[X_1],[X_2],[Y_1],[Y_2]} \omega^{y_1} a_X a_{X_1}a_{X_2}a_{Y_1}a_{Y_2} a_L g^M_{LX} g^N_{X_1X_2} g^X_{Y_1} g^X_{Y_2} g^N_{X_1} g^N_{X_2} g^X_{Y_1} g^X_{Y_2} g^N_{X_1} g^N_{X_2} \]

\[ (y_1 = y_0 - (Y_1, Y_2) = \langle \hat{L}, \hat{M} - \hat{N} \rangle + \langle X_1, X_2 \rangle - \langle Y_1, Y_2 \rangle). \]
By (2.12),
\[
\nu_{Y_2}^{-} \nu_{X_2}^{+} = \sum_{[A],[B]} v^{(Y_2 - B, B - A)}_{Y_2X_2} K_{Y_2 - B}^{+} \nu_{X_2}^{+} \nu_{Y_2}^{-} = \sum_{[A],[B],[C]} v^{(C, B - A)}_{A} a_{A} a_{B} a_{C} g_{C} \nu_{Y_2} \mu_{X_2} K_{A}^{+} L \mu_{Y_2} K_{C}^{+} \nu_{X_2}^{+} \nu_{Y_2}^{-}. 
\]
Thus, LHS =
\[
\sum_{[L],[X_1],[Y_1],[A],[B],[C]} v^{y_2} a_{X} a_{A} a_{C} a_{B} a_{Y_1} g_{LX_1 AC}^{M} g_{LY_1 BC}^{N} K_{L}^{-} \mu_{Y_1} K_{A}^{-} \mu_{X_2} K_{C}^{+} \nu_{X_2}^{+} \nu_{Y_2}^{-} = \sum_{[L],[X_1],[Y_1],[A],[B],[C]} v^{y_2} a_{X} a_{A} a_{C} a_{B} a_{Y_1} g_{LX_1 AC}^{M} g_{LY_1 BC}^{N} K_{L}^{-} \mu_{Y_1} K_{A}^{-} \mu_{X_2} K_{C}^{+} \nu_{X_2}^{+} \nu_{Y_2}^{-} 
\]
(y_2 = y_1 + \langle \hat{C}, \hat{B} - \hat{A} \rangle = \langle \hat{L}, \hat{M} - \hat{N} \rangle + \langle \hat{X}_1, \hat{A} + \hat{C} \rangle - \langle \hat{Y}_1, \hat{B} + \hat{C} \rangle + \langle \hat{C}, \hat{B} - \hat{A} \rangle)
\]
= \sum_{[L],[X_1],[Y_1],[A],[B],[C]} v^{y_3} a_{X} a_{A} a_{C} a_{B} a_{Y_1} g_{LX_1 AC}^{M} g_{LY_1 BC}^{N} K_{L}^{-} \mu_{Y_1} K_{A}^{-} \mu_{X_2} K_{C}^{+} \nu_{X_2}^{+} \nu_{Y_2}^{-} \nu_{Y_1}^{-} 
\]
(y_3 = \langle \hat{L}, \hat{M} - \hat{N} \rangle + \langle \hat{X}_1, \hat{A} + \hat{C} \rangle - \langle \hat{Y}_1, \hat{B} + \hat{C} \rangle + \langle \hat{C}, \hat{B} - \hat{A} \rangle + \langle \hat{L} + \hat{Y}_1, \hat{B} + \hat{C} \rangle).

On the other hand,
RHS := \sum_{[X],[Y],[L]} v^{(L,N-M)} a_{X} a_{Y} a_{L} g_{XLY}^{N} (\mathcal{X}_L^{+}) I(\omega_{\hat{X}}) I(\omega_{\hat{Y}}) 
\]
= \sum_{[X],[Y],[L],[X_1],[X_2],[Y_1],[Y_2]} v^{z_0} a_{X_1} a_{X_2} a_{Y_1} a_{Y_2} a_{L} g_{X_1X_2}^{M} g_{LY_1Y_2}^{N} K_{X_2+L}^{+} \mu_{X_1} \mu_{Y_1} \nu_{X_2}^{+} \nu_{Y_2}^{-} K_{Y_1}^{-} 
\]
(z_0 = \langle \hat{L}, \hat{N} - \hat{M} \rangle + \langle X_1, X_2 \rangle + \langle Y_1, Y_2 \rangle)
\]
= \sum_{[L],[X_1],[X_2],[Y_1],[Y_2]} v^{z_1} a_{X_1} a_{X_2} a_{Y_1} a_{Y_2} a_{L} g_{X_1X_2}^{M} g_{LY_1Y_2}^{N} K_{X_2+L}^{+} \mu_{X_1} \mu_{Y_1} \nu_{X_2}^{+} \nu_{Y_2}^{-} K_{Y_1}^{-} 
\]
(z_1 = z_0 - \langle X_1, X_2 \rangle = \langle \hat{L}, \hat{N} - \hat{M} \rangle + \langle Y_2, Y_1 \rangle - \langle X_2, X_1 \rangle).

By (2.7),
\[
\mu_{X_1} \mu_{Y_1}^{-} = \sum_{[A],[B]} v^{(Y_1 - B, A - B)}_{Y_1X_1} K_{Y_1 - B}^{+} \mu_{B} \mu_{A}^{-} = \sum_{[A],[B],[C]} v^{(C, A - B)}_{A} a_{A} a_{B} a_{C} g_{C} \nu_{X_1} \mu_{Y_1} K_{C}^{+} \mu_{B} \mu_{A}^{-}. 
\]
Thus, RHS =
\[
\sum_{[L],[X_2],[Y_2],[A],[B],[C]} v^{z_2} a_{C} a_{A} a_{X_2} a_{L} a_{Y_2} a_{B} g_{CAX_2L}^{M} g_{LY_2BC}^{N} K_{X_2+L}^{+} \mu_{B} \mu_{A}^{+} \nu_{X_2}^{+} \nu_{Y_2}^{-} K_{B+C}^{-} 
\]
(z_2 = z_1 + \langle \hat{C}, \hat{A} - \hat{B} \rangle = \langle \hat{L}, \hat{N} - \hat{M} \rangle + \langle \hat{Y}_2, \hat{B} + \hat{C} \rangle - \langle \hat{X}_2, \hat{A} + \hat{C} \rangle + \langle \hat{C}, \hat{A} - \hat{B} \rangle)
\]
= \sum_{[L],[X_2],[Y_2],[A],[B],[C]} v^{z_3} a_{C} a_{A} a_{X_2} a_{L} a_{Y_2} a_{B} g_{CAX_2L}^{M} g_{LY_2BC}^{N} K_{X_2+L}^{+} \mu_{B} \mu_{A}^{+} \nu_{X_2}^{+} \nu_{Y_2}^{-} K_{B+C}^{-}
\[(z_3 = \langle \hat{L}, \hat{N} - \hat{M} \rangle + \langle \hat{Y}_2, \hat{B} + \hat{C} \rangle - \langle \hat{X}_2, \hat{A} + \hat{C} \rangle + \langle \hat{C}, \hat{A} - \hat{B} \rangle + (\hat{L} + \hat{X}_2, \hat{A} + \hat{C}) \)).\]

Identifying \(L, X_1, A, C, B, Y_1\) in LHS with \(C, A, X_2, L, Y_2, B\) in RHS, respectively, we obtain that \(y_3 = \langle \hat{C}, \hat{M} - \hat{N} \rangle + \langle \hat{A}, \hat{X}_2 + \hat{L} \rangle - \langle \hat{B}, \hat{Y}_2 + \hat{L} \rangle + \langle \hat{L}, \hat{Y}_2 - \hat{X}_2 \rangle + \langle \hat{B} + \hat{C}, \hat{Y}_2 + \hat{L} \rangle\).

Noting that in RHS \(\hat{M} - \hat{N} = \hat{X} - \hat{Y} = (\hat{X}_1 - \hat{Y}_1) + (\hat{X}_2 - \hat{Y}_2) = (\hat{A} - \hat{B}) + (\hat{X}_2 - \hat{Y}_2)\), we have that

\[
y_3 = \langle \hat{C}, \hat{A} - \hat{B} \rangle + \langle \hat{C}, \hat{X}_2 \rangle - \langle \hat{C}, \hat{Y}_2 \rangle + \langle \hat{A}, \hat{X}_2 \rangle + \langle \hat{A}, \hat{L} \rangle - \langle \hat{B}, \hat{Y}_2 + \hat{L} \rangle + \langle \hat{L}, \hat{Y}_2 - \hat{X}_2 \rangle + \langle \hat{C}, \hat{L} \rangle + \langle \hat{Y}_2, \hat{B} + \hat{C} \rangle + \langle \hat{L}, \hat{B} \rangle
\]

and

\[
z_3 = \langle \hat{L}, \hat{B} \rangle - \langle \hat{L}, \hat{A} \rangle + \langle \hat{L}, \hat{Y}_2 - \hat{X}_2 \rangle + \langle \hat{Y}_2, \hat{B} + \hat{C} \rangle - \langle \hat{X}_2, \hat{A} + \hat{C} \rangle + \langle \hat{C}, \hat{A} - \hat{B} \rangle
\]

\[
+ \langle \hat{C}, \hat{L} \rangle + \langle \hat{L}, \hat{A} \rangle + \langle \hat{X}_2, \hat{A} + \hat{C} \rangle + \langle \hat{A} + \hat{C}, \hat{X}_2 \rangle
\]

\[
= \langle \hat{L}, \hat{B} \rangle + \langle \hat{L}, \hat{Y}_2 - \hat{X}_2 \rangle + \langle \hat{Y}_2, \hat{B} + \hat{C} \rangle + \langle \hat{C}, \hat{A} - \hat{B} \rangle + \langle \hat{C}, \hat{L} \rangle + \langle \hat{A}, \hat{L} \rangle + \langle \hat{A} + \hat{C}, \hat{X}_2 \rangle
\]

\[
y_3.
\]

Hence, LHS = RHS, and we have proved that \(I\) is a homomorphism of algebras.

Since \(D(\mathcal{A}) \cong H^+(\mathcal{A}) \otimes H^-(\mathcal{A})\) as a vector space, and the restriction of \(I\) to the positive (negative) part is injective, we conclude that \(I\) is injective. Therefore, we complete the proof. \(\square\)

4. Applications

In this section, we apply Theorem 3.1 to Bridgeland Hall algebras of \(m\)-cyclic complexes and derived Hall algebras.

4.1. Bridgeland Hall algebras. Assume that \(\mathcal{A}\) has enough projectives, the Bridgeland Hall algebra of 2-cyclic complexes of \(\mathcal{A}\) was introduced in \([1]\). Inspired by the work of Bridgeland, for each nonnegative integer \(m \neq 1\), Chen and Deng \([2]\) introduced the Bridgeland Hall algebra \(\mathcal{DH}_m(\mathcal{A})\) of \(m\)-cyclic complexes. For \(m = 0\) or \(m > 2\), we recall the algebra structure of \(\mathcal{DH}_m(\mathcal{A})\) by \([17]\) as follows:

**Proposition 4.1.** \([17]\) Let \(m = 0\) or \(m > 2\). Then \(\mathcal{DH}_m(\mathcal{A})\) is an associative and unital \(\mathbb{C}\)-algebra generated by the elements in \(\{e_{M,i} \mid [M] \in \text{Iso}(\mathcal{A}), \ i \in \mathbb{Z}_m\}\) and \(\{K_{\alpha,i} \mid \alpha \in\).
$K(A), i \in \mathbb{Z}_m$, and the following relations:

$$K_{\alpha,i}K_{\beta,j} = K_{\alpha+\beta,i}, \quad K_{\alpha,i}K_{\beta,j} = \begin{cases} v^{(\alpha,\beta)} K_{\beta,j} K_{\alpha,i} & i = j + 1, \\ v^{-(\alpha,\beta)} K_{\beta,j} K_{\alpha,i} & i = m - 1 + j, \\ K_{\beta,j} K_{\alpha,i} & \text{otherwise}; \end{cases}$$ (4.1)

$$K_{\alpha,i}e_{M,j} = \begin{cases} v^{(\alpha,\hat{M})} e_{M,j} K_{\alpha,i} & i = j, \\ v^{-(\alpha,\hat{M})} e_{M,j} K_{\alpha,i} & i = m - 1 + j, \\ e_{M,j} K_{\alpha,i} & \text{otherwise}; \end{cases}$$ (4.2)

$$e_{M,i}e_{N,i} = \sum_{[L]} v^{(M,N)} g_{M,N} e_{L,i};$$ (4.3)

$$e_{M,i+1}e_{N,i} = \sum_{[X],[Y]} v^{(\hat{M}-\hat{X},\hat{Y})} \gamma_{X,Y} K_{\hat{M}-\hat{X},i} e_{Y,i} e_{X,i+1};$$ (4.4)

$$e_{M,i}e_{N,j} = e_{N,j}e_{M,i}, \quad i - j \neq 0, 1 \text{ or } m - 1.$$ (4.5)

**Corollary 4.2.** Let $m = 0$ or $m > 2$. Then for each $i \in \mathbb{Z}_m$,

1. there exists an embedding of algebras $\kappa_i : HD(A) \hookrightarrow DH_m(A)$ defined on generators by

$$K_{\alpha}^+ \mapsto K_{\alpha,i+1}, \quad K_{\alpha}^- \mapsto K_{\alpha,i}, \quad \mu_{M}^+ \mapsto e_{M,i+1}, \quad \mu_{M}^- \mapsto e_{M,i};$$

2. there exists an embedding of algebras $\hat{\kappa}_i : \tilde{HD}(A) \hookrightarrow DH_m(A)$ defined on generators by

$$K_{\alpha}^+ \mapsto K_{\alpha,i}, \quad K_{\alpha}^- \mapsto K_{\alpha,i+1}, \quad \nu_{M}^+ \mapsto e_{M,i}, \quad \nu_{M}^- \mapsto e_{M,i+1}.$$

**Proof.** By Proposition [41], the defining relations of $HD(A)$ and $\tilde{HD}(A)$ are preserved under $\kappa_i$ and $\hat{\kappa}_i$, respectively, we obtain that $\kappa_i$ and $\hat{\kappa}_i$ are homomorphisms of algebras. According to [17, Proposition 2.7], we conclude that they are injective. \hfill \Box

As a first application of Theorem [3.1] we have the following

**Theorem 4.3.** Let $m = 0$ or $m > 2$. Then for each $i \in \mathbb{Z}_m$, there exists an embedding of algebras $\psi_i : D(A) \hookrightarrow DH_m(A) \otimes DH_m(A)$ defined on generators by

$$\mathcal{K}_{\alpha}^+ \mapsto K_{\alpha,i+1} \otimes K_{\alpha,i}, \quad \omega_{M}^+ \mapsto \sum_{[M_1],[M_2]} v^{(M_1,M_2)} g_{M_1,M_2}^M a_{M_1,M_2} e_{M_1,i+1} K_{M_2,i+1} \otimes e_{M_2,i};$$

and

$$\mathcal{K}_{\alpha}^- \mapsto K_{\alpha,i} \otimes K_{\alpha,i+1}, \quad \omega_{M}^- \mapsto \sum_{[M_1],[M_2]} v^{(M_2,M_1)} g_{M_1,M_2}^M a_{M_1,M_2} e_{M_1,i} \otimes e_{M_2,i+1} K_{M_1,i+1}.$$
Proof. For each $i \in \mathbb{Z}_m$, by the following commutative diagram

$$
\begin{array}{c}
D(A) \xleftarrow{I} HD(A) \otimes \tilde{HD}(A) \\
\downarrow \psi_i \\
D\mathcal{H}(A) \otimes D\mathcal{H}(A)
\end{array}
$$

we complete the proof. \qed

Remark 4.4. As mentioned in Introduction, there is an isomorphism $\rho : D(A) \to D\mathcal{H}_2(A)$, which is defined on generators by

$$
\begin{align*}
\omega^+_M &\mapsto \frac{E_M}{a_M}, \omega^-_M &\mapsto \frac{F_M}{a_M}, K^+_\alpha &\mapsto K\alpha, K^-_\alpha &\mapsto K^*_\alpha,
\end{align*}
$$

where the notations $E_M$, $F_M$, $K_\alpha$ and $K^*_\alpha$ are the same as those in [16]. Hence, Theorem 4.3 establishes a relation between the Bridgeland Hall algebra of 2-cyclic complexes and that of $m$-cyclic complexes.

4.2. Derived Hall algebras. The derived Hall algebra $D\mathcal{H}(A)$ of the bounded derived category of $A$ was introduced in [12] (see also [14]).

Proposition 4.5. ([12]) $D\mathcal{H}(A)$ is an associative and unital $\mathbb{C}$-algebra generated by the elements in $\{Z^i_M \mid [M] \in \text{Iso}(A), i \in \mathbb{Z}\}$ and the following relations:

$$
\begin{align*}
Z^i_M Z^j_N &= \sum_{[L]} g^{L}_{MN} Z^i_L, \quad (4.6) \\
Z^i_M Z^{i+1}_N &= \sum_{[X],[Y]} q^{-(Y,X)_M} Z^i_Y Z^i_X, \quad (4.7) \\
Z^i_M Z^j_N &= q^{(-1)^{i-j}(N,M)} Z^j_N Z^i_M, \quad i-j > 1. \quad (4.8)
\end{align*}
$$

According to [11], we twist the multiplication in $D\mathcal{H}(A)$ as follows:

$$
Z^i_M \ast Z^j_N = v^{(-1)^{i-j}(M,N)} Z^i_M Z^j_N. \quad (4.9)
$$

The twisted derived Hall algebra $D\mathcal{H}_{\text{tw}}(A)$ is the same vector space as $D\mathcal{H}(A)$, but with the twisted multiplication. In order to relate the modified Ringel–Hall algebra, which is isomorphic to the corresponding Bridgeland Hall algebra if $A$ has enough projectives, to derived Hall algebra, Lin [7] introduced the completely extended twisted derived Hall algebra $D\mathcal{H}_{\text{twce}}(A)$.

Definition 4.6. ([7]) $D\mathcal{H}_{\text{twce}}(A)$ is the associative and unital $\mathbb{C}$-algebra generated by the elements in $\{Z^i_M \mid [M] \in \text{Iso}(A), i \in \mathbb{Z}\}$ and $\{K^i_\alpha \mid \alpha \in K(A), i \in \mathbb{Z}\}$, and the following
relations:

\[
K^{[i]} \alpha K^{[i]} \beta = K^{[i]} \alpha + \beta, \quad K^{[i]} \alpha Z^{[i]}_M = \begin{cases} v(\alpha, \hat{M}) Z^{[i]}_M K^{[i]} \alpha & i = -1, 0, \\ Z^{[i]}_M K^{[i]} \alpha & \text{otherwise}; \end{cases}
\]  \hspace{1cm} (4.10)

\[
K^{[i+1]} \alpha K^{[i]} \beta = v(\alpha, \beta) K^{[i]} \alpha K^{[i+1]} \beta, \quad K^{[i]} \alpha K^{[j]} \beta = K^{[j]} \alpha K^{[i]} \beta, \quad |i - j| > 1;
\]  \hspace{1cm} (4.11)

\[
K^{[i]} \alpha Z^{[i+1]}_M = \begin{cases} v^{-\langle \langle \alpha, M \rangle \rangle} K^{[i]} \alpha Z^{[i+1]}_M & i = -1, 0, \\ Z^{[i+1]}_M K^{[i]} \alpha & \text{otherwise}; \end{cases}
\]  \hspace{1cm} (4.12)

\[
K^{[i]} \alpha Z^{[i-1]}_M = \begin{cases} v^{-\langle \langle \alpha, M \rangle \rangle} K^{[i]} \alpha Z^{[i-1]}_M & i = -1, 0, \\ Z^{[i-1]}_M K^{[i]} \alpha & \text{otherwise}; \end{cases}
\]  \hspace{1cm} (4.13)

For any \(|i - j| > 1|, K^{[i]} \alpha Z^{[j]}_M = \begin{cases} v^{-\langle \langle \alpha, M \rangle \rangle} Z^{[j]}_M K^{[i]} \alpha & i = 0, \quad |j| > 1, \\ v^{-\langle \langle 1, M \rangle \rangle} Z^{[j]}_M K^{[i]} \alpha & i = -1, \quad |j + 1| > 1, \\ Z^{[j]}_M K^{[i]} \alpha & \text{otherwise}. \end{cases}
\]  \hspace{1cm} (4.14)

\[
Z^{[i]}_M Z^{[j]}_N = \sum_{[L]} v^{(M,N)} g_{MN} Z^{[i]}_L; \]

\[
Z^{[i+1]}_M Z^{[j]}_N = \sum_{[X],[Y]} v^{-\langle \langle X,Y \rangle \rangle} \gamma_Z Z^{[i]}_X Z^{[j]}_Y; \]

\[
Z^{[i]}_M Z^{[j]}_N = v^{-\langle \langle i, j \rangle \rangle} Z^{[j]}_N Z^{[i]}_M, \quad |i - j| > 1.
\]  \hspace{1cm} (4.17)

**Remark 4.7.** In Definition [16] we have employed the linear Euler form, not the multiplicative Euler form used in [7]; \(K^{[i]}\) and \(Z^{[i]}\) here are equal to \(K^{[-i]}\) and \(Z^{[-i]}\) in [7], respectively.

Now we reformulate [7] Theorem 5.3, Corollary 5.5 as follows:

**Theorem 4.8.** Assume that \(A\) has enough projectives. Then there exists an isomorphism of algebras \(\phi : DH^{ce}_{tw}(A) \rightarrow DH_0(A)\) defined on generators (with \(n > 0\)) by

\[
Z^{[0]}_M \mapsto e_{M,0}, \quad K^{[n]}_\alpha \mapsto K_{\alpha,n},
\]

\[
Z^{[n]}_M \mapsto v^{n(M,M)} e_{M,n} \prod_{i=1}^n K_{(-1)^i M,-i}, \quad \text{and} \quad Z^{[-n]}_M \mapsto v^{-n(M,M)} e_{M,-n} \prod_{i=0}^{n-1} K_{(-1)^{i+1} M,i-n}.
\]

**Remark 4.9.** (1) The inverse of \(\phi\) in Theorem 4.8 is the homomorphism \(\phi^{-1} : DH_0(A) \rightarrow DH^{ce}_{tw}(A)\) defined on generators (with \(n > 0\)) by

\[
e_{M,0} \mapsto Z^{[0]}_M, \quad K_{\alpha,n} \mapsto K^{[n]}_\alpha,
\]

\[
e_{M,n} \mapsto v^{-n(M,M)} Z^{[n]}_M \prod_{i=0}^{n-1} K_{(-1)^{n-i} M,-i}, \quad \text{and} \quad e_{M,-n} \mapsto v^{n(M,M)} Z^{[-n]}_M \prod_{i=1}^n K_{(-1)^{n-i} M,-i}.
\]
(2) Theorem 4.8 establishes the relation between the Bridgeland Hall algebra of bounded complexes over projectives of $\mathcal{A}$ and the derived Hall algebra of the bounded derived category $D^b(\mathcal{A})$. In other word, one can realize the derived Hall algebra via Bridgeland's construction.

As a second application of Theorem 3.11 we have the following

**Theorem 4.10.** For each $i \in \mathbb{Z}$, there exists an embedding of algebras $\varphi_i : D(\mathcal{A}) \hookrightarrow D_{\mathcal{H}_{\text{tw}}}(\mathcal{A}) \otimes D_{\mathcal{H}_{\text{tw}}}(\mathcal{A})$. Explicitly,

(1) if $i = -1$, $\varphi_i$ is defined on generators by

$$\mathcal{X}^+_\alpha \mapsto K_\alpha^{[0]} \otimes K_\alpha^{[-1]}, \quad \omega^+_M \mapsto \sum_{[M_1],[M_2]} v^{(M,M_2)} \frac{a_{M_1}a_{M_2}}{a_M} z_{M_1}^{[0]} k_{M_2}^{[0]} \otimes z_{M_2}^{[-1]} k_{M_1}^{[-1]},$$

$$\mathcal{X}^-_\alpha \mapsto K_\alpha^{[-1]} \otimes K_\alpha^{[0]}, \quad \omega^-_M \mapsto \sum_{[M_1],[M_2]} v^{(M,M_1)} \frac{a_{M_1}a_{M_2}}{a_M} z_{M_1}^{[-1]} k_{M_2}^{[-1]} \otimes z_{M_2}^{[0]} k_{M_1}^{[0]},$$

(2) if $i = 0$, $\varphi_i$ is defined on generators by

$$\mathcal{X}^+_\alpha \mapsto K_\alpha^{[1]} \otimes K_\alpha^{[0]}, \quad \omega^+_M \mapsto \sum_{[M_1],[M_2]} v^{(M,M_1)} \frac{a_{M_1}a_{M_2}}{a_M} z_{M_1}^{[1]} k_{M_2}^{[0]} \otimes z_{M_2}^{[0]} k_{M_1}^{[0]},$$

$$\mathcal{X}^-_\alpha \mapsto K_\alpha^{[0]} \otimes K_\alpha^{[1]}, \quad \omega^-_M \mapsto \sum_{[M_1],[M_2]} v^{(M,M_2)} \frac{a_{M_1}a_{M_2}}{a_M} z_{M_1}^{[0]} k_{M_2}^{[1]} \otimes z_{M_2}^{[1]} k_{M_1}^{[0]},$$

(3) if $i < -1$, $\varphi_i$ is defined on generators by

$$\mathcal{X}^+_\alpha \mapsto K_\alpha^{[i+1]} \otimes K_\alpha^{[i]}, \quad \mathcal{X}^-_\alpha \mapsto K_\alpha^{[i]} \otimes K_\alpha^{[i+1]},$$

$$\omega^+_M \mapsto \sum_{[M_1],[M_2]} v^{x a_{M_1}a_{M_2}} a_M z_{M_1}^{[-i]} \prod_{j=1}^{-(i+1)} k_{-1}^{[-j]} k_{M_2}^{[-1]} \otimes z_{M_2}^{[-i]} \prod_{j=1}^{-i} k_{-1}^{[j]}$$

$$x = \langle \hat{M}_1, \hat{M}_2 - \hat{M}_1 \rangle - i((\langle M_1, M_1 \rangle + \langle M_2, M_2 \rangle),$$

$$\omega^-_M \mapsto \sum_{[M_1],[M_2]} v^{y a_{M_1}a_{M_2}} a_M z_{M_1}^{[-i]} \prod_{j=1}^{-(i+1)} k_{-1}^{[-j]} k_{M_2}^{[-1]} \otimes z_{M_2}^{[-i]} \prod_{j=1}^{-i} k_{-1}^{[j]}$$

$$y = \langle \hat{M}_2, \hat{M}_1 - \hat{M}_2 \rangle - i((\langle M_1, M_1 \rangle + \langle M_2, M_2 \rangle);$$

(4) if $i > 0$, $\varphi_i$ is defined on generators by

$$\mathcal{X}^+_\alpha \mapsto K_\alpha^{[i+1]} \otimes K_\alpha^{[i]}, \quad \mathcal{X}^-_\alpha \mapsto K_\alpha^{[i]} \otimes K_\alpha^{[i+1]},$$

$$\omega^+_M \mapsto \sum_{[M_1],[M_2]} v^{x a_{M_1}a_{M_2}} a_M z_{M_1}^{[-i]} \prod_{j=0}^{i} k_{-1}^{[-j]} k_{M_2}^{[-1]} \otimes z_{M_2}^{[-i]} \prod_{j=0}^{-i} k_{-1}^{[j]}$$

$$x = \langle \hat{M}_1, \hat{M}_2 - \hat{M}_1 \rangle - i((\langle M_1, M_1 \rangle + \langle M_2, M_2 \rangle),$$
\[ \omega_{M} \mapsto \sum_{[M_1],[M_2]} y^{g_{M_1}g_{M_2}} a_M g_{M_2 M_1} Z[i] M_j \prod_{j=0}^{i-1} K[j]^{(-1)^{i-j-1} M_1} \otimes Z[i+1] M_2 \prod_{j=0}^{i} K[j]^{(-1)^{i-j} M_2} K^{[i+1]}_{M_1} \]

\[ y = \langle \hat{M}_2, \hat{M}_1 - \hat{M}_2 \rangle - i(\langle M_1, M_1 \rangle + \langle M_2, M_2 \rangle) \]

**Proof.** By the following commutative diagram

\[
\begin{array}{ccc}
D(A) & \xrightarrow{\psi_i} & \mathcal{DH}_0(A) \otimes \mathcal{DH}_0(A) \\
\downarrow \phi_i & & \downarrow \phi^{-1} \otimes \phi^{-1} \\
\mathcal{DH}_{tw}^{ce}(A) \otimes \mathcal{DH}_{tw}^{ce}(A) & \cong & \mathcal{DH}_{tw}^{ce}(A) \\
\end{array}
\]

we complete the proof. \[\square\]

**References**

[1] T. Bridgeland, Quantum groups via Hall algebras of complexes, Ann. Math. 177 (2013), 1–21.

[2] Q. Chen and B. Deng, Cyclic complexes, Hall polynomials and simple Lie algebras, J. Algebra 440 (2015), 1–32.

[3] J. A. Green, Hall algebras, hereditary algebras and quantum groups, Invent. Math. 120 (1995), 361–377.

[4] M. Kapranov, Eisenstein series and quantum affine algebras, J. Math. Sci. 84(5) (1997), 1311–1360.

[5] M. Kapranov, Heisenberg doubles and derived categories, J. Algebra 202 (1998), 712–744.

[6] R. M. Kashaev, The Heisenberg double and the pentagon relation, Algebra i Analiz 8(4) (1996), 63–74.

[7] J. Lin, Modified Ringel–Hall algebras, naive lattice algebras and lattice algebras, arXiv:1808.04037v1.

[8] C. M. Ringel, Hall algebras, in: S. Balcerzyk, et al. (Eds.), Topics in Algebra, Part 1, in: Banach Center Publ. 26 (1990), 433–447.

[9] C. M. Ringel, Hall algebras and quantum groups, Invent. Math. 101 (1990), 583–592.

[10] O. Schiffmann, Lectures on Hall algebras, Geometric methods in representation theory II, 1–141, Sémin. Congr., 24-II, Soc. Math. France, Paris, 2012.

[11] J. Sheng and F. Xu, Derived Hall algebras and lattice algebras, Algebra Colloq. 19(03) (2012), 533–538.

[12] B. Toën, Derived Hall algebras, Duke Math. J. 135 (2006), 587–615.

[13] J. Xiao, Drinfeld double and Ringel–Green theory of Hall algebras, J. Algebra 190 (1997), 100–144.

[14] J. Xiao and F. Xu, Hall algebras associated to triangulated categories, Duke Math. J. 143(2) (2008), 357–373.

[15] S. Yanagida, A note on Bridgeland’s Hall algebra of two-periodic complexes, Math. Z. 282(3) (2016), 973–991.

[16] H. Zhang, A note on Bridgeland Hall algebras, Comm. Algebra 46(6) (2018), 2551–2560.

[17] H. Zhang, Bridgeland’s Hall algebras and Heisenberg doubles, J. Algebra Appl. 17(6) (2018).

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