Abstract

The Hamiltonian of the Rabi model is considered. It is shown that the ground state energy of the Rabi Hamiltonian is simple for all values of the coupling strength, which implies the ground state energy does not cross other energy.

1 Introduction

Cavity quantum electrodynamics has supplied us with stronger interaction than the standard quantum electrodynamics (QED) does [HR01, RBH01]. Experimental physicists usually demonstrate the interaction by coupling a two-level atom with a one-mode light (i.e., single-mode laser) in a mirror cavity (i.e., a mirror resonator). The region that the strong interaction in cavity QED belongs to is called the strong coupling regime. At the dawn of the 21st century, the solid-state analogue of the strong interaction in a superconducting system was theoretically proposed in [MSS01, MB01], and it has been experimentally demonstrated in [CBSNHM04, FGBBLBW08, WSBFHMKGS04]. That is, the atom, the light, and the mirror resonator in cavity QED are respectively
replaced by an artificial atom, a microwave, and a microwave resonator on a superconducting circuit. Here, the artificial atom is made by using a superconducting circuit based on some Josephson junctions. This replaced cavity QED is the so-called circuit QED. The circuit QED has been intensifying the coupling strength so that its region is beyond the strong coupling regime. This amazing region of the coupling strength between the artificial atom and the light is called the ultra-strong coupling regime in circuit QED [DGS07, GAHSBLCTLH09, FLMGSHM10, NDHMHSGZHSMG10]. Then, experimental physicists have found some differences in physical phenomena between the two coupling regimes [FLMGSHM10, NDHMHSGZHSMG10]. As one of the striking differences, there is the following. In the strong coupling regime as well as in the weak coupling regime, the Jaynes-Cummings (JC) model is useful to explain the experimental results [HR01, FGBBLBW08]. The Hamiltonian of the JC model is obtained by applying the so-called rotating wave approximation (RWA) to the Rabi Hamiltonian. On the other hand, in the ultra-strong coupling regime, the JC model does not work, and thus, we need a help of the Rabi model [FLMGSHM10, NDHMHSGZHSMG10]. The current cutting-edge technology of circuit QED is beginning to show us the division between the two coupling regimes concretely. We are interested in how physics determines this division.

To see a difference between the Rabi model and the JC model, in this letter we pay particular attention to the energy level crossing. Recently, Braak [Bra11] had given a mathematically intriguing expressions of the eigenenergies of the Rabi model. Then, the following questions arise and are problems of interest to us: (i) are there any energy level crossings among them? If so, (ii) how do they take place? As is shown in [Hir09a, Hir09b], the ground state energy of the JC model consists of many energy level crossings as the coupling strength grows larger and larger. Namely, for the JC model the quantum phase transition in Rey’s sense [Rey09] takes place (see Fig.1 below). For the details on energy level crossing and quantum phase transition, see [Sach99]. On the other hand, as in our numerical computation in Fig.2 we can conjecture that the ground state energy of the Rabi model has no energy level crossing. In this letter we prove this fact with the functional-integral method [HL07, HHL12, HHL12] as a corollary of the fact stating that the ground state energy of the Rabi model is simple (i.e., the ground state is unique). It reveals us that it is in the ultra-strong coupling regime of circuit QED that there is a big qualitative difference as well as quantitative one between the Rabi model and the JC model.
2 Rabi model

2.1 Definition

Let $\sigma_x, \sigma_y, \sigma_z$ be the $2 \times 2$ Pauli matrices:

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (2.1)

In this letter we adopt the natural unit: $\hbar = 1$. The renormalized Hamiltonian of the Rabi model is defined as a self-adjoint operator by

\[
H_{\text{Rabi}} = \sigma_z \Delta + \omega a^\dagger a + g\sigma_x (a + a^\dagger)
\] (2.2)
on $\mathbb{C}^2 \otimes L^2(\mathbb{R})$. Here $\Delta > 0$ and $\omega > 0$ are respectively the atom transition frequency and the cavity resonance frequency, $g \in \mathbb{R}$ stands for a coupling constant, and $a$ and $a^\dagger$ denote the single mode boson annihilation and creation operators satisfying $[a, a^\dagger] = 1$. It is given by

\[
a = \left( \frac{1}{\sqrt{\omega}} \frac{d}{dx} + \sqrt{\omega} x \right)/\sqrt{2} \quad \text{and} \quad a^\dagger = \left( -\frac{1}{\sqrt{\omega}} \frac{d}{dx} + \sqrt{\omega} x \right)/\sqrt{2}.
\]

We are interested in studying spectral properties of eigenenergies of $H_{\text{Rabi}}$, in particular crossing of the ground state energy.

The absence of crossing can be derived from the simplicity of the ground state energy of $H_{\text{Rabi}}$. We will construct a path integral representation of $e^{-tH_{\text{Rabi}}}$ to show that the ground state energy is simple. This is a minor modification of recent papers [HL07, HIL12, HHL12], where the Feynman-Kac type formula with spin is established. In particular the spin-boson model is studied by path measure in [HHL12] and we can apply it in this paper since the Rabi model can be regarded as the single mode photon version of the spin-boson model.

2.2 Two conjectures

Let us here consider the Rabi Hamiltonian $H_R$ before the renormalization:

\[
H_R = H_{\text{Rabi}} + \frac{\omega}{2} = \sigma_z \Delta + \omega \left( a^\dagger a + \frac{1}{2} \right) + g\sigma_x (a + a^\dagger).
\]

In this letter we follow the classification proposed in [CRLGS10], and define the ultra-strong coupling regime by the region in which the dimensionless coupling strength $g/\omega > 0.1$.

Applying the RWA to $H_R$, we have the JC Hamiltonian:

\[
H_{\text{JC}} = \sigma_z \Delta + \omega \left( a^\dagger a + \frac{1}{2} \right) + g \left( \sigma_- a^\dagger + \sigma_+ a \right).
\]
where \(\sigma_\pm = (\sigma_z \pm i\sigma_y)/2\). We denote by \(E_{JC}\) the ground state energy of \(H_{JC}\). The JC model is a completely solvable model, and the eigenstate \(\varphi^{JC}_\nu(g)\) of \(H_{JC}\) and its corresponding eigenenergy \(E^{JC}_\nu(g)\) are given for each \(\nu \in \mathbb{Z}\) in the following procedure: Let \(\varphi_g(x) = (\omega/\pi)^{1/4}e^{i\omega x^2/2}\) be the normalized eigenvector associated with the lowest eigenvalue, 0, of the harmonic oscillator \(\omega a^\dagger a = \frac{1}{2}(\frac{d^2}{dx^2} + \omega^2 x^2 - \omega)\). Then Fock states are defined by \(|n\rangle = \frac{1}{\sqrt{n!}}(\prod a^\dagger)^n \varphi_g\) for the single mode photon, \(n = 0, 1, 2, \cdots\), with \(|0\rangle = \varphi_g\). For the spin ground state \(\nu\) and \(|+\rangle\) by \(|-\rangle \otimes |n\rangle\) and \(|+\rangle \otimes |n\rangle\), respectively, Then,

\[
\begin{align*}
\varphi^{JC}_0(g) &= |-, 0\rangle, \\
\varphi^{JC}_\nu(g) &= \cos(\theta_\nu(g))|+\rangle\nu - |\nu - 1\rangle + \sin(\theta_\nu(g))|-, 0\rangle, \quad \nu \neq 0, \\
\varphi^{JC}_{-\nu}(g) &= -\sin(\theta_\nu(g))|+\rangle\nu - |\nu - 1\rangle + \cos(\theta_\nu(g))|-, 0\rangle, \quad \nu \neq 0,
\end{align*}
\]

where \(\theta_\nu(g) = \frac{1}{2}\tan^{-1}\left(\frac{2g\sqrt{|\nu|}}{2\Delta - \omega}\right)\) if \(2\Delta \neq \omega\); \(\theta_\nu(g) = \pi/4\) if \(2\Delta = \omega\), and

\[
\begin{align*}
E^{JC}_0(g) &= -\frac{(2\Delta - \omega)}{2}, \\
E^{JC}_\pm(g) &= \omega|\nu| \pm \sqrt{(2\Delta - \omega)^2/4 + g^2|\nu|}, \quad \nu \neq 0.
\end{align*}
\]

According to \cite{Hir09a, Hir09b}, the remarkable finding for \(E_{JC}\) is the energy level crossings in the ultra-strong coupling regime: For each \(n = 0, 1, 2, \cdots\), there exists \(g_{n+1} > 0\) such that \(E^{JC}_{-n}(g)\) and \(E^{JC}_{-(n+1)}(g)\) cross each other at \(g = g_{n+1}\), and

\[
\begin{align*}
E_{JC} &= E^{JC}_{-n}(g) \quad & \text{if } g < g_{n+1}, \\
E_{JC} &= E^{JC}_{-n}(g) = E^{JC}_{-(n+1)}(g) \quad & \text{if } g = g_{n+1}, \\
E_{JC} &= E^{JC}_{-(n+1)}(g) \quad & \text{if } g > g_{n+1},
\end{align*}
\]

provided \(2\Delta \geq \omega\). See Fig.1. In other words, as \(g\) gets large, there exists \(\nu_g \in \mathbb{Z}\) such that \(E_{JC} = E^{JC}_{\nu_g}(g)\), and moreover, \(\nu_g\) is strictly decreasing and \(\nu_g \to -\infty\) as \(g \to \infty\). Namely, these energy level crossings take place and make the ground state energy \(E_{JC}\) as the envelop of \(E^{JC}_\nu(g)\), \(\nu = 0, -1, -2, \cdots\), in Fig.1. We also note that the ground state entanglement \cite{PZDS10} for the JC model. Namely, for instance, the ground state is a separable state for \(g < g_1\), but it becomes an entangled state for \(g \geq g_1\). The details on \(g_n\) and \(\nu_g\) are in \cite{Hir09a, Hir09b}.

In Fig.2 there is a numerical computation of the energy levels of \(H_R\). It says that
Figure 1: Energy level crossings among $E^{JC}_\nu(g)$, $\nu = 0, -1, -2, \cdots$, of the JC Hamiltonian. Each color indicates individual index $\nu$ of the energy $E^{JC}_\nu(g)$. (a) $2\Delta = \omega$; (b) $2\Delta = 3\omega$. 
(C1) there is no energy level crossing between the ground state energy and the 1st excited state energy;

(C2) we may say that there are $n$ energy level crossings between the $2n$-th excited state energy and the $(2n + 1)$-th excited state energy, $n = 1, 2, \cdots$.

In this letter, we will prove (C1).

3 Results and proofs

Before going to show the Feynman-Kac formula of $e^{-tH_{\text{Rabi}}}$, we prepare a probabilistic description of $H_{\text{Rabi}}$.

Let $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ be elements of $SU(2)$. The rotation group in $\mathbb{R}^3$ has an adjoint representation on $SU(2)$. Let $n \in \mathbb{R}^3$ be a unit vector and $\theta \in [0, 2\pi)$. Thus we have $e^{(i/2)\theta n \cdot \sigma}$ satisfies that $e^{(i/2)\theta n \cdot \sigma} \sigma_\mu e^{-(i/2)\theta n \cdot \sigma} = (R\sigma)_\mu$, where $R$ denotes $3 \times 3$ matrix representing the rotation around $n$ with angle $\theta$. In particular for $n = (0,1,0)$ and $\theta = \pi/2$, we have $e^{(i/2)\theta n \cdot \sigma_\mu} = \sigma_z$ and $e^{(i/2)\theta n \cdot \sigma_x} e^{- (i/2) \theta n \cdot \sigma} = -\sigma_x$. Set
\[ U = e^{(i\pi/4)\sigma_y} \]. Then
\[ U H_{\text{Rabi}} U^{-1} = \omega a^\dagger a + g\sigma_z (a + a^\dagger) - \sigma_x \Delta. \quad (3.1) \]

Since \( \varphi_g \) is strictly positive, we can define the unitary operator \( U_g : L^2(\mathbb{R}) \to L^2(\mathbb{R}, \varphi_g^2 dx) \) by \( U_g f = \varphi_g^{-1} f \). We set the probability measure \( \varphi_g^2 dx \) on \( \mathbb{R} \) by \( d\mu \). Thus \( U H_{\text{Rabi}} U^{-1} \) is transformed to the operator:
\[ U_g U H_{\text{Rabi}} U^{-1} U_g^{-1} = \frac{1}{2} \left( -\frac{d^2}{dx^2} + \omega x \frac{d}{dx} \right) + g\sigma_z \sqrt{2}\omega x - \sigma_x \Delta \quad (3.2) \]
in \( \mathbb{C}^2 \otimes L^2(\mathbb{R}, d\mu) \). Let us introduce \( Z_2 = \{-1, +1\} \) to redefine the Hamiltonian \( (3.2) \) on a set of scalar functions. We identify \( \mathbb{C}^2 \otimes L^2(\mathbb{R}, d\mu) \) with
\[ \mathcal{H} = L^2(\mathbb{R} \times Z_2, d\mu) = \left\{ f = f(x, \sigma) \left| \sum_{\sigma \in Z_2} \int |f(x, \sigma)|^2 d\mu(x) < \infty \right. \right\} \quad (3.3) \]
by \( \mathbb{C}^2 \otimes L^2(\mathbb{R}, d\mu) \ni \begin{bmatrix} f_+(x) \\ f_-(x) \end{bmatrix} \mapsto f(x, \sigma) \in \mathcal{H} \). Thus under this identification \( (3.2) \) is transformed to the operator \( H \):
\[ Hf(x, \sigma) = \frac{1}{2} \left( -\frac{d^2}{dx^2} + \omega x \frac{d}{dx} \right) + g\sqrt{2}\omega \sigma x \} \quad f(x, \sigma) - \Delta f(x, -\sigma), \quad \sigma \in Z_2 \quad (3.4) \]
in \( \mathcal{H} \). Thus we have the lemma below:

**Lemma 3.1** *The operator \( H_{\text{Rabi}} \) in \( \mathbb{C}^2 \otimes L^2(\mathbb{R}) \) is unitarily equivalent to \( H \) in \( \mathcal{H} \).*

In what follows we deal with \( H \) instead of \( H_{\text{Rabi}} \). Let
\[ h = \frac{1}{2} \left( -\frac{d^2}{dx^2} + \omega x \frac{d}{dx} \right) \]
and \( (X_t)_{t \geq 0} \) be the Ornstein-Uhrenbeck process on some probability space \( (C, \Sigma, P^x) \). We have \( P^x(X_0 = x) = 1 \)
\[
\int d\mu(x) \mathbb{E}_{P^x} [X_t] = 0,
\]
\[
\int d\mu(x) \mathbb{E}_{P^x} [X_t X_s] = \frac{e^{-|t-s|\omega}}{2\omega}.
\]
Here \( \mathbb{E}_Q [\cdots] \) denotes the expectation with respect to a probability measure \( Q \). The generator of \( X_t \) is given by \( -h \) and
\[
(f, e^{-th}g)_{\mathcal{H}} = \int d\mu(x) \mathbb{E}_{P^x} \left[ f(X_0)g(X_t) \right].
\]
The distribution $\rho_t(x, y)$ of $X_t$ under $P^x$ is given by

$$
\rho_t(x, y) = \varphi_h(x)^{-1} K_t(x, y) \varphi_h(y),
$$

where $K_t(x, y)$ denotes the Mehler kernel given by

$$
K_t(x, y) = \frac{1}{\sqrt{\pi(1 - e^{-2t})}} \exp\left(\frac{4xy e^{-t} - (x^2 + y^2)(1 + e^{-2t})}{2(1 - e^{-2t})}\right).
$$

See e.g., [LHB11, 3.10.4] for the detail of Ornstein-Uhlenbeck processes and harmonic oscillators. In order to show the spin part by a path measure we introduce a Poisson process. Let $(N_t)_{t \geq 0}$ be a Poisson process on some probability space $(C', \Sigma', \nu)$ with unit intensity, i.e.,

$$
\mathbb{E}_\nu [\mathbb{1}_{N_t = n}] = \frac{t^n}{n!} e^{-t}, \quad n \geq 0.
$$

We define $\sigma_t = (-1)^{N_t} \sigma, \sigma \in \mathbb{Z}_2$, for $t \geq 0$. Let

$$
\sum_{\sigma \in \mathbb{Z}_2} \int d\mu(x) \mathbb{E}_{P^x} \mathbb{E}_\nu \left[ \cdots \right] = \mathbb{E} \left[ \cdots \right].
$$

**Theorem 3.2** It follows that

$$(\Delta > 0) \quad (f, e^{-tH} g) \varphi = e^t \mathbb{E} \left[ \int f(X_0, \sigma_0) g(X_t, \sigma_t) e^{-g\sqrt{2\omega} \int_0^t \sigma_s X_s ds} \Delta N_t \right],
$$

$$(\Delta = 0) \quad (f, e^{-tH} g) \varphi = e^t \mathbb{E} \left[ \mathbb{1}_{N_t = 0} f(X_0, \sigma_0) g(X_t, \sigma) e^{-g\sqrt{2\omega} \int_0^t \sigma_s X_s ds} \right].
$$

**Proof:** Let $\Delta > 0$. By a minor modification of [HIL12] Theorem 5.10 we can see that

$$
(f, e^{-tH} g) \varphi = e^t \mathbb{E} \left[ \int f(X_0, \sigma_0) g(X_t, \sigma_t) e^{-g\sqrt{2\omega} \int_0^t \sigma_s X_s ds} e^{\int_0^t \log \Delta N_t} \right].
$$

Here $\int_0^t f(N_s) dN_t = \sum_{r, N_r \neq N_r} f(N_r)$. Since $e^{\int_0^t \log \Delta N_t} = e^{\int_0^t \log \Delta N_t} = \Delta N_t$, (3.7) follows. In the case of $\Delta = 0$ only the set $N_t = 0$ contributes to the path integral. Then

$$
(f, e^{-tH} g) \varphi = e^t \mathbb{E} \left[ \int f(X_0, \sigma_0) g(X_t, \sigma_t) e^{-g\sqrt{2\omega} \int_0^t \sigma_s X_s ds} \mathbb{1}_{N_t = 0} \right].
$$

Then (3.8) follows.

Let $E = \inf \sigma(H)$.

**Corollary 3.3** It follows that $\dim \ker (H - E) = 1$, i.e., the ground state of $H_{\text{Rabi}}$ is unique.
Proof: Let $f, g \geq 0$ but not identically zero. Then for sufficiently small $\epsilon > 0$, we see that both $\Omega_f = \{(x, \sigma) \in \mathbb{R} \times \mathbb{Z}_2 | f(x, \sigma) > \epsilon\}$ and $\Omega_g = \{(x, \sigma) \in \mathbb{R} \times \mathbb{Z}_2 | g(x, \sigma) > \epsilon\}$ have positive measures. We have by (3.7),

$$
(f, e^{-tH} g) \geq \epsilon e^t \mathbb{E} \left[ \mathbb{1}_{\Omega_f} (X_0, \sigma_0) \mathbb{1}_{\Omega_g} (X_t, \sigma_t) e^{-g \sqrt{2\omega} \int_0^t \sigma_s X_s ds} \Delta N_t \right].
$$

Since $\Omega_f$ is a subset of $\mathbb{R} \times \mathbb{Z}_2$, we have $\Omega_f = \bigcup_{\sigma \in \mathbb{Z}_2} (\Omega_{\sigma}^f, \sigma)$. Thus either $\Omega^+_f$ or $\Omega^-_f$ ($\subset \mathbb{R}$) have at least a positive measure. Suppose that $\Omega^+_f$ has a positive measure. Similarly we see that $\Omega^+_g = \bigcup_{\sigma \in \mathbb{Z}_2} (\Omega_{\sigma}^g, \sigma)$ and suppose that $\Omega^+_g$ is a positive measure. Let $\Omega$ be the set of paths starting from the inside of $(\Omega^+_f, +)$ and arriving at the inside of $(\Omega^+_g, +)$. We see that

$$
\mathbb{E} \left[ \mathbb{1}_\Omega \right] = \mathbb{E} \left[ \mathbb{1}_{\Omega^+_f} (X_0) \mathbb{1}_{\Omega^+_g} (X_t) \mathbb{1}_{N_t=\text{even}} \right].
$$

By using the distribution $\rho_t$ of $X_t$ we have

$$
\mathbb{E} \left[ \mathbb{1}_\Omega \right] = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} e^{-t} \int_{\Omega^+_f} dx \int_{\Omega^+_g} dy \varphi_0(x) K_t(x, y) \varphi_0(y) > 0.
$$

Hence we conclude that $\Omega$ has a positive measure and

$$
(f, e^{-tH} g) \geq \epsilon e^t \mathbb{E} \left[ \mathbb{1}_\Omega e^{-g \sqrt{2\omega} \int_0^t \sigma_s X_s ds} \Delta N_t \right] > 0.
$$

Thus $e^{-tH}$ is a positivity improving operator. Thus $\dim \ker (H - E) = 1$ follows from the Perron-Frobenius theorem.

$$\text{qed}$$

Corollary 3.4 The ground state energy of $H_{\text{Rabi}}$ has no crossing for all the values of $g$ and $\Delta$.

4 Conclusion

In this letter we have proved the first conjecture (C1) that the numerical computation predicts, while the JC model has many energy level crossings for the ground state energy in the ultra-strong coupling regime of circuit QED though it has no energy level crossing in the weak and strong coupling regimes. It shows that it is in the ultra-strong coupling regime that there is a big qualitative difference as well as quantitative one between the JC model and the Rabi model.
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