EXTREMAL ROOT PATHS OF SCHUR σ-GROUPS
AND FIRST 3-CLASS FIELD TOWERS WITH FOUR STAGES

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Abstract. An extremal property of finite Schur σ-groups $G$ is described in terms of their path to the root in the descendant tree of their abelianization $G/G'$. The phenomenon is illustrated and verified by all known examples of Galois groups $G = \text{Gal}(F_p^\infty(K)/K)$ of 3-class field towers $K = F_p^0(K) < F_p^1(K) < F_p^2(K) < \ldots \leq F_p^\infty(K)$ of imaginary quadratic number fields $K = \mathbb{Q}(\sqrt{d})$, $d < 0$, with elementary 3-group $\text{Cl}_3(K)$ of rank two. Such Galois groups must be Schur σ-groups and the existence of towers with at least four stages is justified by showing the non-existence of suitable Schur σ-groups $G$ with derived length $d(G) \leq 3$. By means of counter-examples, it is emphasized that real quadratic number fields with the same type of 3-class group reveal a totally different behavior, usually without extremal path.

1. Introduction

The statement of our Main Conjecture [4] in this paper requires a few preparatory sections on $p$-class field towers, relation ranks, transfer kernel types, and root paths in descendant trees.

1.1. Galois group $G$ of the $p$-class field tower. For an algebraic number field $K/\mathbb{Q}$, that is a finite extension of the rational number field $\mathbb{Q}$, and a prime number $p \in \mathbb{P}$, the $p$-class field tower $F_p^\infty(K)$ of $K$ is the maximal unramified pro-$p$-extension of $K$. The Galois group $G = \text{Gal}(F_p^\infty(K)/K)$ is a potentially infinite pro-$p$-group with finite abelianization $G/G'$ isomorphic to the $p$-class group $\text{Cl}_p(K)$ of $K$. The derived series of $G$, which is defined recursively by

\begin{equation}
G^{(0)} := G, \quad \text{and} \quad G^{(j)} := [G^{(j-1)}, G^{(j-1)}], \quad \text{for all} \quad j \geq 1,
\end{equation}

determines the various stages of the tower

\begin{equation}
K = F_p^0(K) \leq F_p^1(K) \leq F_p^2(K) \leq \ldots \leq F_p^\infty(K),
\end{equation}

where $F_p^j(K) = \text{Fix}(G^{(j)})$, that is $G^{(j)} = \text{Gal}(F_p^\infty(K)/F_p^j(K))$, for all $j \geq 0$, and

\begin{equation}
G/G^{(j)} = \text{Gal}(F_p^\infty(K)/K)/\text{Gal}(F_p^\infty(K)/F_p^j(K)) \simeq \text{Gal}(F_p^j(K)/K),
\end{equation}

in part. $G/G' = G^{(1)} \simeq \text{Gal}(F_p^1(K)/K) \simeq \text{Cl}_p(K)$, according to Artin’s reciprocity law [2][3].

1.2. Bounds for the relation rank of $G$. The relation rank $d_2(G) := \dim_{\mathbb{F}_p}(H^2(G, \mathbb{F}_p))$ of $G$ frequently admits a decision concerning the length $\ell_p(K)$ of the $p$-class field tower of $K$, based on the Shafarevich cohomology criterion [10]

\begin{equation}
d_1(G) \leq d_2(G) \leq d_3(G) + r + \theta
\end{equation}

in dependence on the generator rank $d_1(G) := \dim_{\mathbb{F}_p}(H^1(G, \mathbb{F}_p))$ of $G$, equal to the $p$-class rank $\text{rk}_p(K)$, the signature $(r_1, r_2)$, and the torsion free Dirichlet unit rank $r = r_1 + r_2 - 1$ of the number

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field $K$. See also [31 Thm. 5.1, p. 28], [35 §§ 1.2–1.3, pp. 75–76], [30]. Here, $\theta$ denotes the flag
\begin{equation}
\theta = \begin{cases} 
1, & \text{if } K \text{ contains the } p\text{-th roots of unity,} \\
0, & \text{otherwise.} 
\end{cases}
\end{equation}

In particular, an imaginary quadratic number field $K$ has signature $(r_1, r_2) = (0, 1)$ and Dirichlet unit rank $r = 0$. If $p$ is an odd prime, and $K$ has a non-cyclic $p$-class group with rank $\text{rk}_p(K) \geq 2$, then $\theta = 0$, and we have the following theorem, according to Koch and Venkov [21] or [1].

**Theorem 1.** The Galois group $G$ of the $p$-class field tower $F_p^\infty(K)$ of an imaginary quadratic number field $K$ with non-cyclic $p$-class group $\text{Cl}_p(K)$ for an odd prime number $p$ must be a so-called Schur $\alpha$-group, that is a pro-$p$-group $G$ with balanced presentation $d_2(G) = d_1(G)$, which possesses an automorphism $\sigma \in \text{Aut}(G)$ acting as inversion $x^\sigma = x^{-1}$ on the abelianization $G/G'$.

**Proof.** This is an immediate consequence of the Formulas (1.4) and (1.5) with $r = \theta = 0$, and the fact that the non-trivial automorphism $\tau \in \text{Gal}(K/\mathbb{Q})$ acts as inversion on $\text{Cl}_p(K) \simeq G/G'$ (since $1 + \tau = \text{Norm}_{K/\mathbb{Q}}$ and $\text{Cl}_p(\mathbb{Q}) = 1$), and has an extension to $\sigma \in \text{Aut}(G)$ with $\sigma|_{G/G'} = \tau$. \hfill \square

1.3. Transfer kernel type of $G$. In the present paper, our focus will be on the smallest odd prime number $p = 3$ and imaginary quadratic base fields $K = \mathbb{Q}(\sqrt{d})$, $d < 0$, with elementary 3-class group $\text{Cl}_3(K) \simeq C_3 \times C_3$ of rank two. The 3-class field tower of such fields has been investigated for the first time by Arnold Scholz and Olga Taussky in 1934 [15]. These authors coined the concept of 3-capitulation type $\kappa(K)$ of $K$, defined by the transfer homomorphisms $T_i : \text{Cl}_3(K) \rightarrow \text{Cl}_3(E_i)$ of 3-classes from $K$ to its four unramified cyclic cubic extensions $E_i$, $1 \leq i \leq 4$, within the Hilbert 3-class field $F_3^1(K)$:
\begin{equation}
\kappa(K) := (\kappa(1), \ldots, \kappa(4)), \quad \text{where } \ker(T_i) = N_{\kappa(i)}.
\end{equation}
Here, $N_i$ denotes the norm class group $\text{Norm}_{E_i/K}(\text{Cl}_3(E_i))$, for each $1 \leq i \leq 4$. According to the Theorem on Herbrand’s quotient $\frac{\#H^1(A, U_i)}{\#H^0(A, U_i)} \equiv [E_i : K] = 3$ of the unit group $U_i := U(E_i)$ as a Galois module over the automorphism group $A := \text{Gal}(E_i/K) \simeq C_3$, all the kernels $\ker(T_i)$ are cyclic of order 3, if $K$ is imaginary quadratic (and thus $\frac{\#H^0(A, U_i)}{\#H^0(A, U_i)}$). Scholz and Taussky arranged the essential 19 types of 3-capitulation (we also speak about the transfer kernel type, TKT) in several sections denoted by upper case letters [15, pp. 34–37]. The designation of individual types by numbers from 1 to 19 is taken from [23, p. 80]:

**Definition 2.** Representatives for the 3-capitulation types are arranged in the following sections:

- Section A: type A.1 $\kappa = (1111)$,
- Section B: types B.2 $\kappa = (1211)$, B.3 $\kappa = (1112)$,
- Section C: types C.15 $\kappa = (1234)$, C.17 $\kappa = (1342)$, C.18 $\kappa = (2341)$,
- Section D: types D.5 $\kappa = (1212)$, D.10 $\kappa = (1123)$,
- Section E: types E.6 $\kappa = (1122)$, E.8 $\kappa = (1231)$, E.9 $\kappa = (1321)$, E.14 $\kappa = (2311)$,
- Section F: types F.7 $\kappa = (2112)$, F.11 $\kappa = (1321)$, F.12 $\kappa = (2131)$, F.13 $\kappa = (2113)$,
- Section G: types G.16 $\kappa = (1243)$, G.19 $\kappa = (2143)$,
- Section H: type H.4 $\kappa = (2111)$.

Fixed point capitulation $\ker(T_i) = N_i$, i.e. $\kappa(i) = i$, is always marked by using boldface font.

Every other 3-capitulation type is equivalent to some of the representatives in Dfn. 2 [23].

**Proposition 3.** The types in sections B and C are generally forbidden for any number field $K$. Type A.1 enforces the extra special 3-group $(27,4)$ of order 27 and exponent 9 as Galois group $G$ and is forbidden for any quadratic field (imaginary and real). The types in all the other sections give rise to Galois groups $G$ which share the extra special 3-group $(27, 3)$ of order 27 and exponent 3 as their common class-2 quotient $G/\gamma_3(G)$. (See [15]. Here, the groups are denoted by their identifiers in the SmallGroups Library [10], and $\gamma_3(G)$ is the third term in Formula (1.7).)
1.4. Root path of $G$. Let $p$ be a prime number. A finite non-abelian $p$-group $G$ gives rise to a root path in the descendant tree of its abelianization $G/G'$. The vertices $V$ of this directed tree are isomorphism classes of finite non-abelian $p$-groups sharing a common abelianization. Two vertices $V$ and $W$ are connected by a directed edge $V \to W$, if $W$ is isomorphic to the quotient of $V$ by the last non-trivial member $\gamma_c(V)$ of its lower central series (or descending central series),

$$\gamma_1(V) = V > \gamma_2(V) = V' > \gamma_3(V) > \ldots > \gamma_c(V) > \gamma_{c+1}(V) = 1,$$

which is defined recursively by

$$\gamma_1(V) := V, \quad \text{and} \quad \gamma_j(V) := [\gamma_{j-1}(V), V], \quad \text{for all} \quad j \geq 2.$$

and becomes trivial for $j > c$ bigger than the nilpotency class of $V$. In this case, $W = \pi(V)$ is called the parent of $V$ and $V$ is called an immediate descendant of $W$. The construction of parents can be iterated, and the root path of $V$ (in the sequel without the last edge) is given by

$$V \to \pi(V) \simeq V/\gamma_c(V) \to \pi^2(V) \simeq V/\gamma_{c-1}(V) \to \ldots \to \pi^{c-1}(V) \simeq V/\gamma_2(V) = V/V'.$$

The nuclear rank $\nu(W)$ of the parent determines the possible step sizes $1 \leq s \leq \nu(W)$ of the edge $V \to W$. The increment of the logarithmic order, nilpotency class and coclass is given by $\log(V) - \log(W) = s, \ cl(V) - cl(W) = 1$ and $cc(V) - cc(W) = s - 1$. Therefore, in a coclass tree (with constant coclass), the step size of all edges is restricted to the minimal value $s = 1$.

We are now in the position to state the Main Conjecture 3 of this paper.

**Conjecture 4. (Extremal root path of $G$)** In the descendant tree $T(C_3 \times C_3)$ of the abelian 3-group $C_3 \times C_3$, the root path $(\pi^{i-1}(G) \to \pi^i(G))_{1 \leq i \leq c-2}$ of the Galois group $G = \text{Gal}(F_3^c(K)/K)$ of the 3-class field tower $F_3^c(K)$ of an imaginary quadratic field $K$ with elementary 3-class group $\text{Cl}_3(K) \simeq C_3 \times C_3$ of rank two consists of edges with maximal step size (=nuclear rank), i.e.

$$\pi^{i-1}(G) \to \pi^i(G) \text{ is an edge of step size } s_i = \nu(\pi^i(G)), \text{ for each } 1 \leq i \leq c-2,$$

where $3 \leq c = \text{cl}(G) < \infty$ denotes the nilpotency class of $G$.

1.5. Layout of this paper. The Main Conjecture 3 has never been violated by any known situation involving Schur $\sigma$-groups. It is our desire to underpin the conjecture with infinite series of parametrized Schur $\sigma$-groups, having proven extremal path property.

At the beginning, in § 2.1 we treat the finitely many metabelian Schur $\sigma$-groups. They have TKTs in Section D.

An infinitude of Schur $\sigma$-groups, arising as quotients of an infinite limit group, was discovered by Bartholdi and Bush §. They are investigated in the present context in § 2.2. The members have TKT II.4 and unbounded derived length.

Other infinite sequences of Schur $\sigma$-groups, arising as quotients of infinite limit groups, were constructed by Newman and ourselves in § 3. Their root paths are checked for the extremal property in § 2.3. All their TKTs are contained in Section E and they share the common derived length 3. Counts of Schur $\sigma$-groups up to order $3^{14}$ are given in § 2.4.

Root paths with extreme complexity are presented in § 2.5. They occur for Schur $\sigma$-groups with TKTs in Section F. In immediate prosecution, the striking novelty of the first proven four-stage towers of 3-class fields over imaginary quadratic fields is heralded as the Main Theorem § and the beginning of a new era of research on maximal unramified pro-$p$-extensions in § 2.6.

In § 2.7 we supplement the exposition with further root paths, not covered by the preceding developments. Counter-examples with real quadratic base fields are provided in § 3.

A summary is given in § 4 and personal historical remarks illuminate the arduous long and winding road towards our present elevated perspective of finite $p$-class field towers in §§ 5 and 7.

2. Proving path extremality of $G$ for infinite series of Schur $\sigma$-groups

Throughout this paper we identify isomorphism classes of finite $p$-groups with the aid of their representative in the SmallGroups Database §, which is implemented in the computer algebra systems GAP § and Magma §. Identifiers (Id) have the format (order, counter).
2.1. Sporadic metabelian Schur $\sigma$-groups $G$. ($dl(G) = 2$)

**Proposition 5.** Among the finite 3-groups $V$ with abelian quotient invariants $V/V' \simeq (3, 3)$ there exist **precisely two metabelian** Schur $\sigma$-groups. Both are of order $ord = 3^5 = 243$ nilpotency class $cl = 3$ and coclass $cc = 2$. The first is $(243, 5)$ with TKT D.10, and the second is $(243, 7)$ with TKT D.5. They are the unique terminal immediate descendants with step size $s = 2$ of the extra special group $(27, 3)$, and thus isolated orphans without parents in the coclass forest $F(2)$.

**Proof.** A search for metabelian Schur groups with balanced presentation $d_2 = d_1 = 2$ additionally yields the extra special group $(27, 4)$ with TKT A.1, but this is not a $\sigma$-group with generator inverting automorphism.

**Remark 6.** According to Nebelung [44, 45], there do not exist other finite metabelian 3-groups $V$ with abelian quotient invariants $V/V' \simeq (3, 3)$ and TKT in Section A or Section D. Since an epimorphism onto a Schur group must be an isomorphism, there cannot exist non-metabelian 3-groups $V$ with abelian quotient invariants $V/V' \simeq (3, 3)$ and TKT in one of these two Sections. Sections A, B, C and D are the only sections with a finite number of associated Schur $\sigma$-groups.

**Theorem 7.** The 3-class field tower of an imaginary quadratic field $K$ with 3-capitulation type in section $D$ is metabelian with length $\ell_3(K) = 2$. The Galois group $G = Gal(F_3^3(K)/K)$ satisfies the extremal path property with $s_1 = 2$.

**Proof.** The length was proven by Scholz and Taussky in [45]. The extremal property is a consequence of Table 1 and a similar table for type D.5 with $(243, 5)$ replaced by $(243, 7)$.

| Ancestor | $\pi(G)$ | $(\nu, \mu)$ | $(N_s/C_s)_{1 \leq s \leq \nu}$ | TKT |
|-----------|---------|--------------|-------------------------------|-----|
| $\nu(G)$  | (27, 3) | (2, 4)        | (4/1, 7/5)                    | a.1 |
| $\mu$     | (243, 5)| (0, 2)       | (3/3)                         | D.10|

Table 1. Root path of $G$ for transfer kernel type D.10

Table 2 shows the statistical distribution of the simplest 3-capitulation types in three ranges of discriminants $d$. The range $-10^6 < d < 0$ was investigated by ourselves in [24, 26]. In [33] we extended to $-10^7 < d < 0$, and the range $-10^8 < d < 0$ is due to Boston, Bush and Hajir [13]. Without doubt, the metabelian Schur $\sigma$-group $(243, 5)$ with TKT D.10, resp. all four cases (see Fig. 1), enjoy the most dense population, covering nearly one third, resp. two thirds, of all cases. Here and in the sequel, we additionally need the second component of the Artin pattern $AP(G) = (\varkappa(G), \tau(G))$. The transfer target type (TTT) $\tau(G)$ consists of the logarithmic abelian type invariants of the four 3-class groups $Cl_3(E_i), 1 \leq i \leq 4$, corresponding to the TKT $\varkappa(G)$, and $\varepsilon$ denotes the number of components of $\tau(G)$ isomorphic to $1^3 \simeq C_3 \times C_3 \times C_3$. (Similarly, $21 \simeq C_9 \times C_3$.)

| Discriminant | Total# | $\varepsilon = 1$ | $\varepsilon = 2$ | $\varepsilon = 3$ | $\varepsilon = 0$ | $\Sigma\%$ |
|--------------|--------|------------------|------------------|------------------|------------------|----------|
| $-10^6 < d < 0$ | 2020 | 33.9% | 13.3% | 14.7% | 4.7% | 65.7% |
| $-10^7 < d < 0$ | 2476 | 31.14% | 14.81% | 14.79% | 4.163% | 64.903% |
| $-10^8 < d < 0$ | 276375 | 30.15% | 14.979% | 14.823% | 3.7724% | 63.7334% |

Table 2. Smallest $G/G^{(2)}$ for imaginary quadratic fields of type $(3, 3)$. 

| $\tau(G)$ | $\varkappa(G)$ | TKT | $G/G^{(2)}$ | $\Sigma\%$ |
|------------|----------------|-----|-------------|----------|
| $(1^3, 21^3)$ | $(1^3)^2, (21)^2$ | D.10 | $(243, 5)$ | $729, 45$ |
| $\pi(G)$ | $(2241)$ | D.5 | $(243, 7)$ | $729, 57$ |
| $\nu(G)$ | (4224) | H.4 | $\langle 729, 45 \rangle$ | $729, 57$ |
Figure 1. Extremal paths to Schur $\sigma$-groups $\langle 5 \rangle$, $\langle 7 \rangle$, log ord 5, in coclass forest $\mathcal{F}(2)$
2.2. Infinite series of Schur $\sigma$-groups $G$ with unbounded derived length. ($\text{dl}(G) \geq 3$)

**Proposition 8.** Among the finite 3-groups $V$ with abelian quotient invariants $V/V' \simeq (3,3)$, transfer kernel type $\text{TKT} \, H.4$ and transfer target type $\tau(V) = ((1^3)^3,21)$, also uniquely characterized by the invariant $\varepsilon = 3$, there exists an infinite sequence of Schur $\sigma$-groups $(S_n)_{n \geq 0}$ with unbounded derived length $\text{dl}(S_n) \geq 3$. (The law for the dependence of $\text{dl}(S_n)$ on $n$ is given in [8].) Their logarithmic order, nilpotency class and coclass are given by the following laws:

\[(2.1) \quad \text{lo}(S_n) = 8 + 3n, \quad \text{cl}(S_n) = 5 + 2n, \quad \text{cc}(S_n) = 3 + n, \text{ for all } n \geq 0.\]

**Proof.** The existence and the unbounded derived length of the infinite sequence was proved by Bartholdi and Bush [8]. The deterministic laws for invariants were deduced by ourselves [29] from the structure of the purged descendant tree of $R := \langle 243, 4 \rangle$, restricted to $\sigma$-groups with generator inverting automorphism. The tree is not a coclass tree, but it has an infinite main trunk with strictly alternating step sizes $s = 1$ and $s = 2$, and thus contains periodic bifurcations to higher coclass. Since $S_n = R(-\#1; a_j - \#2; b_j)_{1 \leq j \leq n+1}$ with certain $1 \leq a_j \leq 4$, $1 \leq b_j \leq 2$, for each $n \geq 0$, it follows that $\text{lo}(S_n) = \text{lo}(R) + 3(n+1) = 5 + 3n + 3 = 8 + 3n, \text{cl}(S_n) = \text{cl}(R) + 2(n+1) = 3 + 2n + 2 = 5 + 2n, \text{and } \text{cc}(S_n) = \text{cc}(R) + (n+1) = 2 + n + 1 = 3 + n. \quad \square$

**Theorem 9.** The 3-class field tower of an imaginary quadratic field with TKT $H.4$ and TTT $\tau = ((1^3)^3,21)$ is non-metabelian with unbounded length $\ell_3(K) \geq 3$. In the simplest case, the Galois group $G$ satisfies the extremal property with $s_1 = 2$, $s_2 = 1$ and $s_3 = 2$. Generally, the extremal path property of all cases is satisfied with strictly alternating step sizes $s = 2$ and $s = 1$.

**Proof.** The unbounded length was proved by Bartholdi and Bush in [8]. The extremal property for all cases was proved by ourselves in [29] [37]. For the simplest case, see Table [3]. \quad \square

| Ancestor | $\text{Id}$ | $\nu, \mu$ | $(N_s/C_s)_{1 \leq s \leq \nu}$ | TKT |
|-----------|-------------|-------------|-------------------------------|-----|
| $\pi^3(G)$ | $(27,3)$    | $(2,4)$     | $(4/1,7/5)$                   | a.1 |
| $\pi^2(G)$ | $(243, 4)$  | $(1,3)$     | $(4/4)$                        | H.4 |
| $\pi(G)$   | $(729, 45)$ | $(2,4)$     | $(4/0, 2/1)$                   | H.4 |
| $G$        | $(6561, 606)$ | $(0,2)$     |                               | H.4 |

**Example 10.** The root path of the smallest Schur $\sigma$-group with TKT $H.4$ and TTT $\tau = ((1^3)^3,21)$ is illustrated with red color in Figure 2. The group is denoted by $S_0 = \langle 6561, 606 \rangle$. In [29] [37] it was proved to be the Galois group $G = \text{Gal}(F_3^\infty(K)/K)$ of the three-stage 3-class field tower of the quadratic fields $\mathbb{Q}(\sqrt{d})$ with discriminants $d \in \{-3896, -25447, -27355\}$.

The root path of the next Schur $\sigma$-group $S_1 = \langle 6561, 605 \rangle - \#1; 2 - \#2; 2$ with order $3^{11} = 177147$ is drawn with red color in Figure 3. In [29] [37] it was proved that the Galois group $G = \text{Gal}(F_3^\infty(K)/K)$ for the quadratic fields $\mathbb{Q}(\sqrt{d})$ with discriminants $d \in \{-6583, -23428, -27991\}$ is certainly not isomorphic to $S_0$. It is currently beyond the reach of actual computations to decide whether $G \simeq S_i$ with $1 \leq i \leq 2$ and $\text{dl}(G) = 3$ or $G \simeq S_i$ with $i \geq 3$ and $\text{dl}(G) \geq 4$.

The root path of the Schur $\sigma$-groups $S_2$, resp. $S_3$, is shown in Figure 4 resp. 5. In all figures, the tree is *purged* in the sense that it is restricted to $\sigma$-groups $V$ with generator inverting automorphism $\sigma \in \text{Aut}(V)$. 

Figure 2. Extremal path to Schur $\sigma$-group $S_0$, log ord 8, on purged tree $T_*(\langle 243, 4 \rangle)$
Figure 3. Extremal path to Schur $\sigma$-group $S_1$, log ord 11, on purged tree $T_4((243,4))$. 

Order

243 $3^5$
729 $3^6$
2187 $3^7$
6561 $3^8$
19683 $3^9$
59049 $3^{10}$
177147 $3^{11}$
531441 $3^{12}$
1594323 $3^{13}$
4782969 $3^{14}$
14348907 $3^{15}$
43046721 $3^{16}$
129140163 $3^{17}$

$\sigma_4 = 1$
$\sigma_3 = 2$
$\sigma_2 = 1$
$\sigma_1 = 2$

metabelianization

$dl(S_0) = 3$
$dl(S_1) = 3$
$dl(S_2) = 3$
$dl(S_3) = 3$

$\langle 243 \rangle$
$\langle 270 \rangle$
$\langle 271 \rangle$
$\langle 272 \rangle$
$\langle 273 \rangle$
$\langle 274 \rangle$
$\langle 275 \rangle$
$\langle 276 \rangle$
$\langle 277 \rangle$
$\langle 278 \rangle$
$\langle 279 \rangle$
$\langle 280 \rangle$

$T_{0,1}$ $T_{0,2}$ $T_{0,3}$ $T_{0,4}$
$T_{1,1}$ $T_{1,2}$ $T_{1,3}$ $T_{1,4}$

$S_0$
$S_1$
$S_2$
$S_3$

1; 2 (not coclass-settled)
1; 2 (not coclass-settled)
1; 2 (not coclass-settled)
1; 2 (not coclass-settled)
Figure 4. Extremal path to Schur $\sigma$-group $S_2$, log ord 14, on purged tree $T_4((243,4))$
Figure 5. Extremal path to Schur $\sigma$-group $S_3$, log ord 17, on purged tree $T_4((243,4))$
2.3. Infinite series of Schur $\sigma$-groups $G$ with derived length three. ($\text{dl}(G) = 3$)

**Proposition 11.** Among the finite 3-groups $V$ with abelian quotient invariants $V/V' \simeq (3,3)$ and transfer kernel type in Section E, there exist precisely six infinite sequences of Schur $\sigma$-groups $\langle S_n^{(t)} \rangle_{n \geq 0}$, $1 \leq t \leq 6$, sharing common derived length $\text{dl}(S_n^{(t)}) = 3$. Their logarithmic order, nilpotency class and coclass are given by the following laws:

$$\text{lo}(S_n^{(t)}) = 8 + 3n, \quad \text{cl}(S_n^{(t)}) = 5 + 2n, \quad \text{cc}(S_n^{(t)}) = 3 + n, \text{ for all } n \geq 0.$$  

The TKT is E.6 for $t = 1$, E.14 for $t = 2, 3$, E.8 for $t = 4$, and E.9 for $t = 5, 6$.

**Proof.** The existence and the constant derived length of the infinite sequences was proved by Newman and ourselves [40]. In fact, $\text{dl} = 3$ is due to the inclusion of the (non-trivial) second derived subgroup in the centre. The deterministic laws for invariants can be deduced in the same way as in the proof of Proposition 8, replacing the root $(243,4)$ of the purged descendant tree by either $R := \langle 243,6 \rangle$ or $R := \langle 243,8 \rangle$. As before, the tree is not a coclass tree, but it has an infinite main trunk with strictly alternating step sizes $s = 1$ and $s = 2$, and thus contains periodic bifurcations to higher coclass [28, 30]. Since $S_n^{(t)} = R(-\#1; a_j - \#2; b_j)_{1 \leq j \leq n+1}$ with certain $1 \leq a_j \leq 4$, $1 \leq b_j \leq 6$, for each $n \geq 0$, we arrive at the same laws as in Proposition 8.

**Theorem 12.** The 3-class field tower of an imaginary quadratic field with 3-capitulation type in section E is non-metabelian with precise length $\ell_3(K) = 3$. The Galois group $G$ of the ground state satisfies the extremal property with $s_1 = 2$, $s_2 = 1$ and $s_3 = 2$. Generally, the extremal path property of all excited states is satisfied with strictly alternating step sizes $s = 2$ and $s = 1$.

**Proof.** The length was proved for the ground state by Boston, Bush and ourselves in [16], and for all excited states by Newman and ourselves in [40]. The extremal property is a consequence of Tables 4 and 5, and analogous tables for the remaining types in section E; for type E.14 with $\langle 6561, 616 \rangle$ replaced by $\langle 6561, 617|618 \rangle$, resp. for type E.9 with $\langle 6561, 622 \rangle$ replaced by $\langle 6561, 620|624 \rangle$. The extremal property for excited states was proved by ourselves in [28, 30].

**Table 4.** Root path of $G$ for the ground state of transfer kernel type E.6

| Ancestor | Id   | $(\nu, \mu)$ | $(N_s/C_s)_{1 \leq s \leq \nu}$ | TKT  |
|----------|------|--------------|---------------------------------|------|
| $\pi^1(G)$ | $\langle 27,3 \rangle$ | $(2,4)$ | $(4/1,7/5)$ | a.1 |
| $\pi^2(G)$ | $\langle 243,6 \rangle$ | $(1,3)$ | $(4/4)$ | c.18 |
| $\pi(G)$ | $\langle 729,49 \rangle$ | $(2,4)$ | $(8/3,6/3)$ | c.18 |
| $G$ | $\langle 6561,616 \rangle$ | $(0,2)$ | | E.6 |

**Table 5.** Root path of $G$ for the ground state of transfer kernel type E.8

| Ancestor | Id   | $(\nu, \mu)$ | $(N_s/C_s)_{1 \leq s \leq \nu}$ | TKT  |
|----------|------|--------------|---------------------------------|------|
| $\pi^1(G)$ | $\langle 27,3 \rangle$ | $(2,4)$ | $(4/1,7/5)$ | a.1 |
| $\pi^2(G)$ | $\langle 243,8 \rangle$ | $(1,3)$ | $(4/4)$ | c.21 |
| $\pi(G)$ | $\langle 729,54 \rangle$ | $(2,4)$ | $(8/3,6/3)$ | c.21 |
| $G$ | $\langle 6561,622 \rangle$ | $(0,2)$ | | E.8 |

Root path of Schur $\sigma$-groups $S_0^{(t)}$, resp. $S_1^{(t)}$, $S_2^{(t)}$, $4 \leq t \leq 6$, is shown in Figure 4 resp. 7.
Figure 6. Extremal paths to Schur $\sigma$-groups, log ord 8, on purged tree $\mathcal{T}_n((243, 8))$

Symmetric topology symbol (ground state):

Transfer kernel types:

$E.8: \sigma_3 = (1231), \ c.21: \sigma_6 = (0231)$

Minimal discriminant TKT $E.8$:

$-34867$

TKT:

$\kappa_1 \kappa_2 \kappa_3 \kappa_0 \kappa_1 \kappa_2 \kappa_3 \kappa_0 \kappa_1 \kappa_2 \kappa_3 \kappa_0 \kappa_1 \kappa_2 \kappa_3 \kappa_0$
Figure 7. Extremal paths to Schur σ-groups, log ord 11, on purged tree $\mathcal{T}_o((243, 8))$

Order

243 $3^5$
2187 $3^7$
729 $3^6$
6561 $3^8$
19,683 $3^9$
59,049 $3^{10}$
177,147 $3^{11}$
531,441 $3^{12}$
1,594,323 $3^{13}$
4,782,969 $3^{14}$

Symmetric topology symbol (1st excited state):

Transf. kernel types:
E.8: $\kappa_3 = (1231)$, c.21: $\kappa_0 = (0231)$

Minimal discriminant TKT E.8:
$-370,740$

TKT:

| $\kappa_1$ | $\kappa_2$ | $\kappa_3$ | $\kappa_0$ | $\kappa_1$ | $\kappa_2$ | $\kappa_3$ | $\kappa_0$ | $\kappa_1$ | $\kappa_2$ | $\kappa_3$ | $\kappa_0$ | $\kappa_1$ | $\kappa_2$ | $\kappa_3$ | $\kappa_0$ |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
|           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
Figure 8. Extremal paths to Schur $\sigma$-groups, log ord 14, on purged tree $T_\ast((243, 8))$

Symmetric topology symbol (2nd excited state):

Transfer kernel types:
E.8: $\kappa_3 = (1231)$, c.21: $\kappa_0 = (0231)$

Minimal discriminant TKT E.8: $-4 087 295$
2.4. Challenges for finding non-metabelian Schur $\sigma$-groups. Why did it take mathematicians so long to discover non-metabelian Schur $\sigma$-groups? In §2.1 we saw that there exist precisely two metabelian Schur $\sigma$-groups $V$ with abelian quotient invariants $V/V' \simeq (3,3)$. They are of logarithmic order $\log(V) = 5$, nilpotency class $\text{cl}(V) = 3$, coclass $cc(V) = 2$, and have transfer kernel types in Section D. These two groups were known to G. Bagnera [7] in 1898 already, and they were rediscovered by means of computer aided investigations in the Ph.D. thesis of J. A. Ascione [1, 5, 6] in 1979 and in the doctoral dissertation of B. Nebelung [44, 45] in 1989. The reason for these discoveries was Bagnera’s search for all $p$-groups of order $p^n$, Ascione’s investigation of 3-groups with coclass two and order up to $3^n$, and Nebelung’s complete classification of all metabelian 3-groups with abelianization $(3,3)$. None of these authors was aware of the balanced presentations.

In §§2.2 and 2.3 we enumerated all non-metabelian Schur $\sigma$-groups $V$ with abelian quotient invariants $V/V' \simeq (3,3)$ and transfer kernel type either in Section E or H.4 under the additional requirement of transfer target type $\tau(V) = ((1^3)^3, 21)$. We saw that they exist for every logarithmic order $\log(V) \equiv 2 \pmod{3}$ setting in with log ord 8, and they arise from periodic bifurcations in the (pruned) descendant trees $T_i((2729, i))$ with either $i \in \{49, 54\}$ or $i = 45$. Since the first bifurcation already causes a transition from coclass two to coclass three, it is clear that non-metabelian Schur $\sigma$-groups were outside of the scope of all three above mentioned authors.

Due to the genesis of Schur $\sigma$-groups $V$ by bifurcations, they never belong to coclass trees, they are rather isolated orphans without proper parents in the sporadic part of coclass forests. They always have nuclear rank $\nu(V) = 0$ and consequently no descendants.

The following theorem was known to Boston, Bush, Hajir [13] in 2012 already, at least partially up to log ord 11.

**Theorem 13.** The exact counts of Schur $\sigma$-groups $V$ of order ord($V$) a power of 3 with abelian quotient invariants $V/V' \simeq (3,3)$ and logarithmic order $\log(V) \in \{5, 8, 11, 14\}$ are given in the following way, classified by transfer kernel types (TKT):

1. There are 2 groups of order $3^5 = 243$:
   one of TKT D.10, (243, 5), and one of TKT D.5, (243, 7) (Fig. 7).

2. There are 7 groups of order $3^8 = 6561$:
   one of TKT H.4, (6561, 606) (Fig. 8).
   one of TKT E.6, (6561, 616), two of TKT E.14, (6561, i) with $i \in \{617, 618\}$,
   one of TKT E.8, (6561, 622), and two of TKT E.9, (6561, i) with $i \in \{620, 624\}$ (Fig. 9).

3. There are 15 groups of order $3^{11} = 177147$: (Fig. 6)
   three of TKT H.4, (6561, 605) − #1; 2 − #2; 2, (6561, i) − #1; 2 − #2; 2 with $i \in \{614, 615\}$,
   one of TKT E.6, (6561, 613) − #1; 1 − #2; 4,
   two of TKT E.14, (6561, 613) − #1; 1 − #2; j with $j \in \{5, 6\}$,
   one of TKT E.8, (6561, 621) − #1; 1 − #2; 2,
   two of TKT E.9, (6561, 621) − #1; 1 − #2; j with $j \in \{4, 6\}$,
   two of TKT G.16, (6561, i) − #1; 4 − #2; 1 with $i \in \{619, 623\}$,
   and four of TKT G.19, (6561, i) − #1; 2 − #2; j with $i \in \{625, 629\}$, $j \in \{1, 2\}$.

4. There are 23 groups of order $3^{14} = 4782969$: three of TKT H.4,
   one of TKT E.6, two of TKT E.14, one of TKT E.8, two of TKT E.9,
   two of TKT G.16 and twelve of TKT G.19 (Fig. 4, 8, 12).

**Proof.** Item (1) was proved in Proposition 5, Item (2), and parts of items (3) and (4), were proved in Propositions 8 and 11. The remainder of items (3) and (4) will be proved in Proposition 20. □

Currently, the number of Schur $\sigma$-groups $V$ with order ord($V$) $= 3^{17}$ and abelian quotient invariants $V/V' \simeq (3,3)$ seems to be unknown. In Theorem 14 however, we shall see that the number of Schur $\sigma$-groups $V$ with order ord($V$) $= 3^{20}$ and merely types in Section F becomes astronomic, compared to the numbers in Theorem 13.
2.5. **Schur σ-groups with TKT in Section F.** Now we come to Schur σ-groups with root paths of high complexity and orders bigger than $3^8 = 5561$, outside of the SmallGroups Library. In this case, it is convenient to characterize descendents of a parent $P$ by the step size $s$ and a counter $c$ in the form $P = \#s; c$, which is used by the ANUPQ package \[17\], implemented in GAP \[18\] and Magma \[22\]. In Table 8 we present a complete root path containing edges with step size $s = 4$, and in Table 9 we communicate a rudimentary root path with edges up to step size $s = 8$, which would continue with an edge of step size $s = 12$ if this were within the reach of actual computations. (Of course, it isn’t.) In both tables, lo (or log ord) denotes the logarithmic order.

We must illuminate the origin of such complicated root paths with inspirations which came from algebraic number theory. Therefore, we start by gathering arithmetical information.

In Table 6 and 7 we present arithmetical information about iterated index-$p$ abelianization data (IPADs), $\tau^{(2)} K = [Cl_3 K; (Cl_3 E_i; (Cl_3 E'_i)_{1 \leq i \leq 4}]]$, of second order for imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with 3-class group $Cl_3 K$ of type $1^2 \neq (3, 3)$, TKT $F$, and a second 3-class group $G_3^2 K$ of coclass 4, which occur in the range $5 \cdot 10^5 < d < 0$ of fundamental discriminants. The negative discriminants were taken from the lower half range of the SmallGroups Library. In Table 8, we present a complete root path containing edges with step size $s = 12$ if this were within the reach of actual computations. (Of course, it isn’t.) In both tables, lo (or log ord) denotes the logarithmic order.

We must illuminate the origin of such complicated root paths with inspirations which came from algebraic number theory. Therefore, we start by gathering arithmetical information.

In Table 6 and 7 we uniformly have metabelianizations with coclass $\text{cc}(G_3^2 K) = 4$.

**Table 6. Moderate ATI of second order for imaginary $K = \mathbb{Q}(\sqrt{d})$**

| Type | $-d$ | $T_1$ | $T_2$ | $T_3$ | $T_4$ |
|------|------|-------|-------|-------|-------|
| F.7  | 225 299 | (31)^3 | (31)^3 | (21)^3, (21)^9 | (21)^3, (21)^9 |
|      | 343 380 | (31)^3 | (31)^3 | (21)^3, (21)^9 | (21)^3, (21)^9 |
|      | 423 476 | (31)^3 | (31)^3 | (21)^3, (21)^9 | (21)^3, (21)^9 |
|      | 486 264 | (31)^3 | (31)^3 | (21)^3, (21)^9 | (21)^3, (21)^9 |
| F.11 | 27 156  | (41)^3 | (31)^3 | (21)^3, (21)^9 | (21)^3, (21)^9 |
|      | 241 160 | (41)^3 | (31)^3 | (21)^3, (21)^9 | (21)^3, (21)^9 |
|      | 477 192 | (41)^3 | (31)^3 | (21)^3, (21)^9 | (21)^3, (21)^9 |
|      | 484 804 | (41)^3 | (31)^3 | (21)^3, (21)^9 | (21)^3, (21)^9 |
| F.12 | 291 220 | (31)^3 | (31)^3 | (21)^3, (21)^9 | (21)^3, (21)^9 |
|      | 167 064 | (41)^3 | (31)^3 | (21)^3, (21)^9 | (21)^3, (21)^9 |
|      | 296 407 | (41)^3 | (31)^3 | (21)^3, (21)^9 | (21)^3, (21)^9 |
|      | 317 747 | (41)^3 | (31)^3 | (21)^3, (21)^9 | (21)^3, (21)^9 |
|      | 401 603 | (41)^3 | (31)^3 | (21)^3, (21)^9 | (21)^3, (21)^9 |
The following theorem was proved in [43] and shows that Schur $\sigma$-groups in the descendant tree of $(243,3)$, resp. $P_7 := (2187,64)$, set in extensively at logarithmic order 20.

**Theorem 14.** Let $M := P_7 - \#2; m$ be a sporadic metabelian 3-group $M$ with type in Section F and coclass $cc(M) = 4$. The following counters concern 1359 pairwise non-isomorphic Schur $\sigma$-groups $S$ of logarithmic order $\log(S) = 20$ and nilpotency class $cl(S) = 9$ such that $S/S'' \simeq M$.

(1) For type F.7, there exist 171, in more detail, 81, resp. 45, 45, Schur $\sigma$-groups $S$ satisfying

$$\tau(2)(S) = [1^2; (32; 2^31, (31^3)^3), (32; 2^31, (31^3)^3), (1^3; 2^31, (2^21)^3, (21^2)^9)]$$

with metabelianization $m = 55$, resp. 56, 58. They all have $\#Aut(S) = 2 \cdot 3^{25}$.

(2) For type F.11, there exist 108 + 324, in more detail,

(a) 54, resp. 54, Schur $\sigma$-groups $S$ satisfying

$$\tau(2)(S) = [1^2; (32; 2^31, (41^3)^3), (32; 2^31, (31^3)^3), (1^3; 2^31, (2^21)^3, (21^2)^9)]$$

with $\#Aut(S) = 2 \cdot 3^{25}$ and metabelianization $m = 36$, resp. 38;

(b) 162, resp. 162, Schur $\sigma$-groups $S$ satisfying Formula 2.4 with $\#Aut(S) = 2 \cdot 3^{26}$ and metabelianization $m = 36$, resp. 38.

(3) For type F.12, there exist 216 + 162, in more detail,

(a) 54, resp. 54, 54, Schur $\sigma$-groups $S$ satisfying

$$\tau(2)(S) = [1^2; (32; 2^31, (n1^3)^3), (32; 2^31, (31^3)^3), (1^3; 2^31, (2^21)^3, (21^2)^9)]$$

with $n = 3$ and metabelianization $m = 43$, resp. 46, 51, 53;

(b) 54, resp. 54, 27, 27, Schur $\sigma$-groups $S$ satisfying Formula 2.5 with $n = 4$ and metabelianization $m = 43$, resp. 46, 51, 53.

They all have $\#Aut(S) = 2 \cdot 3^{25}$.

(4) For type F.13, there exist 216 + 162, in more detail,

(a) 54, resp. 54, 54, 54, Schur $\sigma$-groups $S$ satisfying

$$\tau(2)(S) = [1^2; (32; 2^31, (n1^3)^3), (32; 2^31, (31^3)^3), (1^3; 2^31, (2^21)^3, (21^2)^9)]$$

with $n = 3$ and metabelianization $m = 41$, resp. 47, 50, 52;

(b) 54, resp. 54, 27, 27, Schur $\sigma$-groups $S$ satisfying Formula 2.6 with $n = 4$ and metabelianization $m = 41$, resp. 47, 50, 52.

They all have $\#Aut(S) = 2 \cdot 3^{25}$.

**Proof.** This impressive result goes back to some sparkling emails which we received from Professor Mike F. Newman in January and February 2013. These communications, though being cryptic without explicit identifiers, unambiguously illuminated the way to the rigorous proof. So Theorem 14 without doubt is a joint achievement by Prof. Newman and ourselves. The detailed justification constitutes the dominating part of [43]. □

**Conjecture 15.** (Tower ground state) The imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants $d \in \{-225299, -343380, -423476, -486264\}$ of type F.7, resp. $d \in \{-27156, -241160, -477192, -484804\}$ of type F.11, resp. $d = -291220$ of type F.12, resp. $d \in \{-167064, -296407, -317747, -401003\}$ of type F.13, have 3-class field towers of exact length $\ell_3(K) = 3$ with a suitable Schur $\sigma$-group in Theorem 17.

For all types, F.7, F.11, F.12, F.13, the tower group $S = G_{\sigma}^{\infty}K$ has $\log(S) = 20$, $cl(S) = 9$, $cc(S) = 11$, $\xi_1S = (9,9)$ or $(9,3,3)$, $\gamma_2S = (27,27,9,3,3,3)$ or $(27,9,9,9,3,3)$, and usually $\#Aut(S) = 2 \cdot 3^{25}$, rarely $2 \cdot 3^{26}$.

The extremal root path to the 3-class field tower group $G = \text{Gal}^\infty(K/K) \simeq S$ of $K = \mathbb{Q}(\sqrt{-225299})$ is described in Table 8 and drawn in Figure 9. As opposed to the figures concerning TKT H.4 and Section E, we do not know to which coclass trees the capable vertices on the path belong. We assume $M = S/S'' \simeq P_7 - \#2; 55, S \simeq P_7 - \#4; 196 - \#2; 31 - \#4; 1 - \#1; 2 - \#2; 1.$
2.6. SENSATION: First 3-class field towers with at least four stages. (dl(G) ≥ 4)

Theorem 16. An imaginary quadratic field \( K \) with elementary 3-class group \( \text{Cl}_3(K) \simeq (3,3) \) of rank two, 3-capitulation type F.13, \( \tau = (3143) \), and abelian type invariants of second order, \( \tau^{(2)} K \) (2.7) \( [1^2; (32; 2^31, (31^2)^3)^3, (1^3^2; 2^31, (2^1^2)^3)^3] \)
possesses a 3-class field tower with length \( f_3(K) \geq 4 \).

The field \( K = \mathbb{Q}(\sqrt{d}) \) with fundamental discriminant \( d = -224580 \) (Tbl. 7) is such an example.

Proof. Let \( P_7 := (2187, 64) \) denote the common fork vertex for all \( \sigma \)-groups with transfer kernel types in Section F. We use the \( p \)-group generation algorithm by Newman \[16\] and O’Brien \[37\] (see also \[20\]) in order to construct the relevant part of the descendant tree of the root \( P_7 \).

Since the capitulation of the field \( K \) is of type F.13, \( \tau = (3143) \), (containing a transposition but no fixed point) with three different kernels, 1, 3 and 4, and the logarithmic abelian type invariants of the four unramified cyclic cubic extensions \( E_i/K, 1 \leq i \leq 4 \), are given by \( \tau = (32, 32, 1^3, 1^3) \), there are precisely four possibilities for the metabelianization \( M = G/G'' \), namely \( M \simeq M_{9,m} := P_2 - #2; m \) with \( m \in \{41, 47, 50, 52\} \), and four corresponding possibilities for ancestors of the tower group \( G \) itself, \( V_{11,\ell} := P_2 - #4; \ell \) with \( \ell \in \{9, 15, 18, 20\} \), in the same order, e.g. \( M_{9,41} \simeq V_{11,9}/V_{11,9} \). (So the common fork of the root paths of \( M_{9,m} \) and \( V_{11,\ell} \) is always \( P_2 \).) These are the unique four groups for which all abelian quotients of subgroups of index 9 possess 3-rank 4:

\[
\begin{align*}
(2.8) & \quad [1^2; (32; 2^31, (31^2)^3)^3, (1^3; 2^31, (2^1^2)^3)^3, (1^3; 2^31, (2^1^2)^3, (2^1^2)^3)]
\end{align*}
\]

This will be required in order to arrive at the intended AQI in Formula (2.7).

Each of the four \( V_{11,\ell} \) has derived length 3 and possesses a unique \( \sigma \)-descending among 41 descendants \( V_{11,\ell} - #2; k \) of step size \( s = 2 \), namely \( V_{13,\ell} := V_{11,\ell} - #2; 10 \) for \( \ell \in \{9, 15\} \) (i.e. \( k = 10 \)), and \( V_{13,\ell} := V_{11,\ell} - #2; 2 \) for \( \ell \in \{18, 20\} \) (i.e. \( k = 2 \)).

The further search yields the following results (see Table 4 for the case \( \ell = 20 \)): Each of the four \( V_{13,\ell} \) has 729 descendants of step size \( s = 4 \). For \( \ell \in \{9, 15\} \), resp. \( \ell \in \{18, 20\} \), we find 81, resp. 243, pairs \((j, i)\) with \( 1 \leq j \leq 729 \) and \( 1 \leq i \leq 3281 \) such that \( V_{22,\ell,i} := V_{17,\ell,j} - #5; i \) is the unique \( \sigma \)-descending among 3281 descendants of step size \( s = 5 \) of \( V_{17,\ell,j} := V_{13,\ell} - #4; j \). Among these groups, a veritable abundance of slight variations of the abelian quotient invariants of second order arises, covering all cases in Table 7.

Each of the groups \( V_{22,\ell,j} \) has 19,683 descendants of step size \( s = 8 \). The main difficulty in this proof was the construction of the 265721 descendants of step size \( s = 7 \) for each of the groups \( V_{30,\ell,j,h} := V_{22,\ell,j} - #8; h \) with \( 1 \leq h \leq 19,683 \), which are still of soluble length \( \text{dl}(V_{30,\ell,j,h}) = 3 \). The storage for such a batch of 265721 so-called compact presentations in the computational algebra system Magma \[11\] \[12\] \[22\] occupied 120 GB of RAM, which blew up to 170 GB when additional group theoretic operations were performed on these descendants.
Summarizing the results of this proof, it turned out that all the groups of order $3^{37}$, constructed in the last computationally possible step, were of the common shape $V_{37,\ell,j,h,g} := (2.9) \ P_\ell - #4; \ell - #2; k - #4; j - #5; h - #7; g$, with $dl(V_{37}) = 4$, $\nu(V_{37}) = 12$, $\mu(V_{37}) = 14$. (Recall that $k$ is determined uniquely by $\ell$ and $i$ is determined uniquely by $j$.) This implies that the desired Schur $\sigma$-group $S \simeq G = \text{Gal}(F_3\bar{\Sigma}(K)/K)$ is either infinite or a descendant of one of the groups $V_{37}$, and thus must have derived length $dl(S) \geq 4$. So the 3-class field tower of $K$ has certainly at least four stages.

\begin{proof}
For transfer kernel type F.7 there are three possibilities for the metabelianization $M = G/G''$, namely $M \simeq M_{9,m} := P_\ell - #2; m$ with $m \in \{55,56,58\}$, and three possibilities for the corresponding sibling on the path to $G$, $V_{11,\ell} := P_\ell - #4; \ell$ with $\ell \in \{23,24,26\}$.

For transfer kernel type F.12, there are four possibilities for the metabelianization $M = G/G''$, namely $M \simeq M_{9,m} := P_\ell - #2; m$ with $m \in \{43,46,51,53\}$, and four possibilities for the corresponding sibling on the path to $G$, $V_{11,\ell} := P_\ell - #4; \ell$ with $\ell \in \{11,14,19,21\}$.

And, hypothetically, for transfer kernel type F.11, although no suitable field has been detected up to now, there are only two possibilities for the metabelianization $M = G/G''$, namely $M \simeq M_{9,m} := P_\ell - #2; m$ with $m \in \{36,38\}$, and two possibilities for the corresponding sibling on the path to $G$, $V_{11,\ell} := P_\ell - #4; \ell$ with $\ell \in \{4,6\}$.
\end{proof}
Table 8. Root path of $G$, log ord 20, for the simplest case of transfer kernel type F.7

| Ancestor | Vertex | lo | cl | dl | $(\nu, \mu)$ | $(N_s/C_s)_{1 \leq s \leq \nu}$ | TKT  |
|-----------|--------|----|----|----|----------------|-------------------------------|------|
| $\pi^1(G)$ | $\langle 27, 3 \rangle$ | 3  | 2  | 2  | $(2.4)$        | $(4/1, 7/5)$                   | a.1  |
| $\pi^6(G)$ | $\langle 243, 3 \rangle$ | 5  | 2  | 4  | $(2.4)$        | $(10/6, 15/15)$               | b.10 |
| $\pi^7(G)$ | $\langle 2187, 64 \rangle$ | 7  | 4  | 2  | $(4.6)$        | $(33/2, 453/84, 918/540, 198/198)$ | b.10 |
| $\pi^4(G)$ | $\#4; 196$ | 11 | 2  | 4  | $(2.4)$        | $(20/20, 41/41)$              | F.7  |
| $\pi^3(G)$ | $\#2; 31$ | 13 | 2  | 4  | $(4.6)$        | $(44/0, 204/3, 180/24, 27/27)$ | F.7  |
| $\pi^2(G)$ | $\#4; 1$ | 17 | 2  | 4  | $(1.3)$        | $(5/5)$                       | F.7  |
| $\pi(G)$   | $\#1; 2$ | 18 | 2  | 4  | $(2.4)$        | $(4/0, 1/0)$                  | F.7  |

Table 9. Incomplete root path of $G$ for an extreme case of transfer kernel type F.13

| Ancestor | Vertex | lo | cl | dl | $(\nu, \mu)$ | $(N_s/C_s)_{1 \leq s \leq \nu}$ | TKT  |
|-----------|--------|----|----|----|----------------|-------------------------------|------|
| $\pi^{-2}(G)$ | $\langle 27, 3 \rangle$ | 3  | 2  | 2  | $(2.4)$        | $(4/1, 7/5)$                   | a.1  |
| $\pi^{-3}(G)$ | $\langle 243, 3 \rangle$ | 5  | 3  | 2  | $(2.4)$        | $(10/6, 15/15)$               | b.10 |
| $\pi^{-4}(G)$ | $\langle 2187, 64 \rangle$ | 7  | 4  | 2  | $(4.6)$        | $(33/2, 453/84, 918/540, 198/198)$ | b.10 |
| $\pi^{-5}(G)$ | $\#4; 9$ | 11 | 5  | 3  | $(2.4)$        | $(20/20, 41/41)$              | F.13 |
| $\pi^{-6}(G)$ | $\#2; 10$ | 13 | 6  | 3  | $(4.6)$        | $(108/54, 1674/1674, 3564/3564, 729/729)$ | F.13 |
| $\pi^{-7}(G)$ | $\#4; 144$ | 17 | 7  | 3  | $(5.7)$        | $(227/173, \ldots, 3281/3281)$ | F.13 |
| $\pi^{-8}(G)$ | $\#5; 2516$ | 22 | 8  | 3  | $(8.10)$       | $(3444/54, \ldots, 19683/19683)$ | F.13 |
| $\pi^{-9}(G)$ | $\#8; 1$ | 30 | 9  | 3  | $(7.9)$        | $(\ldots, 265721/265721)$    | F.13 |
| $\pi^{-10}(G)$ | $\#7; 1$ | 37 | 10 | 4  | $(12, 14)$     | $(\ldots, ?/?)$              | F.13 |
Figure 9. Extremal path to Schur $\sigma$-group, log ord 20, with TKT F.7

Topology Symbol:
$F(\frac{3}{2})b(\frac{4}{1})F(\frac{4}{1})F(\frac{4}{1})b(\frac{3}{2})F$
Figure 10. **Incomplete** path, **four-stage** Schur $\sigma$-group, log ord $> 37$, TKT F.13

Topology Symbol:

$$F\left(2^{\cdot}\right)u\left(2^{\cdot}\right)F\left(2^{\cdot}\right)F\left(2^{\cdot}\right)F\left(2^{\cdot}\right)F\left(2^{\cdot}\right)F\left(2^{\cdot}\right)\cdots$$

$M = P_7 = \pi^{c-4} S = P_7 = (3^7, 64)$

$\pi^M = \pi^{c-4} S = P_7 = (3^7, 64)$

$\text{metabelianization}$

$M = P_7 - \#2; 41$

$s_{c-4} = 4$

$\pi^{c-5} S = P_7 - \#4; 9$ (or 15 or 18 or 20)

$s_{c-5} = 2$

$\pi^{c-6} S = \pi^{c-5} S - \#2; 10$

$s_{c-6} = 4$

$\pi^{c-7} S = \pi^{c-6} S - \#4; 144$

$s_{c-7} = 5$

$\pi^{c-8} S = \pi^{c-7} S - \#5; 2516$

$s_{c-8} = 8$

$\pi^{c-9} S = \pi^{c-8} S - \#8; 1$

$dl = 4$

Order $3^n$
2.7. Schur $\sigma$-groups with TKT in other Sections. We have seen that the seven 3-groups with order $3^5 = 243$, class 3 and coclass 2, namely $\langle 243, i \rangle$ with $3 \leq i \leq 9$, which form the stem $\Phi_9(0)$ of Hall’s isoclinism family $\Phi_9$, are crucial roots of various descendant trees or even coclass trees accommodating either Schur $\sigma$-groups, e.g. for $i = 4$ in Proposition 11, or metabelianizations of Schur $\sigma$-groups, e.g. for $i \in \{6, 8\}$ in Proposition 11 whereas for $i \in \{3, 7\}$ they are isolated Schur $\sigma$-groups themselves (Proposition 11). All these vertices are connected with their common parent, the extra special group $(27, 3)$, by edges of step size $s = 2$. We also realized that the descendant $(2187, 64)$ of $\langle 243, 3 \rangle$ with step size $s = 2$ is the fork which is responsible for all $\sigma$-groups with types in Section F, but also for groups with types H.4 and G.16.

Up to this point, we did not consider the root $\langle 243, 9 \rangle$ with TKT G.19. It gives rise to a rather complicated descendant tree containing bifurcations. To our knowledge it is unknown if the tree is infinite and associated with a limit group like $\langle 243, 4 \rangle$. At the first bifurcation $W := \langle 729, 57 \rangle$, there arise six descendants $(6561, i)$, $625 \leq i \leq 630$, with step size $s = 2$ having very different properties, as Figure 11 shows.

**Proposition 20.** There exist precisely four Schur $\sigma$-groups with transfer kernel type G.19 and log ord 11. They are the smallest groups with these properties and can be given in the shape $(6561, i) - \#1; 2 - \#2; j$ with $i \in \{625, 629\}$ and $1 \leq j \leq 2$. The descendant trees of $(6561, i)$ with $i \in \{625, 629\}$ are finite with depth two.

There exist precisely twelve Schur $\sigma$-groups with transfer kernel type G.19 and log ord 14. They can be given in the shape $(6561, 627) - \#1; 2 - \#2; j - \#1; k - \#2; \ell$ with $(j, k) \in \{(1, 3), (2, 2)\}$ and $1 \leq \ell \leq 3$, and six similar descendants of $(6561, 628)$. The descendant tree of $(6561, 627)$ is finite with depth four.

**Proof.** This is the result of a construction of the descendant tree of $\langle 243, 9 \rangle$ up to logarithmic order 14 by means of Magma [22]. See Figure 11 where the descendant subtrees of $(6561, i)$ with even $i$ are not drawn. The subtree of $(6561, 626)$ does not contain many $\sigma$-groups, in particular no Schur $\sigma$-group with log ord up to 14. The subtree of $\Psi := (6561, 628)$ starts similar as the subtree of $\Phi := (6561, 627)$. The subtree of $(6561, 630)$ is the wildest of the six, but does not contain Schur $\sigma$-groups with log ord up to 14. \hfill $\square$

**Example 21.** The imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ with discriminant $d = -114936$ possesses a 3-class field tower with precisely three stages, $\ell_3(K) = 3$, having automorphism group $G = \text{Gal}(F_3^0(K)/K) \simeq (6561, 629) - \#1; 2 - \#2; j$ with $1 \leq j \leq 2$. This was proved by ourselves with the aid of abelian type invariants of second order in [57]. Path see Table 11 and Figure 11.

**Hypothesis 22.** The imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ with discriminant $d = -12067$ (the smallest absolute discriminant with TKT G.19) has probably a 3-class field tower with exactly three stages, $\ell_3(K) = 3$, and Galois group $G = \text{Gal}(F_3^0(K)/K) \simeq (6561, 625) - \#1; 2 - \#2; j$ with $1 \leq j \leq 2$. This is only a conjecture, motivated by the bigger probability of candidates with smallest possible order, but the abelian type invariants of second order of this field also occur for other Schur $\sigma$-groups, which causes uncertainty.
Figure 11. Extremal paths to Schur $\sigma$-groups, log ord 11, purged tree $T_\sigma(\langle 243, 9 \rangle)$
Table 10. Root path of $G$ for the simplest case of transfer kernel type G.19

| Ancestor | Vertex | lo | $(\nu, \mu)$ | $(N_s/C_s)_{1 \leq s \leq \nu}$ | TKT |
|----------|--------|----|--------------|--------------------------------|-----|
| $\pi^3(G)$ | $(27, 3)$ | 3 | $(2, 4)$ | $(4/1, 7/5)$ | a.1 |
| $\pi^2(G)$ | $(243, 9)$ | 5 | $(1, 3)$ | $(2/2)$ | G.19 |
| $\pi^3(G)$ | $(729, 57)$ | 6 | $(2, 4)$ | $(1/0, 6/6)$ | G.19 |
| $\pi^2(G)$ | $(6561, 629)$ | 8 | $(1, 3)$ | $(2/2)$ | G.19 |
| $\pi(G)$ | $-\#1; 2$ | 9 | $(2, 4)$ | $(1/0, 2/0)$ | G.19 |
| $G$ | $-\#2; 1$ | 11 | $(0, 2)$ | | G.19 |

Table 11. Root path of $G$, log ord 20, for the irregular case of transfer kernel type H.4

| Ancestor | Vertex | lo | $(\nu, \mu)$ | $(N_s/C_s)_{1 \leq s \leq \nu}$ | TKT |
|----------|--------|----|--------------|--------------------------------|-----|
| $\pi^5(G)$ | $(27, 3)$ | 3 | $(2, 4)$ | $(4/1, 7/5)$ | a.1 |
| $\pi^6(G)$ | $(243, 3)$ | 5 | $(2, 4)$ | $(10/6, 15/15)$ | b.10 |
| $\pi^5(G)$ | $(2187, 64)$ | 7 | $(4, 6)$ | $(33/2, 453/84, 918/540, 198/198)$ | b.10 |
| $\pi^4(G)$ | $-\#2; 38$ | 13 | $(2, 4)$ | $(20/20, 41/41)$ | H.4 |
| $\pi^3(G)$ | $-\#4; 1$ | 17 | $(1, 3)$ | $(5/5)$ | H.4 |
| $\pi(G)$ | $-\#1; 5$ | 18 | $(2, 4)$ | $(4/0, 1/0)$ | H.4 |
| $G$ | $-\#2; 1$ | 20 | $(0, 2)$ | | H.4 |

**Hypothesis 23.** A possible extremal root path to the 3-class tower group $G = \text{Gal}(F_3^\infty(K)/K) \simeq S$ of $K = \mathbb{Q}(\sqrt{-186483})$ with irregular type H.4 is described in Table 11 and drawn in Figure 12. Except for the additional edge with step size $s = 1$ from the metabelianization $M$, the graph is isomorphic to the graph concerning type F.7 in Figure 9. Similarly as in this preceding figure, we do not know to which coclass trees the capable vertices on the path belong. Here, we assume $M = S/S'' \simeq P_7 - \#2; 34 - \#1; 7$ and $S \simeq P_7 - \#4; 111 - \#2; 38 - \#4; 1 - \#1; 5 - \#2; 1$. The type H.4 is irregular in the sense of B. Nebelung [44, 45], since the commutator subgroup $M'$, resp. $S'$, has abelian type, resp. quotient, invariants $(2^3) = (9, 9, 9, 9)$ instead of the regular $(32^2) = (27, 9, 9, 3)$.

This paradigm shows that Schur $\sigma$-groups $S$ of log ord 20 are expected not only for types in Section F but also for types G.16, G.19, H.4, both, regular and irregular, always under assumption of a sporadic metabelianization $M = S/S''$ of coclass $cc(M) = 4$, either of log ord 9 or 10.
Figure 12. Extremal path to Schur $\sigma$-group, log ord 20, with irregular TKT H.4

Topology Symbol:
\[ H(\downarrow_1)H(\uparrow_1)b(\downarrow_4)H(\downarrow_2)H(\downarrow_4)H(\uparrow_1)H(\uparrow_2)H \]

Order $3^n$
3. Real quadratic counterexamples

According to Formula (1.4), the less restrictive bounds $2 \leq d_2(G) \leq 3$ for the relation rank of the Galois group $G = \text{Gal}(F_3^\infty(K)/K)$ of the 3-class field tower of a real quadratic number field $K = \mathbb{Q}(\sqrt{d})$, $d > 0$, with signature $(r_1, r_2) = (2, 0)$ and Dirichlet unit rank $r = r_1 + r_2 - 1 = 1$ give rise to a totally different behavior, in comparison to imaginary quadratic fields $K$ with elementary 3-class group $\text{Cl}_3(K) \simeq C_3 \times C_3$ of rank two.

Although the Schur $\sigma$-groups $G$ with $d_2(G) = 2$ are still admissible for discriminants $d > 0$, there arises a heavy competition by the much more numerous $\sigma$-groups $G$ with relation rank $d_2(G) = 3$, which also have the advantage of considerably lower order and higher probability.

**Theorem 24.** The Galois group $G = \text{Gal}(F_3^\infty(K)/K)$ of the 3-class field tower of nearly 90% of all the 34631 real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants in the range $0 < d < 10^8$ and elementary 3-class group $\text{Cl}_3(K) \simeq C_3 \times C_3$ of rank two is metabelian (i.e. two-stage) of coclass $cc(G) = 1$, and thus the root path of $G$ does not follow the first possible bifurcation at the common class-2 quotient $G/\gamma_3(G) \simeq (27, 3)$ with nuclear rank $\nu = 2$. Consequently the dominating part of real quadratic fields does not satisfy the extremal path condition Conj. 4.

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**Proof.** The first edge $G \to \pi(G)$ with step size $s = 1 < \nu$ of the root path, which violates the extremal path condition, is drawn in red color for the ground state (GS) of type a.2 ($G \simeq (81, 10)$ with fixed point TKT, for 7104 fields), type a.3* ($G \simeq (81, 7)$ with irregular TTT, for 10244 fields), and type a.3 ($G \simeq (81, 7)$ with regular TTT, for 10514 fields) in Figure 13. Comparison of the percentage of all tower groups $G$ with coclass 1 (ground state and excited states) corresponding to ranges of discriminants $d$ with increasing upper bound $10^{6}$, $10^{7}$, $10^{8}$ in Table 12 reveals a growth from 89\% over 89.40\% and 89.769\% towards clearly dominating 90\%, quod erat demonstrandum.

The information in Table 12 and Figure 13 has been computed by ourselves in 2016, published in [35] presented in key note lectures [39, 39], and refined in [35].

### Table 12. Smallest $G/G^{(2)} \simeq G$ for real quadratic fields of type (3, 3)

| Discriminant | Total# | total | fixed point | regular | irregular | $\Sigma$ | %   |
|--------------|--------|-------|-------------|---------|-----------|---------|-----|
| $0 < d < 10^6$ | 149    | 11    | 35          | 53      | 34        | 133     | 89.3%|
| $0 < d < 10^7$ | 2576   | 151   | 1454        | 698     | 2303      | 31088   | 89.40%|
| $0 < d < 10^8$ | 34631  | 2241  | 7356        | 11247   | 10244     | 31088   | 89.769%|
| $\pi(G)$, GS |        | (32, $1^2$) | (21, $1^2$) | (21, $1^2$) | (1, $1^2$) | 31088   | 89.769%|
| $\pi(G)$ |        | (0000) | (1000)      | (2000)   | (2000)     | 31088   | 89.769%|
| TKT         |        | a.1   | a.2         | a.3      | a.3*       | 31088   | 89.769%|
| GS: $G \simeq$ |        | (729, 99) | (81, 10)   | (81, 8)  | (81, 7)    | 31088   | 89.769%|

**Counterexample 25.** Even when the primary bifurcation at $(27, 3)$ is followed by the root path of $G$ for a real quadratic field $K = \mathbb{Q}(\sqrt{d})$, e.g. by the edge $(243, 9) \to (27, 3)$ from TKT G.19, the 1st, resp. 2nd, bifurcation in the purged tree $T_r((243, 9))$ may be violated by edges of step size $s = 1 < \nu = 2$, as illustrated by Figure 13. For the field with discriminant $d = +214712$ [24] [29], which is the smallest without any total capitulation (discovered by ourselves in January 2006 already), we have $G \simeq (2187, 311)$ of considerably lower log ord 7 instead of 11 in Figure 11 and for the field with discriminant $d = +21974161$ [33, 34] in much higher discriminantal ranges, the edge to $G \simeq Y_1 - \#1; 1$ is abbreviated from log ord 11 to 10 at $Y_1$. 
Figure 13. Non-extremal paths to unbalanced $\sigma$-groups, ord 81, coclass tree $\mathcal{T}_1(9,2)$
Figure 14. Non-extremal paths to unbalanced $\sigma$-groups, log ord 7 and 10

After this terminating figure of a descendant tree, we would like to emphasize the esthetical beauty and the immense value of this ostensive kind of graphical information from the perspective of learning psychology. We have developed this method of representing ancestor-descendant relations in a detailed series of papers [27, 28, 32, 40, 41].
4. Conclusion

In this paper, we have proved the *extremal root path property* of the Galois group of finite 3-class field towers, stated in the Main Conjecture for infinite series of parametrized Schur $\sigma$-groups. The derived length of one of these series was unbounded, whereas the other series had soluble length precisely equal to three. Within the frame of Section F of 3-captitulation types, we provided evidence of root paths with high complexity, and of the first 3-class field tower with length $\ell_3(K) \geq 4$. No counter-examples against the Main Conjecture are known up to now. Our experience suggests the extension of the Main Conjecture to Schur $\sigma$-groups with abelianizations different from $(3, 3)$, and also to primes $p \geq 5$.

In searches for $p$-groups with assigned Artin pattern by means of the strategy of pattern recognition via Artin transfers, considerable amounts of CPU time and RAM storage can be saved by restricting the search paths to the maximal possible step sizes, when the desired group is a Schur $\sigma$-group. Frequently, the descendant number for the maximal step size is smaller than for other step sizes, which may be outside of the reach of current implementations already.

As an impressive application of the technique of extremal root paths, we succeeded in conducting a rigorous proof of the first 3-class field towers with at least four stages over imaginary quadratic number fields. Even when these towers should turn out to be infinite this would be a striking novelty, because it was believed that imaginary quadratic fields with 3-class groups of rank two have towers with finite length.

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6. Historical remarks concerning TKTs in Section F

*Imaginary* quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with 3-class group $\text{Cl}_3 K \simeq C_3 \times C_3$ and transfer kernel type (TKT) in Section F have been detected by Brink in 1984 [14]. The absolute values of their fundamental discriminants $d$ set in with 27 156, outside of the ranges investigated by Scholz and Taussky in 1934 [48], and by Heider and Schmithals in 1982 [19]. However, the computational results in Brink’s Thesis Appendix A, pp. 96–113 were unknown to us until we got a copy via ProQuest in 2006. Their actual extent is not mentioned explicitly in the official paper [15] by Brink and his academic advisor Gold. Therefore, we previously believed to have the priority in discovering the discriminant $d = -27 156$ of a field $K$ with type F.11 in 1989 [23] Tbl., p. 84, and the discriminants $d = -31 908, -67 480, -124 363$ of fields $K$ with types F.12, F.13, F.7 in 2003 [24] Tbl. 3, p. 497], all of them with second 3-class groups $G_3^3 K$ of coclass 4. In 2006, it turned out that our claim must be restricted to $d = -124 363$, which after nearly 20 years eventually provided the first example for type F.7, called the unique undiscovered type by Brink [14], § 7.2, p. 91.

It required further 10 years until we had the courage to study the 3-class tower of number fields with transfer kernel type F, based on abelian type invariants of second order, as developed in [23] [37], and inspired by repeated sparkling ideas of Professor Mike F. Newman in 2013 and 2017.

As opposed to coclass 4, we can definitely claim priority in discovering the discriminant $d = -423 640$ of a complex quadratic field $K = \mathbb{Q}(\sqrt{d})$ with type F.12 in 2010 [24] Tbl. 3, p. 497], and the discriminants $d = -1 677 768, -2 383 059, -4 838 891$ of fields $K$ with types F.7, F.13, F.11 in 2016, all of them with second 3-class groups $G_3^2 K$ of coclass 6.

Similarly, we were the first who found the discriminant $d = 8 321 505$ of a real quadratic field $K = \mathbb{Q}(\sqrt{d})$ with type F.13 in 2010 [24] Tbl. 4, p. 498), and the discriminants $d = 10 165 597, 22 937 941, 66 615 244$ of fields $K$ with types F.7, F.12, F.11 in 2016 [33] Tbl. 4, p. 1291, 33, all of which possess second 3-class groups $G_3^2 K$ of coclass 4.
7. Personal historical remarks

When I began to investigate the 3-capitulation in unramified cyclic cubic extensions $E_i/K$ of imaginary quadratic number fields $K$ with 3-class rank two (it was in autumn 1989, that is, thirty years ago), I had a sound foundation in algebraic number theory and elements of class field theory, but only a very basic knowledge in $p$-group theory. My usage of the concept metabelian group of class two in the resulting paper [23] pp. 73, 79, 84, 86 shows that I was thinking within Helmut Hasse’s scope of metabelian groups with multiple stages, which would be called solvable or soluble nowadays. With class two I rather meant two stages, which is simply metabelian (non-abelian with abelian commutator subgroup, or derived length two) in modern mathematical language.

In [23], my view of the capitulation type $\kappa(K)$ of imaginary quadratic fields $K$ was just a nice new invariant, in addition to class number $h_K$ and $p$-class rank $rk_p(K)$, for various primes $p$. Although I knew that Scholz and Taussky [48] intended to derive information about the metabelian second 3-class group $M := \text{Gal}(F_3^2(K)/K)$ or even the pro-3 Galois group $G := \text{Gal}(F_3^3(K)/K)$ of the entire 3-class field tower of $K$ from the type $\kappa(K)$ of 3-capitulation, I could not get a firm grasp of their claim that the annihilator ideal of the main commutator $s_2 = [y, x]$ of $M = (x, y)$ together with two Schreier polynomials determines the metabelian group $M$ uniquely (see [42]).

My poor horizon concerning the systematic treatment of $p$-groups did not extend until Aissa Derhem (in December 2001) drew my attention to Nebelung’s doctoral thesis [44, 45]. Suddenly I was able to remember numerous finite metabelian 3-groups according to their characteristic positions in descendant trees, more precisely in coclass trees with stable difference $cc = lo - cl$ between logarithmic order and nilpotency class, and I started a big revival of the capitulation problem for imaginary quadratic number fields $K$ with 3-class group $\text{Cl}_3(K) \simeq (3, 3)$ in 2003, now always with the higher goal to identify the corresponding metabelian group $M = \text{Gal}(F_3^3(K)/K)$. In 2006, I extended my investigations to real quadratic fields $K$, for which the capitulation problem was nearly unknown (except for 5 cases in the paper of Heider and Schmithals [19]). This scientific phase got its coronation in a series of papers [24, 25, 26, 27] where the second 3-class groups $M$ were determined for all 4596 quadratic fields $K$ with discriminants $-10^6 < d_K < 10^7$ and $\text{Cl}_3(K) \simeq (3, 3)$. It was the final realization of a dream I had twenty years earlier [23] p. 77.

However, in spite of my success in determining the second stage $M$ of 3-class field towers in the year 2010, I began to realize that neither Nebelung’s theory [44] nor the original work of Scholz and Taussky [48] provided the required techniques for answering the problem concerning the length $\ell_3(K)$ of the tower for the mysterious imaginary quadratic fields $K$ with capitulation types in Section E, which had been raised by Brink and Gold [14] [15] without any conclusive decision. Scholz and Taussky had proved $\ell_3(K) = 2$ for imaginary quadratic fields $K$ with capitulation types in Section D [48] p. 39]. In some hasty remarks they also claimed a two-stage tower for other types, in particular those in Section E [48] pp. 20, 41]. In 1992, Franz Lemmermeyer drew my attention to the doubts of Brink and Gold about these remarks by Scholz and Taussky.

In 2012, a chain of lucky coincidences enabled the rigorous solution of this annoying problem. The dissertation of Tobias Bembom acquainted me with Schur $\sigma$-groups and unpublished drafts by Boston, Bush and Hajir (published much later in [13]) about the role of these groups as pro-3 Galois groups $G = \text{Gal}(F_3^3(K)/K)$ of 3-class field towers of imaginary quadratic fields $K$. In August 2012, I had opportunity to meet these three authors at a conference in Vienna. In a discussion of only one or two hours, Boston, Bush and myself succeeded in strictly proving exact length $\ell_3(K) = 3$ for imaginary quadratic fields $K$ with capitulation types in Section E. The reason why Brink and Gold did not succeed in the definite exclusion of length two in 1987 was their lack of knowledge about Schur $\sigma$-groups, although [40] appeared in 1964 and [21] in 1975.

I shall not become tired of pointing out again and again the fundamental importance of the Shafarevich cohomology criterion for the relation rank $d_2(G)$ of the group $G = \text{Gal}(F_3^3(K)/K)$. In discussions at mathematical conferences, I repeatedly realized that even experts in number theory only know the famous results of Golod and Shafarevich on infinite class field towers, but are not aware of the Shafarevich criterion [49].
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