A KIND OF $KK$-THEORY FOR RINGS

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Abstract. A group equivariant $KK$-theory for rings will be defined and studied in analogy to Kasparov’s $KK$-theory for $C^*$-algebras. It is a kind of linearization of the category of rings by allowing addition of homomorphisms, imposing also homotopy invariance, invertibility of matrix corner embeddings, and allowing morphisms which are the opposite split of split exact sequences. We demonstrate the potential of this theory by proving for example equivalence induced by Morita equivalence and a Green-Julg isomorphism in this framework.

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1. Introduction

Let $G$ be a discrete group. In this paper we consider a variant of $KK$-theory for the class of $G$-equivariant, quadratik rings which is closely related to Kasparov’s original $KK$-theory for $C^*$-algebras [22, 23]. This theory defined in this paper, called $GK^G$-theory, is the universal additive, homotopy invariant, stable and split

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exact category formed from the category of quadratik rings and ring homomorphisms.

It has it roots in Cuntz [9] and Higson’s [19] findings, that Kasparov’s $KK$-theory, when restricted to ungraded separable $C^*$-algebra, is the universal additive, homotopy invariant, split–exact theory formed from the category of separable $C^*$-algebras and $^*$-homomorphisms. See [2, 3] for more on this link.

A ring has the condition to be quadratik if every element of it can be written as a finite sum of products $ab$ for $a, b$ in the ring.

Now in this paper we do homotopy as follows. We at first complexify the two rings at the endpoints, and then do continuous-function homotopy in $\mathbb{C}$-algebras as usual. This may not be viewed as a proper homotopy in rings, as the complexified space is bigger, but it is a convenient equivalence relation suitable for our purposes.

Now stability means the following. We at first consider modules $E$ over a ring $A$ which are equipped with a functional space comprised of $A$-module homomorphisms $\phi : E \rightarrow A$. Such functionals are the analogy of the functionals $E \rightarrow A : \xi \mapsto \langle \eta, \xi \rangle$ induced by the inner product on Hilbert modules $E$ over a $C^*$-algebra $A$. Then we define the ring of compact operators $K_A(E)$ generated by elementary operators $\theta_{\eta, \phi}$ defined by $\theta_{\eta, \phi}(\xi) = \eta \phi(\xi)$ in analogy to such operators in $KK$-theory. Then a corner embedding is the canonical ring homomorphism

$$e : A \rightarrow K_A(E \oplus A)$$

acting by multiplication on the coordinate $A$, and we declare $e$ to be an invertible morphism in $GK^G$-theory. This means “stability”.

If we restrict the class of quadratik rings to the separable $C^*$-algebras, define the functional spaces by the inner product as explained before as usual, restrict $E$ to countably generated Hilbert $A$-modules, and define $K_A(E)$ as before but also taking the norm closure, then we exactly get Kasparov’s theory $KK_G$ for $GK^G$ by remark 3.6 and [2] out.

This theory is also split–exact, meaning that given a short split–exact sequence of rings, splitting at the quotient ring, the opposite split on the ideal-side is declared to be a valid morphism. It is automatically always an additive group homomorphism, but not a ring homomorphism, and so this theory gets the glance of a kind of linearization of the ring category.

The main concerns of this concept are two-folds. Firstly, general rings, even general Banach algebras, do not behave so well as $C^*$-algebras. More precisely, they do not have all the good analysis as $C^*$-algebras do have, like the Kasparov stabilization theorem, approximate units, positivity, to name a view.

A sort of compensate of this is what we do in section 10 loosely called $\nabla$-calculation. An arbitrary complex algebraic expression which is zero in $GK^G$-theory is equivalently reformulated to an expression which is zero on the level (subcategory) of ring homomorphism in $GK^G$-theory by a simple algorithm. The problem that remains is the complicated equivalence relations on this level coming from $GK^G$-theory. Nevertheless, with this method we are able to show that a functor on $GK^G$-theory is faithful if and only if it is faithful on the sublevel of ring homomorphism. See theorem 10.8.

But the second issue is more subtle and appears to us also more severe. If one wants to construct a morphism in $GK^G$-theory involving one synthetical split morphism $\Delta$ axiomatically postulated by the theory, then one ‘automatically’ runs into double-split exact sequences. For example the composition $\pi \Delta$ of $\Delta$ with a
ring homomorphism $\pi$ becomes $\pi' \Delta_{\pi''}$ with respect to a double split-exact sequence by lemmas 7.3 and 7.2. But in practice it is extremely difficult to write down two splitting ring homomorphisms $\pi'$, $\pi''$ of a short exact sequence, such that $\pi' \Delta_{\pi''} \neq 0$. This difficulty besets also Kasparov’s theory in that it might be narrow in applicability. Mostly such double splits are constructed by Clifford algebras, the $K$-theory Bott (or dual-Dirac) element, and the Dirac element induced by the Dirac operator [23].

The next goal of $GK^G$-theory should be clarifying applicability by coming up with examples involving synthetical splits. Of course, one might consider similar examples known from Atiyah-Singer index theory and Kasparov theory by replacing continuous function spaces $C_0$ by differentiable function spaces $C^\infty$. This might be a possible application. Connes’ consideration of $p$-summable operators in non-commutative differential geometry [5] seems to be such a continuous by smooth functions ‘replacement’, very roughly said, and actually such replacements appear in various contexts in the literature.

But for the moment we show some standard results, as roughly speaking, Morita equivalence induces $GK^G$-equivalence in section 13, the establishment of a descent functor in section 14 and an induction functor for subgroups $H$ of $G$ in section 15 and even a Green-Julg isomorphism in section 16. Also the Baum-Connes map might be verbatim generalized to $C^\ast$-algebras, and we check that it is injective if and only if it is injective on the ‘pure’ homomorphism level, by the functor theorem mentioned earlier, see section 17.

There are other generalizations of $KK$-theory to bigger or other classes than $C^\ast$-algebras. Notably, this is Lafforgue's Banach $KK$-theory for Banach algebras [24], and Cuntz’ $kk$-theory for locally convex algebras and diffeomorphic homotopy [10]. Cuntz’ theory has been considered and extended by various authors, see Cuntz [11], Cuntz and Thom [12], Cortiñas and Thom [8], Ellis [13], and Garkusha [14, 15]. Related is also Grensing [17], and we should also mention Weidner [29, 30].

The theory by Cuntz is almost half-exact, meaning that short exact sequences of algebras having linear splits induce, even, long exact and cyclic sequences in $kk$-theory, and so might be even better compared with, the half-exact, $E$-theory by Higson [20], and Connes and Higson [7, 6], which was designed to make, the split-exact, $KK$-theory half-exact. Notice, that our theory here is also split-exact by its very definition, but extremely likely not half-exact.

We remark that all results presented in this paper hold analogously in $KK^G$-theory for $C^\ast$-algebras.

In particular, the $\nabla$-calculation might also be interesting in $KK$-theory, as it shows that an arbitrarily long product of Kasparov products is zero in $KK^G$-theory if and only if an obvious ordinary Kasparov element is zero in $KK^G(A, B)$ without the need of the Kasparov product at all, see for instance example 10.10. However, this is theoretical, because in practice, this element is so complicated, that we need to compute the element classically by computing the Kasparov products to decide the equality with zero by a homotopy, which on the other hand is also theoretical, because in general one needs the axiom of choice for the Kasparov product.

We note that we have chosen to do everything for discrete rings here not because we think this is the best choice or even necessary, but rather to have it easier at first and to make a point. First of all, it works for discrete rings. But, the theory is also understood to be adapted to various situations, and one has to choose appropriate
closures in norm, or locally convex spaces, or Schwartz spaces and so on, and choose
appropriate functional spaces, and also allow $G$ to be a locally compact group.
Notably, the corner embeddings have to be adapted by choosing the appropriate
topological closure, as, notice, the stability axiom is mainly the axiom of $G^KG$-
theory sensitive to the differences, but the homotopy axiom might also be adjusted,
for example by allowing only smooth functions rather than continuous ones.
We also are confident that everything, excepting the Baum-Connes map, may
easily be adapted to the inverse semigroup and not necessarily Hausdorff locally
compact groupoid equivariant setting as in [3]. But to avoid overloading very
moderate technicalities, we abstained from such a general situation the first time.
The brief overview of this paper is as follows. In section 2-3 we explain
$G^KG$-theory. The way one works with double split exact sequences in
$G^KG$-theory is explained in sections 4-9. In section 10 the $\nabla$-calculation is shown. In the final
sections 11-17 the above mentioned standard results will be performed.

2. Rings and Functional Modules

All rings in this paper are neither necessarily commutative nor unital, associa-
tive discrete rings. Throughout, $G$ denotes a discrete group, and all rings $A$ are
equipped with a $G$-action, that is, a group homomorphism $\alpha : G \to \text{Aut}(A)$ into
the automorphism group $\text{Aut}(A)$ of $A$. All ring homomorphisms are $G$-equivariant.
We sometimes say “non-equivariant” ring, homomorphism, module etc. if we want
to ignore any possible $G$-structure. We often write id or 1 for the identity map, for
example in $T \otimes 1$. Likewise we write 1 for the trivial $G$-action. The unitization of
a ring $A$ will be denoted by $A^+$ or $\tilde{A}$.

Given an invertible operator $U$ or a $G$-action $U$, we write $\text{Ad}(U)$ for the map
$T \mapsto U \circ T \circ U^{-1}$ and the $G$-action $(g,T) \mapsto U_g \circ T \circ U_{g^{-1}}$, respectively.

All structures have $G$-action. As soon as we introduce the definition of a $G$-
structure on a module, operator spaces etc. it is understood without saying that
these objects carry $G$-actions.

Functionals on modules are the substitute for inner products on Hilbert mod-
ules. Functional spaces are just auxiliary spaces to define the space of compact
operators. Everything runs in parallel to $C^*$-algebras (now rings), Hilbert modules
(now functional modules) and inner products (now functionals).

Definition 2.1 ($G$-action on module). Let $(A,\alpha)$ be a ring. Let $M$ be a right
$A$-module.

A $G$-action $S$ on $M$ is a group homomorphism $S : G \to \text{AddMaps}(M)$ (additive
maps = abelian group homomorphisms) into the invertible additive maps on $M$
such that $S_1 = 1$ and $S(\xi a) = S(\xi)\alpha(a)$ for all $\xi \in \mathcal{E}, a \in A$.

Definition 2.2 ($G$-action on Module Homomorphisms). Let $(A,\alpha)$ be a ring. Let $(M,S)$ and $(N,T)$ be right $A$-modules. We define $\text{Hom}_A(M,N)$ to be the abelian
group of all right $A$-module homomorphisms $\phi : M \to N$.

We equip $\text{Hom}_A(M,N)$ with the $G$-action $\text{Ad}(S,T)$, that is, we set $g(\phi) :=
T_g \circ \phi \circ S_{g^{-1}} \in \text{Hom}_A(M,N)$ for all $g \in G$ and $\phi \in \text{Hom}_A(M,N)$.

We also write $\text{Hom}_A(M) := \text{Hom}_A(M,N)$ if $(M,S) = (N,T)$. This is a ring
under concatenation and its $G$-action $\text{Ad}(S)$. 
Definition 2.3 (Functionals). Let \((A,\alpha)\) be a ring. Let \(M\) be a right \(A\)-module. Turn \(\text{Hom}_A(M, A)\) to a left \(A\)-module by setting \((a\phi)(\xi) := a\phi(\xi)\) for all \(a \in A, \xi \in M\) and \(\phi \in \text{Hom}_A(M, A)\).

Assume we are given a distinguished functional space \(\Theta_A(M) \subseteq \text{Hom}_A(M, A)\) which is a \(G\)-invariant, left \(A\)-submodule of \(\text{Hom}_A(M, A)\).

Then we call \((M, \Theta_A(M))\) a right functional \(A\)-module.

The functional space \(\Theta_A(M)\) will usually not be notated in \(M\), as it is called anyway always in the same way.

Definition 2.4 (Compact operators). Let \((A, \alpha)\) be a ring. To right functional \(A\)-modules \((M, S)\) and \((N, T)\) is associated the \(G\)-invariant, abelian subgroup (under addition) of compact operators \(K_A(M, N) \subseteq \text{Hom}_A(M, N)\) which consists of all finite sums of all elementary compact operators \(\theta_{\eta,\phi} \in \text{Hom}_A(M, N)\) defined by \(\theta_{\eta,\phi}(\xi) = \eta\phi(\xi)\) for \(\xi \in M, \eta \in N\) and \(\phi \in \Theta_A(M)\).

We write \(K_A(M) \subseteq \text{Hom}_A(M)\) for the \(G\)-invariant subring \(K_A(M, M)\). To observe \(G\)-invariance, we compute \(g(\theta_{\xi,\phi})(\eta) = T_g(\xi\phi(S_{g^{-1}}(\eta))) = \theta_{g(\xi),g(\phi)}(\eta)\).

Definition 2.5 (Multiplier operators). Let \(M\) and \(N\) be right functional \(A\)-modules.

We write \(M(K_A(M, N)) \subseteq \text{Hom}_A(M, N)\) for the \(G\)-invariant, abelian subgroup of multipliers of the compact operators, that is, we set

\[
M(K_A(M, N)) := \{V \in \text{Hom}_A(M, N) \mid V \circ X, Y \circ V \in K_A(M, N)
\forall X \in K_A(M), Y \in K_A(N)\}
\]

The \(G\)-invariant subring \(M(L_A(M, M)) \subseteq \text{Hom}_A(M)\) is denoted by \(M(L_A(M))\).

Clearly, \(K_A(M)\) is a two-sided ideal in \(M(K_A(M))\). We note that the above requirement \(V \circ X \in K_A(M, N)\) is trivially automatic and so superfluous stated. The other requirement \(Y \circ V \in K_A(M, N)\) for all \(Y\) is trivially satisfied when \(\phi \circ V \in \Theta_A(M)\) for all \(\phi \in \Theta_A(N)\), as \(\theta_{\xi,\phi} \circ V = \theta_{\xi,\phi \circ V}\). One might write “\(V^*(\phi) = \phi \circ V\)” and this might justify to sloppily call the elements of \(M(K_A(M, N))\) also ‘adjointable operators’, but we shall not follow this path. In practice there will be little difference between the formal adjointable-operators definition or the multiplier definition, even one can construct counterexamples, see the paragraph after definition 2.13.

The adjointable operators behave slightly better, and we shall exclusively work with them:

Definition 2.6 (Adjointable operators). Let \(M\) and \(N\) be right functional \(A\)-modules.

We write \(L_A(M, N) \subseteq \text{Hom}_A(M, N)\) for the \(G\)-invariant, abelian subgroup of adjointable operators, that is, we set

\[
L_A(M, N) := \{V \in \text{Hom}_A(M, N) \mid \phi \circ V \in \Theta_A(M), \forall \phi \in \Theta_A(N)\}
\]

Definition 2.7 (Module homomorphism between functional modules). A functional module homomorphism between two right functional \(A\)-modules \((E, S, \Theta_A(E))\) and \((F, T, \Theta_A(F))\) is a \(G\)-equivariant, right \(A\)-module homomorphism \(X : E \to F\) together with a \(G\)-equivariant, left \(A\)-module homomorphism \(f : \Theta_A(E) \to \Theta_A(F)\) such that for every \(\phi \in \Theta_A(E)\) and \(\xi \in E\) one has

\[
\tag{1}
f(\phi)(X(\xi)) = \phi(\xi)
\]
Just to demonstrate that the last definition works properly with compact operators, we note:

**Lemma 2.8.** A module homomorphism \((X, f)\) as in the last definition with \(X\) being surjective induces a ring homomorphism \(\sigma : \mathcal{K}_A(E) \to \mathcal{K}_A(F)\), and \(f\) is then automatically injective.

If \((X, f)\) is an isomorphism (i.e. \(X\) and \(f\) bijective) \(\sigma\) is an isomorphism and extends to an isomorphism \(L_A(E) \to L_A(F)\).

**Proof.** Define \(\sigma\) additively and by \(\sigma(\theta_{\xi, \phi}) = \theta_{X(\xi), f(\phi)}\). To see that it is well-defined, assume that \(\sum_i \xi_i \phi_i(\eta) = 0\) for all \(\eta \in E\). Then by the last definition

\[
0 = \sum_i X(\xi_i) \phi_i(\eta) = \sum_i X(\xi_i) f(\phi_i)(X(\eta)) = \sum_i \theta_{X(\xi_i), f(\phi_i)}(X(\eta))
\]

Finally, with \(\theta_{\xi, \phi} \otimes \eta, \psi = \theta_{\xi \otimes \eta, \psi}\) we observe multiplicativity of \(\sigma\). Injectivity of \(f\) follows directly from (\dag\). \(\square\)

If nothing else is said, the tensor product \(\otimes\) means the exterior tensor product of abelian groups, or in other words of \(\mathbb{Z}\)-modules, so \(\otimes\) means \(\otimes_{\mathbb{Z}}\). We note that there is a well-known ring homomorphism

\[
(2) \quad \pi : \text{Hom}_A(E) \otimes \text{Hom}_B(F) \to \text{Hom}_{A \otimes B}(E \otimes F) : \pi(S \otimes T) = S \otimes T
\]

which notice, maps compact operators to compact operators, and adjoint-able operators to adjoint-able ones. However, none of the above ring homomorphisms need either to be injective or surjective.

**Definition 2.9** (Direct sum). Let \((E_i, S_i)_{i \in I}\) be a a family of right functional \(A\)-modules. The algebraic direct sum \(E := \bigoplus_{i \in I} E_i\) of right \(A\)-modules with finite support with respect to \(I\) is a right functional \(A\)-module with diagonal \(G\)-action \(S := \bigoplus S_i\) defined by \(S_g(\bigoplus_{i \in I} E_i) := \bigoplus_{i \in I} S_{i, g}(E_i)\). The functional space \(\Theta_A(E)\) is defined to be the set of functionals \(\phi := \bigoplus_{i \in I} \phi_i\) defined by \(\phi(\bigoplus_{i \in I} E_i) = \sum_{i \in I} \phi_i(E_i)\) for \(\phi_i \in \Theta_A(E_i)\).

**Definition 2.10** (External tensor product). Let \((E, S)\) be a right functional \(A\)-module and \((F, T)\) a right functional \(B\)-module. The **external tensor product** \(E \otimes_{\mathbb{Z}} F\) is a right functional \((A \otimes B)\)-module under the diagonal \(G\)-action \(S \otimes T\) defined by \((S_g \otimes T_g)(\xi \otimes \eta) := S_g(\xi) \otimes T_g(\eta)\). The functional space \(\Theta_{A \otimes B}(E \otimes F)\) is defined to be the set of all sums of all elementary functionals \(\phi \otimes \psi\) defined by \((\phi \otimes \psi)(\xi \otimes \eta) = \phi(\xi) \otimes \psi(\eta)\) for \(\phi \in \Theta_A(E)\) and \(\psi \in \Theta_B(F)\).

**Definition 2.11** (Internal tensor product). Let \(E\) be a right functional \(A\)-module and \(F\) a right functional \(B\)-module and \(\pi : A \to L_B(F)\) a ring homomorphism. Let \(E \otimes_\pi F := (E \otimes F) / \left\{ \sum_i \xi_i \otimes \eta_i \in E \otimes F \mid \sum_i \pi(\phi(\xi_i)) \eta_i = 0, \forall \phi \in \Theta_A(E) \right\}\)

be the **internal tensor product**, which is a right functional \(B\)-module by the \(B\)-module structure of \(F\), equipped with the diagonal \(G\)-action \(S \otimes T\).

We define \(\Theta_B(E \otimes_\pi F)\) to be set of all sums of elementary functionals \(\phi \otimes \psi\) defined by

\[
(\phi \otimes \psi)(\xi \otimes \eta) = \psi \left( \pi(\phi(\xi)) \eta \right)
\]

for \(\phi \in \Theta_A(E)\) and \(\psi \in \Theta_B(F)\).

One may observe that \(a \phi \otimes \psi = \phi \otimes \psi \circ \pi(a)\) for \(a \in A\).
Definition 2.12. A ring $A$ is called quadratik if $A = \sum A^2$ (the two-sided ideal of $A$ comprised of all finite sums of all products $ab$ for $a, b \in A$).

A ring $A$ is called an essential (right) ideal in itself if the ring homomorphism $\pi : A \to \text{Hom}_A(A) : \pi(a)(b) = ab$ is injective.

Definition 2.13. A right functional $A$-module $\mathcal{E}$ is called cofull if $\mathcal{E} = \sum \xi \Theta_A(\mathcal{E})(\mathcal{E})$ (all finite sums of all products $\xi \phi(\eta)$ for $\xi, \eta \in \mathcal{E}$ and $\phi \in \Theta_A(\mathcal{E})$).

As $\mathcal{L}_A(\mathcal{E})$ is unital, it is obviously quadratik and an essential ideal in itself.

Lemma 2.14. If $\mathcal{E}$ is a cofull, right functional $A$-module then $K_A(\mathcal{E})$ is quadratik and an essential ideal in itself.

Proof. Quadratik: Given $\zeta \in \mathcal{E}$ we may write it as $\zeta = \sum_i \xi_i \phi_i(\eta_i)$ and so

$$\theta_{\zeta, \psi} = \sum_i \xi_i \phi_i(\eta_i) \psi = \sum_i \xi_i \phi_i(\eta_i) = \sum_i \theta_{\xi_i, \phi_i, \eta_i} \psi$$

Essential ideal in itself: Assume that $\sum_i \theta_{\eta_i, \psi_i} \neq 0$. Then its evaluation in some $\zeta$ (written as above) is nonzero. Thus

$$\sum_i \theta_{\eta_i, \psi_i}(\zeta) = \sum_i \left( \sum_j \eta_j \psi_i(\xi_i) \phi_i(\eta_i) \right) = \left( \sum_i \eta_i \psi_i \right)(\zeta) = 0$$

$x_i$ obviously defined. Hence at least one $x_i \neq 0$. But this means $\left( \sum_i \theta_{\eta_i, \psi_i} \right) \not= 0$.

Definition 2.15. We turn any ring $(A, \alpha)$ into a right functional $(A, \alpha)$-module $(A, \alpha)$ over itself by setting $\Theta_A(A) := A^+$, where these functionals act by left multiplication in $A$ (i.e. $\phi_a(b) = ab$). If $A$ is quadratik, we prefer to set $\Theta_A(A) := A$.

If $A$ is quadratik, then both functional spaces $A$ and $A^+$ yield obviously the same ring of compact operators $K_A(A)$, and this is what primarily counts for us. But notice, ‘functional-conceptually’ there is a difference: for example, if $T \in L_A(A)$ is a multiplier of $K_A(A)$, then $T$ may be ‘adjoint-able’ with respect to the functional space $A$ but not with respect to $A^+$, as $1 \circ T = T \not\in A^+$ in general.

Also, the functional space of $A \otimes \pi \mathcal{F}$ appears to become different under the two choices $A$ and $A^+$ for the functional space of $A$. So we choose $A$ if $A$ is quadratik, but sometimes allow $1 \in \Theta_A(A)$ for pure notational purposes, for example we shall write $\theta_{a,1}$, knowing it can always be resolved in $\theta_{a,b} = \sum_i \theta_{a,b_i}$ for $a, b_i \in A$.

Lemma 2.16. A ring $A$ is quadratik if and only if the right functional $A$-module $A$ over itself is cofull.

Proof. Cofullness of the module $A$ obviously implies quadratik. For the reverse implication, note that $A = \sum A^n$ for all $n \geq 1$ whenever $A$ is quadratik. Hence $n = 3$ yields.

Lemma 2.17. Let $\pi : A \to B$ is a ring homomorphism between not necessarily quadratik rings $A$ and $B$.

(i) If $A$ is quadratik, then the range $\pi(A)$ is quadratik.

(ii) If $A$ is quadratik and $I$ a two-sided ideal in $A$ then $A/I$ is quadratik.

(iii) If $\mathcal{E}$ is a cofull $A$-modul then $\Theta_A(\mathcal{E})(\mathcal{E})$ is a quadratik subring of $A$.

(iv) If $\mathcal{E}$ is cofull then $\mathcal{E} \otimes \pi B$ is cofull.

(v) If $\mathcal{E}$ is cofull then $K_A(\mathcal{E}) \otimes 1 \subseteq K_A(\mathcal{E} \otimes \pi B)$.
Proof. (i) and (ii) are obvious. (iii) Using cofullness of \( \mathcal{E} \), write \( \phi(\eta) = \sum_i \phi(\eta_{1,i})\phi_{2,i}(\eta_{2,i}) \) for \( \phi \in \Theta_A(\mathcal{E}), \eta \in \mathcal{E} \). (iv) Using this, in \( \mathcal{E} \otimes_B A \) we may write
\[
\xi \phi(\eta) \otimes b = \xi \otimes \pi(\phi(\eta))b = \sum_{i,j} \xi \otimes \pi(\phi(\eta_{1,i}))\pi(\phi_{2,i}(\eta_{2,i,j}))\pi(\phi_{3,i,j}(\eta_{3,i,j}))b
\]
with \( \varrho_{i,j}(d) = \pi(\phi_{2,i}(\eta_{2,i,j}))d \). By cofullness of \( \mathcal{E} \) we are done.

(v) Just compute that \( \theta_{\xi,\phi(\eta),\psi} \otimes 1 = \theta_{\xi \otimes \pi(\phi(\eta)),\psi \otimes 1} \).

\[\Box\]

Lemma 2.18. Let \( B \) be a quadratik ring and \( \mathcal{E} \) a right functional \( B \)-module.
(i) Then \( \mathcal{E} \oplus B \) is cofull if and only if for every \( \xi \in \mathcal{E} \) there are \( \eta_i \in \mathcal{E}, b_i \in B \) such that \( \xi = \sum_i \eta_i b_i \).
(ii) Let \( \mathcal{E}, \mathcal{F} \) and \( \pi \) be as in definition 2.11 and \( \mathcal{F} \) be cofull. Then \( (\mathcal{E} \otimes_B \mathcal{F}) \oplus B \) is cofull.
(iii) If \( \mathcal{E} \oplus B \) is cofull then also \( \mathcal{F} \oplus B \) for any functional \( B \)-submodule \( \mathcal{F} \) of \( \mathcal{E} \).

Proof. (i) Just notice that \( \xi = \sum_i \eta_i b_i = \sum_{i,j} (\eta_i \oplus 0_B)(0 \oplus \phi_{i,j})(0 \phi \oplus y_{i,j}) \) for the functionals \( \phi_{i,j} \in \Theta_B(B) \) defined by \( \phi_{i,j}(a) = x_{i,j}a \), where \( b_i = \sum_j x_{i,j}y_{i,j} \) is by quadratik of \( B \). Cofullness of the \( B \)-summand is by lemma 2.10
(ii) and (iii) are then easily derived from (i).

It might be useful to call \( \mathcal{E} \) weakly cofull if the right condition of lemma 2.18(i) is satisfied. It has the desirable property that it is permanent under taking submodules by the last lemma (if \( B \) is quadratik).

Usually, a \( B \)-submodule \( \mathcal{F} \subseteq \mathcal{E} \) of a right functional \( B \)-module \( \mathcal{E} \) is turned to a functional module by restriction of the functionals, that is, one sets \( \Theta_B(\mathcal{F}) := \{ \phi|_\mathcal{F} | \phi \in \Theta_B(\mathcal{E}) \} \).

There presumably is no point in considering functional modules whose elements are not separated by the functionals. If so, we may call the module separated by functionals, and here is the procedure how we get it:

Lemma 2.19. We may turn a right functional \( A \)-module \( (\mathcal{E}, S) \) to a right functional \( A \)-module
\[ \widehat{\mathcal{E}} := \mathcal{E}/\{ \xi \in \mathcal{E} | \phi(\xi) = 0, \forall \phi \in \Theta_A(\mathcal{E}) \} \]
whose functional space
\[ \Theta_A(\widehat{\mathcal{E}}) := \{ \overline{\phi} | \phi \in \Theta_A(\mathcal{E}) \} \]
where \( \overline{\phi}(\xi) = \phi(\xi) \), separates the points.

(i) Also, there is a ring homomorphism \( \pi : \mathcal{L}_A(\mathcal{E}) \rightarrow \mathcal{L}_A(\widehat{\mathcal{E}}) \) which restricts to a ring homomorphism \( \mathcal{K}_A(\mathcal{E}) \rightarrow \mathcal{K}_A(\widehat{\mathcal{E}}) \).
(ii) Further, \( \mathcal{E} \otimes_A \mathcal{F} = \widehat{\mathcal{E}} \otimes_{\widehat{A}} \mathcal{F} \).
(iii) We also have \( \mathcal{E} \otimes_{\text{id}} A = \widehat{\mathcal{E}} \).

Proof. We also define \( A \)-module structure by \( [\xi]a := [\xi a] \), and a \( G \)-action \( \overline{S} \) by \( \overline{S}_g([\xi]) := [S_g(\xi)] \).
(i) If \( T \in \mathcal{L}_A(\mathcal{E}) \) then we set \( \pi(T)([\xi]) := [T(\xi)] \).
(ii) The outline is that we have maps \( \mathcal{E} \otimes \mathcal{F} \rightarrow \mathcal{E} \otimes_{\pi} \mathcal{F} \rightarrow \mathcal{E} \otimes_{\pi} \mathcal{F} \cong (\mathcal{E} \otimes \mathcal{F})/I \), where \( I \) means intersection of the kernels of all functionals as defined in definition 2.11.
Lemma 2.20. (i) Let $A, B$ be rings and $\pi : A \to B \subseteq \mathcal{L}_B(B)$ a ring homomorphism. Then there is a functional $B$-module isomorphism $X : A \otimes \pi B \to B_0$ to a functional $B$-submodule $B_0$ of $B$.

(ii) If $B$ is quadratik, then $B \otimes_{id} B \cong B$ as functional $B$-modules.

(iii) If $B$ is quadratik, then $B \otimes_{id} (B \times G) \cong B \times G$ as functional $B \times G$-modules.

(iv) If $A$ is unital and $\pi : A \to \mathcal{L}_B(\mathcal{E})$ is a unital ring homomorphism then $A \otimes \pi \mathcal{E} \cong \mathcal{E}$ as right functional $B$-modules.

Proof. (i) We have a $B$-module functional isomorphism $(X, l)$ defined by

$$X : A \otimes \pi B \to B_0 := \sum \pi(A)B \subseteq B : X(a \otimes b) = \pi(a)b$$

$$l : \Theta_B(A \otimes \pi B) \to \Theta_B(B_0) : l(mx \otimes (y)) = m \pi_x (y)$$

where $m_x$ denotes left multiplication operator $(x \in B, y \in A)$ and $\Theta_B(B_0)$ is defined to be the image of $l$, into a (now functional) $B$-submodule $B_0$ of $B$, see also lemma 2.8.

If $X(\sum_i a_i \otimes b_i) = 0$ then also $\sum_i \pi(xa_i)b_i = 0$ for all $x \in A$, and so $\sum_i a_i \otimes b_i = 0$ by definition 2.11 proving injectivity of $X$.

(ii) follows from (i). (iii) is proven similarly.

(iv) The functional $B$-module isomorphism is $p : A \otimes_{\xi, \pi} \mathcal{E} \to \mathcal{E}$ defined by $p(a \otimes \xi) = \pi(a)(\xi)$, and its inverse isomorphism defined by $p^{-1}(\xi) = 1_A \otimes \xi$, together with $f : \Theta_A(\mathcal{E}) \to \Theta_A(A \otimes_{\pi} \mathcal{E})$ defined by $f(\phi) = 1 \otimes \phi$, thereby recalling definition 2.7 and lemma 2.8.

Lemma 2.21. (i) A direct sum $\mathcal{E} = \oplus_i \mathcal{E}_i$ of $A$-modules $\mathcal{E}$ and $\mathcal{E}_i$ is a direct sum of functional $A$-modules if and only if $\mathcal{E}$ is a functional $A$-module and the $A_i$s are functional $A$-submodules of $\mathcal{E}$ obtained by restriction of the functional space of $\mathcal{E}$.

(ii) The exterior and interior tensor products $\mathcal{E} \otimes_{\pi} \mathcal{F}$ commute with direct sums of functional modules in both ‘variables’ $\mathcal{E}$ and $\mathcal{F}$ (provided $\pi(A)$ acts on the summands $\mathcal{F}_i$ of $\mathcal{F}$).

Proof. (i) is an easy check, and (ii) may be deduced from (i) and well-known corresponding isomorphisms of $A$-modules.

For a non-equivariant ring $A$ we write $M_n(A)$ for the ring of $n \times n$-matrices with coefficients in $A$, and also $\overline{M}_\infty(A)$ for arbitrarily big infinite matrices.

Lemma 2.22. Let $(A, \alpha)$ be a quadratik ring. Then there are ring isomorphisms

$$(A, \alpha) \cong (K_A(A), \text{Ad}(\alpha)) \quad (M_n(A), \alpha \otimes 1_{M_n}) \cong K_A((A, \alpha)^n)$$

Also, $M_n(A)$ is quadratik.

Proof. We just discuss the first isomorphism, the other one is done as in the next lemma. By lemmas 2.14 and 2.16 $\pi$ of definition 2.14 is injective. The last claim follows also from these lemmas and the second isomorphism, or completely elementary by suitable matrix multiplication as in the proof of lemma 6.3.

Lemma 2.23. Let $(A, \alpha)$ be a ring and $(\mathcal{E}_i, S_i)$ right $(A, \alpha)$-modules for $1 \leq i \leq n \leq \infty$. Then there are ring isomorphisms

$$(M_n(K_A(\mathcal{E})), \text{Ad}(\oplus_{i=1}^n S_i)) \cong K_A(\oplus_{i=1}^n (\mathcal{E}, S_i))$$

$$(M_n(L_A(\mathcal{E})), \text{Ad}(\oplus_{i=1}^n S_i)) \cong L_A(\oplus_{i=1}^n (\mathcal{E}, S_i))$$
where the $G$-action $\theta := \text{Ad}(\oplus_{i=1}^{n} S_i)$ on the matrix algebras is defined by

$$\theta_g((x_{ij})) = (S_{i,g} \circ x_{ij} \circ S_{j,g^{-1}})$$

Proof. The last isomorphism $\sigma$ is given by $\sigma((x_{ij})_{ij}) = (\sum_{k=1}^{n} x_{ik}(\xi_k))_{ij}$, and one easily checks $G$-equivariance, and that it restricts to $M_n(K_A(\mathcal{E}))$ and recall definition [2.3] for $G$-invariance. □

The following lemma is elementary and well-known, and our sketched proof might appear superfluous.

**Lemma 2.24.** Let $A, B$ rings and $\mathcal{E}, F, M, N$ be modules. Let $(G, 1)$ be a field. Write $\otimes^G$ for the $G$-vector space (i.e. $G$-balanced) tensor product. Then we have isomorphisms

$$(A \otimes B) \otimes G = (A \otimes G) \otimes^G (B \otimes G)$$

$$(\mathcal{E} \otimes_{\pi} F) \otimes G = (\mathcal{E} \otimes G) \otimes^G_{\pi \otimes 1} (F \otimes G)$$

$$(A \times_{\alpha} G) \otimes G \cong (A \otimes G) \times_{\alpha \otimes 1} G$$

$$K_G(G^n) \cong M_n(G), \quad M_n(A) \otimes G \cong M_n(A \otimes G)$$

If $F$ is a subfield of $G$ and all rings and modules here are even vector spaces over $F$ (so $F$-algebras and their modules and homomorphisms, $\otimes$ replaced by $\otimes^F$), then everything above is also true.

Proof. (Sketch) For the first identity we observe that by the universal property of the tensor product an obvious homomorphism and inverse homomorphism are well-defined. These homomorphisms respect certain quotients and so we arrive at the second line. The third line is because the tensor product exchanges with direct sums as abelian groups. The fourth isomorphism is $\theta_{ae_i, bp_j} \mapsto abc_{i,j}$ for $a, b \in G$, $e_i \in G^n$, $e_{i,j} \in M_n(G)$ the canonical basis vectors and $p_j$ the canonical projection onto the $j$th coordinate. □

We impose now the following convention throughout:

From now on, all rings are assumed to be quadratik $G$-rings if nothing else is said. All modules are understood to be right functional $G$-modules and they may not be cofull, but it is implicitly understood that they are whenever they are the underlying module $\mathcal{E}$ of the compact operators $\mathcal{K}_B(\mathcal{E})$, because of lemma 2.14 if nothing else is said.

3. $GK$-theory

In this paper we write compositions of morphisms in a category and compositions of functions from left to right. That is, for instance, if $f : A \to B$ and $g : B \to C$ are maps, then we write $fg$ for $g \circ f$, where composition operator $\circ$ is used in the usual sense from right to left. This will go as far as that we write $fg(x)$ for $g(f(x))$. In spaces of operators like $\mathcal{L}(\mathcal{E})$ we use the multiplication in the usual sense, that is, $ST$ means $S \circ T$ for $S, T \in \mathcal{L}(\mathcal{E})$, but to avoid confusion, we mostly write $S \circ T$.

**Definition 3.1** (Distinguished coordinate). Let $(A, \alpha)$ be a ring. We write $\mathbb{H}_A$ for a direct sum $\mathbb{H}_A := \bigoplus_{i \in I} A$ of copies of $A$, where $I$ may have any cardinality above zero, and where the $G$-action on $\mathbb{H}_A$ is of the form $\alpha \oplus S$ on $\mathbb{H}_A = A \oplus \bigoplus_{i \in I \setminus \{i_0\}} A$, and the direct summand $(A, \alpha)$ on the position $i_0 \in I$ is called the distinguished coordinate of $\mathbb{H}_A$. Thereby, $S$ may be any $G$-action and need not to be a direct sum $G$-action.
This $\mathbb{H}_A$ is merely an auxiliary space. It will become clear, that it is sufficient to take only $\mathbb{H}_A = A$, see remark 3.6 below.

**Definition 3.2 (Corner embedding).** Let $(A, \alpha)$ be a ring. Let $(\mathcal{E}, S)$ be a not necessarily cofull, right functional $A$-module. Let $\mathbb{H}_A = (\bigoplus_{i \in I} A, \alpha \oplus T)$ be a direct sum. Assume that $\mathcal{E} \oplus \mathbb{H}_A$ is cofull. (Cf. lemma 2.15)

The injective ring homomorphism

$$e : (A, \alpha) \rightarrow \mathcal{K}_A((\mathcal{E} \oplus \mathbb{H}_A, S \oplus \alpha \oplus T))$$

defined by $e(a)(\xi \oplus b \oplus \eta) = 0 \oplus ab \oplus 0$ for $\xi \in \mathcal{E}, a \oplus \eta \in \mathbb{H}_A = A \oplus \bigoplus_{i \in I \setminus \{i_0\}} A$ is a corner embedding.

That is, $e(a)$ is the multiplication operator acting on the distinguished coordinate $(A, \alpha)$ of $\mathcal{E} \oplus \mathbb{H}_A$. Note that $e(a) = \theta_{a, \phi}$, where $\phi$ projects onto the distinguished coordinate. We call also ring homomorphisms which are corner embeddings up to ring isomorphism corner embeddings.

There is also a corner embedding $e : A \rightarrow (\mathcal{E} \otimes \mathcal{E}, \alpha \otimes T)$ in the more usual sense defined by $e(a) = \text{diag}(a, 0, 0, \ldots)$ by lemmas 2.22 and 2.23.

Let $C[0, 1] := (C([0, 1], \mathbb{C}), 1)$ be the complex-valued continuous functions on the unit interval, endowed with the trivial $G$-action.

For a ring $(B, \beta)$ we write $B[0, 1] := (B \otimes C[0, 1], \beta \otimes 1)$ for the exterior tensor product with diagonal $G$-action.

Gersten [16] has defined a homotopy theory for rings by declaring polynomial functions as homotopy. For our purposes, logically this is a too strict notion and we rather take continuous functions in the complexified ring. (Replacing $\mathbb{Z}[x]$ in Gersten by $C[0, 1]$ here.)

**Definition 3.3.** Let $(A, \alpha)$ and $(B, \beta)$ be rings. A homotopy is a $G$-equivariant ring homomorphism $f : (A, \alpha) \rightarrow (B \otimes C[0, 1], \beta \otimes 1)$ such that for the evaluation maps $v_t : B \otimes C[0, 1] \rightarrow B \otimes \mathbb{C}, v_t(b \otimes z) = b \otimes z(t)$, the following holds true: the ranges of $fv_0$ and $fv_1$ are in the subring $B \otimes \mathbb{Z}$.

The ring homomorphisms $fv_0 \tau$ and $fv_1 \tau$ from $A$ to $B$ are then called to be homotopic, where $\tau : B \otimes \mathbb{Z} \rightarrow B$ is the canonical ring isomorphism $\tau(b \otimes n) = bn$.

It is often useful to view homotopy as above by complexifying $B$ to $\hat{B} := B \otimes \mathbb{C}$ at first and then do $\mathbb{C}$-algebra homotopy $f : A \rightarrow \hat{B}$ as usual. The choice of continuous homotopies is relatively arbitrary, having some sin and cos would be enough.

**Definition 3.4.** We fix a certain category $R^G$ of $G$-equivariant, quadratik rings (objects) and $G$-equivariant ring homomorphisms (morphisms), closed under all constructions needed in this paper. (In particular, the compact operators, adjointable operators, and corner embeddings.)

We are going to recall the definition of $GK^G$-theory ("Generators and relations $KK$-theory" with "$G$"-equivariance) for which we refer for more details to [2]. The split exactness axiom is slightly but equivalently altered, see [4] Lemma 3.7.

**Definition 3.5.** Let $GK^G$ be the following category. Object class of the class $GK^G$ is the object class of the class $R^G$ (i.e. a given class of $G$-equivariant quadratik rings).

Generator morphism class is the collection of all $G$-equivariant ring homomorphisms $f : A \rightarrow B$ (with obvious source and range objects) from $R^G$ and the collection of the following "synthetical" morphisms:
For every equivariant corner embedding \( e : (A, \alpha) \to (K_A(\mathcal{E} \oplus \mathbb{H}_A), \delta) \) as in definition 3.2 add a morphism called \( e^{-1} : (K_A(\mathcal{E} \oplus \mathbb{H}_A), \delta) \to (A, \alpha) \).

For every equivariant short split exact sequence

\[
\mathcal{S} : 0 \longrightarrow (B, \beta) \xrightarrow{j} (M, \delta) \xrightarrow{f} (A, \alpha) \longrightarrow 0
\]

in \( R^G \) add a morphism called \( \Delta_{\mathcal{S}} : (M, \delta) \to (B, \beta) \) or \( \Delta_s \) if \( \mathcal{S} \) is understood.

Form the free category of the above generators together with free addition and subtraction of morphisms having same range and source (formally this is like the free ring generated by these generator morphisms, but one can only add and multiply if source and range fit together) and divide out the following relations to turn it into the category \( GK^G \):

- (Ring category) Set \( g \circ f = fg \) for all \( f \in R^G(A, B) \) and \( g \in R^G(B, C) \).
- (Unit) For every object \( A \), \( \text{id}_A \) is the unit morphism.
- (Additive category) For all diagrams

\[
\begin{array}{ccc}
A & \xrightarrow{i_A} & A \oplus B \\
\downarrow{p_A} & & \downarrow{p_B} \\
B & \xrightarrow{i_B} & B
\end{array}
\]

(canonical projections and injections) set \( 1_{A \oplus B} = p_Ai_A + p_Bi_B \).

- (Homotopy invariance) For all homotopies \( f : A \to B[0,1] \) in \( R^G(A, B) \) set \( f_0 = f_1 \).

- (Stability) All corner embeddings \( e \) as in definition 3.2 are invertible with inverse \( e^{-1} \).

- (Split exactness) For all split exact sequences (3) set

\[
\begin{align*}
1_B &= j \Delta_s \\
1_M &= \Delta_s j + fs
\end{align*}
\]

For more comments like these confer also [2].

**Remark 3.6.** (i) Set-theoretically \( M = j(B) + s(A) \) in (3), and this is a direct sum by the last point.

(ii) If we have given an additional homomorphism \( u : A \to M \) in (3) then this is a second split for \( f \) if and only if

\[
u(a) - s(a) \in j(B) \quad \forall a \in A
\]

(iii) Given (3), we have \( s \Delta_s = 0 \) because \( s \Delta_s = s \Delta_s j \Delta_s = s(1-fs) \Delta_s = 0 \).

(iv) The \( GK^G \)-theory as in Definition 3.3 remains completely unchanged if we would require stability only for corner embeddings with \( \mathbb{H}_A = \oplus_{i \in I} A \) with only one fixed cardinality \( |I| \geq 1 \). Indeed, if \( |I| = 1 \) then \( \mathcal{E} \oplus \mathbb{H}_A = \mathcal{E} \oplus A \), and so all other cases are obviously included. For the opposite direction write

\[
\begin{array}{ccc}
A & \xrightarrow{e} & K_A(\mathcal{E} \oplus A) \\
\downarrow{f} & & \downarrow{M_\infty(K_A(\mathcal{E} \oplus A))} \\
\end{array}
\]

where \( H_\mathcal{E} := \oplus_{i \in I} \mathcal{E} \) and where \( e \) and \( f \) are corner embeddings in the sense of definition 3.2 (modulo ring isomorphisms). Since \( f \) and \( ef \) are invertible corner embeddings by our required axioms for fixed \( |I| \), \( e \) must also be invertible.

(v) By the analogous reasoning, the above analogous corner embedding \( e \) is also invertible in \( KK^G \)-theory for \( \mathcal{E} \) a countably generated \( A \)-module, because we have \( K_A(H_\mathcal{E} \oplus \mathbb{H}_A) \cong K_A(\mathbb{H}_A) \) by Kasparov’s (equivariantly interpreted) stabilization theorem (cf. [3] lemma 8.5) for \( \mathbb{H}_A \) being the countably infinite direct Hilbert
module sum of copies of $A$, and so $ef$ is also an invertible corner embedding in $K K^G$-theory.

We sometimes refer to the following notion and variants of it (in particular $L_B$ replaced by $K_B$). Typically, corner embeddings might be rotated to other corner embeddings.

**Definition 3.7.** Let $\pi : A \to L_B((\mathcal{E}, S) \oplus (\mathcal{E}, S)) =: X$ be a ring homomorphism. Under a *rotation homotopy* we mean the homotopy

$$\sigma : A \to X[0,1] \cong M_2(L_B(\mathcal{E}) \otimes C[0,1])$$

given by $\sigma(a) = (1_{L_B(\mathcal{E})} \otimes V) \circ (\pi(a) \otimes 1_{C[0,1]}) \circ (1_{L_B(\mathcal{E})} \otimes V^{-1})$ for the complex-valued homotopy $V_t := \left( \begin{array}{cc} \cos t & \sin t \\ -\sin t & \cos t \end{array} \right) \in M_2(C)$ for $t \in [0, \pi/2]$, $V_t^{-1} = V_{t'}$.

**Definition 3.8.** Given $P \in GK^G(A,B)$ we define $P^* : GK^G(B,X) \to GK^G(A,X)$ by $P^*(z) := Pz$ and $P_* : GK^G(X,A) \to GK^G(X,B)$ by $P_*(z) := zP$.

Let us finally remark that we restrict the object class $R^G$ to a set by a particular selection such that $GK^G$ turns to a small category and all Hom-classes $GK^G(A,B)$ are sets. See for example [2].

### 4. Double split exact sequences

We shall also recall in this section and in the next ones a couple of constructions and results from [3], where $GK$-theory for $C^*$-algebras is considered.

Throughout, $(A,\alpha)$ and $(B,\beta)$ are $G$-rings.

**Definition 4.1.** A *double split exact sequence* is a diagram of the form

$$0 \rightarrow (B,\beta) \xrightarrow{j} (M,\gamma) \xrightarrow{f} (A,\alpha) \xrightarrow{s} (M,\delta) \xrightarrow{e_{11}} (M,\theta) \xrightarrow{e_{22}} 0$$

where all morphisms in the diagram are equivariant ring homomorphisms, the first line is a split exact sequence in $R^G$, $t$ is another split in the sense that $tf = 1_A$ in non-equivariant $R^G$ and $e_{ij}$ are the corner embeddings.

**Definition 4.2.** Consider a double split exact sequence as above. We denote by $\mu_\theta$, or $\mu$ if $\theta$ is understood, the morphism

$$\mu : (M,\delta) \to (M,\gamma) : \mu = e_{22}e_{11}^{-1}$$

in $GK^G$. The *morphism in $GK^G$ associated to the double split exact sequence* is $t\mu \Delta_s$.

We use sloppy language and say for example “the diagram is $t\mu \Delta_s$ in $GK$”, or two double split exact sequences are said to be “equivalent” if their associated morphisms are. Throughout, the short notation for the above double split exact sequence will be

$$(B,\beta) \xrightarrow{j} (M,\gamma) \xrightarrow{f} (A,\alpha)$$
Notating such a diagram, it is implicitly understood that this is a double split exact sequence as above if nothing else is said. Often \(s, t\) is stated as \(s_\pm\), which has to be read as \(s_-, s_+\). The \(G\)-action \(\theta\) of definition \(\[\text{12}\]\) will sometimes be called the \("M_2\)-action of the double split exact sequence" for simplicity.

**Lemma 4.3 ([3, lemma 5.4]).** Consider two double split exact sequences which are connected by three morphisms \(b, \Phi, a\) in \(GK^G\) as in this diagram:

\[
\begin{array}{cccccc}
B & \xrightarrow{i} & M & \xrightarrow{f} & A \\
\downarrow{b} & & \downarrow{M_1} & & \downarrow{a} \\
D & \xleftarrow{j} & N & \xleftarrow{g} & C
\end{array}
\]

Here we have defined
\[
\phi := e_{11} \Phi f_{11}^{-1} \quad \psi := e_{22} \Phi f_{22}^{-1}
\]

(i) Then for the commutativity of the left rectangle of the diagram we note
\[
\Delta_s - b = \phi \Delta_t \quad \Leftarrow \quad i \phi = b_j \quad \text{and} \quad f s_\phi = \phi g t_-
\]

(ii) For commutativity within the right big square of the diagram we observe
\[
s_+ \mu \phi = a t_+ \mu \quad \Leftarrow \quad s_+ \psi = a t_+
\]

(iii) Consequently, commutativity of double split exact sequences in this diagram can be decided as
\[
s_+ \mu \Delta_s - b = a t_+ \mu \Delta_t \quad \Leftarrow \quad \text{Conditions of (i) and (ii) hold true}
\]

Let us revisit the last lemma and state for further reference:

**Remark 4.4.** \(\Phi\) will always be of the form \(\Phi = \phi \otimes 1_{M_2}\) for a non-equivariant ring homomorphism \(\phi : M \rightarrow M\). Then this \(\phi\) is the \(\phi\) and \(\psi\) in the above diagram as non-equivariant maps, and both are automatically equivariant as maps as entered in the diagram when \(\Phi\) is. So, to check commutativity of a diagram involving two exact double-split exact sequences with lemma \(\[\text{13}\]\) we only have to show that \(f s_\phi = \phi g t_-, i \phi = b_j\) and \(s_+ \phi = a t_+\), and that \(\Phi\) is \(G\)-equivariant.

**5. The \(M \square A\)-construction**

We shall use the following standard procedure to produce split exact sequences, and this is in fact key:

**Definition 5.1.** Let \(i : (B, \beta) \rightarrow (M, \gamma)\) be an equivariant injective ring homomorphism such that the image of \(i\) is a two-sided ideal in \(M\). Let \(s : (A, \alpha) \rightarrow (M, \gamma)\) be an equivariant ring homomorphism. Then we define the equivariant \(G\)-subring
\[
M \square_s A := \{(s(a) + i(b), a) \in M \oplus A | a \in A, b \in B\}
\]
of \((M \oplus A, \gamma \oplus \alpha)\). We also just write \(M \square A\) if \(s\) is understood. The \(G\)-action on \(M \square A\) is denoted by \(\gamma \square \alpha\). In particular we have a split exact sequence

\[
0 \to B \xrightarrow{j} M \square A \xrightarrow{f} A \xrightarrow{s} 0,
\]

where \(j(b) = (i(b), 0)\), \(f(m, a) = a\) and \((s \square 1)(a) := (s(a), a)\) for all \(a \in A, b \in B, m \in M\).

If we have given a double split exact sequence as in definition 4.1 with \(M\) of the form \(M \square A\) then it is understood that \(j, f\) and \(s \square 1\) are always of the form as in the last definition. Moreover, the construction of \(M \square A\) refers always to the first notated split \(s\), or the split indexed by minus (e.g. \(s \square 1\)) if it appears in a double split exact sequence. We denote elements of \(M \square A\) by \(m \square a := (m, a)\). The operator \(\square\) binds weakly, that is for example, \(m + n \square a = (m + n) \square a\).

Observe that \(M \square_s A = (s \square 1)(A) + B\) is quadratik if \(A\) and \(B\) are.

Non-equivariantly we have

\[
M_2(M \square_s A) \cong M_2(M) \square_s \oplus M_2(A) \subseteq M_2(M) \oplus M_2(A)
\]

with respect to \(i \otimes 1_{M_2}\) and \(s \otimes 1_{M_2}\).

**Definition 5.2.** If we have \(G\)-algebras \((M_2(M), \gamma)\) and \((M_2(A), \delta)\) and \((M_2(M \square A), \theta)\) is canonically a \(G\)-invariant \(G\)-subalgebra of \((M_2(M) \oplus M_2(A), \gamma \oplus \delta)\) then we call \(\theta\) also \(\gamma \square \delta\).

**Remark 5.3.** It will be important to observe, and is understood that the reader is aware of it in checking the validity of double split exact sequences, that the non-equivariant splits of the exact sequence of definition 5.1 are exactly the maps of the form \(t \square 1\), where \(t : A \to M\) is any non-equivariant ring homomorphism such that, for all \(a \in A,\)

\[
t(a) - s(a) \in j(B)
\]

Consequently, also then non-equivariantly \(M \square_s A = M \square_i A\).

We may bring any double split exact sequence to the form of definition 5.1, see for instance [3, lemma 6.3], or lemma 7.3 below, with \(M_2\)-action of the form \(\gamma \square \delta\).

6. **Actions on \(M_2(A)\)**

In this section we want to inspect closer how a \(M_2\)-action of a double split exact sequences looks like. This is a key lemma, and it is just a special case of lemma 7.2 but despite that let us restate it:

**Lemma 6.1** ([3, lemma 7.1]). Let \(S, T\) be two \(G\)-actions on a right functional \((B, \beta)\)-module \(\mathcal{E}\). Then

\[
\alpha_g \left( \begin{array}{cc} x & y \\ z & w \end{array} \right) = \left( \begin{array}{cc} S_g x S_g^{-1} & S_g y T_g^{-1} \\ T_g z S_g^{-1} & T_g w T_g^{-1} \end{array} \right)
\]

defines a \(G\)-action \(\alpha\) on \(M_2(\mathcal{E})\) which leaves the ideal \(M_2(\mathcal{K}_B(\mathcal{E}))\) invariant.

This is actually the inner action \(\text{Ad}(S \oplus T)\) on \(\mathcal{K}_B\left((\mathcal{E}, S) \oplus (\mathcal{E}, T)\right)\).

**Definition 6.2.** The \(\alpha\) of the last lemma is also denoted by \(\text{Ad}(S \oplus T)\) or \(\text{Ad}(S, T)\).
Lemma 6.3 (Cf. [3, lemma 7.3]). Let \((A, \alpha)\) and \((A, \delta)\) be \(G\)-rings.
Let \((M_2(A), \theta)\) be a \(G\)-ring and the corner embeddings \(e_{11} : (A, \alpha) \to (M_2(A), \theta)\)
and \(e_{22} : (A, \delta) \to (M_2(A), \theta)\) be equivariant.
Then \(\theta\) is of the form
\[
\theta_g \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} \alpha_g(x) & \beta_g(y) \\ \gamma_g(z) & \delta_g(w) \end{pmatrix}
\]
Also:
(i) One has the relations
\[
\gamma_g(ax) = \delta_g(a)\gamma_g(x) \quad \gamma_g(xb) = \gamma_g(x)\alpha_g(b)
\]
(ii) \((A, \gamma)\) is a functional Morita equivalence \(((A, \delta), (A, \alpha))\)-bimodule, see definition [14,4] below, where the bimodule structure is multiplication in \(A\), and the right module functional space \(\Theta_{(A,\alpha)}((A,\gamma))\) is given by left multiplication \(\theta_s(b) = ab\),
and the left module functional space \(\Theta_{(A,\delta)}((A,\gamma))\) is given by right multiplication \(g_\delta(a) = ab\).
(iii) Analogously, \((A, \beta)\) is a functional Morita equivalence \(((A, \alpha), (A, \delta))\)-
bimodule, which is inverse to the later one.
(iv) Let \(\chi : A \to \mathcal{L}_A(A)\) be the natural embedding given by \(\chi(a)(b) = ab\).
Then \(\alpha\) and \(\gamma\) are \(G\)-actions on the right \((A, \alpha)\)-module \(A\).
Consequently we have the \(G\)-action \(\text{Ad}(\alpha \oplus \gamma)\) on the matrix algebra \(M_2(\mathcal{L}_{(A,\alpha)}(A))\).
The map \(\chi \otimes 1_{M_2} : (M_2(A), \theta) \to (M_2(\mathcal{L}_{(A,\alpha)}(A)), \text{Ad}(\alpha \oplus \gamma))\)
is a \(G\)-equivariant injective ring homomorphism.
(v) \(\alpha\) and \(\gamma\), or \(\beta\) and \(\gamma\), determine \(\theta\) uniquely and completely.
(vi) \(\theta\) determines \(\alpha, \beta, \gamma\) and \(\delta\) uniquely.
(vii) In general, \(\alpha\) and \(\delta\) do not determine \(\gamma\) and thus not \(\theta\).
(viii) If we drop all assumptions then we may add:
A \(G\)-ring \((A, \alpha)\) and a right functional \((A, \alpha)\)-module action \(\gamma\) on \((A, \Theta_A(A))\)
alone (implicitly meaning also that \(\Theta_A(A)\) becomes \(\text{Ad}(\gamma, \alpha)\)-invariant) ensure the existence of the above \(\theta\) with all assumptions and assertions of this lemma.

Proof. By assumption, the corner rings must be \(G\)-invariant. Hence, by quadratik
we write an element of \(A\) as a sum of products \(abc\) and then by
\[
\begin{pmatrix} 0 & abc \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}
\]
we see that \(\theta_g\) applied to the first matrix has again the form of the first matrix.
(i) One computes expressions like
\[
\begin{pmatrix} \gamma_g(z) & 0 \\ 0 & \delta_g(x) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]
and uses the fact that \(\theta\) is a \(G\)-action on a ring.
(ii) The \(G\)-action property of modules is by line (5), and the \(G\)-invariance of the functional spaces follows from (7).
Lemma 6.5. Analogous proof as in lemma [3, lemma 7.8].

Proof. By (ii), $\alpha$ and $\gamma$ are $G$-actions as claimed, so that the existence of $\text{Ad}(\alpha \oplus \gamma)$ is by lemma [6.1. By relations (i) one can deduce

$$
\begin{pmatrix}
\alpha_g(x)x' \\
\gamma_g(z)z'
\end{pmatrix}
\begin{pmatrix}
\beta_g(y)y' \\
\delta_g(w)w'
\end{pmatrix}
= 
\begin{pmatrix}
\alpha_g(x\alpha_{g^{-1}}(x')) \\
\gamma_g(z\alpha_{g^{-1}}(z'))
\end{pmatrix}
\begin{pmatrix}
\alpha_g(y\gamma_{g^{-1}}(y')) \\
\gamma_g(w\gamma_{g^{-1}}(w'))
\end{pmatrix}
$$

for all $x,\ldots, w' \in A$, which shows $G$-equivariance of $\chi \otimes 1$. In fact, the second matrix line follows directly from [5], and the upper right corner from the first relation of [7].

(v) $\alpha$ and $\delta$ are determined by $\beta$ and $\gamma$ by [7]. The version $\alpha$ and $\gamma$ follows from (iv).

(viii) By (iv), we can construct $\text{Ad}(\alpha \oplus \gamma)$ and aim to define $\theta$ by its restriction. To show that the image of $\chi \otimes 1$ is $G$-invariant, we consider the right hand side of [10] and want to construct identity with the left hand side. For the first column this is clear by the $\gamma$- and $\alpha$-actions. For the upper right corner we use definition 2.3 that $X := \Theta(A,\alpha)(A,\gamma)$ is $G$-invariant. So as left multiplication $m_g$ by $y$ is in $X$, $\alpha_g \circ m_g \circ \gamma_{g^{-1}} = \beta_g(y)$ is also on $X$. For the lower right corner we write $w = w_1w_2$ by quadratik and then

$$
\gamma_g(w\gamma_{g^{-1}}(w')) = \gamma_g(w_1\alpha_g(w_2\gamma_{g^{-1}}(w'))) = \gamma_g(w_1)\beta_g(w_2)w' =: \delta_g(w)w' \quad \text{(under quotation marks)}.
$$

(vii) Take for example $G = \mathbb{Z}/2$, $A = \mathbb{C}$ (or any $A$), $\alpha = \delta$ the trivial action. Then $\gamma_g(x) = x$ and $\gamma_g(x) = (-1)^g x$ are two valid choices. \qed

Corollary 6.4 ([3 corollary 7.5]). Consider the double split exact sequence of definition $4.1$

(i) Then the ideal $M_2(j(B))$ is invariant under the action $\theta$.

(ii) The map $f \otimes 1 : (M_2(M), \theta) \to (M_2(A), \delta)$ is equivariant for the quotient $G$-action $\delta$ on $M_2(A) \cong M_2(M)/M_2(j(B))$.

Lemma 6.5 ([3 lemma 7.8]). Let $S, T$ be two $G$-actions on a $(B, \beta)$-module $E$.

Consider a diagram

$$
\begin{array}{ccc}
K_B(E) & \longrightarrow & L_B(E) \boxtimes A \\
\downarrow s_+ & & \downarrow s_+ \boxtimes 1 \\
A & \longrightarrow & A
\end{array}
$$

which is double split exact except that we have not found a $M_2$-action yet. But we know that $s_-$ is equivariant with respect to $\text{Ad}(S)$ on $L(E)$, and $s_+$ is equivariant with respect to $\text{Ad}(T)$ on $L(E)$.

Equip $M_2(L_B(E) \oplus A) \cong L_B(E \oplus E) \oplus M_2(A)$ with the $G$-action

$$
\text{Ad}(S \oplus T) \oplus (\alpha \otimes 1_{M_2})
$$

Then the following assertions are equivalent:

(i) $s_-(a)(S_gT_{g^{-1}} - S_gS_{g^{-1}})$ and $s_-(a)(T_gS_{g^{-1}} - T_gT_{g^{-1}})$ are in $K_B(E)$ for all $g \in G$, $a \in A$.

(ii) $S_g s_-(a)T_{g^{-1}} - s_-(a)S_gT_{g^{-1}} - s_-(a)S_{g^{-1}} - s_-(a)S_{g^{-1}}$ are in $K_B(E)$ for all $g \in G$, $a \in A$.

(iii) $M_2(L_B(E) \boxtimes A)$ is a $G$-invariant subalgebra.

In case that there is an invertible operator $U \in L(E)$ such that $T_g \circ U = U \circ S_g$ for all $g \in G$, these conditions are also equivalent to

(iv) $G(U) - U) \in K_B(E)$ for all $g \in G$, $a \in A$ (G-action is $\text{Ad}(S)$).

Proof. Analogous proof as in lemma [3 lemma 7.8]. One uses the fact that $S_gkT_{g^{-1}}$ is compact for compact operators $k$, see definition [2.4. The additional occurrences of $T_gS_{g^{-1}}$ are necessary here as we have no involution as in $C^*$-algebras. \qed
7. Computations with double split exact sequences

From now on, if nothing else is said, the $M_2$-action on $M\Box A$ is always understood to be of the form $\gamma \Box (\alpha \otimes 1_{M_2})$ for $G$-algebras $(M_2(M), \gamma)$ and $(A, \alpha)$, cf. the last paragraph of section 5.

Actions on $L_B(E)$ will always be of the form $\text{Ad}(S)$ for a $G$-action $S$ on $E$.

Lemma 7.1 ([3, lemma 8.1]). Given a ring homomorphism $f : A \to B$ we get a double split exact sequence

$$
\begin{array}{ccc}
B & \stackrel{i}{\longrightarrow} & B \oplus A \\
& \stackrel{g}{\longrightarrow} & \Delta \Box 1 \oplus A
\end{array}
$$

with $s_-(a) = (0, a), s_+(a) = (f(a), a)$ and one has $f = s_+ \mu \Delta_{s_-}$ in $GK$.

Lemma 7.2 ([3, lemma 8.2]). Given the first line and an equivariant ring homomorphism $\varphi$ as in this diagram it can be completed to this diagram

$$
\begin{array}{ccc}
B & \stackrel{\Box 0}{\longrightarrow} & M \Box A \\
& \uparrow & \downarrow \phi \\
B & \stackrel{\Box 0}{\longrightarrow} & M \Box X
\end{array}
\quad
\begin{array}{ccc}
A & \longrightarrow & A \\
\varphi & \downarrow & \varphi \\
X & \longrightarrow & X
\end{array}
$$

such that $\varphi(s_+ \Box 1) \mu \Delta_{s_-} = t_+ \mu \Delta_{t_-}$ in $GK$.

We assume here that the $G$-action on $M_2(M \Box A)$ is of the form $\theta \Box (\alpha \otimes 1)$.

Lemma 7.3 ([3, lemma 8.3]). Every double split exact sequence as in the first line is isomorphic to the one of the second line as indicated in this diagram:

$$
\begin{array}{ccc}
B & \stackrel{i}{\longrightarrow} & M \\
& \downarrow \phi & \downarrow \phi \\
B & \stackrel{j}{\longrightarrow} & L_B(B) \Box A \\
& \downarrow \Delta_{t_-} & \downarrow \Delta_{t_-} \\
& \Delta_+ & \Box 1 \\
& \Delta_t & \Box 1
\end{array}
$$

That is, $s_+ \mu \Delta_{s_-} = t_+ \mu \Delta_{t_-}$ in $GK$.

The $G$-action on $M_2(L_B(B) \Box A)$ is of the form $\text{Ad}(S \oplus T) \Box \delta$, where $(M_2(A), \delta)$ and $(M_2(L_B(B)), \text{Ad}(S \oplus T))$ are $G$-algebras.

For useful and important lemmas with respect to homotopy see [3, lemmas 8.10, 8.11], which work analogously also in the ring setting.

Lemma 7.4. Each $\Delta_t$ can be written as a double split exact sequence in $GK^G$.

More precisely, $\Delta_t = (1 \Box 1) \mu \Delta_{st} \Box 1$ with respect to the diagram

$$
\begin{array}{ccc}
J & \stackrel{j}{\longrightarrow} & M \\
& \downarrow \Delta_t & \downarrow \Delta_t \\
J & \stackrel{j}{\longrightarrow} & M \Box M
\end{array}
\quad
\begin{array}{ccc}
M & \stackrel{g}{\longrightarrow} & X \\
& \downarrow t & \downarrow t \\
M & \stackrel{g}{\longrightarrow} & M
\end{array}
$$

Proof. Note that $(1 - gt)(m) \in j(J)$ by the definition of the first line of the above diagram so that the lower line is double split by remark 5.3.
Set \( p(m \square n) = m \) and \( f(m \square n) = n \). By lemma 4.3(i), \( p \Delta_t = \Delta_{gt} \). (Note that \( M \square_{gt} M = M \square_1 M = \{(m + j) \oplus m | m \in M, j \in J\} \) and so \( f(gt \square 1)p = pgt \) on \( M \square_1 M \).) Thus
\[
\Delta_t = (1 \square 1)p \Delta_t = (1 \square 1)\Delta_{gt}
\]
We set the \( M_2 \)-action on the second line of the above diagram to \( (\theta \square \theta) \odot 1_{M_2} \) for \( (M, \theta) \) being the given \( G \)-action of the first line of the above diagram. Note that \( j(B) \) is invariant under the \( \theta \)-action by lemma 6.4 and so the \( M_2 \)-action of the second line is valid. 

The next lemma shows how we may unitize the ‘starter’ ring of a double split exact sequence.

**Lemma 7.5.** Let the left part of the first line of the following diagram be a given double split exact sequence, and \( \pi \) be the identical embedding. Then these data can be completed to this diagram

\[
\begin{array}{ccc}
B & \longrightarrow & M \square A \\
\downarrow \phi & & \downarrow \pi \\
B & \longrightarrow & M \square \tilde{A}
\end{array}
\]

\[
\begin{array}{cccc}
\stackrel{s_+}{\longrightarrow} & (A, \alpha) & \longrightarrow & M_\infty((A, \alpha)) \\
\stackrel{\tilde{s}_+}{\longrightarrow} & (\tilde{A}, \tilde{\alpha}) & \longrightarrow & M_\infty((\tilde{A}, \tilde{\alpha}))
\end{array}
\]

such that \( s_+ \mu \Delta s_+ = \pi \tilde{s}_+ \mu \Delta \tilde{s}_+ \). If \( M \) is unital, then one may replace \( \tilde{M} \) by \( M \) here.

**Proof.** Set
\[
\tilde{s}_+(a + \lambda 1_{\tilde{A}}) = s_+(a) + \lambda 1_{\tilde{M}} \square a + \lambda 1_{\tilde{A}}
\]
and \( \phi \) to be the identical embedding. The \( M_2 \)-action of the second line of the diagram is \( \tilde{\theta} \square \tilde{\alpha} \odot 1_{M_2} \) for being it \( \theta \square \alpha \odot 1_{M_2} \) of the first line. Verify the claim with lemma 4.3 for \( \Phi := \phi \odot 1_{M_2} \), thereby recalling remarks 4.4 and 5.3.

8. **Calculations with corner embeddings**

The next proposition shows how compact operators on a module over compact operators may be regarded as compact operators of a plain module.

**Proposition 8.1.** Let \((B, \beta)\) be a ring, \((E, X)\) a weakly cofull \((B, \beta)\)-module and \((F, Y)\) a cofull \(K_B((E \oplus B, X \oplus \beta))\)-module.

Then there is a ring isomorphism
\[
\pi : K_{K_B(E \oplus B)}(F) \rightarrow K_B(FMB) : \pi(T) = T|_{\mathcal{F}MB}
\]
where \( M_B \subseteq K_B(E \oplus B) \) is the subalgebra of corner multiplication operators \( m_B \) defined by \( m_B(\eta \oplus d) = \eta \odot bd \) for \( b, d \in B, \eta \in E \).

The additive subgroup \( \mathcal{F}MB := \{ \xi m_B \in \mathcal{F} | \xi \in \mathcal{F}, \eta \in M_B \} \) of \( \mathcal{F} \) is turned into a \( B \)-module by setting \( \eta \cdot b := \eta m_B \) for \( b \in B \) and \( \eta \in \mathcal{E}MB \).

Turn the the right \((B, \beta) \cong (M_B, \text{Ad}(\beta))\) module \( \mathcal{F}MB \) into a cofull functional module by defining
\[
\Theta_B(\mathcal{F}MB) := \{ m \circ \phi_{\mathcal{F}MB} | m \in M_B, \phi \in \Theta_{K_B(E \oplus B)}(F) \}
\]
The \( G \)-action on \( \mathcal{F}MB \) is the one induced by restriction of \( Y \).
Proof. (a) We claim that for each $z \in A := K_B(\mathcal{E} \oplus B)$ there are $x_i \in A_1 := K_B(B, \mathcal{E} \oplus B) \subseteq A, y_i \in A_2 := K_B(\mathcal{E} \oplus B, B) \subseteq A, b_i \in B$ such that $z = \sum_{i=1}^n x_i m_{b_i} y_i$.

Indeed, let $z = \theta_{\xi, \phi}$. Write $\xi = \sum_{i=1}^n \zeta_i b_i d_i$ for $\zeta \in \mathcal{E} \oplus B, b_i, d_i \in B$ by cofullness of $\mathcal{E} \oplus B$ and quadratik of $B$. Set $\psi \in \Theta_B(\mathcal{E} \oplus B)$ to be the canonical projection onto the coordinate $B$ (said in the sloppy sense). Then, for $\eta \in \mathcal{E} \oplus B$,

$$z(\eta) = \xi \phi(\eta) = \sum_{i=1}^n \zeta_i b_i d_i \phi(\eta) = \sum_{i=1}^n \theta_{\zeta_i, \psi} \circ m_{\phi_i} \circ \theta_{(\zeta, \psi), \phi}(\eta)$$

The last formula also precises what is meant by $\pi$ as in claim (a)), then, because $z \in M_B$.

(b) Let us view $FM_B$ as a $M_B$-module. As $m\phi(\xi n) = m\phi(\xi)n \in M_B$ for $\phi \in \Theta_B(\mathcal{F}), \xi \in \mathcal{F}, m, n \in M_B$, we see that the functional $m\phi$ of the functional space $FM_B$ maps into $M_B$. To show cofullness of $FM_B$ we write, using cofullness of $\mathcal{F}$ two times and quadratik of $B$ and (a),

$$n = \sum_{i,j=1}^n \xi_i \phi_i(\zeta_j m) = \sum_{i,j=1}^n \xi_i x_{i,j} m_{d_i,j} \phi_i(\zeta_j m)$$

for $\xi, \zeta, \eta, \xi_i, \eta_i \in \mathcal{F}, m_{i,j}, n_{i,j} \in M_B, x_{i,j} \in A_1, y_{i,j} \in A_2$.

(c) Next we prove that $\pi(T)$ is a compact operator as claimed.

By claim (a), the cofullness of $\mathcal{F}$ and quadratik of $B$ we may write a $\xi \in \mathcal{F}$ as $\xi = \sum_{i=1}^n \zeta_i x_i m_{b_i, c_i, y_i}$ for $\zeta_i \in \mathcal{F}, x_i \in A_1, y_i \in A_2, b_i, c_i \in B$.

Then, for $\eta \in \mathcal{F}$ and $\theta_{\xi, \phi} \in \mathcal{K}_A(\mathcal{F}),$

$$\theta_{\xi, \phi}(m b) = \sum_{i=1}^n \zeta_i x_i m_{b_i, c_i, y_i} \phi(\eta) m b = \sum_{i=1}^n \zeta_i \phi_i(\psi m b)$$

for $\zeta_i := \zeta_i x_i m_{b_i}$ and $\phi_i := m_{c_i, y_i} \phi | FM_B$.

(d) We claim that $\pi$ is injective.

Using again claim (a) as before for $\xi \in \mathcal{F}$, assuming $\pi(T) = 0$ gives

$$T(\xi) = \sum_i T(\xi) x_i m_{b_i, y_i} = \sum_i T(\xi) x_i m_{b_i} y_i = 0$$

(e) We claim that $\pi$ is surjective.

Indeed, given $S \in \mathcal{K}_B(\mathcal{F}M_B)$ we try to define its preimage $T \in \mathcal{K}_B(\mathcal{E} \oplus B)(\mathcal{F})$ by

$$T \left( \sum_{i=1}^n \xi_i x_i m_{b_i, y_i} \right) := \sum_{i=1}^n (\sigma \circ S)(\xi_i x_i m_{b_i} y_i)$$

for all $\xi_i \in \mathcal{F}, x_i \in A_1, y_i \in A_2, b_i \in B$. Here, $\sigma : FM_B \to \mathcal{F}$ is the identity embedding.

If we multiply both sides of the above identity with $z = \sum_{j=1}^m p j m d j q j \in A$ ($p_j \in A_1, q_j \in A_2, d_j \in B$ as in claim (a)), then, because $y_ip_j$ is an element in $M_B$, we get

$$T \left( \sum_{i=1}^n \xi_i x_i m_{b_i, y_i} \right) z = \sum_{j=1}^m (\sigma \circ S) \left( \sum_{i=1}^n \xi_i x_i m_{b_i, y_i} p_j m d, q_j \right) m d, q_j = 0$$

Because $z \in A$ was arbitrary, and $\mathcal{K}_B(\mathcal{F}M_B)$ is an ideal in itself by lemma 2.14 and (b), we conclude that $T$ is well-defined. Notice that by claim (a), $T$ is fully defined. □
The next corollary shows that the composition of two corner embeddings is again one.

**Corollary 8.2.** Let the first line of the following diagram be given, where \( e \) and \( f \) are corner embeddings.

\[
\begin{array}{ccc}
B & \xrightarrow{e} & \mathcal{K}_B(\mathcal{E} \oplus \mathbb{H}_B) \\
& \downarrow{g} & \downarrow{\phi} \\
\mathcal{K}_B(\mathcal{Z} \oplus \mathbb{H}_B) & \xrightarrow{\psi} & \mathcal{K}_B\left((\mathcal{F} \oplus \mathbb{H}_{\mathcal{K}_B(\mathcal{E} \oplus \mathbb{H}_B)})M_B\right)
\end{array}
\]

Then one can make the above commuting diagram where \( \phi \) and \( \psi \) are ring isomorphisms and \( g \) is a corner embedding.

**Proof.** Let \( M_B \subseteq A := \mathcal{K}_B(\mathcal{E} \oplus \mathbb{H}_B) \) be the subring of corner multiplication operators as in proposition \([1, \text{lemma } 8.1]\) acting on the distinguished coordinate \( B \) of \( \mathbb{H}_B \).

Then there is a \( B \)-module isomorphism \((B \text{ identified with } M_B)\), recalling also lemma \([2, \text{lemma } 2.5]\),

\[
AM_B \cong \mathcal{K}_B(B, \mathcal{E} \oplus \mathbb{H}_B) \cong \mathcal{K}_B(B, \mathcal{E}) \oplus \mathbb{H}_B
\]

Hence, using this isomorphism for the distinguished first coordinate \( A \) of \( \mathbb{H}_A \) we get a \( B \)-module isomorphism

\[
(\mathcal{F} \oplus \mathcal{A})M_B \cong \mathcal{F}M_B \oplus (\mathcal{H}_A \oplus A)M_B \oplus \mathcal{K}_B(B, \mathcal{E}) \oplus \mathbb{H}_B = \mathcal{Z} \oplus \mathbb{H}_B
\]

for \( \mathcal{Z} \) obviously defined. This yields \( \psi \) and the above commuting diagram. \(\square\)

In the following lemma we show how the inverses of corner embeddings can skip ring homomorphisms.

**Lemma 8.3** (Cf. \([3, \text{lemma } 8.8]\)). Let a corner embedding \( e \) and a ring homomorphism \( \pi \) as in the following diagram be given:

\[
\begin{array}{ccc}
A & \xrightarrow{e} & \mathcal{K}_A(\mathcal{E} \oplus \mathbb{H}_A) \\
& \downarrow{\pi} & \downarrow{\phi} \\
B & \xrightarrow{f} & \mathcal{K}_B((\mathcal{E} \oplus \mathbb{H}_A) \otimes_\pi B \oplus \mathbb{H}_B)
\end{array}
\]

Then we draw the above commuting diagram where \( f \) is a corner embedding.

In particular, \( e^{-1}\pi = \phi f^{-1} \) in \( GK^G \).

**Proof.** Define \( \phi(T) = T \otimes 1 \oplus 0_{\mathbb{H}_B} \). At first we write

\[
\mathcal{K}_B((\mathcal{E} \oplus \mathbb{H}_A) \otimes_\pi B \oplus \mathbb{H}_B) \cong \mathcal{K}_B(\mathcal{E} \oplus \mathbb{H}_A \otimes_\pi B \oplus \mathbb{H}_B)
\]

such that \( e\phi(a) \) acts by \( a \)-multiplication on the summand \((A \otimes_\pi B, \alpha \otimes \beta)\).

By lemma \([2, \text{lemma } 2.6]\)(i) we have a \( B \)-module isomorphism

\[
X : A \otimes_\pi B \to B_0 := \sum \pi(A)B \subseteq B : X(a \otimes b) = \pi(a)b
\]

We have an equivariant ring homomorphism

\[
h : M_2(\pi(A)) \to \mathcal{K}_B(B_0 \oplus B) \subseteq \mathcal{K}_B(B_0 \oplus \mathbb{H}_B)
\]

by matrix-vector multiplication, where the summand \( B \) means here the distinguished first coordinate \((B, \beta)\) of \( \mathbb{H}_B \).
(Here we use cofullness of $A$, i.e. $\pi(a)b = \sum_i \theta_{\pi(a_i),\pi(a'_i)}b_i$.)

Note that $e\phi(a) = h(diag(\pi(a),0))$. We may rotate this to $h(diag(0,\pi(a))) = \pi f(a)$ by a rotation homotopy in the domain of $h$, see definition \ref{def:rotation-homotopy}.

Hence we get $e\phi = \pi f$ in $G^K$ by homotopy.

\[\square\]

**Definition 8.4.** Define $L_0G^K$ (‘level-$0$ morphisms’) to be the smallest additive subcategory of $G^K$ generated by all ring homomorphisms.

So the morphisms in $L_0G^K(A,B)$ are exactly those of the form $\pm f_1 \pm f_2 \pm \ldots \pm f_n$ for some ring homomorphisms $f_i : A \to B$ for all $1 \leq i \leq n$.

The next lemma shows how inverses of corner embeddings can skip such morphisms:

**Lemma 8.5.** If $e \in G^K(A,D)$ is a corner embedding and $z \in L_0G^K(A,B)$ then there is a corner embedding $f \in G^K(B,E)$ and a $w \in L_0G^K(D,E)$ such that

$$ e^{-1}z = wf^{-1} $$

**Proof.** If $z = \pi_1 \pm \ldots \pm \pi_n$ then we do the diagram \ref{diag:z-decomposition} for each $\pi_i$ but with one constant corner embedding $f$ by replacing the lower right corner of diagram \ref{diag:z-decomposition} by $K_B(\oplus \oplus_1 (E \oplus H_A) \oplus \pi_i, B \oplus H_B)$, which works for all $\pi_i$. We obtain then by (the proof of) lemma \ref{lem:corner-lift} that $e^{-1} \pi_i = \phi_i f^{-1}$ for all $1 \leq i \leq n$.

\[\square\]

9. \textsc{Calculations with extended double split exact sequences}

From now on, we make the convention that for double split exact sequences as in definition \ref{def:double-split-sequences} the ring homomorphism $f \otimes 1_{M_2} : (M_2(M),\theta) \to (M_2(A), \alpha \otimes 1_{M_2})$ is $G$-equivariant. In other words, $\delta = \alpha \otimes 1_{M_2}$ in corollary \ref{cor:G-equivalence}

**Definition 9.1.** By an extended double split exact sequence we mean a diagram of the form

$$
\begin{array}{ccc}
(Z,\zeta) & \xrightarrow{\epsilon} & (B,\beta) \\
\downarrow e & & \downarrow j \\
(M,\gamma) & \xrightarrow{\phi} & (A,\alpha)
\end{array}
$$

where $\epsilon$ is a corner embedding and the other part is a double split exact sequence as in definition \ref{def:double-split-sequences}. The associated morphism in $G^K$ to this extended double split exact sequence is $s\mu \Delta e^{-1}$.

We shall also abbreviate $s\mu \Delta e^{-1} = s\nabla_t$ by setting $\nabla_t := \mu \Delta e^{-1}$.

The next lemma shows how we may add or subtract superfluous modules involved in extended double split exact sequences.

**Lemma 9.2.** Let the first line of the following diagram be given and $(\mathcal{F},T)$ be any weakly cofull $B$-module. Then we can draw this diagram

\[\text{(11)}\]

such that $s_+\mu \Delta e^{-1} = t_+\mu \Delta f^{-1}$.
Proof. Set \( \phi(T \square a) = T \otimes 0_{\mathcal{F}} \square a \) and \( \psi(T) = T \otimes 0_{\mathcal{F}} \). Define \( t_{\pm} = s_{\pm} \phi \) and verify the claim with lemma 4.3. The \( M_{2} \)-action of the second line, to be checked by lemma 6.5, is \( \text{Ad}(S \otimes T) \otimes \mathcal{F} \square a \) if of the first line it is \( \text{Ad}(S \otimes \mathcal{F}) \square a \). Check the claim with lemma 7.3 and recall remark 4.4.

Lemma 9.3. Every double split exact sequence can be turned to an extended double split exact sequence, even to the form (11) with \( s_{\pm} \) being of “standard form”, i.e. \( s_{\pm} = s'_{\pm} \oplus 0_{\mathcal{F}} \square 1 \).

Proof. Every double split exact sequence can be turned to an extended one by composing with the inverse of the identity corner embedding \( e : B \to B \cong K_{B}(B) \). But by application of lemmas 7.3 and 7.1, we can also achieve this for corner embeddings of the form \( e : B \to B \cong K_{B}(\mathbb{H}_{B}) \) for bigger direct sums \( \mathbb{H}_{B} \).

Lemma 9.4. A double split exact sequence (11) can be equivalently turned to the same one but with \( E \) replaced by \( \mathcal{F} \). In other words, all involved modules then are separated by the functionals.

Proof. Do an analogous proof as in lemma 9.2, but where in the second line of the diagram \( E \oplus \mathcal{F} \) is replaced by \( \mathcal{F} \), and \( \phi \) and \( \psi \) are induced by the map of lemma 2.19.(i).

In this lemma we show how an extended double split exact sequence may absorb a ring homomorphism from the other side than in lemma 7.2.

Lemma 9.5. Let an extended double split exact sequence as in the first line of the following diagram and a ring homomorphism \( \pi \) as in the following diagram be given:

\[
\begin{array}{ccccccccc}
B & \xrightarrow{e} & K_{B}(E \oplus \mathbb{H}_{B}) & \xrightarrow{\pi} & L_{B}(E \oplus \mathbb{H}_{B}) \square A & \xrightarrow{s_{\pm}} & A \\
\psi & & \downarrow \phi & & \downarrow & & \downarrow \\
D & \xrightarrow{f} & K_{D}(E \oplus \mathbb{H}_{B} \otimes \pi D \oplus \mathbb{H}_{D}) & \xrightarrow{g} & L_{D}(E \oplus \mathbb{H}_{B} \otimes \pi D \oplus \mathbb{H}_{D}) \square A & \xrightarrow{t_{\pm}} & A \\
\end{array}
\]

(i) Then we can complete these data to the above diagram where the second line is extended double split and \( s_{\pm} \mu \Delta_{s_{\pm}} e_{-1}^{-1} \pi = t_{\pm} \mu \Delta_{t_{\pm}} f^{-1} \).

(ii) If \( s_{\pm} \) is standard (see lemma 9.3) then this morphism is also the one of the third line, that is, \( s_{\pm} \mu \Delta_{s_{\pm}} e_{-1}^{-1} \pi = v_{\pm} \mu \Delta_{v_{\pm}} g^{-1} \).

Proof. (i) Set \( \phi(T \square a) = T \otimes 1 \oplus 0_{\mathcal{F}} \square a \), \( \psi(T) = T \otimes 1 \oplus 0_{\mathcal{F}} \), and \( t_{\pm} = s_{\pm} \phi \).

Because \( s_{\pm}(a) - s_{-}(a) \) is in the ideal of compact operators by remark 4.3, \( t_{\pm}(a) - t_{-}(a) \) is in the ideal of compact operators by lemma 2.17.(v), and so the second line of the above diagram is double split.

If the \( M_{2} \)-action of the first line of the diagram is \( \text{Ad}(S \otimes T) \), then on the second line we set it to \( \text{Ad}(S \otimes \delta \otimes R, T \otimes \delta \otimes R) \), where \( D = (D, \delta) \) and \( \mathbb{H}_{D} = (\mathbb{H}_{D}, R) \).

Since for all \( a \in A \)

\[
s_{-}(a)((S_{g} \otimes \delta_{g})(T_{g^{-1}} \otimes \delta_{g^{-1}}) - (S_{g} \otimes \delta_{g})(S_{g^{-1}} \otimes \delta_{g^{-1}})) \in K_{D}(E \oplus \pi D)
\]
by lemmas 6.5 and 2.17 (v), the \( M_2 \)-action of the second line is valid by lemma 6.5 again. The first claim is then verified by lemma 8.3 and lemma 4.3 with \( \Phi = \phi \otimes 1_{M_2} \) and recalling remark 4.4.

(ii) The last claim with the third line simply follows from lemma 9.2 applied to \( F := \mathbb{H}_B \otimes \pi D \).

**Definition 9.6.** The maps \( t_\pm \) of the last lemma (or the \( v_\pm \) if applicable) are denoted by \( s^\pm_\pm := t_\pm \) (or \( s^\mp_\pm := v_\pm \)). So one has, under the assumptions of the last lemma,

\[
s_+ \nabla s_- \pi = s^\mp_+ \nabla s^\pm_-
\]

The next proposition shows that the inverse of a plain corner embedding can be fused with a double split exact sequence.

**Proposition 9.7.** Let \( A \) be unital. Let the first line, an extended double split exact sequence, and the plain matrix embedding \( g \) (i.e. without occurrence of \( E \), but still with any \( G \)-action) of the following diagram be given. Thereby, the cardinalities of the index sets of the direct sums \( \mathbb{H}_A \) and \( \mathbb{H}_B \) may differ. We also need to assume that \( s_-(1) = s_+(1) = 1_\mathcal{E} \).

\[
\begin{array}{cccccc}
B & \xrightarrow{\mathcal{K}_B(\mathcal{E} \oplus \mathbb{H}_B)} & \mathcal{L}_B(\mathcal{E} \oplus \mathbb{H}_B) \mathcal{C} A & \xleftarrow{\phi} & A \\
\downarrow \phi & & \downarrow g & & \\
B & \xrightarrow{\mathcal{K}_B(H_{\mathcal{E}} \oplus \mathbb{H}_B)} & \mathcal{L}_B \big( H_{\mathcal{E}} \oplus \mathbb{H}_B \big) \mathcal{C} K_A(\mathbb{H}_A) & \xrightarrow{t_\pm} & K_A(\mathbb{H}_A)
\end{array}
\]

Then we complete these data to the above diagram such that, in \( GK^G \),

\[
g^{-1}(s_+ \oplus 0) \nabla s_- \oplus 0 = t_+ \nabla t_-
\]

**Proof.** We at first ignore any \( G \)-action.

We use the functional \( B \)-module isomorphism \( p_- : A \otimes s_- \mathcal{E} \to \mathcal{E} \) defined by \( p_-(a \otimes \xi) = s_-(a)(\xi) \), and its inverse isomorphism defined by \( p_-^{-1}(\xi) = 1 \otimes \xi \) of lemma 2.20 (iv), and an analogous \( p_+ \) for \( s_+ \).

Let \( \pi_{\pm} : H_{\mathcal{E}} \to \mathbb{H}_A \otimes s_{\pm} \mathcal{E} \) be the canonical isomorphisms induced by \( p_-^{-1} \) and \( p_+^{-1} \), respectively, where \( H_{\mathcal{E}} := \oplus_{i \in I} \mathcal{E} \).

Set

\[
\phi(T \mathcal{C} a) = T \oplus 0 \oplus 0 \oplus \cdots \, \Delta g(a)
\]

That means, more precisely, \( T \) canonically operates on the right hand side of the identity only at the first summand of \( H_{\mathcal{E}} \) and on \( \mathbb{H}_B \).

We set

\[
t_\pm(T) = \pi_\pm \circ (T \otimes 1_{\mathcal{E}}) \circ \pi_\mp^{-1} \oplus 0_{\mathbb{H}_B} \mathcal{C} T
\]

Let us write \( e_i \) for the canonical generators of \( \mathbb{H}_A \) with entry \( 1_A \) in coordinate \( i \) and otherwise zero, and let \( T(e_i)_j \in A \) denote the \( j \)th coordinate of \( T(e_i) \).

Then

\[
(t_-(T) - t_+(T)) (\oplus_i \xi_i) = \bigoplus_j \left( (s_- - s_+) (T(e_i)_j) \right) (\xi_i) \mathcal{C} 0 = \bigoplus_j \sum_i k_{i,j}(\xi_i) \mathcal{C} 0
\]

for \( \xi_i \in \mathcal{E}, T \in K_A(\mathbb{H}_A) \) and operators \( k_{i,j} \in \mathcal{L}_B(\mathcal{E}, A) \) obviously defined. But they are compact operators, because the first line of the above diagram is split.
exact. Hence the second line of the diagram is verified to be non-equivariantly
double split, because $T$ is a compact operator and so almost all $k_{i, j}$ are zero.

If the $M_2$-action of the first line of the diagram is $\text{Ad}(S \oplus R, T \oplus R) \Box (\alpha \otimes 1)$
then on the second line it is put to

$$\text{Ad}(\pi_-^{-1} \circ (V \otimes S) \circ \pi_-, \pi_+^{-1} \circ (V \otimes T) \circ \pi_+ \oplus R) \Box \text{Ad}(V)$$

where $S, T$ act on $E, R$ on $\mathbb{H}_B$ and $V$ on $\mathbb{H}_A$.

Since $(\mathbb{H}_A, V) = (\mathbb{H}_A, \alpha \oplus W)$ by definition [3.3] the map $\Phi := \phi \otimes 1_{M_2}$ between
the $M_2$-spaces of the first and second line of the above diagram is seen to be $G$-
equivariant, because $\pi_-^{-1} \circ (\alpha \otimes S) \circ \pi_-$ is an identity on the first summand $E$ of $H_E$. Also
recall the formulas of lemma [6.3].

By lemma [6.5] applied to the first line of the above diagram, we have $X_g := T_g T_{g^{-1}} - T_g S_{g^{-1}} \in \mathcal{K}_A(E)$.

If we multiply

\[(12) \pi_-^{-1} \circ (V \otimes S_g) \circ \pi_- \circ \pi_+^{-1} \circ (V \otimes T_{g^{-1}}) \circ \pi_+ \]

from the right hand side with the compact operator $X_g = \pi_+^{-1} \circ (1 \otimes X_g) \circ \pi_+$ then we get [12] minus [12] again, but where $T_g^{-1}$ is replaced by $S_g^{-1}$.

By lemma [6.3] this verifies the validity of the $M_2$-action of the second line of the above diagram.

Set $s_{\pm} := s_+ \oplus 0 \Box 1$. Notice that we have $s_{\pm} \phi = gt_{\pm}$. Finalize the proof by
checking lemma [1.3] with $a := g, b := 1_B, \Phi := \phi \otimes 1_{M_2}, \phi := \psi := \phi, t$ to obtain

$$s_+ \nabla s_- = gt_+ \nabla t_-.$$  

This is the counterpart to [3 lemma 13.1], now with another proof, as the cited
one would not work in our ring setting:

**Corollary 9.8.** The inverse of a plain matrix embedding $e : (A, \alpha) \to M_{\infty}((A, \alpha))$
can be fused with an extended double split exact sequence, i.e. if $s_+ \nabla s_-$ is given
then there is $t_+ \nabla t_-$ such that $e^{-1} s_+ s_- = t_+ t_-$. 

**Proof.** We do the diagram of lemma [7.5] such that

$$e^{-1} s_+ s_- = (\pi \otimes 1) f^{-1} s_+ s_- = (\pi \otimes 1) t_+ t_- = v_+ v_-$$

where the last two identities are by proposition [9.7] and lemma [7.2].

**Lemma 9.9 ([3 lemma 14.1]).** Let two extended double split exact sequences like
in line (12) be given, say for actions $S, T$ and homomorphisms $s_{\pm}, t_{\pm}$. Then we

**can sum up these diagrams to**

$$B \xrightarrow{\mathcal{K}_B(E \oplus F \oplus \mathbb{H}_B) \Box A} \mathcal{L}_B((E, S) \oplus (F \oplus \mathbb{H}_B, T)) \Box A \xrightarrow{s_- \oplus t_-, s_+ \oplus t_+} A$$

and this corresponds to the sum of the associated elements in $GK^G$, i.e.

$$s_+ \mu \Delta s_- e^{-1} + t_+ \mu \Delta t_- e^{-1} = (s_+ \oplus t_+) \mu \Delta s_+ s_- e^{-1}$$

The $M_2$-action is $\text{Ad}(V \oplus W) \Box (\alpha \otimes 1)$ for the two $M_2$-actions $\text{Ad}(V) \Box (\alpha \otimes 1)$
and $\text{Ad}(W) \Box (\alpha \otimes 1)$ of the given diagrams.

(Note that we used the wrong but suggestive notation $s_- \oplus t_- := x \oplus y \Box 1$ for
$s_- = x \Box 1, t_- = y \Box 1$. Also, one has actually different $e$.)
Corollary 9.10 ([3] corollary 14.2). Consider the extended double split exact sequence of line (11). Make its ‘negative’ diagram where we exchange \( s_- \) and \( s_+ \) and transform the \( M_2 \)-action under coordinate flip. Then its associated element in \( GK^G \) is the negative, that is,

\[-s_+ \mu \Delta s_- e^{-1} = s_- \mu \Delta s_+ e^{-1}\]

Lemma 9.11. Every generator morphism (e.g. \( \varphi, f^{-1}, \Delta s \)) of \( GK^G \) can be expressed as an extended double split exact sequence.

Proof. By lemmas 7.1 and 7.4 we can write ring homomorphisms \( \pi \) and synthetical splits \( \Delta s \) as double split exact sequences, and to them we apply lemma 9.3. For a corner embedding we write \( e^{-1} = 1 e^{-1} \) for the double split exact sequence 1 by lemma 7.1. □

10. \( \nabla \)-Calculation

The idea of this section is to take an arbitrarily complicated expression \( x \) in \( GK^G \), and ‘solve’ (decide the truth of) the equation \( x = 0 \) in \( GK^G \) by equivalently reformulating it as \( y = 0 \) in \( GK^G \), where \( y \) is a level-0 morphism.

Definition 10.1. A term of \( N \) elements in a ring (or additive category) \( A \) is an expression in \( A \) using plus, minus and multiplication signs, and no brackets and \( N \) elements \( x_1, \ldots, x_N \) (copies are counted again).

For example, \( x_1x_4 + x_2x_3x_4 - x_5x_6x_7 + x_1x_7x_3x_5 \) is a term of 12 elements.

Lemma 10.2. Every morphism \( z \) in \( GK^G \) can be expressed as a term of extended double split exact sequences. This term may even be chosen to be the product \( z = x_1x_2 \ldots x_n \) of \( n \) extended split exact sequences \( x_1, \ldots, x_n \).

Proof. The first assertion follows simply by lemma 9.11 by expressing \( z \) as a term of generators and then expressing all those by extended double split exact sequences.

By lemma 9.11 write any identity ring homomorphism \( 1_A \) as an extended split exact sequence and then form its negative by corollary 9.10 such that \(-1_A \) is an extended double split exact sequence, so a word. In this way we may rewrite any morphism \( z \) as a positive sum \(+ w_1 + w_2 + \ldots + w_n \) of words \( w_i \) in \( GK^G \). By [4] lemma 3.6] such a morphism can be expressed as a single words \( w \). Now apply lemma 9.11. □

Definition 10.3. A morphism \( P \in GK^G(A, B) \) is called right-invertible in \( GK^G \) (or \( GK^G \)-right-invertible) if there is a morphism \( Q \in GK^G(B, A) \) such that \( PQ = 1 \in GK^G(A, A) \).

Lemma 10.4. Let

\[ x = s_+ \nabla s_- = s_+ \mu \Delta s_- e^{-1} = s_+ e_{22} e_{11}^{-1} \Delta s_- e^{-1} \]

be the associated morphism of an extended double split exact sequence, where \( \mu = e_{22} e_{11}^{-1} \). Then there is a ring homomorphism \( P : B \to D \), which is right-invertible in \( GK^G \), such that

\[ xP = s_+ e_{22} - s_- e_{11} \]

in \( GK^G \). Actually, \( P = e_{s_-} e_{11} \) and \( D = M_2(M) \) is the ‘\( M_2 \)-space’ of \( x \).
Proof. Indeed, \( P = e_{j_+} e_{11} \) is a ring homomorphism with right inverse \( e_{11}^{-1} \Delta_+ e^{-1} \).

Consider the valid ring homomorphism \( f \otimes 1_{M_0} : D \to (M_2(\alpha) \otimes 1_{M_0}) \) by corollary 6.4 and the convention for double split exact sequences that \( \delta = \alpha \otimes 1 \).

Then \( e_{11} f \otimes 1 \) is a (level-0) morphism in \( L \).

By the last filtration this is clear as \( e_{22} e_{11}^{-1} f = f E_{22} E_{11}^{-1} = f' \), the last identity by a rotation homotopy, and where \( f' = f \) non-equivariantly, but with a different source coming from the lower right corner of \( D \).

Consequently,
\[
s_+ e_{22} e_{11}^{-1} \Delta_+ e^{-1} \cdot e_{j_+} e_{11} = s_+ e_{22} e_{11}^{-1} (1 - f s_+) e_{11} = s_+ e_{22} - s_- e_{11}
\]

\( \square \)

**Definition 10.5.** For \( n \geq 1 \) define \( L_n GK^G(A, B) \) (‘level-\( n \) morphisms’) to be the abelian subgroup of the abelian group \( GK^G(A, B) \) which is generated by (so \( \mathbb{Z}\)-linear span of) all elements which are expressible as a product \( x_1 \ldots x_n \) of \( n \) extended double split exact sequences \( x_1, \ldots, x_n \) in \( GK^G \).

As \( 1 \in GK^G(A, B) \) is expressible as a double split exact sequence by lemma 9.11 it is clear that a level-\( n \) morphism \( x_1 \ldots x_n \) can be written as a level-(\( n + 1 \)) morphism \( x_1 \ldots x_{n+1} \), so that we get a filtration of subgroups
\[
L_0 GK^G(A, B) \subseteq L_1 GK^G(A, B) \subseteq L_2 GK^G(A, B) \subseteq \ldots
\]
of \( GK^G(A, B) \) whose union is the whole group \( GK^G(A, B) \) by lemma 10.2

**Lemma 10.6.** If \( L_1 GK^G(A, B) \) is an infinite set then the cardinalities of \( L_1 GK^G(A, B) \) and \( GK^G(A, B) \) coincide.

Proof. By the last filtration this is clear as \( L_n GK^G(A, B) \) has only at most countably many times the number of elements of \( L_1 GK^G(A, B) \).

\( \square \)

**Lemma 10.7.** Let \( z \) be a morphism in \( GK \) which is a term of \( N \) extended split exact sequences.

(i) Then there is a ring homomorphism \( P \), which is right-invertible in \( GK^G \), such that \( zP \) is a term of \( N - 1 \) extended split exact sequences.

(ii) There is also a ring homomorphism \( P \), right invertible in \( GK^G \), such that \( zP \) is a (level-0) morphism in \( L_0 GK^G \).

Proof. (i) Consider at first a term \( z = X_1 \ldots X_N \) which is just a product of extended split exact sequences \( X_i \).

Say, \( X_N = s_+ \mu \Delta_+ e^{-1} \), where \( \mu = e_{22} e_{11}^{-1} \). Set \( P = e_{j_+} e_{11} \). Then by lemma 10.4 we get \( X_N P = s_+ e_{22} - s_- e_{11} \).

If \( N = 1 \) we are done. Otherwise proceed by writing
\[
X_1 \ldots X_{N-2} X_{N-1} (s_+ e_{22} - s_- e_{11}) = X_1 \ldots X_{N-2} (Y - Z) = X_1 \ldots X_{N-2} W
\]
with lemmas 9.5, 9.9 and corollary 9.10 where \( Y, Z, W \) are extended double split exact sequences. We have achieved that \( zP \) is a product of just \( N - 1 \) extended double split exact sequences.

By lemma 10.2 this is already enough, but let us show that if we have an arbitrary term, say \( z = X_1 \ldots X_N + Y_1 \ldots Y_n \) we may do it analogously. We multiply it by \( P \) to eliminate \( X_N \) and fuse \( Y_n P \) to a single double split exact sequence \( Y_n' \) by lemma 9.5. So we get \( zP = X_1 \ldots X_{N-2} W + Y_1 \ldots Y_{n-1} Y_n' \).

(ii) Applying (i) successively \( N \) times we achieve the last claim. So we set \( P = P_N \ldots P_1 \), where \( P_N, \ldots, P_1 \) are the ring homomorphisms repeatedly chosen by (i).
Theorem 10.8. Let $F : GKG \to D$ be a (not necessarily additive) functor into a category $D$. Then $F$ is faithful if and only if $F$ is faithful on the subcategory $L_0GKG$.

Proof. Let $z, w \in GKG(A, B)$ be given such that $F(z) = F(w)$. Choose a ring homomorphism $P$ and a right-inverse $Q$ in $GKG$ such that both $zP, wP$ are level-0 morphisms by lemma 10.4(iii). (Say $P = P_1P_2$ are successively chosen such that $zP_1$ and $wP_2$ are level-0.)

Then $F(z)F(P) = F(zP) = F(wP)$. Hence, by faithfulness of $F$ on $L_0GKG$ we get $zP = wP$. Thus $zPQ = z = w$. □

Example 10.9. We ask whether we could turn the product of two extended double split exact sequences $s_+ \nabla s_-$ and $t_+ \nabla t_-$ to one extended double split exact sequence $u_+ \nabla u_-$ and make the following ansatz and equivalent reformulations to this end (here and below, $e_+ := e_{22}, e_- := e_{11}, E_+ := E_{22}$, etc. for corner embeddings $e, E, F$ into $M_2$-spaces involving $\mu$),

\[
s_+ \nabla_{s_-} t_+ \nabla_{t_-} - u_+ \nabla_{u_-} = 0 \quad \Leftrightarrow \quad s_+ \nabla_{s_-} (t_+ e_+ - t_- e_-) - u_+ \nabla_{u_-} P = 0 \quad \text{by lemma 10.3}
\]

\[
s_+ \nabla_{s_-} (t_+ e_+ - t_- e_-) - u_+ \nabla_{u_-} P = 0 \quad \text{by lemma 10.3}
\]

\[
s_+ \nabla_{s_-} (t_+ e_+ - t_- e_-) - u_+ \nabla_{u_-} P = 0 \quad \text{by corollary 9.10}
\]

\[
(s_+ \nabla_{s_-} + s_+ e_+ \nabla_{u_-}) E_+ - (s_+ e_+ \nabla_{u_-} + s_+ \nabla_{u_-}, E_+ = 0 \quad \text{by lemma 9.9}
\]

All steps done above can be reversed, as $PQ = 1$ for a right-inverse $Q$ in $GKG$, and so all identities are equivalent. Note that the last line is an identity in $L_0GKG$.

Example 10.10. The above pattern repeats. Let us look at another example. By similar arguments as above we have

\[
s_+ \nabla_{s_-} t_+ \nabla_{t_-} u_+ \nabla_{u_-} = 0 \quad \Leftrightarrow \quad s_+ \nabla_{s_-} (t_+ u_+ e_+ - t_- u_+ e_-) E_+ - (t_+ u_+ e_+ + t_+ u_+ e_-) E_- = 0 \quad \text{by lemma 10.3}
\]

\[
(s_+ \nabla_{s_-} (t_+ u_+ e_+ - t_- u_+ e_-) E_+ - (t_+ u_+ e_+ + t_+ u_+ e_-) E_- = 0 \quad \text{by lemma 9.9}
\]

\[
(s_+ \nabla_{s_-} (t_+ u_+ e_+ - t_- u_+ e_-) E_+ - (t_+ u_+ e_+ + t_+ u_+ e_-) E_- = 0 \quad \text{by lemma 9.9}
\]

\[
(s_+ \nabla_{s_-} + s_+ u_+ \nabla_{u_-}) F_+ - (s_+ \nabla_{u_-} + s_+ \nabla_{u_-} e_+ E_+ = 0 \quad \text{by lemma 9.9}
\]

\[
(s_+ \nabla_{s_-} + s_+ u_+ \nabla_{u_-}) F_+ - (s_+ \nabla_{u_-} + s_+ \nabla_{u_-} e_+ E_+ = 0 \quad \text{by lemma 9.9}
\]

\[
(s_+ \nabla_{s_-} + s_+ u_+ \nabla_{u_-}) F_+ - (s_+ \nabla_{u_-} + s_+ \nabla_{u_-} e_+ E_+ = 0 \quad \text{by lemma 9.9}
\]

Lemma 10.11. Let $A$ and $B$ be given rings. Suppose that $GKG(A, B)$ is a countable abelian group. Then there are rings $D_n$ ($n \geq 1$) and abelian group homomorphisms

\[
GKG(A, B) \xrightarrow\varphi \lim_{n \to \infty} L_0GKG(A, D_n) \xrightarrow{f} L_0GKG(A, \lim_{n \to \infty} D_n)
\]

where $\varphi$ is injective.

Proof. (a) Write $X := GKG(A, B) = \{x_1, x_2, x_3, \ldots\}$, where $x_i \in X$.

Set $D_0 := B$ and $P_0 := Q_0 := 1$ in $GKG(B, B)$. By induction by $n \geq 0$, we assume that we have already chosen rings $D_n$ and ring homomorphisms $P_n$

\[
D_0 \xrightarrow{P_0} D_1 \xrightarrow{P_1} D_2 \xrightarrow{P_2} \ldots \xrightarrow{P_n} D_n
\]
and right-inverses \( Q_k \in G^G(D_{k+1}, D_k) \) for \( P_k, \) that is \( P_k Q_k = 1 \) in \( G^G, \) for all \( 1 \leq k \leq n \) such that for \( V_n := P_1 P_2 \ldots P_n \in G^G(B, D_n) \) we have
\[
x_1 V_n, \ldots, x_n V_n \in L_0 G^G(A, D_n).
\]

Then by lemma 11.7(ii) select a ring homomorphism \( P_{n+1} : D_n \to D_{n+1} \)
and a right-inverse \( Q_{n+1} \in G^G(D_{n+1}, D_n) \) for it such that \( x_{n+1} V_n P_{n+1} \in L_0 G^G(A, D_{n+1}) \). This completes the induction step.

(b) Form the abelian groups inductive limit \( M := \lim_{\to} L_0 G^G(A, D_n) \)
induced by the group homomorphisms \( (P_n)_* \). Let \( \varphi_n : L_0 G^G(A, D_n) \to M \) be the standard maps satisfying \( (P_{n+1})_* \varphi_{n+1} = \varphi_n \).

Define the desired \( \varphi \) on the generating set by \( \varphi(x_n) = \varphi_n(x_n V_n) \).

We have \( \varphi(x_n) = \varphi_{n+1}(x_n V_n P_{n+1}) = \varphi_{n+1}(x_n V_{n+1}) \), whence a definition \( \varphi(x_n + x_m) := \varphi(x_n) + \varphi(x_m) \) becomes well-defined. Injectivity: As the connecting maps \( (P_n)_* \) of the direct limit of \( M \) are right-invertible by \( (Q_n)_* \), and so injective, the maps \( \varphi_n \) are injective. Thus, if \( \varphi(z) = \varphi_n(z V_n) = 0 \) then \( z V_n = 0 \), then \( z = V_n Q_n \ldots Q_1 = 0 \).

(c) Form the direct limit ring \( D := \lim_{\to} D_n \) with inductive limit connection ring homomorphism maps \( P_n : D_n \to D_{n+1} \).

Then there is a group homomorphism \( f : M \to G^G(A, D) \)
induced by the maps \( (f_n)_* : G^G(A, D_n) \to G^G(A, D) \).

**Corollary 10.12.** If \( H \subseteq G^G(A, B) \) is a finitely generated abelian subgroup then there is a \( G^G \)-right-invertible ring homomorphism \( P : B \to D \) such that
\[
P_* : H \to L_0 G^G(A, D) : P_*(z) = z P
\]
is an injective group homomorphism.

**Proof.** Assume that \( H \) is the \( \mathbb{Z} \)-linear span of finitely many generators \( \{x_1, \ldots, x_n\} \subseteq G^G(A, B) \). Do the induction as in the last proof and then set the required \( P := V_n \).

\[\square\]

11. Tensor product Functor

Let \((\mathbb{F}, 1)\) be a field. In this section, all rings are assumed to be actually algebras over \( \mathbb{F} \), so the object class of \( R^G \) consists of \( \mathbb{F} \)-algebras only. This affects also all constructions: all ring homomorphisms are assumed to be actually \( \mathbb{F} \)-algebra homomorphisms. All tensor products are \( \mathbb{F} \)-balanced. All modules are \( \mathbb{F} \)-vector spaces, and all module homomorphisms are \( \mathbb{F} \)-linear, etc.

This in particular affects the homotopy axiom of \( G^K \)-theory, as the involved exterior tensor product is now \( \mathbb{F} \)-balanced.

We still shall notoriously say ‘ring’ and ‘ring homomorphism’ but mean ‘algebra’ and ‘algebra homomorphism’ to be in notation in accordance with everything said so far.

**Lemma 11.1.** The \( \mathbb{F} \)-algebra homomorphism (2) restricts to an isomorphism \( \mathcal{K}_A(\mathcal{E}) \otimes \mathcal{K}_B(\mathcal{F}) \to \mathcal{K}_{A \otimes B}(\mathcal{E} \otimes \mathcal{F}) \) and injection \( \mathcal{L}_A(\mathcal{E}) \otimes \mathcal{L}_B(\mathcal{F}) \to \mathcal{L}_{A \otimes B}(\mathcal{E} \otimes \mathcal{F}) \).

**Proof.** The injection (2) is well known from algebra.

As \( \pi(\theta_{\xi,\phi} \otimes \theta_{\eta,\psi}) = \theta_{\xi \otimes \eta, \phi \otimes \psi} \), we see the bijection with respect to the compact operators, see definition 2.10. \[\square\]
Definition 11.2. Let \((D, \delta)\) be a ring (= \(F\)-algebra). Define an additive functor
\[\tau^G : GK^G \to GK^G\]
by
- For an object \((A, \alpha)\) in \(GK^G\) set \(\tau^G((A, \alpha)) := (A \otimes D, \alpha \otimes \delta)\).
- For an equivariant ring homomorphism \(\pi : A \to B\) set \(\tau^G(\pi) = \pi \otimes 1\).
- For a corner embedding \(e \in GK^G\) define
  \[\tau^G(e^{-1}) := \left(\tau^G(e)\right)^{-1}\]
- For a split exact sequence as in (3) set \(\tau^G(\Delta_s) := \Delta_{\tau^G(s)}\) with respect to
  the following split exact sequence
  \[
  0 \longrightarrow B \otimes D \xrightarrow{\tau^G(j)} M \otimes D \xrightarrow{\tau^G(f)} A \otimes D \longrightarrow 0
  \]

Now \(\tau^G(e) = e \otimes 1\) is indeed invertible as it is essentially a corner embedding by
lemmas 2.22 and 11.1:
\[
A \otimes D \xleftarrow{e^{-1}} K_A(E \oplus H_A) \otimes D \xrightarrow{\cong} K_A (E \otimes D \oplus H_{A \otimes D})
\]

12. Descent functor

In this section we are going to define a descent functor in analogy to the descent homomorphism [23, Theorem 3.11].

Definition 12.1. Let \((A, \alpha)\) be a \(G\)-equivariant ring. Define the \textit{crossed product} \(A \rtimes_\alpha G\) to be the set of functions \(f : G \to A\) with finite support, with pointwise addition. Such an \(f\) is written as a formal sum \(f = \sum_{g \in G} f(g) \rtimes g\).

A multiplication in \(A \rtimes G\) is declared by convolution:
\[
\sum_{g \in G} a_g \rtimes g \rtimes (\sum_{h \in G} b_h \rtimes h) := \sum_{g, h \in G} a_g \beta_g(b_h) \rtimes gh,
\]
which turns it to a non-equivariant ring.

Note that if \(A\) is quadratik then also its crossed product. Indeed, if \(a = \sum i b_i c_i\) in \(A\) then \(a \rtimes g = \sum_i (b_i \rtimes g)(g^{-1}(c_i) \rtimes 1)\).

Definition 12.2. Define an additive functor, called \textit{descent functor},
\[j^G : GK^G \to GK\]
by
- For an object \((A, \alpha)\) in \(GK^G\) set \(j^G((A, \alpha)) := A \rtimes_\alpha G\).
- For an equivariant ring homomorphism \(\pi : (A, \alpha) \to (B, \beta)\) set \(j^G(\pi) : A \rtimes_\alpha G \to B \rtimes_\beta G\) to be the non-equivariant ring homomorphism defined by \(j^G(\pi)(a \rtimes g) := \pi(a) \rtimes g\).
- For a corner embedding \(e \in GK^G\) define
  \[j^G(e^{-1}) := \left(j^G(e)\right)^{-1}\]
- For a split exact sequence as in (3) set \(j^G(\Delta_s) := \Delta_{j^G(s)}\) with respect to
  the following split exact sequence
  \[
  0 \longrightarrow B \rtimes \beta G \xrightarrow{j^G(j)} M \rtimes \gamma G \xrightarrow{j^G(f)} A \rtimes_\alpha G \longrightarrow 0
  \]
Instead of $j^G(m)$ we often also write $m \times 1$ for morphisms $m$ in $GK^G$.

All relations of $GK^G$ are effortlessly seen to go through the descent functor, so it is well defined, excepting the corner embedding. Here, the corollary below shows that $j^G(e)$ is indeed invertible for a corner embedding $e$.

**Proposition 12.3.** For any cofull $(B, \beta)$-module $(E, S)$ there is a non-equivariant ring isomorphism

$$\sigma : \mathcal{K}_B(E) \times_{\text{Ad}(S)} G \to \mathcal{K}_{B \rtimes_\beta G}(E \otimes_B (B \rtimes_\beta G))$$

**Proof.** Equip the ring $R := B \rtimes_\beta G$ with the trivial $G$-action and the right $(R, 1)$-module $R$ with the $G$-action

$$V_g(b \rtimes h) = \beta_g(b) \rtimes gh$$

Define now the indicated ring homomorphism by

$$\sigma(T \rtimes g) = (T \otimes 1) \circ (S_g \otimes V_g)$$

where the right hand side are operators acting on $E \otimes_B (B \rtimes_\beta G)$.

By cofullness of $E \otimes_B B$ by lemma 2.17 (iv) we may express $\xi = \sum_i (\eta_i \otimes b_i)(\phi_i \otimes \psi_i)(\xi_i \otimes c_i)$ and so

$$\xi \rtimes g = \sum_i (\eta_i \otimes (b_i \times 1))(\phi_i \otimes (\psi_i \times 1))(\xi_i \otimes (c_i \rtimes g))$$

for $\xi, \eta_i, \xi_i \in E, b_i, c_i \in B$, which shows cofullness of $E \otimes_B (B \rtimes_\beta G)$.

For computing the surjectivity of $\sigma$, take $\phi \in \Theta_B(E)$ and $\psi \in \Theta_{B \rtimes_\beta G}(B \rtimes_\beta G)$.

Suppose $\psi(z) = wz$ for some $w \in R$. Let us say $w = \sum_{n=1}^N a_n \rtimes y_n$. For $\theta = (\psi \otimes_{B \rtimes_\beta G} \phi)$, $\xi, E \in \mathcal{E}$ and $x, F \in B \rtimes_\beta G$ we get

$$\xi \otimes x \theta(E \otimes F) = \xi \otimes x \psi(\pi(\phi(E)) F)$$

$$= \sum_n \xi \otimes x a_n y_n \pi(\phi(E)) F$$

$$= \sum_n \xi \beta_x(a_n) \beta_{xy_n}(\phi(E)) \otimes x y_n F$$

$$= \sum_n \left(S_{xy_n} \otimes V_{xy_n} \right) \left( S_{(xy_n)^{-1}}(\xi \beta_x(a_n)) \phi(E) \otimes F \right)$$

$$= \sum_n \left(S_{xy_n} \otimes V_{xy_n} \right) \circ (T_n \otimes 1)(E \otimes F)$$

$$= \sum_n \sigma\left(xy_n(T_n \rtimes xy_n)(E \otimes F)\right)$$

for the compact operators $T_n \in \mathcal{K}_B(E)$ given by

$$T_n(E) = S_{(xy_n)^{-1}}(\xi \beta_x(a_n)) \phi(E)$$

For the other set inclusion, to show that $\sigma$ maps really into the indicated ring, we read the above computation from the bottom to the top, by setting $N = 1, y_1 = 1$, and letting $T_1$ according to the formula freely given so to say, and thus we may also replace it by $T_1 = x^{-1}(T)$, and by cofullness of $E$ we achieve it for every $T$.

To prove injectivity of $\sigma$, assume that

$$\sigma \left( \sum_{g \in G} T_g \otimes g \right) (\xi \otimes (b \times 1)) = \sum_g T_g S_g(\xi) \otimes (\beta_g(b) \rtimes g) = 0$$
As \( \mathcal{E} \otimes_B (B \times G) \cong \oplus_{g \in G} \mathcal{E} \otimes_B B \cong \oplus_{g \in G} \mathcal{E} \) as abelian (additive) groups by lemma 2.19(iii), we get

\[
T_g S_g(\xi b) = 0 \quad \forall g \in G
\]

in \( \mathcal{E} \). By cofullness of \( \mathcal{E} \) inherited from \( E \), we conclude \( T_g = 0 \).

Corollary 12.4. Let \( e : B \to \mathcal{K}_B((\mathcal{E} \oplus H_B, S)) \) be an equivariant corner embedding.

Then there is a commuting diagram of non-equivariant ring homomorphisms

\[
\begin{array}{ccc}
\mathcal{K}_B(\mathcal{E} \oplus H_B) \times_{\text{Ad}(S)} G & \xrightarrow{\sigma} & \mathcal{K}_B \times_{\beta} G(\mathcal{E} \oplus H_B) \\
\downarrow{e \times 1} & & \downarrow{\pi} \\
B \times_{\beta} G & \xrightarrow{f} & \mathcal{K}_B \times_{\beta} G(\mathcal{E} \oplus_B (B \times_{\beta} G) \oplus H_B \times_{\beta} G)
\end{array}
\]

where \( \sigma \) is an isomorphism, \( \pi \) is the obvious canonical isomorphism by exchanging direct sums with the tensor product, and \( f \) is a corner embedding.

In particular, \( e \times 1 = f \pi^{-1} \sigma^{-1} \) is a corner embedding up to isomorphism.

Proof. It is easy to realize by definition (13) of \( \sigma \) that the diagram commutes. Recall also lemma 2.20(iii).

13. Exactness

The title of this section may be promising too much as we only have little observation with respect to exactness. Recall that \( KK \)-theory for \( C^* \)-algebras is only known to be half-exact if there is a split by a homomorphism or weaker by a completely positive linear map, see [22] and [28]. Exactness may even fail if there is not such a split.

The next lemma shows how exactness of the contravariant functor \( GK^G(-, X) \) can be deduced from exactness of the contravariant functor \( L_0 GK^G(-, X) \).

Lemma 13.1. Given a short exact sequence (or only \( fg = 0 \))

\[
A \xrightarrow{f} B \xrightarrow{g} C
\]

in \( R^G \) we may consider two sequences of abelian groups as follows

\[
\begin{array}{ccc}
GK^G(A, X) & \xrightarrow{f^*} & GK^G(B, X) \\
\downarrow{p_*} & & \downarrow{p_*} \\
L_0 GK^G(A, Y) & \xrightarrow{f^*} & L_0 GK^G(B, Y)
\end{array}
\]

(14) \( \xrightarrow{g^*} \)

\[
GK^G(C, X) & \xrightarrow{g^*} & L_0 GK^G(C, Y)
\]

The vertical arrows have to be ignored. If the lower line of the diagram is exact for every object \( Y \) in \( GK^G \), then the upper line of the diagram is exact for every object \( X \) in \( GK^G \). (Exact means ker \( f^* = \text{im} g^* \).

Proof. Go to line (14). Clearly \( g^* f^* = 0 \). To show ker \( f^* \subseteq \text{im} g^* \), we consider a morphism \( z \) in ker \( f^* \) and choose a ring homomorphism \( P : X \to Y \) with \( GK^G \)-right-inverse \( Q \) such that \( zP \) is a level-0 morphism by lemma 10.7(ii). Clearly, \( f^*(zP) = f zP = 0 \), and so there is a \( w \in L_0 GK^G(C, Y) \) such that \( g^*(w) = zP \) by the assumed exactness of the second line of the above diagram. Hence, \( g^*(wQ) = gwQ = gzPQ = z \).
Tautologically by the split-exactness definition of $GK^G$-theory we record:

**Lemma 13.2.** Both functors $GK^G(\cdot, X)$ and $GK^G(X, \cdot)$ are split exact. That means, given a short split exact sequence \([3]\), the sequence

$$0 \xleftarrow{} GK^G(B, X) \xrightarrow{f^*} GK^G(M, X) \xrightarrow{s^*} GK^G(A, X) \xrightarrow{} 0$$

and its analogy with $GK^G(X, \cdot)$ are split-exact.

**Proof.** (a) If $z \in GK^G(A, X)$ and $f^*(z) = fz = 0$ then $sfz = z = 0$, proving injectivity of $f^*$. If $z \in GK^G(B, X)$ then $z = j\Delta sz = j^*(\Delta sz)$ by split-exactness of definition \([3,5]\), showing surjectivity of $j^*$.

If $z \in \ker(j^*)$, then $j^*(z) = jz = 0$, then $\Delta sz = 0 = (1 - fs)z$ by split-exactness of definition \([3,5]\) then $z = fsz$, so $z \in \text{im } f^*$.

(b) If $z \in \ker(f_*)$, then $f_*(z) = zf = 0$, then $zf s = 0 = z(1 - \Delta sj)$, then $z = z\Delta sj$, so $z \in \text{im } j_*$.

\[\square\]

14. Morita Equivalence

Let $A, B$ be unital general rings. In \([26]\) Morita shows that if a category of right $A$-modules containing $A$ is equivalent to a category of right $B$-modules containing $B$ by a natural transformation $D_1$ and inverse $D_2$, then this transformation can be computed by the formula $M \mapsto M \otimes_A E$, and its inverse by $N \mapsto N \otimes_B F$, for a certain fixed $A, B$-module $E$ and $B, A$-module $F$. Actually $E = D_1(A)$ and $F = D_2(B)$, and the $A$-module structure on $E$ comes from left multiplication in $A$. Now the left multiplication action of $A$ on $B$ is by compact operators $(A \to \mathcal{K}_A(A))$ in our terminology. Ideals of $\text{Hom}_A(A)$ are in one-to-one correspondences to ideals in $\text{Hom}_A(E)$, and perhaps the $A$-action on $E$ is also by compact operators. We have not clarified this, but it should justify our next definition. We take it for granted that it is a suitable definition and it is designed so that our next theorem works.

**Definition 14.1.** Two rings $A$ and $B$ are called **functional Morita equivalent** if there are a right functional $B$-module $E$ and a right functional $A$-module $F$, ring homomorphisms $m : A \to \mathcal{K}_B(E)$, $n : B \to \mathcal{K}_A(F)$ (turning $E, F$ to non-functional bimodules) such that $E \otimes_B F \cong A$ as right functional, left ordinary $A, B$-bimodules, and $F \otimes m E \cong B$ as right functional, left ordinary $B, B$-bimodules.

**Theorem 14.2.** If two rings $A$ and $B$ are functional Morita equivalent then they are $GK^G$-equivalent.

**Proof.** Let $E$ be an $A, B$-bimodule and $F$ a $B, A$-module which realize the Morita equivalence as described in the last definition, including $m$ and $n$.

We may assume that $E$ and $F$ are separated by the functionals. Otherwise we replace them by $\overline{E}$ and $\overline{F}$ of lemma \([2,19]\) and $n$ and $m$ by $\pi \circ n$ and $\pi \circ m$ for $\pi$ of lemma \([2,19]\). Also use lemma \([2,19]\)(ii) to see $\overline{E} \otimes \pi F = \overline{A} = E \otimes m F$.

The $M$ and $N$ in the diagram below are the ring homomorphisms into the canonical corner operators induced by $m$ and $n$. That is, set $M(a)(\xi \oplus \eta) := m(a)(\xi) \oplus 0$ for $\xi \in E, \eta \in H_B$ and $N(b)(\xi \oplus \eta) := n(b)(\xi) \oplus 0$.

In this diagram, $e, f, g, h$ are the obvious corner embeddings. The ring homomorphism $\kappa$ is just the $\phi$ of lemma \([8,8]\) so the upper right rectangle of the above diagram commutes.
The ring homomorphism $\pi$ is that of proposition 8.1. In the third line of the above diagram we have already canceled some $F$s in the domain of $K_A$ because of the occurrence of $M_A$, and also the $K_A(A,H_A)$-summand in the first internal tensor product because the map $N$ cancels it anyway. We have also abbreviated $X := H_B \otimes N K_A(F \oplus H_A)$.

Now consider $K := K_A(A,F)$ as an $B,A$-bimodule by $b \cdot k \cdot a = n(b) \circ k \circ m_a$, where $m_a(x) = ax$, $a,x \in A$, as in proposition 8.1.

Turn it to a right functional module by setting $\Theta_A(K) = K_A(F, A) =: L$ defined by multiplying an element $k \in K$ with an element in $l \in L$ to $l \circ k$.

We get then an $B,A$-bimodule, and even right functional $A$-module isomorphism by

$$ r : \mathcal{F} \to K_A(A,F) : r(\eta b)(a) = \eta ba = \theta_{\eta, \phi_b}(a) $$

$$ f : \Theta_A(\mathcal{F}) \to \Theta_A(K) : f(\psi)(k) = \psi \circ k $$

for $\eta \in \mathcal{F}, a,b \in A, k \in K, \psi \in \Theta_A(\mathcal{F}), \phi_b \in \Theta_A(A), \phi_b(a) = ba$.

Note that $r$ is $A$-linear, as $r(\eta bx)(a) = r(\eta b) \circ m_x(a)$. Injectivity of $r$ follows from $\xi ba = 0$ for all $a \in B$ implies $\phi(\xi b)a = 0$ for all $\phi \in \Theta_B(\mathcal{F}), a \in B$ implies $\xi b = 0$ by lemmas 2.10, 2.11 and that $\mathcal{F}$ separates the functionals.

Hence, by the isomorphism $(r,f)$ we get

$$ \mathcal{E} \otimes_n K_A(A,F) \cong \mathcal{E} \otimes_n \mathcal{F} \cong A $$

arriving at the fourth line of the above diagram by a ring homomorphism $\sigma$.

It is easy to see that $fg\pi\sigma = h$, so that the lower right rectangle of the above diagram commutes.

By commutativity of the above diagram we get

$$ Me^{-1}Nf^{-1} = M\kappa\pi\sigma h^{-1} = 1_A $$

where the last identity we obtain by rotating the corner embedding $h$ acting on the distinguished coordinate of $H_A$ to a corner embedding acting on $\mathcal{E} \otimes_n \mathcal{F} \cong A$.

Analogously we get $Nf^{-1}Me^{-1} = 1_B$, and so $A$ and $B$ are $GK^G$-equivalent. $\square$
15. Induction Functor

**Definition 15.1.** Let $G$ be a discrete group and $H \subseteq G$ a subgroup. Let $(A, \alpha)$ be a $H$-equivariant ring. Let $(\mathcal{E}, \mathcal{S})$ be a $H$-equivariant right functional $A$-module. Define the induced module

$$\text{Ind}^G_H(\mathcal{E}) = \{ f \in c(G, \mathcal{E}) \mid f(gh) = S_{h^{-1}}(f(g)) \forall h \in H, \quad f(gH) = \{0\} \text{ for almost all } gH \in G/H \}$$

(i.e. continuous functions $f : G \to \mathcal{E}$ on the discrete set $G$), where this a is $G$-equivariant right functional $\text{Ind}^G_H(A)$-module, and the ring $\text{Ind}^G_H(A)$ is analogously defined with pointwise ring multiplication. A $G$-action $T$ on $\text{Ind}^G_H(\mathcal{E})$ is defined by $T_g(f)(x) := f(g^{-1}x)$.

The functional space of this module is set to be $\text{Ind}^G_H(\Theta_A(\mathcal{E}))$ (elements of it understood to be pointwise evaluated against the module), defined analogously as above as a left $\text{Ind}^G_H(A)$-module.

**Definition 15.2.** Let $H \subseteq G$ be a subgroup. Define a functor, called induction functor,

$$\text{Ind}^G_H : GK^H \to GK^G$$

by

- For an object $(A, \alpha)$ in $GK^H$ set $\text{Ind}^G_H(A) := \text{Ind}^G_H(A)$ of definition $\text{[15.1]}$.
- For a ring homomorphism $\pi : A \to B$ set $\text{Ind}^G_H(\pi) : \text{Ind}^G_H(A) \to \text{Ind}^G_H(B)$ to be the ring homomorphism defined by $\text{Ind}^G_H(\pi)(f)(g) = \pi(f(g))$.
- For a corner embedding $e \in GK^H(A, \mathcal{K}_B(\mathcal{E} \oplus \mathbb{M}_B))$ define

$$\text{Ind}^G_H(e^{-1}) := \left(\text{Ind}^G_H(e)\right)^{-1}$$

- For a split exact sequence as in $\text{[3]}$ set $\text{Ind}^G_H(\Delta_s) := \Delta_{\text{Ind}^G_H(s)}$ with respect to the following split exact sequence

$$0 \xrightarrow{\text{Ind}^G_H} \text{Ind}^G_H(B) \xrightarrow{\text{Ind}^G_H(j)} \text{Ind}^G_H(M) \xrightarrow{\text{Ind}^G_H(f)} \text{Ind}^G_H(A) \xrightarrow{\text{Ind}^G_H(s)} 0$$

**Definition 15.3.** Let $H \subseteq G$ be a subgroup. Define a restriction functor $\text{Res}^H_G : GK^G \to GK^H$ by restricting $G$-equivariance of rings and ring homomorphisms to $H$-equivariance, and otherwise analogously as the induction functor.

The next lemma shows that induction commutes with operators, and the proof is straightforward.

**Lemma 15.4.** There is a $G$-equivariant ring isomorphism

$$\sigma : \text{Ind}^G_H\left(\text{Hom}_A(\mathcal{E})\right) \to \text{Hom}_{\text{Ind}^G_H(A)}\left(\text{Ind}^G_H(\mathcal{E})\right)$$

defined by $\sigma(f)(\eta)(g) = f(g)(\eta(g))$, where $f$ is in the domain of $\sigma$, $\eta \in \text{Ind}^G_H(\mathcal{E})$ and $g \in G$.

This $\sigma$ restricts to isomorphisms for $\text{Hom}$ replaced by $L$ and $\text{Hom}$ by $K$, respectively.

We shall also work in intermediate steps with the non-quadratik ring $\mathbb{Z}$, and notate $c_0(G/H) = \ell^2(G/H) = \oplus_{\{g\} \in G/H} \mathbb{Z}$ for the direct sum of abelian groups, and $\ell^2(G/H) \otimes B \cong \oplus_{\{g\} \in G/H} B$ for the direct sum functional $B$-module.
Lemma 15.5. Let \((A, \alpha)\) be a \(H\)-equivariant ring and \((B, \beta)\) a \(G\)-equivariant ring. Then there is a \(G\)-equivariant ring homomorphism \(\mu\)

\[
\text{Ind}_H^G(A \otimes \text{Res}_H^G(B)) \xrightarrow{x} \oplus_{g \in G_0} A \otimes B \xrightarrow{y} \text{Ind}_H^G(A) \otimes B
\]

defined by \(\mu(f)(g) = (\text{id}_A \otimes \beta_g)(f(g))\) (sloppily said), where \(f\) is in the domain of \(\mu\) and \(g \in G\).

This also holds for \(A = (Z, 1)\) and then we may identify \(\text{Ind}_H^G(Z) \cong \mathbb{C}_0(G/H)\).

Proof. By the universal property of the tensor product we may easily write down the well-defined inverse map \(\mu^{-1}\) as \(\mu^{-1}(a \otimes b)(g) = a(g) \otimes \beta_{g^{-1}}(b)\). To see that the non-equivariant \(\mu\) is well-defined, we choose an arbitrary complete representation \(G_0 \subseteq G\) of \(G/H\) such that no two elements \(g\) of \(G_0\) are in the same class. Then set \(x(f) := f|_{G_0}\), and

\[
y(\oplus_{g \in G_0} a_g \otimes b_g) = \sum_{g \in G_0} (k \mapsto 1_{\{k = gh \in G\}} h^{-1}(a_g)) \otimes \beta_g(b_g)
\]

and check that \(\mu = xy\), inverse to \(\mu^{-1}\). \(\square\)

Roughly, this type of Frobenius reciprocity and its proof are well-known.

Proposition 15.6. Let \(H \subseteq G\) be a subgroup.

Then the induction functor is left adjoint to the restriction functor with respect to \(G\)-theory. That means we have

\[
G^K G \left(\text{Ind}_H^G(A), B\right) = G^K H \left(A, \text{Res}_G^H(B)\right)
\]

and this identity is natural in \(A\) and \(B\).

Proof. We shall consider the unit and counit of adjunction. First we have the natural transformation \(\iota\) of the identity functor \(\text{id}_{G^K H}\) on \(G^K H\) and the functor \(\text{Res}_G^H\text{Ind}_H^G\) on \(G^K H\) given by the family of ring homomorphisms

\[
\iota_A : A \to \text{Res}_G^H\text{Ind}_H^G(A) : \iota_A(a)(g) = 1_{\{g \in H\}} g^{-1}(a)
\]

Second we have the natural transformation \(\pi\) between the functors \(\text{id}_{G^K G}\) and \(\text{Ind}_H^G\text{Res}_G^H\) on \(G^K G\) defined by the family of ring homomorphisms

\[
\pi_B : \text{Ind}_H^G\text{Res}_G^H(B) \xrightarrow{\mu} \mathbb{C}_0(G/H) \otimes B \xrightarrow{\lambda} \mathcal{L}_B(\ell^2(G/H) \otimes B \oplus \mathbb{H}_B) \xrightarrow{e^{-1}} B
\]

where \(\lambda\) sends to the coordinate-wise diagonal multiplication operators, but to zero on \(\mathbb{H}_B\), and \(e\) is the corner embedding acting on \(\mathbb{H}_B\).

It is sufficient to show that

\[
\pi_{\text{Ind}_G^H(A)} \circ \text{Ind}_H^G(\iota_A) = \text{id}_{\text{Ind}_G^H(A)}\tag{15}
\]

in \(KK^G\) and

\[
\text{Res}_G^H(\pi_B) \circ \iota_{\text{Res}_G^H(B)} = \text{id}_{\text{Res}_G^H(B)}\tag{16}
\]

in \(KK^H\) by [25, IV.1 Theorem 2.(v)].

Define \(P_{\lvert \cdot \rvert}\) to be the projection acting on \(\text{Ind}_H^G(A)\) which leaves an element on \(gH\) unmodified and sets it to zero outside of it.
Then, for $\mu$ of Lemma 15.2 and $a \in \text{Ind}_{H}^{G}(A)$, one has

\begin{equation}
\mu(\text{Ind}_{H}^{G}(\iota A)(a)) = \sum_{[g] \in G/H} 1_{[g]} \otimes P_{[g]}(a) \in c_{0}(G/H) \otimes \text{Ind}_{H}^{G}(A) \tag{17}
\end{equation}

Now $\pi \text{Ind}_{H}^{G}(A) \circ \text{Ind}_{H}^{G}(\iota A) = \text{Ind}_{H}^{G}(\iota A)\mu\lambda e^{-1}$ is a morphism $\text{Ind}_{H}^{G}(A) \rightarrow \text{Ind}_{H}^{G}(A)$. Set $B := \text{Ind}_{H}^{G}(A)$.

We have a direct sum decomposition $X := \ell^{2}(G/H) \otimes \text{Ind}_{H}^{G}(A) \cong M \oplus N$ of $G$-modules, where $M$ is just the set of all elements (18) when $a$ is varied over all $a \in B$.

We have an isomorphism $u : M \rightarrow B : u \left( \sum_{[g] \in G/H} 1_{[g]} \otimes P_{[g]}(a) \right) = a$ of $G$-$\text{Ind}_{H}^{G}(A)$-modules for $a \in B$.

Since by the above computation (17), $z := \text{Ind}_{H}^{G}(\iota A)\mu\lambda$ acts only on $M$, if we compose it with $\text{Ad}(u \oplus 1 \oplus 1)$ it becomes the multiplication operator on $B$, so a corner embedding $f$ and we get $ze^{-1} = fe_{2}^{-1} = 1$ by a rotation homotopy, see the following commuting diagram:

\[
\begin{array}{cccc}
B & \overset{z}{\longrightarrow} & \mathbb{K}_{B}(M \oplus N \oplus \mathbb{H}_{B}) & \overset{e}{\longrightarrow} & B \\
\downarrow & & \downarrow & & \downarrow \\
B & \overset{f}{\longrightarrow} & \mathbb{K}_{B}(B \oplus N \oplus \mathbb{H}_{B}) & \overset{e_{2}}{\longrightarrow} & B
\end{array}
\]

The case (16) is proven similarly. \qed

16. Green-Julg Isomorphism

In this section we shall prove the well-known Green-Julg isomorphism theorem \[21\] in the framework of $GK$-theory.

In this section, if nothing else is said, $G$ denotes a finite, discrete group and we put $n := |G|$.

Moreover, if nothing else is said, $(\mathbb{F}, 1)$ denotes any given commutative, associative field with $\text{char}(\mathbb{F}) \neq n$ (because we shall divide by $n$).

All rings and modules are now $\mathbb{F}$-algebras and $\mathbb{F}$-vector spaces, respectively, as explained in section 11. In particular, all tensor products are $\mathbb{F}$-balanced.

Typically, $\delta_{g}$ for $g \in G$ denote canonical basis elements.

**Definition 16.1.** Let $G$ be a finite group. Write $\ell^{2}(G) := \bigoplus_{g \in G} \mathbb{F}$ for the direct sum functional right (and also left notated) $\mathbb{F}$-module. We also write $\langle \xi, \eta \rangle := \sum_{g \in G} \xi_{g} \eta_{g}$ for the “inner product”.

On it define the right, “right regular” $G$-action $V$ by $V_{g}(\delta_{h}) = \delta_{gh}$ for $\xi \in \ell^{2}(G), g, h \in G$.

Equip $\ell^{2}(G)$ with the left $G$-action $V^{-1}$ defined by $V_{g}^{-1} := V_{g}^{-1}$.

Write $\mathbb{K} := (\mathbb{K}_{\mathbb{F}}(\ell^{2}(G)), \text{Ad}(V^{-1}))$. 

Set $m := \sum_{g \in G} \delta_g \in \ell^2(G)$. A corner embedding $f : \mathbb{F} \to \mathbb{K} = K_F(\mathbb{F}^n)$ is given by

$$f(x)(\xi) = x_m \frac{1}{n} \langle \xi, m \rangle = \theta_{x_m, \phi_f}(\xi)$$

for all $x \in \mathbb{F}, \xi \in \ell^2(G), h \in G$. Here, $\phi_f \in \Theta(\ell^2(G))$ is $\phi_f(\xi) = n^{-1} \langle \xi, m \rangle$.

**Lemma 16.2.** (i) $f$ is indeed a corner embedding.

(ii) If $A$ is a ring then $1 \otimes f : A \cong A \otimes \mathbb{F} \to A \otimes \mathbb{K}$ is a corner embedding.

**Proof.** (i) Observe that $f(xy) = f(x)f(y)$. Choosing a $\mathbb{F}$-vector space basis in $\ker \phi$ we obtain a non-equivariant isomorphism

$$T : \ell^2(G) \cong \mathbb{F} \oplus \ker(\phi_f) \to \mathbb{F} \oplus \mathbb{F}^{n-1} = \mathbb{F}^n$$

of vector spaces, such that $T(m) = m \oplus 0^{n-1}$ for $m$ of definition [16.1]. Select the $G$-action $W$ on $\mathbb{F}^n$ such that $T$ becomes $G$-equivariant.

Observe that $m$ is $V$-invariant (i.e. $V_g(m) = m$), such that $T(m)$ is $W$-invariant, and so the first copy $\mathbb{F}$ of the direct sum $(\mathbb{F}^n, W)$ is a distinguished first coordinate $(\mathbb{F}, 1)$.

Then $(T \circ f(x) \circ T^{-1})(\eta) = x\eta_1 \oplus 0^{n-1}$ for $\eta \in \mathbb{F}^n$, so $f$ is a corner embedding.

(ii) This is just the tensor product functor, but let us observe it one more:

$$(A, \alpha) \otimes K_F((\mathbb{F}^n, W)) \cong K_A(A) \otimes K_F(\mathbb{F}^n) \cong K_A \otimes F((A \otimes \mathbb{F}^n, \alpha \otimes W)) \cong K_A(A^n)$$

where the first copy of $A$ in $A^n$ is a distinguished first coordinate $(A, \alpha)$, and the map $1 \otimes f$ turns to the obvious canonical corner embedding $A \to K_A(A^n)$. \qed

**Definition 16.3.** Let $(A, \alpha)$ be a ring. We often identify

$$(A \otimes K, \alpha \otimes \text{Ad}(V^{-1})) \cong \left(K_A\left(A \otimes \ell^2(G)\right), \text{Ad}(\alpha \otimes V^{-1})\right)$$

and also abbreviate the $G$-action $\rho := \text{Ad}(V^{-1})$ on $K$.

Let $\pi : A \to A \otimes K$ be the ring homomorphism defined by $\pi(a)(b \otimes \delta_g) = \alpha_{g^{-1}}(a)b \otimes \delta_g$.

Let $U : G \to A \otimes K$ be the group homomorphism defined by $U_g(a \otimes \delta_h) = a \otimes \delta_{gh}$.

("Left regular action", but it is not a group action as we have the $G$-action $\alpha$ on $A$.)

Let $\lambda_A : (A \rtimes G, 1) \to (A \otimes \mathbb{K}, \alpha \otimes \rho)$ be the injective, $G$-equivariant ring homomorphism ("left regular representation") given by $\lambda_A(a \rtimes g) = \pi(a) \circ U_g$.

This is the Green-Julg map:

**Definition 16.4.** Let $(A, \alpha)$ be a ring. Define an abelian group homomorphism

$$S : GK^G(\mathbb{F}, A) \to GK(\mathbb{F}, A \rtimes \alpha G)$$

by

$$S(\mathbb{F} \xrightarrow{Y} A) = \mathbb{F} \xrightarrow{M} \mathbb{F} \rtimes G \xrightarrow{Y \rtimes 1} A \rtimes \alpha G$$

where $M$ is the ring homomorphism ("averaging map") given by $M(x) = x_n^{-1} \sum_{g \in G} 1 \times g$.

$M$ is chosen such that $M\lambda_f = 1 \otimes f$, and this is the only reason for its choice.

This is the inverse Green-Julg map:
Definition 16.5. Let \((A, \alpha)\) be a ring. Define an abelian group homomorphism
\[ T : GK(\mathbb{F}, A \rtimes_\alpha G) \to GK^G(\mathbb{F}, A) \]
by
\[ T(F \xrightarrow{L} A \rtimes_\alpha G) = (\mathbb{F}, 1) \xrightarrow{L} (A \rtimes_\alpha G, 1) \xrightarrow{\lambda_A} (A \otimes \mathbb{K}, \alpha \otimes \rho) \xrightarrow{(\langle f \rangle)^{-1}} (A \otimes \mathbb{F}, \alpha \otimes 1) \]

Lemma 16.6. Let \(\varphi : (A, \alpha) \to (B, \beta)\) be a morphism in \(GK^G\). Then the following diagram commutes:

\[ A \rtimes_\alpha G \xrightarrow{\lambda_A} A \otimes \mathbb{K} \xrightarrow{(\langle f \rangle)^{-1}} A \otimes \mathbb{F} \]
\[ \downarrow \varphi \otimes 1 \quad \downarrow \varphi \otimes 1 \]
\[ B \rtimes_\beta G \xrightarrow{\lambda_B} B \otimes \mathbb{K} \xrightarrow{(\langle f \rangle)^{-1}} B \otimes \mathbb{F} \]

Proof. For \(\varphi\) a ring homomorphism this is obvious, for \(\varphi\) the inverse of a corner embedding also, by proving commutativity with the reversed arrows. For \(\varphi = \Delta_s\) one makes obvious diagrams with split exact sequences and verifies with lemma \[4.3\]

Proposition 16.7. We have \(T \circ S = \text{id}_{GK^G(\mathbb{F}, A)}\).

Proof. We have
\[ T \circ S(Y) = F \xrightarrow{M} F \times G \xrightarrow{Y \times 1} A \times G \xrightarrow{\lambda_A} A \otimes \mathbb{K} \xrightarrow{(\langle f \rangle)^{-1}} A \otimes \mathbb{F} \]

Hence, by lemma \[16.6\]
\[ T \circ S(Y) = F \xrightarrow{M} F \times G \xrightarrow{\lambda_Y} F \otimes \mathbb{K} \xrightarrow{(\langle f \rangle)^{-1}} F \otimes \mathbb{F} \xrightarrow{1 \otimes Y} F \otimes A \]

Since \(M \lambda_Y = f \otimes 1\), the proposition is proved.

Lemma 16.8. Let \((A, \alpha)\) be a unital \(C^*\)-algebra with a \(G\)-action. Let \(A^\alpha\) be its fixed point algebra under the \(G\)-action. If \(u \in A^\alpha\) is a unitary element non-equivariantly homotopic to 1 \(\in A^\alpha\) by a unitary path in \(A\) and with spectrum not equal to the torus \(T\), then \(u\) is non-equivariantly homotopic to 1 by a unitary path also in \(A^\alpha\).

Proof. Let \(\nu : A \to A^\alpha : \nu(a) = n^{-1} \sum_{\gamma \in G} \alpha_\gamma(a)\) be the averaging map, which is a linear continuous projection.

Let \((u_t)\) be the given homotopy in \(A\). Since \(\text{sp}(u) \neq T\), there is a fixed scalar \(\lambda \in \mathbb{C}\) (direction through the gap of \(\text{sp}(u)\)) such that the straight line in \(A^\alpha\) connecting \(u\) with \(u + \lambda 1\) consists of only invertible elements. We move further from \(u + \lambda 1\) by the homotopy \(t \mapsto u_t + \lambda 1\) of invertible elements in \(A\) and end at \((\lambda + 1)1\) in \(A^\alpha\). From here, we move by a straight line to 1 \(\in A^\alpha\). If we choose \(|\lambda|\) big enough (by Neumann series), even the image under \(\nu\) of this sketched path in \(A\) from \(u\) to 1 remains to consist of invertible elements only, and is a path in \(A^\alpha\). We may change it to a unitary path in \(A^\alpha\).
Proposition 16.9 (\cite{27} Proposition 4.6). Let \((A, \alpha)\) be a \(F\)-algebra with \(F \subseteq \mathbb{C}\) being a subfield. There is a natural ring isomorphism \(\zeta_A : A \rtimes_\alpha G \rightarrow (A \otimes K)^{\alpha \otimes \rho}\) (which is not \(\lambda_A\)).

In the \(C^*\)-algebra setting this holds also for compact groups \(G\).

Proof. In the \(C^*\)-algebra setting this is \cite{27} and the remark thereafter. Inspection of the proof of \cite{27} shows that no \(C^*\)-typical analysis is involved and consists of relatively canonical homomorphisms, which appear to do not only work for \(\mathbb{C}\)-algebras, but even \(F\)-algebras. \(\square\)

Proposition 16.10. Assume that \(F\) is a subfield of \(\mathbb{C}\).

Then we have \(S \circ T = \text{id}_{GK(A, A, G)}\).

Proof. (a) Let \(\mathbb{C} \xrightarrow{L} A \rtimes_\alpha G\) in \(KK(\mathbb{C}, A \rtimes G)\) be given.

Consider the diagram

\[
\begin{array}{cccccccccc}
\mathbb{C} & \xrightarrow{M} & \mathbb{C} \rtimes_1 G & \xrightarrow{\lambda_1} (A \rtimes_\alpha G) \rtimes_1 G & \xrightarrow{\lambda_A \otimes 1} (A \otimes K) \rtimes_{\alpha \otimes \rho} G & \xrightarrow{(1 \otimes f)^{-1} \otimes 1} A \rtimes_\alpha G \\
\text{C} \otimes K & \xrightarrow{L \otimes 1} & (A \rtimes_\alpha G) \otimes K & \xrightarrow{\alpha \otimes 1} (A \otimes K) \otimes K & \xrightarrow{\lambda_A \otimes \lambda_G} A \otimes K
\end{array}
\]

The top line is exactly \(S \circ T(L)\). It is not ad hoc \(L\), but one may observe that it was if only the two occurring copies of \(G\) would be exchanged.

To make the flip, we go down with the \(\lambda\)'s everywhere as notated in the diagram. The whole diagram commutes by lemma 16.6.

We restrict now the last line of the above diagram to the image spaces of the down-going \(\lambda\)-arrows, so they are now all bijective.

Obviously, again everything in the diagram commutes, where we define the restricted \(L \otimes 1\) as \(\lambda_C^{-1}(L \times 1)\lambda_A \otimes G\).

(b) Assume for the moment that \(F = \mathbb{C}\). On the ring \(B := A \otimes K \otimes K\) the flip ring endomorphism \(F\) defined by \(F(a \otimes k \otimes l) = a \otimes l \otimes k\) is homotopic to the identical ring homomorphism 1 on that space, because \(F = 1 \otimes \text{Ad}(H)\) for the unitary flip operator \(H \in K \otimes K \cong B(\mathbb{C}^n \otimes \mathbb{C}^n)\), and \(H\) is homotopic to 1 by a unitary path \((H_t)\), as is well known.

Set \(u := 1 \otimes H\). Observe that \(u \in B^G := B^{\alpha \otimes \rho \otimes \rho}\) (fixed point algebra), and that \(F = \text{Ad}(u)\) can be restricted to \(B^G\).

By lemma 16.8 applied to \((K \otimes K, \rho \otimes \rho)\) and \(H, F|_{B^G}\) is homotopic to the identity ring homomorphism on \(B^G\) by a homotopy \(F'_t := \text{Ad}(u_t)\) for \(u_t \in B^G\).

(c) Let \(X\) be the range of \(\lambda_A \otimes G\). Now our desired “flip” is

\[
W : X \rightarrow X : W = \lambda_A^{-1} \otimes 1 \zeta \cdot F|_{B^G} \cdot \zeta^{-1} \lambda_A \otimes K,
\]

which is 1 in \(GK\). Here, \(\zeta\) is from proposition 16.9 applied to \(A := (A \otimes K, \alpha \otimes \rho)\).

Let us take for granted, that \(\zeta\) is so “natural” that \(W(a \otimes U_g \otimes U_h) = a \otimes U_h \otimes U_g\) (note that \(U_g\) is \(\rho\)-invariant), which might only be seen by inspection of the proof of \cite{27} Proposition 4.6.\n
Thus, as the image of \(f\) is in the image of \(\lambda_C\) by the first triangle of the above diagram, and the \(\lambda\)'s are realized by the operators \(U\), the operator \(W\) exchanges the coordinates in such a way that \(\lambda_A((1 \otimes f) \otimes 1)W = \lambda_A \otimes f\) in the left area of the above diagram.
We compute now \( S \circ T(L) \) by going the bottom path in the above diagram, and implementing \( W = 1 \) at \( X \). Using the granted ‘naturality’ of \( W \), we thus have

\[
S \circ T(L) = f(L \otimes 1)(\lambda_A \times 1)W(1 \otimes f \otimes 1)^{-1}\lambda_A^{-1} = L
\]

proving the proposition for \( \mathbb{F} = \mathbb{C} \).

(d) Now let \( \mathbb{F} \) be an arbitrary subfield of \( \mathbb{C} \). We do everything as above and need only explain why \( F|_{B^G} \) is still homotopic to \( 1 \) as endomorphisms on \( B^G \). To this end we at first complexify \( B \) and obtain the following \( \mathbb{C} \)-algebra isomorphism by lemma \([2.24]\)

\[
\tau : B \otimes \mathbb{C} = (A \otimes \mathbb{K} \otimes \mathbb{K}) \otimes \mathbb{C} \to \hat{A} \otimes \mathbb{K} \otimes \mathbb{K}
\]

Because \( \mathbb{F} \subseteq \mathbb{C} \) is a subfield, we get \( \mathbb{F} \otimes \mathbb{C} \cong \mathbb{C} \) as algebras over \( \mathbb{C} \), because we use the \( \mathbb{F} \)-balanced tensor product everywhere. Consequently, we get an isomorphism of \( \mathbb{C} \)-algebras

\[
\hat{K} = M_n(\mathbb{F}^n) \otimes \mathbb{C} \cong M_n(\mathbb{F}^n \otimes \mathbb{C}) \cong M_n(\mathbb{C}^n) \cong \mathcal{K}_C(\ell^2(G, \mathbb{C}))
\]

The map \( \tau \) is also equivariant with respect to \( (\alpha \otimes \rho \otimes \rho) \otimes 1 \) and \( \alpha \otimes \rho \otimes \rho \), respectively, whence we can restrict \( \tau \) to the fixed point algebras. We can thus perform the desired and already verified homotopy of endomorphisms on the fixed point algebra of the range of \( \tau \) and go back to endomorphisms on \( B^G \) at 0 and 1 only.

**Corollary 16.11** (Green-Julg Theorem). Let \( G \) be a finite group and \( \mathbb{F} \) a subfield of \( \mathbb{C} \). Then the Green-Julg map \( S \) of definition \([16.4]\) is an isomorphism of abelian groups with inverse isomorphism \( T \) as defined in definition \([16.5]\).

**Lemma 16.12.** The Green-Julg isomorphism \( S \) is functorial, that is, for every morphism \( Z : A \to B \) one has a commuting diagram

\[
\begin{array}{ccc}
KK^G(\mathbb{F}, A) & \xrightarrow{S_A} & KK(\mathbb{F}, A \rtimes G) \\
\downarrow{Z_*} & & \downarrow{(Z \times 1)_*} \\
KK^G(\mathbb{F}, B) & \xrightarrow{S_B} & KK(\mathbb{F}, B \rtimes G)
\end{array}
\]

This is actually a corollary of the above proof:

**Corollary 16.13** (Green-Julg Theorem). Let \( G \) be a compact group. Let \( (A, \alpha) \) be a separable \( G \)-\( C^* \)-algebra. Then there is an isomorphism of abelian groups

\[
S : KK^G(\mathbb{C}, A) \to KK(\mathbb{C}, A \rtimes \alpha G)
\]

with inverse map \( T \), analogously defined as above.

**Proof.** We may identify \( KK^G \cong GK^G \) by \([2]\).

One replaces sums over \( G \) by integrals over \( G \). The vector space \( \ell^2(G) \) is replaced by \( L^2(G) \) (Haar measure) with inner product \( \langle \xi, \eta \rangle = \int_G \xi(g)\overline{\eta(g)}dg \). The averaging factor \( n^{-1} \) can be omitted as the Haar measure of \( G \) is one.

For example, in definition \([16.1]\) one sets \( m := 1 \in L^2(G) \) (constant 1 function) and \( f : \mathbb{C} \to \mathbb{K}, f(x) = x(\xi, m) \).

The function-formulas involving \( \delta_x \) must be one-to-one reformulated in function-evaluation writing. For example, \( U_g(a)(\xi)(h) := a(\xi(g^{-1}h)) \).
The finite basis of the proof of lemma 16.2 has to be replaced by a possibly countably infinite, orthonormal basis. It is well known that the left regular representation of definition 16.3 maps into the compact operators for $G$ compact.

Otherwise the proof is formally completely unchanged. □

We remark that the Green-Julg theorem holds also in a ‘weaker’ theory than $G^K$, where we drop the split-exactness axiom involving the $\Delta$s, because the $\Delta$s are nowhere essentially used in the proof, but do also not disturb.

17. Baum-Connes map

We continue the last section by extending the Green-Julg map to the Baum-Connes map. We let the field be $\mathbb{F} := \mathbb{C}$. To incorporate the $C^*$-case, we also allow $G$ to be a locally compact, second-countable group $G$.

We very briefly recall some notions to define a Baum-Connes map with respect to $G^K$-theory. For more details on these notions see [1], or for instance [18].

**Definition 17.1.** A locally compact space $X$ equipped with a continuous $G$-action is called *proper* if the map $\chi : G \times X \to X \times X$ defined by $\chi(g, x) = (gx, x)$ is proper; that means, if for all compact subsets $K \subseteq X \times X$, $\chi^{-1}(K)$ is compact.

A locally compact $G$-space $X$ is called *$G$-compact* if the quotient space $G \backslash X$ is compact (if and only if $X$ is the $G$-saturation $G \cdot K$ for a compact subset $K \subseteq X$).

**Definition 17.2.** If $X$ is a locally compact, proper $G$-compact $G$-space then there is a cut-off function $c$ for it, that means a continuous function $c : X \to \mathbb{R}$ with compact support such that

$$\int_G c(g^{-1}x)^2 dg = 1 \quad \forall x \in X$$

with respect to the Haar measure on $G$.

A projection $p_X \in C_c(X) \rtimes G$ is defined by

$$p_X(g)(x) = m_G(g)^{-1/2}c(g^{-1}x)c(x)$$

where $m_G$ is the modular function of $G$, and a ring homomorphism $M_X : \mathbb{C} \to C_0(X) \rtimes G$ by $M_X(z) = zp_X$.

Let $EG$ be an universal example for proper actions by $G$, see [1].

**Definition 17.3.** If $X$ is a locally compact, $G$-compact $G$-space then define the assembly map as the abelian group homomorphism

$$\nu_X^{G,A} : GK^G(C_0(X), A) \to GK(\mathbb{C}, A \rtimes G)$$

$$\nu_X^{G,A} : \xymatrix{ C_0(X) \ar[r]^V & A } = \xymatrix{ \mathbb{C} \ar[r]^{M_X} & C_0(X) \rtimes G \ar[r]^{V \times 1} & A \rtimes G },$$

**Definition 17.4.** Define the Baum-Connes assembly map as the abelian group homomorphism

$$\nu^{G,A} : \lim_{X \subseteq EG} GK^G(C_0(X), A) \to GK(\mathbb{C}, A \rtimes G)$$

where the direct limit of abelian groups is indexed over all $G$-compact, second-countable, locally compact $G$-invariant subsets $X \subseteq EG$ with direct limit connecting abelian group homomorphisms $\pi^*$ for all restriction ring homomorphisms
\( \pi : C_0(Y) \to C_0(X) \), \( \pi(f)(x) = f(x) \), whenever \( Y \supseteq X \). The abelian group homomorphism \( \nu_X^{G,A} \) is then defined to be canonically induced by the abelian group homomorphisms \( \nu_X^{G,A} \).

**Lemma 17.5.** The Baum-Connes map is functorial in the coefficient algebra \( A \), that is a completely analogous statement as in lemma\([10,13]\) holds.

More precisely, \( \nu_X^{G,A} \circ (Z \times 1)_* = Z_* \circ \nu_X^{G,B} \) and \( \nu_X^{G,A} \circ (Z \times 1)_* = Z_* \circ \nu_Y^{G,B} \) for all morphisms \( Z : A \to B \).

If we restrict everything on the domain of the Baum-Connes map to level-0 morphisms, then we get a restricted abelian group homomorphism

\[
L_0\nu^{G,A}_Y : \lim_{X \subseteq \mathbb{G}} L_0GK^G(C_0(X), A) \to L_0GK(C_\mathbb{R}, A \times G)
\]

**Proposition 17.6.** The Baum-Connes maps \( \nu^{G,A} \) are injective for all coefficient algebras \( A \) if and only if \( L_0\nu^{G,A} \) is injective for all coefficient algebras \( A \).

**Proof.** Let \( \psi_X^{G,A} \) the abelian group homomorphisms from the domain of \( \nu_X^{G,A} \) to the domain of the Baum-Connes map \( \nu^{G,A} \) associated to the direct limit.

Suppose \( \nu^{G,A}(\psi_X^{G,A}(z)) = \nu_X^{G,A}(z) = 0 \) for a morphism \( z : C_0(X) \to A \). Then \( zP \) is a level-0 morphism for some \( GK^G \)-right-invertible ring homomorphism \( P : A \to D \) with right-inverse \( Q \). Thus, by lemma\([17,5]\)

\[
\nu^{G,D}(\psi_X^{G,D}(zP)) = \nu_X^{G,D}(zP) = \nu_X^{G,D}(z)(P \times 1) = 0
\]

By injectivity of \( L_0\nu^{G,D} \) we conclude that \( \psi_X^{G,D}(zP) = 0 \).

But then \( wzP = 0 \) in \( GK^G \) for some restriction map \( w : C_0(Y) \to C_0(X) \) for \( Y \supseteq X \). Hence \( wz = wzPQ = 0 \). Thus \( \psi_Y^{G,A}(z) = \psi_X^{G,A}(wz) = 0 \). \( \square \)

Note that this holds also with respect to \( C^* \)-algebras and \( KK \)-theory by analogy. That is, the classical Baum-Connes map is injective if and only if it is injective on \( L_0KK^G \), or more precisely, on the abelian subgroup generated by the \( * \)-homomorphisms.

**References**

[1] P. Baum, A. Connes, and N. Higson. Classifying space for proper actions and \( K \)-theory of group \( C^* \)-algebras. Contemp. Math. 167, 241-291 (1994).

[2] B. Burgstaller. The generators and relations picture of \( KK \)-theory. 2016. arXiv:1602.03034v2.

[3] B. Burgstaller. Aspects of equivariant \( KK \)-theory in its generators and relations picture. 2019. arXiv:1912.03275v1.

[4] B. Burgstaller. Some remarks in \( C^* \)- and \( K \)-theory. Operat. Matrices, 15(2):703–720, 2021.

[5] A. Connes. Noncommutative geometry. San Diego, CA: Academic Press, 1994.

[6] A. Connes and N. Higson. Almost homomorphisms and \( KK \)-theory. 1990. Downloadable manuscript.

[7] A. Connes and N. Higson. Déformations, morphismes asymptotiques et \( K \)-théorie bivariante. C. R. Acad. Sci., Paris, Sér. I, 311(2):101–106, 1990.

[8] G. Cortiñas and A. Thom. Bivariant algebraic \( K \)-theory. J. Reine Angew. Math., 610:71–123, 2007.

[9] J. Cuntz. A new look at \( KK \)-theory. \( K \)-Theory, 1(1):31–51, 1987.

[10] J. Cuntz. Bivariante \( K \)-Theorie für lokalconvexe Algebren und der Chern-Connes-Charakter. Doc. Math., 2:139–182, 1997.

[11] J. Cuntz. Bivariant \( K \)-theory and the Weyl algebra. \( K \)-Theory, 35(1-2):93–137, 2005.

[12] J. Cuntz and A. Thom. Algebraic \( K \)-theory and locally convex algebras. Math. Ann., 334(2):339–371, 2006.
[13] E. Ellis. Equivariant algebraic $KK$-theory and adjointness theorems. *J. Algebra*, 398:200–226, 2014.

[14] G. Garkusha. Algebraic Kasparov $K$-theory. I. *Doc. Math.*, 19:1207–1269, 2014.

[15] G. Garkusha. Algebraic Kasparov $K$-theory. II. *Ann. K-Theory*, 1(3):275–316, 2016.

[16] S. M. Gersten. Homotopy theory of rings. *J. Algebra*, 19:396–415, 1971.

[17] M. Grensing. Universal cycles and homological invariants of locally convex algebras. *J. Funct. Anal.*, 263(8):2170–2204, 2012.

[18] E. Guentner, N. Higson, and J. Trout. *Equivariant E-theory for $C^*$-algebras*, volume 703. Providence, RI: American Mathematical Society (AMS), 2000.

[19] N. Higson. A characterization of KK-theory. *Pac. J. Math.*, 126(2):253–276, 1987.

[20] N. Higson. Categories of fractions and excision in KK-theory. *J. Pure Appl. Algebra*, 65(2):119–138, 1990.

[21] P. Julg. $K$-théorie équivariante et produits croisés. *C. R. Acad. Sci., Paris, Sér. I*, 292:629–632, 1981.

[22] G. G. Kasparov. The operator K-functor and extensions of $C^*$-algebras. *Math. USSR, Izv.*, 16:513–572, 1981.

[23] G. G. Kasparov. Equivariant $KK$-theory and the Novikov conjecture. *Invent. Math.*, 91:147–201, 1988.

[24] V. Lafforgue. $K$-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes. *Invent. Math.*, 149(1):1–95, 2002.

[25] S. Mac Lane. *Categories for the working mathematician. 2nd ed*. New York, NY: Springer, 2nd ed edition, 1998.

[26] K. Morita. Duality for modules and its applications to the theory of rings with minimum condition. *Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A.*, 6:83–142, 1958.

[27] M. A. Rieffel. Actions of finite groups on $C^*$-algebras. *Math. Scand.*, 47:157–176, 1980.

[28] G. Skandalis. Exact sequences for the Kasparov groups of graded algebras. *Can. J. Math.*, 37:193–216, 1985.

[29] J. Weidner. KK-groups for generalized operator algebras. I. *K-Theory*, 3(1):57–77, 1989.

[30] J. Weidner. KK-groups for generalized operator algebras. II. *K-Theory*, 3(1):79–98, 1989.

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