Products of conjugacy classes in finite and algebraic simple groups

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Abstract

We prove the Arad–Herzog conjecture for various families of finite simple groups — if \(A\) and \(B\) are nontrivial conjugacy classes, then \(AB\) is not a conjugacy class. We also prove that if \(G\) is a finite simple group of Lie type and \(A\) and \(B\) are nontrivial conjugacy classes, either both semisimple or both unipotent, then \(AB\) is not a conjugacy class. We also prove a strong version of the Arad–Herzog conjecture for simple algebraic groups and in particular show that almost always the product of two conjugacy classes in a simple algebraic group consists of infinitely many conjugacy classes. As a consequence we obtain a complete classification of pairs of centralizers in a simple algebraic group which have dense product. A special case of this has been used by Prasad to prove a uniqueness result for Tits systems in quasi-reductive groups. Our final result is a generalization of the Baer–Suzuki theorem for \(p\)-elements with \(p \geq 5\).

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1. Introduction

In [2, p. 3], Arad and Herzog made the following conjecture:

**Conjecture A (Arad–Herzog).** If $S$ is a finite non-abelian simple group and $A$ and $B$ are nontrivial conjugacy classes of $S$, then $AB$ is not a conjugacy class.

In this paper, we prove this conjecture in various cases. We also consider the analogous problem for simple algebraic groups. Note that the results do not depend on the isogeny class of the group (allowing the possibility of multiplying a class by a central element) and so we work with whatever form is more convenient. Moreover, in characteristic 2, we ignore the groups of type $B$ (the result can be read off from the groups of type $C$).

Here one can prove much more.

**Theorem 1.1.** Let $G$ be a simple algebraic group over an algebraically closed field of characteristic $p \geq 0$. Let $A$ and $B$ be non-central conjugacy classes of $G$. Then $AB$ can never constitute a single conjugacy class. In fact, either $AB$ is the union of infinitely many conjugacy classes, or (up to interchanging $A$ and $B$ and up to an isogeny for $G$) one of the following holds:

1. $G = G_2$, $A$ consists of long root elements and $B$ consists of elements of order 3. If $p = 3$, $B$ consists of short root elements and if $p \neq 3$, $B$ consists of elements with centralizer isomorphic to $\text{SL}_3$.
2. $G = F_4$, $A$ consists of long root elements and $B$ consists of involutions. If $p = 2$, $B$ consists of short root elements and if $p \neq 2$, $B$ consists of involutions with centralizer isomorphic to $B_4$.
3. $G = \text{Sp}_{2n} = \text{Sp}(V), n \geq 2$, $\pm A$ consists of long root elements and $B$ consists of involutions; when $p = 2$ then the involutions $b \in B$ moreover satisfy $(bv, v) = 0$ for all $v \in V$.
4. $G = \text{SO}_{2n+1}, n \geq 2, p \neq 2$, $A$ consists of elements which are the negative of a reflection and $B$ consists of unipotent elements with all Jordan blocks of size at most 2.

The methods rely heavily on closure of unipotent classes. In particular, this gives a short proof for simple algebraic groups of what is referred to as Szep’s conjecture for the finite simple groups (proved in [8])—a finite simple group is not the product of two subgroups with nontrivial centers.

**Corollary 1.2.** Let $G$ be a simple algebraic group over an algebraically closed field of characteristic $p \geq 0$. Let $a, b$ be non-central elements of $G$. Then $G \neq C_G(a) C_G(b)$.

Indeed, we see that $C_G(a) C_G(b)$ is rarely dense in $G$ (it only happens in the exceptional cases in Theorem 1.1)—see Corollary 5.13. In particular, we give a very short proof of the following.

**Corollary 1.3.** If $G$ is a simple algebraic group and $x$ is a non-central element of $G$, then for any $g \in G$, $C_G(x) g C_G(x)$ is not dense in $G$. In particular, $|C_G(x) \setminus G / C_G(x)|$ is infinite.

This was proved independently for unipotent elements by Liebeck and Seitz [25, Chapter 1]. The previous result was used by Prasad [35, Theorem B] to show that any Tits system for a quasi-reductive group satisfying some natural conditions is a standard Tits system (see [35] for more details).

**Conjecture A** is open only for the simple groups of Lie type, where it was known to be true for certain families (cf. [34]), but not for any family of arbitrary rank and field size. Our idea is to show that we can find a small set of irreducible characters $I'$ of $S$ so that for any pair of
