Kinematic analysis of seven-degree-of-freedom exoskeleton rehabilitation manipulator

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Abstract
This article analyzes the forward kinematics and inverse kinematics of the seven-degree-of-freedom exoskeleton rehabilitation manipulator. Denavit–Hartenberg coordinates are used to model the forward kinematics, and the working space of the end effector of the manipulator is analyzed according to the joint motion range of the human arm. In the inverse solution of the seven-degree-of-freedom exoskeleton rehabilitation manipulator, the self-motion angle of the elbow is used. The minimum energy standard is used to calculate the self-motion angle. The minimum energy mainly includes the gravitational potential energy of the upper limbs and the elastic potential energy stored in the muscles. Thus, the inverse solution formula of the seven-degree-of-freedom exoskeleton rehabilitation manipulator is derived. When calculating the angle, an auxiliary parameter is introduced to solve the self-motion manifold of the manipulator. Finally, the theoretical derivation and verification of the forward and inverse kinematics are carried out in this article, and through analysis of the results, it is concluded that the inverse kinematics of this article has some limitations but the theory of inverse kinematics is feasible.

Keywords
Redundant, exoskeleton rehabilitation manipulator, kinematic analysis, self-motion angle, minimum energy

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Introduction
Seven-degree-of-freedom (7-DOF) manipulators are called redundant manipulators, and they can move at any position in space. At present, due to the continuous advancement of science and technology, many exoskeleton rehabilitation robotics have appeared in medical treatment.¹–⁵ Since the current exoskeleton rehabilitation manipulator arms are configured with offset DOFs, these exoskeleton manipulator arms are not only cumbersome but also occupy a lot of limited space and consume a lot of energy during movement. This article mainly analyzes the kinematics of the unbiased 7-DOF exoskeleton rehabilitation manipulator.

For the kinematic analysis of the 7-DOF manipulator, we know that for the tandem manipulator with redundant DOFs, the forward kinematic analysis is relatively simple, and the inverse kinematic analysis is more difficult.

At present, many scholars have proposed many inverse solutions for redundant manipulators.⁶–⁹ Among them, the...
more classic method is based on speed, which is based on the Jacobian matrix. The advantage of the Jacobian matrix method is its strong applicability. However, compared with the method based on the position solution, the disadvantage of the Jacobian matrix is a large amount of calculation and the time-consuming. In 1991, Lee and Bejczy proposed a parametric-based redundant arm motion control method, deduced the closed inverse solution method of the manipulator, and realized the position-based motion control method. This research shows that the calculation efficiency and numerical stability of the position-based motion control method are better. In 2008, Shimizu et al. analyzed the closed inverse kinematics and introduced the arm angle parameter to analyze the self-motion of the redundant manipulator. However, the 3 DOFs are limited in the inverse solution. Further, Yan et al. proposed the two-arm angle parameters and defined a new reference plane, which can avoid the singularity in the solution. Similarly, Gong et al. proposed the use of arm angles and angles that limit 1 DOF to solve the inverse motion, and introduced auxiliary angles to solve the problem of self-moving manifolds. In addition to using arm angles and auxiliary parameters, Faria et al. proposed a calculation method of elbow position based on feasible intervals. At the same time, there are intelligent algorithms for solving inverse kinematic problems, such as neural networks, support vector machines, and particle swarm optimization algorithm. However, these algorithms are usually used to solve inverse kinematic problems off-line. And as the number of iterations becomes progressively larger, the accumulated error will gradually become larger.

For the inverse kinematic solution of the manipulator, previous researchers have provided many excellent methods. In 1986, Wampler proposed to use the damped least-squares method to solve the problem of failure near the singularity in the inverse kinematic solution. Deo and Walker proposed that the damped least-squares method has been used in combination with the redundant resolution scheme to solve the inverse kinematic solution. Jung et al. proposed the optimal robust inverse kinematic solution. Zaplana and Basanez proposed to find the inverse kinematic solution by analyzing the working space of the manipulator. Toz proposed a single-solution optimization algorithm, Vortex Search, to solve the inverse kinematic problem of a biased serial robot manipulator. With the development of the inverse kinematic solution, many scholars are studying the inverse solution of 7-DOF. Wang et al. proposed a new and efficient algorithm for inverse kinematics of 7R 6-DOF robots. It constructs a reasonable conversion between the 7R 6-DOF robot and the well-known equivalent 6R robot, thereby using 6R to solve the 7R problem. Jiang et al. proposed a comprehensive method to solve the inverse kinematic problem of a 7-DOF humanoid arm with an offset wrist. The kinematic simulation verifies the effectiveness of the closed-form solution and integration method. To solve the inverse kinematics of the 7-DOF redundant manipulator, Song et al. proposed a closed-loop framework for the inverse kinematics of the 7-DOF manipulator. Singh et al. proposed kinematic modeling of a 7-DOF spatial hybrid manipulator for medical surgery. And Singla et al. proposed a dimensional synthesis of kinematically redundant serial manipulators for cluttered environments.

Human arm movement is a complex process, so the movement of the human upper limbs needs to be considered when analyzing the kinematic of the rehabilitation manipulator. Lenarcic introduced some characteristics of the humanoid manipulator. The description gives us a clearer understanding of the humanoid manipulator. Almasri and Ouezdou introduced an energy standard, which includes the gravitational potential energy and kinetic energy of the arm, and then analyzed the human-like motion. But he didn’t take into account the energy stored in the muscles. Xie et al. proposed the target arm posture to explain the natural movement of the arm. This method illustrates the relationship between the wrist of the manipulator and the end position of the manipulator. Zhao et al. proposed a standard for optimizing the arm angle of self-motion based on the minimum potential energy standard and also proposed a standard for the size of the potential energy stored in the muscle. From the above research, it can be seen that in the development process of the manipulator, the movement mode of the human upper limb is combined, and the kinetic energy and gravitational potential energy of the arm are also considered. These developments enable the manipulator to better replace the human arm.

There are two main contributions to this article. The first is to establish a minimum energy loss standard to solve the kinematic problem of the 7-DOF exoskeleton rehabilitation manipulator. The second is to combine the self-motion manifold of the elbow and introduce the auxiliary parameter CK for simplification, and also derive the relationship between the end pose of the manipulator and the position of the wrist of the manipulator. The unbiased exoskeleton rehabilitation manipulator established in this article occupies a small space in the application. When the minimum energy standard is used to control it, there will be a variety of options for its energy-providing device, which will make the manipulator more flexible.

**Forward kinematic analysis**

The forward kinematics of the manipulator is solved according to the DOF of each joint. A Denavit–Hartenberg (D-H) coordinate system is established on each DOF of the manipulator, and the homogeneous transformation matrix between adjacent DOFs is calculated. Through the product of the homogeneous transformation matrix, the end pose of the manipulator relative to the space base coordinate system is obtained.
Establish the manipulator coordinate system

In this article, the exoskeleton rehabilitation manipulator has 7 DOFs. The distribution of DOF is S-R-S and unbiased. The seven coordinate systems established by the D-H matrix method are shown in Figure 1. The seven coordinate systems are 7-DOF coordinate systems. The space-based coordinate system $Z_0$ is perpendicular to the ground, and $X_0$ and $Y_0$ are parallel to the ground.

The length of the exoskeleton rehabilitation manipulator rod is based on the data of the standard size of the Chinese male adult arm. For the patient to have a comfortable rehabilitation training environment, the upper arm ($L_0^1$) and forearm ($L_0^2$) need to have an adjustment range. Therefore, the length of the connecting rod ($L_0^1$, $L_0^2$) must be longer than the reference length of the human body ($L_1$, $L_2$). Since the patient performs rehabilitation training by holding the end of the manipulator, the length of the manipulator’s hand ($L_0^3$) is lower than the reference length of the human hand ($L_3$). For the upper arm, take an integer upward on the basis of reference to obtain a size of 500; for the forearm, take the integer up to get the size of 300, and for the hand, take the integer down to get the size of 130 according to the proportion. The length of the exoskeleton rehabilitation manipulator is shown in Table 1.

The geometric shape of the rehabilitation manipulator is shown in Figure 2. We use seven sets of drive systems to achieve rehabilitation training with 7 DOFs for patients. The upper arm plate and the forearm plate mainly play a supporting role. There is a chute at the center of the upper arm plate and the forearm plate, which can adjust the position to allow the patient to perform comfortable rehabilitation exercises. The handheld is the place where the patient grasps during the rehabilitation exercise, and the handle also has a DOF of rotation. The position of the center of mass of each part of the arm of a normal human body: the center of mass of the upper arm is $a_1 = 0.4360$ of the

![Figure 1. Denavit–Hartenberg coordinate system.](image1)

![Figure 2. The geometric shape of rehabilitation manipulator.](image2)

| Name          | Human reference length, $L_i$ (mm) | Length of the manipulator, $L'_0$ (mm) |
|---------------|-----------------------------------|---------------------------------------|
| Upper arm ($L_1$) | 333                               | 500                                   |
| Forearm ($L_2$)     | 253                               | 300                                   |
| Hand ($L_3$)        | 193                               | 130                                   |

Table 1. The length of each part of the manipulator.
length of the upper arm and is close to the shoulder joint. The total weight of the upper arm accounts for about 1/2 of the total weight of the entire upper limb. The position of the center of mass of the human forearm is \( \alpha_2 = 0.430 \) of the length of the forearm and is close to the elbow joint. The weight of the forearm accounts for about 1/3 of the weight of the entire upper limb. The center of mass of the hand is located in the palm. Since the rehabilitation manipulator designed in this article, the patient adopts a hand-holding posture during the rehabilitation exercise, so the center of mass is taken at \( \alpha_3 = 0.5 \) of the length of the hand, and the mass of the hand accounts for about 1/6 of the upper limb. When estimating the motor power, it is assumed that the designed rehabilitation manipulator and the upper limb have an equivalent center of mass. From this, the total mass and position distribution of the upper limb and each part of the manipulator can be obtained as shown in Table 2.

\[
P_{m,i} = \alpha_i L'_i \quad (i = 1, 2, 3)
\]

where \( P_{m,i} \) is the position of the center of gravity of the exoskeleton manipulator; \( \alpha_i \) is the position coefficient of the center of gravity on the arm; \( L'_i \) is the length of the exoskeleton manipulator.

### Using the D-H method to analyze the forward kinematics of the manipulator

The homogeneous transformation matrix \( i^{-1}T \) between adjacent DOFs is

\[
i^{-1}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} - s\alpha_{i-1} - s\alpha_{i-1} d_i & 0 & 0 \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} + c\alpha_{i-1} d_i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

where \( c \) is \( \cos \) and \( s \) is \( \sin \), \( \theta_i, d_i, a_{i-1}, \alpha_{i-1} \) are the D-H parameters.

From Figure 1 and Table 1, we can see the rod length and joint angle of the exoskeleton rehabilitation manipulator, from which we get the D-H parameters as shown in Table 3.

In order to establish the forward kinematic model of the manipulator, suppose the end attitude matrix of the manipulator is \( 7^{-1}T \), as shown in the following formula

\[
7^{-1}T = \begin{bmatrix} 111 & 112 & 113 & p_x \\ 121 & 122 & 123 & p_y \\ 131 & 132 & 133 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

### Inverse kinematic analysis

The inverse solution of the manipulator is based on knowing the end pose of the manipulator and solving the motion angle of each joint. For a manipulator with redundant DOFs, its inverse solution will be more complicated. Therefore, this article uses the minimum energy standard to solve the inverse kinematics with 7 DOFs. The most important parameter used is the self-motion angle \( \phi \) of the elbow. Using the minimum energy criterion will find the unique self-motion angle, and thus the motion angle of each joint. To solve the angle \( \theta_3 \) and self-motion manifold, the auxiliary parameter \( CK \) is introduced.

### Calculation standard of total energy consumed

The main difficulty of the inverse solution is to solve the total energy consumed formula. The difficulty of this formula is to solve the changing height of the gravitational potential energy and the elastic potential energy stored in the human arm. Assuming that the total energy consumed by the human rehabilitation manipulator in the movement process is \( W \), it is converted into the gravitational potential energy of the manipulator and the human arm and the elastic potential energy stored in the human arm without considering the energy loss. Therefore, the formula for calculating the total energy consumption \( W \) is as follows

\[
W = m_g \Delta h_u + m_f g \Delta h_f + m_h g \Delta h_b + \frac{1}{2} k(\pi - \phi)^2
\]

where the values of \( n_{ij} (i, j = 1, 2, 3) \) and \( p_x, p_y, p_z \) are in Appendix 1.

### Table 2. Mass and position distribution of upper limbs and manipulators.

| Name         | Mass (kg) | Position, \( P_{m,i} \) (mm) |
|--------------|-----------|-------------------------------|
| Upper arm \((m_u)\) | 6         | 218 (near the shoulder)       |
| Forearm \((m_f)\)        | 4         | 129 (near the elbow)          |
| Hand \((m_h)\)           | 2         | 65 (middle of hand)           |

### Table 3. D-H parameter table.

| No. | \( \theta_i \) (°) | \( d_i \) (mm) | \( a_{i-1} \) (mm) | \( \alpha_{i-1} \) (°) |
|-----|--------------------|----------------|--------------------|------------------------|
| 1   | \( \theta_1 \)     | 0              | 0                  | -90                    |
| 2   | \( \theta_2 \)     | 0              | 0                  | -90                    |
| 3   | \( \theta_3 \)     | 500            | 0                  | 90                     |
| 4   | \( \theta_4 \)     | 0              | 0                  | 90                     |
| 5   | \( \theta_5 \)     | 300            | 0                  | 0                      |
| 6   | \( \theta_6 \)     | 0              | 0                  | 90                     |
| 7   | \( \theta_7 \)     | 130            | 0                  | 90                     |

\[
0^{-1}T = 7^{-1}T_1^1 7^{-1}T_2^2 7^{-1}T_3^3 7^{-1}T_4^4 7^{-1}T_5^5 7^{-1}T_6^6 7^{-1}T_7^7
\]

where the values of \( n_{ij} (i, j = 1, 2, 3) \) and \( p_x, p_y, p_z \) are in Appendix 1.
The formula of $k$ is as follows:

$$k = 113.02x_w - 23.32y_w - 101.76z_w + 20.32$$  (6)

**Solving inverse kinematics based on the theory of elbow self-motion**

As shown in Figure 3, the self-movement of the elbow assumes that the shoulder and wrist joints are fixed, and the elbow can also move along an ellipse. We assume that the imaginary motion trajectory around the ellipse is $L$. $SE$ is the upper arm, $EW$ is the forearm, and $SW$ is the imaginary axis connecting the shoulder joint and the elbow joint. $SEW$ is the initial plane formed by the upper arm and forearm, perpendicular to the ground. We set $SEW$ as the initial plane after rotating $\phi$ around $SW$, and $\phi$ is the angle between the initial plane and the rotating plane. In this article, the angle $\phi$ is called the self-motion angle. In this article, the value of $\phi$ is $(-180^\circ$ to $180^\circ$), the initial plane rotating clockwise around $SW$ is forward, and counterclockwise is backward.

First, solve the $W$ position: given the end pose coordinate system $(X_H, Y_H, Z_H)$ of the 7-DOF rehabilitation manipulator, use the $H$ point to move to the $W$ point in the space-based coordinate system to obtain the coordinates of the $W$ point.

Split the known homogeneous transformation matrix of the end pose into a rotation matrix and a translation matrix, formulas are as follows

$$0_H R = 0^R = \begin{bmatrix} n_{11} & n_{12} & n_{13} \\
_{21} & n_{22} & n_{23} \\
_{31} & n_{32} & n_{33} \end{bmatrix}$$  (7)

$$0_H P = 0^P = [P_x P_y P_z]^T$$  (8)

To find the position $W$, $0_W P$ is the position of the wrist center point at the space base coordinate system $(X_0, Y_0, Z_0)$. The position of the wrist is also the position of the DOF $\theta_6$, so $0_W P = 0^P$. Suppose we move point $H$ to point $W$, we also need to know $H_W P$, $H_W P = \hat{7}_P$.

As shown in Figure 4, knowing that the space base coordinate systems. So, we assume that the imaginary axis of $SW$ is along the $X_0$ axis of the spatial coordinate system, $W$ at any position in space can be obtained by rotating. Triangle $\Delta SE_1 W_1$ rotates the angle $-\alpha$ around $Y_0$ an axis to get $\Delta SE_2 W_2$, rotates the angle $\beta$ around $Z_0$ the axis to get $\Delta SE_3 W_3$, and then $\Delta SE_3 W_3$ rotates the angle $-\phi$ around the $SW$ to get final $\Delta SEW$. Then we can calculate $\alpha$ and $\beta$. Their calculation formulas are as follows

$$0_W P = 0^P = 0^R 7_P + 0^P = 0^H R_H P + 0^H P$$  (9)

where, $0_W^H P$ is the position of the center of the wrist relative to the endpoint $H$. The distance between the wrist joint point $W$ and the end $H$ of the manipulator is $L_3$, so the formula $0_W^H P$ can be directly obtained

$$0_W^H P = \hat{7}_P = [0, 0, -L_3]^T$$  (10)

As shown in Figure 4, knowing that the space base coordinate systems. So, we assume that the imaginary axis of $SW$ is along the $X_0$ axis of the spatial coordinate system, $W$ at any position in space can be obtained by rotating. Triangle $\Delta SE_1 W_1$ rotates the angle $-\alpha$ around $Y_0$ an axis to get $\Delta SE_2 W_2$, rotates the angle $\beta$ around $Z_0$ the axis to get $\Delta SE_3 W_3$, and then $\Delta SE_3 W_3$ rotates the angle $-\phi$ around the $SW$ to get final $\Delta SEW$. Then we can calculate $\alpha$ and $\beta$. Their calculation formulas are as follows

$$\alpha = \arcsin \frac{0_w z}{L_{SW}}$$  (11)

**Figure 3.** Schematic diagram of elbow self-movement.

**Figure 4.** The relationship between the two coordinate systems.
the new coordinate system 02 is as follows

\[ l_{SW} = \sqrt{\frac{0_x^2}{W_x^2} + \frac{0_y^2}{W_y^2} + \frac{0_z^2}{W_z^2}} \] (13)

A new coordinate \((X_0', Y_0', Z_0')\) is established on the transformed \(\Delta SWE\), as shown in Figure 5. The x-axis of the coordinate is along the \(SW\) direction. Therefore, the relationship between the space base coordinate system 0 and the new coordinate system 02 is as follows

\[
0_{02}R = \begin{bmatrix}
cos\beta & -c\beta\cos\phi + c\phi\beta & -s\beta\sin\phi - c\phi\cos\beta \\
nlsw & n_{SW} & n_{SW} \\
0 & -c\phi & c\phi\end{bmatrix}
\] (14)

As shown in Figure 6, \(\psi\) is the \(\angle WSE\), \(u\) is the gravity center of the \(m_u\), \(f\) is the gravity center of the \(m_f\), \(h\) is the gravity center of the \(m_h\), \(F\) is the foot from points \(E\) to \(SW\). The relationship of the above parameters can be calculated by the following equation

\[
\psi = \arccos \left( \frac{l_{SE}^2 + l_{SW}^2 - l_{EW}^2}{2l_{SE}l_{SW}} \right)
\] (15)

Table 4. Range of joint angles.

| Number | 1   | 2   | 3   | 4   |
|--------|-----|-----|-----|-----|
| \(\theta (\degree)\) | \((-90 to 90)\) | \((-90 to 90)\) | \((-30 to 80)\) | \((0 to 130)\) |
| \(\theta (\degree)\) | \((-45 to 45)\) | \((-20 to 30)\) | \((-10 to 100)\) |

\[
\beta = \arcsin \left( \frac{0_y}{\sqrt{0_x^2 + 0_y^2}} \right)
\] (12)

in the above formula, \(l_{SW}\) can be calculated as follows

\[
l_{SW} = \sqrt{\frac{0_x^2}{W_x^2} + \frac{0_y^2}{W_y^2} + \frac{0_z^2}{W_z^2}}
\]

Figure 5. Schematic diagram of the rotation of point \(W\) in space.

Figure 6. New coordinate system.

\[
l_{EF} = l_{SE}\sin\psi
\] (16)

\[
l_{SF} = l_{SE}\cos\psi
\] (17)

where \(l_{SE}\) is the length of the upper arm, \(l_{SE} = L_1\). \(l_{EW}\) is the forearm length, \(l_{EW} = L_2\). The position of point \(E\) in the new coordinate system \((X_0', Y_0', Z_0')\) can be obtained

\[
0^2E = [l_{SF} 0 -l_{EF}]^T
\] (18)

The position of point \(E\) under the coordinate system \((X_0, Y_0, Z_0)\) can be obtained

\[
0^2E = 0_{02}R^2E
\] (19)

It is assumed that the initial pose of the end is in the negative direction of the axis \((Z_0)\), as shown in Figure 1. As shown in Figure 6, the center of gravity positions of the upper arm, forearm, and arm are known. Set up: \(l_{us} = \mu_1, l_{sw} = \mu_2, l_{sw} = \mu_3\). We can get the position of \(u\) and \(f\) in the new coordinate system 02. Their calculation formulas are as follows

\[
0^2u = [l_{SF} 0 -l_{EF}]^T
\] (20)

\[
0^2f = [(1 - \mu_2) l_{SW} + \mu_2 l_{SE} 0 \ -\mu_2 l_{EF}]^T
\] (21)

Then \(u\) and \(f\) are transformed into the space base coordinate system 0 as follows

\[
0^2u = 0_{02}R^2u
\] (22)

\[
0^2f = 0_{02}R^2f
\] (23)

The increased height of the three arms can be obtained

\[
\Delta h_u = \frac{0}{h} + \mu_1l_{SE}
\] (24)

\[
\Delta h_f = \frac{0}{h} z + l_{SE} + (1 - \mu_2)l_{EW}
\] (25)

\[
\Delta h_h = \frac{0}{h} z + (1 - \mu_3)\frac{0}{h} W_z
\] (26)
The minimum energy consumption standard to solve $\phi$

From the above derivation, the total energy that needs to be consumed during the movement of the rehabilitation manipulator can be obtained as

$$W = m_\text{g}g(0_0 z + \mu_1 l_{SE}) + m_\text{r}g(0_0 z + l_{SE} + (1 - \mu_2)l_{EW}) + m_\text{b}g(u_{31}0_0 z + (1 - \mu_3)u_{01} z) + \frac{1}{2}k(\phi^2 - 2\pi\phi + \pi^2)$$

(27)

Since the total energy $W$ is a quadratic function concerning the angle of self-movement, the total energy $W$ is derivable and has a minimum value at the zero points after the derivation. Let $W' = 0$, the solution is to take the derivative of the total energy formula, solve $\phi$, as follows

$$W' = (\mu_1 m_\text{g}gl_{SE} + \mu_2 m_\text{r}gl_{SE})(\cos \phi) + k \phi - k\pi = 0$$

(28)

From the above formula, can be seen that for solving angle $\phi$, its value has nothing to do with the mass of the rehabilitation manipulator arm and the mass of the patient’s hand.

Calculate the rotation angle of each joint

First, calculate the angle of the first 3 DOFs, so, need to know $\frac{1}{3}T$ and $\frac{3}{5}T$. They can be obtained from the product formula between D-H matrix and matrix

$$\begin{bmatrix} 0 \frac{1}{3} T = 0 \frac{2}{3} T = 0 \frac{1}{5} T^\frac{1}{3} \frac{2}{3} T \end{bmatrix}$$

(29)

$$\begin{bmatrix} 0 \frac{1}{5} T = 0 \frac{2}{5} T = 0 \frac{1}{3} T^\frac{1}{3} \frac{2}{3} T \end{bmatrix}$$

(30)

$$\begin{bmatrix} 0 \frac{1}{3} T = 0 \frac{2}{3} T = 0 \frac{1}{5} T^\frac{1}{3} \frac{2}{3} T \end{bmatrix}$$

(31)

The formula can be obtained

$$500c\theta_1 s\theta_2 = 0_{-E} x$$

$$500c\theta_2 s\theta_4 = 0_{-E} y - 500x\theta_1 s\theta_2 = 0_{-E} z - 300x\theta_2 c\theta_3 c\theta_4$$

$$+ 300c\theta_2 s\theta_4 + 500c\theta_2 c\theta_4 + 500c\theta_1 s\theta_2 = 0_{-E} z - 300c\theta_4 s\theta_3 c\theta_3$$

$$+ 300c\theta_4 c\theta_2 c\theta_3 + 300c\theta_1 s\theta_2 s\theta_4 = 0_{-E} z$$

(32)

$\theta_1, \theta_2, \theta_3$ can be calculated

$$\theta_1 = \arctan \left( \frac{0_{-E} z}{0_{-E} x} \right)$$

(33)

$$\theta_2 = \arctan \left( \frac{0_{-E} y}{0_{-E} x \cos \theta_1} \right)$$

(34)

$$\theta_3 = \arctan \left( a_4, 300\cos \theta_4 \sin \theta_1 \right)$$

(35)

where the values of $a_1$ to $a_4$ are in Appendix 2.

For solving $\theta_4$, we introduce an auxiliary parameter $CK$, as shown in Figure 7. When point $E$ moves above the horizontal plane where $SW$ is located, take $CK = 1$.

$$CK = \begin{cases} -1, & 0_{-E} z > 0_{-E} z \\ 1, & 0_{-E} z \leq 0_{-E} z \end{cases}$$

(36)

Therefore, the formula for solving angle $\theta_4$ is

$$\theta_4 = CK \times \arccos \left( \frac{\alpha_2 - \alpha_3 + \alpha_5}{2\alpha_5} \right)$$

(37)

Because $\frac{1}{4}T, \frac{3}{4}T, \frac{1}{2}T, \frac{3}{2}T$ are known, we can get the three angles by calculation. The calculation formula is as follows

$$\frac{1}{4}T = \frac{1}{5}T - \frac{1}{7}T$$

(38)

The calculation results are as follows

$$130s_5s_6 = \frac{P_{12}}{k_6} - k_{15} + \frac{P_{13}}{k_6} + \frac{P_{14}}{k_{10}}$$

$$130c_5s_6 + 300 = \frac{P_{15}}{k_6} - k_7 + \frac{P_{16}}{k_6} + \frac{P_{17}}{k_{10}}$$

$$-130c_6 = \frac{P_{15}}{k_2} + \frac{P_{16}}{k_4}$$

(39)

$$-s_{67} = \frac{n_{32}k_1}{k_2} + \frac{n_{12}k_3}{k_2} - \frac{n_{22}}{k_{11}}$$

$$s_{6c_7} = \frac{n_{31}k_1}{k_2} + \frac{n_{11}k_3}{k_2} - \frac{n_{21}}{k_{11}}$$

where $c_i$ is cos $\theta_i$, $s_i$ is sin $\theta_i$, the values of $k_1$ to $k_{15}$ are in Appendix 4.

The $\theta_5, \theta_6$, and $\theta_7$ can be solved

$$\theta_5 = \arctan (a_5, a_6)$$

(40)

$$\theta_6 = \arctan (-a_6, a_7 \cos \theta_5)$$

(41)

$$\theta_7 = \arctan (-a_8, a_9)$$

(42)

where the values of $a_5$ to $a_9$ are in Appendix 3.

Therefore, the angles of all DOFs have been calculated.
Forward and inverse kinematic simulation analysis

This section mainly simulates and analyzes forward and reverse movement. The forward kinematic part mainly uses the joint space trajectory function (jtraij) to verify the established D-H matrix and then analyzes the end pose workspace of the 7-DOF human-like rehabilitation manipulator. The inverse kinematic part uses the method proposed in the previous section to perform calculations and then perform theoretical analysis.

Forward simulation analysis

To control the joint motion of the rehabilitation manipulator model established above, we use the jtraij in formula (43) to simulate the joint motion of the manipulator. The jtraij represents the change trajectory of joint coordinates from the initial joint angle to the end joint angle

\[
[Tr, Av, Acc] = jtraij(p_1, p_2, M)
\]

where \(p_1\) and \(p_2\) represent the initial and final positions of the joint movement, and time changes from 0 to 1 in \(M\) steps.

To verify that the manipulator can have a better effect in rehabilitation training, we simulated three sets of training methods. The first group is shoulder adduction and abduction and wrist adduction and abduction, \(p_1 = [0, 0, 0, 0, 0, 0, 0], p_2 = [0, \pi/6, 0, 0, 0, 0, 0], M = 50\). The position trajectory of the end of motion is shown in Figure 8(a), and the joint velocity and joint acceleration are shown in Figure 8(b) and (c), respectively. The second group is shoulder internal/external rotation and elbow flexion/extension, \(p_1 = [0, 0, 0, 0, 0, 0, 0], p_2 = \left[ \pi/2, 0, 0, 0, \pi/2, 0, 0 \right], M = 50\). The position trajectory of the end of motion is shown in Figure 9(a), and the joint velocity and joint acceleration are shown in Figure 9(b) and (c), respectively. The third group is elbow flexion/extension and wrist internal/external rotation, \(p_1 = [0, 0, 0, 0, 0, 0, 0], p_2 = [0, 0, 0, 0, 0, 0, 0], M = 50\). The position trajectory of the end of motion is shown in Figure 10(a), and the joint velocity and joint acceleration are shown in Figure 10(b) and (c), respectively. From the results of the above three sets of simulation training, it can be seen that the speed and acceleration changes of each joint are very smooth, without some sudden changes and other fluctuations, indicating that the designed manipulator model is reasonable, and the joints of the established manipulator model are controllable.

Manipulator end pose working space

According to the range of motion of the human arm joint angle, analyzed the end working space of the rehabilitation manipulator. The range of each angle is shown in Table 4.
In this article, the Monte Carlo method is used to solve the end pose workspace of the 7-DOF manipulator. The Monte Carlo method is a numerical method for solving mathematical problems by random sampling. The Monte Carlo method can be used to solve the workspace of the articulated manipulator. The workspace solved by this method can display the workspace conveniently by using the computer’s graphic display function. The Monte Carlo method does not require the rotation range of each axis, and the error of the space point cloud is also independent of dimension.

The mathematical principle of this method is as follows: using random functions $\text{Rand}(i) (i = 1, 2 \ldots n)$, generate $N$ random values of 0–1, therefore, random step size $(q_{\text{max}_j} - q_{\text{min}_j})\text{Rand}(i)$ is generated. The pseudo-random value of the manipulator is obtained

\[ q_j = q_{\text{min}_j} + (q_{\text{max}_j} - q_{\text{min}_j})\text{Rand}(i) \]

where $q_{\text{max}_j}$ and $q_{\text{min}_j}$ are the maximum and minimum angle variables, and $J (1–7)$ is the number of DOFs.

We use the Monte Carlo method to randomly select 30,000 sets of points $\theta_1–\theta_7$ and calculate the vector coordinates $X$, $Y$, and $Z$ of the end posture of the 7-DOF manipulator. Draw the working space that the manipulator can reach, as shown in Figure 11(a). To show the working space of the 7-DOF robot more clearly, the cross-sectional observation and analysis diagrams of the XOY, XOZ, and YOZ planes of the working space are obtained, as shown in Figure 11(b) to (d).

Without considering the influence of the mechanical mechanism on the working space, the theoretical space of the 7-DOF rehabilitation manipulator is obtained through simulation. It can be seen from Figure 11 that the working space of the manipulator is also affected by the DOF of the joints. It can be seen from the plane projection that the working space of the manipulator conforms to the rotation range of each joint, and the simulation result conforms to the actual space of the manipulator.

**Simulation analysis of inverse kinematic solution**

Assuming that the end pose of the manipulator is known, suppose the Euler angle of $Z, Y, X(\gamma_3, \gamma_2, \gamma_1)$ represents the position of the end pose relative to the space base coordinate system. To avoid the singularity of the manipulator, the increase in Euler angle should not exceed $\pi/2$. The given $x, y,$ and $z$ position change is imitating the rehabilitation movement of the wrist flexion (elbow flexion) of the rehabilitation manipulator

\[
\begin{align*}
x &= 300 + 12 \cos t, \\
y &= 500 + 12 \sin t, \\
z &= 108 + 10t
\end{align*}
\]

\[
\gamma_1 = \frac{t}{24} - \frac{\pi}{2}, \quad \gamma_2 = \frac{t}{24}, \quad \gamma_3 = \frac{t}{24} - \frac{\pi}{2}
\]

The rotation matrix of the Euler angle to represent the desired pose of the end is

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & c_{\gamma_1} & -s_{\gamma_1} \\
0 & s_{\gamma_1} & c_{\gamma_1}
\end{pmatrix}
\]

\[
\begin{pmatrix}
c_{\gamma_2} & 0 & s_{\gamma_2} \\
0 & 1 & 0 \\
-s_{\gamma_2} & 0 & c_{\gamma_2}
\end{pmatrix}
\]

\[
\begin{pmatrix}
c_{\gamma_3} & -s_{\gamma_3} & 0 \\
s_{\gamma_3} & c_{\gamma_3} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

In this article, the time is (0–8$\pi$) seconds. As shown in Figure 12(a) the end pose’s desired trajectory, which is a spiral in space, and (b) is the Euler angle change diagram of the terminal pose.

We use the end posture of the 7-DOF human-like rehabilitation manipulator to obtain the angle change of each joint using the minimum energy standard method established in this article. The time limit is set to (0–8$\pi$), and the distance between points is ($\pi/3$). The results of the joint angle changes are shown in Figure 13.
It can be seen from Figure 13 that within the specified trajectory and time, the minimum energy standard method can obtain the change of the joint angle, and the change is a smooth curve. It can be seen from Figure 14 that the angle value $\phi$ obtained by using the minimum energy loss criterion is very small. This shows that the elbow joint shakes less relative to the virtual connection between the shoulder joint and the wrist joint, which makes the manipulator more stable.

Figure 11. End pose workspace of the manipulator. (a) Three-dimensional workspace. (b) XOY plane workspace. (c) XOZ plane workspace. (d) YOZ plane workspace.

Figure 12. Given known conditions. (a) Desired trajectory of the end effector. (b) Euler angle of end pose.

Figure 13. Changes in joint angle.
We have solved the initial angle and end angle of each joint through the inverse solution method. To verify the above inverse solution method, we use the Cartesian space trajectory function (ctraj) of formula (49) to do straight-line trajectory planning. The ctraj represents the Cartesian trajectory from the attitude $P_1$ to $P_2$, and there are step points of trapezoidal velocity distribution along the path

$$Tc = ctraj(P_1, P_2, N)$$

(49)

where $P_1$ is the calculated starting position of the joint angle, and $P_2$ is the calculated ending position of the joint. $N$ is the number of trapezoidal velocity distribution points along the path.

We use the minimum energy standard method to obtain the joint angle as the input of the inverse solution, and obtain the linear trajectory, joint angle, angular velocity, and angular acceleration of the inverse solution, as shown in Figure 15(a) to (d). The inverse solution method in this article is the same as the linear interpolation method. By comparing the results, it is found that there is a certain error between the obtained linear interpolation trajectory and the set trajectory, which shows that the inverse solution method has certain limitations.

**Conclusion and prospect**

This article refers to the distribution of the length of the human arm and the joint angle to theoretically analyze the kinematics of the 7-DOF rehabilitation manipulator. The forward kinematics of the tandem manipulator is relatively simple, while the inverse solution of the tandem redundant DOF manipulator is more difficult. This article is based on the minimum energy criterion to solve the inverse of the 7-DOF redundant manipulator. The minimum energy standard is used to solve the self-motion angle of the elbow. In the solution, an auxiliary parameter is also introduced to analyze the self-motion manifold of the elbow and solve $\theta_4$.

This article uses Matlab® robotic toolbox to verify the forward kinematics and inverse kinematics of the manipulator. The result shows that the forward kinematic analysis is correct. We use the inverse kinematic analysis method to
calculate the angle of each joint and use the ctraj function to verify the method. The results show that there is a certain error in the method of using the minimum energy standard. In addition, there are some points worth exploring in future research. Questions: (1) Because this method first solves the self-motion angle, but when the weight of the upper arm and forearm changes (the ratio remains the same), the angle of each joint changes differently, and no rule is found; (2) when the motion time changes for a long time, the joint angle calculated by the method in this article will be singular, and the reason for the singularity and the avoiding method has not been found. (3) The obstacle avoidance research of the manipulator is also very important. The next research can consider obstacle avoidance.

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**Appendix 1**

\[ n_{11} = -s_7\left(c_5 \left(s_4 (s_1 s_3 - c_1 c_2 c_3) + c_1 c_4 s_2\right) + s_5 \left(c_4 (s_1 s_3 - c_1 c_2 c_3) - c_1 s_2 s_4\right)\right) - c_7 \left(c_6 \left(c_5 \left(s_4 (s_1 s_3 - c_1 c_2 c_3) - c_1 s_2 s_4\right) - s_5 \left(s_4 (s_1 s_3 - c_1 c_2 c_3) + c_1 c_4 s_2\right)\right) - s_6 (c_3 s_1 + c_1 c_2 s_3)\right) \]

\[ n_{12} = s_7 \left(c_6 \left(c_5 \left(c_4 (s_1 s_3 - c_1 c_2 c_3) - c_1 s_2 s_4\right) - s_5 \left(s_4 (s_1 s_3 - c_1 c_2 c_3) + c_1 c_4 s_2\right)\right) - s_6 (c_3 s_1 + c_1 c_2 s_3)\right) \]

\[ n_{13} = -s_8 \left(c_1 \left(c_4 (s_1 s_3 - c_1 c_2 c_3) - c_1 s_2 s_4\right) - s_5 \left(s_4 (s_1 s_3 - c_1 c_2 c_3) + c_1 c_4 s_2\right)\right) - c_8 (c_3 s_1 + c_1 c_2 s_3) \]

\[ n_{21} = c_7 \left(c_6 \left(c_5 (s_2 s_4 - c_3 c_4 s_2) + s_5 (c_2 c_4 + c_3 s_2 s_4)\right) - s_2 s_3 s_6\right) - s_7 \left(c_5 (c_2 c_4 + c_3 s_2 s_4) - s_5 (c_2 s_4 - c_3 c_4 s_2)\right) \]

\[ n_{22} = c_7 \left(c_5 (s_2 c_4 + c_3 s_2 s_4) + s_5 (c_2 c_4 + c_3 s_2 s_4)\right) - s_7 \left(c_5 (c_2 s_4 - c_3 c_4 s_2) + s_5 (c_2 c_4 + c_3 s_2 s_4)\right) - s_2 s_3 s_6 \]

\[ n_{23} = s_6 \left(c_5 (s_2 c_4 - c_3 c_4 s_2) + s_5 (c_2 c_4 + c_3 s_2 s_4)\right) + c_6 s_2 s_3 \]

\[ n_{31} = -s_7 \left(c_3 \left(s_4 (s_1 s_3 - c_1 c_2 s_3) - c_4 s_1 s_2\right) + s_5 \left(c_4 (s_1 s_3 + c_2 c_3 s_1) + s_1 s_2 s_4\right)\right) - c_7 \left(c_6 \left(c_5 \left(c_4 (s_1 s_3 + c_2 c_3 s_1) + s_1 s_2 s_4\right) - s_5 \left(s_4 (s_1 s_3 + c_2 c_3 s_1) - c_4 s_1 s_2\right)\right) - s_6 (c_1 c_3 - c_2 s_1 s_3)\right) \]

\[ n_{32} = s_7 \left(c_6 \left(c_5 \left(c_4 (s_1 s_3 + c_2 c_3 s_1) + s_1 s_2 s_4\right) - s_5 \left(s_4 (s_1 s_3 + c_2 c_3 s_1) - c_4 s_1 s_2\right)\right) - s_6 (c_1 c_3 - c_2 s_1 s_3)\right) \]

\[ n_{33} = -s_6 \left(c_5 \left(c_4 (s_1 s_3 + c_2 c_3 s_1) + s_1 s_2 s_4\right) - s_5 \left(s_4 (s_1 s_3 + c_2 c_3 s_1) - c_4 s_1 s_2\right)\right) - c_6 (c_1 c_3 - c_2 s_1 s_3) \]
\[ p_x = 500c_1 s_2 - 130s_6 \left( c_5 (c_4 s_1 s_3 - c_1 c_2 c_3) - c_1 s_2 s_4 \right) - s_5 (s_4 (s_1 s_3 - c_1 c_2 c_3) + c_1 c_4 s_2) \]
\[ - 300c_5 (s_1 s_3 - c_1 c_2 c_3) - 130c_6 (c_1 + c_1 c_3 s_1) + 300c_1 s_2 s_4 \]
\[ p_y = 500c_2 + 130s_6 \left( c_5 (c_2 s_4 - c_2 c_4 s_2) + s_5 (c_2 c_4 - c_3 s_2 s_4) \right) + 300c_2 s_4 - 300c_3 c_4 s_2 + 130c_6 s_2 s_3 \]
\[ p_z = -500s_1 s_2 - 130s_6 \left( c_5 (c_4 (c_1 s_3 + c_2 c_3 s_1) + s_1 s_2 s_4) - s_5 (s_4 (c_1 s_3 + c_2 c_3 s_1) - c_4 s_1 s_2) \right) \]
\[ - 300c_4 (c_1 s_3 + c_2 c_3 s_1) - 130c_6 (c_1 c_3 - c_2 s_1 s_3) - 300s_1 s_2 s_4 \]
\[ c_i : \cos \theta_i, \ s_i : \sin \theta_i \]

**Appendix 2**

\[ a_1 = 300 \cos \theta_2 \sin \theta_4 + 500 \cos \theta_2 - \frac{y}{w} \frac{y}{300 \sin \theta_2 \cos \theta_4} \]
\[ a_2 = \frac{500 \cos \theta_1 \sin \theta_2}{a_1} \]
\[ a_3 = \frac{300 \cos \theta_1 \sin \theta_2 \sin \theta_4}{a_1} \]
\[ a_4 = a_2 + 300 \cos \theta_1 \cos \theta_2 \cos \theta_4 + a_3 - \frac{x}{w} \frac{x}{a_1} \]

**Appendix 3**

\[ a_5 = \frac{P_x k_{12}}{k_6} - k_{15} + \frac{P_x k_{11}}{k_6} + \frac{P_x k_{14}}{k_{10}} \]
\[ a_6 = \frac{P_x k_5}{k_6} - k_7 + \frac{P_x k_8}{k_6} + \frac{P_x k_9}{k_{10}} - 300 \]
\[ a_7 = \frac{P_x k_{11}}{k_2} + \frac{P_x k_3}{k_4} \]
\[ a_8 = \frac{n_{21}}{k_2} + \frac{n_{22}}{k_{11}} \]
\[ a_9 = \frac{n_{21}}{k_2} + \frac{n_{22}}{k_{11}} \]

**Appendix 4**

\[ k_1 = c_1 c_2 c_2^2 - s_1 s_3 c_2 + c_1 c_3 s_2^2 \]
\[ k_2 = c_1^2 c_2^2 + c_1^2 c_3^2 + c_2^2 c_2^2 + c_2^2 c_3^2 + c_2^2 c_1^2 + c_3^2 c_2^2 + c_3^2 c_3^2 + c_3^2 c_1^2 \]
\[ k_3 = c_3 s_1 c_2^2 - c_1 s_2 c_2 + c_3 s_1 s_2^2 \]
\[ k_4 = c_1^2 c_3^2 + c_1^2 c_2^2 + c_1^2 c_1^2 + c_2^2 c_3^2 + c_2^2 c_2^2 \]
\[ k_5 = -c_4 s_1 c_2^2 s_3 + c_1 c_4 c_2 c_3 + c_1 s_4 s_3^2 s_2 - c_4 s_1 c_2^2 s_3 + c_1 s_4 s_2^2 s_3^2 \]
\[ k_6 = c_1^2 c_3^2 c_2^2 + c_1^2 c_2^2 c_3^2 + c_1^2 c_1^2 c_3^2 + c_1^2 c_3^2 c_1^2 + c_1^2 c_2^2 c_2^2 + c_1^2 c_2^2 c_1^2 + c_1^2 c_3^2 c_2^2 + c_1^2 c_3^2 c_1^2 + c_1^2 c_1^2 c_2^2 \]
\[ + c_2^2 c_3^2 c_2^2 + c_2^2 c_2^2 c_3^2 + c_2^2 c_2^2 c_1^2 + c_2^2 c_3^2 c_1^2 + c_2^2 c_1^2 c_3^2 + c_2^2 c_1^2 c_1^2 \]
\[ k_7 = \frac{500 s_4}{s_4^2 + c_4} \]
\[ k_8 = c_1 c_4 s_2^2 s_3 + c_4 s_1 c_2 c_3 + s_1 s_4 c_3^2 s_2 + c_1 c_4 s_2^2 s_3 + s_1 s_4 s_2 s_3^2 \]
\[k_9 = c_2 s_4 c_3^2 - c_4 s_2 c_3 + c_2 s_4 s_3^2\]

\[k_{10} = \frac{c_2^2 c_3^2 + c_2^2 c_3^2 + c_2^2 c_4^2 + c_2^2 c_4^2 + c_2^2 c_5^2 + c_2^2 c_5^2 + c_2^2 c_6^2 + c_2^2 c_6^2 + c_2^2 c_7^2 + c_2^2 c_7^2 + c_2^2 c_8^2 + c_2^2 c_8^2}{s_2 s_3}\]

\[k_{11} = \frac{s_2 s_3}{c_2 c_3^2 + c_2^2 c_4^2 + c_2^2 c_5^2 + c_2^2 s_3^2}\]

\[k_{12} = s_1 s_4 c_3^2 s_3 - c_1 s_4 c_2 c_3 + c_1 c_4 c_3^2 s_2 + s_1 s_4 s_2^2 s_3 + c_1 c_4 s_2 s_3^2\]

\[k_{13} = c_1 s_4 c_2^2 s_3 + s_1 s_4 c_2 c_3 - c_4 c_1 s_2^2 s_3 + c_1 s_4 s_2^2 s_3 - c_4 s_1 s_2 s_3^2\]

\[k_{14} = c_2 c_4 c_3^2 + s_2 s_4 c_3 + c_2 c_4 s_3^2\]

\[k_{15} = \frac{500 c_4}{s_4^4 + c_4^2}\]