The KdV equation on a half-line

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Abstract. The initial boundary value problem on a half-line for the KdV equation with the boundary conditions $u|_{x=0} = a \leq 0$, $u_{xx}|_{x=0} = 3a^2$ is integrated by means of the inverse scattering method. In order to find the time evolution of the scattering matrix it turned out to be sufficient to solve the Riemann problem on a hyperelliptic curve of genus two, where the conjugation matrices are effectively defined by initial data.

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1 Introduction

Consider the initial boundary value problem for the KdV equation on the first quadrant

\[ u_t = u_{xxx} - 6uu_x, \quad x > 0, \quad t > 0, \] (1)

\[ u|_{x=0} = a, \quad u_{xx}|_{x=0} = b, \] (2)

\[ u|_{t=0} = u_0(x), \quad u_0(x)|_{x \to +\infty} \to 0. \] (3)

The initial value \( u_0(x) \) is supposed to be consistent with the boundary condition (2) at the corner point: \( u_0(0) = a, \) \( u_{0xx}(0) = b, \) the boundary data \( a \) and \( b \) are chosen to be constants. Besides \( u_0(x) \) should be a smooth function decreasing rapidly enough. Remind that the problem (1)-(3) is correctly posed and uniquely soluble (see, for instance, [1]).

The problem (1)-(3) admits infinitely many integrals of motion. The first three of them are of the form

\[ \int_0^\infty u\,dx = (3a^2 - b)t + \text{const}, \]

\[ \int_0^\infty (u_x^2 + 2u^3)\,dx = -(3a^2 - b)^2t + \text{const}, \]

\[ \int_0^\infty (u_{xx}^2 - 5u^2u_{xx} + 5u^4 + 4(b - 3a^2)u^2)\,dx = \text{const}. \]

When the parameter \( 3a^2 - b \) is different from zero then evidently the first two integrals of motion contain the explicit \( t \)-dependence. It can be proved (see [2]) that the problem (1)-(3) is consistent with an infinite number of higher symmetries and, because of this reason, the KdV equation (1) admits a large class of explicit algebra-geometric solutions satisfying the boundary condition
(2) (see [3]). All these facts allow one to hope that the problem (1)-(3) can be integrated by means of the inverse scattering method. The scattering matrix $s(\xi, t)$ of the associated linear operator $y'' = (u(x, t) - \lambda)y$, $x > 0$, defined in the standard way evaluates in time $t$ by means of the following system of equations

$$s_t = 4i\xi^3[s, \sigma_3] + (u_x\sigma_1 - \frac{4u\xi^2 - u_{xx} + 2u^2}{2\xi}\sigma_2 + \frac{2u^2 - u_{xx}}{2\xi}i\sigma_3)s$$

(4)

containing unknown $u_x(0,t)$ and given $u(0, t) = a$ and $u_{xx}(0, t) = b$ (here and below $\sigma_j$ are the Pauli matrices). The equation (4) shows that the nonlocal change of variables from $u(x, t)$ to $s(\xi, t)$ doesn’t lead to any separation of variables in the problem (1)-(3). Note that such kind obstacles always arise when the initial boundary value problem is studied for integrable equations (see, for instance, [4, 5] where an alternative approach to the problem is discussed).

The system (4) is underdetermined, both the coefficient and the solution are not given (that is why the system is really nonlinear). But it is worth to note that in addition to the system there is an extra condition: for all values of $t$ the solution $s(\xi, t)$ has to preserve its analytical properties, i.e. has to belong to the class of scattering matrices. The latter allows one in some particular cases to simplify essentially the system (4) and ever reduce it to a problem of factorisation. Let us do first a linear change of dependent variables by setting $s(\xi, t) = T(\xi)S(\xi, t)\exp(4i\xi^3\sigma_3t)$. Then the new variable
\( S(\xi, t) \) solves the equation

\[
S_t = (i \nu \sigma_3 + u_x \sigma_1) S, \tag{5}
\]

where

\[
T(\xi) = \begin{pmatrix} q & q \\ -i\nu & i\nu \end{pmatrix}, \quad q = 4\xi^2 + 2a,
\]

and

\[
\nu(\xi) = \sqrt{16\xi^6 + 4\xi^2(b - 3a^2) + 2ab - 4a^3}. \tag{6}
\]

Suppose the function \( P = P(\xi) \) to define an involution of the curve (6) such that \( \nu(\xi) = \nu(P(\xi)) \) for all \( \xi \). Then evidently the identity

\[
S^{-1}(P(\xi), t) S(\xi, t) = S^{-1}(P(\xi), 0) S(\xi, 0) \tag{7}
\]

holds, which may be considered as the first integral of the system (5). The problem arises how to solve the nonlinear equation (7) for \( t > 0 \). Below in the paper we discuss the particular case when the additional constraint \( b = 3a^2, \ a \leq 0 \) is imposed and however we act in a slightly different way (the homogeneous case \( a = b = 0 \) has been studied in [6]). Here we find the solution of the t-equation at the straight-line \( x = 0 \) by gluing up the eigenfunctions of the two operators from the Lax pair and then solving the Riemann problem of linear conjugation for piecewise analytic functions on a six sheeted Riemann surface. The main observation we use below is as the \( x \)- and \( y \)-equations have not only a common fundamental solution but a common solution of the scattering problem for all values of the spectral parameter except some contour on the complex plane.
2 The direct scattering problem

It is well known that the KdV equation is the consistency condition of the following two systems of ordinary linear differential equations:

\[ Y_x = UY, \quad (8) \]
\[ Y_t = VY, \quad (9) \]

where the coefficient matrices are

\[ U = \begin{pmatrix} 0 & 1 \\ u - \lambda & 0 \end{pmatrix}, \quad V = \begin{pmatrix} u_x & -4\lambda - 2u \\ u_{xx} - (4\lambda + 2u)(u - \lambda) & -u_x \end{pmatrix}. \]

Due to the boundary conditions (2) the equation (9) takes the form

\[ Y_t = \begin{pmatrix} u_x & -4\lambda - 2a \\ b - (4\lambda + 2a)(a - \lambda) & -u_x \end{pmatrix} Y \quad (10) \]

along the border \( x = 0 \). In other words the linear systems (8), (9), and (10) give the Lax representation of the initial boundary value problem (1), (2), and (3).

Define matrix-valued solutions of the auxiliary system (8), satisfying the following asymptotic for all real values of the parameter \( \xi = \sqrt{\lambda} \)

\[ Y_1(x, t, \xi) \to T_0(\xi)e^{ix\xi}, \quad x \to +\infty, \quad (11) \]
\[ Y_2(x, t, \xi) \to T_0(\xi), \quad x \to 0, \quad (12) \]

where \( T_0(\xi) = \begin{pmatrix} 1 & 1 \\ i\xi & -i\xi \end{pmatrix} \). The columns \( \psi_k(x, t, \xi) \) and \( \phi_k(x, t, \xi) \), \( k = 1, 2 \) of the matrices \( Y_1 = (\psi_1, \phi_1) \) and \( Y_2 = (\phi_2, \psi_2) \) are known to have analytical
continuations with respect to the parameter $\xi$ from the real axis onto the complex plane. Moreover, $Y_2(x, t, \xi)$ is an entire analytical function and its columns have the following asymptotic in the corresponding half-planes when $|\xi|$ goes to infinity

$$
\phi_2(x, t, \xi) \to e^{ix\xi} \left( \begin{array}{c} 1 \\ i\xi \end{array} \right) \left( 1 + O(\xi^{-1}) \right), \quad \text{Im}\xi \leq 0, \quad (13)
$$

$$
\psi_2(x, t, \xi) \to e^{-ix\xi} \left( \begin{array}{c} 1 \\ -i\xi \end{array} \right) \left( 1 + O(\xi^{-1}) \right), \quad \text{Im}\xi \geq 0. \quad (14)
$$

In general the columns of the other solution $Y_1$ are defined only on the half-planes $\text{Im}\xi < 0$ and $\text{Im}\xi > 0$, respectively, where they have similar asymptotic for $\xi \to \infty$

$$
\phi_1(x, t, \xi) \to e^{-ix\xi} \left( \begin{array}{c} 1 \\ -i\xi \end{array} \right) \left( 1 + O(\xi^{-1}) \right), \quad \text{Im}\xi \leq 0, \quad (15)
$$

$$
\psi_1(x, t, \xi) \to e^{ix\xi} \left( \begin{array}{c} 1 \\ i\xi \end{array} \right) \left( 1 + O(\xi^{-1}) \right), \quad \text{Im}\xi \geq 0. \quad (16)
$$

The scattering matrix of the system (8) given on the half-line is defined as a ratio of two fundamental solutions: $s(\xi) = Y_2^{-1}Y_1$. It is easy to show that entries $s_{11}$ and $s_{21}$ of the matrix $s$ are analytic in the domain $\text{Im}\xi > 0$ and similarly the entries $s_{12}$ and $s_{22}$ – in the domain $\text{Im}\xi < 0$.

Let us compound matrices $\phi$ and $\psi$ by taking the vectors $\phi_i$ and $\psi_i$ as their columns multiplied by scalar factors as follows

$$
\phi(x, \xi) = (\phi_2, \phi_1s_{22}^{-1})(x, \xi)e^{-ix\xi\sigma_3}, \quad (17)
$$

$$
\psi(x, \xi) = (\psi_1s_{11}^{-1}, \psi_2)(x, \xi)e^{-ix\xi\sigma_3}. \quad (18)
$$
By definition the functions $\phi(x, \xi)$ and $\psi(x, \xi)$ are analytic in lower and upper half-planes, respectively. At the real axis $\text{Im} \xi = 0$ they are related to each other by conjugation condition:

$$\phi(x, \xi) = \psi(x, \xi)r(x, \xi). \quad (19)$$

The $x$ dependence of the conjugation matrix $r(x, \xi)$ is given for $x \geq 0$ by:

$$r(x, \xi) = e^{ix\xi\sigma_3}r(0, \xi)e^{-ix\xi\sigma_3}. \quad (20)$$

The asymptotic representations above yield

$$\phi(x, \xi) \rightarrow T_0(\xi) \quad \text{for} \quad \text{Im} \xi \leq 0, \quad \xi \rightarrow \infty \quad \text{and} \quad \psi(x, \xi) \rightarrow T_0(\xi) \quad \text{for} \quad \text{Im} \xi \geq 0, \quad \xi \rightarrow \infty. \quad (21) \quad (22)$$

The eigenfunctions (17), (18) are degenerate at the point $\xi = 0$, since the scattering matrix has generally a pole there. Evidently,

$$\phi(x, 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0, \quad \psi(x, 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0. \quad (23)$$

In the special case when $s(\xi)$ is bounded at $\xi = 0$ the condition (23) is not valid.

The conjugation matrix in (19) is expressed explicitly through entries of the scattering matrix:

$$r(x, \xi) = \begin{pmatrix} 1 & s_{12}s_{22}^{-1}e^{2ix\xi} \\ -s_{21}s_{11}^{-1}e^{-2ix\xi} & s_{11}^{-1}s_{22}^{-1} \end{pmatrix}. \quad (24)$$

A non-degenerate matrix-valued solution $Y(x, \xi)$ of the system (8) is called solution of the scattering problem on the half-line $x \geq 0$ at the point $\xi$, $\text{Im} \xi \neq$
0, if the function $Y(x, \xi)e^{-ix\xi\sigma_3}$ is bounded uniformly for all values of $x \geq 0$. Clearly functions $Y(x, \xi) = \phi(x, \xi)e^{ix\xi\sigma_3}$ and $Y(x, \xi) = \psi(x, \xi)e^{ix\xi\sigma_3}$ are solutions of the scattering problem at the corresponding half-planes except the points $\xi$ where $s_{22}(\xi) = 0$, if $\text{Im}\xi < 0$ and $s_{11}(\xi) = 0$, if $\text{Im}\xi > 0$.

**Remark.** The solution of the scattering problem on a half-line is not unique. It is defined up to the right side matrix-valued multiplier upper triangular when $\text{Im}\xi > 0$ and lower triangular when $\text{Im}\xi < 0$.

We will use this freedom when finding solutions of the similar scattering problem for the second operator which is responsible for the $t$-evolution.

Let us give explicit representations of the matrices $\phi$ and $\psi$ at the end $x = 0$

$$T_0^{-1}(\xi)\phi(0, \xi) = \begin{pmatrix} 1 & s_{12}(\xi) \\ 0 & s_{22}(\xi) \end{pmatrix}, \quad T_0^{-1}(\xi)\psi(0, \xi) = \begin{pmatrix} 1 & 0 \\ s_{21}(\xi) & s_{11}(\xi) \end{pmatrix}. \quad (25)$$

3  **Method of gluing up the eigenfunctions**

In the previous section we constructed eigenfunctions (i.e. solutions of the scattering problem) for the $x$-equation (8). According to our conjecture there should be a common eigenfunction for both $x$- and $t$-equations (8) and (10) at the corner point $(x = 0, t = 0)$ defined for all values of the spectral parameter. But how to find such a solution? The matter is that the scattering problem for these two equations are naturally formulated on two different spectral planes. Really, after the linear change of variables $Y(t, \xi) = T(\xi)\tilde{Y}(t, \xi)$ (11)
takes a form of the well-known Zakharov-Shabat system (cf. (8))

\[ \tilde{Y}_t = (i\nu \sigma_3 + u_x(0, t)\sigma_1) \tilde{Y}, \]  

(26)

where the function

\[ \nu = \sqrt{16\xi^6 + 2a^3} \]  

(27)

is a particular case of (8) under the constraint \( b = 3a^2 \).

To be able to compare functions on \( \xi \) with those on \( \nu \) it is necessary to consider two Riemann surfaces \( \Gamma \) and \( H \) appearing in a natural way such that the function (27) establishes a one-to-one correspondence between them. The first one is a two-sheeted cover of the \( \xi \)-plane glued by its six cuts such that two cuts go along the real axis four others are done to make surface unvariant under rotation by angle \( \frac{\pi}{3} \) (see Pic.1). Denote through \( \Gamma_1 \) and \( \Gamma_2 \) the sheets of the surface. Require that \( \xi \in \Gamma_1 \) if the asymptotic representation \( \nu(\xi) = 4\xi^3 + o(1) \) is valid for \( \xi \to \infty \) and \( 0 < \arg \xi < \frac{\pi}{3} \). Remind that the continuous spectrum of the equation (8) coincides with the real axis \( \text{Im} \xi = 0 \).

By rotating the axis around the origin by angles \( \frac{\pi}{3} \) and \( \frac{2\pi}{3} \), i.e. applying the involution of the curve, one gets two more directed lines. The conjugation contour \( L \), consisting of these three lines taken on both sheets, divides the surface into twelve sectors. Enumerate them in the following way. Denote through \( I_{1,1} \) and \( I_{2,1} \) those lying in the first quadrant on the sheets \( \Gamma_1 \) and \( \Gamma_2 \) respectively. Denote \( I_{2,1} \) and \( I_{2,2} \) the next two sectors that lie partly in the first and partly in the second quadrants on the corresponding sheets and so on.
The function $\xi = \xi(\nu)$ inverse to (27) is evidently single-valued on the six-sheeted cover of the complex plane $H$ with sheets $H_j$, $j = 1, \ldots, 6$ glued by two cuts lying on the imaginary axis and going to infinity from the branch points $\nu_{\pm} = \pm i \sqrt{2|a|^3}$ (see Pic.2). The sheets are enumerated in such a way that the images $\tilde{I}_{k,j} = \nu(I_{k,j})$ are as follows

\[
\tilde{I}_{1,k} = \{ \nu \in H_k, \text{Re}\nu > 0 \} \cup \{ \nu \in H_{k+1}, \text{Re}\nu < 0 \} \cap \{ \text{Im}\nu > 0 \},
\]
\[
\tilde{I}_{2,k} = \{ \nu \in H_k, \text{Re}\nu < 0 \} \cup \{ \nu \in H_{k+1}, \text{Re}\nu > 0 \} \cap \{ \text{Im}\nu < 0 \}.
\]

The sheets are such that when crossing the upper cut from the left to the right one passes from $H_j$ to $H_{j+1}$, where $H_7 = H_1$ and $H_0 = H_6$.

Denote through $Y_{kj}(t,\nu) = \Psi_{kj}(t,\nu)e^{-4i\xi^3\sigma_3}$ solutions of the equation (26) where the multipliers $\Psi_{kj}(t,\nu)$ are regular for large enough $\nu$ from $\tilde{I}_{kj}$. Actually, these multipliers just are to be found. Normalize them by fixing their triangular matrix structure

\[
\Psi_{k,2m}(0,\nu) = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}, \quad \Psi_{k,2m+1}(0,\nu) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.
\] (28)

Comparison of these eigenfunctions with those of the $x$-equation gives rise to the so-called gluing up equations:

\[
\psi(0,\xi)\delta_{kj}(\xi) = T(\xi)\Psi_{k,j}(0,\nu)\Delta_k(\nu)
\] (29)

for $j = 1, 2, 3; k = 1, 2$;

\[
\phi(0,\xi)\delta_{kj}(\xi) = T(\xi)\Psi_{k,j}(0,\nu)\Delta_k(\nu)
\] (30)

for $j = 4, 5, 6; k = 1, 2$. 

The left side of these equations gives an expression of arbitrary solution of the scattering problem for the $x$-equation while the right side gives that of the other one. Here the right side multipliers $\delta_{kj}(\xi)$ are upper (lower) triangular matrix-valued functions defined on $I_{kj}$ for $j = 1, 2, 3$ ($j = 4, 5, 6$, respectively) and $\Delta_{kj}(\nu)$ are upper (lower) triangular for $j = 2, 4, 6$ ($j = 1, 3, 5$) matrices defined on $\tilde{I}_{kj}$. Thus two multipliers $\delta_{kj}$ and $\Delta_{kj}$ are of one and the same matrix structure for only four choices of subindices $(k, j)$: $(1, 2)$, $(2, 2)$, $(5, 1)$, $(5, 2)$. In these cases the equations $(29)$ and $(30)$ are effectively solved by reducing to the problem of triangular factorisation of matrices. For example, when $k = 1$, $j = 2$ one has

$T^{-1}(\xi)\psi(0, \xi) = \Psi_{1,2}(0, \nu(\xi))\Delta_{1,2}(\nu(\xi))\delta_{1,2}^{-1}(\xi), \quad \xi \in I_{1,2}.$

The rest of functions are defined by imposing the following involution

$\Psi_{kj}(\nu(\xi)) = \Psi_{k,j+2}(\nu(\omega\xi)) = \Psi_{k,j+4}(\nu(\omega^2\xi)), \quad (31)$

which is completely consistent with the equations $(29)$, $(30)$ as well as the involutions

$\Psi_{kj}(-\nu(-\xi)) = \sigma_1\Psi_{mn}(\nu(\xi))\sigma_1 \quad (32)$

and

$\Psi_{kj}(-\nu(-\xi)) = \Psi_{mn}(\bar{\nu}(\bar{\xi})), \quad (33)$

where the line over a letter means the usual complex conjugation.

Having all of the functions $\Psi_{kj}(t, \nu)$ at the point $t = 0$ one can find the conjugation matrices of the Riemann-Hilbert-Carleman problem on the Riemann surface $H$ with the contour of jumps $\tilde{L} = \nu(L)$ associated with the
\( t \)-equation from the Lax pair:

\[
\Psi_{kj}(t, \nu) = \Psi_{mn}(t, \nu)R_{kjmtn}(t, \nu). \tag{34}
\]

The conjugation matrices explicitly depend on \( t \):

\[
R_{kjmtn}(t, \nu) = e^{-4i\xi t \sigma_3}R_{jkmn}(0, \nu)e^{4i\xi t \sigma_3} \tag{35}
\]

By solving the Riemann problem one finds the eigenfunctions \( \Psi_{kj}(t, \nu) \) for all \( t > 0 \) and recovers the linear system (26), i.e. its unknown coefficient \( u_x(0, t) \). Afterwards the equation for the time evolution of the scattering matrix (24) becomes linear. On the other hand side the scattering matrix \( s(t, \xi) \) can directly be expressed in terms of the functions \( s(0, \xi) \) and \( \Psi_{kj}(t, \nu) \). And the last step consisting of recovering the potential \( u(x, t) \) for a given \( s(\xi, t) \) called the inverse scattering problem is studied very much (see, for instance, [7]).

To conclude we remark that since the spectrum of \( t \)-equation for \( x = 0 \) lies on a two genus Riemann surface then it is natural to expect that \( u_x(0, t) \) has finite-gap behaviour at infinity.

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Pic.1. The double sheeted Riemann surface $\Gamma$.

Pic.2. The six sheeted Riemann surface $H$.