Behaviour at infinity for solutions of a mixed nonlinear elliptic boundary value problem via inversion

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Abstract

We study a mixed boundary value problem for the quasilinear elliptic equation
\[ \text{div} \mathcal{A}(x, \nabla u(x)) = 0 \]
in an open infinite circular half-cylinder with prescribed continuous Dirichlet data on a part of the boundary and zero conormal derivative on the rest. We prove the existence and uniqueness of bounded weak solutions to the mixed problem and characterize the regularity of the point at infinity in terms of \( p \)-capacities. For solutions with only Neumann data near the point at infinity we show that they behave in exactly one of three possible ways, similar to the alternatives in the Phragmén–Lindelöf principle.

Key words and phrases: continuous Dirichlet data, existence and uniqueness of solutions, mixed boundary value problem, Phragmén–Lindelöf trichotomy, quasilinear elliptic equation, regularity at infinity, Wiener criterion.

Mathematics Subject Classification (2020): Primary: 35J25. Secondary: 35J62, 35B40.

1. Introduction

The Dirichlet problem on a nonempty open set \( \Omega \subset \mathbb{R}^n \) entails finding a function \( u \) which solves a certain partial differential equation in \( \Omega \) with the prescribed boundary data \( u = f \) on \( \partial \Omega \). When the boundary data \( g : \partial \Omega \to \mathbb{R} \) are taken as the normal derivative \( \partial u / \partial \nu = g \), the problem is called a Neumann problem. More general directional derivatives, such as the conormal derivative \( Nu \) considered below, are also possible. In this paper, we study a mixed boundary value problem for the quasilinear equation
\[ \text{div} \mathcal{A}(x, \nabla u(x)) = 0, \]
where a Dirichlet condition is prescribed on a part of the boundary, while the rest carries the zero conormal derivative
\[ Nu(x) := \mathcal{A}(x, \nabla u(x)) \cdot \nu(x) = 0, \]
where \( \nu(x) \) is the unit outer normal at \( x \). Equation (1.1) is considered in an infinite circular half-cylinder and the vector-valued function \( A \) satisfies the standard ellipticity conditions with a parameter \( p > 1 \). The \( p \)-Laplace equation \( \text{div}(|\nabla u|^{p-2} \nabla u) = 0 \) is included as a special case.

We prove the existence and uniqueness of bounded continuous weak solutions to the above mixed boundary value problem with continuous Dirichlet boundary data \( f \) on a closed set \( F \subset \partial G \), which can be bounded or unbounded. More precisely, if \( f \in C(F) \) is such that

\[
\lim_{F \ni x \to \infty} f(x) \quad \text{exists and is finite,}
\]

then there exists a unique bounded continuous weak solution \( u \) of (1.1) with zero conormal derivative on \( \partial G \setminus F \) and such that

\[
\lim_{x \to x_0} u(x) = f(x_0) \quad \text{for all } x_0 \in F \text{ outside a set of } C_p\text{-capacity zero},
\]

see Theorem 5.3 and Remark 5.4. Moreover, \( u \) is Hölder continuous at all points in the Neumann boundary \( \partial G \setminus F \). For Dirichlet data of Sobolev type, existence and uniqueness in the Sobolev sense are proved in Theorem 5.6.

We also characterize the regularity of the point at infinity by a Wiener type criterion. Roughly speaking, if the Dirichlet boundary \( F \) is sufficiently thick at \( \infty \) in terms of a variational \( p \)-capacity adapted to the mixed problem, then for every \( f \in C(F) \) satisfying (1.2), the unique solution of the above mixed problem satisfies

\[
\lim_{x \to \infty} u(x) = f(\infty).
\]

Conversely, thickness of \( F \) at \( \infty \) is also necessary in order for (1.3) to hold for all \( f \in C(F) \), see Theorem 6.5. On the other hand, if \( F \) is bounded, i.e. when only the Neumann condition is used in a proximity of the point at infinity, then we show in Theorem 7.4 that each solution \( u \) of equation (1.1) with zero conormal derivative near \( \infty \) behaves in one of the following three ways as \( x \to \infty \):

(i) The solution has a finite limit \( u(\infty) := \lim_{x \to \infty} u(x) \).

(ii) The solution tends roughly linearly to either \( \infty \) or \( -\infty \).

(iii) The solution changes sign and approaches both \( \infty \) and \( -\infty \), i.e.

\[
\limsup_{x \to \infty} u(x) = \infty \quad \text{and} \quad \liminf_{x \to \infty} u(x) = -\infty.
\]

Similar trichotomy results at \( \infty \) for solutions of the Neumann problem for the linear uniformly elliptic equation \( \text{div}(A(x) \nabla u) = 0 \) were obtained in Ibragimov–Landis [16], [17], Lakhturov [24] and Landis–Panasenko [25]. For (sub/super)solutions of various PDEs in unbounded domains with only Dirichlet data, similar alternative behaviour is often referred to as Phragmén–Lindelöf principle and has been extensively studied. See e.g. Gilbarg [9], Hopf [11], Horgan–Payne [12], [13], Jin–Lancaster [19], [20], Lindqvist [26], Lundström [27], Quintanilla [30], Serrin [31] and Vitolo [32].

Compared with pure Dirichlet and Neumann boundary problems, the literature on mixed boundary value problems is less extensive, especially in unbounded domains. Mixed problems are sometimes called Zaremba problems, mainly in the Russian literature, since they were first considered for the Laplace equation \( \Delta u = 0 \) by Zaremba [35] in 1910. For linear uniformly elliptic equations of the type \( \text{div}(A(x) \nabla u) = 0 \), they were studied by e.g. Ibragimov [14], [15], Kerimov [21], [22] and Novruzov [29].
A mixed problem for linear equations of nondivergence type was considered in Cao–Ibragimov–Nazarov [5] and Ibragimov–Nazarov [18].

Existence of weak solutions for mixed and Neumann problems for linear operators in very general unbounded domains was recently obtained using an exhaustion with bounded domains by Chipot [6] and Chipot–Zube [7]. Wiśniewski [33], [34] studied the decay at infinity of solutions to mixed problems with coefficients approaching the Laplace operator in unbounded conical domains. On the other hand, nonexistence Liouville type results for mixed problems of the form $-\Delta u = f(u)$ in various unbounded domains were obtained in Damascelli–Gladiali [8]. Wiśniewski [33], [34] studied the decay at infinity of solutions to mixed problems with coefficients approaching the Laplace operator in unbounded conical domains. On the other hand, nonexistence Liouville type results for mixed problems of the form $-\Delta u = f(u)$ in various unbounded domains were obtained in Damascelli–Gladiali [8].

We point out that even for $p = 2$, equation (1.1) considered here can be nonlinear. This happens for example when $A(x, q) := a(q/|q|)q$, where $a$ is a sufficiently smooth strictly positive scalar function on the unit sphere; see Example 2.3.

In Kerimov–Maz'ya–Novruzov [23], the regularity of the point at infinity for the Zaremba problem for the Laplace equation $\Delta u = 0$ in an infinite half-cylinder was characterized by means of a Wiener type criterion. A similar problem for certain weighted linear elliptic equations was studied in Björn [3] and more recently for the $p$-Laplace equation in Björn–Mwasa [4]. The results in this paper partially extend the ones in [23], [3] and [4] to general quasilinear elliptic equations of the form (1.1), but we also address other properties of the solutions.

In order to achieve our results, we make use of the change of variables introduced in Björn [3], and later adopted by Björn–Mwasa [4], to transform the infinite half-cylinder $G$ and the quasilinear elliptic equation (1.1) into a unit half-ball and a degenerate elliptic equation

$$\text{div} \, B(\xi, \nabla \tilde{u}(\xi)) = 0,$$

with the $A_p$-weight $w(\xi) = |\xi|^{p-n}$, see Section 3.

After the above transformation, we are able to eliminate the Neumann data by reflecting the unit half-ball and equation (1.4) to the whole unit ball, leaving only the Dirichlet data. This is done in Section 4. In Section 5, we then use tools for Dirichlet problems from Heinonen–Kilpeläinen–Martio [10] and Björn–Björn–Mwasa [2] to prove the existence and uniqueness of continuous weak solutions to the mixed boundary value problem for (1.1) in the cylinder, with zero conormal derivative and continuous Dirichlet data.

In Section 6, we show that regularity at infinity for the mixed problem for (1.1) is equivalent to the regularity of the origin for the Dirichlet problem for (1.4). This can in turn be characterized in terms of weighted $p$-capacities using a Wiener criterion, provided by [10, Theorem 21.30] and Mikkonen [28]. This Wiener criterion for (1.4) is then transferred back to the cylinder to characterize the regularity at infinity for (1.1) by means of a variational $p$-capacity adapted to the cylinder.

Finally, in Section 7, we use estimates from [10, Sections 6 and 7] for capacitory potentials and singular solutions of (1.4) to prove the above Phragmén–Lindelöf type trichotomy (i)–(iii).

Acknowledgement. J. B. was partially supported by the Swedish Research Council grant 2018-04106. A. M. was supported by the SIDA (Swedish International Development Cooperation Agency) project 316-2014 “Capacity building in Mathematics and its applications” under the SIDA bilateral program with the Makerere University 2015–2020, contribution No. 51180060.
2. Notation and preliminaries

Let $G = B' \times (0, \infty) \subset \mathbb{R}^n$ be an open infinite circular half-cylinder, where

$$B' = \{ x' \in \mathbb{R}^{n-1} : |x'| < 1 \}$$

is the unit ball in $\mathbb{R}^{n-1}$. Points in $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, $n \geq 2$, are denoted by $x = (x', x_n) = (x_1, ..., x_{n-1}, x_n)$. Let $F$ be a closed subset of the closure $G'$ of $G$. Assume that $F$ contains the base $B' \times \{0\}$ of $G$. We consider the following mixed boundary value problem

$$
\begin{align*}
\text{div} A(x, \nabla u) &= 0, & \text{in } G \setminus F, \\
uu &= f, & \text{on } F_0 := F \cap \partial(G \setminus F), \text{ (Dirichlet data)}, \\
uu := A(x, \nabla u) \cdot \nu &= 0 & \text{on } \partial G \setminus F, \text{ (generalized Neumann data)},
\end{align*}
$$

where $\nuu$ is the conormal derivative, $\nu$ is the unit outer normal of $G$ and $\partial G$ denotes the boundary of $G$.

Let $1 < p < \infty$ be fixed. The mapping $A : G' \times \mathbb{R}^n \to \mathbb{R}^n$, in (2.1) is assumed to satisfy the standard ellipticity and boundedness conditions:

- $A(\cdot, q)$ is measurable for all $q \in \mathbb{R}^n$,
- $A(x, \cdot)$ is continuous for a.e. $x \in G'$,
- There are constants $0 < \alpha_1 \leq \alpha_2 < \infty$ such that for all $q, q_1, q_2 \in \mathbb{R}^n$, $0 \neq \lambda \in \mathbb{R}$ and a.e. $x \in G'$,

$$
\begin{align*}
A(x, q) \cdot q &\geq \alpha_1 |q|^p, & (2.2) \\
|A(x, q)| &\leq \alpha_2 |q|^{p-1}, & (2.3) \\
A(x, \lambda q) &= \lambda |\lambda|^{p-2} A(x, q) & (2.4)
\end{align*}
$$

and

$$
(A(x, q_1) - A(x, q_2)) \cdot (q_1 - q_2) > 0 \quad \text{when } q_1 \neq q_2. \quad (2.5)
$$

The quasilinear elliptic equation $\text{div} A(x, \nabla u) = 0$ and the conormal derivative in (2.1) will be considered in the weak sense as follows.

**Definition 2.1.** A function

$$
u \in W^{1,p}_{\text{loc}}(G' \setminus F) := \{ u |_{G' \setminus F} : u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n \setminus F) \},$$

is a weak solution of the equation $\text{div} A(x, \nabla u) = 0$ in $G' \setminus F$ if the integral identity

$$
\int_{G' \setminus F} A(x, \nabla u) \cdot \nabla \varphi \, dx = 0 \quad (2.6)
$$

holds for all $\varphi \in C^\infty_0(\mathbb{R}^n \setminus F)$, where $\cdot$ denotes the scalar product in $\mathbb{R}^n$ and $C^\infty_0(\Omega)$ is the space of all infinitely many times continuously differentiable functions with compact support in $\Omega \subset \mathbb{R}^n$.

**Remark 2.2.** If the whole boundary $\partial G$ is contained in $F$, then there is no Neumann condition and the mixed boundary value problem reduces to a purely Dirichlet problem.

As usual, the local space $W^{1,p}_{\text{loc}}(\Omega)$ (and later $H^{1,p}_{\text{loc}}(\Omega, w)$) consists of those functions $u$ which belong to $W^{1,p}(\Omega')$ (resp. $H^{1,p}(\Omega', w)$) for all $\Omega' \in \Omega$, where $\Omega' \in \Omega$ means that $\Omega'$ is a compact subset of $\Omega$.

As mentioned in the introduction, equation (1.1) can be nonlinear even for $p = 2$, as illustrated by the following example.
Example 2.3. For $p = 2$ and $x, q \in \mathbb{R}^n$, let
\[
A(x, q) = \begin{cases} 
0 & \text{if } q = 0, \\
\left(\frac{\alpha}{\cos \alpha}\right)q & \text{if } q \neq 0,
\end{cases}
\]
(2.7)
where the scalar function $\alpha$ is strictly positive and continuous on the unit sphere $\partial B(0, 1)$ in $\mathbb{R}^n$ and such that
\[
\frac{a(\theta')}{a(\theta)} > \frac{1 - \sin \alpha}{1 + \sin \alpha} \quad \text{for any } \theta, \theta' \in \partial B(0, 1) \text{ with } \theta \cdot \theta' = \cos \alpha > 0,
\]
(2.8)
where $\alpha = \alpha(\theta, \theta')$ is the acute angle between $\theta$ and $\theta'$. Then $A$ satisfies the ellipticity conditions (2.2)–(2.5).

Indeed, the only nontrivial verification is that of (2.5). For this, we can clearly assume that $|q| = 1$ and
\[
0 < a(\theta') \leq a(\theta), \quad \text{where } \theta = q \text{ and } \theta' = \frac{q'}{|q'|}.
\]
Condition (2.5) then means that the angle between the vectors $q' - q$ and $\frac{a(\theta')}{a(\theta)}q' - q$ is strictly less than $\frac{\pi}{2}$. Since $a(\theta')/a(\theta) \leq 1$, a geometric consideration shows that this is clearly satisfied if $\alpha \geq \frac{\pi}{2}$ or if
\[
\alpha < \frac{\pi}{2} \text{ and } |q'| \leq \frac{1}{\cos \alpha}.
\]
So assume that $\alpha < \frac{\pi}{2}$ and $|q'| > 1/\cos \alpha$. Applying the cosine theorem to the triangles spanned by the vectors $q$ and $q'$, or by
\[
q \text{ and } q'' := \frac{a(\theta')}{a(\theta)}q',
\]
respectively, as well as to the triangle spanned by $q' - q$ and $q'' - q$, we see that the angle between $q' - q$ and $q'' - q$ is $< \frac{\pi}{2}$ if and only if
\[
|q|^2 + |q'|^2 - 2|q||q'| \cos \alpha + |q|^2 + |q''|^2 - 2|q||q''| \cos \alpha > |q' - q''|^2.
\]
Since $|q| = 1$, this is equivalent to
\[
1 - \left(1 + \frac{a(\theta')}{a(\theta)}\right) |q'| \cos \alpha + \frac{a(\theta')}{a(\theta)} |q'|^2 > 0,
\]
i.e.
\[
\frac{a(\theta')}{a(\theta)} > \frac{|q'| \cos \alpha - 1}{|q'|(|q'| - \cos \alpha)}.
\]
Maximizing the right-hand side over $|q'| > 1/\cos \alpha$, we find that its maximum is attained when
\[
|q'| = \frac{1 + \sin \alpha}{\cos \alpha} \quad \text{and equals } \frac{1 - \sin \alpha}{1 + \sin \alpha}.
\]
This justifies the requirement (2.8).

A concrete example of a function satisfying (2.8) is $a(\theta) = e^{\theta \cdot \phi}$, for any fixed vector $q_0 \in \mathbb{R}^n$ with $|q_0| < 1/\sqrt{2}$. Indeed, with this choice,
\[
\frac{a(\theta')}{a(\theta)} = e^{(\theta' - \theta) \cdot \phi} > 1 - \frac{|\theta' - \theta|}{\sqrt{2}},
\]
while
\[
|\theta' - \theta|^2 = 2 - 2 \theta \cdot \theta' = 2 - 2 \cos \alpha \leq 2 \sin^2 \alpha,
\]
from which (2.8) readily follows.

Clearly, a modification of (2.7) and (2.8) can be made so that $A(x, q)$ and $a(x, \theta)$ also depend on $x \in \mathbb{R}^n$. 
Throughout the paper, unless otherwise stated, $C$ will denote any positive constant whose real value is not important and need not be the same at each point of use. It can even vary within a line. By $a \lesssim b$, we mean that there exists a positive constant $C$, independent of $a$ and $b$, such that $a \leq Cb$. Similarly, $a \gtrsim b$ means $b \lesssim a$, while $a \asymp b$ stands for $a \lesssim b \lesssim a$.

3. Transformation of the half-cylinder

In this section, the quasilinear elliptic operator $\text{div} \ A(x, \nabla u)$ in $G$ is shown to correspond to a weighted quasilinear elliptic operator on the unit half-ball. We will use the following change of variables introduced in Björn [3, Section 3].

Let $\kappa > 0$ be a fixed constant. Define the mapping

$T : \mathbb{R}^n \to T(\mathbb{R}^n) = \mathbb{R}^n \setminus \{(\xi', \xi_n) \in \mathbb{R}^n : \xi' = 0 \text{ and } \xi_n \leq 0\}$

by $T(x', x_n) = (\xi', \xi_n)$, where

$$\xi' = \frac{2e^{-\kappa x_n}x'}{1 + |x'|^2} \quad \text{and} \quad \xi_n = \frac{e^{-\kappa x_n}(1 - |x'|^2)}{1 + |x'|^2}.$$ (3.1)

We will use $x = (x', x_n)$ for points in $\overline{G}$ and $\xi = (\xi', \xi_n) = (\xi_1, \ldots, \xi_{n-1}, \xi_n) \in \mathbb{R}^n$ for points in the space transformed by $T$. Note that

$T(G) = \{\xi \in \mathbb{R}^n : |\xi| < 1 \text{ and } \xi_n > 0\}$,

$T(\overline{G}) = \{\xi \in \mathbb{R}^n : 0 < |\xi| \leq 1 \text{ and } \xi_n \geq 0\}$

are the open and the closed upper unit half-ball, respectively, with the origin $\xi = 0$ removed. From (3.1) it is easy to see that

$$|\xi| = |T(x)| = e^{-\kappa x_n} \to 0 \quad \text{as } x_n \to \infty,$$

that is, the point at infinity for the half-cylinder $G$ corresponds to the origin $\xi = 0$ in $T(G)$.

The mapping $T$ is a smooth diffeomorphism between $\mathbb{R}^n$ and $T(\mathbb{R}^n)$, see Björn–Mwasas [4, Lemma 3.1]. A direct calculation shows that the inverse mapping $T^{-1}$ of $T$ is given by

$$x' = \frac{\xi'}{||\xi|| + \xi_n} \quad \text{and} \quad x_n = \frac{1}{\kappa} \log |\xi|.$$

In the following lemma we show how the operator $\text{div} \ A(x, \nabla u)$ on the half-cylinder is transformed under $T$ to the unit half-ball, cf. [4, Section 3].

**Lemma 3.1.** Let $u, v \in W^{1,p}(\Omega)$ for some open set $\Omega \subset G$ and let $\tilde{u} = u \circ T^{-1}$ and $\tilde{v} = v \circ T^{-1}$. Then for any measurable set $A \subset \Omega$,

$$\int_A A(x, \nabla u) \cdot \nabla v \, dx = \int_{T(A)} B(\xi, \nabla \tilde{u}) \cdot \nabla \tilde{v} \, d\xi,$$ (3.2)

where $B$ is for $\xi = Tx \in T(\overline{G})$ and $q \in \mathbb{R}^n$ defined by

$$B(\xi, q) = |J_T(x)|^{-1}dT(x)A(x, dT^*(x)q).$$ (3.3)

Here, $J_T(x) = \det(dT(x))$ denotes the Jacobian of $T$ at $x$ and $dT^*(x)$ is the transpose of the differential $dT(x)$ of $T$ at $x$, seen as $(n \times n)$-matrices. In (3.3), both $q$ and $A(x, dT^*(x)q)$ are regarded as column vectors for the matrix multiplication to make sense.
Proof. First, we rewrite the scalar product on the left-hand side of (3.2) using matrix multiplication as
\[ \mathcal{A}(x, \nabla u) \cdot \nabla v = (\nabla v)^* \mathcal{A}(x, \nabla u), \]
where both \( \mathcal{A}(x, \nabla u) \) and \( \nabla v \) are seen as column vectors. Using the change of variables \( \xi = T(x) \), together with the chain rule
\[ \nabla u(x) = dT^*(x)\nabla \tilde{u}(\xi), \quad \text{where} \quad \xi = T(x), \]
we get
\[
\int_A (\nabla v)^* \mathcal{A}(x, \nabla u) \, dx = \int_{T(A)} (dT^*(x)\nabla \tilde{v})^* \mathcal{A}(x, dT^*(x)\nabla \tilde{u}) |J_T(x)|^{-1} \, d\xi \\
= \int_{T(A)} |J_T(x)|^{-1} dT(x) \mathcal{A}(x, dT^*(x)\nabla \tilde{u}) \cdot \nabla \tilde{v} \, d\xi \\
= \int_{T(A)} \mathcal{B}(\xi, \nabla \tilde{u}) \cdot \nabla \tilde{v} \, d\xi.
\]
\[ \square \]

In view of the integral identity (2.6), Lemma 3.1 shows that the quasilinear equation (1.1) on \( G \setminus F \) will be transformed by \( T \) into the equation
\[ \text{div} \mathcal{B}(\xi, \nabla \tilde{u}) = 0 \quad \text{on} \quad T(G \setminus F), \tag{3.4} \]
with a proper interpretation of the function spaces and the zero Neumann condition.

To prove the fundamental properties of the transformed operator \( \mathcal{B}(\xi, \nabla \tilde{u}) \), we will need the following estimates from Björn–Mwasa [4, Lemma 3.3].

Lemma 3.2. There exist constants \( C_1, C_2 > 0 \) such that if \( x, y \in B' \times \mathbb{R} \) and \( x_n \leq y_n \), then
\[
C_1 e^{-\kappa x_n} |x - y| \leq |T(x) - T(y)| \leq C_2 e^{-\kappa x_n} |x - y|.
\]
In particular, if \( x \in B' \times \mathbb{R} \) and \( q \in \mathbb{R}^n \) then
\[
|dT^*(x)| \simeq |dT(x)| \simeq e^{-\kappa x_n} |q| \quad \text{and} \quad |J_T(x)| \simeq e^{-\kappa x_n},
\]
where the comparison constants in \( \simeq \) depend on \( \kappa \), but are independent of \( x \) and \( q \).

4. Properties of \( \text{div} \mathcal{B}(\xi, \nabla \tilde{u}) \) and removing the Neumann data

In order to apply the theory of degenerate elliptic equations developed in Heinonen–Kilpeläinen–Martio [10], we need to first show that the mapping \( \mathcal{B} \) in (3.4) satisfies ellipticity assumptions similar to (2.2)–(2.5).

Theorem 4.1. The mapping \( \mathcal{B} : T(\overline{G}) \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), defined by (3.3), satisfies for all \( q \in \mathbb{R}^n \) and a.e. \( \xi \in T(\overline{G}) \) the following ellipticity and boundedness conditions
\[
\mathcal{B}(\xi, q) \cdot q \gtrsim \tilde{w}(\xi) |q|^p \quad \text{and} \quad |\mathcal{B}(\xi, q)| \lesssim \tilde{w}(\xi) |q|^{p-1},
\]
where \( \tilde{w}(\xi) = |\xi|^{p-n} \) is a weight function and the comparison constants in \( \gtrsim \) and \( \lesssim \) are independent of \( \xi \) and \( q \).
Proof. From (3.3), we have that for all \( q \in \mathbb{R}^n \) and a.e. \( \xi = Tx \in T(\mathcal{G}) \),
\[
\mathcal{B}(\xi, q) \cdot q = |J_T(x)|^{-1} dT(x) A(x, dT^*(x)q) \cdot q
= |J_T(x)|^{-1} A(x, dT^*(x)q) \cdot (dT^*(x)q).
\]
Now applying (2.2) together with Lemma 3.2, we get that
\[
\mathcal{B}(\xi, q) \cdot q \geq e^{\kappa x_n} |dT^*(x)q|^p \simeq e^{-\kappa(p-n)x_n} |q|^p = \tilde{w}(\xi)|q|^p,
\]
which completes the proof of the first part. For the second part, we have using (3.3), (2.3) and Lemma 3.2 that
\[
|\mathcal{B}(\xi, q)| = |J_T(x)|^{-1} |dT(x)A(x, dT^*(x)q)| \simeq |J_T(x)|^{-1} e^{-\kappa x_n} |A(x, dT^*(x)q)|
\lesssim e^{(n-1)\kappa x_n} |dT^*(x)q|^{p-1} \simeq e^{-\kappa(p-n)x_n} |q|^{p-1} = \tilde{w}(\xi)|q|^{p-1}.
\]

Theorem 4.2. The mapping \( \mathcal{B} : T(\mathcal{G}) \times \mathbb{R}^n \to \mathbb{R}^n \), defined by (3.3), satisfies for a.e. \( \xi \in T(\mathcal{G}) \) and all \( q_1, q_2 \in \mathbb{R}^n \) the monotonicity condition
\[
(\mathcal{B}(\xi, q_1) - \mathcal{B}(\xi, q_2)) \cdot (q_1 - q_2) > 0 \quad \text{when} \quad q_1 \neq q_2.
\]

Proof. Using (3.3), we have for all \( q_1, q_2 \in \mathbb{R}^n \),
\[
(\mathcal{B}(\xi, q_1) - \mathcal{B}(\xi, q_2)) \cdot (q_1 - q_2)
= |J_T(x)|^{-1}((dT(x)A(x, dT^*(x)q_1) - dT(x)A(x, dT^*(x)q_2)) \cdot (q_1 - q_2))
= |J_T(x)|^{-1}((A(x, dT^*(x)q_1) - A(x, dT^*(x)q_2)) \cdot (dT^*(x)q_1 - dT^*(x)q_2)).
\]
Since \( |J_T(x)|^{-1} > 0 \), we have by (2.5) that the last expression is always nonnegative. Moreover, it is zero if and only if \( dT^*(x)q_1 = dT^*(x)q_2 \), implying that \( q_1 = q_2 \) as \( dT^*(x) \) is invertible.

It is clear from Theorems 4.1 and 4.2, together with the homogeneous condition (2.4), that the assumptions (3.4)–(3.7) in Heinonen–Kilpeläinen–Martio [10] are satisfied for \( \mathcal{B} \) with the weight function \( \tilde{w}(\xi) = |\xi|^{p-n}, \xi \in \mathbb{R}^n \setminus \{0\} \). Moreover, \( \mathcal{B}(\xi, q) \) is measurable in \( \xi \) and continuous in \( q \).

The weight \( \tilde{w}(\xi) \) belongs to the Muckenhoupt \( A_p \) class and the associated measure \( d\mu(\xi) = \tilde{w}(\xi) \ d\xi \) is doubling and supports a \( p \)-Poincaré inequality on \( \mathbb{R}^n \). Such weights are suitable for the study of partial differential equations and Sobolev spaces, see [10, Chapters 15 and 20] for a detailed exposition. We follow [10, Chapters 1 and 3] giving the following definitions.

Definition 4.3. For an open set \( \Omega \subseteq \mathbb{R}^n \), the weighted Sobolev space \( H^{1,p}_0(\Omega, \tilde{w}) \) is the completion of \( C^\infty_0(\Omega) \) with respect to the norm
\[
\|u\|_{H^{1,p}(\Omega, \tilde{w})} := \left( \int_\Omega (|u(\xi)|^p + |\nabla u(\xi)|^p \tilde{w}(\xi)) \ d\xi \right)^{1/p}.
\]
Similarly, \( H^{1,p}(\Omega, \tilde{w}) \) is the completion of the set
\[
\{ \varphi \in C^\infty(\Omega) : \|\varphi\|_{H^{1,p}(\Omega, \tilde{w})} < \infty \}
\]
in the \( H^{1,p}(\Omega, \tilde{w}) \)-norm.

If the weight \( \tilde{w} \equiv 1 \), then the symbol \( \tilde{w} \) is dropped and in this case we have the usual Sobolev space \( W^{1,p}(\Omega) \).
Definition 4.4. A function \( u \in H^{1,p}_{\text{loc}}(\Omega, \tilde{w}) \) in an open set \( \Omega \subset \mathbb{R}^n \) is said to be a weak solution of the equation \( \text{div} \, B(\xi, \nabla u) = 0 \) if for all functions \( \varphi \in C^\infty_0(\Omega) \), the following integral identity holds

\[
\int_{\Omega} B(\xi, \nabla u) \cdot \nabla \varphi \, d\xi = 0. \tag{4.1}
\]

To use the tools developed in Heinonen–Kilpeläinen–Martio [10] for Dirichlet problems, the part of the boundary, \( T(\partial G \setminus F) \), where the zero conormal derivative is prescribed, will be removed. This will be done by a reflection in the hyperplane \( \{ \xi \in \mathbb{R}^n : \xi_n = 0 \} \).

By a reflection, we mean the mapping \( P : \mathbb{R}^n \to \mathbb{R}^n \) defined by

\[ P\xi = P(\xi', \xi_n) = (\xi', -\xi_n). \]

Let \( D \) be the open set consisting of \( T(G \setminus F) \) together with \( PT(G \setminus F) \) and \( T(\partial G \setminus F) \), i.e.

\[ D = B(0,1) \setminus \tilde{F}, \quad \text{where} \quad \tilde{F} = T(F) \cup PT(F) \cup \{0\}. \]

The point at infinity in \( G \) corresponds to the origin \( \xi = 0 \) in \( B(0,1) \). Note that \( \tilde{F} \) is closed and that the base \( B' \times \{0\} \) of \( G \) is mapped onto the upper unit half-sphere \( \{ \xi \in \partial B(0,1) : \xi_n > 0 \} \). By assumption, the base \( B' \times \{0\} \subset F \) and so the whole boundary \( \partial D \subset \tilde{F} \) carries the Dirichlet condition.

Extend \( B(\xi, q) \) from \( T(\tilde{G}) \) to the reflected half-ball \( PT(\tilde{G}) \) by

\[
B(\xi, q) = \begin{cases} 
PB(P\xi, Pq) & \text{if } \xi_n < 0, \\
0 & \text{if } \xi = 0.
\end{cases} \tag{4.2}
\]

The standard ellipticity and monotonicity assumptions for \( \mathcal{B} \) still hold after the extension from \( T(\tilde{G}) \) to \( PT(\tilde{G}) \).

With this reflection, we will now be able to eliminate the Neumann boundary data on \( T(\partial G \setminus F) \) so that only the Dirichlet data on \( \partial D \) remain. In Theorem 4.7, we will show that \( u \in W^{1,p}_{\text{loc}}(\tilde{G} \setminus F) \) is a weak solution of the equation \((1.1)\) in \( G \setminus F \) with zero conormal derivative on \( \partial G \setminus F \) if and only if the symmetric reflection of \( u \circ T^{-1} \) is a weak solution of \((3.4)\) in \( D \). We shall use the function spaces identified in Björn–Mwasa [4].

Lemma 4.5. ([4, Lemma 4.6]) Assume that \( u \in L^1_{\text{loc}}(U) \) with the distributional gradient \( \nabla u \in L^1_{\text{loc}}(U) \) for some open set \( U \subset B'(0,R) \times \mathbb{R} \) and let \( \tilde{u} = u \circ T^{-1} \). Then for any measurable set \( A \subset U \),

\[
\int_A |\nabla u|^p \, dx \simeq \int_{T(A)} |\nabla \tilde{u}|^p \tilde{w} \, d\xi,
\]

\[
\int_A |u|^p e^{-p\kappa x_n} \, dx \simeq \int_{T(A)} |\tilde{u}|^p \tilde{w} \, d\xi,
\]

with comparison constants depending on \( R \) but independent of \( A \) and \( u \).

Proposition 4.6. Let \( u \in W^{1,p}_{\text{loc}}(\tilde{G} \setminus F) \). Then the function

\[
\tilde{u}(\xi', \xi_n) = \begin{cases} 
(u \circ T^{-1})(\xi', \xi_n) & \text{if } \xi \in T(\tilde{G} \setminus F), \\
(u \circ T^{-1})(\xi', -\xi_n) & \text{if } \xi \in PT(G \setminus F),
\end{cases} \tag{4.3}
\]

belongs to \( H^{1,p}_{\text{loc}}(D, \tilde{w}) \). Conversely, if \( \tilde{v} \in H^{1,p}_{\text{loc}}(D, \tilde{w}) \), then \( \tilde{v} \circ T \in W^{1,p}_{\text{loc}}(\tilde{G} \setminus F) \).
Proof. By the definition of \( W^{1,p}_{\text{loc}}(G \setminus F) \), we can assume that \( u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n \setminus F) \).

Then \( u \in W^{1,p}(U) \) for every \( U \in \mathbb{R}^n \setminus F \). To show that \( \tilde{u} \in H^{1,p}_{\text{loc}}(D, \tilde{w}) \), let \( B \subset \subset \mathbb{R}^n \setminus F \) be a ball. Assume without loss of generality that \( B \) is centred in \( T(G \setminus F) \) and set \( U := T^{-1}(B) \). Choose a sequence \( u_j \in C^\infty(\mathbb{R}^n \setminus F) \) such that \( u_j \to u \) in \( W^{1,p}(U) \).

In particular, \( u_j \) are Lipschitz on \( U \). Define \( \bar{u}_j := u_j \circ T^{-1} \) restricted to \( B \cap T(G) \).

By Lemma 3.2, the functions \( \bar{u}_j \) are Lipschitz. Their extensions across the hyperplane \( \xi_n = 0 \) to the whole \( B \), by the reflection \( \bar{u}_j(\xi) = \bar{u}_j(P \xi) \), are still Lipschitz. Lemma 4.5 implies that \( \bar{u} \) can be approximated in the \( H^{1,p}(B, \tilde{w}) \)-norm by these Lipschitz extensions, that is

\[
\| \bar{u}_j - \bar{u} \|_{H^{1,p}(B, \tilde{w})} \lesssim \| u_j - u \|_{W^{1,p}(U)} \to 0 \quad \text{as } j \to \infty.
\]

Hence, \( \tilde{u} \in H^{1,p}(B, \tilde{w}) \). Since \( B \) was arbitrary, we have that \( \tilde{u} \in H^{1,p}_{\text{loc}}(D, \tilde{w}) \).

Conversely, let \( V \in \mathbb{R}^n \setminus F \). Then \( T(V) \in D \) and hence \( \tilde{v} \in H^{1,p}(T(V), \tilde{w}) \).

By Lemma 4.5 and the fact that \( e^{-p|x|^2} \simeq 1 \) on \( V \), we have that \( \tilde{v} \circ T \) belongs to \( W^{1,p}(V) \). Since \( V \) was arbitrary, we get that \( \tilde{v} \circ T \in W^{1,p}_{\text{loc}}(G \setminus F) \).

Theorem 4.7. Let \( u \in W^{1,p}_{\text{loc}}(G \setminus F) \) be a weak solution of the equation

\[
\text{div } A(x, \nabla u) = 0 \quad \text{in } G \setminus F
\]

with zero conormal derivative on \( \partial G \setminus F \), that is, the integral identity (2.6) holds for all \( \varphi \in C^\infty_0(\mathbb{R}^n \setminus F) \). Let \( \tilde{u} \) be as in (4.3). Then \( \tilde{u} \in H^{1,p}_{\text{loc}}(D, \tilde{w}) \) is a weak solution of the equation

\[
\text{div } B(\xi, \nabla \tilde{u}) = 0 \quad \text{in } D.
\]

Conversely, if \( \tilde{u} \in H^{1,p}_{\text{loc}}(D, \tilde{w}) \) is a weak solution of (4.5) such that \( \tilde{u} = \tilde{u} \circ P \), then \( \tilde{u} \in W^{1,p}_{\text{loc}}(G \setminus F) \), with \( \tilde{u} \) restricted to \( T(G \setminus F) \), is a weak solution of the equation (4.4) with zero conormal derivative on \( \partial G \setminus F \).

Proof. The fact that \( \tilde{u} \in H^{1,p}_{\text{loc}}(D, \tilde{w}) \) follows from Proposition 4.6. To prove the first implication, let \( \tilde{\varphi} \in C^\infty_0(D) \) be an arbitrary test function. Clearly, \( \tilde{\varphi} \circ T \mid G \setminus F \) is a restriction of a function from \( C^\infty_0(\mathbb{R}^n \setminus F) \). In Lemma 3.1, replace \( v, \tilde{v} \) and \( A \)

with \( \tilde{\varphi} \circ T, \tilde{\varphi} \) and \( G \setminus F \), respectively. Lemma 3.1 and the integral identity (2.6) then give

\[
\int_{T(G \setminus F)} B(\xi, \nabla \tilde{u}) \cdot \nabla \tilde{\varphi} \, d\xi = \int_{G \setminus F} A(x, \nabla u) \cdot \nabla (\tilde{\varphi} \circ T) \, dx = 0.
\]

Now the change of variables \( \xi = P\xi \), together with (4.2) and the fact that \( \tilde{u} = \tilde{u} \circ P \), yields

\[
\int_{TP(G \setminus F)} B(\xi, \nabla \tilde{u}) \cdot \nabla \tilde{\varphi} \, d\xi = \int_{T(G \setminus F)} B(\xi, \nabla \tilde{u}) \cdot \nabla (\tilde{\varphi} \circ P) \, d\xi,
\]

cf. [4, Lemma 6.1]. Since \( \tilde{\varphi} \circ P \in C^\infty_0(D) \), we see as in (4.6) that the last integral is zero. Adding the left-hand sides of (4.6) and (4.7) shows that \( \tilde{u} \) is a weak solution of (4.5).

Conversely, first we recall from [10, Lemma 3.11] that if \( \tilde{u} \) is a weak solution of (4.5), then the integral identity (4.1) holds for all test functions in \( H^1_0(D, \tilde{w}) \) with compact support in \( D \). Let \( \varphi \in C^\infty_0(\mathbb{R}^n \setminus F) \) be an arbitrary test function. Define

\[
\tilde{\varphi}(\xi', \xi_n) := \begin{cases} 
(\varphi \circ T^{-1})(\xi', \xi_n) & \text{if } \xi \in T(G), \\
(\varphi \circ T^{-1})(\xi', -\xi_n) & \text{if } \xi \in PT(G), \\
0 & \text{otherwise}.
\end{cases}
\]
Note that \( \tilde{\varphi} = \tilde{\varphi} \circ P \) and that \( \tilde{\varphi} \) has compact support in \( D \). Proposition 4.6 therefore implies that \( \tilde{\varphi} \in H^{1,0}_0(D, \tilde{w}) \). Since \( \bar{u} = \tilde{u} \circ P \), the integral identity (4.1), tested with \( \tilde{\varphi} \), gives
\[
\int_{T(G,F)} B(\xi, \nabla \tilde{u}) \cdot \nabla \tilde{\varphi} \, d\xi + \int_{\partial T(G,F)} B(\xi, \nabla \tilde{u}) \cdot \nabla \tilde{\varphi} \, d\xi = \int_D B(\xi, \nabla \bar{u}) \cdot \nabla \bar{\varphi} \, d\xi = 0.
\]

Observe that the integrals on the left-hand side are the same as in (4.7) with \( \tilde{u}, \tilde{\varphi} \) replaced by \( \bar{u}, \bar{\varphi} \) and are thus equal. It follows that
\[
\int_{T(G,F)} B(\xi, \nabla \bar{u}) \cdot \nabla \bar{\varphi} \, d\xi = 0.
\]

Replacing \( u, v, \tilde{\varphi} \) and \( A \) in Lemma 3.1 by \( \tilde{u} \circ T, \varphi, \tilde{\varphi} \) and \( G \setminus F \), respectively, we then get
\[
\int_{G \setminus F} A(x, \nabla (\tilde{u} \circ T)) \cdot \nabla \varphi \, dx = \int_{T(G,F)} B(\xi, \nabla \tilde{u}) \cdot \nabla \tilde{\varphi} \, d\xi = 0.
\]

Since \( \varphi \) was chosen arbitrarily, we conclude that \( \tilde{u} \circ T \) is a weak solution of (4.4) with zero conormal derivative, as in Definition 2.1. Lastly, Proposition 4.6 shows that \( \tilde{u} \circ T \in W^{1,p}_{\text{loc}}(\overline{G \setminus F}) \).

**Remark 4.8.** The weak solution \( \tilde{u} \) can be modified on a set of measure zero, so that it becomes Hölder continuous in \( D \), see Heinonen–Kilpeläinen–Martio [10, Theorems 3.70 and 6.6]. Hence, for the corresponding continuous representative of \( u \), the limit
\[
\lim_{G \setminus F \ni x \to x_0} u(x)
\]
exists and is finite for every \( x_0 \) on the Neumann boundary \( \partial G \setminus F \). Moreover, \( u \) is Hölder continuous at \( x_0 \), by Lemma 3.2.

### 5. Existence and uniqueness of the solutions

In this section, we shall prove the existence and uniqueness of weak solutions to the equation \( \text{div} A(x, \nabla u) = 0 \) in \( G \setminus F \) with zero conormal derivative on \( \partial G \setminus F \) and continuous Dirichlet data. We shall also show that the solution attains its continuous Dirichlet data except possibly on a set of Sobolev \( C_p \)-capacity zero. Recall that \( F \) is a closed subset of \( G \), which contains the base \( B^1 \times \{0\} \), and that the Dirichlet boundary data \( f \) are prescribed on \( F_0 := F \cap \partial(G \setminus F) \).

**Definition 5.1.** Let \( K \subset \mathbb{R}^n \) be a compact set. The Sobolev \((p,\tilde{w})\)-capacity of \( K \) is
\[
C_{p,\tilde{w}}(K) = \inf_v \int_{\mathbb{R}^n} (|v|^p + |\nabla v|^p) \tilde{w} \, dx,
\]
where the infimum is taken over all \( v \in C^\infty_0(\mathbb{R}^n) \) (or equivalently, all continuous \( v \in H^{1,p}_0(\mathbb{R}^n, \tilde{w}) \)) such that \( v \geq 1 \) on \( K \), see Heinonen–Kilpeläinen–Martio [10, Section 2.35 and Lemma 2.36].

The capacity \( C_{p,\tilde{w}} \) can be extended to general sets as a Choquet capacity, see [10, Chapter 2]. In particular, for all Borel sets \( E \subset \mathbb{R}^n \),
\[
C_{p,\tilde{w}}(E) = \sup\{C_{p,\tilde{w}}(K) : K \subset E \text{ compact}\}. \tag{5.1}
\]

We also say that a property holds \( C_{p,\tilde{w}}\)-quasieverywhere if the set where it fails has zero \( C_{p,\tilde{w}} \)-capacity. If \( \tilde{w} \equiv 1 \), then we have the usual Sobolev \( C_p \)-capacity. The following lemma shows that \( T \) preserves sets of zero capacity.
Lemma 5.2. Let $E \subset \overline{G}$. Then $C_p(E) = 0$ if and only if $C_{p,\tilde{w}}(T(E)) = 0$.

Proof. The fact that if $C_{p,\tilde{w}}(T(E)) = 0$ then $C_p(E) = 0$ follows from Lemma 6.6 in Björn–Mwasa [4].

Conversely, assume that $C_p(E) = 0$. Replacing $E$ by a Borel set $E_0 \supset E$ with zero capacity, we can assume that $E$ is a Borel set. Let $K \subset T(E)$ be compact. Then $T^{-1}(K) \subset E$ is compact and hence $C_p(T^{-1}(K)) = 0$. For $\varepsilon > 0$, choose $\varphi \in C_{0}\infty(f^n, \tilde{w})$ such that $\varphi \geq 1$ on $T^{-1}(K)$ and $\|\varphi\|_{W^{1,p}(f^n)} < \varepsilon$. Multiplying $\varphi$ by a suitable cut-off function, we can assume that $\varphi(x) = 0$ when $x_n \leq -1$. Define

$$\tilde{\varphi}((\xi', \xi_n)) := \begin{cases} (\varphi \circ T^{-1})(\xi', \xi_n) & \text{if } \xi \in T(\overline{B'} \times \mathbb{R}), \\ (\varphi \circ T^{-1})(\xi', -\xi_n) & \text{if } \xi \in PT(B' \times \mathbb{R}), \\ 0 & \text{if } \xi = 0. \end{cases}$$

Then $\tilde{\varphi}$ is Lipschitz continuous, by Lemma 3.2, belongs to $H^{1,p}(f^n, \tilde{w})$ and is such that $\tilde{\varphi} \geq 1$ on $K$. Using Lemma 4.5 and the fact that $\varphi(x) = 0$ when $x_n \leq -1$, we have

$$\|\tilde{\varphi}\|_{H^{1,p}(f^n, \tilde{w})} \simeq \|\tilde{\varphi}\|_{H^{1,p}(T(\overline{B'} \times \mathbb{R}), \tilde{w})} \lesssim \|\varphi\|_{W^{1,p}(f^n)} \lesssim \|\varphi\|_{W^{1,p}(f^n)} < \varepsilon,$$

with comparison constants independent of $\varphi$ and $\varepsilon$. Letting $\varepsilon \to 0$, gives that $C_{p,\tilde{w}}(K) = 0$ and hence (5.1) concludes the proof. \hfill \Box

The following existence and uniqueness theorem is the main result of this section. Note that there may also exist unbounded solutions.

Theorem 5.3. Assume that $F_0$ is unbounded and let $f \in C(F_0)$ be such that

$$f(\infty) := \lim_{F_0, D_x \to \infty} f(x) \text{ exists and is finite.}$$

Then there exists a unique bounded continuous weak solution $u \in W^{1,p}_{\text{loc}}(\overline{G} \setminus F)$ of the mixed problem for the equation $\text{div} A(x, \nabla u) = 0$ in $\overline{G} \setminus F$ with zero conormal derivative on $\partial G \setminus F$ and such that

$$\lim_{G \setminus F, D_x \to x_0} u(x) = f(x_0) \quad \text{for } C_p\text{-quasievery } x_0 \in F_0. \quad (5.2)$$

Remark 5.4. The proof below shows that the conclusion of Theorem 5.3 holds also when $F_0$ is bounded. Moreover, since the origin has $(p, \tilde{w})$-capacity zero, the solution $u$ is then independent of the value $f(\infty)$ assigned to $\xi = 0$.

Proof. Define

$$\tilde{f}(\xi) := \begin{cases} (f \circ T^{-1})(\xi) & \text{if } \xi \in T(F_0), \\ (f \circ (PT)^{-1})(\xi) & \text{if } \xi \in PT(F_0 \cap G), \\ f(\infty) & \text{if } \xi = 0. \end{cases} \quad (5.3)$$

Then $\tilde{f} \in C(\partial D)$. By Björn–Björn–Mwasa [2, Theorem 3.12], there exists a unique bounded continuous weak solution $\tilde{u} \in H^{1,p}_{\text{loc}}(D, \tilde{w})$ of the equation (4.5) such that

$$\lim_{D, D_x \to \xi_0} \tilde{u}(\xi) = \tilde{f}(\xi_0) \quad \text{for } C_{p,\tilde{w}}\text{-quasievery } \xi_0 \in \partial D, \quad (5.4)$$

i.e. for all $\xi_0 \in \partial D \setminus Z$ for some set $Z \subset \partial D$ with $C_{p,\tilde{w}}(Z) = 0$. Note that (4.2) holds in $D$ and $\tilde{f} = f \circ P$. So (5.4) gives

$$\lim_{D, D_x \to \xi_0} \tilde{u}(P\xi) = \lim_{D, D_{P_0} \to P_0} \tilde{u}(\xi) = \tilde{f}(P\xi_0) = \tilde{f}(\xi_0)$$
for all $\xi_0 \in \partial D \setminus P(Z)$. That is, $\bar{u} \circ P$ also satisfies (5.4) and $\bar{u} \circ P \in H^{1,p}_{\text{loc}}(D, \bar{w})$. Since $\varphi \circ P \in C^\infty_0(D)$ if and only if $\varphi \in C^\infty_0(D)$, the change of variables $\zeta = P\xi$ together with (4.2) shows that the integral identity (4.1) holds for $\bar{u} \circ P$ as well. Thus $\bar{u} \circ P$ is also a bounded continuous weak solution of (4.5) in $D$, satisfying (5.4).

By the uniqueness in [2, Theorem 3.12], we conclude that $\bar{u} = \bar{u} \circ P$.

Define $u := \bar{u} \circ T$, with $\bar{u}$ restricted to $T(\overline{G} \setminus F)$. Theorem 4.7 shows that $u$ is a continuous weak solution of $\text{div} \mathcal{A}(x, \nabla u) = 0$ in $G \setminus F$ with zero conormal derivative, as in Definition 2.1. Since $\bar{u}$ satisfies (5.4), it then follows that $u$ satisfies (5.2) for every $x_0 \in F_0 \setminus Z_0$, where $Z_0 := T^{-1}(Z \cap T(\overline{G}))$ with $C_p(Z_0) = 0$ by Lemma 5.2.

To prove the uniqueness, suppose that $v \in W^{1,p}_{\text{loc}}(\overline{G} \setminus F)$ is a bounded continuous weak solution of the equation $\text{div} \mathcal{A}(x, \nabla u) = 0$ satisfying (5.2) for all $x_0 \in F_0 \setminus Z_0$ with $C_p(Z_0) = 0$. Let $\bar{v}$ be as in (4.3) with $u$ replaced by $v$. Then by Theorem 4.7, $\bar{v}$ is a continuous weak solution of (4.5).

Since $v$ satisfies (5.2), it follows that $\tilde{v}$ satisfies (5.4) for each $\xi_0 \in \partial D \setminus Z'$, where $Z' := T(Z_0') \cup PT(Z_0') \cup \{0\}$. Now we have that $C_{p,w}(T(Z_0)) = 0$ by Lemma 5.2, and so $C_{p,w}(PT(Z_0)) = 0$, by reflection. The origin 0 has zero $(p, w)$-capacity by [4, Lemma 7.6], and it follows by subadditivity that $C_{p,w}(Z') = 0$.

By [2, Theorem 3.12], the solution of (4.5) satisfying (5.4) is unique, in other words $\tilde{v} = \bar{u}$ and so $v = u$.

The following definition is adopted from Björn–Mwasa [4].

**Definition 5.5.** The space $L^{1,p}_{\text{loc}}(G \setminus F)$ consists of all measurable functions $v$ on $G \setminus F$ such that the norm

$$
\|v\|_{L^{1,p}_{\text{loc}}(G \setminus F)} = \left( \int_{G \setminus F} \left( |v(x)|^p e^{-p|\nabla v(x)|} + |\nabla v(x)|^p \right) dx \right)^{1/p} < \infty,
$$

where $\nabla v = (\partial_1 v, \ldots, \partial_n v)$ is the distributional gradient of $v$. The space $L^{1,p}_{\text{loc}}(G \setminus F)$ is the completion of $C^\infty_0(\mathbb{R}^n \setminus F)$ in the above $L^{1,p}_{\text{loc}}(G \setminus F)$-norm.

Note that the space $L^{1,p}_{\text{loc}}(G \setminus F)$ is contained in $W^{1,p}_{\text{loc}}(\overline{G} \setminus F)$. The following result generalizes [4, Theorem 6.3] to elliptic divergence type equations.

**Theorem 5.6.** Let $f \in L^{1,p}_{\text{loc}}(G \setminus F)$. Then there exists a unique continuous weak solution $u \in L^{1,p}_{\text{loc}}(G \setminus F)$ of the equation $\text{div} \mathcal{A}(x, \nabla u) = 0$ in $G \setminus F$ with zero conormal derivative on $\partial G \setminus F$ and such that $u - f \in L^{1,p}_{\text{loc}}(G \setminus F)$.

**Proof.** Let $\tilde{f}$ be defined as in (4.3), with $u$ replaced by $f$. Then $\tilde{f} \in H^{1,p}(D, \bar{w})$, by [4, Proposition 5.3]. By [10, Theorems 3.17 and 3.70], there exists a unique continuous weak solution $\bar{u} \in H^{1,p}(D, \bar{w})$ of the degenerate equation (4.5) such that $\bar{u} - \tilde{f} \in H^1_0(D, \bar{w})$. Since $B(\xi, Pq) = P B(\xi, q)$ by (4.2), we infer from [4, Corollary 6.2] (with $A(\xi, q)$ in [4] replaced by $B(\xi, q)$) that $\bar{u} = \bar{u} \circ P$. By Theorem 4.7, $u := \bar{u} \circ T$ (with $\bar{u}$ restricted to $T(\overline{G} \setminus F)$) is a weak solution of $\text{div} \mathcal{A}(x, \nabla u) = 0$ in $G \setminus F$ with zero conormal derivative on $\partial G \setminus F$. Moreover, $u \in L^{1,p}_{\text{loc}}(G \setminus F)$ by [4, Proposition 5.3] and $u - f \in L^{1,p}_{\text{loc}}(G \setminus F)$ by [4, Proposition 5.5].

To prove the uniqueness, suppose that $\bar{v} \in L^{1,p}_{\text{loc}}(G \setminus F)$ is a continuous weak solution of the equation $\text{div} \mathcal{A}(x, \nabla u) = 0$ in $G \setminus F$ with zero conormal derivative on $\partial G \setminus F$ and $v - f \in L^{1,p}_{\text{loc}}(G \setminus F)$. Let $\tilde{v}$ be as in (4.3), with $u$ replaced by $v$. Then $\tilde{v}$ is also a bounded continuous weak solution of (4.5). Moreover, by [4, Proposition 5.5] shows that $\tilde{v} - \bar{v} \in H^1_0(D, \bar{w})$. From the uniqueness of solutions to (4.5) we thus get that $\tilde{v} = \bar{u}$, and so $v = u$. □
6. Boundary regularity at infinity

We saw in Remark 4.8 that weak solutions of the equation \( \text{div } A(x, \nabla u) = 0 \) with zero conormal derivative are continuous in \( G \setminus F \) and at the Neumann boundary \( \partial G \setminus F \). Moreover, if the Dirichlet boundary data \( f \) are continuous on \( F_0 \), then the solution is continuous at \( F_0 \), except possibly for a set of Sobolev \( C_p \)-capacity zero. We now study continuity at the point at infinity.

We follow Björn–Mwasa [4, Section 8] giving the following definition.

**Definition 6.1.** Assume that \( F \) is unbounded. We say that the point at \( \infty \) is regular for the mixed problem (2.1) for the equation \( \text{div } A(x, \nabla u) = 0 \) in \( G \setminus F \) with zero conormal derivative on \( \partial G \setminus F \) if for all Dirichlet boundary data \( f \in C(F_0) \) with a finite limit

\[
\lim_{F_0 \ni x \to \infty} f(x) =: f(\infty),
\]

the unique bounded continuous weak solution \( u \) of (2.1), provided by Theorem 5.3, satisfies

\[
\lim_{G \setminus F \ni x \to \infty} u(x) = f(\infty).
\]

Remark 5.4 shows that the point at infinity is always irregular when \( F_0 \) is bounded. Note that due to the conormal derivative condition, the regularity at \( \infty \) in Definition 6.1 differs from the usual notion of boundary regularity for the Dirichlet problem in unbounded domains, as in e.g. Heinonen–Kilpeläinen–Martio [10, Section 9.5]. The following is our first step in characterizing the regularity of the point at infinity for the mixed problem (2.1).

**Proposition 6.2.** The point at \( \infty \) is regular for the mixed problem (2.1) for the equation \( \text{div } A(x, \nabla u) = 0 \) in \( G \setminus F \) if and only if the origin \( 0 \in \partial D \) is regular with respect to the equation

\[
\text{div } B(\xi, \nabla \bar{u}(\xi)) = 0 \quad \text{in } D,
\]

where \( B \) is as in (3.3).

Before the proof of Proposition 6.2, we give the definition below, see [10, Section 9.5].

**Definition 6.3.** A point \( \xi_0 \in \partial D \) is regular for the equation (6.3) if

\[
\lim_{D \ni \xi \to \xi_0} \bar{u}(\xi) = \bar{f}(\xi_0) \quad \text{for all } \bar{f} \in C(\partial D),
\]

where \( \bar{u} \) is the Perron solution of (6.3) with the boundary data \( \bar{f} \), as in [10, Section 9.1].

Note that \( \bar{f} \), being continuous, is resolutive, i.e. the upper and the lower Perron solutions coincide and are equal to the Perron solution \( \bar{u} \), see [10, Theorem 9.25]. Moreover, by Björn–Björn–Mwasa [2, Theorem 3.12], the Perron solution \( \bar{u} \) is the only bounded continuous weak solution of (6.3) that attains the boundary values \( \bar{f} \) \( C_{p,\bar{w}} \)-quasieverywhere on \( \partial D \) in the sense of (5.4).

**Proof of Proposition 6.2.** Assume that \( 0 \in \partial D \) is a regular point with respect to (6.3). Let \( \bar{f} \in C(F_0) \) be such that the limit in (6.1) exists and is finite. Let \( u \) be the unique bounded continuous weak solution of \( \text{div } A(x, \nabla u) = 0 \) in \( G \setminus F \) with zero conormal derivative, provided for \( \bar{f} \) by Theorem 5.3. Define \( \bar{u} \) and \( \bar{f} \) as in (4.3) and (5.3), respectively. Then by Theorem 4.7, \( \bar{u} \) is a bounded continuous weak solution of \( \text{div } B(\xi, \nabla \bar{u}) = 0 \) in \( D \) and attains the boundary values \( \bar{f} \) \( C_{p,\bar{w}} \)-quasieverywhere on \( \partial D \). Note that \( \bar{f} \in C(\partial D) \). Thus, by [2, Theorem 3.12], \( \bar{u} \) is the Perron solution of (6.3) with the boundary data \( \bar{f} \).
Since $0 \in \partial D$ is regular, we have that
\[
\lim_{G \cap F \ni x \to \infty} u(x) = \lim_{D \ni \xi \to 0} \bar{u}(\xi) = \bar{\tilde{f}}(0) = f(\infty).
\]
Thus, $u$ satisfies (6.2) and so $\infty$ is regular for the mixed problem for the equation \( \text{div} \, A(x, \nabla u) = 0 \) in $G \setminus F$.

Conversely, assume that $\infty$ is regular for the mixed problem (2.1) and let $\bar{f} \in C(\partial D)$. The function $\bar{f}$ is not necessarily symmetric, so we consider
\[
\bar{f}_1 = \min\{\bar{f}, \bar{f} \circ P\} \quad \text{and} \quad \bar{f}_2 = \max\{\bar{f}, \bar{f} \circ P\},
\]
which are symmetric, i.e. $\bar{f}_j = \bar{f}_j \circ P$, $j = 1, 2$. Let $\bar{u}$ and $\bar{u}_j$ be the Perron solutions of (6.3) with boundary data $\bar{f}$ and $\bar{f}_j$, respectively. Define $u_j := \bar{u}_j \circ T$, $j = 1, 2$, with $\bar{u}_j$ restricted to $T(G \setminus F)$. Then by Theorem 4.7, $u_j$ are bounded continuous weak solutions of the mixed problem (2.1) satisfying (5.2) with boundary data $f_j := \bar{f}_j \circ T|_{F_0}$. Note that $f_j$ are continuous on $F_0$ and that the limits
\[
\lim_{F_0 \ni x \to \infty} f_j(x) = f_j(\infty) \quad \text{exist and are finite.}
\]
Since $\infty$ is regular for (2.1), the solutions $u_j$ satisfy
\[
\lim_{G \setminus F \ni x \to \infty} u_j(x) = f_j(\infty) = \bar{\tilde{f}}(0).
\]
It now follows that
\[
\lim_{D \ni \xi \to 0} \bar{u}_j(\xi) = \bar{\tilde{f}}(0).
\]
But $\bar{f}_1 \leq \bar{f} \leq \bar{f}_2$ and thus $\bar{u}_1 \leq \bar{u} \leq \bar{u}_2$ by the definition of Perron solutions. We conclude that
\[
\lim_{D \ni \xi \to 0} \bar{u}(\xi) = \bar{\tilde{f}}(0),
\]
and so $0 \in \partial D$ is regular for the equation (6.3). \( \square \)

Regular points with respect to $\text{div} \, B(\xi, \nabla \bar{u}(\xi)) = 0$ in $D$ are characterized by the Wiener criterium, see Heinonen–Kilpeläinen–Martio [10, Theorem 21.30(i)$\Leftrightarrow$(v)] and Mikkonen [28]. Note that the definitions for regularity of a point $\xi \in \partial D$ with respect to (6.3) in terms of both Sobolev and Perron solutions are equivalent, see [10, Theorem 9.20].

Recall that $D = B_1 \setminus \tilde{F}$, where $\tilde{F} = T(F) \cup PT(F) \cup \{0\}$. Note that the ball $B_r := \{\xi \in \mathbb{R}^n : |\xi| < r\}$ in $B_1$ corresponds to the truncated cylinder
\[
G_t := \{x \in \partial G : x_n > t\} = \bar{B}' \times (t, \infty) \quad \text{with} \quad t = -\frac{1}{\kappa} \log r \geq 0
\]
in $\partial G$ and that $G_t$ contains the lateral boundary but not its base $B' \times \{t\}$. With the above notation, we see that $G_{2t}$ and $G_{t-1}$ correspond to $B_{2r}$ and $B_{2r}$, respectively. We follow [4, Section 7] giving the following definition.

**Definition 6.4.** Let $K \subset G_{t-1}$ be a compact set, where $t \geq 1$. The (Neumann) variational $p$-capacity of $K$ with respect to $G_{t-1}$ is
\[
\text{cap}_{p, G_{t-1}}(K) = \inf_v \int_{G_{t-1}} |\nabla v|^p \, dx,
\]
where the infimum is taken over all functions $v \in C_0^{\infty}(\mathbb{R}^n)$ satisfying $v \geq 1$ on $K$ and $v = 0$ on $\partial G \setminus G_{t-1}$. 
Just as $C_{p,\bar{w}}$, the capacity $\text{cap}_{p,G_{t-1}}$ is also a Choquet capacity. This was proved in [4, Section 7], even though we only need $\text{cap}_{p,G_t-1}$ for compact sets here. In [4, Section 8], the Wiener criterion from [10, Section 6.16] is rewritten in terms of $\text{cap}_{p,G_{t-1}}$ and we thus get the following criterion for the boundary regularity at $\infty$ for the mixed boundary value problem (2.1).

**Theorem 6.5.** The point at $\infty$ is regular for the mixed boundary value problem (2.1) in $G \setminus F$ if and only if the following condition holds

$$\int_0^\infty \text{cap}_{p,G_{t-1}}(F \cap (\bar{G}_t \setminus G_{2t}))^{1/(p-1)} dt = \infty.$$  \hspace{1cm} (6.4)

The proof follows from [4, Section 8], with Proposition 6.2 playing the role of Lemma 8.2 in [4]. See Examples 8.7 and 8.8 in [4] for concrete sets satisfying or failing the Wiener condition (6.4).

## 7. General behaviour of the solutions at $\infty$

Our aim in this section is to show a Phragmén–Lindelöf type trichotomy for the solutions of the equation $\text{div} A(x, \nabla u) = 0$ in $G$ with zero conormal derivative $C_p$-quasieverywhere on $\partial G \cap G_t$. We start by showing that sets of Sobolev $C_p$-capacity zero are removable for the solutions. Recall that $F$ is a closed subset of $\bar{G}$.

For compact subsets of $G$, the following removability result is just [10, Theorem 7.36]. Using the transformation $T$, we can remove also parts of the lateral Dirichlet boundary and change them into the Neumann boundary. This will make it possible to study the behaviour of the solutions at $\infty$.

Recall that $G_t := \{ x \in \bar{G} : x_n > t \}$ is the truncated cylinder containing its lateral boundary but not its base $B' \times \{ t \}$.

**Lemma 7.1.** Assume that for some $t \geq 0$, the set $E := F \cap G_t$ satisfies $C_p(E) = 0$. Let $u \in W^{1,p}_0(\bar{G}\setminus F)$ be a continuous weak solution of $\text{div} A(x, \nabla u) = 0$ in $G \setminus F$ with zero conormal derivative on $\partial G \setminus F$. Assume that $u$ is bounded in the set $(G_t \setminus G_r) \setminus F$ for every $\tau > t$.

Then $u$ can be extended to $E$ as a continuous weak solution in $(G \setminus F) \cup E$ with zero conormal derivative on $\partial G \setminus (F \setminus G_t)$.

**Proof.** Set $\tilde{E} := T(E) \cup PT(E)$. By Lemma 5.2, the set $T(E)$ has Sobolev $(p, \tilde{w})$-capacity zero. By reflection, we have $C_{p,\tilde{w}}(\tilde{E}) = 0$. As in (4.3), define

$$\tilde{u}(\xi', \xi_n) = \begin{cases} (u \circ T^{-1})(\xi', \xi_n) & \text{if } \xi \in T(\bar{G}\setminus F), \\ (u \circ T^{-1})(\xi', -\xi_n) & \text{if } \xi \in PT(\bar{G}\setminus F). \end{cases} \hspace{1cm} (7.1)$$

Then by Theorem 4.7, $\tilde{u}$ is a continuous weak solution of $\text{div} B(\xi, \nabla \tilde{u}) = 0$ in $D$, which is bounded in $D \cap (B_r \setminus B_\rho)$ for every $\rho > 0$, where $r = e^{-\kappa t}$. Note that $\tilde{E} \setminus B_\rho$ is relatively closed in $B_r \setminus B_\rho$.

Since $C_{p,\tilde{w}}(\tilde{E}) = 0$, we have by Heinonen–Kilpeläinen–Martio [10, Theorem 7.36] that $\tilde{u}$ can be extended to $\tilde{E}$ so that it is a continuous weak solution in $B_r \setminus B_\rho$ for every $\rho > 0$, and thus also in $D \cup (B_r \setminus \{0\})$. The desired extension of $u$ is then given by $\tilde{u} \circ T$.

Removability and behaviour of the solutions at $\infty$ are addressed in the rest of the section. The following two lemmas provide suitable lower and upper bounds for the solutions.
Lemma 7.2. Assume that for some $t \geq 0$, the set $E := F \cap G_t$ is empty. Let $u \in W^{1,p}_{\text{loc}}(\overline{G}\setminus F)$ be a weak continuous solution of $\text{div} A(x, \nabla u) = 0$ in $G \setminus F$ with zero conormal derivative on $\partial G \setminus F$. Then $u$ is bounded in the set $G_t \setminus G_\tau$ for every $\tau > t' > t$.

If, moreover, $u(x) \leq 0$ when $x_n = t$, then either $u \leq 0$ in $G_t$ or there exist $A > 0$ and $\tau_0 > t$ such that for all $\tau \geq \tau_0$,

$$\max_{x_n = \tau} u(x) \geq A(\tau - t).$$

Proof. Define $\tilde{u}$ as in (7.1). Then by Theorem 4.7, $\tilde{u} \in H^{1,p}_{\text{loc}}(D, \tilde{w})$ is a continuous weak solution of $\text{div} B(x, \nabla \tilde{u}(\xi)) = 0$ in $D$. In particular, since $E$ is empty, $\tilde{u} \in H^{1,p}_{\text{loc}}(B_r \setminus \{0\}, \tilde{w})$ is a continuous weak solution in $B_r \setminus \{0\}$, where $r = e^{-\kappa t}$. This immediately implies the boundedness of $u$ in $G_t \setminus G_\tau$ for every $\tau > t' > t$.

Next, if $u(x) \leq 0$ when $x_n = t$, then $\tilde{u}(\xi_0) \leq 0$ for all $\xi_0 \in \partial B_r$. Hence, by [10, Theorem 7.40], we have that either $\tilde{u} \leq 0$ in $B_r \setminus \{0\}$ or

$$\liminf_{\rho \to 0} (\text{cap}_{p,\tilde{w}}(B_\rho, B_r))^{1/(p-1)} \max_{\partial B_\rho} \tilde{u} > 0,$$

where $\text{cap}_{p,\tilde{w}}$ is the variational capacity associated with the weight $\tilde{w}$, as in [10, Chapter 2]. Inequality (7.2) reveals that there exist constants $a > 0$ and $\rho_0 > 0$ such that for all $0 < \rho < \rho_0$, we have

$$\max_{\partial B_\rho} \tilde{u} \geq a (\text{cap}_{p,\tilde{w}}(B_\rho, B_r))^{1/(1-p)}.$$  \hspace{1cm} (7.3)

By Björn–Mwasa [4, Lemma 7.6], we have for all $\rho < r$,

$$(\text{cap}_{p,\tilde{w}}(B_\rho, B_r))^{1/(1-p)} \gtrsim \log \frac{r}{\rho} = \kappa (\tau - t),$$

where $\tau = -\frac{1}{\kappa} \log \rho > t$

and the constant in $\gtrsim$ depends only on $n$ and $p$. Substituting in (7.3) and using the definition (7.1) of $\tilde{u}$, concludes the proof.

Lemma 7.3. Assume that for some $t \geq 0$, the set $E := F \cap G_t$ is empty. Let $u \in W^{1,p}_{\text{loc}}(\overline{G}\setminus F)$ be a weak continuous solution of $\text{div} A(x, \nabla u) = 0$ in $G \setminus F$ with zero conormal derivative on $\partial G \setminus F$. Assume that $u \geq 0$ in $G_t$. Then there exists $A_0 \geq 0$ such that for all $\tau \geq t + \frac{1}{\kappa} \log 2$,

$$\max_{x_n = \tau} u(x) \leq A_0(\tau - t).$$

Proof. Define $\tilde{u}$ as in (7.1). As in the proof of Lemma 7.2, $\tilde{u}$ is a continuous weak solution of $\text{div} B(\xi, \nabla \tilde{u}(\xi)) = 0$ in $B_r \setminus \{0\}$, where $r = e^{-\kappa t}$. For $0 < \rho < \frac{1}{\kappa} r$, let $\tilde{v}$ be the potential of $\overline{B_\rho}$ in $B_r$, i.e. the continuous weak solution of $\text{div} B(\xi, \nabla \tilde{v}) = 0$ in $B_r \setminus \overline{B_\rho}$ with boundary values 1 on $\overline{B_\rho}$ and 0 on $\partial B_r$. Then by Heinonen–Kilpeläinen–Martio [10, Lemma 6.21], there exists $c > 0$ such that

$$\tilde{v}(\xi) \geq c \left( \frac{\text{cap}_{p,\tilde{w}}(B_\rho, B_r)}{\text{cap}_{p,\tilde{w}}(B_{r/2}, B_r)} \right)^{1/(p-1)}$$

for all $\xi \in \overline{B_r/2}$.

From [4, Lemma 7.6] and [10, Theorem 2.18], we thus have for all $\xi \in \overline{B_r/2}$ that

$$\tilde{v}(\xi) \gtrsim \text{cap}_{p,\tilde{w}}(B_\rho, B_r)^{1/(p-1)} \gtrsim \left( \log \frac{r}{\rho} \right)^{-1} = \frac{1}{\kappa (\tau - t)},$$  \hspace{1cm} (7.4)

where $\tau = -\frac{1}{\kappa} \log \rho > t$ and the constants in $\gtrsim$ depend only on $c$, $n$ and $p$. 
Let $m_p$ be the minimum of $\tilde{u}$ on the sphere $\partial B_p$. Using the boundary values of $\tilde{v}$ and $\tilde{u}$ on both $\partial B_p$ and $\partial B_r$, it follows from the comparison principle [10, Lemma 3.18] and from (7.4) that

$$\tilde{u} \geq m_p \tilde{v} \geq \frac{m_p}{\tau - t} \quad \text{in } B_r/2 \setminus B_p.$$ 

The Harnack inequality for $\tilde{u}$ on the sphere $\partial B_p$ then gives for any fixed $\xi_0 \in \partial B_{r/2}$ that

$$\max_{x_n = \tau} u(x) = \max_{\partial B_r} \tilde{u}(x) \leq m_p \tilde{u}(\xi_0)(\tau - t),$$

where the constants in $\leq$ depend only on $A$, $\kappa$, $n$ and $p$. \hfill \square

We are now ready to conclude the paper with the following trichotomy for the solutions of the mixed problem at $\infty$, when $F$ is negligible near $\infty$.

**Theorem 7.4.** Assume that for some $t \geq 0$, the set $E := F \cap G_\tau$ is such that $C_p(E) = 0$. Let $u \in W^{1,p}_\text{loc}(G \setminus F)$ be a weak continuous solution of $\text{div} \, A(x, \nabla u) = 0$ in $G \setminus F$ with zero conormal derivative on $\partial G \setminus F$. Assume that $u$ is bounded in the set $(G_\tau \setminus G_0) \setminus F$ for every $\tau > t$.

Then there exist constants $\tau_0 > t$, $M$, $M_0$ and $A_0 \geq A \geq 0$, such that exactly one of the following holds.

(i) The solution $u$ is bounded in $G_{\tau_0}$ and the limit

$$\lim_{G \setminus F \ni x \to \infty} u(x) =: u(\infty)$$

exists and is finite. Moreover, for some $\alpha \in (0, 1]$ and all $x \in G_{\tau_0}$,

$$|u(x) - u(\infty)| \leq e^{-\kappa \alpha x_n}. \tag{7.5}$$

(ii) The solution $u$ tends roughly linearly either to $\infty$ or to $-\infty$, i.e. either

$$M + A\tau \leq u(x', \tau) \leq M_0 + A_0\tau \quad \text{for all } x' \in B' \text{ and } \tau \geq \tau_0, \tag{7.6}$$

or (7.6) holds for $-u$ in place of $u$.

(iii) The solution changes sign and approaches both $\infty$ and $-\infty$. More precisely,

$$\max_{x_n = \tau} u(x) \geq M + A\tau \quad \text{and} \quad \min_{x_n = \tau} u(x) \leq M_0 - A\tau \quad \text{for all } \tau \geq \tau_0.$$

**Proof.** By Lemma 7.1, we can assume that $E$ is empty. Define $\tilde{u}$ as in (7.1). Then by Theorem 4.7, the function $\tilde{u}$ is a continuous weak solution of $\text{div} \, E(\xi, \nabla \tilde{u}) = 0$ in $B_r \setminus \{0\}$, where $r = e^{-\kappa t}$. Let $r' \leq \frac{1}{4} r$. By the continuity of the solution $\tilde{u}$ in $B_r \setminus \{0\}$, there exist constants $m'$, $M'$ such that $m' \leq \tilde{u}(\xi) \leq M'$ for all $\xi \in \partial B_{r'}$. Hence, it follows from the definition of $\tilde{u}$ that $m' \leq u(x) \leq M'$ when $x_n = t' := -\frac{1}{\kappa} \log r' > t$.

Applying the second part of Lemma 7.2 to $G_{\tau}$ with $u$ replaced by $u - M'$ and $m' - u$, respectively, shows that there exist $A' > 0$ and $\tau_0 > t'$ such that the following two statements hold:

(a) $u \leq M'$ in $G_{t'}$ or $\max_{x_n = \tau} u(x) \geq M' + A'(\tau - t')$ for all $\tau \geq \tau_0$,

(b) $u \geq m'$ in $G_{t'}$ or $\min_{x_n = \tau} u(x) \leq m' - A'(\tau - t')$ for all $\tau \geq \tau_0$. 

Combining the first two alternatives in (a) and (b) gives (i), while the second alternatives give (iii). Since $C_{p,w}([0]) = 0$, we can in the bounded case (i) use [10, Theorem 7.36] and extend $\tilde{u}$ to 0 so that it becomes a continuous weak solution of \( \text{div} B(\xi, \nabla \tilde{u}) = 0 \) in $B_r$. By [10, Theorem 6.6], $\tilde{u}$ is H"older continuous at the origin, which shows that (7.5) holds.

The remaining alternatives will lead to case (ii). If $u \geq m'$ in $G_{t'}$ and
\[
\max_{x_n = \tau} u(x) \geq M' + A'(\tau - t') \quad \text{for all } \tau \geq \tau_0,
\]
then using the Harnack inequality for $\tilde{u} - m'$ on the sphere $\partial B_\rho$ with $\rho = e^{-\kappa \tau}$, we see that for some $C > 0$ independent of $\tau \geq \tau_0$,
\[
\min_{x_n = \tau} (u - m') \geq C \max_{x_n = \tau} (u - m') \geq C(M' - m' + A'(\tau - t')).
\]
This proves the lower bound in (7.6), while the upper bound follows from Lemma 7.3 applied to $u - m'$. The second case of (ii), i.e. (7.6) for $u$, follows in a similar way by combining $u \leq M'$ with the second alternative in (b).

We finish by giving a concrete example illustrating the cases in Theorem 7.4.

**Example 7.5.** Let $G = (-1,1) \times (0,\infty) \subset R^2$ and $F = [-1,1] \times \{0\}$. The linear function $u(x_1, x_2) = ax_2 + b$, where $a, b \in R$, satisfies $\Delta u = 0$ in $G$ and $\partial u/\partial \nu = 0$ on the lateral boundary $\partial G \setminus F$. The cases (i) and (ii) follow if $a = 0$, $a > 0$ and $a < 0$, respectively. Also, consider the function
\[
u(x_1, x_2) = e^{2x_2} \sin \frac{\pi}{2} x_1,
\]
which is easily verified to satisfy $\Delta u = 0$ in $G$. Then (iii) is achieved when $\sin \frac{\pi}{2} x_1$ attains its maximum and minimum at $[-1,1]$, respectively.

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