A METHOD FOR THE RESOLUTION OF THE JACOBI EQUATION $Y'' + RY = 0$ ON THE MANIFOLD $Sp(2)/SU(2)$.

A. M. Naveira and A. Tarrío *

Abstract

In this paper a method for the resolution of the differential equation of the Jacobi vector fields in the manifold $V_1 = Sp(2)/SU(2)$ is exposed. These results are applied to determine areas and volumes of geodesic spheres and balls.

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Introduction

The resolution of the Jacobi equation on a Riemannian manifold can be quite a difficult task. In the Euclidean space the solution is trivial. For the symmetric spaces, the problem is reduced to a system of differential equations with constant coefficients. In the specialized bibliography, particularly in [11], the explicit solutions of these systems are found as well as their application to the determination of areas and volumes. In [7, 8] a partial solution of this problem for the manifolds $V_1 = Sp(2)/SU(2)$ and $V_2 = SU(5)/Sp(2) \times S^1$ is obtained by I. Chavel. It is well known that these manifolds are nonsymmetric normal homogeneous spaces of rank 1 [2, p.237]. The manifold $V_1$ appears in [2] and in the book of A. L. Besse [4, p.203] as an exceptional naturally reductive homogeneous space. For naturally reductive compact homogeneous spaces, Ziller [20] solves the Jacobi equation working with the canonical connection, which is natural for the nonsymmetric naturally reductive homogeneous spaces; but the solution can be considered of qualitative type, in the sense that it does not allow to obtain in an easy way the explicit solutions of the Jacobi fields for any particular example and for an arbitrary direction of the geodesic. The method used by Chavel, which allows him to solve the Jacobi equation in some particular directions, is based on the use of the canonical connection. Nevertheless, his method does not seem to apply in a simple way to the resolution of the Jacobi equation along a unit geodesic of arbitrary direction. In [7, 8]

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the same author shows the existence of anisotropic Jacobi fields; that is, they do not come from geodesic variations in the isotropic subgroup. Also, the Jacobi equations on a Riemannian manifold appear in a natural way in the theory of Fanning curves [1].

In this paper, always working with the Levi-Civita connection and using an interesting geometric result of Tsukada [18], the Jacobi equation along a unit geodesic of arbitrary direction is solved. Also, the solutions are applied to obtain the area of the geodesic sphere and the volume of the geodesic ball of radius \( t \) in the manifold \( V_1 = Sp(2)/SU(2) \). In §1, for an arbitrary Riemannian manifold, using the induction method, a recurrent formula for the \( i \)-th covariant derivative of the Jacobi operator \( R_t = R(\cdot, \gamma')\gamma'(t) \) along the geodesic \( \gamma \) is given. In §2, using the result of the previous section, the expression of the covariant derivative of the curvature tensor at the point \( \gamma(0) \) is obtained for an arbitrary naturally reductive homogeneous spaces \( M = G/H \), in terms of the brackets of the Lie algebra of \( G \). In order to obtain this result the induction method is used again.

In the following sections, the previous results are applied to the normal homogeneous space \( V_1 \). So, in §3, always working with a unit geodesic \( \gamma \) of arbitrary direction, the values at \( \gamma(0) \) of the Jacobi tensor \( R_0 \), and its covariant derivatives \( R_0^1 \) and \( R_0^2 \) are determined. In Lemma 3.1 it is proved that, for a unit geodesic \( \gamma \),

\[
R_0^3 = -||\gamma'||^2 R_0^1 = -R_0^1
\]

and

\[
R_0^4 = -||\gamma'||^2 R_0^2 = -R_0^2.
\]

This section ends by proving that

\[
R_0^{2n} = (-1)^{n-1} R_0^2
\]

and

\[
R_0^{2n+1} = (-1)^n R_0^1.
\]

Using the Taylor development, at the point \( \gamma(0) \), of the Jacobi operator, it is possible to obtain quite a simple expression of the Jacobi operator \( R_t \) along the geodesic, as well as that of its derivatives. In fact, the explicit expression for \( R_t \) is

\[
R_t = R_0 + R_0^2 + R_0^1 \sin t - R_0^2 \cos t.
\]

It seems interesting to remark that while D’Atri and Nickerson [9, 10] impose conditions on the derivatives of the Jacobi operator, known as Ledger’s conditions of odd order, in our case, conditions are imposed by the geometric properties of the manifold.

In §4 the Jacobi equation with predetermined initial values is solved and the formal expressions of the area and the volume of the geodesic sphere and the ball of radius \( t \) are obtained. In a forthcoming paper the problem of determining the areas of tubular hypersurfaces and the volumes of tubes around compatible submanifolds will be approached. Given its generality, we hope that this method could also be applied to solve the Jacobi equation in several other examples of naturally reductive homogeneous spaces.
1 A formula for the covariant derivative of the Jacobi operator in a Riemannian manifold.

Let $M$ be an $n$-dimensional, connected, real analytical Riemannian manifold, $g = < , >$ its Riemannian metric, $m \in M$, $v \in T_m M$ a unit tangent vector and $\gamma : J \rightarrow M$ a geodesic in $M$ defined on some open interval $J$ of $\mathbb{R}$ with $0 \in J$, $m = \gamma(0)$. For a geodesic $\gamma(t)$ in $M$ the associated Jacobi operator $R_t$ is the self-adjoint tensor field along $\gamma$ defined by

$$R_t := R(\cdot, \gamma')\gamma'(t)$$

for the curvature tensor we follow the notations of [13]. The covariant derivative $R_t^{(i)}$ of the Jacobi operator $R_t$ along $\gamma$ is the self-adjoint tensor field defined by

$$R_t^{(i)} := (\nabla_{\gamma'} \cdot \cdots \nabla_{\gamma'} R)(\cdot, \gamma')\gamma'(t),$$

where $\nabla$ is the Levi-Civita connection associated to the metric. Its value at $\gamma(0)$ will be denoted by

$$R_0^{(i)} := (\nabla_{\gamma'} \cdot \cdots \nabla_{\gamma'} R)(\cdot, \gamma')\gamma'(0)$$

and we denote $R_0^{(0)} = R_t$.

First, we prove two combinatorial lemmas for later use.

**Lemma 1.1** For $i \leq 2k$ we have:

a)

$$\binom{2k+2}{i} = \binom{2k}{i} + 2 \binom{2k}{i-1} + \binom{2k}{i-2};$$

b)

$$\binom{2k+2}{2k+1} = \binom{2k}{2k-1} + 2;$$

c)

$$\binom{2k+2}{2k+2} = \binom{2k}{2k} = 1.$$

The proof is a trivial consequence of some properties of the combinatorial numbers.

**Lemma 1.2**

$$\sum_{j=1}^{i} (-1)^j \binom{k+1}{j} \binom{k-j+1}{i-j} = \binom{k+1}{i}.$$
The proof follows at once by using the formula
\[
\begin{pmatrix}
-x \\
n
\end{pmatrix} = (-1)^n \begin{pmatrix}
x + n - 1 \\
n
\end{pmatrix}
\]
where \(x \in \mathbb{Z}\), and also the Vandermonde’s identity
\[
\begin{pmatrix}
x + y \\
n
\end{pmatrix} = \sum_{j=0}^{n} \begin{pmatrix}
x \\
j
\end{pmatrix} \begin{pmatrix}
y \\
n - j
\end{pmatrix}
\]
with \(x, y \in \mathbb{Z}\).

**Theorem 1.3** For \(n \geq 1\) we have
\[
\nabla_{\gamma'}^{(n)} \cdots \nabla_{\gamma'} R(X, \gamma')\gamma' = \sum_{i=0}^{n} \begin{pmatrix}
n \\
i
\end{pmatrix} R_{i}^{n-i}(\nabla_{\gamma'} \cdots \nabla_{\gamma'} X).
\]

**Proof.** We prove this by induction. For \(n = 1\), we have
\[
\nabla_{\gamma'} R(X, \gamma')\gamma' = \nabla_{\gamma'} R(R_{i}(X, \gamma')\gamma' + R(\nabla_{\gamma'} X, \gamma')\gamma'
\]
that is
\[
\nabla_{\gamma'} R(X, \gamma')\gamma' = R_{i}(X) + R_{0}^{0}(\nabla_{\gamma'} X)
\]
and so the result is true for \(n = 1\). Next, suppose that Theorem 1.3 holds for \(n = k\). Then we have
\[
\nabla_{\gamma'}^{(k)} \cdots \nabla_{\gamma'} R(X, \gamma')\gamma' = \sum_{i=0}^{k} \begin{pmatrix}
k \\
i
\end{pmatrix} R_{i}^{k-i}(\nabla_{\gamma'} \cdots \nabla_{\gamma'} X).
\]

Taking the covariant derivative, we obtain
\[
\nabla_{\gamma'}(\nabla_{\gamma'}^{(k)} \cdots \nabla_{\gamma'} R(X, \gamma')\gamma') = \nabla_{\gamma'}^{(k+1)} \cdots \nabla_{\gamma'} R(X, \gamma')\gamma'
\]
\[
= \nabla_{\gamma'}(\sum_{i=0}^{k} \begin{pmatrix}
k \\
i
\end{pmatrix} R_{i}^{k-i}(\nabla_{\gamma'} \cdots \nabla_{\gamma'} X)).
\]

By applying \(1\) to each term, it is possible to write
\[
\nabla_{\gamma'}^{(k+1)} \cdots \nabla_{\gamma'} R(X, \gamma')\gamma'
\]
\[
= \sum_{i=0}^{k} \begin{pmatrix}
k + 1 \\
i
\end{pmatrix} [R_{i}^{k+1-i}(\nabla_{\gamma'} \cdots \nabla_{\gamma'} X) + R_{i}^{k-i}(\nabla_{\gamma'} \cdots \nabla_{\gamma'} X)]
\]
\[
= \begin{pmatrix}
k + 1 \\
0
\end{pmatrix} R_{i}^{k+1}(X) + \sum_{i=0}^{k-1} \begin{pmatrix}
k \\
i
\end{pmatrix} + \begin{pmatrix}
k \\
i + 1
\end{pmatrix} R_{i}^{k-i}(\nabla_{\gamma'} \cdots \nabla_{\gamma'} X)
\]
\[
+ \begin{pmatrix}
k \\
k
\end{pmatrix} R_{0}^{0}(\nabla_{\gamma'} \cdots \nabla_{\gamma'} X).
\]

4
Now, by applying basic properties of combinatorial numbers we have
\[
\nabla_{\gamma'}^{k+1} \cdot \ldots \nabla_{\gamma'} R(X, \gamma') \gamma' = \sum_{i=0}^{k+1} \binom{k+1}{i} R_{i}^{k+1-i} (\nabla_{\gamma'}^{i} \cdot \ldots \nabla_{\gamma'} X)
\]
and the result follows.

**Corollary 1.4** We have
\[
R_{t}^{n}(X) = \nabla_{\gamma'}^{n} \cdot \ldots \nabla_{\gamma'} R(X, \gamma') \gamma' - \sum_{i=1}^{n} \binom{n}{i} R_{i}^{n-i} (\nabla_{\gamma'}^{i} \cdot \ldots \nabla_{\gamma'} X).
\]

2 An algebraic expression for the covariant derivative of the Jacobi operator on a naturally reductive homogeneous space

Let $G$ be a Lie group, $H$ a closed subgroup, $G/H$ the space of left cosets of $H$, $\pi : G \to G/H$ the natural projection. For $r \in G$ we denote by $\tau$ the induced action of $G$ on $G/H$ given by $\tau(r)(sH) = rsH$, $r, s \in G$. The Lie algebras of $G$ and $H$ will be denoted by $\mathfrak{g}$ and $\mathfrak{h}$, respectively and $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ is a vector space which we identify with the tangent space to $G/H$ at $o = \pi(H)$.

An affine connection on $G/H$ is said to be invariant if it is invariant under $\tau(r)$ for all $r \in G$.

It is well known that it is possible to define in a natural way on $\mathfrak{g}$ an $\text{Ad}$-invariant metric by
\[
\langle u, v \rangle = \text{Tr}(uv^{t}), \quad u, v \in \mathfrak{g}.
\]
Let $\nabla$ be the associated Levi-Civita connection. It is well-known [13, Ch.X, p.186] that there exists an invariant affine connection $D$ on $G/H$ (the canonical connection) whose torsion $T$ and curvature $B$ tensors are also invariant. In the following we always work with $\nabla$.

**Definition 2.1** [8, 13, p.202] $M = G/H$ is said to be a

(a) Reductive homogeneous space if the Lie algebra $\mathfrak{g}$ admits a vector space decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ such that $[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}$. In this case $\mathfrak{m}$ is identified with the tangent space at the origin $o = \pi(H)$.

(b) Riemannian homogeneous space if $G/H$ is a Riemannian manifold such that the metric is preserved by $\tau(r)$ for all $r \in G$.

(c) Naturally reductive Riemannian homogeneous space if $G/H$, with a $H$-invariant Riemannian metric, admits an $\text{Ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ satisfying the condition
\[
\langle [u, v]_{\mathfrak{m}}, w \rangle + \langle v, [u, w]_{\mathfrak{m}} \rangle = 0
\]
for $u, v, w \in \mathfrak{m}$.

(d) Normal Riemannian homogeneous space if the metric on $G/H$ is obtained as follows: there exists a positive definite inner product $\langle, \rangle$ on $\mathfrak{g}$ satisfying
\[
\langle [u, v], w \rangle = \langle u, [v, w] \rangle
\]
for all \( u, v, w \in \mathfrak{g} \). Let \( \mathfrak{m} = \mathfrak{g}/\mathfrak{h} \) be the orthogonal complement of \( \mathfrak{h} \). Then the decomposition \((\mathfrak{g}, \mathfrak{h})\) is reductive, and the restriction of the inner product to \( \mathfrak{m} \) induces a Riemannian metric on \( G/H \), referred to as normal, by the action of \( G \) on \( G/H \).

From now on we will assume that \( G/H \) is a naturally reductive space. If we define \( \Lambda : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m} \) by
\[
\Lambda(u)v = (1/2)[u, v]_{\mathfrak{m}}
\]
for \( u, v \in \mathfrak{m} \), we can identify \( \nabla \) and \( \Lambda \). Evidently, \( \Lambda(u) \) is a skew-symmetric linear endomorphism of \((\mathfrak{m}, <, >)\). Therefore \( e^{t\Lambda(u)} \) is a linear isometry of \((\mathfrak{m}, <, >)\). Since the Riemannian connection is a natural torsion free connection on \( G/H \), we have [16, 13, Vol. II, Ch.X]:

**Proposition 2.2** The following properties hold:

(i) For each \( v \in \mathfrak{m} \), the curve \( \gamma(t) = \tau(\exp tv)(o) \) is a geodesic with \( \gamma(0) = o \), \( \gamma'(0) = v \).

(ii) The parallel translation along \( \gamma \) is given as follows:
\[
\tau(\exp tv) \ast e^{-t\Lambda(v)} : T_o M \rightarrow T_{\gamma(t)} M.
\]

(iii) The \((1,3)\)-tensor \( R_t \) on \( \mathfrak{m} \) obtained by the parallel translation of the Jacobi operator along \( \gamma \) is given as follows:
\[
R_t = e^{t\Lambda(v)} R_0.
\]

Above, \( R_0 \) denotes the Jacobi operator at the origin \( o \) and \( e^{t\Lambda(v)} \) denotes the action of \( e^{t\Lambda(v)} \) on the space \( R(\mathfrak{m}) \) of curvature tensors on \( \mathfrak{m} \).

**Proposition 2.3** [8, 13, Vol. II, p. 202] Let \( \gamma(t) \) be a geodesic with \( \gamma(0) = o \), for \( v = \gamma'(0) \in \mathfrak{m} \). If \( X \) is a differentiable vector field along \( \gamma \), then
\[
R_0(X) = -[[[X, v]_\mathfrak{h}, v] - (1/4)[[X, v]_\mathfrak{m}, v]_\mathfrak{m}.
\]

**Proposition 2.4** Under the same hypothesis that in Proposition 2.3, we have, for \( n > 0 \),
\[
(-1)^{n-2} n R_0^n(X) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} [[[X, v]_\mathfrak{m}, \ldots, v]_{\mathfrak{h}}^{i+1}, \ldots, v]_\mathfrak{m} \tag{2}
\]
where for each term of the sum we have \( n + 2 \) brackets and the exponent \( i + 1 \) means the position of the bracket valued on \( \mathfrak{h} \).

**Proof.** Using Proposition 2.3, Corollary 1.4 and the fact that
\[
\nabla_X Y = (1/2)[X, Y]_\mathfrak{m}, \quad X, Y \in \mathfrak{m},
\]

6
we have immediately that (2) is verified for \( n = 1 \). Next, suppose that this formula holds for \( n = k \); then

\[
(-1)^{k-1} 2k R_0^k(X) = \sum_{i=0}^{k} (-1)^i \left( \begin{array}{c} k \\ i \end{array} \right) [[[X, v]_m, \ldots, v]_{h + 1}^{i+1}, \ldots, v]_m. \tag{4}
\]

Using now Corollary 1.4, we have

\[
R_0^{k+1}(X) = \nabla_v \cdots \nabla_v R(X, v) v - \sum_{i=1}^{k+1} \left( \begin{array}{c} k + 1 \\ i \end{array} \right) R_0^{k+1-i}(\nabla_v \cdots \nabla_v X).
\]

In each term we take into account Proposition 2.3, formulae (3) and (4), so we obtain

\[
R_0^{k+1}(X) = (-1)^k \frac{1}{2k+1} [[[X, v]_h, \ldots, v]_m
\]

\[- \sum_{i=1}^{k+1} \left( \begin{array}{c} k + 1 \\ i \end{array} \right) (-1)^{k-1} \frac{1}{2^{k+1-i}} \left( (-1)^i \frac{1}{2i} \sum_{j=0}^{k+1-i} \left( \begin{array}{c} k + 1 - i \\ j \end{array} \right) (-1)^j [[[X, v]_m, \ldots, v]_{h+j+1}^{i+1}, \ldots, v]_m \right) \]

Let us remark that the sum of the terms with all brackets estimated in \( m \) is 0. By the other hand the terms that have the bracket estimated in \( h \) in the \((i+1)\)-position are

\[
\begin{align*}
&- \left( \begin{array}{c} k + 1 \\ 1 \end{array} \right) \frac{1}{2k} (-1)^{k-1} (-1)^{i-1} \frac{1}{2} (-1)^i \left( \begin{array}{c} k \\ i - 1 \end{array} \right) \\
&- \left( \begin{array}{c} k + 1 \\ 2 \end{array} \right) \frac{1}{2^{k-1}} \frac{1}{2^2} (-1)^{k-2} (-1)^{i-2} \left( \begin{array}{c} k - 1 \\ i - 2 \end{array} \right) \\
&- \cdots - \left( \begin{array}{c} k + 1 \\ i \end{array} \right) \frac{1}{2^{k+1-i}} (-1)^{k-i} \frac{1}{2} (-1)^i \left( \begin{array}{c} k + 1 - i \\ 0 \end{array} \right)
\end{align*}
\]

\[
= (-1)^k \frac{1}{2k+1} (-1)^i \left( \begin{array}{c} k + 1 \\ 1 \end{array} \right) \left( \begin{array}{c} k \\ i - 1 \end{array} \right)
\]

\[- \left( \begin{array}{c} k + 1 \\ 2 \end{array} \right) \left( \begin{array}{c} k - 1 \\ i - 2 \end{array} \right) \\
+ \cdots + (-1)^{i-1} \left( \begin{array}{c} k + 1 \\ i \end{array} \right) \left( \begin{array}{c} k + 1 - i \\ 0 \end{array} \right) \right).
\]

Using Lemma 1.2, the last expression equals

\[
(-1)^k \frac{1}{2k+1} (-1)^i \left( \begin{array}{c} k + 1 \\ i \end{array} \right)
\]

so the formula (2) is true for \( n = k + 1 \) and this finishes the proof.
3 An explicit form for the Jacobi operator on the manifold $V_1 = Sp(2)/SU(2)$.

We consider the Lie group $Sp(2)$ and the subgroup $SU(2)$. It is well known that $V_1 = Sp(2)/SU(2)$ is a normal naturally reductive Riemannian homogeneous space $[2, 7, 8]$. We denote by $sp(2)$ and $su(2)$ the Lie algebras of $Sp(2)$ and $SU(2)$ respectively. Using the notations of $[8]$ it is known that an element of the Lie algebra $sp(2)$ is a skew-Hermitian matrix of the form

$$
\begin{pmatrix}
    a_{11} & a_{12} & a_{13} & a_{14} \\
    -\overline{a}_{12} & -a_{11} & -\overline{a}_{14} & -\overline{a}_{13} \\
    -\overline{a}_{13} & -a_{14} & a_{33} & a_{34} \\
    -\overline{a}_{14} & a_{13} & -\overline{a}_{34} & -a_{33}
\end{pmatrix}
$$

where $a_{11}, a_{33}$ are pure imaginary numbers and the other $a_{ij}$ are arbitrary complex numbers. Let $S_i, i = 1, \ldots, 10$ be the matrices of $sp(2)$ such that

- $S_1 : a_{11} = -a_{22} = i$
- $S_2 : a_{33} = -a_{44} = i$
- $S_3 : a_{12} = -a_{21} = 1$
- $S_4 : a_{12} = a_{21} = i$
- $S_5 : a_{34} = -a_{43} = 1$
- $S_6 : a_{34} = a_{43} = i$
- $S_7 : a_{13} = -a_{31} = a_{24} = -a_{42} = 1$
- $S_8 : a_{13} = a_{31} = a_{24} = a_{42} = i$
- $S_9 : a_{14} = -a_{41} = a_{23} = -a_{32} = 1$
- $S_{10} : a_{14} = a_{41} = a_{23} = a_{32} = i$

the other $a_{ij}$ being zero in all cases. Evidently $\{S_i\}$ is an adapted basis of $sp(2)$. We construct another basis $\{Q_i\}$ as follows:

$$
\begin{pmatrix}
    Q_1 \\
    Q_2 \\
    Q_3 \\
    Q_4 \\
    Q_5 \\
    Q_6 \\
    Q_7 \\
    Q_8 \\
    Q_9 \\
    Q_{10}
\end{pmatrix}
= \begin{pmatrix}
    1/2 & -3/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & \sqrt{5}/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & \sqrt{5}/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & \sqrt{6}/2 & 0 & -\sqrt{2}/2 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & \sqrt{6}/2 & 0 & -\sqrt{2}/2 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & \sqrt{5}/2 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{5}/2 & 0 & 0 \\
    3/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 & \sqrt{3}/2 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1 & 0 & \sqrt{3}/2 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
    S_1 \\
    S_2 \\
    S_3 \\
    S_4 \\
    S_5 \\
    S_6 \\
    S_7 \\
    S_8 \\
    S_9 \\
    S_{10}
\end{pmatrix}.
$$

We have $[8]$

i) If for an inner product on $sp(2)$ we take $< A, B > = -(1/5) \text{Tr}(AB)$, then $\{Q_1, \ldots, Q_{10}\}$ is an orthonormal basis of $sp(2)$. 

8
ii) The inner product is invariant under \( \text{Ad}(Sp(2)) \).

iii) Finally, one can show that \( h = \text{linear span of } \{ Q_8, Q_9, Q_{10} \} \) is Lie diffeomorphic to \( su(2) \) and therefore the group generated by \( h \) is analytically isomorphic to \( SU(2) \).

The previous decomposition is taken from [2] p.234. If we call \( m = sp(2)/su(2) \), we know that \( \{ Q_1, \ldots, Q_7 \} \) is an adapted basis for \( m \). It is immediate to prove that the brackets are given by the following relations

\[
\begin{align*}
[Q_1, Q_2] &= Q_3, & [Q_1, Q_3] &= -Q_2, \\
[Q_1, Q_4] &= -Q_5 - \sqrt{6}Q_{10}, & [Q_1, Q_5] &= Q_4 + \sqrt{6}Q_9, \\
[Q_1, Q_6] &= -Q_7, & [Q_1, Q_7] &= Q_6, \\
[Q_1, Q_8] &= 0, & [Q_1, Q_9] &= -\sqrt{3}Q_5, \\
[Q_1, Q_{10}] &= \sqrt{6}Q_4, & [Q_2, Q_3] &= Q_1 + 3Q_8, \\
[Q_2, Q_4] &= Q_6, & [Q_2, Q_5] &= -Q_7, \\
[Q_2, Q_6] &= -Q_4 + \sqrt{3/2}Q_9, & [Q_2, Q_7] &= Q_5 - \sqrt{3/2}Q_{10}, \\
[Q_2, Q_8] &= -3Q_3, & [Q_2, Q_9] &= -\sqrt{3/2}Q_6, \\
[Q_2, Q_{10}] &= \sqrt{3/2}Q_7, & [Q_3, Q_4] &= Q_7, \\
[Q_3, Q_5] &= Q_6, & [Q_3, Q_6] &= -(\sqrt{2}/2)(\sqrt{2}Q_5 - \sqrt{3}Q_{10}), \\
[Q_3, Q_7] &= -(\sqrt{2}/2)(\sqrt{2}Q_4 - \sqrt{3}Q_9), & [Q_3, Q_8] &= 3Q_2, \\
[Q_3, Q_9] &= -\sqrt{3/2}Q_7, & [Q_3, Q_{10}] &= -\sqrt{3/2}Q_6, \\
[Q_4, Q_5] &= -Q_1 + Q_8, & [Q_4, Q_6] &= Q_2 + \sqrt{5/2}Q_9, \\
[Q_4, Q_7] &= Q_3 + \sqrt{5/2}Q_{10}, & [Q_4, Q_8] &= -Q_5, \\
[Q_4, Q_9] &= -\sqrt{5/2}Q_6, & [Q_4, Q_{10}] &= -2\sqrt{3/2}Q_1 - \sqrt{5/2}Q_7, \\
[Q_5, Q_6] &= Q_3 - \sqrt{5/2}Q_{10}, & [Q_5, Q_7] &= -Q_2 + \sqrt{5/2}Q_9, \\
[Q_5, Q_8] &= Q_4, & [Q_5, Q_9] &= 2\sqrt{3/2}Q_1 - \sqrt{5/2}Q_7, \\
[Q_5, Q_{10}] &= \sqrt{5/2}Q_6, & [Q_6, Q_7] &= -Q_1 + 2Q_8, \\
[Q_6, Q_8] &= -2Q_7, & [Q_6, Q_9] &= \sqrt{3/2}Q_2 + \sqrt{5/2}Q_4, \\
[Q_6, Q_{10}] &= \sqrt{3/2}Q_3 - \sqrt{5/2}Q_5, & [Q_7, Q_8] &= 2Q_6, \\
[Q_7, Q_9] &= \sqrt{3/2}Q_3 + \sqrt{5/2}Q_5, & [Q_7, Q_{10}] &= -\sqrt{3/2}Q_2 + \sqrt{5/2}Q_4, \\
[Q_8, Q_9] &= Q_{10}, & [Q_8, Q_{10}] &= -Q_9, \\
[Q_9, Q_{10}] &= Q_8.
\end{align*}
\]

In order to be able to determine the explicit form of the Jacobi operator along an arbitrary geodesic \( \gamma \) with initial vector \( v \) at the origin \( o \), it is useful to determine the values of \( R^{(i)}_0 \), \( i = 0, 1, 2, 3, 4 \). In the following we always suppose that \( v \in m \) is given by

\[
v = \sum_{i=1}^{7} x_i Q_i, \quad \sum_{i=1}^{7} (x_i)^2 = 1.
\]

We denote \( \{ E_i, i = 1, \ldots, 7 \} \) the orthonormal frame field along \( \gamma \) obtained by parallel translation of the basis \( \{ Q_i \} \) along \( \gamma \).

For the manifold \( V_4 \) the operators \( R^{(i)}_0 \), \( i = 0, 1, 2, 3, 4 \), written in matrix form are given by

\[
R^{(i)}_0 = \begin{pmatrix}
R^{(i)}_{11} & \cdots & R^{(i)}_{17} \\
& \ddots & \end{pmatrix} (0)
\]
where \( R^{(i)}_{j,k}(0) = < R^{(i)}(E_k), E_j > (0) \).

In [18] Tsukada defines the curves of constant osculator rank in the Euclidean space and this concept is applied to naturally reductive homogeneous spaces; see also [15] Vol.IV, Ch.7, Add. 4. For a unit vector \( v \in m \) determining the geodesic \( \gamma \), \( R_t = e^{tA(v)} R_0 \) is a curve in \( R(m) \). Since \( e^{tA(v)} \) is a 1-parameter subgroup of the group of linear isometries of \( R(m) \), the curve \( R_t \) has constant osculating rank \( r \) [18]. Therefore, for the Jacobi operator we have

\[
R_t = R_0 + a_1(t) R^{(1)}_0 + \cdots + a_r(t) R^{(r)}_0.
\]

With the help of Propositions 2.3 and 2.4 we obtain:

**Lemma 3.1** At \( \gamma(0) \) we have:

\[
i) \quad R^{(3)}_0 = -||\gamma'||^2 R^{(1)}_0 = -R^{(1)}_0;
\]

\[
ii) \quad R^{(4)}_0 = -||\gamma'||^2 R^{(2)}_0 = -R^{(2)}_0.
\]

**Proof.** Due to Tsukada’s result about the constant osculator rank of the curvature operator on naturally reductive spaces, we know that there exists \( r \in \mathbb{N} \) such that \( R^{(1)}, \ldots, R^{(r+1)} \) are linearly dependent. Now we are going to prove that \( r = 2 \) in \( V_1 \).

For that we study the relationship between \( R^{(1)} \) and \( R^{(3)} \) (later we shall find another one between \( R^{(2)} \) and \( R^{(4)} \) and so on). In particular, we have to compare \( R^{(1)}_{(i,j)} \) and \( R^{(3)}_{(i,j)} \) for \( i, j = 1, \ldots, 7 \).

Let us show how to proceed, for instance, to make the comparison between \( R^{(1)}_{(1,1)} \) and \( R^{(3)}_{(1,1)} \). The computation on the other 48 elements of \( R^{(1)} \) and \( R^{(3)} \) will be analogous.

From Proposition 2.4 we have

\[
R^{(1)}_0(X) = (1/2)([[[X,v]_h,v]_m,v]_m - [[[X,v]_m,v]_h,v]_m) = (1/2) \sum_{1 \leq i,j,k \leq 7} x_i x_j x_k [[[X,Q_i]_h,Q_j]_m,Q_k]_m - [[[X,Q_i]_m,Q_j]_h,Q_k]_m.
\]

Therefore if we denote

\[ T_{1}[i,j,k,1] = < (\nabla_{Q_k} R)(Q_i,Q_j) , Q_1 > , \]

putting \( X = Q_1 \) and using (6) it follows

\[
R^{(1)}_{(1,1)} = < R^{(1)}_0(Q_1),Q_1 >= (1/2) < [[[Q_1,v]_h,v]_m,v]_m - [[[Q_1,v]_m,v]_h,v]_m , Q_1 > \]

\[
= (1/2) \sum_{1 \leq i,j,k \leq 7} x_i x_j x_k < [[[Q_1,Q_i]_h,Q_j]_m,Q_k]_m - [[[Q_1,Q_i]_m,Q_j]_h,Q_k]_m , Q_1 > \]

\[
= (1/2) \sum_{1 \leq i,j,k \leq 7} x_i x_j x_k T_{1}[i,j,k,1].
\]
Now, using the values of the brackets of the vectors $Q_i$ in (5) we obtain that the non-vanishing components of $T_1$ are

$$
\begin{align*}
T_1[1, 2, 6, 4, 1] &= -3/2, & T_1[1, 2, 7, 5, 1] &= 3/2, & T_1[1, 3, 6, 5, 1] &= -3/2 \\
T_1[1, 3, 7, 4, 1] &= -3/2, & T_1[1, 4, 2, 6, 1] &= 3/2, & T_1[1, 4, 3, 7, 1] &= -3/2 \\
T_1[1, 4, 4, 6, 1] &= -\sqrt{15}/2, & T_1[1, 4, 5, 7, 1] &= -\sqrt{15}/2, & T_1[1, 4, 6, 2, 1] &= -3/2 \\
T_1[1, 4, 6, 4, 1] &= -\sqrt{15}, & T_1[1, 4, 7, 3, 1] &= -3/2, & T_1[1, 4, 7, 5, 1] &= -\sqrt{15} \\
T_1[1, 5, 2, 7, 1] &= -3/2, & T_1[1, 5, 3, 6, 1] &= -3/2, & T_1[1, 5, 4, 7, 1] &= -\sqrt{15}/2 \\
T_1[1, 5, 5, 6, 1] &= \sqrt{15}/2, & T_1[1, 5, 6, 3, 1] &= -3/2, & T_1[1, 5, 6, 5, 1] &= \sqrt{15} \\
T_1[1, 5, 7, 2, 1] &= 3/2, & T_1[1, 5, 7, 4, 1] &= -\sqrt{15}, & T_1[1, 6, 2, 4, 1] &= 3/2 \\
T_1[1, 6, 3, 5, 1] &= 3/2, & T_1[1, 6, 4, 4, 1] &= -\sqrt{15}/2, & T_1[1, 6, 5, 5, 1] &= -\sqrt{15}/2 \\
T_1[1, 7, 2, 5, 1] &= -3/2, & T_1[1, 7, 3, 4, 1] &= 3/2, & T_1[1, 7, 4, 5, 1] &= -\sqrt{15}/2 \\
T_1[1, 7, 5, 4, 1] &= -\sqrt{15}/2
\end{align*}
$$

Finally using (7) and (8) it is a straightforward computation to obtain that

$$
R^{(1)}_{(1,1)} = \sum_{1 \leq i,j,k \leq 7} x_ix_jx_kT_1[1, i, j, k, 1] = -2\sqrt{15}(x_4^2x_6 - x_5^2x_6 + 2x_4x_5x_7). \tag{9}
$$

For $R^{(3)}$, in an analogous way, we have

$$
R^{(3)}_0(X) = \frac{1}{8}([[]][[[X,v]h,v]m,v]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]) + 3([[X]m]v]v]h]v]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v]m]v}n
}
On the other hand, we have that for other sets of indices $I$ different from $A,$ $B$ or $C$ where $A = \{h, h, 4, 6\},$ $B = \{h, h, 5, 6\},$ $C = \{h, h, 4, 5, 7\},$ $h = 1, \ldots, 7,$ the following sum vanishes

$$\sum_{(i,j,k,l,m) \in S(I)} x_i x_j x_k x_l x_m T_3[1, i, j, k, l, m, 1] = 0.$$ 

In consequence

$$\sum_{1 \leq i,j,k,l,m \leq 7} x_i x_j x_k x_l x_m T_3[1, i, j, k, l, m, 1] = \sum_{1 \leq h \leq 7} (x_h^2) 2\sqrt{15}(x_4^2 x_6 - x_5^2 x_6 + 2x_4 x_5 x_7). \quad (11)$$

Then we can conclude from (9) and (11) that

$$R_3^{(1)}(1, 1) = -(x_1^2 + \cdots + x_7^2) R_1^{(1)}(1, 1) = -||\gamma'||^2 R_1^{(1)}(1, 1).$$

The proof of ii) is analogous to i).

Remark. Using Mathematica it is possible to prove that the non-null components we have calculated in the proof are correct.

**Proposition 3.2** At $\gamma(0)$ we have:

i) $R_0^{2n} = (-1)^{n-1} R_0^{2}.$

ii) $R_0^{2n+1} = (-1)^n R_0^{1}.$

**Proof.** We are going to prove i) by induction, ii) may be obtained in a similar way. First, Lemma 3.1 (ii) gives the result for $n = 2.$ Next, suppose that for $n = k$ the result is true, that is,

$$R_0^{2k} = (-1)^{k-1} R_0^{2}.$$ 

Using Proposition 2.4 we have

$$(-1)^{2k+1} 2^{2k+2} R_0^{2k+2}(X) = \sum_{i=0}^{2k+2} (-1)^i \binom{2k+2}{i} [[[X, v]_m, \ldots, v]_{i+1}^{h}, \ldots, v]_{2k+4}^{m}. $$

There are $2k + 3$ terms and each one has $2k + 4$ brackets. If we take into account Lemma 1.1 in the previous expression, we obtain

$$(-1)^{2k+1} 2^{2k+2} R_0^{2k+2}(X) = \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} [[[X, v]_m, \ldots, v]_{i+1}^{h}, \ldots, v]_{2k+2}^{m}, v]_m, v]_m$$

$$-2 \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} [[[X, v]_m, \ldots, v]_{i+2}^{h}, \ldots, v]_{2k+3}^{m}, v]_m$$

$$+ \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} [[[X, v]_m, \ldots, v]_{i+3}^{h}, \ldots, v]_{2k+4}^{m}. $$

12
If we call $X' = [X, v]_m$ and $X'' = [[X, v]_m, v]_m$ and using also Proposition 2.4, it follows
\[
(-1)^{2k+1} 2^{2k+2} R_0^{2k+2}(X)
\]
\[
= (-1)^{2k+1} 2^{2k+2} ([[R_0^{2k}(X), v]_m, v]_m - 2[R_0^{2k}(X'), v]_m + R_0^{2k}(X'')).
\]
Taking into account Lemma 3.1, formula (12) and the values of $X'$ and $X''$ in previous expression, Proposition 3.2 follows.

The next result follows immediately from Proposition 3.2.

**Proposition 3.3** The normal naturally reductive homogeneous space $V_1 = Sp(2)/SU(2)$ is of constant osculator rank 2.

**Corollary 3.4** Along the geodesic $\gamma$ the Jacobi operator can be written as
\[
R_t = R_0 + R_0^{2j} + R_0^{1j} \sin t - R_0^{2j} \cos t.
\]
The proof follows from the Taylor development of $R_t$ at $t = 0$ and by using Proposition 3.2.

**Corollary 3.5** Along the geodesic $\gamma$ the derivatives of the Jacobi operator satisfy:

- $i)$ $R_t^{2n} = (-1)^{n-1} R_t^{2j}$;
- $ii)$ $R_t^{2n+1} = (-1)^n R_t^{1j}$;
- $iii)$ $R_t \cdot R_t^{1j} = R_t^{1j} \cdot R_t$,
- $R_t \cdot R_t^{2j} = R_t^{2j} \cdot R_t$,
- $R_t^{1j} \cdot R_t^{2j} = R_t^{2j} \cdot R_t^{1j}$.

The result is a consequence of Corollary 3.4 and the fact that iii) is true for $t = 0$.

**Remark 3.6** In [6] the authors analyze a class of Riemannian homogeneous spaces which have the property that the eigenspaces of $R_\gamma$ are parallel along $\gamma$. Evidently, this property is not verified in our case. In fact, although, for each $t$, the operators $R_t$ and $R_t^{1j}$ commute and therefore are simultaneously diagonalizable, according to Corollary 3.4 and (ii) of Proposition 2.2 the eigenvectors of these operators are not independent of $t$.

## 4 The solution of the Jacobi equation on the manifold $V_1$. Application to the determination of volumes of geodesic balls.

For a naturally reductive homogeneous Riemannian manifold it is possible to write the Jacobi equation as a differential equation with constant coefficients. In order to do that, the canonical
connection is frequently used. Since this connection and the Levi-Civita connection have the same
differential equation, in an equivalent form, it is possible to write the same equation based on the Levi-Civita
connection \([8]\). In this case, the coefficients are functions of the arc-length along the geodesic. In
order to work with this equation on the manifold \(V_1\) it will be useful to use the simple expression
of the Jacobi operator \(R_t\). We shall now introduce some notation and provide some basic formulae
which will be needed in this section. For more information see \([5, 8, 20]\). Let \(A\) be the Jacobi
tensor field along the geodesic \(\gamma\) (that is, the solution of the endomorphism valued Jacobi equation
\(Y'' + R_tY = 0\) along \(\gamma\)) with initial values
\[
A_0 = 0, \quad A_1^0 = I,
\]
where we consider the covariant differentiation with respect to \(\gamma'\) and \(I\) is the identity transformation
of \(T_{\gamma(0)}M\). Then, the Jacobi’s equation is \(A_1^2 = -R_tA_t\).

In order to be able to obtain the expression of the Jacobi fields with initial conditions \((13)\) at \(\gamma(0)\),
it is enough to know the development in Taylor’s series of \(A_t\) and to apply the initial conditions.
Thus, using Lemma 3.1, in the power series of \(A_t\) only appear \(R_0, R_1^0\) and \(R_2^0\). If \(\{E_i, i = 1, \ldots, 7\}\)
is the orthonormal frame field along \(\gamma\) obtained by parallel translation of the basis \(\{Q_i\}\) along \(\gamma\),
one has \(Y_t = A_tE_t\) or \(Y_{it} = A_i^1E_{ji,t}, 1 \leq i, j \leq 7\), and this is the expression of the Jacobi vector
fields along the geodesic \(\gamma\) with the indicated initial conditions.

**Proposition 4.1** *For the manifold \(V_1\), one has*

\[
A_t = \sum_{k=0}^{\infty} \frac{1}{k!} \beta_k t^k
\]

*where \(\beta_k = \alpha_k + \beta_{k-1}', \quad \alpha_k = \alpha_k' - R_k \beta_{k-1}, k \geq 2\). Moreover, \(\alpha_0 = \beta_0 = 0, \alpha_1 = 0, \beta_1 = I\)
and the coefficients \(\beta_k\) are only functions of \(R_0, R_1^0\) and \(R_2^0\).*

**Proof.** If we successively derive \(A_1^2 = -R_tA_t\), we have

\[
A_t^0 = (\alpha_{t-1}'(t) - R_t \beta_{t-1}(t)) A_t + (\alpha_{t-1}(t) + \beta_{t-1}'(t)) A_t^1,
\]
we can write this expression as

\[
A_t^0 = \alpha_t(t) A_t + \beta_t(t) A_t^1,
\]
where

\[
\alpha_t(t) = \alpha_{t-1}'(t) - R_t \beta_{t-1}(t),
\]
and

\[
\beta_t(t) = \alpha_{t-1}(t) + \beta_{t-1}'(t), \quad i \geq 2;
\]
if \(t = 0\) one has \(A_0^0 = \beta_0(0) = 0, A_0^1 = \beta_1(0) = I, A_0^2 = \beta_2(0) = 0, A_0^3 = \beta_3(0) = -R_0^0 = -R_0,\)
and, in general,

\[
A_0^0 = \alpha_{t-1}(0) + \beta_{t-1}'(0) = \beta_t(0).
\]

If there is no confusion we will identify \(\alpha_t = \alpha_t(0)\) and \(\beta_t = \beta_t(0)\). Now the result follows using the
development in Taylor’s series of \(A_t\).
Let \( m \) be a point of the manifold \( M \) and \( V \) and \( U \) open neighbourhoods of 0 in \( T_m M \) and of \( m \) in \( M \) respectively such that \( \exp_m \) is a diffeomorphism of \( V \) onto \( U \). For all \( v \in V \), \( \theta(v) \) \[3, p. 54\] is a well-defined function, it is defined as the absolute value of a determinant function:

\[
\theta(v) = | \det T_v \exp_m |.
\]

**Definition 4.2** Let \( U_\varepsilon(m) \) be a normal neighbourhood of radius \( \varepsilon > 0 \) of the point \( m \) in \( M \). For each \( t \) such that \( 0 < t < \varepsilon \) and for each \( v \) in \( T_m M \) the function \( t \mapsto \theta(tv) \) is the volume density function at \( m \) in the direction \( v \).

**Lemma 4.3** \[3, p. 90\] Let \( u \in T_o M \) and \( t > 0 \), then for all \( v \in T_o M \), \( T_t u \exp_o(v) \) is the value in \( t \) of the Jacobi field \( Y \) along the geodesic \( \gamma \) (\( \gamma(0) = o \), \( \gamma'(0) = u \)) with initial conditions \( Y(o) = 0 \), \( Y'(o) = v/t \).

**Proposition 4.4** In the manifold \( V_1 \), the volume density function at \( o \) is given by

\[
\theta(tu) = \frac{1}{t^7} | \det A |. \tag{14}
\]

The proof follows in a natural way from the standard methods of \[3, 11, 12\].

**Corollary 4.5** The coefficient of \( t^n \) in the development of \( \det A \) is given by

\[
a_n = \sum_{r_1 + \cdots + r_7 = n} \frac{1}{r_1!} \cdots \frac{1}{r_7!} \sum_{\sigma} \text{sig}(\sigma) \beta_{r_1, \sigma(1)}^1 \cdots \beta_{r_7, \sigma(7)}^7.
\]

**Proof.** The seven columns \( C_j, j = 1, \ldots, 7 \) of \( A \) can be written as

\[
C_j = \beta_j^0 + \cdots + \frac{1}{n!} \beta_j^n t^n + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} \beta_j^k t^k
\]

where the upper index \( j \) shows the \( j^{th} \)-column of the matrix \( \beta_k \) (Proposition 4.1). Taking into account that the determinant is a multilinear function, the coefficient \( a_n \) of \( t^n \) in the development of \( \det A \) is

\[
a_n = \sum_{r_1 + \cdots + r_7 = n} \frac{1}{r_1!} \cdots \frac{1}{r_7!} \det(\beta_{r_1}^1, \ldots, \beta_{r_7}^7).
\]

If we represent the matrix \( \beta_{r_k} \) by \( \beta_{r_k} = (\beta_{r_k,i}^j), i, j = 1, \ldots, 7 \), using the algebraic definition of the determinant it follows that

\[
a_n = \sum_{r_1 + \cdots + r_7 = n} \frac{1}{r_1!} \cdots \frac{1}{r_7!} \sum_{\sigma} \text{sig}(\sigma) \beta_{r_1, \sigma(1)}^1 \cdots \beta_{r_7, \sigma(7)}^7
\]

where \( \sigma \) are the permutations of seven elements and \( \text{sig}(\sigma) \) represents the signature of the corresponding permutation.
Lemma 4.6 \[11\] For the manifold $V_1$,

i) The area of the geodesic sphere with center $o \in V_1$ and radius $t$ is given by

$$S_o(t) = t^6 \int_{\Omega^6(1)} \theta(tu)du$$

where $\Omega^6(1)$ denotes the 6-dimensional Euclidean unit sphere.

ii) The volume of the geodesic ball with center $o \in V_1$ and radius $r$ is given by

$$V_o(r) = \int_0^r S_o(t)dt.$$

Now, using the standard notation for moments \[11, \text{p. 255–258}], we have:

Proposition 4.7 i) The area of the geodesic ball with center $o$ and radius $t$ is given by

$$S_o(t) = \frac{16\pi^3}{105} \sum_{n=3}^{\infty} \frac{1}{2n+1} \left< a_{2n+1} > t^{2n}.$$ 

ii) The volume of the geodesic ball with center $o$ and radius $t$ is given by

$$V_o(t) = \frac{16\pi^3}{105} \sum_{n=3}^{\infty} \frac{1}{2n+1} \left< a_{2n+1} > t^{2n+1}.$$

Proof. i) If we integrate i) of Lemma 4.6 over the sphere we have that the odd powers vanish and then the result follows immediately. For ii) we use that $V_o(r) = \int_0^r S_o(t)dt$.

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Author’s addresses:

A. M. Naveira
Departamento de Geometría y Topología. Facultad de Matemáticas.
Avda. Andrés Estellés, N 1
46100 - Burjassot
Valencia, SPAIN

Phone +34-963544363
Fax: +34-963544571
e-mail: naveira@uv.es

A. D. Tarrío Tobar
E. U. Arquitectura Técnica
Campus A Zapateira. Universidad de A Coruña
15192 - A Coruña, SPAIN

Phone +34-981167000 Ext. 2721, 2713
Fax: +34-981167060
e-mail: madorana@udc.es