Simple quantum cosmology: Vacuum energy and initial state

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Abstract

A static non-singular 10-dimensional closed Friedmann universe of Planck size, filled with a perfect fluid with equation of state \( p = -\frac{2}{3} \varepsilon \), can arise spontaneously by a quantum fluctuation from nothing in 11-dimensional spacetime. A quantum transition from this state can initiate the inflationary quantum cosmology outlined in Ref. 2 [General Relativity and Gravitation 33, 1415, 2001 - gr-qc/0103021]. With no fine-tuning, that cosmology predicts about 60 e-folds of inflation and a vacuum energy density depending only on the number of extra space dimensions (seven), \( G, \hbar, c \) and the ratio between the strength of gravity and the strength of the strong force. The fraction of the total energy in the universe represented by this vacuum energy depends on the Hubble constant \( H_0 \). Estimates of \( H_0 \) from WMAP, SDSS, the Hubble Key Project, and Sunyaev-Zeldovich and X-ray flux measurements range from 60 to 72 km sec\(^{-1}\) Mpc\(^{-1}\). Using a mid-range value of \( H_0 = 65 \text{ km sec}^{-1} \text{ Mpc}^{-1} \), the model in Ref. 2 predicts \( \Omega_\Lambda \approx 0.7 \).

M/string theory, involving eleven spacetime dimensions, is a leading candidate for the theory to describe the four fundamental forces governing the universe. However, the theory is mathematically difficult, and the correct form of the theory applicable to our universe with four large spacetime dimensions and seven small, compact dimensions is not yet known [1]. Polchinski notes that many open puzzles in M/string theory center on issues in cosmology, such as singularities and the cosmological constant [1]. Consequently, it seems worthwhile to consider simpler models, to see if a phenomenological model with four large dimensions and seven small, compact dimensions can reproduce the gross features of our universe. In fact, a simple and surprisingly realistic cosmological model [2] can be developed in 11-dimensional spacetime, using the direct canonical Hamiltonian quantization of the Friedmann equation outlined by Elbaz et al [3] and Novello et al [4].

It is increasingly apparent that the evolution of our universe is dominated by the vacuum
energy density in the inflationary era and in today’s era of accelerated expansion. Therefore, a useful cosmological model must adequately account for the behavior of the vacuum energy density. The model in Ref. 2 predicts a vacuum energy density that depends only on the number of extra space dimensions (seven), $G, \hbar, c$ and the ratio between the strength of gravity and the strength of the strong force. Then, given the Hubble constant $H_0$, the model predicts $\Omega_\Lambda$, the vacuum energy fraction of the total energy density of the universe. Estimates of the Hubble constant, from WMAP, SDSS, the Hubble Key Project, and Sunyaev-Zeldovich and X-ray flux measurements, range from 60 to 72 km sec$^{-1}$ Mpc$^{-1}$ [5, 6, 7]. If a mid-range value of $H_0 = 65$ km sec$^{-1}$ Mpc$^{-1}$ is used, the model in Ref. 2 predicts $\Omega_\Lambda \approx 0.7$. However, if values of $H_0 = 71$ km sec$^{-1}$ Mpc$^{-1}$ [5] and $\Omega_\Lambda = 0.7$ are confirmed, the model in Ref. 2 will not be tenable in its present form, because it predicts $\Omega_\Lambda = 0.6$ if $H_0 = 71$ km sec$^{-1}$ Mpc$^{-1}$.

With no fine-tuning, the model in Ref. 2 predicts about 60 e-folds of inflation. When a quantum fluctuation puts the universe in the unstable initial state of the model, inflation is inevitable, and the model allows an infinite cosmic time for this fluctuation to occur. However, the analysis below indicates the model can also incorporate a more symmetric, static, non-singular initial “doorway state” analogous to a force-symmetric universe that might be described by M/string theory. Inflation would then begin with a quantum fluctuation from this static initial state to the unstable initial state considered in Ref. 2.

Ref. 2 models the universe in a ten-dimensional Euclidean curvature space describing the curvature of a homogeneous ten-dimensional physical space. The coordinate in each dimension of a state in curvature space is the radius of curvature of the corresponding dimension of a ten-dimensional physical space. Initially all forces were equal, and the Planck mass was identical to the proton mass $m_p$. The ratio of the initial strength of gravity $G_i$ to the strength of the strong force was $\frac{G_i m_p^2}{\hbar c} = 1$, and the Planck length was $\delta_i = \sqrt{\frac{\hbar G_i c}{c^3}} = 2.11 \times 10^{-14}$ cm. The required initial state for the cosmological model in Ref. 2 is a closed universe with radius $\frac{\delta_i}{2\pi}$ in all ten space dimensions. A static nonsingular initial state can be modeled with a two-term effective potential in curvature space. One term involves only the radius of curvature $r$ of one of the space dimensions. The other term involves only the radius of curvature $R$ of the remaining nine space dimensions. This follows the idea from M/string theory that one of the ten space dimensions in eleven-dimensional spacetime is a circle and the four forces governing the universe are described by the five equivalent ten-dimensional string theories involving the remaining nine space dimensions.

The complexities of the symmetry breaking of the four forces governing the universe, and the corresponding effects on cosmology, have not been fully worked out in M/string theory. However, general relativity, as a theory of gravity in four-dimensional spacetime, provides useful cosmological models. So, if all forces were initially equal, it seems to be a rea-
sonable phenomenological approach to use the Friedmann equation from ten-dimensional general relativity to model the nine space dimensions involved with the forces in the initial state of the universe. Then, if there is an effective $1/R$ potential in the Friedmann equation, a closed static universe with curvature radius $\frac{8\pi}{3G_i}$ in all ten space dimensions can arise spontaneously by a quantum fluctuation from nothing. The necessary and sufficient condition for a $1/R$ potential in the Friedmann equation is the presence of a perfect fluid with equation of state $p = -\frac{2}{3}\varepsilon$. Incidentally, the winding modes of a 6-brane gas in Type II-A string theory on a nine-dimensional toroidal background space [8] have this equation of state. These winding modes wrap around six of the space dimensions that must eventually collapse (along with the circular tenth space dimension) to generate the three large space dimensions we inhabit.

When the total energy and total angular momentum in curvature space are zero, the Schrödinger equation for the ten-dimensional radius of curvature is [2]

$$-\frac{\hbar^2}{2m} \nabla_\mathbf{R}^2 \Psi + V_\mathbf{R} \Psi = 0$$

where $\mathbf{R}$ is the magnitude of a ten-dimensional vector $\mathbf{R}$ and $m$ is an effective mass. The model assumes $V_\mathbf{R} = V_R + V_r$, so $\Psi = \Psi(R)\Psi(r)$ and

$$\left[ \frac{1}{\Psi(R)} \frac{-\hbar^2}{2m} \nabla_R^2 \Psi(R) + V_R \right] + \left[ \frac{1}{\Psi(r)} \frac{-\hbar^2}{2m} \nabla_r^2 \Psi(r) + V_r \right] = 0 \quad (1)$$

where each bracket is a constant.

When the gravitational constant is $G_i$, the Friedmann equation for the radius of curvature $R$ of a closed ten-dimensional homogeneous isotropic universe is [9]

$$\left( \frac{dR}{dt} \right)^2 - \left( \frac{2\pi G_i}{9} \right) \varepsilon \left( \frac{R}{c} \right)^2 = -c^2 \quad (2)$$

where $\varepsilon$ is the energy density. Multiplied by $\frac{1}{2m}$, equation (2) describes the motion of a fictitious particle with mass $m$ and energy $-\frac{1}{2}mc^2$ in the potential

$$V_R = -\frac{m}{2} \left( \frac{2\pi G_i}{9} \right) \varepsilon \left( \frac{R}{c} \right)^2$$

A quantum mechanical model for the nine space dimensions of a homogeneous universe can be obtained by applying the canonical Hamiltonian quantization procedure of Elbaz et al [3] and Novello et al [4] to equation (2). Setting $\Psi(R) = R^{-4}\psi(R)$ and defining the constants in equation (1) as the curvature energies $-\frac{1}{2}mc^2$ and $\frac{1}{2}mc^2$ results in the following
Schrödinger equation for the Friedmann dimensions

\[-\frac{\hbar^2}{2m} \frac{d^2 \psi(R)}{dR^2} + \left( \frac{6\hbar^2}{mR^2} + V_R \right) \psi(R) = -\frac{1}{2} mc^2 \psi(R)\]

In ten-dimensional spacetime, the energy density \(\epsilon\) of a perfect fluid with equation of state \(p = w\epsilon\) is [9] \(\epsilon = \epsilon_0 R^{-9(1+w)}\). Then, if the Friedmann dimensions are filled with a perfect fluid with \(w = -\frac{2}{3}\), the effective potential is

\[V'_R = \frac{6\hbar^2}{mR^2} - \frac{m}{2} \left( \frac{2\pi G}{9c^2} \right) \epsilon_0 R\]

Near the minimum of \(V'_R\) at \(R = a\), it can be approximated by a harmonic oscillator potential \(V'_R \approx V'_R(a) + \frac{1}{2} \left[ \frac{d^2}{dR^2} V'_R(a) \right] (R - a)^2\). If \(\epsilon_0 = \frac{216}{f G} \left( \frac{\hbar c}{m} \right)^2\) and \(m = \frac{2\pi}{f} \left( \sqrt{12 - 2\sqrt{3}} \right) m_p\), the minimum of \(V'_R\) is at \(f \frac{\delta}{2\pi}\) and the ground state energy of the potential is at \(-\frac{1}{2} mc^2\).

Suppose \(V_r\) has a local minimum at \(r = f \frac{\delta}{2\pi}\), the same effective restoring force around that minimum as the approximate harmonic oscillator potential \(V'_R\), and ground state energy \(\frac{1}{2} mc^2\). Then a universe with \(\sqrt{\langle R^2 \rangle} = \sqrt{\langle r^2 \rangle} = f \frac{\delta}{2\pi}\) can arise by a quantum fluctuation from nothing into the ground states of the effective harmonic oscillator potentials near the minima in \(V'_R\) and \(V_r\). The factor \(f\) is determined from \(\sqrt{\langle R^2 \rangle} = \sqrt{\langle r^2 \rangle} = f \frac{\delta}{2\pi}\). The ground state wavefunctions are \(\psi(x) \approx C e^{-\gamma^2 (f \delta / 2\pi)^2 / (\hbar)^2} \), where \(C\) is a normalization constant, \(\gamma^2 = \frac{f \delta}{2\pi} \sqrt{m \frac{d^2}{dR^2} V_x \left( \frac{f \delta}{2\pi} \right)} = \sqrt{3}\), and \(x\) denotes \(r\) or \(R\). As shown in Ref. 2, \(\sqrt{\langle x^2 \rangle} = f \frac{\delta}{2\pi}\) if

\[f = \left( \frac{-\frac{\gamma^2}{2} + \sqrt{\pi [1 + Erf(\gamma)] \left( 1 + \frac{1}{2\pi} \right)}}{\sqrt{\pi [1 + Erf(\gamma)]}} \right)^{-\frac{1}{2}} = 0.895\]

A quantum transition from the static nonsingular initial state described above can then lead to the unstable state that initiates inflation in the simple quantum cosmology outlined in Ref. 2.

References

1. J. Polchinski, “M Theory: Uncertainty and Unification.” Heisenberg Centennial Symposium, Munich, Dec. 6-7, 2001 [hep-th/0209105]

2. T. R. Mongan, General Relativity and Gravitation 33, 1415, 2001 [gr-qc/0103021]

3. E. Elbaz, M. Novello, J. M. Salim, M. C. Motta da Silva and R. Klippert, General Relativity and Gravitation 29, 481, 1997
4. M. Novello, J.M. Salim, M.C. Motta da Silva and R. Klippert, Phys. Rev. D54, 6202, 1996.

5. S. Eidelman et al, Physics Letters B592, 1, 2004

6. M. Tegmark et al, Phys. Rev. D69, 103501, 2004

7. D. Spergel et al, Astrophys. J. Suppl. 148, 175, 2003

8. S. Alexander, R. Brandenberger and D. Easson, Phys. Rev. D62, 103509, 2000 [hep-th/0005212]

9. D. Youm, Phys. Lett. B531, 276-280, 2002 [hep-th/0201268]