Traveling waves in high energy QCD

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Abstract. Saturation is expected to occur when a high density of partons (mainly gluons)- or equivalently strong fields in Quantum Chromodynamics (QCD) - is realized in the weak coupling regime. A way to reach saturation is through the high-energy evolution of an extended target probed at a fixed hard scale. In this case, the transition to saturation is expected to occur from nonlinear perturbative QCD dynamics. We discuss this approach to saturation, which is mathematically characterized by the appearance of traveling wave patterns in a suitable kinematical representation. A short review on traveling waves in high energy QCD and a first evidence of this phenomenon in deep-inelastic proton scattering are presented.

INTRODUCTION

One of the most intriguing empirical observations about deep-inelastic scattering (DIS) on a proton is the geometric scaling property [1], which states that the $\gamma^* - p$ total cross-section data can be approximately plotted on a one-dimensional curve $\sigma_{\gamma^* - p}(Q^2/Q_s^2(Y))$ where $(Q^2,Y)$ span the virtuality-rapidity kinematical domain of DIS. One finds $Q_s^2(Y) \approx e^{\lambda Y}$.

The question we want to address is whether the observed geometric scaling can be explained in terms of QCD evolution equations describing saturation.

Saturation of parton (mainly gluon) densities at high rapidity may be at the origin of the geometric scaling property. Saturation is expected to occur when a high density of partons is created in the limited geometry of the target. In the expected scenario, the parton wave functions overlap and eventually create a new state of matter, the Color Glass Condensate (CGC) [2]. In the case of DIS on a proton, increasing densities are obtained by high-energy evolution at fixed virtuality $Q^2$, when the corresponding linear QCD evolution, named BFKL after Balitsky, Kuraev, Lipatov, and Fadin [3], gives rise to an exponential growth of the number of gluons. At high enough energy the BFKL formulation gets modified. In the approximation of uncorrelated hard probes, one deals with the Balitsky-Kovchegov (BK) equation [5], where the corrections due to the high density of partons are expressed in terms of a non-linear damping term. We shall investigate whether and how geometric scaling could be due to the nonlinear QCD evolution of the gluon distribution in the target.
FIGURE 1. Schematic picture of the transition to saturation with rapidity. In a region $Y \sim Y_1$: the projectile of size $\sim 1/Q$ is able to probe the number of partons in exponential growth; when $Y \sim Y_2$: the probe counts partons by groups, leading to a nonlinear damping evolution factor in the probed parton density. For $Y > Y_2$: A new phase of partonic matter (CGC) may appear through parton correlations.

GEOMETRIC SCALING AS TRAVELING WAVES

Let us start from the BK equation expressed in momentum space (using the impact-parameter independent formulation)

$$\partial_Y \mathcal{N} = \bar{\alpha} \chi \left( -\partial_L \right) \mathcal{N} - \bar{\alpha} \mathcal{N}^2$$

where $\mathcal{N}(Y, L = 2 \log k)$ is related to the transverse-momentum $k$ distribution of gluons in the target. The (leading order) BFKL kernel is

$$\chi \left( -\partial_L \right) = 2\psi(1) - \psi(-\partial_L) - \psi(1 + \partial_L).$$

The coupling constant will be considered either constant, or running like $\bar{\alpha} = 1/bL$.

From a mathematical point of view, the BK equation can be related to an “universality class” of nonlinear equations for which characteristic properties can be rigorously derived. “Universality” means here that the solutions have general characteristics which are essentially independent from initial conditions. In the case of the BK equation (and also for some extensions beyond BK) this generic feature is the formation of traveling wave patterns when the energy increases. As we shall see, traveling waves are intimately related to geometric scaling. Let us define these traveling waves and give a qualitative explanation of their formation during the rapidity evolution.

Consider first the “diffusive approximation” of the BFKL kernel

$$\chi \left( -\partial_L \right) \sim \chi \left( \frac{1}{2} \right) + \frac{1}{2} \chi'' \left( \frac{1}{2} \right) \times (\partial_L + \frac{1}{2})^2.$$
By a suitable redefinition of the function and variables one can map the BK equation in the diffusive approximation onto the Fisher Kolmogorov-Petrovski-Piskounov (F-KPP) equation \[7\] describing a reaction-diffusion process in space and time:
\[
\partial_t u(t,x) = \partial_x^2 u(t,x) + u(t,x) - u^2(t,x),
\]
In reaction-diffusion language, $\partial_x^2 u(t,x)$ is the diffusion term, $u(t,x)$ is responsible for the exponential growth and $u^2(t,x)$, the damping term.

The “dictionary” between F-KPP and BK can be written as follows

- \textit{Time} $t \rightarrow Y$
- \textit{Space} $x \rightarrow L + \frac{1}{2} \alpha \chi'' \left( \frac{Y}{Y} \right)$
- \textit{Wave Front} $u(x-ct) \rightarrow \mathcal{N}(L-vY)$
- \textit{Traveling Waves} $\rightarrow$ Geometric Scaling.

The key feature of the solutions of the F-KPP equation (and of equations belonging to the same universality class) is the formation of traveling wave patterns, see Fig. (2). It can be rigorously proven that, at large times ($\Rightarrow$ large rapidities), the solutions verify
\[
\frac{u(t,x)}{t \rightarrow \infty} > u(x-m(t)) \Rightarrow \mathcal{N}(Y,L) \xrightarrow{Y \rightarrow \infty} \mathcal{N}(L-L_s(Y))
\]
with
\[
L_s(Y) = \log Q_3^2(Y) = \alpha' \chi'' \left( \frac{\bar{y}}{\bar{y}} \right) Y - \frac{3}{2 \bar{y}} \log Y - \frac{3}{(\bar{y})^2} \sqrt{\frac{2\pi}{\alpha' \chi'' \left( \frac{\bar{y}}{\bar{y}} \right)}} \frac{1}{\sqrt{Y}} + \mathcal{O}(1/Y).
\]
\(\bar{y}\) is the implicit solution of the characteristic equation $\chi(\bar{y})/\bar{y} = \chi'(\bar{y})$ which leads to the asymptotic wave speed $v = \alpha' \chi'' \left( \frac{\bar{y}}{\bar{y}} \right)$. It plays the rôle of a critical QCD anomalous
FIGURE 3. “Pulled vs. Pushed fronts”. The function $u(t,x)$ is represented for the three different classes of initial conditions. Top: “Pushed fronts,” the wave front keeps the initial front profile with a non-universal speed $v_-; \text{ Middle: “Pulled = Pushed”; Bottom: “Pulled front:” both the wave front and the speed acquire universal values at large } t (\sim Y) \text{ in an intermediate kinematical region called the “wave interior”.}

One immediately recognizes the equivalence between geometric scaling and traveling waves solutions of BK. Moreover, quite a detailed information can be obtained from the sole knowledge of the linear kernel, as for the three first terms \cite{6} of the asymptotic expansion of the saturation scale $Q_s^2(Y)$.

ASYMPTOTIC TRAVELING WAVES

In fact the results using the diffusive approximation of BK and its mapping to the F-KPP equation are more general. Instead of entering here in refined (and technically useful) mathematical arguments \cite{8}, let us qualitatively illustrate the reason why traveling wave patterns generically develop during the rapidity evolution from a given initial condition, using the concepts of “pulled fronts”.

Universality features are obtained in the “pulled” front situation, see the bottom plot of Fig.\cite{3}. In this case, one starts with initial conditions corresponding to an anomalous dimension $\gamma_0 > \tilde{\gamma}$, where $\gamma = .6275...$ is for the (leading order) BFKL kernel \cite{6}. The formation of the traveling wave comes from the competition between the exponential growth at small values of the solution $u$, where the linear term dominates the evolution, with the nonlinear damping exercised at larger values of $u$. Constrained by these two opposite trends, the solution is “forced” to adopt an universal behaviour. Fortunately enough, the perturbative QCD initial conditions, through the transparency property ($\gamma_0 = 1 > \tilde{\gamma}$), fall into the universal “pulled” front regime \cite{6}. 


FIGURE 4. The “Reduced” Front profile.

In Fig. 4 the “reduced front” \( \left( \frac{k^2}{Q_s(Y)^2} \right)^n \mathcal{N}(L,Y) \), obtained by factorizing out the wave propagation, is represented for fixed (left) and running (right) coupling constant. It shows the scaling straight line, which envelopes the curves obtained for increasing rapidity. Hence geometric scaling violations are also predicted due to diffusion. One may notice that the diffusive approach to scaling is slower for running coupling, corresponding to “anomalous diffusion” in \( 1/t^{1/3} \sim 1/Y^{1/6} \) (for running coupling \( t \sim Y^{1/2} \)). Normal diffusion \( 1/t^{1/2} \sim 1/Y^{1/2} \) is characteristic of the F-KPP universality class or BK with fixed coupling.

**PARAMETRIC TRAVELING WAVES**

The abovementioned mathematical properties have the major interest of revealing the link between geometric scaling and the nonlinear dynamics of QCD evolution equations at high energy. However, the path from phenomenology to theory is not yet accomplished. On the phenomenological ground, the simulations of BK equation solutions [9] have shown that pre-asymptotics effects may be important and endanger geometrical scaling predictions from QCD in the physical region where it is observed. On the theory side, it has been recently realized [10] that the effect of parton fluctuations and correlations, especially in the dilute (transparency) region which is the driving force of the pulled front, may be responsible for a breaking of geometric scaling. In fact, traveling wave patterns still exist [11] but their event-by-event fluctuations tend to break geometric scaling in the average.

Despite these problems, the numerical simulations [9] also show that patterns behaving like traveling waves exist in a preasymptotic region in energy. Moreover, the mean field BK equation seems to be still valid in this region. In order to explore this possibility and understand its origin, we proposed [12] to attack the problem of traveling wave solutions of the BK equation in a new way.

The initial mathematical idea [13] is to assume and directly insert a traveling wave solution \( u(x,t) \rightarrow U(s = x/c - t) \) into the F-KPP equation, or equivalently to impose a geometric scaling form to a solution of the BK equation. Keeping for simplicity the
space-time language and the FKPP equation (the dictionary to BK being still easy), one gets
\[ U(1 - U) + \frac{dU}{ds} + \frac{1}{c^2} \frac{d^2U}{(ds)^2} = 0. \]
taking into account the previous study showing [7] that the wave speed \( c \geq 2 \), we obtain an iterative solution
\[ U(s) = U_0 + \frac{1}{c^2} U_1 + \sum_{p \geq 2} \frac{1}{c^2p} U_{2p} \]
obeys an exactly solvable [12] hierarchy of equations for the \( U_{2p} \)'s. The first one only (for \( U_0 \)) is nonlinear and can be exactly solved, the other boiling down to a rather simple linear algebra. This mathematical result translates into a parametric form of a geometric scale invariant gluon amplitude \( \mathcal{N}(L - L_s(Y)) \) which can be used in phenomenology. \( \mathcal{N} \) depends only on few parameters determined by the linear kernel. One obtains
\[ \mathcal{N} \propto \frac{1}{1 + \left[ \frac{k^2}{Q_s^2(Y)} \right]^{\mu}} \cdot \frac{1}{c^2} \left( 1 + \left[ \frac{k^2}{Q_s^2(Y)} \right]^{\mu} \right)^2 \log \left[ 4 \left[ \frac{k^2}{Q_s^2(Y)} \right]^{\mu} \right]. \]

This solution (valid also for the running case with an appropriate mapping [12]) is mathematically valid [8] in the “wave interior”, see Fig.[3]. Remarkably, we observed that the corresponding region has a large overlap with the physical region where geometrical scaling is valid at HERA. In order to check this opportunity, we determined the phenomenological observable related to \( \mathcal{N}(L, Y) \) from a MRST parametrization of \( F_2 \) data. The obtained kernel parameters are not far but different from those expected from the leading order BFKL kernel one and thus could require higher order contributions [12]. Interestingly, higher order contributions play an important rôle in a different but related problem: the existence of “backward” traveling waves appearing [14] from the energy evolution starting from an initial high density (and not initial low density as in DIS) state, which could be the case in heavy-ion reactions.
CONCLUSION

Traveling wave patterns have been shown to result from nonlinear QCD equations. The phenomenologically observed geometric scaling in DIS on a proton at large energy is consistent with the traveling wave patterns. It remains to be found whether one can relate the geometric scaling curve with the QCD kernel and more generally to a complete solution of high-energy QCD, including correlation and fluctuation contributions.

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