Star Unfolding from a Geodesic Curve

Stephen Kiazyk · Anna Lubiw

Abstract There are two known ways to unfold a convex polyhedron without overlap: the star unfolding and the source unfolding, both of which use shortest paths from vertices to a source point on the surface of the polyhedron. Non-overlap of the source unfolding is straightforward; non-overlap of the star unfolding was proved by Aronov and O’Rourke (Discrete Comput Geom 8(3):219–250, 1992). Our first contribution is a simpler proof of non-overlap of the star unfolding. Both the source and star unfolding can be generalized to use a simple geodesic curve instead of a source point. The star unfolding from a geodesic curve cuts the geodesic curve and a shortest path from each vertex to the geodesic curve. Demaine and Lubiw conjectured that the star unfolding from a geodesic curve does not overlap. We prove a special case of the conjecture. Our special case includes the previously known case of unfolding from a geodesic loop. For the general case we prove that the star unfolding from a geodesic curve can be separated into at most two non-overlapping pieces.

Keywords Unfolding · Convex polyhedra · Geodesic curve

Mathematics Subject Classification 52B10 · 68Q25
1 Introduction

An unfolding of a polyhedron \( \mathcal{P} \) is obtained by cutting the surface of \( \mathcal{P} \) in such a way that it can be flattened into the plane, forming a single polygon. A main goal is to find unfoldings that are simple, that is, do not self-overlap. If we have an unfolding that does not overlap, we can make a model of the polyhedron from a sheet of paper by cutting the outline of the polygon and gluing appropriate pairs of edges together.

Unfoldings have fascinated people since the time of Dürer’s beautiful examples [7]. A long-standing open question is whether every convex polyhedron has a non-overlapping edge unfolding, where only edges of the polyhedron are cut. However, if we allow cuts that cross faces—which is the model used in this paper—then there are several known methods to unfold convex polyhedra without overlap.

Unfoldings of polyhedra have applications in product manufacturing, for constructing a 3-dimensional object from a sheet of metal or plastic, and also in graphics for applying texture mapping, where 2-dimensional image coordinates must be assigned to points on a 3-dimensional model. Unfolding is also applied as a theoretical tool for the study of shortest paths on the surface of a polyhedron.

There are two main methods known to unfold convex polyhedra without overlap, both defined in terms of shortest paths on the polyhedron surface to a “source” point \( x \). A fundamental property of shortest paths on the surface of a convex polyhedron is that they unfold to straight-line segments. More generally, any geodesic (or locally shortest) path on the surface of a polyhedron unfolds to a straight-line segment.

The star unfolding is obtained by cutting a shortest path from every vertex of the polyhedron to the point \( x \). The cuts form a star at \( x \), hence the name. The source unfolding is obtained by cutting the cut locus (also known as the ridge tree), the locus of points that have more than one shortest path to \( x \). It is easy to see that the source unfolding does not overlap, because all the shortest paths from \( x \) to points on the surface unfold into straight lines radiating from \( x \). The star and source unfoldings are dual in the sense that the pieces of the surface delimited by the cut locus and the vertex-to-source shortest path cuts are joined at “opposite ends”: in the star unfolding the pieces are joined at the cut locus; and in the source unfolding the pieces are joined at the source point \( x \).

Alexandrov thought the star unfolding might overlap ([2] or see [5]); the surprising result that it does not was proved by Aronov and O’Rourke [3]. Their proof is by induction and involves combining two vertices into one in a kind of “Alexandrov surgery.” The proof is long, and Demaine and O’Rourke, in their book, “Geometric Folding Algorithms” [5] call the proof “not straightforward” and omit it.

Our first main result is a new proof of non-overlap of the star unfolding that is more straightforward. We do not modify the polyhedron or appeal to Alexandrov’s gluing theorem.

The star and source unfoldings can be generalized in a natural way by using a simple geodesic curve \( \lambda \) instead of a source point \( x \). For the source unfolding, we cut the cut locus (the locus of points that have more than one shortest path to the curve \( \lambda \)), and for the star unfolding we cut the curve \( \lambda \) itself and a shortest path from every vertex of the polyhedron to \( \lambda \). See Fig. 1. Such generalizations were first introduced by Itoh et al. [9] who proved non-overlap results for the case of closed curves (see also...
Fig. 1 An example star unfolding from a geodesic curve on a rectangular box. a The curve, drawn with a *heavy (red) line*, and shortest paths from the vertices to the curve, drawn with *thinner (blue) lines*. b The unfolding, with *faint dashed lines* indicating some of the original edges of the box.

[8, 11]). Demaine and Lubiw [4, Lem. 1] proved non-overlap of the source unfolding for (open) geodesic curves, and conjectured the same for the star unfolding.

Our second main result is a special case of this conjecture: we prove that the star unfolding from a geodesic curve unfolds without overlap if the curve is “balanced” (as defined in Sect. 3). The balance condition automatically holds for the point star unfolding.

We give two implications of our second result. The first is that the star unfolding does not overlap if the curve is a *geodesic loop*, meaning the curve endpoints $a$ and $b$ are (almost) coincident. In the limit when $a = b$ the unfolding consists of two pieces joined at a point. This gives an alternative to the result of Itoh et al. [9] that the star unfoldings of the inside and the outside of a geodesic loop do not overlap, and that the two pieces may be joined into one non-overlapping piece. Furthermore, their proof that the outside portion unfolds without overlap had a flaw and our result repairs it. We can extend this result (and our other results) from geodesics to *quasigeodesics*, to be defined below.

The second consequence of our result is that every star unfolding from a geodesic curve can be cut into two pieces such that each piece is non-overlapping. The extra cuts consist of shortest paths from a point on the cut locus to the curve.

To conclude this section we mention a few reasons to explore new unfoldings of convex polyhedra. One is to find “nicer” unfoldings. As the number of vertices of a polyhedron increases, the star unfolding from a point becomes very spiky with many sharp angles, for example see [1, Fig. 7]. By contrast, in the star unfolding from a geodesic curve many (or in some cases, all) vertices may have shortest paths to interior points of the curve, resulting in many $90^\circ$ angles and fewer sharp angles. New unfolding methods for convex polyhedra might also shed light on the case of non-convex polyhedra. Having a larger repertoire of unfoldings also opens the door to optimization, e.g., minimize the area of a bounding box of the unfolding, or minimize the total cut length, or maximize the minimum angle.

Geodesic star unfoldings may also have implications for the open question [6] of whether every convex polyhedron has a *general zipper unfolding*, a non-overlapping unfolding formed by cutting a single path on the polyhedron surface. If quasigeodesic star unfoldings do not overlap, then it would suffice to find a quasigeodesic curve that goes through all the vertices.

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1.1 Preliminaries and Definitions for the Point Star Unfolding

**Definition 1** Let $P$ be a convex polyhedron, and let $x$ be a point on $P$. The *star unfolding*, $S_x$, is a 2-dimensional polygon formed by cutting $P$ along a shortest path from every vertex of $P$ to $x$, and flattening the result into the plane. See Fig. 2. Note that there is a choice of cuts if a vertex has more than one shortest path to $x$.

If $P$ has $n$ vertices, then the polygon $S_x$ will have $2n$ vertices in general, and $2(n - 1)$ vertices if $x$ is located at a vertex of $P$. The vertices of $S_x$ alternate around the boundary between *vertex images*, denoted $v_i$, that correspond to the vertices of $P$, and $n$ ‘copies’ of $x$, called *source images* and denoted by $x_i$. See Fig. 2.

The edges of $S_x$ correspond to the shortest path cuts made from each vertex to $x$. Therefore, the two edges incident to any vertex image $v_i$ are always the same length.

The *cut locus* (or “ridge tree”), $T_x$, is the closure of the set of all points on the surface of $P$ that have more than one shortest path to $x$. It is known that the cut locus is a tree [12], and that its edges are shortest paths [1] and thus correspond to straight-line segments in $S_x$. See Fig. 2b. As a corollary to their proof of non-overlap [3, Thm. 10.2], Aronov and O’Rourke proved that the cut locus is a subset of the Voronoi diagram of the images of $x$, but we will not use this result.

Readers interested only in the point case may proceed directly to Sect. 2. In the next subsection we give definitions for the star unfolding from a geodesic curve.

1.2 Preliminaries and Definitions for the Geodesic Star Unfolding

Given a polyhedron $P$, a *geodesic curve* $\lambda$ on $P$ is a curve on the surface of $P$, such that at every interior point $p$ of $\lambda$, the surface angle to either side of $p$ is exactly $\pi$. If the surface angle to either side of $p$ is at most $\pi$ then the curve is called a *quasigeodesic*.
We will only consider [quasi]geodesic curves that are simple, meaning there is no point of self-intersection between any two interior (i.e., non-endpoint) points of the curve.

**Definition 2** Let $P$ be a convex polyhedron, and let $\lambda$ be a simple geodesic curve on the surface of $P$. The geodesic star unfolding $S_\lambda$ is a 2-dimensional polygon formed by cutting $\lambda$, and cutting a shortest path along the surface of $P$ from every vertex $v_i$ of $P$ to $\lambda$, and flattening the result into the plane. See ahead to Fig. 6.

The endpoints of $\lambda$ are labelled $a$ and $b$ and the two sides of $\lambda$ are labelled $s$ and $t$, with the convention that the clockwise order around $\lambda$ on the outside surface of $P$ is $a, s, b, t$. We distinguish shortest paths to $\lambda$ that arrive at (or “report to”) $a$ or $b$, and shortest paths that arrive at interior points of $\lambda$ on side $s$ or $t$. Any shortest path that arrives at an interior point of $\lambda$ forms a right angle with $\lambda$, as shown by the following lemma, which is a special case of [9, Lem. 1].

**Lemma 1** Let $\lambda$ be a [quasi]geodesic curve on a convex polyhedron $P$, with points $x_0$ on $\lambda$ and $x$ not on $\lambda$ such that $xx_0$ is a shortest path from $x$ to $\lambda$. The angles formed between $xx_0$ and $\lambda$ are at least $\frac{\pi}{2}$ to each side.

In the unfolding $S_\lambda$, a source image is either a copy of an endpoint ($a$ or $b$), called a point image, or a sub-segment of $\lambda$ corresponding to side $s$ or $t$, called a segment image. (Note that a segment image might include one of the endpoints of $\lambda$.) Around a clockwise traversal of the boundary of $S_\lambda$ we encounter source images in order $a, s, b, t$—this is because our convention is to draw the inside of the polyhedron’s surface facing up. Each consecutive pair of source images is separated by two edges of equal length joined at a vertex image, such that the edges correspond to the shortest path cut at the vertex.

The cut locus of a geodesic $\lambda$, denoted $T_\lambda$, is the closure of the set of all points on the surface of $P$ that have more than one shortest path to $\lambda$. The cut locus of a geodesic is a tree [13]. (An alternate proof can be found in [4, Lem. 4]. The lemma applies to the outside of any closed convex curve on the surface of $P$—in the present situation, the closed convex curve is the set of points at some fixed distance $\epsilon$ from $\lambda$.) See ahead to Fig. 6.

A key difference from the point case is that the edges of the geodesic cut locus are not necessarily straight-line segments. Every edge of the cut locus is the locus of points that are equidistant from two source images. Thus, when the two sources to either side of an edge are a point image and a segment image respectively, a parabolic cut locus edge will result. The cut locus edges between pairs of point images or pairs of segment images will still be straight.

2 New Proof for the Point Star Unfolding

In this section we give a new proof of non-overlap for the point star unfolding that is simpler than the original proof of Aronov and O’Rourke [3].

The most important idea of our proof is to partition the star unfolding into pairs of congruent triangles. Each vertex of the cut locus has three or more shortest paths to $x$.
on the surface of \( P \). Add all these shortest paths as line segments in the star unfolding. We use the following basic properties of shortest paths: (1) no two shortest paths to \( x \) (or more generally, to a geodesic curve) cross each other [9, Lem. 2] (their lemma is stated for vertices of the polyhedron, but applies more generally); and (2) no shortest path crosses the cut locus [12, Lem. 4.2]. These properties imply that when we add the line segments corresponding to shortest paths between \( x \) and cut locus vertices, each edge of the cut locus will have a triangle to each side. Because all shortest paths from a cut locus vertex to the nearest source images have the same length, the two triangles to either side of a cut locus edge have corresponding sides of equal length, and therefore are congruent. We call each such pair of triangles a \textit{kite}. The kite associated with cut locus edge \( e \) is denoted \( \text{kite}(e) \). The two images of \( x \) to each side of \( e \) are called the \textit{apex vertices} of the kite. Observe that the kites form a partition of \( S_x \). See Fig. 2b.

We define the \textit{source angle} of \( e \), \( \alpha(e) \), to be the interior angle at either apex of \( \text{kite}(e) \). See Fig. 2b. We extend this definition to paths as follows. For a path \( \sigma \) of cut locus edges, where \( \sigma = e_1,\ldots,e_t \), the source angle of \( \sigma \) is \( \alpha(\sigma) = \sum_{i=1}^{t} \alpha(e_i) \). Observe that \( \alpha(\sigma) \leq \pi \) because \( 2\alpha(\sigma) \) measures the total source angle at both apex vertices of every kite on the path, so this is bounded by the total surface angle at the source point \( x \), which is bounded by \( 2\pi \).

**Theorem 1** [3, Thm. 9.1] The star unfolding \( S_x \) does not overlap.

**Proof** We will show that no two kites overlap. Consider two kites, and the path in the cut locus between them. Let \( e_1,\ldots,e_t \) be the edges of the path in the cut locus with \( e_j \) joining cut locus vertices \( p_{i-1} \) and \( p_i \). Let \( k_j = \text{kite}(e_j) \). We will show that \( k_1 \) and \( k_t \) have disjoint interiors. We will define a sequence of regions \( W_1,\ldots,W_{t-1} \), called W-wedges, so that \( W_i \) includes \( k_1,\ldots,k_i \) and excludes \( k_{i+1} \). Then \( W_{t-1} \) includes \( k_1 \) and excludes \( k_t \), which will complete the proof.

The boundary of the W-wedge \( W_i \) consists of four edges, generally shaped like a ‘W’ and defined as follows. The \textit{center point} of \( W_i \) is the point \( p_i \), where \( e_i \) and \( e_{i+1} \) intersect; the \textit{inner legs} are the edges of \( k_i \) that are incident to \( p_i \); and the \textit{outer legs} form an angle with the inner legs (on the side of \( k_i \)) of \( \frac{\pi}{2} + \alpha_i \), where \( \alpha_i = \sum_{j=1}^{i} \alpha(e_j) \). The outer legs extend either to their point of intersection, or as infinite rays if they do not intersect. This boundary divides the plane into two regions and we define \( W_i \) to be the region containing \( k_i \) in a neighbourhood of \( p_i \). We also define \( W_0 \) to have center point \( p_0 \), with inner legs along the edges of \( k_1 \) and outer legs rotated outward (on the opposite side from \( k_1 \)) by \( \frac{\pi}{2} \). See Fig. 4. Note that \( \alpha_i \) is in the range \( [0, \pi] \) as observed above.

We will prove by induction that \( k_i \) is outside \( W_{i-1} \) and that \( W_i \) contains \( k_i \cup W_{i-1} \). At each step, including the base case, we will need the following:

**Lemma 2** Let \( p \) be an endpoint of cut locus edge \( e \), and let \( W \) be a W-wedge centered at \( p \) such that the inner legs of \( W \) are edges of \( \text{kite}(e) \), and the outer legs are rotated out (on the side of \( \text{kite}(e) \)) by an angle \( \theta \) in the range \( [\frac{\pi}{2} + \alpha(e), \frac{3\pi}{2}] \). Then \( \text{kite}(e) \subseteq W \).

**Proof** Consider the two circular sectors \( a_p \) and \( a_q \) centered at the endpoints \( p \) and \( q \) of \( e \) and bounded by the two incident kite edges as radii (see Fig. 3). At each apex of
Fig. 3 Illustration for the proof of Lemma 2. Angle $\theta$ is in the range $[\frac{\pi}{2} + \alpha(e), \frac{3\pi}{2}]$ and thus the outer leg of $W$ lies between the tangents (dashed) to the circular arcs $a_p$ and $a_q$.

Fig. 4 Kite $k_{i-1}$ (shaded) and the corresponding W-wedge $W_{i-1}$ (lightly shaded). In the base case, kite $k_1$ is outside the W-wedge $W_0$ (very lightly shaded), and $W_1$ contains $k_1 \cup W_0$. To prove by induction that $W_i$ contains all previous kites, we expand $W_{i-1}$, first rotating the legs about $p_{i-1}$ to the W-wedge $W'_i$ (dashed line), and then moving the center point to $p_i$ to obtain $W_i$. Note that although the figure shows kites $k_{i-1}$ and $k_i$ sharing only a vertex, in non-degenerate situations cut locus vertices have degree 3, and two consecutive kites will share an edge as well.

We are now ready to prove by induction that $k_i$ is outside $W_{i-1}$ and that $W_i$ contains $k_i \cup W_{i-1}$. For the base case $i = 1$, $k_1$ is outside $W_0$ by the lemma above (applied to the outside of $W_0$), and $W_1$ contains $k_1 \cup W_0$ by the same lemma.

For the induction step we consider $i > 1$. Suppose by induction that $W_{i-1}$ contains $k_{i-1} \cup W_{i-2}$. We will show how to transform $W_{i-1}$ into $W_i$ in a way that makes it clear that $k_i$ is outside $W_{i-1}$ and $W_i$ contains $k_i \cup W_{i-1}$.
Note that the unfolding does not overlap in the neighbourhood of point $p_i$—this is true whether $p_i$ is a point with $2\pi$ surface area, or a vertex, which will be incident to a single cut. Thus the kites $k_i$ and $k_{i-1}$, which are both incident to $p_i$, do not overlap. Rotate each of the inner legs of $W_{i-1}$ about point $p_{i-1}$, away from $k_{i-1}$ to the edge of $k_i$. This is a rotation by $0^\circ$ when an edge of $k_{i-1}$ is coincident with an edge of $k_i$. Keep the angle between inner and outer legs fixed throughout the rotation. Observe that all the kite edges incident to $p_{i-1}$ have the same length, so we really perform a rigid transformation on each half of the W. Call the resulting W-wedge $W'_i$ (shown as a dashed poly-line in Fig. 4). Notice that $W'_i$ contains $W_{i-1}$, because the angle $\alpha_{i-1} + \frac{\pi}{2}$ is in the range $[\frac{\pi}{2}, \frac{3\pi}{2}]$ so the outer legs remain outside the rotation sector of the inner legs. (Here it is crucial that we added the extra $\frac{\pi}{2}$ to the initial angle.) That $k_i$ is outside $W'_i$ follows from applying Lemma 2 to the outside of $W'_i$, noting that $\alpha_{i-1} + \alpha(e_i) \leq \pi$ so the angle $\alpha_{i-1} + \frac{\pi}{2}$ is in the range $[\frac{\pi}{2}, \frac{3\pi}{2} - \alpha(e_i)]$ and therefore the complementary angle is in the range $[\alpha(e_i) + \frac{\pi}{2}, \frac{3\pi}{2}]$.

The second step of the transformation is to move the center point of $W'_i$ from $p_{i-1}$ to $p_i$, while keeping the outer legs fixed. The W-wedge increases until it contains $k_i$. The angle between inner and outer legs increases by $\alpha(e_i)$, to $\alpha_i$. Thus the result is precisely $W_i$, and therefore $W_i$ contains $k_i \cup W_{i-1}$.

Our proof, like Aronov and O’Rourke’s, shows a stronger result that certain regions outside the star unfolding are empty. Aronov and O’Rourke [3] showed that at any vertex $v_i$ adjacent to source images $x_i$ and $x_{i+1}$ in the unfolding, no part of the unfolding enters the circular sector centered at $v_i$ exterior to the unfolding near $v_i$ and bounded by the radii $v_ix_i$ and $v_ix_{i+1}$. See Fig. 5. Our proof shows that larger regions are empty, as formalized in the following lemma.

**Lemma 3** Let $v_i$ be a vertex image adjacent to source images $x_i$ and $x_{i+1}$ on the boundary of a star unfolding $S_x$, and let $W$ be the region bounded by $v_ix_i$, $v_ix_{i+1}$, and the rays extending from $x_i$ and $x_{i+1}$ at right angles from these segments on the exterior side of $v_i$. Then no part of the unfolding intersects the interior of $W$.  

**Proof** In non-degenerate situations, $v_i$ is a leaf of the cut locus. Let $e$ be the cut locus edge incident to $v_i$. For any kite $k = kite(f)$ of the unfolding, consider the path in the cut locus starting with $e$ and ending with $f$, and apply the above proof to this path. Wedge $W$ in the statement of the lemma is wedge $W_0$ in the proof, and the proof shows that kite $k$ does not enter wedge $W_0$.

More generally, $v_i$ may be an internal vertex of the cut locus with more than one incident cut locus edge, and we need a slight generalization of this argument. Consider
any kite $k = \text{kite}(f)$ of the unfolding, and consider the path in the cut locus starting with $v_i$ and ending with $f$. Apply the above proof to this path. The path will use one of the cut locus edges incident to $v_i$. The crucial observation is that wedge $W$ in the statement of the lemma is a subset of wedge $W_0$ in the proof for this path. The proof shows that kite $k$ does not enter wedge $W_0$, and therefore $k$ does not enter $W$. □

### 3 Geodesic Star Unfolding

In this section we will consider the star unfolding from a geodesic curve $\lambda$. By generalizing the proof for the point case, we will establish some situations in which the geodesic star unfolding does not overlap. See Fig. 6.

Recall the definitions from Sect. 1.2. In particular recall that in the unfolding a source image is either a copy of an endpoint ($a$ or $b$), called a point image, or a subsegment of $\lambda$ corresponding to side $s$ or $t$, called a segment image. A segment image might include one of the endpoints of $\lambda$.

As for the point-source case, we will partition the unfolding by adding, for every cut locus vertex $p$, the line segments that correspond to shortest paths from $p$ to the curve $\lambda$. See Fig. 6b. We need a few more line segments in order to deal properly with segment images that include an endpoint of $\lambda$. If $p$ is a point of the cut locus that has a shortest path to an endpoint of $\lambda$ that is included in a segment image, then we will regard $p$ as a cut locus vertex. This means that we will add the line segments that correspond to shortest paths from $p$ to the curve $\lambda$. (Such an example can be found vertically below the leftmost image of $a$ in the figure.)

![Fig. 6](image-url)

Fig. 6 The star unfolding from a geodesic curve. a The polyhedron, the geodesic curve with endpoints $a$ and $b$ and sides $s$ and $t$, and shortest paths from vertices to the curve. (Shortest paths reach interior points of the geodesic curve at right angles, although perspective effects obscure this.) b The corresponding star unfolding with the source images, $a$, $b$, $s$ or $t$, indicated. Images of $a$ [$b$] are drawn as squares [triangles]; segment images of $s$ [$t$] are drawn as heavy dashed [dotted] lines. These images occur in clockwise cyclic order $a$, $s$, $b$, $t$, as indicated by the outer curves (long dashes). Shortest path cuts are coloured according to their destination type. The cut locus is shown in grey. Wing boundaries are indicated by straight dashed segments from the cut locus to the source images. Three wing-pairs are shaded; the darkly shaded one is a hybrid wing-pair with a parabolic cut locus edge. Note that the unfolding shows the inside surface of the polyhedron.
The added line segments partition the unfolding into regions called wings. To justify this, we need two properties of shortest paths to a geodesic curve $\lambda$: (1) no two shortest paths to $\lambda$ cross each other [9, Lem. 2] (their lemma is stated for vertices of the polyhedron, but applies more generally); and (2) no shortest path crosses the cut locus [12, Lem. 4.2] (their result is stated for a point source but the proof carries over). The two wings on either side of a cut locus edge form a wing-pair. All the points inside a wing have shortest paths to $\lambda$ that reach the same source image. A wing with an endpoint source image is a point wing and a wing with a segment source image is a segment wing. A wing-pair may involve two point wings (these are the kites of the previous section), or two segments wings, or one of each, in which case we call it a hybrid wing-pair. The cut locus edge of a hybrid wing-pair is a parabolic segment; all others are straight-line segments.

The source angle of a point wing is the angle at its source image point; the source angle of a segment wing is 0. The source angle of a wing-pair is the sum of the source angles of the two wings. If $e$ is an edge of the cut locus, and $A$ designates one side of this edge, then $\alpha^A(e)$ denotes the source angle of the wing on that side. If $\sigma = e_1, \ldots, e_t$ is a path in the cut locus, with its two sides (arbitrarily) labelled $A$ and $B$, then the source angle of $\sigma$ on side $A$ is $\alpha^A(\sigma) = \sum_{i=1}^t \alpha^A(e_i)$ (and similarly for $B$). The path $\sigma$ is balanced if $\alpha^A(\sigma) \leq \pi$ and $\alpha^B(\sigma) \leq \pi$. We say that the geodesic curve (and the corresponding cut locus) are balanced if every path in the cut locus is balanced. There are examples where the cut locus is not balanced, and in fact it is possible to have all $2\pi$ of source angle to one side of a cut locus path, see Fig. 14.

Our main result in this section is that wing-pairs along a balanced path do not overlap.

**Lemma 4** Let $P$ be a convex polyhedron with a geodesic curve $\lambda$ on its surface. Suppose that $u$ and $v$ are distinct edges of the cut locus such that the path in the cut locus from $u$ to $v$ is balanced. Then the wing-pairs of $u$ and $v$ do not overlap.

Before proceeding with the proof, we note the consequence that the star unfolding from a balanced geodesic curve does not overlap:

**Corollary 1** If $P$ is a convex polyhedron with a balanced geodesic curve $\lambda$ then the geodesic star unfolding from $\lambda$ does not overlap.

**Proof** (Proof of Lemma) We follow the same plan as in the proof of Theorem 1, that is, we will prove that no two wing-pairs in the unfolding overlap, by examining W-wedges along the cut locus path between the two wing-pairs. A quick summary is that there are only two differences in the argument: (1) as we move from W-wedge $W_{i-1}$ to $W_i$ we may increase the angle between inner and outer legs differently on its two sides; (2) when the W-wedge moves past a segment wing, the angle between inner and outer legs does not change, and the inner+outer pair translates. See Fig. 7. We now give the details.

Let $\sigma$ be the path from $u$ to $v$ in the cut locus, with edges $u = e_1, \ldots, e_t = v$. Let $k_i$ be the wing-pair of edge $e_i$. Label the two sides of $\sigma$ by $A$ and $B$. Let $\alpha^A_i = \sum_{j=1}^i \alpha^A(e_j)$ and let $\alpha^B_i = \sum_{j=1}^i \alpha^B(e_j)$. Note that $\alpha^A_i$ and $\alpha^B_i$ are in the range $[0, \pi]$ by the assumption that the path from $u$ to $v$ is balanced. This is the only place where we use the assumption about being balanced.
Fig. 7 Wing-pair $k_i$ (shaded) is a hybrid wing-pair. $W_1$ contains $k_1$ because it contains the circular sector $a$. To prove by induction that $W_i$ contains all previous wing-pairs, we expand $W_{i-1}$ to $W_i$, first rotating the legs about $p_{i-1}$ to the $W$-wedge $W'_i$ (dashed line), and then expanding to include $k_i$. Note that in this example $\alpha_{B_{i-1}} = \alpha_{B_i}$ since the wing to that side is a segment-wing.

We will show that $k_1$ and $k_t$ have disjoint interiors by defining $W$-wedges $W_i$ so that $W_i$ includes $k_1, \ldots, k_i$ and excludes $k_{i+1}$. Then $W_{t-1}$ includes $k_1$ and excludes $k_t$, which will complete the proof. Define $W_i$, for $i = 1, \ldots, t$, to be the $W$-wedge with center point at $p_i$ where $e_i$ and $e_{i+1}$ intersect, with inner legs along the two incident edges of $k_i$, and outer legs rotated out (on the side of $k_i$) by $\alpha_i^A + \frac{\pi}{2}$ on side $A$, and by $\alpha_i^B + \frac{\pi}{2}$ on side $B$. The outer legs extend either to their point of intersection, or as infinite rays if they do not intersect. This boundary divides the plane into two regions and we define $W_i$ to be the region containing $k_i$. We also define $W_0$ to have center point $p_0$ at the beginning of edge $e_1$, with inner legs along the edges of $k_1$ and outer legs rotated outward (on the opposite side from $k_1$) by $\frac{\pi}{2}$. See Fig. 7.

We will prove by induction that $k_i$ is outside $W_{i-1}$ and that $W_i$ contains $k_i \cup W_{i-1}$. At each step, including the base case, we will need the following:

**Lemma 5** Let $p$ be an endpoint of cut locus edge $e$, and let $W$ be a $W$-wedge centered at $p$ such that the inner legs of $W$ are edges of the wings of $e$, and the outer leg on the $A$ side is rotated out by an angle in the range $[\frac{\pi}{2} + \alpha^A(e), \frac{3\pi}{2}]$ and the outer leg on the $B$ side is rotated out by an angle in the range $[\frac{\pi}{2} + \alpha^B(e), \frac{3\pi}{2}]$. Then the wing-pair of $e$ is contained in $W$.

**Proof** Consider the two circular sectors centered at the endpoints $p$ and $q$ of $e$ and bounded by the two incident wing edges as radii (see for example the circular sector marked $a$ centered at $q = p_0$ on wing-pair $k_1$ in Fig. 7). On both the $A$ side and the $B$ side, the angles between the outer leg of $W$ and the sides of the wing of $e$ are at least $\frac{\pi}{2}$. Thus the circular sector at $q$ is inside $W$, and the circular sector at $p$ is outside $W$. This implies that the wing-pair of $e$ is contained in $W$. $\square$
Thus the result is precisely \( \alpha \). Observe that \( W_i \) contains \( k_i \cup W_{i-1} \). For the base case \( i = 1 \), \( k_1 \) is outside \( W_0 \) and \( W_1 \) contains \( k_1 \cup W_0 \) by the lemma above. Suppose by induction that \( W_{i-1} \) contains \( k_{i-1} \cup W_{i-2} \). We will show how to transform \( W_{i-1} \) into \( W_i \) in a way that makes it clear that \( k_i \) is outside \( W_{i-1} \) and \( W_i \) contains \( k_i \cup W_{i-1} \).

Since there is at most 2\( \pi \) surface angle at any point on the surface, the unfolding does not overlap in the neighbourhood of any point. Thus \( k_{i-1} \) and \( k_i \) are disjoint. Rotate the two inner legs of \( W_{i-1} \) about point \( p_{i-1} \), away from \( k_{i-1} \) to the edges of \( k_i \). This is a rotation by \( 0^\circ \) when an edge of \( k_{i-1} \) is coincident with an edge of \( k_i \). Keep the angle between inner and outer legs fixed throughout the rotation. Observe that all the wing edges incident to \( p_{i-1} \) have the same length, so we really perform a rigid transformation on each half of the \( W \). Call the resulting \( W \)-wedge \( W_i' \) (shown as a dashed poly-line in Fig. 7). Notice that \( W_i' \) contains \( W_{i-1} \), because the angles \( \alpha_i^{A-1} + \frac{\pi}{2} \) and \( \alpha_i^{B-1} + \frac{\pi}{2} \) are in the range \( [\frac{\pi}{2}, \frac{3\pi}{2}] \) so the outer legs remain outside the rotation sector of the inner legs. We prove that \( k_i \) is outside \( W_i' \) by applying Lemma 5 to the outside of \( W_i' \). To show that the angle on the \( A \) side is in the required range, observe that \( \alpha_i^{A-1} + \alpha_i^A(e_i) \leq \pi \) so the angle \( \alpha_i^A \) is in the range \( [\frac{\pi}{2}, \frac{3\pi}{2} - \alpha_i^A(e_i)] \) and therefore the complementary angle is in the range \( [\alpha_i^A(e_i) + \frac{\pi}{2}, \frac{3\pi}{2}] \). A similar argument applies on the \( B \) side.

The second step of the transformation is to move \( W_i' \) past \( k_i \) to \( W_i \). We do this separately on the \( A \) side and the \( B \) side. To go past a point wing, we move the inner leg from \( p_{j-1} \) to \( p_j \), while keeping the outer leg fixed. To go past a segment wing, we translate the inner + outer legs so that their common point moves along the segment image; since the segment image is perpendicular to the inner leg, each leg is parallel to its initial version. The resulting \( W \)-wedge contains \( k_i \). On the \( A \) side, the angle between inner and outer legs increases by \( \alpha_i^A \), to \( \alpha_i^A \), and similarly on the \( B \) side. Thus the result is precisely \( W_i \), and therefore \( W_i \) contains \( k_i \cup W_{i-1} \).

### 3.1 Extension to Quasigeodesic Curves

We now show that our geodesic star unfolding result (Lemma 8) carries over to quasigeodesic curves. Recall that a quasigeodesic curve on the surface of a polyhedron \( P \) is a curve such that at each interior point along the curve the surface angle to each side is \( \leq \pi \). So, unlike geodesics, a quasigeodesic can pass through vertices, the only points at which one of the angles will be \( < \pi \) to one or both sides. Consider a quasigeodesic curve \( \lambda \) on the surface of \( P \). We define the quasigeodesic star unfolding in the same way as the geodesic star unfolding, specifically, we cut the curve and we cut a shortest path from every vertex to the curve. See Fig. 8.

To argue about the cut locus of \( \lambda \), we consider the closed curve consisting of the points at some small fixed distance \( \varepsilon \) from \( \lambda \). This curve is composed of circular arcs and geodesic segments joined at angles \( \leq \pi \) (on the side opposite \( \lambda \)). Thus by [4, Lem. 4], its cut locus is a tree. When \( \varepsilon = 0 \), we obtain the cut locus of \( \lambda \) itself. The cut locus of \( \lambda \) may fail to be a tree—specifically, when \( \lambda \) has an interior point \( v \) with surface angle \( < \pi \) on both sides of the curve (e.g., vertex \( v \) in Fig. 8). In this case an edge of the cut locus reaches \( v \) from each side, and a cycle is formed in the cut locus.
This does not harm our analysis however, because we split any such vertex into two copies, corresponding to the $s$ and $t$ sides of the curve, which breaks the cycle in the unfolded cut locus.

Suppose $p$ is an interior point of the quasi-geodesic curve $\lambda$ where the surface angle to one side, say the $s$ side, is $\beta$, where $\beta < \pi$. Necessarily, $p$ is a vertex of the polyhedron, otherwise the surface angle on the other side of the curve would be greater than $\pi$. We do not introduce a cut for this vertex to define the unfolding, since the vertex already lies on $\lambda$.

By Lemma 1, no shortest path cut from any vertex $v$ will report to point $p$ on side $s$, since one of the two surface angles formed would be $\frac{\beta}{2}$. Thus the quasi-geodesic star unfolding from $\lambda$ will have a vertex image with an angle $\beta$ corresponding to the $s$ side of $p$. Let $p_s$ denote this vertex image in the unfolding.

Observe that $p_s$ is a leaf of the unfolded cut locus and that the incident cut locus edge $e$ is a straight segment forming angles $\frac{\beta}{2}$ with the segment images to either side of $p_s$. We can consider $e$ to have two segment wings, albeit degenerate ones, with one of the parallel sides (between $p_s$ and $\lambda$) of length 0. We call this a quasi-wing-pair (see the shaded examples in Fig. 8).

With these observations in hand, we can extend the result of the previous section to quasi-geodesics.

**Lemma 6** Let $P$ be a convex polyhedron with a quasi-geodesic curve $\lambda$ on its surface. Suppose that $u$ and $v$ are distinct edges of the cut locus such that the path in the unfolded cut locus from $u$ to $v$ is balanced. Then the (quasi-)wing-pairs of $u$ and $v$ do not overlap.

**Proof** As noted above, the only difference when we have a quasi-geodesic curve rather than a geodesic curve is the presence of quasi-wing-pairs. A quasi-wing pair can only occur on a cut locus edge incident to a leaf of the unfolded cut locus that corresponds to a vertex on the quasi-geodesic curve. Thus quasi-wing-pairs can only occur as the first or last wing-pairs along a cut locus path.

We follow the induction proof of Lemma 4. The base case of the induction works for a quasi-wing pair—in fact, the W-wedge $W_0$ is empty and $W_1$ is precisely the
 quasi-wing pair. During the induction we never need to grow a W-wedge past a quasi-wing pair because quasi-wing pairs only occur as the first or last wing-pair of a path. Thus the proof carries over.

3.2 Quasigeodesic Loops

In this section, we prove non-overlap of the star unfolding from geodesic (and quasigeodesic) loops. When the two endpoints \(a\) and \(b\) of a geodesic or quasigeodesic curve \(\lambda\) coincide at point \(o\), we call this a [quasi]geodesic loop with loop point \(o\). A [quasi]geodesic loop cuts the surface of the polyhedron into two pieces. One piece must have a surface angle at \(o\) that is \(\leq \pi\), and we call this the inside of the loop and identify it with the \(s\) side of the curve. The other piece is called the outside and will be identified with the \(t\) side of the curve.

Itoh, O’Rourke, and Vîlcu [9] proved that for any quasigeodesic loop: (1) the inside unfolds without overlap; (2) the outside unfolds without overlap; and (3) the two unfolded pieces can be reattached (without overlap) along a common segment of the cut curve. Their proof of (2) relies on a Lemma [9, Lem. 7] about the star unfolding from a point, which they say will be proved in a companion paper, but unfortunately, they discovered\(^1\) that the Lemma is false. The Lemma claims that, for any star unfolding from a point \(x\) and for any polyhedron vertex \(v\), the exterior angle at \(v\) in the unfolding yields an empty wedge. More precisely, if the exterior angle at \(v\) is (counterclockwise) \(x_i, v, x_{i+1}\) then the claim is that the counterclockwise wedge formed by the rays \(v x_i\) and \(v x_{i+1}\) does not contain any part of the unfolding. An example where this fails is shown in Fig. 9.

In this section we examine the star unfolding from a quasigeodesic curve where the two endpoints, \(a\) and \(b\), of the curve are arbitrarily close together. In the limit when \(a = b\) the unfolding consists of two pieces joined at the point \(a = b\) with the angular bisectors at the point \(a = b\) aligned in the unfolding. We call this the conjoined star unfolding from a quasigeodesic loop. See Fig. 10. Our main result of this section is that the conjoined star unfolding from a quasigeodesic loop does not overlap. In particular,

\[^1\] Private communication from J. O’Rourke
this implies that the outside piece unfolds without overlap, which repairs the missing step of Itoh, O’Rourke and Vîlcu’s result.

**Theorem 2**  The conjoined star unfolding from a quasigeodesic loop does not overlap.

We prove Theorem 2 by showing that every path through the cut locus for a geodesic loop is balanced. Then by Lemma 4, the unfolding does not overlap. Furthermore, by Lemma 6, the result extends to quasigeodesics.

**Lemma 7**  Every path through the unfolded cut locus of the conjoined star unfolding from a quasigeodesic loop is balanced.

**Proof**  Recall our assumption that the $s$-side of the curve is inside the loop. By Lemma 1, no vertex inside the loop can report to the loop point $o$, since the surface angle to the interior of the loop is $< \pi$. (In case the angle is equal to $\pi$ and a vertex inside the loop reports to the loop point $o$, we can model this as if the vertex reported to $b$ instead.)

Consider the segment of the cut locus (call it $u$) that lies between $a$ and $b$ on the inside of the loop and touches loop point $o$. Observe what happens in the limit as $a$ and $b$ approach point $o$. Consider the kite-shaped region of the surface delimited by $a$, $o$, $b$, and $p$, where $p$ is the intersection between the rays perpendicular to the geodesic at $a$ and $b$ respectively (see Fig. 11). Call this region $\mathcal{R}$. When $a$ and $b$ are close enough to $o$ there are no vertices or other cut locus edges inside the region $\mathcal{R}$, and therefore some
sub-segment of \( u \) will extend from \( p \) to \( o \) (i.e., perpendicularly bisect \( \overline{ab} \)). Therefore the source angle of each wing of \( u \) is at least \( \frac{\pi}{2} \), and the source angle of \( u \)'s wing-pair is at least \( \pi \). Furthermore, this is the only edge of the cut locus on the inside of the loop that has point-wings reporting to \( a \) or \( b \).

Consider any path \( \sigma \) through the unfolded cut locus. Let \( A \) and \( B \) be the sides of \( \sigma \). We must show that \( \alpha^A(\sigma) \leq \pi \) and \( \alpha^B(\sigma) \leq \pi \). There are three possible cases for the path:

- The path \( \sigma \) does not include \( u \), and remains entirely on the inside of the loop. Because \( u \) is the only cut locus edge inside the loop whose wings have non-zero source angle, therefore \( \alpha^A(\sigma) = \alpha^B(\sigma) = 0 \).
- The path \( \sigma \) does not include \( u \), and remains entirely on the outside of the loop. As noted above, the remaining source angle of all wings along every other possible path (i.e., not including \( u \)) must be \( \leq \pi \). Thus \( \alpha^A(\sigma) \leq \pi \) and \( \alpha^B(\sigma) \leq \pi \).
- The path \( \sigma \) includes \( u \). We must show that \( \alpha^A(\sigma) \leq \pi \) and \( \alpha^B(\sigma) \leq \pi \). Any edge of \( \sigma \) (apart from \( u \)) that lies inside the geodesic loop only has segment wings to either side and these contribute 0 to the source angle of the path. Thus it suffices to look at the portion of \( \sigma \) starting with \( u \) and then following cut locus edges that lie outside the geodesic loop. Call this subpath \( \sigma' \). Cut locus edge \( u \) has a point wing to either side, one reporting to \( a \) and one reporting to \( b \). Suppose that the \( A \) side of the path has the point wing reporting to \( a \). If \( \sigma \) only has point wings that report to \( a \), then its source angle is at most \( \pi \). So suppose that side \( A \) has a point wing that reports to \( b \). See Fig. 12. As we walk around the path \( \sigma' \), in counterclockwise order starting with \( u \) on the \( A \) side, the wings report in order to \( a \), then \( t \), then \( b \). Thus all the wings on the \( B \) side must be point wings that report to \( b \), which implies that the sum of their source angles is at most \( \pi \), i.e., \( \alpha^B(\sigma') \leq \pi \). Since every wing on the \( B \) side is a point-wing reporting to \( b \), every point wing on the \( A \) side must be paired with a point wing on the \( B \) side, and each such pair has equal source angles. Therefore \( \alpha^A(\sigma') \leq \alpha^B(\sigma') \), and we just showed that \( \alpha^B(\sigma') \) is \( \leq \pi \).

\( \square \)

3.3 The Geodesic Star Unfolding as Two Non-overlapping Pieces

Although we have not proved that the geodesic star unfolding never overlaps, in this section we show that it can always be cut into two pieces each of which is non-
overlapping. The extra cuts consist of two shortest paths from a point on the cut locus to the geodesic curve.

**Lemma 8** Let $\mathcal{P}$ be a convex polyhedron and $\lambda$ be a geodesic curve on $\mathcal{P}$. Then there is a point $p$ of the cut locus of $\lambda$ such that cutting two shortest paths on $\mathcal{P}$ from $p$ to the geodesic curve $\lambda$ separates the geodesic star unfolding $S_{\lambda}$ into two pieces each of which is non-overlapping.

**Proof** We will split the cut locus into two subtrees, either at an internal point of an edge or at a cut locus vertex $v$, in which case we split the cut locus edges incident to $v$ into two subsets, each consecutive in the cyclic order of edges around $v$. Call such subtrees *proper*. The geodesic star unfolding $S_{\lambda}$ can then be cut into two pieces as follows: if the two subtrees are joined at an internal point $p$ of a cut locus edge, then we cut the two shortest paths from $p$ to the geodesic curve $\lambda$; and if the two subtrees are joined at vertex $v$ then we cut the two shortest paths from $v$ to $\lambda$ that separate the incident cut locus edges as specified. This ensures that if two wing-pairs are in the same piece of the unfolding, then the cut locus path between them lies in the same subtree. So long as each subtree is balanced, Lemma 4 ensures that no two wing-pairs from the same piece overlap, i.e., that each piece is non-overlapping.

In the remainder of the proof we show how to partition the cut locus into two proper balanced subtrees. Each edge $e$ of the cut locus has an associated source angle of its wing pair, $\sigma(e)$, and the sum of these weights over the whole cut locus is at most $2\pi$. We remark that a weaker form of the lemma with three non-overlapping pieces can be obtained from the result that any edge-weighted tree can be separated at a vertex into three subtrees each of weight at most one half the total weight. To prove the stronger result claimed in the lemma we will rely on properties of the source angles on each side of each cut locus edge, and we will use the freedom to split into subtrees at an internal point of an edge.

Among all proper subtrees of the cut locus, let $T$ be a maximal subtree that is balanced. Let $\overline{T}$ be the complement. We claim that $\overline{T}$ is balanced. Root $T$ at the point it shares with $\overline{T}$.

If the source angle of $T$ is at least $\pi$, then $\overline{T}$ has source angle $\leq \pi$ so it must be balanced as well. Otherwise the source angle of $T$ is $< \pi$. Note that $T$ cannot be rooted at an interior point of an edge otherwise we could move the point further along the edge to increase the source angle of $T$ a small amount without exceeding $\pi$. Therefore $T$ must be rooted at a vertex $v$ of the cut locus. Among the edges incident to $v$ in clockwise order, let $e$ and $f$ be the first and last edges outside $T$. Note that $e \neq f$ (i.e., there is more than one edge incident to $v$ in $\overline{T}$) otherwise we could move the root of $T$ along $e$ to increase the source angle by a small amount. Adding $e$ and its subtree to $T$ gives an unbalanced subtree, so there must be an unbalanced path $\mu_e$ that contains $e$ and a subpath in $T$. Similarly, there must be an unbalanced path $\mu_f$ that contains $f$ and a subpath in $T$.

Note that any two unbalanced path-sides must have a wing in common, otherwise we would have two disjoint sets of wings each with source angle greater than $\pi$. Thus the unbalanced sides of $\mu_e$ and $\mu_f$ must both lie on the clockwise side of $e$ and $f$ or both on the counterclockwise side of $e$ and $f$ (relative to the cyclic ordering of edges at $v$). Suppose the former, without loss of generality. See Fig. 13.
Suppose that \( \overline{T} \) has an unbalanced path \( \mu \). The unbalanced side must share wings with the unbalanced side of \( \mu_e \) and with the unbalanced side of \( \mu_f \), and therefore must include the clockwise sides of both \( e \) and \( f \), which is impossible. Therefore \( \overline{T} \) is balanced, and we can separate the geodesic star unfolding \( S_\lambda \) into two pieces each of which is non-overlapping.

\[ \square \]

### 4 Conclusions

We have given a simple proof that the star unfolding from a point does not overlap, and extended it to some cases of the star unfolding from a geodesic curve. The new idea we use in the proofs is to decompose the unfolding into pieces (kites or wing-pairs) attached to the cut locus and argue that no two pieces overlap in the unfolding because when we walk along the path of the cut locus between the two pieces we can maintain a region (a W-wedge) that contains one piece and not the other.

We leave open the main conjecture that the geodesic star unfolding does not overlap. The most we can say about the general case is that the unfolding can be partitioned into two non-overlapping pieces.
The first author’s thesis [10] contains further results on geodesics that have been “fully extended,” meaning that we take a geodesic segment and extend it in both directions until the endpoints intersect the curve. When the endpoints reach opposite sides of the curve (“S-shaped”) the unfolding does not overlap because it is balanced. When the endpoints reach the same side of the curve (“C-shaped”) the unfolding need not be balanced (see Fig. 14), though some special cases can still be proved to avoid overlap.

The figures in this paper were generated with a custom program written using CGAL, OpenGL, and Cairo. For more information, see the first author’s thesis [10].

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