Using Ramsey’s theorem once

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Abstract
We show that $\text{RT}(2, 4)$ cannot be proved with one typical application of $\text{RT}(2, 2)$ in an intuitionistic extension of $\text{RCA}_0$ to higher types, but that this does not remain true when the law of the excluded middle is added. The argument uses Kohlenbach’s axiomatization of higher order reverse mathematics, results related to modified reducibility, and a formalization of Weihrauch reducibility.

Keywords Ramsey · Weihrauch · Uniform reduction · Higher order · Reverse mathematics · Proof mining

Mathematics Subject Classification 03B30 · 03F35 · 03F50 · 03D30 · 03F60

1 Introduction

One of the questions motivating the exploration of uniform reductions in the article of Dorais et al. [6] was: is it possible to prove Ramsey’s theorem for pairs and four colors from a single use of Ramsey’s theorem for pairs and two colors? Not surprisingly, the answer depends on the base system chosen, as shown in Sect. 4 below. Our approach utilizes a formalization of Weihrauch reducibility described by Hirst at Dagstuhl Seminar 15392 [3], based on higher order reverse mathematics as axiomatized by Kohlenbach [11]. This choice of formalization, along with the choice of different base systems, yields results that differ from those in recent closely related work of Kuyper [13]. We discuss these differences at the end of Sect. 3.
2 Formal Weihrauch reduction

The counting of theorem applications in later sections relies in part on the close connection between proofs in some systems of arithmetic and Weihrauch reduction. This relationship is also central to the arguments of Kuyper [13]. Rather than formalizing Weihrauch reduction by means of indices (as in [13]), we work directly with the functionals, utilizing extensions of reverse mathematics axiom systems [15] to higher types, first formulated by Kohlenbach [11]. These systems have variables for numbers (type 0 objects), functions from numbers to numbers (type 1 objects encoding sets of numbers), and for functions from type 1 functions to numbers, type 1 functions to type 1 functions, and so on. In Kohlenbach’s terminology, $\text{RCA}_0^\omega$ consists of $\text{E-HA}^\omega_1$ plus the law of the excluded middle and $\text{QF-AC}^{1,0}$, a restricted choice scheme. The system $\text{E-HA}^\omega_1$ introduced by Feferman [7], is an axiomatization of intuitionistic Heyting arithmetic in all finite types, with restricted induction and primitive recursion. Equality is a primitive relation only for natural numbers, and an extensionality scheme defines equality for higher order objects in terms of equality for lower types. For full details, see Kohlenbach [12, §3.4]. The choice scheme $\text{QF-AC}^{1,0}$ asserts

$$\forall x \exists n A(x, n) \rightarrow \exists \varphi \forall x A(x, \varphi(x))$$

for $A$ quantifier free, where $x$ is a set variable, $n$ is a number variable, and $\varphi$ is a variable for functions mapping sets to numbers. To make the typography more compact, we will use letters between $i$ and $n$ to denote number variables, letters following $s$ in the alphabet as set variables, and greek letters for various functionals. We also use $i \text{RCA}_0^\omega$ to denote the intuitionistic system arising from omitting the law of the excluded middle from $\text{RCA}_0^\omega$. We refer to Kleene [10] for issues related to classical and intuitionistic predicate calculus. A concise outline of the axioms for $\text{RCA}_0^\omega$ can be found in the article of Hirst and Mummert [9].

Weihrauch reducibility is a computability theoretic approach to measuring relative uniform strength. See Brattka and Gherardi [2] for an extensive survey. We adopt the notion of reduction of problems, as used by Dorais [5] and Dorais et al. [6]. A problem $P$ is a formula of the form $\forall x (p_1(x) \rightarrow \exists y p_2(x, y))$, asserting that whenever $x$ is an instance of the problem then there is a solution $y$ for $x$. Suppose $P : \forall x (p_1(x) \rightarrow \exists y p_2(x, y))$ and $Q : \forall u (q_1(u) \rightarrow \exists v q_2(u, v))$ are problems. We say $Q$ is Weihrauch reducible to $P$, and write $Q \leq_W P$, if there are computable functions $\varphi$ and $\psi$ such that the following hold:

- If $u$ is an instance of $Q$ then $\varphi(u)$ is an instance of $P$, that is:
  $$q_1(u) \rightarrow p_1(\varphi(u))$$

- If $y$ is a solution of $\varphi(u)$, then $\psi(u, y)$ is a solution of $Q$, that is:
  $$p_2(\varphi(u), y) \rightarrow q_2(u, \psi(u, y))$$
Consequently, in the language of \(\text{RCA}_0\) we can formalize \(Q \leq W P\) as:

\[
\exists \varphi \exists \psi \forall u \ (q_1(u) \rightarrow (p_1(\varphi(u)) \land \forall y(p_2(\varphi(u), y) \rightarrow q_2(u, \psi(u, y)))))
\]

We will be working in subsystems of higher order reverse mathematics, so we use \(Q \leq W P\) as an abbreviation for the formula above, despite the fact that the leading quantifiers in the formula are not explicitly restricted to computable functionals. When working in \(i \text{RCA}_0\), for many choices of \(Q\) and \(P\) this is a faithful translation, as shown by Corollary 2. However, in the classical setting, \(\text{RCA}_0 \vdash Q \leq W P\) may not imply \(Q \leq W P\), as shown by the example following Corollary 2.

The concept of uniformity in Weihrauch reducibility is not quite the same as in higher-order arithmetic. In the latter context, we might consider a proof of a theorem \((\forall x)(\exists y)q(x, y)\) to be uniform if the proof constructs a Skolem functional \(G(x)\) with \((\forall x)q(x, G(x))\), as in a previous paper [9].

In this paper, we instead look at a kind of uniformity related to Weihrauch reducibility, arising from the requirement of fixed computable functionals \(\phi\) and \(\psi\) that work for all instances of the problem. We want to investigate when the formal provability of \(Q \leq W P\) in the sense just defined implies the existence of an actual Weihrauch reduction. To obtain a Weihrauch reduction from a formal proof, we will also need to ensure that the reduction functionals we begin with are only applied one time.

It seems appropriate to use systems such as \(i \text{RCA}_0\) for this purpose because they have essentially the minimum expressiveness required to express the functionals in a formalized Weihrauch reduction, while also being well understood as formal analogs of constructive reasoning.

### 3 Counting theorem applications

In this section, we show that formalized Weihrauch reducibility is closely related to the structure of some intuitionistic proofs. The next definition uses the following terminology from sections §19 and §22 of Kleene’s text [10]. A variable \(u\) is said to be varied in a deduction for a given assumption \(q_1(u)\) if \(u\) occurs free in \(q_1(u)\) and the deduction contains an application of Kleene’s Rule 9 or Rule 12 to a formula depending on \(q_1(u)\). Otherwise, we say \(u\) is held constant in the deduction for the assumption \(q_1(u)\). Kleene’s Rule 9 and Rule 12 are contrapositive forms of universal quantifier introduction postulates. In particular, using Rule 9, from \(A(x)\) one can deduce \(\forall x A(x)\). Holding certain variables constant in a deduction is a prerequisite for valid applications of the deduction theorem.

**Definition 1** Suppose \(\mathcal{T}\) is a theory extending \(i \text{RCA}_0\) and \(P: \forall x(p_1(x) \rightarrow \exists y p_2(x, y))\) and \(Q: \forall u(q_1(u) \rightarrow \exists v q_2(u, v))\) are problems. We say \(\mathcal{T}\) proves \(Q\) with one typical use of \(P\) if the following two sentences hold:

1. For a variable \(u\) there is a term \(x_u\) such that using only axioms of \(\mathcal{T}\) and the assumption \(q_1(u)\), and holding \(u\), the free variables of \(q_1(u)\), and the free variables of \(x_u\) constant for \(q_1(u)\), there is a deduction of \(p_1(x_u)\).
(2) For a previously unused variable $y$, there is a term $v_{x_u,y}$ such that using only axioms of $\mathcal{T}$, the assumption $q_1(u)$, and the assumption $p_2(x_u, y)$, and holding $u$, the free variables of $q_1(u)$, and the free variables of $x_u$ constant for $q_1(u)$, and holding $u$, $y$, the free variables of $p_2(x_u, y)$, and the free variables of $x_u$ and $v_{x_u,y}$ constant for $p_2(x_u, y)$, there is a deduction of $q_2(u, v_{x_u,y})$.

Informally, this definition says that given an instance of the problem $Q$, there is an instance $x_u$ of the problem $P$ such that if there is a solution $y$ to $x_u$ then there is a solution $v_{x_u,y}$ to $Q$. Holding the specified variables constant ensures the validity of applications of the deduction theorem in the proof of the following lemma.

**Lemma 1** Suppose $\mathcal{T}$ is a theory that includes intuitionistic predicate calculus, $P: \forall x (p_1(x) \rightarrow \exists y p_2(x, y))$ and $Q: \forall u (q_1(u) \rightarrow \exists v q_2(u, v))$ are problems, and $\mathcal{T}$ proves $Q$ with one typical use of $P$. Then $\mathcal{T}$ proves:

$$\forall u \exists x \forall y \exists v (q_1(u) \rightarrow (p_1(x) \land (p_2(x, y) \rightarrow q_2(u, v))))$$

**Proof** Given a proof of $Q$ in $\mathcal{T}$ with one typical use of $P$, build a new proof as follows. Assume $q_1(u)$ as a hypothesis and, applying sentence (1) of Definition 1, emulate the given proof to construct a term $x_u$ with $p_1(x_u)$. Let $y$ be a new variable and assume $p_2(x_u, y)$ as a hypothesis. By sentence (2) of Definition 1, we can find a term $v_{x_u,y}$ and prove $q_2(u, v_{x_u,y})$. One application of the deduction theorem yields $p_2(x_u, y) \rightarrow q_2(u, v_{x_u,y})$. By $\land$-introduction [10, §19, Ax. 3] followed by the deduction theorem, we have:

$$q_1(u) \rightarrow (p_1(x_u) \land (p_2(x_u, y) \rightarrow q_2(u, v_{x_u,y}))).$$

Note that $x_u$ depends only on $u$ and $v_{x_u,y}$ depends only on $y$ and $x_u$. Alternating applications of $\exists$-introduction [10, §32, fla. 68] and $\forall$-introduction [10, §32, fla. 64] yield

$$\forall u \exists x \forall y \exists v (q_1(u) \rightarrow (p_1(x) \land (p_2(x, y) \rightarrow q_2(u, v))))$$

as desired. \(\square\)

A formula is $\exists$-free if it is built from prime (that is, atomic) formulas using only universal quantification and the connectives $\land$ and $\rightarrow$. Here, the symbol $\perp$ is considered prime, and $\neg A$ is an abbreviation of $A \rightarrow \perp$, so $\exists$-free formulas may include both $\neg$ and $\neg$. Troelstra’s [16] collection $\Gamma_1$ consists of those formulas defined inductively by the following:

- All prime formulas are elements of $\Gamma_1$.
- If $A$ and $B$ are in $\Gamma_1$, then so are $A \land B$, $A \lor B$, $\forall x A$, and $\exists x A$.
- If $A$ is $\exists$-free and $B$ is in $\Gamma_1$ then $\exists x A \rightarrow B$ is in $\Gamma_1$, where $\exists x$ may represent a block of existential quantifiers.
Theorem 1 Suppose $P : \forall x (p_1(x) \to \exists y p_2(x, y))$ and $Q : \forall u (q_1(u) \to \exists v q_2(u, v))$ are problems and the formula $q_1(u) \to (p_1(x) \land [p_2(x, y) \to q_2(u, v)])$, abbreviated as $R(x, y, u, v)$, is in $\Gamma_1$. Then $i\text{RCA}_0^\omega \vdash \forall u \exists x \forall y \exists v R(x, y, u, v)$ if and only if $i\text{RCA}_0^\omega \vdash Q \leq_W P$.

Proof To prove the implication from left to right, suppose $P$, $Q$, and $R$ are as hypothesized, and $i\text{RCA}_0^\omega \vdash \forall u \exists x \forall y \exists v R(x, y, u, v)$. The proof of Lemma 3.9 of Hirst and Mummert [9] also holds for $i\text{RCA}_0^\omega$, so by two applications of that lemma, there are terms $x_u$ and $v_{x_u, y}$, such that $i\text{RCA}_0^\omega \vdash \forall u \forall y R(x_u, y, u, v_{x_u, y})$. $i\text{RCA}_0^\omega$ proves existence of functionals $\varphi(u) = x_u$ and $\psi(u, y) = v_{x_u, y}$, so $i\text{RCA}_0^\omega$ proves:

$$\forall u \forall y (q_1(u) \to (p_1(\varphi(u)) \land [p_2(\varphi(u), y) \to q_2(u, \psi(u, y))]))$$

which is equivalent to $Q \leq_W P$ by intuitionistic predicate calculus via [10, §35, fla. 95], [10, §35, fla. 89], and $\exists$-introduction [10, §32, fla. 68].

Note that the proof of Lemma 3.9 of [9] is based on versions of the soundness theorem for modified realizability, which appears as Theorem 5.8 of Kohlenbach [12] and Theorem 3.4.5 of Troelstra [16], and conversion lemmas for modified reducibility, Lemma 5.20 of Kohlenbach [12] and Lemma 3.6.5 of Troelstra [16]. The conversion lemmas are restricted to formulas in $\Gamma_1$, necessitating the inclusion of this restriction as a hypothesis for this argument. If these background results were based on Gödel’s Dialectica interpretation instead of modified realizability, the extensionality scheme would need to be weakened and the results would be further restricted to the smaller subclass of formulas known as $\Gamma_2$.

To prove the converse, suppose $i\text{RCA}_0^\omega \vdash Q \leq_W P$. Thus, by our formalization adopted in Sect. 2, $i\text{RCA}_0^\omega$ proves the existence of functionals $\varphi$ and $\psi$ satisfying

$$\forall u (q_1(u) \to (p_1(\varphi(u)) \land \forall y [p_2(\varphi(u), y) \to q_2(u, \psi(u, y))]))$$

By intuitionistic predicate calculus [10, §35, fla. 89] and [10, §35, fla. 95], we can move the universal quantifier on $y$ to the front of the formula. Applying appropriate quantifier elimination followed by quantifier introduction yields $\forall u \exists x \forall y \exists v R(x, y, u, v)$. □

As a corollary, we can show a close relationship between intuitionistic proofs and formal Weihrauch reducibility in intuitionistic systems.

Corollary 1 Suppose $P$, $Q$ and $R$ satisfy the hypotheses in Theorem 1. Then $i\text{RCA}_0^\omega$ proves $Q$ with one typical use of $P$ if and only if $i\text{RCA}_0^\omega \vdash Q \leq_W P$.

Proof The forward implication follows immediately from Lemma 1 and Theorem 1. To prove the converse, suppose $i\text{RCA}_0^\omega$ proves the existence of functions $\varphi$ and $\psi$ witnessing $Q \leq_W P$. Then $\varphi(u)$ satisfies sentence (1) of Definition 1, so $p_1(\varphi(u))$. Assume the single use of $P$ given by $p_2(\varphi(u), y)$. Because $Q \leq_W P$, we have $q_2(u, \psi(u, y))$, completing a proof satisfying sentence (2) of Definition 1. □

Theorem 1 also allows us to show that formal Weihrauch reducibility proved in $i\text{RCA}_0^\omega$ is often a faithful representation of actual Weihrauch reducibility.
Corollary 2 Suppose $P$, $Q$ and $R$ satisfy the hypotheses in Theorem 1. If $i\text{RCA}_0^\omega \vdash Q \leq_W P$, then $Q \leq_W P$.

Proof For $P$, $Q$ and $R$ as hypothesized, if $i\text{RCA}_0^\omega \vdash Q \leq_W P$ then by Theorem 1, $i\text{RCA}_0^\omega \vdash \forall u \exists x \forall y \exists v R(x, y, u, v)$. As in the proof of Theorem 1, this means there are terms $x_u$ and $v_{x_u, y}$ in the language of $i\text{RCA}_0^\omega$ such that $i\text{RCA}_0^\omega \vdash \forall u \forall y R(x_u, y, u, v_{x_u, y})$. Thus in any model of $i\text{RCA}_0^\omega$ based on $\omega$ and the power set of $\omega$, where the basic arithmetic function symbols and the combinators have their usual interpretations, the interpretations of the functionals $\lambda u. x_u$ and $\lambda (x_u, y). v_{x_u, y}$ (that is, $\varphi$ and $\psi$ as in the proof of Theorem 1) will be computable functionals witnessing $Q \leq_W P$. □

Corollary 2 does not hold if $i\text{RCA}_0^\omega$ is replaced by $\text{RCA}_0^\omega$. For example, suppose $P$ is the trivial problem defined by using $0 = 0$ for both $p_1$ and $p_2$. Thus every set is an acceptable input for $P$, and every set is a solution of $P$ for any input. To define the problem $Q$, let $T$ be an infinite computable binary tree (all nodes labeled 0 or 1) with no infinite computable path. Viewing an input $u$ as a function from $\mathbb{N}$ to $\mathbb{N}$, we may interpret $u$ as a sequence of zeros and ones by identifying $u(n)$ with 0 if $u(n) = 0$ and identifying $u(n)$ with 1 if $u(n) \neq 0$. Let $q_1$ be 0 if $u(n) = 0$ so every input is acceptable for $Q$. Let $q_2(u, v)$ say that either $v(0) = 0$ and $p(n) = v(n+1)$ is an infinite path in $T$, or $v(0) > 0$ and $\langle u(0), \ldots, u(v(0)) \rangle \notin T$. Since $T$ is $\Delta^0_1$ definable, $q_2(u, v)$ can be written as a $\Pi^0_1$ formula.

Working in $\text{RCA}_0^\omega$, we will prove that $\exists \psi \forall u \exists v q_2(u, \psi(u))$. By the law of the excluded middle, either $T$ has an infinite path or it doesn’t have an infinite path, so either $\exists \psi \forall u \langle \psi(0), \ldots, \psi(n) \rangle \in T$ or $\forall v \exists \psi \langle \psi(0), \ldots, \psi(n) \rangle \notin T$. In the first case, choose an infinite path $p_0$ and define $\psi$ to be the constant functional that maps each input to the sequence 0 followed by $p_0$. In the second case, let $\psi$ map each $u$ to the function that always takes the value $1 + \mu m(u(0), \ldots, u(m)) \notin T$, so for each $u$ and $n$, $\psi(u)(n)$ is a positive witness that $u$ is not an infinite path. In either case, $\forall u q_2(u, \psi(u))$, as desired. Consequently, the identity functional $\psi$ trivially witnesses

$$\forall u \langle 0 = 0 \rightarrow (0 = 0 \land \forall v (0 = 0 \rightarrow q_2(u, \psi(u))))$$,

so $\text{RCA}_0^\omega$ proves that $Q \leq_W P$.

Turning to the computability theoretic framework, we will show that $Q$ is not Weihrauch reducible to $P$. To see this, suppose by way of contradiction that $\varphi$ and $\psi$ are computable functionals witnessing $Q \leq_W P$. Because $P$ is trivial, $\Theta$ (the constant 0 function) is a solution of $\varphi(u)$, so for all $u$, $\psi(u, \emptyset)$ is a solution of $Q$. By König’s Lemma, let $u_0$ be an infinite path through $T$. Because there is no witness that $u_0$ is not an infinite path, we must have $\psi(u_0, \emptyset)(0) = 0$. The functional $\psi$ is computable, so for some finite $k$, if $u$ is any extension of $\langle u_0(0), \ldots, u_0(k) \rangle$, then $\psi(u, \emptyset)(0) = 0$. Choose a computable sequence $s_0$ such that $s_0$ extends $\langle u_0(0), \ldots, u_0(k) \rangle$. Then $\psi(s_0, \emptyset)$ is a solution of $Q$ and $\psi(s_0, \emptyset)(0) = 0$, so $v_0$ defined by $v_0(n) = \psi(s_0, \emptyset)(n+1)$ is an infinite path through $T$. But $v_0$ is computable, contradicting the choice of $T$ and completing the example.

We close this section by comparing Corollary 1 with Theorem 7.1 of Kuyper [13]. In a recent talk at Dagstuhl Seminar 18361 [1], Patrick Uftring [17] presented counterex-
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examples to Kuyper’s Theorem 7.1. More details can be found in Uftring’s forthcoming thesis [18]. Kuyper’s theorem and Corollary 1 are similar in that each states the equivalence of the existence of a formalized Weihrauch reduction with the existence of a restricted resource proof of a related formula. However, the class of pairs of problems \(P\) and \(Q\) such that \(\text{iRCA}_0^\omega\) proves \(Q\) with one typical use of \(P\) is a proper subclass of those for which \((\text{EL}_0 + \text{MP})\text{\kern1pt}^{\exists\alpha}\) proves \(P’ \rightarrow Q’\) (in the sense of Kuyper [13]), as shown by the following theorem. This restriction allows Corollary 1 to avoid the counterexample of Uftring.

Theorem 2 Suppose \(P, Q\), and \(R\) satisfy the hypotheses of Theorem 1. If \(\text{iRCA}_0^\omega \vdash Q \leq_W P\), then there are standard natural number indices \(e_0\) and \(e_1\) such that \(\text{RCA}_0\) proves that \(e_0\) and \(e_1\) witness that \(Q\) Weihrauch reduces to \(P\) as formalized in Theorem 7.1 of Kuyper [13]. The converse of this implication fails.

Proof Suppose \(P, Q\), and \(R\) satisfy the hypotheses of Theorem 1. Note that the formalization of \(Q \leq_W P\) in \(\text{iRCA}_0^\omega\) is in \(\Gamma_1\). By the intuitionistic analog of Lemma 3.9 of Hirst and Mummert [9], there are terms in the language of \(\text{iRCA}_0^\omega\) corresponding to the functionals witnessing \(Q \leq_W P\). The desired indices can be calculated from these terms.

To prove that the converse fails, let \(P\) be the trivial problem \(\forall x (0 = 0 \rightarrow \exists y (0 = 0))\) and let \(Q\) be the problem

\[\forall u(0 = 0 \rightarrow \exists v (\forall n u(n) = 0 \vee \exists n u(n) \neq 0)).\]

Note that \(P, Q\), and the associated formula \(R\) satisfy the hypotheses of Theorem 1. In \(\text{iRCA}_0^\omega\), \(Q \leq_W P\) implies \(\forall u (\forall n u(n) = 0 \vee \exists n u(n) \neq 0))\). Because this conclusion (a form of the Lesser Principle of Omniscience) is not intuitionistically valid, \(\text{iRCA}_0^\omega\) does not prove \(Q \leq_W P\). On the other hand, for any indices \(e_0\) and \(e_1\), the classical system \(\text{RCA}_0\) proves

\[\forall u(0 = 0 \rightarrow (0 = 0 \rightarrow (0 = 0 \wedge \forall y(0 = 0 \rightarrow (\forall n u(n) = 0 \vee \exists n u(n) \neq 0)))))),\]

so for any choice of indices, \(\text{RCA}_0\) proves that \(Q\) Weihrauch reduces to \(P\) in the sense of Theorem 7.1 of Kuyper [13].

In practice, many mathematical theorems of interest are in \(\Gamma_1\) or can be reformulated as \(\Gamma_1\) statements. Looking at the definition, the main obstacle for a theorem of the form

\[(\forall x)[p(x) \rightarrow (\exists y)q(x, y)]\]

(1)
to be in \(\Gamma_1\) is the structure of \(p(x)\). When \(p(x)\) is universal, as is often the case in practice, the naive formalization is typically in \(\Gamma_1\). When \(p(x)\) is more complicated, we can sometimes Skolemize \(p(x)\) to replace the original formula with a different formula of form (1). For examples, see section 4 of Hirst and Mummert [9].

Unlike the results here, the results of Kuyper [13] are not restricted to formulas in \(\Gamma_1\), in part due to his utilization of the Kuroda negative translation. We wonder whether similar methods can extend the results of this paper and our previous results [9] to a wider class of formulas.
4 Ramsey’s theorem

We can use the preceding results to address our question about proofs of Ramsey’s theorem. Let RT(2, 4) denote the following formulation of Ramsey’s theorem for pairs and four colors: If \( f: [\mathbb{N}]^2 \to 4 \), then there is an infinite \( x \subset \mathbb{N} \) and an \( i < 4 \) such that \( f([x]^2) = i \). The set \( x \) is called monochromatic. Similarly, RT(2, 2) denotes Ramsey’s theorem for pairs and two colors.

For any \( k \), we can formalize RT(2, k) as a particularly simple \( \Pi^1_2 \) formula. In the higher order axiom systems described by Kohlenbach [11], all higher order objects are functions, with subsets of \( \mathbb{N} \) being encoded by characteristic functions or by enumerations. Pairs of natural numbers can be encoded by a single natural number, so any function from \( \mathbb{N} \) into \( \mathbb{N} \) are functions, with subsets of \( \mathbb{N} \) being encoded by characteristic functions or by enumerations. Pairs of natural numbers can be encoded by a single natural number, so any function from \( \mathbb{N} \) into \( \mathbb{N} \) (that is, any type 1 object) can be viewed as a function from \( [\mathbb{N}]^2 \) into \( \mathbb{N} \). By composition with a truncation function \( t_n \) defined by \( t_n(m) = m \) if \( m < n \) and \( t_n(m) = 0 \) otherwise, we may view any type 1 function as a map from \( [\mathbb{N}]^2 \) into \( n \). Using these notions, we can formalize RT(2, 4) as

\[
\forall f \exists x \forall m (x(m) < x(m')) \land \forall 0 < i < j < m (t_4(f(x(i), x(j))) = x(0))).
\]

Formalized in this fashion, RT(2, 4) is in \( \Gamma_1 \) and its matrix (the portion beginning with \( \forall m \)) is \( \exists \)-free. If we like, we could write it as \( \forall f (0 = 0 \rightarrow \exists x (\ldots)) \) to coincide with the \( \forall x (p_1 \rightarrow \exists y p_2) \) problem format. Using this formulation for P:RT(2, 2) and Q:RT(2, 4), the predicate \( R \) as in the statement of Theorem 1 is in \( \Gamma_1 \).

Consider the following well-known proof of RT(2, 4) from two applications of RT(2, 2). Given \( f: [\mathbb{N}]^2 \to 4 \), define \( g_1: [\mathbb{N}]^2 \to 2 \) by setting \( g_1(n, m) = 1 \) if \( f(n, m) > 1 \) and \( g_1(n, m) = 0 \) otherwise. Applying RT(2, 2), let \( x = \{x_0, x_1, \ldots\} \) be an infinite monochromatic set for \( g_1 \). Note that \( f([x]^2) \) is either contained in \( \{0, 1\} \) or contained in \( \{2, 3\} \). Define \( g_2: [\mathbb{N}]^2 \to 2 \) by \( g_2(n, m) = 1 \) if \( f(x_n, x_m) \) is odd and \( g_2(n, m) = 0 \) otherwise. Applying RT(2, 2) a second time, let \( y \) be an infinite monochromatic set for \( g_2 \). Then \( z = \{x_m \mid m \in y\} \) is an infinite monochromatic set for \( f \), completing the proof of RT(2, 4). This proof that RT(2, 2) implies RT(2, 4) can be carried out in \( i\text{RCA}_0^\omega \). However, our work from previous sections shows that the second use of RT(2, 2) cannot be eliminated.

**Theorem 3** \( i\text{RCA}_0^\omega \) cannot prove RT(2, 4) with one typical use of RT(2, 2).

**Proof** As noted in the second paragraph of this section, for our formulation of RT(2, 2) and RT(2, 4), P, Q, and R satisfy the hypotheses of Theorem 1. By Corollary 3.4 of Dorais et al. [6], RT(2, 4) \( \not\leq_W \) RT(2, 2). By Corollary 2, \( i\text{RCA}_0^\omega \not\supseteq \) RT(2, 4) \( \leq_W \) RT(2, 2). By Corollary 1, \( i\text{RCA}_0^\omega \) does not prove RT(2, 4) with one typical use of RT(2, 2). \( \square \)

Theorem 3.3 of Hirschfeldt and Jockusch [8] asserts that if \( j, k, n \in \omega \) satisfy the inequalities \( n \geq 1 \) and \( \log_2 n \geq j \geq 2 \), then RT(\( n, k \)) \( \not\leq_W \) RT(\( n, j \)). They note that this result was proved independently by Brattka and Rakotoaiaina [4, Theorem 4.22], and independently proved and strengthened by Patey [14, Corollary 3.14]. Substituting this result for the use of Corollary 3.4 in the proof of Theorem 3 yields the following extension.
Corollary 3 If \( j, k, n \in \omega \) satisfy \( n \geq 1 \) and \( k > j \geq 2 \), then \( i \text{RCA}_0 \) cannot prove \( \text{RT}(n, k) \) with one typical use of \( \text{RT}(n, j) \).

Returning to our original discussion of Ramsey’s theorem for pairs, we next show that \( \text{RT}(2, 4) \) can be proved with one typical use of \( \text{RT}(2, 2) \) in systems such as \( \text{RCA}_0 \) that include the law of the excluded middle. This somewhat counterintuitive result relies on the following definition.

Definition 2 (RCA) Suppose \( f : [\mathbb{N}]^2 \to 4 \). A set \( x \) is 2-mono for \( f \) if there is a set \( \{i, j\} \subseteq \{0, 1, 2, 3\} \) such that \( f([x]^2) \subseteq \{i, j\} \).

Theorem 4 \( \text{RCA}_0 \) can prove \( \text{RT}(2, 4) \) with one typical use of \( \text{RT}(2, 2) \).

Proof The following proof can be carried out in \( \text{RCA}_0 \).

Suppose \( f : [\mathbb{N}]^2 \to 4 \). By the law of the excluded middle, either there is an infinite \( x \subseteq \mathbb{N} \) that is 2-mono for \( f \) or there is no such set.

- Case 1: If there is such a set, define \( j = 0 \), let \( x \) be an increasing enumeration of such a set, and suppose \( f([x]^2) \subseteq \{a_0, a_1\} \).
- Case 2: If there is no such set, define \( j = 1 \) and let \( x \) be an increasing enumeration of \( \mathbb{N} \).

Now define \( g : [\mathbb{N}]^2 \to 2 \) by the following:

\[
g(m, n) = \begin{cases} 
0 & \text{if } j = 0 \text{ and } f(x(m), x(n)) = a_0 \\
1 & \text{if } j = 0 \text{ and } f(x(m), x(n)) = a_1 \\
0 & \text{if } j = 1 \text{ and } f(x(m), x(n)) \leq 1 \\
1 & \text{if } j = 1 \text{ and } f(x(m), x(n)) \geq 2.
\end{cases}
\]

We can prove in \( \text{RCA}_0 \) that there are numbers \( a_0, a_1, j \) and a function \( g \) that fit the construction just specified. By one typical application of \( \text{RT}(2, 2) \), let \( y \) be an infinite monochromatic set for \( g \). If \( j = 1 \), then \( y \) is an infinite 2-mono set for \( f \), contradicting the definition of \( j \). Thus \( j = 0 \), and the set \( z = \{x(m) \mid m \in y\} \) is an infinite monochromatic set for \( f \).

Using similar but more complicated constructions, for each standard integer \( k \) one can show that \( \text{RT}(2, k) \) can be proved with a single application of \( \text{RT}(2, 2) \) in the classical system \( \text{RCA}_0 \). For example, given \( f : [\mathbb{N}]^2 \to 8 \) either \( \mathbb{N} \) contains no infinite 4-mono set, or there is an infinite 4-mono set with no infinite 2-mono subset, or there is an infinite 2-mono set. Define \( g \) based on these possibilities and proceed as above. Furthermore, examination of the proof of Theorem 4 reveals no actual use of the exponent. Consequently, we can extend Theorem 4 as follows.

Corollary 4 Let \( n \) and \( k \) be positive elements of \( \omega \). \( \text{RCA}_0 \) can prove \( \text{RT}(n, k) \) with one typical use of \( \text{RT}(n, 2) \).

This kind of nonconstructive argument leveraging a single typical use of an axiom is not limited to Ramsey’s theorem. For example, for each \( n \in \omega \), \( \text{RCA}_0 \) can prove “every set has an \( n \)th Turing jump” with a single typical use of “every set has a Turing
jump”. Many more examples come to mind, where a single typical use can be used to iterate a principle any finite number of times.

The relationship between Weihrauch reducibility and proofs in intuitionistic systems played an important role in obtaining the results of this section. We did not discover the proof described in Theorem 4 until our work on Corollary 1 indicated the significance of the law of the excluded middle in this setting.

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