Ten-dimensional supersymmetric Janus solutions

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Abstract

The reduced field equations and BPS conditions are derived in Type IIB supergravity for configurations of the Janus type, characterized by an $AdS_4$-slicing of $AdS_5$, and various degrees of internal symmetry and supersymmetry. A generalization of the Janus solution, which includes a varying axion along with a varying dilaton, and has $SO(6)$ internal symmetry, but completely broken supersymmetry, is obtained analytically in terms of elliptic functions. A two-parameter family of solutions with 4 real supersymmetries, $SU(3)$ internal symmetry, a varying axion along with a varying dilaton, and non-trivial $B_{(2)}$ field, is derived analytically in terms of genus 3 hyper-elliptic integrals. This supersymmetric solution is the 10-dimensional Type IIB dual to the $\mathcal{N} = 1$ interface super-Yang-Mills theory with $SU(3)$ internal symmetry previously found in the literature.
1 Introduction

The AdS/CFT correspondence relates string theories on Anti-de Sitter space-times (AdS) to conformal field theories (CFT) on the boundary of the AdS space-time \[^{12}\text{1}^{2}\text{3}\]. The case which is understood perhaps best relates Type IIB string theory on \(AdS_5 \times S_5\) to Yang-Mills theory with \(\mathcal{N} = 4\) supersymmetry and gauge group \(SU(N)\). (For reviews, see \[^{4}\text{5}\].)

The AdS/CFT correspondence is expected to hold as well in situations with less or no supersymmetry, and with space-times which are only asymptotically AdS. Thus, a suitable deformation on the gauge theory side of the correspondence should be dual to an associated deformation on the string theory side and vice versa. Many interesting such cases are known and have been studied extensively. These include the holographic representation of renormalization group-flows \[^{6}\text{6}\], and the solutions of Klebanov-Strassler \[^{7}\text{7}\], Polchinski-Strassler \[^{8}\text{8}\] and Maldacena-Nunez \[^{9}\text{9}\] (first obtained by Chamseddine and Volkov \[^{10}\text{10}\text{, 11}\].)

In \[^{12}\text{12}\], a dilatonic deformation of the Type IIB background \(AdS_5 \times S_5\) was found\[^{1}\]. The Janus solution breaks all supersymmetries, but is nevertheless stable against small and a large class of large perturbations \[^{13}\text{13}\text{, 14}\text{, 15}\]. It can be viewed as a curved dilatonic domain wall \[^{16}\text{16}\]. The holographic dual to the Janus solution is \(\mathcal{N} = 4\) super Yang-Mills theory in 3+1 space-time dimensions, with a planar 2+1 dimensional interface, across which the gauge coupling varies discontinuously. The interface carries no additional degrees of freedom. Conformal invariance in 2+1 dimensions is preserved by the interface at the classical level, and holds in conformal perturbation theory to first non-trivial order as well \[^{17}\text{17}\].

The Janus solution is remarkably simple\[^{2}\]. In fact, in this paper, we shall show that, even when generalized to include a varying axion in addition to a varying dilaton, the Janus solution admits an analytic form in terms of elliptic functions. This raises the hope that correlation functions in the Janus background may be studied using analytic methods, a topic which we plan to address in a later publication.

Furthermore, the remarkable simplicity of the non-supersymmetric Janus solution suggests that Janus may possess supersymmetric generalizations available in analytic form as well. Clearly, such analytic solutions would be valuable as starting points for the analytic study of correlators in the corresponding backgrounds. (Note that, on the one hand, the backgrounds of Klebanov-Strassler \[^{7}\text{7}\] or Maldacena-Nunez \[^{9}\text{9}\] are not asymptotically AdS

\[^{1}\]It was named the Janus solution, after the two-faced Roman god of gates, doors, beginnings, endings and, now, also, string dualities.

\[^{2}\]Other dilatonic deformations found in the literature are singular and their physical interpretation remains more obscure \[^{18}\text{18}\text{, 19}\text{, 20}\].

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while, on the other hand, the Polchinski-Strassler \cite{polchinski_strassler} solution is only known approximately, in an expansion in the strength of the fluxes.)

The fact that interesting supersymmetric generalizations of Janus should exist is further suggested by considering its CFT dual, namely 3+1 dimensional super Yang-Mills theory with a 2+1 dimensional planar interface. In \cite{janus_interface}, it was found that 2 Poincaré supersymmetries can be preserved by adding “interface operators” whose support is confined to the interface. In the conformal limit, 2 conformal supersymmetries emerge as well. The interface operators break the R-symmetry from $SO(6)$ down to $SU(3)$.

In \cite{interface_classification}, a complete classification of supersymmetry restoring interface operators on the gauge theory side is given. In particular, it is established there that, for interface theories with $SU(3)$ internal symmetry, 4 is the maximum number of conformal supersymmetries. The corresponding theory is constructed explicitly, and coincides with the one presented in \cite{janus_interface} in terms of $\mathcal{N} = 1$ off-shell fields. It is further established in \cite{interface_classification} that interface theories with extended supersymmetry exist as well. One interface theory has 8 conformal supersymmetries and $SO(2) \times SU(2)$ internal symmetry, while another has 16 conformal supersymmetries and $SU(2) \times SU(2)$ internal symmetry.

Further evidence for the existence of 10-dimensional supersymmetric generalizations of Janus is provided in \cite{janus_supergravity}, where a supersymmetric Janus solution of five dimensional gauged supergravity was found (building on previous work on curved domain walls in $AdS_5$ \cite{domain_walls_1, domain_walls_2, domain_walls_3, domain_walls_4}). The starting point of \cite{janus_supergravity} was an $SU(3)$ invariant gauging of the universal hypermultiplet of five dimensional $\mathcal{N} = 2$ supergravity \cite{universal_hypermultiplet}. While it is believed that the resulting theory is a consistent truncation of the ten-dimensional supergravity, the details of this truncation, and hence of any possible lift of the solution to ten dimensions, are unknown.

In the present paper, a family of supersymmetric Janus solutions is derived directly in ten-dimensional Type IIB supergravity. We focus here on Janus solutions which provide holographic duals to the supersymmetric interface CFTs of \cite{janus_interface}, with $\mathcal{N} = 1$ interface supersymmetry and 3 chiral multiplets related by $SU(3)$ internal symmetry. Our starting point is the construction of the most general Ansatz for Type IIB supergravity fields which preserves $SO(2,3) \times SU(3)$ symmetry, and which transforms covariantly under the $SL(2,\mathbb{R})$ symmetry of Type IIB supergravity, as well as under the unique $U(1)_\beta \subset SU(4)$ which commutes with $SU(3)$. The reduced field equations are then solved subject to the BPS conditions, namely the conditions for the vanishing of the supersymmetry variations of the dilatino and gravitino fields. The resulting family of solutions contains a subset that is of the Janus type, and the solutions in this subset may be expressed analytically via hyper-elliptic integrals of genus 3.
The Ansatz that will be obtained in this paper for the construction of supersymmetric Janus solutions is based on an $AdS_4$ slicing of $AdS_5$, just as the original non-supersymmetric Janus Ansatz was. If a slicing of $AdS_5$ by 4-dimensional Minkowski space were used instead, one would recover an Ansatz used by Romans [44] to construct $SU(3)$-symmetric compactifications of Type IIB supergravity, but without supersymmetry. (In eleven dimensional supergravity, the corresponding solutions were constructed in [15].) In AdS/CFT such Minkowski-sliced solutions have a natural interpretation in terms of RG flows [6, 49, 47, 33]. Some techniques used for the study of the Minkowski slicings can be applied to the $AdS_4$ slicings, and the resulting reduced field equations are related.

The remainder of the paper is organized as follows.

In Section 2, the field equations of Type IIB supergravity, as well as the supersymmetry variations (both for vanishing fermion fields) are summarized. In Section 3, the original Janus solution is reviewed, extended to include a varying axion along with a varying dilaton, and expressed analytically in terms of elliptic functions. In Section 4, the main results on supersymmetric interface CFT are collected.

In Section 5, the most general Ansatz for Type IIB supergravity fields, subject to $SO(2,3) \times SU(3) \times U(1)_\beta \times SL(2,R)$ symmetry is constructed. The reduced Bianchi identities and field equations for this Ansatz are derived in section 6, where it is shown that these equations may also be obtained from a reduced action, which is computed, and a vanishing Hamiltonian constraint, which amounts to the reduced Wheeler-De Wit equation. In section 7, the supersymmetry variations for the Ansatz are derived. In section 8, it is demonstrated that every supersymmetric solution with varying axion and varying dilaton is actually the $SL(2,R)$ image of a “real” solution with vanishing axion.

In section 9, it is shown that, for supersymmetric solutions, the vanishing condition of the Hamiltonian constraint factorizes into a product of two factors. The vanishing of the first factor leads to a family of degenerate solutions (for which the second factor vanishes as well), and it is these solutions which are obtained analytically in terms of genus 3 hyper-elliptic integrals. In section 10, it is shown that these degenerate solutions are of the Janus type, and thus asymptotically $AdS$. The equations for the non-degenerate solutions are more involved and have not yet been solved analytically. Numerical evidence suggests that these solutions may not be of the Janus type. In section 11, the holographic dual CFT is interpreted in terms of deformations of the $AdS_5 \times S^5$ background, while in section 12, some concluding remarks are offered. Finally, a convenient basis of Dirac matrices is presented in Appendix A.

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3We thank the referee for pointing out this relationship.
2 Type IIB supergravity

In this section, we present the field equations, Bianchi identities, and supersymmetry variations for Type IIB supergravity, which were originally derived in [29, 30]. The metric signature used here is $(- \cdots +)$ in contrast with [29], where the signature is $(+ - \cdots -)$. We restrict to vanishing fermion fields, which will suffice for our analysis.

The bosonic fields of Type IIB supergravity are: the metric $g_{MN}$; the axion-dilaton complex scalar $B$ which takes values in the coset $SU(1,1)/U(1)$; and the antisymmetric tensors $B_{(2)}$ (which is complex) and $C_{(4)}$ (which is real). It is standard to introduce composite fields in terms of which the field equations are expressed simply. They are as follows,

\[ P = f^2 dB, \quad f = \frac{1}{\sqrt{1 - |B|^2}} \]
\[ Q = f^2 \text{Im}(Bd\bar{B}) \]  \hspace{1cm} (2.1)

and the field strengths $F_{(3)} = dB_{(2)}$, and

\[ G = f(F_{(3)} - BF_{(3)}) \]
\[ F_{(5)} = dC_{(4)} + \frac{i}{16} (B_{(2)} \wedge \bar{F}_{(3)} - \bar{B}_{(2)} \wedge F_{(3)}) \]  \hspace{1cm} (2.2)

The scalar field $B$ is related to the axion $\chi$ and dilaton $\phi$ fields by

\[ B = \frac{1 + i\tau}{1 - i\tau} \quad \tau = \tau_1 + i\tau_2 = \chi + ie^{-\phi} \]  \hspace{1cm} (2.3)

In terms of the composite fields $P, Q,$ and $G$, there are “Bianchi identities” given as follows,

\[ dP - 2iQ \wedge P = 0 \]  \hspace{1cm} (2.4)
\[ dG - iQ \wedge G + P \wedge \bar{G} = 0 \]  \hspace{1cm} (2.5)
\[ dQ + iP \wedge \bar{P} = 0 \]  \hspace{1cm} (2.6)
\[ dF_{(5)} - i\frac{1}{8} G \wedge \bar{G} = 0 \]  \hspace{1cm} (2.7)

The field strength $F_{(5)}$ is required to be self-dual,

\[ F_{(5)} = *F_{(5)} \]  \hspace{1cm} (2.8)

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4 Throughout, we shall pass freely between tensor and form notations, with a differential form $\omega$ of rank $n$ associated with tensor components $\omega_{M_1 \cdots M_n}$ by the relation $\omega = \frac{1}{n!} \omega_{M_1 \cdots M_n} dx^{M_1} \wedge \cdots \wedge dx^{M_n}$. 

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The field equations are given by:

\begin{align}
0 &= \nabla^M P_M - 2i Q^M P_M + \frac{1}{24} G_{MNP} G^{MNP} \quad (2.9) \\
0 &= \nabla^P G_{MNP} - i Q^P G_{MNP} - P^P \tilde{G}_{MNP} + \frac{2}{3} i F_{(5)MNPQR} G^{PQR} \quad (2.10) \\
0 &= R_{MN} - P_M P_N - \tilde{P}_M P_N - \frac{1}{6} (F_5^2)_{MN} \\
&\quad - \frac{1}{8} (G_{MNP} \tilde{G}_{NPQ} + \tilde{G}_M^{PQ} G_{NPQ}) + \frac{1}{48} g_{MN} G^{PQR} \tilde{G}_{PQR} \quad (2.11)
\end{align}

The fermionic fields are the dilatino $\lambda$ and the gravitino $\psi_M$, both of which are complex Weyl spinors with opposite 10-dimensional chiralities, given by $\Gamma^{11} \lambda = \lambda$, and $\Gamma^{11} \psi_M = -\psi_M$. The supersymmetry variations of the fermions (still in a purely bosonic background) are

\begin{align}
\delta \lambda &= i P_M \Gamma^M B^{-1} \varepsilon^* - \frac{i}{24} \Gamma^{MNP} G_{MNP} \varepsilon \quad (2.12) \\
\delta \psi_M &= D \varepsilon + \frac{i}{480} F_{(5)NPQRS} \Gamma^{NPQRS} \Gamma_M \varepsilon + \frac{1}{96} (\Gamma^N P Q G_{NPQ} - 9 \Gamma^N P G_{MNP}) B^{-1} \varepsilon^*
\end{align}

where $B$ is the charge conjugation matrix of the ten dimensional Clifford algebra.

Type IIB supergravity is invariant under $SU(1, 1) \sim SL(2, \mathbb{R})$ symmetry, which leaves $g_{\mu \nu}$ and $C_{(4)}$ invariant, acts by Möbius transformation on the field $\tau$, and linearly on $B_{(2)}$,

$$\tau \rightarrow \frac{a \tau + b}{c \tau + d} \quad \text{(Im } B_{(2)} \text{)} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \text{Im } B_{(2)} \\ \text{Re } B_{(2)} \end{pmatrix}$$

with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. In this non-linear realization of $SL(2, \mathbb{R})$, the field $B$ takes values in the coset $SU(1, 1)/U(1) \sim SL(2, \mathbb{R})/U(1)$, and the fermions $\lambda$ and $\psi_\mu$ transform linearly under the isotropy gauge group $U(1)$ with composite gauge field $Q$.

The field equations derive from an action, (we omit the overall prefactor $1/2\kappa_{10}^2$),

$$S = \int dx \sqrt{g} \left\{ R - \frac{1}{2} \partial_M \tau \partial^M \tau - \frac{1}{12} G_{MNP} \tilde{G}^{MNP} - 4 |F_{(5)}|^2 \right\} - i \int C_{(4)} \wedge F_{(3)} \wedge \tilde{F}_{(3)} \quad (2.14)$$

in the following sense. The field equations are derived by first requiring that $S$ be extremal under arbitrary variations of the fields $g_{MN}$, $\tau$, $B_{(2)}$ and $C_{(4)}$; and second by imposing the self-duality condition (2.8) on $F_{(5)}$ as a supplementary equation.

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5. The sign of the term $GG$ in (2.9) has been corrected compared to the original equation in [29]; the need for this correction was noted independently in [31, 32, 33].

6. It is defined by $BB^* = I$ and $\Gamma^M B^{-1} = (\Gamma^M)^*$; see Appendix A for our $\Gamma$-matrix conventions. Throughout, complex conjugation of functions will be denoted by $\text{bar}$, while that of spinors will be denoted by $\star$.
3 The generalized non-supersymmetric Janus solution

In this section, the original Janus solution of [12] is reviewed, extended to include a varying axion along with a varying dilaton, and expressed analytically in terms of elliptic functions. The Ansatz is required to have $SO(2,3) \times SO(6)$ symmetry. Since $AdS_4 \times R \times S^5$ admits no $SO(6)$-invariant 2-forms, the symmetry requires $B_{(2)} = 0$. The metric $g_{MN}$ and 5-form $F_5$ are given by an $AdS_4$-slicing of $AdS_5$, consistent with $SO(2,3) \times SO(6)$ symmetry,

$$ds^2 = h d\mu^2 + hd s^2_{AdS_4} + h_1 d s^2_{S^5}$$

$$F_5 = 2h^{5/2}d\mu \wedge \omega_{AdS_4} + 2h_1^{5/2}\omega_{S^5}$$  \hspace{1cm} (3.1)

where $\omega_{AdS_4}$ and $\omega_{S^5}$ are the canonical volume forms on the corresponding manifolds. The dilaton $\phi$, axion $\chi$ and the functions $h, h_1$ depend on $\mu$ only. Remarkably, the original Janus solution was found by setting $h_1 = 1$, thereby leaving the $S^5$ metric unchanged. Numerical evidence suggests that solutions with varying $h_1$ always have singularities. In section §3.3, we shall present arguments, based on the AdS/CFT correspondence, that such breathing modes which vary $h_1$, must be absent lest the corresponding solution become singular.

The reduced field equations (for $h_1 = 1$) may be expressed in terms of $\tau$ and $h$,

$$\frac{\tau''}{\tau'} + \frac{i}{\tau_2}\tau' + \frac{3}{2} \frac{h'}{h} = 0$$ \hspace{1cm} (3.2)

$$4(h')^2 - 4hh'' + 8h^3 = h^2 \frac{|\tau'|^2}{\tau_2^2}$$ \hspace{1cm} (3.3)

$$12h^2 + (h')^2 + 2hh'' - 16h^3 = 0$$ \hspace{1cm} (3.4)

Since $h$ is real, the imaginary part of \eqref{3.2} is independent of $h$ and may be integrated to give

$$|\tau - p|^2 = r^2 \quad \quad p, r \in \mathbb{R}$$ \hspace{1cm} (3.5)

This means that as $\mu$ varies, $\tau$ evolves along a segment of a geodesic in the upper half plane equipped with the $SL(2, \mathbb{R})$-invariant Poincaré metric $d\tau^2/\tau_2^2$. Integrating also the real part of \eqref{3.2}, we find that $|\tau'|^2/\tau_2^2 = c_0^2/h^3$. for some constant $c_0 \in \mathbb{R}$. This relation gives the velocity of $\tau$ along the geodesic as a function of $h$. Finally, using this result in \eqref{3.3} leads to an equation that is consistent with \eqref{3.4} and has a first integral given by,

$$(h')^2 = 4h^3 - 4h^2 + \frac{c_0^2}{6h}$$ \hspace{1cm} (3.6)

\footnote{Its geodesics are the half circles with arbitrary center $p$ on the real axis and arbitrary radius $r$.}
which is the same equation as the first integral for the original Janus solution [12].

The interpretation of the above results is as follows. In the original Janus solution, the axion field vanished, and the dilaton evolution spanned a rather special geodesic in the upper half plane: a vertical line segment with $\tau_1 = 0$. Since the action of $SL(2, \mathbb{R})$ on the upper half plane is transitive on points as well as on connected geodesic segments of equal length, solutions with varying axion may be obtained as $SL(2, \mathbb{R})$ images of solutions with vanishing axion. The remarkable result obtained above is that all solutions with varying axion may be obtained as $SL(2, \mathbb{R})$ images of solutions with vanishing axion.

![Figure 1: Mapping geodesic segments under $SL(2, \mathbb{R})$ in the dilaton/axion upper half plane.](image)

3.1 Analytical solution in terms of elliptic functions

In [12], the quadrature of (3.6) was obtained numerically. Actually, the general solution may be expressed analytically in terms of elliptic functions. To do so, we consider a coordinate independent object, namely the 1-form,

$$\nu = \frac{d\mu}{\sqrt{h}} = \frac{dh}{\sqrt{4h^4 - 4h^3 + c_0^2/6}}$$

The 1-form $\nu$ is proportional to the holomorphic Abelian differential on a torus. We represent the constant $c_0$ by $c_0^2 = 24\gamma_0^3(1 - \gamma_0)$, and parametrize the Abelian differential $\nu$ and the
function $h$ in terms of $\gamma_0$ and the Weierstrass $\wp$-function, expressed in terms of the canonical coordinate $z$ of the torus. We find the following expressions,

$$\nu = \frac{dz}{\sqrt{\gamma_0}} \quad h(\mu) = \gamma_0 + \frac{\gamma_0(3 - 4\gamma_0)}{\wp(z) + 2\gamma_0 - 1}$$  \hspace{1cm} (3.8)$$

Here, the Weierstrass $\wp$-function has been normalized to standard form, $(\partial_z \wp)^2 = 4\wp^3 - 16\gamma_0(1-\gamma_0)\wp - 4(1-\gamma_0)$. The discriminant of the corresponding curve $\Delta = 64\epsilon^4_0(32\epsilon^2_0 - 81)/27$ vanishes at $\epsilon^2_0 = 0$ and at the critical value $\epsilon^2_0 = 81/32$, identified in [12] as the value where the range of the dilaton begins to diverge. The dilaton axion equation reduces to

$$|d\tau|/\tau^2 = c_0 \frac{\nu}{h}$$  \hspace{1cm} (3.9)$$

which may be integrated by standard elliptic function methods.

### 3.2 Structure of the AdS/CFT dual

The Janus solution can be viewed as a dilatonic domain wall in which the dilaton varies with the coordinate $\mu$, which parameterizes the $AdS_4$ slicing of $AdS_5$. It follows from the dilatino supersymmetry variation (2.12), that no supersymmetries are preserved for the Janus solution with a varying dilaton and vanishing $B_{(2)}$. Nevertheless, in [13, 14, 15] convincing arguments were presented that the Janus solution is stable against all small and a certain class of large perturbations.

![Figure 2: Sketch of the boundary geometry of the Janus solution in global and Poincare coordinates for the $AdS_4$ slices.](image)

The angular coordinate $\mu$ covers a range $\mu \in [-\mu_0, \mu_0]$ where $\mu_0 > \pi/2$. The structure of the boundary of this space can be analyzed using global coordinates for the $AdS_4$ slices.
Near $\mu = \pm \mu_0$ the noncompact part of the metric has the following asymptotic behavior,

$$ds^2 \sim \frac{1}{(\mu \mp \mu_0)^2 \cos^2 \lambda} \left( \cos^2 \lambda d\mu^2 - dt^2 + d\lambda^2 + \sin^2 \lambda d\Omega^2_{S^2} \right)$$

(3.10)

with $0 \leq \lambda < \frac{\pi}{2}$. The constant time section of the boundary is a non-singular geometry, consisting of two halves of $S^3$ at $\mu = \pm \mu_0$, joined at the pole of $S_3$ where $\lambda = \pi/2$. In Poincaré coordinates for the $AdS_4$ slices, the spatial section of the boundary consists of two three dimensional half planes joined by a two dimensional interface. The dilaton varies continuously with $\mu$ and takes two different constant values at the boundary,

$$\lim_{\mu \to \pm \mu_0} \phi(\mu) = \phi^{(0)}_{\pm} + \phi^{(1)}_{\pm} (\mu \mp \mu_0)^4 + O[(\mu \mp \mu_0)^8]$$

(3.11)

The holographic dual gauge theory CFT of the Janus solution was proposed in [12] and analyzed in detail in [17]. The CFT dual is a planar interface theory. The action on both sides of the interface is the standard $\mathcal{N} = 4$ SYM action but the coupling constant varies discontinuously across the interface. The symmetry $SO(2,3)$ of the Janus solution maps to the conformal symmetry of a planar interface on the CFT side. This symmetry is manifest at the classical level, but was also shown to persist at the first non-trivial quantum level [17]. The $SO(6)$ symmetry of the Janus solution maps to an (accidental) internal symmetry on the CFT side. Note that, in contrast to the defect conformal field theories examined earlier in the context of AdS/CFT [34, 35, 36, 37, 38], the CFT dual to the Janus solution is characterized by an interface that carries no degrees of freedom in addition to the ones inherited from the bulk $\mathcal{N} = 4$, whence the name “interface”, as opposed to “defect”.

3.3 The absence of breathing modes

The AdS/CFT dictionary relates the breathing mode of $S^5$, described by the function $h_1$ in the Janus metric of (3.1), to a dimension 8 operator $O^{(20)}_{k=0} = \text{tr}(F^2 F^2)$ (in the notation of [4]). The argument below will show that the breathing mode cannot be excited in the Janus solution, and thus the function $h_1$ must be a constant. On the one hand, a non-vanishing source for the operator $O^{(20)}_{k=0}$ would correspond to a behavior $(\mu - \mu_0)^{-4}$ near the AdS boundary, as $\mu \to \mu_0$. Such a behavior would lead to a singular 10-dimensional metric. Since the Janus solution is regular, this source must be absent. On the other hand, a non-vanishing expectation value would correspond to a behavior $(\mu - \mu_0)^8$ near the boundary. A power series expansion near the boundary of AdS for the Janus solution, and allowing for the presence of a breathing mode, reveals that such a term is forced to vanish. By standard AdS/CFT arguments the breathing mode is therefore exactly zero.
4 Supersymmetric interface CFT

In [17], it was shown that preserving supersymmetry in a planar interface Yang-Mills theory necessarily leads to the breaking of the internal SO(6). In turn, arguments are presented in [17] that 2 Poincaré supercharges may be preserved upon reducing SO(6) to SU(3), and including certain “interface counterterms” which are built out of the fields of the bulk $\mathcal{N} = 4$ super Yang-Mills theory.

Specifically, in [17], the Yang-Mills coupling $g(x^\pi)$ is assumed to be a function of the coordinate $x^\pi$ (the CFT side coordinate corresponding to the coordinate $\mu$ of the AdS side) and to vary across the interface located at $x^\pi = 0$. The Lagrangian is formulated with $\mathcal{N} = 1$ auxiliary fields for a single chiral multiplet and a single gauge multiplet,

\[
\begin{align*}
\mathcal{L}_{\text{chiral}} &= -\partial_{\mu} \bar{\phi} \partial^{\mu} \phi - \frac{i}{2} \bar{\psi} \gamma^{\mu} \partial_{\mu} \psi + \bar{F} F + \left(W' F - \frac{i}{4} W'' \bar{\psi} (1 + \gamma^5) \psi + \text{c.c.} \right) \\
\mathcal{L}_{\text{gauge}} &= -\frac{1}{4 g^2} F_{\mu \nu} F^{\mu \nu} - \frac{i}{2 g^2} \bar{\lambda} \gamma^{\mu} D_{\mu} \lambda + \frac{1}{2 g^2} D^a D^a
\end{align*}
\] (4.1)

Here, $\psi$ and $\lambda^a$ are Majorana spinors, $\phi$ is a complex scalar, $F$ and $D^a$ are auxiliary fields, and $W$ is the superpotential of the chiral multiplet. In $\mathcal{L}_{\text{chiral}}$, all dependence on the coupling $g$ is contained in $W$. Upon adding to $\mathcal{L}_{\text{chiral}}$ and $\mathcal{L}_{\text{gauge}}$ the following “interface counterterms”,

\[
\begin{align*}
\delta \mathcal{L}_{\text{chiral}} &= i \partial_\pi g \left( \frac{\partial W}{\partial g} - \frac{\partial \bar{W}}{\partial g} \right) \\
\delta \mathcal{L}_{\text{gauge}} &= -\partial_\pi \left( g^{-2} \left( \frac{i}{4} \bar{\varepsilon} \gamma^{\pi} \gamma^{\mu} \lambda^a F_{\mu \nu} + \frac{1}{2} \bar{\varepsilon} \gamma^{\pi} \gamma^5 \lambda^a D^a \right) \right)
\end{align*}
\] (4.2)

the combined Lagrangians are invariant under the supersymmetry generated by spinors $\varepsilon$ satisfying the interface projection relation, $1/2(1 + i \gamma^5 \gamma^\pi)\varepsilon = \varepsilon$. For the $\mathcal{N} = 4$ theory, we have $W \sim \varepsilon^{ijk} \Phi^i \Phi^j \Phi^k$, where $\Phi^k$ are 3 complex fields, $\Phi^k = \phi^{2k-1} + i \phi^{2k}$. This theory has SU(3) internal symmetry, which rotates the three chiral multiplets into one another.

Surprisingly, the derivation of the existence of the $\mathcal{N} = 1$ interface supersymmetry presented in [17] seems to depend on the precise normalizations of the chiral and gauge multiplet Lagrangians: canonical for the chiral multiplet, but with the gauge coupling factored out for the gauge multiplet.

In a companion paper [21], the existence of supersymmetry in the $\mathcal{N} = 4$ super Yang-Mills theory with an interface and with “interface counterterms” is solved in generality. We confirm that the model of [17] indeed possesses $\mathcal{N} = 1$ interface supersymmetry and SU(3) global symmetry, independently of any normalization issues. Further models with supersymmetry,
including one with $\mathcal{N} = 4$ interface supersymmetry and $SO(4)$ internal symmetry, are also discovered in [21] along with a complete classification of all possible supersymmetries.

It may be helpful to clarify the counting of the number of preserved supersymmetries. Since the interface field theory has only 2+1 dimensional Poincaré invariance, the counting of supersymmetries is conveniently carried out from a three dimensional point of view. An $\mathcal{N} = 1$ interface supersymmetry corresponds to 2 real Poincaré supercharges. When, in addition, the interface field theory is conformal invariant, the number of supercharges is double the number of Poincaré supercharges. Thus, the $\mathcal{N} = 1$ interface CFT has altogether 4 real supercharges. In the dual supergravity this means that we look for, and find, a two-parameter family of solutions which preserve 4 real supersymmetries.
5 The ten-dimensional Janus Ansatz

In this section, we shall construct the most general Ansatz for Type IIB supergravity fields, consistent with the symmetries of the expected CFT dual theory with \( \mathcal{N} = 1 \) interface supersymmetry and global \( SU(3) \) symmetry relating the 3 chiral multiplets inherited from the parent \( \mathcal{N} = 4 \) theory.

5.1 Symmetries of the Ansatz

For given gauge and interface couplings, the dual CFT has \( SO(2,3) \) conformal and \( SU(3) \) internal symmetry, along with 2 Poincaré and 2 conformal supersymmetries. Therefore, on the AdS side, we shall seek an Ansatz for the supersymmetric generalization of the Janus solution which is invariant under the following bosonic symmetries,

\[
SO(2,3) \times SU(3) \tag{5.1}
\]

The supersymmetries will be achieved later by enforcing the BPS conditions. In analogy with Janus, the solution is expected to depend continuously on at least one parameter (for Janus, this is the constant \( c_0 \)) and include the undeformed \( AdS_5 \times S^5 \) as a limiting case (for Janus, \( c_0 \to 0 \)). Therefore, the topology of the internal space is expected to remain the same and equal to the topology of \( S^5 \).

The interface CFT naturally consists of a family of theories. This is familiar from the parent \( \mathcal{N} = 4 \) theories, which are labeled by the gauge coupling \( g \) and the instanton angle \( \theta \), and are mapped into one another by the standard action of \( SL(2,\mathbb{Z}) \) on \( g \) and \( \theta \). Montonen-Olive duality states that two theories related by \( SL(2,\mathbb{Z}) \) are physically the same. On the AdS side, \( SL(2,\mathbb{Z}) \) degenerates to \( SL(2,\mathbb{R}) \), which constitutes a symmetry of Type IIB supergravity. Therefore, on the AdS side, we shall seek an Ansatz which forms a family on which \( SL(2,\mathbb{R}) \) acts consistently as well. For example, our previous generalization of Janus, which includes the axion and the dilaton, is such a family of Ansätze, while the original Janus Ansatz clearly is not, since \( SL(2,\mathbb{R}) \) does not act consistently on it.

The interface CFT is also naturally a family of theories in terms of its interface couplings. Indeed, the interface couplings of the CFT dual theory of [17] and [21] are mapped into one another by \( SU(4) \). Theories with different interface couplings related in this way by \( SU(4) \) are physically the same. On the AdS side, not all of these \( SU(4) \) transformations can be implemented in a useful way. Once an Ansatz has been forced to be invariant under \( SU(3) \), the embedding of \( SU(3) \) in \( SU(4) \) is fixed, and the only useful transformations of \( SU(4) \) that can be implemented on the \( SU(3) \)-invariant Ansatz are those that commute with \( SU(3) \).
is a single generator, spanning a group $U(1)_\beta$, the notation of which will be motivated later on. To summarize, we shall seek an Ansatz for the supersymmetric generalization of the Janus solution which is invariant under the group

$$SO(2, 3) \times SU(3) \times U(1)_\beta \times SL(2, \mathbb{R})$$

(5.2)

in the sense described above.

The presence of the $SO(2, 3)$ factor requires the Ansatz to be an $AdS_4$ slicing of $AdS_5$, as was already the case for the original Janus. The group $SU(3) \times U(1)_\beta$ must be realized as an isometry of the 5-dimensional internal space. The only 4-dimensional manifold with $SU(3)$ isometry is $CP_2 = SU(3)/S(U(2) \times U(1))$. The product space $CP_2 \times S^1$ thus has the correct isometry $SU(3) \times U(1)_\beta$, but not the correct topology of $S^5$. The topology of $S^5$ is easily recovered by recalling that $S^5$ is the total space of a $S^1$ bundle over $CP_2$ [44, 45, 46, 47]. A natural candidate internal space for our Ansatz is $CP_2 \times _q S^1$, where $\times_q$ produces the $S^1$ bundle over $CP_2$ with integer first Chern class $q \in \mathbb{Z}$. The requirement that this space have the topology of $S^5$ reduces this choice to $q = \pm 1$, so we may simply set $q = 1$. (Note that another candidate internal space $SU(3)/SO(3)$ has the topology of $S^5$, but its isometry does not include the $U(1)_\beta$ factor.)

Based on these symmetry considerations, we conclude that the Ansatz must be constructed on the space

$$\mathbb{R} \times AdS_4 \times CP_2 \times_1 S^1$$

(5.3)

where $CP_2 \times_1 S^1$ is the sphere $S^5$, deformed while preserving $SU(3) \times U(1)_\beta$ isometry.

### 5.2 Invariant Metrics and Frames on $CP_2$

In this section, we summarize important properties of $CP_2$ and its $S^1$ fiber bundle $S^5$, and derive all the invariants needed for the construction of the invariant Ansatz. Recall that both spaces may be viewed as symmetric spaces via the following cosets $S^5 = SO(6)/SO(5)$ and $CP_2 = SU(3)/S(U(2) \times U(1))$. But, $S^5$ may also be viewed as a non-symmetric homogeneous space via the coset $S^5 = SU(3)/SU(2)$, from which it is clear that $S^5$ is the total space of a $S^1 = U(1)$ bundle over $CP_2$.

The space $CP_2$ may be parametrized locally by two complex coordinates $\zeta_1, \zeta_2 \in \mathbb{C}$ or, equivalently, by four real angles $\alpha, \theta, \phi, \psi$, related to one another by

$$\zeta_1 = \tan(\alpha) \cos(\theta/2)e^{i(\psi+\phi)/2}$$
$$\zeta_2 = \tan(\alpha) \sin(\theta/2)e^{i(\psi-\phi)/2}$$

(5.4)

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The Fubini-Study metric is given by
\[ ds^2_{CP^2} = g_{ij} \bar{d} \zeta^i \otimes d \bar{\zeta}^j \]
\[ g_{ij} = \partial_i \partial_j \ln(1 + |\zeta_1|^2 + |\zeta_2|^2) \] (5.5)

It is useful to work with an orthonormal frame \( \hat{e}^a \), \( a = 6, 7, 8, 9 \) on \( CP^2 \) which may be expressed in terms of the angular coordinates,

\[
\begin{align*}
\hat{e}^6 &= d\alpha \\
\hat{e}^7 &= \frac{1}{4} \sin(2\alpha) \sigma_3 \\
\hat{e}^8 &= \frac{1}{2} \sin(\alpha) \sigma_1 \\
\hat{e}^9 &= \frac{1}{2} \sin(\alpha) \sigma_2
\end{align*}
\]
\[ \sigma_3 = d\psi + \cos(\theta)d\phi \]
\[ \sigma_1 = -\sin(\psi)d\theta + \cos(\psi)\sin(\theta)d\phi \]
\[ \sigma_2 = \cos(\psi)d\theta + \sin(\psi)\sin(\theta)d\phi \] (5.6)

The metric may be expressed in terms of the above coordinates and frames by,
\[ ds^2_{CP^2} = d\alpha^2 + \frac{1}{4} \sin^2(\alpha) \left( \sigma_1^2 + \sigma_2^2 + \cos^2(\alpha) \sigma_3^2 \right) \]
\[ = \sum_{a=6}^{9} \hat{e}^a \otimes \hat{e}^a = \hat{e}^{z_1} \otimes \hat{e}^{\bar{z}_1} + \hat{e}^{z_2} \otimes \hat{e}^{\bar{z}_2} \] (5.8)

We shall also need the torsion-free connection associated with this frame, satisfying
\[ d\hat{e}^a + \hat{\omega}^a_{\ b} \wedge \hat{e}^b = 0 \] (5.9)

These connection components are given by,
\[
\begin{align*}
\hat{\omega}^6_7 &= -\frac{1}{2} \cos(2\alpha) \sigma_3 \\
\hat{\omega}^8_9 &= \left( 1 - \frac{1}{2} \cos^2(\alpha) \right) \sigma_3 \\
\hat{\omega}^6_8 &= +\hat{\omega}^7_9 = -\frac{1}{2} \cos(\alpha) \sigma_1 \\
\hat{\omega}^6_9 &= -\hat{\omega}^7_8 = -\frac{1}{2} \cos(\alpha) \sigma_2
\end{align*}
\] (5.10)

\[ ^8 \text{We choose the labels 6, 7, 8, 9 for internal labels, as it is in this manner that the indices will be embedded in 10 dimensional space-time.} \]
5.3 Invariant 2-forms on $CP_2$ and $S^5$

In this subsection, we shall obtain the most general 2-forms on $S^5$, invariant under $SU(3)$, and use them to build an $SU(3) \times U(1)_\beta$-invariant Ansatz for the antisymmetric tensor field $B_{(2)}$ in the subsequent section. It is well-known that the cohomology of $CP_2$ is generated by the Kähler form $K$, which derives from a $U(1)$-connection $A_1$ by $K = dA_1$. In terms of the coordinates of $CP_2$ introduced earlier, these forms are given by,

$$
K = ig_{ij} d\zeta^i \wedge d\bar{\zeta}^j = \frac{1}{2} \sin(2\alpha) d\alpha \wedge \sigma_3 + \frac{1}{2} \sin^2(\alpha) \sigma_1 \wedge \sigma_2
$$

$$
A_1 = -i \frac{\zeta^i d\bar{\zeta}^i - \bar{\zeta}^i d\zeta^i}{2 \sqrt{1 + |\zeta_1|^2 + |\zeta_2|^2}} = \frac{1}{2} \sin^2(\alpha) \sigma_3
$$

In terms of the frames $\hat{e}^a$, we have

$$
K = 2 \hat{e}^6 \wedge \hat{e}^7 + 2 \hat{e}^8 \wedge \hat{e}^9 = i\hat{e}^{z_1} \wedge \hat{e}^{\bar{z}_1} + i\hat{e}^{z_2} \wedge \hat{e}^{\bar{z}_2}
$$

The Kähler form is simply related to the volume form as follows, $K^2 = 8\hat{e}^6 \wedge \hat{e}^7 \wedge \hat{e}^8 \wedge \hat{e}^9$.

The five sphere $S^5$ is constructed as a $S^1$ fibration over $CP_2$. Introducing the $S^1$ coordinate $\beta$, the isometry group $U(1)_\beta$ acts on $S^1$ by shifts of $\beta$. The round metric on $S^5$ is then given in terms of this fibration by,

$$
ds^2_{S^5} = (d\beta + A_1)^2 + ds^2_{CP_2}
$$

where $A_1$ is the Kähler one form defined in (5.11). We introduce a fifth frame component

$$
\hat{e}^5 = d\beta + A_1
$$

Under the action of $SU(3)$, the connection $A_1$ shifts by a non-trivial exact form, which may be compensated for by the opposite shift in $\beta$, so that the combination $d\beta + A_1$ is invariant under $SU(3)$. It is of course also invariant under constant shifts of $\beta$, forming the group $U(1)_\beta$. The frame $\hat{e}^5, \hat{e}^6, \hat{e}^7, \hat{e}^8, \hat{e}^9$ is an orthonormal frame for $S^5$.

Clearly, the Kähler form $K$ on $CP_2$ is invariant under $SU(3)$ and will be a candidate for an invariant 2-form in the Ansatz for the antisymmetric tensor field $B_{(2)}$. To ensure a consistent Ansatz, however, we shall need the most general 2-form invariant under $SU(3)$ on the deformed sphere $S^5$, and this requires an exhaustive study of all $SU(3)$-invariant 2-forms on $S^5$. In this task, we are helped by two theorems (see for example the corollaries 1.18 and 1.14 of [39], as well as [40]), valid for compact, connected $G$ and $H$,..
**Theorem I**  The ring of $G$-invariant $n$-forms on a homogeneous space $G/H$ is obtained from a basis of left-invariant 1-forms $\theta^a$, $a = 1, \ldots, \dim G$ on $G$, by constant tensors $\omega_{a_1 a_2 \cdots a_n}$, which vanish whenever $a_i \in \mathcal{H}$, and are invariant under $\mathcal{H}$, by the expression,

\[ \omega^{(n)} = \omega_{a_1 a_2 \cdots a_n} \theta^{a_1} \wedge \cdots \wedge \theta^{a_n} \]  

(5.15)

**Theorem II**  On a symmetric space $G/H$, every $G$-invariant form is closed.

Given that $S^5 = SO(6)/SO(5)$ is a symmetric space coset, and that $H^2(S^5, \mathbb{R}) = 0$, there are no $SO(6)$-invariant 2-forms on $S^5$, a fact that was used to set $B(2) = 0$ in the original Janus solutions. Given, that the coset $CP_2 = SU(3)/S(U(2) \times U(1))$ is a symmetric space, and that $H^2(CP_2, \mathbb{R})$ is generated by a single element, namely the Kähler form $K$, we conclude that $K$ is the unique $SU(3)$-invariant 2-form on $CP_2$. These facts are standard.

Finally, we wish to obtain all $SU(3)$-invariant 2-forms on $S^5$. To this end, we express $S^5$ as the coset $S^5 = SU(3)/SU(2)$, where $SU(2)$ is embedded in $SU(3)$ in the 2-dimensional representation. This coset is not a symmetric space, so that $SU(3)$-invariant forms need not be closed. It is now straightforward to obtain all such invariant forms, using Theorem I. We need all $SU(2)$-invariant tensors on the left-invariant 1-forms $\theta^a$ on $SU(3)$, which vanish on $SU(2)$. These 1-forms are precisely $\hat{e}^{z_1}, \hat{e}^{z_2}, \hat{e}^{\bar{z}_1}, \hat{e}^{\bar{z}_2}, \hat{e}^5$ constructed earlier. Clearly, we recover the Kähler form of (5.12) following the construction of Theorem I in this manner. There is also the following 2-form (and its complex conjugate),

\[ \hat{e}^{z_1} \wedge \hat{e}^{z_2} = \frac{1}{2} \sum_{i,j=1,2} \varepsilon_{ij} \hat{e}^{\bar{z}_i} \wedge \hat{e}^{\bar{z}_j} \]  

(5.16)

which is not invariant under the action of $SU(3)$ isometries on $CP_2$, because the form is not invariant under the $U(1)$ factor of the isotropy group of $CP_2$. It can be made into a well-defined $SU(3)$-invariant form $\hat{A}_2$ on $S^5$ by compensating for the phase factor,

\[ \hat{A}_2 \equiv \hat{e}^{z_1} \wedge \hat{e}^{z_2} e^{3i\beta} = \frac{i d\zeta^1 \wedge d\zeta^2 e^{3i\beta}}{(1 + |\zeta_1|^2 + |\zeta_2|^2)^{3/2}} \]  

(5.17)

Some further useful properties involving $\hat{A}_2$ are collected below.

\[ d\hat{A}_2 = 3i (d\beta + A_1) \wedge \hat{A}_2 = 3i \hat{e}^5 \wedge \hat{A}_2 \]

\[ \hat{A}_2 \wedge \tilde{\hat{A}}_2 = \frac{1}{2} K^2 = 4 \hat{e}^6 \wedge \hat{e}^7 \wedge \hat{e}^8 \wedge \hat{e}^9 \]  

(5.18)

In particular, the formula for the differential shows that $\hat{A}_2$ is indeed the solution to an $SU(3)$-invariant equation $(d - 3i A_1) \hat{A}_2 = 0$, which is consistent with the fact that $\hat{A}_2$ itself is invariant. By contrast, the form $\hat{e}^{z_1} \wedge \hat{e}^{z_2}$ satisfies a differential equation $(d - 3i A_1)(\hat{e}^{z_1} \wedge \hat{e}^{z_2}) = 0$ which is not invariant. Under $U(1)_\beta$, the form $\hat{A}_2$ transforms with a constant phase factor.
5.4 The Ansatz for the metric

The Ansatz for the metric follows from the symmetry considerations above,

\[ ds^2 = f_4^2 (d\mu^2 + ds_{AdS_4}^2) + f_1^2 (d\beta + A_1)^2 + f_2^2 ds_{CP_2}^2 \]  

(5.19)

Invariance of the metric under \( SO(2,3) \times SU(3) \times U(1)_\beta \) requires that the functions \( f_1, f_2 \) and \( f_4 \) depend only on \( \mu \). According to the transformation rules of Type IIB supergravity, the metric, in the Einstein frame, must be invariant under \( SL(2,R) \), which requires that the functions \( f_1, f_2, \) and \( f_4 \) are invariant under \( SL(2,R) \). The associated orthonormal frame is given by the following set of 1-forms,

\[ e^i = f_4 \hat{e}^i \quad i = 0, 1, 2, 3 \]
\[ e^4 = f_4 d\mu \]
\[ e^5 = f_1 \hat{e}^5 = f_1 (d\beta + A_1) \]
\[ e^a = f_2 \hat{e}^a \quad a = 6, 7, 8, 9 \]  

(5.20)

For \( i = 0, 1, 2, 3 \), the \( \hat{e}^i \) span the orthonormal frame for \( AdS_4 \) and may be chosen as follows,

\[ \hat{e}^0 = r^{-1} dr \quad \hat{e}^i = r^{-1} dx^i \quad i = 1, 2, 3 \]  

(5.21)

For \( a = 6, 7, 8, 9 \), the \( \hat{e}^a \) span the orthonormal frame on \( CP_2 \) of (5.6). The volume form\(^9\) is

\[ e^{0123456789} = f_1 f_2^4 f_4^5 d\mu \wedge \hat{e}^{0123} \wedge \hat{e}^{56789} \]  

(5.22)

5.5 The Ansatz for the antisymmetric tensor fields

Invariance under \( SO(2,3) \times SU(3) \times U(1)_\beta \) requires the self-dual 5-form to be of the form,

\[ F_{(5)} = f_5 \left( -e^{01234} + e^{56789} \right) \]  

(5.23)

where \( f_5 \) is a scalar function that depends only on \( \mu \), by the same arguments as used for the metric. Recall that \( F_{(5)} \) and thus \( f_5 \) must also be invariant under \( SL(2,R) \).

To construct a 2-form \( B_{(2)} \) which is invariant under \( SO(2,3) \times SU(3) \times U(1)_\beta \times SL(2,R) \), we make use of the \( SU(3) \) invariant 2-forms \( K, \hat{A}_2, \) and \( \tilde{A}_2 \). Since \( B_{(2)} \) is complex, we include \( \hat{A}_2 \) and \( \tilde{A}_2 \) with independent complex coefficient functions \( f_3 \) and \( \tilde{g}_3 \),

\[ B_{(2)} = i f_3 \hat{A}_2 - i \tilde{g}_3 \tilde{A}_2 + f_6 K \]  

(5.24)

\(^9\)We shall introduce the following notation, \( e^{i_1 i_2 \cdots i_p} \equiv e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p} \) and use it throughout.
It will turn out that the Type IIB field equation (2.10) for $G_{\mu
u\rho}$ force $f'_6 = 0$ and render this term pure gauge; therefore we shall set $f_6 = 0$ in the sequel. The field strength $F_{(3)}$ is then,

$$F_{(3)} = i \frac{f'_3}{f_4 f_2^2} e^4 \wedge A_2 - i \frac{g'_3}{f_4 f_2^2} e^4 \wedge \tilde{A}_2 \quad (5.25)$$

The associated composite $G$ is given by

$$G = a e^5 \wedge A_2 - b e^4 \wedge A_2 - c e^5 \wedge \tilde{A}_2 - d e^4 \wedge \tilde{A}_2 \quad (5.26)$$

where the coefficient functions are given by

$$a = -\frac{3}{f_1 f_2^2} f(f_3 - B g_3) \quad c = -\frac{3}{f_1 f_2^2} f(g_3 - B \bar{f}_3)$$
$$b = -\frac{1}{f_4 f_2^2} f(f'_3 - B g'_3) \quad d = +\frac{1}{f_4 f_2^2} f(\bar{g}'_3 - B \bar{f'_3}) \quad (5.27)$$

For later convenience, we have here expressed these forms in terms of the frame $e^a$, for which

$$A_2 = f_2^2 \hat{A}_2 = (e^6 + i e^7) \wedge (e^8 + i e^9) e^{3i\beta} \quad (5.28)$$

With $f_3$ and $g_3$ functions only of $\mu$, the Ansatz in (5.24) is invariant under $SO(2, 3) \times SU(3)$. Under $U(1)_{\beta}$, the forms $A_2$ and $\tilde{A}_2$ transform with constant opposite phases. Thus, $U(1)_{\beta}$ is not a symmetry of any one Ansatz, but rather relates one Ansatz to another.

### 5.6 Transformation properties under $SL(2, \mathbb{R})$

Under $SL(2, \mathbb{R})$, the metric (in the Einstein frame) and the 5-form $F_{(5)}$ are invariant. The dilaton/axion field $B$, and the associated function $f$, transform as

$$B^s = \frac{u B + v}{\bar{v} B + \bar{u}} \quad f^s = |\bar{v} B + \bar{u}| f \quad \left( f^2 B' \right)^s = e^{2i\theta} f^2 B' \quad (5.29)$$

where the superscripts $s$ indicate the transformed objects, $u, v \in \mathbb{C}$ and $\bar{u} u - \bar{v} v = 1$. The functions $f_3$ and $g_3$ transform linearly, according to (2.13), and we have,

$$f_3^s = u f_3 + v g_3 \quad f^s(f_3^s - B^s g_3^s) = e^{i\theta} f(f_3 - B g_3)$$
$$g_3^s = \bar{v} f_3 + \bar{u} g_3 \quad f^s(g_3^s - B^s \bar{f}_3^s) = e^{i\theta} f(\bar{g}_3 - B \bar{f}_3) \quad (5.30)$$

where the phase $\theta$ is a field-dependent transformation parameter, given by

$$e^{i\theta} = \left( \frac{v B + u}{\bar{v} B + \bar{u}} \right)^{\frac{1}{2}} \quad (5.31)$$

Since the phases of all terms in $G$ are the same, we get a covariant formula, $G^s = e^{i\theta} G$. 

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6 The Reduced Bianchi identities and Field Equations

In this section, we reduce the Type IIB Bianchi identities and field equations, given in section 2, to the Ansatz constructed in section 5. The Bianchi identities (2.4), (2.5), and (2.6) are automatically satisfied. The Bianchi identity (2.7) for $F_{(5)}$ reduces to

$$f_5' = -4 \frac{f_2'}{f_2} f_5 - \frac{f_1'}{f_1} f_5 + \frac{1}{2} f_4 (a\bar{b} + \bar{a}b + c\bar{d} + \bar{c}d)$$

and is solved by

$$f_5 = \frac{3}{2} \frac{|f_3|^2 - |g_3|^2 + C_1}{f_1 f_2^2}$$

which fixes $f_5$ in terms of $f_1, f_2, f_3, g_3$, and the (real) integration constant $C_1$.

6.1 The reduced field equations for $B$ and $G$

The field equation for the complex scalar $B$ reduces to

$$B'' + B' \left( 3 \frac{f_4'}{f_4} + \frac{f_1'}{f_1} + 4 \frac{f_2'}{f_2} \right) + 2 f_2 B' B' + 2 \frac{f_4^2}{f_2} (ac - bd) = 0$$

where the variables $a, b, c, d$ were introduced in (5.27).

To reduce the field equations of the antisymmetric tensor field $B_{(2)}$, it is convenient to first recast (2.10) in terms of differential forms,

$$* d(*G) + i (i_Q G) + (i_P \bar{G}) - 4i (i_G F_{(5)}) = 0$$

Here $i_V G$ stands for the contraction of $G$ with $V$. To calculate this equation, it is helpful to have the following contractions, $i_{e^5 \wedge A_2} e^{56789} = A_2$, as well as the complex conjugate relation. Identifying terms in $A_2$, we find, after some simplification,

$$f_3'' - B g_3'' - 9 \frac{f_4^2}{f_1} (f_3 - B g_3) + \left( \frac{f_1'}{f_1} + 3 \frac{f_4'}{f_4} \right) (f_3' - B g_3')$$

$$- 2 f_2^2 B' (g_3' - B f_3') - 12 \frac{f_4^2 f_5}{f_1} (f_3 - B g_3) = 0$$

---

10 Our conventions for the Poincaré dual are given via the following pairing relation between two arbitrary rank $p$ differential forms $S_{(p)}$ and $T_{(p)}$, by $S_{(p)} \wedge * T_{(p)} = \frac{1}{p!} S_{(p)}^{a_1 \ldots a_p} T_{(p)}_{a_1 \ldots a_p} e^{0123456789}$. In particular, we have $* S_{(p)} = (-1)^{p+1} S_{(p)}$, and the duals $e^{01234} = -e^{56789}$, and $e^{56789} = e^{012345}$, which will be useful later on.
while in $\tilde{A}_2$, the equation is obtained from (6.5) by letting $f_3 \to \tilde{g}_3$, and $f_5 \to -f_5$, leaving all other functions unchanged. It is actually more convenient to express these field equations in terms of the coefficient functions $a, b, c, d$ of $G$, and we find (with $Q = Q_\mu d\mu$),

\begin{align*}
a' - iQ_\mu a &= -(\frac{f_1'}{f_1} + 2\frac{f_2'}{f_2})a + 3\frac{f_4}{f_1}b - f^2 B'c \\
b' - iQ_\mu b &= -(4\frac{f_4'}{f_4} + 2\frac{f_2'}{f_2} + \frac{f_1'}{f_1})b + 3\frac{f_4}{f_1}a - f^2 B'd + 4f_4f_5a \\
c' - iQ_\mu c &= -(\frac{f_1'}{f_1} + 2\frac{f_2'}{f_2})c - 3\frac{f_4}{f_1}d - f^2 B'a \\
d' - iQ_\mu d &= -(4\frac{f_4'}{f_4} + 2\frac{f_2'}{f_2} + \frac{f_1'}{f_1})d - 3\frac{f_4}{f_1}c + 4f_4f_5c - f^2 B'b \\
\end{align*}

(6.6)

### 6.2 Reducing Einstein’s equations

To reduce the Einstein equations in (2.11), we first obtain the Ricci curvature tensor. It is convenient to carry out all calculations using the orthonormal frame of (5.20), which we shall denote collectively by $e^A$ where $A = (i, 4, 5, a)$ with $i = 0, 1, 2, 3$, and $a = 6, 7, 8, 9$. The torsion-free connection $\omega^A_B$, the associated curvature $\Omega^A_B$, the Riemann tensor $R^A_{BCD}$, and the Ricci tensor (all expressed in frame indices) are then defined by the relations,

\begin{align*}
0 &= de^A + \omega^A_B \wedge e^B \\
\Omega^A_B &= d\omega^A_B + \omega^A_C \wedge \omega_C^B \\
\Omega^A_B &= \frac{1}{2} R^A_{BCD} e^C \wedge e^D \\
R_{BD} &= R^A_{BAD} \\
\end{align*}

(6.7)

where $A, B, C, D = 0, 1, 2, \cdots, 9$. The corresponding objects for the unwarped $AdS_4$ and $CP_2$ geometries will be denoted with hats; they are the frames $\hat{e}^i, \hat{e}^a$, the connections $\hat{\omega}^i_j, \hat{\omega}^a_b$, the curvatures $\hat{\Omega}^i_j, \hat{\Omega}^a_b$, the Riemann tensors $\hat{R}^i_{jkl}, \hat{R}^a_{bcd}$, and the Ricci tensors $\hat{R}_{ij}, \hat{R}_{bd}$. They all obey the equations (6.7) with hatted objects and for the ranges of indices $i, j, k, l = 0, 1, 2, 3$ and $a, b, c, d = 6, 7, 8, 9$. (The unwarped geometry also has structure in the direction 5, where it can be viewed as the $S^1$ fiber of $S^5$ over $CP_2$; we shall treat this direction as separate.) The $AdS_4$ curvature $\hat{\Omega}$ is calculated using the unwarped frame in (5.21), and is given by \[1\]

$$
\hat{\Omega}_{ij} = -\frac{1}{2}(\eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk}) \hat{e}^k \wedge \hat{e}^l
$$

\[1\] The explicit forms of the $AdS_4$ and $CP_2$ unwarped connections will not be needed here.
Finally, for later use, we record also the Ricci scalar,

\[ R_{ijkl} = -(\eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk}) \]
\[ R_{ij} = -3\eta_{ij} \]  

The \( CP_2 \) curvature is calculated using the unwarped frame in (5.6) and is given by

\[ \hat{\Omega}_{ab} = (\delta_{ac}\delta_{bd} + k_{ac}k_{bd} + k_{ad}k_{bd}) \hat{e}^c \wedge \hat{e}^d \]
\[ \hat{R}_{abcd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} + k_{ac}k_{bd} - k_{ad}k_{bc} + 2k_{ab}k_{cd} \]
\[ \hat{R}_{bd} = 6\delta_{bd} \]  

(6.9)

where the tensor \( k \) is anti-symmetric and defined by \( k_{cd}\hat{e}^c \wedge \hat{e}^d = 2K = 2\hat{e}^6 \wedge \hat{e}^7 + 2\hat{e}^8 \wedge \hat{e}^9 \).

The connection components of the full geometry are needed to compute the curvature as well as the supersymmetry variation equations. They are given by \( \omega_{ij} = -\omega_{ji} \) and

\[ \omega_{ij} = \hat{\omega}_{ij} \]  
\[ \omega_{4} = \frac{f'_1}{f_1^2} e^4 \]  
\[ \omega_{5} = \frac{f'_1}{f_1^2} e^5 \]
\[ \omega_{a} = \frac{f'_1}{f_2^2} e^a \]

\[ \omega_{4} = \hat{\omega}_{ab} - \frac{f_1}{f_2^2} k_{ab} e^5 \]  

(6.10)

The components of the curvature form \( \Omega^A_B \) and of the Riemann tensor \( R^A_{BCD} \) are needed only to evaluate the Ricci tensor, and will not be exhibited here. The non-vanishing components of the Ricci tensor are given by,

\[ R_{ij} = -\eta_{ij} \left( \frac{3}{f_1^2} + 2\frac{(f'_1)^2}{f_1^4} + \frac{f''_1 f'_1}{f_1^3} + \frac{f'_1 f''_1}{f_1^3} + 4\frac{f'_1 f'_1}{f_2 f_1^3} \right) \]
\[ R_{44} = -4\frac{f''_1}{f_1^3} + 4\frac{(f'_1)^2}{f_1^4} + \frac{f'_1 f''_1}{f_1^4} - \frac{f''_1 f'_1}{f_1^4} + 4\frac{f''_1 f'_1}{f_2 f_1^3} - 4\frac{f'_1}{f_2 f_1^3} \]
\[ R_{55} = -\frac{f''_1}{f_1^3} - 3\frac{f'_1 f'_1}{f_1^4} - 4\frac{f'_1 f''_1}{f_1^4} + 4\frac{f'_1 f'_1}{f_2 f_1^3} + 4\frac{f'_1}{f_2 f_1^3} \]
\[ R_{ab} = \delta_{ab} \left( \frac{6}{f_2^2} - \frac{f''_1}{f_2 f_1^3} - 3\frac{f''_1 f'_1}{f_2 f_1^3} - \frac{f'_1 f''_1}{f_2 f_1^3} - 2\frac{f'_1}{f_2^2} - 3\frac{(f'_1)^2}{f_2 f_1^3} \right) \]  

(6.11)

Finally, for later use, we record also the Ricci scalar,

\[ R = \frac{4}{f_1^2} \left( 3\frac{(f'_1)^2}{f_1^2} + 3\frac{(f'_1)^2}{f'_1} - 3 + 2\frac{f'_1 f''_1}{f_1 f_2} + 2\frac{f'_1 f'_1}{f_1 f_4} + 8\frac{f''_1 f'_1}{f_2 f_4} - \frac{f''_1 f'_1}{f_2} + \frac{f''_1 f''_1}{f_2} + 6\frac{f'_1}{f_2^2} \right) \]  

(6.12)
which will be used to evaluate the reduced action.

It is straightforward to evaluate the “matter” contributions to Einstein’s equations, which leads to the following reduced Einstein equations.

For \( i, j = 0, 1, 2, 3 \), we have,

\[
3 + 2 \frac{(f_i')^2}{f_i} + \frac{f''_i}{f_i} + \frac{f'_i f'_j}{f_i f_j} + 4 \frac{f'_j f'_i}{f'_i f'_j} - 4 f'_i f'_j = \frac{1}{2} f^2 \left(-|a|^2 - |b|^2 - |c|^2 - |d|^2\right) = 0 \tag{6.13}
\]

For 44, we have,

\[
4 \frac{f''_i}{f_i} - 4 \frac{(f'_i)^2}{f'_i} - \frac{f'_i f'_j}{f'_i f'_j} + \frac{f''_j}{f''_j} - 4 \frac{f'_i f'_j}{f'_i f'_j} + 4 \frac{f''_j}{f''_j} - 4 f''_i f''_j + 2 f^4 |B'|^2 = \frac{1}{2} f^2 \left(-|a|^2 + 3 |b|^2 - |c|^2 + 3 |d|^2\right) = 0 \tag{6.14}
\]

For 55, we have

\[
\frac{f''_i}{f_i} + \frac{3 f'_i f'_j}{f'_i f'_j} + 4 \frac{f'_i f'_j}{f'_i f'_j} - 4 \frac{f''_i}{f''_i} \frac{f''_j}{f''_j} + 4 f''_i f''_j = \frac{1}{2} f^2 \left(3 |a|^2 - |b|^2 + |c|^2 - |d|^2\right) = 0 \tag{6.15}
\]

and finally for \( a, b = 6, 7, 8, 9 \), we have,

\[
- \frac{6 f''_i}{f''_i} + \frac{f''_i}{f''_i} + \frac{3 f'_i f'_j}{f'_i f'_j} + \frac{f'_i f'_j}{f'_i f'_j} + 2 \frac{f''_i}{f''_i} + 3 \frac{(f'_i)^2}{f'_i} + 4 f''_i f''_j
\]

\[
+ \frac{1}{2} f^2 \left(|a|^2 + |b|^2 + |c|^2 + |d|^2\right) = 0 \tag{6.16}
\]

Recall that here, as before, the variables \( a, b, c, d \) are given in terms of the independent functions \( f_3, g_3 \) by the definitions (5.27).

### 6.3 Integral of motion

The 4 Einstein equations possess a single integral of motion, which may be viewed as the (vanishing) Hamiltonian, or Wheeler-De Witt equation. It is obtained by adding the 4 equations with the following multiplicative factors, +4, −1, +1, +4, and is given by

\[
12 + 12 \frac{(f'_i)^2}{f_i} + 12 \frac{(f'_i)^2}{f'_i} + 8 f'_i f'_i + \frac{8 f'_i f'_j}{f'_i f'_j} + \frac{8 f'_i f'_j}{f'_i f'_j} + 32 \frac{f'_i f'_j}{f'_i f'_j} + 4 \frac{f''_i f''_j}{f''_j} - 24 f''_i f''_j + 8 f''_i f''_j - 2 f^4 |B'|^2 + 2 f^4 \left(|a|^2 - |b|^2 + |c|^2 - |d|^2\right) = 0 \tag{6.17}
\]
In general, the above system of reduced equations (including the reduced equations for $B$ and for $a, b, c, d$) does not appear to possess any further first integrals.

### 6.4 Reduced Action Principle

In this subsection, we shall show that the reduced field equations, listed above, may be derived from the Type IIB action (2.14), reduced to our Ansatz. To show this, we shall need one more ingredient in the construction of the Ansatz that was not needed for the field equations, but is needed for the action, namely the anti-symmetric tensor $C^{(4)}$. The starting point is the relation between $C^{(4)}$ and $F^{(5)}$ in (2.2). Using the Ansatz for $F^{(5)}$, $B^{(2)}$ and $F^{(3)}$, as well as the Bianchi identities (6.2), we obtain the following on-shell expression for $dC^{(4)}$,

$$dC^{(4)} = -f_4^* f_5 e^{01234} + \frac{3}{2} C_1 e^{56789} - \frac{i}{4} \left( f_3 f_3' - \tilde{f}_3 f_3' - g_3 \tilde{g}_3' + \bar{g}_3 \bar{g}_3' \right) e^{46789}$$  \hspace{1cm} (6.18)

Therefore, the most general Ansatz for $C^{(4)}$ is as follows,

$$C^{(4)} = g_5 e^{0123} + \frac{3}{2} C_1 \beta e^{6789} + h_5 e^{6789}$$  \hspace{1cm} (6.19)

where $g_5$ and $h_5$ are functions of $\mu$ only. The term proportional to $C_1$ accounts for the term proportional to $C_1$ in (6.18), via the fact that $e^{56789} = d\beta \wedge e^{6789}$.

To obtain an off-shell formulation, for use in the action, we postulate (6.19) as the Ansatz for $C^{(4)}$. This is clearly the most general $SO(2, 3) \times SU(3) \times U(1)_\beta$ invariant 4-form we can write down. Evaluating $F^{(5)}$ now from the invariant Ansatz for $C^{(4)}$ in (6.19), we find an expression which is not necessarily self-dual,

$$F^{(5)} = g_5' e^{01234} + \frac{3}{2} C_1 e^{56789} + h_5' e^{46789}$$

$$+ \frac{i}{4} \left( f_3 f_3' - \tilde{f}_3 f_3' - g_3 \tilde{g}_3' + \bar{g}_3 \bar{g}_3' \right) e^{46789}$$

$$+ \frac{3}{2} \left( |f_3|^2 - |g_3|^2 \right) e^{56789}$$  \hspace{1cm} (6.20)

In turn, self-duality of $F^{(5)}$ requires the following relations,

$$0 = f_4^{-5} g_5' + \frac{3}{2} \frac{|f_3|^2 - |g_3|^2 + C_1}{f_1 f_2^4}$$

$$0 = h_5' + \frac{i}{4} \left( f_3 f_3' - \tilde{f}_3 f_3' - g_3 \tilde{g}_3' + \bar{g}_3 \bar{g}_3' \right)$$  \hspace{1cm} (6.21)

\[12\text{Notice that under the action of } U(1)_\beta, \text{ the form } C^{(4)} \text{ is not strictly invariant, but changes by a gauge transformation since } e^{6789} \text{ is a closed form.} \]
which reproduce the on-shell relations (6.18).

We are now ready to reduce the Type IIB supergravity action of \( (2.14) \). It will be expressed here in terms of our complex fields \( B \) and \( G \), and is given by,

\[
S = \int dx \sqrt{g} \left\{ R - 2 f^4 \partial_M B \partial^M \bar{B} - \frac{1}{12} G_{MNP} G^{MNP} - 4 |F(5)|^2 \right\} - i \int C(4) \wedge G \wedge \bar{G} \tag{6.22}
\]

Since, after variation of the fields, we are to further enforce the self-duality relation, this action is not quite unique. Indeed, we are free to add to the Lagrangian density a term proportional to the square of the self-duality relation \( |F(5) - \ast F(5)|^2 \), as its variation will vanish on self-dual fields. In reducing the action over our Ansatz, it will be convenient to add the term \( 4 |F(5) - \ast F(5)|^2 \) to the Lagrangian density, as this will eliminate terms in the reduction that are quartic in \( f_3 \) and \( g_3 \).

Evaluating this modified action on our Ansatz, omitting the overall volume factors of \( \hat{e}^{0123} \wedge \hat{e}^{56789} \) of \( AdS_4 \times S^5 \), and integrating all second derivatives by part to convert all terms to involve only first derivatives, we obtain

\[
S_{\text{reduced}} = \int d\mu \left\{ 12 (f_4')^2 f_1 f_2 f_3 f_4 + 12 (f_4')^2 f_1 f_2 f_3 f_4 + 8 f_1 f_2 f_3 f_4 + 8 f_1 f_2 f_3 f_4 \\
+ 32 f_1 f_2 f_3 f_4 - 12 f_1 f_2 f_3 f_4 - 4 f_3 f_4 f_4 f_5 + 24 f_1 f_2 f_3 f_4 - 2 f_4 f_4 f_3 f_4 |B'|^2 \\
- 2 f_1 f_4 \left( f^2 |f_3 - B g_3|^2 + f^2 |g_3 - B f_3|^2 \right) \\
- 18 f_4 f_5 \left( f^2 |f_3 - B g_3|^2 + f^2 |g_3 - B f_3|^2 \right) \\
+ 8 f_1 f_2 f_3 f_4 (g_3')^2 + 24 g_3 \left( |f_3|^2 - |g_3|^2 \right) \right\} \tag{6.23}
\]

We have shown that the reduced field equations follow from the action \( S_{\text{red}} \).

The only further relations implied by the self-duality constraint \( (2.8) \) are the value of the constant \( C_1 \) and the expression for the function \( h_5 \) which yields (6.21).

\footnote{Compared to the conventions of [41], we omit an overall factor of \( 1/2 \kappa_{10}^2 \), divide \( C(4) \) by a factor of 4, and divide \( F(5) \) by a factor of 4. The relative normalization of the Chern-Simons term against the \( |F(5)|^2 \) term is checked using the self-duality of \( F(5) \) and the Bianchi identity for \( F(5) \). The absolute normalization of the \( |F(5)|^2 \) term may be checked from Einstein’s equations, while that of the Chern-Simons term may be checked independently against the field equation for \( G \).}
7 Supersymmetry variations

In this section, we reduce the BPS equations \( \delta \lambda = \delta \psi_\mu = 0 \) expressing the vanishing supersymmetry variation of the dilatino and gravitino fields. Fields satisfying these reduced equations with non-vanishing supersymmetry parameter \( \varepsilon \) will exhibit some degree of residual supersymmetry. It will be convenient to express the gravitino equation in differential form notation. The dilatino and gravitino equations are given respectively by,

\[
(\Gamma \cdot P)B^{-1}\epsilon^* - \frac{1}{24}(\Gamma \cdot G)\epsilon = 0 \tag{7.1}
\]

\[
d\varepsilon + \omega \varepsilon + \varphi^{(1)}\varepsilon + \varphi^{(2)}B^{-1}\epsilon^* = 0 \tag{7.2}
\]

where the connection components are as follows\(^\text{14}\)

\[
\omega = \frac{1}{4} \Gamma^{MN} \omega_{MN} \\
\varphi^{(1)} = -\frac{i}{2} Q + \frac{i}{480} (\Gamma \cdot F(5)) e_A \Gamma^A \\
\varphi^{(2)} = -\frac{1}{96} e_A \left( \Gamma^A (\Gamma \cdot G) + 2(\Gamma \cdot G) \Gamma^A \right) \tag{7.3}
\]

Here, \( e^A \) is the frame of (5.20), \( \omega \) its torsion-free connection, \( B \) is the complex conjugation matrix, and we have used the following relation

\[
\Gamma^{MNPQ} G_{NPQ} - 9 \eta^{MN} \Gamma^{PQ} G_{NPQ} = -\Gamma^{M} (\Gamma \cdot G) - 2(\Gamma \cdot G) \Gamma^{M} \tag{7.4}
\]

to relate the gravitino equation of (2.12) to that of (7.2) and (7.3).

7.1 Γ-matrices

It will be useful to choose a well-adapted basis for Γ-matrices\(^\text{15}\)

\[
\Gamma^\mu = \sigma_1 \otimes \gamma^\mu \otimes I_4 \\
\Gamma^m = \sigma_2 \otimes I_4 \otimes \gamma^m \\
\mu = 0, 1, 2, 3, 4 \\
m = 5, 6, 7, 8, 9 \tag{7.5}
\]

for which we have

\[
\Gamma^{11} = \Gamma^{0123456789} = \sigma_3 \otimes I_4 \otimes I_2 \otimes I_2 \\
\mathcal{B} = \Gamma^{2568} = \sigma_3 \otimes \gamma^2 \otimes \sigma_1 \otimes \sigma_2 \tag{7.6}
\]

\(^{14}\)Throughout, we shall use the notation \( \Gamma \cdot T \equiv \Gamma^{M_1 \cdots M_n} T_{M_1 \cdots M_n} \) for the contraction of rank \( n \) totally anti-symmetric Γ-matrices and tensors.

\(^{15}\)A useful set of conventions for the 4 × 4 matrices \( \gamma^A \), as well as some further formulas for combinations of Γ-matrices, such as \( \Gamma^z \), and \( \Gamma^{AB} \), are presented in Appendix B.
7.2 Reducing the dilatino equation

Evaluating the dilatino equation (7.1) on the Ansatz, we find,

$$4\frac{f^2 B'}{f_4} \Gamma^4 B^{-1} \epsilon^* = \left( a\Gamma^5 - ib\Gamma^4 \right) \Gamma^{z_1 \bar{z}_2} \epsilon^{+3i\beta} \epsilon + \left( c\Gamma^5 - id\Gamma^4 \right) \Gamma^{\bar{z}_1 \bar{z}_2} \epsilon^{-3i\beta} \epsilon$$  \hspace{1cm} (7.7)

To solve this equation, we multiply both sides to the left by $\Gamma^4$ and use the fact that $\Gamma^4 \Gamma^5$ commutes with $\Gamma^{z_1 \bar{z}_2}$. The rhs then automatically projects onto the subspace of spinor $s$ corresponding to eigenvalue $+1$ of the Dirac matrix $I_2 \otimes I_4 \otimes \gamma^5$. Since $B$ commutes with this matrix, both $\epsilon$ and $B^{-1} \epsilon^*$ must have eigenvalue $+1$. Introducing a basis of 2-component spinors $u_{\pm}$, with $\sigma_1 u_{\pm} = +u_{\mp}$, and $\sigma_\mp u_{\pm} = 0$, we parametrize the eigenvalue $+1$ subspace for chiral spinors $\epsilon$ satisfying $\Gamma^{11} \epsilon = -\epsilon$, in terms of two 4-dimensional complex spinors $\zeta_{\pm}$, and their complex conjugates $\zeta_{\pm}^*$,

$$\epsilon = u_- \otimes \left( \zeta_+ \otimes u_\mp \otimes u_+ + \zeta_- \otimes u_- \otimes u_- \right)$$

$$B^{-1} \epsilon^* = u_- \otimes \left( i\gamma^2 \zeta_- \otimes u_\mp \otimes u_+ - i\gamma^2 \zeta_+ \otimes u_- \otimes u_- \right)$$  \hspace{1cm} (7.8)

In this basis, the dilatino equation reduces to,

$$(a\gamma^4 + b) \zeta_- = e^{-3i\beta} f^2 B'(f_4)^{-1} \gamma^2 \zeta_-$$

$$(c\gamma^4 + d) \zeta_+ = e^{+3i\beta} f^2 B'(f_4)^{-1} \gamma^2 \zeta_+$$  \hspace{1cm} (7.9)

Notice that the equations for $\zeta_+$ and $\zeta_-$ can be satisfied independently of one another. Non-trivial solutions require the following relations,

$$\zeta_+ \neq 0 \hspace{1cm} (c + d)(\bar{c} - \bar{d}) = f^4 |B'|^2 (f_4)^{-2}$$

$$\zeta_- \neq 0 \hspace{1cm} (a + b)(\bar{a} - \bar{b}) = f^4 |B'|^2 (f_4)^{-2}$$  \hspace{1cm} (7.10)

The $\beta$-dependence of $\zeta_{\pm}$ is readily solved for, since $a, b, c, d, f_4$ and $f^2 B'$ are all $\beta$-independent. The general solution is given by

$$\zeta_{\pm} = e^{\pm\frac{3}{2}i\beta} \hat{\zeta}_{\pm}$$

$$(a\gamma^4 + b) \hat{\zeta}_- = f^2 B'(f_4)^{-1} \gamma^2 \hat{\zeta}_-$$

$$(c\gamma^4 + d) \hat{\zeta}_+ = f^2 B'(f_4)^{-1} \gamma^2 \hat{\zeta}_+$$  \hspace{1cm} (7.11)

where $\hat{\zeta}_{\pm}$ is independent of $\beta$. 

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7.3 Reducing the gravitino equation

We begin by decomposing the gravitino equations (7.2) onto the eigenvalue +1 and −1 of the matrix \( \Gamma_{6789} = -I_8 \otimes \sigma_3 \otimes \sigma_3 \). Since \( \varepsilon \) and \( B^{-1}\varepsilon^\ast \) belong to the subspace with definite eigenvalue −1, this decomposition may be achieved by decomposing the covariant derivative according to components that commute or anti-commute with \( \Gamma_{6789} \). This decomposition is unique since \( \Gamma_{6789} \) is invertible. We have,

\[
\begin{align*}
d\varepsilon + \omega_+ \varepsilon + \varphi_+^{(1)} \varepsilon + \varphi_+^{(2)} B^{-1} \varepsilon^* &= 0 \\
\omega_- \varepsilon + \varphi_-^{(1)} \varepsilon + \varphi_-^{(2)} B^{-1} \varepsilon^* &= 0
\end{align*}
\] (7.12)

Here, the components that commute with \( \Gamma_{6789} \) are given by,

\[
\omega_+ = \frac{1}{4} \omega_{ij} \Gamma^{ij} + \frac{1}{2} \omega_{45} \Gamma^{45} + \frac{1}{4} \omega_{ab} \Gamma^{ab}
\]

\[
\varphi_+^{(1)} = -\frac{i}{2} Q + \frac{i}{480} (\Gamma \cdot F_{(5)}) (e_i \Gamma^i + e_4 \Gamma^4 + e_5 \Gamma^5)
\]

\[
\varphi_+^{(2)} = -\frac{1}{96} \sum_{A=0,1,2,3,4,5} e_A (\Gamma^A (\Gamma \cdot G) + 2 (\Gamma \cdot G) \Gamma^A)
\] (7.13)

while the components that anti-commute with \( \Gamma_{6789} \) are given by,

\[
\omega_- = \frac{1}{2} \omega_{a4} \Gamma^a 4 + \frac{1}{2} \omega_{a5} \Gamma^a 5
\]

\[
\varphi_-^{(1)} = \frac{i}{480} (\Gamma \cdot F_{(5)}) e_a \Gamma^a
\]

\[
\varphi_-^{(2)} = -\frac{1}{96} e_a (\Gamma^a (\Gamma \cdot G) + 2 (\Gamma \cdot G) \Gamma^a)
\] (7.14)

Here, the indices \( i, j \) continue to run over the range 0, 1, 2, 3, while \( a, b \) run over 6, 7, 8, 9.

7.3.1 Reducing the algebraic equation

The second equation in (7.12) is purely algebraic in \( \varepsilon \). Using the expressions for the connection components of (6.10), and the anti-symmetric tensor fields \( F_{(5)} \) of (5.23) and \( G \) of (5.26), we find one equation involving both \( \zeta_+ \) and \( \gamma^2 \xi_+^* \), and another equation involving both \( \zeta_- \) and \( \gamma^2 \xi_-^* \). We proceed by assuming that \( B' \neq 0 \) since we shall be interested in obtaining Janus solutions with varying dilaton field. We may then eliminate \( \gamma^2 \xi_\pm^* \) using the dilatino equation (7.9). The final result is expressed in terms of \( \hat{\zeta}_\pm \), using (7.11), by the following two independent equations,

\[
\left( 2 \frac{f_1'}{f_2} \gamma^4 + \frac{2}{f_2} \frac{f_1 f_4}{f_2^2} - 2 f_5 f_4 - \frac{f_1^2}{f_2^2} B' \gamma^4 (a + b \gamma^4)(c + d \gamma^4) \right) \hat{\zeta}_\pm = 0
\] (7.15)
where $\pm$ is correlated with the subscript of $\hat{\zeta}_\pm$.

The above equations imply that, still under the assumption that $B' \neq 0$, either $\hat{\zeta}_+ \neq 0$, and decompose the equation for $\hat{\zeta}_+$ onto the two $\gamma^4$ chiralities of $\hat{\zeta}_+$. Note that the equations must hold for both chiralities, since the dilatino equation (7.9) chirality,

$$
\begin{align*}
+ 2 \frac{f'_2}{f_2} + 2 \frac{f_1 f_4}{f'_2} - 2 f_5 - \frac{f_1}{f_2^2} B'(a + b) = 0 \\
- 2 \frac{f'_2}{f_2} + 2 \frac{f_1 f_4}{f'_2} - 2 f_5 + \frac{f_1}{f_2^2} B'(a - b) = 0
\end{align*}
$$

where the $\pm$ sign refers to the $\gamma^4$-chirality of $\hat{\zeta}_+$. Using these two relations, required by $\hat{\zeta}_+ \neq 0$ in the equation for $\hat{\zeta}_-$ gives $f_1 f_4 f_2^{-2} \hat{\zeta}_- = 0$. For non-degenerate geometries, $f_1 f_4 f_2^{-2} \neq 0$ and thus we must have $\hat{\zeta}_- = 0$, as announced.

The equations for $\hat{\zeta}_+$ and $\hat{\zeta}_-$ are mapped into one another by parity, which maps $\mu \rightarrow -\mu$, and are equivalent to one another.

### 7.3.2 Reducing the differential equation

To reduce the differential equation in (7.12), we begin by calculating the connection $\omega_{ab} \Gamma^{ab}$, using $\Gamma^{ab} = I_8 \otimes \gamma^{ab}$. Furthermore, we use the relations

$$
\begin{align*}
\gamma^6 u_\pm \otimes u_\pm &= + \gamma^8 u_\pm \otimes u_\pm = \pm i u_\pm \otimes u_\pm \\
\gamma^6 u_\pm \otimes u_\mp &= - \gamma^7 u_\pm \otimes u_\mp = \pm u_\mp \otimes u_\mp \\
\gamma^6 u_\mp \otimes u_\pm &= + \gamma^7 u_\mp \otimes u_\pm = i u_\mp \otimes u_\mp
\end{align*}
$$

and the fact that $\hat{\omega}_{68} = \hat{\omega}_{79}$, and $\hat{\omega}_{69} = -\hat{\omega}_{78}$, as found in (5.10), to derive

$$
\frac{1}{4} \omega_{ab} \gamma^{ab} = \frac{3i}{2} A_1 \sigma_3 - i \frac{f_1}{f_2^2} e^5 \sigma_3
$$

Here, $\sigma_3$ acts on $\zeta_\pm$ by $\sigma_3 \zeta_\pm = \pm \zeta_\pm$. All connection components now commute with $\sigma_3$ and we may decompose the reduced gravitino equation onto its decoupled $\zeta_+$ and $\zeta_-$ components. Finally, the simple $\beta$-dependence of $\zeta_\pm$ found in (7.11) is used to recast the equations in terms of $\hat{\zeta}_\pm$. The differential $d\beta$ picked up in the process combines with the $U(1)$ connection $A_1$ in (7.18) and forms the gauge invariant combination $d\beta + A_1 = \hat{e}^5$, so that we have effectively

$$
\frac{1}{4} \omega_{ab} \gamma^{ab} \rightarrow \frac{3i}{2} f_1 e^5 \sigma_3 - i \frac{f_1}{f_2^2} e^5 \sigma_3
$$

(7.19)
Finally, retaining only the $\hat{\zeta}_+$-component of the differential equation, we obtain

$$0 = d\hat{\zeta}_+ + \left( \frac{1}{4} \hat{\omega}_{ij} \gamma^{ij} + \frac{1}{2} \omega_{ij} \gamma^{ij} - \frac{i}{2} \omega_{45} \gamma^4 + \frac{3i}{2f_1} e^5 - \frac{i}{f_2^2} e^5 \right) \hat{\zeta}_+$$

$$+ \left( -\frac{i}{2} Q + \frac{1}{2} f_5 (e_i \gamma^i + e_4 \gamma^4 - i e_5) \right) \hat{\zeta}_+$$

$$+ \frac{1}{4} \left( e_i \gamma^i (a + b \gamma^4) + e_4 \gamma^4 (a - 3b \gamma^4) + e_5 (-3ia + ib \gamma^4) \right) \gamma^2 \hat{\zeta}_+^*$$

(7.20)

7.3.3 Analyzing the reduced differential equation

Since $\hat{\zeta}_+$ is $\beta$-independent in (7.20), no $\partial_\beta \hat{\zeta}_+ d\beta$ contribution will appear in $d\hat{\zeta}_+$. As a result, we must have $i e_5 d\hat{\zeta}_+ = 0$, which yields a further algebraic equation,

$$\left( \frac{f'_1}{f_1} \gamma^4 + 3 \frac{f_4}{f_1} - 2 \frac{f_1 f_4}{f_2^2} - f_4 f_5 \right) \hat{\zeta}_+ = \frac{1}{2} f_4 (3a - b \gamma^4) \gamma^2 \hat{\zeta}_+^*$$

(7.21)

The decomposition of (7.20) onto the directions $e^4$ and $\hat{e}_i$, yields two differential equations,

$$(\hat{\nabla}_i + \frac{1}{2} f_4' \gamma^i \gamma_4 + \frac{1}{2} f_5 f_4 \gamma^i) \hat{\zeta}_+ + \frac{1}{4} f_4 (a - b \gamma_4) \gamma^i \gamma_2 \hat{\zeta}_+^* = 0$$

(7.22)

$$(\partial_\mu - \frac{i}{2} Q + \frac{1}{2} f_5 f_4 \gamma_4) \hat{\zeta}_+ + \frac{1}{4} f_4 (a \gamma_4 - 3b) \gamma_2 \hat{\zeta}_+^* = 0$$

(7.23)

where $\hat{\nabla}_i$ is the covariant derivative with connection $\hat{\omega}_{ij} \gamma^{ij}/4$. Finally, we proceed to eliminating the spinor $\gamma^2 \hat{\zeta}_+^*$ using the dilatino equation (7.9), and to further simplifying the equations. From the outset, we use the fact that $\hat{\zeta}_- = 0$, as derived earlier. The results for all supersymmetry variation equations are summarized below.

7.4 Summary of the reduced dilatino/gravitino equations

- The implications of the dilatino equations,

$$\left( c + d \right) (\bar{c} - \bar{d}) = f_4' |B'|^2 (f_4)^{-2}$$

(7.24)

- The algebraic integrability equations,

$$\frac{f'_1}{f_2} = \frac{f^2_4}{2f^2 B'} (ac + bd)$$

(7.25)

$$\frac{f_1 f_4}{f_2^2} - f_4 f_5 = \frac{f^2_4}{2f^2 B'} (ad + bc)$$

(7.26)
The $e^5$ component equation,
\[ \frac{f'_1}{f_1} = \frac{f_1^2}{2f^2B'} (3ac - bd) \quad (7.27) \]
\[ 3\frac{f_4}{f_1} - 2\frac{f_1f_4}{f_2} - f_4f_5 = \frac{f_4^2}{2f^2B'} (3ad - bc) \quad (7.28) \]

The $AdS_4$ components’ equation,
\[ \frac{f_2 f_4^2}{f_2^2} - \left( \frac{f'_2}{f_2} + \frac{f'_4}{f_4} \right)^2 = 1 \quad (7.29) \]

This summary of results is obtained as follows. Equation (7.24) is just the original dilatino equation of (7.10) for the case where $\hat{\zeta} = 0$. Equations (7.25) and (7.26) are obtained by taking the sum and the difference of the two algebraic equations (7.16). Equations (7.27) and (7.28) are obtained (7.21) by eliminating $\gamma^2 \hat{\zeta}_+^*$ using the dilatino equation (7.9), and taking the sum and difference of the resulting equations for $\pm 1$ eigenvalues of the $\gamma^4$ matrix. To obtain equation (7.29) requires more work. We begin by eliminating the spinor $\gamma^2 \hat{\zeta}_+^*$ using the dilatino equation (7.9). The resulting equation takes the form,
\[ (\hat{\nabla}_i + \kappa \gamma_i + \lambda \gamma^4) \hat{\zeta}_+ = 0 \quad (7.30) \]
where
\[ \kappa = \frac{1}{2} f_4 f_5 + \frac{f_4^2}{4f^2B'} (ad + bc) = \frac{1}{2} \frac{f_1 f_4}{f_2} \]
\[ \lambda = \frac{1}{2} f'_4 + \frac{f_4^2}{4f^2B'} (ac + bd) = \frac{1}{2} \frac{f'_2}{f_2} + \frac{1}{2} \frac{f'_4}{f_4} \quad (7.31) \]

In deriving the second equalities on the rhs of the equations above, we have eliminated the combination $ad + bc$ using (7.26), and eliminated the combination $ac + bd$ using (7.25). The coefficients $\kappa$ and $\lambda$ depend upon $\mu$ but not upon the variables of $AdS_4$, and may thus be viewed as constants in this differential equations.

The integrability of (7.30) requires $\kappa^2 - \lambda^2 = 1/4$, which yields (7.29). Finally, equation (7.23) is not reproduced in the summary because it only governs the $\mu$-evolution of $\hat{\zeta}_+$, is always integrable, and imposes no new condition of the dynamical variables of the system, namely $f_1, f_2, f_3, f_4$ and $B$.

The solution of equations (7.24) to (7.29) gives a background which preserves four real supercharges. The counting proceeds as follows: In Type IIB supergravity the supersymmetry transformation parameter $\epsilon$ has 32 real components. The compatibility with the $CP_2$
fibration leads to reducing $\epsilon$ to 16 real components parameterized by two complex four dimensional spinors $\zeta_+$ and $\zeta_-$. The analysis of section 7.3 implies that one of the two $\zeta_\pm$ is zero, leaving four complex components. Finally the dilatino supersymmetry condition leads to a reality condition leaving four real unbroken supersymmetries, which is the degree of supersymmetry of our solution. In the sequel, we shall not need to explicitly solve for the spinor $\hat{\zeta}_+$; instead it will suffice to solve the integrability conditions which guarantee the existence of the four real unbroken supersymmetry.
8 Reality properties of supersymmetric solutions

We shall now search for supersymmetric solutions. This requires satisfying the field equations (6.3), (6.6), and (6.13), (6.14), (6.15), (6.16), (one special combination of which is the constraint (6.17)), as well as the supersymmetry conditions (7.24–7.29) of the summary. Assuming that all these equations are simultaneously satisfied guarantees that we will have a true solution to the field equations, which also is supersymmetric. The simultaneous consideration of the susy equations and the field equations leads to many simplifications.

8.1 Solving the equations for $B$ and $\tau$

The difference between (7.25) and (7.27) yields

$$\frac{f_1'}{f_1} - \frac{f_2'}{f_2} = \frac{f_4^2}{f_2 B'} (ac - bd) \quad (8.1)$$

which upon eliminating $ac - bd$ in (6.3), and dividing the resulting equation by $B'$ gives

$$0 = \frac{B''}{B'} + 3 \frac{f_1'}{f_1} + 2 \frac{f_2'}{f_2} + 3 \frac{f_4'}{f_4} + 2 f^2 \bar{B} B' \quad (8.2)$$

Taking the real part of (8.2), and using the fact that $f^2 (\bar{B} B' + B \bar{B}') = (\ln f^2)'$, we get

$$f^2 |B'| = \frac{C_2}{f_3^3 f_2^2 f_4^3} \quad (8.3)$$

where $C_2$ is a constant. Converting the imaginary part of (8.2), into an equation for $\tau = \tau_1 + i\tau_2$, using (2.3), we get precisely the imaginary part of the $\tau$-equation for the non-supersymmetric Janus solution of (3.2),

$$0 = \frac{\tau''}{\tau'} - \frac{\bar{\tau}''}{\bar{\tau}'} + 2i \frac{\tau_1'}{\tau_2} \quad (8.4)$$

Just as in the case of the non-supersymmetric Janus solution, its solution requires the axion/dilaton field $\tau$ to flow along a geodesic in the $\tau$-upper-half-plane,

$$|\tau - p|^2 = r^2 \quad (8.5)$$

where $p, r$ are arbitrary real constants.
8.2 Mapping to real solutions

Since $B$ flows along a geodesic, and $SL(2, \mathbb{R})$ acts transitively on the space of all geodesic segments of the same hyperbolic length, we may use an $SL(2, \mathbb{R})$ transformation to map any one geodesic segment into a geodesic segment of $B$ real, or equivalently $\tau$ purely imaginary. Since $SL(2, \mathbb{R})$ is a symmetry of the Type IIB supergravity equations, the most general solution is obtained by taking $B$ real, and then applying the most general $SL(2, \mathbb{R})$ transformation to the solution. Henceforth, we restrict to $B$ real. The reality of $B$ implies that the following quantities are real,

\[ ac, \ bd, \ ad, \ bc \in \mathbb{R} \quad (8.6) \]

As a result, $a/b$ and $c/d$ are real, and thus have pairwise identical phases. The reality of the products then further imposes the following phase arrangements,

\[
\begin{align*}
  a &= a_r e^{i\theta} \\
  b &= b_r e^{i\theta} \\
  c &= c_r e^{-i\theta} \\
  d &= d_r e^{-i\theta}
\end{align*}
\quad (8.7)
\]

where $a_r, b_r, c_r, d_r, \theta$ are all real. It follows from (7.24) that, unless the dilaton is constant, $c$ cannot vanish identically. Now substitute the above phase relations in the field equation for $c$ and derive that $\theta' = 0$. Hence $\theta$ is a constant phase. Changing this phase is equivalent to making a $U(1)_\beta$ rotation. Thus, we may now take also $a, b, c, d \in \mathbb{R}$. The general solution will be obtained from the real solution by making the inverse $U(1)_\beta \times SL(2, \mathbb{R})$ rotation.

8.3 Reduced equations for real $B, \ a, \ b, \ c, \ d$

The field equations simplify for real field, and we have,

\[
\begin{align*}
  a' &= -\left( \frac{f_1'}{f_1} + 2 \frac{f_2'}{f_2} \right) a + 3 \frac{f_4}{f_1} b - f^2 B' c \\
  b' &= -\left( 4 \frac{f_4'}{f_4} + 2 \frac{f_2'}{f_2} + \frac{f_1'}{f_1} \right) b + 3 \frac{f_4}{f_1} a - f^2 B'd + 4 f_4 f_5 a \\
  c' &= -\left( \frac{f_1'}{f_1} + 2 \frac{f_2'}{f_2} \right) c - 3 \frac{f_4}{f_1} d - f^2 B' a \\
  d' &= -\left( 4 \frac{f_4'}{f_4} + 2 \frac{f_2'}{f_2} + \frac{f_1'}{f_1} \right) d - 3 \frac{f_4}{f_1} c + 4 f_4 f_5 c - f^2 B'b
\end{align*}
\quad (8.8)
\]

\footnote{By a geodesic segment we understand a connected segment of a single geodesic.}
Einstein’s equations are arranged as follows: (6.14) is a linear combination of (6.13), (6.15), (6.16) and (6.17), and will therefore be omitted,

\[ 0 = \frac{f''}{f_4} + 3 + 2 \left( \frac{f''_1}{f_1^4} \right)^2 + \frac{f'_1 f'_4}{f_1 f_4} + 4 \frac{f''_2 f''_1}{f_2 f_2 f_4} - 4 f''_4 f''_4 - \frac{1}{2} f'^2_4 \left( a^2 + c^2 + b^2 + d^2 \right) \] (8.9)

\[ 0 = \frac{f''}{f_2} + \frac{6 f''_2 f''_4}{f_2 f_4} + \frac{f''_1 f''_2}{f_1 f_2} + 2 \frac{f''_1 f''_2}{f_2^2} + 3 \left( \frac{f''_1}{f_2^2} \right)^2 + 4 f''_4 f''_4 + \frac{1}{2} f''_4 \left( a^2 + c^2 + b^2 + d^2 \right) \] (8.10)

The constraint is given by

\[ 0 = 12 \left( \frac{f''_2}{f_2^2} \right)^2 + 12 \left( \frac{f''_4}{f_4^2} \right)^2 + 8 \frac{f''_1 f''_2}{f_1 f_2} + 8 \frac{f''_1 f''_4}{f_1 f_4} + 32 \frac{f''_2 f''_4}{f_2 f_4} - 2 f^4 (B')^2 \]

\[ + 12 + 4 \frac{f''_2 f''_4}{f_2^2} - 24 \frac{f''_2 f''_4}{f_2^2} + 8 f''_4 f''_4 + 2 f''_4 \left( a^2 - b^2 + c^2 - d^2 \right) \] (8.10)

The Bianchi identity for \( F(5) \) is solved in terms of a single constant \( C_1 \),

\[ f_5 = \frac{1}{6} f_1 (a^2 - c^2) + \frac{3}{2} \frac{C_1}{f_1 f_2^3} \] (8.11)

The dilatino equation simplifies as follows,

\[ c^2 - d^2 = f^4 (B')^2 (f_4)^{-2} \] (8.12)

The remaining supersymmetry variation equations continue to be given by (7.25), (7.26), (7.27), (7.28), and (7.29), but now for \( a, b, c, d \) and \( B \) real.
9 Solving the field and supersymmetry equations

We shall now apply the following procedure to the solution of the susy variation equations and the field equations: (1) Take the solution of the $B$-equation from the results above; (2) Solve the susy variation equations and the Hamiltonian constraint (which will restrict the possible initial data); (3) Use those to solve the field equations.

9.1 Solving for real $B, a, b, c, d$

From the linear combinations of $\mathbf{(7.25), (7.26), (7.27), (7.28)}$ that expose the combinations $c \pm d$ we obtain the following equations for $a \pm b$,

\begin{align*}
(a + b)(c + d) &= 2 \frac{f^2 B'}{f_2} \left( \frac{f_2'}{f_2} + \frac{f_1 f_4}{f_2^2} - f_4 f_5 \right) \\
(a - b)(c - d) &= 2 \frac{f^2 B'}{f_4} \left( \frac{f_2'}{f_2} - \frac{f_1 f_4}{f_2^2} + f_4 f_5 \right) 
\end{align*}

(9.1)

The remaining equations may be expressed as follows,

\begin{align*}
3 \frac{f_4}{f_1} - \frac{f_1 f_4}{f_2^2} - 2 f_4 f_5 &= \frac{2 f_4^2}{f^2 B'} ad \\
\frac{f_1'}{f_1} + \frac{f_2'}{f_2} &= \frac{2 f_4^2}{f^2 B'} ac 
\end{align*}

(9.2) (9.3)

They will be important in the next subsection.

9.2 Solving the constraint

From the constraint $\mathbf{(8.10)}$, the combination involving $a, b, c, d$ and $f^2 B'$, is eliminated by using the dilatino equation $\mathbf{(8.12)}$ for $c, d$ and the $f^2 B'$ term, and the product of the two equations in $\mathbf{(9.1)}$ for $a, b$, yielding

\begin{equation}
3 \left( \frac{f_2'}{f_2} + \frac{f_1'}{f_1} \right)^2 + 2 \left( \frac{f_2'}{f_2} + \frac{f_1'}{f_1} \right) \left( \frac{f_1'}{f_1} + \frac{f_2'}{f_2} \right) + 3 - \frac{f_1 f_4}{f_2^2} - 6 \frac{f_2^2}{f_2} + 4 \frac{f_1 f_4^2}{f_2} = 0
\end{equation}

(9.4)

Next, we use $\mathbf{(7.29)}$ to replace the first term,

\begin{equation}
\left( \frac{f_2'}{f_2} + \frac{f_1'}{f_1} \right) \left( \frac{f_1'}{f_1} + \frac{f_2'}{f_2} \right) + \frac{f_1 f_4^2}{f_2^2} - 3 \frac{f_1^2}{f_2^2} + 2 \frac{f_1 f_4^2 f_5}{f_2^2} = 0
\end{equation}

(9.5)

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Using the lhs of (9.2), and the fact that $c \neq 0$ in the equation above, we find,

$$a \left\{ c \left( \frac{f'_2}{f_2} + \frac{f'_4}{f_4} \right) - df \frac{f_1 f_4}{f_2^2} \right\} = 0 \quad (9.6)$$

Thus, the constraint factorizes when reduced to the subspace of supersymmetric solutions. We study the vanishing of each factor in turn.

### 9.3 The Case $a = 0$

The vanishing of $a$ in (9.3) and (9.2) together with the $c \neq 0$ implies that,

$$f_1 f_2 = \rho$$

$$f_5 = \frac{3}{2 f_1} - \frac{1}{2 f_2} \quad (9.7)$$

with $\rho$ a constant, which will be fixed later. The remaining differential equation for $f_2$ is,

$$2c \frac{f'_2}{f_2} = 3d \left( \frac{f_1 f_4}{f_2^2} - \frac{f_4}{f_1} \right) \quad (9.8)$$

The reduced field equation for $a$ in (8.8) for $a = 0$ simplifies to,

$$b = \frac{f_1}{3 f_4} f^2 B' c \quad (9.9)$$

Using this expression to eliminate $b$ from (7.25) and (7.26) yields $c$ and $d$,

$$c = 3 \left( \frac{1}{f_2^2} - \frac{1}{f_1^2} \right)^{\frac{1}{2}} \quad (9.10)$$

$$cd = 6 \frac{f'_2}{f_1 f_2 f_4} \quad (9.11)$$

Combining the result for $c$ with the $f_5$ equation in (8.11), we find that $f_1^2 f_2^2 = \rho^2 = 3C_1/2$.

The field equations for $a, b, c, d$ of (8.8) are now satisfied as follows. The equation for $a$ is automatic using (9.9); the equation for $b$ is automatic using (9.9); the equation for $c$ is automatic using (9.11). Finally, the equation for $d$ is handled as follows. Start with the dilatino equation (8.12) as a definition of $d$, take the derivative and use the field equations in (8.8) for $c, d$. One arrives at

$$4(c^2 - d^2) \frac{f'_2}{f_2} - 8c^2 \frac{f'_4}{f_4} + 8f_4 f_5 c d - 2f^2 B' b d = 0 \quad (9.12)$$
Using (9.9) to eliminate $b$ implies that the first and last terms in (9.12) cancel one another, leaving the following linear relation between $c$ and $d$, after eliminating $f_5$ using (9.7),

\[ \frac{c f_4'}{f_4} = \left( \frac{3 f_4}{2 f_1} - \frac{1}{2} \frac{f_1 f_4}{f_2^2} \right) d \]  

(9.13)

Eliminating now $f_4/f_1$ using (9.8), we get

\[ c \left( \frac{f_2'}{f_2} + \frac{f_1'}{f_1} \right) - d \frac{f_1 f_4}{f_2^2} = 0 \]  

(9.14)

This is the second factor in the relation (9.6).

### 9.4 Exact solution of the case $a = 0$ via hyper-elliptic integrals

Combining the relations (9.14) and (8.12), we may solve completely for $c$ and $d$,

\[ c^2 = \frac{C_2^2}{f_1^4 f_2^8 f_4^6} \quad d^2 = \frac{C_2^2}{f_1^4 f_2^8 f_4^6} \left( 1 - \frac{f_4^4}{f_1^4 f_2^4 f_4^2} \right) \]  

(9.15)

Eliminating $c$ between the expression above and the one already obtained in (9.10), we obtain a polynomial relation between $f_1$, $f_2$, and $f_4$,

\[ \left( f_1^4 f_2^6 - f_2^2 f_4^8 \right) f_4^6 = \frac{1}{9} C_2^2 \]  

(9.16)

The function $f_1$ may be eliminated using (9.7), and the resulting equations (7.29) and (9.16) may be expressed in terms of $f_4$ and the composite $\psi$ defined by $f_2 f_4 = \rho/\psi$, so that

\[ \frac{f_4^4}{\psi^4} = \rho^2 + \frac{C_2^2}{9 \rho^8} \psi^2 \]  

(9.17)

\[ \left( \frac{\psi'}{\psi} \right)^2 = \rho^{-4} f_4^8 \psi^6 - 1 \]  

(9.18)

Eliminating $f_4$ gives a single genus 5 decoupled hyper-elliptic equation for $\psi$, or equivalently gives a genus 3 decoupled equation for $\Psi \equiv \psi^2$,

\[ (\psi')^2 = \left( 1 + \frac{C_2^2}{9 \rho^8} \psi^6 \right)^2 - \psi^2 \]  

(9.19)

\[ \frac{1}{4} (\Psi')^2 = \Psi \left( 1 + \frac{C_2^2}{9 \rho^8} \Psi^3 \right)^2 - \Psi^2 \]  

(9.20)

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9.5 The general Case

The general case is specified by the vanishing of the second factor in (9.6) only. This readily allows for the solution of \( c \) and \( d \) in terms of the functions \( f_1, f_2, f_4 \), and we find,

\[
c = \frac{C_2}{f_1^2 f_2^2 f_4^2} \quad \quad \quad \quad d = vc \quad \quad \quad \quad v = \left(1 - \frac{f_4^2}{f_1^2 f_2^2}\right)^{\frac{1}{2}} \tag{9.21}
\]

The remaining susy variation equations are equivalent to

\[
\begin{align*}
2a \frac{f_1 f_4^2}{f_2^3} &= + \frac{f_1'}{f_1} + \frac{f_2'}{f_2} \\
2b \frac{f_1 f_4^2}{f_2^3} &= - \frac{f_1'}{f_1} + 3 \frac{f_2'}{f_2}
\end{align*}
\]

\[
\begin{align*}
2av \frac{f_1 f_4^2}{f_2^3} &= +3 \frac{f_4}{f_1} - \frac{f_4 f_1}{f_2^2} - 2f_4 f_5 \\
2bv \frac{f_1 f_4^2}{f_2^3} &= -3 \frac{f_4}{f_1} + 5 \frac{f_4 f_1}{f_2^2} - 2f_4 f_5 \tag{9.22}
\end{align*}
\]

Eliminating \( a, b \) gives

\[
\begin{align*}
\frac{1}{v} \left( \frac{f_1'}{f_1} + \frac{f_2'}{f_2} \right) &= +3 \frac{f_4}{f_1} - \frac{f_4 f_1}{f_2^2} - 2f_4 f_5 \\
\frac{1}{v} \left( \frac{f_1'}{f_1} + 3 \frac{f_2'}{f_2} \right) &= -3 \frac{f_4}{f_1} + 5 \frac{f_4 f_1}{f_2^2} - 2f_4 f_5 \tag{9.23}
\end{align*}
\]

Finally, eliminating \( f_5 \) and \( a \) using (8.11) gives the following equations,

\[
\begin{align*}
\left( v + \frac{1}{v} \right) \frac{f_1'}{f_1} + \left( v - \frac{3}{v} \right) \frac{f_2'}{f_2} &= 6 \left( \frac{f_4}{f_1} - \frac{f_4 f_1}{f_2^2} \right) \\
\left( \frac{f_1'}{f_1} + \frac{f_2'}{f_2} + 6 \frac{f_4 f_1^3}{f_2^3} v \right)^2 &= 36 \frac{f_1^2 f_6^4}{f_2^8} - 12 \frac{f_1^2 f_4^4}{f_2^6} + 4 \frac{C_2^2}{f_2^4 f_2^4} f_4^2 - 36 \frac{C_1 f_4^4}{f_2^6} \tag{9.24}
\end{align*}
\]

In attempting to separate these three equations, it appears natural to define the following new combinations of the functions \( f_1, f_2, f_3 \),

\[
\psi_1 \equiv f_2 f_4 \quad \psi_2 \equiv f_1 f_2 \quad \psi_3 \equiv \frac{f_1 f_4}{f_2^2} \quad v = \sqrt{1 - \psi_3^2} \tag{9.25}
\]

Together with equation (7.29), the equations then become,

\[
\begin{align*}
\frac{\psi_1'}{\psi_1} &= \psi_3 v \\
\frac{\psi_2'}{\psi_2} + \frac{1}{\psi_3} \frac{\psi_3'}{\psi_3} &= -5 \psi_3 + 6 \frac{\psi_1}{\psi_2} \\
\left( \frac{\psi_2'}{\psi_2} + 6 \frac{\psi_1 \psi_3^2}{\psi_2^2} v \right)^2 &= 36 \frac{\psi_1^2 \psi_3^4}{\psi_2^2} - 12 \frac{\psi_1 \psi_3^3}{\psi_2} + 4 \frac{C_2^2 \psi_3^2}{\psi_1^4 \psi_2^4} f_4 f_5 - 36 C_1 \frac{\psi_1 \psi_3^3}{\psi_2^3} \tag{9.26}
\end{align*}
\]

So far, we have not succeeded in decoupling these equations.
10 Numerical results

For the special case $a = 0$ it was shown in section 9.3 that the existence of an unbroken supersymmetry is equivalent to the existence of solutions to the equation (9.19) which can be viewed as describing the motion if a particle with coordinate $\psi$ in a potential $V(\psi)$.

$$(\psi')^2 + V(\psi) = 0$$

where the potential is given by

$$V(\psi) = -\left(1 + \frac{C_2^2}{9\rho^8}\psi^6\right)^2 + \psi^2$$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{potential.png}
\caption{The potential $V(\psi)$ for $C_2^2/\rho^8 = .36$}
\end{figure}

The complete form of the solution is determined once $\psi$ is found. Near the boundary of AdS the metric function $f_4$ behaves like $f_4 \sim 1/(\mu \mp \mu_0)$ as $\mu \to \pm \mu_0$. It follows from (9.17) that $\psi$ vanishes as $\psi \sim (\mu \mp \mu_0)$ in this limit. On the other hand the limit $\psi \to \infty$ corresponds to the metric becoming singular. A regular Janus like solution can only exist if the potential is somewhere positive, since in this case there are two allowed regions $0 < \psi < \psi_1$ and $\psi_2 < \psi < \infty$. For the nonsingular Janus solution $\psi$ takes values in the first region. It is easy to see that this implies a condition on the parameters of the potential

$$\frac{C_2^2}{\rho^8} < \frac{5^5}{2^6 3^4}$$

The metric for the undeformed AdS$_5$ corresponds to $f_4(\mu) = 1/\cos(\mu)$ with range $\mu \in [-\pi/2, \pi/2]$. For the Janus solution the range $\mu$ is increasing $\mu \in [-\mu_0, \mu_0]$, where $\mu_0 > \pi/2$. 

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The range depends on $C^2_2/\rho^8$ and diverges when this parameter approaches its critical value (10.3) (See Figure 4 (a)).

In the following we present the results for the solution for three choices of parameters $\rho$ and $C_2$. We have fixed $\rho = 1.4$ and picked $C_2 = 0.445, C_2 = 0.245$ and $C_2 = 0.045$. The dilaton can be determined by numerically integrating (8.3), the plot of the dilaton for the three choices of parameters is given in Figure 4 (b),

![Figure 4](image)

Figure 4: (a) Value of $\mu_0$ as a function of $C^2_2/\rho^8$, (b) dilaton for three values of parameters

The metric function $f_4$ is related to $\psi$ by equation (9.17), it diverges at $\mu = \pm \mu_0$ which corresponds to the two AdS boundary components. The metric function $f_2$ and $f_1$ are determined by the relations $f_2 f_4 = \rho/\psi$ and $f_1 f_2 = \rho$.

![Figure 5](image)

Figure 5: (a) $f_4$ for three values of parameters, (b) $f_1$ and $f_2$ for three values of parameters
Note that since $f_1$ is the scale factor of the $S^1$ fiber and $f_2$ is the scale factor of the $CP_2$ base in the squashed sphere, the fact that $f_1 \to f_2$ as one approaches the AdS boundary components $\mu \to \pm \mu_0$, means that the sphere becomes 'un-squashed' and the $SO(6)$ symmetry is restored at the AdS boundary. The third rank anti-symmetric tensor is determined by $\psi$ via equation (9.15) and the plot for the two function $c$ and $d$ is given by

Figure 6: (a) AST function $c$ for three values of parameters, (b) AST function $d$ for three values of parameters
11 Holographic dual

The AdS/CFT correspondence relates 10-dimensional Type IIB supergravity fields to gauge invariant operators on the $\mathcal{N} = 4$ super Yang-Mills side. In the following we briefly review some aspects of this map. The Poincaré metric of Euclidean $AdS_5$ is given by

$$ds^2 = \frac{1}{z^2} \left( dz^2 + \sum_i dx_i^2 \right)$$

(11.1)

Near the boundary of $AdS_5$, where $z \to 0$, a scalar field $\Phi_m$ of mass $m$ behaves as

$$\Phi_m(z, x) \sim \phi_{\text{non-norm}}(x) z^{4-\Delta} + \phi_{\text{norm}}(x) z^{\Delta}$$

(11.2)

where $m^2 = \Delta(\Delta - 4)$. The non-normalizable mode corresponds to insertion in the Lagrangian of an operator $O_\Delta$ with scaling dimension $\Delta$. The boundary source can be determined from (11.2) by

$$\phi_{\text{non-norm}}(x) = \lim_{z \to 0} z^{\Delta-4} \Phi(z, x)$$

(11.3)

If $\phi_{\text{non-norm}}$ vanishes a nonzero $\phi_{\text{norm}}$ corresponds to a non vanishing expectation value $\langle O_\Delta \rangle = \phi_{\text{norm}}$ of the operators $O_\Delta$ on the Yang-Mills side.

As reviewed in section (3.2) for the Janus solution, the boundary geometry, and hence the holographic dual, are more complicated. The asymptotic behavior of the solution obtained in section (9.3) is readily obtained using power series expansion near the boundary components.

Employing Poincaré coordinates for the $AdS_4$ slices, the asymptotic form of the non compact part of the ten dimensional metric can be obtained by expanding $f_4(\mu)$ near the two boundary components $\mu = \pm \mu_0$

$$ds^2 = \frac{1}{(\mu \mp \mu_0)^2 z^2} \left( dz^2 + \sum_i^3 dx_i^2 + z^2 d\mu^2 \right) + O[(\mu \mp \mu_0)^6]$$

(11.4)

The boundary is reached when $(\mu \mp \mu_0) z \to 0$, and consists of one component with $\mu = \mu_0$ and one component with $\mu = -\mu_0$. The complete boundary corresponds to two 4-dimensional half spaces joined by a $\mathbb{R}^3$ interface located at $z = 0$.

The asymptotic behavior of the dilaton near the boundaries is given by

$$\phi(\mu) = \phi_0^\pm + \frac{C_2}{2\rho^4}(\mu \mp \mu_0)^4 + O[(\mu \mp \mu_0)^6]$$

(11.5)

The dilaton corresponds to a dimension $\Delta = 4$ operator. Hence it follows from (11.3) that there is a source $\phi_0^\pm$ of the operator dual to the dilaton in the two boundary components.
This is interpreted on the Yang-Mills side as a theory where the gauge coupling takes two different values on the half spaces separated by a planar interface.

The asymptotic behavior of the anti-symmetric tensor field components $c$ and $d$ is,

$$c(\mu) = \frac{C_2}{\rho^{9/2}} (\mu_0 \mp \mu)^3 + O[(\mu \mp \mu_0)^5]$$
$$d(\mu) = \frac{C_2}{\rho^{9/2}} (\mu \mp \mu_0)^3 + O[(\mu \mp \mu_0)^5]$$ (11.6)

The $(\mu_0 \mp \mu)^3$ behavior of the functions $c$ and $d$ is in agreement with the fact that the lowest Kaluza-Klein modes on the $S^5$ of the anti-symmetric rank 2 tensor field is associated with dimension $\Delta = 3$ operator [42, 43].

In the light of (11.3), the $c \sim (\mu \mp \mu_0)^\Delta$ behavior of (11.6) seems to suggest that there is no source for the dual operator of the anti-symmetric tensor. However this conclusion is not correct. For the Janus metric the appropriate rescaling of the field to extract the non-normalizable mode is given by

$$c_{\text{non-norm}} = \lim_{\epsilon \to 0} \epsilon^{\Delta-4} c(\mu)$$
$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \frac{C_2}{\rho^{9/2}} (\mu \mp \mu_0)^3$$
$$= \lim_{(\mu \mp \mu_0)z \to 0} \frac{(\mu \mp \mu_0)^2}{z} \frac{C_2}{\rho^{9/2}}$$ (11.7)

where $\epsilon = (\mu \mp \mu_0)z$ was used. For a point on the boundary which is away from the three dimensional interface one has $z \neq 0$ and it follows from (11.7) that the source for the dual operator vanishes away from the interface. However for the interface one has $z = 0$ and $c_{\text{non-norm}}$ in (11.7) diverges. This behavior indicates the presence of a delta function source for the dual $\Delta = 3$ operator on the interface, since the integral over a small disk around the interface $\int d\mu \ dzz \ c_{\text{non-norm}}$ is finite.

The metric functions $f_1$ and $f_2$ behave as

$$f_1(\mu) = \sqrt{\rho} \left( 1 + \frac{C_2^2}{36\rho^8} (\mu \mp \mu_0)^6 + O[(\mu \mp \mu_0)^8] \right),$$
$$f_2(\mu) = \sqrt{\rho} \left( 1 - \frac{C_2^2}{36\rho^8} (\mu \mp \mu_0)^6 + O[(\mu \mp \mu_0)^8] \right)$$ (11.8)

Repeating the analysis of the anti-symmetric tensor for the metric fields reveals that there no additional operator sources turned on by $f_1$ and $f_2$. 
12 Conclusions

In this paper, a supersymmetric generalization of the Janus solution of ten dimensional type IIB superstring theory was found. The solution was constructed by imposing the condition of the existence of a preserved supersymmetry on the most general ansatz compatible with $SO(2,3) \times SU(3) \times U(1)_\beta \times SL(2,\mathbb{R})$ symmetry. The supergravity solution is in agreement with the field theoretical analysis of [17] [21]. The restoration of supersymmetry requires a nontrivial antisymmetric tensor $B_{(2)}$. The dual holographic interpretation of the presence of the $B_{(2)}$ tensor field is given by turning on a dimension $\Delta = 3$ operator on the interface. There are several questions relating to our work which are worth pursuing:

The generalized non-supersymmetric Janus solution we have found can be represented exactly using elliptic functions. Therefore, it may be possible to calculate correlation functions in our background exactly, and compare supergravity and field theory predictions beyond conformal perturbation theory.

In [21] a detailed analysis of the interface field theory found that there are additional possibilities to obtain a larger unbroken supersymmetry. In particular a case with $\mathcal{N} = 4$ interface supersymmetry was found. The internal symmetry of this theory is $SU(2) \times SU(2)$. A possible Ansatz in this case replaces the squashed five sphere used in this paper with the $T^{1,1}$ manifold [44] [48] which realizes the $SU(2) \times SU(2)$ symmetry.

The $AdS_5 \times S_5$ background is famously obtained as the near horizon limit of $N$ D3-branes. Various defect conformal field theories can holographically be obtained via near horizon limits of intersecting D-brane systems. The brane interpretation of the non-supersymmetric Janus, as well as our supersymmetric solution, is unknown at this point. The fact that both the dilaton as well as the AST field are sourced by fivebranes suggests that a brane realization of the supersymmetric Janus solution could be given by the near horizon limit of intersecting D3/D5 branes. However, the only known solutions of this kind treat the five branes in the probe approximation [34] [35]. Whether a fully back reacted solution is related to Janus is a very interesting question, which we plan to investigate in the future.

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A Realization of the \( \Gamma \)-matrices

It will be useful to choose a well-adapted basis for \( \Gamma \)-matrices,

\[
\Gamma^\mu = \sigma_1 \otimes \gamma^\mu \otimes I_4 \quad \mu = 0, 1, 2, 3, 4 \\
\Gamma^m = \sigma_2 \otimes I_4 \otimes \gamma^m \quad m = 5, 6, 7, 8, 9
\]

(A.1)

for which we have

\[
\Gamma_{11} = \Gamma^{0123456789} = \sigma_3 \otimes I_4 \otimes I_2 \otimes I_2 \\
B = \Gamma^{2568} = \sigma_3 \otimes \gamma^2 \otimes \sigma_1 \otimes \sigma_2
\]

(A.2)

The following convention for the \( 4 \times 4 \) matrices \( \gamma^A \) will be adopted,

\[
i \gamma^0 = \sigma_2 \otimes I_2 \\
\gamma^1 = \sigma_1 \otimes I_2 \\
\gamma^2 = \sigma_3 \otimes \sigma_2 \\
\gamma^3 = \sigma_3 \otimes \sigma_1 \\
\gamma^4 = \sigma_3 \otimes \sigma_3 \\
\gamma^5 = \sigma_3 \otimes \sigma_3 \\
\gamma^6 = \sigma_1 \otimes I_2 \\
\gamma^7 = \sigma_2 \otimes I_2 \\
\gamma^8 = \sigma_3 \otimes \sigma_1 \\
\gamma^9 = \sigma_3 \otimes \sigma_2
\]

(A.3)

The \( \Gamma \)-matrices in the complex frame associated with \( CP_2 \) and (5.7), are as follows,

\[
\Gamma^{21} = \Gamma^6 + i \Gamma^7 = 2 \sigma_2 \otimes I_4 \otimes \sigma_+ \otimes I_2 \\
\Gamma^{2\bar{1}} = \Gamma^6 - i \Gamma^7 = 2 \sigma_2 \otimes I_4 \otimes \sigma_- \otimes I_2 \\
\Gamma^{22} = \Gamma^8 + i \Gamma^9 = 2 \sigma_2 \otimes I_4 \otimes \sigma_3 \otimes \sigma_+ \\
\Gamma^{2\bar{2}} = \Gamma^8 - i \Gamma^9 = 2 \sigma_2 \otimes I_4 \otimes \sigma_3 \otimes \sigma_- 
\]

(A.4)

The following combinations will also be useful in evaluating the connection form,

\[
\Gamma^{ij} = I_2 \otimes \gamma^{ij} \otimes I_4 \quad i, j = 0, 1, 2, 3 \\
\Gamma^{i4} = I_2 \otimes \gamma^i \gamma^4 \otimes I_4 \\
\Gamma^{45} = i \sigma_3 \otimes \gamma^4 \otimes \gamma^5 \\
\Gamma^{4a} = i \sigma_3 \otimes \gamma^4 \otimes \gamma^a \\
\Gamma^{5a} = I_2 \otimes I_4 \otimes \gamma^5 \gamma^a \\
\Gamma^{ab} = I_2 \otimes I_4 \otimes \gamma^{ab} \quad a, b = 6, 7, 8, 9
\]

(A.5)
References

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] arXiv:hep-th/9711200.

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428, 105 (1998) arXiv:hep-th/9802109.

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) arXiv:hep-th/9802150.

[4] E. D’Hoker and D. Z. Freedman, “Supersymmetric gauge theories and the AdS/CFT correspondence,” in Strings, Branes, and Extra Dimensions, S.S. Gubser, J.D. Lykken, Eds, World Scientific (2004), arXiv:hep-th/0201253.

[5] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323 (2000) 183 arXiv:hep-th/9905111.

[6] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, “Renormalization group flows from holography supersymmetry and a c-theorem,” Adv. Theor. Math. Phys. 3, 363 (1999) arXiv:hep-th/9904017.

[7] I. R. Klebanov and M. J. Strassler, “Supergravity and a confining gauge theory: Duality cascades and chiSB-resolution of naked singularities,” JHEP 0008 (2000) 052 arXiv:hep-th/0007191.

[8] J. Polchinski and M. J. Strassler, “The string dual of a confining four-dimensional gauge theory,” arXiv:hep-th/0003136.

[9] J. M. Maldacena and C. Nunez, “Towards the large N limit of pure N = 1 super Yang Mills,” Phys. Rev. Lett. 86 (2001) 588 arXiv:hep-th/0008001.

[10] A. H. Chamseddine and M. S. Volkov, “Non-Abelian BPS monopoles in N = 4 gauged supergravity,” Phys. Rev. Lett. 79 (1997) 3343 arXiv:hep-th/9707176.

[11] A. H. Chamseddine and M. S. Volkov, “Non-Abelian solitons in N = 4 gauged supergravity and leading order string theory,” Phys. Rev. D 57 (1998) 6242 arXiv:hep-th/9711181.

[12] D. Bak, M. Gutperle and S. Hirano, “A dilatonic deformation of AdS(5) and its field theory dual,” JHEP 0305, 072 (2003) arXiv:hep-th/0304129.

[13] D. Z. Freedman, C. Nunez, M. Schnabl and K. Skenderis, “Fake supergravity and domain wall stability,” Phys. Rev. D 69, 104027 (2004) arXiv:hep-th/0312055.

[14] I. Papadimitriou and K. Skenderis, “Correlation functions in holographic RG flows,” JHEP 0410, 075 (2004) arXiv:hep-th/0407071.
[15] A. Celi, A. Ceresole, G. Dall’Agata, A. Van Proeyen and M. Zagermann, “On the fakeness of fake supergravity,” Phys. Rev. D 71, 045009 (2005) [arXiv:hep-th/0410126].

[16] J. Sonner and P. K. Townsend, “Dilaton domain walls and dynamical systems,” Class. Quant. Grav. 23, 441 (2006) [arXiv:hep-th/0510115].

[17] A. B. Clark, D. Z. Freedman, A. Karch and M. Schnabl, “The dual of Janus ((<:< < − > (:>)) an interface CFT,” Phys. Rev. D 71, 066003 (2005) [arXiv:hep-th/0407073].

[18] S. S. Gubser, “Dilaton-driven confinement,” arXiv:hep-th/9902155.

[19] D. Bak, M. Gutperle, S. Hirano and N. Ohta, “Dilatonic repulsions and confinement via the AdS/CFT correspondence,” Phys. Rev. D 70, 086004 (2004) [arXiv:hep-th/0403249].

[20] A. Kehagias and K. Sfetsos, “On running couplings in gauge theories from type-IIB supergravity,” Phys. Lett. B 454, 270 (1999) [arXiv:hep-th/9902125].

[21] E. D’Hoker, J. Estes, and M. Gutperle, “Interface Yang-Mills, Supersymmetry, and Janus”, UCLA/06/TEP/03 preprint (2006). [hep-th/0603013]

[22] A. Clark and A. Karch, “Super Janus,” JHEP 0510 (2005) 094 [arXiv:hep-th/0506265].

[23] K. Behrndt and M. Cvetic, “Bent BPS domain walls of D = 5 N = 2 gauged supergravity coupled to hypermultiplets,” Phys. Rev. D 65, 126007 (2002) [arXiv:hep-th/0201272].

[24] A. Ceresole, G. Dall’Agata, R. Kallosh and A. Van Proeyen, “Hypermultiplets, domain walls and supersymmetric attractors,” Phys. Rev. D 64, 104006 (2001) [arXiv:hep-th/0104056].

[25] G. L. Cardoso, G. Dall’Agata and D. Lust, “Curved BPS domain walls and RG flow in five dimensions,” JHEP 0203, 044 (2002) [arXiv:hep-th/0201270].

[26] G. Lopes Cardoso, G. Dall’Agata and D. Lust, “Curved BPS domain wall solutions in five-dimensional gauged supergravity,” JHEP 0107, 026 (2001) [arXiv:hep-th/0104156].

[27] G. L. Cardoso and D. Lust, “The holographic RG flow in a field theory on a curved background,” JHEP 0209, 028 (2002) [arXiv:hep-th/0207024].

[28] A. Ceresole and G. Dall’Agata, “General matter coupled N = 2, D = 5 gauged supergravity,” Nucl. Phys. B 585, 143 (2000) [arXiv:hep-th/0004111].

[29] J. H. Schwarz, “Covariant Field Equations Of Chiral N=2 D = 10 Supergravity,” Nucl. Phys. B 226 (1983) 269.

[30] P. S. Howe and P. C. West, “The Complete N=2, D = 10 Supergravity,” Nucl. Phys. B 238 (1984) 181.
[31] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, “Supersymmetric AdS(5) solutions of type IIB supergravity,” arXiv:hep-th/0510125.

[32] U. Gran, J. Gutowski, G. Papadopoulos and D. Roest, “Systematics of IIB spinorial geometry,” arXiv:hep-th/0507087.

[33] K. Pilch and N. P. Warner, “N = 2 supersymmetric RG flows and the IIB dilaton,” Nucl. Phys. B 594 (2001) 209 arXiv:hep-th/0004063.

[34] A. Karch and L. Randall, “Open and closed string interpretation of SUSY CFT’s on branes with boundaries,” JHEP 0106 (2001) 063 arXiv:hep-th/0105132.

[35] O. DeWolfe, D. Z. Freedman and H. Ooguri, “Holography and defect conformal field theories,” Phys. Rev. D 66, 025009 (2002) arXiv:hep-th/0111135.

[36] O. Aharony, O. DeWolfe, D. Z. Freedman and A. Karch, “Defect conformal field theory and locally localized gravity,” JHEP 0307 (2003) 030 arXiv:hep-th/0303249.

[37] J. Erdmenger, Z. Guralnik and I. Kirsch, “Four-dimensional superconformal theories with interacting boundaries or defects,” Phys. Rev. D 66 (2002) 025020 arXiv:hep-th/020320.

[38] S. Yamaguchi, “AdS branes corresponding to superconformal defects,” JHEP 0306 (2003) 002 arXiv:hep-th/0305007.

[39] B.A. Dubrovin, A.T. Fomenko, S.P. Novikov, Modern Geometry – Methods and Applications, Part III, Introduction to Homology Theory, Graduate Text in Mathematics, Vol 124, Springer-Verlag 1990.

[40] E. D’Hoker, “Invariant effective actions, cohomology of homogeneous spaces and anomalies,” Nucl. Phys. B 451, 725 (1995), arXiv:hep-th/9502162; E. D’Hoker and S. Weinberg, “General effective actions,” Phys. Rev. D 50, 6050 (1994) arXiv:hep-ph/9409402.

[41] J. Polchinski, String Theory, Vol II, Cambridge University Press, 1998, page 90, 91; see corresponding errata on http://www.itp.ucsb.edu/~joep/bigbook.html.

[42] H. J. Kim, L. J. Romans and P. van Nieuwenhuizen, “The Mass Spectrum Of Chiral N=2 D = 10 Supergravity On S**5,” Phys. Rev. D 32 (1985) 389.

[43] M. Gunaydin and N. Marcus, “The Spectrum Of The S**5 Compactification Of The Chiral N=2, D = 10 Supergravity And The Unitary Supermultiplets Of U(2, 2/4),” Class. Quant. Grav. 2 (1985) L11.

[44] L. J. Romans, “New Compactifications Of Chiral N=2 D = 10 Supergravity,” Phys. Lett. B 153 (1985) 392.
[45] C. N. Pope and N. P. Warner, “Two New Classes Of Compactifications Of D = 11 Supergravity,” Class. Quant. Grav. 2 (1985) L1.

[46] J. T. Liu and H. Sati, “Breathing mode compactifications and supersymmetry of the brane-world,” Nucl. Phys. B 605 (2001) 116 [arXiv:hep-th/0009184].

[47] K. Pilch and N. P. Warner, “N = 1 supersymmetric renormalization group flows from IIB supergravity,” Adv. Theor. Math. Phys. 4 (2002) 627 [arXiv:hep-th/0006066].

[48] I. R. Klebanov and E. Witten, “Superconformal field theory on threebranes at a Calabi-Yau singularity,” Nucl. Phys. B 536 (1998) 199 [arXiv:hep-th/9807080].

[49] O. DeWolfe, D. Z. Freedman, S. S. Gubser and A. Karch, “Modeling the fifth dimension with scalars and gravity,” Phys. Rev. D 62 (2000) 046008 [arXiv:hep-th/9909134].