Lagrangian approach to a symplectic formalism
for singular systems

H. Montani and R. Montemayor

Centro Atómico Bariloche, CNEA

and

Instituto Balseiro, Universidad Nacional de Cuyo

8400 - S. C. de Bariloche, Río Negro, Argentina

Abstract

We develop a Lagrangian approach for constructing a symplectic structure for singular systems. It gives a simple and unified framework for understanding the origin of the pathologies that appear in the Dirac-Bergmann formalism, and offers a more general approach for a symplectic formalism, even when there is no Hamiltonian in a canonical sense. We can thus overcome the usual limitations of the canonical quantization, and perform an algebraically consistent quantization for a more general set of Lagrangian systems.
I. INTRODUCTION

There is a canonical recipe for accomplishing the conventional quantization of a dynamical system. From a given Lagrangian we use a Legendre transformation to rewrite the velocities in terms of the momenta and to determine the Hamiltonian, and next from the Poisson brackets between coordinates and momenta we define the quantum commutation relations. This procedure works reasonably well for regular Lagrangians, but it becomes troublesome for the singular ones, which are characterized by the non-invertibility of their Hessian matrix. This not only prevents us from determining all the accelerations in terms of the coordinates and their velocities, but it also confronts us with a singular Legendre transformation when we try to construct the canonical theory. These difficulties reflect the use of an excess of coordinates in the original description, because the true physical degrees of freedom are contained in a subspace of the configuration space. This subspace is defined by a set of Lagrangian constraints, and its complement is a set of physically equivalent configurations or a physically inaccessible region.

Some decades ago, Dirac and Bergmann and colaborators made a very relevant contribution to the subject of singular Lagrangians, and developed a systematic way of constructing a canonical formalism for these systems [1–4]. The Dirac method has played a crucial role in the quantization of many theories of significant physical relevance, as is the case of gauge theories and their gauge fixing procedure. However, there exists some pathological examples where the Dirac method seems to fail [5–8], apparently because the constraint analysis on which it is based is not exhaustive enough [7–8]. In these cases the Dirac construction of the phase space leads to some inconsistencies in relation to the original Euler-Lagrange equations. The meaning of the primary/secondary classification of the constraints is not clear [4–4], and the connection between the constraints and the symmetries of the system is rather vague. This has been manifested by a large body of work speculating on the use of full or partially extended Dirac Hamiltonians, following more or less ad hoc criteria [5–7]. Besides this, for Lagrangians which are not bilinear functions of the velocities, a one to
one relation between both velocities and momenta might not exist. This introduces some additional problems with the use of the Legendre transformation and spoils the construction of a canonical classical formalism and of a consistent canonical quantization \[9,10\]. If we consider the Lagrangian description for a dynamical system as the basic one, these problems are related to the fact that the Dirac approach gives a set of constraints which are not unambiguously defined and with no direct relation with the Lagrangian constraints.

The Dirac approach is an extension of the most usual method of constructing a canonical formalism for regular systems, based on a Legendre transformation. But this is not the only approach to achieve this aim. One possibility, based on the theory of transformations for the Lagrangian formalism, has been explored in Ref. \[11\]. Another possibility is to visualize the canonical formalism for regular systems as a very special first order Lagrangian theory. This first order theory is constructed from the original one by redefining the velocities as new auxiliary variables, which leads to a new configuration space of double dimension. This is the perspective that we exploit in the present work, which allows us to obtain a canonical formalism maintaining a clear relationship with the original Lagrangian, and avoiding the ambiguities involved in the usual construction of the phase space.

Here we develop, in a heuristic language and using local coordinates, such an alternative procedure. It is based on a redefinition of the original velocities as new variables, leading to a Lagrangian with a linear dependence on the new velocities in a configuration space of double dimension. Essentially, it is an extended version for singular systems of the approach discussed by Lanczos \[12\] to accomplish the transition from the Lagrangian to the Hamiltonian form, which is based solely on the method of Lagrangian multipliers. In this context we construct a symplectic structure for singular Lagrangians. The use of Lagrange multipliers allows us to overcome the sometimes not well defined Legendre transformation.

A few years ago, Faddeev and Jackiw \[13\] proposed a method, based on the classical geometrical approach to dynamical systems, in order to identify cyclic variables and the associated constraint functions. They then proceeded to use them to eliminate superfluous degrees of freedom. However, it sometimes happens that such an elimination is not conve-
nient or even possible. To handle these situations, these constraints have been incorporated into the kinetic terms by using velocities as Lagrange multipliers [14]. This procedure has been shown to be equivalent to the Dirac method when the latter is applicable [15], and offers a clear description of the gauge symmetries [16]. Here we apply the schema presented in Ref. [14–16] to develop a transparent analysis of the structure of singular Lagrangians and their constraints in a generalized symplectic context. In particular, this analysis sheds light on the Dirac conjecture, i.e., on the relationship of first class secondary constraints with gauge symmetries [16].

A very important feature of our approach is that the resulting symplectic formalism guarantees the soundness of the original Lagrangian equations of motion. We will deal here with finite-dimensional systems, but the main results can be extended without difficulty to field theory.

The development of the article is as follows. In Section II we discuss how to construct from the original Lagrangian a first order one that leads to a presymplectic structure. If the original Lagrangian is regular, at this stage we already have the Poisson brackets and the symplectic formalism. In the three following sections we follow the analysis for singular Lagrangians and discuss the appearance of constraints. Here we use the derivative Lagrange multipliers to incorporate the constraints to the first order Lagrangian and study the different cases. The resulting schema allows us to give a complete characterization of the symmetries in relation to the constraints, and to construct a well defined symplectic formalism. To illustrate our approach, Section VI contains two examples, already discussed in the literature as presenting a pathological character from the Dirac point of view. Here we show how our approach allows us to treat them. Finally, the last section is devoted to some comments and conclusions.
II. THE PRIMARY FIRST ORDER LAGRANGIAN AND ITS CONSTRAINTS

Let us consider a dynamical system described by a Lagrangian $L(q_i, \dot{q}_i)$, of arbitrary order in the time derivatives $\dot{q}_i$, $i = 1, \ldots, n$. To develop a symplectic form for this system, we start by transforming this Lagrangian to an adequate first order one, by extending the schema of Lanczos [12] to singular Lagrangians. The procedure consists in substituting in the Lagrangian the velocities $\dot{q}_i$ by $n$ auxiliary variables $\alpha_i$ to define

$$\hat{L}(q_i, \alpha_i) = L(q_i, \dot{q}_i)|_{\dot{q}_j = \alpha_j},$$

and in constructing a new Lagrangian $L(0)(q, \dot{q}, \pi, \alpha)$, by adding to $\hat{L}(q, \alpha)$ the constraints $\alpha_i = \dot{q}_i$ through a set of Lagrange multipliers $\pi_i$:

$$L(0)(q, \dot{q}, \pi, \alpha) = \pi_i (\dot{q}_i - \alpha_i) + \hat{L}(q, \alpha).$$

This new Lagrangian leads to exactly the same equations of motion for the $q_i$ variables than the original $L(q_i, \dot{q}_i)$, as can be easily verified by eliminating the cyclic variables ($\pi_i, \alpha_i$) [17]. The structure of $L(0)$ becomes more meaningful when isolating the velocity-independent terms:

$$L(0)(q, \dot{q}, \pi, \alpha) = \pi_i \dot{q}_i - H(q, \pi, \alpha).$$

The function of the extended configuration space

$$H(q, \pi, \alpha) = \alpha_i \pi_i - \hat{L}(q, \alpha)$$

is a constant of motion for the Lagrangian $L(0)(q, \dot{q}, \pi, \alpha)$. The $\alpha_i$ are cyclic variables, and their equations of motion give rise to a set of primary constraints:

$$\Omega_i(1) \equiv -\frac{\partial H(q, \pi, \alpha)}{\partial \alpha_i} = \frac{\partial \hat{L}(q, \alpha)}{\partial \alpha_i} - \pi_i = 0.$$ 

These constraints resemble the definition of the canonical conjugate variables in the usual Legendre transformation, but here they are defined in an extended configuration space, and simply characterize its enlargement. Expression (3) also strongly resembles the usual relation
between the Lagrangian and the Hamiltonian function but here, due to the constraints, the
equations of motion cannot be cast into a canonical form and it is not possible to define a
kind of generalized Poisson brackets.

On the subspace defined by (3) the function $H$ is $\alpha$-independent. If the constraint
$\Omega^i_{(1)} = 0$ allows us to algebraically explicitate $\alpha_i$, it can be used to eliminate this variable
from the Lagrangian $L_{(0)}$, and thus to reduce the dimension of the extended configuration
space. By doing this we show in Appendices A and B how the canonical and the Dirac
Hamiltonian appear.

In the following, to make clear the role of the Lagrangian constraints, we will maintain
the whole set of auxiliary variables. To promote the constraints to Euler-Lagrange equations
they must be incorporated to the Lagrangian $L_{(0)}(q, \dot{q}, \pi, \alpha)$ by means of Lagrange multipli-
ers. We characterize the constraints by their time derivatives, instead of the $\Omega^i_{(1)}$ constraints
themselves [14,15]. This is easily implemented by introducing velocities as Lagrange multi-
pliers, $\dot{\lambda}_{(1)}^i$, in place of the standard ones. This procedure has the advantage that only the
velocity-dependent term is modified, while the velocity-independent part remains the same.
Thus, from here on the resulting function $H(q, \pi, \alpha)$ is completely defined, and the insertion
of the new constraints will only modify the coefficients of the velocities in the equations of
motion. In such a way we obtain a new Lagrangian $L_{(1)}(q, \dot{q}, \pi, \alpha, \dot{\lambda}_{(1)})$:

$$L_{(1)}(q, \dot{q}, \pi, \alpha, \dot{\lambda}_{(1)}) = \pi_i \dot{q}_i + \Omega^i_{(1)} \dot{\lambda}_{(1)}^i - H(q, \pi, \alpha).$$

For the sake of brevity, let us introduce the compact notation $Q^A_{(1)} = (q; \pi; \alpha; \lambda_{(1)})$,
$\Delta^A_{(1)} = \frac{\partial L_{(1)}}{\partial Q^A_{(1)}}$, where the range of the $A$ index is \{1...n; 1...n; 1...n\}, such that $L_{(1)}$ may
be written as

$$L_{(1)}(Q_{(1)}) = \Delta^A_{(1)} \dot{Q}^A_{(1)} - H(Q_{(1)}),$$

with $H(Q_{(1)}) = H(q, \pi, \alpha)$ given by Eq. (4). This Lagrangian leads to the first order
equations of motion:

$$F^{AB}_{(1)} \dot{Q}^B_{(1)} = - \frac{\partial H(Q_{(1)})}{\partial Q^A_{(1)}},$$

6
where \( F^{AB}_{(1)} = \frac{\partial \Delta^B}{\partial Q^A_{(1)}} - \frac{\partial \Delta^A}{\partial Q^B_{(1)}} \) has the explicit form

\[
F_{(1)} = \begin{pmatrix}
  f & 0 & M_{(1)} \\
  0 & 0 & R_{(1)} \\
  -M^T_{(1)} & -R^T_{(1)} & 0
\end{pmatrix}.
\]

(9)

The matrix \( f \) is the symplectic matrix in the \((q, \pi)\) space

\[
f = \begin{pmatrix}
  0 & -I \\
  I & 0
\end{pmatrix},
\]

where \( I \) is the unit matrix, and the remaining components are

\[
M_{(1)} = \begin{pmatrix}
  \frac{\partial \Omega_{(1)}}{\partial q} \\
  \frac{\partial \Omega_{(1)}}{\partial \pi}
\end{pmatrix} = \begin{pmatrix}
  \frac{\partial \Omega_{(1)}}{\partial q} \\
  -I
\end{pmatrix}, \quad R_{(1)} = \begin{pmatrix}
  \frac{\partial \Omega_{(1)}}{\partial \alpha}
\end{pmatrix}.
\]

(10)

We can see that the matrix \( R_{(1)} \), whose elements are \( R^i_{(1)} = \frac{\partial \Omega^i_{(1)}}{\partial \alpha_i} = \frac{\partial^2 \hat{L}(q, \alpha)}{\partial \alpha_i \partial \alpha_j} \), is the Hessian of \( L(q, \dot{q}) \) evaluated in \( \dot{q} = \alpha \).

To make apparent when \( F_{(1)} \) is regular or singular we can perform the following manipulation. Introducing the idempotent matrix

\[
U_{(1)} = \begin{pmatrix}
  I & 0 & f^{-1}M_{(1)} \\
  0 & I & 0 \\
  0 & 0 & -I
\end{pmatrix},
\]

(11)

we construct

\[
W_{(1)} = U^T F_{(1)} U = \begin{pmatrix}
  f & 0 & 0 \\
  0 & 0 & -R_{(1)} \\
  0 & R^T_{(1)} & \omega_{(1)}
\end{pmatrix},
\]

(12)

such that \( \det F_{(1)} = \det W_{(1)} \), where the components of the matrix

\[
\omega_{(1)} = M^T_{(1)} f^{-1} M_{(1)}
\]

(13)
are the Poisson brackets with respect to \((q, \pi)\) of the primary constraints \(\Omega^{(1)}\). Hence, because \(f\) is regular, the regular or singular character of \(F^{(1)}\) is defined by the matrix

\[
\Psi^{(1)} = \begin{pmatrix}
0 & -R^{(1)} \\
R^T & \omega^{(1)}
\end{pmatrix}.
\]  

(14)

Thus, \(F^{(1)}\) will be regular only if the Hessian \(R^{(1)}\) is regular, in which case the symplectic matrix \(F^{-1}\) can be constructed explicitly:

\[
[F^{(1)}]^{-1} = \begin{pmatrix}
f^{-1} & -f^{-1}M^{(1)}R^{-1} & 0 \\
-R^{-1}M^Tf^{-1} & R^{-1}\omega^{(1)}R^{-1} & -R^{-1} \\
0 & R & 0
\end{pmatrix},
\]  

(15)

and we can define the generalized bracket

\[
\{E(Q^{(1)}), G(Q^{(1)})\}^* = \frac{\partial E(Q^{(1)})}{\partial Q^A} [F^{(1)}]^{-1}_{AB} \frac{\partial G(Q^{(1)})}{\partial Q^B}.
\]  

(16)

For dynamical functions that only depend on the \((q, \pi)\) variables it reduces to the Poisson bracket. On the other hand, if the Hessian \(R^{(1)}\) is singular the matrix \(F^{(1)}\) is also singular and we cannot construct such a generalized bracket. To overcome this we must take into account other constraints, as we shall see in the following section.

Here it is convenient to define the classification of the constraints we will use. We call primary constraints the ones discussed above, which are related to the transition to a first order theory. Secondary constraints directly arise from the zero modes of the Hessian of the original Lagrangian, and finally possible additional constraints are characterized as tertiary ones.

III. THE SECONDARY LAGRANGIAN

A non-regular Hessian \(R^{(1)}\) implies a singular \(F^{(1)}\), which is the signature of secondary constraints. These can be made explicit by the application of the \(F^{(1)}\) null eigenvectors on the equations of motion corresponding to the Lagrangian \(L^{(1)}(Q^{(1)})\). Thus, if \(F^{(1)}\) has \(m < n\) zero modes \(v^r_{(1)}, \ r = 1, \ldots, m\), we have \(m\) secondary constraints:

\[
\]
\[ \Omega^r_{(2)}(q, \pi, \alpha, \lambda_{(1)}) \equiv v^r_{(1)A} \frac{\partial H}{\partial Q^A_{(1)}} = 0. \]  

(17)

The zero modes are given by:

\[ v^r_{(1)} = (-v^r_0 M^r_{(1)} f^{-1}, u^r, -v^r_0), \]

(18)

where the first component corresponds to the variables \((q, \pi)\), and \(v^r_0\) and \(u^r\) are the solutions of:

\[ v^r_0 R_{(1)} = 0, \quad u^r_0 R_{(1)} = v^r_0 \omega_{(1)}. \]

(19)

Introducing these vectors into Eq. (17) we get

\[ \Omega^r_{(2)} = v^r_0 \partial H + v^r_0 \frac{\partial \Omega^i_{(1)}}{\partial q^i} + u^r_0 \frac{\partial H}{\partial \alpha_i}, \]

(20)

and using the expressions (4) and (5) for \(H(q, \pi, \alpha)\) and \(\Omega^i_{(1)}\) respectively, we finally have:

\[ \Omega^r_{(2)} = v^r_0 \left\{ \frac{\partial \hat{L}^i(q, \alpha)}{\partial \hat{q}_i} - \frac{\partial^2 \hat{L}(q, \alpha)}{\partial \alpha_i \partial q_j} \alpha_j \right\} + u^r_0 \Omega^i_{(1)}. \]

(21)

The last term may be disregarded because it is weakly null, and the remaining one is precisely a Lagrangian constraint evaluated in \(\dot{q} = \alpha\), which is independent of \(\pi_i\), and thus \(\frac{\partial \Omega^r_{(2)}}{\partial \pi} = 0\).

In the present schema the Lagrangian constraints of the original theory appear as secondary ones, a direct consequence of the singularity of the Hessian of this theory.

Coming back to Eq. (20), and using the Poisson bracket defined by \(f^{-1}\) for the canonically conjugated variables \((q, \pi)\), the constraints \(\Omega^r_{(2)}\) may be written as:

\[ \Omega^r_{(2)} = v^r_0 \{ \Omega^i_{(1)}, H \}. \]

(22)

This shows that, although the primary constraints are considered in an analogous way by our procedure and the Dirac one, there is a relevant difference with respect to the secondary constraints. Our approach leads directly to a maximal set of linearly independent secondary constraints, in contrast with the Dirac method where a rather arbitrary linear combination of the constraints can be considered as a constraint.
Our following step is to incorporate these new constraints into the Lagrangian $L_{(1)}(Q_{(1)})$, using again a new set of velocity Lagrange multipliers, $\dot{\lambda}_{(2)}$. This gives place to a new Lagrangian

$$L_{(2)}(q, \dot{q}, \alpha, \dot{\lambda}_{(1)}, \dot{\lambda}_{(2)}) = L_{(2)}(Q_{(2)}) = \Delta_{(2)}^{A} \dot{Q}_{(2)}^{A} - W(Q_{(1)}) ,$$

where $Q_{(2)} = (q, \pi, \alpha, \dot{\lambda}_{(1)}, \dot{\lambda}_{(2)})$. The symplectic matrix corresponding to this last Lagrangian is:

$$F_{(2)} = \begin{pmatrix} f & 0 & M_{(2)} \\ 0 & 0 & R_{(2)} \\ - M_{(2)}^{T} & - R_{(2)}^{T} & 0 \end{pmatrix} ,$$

with

$$M_{(2)} = \begin{pmatrix} \frac{\partial \Omega_{(1)}}{\partial q} & \frac{\partial \Omega_{(2)}}{\partial q} \\ \frac{\partial \Omega_{(1)}}{\partial \pi} & \frac{\partial \Omega_{(2)}}{\partial \pi} \end{pmatrix} , \quad R_{(2)} = \begin{pmatrix} \frac{\partial \Omega_{(1)}}{\partial \alpha} & \frac{\partial \Omega_{(2)}}{\partial \alpha} \end{pmatrix} .$$

Proceeding as in the previous step we construct the matrix $W_{(2)}$, which has a structure analogous to $W_{(1)}$ in Eq. (12), but now with:

$$\Psi_{(2)} = \begin{pmatrix} 0 & - R_{(2)} \\ R_{(2)}^{T} & \omega_{(2)} \end{pmatrix} .$$

The components of $\omega_{(2)}$ are the Poisson brackets of the primary and secondary constraints, $\Omega_{(1)}$ and $\Omega_{(2)}$, with respect to $(q, \pi)$. Taking into account Eqs. (13) and (25) it can be written:

$$\omega_{(2)} = M_{(2)}^{T} f^{-1} M_{(2)} = \begin{pmatrix} \omega_{(1)} & \left[ \frac{\partial \Omega_{(2)}}{\partial q} \right]^{T} \\ - \left[ \frac{\partial \Omega_{(2)}}{\partial \pi} \right]^{T} & 0 \end{pmatrix} .$$

In contrast with the primary Lagrangian stage, the matrix $R_{(2)}$ is now rectangular. Our main problem is to study when $F_{(2)}$ is singular or regular, which is defined by $\Psi_{(2)}$. This is considered in the two following sections.
IV. A REGULAR $\Psi_{(2)}$ MATRIX: THE RESTRICTED PHASE SPACE AND THE GENERALIZED BRACKETS

In this case $F_{(2)}$ is regular and hence $F_{(2)}^{-1}$ exists. As the matrix $\Psi_{(2)}^{-1}$ exists, we can construct the matrix

$$W_{(2)}^{-1} = U_{(2)}^{-1} F_{(2)}^{-1} U_{(2)}^T = \begin{pmatrix} f^{-1} & 0 \\ 0 & \Psi_{(2)}^{-1} \end{pmatrix},$$

and easily obtain $F_{(2)}^{-1}$ by considering a block decomposition of $\Psi_{(2)}^{-1}$. In fact, if we write

$$\Psi_{(2)}^{-1} = \begin{pmatrix} (\Psi_{(2)}^{-1})_{11} & (\Psi_{(2)}^{-1})_{12} \\ -(\Psi_{(2)}^{-1})_{12} & (\Psi_{(2)}^{-1})_{22} \end{pmatrix},$$

where $(\Psi_{(2)}^{-1})_{11}$ and $(\Psi_{(2)}^{-1})_{22}$ are $n \times n$ and $(n + m) \times (n + m)$ antisymmetric matrices respectively, while $(\Psi_{(2)}^{-1})_{12}$ is a $(n + m) \times n$ rectangular matrix, the expression for $F_{(2)}^{-1}$ becomes

$$F_{(2)}^{-1} = \begin{pmatrix} f^{-1} - f^{-1} M_{(2)} (\Psi_{(2)}^{-1})_{22} M_{(2)}^T f^{-1} & -f^{-1} M_{(2)} (\Psi_{(2)}^{-1})_{12}^T + f^{-1} M_{(2)} (\Psi_{(2)}^{-1})_{22} \\ (\Psi_{(2)}^{-1})_{12} M_{(2)}^T f^{-1} & (\Psi_{(2)}^{-1})_{11}^T - (\Psi_{(2)}^{-1})_{12} \end{pmatrix}$$

Observe that in the block $(1, 1)$ of $F_{(2)}^{-1}$ a structure quite similar to the Dirac bracket can be recognized.

This structure becomes more evident if $\omega_{(2)}$ is regular, in which case one can perform an invertible transformation in order to write $\Psi_{(2)}$ in a block diagonal form. This is supplied by the idempotent matrix

$$S = \begin{pmatrix} I & 0 \\ -\omega^{-1}_{(2)} R_{(2)}^T & I \end{pmatrix},$$

such that:

$$\Psi_{(2)} \rightarrow \hat{\Psi}_{(2)} = S^T \Psi_{(2)} S = \begin{pmatrix} \gamma_{(2)} & 0 \\ 0 & \omega_{(2)} \end{pmatrix},$$

11
where \( \gamma(2) = R(2) \omega^{-1}(2) R^T(2) \). In this case \( \gamma(2) \) defines the regular or singular character of \( F(2) \). It is worth remarking that \( \det \gamma(2) \neq 0 \) is equivalent to the condition of maximal rank for the matrix of the partial derivatives of the Lagrangian constraints with respect to the coordinates and the velocities, which warrants the soundness of the constraints [3]. Consequently, in a well defined constrained Lagrangian theory \( \gamma(2) \) must be regular. This allows us to obtain a more explicit expression for \( \Psi^{-1} \) and then construct \( F^{-1} \) out of the matrix \( R(2) \) and \( M(2) \).

\[
F^{-1}(2) = \begin{pmatrix}
-\gamma^{-1}(2) R(2) \omega^{-1} M^T(2) f^{-1} & -f^{-1} M(2) \omega^{-1} R^T(2) \gamma^{-1}(2) & -f^{-1} M(2) C(2) \\
-f^{-1} M(2) \omega^{-1} R^T(2) \gamma^{-1}(2) & -\gamma^{-1}(2) R(2) \omega^{-1} & -\gamma^{-1}(2) R(2) \omega^{-1} \\
C(2) M^T(2) f^{-1} & \omega^{-1} R^T(2) \gamma^{-1}(2) & C(2)
\end{pmatrix},
\]

(33)

where

\[
C(2) = \omega^{-1}(2) - \omega^{-1}(2) R^T(2) \gamma^{-1}(2) R(2) \omega^{-1}(2),
\]

(34)

which satisfies the relation \( C(2) R^T(2) = 0 \), meaning that \( C(2) \) is constituted by the zero modes of \( R^T(2) \). Therefore \( F^{-1}(2) \) has a structure analogous to the one that defines the symplectic form in the Dirac method, but extended to the whole configuration space, including the auxiliary variables.

Thus, when \( \omega(2) \) is regular we can define a generalized bracket given by:

\[
\{E(Q(2)), G(Q(2))\}^* = \frac{\partial E(Q(2))}{\partial Q^A(2)} [F^{-1}(2)]^{-1}_{AB} \frac{\partial G(Q(2))}{\partial Q^B(2)},
\]

(35)

which in particular for the variables \( \alpha_i \) and \( \lambda_i \) gives:

\[
\{\alpha_i, \alpha_j\}^* = \gamma^{-1}_{ij},
\]

(36)

\[
\{\lambda^k_{(n)}, \lambda^l_{(n)}\}^* = C^{kl}.
\]

(37)

This case includes the systems with only second class constraints according to the Dirac method, and the phase space is a uniquely defined subspace of the primary one.

It is also possible to have \( \Psi(2) \) regular with \( \omega(2) \) singular. In this case we can redefine the constraints such that we can rewrite
where \( r \) is the dimension and \( s \) the rank of \( \omega_{(2)} \). From here on we can follow the preceding procedure, but with \( \tilde{\omega}_{(2)} \) instead of \( \omega_{(2)} \). This will lead to expressions similar to the ones already obtained in the case of \( \omega_{(2)} \) regular.

V. A SINGULAR \( \Psi_{(2)} \) MATRIX

When \( \Psi_{(2)} \) is singular, \( F_{(2)} \) is also singular, which makes it impossible to get a generalized bracket structure at this level. This may happen as a consequence of the existence of some still hidden constraints, which we call tertiary ones, or of a gauge symmetry of the secondary Lagrangian. In this case \( F_{(2)} \) will have the set of orthonormal zero modes \( \{v^{s}_{(2)}\} \)

\[
v^{s}_{(2)} F_{(2)} \simeq 0, \quad s = 1, \ldots, m', m' < m, \quad (38)
\]

where we use the symbol \( \simeq \) to signify that this is a weak equation. The explicit form of \( v^{s}_{(2)} \) is

\[
v^{s}_{(2)} = (z^{s} M_{(2)}^{T} f^{-1}, y^{s}, z^{s}), \quad (39)
\]

where \( z^{s} \) and \( y^{s} \) satisfy

\[
z^{s} R^{T}_{(2)} = 0, \quad z^{s} \omega_{(2)} + y^{s} R_{(2)} = 0. \quad (40)
\]

A. Tertiary constraints

In an analogous way to secondary constraints, we can also have tertiary ones given by

\[
\Omega^{s}_{(3)} \equiv v^{s}_{(2)[\nu]} \frac{\partial H}{\partial Q^{(2)}_{(\nu)}} \equiv z^{s}_{(\nu)} \{\Omega_{(\nu)}, H\} + y^{s} \Omega_{(1)}. \quad (41)
\]

We have new constraints \( \Omega^{s}_{(3)} \) if \( z^{s}_{(\nu)} \{\Omega_{(\nu)}, H\} \) are not linear combinations of the \( \Omega_{(1)} \) and \( \Omega_{(2)} \). In this case we must iterate the procedure to include these constraints in a new Lagrangian.
\[ L_{(3)}(Q_{(3)}) = \Delta_{(3)}^A \dot{Q}_A^A - W(Q_{(1)}) , \tag{42} \]

where \( Q_{(3)} = (q, \dot{q}, \alpha, \dot{\lambda}_1, \dot{\lambda}_2, \dot{\lambda}_3) \).

**B. Gauge Symmetries**

When no new constraints arise from the above equation, the remaining zero modes give rise to the following symmetry transformations for the equations of motion (see Appendix C)

\[ \delta^s Q_{(2)} = \tau_{(2)}^s \epsilon , \tag{43} \]

which in terms of the coordinates of the extended configuration space is:

\[ \delta^s Q = \epsilon z^s M_{(2)}^T f^{-1} = \epsilon z^s_{(\nu)} \{ \Omega_{(\nu)}^r , Q \} , \tag{44} \]

\[ \delta^s \alpha_i = \epsilon y^s , \tag{45} \]

\[ \delta^s \lambda_{(\nu)}^r = \epsilon z^s_{(\nu)} . \tag{46} \]

This is a symmetry transformation of the secondary Lagrangian, but not necessarily a symmetry transformation of the original one. One may wonder under which conditions the above coordinate transformations can be promoted to symmetries of the original action. To answer this question we can consider the variation of the action induced by transformation (43), taking into account that

\[ z^s_{(\nu)} \{ \Omega_{(\nu)} , H \} = U^s_{(\nu)} \Omega_{(\nu)} . \tag{47} \]

Thus we finally get

\[ \delta^s S \simeq \int_{t_1}^{t_2} dt \epsilon \left[ U^s_{(\nu)} \Omega_{(\nu)} - y^s \Omega_{(1)} - z^s_{(\nu)} \left( \frac{\partial \Omega_{(\nu)}}{\partial \pi} \pi - \Omega_{(\nu)} \right) \right] . \tag{48} \]

The surface term in the case of internal symmetries will cancel only if the constraints are first order homogeneous functions of the canonical momenta \( \pi \), as happens in the Dirac
method. It is worth remarking that it is possible to add a term to the time derivative of $\epsilon_s$ in order to cancel the residual terms, which are all of them proportional to the constraints

$$\partial_t \epsilon \rightarrow D^s_t \epsilon = \partial_t \epsilon - z^s \left( U^s (\nu) \Omega_s (\nu) + y^s \Omega_s (1) \right) \epsilon .$$

(49)

Here we have assumed that the zero modes of $R^T (2)$ are orthonormalized and there is no sum on $s$. In this way, by introducing this covariant derivative weakly equivalent to the ordinary one, we obtain a symmetry holding on the whole of the configuration space, whenever Eq.(38) is strongly satisfied.

The relations (40) may be fulfilled in the following two situations.

a) $z^s = 0$ and $y^s$ is a zero mode of $R (2)$. Given that $R (1)$ is singular, this implies that the matrix $(\partial \Omega / \partial \nu)$ has a zero mode. In this case $v^s = (0, y^s, 0)$, and does not lead to new constraints but generates the transformation symmetry

$$\delta Q = 0$$

(50)

$$\delta \alpha = \epsilon_s y^s$$

(51)

$$\delta \lambda ( \nu ) = 0$$

(52)

under which the Lagrangian variation is:

$$\delta L (2) = \delta \alpha_i \partial \Omega / \partial \alpha_i \dot{\lambda} + \delta \alpha_i \partial H / \partial \alpha_i = \epsilon_s v^s (\Omega (1) + R (2) \dot{\lambda} = \epsilon_s v^s \Omega (1) .$$

(53)

b) $z^s \neq 0$ and so there are non-trivial zero modes of $R^T (2)$. If $\omega (2)$ is regular and $\gamma (2)$ is singular, the relations (40) become:

$$z^s = y^s R (2) \omega^{-1} (2) , \ z^s \gamma (2) = 0 ,$$

(54)

so that the zero modes can be written

$$v^s (2) = (v^s R (2) \omega^{-1} (2) M_T f^{-1} , v^s , -v^s R (2) \omega^{-1} (2) ) ,$$

(55)

and the associated constraints are

$$\Omega^s (2) = v^s \Omega (1) + v^s R (2) \omega^{-1} (2) \{ \Omega , H \} .$$

(56)
The first term is a linear combination of the primary constraints, and the second one contains a matrix whose components are the Poisson brackets \( \{ \Omega_{(1)}, H \} \), a linear combination of the secondary constraints, and \( \{ \Omega_{(2)}, H \} \). The primary and secondary constraints have already been considered, and thus the tertiary constraints will only correspond to a given combination of the Poisson brackets \( \{ \Omega_{(2)}, H \} \). In general, in this case we must carry on and include these new constraints in a tertiary Lagrangian, by iterating the preceding construction. Tertiary constraints may arise only when the set of zero modes of \( \gamma_{(2)} \) is bigger than the set of zero modes of \( R_{(2)} \), and whenever the time evolution of the secondary constraints, \( \{ \Omega_{(2)}, H \} \), does not reduce to a linear combination of \( \Omega_{(1)} \) and \( \Omega_{(2)} \). If no new constraint arises, the transformation generated by (43) leaves the equation of motion invariant on the constrained subspace.

In the simplest situation where both sets of zero modes are equal, we have \( v^* R_{(2)} = 0 \) and recover the situation described in the preceding subsection.

VI. SOME EXAMPLES

To clarify the use and possibilities of our method, we consider here two simple examples which have been widely discussed in the literature. Both present pathological features from the point of view of the Dirac approach. The first one is a regular Lagrangian, but with terms of higher degree in the derivatives. This causes a bifurcation in the canonical structure according to the usual approach, which gives place to a problem with multivaluated boundary conditions. Our approach allows us to construct a symplectic formalism with physical sense, which also gives a consistent quantum theory. The second example contains a first class constraint, according to the Dirac classification, which has a weak null bracket with every dynamical function, and thus does not admit a gauge fixing. In this case our approach permits a deeper understanding of the origin of the pathology.
A. 1.- Lagrangian of higher order time derivatives

Let us now consider an example with higher order time derivatives, which has been studied in Ref. [10], defined by the Lagrangian

\[ L(\dot{q}) = \frac{1}{4} (\dot{q})^4 - \frac{1}{2} k (\dot{q})^2 \]  

(57)

The equation of motion in the configuration space is:

\[ (3\dot{q}^2 - k) \ddot{q} = 0 \]

which implies \( \dot{q} = \text{cte.} \), i.e. it describes a one-dimensional free-particle.

When a Legendre transformation is used to construct a canonical formalism, this Lagrangian leads to velocities which are multivaluated functions of the canonical momenta, making the classical motion unpredictable since at any time one can jump from one branch of the Hamiltonian to another. We now analyze this system in the light of our procedure.

Proceeding as proposed in the first section, we introduce the auxiliary variables \( \alpha \) and \( p \) to obtain a first order time derivative Lagrangian \( L_{(0)} \),

\[ L_{(0)}(\dot{q}, p, \alpha) = p(\dot{q} - \alpha) + \frac{1}{4} \alpha^4 - \frac{1}{2} k \alpha^2 . \]  

(58)

Its equations of motion reveal the constraint

\[ \Omega \equiv \frac{\partial L_{(0)}}{\partial \alpha} = \alpha^3 - k \alpha - p . \]  

(59)

By incorporating it to the Lagrangian we get

\[ L_{(1)}(\dot{q}, p, \alpha, \lambda) = p\dot{q} + (\alpha^3 - k \alpha - p) \lambda - H(p, \alpha) , \]  

(60)

where the Hamiltonian is

\[ H(p, \alpha) = -\frac{1}{4} \alpha^4 + \frac{1}{2} k \alpha^2 + p\alpha . \]  

(61)

Arranging the coordinates in a four-dimensional vector \( Q_{(1)}^A \equiv (r = q - \lambda, p, \alpha, \lambda) \), one easily obtains the symplectic matrix \( F_{(1)} \)
\[ F_{(1)} = \begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 3\alpha^2 - k \\
0 & 0 & k - 3\alpha^2 & 0 
\end{bmatrix} \, . \quad (62) \]

Its inverse defines a generalized Poisson bracket

\[
\{ E(Q_{(1)}), G(Q_{(1)}) \}^* = \frac{\partial E(Q_{(1)})}{\partial Q^A_{(1)}} [F_{(1)}]^{-1}_{AB} \frac{\partial G(Q_{(1)})}{\partial Q^B_{(1)}},
\]

so that the equations of motion can be written as

\[
\dot{Q}^A_{(1)} = \{ Q^A_{(1)}, H \}^* . \quad (64)
\]

They lead to the classical solution \( \dot{q}(t) = \text{cte} \), consistent with the original Lagrangian dynamics, but in fact the configuration space is now two-dimensional, without constraints, and therefore we have two degrees of freedom and we need four boundary conditions. This establishes a difference with the usual description for a free particle, given by a Lagrangian quadratic in the velocity. The equations of motion (64) can be displayed as:

\[
\begin{align*}
\dot{r} &= \alpha \\
\dot{\lambda} &= \alpha^3 - k\alpha - p \\
\dot{\alpha} &= 0 \\
\dot{p} &= 0 
\end{align*} \quad (65)
\]

which implies that the four phase-space coordinates are constants of motion, and hence \( \dot{q} = \dot{r} + \dot{\lambda} \) is also a constant of motion. But now it is not sufficient to give, for example, the position and the velocity at a given time:

\[
\begin{align*}
q_0 &= r + \lambda , \\
\dot{q}_0 &= \alpha^3 + (1 - k)\alpha - p ,
\end{align*} \quad (66)
\]

because there is not enough information to determine a point in the phase-space.
There is another remark that should be made. The phase space coordinates \( Q_A \) are not canonical, in the sense that the brackets of the fundamental variables are not equal to one or to zero. If we try to obtain a canonical structure through a change of coordinates it is necessary to consider a non-linear transformation, such as \( \tilde{Q}_A \equiv (r = q - \lambda, \ p, \tilde{\alpha} = \alpha^3 - k\alpha, \lambda) \). These new coordinates satisfy canonical brackets, but this type of transformation also introduces bifurcations in the phase space with a correlated multivaluated Hamiltonian.

Although the \( Q_A \) coordinates are not canonical, the dynamics described in their terms has a symplectic structure. This is given by Eqs. (62, 63), which define the evolution of the system through the generalized Hamiltonian (61) and the equations of motion (64). This symplectic structure can be used to construct a quantum theory by using the canonical procedure given by \( \{ A, B \}^* \rightarrow -i[\hat{A}, \hat{B}] \), where \( \hat{A} \) and \( \hat{B} \) are the quantum operators corresponding to the classical functions \( A \) and \( B \). This procedure leads to:

\[
[\hat{r}, \hat{p}] = i , \tag{67}
\]

\[
[\hat{\alpha}, \hat{\lambda}] = i \left( k - 3\alpha^2 \right)^{-1} , \tag{68}
\]

with all the other fundamental commutators null. A representation for the fundamental operators that satisfies these commutation relations is:

\[
\hat{r} = r , \quad \hat{\alpha} = \alpha , \tag{69}
\]

\[
\hat{p} = -i \frac{\partial}{\partial r} , \quad \hat{\lambda} = -i \left( k - 3\alpha^2 \right)^{-1} \frac{\partial}{\partial \alpha} .
\]

The time derivatives of the fundamental phase space operators are given by:

\[
\dot{r} = -i[\hat{r}, \hat{H}] = \alpha ,
\]

\[
\dot{\alpha} = -i[\hat{\alpha}, \hat{H}] = 0 , \tag{70}
\]

\[
\dot{p} = -i[\hat{p}, \hat{H}] = 0 ,
\]

\[
\dot{\lambda} = -i[\hat{\lambda}, \hat{H}] = -\left( k - 3\alpha^2 \right)^{-1} \left( \hat{p} - \alpha^3 + k\alpha \right) = -\alpha + \frac{\hat{p} + 2\alpha^3}{3\alpha^2 - k} ,
\]

which implies that \( \hat{p}, \ \hat{\alpha}, \frac{d\hat{r}}{dt} \) and \( \frac{d\hat{\lambda}}{dt} \) are constants of motion. For the coordinates of the configuration space, \( r \) and \( \lambda \), and their velocities we have the following commutation relationships:
\[ \left[ \hat{r}, \frac{d\hat{r}}{dt} \right] = [\hat{r}, \hat{\alpha}] = 0, \quad (71) \]

\[ \left[ \hat{\lambda}, \frac{d\hat{\lambda}}{dt} \right] = -i \left( k - 3\alpha^2 \right)^{-1} \left[ \frac{\partial}{\partial \alpha}, \left( -\alpha + \frac{\hat{\rho} + 2\alpha^3}{3\alpha^2 - k} \right) \right] = \frac{i}{k - 3\alpha^2} \left( 1 + \frac{6\alpha}{(3\alpha^2 - k)^2} \left( \hat{\rho} - \alpha (\alpha^2 - k) \right) \right). \quad (72) \]

In particular the velocity of the particle

\[ \frac{d\hat{q}}{dt} = \frac{d\hat{r}}{dt} + \frac{d\hat{\lambda}}{dt} = \hat{p} + 2\alpha^3 \quad (73) \]

is a constant of motion that can take any real value, as is characteristic for a free particle. But the commutator of the position and the velocity is here:

\[ \left[ \hat{q}, \frac{d\hat{q}}{dt} \right] = \left[ \hat{r} + \hat{\lambda}, \frac{\hat{\rho} + 2\alpha^3}{3\alpha^2 - k} \right] = \frac{i}{3\alpha^2 - k} \left[ 1 + \frac{6\alpha}{(3\alpha^2 - k)^2} \left( \hat{\rho} + \alpha \left( 5\alpha^2 - k \right) \right) \right] \neq i. \quad (74) \]

This implies that the uncertainty relationship between the position and the velocity is different from the one resulting when the Lagrangian is quadratic in the velocity. This result clearly shows that we have a new quantum theory, but with the same classical limit than the quadratic one.

**B. 2.- Lagrangian with a pathological Dirac constraint**

The following Lagrangian was proposed by Cawley:

\[ L = \dot{q}_1 \dot{q}_2 + q_3 q_1^2. \quad (75) \]

The construction of a canonical formalism for this system is cumbersome. This and other related examples have motivated several discussions and ad hoc modifications of the Dirac method. The Lagrangian equations of motion give:

\[ \dot{q}_1 = \text{arbitrary constant}, \]

\[ q_2 = 0, \quad (76) \]

\[ q_3 = \text{arbitrary function of } t. \]

If we try to apply the Dirac method we find a primary constraint, \( p_3 = 0 \), and a secondary one, \( q_1^2 = 0 \). This latter has been called ineffective \[\square\] or a fourth class \[\blacksquare\] constraint,
for which it is impossible to construct a gauge fixing constraint. This prevents us from determining a physical subspace of the phase space with a symplectic structure. To overcome this difficulty we could try to use a linearized constraint, $q_2 = 0$, instead of the secondary one. From an algebraic point of view both constraints are equivalent, but this last one leads to an additional constraint, $p_1 = 0$. Thus we would now have three first class constraints, $p_3 = 0$, $q_2 = 0$ and $p_1 = 0$, for which it would be necessary to introduce a gauge fixing. Each gauge fixing would define a configuration characterized by:

$$q_1 = \text{a given constant},$$

$$q_2 = 0,$$  \hspace{1cm} (77)

$$q_3 = \text{a given function of } t,$$

and so it would not reproduce the whole solution space of the Lagrangian equations. For this reason this Lagrangian is considered a pathological one from the point of view of the Dirac method.

Now we will apply our method to this Lagrangian. To simplify the discussion we use the shortcut provided by the algebraic elimination of cyclic variables. The first order Lagrangian is:

$$L_{(0)} = \sum_{i=1}^{3} \pi_i \dot{q}_i - H,$$  \hspace{1cm} (78)

with

$$H = \sum_{i=1}^{3} \pi_i \alpha_i - \alpha_1 \alpha_2 - q_3 q_1^2.$$  \hspace{1cm} (79)

The equations of motion of the $\alpha$ variables give the constraints:

$$\Omega_{(1)}^{1} = \frac{\partial L_{(0)}}{\partial \alpha_1} = \alpha_2 - \pi_1,$$

$$\Omega_{(1)}^{2} = \frac{\partial L_{(0)}}{\partial \alpha_2} = \alpha_1 - \pi_2,$$  \hspace{1cm} (80)

$$\Omega_{(1)}^{3} = \frac{\partial L_{(0)}}{\partial \alpha_3} = -\pi_3.$$
We introduce these constraints into the Lagrangian by means of a set of velocity Lagrange multipliers $\dot{\lambda}$, and thus we obtain the primary Lagrangian $L_{(1)}$

$$L_{(1)} = \sum_{i=1}^{3} \pi_i \dot{q}_i - \sum_{i=1}^{3} \Omega_{(1)}^i \dot{\lambda}_i - H . \quad (81)$$

The Hessian $R_{(1)}$ has a zero mode leading to the secondary constraint:

$$\Omega_{(2)} = \frac{\partial L_{(0)}}{\partial q_3} = -q_1^2 . \quad (82)$$

Again, we incorporate this constraint to the Lagrangian as before:

$$L_{(2)} = \sum_{i=1}^{3} \pi_i \dot{q}_i - \sum_{i=1}^{3} \Omega_{(1)}^i \dot{\lambda}_i - q_1^2 \dot{\eta} - H , \quad (83)$$

and now the rectangular matrix $R_{(2)}$ is

$$R_{(2)} = [R_{(1)} \mid 0_{3\times1}] . \quad (84)$$

It has two zero modes satisfying the Eqs. (84). One of them is basically the same as in the previous step, $z^1 = (0, 0, 1, 0)$, and gives rise again to the constraint $\Omega_{(2)}$

$$\Omega_{(3)}^1 = z_{(\nu)}^1 \{\Omega_{(\nu)}, H\} = \frac{\partial H}{\partial q_3} = -q_1^2 = \Omega_{(2)} \approx 0 . \quad (85)$$

The other zero mode, $z^2 = (0, 0, 0, 1)$, yields the tertiary constraint

$$\Omega_{(3)}^2 = z_{(\nu)}^2 \{\Omega_{(\nu)}, H\} = -\frac{\partial \Omega_{(2)}}{\partial q_1} \frac{\partial H}{\partial \pi_1} = -2 q_1 \alpha_1 \approx 0 , \quad (86)$$

which is weakly zero by virtue of $\Omega_{(2)}$. Hence no new constraints actually arise because of these last two zero modes and we may conclude that these zero modes give rise to a symmetry of the equations of motion.

In fact, the symmetries just found correspond to the last set of equations of motion, which include all the constraints, and do not necessarily correspond to symmetries of the original Lagrangian. The question is now whether it is possible to extend them to the original Lagrangian or not. At this point the symmetry generated by the second zero mode poses a problem. Observe that if we try to carry the expression of $\Omega_{(3)}^2$ into the form of Eq. 22
the coefficients $U_{(\nu)}^s$, given by Eq. (47), become ill defined on the constrained subspace, thus preventing the extension of this symmetry to the whole configuration space, as one can see through the definition of the covariant derivative given by Eq. (49).

In summary, the symmetry generated by the first zero mode can be extended to the whole phase space by constructing an adequate covariant derivative, which enables us to introduce a gauge fixing without altering the original Lagrangian equations of motion. But this is not the case for the symmetry generated by the second zero mode, which is strictly valid only in the constrained subspace. Any additional condition introduced to make the presimplectic matrix regular only corresponds in fact to a gauge fixing in this subspace. Therefore, this additional condition will maintain the equations of motion in the subspace, but will alter them in the whole space, and leads to a set of equations of motion different to the one given by the original Lagrangian.

VII. FINAL REMARKS

In this paper we develop a general theory of singular Lagrangians and their constraints. It leads to a systematic and purely Lagrangian approach for constructing a symplectic structure, thus avoiding the use of a not always well defined Legendre transformation and the rise of many associated pathologies. Our approach is based on an enlargement of the original configuration space (the $q$ space), which makes use of two sets of auxiliary variables. One set allows us to rewrite the original theory as a first order one (the $(\alpha, \pi)$ variables) and the other contains the Legendre multipliers (the $\lambda$ variables), which we use to incorporate all the constraints at the level of Lagrangian equations of motion. These last variables are introduced as velocities in the enlarged Lagrangian, such that they acquire the role of conjugate coordinates of the constraints. In this way we have a well defined first order regular dynamics in an extended configuration space, which warrants that the $q$ coordinates satisfy exactly the same equations of motion than in the original Lagrangian. This procedure allows us to disentangle all the mechanisms hidden in the Dirac-Bergmann approach, and to
identify the source of the pathologies that appear there.

Our construction gives place to three kinds of constraints. In all the cases we have a set of primary constraints, which resemble the definition of the canonical momenta. They characterize the enlargement of the configuration space necessary to pass from the original theory to a first order one. For a regular Lagrangian they are the only constraints that appear and our construction stops here. When the original Lagrangian is singular a new set of constraints appear, which we call secondary constraints. They are associated to the zero modes of the Hessian of the original theory. Finally, there may be additional hidden constraints, not directly related to the zero modes of the Hessian, which we have characterized as tertiary ones.

These constraints can hold in the whole configuration space or in a constrained subspace. According to this they can be classified in two categories. One of them corresponds to what can be called strong constraints, i.e. the set of independent constraints that act on the whole configuration space and give place to a reduction of the extended space. They contain the second class constraints of Dirac.

The other category is constituted by the weak constraints, i.e. constraints that hold only on a constrained subspace. Strictly speaking, the latter are the generators of symmetry transformations in such a constrained subspace. Here there are two possibilities, according to whether they close an algebra or not. In the first case the symmetry can be promoted to be valid on the whole configuration space by defining a covariant time derivative for the parameter of the transformation, weakly equivalent to the ordinary one. Hence we can give a gauge fixing that does not alter the equations of motion of the original coordinates $q$, and we can construct a symplectic form. In the second case, when the constraints do not close an algebra, the symmetry holds only on the constrained subspace and cannot be extended to the whole space. In this case we can give a gauge fixing in the constrained subspace, but it will alter the equations of motion of the original coordinates and it thus becomes impossible to construct a symplectic form without altering the original equations of motion.

In summary, when there are strong constraints or weak ones that close an algebra our
approach allows us to construct a consistent symplectic formalism. Our brackets are defined on the whole extended space, and correspond to the Dirac brackets in the \((q, \pi)\) subspace when they exist. Furthermore, we can perform this construction even when there is no well defined Hamiltonian, as in the example discussed in Section IV-1, which renders our approach more general than the Dirac method.

When there are weak constraints that do not close an algebra, we cannot construct a meaningful symplectic formalism consistent with the original equations of motion. The Lagrangians that lead to this case are pathological within the Dirac method, and also in other approaches.

In contrast with the Dirac method, in our approach the whole set of constraints is unambiguously defined and there is a clear connection between the constraints and the symmetries of the Lagrangian. Some of the main contributions of our work are that it gives a simple and unified framework for understanding the origin of the pathologies that appear in the Dirac-Bergmann formalism, which are explained by the impossibility to promote a given symmetry from the constrained subspace to the whole configuration space, and that it brings a more general approach for constructing a symplectic formalism, even when there is no Hamiltonian at all in a canonical sense. This last point opens the possibility of overpassing the limitations of the canonical quantization, and performing an algebraically consistent quantization on the basis of this non-canonical symplectic algebra.

VIII. ACKNOWLEDGMENTS

This work was partially supported by the Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina.

APPENDIX A: THE CANONICAL HAMILTONIAN

In the very particular case in which all the cyclic variables \(\alpha_i\) can be algebraically eliminated, we are finally faced with a Lagrangian of the form:
\[ L_{(0)}(q, \dot{q}, \pi) = \pi_i \dot{q}_i - H(q, \pi) \]  
(A1)

with the equation of motion given by:

\[ f^{ab} \dot{Q}^b = -\frac{\partial H(Q)}{\partial Q^a} \]  
(A2)

where \( Q^a = (q_i, \pi_i) \) and

\[ f = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \]  
(A3)

a symplectic matrix in the \((q, \pi)\) space. This situation corresponds to a regular Lagrangian, and the equations of motion can be written

\[ \dot{A}(Q) = \{A(Q), H(Q)\} \]  
(A4)

where \( H(Q) \) is the usual canonical Hamiltonian, in terms of the Poisson brackets

\[ \{E(Q), G(Q)\} = \frac{\partial E(Q)}{\partial Q^a} f^{-1}_{ab} \frac{\partial G(Q)}{\partial Q^b} . \]  
(A5)

**APPENDIX B: THE DIRAC HAMILTONIAN**

Let us consider a Lagrangian bilinear in the velocities:

\[ L = \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j - V(q) \]

\[ i = 1, ..., n , \]  
(B1)

where the matrix \( m_{ij} \) is singular. The procedure presented in Section II produces the first order Lagrangian:

\[ L_{(0)} = \pi_i \dot{q}_i - \pi_i \alpha_i + H \]  
(B2)

with

\[ H = \frac{1}{2} m_{ij} \alpha_i \alpha_j - V(q) \]  
(B3)

which leads to the following primary constraints:
\[ \Omega^i_{(1)} = m_{ij} \alpha_j - \pi_i \]  

We assume that the range of \( m_{ij} \) is \( r < n \), and introduce the operator \( U \) that diagonalizes the mass matrix

\[ (U^T m U)_j = \lambda_i \delta_{ij} \]  

and transforms the coordinates according to

\[ \alpha \rightarrow \tilde{\alpha} = U^T \alpha U \]  
\[ \pi \rightarrow \tilde{\pi} = U^T \pi U \]

After this transformation the constraints become

\[ \tilde{\pi}_s = \lambda_s \tilde{\alpha}_s \quad s = 1, ..., r \]  
\[ \tilde{\pi}_p = 0 \quad p = r + 1, ..., n \]

We can use these constraints to eliminate the first \( r \) cyclic variables \( \alpha_s \), but the last \( n - r \) ones \( \alpha_p \) will remain arbitrary. In such a way the Hamiltonian becomes:

\[ \tilde{H} = \frac{1}{2} \lambda_s^{-1} \tilde{\pi}_s \tilde{\pi}_s + V(q) + \tilde{\alpha}_p \tilde{\pi}_p \]  

which in terms of the original variables \( (q_s, \pi_s) \) and the arbitrary functions \( \alpha_p \) can be written

\[ H = \frac{(m^{-1})_{st}}{2} \pi_s \pi_t + V(q) + \alpha_p \pi_p \]

which is the Dirac Hamiltonian.

**APPENDIX C: CONSTRAINTS AND SYMMETRIES**

Given a first order Lagrangian:

\[ L(Q, \dot{Q}) = f(Q)\dot{Q} - H(Q) \]

the form of the corresponding equations of motion is:
\[ F(Q)\dot{Q} - (\nabla_q H)(Q) = 0. \]  

(C2)

If the matrix \( F(Q) \) has zero modes \( v^r \), \( v^r F(Q) = 0 \), from the equations of motion we obtain an associated set of constraints

\[ \Omega^r(Q) = v^r (\nabla_q H)(Q). \]  

(C3)

The transformation of coordinates generated by these zero modes

\[ \delta Q = \bar{v}^r \epsilon_r, \]  

(C4)

where \( \epsilon_r \) are time-dependent parameters, produces the Lagrangian variation

\[ \delta L = \frac{\delta L}{\delta Q} \delta Q = \epsilon_r v^r (F(Q)\dot{Q} - (\nabla_q H)(Q)) = \epsilon_r \Omega^r. \]  

(C5)

The relation (C5) shows that the Lagrangian is weakly invariant under displacements in directions orthogonal to the gradient of the potential. In a Lagrangian formalism the constraints are part of the equations of motion. In this sense this symmetry is valid on a subspace determined by the solutions of the equations of motion, and can be called a ”weak symmetry”, in contrast with the symmetries independent of the solutions of the equations of motion which can be called ”strong symmetries”. When the constraints close an algebra, this symmetry is related to a strong one, obtained by a simple redefinition of the derivatives of the parameter \( \epsilon \).
REFERENCES

[1] P. A. M. Dirac, Can. J. Math. 2 (1950), 129; Proc. Roy. Soc. (London) A246 (1958), 326; J.L. Anderson and P. G. Bergmann, Phys. Rev. 83 (1951), 1018; P. G. Bergmann, Helv. Phys. Acta Suppl. IV (1956), 79.

[2] P. A. M. Dirac, "Lectures on Quantum Mechanics", Belfer Graduated School of Science, Yeshiva University Press, N. Y., 1964; P. Hanson, T. Regge and C. Teitelboim, "Constrained Hamiltonian Systems", Accademia Nazionale dei Lincei 22, 1976.

[3] E. C. G. Sudarshan and N. Mukunda, "Classical Dynamics: A Modern Perspective", Wiley, New York, 1974.

[4] K. Sundermeyer, "Constrained Dynamics", Springer-Verlag, New York, 1982.

[5] R. Cawley, Phys. Rev. Lett. 42 (1979), 413.

[6] A. Frenkel, Phys. Rev. D21 (1980), 2986.

[7] M.J. Gotay, J.M. Nester and G. Hinds, J. Math. Phys. 19 (1978), 2388.

[8] L. Lusanna, Riv. Nuovo Cim. 14 (1991), 1.

[9] S. Hojman and R. Montemayor, Hadronic Journal 3 (1980), 1644.

[10] M. Henneaux and C. Teitelboim, Phys. Rev. A36 (1987), 4417.

[11] R.E. Peierls, Proc. Roy. Soc. (London) A214 (1952), 143; P.J. Bergmann and R. Schiller, Phys. Rev. 89 (1953), 4; P.J. Bergmann, I. Goldberg, A. Janis and E. Newman, Phys. Rev. 103 (1956), 807.

[12] C. Lanczos, "The Variational Principles of Mechanics", University of Toronto Press, Toronto, 1949.

[13] L. Faddeev and R. Jackiw, Phys. Rev. Lett. 60 (1988), 1962.

[14] J. Barcelos Neto and C. Wotzasek, Mod. Phys. Lett. A7 (1992), 1737; Int. J. Mod.
Phys. A7 (1992), 4981.

[15] H. Montani, Int. J. Mod. Phys A8 (1993), 4319.

[16] H. Montani and C. Wotzasek, Mod. Phys. Lett A8 (1993), 3387.

[17] This is not the only possible suitable definition of the auxiliary variables. In this article we restrict ourselves to this particular election. The implications and possibilities of this freedom in the construction of a symplectic formalism will be discussed elsewhere.