Simple Non-Markovian Microscopic Models for the Depolarizing Channel of a Single Qubit

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Abstract. The archetypal one-qubit noisy channels—depolarizing, phase-damping and amplitude-damping channels—describe both Markovian and non-Markovian evolution. Simple microscopic models for the depolarizing channel, both classical and quantum, are considered. Microscopic models which describe phase damping and amplitude damping channels are briefly reviewed.

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1. Introduction

The decoherence process followed by a given physical system is usually modelled by a (generalized) master equation for its density operator. In the literature of open quantum systems, master equations are usually called Markovian if they did not involve an explicit time integration. Otherwise, they are termed non-Markovian. These designations are employed in this paper, albeit they do not agree with the mathematical definitions of Markovian and non-Markovian processes, as was shown long ago [1]. Markovian master equations, which can be derived from system-environment models employing the Born-Markov approximation [2], sometimes present unphysical behavior, because they lead to negative density operators [3, 4]. Master equation of the Lindblad form [5, 6], which preserve the fundamental properties of density operators (non-negativity, unit trace and hermicity), are also markovian. Under special conditions they can also exhibit unphysical behavior, like the unbounded increase of the energy of the system [7].

Markovian master equations are important tools to model open quantum systems, not only due to their mathematical simplicity, but also because they capture the physical behavior of many important systems, such as open QED (Quantum Electrodynamics)
systems [8, 9, 10, 11]. Many Markovian master equations are obtained using the Born-Markov approximation, which relies on a series of assumptions which are not always satisfied [12]: weak system-environment coupling, separable total density operator, bath correlation time much smaller than the relaxation time of the system, and unperturbed transition times of the system much smaller than its relaxation time. The next to last assumption implies that the environment is infinite. Thus, loosely speaking, Markovian behavior is typical of systems which interact weakly with infinite environments.

One of the alternatives to the use of generalized master equations to describe open quantum systems is the Feynman-Vernon influence functional [13]. Other, which was derived by Kraus [14] employing the idea of complete positivity, is known as the “operator-sum representation” of the quantum dynamics of an open system. The operator sum representation and Lindblad-form master equations can be related. In effect, if a closed quantum system $S$, initially in a product state $\rho_S(0) = \rho_A(0) \otimes \rho_B(0)$, comprises two interacting subsystems, the system of interest $A$ and its environment $B$, the exact dynamics of the state any of the subsystems, can be put in Kraus form. If the inequality $\tau_c \ll \tau \ll \tau_H$ is satisfied, where $1/\tau_c$ is the cutoff frequency of the bath density of states, $\tau$ an adequate coarse-graining time scale and $\tau_H$ the characteristic time-scale of the hamiltonian evolution of the system, then it is possible to derive a completely-positive master equation starting from the Kraus operator-sum representation [15, 16], linking both descriptions. The operator-sum representation, used to represent noisy dynamics, is popular in the area of quantum information processing. In particular, Nielsen [17] introduced several archetypal quantum operations on a qubit, including the depolarizing, phase-damping (or phase-flip) and amplitude-damping channels, which are customarily assumed Markovian in theoretical and experimental analyses [18]. However, the description of decoherence in solid state systems, the most promising scalable realizations of quantum processors, often need to be non-Markovian [19, 20, 21, 22].

The purpose of this manuscript is to show, explicitly, that the archetypal one-qubit noisy channels generally describe non-Markovian dynamics. After a brief review of the three channels that we consider in this paper, we present simple (yet non-Markovian) microscopic models for them. The decoherence process is caused by the fluctuation of macroscopic variables which enter in the Hamiltonian or by the establishment of correlation between the system and its environment.

The standard form of Lindblad master equations [5] for the density operator of the system of interest, $\dot{\rho}$, is

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar} \left[ \hat{H}, \rho(t) \right] + \sum_i \left( 2 \hat{L}_i \rho(t) \hat{L}_i^\dagger - \hat{L}_i^\dagger \hat{L}_i \rho(t) - \rho(t) \hat{L}_i^\dagger \hat{L}_i \right), \quad (1)$$

where $\hat{H}$ is a Hamiltonian and $\{\hat{L}_j\}$ are (possibly non-hermitian) operators, usually known as Lindblad operators. For a given physical process the decomposition (1) is not unique. However, it is convenient to interpret the first term of the right hand side of (1) as the unitary evolution in the absence of interaction with the environment, and the second term as the environment-system coupling. On the other hand, the Kraus
representation
\[ \dot{\rho}(t) = \sum_i \hat{E}_i(t) \dot{\rho}(0) \hat{E}_i^\dagger(t), \]  
(2)
where the Kraus operators \( \hat{E}_i(t) \) satisfy the condition of probability conservation
\[ \sum_i \hat{E}_i^\dagger(t) \hat{E}_i(t) = \mathbb{I}, \]
gives the evolution of the density matrix of the system of interest.

We consider a qubit with orthonormal states \( |0\rangle \) and \( |1\rangle \). In this basis the operator
\[ \hat{\sigma}_3 = |0\rangle \langle 0| - |1\rangle \langle 1| \]
is diagonal, and the lowering operator is given by
\[ \hat{\sigma}_- = |1\rangle \langle 0| = \hat{\sigma}_1 - i\hat{\sigma}_2 = \hat{\sigma}_+^\dagger. \]
Several noisy channels have been defined for a single qubit. For example, the depolarizing channel
\[ \dot{\rho}(p) = p \frac{\hat{I}}{2} + (1 - p) \dot{\rho}(0), \]  
(3)
describes a process where the state of the qubit, initially in its state \( \dot{\rho}(0) \), remains in this state with probability \( p \), and changes to the maximally mixed state \( \frac{\hat{I}}{2} \) with probability \( 1 - p \). If a two-qubit mixed state is used, instead of a Bell state, in the standard teleportation protocol, a generalized depolarizing channel is obtained \[23\]. An amplitude damping channel is obtained when a qubit interacts with a large reservoir at zero temperature, or a two-level atom in the electromagnetic vacuum emits spontaneously \[2, 8, 9, 10, 11\].

An exciton confined to a quantum dot and coupled to phonons \[24\], like a spin in a magnetic field whose strength fluctuates in time, suffers a dephasing process. The Markovian master equations corresponding to the depolarizing, amplitude-damping and phase-damping channels are
\[ \frac{d \rho(t)}{dt} = \frac{\Gamma}{8} \sum_i (2 \hat{\sigma}_i \dot{\rho}(t) \hat{\sigma}_i - \hat{\sigma}_i \dot{\rho}(t) \hat{\sigma}_i - \hat{\sigma}_i \dot{\rho}(t) \hat{\sigma}_i), \]  
(4)
\[ \frac{d \rho(t)}{dt} = \frac{\Gamma}{2} (2 \hat{\sigma}_- \dot{\rho}(t) \hat{\sigma}_+ - \hat{\sigma}_+ \dot{\rho}(t) \hat{\sigma}_- - \hat{\sigma}_- \dot{\rho}(t) \hat{\sigma}_+), \]  
(5)
\[ \frac{d \rho(t)}{dt} = \Gamma (2 \hat{\sigma}_+ \dot{\rho}(t) \hat{\sigma}_- - \hat{\sigma}_+ \dot{\rho}(t) \hat{\sigma}_- - \hat{\sigma}_- \dot{\rho}(t) \hat{\sigma}_+), \]  
(6)
respectively. The operator-sum representation \[2\] of these channels depends on the Kraus operators
\[ \hat{E}_0 = \sqrt{1 - \frac{3p}{4}} \hat{I}, \quad \hat{E}_i = \frac{\sqrt{p}}{2} \hat{\sigma}_i, \quad i = 1, 2, 3; \]  
(7)
\[ \tilde{E}_0 = |0\rangle \langle 0| + \sqrt{1 - p} |1\rangle \langle 1| \quad \text{and} \quad \tilde{E}_1 = \sqrt{p} |0\rangle \langle 1|; \]  
(8)
\[ \tilde{E}_0 = \sqrt{1 - p} \hat{I}, \quad \tilde{E}_1 = \sqrt{p} |0\rangle \langle 0|, \quad \tilde{E}_2 = \sqrt{p} |1\rangle \langle 1|; \]  
(9)
where \( p(t) = 1 - e^{-\Gamma t} \). A given physical process can be described by multiple Kraus representations, as can be seen in the phase-damping channel \[9\], which can also be given by the two Kraus operators \[25\]
\[ E_0 = \sqrt{1 - \frac{p}{2}} \hat{I}, \quad E_1 = \sqrt{\frac{p}{2}} \hat{\sigma}_3. \]  
(10)
In the following sections we show that the same Kraus representation can be related to many master equations, generally non-Markovian.
2. Simple quantum microscopic model of a depolarizing channel

Electronic spins in semiconductor quantum dots have been proposed as a prototype of a scalable quantum computer \[26\]. A Caldeira-Leggett microscopic model of decoherence, considered to be a generic phenomenological description of the environment of the spins, have also been advanced. The interaction of the spin with the collection of harmonic oscillators, which describe its environment, is of the form

\[ H_{\text{int}} = \lambda \sum_i \sigma_i \sum_k g_k \left( a_{ik} + a^\dagger_{ik} \right), \]  (11)

where \( a_{ik} (a^\dagger_{ik}) \) is the annihilation (creation) operator of the \( k \)-th harmonic oscillator coupled to the \( i \)-th component of the spin. The interaction is clearly isotropic. It is also natural to represent the degrees of freedom of the environment as spins, corresponding to the hyperfine interaction, or to the interaction of the electronic spins with other electronic spins. Here we consider a simple variant of the spin star model \[27, 28\] in which the dynamics of the spin-bath system is governed by the total Hamiltonian

\[ H = \frac{g}{\hbar} \mathbf{s} \cdot \sum_{k=1}^{N} \mathbf{S}_k, \]  (12)

where \( s_i = \hbar \sigma_i/2 \) is the \( i \)-th component of the electronic spin and \( S_{ki} \) is the \( i \)-th component of the \( k \)-th environmental degree of freedom, which, for the sake of simplicity is assumed to be a one-half spin. The coupling constant \( g \) has units of frequency. It is convenient to write the environmental states in the basis of collective angular momentum. It is well-known that the bath states can be labeled as \(|l,m_l\rangle\) where \( \kappa \leq l \leq N/2 \), \( m_l = -l, -l + 1, \ldots, l \) and \( \kappa = 0 (\kappa = 1/2) \) if \( N \) is even (odd). The degeneracy of the bath momentum \( l \) is given by \[27, 29, 28\]

\[ \nu(N, l) = \left( \frac{N}{2} - l \right) - \left( \frac{N}{2} - l - 1 \right). \]  (13)

In the new basis the spin-bath Hamiltonian is simplified to \( H = \oplus_l \frac{g}{\hbar} \mathbf{s} \cdot \mathbf{S}_l \), where the degeneracy must be taken into account. In the variant we consider here, described by the Hamiltonian

\[ H = \oplus_l \frac{g}{\hbar} \mathbf{s} \cdot \mathbf{S}_l, \]  (14)

not only the degeneracy is not given by (13) but the coupling coefficients can be different.

The initial state of the total system is assumed to be factorized \( \rho_T(0) = \rho_S(0) \otimes \rho_B(0) \), that is, equal to the product of the initial spin state \( \rho_S(0) \) and the initial bath state, \( \rho_B(0) \). The latter is assumed to be the maximally mixed state \( 2^{-N} \mathbb{I}_{N \times N} \). The notation \( \mathbb{I}_{N \times N} \) stands for the identity in \( N \) dimensions. In the new basis we have the interaction between spin 1/2 and spin \( l \), where the initial state is \( \rho_S(0) \otimes (2l + 1)^{-1} \mathbb{I}_{(2l+1) \times (2l+1)} \). For a fixed value of \( l \) it is convenient to make a unitary transformation, from the separated basis, whose states \(|\frac{1}{2}, l; j, m_j\rangle\) are simultaneous eigenstates of \( s^2, S^2_l, s_z \) and \( J_z \), to the coupled basis, whose states \(|\frac{1}{2}, l; j, m_j\rangle\) are
simultaneous eigenstates of $s^2$, $S_z^2$, $J^2$ and $J_z$, where $J = s + S$. In terms of these operators the spin-bath Hamiltonian reads $\otimes \frac{g}{\hbar} (J^2 - s^2 - S_z^2)$. One can go from the separated basis to the coupled one using
\[
\begin{pmatrix}
\frac{1}{2}, l; l + \frac{1}{2}, l + \frac{1}{2} - p \\
\frac{1}{2}, l; l - \frac{1}{2}, l + \frac{1}{2} - p
\end{pmatrix} = \begin{pmatrix}
\cos \alpha_p & \sin \alpha_p \\
-\sin \alpha_p & \cos \alpha_p
\end{pmatrix} \begin{pmatrix}
\frac{1}{2}, l; -\frac{1}{2}, l - p + 1 \\
\frac{1}{2}, l; \frac{1}{2}, l - p
\end{pmatrix},
\] (15)
where the elements of the transition matrix are given by the Clebsh-Gordan coefficients
\[
\cos \alpha_p = \sqrt{\frac{p}{2l + 1}}, \quad \sin \alpha_p = \sqrt{\frac{2l - p + 1}{2l + 1}}.
\]
The dynamical problem is solved, for the initial condition
\[
|\psi_0\rangle = \left(c_+ \frac{1}{2}, \frac{1}{2} + c_- \frac{1}{2}, -\frac{1}{2}\right) \otimes |l, m_l\rangle.
\]
If the initial state is expanded in the coupled basis, evolve the state taking into account that the eigenenergies $\hbar g l / 4$ and $-\hbar g (l + 1) / 4$ correspond to the eigenstates of $S_z^2$ with eigenvalues $l + \frac{1}{2}$ and $l - \frac{1}{2}$, respectively, and transformed back to the separated basis
\[
\begin{align*}
|\psi_t\rangle &= c_1^{m_l} \frac{1}{2}, l; l + \frac{1}{2}, m_l + 1 \rangle + c_2^{m_l} \frac{1}{2}, l; l + \frac{1}{2}, m_l \rangle \\
&\quad + c_3^{m_l} \frac{1}{2}, l; l - \frac{1}{2}, m_l \rangle + c_4^{m_l} \frac{1}{2}, l; l - \frac{1}{2}, m_l - 1 \rangle.
\end{align*}
\] (16)
The time dependent coefficients $c_a^{m_l}$, $a = 1, 2, 3, 4$ are given by
\[
\begin{align*}
c_1^{m_l} &= -2i c_+ e^{-i\frac{gt}{4}} \sqrt{(l - m_l)(l + m_l + 1)} \frac{2l + 1}{2l + 1} \sin \left(\frac{(2l + 1)gt}{4}\right), \\
c_2^{m_l} &= c_+ e^{-i\frac{gt}{4}} \left( \cos \left(\frac{(2l + 1)gt}{4}\right) - i \frac{2m_l + 1}{2l + 1} \sin \left(\frac{(2l + 1)gt}{4}\right) \right), \\
c_3^{m_l} &= c_- e^{-i\frac{gt}{4}} \left( \cos \left(\frac{(2l + 1)gt}{4}\right) + i \frac{2m_l - 1}{2l + 1} \sin \left(\frac{(2l + 1)gt}{4}\right) \right), \\
c_4^{m_l} &= -2i c_- e^{-i\frac{gt}{4}} \sqrt{(l + m_l)(l - m_l + 1)} \frac{2l + 1}{2l + 1} \sin \left(\frac{(2l + 1)gt}{4}\right). 
\end{align*}
\] (17-20)

After finding the total density matrix operator, and tracing out the bath degrees of freedom, the reduced density operator for the spin one-half system is found to be
\[
\rho_{m_l} = \left(|c_1^{m_l}|^2 + |c_2^{m_l}|^2\right) |0\rangle \langle 0| + c_3^{m_l} \overline{c_2^{m_l}} |0\rangle \langle 1| + c_2^{m_l} \overline{c_3^{m_l}} |1\rangle \langle 0| + \left(|c_2^{m_l}|^2 + |c_4^{m_l}|^2\right) |1\rangle \langle 1|,
\]
where the state $\frac{1}{2}, -\frac{1}{2} \left(\langle \frac{1}{2}, \frac{1}{2}\rangle\right)$ have been identified with the state $|0\rangle \langle 1|$). Now, it is possible to take into account that the initial bath spin-$l$ state is $\langle (2j + 1)^{-1} \Pi_{(2j+1)\times(2j+1)} \rangle$, to show that
\[
\rho_l = \frac{1}{2l + 1} \sum_{m_l=-l} \rho_{m_l} = \frac{1}{2} \left( \Pi_{2\times2} + s_l(t) \cdot \sigma \right),
\] (21)
can be calculated exactly. The Bloch vector $s_l(t)$ completely characterizes the quantum state of spin one-half system and is given by
\[
s_l(t) = s_l(0) \frac{4l^2 + 4l + 3 + 8l(l + 1) \cos \left(\frac{(2l+1)gt}{2}\right)}{3(2l + 1)^2},
\] (22)
where $s_i(0)$ is the value of the Bloch vector of the initial state of the system. For large values of $l$, $s_i(t) \approx s_i(0)(1 + 2 \cos \left(\frac{(2l+1)gt}{2}\right))/3$. If we have a fixed value of $l$ and a gaussian distribution of coupling constants, in which the probability density to have a particular value $g$ for the coupling constant is

$$p_G(g) = (2\pi \sigma^2)^{-1/2} \exp(-g^2/(2\sigma^2)), \quad (23)$$

$\sigma$ being the standard deviation of the distribution, then the average Bloch vector

$$s_i(t) = s_i(0) \frac{4l^2 + 4l + 3 + 8l(l + 1)e^{-(2l+1)\sigma^2 t^2/8}}{3(2l + 1)^2}, \quad (24)$$

decreases until its length is reduced for a factor $x = \frac{4l^2+4l+3}{3(2l+1)^2}$, which satisfies the inequality $\frac{1}{3} < x \leq \frac{1}{2}$.

If the depolarizing channel $\mathcal{D}$ is applied to an initial pure state, its Bloch’s vector evolves as $s_D(t) = s_D(0)(1 - p)$. We thereby conclude that we have a depolarizing channel for which

$$p(t) = \frac{8l(l + 1)}{3(2l + 1)^2} \left(1 - e^{-(2l+1)\sigma^2 t^2/8}\right), \quad (25)$$
a gaussian decay very different from the Markovian process, $p(t) = 1 - e^{-\gamma t}$. Moreover, in constrast with the Markovian case, where $p(t)$ varies from zero to one, in the non-Markovian process described by (24), $p(t)$ varies, monotonically, from zero to $p_{\text{inf}}(l) = \frac{8l(l + 1)}{3(2l+1)^2} < \frac{2}{3}$, i.e., this process does not display complete depolarization.

As a second example of the distribution of coupling constants, we assume a Lorentzian distribution $p(g) = a \pi (g^2 + a^2)^{-1}$, where $a$ is the scale parameter which specifies the half-width at half-maximum. The average Bloch vector decreases as in the previous example

$$s_i(t) = s_i(0) \frac{4l^2 + 4l + 3 + 8l(l + 1)e^{-(2l+1)\sigma^2 t^2/2}}{3(2l + 1)^2}, \quad (26)$$

but in an exponential way. The time-dependence of the parameter $p$, given by

$$p(t) = \frac{8l(l + 1)}{3(2l + 1)^2} \left(1 - e^{-(2l+1)\sigma^2 t^2/2}\right), \quad (27)$$
is similar, but different, to that of a Markovian process. Indeed, the corresponding master equation reads

$$\frac{dp(t)}{dt} = \frac{\gamma(t)}{8} \sum_{i=1}^{3} \left(2 \hat{\sigma}_i \hat{\rho}(t) \hat{\sigma}_i - \hat{\sigma}_i \hat{\rho}(t) \hat{\sigma}_i - \hat{\rho}(t) \hat{\sigma}_i \hat{\sigma}_i \right), \quad (28)$$

which does not have the Lindblad form because now $\gamma(t)$ is an explicit function of time

$$\gamma(t) = \Gamma \left(1 - \frac{4l^2 + 4l + 3}{4l^2 + 4l + 3 + 8l(l + 1)e^{-(2l+1)\sigma^2 t^2/2}}\right). \quad (29)$$

In a simpler, but also interesting case, $l$ and $g$ fixed, $p(t)$ varies periodically, and the master equation is given by (28) with a time-dependent $\gamma(t)$,

$$\gamma(t) = \frac{(2l + 1)g}{2} \frac{8l(l + 1) \sin \left(\frac{(2l+1)gt}{2}\right)}{4l^2 + 4l + 3 + 8l(l + 1) \cos \left(\frac{(2l+1)gt}{2}\right)}, \quad (30)$$
which, in contrast to (29), attain negative values.

3. Simple classical microscopic model of a depolarizing channel

In nuclear magnetic resonance experiments and Bose-Einstein condensates the decoherence process is often caused by residual fluctuating magnetic fields, see for example [30], which can be considered classical. In the Hamiltonian description of this process

\[ H = -\mu \cdot B = \hbar g \xi \cdot \sigma, \]

(31)

the dimensionless independent random variables \( \xi_i, i = 1, 2, 3 \), assumed to be gaussian of zero average and standard deviation \( \sigma \), are proportional to the corresponding magnetic field components. The constant \( g \) have units of frequency. If the initial state of the spin system is \( \rho(0) \), its state at time \( t \) is

\[ \rho(t) = e^{-igt\xi \cdot \sigma} \rho(0) e^{igt\xi \cdot \sigma} = \frac{1}{2} (I_{2 \times 2} + s(t) \cdot \sigma), \]

(32)

where

\[ s(t) = s(0) \cos(2gt\xi) + \xi \times s(0) \frac{\sin(2gt\xi)}{\xi} + (\xi \cdot s(0)) \frac{\xi(1 - \cos(2gt\xi))}{\xi^2}, \]

(33)

and \( \xi = \sqrt{\xi \cdot \xi} \). Averaging over the different realizations of the noise variables \( \xi_i, i = 1, 2, 3 \), the state of the two-level system is

\[ \overline{\rho}(t) = \frac{1}{2} (I_{2 \times 2} + \overline{s}(t) \cdot \sigma), \quad s_i(t) = s_i(0) \left( \frac{\cos(gt\xi)}{\xi} + \frac{\xi^2(1 - \cos(gt\xi))}{\xi^2} \right) = s_i(0) f(t), \]

(34)

where the property of vanishing averages of the distributions of \( \xi_i \) were used. Employing the gaussian probability distribution \( p_G(\xi_i), i = 1, 2, 3 \), and transforming to spherical coordinates given by \( r = \xi, \theta = \arccos(\xi_3/\xi), \phi = \arctan(\xi_2/\xi_1) \), where \( r \in [0, \infty), \phi \in [0, 2\pi) \) and \( \theta \in [0, \pi] \), one finds for the polarization factor \( f(t) \)

\[ f(t) = \frac{1}{3} \left( 1 + 2(1 - 4g^2\sigma^2 t^2) e^{-2g^2\sigma^2 t^2} \right). \]

(35)

In figure we have plotted the polarization factor using \( \sigma = 1 \) and time measured in units of \( 1/g \).

We stress that albeit none of the examples considered in the previous section and this section are Markovian, all of them are described by the Kraus representation (7) of the depolarization channel, with different definitions of the parameter \( p \). The time-dependence of \( p \) allows for non-exponential decay and even recoherence, in which case \( \gamma(t) \) is negative.

4. Simple quantum microscopic model of a dephasing channel

The model hamiltonian we consider, given by

\[ H = \frac{\hbar \omega(t)}{2} \sigma_z + \sum_k \hbar \omega_k \left( a_k + \frac{c_k \sigma_z}{\omega_k} \right)^\dagger \left( a_k + \frac{c_k \sigma_z}{\omega_k} \right), \]

(36)
describes the interaction of a two-level system with a collection of oscillators, through an interaction which conserves the system’s observable $\sigma_z$. The creation and annihilation operators $a_k^\dagger$ and $a_k$ satisfy the usual boson commutation relations $[a_k, a_k^\dagger] = \delta_{kk'}$. Hamiltonians of the type we consider here have been explored by many authors \cite{31, 32, 33, 34, 35}. We assume an initial state of the Feynman-Vernon form $\rho(0) = \rho_S(0) \otimes \prod_k \rho_k$, where $\rho_k = \exp(-\beta \hbar \omega_k a_k^\dagger a_k) / Z_k$ is the thermal state of the $k$-th mode at inverse temperature $\beta = 1/(k_B T)$ and $Z_k = \text{tr} \left( \exp(-\beta \hbar \omega_k a_k^\dagger a_k) \right)$. If the action of the unperturbed Hamiltonians is separated from the interaction, the total density operator is
\begin{equation}
\rho(t) = e^{-i\Omega(t)\sigma_z/2} e^{-i \sum_k \omega_k t a_k^\dagger a_k} U_I(t,0) \rho(0) U_I^\dagger(t,0) e^{i \sum_k \omega_k t a_k^\dagger a_k} e^{-i\Omega(t)\sigma_z/2} \tag{37}
\end{equation}
where the interaction evolution operator in the interaction picture, $U_I(t,0)$,

\begin{equation}
U_I(t,0) = \mathcal{T} \exp \left( -i \int_0^t d\tau \sum_k \left( c_k e^{i\omega_k \tau} a_k^\dagger \sigma_z + h.c. \right) \right) = \mathcal{T} e^{-i \int_0^t d\tau \sum_k H_k(\tau)} \tag{38}
\end{equation}
needs the time-ordering prescription, indicated by $\mathcal{T}$, because the commutators $[H_k(\tau), H_k(\tau')]$ do not vanish. However, due to the simple Lie algebra satisfied by the operators appearing in the interaction Hamiltonian, $[a_k \sigma_z, a_k^\dagger \sigma_z] = \delta_{k,l}$, and tracing out the environmental degrees of freedom we can write
\begin{equation}
\rho_S(t) = e^{-i\Omega(t)\sigma_z/2} \text{Tr}_k \left( e^{-i \int_0^t d\tau \sum_k H_k(\tau)} \rho_S(0) \prod_k \rho_k e^{-i \int_0^t d\tau \sum_k H_k(\tau)} \right) e^{-i\Omega(t)\sigma_z/2}. \tag{39}
\end{equation}
Employing algebraic techniques \cite{2} to calculate the trace, the density operator of the system can be written as
\begin{equation}
\rho_S(t) = e^{-i\Omega(t)\sigma_z/2} \left( e^{-2\Gamma(t)(1-\sigma_z \cdot \sigma_z)} \rho_S(0) \right) e^{-i\Omega(t)\sigma_z/2}, \tag{40}
\end{equation}
where the functions $\Omega(t) = \int_0^t d\tau \omega(\tau)$ and $\Gamma(t) = \sum_k |c_k|^2 (1 - \cos(\omega_k t)) \coth(\hbar \omega_k \beta / 2) / \omega_k^2$ have been defined. Taking into account that $\rho_S(0) = \sum_{ij} \rho_{ij} |i\rangle \langle j|$ the expression into brackets in equation (40) can be simplified as follows
\begin{equation}
e^{-2\Gamma(t)(1-\sigma_z \cdot \sigma_z)} \rho_S(0) = e^{-2\Gamma(t)(1-\sigma_z \cdot \sigma_z)} \sum_{ij} \rho_{ij} |i\rangle \langle j| = \sum_{ij} \rho_{ij} e^{-2\Gamma(t)(1-\sigma_z (i) \sigma_z (j))} |i\rangle \langle j|, \tag{41}
\end{equation}
with \( s_z(0) = 1 = -s_z(1) \). If we explicitly write the four terms we have
\[
e^{-2\Gamma(t)(1-s_z\sigma_z)}\rho_S(0) = \rho_{00} \langle 0 | + e^{-4\Gamma(t)}\rho_{01} | 0 | + e^{-4\Gamma(t)}\rho_{10} | 1 | + \rho_{11} | 1 | \langle 1 |
\]
which can be recast as
\[
e^{-2\Gamma(t)(1-s_z\sigma_z)}\rho_S(0) = e^{-4\Gamma(t)}\rho(0) + (1 - e^{-4\Gamma(t)}) (\rho_{00} \langle 0 | + \rho_{11} | 1 | \langle 1 |)
\]
We recognize the form of a phase damping channel with \( p(t) = 1 - e^{-\Gamma(t)} \). Finally, including the effect of the unitary operators \( e^{\pm i\Omega(t)\sigma_z/2} \), we see that, at time \( t \), the reduced density of the system can be written as a phase damping (PD) channel, with 
\[
\tilde{E}_i = e^{-i\Omega(t)\sigma_z/2} E_i, \quad E_0 = \sqrt{1-p}, \quad E_1 = \sqrt{p} \langle 0 |, \quad E_2 = \sqrt{p} | 1 | \langle 1 |
\]
The behavior of \( p(t) \) depends on the number of oscillators of the environment, their frequencies, coupling constants and the temperature of the bath. If the “environment” contains a single oscillator, \( p \) varies periodically, with the unperturbed frequency of this oscillator, between 0 and \( p_M \): the higher the temperature (or the stronger the coupling or the smaller the frequency) the greater the value of \( p_M \). In this case, \( p(t) \) does not have a limit for long times. Moreover the master equation, which can be written as
\[
\frac{d\rho}{dt} = \frac{1}{i\hbar} [\hbar\omega(t)\sigma_z, \rho] + \gamma(t) (2\sigma_z\rho\sigma_z - \sigma_z^2\rho - \rho\sigma_z^2)
\]
is non-Markovian. In particular, \( \gamma(t) \) is negative for an infinite number of time intervals. When the limit to the continuum is taken (\( \sum_k \rightarrow \int d\omega/(2\pi) \)), in the decoherence function \( \Gamma(t) \) at finite temperature a zero-temperature contribution can be isolated,
\[
\Gamma_0(t) = \int_0^\infty \frac{D_f(\omega)}{2\pi\omega} (1 - \cos(\omega t)) \, d\omega.
\]
The behavior of the remaining contribution \( \Gamma_\beta(t) = \Gamma(t) - \Gamma_0(t) \), often differs significantly at large times from that of \( \Gamma_0(t) \), as shown in Figure 2. The examples given before illustrate how \( p(t) \) can oscillate, or grow to one (or to a constant smaller than one) exponentially or not.

**Figure 2.** Dephasing channel. Function \( p(t) \) for a single oscillator with frequency \( 1/(2T) \) and \( |c|^2 \coth(\hbar\omega/2)/\omega^2 = 2 \) (solid line), and for a continuum of oscillators with \( D(\omega) = 2\pi\omega e^{-\omega T} \) for zero temperature (dashed line) and \( \beta = 1/T \) (dotted line). Time is measured in units of \( T \).
5. Simple classical microscopic model of a dephasing channel

A dephasing channel also occurs when the magnitude of a classical magnetic field changes randomly. The Hamiltonian which describes this situation, \( H = \hbar \omega(t) \sigma_z / 2 + \hbar g \xi(t) \sigma_z \), where \( \xi(t) \) is a random variable. The equation of motion for the density operator is easily integrated,

\[
\rho(t) = e^{-i\Omega(t) [\sigma_z, \cdot]} e^{-ig \int_0^t \delta(t) \sigma_z \cdot} \rho(0) = e^{-i\Omega(t) [\sigma_z, \cdot]} \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} \int_0^t dt_1 \cdots \int_0^t dt_n \xi(t_1) \cdots \xi(t_n) [\sigma_z, \cdot]^n \rho(0),
\]

where \( \Omega(t) = \int_0^t \omega(\tau) d\tau \). A gaussian stationary process \( \xi(t) \) with zero average satisfies

\[
\overline{\xi(t_1) \xi(t_2)} = \Phi(t_1 - t_2),
\]

\[
\overline{\xi(t_1) \cdots \xi(t_{2n-1})} = 0,
\]

\[
\overline{\xi(t_1) \cdots \xi(t_{2n})} = \sum_{\text{Perm}} \Phi(t_{i_1} - t_{i_2}) \cdots \Phi(t_{i(2n-1)} - t_{i(2n)}),
\]

where \( n \) is a positive integer and the overline indicates the expected value. The averaged density operator

\[
\overline{\rho}(t) = e^{-i\Omega(t) [\sigma_z, \cdot]} \sum_{n=0}^{\infty} \frac{(-ig)^{2n}}{(2n)!} \int_0^t dt_1 \cdots \int_0^t dt_n \times \sum_{\text{Perm}} \Phi(t_{i_1} - t_{i_2}) \cdots \Phi(t_{i(2n-1)} - t_{i(2n)}) [\sigma_z, \cdot]^{2n} \rho(0),
\]

can be simplified taking into account that there are \( (2n)!/(2^n n!) \) permutations of the \( 2n \) times \( t_i \)

\[
\overline{\rho}(t) = e^{-i\Omega(t) [\sigma_z, \cdot]} \sum_{n=0}^{\infty} \frac{(-ig)^{2n}}{(2n)!} \frac{(2n)!}{2^n n!} \left( \int_0^t dt_1 \int_0^t dt_2 \Phi(t_1 - t_2) \right)^n [\sigma_z, \cdot]^{2n} \rho(0).
\]

The dynamics of the two-level system

\[
\overline{\rho}(t) = e^{-i\Omega(t) [\sigma_z, \cdot]} e^{-\frac{i}{2} \int_0^t dt_1 \int_0^t dt_2 \Phi(t_1 - t_2) [\sigma_z, [\sigma_z, \cdot] \rho(0),
\]

(45)
corresponds to a phase damping channel with \( p(t) = 1 - e^{-\Gamma(t)} \), where \( \Gamma(t) = \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 \Phi(t_1 - t_2) \), which generally is non-Markovian. Only if the two-point correlation function \( \Phi(t_1 - t_2) = \sigma^2 \delta(t_1 - t_2) \) the dynamics is Markovian.

6. Simple quantum microscopic model of an amplitude damping channel

Finally, we consider a qubit interacting with a collection of harmonic oscillators, which models the degrees of freedom of its environment, described by the Hamiltonian

\[
H = \frac{\hbar}{2} |1\rangle \langle 1| + \sum_k \hbar \omega_k a_k^\dagger a_k + \sum_k \left( c_k a_k^\dagger \sigma_+ + c_k^* a_k \sigma_- \right) \quad [37, 12, 38].
\]

Here, \( a_k \) is the annihilation operator of the \( k \)-th mode of the environment of free frequency \( \omega_k \), and \( c_k \)
are interaction constants. If the initial state of qubit and environment is \( |\psi\rangle_S \otimes \prod_k |0\rangle_k \), at time \( t \) it can be written as
\[
|\Psi(t)\rangle = \alpha(t) |1\rangle \otimes \prod_k |0\rangle_k + \beta(t) |0\rangle \otimes \prod_k |0\rangle_k + \sum_l \gamma_l(t) |0\rangle \otimes |1\rangle_l \otimes \prod_{k \neq l} |0\rangle_k . \tag{46}
\]
The use of Schrödinger equation leads to a set of coupled first-order linear equations for the coefficients \( \alpha(t), \beta(t) \) and \( \gamma_l(t) \). Solving the equations for \( \gamma_l(t) \) in terms of \( \alpha(t) \), and replacing into the equation for \( \alpha \) we find the integro-differential equation
\[
\frac{d\alpha(t)}{dt} + i\omega \alpha(t) + \int_0^t d\tau \sum_l |c_l|^2 e^{-i\omega(t-\tau)} \alpha(\tau) = 0 . \tag{47}
\]
The solution of this equation, which is also obtained for the independent oscillator model \[39\], can be written as \( \alpha(t) = \alpha(0) \exp(-\Lambda(t) - i\Omega(t)) \), where \( \Lambda(t) \geq 0 \) and \( \Lambda(t) \) are real functions. The reduced density of the qubit is obtained tracing out the state of the oscillators
\[
\rho_S(t) = \text{tr}_E \ |\Psi(t)\rangle \langle \Psi(t)| = \mu(t) |1\rangle \langle 1| + (1 - \mu(t)) |0\rangle \langle 0| + (\alpha(t)\beta^{\ast}(0) |1\rangle \langle 0| + h.c.) , \tag{48}
\]
where \( \mu(t) = |\alpha(t)|^2 = |\alpha(0)|^2 \exp(-2\Lambda(t)) \). A unitary contribution to the dynamics of the qubit can be isolated,
\[
\rho_S(t) = e^{-i\Omega(1)}(\mu |1\rangle \langle 1| + (1 - \mu) |0\rangle \langle 0| + (\alpha(t)\beta^{\ast}(0) |1\rangle \langle 0| + h.c.)) e^{i\Omega(1)} . \tag{49}
\]
Now, taking into account that
\[
\left( |0\rangle \langle 0| + \sqrt{1-p} |1\rangle \langle 1| \right) \left( \alpha |1\rangle + \beta |0\rangle \right) \left( \alpha^{\ast} |1\rangle + \beta^{\ast} |0\rangle \right) \left( |0\rangle \langle 0| + \sqrt{1-p} |1\rangle \langle 1| \right)
= (1 - p) |\alpha|^2 |1\rangle \langle 1| + |\beta|^2 |0\rangle \langle 0| + \left( \sqrt{1-p} \alpha^{\ast} |1\rangle \langle 0| + h.c. \right) , \quad \text{and} \tag{50}
\]
\[
\sqrt{p} |0\rangle \langle 1| (\alpha |1\rangle + \beta |0\rangle) (\alpha^{\ast} |1\rangle + \beta^{\ast} |0\rangle) \sqrt{p} |1\rangle \langle 0| = p |\alpha|^2 |0\rangle \langle 0| , \tag{51}
\]
where \( |\alpha|^2 + |\beta|^2 = 1 \), we conclude that \( p(t) = 1 - e^{-2\Lambda(t)} \). Moreover, we see that the qubit dynamics corresponds to an amplitude amplitude damping (AD) channel with Kraus operators \( E_i = e^{-i\Omega(1)} |1\rangle \langle 1| E_i, \) and \( \tilde{E}_0 = |0\rangle \langle 0| + \sqrt{1-p} |1\rangle \langle 1| \) and \( \tilde{E}_1 = \sqrt{p} |0\rangle \langle 1| \).

This amplitude damping channel is, as the noisy channels considered in the previous sections, generally non-Markovian and includes cases in which the environment consists of a finite oscillators (even one).

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