A GLOBAL EXISTENCE OF CLASSICAL SOLUTIONS TO THE TWO-DIMENSIONAL VLASOV-FOKKER-PLANCK AND MAGNETOHYDRODYNAMICS EQUATIONS WITH LARGE INITIAL DATA

BINGKANG HUANG
School of Mathematics, Hefei University of Technology
Hefei 230009, China

LAN ZHANG *
School of Mathematics and Statistics, Wuhan University
Wuhan 430072, China

(Communicated by Laurent Desvillettes)

ABSTRACT. We present a two-dimensional coupled system for particles and compressible conducting fluid in an electromagnetic field interactions, which the kinetic Vlasov-Fokker-Planck model for particle part and the isentropic compressible MHD equations for the fluid part, respectively, and these separate systems are coupled with the drag force. For this specific coupled system, a sufficient framework for the global existence of classical solutions with large initial data which may contain vacuum is established.

The purpose of this paper is to provide a global existence theory of classical solutions to the coupled compressible magnetodrodynamic-Vlasov-Fokker-Planck system describing the interaction of the compressible conducting fluid in an electromagnetic field and particles. This kind of coupled system arises out of many practical industrial applications such as the biosprays in medicine [2], sedimentation of solid grain in physics [4, 6], fuel-droplets in combustion theory [36] etc.

In this paper, we specifically consider a coupled kinetic-fluid model of kinetic Vlasov-Fokker-Planck type equation and the compressible isentropic MHD equations. More precisely, let \( f = f(t, x, v) \) be particle distribution function with velocity \( v = (v_1, v_2) \in \mathbb{R}^2 \) at position \( x = (x_1, x_2) \in \mathbb{T}^2 \) at time \( t > 0 \), and let \( \rho(t, x) \) be the density, \( u = u(t, x) = (u_1, u_2)(t, x) \) be the bulk velocity of the compressible fluid and \( H = H(t, x) = (H_1, H_2)(t, x) \) be magnetic field. Then the coupled dynamics of \( [f, \rho, u, H] \) is given by the following kinetic-fluid system:

\[ \frac{\partial}{\partial t} f + \nabla_x \cdot (f v) + \nabla_x \cdot (f v u) + \nabla_x \cdot (f v H) = 0 \]

\[ \frac{\partial}{\partial t} \rho + \nabla_x \cdot (\rho u) = 0 \]

\[ \frac{\partial}{\partial t} u + \nabla_x \cdot (u \otimes u) + \nabla_x \cdot (u H) = -\nabla_x p + \nabla_x \left( \mu \nabla_x u \right) + \nabla_x \left( \frac{\sigma \nabla_x H}{\rho} \right) \]

\[ \frac{\partial}{\partial t} H + \nabla_x \cdot (H \times u) = \nabla_x \left( \frac{\sigma}{\rho} \nabla_x H \right) \]

2010 Mathematics Subject Classification. 35A09, 35Q30, 35Q35, 35Q83, 76N10.

Key words and phrases. Compressible MHD equations, Vlasov-Fokker-Planck equation, global classical solutions, large initial data, vacuum.

This work was supported by a Grant from National Natural Science Foundation of China under Contract 11671309 and “The Fundamental Research Funds for the Central Universities”.

* Corresponding author: Lan Zhang.
\[
\begin{aligned}
  & f_t + v \cdot \nabla_x f = \nabla_v \cdot [(v - u)f + \nabla_v f], \\
  & \rho_t + \nabla_x \cdot (\rho u) = 0, \\
  & (\rho u)_t + \nabla_x \cdot (\rho u) + \nabla_x P(\rho) \\
  & = H \cdot \nabla_x H - \frac{1}{2} \nabla_x |H|^2 + \Gamma_{\rho}u + \int_{\mathbb{R}^2} (v - u)f dv, \\
  & H_t + u \cdot \nabla_x H - H \cdot \nabla_x u + H \nabla_x \cdot u - \nu \Delta_x H = 0, \\
  & \nabla_x \cdot H = 0,
\end{aligned}
\]

subject to initial conditions:

\[
(f(0, x, v), \rho(0, x), u(0, x), H(0, x)) = (f_0(x, v), \rho_0(x), u_0(x), H_0(x)),
\]

\[(x, v) \in \mathbb{T}^2 \times \mathbb{R}^2, \]

where \(P(\rho)\) is the pressure given by

\[P(\rho) = \rho^\gamma, \quad \gamma > 1.\]

For notational simplicity, we denote that the operator \(\Gamma_{\rho}u\) is defined by

\[\Gamma_{\rho}u := \mu \Delta_x u + \nabla_x ((\mu + \lambda(\rho)) \text{div} u),\]

where the shear, bulk viscosity and magnetic diffusive coefficients \(\mu, \lambda\) and \(\nu\) are assumed to satisfy

\[
\begin{align*}
  \mu &= \text{const.} > 0, \\
  \lambda(\rho) &= \rho^\beta, \\
  \beta &= \frac{4}{3}, \\
  \nu &= \text{const.} > 0
\end{align*}
\]

such that \(\Gamma_{\rho}\) is strictly elliptic.

Moreover, \(\rho_f\) and \(u_f\) denote the local mass density, and average local velocity of particle ensemble, respectively:

\[
\rho_f := \int_{\mathbb{R}^2} f dv \quad \text{and} \quad u_f := \begin{cases} 
  \frac{\int_{\mathbb{R}^2} v f dv}{\int_{\mathbb{R}^2} f dv} & \text{if } \rho_f \neq 0, \\
  0 & \text{if } \rho_f = 0.
\end{cases}
\]

The modeling of collective dynamics via a coupled kinetic-fluid system is one of the hottest topics in the field of nonlinear partial differential equations in recent years. When the number of flocking particles is sufficiently large, it is almost impossible to track the motion of each particle. Therefore, we use the corresponding VFP type kinetic equation to describe the motion of particles [35]. On the other hand, the fluid dynamics in an electromagnetic field can be described hydrodynamic models such as compressible MHD equation, please refer to [21, 22] for more details.

Next, we briefly review some earlier results on coupled system. There are many literatures on the coupled system between particles and the compressible flow. In [32, 33], a global weak solution is constructed and the asymptotic analysis has been studied for the coupled system of the VFP equation with the compressible Navier-Stokes (NS) equations in a bounded domain \(\Omega \in \mathbb{R}^3\). In [13], the authors proved the global existence of classical solutions to the coupled system of VFP equation and compressible Euler equations for small initial data in the whole space \(\mathbb{R}^3\). In [27], the global well-posedness of a strong solution to compressible Navier-Stokes-Vlasov-Fokker-Planck system in the three-dimensional whole space is established when the initial data is a small perturbation of some given equilibrium. The global classical solutions to 1D and 2D coupled system of flocking particles and compressible fluids with large initial data have been obtained in [18, 19]. Furthermore, we can refer to [1, 3, 5, 6, 20, 17] for more information on the coupled system.
Especially, when the influence of distribution function $f(t, x, v)$ is neglected, the system (1) becomes isentropic compressible MHD equations. This kind of Vaigant-Kazhikhkov model for compressible NS equations is considered firstly in [37]. Later on, the global existence and uniqueness of classical solutions to Vaigant-Kazhikhkov model of compressible NS equations and compressible MHD equations are established on torus $\mathbb{T}^2$ and the whole space $\mathbb{R}^2$ in [23, 24, 25, 26, 30]. Specifically, Caffarelli-Kohn-Nirenberg inequality is an essential tool for dealing with this system in the whole space $\mathbb{R}^2$.

Up to now, a natural question is raising, can we get the global solvability of coupled system (1) with large initial data? This paper gives a positive answer, even to the initial density $\rho(t, x)$ contains vacuum. We briefly state the main ideas and techniques in the process of proof. Firstly, the friction term in (1) has highly nonlinearity. It requires us deal with the fluid velocity $u(t, x)$ and kinetic distribution function $f(t, x, v)$ in the mean time. The estimate $I_1$ in Lemma 2.2 gives the basic skill, and the high order estimates of friction term have analogous results. Moreover, $L^p$ estimates of the momentum for kinetic part $f(t, x, v)$ in Corollary 9 have been used in several times. Secondly, according to the blow up criterion of the compressible flow in [24], in order to get the existence and continuation in time of flows, the $L^\infty_x$ norm in space variable of density $\rho(t, x)$ need to be bounded. Based on the above fact, we should control the upper bound of density $\rho(t, x)$. The upper bound of density $\rho(t, x)$ is obtained in Lemma 2.8, with the help of introducing the nonlinear functionals $Z(t), \chi(t)$ in Lemma 2.6 and the Riesz transform. Thirdly, we need to deal with the term of $(\rho f)_t$, $(\rho f)_{tt}$ and $(u f \rho f)_t$ in the higher order apriori estimates. Using the equation (1), we convert these terms into taking space derivative on momentum $f(t, x, v)$ in (63).

The rest of paper is organized as follows. In Section 1, we briefly discuss a framework and present our main results. In Section 2, we provide several lemmas to be used later. In Section 3, we derive apriori higher order estimates of the coupled system. In Section 4, a proof of main result is provided.

Notation. Throughout the paper, the definitions of operators are given as following: $\text{div} = \partial_{x_1} + \partial_{x_2}, \nabla_x = (\partial_{x_1}, \partial_{x_2}); \nabla_v = (\partial_{v_1}, \partial_{v_2}); \Delta_x = \partial_{x_1}^2 + \partial_{x_2}^2$ and $\Delta_v = \partial_{v_1}^2 + \partial_{v_2}^2$. Moreover, $C$ denotes a generic positive constant which may change line by line. The small constants to be chosen are denoted by $\varepsilon, \alpha, \delta$. For function spaces, $W^{k, p}(\mathbb{R}^2)$ and $W^{k, p}(\mathbb{R}^2 \times \mathbb{T}^2)$ denote the standard Sobolev spaces with standard norm $\|\cdot\|_{W^{k, p}}$, and $H^k := W^{k, 2}$. $\|\cdot\|_p := (\int_{\mathbb{T}^2} |\cdot|^p dx)^{\frac{1}{p}}$ or $(\int_{\mathbb{T}^2 \times \mathbb{R}^2} |\cdot|^p dx dv)^{\frac{1}{p}}$ with $1 \leq p \leq +\infty$. For notational simplicity, we denote

$$\partial_{\alpha}^\ast f := \partial_{v_1}^{\alpha_1} \partial_{v_2}^{\alpha_2} f, \quad \alpha_\ast = [\alpha_1, \alpha_2], \quad \beta_\ast = [\beta_1, \beta_2],$$

$$\|f\|_{W^{k, p}} := \sum_{|\alpha_\ast| + |\beta_\ast| \leq N} \|v^k \partial_{\beta_\ast} f\|_{L^p(\mathbb{R}^4)}, \quad k \geq 0.$$

Homogeneous Sobolev space $D^{\ell, p}(\mathbb{T}^2)$ is defined by

$$D^{\ell, p}(\mathbb{T}^2) = \{ u \in L^1_{loc}(\mathbb{T}^2) \|\nabla_\ast^\ell u\|_p < +\infty \} \text{ with } \|u\|_{D^{\ell, p}} := \|\nabla_\ast^\ell u\|_p.$$ For the special case $p = 2$, we denote $D^\ell$ as $D^{\ell, 2}$.

1. Main result. Next, we present our main results whose proof will be given in Section 3.

Theorem 1.1. Suppose that the following conditions hold.
1. The parameters $\beta$ and $p$ satisfy
   \[ \beta > \frac{4}{3}, \quad p > 4. \]

2. Initial data $[f_0, \rho_0, u_0, H_0]$ satisfy regularity and integrability:
   \[ f_0 \in L^1(\mathbb{T}^2; L^1(\mathbb{R}^2)), \quad f_0 \in W^3, p \quad f_0(x, v) > 0, \quad \text{for sufficiently large } k, \]
   \[ (\rho_0, P(\rho_0)) \in W^{2, p}(\mathbb{T}^2) \times W^{2, p}(\mathbb{T}^2), \quad (\rho_0, \rho_0^\gamma) \in L^1(\mathbb{T}^2), \quad \rho_0 \geq 0, \]
   \[ P(\rho_0) \in L^1(\mathbb{T}^2), \quad \sqrt{\rho_0}u_0 \in L^2(\mathbb{T}^2), \quad u_0 \in D^1(\mathbb{T}^2) \cap D^2(\mathbb{T}^2), \]
   \[ H_0 \in H^1(\mathbb{T}^2) \cap H^2(\mathbb{T}^2). \]

3. Initial data $[f_0, \rho_0, u_0, H_0]$ satisfy the compatibility condition:
   \[ \Gamma_{\rho_0} u_0 - \nabla_x P(\rho_0) + (\nabla_x \times H_0) \times H_0 - (u_0 - u_{f_0}) \rho_{f_0} = \sqrt{\rho_0} g(x), \]
   \[ \text{where } g \in L^2(\mathbb{T}^2). \]

Then, the Cauchy problem (1)-(2) admits a unique global solution $[f, \rho, u, H]$ satisfying the following regularity and integrability: for any $T \in (0, \infty)$,
   \[ 0 \leq \rho(t, x) \leq C(T), \quad \forall (t, x) \in [0, T] \times \mathbb{T}^2, \quad (\rho, P(\rho)) \in C([0, T]; W^{2, p}(\mathbb{T}^2)), \]
   \[ \rho_0 \in C([0, T]; L^p(\mathbb{T}^2)), \quad (u, H) \in C([0, T]; H^2(\mathbb{T}^2)) \cap L^2(0, T; H^3(\mathbb{T}^2)), \]
   \[ \sqrt{\rho}(u, H) \in L^\infty(0, T; H^3(\mathbb{T}^2)), \quad t(u, H) \in L^\infty(0, T; W^3, q(\mathbb{T}^2)), \]
   \[ (u_t, H_t) \in L^2(0, T; H^1(\mathbb{T}^2)), \quad \sqrt{\rho}(u_t, H_t) \in L^2(0, T; H^2(\mathbb{T}^2)), \]
   \[ t(u_t, H_t) \in L^\infty(0, T; H^2(\mathbb{T}^2)), \quad \sqrt{\rho}(u_{tt}, H_{tt}) \in L^2(0, T; L^2(\mathbb{T}^2)), \]
   \[ t(\sqrt{\rho} u_{tt}, H_{tt}) \in L^\infty(0, T; L^2(\mathbb{T}^2)), \quad t(\nabla_x u_{tt}, \nabla_x H_{tt}) \in L^2(0, T; L^2(\mathbb{T}^2)), \]
   \[ f \in L^\infty(0, T; W^{3, p}(\mathbb{T}^2 \times \mathbb{R}^2)). \]

**Remark 1.**
1. For simplicity, we only require that $k$ is suitably large. Since the weight is used many times throughout the paper, it is really not easy to give an exact expression of $k$. So we do not track the optimal $k$.
2. Here, we only consider the friction term dependent on relative velocity. Furthermore, the condition of fluid velocity density $\rho(t, x)$ is contained is worthy of consideration.
3. By introducing the Caffarelli-Kohn-Nirenberg inequality in [26], the coupled system has similar global solvability in the whole space $\mathbb{R}^2$.

2. **A priori lower-order estimates.** In this section, we present lower-order energy estimates for the coupled system (1), and derive several momentum estimates for $f(t, x, v)$.

2.1. **Propagation of velocity moments.** First, we state elementary energy estimates for the coupled system without proofs.

**Lemma 2.1.** [19, 23, 25, 30] Suppose that initial data $[f_0, \rho_0, u_0, H_0]$ satisfy the conditions (5). For a positive constant $T \in (0, \infty]$, let $[f, \rho, u, H]$ be a smooth solution to system (1)-(2) in $[0, T]$. Then, we have
   \[ \left( \int_{\mathbb{T}^2} (\rho u^2 + \rho \gamma + \rho + H^2) dx + \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} |v|^2 dx dv \right) (t) \]
+ \int_0^t \left( \|\nabla_x u\|^2 + \|\omega\|^2 + (2\mu + \lambda(\rho))^{1/2} \text{div} u \right)^2 + \|\nabla_x H\|^2_2 \\
+ \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (v - u)^2 \, f \, dv \, dx \, dt \leq C(T), \quad \forall t \in (0, T).

(ii) \|f\|_{L^p(0,T;L^p(\mathbb{T}^2 \times \mathbb{R}^2))} + \|\nabla_x (f^\frac{p}{2})\|_{L^2(0,T;L^2(\mathbb{T}^2 \times \mathbb{R}^2))} \leq C(T), \quad 1 \leq p < \infty.

(iii) \|f\|_{L^\infty(0,T;L^\infty(\mathbb{T}^2 \times \mathbb{R}^2))} \leq C(T).

(iv) \|H\|_{L^p(0,T;L^p(\mathbb{T}^2))} \leq C(T), \quad \forall p \geq 2.

Next, we show some momentum (velocity) estimates for the kinetic part \(f(t,x,v)\). For this, we set

\[ m_{k_2} f(x,t) := \int_{\mathbb{R}^2} |v|^k f(x,v,t) \, dv. \]

**Lemma 2.2.** Under the same setting in Lemma 2.1, we have

(i) \( m_{k_2} f(x,t) \leq C(1 + \|f\|_{L^\infty_x}) (m_{k_2} f(x,t))^{\frac{k_1 + 2}{k_2 + 2}}, \quad \forall k_2 > k_1 \geq 0, \)

(ii) \( \sup_{0 \leq t \leq T} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (1 + |v|^k) f \, dv \, dx \)

\[ \leq C(T) \left( \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (1 + |v|^k) f_0 \, dv \, dx + \left( \int_0^T \|u\|_{k+2} \, dt \right)^{k+2} \right), \]

where \( \bar{k} \geq 2 \) is a positive constant.

**Proof.** (i) Note that for \( R > 0 \),

\[ \int_{\mathbb{R}^2} |v|^k \, f \, dv = \int_{|v| \leq R} |v|^k \, f \, dv + \int_{|v| > R} |v|^k \, f \, dv \]

\[ \leq C \left( \|f\|_{L^\infty_x} + R^{k_1 + 2} + \frac{1}{R^{k_2 - k_1}} \int_{\mathbb{T}^2} |v|^{k_2} \, f \, dv \right). \]

We now choose \( R = (\int_{\mathbb{R}^2} |v|^{k_2} \, f \, dv)^{\frac{1}{k_2 + 2}} \) in the above relation to obtain

\[ \int_{\mathbb{R}^2} |v|^k \, f \, dv \leq C(\|f\|_{L^\infty_x} + 1) \left( \int_{\mathbb{R}^2} |v|^{k_2} \, f \, dv \right)^{\frac{k_1 + 2}{k_2 + 2}}. \]

(ii) We multiply (1) by \( (1 + |v|^\bar{k}) \) to have

\[ \frac{d}{dt} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (1 + |v|^\bar{k}) f \, dv \, dx \]

\[ = \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} |v|^\bar{k} \nabla_x \cdot [(v - u) f + \nabla_x f] \, dv \, dx \]

\[ = \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \bar{k} |v|^\bar{k} - 2 v \cdot (u - v) f \, dv \, dx - \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \bar{k} |v|^\bar{k} - 2 v \cdot \nabla_x f \, dv \, dx \]

\[ := \mathcal{I}_{11} + \mathcal{I}_{12}. \]

Below, we estimate the terms \( \mathcal{I}_{1i} (1 \leq i \leq 2) \), separately.
(Estimate of $I_{11}$): Again we apply the Hölder inequality and the result (i) to obtain

$$I_{11} \leq - \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \tilde{k} |v|^k f \, dv \, dx$$

$$+ C \left( \int_{\mathbb{T}^2} \left( \int_{\mathbb{R}^2} |v|^\tilde{k-1} f \, dv \right)^{\frac{k+2}{k+1}} \, dx \right)^{\frac{k+1}{k+2}} \left( \int_{\mathbb{T}^2} |u|^\tilde{k+2} \, dx \right)^{\frac{1}{k+2}}$$

$$\leq C \left( \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (1 + |v|^k) f \, dv \, dx \right)^{\frac{k+1}{k+2}} \left( \int_{\mathbb{T}^2} |u|^\tilde{k+2} \, dx \right)^{\frac{1}{k+2}}.$$  

(Estimate of $I_{12}$): We use integration by parts to get

$$I_{12} = \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \tilde{k}^2 |v|^\tilde{k-2} f \, dv \, dx \leq C \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (1 + |v|^k) f \, dv \, dx.$$  

In (9), we collect all estimates to find

$$\frac{d}{dt} \left( \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (1 + |v|^k) f \, dv \, dx \right)^{\frac{1}{k+2}} \leq C \left( \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (1 + |v|^k) f \, dv \, dx \right)^{\frac{1}{k+2}} + C \|u\|_{\tilde{k+2}}.$$  

Finally, we integrate the above inequality over $[0,t]$ and use the Gronwall lemma to derive the desired estimate.

We apply the operator div to the momentum equation (1.3) to have

$$[\text{div}(\rho u)]_t + \text{div}[\text{div}(\rho \otimes u \cdot H \otimes H)] + (u - u_f) \rho f = \Delta_x F,$$  

where the effective viscous flux $F$ is defined by

$$F := (2\mu + \lambda(\rho)) \text{div} u - P(\rho) - \frac{|H|^2}{2}.$$  

On the other hand, consider the following three elliptic problems on the torus $\mathbb{T}^2$:

$$- \Delta_x \psi = \text{div}(\rho u), \quad \int_{\mathbb{T}^2} \psi(t,x) \, dx = 0,$$

$$- \Delta_x \eta_1 = \text{div}(\rho \otimes u \cdot H \otimes H)), \quad \int_{\mathbb{T}^2} \eta_1(t,x) \, dx = 0,$$

$$- \Delta_x \eta_2 = \text{div}((u - u_f) \rho f), \quad \int_{\mathbb{T}^2} \eta_2(t,x) \, dx = 0.$$  

For equations (11), we can derive the following elliptic estimates in the following lemma. It can be easily established through a similar way as in [25]. So, we omit it here.

**Lemma 2.3.** [25, 30, 37] Let $[\psi, \eta_1, \eta_2]$ be solutions to the elliptic problems in (11). Then, we have

(i) $\|\nabla_x \psi\|_{2m} \leq C m \|\rho\|_{\frac{2m+1}{2}} \|u\|_{2mk}, \quad \forall k > 1, m \geq 1$;

(ii) $\|\nabla_x \psi\|_{2-r} \leq C \|\sqrt{\rho} u\|_{2} \|\rho\|_{\frac{2}{2-r}}, \quad 0 < r < 1$;

(iii) $\|\eta_1\|_{2m} \leq C m (\|\rho\|_{\frac{2m+1}{2m}} \|u\|_{3mk} + \|H\|_{4m}), \quad \forall k > 1, m \geq 1$;

(iv) $\|\nabla_x \eta_2\|_{2m} \leq C m (\|u - u_f\| \rho f\|_{2m}, \quad \forall m \geq 1$.

With Lemma 2.3, we can further derive estimates for $u, \psi, \eta_1, \eta_2$ as follows.
Lemma 2.4. The following estimates hold.

(i) \( \| \psi \|_{2m} \leq C m^{2} \| \nabla_{x} \psi \|_{\frac{2m}{m+1}} \leq C(T) m^{\frac{1}{2}} \| \psi \|_{\frac{2m}{m+1}}, \forall m \geq 1, \)

(ii) \( \| u \|_{2m} \leq C m^{2} \| \nabla_{x} u \|_{2} + 1], \forall m \geq 1, \)

(iii) \( \| \nabla_{x} \psi \|_{2m} \leq C [m^{2} k \| \rho \|_{\frac{2m}{m+1}} + m \| \rho \|_{\frac{2m}{m+1}} ], \forall k > 1, \forall m \geq 1, \)

(iv) \( \| \eta \|_{2m} \leq C(T) [m^{2} k \| \rho \|_{\frac{2m}{m+1}} + m \| \rho \|_{\frac{2m}{m+1}} + m^{2} \phi(t) + m], \)

\( \forall k > 1, \forall m \geq 1, \)

(v) \( \| \eta \|_{2m} \leq C(T) m^{\frac{1}{2}} \left( \int_{T^2} \int_{\mathbb{R}^2} (u - v)^{2} f \, dv \, dx \right)^{\frac{1}{2}} \times \int_{0}^{T} (m^{\frac{1}{2}} \| \nabla_{x} u \|_{2} + 1) \, dt, \)

\( \forall t \in [0, T], \forall m \geq 1, \)

where \( \phi(t) := \int_{T^2} (\mu \omega^2 + (2 \mu + \lambda(\rho)) (\text{div} u)^2 + \nu |\nabla H|^2) \, dx. \)

Proof. Now we only prove the last estimate (v), others can be found in [30].

From (iv) in Lemma 3.3, we have

\[ \| \eta \|_{2m} \leq C m^{\frac{1}{2}} \| \nabla_{x} \eta \|_{\frac{2m}{m+1}} \leq C m^{\frac{1}{2}} \|(u - u_f) \rho_j \|_{\frac{2m}{m+1}} \]  

(12)

On the other hand, we use (ii) in Lemma 2.1, Lemma 2.2 and (ii) in this lemma to have

\[ \|(u - u_f) \rho_j \|_{\frac{2m}{m+1}} \leq C \left( \int_{T^2} \int_{\mathbb{R}^2} (u - v)^{2} f \, dv \, dx \right)^{\frac{1}{2}} \left( \int_{T^2} \int_{\mathbb{R}^2} m \right)^{\frac{1}{2m}} \]

\[ \leq C \left( \int_{T^2} \int_{\mathbb{R}^2} (u - v)^{2} f \, dv \, dx \right)^{\frac{1}{2}} \left( \int_{T^2} \int_{\mathbb{R}^2} |v|^{2m-2} f \, dv \, dx \right)^{\frac{1}{2m}} \]

\[ \leq C(T) \left( \int_{T^2} \int_{\mathbb{R}^2} (u - v)^{2} f \, dv \, dx \right)^{\frac{1}{2}} \int_{0}^{T} \| u \|_{2m} \, dt \]

\[ \leq C(T) \left( \int_{T^2} \int_{\mathbb{R}^2} (u - v)^{2} f \, dv \, dx \right)^{\frac{1}{2}} \int_{0}^{T} (m^{\frac{1}{2}} \| \nabla u \|_{2} + 1) \, dt. \]

(13)

We combine (12) and (13) to derive a desired estimate (v). \qed

It follows from (11) and (10) that

\[ \Delta_{x} \left( \psi_t + \eta_1 + \eta_2 + F - \int_{T^2} F(t, x) \, dx \right) = 0, \]

\[ \int_{T^2} \left( \psi_t + \eta_1 + \eta_2 + F - \int_{T^2} F(t, x) \, dx \right) \, dx = 0, \]

which yields

\[ \psi_t + \eta_1 + \eta_2 + F - \int_{T^2} F(t, x) \, dx = 0. \]

We define

\[ \Lambda(\rho) := \int_{1}^{\rho} \frac{2\mu + \lambda(s)}{s} \, ds = 2\mu \log \rho + \frac{1}{\beta}(\rho^\beta - 1). \]
It follows from the definition of the effective viscous flux $F$ and (1)\textsubscript{2} that

$$(\Lambda(\rho)-\psi)_t+u\cdot\nabla_x(\Lambda(\rho)-\psi)+P(\rho)+\frac{|H|^2}{2}-\eta_1-\eta_2+u\cdot\nabla_x\psi+\int_{\mathbb{T}^2} F(t,x)dx = 0.$$ (14)

Next, we derive the $L^\infty_t L^p_x$ estimate of the density $\rho(t,x)$ by using (14).

**Lemma 2.5.** Let $\beta > \frac{4}{3}$, and assume that the same conditions in Lemma 2.1 hold. Then, we have

$$\sup_{0\leq t \leq T} \|\rho\|_p(t) \leq C(T,p,\beta), \quad p \geq 1.$$ (15)

**Proof.** We set $(h)_+$ to be the positive part of a function $h$, multiply (14) by $\rho(\Lambda(\rho)-\psi)_+^{2m-1}$ with $m \gg 1$ and integrate the resulting relation over $\mathbb{T}^2$ to obtain

$$\frac{1}{2m} \frac{d}{dt} \int_{\mathbb{T}^2} \rho(\Lambda(\rho)-\psi)_+^{2m}dx + \int_{\mathbb{T}^2} \rho P(\rho)(\Lambda(\rho)-\psi)_+^{2m-1}dx

= \int_{\mathbb{T}^2} \rho \eta_1(\Lambda(\rho)-\psi)_+^{2m-1}dx

+ \int_{\mathbb{T}^2} \rho \eta_2(\Lambda(\rho)-\psi)_+^{2m-1}dx

- \int_{\mathbb{T}^2} \rho u \cdot \nabla \psi(\Lambda(\rho)-\psi)_+^{2m-1}dx

+ \int_{\mathbb{T}^2} F(t,x)dx \int_{\mathbb{T}^2} \rho(\Lambda(\rho)-\psi)_+^{2m-1}dx$$

:= \sum_{i=1}^4 \mathcal{I}_{2i}.

For the notational simplicity, we define

$$g(t) := \left( \int_{\mathbb{T}^2} \rho(\Lambda(\rho)-\psi)_+^{2m}dx \right)^{\frac{1}{2m}}$$

and estimate terms $\mathcal{I}_{2i}$ one by one as follows.

- **(Estimate of $\mathcal{I}_{21}$)** We use (iv) in Lemma 2.4 to have

$$|\mathcal{I}_{21}| \leq \int_{\mathbb{T}^2} \rho^{\frac{1}{2m}}|\eta_1||\rho(\Lambda(\rho)-\psi)_+^{2m}dx \leq \|\rho\|^{\frac{1}{2m}}_{2m+\beta+1} \|\eta_1\|_{2m+\beta+1} \|\rho(\Lambda(\rho)-\psi)_+^{2m}dx

\leq C\|\rho\|^{\frac{1}{2m}}_{2m+\beta+1} \left[ (m+\frac{1}{2\beta})^2 \|\rho\|_{2m+\frac{1}{\beta}} \phi(t)

+ (m+\frac{1}{2\beta}) \|\rho\|_{2m+\frac{1}{\beta}} \phi(t) \right]

\leq C\|\rho\|^{1+\frac{1}{2m}}_{2m+\beta+1} g^{2m-1}(t)(m^2 \phi(t) + m),$$

where we have chosen $k = \frac{\beta}{\beta-1}$ in the last inequality.
(Estimate of I_{22}): We use (v) in Lemma 2.4 to have
\[ |I_{22}| \leq \int_{\mathbb{T}^2} \rho^{\frac{m-1}{m}} |\eta_2| (|\rho(\Lambda(\rho) - \psi)_{2m+1}|^2) \frac{2m-1}{2m} \, dx \]
\[ \leq C \rho^{\frac{m}{2m+1}} \|\eta_2\|_{2m+1} \|\rho(\Lambda(\rho) - \psi)_{2m+1}^2 \|_{1} \frac{2m-1}{2m} \]
\[ \leq C \rho^{\frac{m}{2m+1}} \|\eta_2\|_{2m+1} \|\rho(\Lambda(\rho) - \psi)_{2m+1}^2 \|_{1} \frac{2m-1}{2m} \]
\[ \leq C \rho^{\frac{m}{2m+1}} \|\eta_2\|_{2m+1} g^{2m-1}(t) \]
\[ \times \left( \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (u - v)^2 f \, dx \, dv \right)^{\frac{1}{2}} \int_0^t (m^2 \|\nabla u\|_2 + 1) \, dt. \]

(Estimate of I_{23}): We choose
\[ p' = \frac{4m\beta + 2}{m\beta}, \quad q' = \frac{4m\beta + 2}{3m\beta}, \quad k = \frac{\beta}{\beta - \frac{4}{3}}, \]
and use Lemma 2.4 to have
\[ |I_{23}| \leq \int_{\mathbb{T}^2} \rho^{\frac{m}{2m+1}} |u|^{\|\nabla \psi\| (|\rho(\Lambda(\rho) - \psi)_{2m+1}|^2) \frac{2m-1}{2m} \, dx \]
\[ \leq \rho^{\frac{m}{2m+1}} \|\nabla \psi\|_{2m+1} \|u\|_{2m+1} |\rho(\Lambda(\rho) - \psi)_{2m+1}^2 \|_{1} \frac{2m-1}{2m} \]
\[ \leq C \rho^{\frac{m}{2m+1}} \|\nabla \psi\|_{2m+1} (m \phi(t) + 1) \]
\[ \times \rho^{\frac{m}{2m+1}} \|\nabla \psi\|_{2m+1} g^{2m-1}(t) \]
\[ \frac{1}{2} \rho^{\frac{m}{2m+1}} \|\nabla \psi\|_{2m+1} g^{2m-1}(t) \]
\[ \leq C \rho^{\frac{m}{2m+1}} \|\nabla \psi\|_{2m+1} g^{2m-1}(t) (m^2 \phi(t) + 1). \]

(Estimate of I_{24}): We use Lemma 2.1 to have
\[ |I_{24}| \leq \int_{\mathbb{T}^2} \left| (2\mu + \lambda(\rho)) \text{div} u - P(\rho) - \frac{1}{2} |H|^2 \right| \, dx \]
\[ \times \int_{\mathbb{T}^2} \rho^{\frac{m}{2m+1}} \|\rho(\Lambda(\rho) - \psi)_{2m+1}^2 \|_{1} \frac{2m-1}{2m} \, dx \]
\[ \leq \left[ \int_{\mathbb{T}^2} (2\mu + \lambda(\rho))^\frac{1}{2} \rho \, dx \right] \left( \int_{\mathbb{T}^2} (2\mu + \lambda(\rho)) \, dx \right)^{\frac{1}{2}} \]
\[ + \int_{\mathbb{T}^2} P(\rho) \, dx + \frac{1}{2} \int_{\mathbb{T}^2} |H|^2 \, dx \times \rho^{\frac{m}{2m+1}} \|\rho(\Lambda(\rho) - \psi)_{2m+1}^2 \|_{1} \frac{2m-1}{2m} \]
\[ \leq C \rho^{\frac{1}{2}} \phi^{\frac{1}{2}} \left( \int_{\mathbb{T}^2} \rho \, dx \right)^{\frac{1}{2}} g^{2m-1}(t) \]
\[ \leq C \rho^{\frac{1}{2}} \phi^{\frac{1}{2}} \|\rho\|_{2m+1} g^{2m-1}(t). \]

In (16), we collect all estimates to find that
\[ g(t) \leq C(T) \left[ 1 + \int_0^t \phi^{\frac{1}{2}} \|\rho\|_{2m+1} \, d\tau + \int_0^t (m^2 \phi + m) \rho \rho^{\frac{m}{2m+1}} \, d\tau \right] \]
\[ + m^2 \int_0^t \rho^{\frac{1}{2}} \|\rho\|_{2m+1} \rho^{\frac{1}{2}} \left( \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (u - v)^2 \, f \, dx \, dv \, d\tau \right). \]

We set
\[ \Omega_1(t) := \{ x \in \mathbb{T}^2 | \rho(x, t) > 2 \} \quad \text{and} \quad \Omega_2(t) := \{ x \in \Omega_1(t) | \Lambda(\rho)(x, t) - \psi(x, t) > 2 \}. \]
Then, we have
\[
\|\rho\|_{2m\beta+1}^\beta (t)
\leq C \left( \int_{\Omega_1(t)} \rho^{2m\beta+1} dx + \int_{\Omega_2(t)} \rho^{2m\beta+1} dx \right)^{\frac{\beta}{2m\beta+1}} + C
\leq C \left( \int_{\Omega_1(t)} \rho^{2m\beta+1} dx \right)^{\frac{\beta}{2m\beta+1}} + C
\leq C \left( \int_{\Omega_2(t)} \rho |\Lambda(\rho)|^{2m} dx + \int_{\Omega_2(t)} \rho |\psi|^{2m} dx \right)^{\frac{\beta}{2m\beta+1}} + C
\leq C \left( g^{2m}(t) + \int_{\Omega_2(t)} \rho |\psi|^{2m} dx \right)^{\frac{\beta}{2m\beta+1}} + C
\leq C \left[ g(t) + \left( \int_{\Omega_2(t)} \rho |\psi|^{2m} dx \right)^{\frac{\beta}{2m\beta+1}} + 1 \right].
\]

Note that
\[
\left( \int_{\Omega_2(t)} \rho |\psi|^{2m} dx \right)^{\frac{\beta}{2m\beta+1}} \leq \|\rho\|_{2m\beta+1}^\beta \|\psi\|_{2m\beta+1}^{\frac{2m\beta}{2m\beta+1}}
\leq C(T, m) \|\rho\|_{2m\beta+1}^\beta \left[ \left( m + \frac{1}{2\beta} \right)^{\frac{\beta}{2}} \|\rho\|_{\frac{m}{m+\frac{1}{2\beta}}}^{\frac{m}{m+\frac{1}{2\beta}}} \right]^{\frac{2m\beta}{2m\beta+1}}
\leq C(T, m) \left( \|\rho\|_{2m\beta+1}^\beta + 1 \right).
\]

On the other hand, by (i) in Lemma 2.4 and Young’s inequality, we have
\[
\|\rho\|_{2m\beta+1}^\beta (t) \leq C(T, m) \left[ g(t) + \|\rho\|_{2m\beta+1}^\beta (t) + 1 \right]
\leq \frac{1}{2} \|\rho\|_{2m\beta+1}^\beta (t) + C(T, m) (g(t) + 1).
\]

We use (ii) in Lemma 2.1 and (17) to have
\[
\|\rho\|_{2m\beta+1}^\beta (t) \leq C(T, m) [g(t) + 1]
\leq C(T, m) \left( 1 + \int_0^t (m^2 \phi(\tau) + m) \|\rho\|_{2m\beta+1}^{1+\frac{1}{m}} d\tau \right)
\leq C(T, m) \left( 1 + \int_0^t \left( \int_{\Omega_2(t)} \rho |\Lambda(\rho)|^{2m} dx \right)^{\frac{2}{2m\beta+1}} d\tau \right)
+ \int_0^t \|\rho\|_{2m\beta+1}^\beta \int_{\Omega_2(t)} (u-v)^2 f d\nu dx d\tau + \int_0^t \phi^2(\tau) \|\rho\|_{2m\beta+1}^\beta d\tau),
\[
\leq C(T, m) \left[ 1 + \int_0^t \|\rho\|_{2m\beta+1}^2 \, d\tau + \int_0^t (m^2 \phi(\tau) + m) \|\rho\|_{2m\beta+1}^{1+\frac{1}{m}} \, d\tau \\
+ \int_0^t \|\rho\|_{2m\beta+1}^2 \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (u - v)^2 f \, dv \, dx \, d\tau \right]
\leq C(T, m) \left[ 1 + \int_0^t \|\rho\|_{2m\beta+1}^2 \left( 1 + \phi(\tau) + \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (u - v)^2 f \right) \, d\tau \right].
\]

(18)

We now apply the Gronwall Lemma for (18) using the estimate (ii) in Lemma 2.1 to get (15).

Corollary 1. Under the conditions in Lemma 2.1, we have

\((i)\) \(\sup_{0 \leq t \leq T} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (1 + |v|^k) f \, dv \, dx \leq C(T), \quad 2 \leq k < \infty.\)

\((ii)\) \(\sup_{0 \leq t \leq T} \|\rho f\|_p \leq C(T), \quad \sup_{0 \leq t \leq T} \|\rho f u f\|_p \leq C(T), \quad \sup_{0 \leq t \leq T} \|m^2 f\|_p \leq C(T), \quad \forall p \geq 1.\) (19)

Proof. The estimate in (19) are easy consequences of Lemma 2.2 and Lemma 2.5. For \((i)\), it derives from Lemma 2.2, \((ii)\) in Lemma 2.4 and \((i)\) in Lemma 2.1:

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (1 + |v|^k) f \, dv \, dx
\leq C(T) \left( \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (1 + |v|^k) f_0 \, dv \, dx + \left( \int_0^T \|u\|_{k+2} \, dt \right)^{k+2} \right)
\leq C(T) \left( \int_0^T \|\nabla_x u\|^2 \, dt \right)^{k+2}
\leq C(T),
\]

For \((ii)\), we only select one term to estimate, the others can be derived similarly.

In fact, we use \((i)\) in Lemma 2.2 for \(k_1 = 2\) and \(k_2 = 4p - 2\) to have

\[
|m^2 f\|_p \leq C(T) \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} |v|^{4p - 2} f \, dv \, dx.
\]

Then we use \((ii)\) in Lemma 2.2 for \(k = 4p - 2\), \((ii)\) in Lemma 2.4 and Lemma 2.5 to have

\[
\|m^2 f\|_p \leq C(T) \left[ 1 + \left( \int_0^T \|u\|_{4p} \, dt \right)^{4p} \right]
\leq C(T) \left[ 1 + \left( \int_0^T \|\nabla_x u\|^2 \, dt \right)^{4p} \right]
\leq C(T).
\]

\(\square\)
2.2. Estimates on the fluid density. In this subsection, we provide an upper bound of the fluid density by the method of characteristics. With the help of the Brezis-Wainger inequality, we combine the estimate of \(\|\nabla u\|^2\) and \(\|\rho u\|^p\) \((p > 3)\) to derive the upper bound estimate of the fluid density.

We set the material derivative of the fluid velocity by \(\dot{u}\):

\[
\dot{u} = \partial_t u + u \cdot \nabla_x u,
\]

and introduce nonlinear functionals:

\[
Z^2(t) := \int_{\mathbb{T}^2} \left( \omega^2 + \frac{F^2}{2\mu + \lambda(\rho)} + |\nabla_x H|^2 \right) dx + \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (u - v)^2 f dv dx, \\
\chi^2(t) := \int_{\mathbb{T}^2} (\rho|\dot{u}|^2 + |\nabla_x H|^2) dx, \\
\Phi_T := \|\rho\|_{L^\infty_t} + 1.
\]

**Lemma 2.6.** Let \(T\) be a positive constant and suppose that the conditions in Lemma 2.1 hold. Then, for \(t \in [0, T]\) we have

\[
\log \left[ e + Z^2(t) \right] + \int_0^t \frac{\chi^2}{e + Z^2} d\tau \leq C(T)\Phi^{1+\beta_\varepsilon},
\]

where \(\varepsilon > 0\) is a constant which can be arbitrarily small.

**Proof.** We denote the perpendicular gradient by \(\nabla_x^\perp := (\partial_{x_2}, -\partial_{x_1})\). Then, the momentum equation can be rewritten as follows:

\[
\rho \dot{u} = \nabla_x F + \mu \nabla_x^\perp \omega + H \cdot \nabla_x H - (u - u_f)\rho_f.
\]

We multiply the above identity by \(\dot{u}\) to obtain

\[
\int_{\mathbb{T}^2} \rho |\dot{u}|^2 dx = -\int_{\mathbb{T}^2} F \text{div} \dot{u} dx - \mu \int_{\mathbb{T}^2} \omega \nabla_x^\perp \cdot \dot{u} dx + \int_{\mathbb{T}^2} H \cdot \nabla_x H \cdot \dot{u} dx - \int_{\mathbb{T}^2} \dot{u} \cdot (u - u_f)\rho_f dx.
\]

These below relations hold:

\[
\text{div} \dot{u} = (\text{div} u)_t + u \cdot \nabla_x \text{div} u - 2\nabla u_1 \cdot \nabla^\perp u_2 + (\text{div} u)^2, \\
\nabla_x^\perp \cdot \dot{u} = (\omega)_t + u \cdot \nabla_x \omega + \omega \text{div} u.
\]

We should handle the strongly coupled magnetic field with the velocity field to obtain the upper bound of the density.

\[
\int_{\mathbb{T}^2} H \cdot \nabla_x H \cdot \dot{u} dx = \frac{d}{dt} \int_{\mathbb{T}^2} H \cdot \nabla_x H \cdot u dx \\
- \int_{\mathbb{T}^2} (H_t \cdot \nabla_x H \cdot u + H \cdot \nabla_x H_t \cdot u) dx \\
- \int_{\mathbb{T}^2} H \cdot \nabla_x (u \cdot \nabla_x u) \cdot H dx \\
= -\frac{d}{dt} \int_{\mathbb{T}^2} H \otimes H : udx \\
+ \int_{\mathbb{T}^2} (H_t - H \cdot \nabla_x u + u \cdot \nabla_x H) \cdot \nabla_x u \cdot H dx
\]
integration by parts to get

\[ + \int_{\mathbb{T}^2} H \cdot \nabla_x u(H_t + H \text{div} u + u \cdot \nabla_x H) dx \]

\[ = -\frac{d}{dt} \int_{\mathbb{T}^2} H \otimes H : u dx \]

\[ + \int_{\mathbb{T}^2} (\nu \Delta_x H - H \text{div} u) \cdot \nabla_x u \cdot H dx \]

\[ + \int_{\mathbb{T}^2} H \cdot \nabla_x u \cdot (\nu \Delta_x H + H \cdot \nabla_x u) dx. \]

With above, we have

\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^2} \left( \mu \omega^2 + \frac{F^2}{2 \mu + \lambda(\rho)} \right) dx \]

\[ + \frac{d}{dt} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \left( \frac{1}{2} u^2 f - uf' \right) dv dx + \int_{\mathbb{T}^2} \rho |\dot{u}|^2 dx \]

\[ = -\frac{d}{dt} \int_{\mathbb{T}^2} H \otimes H : u dx - \frac{\mu}{2} \int_{\mathbb{T}^2} \omega^2 \text{div} u dx + 2 \int_{\mathbb{T}^2} F \nabla_x u_1 \cdot \nabla_x u_2 dx \]

\[ + \frac{1}{2} \int_{\mathbb{T}^2} F^2 \text{div} u \left[ \rho \left( \frac{1}{2 \mu + \lambda(\rho)} \right)' - \frac{1}{2 \mu + \lambda(\rho)} \right] dx \]

\[ + \int_{\mathbb{T}^2} F \text{div} u \left[ \rho \left( \frac{P(\rho)}{2 \mu + \lambda(\rho)} \right)' - \frac{P(\rho)}{2 \mu + \lambda(\rho)} \right] dx \]

\[ + \frac{1}{2} \int_{\mathbb{T}^2} F H^2 \text{div} u \left[ \rho \left( \frac{1}{2 \mu + \lambda(\rho)} \right)' - \frac{1}{2 \mu + \lambda(\rho)} \right] dx \]

\[ - \int_{\mathbb{T}^2} \frac{F}{2 \mu + \lambda(\rho)} (H \cdot \nabla_x u \cdot H + \nu H \cdot \Delta_x H) dx \]

\[ - \int_{\mathbb{T}^2} u \cdot \nabla_x u (u - u_f) \rho_f dx + \int_{\mathbb{T}^2} (\nu \Delta_x H - H \text{div} u) \cdot u \cdot H dx \]

\[ + \int_{\mathbb{T}^2} H \cdot \nabla_x u \cdot (\nu \Delta_x H + H \cdot \nabla_x u) dx \]

\[ + \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \left( \frac{1}{2} u^2 f_t - u \cdot v f_t \right) dv dx. \]

On the other hand, we multiply (1)\(_1\) by \(\frac{1}{2} v^2\), integrate over \(\mathbb{T}^2\) and use the integration by parts to get

\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} v^2 f dv dx = \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} v \cdot (u - v) f dv dx + \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} f dv dx, \quad (23) \]

and multiply (1)\(_4\) by \(-C_1 \Delta_x H\) and integrate over \(\mathbb{T}^2\) with \(C_1 > 0\) sufficiently large to have

\[ \frac{C_1}{2} \frac{d}{dt} \int_{\mathbb{T}^2} |\nabla_x H|^2 dx + \nu C_1 \int_{\mathbb{T}^2} |\nabla_x^2 H|^2 dx \]

\[ = C_1 \int_{\mathbb{T}^2} (u \cdot \nabla_x H - H \cdot \nabla_x u + H \text{div} u) \Delta_x H dx. \quad (24) \]
We combine (22) – (24) to have

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^2} \left( \mu \omega^2 + \frac{F^2}{2\mu + \lambda(\rho)} + C_1 |\nabla_x H|^2 + H \otimes H \cdot u \right) dx \\
+ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} (u - v)^2 f dv dx + \int_{\mathbb{T}^2} \rho |\dot{u}|^2 dx + \nu C_1 \int_{\mathbb{T}^2} |\nabla_x H|^2 dx \\
= -\frac{\mu}{2} \int_{\mathbb{T}^2} \omega^2 \text{div} u dx + 2 \int_{\mathbb{T}^2} F \nabla_x u_1 \cdot \nabla_x u_2 dx \\
+ \frac{1}{2} \int_{\mathbb{T}^2} F^2 \text{div} u \left[ \rho \left( \frac{1}{2\mu + \lambda(\rho)} \right)' - \frac{1}{2\mu + \lambda(\rho)} \right] dx \\
+ \int_{\mathbb{T}^2} F \text{div} u \left[ \rho \left( \frac{P(\rho)}{2\mu + \lambda(\rho)} \right)' - \frac{P(\rho)}{2\mu + \lambda(\rho)} \right] dx \\
+ \frac{1}{2} \int_{\mathbb{T}^2} FH^2 \text{div} u \left[ \rho \left( \frac{1}{2\mu + \lambda(\rho)} \right)' - \frac{1}{2\mu + \lambda(\rho)} \right] dx \\
- \int_{\mathbb{T}^2} \frac{F}{2\mu + \lambda(\rho)} (H \cdot \nabla_x u \cdot H + \nu H \cdot \Delta_x H) dx - \int_{\mathbb{T}^2} u \cdot \nabla_x u (u - u_f) \rho_f dx \\
+ \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \left( \frac{1}{2} u^2 f_1 - \nu f \right) dv dx + \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (v \cdot (u - v) f + f) dv dx \\
+ \int_{\mathbb{T}^2} (\nu \Delta_x H - H \text{div} u) \cdot u \cdot H dx \\
+ \int_{\mathbb{T}^2} H \cdot \nabla_x u \cdot (\nu \Delta_x H + H \cdot \nabla_x u) dx \\
+ C_1 \int_{\mathbb{T}^2} (u \cdot \nabla_x H - H \cdot \nabla_x u + H \text{div} u) \Delta_x H dx \\
= \sum_{i=2}^{13} I_{3i}.
\]

Now, we estimate the terms $I_{3i}, i = 1, \cdots, 13,$ separately.

- (Estimate of $I_{31}$): From

\[
\int_{\mathbb{T}^2} |H|^2 |\nabla_x u| dx \geq \|\nabla_x u\|_2^2 \|H\|_4^2 \geq \sigma \|\nabla_x u\|_2^2 - C \|\nabla_x H\|^2_2
\]

and $C_1 > 0$ sufficiently large, we can get

\[
|I_{31}| \geq C \int_{\mathbb{T}^2} \left( \omega^2 + \frac{F^2}{2\mu + \lambda(\rho)} + |\nabla_x H|^2 \right) dx.
\]

- (Estimate of $I_{32}$): We use the relations

\[
\Delta_x F = \text{div}(\rho \dot{u} - H \cdot \nabla_x H + (u - u_f) \rho_f), \\
\mu \Delta \omega = \nabla_x \cdot (\rho \dot{u} - H \cdot \nabla_x H + (u - u_f) \rho_f).
\]

(26)
the elliptic estimates, Sobolev inequality and Corollary 1 to get
\[ \| \nabla_x F \|_2 + \| \nabla_x \omega \|_2 \leq C(\| \rho \|_2 + \| H \cdot \nabla_x H \|_2 + \| (u - u_f) \rho_f \|_2) \]
\[ \leq C(\Phi^\frac{1}{2} \chi(t) + \| \nabla_x H \|_2 \chi^\frac{1}{2}(t) + \| u \| s \rho_f \|_2 + \| u_f \rho_f \|_2), \]
\[ \leq C(T)(\Phi^\frac{1}{2} \chi(t) + \| \nabla_x H \|_2 \chi^\frac{1}{2}(t) + \| \nabla_x u \|_2 + 1) \]  
(27)
\[ \| \omega \|_4 \leq C \| \omega \|_2^\frac{1}{2} \| \nabla_x \omega \|_2^\frac{1}{2} \]
\[ \leq C(T)Z^\frac{1}{2}(t)(\Phi^\frac{1}{2} \chi(t) + \| \nabla_x H \|_2 \chi^\frac{1}{2}(t) + \| \nabla_x u \|_2 + 1)^\frac{1}{2}. \]

Then, the above estimates in (27) yield
\[ |\mathcal{I}_{32}| \leq C \| \omega \|_2^\frac{1}{3} \| \text{div}u \|_2 \]
\[ \leq C(T)Z(t)(\Phi^\frac{1}{2} \chi(t) + \| \nabla_x H \|_2 \chi^\frac{1}{2}(t) + \| \nabla_x u \|_2 + 1) \| \nabla_x u \|_2 \]
\[ \leq \frac{1}{8} \chi^2(t) + C(T)(Z^2(t) + 1)(\| \nabla_x H \|_2^2 + \| \nabla_x u \|_2^2 + 1) \Phi_T. \]

- (Estimate of \( \mathcal{I}_{33} \)): By the duality between Hardy \( \mathcal{H}^1 \) and \( \mathcal{BMO} \) spaces, we have
\[ |\mathcal{I}_{33}| \leq C \| F \|_{\mathcal{BMO}} \| \nabla u_1 \cdot \nabla \nabla_x u_2 \|_{\mathcal{H}^1} \]
\[ \leq C \| \nabla_x F \|_2 \| \nabla x u_1 \|_2 \| \nabla_x u_2 \|_2 \]
\[ \leq C(T)(\Phi^\frac{1}{2} \chi(t) + \| \nabla_x H \|_2 \chi^\frac{1}{2}(t) + \| \nabla_x u \|_2 + 1) \| \nabla_x u \|_2^2 \]
\[ \leq \frac{1}{8} \chi^2(t) + C(T)(Z^2(t) + 1)(\| \nabla_x H \|_2^2 + \| \nabla_x u \|_2^2 + 1) \Phi_T. \]

- (Estimate of \( \mathcal{I}_{34} \) and \( \mathcal{I}_{35} \)): We use \( \| F \|_2 \leq \Phi_T^\frac{1}{2}Z(t) \) and (27)\_1 to get
\[ \left\| \frac{F^2}{2\mu + \lambda(\rho)} \right\|_2 \leq C \left\| \frac{F}{\sqrt{2\mu + \lambda(\rho)}} \right\|_2^{1-\varepsilon} \| F \|_2^{1+\varepsilon} \]
(28)
\[ \leq C Z^{1-\varepsilon}(t) \| F \|_2 \| \nabla F \|_2 \]
\[ \leq C(T)Z(t) \Phi^\frac{1}{2}_T \Phi_T^\beta \chi(t) + \| \nabla_x H \|_2 \chi^\frac{1}{2}(t) + \| \nabla_x u \|_2 + 1), \]
for arbitrarily small constant \( \varepsilon > 0 \). Then we have
\[ |\mathcal{I}_{34} + I_{35}| \leq C \int_{\mathbb{R}^2} \left( |F|^2 \| \text{div}u \|_2 \frac{1}{2\mu + \lambda(\rho)} + |F| \| \text{div}u \|_2 \frac{P(\rho)}{2\mu + \lambda(\rho)} \right) \ dx \]
\[ \leq C \| \nabla_x u \|_2 \left( \frac{F^2}{2\mu + \lambda(\rho)} \right)_2 + \| F \|_{2(2+\varepsilon)} \| P(\rho) \|_{2+\varepsilon} \]
\[ \leq C \| \nabla_x u \|_2 \left( \frac{F^2}{2\mu + \lambda(\rho)} \right)_2 + \| F \|_{2+\varepsilon} \| \nabla_x F \|_{2+\varepsilon}^2 \]
\[ \leq C(T) \| \nabla_x u \|_2 \]
\[ \cdot \left( Z(t) \Phi^\beta_T \Phi_T^\frac{1}{2} \chi(t) + \| \nabla_x H \|_2 \chi^\frac{1}{2}(t) + \| \nabla_x u \|_2 + 1 + 1 \right) \]
\[ \leq \frac{1}{8} \chi^2(t) + C(T)(Z^2(t) + 1)(\| \nabla_x u \|_2^2 + \| \nabla_x H \|_2^2 + 1) \Phi_T^{1+\beta_\varepsilon}. \]
Applying the integration by parts on space variables, we use Corollary 1 to have
\[ |I_{36} + I_{37}| \leq C \int_{\mathbb{T}^2} \left| \frac{F}{2\mu + \lambda(\rho)} \right| |H|^2 |\nabla_x u| + |H| |\nabla_x H| \, dx \]
\[ \leq C \| \frac{F}{2\mu + \lambda(\rho)} \|_4 (|H|^2 \| \nabla_x u \|_2 + \| H \|_4 \| \nabla_x^2 H \|_2) \]
\[ \leq C(T) \| \frac{F^2}{2\mu + \lambda(\rho)} \|_\frac{3}{2} (\| \nabla_x u \|_2 + \| \nabla_x^2 H \|_2) \]
\[ \leq C(T) Z^{\frac{1}{2}} \Phi_T^\frac{3}{2} (\Phi_T^\frac{3}{4} \chi(t) + \| \nabla_x H \|_2 \chi^\frac{1}{2} (t) + \| \nabla_x u \|_2 + 1)^{\frac{1}{2}} \]
\[ \cdot (\| \nabla_x u \|_2 + \chi(t)) \]
\[ \leq \frac{1}{8} \chi^2 (t) + C(T) \Phi_T^{1 + \beta_8} (Z^2 (t) + 1)(\| \nabla_x u \|_2^2 + \| \nabla_x H \|_2^2 + 1). \]

(29)

(29)

- (Estimate of $I_{38}$): From (ii) in Lemma 2.4 and Corollary 1, we have
\[ |I_{38}| \leq C \| \rho f \|_4 \| \nabla_x u \|_2 \| u \|_2^2 + \| u \|_8 \| \nabla_x u \|_2 \| u_f \rho f \|_2 \]
\[ \leq C(T) (Z^2 (t) + 1)(\| \nabla_x u \|_2^2 + 1). \]

- (Estimate of $I_{39}$): We have from (1) that
\[ \int_{\mathbb{R}^2} f t \, dv = -\int_{\mathbb{R}^2} v \cdot \nabla_x f \, dv \int_{\mathbb{R}^2} v f \, dv = -\int_{\mathbb{R}^2} \rho f \, dv \int_{\mathbb{R}^2} (u - v) f \, dv. \]

Applying the integration by parts on space variables, we use Corollary 1 to have
\[ |I_{39}| = \left| \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} u \cdot \nabla_x u \cdot v f - \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} \nabla_x u \cdot v \cdot v f - \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} u \cdot (u - v) f \right| \]
\[ \leq C \left( \| u \|_8 \| \nabla u \|_2 \| u_f \rho f \|_2 \|
abla_x u \|_2 \| m_2 f \|_2 + \| u \|_4 \| \rho f \|_2 + \| u \|_2 \| u_f \rho f \|_2 \right) \]
\[ \leq C(T) (Z^2 (t) + 1)(\| \nabla_x u \|_2^2 + 1). \]

- (Estimate of $I_{310}$): we use Corollary 1 to get
\[ |I_{310}| \leq C(T) + \| u \|_2 \| u_f \rho f \|_2 \]
\[ \leq C(T) (1 + \| \nabla_x u \|_2). \]

- (Estimate of $I_{311}$-$I_{313}$):
\[ \left| \sum_{i=11}^{13} I_{3i} \right| \leq C \int_{\mathbb{T}^2} (|H|^2 |\nabla_x u|^2 + |H| |\Delta_x H| |\nabla_x u| + |\nabla_x u| |\nabla_x H|^2 \) \, dx \]
\[ \leq C \| \rho f \|_4 \| \nabla_x u \|_2^2 + \| H \|_4 \| \nabla_x u \|_2 |\nabla^2 H|_2 + \| \nabla_x u \|_2 |\nabla_x H|^2 \]
\[ \leq \frac{1}{8} \chi^2 (t) + C \left( \left\| \frac{F^2}{2\mu + \lambda(\rho)} \right\|_\frac{3}{2} + \| P(\rho) \|_2 \right) \]
\[ + \| H \|_8^2 + \| \omega \|_2^2 + \| \nabla_x u \|_2^2 |\nabla_x H|^2 \]
\[ \leq \frac{1}{8} \chi^2 (t) + C(T) [Z(t) \Phi_T^\frac{3}{2} (\Phi_T^\frac{3}{4} \chi(t) + \| \nabla_x H \|_2 \chi^\frac{1}{2} + \| \nabla_x u \|_2) \]
\[ + \| \nabla_x u \|_2^2 |\nabla_x H|^2 + 1 + 1]. \]
Collecting the estimates of $I_3(1 \leq i \leq 13)$ in (25), we have
\[
\frac{d}{dt} Z^2(t) + \chi^2(t) \leq C(T)(Z^2(t) + 1)(\|\nabla_x u\|_2^2 + \|\nabla_x H\|_2^2 + 1)\Phi_T^{1+\beta_\varepsilon}.
\]
Then we further use Lemma 2.5 to derive (20).

The following $\|\rho u\|_p$ with $p > 3$ will play a crucial role in the estimate of the upper bound of the density as in [23, 24].

**Lemma 2.7.** Assume the conditions in Lemma 2.1 hold. Then it holds that
\[
\|\rho u\|_p \leq C(T)\Phi_T^{1+\frac{p}{2}} (\|\nabla_x u\|_2 + 1)^{1-\frac{2}{p}}. \tag{30}
\]

**Proof.** Firstly, we derive the estimate of $\int_{\mathbb{T}^2} \rho |u|^{2+\alpha'} dx$ with $\alpha' = \frac{\mu^2}{2(\mu+\lambda)} \Phi_T^{-\frac{2}{T}}$. To this end, we multiply the momentum equation by $(2 + \alpha')|u|^{\alpha'} u$, integrate over $\mathbb{T}^2$, and use integration by parts to obtain
\[
\frac{d}{dt} \int_{\mathbb{T}^2} \rho |u|^{2+\alpha'} dx + \mu(2 + \alpha') \int_{\mathbb{T}^2} |u|^{\alpha'} |\nabla_x u|^2 dx
+ \mu(2 + \alpha') \int_{\mathbb{T}^2} \nabla_x \frac{|u|^2}{2} \cdot \nabla_x |u|^{\alpha'} dx
+ (2 + \alpha') \int_{\mathbb{T}^2} (\mu + \lambda(\rho))(\text{div}u)^2 |u|^{\alpha'} dx
+ (2 + \alpha') \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (u - v)^2 f |u|^{\alpha'} dv dx
= (2 + \alpha') \int_{\mathbb{T}^2} P(\rho)\text{div}(u|u|^{\alpha'}) dx
- (2 + \alpha') \int_{\mathbb{T}^2} (\mu + \lambda(\rho))\text{div}uu \cdot \nabla_x |u|^{\alpha'} dx
+ (2 + \alpha') \int_{\mathbb{T}^2} \left( H \cdot \nabla_x H - \nabla_x \frac{|H|^2}{2} \right) \cdot u|u|^{\alpha'} dx
- (2 + \alpha') \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (u - v) \cdot vf |u|^{\alpha'} dv dx
:= \sum_{i=1}^4 I_{4i}. \tag{31}
\]
Now we estimate terms $I_{4i}(1 \leq i \leq 4)$ one by one.

**• (Estimate of $I_{41}, I_{42}$ and $I_{43}$):** Similar to the computation in [23, 30], we use $(ii)$ in Lemma 2.4 to have
\[
|I_{41}| \leq \frac{\mu(2 + \alpha')}{2} \int_{\mathbb{T}^2} |u|^{\alpha'} |\nabla_x u|^2 dx + C(T)(\|\nabla_x u\|_2^2 + 1),
|I_{42}| \leq \frac{(2 + \alpha')}{2} \int_{\mathbb{T}^2} (\mu + \lambda(\rho))(\text{div}u)^2 |u|^{\alpha'} dx + \frac{\mu(2 + \alpha')}{4} \int_{\mathbb{T}^2} |u|^{\alpha'} |\nabla_x u|^2 dx,
|I_{43}| \leq C(T)(\|\nabla_x u\|_2^2 + 1).
\]
Therefore, we combine the above three estimates and (31) to have
\[ |\mathcal{I}_{44}| \leq C \int_{\mathbb{R}^2} ((u^2 + 1)|u|\rho_f| + (|u| + 1)m_2f) \, dx \]
\[ \leq C(1 + \|u|\rho_f\|_4 \|u\|_6^2 + \|u|\rho_f\|_4 \|u\|_4 + \|m_2f\|_2 \|u\|_8 + \|m_2f\|_1) \]
\[ \leq C(T)(\|\nabla_x u\|_2^2 + 1). \]

Therefore, we combine the above three estimates and (31) to have
\[ \frac{d}{dt} \int_{\mathbb{R}^2} \rho|u|^{2+\alpha'} \, dx \leq C(T)(\|\nabla_x u\|_2^2 + 1), \]
which implies
\[ \sup_{0 \leq t \leq T} \int_{\mathbb{R}^2} \rho|u|^{2+\alpha'} \, dx \leq C(T). \]

Thus, for \( q = (1 + \frac{2}{\alpha'})(p - 2) \leq C\Phi_T^2 \), we use interpolation inequality to obtain the result:
\[ \|\rho u\|_p \leq \|\rho u\|_2^{\frac{2}{2+\alpha'}}\|\rho u\|_q^{1-\frac{2}{p}} \]
\[ \leq (\|\rho \frac{\partial \psi}{\partial t} \|_{2+\alpha'} \Phi_T^{1+\frac{2}{2+\alpha'}}) \Phi_T^{1+\frac{2}{2+\alpha'}} [q^\frac{1}{2}(\|\nabla_x u\|_2 + 1)]^{1-\frac{2}{p}} \]
\[ \leq C(T)\Phi_T^{1+\frac{2}{2+\alpha'}}(\|\nabla_x u\|_2 + 1)^{1-\frac{2}{p}}. \]

Now we are in position to prove the upper bound of \( \rho(t, x) \).

**Lemma 2.8.** Suppose that the conditions in Lemma 2.1 hold. Then, we have
\[ 0 \leq \rho(t, x) \leq C(T). \]  

**Proof.** It follows from definition of \( \psi \) and \( \eta_1 \) that we have
\[ u \cdot \nabla_x \psi - \eta_1 = [u_i, R_i R_j](\rho u_j) - [H_i, R_i R_j](H_j). \]
Moreover, it follows from (14) that
\[ \frac{D}{Dt} (\Lambda(\rho) - \psi) + P(\rho) + \frac{|H|^2}{2} + [u_i, R_i R_j](\rho u_j) - [H_i, R_i R_j](H_j) - \eta_2 + \int_{\mathbb{T}^2} F(t, x) \, dx = 0. \]
Consider the trajectory defined by
\[
\begin{cases}
\frac{d\zeta(\tau; t, x)}{d\tau} = u(\tau, \zeta(\tau; t, x)) \\
\zeta(\tau; t, x)|_{\tau=t} = x,
\end{cases}
\]
Then, we have
\[
\frac{d}{d\tau} (\Lambda(\rho) - \psi)(\tau; \zeta(\tau; t, x)) + P(\rho)(\tau; \zeta(\tau; t, x)) + \frac{|H|^2}{2}(\tau; \zeta(\tau; t, x)) \\
- [u_i, R_i R_j](\rho u_j)(\tau; \zeta(\tau; t, x)) \\
+ [H_i, R_i R_j](H_j)(\tau; \zeta(\tau; t, x)) \\
- \int_{\mathbb{T}^2} F(t, x) \, dx + \eta_2(\tau; \zeta(\tau; t, x)).
\]
We integrate the above equation over \([0, t]\) to obtain
\[
2\mu \log \frac{\rho(x, t)}{\rho_0(s_0)} + \frac{1}{\beta}(\rho^\beta(x, t) - \rho_0^\beta(s_0)) - \psi(x, t) + \psi_0(s_0)
\leq - \int_0^t [u_i, R_i R_j](\rho u_j) d\tau + \int_0^t [H_i, R_i R_j](H_j) d\tau + \int_0^t F(t, x) d\tau + \int_0^t \eta_\omega d\tau.
\]
(33)

Now, we estimate the terms on the RHS of (33). For \(\psi\), we use Lemma A.3 to have
\[
\|\psi\|_\infty \leq C(\|\psi\|_{2m} + \|\nabla_x \psi\|_2) \log^{\frac{1}{2}}(e + \|\psi\|_{W^{1,2m}}) + C.
\]
(34)

From the elliptic equation (11), we use (i) in Lemma 2.4 and Lemma 2.7 to get
\[
\|\nabla \psi\|_2 \leq C\|\rho u\|_2 \leq C(T)\Phi^\frac{3}{2},
\]
\[
\|\psi\|_{2m} \leq C(T)m^{\frac{3}{2}}\|\rho\|_m \leq C(T).
\]
(35)

Collecting the estimates (34) and (35), we use (20) to have
\[
\|\psi\|_\infty \leq C(T)\Phi^{1+\beta_\varepsilon}.
\]
(36)

Now we turn to the estimate of the third term in the RHS of (33). To the end, we first use Lemma 2.1 and (27) to have
\[
\|\nabla u\|_4 \leq C(\|\nabla \psi\|_4 + \|\nabla_t\|_4)
\]
\[
\leq C\left(\left\|\frac{F^2}{2\mu + \lambda(\rho)}\right\|^\frac{1}{2} + \|\omega\|_4 + \|H\|_8 + \|P(\rho)\|_4\right)
\]
\[
\leq CZ^\frac{1}{2}(t)\Phi^\beta_T(\Phi^\frac{1}{2}_T \chi^\frac{1}{2}(t) + \|\nabla_x H\|_2^\frac{1}{2} \chi^\frac{1}{2}(t) + \|\nabla_x u\|_2^\frac{1}{2} + 1)
\]
\[
\leq C\Phi^\beta_T Z_1(t) \chi^\frac{1}{2}(t) + C\Phi^\frac{3}{2}_T (Z(t) + 1)
\]
\[
\leq C\Phi^\beta_T Z^\beta T \left(\frac{\chi^\frac{1}{2}(t)}{e + Z^2(t)}\right)^\frac{1}{4}
\]
\[
\times \left(e + \|u\|_2^2 + \|\nabla_x H\|_2^2 + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (u - v)^2 f d\nu dx \right)^\frac{1}{2}
\]
\[
+ C\Phi^{\beta + 2\beta_\varepsilon}_T \left(e + \|\nabla_x u\|_2 + \|\nabla_x H\|_2 + \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (u - v)^2 f d\nu dx \right)^\frac{1}{2}\right),
\]
(37)

where in the last inequality one has used
\[
Z^2(t) \leq C\Phi^\beta_T (\|\nabla_x u\|_2^2 + \|\nabla_x H\|_2^2) + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (u - v)^2 f d\nu dx.
\]

Denoting the commutator \(\vartheta = [u_i, R_i R_j](\rho u_j)\), we use Lemma A.4 and (i) in (30) to have
\[
\|\vartheta\|_\infty \leq C\|\vartheta\|_p^{\frac{1}{\beta}} \|\nabla_x \vartheta\|_p^{\frac{1}{\beta}} \leq C(\|u\|_{BMO} \|\rho u\|_p)^{1-\frac{1}{\beta}} (\|\nabla u\|_4 \|\rho u\|_p)^{\frac{1}{\beta}}
\]
\[
\leq C\Phi^{1+\frac{1}{2\beta}}_T (\|\nabla_x u\|_2 + 1)^{2-\frac{1}{\beta}} \|\nabla_x u\|_4^\frac{1}{\beta}.
\]
Then we choose \( p > 4 \) sufficiently large and use (37) to have

\[
\int_0^t \| \partial_t \|_\infty \, \mathrm{d} \tau \leq C \Phi_T^{1 + \frac{\beta}{4} + \beta \varepsilon}.
\]

On the other hand, we use the Galiardo-Nirenberg inequality and commutator estimates in Lemma A.4 to get

\[
\|[H_i, R_i R_j](H_j)\|_\infty \leq C||[H_i, R_i R_j](H_j)\|_p^{1 - \frac{\beta}{4}} \| \nabla_x [H_i, R_i R_j](H_j)\|_p^{\frac{\beta}{4}}
\]

\[
\leq C(\|H\|_{BMO}\|H\|_p)^{1 - \frac{\beta}{4}} (\|\nabla_x H\|_4 \|H\|_p)^{\frac{\beta}{4}}
\]

\[
\leq C\|\nabla_x H\|_2^{1 - \frac{\beta}{4}} \|\nabla_x H\|_4^{\frac{\beta}{4}}
\]

\[
\leq C(e + Z^2(t))^\frac{\beta}{2} \left( \frac{\chi^2(t)}{e + Z^2(t)} \right) \frac{1}{\beta}
\]

\[
\leq \sigma Z^2(t) + C(T) \left( 1 + \chi^2(t) \right)
\]

\[
\leq \sigma \Phi_T^{\frac{\beta}{2}} (\|\nabla_x u\|_2^2 + \|\nabla_x H\|_2^2) + \sigma \int_{T^2} \int_{\mathbb{R}^2} (u - v)^2 f \, \mathrm{d}v \, \mathrm{d}x
\]

\[
+ C(T) \left( 1 + \chi^2(t) \right)
\]

which implies that

\[
\int_0^t \|[H_i, R_i R_j](H_j)\|_\infty \, \mathrm{d} \tau \leq \sigma \Phi_T^{\frac{\beta}{2}} + C(T) \Phi_T^{1 + \beta \varepsilon}.
\]

It follows from Lemma 2.5 that

\[
\int_0^t \int_{T^2} F(t,x) \, \mathrm{d}x \, \mathrm{d} \tau \leq \int_0^t \int_{T^2} [(2\mu + \lambda(\rho)) \text{div} u + |H|^2] \, \mathrm{d}x \, \mathrm{d} \tau \leq C(T).
\]

Finally, we treat the forth term on the RHS of (33). From the elliptic equation \((11)_3\), we use \((i)\) in Lemma 2.3 and \((\nu)\) in Lemma 2.4 for suitably large \( m \) to have

\[
\int_0^t \| \eta_2 \|_\infty \, \mathrm{d} \tau \leq C \int_0^t (\| \eta_2 \|_{2m} + \| \nabla_x \eta_2 \|_{2m}) \, \mathrm{d} \tau
\]

\[
\leq C(T) \int_0^t \| \nabla_x u \|_2^2 + 1 + \int_{T^2} \int_{\mathbb{R}^2} (u - v)^2 f \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d} \tau \leq C(T).
\]

Finally, we substitute all above estimates into (33) to derive

\[
\Phi_T^{\frac{\beta}{2}} \leq C(T) \Phi_T^{1 + \frac{\beta}{4} + \beta \varepsilon}.
\]

When \( \beta > \frac{4}{3} \), we take positive constant \( \varepsilon \) sufficiently small to have

\[
\sup_{0 \leq t \leq T} \| \rho \|_\infty (t) \leq C(T).
\]

With Lemmas 2.6, 2.8 in hand, we immediately have
Corollary 2. Assume the conditions in Lemma 2.1 hold. Then, $2 \leq p < +\infty$, $\forall \ t \in (0, T)$, it holds that
\[
\left(\|u, H\|_p + \|\nabla_x u, \nabla_x H\|_2 + \|\rho\|_{L^p} + \int_{T_2} \int_{\mathbb{R}^2} (u - v)^2 f d\nu dx\right) (t)
+ \int_0^t (\|\sqrt{\rho} u\|_2^2 + \|\nabla_x^2 H\|_2^2) d\tau \leq C(T).
\] (38)

3. A priori high-order estimates. In this section, we present higher-order estimates. Let $[f, \rho, u, H]$ be a classical solution to the coupled system, and we derive some a priori estimates for the system (1)-(2) with $(x, v) \in T^2 \times \mathbb{R}^2$.

3.1. $W^{1,p}$-estimates with $p > 4$. In this subsection, we derive the $W^{1,p}$ estimates of the classical solution $[f, \rho, u, H]$ to the system (1)-(2): $\|\langle u, v \rangle\|_{W^{1,p}}$ and $\|\langle v \rangle^{k} \nabla x v, f\|_p$ for $4 < p < \infty$.

Lemma 3.1. Suppose that the conditions in Lemma 2.1 hold. Then, we have
\[
\left(\|\sqrt{\rho} u\|_2^2 + \|H_t\|_2^2\right) + \int_0^t \left(\|\nabla_x \dot{u}\|_2^2 + \|\nabla_x H_t\|_2^2 + \|\sqrt{\rho} \dot{u}\|_2^2\right) d\tau \leq C(T). 
\] (39)

Proof. Note that
\[
\dot{u}_j \partial_t (\rho \dot{u}_j) + \text{div}(u \rho \dot{u}_j) = \frac{1}{2} \rho \partial_t (u_j)^2 + \frac{1}{2} \rho u \cdot \nabla_x (u_j)^2.
\]
Then, we apply the operator $\dot{u}_j [\partial_t + \text{div}(u \cdot v)]$ to (1)_{2,j} and use (29) to have
\[
d \frac{d}{dt} \int_{T^2} \rho |\dot{u}|^2 dx + 2 \int_{T^2} \rho f |\dot{u}|^2 dx = -2 \int_{T^2} \dot{u}_j [\partial_j P(\rho) + \text{div}(u \partial_j P(\rho))] dx 
+ 2 \mu \int_{T^2} \dot{u}_j [\partial_t \Delta_x u_j + \text{div}(u \Delta_x u_j)] dx 
+ 2 \int_{T^2} \dot{u}_j [\partial_j ((\mu + \lambda(\rho))\text{div} u) + \text{div}(u \partial_j ((\mu + \lambda(\rho))\text{div} u))] dx 
+ 2 \int_{T^2} \dot{u}_j [\partial_t (H \cdot \nabla_x H_j) + \text{div}(u H \cdot \nabla_x H_j)] dx 
- 2 \int_{T^2} \dot{u}_j [\partial_j (H \cdot H_t) + \text{div}(u H \cdot \partial_j H)] dx 
+ 2 \int_{T^2} \dot{u}_j [u \cdot \nabla_x u_j \rho_f + u_j \text{div}(u_f \rho_f) - \text{div}(u(u_j \rho_f))] dx 
+ 2 \int_{T^2} \dot{u}_j \left[(u_j - v_j)f - v_j (v \cdot \nabla_x f)\right] dx + \text{div}(u(u_f \rho_f)_{ij})] dx 
:= \sum_{i=1}^7 I_{5i},
\] (40)

- (Estimate of $\sum_{i=1}^3 I_{5i}$ and $\sum_{i=4}^5 I_{5i}$): Similar to the corresponding estimates in Lemma 3.9 [23, 30], we have
\[
\sum_{i=1}^3 I_{5i} \leq -\frac{3\mu}{2} \|\nabla_x \dot{u}\|_2^2 - \frac{3\mu}{2} \|\partial_t \text{div} u + u \cdot \nabla_x \text{div} u\|_2^2 + C(T)(1 + \|\nabla_x u\|_4^4).
\]
Similar to the corresponding estimates in Lemma 3.9 [30], we have

\[ \sum_{i=4}^{5} I_{5i} \leq \frac{\mu}{4} \| \nabla x u \|^2 + \frac{\mu}{8} \| \nabla x H_t \|^2 + C(T)(\| H_t \|^2 + \| \nabla^2 H \|^2 + 1). \]

- (Estimate of \( I_{56} \) and \( I_{57} \)): We apply the integration by parts and use (ii) in Lemma 2.4, Corollary 1 and (38) to have

\[ I_{56} \leq C(\| \dot{u} \| s \| u \| s \| \nabla x u \|_2 \| \rho f \|_4 + \| \nabla x u \|_2 \| u \|_s \| u_f \rho f \|_2^2 \]

\[ + \| \dot{u} \|_s \| \nabla x u \|_2 \| u_f \rho f \|_2^2 + \| \nabla x u \|_2 \| u \|_s^2 \| \rho f \|_4) \]

\[ \leq \frac{\mu}{8} \| \nabla x \dot{u} \|^2 + C(T). \]

Similarly, we have

\[ I_{57} \leq C(\| \nabla x \dot{u} \|_s \| u \|_s \| u_f \rho f \|_4 + \| \nabla x \dot{u} \|_2 \| m_2 f \|_2 \]

\[ + \| \dot{u} \|_s \| u \|_s \| \rho f \|_4 + \| \dot{u} \|_4 \| u_f \rho f \|_2) \]

\[ \leq \frac{\mu}{8} \| \nabla x \dot{u} \|^2 + C(T). \]

In order to close the (40), we apply operator \( \partial_t \) to (1.4), multiply the resulting equation by \( H_t \) and integrate over \( \mathbb{T}^2 \) to get

\[ \frac{d}{dt} \int_{\mathbb{T}^2} |H_t|^2 + 2\nu \int_{\mathbb{T}^2} |\nabla x H_t|^2 = \int_{\mathbb{T}^2} (H \cdot \nabla x \dot{u} - \dot{u} \cdot \nabla x H - H \text{div} \dot{u}) \cdot H_t dx \]

\[ + 2 \int_{\mathbb{T}^2} (H_t \partial_x H_{jt} - H_t \partial_x H_{it}) (u \cdot \nabla x u_j) dx \]

\[ + \int_{\mathbb{T}^2} (2H_t \cdot \nabla x u - H_t \text{div} u - 2u \cdot \nabla x H_t) H_t dx. \]

Similar to the corresponding estimates in Lemma 3.9 [30], we have

\[ \frac{d}{dt} \int_{\mathbb{T}^2} |H_t|^2 + 2\nu \int_{\mathbb{T}^2} |\nabla x H_t|^2 \]

\[ \leq \frac{\mu}{8} \| \nabla x \dot{u} \|^2 + C(T)(\| H_t \|^2 + \| \sqrt{\rho} \dot{u} \|^2 + \| \nabla x u \|^4 + 1) \]  \hspace{1cm} (41)

We collect all estimates of \( I_{6i} \) in (40) and (41) to obtain

\[ \frac{d}{dt}(\| \sqrt{\rho} \dot{u} \|^2 + \| H_t \|^2) + \| \nabla x \dot{u} \|^2 + \| \sqrt{\rho} \dot{u} \|^2 + \| \nabla x H_t \|^2 \]

\[ \leq C(T)(1 + \| \nabla x u \|^4 + \| H_t \|^2 + \| \sqrt{\rho} \dot{u} \|^2). \]

Note that

\[ \| \nabla x u \|^4 \leq C(\| \omega \|^4 + \| \text{div} u \|^4) \]

\[ \leq C(T)(\| \omega \|^4 + \| \dot{u} \|^4) \]

\[ \leq C(T)(\| \sqrt{\rho} \dot{u} \|^2 + \| \nabla x \dot{u} \|^2 + \| \nabla x H_t \|^2 + 1) \]

\[ \leq C(T)(\| \sqrt{\rho} \dot{u} \|^2 + 1) \]

by (26) and (38). We can apply the Gronwall inequality and use (15) to further obtain

\[ (\| \sqrt{\rho} \dot{u} \|^2 + \| H_t \|^2) + \int_0^t (\| \nabla x \dot{u} \|^2 + \| \nabla x H_t \|^2 + \| \sqrt{\rho} \dot{u} \|^2) dx \leq C(T). \]  \hspace{1cm} (42)
Lemma 3.2. Suppose that the conditions in Lemma 2.1 hold. Then, we have
\[ \|(\nabla_x \rho, \nabla_x P(\rho))\|_p(t) + \int_0^T \|
abla_x u\|_\infty^2 \, dt \leq C(T), \quad p > 4. \] (43)

**Proof.** We apply the operator \( \nabla_x \) to the continuity equation (1)_2, multiply the resulted equation by \( p|\nabla_x \rho|^{p-2} \nabla_x \rho \), and integrate over \( \mathbb{T}^2 \) to get
\[ \frac{d}{dt} \|\nabla_x \rho\|_p \leq C(T)(\|\nabla_x u\|_\infty \|\nabla_x \rho\|_p + \|\nabla_x^2 u\|_p). \] (44)

- (Estimate of \( \|\nabla_x^2 u\|_p \)): By the interpolation inequality, we use (39) to have
\[ \|\sqrt{p} \dot{u}\|_p \leq C\|\sqrt{p} \dot{u}\|_2^{\frac{2(p-1)}{p-2}} \|\sqrt{p} \dot{u}\|_p^{\frac{p-2}{p-2}} \leq C(T)\|\dot{u}\|_p^p \]
\[ \leq C(T)(\|\nabla_x u\|_2 + 1) \quad \forall p \geq 2, \]
and
\[ \|H\| \|\nabla_x H\|_p \leq \|H\|^{\frac{p}{p-2}} \|\nabla_x H\|_{p^2} \leq C(T)\|\nabla_x H\|_{H^1} \leq C(T)(\|\nabla_x^2 H\|_2 + 1) \]

Then we use (15), Lemma 2.1 and Corollary 2 to have
\[ \|\nabla_x^2 u\|_p \leq C(\|\nabla_x \text{div} u\|_p + \|\nabla_x \omega\|_p) \]
\[ \leq C(\|\nabla_x ((2\mu + \lambda(\rho))\text{div} u)\|_p + \|\text{div} u\|_\infty \|\nabla_x \lambda(\rho)\|_p + \|\nabla_x \omega\|_p) \]
\[ \leq C(T)(\|\nabla_x F\|_p + \|\nabla_x \omega\|_p + \|\nabla_x \rho\|_p + \|\nabla_x H\|_p + \|\text{div} u\|_\infty \|\nabla_x \rho\|_p) \]
\[ \leq C(T)(\|\rho \dot{u}\|_p + \|u\|_2 \|\rho f\|_2 + \|u \rho f\|_p + \|\nabla^2_x H\|_2 \]
\[ + \|\nabla_x \dot{u}\|_2 + \|\text{div} u\|_\infty + 1)(1 + \|\nabla_x \rho\|_p) \]
\[ \leq C(T)(\|\nabla_x \dot{u}\|_2 + \|\nabla_x^2 H\|_2 + \|\text{div} u\|_\infty + 1)(1 + \|\nabla_x \rho\|_p). \]

On the other hand, we use Lemma 2.1, (15)_1 and Corollary 2 to have
\[ \|\text{div} u\|_\infty + \|\omega\|_\infty \leq C(T)(\|F\|_\infty + \|\omega\|_\infty + \|H\|_H^2 + 1) \]
\[ \leq C(T)(\|\nabla_x F\|^{\frac{3}{4}}_4 + \|\nabla_x \omega\|^{\frac{3}{4}}_4 + \|\nabla_x H\|^{\frac{3}{4}}_4 + 1) \]
\[ \leq C(T)(\|\rho \dot{u}\|^{\frac{3}{4}}_4 + \|u\|_2 \|\rho f\|^{\frac{3}{4}}_4 + \|u \rho f\|^{\frac{3}{4}}_4 \]
\[ + \|\nabla_x H\|^{\frac{3}{2}}_2 \|\nabla^2 H\|_2 + 1) \]
\[ \leq C(T)(\|\nabla_x \dot{u}\|_2 + \|\nabla_x^2 H\|_2 + 1). \]

Then we can further estimate \( \|\nabla^2_x u\|_p \) as
\[ \|\nabla_x^2 u\|_p \leq C(T)(\|\nabla_x \dot{u}\|_2 + \|\nabla_x^2 H\|_2 + 1)(1 + \|\nabla_x \rho\|_p). \]

- (Estimate of \( \|u\|_\infty \)): By Lemma A.5, we use the estimate of \( \|\nabla_x^2 u\|_p \) to have
\[ \|\nabla_x u\|_\infty \leq C(\|\text{div} u\|_\infty + \|\omega\|_\infty)(1 + \|\nabla_x^2 u\|_p) + C\|\nabla_x u\|_2 + C \]
\[ \leq C(T)(\|\nabla_x \dot{u}\|_2 + \|\nabla_x^2 H\|_2 + 1)(1 + \|\nabla_x \rho\|_p) \]
\[ + C(T)(\|\nabla_x \dot{u}\|_2 + \|\nabla_x^2 H\|_2 + 1). \] (45)

Collecting the estimates of \( \|\nabla_x^2 u\|_p \) and \( \|\nabla_x u\|_\infty \) in (44), we have
\[ \frac{d}{dt} \|\nabla_x \rho\|_p \leq C(T)(\|\nabla_x \dot{u}\|_2 + \|\nabla_x^2 H\|_2 + 1)(1 + \|\nabla_x \rho\|_p) \]
\[ \leq C(T)(\|\nabla_x \dot{u}\|_2 + \|\nabla_x^2 H\|_2 + 1)(\|\nabla_x \rho\|_p + 1). \]
We apply the Gronwall lemma, and use (15) and Lemma 2.1 to have
\[
\sup_{0 \leq t \leq T} \|\nabla_x \rho\|_p \leq C(T).
\]
The above inequality together with (45) implies
\[
\int_0^t \|\nabla_x u\|_\infty^2 \, dt \leq C(T).
\]

Lemma 3.3. Suppose that the conditions in Lemma 2.1 hold. Then, for \(4 < p < +\infty\) we have
\[
\sup_{0 \leq t \leq T} \left( \|v\|_{L^p}^k \|\partial_x f\|_{L^p}^p + \|v\|_{L^p}^k \|\nabla_v f\|_{L^p}^p \right) (t) + \int_0^t \left( \|v\|_{L^p}^k \|\nabla_v (|\nabla_x f|^{\frac{p}{2}})\|_{L^2}^2 \right. \\
+ \left. \|v\|_{L^p}^k \|\nabla_v (|\nabla_x f|^{\frac{p}{2}})\|_{L^2}^2 \right) \, dt \leq C(T, p).
\]

Proof. We apply the operator \(\partial_x\) to (1) to obtain
\[
\partial_t \partial_x f + v \cdot \nabla \partial_x f = \nabla_v \cdot [(v - u) \partial_x f - \partial_v, f - \partial_x, uf + \partial_x, \nabla_v f].
\]

We multiply the above equation by \(v^{k-2} \partial_x f\), and integrate the resulted equations with respect to \(x, v\) over \(\mathbb{T}^2 \times \mathbb{R}^2\) to give
\[
\frac{d}{dt} \|v\|_{L^p}^k \|\partial_x f\|_{L^p}^p = - \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} v^{k-2} \partial_v, f - v \partial_x, \nabla_v \cdot [(v - u) \partial_x f] \, dv \, dx \\
- \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} v^{k-2} \partial_v, f - v \partial_x, \nabla_v \cdot (\partial_v, uf) \, dv \, dx \\
+ \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} v^{k-2} \partial_v, f - v \partial_x, \nabla_v \Delta_v f \, dv \, dx
\]
\[
:= \sum_{i=1}^3 \mathcal{I}_{6i}.
\]

Now we deal with \(\mathcal{I}_{6i}, i = 1, \cdots, 3\), as below.

- (Estimate of \(\mathcal{I}_{61}\) (\(i = 1, 2\))): We use integration by parts to obtain
  \[
  |\mathcal{I}_{61}| \leq C \|v\|_{L^p}^k \|\partial_x f\|_{L^p}^p, \\
  |\mathcal{I}_{62}| \leq \|\nabla_x u\|_\infty \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} v^{k-2} \partial_v, f - v \partial_x, \nabla_v f \, dv \, dx \\
  \leq C \|\nabla_x u\|_\infty (\|v\|_{L^p}^k \|\partial_x f\|_{L^p}^p + \|v\|_{L^p}^k \|\nabla_v f\|_{L^p}^p).
  \]

- (Estimate of \(\mathcal{I}_{63}\)): Again, we use the integration by parts to have
  \[
  \mathcal{I}_{63} = \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} k p v^{k-2} \nabla_v (|\nabla_x f|^{\frac{p}{2}}) \, dv \, dx - p (p - 1) \|v\|_{L^p}^k \|\nabla_v (|\nabla_x f|^{\frac{p}{2}})\|_{L^2}^2 \\
  \leq -p (p - 1) \|v\|_{L^p}^k \|\nabla_v (|\nabla_x f|^{\frac{p}{2}})\|_{L^2}^2 + C(p) \|v\|_{L^p}^k \|\partial_x f\|_{L^p}^p.
  \]

We collect all estimates of \(\mathcal{I}_{6i}\) in (47) to find
\[
\frac{d}{dt} \|v\|_{L^p}^k \|\nabla_x f\|_{L^p}^p + p (p - 1) \|v\|_{L^p}^k \|\nabla_v (|\nabla_x f|^{\frac{p}{2}})\|_{L^2}^2 \\
\leq C(T, p) (1 + \|\nabla_x u\|_\infty) (\|v\|_{L^p}^k \|\nabla_x f\|_{L^p}^p + \|v\|_{L^p}^k \|\nabla_v f\|_{L^p}^p)).
\]
Similarly, we have
\[ \frac{d}{dt} \| \langle v \rangle \nabla_v f \|_p^p + p(p-1) \| \langle v \rangle \nabla_v (|\nabla_v f|^2) \|_2^2 \leq C(T, p)(\| \langle v \rangle \nabla_v f \|_p^p + \| \langle v \rangle \nabla_v f \|_p^p). \]

Then we combine the above two estimates to have
\[ \frac{d}{dt} (\| \langle v \rangle \nabla_v f \|_p^p + \| \langle v \rangle \nabla_v f \|_p^p) \leq C(T)(1 + \| \nabla_x u \|_\infty)(\| \langle v \rangle \nabla_v f \|_p^p + \| \langle v \rangle \nabla_v f \|_p^p). \]

We further apply the Gronwall inequality and use (43) to derive (46).

**Lemma 3.4.** Suppose that the conditions in Lemma 2.1 hold. Then, we have
\[
\left( \| \sqrt{\rho} u_t \|_2^2 + \| (\rho, P, \rho^2, \lambda(\rho)) \|_{H^1} + \| (\rho, u, H) \|_{H^2} \right) (t) \\
+ \int_0^t \left( \| (u_t, H_t) \|_{H^1}^2 + \| (u, H) \|_{H^2}^2 + \| (\rho, P, \lambda(\rho)) \|_{H^2}^2 \right) d\tau \leq C(T).
\]

**Proof.** By the standard \( L^2 \)-estimates for (1), we use Corollary 2, Lemma 3.1 and Lemma 3.2 to have
\[
\| u \|_{H^2} \leq C(\| \rho u_t \|_2 + \| \nabla_x P(\rho) \|_2 + \| H \|_{H^1}^2 + \| (u-u_f) \|_{H^1}) \\
\leq \frac{1}{4} \| H \|_{H^2} + C(T).
\]

Similarly, we have
\[
\| H \|_{H^2} \leq C(\| u \|_2 + \| H_t \|_2 + \| \nabla_x u \|_2 + \| u \|_{H^2} + \| \nabla_x H \|_2) \\
\leq \frac{1}{2} \| u \|_{H^2} + \frac{1}{4} \| H \|_{H^2} + C(T).
\]

Combine the above together, we get
\[
\| u \|_{H^2} + \| H \|_{H^2} \leq C(T).
\]

By the Sobolev inequalities, we use above estimate to have
\[
\sup_{0 \leq t \leq T} \| u \|_\infty + \| H \|_\infty \leq C(T), \\
\sup_{0 \leq t \leq T} \| (\nabla_x u, \nabla_x H) \|_p \leq C(T).
\]

Then we can further use Corollary 2, Lemma 3.1, 3.2 to obtain
\[
\| \nabla_x u \|_2^2 + \| \nabla_x H \|_2^2 \leq C(T)(\| \nabla_x u \|_2^2 + \| \nabla_x H \|_2^2), \\
\| (u_t, H_t) \|_{H^1} \leq \| u_t \|_{H^1} + \| u \cdot \nabla_x u \|_{H^1} + \| H_t \|_2 + \| \nabla_x H \|_2,
\]

and
\[
\| \sqrt{\rho} u_t \|_2^2(t) + \int_0^t \| (u_t, H_t) \|_{H^1}^2 d\tau \leq C(T).
\]

On the other hand, we apply the operator \( \nabla_x^2 \) to the continuity equation (1)_2 and obtain
\[
\frac{d}{dt} \| \nabla_x^2 \rho \|_2^2 \leq C(T)(\| \nabla_x u \|_\infty + 1) \| \nabla_x^2 \rho \|_2^2 + \| \nabla_x u \|_2^2 + 1).
\]

Similarly, we have
\[
\frac{d}{dt} \| \nabla_x^2 P(\rho) \|_2^2 \leq C(T)(\| \nabla_x u \|_\infty + 1) \| \nabla_x^2 P(\rho) \|_2^2 + \| \nabla_x^3 u \|_2^2 + 1).
\]
Similarly, then we use (32) and Lemma 3.2 to have
\[
\begin{align*}
\| \int_{\mathbb{R}^2} |v|^i \nabla v \mathrm{d}v \|_{p_1}^{p_1} & \leq C \left( \int_{\mathbb{R}^2} \langle v \rangle^{-b \frac{p_1}{p_1 - 1}} \mathrm{d}v \right)^{\frac{p_1 - 1}{p_1}} \int_{\mathbb{R}^2} \langle v \rangle^{(b + i)p_1} |\nabla v|^{p_1} \mathrm{d}v \leq C(T),
\end{align*}
\]
by Lemma 3.3 for \( i = 0, 1, 2, 3 \) and suitably large \( k \).

We set \( m_k \nabla_v f := \int_{\mathbb{R}^2} |v|^k |\nabla_v f| \mathrm{d}v \). Then, by standard elliptic estimates, we use (32), (50) and Lemma 3.2 to have
\[
\begin{align*}
\| \nabla_v u \|_2 & \leq C(\| \nabla_v \nabla u \|_2 + \| \nabla_v \omega \|_2) \\
& \leq C(T) \left( \| \nabla_v F \|_2 + \left\| \nabla_v \left( \frac{H^2}{2} \right) \right\|_2 + \| \nabla_v \omega \|_2 \\
& \quad + \| \nabla_v P(\rho) \|_2 + \| \nabla v \nabla \cdot \nabla \rho \|_2 + \| \nabla \nabla v \cdot \nabla \rho \|_2 \right) \\
& \leq C(T)(\| \nabla_v (\rho \dot{u}) \|_2 + \| u \|_{\infty} \| m_0 \nabla_v f \|_2 + \| \nabla_v u \|_2 \| \rho \|_6 + \| m_1 \nabla_x f \|_2 \\
& \quad + \| \nabla_v \rho \|_4 \| \nabla_v u \|_2^\frac{1}{2} \| \nabla_x u \|_2 \| \nabla^2 \rho \|_2 + \| \nabla_x H \|_4 + \| \nabla^2 H \|_2) \\
& \leq C(T)(\| \nabla_v \|_4 \| \dot{u} \|_4 + \| \rho \|_{\infty} \| \nabla_v \dot{u} \|_2 + \| \nabla_v u \|_{\infty} \| \nabla_v \|_2 + 1) \\
& \leq \frac{1}{2} \| \nabla_v u \|_2 + C(T)(\| \nabla_v u \|_{\infty} \| \nabla_v \rho \|_2 + \| \nabla^2 \rho \|_2 + \| \nabla_v \dot{u} \|_2 + 1).
\end{align*}
\]
Similarly,
\[
\| \nabla_v^2 H \|_2 \leq C(\| \nabla_v H \|_2 + \| \nabla_v u \|_4 \| \nabla_x H \|_4 + \| u \|_{\infty} \| \nabla_v^2 H \|_2 + \| H \|_{\infty} \| \nabla_v^2 u \|_2) \leq C(T)(\| \nabla_v H \|_2 + 1).
\]
We combine (48), (49) and (51) to obtain
\[
\frac{d}{dt} \| \nabla_v \rho, \nabla_v^2 P(\rho) \|_2^2 \leq C([\| \nabla_v u \|_{\infty}^2 + 1])(\| \nabla_v \rho, \nabla_v^2 P(\rho) \|_2^2 + \| \nabla_v \dot{u} \|_2^2 + 1). \tag{53}
\]
We apply the Gronwall inequality and use Lemma 3.1, 3.2 to have
\[
\sup_{0 \leq t \leq T} \| \nabla_v \rho, \nabla_v^2 P(\rho) \|_2^2 + \int_0^t \| (u, H) \|_{H^2}^2 \mathrm{d}t \leq C(T). \tag{54}
\]
By the continuity equation (1)\textsubscript{2}, we have
\[
P(\rho)_t = -u \cdot \nabla_v P(\rho) - \gamma P(\rho) \nabla v, \quad \lambda(\rho)_t = -u \cdot \nabla_v \lambda(\rho) - \beta \lambda(\rho) \nabla v. \tag{55}
\]
Then we use (32) and Lemma 3.2 to have
\[
\| (\rho_0, P(\rho)_t, \lambda(\rho)_t) \|_{H^1} \leq C(\| \rho \|_{\infty}, \| u \|_{\infty})(\| \nabla_v \rho \|_2 + \| \nabla_v u \|_2) \\
\quad + C(\| \rho \|_{\infty}, \| u \|_{\infty})(\| \nabla_v \rho \|_4 \| \nabla_v u \|_4 + \| \nabla^2 \rho \|_2 + \| \nabla_v^2 u \|_2) \leq C(T). \tag{56}
\]
Similarly, we apply the operator $\partial_t$ to (55) to obtain
\[
\int_0^t \left( \|\rho u_t, P(\rho)_{tt}, \lambda(\rho)_{tt}\|_2^2 \right) d\tau \\
\leq C(T) \int_0^t \left( \|u_t\|_2^2 \|\nabla_x \rho\|_2^2 + \|u_t\|_\infty^2 \|\nabla \rho\|_2^2 + \|\rho_t\|_2^2 \|\nabla_x u\|_2^2 + \|\nabla_x u_t\|_2^2 \right) d\tau \\
\leq C(T) \int_0^t \left( \|u_t\|_H^2 + 1 \right) d\tau \leq C(T).
\]
(57)

3.2. Second order estimates. In this subsection, we derive the second order estimates of the classical solution $[f, \rho, u, H]$ to the system (1)-(2): $\|\nabla_x^2 (\rho, u)\|_p$ and $\|\langle u \rangle^k \nabla_x^2 f\|_p$ for $4 < p < +\infty$.

**Lemma 3.5.** Suppose that the conditions in Lemma 2.1 hold. Then, for $4 < p < \infty$ we have
\[
\left( t \|\nabla_x u_t, \nabla_x \dot{u}, H_t\|^2_2 + t\|\sqrt{\rho} u_t\|^2_2 + \|\rho, P(\rho)\|_{W^{2,p}} \right)(t) \\
+ \int_0^t \left( \tau \|\sqrt{\rho} u_t\|^2_2 + \|H_t\|^2_2 + \|\sqrt{\rho} f\|^2_2 + \|(u_t, H_t)\|_{H2} + \|\nabla_x u\|_{W^{2,p}} \right) d\tau \leq C(T).
\]
(58)

**Proof.** We apply the operator $\partial_t$ to (1) to obtain
\[
\rho u_{tt} + \rho u \cdot \nabla_x u_t + \nabla_x P(\rho)_t \\
= \mu \Delta_x u_t + \nabla_x ((\mu + \lambda(\rho)) \text{div} u_t) - \rho u_t - \rho u \cdot \nabla_x u - \rho u_t \cdot \nabla_x u \\
+ \nabla_x (\lambda(\rho) \text{div} u_t) - u_t \rho_f - u(\rho_f)_t + (u_f \rho_f)_t \\
+ (H_t \cdot \nabla) H + (H \cdot \nabla_x) H_t - \nabla_x (H \cdot H_t).
\]
(59)

We multiply the above equation by $u_t$, integrate the resulting equation over $\mathbb{R}^2$ and use the integration by parts to obtain
\[
\|\sqrt{\rho} u_t\|^2_2 + \frac{d}{dt} \int_{\mathbb{T}^2} \left( \frac{1}{2} u_t^2 \rho_f + u \cdot u_t (\rho_f)_t - u_t \cdot (u_f \rho_f)_t \right) dx \\
+ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^2} (\mu \nabla_x u_t)^2 + (\mu + \lambda(\rho)) |\text{div} u_t|^2 \\
+ 2\lambda(\rho) \text{div} u_t \text{div} u_t - 2H \cdot H_t \text{div} u_t) dx \\
= \int_{\mathbb{T}^2} \left( \frac{3}{2} \lambda(\rho)_t |\text{div} u_t|^2 + \lambda(\rho)_{tt} \text{div} u_t \right) dx \\
- \int_{\mathbb{T}^2} (\nabla_x P(\rho)_t + \rho u_{tt} + \rho u \cdot \nabla_x u + \rho u_t \cdot \nabla_x u + \rho u_t \cdot \nabla_x u) \cdot u_t dx \\
+ \int_{\mathbb{T}^2} \left( \frac{3}{2} (u_t^2) (\rho_f)_t + u \cdot u_t (\rho_f)_t - u_t \cdot (u_f \rho_f)_t \right) dx \\
+ \int_{\mathbb{T}^2} [(H_t \cdot \nabla) H + (H \cdot \nabla_x) H_t] \cdot u_t dx \\
- \int_{\mathbb{T}^2} (\|H_t\|^2_2 \text{div} u_t + H \cdot H_{tt}) \text{div} u_t \cdot u_t dx.
\]
(60)
As the corresponding estimates in Lemma 3.12 [30], the first term, the second term, the fourth term and the last term in the RHS of (60) can be estimated as follows:

\[
\int_{\mathbb{T}^2} \left( \frac{3}{2} \lambda(\rho) t |\text{div} u_t|^2 + \lambda(\rho) t |\text{div} u_t| \text{div} u_t \right) dx \\
\leq \frac{1}{8} \| \sqrt{\rho} u_t \|_2^2 + C(T)(\| \nabla_x u_t \|_{\infty}^2 + 1) \| \nabla_x u_t \|_2^2 + C(T)(\| \nabla_x u_t \|_2^2 + \| u_t \|_2^2 + \| \nabla_x u_t \|_{\infty}^2 + \| \nabla_x^2 u_t \|_2^2 + 1),
\]

\[
- \int_{\mathbb{T}^2} (\nabla_x P(\rho) t + \rho_t u_t + \rho u \cdot \nabla_x u + \rho u \cdot \nabla_x u_t + \rho u_t \cdot \nabla_x u) \cdot u_t dx \\
\leq \frac{d}{dt} \int_{\mathbb{T}^2} P(\rho) t |u_t|^2 dx + \frac{1}{8} \| \sqrt{\rho} u_t \|_2^2 \\
+ C(T)(\| \rho_t \|_2^2 + \| P(\rho) t \|_2^2 + \| u_t \|_2^2 + \| \nabla_x u_t \|_2^2 + \| \nabla_x u \|_{\infty}^2 + \| \nabla_x^2 u_t \|_2^2 + 1),
\]

\[
\int_{\mathbb{T}^2} [(H_t \cdot \nabla_x) H + (H \cdot \nabla_x) H_t] \cdot u_t dx \\
- \int_{\mathbb{T}^2} (|H_t|^2 |\text{div} u_t + H \cdot H_t|) \text{div} u_t \cdot u_t dx \\
\leq \frac{d}{dt} \int_{\mathbb{T}^2} H \cdot \nabla_x H_t \cdot u_t dx + \frac{1}{8} \| H_{tt} \|_2^2 + C(T)(\| \nabla_x u_t \|_2^2 + \| \nabla_x H_t \|_2^2).
\]

Now, we turn to treat the third term on the RHS of (60). Rewrite this term as

\[
\int_{\mathbb{T}^2} \left( \frac{3}{2} (u_t)^2 (\rho f) t + u \cdot u_t (\rho f) t - u_t \cdot (u f \rho f) t \right) dx \\
= \int_{\mathbb{T}^2} \frac{3}{2} (u_t) |(\rho f) t|^2 dx + \int_{\mathbb{T}^2} u \cdot u_t (\rho f) t dx - \int_{\mathbb{T}^2} u_t \cdot (u f \rho f) t dx \\
:= \sum_{i=1}^3 \mathcal{I}_7i.
\]

Before the estimation of \( \mathcal{I}_7i \), we first note that

\[
(\rho f) t = - \int_{\mathbb{R}^2} v \cdot \nabla_x f dv,
\]
\[
(\rho f) tt = \int_{\mathbb{R}^2} (v_i v_j \partial^2_{x_i x_j} f - \partial_{x_i} ((u_i - v_i) f)) dv,
\]
\[
(u f \rho f) t = \int_{\mathbb{R}^2} (-vv \cdot \nabla_x f + (u - v) f) dv,
\]
\[
(u f \rho f) tt = \int_{\mathbb{R}^2} \left[ vv_i (v_j \partial^2_{x_i x_j} f + \partial^2_{x_i x_j} ((u_j - v_j) f) - \partial_{x_i} (\Delta_x f)) \right. \\
- (u - v)(v \cdot \nabla_x f + \nabla_v \cdot ((u - v) f)) \right] dv + u_t \rho f.
\]

Applying operator \( \partial_t \) with (1.3), we have

\[
\mu \Delta u_t + \nabla_x ((\mu + \lambda(\rho)) \text{div} u_t) = (\rho \dot{u}) t + \nabla_x R(\rho) t - \nabla_x (\lambda(\rho) \text{div} u) + [(u - u f) \rho f] t, \\
- (H_t \cdot \nabla_x) H - (H \cdot \nabla_x) H_t + \nabla_x (H H_t),
\]
moreover,

\[
\|\nabla_x^2 u_t\|_2 \leq C(\|\sqrt{\rho_t}\|_2 + \|\sqrt{\rho_t u_t}\|_2 + \|\rho_t\|_4\|u_t\|_4 + \|\rho_t\|_4\|u\|_\infty\|\nabla_x u_1\|_4 + \|\nabla_x P(\rho)\|_2 + \|\nabla_x (\rho \nabla_x \rho)\|_2 + \|\Delta_{\nabla_x} \rho\|_2 + \|H_t\|_4 \|\nabla_x H_t\|_4 + \|u_t\|_4 \|\rho_f\|_4 + \|u_f\|_2 \|\rho_f\|_2 + (\|u_1\|_\infty + 1)(\|\rho_f\|_2 + |m_1|_{\nabla_x f\|_2} + |m_2|_{\nabla_x f\|_2})
\]

\[
\leq C(T)(\|\sqrt{\rho_t u_t}\|_2 + \|u_t\|_4 + \|u_t\|_2\|\nabla_x u_t\|_2 + \|\nabla_x^3 u_2\|_2 + 1),
\]

by the estimates in Lemma 3.4.

- (Estimate of \(I_{71}\)): By the integration by parts, we use (ii) in Corollary (1), (63) to have

\[
I_{71} = \int_{T_2} 3u_t \cdot \nabla_x u_t \cdot u_f \rho_f dx \leq C\|u_t\|_4 \|\nabla_x u_t\|_2 \|u_f\|_4 \leq \|u_t\|_4^2 + C(T)\|\nabla_x u_t\|_2^2.
\]

- (Estimate of \(I_{72}\)): We apply the integration by parts and use (ii) in Corollary (1), (63) and (65) to get

\[
I_{72} = \int_{T_2} \int_{\mathbb{R}^2} u \cdot u_t (v_1 v_j \partial_{x_i}^2 f - \partial_{x_i} ((u_t - v_1) f) \leq (\|\nabla_x^2 u_t\|_2 \|u_t\|_4 + \|\nabla_x u_1\|_4 \|\nabla_x u_t\|_2 + \|u\|_\infty \|\nabla_x u_1\|_2 \|\nabla_x^2 u_t\|_2) \|m_2 f\|_2
\]

\[
+ C(T)(\|\nabla_x u_1\|_2 \|u_t\|_4 + \|\nabla_x u_1\|_2 \|u_4\|)(\|u_f\|_4 + \|\rho_f\|_4)
\]

\[
+ C(T)(\|u_t\|_\infty \|\nabla_x u_t\|_2 \|\rho_f\|_2 + C\|\nabla_x u_1\|_4 \|\rho_f\|_4
\]

\[
\leq \frac{1}{16} \|\sqrt{\rho_t u_t}\|_2^2
\]

\[
+ C(T)(\|u_t\|_4^2 + \|\nabla_x^3 u_2\|_2^2 + \|\nabla_x u_2\|_\infty^2 + 1)(\|\nabla_x u_t\|_4^2 + \|\nabla_x H_t\|_2^2 + 1).
\]

- (Estimate of \(I_{73}\)): We rewrite \(I_{73}\) as

\[
I_{73} = -\int_{T_2} \int_{\mathbb{R}^2} u_t (v v_j v_j \partial_{x_i}^2 f + v v_j \partial_{x_i} \rho_f ((u_j - v_j) f) v v_j \partial_{x_i} ((u_t - v_1) f) - v v_j \partial_{x_i} ((u_t - v_1) f)
\]

\[
- \int_{T_2} \int_{\mathbb{R}^2} u_t [-\partial_x (u - v) (v \cdot \nabla_x f + \nabla_v \cdot ((u - v) f)) + u_t f] dx
\]

\[
=: I_{731} + I_{732}.
\]

We apply the integration by parts to use Corollary 1, (63) and (65) to obtain

\[
I_{731} \leq C(T) \|\nabla_x^2 u_t\|_2 \|m_3 f\|_2
\]

\[
+ C(T) \|\nabla_x u_1\|_4 \|\rho_f\|_2 + \|u_f\|_4 \|\rho_f\|_4 + \|m_2 f\|_2
\]

\[
\leq \frac{1}{16} \|\sqrt{\rho_t u_t}\|_2^2 + C(T) \|u_t\|_4^2 + \|\nabla_x^3 u_2\|_2^2
\]

\[
+ (\|\nabla_x u_1\|_\infty^2 + 1)(\|\nabla_x u_t\|_4^2 + \|\nabla_x H_t\|_2^2 + 1)
\]

\[
I_{732} \leq - \frac{1}{16} \|\sqrt{\rho_t u_t}\|_2^2 + C(T) \|\nabla_x u_1\|_2 \|u_f\|_4 \|\rho_f\|_2 + \|u_4\| \|\rho_f\|_4 + \|m_2 f\|_2
\]

\[
+ C(T) \|u_t\|_4 \|\nabla_x u_t\|_2 \|u_f\|_4 \|\rho_f\|_2 + \|u_f\|_4 \|\rho_f\|_2 + \|u_f\|_4 \|\rho_f\|_4 + \|m_2 f\|_4
\]

\[
\leq C(T)(\|\nabla_x u_t\|_2 + \|u_t\|_4).
\]
Collecting above estimates of $I_{731}, I_{732}$ in (68) gives

\[ I_{73} \leq \frac{1}{16} \| \sqrt{\rho} u_t \|_2^2 + C(T) \| u_t \|_4^2 + \| \nabla_x^3 u \|_2^2 + (\| \nabla_x u \|_\infty^2 + 1)(\| \nabla_x u_t, \nabla_x H_t \|_2^2 + 1)]. \]  

(70)

We combine the estimates of $I_{73}(1 \leq i \leq 3)$ and (62) to have

\[
\int_{T^2} \left( \frac{3}{2} (u_t)^2 (\rho_f)_t + u \cdot u_t (\rho_f)_t - u_t \cdot (u f \rho_f)_t \right) dx \\
\leq \frac{1}{8} \| \sqrt{\rho} u_t \|_2^2 + C(T) \| u_t \|_4^2 + \| \nabla_x^3 u \|_2^2 + (\| \nabla_x u \|_\infty^2 + 1)(\| \nabla_x u_t, \nabla_x H_t \|_2^2 + 1)].
\]  

(71)

Note that the standard $L^2$–estimates for elliptic system gives that

\[
\| H_t \|_2 \leq -\nu \frac{d}{dt} \int_{T^2} | \nabla_x H_t |^2 + \| H_t \|_4^2 \| \nabla x u \|_4 \| H_t \|_2 + \| u_t \|_4 \| \nabla x H_t \|_4 \| H_t \|_2 \\
+ \| u_t \|_\infty \| \nabla_x H_t \|_2 \| H_t \|_2 + \| H \|_\infty \| \nabla x u_t \|_2 \| H_t \|_2 \\
\leq -\nu \frac{d}{dt} \int_{T^2} | \nabla_x H_t |^2 + \frac{1}{8} \| H_t \|_2^2 + C(T)(\| \nabla_x H_t \|_2^2 + \| \nabla x u_t \|_2^2).
\]  

(72)

With the estimates (61), (71) and (72) in hands, we can further estimate (60) as

\[
\frac{1}{2} \| \sqrt{\rho} u_t \|_2^2 + \frac{1}{2} \| H_t \|_2^2 + \frac{1}{2} \| \sqrt{\rho} u_t \|_2^2 + \frac{d}{dt} \Pi(t) \\
\leq C(T)(\| (\rho_t, P(\rho)_t, \lambda(\rho)_t) \|_2^2 + \| u_t \|_4^2 + \| \nabla_x^3 u \|_2^2 \\
+ (\| \nabla x u \|_\infty^2 + 1)(\| \nabla x u_t, \nabla x H_t \|_2^2 + 1)],
\]  

(73)

where

\[\Pi(t) = \int_{T^2} \left( \mu | \nabla x u_t |^2 + \nu | \nabla x H_t |^2 + (\mu + \lambda(\rho)) | \text{div} u_t |^2 \\
+ \lambda(\rho) | \text{div} \text{div} u_t - P(\rho)_t \text{div} u_t + \rho_t (u_t)^2 + \rho_t u \cdot \nabla x u \cdot u_t \right) dx \\
+ \int_{T^2} \left( \frac{(u_t)^2}{2} \rho_f + u \cdot u_t (\rho_f)_t - u_t \cdot (u f \rho_f)_t \right) dx \\
- \int_{T^2} ((H_t \cdot \nabla x) H \cdot u_t + (H \cdot \nabla x) H_t \cdot u_t + H \cdot H_t \text{div} u_t) dx.\]  

(74)

Note that

\[
\left| \int_{T^2} (\lambda(\rho)_t \text{div} \text{div} u_t - P(\rho)_t \text{div} u_t) dx \right| \\
\leq \frac{\mu}{8} \| \nabla x u_t \|_2^2 + C(T)(\| \lambda(\rho)_t \|_4^2 \| \text{div} u_t \|_4^2 + \| P(\rho)_t \|_2^2) \\
\leq \frac{\mu}{8} \| \nabla x u_t \|_2^2 + C(T),
\]

\[
\left| \int_{T^2} (\rho_t (u_t)^2 + \rho_t u \cdot \nabla x u \cdot u_t) dx \right| = \left| \int_{T^2} \rho u (\nabla x u_t \cdot u_t + \nabla x (u \cdot \nabla x u \cdot u_t)) dx \right| \\
\leq \| \sqrt{\rho} u_t \|_2 \| \sqrt{\rho} u \|_\infty (\| \nabla x u_t \|_2 + \| \nabla x u \|_4^2 + \| u \|_\infty \| \nabla^2 u \|_2).
\]
To estimate $P$ \( t \to \), we multiply the above inequality by \( C(t) \),
\[
\left| \int_{T^2} (u \cdot u_t(\rho_f)) - u_t \cdot (u_f \rho_f)_t \right| dx
\]
\[
= \left| \int_{T^2} \int_{R^2} -u \cdot u_t v \cdot \nabla_x f + u_t \cdot (v v \cdot \nabla_x f - (u - v)f) dv dx \right|
\leq C(\|\nabla u_t\|_2(\|m_2t\|_2 + \|u\|_4 \|u_f \rho_f\|_4)

+ C(\|\nabla u_t\|_2(\|m_2t\|_2 + \|u\|_4 \|\rho_f\|_2^\frac{1}{3} + \|m_2t\|_2^\frac{1}{3}))
\leq \frac{\mu}{8} \|\nabla u_t\|_2^2 + C(1, \|H_t\|_2 \|\nabla u_t\|_2 \leq \frac{\mu}{8} \|\nabla u_t\|_2^2 + C(T).
\]

We can further obtain
\[
C(T)^{-1}(\|\nabla u_t\|_2^2 + \|\nabla H_t\|_2^2) + \|\nabla f u_t\|_2^2 - 1)
\leq \Pi(t) \leq C(T)(\|\nabla u_t\|_2^2 + \|\nabla H_t\|_2^2 + \|\nabla f u_t\|_2^2 + 1),
\]
\[
\frac{1}{2} \|\nabla u_t\|_2^2 + \frac{1}{2} \|\nabla f u_t\|_2^2 + \|H_t\|_2^2 + \frac{d}{dt} \Pi(t)
\leq C(T)(\|\rho_t, P(\rho)_{tt}, \lambda(\rho)_{tt})\|_2^2 + \|u_t\|_4^4

+ \|\nabla u_t\|_2^2 + (\|\nabla f u_t\|_2^2 + 1)(\Pi(t) + 1)).
\]

We multiply the above inequality by \( t \), integrate the resulting inequality with respect to \( t \) over the interval \([\tau, t_1] \) with \( \tau, t_1 \in [0, T] \), and use the similar arguments as in Lemma 3.12 [30] to obtain
\[
t\|\nabla u_t\|_2^2 \leq t\|\nabla u_t\|_2^2 + t\|u\|_\infty^2 \|\nabla^2 u_t\|_2^2 + t\|\nabla u\|_4^4 \leq C(T),
\]
\[
(t\|\nabla u_t, \sqrt{\rho} u_t\|_2^2

+ \|\nabla H_t\|_2^2) + \int_{0}^{t} \tau(\|\nabla u_t\|_2^2 + \|\nabla f u_t\|_2^2 + \frac{1}{2} \|H_t\|_2^2) d\tau \leq C(T),
\]
\[
t\|\rho_t, P_{tt}, \lambda_{tt}\|_2^2 + \int_{0}^{t} \tau(\|\nabla^2 u_t\|_2^2 + \|\nabla^2 H_t\|_2^2) d\tau \leq C(T), \quad t \in [0, T].
\]

On the other hand, we apply operator \( \nabla^2 u_t \) to the continuity equation (1) \( 1 \) and obtain
\[
\frac{d}{dt} \|\nabla^2 u\|_p \leq C(T)(\|\nabla u\|_\infty \|\nabla^2 \rho\|_p + \|\nabla^2 u\|_{W^{1,p}}).
\]

For \( P(\rho) \), we also have
\[
\frac{d}{dt} \|\nabla^2 P(\rho)\|_p \leq C(T)(\|\nabla u\|_\infty \|\nabla^2 P(\rho)\|_p + \|\nabla^2 u\|_{W^{1,p}}).
\]

To estimate \( \|\nabla^2 u\|_{W^{1,p}} \) on the RHS of (78) and (79), we apply \( \partial_x \) to (1) \( 3 \) and obtain
Then, by Gronwall’s lemma, we have

$$\mu \Delta_x (\partial_x u) + \nabla_x ((\mu + \lambda(\rho)) \text{div} (\partial_x u))$$

$$= \nabla_x (\partial_x \lambda(\rho) \text{div} u) + u_t \partial_x \rho + \rho \partial_x u_t$$

$$+ \rho \partial_x u \cdot \partial_x u + \partial_x u \cdot \nabla_x u$$

$$+ \nabla_x \rho \cdot \partial_x u + \nabla_x \partial_x \rho \cdot \rho \partial_x u + \rho \partial_x \rho$$

$$+ u \partial_x (u \rho) - \partial_x (u \rho_f)$$

$$+ \partial_x (\rho) (\nabla_x H) + \nabla_x (H \cdot \partial_x H).$$

Then, we use (50) to have

$$\text{Lemma 3.6.}$$

$$f$$

mates of

$$388 \quad \text{BINGKANG HUANG AND LAN ZHANG}$$

Similarly,

$$\|\nabla_x u\|_{W^2,p} \leq C(T) \left( \|\nabla_x u\|_{\infty} + 1 \right) \left( \|\nabla_x^2 \rho, \nabla_x^2 P(\rho)\|_{p} \right)$$

$$+ \|\nabla_x u\|_{W^1,p} \|\nabla_x u\|_{p}$$

$$+ \|\partial_x, \rho_f\|_{p} + \|\partial_x (u_f \rho_f)\|_{p}$$

$$+ \|\nabla_x H\|_{W^1,p} + \|\nabla_x^2 H\|_{p} + 1$$

$$\leq C(T) \left( \|\nabla_x u\|_{\infty} + 1 \right) \left( \|\nabla_x^2 \rho, \nabla_x^2 P(\rho)\|_{p} \right)$$

$$+ \|(u, H)\|_{H^3} + \|\nabla_x u\|_{p} + \|\nabla_x^2 H\|_{p} + 1.$$
Proof. We apply $\partial_x^{\alpha} \cdot (|\alpha| = 2)$ to the (1), and multiply the above equation by

$$<(v)^{kp}\partial_x^{\alpha} \cdot f|^{p-2}\partial_x^{\alpha} \cdot f,$$

and integrate the resulted equations with respect to $x$, $v$ over $\mathbb{T}^2 \times \mathbb{R}^2$ to obtain

$$\frac{d}{dt}\|<(v)^{kp}\partial_x^{\alpha} \cdot f|^{p}_p \|
\leq -\int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (v)^{kp}\partial_x^{\alpha} \cdot f|^{p-2}\partial_x^{\alpha} \cdot f| \nabla_v \cdot ((u-v)\partial_x^{\alpha} \cdot f) d v d x
- 2 \sum_{|\alpha'| = 1} \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (v)^{kp}\partial_x^{\alpha} \cdot f|^{p-2}\partial_x^{\alpha} \cdot f| \nabla_v \cdot (\partial_x^{\alpha'} u \partial_x^{\alpha} f - f) d v d x
- \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (v)^{kp}\partial_x^{\alpha} \cdot f|^{p-2}\partial_x^{\alpha} \cdot f| \nabla_v \cdot (\partial_x^{\alpha} u f) d v d x
+ \int_{\mathbb{T}^2} \int_{\mathbb{R}^2} (v)^{kp}\partial_x^{\alpha} \cdot f|^{p-2}\partial_x^{\alpha} \cdot f| \partial_x^{\alpha} \Delta_x f d v d x
\tag{85}
:= \sum_{i=1}^4 I_{8i}.
$$

In the sequel, we estimate the terms $I_{8i}$ separately. We use integration by parts to have

$$|I_{81}| \leq C(p)\|u\|_\infty\|<(v)^{kp}\partial_x^{\alpha} \cdot f|^{p}_p \| \leq C(T)\|<(v)^{kp}\partial_x^{\alpha} \cdot f|^{p}_p \|,$$

$$|I_{82}| + |I_{83}| \leq C(T)(\|\nabla_x u\|_\infty + \|\nabla_x^2 u\|_\infty + \|<(v)^{kp}\partial_x^{\alpha} \cdot f|^{p-1}_p \|
\leq C(T)\|\nabla_x u\|_{W^{2,p}}\|<(v)^{kp}\partial_x^{\alpha} \cdot f|^{p}_p \|$$

$$+ \|<(v)^{kp}\nabla_v f|^{p}_p \| + \|<(v)^{kp}\partial_x^{\alpha} \cdot f|^{p}_p \| + \|<(v)^{kp}\nabla_v^2 f|^{p}_p \|. \tag{86}$$

Similar to the estimate of $I_{83}$, we have

$$I_{84} \leq -p(p-1)\|<(v)^{kp}\nabla_v (\nabla_x^2 f)\|^{2}_p + C(p)\|<(v)^{kp}\partial_x^{\alpha} \cdot f|^{p}_p \|.$$  

We collect the estimates of $I_{8i}$ in (85) to have

$$\frac{d}{dt}\|<(v)^{kp}\partial_x^{\alpha} \cdot f|^{p}_p \| + p(p-1)\|v - v_c\|\|<(v)^{kp}\nabla_v (\nabla_x^2 f)\|^{2}_p \leq C(T,p)(1 + \|\nabla_x^3 u\|_2^2)\|<(v)^{kp}\partial_x^{\alpha} \cdot f|^{p}_p \|_{W^{2,p}(\mathbb{T}^2 \times \mathbb{R}^2)}. \tag{87}$$

Similarly, we can obtain

$$\frac{d}{dt}\|<(v)^{kp}\nabla_x \nabla_v f\|^{p}_p + p(p-1)\|<(v)^{kp}\nabla_x \nabla_x^2 f\|^{2}_p \leq C(T,p)\|<(v)^{kp}\partial_x^{\alpha} \cdot f|^{p}_p \|_{W^{2,p}(\mathbb{T}^2 \times \mathbb{R}^2)}; \tag{88}$$

We combine (87) and (88), use Lemma 3.3 to have

$$\frac{d}{dt}\left(\sum_{|\alpha| + |\beta| = 2} \|<(v)^{kp}\partial_x^{\alpha} \cdot f|^{p}_p \| \right)
\leq C(T)(1 + \|\nabla_x u\|_{W^{2,p}}^2)(1 + \sum_{|\alpha| + |\beta| = 2} \|<(v)^{kp}\partial_x^{\alpha} \cdot f|^{p}_p \|).$$
We further apply the Gronwall inequality and use (83) to have (84).

3.3. Third-order estimates. In this subsection, we derive the third order estimates of the classical solution \([f, \rho, u, H]\) to the system (1)-(2): \(\|\nabla^3_x u\|_p\) and \(\|v^k \nabla^3_x v\|_p\) for \(4 < p < \infty\). The third order estimates for \(f(t, x, v)\) can be obtained as below.

Lemma 3.7. Suppose that the conditions in Lemma 2.1 hold. Then, we have

\[
\sum_{|\alpha_*| + |\beta_*|=3} \|v^k \partial_{\alpha_*}^* f\|_p(t) \leq C(T), \quad 4 < p < \infty. \tag{89}
\]

Proof. We apply \(\partial^\alpha_{\beta_*}\) to (1) with \(|\alpha_*| + |\beta_*|=3\), multiply the equation by \(v^k p |\partial^\alpha_{\beta_*} f|^2 \partial^\alpha_{\beta_*} f\), integrate over \(\mathbb{T}^2 \times \mathbb{R}^2\), sum up \(\alpha_*\), \(\beta_*\) and use the same method in Lemma 3.6 to obtain

\[
\frac{d}{dt} \left( \sum_{|\alpha_*| + |\beta_*|=3} \|v^k \partial_{\alpha_*}^* f\|_p \right) \\
\leq C(T, p)(1 + \|\nabla u\|_{L^2}) \left(1 + \sum_{|\alpha_*| + |\beta_*|=3} \|v^k \partial_{\alpha_*}^* f\|_p \right).
\]

By Gronwall’s lemma, we use (83) to have (89).

In order to obtain the estimate \(t^2 \|\nabla^3_x u(t)\|_p\), \((t > 0)\), we first show the following estimate.

Lemma 3.8. Suppose that the conditions in Lemma 2.1 hold. Then, we have

\[
t^2 \left( \|\sqrt{\rho} u_t\|_2^2 + \|H_{tt}\|_2^2 + \|(u_t, H_t)\|_{H^1}^2 + \|(u, H)\|_{W^{3,2}}^2 \right) \\
+ \int_0^t t^2 \left( \|\nabla_x u_{tt}\|_2^2 + \|\nabla_x H_{tt}\|_2^2 + \|\sqrt{\rho} u_{tt}\|_2^2 \right) dt \leq C(T).
\]

Proof. We apply the operator \(\partial_{tt}\) to (1) to obtain

\[
\rho u_{tt} + \rho u \cdot \nabla_x u_{tt} - \mu \Delta_x u_{tt} - \nabla_x((\mu + \lambda(\rho))\text{div} u_{tt}) \\
= -\nabla_x P(\rho)_{tt} - \rho_{tt}(u_t + u \cdot \nabla_x u) \\
- 2\rho_t(u_t + u \cdot \nabla_x u + u \cdot \nabla_x u_t) \\
- 2\rho_{tt} \cdot \nabla_x u_t - \rho u_t \cdot \nabla_x u + 2\nabla_x(\lambda(\rho)\text{div} u_{tt}) \\
+ \nabla_x(\lambda(\rho)\text{div} u - u_{tt} \rho_f - 2u_t(\rho_f)_{tt}) - u(\rho_f)_{tt} + (u_{tt}, \rho_f)_{tt} \\
+ (H_{tt} \cdot \nabla_x) H + 2(H_t \cdot \nabla_x) H_t + (H \cdot \nabla_x) H_{tt} \\
- \nabla_x(H_{tt}^2 + H \cdot H_{tt}).
\]

We multiply the above equation by \(t^2 u_{tt}\) and integrate the resulted equation with respect to \(x\) over \(\mathbb{T}^2\) to obtain

\[
\frac{1}{2} \frac{d}{dt} \left( t^2 \int_{\mathbb{T}^2} \rho |u_{tt}|^2 dx \right) - t \int_{\mathbb{T}^2} \rho |u_{tt}|^2 dx \\
+ t^2 \int_{\mathbb{T}^2} (\mu |\nabla_x u_{tt}|^2 + (\mu + \lambda(\rho))(\text{div} u_{tt})^2) dx \\
+ t^2 \int_{\mathbb{T}^2} \rho_f |u_{tt}|^2 dx
\]

\[
= t^2 \int_{\mathbb{T}^2} P(\rho)_{tt} \text{div} u_{tt} dx - t^2 \int_{\mathbb{T}^2} \rho_{tt}(u_t + u \cdot \nabla_x u) \cdot u_{tt} dx
\]
In the sequel, we will estimate the terms

\[-2t^2 \int_{T^2} \rho_t(u_{tt} + u_t \cdot \nabla_x u + u \cdot \nabla_x u_t) \cdot u_{tt} \, dx\]

\[-2t^2 \int_{T^2} \rho u_t \cdot \nabla_x u_t \cdot u_{tt} \, dx - t^2 \int_{T^2} \rho u_t \cdot \nabla_x u \cdot u_{tt} \, dx\]

\[-2t^2 \int_{T^2} \lambda(p_t) \text{div} u_t \text{div} u_{tt} \, dx - t^2 \int_{T^2} \lambda(p_t) \text{div} u \text{div} u_{tt} \, dx\]

\[-t^2 \int_{T^2} u \cdot u_{tt}(\rho_f)_{tt} \, dx - 2t^2 \int_{T^2} u_t \cdot u_{tt}(\rho_f)_t \, dx\]

\[+ t^2 \int_{T^2} u_t \cdot (u_f \rho_f)_{tt} \, dx\]

\[-t^2 \int_{T^2} (H_t \cdot \nabla_x) u_{tt} \cdot H + 2(H_t \cdot \nabla_x) u_{tt} \cdot H_t + (H \cdot \nabla_x) u_{tt} \cdot H_{tt} \, dx\]

\[+ t^2 \int_{T^2} (|H_t|^2 + H \cdot H_t) \text{div} u_{tt} \, dx\]

\[:= \sum_{i=1}^{12} I_{9i}. \tag{90}\]

In the sequel, we will estimate the terms $I_{12i}$, separately.

- (Estimates of $I_{9i}$, $1 \leq i \leq 7$): Similar to Lemma 4.6 [25], we have

\[
\sum_{i=1}^{7} |I_{9i}| \leq \frac{\mu}{8} t^2 (\|\text{div} u_{tt}\|_2^2 + \|\nabla_x u_{tt}\|_2^2) \\
+ C(T)t^2 \|\sqrt{\rho} u_{tt}\|_2^2 \left(\|\nabla_x \dot{u}\|_2^2 + \|\nabla_x u\|_\infty^2 + 1\right) \\
+ t^2 \|\nabla_x u_{tt}\|_{H_1}^2 + \|\nabla_x u\|_{H_2}^2 + \|\nabla_x u\|_{H_2}^2 + \|\nabla_x \dot{u}\|_2^2 + 1.\]

- (Estimates of $I_{98}$): We apply the integration by parts and the Sobolev inequalities to use (63) and (50) to get

\[
|I_{98}| \leq C t^2 \int_{T^2} \left(\|\nabla_x u\| \|u_{tt}\| \|m_2 \| + \|\nabla_x u\| \|\nabla_x u_{tt}\| \|m_2 \| \right) \\
+ \|u\| \|\nabla_x u_{tt}\| \|m_2 \| \nabla_x f\|_2 \\
+ \|u\| \|\nabla_x u_{tt}\| \|m_2 \| \nabla_x f\|_2 \\
+ C t^2 \|\nabla_x u\|_\infty \|\nabla_x u_{tt}\|_2 \|m_2 \| \nabla_x f\|_2 \\
+ \|u\|_\infty \|\nabla_x u_{tt}\|_2 \|\rho_f\|_2 \\
+ \|u\|_\infty \|\nabla_x u_{tt}\|_2 \|u_f \rho_f\|_2 + \|u\|_\infty \|\nabla_x u_{tt}\|_2 \|\rho_f\|_2 \\
+ C t^2 \|\sqrt{\rho} u_{tt}\|_2 \left(\|\nabla_x u\|_4 \|m_4 \|_2^\frac{1}{2} \\
+ \|\nabla_x u\|_8 \|u\|_8 \|\rho_f\|_2^\frac{1}{2} + \|\nabla_x u\|_4 \|\rho_f\|_2^\frac{1}{2} \right) \\
\leq \frac{\mu}{8} t^2 \|\nabla_x u_{tt}\|_2^2 + \frac{1}{4} t^2 \|\sqrt{\rho} u_{tt}\|_2^2 + C(T)(\|\nabla_x u\|_2^2 + 1).\]

- (Estimates of $I_{99}$ and $I_{910}$): Similar the estimate of $I_{98}$, we have
\begin{align*}
|I_{99}| & \leq Ct^2 \int_{\Om} (|\nabla_x u_t||u_{tt}| + |u_{tt}||\nabla_x u_{tt}|)|u_f\rho_f|dx \\
& \leq Ct^2\|u_t\|_{\infty}\|\nabla_x u_t\|_2\|u_f\rho_f\|_2 + Ct^2\|\sqrt{\rho_f}u_{tt}\|_2\|\nabla_x u_{tt}\|_4\|m_2f\|_2^{\frac{1}{2}} \\
& \leq \frac{H}{8}t^2\|\nabla_x u_t\|^2_2 + \frac{1}{4}t^2\|\sqrt{\rho_f}u_{tt}\|^2_2 \\
& + C(T)(\|\nabla_x u_t\|^2_{H^1} + \|u_{tt}\|^2_2 + 1),
\end{align*}

\begin{align*}
|I_{910}| & \leq Ct^2 \int_{\Om} (|\nabla_x u_t|(m_3|\nabla_x f| + |u||u_f\rho_f| + |m_2f| + |u_f\rho_f| + |\rho_f|) \\
& + |u_{tt}||(u_f\rho_f) + |\rho_f| + |u||\rho_f| + |\nabla_x u||u_f\rho_f| + |u_{tt}\rho_f|)dx \\
& \leq \frac{H}{8}t^2\|\nabla_x u_t\|^2_2 + \frac{1}{4}t^2\|\sqrt{\rho_f}u_{tt}\|^2_2 \\
& + C(T)(t^2\|\sqrt{\rho_f}u_{tt}\|^2_2 + \|\nabla_x u_t\|^2_{H^1} + 1).
\end{align*}

\textbf{• (Estimates of $I_{911}$ and $I_{912}$):}

\begin{align*}
|I_{911} + I_{912}| & \leq Ct^2 (\|H_t\|_2\|\nabla_x u_t\|_2 \\
& + \|H_t\|^2_2\|\nabla u_{tt}\|_2 + (\|H_t\|^4_2 + \|H_t\|_{L^2})\|\text{div} u\|_2) \\
& \leq \frac{1}{8}t^2\|\nabla_x u_t\|^2_2 + C(T)(t^2\|H_t\|^2_2 + 1).
\end{align*}

We collect the estimates of $I_{9i}$ in (90) to obtain

\begin{align*}
\frac{d}{dt}\left(t^2\int_{\Om} |\rho|u_{tt}|^2dx + t^2\int_{\Om} |H_t|^2dx\right) + t^2\int_{\Om} (|\nabla_x u_{tt}|^2 + |\sqrt{\rho_f}u_{tt}|^2)dx \\
& \leq C(T)t^2(\|\sqrt{\rho_u}_{tt}\|_2^2(\|\nabla_x \tilde{u}\|_2^2 + \|\nabla_x u\|_{\infty}^2 + 1) \\
& + \|\nabla_x u_{tt}\|_{H^1}^2 + \|\sqrt{\rho_f}u_{tt}\|_2^2 + \|\nabla_x u\|_{H^2}^2 + ||P(\rho)_{tt}\|_2^2 \\
& + \|\lambda(\rho)_{tt}\|_2^2\|\nabla_x u\|_{H^2}^2 + \|1 + |x|\|\nabla_x \tilde{u}\|_2^2 + C(T)).
\end{align*}

We integrate the above inequality with respect $t$ over $[\tau, t_1]$, use Lemma 3.1, 3.2 and 3.4 to have

\begin{align*}
&t^2(\|\sqrt{\rho_u}_{tt}\|_2^2 + \|H_t\|_2^2) + \int_{\tau}^{t_1} \int_{\Om} (|\nabla_x u_{tt}|^2 + |\sqrt{\rho_f}u_{tt}|^2)dt \\
& \leq C(T)(1 + \tau^2\|\sqrt{\rho_u}_{tt}\|_2^2(\tau)).
\end{align*}

Since $t\|\sqrt{\rho_u}_{tt}\|_2^2(t) \in L^1(0, T)$, there exists a subsequence $\tau_k$ such that

\begin{align*}
\tau_k & \to 0, \quad \tau_k^2\|\sqrt{\rho_u}_{tt}\|_2^2(\tau_k) \to 0, \quad \text{as} \quad k \to +\infty.
\end{align*}

Taking $\tau = \tau_k$ and $k \to +\infty$, we have

\begin{align*}
&t^2(\|\sqrt{\rho_u}_{tt}\|_2^2 + \|H_t\|_2^2) + \int_{0}^{t} \tau^2(||\nabla_x u_{tt}|^2 + |\sqrt{\rho_f}u_{tt}|^2)\,d\tau \\
& \leq C(T), \quad t \in [0, T].
\end{align*}

By (65) that

\begin{align*}
&t^2\|\nabla_x u|^2_2 \leq C(T)(t^2\|\sqrt{\rho_u}_{tt}\|_2^2 + t^2\|u_{tt}\|^2_{H^1} + t^2\|u\|^2_{H^3} + 1) \leq C(T), \quad t \in [0, T].
\end{align*}

Finally, we use (81) and (82) to obtain

\begin{align*}
&t^2\|\nabla_x u, \nabla_x H\|_{W^{2, \infty}}^2 \leq C(T)(t^2\|(u, H)\|_{H^1}^2 + t^2\|u_{tt}\|^2_{H^2} + 1) \leq C(T), \quad t \in [0, T].
\end{align*}
4. A proof of Theorem 1.1. In this section, we provide the proof of Theorem 1.1. Using linearization, Schauder fixed point theorem and borrowing apriori estimates in Section 4, can give the following local existence of classical solutions to coupled systems as in [13, 30]. We omit the details for simplicity.

**Lemma 4.1.** (Local existence.) Under the assumptions of Theorem 2.1, there exists a small \( T_\ast > 0 \) and a unique classical solution \([f, \rho, u] \) to coupled systems satisfying the regularity properties (7) with replaced by \( T_\ast \).

Firstly, we show that \([f, \rho, u, H] \) is a classical solution to (1) if \([f, \rho, u, H] \) satisfying (5).

Notice that \( u \in L^2(0, T; H^3(\mathbb{T}^2)) \) and \( u_t \in L^2(0, T; H^1(\mathbb{T}^2)) \), so we get

\[
u \in C(0, T; H^2(\mathbb{T}^2)) \rightarrow C(0, T \times \mathbb{T}^2).
\]

From \((\rho, P(\rho)) \in L^\infty(0, T; W^{2,q}(\mathbb{T}^2))\) with \( q > 2 \), \((\rho_t, P(\rho)_t) \in L^\infty(0, T; H^1(\mathbb{T}^2))\), it holds that

\[
(\rho, P(\rho)) \in C(0, T; W^{1,q}(\mathbb{T}^2)) \cap C(0, T; W^{2,q}(\mathbb{T}^2) – \text{weak}).
\]

Combining with Lemma 3.7, it implies that

\[
(\rho, P(\rho)) \in C(0, T; W^{2,q}(\mathbb{T}^2)).
\]

It follows from \( f \in L^\infty(0, T; W^{3,p}(\mathbb{T}^4)) \) with \( p > 4 \) and \( \partial_t f \in L^\infty(0, T; H^1(\mathbb{T}^4)) \) that

\[
f \in C(0, T; W^{2,p}_k(\mathbb{T}^2 \times \mathbb{R}^2)) \cap C(0, T; W^{3,p}_k(\mathbb{T}^2 \times \mathbb{R}^2)) \rightarrow C(0, T; C^{3-\frac{3}{p}}(\mathbb{T}^2 \times \mathbb{R}^2)).
\]

Together with Lemma 3.7, it implies that

\[
f \in C(0, T; W^{3,p}_k(\mathbb{T}^2 \times \mathbb{R}^2)) \rightarrow C([\tau, T] \times \mathbb{T}^2).
\]

By (1)\(_3\), it holds that

\[
\partial_t f = -v \cdot \nabla_x f - \nabla_v \cdot ((u - v)f) + \Delta_v f
\in C([0, T] \times \mathbb{T}^2 \times \mathbb{R}^2)
\]

From

\[
(\nabla_x \rho, \nabla_x P(\rho)) \in C(0, T; W^{1,q}(\mathbb{T}^2)) \rightarrow C([0, T] \times \mathbb{T}^2),
\]

continuity equation (1)\(_1\) and momentum equation (1)\(_2\), it gets that

\[
\rho_t = -(u \cdot \nabla_x \rho + \rho \text{div} u) \in C([\tau, T] \times \mathbb{T}^2)
\]

\[
(\rho u)_t = \mu \Delta_x u + (\mu + \lambda(\rho)) \nabla_x \text{div} u + \text{div} u \nabla_x \lambda(\rho) + \rho u \cdot \nabla_x u + \rho \text{div} u
\]

\[
+ (u \cdot \nabla_x \rho) u - \nabla_x P(\rho) + H \cdot \nabla_x H
\]

\[
- \frac{1}{2} \nabla_x |H|^2 - \int_{\mathbb{R}^2} (u - v)^2 f dv \in C([\tau, T] \times \mathbb{T}^2).
\]

Then the Theorem 2.1 follows from the local existence of classical solution and apriori estimates in Section 3.
Appendix A. Elementary inequalities. In this section, we recall several elementary inequalities without the proofs to be used in this paper. These inequalities play an important role in our proof. First, we state the Garliardo-Nirenberg inequality.

Lemma A.1. \[34\] (1) For any \( h \in (W^{1,m} \cap L^r)(\mathbb{T}^2) \), we have
\[
\|h\|_q \leq C\|\nabla_x h\|_m^{\theta} \|h\|_r^{1-\theta},
\]
where \( \theta = (\frac{1}{r} - \frac{1}{q}) (\frac{1}{r} - \frac{1}{m} + \frac{1}{2})^{-1} \), and if \( m < 2 \), then \( q \) is between \( r \) and \( \frac{2m}{2-m} \), if \( m = 2 \) then \( q \in [r, +\infty) \), if \( m > 2 \), then \( q \in [r, +\infty) \).

(2) For any \( h \in H^1(\mathbb{T}^2) \), we have
\[
\|h\|_p \leq C\|\nabla_x h\|_2^{\frac{p-2}{2}} \|h\|_2^{\frac{1}{2}}, \quad p \in [2, +\infty).
\]

We also need the Sobolev-Poincare inequality:

Lemma A.2. \[16\] For any \( h \in W^{1,m}(\mathbb{T}^2) \) with \( m \in [1, 2) \), we have
\[
\|h\|_m \leq C (2 - m)^{-\frac{1}{2}} \|\nabla_x h\|_m,
\]
where the positive constant \( C \) is independent of \( m \).

The following Brezis-Wainger inequalities and properties of the commutator \([b, R_i R_j](g)\) will be used to derive the upper bound of the density \( \rho \), with \( R_i := (-\Delta_x)^{-\frac{1}{2}} \partial_{x_i} \).

Lemma A.3. \[7, 14\] For \( m > 2 \), there exists a positive constant \( C \) such that every function \( h \in W^{1,m}(\mathbb{T}^2) \) satisfies
\[
\|h\|_{\infty} \leq C \|\nabla_x h\|_2 \log^\frac{1}{2} (e + \|\nabla_x h\|_m) + C (\|h\|_2 + 1).
\]

Lemma A.4. \[11, 12\] Let \( b, g \in C_0^\infty(\mathbb{T}^2) \). Then for \( p \in (1, +\infty) \), there exists a constant \( C(p) \) such that
\[
\|[b, R_i R_j](g)\|_p \leq C(p) \|[b]\|_{BMO} \|g\|_p.
\]
Moreover, for \( q_k \in (1, +\infty) \), \( k = 1, 2, 3 \), with \( \frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3} \), there exists \( C \) depending on \( q_k \), such that
\[
\|\nabla_x [b, R_i R_j](g)\|_{q_1} \leq C \|\nabla_x b\|_{q_2} \|g\|_{q_3},
\]
where \([\cdot, \cdot]\) and \( R_i \) are standard Lie bracket and Riesz transform respectively, that is,
\[
[b, R_i R_j](g) := bR_i \circ R_j (g) - R_i \circ R_j (bg), \quad i, j = 1, 2.
\]

The following Beale-Kato-Majda type inequality will be crucial to derive the \( L^\infty \)-norm of \( \nabla_x u \).

Lemma A.5. \[7\] For \( 2 < q < +\infty \), there exists a positive constant \( C \) may depend on \( q \) such that the following estimate holds for all \( \nabla_x u \in W^{1,q}(\mathbb{T}^2) \),
\[
\|\nabla_x u\|_{\infty} \leq C (\|\text{div} u\|_\infty + \|\omega\|_\infty) \log(e + \|\nabla_x u\|_q) + C \|\nabla_x u\|_2 + C. \tag{92}
\]

Acknowledgments. We would like to thank Prof. Huijiang Zhao for his support and instruction, who is authors’s supervisor. The authors also thank the referees for their valuable comments.
REFERENCES

[1] H.-O. Bae, Y.-P. Choi, S.-Y. Ha and M.-J. Kang, Global existence of strong solution for the Cucker-Smale-Navier-Stokes system, *J. Differential Equations*, 257 (2014), 2225–2255.

[2] C. Baranger, L. Boudin, P.-E Jabin and S. Mancini, A modeling of biospray for the upper airways, *CEMRECS 2004 Mathematics and applications to biology and medicine, ESAIM Proc.*, 14 (2005), 41–47.

[3] C. Baranger and L. Desvillettes, Coupling Euler and Vlasov equations in the context of sprays: The local-in-time, classical solutions, *J. Hyperbolic Differ. Equ.*, 3 (2006), 1–26.

[4] S. Berres, R. Burger, K. H. Karlsen and E. M. Tory, Strongly degenerate parabolic-hyperbolic systems modeling polydisperse sedimentation with compression, *SIAM J. Appl. Math.*, 64 (2003), 41–80.

[5] L. Boudin, L. Desvillettes, C. Grandmont and A. Moussa, Global existence of solution for the coupled Vlasov and Navier-Stokes equations, *Differ. Int. Equations*, 22 (2009), 1247–1271.

[6] L. Boudin, L. Desvillettes and R. Motte, A modelling of compressible droplets in a fluid, *Commun. Math. Sci.*, 1 (2003), 657–669.

[7] H. Brezis and S. Wainger, A note on limiting cases of Sobolev embeddings and convolution inequalities, *Comm. Partial Differential Equations*, 5 (1980), 773–789.

[8] L. Caffarelli, R. Kohn and L. Nirenberg, First order interpolation inequality with weights, *Compos. Math.*, 53 (1984), 259–275.

[9] J. A. Carrillo, R.-J. Duan and A. Moussa, Global classical solutions close to equilibrium to the Vlasov-Euler-Fokker-Planck system, *Kinet. Relat. Models*, 4 (2011), 227–258.

[10] F. Catrina and Z.-Q. Wang, On the Caffarelli-Kohn-Nirenberg inequalities: Sharp constants, existence (and non-existence), and symmetry of extremal functions, *Comm. Pure Appl. Math.*, 54 (2001), 229–258.

[11] R. Coifman, R. Rochberg and G. Weiss, Factorization theorems for Hardy spaces in several variables, *Ann. of Math.*, 103 (1976), 611–635.

[12] R. Coifman and Y. Meyer, On commutators of singular integrals and bilinear singular integrals, *Trans. Amer. Math. Soc.*, 212 (1975), 315–331.

[13] R.-J. Duan and S.-Q. Liu, Cauchy problem on the Vlasov-Fokker-Planck equation coupled with the compressible Euler equations through the friction force, *Kinet. Relat. Models*, 6 (2013), 687–700.

[14] H. Engler, An alternative proof of the Brezis-Wainger inequality, *Comm. Partial Differential Equations*, 14 (1989), 541–544.

[15] E. Feireisl, *Dynamics of Viscous Compressible Fluid*, Oxford University Press Inc., 2004.

[16] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin-New York, 1977.

[17] T. Goudon, L. He, A. Moussa and P. Zhang, The Navier-Stokes-Vlasov-Fokker-Planck system near equilibrium, *SIAM J. Math. Anal.*, 42 (2010), 2177–2202.

[18] S.-Y. Ha, B.-K. Huang, Q.-H. Xiao and X.-T. Zhang, A global existence of classical solutions to the two-dimensional kinetic-fluid model for flocking with large initial data, Submitted.

[19] S.-Y. Ha, B.-K. Huang, Q.-H. Xiao and X.-T. Zhang, Global classical solutions to 1D coupled system of flocking particles and compressible fluids with large initial data, *Math. Models Methods Appl. Sci.*, 28 (2018), 1–60.

[20] K. Hamdache, Global existence and large time behaviour of solutions for the Vlasov-Stokes equations, *Japan J. Indust. Appl. Math.*, 15 (1998), 51–74.

[21] X.-P. Hu and D.-H. Wang, Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows, *Arch. Ration. Mech. Anal.*, 197 (2010), 203–238.

[22] X.-P. Hu and D.-H. Wang, Global solutions to the three-dimensional full compressible magnetohydrodynamic flows, *Comm. Math. Phys.*, 283 (2008), 255–284.

[23] X.-D. Huang and J. Li, Global well-posedness of classical solutions to the Cauchy problem of two-dimensional barotropic compressible Navier-Stokes system with vacuum and large data, arXiv:1207.3746v1.

[24] X.-D. Huang and J. Li, Existence and blowup behavior of global strong solutions to the two-dimensional barotropic compressible Navier-Stokes system with vacuum and large initial data, *J. Math. Pures Appl.*, 106 (2016), 123–154.

[25] Q.-S. Jiu, Y. Wang and Z.-P. Xin, Global well-posedness of 2D compressible Navier-Stokes equations with large data and vacuum, *J. Math. Fluid Mech.*, 16 (2014), 483–521.
Q.-S. Jiu, Y. Wang and Z.-P. Xin, Global well-posedness of the Cauchy problem of two-dimensional compressible Navier-Stokes equations in weighted spaces, *J. Differential Equations*, **255** (2013), 351–404.

F.-C. Li, Y.-M. Mu and D.-H. Wang, Strong solutions to the compressible Navier-Stokes-Vlasov-Fokker-Planck equations: global existence near the equilibrium and large time behavior, *SIAM J. Math. Anal.*, **49** (2017), 984–1026.

P.-L. Lions, Mathematical topics in fluid mechanics. Vol. 1. Incompressible models, Oxford University Press, New York, 1996.

P. L. Lions, Mathematical topics in fluid mechanics. Vol. 2. Compressible models, Oxford University Press, New York, 1998.

Y. Mei, Global classical solutions to the 2D compressible MHD equations with large data and vacuum, *J. Differential Equations*, **258** (2015), 3304–3359.

Y. Mei, Corrigendum to “Global classical solutions to the 2D compressible MHD equations with large data and vacuum”, *J. Differential Equations*, **258** (2015), 3360–3362.

A. Mellet and A. Vasseur, Asymptotic analysis for a Vlasov-Fokker-Planck/compressible Navier-Stokes system of equations, *Comm. Math. Phys.*, **281** (2008), 573–596.

A. Mellet and A. Vasseur, Global weak solutions for a Vlasov-Fokker-Planck/Navier-Stokes system of equations, *Math. Models Methods Appl. Sci.*, **17** (2007), 1039–1063.

A. Novotný and I. Straskraba, *Introduction to the Mathematical Theory of Compressible Flow*, Oxford Lecture Series in Mathematics and its Applications, 27. Oxford University Press, Oxford, 2004.

C. Sparber, J.-A. Carrillo, J. Dolbeault and P.-A. Markowich, On the long-time behavior of the quantum Fokker-Planck equation, *Monatsh. Math.*, **141** (2004), 237–257.

F.-A. Williams, *Combustion Theory*, Benjamin Cummings, 1985.

V.-A. Vaigant and A.-V. Kazhikhov, On the existence of global solutions of two-dimensional Navier-Stokes equations of a compressible viscous fluid, *Sibirsk. Mat. Zh.*, **36** (1995), 1283–1316.

Received March 2018; revised July 2018.

E-mail address: bkhuang@whu.edu.cn/bkhuang92@hotmail.com
E-mail address: zhang.lan@whu.edu.cn