Robinson–Trautman solution with scalar hair

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Explicit Robinson–Trautman solution with minimally coupled free scalar field is derived and analyzed. It is shown that this solution contains curvature singularity which is initially naked but later the horizon envelopes it. We use quasi-local horizon definition and prove its existence in later retarded times using sub- and supersolution method combined with growth estimates. We show that the solution is generally of algebraic type II but reduces to type D in spherical symmetry.

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I. INTRODUCTION

Solutions to Einstein equations with scalar field source provide very useful tool for understanding relativity due to the simplicity of the source. Recently, it becomes progressively plausible that such fields might really exist (LHC) and potentially play a fundamental role in physics. In classical General Relativity they were used to study counterexamples to black hole no-hair theorems and in many other areas. These results were mostly based on highly symmetric solutions and it is therefore important to provide solutions with less or no symmetries to subsequently analyze if those results hold in more generic situations and are not tied to specific symmetry.

Robinson–Trautman spacetimes represent an important class of expanding nontwisting and nonshearing solutions describing non-spherical generalizations of black holes. In general, they do not possess any Killing vectors thus providing important solutions devoid of symmetry. Many properties of this family in four dimensions have been studied, especially in last 25 years. In particular, the existence, asymptotic behaviour and global structure of vacuum Robinson–Trautman spacetimes of type II with spherical topology were investigated by Chruściel and Singleton. Robinson–Trautman solutions were shown to exist for generic, arbitrarily strong smooth initial data for all positive retarded times, and to converge asymptotically to corresponding Schwarzschild metric. Extensions across the “Schwarzschild-like” future event horizon can only be made with a finite order of smoothness. These results were generalized to Robinson–Trautman vacuum spacetimes with cosmological constant. These cosmological solutions settle down to a Schwarzschild–(anti-)de Sitter solution at large times u. Finally, the Chruściel–Singleton analysis was extended to Robinson-Trautman spacetimes including matter, namely pure radiation, showing that they approach the spherically symmetric Vaidya–(anti-)de Sitter metric. So generally the solutions of this family settle down to physically important solutions. The location of the horizon together with its general existence and uniqueness for the vacuum Robinson–Trautman solutions has been studied by Tod. Later, Chow and Lun analyzed some other useful properties of this horizon and made numerical study of both the horizon equation and Robinson–Trautman equation. These results were later extended to nonvanishing cosmological constant. The anisotropy of Robinson–Trautman horizon and its associated asymptotic momentum was also used in the analytic explanation of an “antikick” appearing in numerical studies of binary black hole mergers.

Robinson–Trautman spacetimes (containing aligned pure radiation and a cosmological constant) were also generalized to any dimension. Existence of horizons was subsequently analyzed. Finally, Robinson–Trautman solutions with p-form fields in arbitrary dimension were derived recently. One of the results mentioned therein rules out the existence of aligned scalar field (where alignment refers to the gradient of the field) for generic Robinson–Trautman case.

The solutions for "stringy" Robinson–Trautman spacetime corresponding to Einstein–Maxwell–dilaton system was obtained. Recently, scalar field solutions for Einstein–Maxwell–Lambda system with a conformally coupled scalar field belonging to Plebanski–Demianski family (containing type D solutions of Robinson–Trautman class) were derived.

II. VACUUM ROBINSON–TRAUTMAN METRIC AND FIELD EQUATION

The general form of a vacuum Robinson–Trautman spacetime can be given by the following line element

\[ ds^2 = -2H du^2 - 2 du dr + \frac{r^2}{f^2} (dy^2 + dx^2), \]  

(2.1)
where \( 2H = \Delta(\ln P) - 2r(\ln P)_r - 2m/r - (\Lambda/3)r^2 \),

\[ \Delta \equiv P^2(\partial_{xx} + \partial_{yy}) \]  

and \( \Lambda \) is the cosmological constant. The metric depends on two functions, \( P(x, y, u) \) and \( m(u) \), which satisfy the nonlinear Robinson–Trautman equation

\[ \Delta \Delta(\ln P) + 12m(\ln P)_u - 4m_u = 0. \]  

The spacetime admits a geodesic, shearfree, twistfree and expanding null congruence generated by \( k = \partial_r \). The coordinate \( r \) is an affine parameter along this congruence, \( u \) is a retarded time coordinate, and \( x, y \) are spatial coordinates spanning transversal 2-space with their Gaussian curvature (for \( r = 1 \)) being given by

\[ K(x, y, u) \equiv \Delta(\ln P). \]  

For general fixed values of \( r \) and \( u \), the Gaussian curvature is \( K/r^2 \) so that, as \( r \to \infty \), they become locally flat.

### III. SOLUTION COUPLED TO SCALAR FIELD

We consider the following action, describing a scalar field minimally coupled to gravity,

\[ S = \int d^4x \sqrt{-g}[\mathcal{R} + \nabla_{\mu} \varphi \nabla^{\mu} \varphi] \]  

where \( \mathcal{R} \) is the Ricci scalar for the metric \( g_{\mu\nu} \). The massless scalar field \( \varphi \) is supposed to be real and we use units in which \( c = \hbar = 8\pi G = 1 \). By applying the variation with respect to the metric for the action (3.1), we will get

\[ \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu} \mathcal{R} = T_{\mu\nu} \]  

For the matter of convenience we will be looking for the metric in the following form

\[ ds^2 = -2(H(u, r) + K(u, x, y))du^2 - 2dudr + \frac{R(u, r)^2}{P(x, y)^2}(dx^2 + dy^2) \]  

The scalar field is assumed to be function of \( u \) and \( r \) only (\( \varphi(u, r) \)). The dependence on \( r \) means that the scalar field is not aligned and thus is not ruled out by the results of [10]. The energy momentum tensor for this scalar field is given by

\[ T_{\mu\nu} = \nabla_{\mu} \varphi \nabla_{\nu} \varphi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha \varphi \nabla_\beta \varphi \]  

and it must satisfy corresponding field equation

\[ \Box \varphi(u, r) = 0 \]  

where \( \Box \) is a standard d’Alembert operator for our metric (3.3). The nontrivial components of the Ricci tensor corresponding to this metric are

\[ \mathcal{R}_{uu} = 2 \left( 2\frac{R}{R} \frac{R^r}{R} H + H_{rr} \right) (H + K) + 2\frac{R^r}{R} (H + K)_u - \frac{2}{R} (R_u H_r + R_{u, u}) + \frac{P^2}{R^2} (K_{xx} + K_{yy}) \]  

\[ \mathcal{R}_{rr} = -2\frac{R_{,rr}}{R} \]  

\[ \mathcal{R}_{xu} = \mathcal{R}_{ur} = 2\frac{R}{R} H_r - \frac{R_{,u}}{R} + H_{rr} \]  

\[ \mathcal{R}_{xx} = \mathcal{R}_{yy} = -\frac{1}{P^2} \{ k(x, y) + 2(H + K)(RR)_r + +2RR_r H_r - 2(RR_r)_r \} \]  

where as usual \( ()_{,x^i} = \frac{\partial}{\partial x^i}() \) and

\[ k(x, y) = \Delta(\ln P(x, y)) \]  

where \( \Delta \) is still given by expression (2.2).

We will use the following form of equations equivalent to Einstein equations (3.2) coupled to energy momentum tensor (3.1)

\[ \mathcal{R}_{\mu\nu} = \varphi,_{\mu} \varphi_{,\nu} = \begin{pmatrix} \varphi_{,u}^2 & \varphi_{,u} \varphi_r & 0 & 0 \\ \varphi_{,u} \varphi_r & \varphi_r^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]  

From the above equations describing gravitational field and field equation for the scalar field (3.5) we obtain the following expressions for unknown metric functions and scalar field

\[ H(u, r) = \frac{r}{2}(\frac{\partial U}{\partial u}) \]  

\[ R(u, r) = \sqrt{\frac{U(u)^2 - C_0^2}{U(u)}} \]  

\[ K(u, x, y) = \frac{k(x, y)}{2U(u)} \]  

\[ \varphi(u, r) = \frac{1}{\sqrt{2}} \ln \left\{ \frac{U(u)r - C_0}{U(u)r + C_0} \right\} \]  

\[ \Delta k(x, y) = \alpha^2 \]  

\[ U(u) = \gamma e^{\omega u^2 + \eta u} \]  

in which \( C_0 \neq 0, \alpha, \eta, \omega \) are constants and \( \omega = \frac{a}{2C_0} \). In the following we will assume \( C_0 > 0, \alpha > 0, \eta > 0 \) for simplicity of discussion.

### IV. PROPERTIES OF THE SOLUTION

First, we should ensure that our solution really belongs to the Robinson-Trautman family. This is simply confirmed by studying the properties of a null congruence...
generated by vector $\mathbf{l} = \partial_r$. Such congruence is geodesic, non-twisting, non-shearing and its expansion is given by

$$\Theta_1 = 2 \frac{R_r}{R} = \frac{2 U(u)^2 r}{U(u)2r^2 - C_0^2}. \quad (4.1)$$

Evidently, the above expression is positive only for $r > \frac{C_0}{U(u)}$, which may seem not satisfactory. However by inspecting the Kretschmann scalar

$$\kappa \sim \frac{1}{R(u,r)^8} \quad (4.2)$$

and using (3.11) we immediately see that the geometry has singularities for $r = r_0 = \pm \frac{C_0}{U(u)}$. Naturally, we are led to constrain the range of coordinates to $r \in \left(\frac{C_0}{U(u)}, \infty\right)$. In this range the expansion (4.1) is everywhere positive, diverges at the singularity and approaches zero at infinity (as $r \to \infty$). Also, one can check from the line element that the singularity is a standard pointlike one. Due to the asymptotic behaviour of function $U(u)$ (see (3.11)) the singularity tends to $r_0 = 0$ as $u \to \infty$. The singularity appears due to the divergence of the scalar field and its energy momentum tensor.

When the singularity is present in our solution we will investigate if it is covered by a horizon. Due to dynamical nature of the spacetime it is preferable to use the quasilinear definitions of horizon — apparent horizon [20], trapping horizon [21] or dynamical horizon [22]. The basic local condition is shared by all the standard horizon definitions: these horizons are sliced by marginally trapped surfaces given by

$$M = \mathcal{M}(x,y,u) \quad (4.3)$$

and study the expansion of compact slices of such hypersurface given by $u = u_0 = \text{const.}$ (with $\mathcal{M}(u_0, x, y) = M(x, y)$). We construct null vector fields orthogonal to surface $\mathbf{r} = M(x, y)$

$$\mathbf{l} = \partial_r \quad (4.4)$$

$$\mathbf{k} = \partial_u + \frac{P^2}{2R^2(\mathbf{M}_x^2 + \mathbf{M}_y^2) - (H + K)} \partial_r + \frac{P^2}{R^2}(\mathbf{M}_x \partial_x + \mathbf{M}_y \partial_y) \quad (4.5)$$

that satisfy normalization condition $\mathbf{l} \cdot \mathbf{k} = -1$. From the geometry of the situation one can deduce that congruence $\mathbf{l}$ is outgoing while $\mathbf{k}$ is ingoing. The expansion of the congruence generated by $\mathbf{l}$ is always positive (see (4.11) and discussion beneath) so we are looking for vanishing of expansion related to the other congruence $\mathbf{k}$. These two conditions ($\Theta_1 > 0$ and $\Theta_1 = 0$) mean that we are looking for the past horizon according to the definition given by [21]. The second expansion is given by

$$\Theta_2 = \frac{1}{R^2} \left[ \Delta M - (\ln R),_r (\nabla M \cdot \nabla M) - \quad (4.6) \right.$$

$$\left. -(K + H)(R^2),_r + (R^2),_u \right]$$

where Laplace operator and scalar product denoted by dot correspond to metric $h_{ij}dx^i dx^j = \frac{1}{R^2(x,y)}(dx^2 + dy^2)$ on the space $\Sigma$ spanned by $x, y$. So the horizon is given by the solution of the following quasilinear elliptic partial differential equation

$$\{ \Delta M - (\ln R),_r (\nabla M \cdot \nabla M) - (K + H)(R^2),_r + (R^2),_u \} |_{r=M(x,y), u_k=0} = 0 \quad (4.7)$$

where all dependence on $r$ is replaced by the function $M(x,y)$ and $u$ is evaluated to arbitrary constant value $u_0$.

It is impossible to solve this equation generally but fortunately we can get some useful information about the existence of solution using the technique developed for the case of Robinson–Trautman spacetime in higher dimensions [13]. The proof of existence of the solution to the same type of quasilinear equation ($\Delta u = F(x, u, \nabla u)$) was given there by combining several steps motivated by [23] and using results from [24–26]. The main issues were to provide an estimate for the function $F$ of the form $|F| \leq B(u)(1 + |\nabla u|^2)$ (where $B(u)$ is increasing function on $\mathbb{R}^+$) and to show the existence of a sub- and a supersolution $u^- \leq u^+ \leq C^{1,\beta}(\Sigma) \cap L^\infty(\Sigma)$ (here $C^{1,\beta}(\Sigma)$ are Hölder continuous functions of some suitable index $\beta$). Then we know there is a solution $u \in C^{2,\beta}(\Sigma)$ (for some $\beta$) satisfying $u^- \leq u \leq u^+$.

In case we want to provide an estimate of the form $|M| \leq B(M)(1 + |\nabla M|^2)$ (the norm is taken with respect to the two-dimensional metric $h_{ij}$) for the horizon equation (4.7) when considered in the form $\Delta M = F(x, y, M, \nabla M)$ where

$$F = (\ln R(u_0, M))_r |\nabla M|^2 + k(x,y)M - \frac{C_0^2U'(u_0)}{U^2(u_0)} \quad (4.8)$$

one has to deal with the singular behaviour of $(\ln R)_r$ at $r = \frac{C_0}{U(u)}$. We can do this either by removing the vicinity of singularity from our domain $r \in \mathbb{R}^+ \backslash \left(\frac{C_0}{U(u)}(1 - \delta), \frac{C_0}{U(u)}(1 + \delta)\right)$ or by continuing (with some appropriate smoothing) the divergent function on the problematic interval $\left(\frac{C_0}{U(u)}(1 - \delta), \frac{C_0}{U(u)}(1 + \delta)\right)$ with a constant value it attains on the boundary of the interval. Now, with all the coefficients of the equation finite one can construct the bounding function $B(u)$ easily and thus we can proceed to the construction of sub- and supersolutions $M^\pm$.

First, we note that due to the selection of sign for the free constants made at the end of previous section ($C_0 > 0, \alpha > 0, \eta > 0$) we obtain $U_0 > 0$ if we restrict our attention to retarded time region $u \in (-\frac{C_0^2}{U_0}, \infty)$. We can then understand our solution as being given by initial conditions specified at $u_{ini} = \frac{C_0^2}{U_0}$ which corresponds to usual understanding of Robinson–Trautman solution. As usual, we are looking for constant sub- and supersolutions but we are unable to provide them independently of the value of $u_0$. Generally, we can find the sub- and supersolutions in the following cases:
of explicit methods for determining the algebraic type in [28] that are based on [29]. Namely, when we use invariants
\[ I = \Psi_0 \Psi_4 - 4 \Psi_1 \Psi_3 + 3 \Psi_2^2, \]
we can immediately confirm that \( I^3 = 27J^2 \) is satisfied so that we are dealing with type II or more special. At the same time generally \(IJ \neq 0\) so it cannot be just type III. Moreover, the spinor covariant \( R_{ABCDEF} \) has nonzero components
\[ R_{000000} = \Psi_1 (3 \Psi_0 \Psi_2 - 2 \Psi_4^2) \]
\[ R_{000001} = \frac{1}{2} \Psi_2 (3 \Psi_0 \Psi_2 - 2 \Psi_4^2) \]
which means that generally the spacetime cannot be of type D. So indeed our scalar field solution is of the most general type possible for the Robinson–Trautman vacuum class. Moreover, inspecting the components of the Weyl spinor (5.2) one concludes that in the special case of \( k(x, y) = \text{const} \) (constant Gaussian curvature of two-space spanned by \( x, y \)) the algebraic type becomes D consistent with spherical symmetry.

VI. CONCLUSION AND FINAL REMARKS

We have derived a Robinson–Trautman spacetime with minimally coupled free scalar field. We have shown that it has a singularity for all retarded times created by the divergence of the scalar field therein. This singularity is initially (with respect to retarded time) naked and only later becomes covered by the quasilocal horizon. Note that the energy momentum tensor of the free minimally coupled scalar field trivially satisfies null energy condition (as well as weak and strong ones) and the naked singularity at the beginning of the evolution is probably caused by a slow buildup of effective energy density at the singularity position which is enough to form the singularity but not enough to envelop it in horizon initially. This behaviour suggests similarity with the appearance of a naked singularity in Vaidya spacetime with linear mass function which depends on the speed of growth of such mass [30]. Only later the horizon appears and singularity is no longer naked. From the properties of both null congruences orthogonal to horizon we deduced that we are dealing with past horizon which is natural for standard form of Robinson–Trautman spacetime. Finally, we have proved that our geometry is of algebraic type II and if we restrict to spherically symmetric case it is of type D.

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