Algebraic Quantum Field Theory: A Status Report

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Abstract

Algebraic quantum field theory is an approach to relativistic quantum physics, notably the theory of elementary particles, which complements other modern developments in this field. It is particularly powerful for structural analysis but has also proven to be useful in the rigorous treatment of models. In this contribution a non-technical survey is given with emphasis on interesting recent developments and future perspectives. Topics covered are the relation between the algebraic approach and conventional quantum field theory, its significance for the resolution of conceptual problems (such as the revision of the particle concept) and its role in the characterization and possibly also construction of quantum field theories with the help of modular theory. The algebraic approach has also shed new light on the treatment of quantum field theories on curved spacetime and made contact with recent developments in string theory (algebraic holography).

1 Introduction

In the present year 2000 we are celebrating the 100th birthday of quantum theory and the 75th birthday of quantum mechanics. Thus it took only 25 years from the first inception of the new theory until its final consolidation. Quantum field theory is almost as old as quantum mechanics. But the formulation of a fully consistent synthesis of the principles of quantum theory and classical relativistic field theory has been a long and agonizing process and, as a matter of fact, has not yet come to a satisfactory end, in spite of many successes.

The best approximation to nature in the microscopic and relativistic regime of elementary particle physics which we presently have, the so called Standard Model, does not yet have the status of a mathematically consistent theory. It may be regarded as an efficient algorithm for the theoretical treatment of certain

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specific problems in high energy physics, such as the perturbative calculation
of collision cross sections, the numerical analysis of particle spectra etc. Yet
nobody has been able so far to prove or disprove that the model complies with
all fundamental principles of relativistic quantum physics.

This somewhat embarrassing situation is not so widely known. It is therefore
gratifying that the Clay Mathematics Institute has recently drawn attention to
it by endowing a price of 1.000.000 $ for the mathematical consolidation of an
important piece of the standard model, the Yang–Mills–Theory. So the mathemat-
ical and conceptual problems of relativistic quantum field theory are in many
respects a rewarding field of activity for mathematical physicists.

There are two strategies to make further progress. Either one tries to im-
prove the existing mathematical methods for the treatment of models of physical
interest. This is the approach of constructive quantum field theory [1]. Or one
proceeds from a sufficiently rigid mathematical framework, consistent with all
basic principles of quantum field theory, and aims at new conceptual insights and
constructive ideas by structural analysis.

Such a general framework which is useful for the solution of both, conceptual
puzzles and constructive problems, is Algebraic Quantum Field Theory (AQFT),
frequently also called Local Quantum Physics. It was invented by Rudolf Haag
and Daniel Kastler [2] and has proven to be consistent with the developments in
elementary particle physics for several decades. It is the aim of this contribution
to recall the physical ideas and mathematical structures underlying AQFT and to
outline some recent interesting results which reveal its flexibility for the treatment
of a variety of problems. More systematic recent reviews of this approach can be
found in [3, 4].

2 Foundations of AQFT

The relation between the conventional approach to quantum field theory, based on
the Lagrangian formalism, and algebraic quantum field theory may be compared
with the concrete and abstract approaches to differential geometry. If one is
dealing with concrete (computational) problems in geometry, it is natural to
use coordinates, tensor fields, Christoffel symbols etc, whereas in the general
structural analysis one relies on intrinsic concepts such as the notions of manifold,
fiber bundle, connection etc. Both points of view have their virtues and full insight
is only gained by combining them.

The situation is similar in relativistic quantum field theory. In the concrete
Lagrangian approach, one specifies the field content of the theory acting on the
given space–time manifold $\mathcal{M}$ as well as a corresponding Lagrangian. The dif-
ficult step is “quantization” which is accomplished either by appealing to the
 correspondence principle (canonical quantization) or to general structural results
according to which the problem can be reformulated in terms of classical statisti-
cal field theory (Euclidean approach). If everything has been said and done,
one obtains the vacuum correlation functions of the fields. From them one can

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reconstruct a Hilbert space on which the observables, such as the stress energy
tensor, the currents etc. act as operators. The Lagrangian approach is well suited
for the computational treatment of concrete models, but it is not intrinsic. For
different Lagrangians with different field content may describe the same physics.
This phenomenon of “quantum equivalence” of classical field theories has been
observed in many examples [3–7]. One may therefore ask whether there is a more
intrinsic way of describing relativistic quantum field theories.

A fully satisfactory answer to this question is provided by AQFT. In this
approach the basic objects are the algebras generated by the observables localized
in given space–time regions; fields are not mentioned in this setting and are
regarded as a kind of coordinates of the algebras. The passage from the field
theoretic setting to the algebraic one requires the following steps:

▷ determine the set \( \{ \theta(x) \} \) of observables of the underlying theory for each
space–time point \( x \)

▷ construct for each relatively compact space–time region \( O \subset M \) a corre-
sponding local algebra of observables,

\[
\mathcal{A}(O) \equiv \{ \theta(x) : x \in O \}'',
\]

i.e. the von Neumann algebra (double commutant) generated by the respec-
tive observables in the underlying Hilbert space.

In view of the fact that the observables \( \theta(x) \) are only defined in the sense of
sesquilinear forms (or as operator valued distributions), the latter step is some-
what subtle. But it has been shown to be meaningful in the models which have
been constructed so far [8] and also in the general Wightman setting of quantum
field theory under some very general conditions [9].

The resulting structure is an assignment of algebras to spacetime regions,

\[
O \mapsto \mathcal{A}(O),
\]

which is called a local net in view of its order preserving properties. Any such
net inherits some fundamental properties from the underlying field theory. These
are [10]

▷ locality: operators localized in causally disjoint regions commute

▷ covariance: the space–time symmetries act by automorphisms on the net

▷ stability: there exist distinguished states (expectation functionals) on the
net, describing stable elementary systems such as the vacuum.

Whereas the first two points are well understood and have an obvious physical
interpretation, the mathematical characterization of elementary states on arbi-
trary space–time manifolds is a more difficult issue which is still under discussion,
cf. [11]–[15] and references quoted there. For the class of maximally symmetric
spaces there are no such problems, however.
According to the deep insights of Haag and Kastler, the full physical information of a theory is already contained in the net structure, i.e. the respective map from space–time regions to algebras. Phrased differently, equivalent quantum field theories can be identified by the fact that they generate isomorphic local nets. This assertion may be somewhat surprising at first sight since the passage from the observables, which normally have a specific physical interpretation, to the local algebras seems to be a rather forgetful operation. That no information is lost in this step has been confirmed by now by numerous results [10].

A direct way of seeing this has been established by Fredenhagen and Hertel [16] who proved for the class of Minkowski space theories that the set of basic observables can be recovered from the local net by the formula

\[
\{\phi(x)\} = \bigcap_{O \ni x} \mathcal{A}(O).
\]

The somewhat tricky point in this reconstruction is the need to proceed from the algebras \(\mathcal{A}(O)\) of bounded operators to unbounded sesquilinear forms. It is accomplished by completing these algebras in a suitable locally convex topology, indicated by the bar. It should be noted, however, that the algebraic framework is in some respect more flexible than the field theoretic setting. For the local algebras may also accommodate extended objects, such as Wilson loops or finite Mandelstam strings, which are not built from point like observable fields.

AQFT is thus compatible with the structures found in the quantum field theories of present physical interest and complements them by putting emphasis on their intrinsic features. It may thus be regarded as a minimal setting for the description of the systems appearing in high energy physics. We discuss in the following some issues where the virtues of this approach become manifest.

3 Perturbative AQFT

Guided by insights gained from algebraic quantum field theory, Brunetti and Fredenhagen [17] have recently established a perturbative construction of local nets on arbitrary globally hyperbolic space–times \(\mathcal{M}\). They propose to treat this problem in two steps: first one constructs the nets of local algebras in some convenient Hilbert space representation, thereby fixing the theory. This step requires control of the notorious ultraviolet divergences (renormalization) and can be handled by configuration space methods and microlocal techniques. Infrared problems, related to the so–called adiabatic or infinite volume limit, do not appear in this construction. In a second step one may then turn to the determination of the states of physical interest on this net and to their analysis. This requires the passage to new Hilbert space representations of the algebraic structures and may thus be regarded as a problem in the representation theory of local nets.

In Minkowski space theories both problems are frequently treated simultaneously because of the possibility of characterizing the vacuum state directly by momentum space properties (spectrum condition). But, as indicated above, this
strategy does not work for arbitrary space–time manifolds. Thus the Brunetti–Fredenhagen approach is a very natural way of circumventing these difficulties.

Following [17], we outline this method by discussing the theory of a self-interacting scalar field on a given space–time $\mathcal{M}$. The perturbative construction of the net requires the following steps:

1. consider the free scalar field $\phi_0$ on the space–time $\mathcal{M}$,

   $$(\Box + m^2) \phi_0(x) = 0, \quad [\phi_0(x), \phi_0(y)] = -i \Delta(x, y) 1,$$

   where $\Box$ denotes the D’Alembertian and $\Delta(x, y)$ the causal commutator function on $\mathcal{M}$, in a regular Hilbert space representation induced by some Hadamard state [11].

2. construct Wick powers of the free field for $n \in \mathbb{N}$,

   $$:\phi^n_0:(x).$$

The existence of these operator–valued distributions was first established in [12] by methods of microlocal analysis and shown in [17] to be largely independent of the chosen regular Hilbert space representation. These Wick powers are the building blocks for the construction of interacting fields.

3. define the time ordered exponentials (local $S$–operators)

   $$S(g) \equiv T \exp( i \int d\mu(x) g(x) :\phi^n_0:(x) ),$$

   where $g$ is any test function and $d\mu(x)$ is the volume form on $\mathcal{M}$. (More generally, one considers such exponentials for finite sums of Wick powers.)

The proof that these exponentials are meaningful expressions is the most difficult step in the construction. It has been established in [17] in perturbation theory by defining $S(g)$ as formal power series in $g$. The coefficients in this series suffer from ambiguities due to short distance singularities which require renormalization. This problem is solved by generalizing methods of Epstein and Glaser (causal perturbation theory). If $n \leq 4$, there does not appear a proliferation of these ambiguities with increasing order of perturbation theory (renormalizability).

4. use Bogolubov’s formula

   $$S_g(f) \equiv S(g)^{-1} S(f + g), \quad g \upharpoonright \text{supp} f = q$$

   to define, for given interaction density $q :\phi^n_0:(x)$, local operators for the cutoff density $g(x) :\phi^n_0:(x)$.

Up to this point the construction is akin to the treatment of Minkowski space theories, although the technical details are more involved in view of the absence of space–time symmetries. The adiabatic limit of $S_g(f)$ for $g \to q$ may not exist, however, due to infrared problems caused by the ad hoc choice of a defining representation of the interacting theory. This difficulty can be circumvented by the following important observation [17].
Proposition 3.1 Let $\mathcal{O} \subset \mathcal{M}$ and let $g_1 \upharpoonright \mathcal{O} = g_2 \upharpoonright \mathcal{O} = q$. There exists a unitary operator $V_\mathcal{O}$ such that

$$S_{g_2}(f) = V_\mathcal{O} S_{g_1}(f) V_\mathcal{O}^{-1}, \quad \text{supp} f \subset \mathcal{O}. $$

In view of this result it is meaningful to define local algebras, setting for $\mathcal{O} \subset \mathcal{M}$ and any $g$ with $g \upharpoonright \mathcal{O} = q$

$$\mathcal{A}_g(\mathcal{O}) \equiv \ast\text{–algebra } \{S_g(f) : \text{supp} f \subset \mathcal{O}\}. $$

According to the preceding proposition, these algebras are unique up to isomorphisms $\text{ad} V_\mathcal{O}$, which do not change the physical interpretation. One may thus proceed to an algebraic adiabatic limit by considering the algebras

$$\mathcal{A}(\mathcal{O}) \equiv \ast\text{–algebra } \{(S_g(f))_{g\mid \mathcal{O} = q} : \text{supp} f \subset \mathcal{O}\}. $$

The inclusion (net) structure of this family of algebras is given by their natural embeddings and the algebraic operations of addition, multiplication as well as the $\ast$–operation in $\mathcal{A}(\mathcal{O})$ are pointwise defined for each $g$. In this way one arrives at a (perturbative) net

$$\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$$

for the given interaction.

Thus the algebraic point of view leads to a natural perturbative construction of nets of local algebras in any space–time $\mathcal{M}$. Similar methods have also been applied to the construction of local nets of observables in gauge theories in Minkowski space [18], cf. also [19] and references quoted there.

4 Particle analysis

We turn next to a conceptual problem in Minkowski space quantum field theories, namely the asymptotic particle interpretation. In the gauge theories of physical interest, the basic fields are in general unphysical and not related to stable particles. One has therefore to develop methods to determine the particle features of these theories from the vacuum correlation functions of the local observables. The confinement problem in quantum chromodynamics and the infraparticle problem in quantum electrodynamics are two well known examples illustrating this issue. Here the algebraic point of view led recently to some interesting progress.

The following steps are necessary in the particle analysis of any theory:

- introduce some meaningful particle concept
- establish methods to determine the particle content of the theory
- analyze the properties of these particles
- develop a scattering (collision) theory.
One has a quite satisfactory understanding of these points in theories with short range forces (mass gap). There the natural starting point is Wigner’s particle concept, according to which the possible states of a particle are described by vectors in some irreducible representation of the Poincaré group or its two-fold covering, respectively. It is well known, however, that this approach does not work, for example, in theories with electrically charged particles, cf. for example [20]. So it seems desirable to develop a universal particle concept which applies also in those cases.

Such a more flexible particle concept was introduced in [21,22]. It is based on the notion of particle weight which is a generalization of Dirac’s idea of an improper momentum eigenstate of a particle. There are two possibilities of looking at these improper states.

▷ traditional: improper momentum eigenstates are regarded as vector valued distributions, i.e. maps from a space of wave functions into the physical Hilbert space,

\[ |\rangle : f \mapsto \int dp \ f(p)|p \rangle \in \mathcal{H}. \]

It is anticipated in this approach that the improper states become normalizable by superposition (interference effects).

▷ alternative: improper states of fixed momentum are regarded as linear maps from a space \( L \) of localizing operators into the physical Hilbert space,

\[ |p \rangle : L \mapsto L|p \rangle \in \mathcal{H}. \]

Here the improper states become normalizable by localization.

It is important to notice that the second approach is more general than the first one. It is expected to be applicable even if the superposition principle fails for the improper states. This happens, for example, if the process of localization is inevitably accompanied by particle production (such as in quantum electrodynamics, where infinite clouds of soft photons are produced).

In quantum mechanics, suitable localizing operators would be rapidly decreasing functions of the position operator; but the notion of position operator is not meaningful in a field theoretic setting. Nevertheless, localizing operators \( L \) exist in AQFT in abundance and are easily constructed. Simple examples are all operators of the form

\[ L = \int dx \ f(x) \ A(x). \]

Here \( f \) is any test function whose Fourier transform vanishes in the forward light cone and \( A(x) = U(x)A U(x)^{-1} \), where \( U(x) \) are the unitaries inducing the space–time translations \( x \) and \( A \) is any local observable. The operators \( L \) can be shown to annihilate all states (disregarding vectors of arbitrarily small norm) which do not describe excitations of the vacuum in some (sufficiently large but finite) space–time region \( \mathcal{O}_L \). So, roughly speaking, they “project” onto states which
differ from the vacuum in $\mathcal{O}_L$. In this sense they are localizing operators. It is technically important that these localizing operators form a left ideal $\mathcal{L} \subset \mathcal{A}$ in the C$^*$–algebra $\mathcal{A}$ generated by all local observables.

It is convenient to proceed from the improper states to corresponding positive, linear and non–normalizable functionals on the domain $\mathcal{L}^* \mathcal{L} \subset \mathcal{A}$,

$$\langle p \mid \cdot \mid p \rangle : \mathcal{L}^* \mathcal{L} \to \mathbb{C},$$
called particle weights. These functionals can be characterized in an intrinsic manner. In particular they are extremal and invariant under space–time translations. As $\mathcal{L}^* \mathcal{L}$ is a *–algebra, one can recover by the GNS reconstruction theorem the improper particle states from these particle weights.

After having introduced in AQFT a general particle concept, one has to develop methods to determine the particle content of a theory (described by particle weights). This is accomplished by analyzing the timelike asymptotic properties of the physical states in the vacuum sector $\mathcal{A} \Omega = \{ A \Omega : A \in \mathcal{A} \}$ of the theory. So let $\omega(\cdot)$ be any expectation functional induced by vectors in the subspace $\mathcal{A} \Omega$ of the physical Hilbert space $\mathcal{H}$. One then considers the functionals

$$\rho_t(L^*L) \equiv \frac{1}{2} \int_t^{2t} dx_0 \int dx \, \omega((L^*L)(x)), \quad L \in \mathcal{L}.$$  

These expressions are mathematically meaningful as a consequence of locality and the shape of the energy momentum spectrum [23]. Moreover, the family of functionals $\{\rho_t\}_{t \in \mathbb{R}}$ is equibounded. So this family has limit points

$$\rho_\infty(L^*L) = \lim_{t \to \infty} \rho_t(L^*L), \quad L \in \mathcal{L}.$$  

The functionals $\rho_\infty$ are, by their very construction, invariant under translations, but highly mixed. It is therefore natural to ask whether they can be decomposed into particle weights. An affirmative answer to this question was recently given by Porrmann [24]. More specifically, there holds the following statement.

**Proposition 4.1**

$$\rho_\infty(L^*L) = \int d\mu(p, \iota) \langle p, \iota \mid L^*L \mid p, \iota \rangle, \quad L \in \mathcal{L}$$

where each $\langle p, \iota \mid \cdot \mid p, \iota \rangle$ is a particle weight of momentum $p$ and “internal index” $\iota$ and $d\mu(p, \iota)$ is a measure depending on the initial state $\omega$. (The internal index $\iota$ describes the intrinsic features of a particle, such as its charge quantum numbers and spin.)

Thus the preceding asymptotic construction and subsequent decomposition of functionals provides a general method to determine the stable particle content of any theory, including the charged particles (which appear in the vacuum sector only as pairs of opposite charge, but asymptotically give rise to particle weights contributing to the mixtures $\rho_\infty$). This result is a substantial generalization of
work of Araki and Haag for massive theories with a complete particle interpretation [25]. No \textit{a priori} input about particles is necessary for its derivation; this shows that the concept of particle weight is sufficient for the description of the asymptotic particle features of any theory.

The next step in the analysis is the determination of the possible properties of particle weights. To this end one first constructs for each particle weight the corresponding sector of the physical Hilbert space,

\[ \mathcal{H}_{p,\iota} \equiv \overline{L|p,\iota\rangle} \subset \mathcal{H}. \]

One can then establish the following general results [26].

\begin{itemize}
  \item mass: the energy–momentum \( p \) of the underlying improper state \(|p,\iota\rangle\) and therefore its mass \( m^2 = p^2 \) can be sharply defined in the sector \( \mathcal{H}_{p,\iota} \) by the formula
    \[ U_{p,\iota}(x) \ L|p,\iota\rangle = e^{ipx} L(x)|p,\iota\rangle, \]
    where \( U_{p,\iota}(x) \) is the unitary representation of the translations on \( \mathcal{H}_{p,\iota} \) (which can be shown to exist).
  
  \item spin: if, for given \( p \), there is a finite multiplet of particle weights \( \langle p,\iota| \cdot |p,\iota\rangle \), there exists a unitary representation \( U_{p,\iota} \) of the (covering group of the) stability group \( \mathcal{R} \) of \( p \) such that for \( R \in \mathcal{R} \)
    \[ U_{p,\iota}(R) \ L|p,\iota\rangle = \sum_{\kappa} D_{\iota\kappa}(R^{-1}) L(R)|p,\kappa\rangle, \]
    where \( D \) are matrix representations of \( \mathcal{R} \) and \( L(R) = U(R)LU(R)^{-1} \). Thus if \( m > 0 \), the particle weights can have spin \( s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \), in accordance with the results found by Wigner. However, in contrast to the case of Wigner particles, there may appear for particle weights with mass \( m = 0 \) representations with arbitrary helicity.

  \item coherence: let \( \mathcal{H}_{p,\iota} = \int dq \mathcal{H}_{p,\iota}(q) \) be the decomposition of \( \mathcal{H}_{p,\iota} \) with respect to the spatial momentum and consider the restrictions \( H_{p,\iota} \upharpoonright \mathcal{H}_{p,\iota}(q) \) of the generator \( H_{p,\iota} \) of the time translations to the respective subspaces. For the point spectrum of these restrictions there appear the possibilities
    \[ \sigma_{\text{point}} \{ H_{p,\iota} \upharpoonright \mathcal{H}_{p,\iota}(q) \} \neq \left\{ \emptyset : \text{for all } q \in \mathbb{R}^3 \right\} \cup \left\{ \emptyset : \text{if and only if } q = p \right\}. \]
    The former case corresponds to the familiar situation of Wigner particles, where the improper states \(|q,\iota\rangle, \ q \in \mathbb{R}^3 \), are affiliated with the same sector and thus can coherently be superimposed. In the latter case, where \( \{ H_{p,\iota} \upharpoonright \mathcal{H}_{p,\iota}(q) \} \) has purely continuous spectrum for \( q \neq p \), the superposition principle fails for particle weights of different momenta. This situation is expected to prevail in theories with long range forces.
\end{itemize}
In view of the latter result it seems a rewarding experimental challenge to test the status of the superposition principle for electrically charged particle weights. The theory predicts that there ought to be a substantial difference in the asymptotic coherence properties of electrically neutral and charged particles.

In a final step one has to establish a collision theory for particles described by weights. In general one may not expect that a scattering matrix exists for these entities, but it is always possible to define and compute collision cross sections. A general method to this effect has been outlined in [24].

We conclude this section with the remark that, in contrast to the case of many particle quantum mechanics, the completeness of the particle interpretation in quantum field theory is still an open problem, even in the case of short range forces. A survey of the state of the art can be found in [27].

5 Algebraic holography and transplantation

Triggered by developments in string theory, known under the catchwords of holography or AdS/CFT correspondence (cf. [28] and references quoted there), there has recently emerged interest in the relation between quantum field theories on different space–time manifolds. One speaks of holography if there is a correspondence between a quantum field theory on a space–time \( \mathcal{M}_1 \) and a theory on its boundary \( \mathcal{M}_2 = \partial \mathcal{M}_1 \). In other words, given the theory on \( \mathcal{M}_1 \), one can unambiguously determine the theory on \( \mathcal{M}_2 \) and vice versa. A related notion is transplantation, where such a correspondence exists between theories on space–times \( \mathcal{M}_1, \mathcal{M}_2 \) of equal dimension.

In the conventional field theoretic setting a satisfactory understanding of these issues seems impossible. It is clear from the outset that one cannot identify point fields on \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) in a meaningful way,

\[
\phi_{\mathcal{M}_1}(x_1) \not\sim \phi_{\mathcal{M}_2}(x_2).
\]

With regard to holography, the best one can do is to proceed from a field theory on \( \mathcal{M}_1 \) by restriction to a corresponding theory on \( \mathcal{M}_2 \), cf. [24]. But it is in general impossible to recover from the latter data the original fields. Similar problems hamper also the idea of transplanting fields.

Here the algebraic point of view provides a solution: bearing in mind that a theory is fixed by the underlying net, one realizes that it is sufficient (and frequently possible) to identify local algebras for certain sufficiently rich families of regions \( \{ \mathcal{O}_1 \subset \mathcal{M}_1 \}, \{ \mathcal{O}_2 \subset \mathcal{M}_2 \} \),

\[
\mathcal{A}_{\mathcal{M}_1}(\mathcal{O}_1) \sim \mathcal{A}_{\mathcal{M}_2}(\mathcal{O}_2),
\]

thereby establishing a rigid link between the two theories in question. This insight was first used by Rehren in his analysis of the issue of holography [30]. Similar ideas were applied to the problem of transplantation in [31]. We outline in the following the underlying simple geometrical facts.

Holography:
The simplest example, where the idea of holography can be illustrated, is the
correspondence between quantum field theories on anti–de Sitter space (AdS) and on its Minkowskian boundary. In proper coordinates, AdS can be envisaged as a full cylinder whose tube like boundary is conformal Minkowski space. As indicated above, one has to identify suitable regions in the given space–times in order to establish a correspondence between the respective theories. For the case at hand, Rehren proposed to consider a family of causally complete wedge–shaped regions \( \{ W \subset \text{AdS} \} \). Their intersections with the boundary of AdS are diamond shaped regions, \( \{ C \equiv W \upharpoonright \partial \text{AdS} \} \), cf. Figure 1.

Figure 1: Wedges in AdS and diamonds on its Minkowskian boundary

It is crucial that with this choice there exists a bijection \( \gamma : \{ W \} \mapsto \{ C \} \) between these regions which is

- causal: \( \gamma(W') = \gamma(W)' \), where the prime indicates causal complementation
- symmetric: \( \gamma(gW) = g \gamma(W) \) for \( g \in \text{iso AdS} = \text{conf} \partial \text{AdS} \), where iso and conf indicate the isometry and conformal group of the respective spaces
- order preserving: \( \gamma(W_1) \subset \gamma(W_2) \) if \( W_1 \subset W_2 \).

After these geometrical preparations it is straightforward to establish the desired correspondence between nets on AdS and \( \partial \text{AdS} \) as well as the corresponding unitary representations of the respective symmetry groups and the vacuum states, setting

- \( \mathcal{A}_{\partial \text{AdS}}(C) \equiv \mathcal{A}_{\text{AdS}}(W) \) for \( W = \gamma^{-1}(C) \)
- \( U_{\partial \text{AdS}}(g) \equiv U_{\text{AdS}}(g) \)
- \( \Omega_{\partial \text{AdS}} \equiv \Omega_{\text{AdS}} \).

Starting from a local, covariant and stable net on AdS one obtains in this way a local, conformally covariant and stable net on Minkowski space and vice versa (by reading the defining relations from right to left). So one arrives at Proposition 5.1

**Proposition 5.1** There is a one–to–one correspondence between local, covariant and stable QFT’s on AdS and on \( \partial \text{AdS} = \text{conformal Minkowski space} \).
It seems plausible that the general idea of holography can also be applied to other space–time manifolds with suitable boundaries.

Transplantation:
The possibility of identifying quantum field theories on space–times of equal dimension (transplantation) has been exemplified in [31] for a special class of Robertson–Walker space–times (RW). In this approach one makes use of the fact that these space–times can be conformally embedded into de Sitter space (dS). Given this embedding, one chooses a certain specific family \( \{ \mathcal{C}_{dS} \} \) of diamond shaped regions in dS and, taking the intersection with RW of those regions \( \mathcal{C}_{dS} \) whose edge lies completely in RW, one obtains a corresponding family of regions \( \{ \mathcal{C}_{RW} \} \) in RW,

\[
\{ \mathcal{C}_{RW} \equiv \mathcal{C}_{dS} \cap RW : \text{edge } \mathcal{C}_{dS} \subset RW \}.
\]

This construction is indicated in Figure 2. We mention as an aside that the respective regions can also be characterized in an intrinsic coordinate independent manner.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{Penrose_diagram.png}
\caption{Penrose diagram with diamonds in RW (left) and dS (right)}
\end{figure}

Given these regions, there exists a unique bijection \( \gamma : \{ \mathcal{C}_{dS} \} \rightarrow \{ \mathcal{C}_{RW} \} \) with the property of being

\begin{itemize}
\item causal: \( \gamma(\mathcal{C}_{dS}') = \gamma(\mathcal{C}_{dS})' \)
\item symmetric: \( \dot{g} \gamma(\mathcal{C}_{dS}) = \gamma(g \mathcal{C}_{dS}), \ g \in \text{iso dS} \).
\end{itemize}

Here \( \dot{g} \) denotes the induced action of the elements \( g \) of the isometry group of de Sitter space, \( \text{iso dS} \), on the family of regions \( \{ \mathcal{C}_{RW} \} \). This action can in general not be described by a point transformation on RW. But if \( g \in \text{iso RW} \subset \text{iso dS} \) one has \( \dot{g} = g \).

On the basis of these geometrical facts it is straightforward to establish a one–to–one correspondence between theories on dS and RW, setting

\begin{itemize}
\item \( \mathcal{A}_{RW}(\mathcal{C}_{RW}) \equiv \mathcal{A}_{dS}(\mathcal{C}_{dS}) \) iff \( \mathcal{C}_{RW} = \gamma(\mathcal{C}_{dS}) \)
\item \( U_{RW}(\dot{g}) \equiv U_{dS}(g), \ g \in \text{iso dS} \)
\end{itemize}

An immediate consequence of the geometrical properties of \( \gamma \) is [31]
Proposition 5.2 Each local, covariant $dS$–theory can be mapped to a local, covariant (ultra symmetric) $RW$–theory and vice versa.

Here the term “ultra symmetric” means that the resulting $RW$ theory exhibits, besides the familiar space-time isometries, an additional geometrical symmetry which is not induced by point transformations. This situation differs from the AdS/CFT correspondence, where the Minkowskian theory on the boundary of AdS is conformally invariant.

6 Modular construction of local nets

An intriguing recent result in algebraic quantum field theory is the insight that local nets can be constructed from a few local algebras in suitable “relative positions” [32–35]. It is of interest in this context that the local von Neumann algebras are universal (model independent) objects: they are generically isomorphic to the unique hyperfinite type $III_1$ factor [36].

So the starting point of this novel construction is a concrete and well–studied algebra, denoted by $\mathcal{M}$ in the following; the second ingredient is a standard (cyclic and separating) vector $\Omega$ for $\mathcal{M}$ in the underlying Hilbert space. Given these quantities, one can consistently define a conjugation

$$S_{\mathcal{M}} : M\Omega \mapsto M^*\Omega.$$  

It is an anti–linear closable operator whose closure is denoted by the same symbol and whose polar decomposition has the form

$$S_{\mathcal{M}} = J_{\mathcal{M}} \Delta_{\mathcal{M}}^{1/2}.$$  

Here $J_{\mathcal{M}}$ is an anti–unitary operator, called modular conjugation, and the positive selfadjoint operator $\Delta_{\mathcal{M}}$ is the modular operator. The corresponding unitary group $\{\Delta_{\mathcal{M}}^{-is}\}_{s \in \mathbb{R}}$ is called modular group. Irrespective of the choice of $\mathcal{M}$ and $\Omega$ within the above limitations, there hold the following basic relations established by Tomita and Takesaki:

$$\begin{align*}
\triangleright & \quad \Delta_{\mathcal{M}}^{-is} \mathcal{M} \Delta_{\mathcal{M}}^{is} = \mathcal{M}, \quad s \in \mathbb{R} \\
\triangleright & \quad J_{\mathcal{M}} \mathcal{M} J_{\mathcal{M}}^{-1} = \mathcal{M}'.
\end{align*}$$

Based on these facts, Wiesbrock introduced in [38] the notion of half–sided modular inclusion of a von Neumann algebra $\mathcal{N} \subset \mathcal{M}$, where $\mathcal{N}$ likewise has $\Omega$ as standard vector, by posing the condition

$$\Delta_{\mathcal{M}}^{-is} \mathcal{N} \Delta_{\mathcal{M}}^{is} \subset \mathcal{N}, \quad s \in \mathbb{R}_+.$$  

(*)

The unitary group $U$, obtained by the Trotter product formula

$$U(t) \equiv \lim_{n \to \infty} (\Delta_{\mathcal{M}}^{-it/2\pi n} \Delta_{\mathcal{N}}^{it/2\pi n})^n, \quad t \in \mathbb{R},$$

then has the following properties [32].
Proposition 6.1  Let $\mathcal{N} \subset \mathcal{M}$ be a half–sided modular inclusion. The corresponding modular groups generate a unitary representation of $\mathbb{R}_+ \rtimes \mathbb{R}$ such that

a) $\Delta_{\mathcal{M}}^{-is} U(t) = U(e^{2\pi s} t) \Delta_{\mathcal{M}}^{-is}, \quad s, t \in \mathbb{R}$

b) $J_{\mathcal{M}} U(t) J_{\mathcal{M}}^{-1} = U(-t)$

c) the spectrum of the generator of $U$ is contained in $\mathbb{R}_+$

d) $\mathcal{N} = U(1) \mathcal{M} U(1)^{-1}$.

An analogous result holds if $\mathbb{R}_+$ is replaced by $\mathbb{R}_-$ in (\textasteriskcentered$\ast$). Another useful concept, characterizing the relative position of von Neumann algebras, is the notion of modular intersection [34]: two von Neumann algebras $\mathcal{M}, \mathcal{N}$ are said to have modular intersection if $\mathcal{M} \cap \mathcal{N}$ is half–sided modular in $\mathcal{M}$ and $\mathcal{N}$, respectively. Similarly to the situation discussed in the preceding proposition, the modular groups can be shown to generate a unitary representations of a Lie group in the latter case as well. These general mathematical facts have immediate applications in algebraic quantum field theory.

1. Local nets on $\mathbb{R}$ [33]

Any modular inclusion fixes a local, covariant and stable net of local algebras on the light ray $\mathbb{R}$. One first assigns algebras to half lines, setting

$\mathcal{A}(\mathbb{R}_+ + x) \equiv U(x) \mathcal{M} U(x)^{-1}, \quad \mathcal{A}(\mathbb{R}_- + y) \equiv U(y) \mathcal{M}' U(y)^{-1}$.

The algebras corresponding to arbitrary intervals $I = [x, y] \subset \mathbb{R}$ are given by

$\mathcal{A}(I) \equiv \mathcal{A}(\mathbb{R}_+ + x) \cap \mathcal{A}(\mathbb{R}_- + y)$.

It then follows from the preceding proposition and the basic relations of Tomita–Takesaki–Theory that the net $I \mapsto \mathcal{A}(I)$ on $\mathbb{R}$ is local (operators localized in disjoint intervals commute), $\mathbb{R}_+ \rtimes \mathbb{R}$–covariant and stable ($\Omega$ being a ground state for $U$). Thus, given two algebras in suitable relative position, one can construct a full chiral quantum field theory. (In a similar way one sees that any modular intersection fixes a conformally invariant quantum field theory on the compactified light ray $S^1$.)

2. Local nets on $\mathbb{R}^2$ [33]

In order to obtain in a similar manner local nets on higher dimensional space–times, one has to proceed from a larger set of algebras in appropriate relative positions. In the case of two–dimensional Minkowski space, one starts from three algebras, forming two half–sided modular inclusions $\mathcal{N}_\pm \subset \mathcal{M}$. By the preceding proposition one then has two unitary groups $U_\pm(x_\pm)$, where $x_\pm$ are interpreted as light cone coordinates of $x \in \mathbb{R}^2$. These groups are assumed to commute,

$U(x) \equiv U_+(x_+) U_-(x_-)^{-1} = U_-(x_-) U_+(x_+)$.
In this way one obtains a unitary positive energy representation $U$ of the translations $\mathbb{R}^2$ on which the modular group of $\mathcal{M}$ acts like a Lorentz transformation,

$$\Delta_{\mathcal{M}}^{-is}U(x)\Delta_{\mathcal{M}}^{is} = U_+(e^{2\pi s}x_+)U_-(e^{-2\pi s}x_-) = U(\Lambda(s)x).$$

One then can proceed as in the preceding case and define algebras for wedge shaped regions of the form $\mathcal{W} = \{ x \in \mathbb{R}^2 : x_1 > |x_0| \}$, setting

$$\mathcal{A}(\mathcal{W} + x) \equiv U(x)\mathcal{M}U(x)^{-1}, \quad \mathcal{A}(\mathcal{W}' + y) \equiv U(y)\mathcal{M}'U(y)^{-1}.$$

The algebras associated with diamonds $\mathcal{C}_{x,y} = (\mathcal{W} + x) \cap (\mathcal{W}' + y)$, cf. Figure 3, are obtained by setting

$$\mathcal{A}(\mathcal{C}_{x,y}) \equiv \mathcal{A}(\mathcal{W} + x) \cap \mathcal{A}(\mathcal{W}' + y).$$

In this way one arrives at a local, Poincaré–covariant net on $\mathbb{R}^2$ where $\Omega$ describes the vacuum vector.

Figure 3: A diamond obtained as intersection of wedges

3. Local nets on $\mathbb{R}^d$, $d = 3, 4$ [34, 35]

The construction of local nets from a few algebras was recently extended to three and four–dimensional Minkowski space in [34, 35]. The crucial and difficult step in this approach is the formulation of conditions which guarantee that the modular groups affiliated with the algebras and the underlying vector $\Omega$ generate representations of the Poincaré group $\mathcal{P}_+^\uparrow$.

**Proposition 6.2** Any family of algebras $\mathcal{M}_1, \ldots, \mathcal{M}_{d(d-1)/2+1}$ in suitable modular positions fixes a local, $\mathcal{P}_+^\uparrow$–covariant net on $\mathbb{R}^d$ with vacuum state $\Omega$.

We refrain from giving here the precise conditions on the algebras and only note that, in analogy to the cases discussed before, the corresponding modular groups generate representation of $\mathcal{P}_+^\uparrow$ with positive energy. The algebras $\mathcal{M}_i$ can consistently be assigned to wedge regions in $\mathbb{R}^d$ and the local algebras associated with diamonds are defined by taking intersections. The converse of this statement is a well–known theorem by Bisognano and Wichmann [37], cf. also [38].

These intriguing results should admit a generalization to other space–time manifolds, cf. also [13] for a related approach. Moreover, they seem to be of relevance for the classification of local nets and are possibly a step towards a novel, completely algebraic approach to the construction of local nets.
7 Conclusion

The preceding account of recent results in AQFT illustrates the role of this approach in relativistic quantum field theory: it is a concise framework which is suitable for the development of new constructive schemes, the mathematical implementation of physical concepts and ideas, the elaboration of general computational methods and the clarification of the relation between different theories as well as their structural analysis and classification. So this framework complements the more concrete approaches to relativistic quantum field theory and thereby contributes to the understanding and mathematical consolidation of this important area of mathematical physics.

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