Critical properties of dissipative quantum spin systems in finite dimensions

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Abstract

We study the critical properties of finite-dimensional dissipative quantum spin systems with uniform ferromagnetic interactions. Starting from the transverse field Ising model coupled to a bath of harmonic oscillators with Ohmic spectral density, we generalize its classical representation to classical spin systems with $O(n)$ symmetry and then take the large-$n$ limit to reduce the system to a spherical model. The exact solution to the resulting spherical model with long-range interactions along the imaginary time axis shows a phase transition with static critical exponents coinciding with those of the conventional short-range spherical model in $d + 2$ dimensions, where $d$ is the spatial dimensionality of the original quantum system. This implies that the dynamical exponent is $z = 2$. These conclusions are consistent with the results of Monte Carlo simulations and renormalization group calculations for dissipative transverse field Ising and $O(n)$ models in one and two dimensions. The present approach therefore serves as a useful tool for analytically investigating the properties of quantum phase transitions of the dissipative transverse field Ising and other related models. Our method may also offer a platform to study more complex phase transitions in dissipative finite-dimensional quantum spin systems, which have recently received renewed interest in the context of quantum annealing in a noisy environment.

Keywords: dissipation, quantum phase transitions, critical phenomena, quantum annealing

1. Introduction

The properties of dissipative quantum systems have been attracting attention for many years [1, 2]. A recent surge of interest in quantum annealing has increased the significance of this

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problem because a real quantum annealing machine always operates under the effects of environmental noise [3–6]. In this context it is important to understand how the critical phenomena of the transverse field Ising model representing quantum annealing are modified by its coupling to a heat bath. There exists a large body of investigation on this problem [1, 7–21], and it is generally established that the $d$-dimensional transverse field Ising model with uniform ferromagnetic interactions, coupled to a bath of harmonic oscillators with Ohmic spectral density, has static critical exponents which are equal to those of the $(d + z)$-dimensional classical Ising model with the dynamical critical exponent $z \approx 2$. These results have been derived by several different methods, including Monte Carlo simulations [16, 19–21] and renormalization group calculations [22, 23]. Nevertheless, complementary analytical approaches are desirable, not just to confirm these findings, but also to open a path toward investigations of more difficult cases, including those with first-order phase transitions or with disorder in the interactions of finite-dimensional systems, which are crucially relevant to the performance of quantum annealing in realistic settings [3–6, 24, 25].

The goal of the present paper is to shed new light on the problem of dissipation in quantum spin systems through the exact solution of the spherical model, which is obtained as the $n \rightarrow \infty$ limit of the $O(n)$ generalization of the dissipative transverse field Ising model. Although the spherical model generally has quantitatively different critical exponents than the Ising model, the former is known to successfully capture some of the essential features which are common to the latter, including the values of the upper and lower critical dimensions [26, 27]. It is also important that the spherical model allows us to find the exact solution in finite dimensions—even in the presence of long-range interactions [28]—as this helps us lay a firm basis to come to analytically reliable conclusions for more complex problems that may be out of reach with other methods. A similar idea for replacing Ising spins with spherical variables was used in [8, 29] to study quantum spin glasses with infinite-range interactions.

This paper is organized as follows. The next section introduces the spherical model as the limit of infinitely many components of the $O(n)$ model, the $n = 1$ case of which is the transverse field Ising model coupled to a bath of harmonic oscillators. We derive the exact solution to the spherical model including the critical exponents and the explicit form of the correlation function. The conclusions are described in section 3, and the technical details are delegated to the appendices.

2. Spherical model and its solution

We first introduce the $O(n)$ model with long-range interactions along the imaginary time axis as a generalization of the classical representation of the transverse field Ising model coupled to a bath of harmonic oscillators (to be called the dissipative transverse field Ising model). We then solve the model in the $n \rightarrow \infty$ limit, corresponding to the spherical model, to clarify the critical properties.

2.1. The $O(n)$ model as a generalization of the dissipative transverse field Ising model

Let us start with the Ising model in a transverse field coupled to a bath of harmonic oscillators. The Hamiltonian is $H = H_\text{S} \otimes I_\text{B} + I_\text{S} \otimes H_\text{B} + H_\text{I}$, where

$$H_\text{S} = -A \sum_{i=1}^{N} \sigma_i^z + B H_\text{I}(\{\sigma_i^z\}),$$

(1)
Here, $A$ and $B$ are positive parameters, the three components of $\sigma_i$ ($i = 1, 2, \ldots, N$) denote the Pauli operators, and $b_{ik}^\dagger$ and $b_{ik}$ are the $k$th bosonic operators at site $i$ with frequency $\omega_{ik}$. We keep the parameters $A$ and $B$ constant in this paper since we are focusing our attention on the static phase transitions—although $A$ and $B$ are considered to be time-dependent in the context of quantum annealing [24, 25, 30, 31]. Notice that each site (or qubit) $i$ is independently coupled to a different set of harmonic oscillators. We assume Ohmic-type dissipation with a finite cut-off, i.e., the spectral density is given by $J_G(\omega) = \sum_k k \rho_k \delta(\omega - \omega_k) = \alpha \omega$ [1], where $\alpha$ represents the coupling strength. As for $H_Z$, we consider the Ising model on the $d$-dimensional hypercubic lattice with nearest-neighbor ferromagnetic interactions and periodic boundaries in a uniform magnetic field,

$$H_Z(\sigma_i^z) = -J_0 \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - h_0 \sum_i \sigma_i^z,$$

(4)

where $J_0 > 0$ and $\langle ij \rangle$ denote nearest-neighbor pairs.

Introducing the discrete imaginary time variable $\tau = 1, 2, \ldots, M$ by the Suzuki–Trotter decomposition [32] and integrating out the bosonic degrees of freedom [33, 34], we can express the partition function at an inverse temperature $\beta$ in terms of the classical Ising variables $S_{i,\tau}$. The result is a $(d + 1)$-dimensional classical Ising model with long-range interactions along the imaginary time axis,

$$Z = \lim_{M \to \infty} C \sum_{\{S_{i,\tau} = \pm 1\}} \exp(-H_{\text{eff}}^{\text{Ising}}),$$

(5)

where $C$ is an unimportant constant, and the effective Hamiltonian is given by

$$H_{\text{eff}}^{\text{Ising}} = -K \sum_{\langle ij \rangle, \tau} S_{i,\tau} S_{j,\tau} - K_i \sum_{i,\tau} S_{i,\tau} S_{i,\tau+1} - \frac{\alpha}{2} \sum_{i,\tau, \sigma} \left( \frac{\pi}{M} \right)^2 \sin \frac{\pi (\tau - \sigma)}{M} S_{i,\tau} S_{i,\sigma} - h \sum_{i,\tau} S_{i,\tau} (S_{i,\tau} = \pm 1),$$

(6)

where

$$K = \frac{\beta}{M} B J_0, \quad K_i = -\frac{1}{2} \ln \tanh \frac{\beta A}{M}, \quad h = \frac{\beta}{M} B h_0.$$

(7)

The symbols $i$ and $j$ represent spatial coordinates running from 1 to $N$; $\sigma$ and $\tau$, both running from 1 to $M$, are for the coordinates along the imaginary time axis.

Following [19, 20, 22], we generalize the Ising model (6) to the $O(n)$ model, whose spin has $n$ components at each site with the normalization condition,

$$H_{\text{eff}}^{(n)} = -K \sum_{\langle ij \rangle, \tau} \sum_{a=1}^n S_{i,\tau}^a S_{j,\tau}^a - K_i \sum_{i,\tau} \sum_{a=1}^n S_{i,\tau}^a S_{i,\tau+1}^a - \frac{\alpha}{2} \sum_{i,\tau, \sigma} \left( \frac{\pi}{M} \right)^2 \sin \frac{\pi (\tau - \sigma)}{M} \sum_{a=1}^n S_{i,\tau}^a S_{i,\sigma}^a - h \sum_{i,\tau} \sum_{a=1}^n S_{i,\tau}^a \left( \sum_{a=1}^n (S_{i,\tau}^a)^2 = n, \ \forall \ i, \ \tau \right),$$

(8)
which would physically represent a single electron box, resistively shunted Josephson junctions, or superconductor-to-metal transitions in nanowires [19, 20, 35]. Another important reason for generalization to the $O(n)$ model is that the limit $n \to \infty$ is identical to the spherical model [36, 37],

$$H_{\text{eff}}^{sp} = -K \sum_{\langle j,k \rangle} S_j \tau S_k \tau - K_n \sum_{\langle j,k \rangle} S_j \tau S_k \tau + 1 - \frac{\alpha}{2} \left( \frac{\pi}{M} \right)^2 \sum_{\langle j,k \rangle \neq \sigma} \left( \frac{\sin \frac{\pi |r|}{M} - \sigma}{M} \right)^2 S_j \tau S_k \tau,$$

$$-h \sum_{\langle j,k \rangle} S_j \tau \left( -\infty < S_j \tau < \infty, \sum_{i=1}^{N} \sum_{\tau=1}^{M} (S_i \tau)^2 = NM \right). \quad (9)$$

The spherical model is exactly solvable for arbitrary dimensionality even in the presence of long-range interactions [28]. Since the spherical model is known to have critical properties, some of which are shared by the Ising model [26, 27, 37], we study equation (9) to extract useful information on the original dissipative Ising model in finite dimensions, equation (6). Similar approaches were successfully used to understand the properties of quantum spin glasses with an infinite range of interactions [8, 29]. Throughout this paper, we assume that $M$ and $\beta$ are large but finite, since the critical properties are generally considered to be independent of these parameters—particularly in the present case of the Ohmic bath spectrum [11].

It is straightforward to evaluate the partition function and the free energy of the spherical model in the limit of a large system size $N$ by the standard method [26, 28, 36, 37]:

$$Z \approx e^{-N F(z)}, \quad (10)$$

$$\frac{\beta}{M} F(z) = -\frac{1}{2} \ln 2\pi - z - \frac{h^2}{2z - K} + \frac{1}{2} \int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^d} \frac{1}{M} \sum_{\kappa} \ln(2z - \tilde{K}(k, \kappa)). \quad (11)$$

Here, $\tilde{K}(k, \kappa)$ is the Fourier transform of the coupling constants

$$K(r, \rho) = K \chi [r] \text{ is a nearest-neighbor vector and } \rho = 0$$

$$+ K_n \chi [r = 0 \text{ and } |\rho| = 1]$$

$$+ \alpha \left( \frac{\pi}{M} \frac{\sin(\pi |\rho|/M)}{\sin(\pi |\rho|/M)} \right)^2 \chi [r = 0 \text{ and } \rho = 0], \quad (12)$$

where $r$ and $\rho$ denote the relative coordinates in the spatial and imaginary time directions, respectively, and $\chi [A] = 1$ if $A$ is true and zero otherwise. Additionally, we set $\tilde{K} = \tilde{K}(0, 0)$. The variable $z (-\tilde{K}/2)$ is the solution to the saddle point equation

$$1 - \left( \frac{h}{2z - K} \right)^2 = H(z) = \int_{-\pi}^{\pi} \frac{d^2k}{(2\pi)^d} \frac{1}{M} \sum_{\kappa} \frac{1}{2z - \tilde{K}(k, \kappa)}. \quad (13)$$

When $h = 0$ and $H(\tilde{K}/2) < 1$, the saddle point $z$ sticks to $\tilde{K}/2$, and the system is in the ordered (ferromagnetic) phase. In other words, if there exists a set of parameters $(K, K_n, \alpha)$ such that $H(\tilde{K}/2) = 1$, a second-order phase transition exists. Equation (11) supplemented by equation (13) is the exact formal solution to the dissipative spherical model, equation (9).

2.2. Correlation functions

We are ready to calculate the correlation functions near criticality. Since $K(r, \rho) = K(-r, -\rho)$, the Fourier transform is
\[ \hat{K}(k, \kappa) = \sum_{r \in \mathbb{Z}^d} \sum_{\rho = -\infty}^{\infty} K(r, \rho) \cos(k \cdot r + \kappa \rho), \]  
(14)

where we assume \( M \) to be odd to simplify the analysis, which would not affect the results for sufficiently large \( M \). The wave number along the imaginary time axis \( \kappa \) has the following values:

\[ \kappa = \frac{2\pi \nu}{M}, \quad \nu = -\frac{M - 1}{2}, \ldots, -\frac{M - 3}{2}, -\frac{M - 1}{2}. \]  
(15)

As shown in appendix A, \( \hat{K}(k, \kappa) \) behaves for \( |k| \ll 1 \) as follows:

\[ \hat{K}(k, \kappa) \approx 2 \left[ K \sum_{u=1}^{d} \cos k_u + K_{\kappa} \cos \kappa \pi \alpha \left( \frac{\pi}{3} - |\kappa| \right) \right], \]  
(16)

where \( k = (k_1, \ldots, k_d) \).

Because the terms satisfying \( |k|, |\kappa| \ll 1 \) dominate the integral on the right-hand side of equation (13) near the critical point \( z = \hat{K}/2 \), we can correctly evaluate the asymptotic behavior of the correlation functions near criticality by the approximation of \( \hat{K}(k, \kappa) \) for small \( |k| \) and \( |\kappa| \) as

\[ \hat{K} - \hat{K}(k, \kappa) \approx Kk^2 + K_{\kappa}\kappa^2 + \pi \alpha |\kappa|. \]  
(17)

Then the asymptotic form of the correlation function

\[ G(r, \rho) = \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d M} \sum_{\kappa} \frac{e^{i(k \cdot r + \kappa \rho)}}{2\pi - \hat{K}(k, \kappa)} \approx \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d+1} \frac{e^{i(k \cdot r + \kappa \rho)}}{2\pi - \hat{K}(k, \kappa)} \]  
(18)

can be described by the expression

\[ G(r, \rho) \approx \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d+1} \frac{e^{i(k \cdot r + \kappa \rho)}}{2\pi - \hat{K}(k, \kappa) + K_{\kappa} \kappa^2 + \pi \alpha |\kappa|}, \]  
(19)

where \( u = 2\pi - \hat{K} \). Let us assume that \( u > 0 \), i.e., the system is in the paramagnetic phase. If \( r = |r| \gg \sqrt{\hat{K}/u} = \max \sqrt{\hat{K}/(u + K_{\kappa}\kappa^2 + \pi \alpha |\kappa|)} \), we can evaluate the integral with respect to \( k \) by the Ornstein–Zernike formula as

\[ G(r, \rho) \sim \frac{1}{K} \int \frac{d\kappa}{2\pi} e^{i\kappa \rho} \exp \left( -\frac{u + K_{\kappa}\kappa^2 + \pi \alpha |\kappa|}{K} r \right). \]  
(20)

The fact that \( r \) is sufficiently large and the absolute value of the integrand is maximum at \( \kappa = 0 \) yields

\[ G(r, \rho) \sim \frac{1}{K} e^{-r/\xi} \quad (r \gg \xi), \]  
(21)

where the correlation length is

\[ \xi = \frac{\sqrt{u}}{K}. \]  
(22)

Equation (21) with equation (22) is the exact asymptotic form of the spatial correlation function in the paramagnetic phase \( u > 0 \).
On the other hand, the correlation function along the imaginary time axis obeys a power law:

\[ G(0, \rho) \approx \frac{\alpha}{(\mu \rho)^{25/7}} \left( \rho \gg \frac{\pi}{2K_{c}} \right) \]

which we show in appendix B since the computations are somewhat involved. Because of the power decay of this correlation function, we cannot define the correlation time in the paramagnetic phase—a situation similar to the case of the dissipative transverse field Ising model [16] and the Josephson length [15]. It may, in principle, be possible to calculate the connected correlation function along the imaginary time axis in the ferromagnetic phase, but in practice it is difficult. The connected correlation function in the ferromagnetic phase is expected to decay exponentially, from which the correlation time \( \xi \) is defined and the dynamical exponent \( z \) is consequently extracted from \( \xi \sim \xi^{z} \). We leave it as an open problem to evaluate \( z \) directly along this line of analysis.

2.3. Critical behavior

We next calculate the critical exponents of the spherical model. First, the behavior of \( H(z) \) is investigated in order to extract critical properties using the large-wavelength expression of the integrand of the function \( H(z) \), which is relevant to the critical behavior,

\[ H(z) \approx \int_{-\pi}^{\pi} \frac{dk d\kappa}{(2\pi)^{d+1}} \frac{1}{u + K\kappa^{2} + K_{c}\kappa^{2} + \pi\alpha|\kappa|} \]

\[ \sim \frac{1}{K_{c}} \int_{0}^{\pi} d\kappa k^{d-1} \int_{0}^{\pi} d\kappa \left[ \left( \kappa + \frac{\pi\alpha}{2K_{c}} \right)^{2} - \left( \frac{\pi\alpha}{2K_{c}} \right)^{2} + \frac{K\kappa^{2} + u}{K_{c}} \right]^{-1}. \]

Here the symbol \( \approx \) stands for an approximation which is relevant to the asymptotic behavior, and \( \sim \) denotes a similar approximation but with an unimportant constant dropped. Since the choice of the lattice spacing along the imaginary time axis, \( \beta/M \), does not affect the universality which the Hamiltonian (9) exhibits, we fix \( \beta/M \) to a sufficiently small value to simplify the analysis. This supposition yields \( 1 \gg \alpha/K_{c} \gg K/\alpha \). Noticing that \( (\pi\alpha/2K_{c})^{2} \gg (K\kappa^{2} + u)/K_{c} \), because \( u \approx 0 \) near criticality, we obtain

\[ H(z) \sim \int_{0}^{\pi} d\kappa k^{d-1} \ln \left( \frac{1 + \frac{\alpha}{2K_{c}} - \frac{\kappa}{\pi\alpha}}{1 + \frac{\alpha}{2K_{c}} + \frac{\kappa}{\pi\alpha}} \right) \left( \frac{1}{2} + \frac{K_{c}}{\kappa} \right), \]

where

\[ c(u, \kappa) = \sqrt{\left( \frac{\pi\alpha}{2K_{c}} \right)^{2} - \frac{K\kappa^{2} + u}{K_{c}} \approx \frac{\pi\alpha}{2K_{c}} - \frac{K\kappa^{2} + u}{\pi\alpha}. \]

The use of \( (2cK_{c})^{-1} \sim \alpha^{-1} \) and

\[ \ln \left( \frac{1 + \frac{\alpha}{2K_{c}} - \frac{\kappa}{\pi\alpha}}{1 + \frac{\alpha}{2K_{c}} + \frac{\kappa}{\pi\alpha}} \right) \approx \ln \left( \frac{1 + \frac{K\kappa^{2} + u}{\pi\alpha}}{1 + \frac{K\kappa^{2} + u}{\pi\alpha}} \right) \left( \frac{K_{c}(K\kappa^{2} + u)}{(\pi\alpha)^{2}} \right) \approx -\ln \left( \frac{K_{c}(K\kappa^{2} + u)}{(\pi\alpha)^{2}} \right) \]

\[ (\pi\alpha)^{2} \]

(27)
results in

\[ H(z) \sim \frac{1}{\alpha} \int_0^\pi dk \, k^{d-1} \left( -\ln \frac{K_c (Kk^2 + u)}{(\pi \alpha)^2} \right). \]

(28)

Therefore, we conclude that

\[ H(z = \tilde{K}/2) \sim \frac{1}{\alpha} \int_0^\pi dk \, k^{d-1} \ln \frac{(\pi \alpha)^2 G}{k^2}, \]

where \( G = (KK_c)^{-1} \). This equation shows that \( H(\tilde{K}/2) \) converges if and only if \( d > 0 \).

For \( d > 0 \), we simplify equation (29) using the fact that \( \ln G \) is dominant for small \( \beta/M \):

\[ H(\tilde{K}/2) \sim \frac{\ln G}{\alpha}. \]

(30)

As a result, the system is in the ordered (ferromagnetic) phase if \( \ln G/\alpha \lesssim C_d \iff G^{-1} \gtrsim \exp(-C_d \alpha) \) and is in the disordered (paramagnetic) phase if \( G^{-1} \lesssim \exp(-C_d \alpha) \), where \( C_d \) is a constant which depends only on \( d \). We show the resulting phase diagram in figure 1. The phase boundary \( G^{-1} \sim \exp(-C_d \alpha) \) resembles those in [16, 19, 21] for the dissipative Ising and XY models. One difference is that there is no transition on the line \( G^{-1} = 0 \) in the present case, whereas the dissipative Ising model has one. This would have originated from the continuous nature of the spherical spins [19].

To clarify the critical behavior, we investigate the \( u \)-dependence of \( H(z = \tilde{K}/2) - H(z) \) for \( d > 0 \). Changing variables to \( x = Kk^2/\alpha \) in equation (28) results in

\[ H(\tilde{K}/2) - H(z) \sim \frac{1}{\alpha} \left( \frac{u}{K} \right)^{d/2} \int_0^{Kx^2/\alpha} dx \, x^{(d-2)/2} \ln \left( 1 + \frac{1}{x} \right). \]

(31)

Since the behavior of the integrand is

\[ x^{(d-2)/2} \ln \left( 1 + \frac{1}{x} \right) \approx \begin{cases} -x^{(d-2)/2} \ln x, & x \ll 1, \\ x^{(d-4)/2}, & x \gg 1, \end{cases} \]

(32)
it is only for $d \geq 2$ that the region $x \gg 1$ dominates the integral. Consequently, we have

$$H(\tilde{K}/2) - H(z) \sim \frac{1}{\alpha} \times \begin{cases} \left(\frac{u}{K}\right)^{\frac{d}{2}}, & \quad 0 < d < 2, \\ -\frac{u}{K} \ln \frac{u}{K}, & \quad d = 2, \\ \frac{u}{K}, & \quad 2 < d. \end{cases} \tag{33}$$

This equation coincides with the corresponding equation of the standard classical $(d+2)$-dimensional spherical model with nearest-neighbor interactions [26]. That is, the $d$-dimensional dissipative quantum system corresponds to the $(d+2)$-dimensional classical system.

For $0 < d < 2$, we solve the saddle point equation (13) and derive the critical exponents. If $d > 2$, one should formally insert $d = 2$. In other words, the upper critical dimension is two. We fix the coupling strength $\alpha$ and concentrate on the phase transition which occurs as $G$ changes. This arrangement reflects the situation of quantum annealing. For $h = 0$, equation (13) leads to $H(z) = 1 - H(\tilde{K}/2)$, where $\tilde{K}$ denotes the value of $K$ at the critical point $G = G_c$. Using equations (30) and (33), we obtain

$$\frac{1}{\alpha} \left(\frac{u}{K}\right)^{\frac{d}{2}} \sim \frac{\ln G - \ln G_c}{\alpha} \approx \frac{1}{\alpha} \frac{G - G_c}{G_c} \tag{34}$$

and hence

$$2z - \tilde{K} = u \sim Kg^{2/d}, \tag{35}$$

where $g = (G - G_c)/G_c$. This solution allows us to rewrite physical quantities, like the free energy expressed in terms of $u$, by the physical variable $g$. It is then straightforward to calculate the following critical exponents in a standard manner [26]:

$$\alpha = \frac{d - 2}{d}, \quad \beta = \frac{1}{2}, \quad \gamma = \frac{2}{d}, \quad \delta = \frac{d + 4}{d}. \tag{36}$$

Remember that $\alpha$ and $\beta$ here denote the critical exponents of the specific heat and the magnetization, respectively, which should not be confused with the spin-bath coupling constant and the inverse temperature. Next, equations (22) and (35) yield the correlation length $\xi \sim g^{-1/d}$ and the critical exponent

$$\nu = \frac{1}{d}. \tag{37}$$

We find that exponents (36) and (37) coincide with those for the standard classical $(d+2)$-dimensional spherical model. This result suggests that the dynamical exponent is $z = 2$. In addition, we can derive the exponent $z + \eta = 2$ from equation (19), followed by $\eta = 0$.

These values of the critical exponents are to be compared with the results of the Monte Carlo simulations of the dissipative Ising and $XY$ models in one and two dimensions [16, 19, 21] (see, in particular, table I of [19]). For example, the dissipative transverse field Ising model in one dimension has $\nu = 0.638(3)$, $\eta = 0.015(20)$ and $z = 1.985(15)$ according to the Monte Carlo simulations [19], whereas our dissipative spherical model with $d = 1$ has $\nu = 1$, $\eta = 0$ and $z = 2$. The difference in $\nu$ (0.638(3) and 1) reflects the difference in this exponent between the standard classical Ising model in three dimensions ($\nu = 0.6301(4)$ [38]) and the spherical model in three dimensions ($\nu = 1$ [26, 27]). It is also noticed that the
The $\epsilon$-expansion of the critical exponents of the $O(n)$ model discussed in [19, 23] is in perfect agreement with our results. For example, the $\epsilon$-expansion of the exponent $\nu$ [19, 23] is, in the $n \to \infty$ limit,

$$\nu = \frac{1}{2} + \frac{1}{4} \epsilon + \frac{1}{8} \epsilon^2 + O(\epsilon^3), \tag{38}$$

which is correctly reproduced by equation (37) with $d = 2 - \epsilon$. Our approach and the renormalization group method are complementary to each other, in the sense that the former is exact for any $d$ in the limit $n \to \infty$ whereas the latter is valid for any $n$ as long as $\epsilon$ is not large.

3. Conclusion

We have studied the critical properties of the spherical model with long-range interactions along the imaginary time axis, in addition to the usual nearest-neighbor ferromagnetic interactions in the spatial and imaginary time axes. This model has been introduced as the $n \to \infty$ limit of the $O(n)$ generalization of the transverse field Ising model coupled to a bath of harmonic oscillators with Ohmic spectral density, in which the ultimate goal in mind is to understand the effects of noise on quantum annealing.

The solution to our dissipative spherical model has revealed that the static critical exponents in $d$ spatial dimensions completely coincide with those of the classical spherical model in $d + 2$ dimensions. One of the strengths of our method lies in the exactness of the values of the static critical exponents for the spherical version of the model. If we use the standard argument that the effective dimensionality of a quantum system in $d$ spatial dimensions is $d + z$ [7, 39], where $z$ is the dynamical critical exponent, we conclude that the dynamical exponent is exactly equal to two. The result $z = 2$ is consistent with a naïve power-counting argument [21] using the bare propagator $k^2 + |\kappa|$ of a scalar field theory corresponding to the dissipative transverse field Ising model, in which one compares the $k^2$ term with the linear-$\kappa$ term, requiring $|\kappa|$ to be comparable to $k^z$, i.e., $|\kappa| \approx k^z$ with $z = 2$. It is remarkable that this simple power-counting argument for the bare propagator of the scalar field theory turns out to be essentially exact for the present spherical model, as also seen in equation (19).

We should, of course, be careful to draw too strong a conclusion on the original problem of the dissipative transverse field Ising model. Nevertheless, Monte Carlo simulations and renormalization group calculations of the $d = 1$ and $d = 2$ dissipative transverse field Ising models show that the static critical exponents are close to those of the classical Ising models in $d = 2 \approx 1 + 2$ and $d = 3 \approx 2 + 2$, respectively, and that the dynamical exponent $z$ is close to two [16, 19–23]. Similar results are also found in the dissipative XY model in one dimension [19, 35]. These findings are consistent with our results for the spherical model, which lends its support to the expectation that the analysis of the spherical model is a useful tool for understanding the effects of dissipation on the transverse field Ising model.

From the viewpoint of quantum annealing, the existence of a second-order phase transition does not spell difficulty because the energy gap at such a transition point closes polynomially as a function of the system size [39], which means that the computation time of quantum annealing grows only polynomially; thus, the problem is easy to solve. This implies that the quantitative change of critical exponents by dissipation affects the performance of quantum annealing only quantitatively, not qualitatively, in the present system with uniform ferromagnetic interactions. More serious and practically important are the instances with a
first-order phase transition and/or randomness in the interactions—in particular in finite dimensions—which are the target of our work in progress.

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Appendix A. Large-wavelength behavior of the interaction

Here we derive equation (16). Although the result has been known for years, an explicit presentation of its rigorous derivation seems to be missing in the references; it is thus useful to write out the derivation so that the paper is self-contained.

The first and second terms on the right-hand side of equation (16) produced by the nearest-neighbor interactions are trivial, so that we derive only the following expression:

$$\left(\frac{\pi}{M}\right)^2 \sum_{\rho=1}^{M-1} \cos \frac{\kappa \rho}{M} \approx \frac{\pi}{2} \left(\frac{\pi}{3} - |\kappa|\right),$$

(A1)

which is valid for $|\kappa| \ll 1$. Changing the variable $\kappa$ to $\nu$ with equation (15), the left-hand side of equation (A1) is rewritten as

$$S_{\nu} = \left(\frac{\pi}{M}\right)^2 \sum_{\rho=1}^{M-1} \cos \frac{2\pi \nu \rho}{M}. $$

(A2)

Although $\nu$ can have values such that $\nu^{-1} = o(M^0)$, we show

$$S_{\nu} = \pi^2 \left(\frac{1}{6} - \frac{1}{M^2}\right) + O\left(\frac{1}{M^2}\right)$$

(A3)

for $\nu \in \mathbb{Z}$, which is independent of $M$, because $|\kappa|$ is sufficiently small. Changing $\nu$ back to $\kappa$ yields the right-hand side of equation (A1).

We derive the expression of $S_0$ before that of $S_{\nu}$. Let us divide $S_0$ into two terms:

$$S_0 = \left(\frac{\pi}{M}\right)^2 \sum_{\rho=1}^{M-1} \left(\frac{1}{\sin^2 \frac{\pi \rho}{M}} - \frac{1}{\left(\frac{\pi \rho}{M}\right)^2}\right) + \sum_{\rho=1}^{M-1} \frac{1}{\rho^2}. $$

(A4)

Consider the first term. The fact that the singularity of the function $(\sin x)^{-2} - x^{-2}$ at $x = 0$ is removable allows us to replace the sum with the integral:

$$\left(\frac{\pi}{M}\right)^2 \sum_{\rho=1}^{M-1} \left(\frac{1}{\sin^2 \frac{\pi \rho}{M}} - \frac{1}{\left(\frac{\pi \rho}{M}\right)^2}\right) = \frac{\pi}{M^2} \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{\pi/2} dx \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right) + O\left(\frac{1}{M^2}\right)$$

$$= \frac{\pi}{M^2} \lim_{\varepsilon \downarrow 0} \left(\frac{1}{\tan \varepsilon} - \frac{1}{\varepsilon} + \frac{2}{\pi}\right) + O\left(\frac{1}{M^2}\right)$$

$$= \frac{2}{M} + O\left(\frac{1}{M^2}\right). $$

(A5)
Next, the second term on the right-hand side of equation (A4) is

\[
\sum_{\rho=1}^{\infty} \frac{1}{\rho^2} = \sum_{\rho=1}^{\infty} \frac{1}{\rho^2} - \left( \frac{\pi}{M} \right)^2 \sum_{\rho=1}^{\infty} \frac{1}{\rho} (\frac{\pi}{M})^2 = \frac{\pi^2}{6} - \frac{\pi}{M} \int_{\pi/2}^{\infty} \frac{dx}{x^2} + O\left( \frac{1}{M^2} \right) = \frac{\pi^2}{6} - \frac{2}{M} + O\left( \frac{1}{M^2} \right). \tag{A6}
\]

We thus have

\[
S_0 = \frac{\pi^2}{6} + O\left( \frac{1}{M^2} \right) \tag{A7}
\]

Let us now evaluate the sum

\[
S_0 - S_\nu = 2\left( \frac{\pi}{M} \right)^2 \sum_{\rho=1}^{\infty} \frac{\sin \frac{\pi\rho}{M}}{\sin \frac{\pi\rho}{M}}. \tag{A8}
\]

We can impose the restriction \( \nu \in \mathbb{N} \) without loss of generality since \( S_\nu \) is invariant under the transformation \( \nu \rightarrow -\nu \). We then have

\[
S_0 - S_\nu = 2\frac{\pi}{M} \int_0^{\pi/2} \frac{dx}{\sin x} \left( \frac{\sin \nu x}{\sin x} \right)^2 + O\left( \frac{1}{M^2} \right) = \frac{\pi^2}{M} \nu + O\left( \frac{1}{M^2} \right). \tag{A9}
\]

Subtracting this equation from equation (A7) and using \( S_\nu = S_{|\nu|} \) for \( \nu \in \mathbb{Z} \) lead to equation (A3).

**Appendix B. Inverse-square decay of the imaginary time correlation function**

We evaluate the imaginary time correlation function \( G(0, \rho) \) in equation (19). Introducing an auxiliary variable \( t \) to raise the denominator to the exponent, we obtain

\[
G(0, \rho) \approx \int_0^\infty dt \: e^{-ut} \int_{-\pi}^{\pi} \frac{d^dk}{(2\pi)^d} \exp(-Kik^2) \int_{-\pi}^{\pi} \frac{dk}{2\pi} \exp(-K_0 k^2 - \pi\alpha t|k| + i\rho \kappa). \tag{B1}
\]

The integral with respect to \( k \) is written as

\[
\int_{-\pi}^{\pi} \frac{d^dk}{(2\pi)^d} \exp(-Kik^2) = \left( \frac{\text{erf}(\pi\sqrt{Kt})}{\sqrt{4\pi Kt}} \right)^d, \tag{B2}
\]

where \( \text{erf} = \frac{2}{\sqrt{\pi}} \int_0^z dt \: e^{-t^2} \) denotes the error function. Because the values at \( t \ll u^{-1} \) dominate the integral (B1), we keep only the lowest order in \( \sqrt{t} : \text{erf}(\pi\sqrt{Kt}) \approx \sqrt{4\pi Kt} \). Thus,

\[
\int_{-\pi}^{\pi} \frac{d^dk}{(2\pi)^d} \exp(-Kik^2) \approx 1. \tag{B3}
\]
The integral with respect to $\kappa$ in equation (B1) is reduced to
\[
\int_{-\pi}^{\pi} \frac{d\kappa}{2\pi} \exp(-K_t n \kappa^2 - \pi \alpha t |\kappa| + i\kappa \rho) = 2\pi \int_{0}^{\pi} \frac{d\kappa}{2\pi} \exp[-K_t n \kappa^2 - (\pi \alpha t + i\rho)\kappa]
= \frac{1}{\sqrt{4\pi K_t t}} \Re \left[ \frac{2}{\sqrt{\pi}} \exp\left(\frac{\pi \alpha t + i\rho)^2}{4K_t t}\right) \right.
\times \left. \int_{(\pi \alpha t + i\rho)/\sqrt{4K_t t}}^{\infty} \exp(-x^2) \, dx \right],
\]
(B4)
where we changed variables to $x = \sqrt{K_t t} \left(\kappa + \frac{\pi \alpha t + i\rho}{2K_t t}\right)$. We can write this integral with the scaled complementary error function $\text{erfcx} z = e^x (1 - \text{erf} z) = \frac{2}{\sqrt{\pi}} e^{\pi^2} \int_{1}^{\infty} \frac{e^{-t^2}}{t} \, dt$:
\[
\int_{-\pi}^{\pi} \frac{d\kappa}{2\pi} \exp(-K_t n \kappa^2 - \pi \alpha t |\kappa| + i\kappa \rho)
= \frac{1}{\sqrt{4\pi K_t t}} \Re \left[ \text{erfcx} \left(\frac{\pi \alpha t + i\rho}{\sqrt{4K_t t}}\right) - \exp[-\pi^2 (K_t + \alpha)t - i\kappa \rho]\right.
\times \left. \text{erfcx} \left(\frac{\pi (2K_t + \alpha)t + i\rho}{\sqrt{4K_t t}}\right) \right],
\]
(B5)
Consider the case of $u \rho \gg \pi (2K_t + \alpha)$. Then,
\[
\pi \alpha \sqrt{t} \ll \pi (2K_t + \alpha) \sqrt{t} \ll \frac{\pi (2K_t + \alpha)}{\sqrt{t}} \ll \rho \sqrt{t} \ll \frac{\rho}{\sqrt{t}}
\]
for $t \ll u^{-1}$, and hence we use the series expansion
\[
\Re \text{erfcx} \left(\frac{\pi \alpha t + i\rho}{\sqrt{4K_t t}}\right) = \Re \text{erfcx} \left(\frac{i\rho}{\sqrt{4K_t t}}\right) + \frac{\pi \alpha t}{\sqrt{4K_t t}} \Re \text{erfcx} \left(\frac{i\rho}{\sqrt{4K_t t}}\right) + \ldots.
\]
(B7)
The zeroth order term
\[
\Re \text{erfcx} \left(\frac{i\rho}{\sqrt{4K_t t}}\right) = \exp\left(-\frac{\rho^2}{4K_t t}\right)
\]
is exponentially small for small $t$ and can be neglected, while the first behaves as
\[
\Re \text{erfcx} \left(\frac{i\rho}{\sqrt{4K_t t}}\right) \sim \frac{4K_t t}{\sqrt{\pi} \rho^2}
\]
(B9)
because asymptotically $\text{erfcx} z \sim (\sqrt{\pi} z)^{-1}$ as $z \to \infty$ with $|\text{arg} z| < 3\pi/4$ [40]. We therefore find that
\[
\Re \text{erfcx} \left(\frac{\pi \alpha t + i\rho}{\sqrt{4K_t t}}\right) \approx \frac{\pi \alpha t}{\sqrt{4\pi K_t t}} \Re \text{erfcx} \left(\frac{i\rho}{\sqrt{4K_t t}}\right) \approx \sqrt{\frac{4\pi K_t t}{\rho^2}} \frac{\alpha t}{\rho^2}
\]
and likewise
\[
\Re \text{erfcx} \left(\frac{(2K_t + \alpha)t + i\rho}{\sqrt{4K_t t}}\right) \approx \sqrt{\frac{4\pi K_t t}{\rho^2}} \frac{(2K_t + \alpha)t}{\rho^2}.
\]
(B10)
Remembering $\rho \in \mathbb{Z}$, we obtain

$$
\int_{-\pi}^{\pi} \frac{dk}{2\pi} \exp(-K_t tr^2 - \pi\alpha |k| + i\kappa \rho) 
\approx \frac{\alpha t - (-1)^\rho (2K_t + \alpha)t \exp[-\pi^2(K_t + \alpha)t]}{\rho^2}.
$$

(B12)

Substituting equations (B3) and (B12) into equation (B1) and calculating the integral result in

$$
G(0, \rho) \approx \frac{\alpha}{(\mu\rho)^2} - (-1)^\rho \frac{2K_t + \alpha}{\left[\mu + \pi^2(K_t + \alpha)\rho\right]^2}.
$$

(B13)

Since the first term is the leading contribution near the criticality, we finally arrive at equation (23).

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