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Volume 15, no 2 (2008), p. 135-146.

<http://ambp.cedram.org/item?id=AMBP_2008__15_2_135_0>

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A Note on Free Quantum Groups

Teodor Banica

Abstract

We study the free complexification operation for compact quantum groups, \( G \to G^c \). We prove that, with suitable definitions, this induces a one-to-one correspondence between free orthogonal quantum groups of infinite level, and free unitary quantum groups satisfying \( G = G^c \).

Introduction

In this paper we present some advances on the notion of free quantum group, introduced in [3]. We first discuss in detail a result mentioned there, namely that the free complexification operation \( G \to G^c \) studied in [2] produces free unitary quantum groups out of free orthogonal ones. Then we work out the injectivity and surjectivity properties of \( G \to G^c \), and this leads to the correspondence announced in the abstract. This correspondence should be regarded as being a first general ingredient for the classification of free quantum groups.

We include in our study a number of general facts regarding the operation \( G \to G^c \), by improving some previous work in [2]. The point is that now we can use general diagrammatic techniques from [4], new examples, and the notion of free quantum group [3], none of them available at the time of writing [2].

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Keywords: Free quantum group.
Math. classification: 16W30.
The paper is organized as follows: 1 contains some basic facts about the operation $G \to G^c$, and in 2-5 we discuss the applications to free quantum groups.

1. Free complexification

A fundamental result of Voiculescu [6] states if $(s_1, \ldots, s_n)$ is a semicircular system, and $z$ is a Haar unitary free from it, then $(zs_1, \ldots, zs_n)$ is a circular system. This makes appear the notion of free multiplication by a Haar unitary, $a \to za$, that we call here free complexification. This operation has been intensively studied since then. See Nica and Speicher [5].

This operation appears as well in the context of Wang’s free quantum groups [7], [8]. The main result in [1] is that the universal free biunitary matrix is the free complexification of the free orthogonal matrix. In other words, the passage $O_n^+ \to U_n^+$ is nothing but a free complexification: $U_n^+ = O_n^{+c}$. Moreover, some generalizations of this fact are obtained, in an abstract setting, in [2].

In this section we discuss the basic properties of $A \to \tilde{A}$, the functional analytic version of $G \to G^c$. We use an adaptation of Woronowicz’s axioms in [9].

**Definition 1.1.** A finitely generated Hopf algebra is a pair $(A, u)$, where $A$ is a $C^*$-algebra and $u \in M_n(A)$ is a unitary whose entries generate $A$, such that

\[
\Delta(u_{ij}) = \sum u_{ik} \otimes u_{kj}, \\
\varepsilon(u_{ij}) = \delta_{ij}, \\
S(u_{ij}) = u_{ji}^*
\]

define morphisms of $C^*$-algebras (called comultiplication, counit and antipode).

In other words, given $(A, u)$, the morphisms $\Delta, \varepsilon, S$ can exist or not. If they exist, they are uniquely determined, and we say that we have a Hopf algebra.

The basic examples are as follows:

(1) The algebra of functions $A = C(G)$, with the matrix $u = (u_{ij})$ given by $g = (u_{ij}(g))$, where $G \subset U_n$ is a compact group.
(2) The group algebra $A = C^*(\Gamma)$, with the diagonal matrix $u = \text{diag}(g_1, \ldots, g_n)$, where $\Gamma = \langle g_1, \ldots, g_n \rangle$ is a finitely generated group.

Let $\mathbb{T}$ be the unit circle, and let $z : \mathbb{T} \to \mathbb{C}$ be the identity function, $z(x) = x$. Observe that $(C(\mathbb{T}), z)$ is a finitely generated Hopf algebra, corresponding to the compact group $\mathbb{T} \subset U_1$, or, via the Fourier transform, to the group $\mathbb{Z} = \langle 1 \rangle$.

**Definition 1.2.** Associated to $(A, u)$ is the pair $(\tilde{A}, \tilde{u})$, where $\tilde{A} \subset C(\mathbb{T})^*$ is the $C^*$-algebra generated by the entries of the matrix $\tilde{u} = zu$.

It follows from the general results of Wang in [7] that $(\tilde{A}, \tilde{u})$ is indeed a finitely generated Hopf algebra. Moreover, $\tilde{u}$ is the free complexification of $u$ in the free probabilistic sense, i.e. with respect to the Haar functional. See [2].

A morphism between two finitely generated Hopf algebras $f : (A, u) \to (B, v)$ is by definition a morphism of $\ast$-algebras $A_s \to B_s$ mapping $u_{ij} \to v_{ij}$, where $A_s \subset A$ and $B_s \subset B$ are the dense $\ast$-subalgebras generated by the elements $u_{ij}$, respectively $v_{ij}$. Observe that in order for such a morphism to exist, $u, v$ must have the same size, and that if such a morphism exists, it is unique. See [2].

**Proposition 1.3.** The operation $A \to \tilde{A}$ has the following properties:

1. We have a morphism $(\tilde{A}, \tilde{u}) \to (A, u)$.

2. A morphism $(A, u) \to (B, v)$ produces a morphism $(\tilde{A}, \tilde{u}) \to (\tilde{B}, \tilde{v})$.

3. We have an isomorphism $(\tilde{\tilde{A}}, \tilde{\tilde{u}}) = (\tilde{A}, \tilde{u})$.

**Proof.** All the assertions are clear from definitions, see [2] for details. □

**Theorem 1.4.** If $\Gamma = \langle g_1, \ldots, g_n \rangle$ is a finitely generated group then $\tilde{C}^*(\Gamma) \simeq C^*(\mathbb{Z} \ast \Lambda)$, where $\Lambda = \langle g_i^{-1}g_j | i, j = 1, \ldots, n \rangle$.

**Proof.** By using the Fourier transform isomorphism $C(\mathbb{T}) \simeq C^*(\mathbb{Z})$ we obtain $\tilde{C}^*(\Gamma) = C^*(\tilde{\Gamma})$, with $\tilde{\Gamma} \subset \mathbb{Z} \ast \Gamma$. Then, a careful examination of generators gives the isomorphism $\tilde{\Gamma} \simeq \mathbb{Z} \ast \Lambda$. See [2] for details. □

At the dual level, we have the following question: what is the compact quantum group $G^c$ defined by $C(G^c) = \tilde{C}(G)$? There is no simple answer to this question, unless in the abelian case, where we have the following result.
Theorem 1.5. If $G \subset U_n$ is a compact abelian group then $\tilde{C}(G) = C^*(\mathbb{Z} \ast \hat{L})$, where $L$ is the image of $G$ in the projective unitary group $PU_n$.

Proof. The embedding $G \subset U_n$, viewed as a representation, must come from a generating system $\hat{G} = \langle g_1, \ldots, g_n \rangle$. It routine to check that the subgroup $\Lambda \subset \hat{G}$ constructed in Theorem 1.4 is the dual of $L$, and this gives the result. \hfill $\Box$

2. Free quantum groups

Consider the groups $S_n \subset O_n \subset U_n$, with the elements of $S_n$ viewed as permutation matrices. Consider also the following subgroups of $U_n$:

1. $S'_n = \mathbb{Z}_2 \times S_n$, the permutation matrices multiplied by $\pm 1$.
2. $H_n = \mathbb{Z}_2 \wr S_n$, the permutation matrices with $\pm$ coefficients.
3. $P_n = \mathbb{T} \times S_n$, the permutation matrices multiplied by scalars in $\mathbb{T}$.
4. $K_n = \mathbb{T} \wr S_n$, the permutation matrices with coefficients in $\mathbb{T}$.

Observe that $H_n$ is the hyperoctahedral group. It is convenient to collect the above definitions into a single one, in the following way.

Definition 2.1. We use the diagram of compact groups

$$
\begin{array}{ccc}
U_n & \supset & K_n & \supset & P_n \\
\cup & & \cup & & \cup \\
O_n & \supset & H_n & \supset & S^*_n
\end{array}
$$

where $S^*$ denotes at the same time $S$ and $S'$.

In what follows we describe the free analogues of these 7 groups. For this purpose, we recall that a square matrix $u \in M_n(A)$ is called:

1. Orthogonal, if $u = \bar{u}$ and $u^t = u^{-1}$.
2. Cubic, if it is orthogonal, and $ab = 0$ on rows and columns.
3. Magic', if it is cubic, and the sum on rows and columns is the same.
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(4) Magic, if it is cubic, formed of projections \((a^2 = a = a^*)\).

(5) Biunitary, if both \(u\) and \(u^t\) are unitaries.

(6) Cubik, if it is biunitary, and \(ab^* = a^*b = 0\) on rows and columns.

(7) Magik, if it is cubik, and the sum on rows and columns is the same.

Here the equalities of type \(ab = 0\) refer to distinct entries on the same row, or on the same column. The notions (1, 2, 4, 5) are from [7, 3, 8, 7], and (3, 6, 7) are new. The terminology is of course temporary: we have only 7 examples of free quantum groups, so we don’t know exactly what the names name.

**Theorem 2.2.** \(C(G_n)\) with \(G = OHS^*UKP\) is the universal commu-
tative \(C^*\)-algebra generated by the entries of a \(n \times n\) orthogonal, cubic, magic*, biunitary, cubik, magik matrix.

**Proof.** The case \(G = OHSU\) is discussed in [7, 3, 8, 7], and the case \(G = S'KP\) follows from it, by identifying the corresponding subgroups. \(\square\)

We proceed with liberation: definitions will become theorems and vice versa.

**Definition 2.3.** \(A_g(n)\) with \(g = ohs^*ukp\) is the universal \(C^*\)-algebra generated by the entries of a \(n \times n\) orthogonal, cubic, magic*, biunitary, cubik, magik matrix.

The \(g = ohsu\) algebras are from [7, 3, 8, 7], and the \(g = s'kp\) ones are new.

**Theorem 2.4.** We have the diagram of Hopf algebras

\[
\begin{array}{ccc}
A_u(n) & \rightarrow & A_k(n) & \rightarrow & A_p(n) \\
\downarrow & & \downarrow & & \downarrow \\
A_o(n) & \rightarrow & A_h(n) & \rightarrow & A_{s^*}(n)
\end{array}
\]

where \(s^*\) denotes at the same time \(s\) and \(s'\).

**Proof.** The morphisms in Definition 1.1 can be constructed by using the universal property of each of the algebras involved. For the algebras \(A_{ohsu}\) this is known from [7, 3, 8, 7], and for the algebras \(A_{s'kp}\) the proof is similar. \(\square\)
3. Diagrams

Let $F = \langle a, b \rangle$ be the monoid of words on two letters $a, b$. For a given corepresentation $u$ we let $u^a = u, u^b = \bar{u}$, then we define the tensor powers $u^\alpha$ with $\alpha \in F$ arbitrary, according to the rule $u^{\alpha \beta} = u^\alpha \otimes u^\beta$.

**Definition 3.1.** Let $(A, u)$ be a finitely generated Hopf algebra.

(1) $CA$ is the collection of linear spaces $\{Hom(u^\alpha, u^\beta)|\alpha, \beta \in F\}$.

(2) In the case $u = \bar{u}$ we identify $CA$ with $\{Hom(u^k, u^l)|k, l \in \mathbb{N}\}$.

A morphism $(A, u) \to (B, v)$ produces inclusions

$$Hom(u^\alpha, u^\beta) \subset Hom(v^\alpha, v^\beta)$$

for any $\alpha, \beta \in F$, so we have the following diagram:

$$CA_u(n) \subset CA_k(n) \subset CA_p(n)$$

$$\cap \quad \cap \quad \cap$$

$$CA_o(n) \subset CA_h(n) \subset CA_s^*(n)$$

We recall that $CA_s(n)$ is the category of Temperley-Lieb diagrams. That is, $Hom(u^k, u^l)$ is isomorphic to the abstract vector space spanned by the diagrams between an upper row of $2k$ points, and a lower row of $2l$ points. See [3].

In order to distinguish between various meanings of the same diagram, we attach words to it. For instance $\capab, \capba$ are respectively in $D_s(\emptyset, ab), D_s(\emptyset, ba)$.

**Lemma 3.2.** The categories for $A_g(n)$ with $g = ohsukp$ are as follows:

(1) $CA_o(n) = \langle \cap \rangle$.

(2) $CA_h(n) = \langle \cap, \cap | \cup \rangle$.

(3) $CA_s'(n) = \langle \cap, \cup | \cap, \cap \rangle$.

(4) $CA_s(n) = \langle \cap, \cup | \cap, \cap \rangle$.

(5) $CA_u(n) = \langle \capab, \capba \rangle$.
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(6) \( CA_k(n) = \langle \bigcap_{ab}, \bigcap_{ba}, \bigcup_{ab}, \bigcup_{ba} \rangle. \)

(7) \( CA_p(n) = \langle \bigcap_{ab}, \bigcap_{ba}, \bigcup_{ab}, \bigcup_{ba}, \bigcap_a, \bigcap_b \rangle. \)

Proof. The case \( g = ohs \) is discussed in [3], and the case \( g = u \) is discussed in [4]. In the case \( g = s'kp \) we can use the following formulae:

\[
\bigcup = \sum_{ij} e_{ij} \\
\bigcap = \sum_{i} e_{ii} \otimes e_{ii}
\]

The commutation conditions \( \bigcup, \bigcap \in \text{End}(u \otimes \bar{u}) \) and \( \bigcup, \bigcap \in \text{End}(\bar{u} \otimes u) \) correspond to the cubik condition, and the extra relations \( \bigcup, \bigcap \in \text{End}(u) \) and \( \bigcup, \bigcap \in \text{End}(\bar{u}) \) correspond to the magik condition. Together with the fact that orthogonal plus magik means magic', this gives all the \( g = s'kp \) assertions.

We can color the diagrams in several ways: either by putting the sequence \( xyyxxyyx \ldots \) on both rows of points, or by putting \( \alpha, \beta \) on both rows, then by replacing \( a \to xy, b \to yx \). We say that the diagram is colored if all the strings match, and half-colored, if there is an even number of unmatches.

Theorem 3.3. For \( g = ohs^*ukp \) we have \( CA_g(n) = \text{span}(D_g) \), where:

(1) \( D_s(k, l) \) is the set of all diagrams between \( 2k \) points and \( 2l \) points.

(2) \( D_{s'}(k, l) = D_s(k, l) \) for \( k - l \) even, and \( D_{s'}(k, l) = \emptyset \) for \( k - l \) odd.

(3) \( D_h(k, l) \) consists of diagrams which are colorable \( xyyxxyyx \ldots \).

(4) \( D_o(k, l) \) is the image of \( D_s(k/2, l/2) \) by the doubling map.

(5) \( D_p(\alpha, \beta) \) consists of diagrams half-colorable \( a \to xy, b \to yx \).

(6) \( D_k(\alpha, \beta) \) consists of diagrams colorable \( a \to xy, b \to yx \).

(7) \( D_u(\alpha, \beta) \) consists of double diagrams, colorable \( a \to xy, b \to yx \).

Proof. This is clear from the above lemma, by composing diagrams. The case \( g = ohsu \) is discussed in [3, 4], and the case \( g = s'kp \) is similar. □
Theorem 3.4. We have the following isomorphisms:

1. \( A_u(n) = \tilde{A}_o(n) \).
2. \( A_h(n) = \tilde{A}_k(n) \).
3. \( A_p(n) = \tilde{A}_{s^*}(n) \).

Proof. It follows from definitions that we have arrows from left to right. Now since by Theorem 3.3 the spaces \( \text{End}(u \otimes \bar{u} \otimes u \otimes \ldots) \) are the same at right and at left, Theorem 5.1 in [2] applies, and gives the arrows from right to left. \( \square \)

Observe that the assertion (1), known since [1], is nothing but the isomorphism \( U_n^+ = O_n^{+c} \) mentioned in the beginning of the first section.

4. Freeness, level, doubling

We use the notion of free Hopf algebra, introduced in [3]. Recall that a morphism \( (A, u) \to (B, v) \) induces inclusions \( \text{Hom}(u^\alpha, u^\beta) \subset \text{Hom}(v^\alpha, v^\beta) \).

Definition 4.1. A finitely generated Hopf algebra \( (A, u) \) is called free if:

1. The canonical map \( A_u(n) \to A_s(n) \) factorizes through \( A \).
2. The spaces \( \text{Hom}(u^\alpha, u^\beta) \subset \text{span}(D_s(\alpha, \beta)) \) are spanned by diagrams.

It follows from Theorem 3.3 that the algebras \( A_{ohn^+u,kp} \) are free.

In the orthogonal case \( u = \bar{u} \) we say that \( A \) is free orthogonal, and in the general case, we also say that \( A \) is free unitary.

Theorem 4.2. If \( A \) is free orthogonal then \( \tilde{A} \) is free unitary.

Proof. It is shown in [2] that the tensor category of \( \tilde{A} \) is generated by the tensor category of \( A \), embedded via alternating words, and this gives the result. \( \square \)

Definition 4.3. The level of a free orthogonal Hopf algebra \( (A, u) \) is the smallest number \( l \in \{0, 1, \ldots, \infty\} \) such that \( 1 \in u^\otimes_{2l+1} \).

As the level of examples, for \( A_s(n) \) we have \( l = 0 \), and for \( A_{ohn^+u,kp}(n) \) we have \( l = \infty \). This follows indeed from Theorem 3.3.
Theorem 4.4. If \( l < \infty \) then \( \tilde{A} = C(\mathbb{T}) \ast A \).

Proof. Let \( < r > \) be the algebra generated by the coefficients of \( r \). From \( 1 \in < u > \) we get \( z \in < zu_{ij} > \), hence \( < zu_{ij} > = < z, u_{ij} > \), and we are done. \[\]

Corollary 4.5. \( A_p(n) = C(\mathbb{T}) \ast A_s(n) \).

Proof. For \( A_s(n) \) we have \( 1 \in u \), hence \( l = 0 \), and Theorem 4.4 applies. \[\]

We can define a “doubling” operation \( A \to A_2 \) for free orthogonal algebras, by using Tannakian duality, in the following way: the spaces \( \text{Hom}(u^k, u^l) \) with \( k - l \) even remain by definition the same, and those with \( k - l \) odd become by definition empty. The interest in this operation is that \( A_2 \) has infinite level.

At the level of examples, the doublings are \( A_{ohs^*}(n) \to A_{ohs'}(n) \).

Proposition 4.6. For a free orthogonal algebra \( A \), the following are equivalent:

1. \( A \) has infinite level.
2. The canonical map \( A_2 \to A \) is an isomorphism.
3. The quotient map \( A \to A_s(n) \) factorizes through \( A_{s'}(n) \).

Proof. The equivalence between (1) and (2) is clear from definitions, and the equivalence with (3) follows from Tannakian duality. \[\]

5. The main result

We know from Theorem 3.4 that the two rows of the diagram formed by the algebras \( A_{ohs^*ukp} \) are related by the operation \( A \to \tilde{A} \). Moreover, the results in the previous section suggest that the correct choice in the lower row is \( s^* = s' \). The following general result shows that this is indeed the case.

Theorem 5.1. The operation \( A \to \tilde{A} \) induces a one-to-one correspondence between the following objects:

1. Free orthogonal algebras of infinite level.
2. Free unitary algebras satisfying \( A = \tilde{A} \).
Proof. We use the notations $\gamma_k = abab\ldots$ and $\delta_k = baba\ldots$ ($k$ terms each).

We know from Theorem 4.2 that the operation $A \to \tilde{A}$ is well-defined, between the algebras in the statement. Moreover, since by Tannakian duality an orthogonal algebra of infinite level is determined by the spaces $\text{Hom}(u^k, u^l)$ with $k-l$ even, we get that $A \to \tilde{A}$ is injective, because these spaces are:

$$\text{Hom}(u^k, u^l) = \text{Hom}((zu)^\gamma_k, (zu)^\gamma_l)$$

It remains to prove surjectivity. So, let $A$ be free unitary satisfying $A = \tilde{A}$. We have $CA = \text{span}(D)$ for certain sets of diagrams $D(\alpha, \beta) \subset D_s(\alpha, \beta)$, so we can define a collection of sets $D_2(k, l) \subset D_s(k, l)$ in the following way:

1. For $k - l$ even we let $D_2(k, l) = D(\gamma_k, \gamma_l)$.
2. For $k - l$ odd we let $D_2(k, l) = \emptyset$.

It follows from definitions that $C_2 = \text{span}(D_2)$ is a category, with duality and involution. We claim that $C_2$ is stable under $\otimes$. Indeed, for $k, l$ even we have:

$$D_2(k, l) \otimes D_2(p, q) = D(\gamma_k, \gamma_l) \otimes D(\gamma_p, \gamma_q) \subset D(\gamma_k \gamma_p, \gamma_l \gamma_q) = D(\gamma_{k+p}, \gamma_{l+q}) = D_2(k + p, l + q)$$

For $k, l$ odd and $p, q$ even, we can use the canonical antilinear isomorphisms $\text{Hom}(u^{\gamma_K}, u^{\gamma_L}) \simeq \text{Hom}(u^{\delta_K, \delta_L})$, with $K, L$ odd. At the level of diagrams we get equalities $D(\gamma_k, \gamma_l) = D(\delta_K, \delta_L)$, that can be used in the following way:

$$D_2(k, l) \otimes D_2(p, q) = D(\delta_k, \delta_l) \otimes D(\gamma_p, \gamma_q) \subset D(\delta_k \gamma_p, \delta_l \gamma_q) = D(\delta_{k+p}, \delta_{l+q}) = D_2(k + p, l + q)$$
Finally, for $k, l$ odd and $p, q$ odd, we can proceed as follows:

\[
D_2(k, l) \otimes D_2(p, q) = D(\gamma_k, \gamma_l) \otimes D(\delta_p, \delta_q)
\]

\[
\subset D(\gamma_k \delta_p, \gamma_l \delta_q)
\]

\[
= D(\gamma_{k+p}, \gamma_{l+q})
\]

\[
= D_2(k + p, l + q)
\]

Thus we have a Tannakian category, and by Woronowicz’s results in [10] we get an algebra $A_2$. This algebra is free orthogonal, of infinite level. Moreover, the spaces $\text{End}(u \otimes \bar{u} \otimes u \otimes \ldots)$ being the same for $A$ and $A_2$, Theorem 5.1 in [2] applies, and gives $\hat{A}_2 = \hat{A}$. Now since we have $A = \hat{A}$, we are done. \qed

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