Dynamics of two languages competing on a network: a case study

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Abstract. A language dynamics model on a square lattice, which is an extension of the one popularized by Abrams and Strogatz [1], is analyzed using ODE bifurcation theory. For this model we are interested in the existence and spectral stability of structures such as stripes, which are realized through pulses and/or the concatenation of fronts, and spots, which are a contiguous collection of sites in which one language is dominant. Because the coupling between sites is nonlinear, the boundary between sites containing speaking two different languages is “sharp”; in particular, in a PDE approximation it allows for the existence of compactly supported pulses (compactons). The dynamics are considered as a function of the prestige of a language. In particular, it is seen that as the prestige varies, it allows for a language to spread through the network, or conversely for its demise.

Keywords. ODE bifurcation theory, language competition, prestige

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1. **Introduction**

In their seminal paper Abrams and Strogatz [1] developed a simple ODE model,

\[ \dot{u} = (1 - u)u^p - Au(1 - u)^p, \]

(1.1)

to help understand language competition and the decline in the number of people who speak such historic languages as Welsh, Quechua, and Scottish Gaelic. We will henceforth label (1.1) as the AS model (see Figure 1 for a cartoon representation of this compartment model). The underlying assumptions for this model are that all speakers are monolingual, and the population is highly connected with no spatial or social structure. In equation (1.1) \( u \) represents the proportion of the population which speak language \( U \). If \( v \) is the proportion which speak language \( V \), since all speakers are monolingual, \( v = 1 - u \). The parameter \( p > 0 \) measures volatility. The case \( p = 1 \) is a neutral situation, where transition probabilities from one language to another depend linearly on local language densities. If \( p > 1 \) there is a larger than neutral resistance to changing the language (low volatility), and if \( p < 1 \) there is a lower than neutral resistance to changing the language (high volatility). Experimentally, it is estimated that \( p = 1.31 \pm 0.25 \). The parameter \( A > 0 \) represents the affinity of the general population towards one language or the other. In linguistics terminology the parameter \( A \) can be used to represent the prestige associated with a particular language.

Assume \( p > 1 \), so the volatility is low. The fixed points \( u = 0 \) (language \( V \) is preferred) and \( u = 1 \) (language \( U \) is preferred) are stable, while \( u = B/(1 + B) \) with \( B = A^{1/(p-1)} \) is unstable. If \( A < 1 \) and \( u(0) = 0.5 \) (both languages are initially equally preferred), then \( u(t) \to 1 \) as \( t \to +\infty \), so the population has an affinity for language \( U \). Or, language \( U \) has more prestige in the general population. On the other hand, if \( A > 1 \) and \( u(0) = 0.5 \), then \( u(t) \to 0 \) as \( t \to +\infty \), so language \( V \) has more prestige in the whole population.

\[ \begin{align*}
\dot{u} &= (1 - k)(1 - u)(1 - v)^p - Au(1 - u)^p \\
\dot{v} &= (1 - k)A(1 - v)(1 - u)^p - v(1 - v)^p,
\end{align*} \]

(1.2)

where \( 0 < k < 1 \) represents the ease of bilingualism. In particular, \( k = 0 \) means that conversation is not possible between monolingual speakers, and \( k = 1 \) implies \( U = V \). The larger the value of \( k \), the more similar are the two languages. If \( k = b = 0 \), then model (1.2) reduces to model (1.1). An analysis of the model (1.2) is provided in [8, 25].

An agent-based model associated with the AS model (1.1) when \( p = 1 \),

\[ \dot{u} = (1 - A)u(1 - u), \]

(1.3)
is considered by Stauffer et al. [30]. In particular, on a lattice of dimension $d$ an individual is assumed to feel the influence of $2d$ nearest neighbors. When $A > 1$ the agent-based model results are qualitatively similar to those associated with the solution of (1.3). As expected, the results differ when the macroscopic model fails, $A = 1$. It is not clear how the two models compare when $A < 1$. An agent-based model is also considered by Vazquez et al. [32]. In the fully connected case the dynamics of the associated mean-field model are equivalent to those for the model (1.1).

The AS model has been extended to networks. Each node of the network corresponds to a group whose dynamics are governed by the AS model, and then the dynamics between groups satisfy some other rule. Amano et al. [3] collected and analyzed world-wide data taking into account such things as geographical range size, speaker population size, and speaker growth rate (i.e., changes in the number of speakers) of the world’s languages, and assessed interrelations among these three components to understand how they contribute to shaping extinction risk in languages. The role of population density and how it affects the interaction rates among groups is discussed by Juane et al. [18] in the context of language shift in Galicia, which is a bilingual community in northwest Spain. They model the problem by looking at equations (1.2) on a network, with the strength of the interactions between nodes depending on the population density. The model for $j = 1, \ldots, n$ is,

$$
\dot{u}_j = (1 - k_j)(1 - u_j)(1 - v_j)^p - A_j u_j(1 - u_j)^p + K_j (\bar{u} - u_j),
$$

$$
\dot{v}_j = (1 - k_j)A_j (1 - v_j)(1 - u_j)^p - v_j(1 - v_j)^p + K_j (\bar{v} - v_j).
$$

(1.4)

Here $\bar{y}$ represents the average of the set, $\{y_j\}$. The positive parameter $K_j$ is assumed to be a strictly increasing function of the population density. The authors Vidal-Franco et al. [33] follow a similar strategy, except they assume the nonlinearities are of Lotka-Volterra type. Taking a different approach, Yun et al. [35] assume a diffusion process to take into account spatial effects. Fujie et al. [12] and Zhou et al. [36] consider the problem of competition among more than two languages.

In this paper we consider the language competition problem on a network under the assumption of low volatility, $p > 1$. For example we will primarily work with $p = 2$, but our experience is that other values of $p > 1$ do not affect the results qualitatively. We will assume there is no bilingual subpopulation (see [18, 23, 25] for some work in this area under the assumption of a single group). This may be an unrealistic assumption in terms of language; however, it is less so if one assumes language $U$ actually refers to those who have some type of religious affiliation, and language $V$ represents those who do not [2]. We will assume the existence of $n$ distinct population groups, and let $0 \leq u_j \leq 1$ represent the proportion of those in group $j$ who speak language $U$ ($u_j = 1 - u_j$ speak language $V$). For each $j = 1, \ldots, n$ our model equation is a natural extension of the compartment model illustrated in Figure 1,

$$
\dot{u}_j = \left( \sum_{k=1}^{n} I_{jk} u_k^p \right) \cdot (1 - u_j) - A_j \sum_{k=1}^{n} I_{jk} (1 - u_k)^p \cdot u_j,
$$

(1.5)

where $I_{jk} \geq 0$. We call the matrix $I = (I_{jk})$ the influence matrix, and the term $I_{jk}$ represents the influence group $k$ has on group $j$ through the between-group reaction rate. If we think of the system (1.5) as being a compartment model, then the term $I_{jk} u_k^p$ is the rate constant associated with the influence that the $U$ speakers in group $k$ have on the $V$ speakers in group $j$, and $I_{jk} (1 - u_k)^p$ is the rate constant associated with the influence that the $V$ speakers in group $k$ have on the $U$ speakers in group $j$. Clearly, when $n = 1$ the system (1.5) collapses to the AS model (1.1).

We now compare the systems (1.4) and (1.5). In the case of no bilingual speakers the system (1.4) collapses to,

$$
\dot{u}_j = (1 - u_j)u_j^p - A_j u_j(1 - u_j)^p + K_j (\bar{u} - u_j).
$$

(1.6)

The systems (1.5) and (1.6) have the feature that the on-site dynamics are the same as those for the AS model. However, the coupling between groups is different; in particular, the model (1.6) assumes that group $j$ is influenced by all of the other groups, whereas the model (1.5) allows for each group to be isolated from some of the other groups. Under the assumption that each external group has an equal influence on a given group, $I_{jj} = 1$ and $I_{jk} = K_j/(n - 1)$ for all $k \neq j$, the system (1.5) becomes,

$$
\dot{u}_j = (1 - u_j) \left[ u_j^p + K_j \bar{u} \right] - A_j u_j \left[ (1 - u_j)^p + K_j (1 - u)^{p-1}_j \right],
$$

(1.7)
where we use the notation,
\[
\overline{f}_{j} = \frac{1}{n-1} \sum_{k \neq j} f_k.
\]

The nonlinear coupling term for the model (1.7) is clearly very different than the linear coupling term associated with the model (1.6). It is an open question as to whether this functional difference leads to a qualitative difference in the dynamics.

A simple model such as (1.1) can also be used to model opinion propagation in a population in which it is assumed that people have either opinion \( U \), or opinion \( V \), where we think of \( V \) as being “not \( U \)”. Marvel et al. [21], hereafter referred to as MS, provide a model similar to (1.2) in which it is assumed there are three distinct groups: those who hold opinion \( U \), those who hold opinion \( V \), and the remaining who are undecided.

The underlying assumption in this model is that in order for one who initially holds opinion \( U \) to eventually hold opinion \( V \) (or vice-versa), the person first must become undecided. Wang et al. [34] extended the MS model to allow for several competing opinions. The MS model was extended to networks by Bujalski et al. [7], and the extended model was studied using dynamical systems techniques. Tanabe and Masuda [31] proposed and analyzed an interesting opinion formation model (hereafter labelled TM) in which it was assumed that the population itself breaks down into two groups: congregators, and contrarians. In contrast, the MS model implicitly assumes the entire population is filled with congregators. One conclusion of the TM model is that if a large enough proportion of the population is contrarian, then no majority opinion will be achieved. This is in contrast to the conclusion of those models in which it is assumed there are only congregators, as here a majority opinion is always obtained. The TM model was later refined by Eekhoff [10], and the new model allowed for the effects of peer pressure, and incorporated the influence of zealots. From a qualitative perspective the mean-field models used for opinion dynamics and language death have many similarities. Thus, although we frame our results using the formulation associated with language death, they are also directly applicable to mean-field opinion formation models.

In this paper we are primarily interested in the existence and stability of spatial structures for the network system (1.5). We assume the groups have been arranged on a square lattice. The interactions on this lattice are nearest-neighbor (NN) only. Our experience is that from a qualitative perspective the NN interactions can be expanded without substantively changing the solution behavior as long as the interactions are still somewhat spatially localized (the Implicit Function Theorem provides the theoretical justification). Moreover, there will be no preferential distinction in the reaction rates, \( I_{jk} = I_{kj} \). This is a case study, so we have not fully explored a large set of networks. That work will be left for a future paper. Our goal here is not to do an exhaustive study for all types of influence matrices. Instead, we simply want to get a sense of what is possible for a given type of network.

For this lattice configuration we start by considering the existence and stability of fronts and pulses for the system (1.5). A front is a solution for which \( u_{jk} = U_j \), and \( U_{j} = 0 \) (or \( U_{j} = 1 \)) for \( 1 \leq j \leq n_0 \), and \( U_{j} = 1 \) (or \( U_{j} = 0 \)) for \( j \geq n_0 + \ell \) and some \( \ell \geq 1 \). In other words, to the left of \( n_0 \) language \( V \) is spoken, and to the right of \( n_0 + \ell \) language \( U \) is spoken. A pulse is a solution for which \( U_j = 0 \) for \( j \leq n_0 \) and \( j \geq n_0 + \ell \), and \( U_j > 0 \) for \( n_0 < j < n_0 + \ell \). In other words, on the full lattice there is a stripe of language \( U \) speakers who are surrounded by a group of \( V \) speakers. We will consider when fronts can travel, which implies that language \( U \) is invading language \( V \), or vice-versa. We will also consider when pulses can grow or shrink. A growing pulse can be thought of as the concatenation of two fronts traveling in opposing directions, which implies that language \( U \) eventually takes over the entire network. A shrinking pulse eventually disappears, which means that language \( U \) has gone extinct. As we will see, the prestige associated with speaking \( U \) \((A < 1)\) or \( V \) \((A > 1)\) plays a central role in the analysis. We will conclude with a case study for a spot, which is a contiguous group of sites with \( u_{jk} > 0 \) surrounded by \( u_{jk} = 0 \) - an island of \( U \) in a sea of \( V \).

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2. The model on a square lattice

As already stated, we consider the dynamics of a square lattice with nearest-neighbor interactions only. Here \( u_{jk} \) will represent the proportion of the population at site \((j, k)\) who speak language \(U\). We will henceforth assume that the prestige associated with language \(U\) is uniform throughout the lattice, \(A_{jk} = A\). It is an interesting problem in its own right to allow for a spatially inhomogeneous distribution of the prestige and see how it affects the prevalent dynamics. Moreover, we will assume \(p = 2\). Our numerical experiments indicate that from a qualitative perspective the results presented herein only need \(p > 1\).

Under these assumptions the model \((1.5)\) is,

\[
\dot{u}_{jk} = [\epsilon_0 u_{jk}^2 + \epsilon_1 \left( u_{j+1,k}^2 + u_{j-1,k}^2 + u_{j,k+1}^2 + u_{j,k-1}^2 \right) \left( 1 - u_{jk} \right) \\
- A \left[ \epsilon_0 (1 - u_{jk})^2 + \epsilon_1 \left( (1 - u_{j+1,k})^2 + (1 - u_{j-1,k})^2 + (1 - u_{j,k+1})^2 + (1 - u_{j,k-1})^2 \right) \right] u_{jk}.
\]

Here \(1 \leq j, k \leq n\), and we assume in the model that at the edge of the square there are Neumann boundary conditions, e.g., \(u_{n+1,k} = u_{nk}\). The parameter \(\epsilon_0 > 0\) is the on-site interaction rate, and the parameter \(\epsilon_1 > 0\) is the nearest-neighbor interaction rate. Using the notation for the discrete Laplacian,

\[
\Delta_{\text{dis}} f_{jk} = f_{j+1,k} + f_{j-1,k} + f_{j,k+1} + f_{j,k-1} - 4 f_{jk},
\]

the above ODE takes the more compact form,

\[
\dot{u}_{jk} = (\epsilon_0 + 4\epsilon_1) u_{jk} (1 - u_{jk}) \left[ (1 + A) u_{jk} - A \right] + 2 A \epsilon_1 u_{jk} \Delta_{\text{dis}} u_{jk} + \epsilon_1 \left[ 1 - (1 + A) u_{jk} \right] \Delta_{\text{dis}} u_{jk}^2.
\]  \(2.1\)

If we assume that the interactions between neighbors are strong, i.e., \(\epsilon_1 \gg 1\), then upon setting \(R = \epsilon_0 + 4\epsilon_1 \gg 1\) we have the limiting continuum model,

\[
\partial_t u = Ru(1 - u) \left[ (1 + A) u - A \right] + (1 + A) u(1 - u) \Delta u + \left[ 1 - (1 + A) u \right] |\nabla u|^2.
\]  \(2.2\)

Here \(\Delta\) represents the Laplacian, and \(\nabla\) is the gradient operator. The continuum model incorporates the expected temporal dynamics associated with the original ODE model, but the coupling dynamics between sites is dictated by an effective nonlinear diffusion. The PDE is physical in the following sense: \(u(x, y, t) = 0\) implies \(\partial_t u(x, y, t) \geq 0\), and \(u(x, y, t) = 1\) implies \(\partial_t u(x, y, t) \leq 0\). Note the diffusion coefficient vanishes when the entire population supports one language, \(u = 0\) or \(u = 1\).

When studying the solution structure to the ODE \((2.1)\), or the accompanying PDE \((2.2)\), we will first focus on the existence and spectral stability of time-independent patterns which vary in one direction only. For the ODE \((2.1)\) we will set \(u_{j,k}(t) = U_j\) for all \(j, k\), and \(U_j\) will solve the 1D discrete model,

\[
0 = (\epsilon_0 + 4\epsilon_1) U_j (1 - U_j) \left[ (1 + A) U_j - A \right] + 2 \epsilon_1 A U_j \Delta_j U_j + \epsilon_1 \left[ 1 - (1 + A) U_j \right] \Delta_j U_j^2,
\]  \(2.3\)

where \(\Delta_j f_j = f_{j+1} + f_{j-1} - 2f_j\). For the PDE \((2.2)\) we will set \(u(x, y, t) = U(x)\), and \(U(x)\) will solve the nonlinear ODE,

\[
0 = RU(1 - U) \left[ (1 + A) U - A \right] + (1 + A) U(1 - U) U'' + \left[ 1 - (1 + A) U \right] (U')^2, \quad ' = \frac{d}{dx}.
\]  \(2.4\)

In both cases we will be looking for fronts/pulses, which for the full system will correspond to stripes. These solutions act as transitions between regions where language \(U\) is dominant and language \(V\) is dominant.

Remark 2.1. Even though the derivation is dissimilar, the continuum model \((2.2)\) is remarkably similar to the mean-field model associated with the square lattice as provided for in [32, equation (48)]. The model \((2.2)\) has the additional term, \(\left[ 1 - (1 + A) u \right] |\nabla u|^2\), however, both models have the important feature that the diffusion coefficient is singular. Dynamically, both systems have the feature that small domains tend to shrink, and large domains tend to grow, and the domains tend to evolve in a way that reduces the curvature of the boundary; see also further relevant discussion regarding the dynamics below.
3. **Existence and spectral stability of stripes for the discrete model**

A front solution to (2.3) satisfies $U_j = 0 (1)$ for $j \leq \ell$, and $U_j = 1 (0)$ for $j \geq k$, where $1 < \ell < k < n$. A pulse solution will satisfy $U_j = 0 (1)$ for $j \leq \ell$ and $j \geq k$, and $U_j \sim 1 (0)$ for $\ell < j < k$. The transition between the states 0 and 1 will be monotone. A stripe solution to the full 2D model will be a pulse, or a concatenation of two fronts. As we will see, the concatenation of two fronts provides for a “thicker” stripe. In the same spirit, we can also discuss multi-stripes, which are the concatenation of pulses and/or fronts.

### 3.1. Existence: fronts

If $\epsilon_1 = 0$, the system uncouples, so a front can be constructed analytically. In this limit, for a front we set $U_j = 0 (1)$ for $j = 1, \ldots, \ell$, and $U_j = 1 (0)$ for $j = \ell + 1, \ldots, n$. We will refer to this front as the off-site front. Since each of the fixed points is stable for the scalar AS model, the front will be stable for the full system. By the Implicit Function Theorem the front will persist and be stable for $0 < \epsilon_1 < 1$. We can concatenate these fronts when $\epsilon_1 = 0$ to form stable stripes, and then again apply the Implicit Function Theorem to show the existence and stability for small $\epsilon_1$.

When $\epsilon_1 = 0$ we can construct another front by setting $U_j = 0 (1)$ for $j = 1, \ldots, \ell$, $U_{\ell+1} = A/(1 + A)$, and $U_j = 1 (0)$ for $j = \ell + 2, \ldots, n$. Since all of the fixed points but the one at $j = \ell$ are stable for the scalar AS model, the front will be unstable for the full system with the linearization having one positive eigenvalue. By the Implicit Function Theorem the front will persist and be unstable with one positive eigenvalue for $0 < \epsilon_1 < 1$. We will refer to this front as the on-site front.

When $\epsilon_1 = 0$ the off-site and on-site fronts exist for any value of $A$. However, once there is nontrivial coupling, we expect there will be an interval of $A$ values which contains $A = 1$ for which the fronts will exist. In order to determine this interval we will do numerical continuation using the MATLAB package, Matcont [9]. Using this package will also allow us to numerically continue bifurcation points in parameter space. Setting

$$R_1 = \frac{\epsilon_1}{\epsilon_0},$$

we will numerically explore the $(R_1, A)$-parameter space. Since we analytically know what happens for $R_1 = 0$, we are in a good position to use numerical continuation.

For each fixed $R_1 > 0$ there will be an associated snaking diagram in the parameter $A$. For a particular example, consider the left figure in Figure 2. The horizontal axis is $A$, and the vertical axis is the $L^2$-norm of the front. In this figure the solid (blue) curve corresponds to a stable front (which is off-site when $A = 1$), and the dashed (red) curve corresponds to an unstable front (which is on-site when $A = 1$). These two curves meet at a saddle-node bifurcation point, which is denoted by an open black circle. We see there is an $A_- < 1 < A_+$ for which there are stable fronts for $A_- < A < A_+$, and no stationary fronts (at least as seen via numerical continuation) outside this interval. The values of $A_\pm$ depend on $R_1$. Each of the upward shifts of the stable and unstable branches correspond to waveforms that are shifted by an integer number of lattice nodes to the left (hence the growth in norm). The right panel in Figure 2 shows the functions $A_\pm$ as a function of $R_1$. While we do not show it here, even in the limit $R_1 \to +\infty$ the two curves do not converge to 1; instead, we have $A_+(\infty) \sim 1.0082$, and $A_- (\infty) \sim 0.9918$. Inside the two curves, and for fixed $R_1$, there is a stable stationary front.

### 3.2. Existence: traveling waves

Outside the two curves, $A_\pm(R_1)$, there is a traveling front. Traveling waves will be written as $U(x + ct)$, so $U_j(t) = U(j + ct)$. Setting $\xi = x + ct$, the resulting forward-backward difference equation to which the traveling wave is a solution is,

$$cU' = (\epsilon_0 + 4\epsilon_1) U(1 - U) [(1 + A)U - A]$$

$$+ 2A\epsilon_1 U [U(\xi + 1) + U(\xi - 1) - 2U(\xi)] + \epsilon_1 [1 - (1 + A)U] [U(\xi + 1)^2 + U(\xi - 1)^2 - 2U(\xi)^2].$$
This system is solved using a variant of Newton’s method (see [11, 15, 17] for the details).

We consider in detail the case of $R_1 = 0.6$. Our experience is that from a qualitative perspective the value of $R_1$ is not particularly important. The numerical result is plotted in Figure 3. The points $A_{\pm}$ are marked with a (red) diamond. It should be the case that at these points $c = 0$; unfortunately, the fact that the linearization becomes singular at $A = A_{\pm}$ precludes good convergence of the algorithm near these points. Away from these bifurcation points there is good convergence of the numerical algorithm. Assuming $U_j = 0$ to the left, and $V_j = 0$ to the right, if $c < 0$ language $V$ invades language $U$, whereas if $c > 0$ language $U$ invades language $V$. We see here that if $A > A_{+} \sim 1.0644$, i.e., language $V$ has more prestige, then language $V$ invades language $U$. On the other hand, if $A < A_{-} \sim 0.9395$, i.e., language $U$ has more prestige, then language $U$ invades language $V$. Note that the speed increases as the preferred language becomes more prestigious. Indeed, up to a small correction, and sufficiently far away from $A_{\pm}$, the wave speed follows the formal prediction of the continuum model, equation (4.1). The predicted curve, which is associated with the limit $R_1 \to +\infty$, is given by the black dashed line. This result has been numerically verified for several different values of $R_1$. One can observe the nontrivial effect of discreteness in establishing an interval where the fronts can be stationary. Indeed, the continuum model is found to possess vanishing speed at the isolated point of prestige balance, namely at $A = 1$, while the discrete variant requires a detuning from this value in order to enable such a depinning from the vanishing speed setting.

Remark 3.1. It is an interesting exercise to consider the scaling law for the wave speed as $A \to A_{\pm}$; however, we have not pursued this. The interested reader should consult Anderson et al. [4], Kevrekidis et al. [20] and the references therein for details as to how such a law may be derived.

### 3.3. Existence: pulses

As is the case for fronts, if $\epsilon_1 = 0$ a pulse can be constructed analytically by setting $U_j = 0 \ (1)$ for $1 \leq j \leq \ell$ and $k \leq j \leq n$, and $U_j = 1 \ (0)$ for $\ell < j < k$. Since each of the fixed points is stable for the scalar AS model, the pulse will be stable for the full system. By the Implicit Function Theorem the pulse will persist and be stable for $0 < \epsilon_1 \ll 1$. We can concatenate these pulses when $\epsilon_1 = 0$ to form stable stripes, and then again apply the Implicit Function Theorem to show the existence and stability for small $\epsilon_1$. If so desired,
we can also construct unstable pulses by setting \( U_{\ell+1} = A/(1 + A) \) when \( \epsilon_1 = 0 \), and then using the Implicit Function Theorem for small \( \epsilon_1 \). Assuming the background supports language \( V \), the size of the pulse is the number of adjacent groups which support language \( U \). For small \( \epsilon_1 \) the size is \( k - \ell - 1 \).

Numerically it is seen that if a pulse is of size 4 or larger, then it is realized as a concatenation of a \( V \rightarrow U \) and a \( U \rightarrow V \) stationary front. Consequently, the front dynamics completely determine the pulse dynamics. If the front is stationary, so is the pulse. If the front moves, so will the edge of the pulse. On the other hand, if the pulse is of size 1, 2, or 3, then the dynamics are not related to front dynamics. From a dynamics perspective the pulse ceases to exist after a saddle-node bifurcation occurs.

Using Matcont, the bifurcation point can be traced in \((R_1, A)\)-space. The results are presented in Figure 4. The pulse will exist inside the boundary curve. The cusp point is \((R_1, A) \sim (0.1996, 0.6936)\) for the pulse of size 1, and \((R_1, A) \sim (0.6352, 0.9246)\) for the pulse of size 2. For a pulse of size 3 the cusp point satisfies \( R_1 > 31.77 \) with \( 0 < 1 - A \ll 1 \), and is not shown in the figure. Note that the cusp point converges to \( A = 1 \) as the size of the pulse increases, and satisfies \( A < 1 \). This is due to the fact that language \( U \) has more prestige for \( A < 1 \). If the background was language \( U \) instead of language \( V \), then the cusp point would satisfy \( A > 1 \).

From a dynamics perspective, if \( R_1 \) is less than the cusp point value, and if \( A \) is small enough so that \((R_1, A)\) is below the bottom boundary curve, then the pulse will grow until it can be thought of as a concatenation of two fronts. Once this occurs the edges of the pulse will move according to the front dynamics. The pulse grows because the prestige for language \( U \) is sufficiently large. On the other hand, if \((AR_1, A)\) is above the top boundary curve, then language \( V \) has sufficient prestige so that the background language prevails, and the pulse simply disappears in finite time. See Figure 5 for the corroborating results of a particular simulation.

**Remark 3.2.** If we assume a pulse of language \( V \) sits on a background of language \( U \), then we will get the same curves as in Figure 4. However, the dynamical interpretation leading to Figure 5 will be reversed. In particular, if \( A \) is too small the pulse will disappear, whereas if \( A \) is sufficiently large it will grow.

### 3.4. Multiple stripes via pulse concatenation

We now consider the problem of concatenating individual pulses to form multi-pulses. For the sake of convenience and without loss of generality we assume that background consists of language \( V \). As with the single pulses, each of the multi-pulses will be stable when \( \epsilon_1 = 0 \), and they will persist as stable
Figure 4: (color online) The left panel provides the numerically generated boundary of pulses of size 1 through 3. The boundary is given by a solid (blue) curve for the pulse of size 1, a (red) dashed curve for a pulse of size 2, and a (green) dashed-dotted curve for a pulse of size 3. For a given pulse size, the pulse exists inside the two curves, and ceases to exist outside. The right panel gives an example of each pulse for $R_1 = 0.05$ and $A = 1$. The pulse of size 1 is shown in the upper right panel, the pulse of size 2 in the middle right panel, and the pulse of size 3 in the lower right panel.

structures for sufficiently small $\epsilon_1$. Typically, the construction of multi-pulses would involve a discussion of tail-tail interactions between individual pulses, and an application of the Hale-Lin-Sandstede method (e.g., see [5, 6, 16, 24, 26-29] and the references therein). However, for the system under consideration this is less relevant, as the nonlinear coupling between adjacent sites renders the transition from one state to another to be super-exponential, instead of the exponential rates associated with linear coupling (see Figure 6 for a representative demonstration of this phenomena). Consequently, to leading order one can think of pulses as being compactons (a compactly supported structure), and fronts as being a compactly supported transition between two states.\textsuperscript{1} In this light, to leading order, and as long as the individual pulses are initially sufficiently separated, the dynamics associated with a concatenation of $k$ pulses is really just the dynamics of $k$ uncoupled pulses, each of which evolves according to the rules presented in Section 3.3.

Since this is only a case study, we will focus on the example of the two-pulse, which at the $\epsilon_1 = 0$ limit we label as $j$-$k$-$\ell$. Here $j$ and $\ell$ refer to the size of the pulse which supports language $U$, and $k$ is the intervening pulse of size $k$ which supports language $V$. For example, a 2-1-2 can be thought of when $\epsilon_1 = 0$ as the sequence of $u$-values, $\ldots 001101100 \ldots$.

First consider the 2-1-2 pulse. The boundary for which this solution exists is presented as a solid (blue) curve in Figure 7. The cusp point is $(R_1, A) \sim (0.1478, 1.2954)$. For $(R_1, A)$ values inside the curve the pulse will exist as a stationary solution and be stable, whereas outside the curve it does not exist. From a dynamical perspective, if $R_1 < 0.1478$, and $A$ is chosen so that the point lies below the lower boundary curve, then the solution will quickly become a single pulse of size 5 (i.e., the internal 0 becomes a 1), see the center panel of Figure 8 with $(R_1, A) = (0.1, 1.0)$. As discussed previously, a pulse of this size can be thought of as the concatenation of two fronts. If the value of $A$ is such that the point is also below the lower boundary of the curve presented in the right panel of Figure 2, so that $U$ invades $V$, then both fronts will travel, i.e., expand until the entire lattice is overtaken by language $U$ (see the left panel of Figure 8 with $(R_1, A) = (0.1, 0.6)$). On the other hand, if $R_1 < 0.1478$, and $A$ is chosen so that the point lies above the upper boundary curve, then the solution will quickly decay to a pulse of size zero, i.e., language $V$ is spoken over the entire lattice (see the right panel of Figure 8 with $(R_1, A) = (0.1, 1.7)$).

\textsuperscript{1}We will return to this aspect in more detail in the continuum limit analysis, see Section 4.1.
Figure 5: (color online) The results of a numerical simulation of the full ODE (2.1) where the initial condition satisfies $u_{jk}(0) = u_{j\ell}(0)$ for all $k, \ell$. The color white represents language $V$, and the color black represents language $U$. In the top two figures $R_1 = 0.15$. For the top left figure $A = 0.9$ (so the point is above the boundary for a pulse of size 1), and for the top right figure $A = 0.6$ (so the point is below the boundary for a pulse of size 1). In both figures the initial condition for fixed $k$ is a small perturbation of a pulse of size 1. In the bottom two figures $R_1 = 0.5$. For the bottom left figure $A = 1.0$ (so the point is above the boundary for a pulse of size 2), and for the bottom right figure $A = 0.8$ (so the point is below the boundary for a pulse of size 2). In both figures the initial condition for fixed $k$ is a small perturbation of a pulse of size 2.

Remark 3.3. If $R_1 > 0.1478$, then the pulse no longer exists, and the fate of the perturbation is a more difficult question to answer. This task will be left for a future paper.

Next consider the 2-2-2 pulse. The boundary for which this solution exists is presented as a dashed (red) curve in Figure 7. The cusp point is $(R_1, A) \sim (0.3515, 0.9879)$. The dynamics associated with $(R_1, A)$ points chosen outside of the domain bounded by the curve are exactly as that outlined above. For points below the curve the solution quickly becomes a single pulse of size 6, which again is the concatenation of two fronts. Each front will travel, and $U$ will grow, if $A$ is sufficiently small. For points above the curve the solution again quickly decays to a pulse of size zero.

Finally, consider the 2-$k$-2 pulse for any $k \geq 3$. Here we find this is a true concatenation of two pulses of size 2, so the boundary curve is given by the dashed (red) curve in Figure 4. Moreover, the dynamics of this pulse is initially governed by the dynamics associated with a pulse of size 2 (see the bottom two panels of Figure 5).

While we do not present the corroborating details here, we now have the following rule-of-thumb. If we start with a two-pulse of size $j$-$k$-$\ell$, and if $k \geq 3$, then the resulting dynamics will initially be independently governed by those associated with the pulse of size $j$ and pulse of size $\ell$. The individual pulses “see” each other
only if the gap between the two is one or two adjacent sites. Indeed, this rule holds for any concatenation of pulses. As long as the distance between adjacent pulses is at least 3 sites, the existence boundary curve is exactly that associated with each individual pulse which makes up the entire multi-pulse. Moreover, the dynamics are governed by those associated with the single pulse until the distance between individual pulses is reduced to one or two sites.

3.5. Spectral stability

We have proven stable fronts and pulses exists for small $\epsilon_1$ for the 1D model (2.1). We now remove the assumption that $\epsilon_1$ is small, and assume that a stable front/pulse exists for (2.1). The spectrum for the associated linearized self-adjoint operator, $L_1D$, is then strictly negative, so

$$\langle L_{1D}v_j,v_j \rangle < 0.$$  \hfill (3.1)

We now consider the spectral stability for the original 2D model (2.1). The self-adjoint linearized operator has the form,

$$L_{2D} = L_{1D} + 2(1 + A)\epsilon_1 U_j(1 - U_j)\Delta_k.$$ 

Using a Fourier decomposition for the eigenfunctions in the transverse direction,

$$v_{jk} \mapsto v_j e^{i\xi k}, \quad -\pi \leq \xi < \pi,$$

we find,

$$L_{2D} v_{jk} = [L_{1D} - 4(1 + A)\epsilon_1 (1 - \cos(\xi)) U_j(1 - U_j)] v_j e^{i\xi k}.$$ 

Since the second term in the sum is a nonpositive operator, by using the inequality (3.1) we can conclude that

$$\langle L_{2D}v_{jk},v_{jk} \rangle < 0.$$
Consequently, all the eigenvalues must be strictly negative, so the stable front/pulse for the 1D problem is transversely stable for the 2D problem.

4. Existence and spectral stability of stripes for the continuum model

We now consider the existence and spectral stability of solutions to the continuum model (2.2).

4.1. Existence: compactons

The existence problem is settled by finding solutions to the nonlinear ODE (2.4). Recalling \( R = \epsilon_0 + 4\epsilon_1 \), under the assumption that neither language is more prestigious, \( A = 1 \), there exists the exact compacton solution,

\[
U_c(x) = \frac{1}{2} \left[ 1 + \cos \left( \sqrt{\frac{R}{2}} x \right) \right].
\]

In writing this solutions there is the implicit understanding that the compacton is continuous with \( U_c(x) \equiv 0 \) or \( U_c(x) \equiv 1 \) outside some finite spatial interval. Of course, any spatial translation of the compacton is also a solution. Not only do these compactons define compactly supported pulses, they also define fronts connecting \( u = 0 \) to \( u = 1 \). One front satisfies \( U_c(x) = 0 \) for \( x \leq -\pi \sqrt{2/R} \), and \( U_c(x) = 1 \) for \( x \geq 0 \) (of course, this front can be translated). Another front satisfies \( U_c(x) = 1 \) for \( x \leq 0 \), and \( U_c(x) = 0 \) for \( x \geq \pi \sqrt{2/R} \) (again, this front can be translated). Note that the width of the front/pulse depends upon the reaction rate, \( R \).

Remark 4.1. There is also an explicit compact solution when \( p = 3 \),

\[
U_c(x) = \frac{1}{2} \left[ 1 + \cos \left( \sqrt{\frac{2R}{3}} x \right) \right].
\]

Numerically, we see compactons for any \( p > 1 \).
4.2. Traveling waves

If $A \neq 1$, numerical simulations indicate that the compacton fronts will travel at a constant speed which depends upon $A$. Moreover, the simulations suggest that the shape of the front at a fixed time is roughly that of the compacton for $A = 1$. In order to derive an approximate analytic expression for the wavespeed we plug $U_c(x + ct)$ into the PDE (4.2), multiply the resultant equation by $\partial_x U_c(x + ct)$, and then integrate over the domain where the front is nonconstant. Doing all this leads to the following predictions for the wave-speed,

$$V \to U, \quad c = -\frac{\sqrt{2R}}{\pi} (A - 1); \quad U \to V, \quad c = \frac{\sqrt{2R}}{\pi} (A - 1). \quad (4.1)$$

The notation $j \to k$ corresponds to the front which has value $j$ for $x \ll 0$ and value $k$ for $x \gg 0$. See Figure 9 for the comparison of the theoretical prediction with the results of a numerical simulation of the PDE (4.2). Numerical simulations indicate that these are good predictions for a relatively large range of $A$ for the 1D PDE model; recall the relevant discussion also in Figure 3. Moreover, we find that for $R$ sufficiently large, and away from the saddle-node bifurcation points, these are also good predictions for the wave-speed for the discrete model.

Remark 4.2. If $A < 1$, so that language $U$ is preferred, the front will move so that language $U$ invades language $V$. On the other hand, if $A > 1$, so that $V$ is preferred, $V$ will invade $U$. The standing compacton which exists for $A = 1$ is then seen as a transition between these two invasion fronts.

4.3. Spectral stability: one dimension

Let us now consider the spectral stability of these compactons. The 1D version of the PDE (2.2) is,

$$\partial_t u = Ru(1 - u) [(1 + A)u - A] + (1 + A)u(1 - u)\partial_x^2 u + [1 - (1 + A)u] (\partial_x u)^2. \quad (4.2)$$

Writing $u = U_c + v$, when $A = 1$ the linearized problem for $v$ is,

$$\partial_t v = 2\partial_x [U_c(1 - U_c)\partial_x v] + g(U_c)v, \quad (4.3)$$

where,

$$g(U_c) = R(-6U_c^2 + 6U_c - 1) + 2(1 - 2U_c)\partial_x^2 U_c - 2(\partial_x U_c)^2.$$
Without loss of generality assume the solution in question is the $V \to U$ front, i.e., $U_c(x) = 0$ for $x \leq -\pi \sqrt{2/R}$, and $U_c(x) = 1$ for $x \geq 0$. Outside the interval $[-\pi \sqrt{2/R}, 0]$ the linearized PDE (4.3) becomes an ODE,

$$\partial_t v = -Rv.$$ 

The associated spectral problem is,

$$\lambda v = -Rv \quad \Rightarrow \quad \lambda = -R, \text{ or } v \equiv 0.$$ 

Because of the degeneracy associated with the diffusion coefficient, the essential spectrum for the operator comprises a single point. On the other hand, if then upon using the expression for the compacton the associated spectral problem is the singular Sturm-Liouville problem,

$$\frac{1}{2} \partial_x \left[ \sin^2 \left( \sqrt{\frac{R}{2} x} \right) \partial_x v \right] - \frac{R}{4} \left( 3 \cos^2 \left( \sqrt{\frac{R}{2} x} \right) - 1 \right) v = \lambda v. \quad (4.4)$$

If $\lambda \neq -R$, then for the sake of continuity we need Dirichlet boundary conditions at the endpoints,

$$v \left( -\sqrt{\frac{2}{R}} \pi \right) = v(0) = 0.$$ 

Regarding the interior problem, $x \in [-\pi \sqrt{2/R}, 0]$, due to spatial translation a solution when $\lambda = 0$ is $v_0(x) = \partial_x U_c$. Since the front is monotone, this eigenfunction is of one sign. Consequently, by classical Sturmian theory $\lambda = 0$ is the largest eigenvalue, so the wave is spectrally stable.

Now consider the concatenation of fronts. Since each front is a compacton, there will be no tail-tail interaction leading to small eigenvalues. Consequently, each front will add another eigenvalue associated with the eigenvalue of the original front. The associated eigenfunction will simply be a spatial translation of the associated eigenfunction. In particular, if there are $N$ fronts, then $\lambda = 0$ will be a semi-simple eigenvalue with geometric multiplicity $N$. The multiplicity follows from the fact that each front can be spatially translated without affecting any of the other fronts.
Suppose we have two fronts, so the solution is a flat-topped compacton. As the size of the top is nonzero, there will be two zero eigenvalues, and the rest of the spectrum will be negative. At the limit of a zero length top we have the pulse compacton, 

\[ U_c(x) = \frac{1}{2} \left[ 1 + \cos \left( \sqrt{\frac{R}{2}} x \right) \right], -\sqrt{\frac{2}{R}} \pi \leq x \leq \sqrt{\frac{2}{R}} \pi. \]

Since the diffusion is zero at \( x = 0 \), so the eigenvalue problem is still degenerate, we can still think of this solution as the concatenation of two fronts, a left front and a right front. The eigenvalue at zero will have geometric multiplicity two. One eigenfunction will be \( \partial_x U_c \) of the left front, and zero elsewhere, while another will be \( \partial_x U_c \) of the right front, and zero elsewhere. Using linearity, we note that one eigenfunction is the sum of these two, which is precisely the expected spatial translation eigenfunction of the full compacton, \( \partial_x U_c \).

### 4.4. Spectral stability: two dimensions

A steady-state front solution to the 2D model (4.2) when \( A = 1 \) is the compacton, \( u(x, y) = U_c(x) \). As we saw in Section 4.3, for the 1D model (4.2) the original front is spectrally stable with a simple zero eigenvalue, and a concatenation of \( N \) fronts is spectrally stable with a semi-simple zero eigenvalue of multiplicity \( N \). Let \( U(x) \) represent a spectrally stable concatenation of \( N \) fronts, which is a stripe pattern.

Consider the spectral stability of the stripes for the full 2D problem. Denote the 1D self-adjoint linearization in (4.3) about the concatenation as \( \mathcal{L}_1 \). The linearization about this striped pattern for (2.2) is,

\[ \mathcal{L}_2 = \mathcal{L}_1 + 2U(1-U)\partial_y^2, \]

which is also self-adjoint. Using the Fourier transform to write candidate eigenfunctions, 

\[ w(x, y) = v(x)e^{i\xi y}, \]

we have,

\[ \mathcal{L}_2w = (\mathcal{L}_1 - 2\xi^2U(1-U)) ve^{i\xi y}. \]

We already know \( \mathcal{L}_1 \) is a nonpositive self-adjoint operator. Since \( \xi^2U(1-U) \geq 0 \), we can therefore conclude \( \mathcal{L}_2 \) is a nonpositive self-adjoint operator. Consequently, there are no positive eigenvalues, so the stripe pattern inherits the spectral stability of the concatenation. In particular, it is spectrally stable.

### 5. Spots: a case study

We now consider the existence and spectral stability of spots. A spot is a contiguous set of sites on the lattice which all share language \( U \) (or \( V \)). All other sites share language \( V \) (or \( U \)). For example, a \( 2 \times 3 \) spot will be a rectangle of height 2 and length 3, so there will be 6 total sites which share language \( U \). When \( \epsilon_1 = 0 \) a stable spot of any size and shape can be formed. By the Implicit Function Theorem the spot will persist and be spectrally stable for small \( \epsilon_1 \). Our goal here is to construct a snaking diagram for this spot, and then briefly discuss the dynamics associated with small perturbations of a spot.

#### 5.1. Existence

First consider the snaking diagram associated with a steady-state solution. We will start with the configurations at \( R_1 = 0 \) of a \( 1 \times 1 \) square of \( U \) sitting on a background of \( V \). The results are plotted in Figure 10. The figure on the left gives the snaking diagram, and some stable solutions arising from the snaking are given on the right. For the snaking diagram stable solutions are marked with a (blue) square, and unstable solutions are marked with a (red) dot. The initial \( 1 \times 1 \) configuration grows seemingly without bound. While we do not provide all the pictures here, as the norm of the solution grows the shape of the contiguous \( U \)
Figure 10: (color online) The numerically generated snaking diagram for a square lattice of size $20 \times 20$ when $R_1 = 0.1$ and starting with a $1 \times 1$ spot. The figure on the left is the snaking diagram, and the figures on the right provide stable solutions arising from the diagram. The notation on the vertical axis, $|u|^2$, represents the square of the $\ell_2$-norm of the solution. The upper right panel has $(A, |u|^2) \sim (0.5, 0.6139)$, the next one down has $(A, |u|^2) \sim (0.9276, 7.0127)$, the third one down has $(A, |u|^2) \sim (1.0449, 16.8568)$, and the bottom panel on the right has $(A, |u|^2) \sim (1.0139, 32.2287)$. Each of these points is marked by a large (blue) filled dot on the snaking diagram. For the snaking diagram stable solutions are marked by a (blue) square, and unstable solutions are marked with a (red) dot. While we do not show it here, the growth in terms of the total number of contiguous groups holding language $U$ appears to have no upper bound.

speakers for a stable solution is either a square or something that has roughly a circular geometry. Regarding the transition from stable to unstable solutions, it is generally not a saddle-node bifurcation, e.g., at the transition point the number of unstable eigenvalues will go from zero to two. Moreover, within the curve of unstable solutions there are additional bifurcations where the number of positive eigenvalues either increases or decreases. The solution structure is rich, but we leave a detailed look at it for a different paper.

Remark 5.1. We should point out that as in the case of single stripes being concatenated to form more complicated stripe patterns, we can concatenate single spots to form more complicated structures. All that is required for each spot to essentially be an isolated structure is for the spots to be sufficiently separated. Our experience is that a minimal separation distance between two adjacent spots of three sites is enough.

5.2. Dynamics

Now let us consider the dynamical implications of the snaking diagram. In particular, we shall look at the effect of varying $A$ for fixed $R_1 = 0.1$. Recall that for stripes we saw in Section 3.2 that outside the snaking diagram traveling waves would appear; in particular, if $A < A_-$ then language $U$ would invade language $V$, whereas if $A > A_+$, then language $V$ would invade language $U$. Consequently, we expect a similar behavior for spots; in particular, a spot will grow or die as a function of the prestige. For a particular example we start with a stable solution arising from the $1 \times 1$ initial configuration when $A = 0.8165$. The square of the $\ell_2$-norm of this solution is roughly 11. This solution is contained in the small stable branch shown in Figure 10 with $A_- \sim 0.8139$ and $A_+ \sim 0.8192$.

First suppose that $A = 0.6$. When looking at the snaking diagram, we see that there are no stable steady-state solutions with this value of $A$. The time evolution associated with this initial condition is provided in Figure 11. Of particular interest is the evolution of the square of the norm in the far right panel. We see that the norm is growing up to at least $t = 50$. While we do not show it here, the norm continues to
Figure 11: (color online) The time evolution of an $A = 0.8165$ solution when $A = 0.6$. The panel on the far right shows the evolution of the square of the $\ell^2$-norm of the solution.

Figure 12: (color online) The time evolution of an $A = 0.8165$ solution when $A = 0.75$. The panel on the far right shows the evolution of the square of the $\ell^2$-norm of the solution.

grow until the all the nodes share the common language $U$. The growth in language $U$ is manifested in the square becoming larger and larger as those nodes containing $V$ at the boundary between $U$ and $V$ switch to language $U$.

Next suppose that $A = 0.75$. When looking at the snaking diagram, we see there is a (stable) steady-state solution with this value of $A$ and which also has a larger norm. The time evolution associated with this initial condition is provided in Figure 12. Of particular interest is the evolution of the square of the norm in the far right panel, which in this case achieves a steady-state. The final state at $t = 50$ corresponds to the first stable solution on the snaking diagram where $a = 0.75$, and whose norm is greater than 11. Language $U$ invades language $V$ until a steady-state configuration is reached.

For the next example suppose that $A = 0.9$. When looking at the snaking diagram, we see there is a steady-state solution with this value of $A$ and which also has a smaller norm. The time evolution associated with this initial condition is provided in Figure 13. Of particular interest is the evolution of the square of the norm in the far right panel, which in this case also achieves a steady-state. The final state at $t = 50$ corresponds to the first stable solution on the snaking diagram where $A = 0.9$, and whose norm is less than 11. Language $V$ invades language $U$ until a steady-state configuration is reached. For the last example suppose that $A = 1.1$. When looking at the snaking diagram, we see there is no steady-state solution with
Figure 13: (color online) The time evolution of an $A = 0.8165$ solution when $A = 0.9$. The panel on the far right shows the evolution of the square of the $\ell^2$-norm of the solution.

Figure 14: (color online) The time evolution of an $A = 0.8165$ solution when $A = 1.1$. The panel on the far right shows the evolution of the square of the $\ell^2$-norm of the solution.

this value of $A$ and which also has a smaller norm. The time evolution associated with this initial condition is provided in Figure 14. Of particular interest is the evolution of the square of the norm in the far right panel, which in this case goes to zero. Language $V$ invades language $U$ until the entire lattice shares the common language $V$.

In conclusion, we have the following rule-of-thumb if the initial configuration is near a steady state solution. If the value of $A$ is decreased, so that the prestige of language $U$ increases, then a spot of $U$ in a sea of $V$ will grow until a stable steady-state associated with that value of $A$ is achieved. If no such steady-state exists, then eventually the entire lattice will share language $U$. On the other hand, if the value of $A$ is increased, so that the prestige of language $V$ increases, then a spot of $U$ in a sea of $V$ will shrink in size until a stable steady-state associated with that value of $A$ is achieved. If no such steady-state exists, then eventually the entire lattice will share language $V$. While we do not show it here, this rule was manifested in every numerical simulation that we performed. It would be most interesting to translate this observation into a precise mathematical statement. This is left as an interesting direction for future work.
6. Conclusions & Future Challenges

We have derived an ODE model of language dynamics on a square lattice which is a natural generalization of the AS language model on one lattice site. The model can also be used to discuss, e.g., the spread of an opinion through the lattice, or the growth/decay of religious observance on the lattice. We also looked at the continuum limit of the ODE, which is a PDE which features a degenerate diffusion term. We numerically studied the existence of special spatial structures on the lattice; primarily, stripes and spots. Through a combination of numerics and analysis we analyzed the dynamics associated with small perturbations of these spatial structures. Finally, we provided rules-of-thumb to help understand how languages die and grow in terms of their prestige, and interaction with neighboring communities.

As is already evident from the discussion above, there are numerous directions in this emerging field that are worthwhile of further study. Some are already concerning the model at hand. As highlighted earlier, features such as the bifurcation of traveling solutions from standing ones and their scaling laws, or the more precise identification of the discrete solutions and their tails from a mathematical analysis perspective would be of interest. While it is unclear whether something analytical can be said about the bifurcation diagram of genuinely two-dimensional states such as spots, our numerical observations regarding the model dynamics formulate a well-defined set of conjectures regarding the fate of a spot when the prestige is decreased or increased that may be relevant to further explore mathematically. However, it would also be relevant to consider variations of the model. Here, we selected as a first step of study to explore an ordered two-dimensional square lattice. However, the $I_{jk}$ may be relevant to generalize to more complex networks and modified (influence or) “adjacency matrices” to explore their impact on the findings presented herein. As indicated herein, the role of near-neighbor interactions is expected to maintain some of the key features we considered; yet in a progressively connected world, the consideration of nonlocal, long-range interactions may be of interest in its own right. Another possibility is to insert a spatially heterogeneous prestige $A_{jk}$ and examine how its spatial variation may influence standing and traveling structures. There are numerous variants that can be considered thereafter, e.g., how does a local prestige variation interact with the traveling wave patterns explored herein? Such queries have been considered in other contexts where the interactions bear a linear component recently, e.g., see Hoffman et al. [14], but have yet to be considered in a fully nonlinear setting such as the one herein. Such studies, as applicable, will be reported in future publications.

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