A Reconciliation of Electromagnetism and Gravitation

B.G.Sidharth

Centr for Applicable Mathematics and Computer Sciences
B.M.Birla Science Centre,Hyderabad, 500063,India

Abstract

It is argued that once we consider the underpinning of a Non Commutative geometry, itself symptomatic of extended particles, for example in Quantum Superstring theory, then a reconciliation between gravitation and electromagnetism is possible.

1 Introduction

Despite nearly a century of work, it has not been possible to achieve a unification of gravitation and electromagnetism. It must be borne in mind that the tools used, be it Quantum Theory or General Relativity are deeply entrenched in differentiable space time manifolds (and point particles) - the former with Minkowski space time and the latter with curved space time. The challenge has been, as Wheeler noted\cite{1}, the introduction of Quantum Mechanical spin half into General Relativity on the one hand and the introduction of curvature into Quantum Mechanics on the other.

More recent models including Quantum Superstrings on the contrary deal with extended and not point particles and lead to a non-differentiable space-time and a non commutative geometry (NCG)\cite{2,3,4,5}.

Indeed way back in the 1930s, Einstein himself observed\cite{6} "...it has ben pointed out that the introduction of a space-time continuum may be considered as contrary to nature in view of the molecular structure of everything which happens on a small scale. It is maintained that perhaps the success of
the Heisenberg method points to a purely algebraic method of description of nature that is to the elimination of continuous functions from physics. Then however, we must also give up, by principle the space-time continuum. It is not unimaginable that human ingenuity will some day find methods which will make it possible to proceed along such a path."

Even at the beginning of the twentieth century several Physicists including Poincare and Abraham amongst others were working unsuccessfully with the problem of the extended electron[7, 8]. The problem was that an extended electron appeared to contradict Special Relativity, while on the other hand, the limit of a point particle lead to inexplicable infinities. These infinities dogged physics for many decades. Infact the Heisenberg Uncertainty Principle straightaway leads to infinities in the limit of spacetime points. It was only through the artifice of renormalization that 't Hooft could finally circumvent this vexing problem, in the 1970s[9].

Nevertheless it has been realized that the concept of spacetime points is only approximate. We are beginning to realize that it may be more meaningful to speak in terms of spacetime foam, strings, branes, non commutative geometry, fuzzy spacetime and so on[10]. Indeed non commutativity arises if there is a minimum space time length as shown a long time ago by Snyder[11]. What we will argue below is that once the underlying non commutative nature of the geometry is recognized then it is possible to reconcile electromagnetism and gravitation.

2 NCG

It is well known that once we consider non zero minimum space time intervals or equivalently extended particles as in Quantum Superstrings, then we have the following non commutative geometry (Cf.refs.[2]-[5],[11]):

\[
[x, y] = 0(l^2), [p_x, p_y] \approx \frac{\hbar^20(1)}{l^2}
\]

(and similar equations) where \( l, \tau \) are the extensions of the space time coordinates.

In conventional theory the space time coordinates as also the momenta commute amongst themselves unlike in equation (1). It must be observed that the non commutative relations are self evident, in the sense that \( xy \) or \( yx \) is
each of the order of $l^2$, and so is their difference because of the non commutativity.

The non commutative or in Witten’s words, Fermionic feature is symptomatic of the breakdown of the concept of the spacetime points and point particles at small scales or high energies. As has been noted by Snyder, Witten, and several other scholars, the divergences encountered in Quantum Field Theory are symptomatic of precisely such an extrapolation to spacetime points and which necessitates devices like renormalization. As Witten points out [12], “in developing relativity, Einstein assumed that the space time coordinates were Bosonic; Fermions had not yet been discovered!... The structure of space time is enriched by Fermionic as well as Bosonic coordinates.” Interestingly it has been shown that the commutative relations (1) lead directly to the Dirac equation, on the one hand [13]. On the other hand, it is interesting that a differential calculus over a non commutative algebra uniquely determines a gravitational field in the commutative limit and that there is a unique metric which remains as a classical ”shadow” as shown by Madore [14].

Let us now introduce this effect into the usual distance formula in flat space

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

(2)

Rewriting the product of the two coordinate differentials in (2) in terms of the symmetric and non symmetric combinations, we get

$$g_{\mu\nu} = \eta_{\mu\nu} + kh_{\mu\nu}$$

(3)

where the first term on the right side of (3) denotes the usual flat space time and the second term denotes the effect of the non commutativity, $k$ being a suitable constant.

It must be noted that if $l, \tau \to 0$ then equations (1) and also (3) reduce to the usual formulation.

The effect of the non commutative geometry is therefore to introduce a departure from flat space time, as can be seen from (3).

Infact remembering that the second term of the right side of (3) is small, this can straightaway be seen to lead to a linearized theory of General Relativity [15]. Exactly as in this reference we could now deduce the General Relativistic relation

$$\partial_\lambda \partial^\lambda h^{\mu\nu} - (\partial_\lambda \partial^\nu h^{\mu\lambda} + \partial_\lambda \partial^\mu h^{\nu\lambda})$$
Let us now consider the noncommutative relation (1) for the momentum components. Then, it can be shown using (1) and (3) that
\[ \partial_{\lambda} \partial_{\mu} - \partial_{\mu} \partial_{\lambda} \]
goes over to
\[ \partial_{\lambda} \Gamma_{\lambda}^{\nu} - \partial_{\mu} \Gamma_{\nu}^{\mu} \] (5)

Normally in conventional theory the right side of (5) would vanish. Let us designate this nonvanishing part on the right by
\[ \frac{e}{c} F_{\mu\lambda} \] (6)

(6) can be written as
\[ B l^2 \sim \frac{\hbar c}{e} \] (7)

where \( B \) is the magnetic field, if we are to identify \( F_{\mu\nu} \) with the electromagnetic tensor (16). It will be recognized that (6) gives the celebrated expression for the magnetic monopole, and indeed it has also been shown that a noncommutative space time at the extreme scale throws up the monopole (17, 18).

We have shown here that the noncommutativity in momentum components leads to an effect that can be identified with electromagnetism and in fact from expression (6) we have
\[ A_{\mu} = \hbar \Gamma_{\nu}^{\mu} \] (8)

where \( A_{\mu} \) is the electromagnetic four potential.

Thus noncommutativity as expressed in equations (1) generates both gravitation and electromagnetism.

We can see this in greater detail as follows. The gravitational field equations can be written as (15)
\[ \Box \phi^{\mu\nu} = -k T^{\mu\nu} \] (9)

where
\[ \phi^{\mu\nu} = h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h \] (10)

It also follows, if we use the usual gauge and equation (8) that
\[ \partial_{\mu} h^{\mu\nu} = A^{\nu} \] (11)
in this linearised theory.

Whence, remembering that we have (3), operating on both sides of equation
with $\partial_u$ we get Maxwell’s equations of electromagnetism. Indeed this is not surprising because, as is well known equation (8) is mathematically identical to the formulation of Weyl. However in Weyl’s formulation, the electromagnetic potential was put in by hand. In the above case it is a consequence of the non commutative geometry at small scales, which again is symptomatic of the spinorial behaviour of the electron, as has been discussed in detail elsewhere and in fact (8) deduced from an alternative viewpoint. It is also well known that if equation (8) holds then in the absence of matter the general relativistic field equations (4) reduce to Maxwell equations. In any case, all this provides a rationale for the fact that from (9) we get the equation for spin 2 gravitons (Cf.ref.[15]) while from the Maxwell equations, we have Spin 1 (vector) photons.

3 Discussion

1. The characterization of the metric in equations (2) and (3) in terms of symmetric and non symmetric components is similar to the torsional formulation of General Relativity. However in this latter case, there is no contribution to the differential interval from the torsional (that is non-commutative) effects. The non-commutative contribution is given by (1) and herein comes the extended, rather than point like particle.

In any case the above attempt at unification of electromagnetism and gravitation had made some headway, but unless the underpinning of a non commutative geometry is recognised, the full significance does not manifest itself.

2. We now make the following remarks:

It can be seen from the transition to (3) from (2), that the curvature arises from the non commutativity of the coordinates. Indeed this is the classical analogue of a Quantum Mechanical result deduced earlier that the origin of mass is in the minimum space intervals and the non local Quantum Mechanical amplitudes within them as has been discussed in detail in references cited. In Quantum Superstring theory also, the mass arises out of the tension of the string in this minimum interval. We see here the convergence of the Quantum Mechanical and classical approaches once the extension of particles is recognized.

We also know that the minimum space time intervals are at the Compton scale where the momentum $p$ equals $mc$. For a Planck mass $\sim 10^{-5}gms$, this
is also the Planck scale, as in Quantum Superstring theory.

In Snyder’s original work, the commutation relations like (1) hold good outside the minimum space time intervals, and are Lorentz invariant. This is quite pleasing because in any case, even in Quantum Field Theory, we use Minkowski space time.

3. The above non commutative geometry also holds the key to the mysterious extra dimensions of Quantum Superstrings. This has been discussed in detail in references[5, 18]. But to see in a simple way, we note that equation (1) shows that the coordinates $y$ and $z$ show up as some sort of a momenta, though with a different multiplying constant as the analogue of the Planck constant. That is instead of the single $x$ momentum, $p_x$, we have two extra momenta, this being the same for the $y$ and $z$ momenta also. This leads to the well known 9 + 1 dimensions of Quantum superstrings, though because for all these extra ”momenta”, the multiplying factor, the analogue of the Planck constant is different, so these extra dimensions are supressed or curled up in the Kaluza-Klein sense.

4. A concept which one encounters in Quantum SuperString theory and more generally in the presence of the Non commutative geometry (1) is that of Duality. We will briefly examine this now and see its significance in relation to electrodynamic theory. Infact the relation (1) leads to[18],

$$\Delta x \sim \frac{\hbar}{\Delta p} + \alpha' \frac{\Delta p}{\hbar}$$

(12)

where $\alpha' = l^2$, which in Quantum SuperStrings Theory $\sim 10^{-66}$. This is an expression of the duality relation,

$$R \rightarrow \alpha'/R$$

This is symptomatic of the fact that we cannot go down to arbitrarily small spacetime intervals, below the Planck scale in this case (Cf.ref.[24]).

There is an interesting meaning to the duality relation arising from (12). While it appears that the ultra small is a gateway to the macro cosmos, we could look at it in the following manner. The first term of the relation (12) which is the usual Heisenberg Uncertainty relation is supplemented by the second term which refers to the macro cosmos.

Let us consider the second term in (12). We write $\Delta p = \Delta Nmc$, where $\Delta N$ is the Uncertainty in the number of particles, $N$, in the universe. Also
\[ \Delta x = R, \] the radius of the universe which \( \sim \sqrt{Nl}, \) the famous Eddington relationship. It should be stressed that the otherwise empirical Eddington formula, arises quite naturally in a Brownian characterisation of the universe as has been pointed out earlier (Cf. for example ref.[23]).

We now get,

\[ \Delta N = \sqrt{N} \]

Substituting this in the time analogue of the second term of (12), we immediately get, \( T \) being the age of the universe,

\[ T = \sqrt{N} \tau \]

In the above analysis, including the Eddington formula, \( l \) and \( \tau \) are the Compton wavelength and Compton time of a typical elementary particle, namely the pion. The equation for the age of the universe is also correctly given above. Infact in the closely related model of fluctuational cosmology (Cf. for example ref.[23]) all of the Dirac large number coincidences including the above Eddington formula as also the mysterious Weinberg formula relating the mass of the pion to the Hubble constant, follow as a consequence, and are not empirical. All these relations relating large scale parameters to microphysical constants were shown to be symptomatic of what has been called, stochastic holism (Cf. also ref.[5]), that is a micro-macro connection with a Brownian or stochastic underpinning. Duality, or equivalently, relation (12) is really an expression of this micro-macro link.

We will now see a curious connection between the foregoing analysis with the apparently disparate concept of the Feynman-Wheeler action at a distance theory, which had been quite successful.

Our starting point is the so called Lorentz-Dirac equation[8]:

\[ ma^\mu = F^\mu_{in} + F^\mu_{ext} + \Gamma^\mu \] (13)

where

\[ F^\mu_{in} = \frac{e}{c} F^{\mu\nu}_{in} v_v \]

and \( \Gamma^\mu \) is the Abraham radiation reaction four vector related to the self force and, given by

\[ \Gamma^\mu = \frac{2}{3} \frac{e^2}{c^3} (\dot{a}^\mu - \frac{1}{c} \dot{a}^\lambda a_\lambda v^\mu) \] (14)
Equation (13) is the relativistic generalisation for a point electron of an earlier equation proposed by Lorentz, while equation (14) is the relativistic generalisation of the original radiation reaction term due to energy loss by radiation. It must be mentioned that the mass $m$ in equation (13) consists of a neutral mass and the original electromagnetic mass of Lorentz, which latter does tend to infinity as the electron shrinks to a point, but, this is absorbed into the neutral mass. Thus we have the forerunner of renormalisation in quantum theory.

There are three unsatisfactory features of the Lorentz-Dirac equation (13). Firstly the third derivative of the position coordinate in (13) through $\Gamma^\mu$ gives a whole family of solutions. Except one, the rest of the solutions are run away - that is the velocity of the electron increases with time to the velocity of light, even in the absence of any forces. This energy can be thought to come from the infinite self energy we get when the size of the electron shrinks to zero. If we assume adhoc an asymptotically vanishing acceleration then we get a physically meaningful solution, though this leads to a second difficulty, that of violation of causality of even the physically meaningful solutions. It has been shown in detail elsewhere[7] that these acausal, non local effects take place within the Compton time.

We now come to the Feynman-Wheeler action at a distance theory[26, 27]. They showed that the apparent acausality of the theory would disappear if the interaction of a charge with all other charges in the universe, such that the remaining charges would absorb all local electromagnetic influences was considered. The rationale behind this was that in an action at a distance context, the motion of a charge would instantaneously affect other charges, whose motion in turn would instantaneously affect the original charge. Thus considering a small interval in the neighbourhood of the point charge, they deduced,

$$F^\mu_{\text{ret}} = \frac{1}{2} \{ F^\mu_{\text{ret}} + F^\mu_{\text{adv}} \} + \frac{1}{2} \{ F^\mu_{\text{ret}} - F^\mu_{\text{adv}} \}$$

(15)

The left side of (15) is the usual causal field, while the right side has two terms. The first of these is the time symmetric field while the second can easily be identified with the Dirac field above and represents the sum of the responses of the remaining charges calculated in the vicinity of the said charge. Also here we encounter effects within the Compton scale (Cf. ref.[7]) of the rest of the universe. We thus return to the concept from Quantum Superstring theory, or more generally a theory based on relations like (1) of
extended particles and duality, a manifestation of holism.

5. One could argue that the non commutative relations (1) are an expression of Quantum Mechanical spin. To put it briefly, for a spinning particle the non commutativity arises when we go from canonical to covariant position variables. Zakrzewski[28] has shown that we have the Poisson bracket relation

\[ \{ x^j, x^k \} = \frac{1}{m^2} R^{jk}, \ (c = 1), \]

where \( R^{jk} \) is the spin. The passage to Quantum Theory then leads us back to the relation (1).

Conversely it was shown that the relations (1) imply Quantum Mechanical spin[25]. Another way of seeing this is to observe that (1) implies that \( y = \alpha \hat{\mathbf{p}}_y \), where \( \alpha \) is a dimensional constant viz \([T/M]\) and \( \hat{\mathbf{p}}_y \) is the analogue of the momentum, but with the Planck constant replaced by \( l^2 \). So the spin is given by

\[ |\mathbf{r} \times \hat{\mathbf{p}}| \approx 2x p_y \approx 2\alpha^{-1} l^2 = \frac{1}{2} \left( \frac{\hbar}{m^2 c^2} \right)^{-1} \times \frac{\hbar^2}{m^2 c^2} = \frac{\hbar}{2} \]

as required.

References

[1] C.W. Misner, K.S. Thorne and J.A. Wheeler, ”Gravitation”, W.H. Freeman, San Francisco, 1973, pp.819ff.

[2] Y. Ne’eman, in Proceedings of the First International Symposium, ”Frontiers of Fundamental Physics”, Eds. B.G. Sidharth and A. Burinskii, Universities Press, Hyderabad, 1999, pp.83ff.

[3] E. Witten, Physics Today, April 1996, pp.24-30.

[4] B.G. Sidharth, Chaos, Solitons and Fractals, 11(8), 2000, 1269-1278.

[5] B.G. Sidharth, ”The Chaotic Universe: From the Planck to the Hubble Scale”, Nova Science Publishers, New York, 2001.

[6] A. Einstein in an article in Journal of the Franklin Institute, quoted by M.S. El Naschie in Chaos, Solitons & Fractals, Vol.10 (2/3), 1999, p.163.
[7] B.G. Sidharth, in Instantaneous Action at a Distance in Modern Physics: "Pro and Contra", Eds., A.E. Chubykalo et. al., Nova Science Publishing, New York, 1999.

[8] F. Rohrlich, "Classical Charged Particles", Addison-Wesley, Reading, Mass., 1965.

[9] Gerard 't Hooft, Proceedings of Fourth Frontiers of Fundamental Physics, Kluwer Academic, New York, 2001.

[10] A. Kempf, in "From the Planck Length to the Hubble Radius", Ed. A. Zichichi, World Scientific, Singapore, 2000, p.613-622.

[11] H.S. Snyder, Physical Review, Vol.72, No.1, July 1 1947, p.68-71.

[12] J. Schwarz, M.B. Green and E. Witten, "SuperString Theory", Vol.I, Cambridge University Press, Cambridge, 1987.

[13] B.G. Sidharth, Chaos, Solitons and Fractals 11 (2000), p.1269-1278.

[14] J. Madore, IJMPB, 14 (22 & 23), 2000, pp.2419-2425.

[15] C.H. Ohanian, and R. Ruffini, "Gravitation and Spacetime", New York, 1994, pp.64ff.

[16] B.G Sidharth, "Gravitation and Electromagnetism", to appear in Nuovo Cimento.

[17] T. Saito, Gravitation and Cosmology, 6 (2000), No.22, pp.130-136.

[18] B.G. Sidharth, Proceedings of Fourth Frontiers of Fundamental Physics, Kluwer Academic, New York, 2001.

[19] B.G. Sidharth, Gravitation and Cosmology, 4 (2) (14), 1998, p.158ff.

[20] P.G. Bergmann, "Introduction to the Theory of Relativity", Prentice-Hall (New Delhi), 1969, p248ff.

[21] P.G. Bergmann, "Cosmology and Gravitation", V. De Sabbata (Ed), NATO ADVANCED STUDY INSTITUTES SERIES (B), Plenum Press, New York, 1980.
[22] B.G. Sidharth, Ind.J.Pure and Appl.Phys., Vol.35, July 1997, pp.456-471.

[23] B.G. Sidharth, Int.J.Mod.Phys.A, 13 (15), 1998, p.2599ff.

[24] W. Witten, Physics Today, April 1996, pp.24-30.

[25] B.G. Sidharth, Chaos, Solitons and Fractals, 12(2001), 173-178.

[26] F. Hoyle and J.V. Narlikar, "Lectures on Cosmology and Action at a Distance Electrodynamics", World Scientific, Singapore, 1996.

[27] J.A. Wheeler and R.P. Feynman, Rev. Mod. Phys., 17, 157, 1945.

[28] S. Zakrzewski, "Quantization, Coherent States, and Complex Structures", Ed. J.P. Antoine et al., Plenum Press, New York, 1995, p.249ff.