Kinematics, cluster algebras and Feynman integrals

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We identify cluster algebras for planar kinematics of conformal Feynman integrals in four dimensions, as sub-algebras of that for top-dimensional $G(4,n)$ corresponding to $n$-point massless kinematics. We provide evidence that they encode information about singularities of such Feynman integrals, including all-loop ladders with symbol letters given by cluster variables and algebraic generalizations. As a highly-nontrivial example, we apply $D_3$ cluster algebra to a $n = 8$ three-loop wheel integral, which contains a new square root. Based on the $D_3$ alphabet and three new algebraic letters essentially dictated by the cluster algebra, we bootstrap its symbol, which is strongly constrained by the cluster adjacency. By sending a point to infinity, our results have implications for non-conformal Feynman integrals, e.g. up to two loops the alphabet of two-mass-easy kinematics is given by limit of this generalized $D_3$ alphabet. We also find that the reduction to three dimensions is achieved by folding and the resulting cluster algebras may encode singularities of amplitudes and Feynman integrals in ABJM theory, at least through $n = 7$ and two loops.

I. INVITATION: QUIVERS FOR KINEMATICS

Tremendous progress has been made in unravelling mathematical structures of scattering amplitudes, especially in $\mathcal{N}=4$ supersymmetric Yang-Mills theory (SYM) in the planar limit (c.f. [1, 2]). Cluster algebras [3] have played an important role not only for its all-loop integrals (especially in scattering amplitudes, especially motivated by [34], in this letter we identify cluster algebras for kinematics of such conformal integrals as sub-algebras of that for $G(4,n)$, and investigate possible applications in various contexts.

We first review the planar kinematics of $n$ massless momenta, or equivalently an $n$-gon in (dual) space whose vertices are light-like separated, where the dual points are defined via $p^\mu_i = x^\mu_i - x^\mu_{i+1}$ for $i = 1, \cdots, n$ (mod $n$) satisfying $x^\mu_{i+1} = 0$. Such a kinematics can be described in terms of $n$ momentum twistor variables [37], $Z_{i=1,\cdots,n}$ for $a = 1, \cdots, 4$, one for each massless momentum (null ray of the $n$-gon). These variables follow from expressing a dual point $x^\mu_i \in \mathbb{R}^{4,4}$ as a null projective vector $X^I \in \mathbb{R}^{4,4}$ satisfying $X^2 = 0$ and $X^I \sim T X^I$; after complexifying, such a 6d vector can be viewed as an anti-symmetric tensor $X^{a,b}$ for SL(4, C), whose fundamental representation is the twistor $Z^a$. Each point $x^\mu_i$ corresponds to a line given by the pair $(Z_{i-1}, Z_i)$ and since $Z_i \sim t_i Z_i$, they form homogeneous coordinates of $\mathbb{CP}^3$. The space of dual conformal invariant (DCI) kinematics [20] is simply the Grassmannian mod torus action, $G(4,n)/T$ with dimension $3(n-5)$: conformal invariant quantities are torus-invariant functions of Plücker coordinates $(ijkl) := \det(Z_i Z_j Z_k Z_l)$ (e.g. cross-ratios); in particular, we have $x^2_{i,j} = (i-1j-1)/(i-1i\infty)(j-1j\infty)$, where the factors involving the point (line) at infinity, $I_{\infty}$, drop out in any DCI quantities.

We are interested in positive kinematics, $G_{2,0}(4,n)/T$ where all (ordered) Plückers are non-negative, and it is natural to associate it with a Grassmannian cluster algebra [35]. For $n = 6, 7, 8$, the cluster algebras for $G(4,n)$ are of type $A_3, E_6$ and $E_7^{(1,1)}$ respectively [39]. For ex-
ample, for \( n = 6 \) one can choose the initial quiver shown in Fig. [5] for \( G(4, 6) \), where in addition to \( n = 6 \) frozen variables in blue, we have \( 3(n-5) = 3 \) unfrozen (mutable) \( A \)-coordinates forming an \( A_3 \) Dynkin diagram; by mutating them we have 9 cluster variables for \( A_3 \); in this case the total \( 9 + 6 \) \( A \) coordinates are exactly the 15 Plücker’s \( \langle i,j,k,l \rangle \), which can form 9 DCI combinations (up to multiplication) known as the letters of \( A_3 \) kinematics. Equivalently these 9 DCI letters can be obtained as \( X \) coordinates, expressed in terms of 9 positive polynomials of \( f_1, f_2, f_3 \) (\( X \) coordinates of the initial quiver). This provides a first example of parametrizing the kinematics and symbol letters in terms of cluster variables.

In this letter, we will be working with general planar, conformal kinematics, e.g. for Feynman integrals with “massive” corners (each consists of a pair of legs), which only depends on a subset of the \( n \) dual points. One can arrive at such a kinematics by removing (non-adjacent) dual points from \( x_1, x_2, \ldots, x_n \); e.g. putting \( p_2 \) and \( p_3 \) on a massive corner is equivalent to removing \( x_3 \), and a massive corner with \( p_4, p_5 \) amounts to removing \( x_5 \). The resulting kinematics are labelled by \((x_{i_1}, x_{i_2}, \ldots, x_{i_m})\) (e.g. \((x_1, x_2, x_4, x_6, \ldots)\) in the above example), where \( i_k \) and \( i_{k+1} \) are separated by at most 2, which correspond to \( n \)-point \( m \)-gon with \( n/2 \leq m \leq n \). We denote such a kinematics by \((n, m)\) with \( A, B, \cdots \) distinguishing configurations of massive corners (see appendix). Generically the dimension of a \( n \)-point \( m \)-gon kinematics is given by \( d = n + 2m - 15 \), except for the special, four-mass box case where \( d = 2 \). A priori it is unclear at all if one can find any sub-algebra of the \( G(4, n) \) cluster algebra which parametrizes such a kinematics. We show that it is indeed the case by going through all kinematics with \( n \leq 8 \), which we expect to work for higher \( n \) as well.

The basic idea is to find a suitable quiver of \( G(4, n) \), where we can freeze some mutable variables such that the remaining sub-quiver with \( d \) mutable nodes becomes independent of the removed dual points. We provide details in the appendix and only sketch simple examples here. For kinematics with one massive corner, say, \((23)\), we want to identify a co-dimension 2 sub-algebra independent of dual point \( x_3 \). We would like to “delete” the frozen variables \( \langle 123n \rangle \propto x_{1,3}^2 \) and \( \langle 2345 \rangle \propto x_{2,5}^2 \) that depend on \( x_3 \), which can be achieved if they only connect to other frozen variables; this motivates us to find a quiver where each of them is only connected to a unique mutable node, and the desired sub-quiver is obtained by freezing these two nodes. In the appendix we explicitly show this for \((6, 5)\) kinematics (see Fig. [4]): we choose the quiver of \( G(4, 6) \) as in Fig. [6] and freeze \( \langle 1235 \rangle \) and \( \langle 1346 \rangle \), which are connected to \( \langle 1236 \rangle \) and \( \langle 2345 \rangle \) respectively. The resulting quiver has only one mutable variable \( \langle 146 \rangle \) (and four frozen ones connected to it), which forms an \( A_1 \) sub-algebra of \( A_3 \) with no \( x_3 \) dependence!

In terms of \( \mathcal{X} \)-coordinates, we have \( f_2 = \langle 1235 \rangle \langle 1346 \rangle \) and \( 1 + f_2 = \langle 1245 \rangle \langle 1246 \rangle \langle 1346 \rangle \langle 1245 \rangle \langle 1346 \rangle \langle 1245 \rangle \langle 1346 \rangle \langle 1246 \rangle \), both are independent of \( x_3 \). Similarly in Fig. [8] we identify a \( D_4 \) sub-algebra by freezing \( \langle 1345 \rangle \) and \( \langle 1247 \rangle \), for \((7, 6)\) kinematics in Fig. [4].

In general, if none of the \( n-m \) massive corners are adjacent to each other, it is always possible to find a quiver containing all these \( 2(n-m) \) mutable variables and we reach at our sub-quiver by freezing all of them! For example, to go to \((7, 5)\) A kinematics (see Fig. [4] and Fig. [6]), we just freeze two more from that of \( D_4 \) for \((7, 6)\), \( \langle 3457 \rangle \) and \( \langle 1567 \rangle \), which gives a \( A_2 \) sub-algebra. For such cases, this algorithm of freezing nodes is equivalent to that of \([10]\), where we go to lower-dimensional positroid cells by setting unwanted frozen variables to zero; the sub-quiver we obtain is simply the dual graph of the plabic graph for the cell, whose face variables correspond to \( \mathcal{X} \)-coordinates of the sub-algebra! Such examples include \( D_6 \) sub-algebras of \( G(4, d+3)/T \) for \( d = 4, 5, 6 \), and the \( D_{4,1} \) or affine \( D_4 \) of \( n = 8 \) ((8, 6) A kinematics), which are various hexagon kinematics with non-adjacent massive corners \([10]\). However, for kinematics with adjacent massive corners, we cannot set enough frozen variables to zero to reduce to the correct dimension, and general sub-quiver does not correspond to a positroid cell. In general we have to use the algorithm of freezing nodes of certain \( G(4, n) \) quiver, and indeed we find sub-quivers for all kinematics of \( n = 8 \) (see Table I).

Take \((8, 5)\) A kinematics as an example where we need to remove \( x_3, x_6, x_7 \) (see Fig. [2] and Fig. [7]): while we cannot parametrize this kinematics by a cell \([10]\), we can indeed find a quiver of \( G(4, 8) \) where we freeze 6 nodes: in addition to \( \langle 1345 \rangle \) and \( \langle 1248 \rangle \) for corner \((23)\) (which can be reached via a cell), we freeze 4 more variables as shown in Fig. [7]. The result is a \( D_3 \) sub-algebra (to distinguish from the \( n = 6 \) \( A_3 \)), where the three cross-ratios \( u = x_{1,4}^2 x_{2,5}^2, v = x_{1,5}^2 x_{3,7}^2, w = x_{1,7}^2 x_{2,4}^2 \), are parametrized in terms of the three \( \mathcal{X} \)-coordinates of the initial quiver:

\[
\frac{1}{1 + f_0}, v = \frac{1 + f_0 + f_4 f_6}{(1 + f_4 + f_1 f_4) (1 + f_6)}, w = \frac{1 + f_6}{1 + f_4 + f_1 f_4}.
\]

Note that since \((7, 5)B \) (Fig. [4]) is a degeneration of this kinematics with \( \langle 1234 \rangle \rightarrow 0 \), we can freeze \( (1245) \) in this quiver to get a \( A_2 \) (different from that of \((7, 5)A\)). Finally, as shown in Fig. [7] this \( D_3 \) for \((8, 5)A \) is a sub-algebra of that for \((8, 6)C\); if we do not freeze \( (1345) \) and \( (1248) \), we find a 5-dim sub-algebra, which is a \( D_{4,1} \) (different from that of \((8, 6)A\)): it depends on \( u, v, w \) as well as \( u', v' \)

\[
u' := \frac{x_{1,4}^2 x_{2,5}^2}{x_{1,5}^2 x_{3,7}^2}, v' := \frac{x_{1,7}^2 x_{3,5}^2}{x_{1,5}^2 x_{3,7}^2}, f_1 f_2 f_4 f_7 (1 + f_0) Y
\]

where in addition to \( f_1, f_4, f_6 \) we have two more initial \( \mathcal{X} \) coordinates \( f_2 \) and \( f_7 \), and we have defined \( Y := 1 + f_1 f_2 + f_1 f_7 + f_2 f_4 f_7 + f_1 f_2 f_4 f_7 \).

In practice, for our algorithm of freezing nodes it is more efficient to directly work with \( \mathcal{X} \)-coordinates as opposed to \( A \)-coordinates. After some works in finding suitable initial quivers, it is straightforward to find sub-algebras for \( n = 8 \) cases as we present in Table I. For each sub-quiver of \( n = 8 \) kinematics in the table, we record the
expressions of their \( X \)-coordinates \( f_i \) in terms of Plücker coordinates in the ancillary file \( \text{CA.m} \).

II. CLUSTER ALGEBRAS FOR SINGULARITIES OF CONFORMAL FEYNMAN INTEGRALS

Now we study implications of these cluster algebras for conformal Feynman integrals with the kinematics. For \( n = 6 \) and 7, all conformal integrals computed so far \([22, 24–26, 28, 29]\) are MPLs with symbol letters contained in \( A_3 \) and \( E_6 \) respectively. Even in these simpler cases, it is still non-trivial that the letters of multi-loop integrals with a given kinematics is captured by the corresponding sub-algebra, such as various \( n = 7 \) ladder integrals with \((7, 6) \) and \((7, 5) \) A:B kinematics; the corresponding alphabets, summarized as \( E_6 \supset D_4 \supset (\text{two}) A_2 \) can be found in the appendix.

Beyond \( n = 7 \), cluster variables also successfully predict letters for certain classes of Feynman integrals. For instance, the 9 cluster variables from the \( D_3 \) above are enough for penta-box-ladder integrals (with \((8, 5) \) A kinematics), to arbitrary loops \([28, 31]\). The 9 DCI letters of the \( 9+7 \) \( A \)-coordinates (see appendix) read

\[
u, \ v, \ w, \ 1-u, \ 1-v, \ 1-w, \ 1-uw, \ 1-vw, \ 1-u+v+uvw \tag{3}
\]

which we denote as \( W_1, \cdots , W_9 \); using \([1]\), they are equivalent to 9 positive polynomials, \( f_1, f_4, f_6 \) and:

\[
1+f_1, 1+f_4, 1+f_6, 1+f_4+f_1 f_4, 1+f_1+f_1 f_6, 1+f_6+f_4 f_6.
\]

Generally, most cluster algebras we encounter when \( n \geq 8 \) are infinite type (with some exceptions such as \( D_5 \) and \( D_6 \) mentioned above). On the other hand, general conformal integrals with \( n \geq 8 \), even when they evaluate to MPLs, may contain symbol letters that are algebraic (non-rational) functions of Plücker coordinates, which certainly go beyond cluster variables. At least for \( n = 8 \), there is considerable evidence that these algebraic letters are still controlled by affine cluster algebras, and in particular we may extract them via limit rays of tropicalization/Minkowski sum \([17, 18]\). For \( n = 8 \), such algebraic letters found so far are associated with two cyclically related square roots of four-mass kinematics, or \( A_1, 1 \), which is the “smallest” in this chain of sub-algebras:

\[
E_7^{(1,1)} \supset E_{6,1} \supset D_{4,1} \supset A_{2,1} \supset A_{1,1}.
\]

Algebraic letters for these kinematics can be traced to the simplest ones in \( A_{1,1} \). The four-mass box with dual points \( (x_2, x_4, x_6, x_8) \) is parametrized by \( u = \frac{x_2 x_4 + x_6 x_8}{x_2 x_4 + x_6 x_8} = zz \) and \( v = \frac{x_2 x_4 x_6}{x_2 x_4 + x_6 x_8} = (1-z)(1-\overline{z}) \), where \( (z, \overline{z}, 1-z, 1-\overline{z}) \) are algebraic letters containing the square root of \( \Delta_{2,4,6,8} := (1-v+u)^2 - 4uv \); it is natural to write, in addition to their (rational) products \( u, v \), their ratios (odd letters)

\[
\frac{z}{\overline{z}} \quad \frac{1-z}{1-\overline{z}} \quad \frac{1-z}{1-\overline{z}} \quad \frac{1-z}{1-\overline{z}}.
\]

Note that in positive kinematics \( 0 < u, v < 1 \), not only \( \Delta_{2,4,6,8} \) but also the odd letters are positive; this can be easily seen since \( z, \overline{z} \) are roots of the equation \( r^2 - (1-v+u)r + u = 0 \). For higher sub-algebras containing \( A_{1,1} \), all algebraic letters found in previous computations can be obtained similarly from limit rays, and they take the form \( \frac{r_1}{r_2} \), where \( r_i \) are rational functions (including \( r_i = 0, 1 \)). We have two observations:

(1) the product \( (r_1 - z)(r_1 - \overline{z}) \) always gives a monomial of cluster variables in the corresponding algebra, which guarantees that these letters have definite signs;

(2) there are exactly \( d \) multiplicatively independent ratios (odd letters): for \( A_{2,1}, D_{4,1}, E_{6,1} \) and the sector with \( \Delta_{2,4,6,8} \) of \( E_7^{(1,1)} \), we find exactly 3, 5, 7 and 9 such \( r_i \)'s that give independent ratios. Similarly all the rational letters can be obtained by truncating these affine cluster algebras by the method of \([17, 18]\): for the above cases with \( d = 7, 5, 3 \), by using non-vanishing Plücker coordinates for tropicalization/Minkowski sum we find 100 and 9 rational letters in the truncated sub-algebras. Alternatively we could select those letters that only depend on the kinematics from the truncated \( E_7^{(1,1)} \) alphabet (e.g. with 356 cluster variables \([17]\)): these give 93, 33 and 8 rational letters for these cases. We emphasize that such “truncated” alphabets are not unique, but they already provide highly non-trivial predictions for alphabets (singularities) of corresponding Feynman integrals: e.g. for \( D_{4,1} \) case, such an alphabet with 38 (or 33) rational letters and 5 algebra one has been used to to bootstrap the double-penta-ladder at least through \( L = 4 \) \([31]\). We record all these (truncated) alphabets in the file \( \text{CA.m} \).

A. Bootstrapping \( n = 8 \) wheel: new algebraic letters from \( D_3 \)

It is surprising that even for one of the simplest \( n = 8 \) kinematics, \( D_3 \), certain Feynman integrals contain new square roots and algebraic letters. These are integrals beyond ladder topology, e.g. three-loop wheel integrals; in fact it is intriguing that the \( n = 9 \) wheel with \( D_6 \) kinematics is expected to evaluate to functions beyond MPLs \([27]\)! In this letter we only consider the \( n = 8 \) wheel obtained from its soft limit:

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\text{soft limit} & 4 & \\
9 & 8 & 7 \\
6 & 5 & \\
\end{array}
\]

Direct integrations become difficult due to the presence of a new square root, for which it suffices to compute the (composite) leading singularity and we find \( \text{LS} = \frac{uvw}{\sqrt{\Delta D_3}} \) with \( \Delta D_3 := (u+v)^2 - 4uvw \). Similar to the four-mass box integral, this wheel integral must contain algebraic letters which are odd under \( \sqrt{\Delta D_3} \rightarrow -\sqrt{\Delta D_3} \).
we do not know how to obtain this square root from e.g. tropicalization of $G(4,8)$. It is natural to ask what algebraic letters can appear, and we show that they are essentially dictated by the $D_3$ cluster algebra!

Note that $\Delta_{D_3}$ naturally appears in the following quadratic equation, $Q(r):= r^2 - (u+v)r + uvw = 0$, where the two roots $z, \bar{z}$, satisfy $z + \bar{z} = u + v$ and $z\bar{z} = uvw$. Similar to the four-mass box kinematics, we find that in positive kinematics, not only do we have real roots, i.e. $\Delta_{D_3} > 0$, but again $z/\bar{z}$ and $(1 - z)/(1 - \bar{z})$ are positive since $Q(0) = uvw$ and $Q(1) = W_0$ are monomials of cluster variables! Now we can look for other rational $r_i$’s such that $(r_i - z)/(r_i - \bar{z})$ becomes such a monomial with definite sign. The solutions we find include $r = u, v$ and some more complicated ones denoted as $r_1 = w(1-u)/(1-uvw)$ and $r_2 = r_1|_{u\rightarrow v}$; remarkably the corresponding odd letters satisfy

$$\frac{u-z}{u-\bar{z}} = 1; \quad \frac{r_1-z}{r_1-\bar{z}} = \frac{z}{\bar{z}} = \frac{u-z}{u-\bar{z}} = 1$$

also for $r_2$ in the second relation with $u$ and $v$ swapped. We find exactly 3 multiplicatively independent algebraic letters (in accordance with $d = 3$), and we choose them to be $L_i := (r_i - z)/(r_i - \bar{z})$ for $i = 1, 2, 3$ with $r_3 = 0$.

Based on this alphabet, we are ready to bootstrap the symbol of $n = 8$ wheel. After imposing physical first entries being $u,v,w$, we construct integrable symbols with 9 rational letters and 3 algebraic ones: at weight 6 for $L = 3$, we find 4773 such integrable symbols. Next, note the integral is symmetric in $u,v$, and each term of its symbol must contain odd number of $L_i$’s; with these the number of integrable symbols is reduced to 737. Furthermore, we can impose two boundary conditions: with either $w \to 1$ or $W_0 \to 0$, $\Delta_{D_3}$ becomes a perfect square, and it is straightforward to evaluate the integral in these limits (both $A_2$ functions) \cite{27}. These boundary conditions reduce the number down to 278.

Finally, we impose the Steinmann relations \cite{5,13}, which turns out to be the most constraining: for each pair of incompatible channels, we ask the double discontinuity to vanish. Surprisingly, the constraint from one pair of channels, e.g. $\langle 1267 \rangle$ and $\langle 3481 \rangle$, already fixes all coefficients and we arrive at a unique symbol! In particular, it excludes all terms which contain algebraic letters more than once, and in fact $L_i$’s are constrained to appear only in the last entry:

$$S(I_{n=8}^{wheel}) = \sum_{i=1}^{3} S(F_i([W])) \otimes L_i$$

(4)

where $F_i$ for $i = 1, 2, 3$ are three weight-5 $D_3$ functions! We record the symbol for wheel integral, together with its two special limits in the ancillary file wheel.m.

Steinmann relations for any other pair of channels and all extended Steinmann relations (beyond the first two entries) become very strong cross checks of our result, and it indeed obeys all these constraints. Nicely, we find extended Steinmann relations between $W_i$’s are equivalent to cluster adjacency conditions of $D_3$ in terms of $A$-coordinates; i.e. any two mutable cluster variables cannot appear adjacent in the symbol if they do not show up in the same cluster (one of the 14 for $D_3$) \cite{12}.

III. FOLDING TO KINEMATICS IN THREE DIMENSIONS

In this section, we present a remarkable link between dimension reduction to $D = 3$ and folding of cluster algebras for kinematics. The dimension reduction to $D = 3$ can be achieved by setting Gram determinants to zero: for $x^\mu$ to lie in $D = 3$ ($X^I$ to lie in $D = 5$), any 6 $X$’s must have linearly dependent, which means the $6 \times 6$ matrix made of $X_i \cdot X_j$ must have vanishing determinant \cite{13}. Since $X_i \cdot X_j = x_i^a x_j^b \propto (i-1j-1)$, we ask that for any 6 dual points, $x_1, x_2, \ldots, x_6$, the matrix with entries $G_{a,b} := \langle t_a - x_1 t_b - x_6 \rangle$ (for $1 \leq a, b \leq 6$) must have det $G = 0$. For massless kinematics, exactly $n - 5$ conditions are independent, which thus gives $(2(n-5))$-dim (DCI) kinematics in $D = 3$ as expected.

The key observation is that, for positive kinematics, such Gram-determinant conditions become simple conditions on cluster variables which realize $D = 3$ kinematics as folded cluster algebras! Let us look at the simplest $n = 6$ massless kinematics, where the only Gram determinant is proportional to $\Delta = \sqrt{(1-u_1-u_2-u_3)^2-4u_1 u_2 u_3}$ (recall $u_1 = x_1^2 x_4^2 x_6^2$ and $u_2, u_3$ are obtained by cyclic rotations); it is known \cite{34} that with $\Delta = 0$, the $A_3$ cluster algebra is folded into a $C_2$: only the first 6 of the 9 letters $\{u_1, u_2, u_3\}$ survive since $y_i \to 1$ (recall e.g. $y_1 = 1 + u_1 u_2 u_3$). To see the folding explicitly, we choose a quiver as in Fig 1, where $f_1 = (1234/1256)$, $f_2 = (1246/1345)$ and $f_3 = (1456/2345)$. Very nicely, $\Delta = 0$ is equivalent to $f_1 = f_3$!

Similarly, for the $n = 7$ massless case there are two independent Gram-determinant conditions, which reduce the 42 letters of $E_6$ down to 28 letters of an $F_4$ cluster algebra! In terms of $u_1 = x_{1,2}^2 x_{3,4}^2 x_{5,6}^2$ and $u_4 = u_1 (Z_3 \to Z_{j+1})$, there are 28 parity-even letters that are rational functions, and the remaining parity-odd ones all become 1 when going to $D = 3$, resulting in

$$\{u_1, 1-u_1, 1-u_1 u_4, 1-u_1 u_4 + u_1 + u_4 \} = 1, \ldots, 7.$$ \hspace{1cm} (5)

It is highly non-trivial that these 28 letters (with con-
strains so only $d = 4$) form an $F_4$ cluster algebra, but it can be made manifest with the quiver in Fig. 1 the definition of $f_i$'s can be found in CA.m, and the two conditions become simply $f_2 = f_5$, $f_1 = f_6$. Despite the cluster algebras become infinite type, we have found exactly the same folding for $n = 8, 9$ with $D = 3$, where the quivers can be chosen such that $n - 5$ conditions are simply $f_i = f_{3(n-5) - i}$ for $i = 1, \cdots , n - 5$.

For general kinematics with $m \geq 6$, we expect $m - 5$ independent conditions and the dimension in $D = 3$ becomes $n + m - 10$. Again remarkably in all cases we have studied this corresponds to folding of cluster algebras. For example, the $D_4$ cluster algebras (with $d = 3, 4, 5, 6$) in $D = 3$ become $B_{d-1}$, and even the affine types $E_{6,1}$ and $D_{4,1}$ become $F_{4,1}$ and $B_{3,1}$ respectively!

We can apply these results to conformal integrals in $D = 3$ such as those naturally appeared for ABJM amplitudes. The only data we have found in the literature is the two-loop $n = 6$ amplitude as well as various finite DCI integrals contained in [44]. From these weight-2 functions, we find the 6 letters of $\text{DCI} \int$ contained in it [44]. From these weight-2 functions, we find the 6 letters of $\text{DCI} \int$ contained in it [44]. From these weight-2 functions, we find the 6 letters of $\text{DCI} \int$ contained in it [44]. From these weight-2 functions, we find the 6 letters of $\text{DCI} \int$ contained in it [44].

We conjecture that these 6 + 3 letters may be all we need for higher-loop integrals in ABJM. Although $n = 7$ ABJM amplitude vanishes, we can compute a large class of finite DCI two-loop integrals [45]: at weight 2 we find exactly the first 21 of the 28 letters in $F_4$ (we expect the remaining 7 at higher weights), and 7 odd letters $\chi_i$ for $i = 1, \cdots , 7$ as well! It is an interesting question if these 28 + 7 letters are sufficient for higher-loop integrals. There is evidence that starting $n = 8$, even two-loop ABJM amplitudes contain integrals that evaluate to elliptic functions [45], which are beyond the scope of this letter.

IV. APPLICATIONS FOR NON-CONFORMAL INTEGRALS

Before ending, we apply some of our results to non-conformal Feynman integrals, following the method of [44]. The key observation is that by sending a generic dual point (not light-like separated from any other points, or one that is between two massive corners) to infinity, conformal symmetry is broken and we obtain a non-DCI kinematics from such a DCI one [46].

As a warm-up exercise, by sending $x_6 \to \infty$ for the DCI kinematics (7, 5) B, one arrives at a non-DCI kinematics with three massless leg and a massive one (dual points $x_1, x_2, x_3, x_4$). Define $s = x_{1,3}^2$, $t = x_{2,4}^2$ and $m^2 = x_{1,4}^2$, we see that the 5 letters of $A_2$ cluster algebra is mapped to $z_1 := s/m^2$, $z_2 := t/m^2$, $1 - z_1$, $1 - z_2$ and $1 - z_1 - z_2$.

We have restricted the massive corner to lie between leg 3 and 1, but if we allow other positions, we have an additional letter $(s + t)/m^2 = z_1 + z_2$, which nicely recovers the $C_2$ alphabet for four-point Feynman integrals with a massive leg. This is the same $C_2$ for three-point form factor in $N = 4$ SYM through eight loops (although obtained as folding of $A_3$ in [44]).

Now we move to non-DCI limit of (8, 5)A kinematics: by sending the dual point $x_7$ to infinity, we get non-DCI two-mass-easy box kinematics (dual points $x_1, x_2, x_4, x_5$). In this case $s = x_{2,5}^2$, $t = x_{3,4}^2$, $m_1^2 = x_{2,4}^2$, $m_3^2 = x_{1,5}^2$.

![FIG. 2: (8, 5) A kinematics (left) and its non-DCI limit (right)](image)

and we find that the cross-ratios are mapped to $u = s/m_2^2$, $v = t/m_3^2$, $w = (m_2^2 m_3^2)/(st)$. Therefore, the 9 cluster variables of $D_3$ become 9 ratios of

\[ \{s, t, m_1^2, m_2^3, s-m_2^2, t-m_2^2, s-m_3^3, t-m_3^3, \sqrt{m_1^2 m_2^3} + \sqrt{m_1^2 m_3^3} \} \]

which exactly reproduces the alphabet for (the family of) one-loop two-mass-easy box integral [44] (note that we recover the $A_2$ alphabet above with $m_3^2 \to 0$).

It was known that these 9 letters are insufficient for two-loop integrals with two-mass-easy kinematics; in particular, three additional letters containing a new square root $\Delta_{nc}$ were found: in the notation of [47] they are $\{W_{35}, W_{36}, W_{39}\}$ for the two-mass-easy sector [48]. As we learn from our wheel example, for $D_3$ kinematics we need to supplement the 9 rational letters with three algebraic ones. Remarkably, we find that $\{L_1, L_2, L_3\}$ in the limit spans exactly the same space of $\{W_{35}, W_{36}, W_{39}\}$!

In particular, we find $\Delta_{D_3} \propto (s + t)^2 - 4m_1^2 m_2^2 = \Delta_{nc}$ under the map. This further supports the robustness of our generalized $D_3$ alphabet: in the non-DCI limit, the 9 + 3 letters of $D_3$ kinematics precisely coincide with those for two-mass-easy integrals at two loops!

Finally, the non-DCI kinematics with four massless legs and a massive one can be obtained by sending $x_7 \to \infty$ of our (8, 6)C kinematics [44], where $P_1^2 = m_3^2$ for

![FIG. 3: (8, 6) C kinematics (left) and its non-DCI limit (right)](image)

the massive corner. We have the relations $u = s_{12}/m_1^2$, $v = s_{34}/m_1^2$, $w = (m_2^2 s_{34})/(s_{12} s_{34})$, $u' = s_{23}/m_1^2$ and $v' = s_{45}/m_1^2$, and it is interesting to compare with the 49 relevant letters of [47] (see also [49]). First we note that
\[ \Delta_{D_3}, \Delta_{2,4,6,8} \text{ and the Gram determinant (proportional to } f_2 - f_7) \text{ corresponds to } \Delta_{nc}, W_4, \text{ and } W_9 \text{ respectively; for } \{W_i\}_{i=1,\ldots,49}, \text{ we find that all algebraic letters are reproduced and so do most of the rational letters (depending on precise truncation). However, some rational letters contain factors that are not cluster variables, e.g. } f_2 - f_6, f_6 - f_7, \text{ which in some sense are analog of the Gram determinant. We record the parametrization of } W_i=1,\ldots,49 \text{ using } D_{4,1} \text{ cluster variables in the file } CA.m. \]

V. CONCLUSIONS

We have proposed parametrizing planar kinematics of conformal integrals as sub-algebras of \( G(4, n) \) cluster algebras, and argued that cluster variables and their algebraic generalizations know about singularities of DCI Feynman integrals; in addition to evidence for various ladder integrals, we have demonstrated the power of cluster algebras and adjacency by bootstrapping the three-loop wheel from a generalized \( D_3 \) alphabet. We have also outlined applications to more general, non-DCI Feynman integrals, as well as to those in \( D = 3 \) via folding. There seems to be certain “universality” of cluster algebras in a wide range of integrals and physical quantities.

There are numerous questions raised by our investigations. An important mathematical question is if one could show, for any kinematics, there always exists one (and only one) sub-algebra, which would also make our “freezing” algorithm more systematic. We have seen that DCI Feynman integrals, even when evaluated to MPLs, can have more complicated (e.g. algebraic) singularities than naively expected from the alphabet of cluster algebras, but they are still strongly constrained by the latter, as illustrated by the \( D_3 \) wheel example. It would be highly desirable to see how universal is this phenomenon and understand its origin. Relatedly, it is tempting to ask if singularities of Feynman integrals beyond MPLs (such as the \( D_3 \) \( n = 9 \) wheel [27], or \( n = 10 \) double box integral [50, 51]) may also be constrained by corresponding cluster algebras? So far it seem totally miraculous that cluster algebras (and cluster adjacency) can be applied to form factors, amplitudes/integrals in \( D = 3 \), and non-DCI Feynman integrals etc.. Is it possible to have more systematic understanding, perhaps through the connection of cluster algebras with canonical differential equations satisfied by Feynman integrals of a given kinematics [22]?

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In this appendix we provide some details in finding the sub-quivers for planar kinematics as well as their alphabet. Let us first list possible 6-point and 7-point kinematics except for the fully massless ones in FIG. 4. As discussed in section I and shown in Fig. 5, we find an $A_1$ sub-algebra for $(6,5)$: there are 2 unfrozen variables $\langle 1346 \rangle$, $\langle 1245 \rangle$ and 4 frozen ones $\langle 1234 \rangle$, $\langle 1456 \rangle$, $\langle 1345 \rangle$ and $\langle 1246 \rangle$.

Moving to 7-point kinematics, as we have discussed $(7,6)$ kinematics corresponds to a $D_4$ sub-algebra and $(7,5)$ $A$ corresponds to an $A_2$: here the hexagon is independent of $(7123),(2345)$, and the pentagon further independent of $(3456)$ and $(5671)$. In Fig. 6 we show how to freeze two (four) nodes of $E_6$ to arrive at these sub-algebras. We have $16 + 7 A$-coordinates for the $D_4$ algebra: the unfrozen variables are

$$\{ (1(27)(34)(56)), (4(12)(35)(67)), (5(12)(34)(67)), (6(12)(34)(57)), (7(12)(34)(56)), (1245), (1246), (1256), (1257), (1346), (1347), (1456), (1457), (1467), (3457), (3467) \},$$

and the frozen ones are

$$\{ (1234), (1247), (1267), (1345), (1567), (3456), (4567) \}.$$ 

Among these, alphabet of $(7,5)$ $A$ ($A_2$ algebra) contains

$$\{ (4(12)(35)(67)), (1245), (1347), (1457), (3467) \},$$

as unfrozen variables and 6 frozen variables which are

$$\{ (1234), (1247), (1345), (1467), (3457), (4567) \}.$$
Explicit computation shows that its alphabet contains 5 + 6 $\mathcal{A}$-coordinates: 5 unfrozen variables

\[ \{ (1(27)(34)(56)), (3(17)(24)(56)), (1256), (1347), (2356) \} \]

and 6 frozen ones

\[ \{ (2(17)(34)(56)), (1234), (1567), (1237), (1356), (3456) \}, \]

which form 5 DCI combinations \( \{ u, v, 1 - u, 1 - v, 1 - u - v \} \) with \( u = \frac{x_2^6 + x_3^6 - 1}{x_5^6, x_7^6 - 1}, v = \frac{x_2^2 + x_3^2 - 1}{x_4^2 + x_8^2 - 1} \). One sees that at \( x_6 \to \infty, \frac{x_i^6}{x_j^6} \to 1 \) for arbitrary \( i \) and \( j \). Therefore \( u \to \frac{x_2^2}{x_4^2} := z_2, v \to \frac{x_3^2}{x_4^2} := z_1 \) and we finally arrive at the five non-DCI letters in \( z \) (see section 3).

For \( (8, 5)A \) case, the \( D_3 \) sub-algebra has 9 unfrozen \( \mathcal{A} \)-coordinates

\[ \{ (1(28)(34)(67)), (1(28)(45)(67)), (4(18)(35)(67)), (4(12)(35)(67)), (1267), (3467), (1348), (1458), (1245) \}, \]

and 7 frozen ones

\[ \{ (812(67) \cap 345), (6781), (1467), (4567), (8124), (1234), (1345) \} \].

We list all the sub-algebras and quivers for 8-point kinematics in the following table. We locate their initial quivers as the red parts in the \( G(4, 8) \) quiver on the third column. Note that we choose the same initial quiver for \( (8, 6) \) A and \( (8, 5) \) B, and the same one for \( (8, 6) \) C and \( (8, 5) \) A, to emphasis the inclusion relations \( A_{2,1} \subset D_{4,1}(A) \) and \( D_3 \subset D_{4,1}(C) \).

In the ancillary file \( \text{C.A.m} \), we present definitions of all \( \mathcal{X} \)-coordinates \( f_i \) for the \( n = 8 \) quivers in the table and \( n = 6, 7 \) in Fig.[1] in terms of Plücker coordinates. For each quiver, we also give the corresponding parametrization of momentum twistors \( \mathbf{Z} \) using \( f_i \)'s. Moreover, we also record alphabets for all these cases in terms of \( f_i \), including folding to \( D = 3 \) for \( n = 6, 7 \). For those algebras of infinite type with \( n = 8 \), their alphabets contain both rational and algebraic functions of \( f_i \), and we select rational letters from the 356 letters of \( G(4, 8) \).

In the end of the file, we also provide parametrization of \( \{ W_i \}_{i=1...49} \) from [47] in terms of \( f_i \) of \( (8, 6) \) C.
| (n,m) | kinematics | initial quiver | CA         | folding |
|-------|------------|----------------|------------|---------|
| (8,8) | ![Diag](image1.png) | ![Diag](image2.png) | $E_7^{(1,1)}$ | name unknown |
| (8,7) | ![Diag](image3.png) | ![Diag](image4.png) | $E_{6,1}$  | $F_{4,1}$ |
| (8,6)A | ![Diag](image5.png) | ![Diag](image6.png) | $D_{4,1}$  | $B_{3,1}$ |
| (8,6)B | ![Diag](image7.png) | ![Diag](image8.png) | $D_{5}$    | $B_{4}$  |
| (8,6)C | ![Diag](image9.png) | ![Diag](image10.png) | $D_{4,1}$  | $B_{3,1}$ |
| (8,5)A | ![Diag](image11.png) | ![Diag](image12.png) | $D_{3}$    |         |
| (8,5)B | ![Diag](image13.png) | ![Diag](image14.png) | $A_{2,1}$  |         |

**TABLE I:** Quivers and cluster algebras for 8-point kinematics