Introducing fully UP-semigroups*

Aiyared Iampan†

Department of Mathematics, School of Science
University of Phayao, Phayao 56000, Thailand

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Abstract

In this paper, we introduce some new classes of algebras related to UP-algebras and semigroups, called a left UP-semigroup, a right UP-semigroup, a fully UP-semigroup, a left-left UP-semigroup, a right-left UP-semigroup, a left-right UP-semigroup, a right-right UP-semigroup, a fully-left UP-semigroup, a fully-right UP-semigroup, and find their examples.

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1 Introduction and Preliminaries

In the literature, several researches introduced a new class of algebras related to logical algebras and semigroups such as: In 1993, Jun, Hong and Roh [5] introduced the notion of BCI-semigroups. In 1998, Jun, Xin and Roh [6] renamed the BCI-semigroup as the IS-algebra. In 2006, Kim [7] introduced the notion of KS-semigroups. In 2011, Ahn and Kim [1] introduced the notion of BE-semigroups. In 2015, Endam and Vilela [2] introduced the notion of JB-semigroups. In 2016, Sultana and Chaudhary [8] introduced the notion of BCH-semigroups. In this paper, we introduce some new classes of algebras related to UP-algebras and semigroups, called a left UP-semigroup, a right UP-semigroup, a fully UP-semigroup, a left-left UP-semigroup, a right-left UP-semigroup, a left-right UP-semigroup, a right-right UP-semigroup, a fully-left UP-semigroup, a fully-right UP-semigroup, and find their examples.

Before we begin our study, we will introduce the definition of a UP-algebra.

Definition 1.1. [3] An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra, where $A$ is a nonempty set, $\cdot$ is a binary operation on $A$, and 0 is a fixed element of $A$ (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in A$,

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†Corresponding author. Email: aiyared.ia@up.ac.th
(UP-1) \((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0\),
(UP-2) \(0 \cdot x = x\),
(UP-3) \(x \cdot 0 = 0\), and
(UP-4) \(x \cdot y = y \cdot x = 0\) implies \(x = y\).

In a UP-algebra \(A = (A, \cdot, 0)\), the following assertions are valid (see [3][4]).

\[
\begin{align*}
(\forall x \in A)(x \cdot x = 0), & \quad (1.1) \\
(\forall x, y, z \in A)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0), & \quad (1.2) \\
(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0), & \quad (1.3) \\
(\forall x, y, z \in A)(x \cdot y \cdot z = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0), & \quad (1.4) \\
(\forall x, y \in A)(x \cdot (y \cdot x) = 0), & \quad (1.5) \\
(\forall x, y \in A)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x), & \quad (1.6) \\
(\forall x, y \in A)(x \cdot (y \cdot y) = 0), & \quad (1.7) \\
(\forall a, x, y, z \in A)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0), & \quad (1.8) \\
(\forall a, x, y, z \in A)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0), & \quad (1.9) \\
(\forall x, y, z \in A)((x \cdot y) \cdot z \cdot (y \cdot z) = 0), & \quad (1.10) \\
(\forall x, y \in A)((x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0), & \quad (1.11) \\
(\forall x, y, z \in A)((x \cdot (y \cdot z)) \cdot (x \cdot (y \cdot z)) = 0), & \quad (1.12) \\
(\forall a, x, y, z \in A)((x \cdot (y \cdot z)) \cdot (y \cdot (a \cdot z)) = 0). & \quad (1.13)
\end{align*}
\]

Let \(X\) be a universal set. Define two binary operations \(\cdot\) and \(*\) on the power set of \(X\) by putting, for all \(A, B \in \mathcal{P}(X)\),

\[
\begin{align*}
A \cdot B & = A' \cap B; & \quad (1.14) \\
A * B & = A' \cup B. & \quad (1.15)
\end{align*}
\]

Then \((\mathcal{P}(X), \cdot, \emptyset)\) is a UP-algebra and we shall call it the \textit{power UP-algebra of type 1} [3], and \((\mathcal{P}(X), *, X)\) is a UP-algebra and we shall call it the \textit{power UP-algebra of type 2} [3].

Now, define four binary operations \(\odot, \otimes, \Box\) and \(\nabla\) on the power set of \(X\) by putting, for all \(A, B \in \mathcal{P}(X)\),

\[
\begin{align*}
A \odot B & = X, & \quad (1.16) \\
A \otimes B & = \emptyset, & \quad (1.17) \\
A \Box B & = B, & \quad (1.18) \\
A \nabla B & = A. & \quad (1.19)
\end{align*}
\]

Then \((\mathcal{P}(X), \odot), (\mathcal{P}(X), \otimes), (\mathcal{P}(X), \Box)\) and \((\mathcal{P}(X), \nabla)\) are semigroups. Furthermore, we know that \((\mathcal{P}(X), \cap, X)\) and \((\mathcal{P}(X), \cup, \emptyset)\) are monoids.

**Definition 1.2.** Let \(A\) be a nonempty set, \(\cdot\) and \(*\) are binary operations on \(A\), and \(0\) is a fixed element of \(A\) (i.e., a nullary operation). An algebra \(A = (A, \cdot, *, 0)\) of type \((2, 2, 0)\) in which \((A, \cdot, 0)\) is a UP-algebra and \((A, *, 0)\) is a semigroup is called

1. a \textit{left UP-semigroup} (in short, an \textit{l-UP-semigroup}) if the operation \(*\) is left distributive over the operation \(\cdot\),
2. a \textit{right UP-semigroup} (in short, an \textit{r-UP-semigroup}) if the operation \(*\) is right distributive over the operation \(\cdot\),
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(3) a fully UP-semigroup (in short, an f-UP-semigroup) if the operation “∗” is distributive (on both sides) over the operation “·”,

(4) a left-left UP-semigroup (in short, an (l,l)-UP-semigroup) if the operation “·” is left distributive over the operation “∗” and the operation “∗” is left distributive over the operation “·”,

(5) a right-left UP-semigroup (in short, an (r,l)-UP-semigroup) if the operation “·” is right distributive over the operation “∗” and the operation “∗” is left distributive over the operation “·”,

(6) a left-right UP-semigroup (in short, an (l,r)-UP-semigroup) if the operation “·” is left distributive over the operation “∗” and the operation “∗” is right distributive over the operation “·”,

(7) a right-right UP-semigroup (in short, an (r,r)-UP-semigroup) if the operation “·” is right distributive over the operation “∗” and the operation “∗” is right distributive over the operation “·”,

(8) a fully-left UP-semigroup (in short, an (f,l)-UP-semigroup) if the operation “·” is distributive (on both sides) over the operation “∗” and the operation “∗” is left distributive over the operation “·”,

(9) a fully-right UP-semigroup (in short, an (f,r)-UP-semigroup) if the operation “·” is distributive (on both sides) over the operation “∗” and the operation “∗” is right distributive over the operation “·”,

(10) a left-fully UP-semigroup (in short, an (l,f)-UP-semigroup) if the operation “·” is left distributive over the operation “∗” and the operation “∗” is distributive (on both sides) over the operation “·”,

(11) a right-fully UP-semigroup (in short, an (r,f)-UP-semigroup) if the operation “·” is right distributive over the operation “∗” and the operation “∗” is distributive (on both sides) over the operation “·”, and

(12) a fully-fully UP-semigroup (in short, an (f,f)-UP-semigroup) if the operation “·” is distributive (on both sides) over the operation “∗” and the operation “∗” is distributive (on both sides) over the operation “·”.

In what follows, let A and B denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

The proof of Propositions 1.3, 1.4, 1.5, 1.6, 1.7, and 1.8 can be verified by a routine proof.

Proposition 1.3. (The operations of a UP-algebra \(\mathcal{P}(X)\) is left distributive over the operations of a semigroup \(\mathcal{P}(X)\)) Let X be a universal set. Then the following properties hold: for any \(A, B, C \in \mathcal{P}(X)\),

1. \(A \cdot (B \cap C) = (A \cdot B) \cap (A \cdot C)\),
2. \(A \cdot (B \cup C) = (A \cdot B) \cup (A \cdot C)\),
3. \(A \ast (B \cap C) = (A \ast B) \cap (A \ast C)\),
4. \(A \ast (B \cup C) = (A \ast B) \cup (A \ast C)\),
5. \(A \cdot (B \otimes C) = (A \cdot B) \otimes (A \cdot C)\),
\( (6) \quad A \ast (B \odot C) = (A \ast B) \odot (A \ast C), \)
\( (7) \quad A \cdot (B \boxdot C) = (A \cdot B) \boxdot (A \cdot C), \)
\( (8) \quad A \ast (B \boxtimes C) = (A \ast B) \boxtimes (A \ast C), \)
\( (9) \quad A \cdot (B \boxtimes C) = (A \cdot B) \boxtimes (A \cdot C), \) and
\( (10) \quad A \ast (B \boxtimes C) = (A \ast B) \boxtimes (A \ast C). \)

**Proposition 1.4.** (The operations of a UP-algebra \( \mathcal{P}(X) \) is right distributive over the operations of a semigroup \( \mathcal{P}(X) \)) Let \( X \) be a universal set. Then the following properties hold: for any \( A, B, C \in \mathcal{P}(X), \)
\( (1) \quad (A \boxdot B) \cdot C = (A \cdot C) \boxdot (B \cdot C), \)
\( (2) \quad (A \boxdot B) \ast C = (A \ast C) \boxdot (B \ast C), \)
\( (3) \quad (A \boxtimes B) \cdot C = (A \cdot C) \boxtimes (B \cdot C), \) and
\( (4) \quad (A \boxtimes B) \ast C = (A \ast C) \boxtimes (B \ast C). \)

**Proposition 1.5.** (The operations of a semigroup \( \mathcal{P}(X) \) is left distributive over the operations of a UP-algebra \( \mathcal{P}(X) \)) Let \( X \) be a universal set. Then the following properties hold: for any \( A, B, C \in \mathcal{P}(X), \)
\( (1) \quad A \odot (B \ast C) = (A \odot B) \ast (A \odot C), \)
\( (2) \quad A \odot (B \cdot C) = (A \odot B) \cdot (A \odot C), \)
\( (3) \quad A \boxdot (B \cdot C) = (A \boxdot B) \cdot (A \boxdot C), \) and
\( (4) \quad A \boxdot (B \ast C) = (A \boxdot B) \ast (A \boxdot C). \)

**Proposition 1.6.** (The operations of a semigroup \( \mathcal{P}(X) \) is right distributive over the operations of a UP-algebra \( \mathcal{P}(X) \)) Let \( X \) be a universal set. Then the following properties hold: for any \( A, B, C \in \mathcal{P}(X), \)
\( (1) \quad (A \ast B) \odot C = (A \odot C) \ast (B \odot C), \)
\( (2) \quad (A \cdot B) \odot C = (A \odot C) \cdot (B \odot C), \)
\( (3) \quad (A \cdot B) \boxtimes C = (A \boxtimes C) \cdot (B \boxtimes C), \) and
\( (4) \quad (A \ast B) \boxtimes C = (A \boxtimes C) \ast (B \boxtimes C). \)

**Proposition 1.7.** Let \( X \) be a universal set. Then the following properties hold: for any \( A, B, C \in \mathcal{P}(X), \)
\( (1) \quad (A \cap B) \cdot C = (A \cdot C) \cap (B \cdot C), \)
\( (2) \quad (A \cup B) \cdot C = (A \cdot C) \cup (B \cdot C), \)
\( (3) \quad (A \cap B) \ast C = (A \ast C) \cup (B \ast C), \)
\( (4) \quad (A \cup B) \ast C = (A \ast C) \cap (B \ast C), \)
\( (5) \quad (A \odot B) \cdot C = (A \cdot C) \odot (B \cdot C), \) and
\( (6) \quad (A \odot B) \ast C = (A \ast C) \odot (B \ast C). \)
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Proposition 1.8. Let $X$ be a universal set. Then the following properties hold: for any $A, B, C \in \mathcal{P}(X)$,

1. $(A \cdot B) \odot C = (A \odot C) \ast (B \odot C)$, and
2. $(A \ast B) \odot C = (A \odot C) \cdot (B \odot C)$.

Proposition 1.9. Let $A = (A, \cdot, \ast, 0)$ be an algebra of type $(2, 2, 0)$ in which $(A, \cdot, 0)$ is a UP-algebra and $(A, \ast)$ is a semigroup. Then the following properties hold:

1. if $A$ is an $l$-UP-semigroup, then $x \ast 0 = 0$ for all $x \in A$,
2. if $A$ is an $r$-UP-semigroup, then $0 \ast x = 0$ for all $x \in A$,
3. if the operation “$\cdot$” is right distributive over the operation “$\ast$”, then $x \ast x = x$ for all $x \in A$, and
4. $A = \{0\}$ is one and only one $(r, f)$-UP-semigroup and $(f, f)$-UP-semigroup.

Proof. (1) Assume that $A$ is an $l$-UP-semigroup. Then, by (1.1), we have

$$x \ast 0 = x \ast (0 \cdot 0) = (x \ast 0) \cdot (x \ast 0) = 0$$

for all $x \in A$.

(2) Assume that $A$ is an $r$-UP-semigroup. Then, by (1.1), we have

$$0 \ast x = (0 \cdot 0) \ast x = (0 \ast x) \cdot (0 \ast x) = 0$$

for all $x \in A$.

(3) Assume that the operation “$\cdot$” is right distributive over the operation “$\ast$”. Then, by (UP-3), we have

$$0 = (0 \ast 0) \cdot 0 = (0 \cdot 0) \ast (0 \cdot 0) = 0 \ast 0.$$

Thus, by (UP-2), we have

$$x = 0 \cdot x = (0 \ast 0) \cdot x = (0 \cdot x) \ast (0 \cdot x) = x \ast x$$

for all $x \in A$.

(4) By (UP-2), (1.1), (1) and (2), we have

$$x = 0 \cdot x = (x \ast 0) \cdot x = (x \cdot x) \ast (0 \cdot x) = 0 \ast x = 0$$

for all $x \in A$.

Hence, $A = \{0\}$ is one and only one $(r, f)$-UP-semigroup and $(f, f)$-UP-semigroup.

Example 1.10. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

|   | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 0 | 0 | 2 | 3 |
| 2 | 0 | 0 | 0 | 1 |
| 3 | 0 | 1 | 2 | 0 |

Then $(A, \cdot, \ast, 0)$ is an $f$-UP-semigroup.
Let $X$ be a universal set. Then, by above propositions and an example, we get:

| Types of algebras | Examples |
|-------------------|----------|
| $l$-UP-semigroup | $(P(X), \ast, \odot, X)$ (see Proposition 1.5 (1)) |
|                   | $(P(X), \cdot, \odot, \emptyset)$ (see Proposition 1.5 (2)) |
|                   | $(P(X), \cdot, \boxplus, \emptyset)$ (see Proposition 1.5 (3)) |
|                   | $(P(X), \ast, \boxplus, X)$ (see Proposition 1.5 (4)) |
| $r$-UP-semigroup | $(P(X), \ast, \odot, X)$ (see Proposition 1.6 (1)) |
|                   | $(P(X), \cdot, \odot, \emptyset)$ (see Proposition 1.6 (2)) |
|                   | $(P(X), \cdot, \boxminus, \emptyset)$ (see Proposition 1.6 (3)) |
|                   | $(P(X), \ast, \boxminus, X)$ (see Proposition 1.6 (4)) |
| $f$-UP-semigroup | $(P(X), \ast, \odot, X)$ (see Propositions 1.5 (1) and 1.6 (1)) |
|                   | $(P(X), \cdot, \odot, \emptyset)$ (see Propositions 1.5 (2) and 1.6 (2)) |
|                   | $(A, \cdot, \ast, 0)$ (see Example 1.10) |
| $(l, l)$-UP-semigroup | $(P(X), \cdot, \boxplus, \emptyset)$ (see Propositions 1.5 (3) and 1.3 (7)) |
| $(r, l)$-UP-semigroup | $(P(X), \cdot, \boxplus, \emptyset)$ (see Propositions 1.5 (3) and 1.4 (1)) |
| $(l, r)$-UP-semigroup | $(P(X), \ast, \odot, X)$ (see Propositions 1.6 (1) and 1.3 (6)) |
| $(r, r)$-UP-semigroup | $(P(X), \cdot, \boxminus, \emptyset)$ (see Propositions 1.6 (3) and 1.3 (9)) |
| $(r, f)$-UP-semigroup | $(P(X), \cdot, \boxminus, \emptyset)$ (see Propositions 1.6 (4) and 1.3 (10)) |
| $(f, l)$-UP-semigroup | $(P(X), \cdot, \boxplus, \emptyset)$ (see Propositions 1.5 (3), 1.3 (7), and 1.6 (1)) |
| $(f, r)$-UP-semigroup | $(P(X), \cdot, \boxplus, \emptyset)$ (see Propositions 1.5 (3), 1.3 (5), and 1.6 (2)) |
| $(l, f)$-UP-semigroup | $(P(X), \ast, \odot, X)$ (see Propositions 1.5 (1), 1.3 (6), and 1.6 (1)) |
| $(r, f)$-UP-semigroup | $(P(X), \ast, \odot, X)$ (see Propositions 1.5 (1), 1.3 (10), and 1.4 (4)) |
| $(f, f)$-UP-semigroup | $\{0\}$ is one and only one $(r, f)$-UP-semigroup |
| $(f, f)$-UP-semigroup | $\{0\}$ is one and only one $(f, f)$-UP-semigroup |
Hence, we have the following diagram:

![Diagram showing relationships between different types of UP-semigroups](image)

**Figure 1:** New algebras of type (2,2,0)

**Conclusion**

We have introduced the notions of left UP-semigroups, right UP-semigroups, fully UP-semigroups, left-left UP-semigroups, right-left UP-semigroups, left-right UP-semigroups, right-right UP-semigroups, fully-left UP-semigroups, fully-right UP-semigroups, left-fully UP-semigroups, right-fully UP-semigroups and fully-fully UP-semigroups, and have found examples. We have that right-fully UP-semigroups and fully-fully UP-semigroups coincide, and it is only \( \{0\} \). In further study, we will apply the notion of fuzzy sets and fuzzy soft sets to the theory of all above notions.

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