Abstract

From a group $H$ and a non-trivial element $h$ of $H$, we define a representation $\rho : B_n \to \text{Aut}(G)$, where $B_n$ denotes the braid group on $n$ strands, and $G$ denotes the free product of $n$ copies of $H$. Such a representation shall be called the Artin type representation associated to the pair $(H,h)$. The goal of the present paper is to study different aspects of these representations.

Firstly, we associate to each braid $\beta$ a group $\Gamma_{(H,h)}(\beta)$ and prove that the operator $\Gamma_{(H,h)}$ determines a group invariant of oriented links. We then give a topological construction of the Artin type representations and of the link invariant $\Gamma_{(H,h)}$, and we prove that the Artin type representations are faithful. The last part of the paper is dedicated to the study of some semidirect products $G \rtimes \rho B_n$, where $\rho : B_n \to \text{Aut}(G)$ is an Artin type representation. In particular, we show that $G \rtimes \rho B_n$ is a Garside group if $H$ is a Garside group and $h$ is a Garside element of $H$.

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1 Introduction

Throughout the paper, we shall denote by $B_n$ the braid group on $n$ strands, and by $\sigma_1, \ldots, \sigma_{n-1}$ the standard generators of $B_n$.

Let $H$ be a group, and let $h$ be a non-trivial element of $H$. Take $n$ copies $H_1, \ldots, H_n$ of $H$, and consider the group $G = H_1 \ast \cdots \ast H_n$. We denote by $\phi_i : H \to H_i$ the natural isomorphism and we write $h_i = \phi_i(h) \in H_i$, for all $i = 1, \ldots, n$. For $k = 1, \ldots, n-1$, let $\tau_k : G \to G$ be the automorphism determined by

$$
\tau_k : \left\{\begin{array}{ll}
\phi_k(y) & \mapsto h_k^{-1} \phi_{k+1}(y) h_k \\
\phi_{k+1}(y) & \mapsto h_k \phi_k(y) h_k^{-1} \\
\phi_j(y) & \mapsto \phi_j(y) \quad \text{if} \quad j \neq k, k+1
\end{array}\right.
$$

for $y \in H$. One can easily show the following.

**Proposition 1.1.** The mapping $\sigma_k \mapsto \tau_k$, $k = 1, \ldots, n-1$, determines a representation $\rho : B_n \to \text{Aut}(G)$. \hfill \qed

**Definition 1.2.** The representation of Proposition 1.1 shall be called the Artin type representation of $B_n$ associated to the pair $(H,h)$. 

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If \( H = \mathbb{Z} \) and \( h = 1 \), then \( G = F_n \) is the free group of rank \( n \) and \( \rho \) is the classical representation introduced by Artin in [Art1], [Art2]. Another example which appears in the literature is the case where \( H = \mathbb{Z} \) and \( h \) is a non-zero integer. This case has been introduced by Wada [Wad] in his construction of group invariants of links. Sections 2 and 3 of the present paper are inspired by Wada’s work [Wad].

Our purpose in this paper is to study different aspects of the Artin type representations.

**Definition 1.3.** Let \( \rho : B_n \to \text{Aut}(G) \) be the Artin type representation associated to a pair \((H, h)\). Let \( \beta \in B_n \). Then we denote by \( \Gamma(\beta) = \Gamma_{(H, h)}(\beta) \) the quotient of \( G \) by the relations
\[
g = \rho(\beta)g, \quad g \in G.
\]

For a braid \( \beta \), we denote by \( \widetilde{\beta} \) the oriented link (or more precisely the equivalence class of oriented links) represented by the closed braid of \( \beta \) as defined in [Birm]. Given two braids \( \beta_1 \) and \( \beta_2 \) (not necessarily with the same number of strands), we prove in Section 2 that \( \Gamma(\beta_1) \cong \Gamma(\beta_2) \) if \( \widetilde{\beta}_1 = \widetilde{\beta}_2 \). This allows us to define a group invariant of oriented links, \( \Gamma_{(H, h)} \), by setting \( \Gamma_{(H, h)}(L) \) to be the group \( \Gamma_{(H, h)}(\beta) \) for any braid \( \beta \) such that \( L = \widetilde{\beta} \). Note that, in the case \( H = \mathbb{Z} \) and \( h = 1 \), the invariant \( \Gamma_{(\mathbb{Z}, 1)} \) computes the link group, namely \( \Gamma_{(\mathbb{Z}, 1)}(L) \cong \pi_1(S^3 \setminus L) \) for any link \( L \) in \( S^3 \).

The goal of Section 3 is to give topological constructions of the Artin type representations and of the groups \( \Gamma_{(H, h)}(\beta) \), for \( \beta \in B_n \). If \( H = \mathbb{Z} \) and \( h \) is a non-zero integer, then our constructions coincide with Wada’s constructions (see [Wad], Section 3). In fact, our constructions are straightforward extensions of Wada’s constructions to all Artin type representations.

In Section 4, we prove that Artin type representations are faithful (Proposition 4.1). If \( h \) has infinite order, then the Artin type representation \( \rho : B_n \to \text{Aut}(G) \) contains the classical Artin representation and, therefore, is faithful by [Art1], [Art2]. So, Proposition 4.1 is mostly of interest in the case where \( h \) has finite order. In fact the proof may be easily reduced to the case \( H = \mathbb{Z}/k\mathbb{Z} \) and \( h = 1 \), however we will not need to use any such reduction, as our method applies just as easily in all cases. We note also that the case where \( H \) is cyclic of order 2 follows (by somewhat different methods) from Section 2.3 of [CF]. The proof of Proposition 4.1 is inspired by the proof of Theorem A of [Shp], and it is based on Dehornoy’s work on orders on braids [Deh1], [Deh2].

The remaining sections (Sections 5 and 6) are dedicated to the study of semidirect products \( G \rtimes \rho B_n \), where \( \rho : B_n \to \text{Aut}(G) \) is the Artin type representation associated to a pair \((H, h)\).

If \( H = \mathbb{Z} \) and \( h = 1 \), then \( G \rtimes \rho B_n \) is the Artin group \( A(B_n) \) associated to the Coxeter graph \( B_n \) (not to be confused with the braid group \( B_n \), which is itself an Artin group, of type \( A_{n-1} \)). This result is implicit in [Lam], [CF], and explicit in [CP]. The group \( A(B_n) \) is well-understood. In particular, solutions to the word and conjugacy problems in this group are known (see [Del], [BS]), it is torsion free (see [Bri], [Del]), its center is an infinite cyclic group (see [Del], [BS]), it is biautomatic (see [Chal], [Cha2]), and it has an explicit finite dimensional classifying space (see [De], [Bes]).

A natural next step is to understand the groups \( G \rtimes \rho B_n \) in the case where \( \rho \) is a Wada representation (of type 4), namely, when \( H = \mathbb{Z} \) and \( h \in \mathbb{Z} \setminus \{0\} \). One can readily establish that, for these representations, the group \( G \rtimes \rho B_n \) fails to be an Artin group unless \( h = \pm 1 \). It turns out, however, that these groups do have quite a lot in common with Artin groups: like the Artin groups, they belong to a family of groups known as *Garside groups*. 


Briefly, a Garside group is a group $G$ which admits a left invariant lattice order and contains a so-called Garside element, a positive element $\Delta$ whose positive divisors generate $G$ and such that conjugation by $\Delta$ leaves the lattice structure invariant (there are also conditions placed on the positive cone of $G$, that it be a finitely generated atomic monoid – see Section 5 for more details). The notion of a Garside group was introduced by Dehornoy and the second author [DP] in a slightly restricted sense, and, later, by Dehornoy [Deh5] in the larger sense which is now generally used. The theory of Garside groups is largely inspired by the papers of Garside [Gar], which treated the case of braid groups, and Brieskorn and Saito [BS] which generalised Garside’s work to Artin groups. The Artin groups of spherical (or finite) type which include, notably, the braid groups as well as the groups $A(B_n)$ mentioned above, are motivating examples. Other interesting examples of Garside groups include all torus link groups (see [Pic3]) and some generalized braid groups associated to finite complex reflection groups.

Garside groups have many attractive properties. Solutions to the word and conjugacy problems in these groups are known (see [Deh5], [Pic1], [FG]), they are torsion free (see [Deh4]), they admit canonical decompositions as iterated crossed products of “irreducible” components, and the center of each component is an infinite cyclic group (see [Pic2]), they are biautomatic (see [Deh5]), and they admit finite dimensional classifying spaces (see [DL], [CMW]). Another important property of the Garside groups is that there exist criteria in terms of presentations to detect them (see [DP], [Deh5]).

In Section 6, we prove that, if $H$ is a Garside group, $h$ a Garside element of $H$, and $\rho$ the Artin type representation associated to $(H, h)$, then $G \rtimes_\rho B_n$ is also a Garside group (Theorem 6.1). This result applies in particular to the case $H = \mathbb{Z}$ and $h \in \mathbb{Z} \setminus \{0\}$, but also applies, for example, to the case where $H$ is another braid group, say $H = B_l$, and $h = \Delta^k$ is a non-trivial power of the fundamental element of $B_l$.

The proof of Theorem 6.1 is based on a criterion which is developed in Section 5 for proving that a given group is a Garside group. This criterion rests largely on the “coherence” condition of [DP] which has its roots in the original arguments of Garside [Gar]. It is essentially a variation of other criteria appearing in the literature (see, for example, [Deh5] Prop. 6.14). Our criterion differs from that of Dehornoy [Deh5] just mentioned in that it is not algorithmic. In particular, we do not give any method for finding a Garside element. However, it is relatively easy to apply once one has an appropriate presentation and an expression for a Garside element to hand.

Finally, we add an appendix to our paper in order to answer a question posed by Shpilrain in his study of Wada’s representations [Shp], and which is otherwise a little tangential to the main subject of this paper.

**Definition 1.4.** Let $G$ be a group. Two representations $\rho, \rho' : B_n \to \text{Aut}(G)$ are called equivalent if there exist automorphisms $\phi : G \to G$ and $\mu : B_n \to B_n$ such that

$$\rho'(\mu(\beta)) = \phi^{-1} \circ \rho(\beta) \circ \phi$$

for all $\beta \in B_n$.

**Remark.** If two representations $\rho, \rho' : B_n \to \text{Aut}(G)$ are equivalent, then the groups $G \rtimes_\rho B_n$ and $G \rtimes_{\rho'} B_n$ are isomorphic.

Shpilrain’s question (see [Shp]) was simply to give a classification of Wada’s representations up to equivalence. This classification is given in Proposition A.1.
2 Link invariants

Let $H$ be a group, $h$ a non-trivial element of $H$, and $\rho : B_n \to \text{Aut}(G)$ be the Artin type representation associated to $(H,h)$. Recall that the group $G$ is defined as $G = H_1 \ast \cdots \ast H_n$, where group isomorphisms $\phi_i : H_i \to H$ are given for $i = 1,2,\ldots,n$. The goal of this section is to prove the following.

**Proposition 2.1.** Let $n,m \in \mathbb{N}$, and let $\beta_1 \in B_n$ and $\beta_2 \in B_m$. If $\beta_1 = \beta_2$, then $\Gamma_{(H,h)}(\beta_1) \simeq \Gamma_{(H,h)}(\beta_2)$.

**Definition 2.2.** Let $L$ be an oriented link. We set $\Gamma_{(H,h)}(L) := \Gamma_{(H,h)}(\beta)$, where $\beta$ is any braid (on any number of strings) such that $L = \beta$. By Proposition 2.1, $\Gamma_{(H,h)}$ is a well-defined group invariant of oriented links.

**Proof of Proposition 2.1.** Let $n \in \mathbb{N}$ and let $\beta \in B_n$. We write $\Gamma$ for $\Gamma_{(H,h)}$. By Markov’s theorem (see [Birm, Thm. 2.3]), it suffices to show:

1. $\Gamma(\alpha^{-1}\beta \alpha) \simeq \Gamma(\beta)$ for all $\alpha \in B_n$;
2. $\Gamma(\beta \sigma_n) \simeq \Gamma(\beta)$;
3. $\Gamma(\beta \sigma_n^{-1}) \simeq \Gamma(\beta)$;

where $\beta \sigma_n$ and $\beta \sigma_n^{-1}$ are viewed as braids on $n + 1$ strands.

For a given $n \in \mathbb{N}$, we write $G_{(n)} = H_1 \ast \cdots \ast H_n$. Note that, if $\beta \in B_n$ and $n \leq m$, then the action of $\beta$ via $\rho$ on $G_{(m)}$ agrees with the action via $\rho$ on $G_{(n)} < G_{(m)}$, and is trivial on the free factors $H_{n+1},\ldots,H_m$. We suppress $\rho$ from our notation, writing simply $\beta(g)$ to mean $\rho(\beta)g$, for any $\beta \in B_n$ and $g \in G_{(m)}$.

**Proof of (1):** For $\beta \in B_n$, the group $\Gamma(\beta)$ is defined as the quotient of $G_{(n)}$ by the relations $g = \beta(g)$ for all $g \in G_{(n)}$. Since, for $\alpha \in B_n$, the relation $g = \alpha^{-1}\beta \alpha(g)$ is equivalent to the relation $\alpha(g) = \beta(\alpha(g))$, and $\alpha$ is an automorphism of $G_{(n)}$, it is clear that $\Gamma(\alpha^{-1}\beta \alpha)$ is defined by the same set of relations as $\Gamma(\beta)$. So (1) holds.

**Proof of (2):** The group $\Gamma(\beta \sigma_n)$ may be defined as the quotient of $G_{(n+1)}$ by the family of relations $R(i,x) : \phi_i(x) = \beta \sigma_n(\phi_i(x))$ for all $i = 1,2,\ldots,n+1$ and all $x \in H$. Note that $\sigma_n(\phi_{n+1}(x)) = h_n \phi_n(x) h_n^{-1}$. Therefore the relation $R(n+1,x) : \phi_{n+1}(x) = \beta(h_n \phi_n(x) h_n^{-1})$, where the right hand side is actually an element of $G_{(n)}$. In particular $\Gamma(\beta \sigma_n)$ is generated by the image of $G_{(n)}$. Also,

$$\beta \sigma_n(\phi_n(x)) = \beta(h_n^{-1} \phi_{n+1}(x) h_n) = \beta(h_n^{-1} \phi_{n+1}(x) h_n).$$

So, in view of $R'(n+1,x)$, the relation $R(n,x)$ is now equivalent to the relation $R'(n,x) : \phi_n(x) = \beta(\phi_n(x))$. Finally, since $\sigma_i(\phi_i(x)) = \phi_i(x)$ for all $i < n$, the remaining relations $R(i,x)$ are equivalent to $R'(i,x) : \phi_i(x) = \beta(\phi_i(x))$ for all $i = 1,2,\ldots,n-1$, and all $x \in H$. It now follows that $\Gamma(\beta \sigma_n) \simeq \Gamma(\beta)$. 

Proof of (3): Observe that $\Gamma(\beta^{-1}) \simeq \Gamma(\beta)$, since the relation $g = \beta(g)$ is equivalent to $\beta^{-1}(g) = g$, for all $g \in G_n$. Then

$$\Gamma(\beta \sigma_n^{-1}) \simeq \Gamma(\sigma_n \beta^{-1}) \simeq \Gamma(\beta^{-1} \sigma_n) \simeq \Gamma(\beta^{-1}) \simeq \Gamma(\beta).$$

\[ \square \]

3 Topological construction of the link invariants

Let $X$ be a CW-complex, let $P_0 \in X$ be a basepoint, and let $\alpha : [0,1] \to X$ be a loop based on $P_0$. We assume that $\alpha$ is not homotopic to the constant path. In this section we give a topological realization of the Artin type representation of $B_n$ associated to the pair $(H,h) = (\pi_1(X,P_0),[\alpha])$, and we deduce a topological construction of the link invariant $\Gamma_{(H,h)}$ of the previous section.

Let $D = D(\frac{n+1}{2}, \frac{n+1}{2})$ denote the disk in $\mathbb{C}$ centered at $\frac{n+1}{2}$ of radius $\frac{n+1}{2}$. Now, we construct a space $Y$ obtained from $D$ by making $n$ holes in $D$ and gluing a copy of $X$ into each hole by identifying the circular boundary of the hole to the loop $\alpha$ in $X$.

Choose some small $\varepsilon > 0$ (we require only that $\varepsilon < \frac{1}{8}$). Let

$$Y' = D \setminus \left( \bigcup_{k=1}^{n} D^\circ(k,\varepsilon) \right),$$

where $D^\circ(k,\varepsilon)$ denotes the open disk centered at $k$ of radius $\varepsilon$. Take $n$ copies $X_1, \ldots, X_n$ of $X$, denote by $f_k : X \to X_k$ the natural homeomorphism, and write $\alpha_k = f_k \circ \alpha$ for all $k = 1, \ldots, n$. Then

$$Y = \left( Y' \sqcup \left( \bigcup_{k=1}^{n} X_k \right) \right) / \sim,$$

where $\sim$ is the identification defined by

$$\alpha_k(t) \sim k + \varepsilon e^{2\pi i t}, \quad k = 1, \ldots, n, \quad t \in [0,1].$$

Finally, choose a basepoint $Q_0 \in \partial D$ for $Y$. The following result is a direct consequence of the above construction.

Lemma 3.1. Let $H = \pi_1(X,P_0)$, and let $H_1, \ldots, H_n$ be $n$ copies of $H$. Then $\pi_1(Y,Q_0) \simeq H_1 \ast \cdots \ast H_n$. \[ \square \]

We now show that the braid group $B_n$ acts on $Y$ up to isotopy relative to the boundary of $D$ in such a way that the induced action on $\pi_1(Y)$ is the Artin type representation associated to $(H,h)$, where $h$ is the element of $H = \pi_1(X,P_0)$ represented by $\alpha$. 
Let $\xi \in \mathbb{C}$ and $0 < r < R$. Define the half Dehn twist $T = T(\xi, r, R)$ by

$$T(\xi + \rho e^{i\theta}) = \begin{cases} 
\xi + \rho e^{i(\theta - \pi)} & \text{if } 0 \leq \rho \leq r \\
\xi + \rho e^{i(\theta - \pi)} & \text{if } r \leq \rho \leq R \text{ and } t = \frac{R - \rho}{R - r} \\
\xi + \rho e^{i\theta} & \text{if } \rho \geq R
\end{cases}$$

(see Figure 1).

![Figure 1: A half Dehn twist.](image)

Let $T^D_k : D \to D$ be the homeomorphism defined by

$$T^D_k = T(k, \varepsilon, 2\varepsilon)^{-3} \circ T(k + 1, \varepsilon, 2\varepsilon)^{-1} \circ T(k + \frac{1}{2}, \frac{1}{2} + \varepsilon, \frac{1}{2} + 2\varepsilon).$$

Note that $T^D_k$ leaves invariant the set $\bigcup_{j=1}^n D(j, \varepsilon)$, and therefore restricts to a homeomorphism $T_k : Y' \to Y'$. See Figure 2.

![Figure 2: The homeomorphism $T_k : Y' \to Y'$.](image)

One can verify (with a little effort) that $T_{k+1}T_k$ is isotopic to $T_{k+1}T_kT_{k+1}$ relative to $\partial Y'$ for $k = 1, \ldots, n - 2$, and that $T_kT_l$ is isotopic to $T_lT_k$ relative to $\partial Y'$ for $|k - l| \geq 2$. Moreover, $T_k$ fixes $\partial D$ and transforms the rest of $\partial Y'$ as follows:

$$T_k(j + \varepsilon e^{i\theta}) = \begin{cases} 
j + \varepsilon e^{i\theta} & \text{if } j \neq k, k + 1 \\
k + 1 + \varepsilon e^{i\theta} & \text{if } j = k \\
k + \varepsilon e^{i\theta} & \text{if } j = k + 1.
\end{cases}$$
Therefore, $T_k$ extends to a homeomorphism $T_k : Y \to Y$ by setting, for all $x \in X$,

$$T_k(f_j(x)) = \begin{cases} 
  f_j(x) & \text{if } j \neq k, k+1 \\
  f_{k+1}(x) & \text{if } j = k \\
  f_k(x) & \text{if } j = k+1.
\end{cases}$$

The homeomorphism $T_k$ is the identity on $\partial \mathcal{D}$, $T_kT_{k+1}T_k$ is isotopic to $T_{k+1}T_kT_{k+1}$ relatively to $\partial \mathcal{D}$ for $k = 1, \ldots, n - 2$, and $T_kT_l$ is isotopic to $T_lT_k$ relatively to $\partial \mathcal{D}$ for $|k - l| \geq 2$.

By the above observations, $T_k$ determines an automorphism $\tau_k : \pi_1(Y, Q_0) \to \pi_1(Y, Q_0)$. Moreover,

$$\tau_k\tau_{k+1}\tau_k = \tau_{k+1}\tau_k\tau_{k+1} \quad \text{for } k = 1, \ldots, n - 2$$

$$\tau_k\tau_l = \tau_l\tau_k \quad \text{for } |k - l| \geq 2.$$

So, the mapping $\sigma_k \to \tau_k$ determines a representation $\rho : B_n \to \text{Aut}(\pi_1(Y, Q_0))$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{gamma_k.png}
\caption{The path $\gamma_k$.}
\end{figure}

Set $Q_0 = \frac{n+1}{2} + i\frac{n+1}{2}$. Let $\gamma_k : [0, 1] \to Y$ denote the path which joins $Q_0$ to $f_k(P_0)$ represented in Figure 3. We identify $\pi_1(Y, Q_0)$ with $G = H_1 \ast \cdots \ast H_n$ in such a way that the $k$-th embedding $\phi_k : H = \pi_1(X, P_0) \to H_k \subset G$ is defined by

$$\phi_k([\beta]) = [\gamma_k\beta\gamma_k^{-1}].$$

With this assumption, one can easily show the following.

**Proposition 3.2.** The representation $\rho : B_n \to \text{Aut}(\pi_1(Y, Q_0))$ described above coincides with the Artin type representation of $B_n$ associated to $(H, h)$, where $H = \pi_1(X, P_0)$ and $h$ is the element of $H$ represented by $\alpha$. □

Consider an oriented $m$-component link $L = K_1 \cup \cdots \cup K_m$ in $S^3$. The knot $K_i$ is an embedding $K_i : S^1 \to S^3$, and $K_i(S^1) \cap K_j(S^1) = \emptyset$ for $i \neq j$. Define a tubular neighborhood of $K_i$ to be an embedding $T_i : \mathbb{D}^2 \times S^1 \to S^3$ such that $T_i(0, \xi) = K_i(\xi)$ for all $\xi \in S^1$. Here, $\mathbb{D}^2$ denotes the disk centered at 0 of radius 1 in $\mathbb{C}$. A framing of $L$ is a collection $\{T_i : \mathbb{D}^2 \times S^1 \to S^3\}_{i=1}^m$ of embeddings
such that $T_i$ is a tubular neighborhood of $K_i$, for $i = 1, \ldots, m$, and $T_i(D^2 \times S^1) \cap T_j(D^2 \times S^1) = \emptyset$ for $i \neq j$. The \textit{longitude} of the component $K_i$ is the (oriented) embedding $\lambda_i : S^1 \to S^3$ such that $\lambda_i(\xi) = T_i(1, \xi)$ for all $\xi \in S^1$. The tubular neighborhood of the framing of each component $K_i$ is determined up to isotopy by the homology class of its longitude $\lambda_i$ in the knot complement $S^3 \setminus K_i$.

Given an oriented knot $K$, we identify $H_1(K) := H_1(S^3 \setminus K)$ with $\mathbb{Z}$ in such a way that $1 \in \mathbb{Z}$ is represented by the 1-cycle depicted in Figure 4(a). Let $K_1, K_2$ denote disjoint oriented knots in $S^3$. One defines the \textit{linking number} $\text{lk}(K_1, K_2) \in \mathbb{Z}$ to be the class $[K_1] \in H_1(K_2) = \mathbb{Z}$. The linking number $\text{lk}(K_1, K_2)$ may be measured from any regular projection of the link $K_1 \cup K_2$ by counting with sign the crossings where $K_1$ passes over $K_2$, as indicated in Figure 4(b). (Equally one may choose to count undercrossings with the appropriate sign, and one quickly sees that $\text{lk}(K_1, K_2) = \text{lk}(K_2, K_1)$).

![Figure 4: Sign conventions.](image)

**Notation (Preferred framing).** Let $L = K_1 \cup \cdots \cup K_m$ be an $m$-component oriented link in $S^3$. Up to isotopy, there is a unique framing in which the longitude $\lambda_i$ for each component $K_i$ satisfies the following condition:

$$\sum_{j=1}^{m} \text{lk}(\lambda_i, K_j) = 0.$$ 

Note that, for $j \neq i$, $\text{lk}(\lambda_i, K_j) = \text{lk}(K_i, K_j)$ and is determined by the oriented link $L$. We shall refer to the above framing as the \textit{preferred framing} of $L$.

We now wish to associate to an oriented link $L$ the space $\Omega(L, X)$ obtained by performing a ‘generalised’ surgery on the link $L$ according to the preferred framing just described. More precisely, let $L = K_1 \cup \cdots \cup K_m$ and let $\{T_i : D^2 \times S^1 \to S^3\}_{i=1}^{m}$ be the preferred framing. Let $T_i^\circ$ denote the interior of $T_i(D^2 \times S^1)$ for $i = 1, \ldots, m$, and let

$$\Omega'(L) = S^3 \setminus \left( \bigcup_{i=1}^{m} T_i^\circ \right).$$ 

Take $m$ copies $X_1, \ldots, X_m$ of $X$, denote by $f_i : X \to X_i$ the natural homeomorphism, and write $\alpha_i = f_i \circ \alpha$. Then

$$\Omega(L, X) = \left( \Omega'(L) \sqcup \left( \bigsqcup_{i=1}^{m} (X_i \times S^1) \right) \right) / \sim,$$
where \( \sim \) is the identification defined by

\[
(\alpha_j(t), \eta) \sim T_j(e^{2\pi t}, \eta), \quad j = 1, \ldots, m, \ t \in [0, 1], \ \eta \in S^1.
\]

The following proposition yields a second proof of the fact that \( \Gamma_{(H,h)} \) is a link invariant for any finitely generated group \( H \) and nontrivial element \( h \in H \).

**Proposition 3.3.** Let \( \beta \) be a braid, and let \( \hat{\beta} \) denote the closed braid of \( \beta \). Let \( X \) be a CW-complex with basepoint \( P_0 \) and let \( \alpha \) be a nontrivial loop in \( X \). Then \( \pi_1(\Omega(\hat{\beta}, X)) \) is isomorphic to \( \Gamma_{(H,h)}(\beta) \), where \( H = \pi_1(X, P_0) \) and \( h \) is the element of \( H \) represented by \( \alpha \).

**Proof.** We first remind the reader of the standard construction of the closed braid \( \hat{\beta} \) from a braid \( \beta \) (see [Birm]). Firstly, decompose \( S^3 \) as follows: let \( T_1, T_2 \) be two copies of the solid torus \( D \times S^1 \) and write

\[ S^3 = T_1 \cup_{\kappa: \partial T_1 \to \partial T_2} T_2, \]

where the identifying map \( \kappa \) is a homeomorphism carrying \( \partial D \) to \( S^1 \) and \( S^1 \) to \( \partial D \). The closed braid \( \hat{\beta} \) is the oriented link which is induced by composing the braid \( \beta : \{1, \ldots, n\} \times [0, 1] \to D \times [0, 1] \) with the composition of maps

\[ D \times [0, 1] \xrightarrow{f} D \times S^1 = T_1 \xrightarrow{g} S^3, \]

where \( f(p, t) = (p, e^{2\pi t}) \), for \( p \in D \) and \( t \in [0, 1] \), and \( g \) denotes the inclusion of \( T_1 \) in \( S^3 \). The orientation on \( \hat{\beta} \) is naturally induced from a choice of orientation of the interval \([0, 1] \).

![Figure 5: Braid closure.](image-url)
Figure 6: Choosing a framing for $\hat{\beta} = K_1 \cup \cdots \cup K_m$.

Write $\beta = \sigma_1^{\varepsilon_1} \sigma_2^{\varepsilon_2} \cdots \sigma_r^{\varepsilon_r}$ and define $T^{D}_\beta : \mathcal{D} \to \mathcal{D}$ as the composition of the homeomorphisms $(T^{D}_\beta)^{\varepsilon_j}$ for $j = 1, \ldots, r$. Similarly, define $T_\beta = T^{\varepsilon_1}_1 T^{\varepsilon_2}_2 \cdots T^{\varepsilon_r}_r : \mathcal{Y} \to \mathcal{Y}$. For $j = 1, \ldots, n$, denote by $b_j^{j+\varepsilon}$ the point on $\partial \mathcal{D}(j,\varepsilon)$. This is the point on $\partial \mathcal{Y}'$ to which the basepoint of $X_j$ is attached when forming $\mathcal{Y}$. Since $T^{D}_\beta$ is isotopic to $\text{Id}_D$ relative to $\partial \mathcal{D}$, there is a homeomorphism $U : \mathcal{D} \times [0,1] \to \mathcal{D} \times [0,1]$ such that $U(x,0) = (x,0)$, $U(x,1) = (T^{D}_\beta(x),1)$, for all $x \in \mathcal{D}$, and $U$ fixes $\partial \mathcal{D} \times [0,1]$ pointwise. Moreover, by construction, $U$ carries $\left( \bigsqcup_{j=1}^n \mathcal{D}(j,\varepsilon) \right) \times [0,1]$ to a tubular neighbourhood of (a representative of) the braid $\beta$, and $g \circ f \circ U$ carries the arcs $\{b_j \times [0,1] : j = 1, \ldots, n\}$ to a framing of $\hat{\beta}$ equivalent to that described in Figure 6, namely the preferred framing. Consequently the space $\Omega(\hat{\beta},X)$ is homeomorphic to $T'_1 \cup T_2$ where

$$T'_1 = \mathcal{Y} \times [0,1] / ( (y,0) \sim (T_\beta(y),1) ).$$

We therefore have $\pi_1(T'_1) \cong G \ast \langle t \rangle / (t g t^{-1} \sim \rho(\beta)g \ \forall \ g \in G)$, an HNN-extension. Attaching $T_2$ to $T'_1$ has the effect of simply killing the stable letter $t$. Consequently

$$\pi_1(\Omega(\hat{\beta},X)) \cong G / ( g \sim \rho(\beta)g \ \forall \ g \in G) = \Gamma_{(H,h)}(\beta).$$

\[\square\]

4 Faithfulness

Consider a group $H$ and a non-trivial element $h \in H \setminus \{1\}$, and write $G = H_1 \ast \cdots \ast H_n$, where $H_i$ is a copy of $H$. The aim of this section is to prove the following.

**Proposition 4.1.** Let $\rho : B_n \to \text{Aut}(G)$ be the Artin type representation of $B_n$ associated to $(H,h)$. Then $\rho$ is faithful.

As pointed out in the introduction, the proof of Proposition 4.1 is strongly inspired by the proof of Theorem A of [Shp], and its main ingredient is the following result due to Dehornoy [Deh1], [Deh2].
Proposition 4.2 (Dehornoy). Let $B_{n-1}$ be the subgroup of $B_n$ generated by $\sigma_2, \ldots, \sigma_{n-1}$. Let $\beta \in B_n$. Then either

1. $\beta \in B_{n-1}$; or
2. $\beta$ can be written
   \[ \beta = \alpha_0 \sigma_1 \alpha_1 \sigma_1 \alpha_2 \ldots \sigma_1 \alpha_l, \]
   where $l \geq 1$ and $\alpha_0, \ldots, \alpha_l \in B_{n-1}$; or
3. $\beta$ can be written
   \[ \beta = \alpha_0 \sigma_1^{-1} \alpha_1^{-1} \alpha_2^{-1} \ldots \sigma_1^{-1} \alpha_l, \]
   where $l \geq 1$ and $\alpha_0, \ldots, \alpha_l \in B_{n-1}$. \qed

The following lemma is a preliminary result to the proof of Proposition 4.1.

Lemma 4.3. Let $G' = H_2 \ast \cdots \ast H_n$. Let $u \in G$ such that the normal form of $u$ with respect to the decomposition $G = H_1 \ast G'$ starts with $h_1^{-1}$ and ends with $h_1$.

1. The normal form of $\rho(\sigma_1)(u)$ with respect to the decomposition $G = H_1 \ast G'$ also starts with $h_1^{-1}$ and ends with $h_1$.
2. Let $k \in \{2, \ldots, n-1\}$ and $\varepsilon \in \{\pm 1\}$. The normal form of $\rho(\sigma_k^\varepsilon)(u)$ with respect to the decomposition $G = H_1 \ast G'$ also starts with $h_1^{-1}$ and ends with $h_1$.

Proof. Let $v \in H_1 \ast H_2$. Suppose that the normal form of $v$ is
   \[ v = \phi_1(x_1) \phi_2(y_1) \ldots \phi_1(x_l) \phi_2(y_l), \]
where $x_1, \ldots, x_l, y_1, \ldots, y_{l-1} \in H \setminus \{1\}$, and $y_l \in H$. Then
   \[ \rho(\sigma_1)(v) = h_1^{-1} \cdot \phi_2(x_1) \cdot h_1^2 \phi_1(y_1) h_1^{-2} \cdot \ldots \cdot \phi_2(x_l) \cdot h_1^2 \phi_1(y_l) h_1^{-1}, \]
thus the normal form of $\rho(\sigma_1)(v)$ starts with $h_1^{-1}$.

Similarly, if the normal form of $v$ is
   \[ v = \phi_2(y_1) \phi_1(x_1) \ldots \phi_2(y_l) \phi_1(x_l), \]
where $x_1, \ldots, x_l, y_2, \ldots, y_l \in H \setminus \{1\}$ and $y_1 \in H$, then the normal form of $\rho(\sigma_1)(v)$ ends with $h_1$.

Now, write
   \[ u = v_0 w_1 \ldots w_l v_l \]
where $v_i \in (H_1 \ast H_2) \setminus \{1\}$ and $w_j \in (H_3 \ast \cdots \ast H_n) \setminus \{1\}$, and $l \geq 0$. The hypothesis that $u$ starts with $h_1^{-1}$ implies that $v_0$ starts with $h_1^{-1}$, and the hypothesis that $u$ ends with $h_1$ implies that $v_l$ ends with $h_1$. Both groups, $H_1 \ast H_2$ and $H_3 \ast \cdots \ast H_n$, are invariant by $\rho(\sigma_1)$, and $\rho(\sigma_1)$ is the identity on $H_3 \ast \cdots \ast H_n$. So,
   \[ \rho(\sigma_1)(u) = \rho(\sigma_1)(v_0) \cdot w_1 \cdot \rho(\sigma_1)(v_1) \cdot \ldots \cdot w_l \cdot \rho(\sigma_1)(v_l). \]
By the above observations, $\rho(\sigma_1)(v_0)$ starts with $h_1^{-1}$ and $\rho(\sigma_1)(v_l)$ ends with $h_1$, thus $\rho(\sigma_1)(u)$ starts with $h_1^{-1}$ and ends with $h_1$. 

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Let \( k \in \{2, \ldots, n-1\} \) and \( \varepsilon \in \{\pm 1\} \). Write
\[
u = h_1^{-1} w_1 \ldots v_{l-1} w_l h_1,
\]
where \( v_1, \ldots, v_{l-1} \in H_1 \setminus \{1\} \) and \( w_1, \ldots, w_l \in G' \setminus \{1\} \). Both groups, \( H_1 \) and \( G' \), are invariant by \( \rho(\sigma_k^\varepsilon) \), and \( \rho(\sigma_k^\varepsilon) = 1 \) is the identity on \( H_1 \). So,
\[
\rho(\sigma_k^\varepsilon)(u) = h_1^{-1} \cdot \rho(\sigma_k^\varepsilon)(w_1) \cdot v_1 \cdot \cdots \cdot v_{l-1} \cdot \rho(\sigma_k^\varepsilon)(w_l) \cdot h_1,
\]
thus the normal form of \( \rho(\sigma_k^\varepsilon)(u) \) starts with \( h_1^{-1} \) and ends with \( h_1 \).

\[\begin{proof}
\[\end{proof}\]

**Proof of Proposition 5.1** We argue by induction on \( n \). Assume \( n = 2 \). We have
\[
\rho(\sigma_1^{2l})(h_1) = (h_2 h_1)^{-1} h_1 h_2 h_1 \neq h_1, \quad \text{for } l \in \mathbb{Z} \setminus \{0\}
\]
\[
\rho(\sigma_1^{2l+1})(h_1) = (h_2 h_1)^{-1} h_2 h_1 h_1 \neq h_1, \quad \text{for } l \in \mathbb{Z}
\]
thus the representation \( \rho : B_2 \to \text{Aut}(H_1 \ast H_2) \) is faithful.

Now, assume \( n \geq 3 \). Let \( \beta \in B_n \setminus \{1\} \). By Proposition 4.1, \( \beta \in B_{n-1} \), or \( \beta = \alpha_0 \sigma_1 \ldots \sigma_1 \alpha_l \), where \( l \geq 1 \) and \( \alpha_0, \ldots, \alpha_l \in B_{n-1} \), or \( \beta = \alpha_0 \sigma_1^{-1} \ldots \sigma_1^{-1} \alpha_l \), where \( l \geq 1 \) and \( \alpha_0, \ldots, \alpha_l \in B_{n-1} \).

Suppose \( \beta \in B_{n-1} \). By induction, \( \rho(\beta) \) acts non-trivially on \( G' = H_2 \ast \cdots \ast H_n \), thus \( \rho(\beta) \) acts non-trivially on \( G = H_1 \ast G' \).

Suppose \( \beta = \alpha_0 \sigma_1 \ldots \sigma_1 \alpha_l \), where \( l \geq 1 \) and \( \alpha_0, \ldots, \alpha_l \in B_{n-1} \). Let
\[
u = \rho(\sigma_1 \alpha_l)(h_1) = \rho(\sigma_1)(h_1) = h_1^{-1} h_2 h_1.
\]
By Lemma 4.3, the normal form of \( \rho(\alpha_0 \sigma_1 \ldots \sigma_1 \alpha_{l-1})(\nu) = \rho(\beta)(h_1) \) starts with \( h_1^{-1} \) and ends with \( h_1 \). In particular, \( \rho(\beta)(h_1) \neq h_1 \), thus \( \rho(\beta) \neq \text{Id} \).

Suppose \( \beta = \alpha_0 \sigma_1^{-1} \ldots \sigma_1^{-1} \alpha_l \), where \( l \geq 1 \) and \( \alpha_0, \ldots, \alpha_l \in B_{n-1} \). By the previous case, \( \rho(\beta^{-1}) \neq \text{Id} \), thus \( \rho(\beta) \neq \text{Id} \).

\[\begin{proof}
\[\end{proof}\]

## 5 Garside groups

Our objectives in this section are twofold. Firstly we give a brief presentation of the definition and salient properties of a Garside group. Secondly we establish a criterion (or set of criteria) which allows one to show that a group given by a certain type of presentation is indeed a Garside group, and which we will make use of in the subsequent section. Our presentation of the subject draws in many ways from the work of Dehornoy [Deh3, Deh5] as well as [DF], and, like all treatments of Garside groups, is inspired ultimately by the seminal papers of Garside [Gar], on braid groups, and Brieskorn and Saito [BS], on Artin groups.

**Definition 5.1.** Let \( M \) be an arbitrary monoid. We say that \( M \) is *atomic* if there exists a function \( \nu : M \to \mathbb{N} \) such that
\[
\bullet \ \nu(a) = 0 \text{ if and only if } a = 1;
\]
• $\nu(ab) \geq \nu(a) + \nu(b)$ for all $a, b \in M$.

Such a function $\nu : M \to \mathbb{N}$ is called a *norm* on $M$.

An element $a \in M$ is called an *atom* if it is indecomposable, namely, if $a = bc$ then either $b = 1$ or $c = 1$.

This definition of atomicity is taken from [DP]. See [DP], Proposition 2.1, for a list of further properties all equivalent to atomicity. In the same paper it is shown that any generating set of $M$ contains the set of all atoms. In particular, $M$ is finitely generated if and only if it has only finitely many atoms.

Given that a monoid $M$ is atomic, we may define left and right invariant partial orders $\leq_L$ and $\leq_R$ on $M$ as follows:

• set $a \leq_L b$ if there exists $c \in M$ such that $ac = b$;

• set $a \leq_R b$ if there exists $c \in M$ such that $ca = b$.

We shall call these the *left* and *right divisibility orders* on $M$.

**Definition 5.2.** A *Garside monoid* is a monoid $M$ such that

(i) $M$ is atomic and finitely generated;

(ii) $M$ is cancellative;

(iii) $(M, \leq_L)$ and $(M, \leq_R)$ are lattices;

(iv) there exists an element $\Delta \in M$, which we call a *Garside element*, such that

(a) the set $L(\Delta) := \{x \in M : x \leq_L \Delta\}$ generates $M$, and

(b) the sets $L(\Delta)$ and $R(\Delta) := \{x \in M : x \leq_R \Delta\}$ are equal.

**Remark.** Elsewhere in the literature the condition that $M$ is finitely generated is often incorporated into condition (iv) of the definition by saying that the set $L(\Delta)$ is finite. It seems more natural to state this condition separately. Note that, if $M$ is finitely generated and atomic, then $L(a) = \{x \in M : x \leq_L a\}$ is finite for all $a \in M$.

**Definition 5.3.** For any monoid $M$ one can define the group $G(M)$ which is presented by the generating set $M$ and relations $ab = c$ whenever $ab = c$ in $M$. There is an obvious canonical homomorphism $M \to G(M)$. This homomorphism is not injective in general. The group $G(M)$ is known as the *group of fractions* of $M$. Define a *Garside group* to be the group of fractions of a Garside monoid.

**Remark.**

(1) A Garside monoid $M$ satisfies Ôre’s conditions, thus the canonical homomorphism $M \to G(M)$ is injective. Moreover the partial order $\leq_L$ (resp. $\leq_R$) extends to a left invariant (resp. right invariant) lattice order on $G(M)$ with positive cone $M$. 
A Garside element is never unique. For example, if $\Delta$ is a Garside element, then $\Delta^k$ is also a Garside element for all $k \geq 1$ (see [Deh3], Lemma 2.2).

Let $M$ be a Garside monoid. The lattice operations of $(M, \leq_L)$ are denoted by $\vee_L$ and $\wedge_L$. For $a,b \in M$, we denote by $a \wedge_L b$ the unique element of $M$ such that $a(a \wedge_L b) = a \vee_L b$. Similarly, the lattice operations of $(M, \leq_R)$ are denoted by $\vee_R$ and $\wedge_R$, and, for $a,b \in M$, we denote by $b/ra$ the unique element of $M$ such that $(b/ra)a = a \vee_R b$.

Now, before establishing our criterion for a group to be a Garside group, we briefly explain how to define a biautomatic structure on a given Garside group. By [EpAl], such a structure furnishes solutions to the word problem and to the conjugacy problem, and it implies that the group has quadratic isoperimetric inequalities. We refer to [EpAl] for definitions and properties of automatic groups, and to [Deh3] for more details on the biautomatic structures on Garside groups.

Let $M$ be a Garside monoid, and let $\Delta$ be a Garside element of $M$. For $a \in M$, we write $\pi_L(a) = \Delta \wedge_L a$ and denote by $\partial_L(a)$ the unique element of $M$ such that $a = \pi_L(a) \partial_L(a)$. Using the fact that $M$ is atomic and that $L(\Delta) = \{x \in M : x \leq_L \Delta\}$ contains all the atoms, one can easily show that $\pi_L(a) \not= 1$ if $a \not= 1$, and that there exists some positive integer $k$ such that $\partial_L^k(a) = 1$. Let $k$ be the lowest integer satisfying $\partial_L^k(a) = 1$. Then the expression

$$a = \pi_L(a) \cdot \pi_L(\partial_L(a)) \cdots \pi_L(\partial_L^{k-1}(a))$$

is called the normal form of $a$.

Let $G = G(M)$ be the group of fractions of $M$. Let $c \in G$. Since $G$ is a lattice with positive cone $M$ the element $c$ can be written $c = a^{-1}b$ with $a,b \in M$. Obviously, $a$ and $b$ can be chosen so that $a \wedge_L b = 1$ and, with this extra condition, are unique. Let $a = a_1a_2\ldots a_p$ and $b = b_1b_2\ldots b_q$ be the normal forms of $a$ and $b$, respectively. Then the expression

$$c = a_1^{-1}a_2^{-1}\cdots a_p^{-1}b_1b_2\ldots b_q$$

is called the normal form of $c$.

The following result can be found in [Deh3], Section 3.

**Theorem 5.4 (Dehornoy).** Let $M$ be a Garside monoid and let $G$ be the group of fractions of $M$. Then the normal forms of the elements of $G$ form a symmetric rational language on the (finite) set $L(\Delta)$ which has the fellow traveler property. In particular, $G$ is biautomatic.

We turn now to establish our criterion.

For a finite set $S$, we denote by $S^*$ the free monoid on $S$. The elements of $S^*$ are called words on $S$. The empty word is denoted by $\epsilon$. Let $\equiv$ be a congruence relation on $S^*$, and let $M = (S^*/\equiv)$. For $w \in S^*$, we denote by $\overline{w}$ the element of $M$ represented by $w$, and we call $w$ an expression of $\overline{w}$.

**Definition 5.5.** A complement is a function $f : S \times S \to S^*$ such that $f(x,x) = \epsilon$ for all $x \in S$. To a complement $f : S \times S \to S^*$ we associate the following two monoids.

$$M^f_L = \langle S \mid xf(x,y) = yf(y,x) \text{ for } x,y \in S \rangle^+,$$

$$M^f_R = \langle S \mid f(y,x)x = f(x,y)y \text{ for } x,y \in S \rangle^+.$$

For $u,v \in S^*$, we write $u \equiv_L^f v$ if $u$ and $v$ are expressions of the same element of $M^f_L$, and we write $u \equiv_R^f v$ if $u$ and $v$ are expressions of the same element of $M^f_R$. 
Definition 5.6. A word $w$ in $(S \cup S^{-1})^*$ is $f$-reversible on the left in one step to a word $w'$ if $w'$ is obtained from $w$ by replacing some subword $x^{-1}y$ (with $x, y \in S$) by the corresponding word $f(x, y)f(y, x)^{-1}$. Let $p \geq 0$. We say that $w$ is $f$-reversible on the left in $p$ steps to a word $w'$ if there exists a sequence $w_0 = w, w_1, \ldots, w_p = w'$ in $(S \cup S^{-1})^*$ such that $w_{i-1}$ is $f$-reversible on the left in one step to $w_i$ for all $i = 1, \ldots, p$. The property “$w$ is $f$-reversible on the left to $w'$” is denoted by $w \rightarrow_L^f w'$.

We define the $f$-reversibility on the right in a similar way, replacing subwords $yx^{-1}$ (with $x, y \in S$) by the corresponding words $f(x, y)^{-1}f(y, x)$. The property “$w$ is $f$-reversible on the right to $w'$” is denoted by $w \rightarrow_R^f w'$.

It is shown in [Deh3] that a reversing process is confluent, namely:

Proposition 5.7 (Dehornoy, [Deh3], Lemma 1). Let $f : S \times S \to S^*$ be a complement, and let $w \in (S \cup S^{-1})^*$. Suppose that the word $w$ is $f$-reversible on the left in $p$ steps to a word $w^{-1}v$, with $u, v \in S^*$. Then any sequence of left $f$-reversing transformations starting from $w$ leads in $p$ steps to $wv^{-1}$.

Definition 5.8. Let $f : S \times S \to S^*$ be a complement and let $u, v \in S^*$. Assume that there exist $u', v' \in S^*$ such that $u^{-1}v' \rightarrow_L^f u'(v')^{-1}$. By Proposition 5.7, $u'$ and $v'$ are unique (if they exist). Then we write $u' = C_L^f(u, v)$ and $v' = C_R^f(v', u)$. One has $uC_L^f(u, v) \equiv_L vC_R^f(v', u)$ (see [Deh3, Lem. 2]). If no such words $u', v'$ exist then we write $C_L^f(u, v) = C_R^f(v', u) = \infty$.

Similarly, define the words $C_L^f(u, v)$ and $C_R^f(v', u)$ to be the unique elements of $S^*$ which satisfy $vu^{-1} \rightarrow_R^f vC_R^f(u, v)^{-1}C_R^f(v', u)$, or write $C_R^f(v', u) = \infty$ if no such words exist.

Definition 5.9 (Dehornoy, [Deh3], p.120). Let $f : S \times S \to S^*$ be a complement. We say that $f$ is coherent on the left if, for all $x, y, z \in S$ such that $C_L^f(f(x, y), f(x, z)) \neq \infty$ we have $C_L^f(f(x, y), f(x, z)) \equiv_L^f C_L^f(f(y, x), f(y, z))$.

Similarly, we say that $f$ is coherent on the right if, for all $x, y, z \in S$ such that $C_R^f(f(z, x), f(y, x)) \neq \infty$ we have $C_R^f(f(z, x), f(y, x)) \equiv_R^f C_R^f(f(z, y), f(x, y))$.

A partially ordered set $(X, \leq)$ is said to be a quasi-lattice, or quasi-lattice ordered, if every pair of elements $x, y \in X$ which has a common upper bound ($z$ such that $x \leq z$ and $y \leq z$) has a least upper bound, usually written $x \lor y$.

The proof of the following proposition can be more or less reconstructed from Garside’s original treatment of the braid monoids [Gar], or the similar treatment of Artin monoids in [BS]. The result appears in almost precisely this form (with some notational differences) as Lemma 4 of [Deh3].

Proposition 5.10. Let $M$ be an atomic monoid with generating set $S$, and suppose that $f : S \times S \to S^*$ is a complement which is coherent on the left and such that $M = M_L^f$. Then the following holds:

(LCQL) For all $u, v, x, y \in S^*$ such that $ux \equiv_L^f vy$, there exists $w \in S^*$ such that $x \equiv_L^f C_L^f(u, v)w$ and $y \equiv_L^f C_L^f(v, u)w$.

In particular, (LCQL) implies that $M$ is left cancellative and $(M, \leq_L)$ is a quasi-lattice.
Proof. We refer the reader to [Deh3], Lemma 4, for the proof of the statement (LCQL). The fact that $M$ is left cancellative comes from putting $u = v$ in (LCQL). Also, if $u$ and $v$ represent elements $\mathfrak{u}$ and $\mathfrak{v}$ respectively, and if $\mathfrak{u}$ and $\mathfrak{v}$ have a common upper bound (represented by words $ux \equiv vy$ for some $x, y \in S^*$), then the least upper bound $\mathfrak{u} \lor_L \mathfrak{v}$ is the element represented by $uC^L(u,v) \equiv vC^L(v,u)$. The statement (LCQL) implies that this element divides all common upper bounds of $\mathfrak{u}$ and $\mathfrak{v}$. \qed

Now, Proposition 5.10, together with [DP] and [Deh5], permit the following criterion for a monoid $M$ to be a Garside monoid:

**Criterion 5.11.** Let $M$ be a monoid. Then $M$ is a Garside monoid if and only if it satisfies the following properties:

(C1) $M$ is finitely generated and atomic;

(C2) there exist complements $f : S_1 \times S_1 \to S_1^*$, coherent on the left, and $g : S_2 \times S_2 \to S_2^*$, coherent on the right, such that $M \cong M^L_f$ and $M \cong M^R_g$;

(C3) $M$ possesses a Garside element, namely an element $\Delta \in M$ such that every atom of $M$ left divides $\Delta$ and the sets $L(\Delta) = \{ x \in M : x \leq_L \Delta \}$ and $R(\Delta) = \{ x \in M : x \leq_R \Delta \}$ are equal.

Proof. Let $M$ be a Garside monoid. Clearly, $M$ satisfies (C1) and (C3). So, we just need to show that $M$ satisfies (C2). Choose some finite generating set $S$ for $M$, and consider complements $f : S \times S \to S^*$ and $g : S \times S \to S^*$ such that

$$f(x,y) = x \setminus_L y, \quad g(x,y) = y /_R x,$$

for all $x, y \in S$. Then, by [DP], Theorem 4.1, one has $M = M^L_f = M^R_g$, and, by [Deh3], Lemma 5.2, $f$ is coherent on the left and $g$ is coherent on the right.

Now, recall the statement of [Deh3], Proposition 2.1. Suppose that $M$ is a monoid which satisfies the following properties:

(D1) $M$ is finitely generated and atomic;

(D2) $M$ is left and right cancellative;

(D3) $(M, \leq_L)$ is a quasi-lattice;

(D4) there exists a finite subset $P \subset M$ which generates $M$ and which is closed under $\setminus_L$ (namely, if $a, b \in P$, then $a \setminus_L b \in P$).

Then $M$ is a Garside monoid.

Let $M$ be a monoid which satisfies (C1), (C2), (C3). We wish to show that $M$ is a Garside monoid. By Proposition 5.10, $M$ satisfies (D1), (D2) and (D3). So, it remains to show that $M$ satisfies (D4). Let $P = L(\Delta) = R(\Delta)$. Note that, by hypothesis, $P$ generates $M$. Let $a, b \in P$. Since $a \leq_L \Delta$ and $b \leq_L \Delta$, we have $a \lor_L b \leq_L \Delta$. Let $c \in M$ such that $\Delta = (a \lor_L b)c = a(a \setminus_L b)c$. Then $(a \setminus_L b)c \leq_R \Delta$, thus $(a \setminus_L b)c \leq_L \Delta$ (since $L(\Delta) = R(\Delta)$), therefore $(a \setminus_L b) \leq_L \Delta$, that is $(a \setminus_L b) \in P$. \qed
Remark. In the context of Garside groups, the reversing processes are used not only to determine whether a group is a Garside group, but it is also a very useful tool for solving the word problem and to explicitly compute normal forms. For instance, if $M$ is a Garside monoid, then one can find complements $f : S \times S \to S^*$ and $g : S \times S \to S^*$ such that $M = M_L^f = M_R^g$. If $w$ is in $(S \cup S^{-1})^*$, then any sequence of right $g$-reversing transformations leads to a word $u^{-1}v$ where $u,v \in S^*$, and one has $w = u^{-1}v$. On the other hand, if $u,v \in S^*$, then $u \lor v$ is represented by $uC^f_L(u,v)$, and $u \land v$ can be computed by means of the equality

$$u \land v = (u \lor v)/R((u \lor v) \lor_R (u \lor v))$$

(see [Deh5], Lemma 2.6).

In the next section we shall need the following characterization of a Garside element.

Lemma 5.12 (Garside elements). Let $M$ be a cancellative atomic monoid with atom set $A$, and suppose that $\Delta \in M$ is such that $L(\Delta) := \{ x \in M : x \leq L \Delta \}$ generates $M$. Define $R(\Delta) := \{ x \in M : x \leq R \Delta \}$. Then the following are equivalent:

1. $L(\Delta) = R(\Delta)$;
2. $A \Delta = \Delta A$;
3. $M \Delta = \Delta M$;
4. there exists a monoid automorphism $\tau : M \to M$ such that $w \Delta = \Delta \tau(w)$ for all $w \in M$. (In particular, $\tau(A) = A$. Also $\tau$ is necessarily unique.)

Proof. By cancellativity, there is a well-defined bijection $c : L(\Delta) \to R(\Delta)$ such that $x . c(x) = \Delta$ for all $x \in L(\Delta)$. Suppose that (1) holds. Then $c$ is a bijection $L(\Delta) \to L(\Delta)$ and we may define $\tau = c^2$, also a bijection of $L(\Delta) \to L(\Delta)$. Note that for $x \in L(\Delta)$, we also have $c(x) \in L(\Delta)$, so that $\Delta$ may be written $c(x)c^2(x)$. Therefore, for all $x \in L(\Delta)$, we have $x . \Delta = x . c(x). c^2(x) = \Delta . \tau(x)$. Since $L(\Delta)$ generates $M$, it follows that $\Delta \leq_L w \Delta$ for all $w \in M$ (in fact, if $w = x_1 x_2 ... x_n$ with $x_i \in L(\Delta)$, then $w \Delta = \Delta \tau(x_1) ... \tau(x_n)$). By left cancellativity, there is therefore a unique well-defined function $\tau : M \to M$ such that $w \Delta = \Delta \tau(w)$ for all $w \in M$. By right cancellativity, $\tau$ must be injective. Moreover, given $x,y,z \in M$ such that $z = xy$, we have $\Delta \tau(z) = z \Delta = \Delta \tau(x) \tau(y)$ and, by cancellation, $\tau(z) = \tau(x) \tau(y)$. Thus $\tau$ is a monoid homomorphism. By a similar argument, we may construct the inverse homomorphism $\tau^{-1}$ in order to show that $\tau$ is in fact an automorphism of $M$. Thus (1) implies (4).

Now suppose that (2) holds: $A \Delta = \Delta A$. Then by left and right cancellativity, there is a well-defined bijection $\tau : A \to A$ such that $a \Delta = \Delta \tau(a)$ for all $a \in A$. As in the previous paragraph this extends to an automorphism of $M$ such that $w \Delta = \Delta \tau(w)$ for all $w \in M$. Thus (2) implies (4). By the same reasoning one can show that (3) implies (4). On the other hand, both (2) and (3) are obvious consequences of (4).

Finally, we show that (4) implies (1). Suppose that (4) holds. In particular, we have $\tau(\Delta) = \Delta$. Therefore, $x \leq_L \Delta$ if and only if $\tau(x) \leq_L \Delta$ (since $\tau$ is a monoid automorphism). In other words, $\tau(L(\Delta)) = L(\Delta)$. On the other hand, the equation $x \Delta = \Delta \tau(x)$ shows, by left cancellation, that if $x \in L(\Delta)$ then $\tau(x) \in R(\Delta)$, and, by right cancellation, that if $y = \tau(x) \in R(\Delta)$ then $\tau^{-1}(y) \in L(\Delta)$. Thus $\tau(L(\Delta)) = R(\Delta)$. But then $\tau(L(\Delta)) = L(\Delta) = R(\Delta)$, giving (1). \qed
6 Semi-direct products

We turn back to the Artin type representations. Let $H$ be a group, let $h \in H \setminus \{1\}$, let $G = H_1 \ast \cdots \ast H_n$, where $H_i$ is a copy of $H$, and let $\rho : B_n \to \text{Aut}(G)$ be the Artin type representation associated to $(H,h)$. The aim of this section is to prove the following.

**Theorem 6.1.** Assume that $H$ is the group of fractions of a Garside monoid $M$ and that $h$ is a Garside element. Let $\tilde{G} = G \rtimes \rho B_n$, and let $\tilde{M}$ be the submonoid of $\tilde{G}$ generated by $M_1 = \phi_1(M)$ and the monoid $B^+_n$ of positive braids. Then $\tilde{M}$ is a Garside monoid, $\Delta = (h_1 \sigma_1 \sigma_2 \cdots \sigma_{n-1})^n$ is a Garside element of $\tilde{M}$, and $\tilde{G}$ is the group of fractions of $\tilde{M}$.

The first step in the proof of Theorem 6.1 consists on finding a presentation for $G \rtimes \rho B_n$, namely:

**Proposition 6.2.** Let $H = \langle S \mid R \rangle$ be a presentation for $H$, and let $D \in S^*$ be an expression of $h$. Then $\tilde{G} = G \rtimes \rho B_n$ has a presentation with generators $S \cup \{\sigma_1, \ldots, \sigma_{n-1}\}$, and with relations

- $r \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $r \in R$,
- $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $i = 1, \ldots, n-2$,
- $\sigma_i x = x \sigma_i$ for $x \in S$ and $i = 2, \ldots, n-1$,
- $x \sigma_1 D \sigma_1 = \sigma_1 D \sigma_1 D^{-1} x D$ for $x \in S$.

**Proof.** Let $\tilde{G}_0$ denote the abstract group generated by $S \cup \{\sigma_1, \ldots, \sigma_{n-1}\}$ and subject to the relations given in the statement of Proposition 6.2. Let $X = (\cup_{i=1}^n \phi_i(S)) \cup \{\sigma_1, \ldots, \sigma_{n-1}\}$. With a little effort one can verify that the mapping $\varphi : X \to \tilde{G}_0$ defined by

- $\varphi(\phi_i(x)) = \sigma_i^{-1} \ldots \sigma_{i-1} D^{i-1} x D^{i-1} \sigma_1 \ldots \sigma_{i-1}$ for $i = 1, \ldots, n$ and $x \in S$,
- $\varphi(\sigma_i) = \sigma_i$ for $i = 1, \ldots, n-1$,

determines a homomorphism $\varphi : \tilde{G} \to \tilde{G}_0$, and somewhat more easily that the mapping $\psi : S \cup \{\sigma_1, \ldots, \sigma_{n-1}\} \to \tilde{G}$ defined by

- $\psi(x) = \phi_1(x)$ for $x \in S$,
- $\psi(\sigma_i) = \sigma_i$ for $i = 1, \ldots, n-1$,

determines a homomorphism $\psi : \tilde{G}_0 \to \tilde{G}$. One checks without too much difficulty that $(\psi \circ \varphi)(a) = a$ for all $a \in X$, and $(\varphi \circ \psi)(b) = b$ for all $b \in S \cup \{\sigma_1, \ldots, \sigma_{n-1}\}$, thus $\psi \circ \varphi = \text{Id}_{\tilde{G}}$ and $\varphi \circ \psi = \text{Id}_{\tilde{G}_0}$. □
Proof of Theorem 6.1. Let $\tau : M \to M$ denote the automorphism of $M$ induced by conjugation by $h^{-1}$, so that $xh = h\tau(x)$ for all $x \in M$ (see Lemma 5.12). Let $S$ be a finite generating set for $M$. We may, and do, choose $S$ so that $\tau(S) = S$ (for instance we may simply choose $S$ to be the set of atoms of $M$). Define $f : S \times S \to S^*$ such that $xf(x,y) = yf(y,x) = x \lor_L y$ for all pairs $x,y \in S$. Similarly define $g : S \times S \to S^*$ such that $g(x,y)y = g(y,x)x = x \lor_R y$ for all pairs $x,y \in S$. As pointed out in the proof of Criterion 5.11, one has $M = M_L = M_R$. $f$ is coherent on the left, and $g$ is coherent on the right. We simply write $\sim$ for the congruence relation on $S^*$ defined by the relations in $M$ (namely, $\equiv_L^f$, or equally $\equiv_R^g$). Let $D \in S^*$ be an expression of $h$. Note that for $x \in S$ we have $xD \sim DX(x)$ and $\tau^{-1}(x)D \sim DX(x)$, where $\tau(x)$ and $\tau^{-1}(x)$ also denote elements of the generating set $S$. The last family of relations appearing in Proposition 6.2 may be replaced with $x\sigma_1D\sigma_1 = \sigma_1D\sigma_1\tau(x)$ for all $x \in S$, or equivalently with $\tau^{-1}(x)\sigma_1D\sigma_1 = \sigma_1D\sigma_1x$ for all $x \in S$.

Let $X = S \cup \{\sigma_1, \ldots, \sigma_{n-1}\}$. Let $F : X \times X \to X^*$ be the complement defined by

\[
\begin{align*}
F(x,y) &= f(x,y) \quad \text{for } x,y \in S \\
F(x,\sigma_i) &= \sigma_iD_1 \quad \text{for } x \in S \\
F(\sigma_1, x) &= D_1\tau(x) \quad \text{for } x \in S \\
F(x,\sigma_i) &= \sigma_i \quad \text{for } x \in S \text{ and } i \geq 2
\end{align*}
\]

and let $G : X \times X \to X^*$ be the complement defined by

\[
\begin{align*}
G(x,y) &= g(x,y) \quad \text{for } x,y \in S \\
G(\sigma_1, x) &= \sigma_1D_1 \quad \text{for } x \in S \\
G(x,\sigma_1) &= \tau^{-1}(x)\sigma_1D \quad \text{for } x \in S \\
G(\sigma_i, x) &= \sigma_i \quad \text{for } x \in S \text{ and } i \geq 2
\end{align*}
\]

Let $\tilde{M}_0$ denote the monoid defined by the presentation with generators $X$ and relations as laid out in Proposition 6.2. Then clearly $\tilde{M}_0 \cong M_L^f \cong M_R^g$. We denote by $\approx$ the congruence relation on $X^*$ defined by the relations of $\tilde{M}_0$. (So $\approx$ is the same congruence relation as $\equiv_L^f$ and $\equiv_R^g$). We proceed now to show that $\tilde{M}_0$ satisfies the Criterion 5.11 with complements $F$ and $G$ and Garside element $\Delta = (D\sigma_1\sigma_2 \ldots \sigma_{n-1})^n$. It follows that $\tilde{M}_0$ is a Garside monoid with group of fractions $G$ and is canonically isomorphic to the submonoid $\tilde{M} \subset G$ in the statement of the Theorem.

Clearly $\tilde{M}_0$ is finitely generated. We check that $\tilde{M}_0$ is atomic. Let $\nu : M \to N$ be a norm for $M$. Let $\Sigma = \{\sigma_1, \ldots, \sigma_{n-1}\}$ and define the function $\ell : \Sigma^* \to N$ by $\ell(\sigma_{i_1} \ldots \sigma_{i_l}) = l$. We define a function $\tilde{\nu} : X^* \to N$ as follows. Let $w \in X^*$. Write $w = u_1v_1 \ldots u_lv_l$, where $u_1 \in S^*, \ldots, u_l \in S^* \setminus \{\epsilon\}$, and $v_1, \ldots, v_l \in \Sigma^* \setminus \{\epsilon\}$. Then

\[
\tilde{\nu}(w) = \nu(u_1u_2 \ldots u_l) + \ell(v_1v_2 \ldots v_l).
\]

One can easily verify that $\tilde{\nu}$ is invariant with respect to all of the relations given in Proposition 6.2 and therefore defines a function $\tilde{\nu} : \tilde{M}_0 \to N$. Moreover, it is easily seen that $\tilde{\nu}$ is a norm, and therefore $\tilde{M}_0$ is atomic.

The proof that $F$ is coherent on the left may be deduced from the existence, for each triple $\alpha, \beta, \gamma \in X$, of a certain tiling of the 2-sphere by relations from $M_L^f$ (i.e: relations of the form $\alpha F(\alpha, \beta) \approx \beta F(\beta, \alpha)$ for $\alpha, \beta \in X$) as illustrated in Figures 7 and 8. We illustrate the two most difficult cases, namely when $\{\alpha, \beta, \gamma\} = \{\sigma_1, \sigma_2, x\}$ for some $x \in S$ (Figure 8), and when
Condition (4) of Lemma 5.12, namely that there exists an automorphism \( w \) left divisible by \( \tau \). Recall that \( \beta \)

\[
\tilde{\Delta} \overset{\Delta}{\rightarrow} \Delta
\]

\[ x \in \mathcal{A} \]

\[ \text{This monoid} \]

\[ \sigma \]

\[ \Delta = \Delta \]

\[ \text{for all } x \in \tilde{\mathcal{M}} \]

Finally we show that the word \( \Delta = (D\sigma_1\sigma_2 \ldots \sigma_{n-1})^n \) represents a Garside element of \( \tilde{\mathcal{M}}_0 \). We shall employ Condition (4) of Lemma 5.12. Consider the Artin monoid presentation \[ A^+(B_n) = \langle \beta_1, \beta_2, \ldots, \beta_n \mid \beta_1 \beta_2 \beta_1 \beta_2 = \beta_2 \beta_1 \beta_2 \beta_1, \beta_i \beta_{i+1} \beta_i = \beta_{i+1} \beta_i \beta_{i+1} \text{ for } 2 \leq i \leq n-1, \beta_i \beta_j = \beta_j \beta_i \text{ for } |i-j| \geq 2 \rangle^+. \]

This monoid \( A^+(B_n) \) is well-known as the Artin monoid of type \( B_n \), and has Garside element \( \Delta_B = (\beta_1 \beta_2 \ldots \beta_n)^n \). Clearly there exists a monoid homomorphism \( A^+(B_n) \rightarrow \tilde{\mathcal{M}}_0 \) such that \( \beta_1 \rightarrow D \) and \( \beta_i \rightarrow \sigma_{i-1} \) for \( i = 2, 3, \ldots, n \). Thus any relation which is observed in \( A^+(B_n) \) may be deduced in \( \tilde{\mathcal{M}}_0 \). In particular, the fact that \( \Delta_B \) is a Garside element in \( A^+(B_n) \) implies that \( \Delta \) is left divisible by \( D, \sigma_1, \ldots, \sigma_{n-1} \) and hence is left divisible by every element of \( X \). It remains to verify Condition (4) of Lemma 5.12, namely that there exists an automorphism \( \tilde{\tau} : \tilde{\mathcal{M}}_0 \rightarrow \tilde{\mathcal{M}}_0 \) such that \( w\Delta = \Delta \tilde{\tau}(w) \) for all \( w \in \tilde{\mathcal{M}}_0 \).

We already know that \( \Delta_B \) is central in \( A^+(B_n) \). Thus we have \( \sigma_i \Delta = \Delta \sigma_i \) for all \( i = 1, 2, \ldots, n-1 \). We may also check (by performing the calculation in \( A^+(B_n) \)) that

\[
\Delta \approx DU^{n-1} \quad \text{where } U := \sigma_1D\sigma_1\sigma_2\sigma_3 \cdots \sigma_{n-1}.
\]

Recall that \( \tau \) denotes the automorphism of \( M \) such that, at the level of words, \( xD \sim D\tau(x) \) for all \( x \in S^* \). Observe also that \( xU \approx U\tau(x) \) for all \( x \in S^* \) (or more loosely speaking, for all \( x \in M \).
We now define $\tilde{\tau} : \tilde{M}_0 \to \tilde{M}_0$ such that
\[
\tilde{\tau}(\sigma_i) = \sigma_i \quad \text{for } i = 1, 2, \ldots, n - 1
\]
\[
\tilde{\tau}(x) = \tau^n(x) \quad \text{for all } x \in M.
\]
It is easily seen that $\tilde{\tau}$ is a monoid isomorphism. Moreover, for all $x \in M$,
\[
x\Delta \approx xDU^{n-1}
\approx D\tau(x)U^{n-1}
\approx DU^{n-1}\tau^n(x)
\approx \Delta\tilde{\tau}(x),
\]
and $\sigma_i\Delta \approx \Delta\sigma_i$ for all $i = 1, 2, \ldots, n - 1$. Thus Condition (4) of Lemma 5.12 is satisfied, and $\Delta$ is a Garside element. 

\section{Appendix}
Throughout this section, we shall denote by $F_n$ the free group of rank $n$, and by $x_1, \ldots, x_n$ some fixed basis for $F_n$.

\textbf{Definition.} According to Shpilrain’s terminology \cite{Shp}, a \textit{Wada representation of type (1)} is an Artin type representation associated to $(\mathbb{Z}, h)$, where $h$ is a non-zero integer. Such a representation will be denoted by $\rho_h^{(1)} : B_n \to \text{Aut}(F_n)$. It is determined by
\[
\rho_h^{(1)}(\sigma_k)(x_i) =
\begin{cases}
    x_i & \text{if } i \neq k, k + 1 \\
    x_k^{-h}x_{k+1}x_k^h & \text{if } i = k \\
    x_k & \text{if } i = k + 1
\end{cases}
\]
The Wada representation of type (2) is the representation \( \rho^{(2)} : B_n \to \text{Aut}(F_n) \) determined by

\[
\rho^{(2)}(\sigma_k)(x_i) = \begin{cases} 
  x_i & \text{if } i \neq k, k+1 \\
  x_kx_{k+1}^{-1}x_k & \text{if } i = k \\
  x_k & \text{if } i = k + 1
\end{cases}
\]

and the Wada representation of type (3) is the representation \( \rho^{(3)} : B_n \to \text{Aut}(F_n) \) determined by

\[
\rho^{(3)}(\sigma_k)(x_i) = \begin{cases} 
  x_i & \text{if } i \neq k, k+1 \\
  x_k^2x_{k+1} & \text{if } i = k \\
  x_k^{-1}x_{k-1} & \text{if } i = k + 1
\end{cases}
\]

**Proposition A.1.** (1) Let \( k, l \in \mathbb{Z} \setminus \{0\} \). Then \( \rho^{(1)}_k \) and \( \rho^{(1)}_l \) are equivalent if and only if \( l = \pm k \).

(2) \( \rho^{(2)} \) and \( \rho^{(3)} \) are equivalent.

(3) Let \( k \in \mathbb{Z} \setminus \{0\} \). Then \( \rho^{(2)} \) and \( \rho^{(1)}_k \) are not equivalent.

The following lemmas A.2 and A.3 are preliminary results to the proof of Proposition A.1.

**Lemma A.2.** Consider the action of \( B_n \) on \( F_n \) via the representation \( \rho^{(1)}_h \). For all \( i = 1, \ldots, n-1 \), we have

\[
F_n^{(\sigma_i)} = \langle x_1, \ldots, x_{i-1}, x_i^hx_i^hx_i, x_{i+1}, \ldots, x_n \rangle.
\]

**Proof.** Write \( F_n = C \ast D \), where \( C = \langle x_i, x_{i+1} \rangle \) and \( D = \langle x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_n \rangle \). Both groups, \( C \) and \( D \), are invariant by the action of \( \sigma_i \). Moreover, \( \sigma_i \) is the identity on \( D \) and acts on \( C \) by \( x_i \mapsto x_i^{-h}x_{i+1}x_i^h \), \( x_{i+1} \mapsto x_i \). In particular, \( F_n^{(\sigma_i)} = C^{(\sigma_i)} \ast D \).

Let \( u \in C^{(\sigma_i)} \). Write

\[
u = x_i^{m_1}x_{i+1}^{m_1}\ldots x_i^{m_r}x_{i+1}^{m_r},
\]

where \( r \geq 1 \), \( m_1, \ldots, m_{r-1}, n_2, \ldots, n_r \in \mathbb{Z} \setminus \{0\} \), and \( m_r, n_1 \in \mathbb{Z} \). First, suppose \( n_1 \neq 0 \). Then

\[
\sigma_i(u) = x_i^{-h}x_i^{m_1}x_{i+1}^{m_1}\ldots x_i^{m_r}x_{i+1}^{m_r}x_i^h = u
\]

thus

\[-h = n_1, \quad n_1 = m_1, \quad \ldots, \quad n_r = m_r, \quad m_r + h = 0,\]

hence \( u = (x_i^hx_i^h)^{-r} \). Now, suppose \( n_1 = 0 \). Then

\[
\sigma_i(u) = x_i^{m_1-h}x_i^{m_2}x_{i+1}^{m_2}\ldots x_i^{m_r}x_{i+1}^{m_r}x_i^h,
\]

thus

\[m_1 - h = 0, \quad m_1 = n_2, \quad n_2 = m_2, \quad \ldots, \quad n_r = m_r + h, \quad \text{and} \quad m_r = 0,\]

hence \( u = (x_i^hx_i^h)^{r-1} \).

**Lemma A.3.** Consider the action of \( B_n \) on \( F_n \) via \( \rho^{(1)}_h \). Then \( F_n^{B_n} \) is the cyclic subgroup of \( F_n \) generated by \( x_n^h \ldots x_2^hx_1^h \).
Proof. Let \( u \in F_n^{B_n} \). We have \( u \in F_n^{(\sigma_i)} \) for all \( i = 1, \ldots, n - 1 \), thus, by Lemma A.2, the reduced form of \( u \) satisfies the following properties:

- all the exponents are either equal to \( h \) or equal to \(-h\);
- if \( i \neq 1 \), then \( x_i^h \) is followed by \( x_{i-1}^{-h} \), and, if \( i \neq n \), then \( x_i^h \) is preceded by \( x_{i+1}^h \);
- if \( i \neq n \), then \( x_i^{-h} \) is followed by \( x_{i+1}^{-h} \), and, if \( i \neq 1 \), then \( x_i^{-h} \) is preceded by \( x_{i-1}^{-h} \).

Clearly, these properties hold if and only if \( u \) is of the form \( u = (x_n^h \ldots x_2^h x_1^h)^r \) with \( r \in \mathbb{Z} \). \( \square \)

Proof of Proposition A.1. (1) Let \( k \in \mathbb{Z}\setminus\{0\} \). Let \( \phi : F_n \to F_n \) be the automorphism determined by \( \phi(x_i) = x_i^{-1} \) for all \( i = 1, \ldots, n \). One can easily verify that

\[
\phi^{-1} \circ \rho_k^{(1)}(\sigma_i) \circ \phi = \rho_{-k}^{(1)}(\sigma_i)
\]

for all \( i = 1, \ldots, n - 1 \), thus \( \rho_k \) and \( \rho_{-k} \) are equivalent.

Let \( k, l > 0 \). For a group \( G \), we denote by \( H_1(G) \) the abelianization of \( G \), and, for a subgroup \( H \) of \( G \), we denote by \( \langle \langle H \rangle \rangle \) the normal subgroup of \( G \) generated by \( H \). By Lemma A.3, we have

\[
F_n/\langle \langle F_n^{\rho_k^{(1)}(B_n)} \rangle \rangle \simeq \langle x_1, \ldots, x_n \mid x_n^k \ldots x_2^k x_1^k = 1 \rangle,
\]

hence

\[
H_1(F_n/\langle \langle F_n^{\rho_k^{(1)}(B_n)} \rangle \rangle) \simeq (\mathbb{Z}/k\mathbb{Z}) \times \mathbb{Z}^{n-1}.
\]

So, if \( \rho_k^{(1)} \) and \( \rho_l^{(1)} \) are equivalent, then \( (\mathbb{Z}/k\mathbb{Z}) \times \mathbb{Z}^{n-1} \simeq (\mathbb{Z}/l\mathbb{Z}) \times \mathbb{Z}^{n-1} \), thus \( k = l \).

(2) Write

\[
y_i = x_1^2 \ldots x_{i-1}^2 x_i \quad \text{for } i = 1, \ldots, n.
\]

One can easily verify that

\[
\rho^{(3)}(\sigma_k)(y_i) = \begin{cases} 
    y_i & \text{if } i \neq k, k+1 \\
    y_{k+1} & \text{if } i = k \\
    y_{k+1}y_k^{-1}y_{k+1} & \text{if } i = k + 1
\end{cases}
\]

Let \( \phi : F_n \to F_n \) be the automorphism determined by \( \phi(x_i) = y_{n-i+1} \) for \( i = 1, \ldots, n \), and let \( \mu : B_n \to B_n \) be the automorphism determined by \( \mu(\sigma_i) = \sigma_{n-i} \) for \( i = 1, \ldots, n - 1 \). From the expression of \( \rho^{(3)}(\sigma_k)(y_i) \) given above, follows

\[
\phi^{-1} \circ \rho^{(3)}(\sigma_i) \circ \phi = \rho^{(2)}(\mu(\sigma_i))
\]

for all \( i = 1, \ldots, n - 1 \), thus \( \rho^{(2)} \) and \( \rho^{(3)} \) are equivalent.

(3) Let \( k > 0 \). For \( u \in F_n \), we denote by \([u]\) the element of \( H_1(F_n) \simeq \mathbb{Z}^n \) represented by \( u \). We have

\[
\rho^{(2)}(\sigma_1)(x_1) = (t + 1)[x_1] - t[x_2]
\]

for all \( t \in \mathbb{N} \). On the other hand, \( \rho_k^{(1)}(\beta) \) has finite order as an automorphism of \( H_1(F_n) \), for all \( \beta \in B_n \). This shows that \( \rho^{(2)} \) and \( \rho_k^{(1)} \) are not equivalent. \( \square \)
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