POSITIVITY OF GIBBS STATES ON DISTANCE-REGULAR GRAPHS

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ABSTRACT. We study criteria which ensure that Gibbs states (often also called generalized vacuum states) on distance-regular graphs are positive. Our main criterion assumes that the graph can be embedded into a growing family of distance-regular graphs. For the proof of the positivity we then use polynomial hypergroup theory and translate this positivity into the problem whether for \( x \in [-1, 1] \) the function \( n \mapsto x^n \) has a positive integral representation w.r.t. the orthogonal polynomials associated with the graph. We apply our criteria to several examples. For Hamming graphs and the infinite distance-transitive graphs we obtain a complete description of the positive Gibbs states.

1. Introduction

It is well known that vacuum states on distance-regular graphs lead to interesting models in quantum probability; see e.g. the monograph [12] of Hora and Obata and references there. Besides these classical states one can also study Gibbs states on these graphs. These states, which are also called generalized vacuum states in some references like [12], are related to Gibbs kernels on these graphs; see Section 2.3 of [12]. These Gibbs kernels on a distance-regular graph \( \Gamma = (V, E) \) with vertex set \( V \), edge set \( E \), natural distance function \( d \), and diameter \( D \in \mathbb{N} \cup \{\infty\} \) are defined by \( Q_x(u, v) := x^{d(u,v)} \) \( (u, v \in V) \) for \( x \in [-1, 1] \) (with the convention \( 0^0 = 1 \)). It can be easily seen that states associated with \( Q_x \) are positive in the sense of quantum probability if and only if \( Q_x \) is positive semidefinite, i.e., if

\[
\sum_{i,j=1}^n c_i \overline{c_j} x^{d(u_i, u_j)} \geq 0
\]

for all \( n \in \mathbb{N}, u_1, \ldots, u_n \in V \) and \( c_1, \ldots, c_n \in \mathbb{C} \). While this positivity is obvious for the vacuum kernel \( Q_0 \) as well for \( Q_1 \), the set

\[
P_1 := \{ x \in [-1, 1] : Q_x \text{ positive semidefinite} \}
\]

is unknown for general distance-regular graphs. On the other hand, in some simple cases like \( D = 1 \), \( P_1 \) can be determined easily; see [12]. Moreover, it was shown by Haagerup [11] that \( P_1 = [-1, 1] \) for the infinite homogeneous trees, and by the work of Bozejko [5, 6], it was shown that \( [0, 1] \subset P_1 \) holds for some classes of distance-regular graphs like the Hamming and Johnson graphs; see also [12] for further details.

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In this paper we present a further approach to prove the positivity of some Gibbs kernels which gives some additional informations for further classes of examples. The idea of the approach here is as follow: For each distance-regular graph $\Gamma$ there is a canonical associated (usually finite) sequence of orthogonal polynomials $(P_k)_k$ with an orthogonality measure with some support $S_{\Gamma} \subset [-1,1]$ where $S_{\Gamma}$ is the spectrum of the graph. A standard argument with a Bochner-type theorem for polynomial hypergroups (see [4, 13, 16] for the background) now yields that a kernel $Q: \Gamma \times \Gamma \to \mathbb{R}$ of the form $Q(u,v) = f(d(u,v))$ for some function $f: \{0,1,\ldots\} \to \mathbb{R}$ is positive semidefinite if and only if there is a (necessarily unique) positive measure $\mu$ on $S_{\Gamma}$ with

$$Q(u,v) = f(d(u,v)) = \mu(d(u,v)) = \int_{S_{\Gamma}} P_{d(u,v)}(w) \, d\mu(w) \quad (u, v \in \Gamma).$$

We now assume that for a given distance-regular graph $\Gamma$ there is a sequence $(\Gamma_n)_n$ of distance-regular graphs containing $\Gamma$ such that the coefficients of the three-term-recurrence relations of the associated orthogonal polynomials converge (after a suitable normalization) to certain constants for $n \to \infty$, which means that the polynomials $P_n^k(x)$ associated with the graphs $\Gamma_n$ tend to $x^k$ for all finite $k \leq D$ and $n \to \infty$. We shall call this condition the infinite embedding property in Section 3. With standard arguments on positive semidefiniteness it then follows that all accumulation points $x \in [-1,1]$ of the union of the supports of all $S_{\Gamma_n}$ are contained in $P_{\Gamma}$.

We shall see that this seemingly difficult criterion works quite well for several examples. In particular, we obtain a precise description of $P_{\Gamma}$ in this way for all Hamming graphs in Section 5, and we can also extend the result of Haagerup [11] to a precise description of $P_{\Gamma}$ for all known infinite, locally finite distance-regular graphs in Section 7. Furthermore, for the Johnson graphs we reprove the known fact $[0,1] \subset P_{\Gamma}$ in Section 5, and for the Grassmann graphs over the finite fields $\mathbb{F}_q$ ($q$ a prime power), which are sometimes also called $q$-Johnson graphs, we obtain

$$\{q^{-j} : j \in \mathbb{N}_0 \} \cup \{0\} \subset P_{\Gamma} \quad \text{with} \quad \mathbb{N}_0 := \{0,1,2,\ldots\}.$$

We expect that our criterion can be also applied to further classes of finite distance-regular graphs which are discussed e.g. in [1, 3, 7, 9, 14].

We also point out that the approach of this paper for distance-regular graphs via the associated orthogonal polynomials can be extended to examples of higher rank, i.e., objects like buildings or suitable classes association schemes for which the analogues of associated spherical functions are multivariate orthogonal polynomials; see [3, 22, 23, 24] and references there for a further reading. In this case, however, one first has to agree about canonical extensions of the notions of Gibbs states.

The paper is organized as follows. Section 2 contains some known background material on distance-regular graphs and the associated orthogonal polynomials and polynomial hypergroups. In Section 3 we then study the infinite embedding property and its consequences. Sections 4-7 then are devoted to several classes of examples.

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2. DISTANCE-REGULAR GRAPHS AND ASSOCIATED POLYNOMIAL HYPERGROUPS

In this introductory section we recapitulate some notations and facts on finite and infinite distance-regular graphs and the associated orthogonal polynomials and
We also briefly discuss Gibbs states in this context. The main sources are [12, 14] for Gibbs states on distance-regular graphs, [4, 13] for basics on hypergroups, and [22, 23] for the connections between distance-regular graphs and the associated polynomial hypergroups, where these connections are discussed there for association schemes in a more general context.

We begin with distance-regular graphs:

2.1. **Distance-regular graphs.** Consider an undirected, connected graph $\Gamma = (V, E)$ with an at most countable set $V$ of vertices and a set $E$ of edges. Assume that the graph has no loops and is locally finite, i.e., each vertex has only finitely many neighbors. Let $d : V \times V \to \mathbb{N}_0$ be the usual distance and $D := \sup_{x,y \in V} d(x,y) \in \mathbb{N}_0 \cup \{\infty\}$ the diameter of $\Gamma$. Let $X := \{k \in \mathbb{N}_0 : k \leq D\}$ be the set of all possible distances on $\Gamma$.

The graph $\Gamma = (V, E)$ is called distance-regular if for all $i, j, k \in X$ and $x, y \in V$ with $d(x, y) = k$, the number of all vertices $z \in V$ with $d(x, z) = i$ and $d(y, z) = j$ is independent of the choice of $x, y$. Hence, for $d(x, y) = k$, the numbers $p_{i,j}^k := |\{z \in V : d(x, z) = i, d(y, z) = j\}|$ depend only on $i, j, k \in X$.

For $k \in X$ we now consider the adjacency matrices $A_k$ with entries

$$(A_k)_{x,y} := \begin{cases} 1 & \text{if } d(x, y) = k \\ 0 & \text{otherwise.} \end{cases}$$

In particular $A_0$ is the identity matrix, and all $A_k$ are locally finite, i.e., all rows and columns have only finitely many non-zero entries. Moreover, for all $i, j \in X$

$$A_i \cdot A_j = \sum_{k=|i-j|}^{i+j} p_{i,j}^k A_k$$

where we agree in such sums that $k \in X$ holds. In this way, the $\mathbb{C}$-linear span $\mathcal{A}(\Gamma) := \text{span}(A_i : i \in X)$ becomes a commutative and associative algebra consisting of symmetric matrices where this algebra is generated by $A_1$. In particular, for each $k \in X$, $A_k$ is a polynomial of degree $k$ in $A_1$. For the details see [12].

We now fix some vertex $o \in V$. The associated Gibbs state (or generalized vacuum state) with parameter $q \in [-1, 1]$ is defined as the linear functional $\varphi_q : \mathcal{A}(\Gamma) \to \mathbb{C}$ with

$$\varphi_q(B) := \sum_{x \in V} q^{d(x,o)} B_{x,o} \quad \text{for } B \in \mathcal{A}(\Gamma)$$

where we agree that $0^0 := 1$. With this agreement we have $\varphi_0(B) = B_{o,o} = B_{x,x}$ for $x \in V$, i.e., $\varphi_0$ is the classical vacuum state. Please notice that it not clear (except for $q = 0, 1$) that these Gibbs states are states on $\mathcal{A}(\Gamma)$ in the sense of quantum probability, i.e., that the (not necessarily locally finite) matrices $Q_q$ with

$$(Q_q)_{x,y} := q^{d(x,y)} \quad (x,y \in V)$$

are positive semidefinite, i.e., that for all $n \in \mathbb{N}, x_1, \ldots, x_n \in V$ and $c_1, \ldots, c_n \in \mathbb{C},$

$$\sum_{i,j=1}^n c_i c_j q^{d(x_i,x_j)} \geq 0.$$
also for $q$ in larger compact subintervals of $[-1, 1]$. This is the case, for instance, for the infinite homogeneous trees and the Johnson and Hamming graphs. For these and many further classes of examples we refer to [12].

We now study this positivity in the more general context of “invariant” kernels $Q : V \times V \to \mathbb{C}$ on distance-regular graphs where $Q(x, y)$ depends on $d(x, y)$ only like for the Gibbs states. For this we translate the positivity of $Q$ into the problem whether the associated function $f : X \to \mathbb{C}$ with $Q(x, y) = f(d(x, y))$ is positive definite on $X$ in some hypergroup sense.

To explain this we recapitulate some facts from [23] for commutative association schemes and the associated commutative hypergroups where we restrict our attention here to distance-regular graphs and the associated polynomial hypergroups.

For this, we start with the adjacency matrices $A_k$ ($k \in X$) of a distance-regular graph $\Gamma$. As above we fix $o \in V$ and define

$$\omega_k := |\{ x \in V : d(x, o) = k \}| = p^0_{k, k} \leq \infty \quad (k \in X)$$

as the numbers of vertices with distance $k$ from $o$. We now form the renormalized stochastic matrices

$$\tilde{A}_k := \frac{1}{\omega_k} A_k \quad (k \in X).$$

Then, for $i, j \in X$,

$$\tilde{A}_i \cdot \tilde{A}_j = \sum_{k=|i-j|}^{i+j} \tilde{p}_{i,j}^k \tilde{A}_k \quad \text{with} \quad \tilde{p}_{i,j}^k = \frac{\omega_k}{\omega_i \omega_j} p_{i,j}^k \geq 0 \quad (2.1)$$

where obviously $\sum_{k=|i-j|}^{i+j} \tilde{p}_{i,j}^k = 1$ holds. To get a stochastic interpretation of this identity, we recapitulate the following identity, which is well-known for association schemes (see [3] in the finite case or Lemma 3.5 in [22] in general):

For all $i, j, k \in X$:

$$\omega_k p_{i,j}^k = \omega_i p_{i,j}^k. \quad (2.2)$$

Therefore, for all $i, j, k \in X$,

$$\tilde{p}_{i,j}^k = \frac{1}{\omega_j} p_{i,j}^k \geq 0. \quad (2.3)$$

This means that for fixed $i, j \in X$, the $\tilde{p}_{i,j}^k$ form the distribution for the distance from $o$, when we first make a random jump of size $i$ from $o$, and jump then again in an independent way with size $j$.

With these notations we now define a convolution $\ast$ on the Banach space $M_b(X)$ of all bounded signed measures on $X$. In fact, for point measures $\delta_i, \delta_j$ ($i, j \in X$) we put

$$\delta_i \ast \delta_j := \sum_{k=|i-j|}^{i+j} \tilde{p}_{i,j}^k \delta_k, \quad (2.4)$$

and extend $\ast$ to $M_b(X)$ in a unique bilinear, weakly continuous way (the latter is necessary for $X = \mathbb{N}_0$ only). In this way, $(M_b(X), \ast)$ becomes a commutative Banach-$*$-algebra with the involution $\ast$ with $\mu^*(B) := \mu(B^c)$ ($B \subset X, \mu \in M_b(X)$).

More precisely, $(X, +)$ becomes a polynomial hypergroup, i.e., a special commutative discrete hypergroup. For the convenience of the reader we here briefly recapitulate the definition of these objects and collect a few facts:
Definition 2.1. Let $X \neq \emptyset$ be an at most countable discrete set, and $*$ a weakly continuous, commutative, associative, bilinear product on the Banach space $M_b(X)$ of all bounded signed measures on $X$ with the following properties:

(1) For all $i, j \in X$, $\delta_i \ast \delta_j$ is a probability measure on $D$ with finite support.
(2) There exists a neutral element $e \in E$ with $\delta_i \ast \delta_e = \delta_i \ast \delta_j = \delta_i$ for $i \in X$.
(3) There is an involution $x \mapsto \bar{x}$ on $X$ such that for all $i, j \in X$, $e \in \text{supp}(\delta_i \ast \delta_j)$ holds if and only if $i = j$.
(4) If for $\mu \in M_b(X)$, $\mu^-$ denotes the image of $\mu$ under the involution, then $(\delta_i \ast \delta_j)^- = \delta_j \ast \delta_i^-$ for all $i, j \in X$.

Then $(X, \ast)$ is called a commutative discrete hypergroup. $(X, \ast)$ is called symmetric if the involution is the identity.

We collect some facts on commutative discrete hypergroups from [4, 13].

Definitions and facts 2.2. Let $(X, \ast)$ be a commutative discrete hypergroup.

(1) The identity $e$ and the involution $\ast$ above are unique.
(2) $(M_b(X), \ast)$ is a commutative Banach-$\ast$-algebra with the involution $\mu \mapsto \mu^*$ with $\mu^*(A) := \overline{\mu(A)}$ for $A \subset X$.
(3) $(X, \ast)$ admits a Haar measure $\omega \in M^+(X)$, i.e., a nontrivial positive measure $\omega = \sum_{i \in X} \omega(i) \delta_i$, which satisfies $\omega = \omega \ast \delta_j = \delta_j \ast \omega$ for all $j \in X$. This Haar measure is unique up to a multiplicative constant. It can be defined by

$$\omega(i) := \frac{1}{\delta_i \ast \delta_i^*(\{e\})} \quad \text{for} \quad i \in X. \quad (2.5)$$

From now on we use this Haar measure on $(X, \ast)$.

(4) Let $C(X)$ be the space of all $\mathbb{C}$-valued functions on $X$, and $C_b(X)$ the subspace of all bounded functions on $X$. For $f \in C(X)$, $i, j \in X$ we write

$$f(i \ast j) := (\delta_i \ast \delta_j)(f) := \sum_{k \in X} f(k) \cdot (\delta_i \ast \delta_j)(\{k\}).$$

(5) The spaces of all (bounded) non-trivial multiplicative continuous functions on $(X, \ast)$ are

$$\chi(X, \ast) := \{\alpha \in C(X) : \alpha \neq 0, \alpha(i \ast j) = \alpha(i) \cdot \alpha(j) \quad \text{for all} \quad i, j \in X\}$$

and $\chi_b(X, \ast) := \chi(X, \ast) \cap C_b(X)$. Moreover,

$$\hat{X} := (X, \ast)^\wedge := \{\alpha \in \chi_b(X, \ast) : \alpha(\bar{i}) = \overline{\alpha(i)} \quad \text{for all} \quad i \in X\}$$

is the dual space of $(X, \ast)$. Its elements are called characters. In particular the constant function $1$ is a character.

If all spaces carry the topology of pointwise convergence, then $\chi_b(X, \ast)$ and $\hat{X}$ are compact.

(6) For $f \in L^1(X) := L^1(X, \omega)$ and $\mu \in M_b(X)$, their Fourier(-Stieltjes) transforms are

$$\hat{f}(\alpha) := \int_X f(i)\alpha(i) d\omega(i), \quad \hat{\mu}(\alpha) := \int_X \overline{\alpha(i)} d\mu(i) \quad (\alpha \in \hat{X}).$$

$\hat{f}$ and $\hat{\mu}$ are continuous on $\hat{X}$ with $\|\hat{f}\|_\infty \leq \|f\|_1$, $\|\hat{\mu}\|_\infty \leq \|\mu\|_{TV}$. 


(7) There exists a unique positive measure \( \pi \in \mathcal{M}^+(\hat{X}) \), called the Plancherel measure on \( \hat{X} \), such that the Fourier transform \( \hat{\cdot} : L^1(X) \cap L^2(X) \to C(\hat{X}) \cap L^2(X, \pi) \) is an \( L^2 \)-isometry. The Fourier transform \( \hat{\cdot} \) can be extended uniquely to an isometric isomorphism between \( L^2(X) \) and \( L^2(\hat{X}, \pi) \). Moreover, by (2.3), \( \pi \) is a probability measure.

Notice that, different from abelian groups, the support \( S := \text{supp} \, \pi \) may be a proper closed subset of \( \hat{X} \). In this case, we even have \( 1 \notin S \) for all known examples.

However, if \( (X, \ast) \) is finite, then \( S = \hat{X} \) holds with \( |\hat{X}| = |X| \).

(8) For \( f \in L^1(\hat{X}, \pi) \), \( \mu \in \mathcal{M}_b(\hat{X}) \), their inverse Fourier transforms are given by
\[
\hat{f}(i) := \int_S f(\alpha) \alpha(i) \, d\pi(\alpha), \quad \hat{\mu}(i) := \int_D \alpha(i) \, d\mu(\alpha) \quad (i \in X)
\]
with \( \hat{f} \in C_0(X) \) (i.e. \( \hat{f} \) disappears at \( \infty \)) , \( \hat{\mu} \in C_0(\hat{X}) \) and \( \|\hat{f}\|_\infty \leq \|f\|_1 \), \( \|\hat{\mu}\|_\infty \leq \|\mu\|_{TV} \).

(9) A function \( f \in C_0(X) \) is called positive definite on \( (X, \ast) \) if for all \( n \in \mathbb{N} \), \( x_1, \ldots, x_n \in X \) and \( c_1, \ldots, c_n \in \mathbb{C} \), \( \sum_{k,l=1}^n c_k \bar{c}_l \cdot f(x_k \ast \bar{x}_l) \geq 0 \). Obviously, all characters \( \alpha \in \hat{X} \) are positive definite.

The following theorem of Bochner (see [13]) will be essential for our paper:

**Theorem 2.3.** A function \( f \in C_0(X) \) is positive definite if and only if \( f = \hat{\mu} \) for some positive measure \( \mu \in \mathcal{M}_b^+(\hat{X}) \). This measure is unique, and \( \mu \) is a probability measure if and only if \( \hat{\mu}(e) = 1 \) holds.

In the context of homogeneous trees (i.e., infinite hypergroups \( (X, \ast) \)), we also need the following variant; see [20]:

**Proposition 2.4.** Let \( f \in C_0(X) \) be a positive definite function. Then \( f = \hat{\mu} \) for some \( \mu \in \mathcal{M}_b^+(\hat{X}) \) with \( \text{supp} \, \mu \subset S \) if and only if \( f \) is the pointwise limit of positive definite functions on \( X \) with finite support.

We next turn our attention to polynomial hypergroups:

**Definition 2.5.** Let \( D \in \mathbb{N} \cup \{\infty\} \) and \( X := \{i \in \mathbb{N}_0 : i \leq D\} \). A commutative discrete hypergroup is called a polynomial hypergroup of diameter \( D \), if there are numbers \( a_i, b_i, c_i \geq 0 \) with \( a_i + b_i + c_i = 1 \) \( (i \in X) \) with \( a_0 = 1, b_0 = c_0 = 0, a_i > 0 \) for \( i \in X \) with \( i < D \), \( a_D = 0 \) for \( D < \infty \), such that
\[
\delta_i \ast \delta_i = a_i \delta_{i+1} + b_i \delta_i + c_i \delta_{i-1} \quad (i \in X).
\]

If we compare this definition with the convolution (2.4) above in the context of distance-regular graphs, we see that each distance-regular graph \( \Gamma \) leads to a polynomial hypergroup structure \((X, \ast)\). In particular, by (2.3), the parameters \( a_i, b_i, c_i \geq 0 \) from Definition 2.5 here are given by
\[
a_i = \tilde{p}_{1,i}^{i+1} = \frac{1}{\omega_i} p_{1,i}^{i+1}, \quad b_i = \tilde{p}_{1,i}^{i+1}, \quad c_i = \tilde{p}_{1,i}^{i+1} = \frac{1}{\omega_i} p_{1,i}^{i+1} = \frac{1}{\omega_i} p_{1,i}^{i-1}. \quad (2.6)
\]

We collect some classical facts on polynomial hypergroups; see [3, 16]. Let \( (X, \ast) \) be a polynomial hypergroup with diameter \( D \). Define the polynomials \( (P_i)_{i \in X} \) recursively by
\[
P_0 = 1, \quad P_1(x) = x, \quad P_i \cdot P_i = a_i P_{i+1} + b_i P_i + c_i P_{i-1} \quad (0 < i < D). \quad (2.7)
\]
For \( x \in \mathbb{C} \), the functions \( i \rightarrow P_i(x) \) then form the multiplicative functions on \((X,\ast)\), where we need some additional restriction in the finite case which we discuss below. We first consider the infinite case.

### 2.2. Multiplicative functions on infinite polynomial hypergroups.

In this case, the 3-term-recurrence \([2.6]\) and Favard’s theorem (see e.g. \([8]\)) show that \((P_i)_{i \geq 0}\) is a sequence of orthogonal polynomials. Moreover, if we define the functions \( \alpha_i(x) := P_i(x) \) for \( x \in \mathbb{C}, i \in X = \mathbb{N}_0 \), then \( \alpha_1 = 1 \), \( \chi(X,\ast) = \{ \alpha_x : x \in \mathbb{C} \} \) and

\[
\hat{X} = \{ \alpha_x : x \in \mathbb{R} \text{ with } (P_i(x))_{i \geq 0} \text{ bounded} \}.
\]

If we identify \( \chi(X,\ast) \) and \( \hat{X} \) with \( \mathbb{C} \) and the corresponding subset respectively, then the topology of pointwise convergence agrees with the usual topology. In particular, \( \hat{X} \) may be regarded as a compact subset of \([-1,1]\). We remark that then the Plancherel measure \( \pi \) from Section 2.2(7) is the orthogonality measure of \((P_i)_{i \geq 0}\). It is a probability measure whose support is contained in \( \hat{X} \).

### 2.3. Multiplicative functions on finite polynomial hypergroups.

Here, the 3-term-recurrence \([2.6]\) also leads to a sequence \((P_i)_{i=0,\ldots,D}\) of orthogonal polynomials. If we test whether the functions \( \alpha_x \) as above are multiplicative, we land up with the condition

\[
P_i(x) \cdot P_D(x) = b_D P_D(x) + c_D P_{D-1}(x)
\]

which is in fact solved for precisely \( D + 1 \) different points

\[
-1 < x_D < x_{D-1} < \ldots < x_0 = 1.
\]

In this case we identify \( \chi(X,\ast) = \hat{X} \) with \( \{x_0,\ldots,x_D\} \). The Plancherel measure \( \pi \) then is the orthogonality measure; its support is \( \{x_0,\ldots,x_D\} \).

We now turn to the central positivity result of this section. We state it in the finite and infinite case separately, as the infinite case is more involved. It follows from Theorems 2.6 and 2.7 respectively, and it is shown in Section 6 of \([23]\) in the context of association schemes.

**Theorem 2.6.** Let \( \Gamma = (V,E) \) be a finite distance-regular graph with diameter \( D \). Then for a function \( f : X \rightarrow \mathbb{R} \) the following statements are equivalent:

1. The kernel \( Q_f : V \times V \rightarrow \mathbb{R} \) with \( Q_f(u,v) := f(d(u,v)) \) is positive semidefinite;
2. The function \( f \) is positive definite on the polynomial hypergroup \((X,\ast)\) associated with \( \Gamma \);
3. There is a (unique) positive measure \( \mu \) on \( \hat{X} = \{x_0,\ldots,x_D\} \) with \( f = \hat{\mu} \).

**Theorem 2.7.** Let \( \Gamma = (V,E) \) be an infinite distance-regular graph with associated polynomial hypergroup \((X,\ast)\). Let \( f : X \rightarrow \mathbb{R} \) be a function and \( Q_f \) the associated kernel on \( V \) as before. Then:

1. If \( Q_f \) is positive semidefinite, then \( f \) is a bounded positive definite function on \((X,\ast)\), and there is a (unique) positive measure \( \mu \) on \( \hat{X} \) with \( f = \hat{\mu} \).
2. If \( f \) is the pointwise limit of positive definite functions on \((X,\ast)\) with finite supports, or if \( f = \hat{\mu} \) for some positive measure \( \mu \) on \( \hat{X} \) with \( \text{supp } \mu \subset \text{supp } \pi \), then \( Q_f \) is positive semidefinite.

Please notice that part (2) in Theorem 2.7 is weaker than a complete converse statement of (1). We shall see below that the complete converse statement of (1) is not correct for some examples; see Section 7. We also remark that infinite polynomial hypergroups have unbounded positive definite functions.
Example 2.8. For an integer \( N \geq 2 \) let \( \Gamma \) be the complete graph with \( N \) vertices, i.e., all vertices in \( V := \{1, \ldots, N\} \) are neighbored. Here the convolution \( * \) of the associated polynomial hypergroup \(( X = \{0, 1\}, *) \) satisfies
\[
\delta_0 * \delta_0 = \delta_0, \quad \delta_0 * \delta_1 = \delta_1 * \delta_0 = \delta_1, \quad \delta_1 * \delta_1 = \frac{1}{N-1} \delta_0 + \frac{N-2}{N-1} \delta_1.
\]
Now let \( x \in [-1,1] \). Then the Gibbs kernel \( Q_x \) on \( V \) with \( Q_x(u,v) = x^{d(u,v)} \) is positive semidefinite if and only if the function \( f_x(i) := x^i \) \((i = 0, 1)\) is positive definite on \((X, *)\), and this is the case if and only if the matrix
\[
\begin{pmatrix}
1 & x & x \\
\frac{1}{x} & \frac{1}{1-x} & \frac{x}{1-x} \\
\frac{1}{x} & \frac{x}{1-x} & 1
\end{pmatrix}
\]
is positive semidefinite which is the case by an elementary calculus precisely for \( x \in [-1/(N-1), 1] \).

Example 2.9. The 6 vertices of an octahedron with its 12 edges form a distance-regular graph \( \Gamma = (V,E) \) of diameter \( D = 2 \); see Example 2.17 in [12]. The convolution \( * \) of the associated polynomial hypergroup \((\{0,1,2\}, *)\) with identity 0 satisfies
\[
\delta_2 * \delta_2 = \delta_0, \quad \delta_1 * \delta_2 = \delta_1, \quad \delta_1 * \delta_1 = \frac{1}{4} \delta_0 + \frac{1}{4} \delta_2 + \frac{1}{2} \delta_1.
\]
Therefore, by Theorem 2.6 the Gibbs state \( Q_x(u,v) := x^{d(u,v)} \) is positive semidefinite on \( V \) if and only if the matrix
\[
\begin{pmatrix}
1 & x & x^2 \\
x & \frac{1}{4} + \frac{x}{2} & \frac{x}{2} \\
x^2 & \frac{1}{2}x & 1
\end{pmatrix}
\]
is positive semidefinite. A computation of the principal minors shows that this is the case precisely for \( x \in [-2 + \sqrt{3}, 1] \); see also Example 2.17 in [12].

3. The infinite embedding property and positive Gibbs states

We start with the following simple definition:

Definition 3.1. Let \( \Gamma_n := (V_n, E_n) \) with \( n = 1, 2 \) be distance-regular graphs. \( \Gamma_1 \) is called a subgraph of \( \Gamma_2 \) (for short, \( \Gamma_1 \subset \Gamma_2 \)), if \( V_1 \subset V_2 \), and if the distance function \( d_2 \) on \( V_2 \) restricted to \( V_1 \) is the distance function on \( V_1 \).

Notice that this subgraph property implies that vertices in \( V_1 \) are neighbored in \( V_1 \) if and only if they are in \( V_2 \), but that the converse statement usually does not hold.

The following is obvious for distance-regular graphs \( \Gamma_n := (V_n, E_n) \):

Lemma 3.2. Let \( \Gamma_1 \subset \Gamma_2 \).

1. The diameters \( D_1, D_2 \) and the associated spaces \( X_1, X_2 \) from subsection 2.1 satisfy \( D_1 \leq D_2 \) and \( X_1 \subset X_2 \).
2. If \( Q : V_2 \times V_2 \to \mathbb{R} \) is a positive semidefinite kernel, then its restriction \( Q|_{V_1} : V_1 \times V_1 \to \mathbb{R} \) is also positive semidefinite.

This fact together with Theorems 2.6 and 2.7 implies:
Lemma 3.3. Let $\Gamma_n := (V_n, E_n)$ with $n = 1, 2$ be distance-regular graphs with $\Gamma_1 \subset \Gamma_2$. Let $f \in C_b(X_2)$ be a function on the polynomial hypergroup $(X_2, *_2)$ associated with $\Gamma_2$ such that the associated kernel $Q_f : V_2 \times V_2 \to \mathbb{R}$ with $Q_f(u, v) := f(d_2(u, v))$ is positive semidefinite. Then the restriction $f|_{X_1}$ of $f$ is positive definite on the polynomial hypergroup $(X_1, *_1)$ associated with $\Gamma_1$, and the associated kernel $Q_f|_{X_1} = Q_f|_{V_1}$ on $V_1$ is positive semidefinite.

In particular, by Theorem 2.6.

Proposition 3.4. Let $\Gamma_n := (V_n, E_n)$ with $n = 1, 2$ be finite distance-regular graphs with $\Gamma_1 \subset \Gamma_2$. Let $f \in C_b(X_2)$ be a positive definite function on the polynomial hypergroup $(X_2, *_2)$ associated with $\Gamma_2$. Then $f|_{X_1}$ is positive definite on the polynomial hypergroup $(X_1, *_1)$ associated with $\Gamma_1$, and the associated kernel $Q_f|_{X_1} = Q_f|_{V_1}$ on $V_1$ is positive semidefinite.

Moreover, by Theorem 2.7.

Proposition 3.5. Let $\Gamma_n := (V_n, E_n)$ with $n = 1, 2$ be infinite distance-regular graphs with $\Gamma_1 \subset \Gamma_2$. Let $f \in C_b(X_2)$ be a function on the polynomial hypergroup $(X_2, *_2)$ associated with $\Gamma_2$ such that $f$ is the pointwise limit of finitely supported positive definite functions on $(X_2, *_2)$, or that $f = \tilde{\mu}$ for some positive measure $\mu$ on $X_2$ with supp $\mu \subset$ supp $\pi_2$ (with the Plancherel measure $\pi_2$ on the dual $\hat{X}_2$ of $(X_2, *_2)$). Then $f|_{X_1}$ is positive definite on the polynomial hypergroup $(X_1, *_1)$ associated with $\Gamma_1$, and the associated kernel $Q_f|_{X_1} = Q_f|_{V_1}$ on $V_1$ is positive semidefinite.

The following definition is central for this section:

Definition 3.6. We say that a distance-regular graph $\Gamma := (V, E)$ with diameter $D$ has the infinite embedding property if there is a sequence of distance-regular graphs $(\Gamma_n := (V_n, E_n))_{n \in \mathbb{N}}$ with $\Gamma \subset \Gamma_n$ for $n \in \mathbb{N}$ with the following property:

Let $(X_n, *_n)$ be the polynomial hypergroups associated with the graphs $\Gamma_n$, and let $a_i^{(n)}, b_i^{(n)}, c_i^{(n)} \geq 0$ with $a_i^{(n)} + b_i^{(n)} + c_i^{(n)} = 1$ be the associated 3-term recurrence coefficients as in (2.4) for $i < D$. These coefficients satisfy

$$\lim_{n \to \infty} a_i^{(n)} = 1, \quad \lim_{n \to \infty} b_i^{(n)} = \lim_{n \to \infty} c_i^{(n)} = 0 \quad (i < D). \quad (3.1)$$

We notice, that by (2.0), the condition (3.1) means on the level of the distance-regular graphs $\Gamma_n$ with the numbers $p_{i,j}^{k_1(n), k_2(n)}$, $\omega_{i}^{(n)}$ from subsection 2.1 that

$$\lim_{n \to \infty} p_{i,j}^{k_1(n), k_2(n)} = 1 \quad (i < D). \quad (3.2)$$

We show in the next sections that this infinite embedding property holds for some classical series of distance-regular graphs. We here next discuss some consequences from this property. We first consider the finite case:

Theorem 3.7. Let $\Gamma := (V, E)$ be a finite distance-regular graph with the infinite embedding property with a corresponding sequence of distance-regular graphs $(\Gamma_n := (V_n, E_n))_{n \in \mathbb{N}}$ and associated polynomial hypergroups $(X_n, *_n)$ with the associated dual spaces $\hat{X}_n \subset [-1, 1]$. Let

$$P := \{x \in [-1, 1] : x \text{ is an accumulation point of } \bigcup_{n \in \mathbb{N}} \hat{X}_n\}.$$
Then for all \( x \in P \), the function \( f_x(i) := x^i \) is positive definite on the hypergroup \((X, \ast)\) associated with \( \Gamma \), and the kernel \( Q_{f_x}(u, v) := x^{d(u, v)} \) is positive semidefinite on \( \Gamma \).

**Proof.** Let \( x \in P \). Then there exists a subsequence \((n_l)_{l \in \mathbb{N}} \subset \mathbb{N}\) and a sequence \((x_l)_{l \in \mathbb{N}} \subset [-1, 1]\) with \( \lim_{l \to \infty} x_l = x \) such that for all \( l \), \( x_l \in \tilde{X}_{n_l} \) holds. For each \( l \) consider the orthogonal polynomials \((P_{n_l}^*(i))\), associated with the graph \( \Gamma_{n_l} \). As the character \( f_{x_l}(i) := P_{n_l}^*(i)(x_l) \) is positive definite on \((X_{n_l}, \ast_{n_l})\), Proposition 3.4 implies that that the function \( f_{x_l} \) is positive definite on \((X, \ast)\). On the other hand, \( \lim_{l \to \infty} x_l = x \), and the limit (3.4) for the 3-term recurrence imply that the functions \( f_{x_l} \) tend to the function \( f_x \) from the theorem on \( X \). This yields that \( f_x \) is also positive definite on \((X, \ast)\) as claimed. The last statement follows from Theorem 2.6.

In the infinite case, the situation is slightly more complicated as Theorem 2.7 is weaker than Theorem 2.6. The arguments of the preceding theorem thus only lead to the following weaker result.

**Theorem 3.8.** Let \( \Gamma := (V, E) \) be an infinite distance-regular graph with the infinite embedding property with the corresponding graphs \((\Gamma_n := (V_n, E_n))_{n \in \mathbb{N}}, (X_n, \ast_n)\), \( \tilde{X}_n \subset [-1, 1] \), and \( P \subset [-1, 1] \) as in the preceding theorem. Then for all \( x \in P \), the function \( f_x(i) := x^i \) is positive definite on the hypergroup \((X, \ast)\) associated with \( \Gamma \).

On the other hand, in Section 7 we shall study a class of infinite distance-regular graphs for which we shall determine all \( x \in [-1, 1] \), for which the Gibbs kernels \( Q_{f_x} \) are positive semidefinite. To our knowledge, this class covers all known infinite distance-regular graphs.

In the end of this section we turn to another question. It will turn out for most examples of distance-regular graphs \( \Gamma \) in the next sections that the Gibbs kernels \( Q_{f_x} \) are positive semidefinite for all \( x \in [-1, 1] \). In this context the following result may be of interest, which shows that a weaker result already always ensures that the \( Q_x \) are positive semidefinite for all \( x \in [-1, 1] \). We restrict our attention here to the finite case, as the infinite case will be treated completely in Section 7 without this result.

**Proposition 3.9.** Let \( \Gamma \) be a finite distance-regular graph. Assume that there is a sequence \((x_n)_{n \in \mathbb{N}} \subset [0, 1]\) with \( \lim_{n \to \infty} x_n = 1 \) such that the Gibbs kernels \( Q_{x_n} \) on \( \Gamma \) as above are positive semidefinite for all \( n \). Then \( Q_x \) is positive semidefinite for each \( x \in [0, 1] \).

**Proof.** The set of positive semidefinite kernels on \( \Gamma \) is closed under pointwise limits, nonnegative linear combinations, and pointwise products. This, the power series of the exponential function, and the assumption yield that for each \( n \) and each \( t \geq 0 \) the kernel
\[
\tilde{Q}_{t, n}(u, v) := \exp\left(\frac{t}{1 - x_n}(Q_{x_n}(u, v) - 1)\right) = e^{-x_n \cdot \exp\left(\frac{t}{1 - x_n}(Q_{x_n}(u, v)\right)}
\]
with \( u, v \in \Gamma \) is positive semidefinite. As
\[
\lim_{n \to \infty} \tilde{Q}_{t, n}(u, v) = \exp(-t \cdot d(u, v)) = Q_{e^{-t}}(u, v),
\]
we obtain that \( Q_x \) is positive semidefinite for each \( x \in [0, 1] \). Finally, the case \( x = 0 \) is obvious.
In the next sections we discuss several examples.

4. Hamming graphs and Krawtchouk polynomials

Let \( N \geq 2, D \geq 1 \) be integers. Let \( V := \{1, 2, \ldots, N\}^D \) be equipped with the metric
\[
d(u, v) := |\{i = 1, \ldots, D : u_i \neq v_i\}| \quad (u, v \in V).
\]
It is well known and can be easily seen that the graph \((V, E)\) with \( E = \{\{u, v\} \in V^2 : d(u, v) = 1\}\) is distance-regular with diameter \( D \), and with
\[
\omega_i(p^0, i) = \binom{D}{i} \cdot (N-1)^i, \quad p^1_{i+1} = (D-i)(N-1) \quad (i = 0, \ldots, D); \quad (4.1)
\]
see e.g. Section 5.1 of [12]. We denote this graph as Hamming graph \( H(D, N) \).

For the Hamming graph \( H(D, N) \), the 3-term recurrence coefficients in (2.6) satisfy
\[
a_i = p^1_{i+1} = \frac{D-i}{\omega_1} = \frac{D-i}{D} \quad (i = 0, \ldots, D). \quad (4.2)
\]
Furthermore, for fixed \( N, D, \tilde{D} \) with \( \tilde{D} > D \) we may regard \( H(D, N) \) as subgraph of \( H(\tilde{D}, N) \) in a canonical way. Therefore, by (4.2) we see that all Hamming graphs \( H(D, N) \) have the infinite embedding property when we choose the graphs \( \Gamma_n \) in Definition (3.6) as \( \Gamma_n := H(D + n, N) \) for \( n \in \mathbb{N} \).

We next identify the set \( P \) in Theorem 3.7. For this we recall that the orthogonal polynomials \((P_i)_{i=0, \ldots, D}\) associated with the graphs \( H(D, N) \) are Krawtchouk polynomials up to affine-linear transformations.

For this we recapitulate e.g. from Szegö [18] or Section 5 of [10] that for \( p \in [0, 1[ \) the Krawtchouk polynomials \( K_i(x) = K_i(x; D, p) \) \((i = 0, \ldots, D)\) can be defined by
\[
K_i(x) := K_i(x; D, p) := 2F_1(-i, -1; -D; 1/p) = \sum_{k=0}^{N} \frac{(-i)_k(-x)_k}{(-D)_k k!} \left(\frac{1}{p}\right)^k \quad (4.3)
\]
By [18, 10], these polynomials have the following properties:

(1) \( K_i(x) = K_x(l) \) and \( K_0(x) = K_0(0) = 1 \) for \( x, l \in \{0, 1, \ldots, D\} \);

(2) \( K_1(x) = 1 - x/(Dp) \);

(3) Recurrence relation: For \( l \in \{0, 1, \ldots, D\} \),
\[
K_1 \cdot K_l = \frac{D-l}{D} K_{l+1} + \frac{2p-1}{p} \frac{l}{D} K_l + \frac{1-p}{D} K_{l-1}.
\]

(4) Orthogonality: For \( l, m \in \{0, 1, \ldots, D\} \),
\[
\sum_{x=0}^{D} K_i(x) K_m(x) \cdot \binom{D}{x} p^x (1-p)^{D-x} = \binom{D}{l}^{-1} \left(\frac{1-p}{p}\right)^l \delta_{l,m}.
\]

We now put \( p := (N-1)/N \) for the Krawtchouk polynomials and compare (2), (3) with the recurrence (2.7) for the \((P_i)_{i=0, \ldots, D}\) with the coefficients (4.2). This shows that
\[
K_i(x) = P_i \left(1 - \frac{N_x}{D(N-1)}\right) \quad i = 0, \ldots, D.
\]
In particular, the orthogonality measures of the polynomials \((P_i)_{i=0, \ldots, D}\) have the supports
\[
\left\{1 - \frac{N_x}{D(N-1)} : x = 0, 1, \ldots, D\right\}.
\]
We thus conclude that the set $P$ in Theorem 3.7 here is given by
\[ P = [1 - N/(N - 1), 1] = [-1/(N - 1), 1]. \]
Therefore, Theorems 3.7 and 2.6 lead to $(1) \Rightarrow (2) \iff (3)$ in the following result:

**Theorem 4.1.** For a Hamming graph $H(D, N)$ and $x \in \mathbb{R}$ the following statements are equivalent:

1. $x \in [-1/(N - 1), 1]$;
2. The function $f_i(x) := x^i$ ($i \in X$) is positive definite on the hypergroup $(X, \ast)$ associated with $H(D, N)$;
3. The Gibbs kernel $Q_{f_i}(u, v) := x^{d(u,v)}$ is positive semidefinite on $H(D, N)$.

**Proof.** Assume that (3) holds. As the graph $H(1, N)$ is just the complete graph of valency $N$ considered in Example 2.8 and as $H(1, N)$ is a subgraph of $H(D, N)$, (1) follows from Lemma 5.1.

All further conclusions were already shown above. \(\square\)

5. **Johnson graphs**

We here mainly follow Section 6 of [12]. Let $v \geq 1$, $D \geq 0$ be integers with $D \leq v/2$. Consider the Johnson graph $J(v, D) = (V, E)$ with
\[ V := \{x \subset \{1, \ldots, v\} : |x| = D\}, \quad E := \{\{x, y\} \subset V : |x \cap y| = d - 1\}. \]

It is well known and can be checked by elementary combinatorics that $J(v, D)$ is distance-regular with diameter $D$ with the parameters
\[ \omega_i = \binom{D}{i} \binom{v-D}{v-D-i}, \quad P_i, i+1 = (D - i)(v - D - i) \quad (i = 0, \ldots, D); \quad (5.1) \]
see e.g. Lemmas 6.6 and 6.8 of [12].

For the Johnson graph $J(v, D)$ the 3-term recurrence coefficients $a_i$ in (2.6) thus satisfy
\[ a_i = \frac{P_i, i+1}{\omega_i} = \frac{P_i, i+1}{\omega_i} = \frac{(D - i)(v - D - i)}{D(v-D)} \quad (i = 0, \ldots, D). \quad (5.2) \]

We next show that the Johnson graphs have the infinite embedding property. For this we first observe:

**Lemma 5.1.** Let $v, D$ as above. Then for integers $m, n \geq 0$, the Johnson graph $J(v, D)$ can be regarded as subgraph of $J(v + m + n, D + n)$.

**Proof.** Clearly, $J(v, D)$ may be regarded as subgraph of $J(v + m, D)$, when we only consider sets in the vertex set of $J(v + m, D)$ of size $D$, which only contain elements of $\{1, \ldots, v\}$.

On the other hand, we also can regard $J(v, D)$ as subgraph of $J(v + n, D + n)$. For this we notice that $J(v, D)$ is isomorphic with the Johnson graph $J(v, v - D)$ via taking the complement of a subset of size $D$ in $\{1, \ldots, v\}$ where $J(v, v - D)$ is defined as above (in fact, the condition $D \leq v/2$ was assumed only in order to avoid this doubling of the examples). In this way we obtain
\[ J(v, D) \sim J(v, v-D) \subset J(v+n, v-D) = J(v+n, (v+n)-(D+n)) \sim J(v+n, D+n) \]
which shows that we can regard $J(v, D)$ as subgraph of $J(v + n, D + n)$.

A combination of both arguments now completes the proof of the lemma. \(\square\)
**Proposition 5.2.** Let $v, D$ as above. Then the Johnson graph $J(v, D)$ has the infinite embedding property with the sequence $(\Gamma_n := J(v + 2n, D + n))_n$ of graphs where $J(v, D) \subset J(v + 2n, D + n)$ holds for all $n \geq 1$ as described in Lemma 5.1.

Moreover, the associated set $P$ from Theorem 3.7 is

$$P = [0, 1].$$

**Proof.** By (5.2), the coefficients $a_i^{(n)}$ of the 3-term recurrence of the orthogonal polynomials associated with the graphs $J(v + 2n, D + n)$ according to (2.6) satisfy

$$a_i^{(n)} = p_i^{(n)}(x) = \sum_{j=0}^{n} \frac{\omega_i^{(n)}}{(D + n)(v - D + n)} \rightarrow 1$$

for all $i = 1, \ldots, D - 1$ and $n \to \infty$. This proves the infinite embedding property.

We next identify the set $P$ from Theorem 5.1. For this we recapitulate from the literature (see Section 3.2 of [3] or Section 9.1 of [7]) that the orthogonal polynomials associated the Johnson graph $J(v, D)$ as described in Section 2 have a orthogonality measure which has the support

$$S_{v, D} := \left\{ \frac{D(v - D) - j(v - j + 1)}{D(v - D)} = 1 - \frac{j(v - j + 1)}{D(v - D)} : \ j = 0, \ldots, D \right\}.$$ 

We now consider the numbers in this set for the Johnson graphs $J(v + 2n, D + n)$. Then these numbers are given by

$$x_{j, n} := 1 - \frac{j(v + 2n - j + 1)}{(D + n)(v - D + n)} \quad (j = 0, \ldots, D + n).$$

As

$$x_{j, n} - x_{j+1, n} = \frac{v + 2n - 2j}{(D + n)(v - D + n)} \in \left[0, \frac{v + 2n}{(D + n)(v - D + n)}\right],$$

we see that the distances of the $x_{j, n}$ tend to 0 uniformly in $j$ for $n \to \infty$ with $x_{0, n} = 1$ as largest and $x_{D + n, n}$ as smallest element where $x_{D + n, n} \to 0$ holds. This yields $P = [0, 1]$. \hfill \Box

As in the preceding section, this proposition and Theorem 5.1 lead to the following result on Gibbs states which is well known; see e.g. Proposition 6.27 of [12] where this result is shown via the quadratic embedding test of Bozejko.

**Theorem 5.3.** Let $v, D$ as above and $x \in [0, 1]$. Then the Gibbs kernel $Q_{f_x}(u, v) := x^d(u, v)$ is positive semidefinite on the Johnson graph $J(v, D)$.

Please notice that the interval $[0, 1]$ is usually not optimal; see for instance Example 2.9 which is just the Johnson graph $J(2, 4)$.

6. $q$-analogues of Johnson graphs

These graphs, which are also called Grassmann graphs in [7], are as follows: Let $S$ be a vector space of dimension $v \in \mathbb{N}$ over the finite field $\mathbb{F}_q$ with $q$ some power of a prime. Let $V$ be the finite set of all $D$-dimensional subspaces of $S$ with some positive integer $D \leq v/2$. Let

$$E := \{ \{x, y\} \subset V : \ \text{dim}(x \cap y) = D - 1\}.$$
It is well known (see e.g. Section III.6 of [3] or [7]) that these graphs $J_q(v, D) := (V, E)$ are distance-transitive with diameter $D$, and that the distance of subspaces $x, y \in V$ is given by

$$d(x, y) = D - \dim(x \cap y).$$

The associated parameters in the sense of Section 2 can be also obtained from the literature. In particular, we have

$$\omega_1 = p_{1,1}^0 = \frac{(q^D - 1)(q^{v-D} - 1)}{(q - 1)^2} \cdot q, \quad p_{1,i+1}^i = q^{2i+1} \cdot \frac{(q^{D-i} - 1)(q^{v-D-i} - 1)}{(q - 1)^2},$$

(6.1)

see e.g. Section III.6(ii) of [3]. For the $q$-Johnson graph $J_q(v, D)$ the coefficients $a_i$ in the 3-term-recurrence (2.6) thus satisfy

$$a_i = p_{1,i+1}^i = \frac{q^D - q^i}{\omega_1} \cdot \frac{(q^D - q^i)(q^{v-D} - q^i)}{(q^D - 1)(q^{v-D} - 1)} \quad (i = 0, \ldots, D).$$

(6.2)

We next observe that the $q$-Johnson graphs also have the infinite embedding property like the usual Johnson graphs. As the proof is completely analog to the preceding section, we skip the proof.

**Lemma 6.1.** Let $v, D, q$ as above. Then for integers $m, n \geq 0$, the $q$-Johnson graph $J_q(v, D)$ can be regarded as subgraph of $J_q(v + m + n, D + n)$.

Moreover, $J_q(v, D)$ has the infinite embedding property with the sequence of graphs $(J_q(v + 2n, D + n))_{n \geq 1}$.

Please notice that the last observation follows from the fact that the coefficients $a_i$ from (6.2) for the graphs $J_q(v + 2n, D + n)$ tend to 1 for fixed $i$ for $n \to \infty$.

In order to determine the associated set $P$ from Theorem 3.7, we recapitulate from Theorem 9.3.3 of [7] that the eigenvalues of the adjacency matrix $A_1$ for $J_q(v, D)$ are given by

$$\theta_j = q^{j+1} \frac{(q^D - 1)(q^{v-D-j} - 1)}{(q - 1)^2} - \frac{q^j - 1}{q - 1} \quad (j = 0, \ldots, D).$$

Taking our normalizations from Section 2 into account together with the equation (6.1) for $\omega_1$, we see that the orthogonality measure of the orthogonal polynomials associated with $J_q(v, D)$ is supported by the points

$$\frac{1}{(q^D - 1)(q^{v-D} - 1)} \left( \frac{(q^D - 1)(q^{v-D-j} - 1)}{(q - 1)^2} - \frac{q^j - 1}{q - 1} \right) (q^{j+1} + q^{v-j} - q^D - q^j - 1)/(q - 1)$$

(6.3)

for $j = 0, \ldots, D$. In particular, for $j = 0$, these points are equal to 1, and for $j = D$ equal to

$$\frac{q - 1}{q(q^{v-D} - 1)}.$$

We now consider the parameters $(v + 2n, D + n)$ instead of $(v, D)$ for $n \to \infty$. In this case the expression in the second line of (6.3) behaves like

$$q^{-j} + O(q^{-n}) \quad (n \to \infty)$$

where the terms $O(q^{-n})$ are uniform in $j = 0, \ldots, D$. We thus obtain

$$P = \{q^{-j} : j \in \mathbb{N}_0 \} \cup \{0\}.$$
This and Theorem [5.7] lead to the following result for the Gibbs kernels which was stated in a more general setting and proved by other methods recently in Proposition 3.1 of [14].

**Theorem 6.2.** Let \( v, D, q \) as above and \( x \in \{ q^{-j} : j \in \mathbb{N}_0 \} \cup \{ 0 \} \). Then the Gibbs kernel \( Q_x(u, v) := x^{d(u, v)} \) is positive semidefinite on \( J_q(v, D) \).

This set \( \{ q^{-j} : j \in \mathbb{N}_0 \} \cup \{ 0 \} \) for the \( q \)-Johnson graphs is obviously not optimal.

**Example 6.3.** Consider the \( q \)-Johnson graph \( J_2(2, 4) \). Here \( D = 2 \), and the set \( V \) of all 2-dimensional subspaces of \( \mathbb{F}_2^4 \) has 15 \( \cdot \) 14/6 = 35 elements. Moreover, it can be easily checked that \( \omega_0 = 1, \omega_1 = 18, \omega_2 = 16 \).

The convolution of point measures on \( X = \{ 0, 1, 2 \} \) with 0 as identity has the form
\[
\delta_1 \ast \delta_1 = \frac{1}{\omega_1} \delta_0 + \alpha_1 \delta_1 + \beta_1 \delta_2, \quad \delta_2 \ast \delta_2 = \frac{1}{\omega_2} \delta_0 + \beta_2 \delta_1 + \alpha_2 \delta_2, \quad \delta_1 \ast \delta_2 = \gamma_1 \delta_1 + \gamma_1 \delta_2
\]
with suitable \( \alpha_1, \beta_1, \alpha_2, \beta_2, \gamma_1, \gamma_2 > 0 \). To compute these coefficients, we claim that the spaces
\[
A := \text{span}((1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 0, 1)) \in V
\]
with \( d(A, B) = 1 \) admit 9 spaces \( C \in V \) with \( d(A, C) = d(B, C) = 1 \). In fact there are 5 possibilities of spaces \( C \) of the form \( C = \text{span}((1, 0, 0, 0), u) \in V \) with \( u \in V \setminus (A \cup B) \) and 4 spaces \( C \) of the form \( C = \text{span}(u, v) \in V \) with \( u \in A \setminus (A \cap B) \) and \( v \in B \setminus (A \cap B) \). As \( \delta_1 \ast \delta_1 \) is a probability measure, we conclude that
\[
\delta_1 \ast \delta_1 = \frac{1}{18} (\delta_0 + 9 \delta_1 + 8 \delta_2).
\]

In summary, we can also determine the other convolution products in this combinatorial way. However, the following approach may be more efficient. Our hypergroup \((X, \ast)\) fits into the class of hermitian hypergroups of order 3 considered in Section 4 of Wildberger [24]. By p. 100 of [24] we thus have
\[
\alpha_1 = 1 - \frac{1 + \gamma_1 \omega_2}{\omega_1}, \quad \alpha_2 = 1 - \frac{1 + \gamma_2 \omega_1}{\omega_2}.
\]
These relations and (6.4) now yield \( \gamma_1 \), then \( \gamma_2 \), and finally \( \alpha_2 \) and \( \beta_2 \). In summary, we get
\[
\delta_1 \ast \delta_2 = \frac{1}{2} (\delta_1 + \delta_2), \quad \delta_2 \ast \delta_2 = \frac{1}{16} (\delta_0 + 9 \delta_1 + 6 \delta_2).
\]

Therefore, by Theorem 2.6, the Gibbs kernel \( Q_x(u, v) := x^{d(u, v)} \) is positive semidefinite on \( V \) if and only if the matrix
\[
D := \begin{pmatrix}
1 & x & x^2 \\
x & \frac{1}{18}(1 + 9x + 8x^2) & \frac{1}{2}(x + x^2) \\
x^2 & \frac{1}{2}(x + x^2) & \frac{1}{18}(1 + 9x + 6x^2)
\end{pmatrix}
\]
is positive semidefinite. As
\[
\det \begin{pmatrix}
1 & x \\
x & \frac{1}{18}(1 + 9x + 8x^2)
\end{pmatrix} = \frac{1}{18} (1 - x)(10x - 1)
\]
and
\[
\det D = \frac{1}{288} (1 - x)^2(1 - 2x)(1 + 4x)(16x^2 + 18x + 1),
\]
and as the zeros of $16x^2 + 18x + 1$ are approximately $x_1 \sim -1.06$ and $x_2 \sim -0.059$, we obtain that the Gibbs kernel $Q_x$ is positive if and only if $x \in [x_2, 1/2] \cup \{1\}$ holds. It is quite interesting that the general approach of the preceding sections and Proposition 3.1 of [14] “detect” the upper bound $1/2$ of the interval. This example is also interesting as it is an example of a distance-regular graph where $Q_x$ is not positive semidefinite for some $x \in [0, 1]$.

7. THE INFINITE DISTANCE-TRANSITIVE GRAPHS

The set of all infinite distance-transitive graphs is parametrized by two parameters as follows by Macpherson [17].

Let $a, b \geq 2$ be integers, and $C_b$ the complete graph graph with $b$ vertices. Consider the infinite graph $\Gamma := \Gamma(a, b)$ where precisely $a$ copies of $C_b$ are tacked together at each vertex in a tree-like way. For $b = 2$, $\Gamma$ is the homogeneous tree of valency $a$.

We now consider the associated polynomial hypergroups $(X = \mathbb{N}_0, \ast)$. Some counting shows (see [19]) that the convolution $\ast$ satisfies

$$\delta_m \ast \delta_n = \sum_{k = |m-n|}^{m+n} g_{m,n,k} \delta_k \in M^1(\mathbb{N}_0) \quad (m, n \in \mathbb{N}_0) \quad (7.1)$$

with

$$g_{m,n,m+n} = \frac{a-1}{a} > 0, \quad g_{m,n,|m-n|} = \frac{1}{a(a-1)^{m+n-1}(b-1)^{m+n}} > 0,$$

$$g_{m,n,|m-n|+2k+1} = \frac{b-2}{a(a-1)^{m+n-k-1}(b-1)^{m+n-k}} \geq 0 \quad (k \leq m \land n-1),$$

$$g_{m,n,|m-n|+2k+2} = \frac{a-2}{a(a-1)^{m+n-k-1}(b-1)^{m+n-k-1}} \geq 0 \quad (k \leq m \land n-2).$$

In particular,

$$g_{n,1,n+1} = \frac{a-1}{a}, \quad g_{n,1,n} = \frac{b-2}{a(b-1)}, \quad g_{n,1,n-1} = \frac{1}{a(b-1)}. \quad (7.2)$$

We next define associated orthogonal polynomials $(P_n^{(a,b)})_{n \geq 0}$ according to the general 3-term recurrence (2.7) by

$$P_0^{(a,b)} := 1, \quad P_1^{(a,b)}(x) := x$$

and

$$P_n^{(a,b)} P_{n+1}^{(a,b)} = \frac{1}{a(b-1)} P_{n-1}^{(a,b)} + \frac{b-2}{a(b-1)} P_n^{(a,b)} + \frac{a-1}{a} P_{n+1}^{(a,b)} \quad (n \geq 1). \quad (7.3)$$

These polynomials satisfy

$$P_m^{(a,b)} P_n^{(a,b)} = \sum_{k = |m-n|}^{m+n} g_{m,n,k} P_k^{(a,b)} \quad (m, n \geq 0). \quad (7.4)$$

Please notice that the polynomials $(P_n^{(a,b)})_{n \geq 0}$ differ by some affine-linear transformation from the corresponding notations in [19] [21] [22] [23] [4]. More precisely, the polynomials

$$\tilde{P}_n^{(a,b)}(x) := P_n^{(a,b)}(T(x)) \quad (n \in \mathbb{N}_0) \quad \text{with} \quad T(x) := \frac{2}{a} \sqrt{\frac{a-1}{b-1}} \cdot x + \frac{b-2}{a(b-1)} \quad (7.5)$$
are the polynomials considered e.g. in [22] [23]. Some formulas can be expressed more easily in terms of \((\hat{P}^{(a,b)}_n)_{n \geq 0}\). For instance,
\[
\hat{P}^{(a,b)}_n \left( \frac{z + z^{-1}}{2} \right) = \frac{c(z)z^n + c(z^{-1})z^{-n}}{((a - 1)(b - 1))^{n/2}} \quad \text{for } z \in \mathbb{C} \setminus \{0, \pm 1\} \tag{7.6}
\]
with
\[
c(z) := \frac{(a-1)z-z^{-1}+(b-2)(a-1)^{1/2}(b-1)^{-1/2}}{a(z-z^{-1})}. \tag{7.7}
\]
We define
\[
\tilde{s}_0 := \tilde{s}^{(a,b)}_0 := \frac{2 - a - b}{2\sqrt{(a - 1)(b - 1)}}, \quad s_0 := T(\tilde{s}_0) = -\frac{1}{b-1}
\]
\[
\tilde{s}_1 := \tilde{s}^{(a,b)}_1 := \frac{ab - a - b + 2}{2\sqrt{(a - 1)(b - 1)}}, \quad s_1 := T(\tilde{s}_1) = 1. \tag{7.8}
\]
By [22], the \(\hat{P}^{(a,b)}_n\) fit into the Askey-Wilson scheme (pp. 26–28 of [2]). In particular, by [2], the normalized orthogonality measure \(\tilde{\rho} = \tilde{\rho}^{(a,b)} \in M^1(\mathbb{R})\) of the \(\hat{P}^{(a,b)}_n\) is
\[
\tilde{d}^{(a,b)}_\rho(x) = \frac{b-a}{b} \delta_{\tilde{s}_0} + \begin{cases} \frac{a}{2\pi} \cdot \frac{(1 - x^2)^{1/2}}{\tilde{s}_1 - x}(x - \tilde{s}_0) & \text{for } a \geq b \geq 2 \\ \frac{b-a}{b} \delta_{\tilde{s}_0} & \text{for } b > a \geq 2 \end{cases}
\]
In summary, if we identify the dual space \(\hat{X}\) of \((X, *)\) via the polynomials \(P^{(a,b)}_n\) as in Section 2 with a compact subset of \(\mathbb{R}\), we have the following observations:

1. \(\hat{X} = [T(-\tilde{s}_1), 1]\).
2. The support \(S\) of the Plancherel measure is equal to
\[
\left[ \frac{b - 2}{a(b - 1)} - \frac{2}{a} \cdot \frac{a - 1}{b - 1} \cdot \frac{b - 2}{a(b - 1)} + \frac{2}{a} \cdot \sqrt{\frac{a - 1}{b - 1}} \right]
\]
for \(a \geq b \geq 2\), and for \(b > a \geq 2\) the additional isolated point \(s_0\) appears.
3. \(S = \hat{X}\) holds precisely for \(a = b = 2\).
4. \(s_0 = T(-\tilde{s}_1)\) holds precisely if \(a = 2\) or \(b = 2\).

The following theorem from [22] is central for our considerations:

**Theorem 7.1.** Let \(x \in \mathbb{R}\). Then the kernel
\[
\Gamma \times \Gamma \to \mathbb{R}, \quad (v_1, v_2) \mapsto P^{(a,b)}_{d(v_1,v_2)}(x)
\]
is positive semidefinite if and only if \(x \in [s_0, 1]\) holds.

This theorem has the following consequence which is interesting in view of the differences in the general Theorems 2.6 and 2.7 in the finite and infinite case:

**Corollary 7.2.** Consider a graph \(\Gamma := \Gamma(a, b)\) with parameters \(a, b \geq 2\) as above, and let \((X, *)\) be the associated polynomial hypergroup. Then the following statements are equivalent:

1. \(a = 2\) or \(b = 2\);
(2) If \( f : X \to \mathbb{R} \) is a bounded positive definite function on \((X, *)\), then the
associated kernel \( Q_f : V \times V \to \mathbb{R} \) with \( Q_f(u, v) := f(d(u, v)) \) is positive
semidefinite.

Proof. (2) \( \implies \) (1) is trivial by (4) above and Theorem 7.1. On the other hand,
as each bounded positive definite function on \((X, *)\) has the form \( f = \mu \) for some
positive measure \( \mu \) on \( X \) by Bochner’s theorem 2.3, (1) \( \implies \) (2) also follows from
(4) above and Theorem 7.1. \( \square \)

We are now ready for the main result of this section which generalizes a corre-
sponding result of Haagerup [11] for homogeneous trees.

Theorem 7.3. Consider a graph \( \Gamma(a, b) \) with \( a, b \geq 2 \) as above. Let \( x \in \mathbb{R} \). Then
the Gibbs kernel \( Q_x(u, v) := x^{d(u, v)} \) is positive semidefinite on \( \Gamma(a, b) \) if and only if
\( x \in [\frac{1}{\sqrt{a-1}}, 1] \) holds.

Proof. It follows from (7.2) that \( \Gamma = \Gamma(a, b) \) has the infinite embedding property
with the sequence \((\Gamma(a + n, b))_n\) of graphs with \( \Gamma \subset \Gamma(a + n, b) \). As the parameter
\( s_0 = \frac{1}{\sqrt{a-1}} \) of the graphs \( \Gamma(a + n, b) \) does not depend on \( n \), we see that the set \( P \)
in Theorem 3.7 is \([\frac{1}{\sqrt{a-1}}, 1]\). Hence, by Theorem 3.7, \( Q_x \) is positive semidefinite on
\( \Gamma(a, b) \) for \( x \in [\frac{1}{\sqrt{a-1}}, 1] \).

Now assume that \( Q_x \) is positive semidefinite on \( \Gamma(a, b) \). Then \( Q_x \) is also positive
semidefinite on the complete graph \( C_b \) with \( b \) vertices, and the results in Example
2.8 yield \( x \in [\frac{1}{\sqrt{a-1}}, 1] \). \( \square \)

In the end of the paper we briefly recapitulate some results on the spectral
measures \( \mu_x \in M^1([-1, 1]) \) in the case of homogeneous trees (i.e. \( b = 2 \)) from Letac
[16] which seem to be unknown for a broader audience. We recapitulate that for
fixed valency \( a \geq 2 \) these measures are characterized via

\[
x^n = \mu_x(n) = \int_{-1}^{1} P_n^{(a, 2)}(z) \, d\mu_x(z) \quad (n \in \mathbb{N}_0, x \in [-1, 1]). \tag{7.11}
\]

By p. 136 of [16] we have

\[
d\mu_x(z) = \frac{a}{2\pi} \cdot \frac{1 - x^2}{1 + (a - 1)x^2 - azx} \cdot \frac{\sqrt{4(a-1)/a^2}}{1 - z^2} \, dz \tag{7.12}
\]
for \( |x| \leq \frac{1}{\sqrt{a-1}} \). Notice here that

\[
1 + (a - 1)x^2 - azx > 0 \quad \text{for} \quad |x| < \frac{1}{\sqrt{a-1}}, \quad z \in [-2\sqrt{a-1}/a, 2\sqrt{a-1}/a],
\]
and that \( 1 + (a - 1)x^2 - azx = 0 \) for \( x = \pm \frac{1}{\sqrt{a-1}} \) and \( z = (\text{sign } x) \cdot \frac{2\sqrt{a-1}}{a} \), in which
case this denominator also appears as a factor in the square-root-part of (7.12),
i.e., the density of \( \mu_x \) here has a singularity at one boundary point. Furthermore,
it is mentioned on p. 136 of [16] that for \( |x| \in ]1/\sqrt{a-1}, 1[ \), \( \mu_x \) has also a density
similar to (7.12) with one additional atom. For \( x = \pm 1 \) we clearly have \( \mu_x = \delta_x \).

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