CLASS NUMBER AND REGULATOR COMPUTATION IN CUBIC FUNCTION FIELDS

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Abstract. We present computational results on the divisor class number and the regulator of a cubic function field over a large base field. The underlying method is based on approximations of the Euler product representation of the zeta function of such a field. We give details on the implementation for purely cubic function fields of signatures (3, 1) and (1, 1; 1, 2), operating in the ideal class group and infrastructure of the function field, respectively. Our implementation provides numerical evidence of the computational effectiveness of this algorithm. With the exception of special cases, such as purely cubic function fields defined by superelliptic curves, the examples provided are the largest divisor class numbers and regulators ever computed for a cubic function field over a large prime field. The ideas underlying the optimization of the class number algorithm can in turn be used to analyze the distribution of the zeros of the function field’s zeta function. We provide a variety of data on a certain distribution of the divisor class number that verify heuristics by Katz and Sarnak on the distribution of the zeroes of the zeta function.

1. Introduction and Motivation

One of the more difficult problems in arithmetic geometry is the computation of the order of the group of rational points on the Jacobian of an algebraic curve over a finite field. In this paper, we give results on the application and optimization of a method of Scheidler and Stein [33, 34] to compute the order of the Jacobian of a purely cubic function field over a large base field. We describe details of this implementation for the cases in which the function field has signature (3, 1) and (1, 1; 1, 2), operating in the ideal class group and infrastructure of the function field, respectively. We also provide numerical results for both scenarios.

In general, determining the divisor class number \( h \) of an algebraic function field \( K/\mathbb{F}_q(x) \) of genus \( g \) with \( q = p^n \) large is considered to be a computationally difficult problem. There are several methods available to compute this class number. Some of these methods are general, while others apply only to specific curves. Here, we only highlight the literature that is most closely related to our context. Kedlaya’s \( p \)-adic algorithm, using Monsky-Washnitzer cohomology, computes the zeta function of a hyperelliptic curve over a finite field of odd characteristic [17, 18, 19], and requires \( \tilde{O}(pn^3g^4) \) bit operations, for fixed \( g \) and \( n \). This method has been generalized to superelliptic curves [12, 21], \( C_{ab} \) curves [3], and nondegenerate curves [4]. Minzlaff adapted Kedlaya’s algorithm to compute the zeta function of a superelliptic curve with \( \tilde{O}(p^{0.5}n^{3.5}g^{3.5} + \log(p)n^5g^8) \) bit operations, for fixed \( g \) and \( n \) [24] and found the zeta function of a Picard curve over \( q = 2^{46} + 31 \) in 9.4 hours. While not computed explicitly, the class number of this curve would have been 33 digits.

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For larger genera, index calculus methods have been developed to compute discrete logarithms (and hence group orders) in class groups arising from plane curves in expected time $O(q^{2-2/9})$ bit operations, for fixed genus, as $q \to \infty$ [13], and from plane curves of small degree in expected time $O(q^{2-2/(g-1)})$, for fixed genus, as $q \to \infty$ [7]. In [8], the latter algorithm was tested on a Koblitz $C_{3,4}$ curve (of genus 3) with $q = 2^{31}$ and completed in 9.3 CPU days. Each of these methods requires the place at infinity to ramify completely.

The algorithm of Hess [15], on the other hand, applies to any global function field, provided there is at least one infinite place of degree 1 and computes the structure of the divisor class group in expected time $O \left( \exp \left( \sqrt{2g \log(q)} \log(2g \log(q)) \right) \right)$ bit operations, as $q \to \infty$. This algorithm is implemented in MAGMA and later we will compare the efficiency of this method with ours.

A straightforward method is to search for $h$ in the Hasse-Weil interval $[(\sqrt{7} - 1)^{2g}, (\sqrt{7} + 1)^{2g}]$ using Shanks’ Baby Step-Giant Step [37] or Pollard’s Kangaroo [27] method. In this way, $h$ can be computed deterministically or heuristically, respectively, using $O(q^{(2g-1)/4})$ group operations as $q \to \infty$.

Stein and Williams [33] applied techniques used by Lenstra [23] and Schoof [36] in quadratic number fields to real hyperelliptic function fields to narrow the search space. The basic idea is to find an approximation $E$ of $h$ and value $U$ such that $|h - E| < U$. The new interval $[E - U, E + U]$ is much smaller than the Hasse-Weil interval and in practice much better than expected. As a result, the Stein-Williams algorithm finds the divisor class number and regulator using $O(q^{(2g-1)/5 + \varepsilon(g)})$ infrastructure operations, where $-1/4 \leq \varepsilon(g) \leq 1/2$. The method was generalized to arbitrary hyperelliptic function fields and improved by Stein and Teske [41][40][42], who also applied Pollard’s Kangaroo algorithm to this setting to compute the 29-digit class number and regulator of a real hyperelliptic function field of genus 3.

The algorithm of [13] was generalized to cubic function fields in [33] and to arbitrary function fields in [34]. In this paper, we provide an implementation and numerical examples for purely cubic function fields of signatures $(3,1)$ and $(1,1;1,2)$. For this implementation, one requires efficient arithmetic as well as effective criteria for determining how a place of $F_q(x)$ splits in $K$. These and other algorithmic details can be found in [20]. We show how to explicitly compute an estimate $E$ of $h$ and an upper bound $U$ on the error $|h - E|$ based on the methods of [33][34]. Furthermore, we provide experimental results on the distribution of $h$ in the interval $[E - U, E + U]$ and show how to optimize Pollard’s Kangaroo algorithm on this interval, making improvements to its application to the infrastructure of a purely cubic function field of signature $(1,1;1,2)$. The improved method is applied to compute divisor class numbers up to 28 and 25 digits of function fields of genus 3 and 4, respectively. For the signature $(1,1;1,2)$ examples, we extracted the regulators as well. These are the largest known class numbers and regulators ever computed for a cubic function field over a large base field, with the exception of those computed by Bauer, Teske, and Weng in [2].

A brief discussion of the method of [24], and its more memory-efficient variant of [37], is appropriate here. This technique is specific to function fields defined by a Picard curve, for which it determines the divisor class number using $O(\sqrt{q})$ Jacobian operations. It has generated a divisor class number as large as 55 digits, the largest known class number of any cubic function field. However, this technique is restricted to Picard curves, whereas ours is a general purpose algorithm, and
our implementation applies to any purely cubic function field of signature (3, 1) or (1, 1; 1, 2). Picard fields are a special case of our setting, namely purely cubic function fields of signature (3, 1) for which the curve under consideration must be nonsingular and of genus 3.

The remainder of this paper is organized as follows. We give an overview of cubic function fields, including their class group and (in the signature (1, 1; 1, 2) case) their infrastructure, in Section 2. In Section 3, we describe Pollard’s Kangaroo method, outlining its specific application to the ideal class group of a purely cubic function field of signature (3, 1). We then adapt the Kangaroo method to the infrastructure of a purely cubic function field of signature (1, 1; 1, 2) in Section 4. In Section 5, we review the main results of [33] to compute the divisor class number of a cubic function field and explain the details of our implementation in Section 6. Finally, we provide numerical results obtained by our implementation in Section 7 and conclude with open problems and areas for further research in Section 8.

2. Cubic Function Fields — Class Group and Infrastructure

For a general introduction to function fields, we direct the reader to [14, 11, 29]. Details on cubic function fields and their arithmetic can be found in [32, 30, 31, 1, 33, 20]. Here, we merely highlight the material that is required for our context. Let $F_q$ be a finite field and $F_q(x)$ the field of rational functions in $x$ over $F_q$. Throughout this paper, we assume that char($F_q$) $\geq$ 5. A cubic function field is a separable extension $K/F_q(x)$ of degree 3 with full constant field $F_q$; we write $K = F_q(C)$ with $C : f(x, Y) = 0$, where $f(x, Y) \in F_q[x, Y]$ is an absolutely irreducible monic polynomial of degree 3 in $Y$. Note that we do not require the curve $f(x, Y) = 0$ to be nonsingular. If $C : Y^3 = F$ with $F \in F_q[x]$ cube-free, then $K$ is called a purely cubic function field. In this case, if we write $F = GH^2$ with $G, H \in F_q[x]$, gcd($G, H$) = 1, and both $G$ and $H$ are square-free, then the genus $g$ of $K$ is given by $g = \deg(GH) - 2$ if 3 $| \deg(F)$ and $g = \deg(GH) - 1$ if 3 $| \deg(F)$.

2.1. Divisors and Ideals. Let $D$ denote the group of divisors of $K$ defined over $F_q$, $D_0$ the subgroup of divisors of degree 0 defined over $F_q$, and $\mathcal{P}$ the subgroup of principal divisors defined over $F_q$. Then the (degree 0) divisor class group (or Jacobian) of $K$ is the quotient group $\mathcal{J} = D_0/\mathcal{P}$ and its order $h = |\mathcal{J}|$ is the (degree 0) divisor class number of $K$. Let $S$ be the collection of places of $K$ lying above the place at infinity of $F_q(x)$, $D_0^S = \{D \in D_0 \mid \text{supp}(D) \subseteq S\}$, and $\mathcal{P}^S$ the subgroup of principal divisors in $D_0^S$. Then the order $R_x$ of the quotient group $D_0^S/\mathcal{P}^S$ is the regulator of $K$. Finally, let $D_S = \{D \in D \mid \text{supp}(D) \cap S = \emptyset\}$ and $\mathcal{P}_S = \mathcal{P} \cap D_S$. Then every divisor $D \in D$ can be uniquely written in the form $D = D_S + D^S$ with $D_S \in D_S$ and $D^S \in D^S$.

The maximal order of $K/F_q(x)$ (or coordinate ring of $C$) is the integral closure of $F_q[x]$ in $K$ and is denoted $\mathcal{O}_K$. Let $\mathcal{I} = \mathcal{I}(\mathcal{O}_x)$ be the group of fractional ideals of $\mathcal{O}_x$ and $\mathcal{H} = \mathcal{H}(\mathcal{O}_x)$ the subgroup of principal fractional ideals. The ideal class group of $K$ is the quotient group $Cl(\mathcal{O}_x) = \mathcal{I}/\mathcal{H}$, and its order $h_x = |Cl(\mathcal{O}_x)|$ is the ideal class number of $K$. Let $f_x$ be the greatest common divisor of the inertia degrees of all the places in $S$. By Schmidt [35], there is an exact sequence

$$(0) \rightarrow D_0^S/\mathcal{P}^S \rightarrow \mathcal{J} \rightarrow Cl(\mathcal{O}_x) \rightarrow \mathbb{Z}/f_x \mathbb{Z} \rightarrow (0),$$

so that $f_x h = R_x h_x$. 

There is a well-known isomorphism $\Phi : D_S \to \mathcal{I}$ given by $D \mapsto \{ \alpha \in K^* \mid \text{div}(\alpha)_S \geq D \}$ with inverse $f \mapsto \sum_{p \in D} m_p p$, where $p$ denotes any finite place of $K$, $m_p = \min \{ v_p(\alpha) \mid \alpha \in f \text{ non-zero} \}$, and $v_p$ is the normalized discrete valuation corresponding to $p$. Moreover, $\Phi$ induces an isomorphism from $D_S / P S$ to $Cl(O_x)$.

If $S$ contains an infinite place $\infty_0$ of degree 1, then $\Phi$ can be extended to an isomorphism

$$\Psi : \{ D \in D_0 \mid v_p(D) = 0 \text{ for all } p \in S \setminus \{ \infty_0 \} \} \to \mathcal{I}(O_x)$$

via $\Psi(D_S - \deg(D_S)\infty_0) = \Phi(D_S)$, with inverse map $\Psi^{-1}(f) = \Phi^{-1}(f) - \deg(f)\infty_0$.

2.2. Splitting of Places. Via $\Phi$, there is a one-to-one correspondence between the finite places of $K$ and the prime ideals of $O_x$. If $\mathfrak{P}$ is a finite place of $\mathbb{F}_q(x)$, then $\Phi(\mathfrak{P}) = \langle P \rangle$, the principal ideal generated by some irreducible polynomial $P \in \mathbb{F}_q[x]$, and the splitting behavior of $\mathfrak{P}$ in $K$ is identical to the splitting behavior of $\langle P \rangle$ in $O_x$. Therefore, we may characterize the splitting behavior of the finite places of a purely cubic function field by considering the following criteria.

**Theorem 2.1** (Theorem 3.1 of [30]). Let $K = \mathbb{F}_q(C)$ be a purely cubic function field with $C : Y^3 = F \in \mathbb{F}_q[x]$ cube-free and $\text{char}(K) \geq 5$. If $P \in \mathbb{F}_q[x]$ is an irreducible polynomial, then the principal ideal $\langle P \rangle$ splits into prime ideals in $O_x$ as follows:

1. If $P \mid F$, then $\langle P \rangle = p^3$.
2. If $P \nmid F$ and $q^{\deg(P)} \equiv 2 \pmod{3}$, then $\langle P \rangle = pq$.
3. If $P \nmid F$, $F$ is a cube modulo $P$, and $q^{\deg(P)} \equiv 1 \pmod{3}$, then $\langle P \rangle = pp'p''$.
4. If $P \mid F$ and $F$ is not a cube modulo $P$, then $\langle P \rangle = p$.

An algorithmic realization of this theorem was given in [34].

We also require the splitting behavior of the infinite place $\infty$ of $\mathbb{F}_q(x)$ in $K$. Write $S = \{ \infty_0, \infty_1, \ldots, \infty_r \}$. Let $e_i = e(\infty_i/\infty)$ be the ramification index and $f_i = f(\infty_i/\infty)$ the inertia degree of $\infty_i$ over $\infty$ for $0 \leq i \leq r$. The signature of $K/\mathbb{F}_q(x)$ is the $(2r + 2)$-tuple $\text{sig}(K) = (e_0, e_0, \ldots; e_r, f_r)$ and is given as follows:

**Theorem 2.2** (Theorem 2.1 of [32]). Let $K = \mathbb{F}_q(C)$ be a purely cubic function field with $C : Y^3 = F \in \mathbb{F}_q[x]$ cube-free and $\text{char}(K) \geq 5$. If $\text{sgn}(F)$ denotes the leading coefficient of $F$, then the signature of $K/\mathbb{F}_q(x)$ is given as follows:

1. If $3 \mid \deg(F)$, then $\text{sig}(K) = (3, 1)$.
2. If $3 \nmid \deg(F)$ and $q \equiv 2 \pmod{3}$, then $\text{sig}(K) = (1, 1; 1, 2)$.
3. If $3 \mid \deg(F), \text{sgn}(F) \in \mathbb{F}_q^3$, and $q \equiv 1 \pmod{3}$, then $\text{sig}(K) = (1, 1; 1, 1, 1, 1)$.
4. If $3 \mid \deg(F)$ and $\text{sgn}(F) \notin \mathbb{F}_q^3$, then $\text{sig}(K) = (1, 3)$.

Thus, the signature of a purely cubic function field can be determined quickly. We also note that whenever $K$ has an infinite place of degree 1, there is a straightforward normalization that replaces $C : Y^3 = F$ by an isomorphic curve $C' : Y^3 = F'$ where $F'$ is monic, without changing the coordinate ring or the signature. Hence, in our implementation, we only consider monic polynomials $F$.

For the remainder of this paper, we assume that $S$ contains an infinite place $\infty_0$ of degree 1, so that $f_x = 1$ and $h = R_h x$. Thus, we will no longer consider cubic function fields of signature $(1, 3)$. In fact, our later focus will be on purely cubic function fields of the two signatures $(3, 1)$ and $(1, 1; 1, 2)$. If $\text{sig}(K) = (3, 1)$, then $R_x = 1$ so that $J \cong Cl(O_x)$ and $h = h_x$. In this case, we may conduct our computations to find $h$ in either $J$ or $Cl(O_x)$. If $\text{sig}(K) = (1, 1; 1, 2)$, then...
2.3. Distinguished Divisors and Infrastructure. Let $K$ be a cubic function field with an infinite place $\infty_0$ of degree 1 and maximal order $O_x$. A divisor $D$ of $K$ is said to be finitely effective if $D_S \geq 0$; that is, $\nu_p(D) \geq 0$ for all finite places $p$ of $K$. Following [1] [11] [20], a finitely effective divisor $D$ is said to be distinguished if

1. $D$ is of the form $D = D_S - \deg(D_S)\infty_0$, and
2. if $E$ is any finitely effective divisor that is linearly equivalent to $D$, with $\deg(E_S) \leq \deg(D_S)$ and $E_S \geq D_S$, then $D = E$.

An ideal $a$ of $O_x$ is said to be distinguished if $\Psi^{-1}(a)$ is a distinguished divisor. Note that every distinguished divisor $D$ is uniquely determined by its finite part $D_S$ and hence corresponds to a unique distinguished ideal $\Psi(D)$. Thus, arithmetic of distinguished divisors is reduced to ideal arithmetic. In the context of this paper, if suffices to know that such an arithmetic exists, and that it is efficient.

By [1] Corollaries 5.2 and 5.3, if $\text{sig}(K) = (3, 1)$, then every divisor class of $K$ contains a unique distinguished representative. Analogously, every ideal class of $K$ contains a unique distinguished ideal. The composition $a * b$ of $a$ and $b$ is the distinguished ideal equivalent to the (generally non-distinguished) ideal $ab$. Similarly, the composition of two distinguished divisors $D_1, D_2$ is the unique distinguished divisor that is linearly equivalent to the sum $D_1 + D_2$. Arithmetic in $\text{Cl}(O_x)$ and $\mathcal{J}$ can thus be efficiently conducted via these unique distinguished representatives when $\text{sig}(K) = (3, 1)$. For details, see [1] and [20].

Assume now that $\text{sig}(K) = (1, 1; 1, 2)$. In this case, not every divisor class contains a distinguished divisor; however, if such as divisor exists in any given class, then uniqueness still holds. In this scenario, we operate in the (principal) infrastructure of $K$. A general treatment of infrastructures in function field extensions of arbitrary degree can be found in [32] [39]. The cubic scenario was first presented in [32] [39], and we use the divisor-theoretic description of [20] here. We only discuss the basic infrastructure operations and the notion of distance, again omitting all arithmetic details.

The (finite) set

$$\mathcal{R} = \{D \in D_0 \mid D \text{ is distinguished and } \Psi(D) \text{ is a principal ideal}\}$$

is the (principal) infrastructure of of $K$.

Since $K$ has exactly one infinite place $\infty_0$ of degree 1, there exists a unique embedding of $K$ into the field of Laurent series in $x^{-1}$; that is, into the completion of $K$ at $\infty_0$. This embedding establishes the notion of degree and sign (i.e. leading coefficient) of elements in $K$. Note that $K$ has a fundamental unit $\epsilon$ with $\deg(\epsilon) = 2R_x$ that is unique up to sign. If $D \in \mathcal{R}$ and $a = \Psi(D)$, then there is a unique function $\alpha \in K^*$ such that $a = \langle \alpha \rangle$ and $0 \leq \deg(\alpha) < \deg(\epsilon) = 2R_x$. The distance of $D$ is defined as $\delta(D) = \deg(\alpha)$. Therefore, we may order the divisors in $\mathcal{R}$ increasingly by distance:

$$\mathcal{R} = \{D_0 = 0, D_1, \ldots, D_{l-1}\},$$

with $\delta(D_i) < \delta(D_{i+1})$ for $0 \leq i < l - 1$. There are two main operations on $\mathcal{R}$, the baby step and giant step operations. A baby step maps $D_i$ to $D_{i+1}$ for $0 \leq i < l - 1$. 

$h = R_x h_x$, and we operate in a certain set of divisors called the infrastructure of $K$. To that end, and in order to operate effectively and explicitly in the infrastructure and the class group of $K$, we require the notion of a distinguished divisor.
and \(D_{t-1} \) to \(D_0\). We write \(bs(D_t) = D_{t+1}\). The giant step operation is analogous to composition. As before, if \(D_1, D_2 \in \mathcal{R}\), then \(D_1 + D_2\) is generally not distinguished. However, there is a uniquely defined and efficiently computable function \(\psi \in K^*\) such that

\[
D_1 + D_2 = \Psi^{-1}((\psi) \Psi(D_1 + D_2)) \in \mathcal{R}
\]

and \(-2g \leq \deg(\psi) \leq 0\). We call \(\oplus\) the giant step operation. Under \(\oplus\), \(\mathcal{R}\) is an Abelian group-like structure, failing only associativity. However,

\[
\delta(D_1 \oplus D_2) = \delta(D_1) + \delta(D_2) + \deg(\psi),
\]

so that \(D_1 \oplus D_2\) is "close to" \(D_1 + D_2\) in terms of distance. A third operation that will be required is the computation of the divisor below any integer \(n \in \mathbb{N}\) with \(0 \leq n < 2R_p\). This is the unique divisor \(D(n) \in \mathcal{R}\) such that \(\delta(D(n)) \leq n < \delta(bs(D(n)))\). Details on how to compute a baby step, giant step, and \(D\) steps, which are based on extensive experimental results and plausible theoretical assumptions.

**Heuristic 2.3.** Let \(K\) be a cubic function field of signature \((1, 1; 1, 2)\) and genus \(g\).

1. If \(D \in \mathcal{R}\), then with probability \(1 - 1/q\), we have \(\deg(D_S) = g\).
2. If \(D_1, D_2 \in \mathcal{R}\) and \(\Psi(D_1 \oplus D_2) = (\psi) \Psi(D_1) \Psi(D_2)\), then with probability \(1 - O(1/q)\), we have

\[
\hat{\psi}(g) = \begin{cases} 
-\lfloor g/3 \rfloor & \text{if } g \not\equiv 1 \pmod{3}, \\
-(g + 2)/3 & \text{if } g \equiv 1 \pmod{3},
\end{cases}
\]

where we set \(\hat{\psi}(g) = \deg(\psi)\).

Based on this heuristic, we can conclude that a baby step in \(\mathcal{R}\) has length 2 with probability \(1 - O(1/q)\). Note that this corresponds exactly to the observations made in real hyperelliptic function fields (see [41, 40, 42]).

3. The Kangaroo Method in \(\mathcal{C}(\mathcal{O}_E)\)

If we are given integers \(E, U \in \mathbb{N}\) such that \(h \in [E - U, E + U]\), then the Baby Step-Giant Step and Kangaroo methods may be optimized to compute \(h\) with a deterministic and probabilistic running time of \(O\left(\sqrt{U}\right)\) group operations, respectively. The Baby Step-Giant Step method is generally faster than the Kangaroo method, but it requires the storage of \(O\left(\sqrt{U}\right)\) group elements and cannot be parallelized efficiently. For larger computations, the Kangaroo method is preferable, since variants of this algorithm require very little storage and can be parallelized, hence our reason for describing and using this method. Specifically, we describe the parallelized Kangaroo method of van Oorschot and Wiener [40, 42] and explain important improvements that apply in particular to the problem of computing the divisor class number of a purely cubic function field of signature \((3, 1)\). We always assume that \(2(E - U) > E + U\). In this way, if we determine \(h_0 \in [E - U, E + U]\) to be a multiple of \(h\), then \(h_0/2 < (E + U)/2 < E - U\). Thus, \(h_0 = h\). It is important to note that this is not an unreasonable restriction for the integers \(E\) and \(U\) produced by Scheidler and Stein’s method in [33]; only function fields over very small base fields fail this criterion.
We now describe in detail a modification of the parallelized Kangaroo method using notation similar to that of [41, 42] for hyperelliptic function fields. Let $m$ be the (even) number of available processors. The Kangaroo algorithm uses two herds of kangaroos, a herd $\{T_1, \ldots, T_{m/2}\}$ of tame kangaroos, and a herd $\{W_1, \ldots, W_{m/2}\}$ of wild kangaroos. A kangaroo is a sequence of distinguished ideals. The Kangaroo method requires a collision between a tame and a wild kangaroo to obtain $h$. The tame kangaroos begin their jumps at distinct known points near $E$, and the wild kangaroos at points near $h$ whose location in the interval is unknown, hence their respective names.

The idea of the algorithm is as follows. Let $g$ be a distinguished ideal. Define a set of small (relative to $U$) random positive integers $\{s_1, \ldots, s_{64}\}$, the jump set $J = \{g^{s_1}, \ldots, g^{s_{64}}\}$, and a hash function $\nu : \mathcal{I}(O_E) \to \{1, \ldots, 64\}$. Initialize each tame kangaroo $T_i$ at a distinguished ideal $t_{0,i} \sim g^{E+(i-1)\nu}$ for some small $\nu \in \mathbb{Z}$ and $1 \leq i \leq m/2$, and each wild kangaroo $W_j$ at a distinguished ideal $w_{0,j} \sim g^{(j-1)\nu}$ for $1 \leq j \leq m/2$. The kangaroos jump through $\mathcal{O}(O_E)$ via

$$t_{l+1,i} = t_{l,i} \ast g^{\nu(t_{l,i})}, \quad w_{l+1,j} = w_{l,j} \ast g^{\nu(w_{l,j})}, \quad (l \in \mathbb{N}_0, \quad 1 \leq i, j \leq m/2).$$

The computations $t_{l,i} \to t_{l+1,i}$ and $w_{l,j} \to w_{l+1,j}$ are called (kangaroo) jumps. The distance of the $i$-th tame kangaroo $T_i$ (or $j$-th wild kangaroo $W_j$) at step $l$ is the discrete logarithm of the ideal $t_{l,i}$ (or $w_{l,j}$) with respect to the base ideal $g$ and is denoted by $d_l(T_i)$ and $d_l(W_j)$, for tame and wild kangaroos, respectively. (We note that this definition of distance is different from, and not to be confused with, the notion of infrastructure distance.) Specifically, we initialize $d_0(T_i) = E + (i - 1)\nu$ and $d_0(W_j) = (j - 1)\nu$, for each $1 \leq i, j \leq m/2$, so that

$$d_{l+1}(T_i) = d_l(T_i) + s_{\nu(t_{l,i})}, \quad d_{l+1}(W_j) = d_l(W_j) + s_{\nu(w_{l,j})}, \quad (l \in \mathbb{N}_0).$$

If $t_{l,i} = w_{B,j}$, for some $A, B \in \mathbb{N}_0$ and $1 \leq i, j \leq m/2$, then we have a collision and $g^{d_A(T_i)} = g^{d_B(W_j)}$. If $d_A(T_i) - d_B(W_j) \in [E - U, E + U]$, then we are guaranteed that $h = d_A(T_i) - d_B(W_j)$.

If there is a collision between any two kangaroos, then they will continue on the same path. Therefore, if there is a collision between two kangaroos of the same herd, then we cannot obtain any information about $h$, so we must re-initialize one of the two kangaroos. Without loss of generality, suppose that the two tame kangaroos $T_1$ and $T_2$ collide at the distance $d_A(T_1) = d_B(T_2)$. For a small $c \in \mathbb{N}$, set $t_{A+1,1} = t_{A,1} \ast g^c$ and $d_{A+1}(T_1) = d_A(T_1) + c$, then let $T_3$ continue jumping on its new path as usual. $T_2$ may continue along the same path as before without interruption.

A key feature of the Kangaroo algorithm is that there is no need to store every jump. Using the idea of van Oorschot and Wiener [46], we only store distinguished points. In order to avoid confusion with the concept of distinguished divisors and ideals, such points will be called (kangaroo) traps instead. To this end, define another hash function $\zeta : \mathcal{I}(O_E) \to \{0, \ldots, \theta - 1\}$. Install a trap, that is, store a kangaroo $\mathbf{t}$, if $\zeta(\mathbf{t}) = 0$. In this way, we expect to set a trap every $\theta$ jumps. If $\theta$ is sufficiently large, then the storage requirement is very small. Note that we only detect collisions between traps, but since colliding kangaroos travel along the same path following their first collision, a collision in a trap will eventually be found.

If it is known that there exist integers $a, b \in \mathbb{N}_0$ such that $0 < a < b$ and $h \equiv a \pmod{b}$, then we can make adjustments to the jump set and initializations to only operate within the congruence class $a \pmod{b}$. We change the estimate $E$
to \( E - (E \mod b) \) + a, so that \( E \equiv a \mod b \) for the revised value of \( E \), and choose \( \nu \) and the jump distances such that \( b \mid \nu \) and \( b \mid s_i \), for each \( 1 \leq i \leq 64 \). The remaining initializations are the same.

In Algorithm 3.1 we formalize the procedures described above.

**Algorithm 3.1** Computing \( h \) via the Kangaroo Algorithm - Signature (3, 1)

**Input:** A prime power \( q \); monic, relatively prime, square-free \( G, H \in \mathbb{F}_q[x] \) such that \( 3 \not| \deg(GH^2) \); \( a, b \in \mathbb{N}_0 \) such that \( h \equiv a \mod b \) (or \( b = 1 \) and \( a = 0 \) if no non-trivial \( b \) is known); integers \( E, U \in \mathbb{N} \) such that \( |h - E| < U \) and \( 2(E - U) > E + U \); and an even integer \( m \), the number of processors.

**Output:** The divisor class number \( h \) of \( K = \mathbb{F}_q(C) \), where \( C : Y^3 = GH^2 \).

1. \( g := \deg(GH) - 1 \).
2. Find an estimate, \( \hat{\alpha} := \hat{\alpha}(q, g) \), of the expected value of \( |h - E|/U \) via Table 4.
3. \( \beta := \left(\frac{m}{2}\right)\sqrt{\hat{\alpha}abU} \), \( \nu := \frac{2\beta}{m} - \left(\frac{2\beta}{m}\right) \mod b \).
4. \( \theta := 2\left\lfloor \frac{\beta}{2} \right\rfloor \), \( E := E - (E \mod b) + a \).
5. Choose random integers \( 0 < s_i \leq 2\beta \), for \( 1 \leq i \leq 64 \), with \( Mean(\{s_i\}) = \beta \) and \( b \mid s_i \).
6. Generate a random ideal \( \mathfrak{g} \).
7. Define hash functions \( v : \mathcal{I} \to \{1, \ldots, 64\} \) and \( z : \mathcal{I} \to \{0, \ldots, \theta - 1\} \).
8. for \( i = 1 \) to \( m/2 \) do
9. Initialize the tame kangaroo, \( T_i \): \( t_{0,i} := \mathfrak{g}^{E+(i-1)\nu} \) and \( d_0(T_i) := E + (i-1)\nu \).
10. end for
11. if \( z(t_{0,i}) = 0 \) or \( z(w_{0,i}) = 0 \), for some \( 1 \leq i \leq m/2 \) then
12. Store the respective ideal and its distance.
13. end if
14. j := 0.
16. while A collision between a tame and a wild kangaroo has not been found do
17. for \( i = 1 \) to \( m/2 \) do
18. \( t_{j+1,i} := t_{j,i} \ast \mathfrak{g}^{s_{v(t_{j,i})}} \) and \( d_{j+1}(T_i) := d_{j+1}(T_i) + s_{v(t_{j,i})} \).
19. \( w_{j+1,i} := w_{j,i} \ast \mathfrak{g}^{s_{v(w_{j,i})}} \) and \( d_{j+1}(W_i) := d_{j+1}(W_i) + s_{v(w_{j,i})} \).
20. end for
21. j := j + 1.
22. if \( z(t_{j,i}) = 0 \) or \( z(w_{j,i}) = 0 \), for some \( 1 \leq i \leq m/2 \) then
23. Store the respective ideal and its distance.
24. end if
25. end while
26. if \( t_A,i = w_B,j \) then
27. return \( h := d_A(T_i) - d_B(W_j) \).
28. end if

The following analysis is a generalization of similar ideas in [41, 42]. It justifies our choices of certain variables in Algorithm 3.1 Since it mainly depends on the set-up of the Kangaroo method and not on the underlying function field, the proof is omitted; for details, see [20]. Henceforth, we denote by \([R]\) the nearest integer to a real number \( R \).
Proposition 3.1. Let $K$ be a purely cubic function field of signature $(3, 1)$. Suppose that there exist integers $a$, $b \in \mathbb{N}$ such that $h \equiv a \pmod{b}$. Then the expected heuristic running time, over all cubic function fields over $\mathbb{F}_q(x)$ of genus $g$, to compute $h$ via Algorithm 3.1 is minimized by choosing an average jump distance of $\beta = \left(\frac{m}{2}\right)\sqrt{\alpha bU}$, where $m$ is the (even) number of processors and $\alpha = \alpha(q, g) < 1/2$ is the mean value of $|h - E|/U$ over all cubic function fields $K$ over $\mathbb{F}_q(x)$ of genus $g$. For this choice of $\beta$, the total expected heuristic running time of Algorithm 3.1 for each kangaroo is $(4/m)\sqrt{\alpha U/b} + \theta + O(1)$ ideal compositions as $q \to \infty$, where traps are set on average every $\theta$ jumps.

For further practical considerations, Stein and Teske [45] note that for hyperelliptic function fields, if the jump distances $s_1, \ldots, s_{64}$ are chosen randomly, then the number of useless collisions appears independent of the choice of the initial spacing $\nu$, and suggest using $\nu \lesssim 2\beta/m$. They also recommend choosing $s_i \leq 2\beta$ for $1 \leq i \leq 64$, since such choices yielded results which were slightly better than those using other upper bounds. We expect no difference for cubic function fields and therefore chose $|J| = 64$ to be a power of 2, so that the hash function $\nu$ is fast while still achieving a sufficient level of randomization, and also the space to store the jumps is not too large. Lastly, we expect to store $O\left(\sqrt{U/\theta}\right)$ ideals. Teske [45] suggests in the generic group setting taking $\theta = 2^{\lfloor \log(\beta)/2 \rfloor + c}$ for some small integer $c$. For this choice, we have $\theta = O\left(\sqrt{U}\right)$, which is a reasonable number in practice.

In the next section, we describe appropriate changes to use the Kangaroo method in the infrastructure of a cubic function field of signature $(1, 1; 1, 2)$. These changes correspond to the changes made for real hyperelliptic function fields.

4. THE KANGAROO METHOD IN $\mathcal{R}$

If $K$ has signature $(1, 1; 1, 2)$, then we wish to determine $R_x$ via the computation of some multiple $h_0$ of $R_x$. Under the assumption $2(E - U) > E + U$, if we find $h_0 \in [E - U, E + U]$, then in fact $h_0 = h$. In this case, we adapt the description of the Kangaroo algorithm in Section 3 to operate in $\mathcal{R}$ and show how to take advantage of the faster baby step operation. We formalize these modifications in Algorithm 4.1 and note its running time in Proposition 4.1.

To be consistent with earlier notation, a (tame or wild) kangaroo $Z$ in this context is a sequence of infrastructure divisors, and we write $Z = \{t_0, t_1, \ldots\} \subseteq \mathcal{R}$. If $t_i \in Z$, then the distance of $Z$ at step $l$, $d_l(Z) = \delta(t_l)$, is the distance of $t_l$ as defined for divisors in $\mathcal{R}$. We initialize each tame kangaroo $T_i$ $(1 \leq i \leq m/2)$ at the distinguished divisor $t_0,i = D(2E + (i - 1)\nu) \in \mathcal{R}$. Likewise, each wild kangaroo $W_j$ $(1 \leq j \leq m/2)$ is initialized at $w_{0,j} = D((j - 1)\nu) \in \mathcal{R}$. In this case, the jump set is $J = \{g_1, \ldots, g_{64}\}$, where $g_i = D(s_i)$, for $1 \leq i \leq 64$. However, since we do not necessarily have $\delta(g_i) = s_i$ for all $1 \leq i \leq 64$, we store each distance, $\delta(g_i)$, rather than the random integers $s_i$, for $1 \leq i \leq 64$. Thus, for each step of the algorithm, we have $t_{i+1} = t_i \oplus g_{\delta(t_i)}$, with the distances updated by $d_{l+1}(Z) = d_l(Z) + \delta(g_{\delta(t_l)}) + \delta$, where $\delta = \deg(\psi)$ and $\psi$ is as given in (2.1). In this adaptation, if a tame kangaroo $T_i$ at step $A$ collides with a wild kangaroo $W_j$ at step $B$, then $t_{A+1} = w_{B,j}$. Thus, $\delta_A(T_i) \equiv \delta_B(W_j)$ (mod $2R_x$), so $h_0 = (\delta_i(T_A) - \delta_j(W_B))/2$ is a multiple of $R_x$. We give details on how to determine $R_x$ from $h_0$ in Section 6.4.
In the infrastructure setting, however, we may take advantage of the fact that baby steps are faster than giant steps in $\mathcal{R}$ to speed up the regulator computation by a factor of approximately $\sqrt{\tau_1}/2$, where $\tau_1 = T_G/T_B$ and $T_G$ and $T_B$ are the respective times to compute a giant step and a baby step. The following is a slight change from the idea found in Section 4.1 of [41].

For a real number $\tau \geq 1$, let $\mathcal{S}_\tau \subseteq \mathcal{R}$ such that $|\mathcal{R}|/|\mathcal{S}_\tau| \approx \tau^{\frac{3}{2}}$. After each kangaroo jump (a giant step), we take baby steps until a divisor in $\mathcal{S}_\tau$ is found, then we make the next kangaroo jump. Below, we outline the kangaroo algorithm and give specific choices for $\beta$ and $\tau$ to optimize its running time.

One key difference between the following result to optimize the expected running time of Algorithm 4.1 and Proposition 3.1 is that we express the running time in terms of $T_G$, rather than in terms of the number of kangaroo jumps. The following is a slight improvement of Equation (4.8) of [41] in the hyperelliptic case.

**Proposition 4.1.** If $K/\mathbb{F}_q(x)$ is a purely cubic function field of signature $(1,1;1,2)$, then assuming Heuristic 2.3, the expected heuristic running time, over all cubic function fields over $\mathbb{F}_q(x)$ of genus $g$, to compute a multiple $h_0$ of $R_x$ via Algorithm 4.1 is minimized by choosing $\beta = \left\lfloor m \sqrt{2(2\tau - 1)\alpha U}\right\rfloor - 2(\tau - 1)$, where $m$ is the (even) number of processors, $\tau = T_G/T_B$, $T_G$ and $T_B$ are the respective times required to compute a giant step and a baby step in $\mathcal{R}$, and $\alpha = \alpha(q,g) < 1/2$ is the mean value of $|h - E|/U$ over all cubic function fields over $\mathbb{F}_q(x)$ of genus $g$. With these choices, the expected heuristic running time for each kangaroo is

$$\frac{4}{m} \sqrt{\frac{\alpha U}{2\tau - 1}} + \frac{\theta}{\tau} + O(1) \left(2 - \frac{1}{\tau}\right) T_G,$$

as $q \to \infty$, where traps are set on average every $\theta$ jumps.

As in the signature $(3,1)$ case, we discuss the reasons for the choice of certain other variables in Algorithm 4.1. First, when choosing values for the $s_i$, $1 \leq i \leq 64$, there are a few considerations arising from the reduction required for giant steps, so that the average jump distance is as close to $\beta$ as possible in practice. For each $s_i$, we have $\delta(D(s_i)) = s_i$ with probability roughly $1/2$ and $\delta(D(s_i)) = s_i - 1$ with probability roughly $1/2$, for sufficiently large $q$, by Heuristic 2.3. Moreover, by Part 2 of Heuristic 2.3, we have $(d_{i-1}(Z) + s_{\nu(t_{i-1}))}) - d_i(Z) = \hat{\psi}(g)$ with probability $1 - O(1/q)$. Therefore, to adjust for this “headwind,” as well as the average difference $s_i - \delta(D(s_i))$, we must choose the $s_i$ so that $(s_1 + \cdots + s_{64})/64 = \beta + 1/2 + \hat{\psi}(g)$. Likewise, we choose $s_i \leq 2(\beta + \hat{\psi}(g)) + 1$ so that each jump has distance bounded above by $2\beta$ with probability close to 1. Finally, we cannot have $0 \in J$, otherwise a kangaroo will become permanently stuck at one divisor if it hashes to 0, so we must set a lower bound on the choices of the $s_i$ to avoid this situation. By Theorem 5.3.10 of [20], we have $1 \leq \delta(bs(0)) \leq g + 2$, so $s = g + 2$ is the smallest integer that guarantees that $D(s) \neq 0$. Therefore, we must choose $s_i \geq g + 2$ for all $1 \leq i \leq 64$. With these choices of the $s_i$, the average jump distance is in practice as close to $\beta$ as possible.

Table 1 lists values of $\tau$ for various signature $(1,1;1,2)$ situations of genera $3 \leq g \leq 7$. In each case, we computed the ratios using $10^6$ baby steps and $10^6$ giant

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1If $q$ is prime, then one possible choice for $\mathcal{S}_\tau$ is $\mathcal{S}_\tau = \{D \in \mathcal{R} \mid d(\Psi(-D))(0) < \lfloor q/\tau \rfloor\}$, where $d(f) \in \mathbb{F}_q[x]$ is the denominator of the fractional ideal $f$. 

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steps in a function field \( \mathbb{F}_q(C) \) with \( q = 10^8 + 7 \) and \( C : Y^3 = GH^2 \) where \( G \) and \( H \) were random distinct irreducible polynomials with \( \deg(G) \geq \deg(H) \).

In the next section, we review the method of [33] implemented here to compute the divisor class number of a cubic function field.
Table 1. Giant Step to Baby Step Ratio in $R$

| $g$ | $\deg(G)$ | $\deg(H)$ | $\tau$ | $\deg(G)$ | $\deg(H)$ | $\tau$ |
|-----|-------------|-------------|--------|-------------|-------------|--------|
| 2   | 2           | 2           | 2.96977|             |             |        |
| 3   | 4           | 1           | 2.92374|             |             |        |
| 4   | 6           | 0           | 3.87316| 3           | 3           | 4.11812|
| 5   | 5           | 2           | 5.29813|             |             |        |
| 6   | 7           | 1           | 5.86166| 4           | 4           | 6.10144|
| 7   | 9           | 0           | 7.50799| 6           | 3           | 7.72477|

5. Approximating $h$

5.1. Idea of the Algorithm. Algorithm 5.1 lists the three main phases of the method of [33] to compute the divisor class number $h$ of a cubic function field, with a fourth step in the signature $(1, 1; 1, 2)$ case if the regulator $R_z$ is desired.

Algorithm 5.1 Computing $h$ and/or $R_z$ - The Idea

1: Compute an estimate $E$ of $h$ and an upper bound $U$ on the error $|h - E|$ so that $h \in [E - U, E + U]$.
2: Determine extra information about $h$ such as congruences or divisibility by small primes or distribution of $h$ in the interval $[E - U, E + U]$.
3: Use the Baby Step-Giant Step method or the Kangaroo method to find $h$ in $[E - U, E + U]$ using $O\left(\sqrt{U}\right)$ ideal compositions.
4: If $K$ has signature $(1, 1; 1, 2)$, then factor $h$ and let $R_z$ be the smallest factor $R'$ of $h$ such that $D(2R') = 0$.

5.2. Results and Notation for Phase 1. For full details on the derivation of $E$ and $U$, we refer to [43, 41, 42, 33], and [34] for the cases of quadratic, cubic, and arbitrary function fields, respectively. The idea, however, is to write $h$ as an infinite product over the places of $K$ via the zeta function of $K$; $E$ is determined by computing the product up to a certain degree bound $\lambda$, and $U$ is determined by setting an upper bound on the size of the tail.

Following [33, 34], let

$$(x_1, x_2) = \begin{cases} 
(0, 0) & \text{if } \langle P \rangle = (3, 1), \\
(1, -1) & \text{if } \langle P \rangle = (1, 1; 1, 2), 
\end{cases}$$

and $A(K) = (g + 2)\log(q) - \log((q - x_1)(q - x_2))$. Next, let $\zeta$ be a primitive cube root of unity in some algebraic closure of $\mathbb{F}_q$. If $P \in \mathbb{F}_q[x]$ is a monic irreducible polynomial, then let

$$(z_1(P), z_2(P)) = \begin{cases} 
(0, 0) & \text{if } \langle P \rangle = p^3, \\
(1, -1) & \text{if } \langle P \rangle = pq, \\
(1, 1) & \text{if } \langle P \rangle = pp'p'', \\
(\zeta, \zeta^2) & \text{if } \langle P \rangle = p, 
\end{cases}$$
and

\[ S_\nu(n) = \sum_{\deg(P) = \nu} (z_1^n(P) + z_2^n(P)). \]

From (4.11) and (4.12) of [33], we have

\[ E \]

5.3. Determining \( E \) and \( U \). We review how to explicitly calculate two estimates \( E_1, E_2 \) of \( h \), and three bounds \( V_1, U_2, V_3 \) on the error, \( |h - E| \), as found in [33][34].

For \( i = 1, 2, 3 \) and a fixed \( \lambda \in \mathbb{N} \), we write \( \log(h) = \log(E'_i(\lambda)) + B_i(\lambda) \) so that \( h = E'_i(\lambda)e^{B_i(\lambda)} \). (For ease of notation, we write \( E_2(\lambda) = E_3(\lambda) \) and \( B_2(\lambda) = B_3(\lambda) \).

We then find a sharp upper bound \( \psi_1(\lambda) \) on \( |B_1(\lambda)| \) and define \( E_i(\lambda) := [E'_i(\lambda)] \) and \( U_i(\lambda) := [E'_i(\lambda)(e^{\psi_1(\lambda)} - 1)] \), so that \( |h - E_i(\lambda)| \leq U_i(\lambda) \).

First, we have

\[ \log(E'_1(\lambda)) = A(K) + \sum_{n=1}^{\lambda} \frac{1}{nq^n} \sum_{\nu|n} \nu S_\nu \left( \frac{n}{\nu} \right), \quad B_1(\lambda) = \sum_{n=1}^{\lambda} \frac{1}{nq^n} \sum_{\nu|n} \nu S_\nu \left( \frac{n}{\nu} \right). \]

Then

\[ \psi_1(\lambda) = 2g \left( \log \left( \frac{\sqrt{q}}{\sqrt{q-1}} \right) - \sum_{n=1}^{\lambda} \frac{1}{nq^n/2} \right) + 2 \log \left( \frac{q}{q-1} \right) - 2 \sum_{n=1}^{\lambda} \frac{1}{nq^n} \]

is a sharp upper bound on \( |B_1(\lambda)| \). By moving some terms from \( B_1(\lambda) \) to \( E_1(\lambda) \), we obtain the following second estimate \( E_2(\lambda) \) and error bound. \( E_2(\lambda) \):

\[ \log(E'_2(\lambda)) = A(K) + \sum_{n=1}^{\lambda} \frac{1}{nq^n} \sum_{\nu|n} \nu S_\nu \left( \frac{n}{\nu} \right), \quad B_2(\lambda) = \sum_{n=1}^{\lambda} \frac{1}{nq^n} \sum_{\nu|n} \nu S_\nu \left( \frac{n}{\nu} \right). \]

A sharp upper bound \( \psi_2(\lambda) \) of \( |B_2(\lambda)| \) is then given by

\[ \psi_2(\lambda) = \frac{2}{(\lambda + 1)} \left( gq^{-(\lambda+1)/2} + q^{-(\lambda+1)} \right) + \frac{2g}{(q-1)(\lambda + 1)} q^{-(\lambda+1)} \left( q^{(\lambda+1)/l} - 1 \right) \]

\[ + \frac{2g}{(\lambda + 2)} \left( \sqrt{q} - 1 \right) q^{-(\lambda+2)/2} + \frac{4}{(\lambda + 2)} \frac{q^{(l-1)/l}}{(q-1)(q^{(l-1)/l} - 1)} q^{-(\lambda+2)(l-1)/l} \]

where \( l \) is the smallest prime factor of \( \lambda + 1 \).

Finally, we use extra information to obtain a sharper bound \( \psi_3(\lambda) \) on \( B_2(\lambda) \). Specifically, we can easily calculate \( \nu S_\nu(\lambda + 1/\nu) \) for all \( \nu \mid (\lambda + 1) \) such that
Thus, we obtain

\begin{equation}
\psi_3(\lambda) = \frac{2g}{(\lambda + 1)} q^{-(\lambda+1)/2} + \frac{q^{-(\lambda+1)}}{(\lambda + 1)} \left( 2 + \sum_{\nu\neq \lambda+1, \nu \mid (\lambda+1)} \nu S_\nu \left( \frac{\lambda + 1}{\nu} \right) \right) \\
+ \frac{2g}{(\lambda + 2)} \frac{\sqrt{q}}{(\sqrt{q} - 1)} q^{-(\lambda+2)/2} \\
+ \frac{4}{(\lambda + 2)} q \frac{q^{(l-1)/l}}{(q-1) (q^{(l-1)/l} - 1)} q^{-(\lambda+2)(l-1)/l}.
\end{equation}

5.4. Complexity and Optimization. For complete details on the analysis of the running time of Algorithm 5.1, we refer to [33, 34]. Here, we simply state that as \( q \to \infty \), the optimal choice for \( \lambda \) is

\begin{equation}
\lambda = \begin{cases} 
\left\lfloor \frac{(2g - 1)}{5} \right\rfloor & \text{if } g \equiv 2 \pmod{5}, \\
\left\lfloor \frac{(2g - 1)}{5} \right\rfloor & \text{otherwise}.
\end{cases}
\end{equation}

If \( g \leq 2 \), then \( \lambda = 0 \), so the estimate in Phase 1 is completely determined by the infinite component and runs in polynomial time; there is no asymptotic improvement over using the Hasse-Weil bounds. However, if \( g \geq 3 \), then we have the following result.

**Theorem 5.2.** If \( K \) is a cubic function field of genus \( g \geq 3 \), then the complexity of Algorithm 5.1 is \( O \left( q^{\left\lfloor (2g - 1)/5 \right\rfloor + \varepsilon(g)} \right) \) ideal or infrastructure compositions, as \( q \to \infty \), where

\[ \varepsilon(g) = \begin{cases} 
0 & \text{if } g \equiv 0, 3 \pmod{5}, \\
1/4 & \text{if } g \equiv 1 \pmod{5}, \\
-1/4 & \text{if } g \equiv 2 \pmod{5}, \\
1/2 & \text{if } g \equiv 4 \pmod{5}.
\end{cases} \]

In practice, Step 2 of Algorithm 5.1 requires a negligible amount of time since the information associated with this step is known in advance. Also, Step 4 is faster than Steps 1 and 3 since factoring is asymptotically faster than the overall running time of the algorithm. Therefore, the overall complexity of Algorithm 5.1 is found by balancing the running times of Steps 1 and 3. However, Step 1 requires polynomial arithmetic at each step, whereas Step 3 requires ideal or infrastructure arithmetic at each step, which is much slower. As a result, Step 3 dominates the overall running time in practice.

Next, we discuss practical issues surrounding actual implementations of each step of Algorithm 5.1. We remark that this is the first time that this algorithm has been implemented for cubic function fields.

6. Implementation Details

6.1. Implementation Details for Phase 1. This section presents a number of algorithms and results to apply to the problem of computing both approximations \( E_1 \) and \( E_2 \) of \( h \) in Step 1 of Algorithm 5.1. All the algorithms and derivations in this section are new. Henceforth, we assume that \( q \) is prime. While the following methods can be extended to composite \( q \), we make this restriction to facilitate the execution of Phase 1 since there are very straightforward ways to loop through
the set of all irreducible polynomials up to a fixed degree. In addition, ideal, infrastructure, and polynomial arithmetic is faster using $q$ prime.

First, we note that the splitting behavior of an ideal $\langle P \rangle$ in $\mathcal{O}_x$, where $P \in \mathbb{F}_q[x]$ is irreducible, can be ascertained by computing the cubic power residue symbol $[GH^2/P]_3$ if $q \equiv 1 \pmod{3}$. This can be done via Algorithm 6.2 of [33] and has essentially the same complexity as the Euclidean Algorithm applied to $GH^2$ and $P$. If $q \equiv 2 \pmod{3}$ and $\deg(P)$ is even, we have $[P/Q]_3 \equiv Q(\delta_{\deg(P)} - 1)^{\frac{1}{2}} \pmod{P}$, which is computed using $O(\deg(P) \log(q))$ polynomial operations.

The following equation is the core of Step 1 of Algorithm 5.1, and computes $z_1(P)^n + z_2(P)^n$, where $P \in \mathbb{F}_q[x]$ is irreducible, $n \in \mathbb{N}$, and $z_1(P)$ and $z_2(P)$ are defined as in (6.1). Combining (5.1) with Theorem 2.1 and setting $\chi(P) = [GH^2/P]_3$, we have:

$$(6.1) \quad z_1(P)^n + z_2(P)^n = \begin{cases} 
-1 & \text{if } q^\deg(P) \equiv 1 \pmod{3}, \chi(P) \neq 1, \text{ and } 3 \nmid n, \\
0 & \text{if } P \nmid GH \text{ or } q^\deg(P) \equiv 2 \pmod{3} \text{ and } 2 \nmid n, \\
2 & \text{otherwise}.
\end{cases}$$

Next, we compute the value of $S_\nu(n)$ as given in (5.2) for $1 \leq \nu \leq \lambda$ and all $n \in \mathbb{N}$. From (5.1) and (6.1), we see that $S_\nu(n) = S_\nu(6k + n)$ for all $k, n \in \mathbb{N}$. Moreover, if $q^\deg(P) \equiv 1 \pmod{3}$, then $S_\nu(1) = S_\nu(2) = S_\nu(4) = S_\nu(5)$ and $S_\nu(3) = S_\nu(6)$; and if $q^\deg(P) \equiv 2 \pmod{3}$, then $S_\nu(1) = S_\nu(3) = S_\nu(5) = 0$ and $S_\nu(2) = S_\nu(4) = S_\nu(6)$. For the cases $q^\deg(P) \equiv 2 \pmod{3}$ and $2 \nmid n$, or $q^\deg(P) \equiv 1 \pmod{3}$ and $3 \nmid n$, we have $z_1(P)^n + z_1(P)^n = 0$ if $P \nmid GH$ and $z_1(P)^n + z_1(P)^n = 2$ otherwise.

In light of this, let $I_\nu$ be the number of irreducible polynomials of degree $\nu$ and $F_\nu$ the number of prime divisors of $GH^2$ of degree $\nu$. Using well-known formulas for $I_\nu$, we have

$$(6.2) \quad S_\nu(n) = 2(I_\nu - F_\nu) = 2 \left( \frac{1}{\nu} \sum_{d|\nu} \mu \left( \frac{\nu}{d} \right) q^d - F_\nu \right)$$

for these cases, where $\mu$ is the Möbius function. If $q^\deg(P) \equiv 1 \pmod{3}$, then we must compute $S_\nu(1)$ by determining the splitting behavior of each irreducible polynomial of degree $\nu$. Algorithms 6.1 and 6.2 check each irreducible polynomial for the cases $\deg(P) = 1$ and $\deg(P) = 2$, respectively. We also note that Algorithms 6.1 and 6.2 may be parallelized by letting each processor run on distinct blocks of the interval $0 \leq c < q$.

**Algorithm 6.1 Computing $S_1(1)$**

**Input:** A prime $q$, and monic, relatively prime, square-free $G, H \in \mathbb{F}_q[x]$.

**Output:** $S_1(1)$.

1. $P := x, c := 0, S := 0$.
2. **while** $c < q$ **do**
   3. **Determine** $z := z_1(P)^n + z_2(P)^n$ as in (6.1).
   4. $S := S + z, c := c + 1, P := x + c$.
3. **end while**
4. **return** $S_1(1) = S$.

Finally, we give equations to determine $E_1$ and $E_2$. To compute $E_1$, we simply evaluate the sum given in Section 5.3. To compute $E_2$, we evaluate the sum in
Algorithm 6.2 Computing $S_2(1)$

**Input:** A prime $q$, and monic, relatively prime, square-free $G, H \in \mathbb{F}_q[x]$.

**Output:** $S_2(1)$.

1. $c := 1$, $j := 0$, $S := 0$.
2. while $c < q$ do
3.  if the Legendre symbol $(c/q) \neq 1$ then
4.    while $j < q$ do
5.      $P := (x - j)^2 + c$.
6.      Determine $z := z_1(P)^n + z_2(P)^n$ as in (6.1).
7.      $S := S + z$, $j := j + 1$.
8.  end while
9. end if
10. $j := 0$, $c := c + 1$.
11. end while
12. return $S_2(1) = S$.

Section 5.3 by reversing the order of summation. Let $\nu m = n$ so that

$$\log E'_2(\lambda) = A(K) + \sum_{\nu=1}^{\lambda} \sum_{m=1}^{\infty} \nu S_\nu(m) \nu m \nu q^{\nu m}.$$

Using the identity $\sum_{m \in \mathbb{N}} 1/k m q^{km} = (1/k) \log \left( q^k/(q^k - 1) \right)$, we have

$$\log(E'_2) = A(K) + \sum_{\nu=1}^{\lambda} \left( -S_\nu(1) \log \left( \frac{q^\nu - 1}{q^\nu} \right) ight)$$

$$+ \frac{1}{3} (S_\nu(1) - S_\nu(3)) \log \left( \frac{q^{3\nu} - 1}{q^{3\nu}} \right) \right)$$

if $q \equiv 1 \pmod{3}$. If $q \equiv 2 \pmod{3}$, then

$$\log(E'_2(\lambda)) = A(K) + \sum_{m=1}^{[\lambda/2]} \left( -S_{2m}(1) \log \left( \frac{q^{2m} - 1}{q^{2m}} \right) ight)$$

$$+ \frac{1}{3} (S_{2m}(1) - S_{2m}(3)) \log \left( \frac{q^{6m} - 1}{q^{6m}} \right) \right)$$

$$+ \sum_{m=1}^{[\lambda+1/2]} \left( -S_{2m-1}(1) \log \left( \frac{q^{2m-1} - 1}{q^{2m-1}} \right) ight)$$

$$+ \frac{1}{2} (S_{2m-1}(1) - S_{2m-1}(2)) \log \left( \frac{q^{4m-2} - 1}{q^{4m-2}} \right) \right)$$

(6.4)

6.2. Implementation Details for Phase 2. For Phase 2 of Algorithm 5.1, we gather extra information about $h$ to effectively reduce the size of the interval, $[E - U, E + U]$, determined in Phase 1. One observation is that $h$ is not uniformly distributed in this interval, and tends to be close to the approximation $E$. In Sections 3 and 4, we described how to apply the average $a(q, g) = \text{Mean}(|h - E|/U)$, taken over all cubic function fields over $\mathbb{F}_q(x)$ of genus $g$, to optimize the expected running time of the Kangaroo algorithms. Specifically the Kangaroo algorithm is optimized with a shorter average jump length in order to concentrate our effort on
the middle of the interval $[E - U, E + U]$, thereby obtaining a speed-up by a factor of $(1 + 2\alpha(q, g))/\left(2\sqrt{2\alpha(q, g)}\right)$.

However, values of $\alpha(q, g)$ are very difficult to compute precisely, so in practice we apply an approximation $\hat{\alpha}(q, g)$ of $\alpha(q, g)$ instead. Table 4 in Section 7 lists approximations $\hat{\alpha}(q, g)$ for selected values of $q$ and $g$, based on a large sampling of cubic function fields of characteristic $q$ and genus $g$. For a fixed genus $g$, we assume that there is a limiting value $\alpha(g) = \lim_{q \to \infty} \alpha(q, g)$, as is the case for hyperelliptic function fields [40], so that in practice, we can interpolate or extrapolate as needed when applying these approximations for a given $q$ in Phase 2. As such, the information for this phase is determined in advance. We will discuss the values of $\alpha(q, g)$ in more depth in Section 7.

6.3. Implementation Details for Phase 3. Algorithms 5.1 and 4.1 presented earlier implement Phase 3 of Algorithm 5.1.

A more detailed description of the outline given in Algorithm 5.1 for computing the divisor class number of a purely cubic function field is provided in Algorithm 6.3 below.

**Algorithm 6.3 Class Number Computation for Purely Cubic Function Fields**

**Input:** A prime $q$; monic, relatively prime, square-free $G, H \in \mathbb{F}_q[x]$; and $K = \mathbb{F}_q(C)$, where $C : Y^3 = GH^2$.

**Output:** The divisor class number $h$ of $K$.

1. if $3 \mid \deg(GH^2)$ then
2. $g := \deg(GH) - 2$
3. else
4. $g := \deg(GH) - 1$.
5. end if
6. Set $\lambda$ via (5.5).
7. for $\nu = 1$ to $\lambda$ do
8. if $q^{\nu} \equiv 1 \pmod{3}$ then
9. Compute $S_\nu(1)$ via Algorithm 6.1, 6.2 etc.
10. Compute $S_\nu(3)$ via (6.2).
11. else
12. $S_\nu(1) := 0$
13. Compute $S_\nu(2)$ via (6.2).
14. end if
15. end for
16. if $q \equiv 1 \pmod{3}$ then
17. Compute $E := \left[\exp(\log(E'_2))\right]$ via (6.3), $\psi_3$ via (5.4), and $U := \left[E'_2 (e^{\psi_3} - 1)\right]$.
18. Compute and output $h$ via Algorithm 3.1
19. else
20. Compute $E := \left[\exp(\log(E'_2))\right]$ via (6.4), $\psi_3$ via (5.4), and $U := \left[E'_2 (e^{\psi_3} - 1)\right]$.
21. Compute and output $h$ via Algorithm 4.1
22. end if

Theorem 5.2 implies the following.
Theorem 6.1. With $-1/4 \leq \varepsilon(g) \leq 1/2$ as in Theorem 7.2, the complexity of Algorithm 6.3 is $O\left(q^{(2g-1)/5} + \varepsilon(g)\right)$ ideal operations.

6.4. Implementation Details for Phase 4. Algorithm 6.4 outlines the procedure for the final phase of Algorithm 5.1 for the signature $(1,1;1,2)$ case; that is, determining the regulator $R_x$, given a multiplicative $h_0$. We follow the procedure described in Algorithm 4.4 of [43], making adaptations to the cubic function field case. This technique uses the fact that the regulator $R_x$ is the smallest factor of $h_0$ such that $D(2R_x) = 0$. The algorithm is an infrastructure analogue to determining the order of a group element from the group order.

Algorithm 6.4 Computing the Regulator of a Purely Cubic Function Field of Signature $(1,1;1,2)$: Phase 4

Input: A multiple $h_0$ of $R_x$, a lower bound $l$ of $R_x$, a prime $q$, and monic, relatively prime, square-free $G, H \in \mathbb{F}_q[x]$.

Output: The regulator $R_x$ of $K = \mathbb{F}_q(C)$, where $C : Y^3 = GH^2$.

1: $h^* := 1$.
2: Factor $h_0 = \prod_{i=1}^{k} p_i^{e_i}$.
3: for $i = 1$ to $k$ do
4:  if $p_i < h_0/l$ then
5:    Find $1 \leq e_i \leq a_i$ minimal such that $D(2h_0/p_i^{e_i}) \neq 0$.
6:  end if
7: end for
8: return $R_x := h_0/h^*$

We briefly comment on the running time of Algorithm 6.4 relative to the running time of Algorithm 6.3 especially in light of the factorization in Step 2. First, current heuristic methods to factor the integer $h_0$ require a subexponential number of bit operations in $\log(h_0)$ using the Elliptic Curve Method [24], the Quadratic Sieve [28, 39, 5], or the General Number Field Sieve [22] to achieve this running time. Furthermore, the loop in Steps 3-7 only requires a polynomial number (in $g$ and $\log(q)$) of infrastructure operations. Therefore, determining $R_x$ from $h_0$ does not dominate the overall running time of Algorithm 6.1 The largest divisor class numbers that we found have 28 digits, which required only a few seconds to factor. In fact, we simply used a basic implementation of Pollard’s Rho factoring method [20].

7. Computational Results

In this section, we present results and data obtained from the implementation of Algorithms 6.3 and 6.4 on cubic function fields of signatures $(3,1)$ and $(1,1;1,2)$ and genera $4 \leq g \leq 7$. We first give experimental results that allowed us to obtain constant-time speed-ups of Algorithm 5.1. We then discuss the problem of computing $\alpha_i(q,g) = \text{Mean}(h - E_i)/U_i)$, where the average is considered over all cubic function fields over $\mathbb{F}_q(x)$ of genus $g$. Finally, we list results of divisor class number and regulator computations. For timing and technical considerations, we implemented our algorithms in C++ using NTL, written by Shoup [38], compiled using gcc, and run on Sun workstations with AMD Opteron 148 2.2 GHz processors and 1 GB of RAM running Fedora 7 Linux.
7.1. General Optimization Data. For this section, we applied the Baby Step-Giant Step method to 10,000 function fields of signature $(3, 1)$ of a fixed characteristic $q$ and genus $g$, and organized the data from these computations to optimize implementations of Algorithm 6.3. This data provides means to obtain a constant-time improvement over more straightforward implementations of this algorithm. First, we compared the accuracy of the estimates $E_1$ and $E_2$. We then considered the minimal and maximal values of $|h - E_i|/U_i$ for each $i = 1, 2, 3, q$, and $g$ to provide further analysis of the estimates and compare the sharpness of the bounds $U_i$. Finally, for selected $q$ and for genera $3 \leq g \leq 7$, we list approximations $\hat{\alpha}_i(q, g)$ of $\alpha_i(q, g)$.

In each table of this and later sections, $\lambda$ is the degree bound used to compute the estimates $E_1$ and $E_2$, and $n$ is the number of randomly chosen fields $K$ of the given characteristic $q$ and genus $g$ that we used in each experiment. In Table 2, we compare how well the two estimates $E_1$ and $E_2$ approximate $h$. Here, $\pm gs$ gives the average difference between the respective number of giant steps computed using estimates $E_1$ and $E_2$, $\pm gs\%$ is the average percentage of the giant step time gained or lost by using $E_2$ versus $E_1$, and $P_2$ is the percentage of the trials in which $E_2$ was the better estimate.

| $q$ | $g$ | $\lambda$ | $\pm gs$ | $\pm gs\%$ | $P_2$ | $n$ |
|-----|-----|-----------|----------|-----------|-------|----|
| 997 | 3   | 1         | 0.23     | 0.054\%   | 51.17\% | 10000 |
| 10009 | 3 | 1        | 3.24     | 0.061\%   | 51.52\% | 10000 |
| 100003 | 3 | 1        | 7.52     | 0.014\%   | 52.04\% | 10000 |
| 997 | 4   | 1         | -9.52    | -0.067\%  | 49.53\% | 10000 |
| 10009 | 4 | 1        | 140.01   | 0.033\%   | 49.96\% | 10000 |
| 97  | 5   | 2         | 35.96    | 3.617\%   | 53.86\% | 10000 |
| 997 | 5   | 2         | -106.92  | -0.181\%  | 49.40\% | 10000 |
| 97  | 6   | 2         | 136.51   | 1.280\%   | 52.23\% | 10000 |
| 463 | 6   | 2         | 530.11   | 0.139\%   | 50.56\% | 10000 |
| 19  | 7   | 2         | 68.24    | 5.376\%   | 53.65\% | 10000 |
| 97  | 7   | 2         | 1067.35  | 0.913\%   | 51.27\% | 10000 |

In Table 3, we give the minimum and maximum values, $\min_i$ and $\max_i$, respectively, of $|h - E_i|/U_i$, for $i = 1, 2, 3$, over all the function fields we considered of a fixed $q$ and $g$. This table provides another means to compare $E_1$ and $E_2$ and also to answer the question of which $U_i$ provides the sharper error bound. For every genus and constant field that we tested, there were several examples for which the estimates $E_1$ and $E_2$ yielded extremely accurate estimates. In fact, there were two genus 5 function fields of characteristic 97 for which $E_2 = h$.

In Table 4 we list average values, $\hat{\alpha}_i(q, g) = Mean_n(|h - E_i|/U_i)$, for $i = 1, 2, 3$ ($E_2 = E_3$) and fixed $q$ and $g$, computed from the random sampling of $n = 10000$ function fields.

As with the analogous situation in hyperelliptic function fields (see Section 6 of [40]), we assumed that the limit of the actual averages, $\lim_{q \to \infty} \alpha_i(q, g) = \alpha_i(g)$, exists for each $g$. Again, we can only at best estimate what the actual limits
Table 3. Minimum and Maximum Values of $|h - E_i|/U_i$ for $i = 1, 2, 3$

| $q, g$ | $\min_1$ | $\min_2$ | $\min_3$ | $\max_1$ | $\max_2$ | $\max_3$ |
|--------|---------|---------|---------|---------|---------|---------|
| 997 3  | 0.000125 | 0.000020 | 0.000024 | 0.929573 | 0.694462 | 0.918213 |
| 10009 3 | 0.000070 | 0.000023 | 0.000031 | 0.953275 | 0.713493 | 0.948725 |
| 100003 3 | 0.000012 | 0.000013 | 0.000017 | 0.975737 | 0.730639 | 0.973334 |
| 997 4  | 0.000048 | 0.000054 | 0.000068 | 0.835200 | 0.663961 | 0.825064 |
| 10009 4 | 0.000001 | 0.000006 | 0.000008 | 0.833682 | 0.666135 | 0.831041 |
| 97 5   | 0.000069 | 0.000000 | 0.000000 | 0.813107 | 0.754499 | 0.791369 |
| 997 5   | 0.000012 | 0.000000 | 0.000000 | 0.833248 | 0.826243 | 0.839193 |
| 97 6   | 0.000019 | 0.000072 | 0.000075 | 0.808939 | 0.789328 | 0.821493 |
| 463 6   | 0.000032 | 0.000010 | 0.000011 | 0.755526 | 0.733616 | 0.747999 |
| 19 7   | 0.000033 | 0.000014 | 0.000015 | 0.579609 | 0.546870 | 0.588628 |
| 97 7   | 0.000005 | 0.000001 | 0.000001 | 0.673038 | 0.641696 | 0.664122 |

Table 4. Comparison of the $\hat{\alpha}_i(q, g)$

| $q, g$ | $\lambda$ | $\alpha_1(q, g)$ | $\alpha_2(q, g)$ | $\alpha_3(q, g)$ |
|--------|---------|----------------|----------------|----------------|
| 997 3  | 1       | 0.26832306    | 0.20003340    | 0.26448274    |
| 10009 3 | 1       | 0.27031818    | 0.20234914    | 0.26906175    |
| 100003 3 | 1       | 0.27227076    | 0.20408453    | 0.27187490    |
| 997 4  | 1       | 0.19223965    | 0.15306081    | 0.19019941    |
| 10009 4 | 1       | 0.19252978    | 0.15379110    | 0.19186318    |
| 97 5   | 2       | 0.18195632    | 0.17143328    | 0.17981087    |
| 997 5   | 2       | 0.19188423    | 0.18894457    | 0.19190607    |
| 97 6   | 2       | 0.15246065    | 0.14526827    | 0.15118788    |
| 463 6   | 2       | 0.15992960    | 0.15676849    | 0.15975657    |
| 19 7   | 2       | 0.11428348    | 0.10135344    | 0.10909269    |
| 97 7   | 2       | 0.12684120    | 0.12176623    | 0.12602172    |

are, based on experimental results. Given the behavior of the $\hat{\alpha}_i(q, g)$, the data also suggest that $\alpha_i(q)$ decreases as $q$ increases, as is the case for hyperelliptic function fields [10]. Note also that for $\lambda = 1$, the values for $\hat{\alpha}_2(q, g)$ in Table 3 are noticeably smaller than those for $\hat{\alpha}_1(q, g)$ and $\hat{\alpha}_3(q, g)$, whereas for $\lambda = 2$, the three values $\hat{\alpha}_i(a, g)$ match more closely for $i = 1, 2, 3$. An analogous phenomenon can be observed in Table 3. In the next section, we explain why this behavior is to be expected and also explain the difficulties arising in the computation of each $\alpha_i(q, g)$.

7.2. Analysis of the $\alpha_i(q, g)$. In this section, we take a closer look at the relationship between the averages $\alpha_i(q, g)$ and the error bounds $U_i$. In particular, we explain the obstructions to computing the $\alpha_i(q, g)$ more precisely.

We have $h = \prod_{j=1}^{2g} (1 - \omega_j)$, where $\omega_1, \ldots, \omega_{2g} \in \mathbb{C}$ are the reciprocals of the zeros of the zeta function of $K$. Write $\omega_j = \sqrt{q} e^{i \varphi_j}$, where $i$ is a fixed square root of $-1$ and each $0 \leq \varphi_j < 2\pi$ for $1 \leq j \leq 2g$. It is well-known that the $\varphi_j$ can be arranged
so that \( \omega_j = \pi_j + g \) and \( \varphi_j \equiv -\varphi_j + g \pmod{2\pi} \). We may therefore order the \( \varphi_j \) so that \( 0 \leq \varphi_j \leq \pi \) for \( 1 \leq j \leq g \). Set

\[
G_\lambda(\varphi_1, \ldots, \varphi_g) = \sum_{j=1}^{2g} e^{(\lambda+1)i\varphi_j} = 2 \sum_{j=1}^{g} \cos((\lambda+1)\varphi_j).
\]

The analysis in Section 5 of [34] shows that for large \( q \), we expect that

\[
\alpha_i(q, g) = \text{Mean} \left( \frac{|h - E_i|}{U_i} \right) \approx \frac{|G_\lambda(\varphi_1, \ldots, \varphi_g)| + \eta_1(q, \lambda)}{2g + \eta_2(q, \lambda)}
\]

for \( i = 1, 2, 3 \), where \( \eta_1(q, \lambda) \) and \( \eta_2(q, \lambda) \) are correction terms that depend on \( i \) and vanish for \( i = 1 \). Both \( \eta_1(q, \lambda) = \eta_1(q, \lambda) \) and \( \eta_2(q, \lambda) \) also tend to zero as \( q \) grows, as does \( \eta_2(q, \lambda) = 0 \) for \( \lambda \) even. However, \( \eta_2(q, \lambda) = [K : F_q(x)] - 1 = 2 \) for \( \lambda \) odd.

We therefore see that the averages \( \alpha_i(q, g) \) essentially depend only on the distribution of the values \( \varphi_j \) around the unit circle. For example, if each \( \varphi_j \) is close to either 0 or \( \pi \), then \( \alpha_i(q, g) \approx 1 \) (or \( \approx g/(g+1) \) for \( i = 2 \) and \( \lambda \) odd). On the other hand, if the average of the \( \varphi_j \) is close to \( \pi/2 \), then \( \alpha_i(q, g) \approx 0 \). Based on our experimental results, it is a reasonable assumption that over all cubic function fields over a fixed base field and of fixed genus, the average of the \( \varphi_j \) is distributed symmetrically about \( \pi/2 \). As the genus increases, it becomes less likely for each \( \varphi_j \), for any given function field, to be very close to either 0 or \( \pi \), thereby making it less likely for \( G_\lambda(\varphi_1, \ldots, \varphi_g) \) to be large. Hence, for increasing genus, we expect a decreasingly smaller proportion of function fields with \( \alpha_i(q, g) \) far away from 0, which would explain the lower values of the \( \alpha_i(q, g) \) in Table 4 with increasing genus. This also explains why the minimum and maximum values of the \(|h - E_i|/U_i\) in Table 3 generally decrease with increasing genus.

Note also that for \( \lambda = 1 \), the values for \( \alpha_1(q, g) \) in Table 4 are noticeably smaller than those for \( \alpha_1(q, g) \) and \( \alpha_3(q, g) \), since the denominator in the right-hand side of (7.1) is \( 2g + 2 \) for \( i = 2 \) and closer to \( 2g \) for \( i = 1, 3 \). For \( \lambda = 2 \), this denominator is approximately \( 2g \) for all \( i = 1, 2, 3 \), and thus, the three values \( \alpha_i(a, g) \) match more closely. An analogous phenomenon can be observed in Table 3.

If the values \( \varphi_j \) \( (1 \leq j \leq g) \) were distributed randomly in the interval \([0, \pi]\), over all function fields of a fixed extension degree, genus, and base field, then precise values of the \( \alpha_i(q, g) \) could be obtained for each \( 1 \leq i \leq 3 \). Unfortunately, however, this is not the case, so we cannot make this assumption. In order to determine this distribution, one must know the Haar measure of a subgroup of the symplectic group \( \text{Sp}(2g) \) and the corresponding measure \( \mu_g \). Once this measure is known, precise values of the \( \alpha_i(g) \) may be computed directly via an integral or approximated via Riemann sums. The measure \( \mu_g \) has been derived for elliptic function fields by Birch [29], and for hyperelliptic function fields of genus \( g > 1 \) by Katz, Sarnak, and Weyl [48, 16]. Unfortunately, \( \mu_g \) is not known for any function fields of degree greater than 2. It is however conjectured that we obtain similar results as for hyperelliptic function fields under the same assumptions. Nevertheless, determining \( \alpha_i(g) \) is very difficult, so we must rely on the approximations given in Table 3 to achieve an average running time of Algorithm 6.3 that is close to optimal. For further details, we refer the reader to Section 6 of [40] and Section 5 of [34].

In the following sections, we summarize results on the application of this data to Algorithm 6.3 for large examples for which no faster method is known to exist.
7.3. Families of Curves to Consider. While our algorithm is a general method to compute class numbers of purely cubic function fields of characteristic at least 5, there exist methods which work faster than ours on some special families of cubic function fields, as noted in our introduction. In this section we note which families of curves that our method works fastest on.

For polynomials \( G, H \in \mathbb{F}_q[x] \), the curve \( Y^3 = GH^2 \) is equivalent to \( Y^3 = G^3H \). In light of this and our observation that arithmetic is faster with curves \( Y^3 = GH^2 \) such that \( \deg(H) \leq \deg(G) \), we restricted our attention to such curves. Clearly we did not consider any cubic curves of genus 0, 1, or 2.

Minzlaff’s algorithm [25] is faster than ours on superelliptic curves over a prime field and the algorithm of Castryck, Denef, and Vercauteren [4] should run faster than ours on non-singular curves. Therefore, we did not consider any curves such that \( \deg(H) = 0 \). Moreover, we did not consider any curves that were equivalent to a superelliptic curve over a prime field. More specifically, if \( Y^3 = D = GH^2 \), \( 3 \mid \deg(D) \) and \( a \) is a root of \( D \in \mathbb{F}_q[x] \), then there is a transformation of \( Y^3 = D \) to a superelliptic curve over \( \mathbb{F}_q(a) \). To see this transformation, let \( n = \deg(D) \), \( t = 1/(x-a) \), and \( z = Y/(x-a)^{n/3} \) so that \( z^3 = F(t) \), where \( \deg(F) = n - 2 \). Multiply this through by \( \text{sgn}(F) \), the leading coefficient of \( F \), and finally let \( v = \text{sgn}(F)^{n-1}z \) and \( u = \text{sgn}(F)v \), yielding the superelliptic curve \( v^3 = E(u) \), where \( E \in \mathbb{F}_q(a)[u] \) is monic and squarefree. Therefore, if \( 3 \mid \deg(GH^2) \) (i.e. \( r = 1 \)), then we only considered curves such that \( GH \) had no linear factors.

The following chart organizes which families of curves we considered, by genus and unit rank, based on the degrees of \( G \) and \( H \).

| \( g \) | \( \deg(G) \) | \( \deg(H) \) | \( r = 0 \) | \( \deg(G) \) | \( \deg(H) \) | \( r = 1 \) |
|-------|----------|----------|---------|----------|----------|---------|
| 4     | 3        | 2        |         | 3        | 3        |         |
| 5     | 4        | 2        | 5       | 2        |         |         |
| 6     | 4        | 3        |         | 4        | 4        |         |
| 7     | 5        | 3        |         | 6        | 3        |         |

7.4. Signature \((3, 1)\) Computations. We used the estimate \( E = E_2 \) and the error bound \( U = U_3 \) and applied the values of \( \hat{\alpha}_3(q, g) \), for the largest values of \( q \) in Table [4] to the problem of computing large class numbers of purely cubic function fields of signature \((3, 1)\). We will henceforth denote this value of \( \hat{\alpha}_3(q, g) \) that we use in our computations by \( \hat{\alpha}(g) \). We computed the divisor class numbers of three genus 4 and two genus 5, 6, and 7 purely cubic function fields of signature \((3, 1)\). We parallelized Phases 1 and 3 of each computation, using up to 64 processors, to find class numbers up to 26 digits. In this section, we present the results of these calculations, including timing data and the choices of certain variables. We began with smaller examples in order to test the choices of certain parameters, in
from Proposition 3.1, where \( T \) is a tame and a wild kangaroo ran on the same processor), and “Total” is the sum of estimates the expected time to compute the class number of the specific function \( \beta \) example, the estimate of \( h \) in the computation, \( \lg \) relative phases took to complete, “Jumps” gives the total number of kangaroo jumps listed in Table 6. Here, “Ph. 1” and “Ph. 3” give the times (in seconds) the respective cases; we used a random irreducible polynomial generator supplied by NTL to choose these polynomials. In each case 

The genus 5 curves we used were:

\[
C_4 : Y^3 = (x^5 + 8703x^4 + 5098x^3 + 1571x^2 + 9390x + 9945)x^2, \\
C_5 : Y^3 = (x^5 + 43583x^4 + 40125x^3 + 74978x^2 + 23924x + 38273)x^2.
\]

The genus 6 curves we used were:

\[
C_6 : Y^3 = (x^4 + 212x^3 + 980x^2 + 939x + 282)(x^3 + 271x^2 + 276x + 302)^2, \\
C_7 : Y^3 = (x^4 + 4122x^3 + 698x^2 + 1994x + 4252)(x^3 + 669x^2 + 7328x + 1019)^2.
\]

The genus 7 curves we used were:

\[
C_8 : Y^3 = (x^5 + 59x^4 + 9x^3 + 22x^2 + 30x + 37)(x^3 + 30x^2 + 54x + 80)^2, \\
C_9 : Y^3 = (x^5 + 776x^4 + 117x^3 + 478x^2 + 840x + 747)(x^3 + 402x^2 + 647x + 571)^2.
\]

In each case, we used a constant field \( \mathbb{F}_q \) with prime \( q \equiv 1 \pmod{3} \). We also have \( C_i : Y^3 = G_iH_i^2 \), where \( G_i \) and \( H_i \) are relatively prime and irreducible over the field \( \mathbb{F}_q \) used in the respective cases; we used a random irreducible polynomial generator supplied by NTL to choose these polynomials. In each case The divisor class number \( h \), along with the number of decimal digits in \( h \) and the values \(|h - E|/U\) (with \( E = E_3 \) and \( U = U_3 \)), are given for each example in Table 6.

The divisor class number of \( \mathbb{F}_{10003}(C_5) \) is the largest known divisor class number of a cubic function field of genus at least 4 and signature (3, 1) defined by a singular curve over a large base field.

In Tables 7, 8, and 9 we give results from the computations of the class numbers listed in Table 10 Here, “Ph. 1” and “Ph. 3” give the times (in seconds) the respective phases took to complete, “Jumps” gives the total number of kangaroo jumps in the computation, \( \lg \theta \) indicates our choice of \( \theta \), “Traps” records the number of kangaroo traps that were set, \( m \) is the number of processors used (if \( m = 1 \), then a tame and a wild kangaroo ran on the same processor), and “Total” is the sum of these times. “Exp. 1” gives the quantity, \( (m|h - E|/\beta + 4\beta/m + \theta m)T_{G} \), obtained from Proposition 3.1 where \( T_{G} \) was the time to compose two ideals in the given example, \( \beta = (m/2)\sqrt{\alpha(g)U} \) was the average jump distance in the example, \( E \) was the estimate of \( h \), and \( U \) was the upper bound on the error; the quantity Exp. 1 estimates the expected time to compute the class number of the specific function.
Table 6. Divisor Class Numbers of Cubic Function Fields, Signature $(3,1)$

| Curve | $q$   | $g$ | $d_{ig}$ | $h$                      | $|h - E|/U$ |
|-------|-------|-----|----------|--------------------------|----------------|
| $C_1$ | $10^4 + 9$ | 4   | 17       | 10011509151678732        | 0.1234612     |
| $C_2$ | $10^5 + 3$  | 4   | 21       | 100380717456367838139    | 0.009801      |
| $C_3$ | $10^6 + 3$  | 4   | 25       | 10096470661959167786949  | 0.1887913     |
| $C_4$ | $10^4 + 9$  | 5   | 21       | 102398439790330982469    | 0.3211619     |
| $C_5$ | $10^5 + 3$  | 5   | 26       | 10017258018358358570720475| 0.0889572     |
| $C_6$ | $10^4 + 9$  | 6   | 19       | 1017494771121878691      | 0.0117266     |
| $C_7$ | $10^4 + 9$  | 6   | 25       | 1009516362119878999248876| 0.2450704     |
| $C_8$ | $10^4 + 3$  | 7   | 15       | 117601058790012         | 0.0235252     |
| $C_9$ | $10^3 + 9$  | 7   | 22       | 100242781776498360912   | 0.1006682     |

Table 7. Divisor Class Number Computation Data, Signature $(3,1)$, Genera 5 and 6

| Curve | $q$   | $g$ | $d_{ig}$ | $h$                      | $|h - E|/U$ |
|-------|-------|-----|----------|--------------------------|----------------|
| $C_4$ | $10^4 + 9$ | 4   | 17       | 10011509151678732        | 0.1234612     |
| $C_5$ | $10^5 + 3$  | 4   | 21       | 100380717456367838139    | 0.009801      |
| $C_6$ | $10^6 + 3$  | 4   | 25       | 10096470661959167786949  | 0.1887913     |
| $C_4$ | $10^4 + 9$  | 5   | 21       | 102398439790330982469    | 0.3211619     |
| $C_5$ | $10^5 + 3$  | 5   | 26       | 10017258018358358570720475| 0.0889572     |
| $C_6$ | $10^4 + 9$  | 6   | 19       | 1017494771121878691      | 0.0117266     |
| $C_7$ | $10^4 + 9$  | 6   | 25       | 1009516362119878999248876| 0.2450704     |
| $C_8$ | $10^4 + 3$  | 7   | 15       | 117601058790012         | 0.0235252     |
| $C_9$ | $10^3 + 9$  | 7   | 22       | 100242781776498360912   | 0.1006682     |

Table 8. Divisor Class Number Computation Data, Signature $(3,1)$, Genera 5 and 6

| Curve | $q$   | $g$ | $m$ | Total | Exp. 1 | Exp. 2 |
|-------|-------|-----|-----|-------|--------|--------|
| $C_4$ | $10^4 + 9$ | 4   | 21  | 21.1h | 14.9h  | 11.5h  |
| $C_5$ | $10^5 + 3$  | 64  | 44.3d | 25.0d | 32.5d  |
| $C_6$ | $10^4 + 9$  | 4   | 1.55h | 2.55h | 4.71h  |
| $C_7$ | $10^4 + 9$  | 64  | 37.5d | 59.0d | 43.4d  |

One observation to note is the amount of variation between the actual time to compute certain divisor class numbers and the expected time, Exp. 1, to compute these values using the Kangaroo method. For any given class group, the time to
Table 9. Divisor Class Number Computation Data, Signature (3, 1), Genera 4 and 7

| Curve | $q$ | $m$ | Total | Exp. 1 | Exp. 2 | Jumps | $\lg \theta$ | Traps |
|-------|-----|-----|-------|--------|--------|-------|--------------|-------|
| $C_1$ | $10^4 + 9$ | 4 | 14.6$m$ | 38.2$m$ | 46.2$m$ | 663789 | 14 | 45 |
| $C_2$ | $10^5 + 3$ | 4 | 4.86$h$ | 4.99$h$ | 16.9$h$ | 15961807 | 16 | 283 |
| $C_3$ | $10^6 + 3$ | 64 | 33.6$d$ | 25.9$d$ | 26.5$d$ | 2699578969 | 20 | 2533 |
| $C_8$ | $10^2 + 3$ | 16 | 18.1$m$ | 27.3$m$ | 38.2$m$ | 262189 | 14 | 87 |
| $C_9$ | $10^3 + 9$ | 64 | 10.8$d$ | 13.5$d$ | 16.3$d$ | 408090257 | 18 | 1213 |

compute $h$ depends on the intersection of two kangaroo paths. For a given choice of $\beta$, there are several possible choices for a set of jumps, $\{s_1, \ldots, s_{64}\}$, under the restrictions given in Step 5 of Algorithm 3.1. The number of jumps required to compute $h$ depends uniquely on this choice. Therefore, for one set of jumps, the computation may happen to finish earlier than expected while for another set, the computation may run longer than expected. It is impossible to know in advance how one choice of jump distances will affect the running time.

We also counted the number of useless collisions in each example. The computations for curves $C_1$, $C_2$, and $C_8$ yielded no collisions. At most, we had 4 useless collisions, for curves $C_3$, $C_4$, and $C_5$. A low number of useless collisions was expected, based on the results of Stein and Teske on hyperelliptic function fields [42].

In the next section, we summarize the results of regulator computations in cubic function fields of signature (1, 1; 1, 2).

7.5. Signature (1, 1; 1, 2) Computations. In this section, we tested the practical effectiveness of Algorithms 6.3 and 6.4 to compute the divisor class number and extract the ideal class number and regulator of nine purely cubic function fields of signature (1, 1; 1, 2) of genera 4 through 7. We list the ideal class number $h_x$ the regulator $R_x$ and the ratio $|h - E|/U$ in Table 10 and data from the Kangaroo computations in Tables 11 and 12.

The genus 4 curves we used for the examples in this section were:

$$C_{10} : Y^3 = (x^3 + 2833x^2 + 2425x + 5216) \cdot (x^3 + 6412x^2 + 3035x + 192)^2,$$
$$C_{11} : Y^3 = (x^3 + 18559x^2 + 21371x + 89569) \cdot (x^3 + 1149x^2 + 83421x + 94387)^2,$$
$$C_{12} : Y^3 = (x^3 + 545795x^2 + 378803x + 44676) \cdot (x^3 + 736840x^2 + 529889x + 983699)^2.$$

The genus 5 curves we used were:

$$C_{13} : Y^3 = (x^5 + 7166x^4 + 3769x^3 + 7559x^2 + 5984x + 9826) \cdot (x^2 + 5149x + 8000)^2,$$
$$C_{14} : Y^3 = (x^5 + 85771x^4 + 65270x^3 + 5761x^2 + 36247x + 18059) \cdot (x^2 + 97994x + 77903)^2.$$. 
The genus 6 curves we used were:

\[ C_{15} : Y^3 = (x^4 + 990x^3 + 684x^2 + 159x + 403) \cdot (x^4 + 235x^3 + 621x^2 + 727x + 49)^2, \]
\[ C_{16} : Y^3 = (x^4 + 2267x^3 + 941x^2 + 3751x + 575) \cdot (x^4 + 6786x^3 + 7043x^2 + 9857x + 1472)^2. \]

The genus 7 curves we used were:

\[ C_{17} : Y^3 = (x^6 + 43x^5 + 38x^4 + 9x^3 + 84x^2 + 60x + 16) \cdot (x^3 + 53x^2 + 106x + 104)^2, \]
\[ C_{18} : Y^3 = (x^6 + 54x^5 + 21x^4 + 177x^3 + 64x^2 + 428x + 216) \cdot (x^3 + 63x^2 + 866x + 687)^2. \]

In each case, we used a constant field \( F_q \), with prime \( q \equiv 2 \pmod 3 \). We also have \( C_i : Y^3 = G_iH_i^2 \), where \( G_i \) and \( H_i \) are relatively prime and irreducible over the field \( F_q \) used in the respective cases.

### Table 10. Regulators and Ideal Class Numbers, Signature (1, 1; 1, 2)

| Curve | \( q \) | \( g \) | \( h_x \) | \( R_x \) | \( |h - E|/U \) |
|-------|-------|------|-------|--------|--------|
| \( C_{10} \) | \( 10^4 + 7 \) | 4 | 48 | 2089112952541444 | 0.2552057 |
| \( C_{11} \) | \( 10^5 + 19 \) | 4 | 3 | 3335941882578135923 | 0.2460722 |
| \( C_{12} \) | \( 10^6 + 37 \) | 4 | 3 | 33338313749230954914867 | 0.1998394 |
| \( C_{13} \) | \( 10^4 + 7 \) | 5 | 9 | 11150551526104064200 | 0.0002640 |
| \( C_{14} \) | \( 10^5 + 19 \) | 5 | 3 | 33365166619760496052080 | 0.0005814 |
| \( C_{15} \) | \( 10^4 + 13 \) | 6 | 9 | 1200512183694710111 | 0.0034454 |
| \( C_{16} \) | \( 10^4 + 7 \) | 6 | 108 | 9297527414155973143524 | 0.0022748 |
| \( C_{17} \) | \( 10^2 + 7 \) | 7 | 12 | 13227046636185 | 0.0015069 |
| \( C_{18} \) | \( 10^3 + 13 \) | 7 | 162 | 67655595344411953054 | 0.0082423 |

In Table 11, “BS Jumps” and “GS Jumps” refer to the respective number of baby steps and giant steps computed using the Kangaroo method in each example.

In Table 12 “Coll.” is the number of useless collisions in the given example, “Time” refers to the total time taken in the computation. “Exp. 1” gives the quantity

\[
\left( \frac{2m|\alpha(g)|}{\beta} + \frac{2\beta}{(2\tau - 1)m} + \frac{\theta m}{\tau} \right) \left( 2 - \frac{1}{\tau} \right) T_G,
\]

obtained from Proposition 4.1 and its proof, where \( \tau \) is as given in Table 1, \( \beta = m\sqrt{(2\tau - 1)\hat{\alpha}(g)U - 2(\tau - 1)} \) was the average jump distance in the example, \( E \) was the estimate of \( h \), \( U \) was the upper bound on the error, and \( T_G \) was the time to compute a giant step in the given example; the quantity Exp. 1 estimates the expected time to compute the divisor class number of the specific function field \( F_q(C_i) \) using a single processor, based on the parameters given in Proposition 4.1. “Exp. 2” gives the quantity

\[
\left( 4\sqrt{\hat{\alpha}(g)U/(2\tau - 1) + \theta m/\tau} \right) (2 - 1/\tau) T_G,
\]

which estimates the expected time to compute the divisor class number of a purely cubic function field of the given characteristic \( q \) and genus \( g \), using a single processor.

The remaining columns refer to the same data as in Tables 7 and 9. We omitted
timing data on Phases 1 and 4 since Phase 1 took under 1 second in each case and extracting $R_x$ from $h$ in Phase 4 took at most 6 seconds.

Table 11. Regulator Computation Data, Signature (1, 1; 1, 2)

| Curve | $q$ | $g$ | BS Jumps | GS Jumps | $lg \theta$ | Traps |
|-------|-----|-----|----------|----------|------------|-------|
| $C_{10}$ | $10^4 + 7$ | 4 | 3641959 | 1168906 | 14 | 83 |
| $C_{11}$ | $10^5 + 19$ | 4 | 188417964 | 60450981 | 18 | 233 |
| $C_{12}$ | $10^6 + 37$ | 4 | 13859958890 | 4444797687 | 18 | 16888 |
| $C_{13}$ | $10^4 + 7$ | 5 | 184626263 | 4292390 | 14 | 288 |
| $C_{14}$ | $10^5 + 19$ | 5 | 736255166 | 171327866 | 18 | 711 |
| $C_{15}$ | $10^5 + 13$ | 6 | 5471177 | 1072002 | 18 | 711 |
| $C_{16}$ | $10^4 + 7$ | 6 | 181772316 | 355124175 | 18 | 1484 |
| $C_{17}$ | $10^2 + 7$ | 7 | 2290380 | 344984 | 12 | 143 |
| $C_{18}$ | $10^3 + 13$ | 7 | 511805801 | 76010424 | 18 | 370 |
| $C_{19}$ | $2027$ | 7 | 4838512480 | 718236161 | 20 | 888 |

Table 12. Regulator Computation Data, Signature (1, 1; 1, 2)

| Curve | $q$ | $g$ | $m$ | Coll. | Time | Exp. 1 | Exp. 2 |
|-------|-----|-----|-----|-------|------|-------|-------|
| $C_{10}$ | $10^4 + 7$ | 4 | 1 | – | 41.3 m | 60.6 m | 52.1 m |
| $C_{11}$ | $10^5 + 19$ | 4 | 4 | 0 | 56.9 h | 72.1 h | 43.4 h |
| $C_{12}$ | $10^6 + 37$ | 4 | 64 | 6 | 383.2 d | 125.5 d | 128.2 d |
| $C_{13}$ | $10^4 + 7$ | 5 | 16 | 0 | 14.9 h | 16.3 h | 30.9 h |
| $C_{14}$ | $10^5 + 19$ | 5 | 64 | 0 | 32.1 d | 48.8 d | 89.2 d |
| $C_{15}$ | $10^3 + 13$ | 6 | 8 | 0 | 4.83 h | 12.9 h | 25.2 h |
| $C_{16}$ | $10^4 + 7$ | 6 | 64 | 1 | 65.3 d | 89.5 d | 176.4 d |
| $C_{17}$ | $10^2 + 7$ | 7 | 16 | 2 | 138. m | 64.4 m | 127. m |
| $C_{18}$ | $10^3 + 13$ | 7 | 64 | 0 | 25.9 d | 27.3 d | 51.3 d |

8. Conclusions and Future Work

Using current implementations of the arithmetic of purely cubic function fields of signatures (3, 1) and (1, 1; 1, 2), divisor class numbers up to 26 digits were computed using the method of Scheidler and Stein [33] and the Kangaroo algorithm as a subroutine. In the signature (1, 1; 1, 2) case, we determined regulators up to 25 digits. We computed approximations $\hat{\alpha}(q, g)$ of $\alpha(q, g)$ for a few $q$ and for genera $3 \leq g \leq 7$. This allowed us to achieve a constant-time speed-up in our computations by focusing our effort to find $h$ on the center of the interval $[E - U, E + U]$, where $h$ is more likely to be found. Further speed-ups were obtained in $R$ by computing approximations of the ratio $\tau = T_G/T_B$. The divisor class numbers are the largest such known for any cubic function field of genus greater than 4 constructed from a singular curve over a prime field and the regulators are the largest known regulators...
of any cubic function field. Moreover, the improvement to the Kangaroo algorithm applies to the infrastructure of any signature \((1, 1; 1; 2)\) function field.

An extension of our techniques to the case of signature \((1, 1; 1, 1; 1)\), with appropriate adaptations to the Baby Step-Giant Step and Kangaroo algorithms in a two-dimensional infrastructure, is work in progress. In addition, efficient ideal and infrastructure arithmetic needs to be developed for arbitrary (i.e. not necessarily purely) cubic function fields as well as for characteristic 2 and 3 in order to apply this method to such function fields.

References

1. M. Bauer, *The arithmetic of certain cubic function fields*, Math. Comp. 73 (2004), no. 245, 387–413.
2. M. Bauer, E. Teske, and A. Weng, *Point counting on Picard curves in large characteristic*, Math. Comp. 74 (2005), no. 252, 1983–2005.
3. B. Birch, *How the number of points of an elliptic curve over a fixed prime field varies*, J. London Math. Soc. 43 (1968), 57–60.
4. W. Castryck, J. Denef, and F. Vercauteren, *Computing zeta functions of nondegenerate curves*, Internat. Math. Research Papers 2006 (2006), 1–57, Article ID 72017.
5. S. Contini, *Factoring integers with the self-initializing quadratic sieve*, Master’s thesis, University of Georgia, 1997.
6. J. Denef and F. Vercauteren, *Computing zeta functions of \(C_{ab}\) curves using Monsky-Washnitzer cohomology*, Finite Fields Appl. 12 (2006), no. 1, 78–102.
7. C. Diem, *An index calculus algorithm for plane curves of small degree*, Proc. of ANTS-VII (Berlin) (F. Hess, S. Pauli, and M. Pohst, eds.), Lect. Notes Comput. Sci., vol. 4076, Springer, 2006, pp. 543–557.
8. C. Diem and E. Thomé, *Index calculus in class groups of non-hyperelliptic curves of genus three*, J. Cryptology 21 (2008), no. 4, 593–611.
9. F. Fontein, *Groups from cyclic infrastructures and Pohlig-Hellman in certain infrastructures*, Adv. Math. Comm. 2 (2008), no. 3, 293–307.
10. ———, *The infrastructure of a global field and baby step-giant step algorithms*, Ph.D. thesis, Universität Zürich, Zürich, Switzerland, 2009.
11. S. D. Galbraith, S. M. Paulus, and N. P. Smart, *Arithmetic on superelliptic curves*, Math. Comp. 71 (2002), no. 237, 393–405.
12. P. Gaudry and M. Gürel, *An extension of Kedlaya’s point-counting algorithm to superelliptic curves*, Advances in Cryptology – ASIACRYPT 2001 (Berlin) (C. Boyd, ed.), Lect. Notes Comput. Sci., vol. 2248, Springer, 2001, pp. 480–494.
13. P. Gaudry, E. Thomé, N. Thériault, and C. Diem, *A double large prime variation for small genus hyperelliptic index calculus*, Math. Comp. 76 (2007), no. 257, 475–492.
14. H. Hasse, *Number Theory*, Springer, New York, 1980.
15. F. Hess, *Zur Divisorklassengruppenberechnung in globalen Funktionenkörpern*, Ph.D. thesis, Technische Universität Berlin, 1999.
16. N. Katz and P. Sarnak, *Random Matrices, Frobenius Eigenvalues and Monodromy*, AMS Colloquium Publications, vol. 45, AMS, Providence, RI, 1999.
17. K. Kedlaya, *Counting points on hyperelliptic curves using Monsky-Washnitzer cohomology*, J. Ramanujan Math. Soc. 16 (2001), no. 4, 323–338.
18. ———, *Errata for “Counting points on hyperelliptic curves using Monsky-Washnitzer cohomology”*, J. Ramanujan Math. Soc. 18 (2003), no. 4, 417–418.
19. ———, *Computing zeta functions via p-adic cohomology*, Proc. ANTS-VI (Berlin), Lect. Notes Comput. Sci., vol. 3076, Springer, 2004, pp. 1–17.
20. E. Landquist, *Infrastructure, Arithmetic, and Class Number Computations in Purely Cubic Function Fields of Characteristic at Least 5*, Ph.D. thesis, University of Illinois at Urbana-Champaign, 2009, [http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.163.6450](http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.163.6450)
21. A. Lauder, *Computing zeta functions of Kummer curves via multiplicative characters*, Found. Comp. Math. 9 (2009), no. 3, 273–295.
22. A. Lenstra, H. Lenstra, M. Manasse, and J. Pollard, *The number field sieve*, The Development of the Number Field Sieve (A. Lenstra and H. Lenstra, eds.), Springer, New York, 1993, pp. 11–42.
23. H. Lenstra, *On the computation of regulators and class numbers of quadratic fields*, Journées Arithmétiques 1980 (J. Armitage, ed.), Lond. Math. Soc. Lect. Notes, vol. 56, Cambridge University Press, 1982, pp. 123–150.
24. _, *Factoring integers with elliptic curves*, Ann. of Math. **126** (1987), no. 2, 649–673.
25. M. Minzlaff, *Computing zeta functions of superelliptic curves in larger characteristic*, Mathematics in Computer Science **3** (2010), no. 2, 209–224.
26. J. Pollard, *A Monte Carlo method for factorization*, BIT Num. Math. **15** (1975), no. 3, 331–334.
27. _, *Monte Carlo methods for index computation (mod p)*, Math. Comp. **32** (1978), no. 143, 918–924.
28. C. Pomerance, *Analysis and comparison of some integer factoring algorithms*, Computational Methods in Number Theory, Part I (Jr. H. Lenstra and R. Tijdeman, eds.), vol. 154, Math. Centre Tract, Amsterdam, 1982, pp. 89–139.
29. M. Rosen, *Number Theory in Function Fields*, Grad. Texts Math., vol. 210, Springer, New York, 2002.
30. R. Scheidler, *Ideal arithmetic and infrastructure in purely cubic function fields*, J. Theor. Nombres Bordeaux **13** (2001), no. 2, 609–631.
31. _, *Algorithmic aspects of cubic function fields*, Proc. of ANTS-VI (Berlin) (D. Buell, ed.), Lect. Notes Comput. Sci., vol. 3976, Springer, 2004, pp. 395–410.
32. R. Scheidler and A. Stein, *Voronoi’s algorithm in purely cubic function fields of unit rank 1*, Math. Comp. **69** (2000), no. 231, 1245–1266.
33. _, *Approximating Euler products and class number computation in algebraic function fields*, To appear, Rocky Mountain J. Math., 2008.
34. F. Schmidt, *Analytische Zahlentheorie in Körpern der Charakteristik p*, Math. Zeit. **33** (1931), 668–678.
35. R. Schoof, *Quadratic fields and factorization*, Computational Methods in Number Theory II, Math. Centre Tracts, vol. 155, Math. Centrum, Amsterdam, 1982, pp. 235–286.
36. D. Shanks, *Class number, a theory of factorization and genera*, Proc. Symp. Pure Math. **20** (1971), 415–440.
37. V. Shoup, *NTL: A Library for Doing Number Theory*, New York, NY, 2008, Version 5.4.2.
38. R. Silverman, *The multiple polynomial quadratic sieve method of computation*, Math. Comp. **48** (1987), no. 177, 329–340.
39. A. Stein and E. Teske, *Explicit bounds and heuristics on class numbers in hyperelliptic function fields*, Math. Comp. **71** (2002), no. 238, 837–861.
40. _, *The parallelized Pollard kangaroo method in real quadratic function fields*, Math. Comp. **71** (2002), no. 238, 793–814.
41. A. Weng, *A low-memory algorithm for point counting on Picard curves*, Des. Codes Cryptogr. **38** (2006), no. 3, 383–393.
42. H. Weyl, *Gesammelte Abhandlungen*, vol. II, Springer, Berlin, 1968.
