CONGRUENCES FOR $k$-ELONGATED PLANE PARTITION DIAMONDS

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Abstract. In the eleventh paper in the series on MacMahon’s partition analysis, Andrews and Paule [1] introduced the $k$-elongated partition diamonds. Recently, they [2] revisited the topic. Let $d_k(n)$ count the partitions obtained by adding the links of the $k$–elongated plane partition diamonds of length $n$. Andrews and Paule [2] obtained several generating functions and congruences for $d_1(n)$, $d_2(n)$, and $d_3(n)$. They also posed some conjectures, among which the most difficult one was recently proved by Smoot [11]. Da Silva, Hirschhorn, and Sellers [5] further found many congruences modulo certain primes for $d_k(n)$ whereas Li and Yee [8] studied the combinatorics of Schmidt type partitions, which can be viewed as partition diamonds. In this article, we give elementary proofs of the remaining conjectures of Andrews and Paule [2], extend some individual congruences found by Andrews and Paule [2] and da Silva, Hirschhorn, and Sellers [5] to their respective families as well as find new families of congruences for $d_k(n)$, present a refinement in an existence result for congruences of $d_k(n)$ found by da Silva, Hirschhorn, and Sellers [5], and prove some new individual as well as a few families of congruences modulo 5, 7, 8, 11, 13, 16, 17, 19, 23, 25, 32, 49, 64 and 128.

Key words: Congruence, Generating function, $k$-Elongated partition diamonds

2010 Mathematical Reviews Classification Numbers: Primary 11P83; Secondary 05A17.

1. Introduction and Results

For complex numbers $a$ and $q$ such that $|q| < 1$, we define the infinite $q$-product as

$$(a; q)_\infty := \prod_{j=0}^{\infty} (1 - aq^j).$$

For convenience, we set $f_n := (q^n; q^n)_\infty$ for integers $n \geq 1$.

Andrews and Paule [1] in 2007 introduced a combinatorial object called the $k$-elongated partition diamonds. Recently, they [2] looked back at the $k$-elongated partition diamonds. Let $d_k(n)$ count the partitions obtained by adding the links of the $k$–elongated plane partition diamonds of length $n$. Then the generating function for $d_k(n)$ is given by

$$\sum_{n=0}^{\infty} d_k(n)q^n = \frac{f_2^k}{f_{3k+1}}.$$
Andrews and Paule [2] found some elegant generating functions for $d_1(n)$, $d_2(n)$, and $d_3(n)$. They also proved many Ramanujan type congruences modulo 2, 3, 4, 5, 8, 9, 27, and 243, mainly by using the Mathematica package RaduRK developed by Smoot [11], which uses Radu’s Ramanujan-Kolberg algorithm [10].

Andrews and Paule [2] conjectured some congruences in their paper. For example, they conjectured that for all $n \geq 1$ and $k \geq 1$ such that $8n \equiv 1 \pmod{3^k}$, then
\[ d_2(n) \equiv 0 \pmod{3^k}. \]

By manipulation of a certain ring of modular functions, Smoot recently [12] not only proved but refined (1.1) as
\[ d_2(n) \equiv 0 \pmod{3^{\left\lfloor k/2 \right\rfloor + 1}} \]
for all $n \geq 1$ and $k \geq 1$ such that $8n \equiv 1 \pmod{3^k}$.

Andrews and Paule [2] conjectured some more congruences modulo 81, 243, and 729.

**Conjecture 1.1.** For all $n \geq 0$,
\[ d_2(81n + 44) \equiv 0 \pmod{81}, \]
\[ d_2(81n + j) \equiv 0 \pmod{243}, \text{ where } j \in \{8, 35, 62, 71\}, \]
\[ d_2(243n + 71) \equiv 0 \pmod{729}. \]

**Remark 1.2.** Andrews and Paule conjectured another congruence [2, (7.17)], which is in fact contained in (1.3).

Da Silva, Hirschhorn, and Sellers [5] gave elementary proofs for some of the results of Andrews and Paule [2] and discovered new individual congruences as well as some infinite families of congruences for $d_k(n)$ modulo certain primes. For example, for prime $p \geq 5$, let $r$, $1 \leq r \leq p - 1$, be a quadratic nonresidue modulo $p$. Then for all $n \geq 0$ and $N \geq 1$,
\[ d_{p^{N-1}}(pn + r) \equiv 0 \pmod{p^N}. \]

Additionally, they [5] proved the following overarching theorem, which generalizes the Ramanujan type congruences modulo prime $p$ with arithmetic progression $p$.

**Theorem 1.3.** Let $p$ be a prime, $k \geq 1$, $j \geq 0$, $N \geq 1$ and $r$ be an integer such that $1 \leq r \leq p - 1$. If, for all $n \geq 0$,
\[ d_k(pn + r) \equiv 0 \pmod{p^N}, \]
then for all $n \geq 0$,
\[ d_{p^{N+j+k}}(pn + r) \equiv 0 \pmod{p^N}. \]

Andrews and Paule [2] considered the partition diamonds as the Schmidt type partitions and in [8], Li and Yee found generating functions for Schmidt $k$-partitions and unrestricted Schmidt $k$-partitions in unified combinatorial ways.

In this article, in Section 2 we give an elementary proof of Conjecture 1.1. In Section 3 we extend some individual congruences found by Andrews and Paule [2] and
da Silva, Hirschhorn, and Sellers [5] to their families. In Section 4, we also find some new families of congruences for \( d_k(n) \) modulo 8, 16, 32, 64, and 128. In Section 5, we present a refinement in an existence result for congruences of \( d_k(n) \) found by da Silva, Hirschhorn, and Sellers [5]. Finally in Section 6, we prove some new individual as well as a few families of congruences modulo 5, 7, 11, 13, 17, 19, 23, 25, and 49.

2. Proof of Conjecture 1.1 of Andrews and Paule

First, we recall the following 3-dissections of \( f_1^2/f_2, f_2^2/f_1, 1/f_1^3, \) and \( f_1f_2 \) from [6] (14.3.2), (14.3.3), (39.2.8), and (14.3.1)]

\[
\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3f_{18}^2}{f_6f_9}, \quad (2.1)
\]

\[
\frac{f_2^2}{f_1} = \frac{f_6f_9^2}{f_3f_{18}} + q\frac{f_{18}^2}{f_9}, \quad (2.2)
\]

\[
\frac{f_1}{f_2^2} = \frac{f_3f_9^2}{f_6f_{18}} - q\frac{f_{33}f_{18}^2}{f_6f_9} + q^2\frac{f_3f_{18}^6}{f_6f_9}, \quad (2.3)
\]

\[
\frac{f_3}{f_1} = a(q^3) f_3 - 3qf_3^3, \quad (2.4)
\]

\[
\frac{1}{f_1^3} = a^2(q^3) \frac{f_3^3}{f_{10}^3} + 3q^2a^2(q^3) \frac{f_3f_6^3}{f_{11}^3} + 9q^2 \frac{f_3^9}{f_{12}^3}, \quad (2.5)
\]

\[
f_1f_2 = \frac{f_6f_9^4}{f_3f_{18}^2} - qf_9f_{18} - 2q^2\frac{f_3f_{18}^4}{f_6f_9^2}, \quad (2.6)
\]

where \( a(q) \) is Borwein’s cubic theta function defined by \( a(q) := \sum_{j,k=-\infty}^{\infty} q^{m^2+mn+n^2} \).

We have

\[
\sum_{n=0}^{\infty} d_2(n)q^n = \frac{f_2^2}{f_1} = \frac{f_1^2}{f_2} = \frac{1}{f_1^2}. \quad (2.7)
\]

We apply (2.2) and (2.3) in (2.7) to extract the terms involving \( q^{3n+2} \) to obtain

\[
\sum_{n=0}^{\infty} d_2(3n+2)q^n = 27a^2(q) \frac{f_2f_3^{14}}{f_1^{23}f_6} + 6a^3(q) \frac{f_3^8f_6^2}{f_1^{21}f_6} + 81q^{17}\frac{f_3^{17}f_6^2}{f_1^{24}}. \quad (2.8)
\]

Here, the following identities from [6] Section 22.10 and (21.3.2)] again, come into our use

\[
a(q) = \frac{f_3^3}{f_3} + 9q\frac{f_9^3}{f_3}, \quad (2.9)
\]

\[
a^3(q) = \frac{f_3^3}{f_3} + 27q\frac{f_9^3}{f_3}, \quad (2.10)
\]

\[
8q\frac{f_3^8f_6^5}{f_2f_3} = \frac{f_3^8}{f_3} - \frac{f_3^8}{f_2}. \quad (2.11)
\]
Using (2.9) and (2.10) in (2.8), we have
\[\sum_{n=0}^{\infty} d_2(3n + 2)q^n = 6\frac{f_1^5 f_6^2}{f_1^{12}} + 27\frac{f_2 f_3^{12} f_6}{f_1^{17} f_6} + 243q\frac{f_3^{17} f_6^2}{f_1^{24}} + 486q\frac{f_2 f_3^{12} f_6^3}{f_1^{20} f_6} + 2187q^2 \frac{f_2 f_3^{12} f_6^6}{f_1^{23} f_6^6}.\] (2.12)

Under modulo 729, (2.12) becomes
\[\sum_{n=0}^{\infty} d_2(3n + 2)q^n \equiv 6\frac{f_1^{231} f_6^2}{f_6^{76}} + 27\frac{f_1^{10} f_2 f_3^3}{f_6} + 243q f_3 f_6^2 + 486q f_1 f_2 f_3 f_6^3 \pmod{729}.\] (2.13)

Using (2.4) and (2.6), we find that
\[f_1^{231} \equiv a^{77} \left(q^3\right) f_3^{77} + 12qa^{76} \left(q^3\right) f_3^{76} f_9 + 90q^2 a^{75} \left(q^3\right) f_3^{75} f_9 + 54q^3 a^{74} \left(q^3\right) f_3^{74} f_9^3 + 162q^4 a^{73} \left(q^3\right) f_3^{73} f_9^{12} \pmod{243},\] (2.14)
\[f_1^{10} f_2 \equiv 18q^3 a^2 \left(q^3\right) \frac{f_3 f_9 f_1^4}{f_6} + 9q^2 a^2 \left(q^3\right) f_3^2 f_9 f_1^4 - 2q^2 a^3 \left(q^3\right) \frac{f_3 f_1^4}{f_6} - 9qa^2 \left(q^3\right) \frac{f_3 f_6 f_9^7}{f_1^2} - qa^{3} \left(q^3\right) f_3^{-1} f_9 f_1^4 + a^{3} \left(q^3\right) \frac{f_3^2 f_6 f_9^4}{f_1^2} \pmod{27}.\] (2.15)

Therefore, invoking (2.7), (2.14), and (2.15) in (2.13), and then extracting the terms that involve \(q^{3n+2}\), we have
\[\sum_{n=0}^{\infty} d_2(9n + 8)q^n \equiv 540a^{75} \left(q\right) \frac{f_2 f_3^2 f_6^5}{f_1} - 54a^3 \left(q\right) \frac{f_1^7 f_6^4}{f_2 f_3^2} - 243a^2 \left(q\right) \frac{f_3^5 f_1^4 f_6}{f_2} \pmod{729},\] (2.16)
where we use the fact that \(a(q) \equiv 1 \pmod{3}\), which is clear from (2.9).

From (2.16), we have
\[\sum_{n=0}^{\infty} d_2(9n + 8)q^n \equiv 540\frac{f_2 f_3^{12}}{f_1^{19}} - 54\frac{f_1^7 f_6^4}{f_2 f_3^2} - 54\frac{f_2 f_3^{12} f_6}{f_1 f_3} \left(\frac{f_3^8}{f_6^6} - \frac{f_1^8}{f_2^6}\right) \pmod{243},\]
which, due to \(a^3(q) \equiv 1 \pmod{9}\), reduces to
\[\sum_{n=0}^{\infty} d_2(9n + 8)q^n \equiv 540\frac{f_2 f_3^{12}}{f_1^{19}} - 54\frac{f_1^7 f_6^4}{f_2 f_3^2}.\] (mod 243).
The above identity, on account of (2.11) and (2.1), becomes
\[
\sum_{n=0}^{\infty} d_2(9n+8)q^n \equiv 432q \frac{f_1^3 f_2^3}{f_2 f_3^3} - 864q^2 \frac{f_6^3 f_{18}^2}{f_3^3 f_9} \pmod{243}. \tag{2.17}
\]
Therefore, from the above identity, we evidently have
\[
d_2(27n+8) \equiv 0 \pmod{243} \tag{2.18}
\]
for all \(n \geq 0\).

To prove Theorem 1.1, we require the following 3-dissection of \(a(q)\), which was proved by Hirschhorn, Garvan, and Borwein [7]
\[
a(q) = a\left(q^3\right) + 6q^3 \frac{f_3^3}{f_2}\tag{2.19}
\]
Using (2.2), (2.3), (2.4), and (2.19), we obtain
\[
a^{75}(q) \frac{f_2^2 f_3^5}{f_1^3} \equiv 18q^2 a^{74}\left(q^3\right) f_3^5 f_9 f_{18}^2 + qa^{75}\left(q^3\right) \frac{f_2^5 f_3^2}{f_9} + 18qa^{74}\left(q^3\right) \frac{f_3^2 f_6 f_5^3}{f_{18}}
+ a^{75}\left(q^3\right) \frac{f_3^5 f_6 f_5^2}{f_{18}} \pmod{27}, \tag{2.20}
\]
\[
a^3(q) \frac{f_1^3 f_4^2}{f_2^2 f_3^3} \equiv 9q^4 a^3\left(q^3\right) \frac{f_6^2 f_3^3 f_{18}^3}{f_6^3} - 9q^3 a^2\left(q^3\right) \frac{f_3 f_9 f_{18}^3}{f_6^4} + 12q^3 a^4\left(q^3\right) \frac{f_3^3 f_{18}^6}{f_6^4}
+ 9q^2 a^3\left(q^3\right) \frac{f_9^3 f_6^3}{f_6^4} - 12q^2 a^4\left(q^3\right) \frac{f_3^2 f_9 f_{18}^3}{f_6^5} + q^2 a^5\left(q^3\right) \frac{f_3^3 f_{18}^6}{f_6^5}
+ 12qa^4\left(q^3\right) \frac{f_3^3 f_9 f_{18}^3}{f_6^6} - qa^5\left(q^3\right) \frac{f_3^3 f_{18}^6}{f_6^6} + a^5\left(q^3\right) \frac{f_3^3 f_9 f_{18}^3}{f_6^6} \pmod{27}. \tag{2.21}
\]
Employing (2.1), (2.20), and (2.21) in (2.16) and then extracting the terms containing \(q^{3n+1}\), we find that
\[
\sum_{n=0}^{\infty} d_2(27n+17)q^n \equiv 540a^{75}(q) \frac{f_1^6 f_2^6}{f_3} + 9720a^{74}(q) \frac{f_1^4 f_2 f_3^5}{f_6} - 648a^4(q) \frac{f_1 f_3^6}{f_2^2}
- 243a^3(q) \frac{f_1^6 f_2^6}{f_3} + 54a^5(q) \frac{f_1^3 f_6^3}{f_2^3} - 486a^3(q) \frac{f_2^3 f_3^6}{f_2^2}
\equiv 540a^{75}(q) \frac{f_1^6 f_2^6}{f_3} + 9720f_1 f_2 f_3^6 f_6 - 648a(q) \frac{f_1 f_3^6}{f_2^2}
- 243f_3 f_6^2 + 54a^5(q) \frac{f_1^3 f_6^3 f_3^6}{f_2^2} - 486q^2 f_1^2 f_3^3 f_6^3 \pmod{729}, \tag{2.22}
\]
where we use the facts that \(a(q) \equiv 1 \pmod{3}\) and \(a^3(q) \equiv 1 \pmod{9}\), which is evident from (2.10).
From (2.22) and the fact that \( a(q) \equiv 1 \pmod{3} \), we find that
\[
\sum_{n=0}^{\infty} d_2(27n + 17)q^n \equiv 54 \left( a^{75}(q) \frac{f_1^6 f_3^6}{f_3} + a^5(q) \frac{f_1^3 f_6^3}{f_2^2} \right) \equiv 27f_3f_6^2 \pmod{81}, \tag{2.23}
\]
which immediately proves (1.2).

With the help of (2.3), (2.4), and (2.5), we have
\[
a^{75}(q) \frac{f_1^6 f_3^6}{f_3} \equiv a^{77}(q^3) f_3f_6^2 + 12qa^{76}(q^3) f_6^2 f_3^3 + 9q^2a^{75}(q^3) \frac{f_6^2 f_3^6}{f_3} \pmod{27},
\]
\[
a(q) \frac{f_1^3 f_6^3}{f_2^2} \equiv a(q^3) f_3^3 f_6^3 f_3^6 f_6^6 - qa(q^3) f_3^9 f_6^9 f_3^6 f_6^6 - 6q^2f_3^3 f_6^3 f_3^6 f_6^6 \pmod{9},
\]
\[
a^5(q) \frac{f_1^3 f_6^3}{f_2^2} \equiv a^7(q^3) \frac{f_6^3 f_6^3}{f_3^6 f_3^6} + 6qa^6(q^3) \frac{f_6^3 f_6^3 f_3^1 f_3^1}{f_3^1 f_3^1} - 6a^7(q^3) \frac{f_6^3 f_6^3}{f_3^3 f_3^3} + 18q^2a^6(q^3) \frac{f_6^3 f_6^3 f_3^1 f_3^1}{f_3^1 f_3^1} + 18q^3 a^6(q^3) \frac{f_6^3 f_6^3 f_3^1 f_3^1}{f_3^1 f_3^1} \pmod{27}.
\]

We now apply the above three identities as well as (2.1), (2.3), (2.4), (2.5), and (2.6) in (2.22). Then from the resulting identity, we extract the terms that involve \( q^{3n+2} \) to arrive at
\[
\sum_{n=0}^{\infty} d_2(81n + 71)q^n \equiv -2430 \frac{f_1^8 f_3^3 f_6^3}{f_3^1 f_1^8} + 4860a^{75}(q) \frac{f_2^2 f_3^6}{f_1^2} + 648a^7(q) \frac{f_2 f_3^3 f_6^3}{f_1^8} + 972a^6(q) \frac{f_2^2 f_3^3 f_6^3}{f_1^3} + 9720a(q) \frac{f_1 f_6^4}{f_2^3} - 648a(q) \frac{f_1^1 f_6^4}{f_2^3} - 486q \frac{f_1^1 f_6^4}{f_2 f_3^3} \pmod{729}.
\]
The above identity, again with the aid of \( a(q) \equiv 1 \pmod{3} \), \( a^3(q) \equiv 1 \pmod{9} \), and \( a(q) \equiv f_3^3/f_3 \pmod{9} \), which follows from (2.9), can be rewritten as
\[
\sum_{n=0}^{\infty} d_2(81n + 71)q^n \equiv -2430 \frac{f_1^2 f_3^5 f_6}{f_2} + 5832 \frac{f_2^2 f_3^3 f_6}{f_1} + 648a(q) \frac{f_2 f_3^3 f_6^3}{f_1^8} + 9720 \frac{f_1 f_6^4}{f_2^3} - 648a(q) \frac{f_1 f_6^4}{f_2^3} - 486q \frac{f_1 f_6^4}{f_2 f_3^3} - 486q \frac{f_1^1 f_6^4}{f_2 f_3^3} - 486q \frac{f_1^1 f_6^4}{f_2 f_3^3}.
\]
\[
\equiv -2430 \frac{f_2^3 f_5^3 f_6}{f_2} + 9720 \frac{f_1 f_4^4 f_6}{f_2^2} - 486q \frac{f_2^4 f_6^6}{f_2 f_3^3} \pmod{729}.
\]

From the above identity, we have
\[
d_2(81n + 71) \equiv 0 \pmod{243} \quad (2.24)
\]
for all \(n \geq 0\). Congruences (2.18) and (2.24) together is (1.3).

Employing (2.1) and (2.3) in the identity above, we have
\[
\sum_{n=0}^{\infty} d_2(81n + 71) q^n \equiv -2430 \frac{f_5^3 f_6 f_3^3}{f_{18}} + 9720 \frac{f_3^2 f_6^3}{f_2^2} - 486q \frac{f_6^6 f_3^3}{f_3 f_{18}} + 4860q \frac{f_3^6 f_{18}}{f_9} \\
- 9720q \frac{f_3^3 f_{18}}{f_6^3} + 972q^2 \frac{f_6^8 f_{18}}{f_3^4 f_9} + 9720q^2 \frac{f_4^4 f_6^6}{f_6^4 f_3^3} \pmod{729},
\]
from which, it follows that
\[
\sum_{n=0}^{\infty} d_2(243n + 71) q^n \equiv -2430 \frac{f_5^3 f_2 f_3^2}{f_6} + 9720 \frac{f_1^4 f_3^3}{f_2^2} \equiv -2430 \frac{f_1^4 f_3^3}{f_2^2} + 9720 \frac{f_1^4 f_3^3}{f_2^2} \\
\equiv 7290 \frac{f_2^3 f_3^3}{f_2^2} \pmod{729}.
\]
The above identity clearly implies (1.4). \(\square\)

3. Families for individual congruences of Andrews and Paule [2] and da Silva et. al. [5]

In this section, we extend some of the individual congruences of Andrews and Paule [2] and da Silva, Hirschhorn, and Sellers [5] to certain families of congruences in the following theorem.

**Theorem 3.1.** For all \(n \geq 0\) and \(j \geq 0\),
\[
d_{4j+3}(4n + 2) \equiv 0 \pmod{2} \quad (3.1)
\]
\[
d_{4j+3}(4n + 3) \equiv 0 \pmod{4} \quad (3.2)
\]
\[
d_{8j+7}(4n + 2) \equiv 0 \pmod{4} \quad (3.3)
\]
\[
d_{8j+7}(8n + 5) \equiv 0 \pmod{4} \quad (3.4)
\]
\[
d_{16j+3}(16n + 9) \equiv 0 \pmod{4} \quad (3.5)
\]
\[
d_{8j+7}(4n + 3) \equiv 0 \pmod{8} \quad (3.6)
\]
\[
d_{32j+7}(8n + 4) \equiv 0 \pmod{8} \quad (3.7)
\]
\[
d_{8j+8}(9n + 3) \equiv 0 \pmod{9} \quad (3.8)
\]
\[
d_{27j+2}(9n + 8) \equiv 0 \pmod{27} \quad (3.9)
\]
\[
d_{243j+2}(27n + 8) \equiv 0 \pmod{243}. \quad (3.10)
\]
Using (3.12), we have

Proof. To prove (3.11), we first find the following exact generating function

$$\sum_{n=0}^{\infty} d_{4j+3}(4n+2)q^n$$

$$= 2 \left\{ \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} 2^{2(2k+m)} \left( \frac{6j+5}{4k} \right) \left( \frac{13j+11}{2m+1} \right) q^{k+m} \frac{F_2 F_4}{F_1 F_8} \right\}$$

$$+ \left\{ \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} 2^{2(2k+m)+1} \left( \frac{6j+5}{4k+2} \right) \left( \frac{13j+11}{2m+2} \right) q^{k+m} \frac{F_2 F_4}{F_1 F_8} \right\},$$

(3.11)

where $F_1 := f_{6j}^{5j-8k}$, $F_2 := f_{2}^{30j-24k+4m}$, $F_4 := f_{4}^{53j-16k-12m}$, and $F_8 := f_{8}^{26j-8m}$. From (3.11), (3.11) is evident.

We need the following 2-dissection from [6, (1.9.4)] to establish (3.11).

$$\frac{1}{f_2^2} = \frac{f_5^5}{f_2^5 f_4} + 2q \frac{f_4^2 f_6}{f_2^5 f_4}.$$

(3.12)

Using (3.12), we have

$$\sum_{n=0}^{\infty} d_{4j+3}(n)q^n = \frac{f_2^{4j+3}}{f_1^{12j+10}} = f_2^{4j+3} \left( \frac{1}{f_1^1} \right)^{6j+5} f_2^{4j+3} \left( \frac{f_5^5}{f_2^5 f_4} + 2q \frac{f_4^2 f_6}{f_2^5 f_4} \right)^{6j+5}$$

$$= \sum_{k=0}^{6j+5} 2^k \binom{6j+5}{k} q^k f_2^{2k} f_8^{30j-6k+25} f_2^{12j-4k+10} f_4^4 f_8$$

$$= \frac{1}{f_2^{26j+22}} \left\{ \sum_{k=0}^{3j+2} 2^{2k} \binom{6j+5}{2k} q^{2k+1} \frac{f_4^4 f_8^{30j-12k+25}}{f_2^{12j-8k+10}} \right\}$$

$$+ \sum_{k=0}^{3j+2} 2^{2k+1} \binom{6j+5}{2k+1} q^{2k+1} \frac{f_4^4 f_8^{12j-8k+25}}{f_2^{12j-8k+10}}.$$

Extracting the terms that involve $q^{2n}$ from the above identity, we obtain

$$\sum_{n=0}^{\infty} d_{4j+3}(2n)q^n = \frac{1}{f_1^1} \left( \frac{1}{f_1^1} \right)^{13j+11} \sum_{k=0}^{3j+2} 2^{2k} \binom{6j+5}{2k} q^{k} \frac{f_4^4 f_8^{30j-12k+25}}{f_2^{12j-8k+10}},$$

which again using (3.12) can be written as

$$\sum_{n=0}^{\infty} d_{4j+3}(2n)q^n = \left( \frac{f_5^5}{f_2^5 f_4} + 2q \frac{f_4^2 f_6}{f_2^5 f_4} \right)^{13j+11} \sum_{k=0}^{3j+2} 2^{2k} \binom{6j+5}{2k} q^{k} \frac{f_4^4 f_8^{30j-12k+25}}{f_2^{12j-8k+10}}.$$
Now, we break the right side of the above identity on the parity of $k$ and $m$ as follows.

\[
\sum_{n=0}^{\infty} d_{4j+3}(2n)q^n = \left\{ \sum_{k=0}^{\left\lfloor (3j+2)/2 \right\rfloor} 2^{2k} \left( \frac{6j+5}{4k} \right) q^{2k} \frac{f_4^{30j-12k+25} f_8^{13j+11}}{f_8^{12j-8k+10} f_4^{12j-16k+10}} \right. \\
\left. \quad \times \frac{f_2^{30j-24k+13} f_4^{24k+2} (6j+5)}{f_4^{24k+2} f_8^{24k+2}} \right\} \\
\left\{ \sum_{m=0}^{\left\lfloor (3j+1)/2 \right\rfloor} 2^{2m} \left( \frac{13j+11}{2m} \right) q^{2m} \frac{f_4^{65j-12m+55} f_8^{65j-12m+55} f_4^{65j-12m+55} f_8^{65j-12m+55}}{f_2^{65j-12m+55} f_4^{65j-12m+55} f_8^{65j-12m+55} f_4^{65j-12m+55}} \right. \\
\left. \quad + \sum_{m=0}^{\left\lfloor (3j+10)/2 \right\rfloor} 2^{2m+1} \left( \frac{13j+11}{2m+1} \right) q^{2m+1} \frac{f_4^{65j-12m+49} f_8^{65j-12m+49}}{f_2^{65j-12m+49} f_4^{65j-12m+49} f_8^{65j-12m+49} f_4^{65j-12m+49}} \right\}. \tag{3.13}
\]

From (3.13), extracting the terms involving $q^{2n+1}$, we deduce (3.11).

In a similar way, one can find the following generating functions, from which (3.2), (3.9), and (3.6) are evident, respectively.

\[
\sum_{n=0}^{\infty} d_{4j+3}(4n+3)q^n = \\
4 \left\{ \sum_{k=0}^{\left\lfloor (3j+1)/2 \right\rfloor} \sum_{m=0}^{\left\lceil (3j+1)/2 \right\rceil} 2^{2(2k+m)+1} \left( \frac{6j+5}{4k+3} \right) \left( \frac{13j+11}{2m} \right) q^{k+m} \frac{F_2 F_4}{F_1 F_8} \right. \\
\left. \quad + \sum_{k=0}^{\left\lfloor (3j+10)/2 \right\rfloor} \sum_{m=0}^{\left\lfloor (3j+10)/2 \right\rfloor} 2^{2(2k+m)} \left( \frac{6j+5}{4k+1} \right) \left( \frac{13j+11}{2m+1} \right) q^{k+m} \frac{F_2 F_4}{F_1 F_8} \right\} \tag{3.14}
\]

\[
\sum_{n=0}^{\infty} d_{8j+7}(4n+2)q^n = \\
4 \left\{ \sum_{k=0}^{\left\lceil (3j+1)/2 \right\rceil} \sum_{m=0}^{\left\lfloor (3j+1)/2 \right\rfloor} 2^{4(k+m)} \left( \frac{12j+11}{4k} \right) \left( \frac{13j+12}{2m+1} \right) q^{k+m} \frac{G_2}{G_1 G_4} \right. \\
\left. \quad + \sum_{k=0}^{\left\lfloor (3j+1)/2 \right\rfloor} \sum_{m=0}^{\left\lceil (3j+1)/2 \right\rceil} 2^{4(k+m)} \left( \frac{12j+11}{4k+2} \right) \left( \frac{13j+12}{2m} \right) q^{k+m} \frac{G_2}{G_1 G_4} \right\} \tag{3.15}
\]

where $G_1 := f_1^{18j-8k-8m}, G_2 := f_2^{24j-24k-24m},$ and $G_4 := f_4^{76j-16k-16m}$.

\[
\sum_{n=0}^{\infty} d_{8j+7}(4n+3)q^n
\]
Note that like the above generating functions, the exponents of \( f_1 \) in the generating functions of \( d_{8j+7}(4n+1) \), \( d_{16j+3}(4n+1) \), and \( d_{32j+7}(4n) \) will also involve \( k \). Therefore, the exact generating functions for (3.4), (3.5), and (3.7) can not be found as elegantly as the above exact generating functions. So in the following, we give simple proofs for them as well as for the remaining congruences.

The proofs of (3.4), (3.5), and (3.7) are similar. So, we prove (3.7) only. We have

\[
\sum_{n=0}^{\infty} d_{32j+7}(n)q^n = \frac{f_2^{32j+7}}{f_1^{96j+22}} \equiv \frac{f_1^2}{f_2^{16j+5}} \pmod{8}. \tag{3.17}
\]

Here, we require the following 2-dissection of \( f_1^2 \) and \( 1/f_1^4 \) from [6, (1.9.4) and d (1.10.1)].

\[
f_1^2 = \frac{f_2^5 f_8^5}{f_4^2 f_8^2} - 2q f_2 f_1^6, \tag{3.18}
\]
\[
\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q f_2^{12} f_8 f_4. \tag{3.19}
\]

Employing (3.18) in (3.17), then extracting the terms that involve \( q^{2n} \), and using (3.19), we obtain

\[
\sum_{n=0}^{\infty} d_{32j+7}(2n)q^n \equiv \frac{f_4^5}{f_2^{8j+2} f_8^2} \left( \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q f_2^{12} f_8 f_4 \right) \pmod{8},
\]

which gives

\[
\sum_{n=0}^{\infty} d_{32j+7}(4n)q^n \equiv \frac{f_2^{19}}{f_1^{8j+16} f_3^6} \equiv \frac{1}{f_2^{4j-11} f_4^6} \pmod{8}.
\]

The above identity clearly gives (3.7).

Now, we prove (3.8), (3.9), and (3.10). Using (2.1), we have

\[
\sum_{n=0}^{\infty} d_{3j+8}(n)q^n = \frac{f_2^{3j+8}}{f_1^{27j+25}} \equiv \frac{f_6^{3j+3}}{f_3^{9j+9}} f_2 \equiv \frac{f_6^{3j+3}}{f_3^{9j+9}} \left( \frac{f_9}{f_1^{18}} - 2q f_3 f_8 f_9 \right) \pmod{9}, \tag{3.9}
\]

which gives

\[
\sum_{n=0}^{\infty} d_{3j+8}(3n)q^n \equiv \frac{f_2^{3j+3} f_3^2}{f_1^{9j+9} f_6} \equiv \frac{(f_3^{3j+1})}{f_6} \pmod{9}.
\]
Applying (2.4) in the above identity, then expanding binomially, we find that
\[
\sum_{n=0}^{\infty} d_{9j+8}(3n)q^n \equiv \frac{1}{f_3^{3j+1}f_6} \sum_{k=0}^{j+1} (-3)^k \binom{j+1}{k} q^{2k} (a(q^6) f_6)^{-j-k+1} f_{18}^{3k} \pmod{9}.
\]
Since in the right side of the above identity, there is no term that involve \(q^{3n+1}\), extracting the terms that involve \(q^{3n+1}\) from the above identity, we deduce (3.8).
We have
\[
\sum_{n=0}^{\infty} d_{27j+2}(n)q^n = \frac{f_2^{27j+2}}{f_1^{3j+1}} = \frac{f_6^{9j}}{f_3^{3j}} \cdot \frac{f_2^2}{f_1^3} \cdot \frac{1}{f_1^6} \pmod{27}.
\]
Using (2.2) and (2.5) in the above identity, and then extracting the terms that involve \(q^{3n+2}\), we obtain
\[
\sum_{n=0}^{\infty} d_{27j+2}(3n+2)q^n \equiv \frac{f_2^{9j}}{f_1^{3j+2}} \left( 27 f_2 f_1^3 f_6 + 6a^2(q) f_3^2 f_6^2 f_1^3 + 81 q f_3^3 f_6 f_1^2 \right) 
\equiv 6 \frac{f_3^{3j+2}}{f_1^{3j-2}} \cdot \frac{1}{f_1^3} \pmod{27}.
\]
Now, invoking (2.5) in the above identity, then extracting the terms involving \(q^{3n+2}\), we prove (3.9).

Finally, we prove (3.10) using induction on \(j\). Andrews and Paule [2, (7.12)] proved (3.10) for \(j = 0\). We assume that (3.10) is true for some integer \(j \geq 0\). Now,
\[
\sum_{n=0}^{\infty} d_{243(j+1)+2}(n)q^n = \frac{f_2^{243(j+1)+2}}{f_1^{3(243(j+1)+2)+1}} = \frac{f_6^{243j+2}}{f_3^{243j+2} f_1^3} \cdot \frac{f_2^{243}}{f_1^{243}} = \sum_{n=0}^{\infty} d_{243j+2}(n)q^n \cdot \frac{f_2^{243}}{f_1^{243}} 
\equiv \sum_{n=0}^{\infty} d_{243j+2}(n)q^n \cdot \frac{f_6^{81}}{f_3^{243}} \pmod{243}.
\]
Extracting the terms that involve \(q^{3n+2}\) from both sides of the above identity, we have
\[
\sum_{n=0}^{\infty} d_{243(j+1)+2}(3n+2)q^n \equiv \sum_{n=0}^{\infty} d_{243j+2}(3n+2)q^n \cdot \frac{f_6^{81}}{f_3^{243}} \pmod{243}.
\]
Da Silva, Hirschhorn, and Sellers [5, (21)] showed that \(d_{3j+2}(3n+2) \equiv 0 \pmod{3}\). Therefore, the above identity can be written as
\[
\sum_{n=0}^{\infty} d_{243(j+1)+2}(3n+2)q^n \equiv \sum_{n=0}^{\infty} d_{243j+2}(3n+2)q^n \cdot \frac{f_6^{9}}{f_3^{81}} \pmod{243}.
\]
Again, extracting the terms that involve \(q^{3n+2}\) from both sides of the above identity, we have
\[
\sum_{n=0}^{\infty} d_{243(j+1)+2}(9n+8)q^n \equiv \sum_{n=0}^{\infty} d_{243j+2}(9n+8)q^n \cdot \frac{f_6^{9}}{f_3^{81}} \pmod{243}.
\]
Due to (3.9), the above identity is equivalent to
\[ \sum_{n=0}^{\infty} d_{243(j+1)+2}(9n + 8)q^n \equiv \sum_{n=0}^{\infty} d_{243j+2}(9n + 8)q^n \cdot \frac{f_{5}^{3}}{f_{3}^{27}} \pmod{243}, \]
which gives
\[ \sum_{n=0}^{\infty} d_{243(j+1)+2}(27n + 8)q^n \equiv \sum_{n=0}^{\infty} d_{243j+2}(27n + 8)q^n \cdot \frac{f_{3}^{3}}{f_{27}^{2}} \pmod{243}. \]
Therefore, by the assumption for induction, we see that (3.10) is true for \( j + 1 \) as well. Thus, (3.10) is true for all \( j \geq 0 \). \( \square \)

4. New families of congruences modulo 8, 16, 32, 64, and 128

In this section, we give new families of congruences taking the advantage of (3.15) and (3.16) from Section 3. We also provide an exact generating function for \( d_{16j+15}(4n + 3) \) here and use it to prove new congruences.

**Theorem 4.1.** For all \( n \geq 0 \) and \( j \geq 0 \),

\[ d_{8j+7}(8n + 6) \equiv 0 \pmod{8}, \]  
(4.1)

\[ d_{8j+7}(8n + 7) \equiv 0 \pmod{16}, \]  
(4.2)

\[ d_{16j+7}(8n + 6) \equiv 0 \pmod{16}, \]  
(4.3)

\[ d_{16j+7}(16n + 11) \equiv 0 \pmod{16}, \]  
(4.4)

\[ d_{16j+15}(4n + 3) \equiv 0 \pmod{16}, \]  
(4.5)

\[ d_{16j+15}(8n + 6) \equiv 0 \pmod{16}, \]  
(4.6)

\[ d_{16j+15}(16n + 10) \equiv 0 \pmod{16}, \]  
(4.7)

\[ d_{32j+31}(4n + 3) \equiv 0 \pmod{32}, \]  
(4.8)

\[ d_{16j+15}(8n + 7) \equiv 0 \pmod{64}, \]  
(4.9)

\[ d_{32j+31}(8n + 7) \equiv 0 \pmod{128}. \]  
(4.10)

**Proof.** First, we prove (4.5). Similar to (3.14)–(3.16), using (3.12) and (3.19), one can find that

\[ \sum_{n=0}^{\infty} d_{16j+15}(4n + 3)q^n = \frac{f_{2}^{447}}{f_{1}^{344}f_{4}^{134}} \left\{ \sum_{k=0}^{6j+5} \sum_{m=0}^{13j+12} 2^{4(k+m)} \left( \begin{array}{c} 24j + 23 \\ 4k + 3 \end{array} \right) \left( \begin{array}{c} 26j + 25 \\ 2m \end{array} \right) \right\} q^{k+m} - \frac{f_{2}^{484j-24k-24m}}{f_{1}^{364j-8k-8m}f_{4}^{152j-16k-16m}}. \]
Now, we separate the right side of the above identity with the cases \((k, m) = (0, 0)\) and \((k, m) \neq (0, 0)\) as follows.

\[
\sum_{n=0}^{\infty} d_{16j+15}(4n + 3)q^n
= 8 \frac{f_{16j+15}^{447}}{f_1^{134} f_4^{134}} \left\{ \begin{array}{c}
\sum_{m=0}^{13j+11} 2^{4(m+1)} \left( \begin{array}{c}
24j + 23 \\
3
\end{array} \right) + \left( \begin{array}{c}
24j + 23 \\
1
\end{array} \right) \left( \begin{array}{c}
26j + 25 \\
1
\end{array} \right) \frac{f_{2}^{484j}}{f_1^{364j-8k-336} f_4^{152j-16k-118}} + 8 \sum_{m=0}^{13j+11} 2^{4m} \left( \begin{array}{c}
24j + 23 \\
3
\end{array} \right) f_{16j+15}^{447} \left( \begin{array}{c}
24j + 23 \\
3
\end{array} \right) \left( \begin{array}{c}
26j + 25 \\
1
\end{array} \right) \frac{f_{2}^{484j}}{f_1^{364j-8k-336} f_4^{152j-16k-118}}
\end{array} \right.
\]

On simplifying the above identity, we find that

\[
\sum_{n=0}^{\infty} d_{16j+15}(4n + 3)q^n
= 16 \left\{ \begin{array}{c}
3(24j + 23)(16j + 17)(j + 1) \frac{f_{2}^{484j+447}}{f_1^{364j+447} f_4^{152j+134}} + 8 \sum_{m=0}^{13j+11} 2^{4m} \left( \begin{array}{c}
24j + 23 \\
3
\end{array} \right) f_{16j+15}^{447} \left( \begin{array}{c}
24j + 23 \\
3
\end{array} \right) \left( \begin{array}{c}
26j + 25 \\
1
\end{array} \right) \frac{f_{2}^{484j}}{f_1^{364j-8k-336} f_4^{152j-16k-118}}
\end{array} \right.
\]

Note that \((4.15)\) is evident from \((4.11)\).
Now, we prove (4.10). Replacing \( j \) by \( 2j + 1 \) and taking modulo 128 in (4.11), we obtain

\[
\sum_{n=0}^{\infty} d_{32j+31}(4n+3)q^n \equiv 96(48j+47)(32j+33)(j+1)\frac{f_2^{604j+931}}{f_4^{304j+286}}
\]

\[
\equiv 96(48j+47)(32j+33)(j+1)\frac{f_2^{604j+577}}{f_4^{304j+286}} \pmod{128}.
\]

From the above identity, we clearly have (4.10).

Similar to the proof of (4.10), we can obtain (4.1), (4.3), (4.6), and (4.7) from (3.15), (4.2) and (4.4) from (3.16), and (4.8) and (4.9) from (4.11). \( \square \)

5. AN EXISTENCE RESULT FOR INFINITE FAMILIES OF CONGRUENCES

In this section, we provide the following theorem that refines Theorem 1.3, which was found by da Silva, Hirschhorn, and Sellers [5].

Theorem 5.1. Let \( p \) be a prime, \( k \geq 1, j \geq 0, N \geq 1, M \geq 1, \) and \( r \) be integers such that \( 1 \leq r \leq p^M - 1. \) If for all \( n \geq 0, \)

\[
d_k (p^M n + r) \equiv 0 \pmod{p^N},
\]

then for all \( n \geq 0, \)

\[
d_{p^Mn^{-1}j} (p^M n + r) \equiv 0 \pmod{p^N}.
\]

Proof. Without loss of generality, we may assume that \( r = \sum_{j=0}^{M-1} p^j r_j \) for \( 0 \leq r_j \leq p - 1, \)

because \( p^j r_j \) can take any value between 1 and \( p^M - 1. \) For integers \( M \geq 1 \)

(sufficiently large) and \( N \geq 1, \) we have

\[
\sum_{n=0}^{\infty} d_{p^{M+1}n^{-1}j+k}(n)q^n = \frac{f_2^{p^{M+1}n^{-1}j+k}}{f_4^{3p^{M+1}n^{-1}j+3k+1}} \equiv \frac{f_2^{p^{M+1}n^{-1}j}}{f_4^{3p^{M+1}n^{-1}j-2}} \sum_{n=0}^{\infty} d_k(n)q^n \pmod{p^N}.
\]

Extracting the terms that involve \( q^{pn+r_0} \) from the above identity, we obtain

\[
\sum_{n=0}^{\infty} d_{p^{M+1}n^{-1}j+k}(pn+r_0)q^n = \frac{f_2^{p^{M+1}n^{-1}j}}{f_4^{3p^{M+1}n^{-1}j-2}} \sum_{n=0}^{\infty} d_k(pn+r_0)q^n
\]

\[
\equiv \frac{f_2^{p^{M+1}n^{-1}j}}{f_4^{3p^{M+1}n^{-1}j-2}} \sum_{n=0}^{\infty} d_k(pn+r_0)q^n \pmod{p^N}.
\]

Now, extracting the terms that involve \( q^{pn+r_1} \) from the above identity, we find that

\[
\sum_{n=0}^{\infty} d_{p^{M+1}n^{-1}j+k}(p^2n+r_0+p_1r_1)q^n = \frac{f_2^{p^{M+1}n^{-1}j}}{f_4^{3p^{M+1}n^{-1}j-2}} \sum_{n=0}^{\infty} d_k(p^2n+r_0+p_1r_1)q^n
\]

\[
\equiv \frac{f_2^{p^{M+1}n^{-1}j}}{f_4^{3p^{M+1}n^{-1}j-2}} \sum_{n=0}^{\infty} d_k(p^2n+r_0+p_1r_1)q^n \pmod{p^N}.
\]
From the above identity, we extract the terms that contain $q^{pn+r_2}$, and from the resulting identity, we again extract the terms that contain $q^{pn+r_3}$. It can be seen that after the $M$-th extraction using this iterative scheme, we arrive at

$$
\sum_{n=0}^{\infty} d_{p^{M+N-1-j+k}}(p^M n + r_0 + pr_1 + \cdots + p^{M-1} r_{M-1}) q^n \equiv 0 \pmod{p^N}.
$$

Therefore, if we assume that $d_k(p^M n + r_0 + pr_1 + \cdots + p^{M-1} r_{M-1}) = d_k(p^M n + r) \equiv 0 \pmod{p^N}$, from the above identity, we evidently have

$$
d_{p^{M+N-1-j+k}}(p^M n + r) \equiv 0 \pmod{p^N}.
$$

Thus, we complete the proof Theorem 5.1. \hfill \Box

**Remark 5.2.** Theorem 6.1 is a refinement of Theorem 1.3 in the sense that it extends individual congruences with arithmetic progressions $p^M n + r$, $M \geq 1$ to their respective families, whereas Theorem 1.3 extends individual congruences with arithmetic progressions $pn + r$ only to their respective families. For example, for all $n$, we have

$$
d_7(2n+1) \not\equiv d_7(8n+7) \equiv 0 \pmod{16}.
$$

(5.1)

So, Theorem 1.3 does not provide any information regarding its extension to an infinite family, whereas Theorem 5.1 and (5.1) imply that

$$
d_{64j+7}(8n+7) \equiv 0 \pmod{16}.
$$

6. NEW INDIVIDUAL AND FAMILIES OF CONGRUENCES MODULO 5, 7, 11, 13, 17, 19, 23, 25, AND 49

In this section, we present some new individual as well as a few families of congruences. Here, we use modular identities of the Rogers-Ramanujan continued fraction, which is defined as

$$
R(q) := \frac{1}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}}}
$$

a 7-dissection of $f_1$, series representations of certain $q$-products, and an algorithm developed by Radu [9] to prove the following congruences.

**Theorem 6.1.** For all $n \geq 0$ and $j \geq 0$, we have

$$
d_1(25n + 23) \equiv 0 \pmod{5},
$$

(6.1)

$$
d_1(125n + j) \equiv 0 \pmod{25}, \text{ where } j \in \{23, 123\},
$$

(6.2)

$$
d_2(125n + j) \equiv 0 \pmod{5}, \text{ where } j \in \{97, 122\},
$$

(6.3)
\[ d_1 (49n + j) \equiv 0 \pmod{7}, \quad \text{where } j \in \{17, 31, 38, 45\}, \quad (6.4) \]
\[ d_2 (49n + 43) \equiv 0 \pmod{7}, \quad (6.5) \]
\[ d_3 (49n + 41) \equiv 0 \pmod{7}, \quad (6.6) \]
\[ d_3 (343n + j) \equiv 0 \pmod{49}, \quad \text{where } j \in \{90, 188, 237\}, \quad (6.7) \]
\[ d_4 (343n + j) \equiv 0 \pmod{7}, \quad \text{where } j \in \{39, 235, 284\}, \quad (6.8) \]
\[ d_4 (121n + 96) \equiv 0 \pmod{11}, \quad (6.9) \]
\[ d_5 (121n + 91) \equiv 0 \pmod{11}, \quad (6.10) \]
\[ d_7 (121n + 81) \equiv 0 \pmod{11}, \quad (6.11) \]
\[ d_{13j+3} (13n + 11) \equiv 0 \pmod{13}, \quad (6.12) \]
\[ d_{17j+5} (17n + 13) \equiv 0 \pmod{17}, \quad (6.13) \]
\[ d_6 (289n + j) \equiv 0 \pmod{17}, \quad \text{where } j \in \{52, 69, 137, 171, 188, 205, 222, 239, 273\}, \quad (6.14) \]
\[ d_{19j+3} (19n + 16) \equiv 0 \pmod{19}, \quad (6.15) \]
\[ d_{19j+6} (19n + 9) \equiv 0 \pmod{19}, \quad (6.16) \]
\[ d_{19j+7} (19n + 13) \equiv 0 \pmod{19}, \quad (6.17) \]
\[ d_{23j+8} (23n + 9) \equiv 0 \pmod{23}. \quad (6.18) \]

6.1. Required lemmas. Here, we present some background material on the method of Radu [9]. For integers \(x\), let \([x]_m\) denote the residue class of \(x\) in \(\mathbb{Z}/m\mathbb{Z}\), \(\mathbb{Z}^*\) be the set of all invertible elements in \(\mathbb{Z}_m\), \(S_m\) denote the set of all squares in \(\mathbb{Z}^*_m\), and for integers \(N \geq 1\), we assume that

\[
\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \text{ and } ad - bc = 1 \right\},
\]
\[
\Gamma_\infty := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\},
\]
\[
\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\},
\]
\[
[\Gamma : \Gamma_0(N)] := N \prod_{\ell \mid N} \left(1 + \frac{1}{\ell}\right),
\]

where \(\ell\) is a prime.

For integers \(M \geq 1\), suppose that \(R(M)\) is the set of all the integer sequences \((r_\delta) := (r_{\delta_1}, r_{\delta_2}, r_{\delta_3}, \ldots, r_{\delta_k})\) indexed by all the positive divisors \(\delta\) of \(M\), where \(1 = \delta_1 < \delta_2 < \cdots < \delta_k = M\). For integers \(m \geq 1\), \((r_\delta) \in R(M)\), and \(t \in \{0, 1, 2, \ldots, m - 1\}\), we define the set \(P(t)\) as

\[
P(t) := \left\{ t' \in \{0, 1, 2, \ldots, m - 1\} : t' \equiv ts + \frac{s - 1}{24} \sum_{\delta \mid M} \delta r_\delta \pmod{m} \right\},
\]
for some \([s]_{24m} \in S_{24m}\). \(\quad (6.19)\)

For integers \(N \geq 1\), \(\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\), \((r_\delta) \in R(M)\), and \((r'_\delta) \in R(N)\), we also define

\[
p(\gamma) := \min_{\lambda \in \{0, 1, \ldots, m-1\}} \frac{1}{24} \sum_{\delta \mid M} r_\delta \left(\delta(a + k\lambda c), mc\right)^2, \]

\[
p'(\gamma) := \frac{1}{24} \sum_{\delta \mid N} r'_\delta \left(\delta, c\right)^2. \]

For integers \(m \geq 1\); \(2 \nmid m\), \(M \geq 1\), \(N \geq 1\), \(t \in \{0, 1, \ldots, m-1\}\), \(k := (m^2 - 1, 24)\), and \((r_\delta) \in R(M)\), define \(\Delta^*\) to be the set of all tuples \((m, M, N, t, (r_\delta))\) such that all of the following conditions are satisfied

1. Prime divisors of \(m\) are also prime divisors of \(N\);
2. If \(\delta \mid M\), then \(\delta \mid mN\) for all \(\delta \geq 1\) with \(r_\delta \neq 0\);
3. \(24 \mid kN \sum_{\delta \mid M} r_\delta mN/\delta\);
4. \(8 \mid kN \sum_{\delta \mid M} r_\delta\);
5. \(\frac{24m}{-24kt - k \sum_{\delta \mid M} \delta r_\delta, 24m} \mid N.\)

The following lemma supports Lemma 6.3 in the proof of Theorem 6.1.

**Lemma 6.2.** \([13, \text{Lemma } 4.3]\) Let \(N\) or \(\frac{1}{2}N\) be a square-free integer, then we have

\[
\bigcup_{\delta \mid N} \Gamma_0(N) \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \Gamma_\infty = \Gamma.
\]

We end this section by stating a result of Radu \([9]\), which is especially useful in completing the proof of Theorem 6.1 in the final section.

**Lemma 6.3.** \([9, \text{Lemma } 4.5]\) Suppose that \((m, M, N, t, (r_\delta)) \in \Delta^*, (r'_\delta) := (r'_\delta)_{\delta \mid N} \in R(N), \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subseteq \Gamma\) is a complete set of representatives of the double cosets of \(\Gamma_0(N) \backslash \Gamma / \Gamma_\infty\), \(t_{\text{min}} := \min_{t' \in P(t)} t'\),

\[
\nu := \frac{1}{24} \left(\sum_{\delta \mid M} r_\delta + \sum_{\delta \mid N} r'_\delta\right) \left[\Gamma : \Gamma_0(N)\right] - \sum_{\delta \mid N} \delta r'_\delta - \frac{1}{m} \sum_{\delta \mid M} \delta r_\delta - t_{\text{min}}/m, \quad (6.20)
\]
\[ p(\gamma_j) + p'(\gamma_j) \geq 0 \text{ for all } 1 \leq j \leq n, \text{ and } \sum_{n=0}^{\infty} A(n)q^n := \prod_{\delta \mid M} f_\delta^\alpha. \] If for some integers \( u \geq 1, \) all \( t' \in P(t), \) and \( 0 \leq n \leq [\nu], \) \( A(mn + t') \equiv 0 \pmod{u} \) is true, then for integers \( n \geq 0 \) and all \( t' \in P(t), \) we have \( A(mn + t') \equiv 0 \pmod{u}. \)

6.2. Proof of Theorem 6.1

Proof of (6.1). First, for integers \( \alpha \geq 0 \) and \( \beta, \) we let

\[ P_{\alpha,\beta} := \frac{1}{R(q)^{\alpha+2\beta} R(q^2)^{2\alpha-\beta}} + (-1)^{\alpha+\beta} q^{2\alpha} R(q)^{\alpha+2\beta} R(q^2)^{2\alpha-\beta} \]  

(6.21)

and recall two 5-dissections from [4, pp. 161–165] as follows

\[ f_1 = f_{25} \left( \frac{1}{R(q^5)} - q - q^2 R(q^5) \right), \]  

(6.22)

\[ \frac{1}{f_1} = \frac{f_{25}^5}{f_{5}^6} \left( \frac{1}{R^4(q^5)} + \frac{q}{R^3(q^5)} + \frac{2q^2}{R^2(q^5)} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5 R(q^5) + 2q^6 R^2(q^5) - q^7 R^3(q^5) + q^8 R^4(q^5) \right). \]  

(6.23)

Now,

\[ \sum_{n=0}^{\infty} d_1(n)q^n = f_2 / f_1. \]

Employing the 5-dissections of \( f_2 \) and \( 1 / f_1 \) from (6.22) and (6.23) in the above identity, then extracting the terms that involve \( q^{5n+3} \), and finally with the help of (6.21), we obtain

\[ \sum_{n=0}^{\infty} d_1(5n + 3)q^n = \frac{f_{20}^5 f_{10}^1}{f_{1}^7} \left( -4P_{3,6} + 40P_{3,5} - 105qP_{2,5} - 418qP_{2,4} \right. \]
\[ \left. + 1100qP_{2,3} - 1400q^2P_{1,3} - 1840q^2P_{1,2} + 1200q^2P_{1,1} - 1500q^3P_{0,1} - 1015q^3 \right). \]  

(6.24)

From [3, Lemma 1.3] and [4, (7.4.9)], we have

\[ P_{0,1} = 4q^2 \frac{f_{1}^1 f_{10}^5}{f_{2}^2 f_{5}^5}, \]  

(6.25)

\[ P_{1,1} = \frac{f_{2}^2 f_{5}^5}{f_{1}^1 f_{10}^5} + 2q + 4q^2 \frac{f_{1}^1 f_{10}^5}{f_{2}^2 f_{5}^5} \]  

(6.26)

\[ P_{1,2} = \frac{f_{6}^1}{f_{5}^5} + 11q. \]  

(6.27)
and the following relations hold
\[ P_{1,3} = P_{0,1}P_{1,2} + P_{1,1}, \quad P_{2,3} = P_{1,1}P_{1,2} - q^2P_{0,1}, \]
\[ P_{2,4} = P_{1,2}^2 + 2q^2, \quad P_{2,5} = P_{0,1}P_{2,4} - P_{2,3}, \]
\[ P_{3,5} = P_{1,1}P_{2,4} - q^2P_{1,3}, \quad P_{3,6} = P_{1,2}P_{2,4} + q^2P_{1,2}. \]

Employing (6.25)–(6.27) and the above relations in (6.24), we find that
\[
\sum_{n=0}^{\infty} d_1(5n + 3)q^n = 40f_{10}f_2^{13}f_{13}^{15}f_{14}^{14} - 4f_{10}f_5^2f_5\frac{f_1^6}{f_1^6} - 470qf_{10}f_5^8f_5\frac{f_1^{12}}{f_1^{12}} + 1875qf_{2}f_5^{19}\frac{f_1^{19}}{f_1^{19}} + 15625q^2f_{10}f_5^2f_5\frac{f_1^{25}}{f_1^{25}}f_4^{10} - 8750q^2f_{10}f_5^2f_5\frac{f_1^{14}}{f_1^{14}}f_2^{10} - 260q^2f_{10}f_5^2f_5\frac{f_1^{6}}{f_1^{6}}f_2^{10} - 46875q^3f_{10}f_5^2f_5\frac{f_1^{20}}{f_1^{20}}f_2^{14} - 62500q^4f_{10}f_5^2f_5\frac{f_1^{15}}{f_1^{15}}f_2^{18},
\]
which under modulo 5 gives
\[
\sum_{n=0}^{\infty} d_1(5n + 3)q^n \equiv -4f_{10}f_5^{2}\frac{f_1^{6}}{f_1^{6}} \pmod{5},
\]
\[
\equiv f_{10}f_5^{2}f_1^{2} \pmod{5}.
\]

Invoking the 5-dissection of \(1/f_1\) given by (6.23) in the above identity and then extracting the terms involving \(q^{5n+4}\), we obtain (6.4).

Proof of (6.5). We have
\[
\sum_{n=0}^{\infty} d_2(n)q^n = \frac{f_2^2}{f_1^2} = \frac{f_2^2}{f_7} \pmod{7}. \tag{6.28}
\]
From [6] (10.5.1)], we recall the following 7-dissection of \(f_1\).
\[
f_1 = f_{49} \left( \frac{q^{14}; q^{49}}{q^{2}; q^{49}} - q^{21} \frac{q^{28}; q^{49}}{q^{14}; q^{49}} - q^{5} \frac{q^{7}; q^{49}}{q^{2}; q^{49}} \right).
\]
With the help of the above identity, we use the 7-dissection of \(f_2^2\) in (6.28) and then extract the terms involving \(q^{7n+1}\). This gives
\[
\sum_{n=0}^{\infty} d_2(7n + 1)q^n \equiv \frac{f_{14}^{2}}{f_1} \pmod{7}. \tag{6.29}
\]
Now, if \( p(n) \) counts the unrestricted partitions of an integer \( n \geq 0 \), we have

\[
\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f_1}
\]

and one of Ramanujan’s famous three partition congruences

\[ p(7n + 6) \equiv 0 \pmod{7} \]

for all \( n \geq 0 \).

Therefore, it becomes evident from (6.29) that (6.5) is true. \( \square \)

Proofs of (6.12), (6.13), and (6.16)–(6.18). We have

\[
\sum_{n=0}^{\infty} d_3(n)q^n = \frac{f_3^3}{f_1^{10}} \equiv \frac{f_3^3}{f_2^3} f_1^{10} \pmod{13}.
\]

Using Jacobi’s famous identity [4, (1.3.24)] in the above identity,

\[
\sum_{n=0}^{\infty} d_3(n)q^n \equiv \frac{1}{f_{13}} \sum_{j=0}^{\infty} (-1)^j (2j + 1)q^{j(j+1)/2} \sum_{k=0}^{\infty} (-1)^k (2k + 1)q^{k(k+1)}
\]

\[
= \frac{1}{f_{13}} \sum_{j,k=0}^{\infty} (-1)^{j+k}(2j + 1)(2k + 1)q^{j(j+1)/2+k(k+1)} \pmod{13}.
\]

Now,

\[
8 \left( \frac{j(j+1)}{2} + k(k+1) \right) + 3 = (2j + 1)^2 + 2(2k + 1)^2.
\]

If \( j(j+1)/2 + k(k+1) = 13n + 11 \) for some integer \( n \geq 0 \), the above equality gives

\[
(2j + 1)^2 + 2(2k + 1)^2 \equiv 0 \pmod{13}.
\]

Therefore, \( 2j + 1 \equiv 0 \pmod{13} \) and \( 2k + 1 \equiv 0 \pmod{13} \). Otherwise, we have \( (2j + 1)^2 \equiv 1, 3, 4, 9, 10, 12 \pmod{13} \), which gives \( (2j + 1)^2 + 2(2k + 1)^2 \neq 0 \pmod{13} \). This is a contradiction.

Finally, extracting the terms that involve \( q^{13n+11} \) from (6.30), we find that for all \( n \geq 0 \),

\[
d_3(13n + 11) \equiv 0 \pmod{13}.
\]

(6.31)

Theorem 1.3 and (6.31) ensure (6.12).

Congruences (6.13) and (6.16)–(6.18) can be proved similarly as above. So, we do not go in detail but provide the following product-to-sum identities and a chart useful for their proofs. \( \square \)

\[
f_1^3 = \sum_{j=0}^{\infty} (-1)^j (2j + 1)q^{j(j+1)/2},
\]

(6.32)
\[
\frac{f_2^2}{f_1} = \sum_{j=0}^{\infty} q^{j(j+1)/2}, \quad (6.33)
\]

\[
\frac{f_2^5}{f_1^2} = \sum_{j=-\infty}^{\infty} (-1)^j (3j + 1) q^{3j^2+2j}. \quad (6.34)
\]

| Congruence | Used product-to-sum identities |
|------------|-------------------------------|
| (6.13)     | (6.32), (6.34)                |
| (6.16)     | (6.32)                        |
| (6.17)     | (6.33), (6.34)                |
| (6.18)     | (6.32), (6.34)                |

Proofs of the remaining congruences of Theorem \(6.1\). Proofs of (6.2)–(6.4), (6.6)–(6.11), (6.14), and (6.15) are similar. We elaborate the proof of (6.2) only. We have

\[
\sum_{n=0}^{\infty} d_1(n)q^n = f_2 \equiv f_2^{11} f_2 \equiv f_2^{11} f_2 (\text{mod } 25). \quad (6.35)
\]

Using Conditions 1–5, it is clear that \((m, M, N, t, (r_\delta)) = (125, 10, 10, 23, (21, 1, -5, 0)) \in \Delta^*\). So, by (6.19), we have \(P(t) = \{23, 123\}\). Lemma (6.2) gives that \(\left\{ \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} : \delta \mid N \right\}\) is a complete set of representatives of the double cosets in \(\Gamma_0(N) \backslash \Gamma / \Gamma_\infty\). Using \((r'_\delta) = (18, 0, 0, 0), (6.20)\), and Mathematica, we find that

\[
p \left( \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \right) + p' \left( \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \right) \geq 0 \quad \text{for all } \delta \mid N,
\]

\[
[\nu] = 25,
\]

\[
d_1(125n + j) \equiv 0 \quad (\text{mod } 25)
\]

for \(j \in \{23, 123\}\) are true for all \(0 \leq n \leq [\nu]\). Therefore, by Lemma (6.3) and (6.35), (6.2) is true. The proofs of (6.3), (6.4), (6.6)–(6.11), (6.14), and (6.15) follow analogously from Lemma (6.3) and the chart below.

| Congruence | \((m, M, N, t, (r_\delta))\) and \((r'_\delta)\) | \(P(t)\) | \([\nu]\) |
|------------|---------------------------------|---------|------|
| (6.3)      | (125, 10, 10, 97, (3, 2, -2, 0)) and (30, 0, 0, 0) | \{97, 122\} | 22 |
| (6.4)      | (49, 14, 14, 45, (3, 1, -1, 0)) and (4, 0, 0, 0) | \{45\} | 5 |
|            | (49, 14, 14, 17, (3, 1, -1, 0)) | \{17, 31, 38\} | 6 |
and (4,0,0,0)

(49, 14, 14, 41, (4, 3, −2, 0)) \{41\} 12

and (9,0,0,0)

(343, 14, 14, 90, (39, 3, −7, 0)) \{90,188,237\} 92

and (60,0,0,0)

(343, 14, 14, 39, (1, 4, −2, 0)) \{39,235,284\} 76

and (77,0,0,0)

(121, 22, 22, 96, (9, 4, −2, 0)) \{96\} 31

and (11,0,0,0)

(121, 22, 22, 91, (6, 5, −2, 0)) \{91\} 33

and (14,0,0,0)

(121, 22, 22, 81, (0, 7, −2, 0)) \{81\} 34

and (19,0,0,0)

(289, 34, 34, 205, (15, 6, −2, 0)) \{205\} 77

and (16,0,0,0)

(289, 34, 34, 52, (15, 6, −2, 0)) \{52,69,137,171\} 77

and (16,0,0,0)

(289, 34, 34, 52, (15, 6, −2, 0)) \{188,222,239,273\} 77

and (16,0,0,0)

(19, 38, 38, 16, (9, 3, −1, 0)) \{16\} 29

and (1,0,0,0)

Acknowledgement

The third author was partially supported by Council of Scientific & Industrial Research (CSIR), Government of India under CSIR-JRF scheme. The author thanks the funding agency.

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