Circuit partitions and \#P-complete products of inner products

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Abstract

We present a simple, natural \#P-complete problem. Let $G$ be a directed graph, and let $k$ be a positive integer. We define $q(G; k)$ as follows. At each vertex $v$, we place a $k$-dimensional complex vector $x_v$. We take the product, over all edges $(u, v)$, of the inner product $\langle x_u, x_v \rangle$. Finally, $q(G; k)$ is the expectation of this product, where the $x_v$ are chosen uniformly and independently from all vectors of norm 1 (or, alternately, from the Gaussian distribution). We show that $q(G; k)$ is proportional to $G$’s cycle partition polynomial, and therefore that it is \#P-complete for any $k > 1$.

1 Introduction

Let $x, y \in \mathbb{C}^k$ be $k$-dimensional complex-valued vectors. We denote their inner product as

$$\langle x, y \rangle = \sum_{i=1}^{k} x_i^* y_i.$$  

Now suppose we have a directed graph $G = (V, E)$. Let us associate a vector $x_v \in \mathbb{C}^k$ with each vertex $v$, and consider the product over all edges $(u, v)$ of the inner products of the corresponding vectors:

$$\prod_{(u, v) \in E} \langle x_u, x_v \rangle.$$  

For instance, for the graph in Figure 1 this product is

$$\prod_{(u, v) \in E} \langle x_u, x_v \rangle = \langle x_1, x_2 \rangle \langle x_2, x_3 \rangle \langle x_3, x_1 \rangle |\langle x_3, x_4 \rangle|^2.$$  

The expectation of this product, where each $x_v$ is chosen independently and uniformly from the set of vectors in $\mathbb{C}^k$ of norm 1, is a type of moment, where each $x_v$ appears with order $d_v^\text{in} + d_v^\text{out}$. It is a function of the graph $G$ and the dimension $k$, which we denote as follows:

$$q(G; k) = \text{Exp}_{\{x_v\}} \prod_{(u, v) \in E} \langle x_u, x_v \rangle.$$  

Figure 1: A little directed graph. We use the edge labels in the proof of Theorem 1.

A simple observation is that \( q(G; k) \) is zero unless \( G \) is Eulerian—that is, unless \( d^\text{in}_v = d^\text{out}_v \) for each vertex \( v \). Since \( x_v \) appears in the product \( d^\text{in}_v \) times unconjugated and \( d^\text{out}_v \) times conjugated, multiplying \( x_v \) by \( e^{i\theta} \) multiplies \( q(G; k) \) by \( e^{i\theta(d^\text{in}_v - d^\text{out}_v)} \). But multiplying by a phase preserves the uniform measure, so the expectation is zero if \( d^\text{in}_v \neq d^\text{out}_v \) for any \( v \).

So, let us suppose that \( G \) is Eulerian. In that case, what is \( q(G; k) \)? Does it have a combinatorial interpretation? And how difficult is it to calculate? Our main result is this:

**Theorem 1.** For any \( k \geq 2 \), computing \( q(G; k) \), given \( G \) as input, is \#P-hard under Turing reductions.

If we extend \#P to rational functions in the natural way, then we can replace \#P-hardness in this theorem with \#P-completeness.

Our proof is very simple; we show that \( q(G; k) \) is essentially identical to an existing graph polynomial, which is known to be \#P-hard to compute. Along the way, we will meet some nice combinatorics, and glancingly employ the representation theory of the unitary and orthogonal groups.

2 The circuit partition polynomial

A *circuit partition* of \( G \) is a partition of \( G \)'s edges into circuits. Let \( r_t \) denote the number of circuit partitions containing \( t \) circuits; for instance, \( r_1 \) is the number of Eulerian circuits. The *circuit partition polynomial* \( j(G; z) \) is the generating function

\[
j(G; z) = \sum_{t=1}^{\infty} r_t z^t.
\]

(3)

For instance, for the graph in Figure 1 we have \( j(G; z) = z + z^2 \). This polynomial was first studied by Martin [10], with a slightly different parametrization; see also [1 3 4 6 9 11 12].

Now consider the following theorem.

**Theorem 2.** For any Eulerian directed graph \( G = (V, E) \),

\[
q(G; k) = \left( \prod_{v \in V} \frac{(k-1)!}{(k + d^\text{in}_v - 1)!} \right) j(G; k),
\]

(4)

where \( d_v \) denotes \( d^\text{in}_v = d^\text{out}_v \).
Proof. Given a vector \( x \in \mathbb{C}^k \) and an integer \( d \), the outer product of \( x \otimes d = x \otimes \cdots \otimes x \) with itself is a tensor of rank \( 2d \), or equivalently a linear operator on \( (\mathbb{C}^k)^{\otimes d} \):

\[
\left| x^{\otimes d} \right\langle x^{\otimes d} \mid = \left( | x \rangle \langle x | \right)^{\otimes d}.
\]

In terms of indices, we can write

\[
\left| x^{\otimes d} \right\langle x^{\otimes d} |_{\beta_1, \beta_2, \ldots, \beta_d} = \prod_{\ell=1}^{d} x_{\alpha_{\ell}} x_{\beta_{\ell}}^*. \]

Then \( \prod_{(u,v) \in E} \langle x_u, x_v \rangle \) is a contraction of the product of these tensors, where upper and lower indices correspond to incoming and outgoing edges respectively. For instance, for the graph in Figure 1 we can rewrite the product \( \prod_{(u,v) \in E} \langle x_u, x_v \rangle \) as

\[
\prod_{(u,v) \in E} \langle x_u, x_v \rangle = |x_1 \rangle \langle x_1^\gamma | x_2 \rangle \langle x_2^\beta | x_3 \otimes x_3 \rangle \langle x_3 \otimes x_3^\gamma | x_4 \rangle \langle x_4^\delta |.
\]

Here we use the Einstein summation convention, where any index which appears once above and once below is automatically summed from 1 to \( k \). Now, since the \( x_v \) are independent for different \( v \), we can compute \( q(G;k) \) by taking the expectation over each \( x_v \) separately. This gives a contraction of the tensors

\[
X_d = \text{Exp}_x \left| x^{\otimes d} \right\langle x^{\otimes d} |, \tag{5}
\]

where \( d = d_v \), over all \( v \).

In order to calculate \( X_d \), we introduce some notation. Let \( S_d \) denote the symmetric group on \( d \) elements. We identify a permutation \( \pi \in S_d \) with the linear operator on \( (\mathbb{C}^k)^{\otimes d} \) which permutes the \( d \) factors in the tensor product. That is,

\[
\pi \left( x_1 \otimes x_2 \otimes \cdots \otimes x_d \right) = x_{\pi(1)} \otimes x_{\pi(2)} \otimes \cdots \otimes x_{\pi(d)},
\]

or, using indices,

\[
\pi_{\alpha_1 \alpha_2 \cdots \alpha_d}^{\beta_1 \beta_2 \cdots \beta_d} = \prod_{\ell=1}^{d} \delta_{\alpha_{\pi(\ell)} \beta_{\ell}},
\]

where \( \delta_{ij} \) is the Kronecker delta operator, \( \delta_{ij} = 1 \) if \( i = j \) and 0 if \( i \neq j \). Diagrammatically, \( \pi \) is a gadget with \( d \) incoming edges and \( d \) outgoing edges, wired to each other according to the permutation \( \pi \).

We have the following lemma:

**Lemma 1.** With \( X_d \) defined as in (5), if \( x \) is uniform in the set of vectors in \( \mathbb{C}^k \) of norm 1, then

\[
X_d = \frac{(k-1)!}{(k+d-1)!} \sum_{\pi \in S_d} \pi. \tag{6}
\]
This product works by describing a permutation \( \pi \) of the following product, where 1 denotes the identity permutation, and \( k \) a factor of \( t \) of the first \( t \) objects, composed either with the identity or with a transposition swapping the \( t \)th object with one of the previous \( t - 1 \) objects. This gives \( \dim V_{\text{sym}} = \binom{k + d - 1}{d} \).

To illustrate some ideas that will recur in the next section, we give an alternate proof. First, note that \( \text{tr} \tau \) is the number of ways to label each of \( \tau \)’s cycles with a basis vector ranging from 1 to \( k \), or \( k^{c(\pi)} \) where \( c(\pi) \) denotes the number of cycles (including fixed points). Thus

\[
\text{tr} \sum_{\pi \in S_d} \pi = \sum_{\pi \in S_d} k^{c(\pi)}. \tag{7}
\]

To compute this generating function, we use the fact that each permutation \( \pi \in S_d \) appears once in the following product, where 1 denotes the identity permutation, and \( \tau_{ij} \) denotes the transposition of the \( i \)th and \( j \)th object:

\[
\sum_{\pi \in S_d} \pi = 1(1 + \tau_{1,2})(1 + \tau_{1,3} + \tau_{2,3}) \cdots (1 + \tau_{1,d} + \tau_{2,d} + \cdots + \tau_{d-1,d}). \tag{8}
\]

This product works by describing a permutation \( \pi_t \) of \( t \) objects inductively as a permutation \( \pi_{t-1} \) of the first \( t - 1 \) objects, composed either with the identity or with a transposition swapping the \( t \)th object with one of the previous \( t - 1 \). If we apply the identity, then the \( t \)th object is a fixed point, and \( c(\pi_t) = c(\pi_{t-1}) + 1 \), gaining a factor of \( k \) in (7); but if we apply a transposition, then \( c(\pi_t) = c(\pi_{t-1}) \). Thus (8) becomes

\[
\sum_{\pi \in S_d} k^{c(\pi)} = k(k + 1)(k + 2) \cdots (k + d - 1) \frac{(k + d - 1)!}{(k - 1)!}.
\]

Comparing traces again gives (6). \( \square \)

All that remains is to interpret the operators \( X_{d_v} \), and their contraction, diagrammatically. Lemma I tells us that, for each vertex \( v \) of \( G \), taking the expectation over \( x_v \) converts it to a sum over all \( d_v! \) ways to wire the incoming edges to the outgoing edges. But doing this at each vertex gives us a sum over all cycle partitions of \( G \). Contracting these tensors gives the number of ways to label each cycle in each partition with a basis vector ranging from 1 to \( k \), so each cycle contributes a factor of \( k \). Along with the scaling factor in (9), this completes the proof. \( \square \)

Next we show that the cycle partition polynomial is \#P-hard. To our knowledge, the following theorem first appeared in [7]; we prove it here for completeness.

**Theorem 3.** For any fixed \( z > 1 \), computing \( j(G; z) \) from \( G \) is \#P-hard under Turing reductions.
Proof. Recall that the Tutte polynomial of an undirected graph $G = (V, E)$ can be written as a sum over all subsets $S$ of $E$,

$$T(G; x, y) = \sum_{S \subseteq E} (x - 1)^{c(S) - c(G)} (y - 1)^{c(S) + |S| - n}.$$  

(9)

Here $c(G)$ denotes the number of connected components in $G$. Similarly, $c(S)$ denotes the number of connected components in the spanning subgraph $(V, S)$, including isolated vertices. When $x = y$, we have

$$T(G; x, x) = \sum_{S \subseteq E} (x - 1)^{c(S) + \ell(S) - c(G)},$$

(10)

where $\ell(S) = c(s) + |S| - n$ is the total excess of the components of $S$, i.e., the number of edges that would have to be removed to make each one a tree.

If $G$ is planar, then we can define a directed medial graph $G_m$ as in Figure 2. Each vertex of $G_m$ corresponds to an edge of $G$, edges of $G_m$ correspond to shared vertices in $G$, and we orient the edges of $G_m$ so that they go counterclockwise around the faces of $G$. Each vertex of $G_m$ has $d_{\text{in}} = d_{\text{out}} = 2$, so $G_m$ is Eulerian.

The following identity is due to Martin [10]; see also [11], or [2] for a review.

$$j(G_m; z) = z^{c(G)} T(G; z + 1, z + 1).$$

(11)

We prove this using a one-to-one correspondence between subsets $S \subseteq E$ and circuit partitions of $G_m$. Let $v$ be a vertex of $G_m$, corresponding to an edge $e$ of $G$. Then the circuit partition connects each of $v$’s incoming edge to the outgoing edge on the same side of $e$ if $e \in S$, and crosses over to the other side if $e \notin S$. It is easy to prove by induction that the number of circuits is then $c(S) + \ell(S)$, in which case (10) yields (11).

The theorem then follows from the fact, proven by Vertigan [13], that the Tutte polynomial for planar graphs is $\#P$-hard under Turing reductions, except on the hyperbolas $(x - 1)(y - 1) \in \{1, 2\}$ or when $(x, y) \in \{(1, 1), (-1, -1), (\omega, \omega^*), (\omega^*, \omega)\}$ where $\omega = e^{2\pi i/3}$. Thus computing $j(G; z)$ for any $z > 1$ is $\#P$-hard, even in the special case where $G$ is planar and where every vertex has $d_{\text{in}} = d_{\text{out}} = 2$. 

\[\square\]
3 Real-valued vectors

We can also consider the case where the $x_v$ are real-valued, and are chosen uniformly from the set of vectors in $\mathbb{R}^k$ of norm 1. In this case, the inner product $\langle x_u, x_v \rangle$ becomes symmetric, so the graph $G$ becomes undirected. We might then expect $q(G; k)$ to be related to the circuit partition polynomial for undirected circuits, and indeed this is the case.

We again wish to compute the tensor $X_d = \text{Exp}_x |x^\otimes d\rangle \langle x^\otimes d|$. First, let $M_d$ denote the set of perfect matchings of $2d$ objects; note that

$$|M_d| = (2d - 1)!! = (2d - 1)(2d - 3) \cdots 5 \cdot 3 \cdot 1 = \frac{(2d)!}{2^d d!}.$$  

We can identify each matching $\mu \in M_d$ with a linear operator on $(\mathbb{R}^k)^{\otimes d}$, where the first $d$ objects correspond to upper indices, and the last $d$ correspond to lower indices. However, in addition to permutations that wire upper indices to lower ones with a bipartite matching, we now also have “cups” and “caps” that wire two upper indices, or two lower indices, to each other. For instance, if $d = 2$ then $M_d$ includes three operators, corresponding to the three perfect matchings of 4 objects:

$$\delta^{\alpha_1}_{\beta_1} \delta^{\alpha_2}_{\beta_2} = \bigcup, \quad \delta^{\alpha_1}_{\beta_2} \delta^{\alpha_2}_{\beta_1} = \bigcup, \quad \text{and} \quad \delta^{\alpha_1, \alpha_2}_{\beta_1, \beta_2} = \bigcup. \quad (12)$$

The first two of these operators correspond to the identity permutation and the transposition $\tau_{1,2}$ respectively, as in the previous section. The third one is a cupcap: it is the outer product of the vector $\sum_{i=1}^k e_i \otimes e_i$ with itself, where $e_i$ denotes the $i$th basis vector in $\mathbb{R}^k$. We denote it $\gamma_{1,2}$, and more generally $\gamma_{i,j} = \delta^{\alpha_i}_{\beta_i} \delta^{\alpha_j}_{\beta_j}$.

Now, in the real-valued case, Lemma 2 becomes the following:

**Lemma 2.** If $x$ is uniform in the set of vectors in $\mathbb{R}^k$ of norm 1, then

$$X_d = \frac{1}{k(k+2)(k+4) \cdots (k+2d-2)} \sum_{\mu \in M_d} \mu = \frac{(k-2)!!}{(k+2d-2)!!} \sum_{\mu \in M_d} \mu, \quad (13)$$

where $n!! = n(n-2)(n-4) \cdots 6 \cdot 4 \cdot 2$ if $n$ is even, and $n(n-2)(n-4) \cdots 5 \cdot 3 \cdot 1$ if $n$ is odd.

**Proof.** Analogous to the complex case, $X_d$ is a member of the commutant of the group $O(k)$ of $k \times k$ orthogonal matrices, since these preserve the uniform measure. That is, $X_d$ commutes with $O^{\otimes d}$ for any $O \in O(k)$. The commutant of $O(k)$ is the Brauer algebra; namely, the algebra consisting of linear combinations of the operators $\mu \in M_d$. Thus $X_d$ is of the form $\sum_{\mu \in M_d} a_\mu \mu$.

In addition to being fixed under permutations as in the complex case, $X_d$ is also fixed under partial transposes, which switch some upper indices with some lower ones. Thus $X_d$ is proportional to the uniform superposition $\sum_{\mu \in M_d} \mu$. To find the constant of proportionality, we again compare traces.

As in the case of permutations, the trace of an operator $\mu \in M_d$ is $k^{c(\mu)}$, where $c(\mu)$ is the number of loops in the diagram resulting from joining the upper indices to the lower ones. For instance, for the operators in (12), we have $\text{tr} \ 1 = k^2$, $\text{tr} \ \tau_{1,2} = k$, and $\text{tr} \ \gamma_{1,2} = k$. Thus we wish to calculate

$$\text{tr} \sum_{\mu \in M_d} \mu = \sum_{\mu \in M_d} k^{c(\mu)} . \quad (14)$$


We can write $\sum_{\mu \in M_d}$ as a product, analogous to (8):

$$\sum_{\pi \in S_d} \pi = 1(1 + \tau_{1,2} + \gamma_{1,2})(1 + \tau_{1,3} + \gamma_{1,3} + \tau_{2,3} + \gamma_{2,3}) \cdots (1 + \tau_{1,d} + \gamma_{1,d} + \cdots + \tau_{d-1,d} + \gamma_{d-1,d}).$$

This product describes a matching $\mu_t$ of $2t$ objects inductively as a matching $\mu_{t-1}$ of the first $2(t-1)$ objects, composed either with the identity, or with a transposition or cupcap connecting the $t$th upper object with the $i$th lower one and the $t$th lower object with the $i$th upper one, or vice versa. If we apply the identity, then the $t$th upper object is matched to the $t$th lower one, and $c(\mu_t) = c(\mu_{t-1}) + 1$, gaining a factor of $k$ in (14); but if we apply a transposition or cupcap, then $c(\mu_t) = c(\mu_{t-1})$. Thus (14) becomes

$$\sum_{\pi \in S_d} k^{c(\pi)} = k(k + 2)(k + 4) \cdots (k + 2d - 2) = \frac{(k + 2d - 2)!!}{(k - 2)!!}.$$ 

We again have $\text{tr} X_d = \text{Exp} |x|^{2d} = 1$, and comparing traces gives (13).

As before, $q(G; k)$ is a contraction of the tensors $X_d$. However, now $G$ is undirected, with no distinction between incoming and outgoing edges, so at each vertex of degree $d_v$ the appropriate tensor is $X_{d_v/2}$. Applying Lemma 2 to each $v$ sums over all the ways to match $v$’s edges with each other, and hence sums over all possible partitions of $G$’s edges into undirected cycles. The trace of the resulting diagram is again the number of ways to label each cycle with a basis vector. So, if define a polynomial $j_{\text{undirected}}(G; z)$ as $\sum_{t=1}^{\infty} r_t z^t$, where $r_t$ is the number of partitions with $t$ cycles, then Theorem 2 becomes

**Theorem 4.** For any undirected graph $G = (V, E)$ where every vertex has even degree, if we define $q(G; k)$ by selecting the $x_v$ independently and uniformly from the set of vectors in $R^k$ with norm 1, then

$$q(G; k) = \left( \prod_{v \in V} \frac{(k - 2)!!}{(k + d_v - 2)!!} \right) j_{\text{undirected}}(G; k).$$

(15)

To our knowledge, the computational complexity of $j_{\text{undirected}}(G; z)$ is open, although it seems likely that it is also $\#P$-hard.

### 4 The Gaussian distribution

Our results above assume that each $x_v$ is chosen uniformly from the set of vectors in $C^k$ or $R^k$ of norm 1. Another natural measure would be to choose each component of $x_v$ independently from the Gaussian distribution with variance $1/k$, so that $\text{Exp}[|x_v|^2] = 1$.

For vectors in $C^k$, the probability density of the norm $|x|$ is then

$$p(|x|) = \frac{2k^{k+1}}{k!} |x|^{2k-1} e^{-k|x|^2},$$

Compared to the case where $|x_v| = 1$, each $x_v$ contributes scaling factor of $|x_v|^{2d}$ to the product (1). The even moments of (16) are

$$\text{Exp} \left[ |x_v|^{2d} \right] = \frac{(d + k - 1)!}{k^d (k-1)!}.$$
so in the Gaussian distribution (4) becomes

\[ q(G; k) = \left( \prod_{v \in V} \frac{1}{k^{d_v}} \right) j(G; k) = \frac{1}{k^m} j(G; k), \tag{17} \]

where \( m \) denotes the number of edges.

We could also have derived this directly from the Gaussian analog of Lemma 1. If \( x \) is chosen according to the Gaussian distribution on \( \mathbb{C}^k \), and we again let \( X_d \) denote \( \text{Exp}_x [x^\otimes d] \langle x^\otimes d \rangle \), then

\[ X_d = \frac{1}{k^d} \sum_{\pi \in S_d} \pi. \tag{18} \]

Similarly, in the real-valued case, if we choose each component of \( x \in \mathbb{R}^k \) from the Gaussian distribution on \( \mathbb{R} \) with variance \( 1/k \), then (15) becomes

\[ q(G; k) = \frac{1}{k^m} j_{\text{undirected}}(G; k), \tag{19} \]

since

\[ X_d = \frac{1}{k^d} \sum_{\mu \in M_d} \mu. \tag{20} \]

Both (18) and (20) are consequences of Wick’s Theorem \[8, 15\], that if \( x_1, \ldots, x_{2t} \) obey a multivariate Gaussian distribution with mean zero, then

\[ \text{Exp} \left[ \prod_{i=1}^{2t} x_i \right] = \sum_{\mu \in M_t} \prod_{(i,j) \in \mu} \text{Exp}[x_i x_j]. \]

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References

[1] R. Arratia, B. Bollobás, and G. Sorkin, The interlace polynomial: A new graph polynomial. *Proc. 11th Annual ACM-SIAM Symposium on Discrete Algorithms* 237–245 (2000).

[2] Andrea Austin, The Circuit Partition Polynomial with Applications and Relation to the Tutte and Interlace Polynomials. *Rose-Hulman Undergraduate Mathematics Journal*, 8(2) (2007).

[3] Béla Bollobás, Evaluations of the Circuit Partition Polynomial. *Journal of Combinatorial Theory, Series B* 85, 261–268 (2002)

[4] André Bouchet, Tutte-Martin polynomials and orienting vectors of isotropic systems. *Graphs Combin.* 7(3) 235–252 (1991).

[5] Richard Brauer, On Algebras Which are Connected with the Semisimple Continuous Groups. *Annals of Mathematics*, 38(4) 857–872 (1937).
[6] Joanna A. Ellis-Monaghan, New results for the Martin polynomial. *Journal of Combinatorial Theory, Series B* 74, 326–352 (1998).

[7] Joanna A. Ellis-Monaghan and Irasema Sarmiento, Distance hereditary graphs and the interlace polynomial. *Combinatorics, Probability and Computing* 16(6) 947–973 (2007).

[8] L. Isserlis, On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables. *Biometrika* 12: 134–139 (1918).

[9] F. Jaeger, On Tutte polynomials and cycles of plane graphs. *Journal of Combinatorial Theory, Series B* 44, 127–146 (1988).

[10] P. Martin, Enumérations eulériennes dans les multigraphes et invariants de Tutte-Grothendieck. Thesis, Grenoble 1977.

[11] Michel Las Vergnas, On Eulerian partitions of graphs. *Research Notes in Mathematics* 34, 62–75 (1979).

[12] Michel Las Vergnas, On the evaluation at (3, 3) of the Tutte polynomial of a graph. *Journal of Combinatorial Theory, Series B* 44, 367–372 (1988).

[13] Dirk Vertigan, The Computational Complexity of Tutte Invariants for Planar Graphs. *SIAM J. Comput.*, 35(3) 690–712 (2006).

[14] Hans Wenzl, On the Structure of Brauer’s Centralizer Algebras. *Annals of Mathematics* 128(1) 173–193 (1988).

[15] Gian-Carlo Wick, The evaluation of the collision matrix. *Physical Review* 80(2): 268–272 (1950).