Real-analytic realization of Uniform Circular Systems and some applications

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April 2, 2020

Abstract

Recently Matthew Foreman and Benjamin Weiss showed in a series of papers that smooth ergodic diffeomorphisms of a compact manifold are unclassifiable up to measure-isomorphism. In this paper we show that the uniform circular systems used in the work of Foreman-Weiss admit real-analytic realizations on the torus. As a consequence we obtain the same anti-classification result for real-analytic ergodic diffeomorphisms on the torus. In another application we show the existence of an uncountable family of pairwise non-Kakutani equivalent real-analytic diffeomorphisms on the torus.

Contents

1 Introduction 2
  1.1 Statement of anti-classification results ........................................... 6
  1.2 Outline of the paper ................................................................. 7

2 Preliminaries 8
  2.1 Basics in Ergodic Theory .......................................................... 9
  2.2 Periodic processes ...................................................................... 10
  2.3 Real-analytic diffeomorphisms on the torus .................................. 11

3 Symbolic systems 12
  3.1 Symbolic systems ....................................................................... 12
  3.2 Odometer-based symbolic systems ............................................... 14
  3.3 Circular symbolic systems ............................................................ 15
  3.4 Categories $\mathcal{OB}$ and $\mathcal{CB}$ and the functor $F: \mathcal{OB} \to \mathcal{CB}$ ............. 17

4 Abstract untwisted AbC method 18
  4.1 Notations ..................................................................................... 19
  4.2 The untwisted AbC method .......................................................... 19
  4.3 Special requirements .................................................................... 20

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1 Introduction

In the foundational paper [Ne32] John von Neumann formulated the so-called isomorphism problem for classifying the ergodic measure preserving transformations up to measure theoretic isomorphisms, i.e. it asks to determine when two measure-preserving transformations are isomorphic. It has been a guiding light for directions of research within ergodic theory, and has been solved only for some special classes of transformations, e.g.

- In 1942 von Neumann and Paul R. Halmos showed that the spectrum of the associated Koopman operator is a complete isomorphism invariant for ergodic measure preserving transformations with pure point spectrum (see [HN42]).
- In 1970 Donald Ornstein showed that Bernoulli shifts are completely classified by their entropy (see [Or70]).

Many properties of transformations like mixing of various types or finite rank have been characterized and studied in connection with this problem but the general problem remained intractable.
Starting in the late 1990’s, so-called *anti-classification* results have been established and demonstrated in a rigorous way that classification is not possible. This type of results requires a precise definition of what it means to obtain a classification result.

Informally, a classification is a method of determining isomorphism between transformations, perhaps by computing other invariants for which equivalence is easy to determine. Given a transformation (abstract or smooth or real-analytic) $T$ defined on a measure space (or appropriate manifold), one can ask whether or not it is possible to accurately describe the set $U$ of all other transformations (abstract or smooth or real-analytic) that are measure theoretically isomorphic to $T$. If a classification for such systems is available, then one can expect to proceed by computing invariants and reducing a set containing $T$ to smaller and smaller sets by requiring all elements to have a uniform value of the invariant being considered. For example, one could start with a $T$ and look at the set containing all $S$ with the same entropy as that of $T$ and then in the next step, one could further restrict by requiring their Koopman operator to have the same set of eigenvalues and so on and so forth. So if one wants to prove that certain transformations are not classifiable, one would need to show that no countable (possibly transfinite) protocol, whose basic input is membership in open or closed sets exists that can be used to determine membership in a particular equivalence class. So in other words, the equivalence relation should not be Borel. So, the Borel/non-Borel distinction is a natural and key notion in anti-classification results:

- In 1996 Ferenc Beleznay and Matthew Foreman showed that the class of measure distal transformations used in early ergodic theoretic proofs of Szemeredi’s theorem is not a Borel set [BF96].
- In 2001 Greg Hjorth introduced the notion of *turbulence* and showed that there is no Borel way of attaching algebraic invariants to ergodic transformations that completely determine isomorphism [Hj01].
- Foreman and Benjamin Weiss improved Hjorth’s result by proving that the conjugacy action of the measure preserving transformations is turbulent and hence no generic class can have a complete set of algebraic invariants [FW04].
- Passing from a single transformation to pairs $(S,T)$ of measure preserving Hjorth [Hj01] showed that the equivalence relation defined by isomorphism is not a Borel set. Since his proof uses nonergodic transformations in an essential way, this left open the question what happens if one restricts to ergodic transformations. This was resolved by Foreman, Daniel J. Rudolph and Weiss who showed that the measure-isomorphism relation for ergodic measure preserving transformations is also not a Borel set [FRW11].

Recently, in a series of papers ([FW19a], [FW19b] and [FW3pp]) Foreman and Weiss showed an anti-classification result for $C^\infty$ diffeomorphisms of compact manifolds by proving that the measure-isomorphism relation among pairs of volume preserving ergodic $C^\infty$-diffeomorphisms is not a Borel set with respect to the $C^\infty$-topology. Actually, it is a complete analytic set (see Definitions 1.2 and 1.3).

The goal of this article is to upgrade the aforementioned anti-classification result to the real-analytic category on $\mathbb{T}^2$. At this juncture, we point out that the set of all volume preserving real-analytic diffeomorphisms on a torus is not a metrizable space. In fact it is very difficult to work in this space and we prefer to do all construction on a certain subset $\text{Diff}_\rho(\mathbb{T}^2,\mu)$. For some pre-specified finite number $\rho > 0$ this subset consists of all volume preserving real-analytic
diffeomorphisms homotopic to the identity whose lift to $\mathbb{R}^2$ allows a complexification extending to a band of width $\rho$ in the imaginary direction (see Section 2.3). This set has the structure of a Polish space and we can state the following theorem.

**Theorem A.** For every $\rho > 0$ the measure-isomorphism relation among pairs $(S, T) \in \text{Diff}^\omega_{\rho}(\mathbb{T}^2, \mu) \times \text{Diff}^\omega_{\rho}(\mathbb{T}^2, \mu)$ is a complete analytic set and hence not a Borel set.

Note that the space of all measure preserving real-analytic diffeomorphisms, $\text{Diff}^\omega(\mathbb{T}^2, \mu) := \cup_{\rho > 0} \text{Diff}^\omega_{\rho}(\mathbb{T}^2, \mu)$ can be equipped with the direct limit topology, i.e. a set is $U \subset \text{Diff}^\omega(\mathbb{T}^2, \mu)$ is open iff $U \cap \text{Diff}^\omega_{\rho}(\mathbb{T}^2, \mu)$ is open in $\text{Diff}^\omega_{\rho}(\mathbb{T}^2, \mu)$. See [Ly99, appendix 2] for a detailed discussion of these spaces. However the space obtained in this way in not metrizable. We can see the following result as an immediate consequence of theorem A.

**Theorem B.** The measure-isomorphism relation among pairs $(S, T) \in \text{Diff}^\omega(\mathbb{T}^2, \mu) \times \text{Diff}^\omega(\mathbb{T}^2, \mu)$ is not a Borel set, when $\text{Diff}^\omega(\mathbb{T}^2, \mu)$ is equipped with the direct limit topology.

The Foreman-Weiss anti-classification result and Theorem A are related to another major question in ergodic theory dating back to the pioneering paper [Ne32]: The smooth realization problem asks whether there are smooth versions to the objects and concepts of abstract ergodic theory and whether every ergodic measure-preserving transformation has a smooth model. Here, by a smooth model one means a smooth diffeomorphism of a compact manifold preserving a measure equivalent to the volume element which is isomorphic to the measure-preserving transformation. The only known restriction is due to A. G. Kushnirenko who proved that such a diffeomorphism must have finite entropy. On the other hand, there is a lack on general results on the smooth realization problem.

One way to show that not all finite ergodic measure preserving transformation have a smooth model would be to show that their classification is easier than the general classification result. But the Foreman-Weiss anti-classification result shows that the variety of ergodic transformations that have smooth models is rich enough so that the abstract isomorphism relation restricted to these smooth models is as complicated as it is in general. Theorem A shows that this even holds true for real-analytic diffeomorphisms of $\mathbb{T}^2$.

One of the key steps in the recent work by Foreman and Weiss to adapt the methods from [FRW11] to the case of volume preserving $C^\infty$ diffeomorphisms is to show that a class of symbolic systems, the so-called strongly uniform circular systems (see Section 3.3), can be realized as $C^\infty$ diffeomorphisms. To obtain these smooth realizations the so-called untwisted AbC method is used. This method was introduced by D.V. Anosov and A. Katok in their seminal paper [AK70] and it is also widely known as the approximation by conjugation method or the Anosov-Katok method. In this paper we prove the real-analytic counterpart of the main result in [FW19a] that can be very loosely summarized as the following theorem.

**Theorem C.** Let $T$ be an ergodic transformation on a standard measure space. Then the following are equivalent:

1. $T$ is isomorphic to an real-analytic (untwisted) Anosov-Katok diffeomorphism (satisfying some requirements).

2. $T$ is isomorphic to a (strongly uniform) circular system (with fast growing parameters).
Introduction

For an accurate version of the above theorem stated with all the required technicalities we refer the reader to Theorem G. Since this general realization of uniform circular systems as diffeomorphisms reduces questions about diffeomorphisms to combinatorial questions for symbolic shifts, it bears a lot of flexibility to address questions in the realm of the smooth realization problem.

To exemplify the flexibility and strength of the real-analytic realization of circular systems we construct an uncountable family of real-analytic ergodic diffeomorphisms that are pairwise non-Kakutani equivalent. Recall that two ergodic transformations are said to be Kakutani equivalent if they are isomorphic to measurable cross-sections of the same ergodic flow. Then it immediately follows from Abramov's entropy formula, that two Kakutani equivalent automorphisms must have the same entropy type: zero entropy, finite entropy, or infinite entropy. It was a long-standing open problem whether these three possibilities for entropy completely characterized Kakutani equivalence classes. Until the work of A. Katok [Ka75, Ka77] in the case of zero entropy, and J. Feldman [Fe76] in the general case, no other restrictions were known for achieving Kakutani equivalence. In [Fe76], Feldman showed that there are at least two non-Kakutani equivalent ergodic transformations of entropy zero, and likewise for finite positive entropy and infinite entropy. Ornstein, Rudolph, and Weiss [ORW82] upgraded Feldman’s construction to obtain uncountably many non-Kakutani equivalent ergodic MPT’s of each entropy type. Based on their construction, M. Benhenda [Be15] showed the existence of an uncountable family of pairwise non-Kakutani equivalent zero-entropy $C^\infty$ diffeomorphisms using the AbC method. As already indicated in [Ka03, Chapter 8], this result also follows from the smooth realization of uniform circular systems in general. We present the details in Section 9.1 and combine it with our real-analytic realization of uniform circular systems to obtain the analogue in the real-analytic category.

**Theorem D.** For every $\rho > 0$ there exists an uncountable family of ergodic diffeomorphisms in $\text{Diff}^\omega_{\rho}(\mathbb{T}^2, \mu)$ which are pairwise not Kakutani-equivalent.

In Section 9.2 we use this result to construct real-analytic non-Bernoulli diffeomorphisms with property $K$ on manifolds of dimension greater than 4. Note that a Bernoulli-automorphism (i.e. a measure-preserving invertible transformation of a Lebesgue space isomorphic to a Bernoulli shift) has the $K$-property (i.e. a measure-preserving invertible transformation of a Lebesgue space obeying Kolmogorov’s zero-one law) automatically. Originally, Kolmogorov conjectured that the converse holds also true. The first counterexample in the measurable category was constructed by Ornstein [Or73]. First $C^\infty$ counterexamples were obtained by Katok in dimension greater than 4 [Ka80]. Recently, A. Kanigowski, F. Rodriguez-Hertz, and K. Vinhage have found $C^\infty$ examples in dimension 4 [KRV18]. On the other hand, smooth K-automorphisms in dimension 2 are Bernoulli by Pesin theory [Pe77].

The first real-analytic examples of non-Bernoulli diffeomorphisms with property $K$ were obtained by Rudolph in [Ru88] as follows: Take a hyperbolic toral automorphism $S$ on $\mathbb{T}^2$ given as $\{(\theta, \eta) : 0 \leq \theta, \eta < 1\}$ and suppose $T_t$ is the geodesic flow on the tangent space $TM$ to a compact hyperbolic surface. Now define

$$\hat{S}(\theta, \eta, y) = (S(\theta, \eta), T_{\sin(\theta)}(y)),$$

which is a real analytic $K$-diffeomorphisms and not Bernoulli.

Actually, Theorem D enables us to show the existence of uncountably many real-analytic diffeomorphisms measure theoretically $K$, with the same entropy, and pairwise not isomorphic (not even Kakutani equivalent).
1.1 Statement of anti-classification results

To state our anti-classification results precisely we will need some notions and concepts from Descriptive Set Theory. The main tool is the idea of a reduction.

**Definition 1.1.** Let $X$ and $Y$ be Polish spaces and $A \subseteq X$, $B \subseteq Y$. A function $f : X \to Y$ reduces $A$ to $B$ if and only if for all $x \in X$: $x \in A$ if and only if $f(x) \in B$.

Such a function $f$ is called a Borel (resp. continuous) reduction if $f$ is a Borel (resp. continuous) function.

$A$ being reducible to $B$ can be interpreted as saying that $B$ is at least as complicated as $A$. We note that if $B$ is Borel and $f$ is a Borel reduction, then $A$ is also Borel. Taking the converse of this statement we get that if $A$ is not Borel, then $B$ is not Borel.

**Definition 1.2.** If $S$ is a collection of sets and $B \in S$, then $B$ is called complete for Borel (resp. continuous) reductions if and only if every $A \in S$ is Borel (resp. continuously) reducible to $B$.

Following the interpretation above $B$ is said to be at least as complicated as each set in $S$.

**Definition 1.3.** If $X$ is a Polish space and $B \subseteq X$, then $B$ is analytic if and only if it is the continuous image of a Borel subset of a Polish space. Equivalently, there is a Polish space $Y$ and a Borel set $C \subseteq X \times Y$ such that $B$ is the $X$-projection of $C$.

There are analytic sets that are not Borel. We use a canonical example of such a set: the collection of ill-founded trees.

To introduce those, we consider the set $\mathbb{N}^{<\mathbb{N}}$ of finite sequences of natural numbers. A tree is a set $\mathcal{T} \subseteq \mathbb{N}^{<\mathbb{N}}$ such that if $\tau = (\tau_1, \ldots, \tau_n) \in \mathcal{T}$ and $\sigma = (\sigma_1, \ldots, \sigma_s)$ with $s \leq n$ is an initial segment of $\tau$, then $\sigma \in \mathcal{T}$. If $\sigma$ is an initial segment of $\tau$, then $\sigma$ is a predecessor of $\tau$ and $\tau$ is a successor of $\sigma$. We define the level $s$ of a tree $\mathcal{T}$ to be the collection of elements of $\mathcal{T}$ that have length $s$.

An infinite branch through $\mathcal{T}$ is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have $(f(0), \ldots, f(n-1)) \in \mathcal{T}$. If a tree has an infinite branch, it is called ill-founded. If it does not have an infinite branch, it is called well-founded.

In the following, let $\{\sigma_n : n \in \mathbb{N}\}$ be an enumeration of $\mathbb{N}^{<\mathbb{N}}$ with the property that every proper predecessor of $\sigma_n$ is some $\sigma_m$ for $m < n$. Under this enumeration subsets $S \subseteq \mathbb{N}^{<\mathbb{N}}$ can be identified with characteristic functions $\chi_S : \mathbb{N} \to \{0,1\}$. The collection of such $\chi_S$ can be viewed as the members of an infinite product space $\{0,1\}^{\mathbb{N}^{<\mathbb{N}}}$ homeomorphic to the Cantor space. Here, each function $a : \{\sigma_m : m < n\} \to \{0,1\}$ determines a basic open set

$$\langle a \rangle = \{ \chi : \chi | \{\sigma_m : m < n\} = a \} \subseteq \{0,1\}^{\mathbb{N}^{<\mathbb{N}}}$$

and the collection of all such $\langle a \rangle$ forms a basis for the topology. The collection of trees is a closed (hence compact) subset of $\{0,1\}^{\mathbb{N}^{<\mathbb{N}}}$ in this topology. Moreover, the collection of trees containing arbitrarily long finite sequences is a dense $\mathcal{G}_\delta$ subset. In particular, this collection is a Polish space. We will denote the space of trees containing arbitrarily long finite sequences by $\mathbb{T}$.

Since the topology on the space of trees was introduced via basic open sets giving us finite amount of information about the trees in it, we can characterize continuous maps defined on $\mathbb{T}$ as follows.

**Fact 1.4.** Let $Y$ be a topological space. Then a map $f : \mathbb{T} \to Y$ is continuous if and only if for all open sets $O \subseteq Y$ and all $\mathcal{T} \in \mathbb{T}$ with $f(\mathcal{T}) \in O$ there is $M \in \mathbb{N}$ such that for all $\mathcal{T}' \in \mathbb{T}$ we have: if $\mathcal{T} \cap \{\sigma_n : n \leq M\} = \mathcal{T}' \cap \{\sigma_n : n \leq M\}$, then $f(\mathcal{T}') \in O$. 
As advertised the following classical fact (see e.g. [Ke95]) gives us an example of a complete analytic set.

**Fact 1.5.** Let $\text{Trees}$ be the space of trees.

1. The collection of ill-founded trees is a complete analytic subset of $\text{Trees}$.

2. The collection of trees that have at least two distinct infinite branches is a complete analytic subset of $\text{Trees}$.

We also recall that the *centralizer* of an invertible measure preserving transformation $T : (X, \mu) \to (X, \mu)$ is defined as $C(T) = \{ S : X \to X \text{ invertible m.p. transformation } | S \circ T = T \circ S \}$. Obviously, the powers $T^k$ belong to $C(T)$ for any $k \in \mathbb{Z}$. We also stress that we consider the centralizer in the group $\text{MPT}$ of invertible measure preserving transformations and that $C(T)$ differs from the centralizer inside the group of diffeomorphisms.

Now we can present the precise statement of the main result of the paper that we will prove in Section 8.

**Theorem E.** For every $\rho > 0$ there is a continuous function $F^\rho : \text{Trees} \to \text{Diff}^\omega(T^2, \mu)$ such that for $T \in \text{Trees}$, if $T = F^\rho(T)$:

1. $T$ has an infinite branch if and only if $T \cong T^{-1}$

2. $T$ has two distinct infinite branches if and only if $C(T) \neq \{ T^n | n \in \mathbb{Z} \}$.

Then the classical Fact 1.5 implies the following anti-classification results.

**Theorem F.** For every $\rho > 0$ we have:

1. $\{ T \in \text{Diff}^\omega(T^2, \mu) \mid T \cong T^{-1} \}$ is complete analytic.

2. $\{ T \in \text{Diff}^\omega(T^2, \mu) \mid C(T) \neq \{ T^n | n \in \mathbb{Z} \} \}$ is complete analytic.

Since the map $i(T) = (T, T^{-1})$ is a continuous mapping from $\text{Diff}^\omega(T^2, \mu)$ to $\text{Diff}^\omega(T^2, \mu) \times \text{Diff}^\omega(T^2, \mu)$ and reduces $\{ T \mid T \cong T^{-1} \}$ to $\{ (S, T) \mid S \cong T \}$, this result also yields that the measure-isomorphism relation among pairs $(S, T)$ of ergodic $\text{Diff}^\omega(T^2, \mu)$-diffeomorphisms of $T^2$ is a complete analytic set and, hence, we obtain Theorem A as a consequence.

### 1.2 Outline of the paper

The theme of this paper is to make certain changes to the Foreman-Weiss’ series of paper in order to obtain the same anti-classification result for real-analytic diffeomorphisms of $T^2$. In this connection, we also intend to give a survey of their impressive work.

In their proof of the anti-classification result for ergodic measure-preserving transformations in [FRW11] Foreman, Rudolph and Weiss construct a continuous function from the space $\text{Trees}$ to the invertible measure-preserving transformations assigning to each tree $T$ a transformation $T = F(T)$ of finite entropy such that $T \cong T^{-1}$ just in case $T$ has an infinite branch (see Section 8.3). These transformations have an odometer as a factor and, hence, are *Odometer-based Systems* in up-to-date.
terminology (see Section 3.2). Since it is a persistent open problem to find a smooth realization of transformations with an odometer-factor, Foreman and Weiss circumvent that obstacle by showing that the collection of Odometer-based Systems has the same global structure with respect to joinings as another collection of transformations, the so-called Circular Systems that are extensions of particular circle rotations (see Section 3.3 for a precise description). For this purpose, they show in [FW19b] that there is a functor $F$ between these classes that takes synchronous joinings to synchronous joinings, anti-synchronous joinings to anti-synchronous joinings as well as synchronous and anti-synchronous isomorphisms to synchronous and anti-synchronous isomorphisms (see Section 3.4 for an explanation of the terms synchronous and anti-synchronous).

The definition of these circular systems is inspired by a symbolic representation for circle rotations by certain Liouville rotation numbers found in [FW19a]. Then Foreman and Weiss showed that these circular systems can be realized as volume preserving ergodic $C^\infty$-diffeomorphisms on a torus or a disk or an annulus (under some assumptions on the circular coefficients).

To obtain these smooth realizations the so-called untwisted AbC method is used. In their seminal paper [AK70] D.V. Anosov and A. Katok introduced this method for constructing examples of $C^\infty$-diffeomorphisms. The construction can be carried out on any smooth compact connected manifold that admits an effective circle action $\mathcal{R} = \{R_t\}_{t \in \mathbb{R}}$. It involves inductively constructing measure preserving diffeomorphisms $T_n = H_n \circ R^{\alpha_n} \circ H_n^{-1}$ where the diffeomorphism $H_n$ is obtained as the compositions $H_n = H_{n-1} \circ h_n$ and the rational number $\alpha_n$ is chosen close enough to $\alpha_{n-1}$ to guarantee convergence of the sequence $T_n$ to a diffeomorphism $T$. Usually we want $T$ to satisfy some dynamical property like weak-mixing, minimality, unique ergodicity, etc. and this is achieved by constructing $h_n$ at the $n$-th step in a way so that $T_n$ satisfies some finite version of the targeted property.

Such constructions have resulted in several interesting results in the $C^\infty$-world. Beyond smoothness, the next natural question is the setting of real-analytic realizations. However, there are fewer success stories for real analytic diffeomorphisms due to certain well documented difficulties (see e.g. [FK04 section 7.1] as well as [BK19 Section 6.3]). In recent years new papers have appeared which demonstrate that such constructions are possible with enough flexibility on the torus ([Sa03], [Ba17], [Ku17], [BK19]). A modified version of the construction also shows potential for real-analytic constructions beyond the torus ([FK14]).

In the case of tori $\mathbb{T}^d$, $d \geq 2$, the concept of block-slide type of maps introduced in [Ba17] and their sufficiently precise approximation by volume preserving real-analytic diffeomorphisms allows to find real-analytic counterparts of several Anosov-Katok constructions (see [BK19]). This approach is the important mechanism in the constructions of this paper as well (we emphasize that all constructions in this article are done on the torus and that real-analytic AbC constructions on arbitrary real-analytic manifolds continue to remain an intractable problem). Hereby, we get our real-analytic counterpart of the main result in [FW19a] in Theorem [G] that we already advertised in a vague manner as Theorem [C]. Also, unlike Foreman-Weiss, we use $\mathbb{T}^2$ as our manifold and also as our abstract measure space. This is done to reduce notational complexity.

2 Preliminaries

Here we introduce the basic concepts and establish notations that we will use for the rest of this article. We note that section 2.1 and 2.2 are standard theory presented from [FW19a sections 3.1 and 5] and hence we skip all proofs. Section 2.3 is presented with complete proofs since the theory
is somewhat rare. However one can find similar or identical exposition in [Sa03, FS05, Ba17] and [BK19].

2.1 Basics in Ergodic Theory

We give a concise survey of some concepts regarding measure spaces and measure preserving transformations. Our objective here is not to be comprehensive but rather to introduce some well known concepts in the context of our article and establish certain notations.

Let $X$ be a set, $B$ a Boolean algebra of measurable sets on $X$ and $m$ a measure. Then if the triplet $(X, B, m)$ is a separable non-atomic probability space, we call it a standard measure space. Let $(X, B, m)$ and $(X', B', m')$ be two measure space. Then a map $f$ defined on a set of full $m$ measure of $X$ onto a set of full $m'$ measure of $X'$ is called a measure theoretic isomorphism if $f$ is one-one and both $f$ and $f^{-1}$ are measurable.

An ordered countable partition or simply a partition of $(X, B, m)$ will refer to a sequence $P := \{P_n\}_{n=1}^{\infty}$ such that the following conditions are satisfied:

1. $P_n \in B$ for all $n \in \mathbb{N}$.
2. $P_n \cap P_m = \emptyset$ if $n \neq m$.
3. $m(\cup_{n=1}^{\infty} P_n) = 1$.

Each $P_n$ is called an atom of the partition $P$. Often we deal with partitions where all but finitely many atoms have measure zero and we refer to such a partition as a finite partition. We say that a sequence of partitions $\{P_n\}_{n=1}^{\infty}$ is a generating sequence if the smallest $\sigma$-algebra containing $\cup_{n=1}^{\infty} P_n$ is $B$.

Next we introduce the notion of a distance between two partitions. Let $P = \{P_n\}_{n=1}^{\infty}$ and $Q = \{Q_n\}_{n=1}^{\infty}$ be two partitions, we define

$$D_m(P, Q) := \sum_{i=1}^{\infty} m(P_i \triangle Q_i) \quad (2.1)$$

**Lemma 2.2.** Fix a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Let $(X, B, m)$ and $(X', B', m')$ be two standard measure spaces and $\{T_n\}_{n=1}^{\infty}$ and $\{T'_n\}_{n=1}^{\infty}$ be measure preserving transformations of $X$ and $X'$ converging weakly\(^1\) to $T$ and $T'$ respectively. Suppose $\{P_n\}_{n=1}^{\infty}$ is a decreasing sequence of partitions and $\{K_n\}_{n=1}^{\infty}$ is a sequence of measure preserving transformations such that

1. $K_n : X \to X'$ is an isomorphism between $T_n$ and $T'_n$.
2. $\{P_n\}_{n=1}^{\infty}$ and $\{K_n(P_n)\}_{n=1}^{\infty}$ are generating sequence of partitions for $X$ and $X'$.
3. $D_m(K_{n+1}(P_n), K_n(P_n)) < \varepsilon_n$.

Then the sequence $K_n$ converges in the weak topology to a measure theoretic isomorphism between $T$ and $T'$.

\(^1\) The set of all measure preserving transformation on $X$ has a natural topology where a basic open set is given by the sets $N(T, P, \varepsilon) := \{S : S$ is a measure preserving transformation and $\sum_{A \in P} m(T_A \triangle S_A) < \varepsilon\}$ for a measure preserving transformation $T$, a finite measurable partition $P$ and some $\varepsilon > 0$. This is called the weak topology.
2.2 Periodic processes

Here we recall some relevant facts from the notion of periodic processes. One can refer to [Ka03] for a detailed account. We continue with notations from the previous subsection.

Let $\mathcal{P}$ be a partition of $(\mathcal{X}, \mathcal{B}, m)$ where all the atoms of $\mathcal{P}$ have the same measure. A Periodic process is a pair $(\tau, \mathcal{P})$ where $\tau$ is a permutation of $\mathcal{P}$ such that each cycle has equal length. We refer to these cycles as towers and their length is called height of the tower. We also choose an atom from each tower arbitrarily and call it the base of the tower. In particular if $\tau$ refers to these cycles as towers and their length is called height of the tower. We also choose an atom from each tower arbitrarily and call it the base of the tower. In particular if $\mathcal{P}_1, \ldots, \mathcal{P}_s$ are the towers (of height $q$) of this periodic process with $B_1, \ldots, B_s$ as their respective bases, then any tower $\mathcal{P}_i$ can be explicitly written as $B_i, \tau(B_i), \ldots, \tau^{q-1}(B_i)$. We refer to $\tau^k(B_i)$ as the $k$-th level of the tower $\mathcal{P}_i$. $\tau^{q-1}(B_i)$ is the top level. Next we describe how to compare two periodic processes.

Definition 2.3 ($\varepsilon$-approximation). Let $(\tau, \mathcal{P})$ and $(\sigma, \mathcal{Q})$ be two periodic processes of the measure space $(\mathcal{X}, \mathcal{B}, m)$. We say that $(\sigma, \mathcal{Q})$ $\varepsilon$-approximates $(\tau, \mathcal{P})$ if there exists disjoint collections of $\mathcal{Q}$ atoms $\{S_A : A \in \mathcal{P}, S_A \subset \mathcal{Q}\}$ and a set $D \subset \mathcal{X}$ of measure less than $\varepsilon$ such that the following are satisfied:

1. For every $A \in \mathcal{P}$, we have $\cup \{B : B \in S_A\} \setminus D \subset A$.
2. If $A \in \mathcal{P}$ is not the top level of a tower and $B \in S_A$, we have $\sigma(B) \setminus D \subseteq \tau(A)$
3. For each tower of $\sigma$, the measure of the intersection of $\mathcal{X} \setminus D$ with each level of this tower are the same.

With the above definition in mind, we have the following result regarding the convergence of periodic processes.

Lemma 2.4. Let $\{\varepsilon_n\}_{n \in \mathbb{N}}$ be a summable sequence of positive numbers. Let $\{(\tau_n, \mathcal{P}_n)\}_{n \in \mathbb{N}}$ be a sequence of periodic processes on $(\mathcal{X}, \mathcal{B}, m)$ such that $(\tau_{n+1}, \mathcal{P}_{n+1})$ $\varepsilon_n$-approximates $(\tau_n, \mathcal{P}_n)$ and the sequence $\{\mathcal{P}_n\}_{n \in \mathbb{N}}$ is generating.

Then there exists a unique transformation $T : \mathcal{X} \to \mathcal{X}$ satisfying:

$$\lim_{n \to \infty} m( \bigcup_{A \in \mathcal{P}_n} (\tau_n A \triangle TA)) = 0 \quad (2.5)$$

We call $\{(\tau_n, \mathcal{P}_n)\}$ a convergent sequence of periodic processes.

Lemma 2.6. Let $\{(\tau_n, \mathcal{P}_n)\}_{n \in \mathbb{N}}$ be a sequence of periodic processes converging to $T$ and $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of measure preserving transformations satisfying for each $n$,

$$\sum_{A \in \mathcal{P}_n} \mu(T_n A \triangle \tau_n A) < \varepsilon_n \quad (2.7)$$

Then $\{T_n\}_{n \in \mathbb{N}}$ converges weakly to $T$.

Lemma 2.8. Let $\{\varepsilon_n\}_{n \in \mathbb{N}}$ be a summable sequence of positive numbers. Let $(\mathcal{X}_1, \mathcal{B}_1, m_1)$ and $(\mathcal{X}_2, \mathcal{B}_2, m_2)$ be two standard measure spaces. Let $\{T_n\}_{n \in \mathbb{N}}$ and $\{S_n\}_{n \in \mathbb{N}}$ be two sequences of measure preserving transformations of $\mathcal{X}_1$ and $\mathcal{X}_2$ converging to $T$ and $S$ respectively in the weak topology. Suppose $\mathcal{P}_n$ is a decreasing sequence of partitions and $\{\phi_n\}_{n \in \mathbb{N}}$ be a sequence of measure preserving transformations such that

\[ \text{This is not a necessary requirement for the definition, but is good enough for us.} \]
1. $\phi_n$ is a measure theoretic isomorphism between $T_n$ and $S_n$.

2. The sequence $P_n$ and $\phi_n(P_n)$ generate $B_1$ and $B_2$.

3. $D_{m_2}(\phi_{n+1}(P_n), \phi_n(P_n)) < \varepsilon_n$

Then the sequence $\phi_n$ converges in the weak topology to an isomorphism between $T$ and $S$.

## 2.3 Real-analytic diffeomorphisms on the torus

We will denote the two dimensional torus by

$$ T^2 := \mathbb{R}^2 / \mathbb{Z}^2 $$

We use $\mu$ to denote the standard Lebesgue measure on $T^2$.

We give a description of the space of measure preserving real-analytic diffeomorphisms on $T^2$ that are homotopic to the identity.

Any real-analytic diffeomorphism on $T^2$ homotopic to the identity admits a lift to a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ and has the following form

$$ F(x_1, x_2) = (x_1 + f_1(x_1, x_2), x_2 + f_2(x_1, x_2)) \quad (2.10) $$

where $f_i : \mathbb{R}^2 \to \mathbb{R}$ are $\mathbb{Z}^2$-periodic real-analytic functions. Any real-analytic $\mathbb{Z}^2$ periodic function on $\mathbb{R}^2$ can be extended as a holomorphic (complex analytic) function from some open complex neighborhood $^3$ of $\mathbb{R}^2$ in $\mathbb{C}^2$. For a fixed $\rho > 0$, we define the neighborhood

$$ \Omega_\rho := \{(z_1, z_2) \in \mathbb{C}^2 : |\text{Im}(z_1)| < \rho \text{ and } |\text{Im}(z_2)| < \rho \} \quad (2.11) $$

and for a function $f$ defined on this set, put

$$ \|f\|_\rho := \sup_{(z_1, z_2) \in \Omega_\rho} |f((z_1, z_2))| \quad (2.12) $$

We define $C^\omega_\rho(T^2)$ to be the space of all $\mathbb{Z}^2$-periodic real-analytic functions on $\mathbb{R}^2$ that extends to a holomorphic function on $\Omega_\rho$ and $\|f\|_\rho < \infty$.

We define, $\text{Diff}^\omega_\rho(T^2, \mu)$ to be the set of all measure preserving real-analytic diffeomorphisms of $T^2$ homotopic to the identity, whose lift $F$ to $\mathbb{R}^2$ satisfies $f_i \in C^\omega_\rho(T^2)$ and we also require that the lift $\hat{F}(x) = (x_1 + \hat{f}_1(x), x_2 + \hat{f}_2(x))$ of its inverse to $\mathbb{R}^2$ to satisfy $\hat{f}_i \in C^\omega_\rho(T^2)$.

The metric in $\text{Diff}^\omega_\rho(T^2, \mu)$ is defined by

$$ d_\rho(f, g) = \max\{\hat{d}_\rho(f, g), \hat{d}_\rho(f^{-1}, g^{-1})\} \quad \text{where} \quad \hat{d}_\rho(f, g) = \max_{i=1,2} \inf_{n \in \mathbb{Z}} \|f_i - g_i + n\|_\rho \quad (\text{2.4}) $$

Next, with some abuse of notation, we define the following two spaces

$$ C^\omega_\infty(T^2) := \cap_{n=1}^\infty C^\omega_n(T^2) \quad (2.13) $$

$$ \text{Diff}^\omega_\infty(T^2, \mu) := \cap_{n=1}^\infty \text{Diff}^\omega_n(T^2, \mu) \quad (2.14) $$

We now list some properties of the above spaces that are going to be useful to us.

---

$^3$We identify $\mathbb{R}^2$ inside $\mathbb{C}^2$ via the natural embedding $(x_1, x_2) \mapsto (x_1 + i0, x_2 + i0)$. 

Note that the functions in \(2.13\) can be extended to \(\mathbb{C}^2\) as entire functions.

- \(\text{Diff}_\omega^{\infty}(T^2, \mu)\) is closed under composition. To see this assume that \(f, g \in \text{Diff}_\omega^{\infty}(T^2, \mu)\) and let \(F, G\) be their lifts to \(\mathbb{R}^2\). Then note that \(F \circ G\) is the lift of the composition \(f \circ g\) (with \(\pi : \mathbb{R}^2 \to T^2\) as the natural projection, \(\pi \circ F \circ G = f \circ g \circ \pi\)). Now for the complexification of \(F\) and \(G\) note that the composition \(F \circ G(z) = (z_1 + g_1(z) + f_1(G(z)), z_2 + g_2(z) + f_2(G(z)))\). Since \(g_i \in C_\omega^{\infty}(T^2)\), we have for any \(\rho, \sup_{z \in \Omega_\rho} |\text{Im}(G(z))| \leq \max_i (\sup_{z \in \Omega_\rho} |\text{Im}(z_i) + \text{Im}(g_i(z))|) \leq \max_i (\sup_{z \in \Omega_\rho} |\text{Im}(z_i)| + \sup_{z \in \Omega_\rho} |\text{Im}(g_i(z))|) \leq \max_i (\sup_{z \in \Omega_\rho} |g_i(z)| + \sup_{z \in \Omega_\rho} |f_i(G(z))| < \rho + \text{const} < \rho' < \infty\) for some \(\rho'\). So, \(\sup_{z \in \Omega_{\rho'}} |g_i(z) + f_i(G(z))| \leq \sup_{z \in \Omega_\rho} |g_i(z)| + \sup_{z \in \Omega_\rho} |f_i(G(z))| < \infty\) since \(g_i \in C_\omega^{\infty}(T^2)\), \(G(z) \in \Omega_{\rho'}\) and \(f_i \in C_\rho^{\infty}(T^2)\). An identical treatment gives the result for the inverse.

- If \(\{f_n\}_{n=1}^{\infty} \subset \text{Diff}_\rho^{\infty}(T^2, \mu)\) is Cauchy in the \(d_\rho\) metric, then \(f_n\) converges to some \(f \in \text{Diff}_\rho^{\infty}(T^2, \mu)\). Indeed, uniform convergence guarantees analyticity and consideration of the inverses ensures that the result is a diffeomorphism. So this space is a Polish space.  

Similarly we define the space of all real-analytic functions on \(T^2\) and all measure preserving real-analytic diffeomorphisms of \(T^2\) as follows:

\[
C_\omega^\infty(T^2) := \bigcup_{n=1}^{\infty} C_\omega^n(T^2) \quad (2.15)
\]

\[
\text{Diff}_\omega^{\infty}(T^2, \mu) := \bigcup_{n=1}^{\infty} \text{Diff}_\rho^n(T^2, \mu) \quad (2.16)
\]

The space of all measure preserving real-analytic diffeomorphisms of \(T^2\) is given the corresponding inductive limit topology and it is not a metrizable space. This space is usually very difficult to work with and we will not use this for the rest of this article.

This completes the description of the analytic topology necessary for our construction. Also throughout this paper, the word “diffeomorphism” will refer to a real-analytic diffeomorphism unless stated otherwise. Also, the word “analytic topology” will refer to the topology of \(\text{Diff}_\omega^{\infty}(T^2, \mu)\) described above. See [Sa03] for a more extensive treatment of these spaces.

## 3 Symbolic systems

In this section we introduce the notion of a symbolic system in a way that is most convenient in representing AbC transformations. In particular we write about the uniform circular systems introduced by Foreman and Weiss. We skip proofs and recall the results only for one can refer to [FW19a] sections 3.3 and 4] for the details. We also note that we try to stick to the notations used in [FW19a] but we do make some changes.

### 3.1 Symbolic systems

Let \(\Sigma\) be a finite or countable alphabet endowed with the discrete topology. By \(\Sigma^\mathbb{Z}\) we denote the space of bi-infinite sequences of alphabets from \(\Sigma\) endowed with the product topology. The product topology makes this a totally disconnected separable space that is compact if \(\Sigma\) is finite.

---

4 A Polish space is a separable completely metrizable topological space
In order to get a better description of the of the product topology on this set we define for any $u = \langle u_0, u_1, \ldots, u_{n-1} \rangle \in \Sigma^{<\infty}$,

$$C_k(u) := \{ f \in \Sigma^Z : f|_[k, k+n) = u \}$$

Such sets are known as cylinder sets and they generate the product topology of $\Sigma^Z$. Next we define the (left) shift map

$$\text{sh} : \Sigma^Z \rightarrow \Sigma^Z \quad \text{defined by} \quad \text{sh}(f)(n) = f(n+1) \quad (3.1)$$

If $\mu$ is a shift invariant Borel measure then the system $(\Sigma^Z, B, \mu, \text{sh})$ is called a symbolic system. The closure of the support of $\mu$ is a shift invariant measure preserving system and we call it a symbolic shift or a sub shift.

We recall that we can construct symbolic shifts from an arbitrary measure preserving transformation $(X, \mathcal{B}, \mu, T)$. To see this, fix a partition $\mathcal{P} := \{ A_i : i \in I \}$ for some countable or finite $I$ and an alphabet $\Sigma := \{ a_i : i \in I \}$. We define $\phi : X \rightarrow \Sigma^Z$ by $\phi(x)(n) = a_i \iff T^n x \in A_i$. The $\phi^* \mu$ is an invariant measure and $(\Sigma^Z, C, \phi^* \mu, \text{sh})$ is a factor of $(X, \mathcal{B}, \mu, T)$ with factor map $\phi$. If $\mathcal{P}$ is generating then $\phi$ is an isomorphism.

The above description of symbolic shift and coding is the most straightforward way, but to explicitly understand the resulting symbolic shift conjugate with an AbC transformation we need a step by step inductive procedure for describing symbolic shifts which in certain way emulates the AbC process. So we give an intrinsic definition of the symbolic shifts using the notion of construction sequences.

**Definition 3.2.** A sequence of collection of words $(W_n)_{n \in \mathbb{N}}$, where $\mathbb{N} = \{0, 1, 2, \ldots \}$, satisfying the following properties is called a construction sequence:

1. for every $n \in \mathbb{N}$ all words in $W_n$ have the same length $h_n$,
2. each $w \in W_n$ occurs at least once as a subword of each $w' \in W_{n+1}$,
3. there is a summable sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive numbers such that for every $n \in \mathbb{N}$, every word $w \in W_{n+1}$ can be uniquely parsed into segments $u_0 w_1 u_1 w_1 \ldots u_l w_{l+1}$ such that each $w_i \in W_n$, each $u_i$ (called spacer or boundary) is a word in $\Sigma$ of finite length and for this parsing

$$\frac{\sum_{i=0}^{l+1} |u_i|}{h_{n+1}} < \varepsilon_{n+1}.$$ 

Next we define the following subset of $\Sigma^Z$:

$$\mathcal{K} := \{ x \in \Sigma^Z : \text{ if } x = u w v \text{ for some } u, v \in \Sigma^Z, w \in \Sigma^{<\infty} \text{ then } w \text{ is a contiguous subword of some } w' \in W_n \text{ for some } n \} \quad (3.3)$$

We note that $\mathcal{K}$ is a closed shift invariant subset of $\Sigma^Z$.

Before we talk about measures on $\mathcal{K}$, we need some technical definitions. We say that a construction sequence $W_n$ is uniform if there exists a sequence of functions $\{ d_n : W_n \rightarrow (0, 1) \}_{n=1}^{\infty}$ such that for some summable sequence of positive numbers $\{ \varepsilon_n \}_{n=1}^{\infty}$, we have for any choice of $w \in W_n$ and $w' \in W_{n+1}$,

$$\left| \frac{h_n}{h_{n+1}} \left( \# \{ i : w = w_i \text{ where } w' = u_0 w_0 u_1 w_1 \ldots u_l w_{l+1} \} \right) - d_n(w) \right| < \frac{1}{h_n} \varepsilon_{n+1} \quad (3.4)$$
### 3.2 Odometer-based symbolic systems

We say that $K$ is a **uniform symbolic system** if it is built out of a uniform construction sequence. If it so happens that the number $\# \{ i : w = w_i \}$ depends only on $n$ but not on $w$ or $w'$ then we refer to the construction sequence and $K$ as **strongly uniform**. Note that strong uniformity implies uniformity with $d_n(w) = \# \{ i : w = w_i \}$.

We say that a collection of finite words $W$ is uniquely readable iff whenever $u, v, w \in W$ and $uv = w$ then either $p$ or $s$ is the empty word.

Let $K$ be a uniform symbolic system built out of a construction sequence $\{ W_n \}_{n=1}^{\infty}$ where each $W_n$ is uniquely readable. We define

$$S = \{ x \in K : \exists \text{ natural number sequences } \{a_m\}_{m=1}^{\infty}, \{b_m\}_{m=1}^{\infty} \text{ satisfying } x|_{[a_m,b_m)} \in W_m \} \quad (3.5)$$

and note that $S$ is a dense shift invariant $G_\delta$ set.

**Lemma 3.6.** Let $K$ be a uniform symbolic system built out of the construction sequence $\{ W_n \}_{n=1}^{\infty}$ in a finite alphabet $\Sigma$. Then the following holds:

1. $K$ is the smallest shift invariant closed subset of $\Sigma^\mathbb{Z}$ such that $K \cap C_0(w) \neq \emptyset \quad \forall n \in \mathbb{N} \quad w \in W_n \quad (3.7)$

2. There exists a unique non-atomic shift invariant measure $\nu$ concentrated on $S$ and this measure is ergodic.

3. For any $s \in S \subset K$ and $w \in W_n$, the density of $\{ k : w \text{ occurs in } s \text{ starting at } k \}$ exists and is equal to $\nu(C_0(w))$. Moreover $\nu(C_0(w)) = d_n(w)/h_n$.

### 3.2 Odometer-based symbolic systems

Let $(k_n)_{n \in \mathbb{N}}$ be a sequence of natural numbers $k_n \geq 2$ and

$$O = \prod_{n \in \mathbb{N}} (\mathbb{Z}/k_n\mathbb{Z})$$

be the $(k_n)_{n \in \mathbb{N}}$-adic integers. Then $O$ has a compact abelian group structure and hence carries a Haar measure $\lambda$. We define a transformation $T : O \to O$ to be addition by 1 in the $(k_n)_{n \in \mathbb{N}}$-adic integers (i.e. the map that adds one in $\mathbb{Z}/k_0\mathbb{Z}$ and carries right). Then $T$ is a $\lambda$-preserving invertible transformation called **odometer transformation** which is ergodic and has discrete spectrum.

We now define the collection of symbolic systems that have odometer systems as their timing mechanism to parse typical elements of the system.

**Definition 3.8.** Let $(W_n)_{n \in \mathbb{N}}$ be a uniquely readable construction sequence with $W_0 = \Sigma$ and $W_{n+1} \subseteq (W_n)^{k_n}$ for every $n \in \mathbb{N}$. The associated symbolic shift will be called an **odometer-based system**.

Thus, odometer-based systems are those built from construction sequences $(W_n)_{n \in \mathbb{N}}$ such that the words in $W_{n+1}$ are concatenations of a fixed number $k_n$ of words in $W_n$. Hence, the words in $W_n$ have length $h_n$, where

$$h_n = \prod_{i=0}^{n-1} k_i$$
if $n > 0$, and $h_0 = 1$. Moreover, the spacers in part 3 of Definition 3.2 are all the empty words (i.e. an odometer-based transformation can be built by a cut-and-stack construction using no spacers).

Remark. According to an announcement in [FW19b], any finite entropy system that has an odometer factor can be represented as an odometer-based system.

### 3.3 Circular symbolic systems

We now describe a special class of symbolic shifts called circular symbolic systems. These systems were introduced by Foreman and Weiss in [FW19a, section 4] and are specifically designed to serve as symbolic representations of untwisted AbC systems. We recall the results from their work but skip most proofs.

Consider natural numbers $k, l, p, q$ with $p$ and $q$ mutually prime. We note that for every $i = 0, 1, \ldots, q-1$, there exists a $0 \leq j_i < q$ such that the following holds:

$$pj_i = i \mod q$$

we often use the notation

$$j_i = (p)^{-1}i \mod q$$

for brevity. Note that $q - j_i = j_{q-i}$.

In order to describe uniform circular systems, we start by defining an operator on alphabets. Let $\Sigma$ be any alphabet and $\{b, e\}$ be two letter not in $\Sigma$. Suppose $w_0, \ldots, w_{k-1}$ be a sequence of words constructed out of the alphabet $\Sigma \cup \{b, e\}$. We define the operator:

$$C(w_0, w_1, \ldots, w_{k-1}) := \prod_{i=0}^{q-1} \prod_{j=0}^{k-1} (b^{j_{s+i}}w_j^{l-1}e^{j_i})$$

Note that

- If $|w_i| = q$ for $i = 0, \ldots, k-1$, then $|C(w_0, \ldots, w_{k-1})| = klq^2$.
- For every $e \in C(w_0, \ldots, w_{k-1})$, there exists a $b$ to the left of it.
- If for some $m > n$, there exists a $b$ at the $n$ th position followed by an $e$ at the $m$ th position and additionally we know that neither occurs inside a $w_i$ then there must exist a $w_i$ in between the $m$ th and the $n$ th position.
- The proportion of the word $w$ written in equation 3.11 that belongs to the boundary is $1/l$. Moreover the proportion of the word that is within $q$ letters of the boundary is $3/l$.

We also introduce some notions around this map. Suppose $w = C(w_0, w_1, \ldots, w_{k-1})$. So $w$ consists of blocks with $l - 1$ copies of $w_i$ along with some $b$ s and $e$ s at the ends which are not inside $w_i$. We refer to the portion of $w$ in $w_i$ s to be the interior of $w$ and the $b$ s and $e$ s not in the $w_i$ s to be the boundary of $w$. In a block of the form $w_i^{l-1}$, the first and the last occurrences of $w_i$ is called the boundary portion of the block $w_i^{l-1}$ and the other occurrences are called interior occurrences.
Lemma 3.12. Let $\Sigma$ be a finite or countable alphabet and $u_0, \ldots, u_{k-1}, v_0, \ldots, v_{k-1}, w_0, \ldots, w_{k-1}$ are words of length $q < 1/2$ constructed from the alphabet $\Sigma \cup \{b, e\}$. Put $u = C(u_0, \ldots, u_{k-1}), v = C(v_0, \ldots, v_{k-1})$ and $w = C(w_0, \ldots, w_{k-1})$. If for some words $p$ and $s$ constructed from $\Sigma \cup \{b, e\}$ we have

$$uv = pws$$

then either $p = \text{empty word}$, $u = w$, $v = s$ or $s = \text{empty word}$, $u = p$, $v = w$.

Next we inductively define four sequences of natural numbers $\{k_n\}_{n=1}^\infty, \{l_n\}_{n=1}^\infty, \{p_n\}_{n=1}^\infty$ Define $k_0, l_0$ and $\{q_n\}_{n=1}^\infty$ as follows:

$$p_{n+1} = p_nq_nk_n + 1 \quad q_{n+1} = k_nl_nq_n^2$$

We note that the above relations makes $p_n$ and $q_n$ relatively prime. So we can define for $0 \leq i < q_n$ a natural number $0 \leq j_i < q_n$ such that

$$j_i = (p_n)^{-1}i \mod q_n$$

Now we are ready to build a construction sequence for our symbolic shift. We choose a nonempty finite or countable alphabet $\Sigma$ and choose two letters $b$ and $e$ not in $\Sigma$. We start or induction by putting $W_0 = \Sigma$. Now we assume that the induction has been carried out till the $n$ th step.

At the $n + 1$ th step we we choose a set $P_{n+1} \subset (W_n)^{k_n}$ and put

$$W_{n+1} := \{C(w_0, \ldots, w_{k_n-1}) : (w_0, \ldots, w_{k_n-1}) \in P_{n+1}\}$$

It follows from lemma 3.12 that $W_{n+1}$ is uniquely readable.

Next we introduce the concept of strong unique readability. We can view $W_n$ as a collection of $\Lambda_n$ letters and elements of $P_{n+1}$ can be viewed as words constructed out of $\Lambda_n$. If $P_{n+1}$ is uniquely readable in the alphabet $\Lambda_n$, we say that the construction sequence satisfies the strong unique readability assumption.

A construction sequence which satisfies 3.15 uses parameters satisfying 3.13 and satisfies the strong unique readability assumption is called a circular construction sequence.

Lemma 3.16. If the $\{l_n\}_{n=1}^\infty$ parameters of a circular construction sequence satisfies

$$\sum_{n=1}^\infty \frac{1}{ln} < \infty$$

and for each $n$ there exists a number $f_n$ such that each word $w \in W_n$ occurs exactly $f_n$ times in each word in $P_{n+1}$ then the circular construction sequence is strongly uniform.

Definition 3.18. A symbolic shift $K$ constructed from a circular construction system is called a circular system. A symbolic shift $K$ constructed from a (strongly) uniform circular construction system is called a (strongly) uniform circular system.

Lemma 3.19. Let $K$ be a circular system. Then

1. A shift invariant measure $\nu$ concentrates on $S \subset K$ iff $\nu$ concentrates on the collection of $s \in K$ such that $\{i : s(i) \notin \{b, e\}\}$ is unbounded in both $\mathbb{Z}^+$ and $\mathbb{Z}^-$ direction.
2. If $\mathbb{K}$ is a uniform circular system and $\nu$ is a shift invariant measure on $\mathbb{K}$, then $\nu(S) = 1$. In particular, there is a unique non-atomic shift-invariant measure on $\mathbb{K}$ by Lemma 3.7.

We end this section after defining a canonical factor of a circular system measure theoretically isomorphic to a rotation of the circle. Let $\{k_n\}_{n=1}^{\infty}$ and $\{l_n\}_{n=1}^{\infty}$ be two parameter sequences satisfying $3.17$. Let $\mathbb{K}$ be any circular system constructed out of these parameters. We also construct a second circular system. Let $\Sigma_0 = \{\ast\}$ and we define a construction sequence

\[ W_0 := \Sigma_0, \quad \text{and if} \quad W_n := \{w_n\} \quad \text{then} \quad W_{n+1} = \{C(w_n, \ldots, w_n)\}. \quad (3.20) \]

We denote the resulting circular system by $\mathbb{K}$.

Now we claim that $\mathbb{K}$ is a factor of $\mathbb{K}$. We see that one can construct an explicit factor map as follows

\[ \pi : \mathbb{K} \to \mathbb{K} \quad \text{defined by} \quad \pi(x) := \begin{cases} x(i) & \text{if } x(i) \in \{b, e\} \\ \ast & \text{otherwise} \end{cases} \quad (3.21) \]

We end this section with the following observations:

- $\pi : \mathbb{K} \to \mathbb{K}$ is Lipschitz.
- $\pi \circ \text{sh}^\pm = \text{sh}^\pm \circ \pi$.
- $\pi$ is a factor map of $\mathbb{K}$ to $\mathbb{K}$ and from $\mathbb{K}^{-1}$ to $\mathbb{K}^{-1}$.

In [FW19b, Section 4.3] a specific isomorphism $\natural : \mathbb{K} \to \text{rev}(\mathbb{K})$ is introduced. It is called the natural map and will serve as a benchmark for understanding of maps from $\mathbb{K}$ to $\text{rev}(\mathbb{K})$ (see e.g. Definition 3.22).

### 3.4 Categories OB and CB and the functor $\mathcal{F} : \mathcal{OB} \to \mathcal{CB}$

For a fixed circular coefficient sequence $(k_n, l_n)_{n \in \mathbb{N}}$ we consider two categories $\mathcal{OB}$ and $\mathcal{CB}$ whose objects are odometer-based and circular systems respectively. The morphisms in these categories are (synchronous and anti-synchronous) graph joinings.

**Definition 3.22.** If $\mathbb{K}$ is an odometer based system, we denote its odometer base by $\mathbb{K}^\pi$ and let $\pi : \mathbb{K} \to \mathbb{K}^\pi$ be the canonical factor map. Similarly, if $\mathbb{K}^c$ is a circular system, we let $(\mathbb{K}^c)^\pi$ be the rotation factor $\mathbb{K}$ and let $\pi : \mathbb{K}^c \to \mathbb{K}$ be the canonical factor map.

1. Let $\mathbb{K}$ and $\mathcal{L}$ be odometer based systems with the same coefficient sequence and let $\rho$ be a joining between $\mathbb{K}$ and $\mathcal{L}^{\pm 1}$. Then $\rho$ is called synchronous if $\rho$ joins $\mathbb{K}$ and $\mathcal{L}$ and the projection of $\rho$ to a joining on $\mathbb{K}^\pi \times \mathcal{L}^\pi$ is the graph joining determined by the identity map. The joining $\rho$ is called anti-synchronous if $\rho$ joins $\mathbb{K}$ and $\mathcal{L}^{-1}$ and the projection of $\rho$ to a joining on $\mathbb{K}^\pi \times (\mathcal{L}^{-1})^\pi$ is the graph joining determined by the map $x \mapsto -x$.

2. Let $\mathbb{K}^c$ and $\mathcal{L}$ be circular systems with the same coefficient sequence and let $\rho$ be a joining between $\mathbb{K}^c$ and $(\mathcal{L}^c)^{\pm 1}$. Then $\rho$ is called synchronous if $\rho$ joins $\mathbb{K}^c$ and $\mathcal{L}^c$ and the projection of $\rho$ to a joining on $\mathbb{K} \times \mathcal{L}$ is the graph joining determined by the identity map. The joining $\rho$ is called anti-synchronous if $\rho$ joins $\mathbb{K}^c$ and $(\mathcal{L}^c)^{-1}$ and the projection of $\rho$ to a joining on $\mathbb{K} \times \mathcal{L}^{-1}$ is the graph joining determined by the map $\text{rev}(\cdot) \circ \natural$. 
In [FW19b] Foreman and Weiss define a functor taking odometer-based systems to circular system that preserves the factor and conjugacy structure. To review the definition of the functor we fix a circular coefficient sequence \((k_n, l_n)_{n \in \mathbb{N}}\). Let \(\Sigma\) be an alphabet and \((\mathcal{W}_n)_{n \in \mathbb{N}}\) be a construction sequence for an odometer-based system with coefficients \((k_n)_{n \in \mathbb{N}}\). Then we define a circular construction sequence \((\mathcal{W}_n)_{n \in \mathbb{N}}\) and bijections \(c_n : \mathcal{W}_n \to \mathcal{W}_n\) by induction:

- Let \(\mathcal{W}_0 = \Sigma\) and \(c_0\) be the identity map.
- Suppose that \(\mathcal{W}_n, \mathcal{W}_n,\) and \(c_n\) have already been defined. Then we define

  \[
  \mathcal{W}_{n+1} = \{C_n (c_n (w_0), c_n (w_1), \ldots, c_n (w_{k_n-1})) : w_0 w_1 \ldots w_{k_n-1} \in \mathcal{W}_{n+1}\}
  \]

  and the map \(c_{n+1}\) by setting

  \[
  c_{n+1} (w_0 w_1 \ldots w_{k_n-1}) = C_n (c_n (w_0), c_n (w_1), \ldots, c_n (w_{k_n-1})).
  \]

In particular, the prewords are

\[
P_{n+1} = \{c_n (w_0) c_n (w_1) \ldots c_n (w_{k_n-1}) : w_0 w_1 \ldots w_{k_n-1} \in \mathcal{W}_{n+1}\}.
\]

**Definition 3.23.** Suppose that \(\mathcal{K}\) is built from a construction sequence \((\mathcal{W}_n)_{n \in \mathbb{N}}\) and \(\mathcal{K}^c\) has the circular construction sequence \((\mathcal{W}_n)_{n \in \mathbb{N}}\) as constructed above. Then we define a map \(F\) from the set of odometer-based systems (viewed as subshifts) to circular systems (viewed as subshifts) by

\[
F (\mathcal{K}) = \mathcal{K}^c.
\]

**Remark.** The map \(F\) is a bijection between odometer-based symbolic systems with coefficients \((k_n)_{n \in \mathbb{N}}\) and circular symbolic systems with coefficients \((k_n, l_n)_{n \in \mathbb{N}}\) that preserves uniformity. Since the construction sequences for our odometer-based systems will be uniquely readable, the corresponding circular construction sequences will automatically satisfy the strong readability assumption.

In [FW19b] it is shown that \(F\) gives an isomorphism between the categories \(OB\) and \(CB\). We state the following fact which is part of the main result in [FW19b].

**Fact 3.24.** For a fixed circular coefficient sequence \((k_n, l_n)_{n \in \mathbb{N}}\) the categories \(OB\) and \(CB\) are isomorphic by the functor \(F\) that takes synchronous isomorphisms to synchronous isomorphisms and anti-synchronous isomorphisms to anti-synchronous isomorphisms.

### 4 Abstract untwisted AbC method

We recall the AbC method in a way that is most convenient to us. The exposition is similar to the one in [FW19a, section 6.1] and [Ka03, Part I section 8].

Let \(R\) be the usual action of the circle \(T^1\) on \(T^2\) obtained by translation of the first coordinate. More precisely,

\[
R_t : T^2 \to T^2 \quad \text{defined by} \quad R_t(x_1, x_2) = (x_1 + t, x_2)
\]
4.1 Notations

We introduce some partitions of the circle $\mathbb{T}^1$. For any natural number $q$ we define the partition of the unit circle into half open intervals of length $1/q$ as follows,

$$\mathcal{I}_q := \left\{ \left[ \frac{i}{q}, \frac{i+1}{q} \right) \subset \mathbb{T} : i = 0, 1, \ldots, q-1 \right\}$$  \hspace{1em} (4.2)

Next, given natural numbers $s$ and $q$, we define a partition of the torus $\mathbb{T}^2$ into rectangles of length $1/q$ and height $1/s$ as follows,

$$\xi^s_q := \mathcal{I}_q \otimes \mathcal{I}_s := \left\{ \left[ \frac{i}{q}, \frac{i+1}{q} \right) \times \left[ \frac{j}{s}, \frac{j+1}{s} \right) \subset \mathbb{T} : i = 0, 1, \ldots, q-1; j = 0, 1, \ldots, s-1 \right\}$$  \hspace{1em} (4.3)

In the AbC construction we deal with specific sequences of natural numbers $q_n$ and $s_n$ and we will often use the following notation for convenience,

$$\xi_n := \xi_{q_n, s_n}$$  \hspace{1em} (4.4)

and the respective atoms of the above partition will be denoted by

$$R^n_{i,j} := \left[ \frac{i}{q_n}, \frac{i+1}{q_n} \right) \times \left[ \frac{j}{s_n}, \frac{j+1}{s_n} \right)$$  \hspace{1em} (4.5)

We note that with $\alpha = p/q$, $p$ and $q$ mutually prime, the atoms of $\xi^s_q$ is permuted by the action of $R_\alpha$. This action results in a permutation consisting of $s$ cycles, each of length $q$.

4.2 The untwisted AbC method

Now we give a description of the abstract ‘untwisted’ AbC method. This is an inductive process and we assume that the construction has been carried out till the $n$ th stage. So we have the following information available to us,

1. Sequences of natural numbers $\{k_m\}_{m=1}^{n-1}$, $\{s_m\}_{m=1}^{n-1}$, $\{l_m\}_{m=1}^{n-1}$, $\{p_m\}_{m=1}^{n-1}$, $\{q_m\}_{m=1}^{n-1}$ and a sequence of rational numbers $\{\alpha_m\}_{m=1}^{n}$ satisfying the following relations

$$p_{m+1} = p_m q_m k_m l_m + 1 \quad q_m = k_m l_m q_m^2 \quad \alpha_m = p_m / q_m \quad \beta_m^k \geq s_m + 1$$  \hspace{1em} (4.6)

2. Measure preserving transformations $\{h_m\}_{m=1}^{n}$ of the torus $\mathbb{T}^2$. We assume that $h_m$ is a permutation of $\xi_{k_m-1,q_m-1}$, $h_m$ commutes with $R^{1/q_m-1}$ and $h_m$ leaves the rectangle $\xi_{j=0}^{s_m-1} R^{1}_{1,j}$ invariant.  \footnote{The last condition is why we call this version of Anosov-Katok untwisted.}

3. Measure preserving transformations $\{H_m\}_{m=1}^{n}$ and $\{T_m\}_{m=1}^{n}$ of the torus $\mathbb{T}^2$ satisfying the following conditions,

$$H_m := h_0 \circ h_1 \circ \ldots \circ h_m \quad T_m := H_m \circ R_{\alpha_m} \circ H_m^{-1}$$  \hspace{1em} (4.7)
4.3 Special requirements

Now we describe how the construction is carried out in the \( n + 1 \) the stage of the AbC method. We choose a natural number \( s_{n+1} \) followed by the natural number \( k_n \) so that the following growth condition is satisfied

\[
s_n^{k_n} \geq s_{n+1}
\]  

Next we choose a measure preserving transformation \( h_{n+1} \) of \( \mathbb{T}^2 \) which is a permutation of \( \xi_{k_nq_n,s_{n+1}} \) (and hence a permutation of \( \xi_{n+1} \)). Since we are doing the untwisted version of the AbC method, we additionally ensure that the transformation \( h_{n+1} \) leaves the rectangle \( [0,1/q_n] \times \mathbb{T} \) invariant. We finally choose \( l_n \) to be a large enough natural number so that the conjugacies \( T_n \) and \( T_{n+1} \) are close enough and the sequence \( T_n \) converges in the weak topology. We also note that \( p_{n+1}, q_{n+1} \) and \( \alpha_{n+1} \) are automatically determined by the formulae \( p_{n+1} = p_n q_n k_n l_n + 1, q_{n+1} = k_n l_n q_n^2 \) and \( \alpha_{n+1} = p_{n+1}/q_{n+1} \). This completes the description of our version of the abstract AbC method.

Now we define the following sequence of partitions,

\[
\zeta_n = H_n(\xi_n)
\]

and note that since \( h_{m'} \) is a permutation of \( \xi_m \) for all \( m' \leq m \), we can conclude that \( \zeta_n \) is only a permutation of \( \xi_n \) and hence is a generating sequence of finite partitions if \( \xi_n \) is a generating sequence of partitions.

4.3 Special requirements

Till now we described the abstract untwisted AbC method in its full generality. In our case we would need it to satisfy the following additional requirements:

- **Requirement 1:** The sequence \( s_n \) tends to \( \infty \)

- **Requirement 2:** For each \( R^m_{0,j} \in \xi_n \) and each \( s < s_{n+1} \), we have

\[
\left\{ \frac{s}{k_n} \leq s_{n+1} : h_{n+1}\left( \left[ \frac{t}{k_n q_n}, \frac{t+1}{k_n q_n} \right] \times \left[ \frac{s}{s_{n+1}}, \frac{s+1}{s_{n+1}} \right] \right) \subset R^m_{0,j} \right\} = \frac{k_n}{s_n}
\]  

Note that this assumption allows us to define a map \( s \mapsto (j_0, \ldots, j_{k_n-1})_s \) from \( \{0, 1, \ldots, s_{n+1} - 1\} \) to \( \{0, 1, \ldots, s_{n+1} - 1\}^{k_n} \) so that for any fixed \( s < s_{n+1} \),

\[
h_{n+1}\left( \left[ \frac{t}{k_n q_n}, \frac{t+1}{k_n q_n} \right] \times \left[ \frac{s}{s_{n+1}}, \frac{s+1}{s_{n+1}} \right] \right) \subset R^n_{0,j_t}
\]

- **Requirement 3:** We assume that the map \( s \mapsto (j_0, \ldots, j_{k_n-1})_s \) is one to one.

Note that the above requirements are more than enough to guarantee ergodicity for the limit transformation \( T \). We end this section by stating an obvious lemma which serves as a converse to the above.

**Lemma 4.11.** Let \( w_0, \ldots, w_{s_{n+1}-1} \subset \{0, 1, \ldots, s_n - 1\}^{k_n} \) be words such that each \( i \) with \( 0 \leq i \leq s_n \) occurs \( k_n/s_n \) times in each \( w_j \). Then there is an invertible measure preserving \( h_{n+1} \) commuting with \( R^m_{\xi_{\alpha_n}} \) and inducing a permutation of \( \xi_{k_n q_n, s_{n+1}} \) such that if \( j_t \) is the \( t \) th letter of \( w_s \) then

\[
h_{n+1}\left( \left[ \frac{t}{k_n q_n}, \frac{t+1}{k_n q_n} \right] \times \left[ \frac{s}{s_{n+1}}, \frac{s+1}{s_{n+1}} \right] \right) \subset R^n_{0,j_t}
\]
5 Approximating partition permutations by real-analytic diffeomorphisms

The purpose of this section is to show that any permutation of a partition of $\mathbb{T}^2$ by a rectangular grid can be approximated sufficiently well by real-analytic diffeomorphisms. This is the real-analytic version of the content of [FW19a, section 6.2]. We note that their smooth construction can be done on the torus, annulus or the disk. Unfortunately the lack of bump functions in the real-analytic category makes life harder and our real-analytic constructions are only valid for the torus. For a disk, even very basic questions like the existence of real-analytic ergodic diffeomorphisms remain open. One can refer to [FK04, section 7.1] and [BK19, section 6.3] for a comprehensive analysis of known difficulties for a real-analytic AbC method on arbitrary real-analytic manifolds.

5.1 Block-slide type maps and their analytic approximations

We recall that a step function on the unit interval is a finite linear combination of indicator functions on intervals. We define the following two types of piecewise continuous maps on $\mathbb{T}^2$,

- $h_1 : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $h_1(x_1, x_2) := (x_1, x_2 + s_1(x_1) \mod 1)$ (5.1)
- $h_2 : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $h_2(x_1, x_2) := (x_1 + s_2(x_2) \mod 1, x_2)$ (5.2)

where $s_1$ and $s_2$ are step functions on the unit interval. The first map has the same effect as partitioning $\mathbb{T}^2$ into smaller rectangles using vertical lines and sliding those rectangles vertically according to $s_1$. On the other hand the second map has the same effect as partitioning $\mathbb{T}^2$ into smaller rectangles using horizontal lines and sliding those rectangles horizontally according to $s_2$.

We refer to any finite composition of maps of the above kind as a block-slide type of map on $\mathbb{T}^2$. This is somewhat similar to playing a game of nine without the vacant square on $\mathbb{T}^2$.

The purpose of the section is to demonstrate that a block-slide type of map can be approximated extremely well by measure preserving real analytic diffeomorphisms. This can be achieved because step function and be approximated well by real analytic functions. We have the following lemma where we achieve this approximation and a little more to guarantee periodicity with a pre-specified period.

**Lemma 5.3.** Let $k$ and $N$ be two positive integer and $\alpha = (\alpha_0, \ldots, \alpha_{k-1}) \in [0,1)^k$. Consider a step function of the form

\[ \tilde{s}_{\alpha,N} : [0,1) \to \mathbb{R} \text{ defined by } \tilde{s}_{\alpha,N}(x) = \sum_{i=0}^{kN-1} \tilde{\alpha}_i \chi_{[\frac{i}{kN}, \frac{i+1}{kN})}(x) \]

Here $\tilde{\alpha}_i := \alpha_j$ where $j := i \mod k$. Then, given any $\varepsilon > 0$ and $\delta > 0$, there exists a periodic real-analytic function $s_{\alpha,N} : \mathbb{R} \to \mathbb{R}$ satisfying the following properties:

1. Entirety: The complexification of $s_{\alpha,N}$ extends holomorphically to $\mathbb{C}$.
2. Proximity criterion: $s_{\alpha,N}$ is $L^1$ close to $\tilde{s}_{\alpha,N}$. In fact we can say more,

\[ \sup_{x \in [0,1) \setminus F} |s_{\alpha,N}(x) - \tilde{s}_{\alpha,N}(x)| < \varepsilon, \]

where $F = \bigcup_{i=0}^{kN-1} I_i \subset [0,1)$ is a union of intervals centred around $\frac{i}{kN}$, $i = 1, \ldots, kN-1$ and $I_0 = [0, \frac{k}{kN}) \cup \left[1 - \frac{k}{kN}, 1\right)$ and $\lambda(I_i) = \frac{1}{kN}$ $\forall i$. 

3. Periodicity: $s_{\alpha,N}$ is $1/N$ periodic. More precisely, the complexification will satisfy,
\[ s_{\alpha,N}(z + n/N) = s_{\alpha,N}(z) \quad \forall \, z \in \mathbb{C} \text{ and } n \in \mathbb{Z} \quad (5.5) \]

4. Bounded derivative: The derivative is small outside a set of small measure,
\[ \sup_{x \in [0,1) \setminus F} |s'_{\alpha,N}(x)| < \varepsilon \quad (5.6) \]

Proof. See [Ba17, Lemma 4.7] and [Ku17, Lemma 3.6].

Note that the condition 5.5 in particular implies
\[ \sup_{z : \text{Im}(z) < \rho} s_{\beta,N}(z) < \infty \quad \forall \, \rho > 0. \]

Indeed, for any $\rho > 0$, put $\Omega'_\rho = \{ z = x + iy : x \in [0,1], |y| < \rho \}$ and note that entirety of $s_{\beta,N}$ combined with compactness of $\Omega'_\rho$ implies $\sup_{z \in \Omega'_\rho} |s_{\beta,N}(z)| < C$ for some constant $C$. Periodicity of $s_{\alpha,N}$ in the real variable and the observation $\Omega_\rho = \cup_{n \in \mathbb{Z}} (\Omega'_\rho + n)$ implies that $\sup_{z \in \Omega_\rho} |s_{\beta,N}(z)| < C$. We have essentially concluded that $s_{\beta,N} \in C^\infty_T$.

Now we show that block-slide type of maps on $\mathbb{T}^2$ can be approximated well by entirely extendable real-analytic diffeomorphisms.

**Proposition 5.7.** Let $h : \mathbb{T}^2 \to \mathbb{T}^2$ be a block-slide type of map which commutes with $\phi^{1/q}$ for some natural number $q$. Then for any $\varepsilon > 0$ and $\delta > 0$ there exists a measure preserving real-analytic diffeomorphism $h \in \text{Diff}_{\omega}^\infty(\mathbb{T}^2, \mu)$ such that the following conditions are satisfied:

1. Proximity property: There exists a set $E \subset \mathbb{T}^2$ such that $\mu(E) < \delta$ and $\sup_{x \in \mathbb{T}^2 \setminus E} \| h(x) - h(x) \| < \varepsilon$.

2. Commuting property: $h \circ \phi^{1/q} = \phi^{1/q} \circ h$

In this case we say the the diffeomorphism $h$ is $(\varepsilon, \delta)$-close to the block-slide type map $h$.

Proof. See [BK19, Proposition 2.22]

### 5.2 Approximating partition permutation by diffeomorphisms

Now we describe how partition permutations of $\mathbb{T}^2$ can be approximated well enough by real-analytic diffeomorphisms. At this juncture we point out that for the purpose of the AbC method we would like the approximating diffeomorphism constructed here to commute with $R^{1/q}$ for some given natural number $q$. In the smooth category this is achieved by carrying out the construction on a fundamental domain of $R^{1/q}$ in such a way that this map is identity near the boundary of this fundamental domain and then we glue together $q$ translated copies of this diffeomorphism and the resulting diffeomorphism commutes with $R^{1/q}$ by construction.

In the real-analytic category we do not know how to reproduce the result in a similar fashion and hence we do all construction on the whole of $\mathbb{T}^2$ rather than a fundamental domain of $\mathbb{T}^2$. The problem if we do the construction this way is that we have to keep track that of commutativity all along our construction.
We briefly describe how this construction can be achieved using notations from section 4. We fix natural numbers $k, s, q$ and $\Pi$ a permutation of $\xi_{kq}$ which commutes with $R^{1/q}$.

In [BK19, section 5.1] we proved the subsequent statement on the approximation of arbitrary permutations.

**Proposition 5.8 ([BK19], Theorem E).** Let $k, q, l \in \mathbb{N}$ and $\Pi$ be any permutation of $kql$ elements. We can naturally consider $\Pi$ to be a permutation of the partition $S_{kql}$ of the torus $T^2$. Assume that $\Pi$ commutes with $\phi^{1/q}$. Then $\Pi$ is a block-slide type of map.

We have the approximate real-analytic version of the above theorem as follows:

**Theorem 5.9.** Let $\Pi$ be any permutation of $k \times s$ rectangles which partitions $[0, 1/q) \times T^1$. In particular we can extend this $\pi$ to a permutation of $\xi_{kq}$ which commutes with $\phi^{1/q}$ (see proposition 5.8). Then for any $\varepsilon > 0$, there exists a diffeomorphism $h \in \text{Diff}^\omega(T^2, \mu)$ such that for a set $L \subset T^2$, the following conditions are satisfied:

1. $\mu(L) > 1 - \varepsilon$.
2. For any $x \in L \cap R$, $h(x) \in \Pi(R)$ for any $R \in \xi_{s}$.
3. $\phi^{1/q} \circ h = h \circ \phi^{1/q}$.

We say that $h$ $\varepsilon$-approximates $\Pi$.

**Proof.** Follows using proposition 5.8 followed by proposition 5.7.

We note that the above theorem is the real-analytic counterpart of [FW19a, Theorem 35] and it is valid for the torus only. This is essentially the main difference of our paper with the Foreman-Weiss work. We also remark that one can further improve upon the above result and obtain the real-analytic analogue of [AK70, Theorem 1.2] and a reduced version of [AK70, Theorem 1.1] without the boundary condition. Of course one already has real-analytic version of these theorems (Moser’s theorem is true in the real-analytic category) but the complexification of such diffeomorphisms are not known to be entire in any sense and hence these are not compatible with the AbC method. So the above theorem and the aforementioned generalizations are the best we can hope for at the time.

## 6 Real-analytic AbC method

Our objective in this section is to produce examples of real-analytic diffeomorphisms using the AbC method. We will also show that these real-analytic AbC diffeomorphisms we construct here are isomorphic to the abstract AbC transformations constructed earlier.

### 6.1 Real-analytic AbC method

**Theorem 6.1 (Real-analytic untwisted AbC diffeomorphisms).** Fix a number $\rho > 0$. Suppose $T : T^2 \to T^2$ is a measure preserving transformation built by the abstract AbC method using parameter sequences $\{k_n\}_{n=1}^\infty$ and $\{l_n\}_{n=1}^\infty$.

Then if $\{l_n\}_{n=1}^\infty$ is a sequence of numbers which grows fast enough (see 6.9), there exists a diffeomorphism $T^{(a)} \in \text{Diff}^\omega(T^2, \mu)$ which is measure theoretically isomorphic to $T$. 
6.1 Real-analytic AbC method

Proof. Fix \( \{ \varepsilon_n \}_{n=1}^{\infty} \) such that \( \sum_{n=1}^{\infty} \varepsilon_n < \infty \). Let \( \{ h_n \}_{n=1}^{\infty}, \{ H_n = h_1 \circ \ldots \circ h_n \}_{n=1}^{\infty} \) and \( \{ T_n = H_n \circ R^{\alpha_n} \circ H_n^{-1} \}_{n=1}^{\infty} \) be a sequence of transformations constructed using parameters \( \{ k_n \}_{n=1}^{\infty} \) and \( \{ t_n \}_{n=1}^{\infty} \) via the abstract AbC method.

Using theorem 5.8 we construct diffeomorphisms \( h_n^{(a)} \in \text{Diff}_\omega (\mathbb{T}^2, \mu) \) such that \( h_n^{(a)} \varepsilon_n \)-approximates \( h_n \). We put

\[
H_n^{(a)} := h_1^{(a)} \circ \ldots \circ h_n^{(a)} \quad T_n^{(a)} := H_n^{(a)} \circ R^{\alpha_n} \circ (H_n^{(a)})^{-1}
\]

Now we make the following observation regarding proximity exploiting the commutation relation:

\[
d_{\rho}(T_{n+1}^{(a)}, T_n^{(a)}) = d_{\rho}(H_n^{(a)} \circ R^{\alpha_n} \circ [h_n^{(a)} \circ R^{1/(k_n l_n q_n^2)} \circ (h_n^{(a)})^{-1}] \circ (H_n^{(a)})^{-1}, H_n^{(a)} \circ R^{\alpha_n} \circ (H_n^{(a)})^{-1})
\]

Recall that \( l_n \) is chosen last in the induction step. So, if some choices \( l_n \) to be a large enough natural number then from the continuity of \( d_{\rho} \) with respect to composition we obtain \( d_{\rho}(T_{n+1}^{(a)}, T_n^{(a)}) < \varepsilon_n \). Since we are dealing with real-analytic functions which are often more delicate than smooth functions, we make some more observations to justify this claim. Note that since \( (H_n^{(a)})^{-1} \in \text{Diff}_\omega (\mathbb{T}^2, \mu) \), we can choose some \( \rho' > \rho \) such that \( (H_n^{(a)})^{-1}(\Omega_\rho) \subset \Omega_{\rho'} \). For any \( x \in \Omega_\rho \) we put \( y = (H_n^{(a)})^{-1}(x) \). So, \( R^{1/(k_n l_n q_n^2)}(y) \in \Omega_{\rho'} \). Also note that \( H_n^{(a)} \circ R^{\alpha_n} \in \text{Diff}_\omega (\mathbb{T}^2, \mu) \) and any function in \( \text{Diff}_\omega (\mathbb{T}^2, \mu) \) is uniformly continuous on \( \Omega_{\rho'} \). This implies

\[
\tilde{d}_{\rho'}(T_{n+1}^{(a)}, T_n^{(a)}) \leq \sup_{y \in \Omega_\rho} \| H_n^{(a)} \circ R^{\alpha_n} (R^{1/(k_n l_n q_n^2)}(y)) - H_n^{(a)} \circ R^{\alpha_n}(y) \| < \varepsilon_n / 4
\]

for \( l_n \) sufficiently large. A similar consideration with the inverse gives us the result for \( d_{\rho} \). Hence the sequence \( T_n^{(a)} \) is Cauchy and converges to some \( T^{(a)} \in \text{Diff}_\omega (\mathbb{T}^2, \mu) \).

Next we need to prove that \( T^{(a)} \) is in fact measure theoretically isomorphic to \( T \). Our plan is to use lemma 2.2 for the proof.

In the language of the lemma, we put \( (X, \mathcal{B}, m) = (X', \mathcal{B}', m') = (\mathbb{T}^2, \mathcal{B}, \mu) \). Next we define \( K_n \).

We put

\[
K_n : \mathbb{T}^2 \to \mathbb{T}^2 \quad \text{defined by} \quad K_n := H_n^{(a)} \circ H_n^{-1}
\]

From the definition it follows that \( K_n \) is an isomorphism between \( T_n \) and \( T_n^{(a)} \).

We define the two sequences of partitions \( \mathcal{P}_n := \mathcal{P}_n = H_n(\xi_n) \) and \( \mathcal{P}_n^{(a)} := K_n(\xi_n) = H_n^{(a)}(\xi_n) \) and observe that using lemma 4.11 we can conclude that \( \{ \mathcal{P}_n \}_{n=1}^{\infty} \) is generating. We have to work a little to show that \( \mathcal{P}_n^{(a)} \) is generating.\footnote{Indeed, if \( F \in \text{Diff}_\omega (\mathbb{T}^2, \mu) \), then since \( F \) is \( \mathbb{Z}^2 \)-periodic, we have for any \( \rho'' > 0 \), \( u, v \in \{ z = (z_1, z_2) \in \mathbb{C}^2 : \text{Re}(z_i) \in [0, 1], \text{Im}(z_i) \leq \rho'' \} \) and \( \varepsilon > 0 \), there exists a \( \delta > 0 \) independent of \( u, v \) such that \( \|u - v\| < \delta \Rightarrow \|F(u) - F(v)\| < \varepsilon \) (this follows from the compactness of the domain). In other words \( F \) is uniformly continuous on this restricted domain. For arbitrary \( u, v \in \Omega_{\rho''} \), \( F \) is small, we can find integers \( n_1, n_2, m_1, m_2 \) such that \( u - (n_1, n_2), v - (m_1, m_2) \in \{ z = (z_1, z_2) \in \mathbb{C}^2 : \text{Re}(z_i) \in [0, 1], \text{Im}(z_i) \leq \rho'' \} \). And since \( F \) is \( \mathbb{Z}^2 \)-periodic, \( \|F(u) - F(v)\| = \|F(u - (n_1, n_2)) - F(v - (m_1, m_2))\| \).}
We define the set $L_n$ to be the set of measure more than $1 - \varepsilon_n$ corresponding to $h_n$ as per Theorem 5.9. Consider the following sequence of sets:

$$G_n := L_n \cap \bigcap_{m=n+1}^{\infty} (h_{n+1}^{(a)} \circ \ldots \circ h_m^{(a)})^{-1}(L_m)$$

Note that $G_n$ is an increasing sequence and the Borel-Cantelli lemma guarantees that $\mu(G_n) \not\rightarrow 1$.

We pick a measurable $D \subset \mathbb{T}^2$ and $\delta > 0$. There exists some $n_0$ such that $\mu(G_m) > 1 - \frac{\delta}{2}$ for all $m > n_0$.

We put $D' = (H_{n_0}^{(a)})^{-1}(D)$ and since $\{\xi_n\}_{n=1}^{\infty}$ is a generating sequence, there exists an $m > n_0$ and a collection $C'_m \subset \xi_m$ such that

$$\mu\left( \bigcup_{C \in C'_m} C \right) < \delta/2$$

$$\Rightarrow \mu\left( \bigcup_{C \in C'_m} H_{n_0}^{(a)}(C) \right) < \delta/2$$

On the other hand, note that for any $m > n_0$

$$\mu\left( \bigcup_{R \in \xi_m} h_{n_0+1}^{(a)} \circ \ldots \circ h_m^{(a)}(R) \triangle h_{n_0+1} \circ \ldots \circ h_m(R) \right) < \delta/2$$

$$\Rightarrow \mu\left( \bigcup_{R \in \xi_m} (H_{n_0}^{(a)} \circ h_{n_0+1} \circ \ldots \circ h_m(R) \triangle H_{n_0}^{(a)} \circ h_{n_0+1} \circ \ldots \circ h_m(R)) \right) < \delta/2$$

$$\Rightarrow \mu\left( \bigcup_{R \in \xi_m} (H_m^{(a)}(R) \triangle H_{n_0}^{(a)}(\Pi(R))) \right) < \delta/2$$

Where $\Pi := h_{n_0+1} \circ \ldots \circ h_m$ is a permutation of $\xi_m$. So,

$$\mu\left( \bigcup_{R \in \Pi^{-1}(C'_m)} H_m^{(a)}(R) \triangle D \right)$$

$$\leq \mu\left( \bigcup_{R \in \Pi^{-1}(C'_m)} H_m^{(a)}(R) \triangle H_{n_0}^{(a)}(\Pi(R))) + \mu\left( \bigcup_{R \in \Pi^{-1}(C'_m)} H_{n_0}^{(a)}(\Pi(R)) \triangle D \right) \right)$$

$$\leq \mu\left( \bigcup_{R \in \xi_m} H_m^{(a)}(R) \triangle H_{n_0}^{(a)}(\Pi(R))) + \mu\left( \bigcup_{C \in C'_m} H_{n_0}^{(a)}(C) \triangle D \right) \right)$$

$$\leq \delta/2 + \delta/2$$

$$= \delta$$

This shows that $P'_n$ is a generating sequence of partitions.

Our next objective is to show that $D_\mu(K_{n+1}(P_n), K_n(P_n)) < \varepsilon_n$. So we do the following computations:

$$K_n(P_n) = K_n(\xi_n) = H_n^{(a)} \circ H_{n-1}^{(a)}(H_n(\xi_n)) = H_n^{(a)}(\xi_n) = H_n^{(a)} \circ h_{n+1} \circ h_{n+1}^{-1}(\xi_n)$$
On the other hand,

\[ K_{n+1}(P_n) = K_{n+1}(\zeta_n) = H_{n+1}^{(a)} \circ H_n^{-1}(H_n(\zeta_n)) = H_n^{(a)} \circ h_{n+1}^{(a)} \circ h_n^{-1}(\zeta_n) \]

Put \( Q_n := h_n^{-1}(\zeta_n) \) and note that by construction \( h_{n+1}^{(a)}(Q_n) \) approximates \( h_{n+1}(Q_n) \) and we are done. \( \square \)

6.2 Fast enough in the real-analytic context

We investigate what it means to be fast enough in theorem 6.1. We fix a sequence \( \{\varepsilon_n\}_{n=1}^{\infty} \) and assume that for any \( n \in \mathbb{N} \):

\[ \frac{\varepsilon_n}{4} > \sum_{m=n}^{\infty} \varepsilon_m \quad (6.5) \]

For each choice of sequences \( \{k_n\}_{m=1}^{n}, \{l_n\}_{m=1}^{n-1} \) and \( \{s_n\}_{m=1}^{n+1} \) of natural numbers, we can have finitely many permutations of \( \xi_{k_n}^{n+1} \) and hence finitely many choices of \( h_{n+1} \). For each such choice, there exists a natural number \( l_n := \max_{h_{n+1}}(h_{n+1}, \{k_n\}_{m=1}^{n}, \{l_n\}_{m=1}^{n-1}, \{s_n\}_{m=1}^{n+1}, \rho) \) such that for any \( l \geq l_n \), we can choose \( h_{n+1}^{(a)} \) such that

\[ d_p(T_n^{(a)}, T_n^{(a)}) < \frac{\varepsilon_n}{4} \quad (6.6) \]

Finally to get an uniform estimate, we put

\[ l^*_n = l^*_n(\{k_n\}_{m=1}^{n}, \{l_n\}_{m=1}^{n-1}, \{s_n\}_{m=1}^{n+1}, \rho) := \max_{h_{n+1}}(h_{n+1}, \{k_n\}_{m=1}^{n}, \{l_n\}_{m=1}^{n-1}, \{s_n\}_{m=1}^{n+1}, \rho) \quad (6.7) \]

Now we note that for our construction there is a relation \( s_n = s_{n-1}^{k_n} \). We can define \( b_n = b_n(\{k_n\}_{m=1}^{n-1}) \) such that \( s_n < b_n \). Thus we can have our sequence \( l^*_n \) depend only on \( k_n \) s (after speeding up a little if needed).

In conclusion we obtain a sequence of natural numbers

\[ l^*_n = l^*_n(\{k_n\}_{m=1}^{n}, \{l_n\}_{m=1}^{n-1}, \rho) \quad (6.8) \]

and we say a sequence of natural numbers \( \{l_n\}_{n=1}^{\infty} \) grows fast enough if

\[ l_n \geq l^*_n \quad \forall \ n. \quad (6.9) \]

7 Symbolic representation of AbC systems

The purpose of this section is establish the main result where a symbolic representation of untwisted real-analytic AbC diffeomorphisms is obtained. This section is almost identical to [FW19a, section 7] though our presentation is much succinct.

We begin this section with the intrinsic description of obtaining a symbolic representation of the factor of a periodic process using construction sequences.
7.1 Symbolic representation of periodic processes

Here we exhibit how a periodic process can be viewed as a symbolic system. Everything in this section is done on a standard measure space \((X, \mathcal{B}, m)\).

Let \(\{\varepsilon_n\}_{n \in \mathbb{N}}\) be a summable sequence of positive numbers. Let \(\{(\tau_n, \mathcal{P}_n)\}\) be a sequence of periodic processes converging to some transformation \(T\). We assume that the height of all the towers of \((\tau_n, \mathcal{P}_n)\) is \(q_n\) and \((\tau_{n+1}, \mathcal{P}_{n+1})\) \(\varepsilon\)-approximates \((\tau_n, \mathcal{P}_n)\) with error set \(D_n\). In addition we can assume (after removing some steps in the beginning if needed) that \(\mu(\bigcup D_n) < 1/2\) and also assume \(q_0 = 1\).

We put \(G_n = X \setminus \bigcup_{m \geq n} D_n\). Then note that \(\{G_n\}_{n \in \mathbb{N}}\) is an increasing sequence of sets with \(\mu(G_n) \nearrow 1\). Also, when restricted to \(G_n\), \(\{P_m\}_{m \geq n}\) is a decreasing sequence of partitions. For each tower of \((\tau_n, \mathcal{P}_n)\), the measure of the intersection of \(G_n\) with each level of this tower are all the same.

Let \(\{T_i\}_{i=0}^{q_0-1}\) be the towers of \((\tau_0, \mathcal{P}_0)\). Since \(q_0 = 1\), each \(T_i\) contains a single set and we define \(Q_0\) to be the collection of all these sets.

We inductively define two sequence of sets \(\{B_n\}_{n \in \mathbb{N}}\) and \(\{E_n\}_{n \in \mathbb{N}}\). Let \(B_0 = E_0 = \emptyset\). Next we assume that we have defined \(\{B_m\}_{m=0}^{n-1}\) and \(\{E_m\}_{m=0}^{n-1}\). At the \(n+1\) th stage of the induction process we have to define \(B_{n+1}\) and \(E_{n+1}\). Since \(\tau_{n+1}\) \(\varepsilon\)-approximates \((\tau_n, \mathcal{P}_n)\), we note that each tower of \((\tau_{n+1}, \mathcal{P}_{n+1})\) when restricted to \(G_{n+1}\) contains

1. Contiguous levels: Contiguous sequences of levels of length \(q_n\) contained in towers of \((\tau_n, \mathcal{P}_n)\)

2. Interspersed levels: Levels of the towers of \((\tau_{n+1}, \mathcal{P}_{n+1})\) intersected with \(G_{n+1}\) not in the above contiguous sequences.

Next we divide each of the maximal contiguous portions of the interspersed levels into two contiguous portions arbitrarily. We define \(E_{n+1}\) to be the union of \(E_n\) and the subcollection of levels that comes first and \(B_{n+1}\) to be the union of \(B_n\) and the subcollection that comes second. Also we note the top and the bottom contiguous subcollection of interspersed levels are joined together and is considered to be a single contiguous subcollection for this process.

This completes the construction of the two sequences and we note that \(G_n = Q_0 \cup B_n \cup E_n\).

Our next goal is to find a symbolic representation of a factor of the limiting transformation \(T\) arising from the partition \(Q_0 \cup \{\bigcup_{n \in \mathbb{N}} B_n, \bigcup_{n \in \mathbb{N}} E_n\}\). We do this in two ways: First we describe the standard procedure using \(T\) and then we describe the procedure that uses the aforementioned sequences and construction sequence description of symbolic systems.

Let \(\Sigma\) be an alphabet of size \(q_0\). Let \(b\) and \(e\) be two letters not contained in \(\Sigma\). The letters in \(\Sigma' := \{a_i\}_{i=0}^{n-1}\) are considered to be indexes for the elements of \(Q_0 := \{\mathcal{A}_i\}_{i=0}^{n-1}\). Our symbolic systems is built on the alphabet \(\Sigma' \cup \{b, e\}\). We define a factor map

\[
K : X \to (\Sigma' \cup \{b, e\})^\mathbb{Z} \quad \text{defined by} \quad K(x)(i) := \begin{cases} 
  a_j & \text{if } T^i(x) \in A_j \\
  b & \text{if } T^i(x) \in \bigcup_{n \in \mathbb{N}} B_n \\
  e & \text{if } T^i(x) \in \bigcup_{n \in \mathbb{N}} E_n
\end{cases}
\]  

We give an alternate description of this symbolic system using construction sequences. We inductively define \(W_n\) which are collections of words of length \(q_n\) and surjections \(K_n : \{T \cap G_n : T \text{ is a tower of } (\tau_n, \mathcal{P}_n) \text{ and } G_n \cap T \neq \emptyset\} \to W_n\).

First we put \(W_0 := \Sigma\) and we assume that we have carried out the construction upto the \(n\) th stage. At the \((n+1)\) th stage we define the collection of words \(W_{n+1}\) and \(K_{n+1}\). Let \(T\) be a tower of \((\tau_{n+1}, \mathcal{P}_{n+1})\). Then with \(T \cap G_{n+1}\) we associate a word \(w\) of length \(q_{n+1}\) satisfying:
7.2 Comparison of two periodic processes using a transect - I

1. \( w(j) = v \) if the \( j \)-th level of \( T \) is a subset of the \( k \)-th level of of a tower \( S \) of \((\tau_n, \mathcal{P}_n)\) and the \( k \)-th letter of \( K_\alpha(S) \) is \( v \).

2. \( w(j) = b \) if the \( j \)-th level of \( T \) is a subset of \( B_{n+1} \setminus B_n \).

3. \( w(j) = e \) if the \( j \)-th level of \( T \) is a subset of \( E_{n+1} \setminus E_n \).

We define \( \mathbb{K} \) to be the collection of all \( x \in (\Sigma \cup \{b, e\})^\mathbb{Z} \) such that every contiguous subword of \( x \) is a contiguous subword of some \( w \in \mathcal{W}_n \) for some \( n \). Then \( \mathbb{K} \) is a closed shift invariant set that constitutes the support of \( K^*\mu \) and it is the required symbolic representation of the factor of \( T \) described explicitly earlier (see lemma 3.6).

We note that in the case of the untwisted AbC transformations we consider satisfying requirements 1, 2 and 3 (see section 4.3), \( Q \) will generate the transformation and the resulting symbolic representation will be isomorphic to \( T \).

7.2 Comparison of two periodic processes using a transect - I

Instead of directly comparing two subsequent periodic processes in the AbC method, we start our study slowly with the study of two periodic processes on the circle. This study will shed light on how a tower in the periodic process at the \( n \)-th stage of the AbC method compares with a tower at the \( n+1 \)-th stage. In fact we wish to study how the levels of a periodic process on the circle traverses the levels of another periodic process which it \( \varepsilon \)-approximates. The parameters for the processes are chosen similar to the AbC method.

Let \( p, q \) be two natural numbers and \( \alpha = p/q \). \( I_q := \{I_i^q := [i/q, (i+1)/q)\}_{i=0}^{q-1} \) is the standard partition of \([0,1)\) into \( q \) equal half open intervals. Recall that the atoms of \( I_q \) have two natural orderings:

1. The geometric ordering is the natural ordering of the intervals i.e. \( I_0^q < I_1^q < \ldots < I_{q-1}^q \).

2. The dynamical ordering is the ordering that comes from iteration by \( R_\alpha \) i.e. \( I_0^q < R^\alpha(I_0^q) < \ldots < (R^\alpha)^{q-1}(I_0^q) \).

Recall that with \( j_i = (p)^{-1}i \mod q \), the \( i \)-th interval in the geometric ordering is the \( j_i \)-th interval in the dynamical ordering. Indeed if \((R^\alpha)^{j_i}(I_0^q) = I_i^q \), then \( pj_i \equiv i' \).

Let \( \alpha = p/q \) and \( \alpha' = p'/q' \) where \( p, p', q \) and \( q' \) are natural numbers and \( q' = klq^2 \). We compare the two periodic processes \((R^\alpha, I_q)\) and \((R^{\alpha'}, I_{q'})\).

So if \( J \) is a subinterval of \( I_q^q \), and if \( J \) is not the last subinterval of \( I_q^q \) then \( R^{\alpha'}(J) \) is a subinterval of the \( j_i \)-th subinterval in the dynamical ordering i.e. \( R^\alpha(I_i^q) \). If \( J \) is the last subinterval, then \( R^\alpha(J) \) is geometrically the first subinterval of the \( j_i+1 \)-th subinterval in the dynamical ordering.

With the above observation in mind, we make detailed analysis of how the interval \( J := [0, 1/q') \) traverses the levels of the tower of \((R^\alpha, I_q)\) under the action of \( R^{\alpha'} \). Note that we have to study \( q' \) iterates of \( R^\alpha \) to get a complete picture. So we divide the set \( \{0, 1, \ldots, q' = klq^2 \} \) into contiguous portions of size \( q \).

On the first portion i.e. for \( n = 0, 1, \ldots, klq - 1 \), \((R^{\alpha'})^n(J) \subset (R^\alpha)^n(I_0^q) \) and \((R^{\alpha'})^klq(J) \) is geometrically the first subinterval of \( I_q^q \).

More generally when we study the iterations of \( J \) for \( n = mklq, \ldots, (m+1)klq \), we see that the iterations exhibit the following three types of behavior:
1. The beginning interval \([mkql, \ldots, mkql + q - j_m]\): This is of length \(q - j_m\). Note that \((R^n)^{mkql}(J)\) is geometrically the first subinterval of \(I^n_m\). Then with \(n\) in the beginning interval, \((R^n)^{mkql}(J)\) traverses the interval in places \(j_m, j_m + 1, \ldots, q - 1\) in the dynamical ordering of \((R^n, I_q)\).

2. The middle interval \([mkql + q - j_m, mkql + q - j_m + (kl - 1)q]\): This is of length \(klq - q\). With \(n\) in the middle interval, \((R^n)^{mkql}(J)\) traverses the intervals following the dynamical ordering of \((R^n, I_q)\) starting from \(I^n_0\).

3. The end interval \([mkql + q - j_m + (kl - 1)q, (m + 1)klq]\): This has length \(j_m\). With \(n\) in the end interval, \((R^n)^{mkql}(J)\) traverses the interval in places \(0, 1, \ldots, j_m - 1\) in the dynamical ordering of \((R^n, I_q)\). \((R^n)^{mkql-1}(J)\) is geometrically the last subinterval of \(I^n_m\).

### 7.3 Comparison of two periodic processes using a transect - II

The previous section did shed some light on how two periodic processes compare on the circle when one \(\varepsilon\) approximates the other but we note that in the previous section we did not pay much attention to the fact that the AbC method uses the partition whose projection is \(I_{kq}\) and not \(I_q\). We do the study again, but keeping this finer partition in mind.

Let \(p, q, p', q', \alpha\) and \(\alpha'\) be as before. We divide the subinterval \(I_{kq}\) into \(k\) ordered sets described as follows:

\[
w_j := \{t^k_{j+tk} : t = 0, \ldots, q - 1\}
\]  

(7.2)

So each \(w_j\) is the orbit of \([j/(kq), (j + 1)/(kq)]\) under \(R_n\). It can be viewed as a word of length \(q\) in the alphabet \(I_{kq}\).

Let \(J\) be the subinterval as before (see section 7.2). We track the \(R_n\) iterates of \(J\) through the \(w_j\) s as follows:

0. 0-th interval \(n \in [0, klq]\): Note that any such \(n\) can be written as \(n = mlq + sq + t\) for some appropriately chosen \(0 \leq m < k, 0 \leq s < l, 0 \leq t < q\). So \((R^n)\) is a subinterval of the \(t\)-th element of \(w_m\). So

- \(t \in [0, lq]\): The \(I_{kq}\)-name agrees with the \(w_0\) name repeated \(l\) times
- \(t \in [lq, 2lq]\): The interval crosses the boundary for the \(I_{kq}\) partition and the name changes to \(w_1\) repeated \(l\)-times.
- \(\ldots\).
- \(k - 3\) more times to a total of \(k - 1\) times.
- \(\ldots\).
- \(t \in [(k-1)lq, klq]\): The interval crosses the boundary for the \(I_{kq}\) partition and the name changes to \(w_{k-1}\) repeated \(l\)-times. \(J_{klq-1}\) is geometrically the last subinterval of \(I_1\).

Hence the first \(klq\) letters of the \(I_{kq}\)-name of any point in \(J\) is

\[
w_0^l w_1^l w_2^l \ldots w_{k-1}^l
\]

(7.3)

1. 1-st interval \(n \in [klq, 2klq]\):
7.3 Comparison of two periodic processes using a transect - II

\( t \in [klq, (k + 1)lq] \):
- \( t \in [klq, klq + q - j_1] \): \((R^\alpha)^{klq}(J) = J_{klq} \) is geometrically the first subinterval of \( I_0^q \). We use another \( q - j_1 \) applications of \( R^\alpha \) to bring \( J \) inside \( I_0^q \).
- \( t \in [klq + q - j_1, (k + 1)lq - j_1] \): \( J_{klq+q-j_1} \) is a subinterval of the geometrically first subinterval of \( I_{kq} \). In fact it is in the first subinterval of \( w_1 \). We apply \( R^{\alpha'}(l - 1)q \) times to carry it through \( l - 1 \) copies of \( w_1 \) and it arrives at \( I_0^q \).
- \( t \in [(k + 1)lq - j_1, (k + 1)lq] \): We apply \( R^{\alpha'} \) again \( j_1 \) times to bring it back to \( I_1^q \) inside \( w_2 \).

Resulting name is \( b_0^{q-j_1} w_0^{l_1} e_0^{j_1} \). Here \( b_j^{q-j_1} \) and \( e_j^{j_1} \) are the last \( q - j_1 \) and the first \( j_1 \) elements of \( w_j \).

- \( t \in [(k + 1)lq, (k + 2)lq] \): With an identical argument we get the name \( b_1^{q-j_1} w_1^{l_1} e_1^{j_1} \).

\[ \ldots \]
- \( t \in [(k + k - 1)lq, 2klq] \): With an identical argument we get the name \( b_k^{q-j_1} w_k^{l_1} e_k^{j_1} \). \( J_{2klq-1} \) is the geometrically last subinterval of \( I_1 \).

So the \( klq \) to \( 2klq - 1 \)-th letters of the \( I_{kq} \) name of any point in \( J \) is given by

\[ b_0^{q-j_1} w_0^{l_1} e_0^{j_1} b_1^{q-j_1} w_1^{l_1} e_1^{j_1} \ldots b_{k-1}^{q-j_1} w_{k-1}^{l_1} e_{k-1}^{j_1} \]

(7.4)

m. \( m \)-th interval \( n \in [mkq, (m + 1)klq] \): The \( mkq \) to \( (m + 1)klq - 1 \)-th letters of the \( I_{kq} \) name of any point in \( J \) is given by

\[ b_0^{q-j_m} w_0^{l_1} e_0^{j_m} b_1^{q-j_m} w_1^{l_1} e_1^{j_m} \ldots b_{k-1}^{q-j_m} w_{k-1}^{l_1} e_{k-1}^{j_m} \]

(7.5)

m+1. \( \ldots \)

\[ \ldots \]

q-1. \((q-1)\)-th interval \( t \in [(q - 1)klq, klq] \): An identical argument yields the name

\[ b_0^{q-j_{q-1}} w_0^{l_1} e_0^{j_{q-1}} b_1^{q-j_{q-1}} w_1^{l_1} e_1^{j_{q-1}} \ldots b_{k-1}^{q-j_{q-1}} w_{k-1}^{l_1} e_{k-1}^{j_{q-1}} \]

(7.6)

So in conclusion any point in \( J \) has a \( I_{kq} \)-name as follows:

\[ w = \prod_{i=0}^{q-1-k} \prod_{j=0}^{k-1} (b_j^{q-j} w_j^{l} e_j^{j}) \]

(7.7)

So our periodic process is isomorphic to the symbolic system defined by the above circular operator.
7.4 Back to the regular sequence of periodic processes

Note that the partition $\xi_n$ divides $\mathbb{T}^2$ into $s_n$ identical towers. On each tower, the action is identical and hence when restricted to a single tower of $\mathcal{I}_{k_nq_n} \otimes \mathcal{I}_{s_{n+1}}$ we get the same analysis as before. So we can also copy the previous labeling to a labeling here.

7.5 Symbolic representation of AbC systems

First we describe the general idea behind the representation of the AbC method as a symbolic system. We recall some relevant portions from the abstract untwisted AbC method described in section 4. The limit transformation $T$ is obtained as the weak limit of transformations $T_n := H_n \circ R^{n+1} \circ H_n^{-1}$. At the $n$th stage $T_n$ permutes the partition $\xi_n = H_n(\xi_{n-1})$. With the above in mind, we recall that $(\tau_n, \xi_n)$ is a periodic process where $\tau_n$ is the permutation induced by $T_n$ on $\xi_n$.

If $Q$ is a partition refined by the levels of a periodic process $\tau$, then the $Q$ names of any pointwise realization of $\tau$ are constant on the levels of the tower. Hence this is equivalent to naming the levels in the action of $\tau$ on various towers. We call the resulting collection of names the $\{\tau, Q\}$ names.

Let $Q^*$ be an arbitrary partition of $\mathbb{T}^2$ refined by $\xi_n$. We would like to compare the $Q^*$ names of points under $\tau$ and $\tau_n$. We introduce the partition $(H_n)^{-1}(Q^*)$. So the problem of finding the $(\tau_{n+1}, Q^*)$ name is equivalent to finding the $(h_{n+1} \circ R^{n+1} \circ h_{n+1}, \mathcal{P})$ names of towers whose levels consist of the partition $(H_n)^{-1}(\xi_{n+1}) = \xi_{n+1}$.

Tracking movement of individual rectangles

For notational simplicity we put $k_n = k, q_n = q, l_i = l_i^q, n = 0, 1, \ldots$ Fix $R \in \xi_{n+1}$.

Assume that $R_{t+1,i,j} = h_{n+1}(R) \subset J_i$ for some $i$ and $J_i$ is not the last subinterval of an interval in the partition $\mathcal{I}_{kq}$. In this case both $R^{n+1}(R_{t,i,j})$ and $R^{n+1}(R_{t+1,i,j})$ belong to the same element of $\mathcal{I}_{kq} \times \mathbb{T}^2$.

Since $R^{n+1}$ commutes with $h_{n+1}$ and $h_{n+1}$ permutes the atoms of $\mathcal{I}_{k_nq_n} \otimes \mathcal{I}_{s_{n+1}}$, we have $h_{n+1} \circ R^{n+1} \circ h_{n+1}(R) = h_{n+1} \circ R^{n+1}(R_{t+1,i,j})$ and $h_{n+1} \circ R^{n+1} \circ h_{n+1}(R) = R^{n+1}(R)$ belong to the same atom of $\mathcal{I}_{k_nq_n} \otimes \mathcal{I}_{s_{n+1}}$. So they share the same $\mathcal{P}$-name.

On the other hand if $J_i$ is the last subinterval of an interval in the partition $\mathcal{I}_{kq}$, then $R^{n+1}$ sends $R_{t+1,i,j}$ to the geometrically first subrectangle of a new element $R'$ of $\mathcal{I}_{kq} \otimes \mathcal{I}_{s_{n+1}}$. Thus $h_{n+1} \circ R^{n+1} \circ h_{n+1}(R) = h_{n+1}(R')$.

Tracking movement of levels through a tower

We note that the base of the towers of $(\tau_{n+1}, \xi_{n+1})$ are the rectangles $\{h_{n+1}(R_{0,j}^{n+1})\}_{j=0}^s$ while those for the towers of $(h_{n+1} \circ R^{n+1} \circ h_{n+1}, \mathcal{P})$ are $\{h_{n+1}(R_{0,j}^{n+1})\}_{j=0}^s$.

So with $F_0 := h_{n+1}(R_{0,j}^{n+1})$ as the base, we define $F_t := (h_{n+1} \circ R^{n+1} \circ h_{n+1})^t(h_{n+1}(R_{0,j}^{n+1})) = (h_{n+1} \circ (R^{n+1})^t(R_{0,j}^{n+1})) = h_{n+1}(R_{t+1,i,j})$ where $i_t$ is the dynamical ordering of the $t$-th interval under iterations by $R^{n+1}$.

We recall the labeling we used in section 7.3 using the $u, b$ and $e$ s. So if $t$ is such that $J_{i_t}$ is labeled with a part of $u$, the two transformations $h_{n+1} \circ R^{n+1} \circ h_{n+1}$ and $R^{n+1}$ move $F_t$ to a subrectangle of the same element of $\mathcal{I}_{kq} \otimes \mathcal{I}_{s_{n+1}}$ and hence the same element of $\mathcal{P}$.

---

7 So $\xi_n$ is just $\xi_n$ ordered in a different way.
For $j < k, t < q$ and $s < s_{n+1}$, define:

$$R_{j,t,s} := \left[ \frac{j + tk}{kq}, \frac{j + tk + 1}{kq} \right] \times \left[ \frac{s}{s_{n+1}}, \frac{s + 1}{s_{n+1}} \right]$$

(7.8)

And $u_{j,s}$ be the sequence of $\mathcal{P}$-names for

$$\{h_{n+1}(R_{j,t,s})\}^{q-1}_{t=0}$$

(7.9)

The word $u_{j,s}$ is the sequence of $\mathcal{P}$-names for the levels of the tower $\{(R^{\alpha})^t \circ h_{n+1}(R^{\alpha+1})_{t=0}\}^q_{t=0}$, for any $J_i \subset [j/(kq), (j+1)/(kq)]$.

Using transects, we now describe the $\mathcal{P}$-name of the orbit of $F_0$ under $h_{n+1} \circ R^{\alpha_n+1} \circ h_{n+1}^{-1}$. We use $t$ to denote the iterations of $h_{n+1} \circ R^{\alpha_{n+1}} \circ h_{n+1}^{-1}$ and hence it also denotes the level of the tower.

0. 0-th interval $t \in [0, klq)$:

- $t \in [0, lq)$: The $(h_{n+1} \circ R^{\alpha_{n+1}} \circ h_{n+1}^{-1}, \mathcal{P})$ name agrees with the $R^{\alpha}$-name $u_{0,s}$ repeated $l$-times.
- $t \in [lq, 2lq)$: The interval crosses the boundary for the $I_{kq}$ partition and the name changes to $u_{1,s}$ repeated $l$-times.

\[\ldots\]

- $k - 3$ more times to a total of $k$ times.

\[\ldots\]

- $t \in [(k-1)lq, klq)$: The interval crosses the boundary for the $I_{kq}$ partition and the name changes to $u_{k-1,s}$ repeated $l$-times. $J_{i_{klq-1}}$ is geometrically the last subinterval of $I_1$.

1. 1-st interval $t \in [klq, (k+1)lq)$:

- $t \in [klq, (k+1)lq)$:
  - $t \in [klq, klq + q - j_1)$: $J_{i_{klq}}$ is geometrically the first subinterval of $I_2$. This portion of the transect is labeled with $q - j_1$ copies of $b$.
  - $t \in [klq + q - j_1, (k+1)lq - j_1)$: $J_{i_{klq+q-j_1}}$ is a subinterval of the geometrically first subinterval of $I_{kq}$. So $F_1$ is the first letter of $u_{0,s}$. For the next $q(l - 1)$ iterations the $h_{n+1} \circ R^{\alpha_{n+1}} \circ h_{n+1}^{-1}$ names are the same as the $R^{\alpha}$ names and obtain the name $u_{0,s}$.
  - $t \in [(k+1)lq - j_1, (k+1)lq)$: This segment of the transects is labeled by $j_1$ copies of $c$.

Resulting name is $b^{q-j_1}u_{0,s}^{l-1}e^{j_1}$.

- $t \in [(k+1)lq, (k+2)lq)$: With an identical argument we get the name $b^{q-j_1}u_{1,s}^{l-1}e^{j_1}$.

\[\ldots\]

- $k - 3$ more times to a total of $k$ times.

\[\ldots\]

- $t \in [(k+k-1)lq, 2klq)$: With an identical argument we get the name $b^{q-j_1}u_{k-1,s}^{l-1}e^{j_1}$.

$J_{i_{2klq-1}}$ is the geometrically last subinterval of $I_1$. 

So the name of the 1-st interval is:

\[ b_{q-1} u_{1,1} e_{j_1} \ldots b_{q-1} u_{1,1} e_{j_1} \]  

(7.10)

2. 

3. 

... 

m. m-th interval \( t \in [mklq,(m+1)klq) \): An identical argument yields the name

\[ b_{q-1} u_{1,1} e_{j_1} \ldots b_{q-1} u_{1,1} e_{j_1} \]  

(7.11)

m+1. 

... 

q-1. (q-1)-th interval \( t \in [(q-1)klq,klq] \): An identical argument yields the name

\[ b_{q-1} u_{1,1} e_{j_1} \ldots b_{q-1} u_{1,1} e_{j_1} \]  

(7.12)

Theorem 7.13. Let \( F_0 := h_{n+1}(R_{0,s}) \) be the base of a tower \( \mathcal{T} \) for \( h_{n+1} \circ R^{\alpha^n+1} \circ h_{n+1}^{-1} \). Then the \( \mathcal{P} \)-names of \( \mathcal{T} \) agree with

\[ u := \prod_{i=0}^{q-1} \prod_{j=0}^{k-1} b_{j} u_{j}^{l-1} e_{j} \]  

(7.14)

on the interior of \( u \).

Corollary 7.15. With \( J_i \subseteq [j/(kq), (j+1)/kq - (1/q_{n+1})] \), \( j < kq \) and \( R = R_{t,n}^{n} \), the levels \( \{ (R^{\alpha^n})^i(R) \}_{i=0}^{q-1} \) coincide with the levels \( \{ (h_{n+1} \circ R^{\alpha^n+1} \circ h_{n+1}^{-1})^i h_{n+1}(R) \}_{i=0}^{q-1} \) in the tower for \( \tau_n \). In particular their \( \mathcal{Q}^* \) names agree.

Corollary 7.16. For a set \( x \in X \) having measure at least \( 1 - 3/l_n \), the \( (\tau_n, \mathcal{Q}^*) \) and \( (\tau_n, \mathcal{Q}^*) \) names of \( x \) agree on the interval \([-q,q]\).

Corollary 7.17. Suppose that \( x \in \Gamma_n \) and \( x \) is on level \( t_n \) of a \( \tau_n \)-tower and the level \( t_{n+1} \) of a \( \tau_{n+1} \)-tower. Let \( w_n \) be the \( \mathcal{Q}^* \)-name of \( x \) with respect to \( \tau_n \) and \( w_{n+1} \) be the \( \mathcal{Q}^* \)-name of \( x \) with respect to \( \tau_{n+1} \). Then \( w_{n+1}|_{[t_{n+1} - t_n, t_{n+1} + q_{n+1} - t_n]} = w_n \).
7.5 Symbolic representation of AbC systems

The symbolic representation

We define the following two sets:

\[ B := \{ x \in \mathbb{T}^2 : \text{ for some } m \leq n, \ x \in \Gamma_n \text{ and } H_m^{-1}(x) \in B_m \} \quad (7.18) \]
\[ E := \{ x \in \mathbb{T}^2 : \text{ for some } m \leq n, \ x \in \Gamma_n \text{ and } H_m^{-1}(x) \in E_m \} \quad (7.19) \]

We define the partition

\[ \mathcal{Q} := \{ A_i : i < s_0 \} \cup \{ B, E \} \quad (7.20) \]

where \( \{ A_i \}_{i=0}^{s_0} \) is the partition \( \zeta_0|_{\mathbb{T}^2 \setminus B \cup E} \). Explicitly \( A_i := H_0([0,1) \times [s/s_0, (s+1)/s_0) \setminus B \cup E) \).

We will construct \((T, \mathcal{Q})\)-names for each \( x \in \mathbb{T}^2 \) using the alphabet \( \Sigma \cup \{ b, e \} \) where \( \Sigma := \{ a_i \}_{i=0}^{s_0-1} \).

So the the name of point \( x \in \mathbb{T}^2 \) will be an \( f \in (\Sigma \cup \{ b, e \})^\mathbb{Z} \) with \( f(n) = a_i \iff T^n(x) \in A_i, \)
\( f(n) = b \iff T^n(x) \in B \) and \( f(n) = e \iff T^n(x) \in E. \)

We use induction for a complete description of the \((T, \mathcal{Q})\)-names of points in \( \cup \Gamma_n \). Let \( F_0 := H_{n+1}(R_0^{n+1}) \) for some \( s^* < q_{n+1} \) be the base of a tower \( T \) of \( \tau_{n+1} \).

Inductively we assume that the \((\tau_n, \mathcal{Q})\)-names of the towers with bases \( \{H_n(R_0^{n})\}_{s=0}^{s_n} \) are \( u_0, \ldots, u_{s_n-1} \).

At the \( n + 1 \)-th stage of the induction we start by defining words \( w_0, \ldots, w_{k_n-1} \) by setting

\[ w_j = u_s \iff h_{n+1}\left(\left[\frac{j}{k_n q_n}, \frac{j+1}{k_n q_n}\right] \times \left[\frac{s^*}{s_{n+1}}, \frac{s^*+1}{s_{n+1}}\right]\right) \subseteq R_{0,s}^n \quad (7.21) \]

We say that \((w_0, \ldots, w_{k_n-1})\) is the sequence of \( n \)-words associated with \( T \).

Definition 7.22. We define a circular system by inductively specifying the sequence \( \{W_n\}_{n \in \mathbb{N}} \).
We put \( W_0 := \{a_i\}_{i=0}^{s_0} \). After defining \( W_n \) we define

\[ W_{n+1} := \{C_{n+1}(w_0, \ldots, w_{k_n-1}) : (w_0, \ldots, w_{k_n-1}) \text{ is associated with a tower } T \text{ in } \tau_{n+1}\} \quad (7.23) \]

We say that \( \{W_n\}_{n \in \mathbb{N}} \) is the construction sequence associated with the AbC construction.

Theorem 7.24. Suppose \( T \) is an AbC transformation satisfying Requirements 1, 2, 3 and the \( l_n \) parameters are assumed to grow fast enough. Let \( \{W_n\}_{n \in \mathbb{N}} \) be the construction sequence associated with \( T \) and let \( \mathcal{K} \) be its circular system.

The almost all \( x \in \mathbb{T}^2 \) have \( \mathcal{Q} \)-names in \( \mathcal{K} \). In particular there is a measure \( \nu \) on \( \mathcal{K} \) that makes \((\mathcal{K}, B, \nu, sh)\) isomorphic to the factor of \( \mathbb{T}^2 \) generated by \( \mathcal{Q} \).

Proof. Let \( M \) be any natural number. Then for a.e. \( x \in \mathbb{T}^2 \), there exists an \( N = N(x) \) such that for all \( n > N \) the first or last \( M \) levels of \( \tau_n \) does not contain \( x \).

Let \( x \in \Gamma_{n_0} \subset \cup \Gamma \) be an arbitrary point. Then \( x \) belongs to a tower \( T \) of \( \tau_n \) and we may in light of the previous assumption assume that \( x \) does not belong to the first or last \( M \) levels of \( T \).
Then we can use corollary 7.17 to conclude that if \( x \) is in the \( t_n \)-th level of a \( \tau_n \)-tower, then the \( T \)-name of \( x \) agrees with the \( \tau_n \) name of \( x \) on the interval \([-t_n, q_n - t_n] \).

Thus we can apply theorem 7.13 to conclude that with \( \mathcal{P} = (H_n)^{-1}(\mathcal{Q}) \), we see that a tower \( T \) for \( \tau_n \) has the name:

\[ w = \prod_{i=0}^{q_{n-1} - 1} \prod_{j=0}^{k_{n-1} - 1} (b^y_{n-j} u_j^{l-1} e^j) \quad (7.25) \]
where \( \{w_j\}_{j=0}^{k_n-1} \) is the sequence of words associated with \( T \). For \( x \) in the \( t_n \)-th level of \( T \), the \( Q \)-name of \( x \) on the interval \([-t_n, q_n - t_n]\) is

\[
\prod_{i=0}^{q_n-1} \prod_{j=0}^{k_n-1} (b^{q_n-j_i} u_j^{-1} e^{j_i}) = C_n(w_0, \ldots, w_{k_n-1-1})
\]  

(7.26)

Since \( M < \min(t_n, q_n - t_n) \) we have \( x|_{[-M,M]} \) is a subword of \( W_n \).

It follows that the factor of \((T^2, B, \mu, T)\) corresponding to the partition \( Q \) is a factor of the uniform circular system defined by the constructions sequence \( \{W_n\}_{n \in \mathbb{N}} \).

Conversely, by lemma \( 3.6 \), the uniform circular system \( K \) determined by \( \{W_n\}_{n \in \mathbb{N}} \) is characterized as the smallest shift invariant closed set intersecting every basic open set \( C_0(w) \) in \((\Sigma \cup \{b, e\})^\mathbb{Z}\) determined by some \( w \in W_n \). However each \( w \in W_n \) is represented on \( \Gamma_n \cap T \) for some \( T \), hence each \( C_0(w) \) has non empty intersection with the set of words arising from \((T, Q)\)-names.

**Lemma 7.27.** Let \( a_0, b_0 \in \mathbb{N} \). Then for a.e. \( x \) and large enough \( n, x \in \Gamma_n, x \) does not occur in the first \( a_0 \) levels or last \( b_0 \) levels if a tower of \( \tau_n \). In particular for a.e. \( x \in X \) there are \( a > a_0, b > b_0 \) such that the \( Q \)-name of \( x \) restricted to the interval \([-a, b] \) belongs to \( W_n \).

**Corollary 7.28.** For a.e. \( x \in T^2 \) the \( Q \)-name of \( x \) is in \( S \). In particular if \( \nu \) is the unique non-atomic shift-invariant measure on \( K \), then the factor of \((T^2, B, \mu, T)\) generated by \( Q \) is isomorphic to \((K, B, sh, \nu)\).

**Lemma 7.29.** Suppose that for all \( n \), the map sending a tower \( T \) for \( \tau_n \) to the sequence of \( Q \)-names associated to \( T \) is a one-one function. Then \( Q \) generates the transformation \( T \).

**Proof.** Without loss of generality assume that \( H_0 \) is the identity map. Since \( \{H_n \xi_n\}_{n \in \mathbb{N}} \) is a decreasing and generating sequence of partitions, and \( \mu(\Gamma_n) \searrow 1 \), we conclude that \( \{H_n \xi_n \cap \Gamma_n\}_{n \in \mathbb{N}} \) also generate \( B \). So we will be done once we show that \( H_n \xi_n \cap \Gamma_n \) belongs to the smallest translation invariant \( \sigma \)-algebra generated by \( \{\vee_{n=-N}^{N} T^i (Q \cup \{B, E\})\}_{N \in \mathbb{N}} \).

Each \( P \in H_n \xi_n \) is the \( t \)-th level of some tower \( T \) for \( \tau_n \). Let \( w \in (\Sigma \cup \{b, e\})^\mathbb{N} \) be the \((\tau_n, Q)\)-name of \( T \). So the \( j \)-th letter of \( w \) determines and \( S_{j_i} \in (\{B \cup \{E \cup \{A_i : i < s_0\}) \). Since \((\tau_n, Q)\)-name of \( T \) is correct on \( \Gamma_n \),

\[
P \cap \Gamma_n \subseteq \bigcap_{j=0}^{q_n-1} T^{t-j}(S_{j_i}) \cap \Gamma_n
\]

(7.30)

On the other hand, since the map sending towers to names \( w \) is one to one, we see

\[
P \cap \Gamma_n \supseteq \bigcap_{j=0}^{q_n-1} T^{t-j}(S_{j_i}) \cap \Gamma_n
\]

(7.31)

This completes the proof.

**Lemma 7.32.** If the \( AbC \) construction is done satisfying the Requirements 1, 2 and 3 then \( Q \) generates the transformation \( T \).
7.5 Symbolic representation of AbC systems

Proof. We will show that the Q-names associated with any two $\tau_n$-towers $T$ and $T'$ are different.

For $n = 0$ it is clear by definition. We assume that this is true up to $n$ and we show it for $n + 1$. Let $R_{0,s}^n + 1$ and $R_{0,s'}^n + 1$ be the bases of $T$ and $T'$ respectively. Requirement 3 implies that the $k_n$-tuples $(j_0, \ldots, j_{k_n - 1})$ and $(j'_0, \ldots, j'_n - 1)$ associated with $s$ and $s'$ are distinct. Let $w_i$ be the names of the towers of $\tau_n$ with bases $R_{0,s}^n$. Then all the $w_i$ s are distinct as per our induction hypothesis. Also by the Theorem 7.27, the Q-names of $T$ and $T'$ are given by $\mathcal{C}(w_{j_0}, \ldots, w_{j_n - 1})$ and $\mathcal{C}(w'_{j'_0}, \ldots, w'_{j'_n - 1})$ respectively. Since $(j_0, \ldots, j_{k_n - 1})$ and $(j'_0, \ldots, j'_n - 1)$ are different, $\mathcal{C}(w_{j_0}, \ldots, w_{j_n - 1})$ and $\mathcal{C}(w'_{j'_0}, \ldots, w'_{j'_n - 1})$ are different.

We conclude the proof using lemma 7.32.

\begin{theorem}
Suppose the measure preserving system $(T^2, \mathcal{B}, \mu, T)$ is built by the AbC method using fast growing coefficients and $h_n$ s in the scheme satisfies requirements 1 to 3. Let $\mathcal{Q}$ be the partition defined earlier. Then the $\mathcal{Q}$-names describe a strongly uniform circular construction sequence $\{W_n\}_{n \in \mathbb{N}}$. Let $\mathbb{K}$ be the associated circular system and $\phi : T^2 \rightarrow \mathbb{K}$ be the map sending each $x \in T^2$ to its $\mathcal{Q}$-name. Then $\phi$ is one to one on a set of $\mu$ measure one. Moreover, there is a unique non-atomic shift invariant measure $\nu$ concentrating on the range of $\phi$ and this measure is ergodic. In particular $(T^2, \mathcal{B}, \mu, T)$ is isomorphic to $(\mathbb{K}, \mathcal{B}, \nu, sh)$.
\end{theorem}

Proof. With fast growing $l_n$ parameters, we know that the sequence $W_n$ forms a uniform circular sequence and hence there is a unique shift invariant non-atomic ergodic measure $\nu$ on $S \subset \mathbb{K}$. Using lemma 7.27 we have that the range of $\phi$ is a subset of the set $S$. So the factor determined by $\phi$ is isomorphic to $(S, \mathcal{B}, \mu, sh)$

Since requirements 1,2 and 3 are satisfied, the partition $\mathcal{Q}$ generates $T^2$ and hence $\phi$ is an isomorphism.

\begin{theorem}
Consider three sequence of natural numbers $\{k_n\}_{n \in \mathbb{N}}, \{l_n\}_{n \in \mathbb{N}}, \{s_n\}_{n \in \mathbb{N}}$ tending to infinity. Assume that

1. $l_n$ grows sufficiently fast.
2. $s_n$ divides both $k_n$ and $s_{n+1}$.

Let $\{\{W_n\}\}_{n \in \mathbb{N}}$ be a circular construction sequence in an alphabet $\Sigma \cup \{b, e\}$ such that

1. $W_0 = \Sigma, |W_{n+1}| = s_{n+1}$ for any $n \geq 1$.
2. (Uniform) For each $w' \in W_{n+1}$, and $w \in W_n$, if $w' = \mathcal{C}(w_0, \ldots, w_{k_n - 1})$, then there are $k_n/s_n$ many $j$ with $w = w_j$.

Then,

1. $\{W_n\}_{n \in \mathbb{N}}$ is a uniform construction sequence. If $\mathbb{K}$ is the associated symbolic shift then there is a unique non atomic ergodic measure $\nu$ on $\mathbb{K}$.

2. There is a real-analytic measure preserving transformation $T$ defined on $T^2$ such that the system $(T^2, \mathcal{B}, \mu, T)$ is isomorphic to $(\mathbb{K}, \mathcal{B}, \nu, sh)$.
\end{theorem}

Proof. Suppose that we have defined $\{h_m\}_{m=0}^n$ in the AbC process. From the definition of uniform circular systems (see 3.18), we can find $P_{n+1} \subseteq (W_n)^{k_n}$ such that $W_{n+1}$ is the collection of $w'$ such that for some sequence $(w_0, \ldots, w_{k_n - 1}) \in P_{n+1}$, $w' = \mathcal{C}(w_0, \ldots, w_{k_n - 1})$. We enumerate $P_{n+1} = \{w_0', \ldots, w_{s_{n+1}-1}'\}$. We apply lemma 4.11 to get $h_{n+1}$ from $w_0', \ldots, w_{s_{n+1}-1}'$. 

...
With \( \{h_n\}_{n \in \mathbb{N}} \) constructed as above, we note that requirement 1 is satisfied because \( s_n \uparrow \infty \), requirement 2 and 3 follows from the fact that the words in \( P_{n+1} \) are distinct.

We note that the sequence \( \{P_n\}_{n \in \mathbb{N}} \) determines \( h_n \) in the AbC method which in turn determines a neighborhood in the real-analytic topology in which the resulting AbC diffeomorphism belongs. Conversely, different choices of \( P_n \) gives distant \( h_n \)'s and hence distant \( h_n^{(a)} \) in the analytic topology. We record this as this is going to be crucial in the proof of an anti-classification result.

**Proposition 7.34.** Suppose \( \{U_n\}_{n \in \mathbb{N}} \) and \( \{W_n\}_{n \in \mathbb{N}} \) are two construction sequences for circular systems and \( M \) is such that \( U_n = W_n \forall n \leq M \). If \( S \) and \( T \) are the real-analytic realizations of the circular systems using the AbC method given in this paper, then

\[
d_{\rho}(S, T) < \varepsilon_M \quad (7.35)
\]

**Proof.** The sequences \( \{k_n^T, l_n^T, h_n^T, s_n^T\}_{n \in \mathbb{N}} \) and \( \{k_n^W, l_n^W, h_n^W, s_n^W\}_{n \in \mathbb{N}} \) associated with the two construction sequences determining approximations \( \{S_n\}_{n \in \mathbb{N}} \) and \( \{T_n\}_{n \in \mathbb{N}} \) to diffeomorphisms \( S \) and \( T \) has the property \( k_n^T = k_n^W, l_n^T = l_n^W, h_n^T = h_n^W, s_n^T = s_n^W \) for all \( n \leq M \). So \( S_M = T_M \). It follows from equations 6.5 and 6.6 that

\[
d_{\rho}(S_m, S) < \varepsilon_M/2 \quad (7.36)
\]
\[
d_{\rho}(T_m, T) < \varepsilon_M/2 \quad (7.37)
\]

Combining the two together with the triangle inequality we get the result.

---

**8 Proof of Theorem E**

In this Section we survey the key steps from [FRW11] and [FW3pp] and adapt them to our proof of Theorem E in the real-analytic category.

**8.1 Trees, groups and equivalence relations**

During the construction the following maps will prove useful.

**Definition 8.1.** We define a map \( M : \text{Trees} \to \mathbb{N}^\mathbb{N} \) by setting \( M(\mathcal{T})(s) = n \) if and only if \( n \) is the least number such that \( \sigma_n \in \mathcal{T} \) and \( |\sigma_n| = s \). Dually, we also define a map \( s : \text{Trees} \to \mathbb{N}^\mathbb{N} \) by setting \( s(\mathcal{T})(n) \) to be the length of the longest sequence \( \sigma_m \in \mathcal{T} \) with \( m \leq n \).

**Remark.** When \( \mathcal{T} \) is clear from the context we write \( M(s) \) and \( s(n) \). We also note that \( s(n) \leq n \) and that \( s \) as well as \( M \) are continuous functions when we endow \( \mathbb{N} \) with the discrete topology and \( \mathbb{N}^\mathbb{N} \) with the product topology.

Related to the structure of the tree we will specify several equivalence relations \( \mathcal{Q}_n^\mathcal{T} \) on the words \( \mathcal{W}_n \). For this purpose, we recall some general notions on equivalence relations.

**Definition 8.2.** Let \( X \) be a set and \( \mathcal{Q} \) as well as \( \mathcal{R} \) equivalence relations on \( X \).

- We write \( \mathcal{R} \subseteq \mathcal{Q} \) and say that \( \mathcal{R} \) refines \( \mathcal{Q} \) if considered as sets of ordered pairs we have \( \mathcal{R} \subseteq \mathcal{Q} \).
8.1 Trees, groups and equivalence relations

- We define the **product equivalence relation** \( Q^n \) on \( X^n \) by setting \( x_0 x_1 \ldots x_{n-1} \sim x'_0 x'_1 \ldots x'_{n-1} \) if and only if \( x_i \sim x'_i \) for all \( i = 0, 1, \ldots, n-1 \).

In [FRW11] groups associated to trees are used to approximate conjugacies. These groups are direct sums of \( \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \) and are called **groups of involutions**.

If \( G = \sum_{i \in I} \{ \mathbb{Z}_2 \}_i \) and \( B = \{ r_i : i \in I \} \) is a distinguished basis, then we call the elements \( r_i \in B \) **generators** and we have a well-defined notion of **parity** for elements in \( G \): an element \( g \in G \) is called **even** if it can be written as the sum of an even number of elements in \( B \). Otherwise, it is called **odd**. Parity is preserved under homomorphisms sending the distinguished basis of one group to the distinguished basis of the other. This also yields that for an inverse limit system of groups of involutions \( \{ G_s : s \in \mathbb{N} \} \) (where each group has a distinguished set of generators) with homomorphisms \( \rho_{t,s} : G_t \to G_s \) for \( s < t \), the elements of the inverse limit \( \lim_{\rightarrow} G_s \) have a well-defined parity.

For a tree \( T \subset \mathbb{N}^\mathbb{N} \) we assign to each level \( s \) a group of involutions \( G_s(T) \) by taking a sum of copies of \( \mathbb{Z}_2 \) indexed by the nodes of \( T \) at level \( s \). Moreover, we view these nodes of \( T \) at level \( s \) as the distinguished generators of

\[
G_s(T) = \sum_{\tau \in T, |\tau| = s} (\mathbb{Z}_2)_\tau.
\]

For levels \( s < t \) of \( T \) we have a canonical homomorphism \( \rho_{t,s} : G_t(T) \to G_s(T) \) that sends a generator \( \tau \) of \( G_t(T) \) to the unique generator \( \sigma \) of \( G_s(T) \) that is an initial segment of \( \tau \). We denote the inverse limit of \( \langle G_s(T), \rho_{t,s} : s < t \rangle \) by \( G_\infty(T) \) and we let \( \rho_s : G_\infty(T) \to G_s(T) \) be the projection map.

Since there is a one-to-one correspondence between the infinite branches of \( T \) and infinite sequences \( (g_s)_{s \in \mathbb{N}} \) of generators \( g_s \in G_s(T) \) with \( \rho_{t,s}(g_t) = g_s \) for \( t > s \), we obtain the following characterization.

**Fact 8.3.** Let \( T \subset \mathbb{N}^\mathbb{N} \) be a tree. Then

1. \( G_\infty(T) \) has a nonidentity element of odd parity if and only if \( T \) is ill-founded (i.e. has an infinite branch).

2. \( G_\infty(T) \) has a nonidentity element of even parity if and only if \( T \) has at least two infinite branches.

In order to make the elements of \( G_\infty(T) \) to correspond to conjugacies, one builds symmetries into our construction using the following finite approximations to \( G_\infty(T) \). We let \( G_n(T) \) be the trivial group and for \( s > 0 \) we let

\[
G_s^n(T) = \sum (\mathbb{Z}_2)_\tau, \text{ where the sum is taken over } \tau \in T \cap \{ \sigma_m : m \leq n \}, |\tau| = s.
\]

When \( T \) is clear from the context, we will frequently write \( G^n_s \).

During the course of construction we will consider group actions of \( G^n_s \) on our quotient spaces \( \mathcal{W}_n/\mathcal{Q}_s^n \). Here, we will need to control systems of such group actions on the refining equivalence relations. For that purpose, the following general definitions will prove useful.

**Definition 8.4.** Suppose

- \( Q \) and \( R \) are equivalence relations on a set \( X \) with \( R \) refining \( Q \),

\[ \]
• $G$ and $H$ are groups with $G$ acting on $X/Q$ and $H$ acting on $X/R$,

• $\rho : H \rightarrow G$ is a homomorphism.

Then we say that the $H$ action is subjordinate to the $G$ action if for all $x \in X$, whenever $[x]_R \subset [x]_Q$ we have $h[x]_R \subset \rho(h)[x]_Q$.

**Definition 8.5.** If $G$ acts on $X$, then the canonical diagonal action of $G$ on $X^n$ is defined by

$$g(x_0x_1 \ldots x_{n-1}) = gx_0gx_1 \ldots gx_{n-1}.$$ 

If $G$ is a group of involutions with a distinguished collection of free generators, then we define the skew diagonal action on $X^n$ by setting

$$g(x_0x_1 \ldots x_{n-1}) = gx_{n-1}gx_{n-2} \ldots gx_0$$

for any canonical generator $g$.

**Remark.** We stress that the skew diagonal actions by elements of $G$ of odd parity reverse the orders of the $x_i$'s while group elements of even parity preserve the order.

**Remark.** Recalling the notion of a product equivalence relation from Definition 8.2 we can identify $X^n/Q^n$ with $(X/Q)^n$ in an obvious way. Then we can also extend an action of $G$ on $X/Q$ to the diagonal or skew diagonal actions on $(X/Q)^n$ in a straightforward way.

### 8.2 Specifications and timing assumptions

In [FRWII] the sequences $(\mathcal{W}_n)_{n \in \mathbb{N}}$, equivalence relations $Q^n_s$, and group actions of $G^n_s$ satisfying several specifications are constructed inductively. For each $n \in \mathbb{N}$ with $\sigma_n \in T$ one constructs a set of words $\mathcal{W}_n = \mathcal{W}_n(T)$ in the basic alphabet $\{0,1\}$ and the construction depends only on $T \cap \{\sigma_m \mid m \leq n\}$. To start we set $\mathcal{W}_0 = \{0,1\}$.

1. **(E1)** All words in $\mathcal{W}_n$ have the same length $h_n$ and $|\mathcal{W}_n| := s_n$ is a power of 2.

2. **(E2)** If $\sigma_m$ and $\sigma_n$ are consecutive elements of $T$, then every word in $\mathcal{W}_n$ is built by concatenating words in $\mathcal{W}_m$. Every word in $\mathcal{W}_m$ occurs in each word of $\mathcal{W}_n$ exactly $p^2_n$ many times. Here, $p_n$ is a large prime number chosen when $\sigma_n$ is considered.

3. **(E3)** If $\sigma_m$ and $\sigma_n$ are consecutive elements of $T$, $w \in \mathcal{W}_n$ and $w = pw_1 \ldots w_l s$, where each $w_i \in \mathcal{W}_m$ and $p$ and $s$ are strings in $\{0,1\}$ of length less than the length of the $m$-words, then both $p$ and $s$ are the empty words.

   If $w, w' \in \mathcal{W}_n$, then the first half of $w'$ is not equal to $w$, i.e. if $w = w_1 \ldots w_{h_n/h_m}$ and $w' = w'_1 \ldots w'_{h_n/h_m}$, where $w_i, w'_i \in \mathcal{W}_m$, and $k = \left\lfloor \frac{h_n}{2h_m} \right\rfloor + 1$, we have $w_k \ldots w_{h_n/h_m} \neq w'_k \ldots w'_{h_n/h_m - k - 1}$.

**Remark.** In particular, these specifications say that $\{\mathcal{W}_n \mid \sigma_n \in T\}$ is a uniquely readable and strongly uniform construction sequence.

In the next step, we specify equivalence relations $Q^n_s$ on $\mathcal{W}_n$ which are defined for $s \leq s(n)$. For this purpose, we take a decreasing summable sequence $(\epsilon_n)_{n \in \mathbb{N}}$ and a strictly increasing sequence $\epsilon(n)$ of positive integers. To start, we let $Q^n_0$ be the equivalence relation on $\mathcal{W}_0 = \{0,1\}$ which has one equivalence class, i.e. both elements of $\{0,1\}$ are equivalent.
(Q4) Suppose that \( n = M(s) \). Then any two words in the same \( Q^n_s \) class agree on an initial segment of length at least \( (1 - \epsilon_n) h_n \).

(Q5) For \( n \geq M(s) + 1 \) we can consider words in \( \mathcal{W}_n \) as concatenation of words from \( \mathcal{W}_n^{M(s)} \) and define \( Q^n_s \) as the product equivalence relation of \( Q^{M(s)}_s \).

(Q6) \( Q^n_{s+1} \) refines \( Q^n_s \) and each \( Q^n_s \) class contains \( 2^{r(n)} \) many \( Q^n_{s+1} \) classes.

Remark. Each equivalence relation \( Q^n_s \) will induce an equivalence relation on \( \text{rev}(\mathcal{W}_n) \), which we will also call \( Q^n_s \), as follows: \( \text{rev}(w), \text{rev}(w') \in \text{rev}(\mathcal{W}_n) \) are equivalent with respect to \( Q^n_s \) if and only if \( w, w' \in \mathcal{W}_n \) are equivalent with respect to \( Q^n_s \).

Remark. By (Q5) we can view \( \mathcal{W}_n / Q^n_s \) as sequence of elements \( \mathcal{W}_n^{M(s)} / Q^{M(s)}_s \) and similarly for \( \text{rev}(\mathcal{W}_n) / Q^n_s \). It also follows that \( Q^n_0 \) is the equivalence relation on \( \mathcal{W}_n \) which has one equivalence class.

Remark. We denote the number of \( Q^n_s \) equivalence classes by \( Q^n_s \) and the cardinality of each class by \( C^n_s \). In case that the exponent is not relevant we will refer to the \( Q^n_s \) as \( Q_s \). For \( u \in \mathcal{W}_n \) we write \([u], s \) for its \( Q^n_s \) class.

We now list specifications on the group actions.

(A7) \( G^n_s \) acts freely on \( \mathcal{W}_n / Q^n_s \cup \text{rev}(\mathcal{W}_n / Q^n_s) \) and the \( G^n_s \) action is subordinate to the \( G^n_{s-1} \) action via the natural homomorphism \( \rho_{s,s-1} : G^n_s \to G^n_{s-1} \).

(A8) The canonical generator of \( G^{M(s)}_s \) sends elements of \( \mathcal{W}_n^{M(s)} / Q^{M(s)}_s \) to elements of \( \text{rev}(\mathcal{W}_n^{M(s)}) / Q^{M(s)}_s \) and vice versa.

(A9) Suppose \( M(s) < n \), \( \sigma_m \) and \( \sigma_n \) are consecutive elements of \( T \) and we view \( G^n_n = G^n_m \oplus H \).

Then the action of \( G^n_n \) on \( \mathcal{W}_m / Q^n_m \cup \text{rev}(\mathcal{W}_m / Q^n_m) \) is extended to an action on action on \( \mathcal{W}_n / Q^n_s \cup \text{rev}(\mathcal{W}_n / Q^n_s) \) by the skew diagonal action. If \( H \) is nontrivial, then its canonical generator maps \( \mathcal{W}_n / Q^n_s \) to \( \text{rev}(\mathcal{W}_n / Q^n_s) \).

All these specifications build certain symmetries into the words that allow nodes of the tree to give increasingly precise information about invertible graph joinings. The intent of the following specifications is to give a mechanism for showing that any joining not arising from branches through the tree are independent joinings over a non-trivial factor, i.e. they cannot give an isomorphism.

(J10) Suppose \( \sigma_m \) and \( \sigma_n \) are consecutive elements of \( T \). Let \( u, v \) be elements of \( \mathcal{W}_n \cup \text{rev}(\mathcal{W}_n) \) and let \( 1 \leq t < (1 - \epsilon_n) h_n \). Let \( j_0 \) be a number between \( \epsilon_n h_m \) and \( \frac{h_m}{k_m} - t \). Then for each pair \( u', v' \in \mathcal{W}_m \cup \text{rev}(\mathcal{W}_m) \) such that \( u' \) has the same parity as \( u \) and \( v' \) has the same parity as \( v \), let \( r(u', v') \) be the number of \( j < j_0 \) such that \( (u', v') \) occurs in \( (sh_{hm}(u), v) \) in the \( (j h_m) \)-th position in their overlap. Then

\[
\left| \frac{r(u', v')}{j_0} - \frac{1}{s_0} \right| < \epsilon_n.
\]

(J11) Suppose \( \sigma_m \) and \( \sigma_n \) are consecutive elements of \( T \). Let \( u \in \mathcal{W}_n \) and \( v \in \mathcal{W}_n \cup \text{rev}(\mathcal{W}_n) \). We let \( s = s(u, v) \) be the maximal \( i \) such that there is a \( g \in G^n_i \) such that \( g[u] = [v]_i \). Let \( g = g(u, v) \) be the unique \( g \) with this property and \( (u', v') \in \mathcal{W}_m \times (\mathcal{W}_m \cup \text{rev}(\mathcal{W}_m)) \) such that \( g[u'] = [v']_s \). Let \( r(u', v') \) be the number of occurrences of \( (u', v') \) in \( (u, v) \). Then:

\[
\left| \frac{r(u', v')}{h_n / h_m} - \frac{1}{Q^n_s} \left( \frac{1}{C^n_s} \right)^2 \right| < \epsilon_n.
\]
To satisfy so-called timing assumptions in [FW3pp] Section 7.2.1 for the corresponding circular system one also needs this strengthening of a special case.

(J11.1) Suppose \( \sigma_m \) and \( \sigma_n \) are consecutive elements of \( T \). Let \( u \in W_n \), \( v \in W_n \cup rev(W_n) \), and \( [u]_1 \notin G_1^n[v]_1 \). Let \( j_0 \) be a number between \( \epsilon_n \frac{h_m}{h_n} \) and \( \frac{h_n}{h_n} \). Suppose that \( I \) is either an initial or a tail segment of the interval \( \{0, h_n - 1 \} \cap \mathbb{Z} \) having length \( j_0 \). Then for every pair \( (u', v') \in W_m \times (W_m \cup rev(W_m)) \) such that \( u' \) has the same parity as \( u \) and \( v' \) has the same parity as \( v \), let \( r(u', v') \) be the number of occurrences of \( (u', v') \) in \( (u \mid I, v \mid I) \). Then:

\[
\left| \frac{r(u', v')}{j_0} - \frac{1}{s_n^2} \right| < \epsilon_n.
\]

Under these specifications and the timing assumptions the following existence result for synchronous and anti-synchronous isomorphisms is shown in [FW3pp] Theorem 92.

**Fact 8.6.** Suppose \( K^c \) is a circular system satisfying the timing assumptions and view it as an element \( T \) of MPT. Then:

1. If there is an isomorphism \( \phi : K^c \to K^c \) such that \( \phi \notin \{ T^n \mid n \in \mathbb{Z} \} \), then there is an isomorphism \( \psi : K^c \to K^c \) such that \( \phi \notin \{ T^n \mid n \in \mathbb{Z} \} \) and \( \psi^\pi \) is the identity map.
2. If there is an isomorphism \( \phi : K^c \to (K^c)^{-1} \), then there is an isomorphism \( \psi : K^c \to (K^c)^{-1} \) such that \( \psi^\pi = \pi \).

**8.3 The continuous function \( F : Trees \to MPT \)**

The specifications allow to construct the following continuous map \( F : Trees \to MPT \), which is the main result of [FW11].

**Proposition 8.7.** There is a continuous one-to-one map \( F : Trees \to MPT \) such that for \( T \in Trees \), if \( T = F^s(T) \):

1. \( T \) has an infinite branch if and only if \( T \cong T^{-1} \)
2. \( T \) has two distinct infinite branches if and only if

\[
C(T) \neq \{ T^n \mid n \in \mathbb{Z} \}.
\]

More specifically, the following facts for the map \( F \) hold true.

**Fact 8.8.** The transformations in the range of \( F \) are strongly uniform odometer based transformations and for \( S \) in the range of \( F \) we have \( S \cong S^{-1} \) if and only if there is an anti-synchronous isomorphism between \( S \) and \( S^{-1} \).

**Fact 8.9.** If \( S \) is in the range of \( F \) and \( C(S) \neq \{ S^n \mid n \in \mathbb{Z} \} \), then there is a synchronous isomorphism \( \phi \in C(S) \setminus \{ S^n \mid n \in \mathbb{Z} \} \) such that for some \( n \in \mathbb{N} \) there is a non-identity element \( g \in G_1^n \) satisfying for all generic \( s \in K \) and all large enough \( m \in \mathbb{N} \) that if \( u \) and \( v \) are the principal \( m \)-subwords of \( s \) and \( \phi(s) \), respectively, then \( [v]_1 = [g[u]]_1 \).

**Fact 8.10.** For every \( N \in \mathbb{N} \) there is \( M \in \mathbb{N} \) such that for \( T, T' \in Trees \) with \( T \cap \{ \sigma_n \mid n \leq M \} = T' \cap \{ \sigma_n \mid n \leq M \} \) the first \( N \) steps of the construction sequence for \( F(T) \) are equal to the first \( N \) steps of the construction sequence for \( F(T') \), i.e. \( (\hat{W}_k(T))_{k<N} = (\hat{W}_k(T'))_{k<N} \).
8.4 Numerical requirements

To construct the map \( F : \mathcal{T}rees \to \text{MPT} \) one builds a construction sequence \((W_n(T))_{n \in \mathbb{N}}\) satisfying our specifications for each \( T \in \mathcal{T}rees \). Moreover, this has to be done in such a way that \( W_n(T) \) is entirely determined by \( T \cap \{ \sigma_m : m \leq n \} \). Therefore, in \cite{FRW11} the construction is organized as follows: For each \( n \) and for each subtree \( S \subseteq \{ \sigma_m : m \leq n \} \) and each \( \sigma_m \in S \) we build \( W_m(S) \). By induction we want to pass from stage \( n-1 \) to stage \( n \). So, we assume that we have constructed \((W_m(S))_{\sigma_m \in S}\) for each subtree \( S \subseteq \{ \sigma_m : m \leq n-1 \} \). In the inductive step, we now have to construct \((W_m(S))_{\sigma_m \in S}\) for each subtree \( S \subseteq \{ \sigma_m : m \leq n \} \).

Obviously, for those trees \( S \) with \( \sigma_n \notin S \) there is nothing to do. So, one has to work on the finitely many trees \( S \) with \( \sigma_n \in S \). We list those in arbitrary manner as \( \{ S_1, \ldots, S_E \} \). Assume that \( T \) is the \( \varepsilon \)-th tree on this list and that we have constructed the collections \( W_n(S_i) \) of \( n \)-words for all \( i \in \{ 1, \ldots, e-1 \} \). Let \( \Psi \) be the collection of prime numbers occurring in the prime factorization of any of the lengths of the words that we have constructed so far. Then one wants to construct \( W_n(T), Q^c_n(T) \) and \( G^m_n(T) \). For that purpose, one uses the induction assumption that for \( m \) being the largest number less than \( n \) such that \( \sigma_m \in T \) we have \( W_m = W_m(T), Q^m_s = Q^m_s(T) \), and \( G^m_n = G^m_n(T) \) satisfying our specifications.

During the course of the construction several parameters with numerical conditions about growth and decay rates appear. In \cite{FW3pp} Section 10 all their recursive requirements and interdependencies are listed and their consistency is resolved. In particular, at stage \( n \) the parameter \( l_n \) is chosen last. Hence, we can choose it sufficiently large to guarantee the convergence to a real-analytic diffeomorphism in our Theorem.

8.5 \( F^* = R \circ F \circ F \) is a continuous reduction

Finally, we are ready to prove Theorem. Therefore, we recall that Theorem gives to every strongly uniform circular construction sequence \( \{W_n\}_{n \in \mathbb{N}} \) with \( |W_n| = s_n \to \infty \) and circular coefficients \( (k_n, l_n)_{n \in \mathbb{N}} \), where \( (l_n)_{n \in \mathbb{N}} \) grows sufficiently fast, a diffeomorphism \( T \in \text{Diff}^\omega \mathcal{P}(\mathbb{T}^2, \mu) \) measure theoretically isomorphic to the circular system \( \mathbb{K}\mathbb{C} \). Hereby, we obtain a map from circular systems with fast growing coefficients to \( \text{Diff}^\omega \mathcal{P}(\mathbb{T}^2, \mu) \). We denote this map with our choice of the sequence \( (l_n)_{n \in \mathbb{N}} \) from the previous Subsection by \( R \) and point out that it preserves isomorphism.

Hereby, we define the map \( F^* = R \circ F \circ F : \mathcal{T}rees \to \text{Diff}^\omega \mathcal{P}(\mathbb{T}^2, \mu) \) and show that it is a continuous reduction.

**Lemma 8.11.** \( F^* : \mathcal{T}rees \to \text{Diff}^\omega \mathcal{P}(\mathbb{T}^2, \mu) \) is a continuous function.

**Proof.** Let \( T = F^*(T) \) for \( T \in \mathcal{T}rees \) and \( U \) be an open neighborhood of \( T \) in \( \text{Diff}^\omega \mathcal{P}(\mathbb{T}^2, \mu) \). Let \( T^c = F \circ F(T) \) be the circular system such that \( R(T^c) = T \). By Proposition 7.34 there is \( M \in \mathbb{N} \) sufficiently large such that for all \( S \in \text{Diff}^\omega \mathcal{P}(\mathbb{T}^2, \mu) \) in the range of \( R \) we have the following property: If \((W_n(T^c))_{n \leq M} = (W_n(S^c))_{n \leq M} \), then \( S \in U \). Here, \( S^c \) denotes the circular system such that \( S = R(S^c) \). Moreover, \((W_n(T^c))_{n \in \mathbb{N}} \) and \((W_n(S^c))_{n \in \mathbb{N}} \) denote the construction sequences of \( T^c \) and \( S^c \), respectively. We recall from Subsection 6.4 that \((W_n(T^c))_{n \leq M} \) is determined by the first \( M+1 \) members in the construction sequence of the odometer based system \( F(T) \), i.e. it is determined by \((W_n(T))_{n \leq M} \). By Fact 8.10 there is a basic open set \( V \subseteq \mathcal{T}rees \) containing \( T \) such that for all \( S \in \mathcal{T}rees \) the first \( M+1 \) members of the construction sequences of \( F(T) \) and \( F(S) \) are the same, i.e. \((W_n(T))_{n \leq M} = (W_n(S))_{n \leq M} \). Then it follows that \( F^*(S) \in U \) for all \( S \in V \).

To conclude the proof of Theorem we show that \( F^* \) is a reduction.
Proof of Theorem 8. To see that $F'$ is a reduction, it suffices to check that $F \circ F$ is a reduction because $R$ preserves isomorphism. This follows from [FW3pp, Section 8.1] and we only present the proof of the first part of the Theorem. Here, we let $T = F'F$, $K = F(T)$, and $K' = F \circ F(T)$ for $T \in Trees$.

Suppose $T \in Trees$ has an infinite branch. By Proposition 8.7 and Fact 8.8 there is an anti-synchronous isomorphism $\phi : K \to K^{-1}$. Then we can apply Fact 3.24 on the functor $F$ and obtain that there is an anti-synchronous isomorphism $\phi' : K' \to (K')^{-1}$.

To show the converse direction we suppose that $T \equiv T^{-1}$, which yields $K' \cong (K')^{-1}$. Since the construction sequence $(W_n)_{n \in \mathbb{N}}$ for $K'$ satisfies the timing assumptions, Fact 8.6 shows that there is an anti-synchronous isomorphism $\phi' : K \to (K')^{-1}$. Hence, we can apply Fact 3.24 again and obtain that there is an anti-synchronous isomorphism $\phi : K \to K^{-1}$. By Proposition 8.7 $T$ has an infinite branch.

9 Further applications

9.1 An uncountable family of pairwise non-Kakutani equivalent real-analytic diffeomorphisms

We recall the definition of the $\bar{f}$ distance between strings of symbols introduced by Feldman [Fe76] (a zero-entropy version of this property was introduced independently by Katok [Ka77]).

Definition 9.1. A match between two strings of symbols $a_1a_2\ldots a_n$ and $b_1b_2\ldots b_m$ from a given alphabet $\Sigma$, is a collection $I$ of pairs of indices $(i_s, j_s)$, $s = 1, \ldots, r$ such that $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, $1 \leq j_1 < j_2 < \cdots < j_r \leq m$ and $a_{i_s} = b_{j_s}$ for $s = 1, 2, \ldots, r$. Then

$$\bar{f}(a_1a_2\ldots a_n, b_1b_2\ldots b_m) = 1 - \frac{2\sup\{|I| : I \text{ is a match between } a_1a_2\ldots a_n \text{ and } b_1b_2\ldots b_m\}}{n + m}.$$  

We will refer to $\bar{f}(a_1a_2\ldots a_n, b_1b_2\ldots b_m)$ as the “$\bar{f}$-distance” between $a_1a_2\ldots a_n$ and $b_1b_2\ldots b_m$, even though $\bar{f}$ does not satisfy the triangle inequality unless the strings are all of the same length.

A match $I$ is called a best possible match if it realizes the supremum in the definition of $\bar{f}$.

In the case of zero entropy, the $\bar{f}$ distance allows a simple definition of the loosely Bernoulli property.

Definition 9.2 (Loosely Bernoulli in the case of zero entropy). A measure-preserving process $(T, \mathcal{P}, \nu)$ is zero-entropy loosely Bernoulli if for every $\varepsilon > 0$, there exists a positive integer $K = K(\varepsilon)$ and a collection $\mathcal{G}$ of “good” atoms of $\vee_{1}^{K}T^{-1}\mathcal{P}$ with total measure greater than $1 - \varepsilon$ such that for each pair $A, B$ of atoms in $\mathcal{G}$, $\bar{f}_K(x, y) < \varepsilon$ for $x \in A, y \in B$.

Remark. If this condition is satisfied, then routine estimates show that the $(T, \mathcal{P}, \nu)$ process indeed has zero entropy. Sometimes, zero-entropy loosely Bernoulli transformations are called standard or loosely Kronecker.

The following well-known fact from [Fe76] and [ORW82] as stated in [GKpp, Property 2.4] will prove useful in our arguments.

Fact 9.3. Suppose $a$ and $b$ are strings of symbols of length $n$ and $m$, respectively, from an alphabet $\Sigma$. If $\tilde{a}$ and $\tilde{b}$ are strings of symbols obtained by deleting at most $\lfloor \gamma(n + m) \rfloor$ terms from $a$ and $b$ altogether, where $0 < \gamma < 1$, then

$$\bar{f}(a, b) \geq \bar{f}(\tilde{a}, \tilde{b}) - 2\gamma.$$  

(9.4)
Moreover, if there exists a best possible match between \(a\) and \(b\) such that no term that is deleted from \(a\) and \(b\) to form \(\tilde{a}\) and \(\tilde{b}\) is matched with a non-deleted term, then
\[
\overline{f}(a, b) \geq \overline{f}(\tilde{a}, \tilde{b}) - \gamma.
\] (9.5)

Likewise, if \(\tilde{a}\) and \(\tilde{b}\) are obtained by adding at most \(\lfloor \gamma(n+m) \rfloor\) symbols to \(a\) and \(b\), then (9.5) holds.

Using the notation from Section 3.4 we define the \((n+1)\)-blocks \(w_j^{(n+1)}\), \(j = 1, \ldots, s_{n+1}\), in the odometer-based system as follows:
\[
w_j^{(n+1)} = \left( \begin{array}{c} w_1^{(n)}(l_n-1)^{2j-1} \\ w_2^{(n)}(l_n-1)^{2j-1} \\ \vdots \\ w_{s_n}^{(n)}(l_n-1)^{2j-1} \end{array} \right) (l_n-1)^{2(s_n+j)+1}.
\]

Applying the circular operator \(C_n\) yields the following \((n+1)\)-blocks in the corresponding circular system:
\[
\prod_{i=0}^{q_n-1} \prod_{t=1}^{s_n} \left( y_t^{q_n-j_i} c_n \left( w_t^{(n)} \right) \right) (l_n-1)^{2j-1} (l_n-1)^{2(s_n+j)+1} = n (l_n-1)^{2(s_n+j)+1}.
\]

Since the newly introduced spacers occupy a proportion of at most \(\frac{2}{q_n}\) in substantial substrings of these blocks with at least \(\frac{2n+1}{q_n}\) consecutive symbols, we will ignore them in the following consideration. This might decrease the \(\overline{f}\) distance between substantial substrings by at most \(\frac{4}{l_n}\) according to the aforementioned Fact 9.3. After ignoring the new spacers, the \((n+1)\)-blocks in the circular system look as follows for \(j = 1, \ldots, s_{n+1}\):
\[
\left( \begin{array}{c} c_n \left( w_1^{(n)} \right) \\ c_n \left( w_2^{(n)} \right) \\ \vdots \\ c_n \left( w_{s_n}^{(n)} \right) \end{array} \right) (l_n-1) \to 0 \quad \text{as} \quad n \to \infty.
\]

Hence, we can apply the analysis in [Fe76] Section 5] or [ORW82] Chapter 10 to show that the real-analytic realization \(T \in \text{Diff}_{\rho}^{\omega}(\mathbb{T}^2, \mu)\) of this strongly uniform circular system is ergodic and not loosely Bernoulli.

To show that there even exists an uncountable family of diffeomorphisms in \(\text{Diff}_{\rho}^{\omega}(\mathbb{T}^2, \mu)\) which are pairwise not Kakutani-equivalent we follow the approach in [ORW82] Chapter 12. Let \(a = (a_n)_{n \in \mathbb{N}}\) be an infinite sequence of zeros and ones. If \(a_{n+1} = 0\), we construct the \((n+1)\)-blocks as described above. In case of \(a_{n+1} = 1\), we define the \((n+1)\)-blocks in the construction sequence of the odometer-based system as
\[
w_j^{(n+1)} = \left( \begin{array}{c} w_1^{(n)}(l_n-1)^{2j-1} \\ w_2^{(n)}(l_n-1)^{2j-1} \\ \vdots \\ w_{s_n}^{(n)}(l_n-1)^{2j-1} \end{array} \right) (l_n-1)^{2(s_n+j)+1}.
\]
9.2 Real-analytic non-Bernoulli diffeomorphisms with property $K$

i. e. we run through the types of $n$-blocks in a cycle in decreasing order this time. As above, we consider the corresponding strongly uniform circular system and realize it as a diffeomorphism $T_n \in \text{Diff}^\omega_{\rho}(T^2, \mu)$. As shown in [ORWS82, Chapter 12, Proposition 3.4], if $a_n \neq b_n$ for infinitely many $n \in \mathbb{N}$, then $T_{a_n}$ and $T_{b_n}$ are not Kakutani-equivalent. Certainly, this gives an uncountable family and proves Theorem D.

9.2 Real-analytic non-Bernoulli diffeomorphisms with property $K$

We follow along the lines of [Ka80], where the first $C^\infty$ example of a non-Bernoulli diffeomorphism with property $K$ was constructed. For that purpose, let $A$ be a hyperbolic automorphism of $T^2$ and let $S \in \text{Diff}^\omega_{\rho}(T^2, \mu)$ be an ergodic and not loosely Bernoulli diffeomorphism as constructed in the previous Subsection. We consider its time one suspension $\{S_t\}_{t \in \mathbb{R}}$ on $N = T^2 \times [0,1]/\equiv$, where $(x,1) \equiv (S(x),0)$. Then we define the real-analytic diffeomorphism

$$T : T^2 \times N \to T^2 \times N, \quad T(x,y) = (Ax, S_{\varphi(x)}(y))$$

with a real-analytic map $\varphi : T^2 \to \mathbb{R}$ which is not cohomologous to a constant (i. e. $\varphi$ cannot be written as $\varphi = \psi \circ A - \psi + c$ with measurable $\psi : T^2 \to \mathbb{R}$ and $c \in \mathbb{R}$). Then $T$ has property $K$ by [Ka80, Theorem 1]. To see that $T$ is not Bernoulli we consider the special flow $\{A^e_t\}_{t \in \mathbb{R}}$ on $M^e = \{(x,s) \in T^2 \times \mathbb{R} \mid 0 \leq s \leq \varphi(x)\}$ and the flow $\tilde{T}_t = A^e_t \times S_t$ on $M^e \times N$. Since $S$ is not loosely Bernoulli, $\tilde{T}_t$ is not loosely Bernoulli. Moreover, we see that $T$ is the Poincare map of $\tilde{T}_t$ on $M \times \{0\} \times N$. Hence, $T$ as a section of a non-loosely Bernoulli flow cannot be loosely Bernoulli.

Acknowledgement

The authors of this paper express their gratitude towards Anatole Katok for asking the question this paper answers. He also helped the authors with numerous mathematical discussions and a steady stream of encouragement.

Also this paper would not be possible without the discussions the authors had with Matthew Foreman who patiently explained several intricacies of his delicate work and took an interest towards the current result. He also showed the first authors how to produce diagrams using spreadsheet software. The second author also thanks Marlies Gerber for several discussions in the realm of anti-classification results.
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