A Kawamata–Viehweg type formulation of the logarithmic Akizuki–Nakano vanishing theorem

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Abstract
In this article, we present a Kawamata–Viehweg type formulation of the (logarithmic) Akizuki–Nakano Vanishing Theorem. We give two proofs: one by reduction to an older theorem of Steenbrink via Kawamata’s covering lemma, and another by mod $p$ reduction using results of Deligne–Illusie and Hara. We also include two applications.

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1 Introduction

The main theorem of this article is a vanishing theorem which interpolates between several previously known vanishing results due to Kodaira–Akizuki–Nakano, Esnault–Viehweg, Kawamata–Viehweg, Steenbrink and others. As applications, we show that deformations of log Fanos are unobstructed, and we also give a quick proof of a special case of the theorem of Wei [33, 34] that the zero set of a log 1-form is nonempty if the log canonical bundle is ample.

In order to motivate the statement of our vanishing theorem, we begin by recalling some classical vanishing theorems. Below, we work over the field $\mathbb{C}$ of complex numbers. However, by the Lefschetz principle or flat base change, all the results in this paper are valid over any field of characteristic zero.

Let $X$ be a smooth projective variety, $A$ an integral ample divisor on $X$, and $K_X$ the canonical divisor. In this setting, the classical Kodaira Vanishing Theorem [17] states that

$$H^i(X, \mathcal{O}_X(K_X + A)) = 0 \text{ for } i > 0.$$  

According to the Iitaka philosophy (cf. [23]), we obtain its logarithmic version [26] by adding a simple normal crossings divisor $D = \sum D_i$, called the boundary, on $X$:

$$H^i(X, \mathcal{O}_X(K_X + D + A)) = 0 \text{ for } i > 0. \quad (1.1)$$

The celebrated Kawamata–Viehweg Vanishing Theorem [14, 31] further generalizes (1.1) to the setting where $A$ is a $\mathbb{Q}$-divisor, by allowing the boundary divisor to have a fractional part $F = \sum f_j F_j \quad (0 < f_j < 1)$ such that $F + A$ is integral (i.e., $F = \lceil A \rceil - A$) as well as an integral part $B = \sum_k B_k$. It states that

$$H^i(X, \mathcal{O}_X(K_X + B + F + A)) = 0 \text{ for } i > 0,$$

where $B$ and $F$ share no common components, and $\text{Supp}(B \cup F)$ is a simple normal crossings divisor.

On the other hand, one also has the Akizuki–Nakano Vanishing Theorem [1]:

$$H^i(X, \Omega^j_X(A)) = 0 \text{ for } i + j > \dim X, \quad (1.2)$$

where $A$ is an integral ample divisor on $X$. As before, the Iitaka philosophy suggests that one should obtain a logarithmic version of (1.2) by considering a simple normal crossings boundary divisor $D = \sum D_i$, and replacing the usual sheaf of differential forms with the sheaf of logarithmic differential forms. This leads precisely to the Esnault–Viehweg Vanishing Theorem [7, 8]:

$$H^i(X, \Omega^j_X(\log D)(A)) = 0 \text{ for } i + j > \dim X.$$  

Given the previous discussion, it is natural to ask for an analog of the Kawamata–Viehweg vanishing theorem in this setting.

In [30], Steenbrink proved that

$$H^i(X, \Omega^j_X(\log D)(A - D)) = 0 \text{ for } i + j > \dim X.$$
and the first author [4] proved a “fractional” version of this vanishing (cf. Section 4). It turns out that a slight modification of this fractional version yields, as its special cases, all the aforementioned classical vanishing theorems and this is the main result of this paper.

**Theorem 1.3** Let $X$ be a smooth projective variety, $D$ a simple normal crossings divisor, and $A$ an ample $\mathbb{Q}$-divisor with fractional part $F = \lceil A \rceil - A$. Suppose that $F \leq D$ (i.e. the support of $F$ is contained in $D$) and let $G$ be an integral divisor such that $F \leq G \leq D$. Then we have:

$$H^i(X, \Omega^j_X(\log D)(F + A - G)) = H^i(X, \Omega^j_X(\log D)(\lceil A \rceil - G)) = 0 \text{ for } i + j > \dim X.$$ 

The following relative version easily follows from the above absolute version (see, e.g., an argument in the proof of Theorem 1–2–3 [16]).

**Corollary 1.4** Let $f : X \to Y$ be a projective morphism from a nonsingular variety $X$ to a variety $Y$, $D = \sum D_i$ a simple normal crossings divisor on $X$, $A$ an $f$-ample $\mathbb{Q}$-divisor, and $F := \lceil A \rceil - A$ and let $G$ be an integral divisor such that $F \leq G \leq D$. Then we have:

$$R^i f_* (\Omega^j_X(\log D)(F + A - G)) = R^i f_* (\Omega^j_Y(\log D)(\lceil A \rceil - G)) = 0 \text{ for } i + j > \dim X.$$ 

We present two proofs of the main result. The first proof proceeds by reducing to Steenbrink’s Vanishing Theorem via the Kawamata Covering Lemma. Furthermore, since the logarithmic differential forms have no ramification under the Kummer covers, this proof makes it clear that the process of taking the round up “$\lceil A \rceil$” does not stem from the ramification, but rather from the subtraction of some effective divisor “$G$” appearing in our formulation. We note that this fact is not readily visible in the classical proof of the Kawamata–Viehweg Vanishing Theorem [14, 16, 23] when reducing the fractional case to the integral case via the covering technique. This is due to the fact that the Kawamata–Viehweg Vanishing Theorem only deals with top degree forms, while the Akizuki–Nakano Vanishing Theorem deals also with lower degree differential forms. We emphasize that all the essential ideas of reducing the fractional case to the integral case via covering already appear in [8] as well as in [14].

The second proof uses a simplified version of an argument in [4] to establish a fractional form of the Steenbrink Vanishing Theorem. It uses the method of Deligne–Illusie [6] in positive characteristic. It is also worth noting that instead of the Kawamata Covering Lemma, it uses a lemma of Hara [12] to handle the fractional part. Once the fractional version of the Steenbrink Vanishing is proved, our main result follows immediately as an easy corollary via some “round up” tricks in 3.3.4.

We also recently learned of a nice paper by Huang et al. [13], which proves a similar result by $L^2$-methods in the Kähler setting. Other authors have also considered certain cases in the presence of singularities (e.g. [20]). We do not consider such cases in this article.

Finally, we give the following two simple applications of our vanishing result. Firstly, we show that deformations of log $\mathbb{Q}$-Fano manifolds are unobstructed:

**Theorem 1.5** Assume $(X, B + F)$ is a log $\mathbb{Q}$-Fano manifold. That is to say, $X$ is a smooth projective variety with a boundary divisor $B + F$ where $B$ is an effective and reduced integral divisor and $F$ is a fractional divisor $F = \sum f_j F_j$ ($0 < f_j < 1$), $B$ and $F$ sharing no common components, with $\text{Supp}(B \cup F)$ being a simple normal crossings divisor, such that $-(K_X + B + F)$ is ample. Then the deformations of the pair $(X, B + F)$ are unobstructed.

Secondly, we prove the following special case of the theorem of Wei [33, 34] that the zero set of a log 1-form is nonempty if the log canonical divisor is ample. Wei obtains this result, generalizing previous results due to (in historical order) by Zhang [35], Hacon and Kovács.
[11] and Popa and Schnell [27], as an application of the Kodaira–Saito Vanishing Theorem. However, the proof of the special case obtained here is algebraic.

**Theorem 1.6 (Wei)** Let $X$ be a smooth projective variety, $D$ a simple normal crossings divisor, and suppose that there is a boundary divisor $B + C$ supported on $D$, where $B$ is an effective and reduced integral divisor and $C$ is a fractional $\mathbb{Q}$-divisor having coefficients in $[0, 1)$, such that $K_X + B + C$ is ample. Then any 1-form in $H^0(\Omega_{X}^1(\log D))$ must have a nonempty zero-locus.

In the next section, we explain how to obtain the classical vanishing theorems discussed above as special cases our main vanishing theorem, and also discuss the failure of some naive/stronger versions of our main vanishing result. In the third section, we present the first proof of the main result with some remarks and an alternate argument. In the fourth section, we then present the second proof of the main result. Finally, in the last section, we discuss the applications mentioned above.

### 2 Special cases and stronger versions of the main vanishing result.

In the first subsection, we discuss the failure of a possibly stronger version of our main vanishing result. In the second subsection, we discuss the failure of our vanishing result in general when $A$ is nef and big. In the third subsection, we explain how to obtain classical vanishing results as special cases of our vanishing theorem.

#### 2.1 Failure of a stronger version

Consider the statement of the Kawamata–Viehweg Vanishing Theorem:

$$H^i(X,\mathcal{O}_X(K_X+B+F+A)) = H^i(X,\mathcal{O}_X(K_X+B+[A])) = 0 \text{ for } i > 0.$$ 

Recall that $X$ is a smooth projective variety, $A$ is an ample $\mathbb{Q}$-divisor, where $B = \sum B_k$ is an integral reduced divisor and $F = \sum f_j (0 < f_j < 1)$ is a fractional divisor such that $F + A$ is integral (i.e., $F = [A] - A$). In this case, we observe that the only conditions on $B$ and $F$ are:

(i) $\text{Supp}(B \cup F)$ is a simple normal crossings divisor, and

(ii) $B$ and $F$ share no common components.

**Remark 2.1.1** Suppose $B$ and $F$ had a common component. If this common component is locally defined by $\{x = 0\}$, then $K_X + B + F$ has a local generator of the form $\frac{dx}{x^{1+\delta}} \wedge \cdots$ with $\delta > 0$. However, this violates the standard philosophy that, in an appropriate logarithmic formulation, one should have no worse than simple poles (i.e., $dx/x^1 = d(\log x)$).

Note that setting $j = \dim X$ and $B := D - G$, Theorem 1.3 implies the Kawamata–Viehweg Vanishing Theorem recalled above. In this case, condition (ii) above follows from the condition $F \leq G \leq D$.

On the other hand, still staying in line with the above philosophy, one could imagine the following stronger formulation of Theorem 1.3. Let $X$ be as before, $D = \sum D_i$ a simple normal crossings divisor on $X$, $A$ an ample $\mathbb{Q}$-divisor, and $F := [A] - A$ such that $\text{Supp}(D \cup F)$ is also a simple normal crossings divisor. Let $G$ be an integral divisor such
that $D \cap F \leq G \leq D$, where $D \cap F := \sum_{D_i \subset \text{Supp}(F)} D_i$. Then one is led to consider the following stronger vanishing statement where we do not require $F$ to be contained in $G$ or $D$:

$$H^i(X, \Omega^j_X(\log(D))(F + A - G)) = H^i(X, \Omega^j_X(\log(D))([A] - G)) = 0 \text{ for } i + j > \dim X.$$ 

Note that, if $j = \dim X$, then this stronger formulation is actually equivalent to Theorem 1.3. Moreover, in this case, they are also both equivalent to the Kawamata–Viehweg Vanishing Theorem.

On the other hand, if $j < \dim X$, then the stronger formulation above differs from Theorem 1.3. In fact, if $D = 0$, then the stronger formulation would imply that

$$H^i(X, \Omega^j_X([A])) = 0 \text{ for } i + j > \dim X.$$ 

In view of the following statement of the Kawamata–Viehweg Vanishing (without any integral part $B$ of the boundary divisor)

$$H^i(X, \Omega^j_X([A])) = H^i(X, \mathcal{O}_X(K_X + [A])) = 0 \text{ for } i > 0,$$

this could be interpreted as a Kawamata–Viehweg type formulation of the Akizuki–Nakano Vanishing Theorem. However, this naive formulation, as well as the afore-mentioned stronger formulation, fails to hold!

In fact, one can also consider the following relative version of the stronger formulation with $D = 0$:

$$R^j f_* \Omega^j_X([A]) = 0 \text{ for } i + j > \dim X,$$

where $f : X \to Y$ is a projective morphism, $A$ is an $f$-ample $\mathbb{Q}$-divisor, and where $F = [A] - A$ is a simple normal crossings divisor on $X$. However, the following example demonstrates that this statement fails.

**Example 2.1.2** Let $Y$ be a non-singular 3-fold, $f : X \to Y$ be the blow up of a point $P \in Y$, and $E := f^{-1}(P)$ be the exceptional divisor. Then $A = -\epsilon E$ is an $f$-ample $\mathbb{Q}$-divisor for some sufficiently small and positive rational number $0 < \epsilon << 1$. According to the stronger formulation, we should have

$$R^2 f_* \Omega^2_X([A]) = R^2 f_* \Omega^2_X = 0.$$

On the other hand, we have an exact sequence of coherent $\mathcal{O}_X$-modules

$$0 \to \mathcal{K} \to \Omega^2_X \xrightarrow{\phi} \Omega^2_E \to 0,$$

where $\phi$ is the restriction map and $\mathcal{K}$ is the kernel of the map $\phi$. The associated long exact sequence gives

$$R^2 f_* \Omega^2_X \to R^2 f_* \Omega^2_E \cong H^2(\mathbb{P}^2, \Omega^2_{\mathbb{P}^2}) \to R^3 f_* \mathcal{K} = 0.$$

Here the last term vanishes as the fibers of $f$ have dimension at most 2. Since by the Serre duality

$$H^2(\mathbb{P}^2, \Omega^2_{\mathbb{P}^2}) \cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \cong \mathbb{C} \neq 0,$$

we conclude that

$$R^2 f_* \Omega^2_X \neq 0.$$
2.2 Failure when $A$ is nef and big

The statement of the Kodaira Vanishing holds even if we replace an ample divisor $A$ with a nef and big divisor $L$:

$$H^i(X, \mathcal{O}_X(K_X + L)) = 0 \text{ for } i > 0,$$

where $X$ is a nonsingular projective variety and $L$ is an (integral) nef and big divisor on $X$.

The proof of this statement via the Kawamata–Viehweg Vanishing for a klt pair $(X, \Delta)$ (“klt” is short for “Kawamata log terminal” singularities) goes as follows. Since $L$ is big, by the so-called Kodaira Lemma, we can write $L$ as a $\mathbb{Q}$-divisor

$$L = M + H,$$

where $M$ is an effective divisor and $H$ is an ample divisor. (We note that we do NOT require $H$ to be effective.) For $n \in \mathbb{N}$, we have another equation of $\mathbb{Q}$-divisors:

$$L = \frac{1}{n}L + (n-1)L = \frac{1}{n}M + H + (n-1)L = \frac{1}{n}M + \frac{1}{n}H + (n-1)L.$$

Here $A := \frac{1}{n}H + (n-1)L$ is an ample $\mathbb{Q}$-divisor, and the pair $(X, \Delta = \frac{1}{n}M)$ is klt for $n$ sufficiently large. As an application of the Kawamata–Viehweg Vanishing Theorem for the klt pair $(X, \Delta)$ (cf. Theorem 1-2-5 in [16] Note that klt singularities are weak log-terminal.) we obtain:

$$H^i(X, \mathcal{O}_X(K_X + L)) = H^i(X, \mathcal{O}_X(K_X + \Delta + A)) = 0 \text{ for } i > 0.$$

Note that in the original setting with the SNC divisor $F = \lceil A \rceil - A$, the klt pair we consider is $(X, \Delta = F)$, and that we obtain

$$H^i(X, \mathcal{O}_X(K_X + \lceil A \rceil)) = H^i(X, \mathcal{O}_X(K_X + F + A))$$

$$= H^i(X, \mathcal{O}_X(K_X + \Delta + A)) = 0 \text{ for } i > 0.$$

However, it is well-known that the Akizuki–Nakano Vanishing fails if we replace, in its formulation, an ample divisor $A$ with a nef and big divisor $L$ (cf. 4.3.4 [22], see also examples 2.2.1, 2.2.2 below.). In particular, there is an example where we have

$$H^i(X, \Omega^j_X(L)) \neq 0 \text{ and } i + j > \dim X,$$

where $X$ is a nonsingular projective variety over $\mathbb{C}$ and $L$ is an integral nef and big divisor on $X$. One might consider this to be a “pathology” if one expects that the Akizuki–Nakano Vanishing for a klt pair $(X, \Delta)$ should hold, and hence that one should have for a nef and big divisor $L = \Delta + A$ as above

$$H^i(X, \Omega^j_X(L)) = H^i(X, \Omega^j_X(\Delta + A))$$

$$= H^i(X, \Omega^j_X(\lceil A \rceil)) = 0 \text{ and } i + j > \dim X.$$

However, this is exactly the statement of the stronger version of our main theorem discussed above, which we saw fails to hold. Therefore, in the above sense, we may say that the failure of the Akizuki–Nakano Vanishing for a nef and big divisor and the failure of the stronger version of its Kawamata–Viehweg type formulation share the same origin.
Example 2.2.1 Let \( f : X \to Y \) be the blow up of a point \( P \in Y \) on a nonsingular 3-fold \( Y \) as in Example 2.1.2. Let \( L = \pi^* H \) be the pull-back of an ample divisor \( H \) on \( Y \). In this case, \( L \) is nef and big. Then
\[
R^2 f_* (\Omega^2_X(L)) = R^2 f_* (\Omega^2_X(\pi^* H)) \cong R^2 f_* (\Omega^2_X) \otimes H \neq 0,
\]
since \( R^2 f_* \Omega^2_X \neq 0 \). In particular, this shows the failure of the (relative) Akizuki–Nakano Vanishing, when we replace an ample divisor \( A \) with a nef and big divisor \( L \).

Example 2.2.2 Let \( f : X \to Y \) be as in Example 2.2.1. Consider an ample divisor \( H \) on \( Y \) and, a sufficiently small and positive rational number \( 0 < \epsilon << 1 \) such that \( A = \pi^* H - \epsilon E \) is ample on \( X \). Then looking at the Leray spectral sequence, one immediately sees that
\[
H^2(X, \Omega^2_X([A])) = H^2(X, \Omega^2_X(L)) \neq 0.
\]
This provides a counter-example to the stronger version of our formulation and the Akizuki–Nakano Vanishing for a nef and big line bundle in the absolute setting.

2.3 Special cases of the main vanishing result

We discuss various special cases of Theorem 1.3.

Case 2.3.1 \( A \) is integral and \( G = 0 \). This case yields the Esnault-Viehweg Vanishing Theorem
\[
H^i(X, \Omega^j_X(\log D)(A)) = 0 \text{ for } i + j > \dim X.
\]
When \( j = \dim X \), it yields the logarithmic version of the Kodaira Vanishing Theorem.

Case 2.3.2 \( j = \dim X \) By setting \( B = D - G \), this case yields the Kawamata–Viehweg Vanishing Theorem:
\[
H^i(X, \Omega^d_X(D + [A] - G)) = H^i(X, \mathcal{O}_X(K_X + B + F + A)) = 0 \text{ for } i > 0.
\]
Here \( B \) and \( F \) share no common components because of the condition \( F \leq G \leq D \).

Case 2.3.3 \( G = D \). This case yields
\[
H^i(X, \Omega^d_X(\log D)(F + A - D)) = H^i(X, \Omega^d_X(\log D)([A] - D)) = 0 \text{ for } i + j > \dim X.
\]
This is the fractional version of the Steenbrink Vanishing Theorem, which appears in [4]. We note that when \( j = \dim X \) and \( A \) is integral, we recover the Kodaira Vanishing Theorem, but not its logarithmic version (unless we use the round up trick 3.3.4).

Case 2.3.4 \( D = G = E \), where \( E \) is the support of a projective birational map \( f : X \to Y \). Consider a projective birational map \( f : X \to Y \) from a nonsingular variety \( X \). Then Corollary 1.4 implies
\[
R^i f_* \Omega^j_X(E)([A] - E) = R^i f_* \Omega^j_X(E)(-E) = 0 \text{ for } i + j > \dim X
\]
where \( E = \sum E_i \) is the exceptional divisor (which is assumed to be a simple normal crossings divisor), \( A = \sum -e_i E_i \) is an \( f \)-ample divisor with \([A] = 0\). When \( j = \dim X \), the statement becomes
\[
R^i f_* \omega_X = 0 \text{ for } i > 0,
\]
which is nothing but the Grauert–Riemenschneider Vanishing Theorem.
3 Proof of Theorem 1.3 by Kawamata Covering lemma

In this section, we provide a proof of Theorem 1.3 using the Kawamata Covering Lemma.

3.1 The case when $A$ integral

In this subsection, we prove Theorem 1.3 in the setting where $A$ is integral. We shall further split this case into subcases.

Subcase 3.1.1 $G = D$. In this case, the statement is nothing but the Steenbrink Vanishing Theorem. Note that, when $G = D = 0$, we obtain the Akizuki–Nakano Vanishing Theorem.

Subcase 3.1.2 $G \leq D' = D - D_1 = D_2 + \cdots + D_l < D = D_1 + D_2 + \cdots + D_l$.

In this case, one proceeds via induction on the number of the components in $D$ and the dimension of $X$. Consider the residue sequence

$$0 \rightarrow \Omega^j_X((A - G)) \rightarrow \Omega^j_X(A - G) \xrightarrow{\psi} \Omega^{j-1}_{D_1}(\log(D'|D_1))((A - G)|_{D_1}) \rightarrow 0,$$

where $\psi$ is the residue map.

The corresponding long exact sequence in cohomology gives

$$\cdots \rightarrow H^i(X, \Omega^j_X((A - G))) \rightarrow H^i(X, \Omega^j_X(A - G)) \rightarrow H^i(D_1, \Omega^{j-1}_{D_1}(\log(D'|D_1))((A - G)|_{D_1})) \rightarrow \cdots.$$

If $i + j > \dim X$, then the first term is 0 by induction on the number of the components in $D$ (since the number of the components in $D'$ is one less than that of $D$). On the other hand, if $i + j > \dim X$, the last term is also 0 by induction on the dimension of $X$ (since $\dim D_1 = \dim X - 1$ and since $i + (j - 1) = i + j - 1 > \dim X - 1 = \dim D_1$). Therefore, we conclude that

$$H^i(X, \Omega^j_X((A - G))) = 0$$

if $i + j > \dim X$.

Remark 3.1.3 Note that using the residue sequence above with $G = 0$, one can derive the Esnault-Viehweg Vanishing from the Akizuki–Nakano Vanishing via induction on the number of the components in $D$ and the dimension of $X$. But, it seems that the Steenbrink Vanishing cannot be derived from the Akizuki–Nakano Vanishing via a simple inductive argument using the residue sequence above.

3.2 The case when $A$ fractional

We reduce the case where $A$ is fractional to the case where $A$ is integral, using the following Kawamata Covering Lemma.

Lemma 3.2.1 (Kawamata Covering Lemma ([14, 16, 23])) Let the situation be the same as described in Theorem 1.3. Then there exists a finite morphism $\pi : Y \rightarrow X$ with the extension of the function fields $\mathbb{C}(Y)/\mathbb{C}(X)$ being Galois (and hence $\Gamma := \text{Gal}(\mathbb{C}(Y)/\mathbb{C}(X))$ acts on $Y$ over $X$) such that:
(i) $Y$ is nonsingular projective.
(ii) $\pi^* A$ is integral.
(iii) $\pi$ is ramified only along $D \cup M$, which forms a simple normal crossings divisor for some auxiliary divisor $M$, with $D$ and $M$ sharing no common components.
(iv) There is a sufficiently divisible and large integer $m \in \mathbb{N}$ such that, for any irreducible component $B$ in $D \cup M$, we have
\[ \pi^* B = mB_Y \]
where $B_Y = \pi^{-1}(B)_{\text{red}}$ and that, if $B \subset F$, we have
\[ (\star) \ a_B + \frac{m - 1}{m} \geq \lfloor a_B \rfloor. \]
Here $a_B$ is the coefficient of $B$ in $A$.
(v) For any closed point $P \in X$ there exists a regular system of parameters $(x_1, \ldots, x_l, x_{l+1}, \ldots, x_n)$ such that
\begin{itemize}
  \item $\{\prod_{i=1}^{l} x_\alpha = 0\} = (D \cup M)_P$, where $(D \cup M)_P$ denotes the collection of the components of $D \cup M$ passing through $P$, and
  \item any closed point $Q \in \pi^{-1}(P)$ has a regular system of parameters of the form $(y_1 = x_1^{1/m}, \ldots, y_l = x_l^{1/m}, x_{l+1}, \ldots, x_n)$ (for the same integer “$m$” mentioned in condition (iv)).
\end{itemize}

We make a remark about the notational conventions we use below, extending the ones used in Lemma 3.2.1 above: By $G_Y$ (resp. $D_Y$ or $(D \cup M)_Y$), where we put the subscript $Y$, we mean the pull-back of the divisor $G$ (resp. $D$ or $D \cup M$) to $Y$ with the reduced structure. We note that $G_Y$ (resp. $D_Y$ or $(D \cup M)_Y$) is a simple normal crossings divisor by construction. By $(D \setminus G)_P$ (resp. $(G \setminus F)_P$, $F_P$, or $M_P$), where we put the subscript $P$, we mean the collection of the irreducible components passing through the point $P$ of the divisor $(D \setminus G)$ (resp. $(G \setminus F)$, $F$, or $M$).

Lemma 3.2.2 With the same situation as in Lemma 3.2.1, we have
\[ \left[ \pi_* \{ \Omega^j \log D_Y (\pi^* A - G_Y) \} \right]^{\Gamma} = \Omega^j \log D ([A] - G). \]

Proof First note that
\[ \Omega^j_Y \log D_Y (\pi^* A - G_Y) \subset \Omega^j_Y \log (D \cup M)_Y) \otimes O_Y \mathbb{C}(Y) \]
\[ = \pi^* \{ \Omega^j_X \log (D \cup M) \} \otimes O_Y \mathbb{C}(Y) \]
and hence that
\[ \pi_* \{ \Omega^j_Y \log D_Y (\pi^* A - G_Y) \} \subset \Omega^j_X \log (D \cup M) \otimes O_X \mathbb{C}(Y). \]
The $\Gamma$-action on the left-hand side is induced from the $\Gamma$-action on the right-hand side, where $\Gamma$ acts trivially on the first factor $\Omega^j_X \log (D \cup M)$ and $\Gamma$ acts on the second factor $\mathbb{C}(Y)$ as the Galois group $\text{Gal}(\mathbb{C}(Y)/\mathbb{C}(X))$. Therefore, we conclude
\[ \left[ \pi_* \{ \Omega^j_Y \log D_Y (\pi^* A - G_Y) \} \right]^{\Gamma} \subset \Omega^j_X \log (D \cup M) \otimes O_X \mathbb{C}(X). \]
Our task now, in order to see the equality claimed in Lemma 3.2.2, is to identify the left-hand side of the above inclusion with another subsheaf, which is the left-hand side of the inclusion below
\[ \Omega^j_X \log D ([A] - G) \subset \Omega^j_X \log (D \cup M) \otimes O_X \mathbb{C}(X). \]
For a closed point \( P \in X \), we choose a regular system of parameters
\[
\{(x_s)_{s \in S}, \{x_t\}_{t \in T}, \{x_v\}_{v \in V}, \{x_w\}_{w \in W}, \{x_z\}_{z \in Z}\}
\]
as in condition (v) of the Kawamata Covering Lemma and an affine open neighborhood \( P \in U \) such that
\[
\begin{align*}
\{x_s = 0\}_{s \in S} &= (D \setminus G)_P | U = (D \setminus G)_U \\
\{x_t = 0\}_{t \in T} &= (G \setminus F)_P | U = (G \setminus F)_U \\
\{x_v = 0\}_{v \in V} &= F_P | U = F | U \\
\{x_w = 0\}_{w \in W} &= M_P | U = M | U \\
\{x_z = 0\}_{z \in Z} & \text{ shares no components with } (D \cup M)_P | U \text{ or } (D \cup M)_U,
\end{align*}
\]
and that
\[
\left\{ \bigwedge_{s \in S_0 \subset S} \frac{dx_s}{x_s} \bigwedge_{t \in T_0 \subset T} \frac{dx_t}{x_t} \bigwedge_{v \in V_0 \subset V} \frac{dx_v}{x_v} \bigwedge_{w \in W_0 \subset W} \frac{dx_w}{x_w} \bigwedge_{z \in Z_0 \subset Z} \frac{dx_z}{x_z} \right\},
\]
where the collection of the subsets \( S_0 \subset S, T_0 \subset T, V_0 \subset V, W_0 \subset W, Z_0 \subset Z \) is the one of all those with \#\( S_0 + \#T_0 + \#V_0 + \#W_0 + \#Z_0 = j \), forms a basis of \( \Omega^j_X(\log(D \cup M)) \) as a free \( \mathcal{O}_X \)-module over \( U \), while
\[
\left\{ \bigwedge_{s \in S_0 \subset S} \frac{dy_s}{y_s} \bigwedge_{t \in T_0 \subset T} \frac{dy_t}{y_t} \bigwedge_{v \in V_0 \subset V} \frac{dy_v}{y_v} \bigwedge_{w \in W_0 \subset W} \frac{dy_w}{y_w} \bigwedge_{z \in Z_0 \subset Z} \frac{dx_z}{x_z} \right\}
\]
forms a basis of \( \Omega^j_Y(\log((D \cup M)_Y)) \) as a free \( \mathcal{O}_Y \)-module over \( \pi^{-1}(U) \), while
\[
\left\{ \bigwedge_{s \in S_0 \subset S} \frac{dy_s}{y_s} \bigwedge_{t \in T_0 \subset T} \frac{dy_t}{y_t} \bigwedge_{v \in V_0 \subset V} \frac{dy_v}{y_v} \bigwedge_{w \in W_0 \subset W} \frac{dy_w}{y_w} \bigwedge_{z \in Z_0 \subset Z} \frac{dx_z}{x_z} \right\}
\]
forms a basis of \( \Omega^j_Y(\log((D)_Y)) \) as a free \( \mathcal{O}_Y \)-module over \( \pi^{-1}(U) \).

Take a section
\[
f \in \Gamma(\pi^{-1}(U), \Omega^j_Y(\log((D \cup M)_Y)) \otimes \mathcal{O}_Y \mathcal{C}(Y)) = \Gamma(\pi^{-1}(U), \pi^* \Omega^j_X(\log(D \cup M)) \otimes \mathcal{O}_Y \mathcal{C}(Y))
\]
and write
\[
f = \sum_{\alpha, \beta, \gamma, \delta, \epsilon} \left( \pi^* \left[ \bigwedge_{s \in S_0 \subset S} \frac{dx_s}{x_s} \bigwedge_{t \in T_0 \subset T} \frac{dx_t}{x_t} \bigwedge_{v \in V_0 \subset V} \frac{dx_v}{x_v} \bigwedge_{w \in W_0 \subset W} \frac{dx_w}{x_w} \bigwedge_{z \in Z_0 \subset Z} \frac{dx_z}{x_z} \right] \otimes f_{\alpha, \beta, \gamma, \delta, \epsilon} \right)
\]
with \( f_{\alpha, \beta, \gamma, \delta, \epsilon} \in \mathcal{C}(Y) \).
Observe
\[
f \in \Gamma(U, \pi_*\{\Omega^1_Y/(\log D_Y)(\pi^*A - G_Y)\})
\]
\[
= \Gamma(\pi^{-1}(U), \Omega^1_Y/(\log D_Y)(\pi^*A - G_Y))
\]
\[
\iff \text{div} \left( \frac{f}{\prod_{w \in W_3 \subset W} \pi^*(x_w)^{1/m}} + \pi^*A - G_Y|_{\pi^{-1}(U)} \geq 0, \right.
\]
\[
f_{\alpha, \beta, \gamma, \delta, \epsilon} \in \mathbb{C}(Y), \forall \alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}(Y), \forall \alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}(Y), \forall \alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}(Y), \forall \alpha, \beta, \gamma, \delta, \epsilon.
\]
Therefore, we conclude
\[
f \in \Gamma(U, \left[\pi_*\{\Omega^1_Y/(\log D_Y)(\pi^*A - G_Y)\}\right]^\Gamma)
\]
\[
\iff \text{div} \left( \frac{f}{\prod_{w \in W_3 \subset W} \pi^*(x_w)^{1/m}} + \pi^*A - G_Y|_{\pi^{-1}(U)} \geq 0, \right.
\]
\[
f_{\alpha, \beta, \gamma, \delta, \epsilon} \in \mathbb{C}(Y)^\Gamma = \mathbb{C}(X), \forall \alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}(Y)^\Gamma = \mathbb{C}(X), \forall \alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}(Y)^\Gamma = \mathbb{C}(X), \forall \alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}(Y)^\Gamma = \mathbb{C}(X), \forall \alpha, \beta, \gamma, \delta, \epsilon.
\]
\[
\iff \text{div} \left( \frac{f}{\prod_{w \in W_3 \subset W} \pi^*(x_w)^{1/m}} \right) + A - G + \left\lfloor \frac{m - 1}{m} G \right\rfloor|_U \geq 0,
\]
\[
f_{\alpha, \beta, \gamma, \delta, \epsilon} \in \mathbb{C}(X), \forall \alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}(X), \forall \alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}(X), \forall \alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}(X), \forall \alpha, \beta, \gamma, \delta, \epsilon.
\]
\[
\iff f \in \Gamma(U, \Omega^1/(\log D)(\lceil A \rceil - G)).
\]

We may add the following explanation for the second last equivalence: Let \( B \) vary among all the irreducible components of \( D \cup M \mid_U \). Then the 3rd last condition
\[
\text{div} \left( \frac{f}{\prod_{w \in W_3 \subset W} x_w^{1/m}} \right) + A - G + \left\lfloor \frac{m - 1}{m} G \right\rfloor|_U \geq 0
\]
reads for the component \( B \):
\[
\begin{align*}
\vdots
\iff v_B(f_{\alpha, \beta, \gamma, \delta, \epsilon}) + a_B - 0 + \left\lfloor \frac{m - 1}{m} \right\rfloor \cdot 0 \geq 0 \\
& a_B \in \mathbb{Z} \\
\iff v_B(f_{\alpha, \beta, \gamma, \delta, \epsilon}) + a_B - 0 = v_B(f_{\alpha, \beta, \gamma, \delta, \epsilon}) + [a_B] - 0 \geq 0 \quad \text{if } B \subset D \setminus G \\
\iff v_B(f_{\alpha, \beta, \gamma, \delta, \epsilon}) + a_B - 1 + \left\lfloor \frac{m - 1}{m} \right\rfloor \cdot 1 \geq 0 \\
& a_B \in \mathbb{Z} \\
\iff v_B(f_{\alpha, \beta, \gamma, \delta, \epsilon}) + a_B - 1 = v_B(f_{\alpha, \beta, \gamma, \delta, \epsilon}) + [a_B] - 1 \geq 0 \quad \text{if } B \subset G \setminus F \\
\iff v_B(f_{\alpha, \beta, \gamma, \delta, \epsilon}) + a_B - 1 + \left\lfloor \frac{m - 1}{m} \right\rfloor \cdot 1 \geq 0 \\
& a_B \in \mathbb{Z} \\
= v_B(f_{\alpha, \beta, \gamma, \delta, \epsilon}) + a_B + \left\lfloor \frac{m - 1}{m} \right\rfloor \cdot 1 - 1 \geq 0 \\
\iff v_B(f_{\alpha, \beta, \gamma, \delta, \epsilon}) + [a_B] - 1 \geq 0 \quad \text{if } B \subset F \\
\vdots
\end{align*}
\]
by condition \((\ast)\)
\[
\begin{align*}
\vdots
\iff v_B(f_{\alpha, \beta, \gamma, \delta, \epsilon}) - \left\lfloor \frac{m - 1}{m} \right\rfloor \cdot 0 \geq 0 \\
& a_B \in \mathbb{Z} \\
\iff v_B(f_{\alpha, \beta, \gamma, \delta, \epsilon}) - 1 + a_B - 0 = v_B(f_{\alpha, \beta, \gamma, \delta, \epsilon}) - 1 + [a_B] - 0 \geq 0 \quad \text{if } B \subset M \\
\end{align*}
\]
where \( A = \sum a_B B \).
Now Theorem 1.3 in the fractional case is an immediate consequence of 3.2.2 as follows:

\[ H^i(X, \Omega^j_X(\log D)([A] - G)) = H^i \left( X, \left[ \pi_* \left( \Omega^j_Y(\log D_Y)(\pi^* A - G_Y) \right) \right]^{\Gamma} \right) \]

\[ = H^i(X, \pi_* \left( \Omega^j_Y(\log D_Y)(\pi^* A - G_Y) \right)^{\Gamma}) \]

\[ = H^i(Y, \Omega^j_Y(\log D_Y)(\pi^* A - G_Y))^{\Gamma} = 0, \]

since we have

\[ H^i(Y, \Omega^j_Y(\log D_Y)(\pi^* A - G_Y)) = 0 \text{ for } i + j > \text{dim } X = \text{dim } Y, \]

using the vanishing statement for the case where \( \pi^* A \) is integral. This completes the proof of the main theorem in the case where \( A \) is fractional.

### 3.3 Some remarks on the first proof

#### 3.3.1 Basic Idea of the proof

If we pretend that \( \pi \) is ramified only over \( D \), then the idea of the proof for 3.2.2 is more transparent. Under the pretension, since the logarithmic differential forms do not ramify and \( G_Y = \frac{1}{m} \pi^* G = \frac{m - 1}{m} \pi^* G - \pi^* G \), we have

\[ \Omega^j_Y(\log(D_Y))(\pi^* A - G_Y) = \pi^* \left( \Omega^j_X(\log(D)) \right) \left( \pi^* A + \frac{m - 1}{m} \pi^* G - \pi^* G \right). \]

By taking \( \pi_* \) and the \( \Gamma \)-invariant part, we conclude

\[ \pi_* \left( \Omega^j_Y(\log(D_Y))(\pi^* A - G_Y) \right)^{\Gamma} = \pi_* \left[ \pi^* \left( \Omega^j_X(\log(D)) \right) \left( \pi^* A + \frac{m - 1}{m} \pi^* G - \pi^* G \right) \right]^{\Gamma} \]

\[ = \Omega^j_X(\log(D)) \left( A + \frac{m - 1}{m} G - G \right) \]

\[ = \Omega^j_X(\log(D)) ([A] - G). \]

Here the last equality, replacing \( A + \frac{m - 1}{m} G \) with \([A]\), results from the fact that only the fractional part of \( A \) is affected and, that the coefficients of the components exceed their round ups (by less than one) when we add \( \frac{m - 1}{m} G \).

In the actual proof without the pretension, we have to analyze in more detail how a basis of the free \( O_X \)-module \( \Omega^j_X(\log(D)) \) ramifies over \( M \), when pulled back by \( \pi \), compared to a basis of the free \( O_Y \)-module \( \Omega^j_Y(\log(D_Y)) \) (and conclude that the ramification does not affect the conclusion at all). The basic idea, however, is the same.

#### 3.3.2 Use of the logarithmic forms and subtraction of the divisor \( G \)

In contrast to the logarithmic differential forms, if we use the usual differential forms, the basis of the free \( O_X \)-module \( \Omega^j_X \)

\[ \left\{ \bigwedge_{\{x_\alpha = 0\} \subset (D \cup M)_p} dx_\alpha \bigwedge_{\{x_\beta = 0\} \not\subset (D \cup M)_p} dx_\beta \right\} \]
has varying ramification factors, and gives rise to the following corresponding basis of the free $\mathcal{O}_Y$-module $\Omega^j_Y$

$$\left\{ \frac{1}{\prod_{\{x_\alpha = 0\} \subset (D \cup M)_P} \pi^* x_\alpha^{(m-1)/m}} \cdot \pi^* \left[ \bigwedge_{\{x_\alpha = 0\} \subset (D \cup M)_P} dx_\alpha \bigwedge_{\{x_\beta = 0\} \subset (D \cup M)_P} dx_\beta \right] \right\}.$$  

The varying ramifications cannot be expressed by the twist of a single $(\mathbb{Q}_-)$ divisor. This is why one is led to the use of logarithmic differential forms.

On the other hand, if we use the logarithmic differential forms, since there is no ramification, there is no “push” from the ramification to raise $A$ to $\lceil A \rceil$. This is where the subtraction of the divisor $G$ comes in. The difference between $-G_Y = -\pi^* G + \frac{m-1}{m} \pi^* G$ and $-\pi^* G$, which is $\frac{m-1}{m} \pi^* G$, gives the push to raise $A$ to $\lceil A \rceil$.

### 3.3.3 Comparison with the classical argument

In the classical argument for the proof of the Kawamata–Viehweg Vanishing, where we only have to deal with the top form, the free $\mathcal{O}_X$-module $\Omega^n_X$, where $n = \dim X = \dim Y$, is of rank one, having one generator

$$\bigwedge_{\alpha=1}^l dx_\alpha \bigwedge_{\beta=l+1}^n dx_\beta.$$  

Therefore, it has a unique ramification factor giving rise to the following unique basis of the free $\mathcal{O}_Y$-module $\Omega^n_Y$:

$$\prod_{\alpha=1}^l \frac{1}{\pi^* x_\alpha^{(m-1)/m}} \cdot \pi^* \left[ \bigwedge_{\alpha=1}^l dx_\alpha \bigwedge_{\beta=l+1}^n dx_\beta \right].$$  

Moreover, the reciprocal $\prod_{\alpha=1}^l \pi^* x_\alpha^{(m-1)/m}$ of the ramification factor gives the “push” to raise $A$ to $\lceil A \rceil$. However, the classical argument to look at the usual differential forms would face trouble in the case of lower degree forms since the elements of the basis have varying ramification factors (as discussed in 3.3.2).

Our new argument using the logarithmic forms and subtraction of the divisor $G$ applies to the lower differential forms in the setting dealing with the Kawamata–Viehweg type formulation of the (log) Akizuki–Nakano Vanishing as well as to the top differential form in the setting dealing with the classical Kawamata–Viehweg Vanishing. This also gives a slightly different viewpoint towards the classical argument for the Kawamata–Viehweg Vanishing Theorem.

### 3.3.4 An alternative line of argument

As suggested by Prof. Helmke, one could follow the following line of argument to prove our main vanishing result:

1. Prove the case with $G = D$ of our formulation, i.e.,

$$H^i(X, \Omega^j_X (\log D)(\lceil A \rceil - D)),$$

reducing its verification to the Steenbrink Vanishing Theorem via the Kawamata Covering Lemma as in our argument above.
(2) In order to prove the general case \( F \leq G \leq D \), we set \( A' = A + \epsilon(D - G) \) for a sufficiently small positive number \( 0 < \epsilon << 1 \) so that \( A' \) is again ample with \( F' = \lceil A' \rceil - A' \leq D \). Now, using (1), we conclude

\[
0 = H^i(X, \Omega^j_X(\log D)(\lceil A' \rceil - D)) = H^i(X, \Omega^j_X(\log D)(\lceil A \rceil - G))
\]

as required.

This line of argument avoids the use of the residue sequence. It also makes it clearer that what is essential is

- the Steenbrink Vanishing, and
- its fractional version as in [4].

### 4 Proof of Theorem 1.3 by reduction mod \( p \)

Here we explain how to obtain the fractional version of Steenbrink Vanishing, mentioned above, by reduction mod \( p \). We prove the following theorem, via the results of Deligne–Illusie and Raynaud [6] and a lemma by Hara [12], in characteristic \( p \). The following is a special case of [4, theorem 8.2].

**Theorem 4.1** Let \( X \) be a nonsingular projective variety over an algebraically closed field \( k \) of characteristic \( p > \dim X \). Let \( D = \sum D_i \) be a simple normal crossings divisor such that the pair \((X, D)\) is liftable modulo \( p^2 \). If \( L \) is a line bundle such that \( L(-\Delta) \) is ample for some \( \mathbb{Q} \)-divisor \( \Delta \) supported on \( D \) with coefficients in \( [0, 1) \), then

\[
H^i(X, \Omega^j_X(\log D)(-D) \otimes L) = 0
\]

for \( i + j > \dim X \).

Now by a standard “spreading out” argument, we obtain the following result in characteristic \( 0 \), which implies the main Theorem 1.3 (as explained in 3.3.4). We note that the line bundle \( L \) and the \( \mathbb{Q} \)-divisor \( \Delta \) correspond to \( \lceil A \rceil \) and \( F \) in the notation of §2.

**Corollary 4.2** Let \( X, D, \) and \( L \) be as above, but defined over an algebraically closed field of characteristic \( 0 \). Then

\[
H^i(X, \Omega^j_X(\log D)(-D) \otimes L) = 0
\]

for \( i + j > \dim X \).

We give a short self contained proof (via the results and lemma mentioned above) of 4.1, extracting the ideas from the proofs of [4, theorem 8.2] and [2, theorem 3]. First let us quote the following lemma by Hara [12].

**Lemma 4.3** (Hara [12, 3.3]) Let the situation be the same as described in Theorem 4.1. If \( D' \) is an integral divisor satisfying \( 0 \leq D' \leq (p - 1)D \), then there is a quasi-isomorphism

\[
\Omega^\bullet_X(\log D) \cong \Omega^\bullet_X(\log D)(D')
\]

Now using the results of [6] and the above lemma, we obtain the following.

**Lemma 4.4** Again we keep the same situation as described in Theorem 4.1. Let \( M \) be a line bundle on \( X \). Suppose that \( D' \) is an integral divisor and that \( 0 \leq D' \leq (p - 1)D \). Then the following inequalities hold.
(a) For all $r$,
\[ \sum_{i+j=r} h^i(X, \Omega^j_X(\log D) \otimes M) \leq \sum_{i+j=r} h^i(X, \Omega^j_X(\log D)(D') \otimes M^p) \]

(b) For all $r$,
\[ \sum_{i+j=r} h^i(X, \Omega^j_X(\log D)(-D) \otimes M) \leq \sum_{i+j=r} h^i(X, \Omega^j_X(\log D)(-D - D') \otimes M^p) \]

**Proof** Let $F : X \to X$ denote the absolute Frobenius map. By [6, §4.2], the projection formula, and the previous lemma, we have
\[ H^i\left(X, \bigoplus_j \Omega^j_X(\log D)[-j] \otimes M\right) \cong H^i(X, (F_*\Omega^i_X(\log D)) \otimes M) \]
\[ \cong H^i(X, (F_*\Omega^i_X(\log D) \otimes M^p)) \]
\[ \cong H^i(X, \Omega^i_X(\log D) \otimes M^p) \]
\[ \cong H^i(X, \Omega^i_X(\log D)(D') \otimes M^p) \]

These isomorphisms together with the spectral sequence
\[ E^{ab}_1 = H^b(X, \Omega^a_X(\log D)(D') \otimes M^p) \Rightarrow H^{a+b}(X, \Omega^i_X(\log D)(D') \otimes M^p) \]
prove the first inequality. Note that this is the standard spectral sequence with respect to the bête filtration [5, 1.4.5, 1.4.7]. We obtain the second inequality from the first using the Serre duality.

\[ \square \]

**Lemma 4.5** With the same notation as in the previous lemma for $X$, $D$ and $M$, suppose this time that $D'$ is an integral divisor and that $0 \leq D' \leq (p^n - 1)D$. Then
\[ \sum_{i+j=r} h^i(X, \Omega^j_X(\log D)(-D) \otimes M) \leq \sum_{i+j=r} h^i(X, \Omega^j_X(\log D)(-D - D') \otimes M^p) \]

**Proof** We may write $D' = p^{n-1}D_1 + p^{n-2}D_2' + \ldots$, where $0 \leq D_i \leq (p - 1)D$. Then repeatedly applying lemma 4.4 gives
\[ \sum_{i+j=r} h^i(X, \Omega^j_X(\log D)(-D) \otimes M) \leq \sum_{i+j=r} h^i(X, \Omega^j_X(\log D)(-D) \otimes M^p(-D'_1)) \]
\[ \leq \sum_{i+j=r} h^i(X, \Omega^j_X(\log D)(-D) \otimes M^p(-pD'_1 - D'_2)) \]
\[ \ldots \]

\[ \square \]

**Proof of 4.1** By assumption, $L(-\Delta)$ is ample for some $\Delta = \sum r_i D_i$ with $r_i \in [0, 1) \cap \mathbb{Q}$. Using Kleiman’s ampleness criterion (cf [22]), we can see that $L(-\sum r'_i D_i)$ remains ample, whenever $r'_i$ is sufficiently close to $r_i$. Therefore, we can assume that the coefficients $r_i$ lie in $[0, 1) \cap \mathbb{Z}[\frac{1}{p}]$ for some sufficiently large integer $l$. Thus, $L^{p^n}(-D')$ is ample for some integer $n > 0$ and some integral divisor $0 \leq D' = p^n(\sum r'_i D_i) \leq (p^n - 1)D$. We may also assume, taking $n$ sufficiently large, that
\[ H^i(X, \Omega^j_X(\log D)(-D) \otimes L^{p^n}(-D')) = 0 \]

\[ \square \]
for all $i > 0$ by Serre vanishing. Now 4.1 is a consequence of lemma 4.5, setting $m = l$ and noting that, if $i + j = r > \dim X$, then either $i > 0$ or $i = 0$ with $j > \dim X$ and hence $H^0(X, \Omega^i_X (\log D)(-D) \otimes L) = H^0(X, \Omega^i_X (\log D)(-D) \otimes L^{p^r}(-D')) = 0$. □

5 Applications

The application of the Kawamata–Viehweg Vanishing Theorem in the Minimal Model Program is one of the most remarkable stories in the modern development of the subject of Algebraic Geometry. Applications of the Akizuki–Nakano Vanishing are also well known. Some applications and other extensions of Steenbrink’s vanishing can be found in [3, 9, 10, 19–21]. We list some applications of our vanishing theorem below.

5.1 Unobstructedness of the deformations of log $\mathbb{Q}$-Fano manifolds

Akizuki–Nakano Vanishing can be used to show that deformations of Fano manifolds are unobstructed [25]. We can generalize the argument to the case of log $\mathbb{Q}$-Fano manifolds as follows. Let $(X, B + F)$ be a log $\mathbb{Q}$-Fano manifold. That is to say, it is a pair consisting of a nonsingular projective variety $X$ and a boundary divisor $B + F$ where $B$ is an effective and reduced integral divisor $B = \sum B_k$ and a fractional divisor $F = \sum f_j F_j$ ($0 < f_j < 1$), $B$ and $F$ sharing no common components, with $D = \sum B_k + \sum F_j$ being a simple normal crossings divisor, such that $-(K_X + B + F)$ is ample. Note that the $B_k$ (resp. the $F_j$) are the irreducible components of $B$ (resp. $F$).

A deformation of the pair $(X, B + F)$ over a local Artin $\mathbb{C}$-algebra $A$ is a flat proper scheme $X \rightarrow \text{Spec} A$ with a relative simple normal crossing divisor $B + F$, having the same multiplicities as before, such that $(X, B + F)$ is the special fibre. Standard arguments (c.f. [15, 29]), show that the space of deformations over $\mathbb{C}[\epsilon]/(\epsilon^2)$ is isomorphic to $H^1(X, T_X (− \log D))$, where we write $T_X (− \log D) = \Omega^1_X (\log D)^\vee$. Furthermore, the smoothness obstructions lie in $H^2$ with the same coefficients. More precisely, given an extension of local Artin $\mathbb{C}$-algebras

$$0 \rightarrow I \rightarrow A' \rightarrow A \rightarrow 0$$

with $I^2 = 0$ and a deformation $(X, B + F)$ over $A$, the obstruction to extending it to a deformation over $A'$ lies in $H^2(X, T_X (− \log D) \otimes_A I)$.

To analyze these obstructions, we can reduce to the case where $\dim_{\mathbb{C}} I = 1$, in which case the above group is simply $H^2(X, T_X (− \log D))$.

When the obstructions are all zero, we will say that the deformations of the pair $(X, B + F)$ are unobstructed. This implies that the formal moduli space, or Kuranishi space, is smooth.

**Proof** (Theorem 1.5) We have

$$H^2(X, T_X (− \log D)) \cong H^2(X, \Omega^{\dim X - 1} (\log D))(-(K_X + D))$$

$$= H^2(X, \Omega^{\dim X - 1} (\log D))([-((K_X + B + F)] - G)) = 0,$$

$\square$ Springer
where $G = \sum F_j$ by our Theorem 1.3.

\section*{5.2 Analysis of the zero locus of the (log) 1-forms}

Wei \cite{wei1, wei2} proves that the zero-locus of any global holomorphic log-1-form on a projective log-smooth pair $(X, D)$ of log-general type must be non-empty, using the Kodaira-Saito Vanishing theorem \cite{saito}. This is a generalization of the results proved, in historical order, by Zhang \cite{zhang}, Hacon and Kovács \cite{haconkovacs} and Popa and Schnell \cite{popschnell}. We explain how to deduce a special case of Wei’s result using our vanishing theorem. We first give a variant of \cite[Theorem 2.1]{haconkovacs}.

\begin{prop}
Let $X$ be an $n$ dimensional nonsingular projective variety. Let $D$ be a reduced effective divisor with simple normal crossings on $X$, $A$ an ample $\mathbb{Q}$-divisor with support of the fractional part $F$ (i.e. $F = \lceil A \rceil - A$) contained in $D$, and $F \leq G \leq D$. If $\Omega^1_X(\log D)$ has a nowhere vanishing section $\theta$, then

\[ H^n(X, \mathcal{O}_X(\lceil A \rceil - G)) = 0 \]

\end{prop}

\begin{proof}
Let us write $V = \Omega^1_X(\log D)$ and $L = \mathcal{O}_X(\lceil A \rceil - G)$. Theorem 1.3 implies that

\begin{equation}
H^j(X, i \bigwedge V \otimes L) = 0 \quad \text{when } i + j > n \tag{5.2.2}
\end{equation}

Since the zero-locus of $\theta$ is empty, the Koszul complex

\[ 0 \longrightarrow \mathcal{O}_X \xrightarrow{\theta} V \xrightarrow{\theta} \cdots \xrightarrow{\theta} \bigwedge^n V \longrightarrow 0 \]

must be exact. We can break this into short exact sequences

\[ 0 \longrightarrow B^{i-1} \longrightarrow \bigwedge^i V \longrightarrow B^i \longrightarrow 0 \]

where

\[ B^i = \ker \theta : \bigwedge^{i+1} V \longrightarrow \bigwedge^{i+2} V \]

Tensoring the sequence with $L$, and using the long exact sequence for cohomology together with (5.2.2), we obtain surjections

\[ H^{n-i}(X, B^i \otimes L) \twoheadrightarrow H^{n-i+1}(X, B^{i-1} \otimes L) \rightarrow 0 \]

Since $B^0 = \mathcal{O}_X$ and $B^n = 0$, the proposition follows.

\end{proof}

\begin{proof}[Theorem 1.6]
Suppose $H^0(\Omega^1_X(\log D))$ possessed a section with empty zero-locus. Then applying the proposition with $A = K_X + B + C$ and $G = \lceil B + C \rceil = D$ shows that

\[ H^0(X, \mathcal{O}_X(K_X)) = 0 \]

However, this contradicts $H^0(X, \mathcal{O}) \neq 0$ via Serre duality.

\end{proof}

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References

1. Akizuki, Y., Nakano, S.: Note on Kodaira–Spencer’s proof of Lefschetz’s theorem. Proc. Jpn. Acad. Ser. A 30, 266–272 (1954)
2. Arapura, D.: Kodaira-Saito vanishing via Higgs bundles in positive characteristic. J. Reine Angew. Math. (To appear)
3. Arapura, D.: Local cohomology of sheaves of differential forms and Hodge theory. J. Reine Angew. Math. 409, 172–179 (1990)
4. Arapura, D.: Frobenius amplitude and strong vanishing theorems for vector bundles. With an appendix by Dennis S. Keeler. Duke Math. J. 121(2), 231–267 (2004)
5. Deligne, P.: Théorie de Hodge. II. Inst. Hautes Études Sci. Publ. Math. 40, 5–57 (1971)
6. Deligne, P., Illusie, L.: Relevements modulo $p^2$ et decomposition du complexe de de Rham. Invent. Math. 89, 247–280 (1987)
7. Esnault, H., Viehweg, E.: Logarithmic de Rham complexes and vanishing theorems. Invent. Math. 86(1), 161–194 (1986)
8. Esnault, H., Viehweg, E.: Lectures on Vanishing Theorems, DMV Seminar, vol. 20. Birkhäuser Verlag, Basel (1992)
9. Flenner, H.: Extendability of differential forms on nonisolated singularities. Invent. Math. 94(2), 317–326 (1988)
10. Guillen, F., Navarro, A.V., Pascual, G.P., Puerta, F.: Hyperrésolutions cubiques et descente cohomologique. Papers from the Seminar on Hodge–Deligne Theory Held in Barcelona, Lecture Notes in Mathematics, vol. 1335, p. 1988. Springer, Berlin (1982)
11. Hacon, C., Kovács, S.: Holomorphic one-forms on varieties of general type. Ann. Sci. École Norm. Sup. (4) 38(4), 599–607 (2005)
12. Haras, N.: A characterization of rational singularities in terms of injectivity of Frobenius maps. Am. J. Math. 120, 981–996 (1998)
13. Huang, C., Liu, K., Wan X., Yang, X.: Logarithmic vanishing theorems on compact Kähler manifolds I. arXiv:1611.07671 (2016)
14. Kawamata, Y.: On a generalization of Kodaira–Ramanujam’s vanishing theorem. Math. Ann. 261, 43–46 (1982)
15. Kawamata, Y., Namikawa, Y.: Logarithmic deformations of normal crossing varieties and smoothing of degenerate Calabi–Yau varieties. Invent. Math. 118(3), 395–409 (1994)
16. Kawamata, Y., Matsuda, K., Matsuki, K.: Introduction to the minimal model problem. Adv. Stud. Pure Math. 10, 283–360 (1987)
17. Kodaira, K.: On a differential-geometric method in the theory of analytic stacks. Proc. Natl. Acad. Sci. USA 39, 1268–1273 (1953)
18. Kollár, J.: Vanishing theorems for cohomology groups, Algebraic Geometry Bowdoin 1985. Proceedings of Symposium Pure Math., vol. 46, pp. 233–243 (1987)
19. Kovács, S.: On the number of singular fibers in a family of surfaces of general type. J. Reine Angew. Math. 487, 171–177 (1997)
20. Kovács, S.: Logarithmic vanishing theorems and Arakelov–Parshin boundedness for singular varieties. Compos. Math. 131(3), 291–317 (2002)
21. Kovács, S.: Steenbrink vanishing extended. Bull. Braz. Math. Soc. (N.S.) 45(4), 753–765 (2014)
22. Lazarsfeld, R.: Positivity in Algebraic Geometry I. Springer, Berlin (2004)
23. Matsuki, K.: Introduction to the Mori Program. Universitext, Springer, Berlin (2002)
24. Matsuki, K., Olsson, M.: Kawamata–Viehweg vanishing as Kodaira vanishing for stacks. Math. Res. Lett. 12, 207–217 (2005)
25. Mori, S., Mukai, S.: Classification of Fano 3-folds with $B_2 \geq 2$. Manuscr. Math. 36(2), 147–162 (1981)
26. Norimatsu, Y.: Kodaira vanishing theorem and Chern classes for $\mathcal{G}$- manifolds. Proc. Jpn. Acad. Ser. A Math. Sci. 54(4), 107–108 (1978)
27. Popa, M., Schnell, C.: Kodaira dimension and zeros of holomorphic one-forms. Ann. Math. (2) 179(3), 1109–1120 (2014)
28. Saito, M.: Mixed Hodge modules. Publ. Res. Inst. Math. Sci. 26(2), 221–333 (1990)
29. Sernesi, E.: Deformations of Algebraic Schemes, Grundlehren der Mathematischen Wissenschaften, vol. 334. Springer, Berlin (2006)
30. Steenbrink, J., Vanishing theorems on singular spaces. Differential systems and singularities (Luminy, 1983) Asterisque, No. 130 (1985)
31. Viehweg, E.: Vanishing theorems. J. Reine Angew. Math. 335, 1–8 (1982)
32. Viehweg, E., Zuo, K.: On the isotriviality of families of projective manifolds over curves. J. Algebraic Geom. 10(4), 781–799 (2001)
33. Wei, C.: Fibration of log-general type space over quasi-abelian varieties, arXiv preprint arXiv:1609.03089 (2016)
34. Wei, C.: Logarithmic Kodaira dimension and zeros of holomorphic log-one-forms. arXiv:1711.05854v1 [math], November 15 (2017)
35. Zhang, Q.: Global holomorphic one-forms on projective manifolds with ample canonical bundles. J. Algebraic Geom. 6(4), 777–787 (1997)

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