A NOTE ON EMBEDDING OF ACHIRAL LEFSCHETZ FIBRATIONS

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Abstract. We discuss 4-dimensional achiral Lefschetz fibrations bounding 3-dimensional open books and study their Lefschetz fibration embedding in a bounded 6-dimensional manifold, in the sense of Ghanwat–Pancholi. As an application we give another proof of the fact that every orientable 4-manifold embeds in \( \mathbb{R}^7 \).

1. Introduction

In recent times, various embedding problems for manifolds with extra geometric structures have been investigated. In particular, the study of embeddings in the category of open books and Lefschetz fibrations has seen some progress. The open book case was studied in [2],[12],[13] and [10]. Recently, Ghanwat and Pancholi have proved the existence of embedding of any given oriented closed 4-manifold in \( \mathbb{C}P^3 \). As an application they were able to give a geometric topological proof of an well-known result which says that every orientable closed 4-manifold embeds in \( \mathbb{R}^7 \). In the present article, we prove a relative version of the main Theorem in [7] (see Theorem 2.9).

Theorem 1.1. Let \( V^4 = LF(\Sigma, \phi) \). \( V^4 \) admits a relative LF embedding in \((S^2 \times S^2 \setminus D^4) \times D^2\). Here, \( LF(\Sigma, \phi) \) describes an achiral Lefschetz fibration with fiber \( \Sigma \) and \( \phi \) is a representation of an element in \( \text{MCG}(\Sigma, \partial \Sigma) \). For exact definitions we refer to section 2.2.

Using Theorem 1.1 in section 4 we give another proof of the following fact which was first proved by Fuquan [5] and recently reproved by Ghanwat and Pancholi [7].

Corollary 1.2. Every closed orientable 4-manifold embeds in \( \mathbb{R}^7 \).

Although the main idea behind our proof is mostly similar to that of Ghanwat–Pancholi [7], while applying Theorem 1.1, we use handlebody decomposition of a 4-manifold instead of the notion of broken Lefschetz fibration, used in [7].

Finally, in section 5 we discuss about Lefschetz embeddings in \( D^6 \). Let \( \Sigma_{g,1} \) denote a genus-\( g \) surface with one boundary component. Then \( \text{MCG}(\Sigma_{g,1}, \partial \Sigma_{g,1}) \) is generated by Dehn twists along the set of curves, \( \{a_1, c_1, a_2, c_2, ... a_{g-1}, c_{g-1}, a_g, b_1, b_2\} \), as in Figure 1. These curves are known as the Humphreys generators.

The group generated by all the Humphreys generators, except \( b_2 \), is called the hyperelliptic subgroup of \( \text{MCG}(\Sigma_{g,1}, \partial \Sigma_{g,1}) \). Any element of this group is called hyperelliptic.

Theorem 1.3. Let \( V^4 = LF(\Sigma_{g,1}, \phi) \).

1. If \( V^4 \) admits a proper embedding in \( D^6 \), then \( V^4 \) is spin.
2. If \( \phi \) is hyperelliptic, then \( V^4 \) admits a relative LF embedding in \( D^6 \).

For the definition of relative LF embedding we refer to section 2.10.

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2. Preliminaries

2.1. Open book decomposition. An open book is a decomposition of a manifold into a co-dimension 2 submanifold and a fibration over $S^1$.

Definition 2.1. Let $\Sigma$ be a surface with boundary and let $\phi$ be a diffeomorphism of $\Sigma$ that is identity on a collar neighborhood of $\partial \Sigma$. An open book decomposition of $M$ is a pair $(\Sigma, \phi)$ such that $M$ is diffeomorphic to $\mathcal{MT}(\Sigma, \phi) \cup_{id} \Sigma \times D^2$, where $id$ denotes the identity map of $\partial \Sigma \times S^1$.

Here, $\mathcal{MT}(\Sigma, \phi)$ is the mapping torus of $\phi$. We denote such an open book by $Ob(\Sigma, \phi)$.

For details on open books, we refer to [3] and [6].

2.2. Lefschetz fibration and achiral Lefschetz fibration. The following description of Lefschetz fibration and achiral Lefschetz fibration is mostly taken from [2].

A Lefschetz fibration (LF) of an oriented 4-manifold $X^4$ is a map $\pi : X^4 \rightarrow D^2$, where $D^2$ is a 2-disk and $D\pi$ is surjective at all but finitely many points $p_1, \ldots, p_k$, called singular points, each of which has the following local model. Each point $p_i$ has a neighborhood $U_i$ that is orientation preserving diffeomorphic to an open set $U'_i$ in $\mathbb{C}^2$, and in these local coordinates $\pi$ is expressed as the map $(z_1, z_2) \mapsto z_1 \cdot z_2$. We say $\pi : X^4 \rightarrow D^2$ is an achiral Lefschetz fibration (ALF) if there is a map $\pi : X^4 \rightarrow D^2$ as above, except the local charts expressing $\pi$ as $(z_1, z_2) \mapsto z_1 \cdot z_2$, need not be orientation preserving. Following are some well known facts about (achiral) Lefschetz fibrations.

1. Let $F = D^2 \setminus \pi(\{p_1, \ldots, p_k\})$ and $X' = \pi^{-1}(F)$. Then $\pi|_{X'} : X' \rightarrow F$ is a fibration with fiber some surface $\Sigma$, possibly with boundary.

2. Fix a point $x \in F$ and for each $i = 1, \ldots, k$, let $\gamma_i$ be a path in $D^2$ from $x$ to $\pi(p_i)$ whose interior is in $F$. Then there is an embedded simple closed curve $v_i$ in $F_x = \pi^{-1}(x)$ that is homologically non-trivial in $F_x$ but is trivial in the homology of $\pi^{-1}(\gamma_i)$. The curve $v_i$ is called the vanishing cycle of $p_i$. We will assume that $\gamma_i \cap \gamma_j = \{x\}$ for all $i \neq j$.

3. For each $i$ let $D_i$ be a disk inside $D^2$ containing $\pi(p_i)$ in its interior, disjoint from the $\gamma_j$ for $j \neq i$, and intersecting $\gamma_i$ in a single arc that is transverse to $\partial D_i$. The boundary $\partial \pi^{-1}(D_i)$ is a $\Sigma$-bundle over $S^3$. Identifying a fiber of $\pi^{-1}(\partial D_i)$ with $\Sigma$ using $\gamma_i$, the monodromy of $\pi^{-1}(\partial D_i)$ is given by a positive Dehn twist along $v_i$. 

Figure 1. Humphreys generators of mapping class groups of $\Sigma_{g,1}$

We note that an achiral Lefschetz fibration depends on the presentation of the monodromy of its fiber. Therefore, in statement (2) of Theorem 1.3 the monodromy is assumed to be presented in terms of the Humphrey’s generators.

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(4) Let $D$ be a disk containing $x$ and intersecting each $\gamma_i$ in an arc transverse to $\partial D$. The manifold $X$ can be built from $D^2 \times \Sigma = \pi^{-1}(D)$ by adding a 2–handle for each $p_i$ along $v_i$ sitting in $F_{q_i} = \pi^{-1}(q_i)$ where $q_i = \partial D \cap \gamma_i$ with framing one less than the framing of $v_i$ given by $F_{q_i}$ in $\partial D^2 \times \Sigma$. Conversely, any 4-manifold constructed from $D^2 \times \Sigma$ by attaching 2–handles in this way will correspond to a Lefschetz fibration.

(5) If $\pi : X \to D^2$ is an achiral Lefschetz fibration (ALF), then the above statements are still true but for an achiral singular point the monodromy is a negative Dehn twist and the 2–handle is added with framing one greater than the surface framing.

(6) An ALF determines a factorization of the fiber monodromy.

(7) If $\Sigma$ is a surface with boundary, then the ALF induces an open book decomposition on the boundary manifold with fiber $\Sigma$ and monodromy same as the monodromy of the ALF.

We shall denote an achiral Lefschetz fibration over $D^2$, with fiber $\Sigma$ and a monodromy $\phi$, by $LF(\Sigma, \phi)$.

2.3. A property of $\text{MCG}(\Sigma, \partial \Sigma)$. Let $(\Sigma, \partial \Sigma)$ be an orientable bounded surface with $g$ handles and $m$ boundary components as shown in Figure 2 below.

Let $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ be the simple closed curves along the $g$ handles. If a simple closed curve $\gamma$ intersects $\alpha_i$ for some $i \in \{1, 2, \ldots, g\}$, then we say that $\gamma$ meets the $i^{th}$ handle. Let $\Sigma$ be the closed surface obtained from $\partial \Sigma$ by filling the boundary circles with disks. Let $C$ be any simple closed curve on $\Sigma$. Now, according to Lemma 3 in [9], given any such curve $C$, there exists a diffeomorphism of $\Sigma$ which sends $C$ to a curve which does not meet any handle. The proof of Lemma 3 [9] also holds for surfaces with boundary. This is because the modifications via Dehn twists and isotopies used in Lemma 3 [9], can all be taken to be supported away from the $m$ disks bounding the $\partial \Sigma$s in $\Sigma$.

**Proposition 2.2** (Lickorish [11]). Let $C$ be any simple closed curve on $(\Sigma, \partial \Sigma)$. There exists a diffeomorphism $\phi \in \text{MCG}(\Sigma, \partial \Sigma)$ such that $\phi(C)$ does not meet any handle of $\Sigma$.

2.4. Flexible embedding in standard position. We now recall the notion of flexible embedding and embedding in a standard position from [7]. We shall state the definitions adapted to the relative case which will be useful for our purpose.
Definition 2.3 (Flexible embedding). Let $W^4$ be an orientable bounded smooth manifold and let $(\Sigma, \partial \Sigma)$ be a bounded orientable surface. A smooth proper embedding $h : (\Sigma, \partial \Sigma) \hookrightarrow (W, \partial W)$ is said to be flexible if for all $\phi \in \text{MCG}(\Sigma)$ there exists a relative diffeomorphism $\psi$ of $(W, \partial W)$ isotopic to the identity which maps $(\Sigma, \partial \Sigma)$ to itself and satisfies $\psi \circ h = h \circ \phi$.

Definition 2.4 (Embedding in standard position). A proper embedding $\phi : (\Sigma, \partial \Sigma) \hookrightarrow (W, \partial W)$ is said to be in a standard position if the following properties hold:

1. Every simple closed curve $\gamma$ on $\phi(\Sigma)$ is a boundary of a 2–disk $D^2$ intersecting $\phi(\Sigma)$ only in $\gamma$.
2. There exists a tubular neighborhood $N(D^2)$ of the disk $D^2$ having the boundary $\gamma$ such that $N(D^2)$ is the image of a coordinate chart $\phi_\gamma : \mathbb{C}^2 \to N(D^2)$ satisfying the following:
   
   $\phi_\gamma^{-1}(\phi(\Sigma) \cap N(D^2)) = g^{-1}(1)$, where $g : \mathbb{C}^2 \to \mathbb{C}$ is the polynomial map $g(z_1, z_2) = z_1z_2$.

   Topologically this means that $\phi_\gamma^{-1}(\phi(\Sigma) \cap N(D^2))$ is a Hopf annulus in a 3-sphere around the the origin in $\mathbb{C}^2$.

Next we recall the notion of a separable Hopf link.

Definition 2.5 (Separable Hopf link). We say that a link $l_1 \sqcup l_2$ in a 4–manifold $W$ is a separable Hopf link provided following properties are satisfied:

1. There exist an embedding of a 4–ball $D^4 = D^2 \times D^2$ in $W$ such that $\partial D^2 \times \{0\} \sqcup \{0\} \times \partial D^2 = l_1 \sqcup l_2$.
2. There exists two disjoint properly embedded discs $D_1$ and $D_2$ in $W \setminus (D^2 \times D^2)^0$ such that $\partial D_1 = l_1$ and $\partial D_2 = l_2$.

The following Lemma was proved by Ghanwat–Pancholi (Lemma 4.4,[7]).

Lemma 2.6 ([7]). Let $N$ be a 4–manifold which admits a separable Hopf link. Then there exists an embedding $\phi$ of any closed orientable surface $\Sigma_g$ of genus $g$ in $N$ which satisfies the following:

1. The embedding is flexible.
2. The embedding is in a standard position.

2.5. Lefschetz fibration embedding. We observe that the following relative version of Lemma 2.6 also holds true. The proof is essentially the same as the proof of Lemma 2.6. However, for the sake of convenience, we review the arguments adapted to our setting, i.e., the relative case.

Lemma 2.7. Let $W$ be a 4–manifold with connected boundary which admits a separable Hopf link. Then there exists a proper embedding $\phi$ of any bounded orientable surface $(\Sigma, \partial \Sigma)$ in $(W, \partial W)$ which satisfies the following:

1. $\phi$ is flexible.
2. $\phi$ is in a standard position.

Proof of Lemma 2.7. Let $l_1 \sqcup l_2$ be a separable Hopf link in $W$. So, there exists an embedded 4–ball $D^4 = D^2 \times D^2$ in $W$ such that $\partial D^2 \times \{0\} \sqcup \{0\} \times \partial D^2 = l_1 \sqcup l_2$, and there exist two disjoint properly embedded discs $D_1$ and $D_2$ in $W \setminus (D^2 \times D^2)^0$ such that $\partial D_1 = l_1$ and $\partial D_2 = l_2$. We regard a 4–ball $D^4$ as the 4–ball $B^4(0, 2)$ of radius 2 in $\mathbb{C}^2$ with its center at the origin. We will also regard $S^3 \times [1, 2]$ as the collar $B^4(0, 2) \setminus B^4(0, 1)$ contained in $W$.

Observe that the link $l_1 \times \{\frac{3}{2}\} \sqcup l_2 \times \{\frac{3}{2}\}$ bounds a Hopf annulus, say $\mathcal{H}$, in $S^3 \times \{\frac{3}{2}\}$. We embed $\Sigma$ in $S^3 \times \{\frac{3}{2}\} \subset S^3 \times [1, 2] \subset W$ in the following way. Let $\tilde{\Sigma}_g$ be the closed surface of genus $g$ obtained from $\Sigma$ by killing its $m$ boundary components $\{\partial_1, \ldots, \partial_m\}$, by attaching disks. The standard embedding of $\tilde{\Sigma}$ in $S^3$ is the one that bounds a genus $g$ handlebody. This induces an
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By adding the disjoint cylinders \((l_1 \sqcup l_2 \sqcup \partial_1 \sqcup \cdots \sqcup \partial_m) \times [\frac{3}{2}, 2]\) and two disjoint disc \(D_1, D_2\) to \(\hat{\Sigma}\), we obtain a proper embedding of \(\Sigma\) in \(W\). Let us denote this embedding by \(\phi\). We claim that the embedding \(\phi: (\Sigma, \partial \Sigma) \hookrightarrow (W, \partial W)\) is both flexible and in standard position.

Consider a Dehn twist \(\tau_\gamma\) along a curve \(\gamma\) on \(\Sigma\) embedded in \(W\) via \(\phi\). If \(\phi(\gamma)\), up to isotopy, has a Hopf annulus neighborhood in \(S^3 \times \{\frac{3}{2}\} \subset W\), then \(\tau_\gamma\) can be induced by a diffeomorphism of \(S^3\) that is isotopic to the identity map. As shown in the proof of Lemma 15 in [12], this implies that there exists a diffeomorphism of \((W, \partial W)\), relative isotopic to the identity, which induces \(\tau_\gamma\) on the embedded \(\Sigma\). Note that every Lickorish generator curve has a Hopf annulus neighborhood under the embedding \(\phi\). Therefore, the claim of flexibility follows by successive application of ambient relative isotopies of \((W, \partial W)\) inducing Dehn twists along Lickorish generators. See [12] for details.

We now show that the embedding is in a standard position. By construction, any simple closed curve on \(\phi(\Sigma)\) can be isotoped on the surface \(\phi(\Sigma)\) so that it is contained in \(\phi(\Sigma) \cap S^3 \times \{\frac{3}{2}\}\). We claim that any Lickorish generator on \(\phi(\Sigma)\) as well as any curve which does not meet any handle of \(\phi(\Sigma)\), satisfy both the properties necessary for an embedding to be in standard position. The reasons are the following.

1. All curves mentioned in the claim are unknots in \(S^3 \times \{\frac{3}{2}\}\). Therefore, they bound a disk in \(S^3 \times [1, \frac{3}{2}]\), that meets \(\phi(\Sigma)\) only in the given curve.
2. Any curve \(\gamma\) as in the claim admits a neighborhood \(N(C)\) in \(\phi(\Sigma)\) which is a Hopf band in \(S^3 \times \{\frac{3}{2}\}\).

It follows from both the properties listed above that any curve \(C\), which is either a Lickorish generator or does not meet any handle of \(\Sigma\), satisfies both the properties necessary for a surface to be in standard position.

By Proposition 2.2, given any curve \(C\), there exists a boundary preserving diffeomorphism of \(\phi(\Sigma)\) which sends \(C\) to a curve which does not meet any handle. Since the embedding \(\phi\) of \((\Sigma, \partial \Sigma)\) is flexible in \((W, \partial W)\), given any curve \(C\) that meets some handles can be isotoped so that it does not meet any handle. Hence, the embedding can be assumed to be in a standard position.

We recall the notion of a Lefschetz fibration embeddings from [7]. The map \(\pi_2: N \times \mathbb{CP}^1 \to \mathbb{CP}^1\) corresponds to projection on the second factor.

**Definition 2.8** (Lefschetz fibration embedding). Let \((M, \pi: M \to \Sigma)\) be a Lefschetz fibration, where \(\Sigma\) is 2–disk or \(\mathbb{CP}^1\). An embedding \(f: M \to N \times \mathbb{CP}^1\) of a manifold \(M\) into a manifold...
$N \times CP^1$ is said to be a Lefschetz fibration embedding provided $\pi_2 \circ f = i \circ \pi$, where $i$ is an inclusion of $D^2$ in $CP^1$ when $\partial M \neq \emptyset$, otherwise it is the identity.

**Theorem 2.9** (Ghanwat–Pancholi, [7]). Let $M$ be an orientable smooth 4–manifold. Let $N$ be a 4–manifold which admits a separable Hopf link. If $\pi : M \rightarrow \Sigma$, where $\Sigma$ is either $CP^1$ or a 2–disk $D^2$, is a Lefschetz fibration of $M$ having genus $g$ closed surfaces as fibers with $g \geq 1$, then there exists a Lefschetz fibration embedding of $(M, \pi)$ in $(N \times CP^1, \pi_2)$.

We now state the definition of relative Lefschetz fibration embedding.

**Definition 2.10** (Relative LF embedding). Let $(V^4, \pi : V \rightarrow D^2)$ be an ALF. A proper embedding $f_r : (V, \partial V) \rightarrow (N \times D^2, \partial (N \times D^2))$ is said to be a relative Lefschetz fibration embedding, if $\pi_2 \circ f_r = \pi$.

2.6. Spin structures on ALF with fiber closed surfaces. A smooth manifold $M$ is called spin if its second Stiefel-Whitney class $w_2(M)$ vanishes. Let $\Sigma_g$ denote a closed oriented surface of genus $g$. Suppose that the homology classes of the vanishing cycles of $LF(\Sigma_g, \phi)$ are denoted by $v_1, \cdots, v_k \in H_1(\Sigma_g; \mathbb{Z}_2)$. Stipsicz gave the following criteria for achiral Lefschetz fibrations with fiber $\Sigma_g$, not to be spin.

**Theorem 2.11** (Stipsicz). A Lefschetz fibration $\pi : X \rightarrow D^2$ with fiber $\Sigma_g$, is not spin if and only if there exist vanishing cycles $v_1, \cdots, v_k$ such that their sum $v = \sum_{i=1}^k v_i$ also corresponds to a vanishing cycle, and $k + \sum_{1 \leq i < j \leq k} v_i \cdot v_j \equiv 0 \pmod{2}$.

Here, $v_i \cdot v_j$ denotes the algebraic intersection number between $v_i$ and $v_j$.

3. Existence of codimension 2 proper embedding of $LF(\Sigma, \phi)$

To prove **Theorem 1.1** we need the following Lemma.

**Lemma 3.1.** Let $\pi : V^4 \rightarrow D^2$ be a Lefschetz fibration. Let $(W^4, \partial W^4)$ be a 4–manifold which admits a separable Hopf link. There exists a relative LF embedding of $(V, \pi)$ in $(W \times D^2, \pi_2)$.

**Lemma 3.1** can be seen as a relative version of **Theorem 2.9** and the main arguments behind their proofs are essentially the same. We mostly follow the proof of **Theorem 2.9** (Theorem 4.8, [7]) and refer to [7] for further details.

**Proof of Lemma 3.1.** Let $c_1, c_2, \cdots, c_k$ be the $k$ critical points of the Lefschetz fibration $(V, \pi)$ and let $p_1, p_2, \cdots, p_k$ be their images in $D^2$ under the map $\pi$. Let $\gamma_i$ be the vanishing cycle corresponding to the critical value $p_i$ on a generic fiber $\Sigma$ of the LF. Let $(U_i, z_1, z_2)$ be a complex co-ordinate on a disk neighborhood of $c_i$ in $V^4$ such that $\pi$ is given by $(z_1, z_2) \mapsto z_1 z_2$. Let $D_i = \pi(U_i) \subset D^2$. and let $D_i$ be an open disk containing $p_i$ with $\overline{D_i} \subset \overline{D_i}$ for $i = 1, \cdots, k$. The $\overline{D_i}$s are disjoint.

By **Lemma 2.7** we can take a proper embedding $i$ of $(\Sigma, \partial \Sigma)$ in $(W^4, \partial W^4)$ which is both flexible and is in standard position. Using the flexibility of the embedding $i$, we construct an embedding $\tilde{f}$ of $V \setminus \bigsqcup_{i=1}^k \pi^{-1}(D_i)$ in $W \times (D^2 \setminus \bigsqcup_{i=1}^k D_i)$ such that the following diagram commutes:

\[
\begin{array}{ccc}
V \setminus \bigsqcup_{i=1}^k \pi^{-1}(D_i) & \xrightarrow{\tilde{f}} & W \times (D^2 \setminus \bigsqcup_{i=1}^k D_i) \\
\downarrow \pi & & \downarrow \pi_2 \\
D^2 \setminus \bigsqcup_{i=1}^k D_i & \xrightarrow{Id} & D^2 \setminus \bigsqcup_{i=1}^k D_i.
\end{array}
\]
Since the embedding $\iota$ of $(\Sigma, \partial \Sigma)$ in $(W, \partial W)$ is standard, by Lemma 2.7, there exists an embedding $\Psi$ of the mapping torus $MT(\Sigma, \psi)$ in $M \times S^1$, for all $\psi \in MCG(\Sigma, \partial \Sigma)$, such that the following diagram commutes:

\[
\begin{array}{ccc}
MT(\Sigma, \psi) & \xrightarrow{\Psi} & W \times S^1 \\
\downarrow \pi & & \downarrow \\
S^1 & \xrightarrow{Id} & S^1.
\end{array}
\]

Considering $\partial D_i = S^1$, diagram (2) implies that there is an embedding of $MT(\Sigma, \tau_{\gamma_i})$ in $W \times \partial D_i$, where $\tau_{\gamma_i}$ denotes the Dehn twist along the vanishing cycle $\gamma_i$. Now take arcs connecting a point on $\partial D_i$ to a fixed regular value point $p \in D^2$ of the map $\pi$ as depicted in Figure 4.

Figure 4. The figure depicts a particular case of a Lefschetz fibration $(V, \pi)$ over a disk with 4 critical points, embedded as Lefschetz fibration in the $W \times D^2$. The embedding is such that a generic fiber of $(V, \pi)$ has a flexible embedding in standard position in $W$. The curve $\gamma$ depicts a vanishing.

Note that the Lefschetz fibration $(V, \pi)$ restricted to a regular neighborhood $N$ of $D_i$’s together with arcs connecting them satisfies the following:

1. $\pi^{-1}(\partial D_i)$ is the mapping torus $MT(\Sigma, \tau_{\gamma_i})$.
2. $V$ restricted to $\partial N$ is the mapping torus $MT(\Sigma, \tau_{\gamma_1} \circ \tau_{\gamma_2} \circ \cdots \circ \tau_{\gamma_k})$. 

Thus, we get the required embedding $\hat{f}$ such that diagram (1) commutes.

Next we show how to extend this embedding to produce a relative LF embedding $f$ of $(V, \partial V)$ in $W \times D^2$. For this we need to use the fact that the embedding $\iota$ of $(\Sigma, \partial \Sigma)$ is in standard position. In particular, there exists an embedding $\iota_\gamma : \mathbb{C}^2 \rightarrow W$ which satisfies the second property listed in Definition 2.4. Moreover, for each critical point $c_i$, the following commutative diagram holds:

\[
\begin{array}{c}
U_i \subset V \xrightarrow{\phi_i} \mathbb{C}^2 \xrightarrow{i} \mathbb{C}^2 \times \mathbb{C} \xrightarrow{f_{c_i}} W \times D^2 \\
\downarrow \phi \downarrow \downarrow \downarrow \\
\bar{D}_i \xrightarrow{\phi} \mathbb{C} \xrightarrow{id} \mathbb{C} \xrightarrow{\phi^{-1}} \bar{D}_i,
\end{array}
\]

where the definitions of the maps appearing in the diagram are as follows:

1. $\phi_i : U_i \subset V \rightarrow \mathbb{C}^2$ and $\phi : \bar{D}_i \subset D^2 \rightarrow \mathbb{C}$ are orientation preserving parameterizations around critical point $c_i$ of $\pi$ and $\pi(c_i)$ respectively such that left square commutes in the diagram above,
2. $i : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \times \mathbb{C}$ and $g : \mathbb{C}^2 \rightarrow \mathbb{C}$ are defined as $i(z_1, z_2) = (z_1, z_2, 0)$ and $g(z_1, z_2) = z_1 \cdot z_2$,
3. $f_{c_i} : \mathbb{C}^2 \times \mathbb{C} \rightarrow D^2$ and $P : \mathbb{C}^2 \times \mathbb{C} \rightarrow \mathbb{C}$ are defined as $f_{c_i}(z_1, z_2, z_3) = (\iota_\gamma(z_1, z_2), \phi^{-1}(z_1 \cdot z_2 + z_3))$ and $P(z_1, z_2, z_3) = z_1 \cdot z_2 + z_3$.

The commutativity of the middle square follows from definitions of maps $g, i$ and $P$ and the commutativity of the last square follows from the definition of the map $f_{c_i}$. Note that the commutative diagram allows us to extend the embedding $\hat{f}$ to the embedding $\hat{f}_{c_i}$ of $M_{c_i} = V \setminus (\cup_{i=1}^{k} \pi^{-1}(D_i) \cup U_i)$, because $f$ and $f_{c_i} \circ i \circ \phi_i$ agree on the overlapping region of the domain. Hence, $\hat{f}$ and $f_{c_i} \circ i \circ \phi_i$ together defines a map $\hat{f}_{c_i}$. Moreover, $\hat{f}_{c_i}$ satisfies the following commutative diagram:

\[
\begin{array}{c}
M_{c_i} \xrightarrow{\hat{f}_{c_i}} \hat{f}_{c_i}(M_{c_i}) \subset W \times D^2 \\
\downarrow \pi \\
\pi(M_{c_i}) \subset D^2 \xrightarrow{id} \pi_2(\hat{f}_{c_i}) = \pi(M_{c_i}).
\end{array}
\]

Finally, we observe that by construction the embeddings $\hat{f}_{c_i}$ and $\hat{f}_{c_j}$ agree on $M_{c_i} \cap M_{c_j}$. Since $V = \cup_{i=1}^{k} M_{c_i}$, we get the required proper embedding $f$ of $V$ in $W \times D^2$.

Proof of Theorem 1.1. Let $W = S^2 \times S^2 \setminus D^4$. Note that $W$ contains a separable Hopf link. The proof then follows from Lemma 3.1.

4. Every orientable 4-manifold embeds in $\mathbb{R}^7$

The following result due to Etnyre–Fuller [11](Proposition 12, [1]) is our key ingredient.

**Theorem 4.1** (Etnyre–Fuller). Let $X^4$ be a closed orientable 4–manifold. Then we may write $X = Y_1 \cup Y_2$, where $Y_1$ is a 2–handlebody which admits an achiral Lefschetz fibration over $D^2$ with bounded fibers of the same genus, and with the induced open books on $\partial Y_1 = -\partial Y_2$ coinciding.

Proof of Corollary 1.2. Let $X^4$ be a closed oriented 4–manifold. By Theorem 4.1, there exist two achiral Lefschetz fibrations $Y_i$ and $Y_2$ such that $\partial Y_1 = -\partial Y_2$ and $X^4 = Y_1 \cup Y_2$. By Theorem 1.1 $(Y_i, \partial Y_i)$ admits relative LF embedding in $(S^2 \times S^2 \setminus D^4) \times D^2 \subset S^2 \times S^2 \times D^2$ for $i = 1, 2$. 
Therefore, \( X^4 = Y_1 \cup Y_2 \) admits embedding in \( S^2 \times S^2 \times D^2 \cup_{\partial} S^2 \times S^2 \times D^2 = S^2 \times S^2 \times S^2 \). Since \( S^2 \times S^2 \times S^2 \subseteq \mathbb{R}^7 \), \( X^4 \) embeds in \( \mathbb{R}^7 \).

\[
\begin{align*}
5. \textbf{Relative LF embedding of } LF(\Sigma_{g,1}, \phi) \text{ in } D^6
\end{align*}
\]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{The embedding \( f_0 \) for \( g = 2 \).}
\end{figure}

\textit{Proof of Theorem 1.3.} Recall the Whitney sum formula for direct sum of vector bundles: \( w_k(E_1 \oplus E_2) = \sum_{i+j=k} w_i(E_1) \cup w_j(E_2) \).

1. If \( V^4 \) embeds in \( D^6 \) via \( h \), then the normal bundle of \( h \) is a trivial disk bundle over \( V \). Therefore, \( TV \oplus \epsilon_V^2 = h^*(TD^6) \). Thus, \( w_2(V) = 0 \).

2. Note that \( \Sigma_{g,1} \) is homeomorphic to plumbing of \( 2g \) Hopf annuli in a chain, as in Figure 5. Let us denote this embedding of \( \Sigma_{g,1} \) in \( S^3 \) by \( f_0 \). We can modify this embedding using the notations used in the proof of Lemma 3.1. We embed \( \Sigma_{g,1} \) by \( f_0 \) in \( S^3 \times S^3 \subset S^3 \times [1, 2] \subset D^4 \) and attach the cylinder \( \partial \Sigma_{g,1} \times [\frac{3}{2}, 2] \) to get a proper embedding \( \tilde{f}_0 \) of \( (\Sigma_{g,1}, \partial \Sigma_{g,1}) \) in \( (D^4, \partial D^4) \). We claim that \( \tilde{f}_0 \) is flexible and is in standard position.

The claim is true for the following reason. In the proof of Lemma 2.7, the existence of separable Hopf link was assumed so that every curve that is either a Lickorish generator or does not intersect a handle, should have a neighborhood isotopic to the Hopf annulus in \( S^3 \). Here, we already have a monodromy that factors into Dehn twists along the curves \( \{b_1, a_1, c_1, \cdots, a_g, b_g\} \), and each of these curves has a Hopf annulus neighborhood under \( \tilde{f}_0 \). Thus, \( \tilde{f}_0 \) is both flexible and in standard position. One can then proceed exactly as in the proof of Lemma 3.1 to conclude.

\[
\begin{align*}
5.1. \textbf{Obstruction to LF embedding in } D^6
\end{align*}
\]

The criteria of Stipsciz for non-spin ALF with closed fibers gives obstructions to existence of relative LF embeddings in \( D^6 \). For example, let \( V^4 = LF(\Sigma_{g,1}, \phi) \) be an achiral Lefschetz fibrations that has a separable curve as vanishing cycle. The double of \( \Sigma_{g,1} \) is a closed surface of genus \( 2g \), say \( \Sigma_{2g} \). Let \( \tilde{\phi} \) be the extended monodromy on \( \Sigma_{2g} \) (by taking union). Then by Theorem 2.11, \( LF(\Sigma_{2g}, \tilde{\phi}) \) is not spin. Thus, by statement (1) of Theorem 1.3, \( LF(\Sigma_{2g}, \tilde{\phi}) \) cannot embed in \( \mathbb{R}^6 \). But if \( LF(\Sigma_{g,1}, \phi) \) properly embeds in \( D^6 \), then \( LF(\Sigma_{2g}, \tilde{\phi}) \) embeds in \( S^6 \). Which implies that \( LF(\Sigma_{2g}, \tilde{\phi}) \) embeds in \( \mathbb{R}^6 \), which is a contradiction. Therefore, \( LF(\Sigma_{g,1}, \phi) \) does not admit a relative LF embedding in \( D^6 \).

Similar examples can be found with achiral Lefschetz fibrations having non-separable vanishing cycles. In particular, refer to Figure 1 and consider \( V^4 = LF(\Sigma_{g,1}, \tau_{b_1} \circ \tau_{c_1} \circ \tau_{b_2}) \). Then taking \( k = 2, v = b_2, v_1 = b_1 \) and \( v_2 = c_1 \) as in Theorem 2.11, we see that \( V^4_0 \) does not admit a relative LF embedding in \( D^6 \).


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