C-(k, ℓ)-SUM-FREE SETS

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Abstract. The Minkowski sum of two subsets $A$ and $B$ of a finite abelian group $G$ is defined as all pairwise sums of elements of $A$ and $B$: $A + B = \{a + b : a \in A, b \in B\}$. The largest size of a $(k, ℓ)$-sum-free set in $G$ has been of interest for many years and in the case $G = \mathbb{Z}/n\mathbb{Z}$ has recently been computed by Bajnok. Motivated by sum-free sets of the torus, Kravitz introduces the noisy Minkowski sum of two sets, which can be thought of as discrete evaluations of these continuous sumsets. That is, given a noise set $C$, the noisy Minkowski sum is defined as $A + C = A + B + C$. We give bounds on the maximum size of a $(k, ℓ)$-sum-free subset of $\mathbb{Z}/n\mathbb{Z}$ under this new sum, for $C$ equal to an arithmetic progression with common difference relatively prime to $n$ and for any two element set $C$.

1. Introduction

Given a finite abelian group $G$ of order $n$, the Minkowski sum of two subsets $A, B \subseteq G$ is the set of pairwise sums, defined to be

$$A + B = \{a + b : a \in A, b \in B\}.$$  

We also use $A - B$ to denote the pairwise differences of elements of $A$ and $B$:

$$A - B = \{a - b : a \in A, b \in B\}.$$  

Extending these definitions, for an integer $k \geq 1$, we define $kA = A + \cdots + A$, where there are $k$ copies of $A$ in the summation.

For integers $k, ℓ \geq 1$, we say that $A$ is $(k, ℓ)$-sum-free if $kA \cap ℓA = \emptyset$. Let $\mu_{k, ℓ}(G)$ denote the largest possible size of a $(k, ℓ)$-sum-free subset, that is,

$$\mu_{k, ℓ}(G) = \max\{|A| : A \subseteq G \text{ is } (k, ℓ)\text{-sum-free}\}.$$  

Note that if $k = ℓ$, then $kA = ℓA$, so we may assume that $k > ℓ$.

When $k = 2$ and $ℓ = 1$, we see that $\mu_{2, 1}(\mathbb{Z}/n\mathbb{Z})$ is simply the maximal size of a normal sum-free set in $\mathbb{Z}/n\mathbb{Z}$. The value of $\mu_{2, 1}(\mathbb{Z}/n\mathbb{Z})$ was first calculated by Diamanda and Yap in 1969.

Theorem 1.1 ([3], Lemma 3). For any positive integer $n$,

$$\mu_{2, 1}(\mathbb{Z}/n\mathbb{Z}) = \max_{d|n} \left\{ \left\lfloor \frac{d - 1}{d} \right\rfloor \cdot \frac{n}{d} \right\}.$$  

In 2003, Hamidoune and Plagne extended this result to $(k, ℓ)$-sum-free sets for which $n$ and $k - ℓ$ are relatively prime:

Theorem 1.2 ([4], Theorem 2.6). If $\gcd(n, k - ℓ) = 1$, then

$$\mu_{k, ℓ}(\mathbb{Z}/n\mathbb{Z}) = \max \left\{ \left\lfloor \frac{d - 1}{k + ℓ} \right\rfloor \cdot \frac{n}{d} \right\}.$$  

In 2018, Bajnok finished the problem of determining $\mu_{k, ℓ}(\mathbb{Z}/n\mathbb{Z})$ by calculating $\mu_{k, ℓ}(\mathbb{Z}/n\mathbb{Z})$, even when $n$ and $k - ℓ$ are not relatively prime.

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Theorem 1.3 ([2], Theorem 6). For positive integers \( n, k, \ell \) with \( k > \ell \),

\[
\mu_{k,\ell}(\mathbb{Z}/n\mathbb{Z}) = \max_{d|n} \left\{ \left\lfloor \frac{d - (\delta - r)}{k + \ell} \right\rfloor \cdot \frac{n}{d} \right\},
\]

where \( \delta = \gcd(n, k - \ell) \), \( f = \left\lfloor \frac{d - (\delta - r)}{k + \ell} \right\rfloor \), and \( r \) is the remainder of \( \ell f \) modulo \( \delta \).

We’ll be more interested in a slightly different version of the Minkowski sum, motivated by sum-free sets of the torus \( \mathbb{T} \). In [6], Kravitz suggests the following problem: Consider the map \( \varphi : \mathcal{P}(\mathbb{Z}/n\mathbb{Z}) \to \mathcal{P}(\mathbb{T}) \) from subsets of \( \mathbb{Z}/n\mathbb{Z} \) to subsets of the torus defined by \( \varphi(A) = \bigcup_{i \in A} \left( \frac{i}{n}, \frac{i + 1}{n} \right) \).

Then, since \( \left( \frac{i}{n}, \frac{i + 1}{n} \right) + \left( \frac{j}{n}, \frac{j + 1}{n} \right) = \left( \frac{j + k}{n}, \frac{j + k + 2}{n} \right) \), we have that \( \varphi(A) + \varphi(B) = \varphi(A + B + \{0, 1\}) \).

That is, the normal Minkowski sum of the union of certain intervals in the torus corresponds to a new kind of sum of subsets of \( \mathbb{Z}/n\mathbb{Z} \).

With this motivation, Kravitz defines what we will call the noisy Minkowski sum of two sets with a set \( C \): given sets \( A, B, C \subseteq G \), let

\[
A +_C B = A + B + C = \{a + b + c \mid a \in A, b \in B, c \in C\}.
\]

For a given set \( C \), this operation can be understood as taking the normal Minkowski sum and adding some noise given by \( C \). Then, define \( k *_C A = kA + (k - 1)C \). Note that when \( C = \{0\} \), we recover the normal Minkowski sum.

The quantity we are interested in is

\[
\mu_{k,\ell}^C(G) = \max \{|A| \mid A \subseteq G, \ k *_C A \bigcap \ell *_C A = \emptyset\},
\]

the maximum size of a \((k, \ell)\)-sum-free set of \( G \).

In his paper, Kravitz asks about the value of \( \mu_{k,\ell}^{\{0,1\}}(\mathbb{Z}/n\mathbb{Z}) \). Note that maximal \( \{0,1\}\)-(\(k, \ell\))-sum-free subsets of \( \mathbb{Z}/n\mathbb{Z} \) correspond to maximal \((k, \ell)\)-sum-free subsets of \( \mathbb{T} \), restricting to sets of the form \( \bigcup_{i \in I} \left( \frac{i}{n}, \frac{i + 1}{n} \right) \). Thus, calculating \( \mu_{k,\ell}^{\{0,1\}}(\mathbb{Z}/n\mathbb{Z}) \) answers a less granular version of the largest \((k, \ell)\)-sum-free set in the torus.

In this paper, we address Kravitz’s question about \( \mu_{k,\ell}^{\{0,1\}}(\mathbb{Z}/n\mathbb{Z}) \) as well as give bounds on \( \mu_{k,\ell}^C(\mathbb{Z}/n\mathbb{Z}) \) for some other values of \( C \). In particular, for \( C = \{0,1, \ldots, c - 1\} \), we prove the following bounds:

**Theorem 1.4.** For \( c \geq 2 \) and \( C = \{0,1, \ldots, c - 1\} \), we have

\[
\left\lfloor \frac{n + 2(c - 2) - r}{k + \ell} \right\rfloor - (c - 2) \leq \mu_{k,\ell}^C(\mathbb{Z}/n\mathbb{Z}) \leq \left\lfloor \frac{n + 2(c - 2)}{k + \ell} \right\rfloor - (c - 2),
\]

where \( \delta = \gcd(n, k - \ell) \), and \( r \) is the remainder of \(-k \cdot \left\lfloor \frac{n + 2(c - 2)}{k + \ell} \right\rfloor + (c - 2) \) modulo \( \delta \).

Note that when \( \gcd(n, k - \ell) = 1 \), \( r < \delta = 1 \), so \( r = 0 \), and the two sides of the bound are equal. In this case (and in many others), by taking \( c = 2 \), this theorem gives an explicit answer to Kravitz’s question. When the two sides are not equal, we have that \( r < \delta = \gcd(n, k - \ell) \leq k - \ell < k + \ell \), so the upper and lower bounds can differ by at most 1.

We also consider two element sets \( C \) of the form \( \{0, s\} \). We prove the following bounds on \( \mu_{k,\ell}^{\{0,s\}}(\mathbb{Z}/n\mathbb{Z}) \):

**Theorem 1.5.** For \( k > \ell \), we have

\[
\mu_{k,\ell}^{\{0,s\}}(\mathbb{Z}/n\mathbb{Z}) \geq \max_{e|d} \left\{ \mu_{k,\ell}(\mathbb{Z}/e\mathbb{Z}) \cdot \frac{n}{e} \right\}, \quad \left\lfloor \frac{n + 2(s - 1) - r}{k + \ell} \right\rfloor - (s - 1)
\]

\[
\mu_{k,\ell}^{\{0,s\}}(\mathbb{Z}/n\mathbb{Z}) \leq \max_{e|d} \left\{ \mu_{k,\ell}(\mathbb{Z}/e\mathbb{Z}) \cdot \frac{n}{e} \right\}, \quad \left\lfloor \frac{n}{k + \ell} \right\rfloor,
\]

where \( \delta = \gcd(n, k - \ell) \), \( f = \left\lfloor \frac{n + 2(s - 1) - r}{k + \ell} \right\rfloor \), and \( r \) is the remainder of \(-k \cdot \left\lfloor \frac{n}{k + \ell} \right\rfloor \) modulo \( \delta \).
where \( \delta = \gcd(n, k - \ell) \) and \( r \) is the remainder of \(-k \cdot \left\lfloor \frac{n + 2(p - 1)}{k + \ell} \right\rfloor + (s - 1) \mod \delta \).

When \( \max_{e|d} \left\{ \mu_{k, \ell}(\mathbb{Z}/e\mathbb{Z}) \cdot \frac{d}{e} \right\} \geq \left\lfloor \frac{n}{k + \ell} \right\rfloor \), we have that the two sides are in fact equal, i.e.

\[
\mu^0_{k, \ell}(\mathbb{Z}/n\mathbb{Z}) = \max_{e|d} \left\{ \mu_{k, \ell}(\mathbb{Z}/e\mathbb{Z}) \cdot \frac{n}{e} \right\}.
\]

Another specific case is when \( s = p \) is prime and \( p \) divides \( k - \ell \). Then \( \mu_{k, \ell}(\mathbb{Z}/p\mathbb{Z}) = 0 \), so our inequality simplifies to

\[
\left\lfloor \frac{n + 2(p - 1) - r}{k + \ell} \right\rfloor - (p - 1) \leq \mu^0_{k, \ell}(\mathbb{Z}/n\mathbb{Z}) \leq \left\lfloor \frac{n}{k + \ell} \right\rfloor.
\]

We will see in Section 2 that the bounds we've achieved for \( C = \{0, 1, \ldots, c-1\} \) extend to any set that is an arithmetic progression of length \( c \) with common difference relatively prime to \( n \), and the bounds for \( C = \{0, s\} \) hold for any two element set. In Section 3, we prove Theorem 1.4, and in Section 4, we prove Theorem 1.5. Finally in Section 5, we discuss some open questions and conjectures.

2. Transformations of \( C \)

As there are many choices for \( C \), we may seek to show that several sets are equivalent in the sense that they all have the same size of a maximal \( C-(k, \ell) \)-sum-free set. In this section we introduce two transformations of \( C \) that preserve \( \mu_{k, \ell}^0(\mathbb{Z}/n\mathbb{Z}) \).

**Proposition 2.1.** For any \( g \in \mathbb{Z}/n\mathbb{Z} \), we have that \( \mu_{k, \ell}^C(\mathbb{Z}/n\mathbb{Z}) = \mu_{k, \ell}^{C+g}(\mathbb{Z}/n\mathbb{Z}) \).

*Proof.* We have that

\[
k \ast_C A = kA + (k-1)C \\
= kA + (k-1)(C+\{g\}) + k\{-g\} + \{g\} \\
= k(A + \{-g\} + (k-1)(C+\{g\}) + \{g\} \\
= k \ast_{C+\{g\}} \{A + \{-g\}\} + \{g\},
\]

and similarly, \( \ell \ast_C A = \ell \ast_{C+\{g\}} \{A + \{-g\}\} + \{g\} \), therefore letting \( B = A + \{-g\} \), we have that

\[
k \ast_C A = \ell \ast_C A \iff k \ast_{C+\{g\}} B = \ell \ast_{C+\{g\}} B.
\]

Thus, the proposition follows. \( \square \)

**Proposition 2.2.** For any \( g \in \mathbb{Z}/n\mathbb{Z}^\times \) and set \( A \), let \( A/g = \{a/g : a \in A\} \). Then, \( \mu_{k, \ell}^C(\mathbb{Z}/n\mathbb{Z}) = \mu_{k, \ell}^{C/g}(\mathbb{Z}/n\mathbb{Z}) \).

*Proof.* For any set \( A \subseteq \mathbb{Z}/n\mathbb{Z} \), we have that \( k \ast_C A \cap \ell \ast_C A = \emptyset \iff (k \ast_C A)/g \cap (\ell \ast_C A)/g = \emptyset \), or equivalently, \( k \ast_{C/g} (A/g) \cap \ell \ast_{C/g} (A/g) = \emptyset \). \( \square \)

We will call the operation in Proposition 2.1 *shift* and the operation in Proposition 2.2 *multiplication*, as these are the respective operations the propositions allow us to perform on \( C \). We say that two sets \( C \) and \( D \) are *shift-mult-equivalent* if, by applying a sequence of shifts and multiplications to \( C \), one can attain \( D \). Note since shifts and multiplications are invertible and composable, such sequences define an equivalence relation.

In fact, two sets \( C \) and \( D \) are shift-mult-equivalent iff \( D = g(C+\{h\}) \) for some elements \( h \in \mathbb{Z}/n\mathbb{Z} \) and \( g \in \mathbb{Z}/n\mathbb{Z}^\times \). As a result, \( \{0,1,\ldots,c-1\} \) is shift-mult-equivalent to any length \( c \) arithmetic progression with common difference relatively prime to \( n \), and \( \{0,c\} \) is shift-mult-equivalent to any two element set whose elements have difference \( \Delta \) such that \( \gcd(n, \Delta) = \gcd(n, c) \).
3. Maximal \( \{0, 1, \ldots, c - 1\}\)-(\(k, \ell\))-Sum-Free Sets

In this section we look at \( C = \{0, 1, \ldots, c - 1\} \), with \( c \geq 1 \). Note that when \( c = 0 \), \( C = \{0\} \), which has already been thoroughly investigated. Hence, we consider \( c \geq 2 \).

When \( c = 2 \), we have that \( C = \{0, 1\} \). Note that shifting and multiplying gives us that any set of two elements whose difference is relatively prime to \( n \) is shift-mult-equivalent to \( C \). Using \( (k, \ell) \)-sum-free sets on the torus, we may give a natural upper bound for \( \mu_{k, \ell}^{(0,1)}(\mathbb{Z}/n\mathbb{Z}) \). In order to talk about sets of the torus \( T \), let \( \mu^* \) denote the normalized Haar measure on \( T \) and let \( \mu_{k, \ell}^*(T) = \max\{\mu^*(A) \mid A \subseteq T, kA \cap \ell A = \emptyset\} \). Kravitz proved the following equality for maximal sum-free sets of \( T \):

**Theorem 3.1** ([6], Theorem 1.3). For \( k > \ell \), it holds that \( \mu_{k, \ell}^*(T) = \frac{1}{k + \ell} \).

Using the map \( \phi : \mathcal{P}(\mathbb{Z}/n\mathbb{Z}) \to \mathcal{P}(T) \) we defined in the introduction, we can prove the following upper bound:

**Theorem 3.2.** For \( n, k, \ell \in \mathbb{N} \) with \( k > \ell \), we have that \( \mu_{k, \ell}^{(0,1)} \leq \left\lfloor \frac{n}{k + \ell} \right\rfloor \).

**Proof.** Recall that \( \phi \) is defined on sets \( A \subseteq \mathbb{Z}/n\mathbb{Z} \) by \( \phi(A) = \bigcup_{i \in A} \left( \frac{i}{n}, \frac{i + 1}{n} \right) \) and that \( \phi(A + \{0,1\}B) = \phi(A) + \phi(B) \). Therefore, \( A \) is \( \{0, 1\}\)-(\(k, \ell\))-sum-free iff \( \phi(A) \) is sum-free. Then,

\[
\mu_{k, \ell}^{(0,1)}(\mathbb{Z}/n\mathbb{Z}) \leq n \cdot \mu_{k, \ell}^*(T) = \frac{n}{k + \ell}.
\]

Since \( \mu_{k, \ell}^{(0,1)}(\mathbb{Z}/n\mathbb{Z}) \) is an integer, we in fact have that

\[
\mu_{k, \ell}^{(0,1)}(\mathbb{Z}/n\mathbb{Z}) \leq \left\lfloor \frac{n}{k + \ell} \right\rfloor.
\]

This bound is sharp: for instance, for \( n = 10, k = 2, \) and \( \ell = 1 \), we have that \( \{4, 5, 6\} \) is a size \( \left\lfloor \frac{n}{k + \ell} \right\rfloor = 3 \) \( \{0, 1\}\)-(\(k, \ell\))-sum-free set.

In general, we may prove an upper bound on \( \mu_{k, \ell}^{C}(\mathbb{Z}/n\mathbb{Z}) \) with a different approach, which will align with Theorem 3.2 in the case \( C = \{0,1\} \). The upper bound we give is based on the following result of Kneser.

**Theorem 3.3** (Kneser [5]). Let \( G \) be a finite abelian group. For nonempty \( A, B \subseteq G \) and \( H = \text{stab}(A + B) \) the stabilizer of \( A + B \), then

\[
|A + B| \geq |A + H| + |B + H| - |H|.
\]

In order to apply this theorem, we need the following easy lemma.

**Lemma 3.4.** For sets \( A \) and \( B \), \( \text{stab}(A) \subseteq \text{stab}(A + B) \) as a subgroup inclusion.

**Proof.** We have that

\[
\text{stab}(A) + (A + B) = (\text{stab}(A) + A) + B = A + B.
\]

Recall, by definition, that \( A +_C B = A + B + C \), so this lemma also gives that \( \text{stab}(A) \subseteq \text{stab}(A +_C B) \).

Now, we can lower bound \( |A +_C B| \).

**Lemma 3.5.** When \( C = \{0, 1, \ldots, c - 1\} \) for \( c \geq 2 \), we have that

\[
|A +_C B| \geq \min \{n, |A| + |B| + (c - 2)\}.
\]
Proof. Let $K = \text{stab}(A + C B)$. Then, by Kneser’s result, we have

$$|A + C B| = |A + B + C| \geq |A + B + K| + |\{0, \ldots, c - 1\} + K| - |K|$$

$$\geq |A + B| + |\{0, \ldots, c - 1\} + K| - |K|. \quad (1)$$

If $K = \mathbb{Z}/n\mathbb{Z}$, then this means that $A + C B = \mathbb{Z}/n\mathbb{Z}$ since every element of $\mathbb{Z}/n\mathbb{Z}$ stabilizes $A + C B$. Then, $|A + C B| = n \geq \min\{n, |A| + |B| + c - 2\}$. So, assume $K \neq \mathbb{Z}/n\mathbb{Z}$. Note that if a subgroup $H$ stabilizes a set $X$, we must have that for any $x \in X$, $x + H \subseteq X$, so $X$ is a union of cosets of $H$. Then, if $[\mathbb{Z}/n\mathbb{Z} : K] \leq c$, we have that $A + C B$ is a union of cosets of $K$. However, if $a \in A$ and $b \in B$, then $(a + b, a + b + 1, \ldots, a + b + c - 1) \subseteq A + C B$, so at least one element of each coset of $K$ is in $A + C B$. This implies that $A + C B = \mathbb{Z}/n\mathbb{Z}$, which we’ve assumed is not true.

Now, if $[\mathbb{Z}/n\mathbb{Z} : K] > c$, we have that $|\{0, \ldots, c - 1\} + K| = c|K|$, so Equation 1 can be rewritten as

$$|A + C B| \geq |A + B| + (c - 1)|K|.$$ 

Now, let $H = \text{stab}(A + B)$, so the above equation implies that

$$|A + C B| \geq |A + H| + |B + H| + (c - 1)|K| - |H|$$

$$\geq |A| + |B| + (c - 1)|K| - |H|.$$ 

By Lemma 3.4, $H$ is a subgroup of $K$. In particular, $|H| \leq |K|$. Then, we can write

$$|A + C B| \geq |A| + |B| + (c - 2)|H| \geq |A| + |B| + (c - 2),$$

as desired. \qed

With this result, we can show an upper bound on $\mu^C_{k, \ell}(\mathbb{Z}/n\mathbb{Z})$.

**Theorem 3.6.** We have that $\mu^C_{k, \ell}(\mathbb{Z}/n\mathbb{Z}) \leq \left\lfloor \frac{n+2(c-2)}{k+\ell} \right\rfloor - (c - 2)$.

**Proof.** Take $A$ to be a $C$-$(k, \ell)$-sum-free subset of $\mathbb{Z}/n\mathbb{Z}$. By iteratively applying Lemma 3.5, we find that

$$|k \ast_C A| \geq \min\{n, k|A| + (k - 1)(c - 2)\}$$

$$|\ell \ast_C A| \geq \min\{n, \ell|A| + (\ell - 1)(c - 2)\}.$$ 

Since both $k \ast_C A$ and $\ell \ast_C A$ are nonempty, we must have that $|k \ast_C A|, |\ell \ast_C A| < n$, so

$$|k \ast_C A| \geq k|A| + (k - 1)(c - 2)$$

$$|\ell \ast_C A| \geq \ell|A| + (\ell - 1)(c - 2).$$

Then since $k \ast_C A$ and $\ell \ast_C A$ are disjoint,

$$n \geq |k \ast_C A| + |\ell \ast_C A| \geq (k + \ell)|A| + (k + \ell - 2)(c - 2).$$

Rearranging gives

$$|A| \leq \frac{n + 2(c - 2)}{k + \ell} - (c - 2).$$

Since $|A|$ must be an integer, we must have

$$|A| \leq \left\lfloor \frac{n + 2(c - 2)}{k + \ell} \right\rfloor - (c - 2).$$

We denote this upper bound by $\chi(c, k, \ell) = \left\lfloor \frac{n + 2(c - 2)}{k + \ell} \right\rfloor - (c - 2)$. We will see in the lower bound that this upper bound is achieved or almost achieved by an interval. The analysis of the largest length of a $C$-$(k, \ell)$-sum-free interval will closely follow the methods used by Bajnok in calculating the largest $(k, \ell)$-sum-free sets in $[2]$. 

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\textit{C-$(k, \ell)$-SUM-FREE SETS}
Theorem 3.7. Let $\delta = \gcd(n, k - \ell)$ and $r$ denote the remainder of $-k \cdot \chi(c, k, \ell) - (k - 1)(c - 2)$ modulo $\delta$. Then,

$$
\mu_{k, \ell}(\mathbb{Z}/n\mathbb{Z}) \geq \left\lfloor \frac{n + 2(c - 2) - r}{k + \ell} \right\rfloor - (c - 2).
$$

Proof. We have that an interval $A = [a, \ldots, a + m - 1]$ of length $m$ satisfies

$$
k \ast_{C} A = \{ka, \ldots, ka + km + (c - 2)k - (c - 1)\},$$

$$
\ell \ast_{C} A = \{\ell a, \ldots, \ell a + \ell m + (c - 2)\ell - (c - 1)\},
$$

so

$$
k \ast_{C} A - \ell \ast_{C} A = \{(k - \ell)a - \ell m - (c - 2)\ell + (c - 1), \ldots, (k - \ell)a + km + (c - 2)k - (c - 1)\}.$$

We have that $A$ is $C-(k, \ell)$-sum-free iff $k \ast_{C} A - \ell \ast_{C} A$ does not contain $0$, so $A$ is $C-(k, \ell)$-sum-free iff there exists some $b \in \mathbb{Z}$ for which

$$
bn + 1 \leq (k - \ell)a - \ell m - (c - 2)\ell + (c - 1)$$

$$(b + 1)n - 1 \geq (k - \ell)a + km + (c - 2)k - (c - 1),$$

which can be rearranged to give

$$
\ell m + (\ell - 1)(c - 2) \leq (k - \ell)a - bn \leq n - km - (k - 1)(c - 2)
$$

$$
\iff \frac{\ell m + (\ell - 1)(c - 2)}{\delta} \leq \frac{k - \ell \cdot a - \frac{n}{\delta} \cdot b}{\delta} \leq \frac{n - km - (k - 1)(c - 2)}{\delta}.
$$

Since $\gcd\left(\frac{k - \ell}{\delta}, \frac{n}{\delta}\right) = 1$, any integer can be expressed as $\frac{k - \ell}{\delta} \cdot a - \frac{n}{\delta} \cdot b$ for some choice of $a$ and $b$. Thus, it suffices to show that there exists some integer $z$ for which

$$
\frac{\ell m + (\ell - 1)(c - 2)}{\delta} \leq z \leq \frac{n - km - (k - 1)(c - 2)}{\delta},
$$

or equivalently,

$$
\frac{\ell m + (\ell - 1)(c - 2)}{\delta} \leq \left\lfloor \frac{n - km - (k - 1)(c - 2)}{\delta} \right\rfloor.
$$

Since $\left\lfloor \frac{n}{\delta} \right\rfloor = \left\lfloor \frac{\ell m - (c - 2)}{\delta} \right\rfloor$, we have that any $m$ that satisfies

$$
\frac{\ell m + (\ell - 1)(c - 2)}{\delta} \leq \frac{n - km - (k - 1)(c - 2) - \delta + 1}{\delta}
$$

must also satisfy 2. But the above can be rewritten as

$$
m \leq \frac{n + 2c - \delta - 3}{k + \ell} - (c - 2),
$$

so there is an interval of size $\left\lfloor \frac{n + 2c - \delta - 3}{k + \ell} \right\rfloor - (c - 2) = \left\lfloor \frac{n + 2(c - 2) - (\delta - 1)}{k + \ell} \right\rfloor - (c - 2)$ that is $C-(k, \ell)$-sum-free. Since $\delta = \gcd(n, k - \ell) < k + \ell$, if we let $\gamma(n, k, \ell, c)$ denote the length of the longest $C-(k, \ell)$-sum-free interval, we have that $\gamma(n, k, \ell, c) \in \{\chi(n, k, \ell, c), \chi(n, k, \ell, c) - 1\}$.

Since $r \equiv -k \cdot \chi(n, k, \ell, c) - (k - 1)(c - 2) \pmod{\delta}$ and $n \equiv 0 \pmod{\delta}$, we have that $\gamma(n, k, \ell, c) = \chi(n, k, \ell, c)$ if

$$
\ell \cdot \chi(n, k, \ell, c) + (\ell - 1)(c - 2) \leq n - k \cdot \chi(n, k, \ell, c) - (k - 1)(c - 2) - r
$$

$$
\iff \chi(n, k, \ell, c) \leq \frac{n + 2(c - 2) - r}{k + \ell} - (c - 2).
$$

Since $\chi(n, k, \ell, c)$ is an integer, $\gamma(n, k, \ell, c) = \chi(n, k, \ell, c)$ exactly when

$$
\chi(n, k, \ell, c) \leq \left\lfloor \frac{n + 2(c - 2) - r}{k + \ell} \right\rfloor - (c - 2).
$$
Note that \( \left\lfloor \frac{n+2(c-2)-r}{k+\ell} \right\rfloor - (c-2) \) takes the value \( \chi(n,k,\ell,c) \) exactly when Equation 3 holds and otherwise takes the value \( \chi(n,k,\ell,c) - 1 \), thus \( \gamma(n,k,\ell,c) = \left\lfloor \frac{n+2(c-2)-r}{k+\ell} \right\rfloor - (c-2) \) is the length of the longest \( C-(k,\ell) \)-sum-free interval in \( \mathbb{Z}/n\mathbb{Z} \). In particular, this implies that

\[
\mu_C^{\{0,s\}}(\mathbb{Z}/n\mathbb{Z}) \geq \left\lfloor \frac{n+2(c-2)-r}{k+\ell} \right\rfloor - (c-2).
\]

\[\square\]

Combining the two bounds, and noting that \( r \) is the remainder of \(-k \cdot \chi(n,k,\ell,c) - (k-1)(c-2) = -k \cdot \left\lfloor \frac{n+2(c-2)-r}{k+\ell} \right\rfloor + (c-2) \) modulo \( \delta \), we have proven Theorem 1.4.

4. C of size 2

We now restrict our attention to \( C \) of size 2. First, by shifting, we can write \( C \) as \( \{0,s\} \). When \( s \) is relatively prime to \( n \), we can multiply \( C \) by \( s^{-1} \) so that \( C = \{0,1\} \), which we have examined in the previous section. Therefore, we now consider sets \( C = \{0,s\} \) for which \( d = \gcd(s,n) \neq 1 \).

We will prove the upper and lower bounds of Theorem 1.5 separately. Recall that \( \gamma(n,k,\ell,s+1) = \left\lfloor \frac{n+2(s-1)-r}{k+\ell} \right\rfloor - (s-1) \), so our lower bound will follow immediately from the following theorem:

**Theorem 4.1.** For any \( k > \ell \),

\[
\mu_C^{\{0,s\}}(\mathbb{Z}/n\mathbb{Z}) \geq \max \left\{ \max_{e \mid d} \left\{ \mu_C^{\{0,s\}}(\mathbb{Z}/e\mathbb{Z}) \cdot \frac{n}{e} \right\}, \gamma(n,k,\ell,s+1) \right\}.
\]

**Proof.** For any \( e \mid d \), define \( \psi_{n,e} \) to be the canonical projection from \( \mathbb{Z}/n\mathbb{Z} \) onto \( \mathbb{Z}/e\mathbb{Z} \). That is, \( \psi_{n,e}(a) \) is equal to the remainder of \( a \) modulo \( e \). Suppose we have a maximal \( (k,\ell) \)-sum-free set \( B \) in \( \mathbb{Z}/e\mathbb{Z} \). Then, \( A = \psi_{n,e}^{-1}(B) \) must also be \( (k,\ell) \)-sum-free and in fact \( \{0,s\}-(k,\ell) \)-sum-free. Since \( A \) has size \( \mu_C^{\{0,s\}}(\mathbb{Z}/e\mathbb{Z}) \cdot \frac{n}{e} \), by taking the maximum value over all \( e \mid d \), we have that

\[
\mu_C^{\{0,s\}}(\mathbb{Z}/n\mathbb{Z}) \geq \max_{e \mid d} \left\{ \mu_C^{\{0,s\}}(\mathbb{Z}/e\mathbb{Z}) \cdot \frac{n}{e} \right\}.
\]

Because \( \{0,s\} \subseteq \{0,1,\ldots,s\} \), any \( \{0,1,\ldots,p\}-(k,\ell) \)-sum-free set is also a \( \{0,p\}-(k,\ell) \)-sum-free set in \( \mathbb{Z}/n\mathbb{Z} \). This gives the second lower bound, that \( \mu_C^{\{0,s\}}(\mathbb{Z}/n\mathbb{Z}) \geq \gamma(n,k,\ell,s+1) \).

\[\square\]

In order to handle the upper bound, we need the following lemma, which deals with the case that the stabilizer of \( k \ast_{\{0,s\}} A \) does not contain \( \langle d \rangle \). By \( \langle x \rangle \), we mean the cyclic subgroup of \( \mathbb{Z}/n\mathbb{Z} \) generated by \( x \).

**Lemma 4.2.** For a \( \{0,s\}-(k,\ell) \)-sum-free set \( A \), let \( K = \text{stab}(k \ast_{\{0,s\}} A) \) and suppose that \( K \) does not contain \( \langle d \rangle \), the subgroup of \( \mathbb{Z}/n\mathbb{Z} \) generated by \( d \), (i.e. \( K \neq \langle e \rangle \) for any \( e \mid d \), \( e > 1 \)). Then

\[
|A| \leq \left\lfloor \frac{n}{k+\ell} \right\rfloor.
\]

**Proof.** Our proof of this lemma follows from the following claim: For all \( 1 \leq j \leq k \), \( |j \ast_{\{0,s\}} A| \geq j|A| \).

We prove this claim via induction. For \( j = 1 \), \( |j \ast_{\{0,s\}} A| = |A| \), so the base case is true. Now for \( 2 \leq j \leq k \), suppose the claim is true for \( j - 1 \). Let \( J = \text{stab}(j \ast_{\{0,s\}} A) \). By Kneser, we have that

\[
|j \ast_{\{0,s\}} A| \geq |A + ((j - 1) \ast_{\{0,s\}} A)| + |\{0,s\} + J| - |J|.
\]
By Lemma 3.4, \( J \) is a subgroup of \( K \), so \( J \) also doesn’t contain \( \langle d \rangle \). Then, we have that \( |\{0, s\} + J| = 2|J| \), so we can rewrite the above equation as

\[
|j * \{0, s\} A| \geq |A + ((j - 1) * \{0, s\} A)| + |J| \\
\geq |A| + |(j - 1) * \{0, s\} A| + |J| - |H| \\
\geq |A| + |(j - 1) * \{0, s\} A| \\
= j|A|,
\]

where \( H = \text{stab}(A + ((j - 1) * \{0, s\} A)) \subseteq J \) by Lemma 3.4, so \( |H| \leq |J| \). This completes the proof of the claim.

To finish our proof, we note that \( |k * \{0, s\} A| \geq k|A| \) and \( |\ell * \{0, s\} A| \geq \ell|A| \). Since \( k * \{0, s\} A \cap \ell * \{0, s\} A = \emptyset \), we have that \( n \geq (k + \ell)|A| \implies |A| \leq \left[ \frac{n}{k + \ell} \right] \).

Now, we are ready to prove the upper bound.

**Theorem 4.3.** For \( k > \ell \),

\[
\mu_{k, \ell}^{|0, s\rangle}(\mathbb{Z}/n\mathbb{Z}) \leq \max \left\{ \max_{e | d} \left\{ \mu_{k, \ell}(\mathbb{Z}/e\mathbb{Z}) \cdot \frac{n}{e} \right\} , \left[ \frac{n}{k + \ell} \right] \right\}.
\]

**Proof.** Suppose that \( A \) is a \( \{0, s\}-\langle k, \ell \rangle \)-sum-free set in \( \mathbb{Z}/n\mathbb{Z} \). Once again define the map \( \psi_{n,e} \) to be the canonical projection of \( \mathbb{Z}/n\mathbb{Z} \) onto \( \mathbb{Z}/e\mathbb{Z} \).

First, suppose that \( K = \text{stab}(k * \{0, s\} A) = \langle e \rangle \) for some \( e | d \), so \( k * \{0, s\} A \) is a union of cosets of \( \langle e \rangle \) but is not equal to \( \mathbb{Z}/n\mathbb{Z} \). If \( |\psi_{n,e}(A)| > \mu_{k,\ell}(\mathbb{Z}/e\mathbb{Z}) \), then since \( \psi_{n,e}(k * \{0, s\} A) = k\psi_{n,e}(A) \) and \( k * \{0, s\} A = \psi_{n,e}^{-1}(\psi_{n,e}(k * \{0, s\} A)) \) is a union of cosets, \( k * \{0, s\} A \) and \( \ell * \{0, s\} A \) have nontrivial intersection, contradiction on \( A \) being \( \{0, s\}-k,\ell \)-sum-free. Therefore, \( |\psi_{n,e}(A)| \leq \mu_{k,\ell}(\mathbb{Z}/e\mathbb{Z}) \). In particular, this gives the bound \( |A| \leq \mu_{k,\ell}(\mathbb{Z}/e\mathbb{Z}) \cdot \frac{n}{e} \). Taking the maximum value over all \( e | d \) gives that \( \mu_{k, \ell}^{|0, s\rangle}(\mathbb{Z}/n\mathbb{Z}) \leq \max_{e | d} \left\{ \mu_{k, \ell}(\mathbb{Z}/e\mathbb{Z}) \cdot \frac{n}{e} \right\} \).

Otherwise, \( K \) does not contain \( \langle d \rangle \). Then, by Lemma 4.2, \( \mu_{k, \ell}^{|0, s\rangle}(\mathbb{Z}/n\mathbb{Z}) \leq \left[ \frac{n}{k + \ell} \right] \). Combining the two cases gives the stated result. \( \square \)

5. Further Questions

When \( C = \{0, 1, \ldots, c - 1\} \), the upper and lower bounds given for \( \mu_{k, \ell}^{C}(\mathbb{Z}/n\mathbb{Z}) \) often coincide. When they do not, the two bounds are 1 apart, and we conjecture that the lower bound holds as equality.

**Conjecture 5.1.** For \( c \geq 2 \) and \( C = \{0, 1, \ldots, c - 1\} \),

\[
\mu_{k, \ell}^{C}(\mathbb{Z}/n\mathbb{Z}) = \max \left\{ 0, \left[ \frac{n + 2(c - 2) - r}{k + \ell} \right] - (c - 2) \right\},
\]

where \( \delta = \gcd(n, k - \ell) \), \( \chi(c, k, \ell) = \left[ \frac{n + 2(c - 2)}{k + \ell} \right] - (c - 2) \) and \( r \) is the remainder of \(-k \cdot \chi(c, k, \ell) \mod (k - 1)(c - 2) \) modulo \( \delta \). That is, the largest \( C \cdot (k, \ell) \)-sum-free subset of \( \mathbb{Z}/n\mathbb{Z} \) is achieved by an interval.

Notice that for small enough values of \( n \), it is possible for \( \left[ \frac{n + 2(c - 2) - r}{k + \ell} \right] - (c - 2) \) to be negative, hence we need to compare it to 0.

Consider, for instance, \( n = 40, k = 9 \), and \( \ell = 4 \). For \( C = \{0, 1\} \), the upper bound in Theorem 1.4 is 3, while the lower bound is 2. A simple computer program verifies that the largest \( \{0, 1\} \cdot (9, 4) \)-sum-free set of \( \mathbb{Z}/40\mathbb{Z} \) is 2. For \( C = \{0, 1, 2\} \), the upper and lower bounds once again differ, being 2 and 1 respectively, and a computer program verifies that the maximum \( \{0, 1, 2\} \cdot (9, 4) \)-sum-free subset of \( \mathbb{Z}/40\mathbb{Z} \) has length 1.
We have checked using a computer that for all \(2 \leq c \leq 10\), there are no values of \(n, k,\) and \(\ell\) with \(\ell < 10, k < 20,\) and \(n < 5(k + \ell)\) for which the upper and lower bounds of Theorem 1.4 differ and \(\mu_{k,\ell}^{\{0,1\}}\) is equal to the upper bound, unless the upper bound is equal to 0.

For \(C = \{0, s\}\), our bounds often are wider. We ask for a precise value of \(\mu_{k,\ell}^C(\mathbb{Z}/n\mathbb{Z})\).

**Question 5.2.** For \(\gcd(n, s) \neq 1\), what is the value of \(\mu_{k,\ell}^{\{0,s\}}(\mathbb{Z}/n\mathbb{Z})\)?

Other values of \(C\) may be interesting for study. Note that if \(C\) is shift-mult-equivalent to a set contained in \(\{0, 1, \ldots, x\}\), then \(A\) is \(C-(k,\ell)\)-sum-free if it is \(\{0, 1, \ldots, x\}-(k,\ell)\)-sum-free, so the lower bound of Theorem 1.4 gives a lower bound on \(\mu_{k,\ell}^C(\mathbb{Z}/n\mathbb{Z})\). An upper bound in some cases, with some careful considerations, can be attained following the methods of this paper, or by noting that if \(\{0, x\}\) is a subset of \(C\), then any \(C-(k,\ell)\)-sum-free set must also be \(\{0, x\}-(k,\ell)\)-sum-free, from which we attain an upper bound of \(\left\lfloor \frac{n}{k+\ell} \right\rfloor\) by Theorem 1.5. However, it is not known how to attain an upper bound better than \(\left\lfloor \frac{n}{k+\ell} \right\rfloor\) in all cases.

**Question 5.3.** What can we say about the value of \(\mu_{k,\ell}^C(\mathbb{Z}/n\mathbb{Z})\) for other values of \(C\)?

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