THE MAPPING OF COMPACT INTO THE SET OF ITS CHEBYSHEV CENTRES IS LIPSCHITZ IN THE SPACE $l^\infty$

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In this article the authors prove strong stability of the set of all Chebyshev centres of the bounded closed subset of the metric space. We endow the set of all compacts of the space $l^\infty$ with Hausdorff metric and prove that the map which puts in correspondence to each compact of $l^\infty$ the set of its Chebyshev centres is Lipshitz.

1. Notation and definitions

We assume the notation as follows.
- $\mathbb{R}_+$ denotes the set of all nonnegative real numbers.
- $\mathbb{N}$ stands for the set of all natural numbers.
- $(X, \rho)$ being metric space $X$ endowed with the metric $\rho$.
- $B(X)$ ($B[X]$) is the set of all nonempty bounded (bounded and closed) subsets of the metric space $X$.
- $K(X)$ stands for the set of all compact subsets of $(X, \rho)$.
- $|xy| = \rho(x, y)$, $|xZ| = \inf\{|ux| : u \in Z\}$, $Mx = \sup\{|ux| : u \in M\}$ for $x, y \in X$, $Z \subset X$, $M \in B(X)$.
- $(l^\infty, \| \cdot \|_\infty)$ — Banach space over the field of real numbers endowed with the norm

$$\| < x_1, \ldots, x_n > \|_\infty = \max\{|x_1|, \ldots, |x_n|\}.$$ 

- $\mathbb{P}(\mathbb{R}^n)$ denotes the set of all parallelepipeds that lie in $k$–planes ($1 \leq k \leq n – 1$) of the space $\mathbb{R}^n$, the edges of the parallelepipeds are assumed to be parallel to coordinate axes.
- $B[x, r]$, $B(x, r)$ and $(S(x, r))$ denote respectively closed, open ball and sphere centered in the point $x \in (X, \rho)$ of the radius $r \geq 0$.
- $\Sigma_N$ — set of all nonempty sub sets of $(X, \rho)$, consisting of no more than $N$ points. Element of the set $\Sigma_N$ is called $N$–net [1].
- $\alpha : B[X] \times B[X] \to \mathbb{R}_+$, $\alpha(M, T) = \max\{\sup\{|xT| : x \in M\}, \sup\{|tM| : t \in T\}\}$ — Hausdorff metric on the set $B[X]$ (pseudometric on the set $B(X)$) ([2], pg. 223).
$S(N)$ is the permutation group of $N$ elements.

$X^N/\sim$ denotes factor-space of the space $X^N$ with respect to the equivalence relation: $(x_1, \ldots, x_N) \sim (y_1, \ldots, y_N)$ in case there exists $\sigma \in S(N)$ such that

$$y_1 = x_{\sigma(1)}, \ldots, y_N = x_{\sigma(N)}.$$

$\hat{\alpha} : X^N/\sim \times X^N/\sim \to \mathbb{R}^+$ is a metric on the set $X^N/\sim$. Now using the bijection $f : X^N/\sim \to \Sigma_N$, $f([x_1, \ldots, x_N]) = \{x_1, \ldots, x_N\}$ we introduce metric $\hat{\alpha}$ also on the space $\Sigma_N$.

$D[M]$ — diameter of the set $M \in \Sigma_N$.

$\overline{M}$ — closure of the set $M \subset X$.

$\omega(x, y)$ is the middle set (assumed to be nonempty) of the interval $[x, y]$ such that

$$\omega(x, y) = \{z \in X : 2|xz| = 2|yz| = |xy|\} \text{ for } x, y \in X.$$

In the Euclidean space it is simply the midpoint of the interval $[x, y]$.

Let $M \in \Sigma_N$. $R(M) = \inf\{Mx : x \in X\}$ be a Chebyshev radius of the $N$-net $M$. The point $z \in X$ is called a Chebyshev center if $Mz = R(M)$ [1]. $\text{cheb}(M)$ denotes the set of all Chebyshev centers of $M$.

The mapping $f : (X, \rho) \to (X_1, \rho_1)$ is called Lipschitz one if there exists a constant $L \geq 0$ such that $\rho_1(f(x), f(y)) \leq L\rho(x, y)$ for all $x, y \in X$. Lipschitz mapping with Lipschitz constant 1 is called nonexpanding ([4], pg. 10).

In the norm spaces we use the following notation

$\text{co}(M)$ — convex hull of the set $M$ (i.e. the intersection of all convex sets comprising $M$).

$[x, y]$ — closed interval with endpoints $x, y$.

2. Introduction

In this article the authors investigate behavior of the set of Chebyshev centers of compacts of Euclidean space or those of the space $l^n_\infty$. It is known ([5], [6]) that Chebyshev center of the nonempty bounded subset of the Euclidean or Lobachevskii space is strongly stable, i.e. the mapping $\text{cheb} : B(X, \alpha) \to X, M \mapsto \text{cheb}(M)$, here $B(X)$ denotes the set of all nonempty bounded subsets of the space $X$, is continuous. Here we prove (theorem 2) similar property for the set of all nonempty subsets of the arbitrary metric space. Recall now that the restriction of the mapping $\text{cheb}$
to the set of all balls of the space with inner metric is nonexpanding map [7]. It holds true also for the restriction of this mapping to the set of all $N$–nets of Euclidean line [8] as well as for the restriction of the map to the set of all 2–nets of the space of nonpositive Busemann curvature [9]. At the same time the restriction of the map $\text{cheb}$ to the set of all nonempty closed convex sets of Euclidean plane is not Lipschitz even in the neighbourhood of the closed circle [10]. Note also that if dimension of Euclidean or Lobachevskii space is greater than 1 and $N > 2$ this map is not Lipschitz in the neighbourhood of the space $\Sigma_2(X) \subset (\Sigma_N(X), \alpha)$ [8]. At the same time in Hilbert space the map $\text{cheb} : (\Sigma_N(X) \setminus \Sigma_{N-1}(X), \alpha) \to X$, $M \mapsto \text{cheb}(M)$ stays locally Lipschitz [8] and for any two compacts $M, W$ the inequality $|\text{cheb}(M)\text{cheb}(W)| \leq \sqrt{R(M) + R(W) + \alpha(M, W)\sqrt{\alpha(M, W)}}$ holds true [11]. The authors managed to prove that in the space $X = l_\infty^n$ the situation simplifies as follows: the mapping $\text{cheb} : (K(X), \alpha) \to (P(X), \alpha)$ is Lipschitz with constant 2 (theorem 2).

3. Statements of the results

Let us introduce precise statements of the results of the article. To do this properly we need lemma 0 which is part of lemma 2 from [9].

**Lemma 0** Let the set $\omega(p, x)$ consist of one point for any points $p, x, y$ of the metric space $(X, \rho)$ and the inequality

$$2|\omega(p, x)\omega(p, y)| \leq |xy|$$

hold true.

Then the inequalities

$$|\text{cheb}(M)\text{cheb}(Z)| \leq \alpha(M, Z) \leq |\text{cheb}(M)\text{cheb}(Z)| + (D[M] + D[Z])/2$$

also hold true for all $M, Z \in \Sigma_2(X)$.

The following definition gives us the possibility to characterise Hausdorff metric for separable metric space.

**Definition** Let $(X, \rho)$ be separable metric space, $M, W \in B[X]$, and $M_\ast, W_\ast$ be sequences consisting of elements from $M$ and $W$ respectfully, such that the closure of each sequence coindices with the respective set. Let us define a function $\tilde{\alpha}$ by the formula

$$\tilde{\alpha}(M, W) = \inf \{\sup||x_n y_n| : n \in \mathbb{N} \} : (x_n)_{n \in \mathbb{N}} \in M_\ast, (y_n)_{n \in \mathbb{N}} \in W_\ast\}.$$ 

It is easy to verify both correctness of this definition and metric axioms for $\tilde{\alpha}$. 
Theorem 1 Let \((X, \rho)\) be separable metric space. Then \(\hat{\alpha}(M, W) = \alpha(M, W)\) for all \(M, W \in B[X]\).

Lemma 1 Let \((X, \rho)\) be metric space, \(M, M_n \in (B(X), \alpha)\) for \(n \in \mathbb{N}\),

\[\alpha(M_n, M) \to 0 \quad (n \to \infty), \quad \text{cheb}(M) \neq \emptyset \quad \text{and} \quad \text{cheb}(M_n) \neq \emptyset \quad \text{for} \quad n \in \mathbb{N}.\]

Then \(\lim_{n \to \infty} \text{cheb}(M_n) \subset \text{cheb}(M)\).

Lemma 2 Let \(X\) be uniform convex Banach space and varieties \(M\) and \(W \in B(X)\) be such that \(\text{cheb}(M) \in \overline{\text{co}}(M), \text{cheb}(W) \in \overline{\text{co}}(W), B(\text{cheb}(M), R(M)) \cap B(\text{cheb}(W), R(W)) = \emptyset.\) Then

\[|\text{cheb}(M)\text{cheb}(W)| \leq 2\alpha(M, W).\]

Theorem 2 Let \(n > 1, X = l^n_\infty.\) Then the mapping \(\text{cheb} : (K(X), \alpha) \to (\mathcal{P}(X), \alpha)\) is also Lipschitz with constant 2.

Now using the obvious inequality \(\alpha \leq \hat{\alpha}\) on the set \(\Sigma_N(l^n_\infty) \times \Sigma_N(l^n_\infty)\) we get

Corollary. Let \(n > 1, X = l^n_\infty.\) Then the mapping \(\text{cheb} : (\Sigma_N(X), \hat{\alpha}) \to (\mathcal{P}(X), \alpha)\) is Lipschitz with constant 2.

Note The trivial modification of the proofs for the two preceding statements provides us with one similar to that of theorem 2 for the space \(X = (l_\infty, \| \cdot \|_\infty).\)

Proof of theorem 1 Let \(M, W \in B[X]\) and assume without loss of generality that \(\alpha(M, W) = \sup\{|xW| : x \in M\}.\) Then \(\sup\{|xW| : x \in M\} = \inf\{|x_n|W : n \in \mathbb{N}\} \leq \hat{\alpha}(M, W).\) Thus \(\alpha \leq \hat{\alpha}.\)

Let us prove the converse relation \(\alpha \geq \hat{\alpha}.\) Let \(M^* = \{x_n : n \in \mathbb{N}\},\)

\(W^* = \{y_n : n \in \mathbb{N}\}.\) Then for any \(\varepsilon > 0\) and arbitrary \(n \in \mathbb{N}\) there exist \(m(n), k(n) \in \mathbb{N}\) such that

\[|x_n y_m(n)| \leq \alpha(M, W) + \varepsilon, |y_n x_k(n)| \leq \alpha(M, W) + \varepsilon.\]

Let \(W_1^* = \{y_m(n) : n \in \mathbb{N}\}.\) We construct the set \(W_2^*\) modifying \(W_1^*\) with the help of the following step-by-step procedure (here \(n\) takes values 1, 2, . . .):

If \(y_{m(n+1)} \in \{y_{m(1)}, \ldots, y_{m(n)}\}\) then introduce \(y_{m(n+1), n+1} = y_{m(n+1)}\) and put in correspondence to \(x_{n+1}\) element \(y_{m(n+1), n+1}\) instead of \(y_{m(n+1)}\) and replace element \(y_{m(n+1)}\) by \(y_{m(n+1), n+1}\) in the set \(W_1^*.\) Thus we get a bijection between \(M^*\) and \(W_2^*.\) Let us now put in correspondence to arbitrary element \(y_n \in W_3^* = W^* \setminus W_1^*\) element \(x_{k(n), n} = x_{k(n)}\) of the new set \(M_2^*.\)

Thus we get a bijection between \(M^* \cup M_2^*\) and \(W_3^* \cup W_3^*.\) It is easy to verify that for any \(\varepsilon > 0\) \(\hat{\alpha}(M, W) = \hat{\alpha}(M^* \cup M_2^*, W_3^* \cup W_3^*) \leq \alpha(M, W) + \varepsilon.\)

Since the number \(\varepsilon > 0\) can be arbitrary small we get \(\hat{\alpha}(M, W) \leq \alpha(M, W).\) This completes the proof of theorem 1.
Proof of lemma 1
Recall first that $|R(M_n) - R(M)| \leq \alpha(M_n, M)$ for $n \in \mathbb{N}$ [12].

Let $(y_n)_{n \in \mathbb{N}}$ be a sequence of points $y_n \in \text{cheb}(M_n)$ converging to $y \in X$. Fix arbitrary $u \in M$, then there exists a sequence $(z_n)_{n \in \mathbb{N}}$, $z_n \in M_n$ such that $\alpha(M_n, M) \leq |z_n u_n| \leq 2\alpha(M_n, M)$. Now triangle inequality and definition of Chebyshev center provide us with inequality $|y_n u| \leq |y_n z_n| + |z_n u| \leq R(M_n) + 2\alpha(M_n, M)$. So since $R(M_n) \to R(M)$, $\alpha(M_n, M) \to 0$, $y_n \to y$ ($n \to \infty$) we get the desired inequality $|yu| \leq R(M)$. This completes the proof of lemma 1.

Proof of lemma 2 Note first that clearly $\text{cheb}(M) \in \overline{M}(M)$ for $M \in B(X)$ in real Hilbert and 2-dimensional Banach spaces (cf. [13], [14]). Assume without loss of the generality that $R(W) \leq R(M)$. Then

$$|\text{cheb}(M)\text{cheb}(W)| = |\text{cheb}(M)B[\text{cheb}(W), R(W)]| + R(W) \leq$$

$$\leq \sup\{|xB[\text{cheb}(W), R(W)] : x \in \overline{M}(M)\} + R(W) =$$

$$= \sup\{|xB[\text{cheb}(W), R(W)] : x \in M\} + R(W) \leq 2\alpha(M, W).$$

This completes the proof.

Proof of theorem 2 Recall that the space $(K(X), \alpha)$ is geodesic one [7]. Let $M, W \in K(X)$. Then there exists $j \in \{1, \ldots, n\}$ such that

$$\alpha(\text{cheb}(M), \text{cheb}(W)) = \alpha(\bigcap_{x \in M} B[x, R(M)], \bigcap_{y \in W} B[y, R(W)]) =$$

$$\alpha(\text{pr}_j(\bigcap_{x \in M} B[x, R(M)]), \text{pr}_j(\bigcap_{y \in W} B[y, R(W)]),$$

here $\text{pr}_j$ is projection operator onto $j$-s coordinate. Note that $\bigcap_{x \in M} B[x, R(M)] \neq \varnothing$, $\bigcap_{y \in W} B[y, R(W)] \neq \varnothing$ and the balls of $(X, \| \cdot \|_\infty)$ are convex; hence

$$\text{pr}_j(\bigcap_{x \in M} B[x, R(M)]) =$$

$$= \bigcap_{x \in M} \text{pr}_j(B[x, R(M)]) = \bigcap_{x \in M} [\text{pr}_j(x) - R(M), \text{pr}_j(x) + R(M)].$$

The second expression can be analysed in similar way. Then there exist $x, u \in M$, $y, v \in W$ such that

$$\bigcap_{x \in M} [\text{pr}_j(x) - R(M), \text{pr}_j(x) + R(M)] = [\text{pr}_j(x) - R(M), \text{pr}_j(u) + R(M)],$$
\[ \bigcap_{y \in W} [\text{pr}_j(y) - R(W), \text{pr}_j(y) + R(W)] = [\text{pr}_j(y) - R(W), \text{pr}_j(v) + R(W)]. \]

Now we easily obtain the following inequalities:

\[
\alpha(\text{cheb}(M), \text{cheb}(W)) =
\alpha([\text{pr}_j(x) - R(M), \text{pr}_j(u) + R(M)], [\text{pr}_j(y) - R(W), \text{pr}_j(v) + R(W)]) =
\max\{\max\{\rho((\text{pr}_j(x) - R(M)), [\text{pr}_j(y) - R(W), \text{pr}_j(v) + R(W)]), \rho((\text{pr}_j(u) + R(M)), [\text{pr}_j(y) - R(W), \text{pr}_j(v) + R(W)])\},
\max\{\rho((\text{pr}_j(y) - R(W)), [\text{pr}_j(x) - R(M), \text{pr}_j(u) + R(M)]), \rho((\text{pr}_j(v) + R(W)), [\text{pr}_j(x) - R(M), \text{pr}_j(u) + R(M)])\}\} \leq
\max\{|\text{pr}_j(x) - R(M) - \text{pr}_j(y) + R(W)|, |\text{pr}_j(u) + R(M) - \text{pr}_j(v) - R(W)|\} \leq
\max\{|\text{pr}_j(x) - \text{pr}_j(y)|, |\text{pr}_j(u) - \text{pr}_j(v)|\} + |R(M) - R(W)|.
\]

It is not hard then to achieve the inequality

\[
\max\{|\text{pr}_j(x) - \text{pr}_j(y)|, |\text{pr}_j(u) - \text{pr}_j(v)|\} \leq \alpha(M, W).
\]

At the same time [12] implies that \(|R(M) - R(W)| \leq \alpha(M, W)|. So \(\alpha(\text{cheb}(M), \text{cheb}(W)) \leq 2\alpha(M, W).\)

The last inequality completes the proof of theorem 2.
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