KNOT INVARIANTS FROM DERIVATIONS OF QUANDLES

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ABSTRACT. We introduce some new knot invariants for tame knots using derivations of quandles. The main idea is the notion of action of a quandle on another quandle, which is used to define derivations of quandles. The derivations are then used to introduce some new invariants for tame knots, namely: derivation quandle, total derivation quandle and derivation polynomial. The derivation quandle is a generalisation of the hom quandle invariant of Crans and Nelson [5]. The total derivation quandle contains the hom quandle as an abelian subquandle, and the derivation polynomial is shown to be a stronger invariant than the well-known quandle coloring invariant. Finally, we extend these invariants to tame virtual knots.

1. INTRODUCTION

A quandle is a set with a binary operation that satisfies three axioms motivated by the three Reidemeister moves of diagrams of knots in the Euclidean space \( \mathbb{R}^3 \). These objects first appeared in the work of Joyce [13] under the name quandle, and that of Matveev [20] under the name distributive groupoid. They proved that each oriented diagram \( D(K) \) of a tame knot \( K \) (in fact, a tame link) in \( \mathbb{R}^3 \) gives rise to a quandle \( Q(K) \) called the knot quandle, which is independent of the diagram \( D(K) \). The generators of \( Q(K) \) are connected components of \( D(K) \) and defining relations are of the form \( a \ast b = c \) whenever the arc \( b \) passes over the double point separating arcs \( a \) and \( c \). Further, it has been shown that, if \( K_1 \) and \( K_2 \) are two tame knots with \( Q(K_1) \cong Q(K_2) \), then there is a homeomorphism of \( \mathbb{R}^3 \) mapping \( K_1 \) onto \( K_2 \), not necessarily preserving the orientations. We refer the reader to the survey articles [4, 14, 21] for more on the historical development of the subject and its relationships with other areas of mathematics. In the last decade, quandles and their analogues have been explored extensively to construct newer invariants for knots and links, which is also the aim of this paper.

Though the knot quandle is a strong invariant for tame knots, it is usually not easy to check whether two knot quandles are isomorphic. Thus, we are lead to explore and develop methods to distinguish between knot quandles. Homomorphisms from knot quandles to simpler quandles, called quandle colorings, are sometimes helpful in distinguishing between two knot quandles. The well-known Fox’s \( n \)-colorings are homomorphisms of the knot quandle into the dihedral quandle \( \mathbb{R}_n \) on \( n \) elements [7, 22]. In fact, finite quandles have been used to define invariants of both knots and
links in $\mathbb{R}^3$, and also generalisations such as virtual knots and knotted surfaces in $\mathbb{R}^4$. This makes study of algebraic aspects of quandles, which are not necessarily knot quandles, important.

Automorphisms of quandles, which play an important role in our work, have been investigated in much detail. In [10], automorphism groups of quandles of order less than 6 have been determined. This investigation has been carried forward in [6], wherein the automorphism groups of dihedral quandles are determined. In [11], a description of the automorphism group of Alexander quandles has been determined. In [1], some structural results have been obtained for the group of automorphisms and inner automorphisms of generalised Alexander quandles of finite abelian groups. This work has been extended in [2], wherein some interesting subgroups of automorphism groups of conjugation quandles of groups are determined.

Given a tame knot $K$ with knot quandle $Q(K)$ and a finite quandle $X$, the cardinality of the set $\text{Hom}(Q(K), X)$ of all quandle homomorphisms from $Q(K)$ to $X$ is a well-known knot invariant called the quandle coloring invariant. Recall that, a knot or a link invariant which determines the quandle coloring invariant is called an enhancement. Further, an enhancement is called a proper enhancement if there are examples in which the enhancement distinguishes knots or links which have the same quandle coloring invariant. If $X$ is an abelian quandle, then Crans and Nelson [5] proved that the set $\text{Hom}(Q(K), X)$ is itself an abelian quandle. They further showed that their hom quandle is a proper enhancement of the quandle coloring invariant.

The aim of this paper is to present some new knot invariants using the notion of derivations of quandles. The main idea is the notion of action of a quandle on another quandle, which is in analogy with the usual notion of action of a group on a set. The quandle action is then used to define derivations of quandles, which can be seen as twisted analogues of quandle homomorphisms, where the twisting is by quandle actions. These are then used to define some new invariants, namely: derivation quandle, total derivation quandle and derivation polynomial. The derivation quandle is a generalisation of the hom quandle invariant of Crans and Nelson [5]. The total derivation quandle is a quandle containing the hom quandle as an abelian subquandle. Further, the derivation polynomial is shown to be a stronger invariant than the quandle coloring invariant. In the end, the invariants are extended to tame virtual knots also. Although, the results are stated for tame knots, they hold for tame links as well. Our study again highlights usage of automorphism groups of quandles for constructing newer knot invariants. We note that our approach to derivations of quandles is different from the categorical approach considered by Szymik [25, 26].

The paper is organized as follows: Section 2 collects some basic definitions from the theory of quandles. In Section 3, an action $\phi$ of a quandle $Q$ on another quandle $X$ is defined (Definition 3.1), and the action is further used to define derivations from $Q$ to $X$ (Definition 3.4). In Section 4, derivations are considered with respect to quandle actions on abelian quandles. More precisely, it is shown that if $\phi$ is the action of a quandle $Q$ on an abelian quandle $A$, then the set $\text{Der}_\phi(Q, A)$ of all derivations is itself an abelian quandle (Theorem 4.1). The quandle so obtained is called the derivation quandle, and it is shown to inherit some properties from the abelian quandle $A$ (Propositions 4.3 and 4.4). Section 5 establishes that the derivation quandle of a tame knot with respect to an action of the knot quandle on an abelian quandle is a knot invariant (Theorem 5.4) and is a generalisation of the hom quandle invariant of [5]. Further, a bigger quandle called the total derivation quandle is constructed which contains the hom quandle as an abelian subquandle for constructing newer knot invariants.
(Theorem 5.7). In Section 6, derivations are used to introduce a polynomial invariant called the derivation polynomial (Theorem 6.1), and it is shown to be an enhancement of the quandle coloring invariant (Corollary 6.2. Section 7 consists of computations of various derivation invariants for some tame knots. In particular, Example 7.5 establishes that the derivation polynomial is a stronger invariant than the well-known quandle coloring invariant. The same example shows that the total derivation quandle is a stronger invariant than the hom quandle invariant of [5]. Finally, in Section 8, similar derivation invariants are defined for tame virtual knots.

2. Preliminaries on quandles

We begin with the definition of a quandle.

**Definition 2.1.** A quandle is a non-empty set $X$ with a binary operation $(x, y) \mapsto x * y$ satisfying the following axioms:

1. (Q1) $x * x = x$ for all $x \in X$;
2. (Q2) For any $x, y \in X$ there exists a unique $z \in X$ such that $x = z * y$;
3. (Q3) $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$.

**Example 2.2.** Besides knot quandles associated to tame knots, many interesting examples of quandles come from groups.

- If $G$ is a group, then the set $G$ equipped with the binary operation $a * b = b^{-1}ab$ gives a quandle structure on $G$, called the conjugation quandle, and denoted by $\text{Conj}(G)$.
- If $A$ is an additive abelian group, then the set $A$ equipped with the binary operation $a * b = 2b - a$ gives a quandle structure on $A$, denoted by $T(A)$ and called the Takasaki quandle of $A$. Such quandles first appeared in [27] under the name kei. For $A = \mathbb{Z}/n\mathbb{Z}$, it is called the dihedral quandle, and is denoted by $R_n$.
- If $G$ is a group and we take the binary operation $a * b = ba^{-1}b$, then we get the core quandle, denoted as $\text{Core}(G)$. In particular, if $G$ is additive abelian, then $\text{Core}(G)$ is the Takasaki quandle.
- Let $G$ be a group and $\varphi \in \text{Aut}(G)$. Then the set $G$ equipped with the binary operation $a * b = \varphi(ab^{-1})b$ gives a quandle structure on $G$, called the generalised Alexander quandle of $G$ with respect to $\varphi$.

A quandle $X$ is called trivial if $x * y = x$ for all $x, y \in X$. Unlike groups, a trivial quandle can contain arbitrary number of elements.

A quandle homomorphism is a morphism of quandles. More precisely, if $X$ and $Y$ are quandles, then a quandle homomorphism $f : X \to Y$ is a map satisfying

$$f(x * x') = f(x) * f(x')$$

for all $x, x' \in X$. The set of all quandle homomorphisms from $X$ to $Y$ is denoted by $\text{Hom}(X, Y)$. As a consequence of the first quandle axiom (Q1) it follows that any constant map from one quandle to the other is a quandle homomorphism. This seemingly trivial fact is actually quite useful.

Notice that, the quandle axioms are equivalent to saying that for each $x \in X$, the map

$$S_x : X \to X$$

...
given by
\[ S_x(y) = y \ast x \]
is an automorphism of the quandle \( X \) fixing \( x \). Such an automorphism is called an \textit{inner automorphism} of \( X \), and the group generated by all such automorphisms is denoted by \( \text{Inn}(X) \). A quandle is called \textit{involuntary} if \( S_x \circ S_x = \text{id}_X \) for all \( x \in X \). For example, all Takasaki quandles are involuntary. Motivated by the work of Loos \cite{16, 17} on Riemannian symmetric spaces, Ishihara and Tamaru \cite{12} defined a quandle \( X \) to be \textit{flat} if the group
\[ \langle S_x \circ S_y \mid x, y \in X \rangle \]
is abelian. For example, Takasaki quandles of 2-divisible groups are flat \cite{24}.

As pointed out in \cite{10}, it is convenient to specify quandle structures on a finite set \( X = \{ x_1, x_2, \ldots, x_n \} \) using an \( n \times n \) matrix with integer entries. More precisely, the entry at \( i \)-th row and \( j \)-th column of the quandle matrix of \( X \) is the integer \( k \), where \( x_k = x_i \ast x_j \). This approach is particularly useful for large size quandles, and we use it in Section \ref{section7}.

The following definition is crucial for our work \cite{13}.

\textbf{Definition 2.3.} A quandle \( X \) is said to be \textit{abelian} if
\[ (x \ast y) \ast (z \ast w) = (x \ast z) \ast (y \ast w) \]
for all \( x, y, z, w \in X \).

Abelian quandles are also called \textit{medial quandles} in the literature. For example, if \( A \) is an additive abelian group, then the Takasaki quandle \( T(A) \) is abelian. Also, the quandle \( X \) with matrix given by
\[
\begin{bmatrix}
1 & 3 & 4 & 2 \\
4 & 2 & 1 & 3 \\
2 & 4 & 3 & 1 \\
3 & 1 & 2 & 4
\end{bmatrix}
\]
can easily seen to be abelian.

A quandle \( X \) is said to be \textit{commutative} if
\[ x \ast y = y \ast x \]
for all \( x, y \in X \). Notice that, unlike in group theory, being commutative and being abelian do not mean the same for quandles. For example, the quandle given by \eqref{2.0.1} is abelian but not commutative, since the matrix is not symmetric. In fact, any trivial quandle with more than one element is abelian but not commutative. The dihedral quandle \( R_3 \) on three elements is both abelian and commutative.

\section{Actions and derivations of quandles}

The main idea of this section is the notion of action of a quandle on another quandle, and, in particular, on a set. This is used to introduce derivations of quandles, the main objects of our study in this paper.
Definition 3.1. A quandle action of a quandle $Q$ on a quandle $X$ is a quandle homomorphism

$$\phi : Q \to \text{Conj}(\text{Aut}(X)),$$

where $\text{Aut}(X)$ is the group of all quandle automorphisms of $X$, and $\text{Conj}(\text{Aut}(X))$ its associated conjugation quandle.

We say that the action is trivial if $\text{Im}(\phi) = \{\text{id}_X\}$. Notice that, any set $X$ can be viewed as a trivial quandle. In that case, $\text{Aut}(X) = \Sigma_X$, the group of all bijections of the set $X$, and we obtain the definition of an action of a quandle $Q$ on a set $X$.

Example 3.2. Some examples of quandle actions are:

1. Since a constant map from one quandle to another quandle is obviously a quandle homomorphism, it follows that every quandle $Q$ acts on another quandle $X$. More precisely, if $\alpha \in \text{Conj}(\text{Aut}(X))$ is a fixed element, then the constant map $\phi : Q \to \text{Conj}(\text{Aut}(X))$ sending all elements of $Q$ to $\alpha$ is an action of $Q$ on $X$.

2. If $Q$ is a quandle, then the map $\phi : Q \to \text{Conj}(\text{Aut}(Q))$ given by $q \mapsto S_q$ is a quandle homomorphism. Thus, every quandle acts on itself by inner automorphisms.

3. Let $G$ be a group acting on a set $X$. That is, there is a group homomorphism $\phi : G \to \Sigma_X$. Viewing both $G$ and $\Sigma_X$ as conjugation quandles and observing that a group homomorphism is also a quandle homomorphism between corresponding conjugation quandles, it follows that the quandle $\text{Conj}(G)$ acts on the set $X$.

Let $\phi : Q \to \text{Conj}(\text{Aut}(X))$ be an action of the quandle $Q$ on a quandle $X$. Then the preceding definition implies that there exists a map

$$\tilde{\phi} : X \times Q \to X$$

given by

$$(x, q) \mapsto x^{\phi(q)},$$

where $x^{\phi(q)}$ is the image of $x$ under the automorphism $\phi(q)$ of $X$. Let $q_1, q_2 \in Q$ and $x \in X$. Since, $\phi$ is a quandle homomorphism, we have

$$\phi(q_1 \ast q_2) = \phi(q_1) \ast \phi(q_2) = \phi(q_2)^{-1} \circ \phi(q_1) \circ \phi(q_2)$$

or equivalently

$$\phi(q_2) \circ \phi(q_1 \ast q_2) = \phi(q_1) \circ \phi(q_2).$$

Now evaluating the automorphisms at $x$ give

$$(3.0.1) \quad (x^{\phi(q_2)})^{\phi(q_1 \ast q_2)} = (x^{\phi(q_1)})^{\phi(q_2)}.$$

Remark 3.3. Equation (3.0.1) is the condition required in Rubinsztein’s definition [23 Definition 2.2.] of action of a topological quandle on a topological space. Thus, Definition 3.1 of action of a quandle on a set implies the one due to Rubinsztein in the discrete setting.
However, the converse is not true. Let $Q$ be any quandle and $X = \{1, 2, 3\}$ with quandle matrix 
\[
\begin{bmatrix}
1 & 3 & 1 \\
2 & 2 & 2 \\
3 & 1 & 3 \\
\end{bmatrix}.
\]
Let $\sigma = (1, 2, 3)$ be a permutation of $X$. Notice that $\sigma$ is not a quandle homomorphism of $X$. Now the map $X \times Q \to X$ given by $(i, q) \mapsto \sigma(i)$ for $i \in X, q \in Q$ clearly satisfies (3.0.1). But, for each $q \in Q$, the induced map $X \to X$ is given by $i \mapsto \sigma(i)$, which is not a quandle homomorphism.

**Definition 3.4.** Let $Q$ and $X$ be two quandles and $\phi : Q \to \text{Conj}(\text{Aut}(X))$ a quandle action of $Q$ on $X$. A map $f : Q \to X$ satisfying

\begin{equation}
(3.0.2) \quad f(q_1 \ast q_2) = f(q_1) \ast f(q_2)\phi(q_1)
\end{equation}

for all $q_1, q_2 \in Q$, is called a derivation with respect to the quandle action $\phi$ of $Q$ on $X$.

With the preceding definition, let $\text{Der}_\phi(Q, X) = \{f : Q \to X \mid f$ is a derivation with respect to $\phi\}$ denote the set of all derivations with respect to the quandle action $\phi$ of $Q$ on $X$. Notice that, if $\phi$ is the trivial quandle action, then a derivation is simply a quandle homomorphism, and hence $\text{Der}_\phi(Q, X) = \text{Hom}(Q, X)$, the set of all quandle homomorphisms from $Q$ to $X$. Further, we remark that if $X$ is a trivial quandle, then for any action $\phi$ of $Q$ on $X$, $\text{Der}_\phi(Q, X) = \text{Hom}(Q, X)$. See Example 7.3 for a computation of the derivation set.

Given a non-trivial action $\phi$ of a quandle $Q$ on a non-trivial quandle $X$, it is possible that the set $\text{Der}_\phi(Q, X)$ is non-empty, which we illustrate as follows. Let $S_x \in \text{Inn}(X)$ be an inner automorphism of the quandle $X$ such that $S_x \neq \text{id}_X$. Consider the action $\phi : Q \to \text{Conj}(\text{Aut}(X))$ given by

$\phi(q) = S_x$

for all $q \in Q$. Then the map $f : Q \to X$ defined as

$\quad f(q) = x$

for $q \in Q$, is clearly an element of $\text{Der}_\phi(Q, X)$.

### 4. Derivation Quandles and Properties

In the previous section we defined the set of derivations $\text{Der}_\phi(Q, X)$ with respect to the quandle action $\phi$ of a quandle $Q$ on a quandle $X$. It turns out that if $X$ is an abelian quandle, then $\text{Der}_\phi(Q, X)$ is also a quandle.

**Theorem 4.1.** Let $Q$ and $A$ be quandles such that $A$ is abelian and $\phi : Q \to \text{Conj}(\text{Aut}(A))$ a quandle action. If the set $\text{Der}_\phi(Q, A)$ is non-empty, then it has the structure of an abelian quandle with respect to the binary operation given by

\begin{equation}
(4.0.1) \quad (f \ast g)(q) = f(q) \ast g(q)
\end{equation}

for $f, g \in \text{Der}_\phi(Q, A)$ and $q \in Q$. 
Proof. We verify the quandle axioms one by one. Let \( f \in \text{Der}_\phi(Q, A) \). Then, we have
\[
(f \ast f)(q) = f(q) \ast f(q) = f(q)
\]
for all \( q \in Q \), and hence axiom (Q1) is satisfied in \( \text{Der}_\phi(Q, A) \). Let \( f, g, h \in \text{Der}_\phi(Q, A) \). For each \( q \in Q \), by axiom (Q2) in \( A \), there exists a unique \( a_q \in A \) such that \( a_q \ast f(q) = g(q) \). Define \( h : Q \to A \) by setting \( h(q) = a_q \). Then \( h \ast f = g \), and it remains to show that \( h \in \text{Der}_\phi(Q, A) \). Let \( q_1, q_2 \in Q \). Then,
\[
\begin{align*}
g(q_1) &= a_{q_1} \ast f(q_1), \\
g(q_2) &= a_{q_2} \ast f(q_2), \\
g(q_1 \ast q_2) &= a_{q_1 \ast q_2} \ast f(q_1 \ast q_2).
\end{align*}
\]
Now consider
\[
(4.0.3) \quad g(q_1 \ast q_2) = g(q_1) \ast g(q_2)^\phi(q_1)
\]
\[
= (a_{q_1} \ast f(q_1)) \ast (a_{q_2} \ast f(q_2))^{\phi(q_1)}
\]
\[
= (a_{q_1} \ast f(q_1)) \ast (a_{q_2}^{\phi(q_1)} \ast f(q_2)^{\phi(q_1)}), \text{ since } \phi(q_1) \text{ is a homomorphism}
\]
\[
= (a_{q_1} \ast a_{q_2}^{\phi(q_1)}) \ast (f(q_1) \ast f(q_2)^{\phi(q_1)}), \text{ since } A \text{ is abelian}
\]
\[
= (a_{q_1} \ast a_{q_2}^{\phi(q_1)}) \ast f(q_1 \ast q_2).
\]
Using (4.0.2), (4.0.3) and axiom (Q2), we obtain
\[
a_{q_1 \ast q_2} = a_{q_1} \ast a_{q_2}^{\phi(q_1)}.
\]
Let \( f, g, h \in \text{Der}_\phi(Q, A) \) and \( q \in Q \). Then
\[
((f \ast g) \ast h)(q) = (f(q) \ast g(q)) \ast h(q)
\]
\[
= (f(q) \ast h(q)) \ast (g(q) \ast h(q))
\]
\[
= ((f \ast h) \ast (g \ast h))(q),
\]
and hence \( \text{Der}_\phi(Q, A) \) is a quandle.

Finally, let \( f, g, h, k \in \text{Der}_\phi(Q, A) \) and \( q \in Q \). Then
\[
((f \ast g) \ast (h \ast k))(q) = ((f(q) \ast g(q)) \ast (h(q) \ast k(q))
\]
\[
= (f(q) \ast h(q)) \ast (g(q) \ast k(q)), \text{ since } A \text{ is abelian}
\]
\[
= ((f \ast h) \ast (g \ast k))(q),
\]
and hence \( \text{Der}_\phi(Q, A) \) is abelian. \( \square \)

We refer the quandle \( \text{Der}_\phi(Q, A) \) obtained in the preceding theorem as the derivation quandle with respect to the quandle action \( \phi \) of the quandle \( Q \) on the abelian quandle \( A \).

If the action in the preceding theorem is trivial, then we recover the following result of \([5, \text{ Theorem } 4.1]\), which was our initial motivation.
Theorem 4.2. Let $Q$ be a quandle and $A$ an abelian quandle. Then the set $\text{Hom}(Q, A)$ has the structure of an abelian quandle with respect to the binary operation as given in (1.0.1).

Next we discuss some properties of derivation quandles.

Proposition 4.3. Let $Q$ be a quandle and $A$ an additive abelian group. Let $\phi$ be a quandle action of $Q$ on $T(A)$. Then $\text{Der}_\phi(Q, T(A))$ is an additive abelian group and its quandle structure coincides with the Takasaki quandle of the underlying abelian group.

Proof. It is easy to see that $\text{Der}_\phi(Q, T(A))$ is an additive abelian group with respect to the point-wise addition of functions. Now, for $f, g \in \text{Der}_\phi(Q, T(A))$ and $q \in Q$, we have

\[
(f \ast g)(q) = f(q) \ast g(q) = 2g(q) - f(q) = (2g - f)(q).
\]

Hence, $f \ast g = 2g - f$, which was desired. \qed

Proposition 4.4. Let $Q$ be a quandle and $A$ an abelian quandle. Then the following statements hold:

1. If $A$ is commutative, then $\text{Der}_\phi(Q, A)$ is commutative.
2. If $A$ is involutory, then $\text{Der}_\phi(Q, A)$ is involutory.
3. If $A$ is flat, then $\text{Der}_\phi(Q, A)$ is flat.

Proof. Assertion (1) is straightforward. For assertion (2), let $f, g \in \text{Der}_\phi(Q, A)$ and $q \in Q$. Then

\[
S_f^2(g)(q) = ((g \ast f) \ast f)(q) = (g(q) \ast f(q)) \ast f(q), \text{ since } A \text{ is involutory}
\]

and hence $\text{Der}_\phi(Q, A)$ is involutory.

Let $f_1, f_2, g_1, g_2, h \in \text{Der}_\phi(Q, A)$ and $q \in Q$. Then we have

\[
((S_{f_1} \circ S_{f_2}) \circ (S_{g_1} \circ S_{g_2})(h))(q) = (((h \ast g_2) \ast g_1) \ast f_2) \ast f_1)(q)
\]

and hence $\text{Der}_\phi(Q, A)$ is flat. \qed
5. Derivation quandles as knot invariants

Let $Q_1, Q_2$ be two quandles and $A_1, A_2$ two abelian quandles. Let

$$\phi_1 : Q_1 \rightarrow \text{Conj}(\text{Aut}(A_1))$$

be an action of $Q_1$ on $A_1$, and

$$\phi_2 : Q_2 \rightarrow \text{Conj}(\text{Aut}(A_2))$$

an action of $Q_2$ on $A_2$. A pair of quandle homomorphisms $\sigma : Q_2 \rightarrow Q_1$ and $\tau : A_1 \rightarrow A_2$ is said to be action compatible if

$$\tau(a^{\phi_1(\sigma(q))}) = \tau(a)^{\phi_2(q)}$$

for $q \in Q_2$ and $a \in A_1$. Equivalently, the following diagram commutes

$$\begin{array}{ccc}
A_2 \times Q_2 & \xrightarrow{\sim} & A_2 \\
\tau \uparrow & & \uparrow \tau \\
A_1 \times Q_1 & \xrightarrow{\sim} & A_1.
\end{array}$$

With the preceding definitions, we have

**Theorem 5.1.** Let $Q_1, Q_2$ be two quandles and $A_1, A_2$ two abelian quandles. Let $\phi_1 : Q_1 \rightarrow \text{Conj}(\text{Aut}(A_1))$ and $\phi_2 : Q_2 \rightarrow \text{Conj}(\text{Aut}(A_2))$ be actions of $Q_1, Q_2$ on $A_1, A_2$, respectively. Let $\sigma : Q_2 \rightarrow Q_1$ and $\tau : A_1 \rightarrow A_2$ be action compatible quandle homomorphisms. Then there exists a quandle homomorphism

$$\Phi : \text{Der}_{\phi_1}(Q_1, A_1) \rightarrow \text{Der}_{\phi_2}(Q_2, A_2).$$

Further, if $\sigma$ and $\tau$ are both isomorphisms, then so is $\Phi$. Additionally, if $Q_1, Q_2$ are finitely generated and $A_1, A_2$ are finite, then

$$|\text{Der}_{\phi_1}(Q_1, A_1)| = |\text{Der}_{\phi_2}(Q_2, A_2)|.$$

**Proof.** For each $f \in \text{Der}_{\phi_1}(Q_1, A_1)$, define

$$\Phi : \text{Der}_{\phi_1}(Q_1, A_1) \rightarrow \text{Der}_{\phi_2}(Q_2, A_2)$$

by setting

$$\Phi(f) = \tau \circ f \circ \sigma.$$

Let $q_1, q_2 \in Q_2$. Then

$$\Phi(f)(q_1 * q_2) = \tau \circ f \circ \sigma(q_1 * q_2)$$

$$= \tau(f(\sigma(q_1)) \circ \sigma(q_2))$$

$$= \tau(f(\sigma(q_1)) \circ f(\sigma(q_2))^{\phi_1(\sigma(q_1))})$$

$$= \tau(f(\sigma(q_1)) \circ \tau(f(\sigma(q_2)))^{\phi_1(\sigma(q_1))})$$

$$= \tau(f(\sigma(q_1))) \circ \tau(f(\sigma(q_2)))^{\phi_2(q_1)}, \text{ by condition (5.0.3)}$$

$$= \Phi(f)(q_1) \circ \Phi(f)(q_2)^{\phi_2(q_1)},$$
and hence $\Phi(f) \in \text{Der}_{\phi_2}(Q_2, A_2)$.

Let $f, g \in \text{Der}_{\phi_1}(Q_1, A_1)$ and $q \in Q_2$. Then
\[
\Phi(f * g)(q) = \tau \circ (f * g) \circ \sigma(q) \\
= \tau(f(\sigma(q))) \ast \tau(g(\sigma(q))) \\
= \Phi(f)(q) \ast \Phi(g)(q) \\
= (\Phi(f) \ast \Phi(g))(q),
\]
and hence $\Phi$ is a quandle homomorphism. Finally, if $\sigma$ and $\tau$ are both isomorphisms, then we can define
\[
\Psi : \text{Der}_{\phi_2}(Q_2, A_2) \to \text{Der}_{\phi_1}(Q_1, A_1)
\]
by setting
\[
\Psi(f) = \tau^{-1} \circ f \circ \sigma^{-1}
\]
for $f \in \text{Der}_{\phi_2}(Q_2, A_2)$. An easy check shows that $\Phi$ and $\Psi$ are inverses of each other, and the proof is complete. \hfill \square

Taking $Q_1 = Q_2 = Q$ and $\sigma = \text{id}_Q$ in Theorem 5.1, we obtain

**Corollary 5.2.** Let $Q$ be a quandle and $A_1, A_2$ two abelian quandles. Let $\tau : A_1 \to A_2$ be a quandle homomorphism satisfying $\tau(a^{\phi_1(q)}) = \tau(a)^{\phi_2(q)}$ for all $q \in Q$ and $a \in A_1$. Then there is a quandle homomorphism $\tilde{\tau} : \text{Der}_{\phi_1}(Q, A_1) \to \text{Der}_{\phi_2}(Q, A_2)$.

Further, if $\tau$ is an isomorphism, then so is $\tilde{\tau}$. Additionally, if $Q$ is finitely generated and $A_1, A_2$ are finite, then
\[
|\text{Der}_{\phi_1}(Q, A_1)| = |\text{Der}_{\phi_2}(Q, A_2)|.
\]

Similarly, taking $A_1 = A_2 = A$ and $\tau = \text{id}_A$ in Theorem 5.1 yields

**Corollary 5.3.** Let $Q_1, Q_2$ be quandles and $A$ an abelian quandle. Let $\sigma : Q_2 \to Q_1$ be a quandle homomorphism satisfying $\phi_2 = \phi_1 \circ \sigma$. Then there exists a quandle homomorphism $\tilde{\sigma} : \text{Der}_{\phi_1}(Q_1, A) \to \text{Der}_{\phi_2}(Q_2, A)$.

Moreover, if $\sigma$ is an isomorphism, then so is $\tilde{\sigma}$. In addition, if $Q_1, Q_2$ are finitely generated and $A$ is finite, then
\[
|\text{Der}_{\phi_1}(Q_1, A)| = |\text{Der}_{\phi_2}(Q_2, A)|.
\]

**Theorem 5.4.** Derivation quandles of a tame knot with respect to an abelian quandle are knot invariants.

**Proof.** Let $K_1$ and $K_2$ be two equivalent tame knots with knot quandles $Q(K_1)$ and $Q(K_2)$, respectively. Then, by [13], there is an isomorphism $\sigma : Q(K_2) \to Q(K_1)$. Let $A$ be an abelian quandle and $\phi_1 : Q(K_1) \to \text{Conj} \left( \text{Aut}(A) \right)$ an action of $Q(K_1)$ on $A$. Then $\phi_2 := \phi_1 \circ \sigma$ is an action of $Q(K_2)$ on $A$. By Corollary 5.3, we obtain an isomorphism $\tilde{\sigma} : \text{Der}_{\phi_1}(Q(K_1), A) \cong \text{Der}_{\phi_2}(Q(K_2), A)$. Thus, derivation quandles are knot invariants. \hfill \square
Remark 5.5. If the quandle action in the preceding theorem is trivial, then we recover a result of [5] that the hom quandle of a tame knot with respect to an abelian quandle is a knot invariant.

Remark 5.6. Given a quandle $Q$ and an abelian quandle $A$, we define the derivation quandle multiset of $Q$ with respect to $A$ as

\begin{equation}
\text{Der}(Q, A) = \{ \text{Der}_\phi(Q, A) \mid \phi \in \text{Hom}(Q, \text{Conj(Aut}(A))) \}.
\end{equation}

By Theorem 5.4, the derivation quandle multiset of a tame knot with respect to an abelian quandle is a knot invariant. Thinking of the derivation quandle multiset as a multiset of integral matrices, it is possible to compare the multisets corresponding to two tame knots to see whether they are equivalent.

Given two quandles $(X_1, \star_1)$ and $(X_2, \star_2)$, the disjoint union $X_1 \sqcup X_2$ can be turned into a quandle by defining

\begin{equation}
x \star y = \begin{cases} 
x \star_1 y & \text{if } x, y \in X_1 \\
x \star_2 y & \text{if } x, y \in X_2 \\
x & \text{if } x \in X_1, y \in X_2 \\
x & \text{if } x \in X_2, y \in X_1. 
\end{cases}
\end{equation}

We refer the reader to [2, Proposition 9.2] for a more general construction. It can be easily checked that if $X_1$ and $X_2$ are abelian, then $X_1 \sqcup X_2$ is not abelian in general.

Let $K$ be a tame knot and $A$ an abelian quandle. Taking $X_1 = \text{Hom}(Q(K), A)$ and

\[X_2 = \bigsqcup_{\phi \text{ non-trivial action}} \text{Der}_\phi(Q(K), A),\]

as above, we can construct a non-abelian quandle

\[D(Q(K), A) := X_1 \sqcup X_2,\]

which we refer as the total derivation quandle with respect to the abelian quandle $A$. As a consequence of Theorem 5.4, we obtain the following:

**Theorem 5.7.** The total derivation quandle with respect to an abelian quandle is an invariant of tame knots and contains the hom quandle as an abelian subquandle.

In Example 7.5 we give examples of two knots for which the hom quandles are isomorphic but the total derivation quandles are not so, in fact, they are of different sizes.

### 6. Polynomial invariant using derivations

Let $Q$ be a quandle and $X$ a finite quandle (not necessarily abelian). Define the derivation polynomial of $Q$ with respect to $X$ as the integral polynomial

\begin{equation}
D_X(Q)(u) = |\text{Hom}(Q, X)| + \sum_{\phi \text{ non-trivial action}} u^{|\text{Der}_\phi(Q, X)|+1}.
\end{equation}
Notice that the conclusions of Theorem 5.1, Corollary 5.2 and Corollary 5.3 hold even if the quandles $A$ and $A'$ are not necessarily abelian. In fact, abelian-ness is used only to get a quandle structure on the set of derivations. Thus, if $X$ is a finite quandle (not necessarily abelian), then $\Phi$, $\bar{\tau}$ and $\bar{\sigma}$ are simply maps and bijections between the corresponding sets. Further, an imitation of the proof of Theorem 5.4 shows that the derivation sets of two equivalent tame knots with respect to a finite quandle have the same size. This together with the definition of derivation polynomial yields the following result.

**Theorem 6.1.** The derivation polynomial of a tame knot with respect to a finite quandle is a knot invariant.

Let $K$ be a tame knot and $X$ a finite quandle. We denote the derivation polynomial of $Q(K)$ with respect to $X$ by $D_X(K)(u)$. Notice that $D_X(K)(0) = |\text{Hom}(Q(K), X)|$, the quandle coloring invariant. Thus, we obtain the following.

**Corollary 6.2.** The derivation polynomial of a tame knot is an enhancement of the quandle coloring invariant.

Later, in Section 7, we give an example to illustrate that the derivation polynomial is, in fact, a proper enhancement of the quandle coloring invariant.

**Remark 6.3.** We can extract quite a bit of information from the derivation polynomial of a tame knot with respect to a finite quandle. Let $K$ be a tame knot with the derivation polynomial $D_X(K)(u) = a_0 + a_1 u + \cdots + a_n u^n$ with respect to a finite quandle $X$. Then, as explained above, the constant term $a_0$ gives the quandle coloring invariant, which corresponds to the trivial action of $Q(K)$ on $X$. For each $k \geq 1$, the coefficient $a_k$ counts the number of non-trivial quandle actions $\phi$ of $Q(K)$ on $X$ for which $|\text{Der}_\phi(Q(K), X)| = k - 1$.

### 7. Illustrative examples and computations

In this section, we give computations of various derivation invariants for some tame knots. We give examples of tame knots which are not distinguishable by the quandle coloring invariant as well as the hom quandle invariant of Crans and Nelson [5], but the derivation polynomial distinguishes them. This establishes that the derivation polynomial is a proper enhancement of the quandle coloring invariant. We have used the sub-package Knots and Quandles of the HAP package [8] of GAP [9] for our computations.

**Example 7.1.** Let $K$ be the trefoil knot $3_1$ and $X$ a quandle with matrix

\[
\begin{pmatrix}
1 & 3 & 4 & 2 \\
4 & 2 & 1 & 3 \\
2 & 4 & 3 & 1 \\
3 & 1 & 2 & 4
\end{pmatrix}
\]

Then by a GAP computation, we obtain $D_X(K)(u) = 16 + 15u + 20u^2$. 

Example 7.2. Let $K$ be the figure eight knot 4\textsubscript{1} and $X = R_3$ the dihedral quandle. Then $D_X(K)(u) = 3 + 2u + 3u^2$.

Example 7.3. Consider the trefoil knot 3\textsubscript{1} and the abelian quandle $X$ with matrix

\[
\begin{bmatrix}
1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 2 & 4 & 6 & 8 & 10 & 12 & 14 \\
15 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 1 & 3 & 5 & 7 & 9 & 11 & 13 \\
14 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 2 & 4 & 6 & 8 & 10 & 12 \\
13 & 15 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 1 & 3 & 5 & 7 & 9 & 11 \\
12 & 14 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 2 & 4 & 6 & 8 & 10 \\
11 & 13 & 15 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 1 & 3 & 5 & 7 & 9 \\
10 & 12 & 14 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 2 & 4 & 6 & 8 \\
9 & 11 & 13 & 15 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 1 & 3 & 5 & 7 \\
8 & 10 & 12 & 14 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 2 & 4 & 6 \\
7 & 9 & 11 & 13 & 15 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 1 & 3 & 5 \\
6 & 8 & 10 & 12 & 14 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 2 & 4 \\
5 & 7 & 9 & 11 & 13 & 15 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 1 & 3 \\
4 & 6 & 8 & 10 & 12 & 14 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 2 \\
3 & 5 & 7 & 9 & 11 & 13 & 15 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 1 \\
2 & 4 & 6 & 8 & 10 & 12 & 14 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15
\end{bmatrix}
\]

Then, a GAP computation yields

\[|\text{Hom}(Q(3\textsubscript{1}), X)| = 45\]

and

\[|\text{Hom}(Q(3\textsubscript{1}), \text{Conj}(\text{Aut}(X)))| = 240.\]
Moreover, the derivation polynomial is
\[ DX(3_1)(u) = 45 + 176u + 45u^2 + 3u^6 + 15u^{10}. \]

Since \( X \) is an abelian quandle, by Theorem 4.1, all the derivation sets \( \text{Der}_\phi(Q(3_1), X) \) are also abelian quandles. Let \( \sigma = (2, 12)(3, 8)(5, 15)(6, 11)(9, 14) \in \text{Aut}(X) \). Consider the constant quandle action \( \phi : Q(3_1) \to \text{Conj}(\text{Aut}(X)) \) of \( Q(3_1) \) on \( X \) given as
\[ \phi(a) = \sigma \]
for all \( a \in Q(3_1) \). Then the derivation quandle with respect to the action \( \phi \) is
\[ \text{Der}_\phi(Q(3_1), X) = \{f_1, f_2, f_3, f_4, f_5\}, \]
where \( f_1(a) = 1, f_2(a) = 4, f_3(a) = 7, f_4(a) = 10 \) and \( f_5(a) = 13 \) for all \( a \in Q(3_1) \). Further, the matrix of \( \text{Der}_\phi(Q(3_1), X) \) is
\[ \begin{bmatrix} f_1 & f_3 & f_5 & f_2 & f_4 \\ f_5 & f_2 & f_4 & f_1 & f_3 \\ f_4 & f_1 & f_3 & f_5 & f_2 \\ f_3 & f_5 & f_2 & f_4 & f_1 \\ f_2 & f_4 & f_1 & f_3 & f_5 \end{bmatrix}. \]

The following example illustrates that, sometimes, coloring by the quandle \( \text{Conj}(\text{Aut}(X)) \) can distinguish knots where coloring by the quandle \( X \) itself fails.

**Example 7.4.** Consider the trefoil knot \( 3_1 \) and the figure eight knot \( 4_1 \). Let \( X \) be a quandle with matrix
\[ \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 2 & 4 & 6 & 8 & 10 \\ 11 & 2 & 4 & 6 & 8 & 10 & 1 & 3 & 5 & 7 & 9 \\ 10 & 1 & 3 & 5 & 7 & 9 & 11 & 2 & 4 & 6 & 8 \\ 9 & 11 & 2 & 4 & 6 & 8 & 10 & 1 & 3 & 5 & 7 \\ 8 & 10 & 1 & 3 & 5 & 7 & 9 & 11 & 2 & 4 & 6 \\ 7 & 9 & 11 & 2 & 4 & 6 & 8 & 10 & 1 & 3 & 5 \\ 6 & 8 & 10 & 1 & 3 & 5 & 7 & 9 & 11 & 2 & 4 \\ 5 & 7 & 9 & 11 & 2 & 4 & 6 & 8 & 10 & 1 & 3 \\ 4 & 6 & 8 & 10 & 1 & 3 & 5 & 7 & 9 & 11 & 2 \\ 3 & 5 & 7 & 9 & 11 & 2 & 4 & 6 & 8 & 10 & 1 \\ 2 & 4 & 6 & 8 & 10 & 1 & 3 & 5 & 7 & 9 & 11 \end{bmatrix}. \]

A GAP computation yields
\[ |\text{Hom}(Q(3_1), X)| = |\text{Hom}(Q(4_1), X)| = 11, \]
whereas
\[ |\text{Hom}(Q(3_1), \text{Conj(Aut}(X))))| = 110 \]
and
\[ |\text{Hom}(Q(4_1), \text{Conj(Aut}(X))))| = 330. \]
Finally, the example below shows that the derivation polynomial is a proper enhancement of the quandle coloring invariant. The same example shows that the total derivation quandle is a stronger invariant than the hom quandle.

**Example 7.5.** Consider the figure eight knot 4_1 and the knot 5_2 (see [3]).

![Figure 3. Knot 5_2](image)

Let $X$ be the quandle as in Example 7.4. It can be checked that $X$ is an abelian quandle. Then a GAP computation yields

$$|\text{Hom}(Q(4_1), X)| = |\text{Hom}(Q(5_2), X)| = 11$$

and

$$|\text{Hom}(Q(4_1), \text{Conj(Aut}(X)))| = |\text{Hom}(Q(5_2), \text{Conj(Aut}(X)))| = 330.$$  

Thus, coloring by the quandles $X$ and $\text{Conj(Aut}(X))$ are not able to distinguish the knots 4_1 and 5_2. Since the cardinality of the quandle $X$ is 11, it follows that both the knots have only trivial colorings by $X$, and consequently

$$\text{Hom}(Q(4_1), X) \cong \text{Hom}(Q(5_2), X) \cong X.$$ 

Hence, the *hom quandle* invariant of [5] is also not able to distinguish the knots 4_1 and 5_2.

Our computations show that the derivation polynomials of the knots 4_1 and 5_2 with respect to $X$ are

$$D_X(5_2)(u) = 11 + 120u + 209u^2$$

and

$$D_X(4_1)(u) = 11 + 230u + 99u^2,$$

respectively, and hence distinguishes the knots 4_1 and 5_2. In fact, the total derivation quandles $D(Q(5_2), X)$ and $D(Q(4_1), X)$ have sizes 220 and 110, respectively, and hence also distinguishes the two knots.

Moreover, there exists non-empty derivation sets for some non-constant actions of $Q(5_2)$ on $X$, whereas all the derivation sets corresponding to non-constant actions of $Q(4_1)$ on $X$ are empty. This is also one of the criterion to distinguish between knots using derivations.

**Remark 7.6.** It would be interesting to have examples for non-equivalent knots $K_1, K_2$, a finite abelian quandle $X$ and compatible quandle actions $\phi_1, \phi_2$ of $Q(K_1), Q(K_2)$ on $X$, respectively, such that $\text{Hom}(Q(K_1), X) \cong \text{Hom}(Q(K_2), X)$, $|\text{Der}_{\phi_1}(Q(K_1), X)| = |\text{Der}_{\phi_2}(Q(K_2), X)|$, but $\text{Der}_{\phi_1}(Q(K_1), X) \neq \text{Der}_{\phi_2}(Q(K_2), X)$. 
8. INVARIANTS FOR VIRTUAL KNOTS USING DERIVATIONS

Since the quandle coloring invariant can be extended to virtual knots and links, the ideas of sections 3, 5, and 6 can be easily generalised to the virtual setting. In [15], Kauffman extended the idea of knot quandle to virtual knots. Manturov [18, 19] further generalised it to construct virtual quandles.

Recall from [19] that a virtual quandle, denoted $Q^\alpha$, is a quandle $Q$ together with an automorphism $\alpha \in \text{Aut}(Q)$. If $\alpha = \text{id}_Q$, then we obtain the usual quandle. A homomorphism of virtual quandles $f : Q^\alpha_1 \to Q^\beta_2$ is a quandle homomorphism $f : Q_1 \to Q_2$ such that $\beta \circ f = f \circ \alpha$.

In [19], Manturov associated a virtual quandle to each virtual knot (in fact, virtual link) and proved the following

**Theorem 8.1.** Let $K_1$ and $K_2$ be two equivalent tame virtual knots with virtual quandles $VQ(K_1)^{\alpha_1}$ and $VQ(K_2)^{\alpha_2}$, respectively. Then $VQ(K_1)^{\alpha_1} \cong VQ(K_2)^{\alpha_2}$.

Observe that if $X^\beta$ is a virtual quandle, then its conjugation quandle $\text{Conj}(\text{Aut}(X))$ can be viewed as a virtual quandle by considering the inner automorphism $\hat{\beta}$ of $\text{Aut}(X)$ induced by $\beta$. As in Section 3, we can define action of a virtual quandle on another virtual quandle.

**Definition 8.2.** An action of a virtual quandle $Q^\alpha$ on a virtual quandle $X^\beta$ is a quandle homomorphism $\phi : Q \to \text{Conj}(\text{Aut}(X))$ such that the following diagram commutes

\[
\begin{array}{ccc}
Q & \xrightarrow{\phi} & \text{Aut}(X) \\
\downarrow{\alpha} & & \downarrow{\hat{\beta}} \\
Q & \xrightarrow{\phi} & \text{Aut}(X),
\end{array}
\]

In other words, $\phi$ is a homomorphism of virtual quandles.

Notice that, for $q \in Q$, the preceding definition implies

\[\phi(q) = \beta^{-1} \circ \phi(\alpha^{-1}(q)) \circ \beta.\] (8.0.1)

**Example 8.3.** Examples of virtual quandle actions:

1. Let $Q^\alpha$ and $X^\beta$ be two virtual quandles. Let $f \in C_{\text{Aut}(X)}(\beta)$, the centraliser of $\beta$ in $\text{Aut}(X)$. Then the constant map $\phi : Q \to \text{Conj}(\text{Aut}(X))$ sending all elements of $Q$ to $f$ is a virtual quandle action of $Q^\alpha$ on $X^\beta$.

2. If $Q^\alpha$ is a virtual quandle such that $\alpha^2 = \text{id}_Q$, then the map $\phi : Q \to \text{Conj}(\text{Aut}(Q))$ given by $q \mapsto S_q$ is a virtual quandle action of $Q^\alpha$ on $Q^\alpha$.

Define the set of derivations with respect to the virtual quandle action $\phi$ of $Q^\alpha$ on $X^\beta$ as $\text{Der}_\phi(Q^\alpha, X^\beta) = \{f : Q \to X \mid f \text{ satisfy the derivation condition (3.0.2)} \text{ and } \beta \circ f = f \circ \alpha\}$. 
If $\phi$ is the trivial action, then $\text{Der}_\phi(Q^\alpha, X^\beta) = \text{Hom}(Q^\alpha, X^\beta)$, the set of all virtual quandle homomorphisms from $Q^\alpha$ to $X^\beta$. Further, if we take $X^\beta$ to be a virtual abelian quandle, then we obtain the following

**Theorem 8.4.** Let $Q^\alpha$ and $A^\beta$ be virtual quandles with $A$ abelian and $\phi$ a virtual quandle action of $Q^\alpha$ on $A^\beta$. Then the set $\text{Der}_\phi(Q^\alpha, A^\beta)$ has the structure of a virtual abelian quandle with binary operation

$$\text{Der}(f, g)(q) = f(q) * g(q)$$

for $q \in Q$, and automorphism $\Gamma \in \text{Aut}(\text{Der}_\phi(Q^\alpha, A^\beta))$ given by

$$\Gamma(f) = \beta^{-1} \circ f \circ \alpha^{-1}$$

for $f, g \in \text{Der}_\phi(Q^\alpha, A^\beta)$.

**Proof.** Proving that $\text{Der}_\phi(Q^\alpha, A^\beta)$ is an abelian quandle is a routine check. We outline a proof that $\Gamma \in \text{Aut}(\text{Der}_\phi(Q^\alpha, A^\beta))$. Let $f \in \text{Der}_\phi(Q^\alpha, A^\beta)$. Then we have

$$\beta \circ \Gamma(f) = f \circ \alpha^{-1}$$

$$= \beta^{-1} \circ f$$

$$= \beta^{-1} \circ f \circ \alpha^{-1}$$

$$= \Gamma(f) \circ \alpha.$$ 

Further, for $q_1, q_2 \in Q$, we obtain

$$\Gamma(f)(q_1 * q_2) = \beta^{-1} \circ f \circ \alpha^{-1}(q_1 * q_2)$$

$$= \beta^{-1} \circ f \left(\alpha^{-1}(q_1) * \alpha^{-1}(q_2)\right)$$

$$= \beta^{-1}(f(\alpha^{-1}(q_1))) * \beta^{-1}(f(\alpha^{-1}(q_2)))^{\phi(\alpha^{-1}(q_1))}$$

$$= \beta^{-1}(f(\alpha^{-1}(q_1))) * \beta^{-1}(f(\alpha^{-1}(q_2)))^{\phi(q_1), \text{ by } 8.0.1}$$

$$= \Gamma(f)(q_1) \star \Gamma(f)(q_2)^{\phi(q_1)}.$$ 

Thus, $\Gamma(f) \in \text{Der}_\phi(Q^\alpha, A^\beta)$. It is easy to see that $\Gamma$ is a quandle automorphism with inverse $f \mapsto \beta \circ f \circ \alpha$. 

Let $Q_1^\alpha, Q_2^\alpha$ be two virtual quandles and $A_1^\beta, A_2^\beta$ two virtual abelian quandles. Let

$$\phi_1 : Q_1^\alpha \rightarrow \text{Conj}(\text{Aut}(A_1^\beta))$$

and

$$\phi_2 : Q_2^\alpha \rightarrow \text{Conj}(\text{Aut}(A_2^\beta))$$

be virtual quandle actions of $Q_1^\alpha, Q_2^\alpha$ on $A_1^\beta, A_2^\beta$, respectively. Let $\sigma : Q_2^\alpha \rightarrow Q_1^\alpha$ and $\tau : A_1^\beta \rightarrow A_2^\beta$ be virtual quandle homomorphisms satisfying the compatibility condition

$$\tau(a^{\phi_1(\sigma(q))}) = \tau(a)^{\phi_2(q)}$$

for $q \in Q_2$ and $a \in A_1$. The following result is an analogue of Theorem 5.4 for the virtual case.
Theorem 8.5. Let $Q_1^{\alpha_1}, Q_2^{\alpha_2}$ be two virtual quandles and $A_1^{\beta_1}, A_2^{\beta_2}$ be abelian virtual quandles. Let $\phi_1 : Q_1^{\alpha_1} \to \text{Conj} \left( \text{Aut}(A_1^{\beta_1}) \right)$ and $\phi_2 : Q_2^{\alpha_2} \to \text{Conj} \left( \text{Aut}(A_2^{\beta_2}) \right)$ be virtual actions of $Q_1^{\alpha_1}, Q_2^{\alpha_2}$ on $A_1^{\beta_1}, A_2^{\beta_2}$, respectively. Let $\sigma : Q_2^{\alpha_2} \to Q_1^{\alpha_1}$ and $\tau : A_1^{\beta_1} \to A_2^{\beta_2}$ be action compatible virtual quandle homomorphisms. Then there exists a virtual quandle homomorphism $\Phi : \text{Der}_{\phi_1}(Q_1^{\alpha_1}, A_1^{\beta_1}) \to \text{Der}_{\phi_2}(Q_2^{\alpha_2}, A_2^{\beta_2})$.

Further, if $\sigma$ and $\tau$ are both isomorphisms, then so is $\Phi$. Additionally, if $Q_1, Q_2$ are finitely generated and $A_1, A_2$ are finite, then

$$|\text{Der}_{\phi_1}(Q_1^{\alpha_1}, A_1^{\beta_1})| = |\text{Der}_{\phi_2}(Q_2^{\alpha_2}, A_2^{\beta_2})|.$$

As a consequence of Theorem 8.5 we obtain

Theorem 8.6. Derivation quandles of a virtual tame knot with respect to a virtual abelian quandle are knot invariants.

Given two virtual quandles $(X_1^{\beta_1}, *)_1$ and $(X_2^{\beta_2}, *)_2$, the quandle $X_1 \sqcup X_2$ as defined by (5.0.3) is virtual with respect to the natural automorphism induced by $\beta_1$ and $\beta_2$. Another consequence of Theorem 8.5 is the following

Theorem 8.7. The total derivation quandle with respect to a virtual abelian quandle is an invariant of tame virtual knots and contains the virtual hom quandle as an abelian subquandle.

As in Section 6 we can define the derivation polynomial of a virtual quandle $Q^\alpha$ with respect to a finite virtual quandle $X^\beta$ as

$$D_{X}(Q^\alpha)(u) = |\text{Hom}(Q^\alpha, X^\beta)| + \sum_{\phi \text{ non-trivial action}} u|\text{Der}_{\phi}(Q^\alpha, X^\beta)| + 1. \tag{8.0.4}$$

Notice that $D_{X}(Q^\alpha)(0) = |\text{Hom}(Q^\alpha, X^\beta)|$, the virtual quandle coloring invariant. Thus, we obtain the following

Theorem 8.8. The derivation polynomial of a virtual tame knot with respect to a finite virtual quandle is a knot invariant and is an enhancement of the virtual quandle coloring invariant.

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References

[1] V. G. Bardakov, P. Dey and M. Singh, Automorphism groups of quandles arising from groups, Monatsh. Math. 184 (2017), 519–530.
[2] V. G. Bardakov, T. Nasybullov and M. Singh, Automorphism groups of quandles and related groups, Monatsh. Math. to appear (arxiv.org/abs/1804.01113).
[3] D. Bar-Natan, The knot atlas [http://katlas.org/wiki/Main Page].
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[4] J. Carter, A survey of quandle ideas, Introductory lectures on knot theory, 22–53, Ser. Knots Everything, 46, World Sci. Publ., Hackensack, NJ, 2012.
[5] A. S. Crans and S. Nelson, Hom quandles, J. Knot Theory Ramifications 23 (2014), 1450010, 18 pp.
[6] M. Elhamdadi, J. Macquarrie and R. Restrepo, Automorphism groups of quandles, J. Algebra Appl. 11 (2012), 1250008, 9 pp.
[7] R. Fox, A quick trip through knot theory, 1962 Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961) pp. 120–167 Prentice-Hall, Englewood Cliffs, N.J.
[8] C. Fragnaud and G. Ellis, HAP: Homological Algebra and Programming, http://hamilton.nuigalway.ie/Hap/www/SideLinks/About/aboutQuandles.html.
[9] GAP: Groups, Algorithms, and Programming, Version 4.8.10. Available at http://www.gap-system.org.
[10] B. Ho and S. Nelson, Matrices and finite quandles, Homology Homotopy Appl. 7 (2005), 197–208.
[11] Xiang-dong Hou, Automorphism groups of Alexander quandles, J. Algebra 344 (2011), 373–385.
[12] Y. Ishihara and H. Tamaru, Flat connected finite quandles, Proc. Amer. Math. Soc. 144 (2016), 4959–4971.
[13] D. Joyce, A classifying invariant of knots, the knot quandle, J. Pure Appl. Algebra 23 (1982), 37–65.
[14] S. Kamada, Knot invariants derived from quandles and racks, Invariants of knots and 3-manifolds (Kyoto, 2001), 103–117, Geom. Topol. Monogr., 4, Geom. Topol. Publ., Coventry, 2002.
[15] L. H. Kauffman, Virtual knot theory, European Journal of Combinatorics 20 (7) (1999), 662–690.
[16] O. Loos, Reflexion spaces and homogeneous symmetric spaces, Bull. Amer. Math. Soc. 73 (1967), 250–253.
[17] O. Loos, Spiegelungsräume und homogene symmetrische Räume, Math. Z. 99 (1967), 141–170.
[18] V. O. Manturov, On invariants of virtual links, Acta Appl. Math. 72 (3) (2002), 295–309.
[19] V. O. Manturov and D. P. Ilyutko, Virtual knots. The state of the art, Translated from the 2010 Russian original. With a preface by Louis H. Kauffman. Series on Knots and Everything, 51. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013. xxvi+521 pp.
[20] S. Matveev, Distributive groupoids in knot theory, (Russian) Mat. Sb. (N.S.) 119 (161), 78–88, 160 (1982).
[21] S. Nelson, The combinatorial revolution in knot theory, Notices Amer. Math. Soc. 58 (2011), 1553–1561.
[22] J. Przytycki, 3-coloring and other elementary invariants of knots, Knot theory (Warsaw, 1995), 275–295, Banach Center Publ., 42, Polish Acad. Sci. Inst. Math., Warsaw, 1998.
[23] R. Rubinsztein, Topological quandles and invariants of links, J. Knot Theory and its Ramifications 16 (2007), 789–808.
[24] M. Singh, Classification of flat connected quandles, J. Knot Theory Ramifications 25 (2016), 1650071, 8 pp.
[25] M. Szymik, Alexander-Beck modules detect the unknot, arXiv:1610.08306.
[26] M. Szymik, Quandle homology is Quillen homology, arXiv:1612.06315.
[27] M. Takasaki, Abstraction of symmetric transformation, Tohoku Math. J. 49 (1943), 145-207.

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