Electric field influenced coordinate jump of the guiding centre and magnetotransport

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ABSTRACT

Our studies formulate a classical theory to study the influence of in-plane electric field on electron-impurity scattering process and magnetotransport in a two-dimensional space, in the presence of a strong out-of-plane magnetic field. Our studies connect classical scattering and quantum Landau Level broadening, as will be reviewed in this report. We derived an electric current formula in agreement with the current derived from the Drude theory under a strong magnetic field. Our electric current formula is derived microscopically from the migration of the guiding centres at strong magnetic field regime \(\omega t > 1\). The electron-impurity scattering under electromagnetic field not only shifts the guiding centre coordinates \(X\) and \(Y\), but also changes cyclotron radius \(R\). The change of cyclotron radius \(R\) compensates the change of the electric potential energy during the scattering process. The broadening of cyclotron radius is a classical manifestation of Landau Levels broadening. Our conductivity \(\frac{\sigma}{B^2}\) derived from our current formula in the linear response regime results in the same as the current derived from Kubo current fluctuation theory, providing a special case of the fluctuation-dissipation theorem.

1. Introduction

In condensed matter physics, the central problem surrounds how an out-of-plane magnetic field influences electron transport. However, the issue of how an in-plane electric field influences magnetotransport is equally important. The electric field plays an important role in magnetotransport not only by providing a driving force for electron, but also by breaking the symmetry of a system. Various behaviours, such as the electric field tunable band gap in bilayer or multilayer graphene [1–5], electric field influenced Landau Level broadening [6–11], electric field shifted ferroelectric phase transition [12–15],
electric field induced valley polarisation [16], etc. contain a wealth of situations with symmetry breaking due to electric field.

The electron transport in strong magnetic field can be generally described by the migration of guiding centre, of which the theory is pioneered by Davydov and Pomeranchuk [17]. The derivation of coordinate jump of guiding centre during scattering was pioneered by Kubo current fluctuation theory [18–20], in which the coordinate jump of guiding centre has no electric field dependence, without considering the influence of electric field on scattering process. However, if we do not consider the influence of electric field during the scattering process, the electron is either circularly running around an impurity without scattering, or repeatedly scattered on one impurity. In order to avoid such localisation effect, Kubo’s work implicitly assumes that electrons scatter on impurity for one time. The magnetoconductivity is calculated by the spontaneous fluctuation of guiding centre current in the equilibrium state [18–20]. The average current vanishes, but the fluctuation does not. Our approach, however, is based on Ohm’s law (i.e. the current response to the electric field). We derived our electric current formula with the inclusion of electric field during the scattering process, therefore, the average current is not zero.

In our work, we derived an electric current formula in the presence of an external in-plane electric field and a strong out-of-plane magnetic field, within two dimensional disordered system. In the presence of an out-of-plane electric field, electrons run along a cyclotron trajectory, while the guiding centre approaches an impurity in the direction perpendicular to both electric and magnetic field. The electrons drift away after being scattered without being repeatedly scattered due to the influence of the electric field. Our electric current formula is derived microscopically from the migration of the guiding centres at strong magnetic field regime $\omega \tau > 1$. There are two findings. First, the electron-impurity scattering under electromagnetic field not only shifts the guiding centre coordinates $X$ and $Y$, but also changes cyclotron radius $R$. The change of cyclotron radius $R$ compensates the change of the electric potential energy during the coordinate jump. The broadening of cyclotron radius is a classical manifestation of Landau Levels broadening. Second, our case provides a simple demonstration of the fluctuation-dissipation theorem. We will show that the conductivity $\frac{\sigma}{\rho}$ derived from our current formula in the linear response regime gives the same result with the conductivity derived from Kubo current fluctuation theory [18–20], which is in agreement with the fluctuation-dissipation theorem, i.e. the response of a system to an applied electric field in thermodynamic equilibrium is the same as its response to a spontaneous fluctuation. Specifically, there are three quantities modified by electric field, eigenfunction in $W_Nk, N'k'$, eigenenergy $\epsilon_{N,Y}$ and the energy level density $\delta(\epsilon_{N',Y'} - \epsilon_{N,Y})$. Thus, the Kubo’s current fluctuation corresponds to our linear response current originated from the electric field shifting of Landau Level.
We show the following results. (1) We derive a formula of longitudinal electric current based on Ohm’s law and keep the conductivity in linear response regime. (2) During the scattering, there is a sudden shift of guiding centre coordinates \((X, Y, R, \varphi) \rightarrow (X', Y', R', \varphi')\). (3) The magnetoconductivity has the same form as Drude model under strong magnetic field regime. (4) The cyclotron radius broaden during the scattering due to a change of electric potential energy during the scattering process. (5) The energy level is shifted and broadened during scattering due to the presence of electric field. This corresponds to the Landau Level asymmetric broadening in quantum picture.

2. Coordinate jump of guiding center

We start from the classical picture of electron motion in two-dimensional plane with no electron spin and interaction between electrons. The electric field is in \(y\)-direction and magnetic field is in \(z\)-direction perpendicular to the \(x−y\) plane, where the electrons move, as shown in Figure 1. Each of the impurities is randomly and dilutely distributed, therefore, there is no correlation between impurities.

The electron motion in electromagnetic field can be described as the superposition of a relatively fast circular motion around guiding centre and a relatively slow drift of guiding centre. The guiding centre drift velocity is \(\frac{E \times B}{B^2}\) in \(x\)-direction (perpendicular to both electric and magnetic field), and the relative velocity of electron around the guiding centre is \(v_{cyc}\) (see Figure 1). The guiding centre drift velocity is perpendicular to both the electric field and magnetic field. Unlike a closed circular orbital velocity, the size of electron velocity in lab frame \(v_{lab}\) is changing during the cyclotron motion. \(v_{lab}\) is a summation of the velocity of the guiding centre \(v_{gc} = \frac{E}{B} x\) and the relative velocity of electron around the guiding centre \(v_{cyc}\): \(v_{lab} = v_{cyc} + v_{gc}\).

In order to study the guiding centre motion, we use guiding centre coordinate and spherical coordinate \((X, Y, R, \varphi)\), where \((X, Y)\) are the guiding centre coordinates, \(R\) is the cyclotron radius, and \(\varphi\) is the angle of electron on the circular orbit. Before scattering, the variables \((Y, R)\) stays unchanged.

Figure 1. (Colour online) The cyclotron motion of electron and with electron-impurity scattering in two dimension \(x-y\) plane.
and variables \((X, \varphi)\) change with time: \(X = X_0 + \frac{E}{B} t\), \(\varphi = \varphi_0 + \omega t\), where \(X_0\) and \(\varphi_0\) are \(X(t = 0)\) and \(\varphi(t = 0)\), respectively. During the scattering, there is a sudden shift of all four variables \((X, Y, R, \varphi) \rightarrow (X', Y', R', \varphi')\) (see Figure 2). The shift of the guiding centre coordinate \(\delta X, \delta Y\) is called coordinate jump of guiding centre (see Figure 2).

The shift of cyclotron radius \(\delta R\) is due to the energy conservation law at the presence of electric potential energy during scattering (see Figure 2). With electric field presence at the scattering process, the energy conservation is \(\frac{1}{2}m\omega^2 R^2 + eE \cdot Y = \frac{1}{2}m\omega^2 R'^2 + eE \cdot Y'\). Because the potential energy shifts along with the coordinate jump of guiding centre \(Y\), the kinetic energy \(\frac{1}{2}m\omega^2 R^2\) changes in order to compensate the shift of potential energy \(eE \cdot Y\). The cyclotron radius is explicitly proportional to the kinetic energy, therefore, the cyclotron radius changes by the scattering.

(In order to clarify the meaning of the electric potential energy, we have the following statement. The guiding centre coordinate \(Y\) is the average coordinate of electron over one cyclotron period \(\overline{Ye}\). The electric potential energy \(eE \cdot Y\) is thus the average energy of electron over one cyclotron period.)

While in traditional consideration without electric field in scattering process, the energy conservation is \(\frac{1}{2}m\omega^2 R^2 = \frac{1}{2}m\omega^2 R'^2\), composed of only kinetic energy. Therefore, it requires \(R = R'\).

\[ \begin{align*}
\delta X &= (X - X') \\
\delta Y &= (Y - Y')
\end{align*} \]

**Figure 2.** (Colour online) The cyclotron motion of electron and with electron-impurity scattering in two dimension \(x–y\) plane.
The expression of coordinate jump of guiding centre is

\[ \delta Y = -R' \sin \varphi'_\text{coll} + R \sin \varphi_\text{coll}, \]  
\[ \delta X = -R' \cos \varphi'_\text{coll} + R \cos \varphi_\text{coll}. \]  

(1)

(2)

3. Longitudinal current of guiding center

In this section, we will calculate the steady state current of the system. The original condition is that the guiding centre, with a velocity of \( \frac{E}{B} \), is uniformly distributed in real space. We now introduce the event line. The event line is a tool to count the scattering events. It is assumed that every scattering event takes place when the guiding centre passes through the event line, even though the actual scattering takes place at different scattering point along the edge of the scatterer. This is a valid assumption due to the steady state condition of electrons. Therefore, the event line is a tool to abstract all the scattering events occurring in unit time as taking place on this event line. The event line passes through the centre of scatterer and overlaps with \( x \) axis. The guiding centres are uniformly distributed on the event line during scattering. When the guiding centres are on the event line, the electrons are uniformly distributed in \( Y \) and \( \varphi_0 \) and has \( R \) dependence through Fermi distribution. \( \varphi_0 \) is the cyclotron angle when the guiding centre reaches the event line, i.e. \( \varphi_0 = \varphi_\text{coll} + \omega \Delta t \) (\( \Delta t \) is the time difference between \( \varphi_0 \) and \( \varphi_\text{coll} \)).

In order to calculate the guiding centre current, we sum up the coordinate jump of guiding centre from all the scattering events taking place in unit time.

We start from the derivation of electron distribution function represented by the guiding centre coordinate. The density of electrons is

\[ n = \int f \frac{d^2 k}{(2\pi)^2}, \]  

(3)

where \( p \) is the momentum, and \( f \) is the Fermi distribution. Using spherical coordinate,

\[ n = \int f \frac{k dk d\varphi_0}{(2\pi)^2}. \]  

(4)

We prove that \( \frac{k dk d\varphi_0}{(2\pi)^2} \) is equivalent to \( \left( \frac{eB}{\hbar} \right)^2 R dR d\varphi_0 \) represented by guiding centre coordinate (Appendix 2), therefore,

\[ n = \int \left( \frac{eB}{\hbar} \right)^2 fR dR d\varphi_0 = \int g(R) dR d\varphi_0, \]  

(5)

where we define the quantity \( \left( \frac{eB}{\hbar} \right)^2 fR \equiv g(R) \), which is the Fermi distribution function in the guiding centre coordinate.
The number of guiding centre crossing the event line per unit time in $dRd\varphi_0dY$ is

$$\frac{E}{B}g(R)\,dR\,d\varphi_0\,dY,$$

(6)

where $\frac{E}{B} = \frac{4x}{dt}$, i.e. the guiding centre drift velocity along $x$ direction.

The longitudinal current density along the direction of electric field is

$$j_y = \sum_i -en_i v_i$$

$$= -e \int_0^{2\pi} \int_0^{R_f} g(R)$$

$$\times \int_{-(R+a)}^{(R+a)} \frac{E}{B} n_{im} \delta Y(R, \varphi_0, Y)\,dR\,d\varphi_0\,dY,$$

(7)

where $n_{im}$ is the impurity density, the subscript $i$ denotes $i$th scattering event.

The integral $\frac{E}{B} \int_{-(R+a)}^{(R+a)} \int_0^{2\pi} \int_0^{R_f} g(R) \times \delta Y(R, \varphi_0, Y)\,dR\,d\varphi_0\,dY$ in Equation (7) is the summation of all the coordinate jump occurring in unit time at one impurity. The nonzero longitudinal current density $j_y$ indicates the average coordinate of the guiding centre shifts from $\sum_i Y_i = 0$ to $\sum_i Y_i' \neq 0$ during scattering.

In order to solve Equation (7), we firstly derive $dY(R, \varphi_0, Y)$. However, it is hard to derive $dY$ as a function of $(R, \varphi_0, Y)$ analytically. Instead, we derive $dY$ as a function of $(R, \varphi_{im}, \theta_v)$

$$\delta Y(R, \varphi_{im}, \theta_v)$$

$$= \frac{1}{B\omega_e} [E \cos \theta_v + \sqrt{(-E^2 + B^2v_{yc}^2 + E^2 \cos^2 \theta_v)]}$$

$$\times (-2 \cos (\theta_v - \varphi_{im}) \cos \varphi_{im}),$$

(8)

(9)

where $\varphi_{im}$ is the angle (starting from $x$-axis) on the impurity when the scattering takes place; the $\theta_v$ is the angle of the incident velocity of electron (starting from $x$-axis). The derivation of $\delta Y(R, \varphi_{im}, \theta_v)$ is in Appendix 1.

However, the distribution function of electron is not uniform along $\varphi_{im}$ and $\theta_v$. Therefore, we first calculate the weighting factor of $\varphi_{im}$ and $\theta_v$, respectively. The pair of variables $(\varphi_0, Y)$ can be transformed to pair of variables $(\varphi_{im}, \theta_v)$ by Jacobian determinant

$$\begin{bmatrix}
\frac{\partial \varphi_0}{\partial \varphi_{im}} & \frac{\partial \varphi_0}{\partial \theta_v} \\
\frac{\partial Y}{\partial \varphi_{im}} & \frac{\partial Y}{\partial \theta_v}
\end{bmatrix},$$

which determines the weighting factor.
The Jacobian determinant can be calculated by the following three equations

\[ Y_0 = a \sin \varphi_{im} - R \sin \varphi_{coll}, \]  
\[ \varphi_0 = \varphi_{coll} - \frac{a \cos \varphi_{im} - R \cos \varphi_{coll}}{\Delta X}, \]  
\[ \tan \theta_v = \frac{v_{cyc} \cos \varphi_{coll}}{-v_{cyc} \sin \varphi_{coll} + \frac{\pi}{B}}, \]

where \( \Delta X = \frac{E}{Bo_c} \), which is the drift distance of guiding centre after one cyclotron period. The range of \( \theta_v \) in the integral is restricted by the range \([\varphi_{im} + \frac{\pi}{2}, \varphi_{im} + \frac{3\pi}{2}]\).

Integrating over \( dR \, d\varphi_{im} \, d\theta_v \), the longitudinal current density in Equation (7) becomes

\[ j_y = -en_{im} \frac{E}{B} \int_0^{\frac{2\pi}{3}} \int_0^{\varphi_{im} + \frac{2\pi}{3}} g(R) \, \delta Y(R, \varphi_{im}, \theta_v) \]
\[ \times \left[ \begin{array}{c} \partial \varphi_0 \\ \partial \varphi_{im} \\ \partial \varphi_Y \\ \partial \varphi_{im} \\ \end{array} \right] \, dR \, d\theta_v \, d\varphi_{im}. \]  

By solving Equation (13), the longitudinal current density to the first order of electric field is

\[ j_y = en_{im} \frac{E}{B} \left( \frac{eB}{h} \right)^2 (8a\pi) \frac{1}{3} \left( \frac{h k_f}{eB} \right)^3. \]  

Note that \( \delta Y \) in our theory has electric field dependence (as seen in Equation (9)) and can be expanded with respect to electric field

\[ \delta Y = - \frac{\sqrt{B^2 R_0^2 \omega_c^2 (\cos \theta_v + \cos (\theta_v - 2\varphi_{im}))}}{Bo_c} \]
\[ - \frac{\cos \theta_v (\cos \theta_v + \cos (\theta_v - 2\varphi_{im}))}{Bo_c} \, E + O(E^2). \]

Because the Jacobian determinant has \( E^{-1} \) and \( E^0 \) terms, we keep only \( E^0 \) and \( E^1 \) terms in \( \delta Y \) in order to keep the current \( j_y \) in linear regime. (Note, the \( E^{-1} \) term in Jacobian determinant times \( E^0 \) term in \( \delta Y \) produces \( E^{-1} \) term, which, however, will vanish after integration in Equation (13).)

The conductivity thus is

\[ \sigma_{yy} = \frac{8\pi^2}{3} \frac{e^2}{\hbar} (n_{im}a\lambda_f)(\frac{\hbar}{eB})^2, \]

which can be transformed to

\[ \sigma_{yy} = \frac{mn}{B^2 \tau}, \]  

based on the transport relaxation time \( \frac{1}{\tau} = \frac{8}{3} n_i v_a \) at \( B = 0 \), and the electron density \( n = \frac{k_f^4}{4\pi} \).
4. Transverse current of guiding center

The transverse current density along the direction of electric field is

\[ j_x = \sum_i -en_i v_i \]

\[ = -en_{im} \frac{E}{B} \int_{-(R+a)}^{(R+a)} \int_0^{2\pi} \int_0^{R_F} g(R) \]

\[ \times \delta X(R, \varphi_0, Y) \, d\varphi_0 \, dY. \]

(18)

Same as how we derive \( \delta Y(R, \varphi_{im}, \theta_v) \), we derive \( \delta X(R, \varphi_{im}, \theta_v) \)

\[ \delta X(R, \varphi_{im}, \theta_v) = \frac{1}{B\omega_c} \left[ E \cos \theta_v + \sqrt{(-E^2 + B_0^2 v_{yc}^2 + E^2 \cos^2 \theta_v)} \right] \]

\[ \times 2 \cos (\theta_v - \varphi_{im}) \sin \varphi_{im}, \]

(19)

(20)

The derivation of \( \delta X(R, \varphi_{im}, \theta_v) \) is in Appendix 1.

Integrating over \( dR \, d\varphi_{im} \, d\theta_v \), the transverse current density in Equation (18) becomes

\[ j_x = -en_{im} \frac{E}{B} \int_0^{2\pi} \int_0^{\varphi_{im}} \int_0^{\varphi_{im} + \frac{\pi}{2}} \int_0^{R_F} g(R) \, \delta X(R, \varphi_{im}, \theta_v) \]

\[ \times \left[ \frac{\partial \varphi_0}{\partial \varphi_{im}} \frac{\partial \varphi_0}{\partial \theta_v} \right] \frac{\partial Y}{\partial \varphi_{im}} \frac{\partial Y}{\partial \theta_v} \right] \, dR \, d\theta_v \, d\varphi_{im}. \]

(21)

By solving Equation (21), the transverse current density is

\[ j_x = 0. \]

(22)

There is no anomalous component in transverse current. The only transverse current is from drift current of guiding centre.

5. Discussion – Proof of fluctuation-dissipation theorem

We provide a simple demonstration of fluctuation-dissipation theorem below. The quantum counterpart of our theory (shown below), combining with Kubo’s current fluctuation theory [18–21] is a good demonstration of the fluctuation-dissipation theorem. (The electric field is along \( x \) direction in Kubo’s theory. For consistency, we keep the electric field along \( y \) direction and the notation of longitudinal coordinate jump as \( Y' - Y \) in our manuscript.)

In quantum counterpart of our theory, the longitudinal current is expressed as
\[ j_y = -e \sum_{N,N'} \sum_{Y,Y'} 2f_N W_{Nk,N'k'} (Y' - Y), \]  
\( (23) \)

where \( f_N \) is the Fermi distribution function, \( W_{Nk,N'k'} \) is scattering probability, i.e. \( W_{Nk,N'k'} = \frac{2\pi}{\hbar} |\langle \psi_{N,Y} | U | \psi_{N,Y'} \rangle|^2 \delta (e_{N,Y'} - e_{N,Y}), \) and the factor 2 is due to spin degeneracy considered in our quantum theory, as well as Kubo's theory.

By exchanging \( N, Y \rightarrow N', Y', \)
\[ j_y = -e \sum_{N',N} \sum_{Y,Y'} 2f_N' W_{N'k',Nk} (Y' - Y), \] 
\( (24) \)

the result of the equation remains the same because \( W_{NY,N'Y'} = W_{N'Y',NY}. \) Therefore, by adding Equation (23) and Equation (24) up, the current becomes
\[ j_y = -e \sum_{N,N'} \sum_{Y,Y'} (f_N - f_N') W_{Nk,N'k} (Y' - Y), \] 
\( (25) \)

There are three quantities modified by electric field, eigenfunction in \( W_{Nk,N'k'}, \) eigenenergy \( e_{N,Y} \) and the energy level density \( \delta (e_{N',Y'} - e_{N,Y}). \) We will show that Kubo’s current fluctuation corresponds to our linear response current originated from the electric field shifting of Landau Level.

The Hamiltonian of an electron under an out of plane magnetic field (\( z \) direction) and an in plane electric field (\( y \) direction) in disordered system is
\[ H = \frac{(P + eA)^2}{2m} - eE \cdot y + U. \] 
\( (26) \)

The Schrödinger equation of electron in this system is
\[ \left\{ \frac{(P + eA)^2}{2m} - eE \cdot y + U \right\} \psi(x, y) = 2m \psi(x, y). \] 
\( (27) \)

Assuming that the solution is \( \psi(x, y) = e^{ik_x x} \psi(y), \) and choosing Landau Gauge \( A_x = -By, A_y = A_z = 0, \) substitute this solution to the Schrödinger equation, we get
\[ \left\{ \frac{1}{2m} (P_x - eBy)^2 - eEy + U \right\} \psi(y) = 2m \psi(y). \] 
\( (28) \)

The above equation can be transformed as
\[ \left\{ -\frac{\partial^2}{\partial y^2} + e^2 B^2 \left[ y - \left( \frac{h k_x}{eB} + \frac{mE}{eB^2} \right) \right]^2 \right. \]
\[ -2m \left[ \frac{mE^2}{2B^2} + \frac{Eh k_x}{B} \right] + 2mU \}
\[ \psi(y) = 2m \psi(y). \] 
\( (29) \)
It has been defined that $Y \equiv L k_x$ (where $l$ is the magnetic length, i.e. $l = \sqrt{\frac{\hbar}{m\omega}}$), which is the guiding centre of cyclotron without electric field, and is a good quantum number. In our case, the position of guiding centre is shifted by the inclusion of electric field, i.e. $Y \rightarrow (Y + \frac{mE}{eB})$. The eigenstate is also modified by electric field as $\psi_N(Y + \frac{mE}{eB})$. The eigenenergy becomes $\epsilon_{N,Y} = \hbar \omega (N + \frac{1}{2}) - eEY = \epsilon_N - eEY$. The distribution function $f_N = \frac{1}{e^{\frac{\epsilon_N + \frac{1}{2}}{k_B T}} + 1}$ remains the same because both the eigenenergy and the chemical potential is shifted by $eEY$, which cancels out.

Therefore, Equation (25) becomes

$$j_y = -\frac{e}{2} \sum_{N,N'} \sum_{Y,Y'} 2(f(\epsilon_N) - f(\epsilon_N'))$$

$$\cdot \frac{2\pi}{\hbar} |\langle \psi_{N',Y'}|U|\psi_{N,Y} \rangle|^2 \delta[\epsilon_{N'} - \epsilon_N - eE(Y' - Y)]$$

$$\cdot (Y - Y'),$$

(30)

because of energy conservation, $\epsilon_{N'} = \epsilon_N + eE(Y' - Y)$,

$$f(\epsilon_{N'}) \equiv f(\epsilon_N + eE(Y' - Y)).$$

As long as $E$ is small, in another word, $eE(Y' - Y)/k_B T \ll \hbar \omega$, the distribution function $f(\epsilon_N + eE(Y' - Y))$ can be expanded at $E = 0$, we get

$$f(\epsilon_N + eE(Y' - Y)) = f(\epsilon_N)$$

$$+ \frac{\partial f(\epsilon_N)}{\partial \epsilon_N} eE(Y' - Y) + O(E^2).$$

(32)

Therefore, the longitudinal current becomes

$$j_y = \frac{e}{2} \sum_{N,N'} \sum_{Y,Y'} 2 \frac{\partial f(\epsilon_N)}{\partial \epsilon_N} eE(Y' - Y)^2$$

$$\cdot \frac{2\pi}{\hbar} |\langle \psi_{N',Y'}|U|\psi_{N,Y} \rangle|^2 \delta[\epsilon_{N'} - \epsilon_N - eE(Y' - Y)].$$

(33)

The current density is $j_y/V$, where $V$ is the volume of the material.

This linear response current corresponds to the conductivity derived from fluctuation of current in Kubo’ theory.

In Kubo’s theory, the conductivity is

$$\sigma_{yy} = \frac{2e^2}{V} \sum_{N,Y,p_z} \sum_{N',Y',p_z'} \frac{\partial f(\epsilon_N(p_z))}{\partial \epsilon_N(p_z)}$$

$$\cdot \frac{1}{2} (Y - Y')^2 W_{N',Y',p_z,N,Y,p_z}.$$
Because the system is 2D in our case, by separating $p_z$, we get

$$\sigma_{yy} = \frac{2e^2}{V} \sum_{N,Y} \sum_{N',Y'} \frac{\partial f(e_N)}{\partial e_N} \frac{1}{2} (Y - Y')^2 W_{NN',YY'}.$$ (36)

Both of our theory and Kubo’s theory gives conductivity $\frac{mn}{B^2\tau}$. Combining Our theory and Kubo’s theory, it is an specific example to prove the fluctuation-dissipation theorem, i.e. the response of a system to an applied electric field in thermodynamic equilibrium is the same as its response to a spontaneous fluctuation.

6. Discussion – comparison with Drude theory

Traditional Drude theory considers the electron-impurity scattering as a friction force macroscopically. The equation of motion is

$$m\dot{v} = -e(E + v \times B) - \frac{mv}{\tau},$$ (37)

where $v$ is the average velocity per electron, $\tau$ is the mean time an electron has travelled since the last collision.

It yields the relationship between longitudinal current density $J$ and electric field $E$,

$$J = \frac{ne^2\tau}{1 + (\frac{eB}{m})^2 \tau^2} E.$$ (38)

Under strong magnetic field limit $\omega \tau \gg 1$, the longitudinal current density becomes

$$J = \frac{mn}{B^2\tau} E.$$ (39)

In our theory, we look into each scattering process in detail and provide a microscopic method to calculate the current. We first brought up the strong magnetic field limit. Based on this limit, the electron motion can be represented by guiding centre motion. Then, we figured out that each scattering process can be pictured as a coordinate jump of guiding centre. By accumulating all the coordinate jumps, we derived the current formula Equation (7), which gives the strong field conductivity. Because of the consideration of detailed scattering process, we provide an explicit expression for reverse of relaxation time $\frac{1}{\tau} = \frac{8}{3} n_i na$ and electron density $n = \frac{k_i^2}{4\pi}$.

7. Discussion – change of cyclotron radius during scattering

Counter-intuitively, the cyclotron radius is changed after each collision (illustrated in Figure 1). Macroscopically, it is due to the change of electric potential
during the coordinate jump, as we mentioned in section 'Coordinate Jump of Guiding Centre'.

To be more specific, the value of velocity at the moment of collision is a constant, i.e. \( |v_{in}| = |v_{out}| \) because of the law of energy conservation. However, the cyclotron velocity before and after collision are different. The cyclotron velocity is defined as the velocity of an electron moving on a closed circular orbit and has the same value everywhere along the closed orbit. However, because of the presence of electric field, the orbit is not closed. The instant velocity is different everywhere along the orbit. During the collision, the guiding centre suddenly shifts its position, together with a shift of electrical potential. The position of the guiding centre is the averaged position of the electron within one cycle, therefore the electric potential of the guiding centre is the averaged potential of the electron. Because of the law of energy conservation, the change of electrical potential during scattering will result in a change of the kinetic energy. Therefore, the cyclotron velocity changes, which leads to a change of cyclotron radius after scattering. The microscopic explanation and mathematical derivation will be shown in Appendix 3.

The change of cyclotron radius can be seen as a change of energy level. As we can see in Equation (7), the total coordinate jump in unit time is

\[
\frac{E}{B} \int_{(R+a)}^{(R-a)} \int_{0}^{2\pi} \int_{0}^{R_f} g(R) \times \delta Y(R, \varphi_0, Y) \, dR \, d\varphi_0 \, dY
= \frac{E}{B} \left( \frac{eB}{h} \right)^2 \left( \frac{8a \pi}{3} \right)^1 \left( \frac{h k_f}{eB} \right)^3.
\]

The potential energy changes by

\[
\sum_i eE \cdot \delta Y_i = eE \frac{E}{B} \left( \frac{eB}{h} \right)^2 \left( \frac{8a \pi}{3} \right) \frac{1}{3} \left( \frac{h k_f}{eB} \right)^3.
\]  

(40)

The same goes with the kinetic energy

\[
\sum_i \Delta \varepsilon_{kin} = -eE \frac{E}{B} \left( \frac{eB}{h} \right)^2 \left( \frac{8a \pi}{3} \right) \frac{1}{3} \left( \frac{h k_f}{eB} \right)^3.
\]  

(41)

The kinetic energy spectrum of electron after scattering is not only broadened, but also asymmetrically broadened with respect to the original kinetic energy before scattering, which leads to a shift in its average energy level.

8. Condition for one-time collision

Our theory is valid when the electron only collides once on an impurity before colliding with another impurity. The impurity has the size of an atom, which is \( a \sim 10^{-9}m \); the cyclotron radius is approximately \( 10^{-6}m \) in 2DEG, relatively large compared with the impurity. Because of this, it is reasonable to consider each of the electrons only collides once on one impurity and then scattered away. The \( \Delta X \), that is \( \Delta X = \frac{E}{B \omega_o} \), the distance that the guiding centre moves after one cycle of the cyclotron motion, has to be large enough, in order to have an electron collide only once on one impurity every time before colliding
with another impurity. This generates a lower limit to the range of electric field as shown below.

Specifically, we quantitatively derive the condition for one-time-collision as $\Delta X \geq 10^{-8} m$, or say $E \geq 500 V/m$ when $B \approx 0.5 Tesla$. The derivation is as follows. The area of the guiding centres, in which all possible collisions will take place, is a circle-wise ring within radius $R-a$ and $R+a$, as shown in Figure 3. After the first collision, the guiding centre is inside the ring, and will keep moving in x direction, it will or will not again pass through the area of the ring depending on where the first collision takes place. The sufficient and necessary condition of no second collision is the electron will not be at a certain angle on cyclotron motion which superposes the impurity during its second time passing inside the ring.

![Figure 3](image)

Figure 3. All the possible collisions will take place with the guiding centre confined in a circle wise ring within radius $R-a$ and $R+a$. The condition of one-time collision is that the electron will not be at the certain angle on cyclotron motion which superposes the impurity during its second time passing inside the ring.

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**Disclosure statement**

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As discussed in the previous section, the incident velocity of electron $v_{in}$ and the outgoing velocity $v_{out}$ have the same absolute value.

On the other hand, the direction of outgoing velocity is to turn the incident velocity counter-clockwise by angle $\theta$, where $\theta = \theta_{out} - \theta_{in} = 2\varphi_{im} - 2\varphi_{in} - \pi$. Because the incident velocity of electron is $v_{in} = (-v_{cyc}\sin \varphi_{coll} + \frac{E}{B}, v_{cyc}\cos \varphi_{coll})$, the outgoing velocity can be expressed as

$$v_{out} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} -v_{cyc}\sin \varphi_{coll} + \frac{E}{B} \\ v_{cyc}\cos \varphi_{coll} \end{bmatrix}$$

$$= \begin{bmatrix} -v_{cyc}\sin \varphi_{coll} + \frac{E}{B} \cos \theta - v_{cyc}\cos \varphi_{coll} \sin \theta \\ -v_{cyc}\sin \varphi_{coll} + \frac{E}{B} \sin \theta + v_{cyc}\cos \varphi_{coll} \cos \theta \end{bmatrix}$$

As discussed in the previous section, the outgoing velocity expressed in terms of $v'_{cyc}$ and $\varphi'_{coll}$ is $v_{out} = (-v'_{cyc}\sin \varphi'_{coll} + \frac{E}{B}, v'_{cyc}\cos \varphi'_{coll})$. Combining the two expressions of $v_{out}$, the
relationship between $\varphi'_\text{coll}$ and $\varphi_\text{coll}$, $\theta$ is

$$-\nu'_\text{cyc} \sin \varphi'_\text{coll} + \frac{E}{B} = \left( -\nu_\text{cyc} \sin \varphi_\text{coll} + \frac{E}{B} \right) \cos \theta$$

(A3)

and

$$-\nu'_\text{cyc} \cos \varphi'_\text{coll} \sin \theta,$$

(A4)

As we know $\delta Y = -R' \sin \varphi'_\text{coll} + R \sin \varphi_\text{coll}$, our goal is to express $R'$, $\varphi_\text{coll}$ and $\varphi'_\text{coll}$ in terms of $(\varphi_\text{im}, \theta_v)$, in order to express $\delta Y$ by $(\varphi_\text{im}, \theta_v)$.

From $v_{in} = ( -\nu_\text{cyc} \sin \varphi_\text{coll} + \frac{E}{B}, \nu_\text{cyc} \cos \varphi_\text{coll} )$, there is

$$\cos \theta_v = -\nu_\text{cyc} \sin \varphi_\text{coll} + \frac{E}{B},$$

(A7)

and

$$\sin \theta_v = \frac{v_\text{cyc} \cos \varphi_\text{coll}}{v_{in}},$$

(A8)

where $v_{in} = \sqrt{v^2_\text{cyc} + \frac{E^2}{B^2} - 2\nu_\text{cyc} \frac{E}{B} \sin \varphi_\text{coll}}$. To solve for $\sin \varphi_\text{coll}$ and $\cos \varphi_\text{coll}$, there is

$$v_\text{cyc} \sin \varphi_\text{coll} = \frac{E}{B} - \frac{E}{B} \cos^2 \theta_v$$

(A9)

and

$$\nu_\text{cyc} \cos \varphi_\text{coll} = \sin \theta_v \left[ \sqrt{v^2_\text{cyc} + \frac{E^2}{B^2} \cos^2 \theta_v - \frac{E^2}{B^2}} \right.$$

$$\left. + \sqrt{\frac{E^2}{B^2} \cos^2 \theta_v} \right].$$

(A10)

Combining Equations (A4), (A6), (A10) and (A12), we can reach the goal to express $R'$, $\varphi_\text{coll}$ and $\varphi'_\text{coll}$ in terms of $(\varphi_\text{im}, \theta_v)$.

Therefore, we finally express $\delta Y$ in terms of $(\varphi_\text{im}, \theta_v)$

$$\delta Y(R, \varphi_\text{im}, \theta_v) = -\frac{1}{Bo^2} \left[ E\omega_c \cos \theta_v 
\right.$$

$$\times 2 \cos \theta_v \cos^2 (\theta_v - \varphi_\text{im})$$

$$\left. + \frac{1}{Bo^2} \sqrt{(E\omega_c \cos \theta_v)^2 \right]$$

$$\times 2 \cos \theta_v \cos^2 (\theta_v - \varphi_\text{im})$$

$$+ \frac{1}{Bo^2} \left[ E\omega_c \cos \theta_v \right]$$

$$\times \sin \theta_v \sin (2\theta_v - 2\varphi_\text{im}),$$

(A13)
Appendix 2 Connection between real space integral and the momentum space integral

We will prove that the momentum space integral $k \frac{dk \varphi_0}{(2\pi)^2}$ is equivalent to the real space integral $(\frac{eB}{h})^2 R \, dR \, d\varphi_0$, where $\varphi_0$ is the angle of momentum electron on the cyclotron orbit with respect to $x$ axis.

First, $dx \, dy = J_1 R \, dR \, d\varphi_0$, where

$$J_1 = \frac{\partial x}{\partial R} \frac{\partial y}{\partial R} = \frac{\cos \varphi_0}{\sin \varphi_0} = \frac{-R \sin \varphi_0}{R \cos \varphi_0} = R. \quad (A15)$$

because $x = R \cos \varphi_0$, and $y = R \sin \varphi_0$. Therefore, $dx \, dy = R \, dR \, d\varphi_0$.

Second, it can be proven that $\frac{dk_x}{dk} \frac{dk_y}{dk} = J_2 \frac{dk \varphi_0}{(2\pi)^2}$, where

$$J_2 = \frac{\partial k_x}{\partial k} \frac{\partial k_y}{\partial k} = \frac{-\sin \varphi_0}{\cos \varphi_0} = \frac{-k \cos \varphi_0}{-k \sin \varphi_0} = k, \quad (A16)$$

because $k_x = -k \sin \varphi_0$, and $k_y = -k \cos \varphi_0$. Therefore, $\frac{dk_x}{dk} \frac{dk_y}{dk} = \frac{dk \varphi_0}{(2\pi)^2}$.

In addition, because $k = \frac{mv}{h} = \frac{eBR}{h}$, $k \frac{dk \varphi_0}{(2\pi)^2} = \frac{(eB)^2}{h} R \, dR \, d\varphi_0$.

Appendix 3. Derivation and mathematical proof of the change of cyclotron radius

We prove that the cyclotron radius will be changed after collision.

Microscopically, at the moment of collision, the incident velocity of electron is the sum of guiding centre velocity and the relative velocity of electron $v_{in} = v_{gc} + v_{cyc} = (-v_{cyc} \sin \varphi_{coll} + \frac{E}{B}, v_{cyc} \cos \varphi_{coll})$, where $\varphi_{coll}$ is the angle on the cyclotron orbit (starting from x-axis) at the incident moment.

At the moment after collision, the outgoing velocity of electron is $v_{out} = v_{gc} + v'_{cyc} = (-v'_{cyc} \sin \varphi'_{coll} + \frac{E}{B}, v'_{cyc} \cos \varphi'_{coll})$, where $\varphi'_{coll}$ is the angle on the cyclotron orbit (starting from x-axis) at the moment after collision.

Because of energy conservation, the value of velocity at the moment of collision is a constant, i.e. $|v_{in}| = |v_{out}|$. However, the cyclotron velocity before and after collision are different because

$$v'_{cyc}^2 = \left( v_{out,x} - \frac{E}{B} \right)^2 + v_{out,y}^2 \quad (A17)$$

$$= v_{out,x}^2 + v_{out,y}^2 + \frac{E^2}{B^2} - 2v_{out,x} \frac{E}{B} \quad (A18)$$

$$= v_{in,x}^2 + v_{in,y}^2 + \frac{E^2}{B^2} - 2v_{out,x} \frac{E}{B} = \left( -v_{cyc} \sin \varphi + \frac{E}{B} \right)^2 + v_{cyc}^2 \cos \varphi^2 + \frac{E^2}{B^2} \quad (A19)$$

$$-2v_{out,x} \frac{E}{B} \quad (A20)$$

$$= v_{cyc}^2 + 2 \frac{E^2}{B^2} - 2v_{cyc} \frac{E}{B} \sin \varphi - 2v_{out,x} \frac{E}{B} \quad (A21)$$
Therefore, the difference between the velocity square before and after scattering is
\[ v_{\text{cyc}}'^2 - v_{\text{cyc}}^2 = \frac{2E^2}{B^2} - 2v_{\text{cyc}} \frac{E}{B} \sin \varphi - 2v_{\text{out},x} \frac{E}{B}. \] (A22)

Thus, the change of kinetic energy is
\[ \Delta E_{\text{kin}} = \frac{1}{2} m (v_{\text{cyc}}'^2 - v_{\text{cyc}}^2). \] (A23)