Regular homogeneous T-models with vacuum dark fluid

Kirill Bronnikov\textsuperscript{1,2} and Irina Dymnikova\textsuperscript{3,4}

\textsuperscript{1} VNIIMS, 3-1 M. Ulyanovoy St, Moscow 117313, Russia
\textsuperscript{2} Institute of Gravitation and Cosmology, PFUR, 6 Miklukho-Maklaya St, Moscow 117198, Russia
\textsuperscript{3} Department of Mathematics and Computer Science, University of Warmia and Mazury, Zolnierska 14, 10-561 Olsztyn, Poland
\textsuperscript{4} A F Ioffe Physico-Technical Institute, Politekhnicheskaja 26, St Petersburg, 194021, Russia

E-mail: irina@matman.uwm.edu.pl

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Abstract

We present the class of regular homogeneous T-models with vacuum dark fluid, associated with a variable cosmological term. The vacuum fluid is defined by the symmetry of its stress–energy tensor, i.e., its invariance under Lorentz boosts in a distinguished spatial direction \((\rho_f = -\rho)\), which makes this fluid essentially anisotropic and allows its density to evolve. Typical features of homogeneous regular T-models are: the existence of a Killing horizon; beginning of the cosmological evolution from a null bang at the horizon; the existence of a regular static pre-bang region visible to cosmological observers; creation of matter from anisotropic vacuum, accompanied by very rapid isotropization. We study in detail the spherically symmetric regular T-models on the basis of a general exact solution for a mixture of the vacuum fluid and dust-like matter and apply it to give numerical estimates for a particular model which illustrates the ability of cosmological T-models to satisfy the observational constraints.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Astronomical data provide convincing evidence that our universe is dominated at above 70% of its density by a dark energy responsible for its accelerated expansion due to negative pressure, \(p = w \rho\) with \(w < -1/3\) [1–5]. Observations constrain the equation of state of dark energy within \(-1.45 < w < -0.74\) [4], the best fit \(w = -1\) [6–11] corresponds to a cosmological
constant $\lambda$ related to the vacuum density $\rho_{\text{vac}}$ by $\lambda = 8\pi G \rho_{\text{vac}}$ (for a detailed discussion see [4, 11, 12]).

The Einstein cosmological term $\Lambda g_{\mu\nu} = 8\pi G \rho_{\text{vac}} g_{\mu\nu}$ is plagued with the cosmological constant problem. Quantum field theory estimates $\Lambda$ by the Planck scale [13], so that the resulting zero-point density is incompatible with all observational data [14], and a long-term problem was how to zero it out [13]. A key problem now is why the cosmological constant should be at the scale demanded by observations [12]. Another dynamical aspect is that the inflationary paradigm needs a large value of $\rho_{\text{vac}}$ at the earliest stages to provide a reason for rapid initial expansion [15], typically at the GUT scale [16] (for a review see [17]), observations testify that it is small at the present epoch, whereas the Einstein equations require $\rho_{\text{vac}} = \text{const.}$

A lot of theories and models have been developed to describe dynamically evolving dark energy which today behaves like a cosmological constant (for a recent comprehensive review see [12]). In the isotropic cosmological models with variable $\rho_{\text{vac}}$, its variability is achieved only due to the interaction with the other matter sources ([18–20] and references therein).

On the other hand, the Einstein equations admit a class of solutions with source terms associated with vacuum defined by the symmetry of its stress–energy tensor (SET), including vacuum with reduced symmetry which allows the vacuum density to be evolving and clustering [21, 22].

The Einstein cosmological term $\Lambda g_{\mu\nu}$ is associated with the maximally symmetric vacuum SET (all eigenvalues equal) $T_{\mu\nu} = \rho_{\text{vac}} g_{\mu\nu}$ [23] representing de Sitter vacuum with $p = -\rho_{\text{vac}}$ and $\rho_{\text{vac}} = \text{const.}$ The symmetry of a vacuum SET can be reduced to the case when only one or two of the spacelike eigenvalues of $T_{\mu\nu}$ coincide with its timelike eigenvalue, $T^j_j = T^t_t$ [21, 24, 25], so that the vacuum equation of state holds in the $j$th direction(s), $p_j = -\rho$, and thus a vacuum with reduced symmetry must be evidently anisotropic. Evolution of vacuum density and pressures is governed by the conservation equation $T_{\mu\nu}^\;;\nu = 0$.

The class of GR solutions specified by $p_j = -\rho$, which describe time-dependent and spatially inhomogeneous vacuum energy, presents, in a model-independent way, anisotropic vacuum fluid with continuous density and pressures. It can provide a unified description, based on a spacetime symmetry and introduced in [25], of dark ingredients in the universe by a vacuum dark fluid representing both distributed vacuum dark energy (by an evolving and inhomogeneous cosmological term [21, 26]) and compact self-gravitating objects with interior de Sitter vacuum [22, 27–29].

The variable cosmological term $\Lambda_{\mu\nu} = 8\pi G T_{\mu\nu(\text{vac})}$ has been introduced in the spherically symmetric case [21] when a vacuum $T_{\mu\nu(\text{vac})}$ is invariant under radial Lorentz boosts [24], while the full symmetry is restored asymptotically at the centre and at infinity [22, 24, 27, 30].

It provides a description of a smooth evolution of the vacuum from $\Lambda g_{\mu\nu}$ to $\lambda g_{\mu\nu}$ with $\lambda < \Lambda$. In the spherically symmetric case, $T_{\mu\nu(\text{vac})}$ generates globally regular spacetimes with a de Sitter centre [22]. Depending on the parameters and on the choice of a reference frame, they describe localized objects with de Sitter vacuum trapped inside (nonsingular black and white holes [21, 24, 31, 32], self-gravitating vacuum structures without horizons [27] called G-lumps [22, 25]), and regular cosmological models with variable vacuum density and pressures considered and classified in our previous paper [26].

In the cosmological context, the spatial direction $x^j$ in which the vacuum symmetry of $\Lambda_{\mu\nu} = 8\pi G T_{\mu\nu(\text{vac})}$ is preserved, is necessarily distinguished by the expansion anisotropy. In cosmologies guided by $\Lambda_{\mu\nu}$, an anisotropic stage is generic [26, 31]. A further early evolution is the most transparent in homogeneous T-models, and this is the question addressed in the present paper.

T-models are defined as cosmological models representing T-regions of spacetimes which contain Killing horizons. T-regions were originally found by Novikov as essentially non-static
regions beyond the horizon in the Schwarzschild spacetime [33]. In the spherically symmetric case, \( ds^2 = e^{\nu(q,t)} dt^2 - e^{\mu(q,t)} dq^2 - r^2(q,t) d\Omega^2 \), they are specified by the invariant quantity \( \Delta = g^{ik} r,ir,k = e^{-\nu}(\partial r/\partial t)^2 - e^{-\mu}(\partial r/\partial q)^2 \) [34, 35]. In the regions where \( \Delta < 0 \), called R-regions, the surfaces of constant \( r \) are timelike, so that static observers can exist there. When \( \Delta > 0 \), the normal \( N_i = r,i \) to the surface \( r = \text{const} \) is timelike, so that this surface is spacelike, and orbits of constant \( r \) and hence static observers do not exist in principle. T-regions are further specified as T--regions when \( \dot{r} < 0 \) and T+-regions when \( \dot{r} > 0 \) (with \( \Delta > 0 \), \( \dot{r} \) cannot vanish, and this condition is invariant) [35]. At the Killing horizons \( \Delta = 0 \).

Spherically symmetric T-models were introduced by Ruban [39] as modification of vacuum T-regions to the case of dust-like matter considered in a synchronous co-moving reference frame with the metric representing an inhomogeneous generalization, \( ds^2 = dr^2 - e^{\alpha(t,q)} dq^2 - \mathcal{R}^2(t) d\Omega^2 \), of the anisotropic cosmological models of semi-closed type [40] with hypercylindrical spatial sections \( V_3 = S^2 \otimes R^1 \). Actually, Ruban deeply revised and generalized (to the case of nonzero \( \lambda \)) Datt’s solution [41] qualified by the author as being ‘of little physical significance’ [42]. The particular case \( \alpha(t,q) = \alpha(t) \) is known as the Kantowski–Sachs model [43]. A remarkable feature of T-models discovered by Ruban is a gravitational mass defect maximally possible in GR, equal to the total rest mass of the matter which appears to be entirely gravitationally bound [39].

T-models were further extended to the case of a perfect fluid with nonzero pressure [44–48], and the class of spherically symmetric T-models with isotropic perfect fluid was completed by Ruban in 1983 [49].

Amazingly, Ruban excluded from his consideration the case \( T^0_0 (\rho_r = -\rho) \) as being ‘of no interest for cosmology and relativistic astrophysics’ [50]. On the other hand, the requirement of regularity leads in this case to globally regular spacetimes with an obligatory de Sitter centre [22] which represent distributed and clustered vacuum dark energy and describe both black (white) nonsingular holes and regular cosmological models dominated by anisotropic vacuum fluid, including regular modifications of Kantowski–Sachs models [26]. In these models, the cosmological evolution starts from horizons with a highly anisotropic null bang, but information on the pre-bang history in the de Sitter region in the remote past is available to Kantowski–Sachs observers [26, 51].

The latter circumstance provides a solution to one more long-standing problem, that of the initial cosmological singularity.

Homogeneous regular cosmological T-models with anisotropic vacuum fluid can be seen as T-regions of spacetimes with 3-parameter isometry groups acting transitively on spatial 2-surfaces, i.e., those with spherical, planar and pseudospherical symmetries. Among them spherically symmetric T-models related to Kantowski–Sachs (KS)-type regular models are of greatest interest.

The aim of this paper is to introduce and to study spherically symmetric homogeneous regular cosmological T-models with anisotropic vacuum fluid and dust-like matter, bearing in mind that the material content of the modern universe is almost entirely represented by dark energy and pressureless matter.

The paper is organized as follows. In section 2, we consider T-models of KS type filled with dust and an anisotropic vacuum fluid described by \( \Lambda_{\mu\nu} \). For non-interacting dust and vacuum, we present a general solution without specifying the particular density profile of \( \Lambda_{\mu\nu} \).

In section 3, we argue that regular models on the basis of the above solutions must have a

[5] There are exceptions to this rule, represented by T-regions with \( r(t) \) having a minimum. This happens, e.g., in regular black-hole solutions of brane gravity [36] and in recent solutions with nonminimally coupled Yang–Mills fields [37].
static, purely vacuum, asymptotically de Sitter region in the remote (pre-bang) past, and briefly
describe the properties of such regular models. In section 4, we give a detailed description
of a particular model illustrating the ability of regular T-models to satisfy the observational
constraints. In section 5, we summarize and discuss the results.

2. General solution

We begin with the KS metric in the general form\(^6\)

\[
d s^2 = e^{2\gamma} d\tau^2 - e^{2\alpha} dx^2 - e^{2\beta} r^2 d\Omega^2,
\]

where \(\alpha, \beta, \gamma\) are functions of the time coordinate \(t\) and \(d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2\). Then the
nonzero Ricci tensor components are

\[
R_t^t = e^{-2\gamma} \left[ \ddot{\alpha} + 2 \dot{\alpha} \dot{\beta} - \dot{\gamma} (\dot{\alpha} + 2 \dot{\beta}) \right],
\]

\[
R_x^x = e^{-2\gamma} \left[ \ddot{\alpha} + \dot{\alpha} (\dot{\alpha} + 2 \dot{\beta} - \dot{\gamma}) \right],
\]

\[
R_{\theta}^{\theta} = R_{\phi}^{\phi} = e^{-2\beta} + e^{-2\gamma} \left[ \ddot{\beta} + \dot{\beta} (\dot{\alpha} + 2 \dot{\beta} - \dot{\gamma}) \right],
\]

(1) (2) (3) (4) (5)

The Einstein equations read

\[
G_{\mu}^{\nu} \equiv R_{\mu}^{\nu} - \frac{1}{2} \delta_{\mu}^{\nu} R = -8\pi G T_{\mu}^{\nu}.\]

(8)

For our purposes, it is helpful to choose the time coordinate \(t\) in a way similar to choosing
the curvature coordinates in static spherical symmetry, namely to put \(t = r = e^{\beta}\), so that the
metric is rewritten as

\[
d s^2 = e^{2\gamma(r)} dr^2 - e^{2\alpha(r)} dx^2 - r^2 d\Omega^2.
\]

(9)

Then the \(i_1\) component of (8) and the difference \(i_0 - i_1\) read

\[
\frac{1}{r^2} [r (e^{-2\gamma} + 1)]' = 8\pi G T_{i}^i,
\]

(10)

\[
\frac{2}{r} e^{-2\gamma} (\alpha' + \gamma') = 8\pi G (T_{i}^r - T_{r}^i),
\]

(11)

where the prime denotes \(d/dr\). In full analogy with static spherical symmetry, equations (10)
and (11) may be rewritten in an integral form

\[
e^{-2\gamma} = -1 + \frac{8\pi G}{r} \int T_i^r r^2 dr,
\]

(12)

\[
\alpha = -\gamma + 4\pi G \int r e^{2\gamma} (T_i^r - T_r^i) dr.
\]

(13)

\(^6\) Our conventions are: the metric signature \((+ -- - -)\); the curvature tensor \(R_{\mu}^{\nu} = \partial_{\nu} \Gamma_{\mu}^{\rho} - \partial_{\mu} \Gamma_{\nu}^{\rho} - \cdots ; R_{\mu}^{\nu} = R_{\nu}^{\mu} \) so that the Ricci scalar \(R > 0\) for de Sitter spacetime and the matter-dominated cosmological epoch; the system of units \(c = \hbar = 1\).
If $T^r_r$ is known as a function of $r$, then equation (12) is a solution to (10) that yields the lapse function $e^{2\gamma(r)}$, and the other metric function $\alpha$ is found from (13) if, in addition, the difference $T^r_r - T^r_r$ is known as a function of $r$. It is really so in many important cases. In a general setting this question is studied in [38].

Now, let us take the SET of matter as a sum of SETs of dust and vacuum. Dust, with the density $\rho_d(r)$, is at rest in a T-model with the metric (2), while for a spherically symmetric vacuum fluid we have $T^\mu_\mu = \text{diag}(\rho_v, \rho_v, -p_v \perp, -p_v \perp)$. The full SET reads

$$T^\mu_\mu = \text{diag}(\rho_d + \rho_v, \rho_v, -p_v \perp, -p_v \perp).$$  \hspace{1cm} (14)

Evidently, if we know the vacuum density profile $\rho_v(r)$, equation (12) gives us $e^{2\gamma(r)}$:

$$e^{-2\gamma} = -1 + \frac{2\mathcal{M}(r)}{r}, \quad \mathcal{M}(r) := 4\pi G \int \rho_v r^2 \, dr.$$  \hspace{1cm} (15)

The function $\mathcal{M}(r)$ is similar to the mass function conventionally introduced in the description of static, spherically symmetric configurations.

Note that this result is obtained without using the conservation law for dust and vacuum and is even valid when they are interacting.

It remains to find the dust density $\rho_d$ and the second metric function $\alpha(r)$. This can be done with the aid of the conservation law component $\nabla_\mu T^\mu_0 = 0$ which in our case reads

$$\rho'_d + \rho_d \left(\alpha' + \frac{2}{r}\right) + \rho'_v + \frac{2}{r}(\rho_v + p_v \perp) = 0,$$  \hspace{1cm} (16)

and equation (11) where $T^r_r - T^r_r = \rho_d$. At stages of the cosmological evolution when there is no interaction between dust and vacuum, we can write the conservation law separately for each of them. Then, for dust we have

$$\rho'_d + \rho_d \left(\alpha' + \frac{2}{r}\right) = 0 \quad \Rightarrow \quad \rho_d = \mu_0 e^{-\alpha/r^2}, \quad \mu_0 = \text{const},$$  \hspace{1cm} (17)

while the corresponding relation for vacuum

$$\rho'_v + \frac{2}{r}(\rho_v + p_v \perp) = 0$$  \hspace{1cm} (18)

expresses $p_v \perp$ in terms of $\rho_v$. Substituting (17) into (11), we can rewrite it as follows:

$$e^{\alpha+y}(\alpha' + y') = \frac{4\pi G \mu_0}{r} e^{3y},$$  \hspace{1cm} (19)

which after integration gives

$$e^\alpha = 4\pi G \mu_0 e^{-\gamma} \int \frac{dr}{r} e^{3\gamma},$$  \hspace{1cm} (20)

with $e^\gamma$ being given by (15). This completes the solution. The physical time $\tau$ is obtained by integration,

$$\tau = \int e^{\gamma(r)} \, dr,$$  \hspace{1cm} (21)

and all unknowns are expressed as functions of $\tau$ in a parametric form.

Thus we have solved by quadratures the Einstein equations for KS cosmology with a non-interacting mixture of dust and $\Lambda_{\mu\nu}$-vacuum. The solution contains one essential integration constant $\mu_0$ and one arbitrary function, the vacuum density profile $\rho_v(r)$ (or equivalently the mass function $\mathcal{M}(r)$). This is a new solution for the general case $p_v \perp \neq -\rho_v$. In the particular case of the conventional cosmological constant $\lambda = 8\pi G \rho_v = -8\pi G p_v \perp = \text{const}$, our solution agrees with Ruban’s homogeneous solution [39].

For interacting dust and vacuum, a further solution depends on the particular form of the interaction. Though, if we know (or have a reason to prescribe) the function $\rho_d(r)$, then equation (13) immediately yields $\alpha$ as a quadrature.
3. Nonsingular models and the static core in the pre-bang past

Let us now discuss the possibility of obtaining globally regular models of the universe on the basis of the above KS metrics. The following three points will be argued:

(i) In regular expanding universe models, the cosmological evolution should begin from a Killing horizon.

(ii) The region beyond the horizon should be purely vacuum, i.e., dust is absent there.

(iii) Dust appears in the cosmological region due to the interaction with the vacuum fluid.

To prove item (i), we will use, in the metric (2), a new time variable specified by the coordinate condition
\[ \alpha + \gamma = 0, \]
so that (2) takes the form
\[ ds^2 = a^{-2}(t) dt^2 - a^{-2}(t) dx^2 - r^2(t) d\Omega^2. \]
Then, instead of (11), we obtain the equation
\[ 2a^2 \frac{\ddot{r}}{r} = -8\pi G \rho_d. \] (22)
Assuming \( \rho_d \geq 0 \), we immediately conclude that \( \dot{r} \leq 0 \). Consequently, if there is expansion \( \dot{r} > 0 \) at some instant \( t_1 \), we can assert that \( r(t) \) reached zero at some finite earlier instant \( t_s < t_1 \). Meanwhile, \( r = 0 \) is a curvature singularity of the metric (2).

The only way out is to assume that the cosmological evolution started \textit{later} than the would-be singularity, which is only possible if there was a Killing horizon at some \( t_h > t_s \), where \( a(t_h) = 0 \). Then there is a hope to find a static regular region beyond such a horizon. This reasonable assumption is additionally justified by the consideration below concerning (ii) and by the fact that all related regular vacuum models without dust [26] do have horizons and regular R-regions with a de Sitter core in their remote past.

As to item (ii), there is a serious physical argument. If we imagine that cosmological dust is continued to the static region where the cosmological time \( t \) becomes a radial coordinate and vice versa, then the density \( \rho_d = T_{\ell d} \) is converted there to radial pressure, while the pressure \( -T_r = 0 \) is converted to zero density, and we obtain quite an implausible kind of matter with zero density but nonzero pressure.

Moreover, it can be strictly shown that \( \rho_d = 0 \) at the horizon. Indeed, at \( t = t_h \), according to (22), \( \rho_d = 0 \) due to \( a = 0 \), unless \( \dot{r} \to \infty \) as \( t \to t_h \). But the latter opportunity is unacceptable for a horizon as a regular surface, where \( r^2(t) \) should be an analytic function. Thus \( \rho_d = 0 \) at the horizon, and it is natural to conclude that dust did not exist to the past of it (item (ii)).

Returning to the cosmological region, we note that non-interacting dust is incompatible with a horizon: it would then have an infinite density according to (17), \( \rho_d \sim \frac{|a(t)r^2(t)|^{-1}}{a(t)} \). Since \( \rho_d = 0 \) at the horizon, \( \rho_d > 0 \) at later times can only emerge from the vacuum fluid due to interaction—and this is our item (iii).

Let us now consider the region beyond \( t = t_h \) filled with vacuum fluid and suppose that there is a regular centre \( r = 0 \). Such spacetimes were discussed in detail in [26]; they can be described by metrics of the form (9) with \( e^{-2\nu} = e^2a = -A(r) \) (so that \( r \) is a spatial coordinate while \( x \) is a temporal one), i.e.,
\[ ds^2 = A(r) dt^2 + \frac{dr^2}{A(r)} - r^2 d\Omega^2, \] (23)

\textsuperscript{7} The coordinate defined by the condition \( \alpha + \gamma = 0 \) varies near a horizon (up to a nonzero constant factor) like manifestly well-behaved null coordinates of Kruskal type used for analytic continuation of the metric [52, 53]. This implies the analyticity requirement for \( r^2(t) \).
with \( A(r) = -e^{-2\gamma} \) given by equation (15). Taking at the centre \( A(0) = 1 \), we can write
\[
A(r) = 1 - \frac{2M(r)}{r}, \quad M(r) = 4\pi G \int_0^r \rho_v r^2 \, dr, \tag{24}
\]
so that \( M \) is the conventional mass function.

The function \( A(r) \) may have different number and orders of zeros (depending on the particular density profile \( \rho_v(r) \)), corresponding to Killing horizons in spacetime [26].

In any spacetime with the metric (23), the geodesics equations have the following first integral:
\[
\left( \frac{dr}{d\tau} \right)^2 + kA(r) + \frac{L^2}{r^2} A(r) = E^2, \tag{25}
\]
where \( E \) and \( L \) are the constants of motion associated with the particle energy and angular momentum, while the constant \( k \) takes the value \( k = 1 \) for timelike geodesics and \( k = 0 \) for null geodesics; the affine parameter \( \tau \) has the meaning of proper time along a geodesic in the case \( k = 1 \). Trajectories of the Kantowski–Sachs (KS) co-moving observers are time lines in the T-region, \( k = 1, L = 0, E = 0 \). According to (25), in the case of double or higher-order horizons, a null bang has occurred in the infinitely remote past of KS observers. On the other hand, in the case \( E \neq 0 \) one has \( dr/d\tau \neq 0 \) for all geodesics, whence it follows that \( |\tau| < \infty \), irrespective of the order of the horizon. As a result, in any case KS observers can receive pre-bang information (in the case of double or higher-order horizon from their infinitely remote past) brought by particles and photons following geodesics (25) and crossing the horizon at their finite proper time [26].

In spacetimes with horizons, static observers exist in R-regions (if any), while KS observers exist in T-regions. Both static and KS observers have problems with horizons which are singular surfaces in their coordinate mappings. Connecting coordinates with radial geodesics in the way similar to that applied by Lemaître for the Schwarzschild geometry (\( L = 0 \) in equation (25)), we introduce observers to whom more extended parts of a manifold are available, and who can tell a KS observer his pre-bang story. As we see in figure 1 [51], Kantowski–Sachs observers reside in \( T_+ \) regions specified in figure 1 as \( CC_1 \). A Lemaître observer starting with \( E^2 > 1 \) from the R-region \( RC_1 \), travels through a white hole \( WH \) and R-region \( U_1 \), and arrives at \( T_+ \) region \( CC_1 \) where he can meet a Kantowski–Sachs observer.

Let us here discuss the simplest generic structure with a single simple (first-order) horizon. The horizon radius \( r = r_h \) corresponds to \( r_h = 2M(r_h) \) and can be found if \( \rho_v(r) \) is specified. There are two types of one-horizon configurations shown in figure 2 [51]. The metric function is plotted as a function of the luminosity distance \( r \) which is spacelike radial coordinate in R-regions, where \( A(r) \) is given by (24). In both cases the global structure of spacetime is the same as in the de Sitter case. Here we are interested in the type presented by the lower curve in figure 2, with rich dynamical possibilities in the T-region.

The static R-region between \( r = 0 \) and \( r = r_h \) represents a regular ‘core’ in the remote past of an expanding KS universe in which the cosmological evolution starts with a null bang (i.e., contains a horizon instead of a singularity).

It is important that equation (15) with \( M(r) \) given in (24) is now valid for all values of \( r \), in the whole spacetime, whereas for the other metric coefficient \( e^{2\alpha} \) there are different expressions: \( e^{2\alpha} = -A(r) \) for \( r \leq r_h \) (compare (9) and (23)); furthermore, equation (20) holds for the cosmological epoch without interaction, and there is a yet unknown expression for \( e^{-2\alpha} \) in the interaction epoch.

8 Note that the time coordinate \( t_* \) of the static region has nothing to do with the cosmological time in the metric (2): it is rather related to the spatial coordinate \( x \).
Since the horizon is crossed at a purely vacuum stage, a nonzero matter density $\rho_d$ appears due to the interaction between dust and vacuum in a certain period after the null bang. This perfectly conforms to the well-known prediction of QFT in curved spacetime that nonstationary spacetimes, especially anisotropic ones, create particles [40, 54, 55], and a decaying vacuum density accompanied by growing $\rho_d$ is a phenomenological description of this process.

In what follows we will give some estimates for a specific choice of the vacuum decay law.

4. T-model with vacuum density $\rho_v = \rho_c e^{-r^3/r^3_*} + \rho_\lambda$

Let us choose the profile $\rho_v(r)$ in the form

$$\rho_v = \rho_c e^{-r^3/r^3_*} + \rho_\lambda, \quad \rho_\lambda = \frac{\lambda}{8\pi G}, \quad (26)$$
The first term results from a simple semiclassical model for vacuum polarization in spherically symmetric gravitational fields [22, 24, 27]. The second one refers to a background cosmological term which is at the scale suggested by observations. An additional indirect justification of the ansatz (26) comes from the minisuperspace model of quantum cosmology. In a certain gauge, the cosmological constant is quantized as an eigenvalue of the appropriate operator in the Wheeler–DeWitt equation [56]. The form of the potential corresponds to a nonzero positive quantum value of the cosmological constant at each finite value of the time-dependent scale factor.

We thus have three distinct length scales: \( r_c = \sqrt{3/(8\pi G \rho_c)} \), a de Sitter radius corresponding to the central density \( \rho_c \), the scale \( r_* \sim (r_s r_g)^{1/3} \) [24, 57] characterizing the vacuum decay rate in the spherically symmetric gravitational field \( r_g \) is the Schwarzschild radius related to the total gravitational mass of the decaying vacuum), and the scale \( r_\lambda \) related to the background cosmological constant \( \lambda \), assumed to be of the present Hubble order of magnitude, \( \lambda \sim 10^{-56} \text{ cm}^{-2}, r_\lambda \sim 10^{28} \text{ cm} \).

Let us assume that the central density \( \rho_c \) is of the GUT scale,

\[
\rho_c \approx (10^{15} \text{ GeV})^4 \approx 2.2 \times 10^{77} \text{ g cm}^{-3} \Rightarrow (28)
\]

and \( r_* \approx 4.6 \times 10^{-28} \text{ cm} \) for the scale \( r_* \) given in [24].

### 4.1. Horizon radius

Since \( r_* \gg r_c \), evidently, the density \( \rho_v \) is nearly constant and the metric is approximately de Sitter at radii \( r \lesssim r_c \), and, with good accuracy, we have a cosmological horizon at \( r \approx r_c \). The same is true for any profile \( \rho_v(r) \) with a slow enough decay rate from \( \rho_c \) corresponding to a regular centre. As a result, in all such cases, including (26),

\[
r_h \approx r_c, \quad \rho_v(r_h) \approx \rho_c. (30)
\]

More precisely, with (26) and (28), we have \( r_h = r_c [1 + O(b^{-3})] \) where \( b = r_*/r_c \approx 5.6 \times 10^{17} \).

### 4.2. Vacuum decay time

Next, let us estimate the vacuum decay time, taking \( \tau = 0 \) at crossing the horizon and seeking \( \tau_f \), the instant when the first, exponential term in (26) becomes equal to the second term, \( \rho_v \). So,

\[
\tau_f = \int_{r_0}^{r_f} e^{\gamma(r)} dr,
\]

where \( r_f \) is found from the condition

\[
\rho_v e^{-r_f/r_*^2} = \rho_c \quad \Rightarrow \quad r_f \approx 6.3 r_* . \quad (32)
\]

The integrand \( e^{\gamma} \) is expressed from the relation due to (24)

\[
e^{-2\gamma} = -1 + \frac{b^2}{x} (1 - e^{-x}) + \frac{\lambda}{3} x^2, \quad x := \frac{r}{r_*}. \quad (33)
\]

The integral (31) is taken over the interval \((1/b, 6.3)\). It is easily seen that, on this interval, the term containing \( \lambda \) is more than 60 orders of magnitude smaller than the others and can be omitted.
Near the horizon, where $x \ll 1$, the integral may be calculated analytically:

$$
\tau_1 = \tau_f [r_1 - r_h] \approx \int_{r_h}^{r_1} \frac{dx}{\sqrt{b^2 x^2 - 1}},
$$

where $r_1$ is some small enough value of $r$. For $r_1 = 0.1 r_*$ we have (with a relative error smaller than $10^{-3}$) $\tau_1 \approx 39.3 r_*$. In finding the remaining integral, we may leave in equation (33), with high precision, only the term with $b^2$ and get numerically

$$
\tau_2 = r_* \int_{0.1}^{6.3} \frac{\sqrt{x} dx}{\sqrt{1 - e^{-x^3}}} \approx 12.3 r_*,
$$

which finally gives

$$
\tau_f = \tau_1 + \tau_2 \approx 51.6 r_* \approx 0.4 \times 10^{-21} \text{cm},
$$

or, in seconds, $\tau_f \approx 1.3 \times 10^{-34}$ s. This value is rather close to the GUT parameters, while by this time the scale factor $r$ has already inflated from $r = r_h \approx 0.8 \times 10^{-25}$ cm to $r_f \approx 6.3 b r_* \approx 2.8 \times 10^{-7}$ cm. The other scale factor, $a(r)$, has inflated from zero at the horizon to some finite value.

The inflation of $r$ corresponds to approximately 43 e-foldings which is not regarded as sufficient in conventional inflationary cosmology. However, in our class of models inflation as such is not necessary because, due to the existence and observability of a static core in the remote past, all parts of our model universe are causally connected.

4.3. Solution in the regime of constant $\lambda$

In the period $\tau > \tau_f$ we can neglect the variable part of the vacuum SET, remaining with the usual cosmological term, $\rho_\Lambda = -p_{\perp \Lambda} = \lambda/(8\pi G) > 0$. Then, according to (15), the lapse function is given by

$$
e^{-2\gamma} = -1 + \frac{2M}{r} + \frac{1}{3} \lambda r^2,
$$

where $M$ is an integration constant expressing the total contribution of the decaying vacuum component to the mass function (this corresponds to dropping $e^{-x^3}$ in (33), and we have then $2M = b^2 r_* \approx 1.4 \times 10^{28}$ cm).

Integration in equation (20) then leads to a very cumbersome expression in terms of elliptic functions, and we will not present it here. We note that, in the absence of dust, the integral in (20) reduces to an arbitrary constant, and after simple rescaling of $t$ (or directly from equation (11)) we obtain the conventional Schwarzschild–de Sitter metric in its T-region, i.e., $\alpha = -\gamma$ with $\gamma$ given by (35).

4.4. Matter-dominated epoch

In our model, there is a large period when

$$r_f \ll r \ll M \sim 10^{28} \text{ cm},
$$

in which $\rho_d \gg \rho_\Lambda$ and we can write $e^{-2\gamma} \approx 2M/r$. This corresponds to the epoch when all matter has already been created but the influence of the cosmological constant $\lambda$ is still negligible. In this period, the integral in (20) is easily calculated, and both $r$ and $\alpha$ may be expressed in terms of the cosmic time $\tau$:

$$
a(r) = e^\alpha = \frac{4\pi G\mu_0}{3M} \left( r + \frac{c_1}{\sqrt{r}} \right),
$$

(37)
where \( c_1 \) and \( \tau_0 \) are integration constants. This very simple expression gives a reasonable approximation to the corresponding exact dust solution \([49]\).

The values of \( c_1 \) and \( \tau_0 \) are determined by the details of the interaction period \( r < r_f \). If we admit for certainty that the matter-dominated epoch begins at \( r = r_f \) and, as before, put \( \tau = 0 \) at the horizon, we must require \( r(\tau_f) = r_f \), which leads to \( \tau_f - \tau_0 \approx 0.8 \times 10^{-24} \) cm, and using (31) we obtain \( \tau_0 \approx 3.2 \times 10^{-24} \) cm.

The value of \( c_1 \) is severely restricted by the conditions \( a > 0 \) and \( a' > 0 \) at \( r > r_f \): \n\[
-r_f^{3/2} < c_1 < 2r_f^{3/2}.
\]

### 4.5. Isotropization

The condition that a KS universe expands isotropically is that the directional Hubble parameters, defined as \([58]\)
\[
H_a := e^{-\gamma} \dot{a} \equiv d\alpha / d\tau,
\]
\[
H_r := e^{-\gamma} \dot{\beta} \equiv d\beta / d\tau = (1/r) (dr / d\tau),
\]
are equal. Entire isotropy would mean that \( R_x^x = R_\theta^\theta \), which, according to (4) and (5), would lead (in the proper time gauge \( t = \tau \)) to
\[
\ddot{\alpha} + 3\dot{a}^2 = \ddot{\alpha} + 3\dot{a}^2 + 1/r^2.
\]
This shows that any KS cosmology is necessarily anisotropic for topological reasons and can only isotropize asymptotically at large \( r \). The topological anisotropy, i.e., the non-equivalence between the \( x \) direction which is open and the angular directions which are closed, can in principle be revealed observationally by detecting the same source of signal twice, in opposite angular directions.

The degree of anisotropy may be characterized by the ratio \( A = \sigma / H \), where
\[
\sigma = (H_a - H_r) / \sqrt{3}, \quad H = (H_a + 2H_r) / 3
\]
are the shear scalar and the mean Hubble parameter, respectively. Since \( H_a \) and \( H_r \) can in principle be equal, the observable parameter \( A \) can be zero despite the topological anisotropy.

Observations of the cosmic microwave background show that our universe is highly isotropic \([59]\): at the recombination epoch, at which the electromagnetic radiation has decoupled from matter (at redshifts \( z \approx 1000 \)), the ratio \( \sigma / H \) could not exceed \( 10^{-6} \), whence it follows that at present it is at most of order \( 10^{-9} \) \([58]\). This, in principle, strongly constrains the parameters of any cosmological model based on the KS metric. We shall see, however, that our model is automatically sufficiently isotropic.

In terms of the quantities \( r \) and \( a(r) \), the anisotropy parameter reads
\[
A = \sqrt{3} r a'(r) - 1 / r a'(r) + 2.
\]
For the matter-dominated epoch we obtain from (38)
\[
A = - \frac{\sqrt{3} c_1}{2 r_f^{3/2} + c_1}.
\]
From the restriction (39) it is evident that our model universe expands almost isotropically very soon after the end of the interaction epoch, and at \( r \gg r_f \) we have simply
\[
A \lesssim (r_f / r)^{3/2}.
\]
For instance, in the recombination epoch \( r \sim 10^{25} \text{ cm} \), still belonging to the matter-dominated period, we obtain \( A \lesssim 10^{-48} \), a very small degree of (global) anisotropy as compared with the observational constraint. (This estimate concerns a background anisotropy and certainly does not concern the observable small-scale CMB anisotropy described by perturbations of the homogeneous spacetime, whose study is beyond the scope of this paper.)

4.6. Matter density after creation

It is reasonable to suppose that, by the time \( \tau = \tau_f \), when the rapidly decaying vacuum component almost completely vanishes, the whole amount of matter particles had already been created, and the further evolution proceeded with each of the conservation laws (18) and (17) for dust and vacuum being valid separately, hence we have the expression (20) for \( a(r) = e^{\alpha(r)} \).

Let us obtain an order-of-magnitude estimate of matter density at \( \tau = \tau_f \), using the conservation law (17), assuming that at present, when we can put \( r = a = r_0 \approx 10^{28} \text{ cm} \) the dust density corresponds to the total (visible plus dark) matter density of the universe, i.e.,

\[
\rho_d \approx m_0/r_0^3 \approx 2 \times 10^{-30} \text{ g cm}^{-3}, \quad m_0 = \text{const} \approx 2 \times 10^{54} \text{ g}.
\]

Neglecting the anisotropy (which, as we saw, is really negligible soon after \( \tau = \tau_f \)), i.e., assuming \( a(r) \approx r \) at all \( \tau > \tau_f \), we obtain

\[
\rho_d(\tau_f) \approx 0.8 \times 10^{12} \text{ g cm}^{-3} \approx \frac{1}{2600} \rho_c.
\]

This seems to be a reasonable figure for the state of the universe right after the vacuum decay and matter creation.

5. Conclusion

A general solution for homogeneous T-models has been found for a mixture of vacuum dark fluid and dust-like matter. It presents the class of T-models specified by the density profile of the vacuum fluid \( \rho_v \). The solution contains one arbitrary integration constant related to the dust density.

We have proved the existence of regular homogeneous T-models with the following structure:

(i) A regular static R-region in the pre-bang past, which is asymptotically de Sitter as \( r \to 0 \).
(ii) Killing horizon(s) separating static R-region(s) from KS T-region(s).
(iii) A null bang from a Killing horizon, followed by matter creation from an anisotropic vacuum, accompanied by rapid isotropization.
(iv) Further evolution in the regime without interaction between matter and vacuum.

Let us emphasize that the rapid isotropization was not imposed by hand. For our regular T-models rapid isotropization is generic, since the cosmological evolution starts from the Killing horizon, in this case \( a > 0, \dot{a} > 0 \), which immediately leads to tight constraints on the anisotropy factor.

The most interesting feature is that information about pre-bang history is available, and hence observable for the co-moving observers in T-regions, and can influence their dynamics.

Let us also emphasize that regular homogeneous T-models with anisotropic vacuum dark fluid do not need an inflationary stage since there are no causally disconnected spatial regions. Still, in this class of models, there is a certain amount of inflation, which happens simultaneously with isotropization, right after the null bang: the scale factor \( a(\tau) \) grows from
zero to a finite value while the other scale factor $r(\tau)$ rapidly grows from one finite value to another being driven by the (currently large but decaying) variable $\Lambda$-term.

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