Duality in Fuzzy Sigma Models

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Abstract

Nonlinear ‘sigma’ models in two dimensions have BPS solitons which are solutions of self- and anti-self-duality constraints. In this paper, we find their analogues for fuzzy sigma models on fuzzy spheres which were treated in detail by us in earlier work. We show that fuzzy BPS solitons are quantized versions of ‘Bott projectors’, and construct them explicitly. Their supersymmetric versions follow from the work of S. Kürkçüoğlu.
1. INTRODUCTION

Nonlinear field theories such as $\mathbb{C}P^N$-models in two dimensions are of theoretical interest. For large $N$, $\mathbb{C}P^N$-models for example are asymptotically free and show features which resemble QCD. They can also contain solitonic solutions and can thus serve as relatively simple quantum field theories (QFT’s) for examining asymptotic freedom and solitons.

In Euclidean QFT’s in 2-d, spacetime can be compactified to the two-sphere $S^2$. In turn $S^2$ can be discretized to the fuzzy sphere $S^2_F$ by quantization \[1\]. Such a discretization, besides its novelty, has other meritorious features such as preserving rotational invariance and supersymmetry on $S^2$ \[2\], \[3\], avoiding fermion doubling \[4\], and having a precise instanton, monopole and index theory \[5\]. Numerical work on certain QFT’s on $S^F_2$ such as $(\phi^4)_2$ has also been completed \[6\]. They do have correct limits to QFT’s on $S^2$ and are thus alternatives to lattice regularization.

In previous work \[7\], \[8\], fuzzy nonlinear models, such as fuzzy $\mathbb{C}P^N$ and Grassmanian models were constructed on $S^F_2$. Their supersymmetric generalizations were also found \[3\].

$\mathbb{C}P^1$ models on $S^2$ are of particular theoretical interest. As the target space is $S^2$, they are models of ferromagnets. As shown by Belavin and Polyakov \[9\], their solitons can be self-dual or anti-self-dual. These solutions saturate a topological bound on actions and exactly solve the field equations. They are the 2-d analogues of 4-d instantons.

In our work \[7\], we did not properly discuss fuzzy analogues of these self- and anti-self-dual solutions. In this paper, we resolve this lack of completeness. We establish that the fuzzy $\sigma$-fields based on Bott projectors are the fuzzy analogues of $S^2$-fields with a duality invariance. As Kurkçuoğlu’s work \[3\] is based on a supersymmetric generalization of Bott projectors, we now also have a supersymmetric version of these solitons.

2. PREVIOUS WORK.

We will briefly recall our previous work \[7\] on $\sigma$-models here.

Let $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2 \setminus \{0\}$. Then

$$S^3 = \{ z = \frac{\xi}{||\xi||}, \xi \in \mathbb{C}^2 \setminus \{0\}, ||\xi|| = \left( \sum |\xi_a|^2 \right)^{1/2} \}.$$ (2.1)

A point $\vec{x}$ ($\vec{x} \cdot \vec{x} = 1$) on the sphere $S^2$ is related to $z$ by

$$x_i = \xi^\dagger \tau_i z, \quad \tau_i = \text{Pauli matrices.}$$ (2.2)

$S^2$ is the complex manifold $\mathbb{C}P^1$, with a complex structure inherited from that of $\mathbb{C}^2 \setminus \{0\}$ with its complex variables $z$: the holomorphic coordinates on $\mathbb{C}P^1$ are obtained from $\mathbb{C}^2 \setminus \{0\}$ by the projective maps $\xi \rightarrow \frac{\xi_1}{\xi_2} = \frac{z_1}{z_2} = \frac{z_1 - iz_2}{1 - x_3}$ (if $z_2 \neq 0$, i.e. away from the north pole), and $\rightarrow \frac{\xi_2}{\xi_1} = \frac{z_2}{z_1} = \frac{z_1 + iz_2}{1 + x_3}$ (if $z_1 \neq 0$, i.e. away from the south pole).
A winding number $\kappa$ map to the target $S^2$ can be constructed as follows. Let $\kappa > 0$ first, and the ‘partial isometry’ $v_\kappa$ be defined as

$$v_\kappa : S^3 \to S^3 \ , \ v_\kappa(z) = \frac{1}{\sqrt{|z_1|^{2\kappa} + |z_2|^{2\kappa}}} \left( \begin{array}{c} z_1^\kappa \\ z_2^\kappa \end{array} \right) .$$  

(2.3)

Note that as $z \neq 0$, not both $z_\alpha$ can be zero and hence $\frac{1}{\sqrt{|z_1|^{2\kappa} + |z_2|^{2\kappa}}}$ is well-defined. $v_\kappa(z)$ is of degree $\kappa$ under $z \to e^{i\theta} z$, meaning that $v_\kappa(z) \to v_\kappa(z) e^{i\kappa \theta}$.

The field $n^{(\kappa)}$ on $S^2$ associated with $v_\kappa$ has components $n^{(\kappa)}_a$, that at the point $\vec{x}$ take the values

$$n^{(\kappa)}_a(\vec{x}) = v_\kappa(\vec{x})^\dagger \tau_a v_\kappa(\vec{x}) .$$  

(2.4)

The invariance of the R.H.S. under $z \to ze^{i\theta}$ means that they depend just on $\vec{x}$. $v_{\kappa|\kappa}(\vec{z}) = \overline{v_{\kappa|\kappa}(\vec{z})}$ has degree $-|\kappa|$. The construction of $n^{(\kappa)}$ for $\kappa = -|\kappa|$ is carried out using $\overline{v_{\kappa|\kappa}}$ for $v_{\kappa|\kappa}$.

(2.4) and its analogue for $\kappa < 0$ give particular maps $S^2 \to S^2$ with winding number $\kappa$. The general map $N^{(\kappa)}$ for either sign of $\kappa$ is got by replacing $v_\kappa(z), \ v_\kappa(z)^\dagger$ in (2.4) by

$$\mathcal{V}^{(\kappa)}(z) = U(\vec{x}) v_\kappa(z) \ , \ \mathcal{V}^{(\kappa)}(z)^\dagger = v_\kappa(z)^\dagger U(\vec{x})^\dagger ,$$

(2.5)

where $U(\vec{x})$ is any $2 \times 2$ unitary matrix which depends only on $\vec{x}$. In our previous paper it was shown that the winding number of a map can be calculated by

$$\kappa = \frac{1}{8\pi} \int_{S^2} \epsilon_{abc} \mathcal{N}^{(\kappa)}_a d\mathcal{N}^{(\kappa)}_b d\mathcal{N}^{(\kappa)}_c = \frac{1}{2\pi i} \int_{S^2} d(\mathcal{V}^{(\kappa)}_\kappa d\mathcal{V}^{(\kappa)}_\kappa) ,$$

(2.6)

so long as $U$ is a function of $\vec{x}$, the R.H.S. is equal to $\kappa$.

Fuzzy models of these maps are obtained from (2.3) by replacing $z_\alpha, \bar{z}_\beta$ by annihilation and creation operators $a_\alpha, a_\beta^\dagger : [a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}$ etc.. Then $v_\kappa \to \hat{v}_\kappa$, where for $\kappa > 0$

$$\hat{v}_\kappa = \left( \begin{array}{c} a_1^\kappa \\ a_2^\kappa \end{array} \right) , \ \hat{v}_\kappa^\dagger = \frac{1}{\sqrt{Z_\kappa}} \left( (a_1^\dagger)^\kappa , (a_2^\dagger)^\kappa \right) ,$$

(2.7)

$$\hat{Z}_\kappa = \hat{Z}_{\kappa}^{(1)} + \hat{Z}_\kappa^{(2)} \ , \ \hat{Z}_\kappa^{(a)} = \hat{N}_a (\hat{N}_a - 1)...(\hat{N}_a - \kappa + 1) \ , \ \hat{N}_a = a_\alpha^\dagger a_\alpha ,$$

while for $\kappa < 0$, we change $\hat{v}_\kappa$ to

$$\hat{v}_{|\kappa|} = \left( \begin{array}{c} (a_1^\dagger)^{|\kappa|} \\ (a_2^\dagger)^{|\kappa|} \end{array} \right) , \ \hat{v}_{|\kappa|}^\dagger = \frac{1}{\sqrt{Z_{|\kappa|}}} \left( a_1^{|\kappa|} , a_2^{|\kappa|} \right)$$

(2.8)

$$\hat{Z}_{|\kappa|} = \hat{Z}_{|\kappa|}^{(1)} + \hat{Z}_{|\kappa|}^{(2)} \ , \ \hat{Z}_{|\kappa|}^{(a)} = (\hat{N}_a + |\kappa|)(\hat{N}_a + |\kappa| - 1)...(\hat{N}_a + 1) \ , \ \hat{N}_a = a_\alpha^\dagger a_\alpha .$$

The particular expressions chosen for $\hat{Z}_\kappa$ ensure the normalization $\hat{v}_\kappa^\dagger \hat{v}_\kappa = 1$.

With this definition we can use (2.4, 2.8) for either sign of $\kappa$.

The quantized $\hat{x}$ and $\hat{n}^{(\kappa)}$ for either sign of $\kappa$ are

$$\hat{x} = \hat{v}_1^\dagger \tau_1 \ , \ \hat{n}^{(\kappa)} = \hat{v}_\kappa^\dagger \tau_\kappa \hat{v}_\kappa .$$

(2.9)
The general fuzzy $\sigma$-field $\mathcal{N}(\kappa)$ is obtained by transforming $\hat{\kappa}$ by a $2 \times 2$ unitary matrix $\hat{U}$, where $\hat{U}_{\alpha\beta}$ is a function of $a_{\alpha}^1 a_{\beta}$. That gives

$$\hat{\kappa} = \hat{U} \kappa, \quad \mathcal{N}(\kappa) = (\hat{\kappa})^\dagger \tau_{\alpha} \hat{\kappa}$$

(2.10)

Now $\hat{x}$, $\mathcal{N}(\kappa)$ commute with the number operator $\hat{N} = \hat{N}_1 + \hat{N}_2$ and hence can be restricted to the subspace $\hat{N} = n$ ($> 0$). This subspace has dimension $n + 1$. That gives us a finite dimensional matrix model for fuzzy solitons.

3. A GENERALIZATION.

To motivate a generalization of (2.3), let us note that taking as complex coordinates $\zeta = \frac{z_1}{z_1}$ in a patch of $S^2$, and $\phi = \frac{\alpha + i\beta}{1 + \alpha \beta}$ in a patch of target $S^2$, one can express the 'energy density' of a field configuration by

$$-\mathcal{L} \kappa L \kappa = -\frac{4 \mathcal{L} \phi \kappa L \phi}{(1 + \phi)^2} = 2 \left( 1 + \frac{1}{1 + \phi} \right)^2 \left( \frac{\partial \phi \partial \phi}{\partial \zeta \partial \zeta} + \frac{\partial \phi \partial \phi}{\partial \zeta \partial \zeta} \right), \quad \mathcal{L} = -i \epsilon_{ijk} x_j \partial_k.$$

(3.1)

The 'energy density' of the field configuration (2.3), which has $\phi(\zeta) = \zeta^\kappa$, is therefore concentrated around the north pole $x_3 = 1$ of the sphere. For a soliton of variable width and height localized at

$$\zeta' = \frac{x_1' + i x_2'}{1 + x_3'},$$

we can change (2.3) to

$$v_k(z, \zeta', \lambda) = \frac{1}{\sqrt{|\lambda z_1|^{2\kappa} + |z_2 - \zeta' z_1|^{2\kappa}}} \left( \frac{(\lambda z_1)^\kappa}{(z_2 - \zeta' z_1)^\kappa} \right), \lambda \neq 0$$

(3.3)

for $\kappa > 0$. More generally, a multisoliton field configuration with winding number $\kappa$ can be obtained replacing the partial isometry (2.3) with one of the form

$$v_k(z, c) = \frac{1}{\sqrt{|P_{1k}(z)|^2 + |P_{2k}(z)|^2}} \left( \frac{P_{1k}(z)}{P_{2k}(z)} \right), \quad P_{\alpha}(z) = \sum_{\kappa=0}^\kappa c_{\alpha h} z_1^{\kappa-h} z_2^h,$$

(3.4)

with arbitrary complex coefficients $c_{\alpha h}$, $\alpha = 1, 2$, $h = 0, ..., \kappa$, such that $P_{1k}(z)$, $P_{2k}(z)$ have no common zeroes on $S^2$ (so that the denominator is not 0 on $S^2$). This partial isometry can be obtained from the $v_k(z)$ defined in (2.3) applying to it $V(\vec{x}) = v_k(z, c) v_k(z)^\dagger$, which is a function of $\vec{x}$ as indicated. Therefore, if we use $v_k(z, c)$ and then a unitary $U(\vec{x})$ to construct $n^{(\kappa)}(\vec{x}, c)$, $V_k(z, c)$ and $\mathcal{N}(\kappa)(\vec{x}, c)$, we will find that

- $n^{(\kappa)}(\vec{x}, c)$ is still invariant under $z \rightarrow e^{i \theta} z$, and hence depends only on $\vec{x}$.
- By the argument indicated after (2.3), the winding number of $n^{(\kappa)}(\vec{x}, c)$ is indeed $\kappa$. 

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• By the same argument, as long as \( U \) is a function of \( \vec{x} \), \( \mathcal{N}^{(\kappa)}(\vec{x}, c) \) depends just on \( \vec{x} \) and has winding number \( \kappa \).

If \( c_{\alpha \kappa} \neq 0 \) for example, the coordinate \( \phi \) introduced above for a patch of target \( S^2 \) can be written for this type of configurations in the form

\[
\phi(\zeta, c) = c_1 \frac{(\zeta - a_1 - \cdots - \zeta - a_\kappa)}{(\zeta - b_1 - \cdots - \zeta - b_\kappa)} , \quad c = \frac{c_{2\kappa}}{c_{1\kappa}} \neq 0 , \quad (3.5)
\]

which is the one given in [9], [10].

For \( \kappa < 0 \) we let \( v_{-|\kappa|}(z, c) = v_{|\kappa|}(\bar{z}, c) \), and then construct the rest in an obvious manner.

For quantization, we let \( z_\alpha \rightarrow a_\alpha, \bar{z}_\alpha \rightarrow a_\alpha^\dagger \), but keep \( c_{\alpha h} \) as complex numbers. Then for example for \( \kappa > 0 \),

\[
\hat{v}_\kappa(c) = \frac{1}{(\hat{w}^{(\kappa)}(c)(\hat{w}^{(\kappa)}(c))^{1/2}} , \quad \hat{w}^{(\kappa)}(c) = \left( \sum c_{1h} a_{\kappa-h} c_{2h} a_{\kappa-h} \right) . \quad (3.6)
\]

In this way we get all the fuzzy solutions with \( c_{\alpha h} \)-dependence.

Note that we can study fuzzy solitons using the expressions in (3.6), which are well-defined. There is no need to find analogues of (3.5), which at best would be messy.

4. **Duality for Commutative CP\(^1\) → CP\(^1\).**

In the CP\(^1\)-model on \( S^2 \), without any discretization, self–duality and anti–self–duality are the conditions

\[
\mathcal{L}_i \mathcal{N}^{(\kappa)}_a + \epsilon_{ijk} x_k \epsilon_{abc} \mathcal{N}^{(\kappa)}_c \mathcal{L}_j \mathcal{N}^{(\kappa)}_b = 0 . \quad (4.1)
\]

It requires some work to show that

• Self–dual solutions require \( \kappa > 0 \) and are given by the choice \( U(\vec{x}) = 1, \mathcal{N}^{(\kappa)}_a(\vec{x}, c) = n_a^{(\kappa)}(\vec{x}, c) \).

• Anti–self–dual solutions require \( \kappa < 0 \) and are given by the choice \( U(\vec{x}) = 1, \mathcal{N}^{(\kappa)}_a(\vec{x}, c) = n_a^{(\kappa)}(\vec{x}, c) \).

Belavin and Polyakov recognized that the self-duality or anti-self-duality conditions are equivalent to Cauchy-Riemann equations on the world sheet \( \mathbb{R}^2 \). Summarizing their argument, one may use the complex coordinate \( \phi \) for the target \( S^2 \) introduced in the previous section to find that (4.1) implies

\[
\mathcal{L}_i \phi = \pm i \epsilon_{ijk} x_j \mathcal{L}_k \phi , \quad (4.2)
\]

and that, in terms of the complex coordinate \( \zeta \), this is equivalent to

\[
\frac{\partial \phi}{\partial \zeta} = 0 \quad \text{for the upper sign}, \quad \frac{\partial \phi}{\partial \zeta} = 0 \quad \text{for the lower sign}. \quad (4.3)
\]

To translate the conditions to the fuzzy case we need however a statement which does not involve ratios or local coordinates. This we provide in \( \S5 \).
5. ANALYTICITY AND DUALITY.

The analyticity properties of duality equations can be partly attributed to a certain scale invariance of the latter. We shall first introduce a formalism which explicitly brings out this invariance.

Let

\[ D^c SO(3) = \{ s \in Mat_2(\mathbb{C}) : s^3 s = \Delta I, \Delta > 0 \} \, . \]  

(5.1)

It is clear that if \( \Delta \neq 0 \), then \( \Delta > 0 \), and that \( D^c SO(3) \) is a group. It is the central extension of \( SO(3) \) by complex dilatations \( D^c \). We can show this as follows. First we quotient \( D^c SO(3) \) by the connected component of real dilatations

\[ \Delta = \left\{ \lambda I : \lambda > 0 \right\} \, , \]  

(5.2)

to get \( U(2) \), the homomorphism \( D^c SO(3) \rightarrow U(2) \) being

\[ s \rightarrow s \left( \frac{1}{|s|} \right) \, , \quad |s| = (s^\dagger s)^{1/2} > 0 \, . \]  

(5.3)

Its kernel consists of positive multiples of \( I \), that is of \( \Delta^R \). But \( D^c SO(3) \) contains also \( U(1) = \{ \alpha I : |\alpha| = 1 \} \). On quotienting \( U(2) \) by \( U(1) \) we get \( SO(3) \).

A map from \( \mathfrak{u}^2 \backslash \{ 0 \} \) to \( D^c SO(3) \), and then to \( S^2 \) is given by

\[ \mathfrak{u}^2 \backslash \{ 0 \} \ni \xi \rightarrow s = \begin{pmatrix} \xi_1 & -\xi_2 \\ \xi_2 & \xi_1 \end{pmatrix} \in D^c SO(3) \, , \]  

(5.4)

where \( Ad u(s) \) is the matrix of \( u(s) \) in the adjoint representation. Note that since \( s \tau_3 s^{-1} = u(s) \tau_3 u(s)^{-1} \), \( Ad s \) is equal to \( Ad u(s) \).

Under infinitesimal rotations \( \xi \rightarrow \xi' = (I + \frac{i}{2} \alpha_i \tau_i) \xi \), \( s \rightarrow s' = (I + \frac{i}{2} \alpha_i \tau_i) s \), \( x_i \rightarrow x_i' = x_i - \alpha_k \epsilon_{kij} x_j \). Functions on \( S^2 \) can be pulled back to functions on \( D^c SO(3) \), and by comparing the actions of rotations, we find that the angular momentum generators \( \mathcal{L}_i \) can be lifted to the negative of the left acting \( L_i \)'s, defined by

\[ (e^{i \alpha_j L_j} f)(r) = f(e^{-i \alpha_j \tau_j/2} s) \, . \]  

(5.5)

Consider maps \( \mathcal{W}_\kappa : \mathfrak{u}^2 \backslash \{ 0 \} \rightarrow \mathfrak{u}^2 \backslash \{ 0 \} \) which give \( \mathcal{W}_\kappa \) on normalization. For \( \kappa > 0 \) we have taken

\[ \mathcal{W}_\kappa(\xi) = U(\bar{\xi}) w_\kappa(\xi) \, , \quad w_\kappa(\xi) = \begin{pmatrix} \xi_1^\kappa \\ \xi_2^\kappa \end{pmatrix} \]  

(5.6)

while for \( \kappa < 0 \), we must complex conjugate the \( \xi_i \)'s in \( w_\kappa(\xi) \). Now \( \mathcal{W}_\kappa \) gives us a set of maps \( \mathcal{G} = \{ g \} \) from \( \mathfrak{u}^2 \backslash \{ 0 \} \) to \( D^c SO(3) \):

\[ g(\xi) = \begin{pmatrix} w_{\kappa_1} & -\overline{w}_{\kappa_2} \\ w_{\kappa_2} & \overline{w}_{\kappa_1} \end{pmatrix} (\xi) \, , \quad g^\dagger g = \mathcal{W}_{\kappa}^\dagger \mathcal{W}_\kappa I \, . \]  

(5.7)
\( g(\xi) \) is indeed valued in \( D^cSO(3) \) since \( \mathcal{W}_{\kappa,1,2} \) have no common zeroes in \( \xi \) and hence \( \mathcal{W}_{\kappa}^1 \mathcal{W}_{\kappa} > 0 \) for all \( \xi \). These maps are such that

\[
g \tau_3 g^{-1} = \tau_a \mathcal{N}^{(\kappa)}_a \quad (5.8)
\]

Associating an \( \mathcal{R} \in SO(3) \) to \( g \) in the usual way by

\[
g \tau_a g^{-1} = \tau_b \mathcal{R}_{ba} \quad , \quad (5.9)
\]

we see that \( \mathcal{N}^{(\kappa)}_a = \mathcal{R}_{a3} \).

Eq. \((5.9)\) is invariant under scale transformations of \( g \). Now \((5.1)\) can be expressed in terms of \( g \), as we shall show below. Thus it can depend only on ratios of components of \( g \), an important result (cf. \((3.5)\)). The formalism using \( g \) is convenient to express this scale invariance.

Indicating by \( \theta_a, (\theta_a)_{ij} = -i \epsilon_{aij} \), the components of spin 1 angular momentum, we have from \((5.4), (5.8)\) that:

\[
(Ad s) \theta_3 (Ad s)^{-1})_{ij} = -i \epsilon_{ijk} x_k \quad , \quad (\mathcal{R} \theta_3 \mathcal{R}^{-1})_{ab} = -i \epsilon_{abc} \mathcal{N}^{(\kappa)}_c
\]

\((5.10)\)

We may therefore rewrite \((4.1)\) in the form

\[
(Ad s)^{-1}_{ij} L_j = -L_i^R \quad , \quad (Ad s)^{-1}_{ij} L_j = -L_i^R \quad ,
\]

\((5.11)\)

Next we can go from the left-acting \( L_i \) to the right acting \( L_i^R \), defined by

\[
(e^{i\alpha_j L_j^R} f)(s) = f(s e^{i\alpha_j \tau_j/2}) \quad , \quad (5.12)
\]

using the relation

\[
(Ad s)^{-1}_{ij} L_j = -L_i^R \quad , \quad (5.13)
\]

thus turning \((5.11)\) to

\[
(\mathcal{R}^{-1} L_i^R \mathcal{R})_{a3} = \mp (\theta_3)_{ij} (\theta_3 \mathcal{R}^{-1} L_j^R \mathcal{R})_{a3} \quad .
\]

\((5.14)\)

Taking successively \( i = 3, 1, 2 \), we see that the independent relations we have are

\[
(\mathcal{R}^{-1} L_i^R \mathcal{R})_{a3} = (\mathcal{R}^{-1} L_i^R \mathcal{R})_{13} = i(\mathcal{R}^{-1} L_i^R \mathcal{R})_{23} = 0 \quad ,
\]

\((5.15)\)

where \( L_i^R = L_i^R - iL_i^R \). The real and imaginary parts of \((5.15)\) give \((5.14)\), because \( \mathcal{R}_{ab} \) is real, and \((L_j^R \mathcal{R})_{ab} \) is pure imaginary. The matrices \((\mathcal{R}^{-1} L_i^R \mathcal{R})\) are therefore antisymmetric, and since the \( \theta_a \) are pure imaginary, these equations are equivalent to:

\[
\text{Tr} \left((\theta_1 - i\theta_2) \mathcal{R}^{-1} L_3^R \mathcal{R} \right) = \text{Tr} \left((\theta_1 \mp i\theta_2)(\mathcal{R}^{-1} L_3^R \mathcal{R}) \right) = 0 \quad .
\]

\((5.16)\)

These relations must hold in any representation of the Lie algebra, and therefore

\[
\text{Tr} \left((\tau_1 - i\tau_2) g^{-1} L_3^R g \right) = \text{Tr} \left((\tau_1 \mp i\tau_2) g^{-1} L_3^R g \right) = 0 \quad .
\]

\((5.17)\)
In this way we have succeeded in expressing the self-duality - anti-self-duality conditions (4.1) directly in terms of the $W_{\kappa \alpha}$, $\overline{W}_{\kappa \alpha}$. Explicitly, the relations we have obtained are:

$$
\overline{W}_{\kappa 1} L_3 R \overline{W}_{\kappa 2} - \overline{W}_{\kappa 2} L_3 R \overline{W}_{\kappa 1} = 0 \quad (5.18)
$$

upper sign: $- \overline{W}_{\kappa 1} L_3 R \overline{W}_{\kappa 2} + \overline{W}_{\kappa 2} L_3 R \overline{W}_{\kappa 1} = 0$

lower sign: $- W_{\kappa 2} L R \overline{W}_{\kappa 1} + W_{\kappa 1} L R \overline{W}_{\kappa 2} = 0 \quad (5.19)$

First consider (5.19), and for definiteness the 'upper sign' (self-duality) equation. What it means is that the ratio $\frac{\overline{W}_{\kappa 2}}{W_{\kappa 1}}$ is annihilated by $L R$. Now on functions $f$ on $D^c SO(3)$,

$$(L_R f(s)) = (\overline{\xi}_1 \frac{\partial}{\partial \xi_2} - \overline{\xi}_2 \frac{\partial}{\partial \xi_1}) f(s). \quad (5.20)$$

Hence $\frac{\overline{W}_{\kappa 2}}{W_{\kappa 1}}$ depends only on $(\overline{\xi}_1, \overline{\xi}_2)$. This implies that the dependance of $(W_{\kappa 1}, W_{\kappa 2})$ on $(\overline{\xi}_1, \overline{\xi}_2)$ factors out when we take the ratio, or can be eliminated by a rescaling. Suppose we do this rescaling, so that we may represent the $W_{\kappa \alpha}$’s in the form $W_{\kappa \alpha} = \sum_{kn} c_{\kappa kn} \xi_1^{k-n} \xi_2^n$. Then, since we may express $L_R^3$ as

$$(L_R^3 f(s)) = \frac{1}{2} \sum_{\alpha=1,2} (\xi_\alpha \frac{\partial}{\partial \xi_\alpha} - \overline{\xi}_\alpha \frac{\partial}{\partial \overline{\xi}_\alpha}) f(s), \quad (5.21)$$

(5.18) implies that the sum must be restricted to a single value of $k$, the same for both values of $\alpha$. It follows that for self-duality we must have

$$W_{\kappa 1} = \sum_{n=0}^{\kappa} c_{1n} \xi_1^{\kappa-n} \xi_2^n, \quad W_{\kappa 2} = \sum_{n=0}^{\kappa} c_{2n} \xi_1^{\kappa-n} \xi_2^n. \quad (5.22)$$

for some integer $\kappa > 0$ and coefficients $c_{\alpha n}$.

We can interpret (5.22) by saying that $W_{\kappa \alpha}$ are highest weight vectors with angular momentum $\kappa/2$ for the $SU(2)$ Lie algebra generated by $L_i^R$. They are holomorphic (being polynomials) in $\xi_\alpha$.

A similar discussion can be made for the lower sign in (5.19) (anti-self-duality).

Summarizing, (4.11) expresses duality using unit vectors of world sheet and target $S^2$. That is not the best way for fuzzy physics. For the latter, it is better to rewrite it after scaling and using the holomorphic coordinates of $\mathbb{C}^2 \setminus \{0\}$.

6. FUZZIFICATION OF DUALITY.

The dual solutions for $\mathbb{C}P^1 \rightarrow \mathbb{C}P^1$ are $v_{\kappa}(z, c)$ and their derived structures. They are very easy to quantize: replace $\xi_\alpha$ by $a_\alpha$ and $\overline{\xi}_\alpha$ by $a_\alpha^\dagger$. That gives a fuzzy version of BPS states.

In the commutative case, BPS solutions saturate the lower bound on the energy functional [9]. Such a result is not quite correct in the fuzzy case [7].
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