Quantum KdV hierarchy and quasimodular forms

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Abstract

Dubrovin \cite{Dubrovin10} has shown that the spectrum of the quantization (with respect to the first Poisson structure) of the dispersionless Korteweg–de Vries (KdV) hierarchy is given by shifted symmetric functions; the latter are related by the Bloch–Okounkov Theorem \cite{Bloch1} to quasimodular forms on the full modular group. We extend the relation to quasimodular forms to the full quantum KdV hierarchy (and to the more general quantum Intermediate Long Wave hierarchy). These quantum integrable hierarchies have been defined by Buryak and Rossi \cite{Buryak6} in terms of the Double Ramification cycle in the moduli space of curves. The main tool and conceptual contribution of the paper is a general effective criterion for quasimodularity.

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1 Introduction

1.1 Differential polynomials and \( q \)-series.

Let \( \Lambda := \mathbb{Q}[p] \) be the ring of polynomials with rational coefficients in the variables \( p_j \), for \( j \geq 1 \), collectively denoted by \( p := (p_1, p_2, p_3, \ldots) \). Assigning weight \( k \) to \( p_k \) we have the grading \( \Lambda = \bigoplus_{n \geq 0} \Lambda_n \), where \( \Lambda_n \) consists of polynomials of homogeneous weight \( n \). For any linear operator \( G \in \text{End}(\Lambda) \) such that\footnote{Throughout this paper we denote by \( [A, B] := AB - BA \) the commutator of \( A \) and \( B \).}

\[
\left[ G, \sum_{k \geq 1} k p_k \frac{\partial}{\partial p_k} \right] = 0, \tag{1.1}
\]

i.e., such that \( G \) restricts to a linear operator on \( \Lambda_n \) for any \( n \geq 0 \), we introduce the \( q \)-series

\[
\{G\}_q := \frac{\sum_{n \geq 0} q^n \text{tr}_{\Lambda_n} G}{\sum_{n \geq 0} q^n \text{dim} \Lambda_n}. \tag{1.2}
\]

Equivalently, \( \{G\}_q = q^{-1/24} \eta(q) \sum_{n \geq 0} q^n \text{tr}_{\Lambda_n} G \), where \( \eta(q) = q^{1/24} \prod_{k \geq 1} (1 - q^k) \) is the Dedekind eta function.

In certain cases, the \( q \)-series (1.2) specializes to the well-studied \( q \)-bracket, introduced in \cite{Borcherds}. The latter is attached to a function \( f : \mathcal{P} \to \mathbb{Q} \) from the set \( \mathcal{P} \) of partitions to the rationals and defined by

\[
\langle f \rangle_q := \frac{\sum_{\lambda \in \mathcal{P}} q^{\lambda} f(\lambda)}{\sum_{\lambda \in \mathcal{P}} q^{\lambda}} = q^{-1/24} \eta(q) \sum_{\lambda \in \mathcal{P}} q^{\lambda} f(\lambda), \tag{1.3}
\]
where \( |\lambda| := \lambda_1 + \cdots + \lambda_k \) denotes the integer \( \lambda = (\lambda_1, \ldots, \lambda_k) \) is a partition of. (Namely, if \( G \) is the diagonal operator acting as multiplication by \( f(\lambda) = p_{\lambda_1} \cdots p_{\lambda_k} \), then \( \{ G \}_q = \langle f \rangle_q \).

In particular, Bloch and Okounkov [1] proved that for the class of shifted symmetric functions of partitions, the \( q \)-bracket (1.3) is quasimodular of homogeneous weight. This class consists of homogeneous polynomials of certain basic functions \( Q_k : \mathcal{P} \to \mathbb{Q} \), for \( k \geq 0 \), where the weight of \( Q_k \) is defined to be \( k \). The basic functions are defined by \( Q_0(\lambda) := 1 \) and

\[
Q_k(\lambda) := \frac{1}{(k-1)!} \sum_{i \geq 1} \left[ (\lambda_i - i + \frac{1}{2})^{k-1} - (-i + \frac{1}{2})^{k-1} \right] + \beta_k
\]

for \( k \geq 1 \), where \( \beta_k = \left( \frac{1}{2\pi i} - 1 \right) \frac{B_k}{k!} \) and \( B_k \) denotes the \( k \)th Bernoulli number. The central characters of the symmetric group are shifted symmetric functions [20] and hence these functions appear in the study of asymptotic properties of partitions [19, 26], as well as in many works in enumerative geometry, e.g., in the Hurwitz/Gromov–Witten theory of an elliptic curve [9, 13, 14, 16, 24, 25], or in the determination of the Siegel–Veech constants of the moduli space of flat surfaces [14, 7, 8].

The appearance of shifted symmetric functions in the study of integrable hierarchies is part of an interesting story. As explained in [10], Eliashberg [11] solved the quantization problem for the classical Hopf hierarchy by using ideas coming from Symplectic Field Theory. Concretely, he constructed a commuting family of quantum Hamiltonians \( G_k^{\text{Hopf}} \) (\( k \geq -2 \)), which are operators on \( \Lambda \) depending on a constant \(^2 c \), obtained from differential polynomials by the procedure explained in the next paragraph. In particular, after inserting a quantization parameter \( \hbar \) in these differential polynomials (see Remark 3.1), in the limit \( \hbar \to 0 \) they reduce to the Hamiltonian densities of the classical Hopf hierarchy. Rossi [27] showed that the operators \( G_k^{\text{Hopf}} \), under the boson-fermion correspondence (see, e.g., [23]), are quadratic in fermions—a fact that was exploited by Dubrovin to diagonalize these operators (to then provide applications to the symplectic field theory of the disk).

Namely, [10, Theorem 1.4]

\[
G_k^{\text{Hopf}} s_\lambda(p) = E_k^{[0]}(\lambda) s_\lambda(p), \quad \lambda \in \mathcal{P}, \quad k \geq -2,
\]

where the eigenvalues \( E_k^{[0]} : \mathcal{P} \to \mathbb{Q} \) are shifted symmetric functions, given explicitly by

\[
E_k^{[0]} = \sum_{j=0}^{k+2} \frac{e^{k+2-j}}{(k+2-j)!} Q_j,
\]

and the eigenvectors \( s_\lambda(p) \) are the Schur functions\(^3 [22] \), defined by

\[
\begin{align*}
s_\lambda(p) & := \det [h_{\lambda_i-1+j}(p)]_{i,j=1}^{(\lambda)}, \\
& = \exp \left( \sum_{k \in \mathbb{Z}} \frac{p_k}{k} y^k \right).
\end{align*}
\]

It follows immediately from (1.5) and the Bloch–Okounkov Theorem that

\[
\{ G_k^{\text{Hopf}} \}_q = \sum_{j=0}^{k+2} \frac{e^{k+2-j}}{(k+2-j)!} \{ Q_j \}_q
\]

is a polynomial in \( c \) with quasimodular coefficients. Note that this expression is of homogeneous weight \( k + 2 \), provided we assign weight +1 to \( c \), and consider the quasimodular coefficients with the corresponding quasimodular weight.

In the present work we study quasimodularity properties of a deformation \( G_k^{\text{KdV}}(c) \) (constructed by Buryak and Rossi [6]) of the operators \( G_k^{\text{Hopf}} \), depending (polynomially) on an additional parameter \( \epsilon \) and satisfying \( G_k^{\text{KdV}}(0) = G_k^{\text{Hopf}} \). These are the Hamiltonian operators of the quantum

\(^2\)This constant is denoted \( u_0 \) in [10].

\(^3\)Expressed in terms of the power sum polynomials.
Korteweg–de Vries (KdV) hierarchy\(^4\), as the corresponding densities (again, after properly introducing the parameter \(\hbar\)), reduce in the limit \(\hbar \to 0\) to the Hamiltonian densities of the classical KdV hierarchy. (Incidentally, the construction in op. cit., which we briefly review in Section 3.1, is much more general, and produces a quantum integrable hierarchy attached to any Cohomological Field Theory. The KdV case corresponds to the trivial Cohomological Field Theory.)

Our first goal is to study whether the quasimodularity of \(q\)-series associated with the Hopf hierarchy, expressed by (1.8), survives under the deformation in \(\epsilon\) (as anticipated in [28]). We answer this in the affirmative (Theorem 1.2), by providing a general criterion for quasimodularity (Theorem 1.3). Moreover, as a byproduct of the general criterion, we obtain a simplification of the quantum Hamiltonian operators which we expect to be useful in the study and classification of quantum integrable hierarchies of rank 1 [4].

We now move on to a more detailed explanation of our findings.

From differential polynomials to operators. In this paper, we shall be concerned with quasimodular properties of \(\{G\}_q\) for operators \(G\) on \(\Lambda\) which are obtained out of a polynomial \(g \in \mathbb{Q}[u]\) by the following quantization procedure. Here \(\mathbb{Q}[u]\) is the ring of polynomials with rational coefficients in the variables \(u_j\), for \(j \geq 0\), collectively denoted by \(u := (u_0, u_1, u_2, \ldots)\).

First, for \(j \in \mathbb{Z}_{\geq 0}\), we introduce the Fourier series\(^5\) \(v_j(x) := \sum_{\ell \in \mathbb{Z}} (i \ell)^j \omega_\ell e^{i \ell x}\), where we assign weight \(\ell\) to \(\omega_\ell\). Note that \(v_j(x) = \partial^j_x v_0(x)\) for \(j \geq 1\). Next, given \(g \in \mathbb{Q}[u]\), define a formal power series (of homogeneous weight 0) in the variables \(\omega_\ell, \ell \in \mathbb{Z}\), by

\[
\int_0^{2\pi} g(\nu(x)) \frac{dx}{2\pi} \quad \nu(x) := (v_0(x), v_1(x), \ldots).
\] (1.9)

Write all monomials in this series as a product of \(\omega_\ell\)'s where all the \(\omega_\ell\)'s with \(\ell \geq 0\) appear to the left of all \(\omega_\ell\)'s with \(\ell < 0\) (normal ordering). In this expression, we finally replace \(\omega_\ell\) with the operator \(\hat{\omega}_\ell \in \text{End}(\Lambda)\), defined by

\[
\hat{\omega}_\ell f := \begin{cases} \rho_\ell f & \ell \geq 1, \\ c f & \ell = 0, \\ -\ell \frac{\partial f}{\partial \rho_{-\ell}} & \ell \leq -1, \end{cases} \quad (\ell \in \mathbb{Z}, f \in \Lambda) \tag{1.10}
\]

with \(c \in \mathbb{Q}\) an arbitrary constant. We denote by \(\text{Op}_c(g)\) the operator on \(\Lambda\) obtained from \(g \in \mathbb{Q}[u]\) in this way. In other words, we give the following definition.

**Definition 1.1.** For \(g \in \mathbb{Q}[u]\), denote by \(\text{Op}_c(g) : \Lambda \to \Lambda \oplus \Lambda i\) the operator, depending on \(c \in \mathbb{Q}\), given by

\[
\text{Op}_c(g) := \int_0^{2\pi} g(\bar{\nu}(x)) \frac{dx}{2\pi},
\]

where \(\cdots\) denotes the normal ordering, \(\bar{\nu}(x) = (\bar{v}_0(x), \bar{v}_1(x), \ldots)\) with \(\bar{v}_j(x) := \sum_{\ell \in \mathbb{Z}} (i \ell)^j \hat{\omega}_\ell e^{i \ell x}\) and \(\hat{\omega}_\ell\) as defined above.

It is worth noting that \(\text{Op}_c(g)\) might be complex-valued. More precisely, let \(\mathbb{Q}[u] = \mathbb{Q}[u]^{\text{even}} \oplus \mathbb{Q}[u]^{\text{odd}}\), where \(\mathbb{Q}[u]^{\text{even}}\) (respectively, \(\mathbb{Q}[u]^{\text{odd}}\)) is the span of monomials which are even (respectively, odd) with respect to the weight operator \(\sum_{j \geq 1} j u_j \frac{\partial}{\partial u_j}\). Then, \(\text{Op}_c(g)\) is purely real for \(g \in \mathbb{Q}[u]^{\text{even}}\), and purely imaginary for \(g \in \mathbb{Q}[u]^{\text{odd}}\).

Let us give a few examples:

- \(\text{Op}_c(g) = 0\) when \(g\) is in the image of the operator \(\partial_x := \sum_{j \geq 0} u_{j+1} \frac{\partial}{\partial u_j}\),
- \(\text{Op}_c(u_0) = c\),

\(^4\)The quantization is with respect to the first Poisson structure, cf. [6, 10].

\(^5\)With the convention \(0^0 = 1\).
\[\cdot \mathbf{O}_p(u_0^2) = c^2 + 2 \sum_{j \geq 1} j p_j \frac{\partial}{\partial p_j},\]
\[\cdot \mathbf{O}_p(u_0^3) = c^3 + 6c \sum_{j \geq 1} j p_j \frac{\partial}{\partial p_j} + 6\Delta,\] where
\[
\Delta = \frac{1}{2} \sum_{j,k \geq 1} \left( (j+k) p_j p_k \frac{\partial}{\partial p_{j+k}} + j k p_{j+k} \frac{\partial^2}{\partial p_j \partial p_k} \right)
\] (1.11)
is the cut-and-join operator [15].

If \( g \in \mathbb{Q}[u]^{\text{even}} \), the operator \( \mathbf{O}_p(g) \) is symmetric (see [28, Lemma 2.3]), i.e., \( (v, \mathbf{O}_p(g)w) = (\mathbf{O}_p(g)v, w) \) for all \( v, w \in \Lambda \) with respect to the standard scalar product \( \langle \ , \ \rangle \) on \( \Lambda \) (see [22]). The latter can be defined by
\[
\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda, \mu},
\] (1.12)
where
\[
p_\lambda := p_{\lambda_1} \cdots p_{\lambda_r}
\] (1.13)
is the monomial basis of \( \Lambda \) indexed by partitions \( \lambda = (\lambda_1, \ldots, \lambda_r) \), and \( z_\lambda := \prod_{m \geq 1} r_m(\lambda)! m^{r_m(\lambda)} \), where \( r_m(\lambda) := \# \{ i \mid \lambda_i = m \} \).

1.2 Quantum Korteweg–de Vries hierarchy.

A construction by Buryak and Rossi [3, 5, 6], also inspired by previous works in Symplectic Field Theory [11, 12], provides an effective construction of quantum integrable hierarchies associated with an arbitrary Cohomological Field Theory (CohFT). Even though the construction is completely general, in this work we restrict to the case of rank 1 CohFTs only.

The output of this construction, which is briefly reviewed in Section 3.1, is a family of Hamiltonian densities \( g_k(u; c) \in \mathbb{Q}[u]^{\text{even}} \otimes \mathbb{Q}[c] \), for \( k \geq -2 \), possibly depending on the parameters of the CohFT (more details below). They are determined by either an explicit formula in terms of the Double Ramification cycles in the moduli space of curves, see (3.3), or (more effectively) by a recurrence relation of order one, see (3.5) and (3.6).

One of the main properties of the Hamiltonian operators \( G_k(c) := \mathbf{O}_p(g_k(u; c)) \in \text{End} (\Lambda) \otimes \mathbb{Q}[c] \) is that they enjoy the commutativity\(^6\)
\[
[G_j(c), G_k(c)] = 0, \quad j, k \geq -2.
\] (1.14)

The relevance of this construction to the theory of integrable systems stems from the fact that, after suitably introducing a quantization parameter \( \hbar \) (see Remark 3.1), the Hamiltonian densities reduce, in the limit \( \hbar \rightarrow 0 \), to those of classical integrable hierarchies, well-studied in the literature. We refer to the aforementioned literature, particularly to the Introduction of [10], for more details on this point. Since in this work we are mainly interested in the quantum Hamiltonian densities, we opted to simplify the exposition by dropping the parameter \( \hbar \) (which can be reinstated at any time by the transformations described in Remark 3.1).

The simplest instance of this construction is provided by the quantum Korteweg–de Vries (KdV) hierarchy (associated with the trivial CohFT), a prototypical example of an integrable system. In

\(^6\) It is appropriate to remark that the setting of [6] is more general, and the only requirement is the commutation relation \( [\tilde{\omega}_a, \tilde{\omega}_b] = -a \delta_{a+b,0} \). For our purposes it is convenient to fix the representation (1.10) (see also [10]).
In this case, the first few densities read
\[
\begin{align*}
    g_{-2}^{KdV}(u; \epsilon) &= 1, & g_{-1}^{KdV}(u; \epsilon) &= u_0, & g_0^{KdV}(u; \epsilon) &= \frac{\epsilon^2}{2} - \frac{1}{24} + \frac{\epsilon}{24} u_2, \\
    g_1^{KdV}(u; \epsilon) &= \frac{u_0^2}{6} - \frac{u_0}{24} - \frac{u_2}{24} + \frac{\epsilon}{24}(u_0 u_2 - \frac{1}{120}) + \left(\frac{\epsilon}{24}\right)^2 u_4 - \frac{1}{24} u_2, \\
    g_2^{KdV}(u; \epsilon) &= \frac{u_0^4}{24} - \frac{u_0 u_2}{48} - \frac{u_4 u_2}{24} + \frac{7}{5760} + \frac{\epsilon}{24}\left(\frac{u_0^2 u_2}{2} - \frac{u_4}{30} - \frac{u_2}{24} - \frac{u_0}{120}\right) \\
        &\quad + \left(\frac{\epsilon}{24}\right)^2\left(\frac{7 u_0^2}{10} + \frac{u_0 u_4}{2} - \frac{1}{210}\right) + \left(\frac{\epsilon}{24}\right)^3 u_6. 
\end{align*}
\]
\tag{1.15}

(For the recursion determining them see (3.5) and (3.6) below, and for further properties see Appendix A.)

The Hamiltonian operators \(G_k^{[\text{Hopf}]} := \text{Op}_c(g_k^{KdV}(u; \epsilon))\) are polynomials of degree \(k\) in \(\epsilon\). As already mentioned in Section 1.1, their constant term \(G_k^{[0]} := [\epsilon^0]G_k^{KdV}(\epsilon)\) coincides with the Hamiltonian operators of the Hopf hierarchy, i.e., \(G_k^{[0]} = G_k^{\text{Hopf}}\), and the quasimodularity of \(\{G_k^{[0]}\}_q\) follows by Dubrovin’s result (1.5) along with the Bloch–Okounkov theorem. On the other hand, their leading coefficient \(G_k^{[\infty]} := [\epsilon^k]G_k^{KdV}(\epsilon)\) is given by (see Corollary A.4)
\[
G_k^{[\infty]} = \frac{c^2}{2} \delta_{k,0} + \frac{L_{2k+2}}{(-4)^{k}(2k + 1)!}, \quad k \geq 0, 
\tag{1.16}
\]
where the operator \(L_k\) is given by
\[
L_k := -\frac{c^2}{2} \delta_{k,2} - \frac{B_k}{2k} - \frac{j^k}{2} \text{Op}_c(u_0 u_{k-2}) = -\frac{B_k}{2k} + \sum_{j \geq 1} j^{k-1} \frac{\partial}{\partial p_j}. 
\tag{1.17}
\]

Therefore, \(G_k^{[\infty]}\) are diagonal on the monomial basis (1.13) of \(\Lambda\),
\[
G_k^{[\infty]} p_\lambda = E_k^{[\infty]}(\lambda) p_\lambda, \quad \lambda \in \mathcal{P}, \quad k \geq -2, 
\tag{1.18}
\]
where \(E_k^{[\infty]} : \mathcal{P} \to \mathbb{Q}\) are given in terms of the moment functions \(E_k^{[\infty]} : \mathcal{P} \to \mathbb{Q}\) are given in terms of the moment functions \(E_k^{[\infty]}(\lambda) := -\frac{B_k}{2k} + \sum_{i=1}^{\ell(\lambda)} \lambda_i^{k-1}, \quad k \geq 1, \quad \ell(\lambda) := -1\)
by
\[
E_k^{[\infty]} = \frac{c^2}{2} \delta_{k,0} + \frac{S_{2k+2}}{(-4)^{k}(2k + 1)!}, \quad k \geq 0. 
\tag{1.20}
\]

It was observed in [32] that the \(q\)-bracket of \(S_{2k+2}\) is an Eisenstein series which is quasimodular of weight \(2k + 2\).

Let us denote by \(\tilde{\mathcal{M}}\) the ring of quasimodular forms (with rational coefficients) on the full modular group \(\text{SL}_2(\mathbb{Z})\) (see e.g., [31, Section 5.3]), containing, for each even \(k \geq 2\) the Eisenstein series
\[
G_k = -\frac{B_k}{2k} + \sum_{m, r \geq 1} m^{k-1} q^{mr}. 
\tag{1.21}
\]

Then \(\tilde{\mathcal{M}} = \bigoplus \tilde{\mathcal{M}}_k\) is a graded ring freely generated over the rationals by the Eisenstein series \(G_2, G_4\) and \(G_6\), where the weight of \(G_k\) is defined to be \(k\). Moreover, let \(\tilde{\mathcal{M}}[c, \epsilon] := \tilde{\mathcal{M}} \otimes \mathbb{Q}[c, \epsilon] := \bigoplus_k \tilde{\mathcal{M}}[c, \epsilon]_k\), where we assign weight +1 to \(c\) and −1 to \(\epsilon\).
Theorem 1.2. For the quantum KdV Hamiltonian operators
\[ G_k^{\text{KdV}}(\epsilon) := \text{Op}_c(g_k^{\text{KdV}}(u; \epsilon)) \in \text{End}(\Lambda) \otimes \mathbb{Q}[c, \epsilon], \quad (k \geq -2) \] (1.22)
we have
\[ \{ G_k^{\text{KdV}}(\epsilon) \}_q \in \tilde{M}[c, \epsilon]_{k+2}. \] (1.23)

The proof is given in Section 3.2, for the more general case of the quantum Intermediate Long Wave hierarchy, which is a generalization of the KdV hierarchy [6] (see Theorem 3.8). The key step in the proof is a general criterion (Theorem 1.3 below) for quasimodularity of homogeneous weight which applies to operators of the form \( \text{Op}_c(g) \) for \( g \in \mathbb{Q}[u] \). Explicit examples for \( k \leq 3 \) in terms of Eisenstein series are included in Appendix B.

We expect that Theorem 1.2 holds true for all rank 1 quantum Double Ramification integrable hierarchies. Namely, the quantum KdV Hamiltonian densities are the special case \( s_i = 0 \) of a more general hierarchy of Hamiltonian densities \( g_k(u; \epsilon, s) \), depending on parameters \( s = (s_1, s_3, s_5, \ldots) \) (see, Section 3.1). There exist different possible normalizations for these Hamiltonian densities (see, [4, eq. 5.3]), and by Theorem 1.3 below in all normalizations \( \{ \text{Op}_c(g_k(\epsilon, s)) \}_q \) belongs to \( \tilde{M}[c][s, \epsilon] \).

We expect that there exists a convenient normalization such that \( \{ \text{Op}_c(g_k(\epsilon, s)) \}_q \) is quasimodular of homogeneous weight (where \( s_k \), for \( k \) odd, is assigned weight \( k \)).

Moreover, as suggested to us by Don Zagier, we expect that the \( q \)-series (1.2) associated to arbitrary compositions of the quantum KdV operators give rise to quasimodular forms as well, or, even stronger, that the eigenvalues of the quantum KdV operators are shifted symmetric functions of homogeneous weight. Namely, by [28] there exists a simultaneous basis \( r_\lambda(p; \epsilon) \in \Lambda_{[\lambda]}[\epsilon] \) of eigenfunctions \( E_k(\lambda; \epsilon) \) for \( G_k^{\text{KdV}}(\epsilon) \) for all \( \lambda \in \mathcal{P} \). Then, we expect that
\[ E_k(\lambda; \epsilon) \in \mathbb{Q}[c, Q_2, Q_3, \ldots][\epsilon]_{k+2}, \] (1.24)
where \( \epsilon \) is assigned weight \(-1\), \( c \) weight \(+1\), and \( Q_k \) weight \( k \). Note \( E_k(\lambda; 0) = E_k^{(0)}(\lambda) \) is the shifted symmetric function in (1.6) and, as a consequence of Theorem 1.2, we have \( (E_k(\lambda; \epsilon))_q = \{ G_k^{\text{KdV}}(\epsilon) \}_q \in \tilde{M}[c, \epsilon]_{k+2} \). We have numerical evidence for (1.24) in a few instances, and we hope to return to it in a later publication.

1.3 A criterion for homogeneous quasimodularity

We will show that the \( q \)-series \( \{ \text{Op}_c(g) \}_q \) (for \( g \in \mathbb{Q}[u] \)) is always a quasimodular form of mixed weight. Moreover, we provide a criterion for the quasimodularity of homogeneous weight for the \( q \)-series \( \{ \text{Op}_c(g) \}_q \). To state it, we assign weight \( k+1 \) to \( u_k \), so that \( \mathbb{Q}[u] \) (as well as its subspaces \( \mathbb{Q}[u]^{\text{even}} \) and \( \mathbb{Q}[u]^{\text{odd}} \) defined at page 3) become graded algebras. Moreover, let \( \tilde{M}[c] := \tilde{M} \otimes \mathbb{Q}[c] =: \bigoplus_k \tilde{M}[c]_k \), be the polynomial ring in \( c \) and \( \epsilon \) over the graded ring of quasimodular forms, graded by the quasimodular weight and by assigning weight \(+1\) to \( c \).

Moreover, let \( \mathcal{B} \) be the linear operator on \( \mathbb{Q}[u] \) defined by
\[ \mathcal{B} := \exp \left( \frac{1}{2} \sum_{i,j \geq 0} (-1)^{i+j} \frac{B_{i+j+2}}{i+j+2} \partial u_i \partial u_j \right), \quad B_k = k\text{th Bernoulli number.} \] (1.25)

\[ \text{It might seem more natural to write } u_{k+1} \text{ for what is called } u_k \text{ here, but in order to adhere to standard notation for the Hamiltonian densities introduced in the next section, we do not do so. In fact, recall that in the background there is a second weight operator, which gives rise to the even and odd part of } \mathbb{Q}[u], \text{ and assigns weight } k \text{ to } u_k. \]
Theorem 1.3.
(i) For any \( g \in \mathbb{Q}[u] \) we have \( \{ \mathcal{O}_p(g) \}_q \in \tilde{\mathcal{M}}[c] \).
(ii) For any \( g \in \mathbb{Q}[u]^{\text{odd}} \), we have \( \{ \mathcal{O}_p(g) \}_q = 0 \).
(iii) The mapping \( \mathbb{Q}[u] \to \tilde{\mathcal{M}}[c] \)
\[
g \mapsto \{ \mathcal{O}_p(\mathcal{B} g) \}_q
\]
(1.26)
is a morphism of graded vector spaces.

The proof is given in Section 2 and builds on the previous work [18] of the first author. In the special case \( c = 0 \), the theorem states the following.

Corollary 1.4. Let \( g \in \mathbb{Q}[u] \) be such that \( \mathcal{B}^{-1} g \) is of homogeneous weight. Then, \( \{ \mathcal{O}_p(g) \}_q \) is quasimodular of homogeneous weight.

Holomorphic anomaly equation. Just as in [17] we answer the question when \( \{ \mathcal{O}_p(g) \}_q \) is actually modular (rather than quasimodular). Namely, if we, instead, consider \( c \) to be a formal variable, the holomorphic anomaly equation of \( \{ \mathcal{O}_p(g) \}_q \) (determining the failure of modularity) can be expressed as
\[
-2 \mathfrak{d} \{ \mathcal{O}_p(g) \}_q = \frac{\partial^2}{\partial q^2} \{ \mathcal{O}_p(g) \}_q,
\]
(cf. Proposition 2.3) where \( \mathfrak{d} \) is the unique derivation on quasimodular forms which vanishes on modular forms and for which \( \mathfrak{d}(G_2) = -\frac{1}{2} \), where \( G_2 = -\frac{1}{24} + \sum m, r \geq 1 m q^{mr} \) is the Eisenstein series of weight 2. Together with the differential operator \( q \frac{\partial}{\partial q} \) and the weight operator, this derivation \( \mathfrak{d} \) gives an action of \( \mathfrak{sl}_2 \) on quasimodular forms. Note that (1.27) can equivalently be written as
\[
-2 \mathfrak{d} \{ \mathcal{O}_p(g) \}_q = \{ \mathcal{O}_p(\partial^2 g/\partial u_0^2) \}_q.
\]
(1.27)
Hence, \( \{ \mathcal{O}_p(g) \}_q \) is modular precisely if \( \{ \mathcal{O}_p(\partial^2 g/\partial u_0^2) \}_q = 0 \).

Remark 1.5. By (1.6) the holomorphic anomaly equation (1.27) can be explicitly checked in the limit \( \epsilon \to 0 \), because we know by [32, Theorem 3] that \( \mathfrak{d}(Q_j)_q = -\frac{1}{2} (Q_{j-2})_q \) for all \( j \geq 2 \). In the limit \( \epsilon \to \infty \), the holomorphic anomaly equation (1.27) is consistent with (1.20), because, as mentioned earlier, in [32] it is shown that the \( q \)-bracket of \( S_{2k+2} \) is an Eisenstein series of weight \( 2k + 2 \), and every Eisenstein series of weight \( 2k + 2 \) with \( k > 0 \) is actually modular.

Functions on partitions having the same \( q \)-bracket. The \( q \)-bracket (1.3) is not only a specialization of the \( q \)-series (1.2); it is also related to this \( q \)-series by the following construction. For any basis \( b_\lambda(p) \) of \( \Lambda \), indexed by partitions \( \lambda \in \mathcal{P} \) and satisfying \( b_\lambda \in \Lambda_{|\lambda|} \), and for any \( G \in \text{End}(\Lambda) \) satisfying (1.1) we have the equality \( \{ G \}_q = \{ f \}_q \) where \( f(\lambda) \) is defined to be the \( \lambda \)-th diagonal entry of the matrix representation of \( G \), i.e., \( f(\lambda) := (b_\lambda, G b_\lambda)/(b_\lambda, b_\lambda) \), where the inner product is defined by (1.12). Hence, by studying \( \{ G \}_q \) for \( G \in \text{End}(\Lambda) \), we study the \( q \)-brackets \( \{ f \}_q \) for many functions \( f \) at the same time.

Incidentally, this reflects the fact that different functions \( f: \mathcal{P} \to \mathbb{Q} \) can have the same \( q \)-bracket. For example, as observed in [7, Section 13] the moment functions (1.19) and the shifted symmetric functions\(^*\) (in terms of the generators \( Q_k \), defined by (1.4))
\[
T_k(\lambda) := \frac{(k - 2)!}{2} \sum_{i=0}^{k} (-1)^i Q_i(\lambda) Q_{k-i}(\lambda), \quad k \geq 2,
\]
(1.28)
\(^*\)These shifted symmetric functions have a nice interpretation as the moments of the hook-lengths of partitions, see [7, Section 19].
are two instances of functions on partitions for which, in case $k$ is even, the $q$-bracket equals the Eisenstein series (1.21). This particular example can be explained as $T_k$ is the so-called Möller transform of [32] of $S_k$. Now, the Möller transform corresponds to the change of coordinates between the monomial basis $p_\lambda$ and the Schur basis $s_\lambda$ of $\Lambda$. Indeed, we have\footnote{The second equality follows from [7, Section 13] after expanding $s_\lambda$ in the monomial basis of $\Lambda$.}

$$S_k(\lambda) = \frac{(p_\lambda, L_k p_\lambda)}{(p_\lambda, p_\lambda)}, \quad T_k(\lambda) = \frac{(s_\lambda, L_k s_\lambda)}{(s_\lambda, s_\lambda)}, \quad k \geq 2 \text{ even}, \quad (1.29)$$

where $L_k$ is the operator defined in (1.17).

Outline of the rest of the paper.

In Section 2 we prove Theorem 1.3 and the holomorphic anomaly equation (1.27). In Section 3.1 we review the construction of Double Ramification quantum integrable hierarchies. In Section 3.2 we prove Theorem 3.8 which is a generalization of Theorem 1.2. In Appendix A we give explicit formulas for the limits $\epsilon \to 0, \infty$ of the quantum KdV Hamiltonian densities. In Appendix B we illustrate our two main theorems by giving tables of quasimodular forms obtained as \{${\cal G}^{KdV}(\epsilon)$\}$_q$ and as \{Op$_c(g)$\}$_q$ for some $g \in \mathbb{Q} [u]$.

2 Partitions and quasimodular forms

2.1 Proof of Theorem 1.3

First of all, we refine the $q$-bracket (1.3) as follows. Define the $\underline{\cdot}$-bracket\footnote{In [18] this was called the $\underline{\cdot}$-bracket.} of a function $f : \mathcal{P} \to \mathbb{Q}$ by

$$\langle f \rangle_{\underline{\cdot}} = \frac{\sum_{\lambda \in \mathcal{P}} f(\lambda) x_\lambda}{\sum_{\lambda \in \mathcal{P}} x_\lambda} \in \mathbb{Q}[x_1, x_2, x_3, \ldots] \quad (x_\lambda = x_{\lambda_1} x_{\lambda_2} \cdots). \quad (2.1)$$

Then, one has $\langle f \rangle_q = \langle f \rangle_{(q,q^2,q^3,\ldots)}$. Observe that the $\underline{\cdot}$-bracket defines an isomorphism of vector spaces (but not of algebras!)

$$\mathbb{Q}^{\mathcal{P}} \xrightarrow{\sim} \mathbb{Q}[[x_1, x_2, x_3, \ldots]], \quad f \mapsto \langle f \rangle_{\underline{\cdot}}. \quad (2.2)$$

Given $f, g \in \mathbb{Q}^{\mathcal{P}}$ we define their induced product $f \odot g$ by

$$\langle f \odot g \rangle_{\underline{\cdot}} = \langle f \rangle_{\underline{\cdot}} \langle g \rangle_{\underline{\cdot}}, \quad (2.3)$$

where the product of $\langle f \rangle_{\underline{\cdot}}$ and $\langle g \rangle_{\underline{\cdot}}$ is the usual product of power series. In particular, $\langle f \odot g \rangle_q = \langle f \rangle_q \langle g \rangle_q$.

Now, given a partition $\lambda$, write $r_m(\lambda) = \# \{ i \mid \lambda_i = m \}$, and similarly for $k \in (\mathbb{Z}_{>0})^n$ we write $r_m(k) = \# \{ i \mid k_i = m \}$. Note (for example, using [18, Proposition 3.1.4]) that

$$\langle r_m \rangle_{\underline{\cdot}} = \sum_{r=1}^{\infty} x_r^m = \frac{x_m}{1 - x_m}. \quad (2.4)$$

In the proof of Theorem 1.3, we will need the following result.

Lemma 2.1. Given $n \geq 0$ and $k \in (\mathbb{Z}_{>0})^n$, we have

$$\langle \prod_{m \geq 1} \left( r_m(\lambda) / r_m(k) \right) \rangle_q = \prod_{k \in \mathbb{K}} \frac{q^k}{1 - q^k}. \quad (2.5)$$
Proof. By [18, Proposition 7.2.3(ii)] we have
\[ \prod_{m \geq 1} \frac{(r_m(\lambda))}{r_m(k)} = \bigcirc_{k \in k} r_k(\lambda). \] (2.6)

The result follows from (2.3) and (2.4).

Proof of Theorem 1.3. Recall that the \( p_\lambda = p_{\lambda_1} \cdots p_{\lambda_r} \) for \( \lambda \in \mathcal{P} \) form a basis for \( \Lambda \). The main observation is that with respect to this basis, for all \( k \in \mathbb{N}^r \), \( l \in \mathbb{N}^s \) and \( \lambda \in \mathcal{P} \) we have
\[ [p_\lambda] \frac{p_{k_1} \cdots p_{k_r}}{p_{l_1} \cdots p_{l_s}} = \begin{cases} \prod_{m}^{(r_m(\lambda))} r_m(l)! & \text{if } k \text{ is a permutation of } l, \\ 0 & \text{else,} \end{cases} \] (2.7)
where \([p_\lambda]\) indicates we extract the coefficient of \( p_\lambda \). Here, \( r_m(\lambda) \) and \( r_m(l) \) are as defined above.

Given a monomial \( u_a = u_{a_1} \cdots u_{a_n} \in \mathbb{Q}[u] \), consider
\[ \text{Op}_c(u_a) = \sum_{(ik)^n} c^{\#(j|k_j>0)} \prod_{j|k_j>0} p_{k_j} \prod_{j|k_j<0} \Big( -k_j \frac{\partial}{\partial p_{-k_j}} \Big), \] (2.8)
where \((ik)^n = \prod_{j=1}^{n} (ik_j)^{a_j} \). As the diagonal contribution of this operator is zero unless we have the equality of multisets \( \{k_j \mid k_j > 0\} = \{ -k_j \mid k_j < 0\} \), we can partition the set of indices in pairs.

That is, write \( \Pi_2(n,m) \) for the partitions of all \( B \subset \{1,2,\ldots,n\} \) in \( m \) sets of two elements, i.e., \( \pi \in \Pi_2(n,m) \) can be written as \( \pi = \{A_1,\ldots,A_m\} \) with \( |A_i| = 2 \) for all \( i \) and \( \bigcup A_i = B \) for some \( B \subset \{1,\ldots,n\} \) with \( |B| = 2m \). Often, we write \( B(\pi) \) for \( B \) to stress the dependence of \( B \) on \( \pi \). Observe that \( |\Pi_2(n,m)| = \binom{n}{2m} \cdot (2m - 1)! \). For a set \( S \), write \( a_S = \sum_{i \in S} a_i \) and \( s(a,S) = i^a \cdot \sum_{i \in S} (-1)^{a_i} \).

We write \( a \) for \( a_{\{1,\ldots,n\}} = \sum_{i=1}^{n} a_i \). Also, for \( k \in \mathbb{Z}^m \) write \( s_k = \prod_{m} r_m(k)! \prod_{m} r_m(k) \). Then,
\[ [p_\lambda] \text{Op}_c(u_a) p_\lambda = [p_\lambda] \sum_{m=0}^{[n/2]} c^{n-2m} \sum_{\pi \in \Pi_2(n,m)} \sum_{a_{\pi(s)}=a} 1 \prod_{A \in \pi} s(a,A) k_A^{a_A+1} p_{k_A} \prod_{A \in \pi} \frac{\partial}{\partial p_{k_A}} p_\lambda \]
\[ = \sum_{m=0}^{[n/2]} c^{n-2m} \sum_{\pi \in \Pi_2(n,m)} \sum_{a_{\pi(s)}=a} 1 \prod_{A \in \pi} s(a,A) k_A^{a_A+1} \prod_{m} \frac{r_m(\lambda)}{r_m(k)}. \] (2.9)

From Lemma 2.1, it follows that
\[ \{ \text{Op}_c(u_a) \} = \sum_{m=0}^{[n/2]} c^{n-2m} \sum_{\pi \in \Pi_2(n,m)} \sum_{a_{\pi(s)}=a} \prod_{A \in \pi} s(a,A) k_A^{a_A+1} \frac{q^{k_A}}{1-q^{k_A}} \]
\[ = \sum_{m=0}^{[n/2]} c^{n-2m} \sum_{\pi \in \Pi_2(n,m)} \prod_{A \in \pi} s(a,A) \left( \frac{B_k}{2k} + \mathbb{G}_{a_A+2} \right), \] (2.10)
where \( \mathbb{G}_k \) for \( k \geq 2 \) is the holomorphic Eisenstein series \( (1.21) \) of weight \( k \).

Observe that if \( a = (i,j) \) for some \( i,j \in \mathbb{Z} \), then
\[ s(a,\{1,2\}) = (-1)^{i+j}((-1)^i + (-1)^j) = \begin{cases} 2(-1)^{i+j} & i \equiv j \mod 2 \\ 0 & \text{else.} \end{cases} \] (2.11)
Hence, \( s(a,A) \) vanishes if \( a_A + 2 \) is odd. As the Eisenstein series \( \mathbb{G}_k \) are quasimodular for even \( k \), we have shown that \( \{ \text{Op}_c(g) \} \in \mathcal{M}[c] \) for all \( g \in \mathbb{Q}[u] \).
Next, assume \( u_a \in \mathbb{Q}[u]^{\text{odd}} \), i.e., \(|a|\) is odd. Then, in (2.10) we see that every summand of \( \{ \text{Op}_c(g) \}_q \) contains a factor \( s(a, A) \) for which \( a_A \) is odd. Hence, such a factor \( s(a, A) \) vanishes. That is, \( \{ \text{Op}_c(u_a) \}_q = 0 \). Therefore, by linearity, \( \{ \text{Op}_c(g) \}_q = 0 \) for all \( g \in \mathbb{Q}[u]^{\text{odd}} \).

By definition of \( B \) we deduce from (2.10) that
\[
\{ \text{Op}_c(B u_a) \}_q = \sum_{m=0}^{n/2} c^{n-2m} \sum_{\pi \in \Pi_2(n, m)} \prod_{A \in \pi} s(a, A) G_{a_A+2},
\]
(2.12)
which is of homogeneous weight \(|a| + n\) (where \( n \) is the length of the vector \( a \)). Therefore, it follows that \( g \mapsto \{ \text{Op}_c(B g) \}_q \) is a morphism of graded vector spaces. \( \square \)

Observe that as every quasimodular form has a (highly non-unique) representation as a polynomial in Eisenstein series, it follows that the mapping \( g \mapsto \{ \text{Op}_c(B g) \}_q \) from \( \mathbb{Q}[u] \) to \( \mathcal{M} \) is surjective.

2.2 Holomorphic anomaly equation

A quasimodular form is a holomorphic function \( f = f(\tau) \) of \( \tau \) in the complex upper half plane, admitting a Fourier series (\( q \)-series, \( q = e^{2\pi i \tau} \)) at infinity and such that for all \( (a \ b \ c \ d) \in \text{SL}_2(\mathbb{Z}) \)
\[
(c \tau + d)^{-k} f \left( \frac{a \tau + b}{c \tau + d} \right) = \sum_{j=0}^{p} \left( \frac{b^j f(\tau)}{j!} \right) \left( \frac{1}{2\pi i (c \tau + d)} \right)^j, \quad \text{Im} \tau > 0,
\]
(2.13)
where \( \partial \) is the derivation defined in the introduction. The transformation (2.13) of the quasimodular forms in Theorem 1.3 is determined by the holomorphic anomaly equation expressed in Proposition 2.3 below.

We first have a simple lemma.

Lemma 2.2. For any \( g \in \mathbb{Q}[u] \) we have \( \frac{\partial}{\partial c} \text{Op}_c(g) = \text{Op}_c \left( \frac{\partial g}{\partial u_0} \right) \).

Proof. For any \( g(u) \) the operator \( \text{Op}_c(g(u_0 - c, u_1, \ldots )) \) is independent of \( c \) by construction. Hence
\[
\text{Op}_c(g) = \sum_{s \geq 0} \frac{c^s}{s!} \text{Op}_c((\partial u_0)^s g(u_0 - c, u_1, \ldots ))
\]
(2.14)
and so
\[
\frac{\partial}{\partial c} \text{Op}_c(g) = \sum_{s \geq 1} \frac{c^{s-1}}{(s-1)!} \text{Op}_c((\partial u_0)^s g(u_0 - c, u_1, \ldots ))
\]
\[
= \sum_{s \geq 0} \frac{c^s}{s!} \text{Op}_c((\partial u_0)^s (\partial u_0 g)(u_0 - c, u_1, \ldots )) = \text{Op}_c \left( \frac{\partial g}{\partial u_0} \right),
\]
(2.15)
as claimed. \( \square \)

Proposition 2.3. For all \( g \in \mathbb{Q}[u] \), we have
\[
-2 \partial \{ \text{Op}_c(g) \}_q = \frac{\partial^2}{\partial c^2} \{ \text{Op}_c(g) \}_q = \left\{ \text{Op}_c \left( \frac{\partial^2 g}{\partial u_0^2} \right) \right\}_q.
\]
(2.16)

Proof. Recall that by (2.12) we have
\[
\{ \text{Op}_c(B u_a) \}_q = \sum_{m=0}^{n/2} c^{n-2m} \sum_{\pi \in \Pi_2(n, m)} \prod_{A \in \pi} s(a, A) G_{a_A+2}.
\]
(2.17)
For the first equation, it suffices to show that $-2 \partial$ and $\frac{\partial^2}{\partial c^2}$ agree on $\{\text{Op}_c(Bu_a)\}_q$. We compute

$$-2 \partial \{\text{Op}_c(Bu_a)\}_q = 2 \sum_{m=0}^{[n/2]} c^{n-2m} \sum_{\pi \in \Pi_2(n,m)} \sum_{A' \in \pi} A \in \pi' A_{B(\pi)} = |a| A_{A(\pi)} = 0 \neq A' \pi' \sum_{a_{B(\pi)}} s(a, A) G_{a_{A(\pi)} + 2},$$

where we used that $\partial$ is a derivation which annihilates $G_{a_{A(\pi)} + 2}$ except if $a_A = 0$. Now, let $\pi' \in \Pi_2(n, m - 1)$ be the partition after removing $A'$ from $\pi$. Note that there are $(n-2m+2)/2$ partitions $\pi'$ yielding $\pi'$. Hence, the first equality follows from the computation

$$-2 \partial \{\text{Op}_c(Bu_a)\}_q = 2 \sum_{m=0}^{[n/2]} c^{n-2m} \left(\frac{n - 2m + 2}{2} \right) \sum_{\pi' \in \Pi_2(n, m - 1)} s(a, A) G_{a_{A(\pi)} + 2} A' \in \pi' A_{B(\pi)} = |a| \pi' \sum_{a_{B(\pi)}}$$

$$= \frac{\partial^2}{\partial c^2} \sum_{m=1}^{[n/2]} c^{n-2m+2} \sum_{\pi' \in \Pi_2(n, m - 1)} s(a, A) G_{a_{A(\pi)} + 2} A' \in \pi' A_{B(\pi)} = |a| \sum_{a_{B(\pi)}}.$$

(2.19)

The second equality follows from Lemma 2.2.

3 Applications to quantum integrable hierarchies

3.1 Double Ramification quantum integrable hierarchies

A Cohomological Field Theory (CohFT) [21] consists of

- a finite-dimensional $\mathbb{Q}$-vector space $V$, equipped with a non-degenerate symmetric two-form $\eta \in \text{Sym}^2(V^*)$ and with a distinguished element $1 \in V$, and
- linear maps $c_{g,n} : V^\otimes n \to H^{2g}(\mathcal{M}_{g,n}, \mathbb{Q})$, indexed by $g, n \geq 0$ such that $2g - 2 + n \geq 0$.

Here, $\mathcal{M}_{g,n}$ is the Deligne–Mumford moduli space of stable curves of genus $g$ with $n$ marked points and $H^{2g}(\mathcal{M}_{g,n}, \mathbb{Q})$ is the even part of its rational cohomology ring. The maps $c_{g,n}$ have to satisfy a number of axioms prescribing their behaviour under natural maps between the moduli spaces, i.e., under permutation of marked points, forgetting of marked points, and glueing of curves. For more details see [3, Section 3] and references therein.

In [6] a family of Hamiltonian densities $g_k(u; \epsilon)$ is defined starting from an arbitrary CohFT. We shall consider here only the case of one-dimensional CohFTs, namely $V = \mathbb{Q}1$, under the additional assumption $\eta(1 \otimes 1) = 1$ and, by a slight abuse of notation, we shall denote $c_{g,n}$ the value at $1^\otimes n$. A result of Teleman [30] implies that all such CohFTs are given by

$$c_{g,n} = \exp \left( \sum_{j \geq 1} s_{2j-1} \chi_{2j-1}(\mathbb{E}) \right),$$

(3.1)

in terms of parameters $s_k$, for $k \geq 1$ and odd, where $\chi_k(\mathbb{E}) \in H^k(\mathcal{M}_{g,n}, \mathbb{Q})$ are the Chern characters of the Hodge bundle $\mathbb{E}$ over $\mathcal{M}_{g,n}$. The densities are defined by\[11\]

$$g_{-2}(u; \epsilon) := 1, \quad g_{-1}(u; \epsilon) := u_0,$$

(3.2)

and $g_k(u; \epsilon)$ for $g \geq 0$ is defined in terms of the Fourier series (already used in the introduction)

$$v_j(x) = \sum_{\ell \in \mathbb{Z}} (\ell i)^j \omega \epsilon^{\ell x},$$

for $j \geq 0$, by requiring that

$$g_k(v(x); \epsilon) := \sum_{n \geq 0} \sum_{(a_1, \ldots, a_n) \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \psi_{\mathbb{E}}^k A_g(\epsilon) c_{g,n+1} \omega_{a_1} \ldots \omega_{a_n} \epsilon^{(|a|x)}.$$
• $\text{DR}_g(a_0, a_1, \ldots, a_n) \in H_{2g-2+n}(\overline{M}_{g,n+1}, \mathbb{Q})$ is the Double Ramification cycle (roughly speaking, defined as the locus in $\overline{M}_{g,n+1}$ of stable curves with marked points $(C; p_0, \ldots, p_n)$ such that $\mathcal{O}_C \simeq \mathcal{O}_C(\sum_{i=0}^n a_ip_i)$).

• $\psi_1 \in H^2(\overline{M}_{g,n+1}, \mathbb{Q})$ is the Chern class of the cotangent line bundle at the first marked point, and

• $\Lambda_g' = 1 - \varepsilon \lambda_1 + \cdots + (-\varepsilon)^g \lambda_g$, where $\lambda_i \in H^2(\overline{M}_{g,n+1}, \mathbb{Q})$ are the Chern classes of the Hodge bundle.

It is worth pointing out that all densities $g_k(u; \varepsilon)$ are sums of even monomials with respect to the weight operator $\sum_{j \geq 0} u_j \frac{\partial}{\partial u_j}$, i.e., $g_k(u; \varepsilon) \in \mathbb{Q}[u]_{\text{even}}[\varepsilon]$ (see [6, Appendix B]).

**Remark 3.1.** To simplify the exposition we have omitted the quantization parameter $\hbar$ of [6], as the normalization of op. cit. can be recovered from (3.3) by the transformations

$$
\varepsilon \mapsto \varepsilon(i\hbar)^{-\frac{1}{2}}, \quad s_k \mapsto s_k(i\hbar)^{\frac{1}{2}}, \quad u_k \mapsto u_k(i\hbar)^{-\frac{1}{2}}, \quad g_k \mapsto (i\hbar)^{\frac{k+2}{2}} g_k.
$$

(3.4)

This follows directly from the dimensional constraints of the integrals over the Double Ramification cycle in (3.3). Moreover, the parameter $\varepsilon$ in this paper corresponds to $\varepsilon^2$ in [6]. To compare with the normalization of [10], where $\varepsilon = 0$, we need to replace $i\hbar$ with $\hbar$.

It is also follows from [6, Theorem 3.5 and Lemma 3.7], combined with Remark 3.1 (see also [28, Section 4]), that the densities $g_k(u; \varepsilon)$ in (3.3) can be determined from $g_{-1}(u; \varepsilon) = u_0$ by the recursion

$$
\frac{\partial g_{k+1}(u; \varepsilon)}{\partial u_0} = g_k(u; \varepsilon),
$$

(3.5)

$$(k + 2 + \mathcal{D}) \partial_x g_{k+1}(u; \varepsilon) = \left[ g_k(u; \varepsilon), g_1(u; \varepsilon) \right], \quad k \geq -1,
$$

(3.6)

provided one has computed $g_1(u; \varepsilon)$ in advance, at least up to constant terms in $u$, cf. (3.8). Here

$$
\partial_x := \sum_{i \geq 0} u_{i+1} \frac{\partial}{\partial u_i}, \quad \mathcal{D} := \frac{\partial}{\partial \varepsilon} + \sum_{i \geq 1} (2i - 1)s_{2i-1} \frac{\partial}{\partial s_{2i-1}}.
$$

(3.7)

and the expression $[f, g]$ is defined for $f, g \in \mathbb{Q}[u][\varepsilon]$ by (cf. [6, Equation 2.2])

$$
[f, g] = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \sum_{r_1, \ldots, r_n \geq 0 \atop s_1, \ldots, s_n \geq 0} \frac{\partial^n f}{\partial u_{r_1} \cdots \partial u_{r_n}} (-1)^{|r|} P_{r_1 + s_1 + 1, \ldots, r_n + s_n + 1} (\partial_x) \frac{\partial^n g}{\partial u_{r_1} \cdots \partial u_{r_n}},
$$

(3.8)

and does not depend on constant terms of $f$ and $g$ in $u$. In (3.8), $|r| = \sum_{i=1}^n r_i$ and $P_{\ell_1, \ldots, \ell_n}(\xi)$ are polynomials in $\xi$ defined by the sequence of their coefficients, namely,

$$
[\xi^j] P_{\ell_1, \ldots, \ell_n}(\xi) = \begin{cases} (-1)^{n-j+|\xi|} [\xi^j] \tilde{P}_{\ell_1, \ldots, \ell_n}(\xi) & n - 1 + j + |\xi| \text{ is even,} \\ 0 & \text{otherwise} \end{cases}
$$

(3.9)

and $\tilde{P}_{\ell_1, \ldots, \ell_n}(\xi)$ are polynomials in $\xi$ determined by their values at positive integers $\xi$,

$$
\tilde{P}_{\ell_1, \ldots, \ell_n}(\xi) = \sum_{a_1, \ldots, a_n \geq 0 \atop a_1 + \cdots + a_n = \xi} a_1^{\ell_1} \cdots a_n^{\ell_n}.
$$

(3.10)

Later we shall need the following particular cases.

**Lemma 3.2.** We have $P_{\ell}(\xi) = \xi^\ell$ and

$$
P_{\ell, m}(\xi) = \frac{\ell!m!}{(\ell + m + 1)!} \xi^{|\ell+m+1|} + \sum_{i \geq 0} \frac{B_{2i+2}}{2i+2} (-1)^{\ell+i} \left( \frac{\ell}{2i+1-m} \right) + (-1)^{m+i} \left( \frac{m}{2i+1-\ell} \right) \xi^{m-2i-1}.
$$

(3.11)
Proof. (See also [18, Lemma 6.1.2,].) Only (3.11) needs a proof. We compute the generating series (for $\xi$ integer)
\[
P(u,v;\xi) := \sum_{\ell, m \geq 0} \tilde{P}_{\ell,m}(\xi) \frac{u^{ \ell} v^{ m}}{\ell! \, m!} = \sum_{a=0}^{\xi} e^{a \xi} e^{(\xi + \epsilon) v} = e^{\xi v} \sum_{a=0}^{\xi} e^{a(u-v)} = \frac{e^{\xi(\xi+1)} - e^{\xi(1+1)}}{e^{\xi} - e^{v}}. \tag{3.12}
\]

The proof is complete by Taylor expanding $P(u,v;\xi) = \frac{e^{\xi-1}}{e^{v-1}} + \frac{e^{\xi-1}}{e^{v-1}}$ using $\frac{z}{e^{z-1}} = \sum_{j \geq 0} B_{j} \frac{z^{j}}{j!}$ and by (3.9).

\[\text{Remark 3.3. It is explained in [6, Section 3.5] that the recursion equations (3.5) and (3.6) uniquely determine the } g_k \text{'s, for all } k \geq -1, \text{ using as initial data } g_{-1} = u_0 \text{ and } g_1 \text{ (the latter up to additive constants which are fixed by the recursion). Indeed, one first uses (3.6) to determine the } g_k \text{'s for } k \geq -1 \text{ up to a constant depending on } k, \epsilon, s \text{ only. Note that this yet undetermined constant does not affect the right-hand side of (3.6) so that the recursion works. These constants are finally determined by (3.5).}
\]

### 3.2 Quantum Intermediate Long Wave hierarchy

The quantum Intermediate Long Wave hierarchy corresponds to the construction of Buryak and Rossi for the Hodge CohFT (see also [2])
\[
e_{g,n} = 1 + \mu \lambda_1 + \cdots + \mu^g \lambda_g, \tag{3.13}
\]
where $\mu$ is a parameter and $\lambda_k \in H^{2k}(\mathcal{M}_{g,n}, \mathbb{Q})$ are the Chern classes of the Hodge bundle. In terms of the parameters $s$ in (3.1) we have $s_{2i-1} = (2i - 2)! \mu^{2i-1}$, whence (3.7) reduces to
\[
\mathcal{D} = \epsilon \frac{\partial}{\partial \epsilon} + \mu \frac{\partial}{\partial \mu}. \tag{3.14}
\]

Let us denote $g_{k}^{\text{ILW}}(u; \epsilon, \mu)$ and $G_{k}^{\text{ILW}}(\epsilon, \mu) := \text{Op}_c(g_{k}^{\text{ILW}}(u; \epsilon, \mu))$ the densities and operators for this hierarchy. We know from [6, Lemma 4.2] that
\[
G_{1}^{\text{ILW}}(\epsilon, \mu) = \text{Op}_c\left(\frac{u_{0}^{3}}{6} - \frac{u_{0}}{24} + (\epsilon - \mu) \sum_{g \geq 1} (\epsilon \mu)^{g-1} \frac{1!}{2!} \frac{1!}{(2g)!} \left(\frac{B_{2g}}{2^{2g}} - \frac{|B_{2g+2}|}{2g+2}\right)\right). \tag{3.15}
\]

We start by making the recursion (3.6) more explicit in this case.

\[\text{Lemma 3.4. The operator}
\]
\[
\mathcal{R}^{\text{ILW}} : g(u; \epsilon, \mu) \mapsto \left[g(u; \epsilon, \mu), g_{1}^{\text{ILW}}(u; \epsilon, \mu)\right] \tag{3.16}
\]
\[\text{can be spelled out as}
\]
\[
\mathcal{R}^{\text{ILW}} = \mathcal{R}_{1}^{\text{ILW}} + \mathcal{R}_{2}^{\text{ILW}}, \tag{3.17}
\]
\[\text{where}
\]
\[
\mathcal{R}_{1}^{\text{ILW}} = \sum_{i \geq 0} \partial_{x}^{i+1} \left(\frac{u_{0}^{3}}{2} + (\epsilon - \mu) \sum_{g \geq 1} (\epsilon \mu)^{g-1} \frac{1!}{2!} \frac{1!}{(2g)!} \left(\frac{B_{2g}}{2^{2g}} - \frac{|B_{2g+2}|}{2g+2}\right) \frac{\partial}{\partial u_{i}}\right) - \frac{1}{2} \sum_{i,j \geq 0} (i + 1)! (j + 1)! \frac{u_{i+j+1}}{(i+j+3)!} \frac{\partial^{2}}{\partial u_{i} \partial u_{j}}. \tag{3.18}
\]
\[
\mathcal{R}_{2}^{\text{ILW}} = \sum_{i,j \geq 0} \frac{B_{2l+2}}{2l+2} \left((-1)^{i+l} \left(\frac{i+1}{2l-j} + (-1)^{j+l} \left(\frac{j+1}{2l-i}\right)\right) u_{i+j+1-2l} \frac{\partial^{2}}{\partial u_{i} \partial u_{j}}. \tag{3.19}
\]

\[\text{Proof. It follows directly from (3.8) and Lemma 3.2.}\]
Introduce the reduced ILW densities
\[
\tilde{g}^{\text{ILW}}_k := B^{-1} g^{\text{ILW}}_k,
\]
where $B$ is the operator given in (1.25). Remarkably, they satisfy a similar but slightly simpler recursion.

**Lemma 3.5.** The reduced ILW densities $\tilde{g}^{\text{ILW}}_k(u; \epsilon, \mu)$ are uniquely determined from $\tilde{g}^{\text{ILW}}_{-1}(u; \epsilon, \mu) = u_0$ and $\tilde{g}^{\text{ILW}}_{-2}(u; \epsilon, \mu) = u_0$ by the recursion
\[
\frac{\partial \tilde{g}^{\text{ILW}}_{k+1}(u; \epsilon, \mu)}{\partial u_0} = \tilde{g}^{\text{ILW}}_k(u; \epsilon, \mu),
\]
\[
(k + 2 + \mathcal{D}) \partial_x \tilde{g}^{\text{ILW}}_{k+1}(u; \epsilon, \mu) = \mathcal{R} g^{\text{ILW}}_k(u; \epsilon, \mu), \quad k \geq -1,
\]
where $\mathcal{R}^{\text{ILW}}$ is given in (3.18).

**Proof.** For the sake of clarity, let us drop the superscript ILW and the dependence on $u, \epsilon, \mu$ in this proof. The equation (3.21) follows from the chain of equalities
\[
\frac{\partial}{\partial u_0} \tilde{g}_{k+1} = B^{-1} \frac{\partial}{\partial u_0} g_{k+1} = B^{-1} g_k = \tilde{g}_k,
\]
where we use $[B, \partial_{u_0}] = 0$. Next, for any power series $\Phi(\xi_0, \xi_1, \ldots)$ we have $[\Phi(\partial_{u_0}, \partial_{u_1}, \ldots), u] = (\partial_{\xi_j} \Phi)(\partial_{u_0}, \partial_{u_1}, \ldots)$. In particular when $\Phi(\xi_0, \xi_1, \ldots) = \exp(\pm \frac{1}{2} \sum_{i,j \geq 0} \nu_{i,j} \xi_i \xi_j)$, with
\[
\nu_{i,j} := (-1)^{i+j} \frac{B_{i+j+2}}{i+j+2},
\]
we get
\[
[B^\pm, u_j] = \mp \sum_{i \geq 0} \nu_{i,j} \frac{\partial}{\partial u_i} B^\pm.
\]
We claim that $[B, (k + 2 + \mathcal{D}) \partial_x] = 0$. Indeed, by (3.25) we have
\[
[B, \partial_x] = -\sum_{i,j \geq 0} \nu_{i,j+1} \frac{\partial^2}{\partial u_i \partial u_j} B = 0,
\]
because $\nu_{i,j+1} = -\nu_{j,i+1}$, and $[B, \mathcal{D}] = 0$. Therefore,
\[
(k + 2 + \mathcal{D}) \partial_x \tilde{g}_{k+1} = B^{-1} (k + 2 + \mathcal{D}) \partial_x g_{k+1} = B^{-1} \mathcal{R} g_k = B^{-1} \mathcal{R} B \tilde{g}_k
\]
The proof is complete once we show the identity $B^{-1} \mathcal{R} B = \mathcal{R}_1$, or, equivalently,
\[
[B, \mathcal{R}_1] = \mathcal{R}_2 B.
\]
The operator $\mathcal{R}_1$ consists of three parts, namely, $\mathcal{R}_1 = \mathcal{R}_1^{(a)} + \mathcal{R}_1^{(b)} + \mathcal{R}_1^{(c)}$ with
\[
\mathcal{R}_1^{(a)} = \sum_{\ell \geq 0} (\ell - \mu) \sum_{g \geq 1} (\epsilon \mu)^{g-1} \frac{B_{2g}}{(2g)!} u_{\ell+2g+1} \frac{\partial}{\partial u_\ell},
\]
\[
\mathcal{R}_1^{(b)} = \frac{1}{2} \sum_{\ell \geq 0} \sum_{k=0}^{\ell+1} \frac{(\ell + 1)!}{k!} u_k u_{\ell+1-k} \frac{\partial}{\partial u_\ell},
\]
\[
\mathcal{R}_1^{(c)} = \frac{1}{2} \sum_{\ell,m \geq 0} \frac{(\ell + 1)! (m + 1)!}{(\ell + m + 3)!} u_{\ell+m+3} \frac{\partial^2}{\partial u_\ell \partial u_m}.
\]
We compute separately each contribution, using (3.25).
(a) We have
\[ [\mathcal{B}, \mathcal{R}_1^{(a)}] = (\epsilon - \mu) \sum_{\ell \geq 0} \sum_{g \geq 1} (\epsilon \mu)^{g-1} \frac{|B_{2g}|}{(2g)!} [\mathcal{B}, u_{\ell+2g+1}] \frac{\partial}{\partial u_{\ell}} \]
\[ = (\epsilon - \mu) \sum_{i, \ell \geq 0} \sum_{g \geq 1} (\epsilon \mu)^{g-1} \frac{|B_{2g}|}{(2g)!} \nu_{i, \ell+2g+1} \frac{\partial^2}{\partial u_i \partial u_{\ell}} \mathcal{B} = 0, \quad (3.32) \]
because \( \nu_{i, \ell+k} = -\nu_{i, j+k} \) for any odd \( k \).

(b) We first compute
\[ [\mathcal{B}, u_{j_1} u_{j_2}] = u_{j_1} [\mathcal{B}, u_{j_2}] + [\mathcal{B}, u_{j_1}] u_{j_2} \]
\[ = - \sum_{i \geq 0} \left( \nu_{i, j_2} u_{j_1} \frac{\partial}{\partial u_i} \mathcal{B} + \nu_{i, j_1} \frac{\partial}{\partial u_i} \mathcal{B} u_{j_2} \right) \]
\[ = - \sum_{i \geq 0} \left( \nu_{i, j_2} u_{j_1} \frac{\partial}{\partial u_i} \mathcal{B} + \nu_{i, j_1} \frac{\partial}{\partial u_i} \mathcal{B} u_{j_2} \right) - \sum_{i \geq 0} \nu_{i, j_1} \frac{\partial}{\partial u_i} [\mathcal{B}, u_{j_2}] \]
\[ = - \sum_{i \geq 0} \left( \nu_{i, j_2} u_{j_1} \frac{\partial}{\partial u_i} + \nu_{i, j_1} u_{j_2} \frac{\partial}{\partial u_i} \right) + \nu_{j_1, j_2} + \sum_{i, j_2 \geq 0} \nu_{i, j_1} u_{j_2} \frac{\partial^2}{\partial u_i \partial u_{j_2}} \mathcal{B}. \quad (3.33) \]
Therefore,
\[ [\mathcal{B}, \mathcal{R}_1^{(b)}] \mathcal{B}^{-1} = \frac{1}{2} \sum_{\ell \geq 0} \sum_{k=0}^{\ell+1} \left( \frac{\ell + 1}{k} \right) [\mathcal{B}, u_{k u_{\ell+1-k}}] \mathcal{B}^{-1} \frac{\partial}{\partial u_{\ell}} \]
\[ = - \frac{1}{2} \sum_{\ell \geq 0} \sum_{k=0}^{\ell+1} \left( \frac{\ell + 1}{k} \right) \nu_{i, \ell+1-k} u_{k} + \nu_{i, k u_{\ell+1-k}} \frac{\partial^2}{\partial u_i \partial u_{\ell}} + \]
\[ - \frac{1}{2} \sum_{\ell \geq 0} \sum_{k=0}^{\ell+1} \left( \frac{\ell + 1}{k} \right) \nu_{k, \ell+1-k} \frac{\partial}{\partial u_{\ell}} + \]
\[ + \frac{1}{2} \sum_{i, j_1, j_2 \geq 0} \sum_{k=0}^{\ell+1} \left( \frac{\ell + 1}{k} \right) \nu_{i, j_1} u_{j_2} \frac{\partial^3}{\partial u_i \partial u_{j_1} \partial u_{\ell}}. \quad (3.34) \]
Since
\[ \sum_{k=0}^{\ell+1} \left( \frac{\ell + 1}{k} \right) \nu_{k, \ell+1-k} = (-1) \frac{\ell+1}{2} \frac{B_{\ell+3}}{\ell+3} \sum_{k=0}^{\ell+1} \left( \frac{\ell + 1}{k} \right) (-1)^k = 0, \quad \ell \geq 0, \quad (3.35) \]
the second sum in (3.34) vanishes for all \( \ell \geq 0 \).

(c) Finally, we have
\[ [\mathcal{B}, \mathcal{R}_1^{(c)}] = \frac{1}{2} \sum_{i, \ell, m \geq 0} \frac{(\ell + 1)! (m + 1)!}{(\ell + m + 3)!} \nu_{i, \ell+\mu+3} \frac{\partial^3}{\partial u_i \partial u_{\ell} \partial u_m} \mathcal{B}. \quad (3.36) \]
Combining these three computations we obtain, denoting for convenience \( b_k := B_k/k \),
\[ [\mathcal{B}, \mathcal{R}_1] \mathcal{B}^{-1} = - \frac{1}{2} \sum_{i, \ell, m \geq 0} \sum_{k=0}^{\ell+1} \left( \frac{\ell + 1}{k} \right) \left( -1 \right) \frac{i}{2} b_{i+k+\ell+3} u_k + \left( -1 \right) \frac{i}{2} b_{i+k+2 u_{\ell+1-k}} \frac{\partial^2}{\partial u_i \partial u_{\ell+1-k}} + \]
\[ + \frac{1}{2} \sum_{i, j_1, j_2 \geq 0} \sum_{k=0}^{\ell+1} \left( \frac{\ell + 1}{k} \right) \left( -1 \right) \frac{i}{2} b_{i+k+2} u_{j_2} \frac{\partial^3}{\partial u_i \partial u_{j_2} \partial u_{\ell}} + \]
\[ + \frac{1}{2} \sum_{i, \ell, m \geq 0} \frac{(\ell + 1)! (m + 1)!}{(\ell + m + 3)!} \left( -1 \right) \frac{i}{2} b_{i+\ell+5} \frac{\partial^3}{\partial u_i \partial u_{\ell} \partial u_m}. \quad (3.37) \]
It is easy to check that the first line on the right-hand side of (3.37) equals $R_2$ (see (3.19)). To complete the proof we need to show that the last two lines in (3.37) cancel each other. To this end, we first recall the following theorem by Skoruppa.

**Theorem 3.6** ([29]). Suppose that the symmetric homogeneous bivariate polynomial $H(x, y) = \sum_{\nu=0}^{\nu=n} h_{\nu} x^\nu y^{n-\nu}$ of even positive degree $n$ satisfies $H(x, y) = H(y, y-x)$. Then

$$\sum_{0<\nu<n} h_{\nu} G_{\nu+1} G_{n+1-\nu} = h \, G_{n+2} - \frac{h_1}{n} \frac{d}{dq} G_n,$$  

(3.38)where $h := -\frac{1}{2} \int_0^1 H(1, y) dy$ and $G_n$ is the Eisenstein series (1.21).

Since $G_n$ has a Fourier series expansion in the upper half plane whose constant term is $-b_n/2$, we get

$$\sum_{0<\nu<n} h_{\nu} b_{\nu+1} b_{n+1-\nu} = -2 h b_{n+2},$$  

(3.39)with the notations of Theorem 3.6. We need the following specialization: given positive integers $a_1, a_2, a_3$ with $a_1 + a_2 + a_3$ even, the polynomial

$$H(x, y) := \sum_{\pi \in S_3} x^{a_{\pi}(1)} (-y)^{a_{\pi}(2)} (y-x)^{a_{\pi}(3)}$$

$$= \sum_{\pi \in S_3} \sum_{k=0}^{a_{\pi}(3)} \binom{a_{\pi}(3)}{k} (-1)^{a_{\pi}(2)+k} x^{a_{\pi}(1)+k} y^{a_{\pi}(2)+a_{\pi}(3)-k}$$

(3.40)satisfies the condition of Theorem 3.6, because

$$H(y, y-x) = \sum_{\pi \in S_3} y^{a_{\pi}(1)} (x-y)^{a_{\pi}(2)} (-x)^{a_{\pi}(3)}$$

$$= (-1)^{a_1+a_2+a_3} \sum_{\pi \in S_3} (-y)^{a_{\pi}(1)} (y-x)^{a_{\pi}(2)} (x)^{a_{\pi}(3)} = H(x, y).$$

(3.41)

Hence, by (3.39)

$$\sum_{\pi \in S_3} \sum_{k=0}^{a_{\pi}(3)} \binom{a_{\pi}(3)}{k} (-1)^{a_{\pi}(2)+k} b_{a_{\pi}(1)+k+1} b_{a_{\pi}(2)+a_{\pi}(3)-k+1}$$

$$= b_{a_1+a_2+a_3+2} \sum_{\pi \in S_3} (-1)^{a_{\pi}(1)} \frac{a_{\pi}(2)! a_{\pi}(3)!}{(a_{\pi}(2) + a_{\pi}(3) + 1)!}.$$  

(3.42)

Note that $a_{\pi}(2) + a_{\pi}(3) - k + 1$ is even in the left-hand side of the last identity and so, multiplying both sides by $(-1)^{\frac{a_1+a_2+a_3}{2}}$, we may write it as

$$\sum_{\pi \in S_3} \sum_{k=0}^{a_{\pi}(3)} \binom{a_{\pi}(3)}{k} (-1)^{\frac{a_{\pi}(1)+a_{\pi}(2)-a_{\pi}(3)}{2}} b_{a_{\pi}(1)+k+1} b_{a_{\pi}(2)+a_{\pi}(3)-k+1}$$

$$= -b_{a_1+a_2+a_3+2} \sum_{\pi \in S_3} (-1)^{\frac{a_{\pi}(1)+a_{\pi}(2)-a_{\pi}(3)}{2}} \frac{a_{\pi}(2)! a_{\pi}(3)!}{(a_{\pi}(2) + a_{\pi}(3) + 1)!}.$$  

(3.43)

It is clear by this identity that the two cubic operators in the $\partial/\partial u_i$’s, which appear in the second and third line of (3.37), cancel each other. □

**Corollary 3.7.** For all $k \geq -2$, the reduced density $g_{k}^{RW}(u; \epsilon)$ is homogeneous of weight $k + 2$ if we assign weight $i+1$ to $u_i$, $+1$ to $c$, and $-1$ to $\epsilon$ and $\mu$. 

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Proof. Let \( W := \sum_{i \geq 0} (i+1)u_i \frac{\partial}{\partial u_i} - \epsilon \frac{\partial}{\partial \epsilon} - \mu \frac{\partial}{\partial \mu} \) be the grading operator with respect to the weights of the statement. We need to prove that \( W g^{ILW}_k = (k+2)\tilde{g}^{ILW}_k \) for \( k \geq -1 \) (the case \( k = -2 \) being trivial). It is straightforward to verify the commutation relations

\[
[W, \partial_{u_0}] = -\partial_{u_0}, \quad [W, (k+2 + D)\partial_x] = (k+2 + D)\partial_x, \quad [W, R^{ILW}_k] = 2R^{ILW}_1, \tag{3.44}
\]

where \( D \) is given in (3.14). It follows from these relations and (3.21)-(3.22) that the polynomials \( f_k := W \tilde{g}^{ILW}_k \) satisfy the recursion

\[
\frac{\partial}{\partial u_0} f_{k+1} = f_k + \tilde{g}^{ILW}_k, \tag{3.45}
\]

\[
(k+2 + D)\partial_x f_{k+1} = R^{ILW}_k (f_k + \tilde{g}^{ILW}_k), \quad k \geq -1, \tag{3.46}
\]

with initial condition \( f_{-1} = u_0 \). This recursion uniquely determines (by an argument parallel to that in Remark 3.3) all \( f_k \)’s, for \( k \geq -1 \). On the other hand, this recursion is satisfied by \( f_k = (k+2)\tilde{g}^k \), and the proof is complete. \( \square \)

Let \( \tilde{M}[c, \epsilon, \mu] := \tilde{M} \otimes \mathbb{Q}[c, \epsilon, \mu] =: \bigoplus_{k} \tilde{M}[c, \epsilon, \mu]_{k+2} \), graded by the quasimodular weight and by assigning weight \(+1\) to \( c \) and \(-1\) to \( \epsilon \) and \( \mu \). The central result of this section now follows directly from Theorem 1.3 and Corollary 3.7.

**Theorem 3.8.** For all \( k \geq -2 \), we have

\[
\left\{ G_k^{ILW}(\epsilon, \mu) \right\}_q \in \tilde{M}[c, \epsilon, \mu]_{k+2}. \tag{3.47}
\]

The quantum KdV hierarchy mentioned in Section 1 corresponds to the special case \( \mu = 0 \) of the ILW hierarchy, namely

\[
g_k^{KdV}(u; \epsilon) = g_k^{ILW}(u; \epsilon, \mu)|_{\mu = 0}, \quad G_k^{KdV}(\epsilon) = G_k^{ILW}(\epsilon, \mu)|_{\mu = 0}. \tag{3.48}
\]

Therefore, Theorem 1.2 is a direct corollary of Theorem 3.8.

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## A Closed formulas for the quantum KdV hierarchy for \( \epsilon \rightarrow 0 \) and \( \epsilon \rightarrow \infty \)

### A.1 Dispersionless limit \( \epsilon \rightarrow 0 \)

An explicit generating function for the Hamiltonian densities in the case \( \epsilon = 0 \) is due to Eliashberg, see [11] and [6, Proposition 4.1]. Namely, it is known that

\[
\sum_{k \geq -2} y^{k+2} g_k^{KdV}(u; \epsilon = 0) = \exp(y S(iy \partial_x) u_0), \quad S(y) := \frac{\sinh(y/2)}{y/2} = \sum_{k \geq 0} \frac{y^{2k}}{4^k (2k+1)!}, \tag{A.1}
\]

where \( \partial_x \) is defined by (3.7). We observe here the simplification of this formula when we consider the reduced densities instead.
Proposition A.1. Denoting \( \tilde{g}_k^{KdV} := \mathcal{B}^{-1} g_k^{KdV} \), we have

\[
\sum_{k \geq -2} y^{k+2} \tilde{g}_k^{KdV}(u; \epsilon = 0) = \exp(y S(\imath y \partial_x) u_0). \tag{A.2}
\]

Remark A.2. By assigning degree \(-1\) to \( y \) we see directly that the right-hand side in (A.2) is of homogeneous weight zero (provided \( u_k \) has weight \( k + 1 \)), hence we recover that \( \tilde{g}_k^{KdV}(u; \epsilon = 0) \) has degree \( k + 2 \). Actually, since by the definition of the operator \( \mathcal{B} \) we have \( g_k^{KdV} = \tilde{g}_k^{KdV} + \) lower degree terms, it is not difficult to derive (A.2) from (A.1).

A.2 Limit \( \epsilon \to \infty \).

We also provide formulas for the (sub)leading terms in \( g_k^{KdV}(u; \epsilon) \) as \( \epsilon \to \infty \).

Proposition A.3. For all \( k \geq -1 \), the density \( g_k^{KdV}(u; \epsilon) \) is a polynomial in \( \epsilon \) of degree \( k + 1 \), whose leading and subleading terms are given by

\[
[e^{k+1}] g_k^{KdV}(u; \epsilon) = \frac{u_{2k+2}}{24^{k+1}(k + 1)!}, \quad k \geq -1, \tag{A.3}
\]
\[
[e^k] g_k^{KdV}(u; \epsilon) = -\frac{1}{(-4)^k (2k+1)!!} \frac{B_{2k+2}}{4k+4} + \frac{1}{24^k} \sum_{j=0}^{2k} d(j, k) u_j u_{2k-j}, \quad k \geq 0, \tag{A.4}
\]

where the coefficients \( d(j, k) \) (for \( j, k > 0 \)) are given by the generating series

\[
\sum_{j,k \geq 0} (2k+1)!! d(j, k) y^j x^k = \frac{1}{2 \sqrt{1 - 2x(1+y)^2(1 - 2x(1-y+y^2))}} \tag{A.5}
\]

and in particular satisfy \( d(j, k) = 0 \) for \( j > 2k \) and \( d(j, k) = d(2k-j, k) \) for \( j, k \geq 0 \) with \( j \leq 2k \).

Proof. It is straightforward from the definition of \( \mathcal{B} \) and the identity

\[
\sum_{j=0}^{2k} (-1)^j d(j, k) = \frac{1}{(2k+1)!!} \left[ x^k \right] \frac{1}{2(1-6x)} = \frac{6^k}{2(2k+1)!!} \tag{A.6}
\]

to check that the statement is equivalent to the fact that the reduced density \( \tilde{g}_k^{KdV} := \mathcal{B}^{-1} g_k^{KdV} \) is a polynomial in \( \epsilon \) of degree \( k + 1 \) for \( k \geq -1 \) whose leading and subleading terms are given by

\[
[e^{k+1}] \tilde{g}_k^{KdV}(u; \epsilon) = \frac{u_{2k+2}}{24^{k+1}(k + 1)!}, \quad k \geq -1, \tag{A.7}
\]
\[
[e^k] \tilde{g}_k^{KdV}(u; \epsilon) = \frac{1}{24^k} \sum_{j=0}^{2k} d(j, k) u_j u_{2k-j}, \quad k \geq 0. \tag{A.8}
\]

Therefore, it suffices to show (A.7) and (A.8). The \( \tilde{g}_k \)’s are determined by (3.21) and (3.22) with \( \mu = 0 \), namely,

\[
(k + 2 + \epsilon \frac{\partial}{\partial \epsilon}) \partial_x \tilde{g}_k^{KdV} = (\mathcal{R}_0 + \epsilon \mathcal{R}_1) \tilde{g}_k^{KdV}, \quad \frac{\partial \tilde{g}_k^{KdV}}{\partial u_0} = \tilde{g}_k^{KdV}, \quad k \geq -1, \tag{A.9}
\]

with initial condition \( \tilde{g}_{-1} = u_0 \), where

\[
\mathcal{R}_0 := \frac{1}{2} \sum_{i,j \geq 0} \left( \frac{(i+1)!}{(i+1-j)! j!} u_{i+1-j} \frac{\partial}{\partial u_i} - \frac{(i+1)!(j+1)!}{(i+j+3)!} u_{i+j+3} \frac{\partial^2}{\partial u_i \partial u_j} \right), \tag{A.10}
\]
\[
\mathcal{R}_1 := \frac{1}{12} \sum_{i \geq 0} u_{i+3} \frac{\partial}{\partial u_i}. \tag{A.11}
\]
It follows that $\tilde{g}_k$ is a polynomial of degree at most $k + 1$ in $\epsilon$. Moreover, the leading term satisfies the recursion
\begin{equation}
(2k + 4) \partial_x ([e^{k+2}] g_{k+4}) = \mathcal{R}_1([e^{k+1}] g_{k+4}), \quad k \geq -1, \tag{A.12}
\end{equation}
with $[e^0] g_{-1} = u_0$. Hence, for $k \geq -1$ it follows that $[e^{k+1}] g_{k+4} = \frac{u_{2k+2}}{24k+1} + c_k$ for some constants $c_k$ depending on $k$ only. By Corollary 3.7, $[e^{k+1}] g_{k+4}$ must be of homogeneous weight $2k + 3$, hence $c_k = 0$. Then the subleading term is determined by the recursion
\begin{equation}
(2k + 3) \partial_x ([e^{k+1}] g_{k+1}) = \frac{\mathcal{R}_0(u_{2k+2})}{24k+1} + \mathcal{R}_1([e^{k}] g_{k+1}), \quad k \geq 0 \tag{A.13}
\end{equation}
with $[e^0] g_{0} = u_0^2/2$. Therefore, $[e^{k}] g_{k+1} = 24^{-k} \sum_{j=0}^{2k} d(j,k) u_j u_{2k-j}$ for some coefficients $d(j,k)$. This is in principle only true up to a constant depending on $k$ only, however Corollary 3.7 again implies that this constant vanishes. Here the coefficients are assumed to satisfy $d(j,k) = d(2j - k, j)$ and, as a consequence of (A.13), are subject to the recurrence
\begin{equation}
(2k+3)(d(j, k+1)+d(j-1, k+1)) = \frac{1}{2(k+1)!} \binom{2k+3}{j} + 2(d(j-3, k)+d(j, k)), \quad j, k \geq 0, \tag{A.14}
\end{equation}
where $d(j, k) = 0$ for $j < 0$ or $j > 2k$, with initial condition $d(0, 0) = 1/2$. Therefore
\begin{equation}
(2k+3)(1+y) \Delta_{k+1}(y) = \frac{(1+y)2k+3}{2(k+1)!} + 2(1+y^3) \Delta_k(y), \quad \Delta_k(y) := \sum_{j=0}^{2k} d(j,k) y^j. \tag{A.15}
\end{equation}
Dividing by $(1+y)$, multiplying by $x^{k+1}(2k+1)!!$, and summing over $k \geq 0$ we obtain
\begin{equation}
D(x, y) - \frac{1}{2} = \frac{1}{2} \left( \frac{1}{\sqrt{1 - 2x(y+1)^2}} - 1 \right) + 2x(1 - y + y^2)D(x, y), \tag{A.16}
\end{equation}
where $D(x, y) := \sum_{k \geq 0} (2k+1)!! \Delta_k(y) x^k$, and (A.8) follows. \hfill \square

In particular, we obtain the following immediate consequence, whose proof is omitted.

**Corollary A.4.** For $k \geq 0$, $G_k^{\text{KdV}}(\epsilon) = \text{Op}_c(g_k^{\text{KdV}}(u; \epsilon))$ is a polynomial in $\epsilon$ of degree $k$ whose leading coefficient is
\begin{equation}
[e^k] G_k^{\text{KdV}}(\epsilon) = \frac{c^2}{2} \delta_{k,0} + \frac{1}{(-4)^k (2k+1)!!} \left( \frac{B_{2k+2}}{4k+4} + \sum_{j \geq 1} j^{2k+1} p_j \frac{\partial}{\partial p_j} \right). \tag{A.17}
\end{equation}

**B** Tables of quasimodular forms

To illustrate our main theorems, we provide the following examples. As above, $G_k$ is the Eisenstein series (1.21). Note that $G_8 = 120G_2^2$.

**B.1** Table of $q$-series associated with the first few quantum KdV Hamiltonian operators.

| $k$ | $\{G_k^{\text{KdV}}(\epsilon)\}_q$ |
|-----|---------------------------------|
| $-2$ | 1 |
| $-1$ | $c$ |
| 0   | $G_2 + \frac{c^2}{2}$ |
| 1   | $cG_2 + \frac{c^3}{3} - \frac{c}{2} (2G_4)$ |
| 2   | $\frac{1}{6} G_2^2 + \frac{1}{12} G_4 + \frac{c^2}{2} G_2 + \frac{c^4}{4} - \frac{c}{2} (2c G_4) + \left( \frac{c}{2} \right)^2 \frac{16}{3} G_6$ |
| 3   | $\frac{5}{6} G_2^2 + \frac{c^2}{6} G_4 + \frac{c^6}{6} G_2 + \frac{c^6}{60} - \frac{c}{2} (8c G_6 + 2G_4 G_2 + c^2 G_4) + \left( \frac{c}{2} \right)^2 \frac{64}{15} c G_6 - \left( \frac{c}{2} \right)^3 \frac{32}{45} G_8$ |
Table of non-zero \( g \)-series of the form \( \{ \text{Op}_c(Bg) \}_q \) for monomials \( g \in \mathbb{Q}[u] \) of weight up to 9 (recalling that \( u_k \) has weight \( k + 1 \)).

| \( g \) | \( \{ \text{Op}_c(Bg) \}_q \) | \( g \) | \( \{ \text{Op}_c(Bg) \}_q \) |
|---|---|---|---|
| \( u_0^2 \) | \( 2G_2 + c^2 \) | \( u_0^4u_2 \) | \( -120G_4G_2^3 - 120G_4G_2c^2 - 10G_4c^4 \) |
| \( u_0^4 \) | \( 12G_2^3 + 12G_2c^2 + c^4 \) | \( u_0^2u_4^2 \) | \( 24G_4G_2^2 + 24G_4G_2c^2 + 2G_4c^4 \) |
| \( u_0^2u_2 \) | \( -2G_4 \) | \( u_0^3u_4 \) | \( 12G_6G_2 + 6G_6c^2 \) |
| \( u_2^2 \) | \( 2G_4 \) | \( u_2^3u_4 \) | \( -4G_6G_2 - 2G_6c^2 \) |
| \( u_0^6 \) | \( 120G_4^2 + 180G_2^2c^2 + 30G_2c^4 + c^6 \) | \( u_0^4u_2 \) | \( 8G_4^2 + 4G_6G_2 + 2G_6c^2 \) |
| \( u_0^2u_2^2 \) | \( -12G_4G_2 - 6G_4c^2 \) | \( u_0u_6 \) | \( -2G_8 \) |
| \( u_0^2u_4 \) | \( 2G_6 \) | \( u_1u_5 \) | \( 2G_8 \) |
| \( u_1u_3 \) | \( -2G_6 \) | \( u_1^4 \) | \( 12G_4^2 \) |
| \( u_2^2 \) | \( 2G_6 \) | \( u_2u_4 \) | \( -2G_8 \) |
| \( u_0^8 \) | \( 1680G_4^2 + 3360G_2^2c^2 + 840G_2c^4 + 56G_2c^6 + c^8 \) | \( u_0^4 \) | \( 2G_8 \) |

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