ON KERNEL THEOREMS FOR FRÉCHET AND DF SPACES

A. G. SMIRNOV AND M. A. SOLOVIEV

Abstract. A convenient technique for calculating completed topological tensor products of functional Fréchet or DF spaces is developed. The general construction is applied to proving kernel theorems for a wide class of spaces of smooth and entire analytic functions.

Let $X$ and $Y$ be sets and $F$, $G$, and $H$ be locally convex spaces consisting of functions defined on $X$, $Y$, and $X \times Y$ respectively. If the function $(x, y) \rightarrow f(x)g(y)$ belongs to $H$ for any $f \in F$ and $g \in G$, then $F \otimes G$ is identified with a linear subspace of $H$. In applications, it is often important to find out whether $H$ can be interpreted as the completion of $F \otimes G$ with respect to some natural tensor product topology. Results of this type (or, more often, their reformulations in terms of one or other representations for continuous bilinear forms defined on $F \times G$) for concrete functional spaces are known as kernel theorems. For example, Schwartz’s kernel theorem for tempered distributions amounts to the statement that the space $S(\mathbb{R}^{k_1+k_2})$ of rapidly decreasing smooth functions on $\mathbb{R}^{k_1+k_2}$ is identical to the completion $S(\mathbb{R}^{k_1}) \hat{\otimes}_\pi S(\mathbb{R}^{k_2})$ of $S(\mathbb{R}^{k_1}) \otimes S(\mathbb{R}^{k_2})$ with respect to the projective topology. Very general conditions ensuring the algebraic coincidence of $H$ with $F \hat{\otimes}_\pi G$ can be derived from the description of completed tensor products of functional spaces given by Grothendieck ([1], Chapitre 2, Théorème 13). Namely, suppose both $F$ and $G$ are Hausdorff and complete, $F$ is nuclear and representable as an inductive limit of Fréchet spaces, and the topology of $F$ is stronger than that of simple convergence. Then $F \hat{\otimes}_\pi G$ is algebraically identified with $H$ if and only if $H$ consists exactly of all functions $h$ on $X \times Y$ such that $h(x, \cdot) \in G$ for every $x \in X$ and the function $h_v(x) = \langle v, h(x, \cdot) \rangle$ belongs to $F$ for all $v \in G'$ (for a locally convex space $G$, we denote by $G'$ and $\langle \cdot, \cdot \rangle$ the continuous dual of $G$ and the canonical bilinear form on $G' \times G$ respectively). In practice, however, the space $H$ usually carries its own topology and one needs to prove the topological coincidence of $H$ and $F \hat{\otimes}_\pi G$. In this paper, we find a convenient criterion (Theorem 3 below) ensuring such a coincidence under the assumption that $F$, $G$, and $H$ are either all reflexive Fréchet spaces or all reflexive DF spaces. This criterion allows us to obtain simple proofs of kernel theorems for a generalization of well known spaces $K(M_p)$ of smooth functions and for new classes of entire analytic functions which

2000 Mathematics Subject Classification. Primary 46A32, 46E10; Secondary 46A04.

The research was partially supported by the grants RFBR 02-01-00556 and LSS-1578.2003.2., the first author was also supported by the grant INTAS 03-51-6346.

1Recall that the projective (inductive) topology on $F \otimes G$ is the strongest locally convex topology on $F \otimes G$ such that the canonical bilinear mapping $(f, g) \rightarrow f \otimes g$ is continuous (resp., separately continuous). In general, one must carefully distinguish between the inductive and projective topologies. However, these topologies always coincide for Fréchet and barrelled DF spaces which are considered in this paper. For definiteness, we speak everywhere about the projective topology.
arise in quantum field theory (see, e.g., [8] and [9] for a detailed discussion of this application).

Given a locally convex space $F$, we denote by $F_\sigma$ and $F'_\sigma$ the space $F$ endowed with its weak topology $\sigma(F,F')$ and the space $F'$ endowed with its weak* topology $\sigma(F',F)$ respectively. The dual of $F$ endowed with the strong topology will be denoted just by $F'$ without any indices. We refer the reader to [11] for the definition and properties of DF spaces. Here we only mention that the strong dual of a Fréchet space (resp., DF space) is a DF space (resp., Fréchet space).

We shall derive our main result concerning functional spaces from the following more general theorem describing tensor products of abstract Fréchet and DF spaces.

**Theorem 1.** Let $F$, $G$, and $H$ be either all reflexive Fréchet spaces or all reflexive DF spaces and let at least one of the spaces $F$ or $G$ be nuclear. Suppose $\Phi: F \times G \to H$ and $\Psi: F' \times G' \to H'$ are bilinear mappings such that

(i) The bilinear forms $(f,g) \mapsto (w,\Phi(f,g))$ and $(u,v) \mapsto (\Psi(u,v),h)$ on $F \times G$ and $F' \times G'$ respectively are separately continuous for all $h \in H$ and $w \in H'$.

(ii) $(\Psi(u,v),\Phi(f,g)) = (u,f)(v,g)$ for all $f \in F$, $g \in G$, $u \in F'$, and $v \in G'$.

(iii) Either the linear span of $\Phi(F,G)$ is dense in $H$ or the linear span of $\Psi(F',G')$ is dense in $H'$.

Then the bilinear mappings $\Phi$ and $\Psi$ are continuous and induce the topological isomorphisms $F \hat{\otimes}_\pi G \simeq H$ and $F' \hat{\otimes}_\pi G' \simeq H'$ respectively.

The proof of Theorem 1 is based on the following lemma.

**Lemma 2.** Let $F$, $G$, and $H$ be Hausdorff complete locally convex spaces such that $F$ and $G$ are semireflexive, $H$ is barrelled and at least one of $F$ or $G$ is nuclear. Let $\Phi: F \times G \to H$ and $\Psi: F' \times G' \to H'$ be continuous bilinear mappings satisfying condition (ii). Assume that the linear span of $\Psi(F',G')$ is dense in $H'$. Then $\Phi$ induces the topological isomorphism $F \hat{\otimes}_\pi G \simeq H$.

**Proof.** Let $\Phi_*: F \otimes_\pi G \to H$ be the continuous linear mapping determined by $\Phi$. We have to show that its extension $\hat{\Phi}_*: F \hat{\otimes}_\pi G \to H$ is a topological isomorphism. As usual, let $\mathcal{B}_e(F'_\sigma,G'_\sigma)$ denote the space of separately continuous bilinear forms on $F'_\sigma \times G'_\sigma$ equipped with the biequicontinuous convergence topology $\tau_e$ (i.e., the topology of the uniform convergence on the sets of the form $A \times B$, where $A$ and $B$ are equicontinuous sets in $F'$ and $G'$ respectively). Let $S$ be the natural continuous linear mapping $F \otimes_\pi G \to \mathcal{B}_e(F'_\sigma,G'_\sigma)$ which takes $f \otimes g$ to the bilinear form $(u,v) \mapsto (u,f)(v,g)$. Since $F$ and $G$ are assumed complete and one of them is nuclear, the extension $\hat{S}$ to the completions is a topological isomorphism (see [11], Chapitre 2, Théorème 6 or [12]) and so $F \hat{\otimes}_\pi G$ and $\mathcal{B}_e(F'_\sigma,G'_\sigma)$ are identified. The space $H$ also can be mapped into $\mathcal{B}_e(F'_\sigma,G'_\sigma)$. Namely, for $h \in H$, we denote by $b_h$ the bilinear form on $F' \times G'$ defined by $b_h(u,v) = (\Psi(u,v),h)$. The continuity of $\Psi$ ensures that $b_h$ is continuous with respect to the strong topologies of $F'$ and $G'$ and the semireflexivity of $F$ and $G$ implies that $b_h \in \mathcal{B}_e(F'_\sigma,G'_\sigma)$. The mapping $T: h \mapsto b_h$ is continuous under the topology $\tau_e$ because $\Psi(A,B)$ is an equicontinuous subset of $H'$ for every pair $A,B$ of equicontinuous subsets of $F'$ and $G'$ (indeed, $\Psi(A,B)$ is strongly bounded and hence equicontinuous since $H$ is barrelled). Furthermore, $T$ is an injection because the linear span of $\Psi(F',G')$ is dense in $H'$. Condition (ii) implies that $S = T\Phi_*$. On extending to the completion of $F \hat{\otimes}_\pi G$, we obtain $T\hat{\Phi}_* = \text{id}$, which completes the proof. \(\square\)
Proof of Theorem 1. Since we are dealing with reflexive Fréchet spaces and their strong duals, all the spaces $F$, $G$, $H$, $F'$, $G'$, and $H'$ are bornological, complete, and reflexive, see [7]. Moreover, either $F'$ or $G'$ is nuclear because strong duals of nuclear $F$ and DF spaces are nuclear ([3], Chapitre 2, Théorème 7). For $u \in F'$, let $\Psi_u$ be the linear mapping $v \to \Psi(u, v)$ from $G'$ to $H$. It follows from (i) that $\Psi_u$ takes bounded sets in $G'$ to $\sigma(H', H)$-bounded sets in $H$. The reflexivity of $H$ implies that they are strongly bounded as well. Since $G'$ is bornological, it follows that $\Psi_u$ is continuous for all $u \in F'$. Analogously, the mapping $u \to \Psi(u, v)$ from $F'$ to $H'$ is continuous for all $v \in G'$. Thus, $\Psi$ is separately continuous. By the same arguments, $\Phi$ also has this property. Moreover, separate continuity is equivalent to continuity for bilinear mappings defined on Fréchet or barrelled DF spaces (see [7], Theorem III.5.1 and [4], Corollaire du Théorème 2). Suppose now that the linear span of $\Phi(F', G')$ is dense in $H'$. Then Lemma 2 shows that $\Phi$ induces the topological isomorphism $F \hat{\otimes}_\pi G \simeq H$. This means, in particular, that the linear span of $\Phi(F, G)$ is dense in $H$. Therefore, we can apply Lemma 2 to the dual spaces and conclude that $\Psi$ induces the topological isomorphism $F' \hat{\otimes}_\pi G' \simeq H'$. If we start from the assumption that the linear span of $\Phi(F, G)$ is dense in $H$, then Lemma 2 should be applied in the inverse order.

\[\Box\]

Theorem 3. Let $X$ and $Y$ be sets and $F$, $G$, and $H$ be either all reflexive Fréchet spaces or all reflexive DF spaces consisting of scalar functions defined on $X$, $Y$, and $X \times Y$ respectively. Let at least one of the spaces $F$ or $G$ be nuclear and the topologies of $F$, $G$, and $H$ be stronger than that of simple convergence. Suppose the following conditions are satisfied:

\begin{enumerate}
\item[(a)] For every $f \in F$ and $g \in G$, the function $(x, y) \to f(x)g(y)$ on $X \times Y$ belongs to $H$ and the bilinear mapping $\Phi: F \times G \to H$ taking $(f, g)$ to this function is separately continuous.
\item[(b)] If $h \in H$, then $h(x, \cdot) \in G$ for every $x \in X$ and the function $h_v(x) = \langle v, h(x, \cdot) \rangle$ belongs to $F$ for every $v \in G'$.
\item[(c)] The mapping $h \to h(x, \cdot)$ from $H$ to $G$ is continuous for every $x \in X$.
\end{enumerate}

Then $\Phi$ is continuous and induces the topological isomorphism $F \hat{\otimes}_\pi G \simeq H$.

Proof. Let $h \in H$ and $S_h: G' \to F$ be the linear mapping taking $v \in G'$ to $h_v$. We claim that the graph $\hat{G}$ of $S_h$ is closed and, therefore, $S_h$ is continuous ([7], Theorem IV.8.5; note that all considered spaces are barrelled and B-complete). It suffices to show that if an element of the form $(0, f)$ belongs to the closure $\hat{G}$ of $G$, then $f = 0$. Suppose the contrary that there is $f_0 \in F$ such that $f_0 \neq 0$ and $(0, f_0) \in \hat{G}$. Let $x_0 \in X$ be such that $f_0(x_0) \neq 0$ and let the neighborhood $U$ of $f_0$ be defined by the relation $U = \{f \in F : |\langle \delta_{x_0}, f - f_0 \rangle| < |f_0(x_0)|/2\}$ (if $x \in X$, then $\delta_x$ is the linear functional on $F$ such $\langle \delta_x, f \rangle = f(x)$; it is continuous because the topology of $F$ is stronger than the topology of simple convergence).

Let $V = \{v \in G' : |\langle v, h(x_0, \cdot) \rangle| < |f_0(x_0)|/2\}$. If $f \in U$ and $v \in V$, then we have

\begin{equation}
|h_v(x_0)| < |f_0(x_0)|/2 < |f(x_0)|.
\end{equation}

Hence the neighborhood $V \times U$ of $(0, f_0)$ does not intersect $\hat{G}$. This contradicts to the assumption that $(0, f_0) \in \hat{G}$, and our claim is proved.

Further, let $v \in G'$ and $T_v: H \to F$ be the linear mapping taking $h$ to $h_v$. Suppose there is $f_0 \in F$ such that $f_0 \neq 0$ and $(0, f_0)$ belongs to the closure of the graph of $T_v$. Let $x_0$ and $U$ be as above and $W = \{h \in H : |\langle v, h(x_0, \cdot) \rangle| < |f_0(x_0)|/2\}$.
\(|f_0(x_0)|/2\). It follows from \((\gamma)\) that \(W\) is a neighborhood of the origin in \(H\). For \(f \in U\) and \(h \in W\), inequalities \((\Pi)\) are again satisfied and, therefore, \(W \times U\) does not intersect the graph of \(T_v\). The obtained contradiction shows that \(T_v\) has a closed graph and, hence, is continuous.

The required statement will be proved if we construct a bilinear mapping \(\Psi : F' \times G' \rightarrow H'\) such that conditions (i), (ii), and (iii) of Theorem \((\Pi)\) are satisfied. We define \(\Psi\) by the relation

\[
\langle \Psi(u, v), h \rangle = \langle u, h_v \rangle, \quad u \in F', \ v \in G', \ h \in H.
\]

The continuity of \(T_v\) implies that \(\Psi(u, v)\) is a continuous functional on \(H\) and the continuity of \(S_h\) ensures that \(\Psi\) satisfies (i). Since \(\Phi\) is separately continuous, it also satisfies (i) and the fulfillment of (ii) follows immediately from the definitions of \(\Phi\) and \(\Psi\). Further, for every \(x \in X\) and \(y \in Y\), we have \(\Psi(\delta_{x}, \delta_{y}) = \delta_{(x, y)}\). By the reflexivity of \(H\), the linear span of \(\delta\)-functionals is dense in \(H'\). Therefore, condition (iii) is satisfied and the theorem is proved. \(\square\)

**Remark 4.** In contrast to the works \([10]\) and \([5]\), where the density of \(\mathbb{F} \otimes \mathbb{G}\) in \(H\) (for concrete functional spaces) was proved “by hand”, we have obtained this density automatically as a consequence of nuclearity and the density of \(\delta\)-functionals in dual spaces. In some cases (especially for spaces of analytic functions), a direct check of the density of \(\mathbb{F} \otimes \mathbb{G}\) in \(H\) may present considerable difficulty.

We now apply Theorem \((\Box)\) to proving kernel theorems for some spaces of smooth functions. In what follows, we use the standard multi-index notation:

\[|\mu| = \mu_1 + \ldots + \mu_k, \quad \partial^\mu f(x) = \frac{\partial^{|\mu|} f(x)}{\partial x_1^{\mu_1} \ldots \partial x_k^{\mu_k}} \quad (\mu \in \mathbb{Z}_+^k).\]

**Definition 5.** Let \(M = \{M_{\gamma}\}_{\gamma \in \Gamma}\) be a family of nonnegative measurable functions on \(\mathbb{R}^k\) which are bounded on every bounded subset of \(\mathbb{R}^k\) and satisfy the following conditions:

(a) For every \(\gamma_1, \gamma_2 \in \Gamma\), one can find \(\gamma \in \Gamma\) and \(C > 0\) such that \(M_{\gamma} \geq C(M_{\gamma_1} + M_{\gamma_2})\).

(b) There is a countable set \(\Gamma' \subset \Gamma\) with the property that for every \(\gamma \in \Gamma\), one can find \(\gamma' \in \Gamma'\) and \(C > 0\) such that \(CM_{\gamma} \leq M_{\gamma'}\).

(c) For every \(x \in \mathbb{R}^k\), one can find \(\gamma \in \Gamma\), a neighborhood \(O(x)\) of \(x\), and \(C > 0\) such that \(M_{\gamma}(x') \geq C\) for all \(x' \in O(x)\).

The space \(\mathcal{K}(M)\) consists of all smooth functions \(f\) on \(\mathbb{R}^k\) having the finite seminorms

\[
\|f\|_{\gamma, m} = \sup_{x \in \mathbb{R}^k, |\mu| \leq m} M_{\gamma}(x)|\partial^\mu f(x)|
\]

for all \(\gamma \in \Gamma\) and \(m \in \mathbb{Z}_+.\) The space \(\mathcal{K}_p(M), p \geq 1,\) consists of all smooth functions \(f\) on \(\mathbb{C}^k\) having the finite seminorms

\[
\|f\|_{\gamma, m}^p = \left(\int_{\mathbb{R}^k} [M_{\gamma}(x)]^p \sum_{|\mu| \leq m} |\partial^\mu f(x)|^p \, dx \right)^{1/p}.
\]

The spaces \(\mathcal{K}(M)\) and \(\mathcal{K}_p(M)\) are endowed with the topologies determined by seminorms \((\Pi)\) and \((\Box)\) respectively.
We shall say that \( M = \{ M_{\gamma} \}_{\gamma \in \Gamma} \) is a defining family of functions on \( \mathbb{R}^k \) if it satisfies all requirements of Definition [12]. Note that if all \( M_{\gamma} \) are strictly positive and continuous, then condition (c) holds automatically. Condition (b) ensures that \( K(M) \) and \( K_p(M) \) possess a countable fundamental system of neighborhoods of the origin. It is easy to see that \( K(M) \) is actually a Fréchet space. Indeed, let \( f_n \) be a Cauchy sequence in \( K(M) \). Then it follows from (c) that \( \partial^\mu f_n(x) \) converge uniformly on every compact subset of \( \mathbb{R}^k \) for every multi-index \( \mu \). This implies that \( f_n \) converge pointwise to a smooth function \( f \). For \( \varepsilon > 0, \gamma \in \Gamma \), and \( m \in \mathbb{Z}_+ \), choose \( n_0 \) such that \( \| f_{n+l} - f_n \|_{\gamma, m} < \varepsilon \) for all \( n \geq n_0 \) and \( l \in \mathbb{Z}_+ \). Then \( M_{\gamma}(x) |\partial^\mu f_{n+l}(x) - \partial^\mu f_n(x)| < \varepsilon \) for every \( x \in \mathbb{R}^k \) and \( |\mu| \leq m \). Passing to the limit \( l \to \infty \), we obtain \( M_{\gamma}(x) |\partial^\mu f(x) - \partial^\mu f_n(x)| < \varepsilon \), i.e., \( \| f - f_n \|_{\gamma, m} < \varepsilon \) for \( n \geq n_0 \). Hence it follows that \( f \in K(M) \) and \( f_n \to f \) in this space.

**Lemma 6.** Let \( M = \{ M_{\gamma} \}_{\gamma \in \Gamma} \) be a defining family of functions on \( \mathbb{R}^k \). The space \( D(\mathbb{R}^k) \) of smooth functions with compact support is dense in \( K_p(M) \) for \( p \geq 1 \).

**Proof.** Let \( f \in K_p(M) \) and \( \varphi \in D(\mathbb{R}^k) \) be such that \( \varphi(x) = 1 \) for \( |x| \leq 1 \) (\( | \cdot | \) is a norm on \( \mathbb{R}^k \)). For \( n = 1, 2, \ldots \), we define \( \varphi_n \in D(\mathbb{R}^k) \) by the relation \( \varphi_n(x) = \varphi(x/n) \) and set \( \psi_n = 1 - \varphi_n \). To prove the statement, it suffices to show that \( \varphi_n f \to 0 \) (or, which is the same, that \( \psi_n f \to 0 \)) in \( K_p(M) \) as \( n \to \infty \). Let \( \gamma \in \Gamma, m \in \mathbb{Z}_+, \) and \( \mu \) be a multi-index such that \( |\mu| \leq m \). An elementary estimate using the Leibniz formula gives \( |\partial^\mu(\psi_n f)(x)| \leq A 2^m \sum_{|\nu| \leq m} |\partial^\nu f(x)| \), where \( A = 1 + \sup_{x, |\nu| \leq m} |\partial^\nu \varphi(x)| \). Since \( \psi_n \) vanishes for \( |x| \leq n \), it hence follows that

\[
\| \psi_n f \|_{p, m}^p \leq A 2^m q(m) \left( \int_{|x|>n} |M_{\gamma}(x)|^p \sum_{|\nu| \leq m} |\partial^\nu f(x)|^p \, dx \right)^{1/p},
\]

where \( q(m) \) is the number of multi-indices whose norm does not exceed \( m \). Since \( \| f \|_{p, m}^p < \infty \), the integral in the right-hand side tends to zero as \( n \to \infty \). \( \square \)

**Lemma 7.** Let \( M = \{ M_{\gamma} \}_{\gamma \in \Gamma} \) be a defining family of functions on \( \mathbb{R}^k \) satisfying the following conditions:

(I) For every \( \gamma \in \Gamma \), there are \( \gamma' \in \Gamma \) and a summable nonnegative function \( L_{\gamma \gamma'} \) on \( \mathbb{R}^k \) such that \( M_{\gamma} \leq L_{\gamma \gamma'} M_{\gamma'} \) and \( L_{\gamma \gamma'}(x) \to 0 \) as \( |x| \to \infty \).

(II) For every \( \gamma \in \Gamma \), there are \( \gamma' \in \Gamma \), a neighborhood of the origin \( B \) in \( \mathbb{R}^k \), and \( C > 0 \) such that \( M_{\gamma}(x) \leq C M_{\gamma'}(x + y) \) for any \( x \in \mathbb{R}^k \) and \( y \in B \).

Then the space \( K(M) \) is nuclear and coincides, both as a set and topologically, with \( K_p(M) \) for all \( p \geq 1 \).

**Proof.** Let \( f \in K(M) \), \( \gamma \in \Gamma \), and \( m \in \mathbb{Z}_+ \). Choosing \( \gamma' \) and \( L_{\gamma \gamma'} \) such that (I) is satisfied, we obtain

\[
\| f \|_{p, m}^p \leq A \| f \|_{p, \gamma', m}^p,
\]

where \( A = \left( q(m) \int L_{\gamma \gamma'}(x)^p \, dx \right)^{1/p} < \infty \) (as above, \( q(m) \) is the number of multi-indices with the norm \( \leq m \)). Hence we have a continuous inclusion \( K(M) \subset K_p(M) \).

We now prove that the topology induced on \( K(M) \) from \( K_p(M) \) coincides with the original topology of \( K(M) \). In other words, given \( \gamma \in \Gamma \) and \( m \in \mathbb{Z}_+ \), we have to find \( \tilde{\gamma} \in \Gamma, \tilde{m} \in \mathbb{Z}_+ \), and \( A > 0 \) such that

\[
\| f \|_{\gamma, m} \leq A \| f \|_{\tilde{\gamma}, \tilde{m}}^p
\]

for every \( f \in K(M) \). By (II), there are \( \gamma', \gamma'' \in \Gamma \), a neighborhood of the origin \( B \subset \mathbb{R}^k \), and \( C > 0 \) such that \( M_{\gamma}(x) \leq C M_{\gamma'}(x + y) \) and \( M_{\gamma'}(x) \leq C M_{\gamma''}(x + y) \).
for any \( x \in \mathbb{R}^k \) and \( y \in B \). Let \( \psi \) be a smooth nonnegative function such that \( \int \psi(x)dx = 1 \) and \( \text{supp} \psi \subset B \). Set \( M(x) = \int M_{\gamma'}(x + x')\psi(x')dx' \). Then \( M \) is a smooth function on \( \mathbb{R}^k \) and we have

\[
M_{\gamma}(x) = \int M_{\gamma}(x)\psi(x')dx' \leq C \int M_{\gamma'}(x + x')\psi(x')dx' = CM(x),
\]

where \( C = \int |\partial^\mu M(x)|dx \). In view of condition (I) inequality (6) implies that \( |\partial^\mu M(x)|dx \to 0 \) as \( |x| \to \infty \) for all multi-indices \( \mu \) and \( \nu \) and every \( f \in K(M) \).

For \( |\mu| \leq m \), it hence follows from (4) and (6) that

\[
M_{\gamma}(x)|\partial^\mu f(x)| \leq CM(x)|\partial^\mu f(x)| = C',
\]

where \( C' = C \sum_{|\mu| \leq k} C \mu \). Let \( \tilde{m} = m + k \) and \( \tilde{\gamma} \) be such that \( M_{\gamma''} \leq L_{\gamma'', \tilde{\gamma}} M_{\tilde{\gamma}} \), where \( L_{\gamma'', \tilde{\gamma}}(x) \) is integrable and tends to zero as \( |x| \to \infty \). Estimating \( \|f\|_{1, \gamma''}^{\gamma''} \) by the Hölder inequality, we conclude from (7) that (8) holds with

\[
A = C' \left( q(\tilde{m}) \int |L_{\gamma'', \tilde{\gamma}}(x)|^{p/(p-1)}dx \right)^{(p-1)/p}.
\]

Since \( D(\mathbb{R}^k) \subset K(M) \), it follows from Lemma 6 that \( K(M) \) is a dense subspace of \( K_p(M) \). At the same time, the completeness of \( K(M) \) implies that it is closed in \( K_p(M) \). Hence, we have \( K(M) = K_p(M) \).

To prove the nuclearity of \( K(M) \), we shall use the following criterion obtained by Pietsch [6].

**Lemma 8.** A locally convex space \( F \) is nuclear if and only if some (every) fundamental system \( U \) of absolutely convex neighborhoods of the origin has the following property:

For every neighborhood of the origin \( U \in U \), there is a neighborhood of the origin \( V \in U \) and a positive Radon measure\(^2\) \( \tau \) on \( V^\circ \) such that

\[
p_{\gamma}(x) \leq \int_{V^\circ} |\langle u, f \rangle| d\tau(u)
\]

for every \( f \in F \) (\( p_U \) is the Minkowski functional of the set \( U \)).

For \( \gamma \in \Gamma \) and \( m \in \mathbb{Z}_+ \), we set \( U_{\gamma, m} = \{ f \in H(M) : \|f\|_{\gamma, m} \leq 1 \} \). By condition (a) of Definition 5 the scalar multiples of \( U_{\gamma, m} \) form a fundamental system of neighborhoods of the origin in \( K(M) \), and we have \( p_{U_{\gamma, m}}(f) = \|f\|_{\gamma, m} \). Fix \( \gamma \) and \( m \) and choose \( \tilde{\gamma}, \tilde{m}, A \) such that inequality (1) with \( p = 1 \) is satisfied for all \( f \in K(M) \). Let \( \tilde{\gamma}' \) be such that \( M_{\tilde{\gamma}'} \leq L_{\tilde{\gamma}', \tilde{\gamma}} M_{\tilde{\gamma}} \), where \( L_{\tilde{\gamma}', \tilde{\gamma}} \) is integrable. Acting as in the derivation of formulas (3) and (6), we find a smooth function \( \tilde{M} \), an index \( \tilde{\gamma}'' \), and \( C > 0 \) such that \( M_{\tilde{\gamma}''} \leq CM_{\tilde{\gamma}'} \) and \( \tilde{M} \leq CM_{\tilde{\gamma}''} \). For every \( x \in \mathbb{R}^k \) and multi-index \( \mu \), we define the functional \( \varepsilon_{\mu}^x \in K(M) \) by the relation

\[
\langle \varepsilon_{\mu}^x, f \rangle = \tilde{M}(x)\partial^\mu f(x)/C, \quad f \in K(M).
\]

\(^2\)Recall that a Radon measure on a compact set \( K \) is, by definition, a continuous linear form on the space \( C(K) \) of continuous functions on \( K \). Recall also that the polar set \( V^\circ \) of a neighborhood of the origin \( V \) in a locally convex space is weakly compact.
If $|\mu| \leq \tilde{m}$, then we obviously have $|\langle \varepsilon^\mu, f \rangle| \leq 1$ for $\|f\|_{\gamma', \tilde{m}} \leq 1$, i.e., $\varepsilon^\mu \in U_{\gamma', \tilde{m}}$. Moreover, the mapping $x \to \varepsilon^\mu_x$ from $\mathbb{R}^k$ to $K'(M)$ is weakly continuous. Hence the function $\varphi(\varepsilon^\mu_x)$ is bounded and continuous on $\mathbb{R}^k$ for every continuous function $\varphi$ on the weakly compact set $U_{\gamma', \tilde{m}}$. Therefore, the formula

$$\tau(\varphi) = AC^2 \int_{\mathbb{R}^k} L_{\gamma, \gamma'}(x) \sum_{|\mu| \leq \tilde{m}} \varphi(\varepsilon^\mu_x) \, dx,$$

defines a positive Radon measure $\tau$ on $U_{\gamma', \tilde{m}}$. It follows from this definition that

$$\|f\|_{\gamma, m} \leq AC^2 \int_{\mathbb{R}^k} L_{\gamma, \gamma'}(x) \sum_{|\mu| \leq \tilde{m}} |\langle \varepsilon^\mu, f \rangle| \, dx = \int_{U_{\gamma', \tilde{m}}} |\langle u, f \rangle| \, d\tau(u)$$

for every $f \in K(M)$. In view of Lemma 8 this estimate shows that $K(M)$ is nuclear. Lemma 9 is proved.

**Examples.** 1. Let $\Gamma$ be the set of all compact subsets of $\mathbb{R}^k$ and $\chi_{\gamma}$ be the characteristic function of $\gamma$. Then $K(M)$ is the space $C^\infty(\mathbb{R}^k)$ endowed with its standard topology. Conditions (I) and (II) are obviously satisfied.

2. Let $\Gamma = Z_+$ and $M_1 = (1 + |x|)^f$. Then $K(M)$ is the Schwartz space $S(\mathbb{R}^k)$ of rapidly decreasing functions. Conditions (I) and (II) are obviously satisfied.

3. Let $\alpha > 0$, $A \geq 0$, $\Gamma$ be the interval $(A, \infty)$, and $M_\alpha(x) = \exp(|x/A'|^{1/\alpha})$ for every $A' > A$. Then $K(M)$ coincides with the Gelfand–Shilov space $S_{\alpha, \tilde{A}}$, where $\tilde{A} = (\alpha/e)\alpha A$ (see [2], Section IV.3). Conditions (I) and (II) are obviously satisfied and, therefore, the space $S_{\alpha, \tilde{A}}$ is nuclear for any $A \geq 0$.

**Remark 9.** The spaces $K(M)$ are similar to the spaces $K(M_p)$ introduced in the classical book [2]. Theorem 1.7 of [2] asserts that condition (I) of Lemma 4 (which is called condition (N) there) is sufficient for the nuclearity of $K(M_p)$. However, the proof of this theorem contains an error (an estimate of type [4] is obtained with a constant $A$ depending implicitly on the function $f$). Moreover, the condition $M_p(x) \geq 1$ included in the definition of $K(M_p)$ is actually redundant being an artifact of this erroneous proof.

Let $M = \{M_\gamma\}_{\gamma \in \Gamma}$ and $N = \{N_\omega\}_{\omega \in \Omega}$ be defining families of functions on $\mathbb{R}^{k_1}$ and $\mathbb{R}^{k_2}$ respectively. We denote by $M \otimes N$ the family formed by the functions

$$(M \otimes N)_{\gamma \omega}(x, y) = M_\gamma(x)N_\omega(y), \quad (\gamma, \omega) \in \Gamma \times \Omega.$$

Clearly, $M \otimes N$ is a defining family of functions on $\mathbb{R}^{k_1+k_2}$.

**Lemma 10.** Let $M = \{M_\gamma\}_{\gamma \in \Gamma}$ and $N = \{N_\omega\}_{\omega \in \Omega}$ be defining families of functions on $\mathbb{R}^{k_1}$ and $\mathbb{R}^{k_2}$ respectively and let $h \in K(M \otimes N)$. Suppose $N$ satisfies the conditions (I) and (II) of Lemma 4. Then $h(x, \cdot) \in K(N)$ for every $x \in \mathbb{R}^{k_1}$ and the function $h_v(x) = \langle v, h(x, \cdot) \rangle$ belongs to $K(M)$ for all $v \in K'(N)$. Moreover, for every multi-index $\mu \in \mathbb{Z}^{k_1}_+$, we have

$$\partial^\mu h_v(x) = \langle v, \partial^\mu_h h(x, \cdot) \rangle.$$

**Proof.** By Lemma 4, $K(N)$ is a nuclear Fréchet space. This implies, in particular, that it is reflexive. Let $Q(M)$ be the space consisting of the sequences
\[ \psi = \{ \psi^\mu \}_{\mu \in \mathbb{Z}_+^k} \] of functions on \( \mathbb{R}^k_1 \) having the finite norms

\[ \|\psi\|_\gamma,m = \sup_{x \in \mathbb{R}^k_1, |\mu| \leq m} |\psi^\mu(x)| M_\gamma(x) \]

for all \( \gamma \in \Gamma \) and \( m \in \mathbb{Z}_+ \). Let \( T : \mathcal{K}(M) \to Q(M) \) be the mapping taking \( f \in \mathcal{K}(M) \) to the sequence \( \{ \partial^\mu f \} \). Obviously, \( T \) maps \( \mathcal{K}(M) \) isomorphically onto its image, and since \( \mathcal{K}(M) \) is complete, \( \text{Im} T \) is a closed subspace of \( Q(M) \). For \( v \in \mathcal{K}'(N) \) and \( \mu \in \mathbb{Z}_+^k \), we set \( \psi^\mu_v(x) = \langle v, \partial^\mu h(x, \cdot) \rangle \). Since \( h_v = \psi^\mu_v \) for zero \( \mu \), it suffices to show that the sequence \( \psi_v = \{ \psi^\mu_v \} \) belongs to \( \text{Im} T \). For every \( \omega \in \Omega \) and \( n \in \mathbb{Z}_+ \), we set \( B_{\omega,n} = \{ v \in \mathcal{K}'(N) : |\langle v, g \rangle| \leq \|g\|_{\omega,n} \forall g \in \mathcal{K}(N) \} \). If \( \gamma \in \Gamma, m \in \mathbb{Z}_+ \), \( |\mu| \leq m \), and \( v \in B_{\omega,n} \), then we have

\[ |\langle v, \partial^\mu h(x, \cdot) \rangle| M_\gamma(x) \leq \|\partial^\mu h(x, \cdot)\|_{\omega,n} M_\gamma(x) \leq \|h\|_{(\gamma,\omega),m,n}, \quad x \in \mathbb{R}^k_1. \]

Hence, \( \|\psi_v\|_\gamma,m \leq \|h\|_{(\gamma,\omega),m+n} \) for \( v \in B_{\omega,n} \). Thus, \( \psi_v \) belongs to the space \( Q(M) \) for any \( v \in \mathcal{K}'(N) \) and the image of \( B_{\omega,n} \) under the mapping \( v \to \psi_v \) is bounded in \( Q(M) \). The scalar multiples of \( B_{\omega,n} \) form a fundamental system of bounded subsets in the space \( \mathcal{K}'(N) \), which is bornological as the strong dual of a reflexive Fréchet space (\( \mathbb{R}^k_1 \)). Therefore, the mapping \( v \to \psi_v \) from \( \mathcal{K}'(N) \) to \( Q(M) \) is continuous. If \( v = \delta_y \) for some \( y \in \mathbb{R}^k_2 \), then \( \psi_v \) obviously belongs to \( \text{Im} T \). This implies that \( \psi_v \in \text{Im} T \) for all \( v \in \mathcal{K}'(N) \) because \( \text{Im} T \) is closed in \( Q(M) \), the linear span of \( \delta \)-functionals is dense in \( \mathcal{K}'(N) \) by the reflexivity of \( \mathcal{K}(N) \), and the image of the closure of a set under a continuous mapping is contained in the closure of the image of this set.

**Theorem 11.** Let \( M \) and \( N \) be defining families of functions on \( \mathbb{R}^k_1 \) and \( \mathbb{R}^k_2 \) respectively satisfying conditions (I) and (II) of Lemma 4. Let the bilinear mapping \( \Phi : \mathcal{K}(M) \times \mathcal{K}(N) \to \mathcal{K}(M \otimes N) \) be defined by the relation \( \Phi(f, g)(x, y) = f(x)g(y) \). Then \( \Phi \) induces the topological isomorphism \( \mathcal{K}(M) \otimes^\pi \mathcal{K}(N) \simeq \mathcal{K}(M \otimes N) \).

**Proof.** It is easy to check that \( M \otimes N \) satisfies (I) and (II) if both \( M \) and \( N \) satisfy these conditions. Hence Lemma 4 implies that \( \mathcal{K}(M), \mathcal{K}(N), \) and \( \mathcal{K}(M \otimes N) \) are nuclear Fréchet spaces (and, in particular, reflexive spaces). The required statement therefore follows from Theorem 3 because the fulfilment of (\( \alpha \)) and (\( \gamma \)) is obvious and (\( \beta \)) is assured by Lemma 10.

We now consider the spaces of entire analytic functions. We say that a family \( M = \{ M_\gamma \}_{\gamma \in \Gamma} \) of functions on \( \mathbb{C}^k \) is a defining family of functions on \( \mathbb{C}^k \) if \( M \) is a defining family of functions on the underlying real space \( \mathbb{R}^k_2 \). In what follows, we identify defining families of functions on \( \mathbb{C}^k \) with the corresponding defining families of functions on \( \mathbb{R}^k_2 \). In particular, if \( M \) is a defining family of functions on \( \mathbb{C}^k \), then \( \mathcal{K}(M) \) will denote the corresponding space of \( C^\infty \)-functions on \( \mathbb{R}^k_2 \) and the statement that \( M \) satisfies conditions (I) and (II) of Lemma 4 will mean that (I) and (II) are fulfilled if \( M \) is viewed as a family of functions on \( \mathbb{R}^k_2 \). If \( M \) and \( N \) are defining families of functions on \( \mathbb{C}^{k_1} \) and \( \mathbb{C}^{k_2} \) respectively, then \( M \otimes N \) will be interpreted as a defining family of functions on \( \mathbb{C}^{k_1+k_2} \).

**Definition 12.** Let \( M = \{ M_\gamma \}_{\gamma \in \Gamma} \) be a defining family of functions on \( \mathbb{C}^k \). The space \( \mathcal{H}(M) \) consists of all entire analytic functions \( f \) on \( \mathbb{C}^k \) having the finite seminorms

\[ \|f\|_\gamma = \sup_{z \in \mathbb{C}^k} M_\gamma(z)|f(z)|. \]
For \( p \geq 1 \), the space \( \mathcal{H}_p(M) \) consists of all entire analytic functions \( f \) on \( \mathbb{C}^k \) having the finite seminorms

\[
\|f\|_p^\gamma = \left( \int_{\mathbb{C}^k} |M_\gamma(z)|^p |f(z)|^p \, d\lambda(z) \right)^{1/p},
\]

where \( d\lambda \) is the Lebesgue measure on \( \mathbb{C}^k \). The spaces \( \mathcal{H}(M) \) and \( \mathcal{H}_p(M) \) are endowed with the topologies determined by the seminorms \( (10) \) and \( (11) \) respectively.

The same arguments as in the case of \( \mathcal{K}(M) \) show that \( \mathcal{H}(M) \) is a Fréchet space for any defining family of functions \( M \).

**Lemma 13.** Let \( M = \{M_\gamma\}_{\gamma \in \Gamma} \) be a defining family of functions on \( \mathbb{C}^k \) satisfying conditions (I) and (II) of Lemma 7. Then the space \( \mathcal{H}(M) \) is nuclear and coincides, both as a set and topologically, with \( \mathcal{H}_p(M) \) for all \( p \geq 1 \).

**Proof.** Let \( \tilde{\mathcal{H}}(M) \) be the subspace of \( \mathcal{K}(M) \) consisting of all elements of \( \mathcal{K}(M) \) which are entire analytic functions. Since a subspace of a nuclear space is nuclear, it follows from Lemma 7 that \( \tilde{\mathcal{H}}(M) \) is a nuclear space. Therefore, to prove the nuclearity of \( \mathcal{H}(M) \), it suffices to show that \( \tilde{\mathcal{H}}(M) = \mathcal{H}(M) \). We obviously have the continuous inclusion \( \tilde{\mathcal{H}}(M) \subset \mathcal{H}(M) \). Conversely, let \( f \in \mathcal{H}(M) \), \( \gamma \in \Gamma \), and \( \mu, \nu \in \mathbb{Z}^k_+ \) be multi-indices. By (II), we can find \( \gamma' \in \Gamma \), a neighborhood of the origin \( B \subset \mathbb{C}^k \), and \( C > 0 \) such that \( M_{\gamma'}(z) \leq CM_{\gamma'}(z + z') \) for any \( z \in \mathbb{C}^k \) and \( z' \in B \). For \( r > 0 \) and \( z \in \mathbb{C}^k \), let \( D_r(z) \) denote the polydisk with the radius \( r \) centered at \( z \). If \( \zeta - z \in B \), then we have \( M_\gamma(z)|f(\zeta)| \leq C\|f\|_{\gamma'} \). Therefore, choosing \( r > 0 \) so small that \( D_r(0) \subset B \) and using the Cauchy formula, we obtain

\[
|\partial_\gamma^\mu \partial_{\gamma'}^\nu f(x + iy)|M_\gamma(x + iy) = |\partial_\gamma^\mu f(z)|M_\gamma(z) \leq C(m + \nu)!\|f\|_{\gamma'} \leq C(m + \nu)!r^{-|\mu + \nu|}M_\gamma(z),
\]

where \( z = x + iy \) and \( \nu \) is the multi-index \((1, \ldots, 1)\). This inequality implies that \( \|f\|_{\gamma, m} \leq Cm!r^{-m}\|f\|_{\gamma'} \) for any \( m \in \mathbb{Z}_+ \) (\( \|f\|_{\gamma, m} \) is given by (2)). Hence \( \mathcal{H}(M) \) is continuously embedded in \( \tilde{\mathcal{H}}(M) \) and, therefore, \( \tilde{\mathcal{H}}(M) = \mathcal{H}(M) \).

We now prove the coincidence of \( \mathcal{H}(M) \) and \( \mathcal{H}_p(M) \). Let \( f \in \mathcal{H}(M) \) and \( \gamma \in \Gamma \). By (I), we can find \( \gamma' \in \Gamma \) such that \( M_\gamma \leq L_{\gamma\gamma'}M_{\gamma'} \), where \( L_{\gamma\gamma'}(z) \) is integrable and tends to zero as \( |z| \to \infty \). Then we obtain \( \|f\|_{\gamma, m} \leq A\|f\|_{\gamma'} \), where \( A = (\int |L_{\gamma\gamma'}(z)|^m \, d\lambda(z))^{1/p} < \infty \). Hence we have the continuous inclusion \( \mathcal{H}(M) \subset \mathcal{H}_p(M) \). Conversely, let \( f \in \mathcal{H}_p(M) \) and \( \gamma \in \Gamma \). Let \( \gamma', B, C, \) and \( r \) be as in the preceding paragraph. It follows from the Cauchy formula that \( f(z) = (\pi r^2)^{-k} \int_{D_r(z)} f(\zeta)\, d\lambda(\zeta) \) for every \( z \in \mathbb{C}^k \). Multiplying both parts of this relation by \( M_\gamma(z) \) and using (II) and the Hölder inequality, we obtain

\[
|f(z)|M_\gamma(z) \leq C \left( \frac{1}{(\pi r^2)^k} \int_{D_r(z)} |M_{\gamma'}(\zeta)|^p |f(\zeta)|^p \, d\lambda(\zeta) \right)^{1/p}.
\]

Extending integration in the right-hand side to the whole of \( \mathbb{C}^k \) and passing to the supremum in the left-hand side, we find that \( \|f\|_{\gamma} \leq C(\pi r^2)^{-k/p}\|f\|_{\gamma'}^\gamma \). Hence \( \mathcal{H}_p(M) \) is continuously embedded in \( \mathcal{H}(M) \) and, therefore, \( \mathcal{H}(M) = \mathcal{H}_p(M) \). \( \square \)
Lemma 14. Let $M = \{M_\gamma\}_{\gamma \in \Gamma}$ and $N = \{N_\omega\}_{\omega \in \Omega}$ be defining families of functions on $\mathbb{C}^{k_1}$ and $\mathbb{C}^{k_2}$ respectively and let $h \in \mathcal{H}(M \otimes N)$. Suppose $N$ satisfies conditions (I) and (II) of Lemma 4. Then $h(z, \cdot) \in \mathcal{H}(N)$ for every $z \in \mathbb{C}^{k_1}$ and the function $v(z) = \langle v, h(z, \cdot) \rangle$ belongs to $\mathcal{H}(M)$ for all $v \in \mathcal{H}'(N)$.

Proof. Let $v \in \mathcal{H}'(N)$. As shown in the proof of Lemma 10, $\mathcal{H}(N)$ is the subspace of $\mathcal{K}(N)$ consisting of those elements of $\mathcal{K}(N)$ that are entire analytic functions. By the Hahn–Banach theorem, $v$ has a continuous extension $\hat{v}$ to $\mathcal{K}(N)$. Then $h_v(z) = \langle \hat{v}, h(z, \cdot) \rangle$ and Lemma 10 implies that $h_v \in \mathcal{K}(M)$. Moreover, it follows from (10) that $h_v$ satisfies the Cauchy–Riemann equations and, therefore, is an entire analytic function. Hence $h_v \in \mathcal{H}(M)$ and the lemma is proved.

Theorem 15. Let $M$ and $N$ be defining families of functions on $\mathbb{C}^{k_1}$ and $\mathbb{C}^{k_2}$ respectively satisfying conditions (I) and (II) of Lemma 4. Let the bilinear mapping $\Phi: \mathcal{H}(M) \times \mathcal{H}(N) \to \mathcal{H}(M \otimes N)$ be defined by the relation $\Phi(f,g)(x,y) = f(x)g(y)$. Then $\Phi$ induces the topological isomorphism $\mathcal{H}(M) \otimes_\pi \mathcal{H}(N) \simeq \mathcal{H}(M \otimes N)$.

Proof. It is easy to check that $M \otimes N$ satisfies (I) and (II) provided that both $M$ and $N$ satisfy these conditions. Hence Lemma 6 implies that $\mathcal{H}(M)$, $\mathcal{H}(N)$, and $\mathcal{H}(M \otimes N)$ are nuclear Fréchet spaces (and, in particular, reflexive spaces). The required statement therefore follows from Theorem 3 because the fulfilment of (α) and (γ) is obvious and (β) is ensured by Lemma 14.

In conclusion, it is worth noting that the obtained results are also applicable to the treatment of the tensor products of regular inductive limits of Fréchet spaces, but this development is beyond the scope of the present work.

References

[1] I. M. Gelfand and G. E. Shilov, Generalized functions, Vol. 2, Academic Press, New York–London, 1967.
[2] I. M. Gelfand and N. Ya. Vilenkin, Generalized functions, Vol. 4, Academic Press, New York–London, 1968.
[3] A. Grothendieck, Produits tensoriels topologiques et espaces nucléares, Mem. Amer. Math. Soc. 16 (1955).
[4] A. Grothendieck, Sur les espaces (F) et (DF), Summa Bras. Math., 3 (1954) 57–123.
[5] H. Komatsu, Ultradistributions, II. The kernel theorem and ultradistributions with support in a submanifold, J. Fac. Sci. Univ. Tokio (Sec. 1A Math.), 24 (1977) 607–628.
[6] A. Pietsch, Nuclear locally convex spaces, Springer, Berlin–Heidelberg–New York, 1972.
[7] H. Schaefer, Topological Vector Spaces, Springer, Berlin–Heidelberg–New York, 1981.
[8] A. G. Smirnov, Towards Euclidean theory of infrared singular quantum fields, J. Math. Phys. 44 (2004) 2056–2070.
[9] M. A. Soloviev, An extension of distribution theory and of the Paley-Wiener-Schwartz theorem related to quantum gauge theory, Commun. Math. Phys. 184 (1997) 579–596.
[10] F. Treves, Topological vector spaces, distributions, and kernels, Academic Press, New York–London, 1967.