Neuron-like spiking dynamics in the asymmetrically-driven dissipative photonic Bose-Hubbard dimer

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We demonstrate neuron-like spiking dynamics in the asymmetrically driven dissipative photonic Bose-Hubbard dimmer model which describes two coupled nonlinear passive Kerr cavities. Spiking dynamics appear due to the excitability of the system. In this context, excitations in the phase space correspond to spikes in the temporal evolution of the field variables. In our case, excitability is mediated by the destruction of an oscillatory state in a global homoclinic bifurcation. In this type of excitability (known as type-I) the period of the oscillatory state diverges when approaching the bifurcation. Beyond this point, the system exhibits excitatory dynamics that are suitable for the application of perturbations. We have also characterized the effect that additive Gaussian noise has on the excitation, showing that the system undergoes a coherence resonance for a given value of the noise strength.

I. INTRODUCTION

Spiking-like dynamics were observed for the first time in biology in the context of neuronal and heart cells [1–3]. Spiking dynamics is related to the concept of excitability. The main features of an excitable system are related to how the system responds to external perturbations from a stable steady resting state. The nature of this response depends on the amplitude of the perturbation compared to a certain threshold. For perturbations below the threshold, the system decays very fast to the resting state [see Fig. 1(a)], while for perturbations above such threshold the system experiences a non-trivial trajectory (excursion) in the phase space before returning to the resting state [see Fig. 1(b)]. This large excursion is independent of the parameters of the system and corresponds to the so-called spike observed in the temporal trace. One essential feature of excitability is that once a spike is triggered, the system is unable to initiate another excursion during the recovery time (or refractory period). [see Figs. 1(c),(d)]. This dynamical process is what makes neurons able to perform computations and process external inputs [2–4].

In general, excitability requires the destruction of a permanent oscillatory state (i.e., a limit cycle) of finite-amplitude as a suitable parameter of the system is modified. Thus, an excitable excursion would follow the remnants of the cycle before returning to the resting state [2, 4].

Since its discovery, excitability has been observed in a wide range of natural systems, which include, chemical reactions [5, 6], biology [7–10], Josephson junctions [11], and laser physics [12], to only cite a few. In the latter, some examples include semiconductor lasers [13–16], lasers with saturable absorbers [17, 18], lasers with optical feedback [19] silicon-on-insulator microrings [20–23], and microdisk lasers [24]. The main goal of these studies is to propose an effective optical spiking neuron design on a chip.

Excitability may also arise in systems with some extended dimension, i.e. in the presence of spatial coupling. In this context, excitability waves can emerge in extended systems which are locally excitable [7, 25, 26]. However, the spatial coupling can be responsible for the coherent structures emerging from the spiking dynamics even in systems that are non locally excitable [27, 28], and for excitable-like behaviors which stem from front interactions [29]. Recent works have also focused on the characterization of travelling pulses in type-I excitable media [30, 31].

In this paper, we demonstrate, for the first time to our knowledge, the emergence of spiking dynamics in the context of two coupled passive cavities like those shown schematically in Figs. 1(e-g). In what follows, we refer to these systems as driven dissipative photonic Bose-Hubbard (DDBH) dimmer, in analogy with the open quantum boson system [33]. Furthermore, this model can also describe the dynamics between interacting bosons for open quantum systems [33]. In its simplest realization, two macroscopic phase coherent wave functions are coupled to form a Bose-Hubbard dimer, which can be also referred to as a bosonic Josephson junction [11, 34].

Here we show that the asymmetrically driven dissipative Bose-Hubbard dimer possesses all the ingredients required for the emergence of excitable behavior: a stable steady resting state and the destruction of a limit cycle occurring close to it. To do so, we perform a detailed bifurcation analysis which allows us to identify global bifurcations which are responsible for this kind of dynamics.

The paper is organized as follows. First, we will shortly introduce the mean-field model that we study (see Sec. II). After that, we present the bifurcation structure of the system and identify the most relevant bifurcations.
Figure 1. Panels (a)-(d) show how different kinds of inputs (left column) cause different kinds of responses (right column). An input below threshold (dashed line) induces a fast decaying to the resting state [see (a)]; (b) an input above threshold trigger a spike; (c) two impulses generated within the refractory time yield a single spike, however for time separations larger than the refractory time two spikes can be generated [see (d)]. Panel (e) shows a schematic representation of an asymmetrically-driven dissipative photonic Bose-Hubbard dimer, (f) corresponds to a schematic of polaritons in nanopillars and (g) to a photonic crystal cavity.

(see Sec. III). Section IV is devoted to the characterization of the excitable dynamics, and in Sec. V we show how noise can affect such type of behavior. Finally, in Sec. VI we draw some final conclusions.

II. ASYMMETRICALLY-DRIVEN DISSIPATIVE BOSE-HUBBARD MODEL

In the absence of dispersion, photonic dimers, like the one shown in Fig. 1(e), can be described by the mean-field model

\[
\frac{d\psi_1}{dt} = [-1 + i(|\psi_1|^2 - \Delta_1)]\psi_1 + iC\psi_2 + S
\]

\[
\frac{d\psi_2}{dt} = [-1 + i(|\psi_2|^2 - \Delta_2)]\psi_2 + iC\psi_1,
\]

where \(t\) is the time, \(\Delta_j\) the detuning from the closest (single cavity) resonance, \(C\) the coupling parameter between the two cavities, \(\psi_j\) \((j = 1, 2)\) the complex fields and \(S\) the driving power. These parameters and variables are dimensionless.

In a previous work, we have analyzed the temporal dynamics arising in this system [35, 36]. Applying bifurcation analysis and path-continuation techniques [37, 38], through the free software package AUTO-07p [39], we were able to classify the region of existence of the different dynamical regimes such as self-pulsing oscillations and chaos [36]. This analysis predicted that for a coupling constant value \(C \approx 1\) the continuous-wave state becomes unstable for \(\Delta_1 \gtrsim 2.1\), leading to stable permanent oscillations. As oscillatory dynamics is needed for excitability, in what follows we fix \(C = 1.1\) and \(\Delta_1 = 3.1\).

Figure 2. (a) Phase diagram in the \((\Delta_2, S)\) parameter space for \(\Delta_1 = 3.1\). The SN\(_j\) lines refer to saddle-node bifurcations, bistability occurs inside these regions. The Hopf bifurcations \(H_a, b\) (red lines) mark the boundaries of the self-pulsing region and the Hom bifurcation (black line) marks the boundaries of the area where there is no self-pulsing. (b) Normalized intracavity power \(|\psi_1|^2\) as a function of \(\Delta_2\) at a driving value of \(S = 3.0\). Red lines show the maximum and minimum oscillation amplitudes. Note that \(H_b\) is located at SN\(_4\). (c) \(L_2\)-norm as a function of \(\Delta_2\). The inset shows a close-up view of the region where excitability may occur (gray). The dashed vertical \((\Delta_2 = 2.785)\) line corresponds to the value where Fig. 4 is done.
III. PHASE DIAGRAM AND TEMPORAL DYNAMICS

Fig. 2(a) shows the dynamical regions in the \((\Delta_2, S)\)-phase diagram [35]. The red shadowed region bounded by the Hopf bifurcation \(H\) corresponds to self-pulsing oscillations. The blue and green regions show the coexistence of different homogeneous states which are the equilibrium points of Eq. (1).

To better understand the dynamical organization of the system it is useful to slice Fig. 2(a) by fixing either \(\Delta_2\) or \(S\). Figure 2(b) shows one such slice for \(S = 3.1\), where the intensity of the homogeneous state in the first cavity, \(|\psi_1|^2\), is plotted as a function of \(\Delta_2\). This curve corresponds to the nonlinear resonance of the cavity. For this value of \(\Delta_2\), \(H\) is intersected at two different points that we label \(H_{\text{a},\text{b}}\).

On the left of the resonance, the stable equilibria \(\psi_e^a\) and \(\psi_e^c\) coexist in-between \(\text{SN}_1\) and \(\text{SN}_2\), and are linked through the unstable state \(\psi_e^b\). Increasing \(\Delta_2\), \(\psi_e^c\) encounters \(H_a\), losing stability in favor of autonomous oscillations. In dynamical systems terms, this state is known as a limit cycle, and we label it \(\Gamma\). \(\Gamma\) arises from \(H_a\) supercritically [see red curve in Fig. 2(b)], and the amplitude of the oscillations increases with \(\Delta_2\). Eventually, \(\Gamma\) dies at a global homoclinic bifurcation \(\text{Hom}_a\). This bifurcation corresponds to the Hom black line plotted in Fig. 2(a). Thus, in the gray region bounded by this line, self-sustained oscillations are absent.

It is easier to see how the cycle touches the unstable equilibrium \(\psi_e^c\) by plotting the \(L2\)-norm

\[
|\psi| = \sqrt{T^{-1} \int_0^T (|\psi_1(t)|^2 + |\psi_2(t)|^2) dt},
\]

as a function of \(\Delta_2\), where \(T\) is the period of the oscillatory state. With this visualization, we can easily see how the cycle disappears at \(\text{Hom}\). Above this point, the system falls into the stable homogeneous equilibrium \(\psi_e^c\). From the right of the resonance, periodic oscillations also emerge from \(H_b\), but soon after that, they die at \(\text{Hom}\).

This homoclinic bifurcation involves the collision of a limit cycle and a saddle-node type equilibrium and is known as tame Shilnikov homoclinic bifurcation [36, 40]. These kind of bifurcations are characterized by the exponential divergence of the cycles period \(T\) when approaching \(\text{Hom}\). This behavior is depicted in Fig. 3(a), here the divergence (see red segment) follows the scaling law given by

\[
T \propto -\lambda_u^{-1} \ln(\Delta_2^2 - \Delta_2),
\]

where \(\lambda_u > 0\) is the (unstable) real eigenvalue associated with \(\psi_e^c\), and \(\Delta_2^c\) is the value where \(\text{Hom}\) occurs. The fit shown in inset of Fig. 3(a) leads a slope of 3.11 that is in close agreement with the theoretical value of \(1/\lambda_u = 3.12\). The evolution of \(T\) is illustrated in Figs. 3(i)-(iv) for \(T = 15\), 20, 30 and 40 (see crosses in Fig 3(a)).

IV. EXCITABILITY

The presence of homoclinic bifurcations [see Fig. 2(c)] may lead to excitability of type-I and spiking dynamics [4]. Here, the stable fixed point (resting state) is \(\psi_e^c\) while \(\psi_e^c\) plays the role of the perturbation threshold. The region where excitability may occur is delimited by the \(\text{Hom}\) and the \(\text{SN}_i\) bifurcations (see gray region in inset).

Figure 4 illustrates this dynamical behaviour. In Fig. 2(a) we show the \((|\psi_1|^2, |\psi_2|^2)\)-phase plane where the three equilibria \(\psi_e^{a,b,c}\) are depicted. The nature of these points is determined by their associated eigenvalues \(\lambda_i\), which are plotted in Fig. 5. Thus, \(\psi_e^c\) is a 1-spiral sink, and therefore stable, [see Fig. 5(i)], \(\psi_e^b\) is a spiral-3:1 saddle [see Fig. 5(ii)], and \(\psi_e^a\) is a 2:spiral-2 saddle [see Fig. 5(iii)] [41]. The two unstable fixed points are different types of saddle-focus equilibria [42].

In order to excite the system and trigger a spike, we apply to \(\psi_e^c\) the perturbation \(\eta r_e\), where \(r_e\) corresponds to the unstable eigenvector of \(\psi_e^c\). For \(\eta = 2\) [see Fig. 4(b)] the perturbation is not large enough to cross the threshold, and no excitable excursion takes place. This excursion corresponds to the orange trajectory shown in the \((|\psi_1|^2, |\psi_2|^2)\)-phase plane of Fig. 4(a). Increasing the amplitude of the perturbation to \(\eta = 3\) is enough to over-
The encountered excitability is of type-I as it is characterized by the divergence of the period of the oscillations occurring close to the bifurcation where is destroyed and for the presence of a real perturbation threshold. In contrast, type-II excitability (not found here) is mediated by Hopf bifurcations, and a well defined threshold does not exist. This is the type of excitability arising in the Fitzhugh-Nagumo models [4].

V. EFFECT OF NOISE AND COHERENT RESONANCE

The presence of noise in any real experimental setup is unavoidable, and it could have interesting implications in the context of excitability. The interplay between noise and excitability has been studied in different contexts [29, 43, 44] and may randomly trigger excitable excursions, even for subthreshold perturbations. The emergence of oscillations due to the presence of noise is known as coherence resonance or internal stochastic resonance [43–45], and has been analyzed in detail in the context of noise-driven excitable systems [43].

The dynamics of our system in the presence of stochastic terms is described by the Langevin type equations [46]

\[
\begin{align*}
\dot{\psi}_{1,2}(t) & = (\kappa_{1,2}(t) + iC\psi_{2,1}(t))dt + d\xi_{1,2}(t) \\
\kappa_{j}(t) & = [-1 + i(\psi_{j}(t)^2 - \Delta_j)]\psi_j(t) + S_j
\end{align*}
\]

where \(\xi_j = \xi_j(t), j = 1, 2\), and \(S_2 = 0\) since only one cavity is driven.

To illustrate the impact of noise on the system we consider an additive complex Gaussian white noise of amplitude \(\sqrt{D}\), satisfying

\[
\begin{align*}
\langle \xi_j(t) \rangle & = 0, \\
\langle \xi_j^*(t_a)\xi_j(t_b) \rangle & = \delta_{j,j'}\delta(t_a - t_b)D
\end{align*}
\]

(5)

To solve the stochastic system (4), we integrate it applying a Heun method [47].

The numerical simulation leads to the stochastic dynamics shown in Fig. 6(a), for \(\Delta_2 = 2.7112\) and a noise amplitude \(\sqrt{D} = 0.005\), where we plot the intensities of the field in both cavities \((|\psi_{1,2}|^2)\) as a function of time. As can be seen in Fig. 6(a), this noise level is able to randomly excite the system, triggering the spiking behavior. However, spikes are rarely triggered, and the intervals between spikes, that hereafter we refer to as \(\tau\), are very long and irregular. Increasing a bit \(\sqrt{D}\), i.e., for a moderate noise [see Fig. 6(b)] the spiking is more regular. This means that, in general, the interspike time interval \(\tau\) does not differ that much. For even larger values of \(\sqrt{D}\) [see Fig. 6(c)], the spiking is more frequent. Above a certain value of \(\sqrt{D}\), the frequency of spiking continues increasing, although the time between spikes is more irregular again (not shown here).

The variability of the inter-spiking time \(\tau\), and therefore the coherence resonance itself, can be quantified using the coefficient of variation

\[
R = \sqrt{\langle \Delta\tau^2 \rangle}/\langle \tau \rangle,
\]

(6)
where $\langle \tau \rangle$ is the mean and $\langle \Delta \tau^2 \rangle$ is its variance [43–45]. Periodic spiking is characterized by $R = 0$, and a Poisson process (i.e., random spiking) has $R = 1$. The value of $R$ goes through a minimum for intermediate values of noise, indicating that spiking takes place on a more regular basis. The existence of a minimum indicates the occurrence of the coherence resonance [43]. Figure 7 shows the coefficient of variation as a function of the noise intensity $\sqrt{D}$ for different values of $\Delta_2$. The red curve ($\Delta_2 = 2.7112$) corresponds to the time traces depicted in Fig. 6, the spiking dynamics corresponding to its minimum is shown in Fig. 6(c). For other values of $\Delta_2$, i.e., for different separations from the Homoclinic bifurcation, the coherence resonance indicator behaves similarly. We have verified that when the noise is directly added to the detuning through the term $i\psi_j(t)\xi_j(t)$ the coherence resonance is similar to the one showed in Fig. 7.

This phenomenon differs from the stochastic resonance, where the characteristic time scale comes from external driving. Here, the noise activates a hidden characteristic time scale (time need to excite the system plus the refractory time) of the system due to its excitable nature.

VI. CONCLUSIONS

We have characterized the neuron-like spiking behavior emerging in asymmetrically-driven dissipative Bose-Hubbard model describing photonics dimers in the form of two coupled nonlinear Kerr passive cavities [see Sec. II]. Spiking dynamics appear because the system is excitable, i.e., an external perturbation may cause a different response on the system depending on the amplitude of the perturbation. In this context, a spike corresponds to a long transient response of the system once a certain threshold is overpassed. This long response is commonly known as an excitatory excursion. Excitable behavior is related to the appearance/destruction of a limit cycle (i.e., periodic oscillations) when varying a suitable parameter.

Applying well-known results of dynamical systems and bifurcation theory we have shown the emergence of spiking dynamics in our system. In our case, the emergence of excitability is related to the presence of a homoclinic bifurcation where the oscillatory state is de-
stroyed (Sec. III). Excitability mediated by homoclinic bifurcation is known as type-I and is characterized by the presence of a real threshold for the perturbations and by the divergence of the limit cycle period involved in the process as approaching its destruction [4]. Excitability and spiking behavior are described in Sec. IV, where spike trajectory in a reduced phase space is compared with a perturbation below a threshold. Finally, we have also studied the implications that additive Gaussian noise may have on the spiking dynamics (Sec. V). We show that a weak noise is able to excite the system triggering spikes randomly and irregularly. As the noise increases, the spike frequency increases and the interspike time reduces, leading to the appearance of regular oscillations for an optimal noise value. This process is known as coherence resonance [45].

This model predicts the presence of excitability in different platforms of photonic dimers that can find applications as integrated networks of nonlinear optical cells [48]. Also, spiking and neuron-like behaviors can find applications in brain-inspired softwares and hardwares. [49, 50]

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