Particle-hopping Models of Vehicular Traffic: Distributions of Distance Headways and Distance Between Jams

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Abstract
We calculate the distribution of the distance headways (i.e., the instantaneous gap between successive vehicles) as well as the distribution of instantaneous distance between successive jams in the Nagel-Schreckenberg (NS) model of vehicular traffic. When the maximum allowed speed, $V_{\text{max}}$, of the vehicles is larger than unity, over an intermediate range of densities of vehicles, our Monte Carlo (MC) data for the distance headway distribution exhibit two peaks, which indicate the coexistence of “free-flowing” traffic and traffic jams. Our analytical arguments clearly rule out the possibility of occurrence of more than one peak in the distribution of distance headways in the NS model when $V_{\text{max}} = 1$ as well as in the asymmetric simple exclusion process. Modifying and extending an earlier analytical approach for the NS model with $V_{\text{max}} = 1$, and introducing a novel transfer matrix technique, we also calculate the exact analytical expression for the distribution of distance between the jams in this model; the corresponding distributions for $V_{\text{max}} > 1$ have been computed numerically through MC simulation.

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1. INTRODUCTION:

The motion of each vehicle in traffic is influenced by others around it; the statistics and dynamics of traffic flow [1-5] crucially depend on these "interactions" between the vehicles. However, the dynamics of vehicles is more complicated than that of interacting particles [6] because of the dependence on human behaviour; usually, the driver has the aim of reaching his destination safely in the shortest time and can also learn from experience. Nevertheless, significant progress has been made in modelling traffic by extending the concepts and techniques of statistical mechanics [4,5] by capturing some of the common and essential features of the driving habits of the individual drivers.

The theoretical models of traffic can be broadly divided into two classes: (i) continuum models, and (ii) car-following models; the former is the analogue of the "hydrodynamic" models of fluids while the latter is the analogue of the "microscopic" models in statistical mechanics. The older version of the car-following theory [7] was formulated mathematically using differential equations for describing the time evolution of the positions and speeds of the individual vehicles in time, which was treated as a continuous real variable. In the modern versions, namely, the particle-hopping models, not only the position and speed of the vehicles but time also is assumed to be discrete. Some of these particle-hopping models [8] may be regarded as stochastic cellular automata [9].

The distance headway is one of the most important characteristics of vehicular traffic. It is defined as the distance from a selected point on the lead vehicle to the same point on the following vehicle. Usually, the front edges or bumpers are selected [3]. One of the aims of this paper is to calculate the distributions of the distance headways in the steady state, using a class of particle-hopping models, which are known to capture some of the essential features of traffic flow on highways. We also report analytical as well as numerical results on the distribution of instantaneous distance between successive vehicular jams in the steady state of these models.

We define the models and the characteristic quantities of interest in section 2. We compute the distance headway distribution in section 3 and the distribution of distance between jams in section 4. We summarize the results and conclusion in section 5.

2. THE MODELS AND CHARACTERISTIC QUANTITIES OF INTEREST:

In all the "microscopic" models under our consideration, a lane is represented by a one-dimensional lattice of $L$ sites (i.e., a chain of $L$ equispaced points). Each of the lattice sites can be either empty or occupied by a "vehicle". There is an "exclusion principle", namely, that no two vehicles can occupy the same lattice site simultaneously.

The ratio of the number of occupied lattice sites to the total number of lattice sites is defined as the density of the vehicles on the lane. The density is time-dependent for a finite lattice with open boundaries where vehicles enter from one end and leave from the other. In this paper we shall consider only
idealized single-lane highways with periodic boundary conditions. Therefore, in this geometry of the highway, the time-independent density \( c \) of the vehicles is \( N/L \) where \( N(\leq L) \) is the total number of vehicles in the traffic.

### 2.1. The Asymmetric Simple Exclusion Process:

One of the simplest dynamical models of interacting particle systems is the so-called asymmetric simple exclusion process (ASEP) which may be regarded as a caricature of vehicular traffic. The rules for updating the positions of the particles in the ASEP are as follows: one particle is picked at random and moved forward by one lattice site if the new site is empty. In this random sequential update, it is the random picking that introduces stochasticity (noise) into the model. Although speeds of the particles do not explicitly enter into the rules, the effective speed of a particle in the ASEP can take only two values, namely, 0 and \( V_{\text{max}} = 1 \).

### 2.2. The Nagel-Schreckenberg Model:

We now recall the "rules" describing the dynamics of vehicles in the Nagel-Schreckenberg (NS) model [8] where the speed \( V \) of each vehicle can take one of the \( V_{\text{max}} + 1 \) allowed integer values \( V = 0, 1, \ldots, V_{\text{max}} \). At each discrete time step \( t \to t+1 \), the arrangement of \( N \) vehicles is updated in parallel according to the following rules. Suppose, \( V_n \) is the speed of the \( n \)-th vehicle at time \( t \).

- **Step 1: Acceleration.** If, \( V_n < V_{\text{max}} \), the speed of the \( n \)-th vehicle is increased by one, i.e., \( V_n \to V_n + 1 \).
- **Step 2: Deceleration (due to other vehicles).** If \( d \) is the gap in between the \( n \)-th vehicle and the vehicle in front of it, and if \( d \leq V_n \), the speed of the \( n \)-th vehicle is reduced to \( d - 1 \), i.e., \( V_n \to d - 1 \).
- **Step 3: Randomization.** If \( V_n > 0 \), the speed of the \( n \)-th vehicle is decreased randomly by unity (i.e., \( V_n \to V_n - 1 \)) with probability \( p \) (\( 0 \leq p \leq 1 \)); \( p \), the random deceleration probability, is identical for all the vehicles and does not change during the updating.
- **Step 4: Vehicle movement.** Each vehicle is moved forward so that \( X_n \to X_n + V_n \) where \( X_n \) denotes the position of the \( n \)-th vehicle at time \( t \).

In the literature on traffic flow, the relation between the average flux of the vehicles and their density is usually referred to as the "fundamental relation". The qualitative features of the fundamental relation of the NS model are very similar to those of the empirically derived fundamental relation for real traffic. \( c_m \), the density corresponding to the maximum flux, is usually called the optimum density.

### 2.3. Definitions of Distance Headway and Distance Between Jams:

The number of empty lattice sites in front of a vehicle is taken to be the measure of the corresponding distance headway. If the instantaneous speed of a vehicle is zero, then it is part of a traffic jam. The number of lattice sites in between two such "jammed" vehicles is taken as the measure of the distance between the two corresponding instantaneous jams to which they belong, provided none of the sites between these two sites is occupied by any other "jammed"
vehicle (the sites in between may, of course, be either empty or occupied by vehicles with non-zero instantaneous speed).

We have calculated both the distributions for the NS model analytically when \( V_{\text{max}} = 1 \). However, for the NS model with \( V_{\text{max}} > 1 \) our results in this paper have been obtained by computer simulation as analytical calculations become too complicated to carry through.

3. RESULTS ON DISTANCE HEADWAY DISTRIBUTION:

3.1. Distance Headway Distribution for \( V_{\text{max}} > 1 \):

NS claimed that \( V_{\text{max}} = 5 \) is a very realistic choice and, therefore, most of the subsequent works on the NS model have been carried out for this particular value of \( V_{\text{max}} \). Besides, the most common choice for \( p \) is 0.5. Therefore, we begin presentation of our results by plotting the distribution of distance headways in the NS model in fig.1(a) for \( V_{\text{max}} = 5 \), \( p = 0.5 \). For small values of the density \( c \), the gap distribution exhibits a single peak; interestingly, the gap distribution vanishes for gap sizes smaller than the corresponding value of \( V_{\text{max}} \). This implies that, at low densities, the vehicles arrange themselves in such a manner as to maintain a gap, which is at least as large as \( V_{\text{max}} \), so that free flow of the vehicles can take place. With the increase of the density of the vehicles the distribution still exhibits a single maximum, although the most probable gap becomes smaller, provided the density is sufficiently low. However, as \( c \) approaches \( c_m \), a second peak appears at a gap of unit lattice spacing; this new peak is a consequence of the congestion of the vehicles. With the further increase of density the new peak becomes higher at the cost of that at the larger gap and, eventually, as \( c \to 1 \), the distribution approaches a \( \delta \)-function at unity.

Are the characteristics of the distribution in fig.1(a) merely special feature of the chosen value \( V_{\text{max}} = 5 \) or generic features of the NS model irrespective of the specific value of \( V_{\text{max}} \)? In order to answer this question let us plot the distribution of distance headways in the NS model for another value of \( V_{\text{max}} \), say, \( V_{\text{max}} = 3 \) in fig.1(b) where \( p \) is chosen to be 0.5 as in fig.1(a). The characteristics of the distribution and the trend of its variation with density are very similar to those in fig.1(a).

Now we shall show that the occurrence of two peaks in the distribution of distance headways depends on \( p \). For example, for the same \( p \) as in fig.1, namely, \( p = 0.5 \), the distribution does not show the occurrence of two peaks for any value of the density \( c \) when \( V_{\text{max}} \) is 2 (see fig.2(a)). Nevertheless, the two-peak structure of the distribution over a regime of density is observed even for \( V_{\text{max}} = 2 \) provided \( p \) is large enough (see fig.2(b)).

3.2. Distance Headway Distribution for \( V_{\text{max}} = 1 \):

Does the distance headway distribution in the NS model exhibit two peaks also for \( V_{\text{max}} = 1 \)? In order to establish the absence of any such two-peak structure in the gap distributions, irrespective of the density of the vehicles, in the NS model for \( V_{\text{max}} = 1 \) and in the ASEP we next derive the corresponding exact analytical expressions.

For the convenience of their analytical calculations, Schreckenberg et al.[10]
changed the order of the steps in the update rules in such a manner that it does not influence the steady-state properties of the model. They assumed the sequence of steps $2 - 3 - 4 - 1$, instead of $1 - 2 - 3 - 4$; the advantage is that there is no vehicle with $V = 0$ immediately after the acceleration step and this reduces the number of possible states of a site by one. For example, if $V_{max} = 1$, $V$ can take only the value 1 at the end of a sequence of steps $2 - 3 - 4 - 1$ and Schreckenberg et al.\cite{10} could use a binary site variable $\sigma$ to describe the state of each site; $\sigma = 0$ represented an empty site and $\sigma = 1$ represented a site occupied by a vehicle (with speed $V = 1$). In this notation, the probability of finding an $n$ - cluster in the configuration $(\sigma_1, \cdots, \sigma_n)$ in the steady state can be denoted by the symbol $P_n(\sigma_1, \cdots, \sigma_n)$.

Although the sequence of steps $2 - 3 - 4 - 1$ and the description of the configurations in terms of the binary variable $\sigma$ can be used also for our analytical calculation of the distance headway distribution in the NS model with $V_{max} = 1$ these, however, would be inadequate for our analytical calculation of the distribution of distance between jams in the same model, as we shall explain in the next section. Therefore, for the convenience of extension of the approach in the next section we now slightly modify the notation. We use the site variable $s$, instead of $\sigma$, to denote the state of each site; $s = -1$ represents an empty site whereas $s = 1$ represents a site occupied by a vehicle (with speed $V = 1$). But, for the calculation of the distance headway distribution, we adopt the same sequence as in ref.\cite{10}, i.e., $2 - 3 - 4 - 1$. Therefore, in this section, the site variable $s$ can take either of the only two allowed values, namely, +1 and −1.

Suppose, $P(j)$ is the probability of finding a gap of size $j$ in front of a vehicle. Stated more precisely, $P(j)$ is the conditional probability of finding $j$ empty sites in front of a site which is given to be occupied by a vehicle, i.e.,

\[
P(j) = P(\underbrace{1 - 1 - 1 \cdots - 1}_{j})
\]

where $P$ is the probability of occurrence of the argument configuration where the underlined part is fixed.

In the mean field approximation we neglect all correlations between the vehicles. Therefore, in this approximation, the probability on the right hand side of equation (1) can be factorized and, hence, the gap distribution is

\[
P_{mfa}(j) = c(1 - c)^j \quad \text{for } j = 0, 1, 2, ...
\]

Mean-field approximation is known to be exact for the ASEP \cite{11-13} and, therefore, the right hand side of equation (1) is the exact analytical expression for gap distribution between the particles in the ASEP; note that $P_{mfa}(j)$ does not exhibit two peaks simultaneously at any value of $c$ (see fig.3).

Mean field results can be improved by taking into account the correlation between the vehicles. In general, this is achieved by dividing the lattice into clusters of length $n$ which amounts to taking the effect of correlation upto $n$
nearest neighbours [10]. In the $n$-cluster approximation, we divide the lattice into segments or clusters of length $n$, such that two neighbouring clusters have $(n-1)$ sites in common. For the case of $V_{max} = 1$ one expects strong correlation only between vehicles at nearest-neighbour lattice sites and, therefore, from now onwards, we carry out all our analytical calculations for this case only in the 2-cluster approximation.

In the 2-cluster approximation [10], the probability of finding a gap of $j$ sites, can now be written as [14]

$$P_{2c}(j) = P_2(+1|+1) \quad \text{for} \quad j = 0,$$

$$P_{2c}(j) = P_2(+1|-1)P_2(-1|+1), \quad \text{for} \quad j = 1,$$

$$P_{2c}(j) = P_2(+1|-1)(P_2(-1|-1))^{j-1}P_2(-1|+1), \quad \text{for} \quad j \geq 1,$$

where the conditional probabilities $P_2(s_{i-1}|s_i)$ and $P_2(s_{i+1}|s_{i+2})$ are given, in terms of the 2-cluster probabilities $P_2(s_{i-1}, s_i)$ and $P_2(s_{i+1}, s_{i+2})$, by the relations

$$P_2(s_{i-1}|s_i) = \frac{P_2(s_{i-1}, s_i)}{P_2(s_{i-1}, s_i) + P_2(1, s_i)}$$

and

$$P_2(s_{i+1}|s_{i+2}) = \frac{P_2(s_{i+1}, s_{i+2})}{P_2(s_{i+1}, -1) + P_2(s_{i+1}, 1)}$$

Schreckenberg et al. [10] pointed out that only one of the four 2-cluster probabilities $P_2(+1,+1), P_2(+1,-1), P_2(-1,+1), P_2(-1,-1)$ is independent. Solving the master equation for $P_2(+1,-1)$ they found that

$$P_2(+1,-1) = \left( \frac{1}{2q} \right) \left[ 1 - \{1 - 4qc(1 - c)\}^{1/2} \right] = y$$

where $q = 1 - p$. The three other 2-cluster probabilities can be obtained from the equations

$$P_2(+1,+1) = c - y$$

$$P_2(-1,+1) = y$$

$$P_2(-1,-1) = 1 - c - y$$

Using the expressions (6)-(11) in the equations (3)-(5) we get [14]

$$P_{2c}(j) = 1 - (y/c) \quad \text{for} \quad j = 0,$$

$$= \left( \frac{y^j}{c^{j-1}} \right) \left( 1 - \frac{y}{(1-c)} \right)^{j-1} \quad \text{for} \quad j = 1, 2, 3, ...$$

where $y = P_2(+1,-1)$. The distribution (12) is the exact analytical expression for the distance headway distribution in the NS model when $V_{max} = 1$ and it also does not exhibit two peaks simultaneously at any value of $c$ (see fig.4). The differences between the exact gap distributions for the ASEP and the NS model
with $V_{\text{max}} = 1$ arise from the fact that parallel updating is used in the NS model in contrast to the random sequential updating in the ASEP [8,10].

The two-peak structure of the gap distribution has been interpreted as a signature of "two-phase coexistence" in a recent study of a continuum model [15]. We believe that an identical interpretation can be given also for the occurrence of two-peak structure in the gap distribution of the discrete NS model; the two coexisting phases being the "free-flowing phase" and a "jammed" phase. In the context of particle-hopping models, the two-peak structure of the gap distribution was first discovered during a computer simulation of a two-lane model [16] but no analytical calculations were attempted. Our comparison here with the exact analytical expressions for $V_{\text{max}} = 1$ clearly establishes qualitative differences between the distributions for $V_{\text{max}} = 1$ and those for $V_{\text{max}} > 1$; there is only one peak in the gap distribution for all densities when $V_{\text{max}} = 1$ whereas for $V_{\text{max}} > 1$ two peaks appear in an intermediate regime of density of vehicles.

In this paper we have indirectly inferred the coexistence of the two dynamical phases in the NS model with $V_{\text{max}} > 1$ over a $p$-dependent range of densities by computing the distance headway distribution which is, traditionally, one of the important characteristics of vehicular traffic. Very recently, Lübeck et al.[17] have proposed a new method, based on the "local density distribution", in an attempt to characterize the free-flowing and jammed phases quantitatively and to locate the region of coexistence of these phases on a "phase diagram" of the NS model.

4. RESULTS ON THE DISTRIBUTION OF DISTANCE BETWEEN JAMS:

4.1. Distribution of Distances between Jams for $V_{\text{max}} = 1$:

According to our definition, each of the jammed vehicles has a speed $V = 0$. Therefore, for the calculation of the distribution of distance between the jams the site variable $s$ must be allowed to take one of the three allowed values, namely, $s = -1$ if the site is empty, $s = 1$ if the site is occupied a vehicle whose instantaneous speed is $V = 1$ and $s = 0$ if the site is occupied by a vehicle which is instantaneously jammed, i.e., has a speed is $V = 0$. Since $s$ can now take three values, namely, $1, 0, -1$, we have a total of nine 2-cluster probabilities. Therefore, in the case of the sequence $2 - 3 - 4 - 1$ of the steps in the NS model, we have the equations

$$P_2(0, 1) = P_2(0, 0) = P_2(0, -1) = P_2(1, 0) = P_2(-1, 0) = 0. \quad (13)$$

in addition to the equations (8)-(11).

Since there is no vehicle with speed $V = 0$ after the step (1) of the NS model, we cannot use the sequence $2 - 3 - 4 - 1$ for the calculation of the distribution of distance between jams. Instead, we use the sequence $4 - 1 - 2 - 3$, i.e., we calculate the required two cluster probabilities after the step 3 of the NS model [18]. In order avoid any confusion between the two-cluster probabilities $P_2(s_i, s_{i+1})$ associated with the sequence $2 - 3 - 4 - 1$, which we considered in
section 3, and those associated with the sequence 4 − 1 − 2 − 3 we denote the latter by the symbol $p_2(s_i, s_{i+1})$; we shall use the expressions for $P_2(s, s')$ to calculate those for $p_2(s, s')$.

In the 2-cluster approximation, the probability of a distance $k$ between two successive jams is given by

$$P_{2c}(k) = \sum_{s_i=\pm 1} p_2(0|s_1) p_2(s_1|s_2) \cdots p_2(s_k|0) \text{ for } k \geq 1$$

and

$$P_{2c}(k) = p_2(0|0) \text{ for } k = 0. \quad (15)$$

What is the physical meaning of the probability of gap $k = 0$ between two jams? Shouldn’t two jammed vehicles with vanishing gap be regarded part of the same jam? Indeed, the equation (15) clearly shows that $P_{2c}(k = 0)$ is the total fraction of the vehicles which are simultaneously in the jammed state.

The required 2-cluster probabilities $p_2(a, b)$ can be calculated from the expressions for $P_2(c, d)$, as expressed by the following equations [18]

$$p_2(-1, -1) = P_2(-1, -1) \quad (16)$$
$$p_2(-1, 0) = pP_3(-1, +1| -1) + P_3(-1, +1| +1) \quad (17)$$
$$p_2(-1, +1) = qP_3(-1, +1| -1) \quad (18)$$
$$p_2(0, -1) = pP_2(+1, -1) \quad (19)$$
$$p_2(0, 0) = pP_3(+1, +1| -1) + P_3(+1, +1| +1) \quad (20)$$
$$p_2(0, +1) = qP_3(+1, +1| -1) \quad (21)$$
$$p_2(+1, -1) = qP_2(+1, -1) \quad (22)$$
$$p_2(+1, 0) = 0 \quad (23)$$
$$p_2(+1, +1) = 0 \quad (24)$$

The new conditional probabilities defined by

$$p_2(s_{i-1}|s_i) = \frac{p_2(s_{i-1}, s_i)}{p_2(-1, s_i) + p_2(0, s_i) + p_2(1, s_i)} \quad (25)$$

and

$$p_2(s_{i+1}|s_{i+2}) = \frac{p_2(s_{i+1}, s_{i+2})}{p_2(s_{i+1}, -1) + p_2(s_{i+1}, 0) + p_2(s_{i+1}, 1)} \quad (26)$$

can be obtained from the equations (16)-(24) using 2-cluster approximations for the 3-cluster probabilities, i.e.,

$$P_3(s, s'|s'') = P_2(s, s') P_2(s'|s'') = P_2(s, s') \frac{P_2(s', s'')}{P_2(s', -1) + P_2(s', +1)} \quad (27)$$
and the expressions (8)-(11), (13), for the 2-cluster probabilities \( P_2(s, s') \). Thus, we get

\[
p_2(0|0) = \frac{py}{c - qy} \quad (28)
\]

\[
p_2(0|0) = 1 - \frac{y}{c} \quad (29)
\]

\[
p_2(0 + 1) = \frac{qy(c - y)}{c(c - qy)} \quad (30)
\]

\[
p_2(-1|1) = \frac{1 - c - y}{1 - c} \quad (31)
\]

\[
p_2(-1|0) = \frac{yc - qy}{c(1 - c)} \quad (32)
\]

\[
p_2(-1) + 1 = \frac{qy^2}{c(1 - c)} \quad (33)
\]

\[
p_2(\pm 1|0) = 1 \quad (34)
\]

\[
p_2(\pm 1|0) = 0 \quad (35)
\]

\[
p_2(\pm 1|0) = 0 \quad (36)
\]

We shall use these expressions for the conditional probabilities in equations (14) and (15) to derive the distribution of the distance between the jams.

We now define the \( 2 \times 2 \) transfer matrix \( T \) whose elements \( T[\alpha, \beta] \) \((\alpha, \beta = 1, 2)\) are given by [18]

\[
T[1, 1] = p_2(\overline{1}|1) = 0 \quad (37)
\]

\[
T[1, 2] = p_2(\overline{1}|1) = 1 \quad (38)
\]

\[
T[2, 1] = p_2(-\overline{1}|1) = \frac{qy^2}{c(1 - c)} \quad (39)
\]

\[
T[2, 2] = p_2(-\overline{1}|1) = \frac{1 - c - y}{1 - c} \quad (40)
\]

Hence, within the 2-cluster approximation,

\[
\tilde{P}_{2c}(k) = p_2(\overline{1}|1)T^{k-1}[1, 1]p_2(\overline{1}|0) + p_2(\overline{1}|1)T^{k-1}[2, 1]p_2(\overline{1}|0) + p_2(\overline{1}|1)T^{k-1}[1, 2]p_2(-\overline{1}|0) + p_2(\overline{1}|1)T^{k-1}[2, 2]p_2(-\overline{1}|0) \quad (41)
\]

where \( T^{k-1}[\alpha, \beta] \) refers to the \([\alpha, \beta] \) element of the matrix \( T^{k-1} \). Now

\[
T^{k-1}[1, 1] = \frac{\lambda_1 \lambda_2^{k-1} - \lambda_2 \lambda_1^{k-1}}{\lambda_1 - \lambda_2} \quad (42)
\]

\[
T^{k-1}[1, 2] = \frac{\lambda_1^{k-1} - \lambda_2^{k-1}}{\lambda_1 - \lambda_2} \quad (43)
\]
\[ T^{k-1}[2,1] = \frac{\lambda_1 \lambda_2^k - \lambda_2 \lambda_1^k}{\lambda_1 - \lambda_2} \]  
\[ T^{k-1}[2,2] = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2} \]

where

\[
\lambda_1 = \left( 1 - \frac{y}{1-c} \right) + \sqrt{\left( 1 - \frac{y}{1-c} \right)^2 + 4 \left( \frac{y}{c(1-c)} - 1 \right)} / 2 \] 
(46)

and

\[
\lambda_2 = \left( 1 - \frac{y}{1-c} \right) - \sqrt{\left( 1 - \frac{y}{1-c} \right)^2 + 4 \left( \frac{y}{c(1-c)} - 1 \right)} / 2 \] 
(47)

are the two eigenvalues of the transfer matrix \( T \). Hence, finally, we get [18]

\[
\tilde{P}_{2c}(k) = \frac{qy(c-y)\left( \lambda_1^{k-1} - \lambda_2^{k-1} \right)}{c(1-c)} \frac{y(c-qq)}{\lambda_1 - \lambda_2} + \frac{py\left( \lambda_1^k - \lambda_2^k \right)}{(c-qq)\left( \lambda_1 - \lambda_2 \right)} \frac{y(c-qq)}{c(1-c)} \quad \text{for} \quad k \geq 1 \]
(48)

i.e.,

\[
\tilde{P}_{2c}(k) = \frac{[py^2c(\lambda_1^k - \lambda_2^k)] + [qq^2(c-y)(\lambda_1^{k-1} - \lambda_2^{k-1})]}{c^2(1-c)(\lambda_1 - \lambda_2)} \quad \text{for} \quad k \geq 1 \]
(49)

and

\[
\tilde{P}_{2c}(k) = 1 - \frac{y}{c} \quad \text{for} \quad k = 0. \]
(50)

The distribution (49)-(50) is in excellent agreement with the corresponding results of MC simulation (see figs.5(a) and (b)).

**4.2. Distribution of Distance between Jams for \( V_{\text{max}} > 1 \):**

We have computed the distribution of the distance between jams numerically through MC simulation for several values of \( V_{\text{max}} > 1 \); some typical distributions for \( V_{\text{max}} = 5 \) are shown in fig.6. The lower is the density the more significant is the difference between the distributions for \( V_{\text{max}} = 1 \) (fig.5) and for \( V_{\text{max}} = 5 \) (fig.6). This, of course, is a consequence of the fact that the higher is the density the fewer are the vehicles which can attain speeds larger than unity.

**5. SUMMARY AND CONCLUSION:**

In this paper we have calculated the distance headway distribution and the distribution of the distance between the jams in the NS model of vehicular traffic on single-lane highways. We have derived analytical expressions for these distributions in the special case \( V_{\text{max}} = 1 \); but obtained only numerical results for \( V_{\text{max}} > 1 \) by carrying out computer simulation. Our results indicate the coexistence of "free- flowing" and jammed traffic over a \( p \)-dependent range of densities provided \( V_{\text{max}} > 1 \).
In the analytical approach followed in this paper one stores the information on the state of occupation of each site; therefore, this formalism is sometimes called a "site-oriented" approach. Very recently a "car-oriented" formalism of analytical calculations for the NS model has been developed. Since this new "car-oriented" approach takes into account longer range correlations than in the "site-oriented" approach, it would be interesting to recalculate analytically the distance headway distribution and the distribution of distance between the jams in the NS model using this new formalism.

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References
[1] D.L. Gerlough and M.J. Huber, Traffic Flow Theory (National Research Council, Washington, D.C. 1975)
[2] W. Leutzbach, Introduction to the Theory of Traffic Flow (Springer, Berlin, 1988)
[3] A.D. May, Traffic Flow Fundamentals (Prentice-Hall, Englewood Cliffs, 1990)
[4] I. Prigogine and R. Herman, Kinetic Theory of Vehicular Traffic (Elsevier, Amsterdam, 1971)
[5] D.E. Wolf, M. Schreckenberg and A. Bachel eds.) Traffic and Granular Flow (World Scientific, Singapore, 1996)
[6] H. Spohn, Large scale dynamics of interacting particles (Springer, Berlin 1991).
[7] D.C. Gazis, Science, 157, 273 (1967) and references therein.
[8] K. Nagel, in Physics Computing ’92, R.A. de Groot and J. Nadrchal, eds. (World Scientific, Singapore, 1993); K. Nagel and M. Schreckenberg, J. Physique I, 2, 2221 (1992).
[9] S. Wolfram, Theory and Applications of Cellular Automata (World Scientific, Singapore, 1986)
[10] M. Schreckenberg, A. Schadschneider, K. Nagel and N. Ito, Phys. Rev. E, 51, 2939 (1995).
[11] R.B. Stinchcombe, unpublished.
[12] B. Derrida, M.R. Evans, V. Hakim and V. Pasquier, J. Phys. A, 26, 1493 (1993)
[13] R.B. Stinchcombe and G.M. Schutz, Phys. Rev. Lett. 75, 140 (1995)
[14] A. Majumdar, Masters Project (Theory) Report, IIT, Kanpur, 1997 (unpublished).
[15] S. Krauss, P. Wagner and C. Gawron, Phys. Rev. E 54, 3707 (1996); see also K. Nagel, Phys. Rev. E 53, 4655 (1996).
[16] D. Chowdhury, D.E. Wolf and M. Schreckenberg, Physica A, 235, 417 (1997)
[17] S. Lübeck, M. Schreckenberg and K.D. Usadel, Uni-Duisburg Preprint (1997)
[18] S. Sinha, Masters Project (Theory) Report, IIT, Kanpur, 1997, (unpublished).
[19] A. Schadschneider and M. Schreckenberg, J.Phys.A 30, L69 (1997).
Figure Captions:

**Fig.1:** Computer simulation results on the probability distribution of the gaps in front of the vehicles in the NS model when (a) $V_{\text{max}} = 5$ and (b) $V_{\text{max}} = 3$; $p = 0.5$ in both (a) and (b). In (a) the symbols $+$, $\Box$, $\diamond$, $\times$ correspond to $c = 0.05, 0.08, 0.10$ and $0.20$, respectively. In (b) the symbols $+$, $\diamond$, $Box$ and $\times$ correspond to $c = 0.10, 0.15, 0.18$ and $0.30$, respectively. The lines merely connect the successive data points and serve as a guide to the eye.

**Fig.2:** Computer simulation results on the probability distribution of the gaps in front of the vehicles in the NS model with $V_{\text{max}} = 2$ when (a) $p = 0.5$ and (b) $p = 0.9$. In (a) the symbols $\diamond$, $\Box$, $\times$, $\triangle$ correspond to the densities $c = 0.20, 0.25, 0.30, 0.40$ and $0.60$, respectively. In (b) the symbols $+$, $\Box$, $\diamond$ and $\times$ correspond to $c = 0.04, 0.06, 0.07$ and $0.09$, respectively. The lines merely connect the data points and serve as a guide to the eye.

**Fig.3:** Mean-field results on the probability distribution of the gaps in front of the vehicles in the NS model with $V_{\text{max}} = 1$. The symbols $+$, $\Box$, $\diamond$, $\times$ correspond to $c = 0.1, 0.2, 0.4$ and $0.6$, respectively.

**Fig.4:** Results of 2-cluster theory of the probability distribution of the gaps in front of the vehicles in the NS model with $V_{\text{max}} = 1$. The densities (and symbols) are same as in fig.3 and $p = 0.5$.

**Fig.5(a):** Results of 2-cluster theory of the probability distribution of the instantaneous distance between successive jams in the NS model with $V_{\text{max}} = 1$ when $p = 0.5$. The analytical results are plotted as dots connected by lines (solid line, dashed line and dashed-dotted line for $c = 0.1, 0.2$ and $0.4$, respectively) whereas the results of computer simulation are denoted by the discrete symbols ($\diamond$, $+$, $\Box$ for $c = 0.1, 0.2, 0.4$, respectively).

**Fig.5(b):** Same as in fig.5(a), except that the symbols $\diamond$, $+$ and $\Box$ correspond to $c = 0.5, 0.7$ and $0.9$, respectively.

**Fig.6:** Computer simulation results on the probability distribution of the instantaneous distance between successive jams in the NS model with $V_{\text{max}} = 5$ when $p = 0.5$. The symbols $\diamond$, $+$ and $\Box$ correspond to $c = 0.05, 0.10$ and $0.90$, respectively. The lines merely connect the successive data points and serve as a guide to the eye.