SYMMETRIC ORTHOGONAL TENSOR DECOMPOSITION IS TRIVIAL∗

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Abstract. We consider the problem of decomposing a real-valued symmetric tensor as the sum of outer products of real-valued, pairwise orthogonal vectors. Such decompositions do not generally exist, but we show that some symmetric tensor decomposition problems can be converted to orthogonal problems following the whitening procedure proposed by Anandkumar et al. (2012). If an orthogonal decomposition of an m-way n-dimensional symmetric tensor exists, we propose a novel method to compute it that reduces to an n × n symmetric matrix eigenproblem. We provide numerical results demonstrating the effectiveness of the method.

1. Introduction. Let \( \mathbf{A} \) be an m-way n-dimensional real-valued symmetric tensor. Let \( \mathbf{X} \) represent an m-way, n-dimensional symmetric outer product tensor such that \( \mathbf{x}^m \). Comon et al. [2] showed there exists a decomposition of the form

\[
\mathbf{A} = \sum_{k=1}^{p} \lambda_k \mathbf{x}_k^m,
\]

where \( \lambda = [\lambda_1 \cdots \lambda_p]^T \in \mathbb{R}^p \) and \( \mathbf{X} = [\mathbf{x}_1 \cdots \mathbf{x}_p] \in \mathbb{R}^{n \times p} \); see Figure 1.1. Without loss of generality, we assume each \( \mathbf{x}_k \) has unit norm, i.e., \( \|\mathbf{x}_k\|_2 = 1 \). The least value \( p \) such that (1.1) holds is called the symmetric tensor rank. Finding real-valued symmetric tensor decompositions has been the topic of several recent papers, e.g., [4].

If we can discover an \( \mathbf{X} \) with orthogonal columns, i.e., \( \mathbf{X}^\top \mathbf{X} = \mathbf{I}_p \), then we say that \( \mathbf{A} \) has an orthogonal symmetric tensor decomposition. Generally, orthogonal decompositions do not exist; Robeva [7] classifies the tensors that have such decompositions. Nevertheless, we consider the problem of how to compute orthogonal symmetric tensor decompositions, as has been recently considered by Anandkumar et al. [1]. They show that the pairs \( (\lambda_k, \mathbf{x}_k) \) are Z-eigenpairs of \( \mathbf{A} \) for \( k = 1, \ldots, p \), and propose solving the problem via an iterative power method. We show that this problem can instead be solved via a symmetric matrix eigenproblem on an \( n \times n \) matrix, including determining the rank. This is can be interpreted as a special case of the simultaneous matrix diagonalization approach proposed by De Lathauwer [3].

∗This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Advanced Scientific Computing Research, Applied Mathematics program. Sandia National Laboratories is a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy’s National Nuclear Security Administration under contract DE-AC04-94AL85000.
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Although orthogonal symmetric tensor decompositions do not generally exist, Anandkumar et al. [1] showed that certain symmetric tensor decompositions problems can be transformed via whitening to orthogonal problems. We generalize their results to show that such a transformation is possible whenever \( X \) has full column rank and there exists a linear combination of two-dimensional slices of \( \mathbf{A} \) that is positive definite. The second condition is always satisfied, for example, if \( \mathbf{A} \) is positive definite.

2. Background. A tensor is a multidimensional array. The number of ways or modes is called the order of a tensor. For example, a matrix is a tensor of order two. Tensors of order three or greater are called higher-order tensors.

We use the notation \( \mathbf{I}_n \) to denote the \( n \times n \) identity matrix.

2.1. Symmetry. A tensor is symmetric if its entries do not change under permutation of the indices. Formally, we let \( \pi(m) \) denote the set of permutations of length \( m \). We say a real-valued \( m \)-way \( n \)-dimensional tensor \( \mathbf{A} \) is symmetric [2] if

\[
a_{i_{p(1)}\ldots i_{p(m)}} = a_{i_1\ldots i_m} \quad \text{for all } i_1, \ldots, i_m \in \{1, \ldots, n\} \text{ and } p \in \pi(m).
\]

The tensor-vector product \( \mathbf{A} \mathbf{x}^{(m-1)} \) produces a vector in \( \mathbb{R}^n \) such that

\[
(\mathbf{A} \mathbf{x}^{(m-1)})_{i_1} = \sum_{i_2, \ldots, i_m = 1}^{n} a_{i_1\ldots i_m} x_{i_2} \cdots x_{i_m} \quad \text{for } i_1 \in \{1, \ldots, n\}.
\]

The tensor-vector product \( \mathbf{A} \mathbf{x}^{m} \) produces a scalar such that

\[
\mathbf{A} \mathbf{x}^{m} = \mathbf{x}^\top (\mathbf{A} \mathbf{x}^{(m-1)}) = \sum_{i_1, \ldots, i_m = 1}^{n} a_{i_1\ldots i_m} x_{i_1} \cdots x_{i_m}.
\]

A symmetric tensor \( \mathbf{A} \) is positive definite if \( \mathbf{A} \mathbf{x}^{m} > 0 \) for all \( \mathbf{x} \neq 0 \).

2.2. Tensor-vector Products. Let \( \mathbf{V} \) be an \( p \times n \) matrix. Then the tensor-matrix product \( \mathbf{A}(\mathbf{V}, \ldots, \mathbf{V}) \) indicates multiplication of the tensor \( \mathbf{A} \) in each mode by the matrix \( \mathbf{V} \). The result is a symmetric tensor that is the same order as \( \mathbf{A} \) but now \( p \)-dimensional such that

\[
(\mathbf{A}(\mathbf{V}, \ldots, \mathbf{V}))_{i_1\ldots i_m} = \sum_{j_1, \ldots, j_m = 1}^{n} a_{j_1\ldots j_m} v_{i_1,j_1} \cdots v_{i_m,j_m} \quad \text{for } i_1, \ldots, i_m \in \{1, \ldots, p\}.
\]

2.3. Tensor-matrix Products. Let \( \mathbf{V} \) be a real-valued \( n \times n \) matrix. Then the tensor-vector product \( \mathbf{A} \mathbf{V} \) indicates multiplication of the tensor \( \mathbf{A} \) in each mode by the matrix \( \mathbf{V} \). The result is a symmetric tensor that is the same order as \( \mathbf{A} \) but now \( p \)-dimensional such that

\[
(\mathbf{A}(\mathbf{V}, \ldots, \mathbf{V}))_{i_1\ldots i_m} = \sum_{j_1, \ldots, j_m = 1}^{n} a_{j_1\ldots j_m} v_{i_1,j_1} \cdots v_{i_m,j_m} \quad \text{for } i_1, \ldots, i_m \in \{1, \ldots, p\}.
\]

2.4. Tensor Rank. Recall that the rank of a tensor \( \mathbf{A} \), denoted \( \text{rank}(\mathbf{A}) \), is the smallest number of rank-one tensors that sums to the original tensor [5]. The symmetric tensor rank, denoted \( \text{symrank}(\mathbf{A}) \), is the smallest number of symmetric rank-one tensors that sums to the original tensor [2]. In the case of the tensor \( \mathbf{A} \) in (1.1) with \( \mathbf{X} \) having orthogonal columns, it is easy to show that the symmetric tensor rank of \( \mathbf{A} \) is equal to the tensor rank of \( \mathbf{A} \) which is equal to the matrix rank of \( \mathbf{X} \) which is equal to \( p \), i.e.,

\[
\text{symrank}(\mathbf{A}) = \text{rank}(\mathbf{A}) = \text{rank}(\mathbf{X}) = p,
\]

via unfolding arguments.
3. **Orthogonal Symmetric Decomposition.** Given $\mathcal{A}$, our goal is to find $\lambda$ and $X$ that satisfies (1.1), under the assumption that $X$ is known to have orthogonal columns.

We define $B$ to be an arbitrary linear combination of slices of $\mathcal{A}$:

$$B = \sum_{i_3, \ldots, i_m} \beta_{i_3 \ldots i_m} A(:, i_3, \ldots, i_m) = \sum_{k=1}^p \sigma_k x_k x_k^T = X\Sigma X^T$$

(3.1)

where the $\beta$-values define the linear combination and

$$\sigma_k = \lambda_k \sum_{i_3, \ldots, i_m} \beta_{i_3 \ldots i_m} x_{i_3 k} \cdots x_{i_m k} \quad \text{for} \quad k = 1, \ldots, p.$$  

Observe that $(\sigma_k, x_k)$ is an eigenpair of $B$.

**3.1. Generic case.** If $X$ is a generic matrix (with orthogonal columns) and the $\beta$-values are arbitrary, then $\lambda_k \neq 0$ generically implies $\sigma_k \neq 0$ for $k = 1, \ldots, p$. It follows that the tensor rank of $\mathcal{A}$ is generically equal to the matrix rank of $B$, i.e.,

$$\text{rank}(\mathcal{A}) = \text{rank}(B).$$

Additionally, the $\sigma$-values are generically distinct, so the eigenvectors of $B$ are unambiguous and equal to the vectors in the decomposition of $\mathcal{A}$. Therefore, we can compute $X$ from $B$ via an eigenvalue decomposition and recover the $\lambda$-values via

$$\lambda_k = \mathcal{A}x_k^m \quad \text{for} \quad k = 1, \ldots, p.$$  

(3.2)

**3.2. Non-generic Case.** In the generic case, the matrix rank of $B$ is equal to the symmetric tensor rank of $\mathcal{A}$ and the nonzero eigenvalues of $B$ are distinct. In order to guard against the non-generic case, randomly project the tensor as follows. Let $V \in \mathbb{R}^{n \times n}$ be a random orthonormal matrix. Then compute,

$$\hat{\mathcal{A}} = \mathcal{A}(V, \ldots, V) = \sum_{k=1}^p \lambda_k (Vx_k)^m = \sum_{k=1}^p \lambda_k \hat{x}_k^m.$$  

Apply the procedure outlined above to obtain the decomposition of $\hat{\mathcal{A}}$ in terms of $\hat{\lambda}$ and $\hat{X}$. Then the decomposition of $\mathcal{A}$ is given by

$$\lambda = \hat{\lambda} \quad \text{and} \quad X = V^T \hat{X}.$$  

**3.3. Algorithm.** The algorithm for computing the orthogonal symmetric decomposition of $\mathcal{A}$ using optional randomization is given in Algorithm 1.

**4. Whitening.** Although most tensors do not have symmetric decompositions, Anandkumar et al. [1] show how whitening may be used in a special case of nonorthogonal symmetric tensor decomposition. We generalize their result.

Given $\mathcal{A}$, our goal is to find $\lambda$ and $X$ that satisfies (1.1), under the assumption that $X$ is known to have full column rank but may not have orthogonal columns.
Algorithm 1 Orthogonal Symmetric Decomposition
Input: Let $\mathcal{A}$ be a symmetric $m$-way, $n$-dimensional real-valued tensor that is known to have an orthogonal symmetric tensor decomposition.

1: if apply randomization then
2: $V \leftarrow$ random $n \times n$ orthonormal matrix
3: else
4: $V \leftarrow I_n$
5: end if
6: $\hat{A} \leftarrow \mathcal{A}(V, \ldots, V)$
7: $\beta \leftarrow$ arbitrary $(m-2)$-way real-valued tensor of dimension $n$
8: $B \leftarrow \sum_{i_3, \ldots, i_m} \beta_{i_3 \ldots i_m} \hat{A}(i_1, i_3, \ldots, i_m)$
9: $\{ \sigma_k, \hat{x}_k \}_{k=1}^p \leftarrow$ eigenpairs of $B$ with $\sigma_k \neq 0$
10: for $k = 1, \ldots, p$ do
11: $x_k \leftarrow V^T \hat{x}_k$
12: $\lambda_k \leftarrow \mathcal{A}x_k^n$
13: end for

4.1. Transformation to orthogonal problem. Let $C$ be an arbitrary linear combination of slices:

$$C = \sum_{i_3, \ldots, i_m} \gamma_{i_3 \ldots i_m} A(i_1, i_3, \ldots, i_m),$$

where the $\gamma$-values define the linear combination. If $C$ is positive semi-definite (p.s.d.) and has the same rank as $\mathcal{A}$, then we can apply whitening as follows. Let

$$UDU^T = C$$

be the “skinny” eigendecomposition of $C$ where $U$ is an orthogonal matrix of size $n \times p$ and $D$ is a diagonal matrix of size $p \times p$. Define the whitening matrix as

$$W = D^{-1/2}U^T$$

so that

$$WCW^T = (WUD^{1/2})(WUD^{1/2})^T = I_p.$$

We use $W$ to whiten the tensor as

$$\bar{A} = \mathcal{A}(W, \ldots, W) = \sum_{k=1}^p \lambda_k Wx_k = \sum_{k=1}^p \lambda_k \bar{x}_k$$

Now, $\bar{X} = WX \in \mathbb{R}^{p \times p}$ is a matrix with orthogonal columns. Moreover, the size of the problem is reduced because $\bar{A}$ is an $m$-way $p$-dimensional tensor.

We compute the orthogonal tensor decomposition of $\bar{A}$ via the procedure outlined above to get $\bar{\lambda}$ and $\bar{X}$. The final decomposition is given by

$$\lambda = \bar{\lambda} \quad \text{and} \quad X = W^\dagger \bar{X}.$$ 

Here $W^\dagger = UD^{1/2}$ represents the pseudoinverse of $W$.

4.2. Failure of the Method. If the algorithm cannot find a p.s.d. matrix $C$, then the algorithm fails. Additionally, if rank$(C) < p$, then the algorithm has a soft failure.
Algorithm 2 Whitening for Orthogonal Symmetric Decomposition

Input: Let $\mathcal{A}$ be an $m$-way, $n$-dimensional real-valued tensor.

1: if apply random orthogonal matrix then
2: \quad $V \leftarrow$ random $n \times n$ orthonormal matrix
3: else
4: \quad $V \leftarrow I_n$
5: end if
6: $\hat{\mathcal{A}} \leftarrow \mathcal{A}(V, \ldots, V)$
7: repeat
8: \quad $\gamma \leftarrow$ arbitrary $(m - 2)$-way real-valued tensor of dimension $n$
9: \quad $C \leftarrow \sum i_3, \ldots, i_m \gamma_{i_3 \ldots i_m} \hat{\mathcal{A}}(:,:,i_3, \ldots, i_m)$
10: until $C$ is p.s.d. or exit with failure
11: $UDU^T \leftarrow \text{"skinny" eigendecomposition of } C$
12: $W \leftarrow D^{-1/2}U$
13: $\bar{\mathcal{A}} \leftarrow \mathcal{A}(W, \ldots, W)$
14: $\beta \leftarrow$ arbitrary $(m - 2)$-way real-valued tensor of same dimension as $\hat{\mathcal{A}}$
15: $B \leftarrow \sum i_3, \ldots, i_m \beta_{i_3 \ldots i_m} \hat{\mathcal{A}}(:,:,i_3, \ldots, i_m)$
16: $\{ \sigma_k, x_k \}_{k=1}^p \leftarrow$ eigенpairs of $B$ with $\sigma_k \neq 0$
17: for $k = 1, \ldots, p$ do
18: \quad $x_k \leftarrow V^T UD_{1/2} x_k$
19: \quad $\lambda_k \leftarrow \bar{\mathcal{A}} x_k^m$
20: end for

4.3. Algorithm. The algorithm for computing the symmetric decomposition of $\mathcal{A}$ using whitening is given in Algorithm 2.

5. Numerical Results. We consider the results of applying the algorithms to numerical examples. All experiments are done in MATLAB, Version R2014b. Numerically, we say

- an eigenvalue $\sigma$ of $B$ is nonzero if $|\sigma| > 10^{-10}$,
- $C$ is p.s.d. if its smallest eigenvalue satisfies $d > -10^{-10}$, and
- the skinny decomposition of $C$ uses only eigenvalues such that $d > 10^{-10}$.

We choose both $\beta$ and $\gamma$ values from $\mathcal{U}[0, 1]$, i.e., uniform random on the interval $[0, 1]$ and then normalize so the values sum to one. Random orthogonal matrices are generated via the MATLAB code RANDORTHMAT by Olef Shilon.

We generate artificial data as follows. For a given $\lambda^* \in \mathbb{R}^p$ and $X^* \in \mathbb{R}^{n \times p}$, the noise-free data tensor is given by

$$\mathcal{A}^* = \sum_{k=1}^p \lambda_k^* (x_k^*)^m.$$  \hspace{1cm} (5.1)

The data tensor $\mathcal{A}$ may also be contaminated by noise as controlled by the parameter $\eta \geq 0$, i.e.,

$$\mathcal{A} = \mathcal{A}^* + \eta \frac{\|\mathcal{A}^*\|}{\|\mathcal{N}\|} \mathcal{N} \quad \text{where} \quad n_{i_3, \ldots, i_m} \sim \mathcal{N}(0, 1).$$  \hspace{1cm} (5.2)

Here $\mathcal{N}$ is a noise tensor such that each element is drawn from a normal distribution, i.e., $n_{i_3, \ldots, i_m} \sim \mathcal{N}(0, 1)$. The parameters $m, n, p$ control the size of the problem.

In our randomized experiments, we consider three sizes:
• $m = 3, n = 4, p = 2$;
• $m = 4, n = 25, p = 3$; and
• $m = 6, n = 6, p = 4$.

For each size, we also consider two noise levels: $\eta \in \{0, 0.01\}$, i.e., no noise and a small amount of noise.

The output of each run is a rank $p$, a weight vector $\lambda$, and a matrix $X$. The relative error measures the proportion of the observed data that is explained by the model, i.e.,

$$\text{relative error} = \left\| A - \sum_{k=1}^{p} \lambda_k x_k^* \right\| / \| A \|.$$ 

In the case of no noise, the ideal relative error is zero; otherwise, we hope for something near the noise level, i.e., $\eta$.

To compare the recovered solution $\lambda$ and $X$ with the true solution $\lambda^*$ and $X^*$, we compute the solution score as follows. Without loss of generality, we assume both $X$ and $X^*$ have normalized columns. (If $\|x_k\|_2 \neq 1$, then we rescale $\lambda_k = \lambda_k \sqrt{\|x_k\|}$ and $x_k = x_k / \|x_k\|$.). There is a permutation ambiguity, but we permute the computed solution so as to maximize the following score:

$$\text{solution score} = \frac{1}{p} \sum_{k=1}^{p} \left( 1 - \frac{|\lambda_k - \lambda_k^*|}{\max\{\|\lambda_k\|, \|\lambda_k^*\|\}} \right) |x_k^T x_k^*|.$$ 

A solution score of 1 indicates a perfect match. If $X$ has more columns than $X^*$, we choose the $p$ columns that maximize the score.

5.1. Orthogonal Example Showing Impact of $\beta$-values. We discuss why we recommend a linear combination of slices instead of a single slice. Consider the following example. Let $m = 3$, $n = 3$, and $p = 3$. Further, supposed $X^* = I_n$ and $\lambda^*$ is an arbitrary vector. Assume no noise so that $A = A^*$. Each slice of the tensor $A$ is a rank-1 matrix. For instance,

$$A(:, :, 1) = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

If we only select the first slice, which corresponds to $\beta = [1 \ 0 \ 0]$, will not yield a matrix $B$ that has rank equal to $p$. But, using a random linear combination remedies this problem. Alternatively, using the randomization to make the problem generic will also correct the problem.

5.2. Random Orthogonal Examples. In this case, we generate tensors such that $X^*$ is a random orthogonal matrix and $\lambda^*$ is the all ones vector. (Note that a matrix that had repeated eigenvalues would not have a unique factorization, but tensors with repeated eigenvalues are not impacted in the same way.) We apply Algorithm 1 with no randomization ($V = I_n$). The results are shown in Table 5.1.

We generated 100 random instances for each size, and then we ran the code 10 timers per instance (each run uses a different random choice for $\beta$). In the noise-free case ($\eta = 0$), the method works perfectly: the rank is perfectly predicted and the exact solution is found. In the noisy case ($\eta = 0.1$), the method is less reliable. The rank is never predicted correctly; instead, the $B$ matrix is nearly always full rank.
Nevertheless, the relative error is usually less than $10 \eta$ and the solution score is nearly always $\geq 0.99$. So, we can presumably threshold the small $\lambda$-values in the noisy cases to recover a good solution.

| Size  | No noise $\eta = 0$ | Noise $\eta = 0.01$ |
|-------|---------------------|---------------------|
| $m$  | $n$  | $p$  | Rank = $p$ | Rel. Error $\leq 1e-10$ | Soln. Score $\geq 0.99$ | $m$  | $n$  | $p$  | Rank = $p$ | Rel. Error $\leq 0.1$ | Soln. Score $\geq 0.99$ |
| 3 4 2 | 1000 100 2.000 | 1000 100 0.0000 | 1000 100 1.0000 | 0 0 4.000 | 921 100 0.0428 | 925 100 0.9903 |
| 4 25 3 | 1000 100 3.000 | 1000 100 0.0000 | 1000 100 1.0000 | 0 0 24.999 | 818 100 0.0716 | 864 100 0.9856 |
| 6 6 4 | 1000 100 4.000 | 1000 100 0.0000 | 1000 100 1.0000 | 0 0 6.000 | 393 92 0.1916 | 498 99 0.9530 |

Table 5.1

Random orthogonal examples. We create 100 random instances and run the method 10 times per instance for a total of 1000 runs. For each metric, we report the total number of runs where the metric meets the desired criteria, the number of instances where at least one run meets the desired criteria, and the mean value of the metric.

5.3. Non-orthogonal Example. In Example 5.5(i) of [6], Nie considers an method for determining the rank of a tensor. The example tensor is of order $m = 4$ and defined by

$$\lambda^* = \begin{bmatrix} 676 \\ 196 \end{bmatrix}$$

and

$$X^* = \begin{bmatrix} 0 & 3/\sqrt{14} \\ 1/\sqrt{26} & 2/\sqrt{14} \\ -5/\sqrt{26} & -1/\sqrt{14} \end{bmatrix} \approx \begin{bmatrix} 0.00 & 0.80 \\ 0.20 & 0.53 \\ -0.98 & -0.27 \end{bmatrix}.$$ 

The matrix $X^*$ is not orthogonal but is full column rank. We apply algorithm Algorithm 2 one hundred times. We do not apply randomization (i.e., $V = I_n$). For every run, the predicted rank is 2 and the solution score is 1 (perfect match). The average number of attempts (i.e., choosing a random set of $\gamma$-values) to find a p.s.d. $C$ is 1.26, and the maximum is 4.

5.4. Random Non-orthogonal Examples. In this case, we generate tensors such that $X^*$ comes from a matrix with entries drawn from the standard norm distribution whose columns are normalized (i.e., $\|x_k\|_2 = 1$ for $k = 1, \ldots, p$) and $\lambda^*$ is the all ones vector. We apply Algorithm 2 with no randomization ($V = I_n$). The method fails if it cannot find a p.s.d. $C$ after 100 attempts. The results are shown in Table 5.2.

In the case of no noise ($\eta = 0$), the method is surprising effective. Every problem is solved exactly for the even-order tensors ($m = 4$ and $m = 6$), which are constructed so that they are positive definite, guaranteeing that a p.s.d. $C$ exists. For the odd-ordered tensor ($m = 3$), the method is able to find a p.s.d. $C$ for 77 out of 100 instances. When it successfully finds the transformation, the problem is solved exactly.

For the noisy case ($\eta = 0.01$), the impact is dramatic. In the smallest example ($m = 3, n = 4, p = 2$), only 32 instances can find a p.s.d. $C$ matrix, and only 10 instances have a solution score of 0.99 or higher. For the case $m = 4, n = 25, p = 3$, the algorithm fails to find a p.s.d. $C$ in every instance. For the case $m = 6, n = 6, p = 4$, the algorithm find a p.s.d. $C$ in a handful of instances but ultimately fails to solve
the problem. We hypothesize that the problem stems from the fact that the noisy version of tensor has a rank that is higher than \( p \), so the \( X \) that corresponds to the noisy tensor does not have full column rank.

| Size  | No noise \( \eta = 0 \) | Noise \( \eta = 0.01 \) |
|-------|-------------------------|-------------------------|
| \( m \) \( n \) \( p \) | p.s.d. \( C \)? | Rank = \( p \) | Rel. Err. \( \leq 10^{-10} \) | Soln. Sc. \( \geq 0.99 \) | p.s.d. \( C \) | Rank = \( p \) | Rel. Err. \( \leq 0.1 \) | Soln. Sc. \( \geq 0.99 \) |
| 3 4 2 | 700 77 11.6 | 700 77 2.0 | 700 77 0.0 | 700 77 1.0 | 238 32 20.1 | 0 0 4.0 | 76 22 0.4 | 17 10 0.8 |
| 4 25 3 | 1000 100 2.5 | 1000 100 3.0 | 1000 100 0.0 | 1000 100 1.0 | 0 0 — | 0 0 — | 0 0 — | 0 0 — |
| 6 6 4 | 1000 100 4.4 | 1000 100 4.0 | 1000 100 0.0 | 1000 100 1.0 | 256 35 16.9 | 0 0 6.0 | 0 0 16.8 | 0 0 0.4 |

Table 5.2
Random non-orthogonal examples. We create 100 random instances and run the method 10 times per instance for a total of 1000 runs. For each metric, we report the total number of runs where the metric meets the desired criteria, the number of instances where at least one run meets the desired criteria, and the mean value of the metric for all successful attempts. For the p.s.d. \( C \) column, the final number is the mean number of attempts needed to find a p.s.d. \( C \) whenever it occurs.

6. Conclusions. If a symmetric tensor \( A \) is known to have a factor matrix \( X \) with orthogonal columns, we show that it is possible to solve the symmetric orthogonal tensor decomposition algorithm via a straightforward matrix eigenproblem. The method appears to be effective even in the presence of a small amount of noise. This is an improvement over previous work [1] that proposed solving the problem iteratively using the tensor eigenvalue power method and deflation.

We also consider the application of whitening as proposed by [1] in the case where a symmetric tensor \( A \) is known to have a factor matrix \( X \) that is full rank, but does not have orthogonal columns. In the noise-free case, the methods works extremely well. In the case that a small amount of noise is added, however, the current method is much less effective. Improving performance in that regime is a potential topic of future study.

Acknowledgments. I am grateful to Anima Anandkumar (UC Irvine) for motivating this work with her talk and at the Fields Institute Workshop on Optimization and Matrix Methods in Big Data. I am indebted to my Sandia colleagues Grey Ballard and Jackson Mayo for helpful feedback on this manuscript.

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