Dissipative systems: uncontrollability, observability and RLC realizability

Karikalan Selvaraj, Madhu N. Belur and Rihab Abdulrazak

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Abstract

The theory of dissipativity has been primarily developed for controllable systems/behaviors. For various reasons, in the context of uncontrollable systems/behaviors, a more appropriate definition of dissipativity is in terms of the dissipation inequality, namely the existence of a storage function. A storage function is a function such that along every system trajectory, the rate of increase of the storage function does not exceed the power supplied. While the power supplied is always expressed in terms of only the external variables, whether or not the storage function should be allowed to depend on only the external variables and their derivatives or also unobservable/hidden variables has various consequences on the notion of dissipativity: this paper thoroughly investigates the key aspects of both cases, and also proposes another intuitive definition of dissipativity.

We first assume that the storage function can be expressed in terms of the external variables and their derivatives only and prove our first main result that, assuming the uncontrollable poles are unmixed, i.e. no pair of uncontrollable poles add to zero, and assuming a strictness of dissipativity at the infinity frequency, the dissipativities of a system and its controllable part are equivalent; in other words once the autonomous subsystem satisfies a Lyapunov equation solvability-like condition, it does not interfere with the dissipativity of the system. We also show that the storage function in this case is a static state function. This main result proof involves new results about solvability of the Algebraic Riccati Equation, and uses techniques from Indefinite Linear Algebra and Hamiltonian matrix properties.

We then investigate the utility of unobservable/hidden variables in the definition of storage function: we prove that lossless uncontrollable behaviors are ones which require storage function to be expressed in terms of variables that are unobservable from the external variables.

We next propose another intuitive definition: a behavior is called dissipative if it can be embedded in a controllable dissipative super-behavior. We show that this definition imposes a constraint on the number of inputs and thus explains unintuitive examples from the literature in the context of lossless/orthogonal behaviors. These results are finally related to RLC realizability of passive networks, specifically to the nonrealizability of the nullator one-port circuit using RLC elements.

1 Introduction

The theory of dissipativity for linear dynamical systems helps in the analysis and design of control systems for several control problems, for example, LQR/LQG control, $H_\infty$, synthesis of passive systems, and
optimal estimation problems. When dealing with LTI systems, it is straightforward to define dissipativity for controllable systems due to a certain property of such systems that their compactly supported system trajectories are, loosely speaking, ‘dense’ in the set of all allowed trajectories. However, this is not the case for uncontrollable systems, and this situation is the central focus of this paper. We elaborate more on this point when we define dissipativity and review equivalent conditions for controllable systems in Section 2.

In this paper, we use a less-often-used definition of dissipativity for systems, possibly uncontrollable, and generalize key results using some techniques from indefinite linear algebra (see [10]) for solving Algebraic Riccati Inequalities in the context of an uncontrollable state space system. Like in [7, 12], we define a system as dissipative if there exists a storage function that satisfies the dissipation inequality for all system trajectories. The existential aspect of this definition raises key issues that this paper deals with.

The main result we show is that if the uncontrollable poles of an LTI system are such that no two of them add to zero, and if the controllable subsystem strictly dissipates energy at frequency equal to infinity, then the dissipativity of the controllable subsystem is equivalent to the system’s dissipativity. We also show that, using the concatenability axiom of the state, the energy stored in a system is a static function of the state variables. Further, we also show that this state is ‘observable’ from the external variables, i.e. the state is a linear combination of the external variables and possibly their derivatives. This is intuitively expected in view of the fact that energy exchange between the system and its ambience takes place through the external variables. However, it appears that this may not be the case for lossless systems, i.e. systems that don’t dissipate any energy, nor contain a source within.

2 Preliminaries

In this section we include various definitions about the behavioral framework for studying dynamical systems (Subsection 2.1) and then introduce background results about dissipative systems (Subsection 2.2). Subsection 2.3 contains brief notation about indefinite linear algebra from [10]. For this paper, \( \mathbb{R} \) denotes the set of all real numbers and \( \mathbb{R}[\xi] \) the set of polynomials in the indeterminate \( \xi \) and real coefficients; matrices and polynomial matrices are denoted the obvious way. We use \( \bullet \) to leave a row dimension unspecified, for example, \( \mathbb{R}^{\bullet \times w}[\xi] \). The space of infinitely often differentiable functions from \( \mathbb{R} \) to say \( \mathbb{R}^n \) is denoted by \( \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^n) \) and \( \mathcal{D} \) denotes the set of compactly supported functions within this space.

2.1 The behavioral approach

When dealing with linear differential systems, it is convenient to use polynomial matrices for describing a differential equation. Suppose \( R_0, R_1 \ldots R_N \) are constant matrices of the same size such that

\[
R_0 w + R_1 \frac{d}{dt} w + R_2 \frac{d^2}{dt^2} w \ldots R_N \frac{d^N}{dt^N} w = 0
\]

is a linear constant coefficient ordinary differential equation in the variable \( w \). We define the polynomial matrix \( R(\xi) := R_0 + R_1 \xi + \ldots R_N \xi^N \), and represent the above differential equation as \( R(\frac{d}{dt}) w = 0 \).
A linear differential behavior, denoted by $\mathcal{B}$, is defined as the set of all infinitely often differentiable trajectories that satisfy a system of ordinary linear differential equations with constant coefficients, i.e.,

$$\mathcal{B} := \{ w \in C^\infty(\mathbb{R}, \mathbb{R}^m) \mid R\left(\frac{d}{dt}\right)w = 0\},$$

where $R(\xi)$ is a polynomial matrix having $w$ number of columns, i.e., $R(\xi) \in \mathbb{R}^{\bullet \times w}[\xi]$. We denote the set of all such linear differential behaviors with $w$ number of variables by $\mathcal{L}^w$. The linear differential behavior $\mathcal{B} \in \mathcal{L}^w$ given by the above definition can also be represented by $\mathcal{B} = \ker R\left(\frac{d}{dt}\right)$. One views the rows of $R$ as differential equations that the variable $w$ has to satisfy in order for a trajectory $w(t)$ to be in the behavior $\mathcal{B}$. The matrix $R$ is not unique and one can use elementary row operations to modify $R$ and this does not change the set of solutions $\mathcal{B}$: this thus allows assuming without loss of generality that $R$ has full row rank (see [24]). The number of inputs of $\mathcal{B}$ is defined as $w - \text{rank } (R)$ and is called the input cardinality. This integer depends only on $\mathcal{B}$ and not on the $R$ used to define it; $m(\mathcal{B})$ denotes the number of inputs.

An important fundamental concept is controllability of a system. A behavior $\mathcal{B} = \ker R\left(\frac{d}{dt}\right)w$ is said to be \textit{controllable}, if for every $w_1$ and $w_2 \in \mathcal{B}$, there exist $w_3 \in \mathcal{B}$ and $\tau > 0$ such that

$$w_3 = \begin{cases} w_1(t) \text{ for all } t \leq 0, \\ w_2(t) \text{ for all } t \geq \tau. \end{cases}$$

The set of all \textit{controllable} behaviors with $w$ variables is denoted as $\mathcal{L}_{w_{cont}}^w$. This patchability definition of controllability is known to have the traditional Kalman state-space definition of controllability as a special case in [24]. Further, it is shown that $\mathcal{B} = \ker R\left(\frac{d}{dt}\right)w$ is controllable if and only if $R(\lambda)$ has full rank for all $\lambda \in \mathbb{C}$. It is also shown in [24] that $\mathcal{B}$ is controllable if and only if it can be defined as

$$\mathcal{B} := \{ w \in C^\infty(\mathbb{R}, \mathbb{R}^m) \mid \text{there exists an } \ell \in C^\infty(\mathbb{R}, \mathbb{R}^n) \text{ such that } w = M\left(\frac{d}{dt}\right)\ell\},$$

where $M(\xi) \in \mathbb{R}^{\ell \times m}[\xi]$. This representation of $\mathcal{B} \in \mathcal{L}_{w_{cont}}^w$ is known as an \textit{image representation}. The variable $\ell$ is called a \textit{latent} variable: these are auxiliary variables used to describe the behavior; we distinguish the variable $w$ as the manifest variable, the variable of interest. It is known that $\mathcal{B} \in \mathcal{L}_{w_{cont}}^w$ always allows an image representation with $M(\xi)$ such that $M(\lambda)$ has full column rank for every $\lambda \in \mathbb{C}$. This kind of image representation is known as an \textit{observable} image representation. In this paper, unless otherwise stated explicitly, we assume the image representations are observable. The use of the term ‘observable’ is motivated by the fact that the variable $\ell$ is \textit{observable} from the variable $w$. This notion is defined as follows.

For a behavior $\mathcal{B}$ with variables $w$ and $\ell$, we say $\ell$ is observable from $w$ if whenever $(w, \ell_1)$ and $(w, \ell_2)$ both are in $\mathcal{B}$, we have $\ell_1 = \ell_2$. Observability of $\ell$ from $w$ in a behavior $\mathcal{B}$ implies that there exists a polynomial matrix $F(\xi)$ such that $\ell = F\left(\frac{d}{dt}\right)w$ for all $w$ and $\ell$ in the behavior.

We now define relevant notions in the context of uncontrollable behaviors. For a behavior $\mathcal{B}$, possibly uncontrollable, the largest controllable behavior contained in $\mathcal{B}$ is called the \textit{controllable} part of $\mathcal{B}$, and denoted by $\mathcal{B}_{cont}$. An important fact about the controllable part of $\mathcal{B}$ is that $m(\mathcal{B}_{cont}) = m(\mathcal{B})$. The set of complex numbers $\lambda$ for which $R(\lambda)$ loses rank is called the set of \textit{uncontrollable modes} and is denoted by $\Lambda_{un}$. For a detailed exposition on behaviors, controllability and observability we refer the reader to [24].
2.2 Quadratic Differential Forms and dissipativity

The concept of Quadratic Differential Forms (QDF) (see [20]) is central to this paper. Consider a two variable polynomial matrix with real coefficients, \( \Phi(\zeta, \eta) := \sum_{j,k} \Phi_{jk} \zeta^j \eta^k \in \mathbb{R}^{w \times w}[\zeta, \eta] \), where \( \Phi_{jk} \in \mathbb{R}^{w \times w} \).

The QDF \( Q_\Phi \) induced by \( \Phi \) is a map \( Q_\Phi : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^\ell) \rightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \) defined as
\[
Q_\Phi(w) := \sum_{j,k} \left( \frac{d^jw}{dt^j} \right)^T \Phi_{jk} \left( \frac{d^kw}{dt^k} \right).
\]

When dealing with quadratic forms in \( w \) and its derivatives, we can assume without loss of generality that \( \Phi(\zeta, \eta) = \Phi^T(\eta, \zeta) \): such a \( \Phi \) is called a symmetric two variable polynomial matrix. A quadratic form induced by a real symmetric constant matrix \( S \in \mathbb{R}^{w \times w} \) is a special QDF. We frequently need the number of positive and negative eigenvalues of a nonsingular, symmetric matrix \( S \): they are denoted by \( \sigma_+(S) \) and \( \sigma_-(S) \) respectively.

For a two variable polynomial matrix \( \Phi(\zeta, \eta) \in \mathbb{R}^{w \times w}[\zeta, \eta] \), we define the single variable polynomial matrix \( \partial \Phi \) by \( \partial \Phi(\xi) = \Phi(-\xi, \xi) \).

Consider \( \Sigma \in \mathbb{R}^{w \times w} \), a symmetric nonsingular matrix. A behavior \( \mathcal{B} \in \mathcal{L}^\omega \) is said to be dissipative with respect to the supply rate \( \Sigma \) (or \( \Sigma \)-dissipative) if there exists a QDF \( Q_\Psi \) such that
\[
\frac{d}{dt}Q_\Psi(w) \leq w^T \Sigma w \quad \text{for all} \quad w \in \mathcal{B} \quad (1).
\]

The QDF \( Q_\Psi \) is called a storage function: it signifies the energy stored in the system at any time instant. The above inequality is called the dissipation inequality. A behavior \( \mathcal{B} \) is called \( \Sigma \)-lossless if the above inequality is satisfied with an equality for some QDF \( Q_\Psi \). Notice that the storage function plays the same role as that of Lyapunov functions in the context of autonomous systems; the notion of storage functions is a generalization to non-autonomous systems of Lyapunov functions, as pointed in [20]. The following theorem from [20] applies to controllable behaviors.

**Proposition 2.1** Consider \( \mathcal{B} \in \mathcal{L}^\omega_{\text{cont}} \) and let \( w = M \left( \frac{d}{dt} \right) \ell \) be an observable image representation. Suppose \( \Sigma \in \mathbb{R}^{w \times w} \) is symmetric and nonsingular. Then, the following are equivalent.

1. There exists a QDF \( Q_\Psi \) such that inequality (1) is satisfied for all \( w \in \mathcal{B} \).
2. \( \int_{\mathbb{R}} w^T \Sigma wd\ell \geq 0 \) for all \( w \in \mathcal{B} \cap \mathcal{D} \), the compactly supported trajectories in \( \mathcal{B} \).
3. \( M^T(-j\omega) \Sigma M(j\omega) \geq 0 \) for all \( \omega \in \mathbb{R} \).

The significance of the above theorem is that, for controllable systems, it is possible to verify dissipativity by checking non-negativity of the above integral over all compactly supported trajectories: the compact support signifying that we calculate the ‘net power’ transferred when the system starts ‘from rest’ and ‘ends at rest’. The starting and ending ‘at rest’ ensures that for linear systems there is no internal energy at this time. The absence of internal energy allows ruling out the storage function from this condition: in fact, this is used as the definition of dissipativity for controllable systems. The same cannot be done for uncontrollable systems due to the compactly supported trajectories not being dense in the behavior (see [13]). An extreme case is an autonomous behavior, i.e. a behavior which has \( m(\mathcal{B}) = 0 \): while the zero trajectory is the only compactly supported trajectory, the behavior consists of exponentials corresponding to the uncontrollable poles of \( \mathcal{B} \). The issue of existence of storage functions is elaborated in [20 Remark 5.9] and in text following [12 Proposition 3.3].
2.3 States

The state variable is defined as a latent variable that satisfies the property of state, that is, if \((w_1, x_1), (w_2, x_2) \in \mathcal{B}_f\) and \(x_1(0) = x_2(0)\), then the new trajectory \((w, x)\) formed by concatenating \((w_1, x_1)\) and \((w_2, x_2)\) at \(t = 0\), i.e.,

\[
(w, x)(t) = \begin{cases} (w_1, x_1)(t) & \text{for all } t \leq 0 \\ (w_2, x_2)(t) & \text{for all } t > 0, \end{cases}
\]

also satisfies the system equation of \(\mathcal{B}_f\) in a distributional sense \cite{21}. It is intuitively expected that a variable \(x\) has the state property if and only if \(w\) and \(x\) satisfy an equation that is at most first order in \(x\) and zeroth order in \(w\): see \cite{15} for precise statement formulation and proof. When \(w\) is partitioned into \(w = (w_1, w_2)\), with \(w_1\) as the input and \(w_2\) as the output, then \(\mathcal{B}_f\) admits the more familiar input/state/output (i/s/o) representation as

\[
\frac{d}{dt} x = Ax + Bw_1, \quad w_2 = Cx + Dw_1. \tag{2}
\]

One can ensure that \(x\) is observable from \(w\); this is equivalent to conventional observability of the pair \((C, A)\). While such a state space representation is admitted by any \(\mathcal{B}\), the pair \((A, B)\) is controllable (in the state space sense) if and only if \(\mathcal{B}\) is controllable (in the behavioral sense defined above).

2.4 Indefinite linear algebra

In this paper, we use certain properties of matrices that are self-adjoint with respect to an indefinite inner product. We briefly review self-adjoint matrices and neutral subspaces (see \cite{10}). Let \(P \in \mathbb{C}^{n \times n}\) be an invertible Hermitian matrix. This defines an indefinite inner product on \(\mathbb{C}^n\) by \((Px, x) := x^*Px\), where \(x^*\) is the complex conjugate transpose of the vector \(x \in \mathbb{C}^n\). For a complex matrix \(A\), the complex conjugate transpose of \(A\) is denoted by \(A^*\).

Consider matrices \(A\) and \(P \in \mathbb{C}^{n \times n}\) with \(P\) invertible and Hermitian. The \(P\)-adjoint of the matrix \(A\), denoted by \(A^{[P]}\), is defined as \(A^{[P]} = P^{-1}A^*P\). The matrix \(A\) is said to be \(P\)-self-adjoint if \(A = A^{[P]}\), i.e., \(A = P^{-1}A^*P\).

Since \(P\) is not sign-definite in general, the sign of \((Px, x)\) is zero, positive or negative depending on the vector \(x\). A subspace \(M \subseteq \mathbb{C}^n\) is said to be \(P\)-neutral if the inner product \((Px, x) = 0\) for all \(x \in M\).

3 Dissipativity of uncontrollable behaviors

In this paper, we deal with systems which satisfy the dissipativity property, i.e. net energy is directed inwards along every system trajectory. As elaborated below, the ‘total’ aspect of the energy involves an integral, thus bringing in the initial and final conditions of the trajectory being integrated. The convenience of starting-from-rest and ending-at-rest applies to only controllable systems as we will review soon. The notion of storage function helps in formulating the dissipation property as an inequality to be satisfied at each time-instant. A central issue in this paper is what variables should the storage function be allowed to depend on. We explore dependencies on a latent variable \(\ell\), on a state variable \(x\), or on the manifest variable \(w\): this is indicated in the storage function subscript; the following definition has appeared in several works, see \cite{20,22,24}, for example.
Definition 3.1 Let $\Sigma \in \mathbb{R}^{w \times w}$ be a nonsingular symmetric matrix, inducing the supply rate $w^T \Sigma w$. Consider a behavior $\mathcal{B} \in \mathcal{L}^w$ with manifest variables $w$ and latent variable $\ell$, with the corresponding full behavior $\mathcal{B}^\ell_{\text{full}}$. For the behavior $\mathcal{B}$, let $x$ be a state variable with the corresponding full behavior $\mathcal{B}^x_{\text{full}}$. With respect to the supply rate $\Sigma$, the behavior $\mathcal{B}$ is said to be dissipative if there exists a quadratic differential form $Q_\psi$ such that
\[
\frac{d}{dt} Q_\psi(\ell) \leq w^T \Sigma w \text{ for all } (w, \ell) \in \mathcal{B}^\ell_{\text{full}}.
\] (3)

1. The function $Q_\psi$, a quadratic function of $\ell$ and its derivatives, is called a storage function.

2. A storage function $Q_\psi$ is said to be an observable storage function if the latent variable $\ell$ is observable from the manifest variable $w$. In this case, there exists a storage function $Q_{\psi_w}$ such that $\frac{d}{dt} Q_{\psi_w}(w) \leq w^T \Sigma w$ for all $w \in \mathcal{B}$.

3. A storage function $Q_{\psi_x}(x)$ is said to be a state function if $Q_{\psi_x}(x)$ is equal to $x^T K x$ for some constant matrix $K$.

In this paper, we study dissipativity with respect to a constant, nonsingular, symmetric $\Sigma \in \mathbb{R}^{w \times w}$. It is known (see [20, Remark 5.11] and [21, Proposition 2]) that $\Sigma$-dissipativity of a behavior $\mathcal{B}$ implies that the input cardinality of $\mathcal{B}$ cannot exceed $\sigma_+(\Sigma)$, i.e., $m(\mathcal{B}) \leq \sigma_+(\Sigma)$. In this context it is helpful to perform a coordinate transformation in the variables $w$ so that $\Sigma$ is a diagonal matrix consisting of only +1’s and −1’s along the diagonal. Moreover, there exists an input/output partition such that all the inputs correspond to +1’s only and such that the transfer function from these inputs to all other variables is proper (see [20, Remark 5.11]). In view of these facts and the inequality $m(\mathcal{B}) \leq \sigma_+(\Sigma)$, we assume without loss of generality
\[
\Sigma = \begin{bmatrix} I_m & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & -I_p \end{bmatrix}
\text{ and define } J_{pq} = \begin{bmatrix} I_q & 0 \\ 0 & -I_p \end{bmatrix}
\] (4)
where $m$ is the number of inputs in $\mathcal{B}$.

3.1 Dissipativity of uncontrollable behaviors: main results

The following theorem assumes an unmixing condition - no pair of uncontrollable poles are symmetric with respect to the imaginary axis. If this unmixing condition is satisfied for the uncontrollable poles, then the controllable part of a behavior being dissipative is equivalent to the dissipativity of the whole behavior.

Theorem 3.2 Consider a behavior $\mathcal{B} \in \mathcal{L}^w$ and a nonsingular, symmetric $\Sigma \in \mathbb{R}^{w \times w}$ with the input cardinality of $\mathcal{B}$ at-most the positive signature of $\Sigma$, i.e., $m(\mathcal{B}) \leq \sigma_+(\Sigma)$. Assume that the uncontrollable poles are such that $\Lambda_{\text{un}} \cap -\Lambda_{\text{un}} = \emptyset$. Let $\mathcal{B}$ have an observable image representation, $\mathcal{B} = \text{Image } M(\frac{d}{dt})$, where $M(\xi)$ is partitioned as
\[
M(\xi) = \begin{bmatrix} W_1(\xi) \\ W_2(\xi) \end{bmatrix}; \quad W_1 \in \mathbb{R}^{m \times n}[\xi], W_2 \in \mathbb{R}^{(p + q) \times n}[\xi].
\] (5)
Let $G(\xi) := W_2(\xi) W_1(\xi)^{-1}$ and $D := \lim_{\omega \to \infty} G(j \omega)$. Assume $\mathcal{B}_{\text{cont}}$ is such that $(I_m + D^T J_{pq} D) > 0$. Then, $\mathcal{B}$ is $\Sigma$-dissipative if and only if its controllable part $\mathcal{B}_{\text{cont}}$ is $\Sigma$-dissipative.
The above result gives conditions under which the autonomous part of a behavior plays no hindrance to dissipativity of the behavior after the controllable part is dissipative. One of the conditions for this is that the uncontrollable poles are not ‘mixed’, meaning no two of the uncontrollable poles add to zero. For autonomous LTI systems, this condition is a necessary and sufficient condition for solvability of the Lyapunov equation. Of course, storage functions are just generalizations of Lyapunov functions to non-autonomous systems. The other condition: \((I_m + D^T J_{pq} D) > 0\) on the controllable part is a kind of strictness of dissipativity ‘at the infinity frequency’. While this condition allows the use of Hamiltonian matrices in the proofs, this condition also rules out consideration of lossless systems from the above result. More significance of the assumption is noted in the remark below.

**Remark:** While a behavior \(B \in L^w\) admits many i/o partitions, and for each i/o partition, admits many i/s/o representations, the matrix \(I + D^T J_{pq} D\) depends only on the behavior, and in particular, only on \(B_{\text{cont}}\) the controllable part. In other words, this matrix can be found from any image representation of \(B_{\text{cont}}\). This matrix being positive semi-definite is a necessary condition for dissipativity of \(B_{\text{cont}}\), and this denotes dissipativity at very high frequencies, i.e. as \(\omega \to \infty\). Positive definiteness of \(I + D^T J_{pq} D\) may hence be termed as ‘strict dissipativity at the infinity frequency’. This assumption helps in the existence of a Hamiltonian matrix, our proofs use the Hamiltonian matrix properties and techniques from indefinite linear algebra (see [10]). Positive definiteness of this matrix is guaranteed, for example, by strict dissipativity of a behavior; on the other hand, lossless controllable behaviors have this matrix as zero. Another interpretation of the condition \(I + D^T J_{pq} D > 0\) is that the ‘memoryless’ part of the behavior is strictly dissipative; the memoryless/static part of a behavior was defined in [21], and we don’t require this notion in this paper.

We now review some existing results and formulate/prove new results about Hamiltonian matrices in the context of dissipativity of controllable and uncontrollable systems. These are required for the proof of the main result.

To prove that the behavior \(B\) is \(\Sigma\)-dissipative, we use the proposition below and show that the existence of a symmetric solution to the dissipation LMI is sufficient for \(\Sigma\)-dissipativity. Let the behavior have an input/output partition \(w = (w_1, w_2)\), \(w_1 \in C^\infty(\mathbb{R}, \mathbb{R}^m)\) as the input and \(w_2 \in C^\infty(\mathbb{R}, \mathbb{R}^{p+q})\) as the output. Let the input/state/output representation of the behavior \(B\) be,

\[
\frac{d}{dt} x = Ax + Bw_1, \quad w_2 = Cx + Dw_1.
\]

(6)

with \((C, A)\) observable. The following result is well-known: see [10, 17, 5].

**Proposition 3.3** Let a behavior \(B \in L^w\) have an i/s/o representation for an input/output partition \(w = (w_1, w_2)\) with \(A, B, C, D\) as state space matrices and \((A, B)\) possibly uncontrollable. The behavior \(B\) is \(\Sigma\)-dissipative if there exists a symmetric \(K \in \mathbb{R}^{n \times n}\) which solves the following LMI

\[
\begin{bmatrix}
(KA + A^T K - C^T J_{pq} C) & (KB - C^T J_{pq} D) \\
(KB - C^T J_{pq} D)^T & -(I_n + D^T J_{pq} D)
\end{bmatrix} \leq 0.
\]

(7)

Further, under the condition that \((A, B)\) is controllable, existence of \(K\) solving the above LMI is necessary and sufficient for \(\Sigma\)-dissipativity of \(B\).

\(^1\)The feed-through term \(D\) is finite, since the transfer function is proper; see text before Equation (1).
As \((I_n + D^T J_{pq} D)\) is invertible, the Schur complement of \((I_n + D^T J_{pq} D)\) in the above LMI gives the Algebraic Riccati Inequality (ARI)

\[
K(A - B(I_n + D^T J_{pq} D)^{-1} D^T J_{pq} C) \\
+ (A - B(I_n + D^T J_{pq} D)^{-1} D^T J_{pq} C)^T K \\
+ KB(I_n + D^T J_{pq} D)^{-1} B^T K - C^T (J_{pq} + DD^T)^{-1} C A \\
\leq 0
\] (8)

The corresponding equation is the Algebraic Riccati Equation (ARE) and we use properties of this ARE in proving Theorem 3.2. We rewrite the above ARE as

\[
\tilde{K} \tilde{A} + \tilde{A}^T K + K \tilde{D} K - \tilde{C} = 0
\] (9)

where

\[
\tilde{A} := (A - B(I_n + D^T J_{pq} D)^{-1} D^T J_{pq} C),
\]

and

\[
\tilde{D} := B(I_n + D^T J_{pq} D)^{-1} B^T, \quad \tilde{C} := C^T (J_{pq} + DD^T)^{-1} C.
\]

Also define the Hamiltonian matrix corresponding to the ARE:

\[
H = \begin{bmatrix}
\tilde{A} & \tilde{D} \\
\tilde{C} & -\tilde{A}^*
\end{bmatrix}.
\] (10)

We define the following matrices; they play a crucial role in the results we use from [10] and in our proofs.

\[
M := jH, \quad P := \begin{bmatrix}
-\tilde{C} & \tilde{A}^* \\
\tilde{A} & \tilde{D}
\end{bmatrix} \quad \text{and} \quad \tilde{P} := j \begin{bmatrix}
0 & I \\
-I & 0
\end{bmatrix}.
\] (11)

It is well-known that a symmetric solution to ARE can be obtained from an \(n\)-dimensional, \(M\)-invariant, \(P\)-neutral subspace and we state this in the proposition below. Let \(K \in \mathbb{C}^{n \times n}\) be a Hermitian matrix. The graph subspace corresponding to matrix \(K\) is defined as,

\[
\mathcal{G}(K) := \text{Im} \begin{bmatrix}
I \\
K
\end{bmatrix}.
\]

Notice that \(\mathcal{G}(K)\) is an \(n\)-dimensional subspace of \(\mathbb{C}^{2n}\). The following proposition from [10] states solvability of the ARE in terms of \(P\)-neutrality with respect to the above \(P\).

**Proposition 3.4** (See [10]) Consider the ARE (9) with \((\tilde{A}, \tilde{D})\) possibly uncontrollable. Then \(K \in \mathbb{C}^{n \times n}\) is a Hermitian solution of the ARE if and only if the graph subspace of \(K\) is \(M\)-invariant and \(P\)-neutral.

The proposition below allows use of simplified \(A\) and \(B\) for all later purposes when dealing with uncontrollable systems.

**Proposition 3.5** (See [25]) Consider the behavior \(\mathfrak{B} \in \mathfrak{L}^a\) with an input/state/output representation

\[
\frac{d}{dt} x = Ax + Bw_1, \quad w_2 = Cx + Dw_1, \quad \text{where} \quad w = (w_1, w_2).
\]

Then there exists a nonsingular matrix \(T \in \mathbb{R}^{n \times n}\) such that

\[
T^{-1} A T = \begin{bmatrix}
A_c & A_{cp} \\
0 & A_u
\end{bmatrix}, \quad T^{-1} B = \begin{bmatrix}
B_c \\
0
\end{bmatrix} \quad \text{and} \quad CT = \begin{bmatrix}
C_c & C_u
\end{bmatrix}.
\]
Further,\[ \frac{d}{dt}x = A_c x + B_c w, \quad w_2 = C_c x + D w_1. \]
gives an i/s/o representation for the controllable part \( \mathcal{B}_{\text{cont}} \).

Let \( H_c \) and \( M_c \) denote the corresponding matrices for the controllable part \( \mathcal{B}_{\text{cont}} \) as defined in (11) with \( \tilde{A}_c, \tilde{D}_c \) and \( \tilde{C}_c \) defined accordingly in (10).

**Lemma 3.6** Suppose \( \mathcal{B}_{\text{cont}} \in \mathcal{L}_{\text{cont}}^w \) satisfies the assumption that \( (I_m + D^T J_{pq} D) > 0 \). If \( \mathcal{B}_{\text{cont}} \) is \( \Sigma \)-dissipative, then the partial multiplicities corresponding to the real eigenvalues of \( M_c \), if any, are all even.

In order to prove Lemma 3.6, we use a result from [10] concerning the partial multiplicities of real eigenvalues of \( M_c \). Let the generalized eigenspace of a matrix \( A \) corresponding to an eigenvalue \( \lambda_0 \) be denoted by \( \mathcal{R}_{\lambda_0}(A) \) and the controllable subspace of the pair \( (\tilde{A}_c, \tilde{D}_c) \) be denoted by \( \mathcal{C}_{\tilde{A}_c, \tilde{D}_c} \).

**Proposition 3.7** (See [10]) Consider the behavior \( \mathcal{B}_{\text{cont}} \in \mathcal{L}_{\text{cont}}^w \). Assume \( \tilde{D}_c \geq 0 \) and \( \tilde{C}_c^* = \tilde{C}_c \) and there exists a Hermitian solution \( K \in \mathbb{C}^{n \times n} \) to the ARE (9). Suppose \( \mathcal{R}_{\lambda_0}(\tilde{A}_c + \tilde{D}_c K) \subseteq \mathcal{C}_{\tilde{A}_c, \tilde{D}_c} \) for every purely imaginary eigenvalue \( \lambda_0 \) of \( (\tilde{A}_c + \tilde{D}_c K) \). Then, the partial multiplicities of corresponding real eigenvalues of \( M_c \) are all even and are twice the partial multiplicities of those corresponding to the purely imaginary eigenvalues of \( (\tilde{A}_c + \tilde{D}_c K) \).

**Proof of the Lemma 3.6** As the controllable part is \( \Sigma \)-dissipative, there exists a symmetric solution \( K \) to the ARE by Proposition 3.3. Since the behavior is controllable, the following is true: \( \mathcal{C}_{\tilde{A}_c, \tilde{D}_c} = \mathbb{C}^n \) and hence \( \mathcal{R}_{\lambda_0}(\tilde{A}_c + \tilde{D}_c K) \subseteq \mathcal{C}_{\tilde{A}_c, \tilde{D}_c} \) for every purely imaginary eigenvalue \( \lambda_0 \) of \( (\tilde{A}_c + \tilde{D}_c K) \). Thus, using Proposition 3.7, the partial multiplicities of all real eigenvalues of \( M_c \), if any, are all even. This completes the proof.

We define a set called c-set as in [10]. Such a c-set, if exists, guarantees the existence of a unique \( P \)-neutral, \( M \)-invariant subspace \( N \) under certain conditions. This is made precise in the proposition below.

**Definition 3.8** (See [10]) Let \( M \in \mathbb{C}^{n \times n} \) and let \( \mathcal{C} \) be a finite set of non-real complex numbers. \( \mathcal{C} \) is called a c-set of \( M \) if it satisfies the following properties

1. \( \mathcal{C} \cap \overline{\mathcal{C}} = \emptyset \)
2. \( \mathcal{C} \cup \overline{\mathcal{C}} = \sigma(M) \setminus \mathbb{R} \), the set of all non-real eigenvalues of \( M \).

Let \( N \) be an invariant subspace of \( M \) and denote the restriction of \( A \) to \( N \) as \( A|_N \).

**Proposition 3.9** (See [10]) Let \( M \in \mathbb{C}^{2n \times 2n} \) be a \( P \)-self-adjoint matrix such that the sizes of the Jordan blocks of \( M \), say: \( m_1, m_2, \ldots, m_r \), corresponding to real eigenvalues of \( M \) are all even. Then for every c-set \( \mathcal{C} \) there exists a unique \( P \)-neutral \( M \)-invariant subspace \( N \) of dimension \( n \) and \( \sigma(M|_N) \setminus \mathbb{R} = \mathcal{C} \), and the sizes of the Jordan blocks of \( M|_N \) corresponding to the real eigenvalues are \( \frac{1}{2} m_1, \frac{1}{2} m_2, \ldots, \frac{1}{2} m_r \).
It is easy to verify that $M$ and $P$ satisfy the following relation, $PM = M^*P$. Hence $M$ is $P$-self-adjoint and Proposition 3.9 can be used.

Following proposition, which is a reformulation and combination of Theorems A.6.1, A.6.2 and A.6.3 in [10], states that the partial multiplicities of an eigenvalue are unaffected by pre-multiplying and/or post-multiplying by an unimodular matrix. This result is used in the proof of the Theorem 3.2.

**Proposition 3.10** Consider $S_1(\xi)$ and $S_2(\xi) \in \mathbb{R}^{w \times w} [\xi]$ and let $p_j^1(\xi)$ and $p_j^2(\xi) \in \mathbb{R}[\xi]$ for $i = 1, \ldots, w$ be the invariant polynomials of $S_1(\xi)$ and $S_2(\xi)$ respectively. Suppose $S_1(\xi) = T_1S_2(\xi)T_2$, for invertible $T_1, T_2 \in \mathbb{R}^{w \times w}$. Let $\lambda \in \mathbb{C}$ and $\beta_1, \ldots, \beta_w$ be the maximum integers, for $j = 1, 2$, such that $(\xi - \lambda)^{\beta_j}$ divides $p_j^i(\xi)$ for $j = 1, \ldots, w$. Then $\beta_j^1 = \beta_j^2$ for $j = 1, \ldots, w$. In particular, if

\[
S_1(\xi) = \begin{bmatrix} \xi I - P & 0 \\ 0 & Q(\xi) \end{bmatrix}
\]

and

\[
S_2(\xi) = \begin{bmatrix} \xi I - P & 0 \\ 0 & I \end{bmatrix}
\]

and $\det Q(\lambda) \neq 0$, then the partial multiplicities of $S_1(\xi)$ and $S_2(\xi)$ corresponding to $\lambda$ are equal.

Before proving the main result Theorem 3.2 we state and prove another useful result. The following lemma relates the partial multiplicities of purely imaginary eigenvalues of the Hamiltonian matrix corresponding to the controllable part to that of the uncontrollable behavior.

**Lemma 3.11** Consider the behavior $\mathcal{B} \in \mathcal{L}^w$ with the set of uncontrollable modes $\Lambda_{un}$ satisfying $\Lambda_{un} \cap -\Lambda_{un} = \emptyset$. Let $\mathcal{B}$ have an observable i/s/o representation as $\frac{d}{dt}x = Ax + Bw_1$, $w_2 = Cx + Dw_1$, induced by $w = (w_1, w_2)$, such that $(I_m + D^T J_{pq} D) > 0$. Further, let $\mathcal{B}_{cont} = \text{im} M(\frac{d}{dt})$. Define $\Phi(\zeta, \eta) = M^T(\zeta) \Sigma M(\eta)$ and construct the Hamiltonian matrix, $H$ as in (10). Then

1. $\sigma(H) = \text{roots} (\det \partial \Phi(\xi)) \cup \Lambda_{un} \cup (-\Lambda_{un})$.

2. If the controllable part $\mathcal{B}_{cont}$ is $\Sigma$-dissipative, then the partial multiplicities corresponding to the purely imaginary eigenvalues of $H$, if any, are all even.

**Proof of Lemma 3.11**

**Statement 1:** See [12].

**Statement 2:** We use the fact that if the controllable part is dissipative, then the partial multiplicities corresponding to purely imaginary eigenvalues of $H_c$ are even. Without loss of generality the following i/s/o representation for $\mathcal{B}$ is assumed

\[
\frac{d}{dt} \begin{bmatrix} x_c \\ x_u \end{bmatrix} = \begin{bmatrix} \hat{A}_c & \hat{A}_{cp} \\ 0 & \hat{A}_u \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + \begin{bmatrix} \hat{B}_c \\ 0 \end{bmatrix} w_1
\]

\[
w_2 = \begin{bmatrix} \hat{C}_c & \hat{C}_u \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + \hat{D} w_1
\]

with $(\hat{A}_c, \hat{B}_c)$ controllable.

Then, the Hamiltonian matrix gets the following form
Now, since \( \Lambda_{un} \cap j\mathbb{R} = \emptyset \), for \( \lambda \in \sigma(H_c) \cap j\mathbb{R} \), the matrix blocks \( \lambda I_{nu} - A_u \) and \( \lambda I_{nu} + A_u^T \) are invertible.

Pre-multiplying \( H_1(\xi) \) by \( E_2 \) and post-multiplying by \( E_3 \), we get

\[
H_2(\xi) := E_2 H_1(\xi) E_3 = \begin{bmatrix}
\xi I_{nc} - A_c & -B_c B_c^T & 0 & 0 \\
-C_c^T C_c & \xi I_{nc} + A_c^T & -C_c^T C_u & 0 \\
0 & 0 & \xi I_{nu} - A_u & 0 \\
-C_u^T C_c & A_c^T & -C_u^T C_u & \xi I_{nu} + A_u^T \\
\end{bmatrix}
\]
where

\[
E_2 := \begin{bmatrix}
I_{n_c} & 0 & -T_3T_1^{-1} & 0 \\
0 & I_{n_c} & -T_4T_1^{-1} & 0 \\
0 & 0 & I_{n_u} & 0 \\
0 & 0 & 0 & I_{n_u}
\end{bmatrix},
\]

and

\[
E_3 := \begin{bmatrix}
I_{n_c} & 0 & 0 & 0 \\
0 & I_{n_c} & 0 & 0 \\
0 & 0 & I_{n_u} & 0 \\
-T_2^{-1}T_4^T & T_2^{-1}T_3^T & 0 & I_{n_u}
\end{bmatrix}
\]

and \(T_1 := \lambda I_{n_u} - A_u\), \(T_2 := \lambda I_{n_u} + A_u^T\), \(T_3 := -A_{cp}\) and \(T_4 := -C_c^T C_u\).

Thus

\[
H_2(\xi) = E_2E_1H(\xi)E_1E_2 = \begin{bmatrix}
\xi I_{2n_c} - H_c & 0 \\
0 & Q_u(\xi)
\end{bmatrix}
\]

where

\[
Q_u = \begin{bmatrix}
T_1 & 0 \\
-C_u^T C_u & T_2
\end{bmatrix}
\]

Thus by using Proposition 3.10, the partial multiplicities of purely imaginary eigenvalues of \(H\) are even. This completes the proof of Lemma 3.11.

4 Proof of the main result: Theorem 3.2

Proof: If part: Assume that the controllable part \(\mathfrak{B}_{\text{cont}}\) is \(\Sigma\)-dissipative and the assumption in the theorem is satisfied. By Propositions 3.3 and 3.4 to prove that the behavior \(\mathfrak{B}\) is dissipative, it suffices to show the existence of a \(K \in \mathbb{C}^{n \times n}\) such that the corresponding graph subspace is an \(n\)-dimensional, \(M\)-invariant, \(P\)-neutral subspace. To show the existence of such a \(K\), we use Lemma 3.9 to construct a \(c\)-set such that the corresponding \(n\)-dimensional \(M\)-invariant, \(P\)-neutral subspace is also a graph subspace of \(K\). This \(K\) is the solution to the ARE and storage function for the whole behavior would then be defined as \(x^TKx\), thus completing the proof.

As the unmixing assumption on uncontrollable modes is assumed, \(\lambda \in j\mathbb{R} \cap \sigma(H)\) means that \(\lambda \notin \Lambda_{un}\) and \(\lambda \in \sigma(H_c)\). As we have assumed that the controllable part is \(\Sigma\)-dissipative, from Lemma 3.11 the partial multiplicities of real eigenvalues of \(M(= iH)\) are all even. Using this fact, Lemma 3.9 can be used to infer that there exists a unique \(n\)-dimensional \(M\)-invariant, \(P\)-neutral subspace for every \(c\)-set.

Now, it remains to show the existence of a \(c\)-set such that the corresponding \(n\)-dimensional, \(M\)-invariant, \(P\)-neutral subspace is also a graph subspace.

We choose a \(c\)-set \(\mathcal{C}\) such that \(j\Lambda_{un} \subseteq \mathcal{C}\) and show that the corresponding \(n\)-dimensional, \(M\)-invariant, \(P\)-neutral subspace is a graph subspace. Let \(\mathcal{L}\) be the \(n\)-dimensional, \(P\)-neutral, \(M\)-invariant subspace of \(\mathbb{C}^{2n}\) corresponding to the \(c\)-set \(\mathcal{C}\) and suppose

\[
\mathcal{L} = \text{Im} \begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
\]

(16)
for matrices $X_1$ and $X_2 \in \mathbb{C}^{n \times n}$. In order to prove that $\mathcal{L}$ is a graph subspace it is enough to prove that $X_1$ is invertible. This is proved using contradiction: we assume $X_1$ is singular and show that we get a contradiction to the unmixing assumption on $\Lambda_{un}$. This constitutes the rest of the proof of the 'if part'.

We prove this along the lines of [10].

$M$-invariance of $\mathcal{L}$ implies that

$$
\begin{bmatrix}
\tilde{A} & \tilde{D} \\
\tilde{C} & -\tilde{A}^*
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
= 
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
T
$$

for $T \in \mathbb{C}^{n \times n}$. In other words,

$$
\begin{equation}
\begin{aligned}
j(\tilde{A}X_1 + \tilde{D}X_2) &= X_1T, \\
j(\tilde{C}X_1 - \tilde{A}^*X_2) &= X_2T.
\end{aligned}
\label{eq:17}
\end{equation}
$$

Then as $\mathcal{L}$ is $P$-neutral

$$
\begin{bmatrix}
X_1^* & X_2^*
\end{bmatrix}
\begin{bmatrix}
-\tilde{C} & \tilde{A}^* \\
\tilde{A} & \tilde{D}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
= 0,
\label{eq:18}
$$

i.e. $X_2^*\tilde{D}X_2 + X_1^*\tilde{A}^*X_2 + X_2^*\tilde{A}X_1 - X_1^*\tilde{C}X_1 = 0$.

Now suppose $X_1$ is singular. Let $\mathcal{K} = \ker X_1$. From (18), for every $x \in \mathcal{K}$ we have,

$$
x^*X_2^*\tilde{D}X_2x + x^*X_1^*\tilde{A}^*X_2x + x^*X_2^*\tilde{A}X_1x - x^*X_1^*\tilde{C}X_1x = 0,
$$

which implies

$$
x^*X_2^*\tilde{D}X_2x = 0.
\label{eq:19}
$$

Since $\tilde{D} \succeq 0$, $X_2x \in \ker \tilde{D}$, i.e., $X_2\mathcal{K} \subseteq \ker \tilde{D}$. Now, for every $x \in \mathcal{K}$, from equations (17) we have,

$$
X_1Tx = j\tilde{A}X_1x + j\tilde{D}X_2x = 0,
$$

that is,

$$
T\mathcal{K} \subseteq \mathcal{K}.
\label{eq:20}
$$

This implies $\mathcal{K}$ is $T$-invariant. Hence there exists a non-zero $v$ which is an eigenvector of $T$ such that $X_1v = 0$ corresponding to eigenvalue, say $\lambda$. We claim that $\lambda$ cannot be a real eigenvalue and prove this below.

Post-multiplying the second equation of (17) by $v$ we get,

$$
j\tilde{C}X_1v - j\tilde{A}^*X_2v = X_2Tv.
\label{eq:21}
$$

$$
-j\tilde{A}^*X_2v = \lambda X_2v.
\label{eq:22}
$$

This implies that $X_2v$ is a left eigenvector of $\tilde{A}$ with eigenvalue $-j\bar{\lambda}$ and from equation (19) we have $B^TX_2v = 0$. This means that $-j\bar{\lambda}$ is an uncontrollable eigenvalue of $\tilde{A}$ and $-j\bar{\lambda} \in \Lambda_{un}$, i.e., $\bar{\lambda} \in j\Lambda_{un}$.

Now, if $\lambda$ were a real eigenvalue, then $-j\bar{\lambda}$ and $j\bar{\lambda}$ belong to $\Lambda_{un}$ and contradicts the unmixing assumption $\Lambda_{un} \cap \Lambda_{un} = \emptyset$.

Since $\sigma(T) \setminus \mathbb{R} = \mathcal{C}$ (by Proposition 3.9), $\lambda \in \mathcal{C}$. We now have $\lambda \in \mathcal{C}$ but $\bar{\lambda}$ cannot belong to $\mathcal{C}$. But we have $\bar{\lambda} \in j\Lambda_{un}$ and thus $\bar{\lambda} \in \mathcal{C}$ which is a contradiction by definition of $c$-set. This completes the proof for the if part.
**Only if part:** Assume $\mathcal{B}$ is $\Sigma$-dissipative. Then there exists a storage function $Q_\Psi(w)$ such that
\[
\frac{d}{dt} Q_\Psi(w) \leq Q_\Sigma(w), \quad \forall w \in \mathcal{B}
\]
Integrating both sides for every $w \in \mathcal{B} \cap \mathcal{D}$,
\[
\int_{\mathbb{R}} Q_\Sigma(w) dt \geq 0
\]
which implies that $\mathcal{B}_{\text{cont}}$ is $\Sigma$-dissipative. This completes the proof of Theorem 3.2.

The above proof is constructive in the sense that if a behavior $\mathcal{B} \in \mathcal{L}^w$ satisfies the three conditions:

- uncontrollable poles are unmixed, i.e. no two of them add to zero
- the controllable part $\mathcal{B}_{\text{cont}}$ is dissipative,
- the controllable part $\mathcal{B}_{\text{cont}}$ is ‘strictly dissipative’ at infinity, i.e. $(I_m + D^T J_{pq} D) > 0$ where $D$ is the feed-through term of the transfer function,

then we construct a storage function that satisfies the dissipation inequality for the whole behavior $\mathcal{B}$. Further, the storage function we construct is equal to $x^T K x$ where $K$ is a solution to the corresponding Algebraic Riccati Equation. Further, we constructed the storage function by starting with a state representation in which the state $x$ is observable from the manifest variable $w$. These facts lead to the following important corollary.

**Corollary 4.1** Let $\mathcal{B} \in \mathcal{L}^w$ be an uncontrollable behavior whose uncontrollable poles $\Lambda_{un}$ are unmixed, i.e. $\Lambda_{un} \cap -\Lambda_{un} = \emptyset$. Consider $\Sigma$ partitioned in accordance with the input cardinality of $\mathcal{B}$ as
\[
\Sigma = \begin{bmatrix}
I_m & 0 & 0 \\
0 & I_q & 0 \\
0 & 0 & -I_p
\end{bmatrix}.
\]
Suppose $\mathcal{B}_{\text{cont}}$, the controllable part of $\mathcal{B}$, has an observable image representation, $w = M(\frac{d}{dt}) \ell$, where $M(\xi)$ is partitioned as
\[
M(\xi) = \begin{bmatrix}
W_1(\xi) \\
W_2(\xi)
\end{bmatrix}; \quad W_1 \in \mathbb{R}^{m\times m}[\xi], W_2 \in \mathbb{R}^{(p+q)\times m}[\xi].
\]
Let $G(s) := W_2(s) W_1(s)^{-1}$ and $D := \lim_{s \to \infty} G(s)$. Assume $\mathcal{B}_{\text{cont}}$ is such that $(I_m + D^T J_{pq} D) > 0$. Then, the following are equivalent.

1. $\mathcal{B}_{\text{cont}}$ is dissipative.
2. There exists a $\Psi \in \mathbb{R}^{w\times w}[\zeta, \eta]$ such that $Q_\Psi(w)$ is a storage function, i.e. $\frac{d}{dt} Q_\Psi(w) \leq w^T \Sigma w$ for all $w \in \mathcal{B}$.
3. There exists a matrix $K$ and an observable state variable $x$ such that $\frac{d}{dt} x^T K x \leq w^T \Sigma w$ for all $w \in \mathcal{B}$.

Statement 2 tells that the storage function can be expressed as a quadratic function of the manifest variables $w$ and their derivatives. Statement 3 says that the storage function is a ‘state function’, i.e. a static function of the states, and hence storage of energy requires no more memory of past evolution of trajectories than required for arbitrary concatenation of any two system trajectories.
In this section we discuss two examples of uncontrollable systems that are dissipative. The Riccati equations encountered in these cases are solvable by the methods proposed in this paper; we also give solutions to the Riccati equations.

The first example is of an uncontrollable system with uncontrollable modes satisfying the unmixing assumption, i.e. no two of the uncontrollable poles add to zero. However, the Hamiltonian matrix has eigenvalues on the imaginary axis.

**Example 5.1** Consider the behavior $\mathcal{B}$ whose input/state/output representation is given by the following $A, B, C$ and $D$ matrices

\[
A = \begin{bmatrix} 0 & -0.5 \\ 1 & -1.5 \end{bmatrix}, \quad B = \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -0.5 \end{bmatrix}, \quad D = 0.5
\]

with $\sigma(A) = \{-\frac{1}{2}, -1\}$. Here $\Lambda_{un} = \{-1\}$ which satisfies the unmixing assumption. An equivalent kernel representation of the behavior is given by

\[
\begin{bmatrix} (\xi^2 + 2\xi + 1) & -(2\xi^2 + 3\xi + 1) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0
\]

In this case $\Sigma = \text{diag}(1, -1)$ and hence $\sigma_+(\Sigma) = \mathcal{m}(\mathcal{B})$, it can be checked that the controllable part $\mathcal{B}_{cont} = \ker \begin{bmatrix} (\xi + 1) & -(2\xi + 1) \end{bmatrix}$ is $\Sigma$-dissipative. And $(I_m + D^T D) = 3/4 > 0$. Thus from Theorem 3.2, $\mathcal{B}$ is $\Sigma$-dissipative.

The following real symmetric matrix induces a storage function that satisfies the dissipation inequality

\[
K = \frac{1}{6} \begin{bmatrix} 7 & -1 \\ -1 & 1 \end{bmatrix}
\]

The 2-dimensional, $M$-invariant, $P$-neutral subspace which gives the solution is

\[
\begin{bmatrix} j & 2 \\ j & 8 \\ j & 1 \\ 0 & 1 \end{bmatrix}
\]

The next example is an RLC circuit shown in Figure 1.

**Example 5.2** Consider the RLC circuit system whose input $u$ is the current flowing into the circuit, and output $y$ is the current through the inductor. A state space representation of the system is found using the following definition of the states. The state variables are $x_C$, voltage across the capacitor, and $x_L$, the current through the inductor. Assume $R_L = R_C =: R$.

\[
A = \begin{bmatrix} 0 & -1/C \\ 1/L & -2R/L \end{bmatrix}, \quad B = \begin{bmatrix} 1/C \\ R/L \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = R,
\]

The system becomes uncontrollable when $L = R^2C$. Let $R = 0.5$, $C = 1$ and $L = 0.25$. The state representation of the system is
Figure 1: An RLC circuit system with input $u$ and output $y$

$$A = \begin{bmatrix} 0 & -1 \\ 4 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0,$$

Poles of the system are $-2, -2$ and one of them is uncontrollable. The controllable part is dissipative and the corresponding Hamiltonian matrix has eigenvalues on the imaginary axis. A solution to the ARE is given by the following symmetric matrix

$$K = \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.25 \end{bmatrix}$$

which induces the storage function $3x_C^2 - x_Cx_L + x_L^2/4$.

6 Behaviors with static controllable part

In this section we consider the case when all the states of the system are uncontrollable, in other words, when the only controllable part is static/memoryless. In a state space realization, this means that the matrix $B$ is zero. The case of autonomous behaviors is clearly a special case: $D$ also is zero and the assumptions in the lemma are satisfied.

Lemma 6.1 Consider a behavior $\mathcal{B}$ with static controllable part. Let the behavior have a state representation,

$$\frac{dx}{dt} = Ax, \quad w_2 = Cx + Dw_1.$$  \hfill (26)

with the pair $(C, A)$ observable. Assume $(I_m + D^TJ_{pq}D) > 0$ and $\Lambda_{un} \subset i\mathbb{R}$. Then there does not exist a symmetric solution to the corresponding ARE.

Proof: The Hamiltonian matrix takes the form,

$$H = \begin{bmatrix} \tilde{A} & 0 \\ \tilde{C} & -\tilde{A}^* \end{bmatrix}$$

Here $\tilde{A} = A$, $\tilde{C} = C^T(J_{pq} + DD^T)^{-1}C$.

Next we use the following proposition to say that partial multiplicities of purely imaginary eigenvalues of the Hamiltonian matrix are even.
Proposition 6.2 Consider the matrix

\[ N = \begin{bmatrix} \hat{A} & \hat{D} \\ 0 & -\hat{A}^* \end{bmatrix}. \]  

Then for every purely imaginary \( \lambda_0 \in \sigma(N) \) such that \( R_{\lambda_0}(N) \subseteq \sigma(\hat{A}, \hat{D}) \), the partial multiplicities of such \( \lambda_0 \) are even. In fact, they are twice the partial multiplicities of \( \lambda_0 \) as an eigenvalue of \( A \).

Consider,

\[ H^T = \begin{bmatrix} \hat{A}^T & \hat{C}^T \\ 0 & -\hat{A} \end{bmatrix}. \]  

As the pair \((\hat{C}, \hat{A})\) is observable, the pair \((\hat{A}^T, \hat{C}^T)\) is controllable. From Proposition 6.2, the partial multiplicities of purely imaginary eigenvalues are twice the partial multiplicities of purely imaginary eigenvalues of \( A \).

7 Lossless autonomous behaviors

In this section we investigate the requirement of unobservable variables in the definition of the storage function. As has been studied/shown so far, for controllable dissipative systems (\cite{20}), the storage function need not depend on unobservable variables. It was later shown in \cite{12} that for the case of strict dissipativity of uncontrollable systems, observable storage functions are enough under unmixing and maximum input cardinality conditions. Theorem 3.2 shows that this is true for a more general scenario, i.e., dissipativity (including non-strict dissipativity) for all input cardinality conditions under unmixing assumption. On the other hand, when relaxing the unmixing assumption elsewhere except on the imaginary axis, under certain conditions solutions to the ARI exists though ARE does not have a solution (see \cite{9}). In this section, we investigate the need for unobservable storage functions for uncontrollable systems whose uncontrollable poles lie entirely on the imaginary axis. We discuss the case for autonomous behaviors below.

Lemma 7.1 Consider an autonomous behavior \( \mathcal{B}_{aut} \)

\[ \frac{d}{dt} x = Ax, \quad w = Cx, \]

with \( \sigma(A) \cap i\mathbb{R} \neq \emptyset \). Let the supply rate be \( Q_\Sigma(w) = -w^T w \). Then the following is true.

If there exists a storage function \( Q_\psi(w) \) satisfying the inequality

\[ \frac{d}{dt} Q_\psi(w) \leq Q_\Sigma(w), \quad \forall w \in \mathcal{B}_{aut} \]  

(30)

then any \( \lambda \in \sigma(A) \cap i\mathbb{R} \) is \( C \)-unobservable.

Proof: Suppose if there exists a storage function which is a state function \( x^T K x \) satisfying the dissipation LMI, then the dissipation LMI (7) is equivalent to the Lyapunov inequality

\[ KA + A^* K + C^* C \leq 0. \]  

(31)

Now for every eigenvector of \( A \) corresponding to eigenvalue \( \lambda \in i\mathbb{R} \), we have

\[ x^* (KA + A^* K + C^* C) x \leq 0, \]  

(32)
which gives
\[ \lambda x^* K x + \bar{\lambda} x^* K x + x^* C^* C x \leq 0 \] (33)
or
\[ x^* C^* C x \leq 0. \] (34)

But, as \( x^* C^* C x \geq 0 \), we have \( C x = 0 \) for every eigenvector of \( A \) corresponding to \( \lambda \in \sigma(A) \cap i \mathbb{R} \). This implies that the any \( \lambda \in \sigma(A) \cap i \mathbb{R} \) is \( C \)-unobservable.

This observation tells that for dissipativity of autonomous systems having eigenvalues on the imaginary axis, it is necessary to allow storage functions to depend on unobservable variables also.

### 8 Orthogonality and uncontrollable behaviors

In this section we investigate the property of orthogonality of two behaviors in the absence of controllability. We propose a definition that is intuitively expected and show that by relating this definition to lossless uncontrollable behaviors, we encounter a situation that suggests an exploration whether dissipativity should be defined for behaviors for which the input-cardinality condition is not satisfied. We first review a result about orthogonality of controllable behaviors.

**Proposition 8.1** Let \( \Sigma \in \mathbb{R}^{w \times w} \) be nonsingular, and suppose \( \mathcal{B}_1, \mathcal{B}_2 \in \mathcal{L}_{\text{cont}}^w \). The following are equivalent.

1. \( \int_{\mathbb{R}} w_1^T \Sigma w_2 dt = 0 \) for all \( w_1 \in \mathcal{B}_1 \cap \mathcal{D} \) and for all \( w_2 \in \mathcal{B}_2 \cap \mathcal{D} \).

2. \( \mathcal{B}_1 \times \mathcal{B}_2 \) is lossless with respect to
\[ \begin{bmatrix} 0 & \Sigma \\ \Sigma^T & 0 \end{bmatrix}. \]

3. There exists a bilinear differential form \( L_\Psi \), induced by \( \Psi \in \mathbb{R}^{w \times w}[\zeta, \eta] \) such that
\[ \frac{d}{dt} L_\Psi(w_1, w_2) = w_1^T \Sigma w_2. \]

Statement 1 above is taken as the definition of orthogonality between two controllable behaviors \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) in [20]. Keeping in line with Definition 3.1 for dissipativity, we could take Statement 3 above as the definition of orthogonality for behaviors not necessarily controllable. The drawback of this approach is elaborated later below in this section. We pursue a different direction as follows. Notice that if \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) satisfy the integral condition in Statement 1, then this integral condition is satisfied for every respective sub-behaviors \( \mathcal{B}_1' \) and \( \mathcal{B}_2' \) also. Of course, restricting to compactly supported trajectories in the integration implies only controllable parts of respectively \( \mathcal{B}_1' \) and \( \mathcal{B}_2' \) satisfy orthogonality. The following definition builds on this property.

**Definition 8.2** Consider a nonsingular \( \Sigma \in \mathbb{R}^{w \times w} \) and let \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \in \mathcal{L}^w \). Behaviors \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) are said to be \( \Sigma \)-orthogonal (and denoted by \( \mathcal{B}_1 \perp_{\Sigma} \mathcal{B}_2 \)) if there exist \( \mathcal{B}_1^\epsilon \) and \( \mathcal{B}_2^\epsilon \in \mathcal{L}_{\text{cont}}^w \) such that

- \( \mathcal{B}_1 \subseteq \mathcal{B}_1^\epsilon, \)
- \( \mathcal{B}_2 \subseteq \mathcal{B}_2^\epsilon, \) and
- \( \int_{\mathbb{R}} w_1^T \Sigma w_2 dt = 0 \) for all \( w_1 \in \mathcal{B}_1^\epsilon \cap \mathcal{D} \) and for all \( w_2 \in \mathcal{B}_2^\epsilon \cap \mathcal{D} \).
While this definition is not existential in the storage function, it is existential in \( B_1^c \) and \( B_2^c \), raising questions about how to check orthogonality. Note that if \( B \) is an uncontrollable behavior, then any controllable \( B^c \in \mathcal{L}_w^{\text{cont}} \) such that \( B \subseteq B^c \) satisfies \( m(B) < m(B^c) \), and \( B \subseteq B^c \).

The question arises as to how much larger a controllable \( B^c \) would have to be for it to contain \( B \). This problem is addressed in the following subsection.

8.1 Smallest controllable superbehavior

Due to its significance for determining whether two uncontrollable behaviors are orthogonal, in this subsection we study the following problem:

**Problem 8.3** Let \( B_1 \in \mathcal{L}_w \). Find \( B_2 \in \mathcal{L}_w \) such that

1. \( B_2 \supseteq B_1 \)
2. \( B_2 \in \mathcal{L}_w^{\text{cont}} \) i.e., a controllable behavior
3. \( B_2 \) is a behavior with the smallest input cardinality satisfying Properties 1 and 2.

The following theorem answers this question.

**Theorem 8.4** For \( B \in \mathcal{L}_w \), the following statements are true.

1. There exists \( B_2 \in \mathcal{L}_w^{\text{cont}} \) satisfying the requirements in Problem 8.3.
2. The behavior \( B_2 \) is unique if and only if \( B_1 \) is controllable, and in that case \( B_1 = B_2 \).
3. Assume \( B_1 \) is uncontrollable. The input cardinality of \( B_2 \), \( m(B_2) \) satisfies \( m(B_2) > m(B_1) \). More precisely, \( m(B_2) = m(B_1) + k \) where \( k := \max_{\lambda \in \mathbb{C}} \text{rank} (R_1(\lambda)) - \min_{\lambda \in \mathbb{C}} \text{rank} (R_1(\lambda)) \) where \( R_1 \) is a kernel representation of \( B_1 \).

**Proof.** (1): This is shown by constructing such a \( B_2 \). Let \( \ker R_1(\xi) \) and \( \ker R_2(\xi) \) be the kernel representations of \( B_1 \) and \( B_2 \) respectively where \( R_1 \in \mathbb{R}^{p_1 \times w} \), \( R_2 \in \mathbb{R}^{p_2 \times w} \) and there exists an \( F \in \mathbb{R}^{p_2 \times p_1} \) such that \( FR_1 = R_2 \). Let \( A_{un} \) be the set of uncontrollable modes of \( B_1 \). Without loss of generality, we assume \( R_1 \) to be

\[
R_1 = \begin{bmatrix} S & 0 \end{bmatrix}
\]

where \( S \) is of the form

\[
\begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}
\]

such that \( I \in \mathbb{R}^{(p-k) \times (p-k)} \) is identity matrix and \( D \in \mathbb{R}^{k \times k}[\xi] \) is a diagonal matrix with \( d_1, d_2, \ldots, d_k \) along its diagonal and satisfying the divisibility property: \( d_1 | d_2, d_2 | d_3, \ldots, d_{k-1} | d_k \), with degree of \( d_1 \) at least one, and with

\[
k = \max_{\lambda \in \mathbb{C}} \text{rank} (R_1(\lambda)) - \min_{\lambda \in \mathbb{C}} \text{rank} (R_1(\lambda)).
\]

Partitioning \( F = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \) conforming to the row partition of \( R_1 \), we have

\[
R_2 = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & D & 0 \end{bmatrix}
\]

19
This simplifies to
\[ R_2 = \begin{bmatrix} F_1 & F_2D & 0 \end{bmatrix} \] (37)

Let \( \mathcal{B}_2 \) be the behaviour defined by the kernel representation of \( R_2 \). For \( \mathcal{B}_2 \) to be controllable, \( R_2(\lambda) \) needs to have full row rank for all \( \lambda \in \mathbb{C} \). As \( F_2P \) loses rank for \( \lambda \in \Lambda_{\text{un}} \), \( F_1 \) should have full row rank for every \( \lambda \in \mathbb{C} \) so that \( \mathcal{B}_2 \) is controllable. This proves the existence of \( \mathcal{B}_2 \) satisfying properties 1 and 2 of 8.3. In order to satisfy property 3 in Problem 8.3 i.e. \( m(\mathcal{B}_2) \) has to be the least, we have to choose a unimodular \( F_1 \) and free \( F_2 \) such that \( \mathcal{B}_2 \) satisfies the three conditions. \( F_2 \) can be freely chosen because the choice of \( F_2 \) does not affect the input cardinality of \( \mathcal{B}_2 \).

(2): Let \( \mathcal{B}_1 \) be controllable. Then, in the above, \( S = I \) and \( D \) does not exist. This means that \( F_2 \) does not exist. Thus the kernel representation matrix of \( \mathcal{B}_2 \) would be given by \( [F_1 \ 0] \) where \( F_1 \) is unimodular. Therefore, \( \mathcal{B}_2 \) is unique. Let \( \mathcal{B}_1 \) be uncontrollable. Then \( D \) exists and as shown above, free \( F_2 \) can be chosen. This makes the behaviour \( \mathcal{B}_2 \) non-unique.

(3): The input cardinality of \( \mathcal{B}_2 \) is given by
\[
m(\mathcal{B}_2) = w - p_2
\]
\[= w - (p_1 - k)
\]
\[= m(\mathcal{B}_1) + k \] (38)

Here, \( k = \max_{\lambda \in \mathbb{R}} \text{rank} (R_1(\lambda)) - \min_{\lambda \in \mathbb{R}} \text{rank} (R_1(\lambda)) \) which is determined by the size of the \( D \) matrix. \( \square \)

8.2 Superbehaviors and orthogonality

We saw above in this section that orthogonality of two uncontrollable behaviors is defined by requiring these uncontrollable behaviors to be sub-behaviors of two orthogonal controllable behaviors. Using the result on existence of super-behaviors that are controllable, and their non-uniqueness even if the are the smallest controllable superbehavior, we formulate the question of whether two uncontrollable behaviors are orthogonal as a question of finding a pair of smallest controllable superbehaviors that are mutually orthogonal. The requirement of them being smallest is motivated by the fact that orthogonality of two controllable behaviors imposes an upper bound on their input cardinalities: this is reviewed below. For a behavior \( \mathcal{B} \in \mathcal{L}^w \) and a nonsingular matrix \( \Sigma \), the set \( \Sigma \mathcal{B} \) is defined as follows
\[
\Sigma \mathcal{B} := \{ w \in C^\infty(\mathbb{R}, \mathbb{R}^w) | \text{there exists } v \in \mathcal{B} \text{ such that } w = \Sigma v \}.
\]

It is straightforward that \( \Sigma \mathcal{B} \) is also a behavior, its controllability is equivalent to that of \( \mathcal{B} \), and the input cardinalities are equal.

**Proposition 8.5** Let \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \in \mathcal{L}_w^{\text{cont}} \) and suppose \( \Sigma \in \mathbb{R}^{w \times w} \) is nonsingular. Then, the following are true.

1. \( \mathcal{B}_1 \bot_\Sigma \mathcal{B}_2 \iff \mathcal{B}_1 \bot (\Sigma \mathcal{B}_2) \).
2. \( \mathcal{B}_1 \bot \mathcal{B}_2 \Rightarrow m(\mathcal{B}_1) + m(\mathcal{B}_2) \leq w. \)
3. $B_1 \perp_{\Sigma} B_2 \Rightarrow m(B_1) + m(B_2) \leq w$.

Due to the above inequality constraint on the input-cardinalities of orthogonal controllable behaviors, the uncontrollable behaviors too have a necessary condition to satisfy for mutual orthogonality.

Lemma 8.6 Suppose $B_1$ and $B_2 \in \mathcal{L}^w$ with at least one of them uncontrollable and let $\Sigma \in \mathbb{R}^{w \times w}$ be nonsingular. Assume $B_1 \perp_{\Sigma} B_2$. Then $m(B_1) + m(B_2) < w$.

Example 8.7 Consider the pair of ‘seemingly’ orthogonal behaviors studied in [22, page 360]. Define $B_1$ and $B_2 \in \mathcal{L}^w$ by

$$\frac{d}{dt}x = Ax, \quad w_1 = Cx, \quad w_2 \text{ free, i.e. } B_2 = C^{\infty}(\mathbb{R}, \mathbb{R}^w)$$

and the supply rate $w_1^Tw_2$. Thus $B_1$ is autonomous. Consider the following non-observable latent variable representation for $B_2$: $\frac{d}{dt}z = -A^Tz + C^Tw_2$. It can be checked that the ‘storage function’ $x^Tz$ satisfies $\frac{d}{dt}x^Tz = w_1^Tw_2$. Existence of such a storage function is, in fact, reasonable for a (different) definition of orthogonality of the two behaviors $B_1$ and $B_2$. In fact, any autonomous behavior $B_1 \in \mathcal{L}^w$ is then ‘orthogonal’ to $B_2 = C^{\infty}(\mathbb{R}, \mathbb{R}^w)$! However, the ‘embeddability’ definition we have used above rules out this example for an orthogonal pair of behaviors since the necessary condition of the above lemma is not satisfied. In other words, there doesn’t exist a controllable behavior $B^c_1$ such that $B^c_1$ contains $B_1$ and $B^c_1 \perp_{\Sigma} B_2$. Thus $B_1$ and $B_2$ are not $I$-orthogonal. □

9 Dissipative sub-behaviors/superbehaviors

In this section we look into the input cardinality condition for dissipative behaviors. Recall that a behavior $B \in \mathcal{L}^w$ which is dissipative with respect to the supply rate $\Sigma$ (constant, symmetric, nonsingular matrix) satisfies the condition $m(B) \leq \sigma_+(\Sigma)$. We now look into the possibility of embedding a behavior $B$ in a controllable superbehavior that is $\Sigma$-dissipative, and into the drawback of using this as the definition of dissipativity, along the lines of orthogonality defined in the previous section.

Problem 9.1 Given a nonsingular, symmetric and indefinite $\Sigma \in \mathbb{R}^{w \times w}$, find conditions for existence of a behavior $B \in \mathcal{L}^w$ such that

- there exist $B_+ \text{ and } B_- \in \mathcal{L}^w_{\text{cont}}$ with $B = B_+ \cap B_-$.  
- $B_+$ is strictly $\Sigma$ dissipative
- $B_-$ is strictly $-\Sigma$ dissipative.

The significance of the above problem is that if a nonzero behavior $B$ satisfying above conditions exists, then clearly such a behavior would be both strictly $\Sigma$ and strictly $-\Sigma$ dissipative, raising concerns about whether embeddability in a dissipative controllable superbehavior is a reasonable definition of dissipativity (when dealing with uncontrollable behaviors). The following theorem states that nonzero autonomous behaviors can indeed exist satisfying above condition.

Theorem 9.2 Let $\Sigma \in \mathbb{R}^{w \times w}$ be nonsingular, symmetric and indefinite. Then there exists a nonzero $B \in \mathcal{L}^w$ such that requirements in Problem 9.1 are satisfied. Further, any such $B$ satisfies $m(B) = 0$, i.e. $B$ is autonomous.
Proof. We first show that given any nonsingular, symmetric and indefinite \( \Sigma \in \mathbb{R}^{w \times w} \), a behavior \( \mathcal{B} \) satisfying above properties exists. Without loss of generality, let

\[
\Sigma = \begin{bmatrix}
I_+ & 0 \\
0 & -I_-
\end{bmatrix}
\]

with sizes of the identity matrices \( I_+ \) and \( I_- \) equal to \( \sigma_+ \) and \( \sigma_- \). Due to indefiniteness of \( \Sigma \), \( \sigma_+, \sigma_- \geq 1 \) and, due to the nonsingularity, they sum up to \( w \). Choose any\(^2\) nonzero polynomial matrix \( M_+ \in \mathbb{R}^{w \times m_+}[\xi] \) with \( M_+^T(-j\omega)\Sigma M_+(j\omega) \geq \epsilon_+ I_w \) for all \( \omega \in \mathbb{R} \) for some \( \epsilon_+ > 0 \). Similarly, choose \( M_- \in \mathbb{R}^{w \times m_-}[\xi] \) such that \( M_-^T(-j\omega)\Sigma M_-(j\omega) \leq -\epsilon_- I_w \) for all \( \omega \in \mathbb{R} \) for some \( \epsilon_- > 0 \). Now define \( \mathcal{B}_+ \) and \( \mathcal{B}_- \) by image representations \( w = M_+(\frac{d}{dt})\ell \) and \( w = M_-(\frac{d}{dt})\ell \) respectively. Define \( \mathcal{B} := \mathcal{B}_+ \cap \mathcal{B}_- \), thus proving\(^3\) existence of \( \mathcal{B} \) as stated in the theorem.

We now show that \( \mathcal{B} \) is autonomous. Let \( R_+ \) and \( R_- \) be minimal kernel representation matrices of \( \mathcal{B}_+ \) and \( \mathcal{B}_- \) respectively. Then \( \mathcal{B} = \mathcal{B}_+ \cap \mathcal{B}_- \) is described by the kernel representation matrix \( R \in \mathbb{R}^{\bullet \times w}[\xi] \) with \( R := \begin{bmatrix} R_+ & R_- \end{bmatrix} \). Clearly, \( \text{rank} (R) \leq w \). Suppose \( \text{rank} (R) < w \). Then, there exists \( p \in \mathbb{R}^w[s] \) and \( p \neq 0 \) such that \( R_+ p = 0 \) and \( R_- p = 0 \). This implies \( \text{Image} (p(\frac{d}{dt})) \in \mathcal{B}_+ \) and \( \text{Image} (p(\frac{d}{dt})) \in \mathcal{B}_- \).

Taking \( w = p(\frac{d}{dt})\ell \) with \( \ell \in \mathcal{D}(\mathbb{R}, \mathbb{R}) \) and \( \ell \neq 0 \), it follows that \( w \in \mathcal{D}(\mathbb{R}^w, \mathbb{R}) \). Further, \( p \neq 0 \), hence \( w \neq 0 \) because \( \ell \) is non-zero and of compact support. Further, we have \( \epsilon_+, \epsilon_- > 0 \) such that

\[
\int_{-\infty}^{\infty} w^T \Sigma w dt \geq \epsilon_+ \|w\|_{L_2}^2 \quad \text{and} \quad \int_{-\infty}^{\infty} w^T \Sigma w dt \leq -\epsilon_- \|w\|_{L_2}^2
\]

Both the above conditions cannot be satisfied simultaneously for \( w \neq 0 \). Thus, \( \text{rank} (R) < w \) gives a contradiction. This proves \( \text{rank} (R) = w \) and hence autonomy of \( \mathcal{B} \).

We illustrate the above theorem using an example.

\textbf{Example 9.3} Let \( \Sigma = \text{diag} \ (1, -1) \). Define \( \mathcal{B}_+ \) and \( \mathcal{B}_- \) by image representations \( w = M_+(\frac{d}{dt})\ell \) and \( w = M_-(\frac{d}{dt})\ell \) respectively, with

\[
M_+(\xi) = \begin{bmatrix}
\xi + 4 \\
3
\end{bmatrix} \quad \text{and} \quad M_-(\xi) = \begin{bmatrix}
2 \\
\xi + 5
\end{bmatrix}.
\]

Strict dissipativities is easily verified. Calculating the kernel representations, we get a kernel representation for \( \mathcal{B} := \mathcal{B}_+ \cap \mathcal{B}_- \) as \( R(\frac{d}{dt})w = 0 \)

\[
R(\xi) = \begin{bmatrix}
-3 & \xi + 4 \\
\xi + 5 & -2
\end{bmatrix}.
\]

Clearly, \( R \) is nonsingular and hence \( \mathcal{B} \) is autonomous. \( \square \)

For non-strict dissipativity case, we have the following problem and theorem.

\(^2\)Such matrices are plenty due to existence of sufficiently many controllable strictly dissipative behaviors for every supply rate \( \Sigma \) satisfying the stated conditions.

\(^3\)It is not difficult to show that if \( M_+ \) and \( M_- \) were nonconstant polynomial matrices, then \( \mathcal{B} \) is not the zero behavior. An example following the proof makes this easier to see.
**Problem 9.4** Given a nonsingular, symmetric and indefinite $\Sigma \in \mathbb{R}^{w \times w}$, find conditions for existence of a behavior $\mathcal{B} \in \mathcal{L}^w$ such that

- there exist $\mathcal{B}_+$ and $\mathcal{B}_- \in \mathcal{L}^w_{\text{cont}}$ such that $\mathcal{B} = \mathcal{B}_+ \cap \mathcal{B}_-$.
- $\mathcal{B}_+$ is $\Sigma$ dissipative
- $\mathcal{B}_-$ is $-\Sigma$ dissipative.

**Theorem 9.5** Let $\Sigma \in \mathbb{R}^{w \times w}$ be nonsingular, symmetric and indefinite. Then there exists $\mathcal{B} \in \mathcal{L}^w$ such that requirements in Problem 9.4 are satisfied. Any such $\mathcal{B}$ satisfies $m(\mathcal{B}) \leq \min(\sigma_+(\Sigma), \sigma_-(\Sigma))$. In case $\mathcal{B}$ is uncontrollable, $m(\mathcal{B}) < \min(\sigma_+(\Sigma), \sigma_-(\Sigma))$

If $m(\mathcal{B}) \geq 1$, then neither $\mathcal{B}_+$ nor $\mathcal{B}_-$ can be strictly dissipative.

**Proof.** The proof proceeds in the same way as the proof for the previous theorem, except for the strictness of the dissipativities. Construct $\mathcal{B}_+$ and $\mathcal{B}_-$ as in the previous proof, but with $\epsilon_+$ and $\epsilon_-$ equal to zero. We have

$$\int_{-\infty}^{\infty} w^T \Sigma w dt \geq 0 \text{ for all } w \in \mathcal{B}_+ \cap \mathcal{D} \quad \text{and} \quad \int_{-\infty}^{\infty} w^T \Sigma w dt \leq 0 \text{ for all } w \in \mathcal{B}_- \cap \mathcal{D}$$

The above two equations imply $\int_{-\infty}^{\infty} w^T \Sigma w dt = 0$ for all $w \in \mathcal{B} \cap \mathcal{D}$. Since $\mathcal{B} = \mathcal{B}_+ \cap \mathcal{B}_-$, the behavior $\mathcal{B}$ is dissipative with respect to both $\Sigma$ and $-\Sigma$. Dissipativity with respect to $\Sigma$ implies $m(\mathcal{B}) \leq \sigma_+(\Sigma)$. Similarly, dissipativity with respect to $-\Sigma$ implies $m(\mathcal{B}) \leq \sigma_-(\Sigma)$. This implies

$$m(\mathcal{B}) \leq \min(\sigma_+, \sigma_-) \quad (39)$$

If $\mathcal{B}$ is uncontrollable, then from Theorem 8.4, then the two inequalities leading to the inequality (39) are both strict. Hence, the input cardinality of $\mathcal{B}$ has to be strictly less than that of $\mathcal{B}_+$ and $\mathcal{B}_-$. This implies

$$m(\mathcal{B}) < \min(\sigma_+, \sigma_-) \quad (40)$$

If $m(\mathcal{B}) \geq 1$, then from Theorem 9.2, $\mathcal{B}_+$ and $\mathcal{B}_-$ cannot be strictly dissipative with respect to $\Sigma$ and $-\Sigma$ respectively. This completes the proof. \[\square\]

As one of the consequences of the above theorem, if the input cardinality condition is satisfied for an uncontrollable behavior, i.e. $m(\mathcal{B}) = \sigma_+(\Sigma)$ or $m(\mathcal{B}) = \sigma_-(\Sigma)$, then such a behavior cannot be embedded into both a $\Sigma$-dissipative controllable behavior and a $-\Sigma$-dissipative controllable behavior. However, an observable storage function for such a situation exists when the controllable part is strictly dissipative at $\infty$ and when the uncontrollable modes satisfy the unmixing condition (see 3.2 and [16, 12] for this situation in presence of more assumptions), A situation when $m(\mathcal{B}) = \sigma_+(\Sigma)$ is very familiar: we deal with RLC circuits in the next section.

6. We use this method of defining orthogonality of two behaviors to explore further the definition of dissipativity of a behavior. Here we bring out a fundamental significance of the so-called input-cardinality condition: the condition that the number of inputs to the system is equal to the positive signature of the matrix that induces the power supply. We show that when this condition is not satisfied, then a behavior could be both supplying and absorbing net power, and is still not lossless.
10 RLC realizability

In this brief section we revisit a classical result: a rational transfer function matrix being positive real is a necessary and sufficient condition for that transfer matrix to be realizable using only resistors, capacitors and inductors (see [6] and also [4] for the case with transformers). Note that the transfer matrix captures only the controllable part of the behavior and positive realness of the transfer matrix is nothing but dissipativity with respect to the supply rate \( v^T i \). This is made precise below.

Consider an \( n \)-port electrical network (with each port having two terminals) and the variable \( w = (v, i) \), where \( v \) is the vector of voltages across the \( n \)-ports and \( i \) is the vector of currents through these ports, with the convention that \( v^T i \) is the power flowing into the network. Behaviors that are dissipative with respect to the supply rate \( v^T i \) are also called 'passive'. Given a controllable behavior \( \mathcal{B} \) whose transfer matrix \( G \) with respect to a specific input/output partition, say current \( i \) is the input and voltage \( v \) is the output, is positive real, one can check that this behavior is passive. \( G \) in this case is the impedance matrix and is square, i.e. the number of inputs is equal to the number of outputs. One can introduce additional laws that the variables need to satisfy, thus resulting in a sub-behavior \( \mathcal{B}_{\text{sub}} \subseteq \mathcal{B} \) which has a lesser number of inputs; consider for example these additional laws as putting certain currents equal to zero: due to opening of certain ports. However, the transfer matrix for \( \mathcal{B}_{\text{sub}} \) with respect to the input/output partition: input as currents through the non-open ports and output as the voltages across all the ports, is clearly not square, and in fact, tall, i.e. has strictly more rows than columns. Let \( G_{\text{sub}} \) denote this transfer function. Since \( \mathcal{B}_{\text{sub}} \subseteq \mathcal{B} \), the behavior \( \mathcal{B}_{\text{sub}} \) is also passive. Of course, \( \mathcal{B}_{\text{sub}} \) need not be controllable, even if \( \mathcal{B} \) is assumed to be controllable. As an extreme case, suppose all the currents are equal to zero, and we obtain an autonomous \( \mathcal{B}_{\text{sub}} \). While RLC realization of such transfer matrices which are tall, and further of autonomous behaviors obtained by, for example, opening all ports, has received hardly any attention, we remark here one very well-studied sub-behavior of every passive behavior: the zero behavior.

Consider the single port network with \( v = 0 \) and \( i = 0 \). This port which is called a nullator (also nullor, in some literature) behaves as both the open circuit and shorted circuit: see [2, page 75] and [3]. The significant fact about a nullator is that a nullator cannot be realized using only passive elements, and moreover any realization (necessarily active) leads to the realization of both a nullator and its companion, the norator; a norator is a two-terminal port that allows both the voltage across it and current through it to be arbitrary.

11 Concluding remarks

We briefly review the main results in this paper. We first used the existence of an observable storage function as the definition of a system’s dissipativity and proved that the dissipativities of a behavior and its controllable part are equivalent assuming the uncontrollable poles are unmixed and the dissipativity at infinity frequency is strict. This result’s proof involved new results in the solvability of ARE and used indefinite linear algebra results.

We showed that for lossless autonomous dissipative systems, the storage function cannot be observable,

\footnote{It is also common to require the storage function in the context of this dissipativity to satisfy non-negativity. The sign-definiteness of the storage function is not the focus of this paper, hence we ignore this aspect.}
thus motivating the need for unobservable storage functions. We then studied orthogonality/lossless behaviors in the context of using the definition of existence of a controllable dissipative superbehavior as a definition of dissipativity. In addition to results about the smallest controllable superbehavior, we showed necessary conditions on the number of inputs for embeddability of lossless/orthogonal behaviors in larger controllable such behaviors.

In the context of embeddability as a definition of dissipativity, we showed that one can always find behaviors that can be embedded in both a strictly dissipative behavior and a strictly ‘anti-dissipative’ behavior, thus raising a question on the embeddability definition. We related this question to the well-known result that the nullator one-port circuit is not realizable using only RLC elements.

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