Boundaries of reduced C*-algebras of discrete groups

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$$\lambda : \ell^\infty(G) \to \mathbb{C},$$

i.e. a unital positive $G$-invariant linear map.
Definition

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i.e. a unital positive $G$-invariant linear map.

In this case, $\lambda$ is a unital positive $G$-equivariant projection.
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A discrete group $G$ is **amenable** if there is a unital positive $G$-equivariant projection

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Therefore, $G$ is non-amenable if $\mathbb{C}$ is “too small” to be the range of a unital positive $G$-equivariant projection on $\ell^\infty(G)$. 
Consider the minimal C*-subalgebra $A_G$ of $\ell^\infty(G)$ such that there is a unital positive $G$-equivariant projection

$$P : \ell^\infty(G) \rightarrow A_G.$$
| Idea |
|------|
| Consider the minimal C*-subalgebra $A_G$ of $\ell^\infty(G)$ such that there is a unital positive $G$-equivariant projection $P : \ell^\infty(G) \to A_G$. |

The size of $A_G$ should somehow “measure” the non-amenability of $G$. 
Theorem (Kalantar-K 2014)

There is a unique minimal C*-algebra $A_G$ arising as the range of a unital positive $G$-equivariant projection $P : \ell^\infty(G) \to A_G$.

The algebra $A_G$ is isomorphic to the algebra $C(\partial F G)$ of continuous functions on the Furstenberg boundary $\partial F G$ of $G$. 
Motivation
Kirchberg proved that every exact C*-algebra can be embedded into a nuclear C*-algebra.
Kirchberg proved that every exact $C^*$-algebra can be embedded into a nuclear $C^*$-algebra.

In the separable case, Kirchberg and Phillips proved the nuclear $C^*$-algebra can be taken to be the Cuntz algebra on two generators.
Ozawa conjectured the existence of what he calls a “tight” nuclear embedding.

Conjecture (Ozawa 2007)

Let $\mathcal{A}$ be an exact C*-algebra. There is a canonical nuclear C*-algebra $N(\mathcal{A})$ such that

$$\mathcal{A} \subset N(\mathcal{A}) \subset I(\mathcal{A}),$$

where $I(\mathcal{A})$ denotes the injective envelope of $\mathcal{A}$. 
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The algebra $\mathcal{N}(\mathcal{A})$ will inherit many properties from $\mathcal{A}$, for example simplicity and primality.
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**Theorem (Ozawa 2007)**

Let $C^*_r(\mathbb{F}_n)$ denote the reduced C*-algebra of $\mathbb{F}_n$ for $n \geq 2$. There is a canonical nuclear C*-algebra $N(C^*_r(\mathbb{F}_n))$ such that

$$C^*_r(\mathbb{F}_n) \subset N(C^*_r(\mathbb{F}_n)) \subset I(C^*_r(\mathbb{F}_n)),$$

where $I(C^*_r(\mathbb{F}_n))$ denotes the injective envelope of $C^*_r(\mathbb{F}_n)$. 
Ozawa proved this conjecture for the reduced C*-algebra of the free group $F_n$ on $n \geq 2$ generators.

**Theorem (Ozawa 2007)**

Let $C^*_r(F_n)$ denote the reduced C*-algebra of $F_n$ for $n \geq 2$. There is a canonical nuclear C*-algebra $N(C^*_r(F_n))$ such that

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where $I(C^*_r(F_n))$ denotes the injective envelope of $C^*_r(F_n)$.

Note that $C^*_r(F_n)$ is exact since $F_n$ is an exact group.
Ozawa takes $\mathcal{N}(C^*_r(F_n)) = C(\partial F_n) \rtimes_r F_n$, where $\partial F_n$ denotes the hyperbolic boundary of $F_n$. 
Ozawa takes $\mathcal{N}(C^*_r(\mathbb{F}_n)) = C(\partial \mathbb{F}_n) \rtimes_r \mathbb{F}_n$, where $\partial \mathbb{F}_n$ denotes the hyperbolic boundary of $\mathbb{F}_n$.

**Key Proposition (Ozawa 2007)**

Let $\mu$ be a quasi-invariant doubly ergodic measure on $\partial G$. If

$$\varphi : C(\partial \mathbb{F}_n) \to L^\infty(\partial G, \mu)$$

is a unital positive $\mathbb{F}_n$-equivariant map, then $\varphi = \text{id}$. 
Equivariant Injective Envelopes
An operator system is a unital self-adjoint subspace of a C*-algebra.
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A $G$-operator system is an operator system equipped with the action of a group $G$, i.e. a unital homomorphism from $G$ into the group of order isomorphisms on $S$. 
Let $C$ be a category consisting of objects and morphisms. An object $I$ is injective in $C$ if, for every pair of objects $E \subset F$ and every morphism $\varphi : E \to I$, there is an extension $\tilde{\varphi} : F \to I$. When the objects are operator systems and the morphisms are unital completely positive maps, we get injectivity. When the objects are $G$-operator systems and the morphisms are $G$-equivariant unital completely positive maps, we get $G$-injectivity.
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The *$G$-injective envelope* of a $G$-operator system $S$ is the minimal $G$-injective operator system containing $S$. 
Theorem (Hamana 1985)

If $S$ is a $G$-operator system, then $S$ has a unique $G$-injective envelope $I_G(S)$. Every unital completely isometric $G$-equivariant embedding

$$\varphi : S \to \mathcal{T},$$

extends to a unital completely isometric $G$-equivariant embedding

$$\tilde{\varphi} : I_G(S) \to \mathcal{T}.$$
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Since there is a unital completely isometric $G$-equivariant embedding of $S$ into $\ell^\infty(G,S)$ there are unital completely isometric $G$-equivariant embeddings

$$S \subset I_G(S) \subset \ell^\infty(G,S).$$
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and a unital positive $G$-equivariant projection $P : \ell^\infty(G, S) \to I_G(S)$.
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and a unital positive $G$-equivariant projection $P : \ell^\infty(G, S) \to I_G(S)$.

The $G$-injective envelope $I_G(S)$ has a natural C*-algebra structure (induced by the Choi-Effros product).
Corollary

Let $G$ be a discrete group acting trivially on $\mathbb{C}$ and let $I_G(\mathbb{C})$ denote the $G$-injective envelope of $\mathbb{C}$. Then

$$\mathbb{C} \subset I_G(\mathbb{C}) \subset \ell^\infty(G),$$

and there is a unital positive $G$-equivariant projection

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The $G$-injective envelope $I_G(\mathbb{C})$ is a commutative C*-algebra equipped with a $G$-action, so there is a compact $G$-space space $\partial_H G$ such that $I_G(\mathbb{C}) \simeq C(\partial_H G)$.

We call $\partial_H G$ the Hamana boundary of $G$. 
The Furstenberg Boundary
Definition

Let $X$ be a compact $G$-space.

1. The $G$-action on $X$ is \textit{minimal} if the $G$-orbit

$$Gx = \{sx \mid s \in G\}$$

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2. The $G$-action on $X$ is *strongly proximal* if, for every probability measure $\nu$ on $X$, the weak*-closure of the $G$-orbit

$$G\nu = \{s\nu \mid s \in G\}$$

contains a point mass $\delta_x$ for some $x \in X$. 
Definition (Furstenberg 1972)

A compact $G$-space $X$ is a *boundary* if it is minimal and strongly proximal.
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**Key Property**

If $X$ is a boundary, then for every probability measure $\nu$ on $X$, the weak*-closure of the $G$-orbit $G\nu$ contains all of $X$.

Here $x \in X$ is identified with the point mass $\delta_x$ on $X$. 
Theorem (Kalantar-K 2014)

The Hamana boundary $\partial_H G$ is a boundary in the sense of Furstenberg.
Theorem (Furstenberg 1972)

Every group $G$ has a unique boundary $\partial_F G$ that is universal, in the sense that every boundary of $G$ is a continuous $G$-equivariant image of $\partial_F G$. 

We refer to $\partial_F G$ as the Furstenberg boundary of $G$.
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**Theorem (Kalantar-K 2014)**

Let $G$ be a discrete group and let $\partial_F G$ denote the Furstenberg boundary of $G$. Then the $C^*$-algebra $C(\partial_F G)$ is $G$-injective. Moreover, we have the following rigidity results:

1. Every unital positive $G$-equivariant map from $C(\partial_F G)$ is completely isometric.
2. The only positive $G$-equivariant map from $C(\partial_F G)$ to itself is the identity map.
3. If $M$ is a minimal $G$-space, then there is at most one unital $G$-equivariant map from $C(\partial_F G)$ to $C(M)$, and if such a map exists, then it is a unital injective *-homomorphism.
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Exactness and Nuclear Embeddings
A discrete group $G$ is exact if the reduced C*-algebra $C^*_r(G)$ is exact.
Ozawa proved that a discrete group $G$ is exact if and only the $G$-action on its Stone-Cech compactification $\beta G$ is amenable.
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Applying a result of Anantharaman-Delaroche gives the following corollary.
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**Theorem (Kalantar-K 2014)**

*Let $G$ be a discrete group. Then $G$ is exact if and only if the $G$-action on the Furstenberg boundary $\partial_F G$ is amenable.*

Applying a result of Anantharaman-Delaroche gives the following corollary.

**Corollary**

*If $G$ is a discrete exact group, then the reduced crossed product $C(\partial_F G) \rtimes_r G$ is nuclear.*
Theorem (Kalantar-K 2014)

Let $G$ be a discrete exact group. Then there is a canonical nuclear $C^*$-algebra $N(C^*_r(G))$ such that

$$C^*_r(G) \subset N(C^*_r(G)) \subset I(C^*_r(G)),$$

where $I(C^*_r(G))$ denotes the injective envelope of $C^*_r(G)$. 

Note: This is non-separable in general, but can be replaced by a separable nuclear $C^*$-algebra at the expense of no longer being canonical.
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C*-Simplicity
### Open Problem

Let $G$ be a discrete group. When is $G$ C*-simple, i.e. when is the reduced group C*-algebra $C^*_r(G)$ simple?

Day showed in 1957 that every discrete group $G$ has a largest amenable normal subgroup $R_a(G)$ called the amenable radical of $G$. If $G$ is C*-simple, then $R_a(G)$ is necessarily trivial.

Conjecture (de la Harpe, ?) The reduced group C*-algebra $C^*_r(G)$ is simple if and only if the amenable radical $R_a(G)$ is trivial.
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**Conjecture (de la Harpe, ?)**

The reduced group C*-algebra $C^*_r(G)$ is simple if and only if the amenable radical $R_a(G)$ is trivial.
**Definition**

Let $G$ be a discrete group with identity element $e$. The $G$-action on a compact $G$-space $X$ is *topologically free* if, for every $s \in G$, the set

$$X \setminus X^s = \{x \in X \mid sx \neq x\}$$

is dense in $X$. 
The property of the $G$-action on the Furstenberg boundary $\partial_F G$ being topologically free is an intermediate property between $\text{C}^*$-simplicity and triviality of the amenable radical $R_a(G)$.
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**Theorem (Kalantar-K 2014)**

Let $G$ be a discrete group.

1. If the $G$-action on $\partial_F G$ is topologically free, then $R_a(G)$ is trivial.
2. If $G$ is exact, and the reduced $\mathrm{C}^*$-algebra $\mathcal{C}_r(G)$ is simple, then the $G$-action on $\partial_F G$ is topologically simple.
**Figure:** Implications for an arbitrary discrete group $G$. 
Figure: Implications for a discrete exact group $G$. 
A Tarski monster group is a finitely generated group with the property that every nontrivial subgroup is cyclic of order $p$, for some fixed prime $p$. 

Theorem (Olshanskii 1982) Tarski monster groups exist for every prime $p > 10^75$. This answered a question of von Neumann about the existence of non-amenable groups which do not contain non-abelian free groups.
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**Theorem (Kalantar-K 2014)**

*If G is a Tarski monster group, then the G-action on the Furstenberg boundary $\partial_F G$ is topologically free.*
Rigidity of Maps
Theorem (Kalantar-K 2014)

Let $G$ be a non-amenable hyperbolic group, and let $\mu$ be an irreducible probability measure on $G$ with finite first moment. Let $\nu$ be a $\mu$-stationary probability measure on the hyperbolic boundary $\partial G$. If 

$$\varphi : C(\partial G) \to L^\infty (\partial G, \nu)$$

is a unital positive $G$-equivariant map, then $\varphi = \text{id}$. 

We apply Jaworski's theory of strongly approximately transitive measures, combined with a uniqueness result of Kaimanovich for stationary measures.
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Corollary

Let $G$ be as above, and let $\partial_F G$ denote the Furstenberg boundary of $G$. Then

$$I_G(C(\partial G)) = C(\partial_F G),$$

where $I_G(C(\partial G))$ denotes the $G$-injective envelope of $C(\partial G)$.
Corollary

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\[ I_G(C(\partial G)) = C(\partial_F G), \]

where $I_G(C(\partial G))$ denotes the $G$-injective envelope of $C(\partial G)$.

The Furstenberg boundary $\partial_F G$ can be thought of as a “projective cover” of the hyperbolic boundary $\partial G$. 
Quantum Groups
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Many of our results hold in this setting. We intend to pursue this further...
Thanks!