Universality of the critical conductance distribution in various dimensions

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Abstract

We study numerically the metal-insulator transition in the Anderson model on various lattices with dimension 2 < d ≤ 4 (bifractals and Euclidian lattices). The critical exponent ν and the critical conductance distribution are calculated. We confirm that ν depends only on the spectral dimension. The other parameters - critical disorder, critical conductance distribution and conductance cummulants - depend also on lattice topology. Thus only qualitative comparison with theoretical formulae for dimension dependence of the cummulants is possible.

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It is commonly accepted, though not proved, that metal-insulator transitions (MIT) can be described by one-parameter scaling theory. The critical exponent ν which describes the divergence of correlation length at MIT depends only on the system dimension for a chosen universality class. Microscopic details of models do not affect it. This was confirmed by numerical analysis of quasi-one-dimensional (Q1D) systems. Theoretical dependence of ν on dimension d = 2 + ε was found in [2] for small ε. Numerically, ν(ε) was studied on bifractals [3].

The conductance g was originally chosen as the order parameter in the scaling theory [4]. Soon it became clear, that the absence of self-averaging of g in the critical region must be taken into account [5]. The shape of the critical conductance distribution P(g) in 3D models was numerically analysed in detail [6, 7]. Contrary to the critical exponent, P(g) is not universal. Its shape depends not merely on the dimension [8] and physical symmetry [9] but also on boundary conditions [10] and even anisotropy [11]. Nevertheless, for a given physical model the mean conductance and resistance follow one parameter scaling [12].

Analytical theory of MIT is restricted to systems with dimension close to the lower critical dimension: 2 + ε with ε ≪ 1 [13]. In spite of predicted non-universality of higher order conductance cummulants ⟨δgn⟩,

$$\langle \delta g^n \rangle = \begin{cases} \varepsilon^{n-2} & n < n_0 = \varepsilon^{-1} \\ \sim L \varepsilon^{n-2} & n > \varepsilon^{-1} \end{cases}$$

(1)

the distribution P(g) should be universal in the infinite system size limit [14]. For small ε, the bulk of the distribution is approximately Gaussian near the mean value ⟨g⟩. The parameters of Gaussian peak,

$$\langle g \rangle \sim \varepsilon^{-1} \quad \text{and} \quad \text{varg} = \langle g^2 \rangle - \langle g \rangle^2 \sim \varepsilon^0. \quad (2)$$

agree with the estimation of the first cummulants [14].

In this Letter we present the critical exponent and the critical conductance distribution for Anderson model obtained numerically on three bifractal lattices with dimension 2 < d < 3. They all possess the same fractal dimension df = log 3/ log 2 + 1. Two of these lattices have the same spectral dimension ds. Their critical exponents are identical within error bars. This confirms the universality of MIT. The shape of the P(g), and the value of the critical disorder, depend not only on ds but also on the lattice topology. This novel non-universality of P(g) is found also by numerical analysis of two different 3D lattices.

Topological dependence of the P(g) and of the critical disorder disables the verification of theoretical formulæ for conductance cummulants. Known theoretical formulæ may be valid only for d-dimensional hyper-cubes which can be numerically simulated only for integer d. Thus we calculated the critical parameters for four dimensional (4D) lattice and compared them with known results for 3D lattice. Ratios of the first two conductance cummulants are P(g)3D/⟨g⟩4D ≈ 2 and var g3D/ var g4D, in full agreement with relations [14].

We consider the Anderson Hamiltonian

$$\mathcal{H} = \sum_n \varepsilon_n |n\rangle \langle n| + \sum_{|nn'|} |n\rangle \langle n'| + |n'| \langle n|.$$  \quad (3)

Random energies \varepsilon_n are uniformly distributed from \langle -W/2, W/2 \rangle. Parameter W measures the disorder strength. Fermi energy equals to zero. n numbers lattice sites, and \{nn'\} are two nearest-neighbor sites.

All systems under consideration are linear in the z-direction. In the plane perpendicular to the current direction we construct fractals A-C according to figure 1. A combination of a d-dimensional fractal with linear chain
in z direction produces a bifractal with dimension \(d+1\) both for \(d = d_s\) and \(d = d_f\). The length of the system in z direction is \(L = 2^n\) and the number of lattice sites in the slice grows as \(N = 3^n\) for the nth generation of bifractal. We study also two 3D models with triangular and honeycomb 2D lattice in the xy plane (referred as 3t and 3h) and compare the obtained results with known data for 3D cubes (3s) \([4, 21]\).

In \([4]\) it was supposed that critical parameters are completely determined by the spectral dimension \(d_s\) of the lattice \([10]\). Following \([4, 21]\) we find the analytical values for the spectral dimension for fractals A-C \([4, 22]\):

\[d_A^s = d_B^s = 2 \log 3 / \log 5\] and \(d_C^s = 2 \log 3 / \log 6\). The last value differs slightly from the one obtained by numerical simulation of a random walker \([18]\).

Linear form of all systems in z direction enables us to apply the standard numerical procedure for calculation of the critical disorder \(W_c\) and critical exponents \(\nu\) in Q1D systems \([24]\) and the Landauer formula for conductance (in units \(e^2/h\)) \([21]\):

\[g = 2Tr t^t t .\]  \hspace{1cm} (4)

The same spectral dimension of fractals A and B requires \(\nu_A = \nu_B\). Although recently reported data, \(\nu_A = 2.2 \pm 0.2\) \([17]\) and \(\nu_B \approx 2.5 \pm 0.25\) \([3]\), could be regarded as equal to each other within error bars, we wanted to check their equivalence more accurately. In our analysis we consider first five generations \((L \leq 32)\) of fractals. The smallest Lyapunov exponent \(z_1\) was calculated with accuracy of 0.1% for \(L = 4, 8, 16\) and 0.5% for \(L = 32\). Our result indeed confirms \(\nu_A = \nu_B\) (Table 1).

We tested the universality of \(\nu\) also for 3D lattices with various topology. Q1D systems up to \(14^2\) lattice sites in the xy plane were considered. The accuracy of the first Lyapunov exponent was 0.1% for small crossections, and decreases as \(L\) increases, being only 0.5-1% for \(L = 14\). Our results confirm the universality of \(\nu\) as expected.

For 4D hyper-cubes we involve systems up to \(7^3 \times \infty\). Our resulting \(\nu \approx 1.1\) is in a very good agreement with the previous ones \([4, 22]\). The obtained critical parameters are presented in Table 2.

The critical conductance distribution \(P(g)\) for lattices A, B and C is presented in figure 2. For bifractals A and B we can approve the system-size independence of \(P(g)\). For C, we do not reach the limiting form of \(P(g)\) because of finite-size effects which are much stronger at lower \(d_s\).

As supposed, the mean conductance \(\langle g \rangle\) increases as \(d_s\) decreases. We cannot, however, compare our data with the theory, since the theoretical analysis has been performed only for \(d\)-dimensional hyper-cubes. We can only describe some general features of the critical distribution: (i) \(P(g)\) converges to Gaussian as \(d_s\) decreases to 2, as predicted in \([4]\). (ii) We found no evidence of the power law decrease of \(P(g)\) for \(g \gg 1\). This could be caused by the small statistical ensembles (we have only \(\sim 1.000\) samples in the 6th generation, \(L = 64\)). We note that the qualitative arguments against the power-law decrease \([8]\) may not be valid in the limit \(d \rightarrow 2^+\) because the differences between Lyapunov exponents \(z_{i+1} - z_i\) become very small for \(\varepsilon \rightarrow 0\).

Different forms of \(P(g)\) for bifractals A and B indicate that the critical conductance distribution and also conductance cummulants depend on the lattice topology. A more convincing proof of this statement is in figure 3 where we present the critical conductance distributions for 3D systems with honeycomb and triangular lattice in the xy plane and compare them with \(P(g)\) for 3D cubic lattice.

Figure 4 presents \(P(\log g)\) for 4D cubic lattice (with fixed boundary conditions). The mean conductance is half of that in 3D cubic lattice. The shape of \(P(g)\) and \(P(\log g)\) is similar to the one for 3D (figure 3) and it can be analyzed by standard methods \([10, 24]\).

Finally, figure 5 summarizes the dimension dependence of the critical parameters. The main difference between our numerical data for \(\nu\) and those published in \([4]\) is the estimation of the spectral dimension which is more accurate in the present work. Contrary to Ref. \([4]\), we do not try to fit the dimension-dependence of our data to any simple function. As pointed out in \([4]\), \(\varepsilon\)-dependence of \(\nu\) is non-trivial. The leading term \(\nu = 1/\varepsilon\) would be observable only on lattices with much smaller spectral dimension. As an exact knowledge of \(d_s\) is crucial for numerical estimation of the \(\varepsilon\)-dependence of \(\nu\), we did not analyse statistical fractals, for which \(d_s\) must be estimated by numerical simulations. Another disadvantage of statistical fractals is that due to the topology dependence of \(P(g)\) we do not suppose that system-size invariant critical conductance distribution can be found on available scales.

The dependence of the conductance cummulants on the lattice topology disables quantitative comparison of numerical and theoretical data. We can only conclude that the mean conductance increases as \(\varepsilon\) decreases. To our surprise, \(\langle g \rangle\) and \(\varepsilon\) for 3D and 4D seem to follow \([10]\).

Does it mean that the relation \(n_0 = 1/\varepsilon\) in \([10]\) underestimates the upper bound \(n_0\)? A positive answer could explain the absence of the power law decrease of \(P(g)\) for large \(g\).

In conclusion, we have analyzed the critical parameters of the metal-insulator transition on lattices with different dimension and different topology. We confirm that the critical exponent \(\nu\) depends only on the spectral dimension of lattice. This confirms universality of the MIT. Our result for \(\nu\) in 4D systems agrees with the previous numerical estimations \([10, 24]\), but differs considerably from theoretical expectation \(\nu = 1/2\) \([24]\).

We present the critical conductance distribution on the lattices of dimension \(2.226 \leq d_s \leq 4\) and compare them with theory. We prove that two lattices with the same spectral dimension but different lattice topology have a different critical conductance distribution. This prevents a quantitative comparison of numerical data for the conductance cummulants and for the critical disorder with theoretical formulae. In agreement with \([10, 24]\), \(P(g)\) converges to the Gaussian when dimension decreases toward
The lower critical dimension $d_s = 2$. We found no power-law tail of the distribution, maybe due to the restricted size of the statistical ensembles.

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We remind that the spectral dimension determines the low-frequency form of the phonon density as $\rho(\omega) \propto \omega^{d_s}$. It larges eigenvalue is 6 and $d_s^C = 2 \log 3 / \log 6 \approx 1.226$.

Figure 1: The structure of three fractal lattices considered in this Letter. Top: the second generation (N=9), bottom: the third one (N=27). All fractals have the same fractal dimension $d_f = \log 3 / \log 2 \approx 1.58$, A and B have also the same spectral dimension (Table 1) [20]. A is doubled Szierpinski gasket (points connected by dotted lines are identical). B and C are modifications of Szierpinski gasket. Note different number of nearest-neighbor lattice sites: it is 4 for A, 3 for B and 1, 2 or 3 for C. Note also the absence of closed loops in the C lattice.

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It larges eigenvalue is 6 and $d_s^C = 2 \log 3 / \log 6 \approx 1.226$.
Figure 2: Critical conductance distribution $P(g)$ for bifractals A, B and C. The number of samples in statistical ensembles decreases from $10^5$ for smaller systems to $\sim 1.000$ for $L = 64$. The last figure shows the system size dependence of mean conductance (open symbols) and var $g$ (full symbols). Note that finite-size effects increase as critical disorder decreases.

Figure 3: Critical conductance distribution for three dimensional systems with triangular and honeycomb structure in the $xy$ plane. For comparison, $P(g)$ for the cubic lattice is also presented (dotted line).

Figure 4: Critical conductance distribution of $\log g$ for 4D hyper-cubes with fixed boundary conditions. The parameters of the statistical ensembles are presented in the legend: $L$, $\langle \log g \rangle$, var $\log g$ and number of samples in the statistical ensemble. Inset: The first Lyapunov exponent $z_1$ as a function of disorder for the Q1D lattice $L^3 \times L_z$ and $4 \leq L \leq 7$. Solid lines are linear fits $z_1(W, L) = z_1^{(0)}(L) + W z_1^{(1)}(L)$.

Figure 5: Dimension dependence ($\varepsilon = d - 2$) of critical exponent (left) and of the mean and variance of conductance (right). Systems with the same spectral dimension $d_s$ have different mean and variance of conductance, but they possess the same critical exponent $\nu$. For details see Tables 1 and 2.
Table 1: Critical parameters of bifractals A, B and C (Figure 1). $d_s$ is spectral dimension [20]. Critical disorder and critical exponent were found from fit of the numerical data by a simple one-parameter scaling formula. Accuracy of critical parameters was estimated by comparison of numerical fits for system size $L = 4 \div 32$ and $L = 8 \div 32$ and for various intervals of $W$. Limiting values of the moments of conductance were estimated by the fit $X_L = X_\infty + \text{const}/L$ with $L = 8, 16, 32$ and 64 (figure 2).

| Parameter | A     | B     | C     |
|-----------|-------|-------|-------|
| $d_s$     | 2.365 | 2.365 | 2.226 |
| $W_c$     | 9.7(1) | 7.98(02) | 5.77(02) |
| $\nu$     | 2.27(06) | 2.29(08) | 2.82(05) |
| $z_1$     | 2.04(04) | 1.59(01) | 0.91(01) |
| $\langle g \rangle$ | 2.00 | 2.82 | 7.99 |
| $\text{var } g$ | 0.84 | 1.24 | 2.53 |
| $\langle \log g \rangle$ | 0.58 | 0.96 | 2.04 |
| $\text{var } \log g$ | 0.31 | 0.22 | 0.56 |

Table 2: Critical parameters of three dimensional systems with square (3s) [2, 12, 23], honeycomb (3h) and triangular (3t) lattice in $xy$ plane and of 4D hyper-cube. Periodic boundary conditions (b.c.) were used for Q1D simulations. Fixed b.c. were used for studies of conductance statistics.

| Lattice | 3s     | 3h     | 3t     | 4      |
|---------|--------|--------|--------|--------|
| $d_s$   | 3      | 3      | 3      | 4      |
| $W_c$   | 16.5   | 13.5(2) | 19.9(2) | 34.3(2) |
| $\nu$   | 1.54(03) | 1.58(04) | 1.53(03) | 1.1(1) |
| $z_1$   | 3.45(01) | 2.7(3) | 4.2(2) | 5.4(1) |
| $\langle g \rangle$ | 0.57 | 0.96 | 0.30 | 0.27 |
| $\text{var } g$ | 0.17 | 0.58 | 0.19 | 0.17 |
| $\langle \log g \rangle$ | -1.93 | -0.44 | -2.34 | -2.39 |
| $\text{var } \log g$ | 1.76 | 1.00 | 3.02 | 2.61 |