A presentation of general multipersistence modules computable in polynomial time*

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Abstract

Multipersistence homology modules were introduced by G.Carlsson and A.Zomorodian [1] which gave, together with G.Singh [4], an algorithm to compute their Gröbner bases. Although their algorithm has polynomial complexity when the chain modules are free, i.e. in the one-critical case, it might be exponential in general. We give a new presentation of multipersistence homology modules, which allows us to design an algorithm to compute their Gröbner bases always in polynomial time by avoiding the mapping telescope.

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1 Introduction

The theory of persistent homology builds a bridge between computational algebraic topology and data analysis using homology as an effective tool to associate a computable invariant to a point cloud. Other applications of the theory range from coverage problems in sensor networks to complex network theory. The main idea is to approximate the point cloud embedded in a metric space with an increasing sequence of simplicial complexes (filtration), see [2, 8]. By analysing the persistent, i.e. long living, homological features in the filtration, the shape of the point cloud can be inferred.

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Multipersistent homology is a generalization of this theory in which the homology of a filtration of simplicial complexes indexed by $\mathbb{N}^r$ (multifiltration) is analysed. A multifiltration represents a simplicial complex evolving along $r$ different directions.

Multipersistent homology has been introduced by Carlsson and Zomorodian in the seminal paper [1], where the authors study the classification problem for multipersistence homology modules. From then, several papers have considered these modules: the same authors and Singh in [4] developed an algorithm for computing their Gröbner bases [7, 9] and Chacholski et al. [6] gave a combinatorial multigraded resolution for multipersistence homology modules. The classification problem has been studied in [11].

A particular type of multifiltration is highlighted in [4] and called one-critical. Intuitively, in the one critical case, simplices enter the filtration in one single instance rather than in non comparable states. The algorithm in [4] works on one-critical multifiltrations optimally and on general ones, reducing them to the one-critical case by using the mapping telescope.

As observed by the authors in [4] the mapping telescope increases the size of the multifiltration exponentially in the worst case.

Therefore, although the algorithm in [4] has polynomial complexity in the one-critical case, the use of the mapping telescope introduces a bottleneck in the general case. In this paper, we give a new presentation of multipersistence modules. By using this, we are able to extend a version of the algorithm presented in [4] to all multipersistence modules avoiding the use of the mapping telescope and therefore eliminating the bottleneck of the previous approach. Our algorithm has polynomial complexity for all multifiltrations and in the one-critical case it essentially coincides with the one in [4].

We will now sketch the structure of the paper. Section 2 introduces a construction that generates a non one-critical multifiltration from a point cloud, motivating our algorithm. In section 3 we lay the theoretical basis: we show in fact how the modules of cycles and boundaries are isomorphic to submodules of a finitely generated free module. Embedding cycles and boundaries in the free module we can adapt the algorithm in [4] to the general case. The algorithm presented by Carlsson et al. is described in section 4 along with some definitions from commutative algebra. Section 5 presents our algorithm for the computation of Gröbner bases in the general case.

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2 Motivating Construction

Multifiltrations arise from data sets in many ways. For instance, in analyzing the structure of natural images the data is filtered according to density, [5]. The following generalization of the Rips Vietoris complex determines a non one-critical multifiltration from a data set.

Let \( S \subset \mathbb{R}^2 \) be a point cloud. Consider the ellipses centered in the points of \( S \) with semi-axis of lenght \( a \) and \( b \) respectively. Fixed a direction \( v \in \mathbb{R}^2 \), we assume that the semi-axis of length \( a \) is parallel to \( v \). Fixing the values of \( a \) and \( b \), we can build a simplicial complex on \( S \) by following the same procedure used for the Rips-Vietoris complex. Given \( n \) points in \( S \), they define a \( (n-1) \)-face of the complex if the corresponding ellipses intesect two by two. According to this construction, the critical coordinates for \( (n-1) \)-faces correspond to couples \((a,b)\) such that \( n \) ellipses of semi-axis \( a \) and \( b \) are tangent two by two. As \( a \) and \( b \) vary in \([0, \infty)\) we obtain a non one-critical multifiltered complex. Given two points in \( S \) there can be non comparable values of the semi-axes such that the corresponding ellipses are tangent. In fact there are infinitely many (see example 2.1). Note that for \( a = b \) the construction is the Rips-Vietoris complex and that this multifiltration shows if some points are aligned in direction \( v \).

Example 2.1. Consider the origin \( 0 \) and a point on the unitary circonference centered in the origin \((\cos(t), \sin(t))\) \( t \in [0, 2\pi) \). We assume \( v \) is the \( x \)-axis. For \( t \) varying in \((0, \pi/2)\), the implicit function between \( a \) and \( b \) in the tangency point of the ellipses centered in the two points is a branch of hyperbola, see figure [2.1]. The hyperbola degenerates continuously to the vertical and horizontal axes for \( t = 0 \) and \( \pi/2 \) respectively. The example is quite general: the function is symmetric with respect to the \( x \) and \( y \) axes.

3 Theoretical setting

Let us consider \( \mathbb{N}^r \) equipped with the product order, i.e. \( v = (v_1 \ldots v_r) \preceq w = (w_1 \ldots w_r) \) iff \( v_i \leq w_i \) for all \( i \). We denote by \( R = k[x_1 \ldots x_r] \) the polynomial ring in \( r \) indeterminates over a field \( k \) and \( x^v \) is the monomial \( x_1^{v_1} \ldots x_r^{v_r} \). Unless otherwise stated, we will call module a \( R \)-module and vector space a \( k \)-vector space.

A simplicial complex \( X \) is called multifiltered if we are given a family of subcomplexes \( \{X_v\} \) with \( v \in \mathbb{N}^r \), such that \( X_v \subseteq X_w \) whenever \( v \preceq w \). The family \( \{X_v\} \) is called a multifiltration of \( X \). If \( X \) is a finite simplicial complex, every multifiltration of \( X \) stabilizes, i.e. there exists a multi-index
Figure 1: Implicit function between the semi-axes $a$ and $b$ in the tangency point of the ellipses.

$v' = (v'_1 \ldots v'_r)$ such that $X_{w+e_i} \simeq X_w$ for all $w = (w_1 \ldots v'_i \ldots w_r) \in \mathbb{N}^r$ and $1 \leq i \leq r$, where $e_i$’s are, as usual, the vectors in $\mathbb{N}^r$ having one in the $i$–th entry and zero elsewhere.

We fix a filtered $d$-dimensional finite simplicial complex $X$ with multifiltration $\{X_v\}$. The vector space $C_n(X_v)$ of $n$-chains in $X_v$ will be denoted by $C_n(v)$. This is the vector space with linear basis $B_n(v)$, the set of $n$-faces in $X_v$. The $n$-faces of $X_{v'} \simeq X$ will be ordered and denoted by $\sigma^n_1, \ldots, \sigma^n_{d_n}$ (or simply by $\sigma_i$ when $n$ is fixed) where $d_n$ is the rank of $C_n(v')$.

The multipersistence $n$-chain module (first defined in [4] and described in [6]) is the multigraded vector space

$$C_n = \bigoplus_v C_n(v)$$  \hspace{1cm} (3.1)

equipped with the left $R$-action defined by the linear maps $z^u : C_n(\bullet) \to C_n(\bullet + u)$ induced by the inclusions $X_\bullet \to X_{\bullet+u}$. The set of bases $\{B_n(v)\}$ is closed under the module action. Indeed, $z^w B_n(\bullet) \subseteq B_n(\bullet + w)$ for all $w \in \mathbb{N}^r$.

**Definition 3.1.** The module $D_n := \bigoplus_w C_n(v' + w)$.

**Remark 1.** The module $D_n$ inherits the grading from $C_n$ in the obvious way and it is a free module, isomorphic to $R^{d_n}$, with $R$-basis $B_n(v')$. All the generators have multidegree zero.
The \textit{multipersistence homology modules} are the homology modules of the multigraded module chain complex (see [6])

\[ C_\bullet: 0 \to C_d \xrightarrow{\partial_d} C_{d-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0 \]

obtained as the direct sum of the simplicial chain complexes of \( X_v \) for all \( v \in \mathbb{N}^r \). As usual, we denote by \( Z_n \subseteq C_n \) the kernel of \( \partial_n \) and with \( B_{n-1} \subseteq C_{n-1} \) the image of \( \partial_n \). Since multiplication by monomials is injective in \( C_n \), the chain complex \( C_\bullet \) is isomorphic to the shifted chain complex

\[ x^{v'} C_\bullet: 0 \to x^{v'} C_d \xrightarrow{\partial_d} x^{v'} C_{d-1} \to \cdots \to x^{v'} C_1 \xrightarrow{\partial_1} x^{v'} C_0 \to 0. \]

Note that in general \( x^{v'} C_n \) is a proper submodule of \( D_n \). It is indeed enough to observe that \( x^{v'} C_n(\langle 0, \ldots, 0 \rangle) \subseteq C_n(v') \).

Since \( \partial_n(D_n) \subseteq D_{n-1} \), as it easy to check, the following is a chain complex of multigraded modules

\[ D_\bullet: 0 \to D_d \xrightarrow{\partial_d} D_{d-1} \to \cdots \to D_1 \xrightarrow{\partial_1} D_0 \to 0. \]

where the boundaries are obtained by restriction. We see \( x^{v'} C_\bullet \) as a subcomplex of \( D_\bullet \).

\textbf{Definition 3.2.} Let \( \sigma \in B_n(v') \) be a basis element of \( D_n \), a critical coordinate for \( \sigma \) is a minimal \( v \in \mathbb{N}^r \) such that there exists \( \tau \in B_n(v) \) with \( x^{v'-v} \tau = \sigma \). The element \( \tau \) is called a fundamental element associated to \( \sigma \).

In general, critical coordinates and fundamental elements for \( \sigma \in B_n(v') \) are not unique. We denote by \( F_\sigma \) the set of fundamental elements associated to \( \sigma \in B_n(v') \) and by \( C_\sigma \) the set of corresponding critical coordinates i.e. the multi-degrees of elements in \( F_\sigma \). The cardinality of these sets is denoted by \( k_\sigma := |F_\sigma| = |C_\sigma| \). The set \( F_n = \bigcup_{\sigma \in B_n(v')} F_\sigma \) generates the module \( C_n \).

A presentation of \( C_n \) is given by the short exact sequence

\[ 0 \to \ker \varphi_n \to R^{d_n} \xrightarrow{\varphi_n} C_n \to 0 \]

where \( R^{d_n} \) is the free module with basis the fundamental elements. We define maps \( \tilde{\partial}_n: R^{d_n} \to D_{n-1} \) such that the following diagrams are commutative.
The maps $\tilde{\partial}_n$ are linear maps between free modules and they can thus be represented by matrices with coefficients in $R$. The columns of such matrices are the boundaries of $n$-dimensional fundamental elements, shifted in $D_{n-1}$. Each column is given by monomial entries and two elements can differ only by scalar coefficients, being the columns homogeneous elements of $D_{n-1}$.

**Theorem 3.0.1.** The modules $Z_n$ and $B_n$ can be calculated from matrices:

$$B_n \simeq x^{v'} B_n = \text{Im} \tilde{\partial}_{n+1} \quad \text{and} \quad Z_n \simeq x^{v'} Z_n = x^{v'} \circ \varphi_n(\ker \tilde{\partial}_n).$$

**Proof.** The modules $Z_n$ and $B_n$ are isomorphic to $x^{v'} Z_n$ and $x^{v'} B_n$ respectively, by the injectivity of multiplication by $x^{v'}$. Considering also that diagram (3.2) is commutative and that $\varphi_n$ is surjective, the following equalities are satisfied.

$$x^{v'} B_n = x^{v'} \circ \partial_n(C_n) = \partial_n \circ x^{v'}(C_n) = \tilde{\partial}_n(R^{d_n}).$$

$$x^{v'} Z_n = x^{v'} \ker(\partial_n) = x^{v'} \ker(x^{v'} \circ \partial_n) = x^{v'} \circ \varphi_n(\ker \tilde{\partial}_n).$$

**Example 3.1.** Consider the multifiltered finite simplicial complex with four vertices, five edges and one 2-face in figure 2.

The four vertices have critical coordinates $C_{v_1} = \{(0, 0)\}$, $C_{v_2} = \{(1, 0), (0, 1)\}$, $C_{v_3} = \{(2, 0), (1, 2)\}$, $C_{v_4} = \{(3, 0), (0, 1)\}$. The critical coordinates for the five edges are $C_{v_2 v_1} = \{(0, 2), (2, 0)\}$, $C_{v_3 v_2} = \{(2, 0)\}$, $C_{v_4 v_2} = \{(0, 2), (3, 0)\}$, $C_{v_1 v_3} = \{(0, 2), (1, 1)\}$, $C_{v_3 v_4} = \{(3, 0)\}$. There is just one 2-face with critical coordinate $(2, 2)$. The boundary operators of the simplicial chain complex associated to $X_{(3,2)}$ are represented by the matrices

$$\partial_1 = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix} \quad \partial_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$
and the matrices of the maps $\tilde{\partial}$ are

$$\tilde{\partial}_1 = \begin{pmatrix} x^2 & y^2 & 0 & 0 & 0 & -y^2 & -xy & 0 \\ -x^2 & -y^2 & x^2 & y^2 & x^3 & 0 & 0 & 0 \\ 0 & 0 & -x^2 & 0 & 0 & 0 & 0 & -x^3 \\ 0 & 0 & 0 & -y^2 & -x^3 & y^2 & xy & x^3 \end{pmatrix} \quad \tilde{\partial}_2 = \begin{pmatrix} x^2 y^2 \\ 0 \\ x^2 y^2 \\ x^2 y^2 \\ 0 \end{pmatrix}$$

### 3.1 One-critical case

Following [4], a multifiltration such that there exists a unique critical coordinate for each $\sigma \in B_n(v')$ and for all $n$ is called one-critical. The module $C_n$ is free if and only if the corresponding multifiltration is one-critical. In this case,

$$C_n \simeq R^{|F_n|} \simeq R^{d_n}$$

and the boundary operator $\partial_n$ can be represented as a matrix with coefficients in $R$. It is also true that in the one-critical case the modules $Z_n$ and $B_n$ are naturally submodules of the free module $C_n$, thus it is possible to compute the quotient $H_n = Z_n/B_n$ in $C_n$. Exploiting these properties, Carlsson et al. in [4] present an algorithm to compute a Gröbner basis for multipersistence homology modules in the one-critical case. In general there is not a natural choice for an ambient free module, but by shifting $Z_n$ and $B_n$ in $D_n$ it is possible to compute the quotient $x^{v'} H_n \simeq H_n$ in $D_n$. Us-

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**Figure 2:** A non one-critical multifiltration
ing this device, in section 5 we will generalize Carlsson’s algorithm to all multifiltrations.

**Remark 2.** In the one critical case $\mathfrak{g}^{n'} : C_n \to D_n$, is an isomorphism. Therefore, in this case, we have an identification of $\partial_n$ and $\partial_n$ up to isomorphisms, as one can check from diagram $\mathfrak{I}_2$. This gives the exact relationship between our approach and the one developed in $[4]$.

### 4 One-critical case algorithm

In this section we will recall the basic steps of the algorithm for the one-critical case $[4]$. For background information on Gröbner basis theory we refer the reader to $[7, 9]$.

Let $R^N$ be a finitely generated free module with canonical basis $\{e_1, \ldots, e_N\}$. Each element $f \in R^N$ is a linear combination of monomials $x^n e_i$. The module generated by elements $\{f_1, \ldots, f_t\}$ in $R^N$ will be denoted by $\langle f_1, \ldots, f_t \rangle$.

We fix a monomial order $[7, 9]$ on the monomials of $R^N$. The leading monomial $\text{LM}(f)$ and leading coefficient $\text{LC}(f)$ of $f \in R^N$ are respectively the greatest monomial of $f$ and its corresponding coefficient. If $F$ is a finitely generated submodule of $R^N$, then $\text{LM}(F)$ is the submodule of $R^N$ generated by the leading monomials of the elements of $F$.

**Definition 4.1.** A finite set of generators $\{f_1, \ldots, f_t\}$ for a module $F \subset R^N$ is a Gröbner basis of $F$ if

$$\text{LM}\left(\langle f_1, \ldots, f_t \rangle \right) = \langle \text{LM}(f_1), \ldots, \text{LM}(f_t) \rangle$$

Using Buchberger algorithm, shown in listing $[4]$ a Gröbner basis of $F \subset R^N$ can be computed from any finite set of generators.

**Definition 4.2.** For each set of elements $\{f_1, \ldots, f_m\} \in R^N$, the syzygy module $\text{Syz}(f_1, \ldots, f_m)$ is the kernel of the map $\psi : R^m \to R^N$ sending the canonical basis elements of $R^m$ to $\{f_1, \ldots, f_m\}$.

A Gröbner basis for the syzygy module $\text{Syz}(f_1, \ldots, f_m)$ exists by Schreyer’s theorem (see page 334 in $[9]$). We can compute such basis using the Wall algorithm described in $[12]$ or the Schreyer algorithm described in $[4]$.

Buchberger and Wall (or Schreyer) algorithms share most of their logic. The Gröbner bases for $\langle f_1, \ldots, f_m \rangle$ and $\text{Syz}(f_1, \ldots, f_m)$ can indeed be computed concurrently as shown in listing $[2]$ (using the Wall algorithm).

The algorithm described by Carlsson et al. in $[4]$ uses these algorithms to compute Gröbner bases for all $B_n$, $Z_n$ and $H_n$ in the one-critical case.
The input of the algorithm are the matrices of the maps $\partial_n$. The steps of the algorithm are the following:

1. Compute the (reduced) Gröbner basis of $B_n$ using Buchberger algorithm on the columns of $\partial_{n+1}$.

2. Compute the (reduced) Gröbner basis of $Z_n$ as the syzygy module of the elements corresponding to the columns of $\partial_n$.

3. Compute the quotient $H_n = B_n/Z_n$ using the multivariate division algorithm.

The Gröbner bases of $B_n$ and $Z_{n+1}$ can be computed concurrently using the algorithm we have described above.

**Remark 3.** As stated in the previous section, the differential operators $\partial_n$ are matrices if and only if the multifiltration is one-critical. The previous algorithm therefore cannot be directly applied to all multifiltrations.

In [4], Carlsson et al. observed that the entries of the matrices of $\partial_n$ are terms whose monomials only depend on the position in the matrix. They called the matrices with this property homogeneous, highlighting the fact that the image of $\partial_n$ is given by homogeneous elements of $C_{n-1}$.

### 5 General Multipersistence Algorithm

With our algorithm we compute, starting from general multifiltrations, reduced Gröbner bases of $x^\nu B_n$, $x^\nu Z_n$ and $x^\nu H_n \simeq H_n$ for all $n \in \{0 \ldots d\}$. To compute a Gröbner basis for $\underline{x}^\nu B_n$ we can use the Buchberger algorithm (as in step 1 of [4]) on the columns of $\widetilde{\partial}_{n+1}$.

To compute a Gröbner basis for $\underline{x}^\nu Z_n$ we first compute the syzygy module of the elements corresponding to the columns of the matrix $\widetilde{\partial}_n$ (as in step 2 of [4]). This is however a submodule of $R^{F_n}$. To obtain a Gröbner basis of $\underline{x}^\nu Z_n$ as desired, it is thus necessary to map the result in $D_n$.

The multivariate division algorithm (step 3 of [4]) can be used without any modifications to compute a Gröbner basis of the quotient $x^\nu Z_n/x^\nu B_n$.

**Remark 4.** Observe that the matrices $\widetilde{\partial}_n$ are homogenous matrices, this implies that the computational complexity of our algorithm is polynomial. The algorithm in [4] is in fact polynomial because the matrices are homogeneous and we perform the same operations, adding only multiplication by a matrix that is polynomial.
Example 5.1. We will now compute Gröbner bases for the significant modules arising from the multifiltration in figure 2. In this example the only non trivial homologies are $H_0$ and $H_1$. The modules $H_0$ and $H_1$ are isomorphic to $Z_0$ and $Z_1$ respectively. The information given by the couple $(Z_n, B_n)$ is anyway more informative than the quotient $H_n$ to understand the birth and death of generators in terms of multi-degree. As already stated, it is necessary to embed $Z_n$ and $B_n$ in $D_n$ to compute Gröbner bases.

The reduced Gröbner bases for $x^3y^2Z_0$ and $x^3y^2B_0$ are

$$
Gb(x^3y^2Z_0) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}
$$

and

$$
Gb(x^3y^2B_0) = \left\{ \begin{pmatrix} 0 \\ x^2y \\ x^3 \\ x^2 \\ x^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ y^2 \\ x^2 \\ y^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ x \end{pmatrix} \right\}
$$

The reduced Gröbner bases for $x^3y^2Z_1$ and $x^3y^2B_1$ are

$$
Gb(x^3y^2Z_1) = \left\{ \begin{pmatrix} x^3y \\ 0 \\ 0 \\ x^3y \\ 0 \end{pmatrix}, \begin{pmatrix} y^2 \\ 0 \\ 0 \\ y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^3 \\ x^3 \\ 0 \\ x^3 \end{pmatrix} \right\}
$$

and

$$
Gb(x^3y^2B_1) = \left\{ \begin{pmatrix} x^2y^2 \\ 0 \\ 0 \\ x^2y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} x^2y^2 \\ 0 \\ 0 \\ x^2y^2 \\ 0 \end{pmatrix} \right\}
$$
Algorithm 1 Buchberger algorithm

function Buchberger$(f_1, \ldots, f_m)$

\[ G \leftarrow \{ f_1, \ldots, f_m \}, \quad P \leftarrow \{ \text{Svector}(f_i, f_j) \neq 0 \mid 0 \leq i < j \leq s \} \]

for all \( p \in P \) do

\[ P \leftarrow P - \{ p \} \]

if \( h = \text{Reduce}(p, G) \neq 0 \) then

\[ G \leftarrow G \cup \{ h \}, \quad P \leftarrow P \cup \{ \text{Svector}(g, h) \neq 0 \mid g \in G \} \]

end if

end for

return \( G \)

end function

function Svector$(f_1, f_2)$

\[ c \leftarrow \text{LC}(f_1) / \text{LC}(f_2), \quad s_{ij} = \text{lcm}(\text{LM}(f_i), \text{LM}(f_j)) / \text{LM}(f_j) \]

return \( s_{21} f_1 - c s_{12} f_2 \)

end function

function Reduce$(f, \{ g_1, \ldots, g_t \})$

while there exists a \( g_i \) such that \( \text{LM}(g_i) \) divides \( \text{LM}(f) \) do

\[ c \leftarrow (\text{LC}(f) \text{LM}(f)) / (\text{LC}(g_i) \text{LM}(g_i)), \quad f \leftarrow f - c g_i \]

end while

return \( f \)

end function
Algorithm 2 Buchberger algorithm with syzygy computation

function \textsc{BuchbergerWithSyzygy}(f_1, \ldots, f_m)
\begin{align*}
&G \leftarrow \{(f_1, \epsilon_1), \ldots, (f_m, \epsilon_m)\}, \quad S \leftarrow \emptyset \\
&P \leftarrow \{ \textsc{Svector}((f_i, \epsilon_i), (f_j, \epsilon_j)) \neq 0 \mid 0 \leq i < j \leq s\} \\
&\text{for all } p \in P \text{ do} \\
&\quad P \leftarrow P - \{p\} \\
&\quad \text{if } (h, s) = \textsc{Reduce}(p, G) = (0, s) \text{ then} \\
&\quad \quad S \leftarrow S \cup \{s\} \\
&\quad \text{else} \\
&\quad \quad G \leftarrow G \cup \{h\}, \quad P \leftarrow P \cup \{ \textsc{Svector}(g, h) \neq 0 \mid g \in G\} \\
&\quad \text{end if} \\
&\text{end for} \\
&G' \leftarrow \{g \mid (g, s) \in G\} \\
&\text{return } (G, S) \\
\end{align*}
end function

function \textsc{Svector}((f_1, s_1), (f_2, s_2))
\begin{align*}
&c \leftarrow \frac{\text{LC}(f_1)}{\text{LC}(f_2)} \quad s_{ij} = \text{lcm}(\text{LM}(f_i), \text{LM}(f_j)) / \text{LM}(f_j) \\
&\text{return } (s_{21} f_1 - c s_{12} f_2, s_{21} s_1 - c s_{12} s_2) \\
\end{align*}
end function

function \textsc{Reduce}((f, s), \{(g_1, s_1), \ldots, (g_t, s_t)\})
\begin{align*}
&\text{while there exists a } g_i \text{ such that } \text{LM}(g_i) \text{ divides } \text{LM}(f) \text{ do} \\
&\quad c \leftarrow \frac{\text{LC}(f)}{\text{LC}(g_i)} \quad s_{i} \leftarrow s - c s_i \\
&\quad f \leftarrow f - c g_i \\
&\text{end while} \\
&\text{return } (f, s) \\
\end{align*}
end function
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