Bounce Models within Teleparallel modified gravity

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Abstract

In this paper, working in a Friedman-Lemaitre-Robertson-Walker (FLRW), first, in the flat case, we recover the generalized Friedman equation of Quantum Loop cosmology, and therefore the cosmological bounce, in the framework of modified teleparallel gravity $f(T)$-model, $T$ being the torsion scalar introduced in teleparallel gravity approach, without invoking unconventional exotic matter. Furthermore we study the associated perturbations again in a flat FLRW space-time. Then, we generalize the results to the curved FLRW space-time, where some issues related to the choice of tetrad exist, by using an appropriate formulation. In this context, the results of Born-Infeld model are also investigated.

1 Introduction

General Relativity (GR) with the presence of a suitable cosmological constant and the addition of cold dark matter, the so called $\Lambda$CDM model, is able to describe a large part of the cosmic history of our universe. The cosmic acceleration or dark energy dominated era which our universe undergoes today is well supported by the cosmological constant which drives the de Sitter expansion, and recently the $\Lambda$CDM model has been tested with high accuracy [1, 2].

However, it is well known that GR admits singular space-time solutions, where scalar curvature invariants become singular and geodesic incomplete metrics exist. In radiation/matter dominated universe the Big Bang singularity occurs and only under the assumption of unconventional exotic matter it can be avoided and eventually replaced by a bounce, where a contraction phase is followed by an expansion and the scale factor reaches a minimum but finite value (see Ref. [3] for a review).

In Quantum Loop Cosmology (QLC) an effective modified Friedman equation has been obtained and the solution in a flat Friedmann–Lemaître–Robertson–Walker (FLRW) space-time admits the cosmological bounce [4, 5, 6, 7, 8, 9, 10, 11].

In this paper, following [12, 13] we present a different framework with respect to GR which allows to extend QLC-like results to a spatially curved case. In particular we will analyze modified teleparallel gravity models.

Teleparallel gravity is an approach to gravity in which, instead of making use of the Levi-Civita connection, the so called Weitzenbock connection [14, 15, 16] is considered. This choice of the connection leads to a vanishing curvature tensor, but to a non vanishing torsion tensor. In the modified teleparallel framework the Lagrangian of the theory is a function $f(T)$ of the torsion scalar $T$, which is a suitable combination of quadratic scalars depending on the...
torsion tensor. Due to this choice, it is remarkable that the field equations remain at second order even in the modified teleparallel framework, making these theories less problematic than others where the modifications are proposed depending on curvature invariants, as in \( F(R) \) models.

However we should not forget that teleparallel gravity is a variant of Riemann-Cartan geometry where a spin connection is present. In the Weitzenbock gauge, the spin connection is chosen to be vanishing, and some issues related to the breaking of the local Lorentz invariance are present. In the Cartesian formulation of flat FLRW space-time we can safely use the Weitzenbock gauge, but in the curved case the choice of tetrad may be problematic. For this reason we limit our discussion to highly symmetric space-times, namely FLRW space-times, where the issue has been clarified. We focus on solutions without singularities in flat and curved FLRW space-times. The possible relevance of curved FLRW space-times has been recently pointed out in [17, 18].

The paper is organized as follows. In Section 2 we briefly review the formalism of teleparallel gravity and the teleparallel formalism of GR. In Section 3, we discuss a \( f(T) \) model which reproduces the QLC results in a flat FLRW space-time. A Born-Infeld model is also revisited. The cosmological perturbations of these models are studied in Section 3.2, where we show that the theory is not affected by superluminarity effects or gradient instabilities. In Section 3.3 the curved case in investigated. Using the mini-superspace approach, the field equations of the theory are derived. Finally, conclusions are given in Section 4.

2 Teleparallel gravity review

In this section we review the Riemann-Cartan and Weitzenbock Teleparallel geometry and the main difference with respect to textbook’s GR. In fact, in GR one usually considers the Levi-Civita connection, here denoted by \( \hat{\Gamma}_{\alpha\beta}^\gamma \). This connection satisfies the metric compatibility condition, i.e. the associated covariant derivative of the tensor metric vanishes. Furthermore it is symmetric in the lower indices. As a result, the theory is torsion free.

In general, a generic metric compatible connection, or Cartan connection, has a non trivial anti-symmetric part, and the torsion tensor is not vanishing. In the following, we shall present a short review of the formalism we will use throughout the paper.

The geometry which deal with a Cartan or spin connection is called Riemann-Cartan geometry. In this context, the dynamical variables are one-forms \( e^a \), which constitute the tetrad (or vierbein in \( d = 4 \)), and the spin connection one-forms \( \omega^a_b \). Introducing the natural basis one-form \( dx^\mu \) by means of \( e^a = e^a_\mu dx^\mu \), and \( \omega^a_b = \omega^a_{\mu b} dx^\mu \), the metric tensor is related to the tetrads via

\[
g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu, \tag{1}
\]

where \( \eta_{ab} \) is the Minkowsky metric tensor of the tangent space. From this we note that the metric determinant which effectively appears in the Lagrangian is \( \sqrt{-g} = \text{det}(e^a_\mu) \).

The related Riemann-Cartan and torsion two-forms are given by Cartan equations

\[
R^a_b = de^a_b + \omega^a_c \wedge \omega^c_b, \tag{2}
\]
\[
T^a = de^a + \omega^a_b \wedge e^b. \tag{3}
\]

If we denote by \( L = L(x) \) a local Lorentz transformation, namely \( \eta = L^T \eta L \), one has

\[
e^a' = L e^a, \quad g' = g, \quad \omega' = L \omega B + L dB \quad \text{where} \quad B = L^{-1}, \tag{4}
\]

where \( g \) is the metric tensor. If we impose a vanishing Riemann-Cartan two-form, we obtain the so called Weitzenbock geometry, and the related connection is called the Weitzenbock connection, here denoted by \( \Gamma^a_{\alpha\beta} \). It is possible to show that the most general solution of \( R^a_b = 0 \) is \( \omega = L dB = -dLB \), namely

\[
\omega^a_b = L^a_c dB^c_b. \tag{5}
\]

Thus, the non vanishing related torsion two-form reads

\[
T^a = de^a + L^a_c dB^c_b \wedge e^b. \tag{6}
\]
Recalling that $e^a = e^a_\mu dx^\mu$ and $\omega^a_\mu = \omega^a_\mu dx^\mu$, we get the torsion tensor as

$$T^a_{\mu\nu} = \frac{e^a_\mu}{2} \left( \partial_\nu e^a_\mu + \omega^b_\nu e^a_\mu \right) - \left( \mu \leftrightarrow \nu \right). \quad (7)$$

Given a Cartan connection, we can introduce the contorsion tensor $K^a_{\mu\beta}$ by means

$$\Gamma^a_{\mu\beta} = \hat{\Gamma}^a_{\mu\beta} + K^a_{\mu\beta}. \quad (8)$$

Therefore the definition of $K^a_{\mu\beta}$ from the torsion tensor is

$$2K^a_{\mu\nu} = T^a_{\mu\nu} + T^a_{\nu\mu} - T^a_{\mu\nu}. \quad (9)$$

We also have

$$\Gamma^a_{\mu\beta} = e^a_\mu \partial_\nu e^a_\nu + \omega^a_\mu e^b_\nu e^b_\mu. \quad (10)$$

Within the Weitzenbock geometry (see for example the recent paper in Ref. [19] and references therein), it is possible to choose $\omega^a_\mu = 0$, dubbed Weitzenbock gauge, which obviously leads to vanishing Cartan-Riemann curvature. Thus the Weitzenbock connection depends only on the tetrad, and the related inverse $e^a_\mu$, and it reads

$$\Gamma^a_{\mu\beta} = e^a_\mu \partial_\nu e^a_\nu. \quad (11)$$

However, the choice $\omega^a_\mu = 0$ leads to a breaking of the Local Lorentz Invariance (LLI), since $dT^a$ is no longer associated with a tensor 2-form. Furthermore, once we choose the coordinates for the metric, a non careful choice of the tetrad may lead to contradictions in the formalism (see Ref. [20]). A different approach to this issue is presented in Refs. [21, 22].

Thus, once we choose the set of coordinates, and given the metric (space-time geometry), the choice of the tetrad may be problematic. A typical example, as we will see below, is the flat Friedmann-Lemaître-Robertson-Walker (FLRW) space-time with Cartesian spatial coordinates. If one makes use of spherical spatial coordinates, one may make use of non trivial choice of the tetrad [20], or one should consider a suitable non vanishing spin connection [21, 22].

Finally, we introduce the torsion scalar, the only scalar, which being a particular combination of the three independent quadratic torsion scalars, leads to second order differential equations of motion,

$$T = T_{\mu\nu} S^{\mu\nu}, \quad (12)$$

where

$$S^{\mu\nu} = K^{\mu\nu} + g^{\mu\sigma} T^{\nu\beta} - g^{\nu\sigma} T^{\mu\beta}. \quad (13)$$

### 2.1 Teleparallel formulation of General Relativity

In this section we review the teleparallel formulation of GR. If we consider the Weitzenbock connection, one has the important identity

$$\sqrt{-g} \hat{R} = -\sqrt{-g} T - \partial_\mu (2T^{\beta\mu}) \quad (14)$$

where $\hat{R}$ is the standard GR Ricci tensor obtained with the Levi-Civita connection. Since the second term of equation [14] is a surface term, the action

$$I = \frac{-1}{2} \int dx^4 \sqrt{-g} T + I_m \quad (15)$$

leads to equation of motion which are equivalent to the GR ones.

As an important example, consider the flat FLRW space-time. Here one is dealing with high symmetric space-time, namely

$$ds^2 = -dt^2 + a(t)^2 dx^2 \quad (16)$$

where $a(t)$ is the scale factor and depends on the cosmological time only. If we use the Cartesian spatial coordinates $x^i$, one has $ds^2_0 = \delta_{ij} dx^i dx^j$, and a suitable choice for the tetrad is the naive diagonal one, namely

$$e^a_\mu = \text{diag}(1, a(t), a(t), a(t)). \quad (17)$$
In this case we can safely use the Weitzenbock connection $\omega_a^b = 0$. The torsion scalar $T$ reads

$$T = 6H^2,$$

and the above teleparallel action leads to the GR Friedmann equation

$$3H^2 = \rho.$$ 

If we instead consider spherical coordinates $ds^2_0 = dp^2 + \rho^2 dS^2$, and if we try to enforce the Weitzenbock connection $\omega_a^b = 0$, the naive choice for the tetrad leads to contradictions. In this case such naive choice is the "bad" one [23], see also [20] and Ref. [22]. A suitable non diagonal choice for the tetrad leads again to $T = 6H^2$, and thus again to the GR Friedmann equation.

## 3 Modified Teleparallel Gravity

It is well known that a possible description of Dark Energy (DE) can be achieved by the modified gravity models based on $f(R)$-gravity, where $R$ is the Ricci scalar. However, this approach involves fourth order differential equations of motion, possibly leading to Ostrogradskij instabilities. In this section we want to discuss a similar theory, where the function is a function of the scalar torsion $T$ instead of the Ricci scalar. In fact, one may introduce the modified teleparallel gravity considering a Weitzenbock geometry with a Lagrangian depending only on the scalar torsion $T$, namely $f(T)$-gravity [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. A review on this approach can be found on Ref. [36].

The action of these models is defined as

$$I = -\frac{1}{2} \int dx^4 \sqrt{-g} f(T) + I_m.$$ (20)

It is possible to show that, with this choice of the tensor scalar $T$, the related equations of motion are second-order partial differential equations.

As an example, we verify this important feature of the $f(T)$ models considering again the flat FRW space-time. In fact, in this relevant case, the Cartesian and spherical coordinates lead to the same expression for the scalar tensor, namely $T = 6H^2$. Moreover, the generalized Friedmann equation is [36]

$$\rho = \frac{f(T)}{2} - Tf_T(T),$$ (21)

or, using the value of the torsion tensor in this space-time $T = 6H^2$,

$$\rho = \frac{f(H)}{2} - Hf'_H = \frac{Hf''}{2},$$ (22)

where $\rho$ is the standard matter energy density and $f''(H) = \frac{df}{dH}$. The diffeormorphism invariance leads to

$$\dot{\rho} = -3H(\rho + p),$$ (23)

and the second Friedmann equation follows as usual by the above two equations [24]–[26], namely

$$-p = \frac{f(H)}{2} - Hf'_H - \frac{1}{6} Hf''(H),$$ (24)

where the dot denotes the time derivative and $p$ the standard matter pressure.

### 3.1 Cosmological bounce models

In this section we propose two bounce models for the early Universe in the context of teleparallel gravity.
3.1.1 A model proposal

We introduce the following expression for the function of the torsion scalar

$$f(T) = \frac{12}{\alpha^2} \left[ 1 - \sqrt{1 - \alpha^2\frac{T}{6}} - \alpha \sqrt{T/6} \arcsin \left( \alpha \frac{T}{6} \right) \right],$$

which can be rewritten, using $$T = 6H^2$$,

$$f(H) = \frac{12}{\alpha^2} \left[ 1 - \sqrt{1 - \alpha^2H^2} - \alpha H \arcsin (\alpha H) \right],$$

where $$\alpha$$ is a dimensional positive parameter. Note that $$f(H) = -6H^2 + O(\alpha^2)$$ when $$|\alpha| \to 0$$, confirming that we are studying a correction to Einstein gravity which can be controlled with the the value of the parameter $$\alpha$$.

The first Friedmann equation is

$$\frac{6}{\alpha^2} \left[ 1 - \sqrt{1 - H^2\alpha^2} \right] = \rho,$$

which is equivalent to the QLC Friedmann equation

$$3H^2 = \rho \left[ 1 - \frac{\rho}{\rho_c} \right], \quad \text{where} \quad \rho_c = \frac{12}{\alpha^2},$$

where $$\rho_c$$ is called critical density. Thus, by assuming $$p = \omega \rho$$, with $$\omega \neq -1$$, when $$\rho_c \to \infty$$, we recover Einstein’s gravity. It is well known that the above equation leads to a cosmological bounce solution. For example the density reads [13]

$$\rho = \frac{1}{\alpha^2 + \frac{3(1+\omega)^2}{4}t^2}$$

As a result, the correction to GR avoids the Big bang singularity. In this case, the model admits a bounce solution with $$H = 0$$. On the other hand, if $$\omega = -1$$, namely $$\rho = \rho_0$$ where $$\rho_0$$ is a constant, in general we get a flat de Sitter solution.

3.1.2 Modified Born-Infeld model

In the literature several other modified teleparallel gravity bounce models have been proposed. In this section we revisit one of them. In particular, we study a Born-Infeld type model based on the choice suggested in [37]. For other models see [38]

$$f(T) = \frac{12}{\alpha^2} \left[ \sqrt{1 - \alpha^2\frac{T}{6}} - 1 \right],$$

which can be rewritten as

$$f(H) = \frac{12}{\alpha^2} \left[ \sqrt{1 - \alpha^2H^2} - 1 \right].$$

The related equations of motion lead to

$$\left( 1 + \frac{\alpha^2 \rho}{6} \right) \left[ \sqrt{1 - \alpha^2H^2} \right] = 1,$$

from which we can compute the modified Friedmann equation

$$3H^2 = \rho \left[ 1 + \frac{\alpha^2 \rho}{12} \right] \left( 1 + \frac{\alpha^2 \rho}{6} \right)^{-2}.$$ 

From the matter conservation law and a barotropic fluid $$p = \omega \rho$$, we obtain

$$\dot{\rho} = -\sqrt{3}(1 + \omega) \left[ \rho^3 + \frac{\alpha^2 \rho}{12} \right]^{1/2} \left( 1 + \frac{\alpha^2 \rho}{6} \right)^{-1},$$
We can explicitly write the various components of the perturbed equations of motion. Recall that this equation above is a transcendental equation for \( \rho \) which in general is difficult to solve analytically. However, we may find an approximated solution for small values of \( |\alpha| \),

\[
\rho(t) = \frac{1}{3(1 + \omega)^2 t^2 + \frac{2}{\omega}} + O(\alpha^4) .
\]

Therefore also in this model we can remove the GR Big bang singularity at \( t = 0 \), which can be retrieved in the limit \( |\alpha| \to 0 \).

## 3.2 Cosmological perturbations

In this section we study the cosmological linear perturbations of modified teleparallel gravity in a flat FLRW space-time. We consider the Newtonian gauge, and the following metric

\[
ds^2 = -(1 + 2\Phi) dt^2 + a^2(1 - 2\Psi) (\delta_{ij} + h_{ij}) dx^i dx^j, \tag{37}
\]

where \( \Phi \equiv \Phi(t, x) \) and \( \Psi \equiv \Psi(t, x) \) are the Newtonian potentials, and \( h_{ij}(t, x) \) denotes the tensor perturbations. Note that \( h_{ij} \) is traceless, \( \delta^{ij} h_{ij} = 0 \), and \( \delta^{ij} h_{ij,k} = 0 \).

### 3.2.1 Scalar perturbations

In order to study the scalar perturbations, we start from the vierbein perturbations, from which we can obtain the perturbed metric 39, 40.

Denoting with \( e_\mu^A \) the perturbed quantity and with \( \bar{e}_\mu^A \) the unperturbed one, the scalar vierbein perturbations write,

\[
\bar{e}_\mu^A = e_\mu^A + t_\mu^A , \tag{38}
\]

where

\[
\bar{e}_0^0 = \delta_0^0 , \quad e_0^0 = \delta_0^a \bar{e}_a^0 , \quad \bar{e}_0^a = \delta_0^a , \quad \bar{e}_a^0 = \delta_a^a , \tag{39}
\]

\[
t_0^0 = \delta_0^0 \Phi , \quad t_0^a = -\delta_0^a \Psi , \quad t_a^0 = -\delta_0^a \Phi , \quad t_a^a = \delta_a^a \Psi . \tag{40}
\]

In the above expressions, the various coefficients are chosen in order to retrieve the metric scalar perturbation in the Newtonian gauge in the form of equation 37.

Using the above equations, we obtain the torsion scalar perturbation

\[
\delta T = 12H(H\Psi + \Phi) . \tag{41}
\]

We can explicitly write the various components of the perturbed equations of motion. Recalling that \( \dot{T} = 6H^2 \), at the linear order in the perturbation quantities, we obtain the equations

\[
- \frac{f'(H)}{6H} \frac{\nabla^2}{a^2} \Psi + \frac{H}{2}(\psi + \Phi H) f''(H) = \delta \rho , \tag{42}
\]

\[
- \frac{f'(H)}{6}(\dot{\psi} + \Phi \dot{H}) = \delta u , \tag{43}
\]

\[
- \frac{f'(H)}{12H}(\psi - \Phi) = \delta s , \tag{44}
\]

\[
- \frac{f''(H)}{6}(\dot{\psi} - \Phi - \dot{H}) - \left( \frac{f'(H)}{2} H + \frac{f''(H)}{6} \dot{H} \right) \Psi + \left( \frac{f'(H)}{2} \dot{H} + \frac{H}{3} f''(H) + \frac{f''(H)}{6} \dot{H} \dot{H} \right) \Phi - \frac{f'(H)}{12H} \frac{\nabla^2}{a^2} (\psi - \Phi) = \delta \rho , \tag{45}
\]

where the functions \( \delta \rho, \delta p, \delta u, \delta s \) are the fluctuations of energy density, pressure, fluid velocity and anisotropic stress, respectively.

If we consider a vanishing anisotropic stress, from Eq. 44 we have \( \Phi = \Psi \). Furthermore, for adiabatic perturbations it is possible to relate the pressure and the energy density, namely
\[ \delta p = c_s^2 \delta \rho. \] Combining these two relations and the Eqs. \((42, 45)\), one obtains a closed equation for \(\Phi\),

\[ \ddot{\Phi} + \Gamma \dot{\Phi} + \beta \Phi - \omega \nabla^2 a^2 \Phi = 0, \tag{46} \]

where

\[ \Gamma = 3H \left( \frac{4}{3} + c_s^2 \right) + \frac{f'''(H)}{f''(H)} \dot{H}, \tag{47} \]

\[ \beta = 3H^2 (1 + c_s^2) + 2\dot{H} + \frac{f''(H)}{f''(H)} H \dot{H}, \tag{48} \]

and

\[ \omega = \frac{f'(H)}{Hf''(H)} c_s^2. \tag{49} \]

The latter expression shows that \(\omega\) can be considered a correction of the standard GR sound speed given by modified teleparallel gravity. In fact, in the GR limit \(f(H) = -6H^2\), \(\omega = c_s^2\). Thus it is interesting to compute the quantity \(\omega\) for the two models considered in section 3.1. From the model \((26)\), we obtain

\[ \omega = \sqrt{1 - \alpha^2 H^2} \arcsin \frac{\alpha H}{c_s^2}, \tag{50} \]

Thus, for small values of \(\alpha^2\), one has

\[ \omega = \left( 1 - \frac{\alpha^2 H^2}{2} \right) c_s^2, \tag{51} \]

namely \(\omega < c_s^2\) and superluminarity effects are avoided. In the Born-Infeld type model \((31)\), we obtain

\[ \omega = (1 - \alpha^2 H^2) c_s^2, \tag{52} \]

and superluminarity is again not present.

### 3.2.2 Tensor perturbations

The tensor perturbations obey the following equation \((39)\).

\[ \left( \ddot{h}_{ij} + 3Hh_{ij} - \frac{\nabla^2}{a^2} h_{ij} \right) - \dot{H} \left( \frac{f''(H)}{f'(H)} - \frac{1}{H} \right) \dot{h}_{ij} = 0. \tag{53} \]

Although we have a new friction term, there are no new mass terms. We can therefore conclude that in general \(f(T)\) theories do not introduce massive gravitons. Or, in other words, the velocity of tensor waves is equal to the speed of light at any time. Finally, similarly to the scalar case, the limit \(\alpha \to 0\) recovers the behaviour of GR perturbations at linear order.

### 3.3 Non-flat FLRW case

In this section we review the teleparallel gravity in a non spatially flat FLRW space-time. In fact, the generalization of what we previously obtained in the flat to non flat FLRW case is not straightforward, due to the subtleties associated with the evaluation of the torsion scalar \(T\). Once \(T\) is computed, since we are considering a dynamical highly spherical symmetric space-time, we may apply the superspace methods (see for example Ref. \(\{41\}\). For a more general approach, see for example \(\{42\}\) and reference therein.

Consider the spatially curved FLRW, written in spherical coordinates

\[ ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \tag{54} \]

where we units such that the curvature scalar \(|k| = 1\). The key point is the evaluation of the scalar tensor \(T\) on this space-time. In a non spatially flat FLRW in spherical coordinates, the naive choice for the tetrad requires a suitable non vanishing spin connection, or alternatively,
in the gauge $\omega = 0$, one has to consider a non trivial suitable choice for the tetrad (see for example Refs. [20, 43, 44]). In both approaches, the result is the same and reads

$$T = 6 \left[ H^2(t) - \frac{k}{a^2(t)} \right].$$

(55)

However, this is not the unique expression for the scalar $T$ in a spatially curved space-time. In fact, according to [45], for $k < 0$ we may also have

$$T = 6 \left[ H(t) \pm \frac{1}{a(t)} \right]^2.$$  

(56)

In all cases, applying the superspace method, the action can be written, up to a trivial multiplicative time independent factor, as

$$I = \int a^3 f(T(a, \dot{a})) dt + I_m,$$

(57)

where $I_m$ is the usual matter Lagrangian.

Firstly we deal with the choice of the torsion scalar (55). We can consider $a(t)$ as Lagrangian coordinate, and from the variation of the action with respect to it we obtain the second Friedmann equation

$$-p = \frac{f(T)}{2} - f_T \left( 6H^2 + 2\dot{H} - 2\frac{k}{a^2} \right) - 24H^2 \left( \dot{H} + \frac{k}{a^2} \right) f_{TT}.$$  

(58)

The first Friedmann equation follows from the above equation and the standard matter conservation law, ensured by the diffeomorphism invariance of the model, and reads

$$\rho = \frac{f(T)}{2} - 6H^2 f_T.$$  

(59)

For the second choice of the torsion scalar (56), the second Friedmann equation reads

$$-p = \frac{f(T)}{2} - 2f_T \left( 3H^2 + \dot{H} + 2\frac{1}{a^2} \pm 3\frac{H}{a} \right) - 24 \left( H \pm \frac{1}{a} \right)^2 \left( \dot{H} \mp \frac{H}{a} \right) f_{TT},$$  

(60)

while the first Friedmann equation is

$$\rho = \frac{f(T)}{2} - 6H \left( H \pm \frac{1}{a} \right) f_T,$$

(61)

in agreement with the results of [45]. As a simple consistency check, in the GR case $f(T) = -T$, and it both cases we obtain the GR result

$$3H^2 + \frac{3k}{a^2} = \rho.$$  

(62)

In the next sections we will consider two applications of teleparallel gravity in the non flat case. In particular, we will study the models [25] and [50] in the non-flat case.

### 3.3.1 Example 1

Consider again the expression used in (25). In this example we consider only the first class of models, where $T$ is given by (55). The other choice leads to similar results. From Eq. (59) we obtain

$$\frac{6}{a^2} \left( 1 - \sqrt{1 - \frac{k}{a^2} T} \right) = \rho - \frac{6k}{a^2} \left[ \frac{1}{\sqrt{a^2T/6}} \arcsin \left( \frac{T}{a^2} \right) \right], \quad T = 6 \left( H^2 - \frac{k}{a^2} \right),$$  

(63)

which can be rewritten as

$$3 \left( H^2 - \frac{k}{a^2} \right) = \rho - \frac{6k}{a^2} \psi_k(T) - \frac{a^2}{12} \left[ \rho - \frac{6k}{a^2} \psi_k(T) \right]^2,$$  

(64)
where
\[ \psi_k(T) \equiv \frac{1}{\alpha \sqrt{T/6}} \arcsin \left( \frac{T}{6} \right). \] (65)

At the first order in \( \alpha^2 \) the equation above become
\[ 3 \left( H^2 - \frac{k}{a^2} \right) = \left( \rho - \frac{6k}{a^2} \right) \left( 1 - \frac{\rho - 6k}{\rho_c} \right), \quad \rho_c = \frac{12}{\alpha^2}, \] (66)
or
\[ 3 \left( H^2 + \frac{k}{a^2} \right) = \rho - \left( \frac{\rho - 6k}{\rho_c} \right)^2, \quad \rho_c = \frac{12}{\alpha^2}. \] (67)
The case is analog to the one discussed in [13], where the Big-Bang singularity at \( t = 0 \) is absent. An exact solution can be found if we take a barotropic equation of state \( p = \rho/3 \), namely \( \omega = -1/3 \) and \( \rho(t) = \rho_0 a(t)^{-2} \). We can rewrite equation the above equation introducing a new variable \( y = a(t)^2 \)
\[ \frac{3}{4} y^2 = (\rho_0 - 3k)y - \frac{(\rho_0 - 6k)^2}{\rho_c}, \quad y = a^2(t), \] (68)
whose solution is
\[ y(t) \equiv a^2(t) = \frac{(\rho_0 - 6k)^2}{\rho_c} + \frac{\rho_0 - 3k}{3} t^2, \] (69)
where we assume \( \rho_0 > 3k \). This (bounce) solution is regular at \( a(0) \neq 0 \). When \( \rho_c \) goes to infinity, we recover the GR solution with the Big Bang singularity.

Moreover, for the case of a generic constant value of \( \omega \), we can prove the absence of singularities. In fact, considering a perfect fluid with \( p = \omega \rho \) and \( \rho = \rho_0 a^{-3(1+\omega)} \), with a constant \( \rho_0 \), we can rewrite Eq. (67) as
\[ \int \frac{dy}{Y(y)} = t, \quad y = a(t)^2, \] (70)
where \( Y(y) \) is
\[ Y(y) = \frac{4}{3} \left( \rho_0 y \frac{\omega}{2} \right) - 3ky - \frac{\rho_0^2 y^{-1-3\omega} + 36k^2 - 12k^3 a^{-3(1+\omega)}}{\rho_c} \] (71)
We can find an approximate solution, valid for small \( t \) around the critical point \( Y(y) = 0 \)
\[ y(t) \approx y_* + \frac{Y'(t^2)}{4}, \] (72)
where \( t_* \) is the solution of the transcendental equation
\[ \rho_0 y_* \left( y_* \frac{1-3\omega}{2} - \frac{3k}{\rho_0} \right) - \frac{\rho_0^2 y_*^{-1-3\omega} + 36k^2 - 12k^3 a^{-3(1+\omega)}}{\rho_c} = 0. \] (73)
Therefore only in the GR limit \( \rho_c \to \infty \) (i.e. \( \alpha \to 0 \)) we have a singular solution for the flat case with \( k = 0 \), otherwise \( y_* \neq 0 \) and the bounce occurs, independently on the space curvature. For example, considering \( k = 0 \), Eq. (67) becomes
\[ 1 - \frac{\rho_0 y_*^{-3(1+\omega)}}{\rho_c} = 0, \] (74)
whose solution reads
\[ y_* \equiv a^2(t_*) = \left( \frac{\rho_c}{\rho_0} \right) \frac{1}{1+\omega}, \] (75)
which corresponds to the minimum value of the scale factor on the cosmological bounce.
3.3.2 Example 2

Consider the model in non-flat FLRW space-time. Again, we use the case where the torsion scalar is given by (55). The first Friedmann equation becomes

\[ 3 \left( H^2 - \frac{k}{a^2} \right) = \rho \left( 1 + \frac{\alpha^2}{12} \right) - \frac{6k}{a^7} - \frac{3\alpha^2 k^2}{a^2} \left( \frac{2a^2}{6} + 1 \right)^2. \] (76)

For a perfect fluid we can write

\[ \int \frac{dy}{\sqrt{Y(y)}} = t, \quad y = a(t)^2, \] (77)

where, at the first order of \( \alpha^2 \), \( Y(y) \) is

\[ Y(y) = \frac{2}{27} \alpha^2 \rho_0^2 (3k - 5y)y^{-3\omega/2} + \frac{8}{3} \alpha^2 k \rho_0 y^{-1/2} + \frac{3}{4} \left( \alpha^2 k + y \right) + \frac{4}{3} \rho_0 y^{1/2 - \omega/2} = 0. \] (78)

Therefore an approximate solution, valid for small \( t \) around the critical point \( Y(y_\ast) = 0 \), can be found and reads

\[ y(t) \approx y_\ast + \frac{Y(y_\ast)}{4}, \] (79)

where \( y_\ast \) is the solution of

\[ \frac{2}{27} \alpha^2 \rho_0^2 (3k - 5y_\ast)y_\ast^{-3\omega/2} + \frac{8}{3} \alpha^2 k \rho_0 y_\ast^{-1/2} + \frac{3}{4} \left( \alpha^2 k + y_\ast \right) + \frac{4}{3} \rho_0 y_\ast^{1/2 - \omega/2} = 0. \] (80)

Thus in the limit \( \alpha \to 0 \) and \( k = 0 \) we recover the Big bang singularity at \( y_\ast = 0 \), otherwise the bounce appears.

4 Conclusions

In this paper we have investigated a derivation of Friedmann equations of QLC-like cosmology in the framework of modified teleparallel gravity, which offers a very interesting alternative approach with respect to modified gravitational theories based on metric formulation as \( F(R) \) modified models. In fact, instead of using the Levi-Civita connection, teleparallel gravity is based on the so called Weitzenbock connection with a related vanishing curvature tensor, but non vanishing torsion tensor. All the GR results can be found in the equivalent teleparallel GR formulation. Furthermore, the extension of the theory to modified teleparallel gravity preserves the field equations at the second order: this is one of the reasons why modified teleparallel gravity is often used to investigate a wide variety of solutions in a cosmological context.

We have also proposed a model firstly in flat FLRW space-time, which features a non linear correction to GR equations that depends on a critical density and it is sufficient to avoid the Big Bang singularity, namely the model admits the cosmological bounce of QLC cosmology. We also argued that we can recover the GR limit, when the critical density goes to infinity. Moreover the analysis of scalar and tensor perturbations shows the viability of the theory and the lack of superluminal effects.

We have finally generalized the results to the curved FLRW space-time. Even though the non-flat case is more complex due to the nature of teleparallel theories, a direct evaluation of the field equations from the on-shell form of the Lagrangian, once the torsion scalar has been determined, is easily achieved. This is possible because we are working on highly symmetric space-times. Despite to the fact that exact solutions can be found only for some specific matter choices, we have investigate approximate solutions near \( t = 0 \), and proved, using the critical points of the theory that the models do not contain singularities and provide cosmological bounce solutions. Finally, we have generalized the Born-Infeld model to spatially FLRW curved space.
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