σ-Porosity of the set of strict contractions in a space of non-expansive mappings

Christian Bargetz
joint work with Michael Dymond

Relations Between Banach Space Theory
and Geometric Measure Theory
8–12 June 2015
The setting

Let $X$ be a Banach space and $C \subseteq X$ a closed, convex and bounded set. We consider the space

$$M = \{ f : C \to C : \forall x, y \in C : \| f(x) - f(y) \| \leq \| x - y \| \}$$

equipped with the metric

$$d(f, g) = \sup_{x \in C} \| f(x) - g(x) \|$$

of uniform convergence.
$M$ is a complete metric space.
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$\mathcal{M}$ is a complete metric space.
How small is the set of strict contractions?

We consider the set

\[ \mathcal{N} = \{ f \in \mathcal{M} : \text{Lip}(f) < 1 \} \]

of strict contractions.

**Question**

“How small” is the set of strict contractions in \( \mathcal{M} \)?
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Question

“How small” is the set of strict contractions in $\mathcal{M}$?
σ-porous sets

A subset $A \subset M$ is said to be porous at $x \in A$ if there are constants $\alpha > 0$ and $\varepsilon_0 > 0$ with the following property: For all $\varepsilon \in (0, \varepsilon_0)$ there is a point $y \in M$ with $\|y - x\| \leq \varepsilon$ and $B(y, \alpha \varepsilon) \cap A = \emptyset$. The set $A$ is called porous if it is porous at all of its points.
A subset $A \subset M$ is said to be *porous at* $x \in A$ if there are constants $\alpha > 0$ and $\varepsilon_0 > 0$ with the following property: For all $\varepsilon \in (0, \varepsilon_0)$ there is a point $y \in M$ with $\|y - x\| \leq \varepsilon$ and $B(y, \alpha \varepsilon) \cap A = \emptyset$. The set $A$ is called *porous* if it is porous at all of its points.
\(\sigma\)-porous sets

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The set $A$ is called $\sigma$-*porous* if it is a countable union of porous sets.
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Note that σ-porous sets are of first category in the sense of the Baire category theorem.
The Hilbert space case

**Theorem (De Blasi and Myjak, 1989)**

*If* $X$ *is a Hilbert space the set* $\mathcal{N}$ *of strict contractions on* $C$ *is a* $\sigma$-*porous subset of* $\mathcal{M}$.

**Proof sketch.**

Take a sequence $\left( L_k \right)_{k \in \mathbb{N}}$ with $L_n \uparrow 1$ and set

$$\mathcal{N}_k = \{ f \in \mathcal{M} : \text{Lip}(f) \leq L_k \}.$$  

Given $f \in \mathcal{N}_k$ and $\varepsilon > 0$ set $g$ to the identity on a small ball around the fixed point $x^*$ of $f$ and to $f$ outside a bigger ball around $x^*$ then use Kirszbraun’s extension theorem to get a close enough midpoint of a ball outside $\mathcal{N}_k$. 

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The general case

**Theorem (B. and Dymond, 2015)**

Let $X$ be a Banach space and $C \subset X$ a closed, convex and bounded set. Then the set $\mathcal{N}$ of all strict contractions is a $\sigma$-porous subset of $\mathcal{M}$. 
Sketch of the proof

We define

\[ \mathcal{N}^{p}_{a,b} = \left\{ f \in \mathcal{N} : a < \text{Lip}(f, \Gamma) \leq b, \ \text{Lip}(f) \leq 1 - \frac{1}{p} \right\} . \]

Fix \( f \in \mathcal{N}^{p}_{a,b} \). Choose \( x_0 \in \Gamma \) such that

\[ \liminf_{t \to 0^+} \frac{\|f(x_0 + te) - f(x_0)\|}{t} \geq a. \]
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\[ N_{a,b}^p = \left\{ f \in N : a < \text{Lip}(f, \Gamma) \leq b, \ \text{Lip}(f) \leq 1 - \frac{1}{p} \right\}. \]

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Fix $\alpha > 0$, $\varepsilon > 0$. Now set

$$g(x) = f(x + \sigma \phi_{\varepsilon}(e^*(x - x_0))(e - (x - x_0))).$$

where $\sigma \phi_{\varepsilon}(e^*(x - x_0))(e - (x - x_0))$ stretches along $\Gamma$ to increase the Lipschitz constant.

Setting $R = \text{diam}(C)$, if $b - a$ is small enough, we obtain

- $g \in \mathcal{M}$
- $d(f, g) \leq R\varepsilon$
- $B(g, \alpha R\varepsilon) \cap \mathcal{N}^p_{a, b} = \emptyset$.

Hence $\mathcal{N}^p_{a, b}$ is porous.
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Sketch of the proof, part II

Fix \( \alpha > 0, \varepsilon > 0 \). Now set

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Hence $\mathcal{N}_{a,b}^p$ is porous.
Additionally to the condition that \( b - a \) has to be small enough, it has to be big enough so that we can cover the whole interval \((0, 1)\). Writing

\[
\mathcal{N} = \left( \bigcup_{k,p} \mathcal{N}^{p}_{a_k,p,b_k,p} \right) \cup \left( \bigcup_{k,p} \mathcal{N}^{p}_{a'_k,p,b'_k,p} \right) \cup \mathcal{N}_0.
\]

for suitable sequences \((a_k,p)_{k,p}\), \((b_k,p)_{k,p}\), \((a'_k,p)_{k,p}\) and \((b'_k,p)_{k,p}\) and

\[
\mathcal{N}_0 = \{ f \in \mathcal{M} : f|_\Gamma = \text{const.} \}
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finishes the proof.
Sketch of the proof, part III

Additionally to the condition that $b - a$ has to be small enough, it has to be big enough so that we can cover the whole interval $(0, 1)$. Writing

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for suitable sequences $(a_{k,p})_{k,p}$, $(b_{k,p})_{k,p}$, $(a'_{k,p})_{k,p}$ and $(b'_{k,p})_{k,p}$ and

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finishes the proof.
The case of separable Banach spaces

If $X$ is a separable Banach space we get the following stronger result:

**Theorem (B. and Dymond, 2015)**

Let $X$ be a separable Banach space. Then there exists a $\sigma$-porous set $\tilde{N} \subset M$ such that for every $f \in M \setminus \tilde{N}$, the set

$$R(f) = \{ x \in C : \text{Lip}(f, x) = 1 \}$$

is a residual subset of $C$.

Put differently, this Theorem says that outside of a negligible set, all mappings in the space $M$ have the maximal possible Lipschitz constant 1 at typical points of their domain $C$. 
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Put differently, this Theorem says that outside of a negligible set, all mappings in the space $M$ have the maximal possible Lipschitz constant 1 at typical points of their domain $C$. 
Denote by $g_\varepsilon$ the function

$$g_\varepsilon: C \to C, x \mapsto f(x + \sigma \phi_\varepsilon(e^*(x - x_0))(e - (x - x_0))).$$

The curve

$$[0, \varepsilon_0) \to C(X; X), \varepsilon \mapsto g_\varepsilon$$

is Lipschitz.

**Question**

Can such a curve be chosen differentiable, to get information on the directions from which the midpoints $g_\varepsilon$ are approaching $f$?
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Outlook

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References

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C. Bargetz and M. Dymond.
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