Signature-Based Abduction with Fresh Individuals and Complex Concepts for Description Logics (Extended Version)

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Abstract
Given a knowledge base and an observation as a set of facts, ABox abduction aims at computing a hypothesis that, when added to the knowledge base, is sufficient to entail the observation. In signature-based ABox abduction, the hypothesis is further required to use only names from a given set. This form of abduction has applications such as diagnosis, KB repair, or explaining missing entailments. It is possible that hypotheses for a given observation only exist if we admit the use of fresh individuals and/or complex concepts built from the given signature, something most approaches so far do not support or only support with restrictions. In this paper, we investigate the computational complexity of this form of abduction—allowing either fresh individuals, complex concepts, or both—for various description logics, and give size bounds on the hypotheses if they exist.

1 Introduction
Description logics (DLs) are a powerful formalism to describe knowledge bases (KBs) containing both general domain knowledge from a DL ontology and a set of facts (the ABox). Using a DL reasoner, we can then infer information that is implicit in the data, and can be logically deduced based on the ontology [Baader et al., 2017]. Sometimes it is useful to not only reason about what logically follows from a DL KB, but to reason also about what does not follow. In abduction, we are given a KB as background knowledge, in combination with a set of facts (the observation) that cannot be deduced from the background knowledge. We are then looking for the missing piece in the background knowledge (the hypothesis) that is needed to make the observation logically entailed [Elsenbroich et al., 2006]. This form of reasoning has many applications: 1) it can be used to explain why something cannot be deduced [Calvanese et al., 2013], 2) it can be used for diagnosis tasks, giving the hypothesis as possible explanation for an unexpected observation [Obeid et al., 2019], and 3) it can be used in KB repair to give hints on how to fix missing entailments [Wei-Kleiner et al., 2014].

As a simplified application example from the geology domain, assume we have observed that in an area near a canal, holes appeared in the street as a result of subsidence due to an unstable ground. A possible explanation could involve the presence of a formation of so-called evaporite below the street, which dissolves when in contact with water [Fidelibus et al., 2011]. Our background knowledge consists of a geology ontology together with data about the area. Among others, it contains the following abbreviated axioms:
1. EvaFor ∩ bord.(Wat ∩ ¬lin.WatPro) ⊑ aff.Dis
2. EvaFor ∩ aff.Dis ⊑ xabov.Unst
3. (Wat ∩ Str) ∩ EvoFo ⊑ ⊥
4. Wat(can) ⊑ Str(str)
which state that 1. an Evaporite Formation which borders to a Waterway without Water-Proof lining will be affected by Dissolution; 2. All ground above an evaporite formation affected by dissolution is Unstable; 3. waterways and Streets are not evaporite formations; 4. can is a waterway; 5. str is a street. Our observation would be that the street is unstable: Unst(a1), and a hypothesis based on our background knowledge would be
\[ \mathcal{H} = \{ \text{EvaFor}(e), \text{abov}(e, \text{str}), \text{bord}(e, \text{can}), \forall \text{lin}.\bot(\text{can}) \} \]

stating that there is an evaporite formation \( e \) below the street that borders with the canal, and that the canal has no lining. A team of geologists can then verify the hypothesis by analysing the canal and the ground below the street. We highlight two aspects of this hypothesis: 1) it refers to a previously unknown individual, the evaporite formation, and 2) it uses a complex (composed) DL concept (\( \forall \text{lin}.\bot(\text{can}) \)). We are interested in hypotheses like this for signature based abduction, where we are additionally given a signature \( \Sigma \) of abducibles—a vocabulary of names to be used within the hypothesis [Koopmann et al., 2020]. The aim of \( \Sigma \) is to restrict hypotheses that have explanatory character. In the present example, we would exclude aff and Dis from \( \Sigma \), as the dissolution alone would be a too shallow explanation, and Wat, because we already know the waterways in the area. Furthermore, we are looking at ABox abduction, in which observations and hypotheses are ABoxes, in contrast to TBox abduction [Du et al., 2017; Wei-Kleiner et al., 2014], KB abduction [Koopmann et al., 2020; Elsenbroich et al., 2006] or concept abduction [Bienvenu, 2008].

While there are practical approaches to ABox abduction without signature-restriction [Klarman et al., 2011; Halland and Britz, 2012; Pukancová and Homola, 2017],
works on signature-based ABox abduction often restrict hypotheses to flat ABoxes with a given set of individuals [Ceylan et al., 2020; Du et al., 2012]—which essentially means that statements in hypothesis can be picked from a finite set—or they restrict to rewritable DLs which have limited expressivity [Du et al., 2014; Calvanese et al., 2013]. As with DLs, we usually have the open-world semantics, in which not all individuals are known, and DLs offer much more expressivity, it is a natural next step to look also at abduction allowing for fresh individuals and complex concepts in the result. This changes the nature of the abduction problem drastically as there is now an unbounded set of axioms that may occur in a hypothesis. Problems such as “Does axiom α belong to some/every/an optimal solution?” [Calvanese et al., 2013; Ceylan et al., 2020] become less helpful while new questions become interesting, such as whether we can give bounds on the number of individuals in a hypothesis, or on the overall size of the hypothesis.

Without understanding the theoretical properties yet, practical methods for signature-based abduction that admit expressive DL concepts in the hypothesis are presented in [Koopmann et al., 2020; Del-Pinto and Schmidt, 2019]. The authors consider hypotheses that we would call complete in the sense that they cover all hypotheses at the same time. To make this possible, solutions are represented in a very expressive DL using non-classical operators such as fixpoints and axiom disjunction. In this setting, ABox abduction can be reduced to uniform interpolation, which however may produce solutions that are triple exponentially large [Zhao and Schmidt, 2017; Lutz and Wolter, 2011]. A natural question is whether this blow-up is really necessary, or whether we can obtain smaller or simpler hypotheses if we drop the requirement of completeness and look for hypotheses in a classical DL that is sufficient to entail the observation.

Unfortunately, our results indicate that this is not the case: if we only allow for fresh individuals but not for complex concepts, hypotheses may require exponentially many assertions for DLs between $\mathcal{EL}_\bot$ and $\mathcal{ALC}$, while for $\mathcal{ALC}_F$ there does not even exist a general bound. If in addition, we allow for complex concepts, we are able to explain more observations, but the explanations may become triple exponentially large in comparison to the input. Motivated by this, we also consider a variant of the abduction problem in which we are additionally given a bound on the size of the hypothesis. To summarize, our contributions are the following.

1. we investigate signature-based ABox abduction for DLs ranging from $\mathcal{EL}$ to $\mathcal{ALC}_Q$ where hypotheses may use fresh individuals, complex concepts or both,
2. we give tight bounds on the size of hypotheses if they exist,
3. we analyse the computational complexity of deciding whether a hypothesis exists, and
4. we analyse the complexity of deciding whether a hypothesis of bounded size exists.

Proof details are provided in the appendix.

## 2 Description Logics and ABox Abduction

We recall the DLs relevant to this paper [Baader et al., 2017] and provide the formal definition of the abduction problem we consider in this paper.

Let $\mathcal{N}_C$, $\mathcal{N}_R$ and $\mathcal{N}_I$ be three pair-wise disjoint sets of respectively concept, role and individual names. A role $R$ is either a role name $r$ or an inverse role $r^\bot$, where $r \in \mathcal{N}_R$. $\mathcal{EL}$ concepts are built according to the following syntax rule, where $A \in \mathcal{N}_C$ and $R \in \mathcal{N}_R$:

$$C ::= T \mid A \mid C \cap C \mid \exists R.C$$

More expressive DLs allow for the following additional concepts, where $n \in \mathcal{N}$, and in brackets, we give the name of the corresponding DL:

$$\perp (\mathcal{EL}_\bot) \quad \neg C (\mathcal{ALC}) \leq 1 R.T (\mathcal{ALC}_F) \leq n.R.C(\mathcal{ALC}_Q)$$

In each case, all previous constructs are allowed in the DL as well. Using the letter $T$ in the DL name we express that in the above, $R$ may also be an inverse role. For example, $(\geq mr^{-1}.C)$ is an $\mathcal{ALC}_Q$ concept but not an $\mathcal{ALC}_Q$-concept, and $\forall r.\bot$ is an $\mathcal{EL}_\bot$ concept. Additional operators are introduced as abbreviations: $\forall r.D = \neg(\forall r.D)$ and $\exists r.C = \neg(\exists r.C)$.

A $\mathcal{KB}$ is a set of axioms, that is, concept inclusions (CIs) $C \subseteq D$, concept assertions $C(a)$ and role assertions $r(a, b)$, where $C, D$ are concepts, $a, b \in \mathcal{N}_I$ and $r \in \mathcal{N}_R$. If a $\mathcal{KB}$ contains only concept and role assertions, it is called ABox, and if every concept assertion is of the form $A(a)$, where $A \in \mathcal{N}_C$, flat ABox. Given a concept/axiom/KB/ABox $E$, we denote by $\text{sub}(E)$ the set of (sub-)concepts occurring in $E$, by $\text{sig}(E)$ the set of concept and role names occurring in $E$, and by $\text{ind}(E)$ the set of individual names in $E$. By $\text{size}(E)$, we denote the number of symbols required to write $E$ down, where operators, as well as concept, role and individual names count as one, numbers are encoded in binary and the introduced abbreviations can be used.

The semantics of DLs is defined based on interpretations, which are tuples $I = (\Delta^I,^-I)$ of a set $\Delta^I$ of domain elements and an interpretation function $\cdot^I$ which maps every $a \in \mathcal{N}_I$ to some $a^I \in \Delta^I$, every $A \in \mathcal{N}_C$ to some $A^I \subseteq \Delta^I$, every $r \in \mathcal{N}_R$ to some $r^I \subseteq \Delta^I \times \Delta^I$, satisfies $(r^-)^I = (r^I)^{-1}$, and is extended to concepts as follows, where $#S$ denotes the cardinality of the set $S$:

$$\perp^I = \emptyset \quad (C \cap D)^I = C^I \cap D^I \quad (\neg C)^I = \Delta^I \setminus C^I$$

$$(\exists R.C)^I = \{d \in \Delta^I \mid \langle d, e \rangle \in R^I, e \in C^I \}$$

$$(\leq n.R.C)^I = \{d \in \Delta^I \mid \#\{\langle d, e \rangle \in R^I \mid e \in C^I \} \leq n\}$$

$I$ satisfies an axiom $\alpha$, in symbols $I \models \alpha$, if $\alpha = C \subseteq D$ and $C^I \subseteq D^I$; $\alpha = C(a)$ and $a^I \in C^I$; or $\alpha = r(a, b)$ and $\langle a^I, b^I \rangle \in r^I$. If $I$ satisfies all axioms in a KB $K$, we write $I \models K$ and call $I$ a model of $K$. A KB entails an axiom/KB $E$, in symbols $K \models E$, if $I \models E$ for every model $I$ of $K$. If $K$ has no model, it is inconsistent and we write $K \models \perp$.

We can now define the main reasoning problem we are concerned with in this paper. We consider signature-based ABox abduction problems, which for convenience, we just call abduction problems from here on.
We set EL concept names is a solution to the abduction problem, then neither would any other ABox. Consequently, if there
Clearly, Theorem 1.
The only way to produce this chain of $2^n$ elements is using $2^n−1$ role assertions, which establishes our lower bound.

This bound remains tight if we add expressivity up to $\text{ALCI}$, while we lose any bound on the size once we additionally allow concepts of the form $\bot$. □

Theorem 2. If there exists a hypothesis for a flat L abduction problem $\mathfrak{A}$, then there exists one of size
1. polynomial in the size of $\mathfrak{A}$ if $L = \mathcal{EL}$,
2. exponential in the size of $\mathfrak{A}$ if $L = \text{ALCI}$, and
3. if $L = \text{ALCF}$, no general upper bound based on $\mathfrak{A}$ can be given.

Proof sketch. We already established the bound for $L = \mathcal{EL}$. For $\text{ALCI}$, we assume there exists some hypothesis $\mathcal{H}_0$, based on which we build one of bounded size. For this, we pick any model $\mathcal{I}$ of $\mathcal{H}_0 \cup \mathcal{K}$, which allows us to identify individual names $a \in \text{ind}(\mathcal{H}_0)$ using at most exponentially many types $\text{tp}(a) = \text{tp}(a^{T})$, where
\[
\text{tp}(d)_{\mathcal{I}} = \{C \in \text{sub}(\mathcal{K} \cup \Phi) \mid d \in C^{T}\}.
\]
We associate to every type $\text{tp}(a)$ an individual name $b_{\text{tp}(a)}$, and define $h : \text{ind}(\mathcal{H}_0) \rightarrow \mathbb{N}$ by $h(a) = a$ if $a \in \text{ind}(\mathcal{K} \cup \Phi)$ and $h(a) = b_{\text{tp}(a)}$ otherwise. The hypothesis $\mathcal{H}$ is then:
\[
\{A(h(a)) \mid A(a) \in \mathcal{H}_0 \} \cup \{r(h(a), h(b)) \mid r(a, b) \in \mathcal{H}_0\}.
\]
Based on $\mathcal{I}$ and $h$, one can construct a model for $\mathcal{K} \cup \mathcal{H}$, so that $\mathcal{H}$ satisfies $\text{A1}$. Because $\mathcal{H}_0$ satisfies $\text{A3}$, by construction, so does $\mathcal{H}$. Finally, using the fact that $h$ is a homomorphism from $\mathcal{K} \cup \mathcal{H}_0$ into $\mathcal{K} \cup \mathcal{H}$ s.t. for every $a \in \text{ind}(\mathcal{K} \cup \Phi)$ $h(a) = a$, we can show that $\mathcal{K} \cup \mathcal{H} \models \Phi$, and thus $\text{A2}$.

For $L = \text{ALCF}$, we note that if there was a bound on the size of hypotheses, we could decide the instance query emptiness problem for $\text{ALCF}$ by iterating over all candidates, contradicting that this problem is undecidable for $\text{ALCF}$ [Baader et al., 2016]. □

The proof of Theorem 2 indicates how types can be used to perform abduction, which is used in the following theorem.

Theorem 3. Flat L ABox abduction is
\begin{itemize}
  \item $\text{P}$-complete for $L = \mathcal{EL}$,
  \item $\text{EXPTIME}$-complete for $L = \mathcal{EL}_{\perp}$,
  \item $\text{coNEXPTIME}$-complete for $L = \text{ALCI}$,
  \item undecidable for $L = \text{ALCF}$.
\end{itemize}
4 Abduction with Complex Concepts

As illustrated in the introduction, abduction may only be successful if we also admit complex concepts in the hypothesis. Determining such hypotheses turns out to be more challenging than for flat hypotheses, and we cannot find a correspondence to a known problem as for flat abduction. Indeed, one might assume such a relation to uniform interpolation: given a KB \( K \) and a signature \( \Sigma \), the uniform interpolant of \( K \) for \( \Sigma \) is a \( \Sigma \) ontology that captures all entailments of \( K \) within \( \Sigma \) [Koopmann and Schmidt, 2015]. By negating the observation, this can be used to perform complete abduction [Koopmann et al., 2020; Del-Pinto and Schmidt, 2019], that is, to compute a hypothesis that would be entailed by any other hypothesis. However, if we are interested just in computing any hypothesis rather than a complete one, this correspondence falls short, as uniform interpolants have stronger requirements than hypotheses, and the reasons for non-existence are different: namely, capturing all entailments of \( K \) in \( \Sigma \) in the uniform interpolant, using only names from \( \Sigma \), may require infinitely many axioms in case of cyclic axioms. In contrast, for abduction, non-existence is always due to Condition A1.

We consider abduction for ALC, EL, and EL_\perp, starting with the latter. In EL and EL_\perp, complex concepts do not bring much benefit compared to fresh individuals: an EL_\perp concept can only state the existence of role successors, which we can also do in flat ABoxes. In fact, for EL_\perp, if we allow complex concepts instead of fresh individuals, hypotheses even get more complex.

**Theorem 4.** There exists a family of EL_\perp abduction problems for which every hypothesis without fresh individuals is at least of double exponential size. If there exists such a hypothesis, there always exists one of at most double exponential size, whose existence can be decided in exponential time.

**Proof sketch.** The family of abduction problems is obtained similarly as in the proof for Theorem 1, only that we now use two roles \( r \) and \( s \) To get the corresponding upper bound, we first flatten an existing hypothesis \( \mathcal{H}_0 \) and again simplify the ABox based on the types in some model, however this time making sure the resulting flat ABox can be translated back into a complex one without fresh individuals.

The same care has to be taken when we modify the method used for Theorem 3. Specifically, we have to make sure that the hypothesis \( \mathcal{H}_o \) that we generate for a given mapping \( s : \text{ind}(K \cup \Phi) \to T_{K \cup \Phi} \) does not contain cycles between fresh individuals, so that it can be translated into a hypothesis without fresh individuals. Our fresh individuals are now of the form \( b_{a,t,s} \), where \( a \in \text{ind}(K \cup \Phi) \), \( t \in T = T_{K \cup \Phi} \), and \( t \in [1,2^{2^{|T|}}] \). Set \( b_{a,s(a),a} = a \). \( \mathcal{H}_s \) is then defined as:

\[
\mathcal{H}_s = \{ A(b_{a,t,k}) \mid a \in \text{ind}(\mathcal{A}), t \in T, k \in [0,2^{2^{|T|}}], A \in (T \cap \Sigma) \}
\cup \{ r(a,b) \mid a \in \text{ind}(K \cup \Phi), t \in T, r \in \Sigma, t_2 \in S^r_T(v_1), k \in [0,2^{2^{|T|}} - 1] \}
\cup \{ r(a,b) \mid s(b) \in S^r_T(s(a)) \}
\]

The hardness result requires again \( \perp \): for \( \mathcal{E}_\perp \), we can always use a flat solution as in the last section. In contrast, with more expressivity, the problem becomes even harder, even if we do admit fresh individuals. The reason is that for concepts of the form \( \forall r.C \), fresh individuals cannot come to the rescue anymore, and disjunctions may become necessary. The following theorem is shown by a modification of a construction in [Ghilardi et al., 2006].

**Theorem 5.** There is a family of ALC abduction problems for which the smallest (non-flat) ABox hypotheses are triple exponential in size.

We can show that this bound is tight.

**Theorem 6.** Let \( \mathcal{A} \) be an ALC abduction problem. Then, there exists a hypothesis for \( \mathcal{A} \) iff there exists a hypothesis of triple exponential size.

To show this theorem, we use a technique similar as for Theorem 2: we take an arbitrary hypothesis, and transform it into one of triple exponential size. However, this time, a construction based on a single model is not sufficient, and we have to take into account an appropriate abstraction of several models of \( K \cup \mathcal{H}_0 \). We thus proceed as follows:

1. we abstract the KB \( K \cup \mathcal{H}_0 \) into a model abstraction,
2. we reduce the size of this abstraction,
3. based on which we construct a hypothesis \( \mathcal{H} \) of triple exponential size.

In the model abstraction, elements are represented as nodes \( v \in V \) that are labeled with a set \( \lambda(v) \) of types with the intuitive meaning “this element may have one of the types in \( \lambda(v) \)”. Role relations are represented using tuples \((v_1,t,r,v_2)\) which are read as: if the node \( v_1 \) has type \( t \), then it has an \( r \)-successor corresponding to \( v_2 \). Roughly, from a model abstraction, we can obtain a model using the following inductive procedure: 1) start with the nodes that represent individuals, 2) assign to each node a type from its label set, 3) if for those types, the node requires successor nodes, add those
and continue in 2). To allow for unbounded paths in models for finite acyclic model abstractions, we further have “open” nodes whose role successors are only restricted by the TBox.

**Definition 3.** An interpretation abstraction for \( \langle K, \Phi, \Sigma \rangle \) is a tuple \( \mathcal{I} = \langle V, \lambda, s, \mathcal{R}, F \rangle \), where

- \( V \) is a set of nodes,
- \( \lambda : V \to 2^{T \lor \land} \) maps each node to a set of types,
- partial function \( s : N_l \to V \), \( N_l \) assigns individuals to nodes
- \( \mathcal{R} \subseteq (V \times T \lor \land \times (\Sigma \lor \land N_R) \times V) \) is the role assignment,
- and \( F \subseteq V \) is the set of open nodes.

\( \mathcal{I} \) abstracts an interpretation \( \mathcal{I} \) if there is a subset \( \Delta' \subseteq \Delta^2 \) and a function \( h : \Delta' \to V \) s.t. for every \( d \in \Delta' \) and \( e \in \Delta' \) and \( \{d,e\} \in \mathcal{R} \), \( h) \) \( e \) \( ) \) s.t. \( \mathcal{I} \) is cyclic and its length is its number of nodes.

**Definition 4.** \( \mathcal{I} = \langle V, \lambda, s, \mathcal{R}, F \rangle \) is called ALC-conform if

- **D1** there is no cyclic path between outgoing nodes,
- **D2** for every internal node \( v \), if \( \langle v, t, r, v' \rangle \in \mathcal{R} \), then \( \langle v, t', r, v'' \rangle \in \mathcal{R} \) for every \( t' \in \lambda(v') \), and
- **D3** for every \( \langle v_1, t, r, v_2 \rangle \in \mathcal{R} \), where \( v_2 \) is internal, there exists \( \langle v_1, t, r, v' \rangle \in \mathcal{R} \) s.t. \( v_2 \) is outgoing.

We say that \( \mathcal{I} \) is \( \Sigma \)-complete if

- **D4** for every \( v \in V \), and \( t_1 \in \lambda(v) \), \( \lambda(v) \) contains every type \( t_2 \in T_{\lor \land} \) s.t. \( \lambda(t_1) \cap \Sigma = \lambda(t_2) \cap \Sigma \), and
- **D5** for every \( \langle v_1, t, r, v_2 \rangle \in \mathcal{R} \) and \( t' \in T_{\lor \land} \) s.t. \( t \cap \Sigma = t' \cap \Sigma \), also \( \langle v_1, t', r, v_2 \rangle \in \mathcal{R} \).

\( D1 \) ensures that we can represent the outgoing paths from a node \( s(a) \) in a single assertion \( C(a) \). \( D2 \) expresses that the relations between internal nodes is independent of the type, which allows to represent them using role assertions. \( D3 \) is needed to capture allowed paths using universal role restrictions. \( D4 \) and \( D5 \) ensure that \( \mathcal{I} \) can be captured using only names in \( \Sigma \).

From here on, we fix an abduction problem \( \mathcal{A} = \langle K, \Phi, \Sigma \rangle \). We say that \( \mathcal{I} \) explains \( \Phi \) iff some model of \( K \) is abstracted by \( \mathcal{I} \) and every model of \( K \) that is abstracted by \( \mathcal{I} \) entails \( \Phi \). If \( \mathcal{I} \) is ALC-conform, \( \Sigma \)-complete, and explains \( \Phi \), we call it a hypothesis abstraction.

**Lemma 1.** Every hypothesis \( \mathcal{H} \) for \( \mathcal{A} \) can be translated into a hypothesis abstraction.

Now that we can translate hypotheses into an alternative representation, the next step is to decrease their size while making sure they still explain the observation. For this, we can now use similar techniques as in Theorems 2 and 4, where we now identify nodes \( v \in V \) based on the their label \( \lambda(v) \) instead of on a single type.

**Lemma 2.** Let \( \mathcal{F} \) be a hypothesis abstraction. Then, \( \mathcal{F} \) can be transformed into a hypothesis abstraction \( \mathcal{F}' = (V', X', R', F') \) where, for \( t = \{ T_{\lor \land} \} \cdot 2^{T_{\lor \land}} \), \( v \) \( t \) contains at most \( t \) internal nodes, \( v \) \( t \) contains at most \( t \) successors in \( R' \), and \( (v \cap \ell_0, v) \) \( t \) every path of outgoing nodes contains at most \( t \) nodes.

The final ingredient to establish Theorem 6 is the following lemma.

**Lemma 3.** For every ALC-conform, \( \Sigma \)-complete interpretation abstraction \( \mathcal{I} = \langle V, \lambda, s, \mathcal{R}, F \rangle \), there exists an ABox \( \mathcal{H} \) s.t. i) the models of \( \mathcal{K} \cup \mathcal{H} \) are exactly the models of \( K \) accepted by \( \mathcal{I} \), ii) \( |\mathcal{R}(\mathcal{I})| \leq |V| \), and iii) for every \( a \in \mathcal{R}(\mathcal{I}) \), \( \mathcal{H} \) contains a role assertion \( C(a) \), with size \( C \) exponentially bounded by the path lengths between outgoing nodes in \( \mathcal{I} \).

Unfortunately, interpretation abstractions cannot be as easily constructed by a deterministic procedure as we did for Theorems 3 and 4, as there can be non-trivial interactions between connected internal nodes, and we only have a double exponential upper bound on their number. To decide ALC abduction, we can however guess an interpretation abstraction within the bounds of Lemma 2, and then guess assignments of types to nodes to obtain its models. We thus obtain the following theorem.

**Theorem 7.** ALC ABox abduction is in \( N2\text{ExpTime}^{\text{NP}} \).

**5 Size-Restricted Abduction**

Because hypotheses can become very large, a natural requirement is to compute hypotheses of minimal or bounded size. We here obtain the following complexities.

**Theorem 8.** Size restricted \( \mathcal{L} \) ABox abduction is

- \( \text{NP-complete for } \mathcal{L} = \mathcal{L} \),
- \( \text{NExpTime-complete for } \mathcal{L} = \mathcal{L}_{\mathcal{L}} \),
- \( \text{NExpTime}^{\text{NP}} \)-complete for the flat variant and \( \mathcal{L} \in \{ \mathcal{ALC}, \mathcal{ALCQI} \} \), and
- \( \text{in } 2\text{ExpTime} \) for \( \mathcal{L} = \mathcal{ALCQ} \).

The upper bounds are based on guess-and-check algorithms. For \( \mathcal{L} \), we exploit the fact that, by Theorem 2, we can always find a solution of polynomial size. For \( \mathcal{L}_{\mathcal{L}} \), we note that the size of the hypothesis is exponentially bounded by the number of bits used for the size bound \( k \). The \( \text{NExpTime}^{\text{NP}} \)-upper bound can be obtained by a refinement of the procedure used in the proof for Theorem 3. For the double exponential upper bound, we iterate over the double exponentially many possible KBs within the size and signature bounds—indeed, independent on whether we are interested in flat or complex solutions—and then check for entailment in time exponential in the size of the current solution. The lower bounds are provided by the following lemmas.
Lemma 4. Size-restricted $\mathcal{EL}$ abduction is NP-hard.

Proof sketch. We reduce the NP-complete problem CNF-SAT to deciding whether a given signature-based problem has a hypothesis of size at most $k$. Let $\phi = c_1 \land \ldots \land c_n$ be a CNF formula over propositional variables $p_1, \ldots, p_m$. $\mathcal{K}$ contains the following axioms:

- $\text{True} \subseteq P$  $\text{False} \subseteq P$  $\exists r. \text{True} \subseteq C$  $\exists s. \text{False} \subseteq C$
- $r(c_i, p_j)$ for every $i \in [1,n], j \in [1,m]$, if $p_j \in c_j$
- $s(c_i, p_j)$ for every $i \in [1,n], j \in [1,m]$, if $-p_j \in c_j$
- $\Phi$ contains $P(p_i)$ for every $i \in [1,m]$ and $C(c_i)$ for every $i \in [1,n]$.
- Finally, $\Sigma = \{ \text{True}, \text{False} \}$.  $(\mathcal{K}, \Phi, \Sigma)$ has a hypothesis of size at most $2m$ iff $\phi$ is satisfiable.

Lemma 5. Size restricted $\mathcal{EL}_\bot$ abduction is $\text{NExpTime}$-hard.

Proof sketch. The hardness follows from a reduction from the $\text{NExpTime}$-complete exponential tiling problem, which is given by a tuple $(T, T_1, t_e, V, H, n)$ of a set $T$ of tile types, a sequence $T_1 \in T^T$ of initial tiles, a final tile $t_e$, vertical and horizontal tiling conditions $V, H \subseteq T \times T$, and a number $n$ in unary encoding. A solution to this problem is then a tiling, as a function $f : [1, 2^n] \times [1, 2^n] \rightarrow T$ assigning tiles to coordinates, s.t. the first tiles are as in $T_1$, $f(2^n, 2^n) = t_e$, and that obeys the vertical and horizontal tiling conditions [van Emde Boas, 1997].

In the reduction, concept names Start and End respectively mark the initial and the final tile. We implement two binary counters $X$ and $Y$ as for Theorem 1 which are decremental over the roles $x$ and $y$, and encode the coordinates of the tiles. Each tile type $t \in T$ is represented by a concept name $A_t$. We enforce the horizontal tiling conditions using CIs

$$\exists r. A_t \cap A_{t'} \subseteq \bot$$

for each $(t, t') \in (T \times T) \setminus H$ and correspondingly for the vertical conditions. The (hidden) concept name $B \not\in \Sigma$ is used to ensure that the hypothesis contains at least one individual per coordinate. This name is initialised by the individual satisfying Start, and then propagated in $x$ and $y$ direction, provided that a tiling type is associated. The observation to be explained is End$(a)$, where End occurs in the following CI:

$$\prod_{i=1}^{n} X_i \cap \prod_{i=1}^{n} Y_i \cap B \cap A_t \subseteq \text{End}$$

and the abducibles are

$$\Sigma = \{ \text{Start}, x, y, z \} \cup \{ A_t \mid t \in T \}.$$ 

Without the size restriction, a valid hypothesis corresponds to a binary tree with tile types associated to each node, and tiling conditions ensured along the $x$- and $y$-successors. To make sure it forms a $2^n \times 2^n$ grid, we choose the size $k$ appropriately in a way that every coordinate can be used at most once. Valid hypotheses of size $k$ then correspond to solutions to the tiling problem.

To present the proof idea for the $\text{NExpTime}$-hardness result more concisely, we introduce a new tiling problem.

Definition 5. A $\text{NExpTime}^{\text{NP}}$-tiling problem is given by a tuple $(T, T_1, t_e, V_1, H_1, V_2, H_2, n)$, where $(T, T_1, t_e, V_1, H_1, V_2, H_2, n)$ is an exponential tiling problem, $H_2, V_2 \subseteq T \times T$ are additional tiling conditions, and for which we want to decide the existence of a valid tiling $f : [1, 2^n] \times [1, 2^n] \rightarrow T$ for the tiling problem $(T, T_1, t_e, V_1, H_1, n)$, s.t. for no $i \in [1, 2^n]$, there exists a valid tiling for the tiling problem $(T, f(i), t_e, V_2, H_2, n)$, where $f(i)$ denotes the $i$th row of the tiling $f$.

In other words, we have to find a tiling using conditions $H_1$ and $V_1$, while avoiding any rows that can be first row of any tiling for conditions $H_2$ and $V_2$.

Lemma 6. The $\text{NExpTime}^{\text{NP}}$-tiling problem is $\text{NExpTime}^{\text{NP}}$-hard.

Proof sketch. We modify the construction for Lemma 5 to encode the $\text{NExpTime}^{\text{NP}}$-tiling problem. We now use 3 roles $x$, $y$, and $z$ and corresponding binary counters so that, together with the size restriction, each hypothesis will have the shape of a cube. The bottom side of this cube has to correspond to a tiling for $(T, T_1, t_e, V_1, H_1)$, which can be achieved using similar axioms as for Lemma 5. For nodes outside of the bottom side of the cube, we require the use a different set of concept names for the tile types, which are of the form $A_t^*$, and for which we have the axiom $T^* \sqsubseteq \bigcup_{i \in T} A_i^*$. We use

$$\Sigma = \{ \text{Start}, x, y, z, T^* \} \cup \{ A_t^* \mid t \in T \} \cup \{ A_t^* \}$$

and again require every coordinate to be assigned some tile type. For the coordinates outside the bottom side, we have to use the concept name $T^*$ to assign tile types, which leaves the precise selection of the tile type to the different models of the hypothesis. We detect tiling errors in the different $x \times z$-squares with the following axioms

$$\exists r. A_t^* \cap A_{t'}^* \subseteq B_3 \quad \text{for } (t, t') \in (T \times T) \setminus H_2$$

and correspondingly for $V_2$. This information is propagated along the succeeding coordinates so that the observation End$(a)$ is only entailed if every model of the hypothesis encodes a tiling error on each of the $x \times z$ squares.

6 Outlook

We believe that our results for complex abduction in $\mathcal{ALC}$ can be extended to $\mathcal{ALCI}$, and that the bound for size restricted $\mathcal{ALCQI}$ abduction is tight. A question is whether we can improve the $\text{N2ExpTime}^{\text{NP}}$-bound for the most general variant of our abduction problem. Apart from that, we want to investigate our setting for observations formulated as conjunctive queries, which would allow us to explain negative query answers [Calvanese et al., 2013]. Another interesting question is what happens if we allow fresh individual names for abduction with ontologies formulated using existential rules. For the $\mathcal{EL}_\bot$-variant, we are currently working on a practical method for computing size-minimal flat hypotheses.

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Proof Details

A Flat ABox Hypotheses

Theorem 2. If there exists a hypothesis for a flat L abduction problem A, then there exists one of size

1. polynomial in the size of A if L = E L,
2. exponential in the size of A if L = ALCI, and
3. if L = ALC F, no general upper bound based on A can be given.

Proof. Let H 0 be a hypothesis for (K, Φ, Σ). We first complete the argument in the main text for L = ALCI, and then for L = E L.

For L = ALCI, we construct H and h as in the main text based on an arbitrary hypothesis H 0 and an arbitrary model I of K ∪ H 0. Note that there can be at most exponentially many distinct types occurring in I (one for each subset of sub(K ∪ Φ)). Consequently, H contains at most exponentially many individual names, and, because it is flat, at most exponentially many assertions. It remains to show that H satisfies A1 and A2.

For A2, we need to show that K ∪ H ⊨ ⊥. For this, we construct the interpretation J based on the types occurring in I. The domain of J contains an element for every individual in K ∪ Φ and additionally types occurring in I. We first define a superset of the domain:

\[ \Delta = \text{ind}(K ∪ Φ) \cup \{tp(d)| d ∈ Δ^I}\]

We define a function \( g : Δ^I → Δ \) by setting:

\[ g(d) = \begin{cases} a & \text{if } d = a^x \text{ and } a ∈ \text{ind}(K ∪ Φ) \\ tp(d) & \text{otherwise} \end{cases} \]

The interpretation J is now defined as follows for all a ∈ N I, A ∈ N C and r ∈ N R:

- \( Δ^J = \{g(d)| d ∈ Δ^I\}\)
- \( a^J = \begin{cases} a & \text{if } a ∈ \text{ind}(K ∪ Φ) \\ t & \text{if } a ∈ \text{ind}(H) \setminus \text{ind}(K ∪ Φ) \\ \text{arbitrary otherwise} \end{cases} \)
- \( A^J = \{g(d)| d ∈ A^I\}, \)
- \( r^J = \{(g(d), g(e)) | (d, e) ∈ r^I\} \)

It can be shown by structural induction that for every concept C ∈ sub(K ∪ Φ) and d ∈ Δ^I, d ∈ C^J iff g(d) ∈ C^J, which implies that J is a model of K. Note however that this would not work for L = ALC F, since the transformation can change the number of role-successors.

We can also show that J = H. First observe that for every a ∈ ind(K ∪ H 0) and b ∈ ind(K ∪ H), h(a) = b iff g(a) = b^J, if a ∈ ind(K ∪ Φ), we have by definition h(a) = a, g(a^x) = a and a = a^J. Otherwise, we have h(a) = btp(a^x) and g(a^x) = tp(a^x) and btp(a^x) = tp(a^x). In both cases, we have h(a) = b iff g(a^x) = b^J.

Assume A(b) ∈ H. Then b = h(a) for some a ∈ ind(K ∪ H 0) and A(a) ∈ H 0. We have g(a^x) = b^J by the above observation, and we have already pointed out that this implies for every C ∈ sub(K ∪ Φ) that a^x ∈ C^J iff b^x ∈ C^J. Since I = H 0, we have a^x ∈ A^I, and thus b^J ∈ A^J, so that J = A(b).

We have shown J = H, and thus that K ∪ H is satisfiable, that is, that H satisfies A1.

It remains to show A2, that is, K ∪ H ⊨ Φ. Assume K ∪ H ⊨ Φ, that is, there exists some assertion α ∈ Φ s.t. K ∪ H ⊨ α. If α is of the form r(a, b), then this axiom must explicitly occur in either K or H 0, since ALCI is not expressive enough to enforce such entailments in a different way). If r(a, b) ∈ K, trivially K ∪ H |= r(a, b). If r(a, b) ∈ H 0, by construction also r(a, b) ∈ H.

Assume α is of the form C(a), and there exists a model J of K ∪ H s.t. J = C(a). Using the homomorphism h : ind(H 0) → ind(H), we can transform J into a model J′ of K ∪ H 0 s.t. J′ = C(a). Specifically, we define J′ by

- \( J^J = J^I \cup \{a^J | a ∈ \text{ind}(H)\}\)
- \( A^J = A^I \cup \{a^J | a ∈ \text{ind}(H), h(a)^J = A\} \) for all A ∈ N C, and
- \( r^J = r^I \cup \{a^J, b^J | a, b ∈ \text{ind}(H), \langle h(a)^J, h(b)^J \rangle ∈ r^I\} \) for all r ∈ N R.

J′ can be seen as the result of duplicating some elements from J based on h, while keeping their concept names and role successors. Thus, it is easy to see that the transformation is type-preserving, that is, for every a ∈ ind(H) and b ∈ ind(H 0) s.t. h(b) = a, we have tp(a^J) = tp(b^J). As a consequence, J′ = K and J′ = C(a). Furthermore, by construction, J′ = H 0. We obtain that K ∪ H ⊨ Φ, and thus contradicting that H 0 was a hypothesis for Φ to begin with. Consequently, there cannot be such a model J of K ∪ H, and we must have K ∪ H ⊨ Φ.

We have shown that H satisfies Conditions A1–A2, and thus that it is an exponentially sized hypothesis for (K, Φ, Σ).

Now assume L = E L. We then construct a polynomially sized hypothesis by

\[ H = \{A(a) | A ∈ Σ ∩ N C, a ∈ \text{ind}(K ∪ Φ)\} \]

∪ \{r(a, b) | r ∈ Σ ∩ N R, a, b ∈ \text{ind}(K ∪ Φ)\}

Again we can define a homomorphism h from H 0 onto H: this time, we map every individual name not in ind(K ∪ Φ) to some randomly selected individual name. We have sig(H) ⊆ Σ by construction, and K ∪ H ⊨ Φ since every E L KB is consistent. K ∪ H ⊨ Φ can be shown as for L = ALCI using the homomorphism h. It follows that H is a polynomially sized hypothesis for (K, Φ, Σ).

Finally, for L = ALC F, the existence of any upper bound would allow us to decide existence of abductive solutions by a simple guess-and-check algorithm. In particular, this would...
allow to decide the IQ-query emptiness problem which asks, given a TBox $\mathcal{T}$, a signature $\Sigma$ and a concept $C$, whether there exist any flat ABoxes in $\Sigma$ that entail $C(a)$ for some individual name $a \in N_0$. As shown in [Baader et al., 2016], this problem is undecidable for $\mathcal{ALCF}$ TBoxes, however, it can be easily reduced to flat $\mathcal{ALCF}$ abduction.

**Theorem 3.** Flat $\mathcal{L}$ ABox abduction is

- P-complete for $\mathcal{L} = \mathcal{E}\mathcal{L}$,
- $\text{ExpTime}$-complete for $\mathcal{L} = \mathcal{E}\mathcal{L}_\perp$,
- co$\text{ExpTime}$-complete for $\mathcal{L} = \mathcal{ALCI}$,
- undecidable for $\mathcal{L} = \mathcal{ALCF}$.

**Proof.** We only need to consider the upper bounds for the cases $\mathcal{L} = \mathcal{E}\mathcal{L}_\perp$ and $\mathcal{L} = \mathcal{ALCI}$.

We first describe how to compute a maximal set $T_{K\cup \Phi}$ of types for $K \cup \Phi$ using type elimination. This construction is fairly standard and only included here for self-containment. For a role $R$, we set $\text{Inv}(r) = r^-*$ and $\text{Inv}(r^-) = r$.

We start with the set $T_0 = \{t \mid t \subseteq \text{sub}(K \cup \Phi)\}$ from which we construct a sequence $T_0, T_1, \ldots, T_n$ of such sets. For a given such set $T_i$, a type $t \subseteq \text{sub}(K \cup \Phi)$ and a role $R$, we denote by $S^R_i(t)$ the set of all types $t' \in T_i$ s.t. for every $\exists R.C \in \text{sub}(K \cup \Phi \setminus t)$, $C \neq t'$, and for every $\exists \text{Inv}(R).C \in \text{sub}(K \cup \Phi \setminus t')$, $C \neq t$. $S^R_i(t)$ contains the $R$-successor candidates of $t$ within $T_i$, that is, types from $T_i$ that an $R$-successor of an instance of $t$ could have. For each $i \in [1, n]$, $T_i$ is obtained from $T_{i-1}$ by removing an element $t$ that fails to satisfy any of the following conditions:

1. $\bot \notin t$,
2. for every $C \cap D \in \text{sub}(K \cup \Phi)$, $C \cap D \in t$ iff $\{C, D\} \subseteq t$,
3. for every $\neg C \in \text{sub}(K \cup \Phi)$, $C \in t$ iff $\neg C \notin t$,
4. if $C \in t$ and $C \subseteq D \in K$, then $D \in t$, and
5. if $\exists R.C \in t$, then there exists $t' \in S^R_{i-1}(t)$ s.t. $C \subseteq t'$.

We set $T_{K\cup \Phi} = T_n$. Since $T_0$ contains exponentially many elements, $T_{K\cup \Phi} < T_{K\cup \Phi}$ is obtained after at most exponentially many steps, each taking polynomial time.

We define a selector function to be any function $s : \text{ind}(K \cup \Phi) \rightarrow T_{K\cup \Phi}$. We note that there are at most exponentially many such selector functions, specifically,

\[
|T|^{\text{ind}(K \cup \Phi)} \leq \left(2|\text{sub}(K \cup \Phi)|\right)^{\text{ind}(K \cup \Phi)} \leq 2^{\text{size}(K \cup \Phi)^2}
\]

We call such a selector function compatible with $K$ if for every $A(a) \in K$, $A \in \text{sub}(a)$, and for every $r(a, b) \in K$, $s(b) \in \text{sub}(A(a))$.

For every selector function $s$ compatible with $K$, we build a hypothesis candidate $H_s$ as follows. To every type $t \in T_{K\cup \Phi}$, we assign an fresh individual name $b_t$. The set

\[
I = \text{ind}(K \cup \Phi) \cup \{b_t \mid t \in T_{K\cup \Phi}\}
\]

contains the individual names in our hypothesis, to which we extend $s$ by setting $s(b_t) = t$. $H_s$ is then defined as

\[
H_s = \{A(a) \mid a \in I, A \in \text{sub}(a) \cap \Sigma\}
\]

\[
\cup \{r(a, b) \mid a, b \in I, r \in \Sigma, s(b) \in S_{T_{K\cup \Phi}}(s(a))\}
\]

**Claim 1.** If $s$ is a selector function compatible with $K$, then $K \cup H_s \not\models \bot$.

**Proof of claim.** We construct a model $I$ for $K \cup H_s$ based on $T_{K\cup \Phi}$ and $s$:

1. $\Delta^I = I$,
2. for all $a \in I$: $a^I = a$,
3. $A^I = \{a \in I \mid A \in \text{sub}(a) \cap N_C\}$,
4. $r^I = \{(a, b) \in I \times I \mid s(b) \in S_{T_{K\cup \Phi}}(s(a))\}$

It is now standard to show by induction on the structure of concepts that for every $a \in N_0$ and $C \in \text{sub}(a)$, $a \in C^I$. From the construction of the type set $T_{K\cup \Phi}$, it follows that $I$ satisfies all GCIs in $K$. Since $s$ is compatible with $K$, it follows that $I$ also satisfies all assertions in $K$. Finally, by construction $I \models H_s$.

**Claim 2.** If there exists a flat hypothesis for $(K, \Phi, \Sigma)$, then there exists a selector function $s$ s.t. $H_s$ is such a hypothesis.

**Proof of claim.** Assume there exists a flat hypothesis $H_0$ for $(K, \Phi, \Sigma)$. Since for every selector function $s$ compatible to $K$, $H_s$ satisfies $A1$ by Claim 1 and $A3$ by construction, we only need to show that there exist some such $s$ s.t. $H_s$ satisfies $A2$. We use the construction from the proof for Theorem 2 to obtain, based on $H_0$, an assignment of types $t(p)$ to every $a \in \text{ind}(K \cup \Phi)$, as well as the exponentially bounded hypothesis $H$. Define $s$ s.t. $s(a) = t(p(a^2))$ for every $a \in \text{ind}(K \cup \Phi)$. $s$ is compatible with $K$, and $H \subseteq H_s$. Consequently, $K \cup H_s \models \Phi$ and $H_s$ satisfies $A1$–$A2$.

Claim 2 allows to obtain an ExpTime-decision procedure for the case where $\mathcal{L} = \mathcal{E}\mathcal{L}_\perp$: specifically, we first compute $T_{K\cup \Phi}$ in exponential time, and then iterate over the exponentially many possible selector functions and check whether $H_s$ satisfies $A1$ and $A2$. Since entailment checking in $\mathcal{E}\mathcal{L}_\perp$ can be decided in polynomial time and each $H_s$ is exponentially large, the entire procedure runs in deterministic exponential time.

For $\mathcal{L} = \mathcal{ALCI}$, entailment checking is ExpTime-hard, so that this procedure would only yield in 2ExpTime-upper bound. To obtain a coExpTime upper bound, we devise a non-deterministic procedure to determine whether there is no flat hypothesis for $(K, \Phi, \Sigma)$. For this, by Claim 1 and 2, we need to show that for every selector function $s$, there exists some model $I$ of $K \cup H_s$ s.t. $I \not\models \Phi$. For this, we first check whether for some $r(a, b) \in \Phi$, $r(a, b) \not\in H_s \cup K$. If this test fails, we guess a function $s_e : \text{ind}(H_s) \rightarrow T_{K\cup \Phi}$ and check whether

1. $s_e$ is compatible with $K \cup H_s$ in the sense above, and
2. for some $C(a) \in \Phi$, $C \not\models s_e(a)$.

If for every selector $s$ function, one of these checks is successful, we accept, and otherwise we reject. Since this non-deterministic procedure requires exponentially many steps units, we obtain the required coExpTime upper bound, provided that the algorithm is correct.
Claim 3. \( \mathcal{K} \cup \mathcal{H}_s \models \Phi \text{ iff for every } (r, a, b) \in \Phi, (r, a, b) \in \mathcal{H}_s \text{ and no function } s_c : \text{ind}(\mathcal{H}_s) \to T_{\mathcal{K} \cup \Phi} \text{ satisfies Conditions } s1 \text{ and } s2).$

Proof of claim. Assume \( \mathcal{K} \cup \mathcal{H}_s \models \Phi \) and for every \( (r, a, b) \in \Phi, (r, a, b) \in \mathcal{H}_s \). Then there exists a model \( \mathcal{I} \) of \( \mathcal{K} \cup \mathcal{H}_s \) s.t. \( \mathcal{I} \not\models \Phi \). Define \( s_c \) by setting \( s_c(a) = \text{tp}(a^T)_{\mathcal{I}} \), and we obtain from the fact that \( \mathcal{I} \) is a model of \( \mathcal{K} \cup \mathcal{H}_s \) that \( s_c \) satisfies Condition s1, and from the fact that \( \mathcal{I} \not\models \Phi \), that there is some \( C(a) \in \Phi \) s.t. \( a^T \not\models C^T \), which implies \( C \not\models s_c(a) \) and thus Condition s2.

For the other direction, first observe that if \( (r, a, b) \not\in \mathcal{H}_s \cup \mathcal{K} \) for some \( (r, a, b) \in \Phi \), we can always find a model \( \mathcal{I} \) of \( \mathcal{K} \cup \mathcal{H}_s \) s.t. \( \mathcal{I} \not\models \Phi \). Assume for every \( (r, a, b) \in \Phi \), we have also \( (r, a, b) \in \mathcal{H}_s \cup \mathcal{K} \), and there exists a function \( s_c : \text{ind}(\mathcal{H}_s) \to T_{\mathcal{K} \cup \Phi} \) that satisfies both Condition s1 and s2. We define

\[
s_c^* : \text{ind}(\mathcal{H}_s) \cup T_{\mathcal{K} \cup \Phi} \to T_{\mathcal{K} \cup \Phi}
\]

by setting \( s_c^*(a) = s_a(a) \) if \( a \in \text{ind}(\mathcal{H}_s) \) and \( s_c^*(t) = t \) if \( t \in T_{\mathcal{K} \cup \Phi} \). We define an interpretation \( \mathcal{I} \) as follows:

- \( \Delta^T = \{ a^T \mid a \in \text{ind}(\mathcal{H}_s) \} \cup T_{\mathcal{K} \cup \Phi} \)
- for every \( A \in \mathcal{N}_e \), \( A^T = \{ d \in \Delta^T \mid A \in s_c^*(d) \} \)
- for every \( r \in \mathcal{N}_r \), \( r^T = \{ (d, e) \in \Delta^T \times \Delta^T \mid s_c^*(e) \in S^T_{\mathcal{K} \cup \Phi}(s_c^*(d)) \} \)

One easily verifies that for every \( d \in \Delta^T \), \( \text{tp}(d)_{\mathcal{I}} = s_c^*(t) \). This implies that \( \mathcal{I} \) satisfies every GCI in \( \mathcal{K} \), and by Condition s2 that \( \mathcal{I} \models \Phi \) for some \( (r, a, b) \in \Phi \). Using Condition s1, one also verifies that \( \mathcal{I} \) is a model of every assertion in \( \mathcal{K} \cup \mathcal{H}_s \).

We obtain that our non-deterministic procedure is correct, and correspondingly that for \( \mathcal{L} = A\mathcal{CIT} \), existence of flat hypotheses is in \text{CONEXP}TIME. 

\[\Box\]

\section*{B Complex ABox Hypotheses for \( \mathcal{E}\mathcal{L}_\perp \)}

We show Theorem 4 using three separate lemma for the lower bound of the size, the upper bound of the size, as well as for the decision procedure.

Lemma 8. \textit{There exists a family of \( \mathcal{E}\mathcal{L}_\perp \) TBoxes (\( \mathcal{T}_n \)) \( \forall n \geq 1 \) s.t. each \( \mathcal{T}_n \) is of size polynomial in \( n \) and every hypothesis for \( \langle \mathcal{T}_n, \mathcal{A}(a), \{ B, r, s \} \rangle \) that uses only \( a \) as individual name is of double exponential size.}

Proof. \( \mathcal{T}_n \) can be seen in Figure 1 and works as described in the main text. \[\Box\]

Lemma 9. \textit{Let \( \langle \mathcal{K}, \Phi, \Sigma \rangle \) be a signature based abduction problem s.t. \( \mathcal{K} \) and \( \Phi \) are formulated in \( \mathcal{E}\mathcal{L}_\perp \). Then, there exists a solution to the abduction problem without fresh individual names iff there exists such a solution of double exponentially bounded size.}

Proof. Let \( \mathcal{H}_0 \) be a hypothesis for \( \langle \mathcal{K}, \Phi, \Sigma \rangle \). We first create a flattened version of \( \mathcal{H}_0 \), where we use a mapping \( \text{root} : \mathcal{N}_s \to \mathcal{N}_t \) to keep track of a “root individual” for each individual added. Initially, we set \( \text{root}(a) := a \) for each \( a \in \text{ind}(\mathcal{H}_0) \). We then exhaustively apply the following rules:

1. \( ( \mathcal{C} \cap \mathcal{D} )(a) \implies \mathcal{C}(a), \mathcal{D}(a) \)

2. \( (\exists r.C)(a_1) \implies r(a_1, a_2), C(a_2) \), where \( a_2 = b_{\text{root}(a_1),C} \) is fresh

Denote the resulting ABox by \( \mathcal{H}_1 \) and note that \( \mathcal{H}_1 \) is a flat hypothesis for \( \langle \mathcal{K}, \Phi, \Sigma \rangle \). Moreover, \( \mathcal{H}_1 \) is forest-like in the sense that the fresh role-descendants of any fixed individual name \( a \) from \( \mathcal{H}_0 \) form a tree. Specifically, for every fresh individual name \( b_{a,C} \) there exists exactly one role assertion of the form \( r(a', b_{a,C}) \).

Similar to the proof for Theorem 2, we take a random model \( \mathcal{I} \) of \( \mathcal{K} \cup \mathcal{H}_1 \), which allows us to assign a type \( \text{tp}(a^T) \) to every individual in \( \mathcal{H}_1 \).

We use this type to reduce the path length from any individual from \( \mathcal{H}_0 \) to an individual in \( \mathcal{H}_1 \) by removing repeated types on each path. To keep track of our changes, we use a mapping \( h \) which is initialised with \( h(a) = a \) for every \( a \in \text{ind}(\mathcal{H}_1) \) and updated during the construction. We then repeat the following steps exhaustively:

1. If for two individual names \( a, b \), \( \text{root}(a) = \text{root}(b) \), \( \text{tp}(a^T) = \text{tp}(b^T) \) and there is a non-empty path from \( a \) to \( b \), then replace the role assertion of the form \( r(a', a) \) by \( r(a', b) \), and remove all role assertions involving \( a \).

2. If for any two role assertions of the form \( r(a, b), r(a, c) \), we have \( \text{root}(b) = \text{root}(c) \) and \( \text{tp}(b^T) = \text{tp}(c^T) \), then remove all role assertions involving \( c \).

Denote the resulting ABox by \( \mathcal{H}_2 \). Different to the construction for Theorem 2, \( h \) is not a homomorphism. We note that the operation preserves the forest shape, and moreover makes sure that every path from an individual name \( a \in \text{ind}(\mathcal{H}_0) \) to an individual name \( b_{a,C} \) is exponentially bounded, and for every individual name \( a \), there are at most exponentially many role assertions of the form \( r(a, b) \). As a result, for every individual name \( a \in \text{ind}(\mathcal{H}_0) \), there are \( 2^n \) individual names \( b \) with \( \text{root}(a) = b \). We can thus build a double exponentially bounded hypothesis candidate by “rolling up” the ABox again, or exhaustively applying the following operation:

- If \( a \) has no role successor and \( \text{root}(a) \neq a \), replace the role assertion \( r(b, a) \) by \( \exists r \{ (\langle a^T, b \rangle) | \langle a^T, b^T \rangle \in r^T \} \cup \{ (d, e) | e \neq a^T \text{ for all } a \in \text{ind}(\mathcal{H}_0) \} \).
It can be shown by induction on the structure of concepts that for every concept \( C \in \text{sub}(\mathcal{K} \cup \Phi) \) and \( d \in \Delta^T, d \in C^T \) iff \( d \in C^J \), which implies \( \text{tp}(d)_T = \text{tp}(e)_T \) and that \( J \) is a model of \( \mathcal{K} \cup \mathcal{H}_2 \).

For Condition A2, assume there is a model \( I_2 \) of \( \mathcal{K} \cup \mathcal{H}_2 \) s.t. \( I_2 \not\models \Phi \). Based on \( I_2 \), we construct a model \( I_1 \) of \( \mathcal{K} \cup \mathcal{H}_1 \). Define \( m : \Delta^T \cup \text{ind}(\mathcal{K} \cup \mathcal{H}) \rightarrow \Delta^{I_1} \) by

\[
m(d) = \begin{cases} 
  d & \text{if } d \in \Delta^{I_2} \\
  h(d)^{I_2} & \text{if } d \in \text{ind}(\mathcal{K} \cup \mathcal{H}).
\end{cases}
\]

Using \( m \), we define \( I_1 \) as follows

- \( \Delta^{I_1} = \Delta^{I_2} \cup \text{ind}(\mathcal{K} \cup \mathcal{H}) \),
- \( a^{I_1} = a \) for all \( a \in \text{ind}(\mathcal{K} \cup \mathcal{H}) \),
- \( A^{I_1} = \{ d \in \Delta^{I_1} \mid m(d) \in A^{I_2} \} \) for all \( A \in \mathcal{N}_C \), and
- \( r^{I_1} = \{ (d, e) \mid (m(d), m(e)) \in r^{I_2} \} \) for all \( r \in \mathcal{N}_R \).

Again it is straightforward to prove that the construction is type preserving, a model of \( \mathcal{K} \cup \mathcal{H}_1 \), and \( I_1 \not\models \Phi \) for the same reason as for \( I_2 \). It follows that there cannot be a model \( I_2 \) of \( \mathcal{K} \cup \mathcal{H}_2 \) s.t. \( I_2 \not\models \Phi \), and thus that \( \mathcal{K} \cup \mathcal{H}_2 \) satisfies Condition A2.

\[ \square \]

**Lemma 10.** It can be decided in exponential time whether a given \( \mathcal{EL}_\bot \) abduction problem has a solution without fresh individuals.

**Proof.** We use a similar procedure as in Theorem 3 using the in exponential time computed set \( T_{\mathcal{K} \cup \Phi} \) of types for \( \mathcal{K} \cup \Phi \) and the guessed selector function \( s : \text{ind}(\mathcal{K} \cup \Phi) \rightarrow T_{\mathcal{K} \cup \Phi} \).

However, this time, the hypothesis is constructed differently by making sure that we do not introduce cycles between introduced individuals, so that the flat ABox can always be "rolled up" into a complex ABox without fresh individual names. To be able to do this, our introduced individuals are of the form \( b_{a,t,i} \), where \( a \in \text{ind}(\mathcal{K} \cup \Phi), t \in T_{\mathcal{K} \cup \Phi}, \) and \( i \in [1, 2^{|T_{\mathcal{K} \cup \Phi}|}] \). Note that still, there are only exponentially many individuals of this form. For convenience, we set \( b_{a,s(a),0} = a \).

The ABox \( \mathcal{H}_s \) is then defined as follows, where \( T = T_{\mathcal{K} \cup \Phi} \):

\[
\mathcal{H}_s = \{ A(b_{a,t,k}) \mid a \in \text{ind}(\mathcal{K} \cup \Phi), t \in T, k \in \left[0, 2^{|T|}\right] \},
\]

\[
A \in (t \cap \Sigma)
\]

\[
\cup \{ r(b_{a,t_1,k}, b_{a,t_2,k+1}) \mid a \in \text{ind}(\mathcal{K} \cup \Phi), t_1 \in T,
\]

\[
r \in \Sigma, t_2 \in S^+_T(t_1), k \in \left[0, 2^{|T|} - 1\right]
\]

\[
\cup \{ (a, b) \mid s(b) \in S^+_T(s(a)) \}.
\]

Since \( \mathcal{H}_s \) is of exponential size, we can verify \( \mathcal{K} \cup \mathcal{H} \models \Phi \) in exponential time, which means that, since we have to consider exponentially many selector functions, the entire procedure runs in exponential time. One can show similarly as for Theorem 3 that for every selector function \( s \) that is compatible with \( \mathcal{K}, \mathcal{H}_s \) satisfies A1 and A3. It remains to show that for every hypothesis \( \mathcal{H}_0 \) without fresh individuals of the current abduction problem, there is a corresponding compatible selector function \( s \) s.t. \( \mathcal{H}_s \) satisfies also A2, that is, \( \mathcal{K} \cup \mathcal{H}_s \models \Phi \). For this, we first transform \( \mathcal{H}_0 \) as in Lemma 9 into a flat, forest-like hypothesis \( \mathcal{H}_2 \). Recall that this construction assigns every individual \( a \) a type \( \text{tp}(a^2)_T \) based on an arbitrary interpretation \( T \), and is such that, on every path from a named individual to a fresh individual, no type is repeated. Choose s.t. \( s(a) = \text{tp}(a^2)_T \). Using that \( s \) is constructed from a model of \( \mathcal{K} \), one can show that \( s \) is compatible with \( \mathcal{K} \). We show that also \( \mathcal{K} \cup \mathcal{H}_2 \models \Phi \) iff \( \mathcal{K} \cup \mathcal{H}_s \models \Phi \). For this, we define a mapping \( m : \text{ind}(\mathcal{H}_2) \rightarrow \text{ind}(\mathcal{H}_s) \) inductively as follows:

- for every \( a \in \text{ind}(\mathcal{K} \cup \Phi) \), \( m(a) = a = b_{a,s(a),0} \).
- for every \( r(a, b) \in \mathcal{H}_2 \) s.t. \( b \not\in \text{ind}(\mathcal{K} \cup \Phi) \) and \( m(a) = b_{a,t,i} \), set \( m(b) = b_{a,t',i+1} \), where \( t' = \text{tp}(b^2)_T \).

The induction terminates because \( \mathcal{H}_2 \) has the "forest-shape", that is, for every individual name \( a \in \text{ind}(\mathcal{K} \cup \Phi) \), the fresh ancestors of \( a \) form a tree. By checking the definition of \( \mathcal{H}_s \), one can see that for every \( A(a) \in \mathcal{H}_2 \), we have \( A(m(a)) \in \mathcal{H}_s \), and for every \( r(a, b) \in \mathcal{H}_2 \), we have \( r(m(a), m(b)) \in \mathcal{H}_2 \). \( m \) is thus a homomorphism from \( \mathcal{H}_2 \) into \( \mathcal{H}_s \), which can be used as in the proof for Theorem 2 to show that \( \mathcal{H}_2 \models \Phi \) implies that also \( \mathcal{H}_s \models \Phi \). \( \square \)
\[
\begin{align*}
\text{T} & \equiv \exists r.T \sqcap \exists s.T & \quad (8) \\
\forall (r \cup s). (B \sqcup B') & \sqsubseteq B' & \quad (9) \\
B & \equiv \text{Init} \sqcap \neg \text{Bit} & \quad (10) \\
\forall (r \cup s). (\text{Init} \sqcap \bigcap_{i=1}^{n} Y_i) & \sqsubseteq \text{Init} \sqcap \neg \text{Bit} & \quad (11) \\
\bigcap_{i=1}^{n} X_i & \equiv \text{Flip} & \quad (12) \\
\forall (r \cup s). (\text{Flip} \sqcap B) & \sqsubseteq \text{Flip} & \quad (13) \\
\bigcup_{i=1}^{n} X_i \cap \exists (r \cup s). (\neg \text{Flip} \sqcup \neg \text{Bit}) & \sqsubseteq \neg \text{Flip} & \quad (14) \\
\bigcap_{i=0}^{n} \text{Flip} \cap B' & \sqsubseteq \text{Error} \sqcup \text{Goal} & \quad (15) \\
\text{Error} & \equiv \neg \text{Init} \sqcap \exists (r \cup s). (\text{Error} \sqcup E_0) & \quad (16) \\
E_0 & \equiv \bigcap_{i=1}^{n} Y_i \cap E \cap (\text{NBit} \leftrightarrow \text{Bit}) & \quad (17) \\
E \sqcap \text{NBit} & \equiv \exists (r \cup s). (E \sqcap \text{NBit}) & \quad (18) \\
E \neg \text{NBit} & \equiv \exists (r \cup s). (E \neg \text{NBit}) & \quad (19) \\
\bigcup_{i=1}^{n} Y_i \cap E & \equiv \exists r. E_f \sqcup \exists s. E_f & \quad (20) \\
E_f \cap \text{Flip} & \sqsubseteq (\text{Bit} \leftrightarrow \text{NBit}) & \quad (21) \\
E_f \neg \text{Flip} & \sqsubseteq (\text{Bit} \leftrightarrow \neg \text{NBit}) & \quad (22)
\end{align*}
\]

Figure 2: Part of the background knowledge base requiring a double exponentially large hypothesis.

C Complex ABox Abduction for ALC

Theorem 5. There is a family of ALC abduction problems for which the smallest (non-flat) ABox hypotheses are triple exponential in size.

Proof. The idea is, similar as for the bound in Theorem 4, to enforce the hypothesis to represent a binary tree with the concept name \( B \) in its leafs. However, this time, instead of using an \( n \)-bit counter as in Theorem 1, we use a trick from [Ghilardi et al., 2006] to implement a \( 2^n \)-bit counter. Here, a single counter value is encoded by a chain of \( 2^n \) elements satisfying either \( \text{Bit} \) or \( \neg \text{Bit} \), depending on whether the corresponding bit has the value 1 or 0. The counting then produces a chain of \( 2^{2^n} \cdot 2^n \) elements, where each consecutive \( 2^n \) elements represent the next number in the sequence.

The hypothesis for our problem will be of the form \( C(a) \), where \( C \) only uses universal restriction, conjunction, and negation on concept names, to express that every path of \( r \)- and \( s \)-successors must lead to an instance of \( B \), and reading each path backwards from the \( B \)-instance to \( a \), it must encode contain a sequence of \( 2^{2^n} \) increasing counter values. The signature is \( \Sigma = \{ r, s, \text{Bit}, B \} \) and the observation is \( \text{Goal}(a) \). To disallow solutions such as \( (\forall r. \bot \sqcap \forall s. \bot)(a) \), we add \( \text{T} \equiv \exists r. \text{T} \sqcap \exists s. \text{T} \) to \( \mathcal{K} \). A concept \( B' \) is used to mark elements from which every path of \( s \)- and \( r \)-successors leads to an instance of \( B \). The concept name \( \text{Bit} \) is used for our counter. To identify bit positions in the current counter value, we use an \( n \)-bit-counter as for Theorem 1 which is initialised at instances of \( B \).

The construction is similar to that given in [Ghilardi et al., 2006] to give a triple exponential lower bound for witness concepts of conservative extensions in ALC, but slightly adapted to be applicable to the abduction problem, as well as improved in its representation.

The background knowledge base \( \mathcal{K} \) contains the axioms in Figure 2, together with Axioms (1)–(6) from Figure 1 as they are, as well as Axioms (2)–(6) from Figure 1 with \( X_i / Y_i \), replaced by \( Y_i / Y_i \). In Figure 2, we use the construct \( \forall (r \cup s). C \) as an abbreviation for \( \forall r. C \sqcap \forall s. C \), \( \exists (r \cup s). C \) as abbreviation for \( \exists r. C \sqcup \exists s. C \), and \( C \leftrightarrow D \) as abbreviation for \( (C \sqcap D) \sqcup (\neg C \sqcap \neg D) \).

The set of abducibles is \( \Sigma = \{ s, r, \text{Bit}, B \} \), and our observation is \( \Phi = \{ \text{Goal}(a) \} \). We explain the axioms in \( \mathcal{K} \) one after the other.

- The axioms taken from Figure 1 implement binary counters—one using concept names \( X_0, \ldots, X_n \), to represent positive bit values, and one using concept names \( Y_0, \ldots, Y_n \) to represent positive bit values. We call those two single exponential counters \( X \)-counter and \( Y \)-counter respectively. The \( X \)-counter is initialised at instances of \( B \), while we do not state yet where the \( Y \)-counter is initialised. The \( X \)-counter is used to keep track of the current bit position in the double exponential counter.
- (8) ensures that no hypothesis can limit the length of \( r \) or \( s \)-paths in an interpretation, and is needed to avoid the use of concepts \( \forall r. \bot \) and \( \forall s. \bot \) in hypotheses.
(9) uses the concept $B'$ to mark elements from which every path along r- and s-edges leads to an instance of $B$.

(10)–(11) make sure that any element from which there are only paths of less than $2^n$ elements to an instance of $B$ satisfies $\text{Bit}$ and $\neg \text{Bit}$. Those first paths of length $2^n$ represent the initial value of 0 of the double exponential counter.

(12)–(14) use the concept name $\text{Flip}$ to mark elements which correspond to a bit that should flip for the next double exponential counter value.

(15) now gives the conditions for the observation to be entailed. With the previous axioms, it should be clear that $\prod_{i=0}^{n} X_i \cap \text{Fl}^i$ is satisfied by instances from which every element on every path of length $2^n$ satisfies $\text{Bit}$, or, if we are on the last bit of the double exponential counter with the current value of $2^2^n - 1$. Axiom (15) states now: if we are in such a situation, and we additionally satisfy $B''$, which is, every path of r- and s-successors leads eventually to an instance of $B$, then we satisfy either $\text{Error}$ or $\text{Goal}$. Consequently, the hypothesis has to make sure that $\text{Error}$ is not satisfied in order to entail the observation. Intuitively, $\text{Error}$ can only be satisfied if the double exponential counter made an error on one of the paths leading to a $B$-instance. Consequently, to avoid $\text{Error}$ from being satisfied, all paths to the next $B$-instance have to use the double exponential counter correctly, that is, encode all numbers from 0 to $2^2^n - 1$ in decreasing order.

(16)–(22) now implement this error-detection mechanism via the concept $\text{Error}$. Specifically, if we have an instance of $\text{Error}$, then there must be a path to some instance of $E_0$ before we reached the initial value of the double exponential counter (marked using the concept name $\text{Init}$). $E_0$ marks the first bit where we observe the error: it initialises the Y-counter (which uses the concept names $Y_i$), and stores the current bit value using in the concept name $\text{NBit}$. The value of $\text{NBit}$ is now transported, together with the concept name $E$, along some path of r- and s-successors of length $2^n$ until the Y-counter reaches its maximum value. We have then reached the corresponding bit position in the previous value of the double exponential counter at which the error is supposed to happen, which we mark with the concept $E_f$. Now we compare the stored bit-value in $\text{NBit}$ with the current Bit-value and verify that a counting error has indeed happened.

For a hypothesis to entail $\text{Goal}(a)$, it has to be of the form $C(a)$, where $C$ expresses that every path along r- and s-successors eventually leads to an instance of $B$, and that, when following those paths down from these $B$-instances to $a$, they encode the sequence of all consecutive $2^n$-bit values via the concept $\text{Bit}$. This can only be done by a concept of triple exponential size. Specifically, let $\pi$ be a string of 0s and 1 that is the result of concatenating all $2^n$-bit values, starting from the lowest value, and ending with the largest, and let $\pi[i]$ denote to the $i$th bit on that sequence. Note that $\pi$ has length $\ell = 2^{2^n} \cdot 2^n$. Define $\text{Bit}[i]$ by $\text{Bit}[i] = \text{Bit}$ if $\pi[i] = 1$ and $\text{Bit}[i] = \neg \text{Bit}$ if $\pi[i] = 0$. A hypothesis for $\text{Goal}(a)$ would be the concept $C(a)$ recursively defined as follows:

- $C_0 = B \cap \text{Bit}[0]$
- For $i \in [1, \ell - 1]$, $C_i = \text{Bit}[i] \cap \forall (r \cup s).C_{i-1}$
- $C = C_\ell$.

This finishes the proof.

**Lemma 1.** Every hypothesis $\mathcal{H}$ for $\mathcal{A}$ can be translated into a hypothesis abstraction.

**Proof.** Let $I$ be the set of all models of $\mathcal{K} \cup \mathcal{H}$. We first construct an interpretation abstraction $\mathcal{I}_1$ based on $I$, which is then modified to become $\Sigma$-complete. $\mathcal{I}_1 = (V, \lambda_1, s, \mathcal{R}_1, F)$ is constructed together with a function $I$ that assigns to every $v \in V$ a set $I(v)$ of pairs $(\mathcal{I}, d')$ of an interpretation $\mathcal{I} \in I$ and a domain element $d' \in \Delta^\mathcal{I}$.

For every individual name $a \in \text{ind}(\mathcal{K} \cup \mathcal{H})$, we add a node $v_a$ with $s(a) = v_a$ and $\lambda_1(v_a) = \{\text{tp}(a^2\mathcal{I}) \mid \mathcal{I} \in I\}$. We set $I(v) = \{(\mathcal{I}, a^2\mathcal{I}) \mid \mathcal{I} \in I\}$, and to capture the role assertions, for every $r(a, b) \in \mathcal{K} \cup \mathcal{A}$ and $\mathcal{I} \in I$, we add the role assignment $(v_a, \text{tp}(a^2\mathcal{I})_x, r, v_b)$ to $\mathcal{R}_1$.

We add further outgoing nodes based on the following inductive construction. For two connected nodes $v_1$, $v_2$ we denote their distance as the number of nodes occurring in the shortest path connecting them. Let $n$ be the maximal nesting depth in $\mathcal{A}$, by which we mean nesting of existential or universal role restrictions. For every node $v$ whose distance to the next named node is less than $n$, every type $t \in \lambda_1(v)$, and every $\exists r.C \in t$, we add a node $v'$ unique to $(v, t, \exists r.C)$ and set

$$I'(v') = \{(\mathcal{I}, d) \mid (\mathcal{I}, d_1) \in I(v), \text{tp}(d_1)_x = t,
\{d_1, d\} \in \Delta^2, d \in C^2\}$$

$$\lambda_1(v') = \{\text{tp}(d)_x \mid (\mathcal{I}, d) \in I(v')\}$$

$v, t, v'$ to $\lambda_1$.

Every outgoing node $v$ whose distance to the next named individual is $n$ is added to $F$.

The construction directly ensures that $\mathcal{I}_1$ satisfies Conditions D1 and D2. It is also not hard to understand that $\mathcal{I}_1$ abstracts at least one interpretation from $I$, and thus a model of $\mathcal{K}$. We can show that it also explains $\Phi$, that is, that every model of $\mathcal{K}$ that is abstracted by $\mathcal{I}_1$ is also a model of $\Phi$. Specifically, let $\mathcal{I}$ be such a model of $\mathcal{K}$, and $h : \Delta^\mathcal{I} \rightarrow V$ be the mapping that witnesses this. Since $\mathcal{K} \cup \mathcal{H} \models \Phi$, for every $C(a) \in \Phi$, we must have $a^2 \in C^\mathcal{I}$ for all $\mathcal{I} \in I$, and thus $C \in t$ for every $t \in \lambda_1(s(a))$. By Condition I1 and I2, it follows that $\mathcal{I} \models C(a)$ for every $C(a) \in \Phi$. Now assume $r(a, b) \in \Phi$. We then must have $r(a, b) \in \mathcal{K} \cup \mathcal{H}$. If $r(a, b) \in \mathcal{K}$, $\mathcal{I} \models r(a, b)$ follows from the fact that $\mathcal{I}$ is a model of $\mathcal{K}$. Otherwise, $r(a, b) \in \mathcal{H}$, and our construction explicitly added $(v_a, t, r, v_b)$ to $\mathcal{R}_1$ for every $t \in \lambda_1(v_a)$, so that by Conditions I1 and I3, also $\mathcal{I} \models r(a, b)$. It follows that $\mathcal{I} \models \Phi$, and thus that $\mathcal{I}_1$ explains $\mathcal{I}$.

It remains to ensure $\Sigma$-completeness. For this, we modify $\mathcal{I}_1$ into an interpretation abstraction $\mathcal{I}$ that is $\Sigma$-complete, while still abstracting the same models of $\mathcal{K}$. This way,
we make sure that $\mathcal{J}$ also explains $\Phi$. Specifically, $\mathcal{J} = < V, \lambda, s, \mathcal{R}, F >$ is obtained from $\mathcal{J}_1 = < V, \lambda_1, s, \mathcal{R}_1, F >$ by setting:

- for every $v \in V$, $\lambda(v) = \{ t \in T_{\mathcal{K}, \Phi} | t' = \lambda_1(v), t \cap \Sigma = t' \cap \Sigma \}$,
- $\mathcal{R} = \mathcal{R}_1 \cup \{ (v_1, t, r, v_2) | \langle v_1, t', r, v_2 \rangle \in \mathcal{R}_1, t \in T_{\mathcal{K}, \Phi}, t \cap \Sigma = t' \cap \Sigma \}$

Since we only added types, every model abstracted by $\mathcal{J}_1$ is also abstracted by $\mathcal{J}$. We show that also every model of $\mathcal{K}$ that is abstracted by $\mathcal{J}$ is abstracted by $\mathcal{J}_1$. Assume $\mathcal{I}$ is a model of $\mathcal{K}$ that is abstracted by $\mathcal{J}$ via $h : \Delta' \rightarrow V$. We show that $\mathcal{I}$ then is a model of $\mathcal{H}$, and thus $\mathcal{I} \in \mathcal{I}$, from which it follows from our construction that for every $d \in \Delta'$, $\text{tp}(d)_{\mathcal{I}} \in \lambda_1(h(d))$, and thus that $\mathcal{I}$ is also abstracted by $\mathcal{J}_1$.

By construction, for every $r \in \mathcal{R}$, there exists some role $\delta_\mathcal{I} \in \mathcal{I}$, $a \in \mathcal{R}$, $a^X \in \mathcal{C}^2$ iff $a^{\delta_{\mathcal{I}}} \in \mathcal{C}^1$. We show in Proposition 1.1 that this property directly without introducing the notion of bisimulation.

- For a node $v \in V$, denote by $\text{dist}(v)$ the distance to the next internal node (or 0 if $v$ is internal). For a concept $C$, by $\text{nd}(C)$ the nesting depth of role restrictions of $C$. We extend dist to $\Delta'$ by setting dist$(d) = \text{dist}(h(d))$. Recall that $n$ is the maximal nesting depth of any concept in $\mathcal{H}$, and that our instruction insures that for any outgoing node, dist$(d) \leq n$. Consequently, every path of outgoing nodes has at most length $n - \text{dist}(v)$. We show by structural induction that for every $C \in \text{Sub}(\mathcal{H})$ and $d \in \Delta'$, st. $(n - \text{dist}(d)) \leq \text{nd}(C)$, $d \in C^2$ iff $d^1 \in C^2$. We distinguish the cases based on the syntactical shape of $C$.

- For $C = T$, the claim follows trivially.
- For $C = A \in \mathcal{N}_C$, we note that by construction, $\text{tp}(d)_{\mathcal{I}} \cap \Sigma = \text{tp}(d^1)_{\mathcal{I}} \cap \Sigma$, which implies that $A \in \Delta'$ iff $d^1 \in A^1$.
- For $C = \neg D$, by inductive hypothesis, $d \in D^1$ iff $d^1 \in D^2$, so that also $d \in (\neg D)^1$ iff $d^1 \in (\neg D)^2$.
- Let $C = C_1 \cap C_2$. We have $d \in (C_1 \cap C_2)^2$ iff $d \in C_1^2$ and $d \in C_2^2$. By inductive hypothesis, this holds iff $d^1 \in C_1^2$ and $d^1 \in C_2^2$, which holds iff $d^1 \in (C_1 \cap C_2)^1$.
- Finally, let $C = \exists r. D$. If $d \in C^2$, there must be $(d, e) \in r^2$, which by $\mathcal{J}_1$ means that $(h(d), \text{tp}(d)_{\mathcal{I}}, r, h(e)) \in \mathcal{R}_1$. By construction, we have $\langle h(d), f(d), r, h(e) \rangle \in \mathcal{R}_1$, which means by construction of $\mathcal{J}$ that we have $(d^1, e^1) \in r^1$. Note that dist$(e) = \text{dist}(d) + 1$, and nd$(D) = \text{nd}(C) - 1$, so that we can apply the inductive hypothesis. We obtain $e^2 \in D^2$, and thus $d^2 \in (\exists r. D)^2$.

Now assume $d^2 \in C^2$. By construction, we then must have $(h(d), \text{tp}(d^1)_{\mathcal{I}}, r, v') \in \mathcal{R}_1$ s.t. $e \in C^2$, for all $(\exists r. e) \in I(v')$, and by construction, we have $(h(d), \text{tp}(d), r, v') \in \mathcal{R}_1$. By $\mathcal{J}_1$, there must then be $e \in \Delta'$ s.t. $h(e) = v'$. Since thus $(\exists r. e) \in I(v')$, we obtain that $e^2 \in D^2$, which by inductive hypothesis means that $e^2 \in D^2$. It follows that $d^2 \in C^2$.

It follows now in particular that for any $a \in \text{ind}(\mathcal{H} \cup \mathcal{K})$ and $C \in \mathcal{H}$, $a^{\delta_{\mathcal{I}}} \in C^2$ iff $a^2 \in C^2$. Since $\mathcal{I}_1 \models \mathcal{H}$, and thus for all $(a, \gamma) \in \mathcal{H}$, $\mathcal{I}_1 \models (a, \gamma)$. As a consequence, we obtain $\mathcal{I} \models \gamma(a)$, and thus $\mathcal{I} \models \mathcal{J}$ is abstracted also by $\mathcal{J}_1$.

Lemma 2. Let $\mathcal{J}$ be a hypothesis abstraction. Then, $\mathcal{J}$ can be transformed into a hypothesis abstraction $\mathcal{J}' = (\mathcal{V'}, \lambda', \mathcal{R}', F')$ where, for $\ell = |T_{\mathcal{K}, \Phi}| \cdot |T_{\mathcal{K}, \Phi}|$, i) $\mathcal{V'}$ contains at most $\ell$ internal nodes, ii) every $v \in \mathcal{V'}$ has at most $\ell$ successors in $\mathcal{R}'$, and iii) every path of outgoing nodes contains at most $\ell$ nodes.

Proof. To show that $\mathcal{J}$ can be reduced into an interpretation abstraction of the required size, we use the facts that

- n1 nodes that have no connection to a named node can be removed,
- n2 we can identify internal nodes with the same label,
- n3 if there are two role assignments $\langle v_1, t, r, v_2 \rangle$ and $\langle v_1, t', r, v_2' \rangle \in \mathcal{R}$ and $\lambda(v_2) = \lambda(v_2')$, then the latter can be removed, and
- n4 the length of any path from or to a named node that does not contain another named node can be restricted to $\ell$.

Point n1 does not require further explanation, while the others need more of an argument.

For n2, assume $v_1, v_2 \in V$ are two internal nodes such that $\lambda(v_1) = \lambda(v_2)$. Define a mapping $m : V \rightarrow (V \setminus \{ v_2 \})$ by setting $m(v) = v$ for $v \neq v_2$ and $m(v_2) = v_1$. We construct a new interpretation abstraction $\mathcal{J}_1 = (V_1, \lambda_1, s, \mathcal{R}_1, F_1)$ by setting

- $V_1 = V \setminus \{ v_2 \}$,
- $\lambda_1(v) = \lambda(v)$ for all $v \in V_1$,
- $\mathcal{R}_1 = \{ (m(v), t, r, m(v')) | \langle v, t, r, v' \rangle \in \mathcal{R} \}$, and
- $F_1 = F \cap V_1$.

One easily verifies that $\mathcal{J}_1$ satisfies Conditions D1–D5, and thus is an $\mathcal{ALC}$-conform and $\mathcal{S}$-complete. To show that it explains $\Phi$, we need to show that $\mathcal{J}_1$ abstracts at least one model of $\mathcal{K}$, and every model of $\mathcal{K}$ abstracted by $\mathcal{J}_1$ entails $\mathcal{H}$. Let $\mathcal{I}$ be a model of $\mathcal{K}$ abstracted by $\mathcal{J}$ via $h : \Delta' \rightarrow V$. Let $a_{v_1}$ be such that $s(a_{v_1}) = v_1$, and $a_{v_2}$ be such that $s(a_{v_2}) = v_2$. We construct a new interpretation $\mathcal{I}_1$ which has as domain $\Delta^2 = \{ d \in \Delta^2 | h(d) \neq v_2 \}$. Define a mapping $m : \Delta^2 \rightarrow D^2$ by setting $m(d) = d$ if $d \in \Delta'$ or $h(d) \neq v_2$, and $m(d) = d^2$ otherwise. Construct $\mathcal{I}_1$ as

- $A^2 = A^2 \cap A^2$ for all $A \in \mathcal{N}_C$, and
- $r^2 = \{ (m(d), m(e)) | \langle d, e \rangle \in r^2 \}$ for all $r \in \mathcal{N}_R$.
The transformation is type-preserving, and since $\mathcal{I} \models \mathcal{K}$, so does $\mathcal{I}_1$. We can obtain a function $h' : (\Delta' \cap \Delta_{1}) \to V'$ witnessing abstraction by $\mathcal{J}_1$ by setting $h'(d) = h(m(d))$, for which Conditions $11\text{--}13$ are easily verified. Consequently, $\mathcal{J}_1$ abstracts a model of $\mathcal{K}$. Now let $\mathcal{I}$ be a model of $\mathcal{K}$ abstracted by $\mathcal{J}_1$. By duplicating the individual $d$ s.t. $h(d) = v_i$ to an individual $d'$ with $h(d') = v_i$, we can in a similar fashion construct a $\mathcal{I}_2$ that is abstracted by $\mathcal{J}_1$ and that entails $\mathcal{K} \cup \Phi$ iff so does $\mathcal{I}$.

It follows that every model of $\mathcal{K}$ abstracted by $\mathcal{J}_1$ entails $\mathcal{H}$, so that $\mathcal{J}_1$ explains $\Phi$. $\mathbf{n3}$ can be shown in the same way.

For Point $\mathbf{n4}$, we only consider the case of a path from an named node. The other case is similar. Assume there is such a path 

$$v_1, t_1, r_1, v_2, t_2, r_2, \ldots, r_{n-1}, t_{n-1}, v_n$$

where $n > \ell$. We show that $\mathcal{J}$ can then be modified in a way that preserves all properties required by the Lemma so that this path becomes shorter. We ignore the types for now, and consider $\pi = v_1, r_1, \ldots, r_{n-1}, v_n$. We say that an interpretation $\mathcal{I}$ abstracted by $\mathcal{J}$ by $h$ implements $\pi$ if there exists $d_i \in \Delta'$ with $h(d_i) = v_i$ for all $i \in [1, n]$ s.t. $\langle d_i, d_{i+1} \rangle \in r_i$ for all $i \in [1, n - 1]$. If there is no interpretation $\mathcal{I}$ that implements $\pi$, let $i$ be the largest element in $[1, n]$ s.t. the path $v_1, r_1, \ldots, r_{i-1}, v_i$ is implemented. Clearly, we can then remove all role assignments $\langle v_i, t, r, v_{i+1} \rangle$, since they do not affect the interpretations abstracted by $\mathcal{J}$. Otherwise, assume the interpretation $\mathcal{I}$ implements $\pi$, and $h$ is the function by which $\mathcal{I}$ is abstracted by $\mathcal{J}$. For $i \in [1, n]$, let $d_i \in \Delta'$ be s.t. $h(d_i) = v_i$. Because of $13$ in Definition $3$, for every $i \in [1, n - 1]$ there exists $\langle v_i, t(d_i)_x, r_i, v_{i+1} \rangle \in \mathcal{R}$, so that we can assume that $t_i = t(d_i)_x$. Because there are at most $\ell$ pairs $\langle t, T \rangle$ of a type $T$ a set of types, by the pigeon hole principle, there must be $i, j \in [1, n]$ s.t. $i < j$, $\lambda(v_i) = \lambda(v_j)$ and $t_i = t_j$. We modify $\mathcal{H}$ so that it “skips” the elements between $v_i$ and $v_j$ by replacing $\langle v_{i-1}, t_{i-1}, r_{i-1}, v_i \rangle$ with $\langle v_{i-1}, t_{i-1}, r_{i-1}, v_j \rangle$.

Denote by $\mathcal{J}_1 = \{ V_1, \lambda_1, s, \mathcal{R}_1 \}$ the result of applying this operation. $\mathcal{J}_1$ is $\mathcal{ALC}$-conform and $\Sigma$-complete if so is $\mathcal{J}$. We show that furthermore, $\mathcal{J}_1$ still explains $\Phi$, that is, that for every model $\mathcal{I}_1$ of $\mathcal{K}$ that is abstracted by $\mathcal{J}_1$ we have $\mathcal{I}_1 \models \Phi$. Let $\mathcal{I}_2$ be a model of $\mathcal{K}$ that is abstracted by $\mathcal{J}_1$ via the mapping $h_1 : \Delta' \to V_1$. If $\mathcal{I}_2$ does not use the new role assignment $\langle v_{i-1}, t_{i-1}, r_{i-1}, v_i \rangle$, then it is also abstracted by $\mathcal{J}_1$. Otherwise, there are $d_{i-1}', d_i' \in \Delta'$ s.t. $h_1(d_{i-1}') = v_{i-1}$, $h_1(d_i') = v_i$ and $\langle d_{i-1}', d_i' \rangle \in r_{i-1}$ Based on $\mathcal{I}_1$ and the model $\mathcal{I}_2$ used before, we construct a model $\mathcal{I}_2$ that is abstracted by $\mathcal{J}$ s.t. $\mathcal{I}_2 \models \mathcal{K}$ and $\mathcal{I}_2 \not\models \Phi$.

Intuitively, we adapt $\mathcal{I}_1$ by putting the path connecting $d_i$ and $d_j$ in $\mathcal{I}$ between $d_{i-1}$ and $d_i$. Specifically, we define $\mathcal{I}_2$ as follows:

$$\Delta_2 = \Delta_{1} \cup \Delta_{1} \setminus \{ d_j' \},$$

$$A_{2} = A_{1} \cup A_{1}$$

for every $A \in \text{N}_{C}$,

$$r_{2} = \{ \langle d, d' \rangle \in r_{1} \mid d \neq d' \}$$

for every $r \in \text{N}_{R}$,

$$a_{2} = a_{1}$$

for every $a \in \text{N}_{I}$.

Because $\text{tp}(d_{j})_{I} = \text{tp}(d_{j})_{I_{1}}$, the construction is type-preserving, which implies $\mathcal{I}_2 \models \mathcal{K}$, but also $\mathcal{I}_2 \not\models \Phi$. Furthermore, $\mathcal{I}_2$ is abstracted by $\mathcal{J}$.

Lemma 3. For every $\mathcal{ALC}$-conform, $\Sigma$-complete interpretation abstraction $\mathcal{J} = \{ \mathcal{V}, \lambda, s, \mathcal{R}, \mathcal{F} \}$, there exists an $\text{ABox}$ $\mathcal{H}$ s.t. i) the models of $\mathcal{K} \cup \mathcal{H}$ are exactly the models of $\mathcal{K}$ accepted by $\mathcal{J}$, ii) $|\text{ind}(\mathcal{H})| \leq |\mathcal{V}|$, and iii) for every $a \in \text{ind}(\mathcal{H})$, $\mathcal{H}$ contains one assertion $C(a)$, with $\text{size}(C)$ exponentially bounded by the path lengths between outgoing nodes in $\mathcal{J}$.

Proof. For every internal node $v \in \mathcal{V}$, let $a_v$ be the individual s.t. $s(a_v) = v$. We assign to every node $v \in \mathcal{V}$ and $i \geq 0$ a depth-bounded concept $C_v^{(i)}$, and to every $v \in \mathcal{V}$ a concept $C_v$. $C_v^{(0)}$ is defined by

$$C_v^{(0)} = \bigcup_{\ell \in \Lambda(v)} \left( \bigcap_{A \in \text{N}_{C} \cap \Sigma} A \cap \bigcap_{A \in \text{N}_{C} \setminus \ell} \neg A \right)$$

For $i > 0$, if $v \in \mathcal{V}$, $C_v^{(i)} = C_v^{(i-1)}$, and otherwise

$$C_v^{(i)} = \bigcup_{\ell \in \Lambda(v)} \left( \bigcap_{A \in \text{N}_{C} \cap \Sigma} A \cap \bigcap_{A \in \text{N}_{C} \setminus \ell} \neg A \right)$$

$$\cap \prod_{r \in \text{N}_{R}} \left( \left[ C_{v'}^{(i)} \mid \langle v, t, r, v' \rangle \in \mathcal{R}, v' \text{ is outgoing} \right] \right)$$

$$\cap \prod_{r \in \text{N}_{R}} \forall v. \left[ C_{v'}^{(0)} \mid \langle v, t, r, v' \rangle \in \mathcal{R} \right]$$

We then set $C_v = C_v^{(n)}$, where $n$ is the maximal path length of any path of outgoing edges. As the outgoing nodes cannot form a cycle due $\mathbf{D1}$, the induction on the existential role restrictions always terminates. The depth-bound is used to ensure that the induction also terminates for the universal restrictions. These universal restrictions are used to restrict the possible successors in accordance with $\mathbf{I3}$, without conflicting with the internal nodes which may form cycles.

$\mathcal{H}$ contains the concept assertion $C_v(a_v)$ for every internale node $v$, and the role assertion $r(a_v, a_{v'})$ for every $\langle v, t, r, v' \rangle \in \mathcal{R}$ for which $v$ and $v'$ are internal. One can verify that $\mathcal{H}$ satisfies the size restrictions given in the lemma.

It remains to show that every model of $\mathcal{K}$ abstracted by $\mathcal{J}$ is a model of $\mathcal{H}$, and that every model of $\mathcal{K} \cup \mathcal{H}$ is abstracted by $\mathcal{J}$. 


Assume $I$ is abstracted by $J$ via $h : \Delta' \to V$. We show by induction on the nesting depth that for every node $v$, $h(v) \in C_v^I$: For $v \in F$, by I2, we have $tp(h(v)) \in \lambda(v)$, and thus
\[
h(v) \in \left( \bigcap_{A \in \Sigma(Nc)} A \cap \bigcap_{A \in \Sigma(Nc)} \neg A \right)^I.
\]

For the induction step, assume that for every role successor $v'$ of $v$ we have $h(v') \in C_v^I$. Then it follows directly from the definition that then $h(v) \in C_v^I$. Consequently, $I \models C(a_v)$ for every $C(a_v) \in \mathcal{H}$. Let $r(a_v, a_w) \in \mathcal{H}$. By Condition D2, because $v$ and $v'$ are internal, we have $(v, t, r, v') \in \mathcal{I}$ for every type $t \in \lambda(v)$. It follows by I3 that $(h(v), h(v')) \in r^2$, and thus $I \models r(a_v, a_w)$. We have shown that $I \models \mathcal{H}$.

Now assume $I \not\models \mathcal{H}$. We recursively build a function $h : \Delta' \to V$ that satisfies conditions I1–I3, and thus witnesses that $I$ is abstracted by $J$. For each internal node $v$, we have a named individual $a_v$, and we set $h(v) = a_v^*$. Outgoing nodes are assigned inductively based on the definition of $C_v$. If $h(d) = v$ and $d \in C_v^I$, outgoing nodes are assigned inductively based on the definition of $C_v$. If $h(d) = v$ and $d \in C_v^I$, then there must be $(d', d'') \in r^2$ such that $d' = v$. As a result, we obtain a mapping $h : \Delta' \to V$ for some $\Delta' \subseteq \Delta^2$ s.t. $h(d) \in C_v^I$. We then obtain a mapping $h : \Delta' \to V$ for every $d \in \Delta'$. We have shown that $I$ is abstracted by $J$.

\[\text{Theorem 7. ALC ABox abduction is in N2ExpTime}^\mathbf{NP}.\]

**Proof.** We guess an ALC-conform, $\Sigma$-complete interpretation abstraction $J$ within the size bounds of Lemma 2. Because we can reuse outgoing nodes with the same label, it suffices to guess one such interpretation abstraction that is of at most double exponential size, similar as we did for Lemma 10. We then verify it abstracts some model of $K$ by guessing for each node $v$ a type $t \in \lambda(v)$ and checking whether those types can be implemented by a model of $K$. This check is possible in polynomial time by just considering the node, its assigned type, and its successors. Finally, we verify that it explains the observation by making another such guess to verify whether there exists a model of $K$ s.t. $K \not\models \Phi$.

\[\text{D Size-Restricted Abduction}\]

Regarding the upper bounds, only the argument for flat solutions in ALC requires some additional argument.

**Lemma 11.** Existence of hypotheses for size restricted ALC abduction problems can be decided in NExpTime$^\mathbf{NP}$. 

**Proof.** Let $\mathcal{A} = \langle K, \Phi, \Sigma, k \rangle$ be the size-restricted abduction problem. We modify the coNExpTime-procedure in the proof for Theorem 3 as follows: after we computed in deterministic exponential time the set $T_{K,\Phi}$ of types, we guess a flat ABox $\mathcal{H}$ of size at most $k$ s.t. for every $r(a, b) \in \Phi$, $r(a, b) \in \mathcal{H}$. Since $k$ is encoded in binary, $\mathcal{H}$ may be up to exponential in size. To decide whether $\mathcal{H}$ is a hypothesis, we use the NP-oracle to decide whether for every selector function $s_e : \text{ind} \mathcal{H} \to T_{K,\Phi}$, either $s_e$ is not compatible with $K \cup \mathcal{H}$, or for some $C(a) \in \Phi$, $C \not\models s_e(a)$. This can be easily verified by a non-deterministic, polynomially bounded oracle which as takes in as input the pairs $(T_{K,\Phi}, \mathcal{H})$, guesses the function $s_e$, and accepts if $s_e$ is compatible with $K \cup \mathcal{H}$ and for some $C(a) \in \Phi$, $C \not\models s_e(a)$. 

**Lemma 4.** Size-restricted $\mathcal{EL}$ abduction is NP-hard.

**Proof.** Let $\mathcal{L}, \mathcal{K}, \Phi$, and $\Sigma$ be as in the proof sketch of the main text. We need to show that $(K, \Phi, \Sigma)$ has a hypothesis of size at most $2m$ if $\phi$ is satisfiable. If $\phi$ is satisfiable by a truth assignment $\alpha : \{p_1, \ldots, p_m\} \to \{0, 1\}$, we can build a hypothesis of size $2m$ by setting
\[
\mathcal{H} = \{\text{True}(p_i) \mid \alpha(p_i) = 1\} \cup \{\text{False}(p_i) \mid \alpha(p_i) = 0\}.
\]

Now assume $(K, \Phi, \Sigma)$ has a hypothesis $\mathcal{H}$ of size at most $2m$. For every variable $p_i$, the only way to entail $T(p_i)$ using only names from $\Sigma$ is to add either True($p_i$) or False($p_i$) to the hypothesis. Each such assertion adds 2 to the size of $\mathcal{H}$, and as $\mathcal{H}$ is not allowed to have size greater than $2m$, it must thus contain exactly one such axiom for each variable. The satisfying truth assignment $\alpha : \{p_1, \ldots, p_m\} \to \{0, 1\}$ is then obtained by setting $\alpha(p_i) = 0$ if False($p_i$) $\in \mathcal{H}$ and $\alpha(p_i) = 1$ if True($p_i$) $\in \mathcal{H}$. Since $K \cup \mathcal{H} \models C(c_i)$ for every clause $c_i$, we can see that $\alpha$ makes every clause in $\phi$ true, and is thus a satisfying assignment.

**Lemma 5.** Size restricted $\mathcal{EL}_\perp$ abduction is NExpTime-hard.

**Proof.** We first give a more formal definition of the existential tiling problem. The problem is given by a tuple $(T, T_1, t, V, H, n)$ of a set $T$ of tile types, a sequence $T_1 = t_1 \ldots t_m = T^n$ of initial tiles, a final tile $t_e$, vertical and horizontal tiling conditions $V, H \subseteq T \times T$, and a number $n$ in binary encoding. We call a mapping
\[
f : \{1, 2^n\} \times \{1, 2^n\} \to T
\]
a tiling, which is valid iff $f(2^n, 2^n) = t_e$, for all $i \in [1, m]$, $f(i, 1) = t_i$, and for all $i \in [1, 2^n - 1]$ and $j \in [1, 2^n]$,\[
\langle f(i, j), f(i + 1, j) \rangle \in H
\]
and
\[
\langle f(j, i), f(j, i + 1) \rangle \in V.
\]
Based on $T$, $H$, $V$ and $n$, we construct an $\mathcal{EL}_\perp$ KB $K$, an ABox $\Phi$, and a signature $\Sigma$ s.t. $(K, \Phi, \Sigma)$ has an ABox hypothesis of size at most
\[
k = 2 \cdot (2^{2n} - m) + 3 \cdot 2 \cdot (2^{2n} - 2^n) + 2
\]
iff the tiling problem has a solution.
We define the signature as
\[ \Sigma = \{ \text{Start}, x, y \} \cup \{ A_t \mid t \in T \}, \]
where the concept name Start will mark the first tile, the role names \( x \) and \( y \) will denote the horizontal and vertical neighbourhood in the tiling, and each tile type \( t \in T \) has a corresponding concept name \( A_t \). The observation is
\[ \Phi = \{ \text{End}(a) \}. \]

\( \mathcal{K} \) uses additional concept names \( C_1, C_1, \ldots, C_n, C_n \) for \( C \in \{ X, Y \} \) to encode a binary counter for both coordinates, and a special concept name \( B \) that will ensure that every tile is represented in the hypothesis. Together with a tile, the concept name Start initialises the counter:
\[
\text{Start} \sqcap A_t \sqsubseteq \overline{X}_1 \sqcap \ldots \sqcap \overline{X}_n \sqcap \overline{Y}_1 \sqcap \ldots \sqcap \overline{Y}_n \quad \text{for every } t \in T.
\]
To entail the observation, both counters must reach \( 2^n \) on an individual name marked with \( B \) and some type tile:
\[
X_1 \sqcap \ldots \sqcap X_n \sqcap Y_1 \sqcap \ldots \sqcap Y_n \sqcap B \sqcap A_t \sqsubseteq \text{End}
\]
Every domain element represents at most one counter value:
\[
C_i \sqcap \overline{C}_i \sqsubseteq \bot \quad \text{for every } i \in [1, n], C \in \{ X, Y \}.
\]
The counter values are incremented along the \( x \) and \( y \)-predicess using the following axioms for \( (C, c) \in \{ (X, x), (Y, y) \} \):
\[
\begin{align*}
\exists c.(C_i \sqcap C_{i-1} \ldots \ldots C_1) & \sqsubseteq C_i \quad \text{for } i \in [1, n] \\
\exists c.(C_i \sqcap C_{i-1} \ldots \ldots C_1) & \sqsubseteq C_i \quad \text{for } i \in [1, n] \\
\exists c.(C_i \sqcap \overline{C}_j) & \sqsubseteq C_i \quad \text{for } i, j \in [1, n], j < i \\
\exists c.(C_i \sqcap \overline{C}_j) & \sqsubseteq Z_i \quad \text{for } i, j \in [1, n], j < i.
\end{align*}
\]
The vertical and horizontal tiling conditions are ensured using the following axioms:
\[
\begin{align*}
\exists x. A_t \sqcap A_{t'} \sqsubseteq \bot & \quad \text{for every } (t, t') \in (T \times T) \setminus H \\
\exists y. A_t \sqcap A_{t'} \sqsubseteq \bot & \quad \text{for every } (t, t') \in (T \times T) \setminus V
\end{align*}
\]
The initial tilings from \( T_I \) are specified as follows
\[
\begin{align*}
\text{Start} & \sqsubseteq I_1 \\
I_t & \sqsubseteq \forall x. I_{t+1} \quad \text{for } i \in [1, m - 1] \\
I_t & \sqsubseteq A_{t_i} \quad \text{for } i \in [1, m]
\end{align*}
\]
Finally, the special concept name \( B \) is used to ensure that every hypothesis contains an individual name for every tile. This name is initialised by the individual satisfying \( \text{Start} \), and then propagated in \( x \) and \( y \) direction, provided that both the \( x \) and \( y \) successor satisfy it, or one of the counter values is 0 (in which case we do not require a successor in the corresponding direction).
\[
\begin{align*}
\text{Start} & \sqsubseteq B \\
\overline{X}_1 \sqcap \ldots \sqcap \overline{X}_n \sqcap \exists y. (A_t \sqcap B) & \sqsubseteq B \quad \text{for every } t \in T \\
\overline{Y}_1 \sqcap \ldots \sqcap \overline{Y}_n \sqcap \exists x. (A_t \sqcap B) & \sqsubseteq B \quad \text{for every } t \in T \\
\exists x. (A_t \sqcap B) \sqcap \exists y. (A_{t'} \sqcap B) & \sqsubseteq B \quad \text{for every } t, t' \in T
\end{align*}
\]
This completes the construction.

First, assume the tiling problem has a solution \( f \). Based on \( f \), we construct a hypothesis for \( \langle K, \Phi, \Sigma \rangle \) of size \( k \). We use \( 2^{2n} \) individual names \( a_{i,j} \) assigned with coordinates \( i, j \in [1, 2^n] \), where for convenience, we let \( a_{2^n, 2^n} = a \). The hypothesis is then:
\[
\mathcal{H} = \{ \{ A_t(a_{i,j}) \mid f(i, j) = t \} \cup \{ A_t(a_{i,j}) \mid 1 \in [1, m] \} \}
\cup \{ x(a_{i+1,j}, a_{i,j}) \mid i \in [1, 2^n - 1], j \in [1, 2^n] \}
\cup \{ y(a_{i,j+1}, a_{i,j}) \mid i \in [1, 2^n], j \in [1, 2^n - 1] \}
\cup \{ \text{Start}(a_{1,1}) \}
\]
It is standard to verify that \( \text{size}(\mathcal{H}) = k, K \cup \mathcal{H} \models \text{B}(a_{i,j}) \) for every \( i, j \in [1, 2^n] \), as well as \( K \cup H \models \Phi \).
Now assume \( \langle K, \Phi, \Sigma \rangle \) has a hypothesis \( \mathcal{H} \) of size \( k \). Since \( K \cup H \models \Phi, \mathcal{H} \) has to ensure the following:
\begin{enumerate}
\item \( K \cup \mathcal{H} \models (C_1 \cap \ldots \cap C_n)(a) \) for both \( C \in \{ X, Y \} \). Since Start is the only name in \( \Sigma \) that can trigger the assignment of any counter value, there must be some assertion \( \text{Start}(a_0) \in \mathcal{H} \) as well as a path of length \( 2^{n+1} \) along \( x \) and \( y \)-successors connecting \( a \) to \( a_0 \).
\item \( K \cup \mathcal{H} \models \text{B}(a) \). While the assertions mentioned so far ensure \( K \cup \mathcal{H} \models \text{B}(a_0) \), any element on the path connecting \( a \) to \( a_0 \) must entail \( B \) as well. This means that either 1) their \( X \)-counter corresponds to 0 and they have a \( y \)-successor satisfying \( B \) and \( A_t \) for some \( t \in T \), or 2) their \( Y \)-counter corresponds to 0 and they have an \( x \)-successor satisfying \( B \) and \( A_t \) for some \( t \in T \), or 3) they have both an \( x \)-successor and a \( y \)-successor satisfying \( B \) and some tile.
\end{enumerate}
These conditions, together with the fact that every individual can have only one counter value assigned, require that for every coordinate \( (x, y) \in [1, 2^n] \times [1, 2^n] \) there must exist at least one individual satisfying the corresponding counter values, for which \( \mathcal{H} \) explicitly states the tile type and its \( x \) and \( y \)-successor.
Except for the \( m \) initial tiles, we thus need one concept assertion per tile type, plus the assertion \( \text{Start}(a_{1,1}) \). Therefore, we need at least \( 2^{2n} - m + 1 \) concept assertions (for the tile types) and \( 2 \cdot (2^{2n} - 2^n) \) role assertions (for the neighbourhood). Each concept assertion has at least size 2, and every role assertion has size 3, which means that
\[
\text{size}(\mathcal{H}) \geq 2 \cdot (2^{2n} - m) + 3 \cdot 2 \cdot (2^{2n} - 2^n) - 2 = k.
\]
Since also \( \text{size}(\mathcal{H}) \leq k \), we obtain that \( \text{size}(\mathcal{H}) = k \), and that \( H \) is the smallest ABox that satisfies all these conditions. We obtain that every coordinate \( (i, j) \in [1, 2^n] \times [1, 2^n] \) is represented by exactly one individual name which has a tile type assigned in \( \mathcal{H} \). As \( K \) furthermore enforces that the tiling conditions are met, we obtain that \( \mathcal{H} \) can be used to construct a solution to the tiling problem.

\[ \square \]

**Lemma 6.** The \text{NEXPTIME}^{NP}-tiling problem is \text{NEXPTIME}^{NP}-hard.

**Proof.** We show that the \text{NEXPTIME}^{NP} tiling problem is \text{NEXPTIME}^{NP} hard by a reduction from the word acceptance problem for non-deterministic exponentially time bounded Turing machines with access to an NP-oracle.
For convenience, we assume the Turing machine $T$ to have a slightly different behaviour than usual: while there is a special state $q_T$ to trigger oracle queries, we do not use dedicate states for the query outcome. Instead, when in query state $q_T$, the machine immediately fails if the oracle $\Sigma_o$ would accept the word, and otherwise goes into another state as specified by the transition relation. This is without loss of generality, as we can simulate regular oracle calls as follows: 1) guess the outcome of the oracle call, 2) if the guess is yes, verify this by performing the computation in $\Sigma$, 3) if the guess is no, verify this using the oracle $\Sigma_o$.

Let

$$\Sigma_o = \langle Q_o, \Sigma_o, \Gamma_o, \delta_o, q_{o0}, F_o, \Delta_o \rangle$$

be the non-deterministic Turing machine to be used by the oracle, where

- $Q_o$ are the states,
- $\Sigma_o$ is the input alphabet
- $\Gamma_o$ with $\Sigma_o \subseteq \Gamma_o$ is the tape alphabet
- $\delta_o \in (\Gamma_o \setminus \Sigma_o)$ is the blank symbol,
- $q_{o0} \in Q_o$ is the initial state,
- $F_o$ contain the accepting states, and
- $\Delta_o \subseteq Q_o \times \Gamma_o \times Q_o \times \Gamma_o \times \{-1, +1\}$ is the transition function, where a tuple $(q_1, a_1, q_2, a_2, D)$ states that, when the machine is in state $q_1$ and reads an $a_1$ at the current tape position, then it moves to state $q_2$, writes $a_2$ at the current tape position, and moves the tape head according to the value of $D$.

The non-deterministic oracle Turing machine using this oracle is defined as

$$\Sigma = \langle Q, \Sigma, \Gamma, \delta, \Gamma_o, q_0, F, \Delta, \delta_o \rangle$$

where

- $Q$ are the states,
- $\Sigma$ is the input alphabet,
- $\Gamma$ with $\Sigma \subseteq \Gamma$ is the tape alphabet,
- $\delta \in (\Gamma \setminus \Sigma)$ is again the blank symbol
- $\Gamma_o$ is the oracle input tape alphabet,
- $q_0$ is the initial state,
- $F \subseteq Q$ is the set of accepting states,
- $\Delta \subseteq (Q \times \Gamma \times \Gamma_o) \times (Q \times \Gamma \times \{-1, 0, +1\} \times \Gamma_o \times \{-1, 0, +1\})$ is the transition relation, (which works on two tapes, the standard tape and the oracle tape), and
- $q_T \in (Q \setminus F)$ is the query state to query the oracle.

We further assume $\Sigma_o$ to be polynomially bounded, that is, there exists a polynomial $p_1$ s.t. for input words $w$, each run on $\Sigma_o$ uses at most $p_1(|w|)$ steps, and $\Sigma$ to be exponentially bounded, that is, there exists a polynomial $p_2$ s.t. for input words $w$, every run of $\Sigma$ requires at most $2^{p_2(|w|)}$ steps. Note that as a result, we can bound both the tape and the length of runs on both Turing machines by $2^{p(|w|)}$ for some polynomial $p$.

Given $\Sigma$, $\Sigma_o$ and the input word $w = a_1, \ldots, a_n$, we construct a NEXPTIME${}^{NP}$-tiling problem. Without loss of generality, we assume $n > 2$. Our set $T$ of tile types contains the special tiling type $t_o$ that also serves as final tile, and all other tile types are pairs $t = (t_1, t_2)$, where

$$t_1 \in Q \times \Gamma \times \{l, h, r, -, 0\}$$

and

$$t_2 \in Q_o \times \Gamma_o \times \{l, h, r, -, 0\}$$

The two components of the tile type are used to define the different tape contents, the first component being the tape of $\Sigma$, and the second the tape of $\Sigma_o$. For a tile type $t = (t_1, t_2)$, $t_1 = (q, a, D)$ denotes that $\Sigma$ is in state $q$, the current tape cell stores $a$, and $D$ refers to the relative position to the tape head: left, here, right, or elsewhere ($\cdot$). The value of $D = 0$ is furthermore used to mark the first configuration.

The initial tiles are then

$$T_1 = \langle t_1^{(1)}, t_2^{(1)} \rangle, \ldots, \langle t_1^{(n+1)}, t_2^{(n+1)} \rangle$$

where the first components are defined by

- $t_1^{(1)} = (q_0, a_1, h)$,
- $t_1^{(2)} = (q_0, a_2, l)$,
- $t_1^{(i)} = (q_0, a_i, 0)$ for $i \in [3, n]$, and
- $t_1^{(n+1)} = (q_0, \delta, 0)$,

and the second components are defined by

- $t_2^{(1)} = (q_{o0}, \delta, h)$,
- $t_2^{(2)} = (q_{o0}, \delta, l)$, and
- $t_2^{(i)} = (q_{o0}, \delta, 0)$ for $i \in [3, n+1]$.

The horizontal tiling condition $H_1$ contains all tuples $\langle (t_1, t_2), (t'_1, t'_2) \rangle$, s.t. for $i \in \{1, 2\}$, $t_i = (q, a, D)$ and $t'_i = (q', a', D')$:

- $q = q'$,
- if $D = 0$ and $a = \delta$, then $D' = D$ and $a' = \delta$,
- if $D = l$, then $D' = h$,
- if $D = h$, then $D' = r$,
- if $D = r$, then $D' \in \{0, -\}$, and
- if $D = -$, then $D' \in \{0, -l\}$.

This makes sure that in the first row of the tiling, all tiles following the initial tiles represent the situation where both tapes have a blank symbol, and that in every row, the state associated to a tile must be the same.

The vertical tiling condition $V_1$ encodes the transitions from one configuration to another. $V_1$ contains all tuples $\langle (t_1, t_2), (t'_1, t'_2) \rangle \in V_1$, where for $i \in \{1, 2\}$, $t_i = (q, a, D)$ and $t_i = (q', a', D')$, we have

- $D' \neq 0$,  
- if $D \neq h$, then $a = a'$,
- for $i = 2$, $q = q' = q_{o0}$,
- for $i = 1$, $\Sigma$ has a transition from $q$ to $q'$.
• if \( D = h \), then the transition from \( q \) to \( q' \) reads \( a \) and writes \( a' \) (on the normal tape if \( i = 1 \), and the oracle tape if \( i = 2 \)),
• if \( D = l \), then \( D' = h \) if the transition moves the head on the corresponding tape to the left, and otherwise \( D' = - \),
• if \( D = r \), then \( D' = h \) if the transition moves to the right, and otherwise \( D' = - \), and
• if \( D = h \), then \( D' = r \) if the transition moves to the left, and \( D' = l \) if the transition moves to the right.

In addition, \( V_1 \) contains \( \langle t_1, t_2, t_e \rangle \) for every \( t_1 \) with associated state \( q_f \in F \), to make sure the final tile is reached once the Turing machine enters a final state.

Intuitively, the tiling problem \( \langle H_1, V_1, T_1, t_e, p(n) \rangle \) encodes the behavior of \( \mathcal{T} \) without the oracle. More formally: if we consider the variant of \( \mathcal{T} \) where \( q_f \) acts just like a normal state, this Turing machine accepts \( w \) iff the tiling problem has a solution.

To now take into account the oracle, it remains to specify the tiling conditions for \( H_2 \) and \( V_2 \), which are similar to those for \( H_1 \) and \( V_1 \). Note that we want the rows of the first tiling \( \mathcal{T} \) to be initial rows of a tiling for \( \langle H_2, V_2, f(i), t_e, p(n) \rangle \), unless the encoded state is \( q_f \) and they encode the initial configuration of an accepting run for \( \mathcal{T} \). Since \( \mathcal{T} \) has only one tape, \( H_2 \) and \( V_2 \) now only consider the state of the first component of the tile type, which has to be \( q_f \), and otherwise only consider the second component. Regarding the second component, they are defined just like \( H_1 \) and \( V_1 \) accept that now, they consider transitions in \( \mathcal{T} \), and modify the state in the second component and not in the first.

**Lemma 7.** Size-restricted \( \text{ALC} \) abduction is \( \text{NEXPTIME}^{\text{NP}} \)-hard.

**Proof.** The background KB \( \mathcal{K} \) can be seen in Figure 3, and works like it is described in the main text. The signature of abducibles is
\[
\Sigma = \{ \text{Start}, x, y, T^* \} \cup \{ A_t \mid t \in T \}
\]
and the observation to be explained is \( \text{End}(a) \). We describe the different parts of the knowledge base. In the following, we denote by “base tiling” the tiling on the bottom of the generated cube, which is going to be the solution of the tiling problem \( \langle T, T_1, t_e, V_1, H_1, n \rangle \), while we call the other tilings \( x \times z \)-tilings.

• (26) puts the first tile and initialises all 3 counters.
• (27) gives the conditions for the observation to be entailed: all counters must reach their maximum value, and additionally the concepts \( B_2 \) and \( B_4 \) must be satisfied. Intuitively, \( B_2 \) signals that every coordinate always has a tile associated, and \( B_4 \) that for none of the rows in the base tiling, a tiling compatible to \( H_2 \) and \( V_2 \) can be found.
• (28)–(30) make sure the initial tiles in \( T_1 = \{ t_1, \ldots, t_m \} \) are placed.
• (31)–(35) implement the binary counters for the three coordinates.
• (39)–(39) specify the behaviour of the “hidden tiles” that should not be explicit in the hypothesis, to allow for different tilings to be tested in the models of the hypothesis: the abducible \( T^* \) states the existence of a hidden tile, only one hidden tile can be used at the same time, and hidden tiles can only be used outside of the ground plane of the cube.
• (40)–(43) ensure that every coordinate is assigned a (hidden or explicit) tile type. Specifically, \( B_1 \) is entailed at coordinates for which every preceeding coordinate has a tile type associated, \( B_2 \) is entailed if additionally, the current coordinate has a tile type associated, and for \( c \in \{x, y, z\} \), \( B_c \) states the \( c \)-coordinate has either value 0, or the next lower \( c \)-neighbour satisfies \( B_2 \). Consequently, we require either \( \text{Start} \) or \( B_x \cap B_y \cap B_z \) to entail \( B_1 \).
• (44) and (45) are as in the proof for Lemma 5, and make sure that the base tiling does not break the tiling conditions \( H_1 \) and \( V_1 \).
• (46) and (47) test for the tiling conditions on the hidden tilings, and mark errors using the concept name \( B_3 \), which is then propagated towards all \( x \)- and \( y \)-predecessors through (48).
• Finally, (49) makes sure that the concept name \( B_4 \)—required for the observation—is propagated along the \( y \)-axis to check whether all tiling attempts on the different \( H_2 \)-\( V_2 \)-tilings fail.

Similar as for Lemma 5, we can find a bound \( k \) that ensures that for every coordinate at most one individual is used, so that hypotheses consists of assertions forming a cube, where the bottom side corresponds to a valid tiling, while the remaining coordinates have the concept name \( T^* \) assigned. The observation \( \text{End}(a) \) can only be entailed by such a cube if the \( \text{NEXPTIME}^{\text{NP}} \)-tiling problem has a solution. \( \square \)
\[
\begin{align*}
\text{Start} \cap A_t & \subseteq \bigcap_{i=1}^{n} (X_i \cap Y_i \cap Z_i) \\
\bigcap_{i=1}^{n} (X_i \cap Y_i \cap Z_i) \cap B_2 \cap B_4 & \subseteq \text{End} \\
\text{Start} & \subseteq B_1 \\
I_i & \subseteq \forall x. I_{i+1} \\
I_i & \subseteq A_{t_i} \\
C_i \cap \overline{C}_i & \subseteq \bot \\
\exists c. (C_i \cap C_{i-1} \cap \ldots \cap C_1) & \subseteq C_i \\
\exists c. (C_i \cap C_{i-1} \cap \ldots \cap C_1) & \subseteq \overline{C}_i \\
\exists c. (C_i \cap \overline{C}_j) & \subseteq \overline{C}_i \\
\exists c. (C_i \cap \overline{C}_j) & \subseteq C_i \\
T^* & \subseteq \bigcup_{t \in T} A_t^* \\
A_t^* \cap A_t^* & \subseteq \bot \\
\bigcap_{i=1}^{n} Z_i \cap \bigcup_{t \in T} A_t^* & \subseteq \bot \\
\bigcup_{i=1}^{n} Z_i \cap \bigcup_{t \in T} A_t & \subseteq \bot \\
\text{Start} & \subseteq B_1 \\
B_1 \cap (A_t \cup A_t^*) & \subseteq B_2 \\
\bigcap_{i=1}^{n} \overline{C}_i & \subseteq \exists c. B_2 \subseteq B_c \\
B_2 \cap B_c \cap B_2 & \subseteq B_1 \\
\exists x. A_t \cap A_t^* & \subseteq \bot \\
\exists y. A_t \cap A_t^* & \subseteq \bot \\
(\exists x. A_t \cap A_t^*) \cup (\exists c. A_t^* \cap A_t^*) & \subseteq B_3 \\
\exists z. A_t^* \cap A_t^* & \subseteq B_3 \\
\exists z. B_3 \cup \exists z. B_3 & \subseteq B_3 \\
\bigcap_{i=1}^{n} (X_i \cap Z_i) \cap (B_3 \cup \neg A_t) & \subseteq B_4
\end{align*}
\]

for every \( t \in T \)  
for every \( t \in T \)  
for every \( i \in [1, m-1] \)  
for every \( i \in [1, m] \)  
for every \( i \in [1, n] \), \( C \in \{X, Y, Z\} \)  
for every \( i \in [1, n] \), \( (C, c) \in \{(X, x), (Y, y), (Z, z)\} \)  
for every \( i \in [1, n] \), \( (C, c) \in \{(X, x), (Y, y), (Z, z)\} \)  
for every \( i, j \in [1, n] \), \( j < i \), \( (C, c) \in \{(X, x), (Y, y), (Z, z)\} \)  
for every \( i, j \in [1, n] \), \( j < i \), \( (C, c) \in \{(X, x), (Y, y), (Z, z)\} \)  
for every \( t, t' \in T \) s.t. \( t \neq t' \)  
for every \( (C, c) \in \{(X, x), (Y, y), (Z, z)\} \)  
for every \( (C, c) \in \{(X, x), (Y, y), (Z, z)\} \)  
for every \( (t, t') \in (T \times T) \setminus H_1 \)  
for every \( (t, t') \in (T \times T) \setminus V_1 \)  
for every \( (t, t') \in (T \times T) \setminus H_2 \)  
for every \( (t, t') \in (T \times T) \setminus V_1 \)  

\text{Figure 3: Background knowledge for the reduction from the NEXPTIME}^{\text{NP}}\text{-tiling problem.}