ON SOME FRACTIONAL ORDER BINOMIAL SEQUENCE SPACES WITH INFINITE MATRICES

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ABSTRACT. The main purpose of this article is to introduce some new binomial difference sequence spaces of fractional order \( \tilde{\alpha} \) along with infinite matrices. Some topological properties of these spaces are considered along with the Schauder basis and \( \alpha- \), \( \beta- \) and \( \gamma- \)duals of the spaces.

1. INTRODUCTION AND PRELIMINARIES

By \( \omega \), we denote the space of all real valued sequences and any subspace of \( \omega \) is called a sequence space. Let \( c_0, c \) and \( l_\infty \) be the spaces of all null, convergent and bounded sequences respectively which are normed by \( \|x\|_\infty = \sup_k |x_k| \). Again \( l_1, cs, bs \) denote the spaces of absolutely summable, convergent series and bounded series respectively. The space \( l_1 \) is normed by \( \sum_k |x_k| \) and the spaces \( cs, bs \) are normed by \( \sup_{n} |\sum_{k=0}^{n} x_k| \), throughout this paper the summation without limit runs from 0 to \( \infty \) and \( n \in \mathbb{N}^+ = \{0, 1, 2, ...\} \).

The notion of difference sequence spaces was introduced by Kizmaz [16] by considering \( X(\Delta) = \{x = (x_k) : \Delta (x_k) \in X\} \) which is further generalized to \( m^{th} \) order difference sequence space as

\[ \Delta^m(X) = \{x = (x_k) : \Delta^m(x) \in X\} \quad \text{for} \quad X = \{c_0, c, l_\infty\} \]

by Et and Çolak [13] where, \( m \) be a non-negative integer.

Also \( \Delta^m(x) = \Delta^{m-1}(x_k) - \Delta^{m-1}(x_{k+1}) \)

\( \Delta^0(x) = x_k \) and

\[ \Delta^m(x_k) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i} \]

The spaces \( l_\infty, c, c_0 \), are Banach spaces normed by

\[ \|x\|_\Delta = \sum_{i=1}^{n} |x_i| + \sup_k |\Delta^m(x_k)| . \]

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Altay and Basar [2] and Altay et al [3] introduced the following Euler sequence spaces

\[
e^r_0 = \left\{ x = (x_k) \in \omega : \lim_{n \to \infty} \sum_k \binom{n}{k} (1-r)^{n-k} x_k = 0 \right\}
\]

\[
e^r_c = \left\{ x = (x_k) \in \omega : \lim_{n \to \infty} \sum_k \binom{n}{k} (1-r)^{n-k} x_k \text{ exists} \right\}
\]

\[
e^r_{\infty} = \left\{ x = (x_k) \in \omega : \sup_{n \in \mathbb{N}} \left| \sum_k \binom{n}{k} (1-r)^{n-k} x_k \right| < \infty \right\}
\]

where the \(r^{th}\) order Euler mean matrix \(E^r\) is defined as \(E^r = (e^r_{nk})\), with \(0 < r < 1\) and

\[
e^r_{nk} = \begin{cases} \binom{n}{k} (1-r)^{n-k} k, & \text{if } 0 \leq k \leq n \\ 0, & \text{if } k > n \end{cases}
\]

Altay and Polat [1] introduced and studied the Euler difference sequence spaces \(Z(\Delta)\), for \(Z \in \{e^r_0, e^r_c, e^r_{\infty}\}\). Polat and Başar [18] further extended those space to \(Z(\Delta^{(m)}) = \{x = (x_k) : (\Delta^{(m)}(x_k)) \in Z, m \in \mathbb{N}\}\)
where \(Z \in \{e^r_0, e^r_c, e^r_{\infty}\}\) and \(\Delta^{(m)} = \delta^{(m)}_{nk}\) is a triangle defined by

\[
\delta^{(m)}_{nk} = \begin{cases} (-1)^{n-k} \binom{m}{n-k}, & \text{if } \max\{0, n-m\} \leq k \leq n \\ 0, & \text{if } 0 \leq k \leq \max\{0, n-m\} \text{ or } k > n \end{cases}
\]

The difference sequence spaces still attracted various mathematicians. In 2016 Bisgin [8, 9] introduced the Binomial sequence spaces

\[
b^r_s = \left\{ x = (x_k) \in \omega : \lim_{n \to \infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} x_k = 0 \right\}
\]

\[
b^r_c = \left\{ x = (x_k) \in \omega : \lim_{n \to \infty} \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} x_k \text{ exists} \right\}
\]

\[
b^r_{\infty} = \left\{ x = (x_k) \in \omega : \sup_{n \in \mathbb{N}} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} x_k \right| < \infty \right\}
\]

with introducing the binomial matrix \(B^r_{nk} = (b^r_{nk})\), defined by:

\[
b^r_{nk} = \begin{cases} \frac{1}{(s+r)^n} \binom{n}{k} s^{n-k} k, & \text{if } 0 \leq k \leq n \\ 0, & \text{if } k > n \end{cases}
\]

where \(r, s \in \mathbb{R}\) and \(r + s \neq 0\)

For \(r + s = 1\), the Binomial matrix reduces to Euler matrix \(E^r\).

Further Meng and Song [17] introduced the sequence space

\[
Z(\nabla^m) = \{x = (x_k) : (\nabla^{(m)}(x_k)) \in Z\},
\]

of order \(m\), for \(Z \in \{b^r_0, b^r_c, b^r_{\infty}\}\) and \(\nabla^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k-i}\).

For a positive fraction \(\tilde{a}\), Baliarsingh & Dutta ([4, 5, 6, 7, 10, 11]) they introduced
the difference operator \( \Delta^{\tilde{\alpha}} \)
\[
\Delta^{\tilde{\alpha}}x_k = \sum_i (-1)^i \frac{\Gamma (\tilde{\alpha} + 1)}{i!\Gamma (\tilde{\alpha} - i + 1)} x_{k-i}
\tag{1.1}
\]
with its inverse as
\[
\Delta^{-\tilde{\alpha}}x_k = \sum_i (-1)^i \frac{\Gamma (-\tilde{\alpha} + 1)}{i!\Gamma (-\tilde{\alpha} - i + 1)} x_{k-i}
\tag{1.2}
\]
\( \Delta^{(\tilde{\alpha})} \) can be expressed as a triangle
\[
(\Delta^{\tilde{\alpha}})_{nk} = \begin{cases}
(\lambda_{n-k}-\lambda_{n-k-1}) & \text{if } 0 \leq k \leq n \\
0 & \text{if } k > n
\end{cases}
\]
where \( \Gamma (m) \) is a Gamma function of all real numbers \( m \notin \{0, -1, -2, \ldots\} \), with
\[
\Gamma (m) = \int_{0}^{\infty} e^{-x} x^{m-1} dx
\tag{1.3}
\]
Now let \( \lambda = (\lambda_k)_{k=0}^{\infty} \) be a strictly increasing sequence of positive reals tending to infinity, that is \( 0 < \lambda_0 < \lambda_1 < \ldots \) and \( \lambda_k \to \infty \) as \( k \to \infty \). Mursaleen and Noman [14] introduced the sequence spaces \( l_{p}^{\lambda} \) and \( l_{\infty}^{\lambda} \) of non-absolute type as the spaces of all sequences whose \( \Lambda \)-transforms are in the spaces \( l_{p} \) and \( l_{\infty} \) respectively.

where
\[
\lambda_{nk} = \begin{cases}
\frac{\lambda_{k}-\lambda_{k-1}}{\lambda_{n}} & \text{if } 0 \leq k \leq n \\
0 & \text{if } k > n
\end{cases}
\]

2. Main Results

Now we define the product matrix \( \Lambda \left( B^{r,s} \left( \Delta^{\tilde{\alpha}} \right) \right) \) and obtain their inverses and introduce binomial difference sequence spaces of fractional order \( \tilde{\alpha} \), \( [c_0]_{\Lambda(B^{r,s}(\Delta^{\tilde{\alpha}}))} \), \( [c]_{\Lambda(B^{r,s}(\Delta^{\tilde{\alpha}}))} \), \( [l_{\infty}]_{\Lambda(B^{r,s}(\Delta^{\tilde{\alpha}}))} \) and give some topological properties of the spaces. Combining the infinite matrix \( \Lambda \), binomial matrix \( B^{r,s} \) and the difference operator \( \Delta^{\tilde{\alpha}} \), the product matrix is defined as
\[
(\Lambda \left( B^{r,s} \left( \Delta^{\tilde{\alpha}} \right) \right))_{nk} = \begin{cases}
\sum_{i=k}^{n} \frac{1}{x_n} \frac{\Gamma (\tilde{\alpha} + 1)}{(i-k)!\Gamma (\tilde{\alpha} - i + k + 1)} s^{i} x_{n-i}, & \text{if } 0 \leq k \leq n \\
0, & \text{if } k > n
\end{cases}
\tag{2.1}
\]
Equivalently
\[(\Lambda (B^{r,s} (\Delta \bar{a})))^{-1} = \begin{pmatrix}
\frac{\lambda_0-\lambda_1}{\lambda_0} & 0 & 0 & \ldots \\
\frac{\lambda_0-\lambda_1}{\lambda_0} \frac{(s-\tilde{a}r)}{(s+r)} & \frac{\lambda_1-\lambda_0}{\lambda_1} & 0 & \ldots \\
\frac{\lambda_0-\lambda_1}{\lambda_2} \frac{1}{(s+r)^2} (s^2 - 2\tilde{a}sr + \frac{\tilde{a}(\tilde{a}-1)}{2} r^2) & \frac{\lambda_1-\lambda_0}{\lambda_2} \frac{1}{(s+r)^2} (2sr - \tilde{a}r^2) & 0 & \ldots \\
\vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\] (2.2)

where * means \(\frac{\lambda_2-\lambda_k}{\lambda_2} \frac{1}{(s+r)^2 r^2}\)

Now we are interested the inverse of above mentioned matrices.

Before proving certain theorems we now quote certain lemmas which will be used in sequel.

**Lemma 2.1.** The inverse of the difference matrix \(\Delta \bar{a}\) is given by the triangle
\n\[(\Delta^{-\bar{a}})_{nk} = \begin{cases} (-1)^{n-k} \frac{\Gamma(-\tilde{a}+1)}{(n-k)! \Gamma(-\tilde{a}-n+k+1)} & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k > n \end{cases}
\]

**Lemma 2.2.** The inverse of the binomial matrix \(B^{r,s}\) is given by the triangle
\n\[(B^{r,s})^{-1}_{nk} = \begin{cases} (-1)^{n-k} (s+r)^k \binom{n}{k} s^{n-k} r^{-n} & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k > n \end{cases}
\]

**Proof.** The proof can be obtained easily using the method as provided in [7, 10] and hence omitted.

**Lemma 2.3.** The infinite matrix inverse
\n\[\Lambda^{-1} = \lambda^{-1}_{nk} = \begin{cases} (-1)^{n-k} \frac{\lambda_0}{\lambda_0-\lambda_{k-1}} & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k > n \end{cases}
\]

**Theorem 2.4.** The inverse of the product matrix \((\Lambda (B^{r,s} (\Delta \bar{a})))\) is given by
\n\[(\Lambda (B^{r,s} (\Delta \bar{a})))^{-1}_{nk} = \begin{cases} (s+r)^k \sum_{i=k}^{n} (-1)^{n-k} \frac{\lambda_0}{\lambda_0-\lambda_{k-1}} \binom{i}{k} \frac{\Gamma(-\tilde{a}+1)}{(n-i)! \Gamma(-\tilde{a}-n+i+1)} r^{-i} s^{i-k} & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k > n \end{cases}
\]

**Proof.** The result can be easily obtained by using lemma 2.1, lemma 2.2 and lemma 2.3.

Now we define the sequence spaces
\n\[(\Lambda (B^{r,s} (\Delta \bar{a})))_0, \Lambda (B^{r,s} (\Delta \bar{a})): (\Lambda (B^{r,s} (\Delta \bar{a})))_\infty, \tilde{I}(p) as follows :
\]
\n\[\Lambda (B^{r,s} (\Delta \bar{a})))_0 = [c]_{\langle \Lambda (B^{r,s} (\Delta \bar{a}))) \rangle} = \left\{ x = (x_k) \in \omega : \lim_{n \to \infty} \sum_{k=0}^{n} i=k \frac{1}{\lambda_0} (\lambda_0-\lambda_{k-1}) (-1)^{i-k} \frac{1}{(s+r)^n} \binom{n}{r-i} s^{n-i} x_k = 0 \right\}
\]

\n\[\Lambda (B^{r,s} (\Delta \bar{a})))_\infty = [c]_{\langle \Lambda (B^{r,s} (\Delta \bar{a}))) \rangle} = \left\{ x = (x_k) \in \omega : \lim_{n \to \infty} \sum_{k=0}^{n} i=k \frac{1}{\lambda_0} (\lambda_0-\lambda_{k-1}) (-1)^{i-k} \frac{1}{(s+r)^n} \binom{n}{r-i} s^{n-i} x_k \right\}
\]
Define the sequence \( y \) and \( \lambda \)

\[
\|x\|_{X(\Delta)} = \|y\|_{\Delta},
\]

\[
\|x\|_{X(\Delta)} = \|y\|_{\Delta},
\]

where \( X \in \{ (\Lambda (B^{r,s}(\Delta)), \Lambda (B^{r,s}(\Delta)))_0, (\Lambda (B^{r,s}(\Delta)), (\Lambda (B^{r,s}(\Delta)))_\infty \} \).

**Proof.** The proof is a routine verification and hence omitted.

**Theorem 2.6.** The sequence spaces \( (\Lambda (B^{r,s}(\Delta)), \Lambda (B^{r,s}(\Delta)))_0, (\Lambda (B^{r,s}(\Delta)), (\Lambda (B^{r,s}(\Delta)))_\infty \) are linearly isomorphic to \( c_0, c, l_\infty \), respectively.

**Proof.** The result will be proved for the space \( (\Lambda (B^{r,s}(\Delta)), \Lambda (B^{r,s}(\Delta)))_0 \).

For other spaces the results can follow in a similar manner.

Now define a mapping \( T : (\Lambda (B^{r,s}(\Delta)), \Lambda (B^{r,s}(\Delta)))_0 \to c_0 \) by \( x \to y = Tx \).

Clearly whenever \( Tx = 0 \), \( T \) is linear.

Which implies \( T \) is injective.

Let \( y \in c_0 \), we define a sequence \( x = (x_k) \) by

\[
x_k = \sum_{i=0}^{n} (s+r)^i \sum_{j=1}^{n} \left( -1 \right)^{k-i} \frac{\lambda_k}{\lambda_i - \lambda_{i-1}} \left( \frac{j}{i} \right) \frac{\Gamma (-\tilde{\alpha} + 1)}{(k-j)!\Gamma (-\tilde{\alpha} - k + j + 1)} r^{-j} s^{k-i} y_i
\]

using theorem 2.4. Then we have

\[
\lim_{n \to \infty} \left( \Lambda (B^{r,s}(\Delta)) x \right)_n = \lim_{n \to \infty} \sum_{j=0}^{n} \sum_{i=j}^{n} \left( -1 \right)^{i-j} \frac{1}{\lambda_n} \left( \frac{1}{s+r} \right)^i \left( \frac{n}{n-i} \right) \frac{\Gamma (\tilde{\alpha} + 1)}{(i-j)!\Gamma (\tilde{\alpha} + j + 1)} r^{-j} s^{k-i} x_j
\]

\[
= \lim_{n \to \infty} y_n = 0.
\]

Therefore

\[
x \in (\Lambda (B^{r,s}(\Delta)))_0
\]

and \( y = Tx \)

Which implies \( T \) is surjective and norm preserving.
i.e. \( (\Lambda(B^{r,s}(\Delta^s)))_0 \cong c_0 \).

3. Schauder Basis

This section deals with Schauder basis for the sequence spaces \( (\Lambda(B^{r,s}(\Delta^s)))_0 \) and \( (\Lambda(B^{r,s}(\Delta^s))) \). A sequence \((x_k)\) of a normed space \((X, \| \cdot \|)\) is called a Schauder basis if for every \( u \in X \) there exist an unique sequence of scalars \((a_k)\) such that

\[
\lim_{n \to \infty} \left\| u - \sum_{k=0}^{n} a_k x_k \right\| = 0
\]

Define the sequence \((k)\theta^{r,s} = (\theta^{r,s}_n)_{n \in \mathbb{N}}\) by

\[
(k)\theta^{r,s}_n = \begin{cases} (s + r)^k \sum_{i=k}^{n} (-1)^{n-k} \frac{\lambda_i - \lambda_{k-1}}{\lambda_k} \binom{i}{k} \frac{\Gamma(-\alpha+1)}{\Gamma(-\alpha+n+1)} r^{-i} s^{i-k} & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k > n \end{cases}
\]

for each \( k \in \mathbb{N} \).

**Theorem 3.1.** The sequence \((k)\theta^{r,s}\) is a Schauder basis for the sequence space \( (\Lambda(B^{r,s}(\Delta^s)))_0 \) and every \( x \in (\Lambda(B^{r,s}(\Delta^s)))_0 \) has unique representation of the form

\[
x = \sum_{k} \sigma^{r,s(k)} k \theta^{r,s}_k
\]

where \( \sigma^{r,s}_k = [\Lambda(B^{r,s}(\Delta^s)) x]_k \) for each \( k \in \mathbb{N} \).

**Proof.** Using the definition of \( (\Lambda(B^{r,s}(\Delta^s))) \) and \((k)\theta^{r,s}\), we can easily verify that

\[
((\Lambda(B^{r,s}(\Delta^s))) (k)\theta^{r,s}) = \epsilon^{(k)} \in c_0,
\]

where \( \epsilon^{(k)}\) is a sequence with 1 in the \( k \)th place and zeros elsewhere. So, the inclusion \((k)\theta^{r,s} \in [\Lambda(B^{r,s}(\Delta^s)) x]_0\).

The set \( \{\epsilon^{(k)} : k \in \mathbb{N}\}\) is the Schauder basis for the space \( c_0 \). Because the isomorphism \( T \) defined by \( x \to y = Tx \) by (see Theorem 2.6 from the space \( (\Lambda(B^{r,s}(\Delta^s)))_0 \) to \( c_0\) is onto, therefore the inverse image of the basis of space \( c_0 \) forms the basis of \( (\Lambda(B^{r,s}(\Delta^s)))_0 \), i.e.

\[
\lim_{n \to \infty} \left\| x - \sum_{k=0}^{n} \sigma^{r,s(k)} k \theta^{r,s}_k \right\| = 0, \quad x \in (\Lambda(B^{r,s}(\Delta^s)))_0.
\]

To verify the uniqueness of the representation assume that \( x = \sum_{k} \mu^{r,s(k)} k \theta^{r,s}_k \) then we have

\[
[\Lambda(B^{r,s}(\Delta^s)) x]_k = \sum_{k} \mu^{r,s(k)} (\Lambda(B^{r,s}(\Delta^s))(k)\theta^{r,s})_k = \sum_{k} \mu^{r,s(k)} \epsilon^{(k)} = \mu^{r,s}_k.
\]

This contradict to our assumption that \( \mu^{r,s}_k = [\Lambda(B^{r,s}(\Delta^s)) x]_k \) for each \( k \in \mathbb{N} \). \( \square \)
Theorem 3.2. Define $\eta = \eta_k$ by

$$\eta_k = \sum_{i=0}^{k} (s + r)^i \sum_{j=i}^{k} (-1)^{n-i} \binom{j}{i} \frac{\Gamma(-\tilde{\alpha} + 1)}{(n-j)!(\tilde{\alpha} - n + j + 1)} r^{-j} s^{j-i}, ~ k, n \in \mathbb{N}$$

and $\lim_{k \to \infty} \sigma_k^{r,s} = l$. Then the set $\{\eta, \theta^{r,s}, \theta^{r,s}, \ldots\}$ is a Schauder basis for the space $(\Lambda \left(B^{r,s} \left(\Delta^{\tilde{\alpha}}\right)\right))_c$ and every $x \in (\Lambda \left(B^{r,s} \left(\Delta^{\tilde{\alpha}}\right)\right))_0$ has a unique representation of the form

$$x = l\eta + \sum_{k} (\sigma_k^{r,s} - l)^{(k)} \theta^{r,s}.$$

Proof. The proof as it is similar to previous theorem.  

Corollary 3.3. The sequence spaces $(\Lambda \left(B^{r,s} \left(\Delta^{\tilde{\alpha}}\right)\right))_c$ and $(\Lambda \left(B^{r,s} \left(\Delta^{\tilde{\alpha}}\right)\right))_0$ are separable.

Proof. The result follows from the theorems 2.5, 3.1, 3.2.  

4. $\alpha, \beta$ AND $\gamma$ DUALS

This section deals with $\alpha-$, $\beta-$ and $\gamma-$duals of $(\Lambda_0 \left(B^{r,s} \left(\Delta^{\tilde{\alpha}}\right)\right))$ and $\left(\Lambda_c \left(B^{r,s} \left(\Delta^{\tilde{\alpha}}\right)\right)\right)$. For the sequence spaces $X$ and $Y$, define multiplier sequence space $M(X, Y)$ by

$$M(X, Y) = \{u = (u)_k \in \omega : ux = (u_k x_k) \in Y, \text{ whenever } x = (x_k) \in X\}$$

Let $\alpha-$, $\beta-$ and $\gamma-$duals be denoted by

$X^\alpha = M(X, l_1)$, $X^\beta = M(X, cs)$, $X^\gamma = M(X, bs)$ respectively.

Throughout $\tau$ will denote the collection of all finite subsets of $\mathbb{N}$. We consider $K \in \tau$.

We now quote the following results which will be used for finding the duals.

\[
\sup_{k \in \tau} \sum_{n} \left| \sum_{k \in K} a_{n,k} \right| < \infty \tag{4.1}
\]

\[
\sup_{n \in N} \sum_{k} |a_{n,k}| < \infty \tag{4.2}
\]

\[
\lim_{n \to \infty} a_{n,k} = a_k, \quad \text{for each } k \in N \tag{4.3}
\]

\[
\lim_{n \to \infty} \sum_{k} a_{n,k} = a \tag{4.4}
\]

\[
\lim_{n \to \infty} \sum_{k} |a_{n,k}| = \sum_{k} \lim_{n \to \infty} |a_{n,k}| \tag{4.5}
\]

Lemma 4.1. [19] Let $A = (a_{n,k})$ be an infinite matrix, then

1. $A \in (c_0 : l_1) = (c : l_1) = (l_\infty : l_1)$ iff (4.1) holds.
2. $A \in (c_0 : c)$ iff (4.2), (4.3) hold.
3. $A \in (c : c)$ iff (4.2), (4.3), (4.4) hold.
4. $A \in (l_\infty : c)$ iff (4.3) and (4.5) hold.
5. $A \in (c_0 : l_\infty) = (c : l_\infty) = (l_\infty : l_\infty)$ iff (4.2) holds.
Therefore we deduce that
$$D_{1}^{r,s} = \left\{ d = (d_k) \in \omega : \sup_{k \in \mathbb{N}} \left( \sum_{i \in K} (s + r)^k(-1)^{n-k} \frac{\lambda_k}{\lambda_i - \lambda_{i-1}} \sum_{j=k}^{n} \left( \begin{array}{c} j \\ k \end{array} \right) \frac{\Gamma(-\bar{\alpha} + 1)}{(n-j)!\Gamma(-\bar{\alpha} - n + j + 1)} s^{j-k} r^{j-k} d_j \right) \right\}$$

Theorem 4.2. The $\alpha-$duals of the spaces $(\Lambda_0 \left( B^{r,s} (\Delta^{(\bar{\alpha})}) \right), (\Lambda_c \left( B^{r,s} (\Delta^{(\bar{\alpha})}) \right))$ and $(\Lambda_{\infty} \left( B^{r,s} (\Delta^{(\bar{\alpha})}) \right))$ is the set

$$D_{1}^{r,s} = \left\{ (\Lambda_0 \left( B^{r,s} (\Delta^{(\bar{\alpha})}) \right)), (\Lambda_c \left( B^{r,s} (\Delta^{(\bar{\alpha})}) \right)) \right\}$$

Proof. Considering $x = (x_k)$ as in 2.4, let $d = (d_k) \in \omega$ define

$$d_{n,x} = \sum_{k=0}^{n} (s + r)^k(-1)^{n-i} \frac{\lambda_k}{\lambda_i - \lambda_{i-1}} \sum_{j=i}^{n} \left( \begin{array}{c} j \\ i \end{array} \right) \frac{\Gamma(-\bar{\alpha} + 1)}{(n-j)!\Gamma(-\bar{\alpha} - n + j + 1)} s^{j-i} d_n y_i$$

where $D_{r,s} = (d_{r,s})$ is defined by

$$d_{r,s} = \left\{ \begin{array}{ll}
(s + r)^k \frac{\lambda_k}{\lambda_i - \lambda_{i-1}} (-1)^{n-k} \sum_{j=i}^{n} \left( \begin{array}{c} j \\ k \end{array} \right) \frac{\Gamma(-\bar{\alpha} + 1)}{(n-j)!\Gamma(-\bar{\alpha} - n + j + 1)} s^{j-k} r^{j-k} d_n, & \text{if } 0 \leq k \leq n \\
0, & \text{if } k > n
\end{array} \right.$$

Therefore we deduce that

$$d_k = (d_{n,x}) \in \mathbb{L}_1$$

whenever $x \in (\Lambda_0 \left( B^{r,s} (\Delta^{(\bar{\alpha})}) \right))$ or $x \in (\Lambda_c \left( B^{r,s} (\Delta^{(\bar{\alpha})}) \right))$

or $x \in (\Lambda_{\infty} \left( B^{r,s} (\Delta^{(\bar{\alpha})}) \right))$ if and only if $D^{r,s} y \in \mathbb{L}_1$ whenever $y \in \mathbb{C} \& l_{\infty}$, which implies that $d = (d_n) \in \left[ \Lambda_0 \left( B^{r,s} (\Delta^{(\bar{\alpha})}) \right) \right]^{\hat{\alpha}} = \left[ \Lambda_c \left( B^{r,s} (\Delta^{(\bar{\alpha})}) \right) \right]^{\hat{\alpha}}

\begin{equation}
= \left[ \Lambda_{\infty} \left( B^{r,s} (\Delta^{(\bar{\alpha})}) \right) \right]^{\hat{\alpha}}
\end{equation}

if and only if $D^{r,s} \in (c_0 : l_1) = (c : l_1) = (l_{\infty} : l_1)$ by lemma 4.1(1), we obtain

$$\sup_{k \in \mathbb{N}} \left| \sum_{i=0}^{n} (s + r)^k(-1)^{n-i} \frac{\lambda_k}{\lambda_i - \lambda_{i-1}} \sum_{j=k}^{n} \left( \begin{array}{c} j \\ k \end{array} \right) \frac{\Gamma(-\bar{\alpha} + 1)}{(n-j)!\Gamma(-\bar{\alpha} - n + j + 1)} s^{j-k} r^{j-k} d_k \right| < \infty$$

Thus we have

$$\left( \Lambda_0 \left( B^{r,s} (\Delta^{(\bar{\alpha})}) \right) \right) = \left( \Lambda_c \left( B^{r,s} (\Delta^{(\bar{\alpha})}) \right) \right) \right\}^{\hat{\alpha}} = \left( \Lambda_{\infty} \left( B^{r,s} (\Delta^{(\bar{\alpha})}) \right) \right) = \mathbb{D}^{r,s}_{1}$$

$\square$

Theorem 4.3. Now we define the sets $D_{2}^{r,s}, D_{3}^{r,s}$ and $D_{4}^{r,s}$ by

$$D_{2}^{r,s} = \left\{ d = (d_k) \in \omega : \sup_{n \in \mathbb{N}} \sum_{K} |t_{n,k}^{r,s}| < \infty \right\}$$

$$D_{3}^{r,s} = \left\{ d = (d_k) \in \omega : \lim_{n \to \infty} t_{n,k}^{r,s} \text{ exists for all } k \in \mathbb{N} \right\}$$

$$D_{4}^{r,s} = \left\{ d = (d_k) \in \omega : \lim_{n \to \infty} \sum_{K} t_{n,k}^{r,s} \text{ exists} \right\}$$

where

$$t_{n,k}^{r,s} = \left\{ \begin{array}{ll}
\sum_{i=k}^{n} (s + r)^k(-1)^{n-k} \frac{\lambda_k}{\lambda_i - \lambda_{i-1}} \sum_{j=k}^{n} \left( \begin{array}{c} j \\ k \end{array} \right) \frac{\Gamma(-\bar{\alpha} + 1)}{(n-j)!\Gamma(-\bar{\alpha} - n + j + 1)} s^{j-k} r^{j-k} d_k, & \text{if } 0 \leq k \leq n \\
0, & \text{if } k > n
\end{array} \right.$$
Theorem 4.4. Let \( \lambda_0 (B^{r,s} (\Delta (\bar{a}))))^\beta = D_2^{r,s} \cap D_3^{r,s} \), \( \lambda_c (B^{r,s} (\Delta (\bar{a}))))^\beta = D_2^{r,s} \cap D_3^{r,s} \cap D_4^{r,s} \), and \( \lambda_\infty (B^{r,s} (\Delta (\bar{a}))))^\beta = D_3^{r,s} \cap D_4^{r,s} \).

Proof. (i) Let \( d = (d_k) \in \omega \) and \( x = (x_k) \) is defined as in (2.4). Then

\[
\sum_{k=0}^n d_k x_k = \sum_{k=0}^n d_k \sum_{i=0}^k (s+r)^i y_i \frac{(j)^i}{i!} \sum_{j=0}^k \frac{\lambda_i}{\lambda_j} \sum_{j=k}^i \binom{j}{k} \frac{\Gamma(-\hat{\alpha}+1)}{(i-j)!\Gamma(-\hat{\alpha}+j+1)} r^{-j} s^{j-k} y_k
\]

Hence \( dx = (d_k x_k) \in c\omega \) whenever \( x \in (A_0 (B^{r,s} (\Delta (\bar{a}))))^\beta \) and only if \( T^{r,s} y \in c\omega \) whenever \( y \in c_0 \), which implies that \( d = (d_k) \in [A_0 (B^{r,s} (\Delta (\bar{a}))))^\beta \) if and only if \( T^{r,s} \in (c_0 : c) \).

By lemma 4.2, we obtain

\[
[A_0 (B^{r,s} (\Delta (\bar{a}))))]^\beta = D_2^{r,s} \cap D_3^{r,s}.
\]

Then proves for (ii) and (iii) can be obtained in similar manner. \( \square \)

Theorem 4.4. The \( \gamma - duality \) of the spaces \( (A_0 (B^{r,s} (\Delta (\bar{a})))) \), \( (A_c (B^{r,s} (\Delta (\bar{a})))) \), and \( (A_\infty (B^{r,s} (\Delta (\bar{a})))) \) in the set \( D_2^{r,s} \).

Proof. As it is a routine verification we omit the proof. \( \square \)

Now using Geometric sequence spaces formulas in \( (A (B^{r,s} (\Delta (\bar{a})))) \) its written as \( (A (B^{r,s} (\Delta (\bar{a}))))_G \) as

\[
(A (B^{r,s} (\Delta (\bar{a}))))_G = \left( \begin{array}{cccc}
\frac{e}{\lambda_0^2} & 1 & 1 \\
\frac{e}{\lambda_0^3} & 1 & 1 \\
\frac{e}{\lambda_0^4} & 1 & 1 \\
\vdots & \vdots & \vdots \\
\end{array} \right)
\]

(4.6)

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