Non-trivial Steenrod Squares on Prime Knots

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Abstract

We use the work of Lipshitz, Sarkar [9, 12] and Lawson, Lipshitz, Sarkar [6] as well as Wilson and Levine-Zemke [7, 15] to prove that there are prime knots with arbitrarily high Steenrod squares on their (reduced and unreduced) Khovanov homology. On the way, we introduce a natural extension of the Wilson and Levine-Zemke result about ribbon concordances inducing split injections on reduced Khovanov homology.

1 Introduction

In 2014, Lipshitz and Sarkar introduced a stable homotopy refinement of Khovanov homology [10]. For each fixed $j$ it takes the form of a suspension spectrum $X^j$. The cohomology $H^*(X^j)$ of this spectrum is isomorphic to the Khovanov homology $Kh^{*,j}$. In subsequent work (e.g. [11]) they used this refinement to define stable cohomology operations on Khovanov homology. This lead to a refinement of Rasmussen’s $s$-invariant for each non-trivial cohomology operation, and in particular, for the Steenrod squares [11]. In this short note we offer a solution to the following question posed in Lipshitz-Sarkar [12, Question 3]: Are there prime knots with arbitrarily high Steenrod squares on their Khovanov homology? Explicitly, we prove the following theorem:

**Theorem 1.** Given any $n$, there exists a prime knot $P_n$ so that the operation

$$Sq^n : \tilde{Kh}^{i,j}(P_n) \to \tilde{Kh}^{i+n,j}(P_n)$$

is non trivial for some $(i,j)$. Here $\tilde{Kh}$ denotes reduced Khovanov homology.

**Corollary 1.** Given any $n$, there exists a prime knot $P_n$ so that the operation

$$Sq^n : Kh^{i,j}(P_n) \to Kh^{i+n,j}(P_n)$$

is non-trivial, on unreduced Khovanov homology, for some $(i,j)$.

We will use the following analogue for reduced Khovanov homology of a theorem of Wilson and Levine-Zemke [7,15]

**Theorem 2.** Suppose $C$ is a ribbon concordance between knots $K$ and $K'$. Then the induced map $F_C : \tilde{Kh}(K) \to \tilde{Kh}(K')$ is injective.

The strategy of proof is the following. In Section 2, we review the results of Wilson and Levine and Zemke [7,15] showing that ribbon concordances induce split injections on Khovanov homology. In Section 3, we prove the analogue of this theorem for reduced Khovanov homology. In Section 4, we show that any knot is ribbon concordant to a prime knot [5,8]. In Section 5, we collect various results about the naturality of Steenrod squares with respect to births, Reidemeister moves and saddle maps and the behavior of Khovanov stable homotopy type under connected sums. In Section 6 we show that the non triviality of Steenrod squares on composite knots constructed by Lipshitz-Sarkar [6, Corollary 1.4] and [12, Corollary 3.1] propagates to the non-triviality of Steenrod squares on the Khovanov homology of prime knots.

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2 Khovanov Homology and Ribbon Concodances

In this section we review the behavior of Khovanov Homology under ribbon concordances. See [7, 15] for more detail. For background on Khovanov homology see [1]. Unless explicitly stated otherwise, throughout this paper we write $Kh(K)$ to mean $Kh(K;\mathbb{F}_2)$.

**Definition 1.** Let $K_0$ and $K_1$ be links in $S^3$. A concordance from $K_0$ to $K_1$ is a smoothly embedded cylinder in $[0,1] \times S^3$ with boundary $-((0) \times K_0) \cup (\{1\} \times K_1)$. A concordance $C$ is said to be ribbon if $C$ has only index 0 and 1 critical points with respect to the projection $[0,1] \times S^3 \to [0,1]$.

Throughout this paper, we will use the notation $\overline{C}$ to denote the ribbon concordance $C$ upside-down.

**Theorem 3.** [7,15] If $C$ is a ribbon concordance from $K_0$ to $K_1$, then the induced map

$$Kh(C) : Kh(K_0) \to Kh(K_1)$$

is injective, with left inverse $Kh(\overline{C})$. In particular, for any bigrading $(i,j)$ the group $Kh^{i,j}(K)$ is a direct summand of the group $Kh^{i,j}(K_1)$.

The proof of this theorem involves decomposing the cobordism $D := \overline{C} \circ C$ as the disjoint union of the identity cobordism (a cylinder) and sphere components joined to the cylinder by tubes (formed from the ribbons and their duals). For details, consult [7] or [15]. In the next section, we present an analogue of Theorem 3 for reduced Khovanov homology, after reviewing the necessary definitions.

3 The Base-point action and Reduced Khovanov Homology

In this section we review the necessary definitions concerning the reduced Khovanov homology. Our main result of this section is the analogue of Theorem 3 for reduced Khovanov homology. For grading conventions, see [14].

We begin with the definition of the base-point action on Khovanov homology.

**Definition 2.** Fix a diagram of the knot $K$ and pick a base-point $q \in K$ not on any of the crossings. Then we make the Khovanov complex $C_{Kh}(K)$ of $K$ into a module over $\mathbb{F}_2[X]/X^2$ as follows. Generators of the chain groups are complete resolutions of $K$ and a choice of 1 or $X$ for each component of the complete resolution. Multiplication by $X$ is zero if the generator labels the circle containing the base-point with an $X$ and it changes the label of the circle containing 1 from 1 to $X$ if the generator labels the circle containing $q$ by 1. Note that multiplication by $X$ has bidegree $(0,-2)$. That is

$$X : Kh^{i,j}(K) \to Kh^{i+1,j}(K)$$

**Definition 3.** Let $\mathbb{F}$ be the $\mathbb{F}_2[X]/X^2$ module $\mathbb{F}_2$ where $X$ acts trivially. Then define

$$\widetilde{C}_{Kh}(K) := C_{Kh}(K) \otimes_{\mathbb{F}_2[X]/X^2} \mathbb{F}.$$ 

The homology of the complex $\widetilde{C}_{Kh}(K)$ is called reduced Khovanov homology and denoted $\widetilde{Kh}(K)$.

We have the following two results of Shumakovitch [14, Corollaries 3.2.B and 3.2.C]

**Theorem 4.** The action of $X$ on $C_{Kh}(K)$ commutes with the Khovanov differential, so induces a map (also called $X$) on homology. Further,

1. The following sequence is exact:

$$\cdots \xrightarrow{X} Kh^{i,j+2}(K) \xrightarrow{X} Kh^{i,j+1}(K) \xrightarrow{X} Kh^{i,j}(K) \xrightarrow{X} Kh^{i,j-2}(K) \xrightarrow{X} \cdots$$

2. The reduced Khovanov homology over $\mathbb{F}_2$ is isomorphic to the kernel of $X$ (which is the image of $X$ by part 1), and we have the direct sum decomposition

$$Kh^{i,j}(K) \cong \widetilde{Kh}^{i,j-1}(K) \oplus \widetilde{Kh}^{i,j+1}(K)$$
With these preliminaries in mind, we prove Theorem 2 from the introduction.

**Proof of Theorem 2.** By Theorem 3 we know that the map $F_C : Kh(K_0) \to Kh(K_1)$ is a split injection with left inverse $F_C$. By the above remarks, for $a \in \{0, 1\}$,

$$\widetilde{K}h(K_a) \cong \text{Ker}(X : Kh(K_a) \to Kh(K_a)) \cong \text{Im}(X : Kh(K_a) \to Kh(K_a)).$$

Therefore, it is enough to show that the map $F_C$ is a map of $F_2[X]/X^2$ module maps. Indeed, then $F_C|_{\text{Ker}}$ maps $\text{Ker}(X : Kh(K_0) \to Kh(K_0))$ to $\text{Ker}(X : Kh(K_1) \to Kh(K_1))$ and $F_C|_{\text{Ker}}$ maps $\text{Ker}(X : Kh(K_1) \to Kh(K_1))$ to $\text{Ker}(X : Kh(K_0) \to Kh(K_0))$. Further, $F_C|_{\text{Ker}} \circ F_C|_{\text{Ker}} = \text{id}|_{\text{Ker}}$. Therefore $F_C|_{\text{Ker}}$ is a split injection.

Now, any ribbon concordance can be decomposed into births (0-handles) and saddle moves (1-handle attachments). So, to show that the maps induced on Khovanov homology by ribbon concordances and upside-down ribbon concordances respect the $X$ action, it suffices to verify the following.

1. Births and deaths respect the module structure with respect to a base-point not on the circle dying or being born.

2. The isomorphisms of Khovanov homology associated to Reidemeister moves respect the module structure.

3. The maps associated with saddles respect the module structure

Item 1 is clear from the definition of the $X$ action, provided we chose a base-point on the original knot diagram, away from where the births and deaths occur.

Item 2 follows from Proposition 2.2 of Hedden-Ni [4]. Evidently, the homotopy equivalences induced from Reidemeister moves commute with the $X$ action if the Reidemeister moves does not involve a strand moving across a base-point. Therefore it suffices to show that moving a strand across the base-point does not change the action of $X$ on homology. This follows by writing down an explicit chain homotopy between the different base-point actions associated with choosing two marked points, on the same component, on opposite sides of a crossing. These homotopy equivalences appear in [4, Lemma 2.3].

Item 3 reduces to a local calculation in a complete resolution. Either the saddle cobordism merges two components, or splits one component into two. In either case, it is easy to check that the maps involved commute with the $X$ action.

### 4 Knots and Prime Tangles

The main theorem of this section states that any knot is ribbon concordant to a prime knot. We begin with a definition and a convention [2,5,8]

**Definition 4.** Tangle here means a 4-ended (2-strand) tangle embedded in the 3-ball $B = B^3$ with no closed components. We denote such a tangle by $(B, T)$ or just $T$. A tangle $(B, T)$ is prime if

1. Any 2-sphere embedded in $B$ that intersects the knot transversely at two points bounds on one side a three ball $A$ so that $A \cap T$ is homeomorphic to the standard ball arc pair $(D^2 \times [0, 1], 0 \times [0, 1])$.

2. $(B, T)$ is not a rational tangle. Equivalently, $(B, T)$ does not contain any separating disks.

One motivation for the name prime tangle and illustration of their use is indicated by the following:

**Theorem 5.** [8, Lemmas 1, 2] The sum of two prime tangles is a prime knot. The partial sum of two prime tangles is a prime tangle.
For the proof, see [8]. In this paper, we use the notation \( +_p \) for the partial sum of two tangles and the notation \( T_1 + T_2 \) for the sum of two tangles. These operations depend on a choice of which endpoints are identified. In the present work, the operations \( +_p \) and \( + \) mean the operations defined below.

Our interest, in the present paper, in prime tangles stems from the following Theorem.

Theorem 6. \( \[5,8\] \) Any knot is concordant to a prime knot.

We prove this theorem in a sequence of Lemmas.

Lemma 1 (See [5], [2]). For any non-trivial knot \( K \) in \( S^3 \) there is an embedded \( S^2 \) meeting \( K \) transversely in four points separating \( S^3 \) into two three balls \( A \) and \( B \) so that

1. \( (A, A \cap K) \) is a ball with two standard spanning arcs, so is a trivial tangle
2. \( (B, B \cap K) \) is a prime tangle

Lemma 2. The clasp tangle \( Cl \) is a prime tangle.

Proof. Since each of the individual strings that compose the clasp tangle are unknotted, condition 1 in the definition of a prime tangle is automatically satisfied. We just need to verify that the clasp is not a rational tangle. Suppose for the sake of contradiction that it is. Recall that a knot built out of two rational tangles is a two bridge knot. It is a classical fact (originally proved by Schubert, see J. Schultens [13] for a modern proof) that the bridge number of a knot, \( b(K) \), satisfies \( b(K#K') = b(K) + b(K') - 1 \). Further, the only knot with bridge number 1 is the unknot. These two facts together imply that two-bridge knots are prime. However, the numerator closure of the clasp tangle is clearly a connected sum \( 3_1 \# 3_1 \).

Proof of Theorem 6. Since any knot can be decomposed as a connected sum of prime knots and the concordance respects this decomposition, it suffices to prove the result for a knot \( K = K_1\#K_2 \) where \( K_i \) are prime. By Lemma 1, we can find two disjoint three balls \( A_1 \) and \( A_2 \) so that \( T_i = A_i \cap K_i \) is a prime tangle and
$(S^3 \setminus A_i) \cap K_i$ is an untangle. We then position $T_1$ and $T_2$ as shown in figure 1, joining $T_1$ to $T_2$ with a band, obtaining the partial tangle sum $T_1 +_p T_2$. This tangle is prime by Theorem 5. The denominator closure of the resulting tangle is $K_1 \# K_2$. The tangle sum $K_1 +_p K_2 + Cl$ is then a prime knot by Theorem 5. The ribbon concordance, shown below, between the untangle and the clasp tangle $Cl$ establishes the result. $\square$

![Figure 5: Denominator closure of the clasp tangle](image1)

![Figure 6: Denominator closure of $T_2 +_p T_2$](image2)

5 Steenrod Operations and Stable Homotopy Type

In this section we review, in bare bones fashion, the necessary facts about Khovanov stable homotopy type needed in establishing Theorem 1.

We begin with a theorem, which explains how the Khovanov stable homotopy type behaves under the operation of connected sum.

**Theorem 7.** [6, Theorem 2] Let $L_1$ and $L_2$ be based links and let $L_1 \# L_2$ denote their connected sum, the connected sum being taken near the base-point. Then

$$\tilde{X}^j_{Kh}(L_1 \# L_2) \simeq \bigvee_{j_2 + j_2 = j} \tilde{X}^{j_1}(L_1) \wedge \tilde{X}^{j_2}(L_2)$$

Next, we recall the precise naturality statement enjoyed by stable cohomology operations.

**Theorem 8.** [9, Theorem 4] Let $S$ be a smooth cobordism in $[0,1] \times S^3$ from $L_1$ to $L_2$, and let $F_S : Kh^{*,*}(L_1) \to Kh^{*,*+\chi(S)}(L_2)$ be the map associated to $S$. Let $\alpha : H^*(\cdot; \mathbb{F}) \to H^{*+n}(\cdot; \mathbb{F})$ be a stable cohomology operation. Then the following diagram commutes up to sign:

$$
\begin{array}{ccc}
K_h^{i,j}(L_1; \mathbb{F}) & \xrightarrow{\alpha} & K_h^{i+n,j}(L_1; \mathbb{F}) \\
F_2 \downarrow & & \downarrow F_2 \\
K_h^{i,j+\chi(S)}(L_2; \mathbb{F}) & \xrightarrow{\alpha} & K_h^{i+n,j+\chi(S)}(L_2; \mathbb{F})
\end{array}
$$

Recall that the $X$ action of $Kh$ can also be viewed as induced from a merge cobordism $U \sqcup K \to K$ where the unknot is placed near the basepoint. Then, by Theorem 4 the following diagram commutes:

$$
\begin{array}{ccc}
Kh(U \sqcup K; \mathbb{F}) \cong F_2[X]/X^2 \otimes Kh(K; \mathbb{F}) & \xrightarrow{\alpha} & Kh(U \sqcup K; \mathbb{F}) \cong F_2[X]/X^2 \otimes Kh(K; \mathbb{F}) \\
\downarrow m & & \downarrow m \\
Kh(K; \mathbb{F}) \xrightarrow{\alpha} & & \xrightarrow{\alpha} Kh(K; \mathbb{F})
\end{array}
$$

Commutativity of the above diagram is the statement that any stable cohomology operation is a map of $F_2[X]/X^2$ modules. It follows that the analogous diagram to the one in Theorem 8 commutes with Khovanov homology replaced by reduced Khovanov homology.
Lemma 3. [6, Corollary 1.4] For any $n$ there is a knot $K_n$ so that the operations

$$Sq^n : \tilde{Kh}^{i,j}(K_n) \to \tilde{Kh}^{i+n,j}(K_n)$$

and

$$Sq^n : Kh^{i,j}(K_n) \to Kh^{i+n,j}(K_n)$$

are non-zero, for some $(i,j)$.

Proof Sketch. Fix a positive integer $n$ and consider the knot $K = 15_{1127}$. There is (see the proof of Corollary 1.4 [6]) a class $\alpha \in \tilde{Kh}^{-1,0}(K;\mathbb{Z}/2\mathbb{Z})$ so that $Sq^1(\alpha) \neq 0 \in Kh^{0,0}(K;\mathbb{Z}/2\mathbb{Z})$ and $Sq^i(\alpha) = 0$ for $i > 1$.

Then, letting $K_n = K \# K \# \cdots \# K$ ($n$ summands), it follows from Theorem 5 that the class $\beta = \alpha \wedge \cdots \wedge \alpha$ is a class in $\tilde{Kh}^{-n,0}(K_n)$. By the Cartan Formula [3],

$$Sq^n(\beta) = Sq^n(\alpha \wedge \cdots \wedge \alpha) = \sum_{i=0}^n Sq^i(\alpha) \wedge Sq^i(\hat{\alpha} \wedge \cdots \wedge \alpha) = \cdots = Sq^1(\alpha) \wedge \cdots \wedge Sq^1(\alpha) \neq 0$$

Since $Sq^i(\alpha) = 0$ for $i > 1$.

The argument that $Sq^n$ is non-trivial on unreduced Khovanov homology is given in Corollary 1.4 of [6]. We omit the details here and include some of them in the proof of Corollary 1.

Note that the knot $K_n$ in the above theorem is the knot $K$ connect summed with itself $n$ times. As in section 3, we view the connected sum $K_n$ as the denominator closure of the partial tangle sum $K +_p \cdots +_p K$ (see Figure 7).

Figure 7: $K_2 = K \# K$ is the denominator closure of $T_2 +_p T_2$

Figure 8: $(T_2 +_p T_2) \sqcup Unknot$ after isotopy

Figure 9: Another isotopy.

Figure 10: The final stage of the ribbon concordance between $K \# K$ and $(T_2 +_p T_2) + Cl$. Compare with Figure 2.
6 Proof of Theorem 1

In this section, we collect the results from the previous sections together to construct a proof of Theorem 1.

**Proof of Theorem 1.** By Lemma 1, the knot $K = 15_4^{1127}$ can be decomposed as a prime tangle $T_2$ and an untangle $T_1$, so that the denominator closure of $T_2$ is $K$ (this is “K with ears” see [2]). Then, the knot $K_n = K \cdot \cdot \cdot \cdot K$ is the denominator closure of the (prime) tangle $T_2 + p_1 T_2 + p_2 \cdot \cdot \cdot + p_n T_2$, where recall $+p$ denotes the partial sum of tangles. Consider the ribbon concordance $C$ given in Theorem 6, from $K_n$ to $P_n := (T_2 + p_1 T_2 + p_2 \cdot \cdot \cdot + p_n T_2) + Cl$. This is illustrated for $n = 2$ in the sequence Figure 7-Figure 10. By Theorem 5 and Lemma 2, $P_n$ is a prime knot.

By Theorem 2 $F_C : \tilde{Kh}(K_n) \to \tilde{Kh}(P_n)$ is injective with left inverse given by $F_C^\text{upside-down}$ where $C$ is the concordance $C$ upside-down. Therefore $\tilde{Kh}(P_n) = \tilde{Kh}(K_n) \oplus G$ for some complement $G$. Since $\tilde{Kh}(K_n)$ has non trivial $Sq^n$ by Lemma 3, so does $\tilde{Kh}(P_n)$.

Explicitly, Theorem 8 implies that the following commutes (note that the Euler characteristic of a ribbon concordance is 0):

$$
\begin{array}{ccc}
\tilde{Kh}^{-n,0}(K_n) & \xrightarrow{Sq^n} & \tilde{Kh}^{0,0}(K_n) \\
\downarrow F_C & & \downarrow F_C \\
\tilde{Kh}^{-n,0}(P_n) & \xrightarrow{Sq^n} & \tilde{Kh}^{0,0}(P_n)
\end{array}
$$

This immediately implies Theorem 1, since the vertical maps are injective.

Finally, we prove Corollary 1 from the introduction.

**Proof of Corollary 1.** This proof follows closely the proof of [6, Corollary 1.4]. There is a long exact sequence in Khovanov homology induced from the cofiber sequence:

$$\tilde{X}^{i-1}(P_n) \to \tilde{X}^i(P_n) \to \tilde{X}^{i+1}(P_n).$$

The long exact sequence takes the form:

$$\cdots \to \tilde{Kh}^{-i,j+1}(P_n) \to \tilde{Kh}^{i,j}(P_n) \xrightarrow{\pi} \tilde{Kh}^{-i,j-1}(P_n) \to \tilde{Kh}^{-i+1,j+1}(P_n) \to \cdots$$

Since over the field $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$ the Khovanov homology of any knot $K$ is isomorphic to the direct sum $\tilde{Kh}^{-i,j+1}(K; \mathbb{F}) \oplus \tilde{Kh}^{-i,j-1}(K; \mathbb{F})$ of the shifted reduced homology, the map $\pi$ above is surjective. So, there is a class $\gamma \in \tilde{Kh}^{-n,1}(P_n)$ so that $\pi(\gamma) = \beta$, where the class $\beta$ is as in the proof of Theorem 1. Naturality of the Steenrod squares establishes the result.

Remarks: The above proof applies to any stable homotopy refinement of Khovanov homology that satisfies the analogue of Theorems 7 and 8. The idea of the proof also offers an obstruction to ribbon concordance between two knots. If $P$ and $Q$ are knots with a ribbon concordance between them, the Khovanov homology of $P$ is a summand of the Khovanov homology of $Q$ with the same stable cohomology operations as the Khovanov homology of $Q$. 

\[ \text{7} \]
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