1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the $C^*$-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. As usual, we reserve $m, M$ for scalars and $I$ for the identity operator on $\mathcal{H}$. A self-adjoint operator $A$ is said to be positive (written $A \geq 0$) if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$, while it is said to be strictly positive (written $A > 0$) if $A$ is positive and invertible. If $A$ and $B$ are self-adjoint, we write $B \geq A$ in case $B - A \geq 0$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$-isomorphism between the $C^*$-algebra $C(\sigma(A))$ of continuous functions on the spectrum $\sigma(A)$ of a self-adjoint operator $A$ and the $C^*$-algebra generated by $A$ and the identity operator $I$. This is called the functional calculus of $A$. If $f, g \in C(\sigma(A))$, then $f(t) \geq g(t)$ ($t \in \sigma(A)$) implies $f(A) \geq g(A)$ (see [11, p. 3]).

A linear map $\Phi: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It’s said to be unital if $\Phi(I) = I$.

For any strictly positive operator $A, B \in \mathcal{B}(\mathcal{H})$ and $0 \leq v \leq 1$, we write

$$A^v B = \left( A^{-1} V_s B^{-1} \right)^v, \quad A^{\#} B = A^{\frac{1}{2}} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\frac{1}{2}}, \quad A V B = (1 - v) A + v B.$$  

For the case $v = \frac{1}{2}$, we write $!$, $\#$, and $\nabla$, respectively. We use the same notions for scalars.

It is well-known that the arithmetic–geometric mean inequality (in short, AM-GM inequality), with respect to operator order, says that

$$A^\# B \leq A V B.$$  

The Löwner–Heinz theorem [11, Theorem 1.8] says that if $A, B \in \mathcal{B}(\mathcal{H})$ are positive, then for $0 \leq p \leq 1$,

$$A \leq B \Rightarrow A^p \leq B^p.$$  

In general (2) is not true for $p > 1$.

Lin [7] nicely reduced the study of squared operator inequalities to that of some norm inequalities. Actually, he found that a reverse of operator AM-GM inequality can be squared

$$(AVB)^2 \leq K(h)^2 (A\#B)^2,$$  

References

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Corresponding author: Ali Farokhinia

Email addresses: ra.safshekan@yahoo.com (Rahim Safshekan), farokhinia@iaushiraz.ac.ir (Ali Farokhinia)
whenever $A, B \in B(H)$ are two positive operators satisfying $mI \leq A, B \leq MI$ for some scalars $0 < m < M$, and $K(h) = \left( \frac{(h+1)^2}{4h} \right)$ with $h = \frac{M}{m}$. It follows from (2) and (3) that

\[(AVB)^p \leq K(h)^p (A\#B)^p \quad (0 < p \leq 2). \tag{4}\]

It is natural to ask whether inequality (4) is true for $p \geq 2$? Recently, an affirmative answer to this question has been given by Fu and He [2], where it has been proved that

\[(AVB)^p \leq \left( \frac{(M + m)^2}{4^p M^p} \right) (A\#B)^p. \]

The problem of squaring operator inequalities has been studied extensively in the literature. We refer the reader to [4, 8–10, 12] as sample of this work.

As mentioned above, Lin’s method was based on some observations about the operator norm and an arithmetic-geometric mean inequality of Bhatia and Kittaneh (see [7, Lemma 2.2]). This paper intends to square a reverse of operator AM-GM inequality in a different way. Moreover, we square the operator Pólya–Szegő inequality [6, 10].

2. Main Results

To prove our generalized operator AM-GM inequalities, we need several well known lemmas. The first lemma is a simple consequence of the Jensen inequality concerning the convexity of certain power function [11, Theorem 1.4].

**Lemma 2.1.** (Hölder–McCarthy inequality) Let $A \in B(H)$ be a positive operator. Then for any unit vector $x \in H$,

\[\langle Ax, x \rangle^p \leq \langle A^p x, x \rangle \quad (p \geq 1).\]

The second lemma is the converses of Hölder–McCarthy inequality [11, Theorem 1.29].

**Lemma 2.2.** Let $A \in B(H)$ such that $mI \leq A \leq MI$ for some scalars $0 < m < M$. Then for any unit vector $x \in H$,

\[\langle A^2 x, x \rangle \leq K(h) \langle Ax, x \rangle^2,\]

where $K(h) = \left( \frac{(h+1)^2}{4h} \right)$ with $h = \frac{M}{m}$.

The third lemma is a reverse of operator AM–GM inequality, which has been proved in [5, Theorem 1].

**Lemma 2.3.** Let $A, B \in B(H)$ such that $mI \leq A, B \leq MI$ for some scalars $0 < m < M$. Then for any $0 \leq v \leq 1$,

\[AV_v B \leq \frac{mV_\lambda M}{m^2 \lambda M} A\#_v B,\]

where $\lambda = \min \{v, 1 - v\}$. In particular,

\[AVB \leq \sqrt{K(h)} A\#B,\]

where $K(h) = \left( \frac{(h+1)^2}{4h} \right)$ with $h = \frac{M}{m}$.

Our first result is a generalization of the inequality (3).

**Theorem 2.4.** Let $A, B \in B(H)$ such that $mI \leq A, B \leq MI$ for some scalars $0 < m < M$. Then for any $0 \leq v \leq 1$,

\[(AV_v B)^2 \leq K(h) \left( \frac{mV_\lambda M}{m^2 \lambda M} \right)^2 (A\#_v B)^2, \tag{5}\]

where $K(h) = \left( \frac{(h+1)^2}{4h} \right)$ with $h = \frac{M}{m}$, and $\lambda = \min \{v, 1 - v\}$. In particular,

\[(AVB)^2 \leq K(h)^2 (A\#B)^2.\]
Proof. One can see that Lemma 2.3 implies
\[ \langle AV_x Bx, x \rangle \leq \left( \frac{m \nabla M}{m^\# M} A^\#_x Bx, x \right) \]
for any unit vector \( x \in \mathcal{H} \). Taking the square in (6), we have
\[ \langle AV_x Bx, x \rangle^2 \leq \left( \frac{m \nabla M}{m^\# M} A^\#_x Bx, x \right)^2 \]
\[ = \left( \frac{m \nabla M}{m^\# M} \right)^2 \langle A^\#_x Bx, x \rangle^2 \]
\[ \leq \left( \frac{m \nabla M}{m^\# M} \right)^2 \langle (A^\#_x B)^2 x, x \rangle \] (by Lemma 2.1).

On the other hand, \( ml \leq A, B \leq MI \) implies
\[ (1 - v) ml \leq (1 - v) A \leq (1 - v) MI, \]
and
\[ vmI \leq vB \leq vMI. \]
It follows from (8) and (9) that
\[ ml \leq AV_x B \leq MI. \]
By applying Lemma 2.2, we get
\[ \frac{1}{K(h)} \langle (AV_x B)^2 x, x \rangle \leq \langle AV_x B, x \rangle^2. \]
Combining (7) and (10) we infer
\[ \langle (AV_x B)^2 x, x \rangle \leq K(h) \left( \frac{m \nabla M}{m^\# M} \right)^2 \langle (A^\#_x B)^2 x, x \rangle \]
for any unit vector \( x \in \mathcal{H} \). This completes the proof. \( \square \)

Another result of this type is the following one:

Corollary 2.5. Let \( A, B \in \mathcal{B}(\mathcal{H}) \) such that \( ml \leq A, B \leq MI \) for some scalars \( 0 < m < M \). Then for any \( 0 \leq v \leq 1, \)
\[ (AV_x B)^2 \leq \left( K(h) \left( \frac{m \nabla M}{m^\# M} \right)^2 - 1 \right) M^2 I + (A^\#_x B)^2, \]
where \( \lambda = \min \{v, 1 - v\}. \)

Proof. It follows from (5) that
\[ (AV_x B)^2 - (A^\#_x B)^2 \leq \left( K(h) \left( \frac{m \nabla M}{m^\# M} \right)^2 - 1 \right) (A^\#_x B)^2 \]
\[ \leq \left( K(h) \left( \frac{m \nabla M}{m^\# M} \right)^2 - 1 \right) M^2 I, \]
where the second inequality follows from the fact that
\[ A, B \leq MI \quad \Rightarrow \quad A^\#_x B \leq MI^\#_x MI = MI. \]
\( \square \)
As pointed out by Fujii and Nakamura in their paper [3, Theorem 2], if \( A \in \mathcal{B}(\mathcal{H}) \) is a positive operator such that \( mI \leq A \leq MI \) for some scalars \( 0 < m < M \), and \( x \in \mathcal{H} \) is a unit vector, then for any \( p \geq 1 \),

\[
\langle A^p x, x \rangle \leq K(m, M, p) \langle Ax, x \rangle^p,
\]

(11)

where

\[
K(m, M, p) = \frac{mM^p - Mm^p}{(p - 1)(M - m)} \left( \frac{p - 1}{p} \cdot \frac{M^p - m^p}{mM^p - Mm^p} \right)^p.
\]

(12)

We note that \( K(m, M, -1) = K(m, M, 2) = \frac{(M+m)^2}{4Mm} \) is the original Kantorovich constant.

Now, by employing (11) and applying a same arguments as in the proof of Theorem 2.4 we reach the following result.

**Corollary 2.6.** Let \( A, B \in \mathcal{B}(\mathcal{H}) \) such that \( mI \leq A, B \leq MI \) for some scalars \( 0 < m < M \). Then for any \( p \geq 1 \) and \( 0 \leq v \leq 1 \),

\[
(A^p v B)^v \leq K(m, M, p) \left( \frac{m\nabla_v M}{m^p M} \right)^v (A^p v B)^v,
\]

where \( K(m, M, p) \) is defined as in (12), and \( \lambda = \min \{v, 1 - v\} \).

Since \( A^{-1} \nabla_v B^{-1} \leq \frac{m\nabla_v M}{m^p M} A^{-1} v B^{-1} \) and \( (A^{-1} v B^{-1})^{-1} = A^p v B \), it follows that

\[
A^p v B \leq \frac{m\nabla_v M}{m^p M} A^p B.
\]

(13)

The following result concerning (13) may be stated:

**Corollary 2.7.** Let \( A, B \in \mathcal{B}(\mathcal{H}) \) such that \( mI \leq A, B \leq MI \) for some scalars \( 0 < m < M \). Then for any \( p \geq 1 \) and \( 0 \leq v \leq 1 \),

\[
(A^p v B)^v \leq K(m, M, p) \left( \frac{m\nabla_v M}{m^p M} \right)^v (A^p v B)^v,
\]

where \( K(m, M, p) \) is defined as in (12), and \( \lambda = \min \{v, 1 - v\} \).

The following result is interesting in itself as well.

**Proposition 2.8.** Let \( A \in \mathcal{B}(\mathcal{H}) \) be a positive operator with the spectra contained in the interval \( J \). If \( f : J \to \mathbb{R} \) is a convex function, then for any unit vector \( x \in \mathcal{H} \)

\[
\langle A^p x, x \rangle \leq \langle Ax, x \rangle^p + px,
\]

where

\[
\alpha = \sup_{x \in \mathcal{H}} \left\{ \langle A^p x, x \rangle - \langle Ax, x \rangle \langle A^{p-1} x, x \rangle \right\}.
\]

Proof. It is well known that if \( f \) is a convex function on an interval \( J \), then for each point \( (s, f(s)) \), there exists a real number \( C_s \) such that

\[
f(s) + C_s (t - s) \leq f(t)
\]

for all \( t \in J \). If \( f \) is a differentiable at \( s \), then

\[
f(s) + f'(s)(t - s) \leq f(t).
\]

(14)
Since \( f(t) = t^p \) \((p \geq 1)\) is a convex and differentiable function, then from (14) we obtain
\[
s^p + ps^{p-1}(t-s) \leq t^p.
\]
Applying functional calculus we get
\[
A^p + \left( ptA^{p-1} - pA^p \right) \leq t^p I.
\]
Hence for any unit vector \( x \in \mathcal{H} \),
\[
\langle A^p x, x \rangle + \left( pt \langle A^{p-1} x, x \rangle - p \langle A^p x, x \rangle \right) \leq t^p.
\]
(15)
Now, since the spectra of \( \langle Ax, x \rangle \), contained in the interval \( J \), by replacing \( t \) by \( \langle Ax, x \rangle \) in (15), we infer (see also [1, Theorem 2.1])
\[
\langle A^p x, x \rangle + \left( p \langle Ax, x \rangle \langle A^{p-1} x, x \rangle - p \langle A^p x, x \rangle \right) \leq \langle Ax, x \rangle^p.
\]
On the other hand, by Lemma 2.1, we know that
\[
\langle A^p x, x \rangle - \langle Ax, x \rangle \langle A^{p-1} x, x \rangle \geq 0.
\]
Therefore,
\[
\langle A^p x, x \rangle - \langle Ax, x \rangle \langle A^{p-1} x, x \rangle \leq \sup_{x \in \mathcal{H}, \|x\| = 1} \left\{ \langle A^p x, x \rangle - \langle Ax, x \rangle \langle A^{p-1} x, x \rangle \right\}
\]
Consequently,
\[
\langle A^p x, x \rangle \leq \langle Ax, x \rangle^p + px
\]
which completes the proof of this proposition.  \( \Box \)

We now present our next main result.

**Theorem 2.9.** Let \( A, B \in \mathcal{B}(\mathcal{H}) \) such that \( mI \leq A, B \leq MI \) for some scalars \( 0 < m < M \). Then for any \( p \geq 1 \) and \( 0 \leq v \leq 1 \),
\[
(A\nabla_v B)^p \leq \left( \frac{m\nabla_1 M}{m\nabla_1 M} \right)^p (A\nabla_v B)^p + p\beta I,
\]
where \( \lambda = \min \{v, 1-v\} \) and
\[
\beta = \sup_{x \in \mathcal{H}, \|x\| = 1} \left\{ \langle (A\nabla_v B)^p x, x \rangle - \langle A\nabla_v B x, x \rangle \langle (A\nabla_v B)^p x^{-1} x, x \rangle \right\}.
\]
Furthermore,
\[
(A\nabla_v B)^p \leq \left( \frac{m\nabla_1 M}{m\nabla_1 M} \right)^p (A\nabla_v B)^p + \gamma p I,
\]
where
\[
\gamma = \sup_{x \in \mathcal{H}, \|x\| = 1} \left\{ \langle (A\nabla_v B)^p x, x \rangle - \langle A\nabla_v B x, x \rangle \langle (A\nabla_v B)^p x^{-1} x, x \rangle \right\}.
\]
Proof. Employing Proposition 2.8 for two inequalities

\[
\langle AV_x x, x \rangle^p \leq \left( \frac{mV_1 M}{m^{\frac{1}{m}} M} \right)^p \langle (A_{\#} B)^p x, x \rangle,
\]

and

\[
\langle A_{\#} B x, x \rangle^p \leq \left( \frac{mV_1 M}{m^{\frac{1}{m}} M} \right)^p \langle (A_{\#} B)^p x, x \rangle.
\]

\]

It has been shown in [6], that if \( A, B \in \mathcal{B}(H) \) are two positive operators satisfying \( mI \leq A, B \leq MI \) for some scalars \( 0 < m < M \), and \( \Phi : \mathcal{B}(H) \to \mathcal{B}(H) \) is a unital positive linear map, then

\[
\Phi(A) \Phi(B) \leq \sqrt{K(h)} \Phi(A \# B).
\] (16)

Remark 2.10. We give a simple proof of (16). One can write

\[
\Phi(A) \Phi(B) \leq \Phi(A) \nabla \Phi(B) \quad \text{(by (1))}
\]

\[
= \Phi(AV_B)
\]

\[
\leq \Phi(\sqrt{K(h)} A \# B) \quad \text{(by Lemma 2.3)}
\]

\[
= \sqrt{K(h)} \Phi(A \# B)
\]

i.e.,

\[
\Phi(A) \Phi(B) \leq \Phi(A) \nabla \Phi(B) \leq \sqrt{K(h)} \Phi(A \# B)
\]

which actually refines the inequality (16).

We next present the generalizations of (16).

Theorem 2.11. Let \( A, B \in \mathcal{B}(H) \) such that \( mI \leq A, B \leq MI \) for some scalars \( 0 < m < M \). Then for any \( p \geq 1 \),

\[
\langle \Phi(A) \Phi(B) x, x \rangle^p \leq K(h)^{\frac{1}{2}} K(m, M, p) \Phi(A \# B)^p,
\] (17)

and

\[
\langle \Phi(A) \Phi(B) x, x \rangle^p - \Phi(A \# B)^p \leq (K(h)^{\frac{1}{2}} K(m, M, p) - 1) M^p I,
\] (18)

where \( K(m, M, p) \) is defined as in (12).

Proof. It follows from (16) that

\[
\langle \Phi(A) \Phi(B) x, x \rangle^p \leq K(h)^{\frac{1}{2}} \langle \Phi(A \# B) x, x \rangle^p
\]

\[
\leq K(h)^{\frac{1}{2}} \langle \Phi(A \# B)^p x, x \rangle
\] (19)

for any unit vector \( x \in H \).

Since \( mI \leq A, B \leq MI \) and \( \Phi \) is a unital positive linear mapping, then \( mI \leq \Phi(A), \Phi(B) \leq MI \). Thus, \( mI \leq \Phi(A) \Phi(B) \leq MI \). Hence from (11),

\[
\langle (\Phi(A) \Phi(B))^p x, x \rangle \leq K(m, M, p) \langle (\Phi(A) \Phi(B))^p x, x \rangle
\] (20)
for any unit vector $x \in \mathcal{H}$. Combining (19) and (20), we get the desired inequality (17). For (18),

$$\left( \Phi (A) \# \Phi (B) \right)^{p} - \Phi(A \# B)^{p} \leq \left( K(h)^{2}K(m, M, p) - 1 \right) \Phi(A \# B)^{p} \leq K(h)^{2}K(m, M, p) - 1 \ M^{p}I. $$

where we have used the fact that

$$mI \leq \Phi(A \# B) \leq MI.$$

It is immediate to see from (17) and (18) that

$$\left( \Phi (A) \# \Phi (B) \right)^{2} \leq K(h)^{2} \Phi(A \# B)^{2},$$

and

$$\left( \Phi (A) \# \Phi (B) \right)^{2} - \Phi(A \# B)^{2} \leq \left( K(h)^{2} - 1 \right) M^{2}I.$$

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