EEG/MEG forward simulation through h- and p-type finite elements

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Abstract. Electro/Magnetoencephalography (EEG/MEG) is a non-invasive imaging modality, in which a primary current density generated by the neural activity in the brain is to be reconstructed from external electric potential/magnetic field measurements. This work focuses on effective and accurate simulation of the EEG/MEG forward model through the h- and p-versions of the finite element method (h- and p-FEM). The goal is to compare the effectiveness of these two versions in forward simulation. Both h- and p-type forward simulations are described and implemented, and the technical solutions found are discussed. These include, for example, suitable ways to generate a finite element mesh for a real head geometry through the use of different element types. Performances of the two implemented forward simulation types are compared by measuring directly the forward modeling error, as well as by computing reconstructions through a regularized FOCUSS (FOCal Underdetermined System Solver) algorithm. The results obtained suggest that the p-type performs better in terms of the forward modeling error. However, both types perform well in regularized FOCUSS reconstruction.

1. Introduction
Electroencephalography/Magnetoencephalography (EEG/MEG) [6] is a non-invasive imaging modality in which a primary current density generated by the neural activity in the brain is to be reconstructed from external potential/magnetic field measurements. Recovery of the primary current density is an ill-posed inverse problem, and therefore, use of effective inversion [2, 9, 13] and forward modeling [7, 11, 18, 19, 20, 17] methods is necessary in the reconstruction process.

This paper focuses on effective and accurate simulation of the EEG/MEG forward model through the h- and p-versions of the finite element method (h- and p-FEM) [10, 15]. The goal is to compare the effectiveness of these two types of finite elements in forward simulation. The motivation for this study is that, according to the forward model, the electromagnetic field generated by the neural activity is likely to be smooth within the computational domain, and the p-FEM is very effective in discretization of smooth functions.

In the p-FEM, accuracy of the discretization can be controlled by varying the polynomial degree p (smoothness) of the finite element basis functions. In the h-FEM, in contrast, only the element size is varied while the polynomial degree is constant and relatively low. However, despite its effectiveness, the p-FEM is relatively seldom used in practical applications, since it is rather laborious to be implemented; a numerically stable implementation requires use of high-order elements with hierarchic shape functions (local basis functions) as well as specific mesh...
design.

In this work, both \( h \)- and \( p \)-FEM type forward simulations are described and implemented, and the technical solutions found are discussed. Performances of the two implemented forward simulation types are compared by measuring directly the forward modeling error, as well as by computing reconstructions through a regularized FOCUSS (FOCal Underdetermined System Solver) algorithm.

This paper is organized as follows. Section 2 briefly reviews the EEG/MEG inverse problem, the corresponding forward model, forward simulation, and regularized FOCUSS. Section 3 describes the applied finite element types. Section 4 reports the numerical experiments. Section 5 summarizes the results and discusses possible directions for the future work. Finally, section 6 gives the conclusions.

2. EEG/MEG inverse problem
The present EEG/MEG inverse problem \([6]\) is to recover a primary current density \( J^p \) in an open three-dimensional domain \( \Omega \), given an array of noisy external electric potential/magnetic field \((u/B)\) measurements. This is a linear inverse problem; its discrete counterpart is to find a vector \( x \) satisfying an underdetermined and ill-conditioned linear system of the form

\[
y = Lx + e,
\]

(1)

where \( L \) is a so-called lead-field matrix, the vector \( y \) contains the measured data, and \( e \) is an error term caused by the noise in the measurements.

2.1. EEG/MEG forward model
In the standard EEG/MEG forward model, the electric field is assumed to be of the form \( E = -\nabla u \) and the total current density is given by

\[
J = J^p + J^s = J^p - \sigma \nabla u,
\]

(2)

where \( \sigma \) is a given conductivity distribution, and \( J^s = -\sigma \nabla u \) is called a secondary or a volume current density. In this paper, the complete electrode model (CEM) \([16, 12]\) is applied for computation of \( u \). The CEM assumes that a set of contact electrodes \( e_1, e_2, \ldots, e_L \) is attached on the boundary \( \partial \Omega \). The electrode potential values form a voltage vector \( U = (U_1, U_2, \ldots, U_L) \) and the electric potential field \( u \) satisfies the equation

\[
\nabla \cdot (\sigma \nabla u) = \nabla \cdot J^p, \quad \text{in } \Omega,
\]

(3)

as well as the boundary conditions

\[
\sigma \frac{\partial u}{\partial n} \bigg|_{\partial \Omega \setminus \bigcup_{i \neq \ell} e_i} = 0, \quad \int_{e_\ell} \sigma \frac{\partial u}{\partial n} \, ds = 0, \quad \left( u + z_\ell \sigma \frac{\partial u}{\partial n} \right) \bigg|_{e_\ell} = U_\ell,
\]

(4)

with \( \ell = 1, 2, \ldots, L \). Additionally, the Kirchoff’s voltage law \( \sum_{\ell=1}^L U_\ell = 0 \) is assumed to hold.

The weak form of (3) and (4) can be formulated by requiring that \( u \in H^1(\Omega) = \{ w \in L^2(\Omega) : \partial w/\partial r_i \in L^2(\Omega), \; i = 1, 2, 3 \} \) and \( J^p \in H(\text{div}; \Omega) = \{ w \in L^2(\Omega)^3 : \nabla \cdot w \in L^2(\Omega) \} \). According to the general theory of partial differential equations \([4]\), the potential field \( u \), a function that belongs to the most commonly used Sobolev space \( H^1(\Omega) \), is likely to be smooth.

The magnetic field point values, that are measured in MEG, can be obtained through a straightforward differentiation and integration procedure given the electric potential. Namely, the Ampere-Laplace law combined with (2) states that the magnetic field at the point \( r_0 \) is given by the formula

\[
B(r_0) = \int_{\Omega} \left( J^p - \sigma \nabla u \right) \times \frac{r_0 - r}{|r_0 - r|^3} \, dr.
\]

(5)
2.2. Forward simulation
In this paper, the discretized fields corresponding to \( u \in H^1(\Omega) \) and \( J^\nu \in H(\text{div};\Omega) \) as well as their coordinate vectors are defined by

\[
 u_T = \sum_{i=1}^{N_u} z_i \psi_i, \quad J_T^i = \sum_{i=1}^{N_J} x_i \psi_i, \quad \text{and} \quad z = (z_1, z_2, \ldots, z_{N_u})^T, \quad x = (x_1, x_2, \ldots, x_{N_J})^T, \quad (6)
\]

respectively. Here, the functions \( \psi_1, \psi_2, \ldots, \psi_{N_J} \in H^1(\Omega) \) and \( \psi_1, \psi_2, \ldots, \psi_{N_u} \in H(\text{div};\Omega) \) are scalar and vector valued finite element basis functions defined on a shape regular finite element mesh \( T \). Furthermore, since in the CEM the sum of the electrode potentials is assumed to be zero, it is defined that

\[
 U = (U_1, U_2, \ldots, U_L)^T = R \tilde{z}, \quad (7)
\]

where \( R \in \mathbb{R}^{L \times (L-1)} \) is a matrix, whose entries are given by \( R_{i,j} = -R_{j+1,i} = 1 \) for \( j = 1, 2, \ldots, L-1 \), and otherwise \( R_{i,j} = 0 \). The vectors \( x, z, \) and \( \tilde{z} \) are linked through the linear system

\[
 \begin{pmatrix}
 B & C \\
 C^T & G
 \end{pmatrix}
 \begin{pmatrix}
 z \\
 \tilde{z}
 \end{pmatrix} =
 \begin{pmatrix}
 M x \\
 0
 \end{pmatrix},
 \quad (8)
\]

in which the submatrix entries are given by

\[
 M_{i,k} = \int_{\Omega} (\nabla \cdot \psi_k) \psi_i d\Omega, \quad (9)
\]

\[
 B_{i,j} = \int_{\Omega} \sigma \nabla \psi_i \cdot \nabla \psi_j d\Omega + \sum_{\ell} \int_{\ell} \psi_i \psi_j dS, \quad (10)
\]

\[
 C_{i,j} = -\frac{1}{z_1} \int_{e_1} \psi_i dS + \frac{1}{z_{j+1}} \int_{e_{j+1}} \psi_i dS, \quad (11)
\]

\[
 G_{i,j} = \frac{1}{z_j} \int_{e_j} dS + \frac{\delta_{i,j}}{z_{j+1}} \int_{e_{j+1}} dS, \quad (12)
\]

where \( \delta_{i,j} \) denotes the Kronecker delta. The system (8) arises from the Ritz-Galerkin discretization of the weak form of (3) and (4). The vector \( \tilde{z} \) can be solved from (8) through the use of the Schur complement [5] and the electrode potentials can be then obtained by evaluating (7). These two steps can be written as a matrix

\[
 L = -R(G - C^T B^{-1} C)^{-1} C^T B^{-1} M, \quad \text{that satisfies} \quad U = L x. \quad (13)
\]

This is the lead-field matrix for the present EEG inverse problem. In MEG, computation of the lead-field matrix requires, additionally, the use of (5), and the result is somewhat more complicated.

2.3. Regularized FOCUSS reconstruction
The regularized FOCUSS algorithm, that is used in this paper for the inverse problem (1), is a simple iteration defined by \( x^{(0)} = (1,1,\ldots,1)^T \), \( W_k = \text{diag}(|x_1^{(k)}|, \ldots, |x_{N_J}^{(k)}|)^{1/2}, \) \( L_k = LW_k \), and

\[
 x^{(k+1)} = W_k (L_k^T L_k + \frac{\lambda}{2} I)^{-1} L_k^T y, \quad k = 0, 1, \ldots, n, \quad (14)
\]

where \( n \in \mathbb{N} \) and \( \lambda > 0 \) are parameters to be chosen by the user. This algorithm has been described and studied e.g. in [14]. It finds an estimate for the minimizer of the function

\[
 \Psi(x) = ||y - Ax||_2^2 + \lambda ||x||_1. \quad (15)
\]
Regularized FOCUSS finds an estimate, in which the amount of nonzero entries is small. Therefore, it is a suitable reconstruction strategy for finding sparsely distributed and well-localized sources. In this work, a sparse distribution of dipoles is sought using (14).

3. Finite elements

An appropriate implementation of the discretization (6) requires the use of both $H^1(\Omega)$- and $\mathbf{H}(\text{div}; \Omega)$-conforming elements. In this work, tetrahedral meshes and the element types described in the following are applied.

3.1. Lowest order Raviart-Thomas elements

The discretization of the primary current density $\mathbf{J}_T^p \in \mathbf{H}(\text{div}; \Omega)$ is established using the lowest order Raviart-Thomas element [3]. In this element, the $i$th shape function $\varphi_i$ is linear, vanishes precisely at one of the $i$th vertex of the element, and the direction of $\varphi_i$ is constant and given by the normal vector of the face opposite to the $i$th vertex. The resulting global finite element space is a subspace of $\mathbf{H}(\text{div}; \Omega)$, i.e. this element is $\mathbf{H}(\text{div}; \Omega)$-conforming.

3.2. $H^1(\Omega)$-conforming elements with hierarchic shape functions

Computation of the discretized electric potential field $u_T \in H^1(\Omega)$ through $p$-FEM requires the use of higher-order elements, i.e. elements with shape functions of relatively high polynomial degree. Because numerical instability can occur in connection with higher-order elements, it is generally agreed that the shape functions should be hierarchic. In this context, hierarchic means that the shape functions of maximal degree $p$ are also shape functions of maximal degree $p + 1$, and that the number of shape functions not vanishing at the vertices and the sides of the elements is minimal. Hierarchic shape functions are constructed using Legendre polynomials

$$p_k(x) = \frac{1}{2^k k!} \frac{d^k}{dx^k}(x^2 - 1)^k, \quad k \geq 0. \quad (16)$$

This paper adopts a notation [1] in which the vector $\mathbf{t} = [t_1, t_2, t_3, t_4]$ denotes a tetrahedron with oriented edges that belongs to a tetrahedral finite element mesh $T$. The indices $t_1, t_2, t_3,$ and $t_4$ denote the global vertex indices and the sets that contain the vertex, edge, and face indices of $\mathbf{t}$ are denoted by $\mathcal{V}(\mathbf{t})$, $\mathcal{E}(\mathbf{t})$, and $\mathcal{F}(\mathbf{t})$, respectively. These sets can be written explicitly as follows: $\mathcal{V}(\mathbf{t}) = \{t_1, t_2, t_3, t_4\}$, $\mathcal{E}(\mathbf{t}) = \{[t_1, t_2], [t_1, t_3], [t_1, t_4], [t_2, t_3], [t_2, t_4], [t_3, t_4]\}$, and $\mathcal{F}(\mathbf{t}) = \{[t_1, t_2, t_3], [t_1, t_2, t_4], [t_1, t_3, t_4], [t_2, t_3, t_4]\}$.

Hierarchic shape functions can be constructed using two reference tetrahedra $\mathbf{t}_1 = [1, 2, 3, 4]$ (figure 1) and $\mathbf{t}_2 = [1, 3, 2, 4]$ (figure 2) as proposed in [1]. Each of these share the same vertices $\mathbf{v}_1 = (-1, 0, 0)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (0, \sqrt{3}, 0)$, and $\mathbf{v}_4 = (0, 1/\sqrt{3}, 2\sqrt{2}/\sqrt{3})$, but their edges and faces are oriented differently. The reason for the use of more than the standard one reference element is to minimize the difficulty of enforcing the conformity (global continuity), which can be problematic with hierarchic shape functions. Any tetrahedron with oriented edges in an unstructured tetrahedral finite element mesh can be reduced to either of the present two reference tetrahedra through cyclic rotations of the first three or the last three vertex indices, which are equivalent to rotations of two different faces [1].

Hierarchic shape functions of $H^1(\Omega)$-conforming tetrahedral elements are divided into vertex, edge, face, and internal functions. The first three function types vanish at all but one vertices, edges, or faces, respectively, and the internal functions vanish on element boundaries. The vertex functions are first degree polynomials, and the other function types, also known as bubbles, are of higher degree.

The vertex functions of a tetrahedron $\mathbf{t} = [t_1, t_2, t_3, t_4] \in T$ are defined as $\varphi^{(v)} = \lambda_v$ for $v \in \mathcal{V}(\mathbf{t})$, where $\lambda_v$ denotes a barycentric coordinate, i.e. it is a linear polynomial that obtains
the value one at the vertex $i$ and vanishes on the opposite face. The corresponding edge, face, and internal modes of maximal degree $p$ are, respectively, defined as

$$\varphi_j^{(e)} = \lambda_{e_1} \lambda_{e_2} p_j (\lambda_{e_2} - \lambda_{e_1}),$$

(17)

$$\varphi_{k_1 k_2}^{(f)} = \lambda_{f_1} \lambda_{f_2} \lambda_{f_3} p_{k_1} (\lambda_{f_2} - \lambda_{f_1}) p_{k_2} (\lambda_{f_3} - \lambda_{f_1}),$$

(18)

$$\varphi_{\ell_1 \ell_2 \ell_3}^{(t)} = \lambda_{t_1} \prod_{d=1}^3 \left[ \lambda_{t_d+1} p_{\ell_d} (\lambda_{t_{d+1}} - \lambda_{t_1}) \right].$$

(19)

Here, the superscript $e$ denotes an edge $[e_1, e_2] \in \mathcal{E}(t)$ and $f$ denotes a face $[f_1, f_2, f_3] \in \mathcal{F}(t)$, and the subscripts determining the Legendre polynomial degrees are positive or zero integers such that $j \leq p - 2$, $k_1 + k_2 \leq p - 3$, and $\ell_1 + \ell_2 + \ell_3 \leq p - 4$.

A scalar shape function $\varphi$ supported on $t$ and the corresponding reference element shape function $\hat{\varphi}$ are related through the transformation $\varphi = \hat{\varphi} \circ F_t^{-1}$, where $F_t$ is an invertible affine mapping from the corresponding reference tetrahedron onto $t$. Using this transformation, the $H^1(\Omega)$-conforming hierarchic basis functions can be constructed in the usual way using the reference element shape functions.

4. Numerical experiments

Numerical EEG imaging experiments were performed using a head model based on a real human head with 31 electrodes attached on the skin (figures 3–5). This head model consisted of 108914 tetrahedral elements. The conductivity distribution was assumed to consist of four layers: the scalp, skull, cerebrospinal fluid, and the brain. The conductivity distribution inside the head is illustrated in figures 6–8 and the conductivities in the different layers are listed in table 1.

In generation of the mesh, the skull and brain layers were generated first using a tetrahedral mesh generator based on heuristic algorithms. After that, both the scalp and the cerebrospinal

| Layer                  | Conductivity |
|------------------------|--------------|
| scalp                  | 0.33         |
| skull                  | 0.0042       |
| cerebrospinal fluid    | 1            |
| brain                  | 0.33         |

| Table 1. The layers inside the head and their conductivities. |
fluid layers were generated by first adding a layer of prisms to the mesh and then dividing each prism into three tetrahedra. The primary conductivity density inside the head was simulated using the lowest order Raviart-Thomas elements, and the contact impedances were assumed all to be equal to one, i.e. $z_1 = z_2 = \cdots = z_L = 1$.

The first approach used to study the accuracy of the forward simulation was to measure the following relative error:

$$\text{relative error} = \frac{||L_{\text{ref}} - L||_{\text{fro}}}{||L_{\text{ref}}||_{\text{fro}}},$$

where $L$ is a lead-field matrix of the form (13), $L_{\text{ref}}$ is a reference lead-field matrix, and $||L||_{\text{fro}} = (\sum_{i,j} |L_{i,j}|^2)^{1/2}$. The accuracy of the forward simulation was varied using different $h$- and $p$-values for the $H^1(\Omega)$-conforming basis functions. Here, processes where either the mesh is refined or the polynomial degree of the basis functions is increased are called extensions.

| Name | Description | Finite element space dimensions |
|------|-------------|---------------------------------|
| (I)  | $h = h_0, \frac{1}{2} h_0, \ p = 1$ | 20639 154481 |
| (II) | $h = h_0, \ p = 1, 2, 3, 4$ hierarchic | 20639 154481 510449 1197457 |
| (III)| $h = h_0, \ p = 1, 2, 3$ with internal functions | 129553 263395 619363 |
Three extensions (I)–(III) were executed. These are listed in table 2. In the extension (I), the polynomial degree was \( p = 1 \) and the mesh size \( h \) was given the values \( h = h_0 \) and \( h = h_0/2 \). In the extension (II), the polynomial degree was given the values \( p = 1, 2, 3, 4 \) and the mesh size was \( h = h_0 \). And in (III), polynomials of degree \( p = 1, 2, 3 \) were used together with fourth order internal functions, and the mesh size was again \( h = h_0 \). The last lead-field matrix of the extension (II) was used as the reference lead-field matrix \( L_{\text{ref}} \).

The second approach was to compute regularized FOCUSS reconstructions using the algorithm (14). The data \( y \) for the reconstruction was generated according to \( y = L_{\text{ref}}x_{\text{ref}} + e \), where \( x_{\text{ref}} \) is a reference primary current density (figures 10–12) to be reconstructed, and \( e \) is obtained by drawing a random sample from a Gaussian white noise distribution and scaling the sample so that either \( \| e \|_2/\| y \|_2 \approx 0.015 \) (lower noise level) or \( \| e \|_2/\| y \|_2 \approx 0.030 \) (higher noise level). The parameters in (14) were chosen as \( \lambda = 2 \cdot 10^{-8} \) and \( n = 20 \). The reconstructions computed, (a)–(d), are described in detail in table 3. Lower and higher noise levels were used in computation of (a), (b) and (c), (d), respectively. Reconstructions (a), (c) correspond to \( h = h_0, p = 1 \) and (b), (d) to \( h = h_0, p = 3 \) (hierarchic). In each case, the system to be solved in (14) was solved iteratively using the conjugate gradient [3] iteration.

### 4.1. Results

Behavior of the relative error (20) in the extensions (I)–(III) as a function of the finite element space dimension is illustrated in figure 9 on a \( \log_{10} - \log_{10} \) scale. Based on figure 9, it is clear that increasing the value of \( p \) provides, within this numerical framework, a more effective way to enhance the accuracy of the forward simulation than decreasing \( h \); the relative error in the extensions (II) and (III) converges more rapidly than in the extension (I). The best trade-off between the relative error and the dimension of the finite element space is obtained in the extension (II).

The reference primary current density \( x_{\text{ref}} \), a sparse distribution of dipoles, is visualized in figures 10–12. The reconstructions (a)–(d) are illustrated in figures 13–24. In these figures, the cross section area of the arrows is linearly proportional to the intensity of the primary current density. Based on visual inspection, the differences between the reconstructions (a) and (b), computed using the lower noise level, are small but visible. Compared to these, artifacts are more pronounced in the higher noise level case reconstructions (c) and (d), which again differ from each other slightly but visibly.

### 5. Discussion

Numerical experiments, in which the accuracy of the EEG forward simulation was measured using two approaches, were reported in this paper. The performances of the \( h \)-FEM and \( p \)-FEM forward simulations were compared. The first approach was to measure the relative discretization error as described in (20), and the second approach was to compute regularized FOCUSS reconstructions.
through the iteration (14).

The results of the numerical experiments suggest that the performance of the $p$-FEM in forward simulation, when measured as in (20), is better than that of the $h$-FEM. Namely, the extensions (II) and (III) were found to perform better than the extension (I) in terms of converge rate of the relative error. The best trade-off between the relative error and the dimension of the finite element space was obtained in the extension (II), which suggests that the use of fourth order internal functions in the extension (III) is not advantageous.

Based on the results concerning the relative discretization error, it seems that an appropriate discretization of the form (6) can be found by deciding the mesh size based on the a priori knowledge [6] on what is a good choice for dipole size as well as on the knowledge on the applied $H(\text{div}; \Omega)$-conforming basis functions; after that the accuracy of the forward simulation can be controlled effectively by controlling the polynomial degree of the $H^1(\Omega)$-conforming basis functions. EEG reconstruction with real EEG data and with this kind of mesh generation technique could be an interesting target for future work.

It was found that at both applied noise levels, the difference between the reconstructions corresponding to $p = 1$ and $p = 3$ forward simulations were small but visible. The reason for the similarity of the reconstructions at each noise level is that finding just the location of a dipole is a less ill-conditioned inverse problem than e.g. finding the dipole moment and depth at the same time [6]. Comparison of the reconstructions (a) and (b) as well as of (c) and (d) shows that when the level of measurement noise is given the locations of the dipoles found with $p = 1$ or with $p = 3$ are rather similar; however, there is a visible difference in how one of the dipoles in the reference primary current density $\mathbf{x}_{\text{ref}}$ is reconstructed in terms of the dipole moment. For these reasons, another interesting topic for future studies could be to examine whether $p$-FEM provides a suitable way to recover both dipole moment and depth at the same time. Also, adaptive $hp$-FEM simulation techniques, in which $h$- and $p$-values can vary in different parts of the computational domain, could be applied for the forward model in the future.

Unlike many other studies, the present study applied the complete electrode model [16, 12] in numerical experiments of the EEG forward simulation. The complete electrode model provides a way to include the effect of the electrode contact impedance in the forward simulation. Whether this is necessary in some applications and whether there are some real advantages in the use of the complete electrode model are at least partially open questions to be studied in more depth in the future.

Finally, one of the findings of this study was that generation of the thin mesh layers (skin and cerebrospinal fluid) can be done successfully by first adding a layer of prisms to the mesh and then dividing each prism into three tetrahedra. A more sophisticated solution would be to use prism elements in these layers. With such a solution the number of elements would be smaller, which would be advantageous regarding $p$-FEM. With the present solution, it was enough to use

![Figure 9](image.jpg)

**Figure 9.** The relative error (20) in the executed extensions (I)–(III) plotted against the dimension of the finite element space on a $\log_{10}$-$\log_{10}$ scale. The dashed line with diamond markers is the extension (I), the solid line with square markers is the extension (II), and the dash-dot line with circle markers is the extension (III).
only tetrahedral elements and the mesh was still relatively sparse. If a heuristic mesh generator would have been applied for the thin layers, the number of elements would have been much higher.

6. Conclusions
This work compared the performances of $h$- and $p$-type $H^1(\Omega)$-conforming finite elements in EEG/MEG forward simulation. Both types of forward simulation were described and implemented. The complete electrode model was successfully applied. The performance comparison was done by measuring directly the relative discretization error and by computing reconstructions through the regularized FOCUSS algorithm. It was found that in terms of the relative discretization error the performance of the $p$-type FEM was superior to that of the $h$-type. Comparison of the reconstructions revealed only small differences. One important topic for future work is to study mesh generation for $p$-FEM, based on the a priori knowledge on
what is a good choice for dipole size. Another topic is to investigate whether $p$-FEM forward simulation provides a suitable way to recover both dipole moment and depth at the same time. Adaptive $hp$-FEM, the effect of the complete electrode model in forward simulation as well as implementation of prism elements are also interesting targets for future work.

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