Local well posedness for a linear coagulation equation.

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1 Introduction

This paper is part of a program to study the well posedness of classical solutions after the gelation time for a general class of coagulation equations that behave asymptotically as homogeneous kernels.

The classical coagulation equation reads:

$$\partial_t f(x,t) = \frac{1}{2} \int_0^x K(x-y,y)f(x-y,t)f(y,t)dy - \int_0^{\infty} K(x,y)f(x,t)f(y,t)dy$$ (1.1)

This equation can be though as describing the distribution of sizes for a set of particles with size $x$ that aggregate with particles of size $y$, independently distributed, with a rate $K(x,y)$.

It is well known that for kernels $K(x,y)$ that behave asymptotically for large $x,y$ as $(xy)^{\lambda}$ with $1 < \lambda < 2$, the solutions of (1.1) exhibit the phenomenon known as “gelation”, that means that the first moment of $f$, that is formally preserved for the solutions of (1.1), is not any longer preserved after some finite time $t^*$, due to the fact that a macroscopic fraction of particles “escapes” to infinitely large sizes (cf. [7]).

A detailed description of the asymptotics of the function $f(x,t)$ as $x \to \infty$ for solutions exhibiting gelation behaviour is still lacking, except for the case $K(x,y) = x \cdot y$ where (1.1) can be explicitly solved using integral transform (cf. [3], [9], [10]). In order to obtain more information on the asymptotics of the solutions of (1.1) for more general kernels, we have studied in a detailed way in [4] the fundamental solution of the linearized problem obtained considering the evolution of functions $f$ that are close to $\tilde{f}(x) = Ax^{-\frac{1+\lambda}{\lambda}}$ with kernel $K(x,y) = (xy)^{\frac{\lambda}{2}}$, $1 < \lambda < 2$. The reason for considering the evolution near such
power law, is that the interpretation of this distribution in the particle setting above corresponds to a continuous transport of particles emanating from the origin and being transported towards \( x = \infty \) at a constant rate (cf. [4]). This power law was obtained in [11] where explicit particular solutions of the discrete coagulation equations yielding loss of mass have been constructed.

In order to understand solutions of the coagulation equation (1.1) yielding a “flux” of particles towards infinity, it is natural to linearize around \( \bar{f}(x) \), in order to clarify how such transport of particles could take place. We will denote the linearized problem considered in [4] as:

\[
g_t = L[g] \quad g(0) = g_0
\]

where

\[
L(g) = \int_0^{x/2} \left( (x - y)^{-3/2} - x^{-3/2} \right) y^{\lambda/2} g(y) dy
+ \int_0^{x/2} \left( (x - y)^{\lambda/2} g(x - y) - x^{\lambda/2} g(x) \right) y^{-3/2} dy
- x^{-3/2} \int_{x/2}^{\infty} y^{\lambda/2} g(y) dy - 2\sqrt{2}x^{(\lambda-1)/2} g(x).
\] (1.3)

A technical difficulty that arises in the study of the linearization of (1.1) around \( \bar{f}(x) \) is due to the fact that this function is singular near \( x = 0 \). This has several relevant consequences. First, the resulting linearized operator becomes singular near \( x = 0 \), and as a consequence, it has regularizing effects that cannot be expected to take place for the original problem (1.1). As a matter of fact, the linearized operator considered in [4] behaves, locally near a given value of \( x \), as:

\[
g_t = -\left(-D_{xx}\right)^{1/4} g + \text{higher order terms}
\]

In particular, this problem can be considered a nonlocal parabolic equation whose generator has Fourier symbol \(-2\sqrt{2}k\). Clearly, the regularity properties associated to this problem can be expected to be very different from the ones associated to (1.1).

On the other hand, from the physical point of view, the solution \( \bar{f}(x) \) is associated to the presence of a constant flux of particles leaving from the origin, as it can be seen in [4]. Bounded solutions of (1.1) do not have such a constant flux of particles with size \( x = 0 \).

However, in spite of these differences between the linearized problem (1.2) and (1.1), there are good reasons for studying (1.2) in order to understand particle fluxes towards infinity for the nonlinear equation (1.1). The main one is that solutions of (1.1) yielding particle fluxes towards \( x = \infty \) can be expected to behave as one of the solutions \( \bar{f} \) as indicated above for each given time. Moreover, the problem (1.2) can be solved explicitly using the methods in [1] due to its good properties under rescalings. Moreover it is possible to derive detailed estimates of the corresponding fundamental solution in all the regions of the space-time \((x,t)\) (cf. [4]).

Nevertheless, in order to avoid the shortcomings of (1.2) as an approximation of (1.1), it would be more convenient to study the linearization of (1.1) near a smooth bounded
function $f_0(x)$ that behaves asymptotically as $Ax^{-\frac{3q}{2}}$ as $x \to \infty$. The resulting problem would be:

$$g_t = \mathcal{L}(g) , \quad g(0) = g_0$$

with:

$$\mathcal{L}(g) = \int_0^x (x - y)^\frac{q}{2} f_0(x - y) x^\frac{q}{2} g(y) dy - x^\frac{q}{2} f_0(x) \int_0^\infty y^\frac{q}{2} g(y) dy - x^\frac{q}{2} g(x) \int_0^\infty y^\frac{q}{2} f_0(y) dy$$

Problem (1.4) is in some sense closer to (1.1) than (1.2). Indeed, (1.4) does not have regularizing properties at any $x > 0$. On the other hand, bounded solutions of (1.4) yield a zero flux of particles from the origin.

Unfortunately, the solution of (1.4) cannot be obtained explicitly as it has been made in [4]. Moreover, to prove even local solvability in time of (1.4) is not an easy task due to the presence of the integral term $\int_0^\infty y^\frac{q}{2} g(y) dy$. In the absence of this term the local solvability of (1.4) could be easily obtained using a fixed point argument. However, the presence of this integral term makes this problem much harder to solve.

The key idea that will be used in this paper is to solve (1.4) approximating it by means of (1.2) for $x \to \infty$. The operator on the right hand side can be thought as an operator having half derivative at $x = \infty$. As a consequence, the equation (1.4) has some kind of “smoothing effects” for $x = \infty$. The presence of these regularizing effects is more clear in the equation (1.2). Nevertheless, this last equation has regularizing effects for all the values of $x$. Therefore, to approximate the regularizing effects of (1.4) by means of those of (1.2) is something that must be given a precise meaning and it will be given in this paper. Regularising effects in kinetic equations with singular kernels have been previously obtained for Boltzmann equation cf. [2] and [12].

There is another feature of the approximation of (1.4) using (1.2) that is worth mentioning. As indicated above, the function $\tilde{f}(x)$ can be thought as the source of a flux of particles coming from $x = 0$ that are transported towards $x = \infty$. On the contrary, the function $f_0(x)$ does not provide any flux of particles from the origin, although it is associated to a flux of particles transported towards $x = \infty$. If we rewrite these two functions using the change of variables $F = R^\frac{q+1}{2} \tilde{f}(R\xi)$, with $\xi$ of order one and $R \to \infty$, it follows that $\tilde{f}(x) = Ax^{-\frac{3q}{2}}$ and $f_0(x)$ becomes $F_0(\xi) = R^\frac{q+1}{2} \tilde{f}(R\xi)$. Notice that $F_0(\xi) \to \tilde{F}(\xi)$ as $R \to \infty$, for all $\xi > 0$. Such a convergence fails for $\xi \to 0$ or, more precisely, for $x$ of order one. Actually that is the region where (1.4) cannot be approximated by (1.2). This region can be considered as containing a “boundary layer” where the boundedness of $f_0$ plays a role, and where the absence of particle fluxes and regularizing effects for (1.4) are seen. The analysis of this paper can be thought as the development of the mathematical techniques to handle such a boundary layer effects, as well as the proof of the fact that the dynamics of (1.4) can be approximated by means of the singular problem (1.2) at least for times of order one.

Let us remark that to solve the problems (1.2), (1.4) is equivalent to the solution of suitable problems with sources and vanishing initial data. Indeed, suppose that $\tilde{g} =$
\( g(x,t) \), is a smooth function satisfying \( \tilde{g}(x,0) = g_0(x) \). Let us define \( g = h - \tilde{h} \). Then:

\[
\begin{align*}
ht &= L[h] + \tilde{\mu} , \quad h(0) = 0 \\
\ht &= L[\tilde{g}] + \mu , \quad h(0) = 0
\end{align*}
\]

with \( \tilde{\mu} = L[\tilde{g}] - \tilde{g}_t \), \( \mu = L[g] - g_t \).

The method that we will use in this paper to solve (1.4) makes use of a classical continuation method. More precisely, we will embed (1.6) into the family of problems:

\[
\begin{align*}
h &= (1 - \theta) L[h] + \theta L(h) + \mu , \quad h(0) = 0 , \quad \theta \in [0, 1]
\end{align*}
\]

(1.7)

The problem (1.7) can be explicitly solved for \( \theta = 0 \) using the fundamental solution in [4]. Suppose that (1.7) can be solved for \( \theta = \theta^* \in [0, 1) \). We will show that (1.7) can be solved for \( \theta > \theta^* \) with \( (\theta - \theta^*) \) small enough. This will allow to extend by continuity the solution of (1.7) from \( \theta = 0 \) to \( \theta = 1 \), and then to obtain a solution of (1.6).

Similar continuity methods have been extensively used in the analysis of PDE’s (cf. [5, 6, 13]).

The plan of the paper is the following. In Section 2 we define our functional framework and state the main results of this paper. In Section 3 we obtain some technical auxiliary results that are needed in the proofs of the interior regularity estimates for the operator \( \mathcal{L} \). These are later obtained in Section 4 and will provide the essential smoothness required in this paper. Section 5 contains some estimates that provide a precise meaning to the approximation of the operator \( \mathcal{L} \) by means of \( L \). Finally, Section 6 provides the proof of the main result of the paper, namely the local well-posedness of (1.6).

## 2 Functional Framework and statement of the main results.

We introduce now the set of initial data, \( f_0 \in C^{1,\gamma}(\mathbb{R}^+) \), \( \gamma \in (0,1) \), that we shall consider in this paper. We will assume that the function \( f_0 \) is close to the function \( Ax^{-(3+\lambda)/2} \) for some constant \( A \in \mathbb{R} \). To this end define

\[
h_0(x) = f_0(x) - Ax^{-(3+\lambda)/2} \xi(x)
\]

(2.1)

where \( \xi \) is a smooth cutoff function such that \( \xi(x) = 1 \) for \( x \geq 1 \) and \( \xi(x) = 0 \) if \( 0 \leq x \leq 1/2 \). We then require in all this paper that for some positive constants \( B \) and \( \delta \), the following condition holds:

\[
y^{2+\lambda + \delta}|h_0(y)| + y^{\frac{2+\lambda + 1+\delta}{2}}|h_0'(y)| + \sup_{R \geq 1} R^{\frac{2+\lambda + \gamma + \delta}{2}} [h_0]_{C^{\gamma}(0;R/2,2R)} + [h_0']_{C^{\gamma}(0,1)} \leq B.
\]

(2.2)

All the estimates in the rest of paper will depend on the constants \( A, B, \gamma \) and \( \delta \). For the sake of shortedness this dependence will not be indicated.
The operator $L$ is given by:

$$
L(g) = \int_0^x (x-y)^{\lambda/2} f_0(x-y) y^{\lambda/2} g(y) \, dy - x^{\lambda/2} f_0(x) \int_0^\infty y^{\lambda/2} g(y) \, dy - x^{\lambda/2} g(x) \int_0^\infty y^{\lambda/2} f_0(y) \, dy
$$

(2.3)

$$
\frac{\partial h}{\partial t} = L(h) + \mu(\tau, x)
$$

(2.5)

$$
h(0, x) = 0
$$

(2.6)

Our first goal is to study the solutions of the Cauchy problem:

$$
\begin{align*}
\frac{\partial h}{\partial t} &= L(h) + \mu(\tau, x) \\
h(0, x) &= 0
\end{align*}
$$

(2.4)

for some initial data $h_0$ and non-homogeneous term $\mu$.

We shall also use repeatedly the following “localised version” of this equation. To this end, for all $R > 1$ fixed, let $\chi(x) \in C_0^\infty(0, +\infty)$ be such that:

$$
\chi(x) = \begin{cases} 
1 & \text{if } x \in (R - R^4, R + R^4) \\
0 & \text{if } x \notin (R - R^4, R + R^4)
\end{cases}
$$

(2.7)

If we multiply the equation (2.5) by $\chi(x)$ and call $\tilde{g} = \chi(x) g(x)$ we obtain:

$$
\begin{align*}
\frac{\partial \tilde{g}}{\partial t} &= L(\tilde{g}) + R(g) \\
R(g) &= \chi(x) \int_0^{x/2} \left( (x-y)^{\lambda/2} f_0(x-y) - x^{\lambda/2} f_0(x) \right) y^{\lambda/2} g(y) \, dy \\
&\quad - x^{\lambda/2} \tilde{g}(x) \int_{x/2}^\infty y^{\lambda/2} f_0(y) \, dy - x^{\lambda/2} f_0(x) \chi(x) \int_{x/2}^\infty y^{\lambda/2} g(y) \, dy \\
&\quad + \int_0^{x/2} (\chi(x) - \chi(x-y)) (x-y)^{\lambda/2} g(x-y) y^{\lambda/2} f_0(y) \, dy.
\end{align*}
$$

(2.9)

For any $p \geq 1$, $L^p$ will denote the usual Lebesgue space. For any $\sigma > 0$ and any interval $I \subset (0, +\infty)$ we denote $H^\sigma(I)$ the usual Sobolev space $W^{\sigma,2}(I)$. The corresponding norms will be denoted $\| \cdot \|_{L^p}$ and $\| \cdot \|_{H^\sigma}$. When dealing with functions depending on variables $x$ and $t$ we will write $H^\sigma_x$ or $L^p_t$ in order to indicate the argument with respect to which the norm is taken.

In order to define the functional spaces that will be needed we first introduce

$$
N_\infty(h; t_0, R) = \left( R^{\frac{\lambda-1}{2}} \int_{t_0}^{\min(t_0 + R^{-1/2}T, T)} ||h(t)||_{L^\infty(R/2, 2R)}^2 \, dt \right)^{1/2}
$$

(2.10)

$$
N_{2; \sigma}(h; t_0, R) = \left( R^{\frac{\lambda-1}{2} + 2\sigma - 1} \int_{t_0}^{\min(t_0 + R^{-1/2}T, T)} ||D^\sigma_x h(t)||_{L^2(R/2, 2R)}^2 \, dt \right)^{1/2}
$$

(2.11)

$$
M_\infty(h; R) = \left( \int_0^T ||h(t)||_{L^\infty(R/2, 2R)}^2 \, dt \right)^{1/2}
$$

(2.12)
Figure 1: Domain decomposition for $\lambda = 1.5$, $t_0 = n R^{-(\lambda-1)/2}$ and $R = 2^n$.

$$M_{2;\sigma}(h; R) = \left( R^{2\sigma-1} \int_0^T \| D_x^\sigma h(t) \|_{L^2(R/2,2R)}^2 dt \right)^{1/2}$$

(2.13)

Then, we define the spaces:

$$\| f \|_{X_{\varphi, p}(T)} = \sup_{0 < R < 1} R^q M_{\infty}(f; R) + \sup_{0 \leq t_0 \leq T} R^p N_{\infty}(f; t_0, R)$$

(2.14)

$$X_{p,q}(T) = \{ f; \| f \|_{X_{\varphi, p}(T)} < \infty \}$$

(2.15)

$$\| f \|_{Y_{\phi, \sigma}(T)} = \sup_{0 < R \leq 1} R^q M_{2;\sigma}(f; R) + \sup_{0 \leq t_0 \leq T} R^p N_{2;\sigma}(f; t_0, R)$$

(2.16)

$$Y_{\phi, \sigma}(T) = \{ f; \| f \|_{Y_{\phi, \sigma}(T)} < \infty \} .$$

(2.17)

We also use the following norms defined for functions $\varphi = \varphi(x)$:

$$||| \varphi |||_{q, p} = \sup_{0 \leq x \leq 1} \{ x^q |\varphi(x)| \} + \sup_{x > 1} \{ x^p |\varphi(x)| \}$$

and the next one defined for functions $\psi = \psi(\cdot, t)$ and any $T > 0$:

$$||| \psi |||_{\sigma} = \sup_{0 \leq t \leq T} \{ ||| \psi(\cdot, t) |||_{3/2,(3+\lambda)/2} + ||| \psi \|_{Y_{\phi, \sigma}(T)}\}$$

$$Y_{\phi, \sigma}(T) = \{ f; \| f \|_{\sigma} < \infty \}$$

We define the space $\mathcal{E}_{T,\sigma}$ as:

$$\mathcal{E}_{T,\sigma} = \{ f; \| f \|_{\sigma} < \infty \}$$
endowed with the norm $||| \cdot |||_\sigma$. We assume in all the paper that $\sigma$ is a fixed number satisfying
\[
\sigma \in (1/2, 1).
\tag{2.18}
\]

In order to discharge the notation we will not write explicitly the dependence of the space $\mathcal{E}_{T,\sigma}$ and the norm $||| \cdot |||_\sigma$ on $\sigma$ unless it is needed. We will then write:
\[
\mathcal{E}_{T,\sigma} \equiv \mathcal{E}_T; \quad ||| \cdot |||_\sigma \equiv ||| \cdot |||
\tag{2.19}
\]

We introduce a functional seminorm that measures in a natural way the regularising effect of the operator $L$ as $x \to \infty$. Consider a cutoff function $\eta(x)$ defined as
\[
\eta(x) = \begin{cases} 
1 & \text{if } x \in \left(\frac{3}{4}, \frac{5}{4}\right), \\
0 & \text{if } x \notin \left(\frac{1}{8}, 4\right).
\end{cases}
\tag{2.20}
\]

Given then a function $f \in \mathcal{E}_{T,\sigma}$, for all $R > 0$ and $t_0 \in [0, T]$ we define
\[
F_{R, t_0}(X, \tau) = \eta(X) f \left( RX, t_0 + \tau R^{-(\lambda - 1)/2} \right)
\tag{2.21}
\]
\[
[f] = \sup_{R \geq 1} \sup_{10 \leq t_0 \leq T} R^{(3+\lambda)/2} \times 
\int_{\min(t_0 + R^{-(\lambda - 1)/2}, T)}^{\min(t_0 + R^{-\lambda/2}, T)} \int_{\mathbb{R}} \left| \hat{F}_{R, t_0}(k, \tau) \right|^2 \left( 1 + |k|^{2\sigma} \min\{|k|, R\} \right) dk dt \right)^{1/2}.
\tag{2.22}
\]

The main results of this paper are the following

**Theorem 2.1** For any $\sigma \in (1/2, 1)$, $\delta > 0$ and for any $f_0$ satisfying (2.1) and (2.2), there exists $T > 0$ such that for all $\mu \in Y_{3/2,2+\delta}^\sigma$ the Cauchy problem (2.5) (2.6) has a unique solution $h$ in $\mathcal{E}_{T,\sigma}$. Moreover,
\[
||| h ||| \leq C ||| \mu |||_{Y_{3/2,2+\delta}^\sigma}
\]
for some positive constant $C$ depending on $T$, $\sigma$, $\delta$ as well as $A$, $B$ and $\gamma$ in (2.1) and (2.2) but not on $\mu$.

**Theorem 2.2** For any $\sigma \in (1/2, 1)$, $\delta > 0$ and for any $f_0$ satisfying (2.1) and (2.2), the solution of the Cauchy problem (2.5) (2.6) satisfies
\[
[h] \leq C ||| \mu |||_{Y_{3/2,2+\delta}^\sigma}
\]
for some positive constant $C$ depending on $T$, $\sigma$, $\delta$ as well as $A$, $B$ and $\gamma$ in (2.1) and (2.2) but not on $\mu$.
Theorem 2.2 is a regularising effect for the solutions of (2.5), (2.6). The operator $L$ can be thought as half a derivative as $x \to \infty$. However the solutions of (2.5), (2.6) do not gain any regularity for any finite value of $x$. The norm (2.23) can be thought heuristically as a measure of

$$R^{\frac{3+\lambda}{2}} \left| f(x + \varepsilon) - f(x) \right| \varepsilon^{1/2}$$

with $\varepsilon \geq 1/R$. Theorem 2.2 then states that for the function $h$ this quantity may be estimated by $\|\mu\|_{Y_{\gamma/2,2+\delta}}^\varepsilon$.

We end this Section with two warning remarks. The first one is that all along the paper we are going to use freely the letters $I_1, I_2, \cdots$ and $J_1, J_2, \cdots$ to denote different integrals. These letters will be used in different arguments. They will be used consistently within each argument. The second remark is that, in several arguments, we shall need to extend to a given interval suitable regularity estimates that have already been proved in smaller intervals. This is done following a standard and well known procedure involving decomposition of the identity and is not detailed in the paper.

3 Interior regularity estimates for $L$. Some technical results.

In order to study the regularity properties of the solutions to the equation (1.7) we define, for all $\varepsilon$ such that $0 \leq \varepsilon \leq 1$:

$$T_{\varepsilon,R}(f)(x) = \int_0^\infty (f(x) - f(x - y)) \Phi(y, R, \varepsilon) dy$$

(3.1)

$$\Phi(y, R, \varepsilon) = \frac{\varepsilon}{y^{3/2}} + (1 - \varepsilon) R^{(3+\lambda)/2} y^{\lambda/2} f_0(Ry)$$

(3.2)

$$\left( M_{\lambda/2} f \right)(x) = x^\frac{3}{2} f(x).$$

(3.3)

This family of operators provides an interpolation between a half derivative operator (for $\varepsilon = 1$) and the operator that we are interested in (for $\varepsilon = 0$). We will also use the operator

$$\Lambda \varphi(\xi) = -\sqrt{2\pi} |\xi|^{1/2} \hat{\varphi}(\xi).$$

(3.4)

We study now the interior regularity properties of the linear semigroup generated by the operator $T_{\varepsilon,R} \circ M_{\lambda/2}$.

**Theorem 3.1** (i) Suppose that $Q \in L^2(0,1;H^\sigma_x(1/2,2))$, $P \in L^2(0,1;H^{\sigma-1/2}_x(1/2,2))$ with $\sigma \in (1/2,2)$, $\kappa \in (0,1]$ and $f \in L^\infty((1/4,2) \times (0,1)) \cap L^2(0,1;H^{1/2}(1/4,2)) \cap H^1(0,1;L^2(1/4,2))$ is such that $f = 0$ if $x < 1/8$ or $x > 7$ and satisfies

$$\frac{\partial f}{\partial t} = \kappa T_{\varepsilon,R} \left( M_{\lambda/2} f \right) + Q + P$$

(3.5)

for all $x \in (1/4,2)$, $t \in (0,1)$ and $f(x,0) = 0$. Then:

$$||f||_{L^2_t(0,1;H^\sigma_x(3/4,5/4))} \leq C \left( ||Q||_{L^2_t(0,1;H^{\sigma-1/2}_x(1/2,2))} + \frac{1}{\varepsilon \kappa} ||P||_{L^2_t(0,1;H^{\sigma-1/2}_x(1/2,2))} \right) + \left( \frac{1}{\varepsilon \kappa} ||f||_{L^\infty((1/4,2) \times (0,1))} \right)$$

(3.6)
for some positive constant $C$ independent of $\varepsilon$ and $R$.

(ii) Suppose moreover that, for some $T_{\text{max}} > 0$, $Q \in L^2_t(0, T_{\text{max}}; H_x^\sigma(1/2, 2))$, $P \in L^2_t(0, T_{\text{max}}; H^{\sigma-1/2}_x(1/2, 2))$, $f \in L^\infty((1/4, 2) \times (0, T_{\text{max}})) \cap C^1_t(0, T_{\text{max}}; H^{1/2}_x(1/4, 2)))$ is such that $f = 0$ if $x < 1/8$ or $x > 4$ and satisfies

$$
\frac{\partial f}{\partial t} = T_{\varepsilon,R} \left( M_{\lambda/2} f \right) + Q + P - a(x, t) f, \quad x \in (1/4, 2), \; t > 0
$$

(3.7)

$$
f(x, 0) = 0
$$

(3.8)

for some function $a \in L^\infty(0, T_{\text{max}}; H^\sigma(1/2, 2))$, $a \geq A > 0$. Then, for all $t \in [0, T_{\text{max}} - 1]$:

$$
\sup_{0 \leq T \leq T_{\text{max}}} \left( \int_T^{\min(T+1, T_{\text{max}})} ||f(t)||^2_{H^\sigma(3/4, 5/4)} dt \right)^{1/2} \leq C \sup_{0 \leq T \leq T_{\text{max}}} \left( \int_T^{\min(T+1, T_{\text{max}})} ||Q(t)||^2_{H^\sigma(1/2, 2)} dt \right)^{1/2} + C \varepsilon \sup_{0 \leq T \leq T_{\text{max}}} \left( \int_T^{\min(T+1, T_{\text{max}})} ||P(t)||^2_{H^{\sigma-1/2}(1/2, 2)} dt \right)^{1/2}
$$

(3.9)

(iii) Suppose that for some $T_{\text{max}} > 0$, $Q \in L^2_t(0, T_{\text{max}}; H_x^\sigma(1/2, 2))$, $f \in L^\infty((1/4, 2) \times (0, T_{\text{max}})) \cap C^1_t(0, T_{\text{max}}; H^{1/2}_x(1/4, 2)))$ is such that $f = 0$ if $x < 1/8$ or $x > 4$ and satisfies

$$
\left( \int_T^{\min(T+1, T_{\text{max}})} \int_{\mathbb{R}} |\hat{F}(k, t)|^2 |k|^{2\sigma} \min\{\{k, R\} \} dk \right)^{1/2} \leq C \sup_{0 \leq T \leq T_{\text{max}}} \left( \int_T^{\min(T+1, T_{\text{max}})} ||Q(t)||^2_{H^\sigma(1/2, 2)} dt \right)^{1/2} + C ||f||_{L^\infty((1/4, 2) \times (0, T_{\text{max}}))}
$$

(3.10)

where $F(x, t) = \eta(x) f(x, t)$, $\eta$ is defined in (3.20) and $C$ is independent of $R$.

**Remark 3.2** It will be used repeatedly in the paper that the condition $\sigma > 1/2$ ensures that the space $H_x^\sigma$ is an algebra under the multiplication.

**Remark 3.3** Roughly speaking, the part (i) of Theorem [3.1] provides regularity estimates for times $t$ of order one. While part (ii) provides regularity estimates for arbitrary long times. It is important to notice that in the Theorem 3.1, the time $T_{\text{max}}$ can be arbitrarily large.

The proof of Theorem 3.1 is based on the classical freezing coefficients method that reduces the problem to the case of a constant coefficient operator. Let us then define, for all $x_0 \in \mathbb{R}^+$ the operator:

$$
S_{\varepsilon,R}(t) = \exp \left[ t x_0^\lambda \right].
$$

(3.11)
We also define the operators:

\[ T_1(\varphi)(\xi) = \text{Re} W(\xi, \varepsilon, R) \hat{\varphi}(\xi) \]  
\[ T_2(\varphi)(\xi) = i \text{Im} W(\xi, \varepsilon, R) \hat{\varphi}(\xi) \]  
\[ W(\xi, \varepsilon, R) = \int_0^\infty \Phi(y, \varepsilon, R) \left( e^{-iy\xi} - 1 \right) dy \]  
\[ \tilde{W}(\xi) = \int_0^\infty f_0(y) y^{\lambda/2} \left( e^{-iy\xi} - 1 \right) dy \]

We now collect several estimates on the operators \( T_1 \) and \( T_2 \) which are used in order to obtain bounds on the operator \( S_{\varepsilon, R} \).

**Lemma 3.4** The function \( W(\xi, \varepsilon, R) \) defined in \((3.14)\) may be rewritten as follows:

\[ W(\xi, \varepsilon, R) = -\varepsilon \sqrt{2} \Gamma(1/2) (1 + i \text{sign}(\xi)) |\xi|^{1/2} + (1 - \varepsilon) \sqrt{R} \tilde{W}(\xi/R), \]

where the function \( \tilde{W} \) satisfies:

\[ \text{Re} \tilde{W} \leq 0 \text{ with } \text{Re} \tilde{W} = 0 \text{ if and only if } \xi = 0, \]  
\[ \lim_{z \to 0} \tilde{W}(z) = -\sqrt{2} \Gamma(1/2), \]  
\[ \lim_{z \to +\infty} \tilde{W}(z) = -\int_0^\infty y^{\lambda/2} f_0(y), \]  
\[ |\tilde{W}'(\xi)| \leq \frac{C}{1 + |\xi|^{1+\gamma}} \text{ for all } \xi > 0. \]

As a consequence of these properties, the function \( W \) satisfies:

\[ \text{Re} W \leq 0 \text{ with } \text{Re} W = 0 \text{ if and only if } \xi = 0, \]  
and is such that, for all \( \varepsilon > 0 \) and \( \xi \) fixed,

\[ \lim_{R \to +\infty} W(\xi, \varepsilon, R) = -\sqrt{2}/2(1 + i \text{sign}(\xi)) |\xi|^{1/2}. \]

**Proof of Lemma 3.4** Using formulas \((3.2), (3.14)\) and \((3.15)\) properties \((3.17)-(3.19)\) follow. In order to prove \((3.20)\) we may write:

\[ \tilde{W}'(\xi) = -i \int_0^\infty y^{\lambda/2+1} f_0(y) e^{-iy\xi} dy = i \int_0^\infty \partial_y \left( y^{\lambda/2+1} f_0(y) \right) e^{-iy\xi} dy \]

\[ = i \xi \int_0^\infty h(y) e^{-iy\xi} dy \quad \text{with } h(y) = \partial_y \left( y^{\lambda/2+1} f_0(y) \right). \]

Writing now

\[ \int_0^\infty h(y) e^{-iy\xi} dy = \sum_{n=0}^{\infty} \int_{2\pi n/\xi}^{2\pi (n+1)/\xi} h(y) e^{-iy\xi} dy \]

\[ = \sum_{n=0}^{\infty} \int_{2\pi n/\xi}^{2\pi (n+1)/\xi} h \left( \frac{2\pi n}{\xi} \right) + O \left( \frac{1}{|\xi|^\gamma (1 + |y|^{3/2+\gamma})} \right) dy \]

\[ = \sum_{n=0}^{\infty} \int_{2\pi n/\xi}^{2\pi (n+1)/\xi} O \left( \frac{1}{|\xi|^\gamma (1 + |y|^{3/2+\gamma})} \right) dy \leq C \frac{1}{|\xi|^\gamma}. \]
Finally, properties (3.21) and (3.22) directly follow from (3.17)–(3.20). □

We collect now some regularising properties of the semigroups generated by the operators $x_0^\lambda/2\mathcal{T}_1$ and $S_{\varepsilon,R}$.

**Proposition 3.5** For all $\sigma > 0$ and $\kappa \in (0, 1]$:

$$
\int_0^1 \left\| \int_0^t \kappa \mathcal{T}_1 e^{x_0^\lambda/2 \mathcal{T}_1 (t-s)} \eta \right\|^2_{H^\sigma(\mathbb{R})} ds \leq C \int_0^1 \left\| h(s) \right\|^2_{H^\sigma(\mathbb{R})} ds
$$

(3.23)

Moreover, for all $\beta \in (0, 1]$ and $\eta$ a $C^\infty$ function of compact support, there exists $0 < \rho < \min(\sigma, \beta/2)$ such that

$$
\left\| S_{\varepsilon,R}(t) \right\|_{H^\rho(\mathbb{R})} \leq C t^{-\beta} \left\| h \right\|_{H^{\sigma-\rho}(\mathbb{R})},
$$

(3.26)

where $C$ denotes a generic positive constant independent of the function $h$ of $R$ and $\varepsilon$ but depending on $\sigma$, $\beta$, $\rho$ and $\eta$.

In the proof of Proposition 3.5, we will use the following result.

**Lemma 3.6** There exists a positive constant $C$ such that for all $R > 1$, $\varepsilon > 0$ and $\alpha \geq 0$, $\beta \geq 0$ satisfying $\alpha + \beta = 1$:

$$
\left| W(\xi, \varepsilon, R) - W(z, \varepsilon, R) \right| \leq C \frac{\left| \xi - z \right|}{\left| z \right|^\alpha \left| \xi \right|^\beta} (1 + \left| W(z, \varepsilon, R) \right|)^\alpha (1 + \left| W(\xi, \varepsilon, R) \right|)^\beta
$$

(3.28)

for all $z \in \mathbb{R}$ and $\xi \in \mathbb{R}$ such that $|z| \geq 1$ and $|\xi| \geq 1$.

**Proof of Lemma 3.6**

$$
W(\xi, \varepsilon, R) = -\varepsilon \sqrt{2 \pi} (1 + isign(\xi)) |\xi|^{1/2} + (1 - \varepsilon) \sqrt{R} \tilde{W}(\xi/R).
$$

The following estimate can be readily obtained studying separately the cases $sign(\xi) = sign(z)$, $sign(\xi) = -sign(z)$

$$
\left| sign(\xi) |\xi|^{1/2} - sign(z) |z|^{1/2} \right| \leq 2 \frac{\left| \xi - z \right|}{\left| \xi \right|^{1/2} + \left| z \right|^{1/2}}
$$

Therefore:

$$
\left| \varepsilon \sqrt{2 \pi} (1 + isign(\xi)) |\xi|^{1/2} - \varepsilon \sqrt{2 \pi} (1 + isign(z)) |z|^{1/2} \right| \leq C \frac{\left| \xi - z \right|}{\left| \xi \right|^{1/2} + \left| z \right|^{1/2}}.
$$
Using then $|W(z)| \geq 2 \sqrt{\pi} \varepsilon |z|^{1/2}$ we obtain:

$$\frac{|\varepsilon \sqrt{2 \pi} (1 + isign(\xi))| |\xi|^{1/2} - \varepsilon \sqrt{2 \pi} (1 + isign(z)) |z|^{1/2}|}{(1 + |W(z)|^\alpha (1 + |W(\xi)|)^\beta) \leq C \frac{|\xi - z|}{|z|^\alpha |\xi|^\beta}.$$ 

In order to prove a similar estimate for $\tilde{W}$ we consider the following cases:

(i) $|\xi| \leq 2R$ and $|z| \leq 2R$,

(ii) $|\xi| \geq R/2$ and $|z| \geq R/2$

(iii) $|\xi| \geq 2R$ and $|z| \leq R/2$

(iv) $|z| \geq 2R$ and $|\xi| \leq R/2$.

In the case (i) we have:

$$C_1 \sqrt{\xi} \leq |\sqrt{R} \tilde{W} \left( \frac{\xi}{R} \right)| \leq C_2 |\xi|^{1/2}$$

$$C_1 \sqrt{|\xi|} \leq \left| \frac{\partial}{\partial \xi} \tilde{W} \left( \frac{\xi}{R} \right) \right| \leq C_2 |\xi|^{-1/2}$$

and similar estimates also for $z$. Defining $g = \sqrt{R} \tilde{W} \left( \frac{\xi}{R} \right)$ we have by Taylor’s theorem:

$$|g^2(\xi) - g^2(z)| \leq \int_z^\xi |g(\eta)||g'(\eta)| d\eta \leq C |\xi - z|.$$ 

Then

$$|g(\xi) - g(z)| \leq \frac{|\xi - z|}{g(\xi) + g(z)}$$

whence:

$$\sqrt{R} \left| \tilde{W} \left( \frac{\xi}{R} \right) - \tilde{W} \left( \frac{z}{R} \right) \right| \leq C \frac{|\xi - z|}{|\xi|^{1/2} + |z|^{1/2}}$$

and the conclusion follows as above.

If condition (ii) holds, suppose first that $\text{sign} \xi = -\text{sign} z$. Then, $|\xi - z| = |\xi| + |z|$ and

$$\left| \sqrt{R} \tilde{W} \left( \frac{\xi}{R} \right) - \sqrt{R} \tilde{W} \left( \frac{z}{R} \right) \right| \leq C \sqrt{R}.$$ 

Using Young’s inequality:

$$\frac{|\xi| + |z|}{|\xi|^\beta |z|^\alpha} \geq C > 0.$$
for a positive constant \( C = C(\alpha, \beta) \), we deduce:

\[
\left| \sqrt{R} \tilde{W} \left( \frac{\xi}{R} \right) - \sqrt{R} \tilde{W} \left( \frac{z}{R} \right) \right| \leq C R^{\alpha/2} R^{\beta/2} \frac{|\xi| + |z|}{|\xi|^\beta |z|^\alpha} \\
\leq C \frac{|\xi - z|}{|\xi|^\beta |z|^\alpha} (1 + |W(z)|)^\alpha (1 + |W(\xi)|)^\beta.
\]

Suppose now, still under assumption (ii), that \( \text{sign} \xi = \text{sign} z \). Then, using (3.20) we have

\[
\left| \frac{\partial}{\partial \xi} \left( \sqrt{R} \tilde{W} \left( \frac{\eta}{R} \right) \right) \right| \leq C \frac{R^{1/2 + \alpha}}{|\eta|^\alpha + 1}
\]

for all \( \eta \geq R/2 \). Using Taylor’s theorem it then follows that:

\[
\left| \sqrt{R} \tilde{W} \left( \frac{\xi}{R} \right) - \sqrt{R} \tilde{W} \left( \frac{\eta}{R} \right) \right| \leq R^\alpha \int_\xi^\eta \frac{d\eta}{\eta^{1+\alpha}}.
\]

Therefore, if \( |\xi|/2 \leq |z| \leq 2|\xi| \) then,

\[
R^\alpha \int_\xi^\eta \frac{d\eta}{\eta^{1+\alpha}} \leq R^\alpha \frac{|\xi - z|}{|z|^{\alpha+1}} \leq \frac{|\xi - z|}{|z|^\alpha |\xi|^\beta} \leq C \frac{|\xi - z|}{|z|^\alpha |\xi|^\beta}.
\]

Otherwise,

\[
R^\alpha \int_\xi^\eta \frac{d\eta}{\eta^{1+\alpha}} \leq R^\alpha \int_{R/2}^\infty \frac{d\eta}{\eta^{1+\alpha}} = \frac{2^\alpha}{\alpha} \leq C \frac{|\xi - z|}{|z|^\alpha} \leq C \frac{|\xi - z|}{|z|^\alpha |\xi|^\beta}.
\]

The cases (iii) and (iv) can be treated equivalently. In both cases we have

\[
\frac{|\xi - z|}{|z|} \geq C_1 > 0.
\]

Moreover, in the case (iv):

\[
\left| \sqrt{R} \tilde{W} \left( \frac{\xi}{R} \right) - \sqrt{R} \tilde{W} \left( \frac{\eta}{R} \right) \right| \leq C \frac{R^{\alpha/2}}{|\xi|^\alpha/2} \leq C \frac{|z|^{\alpha/2}}{|\xi|^\alpha/2} \\
\leq C \frac{|z|^{\alpha/2}}{|\xi|^\alpha} \leq C \frac{|z - \xi|}{|z|^{1-\alpha} |\xi|^\alpha} \leq C \frac{|z - \xi|}{|z|^{1-\alpha} |\xi|^\alpha}.
\]

And this ends the proof of Lemma 3.6. \( \square \)

**Proof of Proposition 3.5**

\[
\left\| \int_0^t S_{\varepsilon, R}(t-s) h(s) \, ds \right\|_{H^\alpha(\mathbb{R})}^2 =
\]

\[
C \int_\mathbb{R} \int_0^t \int_0^t \left( 1 + |\xi|^3 \right) e^{\lambda/2 (t-s_1)} T_1(\xi) e^{\lambda/2 (t-s_1)} T_2(\xi) \tilde{h}(\xi, s_1) \times
\]

\[
x e^{\lambda/2 (t-s_2)} T_1(\xi) e^{-\lambda/2 (t-s_2)} T_2(\xi) \tilde{h}(\xi, s_2) \, ds_1 \, ds_2 \, d\xi
\]

\[
\leq C \int_\mathbb{R} \left( 1 + |\xi|^3 \right) \left( \int_0^t e^{\lambda/2 (t-s_1)} \tilde{h}(\xi, s_1) \, ds_1 \right)^2 \, d\xi \leq C \int_0^t \left\| h(s) \right\|_{L^2(\mathbb{R})}^2 \, ds_1,
\]

13
which proves (3.23).

We prove now (3.24). To this end let us define the function

$$
\varphi(x, t) = \int_0^t e^{x_0 \lambda / 2 \kappa T_1(t-s)} h(s) \, ds
$$

which verifies:

$$
\begin{align*}
\frac{\partial \varphi}{\partial t} &= x_0^{\lambda / 2} \kappa T_1(\varphi) + h(x, t), \quad t > 0, \ x > 0, \\
\varphi(x, 0) &= 0.
\end{align*}
$$

Multiplying this equation by $-\kappa T_1 M^2$ in $L^2(\mathbb{R})$ where $M$ is the multiplier operator associated to the symbol $|\xi|$ we obtain:

$$
\frac{\kappa}{2} \frac{d}{dt} ||M^\sigma (-T_1)^{1/2} \varphi||_{L^2(\mathbb{R})}^2 + \kappa^2 x_0^{\lambda / 2} ||M^\sigma (T_1 \varphi)||_{L^2(\mathbb{R})}^2 \leq \kappa ||M^\sigma T_1(\varphi)||_{L^2(\mathbb{R})} ||M^\sigma h||_{L^2(\mathbb{R})} \leq \frac{1}{2} \kappa x_0^\lambda \kappa T_1(\varphi)
$$

whence:

$$
\frac{\kappa}{2} \frac{d}{dt} ||M^\sigma (-T_1)^{1/2} \varphi||_{L^2(\mathbb{R})}^2 + \kappa^2 x_0^{\lambda / 2} ||M^\sigma (T_1 \varphi)||_{L^2(\mathbb{R})}^2 \leq \frac{1}{2} \kappa x_0^\lambda ||M^\sigma h||_{L^2(\mathbb{R})}^2.
$$

The result follows integrating in time and adding the corresponding inequality for $\sigma = 0$. The proof of (3.25) is similar. We multiply the equation by $-M^{2(\sigma-1/2)} \Lambda$ in $L^2(\mathbb{R})$ to obtain:

$$
\frac{1}{2} \frac{d}{dt} ||(-\Lambda)^{1/2} \varphi||_{H^{\sigma-1/2}(\mathbb{R})}^2 + \varepsilon ||\Lambda \varphi||_{H^{\sigma-1/2}(\mathbb{R})}^2 \leq ||h||_{H^{\sigma-1/2}(\mathbb{R})} ||\Lambda \varphi||_{H^{\sigma-1/2}(\mathbb{R})}.
$$

Using Young’s inequality and integrating in time we obtain (3.25).

We prove now (3.26). By definition:

$$
[\eta, \overrightarrow{T_{\varepsilon,R}}] \varphi(\xi) = \int_{\mathbb{R}} K(\xi - z) (W(z, \varepsilon, R) - W(\xi, \varepsilon, R)) \hat{\varphi}(z) \, dz
$$

where $K(z) = \hat{\eta}(z)$. Since the function $\eta$ is $C^\infty$, for any $m > 0$ there is a constant $C_m$ such that:

$$
|K(\xi)| \leq \frac{C_m}{1 + |\xi|^m} \text{ for all } \xi \in \mathbb{R}. \quad (3.29)
$$

Therefore:

$$
\begin{align*}
||S_{\varepsilon,R}(t) [\eta, \overrightarrow{T_{\varepsilon,R}}] h||_{H^\sigma(\mathbb{R})}^2 &= \\
\int_{\mathbb{R}} e^{-t|\text{Re}W(\xi)|} (1 + |\xi|^2)^2 \left| \int_{\mathbb{R}} K(\xi - z) (W(\xi) - W(z)) \hat{\varphi}(z) \, dz \right|^2 \, d\xi
\end{align*}
$$
We split the integral in two pieces:

$$||S_{e,R}(t) [\eta, T_{e,R}] h||_{H^s(\mathbb{R})}^2 = \int_{|\xi| \leq 1} [\cdots] d\xi + \int_{|\xi| \geq 1} [\cdots] d\xi$$

The first term is estimated by:

$$\left| \int_{|\xi| \leq 1} [\cdots] d\xi \right| \leq C \int_{|\xi| \leq 1} \left( \int_{\mathbb{R}} \frac{C_m}{1 + |\xi - z|^m} (1 + |z|^{1/2}) |\hat{\varphi}(z)| dz \right)^2 d\xi \leq C \left( \int_{\mathbb{R}} \frac{C_m}{1 + |\xi - z|^m} |\hat{\varphi}(z)| dz \right)^2 \leq C ||\varphi||_{L^2}^2. \tag{3.30}$$

In the second term we have:

$$\int_{|\xi| \geq 1} [\cdots] d\xi \leq 2 \int_{|\xi| \geq 1} e^{-2t |ReW|} (1 + |\xi|)^2 \left( \int_{|z| \leq 1} [\cdots] dz \right)^2 d\xi + 2 \int_{|\xi| \geq 1} e^{-2t |ReW|} (1 + |\xi|)^2 \left( \int_{|z| \geq 1} [\cdots] dz \right)^2 d\xi = J_1 + J_2.$$

We estimate $J_1$ follows:

$$J_1 \leq C \int_{|\xi| \geq 1} (1 + |\xi|)^2 \left( \int_{|z| \leq 1} \frac{C_m}{1 + |\xi - z|^m} (1 + |\xi|^{1/2}) |\hat{\varphi}(z)| dz \right)^2 d\xi \leq C \int_{|\xi| \geq 1} \frac{1}{(1 + |\xi|^{2m-1-2\sigma})} \left( \int_{|z| \leq 1} |\hat{\varphi}(z)| dz \right)^2 d\xi \leq C ||\varphi||_{L^2}^2. \tag{3.31}$$

It only remains to estimate $J_2$.

$$J_2 \leq \int_{|\xi| \geq 1} e^{-2t |ReW|} (1 + |\xi|)^2 \left( \int_{|z| \geq 1} K(\xi - z) (W(\xi) - W(z)) \hat{\varphi}(z) dz \right)^2 d\xi \leq C_m ||\varphi||_{H^s-\rho}^2 \times \int_{|\xi| \geq 1} e^{-2t |ReW|} (1 + |\xi|)^2 \left( \int_{|z| \geq 1} \frac{|W(\xi) - W(z)|^2}{(1 + |\xi - z|^m)^2} \frac{dz}{(1 + |z|^{\sigma-\rho})^2} \right) d\xi$$

Using Lemma 3.6

$$J_2 \leq C ||\varphi||_{H^s-\rho} \int_{|\xi| \geq 1} e^{-2t |ReW|} (1 + |\xi|)^2 \left( \int_{|z| \geq 1} \frac{|\xi - z|^2}{(1 + |\xi - z|^m)^2} \times \frac{|W(\xi)|^{2\beta} |W(z)|^{2\alpha}}{|z|^{2\alpha |\xi|^{2\beta}} (1 + |z|^{\sigma-\rho})^2} \right) d\xi \leq C ||\varphi||_{H^s-\rho} t^{-2\beta} \times \int_{|\xi| \geq 1} |\xi|^{2\alpha} \left( \int_{|z| \geq 1} \frac{1}{(1 + |\xi - z|^m)^2} \frac{|W(z)|^{2\alpha}}{|z|^{2\alpha |\xi|^{2\beta}} (1 + |z|^{\sigma-\rho})^2} d\xi \right)$$

where we have used that, for all $\xi \in \mathbb{R}$ and all $t > 0$:

$$e^{-2t |ReW|} |w(\xi)|^{2\beta} \leq \frac{C}{t^{\beta \alpha}}. \tag{3.32}$$
Using now that $|W(z)|/|z| \leq |z|^{-1/2}$ we deduce

$$J_2 \leq C\|\varphi\|_{H^{\sigma-\rho}L^{2\beta}} \int_{|\xi|\geq 1} \frac{|\xi|^{2(\sigma-\beta)}}{(1 + |\xi - z|^{|m-1|})^2} \left( \int_{|z|\geq 1} \frac{1}{|1 + |\xi - z|^{|m-1|}} \frac{dz}{|z|^\alpha (1 + |z|^{-\rho})^2} \right) d\xi.$$

We change the order of integration and rewrite the resulting integral as

$$\int_{|\xi|\geq 1} \frac{|\xi|^{2(\sigma-\beta)}}{(1 + |\xi - z|^{|m-1|})^2} = \int_{|\xi|\geq 1, |\xi| \leq 8|z|} \cdots \, d\xi + \int_{|\xi|\geq 1, |\xi| \geq 8|z|} \cdots \, d\xi = I_1 + I_2.$$ 

In the second integral we have $|\xi - z| \geq C|\xi|$ and therefore:

$$\int_{|\xi|\geq 1, |\xi| \geq 8|z|} \frac{|\xi|^{2(\sigma-\beta)}}{(1 + |\xi - z|^{|m-1|})^2} \leq C \int_{|\xi|\geq 1, |\xi| \geq 8|z|} \frac{|\xi|^{2(\sigma-\beta)}}{|\xi|^{2(|m-1|)}} \leq C|z|^{2\sigma - 2\beta - 2m + 3} \leq C|z|^{2(\sigma - \beta)},$$

assuming that $m$ is large. In the first integral

$$\int_{|\xi|\geq 1, |\xi| \leq 8|z|} \frac{|\xi|^{2(\sigma-\beta)}}{(1 + |\xi - z|^{|m-1|})^2} \leq C|z|^{2(\sigma - \beta)} \int_{|\xi|\leq 1, |\xi| \leq 8|z|} \frac{d\xi}{(1 + |\xi - z|^{|m-1|})^2}$$

Then

$$I_1 + I_2 \leq C|z|^{2(\sigma - \beta)}$$

and

$$\int_{|\xi|\geq 1} |\xi|^{2(\sigma-\beta)} \left( \int_{|z|\geq 1} \frac{1}{|1 + |\xi - z|^{|m-1|}} \frac{dz}{|z|^\alpha (1 + |z|^{-\rho})^2} \right) d\xi \leq C \int_{|z|\geq 1} |z|^{-1 - \beta + 2\rho} \, dz.$$ 

This integral is bounded as soon as $2\rho < \beta$. This concludes the proof of (3.26).

In order to prove (3.27) we estimate its left hand side as:

$$\int_0^t dt \int_0^t ds \int_{\mathbb{R}} e^{-2(t-s)ReW(\xi)} (1 + |\xi|^{2\sigma})|W(\xi)|^2 \left| \int_{\mathbb{R}} K(\xi - z)(W(\xi) - W(z))h(z, s) \, dz \right|^2 d\xi$$ 

arguing as in the proof of (3.30) and (3.31) we obtain that

$$\int_0^t dt \int_0^t ds \int_{\mathbb{R}} e^{-2(t-s)ReW(\xi)} (1 + |\xi|^{2\sigma})|W(\xi)|^2 \times$$

$$\times \left| \int_{\mathbb{R}} \mathbf{1}_{\min(|\xi|, |z|) \leq 1}(\xi, z)K(\xi - z)(W(\xi) - W(z))h(z, s) \, dz \right|^2 d\xi \leq C \int_0^1 \|h(s)\|^2_{L^2(\mathbb{R})} ds.$$

(3.33)
On the other hand,
\[
\int_0^1 dt \int_0^t ds \int_{|\xi| \geq 1} \left| e^{-2(t-s)\text{Re} W(\xi)} (1 + |\xi|^{2\sigma}) |W(\xi)|^2 \times 
\right.
\]
\[
\left. \times \left| K(\xi - z)(W(\xi) - W(z))h(z, s) \right| d\xi \right|^{2} d\xi \leq \int_0^1 dt \int_0^t ds \int_{|\xi| \geq 1} \left| e^{-2(t-s)\text{Re} W(\xi)} (1 + |\xi|^{2\sigma}) \times 
\right.
\]
\[
\left. \times \left| K(\xi - z)|W(\xi) - W(z)||W(z)||h(z, s) \right| d\xi \right|^{2} d\xi + 
\]
\[
+ \int_0^1 dt \int_0^t ds \int_{|\xi| \geq 1} \left| e^{-2(t-s)\text{Re} W(\xi)} (1 + |\xi|^{2\sigma}) \times 
\right.
\]
\[
\left. \times \left| K(\xi - z)|W(\xi) - W(z)|^2 h(z, s) \right| d\xi \right|^{2} d\xi = I_1 + I_2.
\]

Arguing as in the derivation of (3.26) we obtain
\[
I_1 \leq C \int_0^1 dt \int_0^t ds \int_{|\xi| \geq 1} \left| T_1 h(s) \right|^{2}_{H^{\sigma}(\mathbb{R})} \leq C \int_0^1 \left| T_1 h(s) \right|^{2}_{H^{\sigma}(\mathbb{R})} ds. \tag{3.34}
\]

On the other hand, in $I_2$ we use (3.28) and formula (3.32), to obtain:
\[
I_2 \leq \int_0^1 dt \int_0^t ds \int_{|\xi| \geq 1} \left| T_1 h(s) \right|^{2}_{H^{\sigma}(\mathbb{R})} \left( e^{-2(t-s)\text{Re} W(\xi)} |\xi|^{2\sigma} d\xi \times 
\right.
\]
\[
\left. \times \left( \int_{|\xi| \geq 1} \frac{|\xi - z|^4}{(1 + |\xi - z|^n)^2} \frac{|W(\xi)|^{4\beta}}{|\xi|^{4\beta}} \frac{|W(z)|^{4\alpha - 2}}{|z|^{4\alpha - 2}} \frac{dz}{dx} \right) \right|^{2} d\xi \leq \int_0^1 dt \int_0^t ds \int_{|\xi| \geq 1} \left| T_1 h(s) \right|^{2}_{H^{\sigma}(\mathbb{R})} \left( e^{-2(t-s)\text{Re} W(\xi)} |\xi|^{2\sigma} d\xi \times 
\right.
\]
\[
\left. \times \left( \int_{|\xi| \geq 1} \frac{1}{(1 + |\xi - z|^n)^2} \frac{1}{|\xi|^{4\beta}} \frac{1}{|z|^{4\alpha + 2\sigma - 2\beta}} \frac{dz}{dx} \right) \right|^{2} d\xi \leq \int_0^1 dt \int_0^t ds \int_{|\xi| \geq 1} \left| T_1 h(s) \right|^{2}_{H^{\sigma}(\mathbb{R})} \left( e^{-2(t-s)\text{Re} W(\xi)} |\xi|^{2\sigma} d\xi \times 
\right.
\]
\[
\left. \times \left( \int_{|\xi| \geq 1} \frac{1}{(1 + |\xi - z|^n)^2} \frac{1}{|\xi|^{4\beta}} \frac{1}{|z|^{4\alpha + 2\sigma - 2\beta}} \frac{dz}{dx} \right) \right|^{2} d\xi \leq C \int_0^1 \left| T_1 h(s) \right|^{2}_{H^{\sigma}(\mathbb{R})} ds. \tag{3.35}
\]

Combining (3.33), (3.34) and (3.35), (3.27) follows and then Proposition 3.5.

We will also use the following Lemma.

**Lemma 3.7** Let $\alpha \in C_0^{\infty}(0, +\infty)$ and $\varepsilon_0 > 0$ such that $\text{supp} \alpha \subset (x_0 - \varepsilon_0, x_0 + \varepsilon_0)$ and
\[
\varepsilon_0^n \left| \frac{d^n \alpha(x)}{dx^n} \right| \leq C_n \varepsilon_0, \text{ for all } x > 0 \tag{3.36}
\]
for some positive constant $C_n$ independent of $\varepsilon_0$. Then, there exists positive constants $K$ and $C_{\varepsilon_0}$, with $K$ independent of $\varepsilon_0$, such that

\[
\|\alpha f\|_{H^s(\mathbb{R}^+)} \leq K\varepsilon_0 \|f\|_{H^s(\mathbb{R}^+)} + C_{\varepsilon_0} \|f\|_{L^\infty(\mathbb{R}^+)} \tag{3.37}
\]

\[
\|T_1\alpha f\|_{H^s(\mathbb{R}^+)} \leq K\varepsilon_0 \|T_1 f\|_{H^s(\mathbb{R}^+)} + C_{\varepsilon_0} \|f\|_{L^\infty(\mathbb{R}^+)}.
\tag{3.38}
\]

**Proof of Lemma 3.7**

Let us consider the function $m(k)$ that will equal to one in the proof of (3.37) and $|\text{Re}(W(k, \varepsilon, R))|$ in the proof of (3.38) where $W(k, \varepsilon, R)$ is defined in (3.16). Due to the hypothesis (3.36):

\[
|\hat{\alpha}(k)| \leq \frac{C_n \varepsilon_0^2}{1 + |k\varepsilon_0|^\sigma} \quad \text{for all } k \in \mathbb{R} \tag{3.39}
\]

We proceed to estimate

\[
J = \int_\mathbb{R} |m(k)|^2 |k|^{2\sigma} \left| \hat{\alpha}(k - \xi) \hat{f}(\xi) \right|^2 dk \tag{3.40}
\]

\[
= \int_{|k| \leq 1} [\cdots] dk + \int_{|k| \geq 1} [\cdots] dk = J_1 + J_2 \tag{3.41}
\]

The term $J_1$ is estimated as follows

\[
|J_1| \leq C\|f\|_{L^2(\mathbb{R})}^2 \|\hat{\alpha}\|_{L^1} \leq C\varepsilon_0 \|f\|_{L^2(\mathbb{R})}^2 \tag{3.42}
\]

for some positive constant $C$ independent on $\varepsilon_0$. On the other hand, we split $J_2$ as follows:

\[
|J_2| \leq J_{21} + J_{22} \tag{3.43}
\]

\[
J_{21} = \int_{|k| \geq 1} |m(k)|^2 |k|^{2\sigma} \left| \int_{|\xi| \leq 1} \hat{\alpha}(k - \xi) \hat{f}(\xi) d\xi \right|^2 dk \tag{3.44}
\]

\[
J_{22} = \int_{|k| \geq 1} |m(k)|^2 |k|^{2\sigma} \left| \int_{|\xi| \geq 1} \hat{\alpha}(k - \xi) \hat{f}(\xi) d\xi \right|^2 dk. \tag{3.45}
\]

To estimate $J_{21}$ we use that, for $|k| \geq 1$ and $|\xi| \leq 1$, one has $|m(k) - m(\xi)| \leq C(1 + m(k - \xi))$. We deduce that in the same range of $k$ and $\xi$:

\[
|m(k)| |k|^\sigma - m(\xi)| |\xi|^\sigma| \leq C(1 + |k - \xi|^\sigma)(1 + m(k - \xi)).
\]

Then, since $|m(k) - \xi|^\sigma \leq C(1 + \sqrt{|k - \xi|})$ we obtain:

\[
J_{21} \leq C_{\varepsilon_0} \int_{|k| \geq 1} \left( \int_{|\xi| \leq 1} \frac{|\hat{f}(\xi)|}{|k - \xi|^\sigma} d\xi \right)^2 \leq C_{\varepsilon_0}' \|f\|_{L^2(\mathbb{R})} \tag{3.46}
\]

where $C_{\varepsilon_0}$ and $C_{\varepsilon_0}'$ are constants depending on $n$ and $\varepsilon$ and using Young’s inequality in the last step. Consider finally $J_{22}$. To this end we notice that, using (3.28) for the case when $m(k) = |\text{Re}(W(k, \varepsilon, R))|$

\[
|||k|^\sigma m(k) - |\xi|^\sigma m(\xi)| \leq |k|^\sigma|m(k) - m(\xi)| + ||k|^\sigma - |\xi|^\sigma| m(\xi)
\]

\[
\leq |k|^\sigma \frac{|\xi - k|}{|\xi|} + |k - \xi|^\sigma m(\xi)
\]

18
whence, using once again $|k| \leq |k - \xi| + |\xi|$, 
\[
||k|^\sigma m(k) - |\xi|^\sigma m(\xi)| \leq C \left( \frac{|k - \xi|^\sigma + |k - \xi| |\xi|^\sigma}{|\xi|} + |k - \xi|^\sigma \right) m(\xi)
\]
and then,
\[
J_{2,2} \leq \int_{|k| \geq 1} \int_{|\xi| \geq 1} \hat{\alpha}(k - \xi) m(\xi) |\xi|^\sigma \hat{f}(\xi) d\xi \right|^2 dk \\
+C_{c_0,\nu'} \int_{|k| \geq 1} \int_{|\xi| \geq 1} \frac{1 + |\xi|^{\sigma - 1}}{1 + |k - \xi|\nu'} m(\xi) |\xi|^\sigma \hat{f}(\xi) d\xi \right|^2 dk.
\]
Using Young’s inequality we obtain
\[
J_{2,2} \leq K\varepsilon_0 ||T_1 f||_{H^\sigma(\mathbb{R})} + C_{c_0} ||T_1 f||_{H^{(\sigma-1)+}(\mathbb{R})} \quad \text{if} \ m(k) = |ReW| \quad (3.47)\\nJ_{2,2} \leq K\varepsilon_0 ||f||_{H^\sigma(\mathbb{R})} + C_{c_0} ||f||_{H^{(\sigma-1)+}(\mathbb{R})} \quad \text{if} \ m(k) = 1. \quad (3.48)
\]
Combining (3.42), (3.46), (3.47), (3.48), and a classical interpolation argument to estimate the norm $H^{(\sigma-1)+}(\mathbb{R})$ by the $L^\infty$ and $H^\sigma(\mathbb{R})$ norms the Lemma follows. \hfill \Box

**Lemma 3.8** Let $\eta$ be a $C^\infty$ compactly supported function in $\mathbb{R}^+$. Then, for any $\sigma > 0$ there exists a positive constant $C$ such that for any $h \in H^\sigma(\mathbb{R})$, for any $R > 0$ and any $\varepsilon > 0$:
\[
\left\| \int_0^\infty h(x - y) (\eta(x) - \eta(x - y)) \Phi(y,R,\varepsilon) dy \right\|_{H^{\sigma+1/2}(\mathbb{R})} \leq C \|h\|_{H^\sigma(\mathbb{R})},
\]
where $\Phi(y,R,\varepsilon)$ is defined by (3.2).

**Proof of Lemma 3.8** We define three functions $M(x,y)$, $P(x,y,R,\varepsilon)$ and $Q(x,R,\varepsilon)$ as follows
\[
Q(y) = y \Phi(y,R,\varepsilon) \\
M(x,y) = \frac{\eta(x) - \eta(x - y)}{y} \\
P(x,y) = (\eta(x) - \eta(x - y)) \Phi(y,R,\varepsilon) = M(x,y) Q(y).
\]
Where the dependence of $P$ and $Q$ on $R$ and $\varepsilon$ is not explicitly written by shortedness. Notice that $M(x,y) \in C^\infty(\mathbb{R} \times \mathbb{R})$. If we suppose that the support of $\eta$ is contained in an interval $I \subset \mathbb{R}^+$, then the support of $m$ is such that:
\[
\text{supp} \ (M) \subset I \times \mathbb{R}^+ \cup \{(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+; \ x - y \in I \}.
\]
Our goal is then to estimate estimate the the $H^{\sigma+1/2}$ norm of
\[
B(h) := \int_0^\infty h(x - y,s) (\eta(x) - \eta(x - y)) \Phi(y,R,\varepsilon) dy
\]
\[
19
\]
which we write:

\[ ||B(h)||^2_{H^{\sigma+1/2} (\mathbb{R})} = \int_{\mathbb{R}} \left( 1 + |\xi|^{\sigma+1/2} \right)^2 |\widehat{B(h)}(\xi)|^2 \, d\xi \]

\[ \widehat{B(h)}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{0}^{\infty} e^{i\eta(x-y)} M(x,y) Q(y) e^{-i\xi \hat{h}(\eta)} \, dy \, d\eta \, dx \]

\[ = \int_{\mathbb{R}} \hat{P}(\xi - \eta, \eta) \hat{h}(\eta) \, d\eta \]

where

\[ \hat{P}(\zeta_1, \zeta_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i(x \zeta_1 + y \zeta_2)} P(x,y) \, dx \, dy \] (3.51)

is the Fourier of the function \( P \) with respect to the two variables \( x \) and \( y \).

Notice that:

\[ ||B(h)||^2_{H^{\sigma+1/2} (\mathbb{R})} = \int_{\mathbb{R}} \left( 1 + |\xi|^{\sigma+1/2} \right)^2 \left| \int_{\mathbb{R}} \hat{P}(\eta - \xi, -\eta) \hat{h}(\eta) \, d\eta \right|^2 \, d\xi \]

We now proceed to estimate the function \( M \). For any \( m = 0, 1, \cdots \) there is a positive constant \( C_m \), independent of \( R \) and \( \varepsilon \), such that

\[ \left| \frac{\partial^m M(x,y)}{\partial x^m} \right| \leq \frac{C_m}{1 + |y|} \quad \text{for all } (x,y) \in \text{supp}(M). \] (3.52)

On the other hand, there exists a positive constant \( C \) independent on \( R \) and \( \varepsilon \) such that for all \( y \in \mathbb{R}^+ \):

\[ |Q(y,R,\varepsilon)| \leq \frac{C}{\sqrt{|y|}} \] (3.53)

Combining (3.49), (3.51), (3.52) and (3.53) we deduce that \( P(x,y) \) is integrable in \( \mathbb{R}^2 \) and then \( \hat{P} \) is a well defined and bounded function on \( \mathbb{R}^2 \).

Moreover, we can also deduce decay estimate for \( \hat{P} \) for \( |\zeta_1| + |\zeta_2| \to +\infty \). To this end we integrate by parts in formula (3.51)

\[ \hat{P}(\zeta_1, \zeta_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i(x \zeta_1 + y \zeta_2)} P(x,y) \, dx \, dy \] (3.54)

\[ \hat{P}(\zeta_1, \zeta_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i(x \zeta_1 + y \zeta_2)} P(x,y) \, dx \, dy \] (3.55)

where

\[ S_n(\zeta_1, y) = \int_{\mathbb{R}} e^{-i\zeta_1 x} \frac{\partial^n M(x,y)}{\partial x^n} \, dx. \] (3.56)

Differentiating (3.54) with respect to \( y \) and integrating by parts, it easily follows that the function \( S_n \) are such that, for all \( m = 0, 1, \cdots, k = 0, 1, \cdots \) there is a positive constant \( C_{k,m,n} \), independent on \( R \) and \( \varepsilon \) satisfying, for all \( \zeta_1 \in \mathbb{R}^+ \) and \( y \in \mathbb{R}^+ \):

\[ \left| \frac{\partial^k S_n(\zeta_1, y)}{\partial y^k} \right| \leq \frac{C_{k,m,n}}{(1 + |y|) (1 + |\zeta_1|)^m} \] (3.57)
Let us consider the behaviour of $\hat{P}$ with respect to $\zeta$. Using (3.54)

$$\hat{P}(\zeta_1, \zeta_2) = \frac{\varepsilon}{2\pi i^n \zeta_1^n} \int_0^1 e^{-i\zeta_2 y} \frac{1}{y^{1/2}} S_n(\zeta_1, y) dy +$$

$$+ \frac{(1 - \varepsilon)}{2\pi i^n \zeta_1^n} \int_0^1 e^{-i\zeta_2 y} R^{(3+\lambda)/2} y^{1+\lambda/2} f_0(Ry) S_n(\zeta_1, y) dy$$

$$+ \frac{1}{2\pi i^n \zeta_1^n} \int_1^\infty e^{-i\zeta_2 y} Q(y, R, \varepsilon) S_n(\zeta_1, y) dy = \frac{1}{2\pi i^n \zeta_1^n} (J_1 + J_2 + J_3). (3.58)$$

In order to estimate the term $J_1$ we rewrite it as follows:

$$J_1 = \int_0^1 e^{-i\zeta_2 y} \frac{\varepsilon}{y^{1/2}} S_n(\zeta_1, 0) dy + \int_0^1 e^{-i\zeta_2 y} \frac{\varepsilon}{y^{1/2}} (S_n(\zeta_1, y) - S_n(\zeta_1, 0)) dy$$

$$= \varepsilon \frac{S_n(\zeta_1, 0)}{\zeta_2^{1/2}} \int_0^{\zeta_2} e^{-iz} \frac{dz}{z^{1/2}} + \int_0^1 e^{-i\zeta_2 y} \frac{\varepsilon}{y^{1/2}} (S_n(\zeta_1, y) - S_n(\zeta_1, 0)) dy.$$

The integral $\int_0^{\zeta_2} e^{-iz} \frac{dz}{z^{1/2}}$ is uniformly bounded for $\zeta_2 \in \mathbb{R}$. On the other hand, due to (3.57) we have that

$$\left| \frac{\partial}{\partial y} \left( \frac{1}{y^{1/2}} (S_n(\zeta_1, y) - S_n(\zeta_1, 0)) \right) \right| \leq C\sqrt{y}.$$

Integrating by parts we obtain the existence of a constant $C$ such that for all $\zeta_2 \in \mathbb{R}$:

$$\left| \int_0^1 e^{-i\zeta_2 y} \frac{\varepsilon}{y^{1/2}} (S_n(\zeta_1, y) - S_n(\zeta_1, 0)) dy \right| \leq \frac{C}{1 + |\zeta_2|} \text{ for all } \zeta_2 \in \mathbb{R}.$$

Therefore:

$$|J_1| \leq \frac{C}{1 + |\zeta_2|^{1/2}}.$$

We use similar arguments to estimate $J_2$ that we write as follows

$$J_2 = S_n(\zeta_1, 0) \int_0^1 e^{-i\zeta_2 y} \left( R^{(3+\lambda)/2} y^{1+\lambda/2} f_0(Ry) \right) dy$$

$$+ \int_0^1 e^{-i\zeta_2 y} \left( R^{(3+\lambda)/2} y^{1+\lambda/2} f_0(Ry) \right) (S_n(\zeta_1, y) - S_n(\zeta_1, 0)) dy$$

$$= I_1 + I_2.$$

The term $I_2$ may be estimated as above since (3.57) gives:

$$\left| \frac{\partial}{\partial y} \left( R^{(3+\lambda)/2} y^{1+\lambda/2} f_0(Ry) \right) (S_n(\zeta_1, y) - S_n(\zeta_1, 0)) \right| \leq C y^{-1/2}, \ y \in [0, 1]$$

using that

$$\frac{\partial}{\partial y} [R^{(3+\lambda)/2} y^{1+\lambda/2} f_0(Ry)] (S_n(\zeta_1, y) - S_n(\zeta_1, 0))$$

$$= (S_n(\zeta_1, y) - S_n(\zeta_1, 0)) \frac{\partial}{\partial y} \left[ R^{(3+\lambda)/2} y^{1+\lambda/2} f_0(Ry) \right]$$

$$+ \left( R^{(3+\lambda)/2} y^{1+\lambda/2} f_0(Ry) \right) \frac{\partial}{\partial y} [(S_n(\zeta_1, y) - S_n(\zeta_1, 0))].$$
as well as the bounds on $f_0$ and $f'_0$. 

$$|I_2| \leq \frac{C}{1 + |\zeta_2|}$$

We use in $I_1$ the change of variables $\eta = \zeta_2 y$ and the auxiliary function $g(y) = y^{1+\lambda/2} f_0(y)$ to obtain

$$I_1 = \frac{R^{1/2}}{\zeta_2} \int_0^{\zeta_2} e^{-i\eta g} \left( \frac{R\eta}{\zeta_2} \right) d\eta = \frac{R^{1/2}}{i\zeta_2} \int_0^{\zeta_2} (e^{-i\eta} - 1) \frac{R g'}{\zeta_2} \left( \frac{R\eta}{\zeta_2} \right) d\eta$$

after integrating by parts. Then, there is a positive constant $C$ independent on $R$ and $\varepsilon$ such that for all $\zeta_2 \in \mathbb{R}$

$$|I_1| \leq \frac{C}{1 + |\zeta_2|^{1/2}},$$

whence, combining the estimates for $I_1$ and $I_2$:

$$|J_2| \leq \frac{C}{1 + |\zeta_2|^{1/2}}.$$

We estimate $J_3$ integrating by parts and using (3.57):

$$|J_3| \leq C \frac{1}{1 + |\zeta_2|^{1/2}},$$

whence

$$|\hat{P}(\zeta_1, \zeta_2)| \leq \frac{C}{(1 + |\zeta_1|^m)(1 + |\zeta_2|^{1/2})}.$$

To conclude the proof of Lemma 3.8 we bound the norm of $B(h)$ in $H^{\sigma+1/2}(\mathbb{R})$ as follows:

$$||B(h)||_{H^{\sigma+1/2}(\mathbb{R})}^2 \leq \int_{\mathbb{R}} \left( 1 + |\xi|^{\sigma+1/2} \right) \left| \int_{\mathbb{R}} \hat{P}(\xi - \eta, \eta) \hat{h}(\eta) d\eta \right|^2 d\xi$$

$$\leq \int_{\mathbb{R}} \left( 1 + |\xi|^{\sigma+1/2} \right) \left| \int_{\mathbb{R}} \frac{\hat{h}(\eta)}{1 + |\eta - \xi|^m (1 + |\eta|^{1/2})} d\eta \right|^2 d\xi$$

$$\leq C \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \frac{\hat{h}(\eta)}{1 + |\eta - \xi|^m} d\eta \right|^2 d\xi + C \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \frac{\hat{h}(\eta)}{1 + |\eta - \xi|^m} d\eta \right|^2 d\xi$$

$$\leq C ||h||_{L^2(\mathbb{R})}^2 ||(1 + |\cdot|^{-m'})||_{L^1(\mathbb{R})}^2 + C ||h||_{H^\sigma(\mathbb{R})}^2 ||(1 + |\cdot|^{-m'})||_{L^1(\mathbb{R})}^2$$

where we have used Young’s inequality in the last step. □
4 Interior regularity theory for the operator $\mathcal{L}$.

We start with the proof of Theorem 3.1.

Proof of (i) of Theorem 3.1. We apply now the classical method of freezing coefficients. To this end let us call $\chi$ a $C^\infty$ function such that

$$
\chi(x) = \begin{cases} 
1 & \text{if } x \in (5/8, 11/8), \\
0 & \text{if } x \not\in (1/2, 2).
\end{cases}
$$

We define $\tilde{f}(x) = \chi(x) f(x)$. Then, for all $x \in \mathbb{R}$:

$$
\frac{\partial \tilde{f}}{\partial t} = \kappa T_{\varepsilon,R}(M_{\lambda/2} \tilde{f}) + \kappa \int_0^\infty (x-y)^{\lambda/2} f(x-y) \left( \chi(x) - \chi(x-y) \right) \Phi(y, R, \varepsilon) \, dy + \\
+ \chi(x) Q + \chi(x) P
$$

We define $\tilde{Q}(x) = \chi(x) f(x)$. Then, for all $x \in \mathbb{R}$:

$$
\tilde{Q}_1 = \kappa \int_0^\infty (x-y)^{\lambda/2} f(x-y) \left( \chi(x) - \chi(x-y) \right) \Phi(y, R, \varepsilon) \, dy
$$

It is readily seen that

$$
\|\tilde{Q}_1\|_{L^\infty((0,1);W^{1,\infty}(\mathbb{R}))} \leq C \kappa \|f\|_{L^\infty((1/4,2) \times (0,1))} \tag{4.2}
$$

Equation (4.3) may be written as

$$
\frac{\partial \tilde{f}}{\partial t} = x_0^{\lambda/2} \kappa T_{\varepsilon,R} \left( \tilde{f} \right) + \kappa T_{\varepsilon,R} \left( (M_{\lambda/2} - M_{\lambda/2,0}) \tilde{f} \right) + \tilde{Q} + \tilde{P} \tag{4.3}
$$

where $M_{\lambda/2,0} \tilde{f}(x) = x_0^{\lambda/2} \tilde{f}(x)$.

Fix now a new cutoff function $\eta$ such that

$$
\eta(x) = \begin{cases} 
1 & \text{if } |x - x_0| \leq \delta, \\
0 & \text{if } |x - x_0| \geq 2\delta.
\end{cases}
$$

with $\delta$ such that

$$
|x^{\lambda/2} - x_0^{\lambda/2}| \leq \varepsilon_0, \quad \text{for } |x - x_0| \leq 2\delta.
$$

with $\varepsilon_0$ small enough to be chosen later. If we multiply the equation (4.3) by $\eta$ and denote $\tilde{f} = \eta \tilde{f}$ we obtain:

$$
\frac{\partial \tilde{f}}{\partial t} - x_0^{\lambda/2} \kappa T_{\varepsilon,R} \left( \tilde{f} \right) = \kappa \eta(x) T_{\varepsilon,R} \left( (M_{\lambda/2} - M_{\lambda/2,0}) \tilde{f} \right) + \eta(x)(\tilde{Q} + \tilde{P}) + \\
+ \kappa x_0^{\lambda/2} \int_0^\infty \tilde{f}(x-y) (\eta(x) - \eta(x-y)) \Phi(y, R, \varepsilon) \, dy \tag{4.5}
$$
We have the following representation formula for the solution $\tilde{f}$ of (4.5) in $L^\infty((1/4, 2) \times (0, 1)) \cap L^2(0, 1; H^{1/2}(1/4, 2)) \cap H^1(0, 1; L^2(1/4, 2))$ is:

\[
\begin{align*}
\tilde{f}(x, t) &= \int_0^t S_{\varepsilon, R}(\kappa(t - s))\eta(x) \left( \tilde{Q}(s) + \tilde{P}(s) \right) ds + \\
&\quad + \kappa \int_0^t S_{\varepsilon, R}(\kappa(t - s)) \left[ \eta(x) T_{\varepsilon, R} \left( (M_{\lambda/2} - M_{\lambda/2,0})\tilde{f} \right) (s) \right] ds \\
&\quad + \kappa x_0^{\lambda/2} \int_0^t S_{\varepsilon, R}(\kappa(t - s)) \int_{\infty}^s \tilde{f}(x - y, s) (\eta(x) - \eta(x - y)) \Phi(y, \varepsilon, R) dy ds \\
&= \tilde{T}_1(x, t) + \tilde{T}_2(x, t) + \tilde{T}_3(x, t) \tag{4.7}
\end{align*}
\]

for all $x \in \mathbb{R}$. This follows from the fact that the unique solution $f$ in the space $L^\infty((1/4, 2) \times (0, 1)) \cap L^2(0, 1; H^{1/2}(1/4, 2)) \cap H^1(0, 1; L^2(1/4, 2))$ of:

\[
\begin{align*}
\frac{\partial f}{\partial t} - x_0^{\lambda/2} \kappa T_{\varepsilon, R}(f) &= G(x, t) \\
f(0, x) &= 0,
\end{align*}
\]

where $G \in L^\infty((0, 1) \times (0, +\infty)$ and $f$ and $G$ compactly supported in $(1/4, 2) \times (0, 1)$, is given by Duhamel’s formula. The uniqueness of $f$ can be obtained by taking the difference of two such solutions and taking the scalar product of (4.8) with that difference in $L^2$.

Such computations are possible by the regularity that is assumed on the solutions.

\[
\begin{align*}
\|\tilde{T}_1\|_{H^r(\mathbb{R})}^2 &\leq C \int_0^t \|\eta \tilde{Q}\|_{H^r(\mathbb{R})}^2 ds_1 + C \int_0^t S_{\varepsilon, R}(\kappa(t - s))\eta(x) \tilde{P}(s) ds \bigg\|_{H^r(\mathbb{R})}^2 \tag{4.9}
\end{align*}
\]

Let us estimate the second term in the right hand side of (4.9). Formulas (3.1) and (3.4) imply:

\[
\begin{align*}
\left\| \int_0^t S_{\varepsilon, R}(\kappa(t - s))\eta(x) \tilde{P}(s) ds \right\|_{H^r(\mathbb{R})}^2 &\leq \\
&\left\| \int_0^t e^{-\sqrt{2\pi\varepsilon} \kappa |\xi|^{1/2}(t-s)} \tilde{P}(\xi, s) ds(1 + |\xi|^\sigma) \right\|_{L^2(\mathbb{R})}^2 = \\
&= \left\| \int_0^t e^{-\varepsilon \kappa \lambda(t-s)} M \left( \eta \tilde{P} \right) ds \right\|_{H^r(\mathbb{R})}^2.
\end{align*}
\]

Integration in time and (3.25) yields:

\[
\int_0^1 \|\tilde{T}_1(t)\|_{H^r(\mathbb{R})}^2 dt \leq C \int_0^1 \|\eta \tilde{Q}(t)\|_{H^r(\mathbb{R})}^2 dt + \frac{C}{\varepsilon^2 \kappa^2} \int_0^1 \|\eta \tilde{P}(t)\|_{H^r-1/2}^2 \tag{4.10}
\]

In order to estimate the term corresponding to $\tilde{T}_2$ we first write

\[
\eta(x) \tilde{T}_{\varepsilon, R} \left( (M_{\lambda/2} - M_{\lambda/2,0})\tilde{f} \right) = T_{\varepsilon, R} \left( \eta(x) (M_{\lambda/2} - M_{\lambda/2,0})\tilde{f} \right) + [\eta, T_{\varepsilon, R}] \left( (M_{\lambda/2} - M_{\lambda/2,0})\tilde{f} \right)
\]

where $[\eta, T_{\varepsilon, R}]$ is the commutator of $T_{\varepsilon, R}$ and the multiplication by $\eta$ 

\[
[\eta, T_{\varepsilon, R}] (\varphi)(x) = \eta(x) T_{\varepsilon, R}(\varphi)(x) - T_{\varepsilon, R}(\eta \varphi)(x)
\]
Therefore
\[ T_2 = \kappa \int_0^t S_{\epsilon,R}(\kappa(t - s)) \left[ \eta(x)T_{\epsilon,R} \left( (M_{\lambda/2} - M_{\lambda/2,0})\tilde{\mathcal{f}} \right)(s) \right] \, ds \]
\[ = \kappa \int_0^t S_{\epsilon,R}(\kappa(t - s)) \left[ T_{\epsilon,R} \left( (M_{\lambda/2} - M_{\lambda/2,0})\mathcal{T} \right)(s) \right] \, ds \]
\[ + \kappa \int_0^t S_{\epsilon,R}(\kappa(t - s)) \left[ \eta, T_{\epsilon,R} \left( (M_{\lambda/2} - M_{\lambda/2,0})\tilde{f} \right) \right] \, ds \]
\[ = \mathcal{T}_{2,1} + \mathcal{T}_{2,2} \quad (4.11) \]
where we have used that \( \eta\tilde{f} = \mathcal{T} \). Let us denote
\[ \Psi(x,s) = (M_{\lambda/2} - M_{\lambda/2,0})\mathcal{T}(s) \quad (4.12) \]
and define the operator \( M(x) \) as:
\[ M(\varphi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\hat{\varphi}(\xi)| e^{ix\xi} \, d\xi. \]
Then:
\[ \left| \mathcal{T}_{2,1}(\xi) \right|^2 \leq C \kappa^2 \int_0^t \int_0^t e^{\lambda/2(\kappa(t-s))T_1(\xi)} e^{\lambda/2(\kappa(t-s))T_2(\xi)} \left| \mathcal{T}(\Psi)(\xi,s) \right| \, ds \, ds_2 \]
\[ \leq C \left| \kappa \int_0^t e^{\lambda/2(\kappa(t-s))T_1(\xi)} \left| \mathcal{T}(\Psi)(\xi,s) \right| \, ds \right|^2 \]
\[ \leq C \left| \kappa \int_0^t e^{\lambda/2(\kappa(t-s))T_1(\xi)} T_1(M(\Psi))(s) \, ds \right|^2. \]
where we have used that \( |W(\xi,\varepsilon,R)| \leq C|\text{Re}W(\xi,\varepsilon,R)| = C T_1(\xi) \). Therefore using \((3.24)\):
\[ \int_0^1 \left\| \mathcal{T}_{2,1} \right\|^2_{H^s(\mathbb{R})} \, dt \leq C \int_0^1 \left\| \int_0^t e^{\lambda/2(\kappa(t-s))T_1(\xi)} T_1(M(\Psi))(s) \, ds \right\|^2_{H^s(\mathbb{R})} \, dt \]
\[ \leq C \int_0^1 \left\| M(\Psi) \right\|^2_{H^s(\mathbb{R})} \, ds = C \int_0^1 \left\| \Psi \right\|^2_{H^s(\mathbb{R})} \, ds. \]
The function \( \Psi \) may be written as \( \Psi(x,t) = \alpha(x) \mathcal{T}(x,t) \) with \( \alpha(x) = \tilde{\eta}(x) (x^{\lambda/2} - x_0^{\lambda/2}) \)
where \( \tilde{\eta} \) is a cutoff supported in the interval \( |x - x_0| \leq \varepsilon_0 \) and \( \tilde{\eta}(x) = 1 \) in \( |x - x_0| \leq 2\delta \)
where \( \delta \) is given in \((4.4)\). Notice that \( \alpha \) may be assumed to satisfy condition \((3.36)\).
Lemma \(3.7\) then implies:
\[ \left\| \Psi \right\|_{H^s(\mathbb{R})} \leq K \varepsilon_0 \left\| \mathcal{T} \right\|_{H^s} + C \left\| \mathcal{T} \right\|_{L^\infty(\mathbb{R} \times (0,1))} \quad (4.13) \]
where the constant \( C \) here and until the end of the Proof of Theorem \(3.1\) may depend on \( \varepsilon_0 \) but \( K \) independent on it. We have then obtained:
\[ \int_0^1 \left\| \mathcal{T}_{2,1} \right\|^2_{H^s(\mathbb{R})} \, dt \leq K \varepsilon_0 \int_0^1 \left\| \mathcal{T}(s) \right\|^2_{H^s(\mathbb{R})} \, ds + C \left\| \mathcal{T} \right\|^2_{L^\infty} \quad (4.14) \]
We consider now $\mathcal{F}_{2,2}$. Using (5.20) we have:

$$||\mathcal{F}_{2,2}||_{H^s(\mathbb{R})} \leq C \kappa^{1-\beta} \int_0^t (t-s)^{-\beta} \| (M_{\lambda/2} - M_{0,\lambda/2}) \tilde{f} \|_{H^s-\rho(\mathbb{R})} ds$$

and by (4.13) with $\sigma$ replaced by $\sigma - \rho$:

$$||\mathcal{F}_{2,2}||_{H^s(\mathbb{R})} \leq C \kappa^{1-\beta} \int_0^t (t-s)^{-\beta} \| \tilde{f} \|_{H^s-\rho(\mathbb{R})} ds + C \kappa^{1-\beta} \| \tilde{f} \|_{L^\infty(\mathbb{R})}.$$  \hspace{1cm} (4.15)

Squaring and integrating (4.15) and adding to the results for $\beta$ small estimate, we obtain:

$$\int_0^1 ||\mathcal{F}_2(s)||_{H^s(\mathbb{R})}^2 ds \leq \epsilon_0 \int_0^1 ||\mathcal{F}(s)||_{H^s(\mathbb{R})}^2 ds + C \int_0^1 ||\tilde{f}(s)||_{H^s-\rho(\mathbb{R})} ds$$

$$+ C ||\tilde{f}||_\infty^2.$$ \hspace{1cm} (4.16)

The last term $\mathcal{F}_3$ is estimated as follows. Using Lemma 3.8 we obtain:

$$|| \int_0^\infty \mathcal{F} (x-y, s) (\eta(x) - \eta(x-y)) \Phi(y, \epsilon) dy ||_{H^s(\mathbb{R})} \leq C ||\tilde{f}||_{H^{(\sigma-1/2)}_s(\mathbb{R})}.$$ \hspace{1cm} (4.17)

Then, (3.22) in Lemma 3.5 and an interpolation argument yield:

$$||\mathcal{F}_3||_{H^s(\mathbb{R})} \leq C \kappa \int_0^t ||\tilde{f}(s)||_{H^{(\sigma-1/2)}_s(\mathbb{R})} ds,$$

whence:

$$\int_0^1 ||\mathcal{F}_3||_{H^s(\mathbb{R})}^2 ds \leq C \kappa^2 \int_0^1 ||\tilde{f}(s)||_{H^{(\sigma-1/2)}_s(\mathbb{R})}^2 ds + C \kappa^2 ||\tilde{f}||_\infty^2.$$ \hspace{1cm} (4.18)

Adding (4.10), (4.16) and (4.18) and using $\rho < 1/2$, we deduce:

$$\int_0^1 ||\mathcal{F}(s)||_{H^s(\mathbb{R})}^2 ds \leq \epsilon_0 \int_0^1 ||\mathcal{F}(s)||_{H^s(\mathbb{R})}^2 ds + C \int_0^t ||\tilde{f}(s)||_{H^s-\rho(\mathbb{R})} ds$$

$$+ C \int_0^1 ||\eta \tilde{Q}||_{H^s(\mathbb{R})}^2 ds + C ||\tilde{f}||_L^2 ||\mathcal{F}(t)||_{H^{s-1/2}}.$$  \hspace{1cm}

Choosing $\epsilon_0$ small enough:

$$\int_0^1 ||\mathcal{F}(s)||_{H^s(\mathbb{R})}^2 ds \leq C \int_0^t ||\tilde{f}(s)||_{H^s-\rho(\mathbb{R})} ds + C \int_0^1 ||\eta \tilde{Q}||_{H^s(\mathbb{R})}^2 ds + C ||\tilde{f}||_L^2 ||\mathcal{F}_1(t)||_{H^{s-1/2}}$$

Using a partition of the unity $(\eta_i)_{i \in \mathbb{N}}$ of the interval $(1/2, 2)$, and adding the contributions of all the terms we obtain:

$$\int_0^1 ||\mathcal{F}(s)||_{H^s(\mathbb{R})}^2 ds \leq C \int_0^t ||\tilde{f}(s)||_{H^s-\rho(\mathbb{R})} ds + C \int_0^1 ||\tilde{Q}||_{H^s(\mathbb{R})}^2 ds + C ||\tilde{f}||_L^2 ||\mathcal{F}_1(t)||_{H^{s-1/2}}$$

\hspace{1cm} (4.19)

26
where the constants $C$ depend on $\delta$. An interpolation argument then implies:

$$
\int_0^1 ||\tilde{f}(s)||^2_{H^s(\mathbb{R})} \leq \varepsilon \int_0^t ||\tilde{f}(s)||^2_{H^s(\mathbb{R})} ds + C \int_0^1 ||\tilde{Q}(s)||^2_{H^s(\mathbb{R})} ds + C||\tilde{f}||^2_{L^\infty(\mathbb{R} \times (0,1))}
$$

where we have defined:

$$
\omega(t, s) = e^{-\int_s^t a(\lambda) d\lambda}
$$

We estimate first the term with $\tilde{f}_1$. If $T \leq 1$, then the same argument of the proof of point (i) shows that

$$
\left( \int_T^{T+1} ||\tilde{f}_1(t)||^2_{H^2} dt \right)^{1/2} \leq C \left( \int_0^1 ||\tilde{Q}(t)||^2_{H^2} dt \right)^{1/2} + \frac{C}{\varepsilon^2} \left( \int_0^1 ||\tilde{P}(t)||^2_{H^2} dt \right)^{1/2}.
$$

whence part (i) of Theorem 5.1 follows.

**Remark 4.1** Notice that in (4.14), we estimate the $H^s$ norm of $\tilde{f}_{2,1}$ in terms of the $H^s$ norm of $\tilde{T}$, not of $\tilde{f}$.

**Remark 4.2** In the estimates of $\tilde{T}_j$, $j = 1, 2, 3$ the term $\tilde{T}_{2,1}$ is the only one where we are using the continuity in the freezing coefficients argument to obtain (4.14).

**Proof of (ii) of Theorem 3.1.** In order to prove part (ii) we first notice that the equation satisfied by $\tilde{T}$ is:

$$
\frac{\partial \tilde{T}}{\partial t} - x_0^{\lambda/2} T_{e,R}(\tilde{T}) + a(t) \tilde{T} = \eta(x) T_{e,R} \left( (M_{\lambda/2} - M_{\lambda/2,0}) \tilde{f} \right) + \eta(x) \tilde{Q} + \eta(x) \tilde{P} + x_0^{\lambda/2} \int_0^\infty \tilde{f}(x-y) (\eta(x) - \eta(x-y)) \Phi(y, R, \varepsilon) dy - (a(x, t) - a(t)) \tilde{T}
$$

(4.20)

where $x_0$ and $\eta$ have been chosen as before and where $a(t) = a(x_0, t)$. Then:

$$
\tilde{T}(x, t) = \int_0^t \omega(t, s) S_{e, R}(t-s) \left( \eta(x) \tilde{Q}(s) + \eta(x) \tilde{P}(s) \right) ds +
$$

$$
\int_0^t \omega(t, s) S_{e, R}(t-s) \left[ \eta(x) T_{e, R} \left( (M_{\lambda/2} - M_{\lambda/2,0}) \tilde{f} \right) \right] ds +
$$

$$
+ x_0^{\lambda/2} \int_0^t \omega(t, s) S_{e, R}(t-s) \int_0^\infty \tilde{f}(x-y, s) (\eta(x) - \eta(x-y)) \Phi(y, R, \varepsilon) dy ds
$$

$$
- \int_0^t \omega(t, s) S_{e, R}(t-s)(a(x, t) - a(t)) \tilde{T}(s) ds
$$

$$
= \tilde{T}_1(x, t) + \tilde{T}_2(x, t) + \tilde{T}_3(x, t) + \tilde{T}_4(x, t)
$$

where we have defined:

$$
\omega(t, s) = e^{-\int_s^t a(\lambda) d\lambda}
$$

(4.21)
If $T > 1$, using the change of variables: $t = (T - 1) + \tau$ we write

$$
\left( \int_T^{T+1} \frac{\| \int_0^t \omega(t,s)S_{\varepsilon,R}(t-s)\eta(x) \left( \bar{Q}(s) \right. \left. + P(s) \right) ds\|^2_{H^{s}}} {dt} \right)^{1/2}
$$

$$
\leq \sum_{n=1}^{[T]} \left( \int_T^{T+1} \frac{\| \int_{n-1}^n \omega(t,s)S_{\varepsilon,R}(t-s)\eta(x) \left( \bar{Q}(s) \right. \left. + P(s) \right) ds\|^2_{H^{s}}} {dt} \right)^{1/2}
$$

$$
+ \left( \int_T^{T+1} \frac{\| \int_{[T]}^t \omega(t,s)S_{\varepsilon,R}(t-s)\eta(x) \left( \bar{Q}(s) \right. \left. + P(s) \right) ds\|^2_{H^{s}}} {dt} \right)^{1/2}
$$

$$
= I_1 + I_2.
$$

The estimate of the term $I_2$ follows as in the proof of point (i) of the Theorem and gives

$$
I_2 \leq C \left( \int_T^{T+1} \frac{\| \bar{Q}(s) \|^2_{H^{s}}} {dt} \right)^{1/2} + \frac{C}{\varepsilon} \left( \int_T^{T+1} \frac{\| P(s) \|^2_{H^{s-1/2}}} {dt} \right)^{1/2}.
$$

To estimate $I_1$ we argue as follows. Changing the time variable $t$ as $t = \tau + (T - n)$ and obtain:

$$
I_1 = \sum_{n=1}^{[T]} \left( \int_n^{n+1} \frac{\| \int_{n-1}^n \omega(\tau + (T - n),s)S_{\varepsilon,R}((T - n) + \tau - s)\times} {\eta(x) \left( \bar{Q}(s) \right. \left. + P(s) \right) ds\|^2_{H^{s}}} d\tau \right)^{1/2}
$$

$$
\leq \sum_{n=1}^{T} \left( \int_n^{n+1} \frac{\| \int_{n-1}^n \omega(\tau + (T - n),s)S_{\varepsilon,R}(\tau - s)\eta(x) \left( \bar{Q}(s) \right. \left. + P(s) \right) ds\|^2_{H^{s}}} {d\tau} \right)^{1/2}
$$

since $\|S_\varepsilon(T - n) h\|_{H^{s}} \leq \|h\|_{H^{s}}$ because $T - N \geq 0$. We use now that for each $n$

$$
\int_n^{n+1} \frac{\| \int_{n-1}^n \omega(\tau + (T - n),s)S_{\varepsilon,R}(\tau - s)\eta(x) \left( \bar{Q}(s) \right. \left. + P(s) \right) ds\|^2_{H^{s}}} {d\tau}
$$

$$
\leq \int_{n-1}^{n+1} \frac{\| \int_{n-1}^\tau \omega(\tau + (T - n),s)\eta(x) \times} {\omega(n,s)S_{\varepsilon,R}(\tau - s)\eta(x) \times} \frac{1}{(n-1,n)}(s) \left( \bar{Q}(s) \right. \left. + P(s) \right) ds\|^2_{H^{s}}} d\tau
$$

$$
\leq Ce^{-2A(T-n)} \left( \int_{n-1}^{n+1} \frac{1}{(n-1,n)}(s) \omega(n,s)\| \bar{Q}\|^2_{H^{s}}} ds +
$$

$$
+ \frac{1}{\varepsilon^2} \int_{n-1}^{n+1} \frac{1}{(n-1,n)}(s) \omega(n,s)\| \bar{P}\|^2_{H^{s-1/2}}} ds
$$

$$
\leq Ce^{-2A(T-n)} \int_{n-1}^{n} \| \bar{Q}\|^2_{H^{s}}} ds + \frac{Ce^{-2A(T-n)}}{\varepsilon^2} \int_{n-1}^{n} \| \bar{P}\|^2_{H^{s-1/2}}} ds,
$$

28
whence:

\[
I_1 + I_2 \leq C \sum_{n=1}^{[T]} e^{-A(T-n)} \left( \int_{n-1}^{n} ||\tilde{Q}||_{H^s}^2 \, ds \right)^{1/2} + C \left( \int_{[T]}^{T+1} ||\tilde{Q}(s)||_{H^s}^2 \, ds \right)^{1/2} + \\
+ \frac{C}{\varepsilon} \sum_{n=1}^{[T]} e^{-A(T-n)} \left( \int_{n-1}^{n} ||\tilde{P}||_{H^{s-1/2}}^2 \, ds \right)^{1/2} + \frac{C}{\varepsilon} \left( \int_{[T]}^{T+1} ||\tilde{P}(s)||_{H^{s-1/2}}^2 \, ds \right)^{1/2}
\]

\[
\leq C \sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{T+1} ||\tilde{Q}||_{H^s}^2 \, ds \right)^{1/2} + \frac{C}{\varepsilon} \sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{T+1} ||\tilde{P}||_{H^{s-1/2}}^2 \, ds \right)^{1/2}
\]

and

\[
\leq C \sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{\min(T+1,T_{\max})} ||\tilde{Q}||_{H^s}^2 \, ds \right)^{1/2} + \\
+ \frac{C}{\varepsilon} \sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{\min(T+1,T_{\max})} ||\tilde{P}||_{H^{s-1/2}}^2 \, ds \right)^{1/2}
\]

(4.23)

The term \( T_2 \) is written as \( T_2 = T_{2,1} + T_{2,2} \) where \( T_{2,1} \) and \( T_{2,2} \) are defined as in (4.11). We first estimate \( T_{2,1} \). Consider then

\[
\left( \int_{T}^{T+1} \left\| \int_{0}^{t} \omega(t,s) S_{\varepsilon,R}(t-s) [T_{\varepsilon,R} \Psi(x,s)] \, ds \right\|_{H^s}^2 \, dt \right)^{1/2} \leq \\
\sum_{n=1}^{[T]} \left( \int_{T}^{T+1} \left\| \int_{n-1}^{n} \omega(t,s) S_{\varepsilon,R}(t-s) [T_{\varepsilon,R} \Psi(x,s)] \, ds \right\|_{H^s}^2 \, dt \right)^{1/2} + \\
\left( \int_{T}^{T+1} \left\| \int_{T}^{t} \omega(t,s) S_{\varepsilon,R}(t-s) [T_{\varepsilon,R} \Psi(x,s)] \, ds \right\|_{H^s}^2 \, dt \right)^{1/2}
\]

\[= I_1 + I_2.\]

Arguing as in the derivation of (4.13) we obtain that there exists a positive constant \( \varepsilon_0 \) that can be chosen arbitrarily small if \( \delta \) is small enough, and such that:

\[
I_2 \leq \varepsilon_0 \left( \int_{T}^{\min(T+1,T_{\max})} ||\tilde{T}(s)||_{H^s}^2 \, ds \right)^{1/2} + C ||\tilde{T}||_{L^\infty}^2. \quad (4.24)
\]

In the first term \( I_1 \), we change the time variable \( t \) as \( t = \tau + (T - n) \) and obtain:

\[
I_1 \leq \sum_{n=1}^{[T]} \left( \int_{n}^{n+1} \left\| \int_{n-1}^{n} \omega(\tau + (T - n),s) S_{\varepsilon,R}(\tau + (T - n) - s) [T_{\varepsilon,R} \Psi(x,s)] \, ds \right\|_{H^s}^2 \, d\tau \right)^{1/2}
\]

\[\leq \sum_{n=1}^{[T]} \left( \int_{n}^{n+1} \left\| \int_{n-1}^{n} \omega(\tau + (T - n),s) S_{\varepsilon,R}(\tau - s) [T_{\varepsilon,R} \Psi(x,s)] \, ds \right\|_{H^s}^2 \, d\tau \right)^{1/2}.
\]
Arguing again as in the derivation of (4.14) we obtain, for \( \varepsilon_0 \) defined as above:

\[
\int_{n}^{n+1} \left| \int_{n-1}^{n} \omega(\tau + (T - n), s) S_{\varepsilon,R}(\tau - s) \left[ T_{\varepsilon,R} \Psi(x, s) \right] ds \right|^2 d\tau
\]

\[
= \int_{n-1}^{n+1} \left| \int_{n-1}^{n} \omega(\tau + (T - n), s) S_{\varepsilon,R}(\tau - s) \left[ T_{\varepsilon,R} \left( \mathbf{1}_{(n-1,n)}(s) \Psi(x, s) \right) \right] ds \right|^2 d\tau
\]

\[
\leq e^{-2A(T-n)} \left( \varepsilon_0 \int_{n-1}^{n} \left| \mathbf{f} \right|^2_{H^{\sigma}} ds \right)^{1/2} + C \left| \mathbf{f} \right|^2_{L^{\infty}}.
\]

Therefore

\[
I_1 \leq \varepsilon_0 \sum_{n=1}^{\lceil T \rceil} e^{-A(T-n)} \left( \int_{n-1}^{n} \left| \mathbf{f} \right|^2_{H^{\sigma}} ds \right)^{1/2} + C \sum_{n=1}^{\lceil T \rceil} e^{-A(T-n)} \left| \mathbf{f} \right|^2_{L^{\infty}}
\]

\[
\leq \varepsilon_0 \sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{\min(T+1,T_{\max})} \left| \mathbf{f} \right|^2_{H^{\sigma}} ds \right)^{1/2} + C \left| \mathbf{f} \right|^2_{L^{\infty}},
\]

whence

\[
I_1 + I_2 \leq \varepsilon_0 \sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{\min(T+1,T_{\max})} \left| \mathbf{f} \right|^2_{H^{\sigma}} ds \right)^{1/2} + C \left| \mathbf{f} \right|^2_{L^{\infty}}.
\]

We then obtain the estimate:

\[
\sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{\min(T+1,T_{\max})} \left| \mathbf{f}_{2,1}(s) \right|^2_{H^{\sigma}} ds \right)^{1/2}
\]

\[
\leq \varepsilon_0 \sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{\min(T+1,T_{\max})} \left| \mathbf{f} \right|^2_{H^{\sigma}} ds \right)^{1/2} + C \left| \mathbf{f} \right|^2_{L^{\infty}}. \quad (4.25)
\]

A similar argument using the contractivity of \( S_{\varepsilon,R} \) in the spaces \( H^{\sigma} \) gives for \( \mathbf{f}_{2,2} \) and \( \mathbf{f}_{3} \):

\[
\sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{\min(T+1,T_{\max})} \left| \mathbf{f}_{2,2}(t) \right|^2_{H^{\sigma}} dt \right)^{1/2}
\]

\[
\leq C \sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{\min(T+1,T_{\max})} \left| \tilde{\mathbf{f}} \right|^2_{H^{\sigma-\rho}} ds \right)^{1/2} + C \left| \mathbf{f} \right|^2_{L^{\infty}} \quad (4.26)
\]

\[
\sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{\min(T+1,T_{\max})} \left| \mathbf{f}_{3}(t) \right|^2_{H^{\sigma}} dt \right)^{1/2}
\]

\[
\leq C \sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{\min(T+1,T_{\max})} \left| \tilde{\mathbf{f}} \right|^2_{H^{\sigma-1/2}} ds \right)^{1/2} + C \left| \tilde{\mathbf{f}} \right|^2_{L^{\infty}}. \quad (4.27)
\]
We now estimate \( \mathcal{F}_4 \):

\[
\left( \int_T^{T+1} ||\mathcal{F}_4(s)||^2_{H^\sigma} ds \right)^{1/2} \\
\leq \sum_{n=1}^{[T]} \left( \int_T^{T+1} || \int_{t-n}^{t-n+1} \omega(t,s)S_{\varepsilon,R}(t-s)(a(x,t) - a(t))T(s)ds||^2_{H^\sigma} ds \right)^{1/2} \\
+ \left( \int_T^{T+1} || \int_T^{t} \omega(t,s)S_{\varepsilon,R}(t-s)(a(x,t) - a(t))T(s)ds||^2_{H^\sigma} \right)^{1/2} \\
= I_1 + I_2.
\]

We use now the continuity of the semigroup \( S_{\varepsilon,R} \) in \( H^\sigma \), the fact that \( a \in H^\sigma(\mathbb{R}) \) and since \( \sigma > 1/2 \), the imbedding of \( H^\sigma \) into \( C(\mathbb{R}) \) is continuous to obtain the existence of a positive constant \( \varepsilon_0 \), which can be made arbitrarily small if \( \delta \) is sufficiently small, such that

\[
||S_{\varepsilon,R}(t-s)(a(x,t) - a(t))T(s)||_{H^\sigma} \leq ||(a(x,t) - a(t))T(s)||_{H^\sigma} \\
\leq (\varepsilon_0 ||T||_{H^\sigma} + C ||T||_{L^\infty}) ||a||_{H^\sigma}.
\]

Arguing as in the derivation of (4.25) we obtain

\[
\sup_{0 \leq T \leq T_{\max}} \left( \int_T^{\min(T+1,T_{\max})} ||\mathcal{F}_4(s)||^2_{H^\sigma} ds \right)^{1/2} \\
\leq \varepsilon_0 \sup_{0 \leq T \leq T_{\max}} \left( \int_T^{\min(T+1,T_{\max})} ||T||^2_{H^\sigma} ds \right)^{1/2} + C ||T||_{L^\infty}, \quad (4.28)
\]

Adding formulas (4.23), (4.25)–(4.28) we obtain:

\[
\sup_{0 \leq T \leq T_{\max}} \left( \int_T^{\min(T+1,T_{\max})} ||\tilde{T}(t)||^2_{H^\sigma} dt \right)^{1/2} \\
\leq \varepsilon_0 \sup_{0 \leq T \leq T_{\max}} \left( \int_T^{\min(T+1,T_{\max})} ||\bar{T}||^2_{H^\sigma} ds \right)^{1/2} + C ||\tilde{T}||_{L^\infty} \\
+ C \sup_{0 \leq T \leq T_{\max}} \left( \int_T^{\min(T+1,T_{\max})} ||\bar{Q}||^2_{H^{\sigma-\rho}} ds \right)^{1/2} + C \varepsilon \sup_{0 \leq T \leq T_{\max}} \left( \int_T^{T+1} ||\bar{P}||^2_{H^{\sigma-1/2}} ds \right)^{1/2}
\]
where we have used that \( ||f||_{L^\infty} \leq ||\tilde{f}||_{L^\infty} \) and \( \rho \leq 1/2 \). Then,
\[
\sup_{0 \leq T \leq T_{\max}} \left( \int_T^{\min(T+1,T_{\max})} ||\tilde{f}(t)||^2_{H^\sigma} \, dt \right)^{1/2} \leq C \sup_{0 \leq T \leq T_{\max}} \left( \int_T^{\min(T+1,T_{\max})} ||\tilde{f}||^2_{H^\sigma-\rho} \, ds \right)^{1/2}
\]
\+
\sup_{0 \leq T \leq T_{\max}} \left( \int_T^{\min(T+1,T_{\max})} ||\tilde{Q}||^2_{H^\sigma} \, ds \right)^{1/2}
\]
\+
\sup_{0 \leq T \leq T_{\max}} \left( \int_T^{T+1} ||\tilde{P}||^2_{H^{\sigma-1/2}} \, ds \right)^{1/2}.
\]
(4.29)

Using a partition of unity as in the derivation of (4.19), we arrive at
\[
\sup_{0 \leq T \leq T_{\max}} \left( \int_T^{\min(T+1,T_{\max})} ||\tilde{f}(t)||^2_{H^\sigma} \, dt \right)^{1/2} \leq C \sup_{0 \leq T \leq T_{\max}} \left( \int_T^{\min(T+1,T_{\max})} ||\tilde{f}||^2_{H^\sigma-\rho} \, ds \right)^{1/2}
\]
\+
\sup_{0 \leq T \leq T_{\max}} \left( \int_T^{\min(T+1,T_{\max})} ||\tilde{Q}||^2_{H^\sigma} \, ds \right)^{1/2}
\]
\+
\sup_{0 \leq T \leq T_{\max}} \left( \int_T^{T+1} ||\tilde{P}||^2_{H^{\sigma-1/2}} \, ds \right)^{1/2}.
\]
(4.30)

where the constants \( C > 0 \) depend on \( \delta \). An interpolation argument yields
\[
\sup_{0 \leq T \leq T_{\max}} \left( \int_T^{\min(T+1,T_{\max})} ||\tilde{f}(t)||^2_{H^\sigma} \, dt \right)^{1/2} \leq C \sup_{0 \leq T \leq T_{\max}} \left( \int_T^{\min(T+1,T_{\max})} ||\tilde{Q}(t)||^2_{H^\sigma} \, dt \right)^{1/2} + C ||\tilde{f}||_{L^\infty}
\]
\+
\sup_{0 \leq T \leq T_{\max}} \left( \int_T^{T+1} ||\tilde{P}||^2_{H^{\sigma-1/2}} \, ds \right)^{1/2}.
\]
(4.31)

Using that \( \chi = 1 \) in the interval \((5/8, 11/8)\) we have:
\[
||f||_{H^\sigma(3/4,5/4)} \leq ||\tilde{f}||_{H^\sigma}.
\]
(4.32)

Part (ii) of Theorem 3.1 then follows combining (4.31) and (4.32).

**Proof of part (iii) of Theorem 3.1**  The equation satisfied by \( \tilde{f} \) is now (4.20) with \( \tilde{P} = 0 \) and \( \varepsilon = 0 \). Then:
\[
\tilde{f}(x,t) = \int_0^t \omega(t,s)S_{\varepsilon,R}(t-s)\eta(x)\tilde{Q}(s) \, ds
\]
\+
\int_0^t \omega(t,s)S_{\varepsilon,R}(t-s) \left[ \eta(x)T_{\varepsilon,R}((M_{\lambda/2} - M_{\lambda/2,0})\tilde{f}) \right] \, ds
\]
\+
\int_0^x \frac{\rho}{2} \left[ \int_0^t \omega(t,s)S_{\varepsilon,R}(t-s) \int_0^\infty \tilde{f}(x-y,s) (\eta(x) - \eta(x-y)) \Phi(y,R) \, dy \, ds \right]
\]
\+
\int_0^t \omega(t,s)S_{\varepsilon,R}(t-s)(a(x,t) - a(t)) \tilde{Q}(s) \, ds
\]
= \tilde{f}_1(x,t) + \tilde{f}_2(x,t) + \tilde{f}_3(x,t) + \tilde{f}_4(x,t)
\]
(32)
where $\omega$ is given by (4.21). The term $\mathcal{F}_1$ is estimated using (3.24) for $T \leq 1$. Then,

$$
\left( \int_T^{T+1} \| T_1(\mathcal{F}_1)(t) \|_{H^s}^2 dt \right)^{1/2} \leq \left( \int_0^2 \| T_1(\mathcal{F}_1)(t) \|_{H^s}^2 dt \right)^{1/2} \\
\leq C \left( \int_0^2 \| \tilde{Q}(t) \|_{H^s}^2 dt \right)^{1/2}.
$$

(4.33)

If $T > 1$, using the change of variables: $t = (T - 1) + \tau$ we write

$$
\left( \int_T^{T+1} \| T_1 \left( \int_0^T \omega(t, s) S_{\varepsilon, R}(t-s) \eta(x) \tilde{Q}(s) ds \right) \|_{H^s}^2 dt \right)^{1/2} \\
\leq \sum_{n=1}^{[T]} \left( \int_T^{T+1} \| \int_{n-1}^n \omega(t, s) T_1 \left( S_{\varepsilon, R}(t-s) \eta(x) \tilde{Q}(s) \right) ds \|_{H^s}^2 dt \right)^{1/2} \\
+ \left( \int_T^{T+1} \| \int_T^t \omega(t, s) T_1 \left( S_{\varepsilon, R}(t-s) \eta(x) \tilde{Q}(s) \right) ds \|_{H^s}^2 dt \right)^{1/2} = I_1 + I_2.
$$

The estimate of the term $I_2$ can be made as in (4.33):

$$
I_2 \leq C \left( \int_T^{T+1} \| \tilde{Q}(s) \|_{H^s}^2 \right)^{1/2}.
$$

To estimate $I_1$ we argue as follows: we change the time variable $t$ as $t = \tau + (T - n)$ and obtain:

$$
I_1 = \sum_{n=1}^{[T]} \left( \int_n^{n+1} \| \int_{n-1}^n \omega(\tau + (T - n), s) T_1 S_{\varepsilon, R}((T - n) + \tau - s) \left( \eta(x) \tilde{Q}(s) \right) ds \|_{H^s}^2 d\tau \right)^{1/2} \\
\leq \sum_{n=1}^T \left( \int_n^{n+1} \| \int_{n-1}^n \omega(\tau + (T - n), s) T_1 S_{\varepsilon, R}(\tau - s) \left( \eta(x) \tilde{Q}(s) \right) ds \|_{H^s}^2 d\tau \right)^{1/2}
$$

where we have used $\| S_{\varepsilon}(T - n) h \|_{H^s} \leq \| h \|_{H^s}$. We notice now that, for each $n$

$$
\int_{n-1}^{n+1} \| \int_{n-1}^n \omega(\tau + (T - n), s) T_1 S_{\varepsilon, R}(\tau - s) \left( \eta(x) \tilde{Q}(s) \right) ds \|_{H^s}^2 d\tau \\
\leq \int_{n-1}^{n+1} \| \int_{n-1}^n \omega(\tau + (T - n), s) T_1 S_{\varepsilon, R}(\tau - s) \left( \eta(x) \mathbf{1}_{(n-1,n)}(s) \tilde{Q}(s) \right) ds \|_{H^s}^2 d\tau \\
= \int_{n-1}^{n+1} \| \int_{n-1}^n \omega(\tau + (T - n), s) \left( \omega(n, s) T_1 S_{\varepsilon, R}(\tau - s) \left( \eta(x) \mathbf{1}_{(n-1,n)}(s) \tilde{Q}(s) \right) ds \|_{H^s}^2 d\tau \\
\leq C e^{-2A(T-n)} \left( \int_{n-1}^{n+1} \mathbf{1}_{(n-1,n)}(s) \omega(n, s) \| \tilde{Q}(s) \|_{H^s}^2 ds \right) \\
\leq C e^{-2A(T-n)} \int_{n-1}^{n} \| \tilde{Q} \|_{H^s}^2 ds,
$$

33
whence:

\[ I_1 + I_2 \leq C \sum_{n=1}^{[T]} e^{-A(T-n)} \left( \int_{n-1}^{n} \| \tilde{Q} \|_{H^s}^2 \, ds \right)^{1/2} + C \left( \int_{[T]}^{T+1} \| \tilde{Q}(s) \|_{H^s}^2 \, ds \right)^{1/2} \]

\[ \leq C \sup_{0 \leq T \leq T_{\text{max}}} \left( \int_{T}^{T+1} \| \tilde{Q} \|_{H^s}^2 \, ds \right)^{1/2} \]

and

\[ \sup_{0 \leq T \leq T_{\text{max}}} \left( \int_{T}^{\min(T+1,T_{\text{max}})} \| T_1(\tilde{f}_1)(s) \|_{H^s}^2 \, ds \right)^{1/2} \]

\[ \leq C \sup_{0 \leq T \leq T_{\text{max}}} \left( \int_{T}^{\min(T+1,T_{\text{max}})} \| \tilde{Q} \|_{H^s}^2 \, ds \right)^{1/2}. \]  \hspace{1cm} (4.34)

The term \( \tilde{f}_2 \) is written as \( \tilde{f}_2 = \tilde{f}_{2,1} + \tilde{f}_{2,2} \) where \( \tilde{f}_{2,1} \) and \( \tilde{f}_{2,2} \) are defined as in (4.11). We first estimate \( \tilde{f}_{2,1} \). Consider then

\[ \left( \int_{T}^{T+1} \| T_1 \int_{0}^{t} \omega(t,s)S_{\epsilon,R}(t-s) [T_{\epsilon,R}\Psi(x,s)] \, ds \|_{H^s}^2 \, dt \right)^{1/2} \]

\[ \leq \sum_{n=1}^{[T]} \left( \int_{T}^{T+1} \| \int_{n-1}^{n} \omega(t,s)T_1 S_{\epsilon,R}(t-s) [T_{\epsilon,R}\Psi(x,s)] \, ds \|_{H^s}^2 \, dt \right)^{1/2} \]

\[ + \left( \int_{T}^{T+1} \| \int_{[T]}^{T+1} \omega(t,s)T_1 S_{\epsilon,R}(t-s) [T_{\epsilon,R}\Psi(x,s)] \, ds \|_{H^s}^2 \, dt \right)^{1/2} \]

\[ = I_1 + I_2. \]

Arguing as in the derivation of (4.14), but using (3.38) instead of (3.37) we obtain that there exists a positive constant \( \varepsilon_0 \) that can be chosen arbitrarily small such that:

\[ I_2 \leq \varepsilon_0 \left( \int_{T}^{\min(T+1,T_{\text{max}})} \| T_1(\tilde{f}_1)(s) \|_{H^s}^2 \, ds \right)^{1/2} + C \| T \|_{L_{\infty}}. \]  \hspace{1cm} (4.35)

In the first term \( I_1 \), we change the time variable \( t \) as \( t = \tau + (T-n) \) and obtain:

\[ I_1 \leq \sum_{n=1}^{[T]} \left( \int_{n}^{n+1} \int_{n-1}^{n} \omega(\tau + (T-n),s)T_1 S_{\epsilon,R}(\tau + (T-n) - s) [T_{\epsilon,R}\Psi(x,s)] \, ds \|_{H^s}^2 \, d\tau \right)^{1/2} \]

\[ \leq \sum_{n=1}^{[T]} \left( \int_{n}^{n+1} \int_{n-1}^{n} \omega(\tau + (T-n),s)T_1 S_{\epsilon,R}(\tau - s) [T_{\epsilon,R}\Psi(x,s)] \, ds \|_{H^s}^2 \, d\tau \right)^{1/2}. \]
Arguing again as in the derivation of (4.14) we obtain, for \( \varepsilon_0 \) defined as above,

\[
\int_n^{n+1} \int_{n-1}^{n} \omega(\tau + (T - n), s) T_1 S_{\varepsilon,R}(\tau - s) [T_{\varepsilon,R} \Psi(x, s)] \, ds ||| T_{1,1} \, d\tau \\
\leq \exp[-A(T-n)] \left( \varepsilon_0 \int_{n-1}^{n} 1_{(n-1,n)}(s) ||| T_1 \, ds + C ||| T \right) \\
= \exp[-A(T-n)] \left( \varepsilon_0 \int_{n}^{n+1} ||| T_1 \, ds + C ||| T \right).
\]

Therefore

\[
I_1 \leq \varepsilon_0 \sum_{n=1}^{[T]} e^{-A(T-n)} \left( \int_{n-1}^{n} ||| T_1 \, ds \right)^{1/2} + C \sum_{n=1}^{[T]} e^{-A(T-n)} ||| T \right) \\
\leq \varepsilon_0 \sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{\min(T+1,T_{\max})} ||| T_1 \, ds \right)^{1/2} + C ||| T \right),
\]

whence

\[
I_1 + I_2 \leq \varepsilon_0 \sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{\min(T+1,T_{\max})} ||| T_1 \, ds \right)^{1/2} + C ||| T \right)
\]

and therefore

\[
\sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{\min(T+1,T_{\max})} ||| T_1 \, ds \right)^{1/2} \\
\leq \varepsilon_0 \sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{\min(T+1,T_{\max})} ||| T_1 \, ds \right)^{1/2} + C ||| T \right). \tag{4.36}
\]

A similar argument using the contractivity of \( S_{\varepsilon,R} \) in the spaces \( H^\sigma \) and formula (3.27) gives

\[
\sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{\min(T+1,T_{\max})} ||| T_1 \, ds \right)^{1/2} \\
\leq C \sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{\min(T+1,T_{\max})} ||| T_1 \, ds \right)^{1/2} + C ||| T \right). \tag{4.37}
\]

To estimate \( f_3 \) we combine (3.24) and (4.17) to obtain

\[
\sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{\min(T+1,T_{\max})} ||| T_1 \, ds \right)^{1/2} \\
\leq C \sup_{0 \leq T \leq T_{\max}} \left( \int_{T}^{\min(T+1,T_{\max})} ||| T_1 \, ds \right)^{1/2} + C ||| T \right). \tag{4.38}
\]

35
We now estimate $I_4$:

$$
\left( \int_T^{T+1} \| T_1 \mathcal{F}_4(s) \|_{H^\sigma}^2 \, ds \right)^{1/2}
\leq \sum_{n=1}^{[T]} \left( \int_T^{T+1} \| \int_{n-1}^n \omega(t,s) T_1 S_{\epsilon,R}(t-s)(a(x,t) - a(t)) \mathcal{F}(s) \, ds \|_{H^\sigma}^2 \, dt \right)^{1/2}
+ \left( \int_T^{T+1} \| \int_T^t \omega(t,s) T_1 S_{\epsilon,R}(t-s)(a(x,t) - a(t)) \mathcal{F}(s) \, ds \|_{H^\sigma}^2 \right)^{1/2}
= I_1 + I_2.
$$

Using (3.24) we get

$$
I_2 \leq \left( \int_T^{T+1} \| (a(x,t) - a(t)) \mathcal{F} \|_{H^\sigma(\mathbb{R})}^2 \right)^{1/2}
$$

Since $a \in H^{\sigma+1} \subset C^{1,\alpha}$ for some $\alpha > 0$, we obtain, for $\delta$ sufficiently small

$$
I_2 \leq C \left( \int_T^{T+1} \left( \varepsilon_0 \| \mathcal{F}(t) \|_{H^\sigma(\mathbb{R})}^2 + C \| \mathcal{F}(t) \|_{L^\infty} \right) \, dt \right)^{1/2}
$$

The term $I_1$ can be estimated similarly using the exponential decay of $\omega(t,s)$ as in the previous cases. Then

$$
\sup_{0 \leq T \leq T_{max}} \left( \int_T^{\min(T+1,T_{max})} \| T_1 \mathcal{F}_4(s) \|_{H^\sigma}^2 \, ds \right)^{1/2}
\leq \varepsilon_0 \sup_{0 \leq T \leq T_{max}} \left( \int_T^{\min(T+1,T_{max})} \| \mathcal{F}_s \|_{H^\sigma}^2 \, ds \right)^{1/2} + C \| \mathcal{F} \|_{L^\infty}.
$$

Adding formulas (3.23), (3.25)–(3.28) we obtain:

$$
\sup_{0 \leq T \leq T_{max}} \left( \int_T^{\min(T+1,T_{max})} \| T_1 \mathcal{F}(t) \|_{H^\sigma}^2 \, dt \right)^{1/2}
\leq \varepsilon_0 \sup_{0 \leq T \leq T_{max}} \left( \int_T^{\min(T+1,T_{max})} \| T_1 \mathcal{F} \|_{H^\sigma}^2 \, ds \right)^{1/2}
+ C \sup_{0 \leq T \leq T_{max}} \left( \int_T^{\min(T+1,T_{max})} \| \mathcal{F} \|_{H^\sigma}^2 \, ds \right)^{1/2}
+ C \| \mathcal{F} \|_{L^\infty}
$$

where we have used that $\| \mathcal{F} \|_{L^\infty} \leq \| \tilde{\mathcal{F}} \|_{L^\infty}$ and $\rho \leq 1/2$. Then,

$$
\sup_{0 \leq T \leq T_{max}} \left( \int_T^{\min(T+1,T_{max})} \| T_1 \mathcal{F}(t) \|_{H^\sigma}^2 \, dt \right)^{1/2}
\leq C \sup_{0 \leq T \leq T_{max}} \left( \int_T^{\min(T+1,T_{max})} \| \mathcal{F} \|_{H^{\sigma-\rho}}^2 \, ds \right)^{1/2}
+ C \| \mathcal{F} \|_{L^\infty} + C \sup_{0 \leq T \leq T_{max}} \left( \int_T^{\min(T+1,T_{max})} \| \tilde{Q} \|_{H^\sigma}^2 \, ds \right)^{1/2}.
$$

(4.40)
Using a partition of unity as for the derivation of (4.19), we arrive at
\[
\sup_{0 \leq T \leq T_{\max}} \left( \int_T^{\min(T+1, T_{\max})} |T_1 \tilde{f}(t)|^2_{H^\sigma_T} dt \right)^{1/2} \leq C \sup_{0 \leq T \leq T_{\max}} \left( \int_T^{\min(T+1, T_{\max})} ||\tilde{f}||^2_{H^\sigma_{T - \rho}} ds \right)^{1/2} + C||\tilde{f}||_{L^\infty} + C \sup_{0 \leq T \leq T_{\max}} \left( \int_T^{\min(T+1, T_{\max})} \|Q\|_{H^\nu_T}^2 ds \right)^{1/2}
\]
where the constants $C > 0$ depend on $\delta$. An interpolation argument yields
\[
\sup_{0 \leq T \leq T_{\max}} \left( \int_T^{\min(T+1, T_{\max})} ||T_1 \tilde{f}(t)||^2_{H^\sigma_T} dt \right)^{1/2} \leq C \sup_{0 \leq T \leq T_{\max}} \left( \int_T^{\min(T+1, T_{\max})} \|Q\|_{H^\nu_T}^2 dt \right)^{1/2} + C||\tilde{f}||_{L^\infty}
\]
Using that $\chi = 1$ in the interval $(5/8, 11/8)$, we obtain
\[
||T_1 f||_{H^\sigma(3/4, 5/4)} \leq C||T_1 \tilde{f}||_{H^\sigma}. \tag{4.42}
\]
On the other hand,
\[
|W(k, R, 0)| \leq C \min\{|k|, R\}
\]
Therefore estimate (3.10) holds. This concludes the proof of part (iii) of Theorem 3.1.

We state now the main result of this Section.

**Theorem 4.3** Suppose that $\sigma \in (1/2, 2)$, $\nu \in L^2(0, 1; H^0_{x}((1/4, 4)))$, $\varepsilon \in [0, 1]$, $K \in L^\infty((1/4, 4) \times (0, 1)) \cap L^2(0, 1; H^0_{x}((1/4, 4)))$, and $h \in L^\infty((1/8, 4) \times (0, 1)) \cap L^2(0, 1; H^{1/2}(1/4, 2)) \cap H^1(0, 1; L^2(1/4, 2))$, $W \in L^2(0, 1; H^{\sigma - 1/2}_{x}(1/4, 4))$ satisfies:

\[
\frac{\partial h}{\partial t} = \varepsilon \int_0^{x/2} \frac{(x - y)^{\lambda/2} h(x - y) - x^{\lambda/2} h(x)}{y^{\lambda/2}} + (1 - \varepsilon) R^{\lambda/2} \int_0^{x/2} (x - y)^{\lambda/2} h(x - y) - x^{\lambda/2} h(x) (Ry)^{\lambda/2} f_0(Ry) dy + K(x, t) h(x, t) + \nu(x, t) + W(x, t),
\]

for all $x \in (1/4, 4)$ and $R > 1$ and $h(x, 0) = 0$. Then for any $T \in [0, 1]:$

\[
||h||_{L_t^2(0, T; H^\sigma_x(7/8, 9/8))} \leq C \left( ||\nu||_{L_t^2(0, 1; H^0_{x}((1/4, 4)))} + ||h||_{L^\infty((1/8, 4) \times (0, 1))} + \frac{1}{\varepsilon} ||W||_{L_t^2(0, 1; H^{\sigma - 1/2}_{x}(1/4, 4))} \right)
\]

where the constant $C$ is independent of $\varepsilon$ and $R$ but depends on $||K||_{L^\infty((1/2, 2) \times (0, 1))}$ and $||K||_{L_t^2(0, 1; H^0_{x}((1/4, 4)))}$. 

37
Proof of Theorem 4.3. Let \( \chi \) be a \( C^\infty \) function such that
\[
\chi(x) = \begin{cases} 1 & \text{if } x \in (1/2, 2), \\ 0 & \text{if } x \notin (1/4, 4). \end{cases}
\] (4.43)

We define \( \tilde{h}(x, t) = \chi(x) h(x) \). Then, for all \( x \in \mathbb{R} \) the function \( \tilde{h} \) satisfies
\[
\frac{\partial \tilde{h}}{\partial t} = \int_0^{x/2} \left( (x-y)^{\lambda/2} \tilde{h}(x-y) - x^{\lambda/2} \tilde{h}(x) \right) \Phi(y, R, \varepsilon) dy \\
+ \int_0^{x/2} (x-y)^{\lambda/2} h(x-y) (\chi(x) - \chi(x-y)) \Phi(y, R, \varepsilon) dy \\
+ K(x, t) \tilde{h}(x, t) + \chi(x) \nu + \chi(x) W \\
= \int_0^{+\infty} \left( (x-y)^{\lambda/2} \tilde{h}(x-y) - x^{\lambda/2} \tilde{h}(x) \right) \Phi(y, R, \varepsilon) dy \\
- \int_{x/2}^{+\infty} \left( (x-y)^{\lambda/2} \tilde{h}(x-y) - x^{\lambda/2} \tilde{h}(x) \right) \Phi(y, R, \varepsilon) dy \\
+ \int_0^{x/2} (x-y)^{\lambda/2} h(x-y) (\chi(x) - \chi(x-y)) \Phi(y, R, \varepsilon) dy \\
+ K(x, t) \tilde{h}(x, t) + \chi(x) \nu + \chi(x) W. \tag{4.44}
\]

where the function \( \Phi(y, R, \varepsilon) \) has been defined in (3.2).

We write the equation (4.44) in terms of the new function \( \overline{h} \) defined as:
\[
\tilde{h}(x, t) = e^{\int_0^t K(x, s) ds + c_0(\varepsilon, R, x) t} \overline{h}(x, t)
\]
with,
\[
c_0(\varepsilon, R, x) = x^{\lambda/2} \int_{x/2}^{\infty} \Phi(y, R, \varepsilon) dy
\]
\[
\frac{\partial \overline{h}}{\partial t} = T_{\varepsilon, R} (M_{\lambda/2} \overline{h}) + Q_1 + Q_2 + Q_3 + Q_4
\]
where \( T_{\varepsilon, R} \) has been defined in (3.1) and
\[
Q_1 = -\int_{x/2}^{+\infty} (x-y)^{\lambda/2} \overline{h}(x-y) \Phi(y, R, \varepsilon) dy \\
Q_2 = e^{-(\int_0^t K(x, s) ds + 2\sqrt{2\varepsilon} \nu(x) t)} \int_0^{x/2} (x-y)^{\lambda/2} h(x-y) (\chi(x) - \chi(x-y)) \Phi(y, R, \varepsilon) dy \\
Q_3 = e^{-(\int_0^t K(x, s) ds + c_0(\varepsilon, R, x) t)} \chi(x) \nu \\
Q_4 = e^{-(\int_0^t K(x, s) ds + c_0(\varepsilon, R, x) t)} \chi(x) W.
\]
These terms are estimated as follows:

\[
\|Q_1\|_{L^\infty(0,1;W^{1,\infty}((1/4,4)))} + \|Q_2\|_{L^\infty(0,1;H^\sigma_x((1/4,4)))} \leq C \|h\|_{L^\infty((1/8,4)\times(0,1))}
\]

\[
\|Q_3\|_{L^2(0,1;H^\sigma_x(1/4,4))} \leq C \|\nu\|_{L^2(0,1;H^\sigma_x(1/4,4))},
\]

\[
\|Q_4\|_{L^2(0,1;H^{\sigma_x-1/2}(1/4,4))} \leq C \|\nu\|_{L^2(0,1;H^{\sigma_x-1/2}(1/4,4))}.
\]

where \(\tilde{\sigma} = \min\{\sigma, 1\} \).

If \(1/2 < \sigma \leq 1\), Theorem 3.1 immediately yields:

\[
\|\tilde{h}\|_{L^2(0,1;H^\sigma_x(3/4,5/4))} \leq C \left(\|h\|_{L^\infty((1/8,4)\times(0,1))} + \|\nu\|_{L^2(0,1;H^\sigma_x(1/4,4))} + \|\nu\|_{L^\infty((1/8,4)\times(0,1))}\right) + 
\]

\[
+ \frac{C}{\varepsilon} \|W\|_{L^2(0,1;H^{\sigma_x-1/2}(1/4,4))}
\]

\[
\leq C \left(\|\nu\|_{L^2(0,1;H^\sigma_x(1/4,4))} + \|\nu\|_{L^\infty((1/8,4)\times(0,1))}\right) + 
\]

\[
+ \frac{C}{\varepsilon} \|W\|_{L^2(0,1;H^{\sigma_x-1/2}(1/4,4))}. \tag{4.45}
\]

If \(\sigma > 1\) we apply Theorem 3.1 with \(\sigma = 1\) to obtain:

\[
\|\tilde{h}\|_{L^2(0,1;H^\sigma_x(3/4,5/4))} \leq C \left(\|h\|_{L^\infty((1/8,4)\times(0,1))} + \|\nu\|_{L^2(0,1;H^\sigma_x(1/4,4))} + \|\nu\|_{L^\infty((1/8,4)\times(0,1))}\right) + 
\]

\[
+ \frac{C}{\varepsilon} \|W\|_{L^2(0,1;H^{\sigma_x-1/2}(1/4,4))}
\]

\[
\leq C \left(\|\nu\|_{L^2(0,1;H^\sigma_x(1/4,4))} + \|\nu\|_{L^\infty((1/8,4)\times(0,1))}\right) + 
\]

\[
+ \frac{C}{\varepsilon} \|W\|_{L^2(0,1;H^{\sigma_x-1/2}(1/4,4))}. \tag{4.45}
\]

Since \(Q_1\) and \(Q_2\) involve integrals of the function \(\tilde{h}\), Theorem 3.1 provides better estimates on \(Q_1\) and \(Q_2\) although on the smaller interval \((3/4, 5/4)\):

\[
\|Q_1\|_{L^\infty(0,1;H^\sigma_x((3/4,5/4)))} + \|Q_2\|_{L^\infty(0,1;H^\sigma_x((3/4,5/4)))} \leq C \|\tilde{h}\|_{L^2(0,1;H^\sigma_x(3/4,5/4))}
\]

\[
\leq C \left(\|\nu\|_{L^2(0,1;H^\sigma_x(1/4,4))} + \|\nu\|_{L^\infty((1/8,4)\times(0,1))}\right). \tag{4.45}
\]

Using again the Theorem 3.1 with \(\sigma < 2\):

\[
\|\tilde{h}\|_{L^2(0,1;H^\sigma_x(7/8,9/8))} \leq C \left(\|h\|_{L^\infty((1/8,4)\times(0,1))} + \|\nu\|_{L^2(0,1;H^\sigma_x(1/4,4))} + 
\]

\[
+ \|\nu\|_{L^\infty((1/4,2)\times(0,1))}\right) + \frac{C}{\varepsilon} \|W\|_{L^2(0,1;H^{\sigma_x-1/2}(1/4,4))}
\]

\[
\leq C \left(\|\nu\|_{L^2(0,1;H^\sigma_x(1/4,4))} + \|\nu\|_{L^\infty((1/8,4)\times(0,1))}\right) + 
\]

\[
+ \frac{C}{\varepsilon} \|W\|_{L^2(0,1;H^{\sigma_x-1/2}(1/4,4))}. \tag{4.45}
\]

This ends the proof of Theorem 3.1. \(\square\)

We end this Section with the following property of the operator \(\mathcal{L} - L\).
Lemma 4.4 Consider the operators $\mathcal{W}_R$ and $\mathcal{W}_\infty$ defined as:

$$\mathcal{W}_R(h) = R^{(3+\lambda)/2} \int_0^{x/2} \left( (x-y)^{\lambda/2}h(x-y) - x^{\lambda/2}h(x) \right) y^{\lambda/2} f_0(Ry) dy$$  \hspace{1cm} (4.46)

$$\mathcal{W}_\infty(h) = \int_0^{x/2} \left( (x-y)^{\lambda/2}h(x-y) - x^{\lambda/2}h(x) \right) y^{-3/2} dy$$ \hspace{1cm} (4.47)

$$\mathcal{W}_\infty,\varepsilon(h) = \int_0^{x/2} \left( (x-y)^{\lambda/2}h(x-y) - x^{\lambda/2}h(x) \right) \frac{dy}{y^{3/2} + \varepsilon^{3/2}x^{3/2}}$$ \hspace{1cm} (4.48)

Then, for any $\eta \in C^\infty(\mathbb{R})$ of compact support contained in $(1/2, 3/2)$ such that $\eta = 1$ on $(3/4, 5/4)$, and for all $\sigma \geq 1/2$, there exists a positive constant $C$, depending only on the function $\eta$ and its derivatives, such that for all $\psi \in H^s(\mathbb{R})$:

$$||\mathcal{W}_\infty(\eta \psi)||_{H^{s-1/2}(\mathbb{R})} + ||\mathcal{W}_R(\eta \psi)||_{H^{s-1/2}(\mathbb{R})} + ||\mathcal{W}_\infty,\varepsilon(\eta \psi)||_{H^{s-1/2}(\mathbb{R})} \leq C||\eta \psi||_{H^s(\mathbb{R})}.$$ \hspace{1cm} (4.49)

Moreover, for all $h \in H^s(\mathbb{R})$ fixed:

$$\lim_{\varepsilon \to 0} ||\eta \cdot (\mathcal{W}_\infty,\varepsilon - \mathcal{W}_\infty)(\eta h)||_{H^{s-1/2}(\mathbb{R})} = 0.$$ \hspace{1cm} (4.50)

Proof of Lemma 4.4 The function $\mathcal{W}_R(\eta \psi)$ can be written as follows

$$R^{(3+\lambda)/2} \int_0^{x/2} \left( (x-y)^{\lambda/2}\eta(x-y) \psi(x-y) - x^{\lambda/2}\eta(x)\psi(x) \right) y^{\lambda/2} f_0(Ry) dy$$

$$= T_{0,R} \circ M_{\lambda/2}(\eta \psi) + \mathcal{Z}$$

with

$$||\mathcal{Z}||_{H^s} \leq C||\eta \psi||_{H^s}.$$  

Using now the fact that the operator $T_{0,R}$ is the multiplier by a function bounded by $|\xi|^{1/2}$ and $M_{\lambda/2} h$ is the product of $h$ by $x^{\lambda/2}$ which is a smooth function in the interval $(1/2, \pi/2)$ the result follows. The same argument yields the estimate for $\mathcal{W}_\infty$:

$$||\mathcal{W}_\infty(\eta \psi)||_{H^{s-1/2}(\mathbb{R})} \leq C||\eta \psi||_{H^s}.$$ \hspace{1cm} (4.51)

The thirdoperator $\mathcal{W}_\infty,\varepsilon$ may be written as a pseudo differential operator with symbol

$$P_{\varepsilon}(x,k) = \int_0^{\infty} \frac{(e^{-iky} - 1)}{y^{3/2} + \varepsilon^{3/2}x^{3/2}} dy.$$ \hspace{1cm} (4.52)

Therefore,

$$||\eta \cdot (\mathcal{W}_\infty,\varepsilon - \mathcal{W}_\infty)(\eta h)||_{H^{s-1/2}(\mathbb{R})}^2 = \int_{\mathbb{R}} dk_1 (1 + |\hat{k}|^2)^{\sigma - 1/2} \int_{\mathbb{R}} dk_2 \hat{\psi}(k_1) \times$$

$$\times \int_{\mathbb{R}} dk_2 \hat{\psi}(k_2) Z_\varepsilon(k_1, k_2, \hat{k})$$
We now show:

To this end we notice that we may write:

\[ \int dx_1 \int dx_2 [P_\varepsilon(x_1, k_1) - P_0(x_1, k_1)] \times \]

\[ \times [P_\varepsilon(x_2, k_2) - P_0(x_2, k_2)] e^{-i(k_1 - \tilde{k})x_1} e^{-i(k_2 - \tilde{k})x_2} \eta(x_1) \eta(x_2) \]

We now show:

\[ |Z_\varepsilon(k_1, k_2, \tilde{k})| \leq C_m \frac{|k_1|^{1/2}|k_2|^{1/2}}{(1 + |k - k_1|^m)(1 + |k - k_1|^m)}. \]  \hfill (4.53)

To this end we notice that we may write:

\[ P_\varepsilon(x, k) - P_0(x, k) = \int_0^\infty dy \left( e^{-iky} - 1 \right) \frac{\varepsilon^{3/2}x^{3/2}}{y^{3/2}(y^{3/2} + \varepsilon^{3/2}x^{3/2})} \]

whence,

\[ \int \int e^{-i(k - \tilde{k})x} (P_\varepsilon(x, k) - P_0(x, k)) \eta(x) dx = \int \int e^{-i(k - \tilde{k})x} \int_0^\infty e^{-iky} - 1 \frac{1}{y^{3/2}} R \left( \frac{y}{\varepsilon x} \right) \eta(x) dx dy \]

\[ R(\xi) = \frac{1}{\xi^{3/2} + 1} \]

For \(|k - \tilde{k}| \leq 1\) we immediately obtain from (4.54) that, for some positive constant \(C\) independent of \(\varepsilon\):

\[ \left| \int \int e^{-i(k - \tilde{k})x} (P_\varepsilon(x, k) - P_0(x, k)) \eta(x) dx \right| \leq C \]  \hfill (4.55)

On the other hand, using:

\[ e^{-i(k - \tilde{k})x} = \frac{i}{k - \tilde{k}} \frac{\partial}{\partial x} \left( e^{-i(k - \tilde{k})x} \right) \]

and integrating by parts \(m\) times in the right hand side of (4.54) we obtain that for any \(m \in \mathbb{N}\) there exists a positive constant \(C_m\) such that

\[ \left| \int_0^\infty e^{-i(k - \tilde{k})x} R \left( \frac{y}{\varepsilon x} \right) \eta(x) dx \right| \leq \frac{C_m}{1 + |k - k_1|^m}. \]  \hfill (4.56)

In the derivation of (4.56) we have used:

\[ \frac{\partial}{\partial x} R \left( \frac{y}{\varepsilon x} \right) = -\frac{1}{x} \xi R' (\xi), \quad \xi = \left( \frac{y}{\varepsilon x} \right) \]

the function \(\xi R' (\xi)\) has the same structure than \(R(\xi)\): it is a rational function of \(\xi^{3/2}\) decreasing as \(\xi \to \infty\) like \(\xi^{-3/2}\). This is also true for all the derivatives of higher order. Moreover, since \(\text{supp}(\eta) \subset (1/2, 2)\), the term \(\eta(x)/x\) is uniformly bounded in \(\mathbb{R}\).

Define now the function

\[ M(y, k - \tilde{k}) = \frac{1}{y^{3/2}} \int_0^\infty e^{-i(k - \tilde{k})x} R \left( \frac{y}{\varepsilon x} \right) \eta(x) dx \]
An integration by parts yields:
\[
\int_0^\infty (e^{-iky} - 1)M(y, k - \tilde{k})dy = -ik \int_0^\infty e^{-iky} \int_\Gamma M(\sigma, k - \tilde{k})d\sigma dy. \tag{4.57}
\]
This identity still holds in the straight lines \(\Gamma\) of the complex plane defined by
\[
|Im(y) = \varepsilon_0|Re(y)|, \quad sign(Im(y)) = -sign(k)
\]
Using then (4.56) we obtain:
\[
\left| ik \int_\Gamma e^{-iky} \int_y^\infty M(\sigma, k - \tilde{k})d\sigma dy \right| \leq \frac{C_m |k|}{1 + |k - k'|^m} \int_\Gamma |y|^{1/2} dy \leq \frac{C_m^\prime |k|^{1/2}}{1 + |k - k'|^m}. \tag{4.58}
\]
Using (4.58) twice, estimate (4.53) follows. Therefore,
\[
\left| \int_\mathbb{R} dk (1 + |k|^2)^{\sigma - 1/2} \int_\mathbb{R} dk_1 \bar{\psi}(k_1) \int_\mathbb{R} dk_2 \hat{\psi}(k_2) \hat{Z}_\varepsilon(k_1, k_2, \tilde{k}) \right| \leq \frac{|k_1|^{1/2}|k_2|^{1/2}}{(1 + |k_1 - k|)^m (1 + |k_2 - \tilde{k}|)^m}. \tag{4.59}
\]
Using that \(|\tilde{k}| \leq |k_1| + |\tilde{k} - k_1|\) we have:
\[
\int_\mathbb{R} \frac{(1 + |\tilde{k}|^2)^{\sigma - 1/2} dk}{(1 + |k_1 - k|)^m (1 + |k_2 - \tilde{k}|)^m} \leq C \frac{(1 + |k_1|)^{2\sigma - 1}}{(1 + |k_2 - k_1|)^m}. \tag{4.60}
\]
for some \(m' < m\). Using (4.60) in (4.59) and Cauchy-Schwartz’s inequality we obtain:
\[
\left| \int_\mathbb{R} dk (1 + |\tilde{k}|^2)^{\sigma - 1/2} \int_\mathbb{R} dk_1 \bar{\psi}(k_1) \int_\mathbb{R} dk_2 \hat{\psi}(k_2) \hat{Z}_\varepsilon(k_1, k_2, \tilde{k}) \right| \leq \frac{||\psi||_{H^\sigma(\mathbb{R})} \int_\mathbb{R} dk_1 \int_\mathbb{R} dk_2 |k_2|^{1/2} (1 + |k_1|)^{\sigma - 1/2} |\hat{\psi}(k_2)|^2}{(1 + |k_1 - k_2|)^{m'}} \leq C ||\psi||_{H^\sigma(\mathbb{R})}^2 \tag{4.61}
\]
for some \(m'' < m'\). Young’s inequality then emplies:
\[
\left| \int_\mathbb{R} dk (1 + |\tilde{k}|^2)^{\sigma - 1/2} \int_\mathbb{R} dk_1 \bar{\psi}(k_1) \int_\mathbb{R} dk_2 \hat{\psi}(k_2) \hat{Z}_\varepsilon(k_1, k_2, \tilde{k}) \right| \leq C ||\psi||_{H^\sigma(\mathbb{R})}^2, \tag{4.62}
\]
and therefore
\[
||\varepsilon \cdot (\mathcal{W}_\infty, \varepsilon - \mathcal{W}_\infty)(\varepsilon h)||_{H^{\sigma - 1/2}(\mathbb{R})} \leq C ||\psi||_{H^\sigma(\mathbb{R})}. \tag{4.63}
\]
Combining (4.51) and (4.63) we obtain the estimate for \(\mathcal{W}_\infty, \varepsilon \cdot \eta h\) in (4.49).

It remains to prove that (4.50) holds true. By the estimate (4.53) in \(\hat{Z}_\varepsilon(k_1, k_2, \tilde{k})\) this is reduced to prove that for any \(k_1, k_2\) and \(\tilde{k}\), \(\hat{Z}_\varepsilon(k_1, k_2, \tilde{k}) \to 0\) as \(\varepsilon \to 0\). This follows
from the fact that the support of $\eta$ is compact and that $P_\varepsilon(x, k) \to P_0(x, k)$ as $\varepsilon \to 0$ as it follows from the explicit expressions (4.52).

If $y = \varepsilon t$, we obtain,

$$P_\varepsilon(x, k) = \frac{1}{\sqrt{\varepsilon}} \int_0^\infty \frac{(e^{-ikt} - 1)}{t^{3/2} + x^{3/2}} dt.$$

Therefore it follows that, for some positive constant $C$ independent of $\varepsilon > 0$ and $x > 0$:

$$|P_\varepsilon(x, k)| \leq C |k|^{1/2}, \forall \varepsilon > 0, \forall x > 0$$

and (4.48) follows. $\square$

5 Estimating the difference between $L$ and $L$.

In this Section we estimate the operator $L - L$ which appear in the equation (1.7).

$$(L - L)(\varphi)(x, t) = A_1 + A_2,$$

$$A_1(x) = \int_0^{x/2} (H(x - y) - H(x)) y^{3/2} \varphi(y, t) dy$$

$$- H(x) \int_{x/2}^\infty y^{3/2} \varphi(y, t) dy - x^{3/2} \varphi(x, t) \left( \int_{x/2}^\infty H(y) dy \right)$$

(5.1)

$$A_2(x) = \int_0^{x/2} \left( (x - y)^{3/2} \varphi(x - y, t) - x^{3/2} \varphi(x, t) \right) H(y) dy$$

(5.2)

$$H(y) = y^{3/2} f_0(y) - y y^{-3/2}. $$

(5.3)

Since it will be needed in the Section 6, we shall actually estimate more general operators where the function $A_2$ has the more general form:

$$A_{2, \varepsilon}(x) = \int_0^{x/2} \left( (x - y)^{3/2} \varphi(x - y, t) - x^{3/2} \varphi(x, t) \right) H_\varepsilon(x, y) dy$$

(5.4)

$$H_\varepsilon(x, y) = y^{3/2} f_0(y) - \frac{1}{y^{3/2} + \varepsilon^{3/2} x^{3/2}}.$$  

(5.5)

Notice that $\varepsilon = 0$ corresponds to the functions $A_2$ and $H$ defined in (5.2) and (5.3).

In the two following Lemmas we estimate the two terms $A_1$ and $A_{2, \varepsilon}$ assuming some conditions of the function $f_0$.

Lemma 5.1 Suppose that $f_0$ satisfies conditions (2.1), (2.2) and $|||\varphi|||_{3/2, (3+\lambda)/2} < \infty$. Then

$$|||A_1|||_{3/2, 2+\delta} \leq C |||\varphi|||_{3/2, (3+\lambda)/2}.$$
Proof of Lemma 5.1. The estimate on $A_1(x)$ for $0 < x < 1$ is immediate:

\[
\int_0^{x/2} (H(x - y) - H(x)) y^{\lambda/2} \varphi(y) dy \leq \|\varphi\|_{3/2,(3+\lambda)/2} \int_0^{x/2} |H(x - y) + H(x)| y^{(-3+\lambda)/2} dy \leq C \|\varphi\|_{3/2,(3+\lambda)/2} x^{-3/2}
\]  

(5.6)

\[
\left| H(x) \int_{x/2}^\infty y^{\lambda/2} \varphi(y) dy \right| \leq C \|\varphi\|_{3/2,(3+\lambda)/2} x^{-3/2},
\]  

(5.7)

\[
x^{\lambda/2} \left( \int_{x/2}^\infty H(y) dy \right) \varphi(x) \leq C \|\varphi\|_{3/2,(3+\lambda)/2} x^{\lambda/2 - 2} \leq C \|\varphi\|_{3/2,(3+\lambda)/2} x^{-3/2}.
\]  

(5.8)

Let us consider the case when $x > 1$. In order to estimate the first term in the right hand side of (5.1) we write:

\[ H(x - y) - H(x) = y \int_0^1 H'(x - \theta y) d\theta \]

where

\[ H'(z) = \frac{\lambda}{2} z^{(\lambda - 2)/2} (f_0(z) - z^{-(3+\lambda)/2}) + z^{\lambda/2} (f'_0(z) + \frac{3 + \lambda}{2} z^{-(3+\lambda)/2 - 1}) \]

By assumptions (2.1) (2.2), for all $z > 1$:

\[ |H'(z)| \leq \left( 1 + \frac{\lambda}{2} \right) z^{-5/2 - \delta} \]

In particular, for all $y < x/2$ and $0 < \theta < 1$ we have $x - \theta y > x/2$ and so, if $x > 2$:

\[ |H(x - y) - H(x)| = \left| y \int_0^1 H'(x - \theta y) d\theta \right| \leq C y x^{-5/2 - \delta} \]

and

\[
\int_0^{x/2} (H(x - y) - H(x)) y^{\lambda/2} \varphi(y) dy \leq C x^{-5/2 - \delta} \int_0^{x/2} y^{1 + \lambda/2} |\varphi(y)| dy \leq C \|\varphi\|_{3/2,(3+\lambda)/2} x^{-2 - \delta} \]

(5.9)

In order to estimate the second term in (5.1) we use:

\[ |H(x)| = x^{\lambda/2} |f_0(x) - G(x)| \leq C x^{\lambda/2} x^{-(3+\lambda)/2 - \delta} \text{ for } x > 1 \]

whence, for $x > 1$:

\[
\left| H(x) \int_{x/2}^\infty y^{\lambda/2} \varphi(y) dy \right| \leq x^{-3/2 - \delta} \int_{x/2}^\infty y^{\lambda/2} \varphi(y) dy \leq C \|\varphi\|_{3/2,(3+\lambda)/2} x^{-2 - \delta}.
\]  

(5.10)

The third term of (5.1) is bounded by

\[ x^{\lambda/2} |\varphi(x)| \int_{x/2}^\infty y^{-3/2 - \delta} dy = C \|\varphi\|_{3/2,(3+\lambda)/2} x^{-2 - \delta} \text{ for } x > 1. \]  

(5.11)

Lemma 5.1 then follows combining (5.6)-(5.11).
Lemma 5.2  Let $0 \leq T \leq 1$. Then, there exists a constant $C > 0$ such that, for any $\varphi \in \mathcal{E}_{T,\sigma}$ and for all $t_0 \in (0, T)$:

$$
R^2 N_{2;\sigma} (A_1; R, t_0) \leq C \||\varphi\||, \quad \forall R > 1,
$$

$$
R^{2-\lambda/2} M_{2;\sigma} (A_1; R) \leq C \||\varphi\||, \quad \forall 0 < R < 1.
$$

Proof of Lemma 5.2  For $R > 1$ we write

$$
\varphi(x, t) = \sum_{n=0}^{\infty} \chi(x/2^n) \varphi(x, t)
$$

where $\chi \in C_0^\infty$, supp $\chi \subset (1/2, 2)$. Let us consider $R = 2^{n_0}$, $x \in (R/2, 2R)$ and rescale $x = RX$, $y = RY$, $\tau = (t - t_0)R^{(\lambda - 1)/2}$, $\varphi(x, t) = R^{-(3+\lambda)/2}\psi(X, \tau)$, $A_1(X, \tau) = A_1(x, t)$ to obtain:

$$
R^2 |A_1(X, \tau)| = \int_0^{X/2} \left[ H_R(X - Y) - H_R(X) \right] Y^{\lambda/2} \psi(Y, \tau) dY - \int_X^\infty H_R(X) dY
$$

(5.12)

where the function $H_R$ is defined as follows:

$$
H_R(X) = R^{(3+\lambda)/2}X^{\lambda/2}f_0(RX) - X^{-3/2}.
$$

(5.13)

Since $\||\varphi\||_{3/2,(3+\lambda)/2} < \infty$, we have the following bound on $\Psi(X, \tau)$

$$
|\Psi(X, \tau)| \leq C \min \left\{ \frac{R^{\lambda/2}}{X^{3/2}}, \frac{1}{X^{(3+\lambda)/2}} \right\} \||\varphi(t)||_{3/2,(3+\lambda)/2}
$$

(5.14)

for all $X \geq 0$ and $\tau \in (0, T R^{(\lambda - 1)/2})$. Using this estimate it then follows that the integrals in the right hand side of (5.12) are convergent. Moreover, using conditions (2.1) and (2.2) we obtain:

$$
R^2 \left( \int_0^1 d\tau \int_{1/2}^2 |D_x^\alpha A_1(X, \tau)|^2 dX d\tau \right)^{1/2} \leq C \||\varphi\||, \quad \forall R > 1.
$$

(5.15)

For $R \in (0, 1)$ we scale the variables $x \in (R/2, 2R)$ and $\varphi$ as $x = RX$, $y = RY$, $\varphi(x, t) = R^{-3/2}\psi(X, t)$, $A_1(X, t) = A_1(x, t)$ to obtain in this case:

$$
R^{2-\lambda/2} |A_1(X, t)| = \int_0^{X/2} \left[ H_R(X - Y) - H_R(X) \right] Y^{\lambda/2} \psi(Y, t) dY - \int_X^\infty H_R(X) dY
$$

(5.16)

where the function $H_R$ is defined as follows:

$$
H_R(X) = R^{(3+\lambda)/2}X^{\lambda/2}f_0(RX) - X^{-3/2}.
$$

(5.17)

Using again (2.1), (2.2) and (5.14) we deduce

$$
R^{2-\lambda/2} \left( \int_0^1 d\tau \int_{1/2}^2 |D_x^\alpha A_1(X, \tau)|^2 dX d\tau \right)^{1/2} \leq C \||\varphi\||, \quad \forall R \in (0, 1).
$$

(5.17)

Lemma 5.2 follows from (5.15) and (5.17). $\square$

The following technical Lemma will be needed in order to estimate $A_{2,\varepsilon}$.
Lemma 5.3 For any given function $h \in H^\sigma(\mathbb{R})$ supported in $(1/2, 2)$, and $\delta \in [0, \min(1/2, 1 - \sigma))$ there holds:

$$\int_0^{5/8} |h(X - Y) - h(X)| Y^{-3/2 - \delta} dY \leq C\|h\|_{H^\sigma}$$

Proof of Lemma 5.3 Using (2.10) - (2.13).

Suppose that Lemma 5.4 $A$ and $\delta \leq C$.

For any given function $\sigma \geq 0$ since $\delta > 0$, we obtain

$$\int_0^{5/8} |h(X - Y) - h(X)| Y^{-3/2 - \delta} dY \leq C \int_0^{1/2} \left( \int_{\mathbb{R}} \left| \frac{\partial}{\partial \xi} h(\xi) e^{-i\xi Y} - 1 \right| Y^{-3/2 - \delta} d\xi dY \right)^{1/2} \left( \int_{\mathbb{R}} \left| \frac{\partial}{\partial \xi} h(\xi) e^{-i\xi Y} - 1 \right| Y^{-3/2 - \delta} d\xi dY \right)^{1/2}.$$

Using the change of variables $\xi y = z$ we arrive at

$$\int_0^{5/8} \left| e^{-i\xi Y} - 1 \right| Y^{-3/2 - \delta} dY \leq C \xi^{1/2 + \delta} \int_0^{5/8} \left| e^{-i\xi Y} - 1 \right| \frac{d\xi}{1 + |\xi|^{2\sigma}} \leq C \xi^{1/2 + \delta},$$

and

$$\int_{\mathbb{R}} \left| \frac{\partial}{\partial \xi} h(\xi) e^{-i\xi Y} - 1 \right| Y^{-3/2 - \delta} d\xi dY \leq \int_{\mathbb{R}} |\xi|^{1 + 2\delta} \frac{d\xi}{1 + |\xi|^{2\sigma}} < \infty$$

since $\sigma > 1 + \delta$. \qed

We have the following estimate for $A_{2,\epsilon}$ in (5.4).

Lemma 5.4 Suppose that $f_0$ satisfies conditions (2.1) - (2.3) and $\sigma > 1 + \delta$, then

$$\sup_{t_0 \in (0,T)} \sup_{R > 1} R^{2 + \delta} N_\infty(A_{2,\epsilon}; t_0, R) \leq C\|\varphi\|, \quad (5.18)$$

$$\sup_{t_0 \in (0,T)} \sup_{R > 1} R^2 N_{2; \sigma - \frac{1}{2}}(A_{2,\epsilon}; t_0, R) \leq C\|\varphi\|, \quad (5.19)$$

$$\sup_{0 < R < 1} R^{2 - \lambda/2} M_\infty(A_{2,\epsilon}; R) \leq C\|\varphi\|, \quad (5.20)$$

$$\sup_{0 < R < 1} R^{2 - \lambda/2} M_{2; \sigma - \frac{1}{2}}(A_{2,\epsilon}; R) \leq C\|\varphi\|, \quad (5.21)$$

where the functions $N_\infty(\cdot; t_0, R)$, $N_{2; \sigma}(\cdot; t_0, R)$, $M_\infty(\cdot; R)$ and $M_{2; \sigma}(\cdot; R)$ are defined in (2.10) - (2.13).

Proof of Lemma 5.4 For $R > 1$ we write

$$\varphi(x, t) = \sum_{n=0}^{\infty} \chi(x/2^n) \varphi(x, t)$$

46
Therefore, and (5.18) follows. whence:

\[ |A_{2,\varepsilon}(x, t)| \leq \sum_{n=n_0-2}^{n_0+1} \int_0^{x/2} (x - y)^{\lambda/2} \varphi(x - y, t) \chi \left( \frac{x - y}{2^{\ell_n}} \right) - x^{\lambda/2} \varphi(x, t) \chi \left( \frac{x}{2^{\ell_n}} \right) y^{-3/2 - \delta} dy. \]

Let us consider \( R = 2^{n_0}, x \in (R/2, 2R) \) and rescale \( x = RX, y = RY, \tau = (t-t_0)R^{(\lambda-1)/2}, \varphi(x, t) = R^{-(3+\lambda)/2} \psi(X, \tau), A_{2,\varepsilon}(X, \tau) = A_{2,\varepsilon}(x, t) \) to obtain:

\[
|A_{2,\varepsilon}(X, \tau)| \leq R^{-2 - \delta} \times \sum_{\ell = -2}^1 \int_0^{X/2} (X - y)^{\lambda/2} \psi(X - y, \tau) \chi \left( \frac{X - y}{2^\ell} \right) - X^{\lambda/2} \psi(X, \tau) \chi \left( \frac{X}{2^\ell} \right) y^{-3/2 - \delta} dY. 
\]

Using Lemma 5.3, we deduce, for \( X \in (3/4, 5/4) \):

\[
|A_{2,\varepsilon}(X, \tau)| \leq R^{-(2+\delta)} C \left( ||\varphi(t)||_{L^\infty(1/8, 8)} + ||\psi(\tau)||_{H^s(1/8, 8)} \right)
\]

whence:

\[
R^{2+\delta} \left( \int_0^{\min(1, R^{(\lambda-1)/2}(T-t_0))} ||A_{2,\varepsilon}(t)||^2_{L^\infty(3/4, 5/4)} dt \right)^{1/2} \leq C \sup_{0 \leq \tau \leq \min(1, R^{(\lambda-1)/2}(T-t_0))} ||\varphi(t)||_{L^\infty(1/8, 8)} + \]

\[
+ C \left( \int_0^{\min(1, R^{(\lambda-1)/2}(T-t_0))} ||\psi(\tau)||^2_{H^s(1/8, 8)} dt \right)^{1/2}.
\]

Therefore,

\[
R^{2+\delta} N_\infty(A_{2,\varepsilon}; t_0, R) \leq C R^{(3+\lambda)/2} \left[ \sup_{t_0 \leq t \leq t_0 + R^{-(\lambda-1)/2}(T)} ||\varphi(t)||_{L^\infty(R, 8R)} + \right. 
\]

\[
\left. + \sum_{\ell = -3}^3 N_{2,\sigma}(\varphi; t_0, 2^\ell R) \right] \leq C ||\varphi||, 
\]

and (5.18) follows.

We now prove (5.19). To this end notice that:

\[
A_{2,\varepsilon}(X, \tau) = R^{-2} \sum_{\ell = -2}^1 \int_0^{X/2} (X - y)^{\lambda/2} \psi(X - y, \tau) \chi \left( \frac{X - y}{2^\ell} \right) - X^{\lambda/2} \psi(X, \tau) \chi \left( \frac{X}{2^\ell} \right) \mathcal{H}_\varepsilon(X, Y) dY
\]

where \( \mathcal{H}_\varepsilon(X, Y) = R^{3/2}H_\varepsilon(x, y) \), using Lemma 14 we obtain:

\[
R^2 \left( \int_0^{\min(1, R^{(\lambda-1)/2}(T-t_0))} ||A_{2,\varepsilon}(\tau)||^2_{H^{s-1/2}(3/4, 5/4)} dt \right)^{1/2} \leq \]

\[
\leq C \left( \int_0^{\min(1, R^{(\lambda-1)/2}(T-t_0))} ||\psi(\tau)||^2_{H^s(1/8, 8)} dt \right)^{1/2} + \]

\[
+ C \sup_{0 \leq \tau \leq \min(1, R^{(\lambda-1)/2}(T-t_0))} ||\psi(\tau)||_{L^\infty(1/8, 8)}. 
\]
Therefore
\[
R^{\sigma+1} N_{2,\sigma-\frac{1}{2}}(A_2,\epsilon; t_0, R) \leq C R^{3+\lambda/2} \left[ \sum_{\ell=-3}^{3} N_{2,\sigma}(\varphi; t_0, 2^\ell R) + \sup_{t_0 \leq t \leq \min(t_0 + R^{-(\lambda-1)/2}, T)} \|\varphi(t)\|_{L^\infty(R/8,8R)} \right].
\]
whence (5.19) follows.

We consider now the case where \(0 < R \leq 1\). The arguments are very similar to those used in the previous case. In order to prove (5.19) we write
\[
\varphi(x,t) = \sum_{n=0}^{\infty} \chi(2^n x) \varphi(x,t)
\]
where \(\chi \in C_0^\infty\), \(\text{supp} \chi \subset (1/2, 2)\)
\[
|A_2(\epsilon, x, t)| \leq C \sum_{n=0}^{n_0+1} \int_0^{x/2} (x-y)^{\lambda/2} \varphi(x-y, t) \chi(2^n(x-y)) - x^{\lambda/2} \varphi(x, t) \chi(2^n x) y^{-3/2} dy.
\]
Let us consider \(R = 2^{n_0}\), \(x \in (R/2, 2R)\) and rescale \(x = RX\), \(y = RY\), \(\varphi(x,t) = R^{-3/2} \psi(X,t)\), \(A_2(\epsilon, X, t) = A_2(\epsilon, x, t)\) to obtain:
\[
|A_2(\epsilon, X, t)| \leq R^{\lambda/2-2} \sum_{\ell=-2}^{1} \int_0^{X/2} (X-Y)^{\lambda/2} \psi(X-Y, t) \chi(2^\ell(X-Y)) - X^{\lambda/2} \psi(X, t) \chi(2^\ell X) Y^{-3/2} dY.
\]
Using Lemma 5.3 we deduce that for \(X \in (3/4, 5/4)\):
\[
|A_2(\epsilon, X, t)| \leq R^{-2+\frac{\lambda}{2}} C \left( \|\psi(t)\|_{L^\infty(1/8,8)} + \|\psi(t)\|_{H^\sigma(1/8,8)} \right)
\]
whence,
\[
R^{2-\lambda/2} \left( \int_0^T \|A_2(\epsilon, t)\|_{L^\infty(3/4,5/4)}^2 dt \right)^{1/2} \leq C \sup_{0 \leq t \leq T} \|\psi(t)\|_{L^\infty(1/8,8)} + \int_0^T \|\psi(t)\|_{H^\sigma(1/8,8)}^2 dt \right)^{1/2}.
\]
Therefore
\[
R^{2-\lambda/2} M_\infty(A_2, \epsilon, R) \leq C R^{3/2} \left[ \sup_{0 \leq t \leq T} \|\varphi(t)\|_{L^\infty(R/8,8R)} + \sum_{\ell=-3}^{3} M_{2,\sigma}(\varphi, 2^\ell R) \right]
\]
\[
\leq C \|\varphi\|,
\]
48
and (5.20) follows.

We now prove (5.21). Since:

\[ A_{2,\epsilon}(X, t) = R^{-\lambda/2} \sum_{\ell=-2}^{1} \int_{0}^{X/2} (X-Y)^{\lambda/2} \psi(X-Y, t) \chi \left( 2^{\ell} (X-Y) \right) - X^{\lambda/2} \psi(X, t) \chi \left( 2^{\ell} X \right) \right) \mathcal{H}_{\epsilon}(X, Y) dY \]

where \( \mathcal{H}_{\epsilon}(X, Y) = R^{3/2} H_{\epsilon}(x, y) \), using Lemma 4.4 we obtain:

\[ R^{2-\lambda/2} \left( \int_{0}^{T} ||A_{2,\epsilon}(t)||^2_{H^{\sigma-1/2}(3/4, 5/4)} dt \right)^{1/2} \leq C \left( \int_{0}^{T} ||\psi(t)||^2_{H^{\sigma(1/8, 1/8)}} dt \right)^{1/2} + C \sup_{0 \leq t \leq T} ||\psi(t)||_{L^{\infty}(1/8, 1/8)}. \]

Therefore

\[ R^{2-\lambda/2} M_{2, \sigma-1/2}(A_{2,\epsilon}; R) \leq C R^{3/2} \left[ \sum_{\ell=-3}^{3} M_{2, \sigma}(\varphi; 2^{\ell} R) + \sup_{0 \leq t \leq T} ||\varphi(t)||_{L^{\infty}(R/8, 8R)} \right]. \]

whence (5.21) follows. \( \square \)

The following result has been proved in [4]:

**Proposition 5.5** The fundamental solution \( g(t, x, x_0) \) of the operator \( L \) defined in (1.3) such that \( g(0, x, x_0) = \delta(x - x_0) \) satisfies:

\[ g(t, x, x_0) = \frac{1}{x_0} g \left( t \frac{x_0^{(\lambda-1)/2}}{x_0}, \frac{x}{x_0}, 1 \right) \]  

(5.22)

\[ |g(t, x, 1)| \leq C t x^{-3/2}, \quad \text{for all } 0 \leq t \leq 1, \, 0 < x \leq 1/2, \]  

(5.23)

\[ |g(t, x, 1)| \leq C t x^{-(3+\lambda)/2}, \quad \text{for all } 0 \leq t \leq 1, \, x \geq 3/2, \]  

(5.24)

\[ |g(t, x, 1)| \leq C t^{-2} \Phi \left( \frac{x-1}{t^2} \right) x^{-3/2}, \quad \text{for all } 0 \leq t \leq 1, \, 1/2 \leq x \leq 3/2, \]  

(5.25)

where,

\[ \Phi(\xi) = \frac{1}{1 + |\xi|^{3/2-\sigma}}. \]

(5.26)

Moreover

\[ g(t, x, x_0) \leq C t^{2/(\lambda-1)} \sigma^{-3/2}, \quad \text{for all } t \geq 1, \, 0 < \sigma \leq 1, \]  

(5.27)

\[ |g(t, x, 1)| \leq C t^{2/(\lambda-1)} \sigma^{-(3+\lambda)/2}, \quad \text{for all } t \geq 1, \, \sigma \geq 1, \]  

(5.28)

with

\[ \sigma = t^{2/(\lambda-1)} x. \]

(5.29)
Lemma 5.6 For $T \in (0, 1]$ there is a constant $C > 0$ such that, for all $\|\nu\|_{X^{3/2, 2-\delta}(T)} < \infty$:

$$
\sup_{0 \leq t \leq T} \left\| \int_0^t G(t-s)\nu(s)\,ds \right\|_{3/2, 2-\delta} \leq C T^\beta \|\nu\|_{X^{3/2, 2-\delta}(T)}
$$

where

$$
\beta = \min \left( 1, \frac{2\delta}{\lambda - 1} \right). \quad (5.30)
$$

Proof of Lemma 5.6 We assume first that

$$
R^{-(\lambda-1)/2} \leq t. \quad (5.31)
$$

Let us suppose that

$$
x \in \left( \frac{3R}{4}, \frac{5R}{4} \right). \quad (5.32)
$$

Using Proposition 5.5,

$$
\begin{align*}
\int_0^t G(t-s)\nu(s,y)\,ds &= \int_0^t ds \int_0^\infty dy \nu(s,y) g \left( (t-s)y^{\lambda-1}, \frac{x}{y} \right) \frac{dy}{y} \\
&\leq \int_{t-R}^t ds \int_{|x-y| \leq R/2} \nu(s,y) g \left( (t-s)y^{\lambda-1}, \frac{x}{y} \right) \frac{dy}{y} \\
&\quad + \int_{t-R}^t ds \int_{|x-y| \leq R/2} \nu(s,y) g \left( (t-s)y^{\lambda-1}, \frac{x}{y} \right) \frac{dy}{y} \\
&\quad + \int_0^t ds \int_{|y| \leq R/2} \nu(s,y) g \left( (t-s)y^{\lambda-1}, \frac{x}{y} \right) \frac{dy}{y} \\
&\quad + \int_0^t ds \int_{|y| \geq 2R} \nu(s,y) g \left( (t-s)y^{\lambda-1}, \frac{x}{y} \right) \frac{dy}{y} \\
&= I_1 + I_2 + I_3 + I_4. \quad (5.33)
\end{align*}
$$

To estimate $I_1$ we use the fact that $\ref{4.20}$ implies:

$$
g(s, z) \leq \frac{C}{s^2} \Phi \left( \frac{z-1}{s^2} \right) \quad (5.34)
$$

for $0 \leq s \leq 1, z \in (1/2, 3/2)$.

$$
|I_1| \leq C \int_{t-R}^t ds \int_{|x-y| \leq R/2} \frac{\nu(s,y)}{(t-s)^2 y^{\lambda-1}} \Phi \left( \frac{z-1}{(t-s)^2 y^{\lambda-1}} \right) \frac{dy}{y} \\
\leq C \int_{t-R}^t \|\nu(s)\|_{L^\infty(R/2, 2R)} ds \int_{|x-y| \leq R/2} \frac{1}{(t-s)^2 y^{\lambda}} \Phi \left( \frac{x-y}{(t-s)^2 y^{\lambda}} \right) dy \\
\leq C \int_{t-R}^t \|\nu(s)\|_{L^\infty(R/2, 2R)} ds.
$$

50
where we have used (5.31) in the last inequality. Using Hölder’s inequality we deduce:

\[ |I_1| \leq CR^{-(\lambda-1)/2}N_\infty(\nu; t_0, R) \]

whence:

\[ |I_1| \leq CR^{-(\lambda+\lambda)/2}R^{-\delta} \left[ R^{2+\delta}N_\infty(\nu; t_0, R) \right] \leq CR^{-(\lambda+\lambda)/2}t^{2/\delta(\lambda-1)}|||\nu|||_{X_{3/2,2+\delta}}. \quad (5.35) \]

We consider now the term \( I_2 \):

\[ I_2 = \int_0^{t-R} \frac{ds}{|x-y|^{\lambda-1}} \int_{|x-y| \leq R/2} \nu(y)g \left( (t-s)y^\frac{\lambda-1}{\lambda}, \frac{x}{y} \right) dy \]

In the region of integration we have \( (t-s)y^\frac{\lambda-1}{\lambda} \geq 1 \). Using then (5.28) we deduce

\[ g \left( (t-s)y^\frac{\lambda-1}{\lambda}, \frac{x}{y} \right) \leq C (t-s)^{-\frac{\lambda-1}{\lambda(\lambda+1)}} \frac{y}{x^{(\lambda+1)/2}} \quad (5.36) \]

for \( s \geq 1 \) and \( 1/7 \leq |z| \leq 7 \). Therefore:

\[ |I_2| \leq R^{-(\lambda+\lambda)/2} \int_0^{t-R} \frac{ds}{|x-y|^{\lambda-1}} \int_{|x-y| \leq R/2} |||\nu(t-s)|||_{L^\infty(R/2,2R)} s^{\frac{\lambda-1}{\lambda-2}} ds \]

\[ = R^{-(\lambda+\lambda)/2} \sum_{n=1}^{\lceil R^{(\lambda-1)/2} \rceil} \int_{nR^{-(\lambda-1)/2}}^{\min\{n+1R^{-(\lambda-1)/2}, t\}} |||\nu(t-s)|||_{L^\infty(R/2,2R)} s^{\frac{\lambda-1}{\lambda-2}} ds \]

\[ \leq CR^{-(\lambda+\lambda)/2} \sum_{n=1}^{\lceil R^{(\lambda-1)/2} \rceil} R^{(\lambda+1)/2}n^{-(\lambda+1)/(\lambda-1)} R^{-(\lambda-1)/2}N_\infty(\nu; nR^{-(\lambda-1)/2}, R) \]

\[ \leq CR^{-(\lambda+\lambda)/2} R^{-1-\delta}|||\nu|||_{X_{3/2,2+\delta}} \leq CR^{-(\lambda+\lambda)/2}t^{2(1+\delta)/(\lambda-1)}|||\nu|||_{X_{3/2,2+\delta}}, \quad (5.37) \]

where we have used (5.31) in the last step.

We next consider the term \( I_3 \).

\[ I_3 = \int_0^t ds \int_{|y| \leq R/2} \nu(y)g \left( (t-s)y^\frac{\lambda-1}{\lambda}, \frac{x}{y} \right) dy = \quad (5.38) \]

\[ = \int_0^t ds \int_0^{t-2/(\lambda-1)} \nu(y)g \left( (t-s)y^\frac{\lambda-1}{\lambda}, \frac{x}{y} \right) dy + \]

\[ + \int_0^t ds \int_{t-2/(\lambda-1)}^{R/2} \nu(y)g \left( (t-s)y^\frac{\lambda-1}{\lambda}, \frac{x}{y} \right) dy = I_{3,1} + I_{3,2}. \quad (5.39) \]

We can use (5.24) in the region of integration of \( I_{3,1} \). Therefore:

\[ g \left( (t-s)y^\frac{\lambda-1}{\lambda}, \frac{x}{y} \right) \leq C(t-s)x^{-(3+\lambda)/2} y^{\lambda+1}. \quad (5.40) \]

51
Then:

\[
|I_{3,1}| \leq C \times^{-(3+\lambda)/2} \int_0^t ds(t-s) \int_0^{t-2/(\lambda-1)} |\nu(y)|y^\lambda dy
\]

\[
= C \times^{-(3+\lambda)/2} \int_0^t ds(t-s) \left( \int_0^1 |\nu(y)|y^\lambda dy + \int_1^{t-2/(\lambda-1)} |\nu(y)|y^\lambda dy \right)
\]

\[
= I_{3,1,1} + I_{3,1,2}.
\]

(5.41)

\[
I_{3,1,1} \leq C \times^{-(3+\lambda)/2} \sum_{n=0}^{\infty} 2^{-n(\lambda+1)} \int_0^t ds(t-s) \|\nu(s)\|_{L^\infty(2^{-(n+1)}, 2^{-n})}
\]

\[
\leq C \times^{-(3+\lambda)/2} 2^{\lambda/2} \sum_{n=0}^{\infty} 2^{-n(\lambda+1)} M_\infty(\nu; 2^{-n})
\]

\[
\leq C \times^{-(3+\lambda)/2} 2^{\lambda/2} \|\nu\|_{X_{3/2+\delta}} \sum_{n=0}^{\infty} 2^{-n(\lambda-1/2)}
\]

(5.42)

\[
I_{3,1,2} \leq C \times^{-(3+\lambda)/2} t \sum_{0 \leq 2^n \leq t-2/\lambda-1} \left[ t(2^n)^{(\lambda-1)/2} \right] \sum_{\ell=1}^{\infty} \int_{2^{-\lambda-1/2} \ell}^{t-2^{-\lambda-1/2} (\ell+1)} \|\nu(s)\|_{L^\infty(2^n, 2^{n+1})} ds \times
\]

\[
\times \int_{2^n}^{2^{n+1}} y^\lambda dy
\]

\[
\leq C \times^{-(3+\lambda)/2} t \sum_{0 \leq 2^n \leq t-2/\lambda-1} \left[ t(2^n)^{(\lambda-1)/2} \right] \sum_{\ell=1}^{\infty} \int_{2^{-\lambda-1/2} \ell}^{t-2^{-\lambda-1/2} (\ell+1)} \|\nu(s)\|_{L^\infty(2^n, 2^{n+1})} ds \times
\]

\[
\times 2^{n(\lambda+1)}
\]

\[
\leq C \times^{-(3+\lambda)/2} t \sum_{0 \leq 2^n \leq t-2/\lambda-1} \left[ t(2^n)^{(\lambda-1)/2} \right] 2^{-n(\lambda-1)/2} \times N_\infty(\nu; 2^{-n(\lambda-1)/2} \ell, 2^n) ds 2^{\ell(\lambda+1)}
\]

\[
\leq C \times^{-(3+\lambda)/2} t^2 \|\nu\|_{X_{3/2+\delta}} \sum_{0 \leq 2^n \leq t-2/\lambda-1} \frac{(2^n)^{\lambda-1+\delta}}{2^{n(\lambda+1)}}
\]

\[
\leq C \times^{-(3+\lambda)/2} t^2 \|\nu\|_{X_{3/2+\delta}} (t-2/(\lambda-1))^{\lambda-1+\delta} = C \frac{2^{5/(\lambda-1)}}{x^{(3+\lambda)/2}} \|\nu\|_{X_{3/2+\delta}}.
\]

(5.43)

On the other hand:

\[
I_{3,2} = \int_0^t ds \int_{t-2/(\lambda-1)}^{R/2} \nu(y, s) g \left( (t-s) y^{-\lambda+1}, \frac{x}{y} \right) dy
data +
\]

\[
+ \int_{t-2/(\lambda-1)}^{R/2} dy \int_0^{t-y^{-\lambda+1}/2} \nu(y, s) g \left( (t-s) y^{-\lambda+1}, \frac{x}{y} \right) ds
\]

\[
= I_{3,2,1} + I_{3,2,2}.
\]

(5.44)
In the term $I_{3,2,1}$ we use \(|5.36|\) that gives:

\[
|I_{3,2,1}| \leq x^{-(3+\lambda)/2} \int_{1-2/(\lambda-1)}^{R/2} dy \int_{t-y^-((\lambda-1)/2)}^{t} (t-s)^{-(\lambda+1)/(\lambda-1)} |\nu(y,s)| ds \\
\leq C x^{-(3+\lambda)/2} \sum_{0 \leq 2^n \leq 1-2/(\lambda-1)} \int_{2^n}^{2^{n+1}} dy \int_{2^{-n/(\lambda-1)/2}}^{2^{-n/(\lambda-1)/2} (\ell+1)} ds \\
(2^{-n/(\lambda-1)/2} \ell)^{-(\lambda+1)/(\lambda-1)} 2^{-n/(\lambda-1)/2} N_{\infty}(\nu; t - 2^{-n/(\lambda-1)/2} \ell, 2^n) \\
\leq C x^{-(3+\lambda)/2} |||\nu|||_{X_{3/2,2+\delta}} \sum_{0 \leq 2^n \leq 1-2/(\lambda-1)} (2^n)^{-\delta} \sum_{\ell=1}^{2^{n+1}} \ell^{-(\lambda+1)/(\lambda-1)} \\
= C x^{-(3+\lambda)/2} |||\nu|||_{X_{3/2,2+\delta}} \frac{2^n}{(\lambda-1)} 
\] (5.45)

In the term $I_{3,2,2}$, we use \(|5.40|\) which gives:

\[
|I_{3,2,2}| \leq x^{-(3+\lambda)/2} \int_{1-2/(\lambda-1)}^{R/2} \frac{dy}{y} \int_{t-y^-((\lambda-1)/2)}^{t} (t-s) |\nu(y,s)| ds \\
\leq C x^{-(3+\lambda)/2} \sum_{1-2/(\lambda-1) \leq 2^n \leq R/2} \int_{2^n}^{2^{n+1}} y^{\lambda+1} dy \int_{t-2^{-n/(\lambda-1)/2}}^{t} (t-s) |\nu(y,s)| ds \\
\leq C x^{-(3+\lambda)/2} \sum_{1-2/(\lambda-1) \leq 2^n \leq R/2} \int_{2^n}^{2^{n+1}} y^{\lambda} dy 2^{-n/(\lambda-1)/2} 2^{-n/(\lambda-1)/2} \\
\times 2^n \int_{t-2^{-n/(\lambda-1)/2}}^{t} |\nu(y,s)| ds \\
\leq C x^{-(3+\lambda)/2} \sum_{1-2/(\lambda-1) \leq 2^n \leq R/2} \int_{2^n}^{2^{n+1}} dy 2^n N_{\infty}(\nu; t - 2^{-n/(\lambda-1)/2}, 2^n) \\
\leq C x^{-(3+\lambda)/2} |||\nu|||_{X_{3/2,2+\delta}} \sum_{1-2/(\lambda-1) \leq 2^n \leq R/2} 2^{-n\delta} \\
\leq C x^{-(3+\lambda)/2} |||\nu|||_{X_{3/2,2+\delta}} \frac{2^n}{(\lambda-1)}. 
\] (5.46)

Estimates \(|5.45|\) and \(|5.46|\) yield:

\[
|I_{3,2}| \leq C x^{-(3+\lambda)/2} \int_{\frac{2^n}{(\lambda-1)}} \frac{2^n}{(\lambda-1)}. 
\] (5.47)

53
Then, using also (5.39) and (5.41), we deduce that
\[ |I_3| \leq C x^{-(3+\lambda)/2} t^{2\delta} \|\nu\|_{X^{3/2,2+\delta}}. \] (5.48)

We estimate now the term \( I_4 \). To this end we have:
\[
\int_0^t ds \int_{y \geq 2R} \nu(y) g \left( (t-s) y^{2 \lambda \delta}, x/y \right) \frac{dy}{y} \leq \int_{y \geq 2R} \frac{dy}{y} \int_0^{t-y-(\lambda-1)/2} \cdots \] ds + \int_{y \geq 2R} \frac{dy}{y} \int_{y-x-(\lambda-1)/2}^{t-y-(\lambda-1)/2} \cdots \] ds = I_{4,1} + I_{4,2}.

We split \( I_{4,1} \) in two pieces as follows:
\[
I_{4,1} = \int_{y \geq 2R} \frac{dy}{y} \int_0^{t-x-(\lambda-1)/2} \cdots \] ds + \int_{y \geq 2R} \frac{dy}{y} \int_{y-x-(\lambda-1)/2}^{t-y-(\lambda-1)/2} \cdots \] ds = I_{4,1,1} + I_{4,1,2}. \] (5.49)

In the term \( I_{4,1,1} \) we are in the region where (5.28) holds. Then, we use (5.36) to obtain:
\[
|I_{4,1,1}| \leq C x^{-(3+\lambda)/2} \int_{y \geq 2R} dy \int_{x-(\lambda-1)/2}^t s^{-(\lambda+1)/(\lambda-1)} |\nu(y, (t-s))| ds \leq C x^{-(3+\lambda)/2} \sum_{2^n \geq 2R} \int_{2^n}^{2^{n+1}} dy \sum_{2^{-(\lambda-1)/2} \leq 2^{-(\lambda-1)/2} \ell \leq t} 2^{-n-(\lambda-1)/2(\ell+1)} s^{-(\lambda+1)/(\lambda-1)} |\nu(y, (t-s))| ds \leq C x^{-(3+\lambda)/2} \sum_{2^n \geq 2R} 2^{n+1} \sum_{2^{-(\lambda-1)/2} \leq 2^{-(\lambda-1)/2} \ell \leq t} 2^{-n-(\lambda-1)/2 \ell} N_{\infty}(\nu; t - 2^{-(\lambda-1)/2} \ell, 2^n) \leq C x^{-(3+\lambda)/2} \|\nu\|_{X^{3/2,2+\delta}} \sum_{2^n \geq 2R} 2^{-n\delta} \leq C x^{-(3+\lambda)/2} \|\nu\|_{X^{3/2,2+\delta}} R^{-2\delta} \leq C x^{-(3+\lambda)/2} \|\nu\|_{X^{3/2,2+\delta}} t^{2\delta/(\lambda-1)}. \] (5.50)

In the integral \( I_{4,1,2} \) we use (5.27) to obtain:
\[
|g \left( (t-s) y^{\lambda \delta}, x/y \right) | \leq C (t-s)^{-1/(\lambda-1)} y^{-3/2}. \] (5.51)

This yields,
\[
|I_{4,1,2}| \leq C x^{-3/2} \int_{y \geq 2R} dy \int_{y-x-(\lambda-1)/2}^{t-y-(\lambda-1)/2} (t-s)^{-1/(\lambda-1)} |\nu(y, s)| ds = C x^{-3/2} \int_{y \geq 2R} dy \int_{y-(\lambda-1)/2}^{y-x-(\lambda-1)/2} s^{-1/(\lambda-1)} |\nu(y, (t-s))| ds \leq C x^{-3/2} \int_{2^n \geq 2R} dy \sum_{2^{n+1}} \int_{2^n} dy \cdot \sum_{1 \leq \ell \leq 2^{-(\lambda-1)/2}} 2^{-n-(\lambda-1)/2(\ell+1)} s^{-1/(\lambda-1)} |\nu(t-s)|_{L^\infty(2^n,2^{n+1})} ds \leq C x^{-3/2} \|\nu\|_{X^{3/2,2+\delta}} t^{2\delta/(\lambda-1)}.
\]
Then, for $R$ which, combined with (5.35), (5.37) and (5.48) yields,

$$
\text{Estimates (5.50), (5.52) and (5.54) give}
\leq C x^{-3/2} R^{-\nu/2} \sum_{2^n \geq 2R} 2^{-n\delta} \leq C x^{-(3+\lambda)/2} |||\nu||| X_{3/2,2+\delta} t^{2\delta/(\lambda-1)}
$$

(5.52)

using (5.31) in the last step.

In the term $I_{4,2}$ we use (5.23) to obtain,

$$
|g \left((t-s)y^{\frac{\lambda+1}{2}}, \frac{x}{y}\right)| \leq C (t-s) y^{(\lambda+2)/2} x^{-3/2}
$$

(5.53)

and then

$$
|I_{4,2}| \leq C x^{-3/2} \int_{y \geq 2R} y^{(\lambda+2)/2} \frac{dy}{y} \int_{t-y-(\lambda-1)/2}^{t} (t-s) |\nu(y,s)| ds
$$

\leq C x^{-3/2} \sum_{2^n \geq 2R} 2^{n+1} \int_{y \geq 2R} y^{\lambda/2} \frac{dy}{y} \int_{t-y-(\lambda-1)/2}^{t} (t-s) |\nu(y,s)| ds
$$

\leq C x^{-3/2} \sum_{2^n \geq 2R} 2^{-n(\lambda-1)} 2^{\lambda/2} 2^n N_\infty (\nu, t-2^{-n(\lambda-1)/2}, 2^n)
$$

\leq C x^{-(3+\lambda)/2} t^{2\delta/(\lambda-1)} |||\nu||| X_{3/2,2+\delta},
$$

(5.54)

where we have used (5.31) in the last inequality.

Estimates (5.50), (5.52) and (5.54) give

$$
|I_4| \leq C R^{-(3+\lambda)/2} t^{2\delta/(\lambda-1)} |||\nu||| X_{3/2,2+\delta},
$$

(5.55)

which, combined with (5.35), (5.37) and (5.48) yields,

$$
R^{(3+\lambda)/2} \left|\int_0^t G(t-s) \nu(s) ds\right|_{L^\infty (R/2, 2R)} \leq C t^{2\delta/(\lambda-1)} |||\nu||| X_{3/2,2+\delta}
$$

(5.56)

for $R \geq t^{-2/(\lambda-1)}$.

We assume now:

$$
1 \leq R \leq t^{-2/(\lambda-1)}
$$

(5.57)

Then,

$$
\int_0^t G(t-s) \nu(s,y) ds = \int_0^t ds \int_0^\infty dy \nu(s,y) \left((t-s)y^{\frac{\lambda+1}{2}}, \frac{x}{y}\right) \frac{dy}{y}
$$

\leq \int_0^t ds \int_{|x-y| \leq R/2} \nu(s,y) \left((t-s)y^{\frac{\lambda+1}{2}}, \frac{x}{y}\right) \frac{dy}{y}
$$

+ \int_0^t ds \int_{y \leq 1} \nu(s,y) \left((t-s)y^{\frac{\lambda+1}{2}}, \frac{x}{y}\right) \frac{dy}{y}
$$

+ \int_0^t ds \int_{1 \leq y \leq 5R/4} \nu(s,y) \left((t-s)y^{\frac{\lambda+1}{2}}, \frac{x}{y}\right) \frac{dy}{y}
$$

+ \int_0^t ds \int_{|y| \geq 5R/4} \nu(s,y) \left((t-s)y^{\frac{\lambda+1}{2}}, \frac{x}{y}\right) \frac{dy}{y}
$$

= J_1 + J_2 + J_3 + J_4.
$$

(5.58)
In the term $J_1$ we use again (5.34) to obtain

$$|J_1| \leq C \int_0^t ds \int |x-y| \leq R/2 \left( \frac{\nu(s,y)}{(t-s)^2 y^{1/2}} \right)^2 \Phi \left( \frac{x}{y} - 1 \right) \frac{dy}{y} \leq C \int_0^t ds \| \nu(s) \|_{L^\infty(R/2,2R)} \leq C \sqrt{t} \left( \int_0^t ds \| \nu(s) \|_{L^\infty(R/2,2R)}^2 \right)^{1/2} \leq C \sqrt{t} R^{-(\lambda-1)/4} \left( R^{(\lambda-1)/2} \int_0^{R^{-(\lambda-1)/2}} ds \| \nu(s) \|_{L^\infty(R/2,2R)}^2 \right)^{1/2} \leq C R^{-(3+\lambda)/2} \left[ \sqrt{t} R^{(\lambda-1)/4} R^{-\delta} \right] \| \nu \|_{X_{3/2,2+\delta}}.$$  

(5.59)

In the term $J_2$ we have again (5.40) and therefore:

$$|J_2| \leq C x^{-(3+\lambda)/2} \int_0^t ds \sum_{n=0}^\infty \| \nu(s) \|_{L^\infty(2^{-(n+1)},2^{-n})} (t-s) \int_2^{2^{-n}} y^{\lambda} dy \leq C x^{-(3+\lambda)/2} \sum_{n=0}^\infty \int_0^t ds \| \nu(s) \|_{L^\infty(2^{-(n+1)},2^{-n})} 2^{-n(\lambda+1)} \leq C x^{-(3+\lambda)/2} \sum_{n=0}^\infty 2^{-n(\lambda+1)} \sqrt{t} M_{\infty}(\nu; 0, 2^{-n}) \leq C x^{-(3+\lambda)/2} t^{3/2} \| \nu \|_{X_{3/2,2+\delta}} \sum_{n=0}^\infty 2^{-n(\lambda-1)/2}. \quad (5.60)$$

We consider now $J_3$ where we still have (5.40) and then,

$$|J_3| \leq C x^{-(3+\lambda)/2} \int_0^t ds (t-s) \int_1^{SR/4} y^{\lambda} |\nu(y,s)| dy \leq C x^{-(3+\lambda)/2} \sum_{1 \leq n \leq 5R/4} \int_0^t ds (t-s) \| \nu(s) \|_{L^\infty(2n,2^{n+1})} \int_{2^n}^{2^{n+1}} y^{\lambda} dy \leq C x^{-(3+\lambda)/2} \sum_{1 \leq n \leq 5R/4} 2^{n(\lambda+1)} t^{1/2} \left( \int_0^{R^{-(\lambda-1)/2}} ds \| \nu(s) \|_{L^\infty(2n,2^{n+1})}^2 \right)^{1/2} \leq C x^{-(3+\lambda)/2} t^{3/2} \sum_{1 \leq n \leq 5R/4} 2^{n(\lambda+1)} 2^{-n(\lambda-1)/4} N_{\infty}(\nu; 0, 2^n) \leq C x^{-(3+\lambda)/2} t^{3/2} \| \nu \|_{X_{3/2,2+\delta}} \sum_{1 \leq n \leq 5R/4} (2^n)^{\frac{\lambda}{4}(\lambda-1)-\delta} \leq C x^{-(3+\lambda)/2} t^{3/2} \| \nu \|_{X_{3/2,2+\delta}} R^\frac{\lambda}{4}(\lambda-1)-\delta \leq C x^{-(3+\lambda)/2} \| \nu \|_{X_{3/2,2+\delta}} t^{\frac{\lambda}{4}}. \quad (5.61)$$
In the term $\mathcal{J}_4$,

$$\mathcal{J}_4 = \int_{[y] \geq 5R/4} \frac{dy}{y} \left( t - y^{-(\lambda - 1)/2} \right) ds \nu(s, y) g \left( (t-s)y^{\frac{\delta}{2}}, \frac{x}{y} \right) + \int_{[y] \geq 5R/4} \frac{dy}{t} \left( t - y^{-(\lambda - 1)/2} \right) ds \nu(s, y) g \left( (t-s)y^{\frac{\delta}{2}}, \frac{x}{y} \right) = \mathcal{J}_{4,1} + \mathcal{J}_{4,2}.$$  \hspace{1cm} (5.62)

In the first term at the right hand side of (5.62) we are in the region where (5.27) holds and then we have (5.51) to obtain:

$$|\mathcal{J}_{4,1}| \leq C x^{-3/2} \int_{[y] \geq 5R/4} dy \int_{0}^{t} \left( t - y^{-(\lambda - 1)/2} \right) ds \nu(s, y) \left| (t-s)^{-1/\lambda} |\nu(y, s)| \right| ds.$$  

Notice that this integral is nonzero if and only if $y \geq t^{-2/(\lambda - 1)}$. In that case:

$$\begin{align*}
|\mathcal{J}_{4,1}| &\leq C x^{-3/2} \sum_{2^n \geq t^{-2/(\lambda - 1)} \geq R} \int_{2^n}^{2^{n+1}} dy \int_{y^{-(\lambda - 1)/2}}^{t} s^{-1/(\lambda - 1)} ||\nu(t-s)||_{L^\infty(2^n,2^{n+1})} ds \\
&\leq C x^{-3/2} \sum_{2^n \geq t^{-2/(\lambda - 1)} \geq R} \times \sum_{\ell = 1, 2^{-n(\lambda - 1)/2} \leq t}^{2^{n+1}} \int_{2^{-n(\lambda - 1)/2}}^{2^{-n(\lambda - 1)/2} t} dy \int_{2^{-n(\lambda - 1)/2} \ell}^{2^{-n(\lambda - 1)/2} (\ell+1)} s^{-1/(\lambda - 1)} ||\nu(t-s)||_{L^\infty(2^n,2^{n+1})} ds \\
&\leq C x^{-3/2} \sum_{2^n \geq t^{-2/(\lambda - 1)} \geq R} \sum_{1 \leq \ell \leq 2^{-n(\lambda - 1)/2}} 2^n 2^{-n(\lambda - 1) \ell^{-1/\lambda - 1}} 2^{n/2} \times \times N_{\infty}(\nu; t - 2^{-n(\lambda - 1)/2} \ell, 2^n) \hspace{1cm} (5.63)
\end{align*}$$

$$\begin{align*}
&\leq C x^{-3/2} ||\nu||_{X_{3/2,2+\delta}} \sum_{2^n \geq t^{-2/(\lambda - 1)} \geq R} 2^{-n \lambda / 2} 2^{-n \delta} \\
&\leq C x^{-3/2} ||\nu||_{X_{3/2,2+\delta}} t^{2\delta/(\lambda - 1)} R^{-\lambda/2} \leq C ||\nu||_{X_{3/2,2+\delta}} t^{2\delta/(\lambda - 1)} R^{-(3+\lambda)/2}. \hspace{1cm} (5.64)
\end{align*}$$

In the integral $\mathcal{J}_{4,2}$, we are in a region where (5.24) holds true. Then we may use (5.33) $(t-s) y^{(\lambda + 2)/2} x^{-3/2}$ to get:

$$|\mathcal{J}_{4,2}| \leq C x^{-3/2} \int_{[y] \geq 5R/4} y^{\lambda/2} dy \int_{(t-y^{-(\lambda - 1)/2})}^{t} ds (t-s) |\nu(y, s)|.$$
The last integral is bounded as follows:

\[ |\mathcal{J}_{4,2}| \leq C x^{-3/2} \int_{5R/4}^{R-2/(\lambda-1)} y^{\lambda/2} dy \int_0^t ds \, (t-s) |\nu(y,s)| + \]

\[ + C x^{-3/2} \int_{t-2/(\lambda-1)}^{\infty} y^{\lambda/2} dy \int_0^t ds \, (t-s) |\nu(y,s)| \]

\[ \leq C x^{-3/2} t^{3/2} \sum_{R \leq 2^n \leq t-2/(\lambda-1)} (2^n)^{\lambda/2 + 1/2} 2^{-n(\lambda-1)/2} \int_0^{2-n(\lambda-1)/2} ds \, |\nu(s)| 2^{r_{(\lambda-1)/2} \frac{2}{L_\infty (2^n, 2^{n+1})} \frac{1}{2}} + \]

\[ + C x^{-3/2} \sum_{2^n \geq t-2/(\lambda-1)} 2^{n-\lambda/2} R_n^2 N_\infty (\nu; 0, 2^n) + \]

\[ \leq C x^{-3/2} t^{3/2} \sum_{R \leq 2^n \leq t-2/(\lambda-1)} (2^n)^{\lambda/2 + 1/2} 2^{-n(\lambda-1)/4} N_\infty (\nu; 0, 2^n) + \]

\[ + C x^{-3/2} \sum_{2^n \geq t-2/(\lambda-1)} 2^{n-\lambda/2} R_n^2 N_\infty (\nu; 0, 2^n) \]

\[ \leq C x^{-3/2} t^{3/2} R^{-\lambda/2} \frac{\gamma^2}{\lambda} \|\nu\|_{X_{3/2, 2+\delta}} t^{-\frac{2}{\lambda-1} \frac{3(\lambda-1)}{2}} + \]

\[ + C x^{-3/2} \|\nu\|_{X_{3/2, 2+\delta}} R^{-\lambda/2} t^{3/2} \sum_{2^n \geq t-2/(\lambda-1)} 2^{2n-\lambda/2} = C R^{-3(\lambda+1)/2} \|\nu\|_{X_{3/2, 2+\delta}} t^{2\delta/(\lambda-1)}. \]

This yields,

\[ |\mathcal{J}_{4,2}| \leq C R^{-3(\lambda+1)/2} t^{2\delta/(\lambda-1)} \|\nu\|_{X_{3/2, 2+\delta}}, \quad (5.65) \]

which, combined with (5.65) gives

\[ |\mathcal{J}_{4}| \leq C R^{-3(\lambda+1)/2} t^{2\delta/(\lambda-1)} \|\nu\|_{X_{3/2, 2+\delta}}, \quad (5.66) \]

Adding (5.56), (5.60), (5.61) and (5.66):

\[ R^{(\lambda+1)/2} \|\nu\|_{L_\infty (R/2, 2R)} \leq C t^{2\delta/(\lambda-1)} \|\nu\|_{X_{3/2, 2+\delta}}, \quad (5.67) \]

for all \( R \geq t-2/(\lambda-1) \). Adding (5.56) and (5.67) yields, for all \( R > 1 \):

\[ R^{(\lambda+1)/2} \left\| \int_0^t G(t-s) \nu(s) \, ds \right\|_{L_\infty (R/2, 2R)} \leq C t^{2\delta/(\lambda-1)} \|\nu\|_{X_{3/2, 2+\delta}}, \quad (5.68) \]

We now consider the region where \( 0 < R < 1 \). Then, for \( |x-R| \leq R/8 \):

\[ \int_0^t G(t-s) \nu(s, y) dy \leq \int_0^t ds \int_{y \leq 3R/4} |\nu(s, y)| g \left( (t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y} \right) \frac{dy}{y} \]

\[ + \int_0^t ds \int_{y \geq 5R/4} |\nu(s, y)| g \left( (t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y} \right) \frac{dy}{y} \]

\[ + \int_0^t ds \int_{|x-y| \leq R/2} |\nu(s, y)| g \left( (t-s)y^{\frac{\lambda-1}{2}}, \frac{x}{y} \right) \frac{dy}{y} \]

\[ = K_1 + K_2 + K_3 \]

(5.69)
The last integral in the right hand side of (5.69) is estimated as follows. Since in that term (5.25) holds we still have (5.34) and then

\[
|K_3| \leq C \int_0^t ds \int_{|x-y| \leq R/2} \frac{\nu(s, y)}{(t-s)^{\frac{\lambda-1}{2}}} \Phi \left( \frac{x-y}{(t-s)^{\frac{\lambda-1}{2}}} \right) dy / y
\]

\[
\leq C \int_0^t \|\nu(s)\|_{L^\infty(R/2, 2R)} ds \int_{|x-y| \leq R/2} \frac{1}{(t-s)^{\lambda y}} \Phi \left( \frac{x-y}{(t-s)^{\lambda y}} \right) dy
\]

\[
\leq C \int_0^t \|\nu(s)\|_{L^\infty(R/2, 2R)} \leq C\sqrt{t} M_\infty(\nu; R) \leq C\sqrt{t} R^{-3/2} \|\nu\|_{X_{3/2, 2+\delta}} \cdot (5.70)
\]

Using (5.24) we deduce that, in the integral $K_1$ the following estimate holds:

\[
g \left( (t-s)y^{(\lambda-1)/2}, \frac{x}{y} \right) \leq C(t-s)y^{\lambda+1}x^{-(3+\lambda)/2}.
\]

Using this estimate we deduce:

\[
|K_1| \leq C x^{-3+\lambda/2} \sum_{2^{-n} \leq R} \int_0^t dy \int_{2^{-(n+1)}}^{2^{-n}} y^{\lambda} \|\nu(s)\|_{L^\infty(2^{-(n+1)}, 2^{-n})} (t-s) dy
\]

\[
\leq C x^{-3+\lambda/2} t \sum_{2^{-n} \leq R} 2^{-n(\lambda+1)} \sqrt{t} M_\infty(\nu, 2^{-n})
\]

\[
\leq C x^{-3+\lambda/2} t^{3/2} \|\nu\|_{X_{3/2, 2+\delta}} \sum_{2^{-n} \leq R} 2^{-n(\lambda+1-3/2)}
\]

\[
\leq C x^{-3+\lambda/2} t^{3/2} \|\nu\|_{X_{3/2, 2+\delta}} R^{-1/2} \leq C x^{-3+\lambda/2} t^{3/2} \|\nu\|_{X_{3/2, 2+\delta}} \cdot (5.71)
\]

for $x \in (R/2, 2R)$. We are then left with the term $|K_2|$. 

\[
K_2 = \int_{R/4 \leq y \leq 2} \frac{dy}{y} \int_0^t ds \nu(s, y) g \left( (t-s)y^{\frac{\lambda+1}{2}}, \frac{x}{y} \right)
\]

\[
+ \int_{y \geq 2} \frac{dy}{y} \int_0^{(t-y)^{-(\lambda-1)/2}+} ds \nu(s, y) g \left( (t-s)y^{\frac{\lambda+1}{2}}, \frac{x}{y} \right)
\]

\[
+ \int_{y \geq 2} \frac{dy}{y} \int_0^t ds \nu(s, y) g \left( (t-s)y^{\frac{\lambda+1}{2}}, \frac{x}{y} \right)
\]

\[
= K_{2,1} + K_{2,2} + K_{2,3}. \quad (5.72)
\]
In the term $K_{2,1}$, we may use (5.53) to obtain:

$$|K_{2,1}| \leq C x^{-3/2} \int_{5R/4 \leq y \leq 2} dy \lambda^{\gamma/2} \int_0^t ds |\nu(s,y)|(t-s)$$

$$\leq C x^{-3/2} t \sum_{n=0, R \leq 2^{-n}} 2^{-n(1+\lambda/2)} \sqrt{t} \left( \int_0^t ds \left| \nu(s) \right| L^\infty_{(2-(n+1), 2^{n+1})} \right)^{1/2}$$

$$\leq C x^{-3/2} t^{3/2} \sum_{n=0, R \leq 2^{n+1}} 2^{-n(1+\lambda/2)} M_\infty(\nu, 2^{-n})$$

$$\leq C x^{-3/2} t^{3/2} ||\nu||_{X_{3/2, 2+\delta}} \sum_{n=0, R \leq 2^{-n}} 2^{-n(\lambda-1)/2}$$

$$\leq C x^{-3/2} t^{3/2} ||\nu||_{X_{3/2, 2+\delta}}.$$  \hspace{1cm} (5.73)

In $K_{2,2}$, (5.51) holds and then,

$$|K_{2,2}| \leq C x^{-3/2} \int_{y \geq 2} dy \int_0^{(t-y^{-(\lambda-1)/2})} ds \ |\nu(s,y)|(t-s)^{-1/(\lambda-1)}$$  \hspace{1cm} (5.74)

We notice also here that the last integral in the right hand side of (5.74) is nonzero only if $y \geq t^{-2/(\lambda-1)}$. Therefore

$$|K_{2,2}| \leq C x^{-3/2} \sum_{n=1, 2^n \geq t^{-2/(\lambda-1)}} 2^n \int_{y^{-(\lambda-1)/2}}^{t} ds \ s^{-1/(\lambda-1)} ||\nu(s)|| L^\infty(2^n, 2^{n+1})$$

$$\leq C x^{-3/2} \sum_{n=1, 2^n \geq t^{-2/(\lambda-1)}} 2^{3n/2} \sum_{\ell=1, 2^{-n(\lambda-1)/2} \ell \leq t} \ell^{-1/(\lambda-1)} 2^{-n(\lambda-1)/2} \times N_\infty(\nu; t-2^{-n(\lambda-1)/2} \ell, 2^n)$$

$$\leq C x^{-3/2} ||\nu||_{X_{3/2, 2+\delta}} \sum_{n=1, 2^n \geq t^{-2/(\lambda-1)}} 2^{-n(\lambda+2+\delta)}$$

$$\leq C x^{-3/2} ||\nu||_{X_{3/2, 2+\delta}} t^{1/2 - \lambda/2 + \delta}. \hspace{1cm} (5.75)$$

In $K_{2,3}$, we may use (5.53), whence

$$|K_{2,3}| \leq C x^{-3/2} \int_{y \geq 2} y^{\lambda/2} \int_0^t \int_{(t-y^{-(\lambda-1)/2})}^{(t-y^{-(\lambda-1)/2})} ds \ |\nu(y,s)|(t-s) ds$$

$$\leq C x^{-3/2} t \sum_{n=1}^{\infty} 2^{n(\lambda/2+1)/2} 2^{-n(\lambda-1)/2} \int_{(t-2^n(2n)^{2n})}^{(t-2^n(2n)^{2n})} ||\nu(s)|| L^\infty(2n, 2^{n+1}) ds$$

$$\leq C x^{-3/2} t \sum_{n=1}^{\infty} 2^{n(\lambda/2+1)/2} 2^{-n(\lambda-1)/2} 2^{n} N_\infty(\nu; t-2^{-n(\lambda-1)/2} 2^n)$$

$$\leq C x^{-3/2} t ||\nu||_{X_{3/2, 2+\delta}} \sum_{n=1}^{\infty} 2^{-n(1/2+\delta)} \leq C x^{-3/2} t ||\nu||_{X_{3/2, 2+\delta}}.$$  \hspace{1cm} (5.76)

By (5.72), (5.73), (5.75) and (5.76) we have

$$|K_2| \leq C x^{-3/2} \left( t + t^{2\delta + \lambda/(\lambda-1)} \right) ||\nu||_{X_{3/2, 2+\delta}}. \hspace{1cm} (5.77)$$
Adding (5.70), (5.71) and (5.77) we obtain the following estimate for $0 < R \leq 1$:

$$R^{3/2} \left\| \int_0^t G(t - s) \nu(s) \, ds \right\|_{L^\infty(R/2, 2R)} \leq Ct \| \nu \|_{X_{3/2, 2+\delta}}. \quad (5.78)$$

The Lemma follows combining (5.68) and (5.69).

Lemma 5.7 For all $\varphi \in Y_{3/2, (3+\lambda)/2}^\sigma (T)$ with $\sigma > 1 + \delta$, $\varepsilon \geq 0$ and $0 < T < 1$:

$$\sup_{0 \leq t \leq T} \left\| \int_0^t G(t - s)(L - L_\varepsilon)\varphi(s) \, ds \right\|_{3/2, (3+\lambda)/2} \leq CT^\beta \| \varphi \|$$

for some constant $C > 0$ independent of $T$, of $\varepsilon$ and $\varphi$.

Proof of Lemma 5.7

By Lemma 5.6

$$\sup_{0 \leq t \leq T} \left\| \int_0^t G(t - s)A_1(s) \, ds \right\|_{3/2, (3+\lambda)/2} \leq CT^\beta \| A_1 \|_{X_{3/2, (3+\lambda)/2}^\sigma (T)}.$$

Moreover, for all $h(t, x), q, p$:

$$\| h \|_{X_{q,p}(T)} \leq C \sup_{0 \leq t \leq T} \| h(t) \|_{q,p}$$

Then, using also Lemma 5.1 and the definition of the norm $\| \|_{\| \|}$

$$T^\beta \| A_1 \|_{X_{3/2, (3+\lambda)/2}^\sigma (T)} \leq T^\beta \| \varphi \|_{3/2, (3+\lambda)/2} \leq C \| \varphi \|$$

And then

$$\sup_{0 \leq t \leq T} \left\| \int_0^t G(t - s)A_1(s) \, ds \right\|_{3/2, (3+\lambda)/2} \leq CT^\beta \| \varphi \|. \quad (5.79)$$

A similar argument is used for the term $A_{2, \varepsilon}$. First, by Lemma 5.6

$$\sup_{0 \leq t \leq T} \left\| \int_0^t G(t - s)A_{2, \varepsilon}(s) \, ds \right\|_{3/2, (3+\lambda)/2} \leq CT^\beta \| A_{2, \varepsilon} \|_{X_{3/2, (3+\lambda)/2}^\sigma (T)}.$$

Then, by Lemma 5.4

$$\| A_{2, \varepsilon} \|_{X_{3/2, (3+\lambda)/2}^\sigma (T)} \leq C \| \varphi \|$$

whence, using the definition of the norm $\| \|_{\| \|}$:

$$\sup_{0 \leq t \leq T} \left\| \int_0^t G(t - s)A_{2, \varepsilon}(s) \, ds \right\|_{3/2, (3+\lambda)/2} \leq CT^\beta \| \varphi \|. \quad (5.80)$$

and Lemma 5.7 follows from (5.79) and (5.80). □
Lemma 5.8 There exists a positive constant $C$ such that, for all $0 < T^* < 1$, for all $\theta \in [0, 1]$, for all $\nu \in Y^{\sigma}_{3/2, 2+\delta}(T)$ and all $\varphi$ satisfying $|||\varphi||| < +\infty$ and solving:

\[
\frac{\partial \varphi}{\partial t} = L(\varphi) + \theta (L - L)(\varphi) + \nu, \quad x > 0, \quad t \in (0, T^*)
\]  

we have:

\[
|||\varphi||| \leq C |||\nu|||_{Y^{\sigma}_{3/2, 2+\delta}(T^*)}.
\]

Remark 5.9 The result of Lemma 5.8 remains true if the space $Y^{\sigma}_{3/2, 2+\delta}(T^*)$ is replaced by $Y^{\sigma}_{3/2, 2}(T^*)$. However, a solution of (5.81) satisfying $|||\varphi||| < +\infty$ does not exists in general if $\nu \in Y^{\sigma}_{3/2, 2}(T^*)$.

Proof of Lemma 5.8 We first rewrite the equation (5.81) as follows:

\[
\frac{\partial \varphi}{\partial t} = (1 - \theta)L(\varphi) + \theta L(\varphi) + \nu
\]

Then, for $x \in (3R/4, 5R/5)$ and $R > 1$ we define the new variables: $x = XR, \ y = YR$, $t = (\tau/R^{(\lambda-1)/2})$ and $\varphi(x,t) = R^{-(3+\lambda)/2} \Psi(X,\tau)$. Since $t \in (0, T_*), \ \tau \in (0, T_* R^{(\lambda-1)/2})$.

\[
\frac{\partial \Psi}{\partial \tau} = (1 - \theta) L(\Psi) + \theta \left[ R^{3/2} \int_{0}^{X/2} \left( (X - Y)^{\lambda/2} \Psi(X - Y) - X^{\lambda/2} \Psi(X) \right) (RY)^{\lambda/2} f_0(RY) \, dY \right] - \theta X^{\lambda/2} \Psi(X) \int_{X/2}^{\infty} Y^{\lambda/2} f_0(RY) \, dY + \tilde{\nu}_1
\]

\[
\tilde{\nu}_1 = R^2 \nu(RX, \tau R^{-(\lambda-1)/2}) + \theta R^{(3+\lambda)/2} \times \int_{0}^{X/2} \left( (X - Y)^{\lambda/2} f_0(R(X - Y)) - X^{\lambda/2} f_0(RX) \right) Y^{\lambda/2} \Psi(Y) \, dY
\]

\[
- \theta X^{\lambda/2} R^{(3+\lambda)/2} f_0(RX) \int_{X/2}^{\infty} Y^{\lambda/2} \Psi(Y) \, dY
\]  

(5.82)

Using the expression of the operator $L$ given in (1.3)

\[
\frac{\partial \Psi}{\partial \tau} = (1 - \theta) \int_{0}^{X/2} \left( (X - Y)^{\lambda/2} \Psi(X - Y) - X^{\lambda/2} \Psi(x) \right) Y^{\lambda/2} Y^{-3/2} \, dy
\]

\[
+ [R^{(3+\lambda)/2}] \left[ R^{3/2} \int_{0}^{X/2} \left( (X - Y)^{\lambda/2} \Psi(X - Y) - X^{\lambda/2} \Psi(X) \right) (RY)^{\lambda/2} f_0(RY) \, dY \right] - 2(1 - \theta) \sqrt{2} X^{(\lambda-1)/2} \Psi(X) - \theta R^{(3+\lambda)/2} X^{\lambda/2} \Psi(X) \int_{X/2}^{\infty} Y^{\lambda/2} f_0(RY) \, dY
\]

\[
+ \tilde{\nu}_1 + \tilde{\nu}_2
\]

\[
\tilde{\nu}_2 = (1 - \theta) \int_{0}^{X/2} \left( (X - Y)^{-3/2} - X^{-3/2} \right) Y^{\lambda/2} \Psi(Y) \, dY - (1 - \theta) X^{-3/2} \int_{X/2}^{\infty} Y^{\lambda/2} \Psi(Y) \, dY
\]  

(5.83)
We can rewrite the equation as
\[
\frac{\partial \Psi}{\partial \tau} = T_{1-\theta,R} \left( M_{\lambda/2} \Psi \right) - a(X,t) \Psi + Q
\] (5.84)
\[
a(X,t) = 2(1 - \theta) \sqrt{2} X^{(\lambda-1)/2} + \theta R^{(3+\lambda)/2} X^{\lambda/2} \int_{X/2}^{\infty} Y^{\lambda/2} f_0(RY) dY
\] (5.85)
\[
Q = \nu_1 + \nu_2.
\] (5.86)

Since \( ||\varphi||_{3/2,(3+\lambda)/2} \) is finite, we can combine (5.71) in Theorem 5.1 with (5.11) to obtain:
\[
\sup_{0 \leq T \leq R^{(\lambda-1)/2}} \left( \int_{T}^{\min(T+1,T^* R^{(\lambda-1)/2})} ||\Psi(s)||_{H^s(3/4,5/4)}^2 ds \right)^{1/2} \leq C \sup_{0 \leq t \leq T^*} ||\varphi(t)||_{3/2,(3+\lambda)/2} + C ||\Psi(s)||_{H^{(2\lambda-1)+(1/2,2)}}
\]

Moreover, in order to estimate the norm of \( Q(s) \) we first notice that, using (5.11):
\[
||\varphi||_{3/2,(3+\lambda)/2} \) is finite, we can combine (5.71) in Theorem 5.1 with (5.11) to obtain:
\[
\sup_{0 \leq T \leq R^{(\lambda-1)/2}} \left( \int_{T}^{\min(T+1,T^* R^{(\lambda-1)/2})} ||\Psi(s)||_{H^s(3/4,5/4)}^2 ds \right)^{1/2} \leq C \sup_{0 \leq t \leq T^*} ||\varphi(t)||_{3/2,(3+\lambda)/2} + C ||\Psi(s)||_{H^{(2\lambda-1)+(1/2,2)}}
\]

The same estimate holds trivially for the term \( \theta X^{\lambda/2} R^{(3+\lambda)/2} f_0(RX) \int_{X/2}^{\infty} Y^{\lambda/2} \Psi(Y) dY \) in \( \nu_1 \). We are then left with the term
\[
\int_{X/2}^{\infty} \left( (X - Y)^{-3/2} - X^{-3/2} \right) Y^{\lambda/2} \Psi(Y) dY.
\]

Using that
\[
||\Psi(Y)|| \leq \frac{1}{Y^{(3+\lambda)/2}}
\]
we deduce:
\[
|| \int_{0}^{X/2} \left( (X - Y)^{-3/2} - X^{-3/2} \right) Y^{\lambda/2} \Psi(Y) dY \mid_{H^{s}(1/2,2)} \mid \leq C ||\varphi(t)||_{3/2,(3+\lambda)/2} + C ||\Psi(s)||_{H^{(2\lambda-1)+(1/2,2)}}
\]

This gives
\[
\sup_{0 \leq T \leq T^* R^{(\lambda-1)/2}} \left( \int_{T}^{\min(T+1,T^* R^{(\lambda-1)/2})} ||\Psi(s)||_{H^s(3/4,5/4)}^2 ds \right)^{1/2} \leq C \sup_{0 \leq t \leq T^*} ||\varphi||_{3/2,(3+\lambda)/2} + C ||\varphi||_{3/2,(3+\lambda)/2} + C \sup_{0 \leq t \leq T^*} ||\varphi||_{3/2,(3+\lambda)/2} + C ||\varphi||_{3/2,(3+\lambda)/2}
\]

A bootstrap argument then yields:
\[
\sup_{0 \leq T \leq T^* R^{(\lambda-1)/2}} \left( \int_{T}^{\min(T+1,T^* R^{(\lambda-1)/2})} ||\Psi(s)||_{H^s(3/4,5/4)}^2 ds \right)^{1/2} \leq C \sup_{0 \leq t \leq T^*} ||\varphi||_{3/2,(3+\lambda)/2} + C ||\varphi||_{3/2,(3+\lambda)/2}
\] (5.87)
(actually in an interval slightly smaller than \((3/4, 5/4)\), for example: \((7/8, 9/8)\)). We deduce,

$$
\sup_{0 \leq t_0 \leq T} \sup_{R > 1} \left( R^{(3+\lambda)/2} N_{2; \sigma} (\varphi; R, t_0) \right) \leq C \sup_{0 \leq t \leq T} ||| \varphi(t) |||_{3/2, (3+\lambda)/2} + C ||| \nu |||_{Y^{\infty}_{3/2, 2+\delta}(T)}.
$$

We consider now the case where \(0 < R \leq 1\). We rescale the equation for \(x \in (3R/4, 5R/4)\) and \(R < 1\). The new variables are now \(x = XR\), \(y = YR\), and \(\varphi(x, t) = R^{-3/2} \Psi(X, t)\). Arguing as above, the function \(\Psi\) satisfies now:

$$
\frac{\partial \Psi}{\partial \tau} = R^{\lambda-1} T_{1-\theta, R} (M_{\lambda/2}) + Q
$$

\[ Q = R^{3/2} \nu (RX, \tau) + R^{(\lambda-1)/2} \left( (1 - \theta) \int_0^{X/2} ((X - Y)^{-3/2} - X^{-3/2}) Y^{\lambda/2} \Psi(Y) dY - (1 - \theta) X^{-3/2} \int_{X/2}^{\infty} Y^{\lambda/2} \Psi(Y) dY - 2(1 - \theta) \sqrt{2} X^{(\lambda-1)/2} \Psi(X) - (1 - \theta) X^{-3/2} \int_{X/2}^{\infty} Y^{\lambda/2} \Psi(Y) dY \right) + \theta R^{\lambda+1} \int_0^{X/2} ((X - Y)^{\lambda/2} \Psi(X - Y) - X^{\lambda/2} \Psi(X)) y^{\lambda/2} f_0 (RY) dY + \theta R^{\lambda+1} \int_0^{X/2} ((X - Y)^{\lambda/2} f_0 (RX) - X^{\lambda/2} f_0 (RX)) Y^{\lambda/2} \Psi(Y) dY - \theta R^{\lambda/2} X^{\lambda/2} \Psi(X) \int_0^{\infty} y^{\lambda/2} f_0 (y) dy + \theta R^{\lambda/2} X^{\lambda/2} \Psi(X) \int_0^{X/2} Y^{\lambda/2} f_0 (RY) dY - \theta R^{\lambda/2} f_0 (RX) X^{\lambda/2} \int_0^{\infty} y^{\lambda/2} \varphi(y, t) dy - (1 - \theta) X^{-3/2} \int_0^{\infty} \varphi(y, t) y^{\lambda/2} dy + R^{\lambda-1} X^{-3/2} \int_0^{X/2} \Psi(Y) Y^{\lambda/2} dY. \quad (5.89)
\]

Where we have used that:

$$
x^{-3/2} \int_{x/2}^{\infty} g(y) y^{\lambda/2} dy = x^{-3/2} \int_0^{\infty} g(y) y^{\lambda/2} dy - x^{-3/2} \int_0^{x/2} g(y) y^{\lambda/2} dy = a(t) x^{-3/2} - x^{-3/2} \int_0^{x/2} g(y) y^{\lambda/2} dy.
$$

By Theorem 3.3 with \(\kappa = R^{(\lambda-1)/2}\)

$$
|||\Psi|||_{L^2_t(0, T; \mathcal{H}^s(3/4, 5/4))} \leq C |||Q|||_{L^2_t(0, T; \mathcal{H}^\sigma_2(1/2, 2))}.
$$

We now have:

$$
|||Q|||_{\mathcal{H}^s(3/4, 5/4)} \leq C |||\Psi|||_{\mathcal{H}^{(s-1)+} (1/2, 2)} + \sup_{0 < t < T} |||\varphi(t)|||_{3/2, (3+\lambda)/2} + C |||\nu|||_{Y^{\infty}_{3/2, 2+\delta}(T)}.
$$

64
As before, a bootstrap argument as in the case $R > 1$ gives

$$||\Psi||_{L^2_T(0, T; H^\nu((3/4, 5/4)))} \leq C \sup_{0 \leq t \leq T^*} ||\varphi(t)||_{L^2_{(3/2, (3+\lambda)/2)}} + C ||\nu||_{Y^{\alpha}_{3/2, 2+\beta}(T^*)} \leq C \sup_{0 \leq t \leq T^*} ||\varphi(t)||_{L^2_{(3/2, (3+\lambda)/2)}} + C ||\nu||_{Y^{\alpha}_{3/2, 2+\beta}(T^*)}$$

and then, rewriting this estimate in the original variables

$$R^{3/2} M_{2; \sigma}(\varphi, R) \leq C \left( \sup_{0 \leq t \leq T^*} ||\varphi(t)||_{L^2_{(3/2, (3+\lambda)/2)}} + ||\nu||_{Y^{\alpha}_{3/2, 2+\beta}(T^*)} \right). \quad (5.90)$$

Combined with (5.88) we deduce

$$\sup_{0 \leq t \leq T^*} \left( R^{3/2} M_{2; \sigma}(\varphi, R) \right) \leq C \left( \sup_{0 \leq t \leq T^*} ||\varphi(t)||_{L^2_{(3/2, (3+\lambda)/2)}} + ||\nu||_{Y^{\alpha}_{3/2, 2+\beta}(T^*)} \right), \quad (5.91)$$

and then

$$||\varphi|| \leq C \sup_{0 \leq t \leq T^*} ||\varphi(t)||_{L^2_{(3/2, (3+\lambda)/2)}} + C ||\nu||_{Y^{\alpha}_{3/2, 2+\beta}(T^*)}. \quad (6.1)$$

We use now

$$\varphi(t) = \theta \int_0^t G(t - s) (\mathcal{L} - L) (\varphi)(s) \, ds + \int_0^t G(t - s) \nu(s) \, ds$$

which yields

$$||\varphi(t)||_{L^2_{(3/2, (3+\lambda)/2)}} \leq C \int_0^t ||G(t - s) (\mathcal{L} - L) (\varphi)(s)||_{L^2_{(3/2, (3+\lambda)/2)}} \, ds + \int_0^t ||G(t - s) \nu(s)||_{L^2_{(3/2, (3+\lambda)/2)}} \, ds.$$

By Lemma 5.6 and Lemma 5.7

$$||\varphi(t)||_{L^2_{(3/2, (3+\lambda)/2)}} \leq C T^* ||\varphi||_{L^2_{(3/2, (3+\lambda)/2)}} + (T^* + T^* 2((\lambda-1)/2)) ||\nu||_{Y^{\alpha}_{3/2, 2+\beta}(T^*)}$$

and the result follows taking $T^*$ small enough. \hfill \Box

### 6 Proof of Theorem 2.1

We introduce the auxiliary operators $L_\epsilon$, for $\epsilon > 0$, defined as follows:

$$L_\epsilon(g) = \int_0^{x/2} \left( (x - y)^{-3/2} - x^{-3/2} \right) y^{\lambda/2} g(y) \, dy$$

$$+ \int_0^{x/2} \left( (x - y)^{\lambda/2} g(x - y) - x^{\lambda/2} g(x) \right) \frac{dy}{y^{3/2} + \epsilon^{3/2} x^{3/2}},$$

$$- x^{-3/2} \int_{x/2}^\infty y^{\lambda/2} g(y) \, dy - 2\sqrt{2} x^{(\lambda-1)/2} g(x). \quad (6.1)$$

For all $\epsilon > 0$ the operator $L_\epsilon$ is more regular than $L$. Notice in particular that $g$ and $L_\epsilon g$ have the same regularity.
Lemma 6.1 Let $0 \leq T \leq 1$. Then, there exists a constant $C > 0$ such that, for any $\varphi \in \mathcal{E}_{T,\sigma}$, for all $t_0 \in (0, T)$ and $\varepsilon \geq 0$:

\[
R^2 N_{2,\sigma-1/2}((\mathcal{L} - L_\varepsilon)(\varphi); R, t_0) \leq C \|||\varphi|||, \quad \forall R > 1, \quad (6.2)
\]

\[
R^{2-\lambda/2} M_{2,\sigma-1/2}((\mathcal{L} - L_\varepsilon)(\varphi); R) \leq C \|||\varphi|||, \quad \forall \ 0 < R < 1. \quad (6.3)
\]

Proof of Lemma 6.1 Notice that the operator $(\mathcal{L} - L_\varepsilon)$ may be written as $A_1 + A_{2,\varepsilon}$ where $A_1$ and $A_{2,\varepsilon}$ are defined in (5.1) and (5.4)-(5.5). Lemma 6.1 then follows using Lemma 6.2 and Lemma 5.4.

\]

Lemma 6.2 (i) There exists a constant $C > 0$ such that, for all $\varepsilon \in (0, 1]$, $\theta \in [0, 1)$, $\varphi \in \mathcal{E}_{T,\sigma}$ and $u \in \mathcal{E}_{T,\sigma}$ satisfying:

\[
\partial_t \varphi = (1 - \theta) L(\varphi) + \theta \mathcal{L}(\varphi) + (\mathcal{L} - L_\varepsilon) (u)
\]

there holds:

\[
|||\varphi||| \leq C \sup_{0 \leq t \leq T} \|||\varphi|||_{3/2,(3+\lambda)/2} + \frac{C}{1 - \theta}|||u|||.
\]

Proof of Lemma 6.2 The proof of this Lemma is similar to that of Lemma 5.8. The difference comes from the fact that we must use the regularising effect of the operator $T_{1-\sigma,R}$ of Theorem 3.1. We then start by scaling the variables.

In the case $R > 1$ and for $x \in (3R/4, 5R/5)$ we define the new variables: $x = XR$, $y = YR$, $t = (\tau/R^{(\lambda-1)/2})$ and $\varphi(x,t) = R^{-3(3+\lambda)/2} \psi(X, \tau)$. Since $t \in (0, T_\varepsilon)$, $\tau \in (0, T_\varepsilon) R^{(\lambda-1)/2}$. The function $\psi(X, \tau)$ satisfies equations (5.81)-(5.86) with $\tilde{\nu}_1$ and $\tilde{\nu}_2$ are defined as in (5.82), (5.83) but where $\nu$ is now given by

\[
\nu = (\mathcal{L} - L_\varepsilon) (u).
\]

Using Lemma 6.1 and Theorem 3.1 with $\varepsilon = 1 - \theta$, we obtain, arguing as in the proof of (6.2),

\[
\sup_{0 \leq t \leq T} \left( \int_T^{\min(T+1, T_\varepsilon)} \|||\psi(s)|||^2_{H^1_\sigma(3/4,5/4)} ds \right)^{1/2} \leq C \sup_{0 \leq t \leq T} \|||\varphi|||_{3/2,(3+\lambda)/2} + \frac{C}{1 - \theta}|||u|||.
\]

(6.5)

Notice that the only difference between the proof of (6.5) and that of (5.87) comes from the control of the term $\nu$ defined in (6.1). However that term is estimated as the term $P$ in (3.5) with $\kappa = 1$, and $\varepsilon = 1 - \theta$ combined with (6.2).

We consider now the range $R \in (0, 1)$ and rescale the equation for $x \in (3R/4, 5R/5)$. The new variables are now $x = XR$, $y = YR$, $\varphi(x,t) = R^{-3/2} \psi(X, t)$ and $u(x,t) = R^{-3/2} U(X,t)$. Arguing as above, the function $\psi$ satisfies now the same equation (6.89) where the term $Q$ is defined in (6.89) where here again $\nu$ is given by (6.4). The term $R^{3/2} \nu(R, X, t)$ in (6.89) is rewritten using (6.1) as follows:

\[
R^{3/2} \nu(R, X, t) = R^{(\lambda-1)/2} (\mathcal{L} - L_\varepsilon) (U)
\]

\[
= Q_0(X, t) + Q_1(X, t)
\]
where,

\[
Q_0(X,t) = R^{(\lambda-1)/2} \int_0^{X/2} \left( (X-Y)^{\lambda/2} U(X-Y) - X^{\lambda/2} U(X) \right) \frac{dY}{Y^{3/2} + \varepsilon^{3/2} X^{3/2}}
\]

\[
Q_1(X,t) = R^{(\lambda-1)/2} \int_0^{X/2} \left( (X-Y)^{-3/2} - X^{-3/2} \right) Y^{\lambda/2} U(Y,t) dY
\]

\[ - X^{-3/2} \int_{R X/2}^{\infty} y^{\lambda/2} u(y,t) dy - 2\sqrt{2} R^{(\lambda-1)/2} X^{(\lambda-1)/2} U(X), \]

and \( Q_1(X,t) \) satisfies,

\[ |M_{2,\sigma}(Q_1; 1)| \leq C \|u\|. \]

Using now Theorem 3.1 with \( \varepsilon = 1 - \theta \) and \( \kappa = R^{(\lambda-1)/2} \) and estimating all the remaining terms as in the proof of (5.90) we obtain

\[ R^{3/2} M_{2,\sigma}(\varphi; R) \leq C \sup_{0 \leq t \leq T^*} \|\varphi\|_{3/2, (3+\lambda)/2} + \frac{C}{1 - \theta} \|u\|. \] (6.6)

Combining (6.5) and (6.6) the Lemma follows. \( \Box \)

**Lemma 6.3** Let \( 0 \leq T \leq 1 \). Then for any \( \varphi \in \mathcal{E}_{T,\sigma} \), for all \( t_0 \in (0, T) \):

\[ \lim_{\varepsilon \to 0} N_{2,\sigma-1/2} ((L - L_\varepsilon)(\varphi); R, t_0) = 0, \quad \forall R > 1, \]

\[ \lim_{\varepsilon \to 0} M_{2,\sigma-1/2} ((L - L_\varepsilon)(\varphi); R) = 0, \quad \forall 0 < R < 1. \]

**Proof of Lemma 6.3.** After rescaling the variables \( x = R X, t = t_0 + R^{-(\lambda-1)/2} \tau \) and \( \varphi(x,t) = \psi(X,\tau) \), the two identities reduce to:

\[ \lim_{\varepsilon \to 0} \int_0^{\tau^*} \| (L - L_\varepsilon)(\psi)(\tau) \|^2_{H^1_X(1/2,2)} d\tau = 0 \] (6.7)

with \( 0 < \tau^* < 1 \). Using (1.3) and (6.1) we have

\[ (L - L_\varepsilon)(\psi) = (\mathcal{W}_\infty - \mathcal{W}_{\infty,\varepsilon})(\psi) \]

where \( \mathcal{W}_\infty \) and \( \mathcal{W}_{\infty,\varepsilon} \) are defined in (4.47) and (4.48). Therefore the Lemma follows combining (4.50) and the Lebesgue convergence Theorem. \( \Box \)

**End of the proof of Theorem 2.1.**

Our goal is to solve (1.7) for \( \theta = 1 \). To this end we use a continuation argument starting at \( \theta = 0 \).

For \( \theta = 0 \) equation (1.7) has a solution \( \varphi \in \mathcal{E}_{T,\sigma} \). This is a consequence of the results of [4] and of Lemma 5.8 in Section 3 with \( \theta = 0 \).

Then, we define:

\[ \theta^* = \sup \left\{ \theta \geq 0; \text{for all } \nu \in Y_{3/2,2+\delta}^\sigma(T), \text{there exists } \varphi \in \mathcal{E}_{\nu,\sigma} \text{ solution of (1.7)} \right\} \] (6.8)
The Lemmas [6.7] and [5.8] show that there exists a constant $C > 0$ such that, for any $\theta < \theta^*$ and for all $\nu \in Y_{\sigma}^{3/2,2+\delta}(T)$ there exists a function $\varphi \in \mathcal{E}_{T,\sigma}$ such that

$$|||\varphi||| \leq C|||\nu||| Y_{\sigma}^{3/2,2+\delta}(T).$$

Suppose that $\theta^* < 1$. We will show that for all $\theta > \theta^*$ with $\theta - \theta^*$ sufficiently small and all $\nu \in Y_{\sigma}^{3/2,2+\delta}(T)$ there exists a function $\varphi \in \mathcal{E}_{T,\sigma}$ and solving (1.7). This would give a contradiction.

To this end we use a fixed point argument.

Given $\tilde{\varphi} \in \mathcal{E}_{T,\sigma}$ and $\nu \in Y_{\sigma}^{3/2,2+\delta}(T)$ we define $\varphi_{\varepsilon,n} \in \mathcal{E}_{T,\sigma}$ as the solution of

$$\partial_t \varphi_{\varepsilon,n} = (1 - \theta_n) L(\varphi_{\varepsilon,n}) + \theta_n L(\varphi_{\varepsilon,n}) + (\theta - \theta_n) (L - L_\varepsilon)(\tilde{\varphi}) + \nu$$

where $\theta_n$ is a sequence such that $\theta_n < \theta^*$, $\theta_n \to \theta^*$ as $n \to +\infty$. The functions $\varphi_{\varepsilon,n}$ are well defined since $\theta_n < \theta^*$ and $(L - L_\varepsilon)(\tilde{\varphi}) \in Y_{\sigma}^{3/2,2+\delta}(T)$. Combining Lemma 5.8 and Lemma 6.2 we obtain:

$$|||\varphi_{\varepsilon,n}||| \leq C(\theta - \theta_n) \sup_{0 \leq t \leq T^*} |||\varphi||| Y_{\sigma}^{3/2,2+\lambda}/2 + C \frac{\theta - \theta_n}{1 - \theta_n} |||\tilde{\varphi}||| + |||\nu||| Y_{\sigma}^{3/2,2+\delta}.$$  

(6.10)

Since $\varphi_{\varepsilon,n}$ satisfies equation (6.9) we have:

$$\varphi_{\varepsilon,n} = \int_0^t G(t-s) [\theta_n(L - L)(\varphi_{\varepsilon,n}) + (\theta - \theta_n)(L - L_\varepsilon)(\tilde{\varphi}) + \nu] ds.$$  

(6.11)

Using now Lemma 5.6 and Lemma 5.7 we obtain:

$$\sup_{0 \leq t \leq T} |||\varphi_{\varepsilon,n}||| Y_{\sigma}^{3/2,2+(\lambda+2)/2} \leq CT^{3/2} \left(|||\varphi_{\varepsilon,n}||| + (\theta - \theta_n) |||\tilde{\varphi}||| + |||\nu||| Y_{\sigma}^{3/2,2+\delta} \right).$$  

(6.12)

Therefore, using (6.10) and (6.12) for $T$ small we obtain

$$|||\varphi_{\varepsilon,n}||| \leq C \frac{\theta - \theta_n}{1 - \theta_n} |||\tilde{\varphi}||| + |||\nu||| Y_{\sigma}^{3/2,2+\delta}.$$  

(6.13)

Moreover, given $\tilde{\varphi} \in \mathcal{E}_{T,\sigma}$, $\tilde{\varphi}' \in \mathcal{E}_{T,\sigma}$ and denoting the corresponding solutions as $\varphi_{\varepsilon,n}$ and $\varphi_{\varepsilon,n}'$ a similar argument yields:

$$|||\varphi_{\varepsilon,n} - \varphi_{\varepsilon,n}'||| \leq C \frac{\theta - \theta_n}{1 - \theta_n} |||\tilde{\varphi} - \tilde{\varphi}'|||.$$  

(6.14)

By Lemma 6.3 we deduce that

$$\lim_{\varepsilon, \varepsilon' \to 0} N_{2;\sigma - 1/2} \left( (L_{\varepsilon'} - L_{\varepsilon})(\tilde{\varphi}); R, t_0 \right) = 0, \ \forall R > 1, \ \forall t_0 \in (0,T),$$

$$\lim_{\varepsilon, \varepsilon' \to 0} M_{2;\sigma - 1/2} \left( (L_{\varepsilon'} - L_{\varepsilon})(\tilde{\varphi}); R \right) = 0, \ \forall \ 0 < R < 1, \ \forall t_0 \in (0,T).$$

We use now the regularising effects obtained in Theorem 3.1 combined with the rescaling argument that have already been used in the proof of Lemma 5.8 to obtain:

$$\lim_{\varepsilon, \varepsilon' \to 0} N_{2;\sigma} \left( \varphi_{\varepsilon,n} - \varphi_{\varepsilon,n}'; R, t_0 \right) = 0, \ \forall R > 1, \ \forall t_0 \in (0,T),$$

$$\lim_{\varepsilon, \varepsilon' \to 0} M_{2;\sigma} \left( \varphi_{\varepsilon,n} - \varphi_{\varepsilon,n}'; R \right) = 0, \ \forall \ 0 < R < 1, \ \forall t_0 \in (0,T)$$

$$\lim_{\varepsilon, \varepsilon' \to 0} |||\varphi_{\varepsilon,n} - \varphi_{\varepsilon,n}'|||_{L^\infty([0,T]\times[R/2,2R])} = 0 \ \forall R > 0 \ \forall t \in (0,T).$$
By (6.13) we have:

\[
\lim_{\varepsilon, \varepsilon' \to 0} N_{2; \sigma} (\varphi_{n} - \varphi; R, t_{0}) = 0, \quad \forall R > 1, \quad \forall t_{0} \in (0, T),
\]

(6.15)

\[
\lim_{\varepsilon, \varepsilon' \to 0} M_{2; \sigma} (\varphi_{n} - \varphi; R) = 0, \quad \forall 0 < R < 1, \quad \forall t_{0} \in (0, T),
\]

(6.16)

\[
\lim_{\varepsilon, \varepsilon' \to 0} ||\varphi_{n} - \varphi||_{L^\infty([0,T] \times [R/2, 2R])} = 0 \quad \forall R > 0, \quad \forall t \in (0, T).
\]

(6.17)

By (6.13) we have:

\[
R^{(3+\lambda)/2} N_{2; \sigma} (\varphi_{n}; R, t_{0}) \leq C \frac{\theta - \theta_{n}}{1 - \theta_{n}} |||\tilde{\varphi}||| + C |||\nu|||_{Y_{3/2,2+\delta}^{\gamma}}, \quad \forall R > 1, \quad \forall t_{0} \in (0, T),
\]

(6.18)

\[
R^{3/2} M_{2; \sigma} (\varphi_{n}; R, t_{0}) \leq C \frac{\theta - \theta_{n}}{1 - \theta_{n}} |||\tilde{\varphi}||| + C |||\nu|||_{Y_{3/2,2+\delta}^{\gamma}}, \quad \forall 0 < R < 1, \quad \forall t_{0} \in (0, T),
\]

(6.19)

\[
\max \left\{ x^{3/2}, x^{(3+\lambda)/2} \right\} |||\varphi_{n}(x, t)||| \leq C \frac{\theta - \theta_{n}}{1 - \theta_{n}} |||\tilde{\varphi}||| + C |||\nu|||_{Y_{3/2,2+\delta}^{\gamma}}, \quad \forall x \in (R/2, 2R),
\]

(6.20)

\[
\forall R > 0 \quad \forall t \in (0, T).
\]

Taking limits as \( \varepsilon \to 0 \):

\[
R^{(3+\lambda)/2} N_{2; \sigma} (\varphi; R, t_{0}) \leq C \frac{\theta - \theta_{n}}{1 - \theta_{n}} |||\tilde{\varphi}||| + C |||\nu|||_{Y_{3/2,2+\delta}^{\gamma}}, \quad \forall R > 1, \quad \forall t_{0} \in (0, T),
\]

(6.21)

\[
R^{3/2} M_{2; \sigma} (\varphi; R, t_{0}) \leq C \frac{\theta - \theta_{n}}{1 - \theta_{n}} |||\tilde{\varphi}||| + C |||\nu|||_{Y_{3/2,2+\delta}^{\gamma}}, \quad \forall 0 < R < 1, \quad \forall t_{0} \in (0, T),
\]

(6.22)

\[
\max \left\{ x^{3/2}, x^{(3+\lambda)/2} \right\} |||\varphi(x, t)||| \leq C \frac{\theta - \theta_{n}}{1 - \theta_{n}} |||\tilde{\varphi}||| + C |||\nu|||_{Y_{3/2,2+\delta}^{\gamma}}, \quad \forall x \in (R/2, 2R),
\]

(6.23)

\[
\forall R > 0 \quad \forall t \in (0, T).
\]

whence \( \varphi_{n} \in \mathcal{E}_{T; \sigma} \) and,

\[
|||\varphi_{n}||| \leq C \frac{\theta - \theta_{n}}{1 - \theta_{n}} |||\tilde{\varphi}||| + |||\nu|||_{Y_{3/2,2+\delta}^{\gamma}}.
\]

A similar argument yields

\[
|||\varphi_{n} - \varphi'_{n}||| \leq C \frac{\theta - \theta_{n}}{1 - \theta_{n}} |||\tilde{\varphi} - \tilde{\varphi}'|||.
\]

Notice that \( \varphi_{n} \in L^{2}(0, T; H_{loc}^{\sigma}(\mathbb{R}^{+})) \). Moreover, passing to the weak limit in the equation (6.9) as \( \varepsilon \to 0 \) we obtain that \( \varphi_{n} \) solves

\[
\partial_{t} \varphi_{n} = (1 - \theta_{n}) L(\varphi_{n}) + \theta_{n} \mathcal{L}(\varphi_{n}) + (\theta - \theta_{n}) (\mathcal{L} - L) (\tilde{\varphi}) + \nu
\]

in the sense of distributions. Then, \( \varphi_{n} \in H^{1}(0, T; H_{loc}^{\sigma}(\mathbb{R}^{+})) \).

Formula (6.13) implies that the application \( \tilde{\varphi} \mapsto \varphi_{n} \) has a fixed point for any \( \nu \in Y_{3/2,2+\delta}^{\gamma} \), \( n \) sufficiently large and \( \theta - \theta^{*} > 0 \) sufficiently small. Let us denote by \( \varphi \) such a fixed point that satisfies:

\[
\partial_{t} \varphi = (1 - \theta_{n}) L(\varphi) + \theta_{n} \mathcal{L}(\varphi) + (\theta - \theta_{n}) (\mathcal{L} - L) (\varphi) + \nu
\]

(6.19)
whence,
\[ \partial_t \varphi = (1 - \theta) L(\varphi) + \theta L(\varphi) + \nu \]  
(6.20)
and since \( \theta > \theta^* \) this yields a contradiction. It then follows that \( \theta^* = 1 \). We prove now
the solvability of the equation for \( \theta = 1 \).
To this end we consider a sequence \( \theta_k \to 1 \) and the corresponding sequence of solutions
\( \varphi_k \in E_{T, \sigma} \).
\[
\partial_t \varphi_k = (1 - \theta_k) L(\varphi_k) + \theta_k L(\varphi_k) + \nu. 
\]  
(6.21)
By Lemma 5.8 we have
\[
|||\varphi_k||| \leq C|||\nu|||_Y^\sigma_y \|_{3/2, 2+\delta}. 
\]  
(6.22)
Therefore,
\[
R^{(3+\lambda)/2} N_{2, \sigma} (\varphi_k; R, t_0) \leq C|||\nu|||_Y^\sigma_y \|_{3/2, 2+\delta}, \quad \forall R > 1, \forall t_0 \in (0, T), 
\]  
(6.23)
\[
R^{3/2} M_{2, \sigma} (\varphi_k; R, t_0) \leq C|||\nu|||_Y^\sigma_y \|_{3/2, 2+\delta}, \quad \forall 0 < R < 1, \forall t_0 \in (0, T) 
\]  
(6.24)
\[
\max \left\{ x^{3/2}, x^{(3+\lambda)/2} \right\} |\varphi_k(x, t)| \leq C|||\nu|||_Y^\sigma_y \|_{3/2, 2+\delta}, \quad \forall x \in (R/2, 2R), \forall R > 0 \forall t \in (0, T). 
\]  
(6.25)
The sequence \( \{\varphi_k\}_{k \in \mathbb{N}} \) is then weakly compact in \( L^2(t_0, t_0 + R^{-(\lambda-1)/2} T; H^\sigma (R/2, 2R)) \)
for all \( R > 0 \) and \( t_0 \in (0, T) \). Therefore, using a diagonal procedure, there exists a subsequence, still denoted
\( \{\varphi_k\}_{k \in \mathbb{N}} \), and a function \( \varphi \) defined in all \( \mathbb{R}^+ \times (0, T) \) such that
\( \varphi_k \) converges to \( \varphi \) weakly in \( L^2((0, T); H^\sigma (R_1, R_2)) \) for all \( R_2 > R_1 > 0 \). Since the left
hand sides in the inequalities \( (6.23)-(6.25) \) are all of them convex functions of \( \varphi_k \), these
inequalities are preserved under weak limits. Therefore
\[
R^{(3+\lambda)/2} N_{2, \sigma} (\varphi; R, t_0) \leq C|||\nu|||_Y^\sigma_y \|_{3/2, 2+\delta}, \quad \forall R > 1, \text{ for a.e. } t_0 \in (0, T), 
\]  
(6.26)
\[
R^{3/2} M_{2, \sigma} (\varphi; R, t_0) \leq C|||\nu|||_Y^\sigma_y \|_{3/2, 2+\delta}, \quad \forall 0 < R < 1, \text{ for a.e. } t_0 \in (0, T), 
\]  
(6.27)
\[
\max \left\{ x^{3/2}, x^{(3+\lambda)/2} \right\} |\varphi(x, t)| \leq C|||\nu|||_Y^\sigma_y \|_{3/2, 2+\delta}, \quad \forall x \in (R/2, 2R), \forall R > 0, \text{ for a.e. } t \in (0, T). 
\]  
(6.28)
whence \( \varphi \in E_{T, \sigma} \).
On the other hand, it is possible to pass to the limit in the equation \( (6.21) \) in the weak
sense of \( L^2(0, T; H^\sigma (R_1, R_2)) \) for any \( R_2 > R_1 > 0 \) to obtain that \( \varphi \in L^2(0, T; H^\sigma_{loc}(\mathbb{R}^+)) \) \( \cap \)
\( H^1(0, T; H^\sigma_{loc}^{-1/2}(\mathbb{R}^+)) \) is a solution of
\[
\partial_t \varphi = L(\varphi) + \nu. 
\]  
(6.29)
in the sense of distributions.
Finally, in order to prove uniqueness let us assume that \( \varphi_1 \) and \( \varphi_2 \) are two solutions
of \( (6.29) \). Then, the function \( \psi = \varphi_1 - \varphi_2 \) satisfies,
\[
\partial_t \psi = L(\psi) \\
\psi(x, 0) = 0. 
\]  
and Lemma 5.8 for \( \theta = 1 \) and \( \nu = 0 \) shows that \( \psi = 0 \) and uniqueness holds. \( \square \)
7 Proof of Theorem 2.2

Consider the function $F_{R,t_0}(X,\tau)$ defined in (2.21). The function $\Psi(X,\tau) = R^{(3+\lambda)/2}F_{R,t_0}(X,\tau)$ satisfies equation (5.85) with $\theta = 1$. Then, using (3.10) we obtain

$$R^{(3+\lambda)/2}\left(\int_{t_0}^{\min(t_0+R-(\lambda-1)/2,T)} \left|\hat{F}_{R,t_0}(k,\tau)\right|^2 |k|^{2\sigma} \min\{|k|, R\} \, dk \, dt\right)^{1/2} \leq C \left(||\varphi|| + ||\nu||_{Y^{3/2,2+\delta}}\right)$$

whence Theorem 2.2 follows. \qed

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