ACCELERATION-EXTENDED GALILEAN SYMMETRIES WITH CENTRAL CHARGES AND THEIR DYNAMICAL REALIZATIONS*

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Abstract

We add to Galilean symmetries the transformations describing constant accelerations. The corresponding extended Galilean algebra allows, in any dimension $D = d + 1$, the introduction of one central charge $c$ while in $D = 2 + 1$ we can have three such charges: $c, \theta$ and $\theta'$. We present nonrelativistic classical mechanics models, with higher order time derivatives and show that they give dynamical realizations of our algebras. The presence of central charge $c$ requires the acceleration square Lagrangian term. We show that the general Lagrangian with three central charges can be reinterpreted as describing an exotic planar particle coupled to a dynamical electric and a constant magnetic field.

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1 Introduction

In our recent paper \[1\] we studied conformal extensions of Galilean symmetries in \(D = 2 + 1\) dimensions. Our extension was performed by the addition of four generators:

i) two generators - dilation \(D\) and expansion \(K\) - extending the Galilean algebra to the Schrödinger algebra \([2-4]\)

ii) two new generators \(F_i, (i = 1, 2)\) describing constant accelerations.

It is easy to see that such an extension can be made for any nonrelativistic dimension \(d\); in fact, it can be obtained as a nonrelativistic \(c \to \infty\) limit of the relativistic \(D\)-dimensional \((D = d + 1)\) conformal algebra described by the Lie algebra \(O(d, 2)\). In such a derivation the generators \(F_i (i = 1\ldots d)\) are provided by the nonrelativistic limits of space components of special conformal generators and describe, in a nonrelativistic theory, constant accelerations. Thus, such acceleration-extended Galilean symmetries, in \(d\) spacial dimensions, are described by \(\frac{1}{2} d(d-1) + 3d + 1\) generators, namely \((i = 1 \ldots d)\)

- \(J_{ij} = -J_{ji}\) (space rotations \(\alpha_{ij}\))
- \(P_i\) (space translations \(a_i\))
- \(K_i\) (constant velocity motions, or Galilean boosts \(v_i\))
- \(F_i\) (constant acceleration motions, or Galilean accelerations \(b_i\))
- \(H\) (time translations \(t' = t + a\)).

These symmetries taken in their infinitesimal form can be realised in nonrelativistic space-time as

\[
\delta x_i = -\delta a_i - \delta v_i t - \delta b_i t^2 + \delta \alpha_{ij} x_j,
\]

\[
\delta t = \delta a.
\]

The acceleration-extended Galilean algebra has the following nonvanishing commutators:

\[
[J_{ij}, J_{kl}] = \delta_{ik} J_{jl} - \delta_{jl} J_{ik} + (i \leftrightarrow j, k \leftrightarrow l),
\]

\[
[J_{ij}, A_k] = \delta_{ik} A_j - \delta_{jk} A_i \quad (A_i = P_i, K_i, F_i),
\]

\[
[H, K_i] = P_i,
\]

\[1\]In \[1\] such a limit was mentioned for \(D = 2 + 1\)
The aim of this paper is to study central extensions of the algebra (2·3) and its dynamical representations.

First we recall that the standard Galilean algebra (relations (2) for \( J_{ij}, P_i, K_i \) and \( H \)) has two central extensions:

i) For arbitrary \( D \) one can introduce one central extension \([5]\)

\[
[P_i, K_j] = m \delta_{ij},
\]  
(4)

where \( m \) describes the nonrelativistic mass parameter.

ii) In \( D = 2 + 1 \) we can introduce a second charge \( \theta \), called exotic \([6],[7]\)

\[
[K_i, K_j] = \theta \epsilon_{ij},
\]  
(5)

which can be related to the noncommutativity of the space components of the nonrelativistic \( D = 2 + 1 \) space-time \([8]\).

A dynamical realisation of the \( D = 2 + 1 \) exotic algebra can be deduced from the following nonrelativistic model \([8]\)

\[
L = \frac{1}{2} m \ddot{x}_i^2 - \frac{\theta}{2} \epsilon_{ij} \dddot{x}_i \dddot{x}_j.
\]  
(6)

One can show by considering the Jacobi identity for the three generators \( H, P_i, F_j \) and using (3), (4) \( 0 \equiv [H, [P_i, F_j]] + [P_i, F_j, H] + [F_j, H, P_i] = -2[P_i, K_j] = -2m \delta_{ij} \) that in acceleration - extended Galilei algebra we must put \( m = 0 \). In such a case one arrives at the following central extensions

- For arbitrary \( D \) one can introduce a single central extension

\[
[K_i, F_j] = 2c \delta_{ij}.
\]  
(7)

- In \( D = (2+1) \), in addition to relations (5) and (7) one can have a third central charge

\[
[F_i, F_j] = \theta' \epsilon_{ij}.
\]  
(8)

The algebraic consistency requires also that

\[
[P_i, F_j] = -2\theta \epsilon_{ij}.
\]  
(9)

The paper is organised as follows. In the next section we consider a dynamical model in any dimension \( D \) with the Lagrangian being given by the square of the accelerations of the particle and the Noether charges satisfying (7). In Section 3 we discuss those properties of the acceleration-extended \( D = 2 + 1 \) Galilean algebra which are model independent, \( ie \) have a purely
We consider the enveloping algebra $U(\hat{g}_\theta)$ with one central charge $\theta$ (see (5) and (9)) and we show that the symmetry algebras with three central charges can be embedded in $U(\hat{g}_\theta)$. We also present the Casimirs in the presence of central charges and discuss the possible enlargements of the symmetry algebra. A particular case here is the extension of the Galilean conformal algebra by the addition of two further generators $D$ and $K$. We find that when all three central parameters ($\theta, \theta'$ and $c$) are nonzero, the commutators in the Galilean conformal algebra involving the generators $D$ and $K$ are deformed. In Section 4 we introduce $D = 2 + 1$ Lagrangian models which realize this three-fold centrally extended symmetry algebra. In particular, in Section 4.3, we show that the general planar model with three central charges can be reinterpreted as describing the motion of a planar noncommutative particle interacting with dynamic electric and constant magnetic fields. Section 5 contains some concluding remarks.

2 An acceleration square Lagrangian and a new central charge $c$ (arbitrary $D = d + 1$)

The nonrelativistic kinetic term which is quasi-invariant under the acceleration extended Galilei symmetry (1) has the form ($\ddot{x}^2 = \ddot{x}_i \ddot{x}_i; i = 1,...,d$)

$$L = \frac{c \ddot{x}^2}{2}. \quad (10)$$

Indeed, performing the transformation (1) one obtains

$$\delta L = -2c \delta b_i \ddot{x}_i = \frac{d}{dt}(-2c \delta b_i \dot{x}_i). \quad (11)$$

Introducing $y_i = \dot{x}_i$ as an independent coordinate the Lagrangian (10) can be put into the following equivalent first order form

$$L = p_i(\ddot{x}_i - y_i) + q_i \dot{y}_i - \frac{1}{2c} q_i^2. \quad (12)$$

Using the Faddeev-Jackiw procedure [9]-[10] we obtain the following nonvanishing Poisson brackets

$$\{x_i, p_j\} = \delta_{ij}, \quad \{y_i, q_j\} = \delta_{ij}. \quad (13)$$

The acceleration-extended Galilean transformations of the additional variables $(p_i, y_i, q_i)$, which are consistent with the field equations

$$y_i = \dot{x}_i, \quad \dot{p}_i = 0, \quad (14)$$

$$\ddot{x}_i = 2c q_i.$$
\[ \dot{y}_i = \frac{1}{c} q_i, \quad \dot{q}_i + p_i = 0 \]

take the form
\[ \delta p_i = \delta \alpha_{ij} p_j \]
\[ \delta y_i = -\delta v_i - 2t \delta b_i - \frac{\delta a}{c} q_i + \delta \alpha_{ij} y_j, \quad (15) \]
\[ \delta q_i = -2c \delta b_i + \delta a p_i + \delta \alpha_{ij} q_j \]
and also
\[ \delta x_i = -\delta a_i - t \delta v_i - t^2 \delta b_i + \delta \alpha_{ij} x_j - y_i \delta a. \quad (16) \]

We note that the variation (15-16) implies that the Lagrangian is quasi-invariant
\[ \delta L = \frac{d}{dt} (\delta a p_i y_i - 2c \delta b_i y_i). \quad (17) \]

The transformations (15-16) are generated by the Poisson brackets (13) in the standard way\(^2\) by the following realization of the generators of the acceleration extended algebra with one central charge \(c\) (see (7)):
\[ P_i = p_i, \quad H = p_i y_i + \frac{1}{2c} q_i^2, \quad (18) \]
\[ J_{ij} = \frac{1}{2} \left( x_i [p_j] + y_i [q_j] \right), \quad (19) \]
\[ K_i = q_i + p_i t, \quad F_i = -2cy_i + 2t q_i + t^2 p_i, \quad (20) \]
where the second term in the energy generator \(H\) is consistent with Souriau general theorem on the barycentric decomposition of nonrelativistic systems in phase space \([11,12]\). It is easy to check, using (13), that (7) is satisfied.

In quantum theory the quantized Poisson brackets (13) take the form
\[ [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad [\hat{y}_i, \hat{q}_j] = i\hbar \delta_{ij} \quad (21) \]
and can be realised in the standard way on the wave functions \(\Psi(x_i, y_j)\) via the Schrödinger representation:
\[ \hat{x}_i = x_i, \quad \hat{p}_i = \hbar \frac{\partial}{i \partial x_i}, \quad \hat{y}_i = y_i, \quad \hat{q}_i = \hbar \frac{\partial}{i \partial y_i}. \quad (22) \]

The Schrödinger equation, in the first quantized theory, corresponding to the Lagrangian (10) and the Hamiltonian (18) takes the form
\[ \left( -\frac{\hbar^2}{2c} \frac{\partial^2}{\partial y_i \partial y_i} + \frac{\hbar}{i} y_i \frac{\partial}{\partial x_i} \right) \Psi = E \Psi. \quad (23) \]

\(^2\)The variation of the phase space variables \(Y_k\), generated by \(G_r\), is given by \(\delta Y_i = \delta \alpha_r \{G_r, Y_i\}\) (\(\alpha_r\) - symmetry parameters).
3 Properties and Enlargements of the $D = 2 + 1$ acceleration-extended Galilean algebra

The space of dimension $d = 2$ is special due to the existence of an antisymmetric covariant constant tensor $\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If, for $d = 2$, we extend the Galilei algebra by adding to it two generators $F_i$, describing constant acceleration motions, due to the existence of the tensor $\epsilon_{ij}$, we can introduce, through relations (5) and (8) two independent central charges $\theta$ and $\theta'$. In the general case, for $d = 2$, we can introduce three central charges: $\theta$, $\theta'$ and $c$.

3.1 How do we go from one central charge to three

Let us observe that the status of all three central charges is not the same. To see this we introduce the universal enveloping algebra $U(\hat{g}_\theta)$, where $\hat{g}_\theta$ denotes the acceleration-extended Galilei algebra with only one central charge $\theta$. Then we observe that the following modification of the relations

$$[K_i, F_j] = 0, \quad \rightarrow \quad [K'_i, F'_j] = 2c\delta_{ij}$$
$$[F_i, F_j] = 0, \quad \rightarrow \quad [F'_i, F'_j] = \theta'\delta_{ij}$$

(24)

can be achieved by the linear change of basis inside the enveloping algebra $U(\hat{g}_\theta)$

$$K'_i = K_i + \frac{c}{2\theta} \epsilon_{ij} P_j$$
$$F'_i = F_i + \frac{c^2}{\theta} \epsilon_{ij} K_j + \frac{1}{4} \left( \frac{c^2}{\theta^2} - \frac{\theta'}{\theta} \right) P_i.$$ 

(25)

It can be checked that under this transformation all Lie algebra relations of $\hat{g}_\theta$ besides (24) remain unchanged. Thus we see that the two acceleration extended Galilean algebra with a one parameter central extension (by the parameter $\theta$) and the algebra with three central extension parameters ($\theta, c, \theta'$) have the same enveloping algebras. We can recall here the well known analogy: in $D = 2 + 1$ spacetime the exotic Galilean algebra with two central charges ($m, \theta$) can be obtained through the transformation [13], [8]

$$K'_i = K_i - \frac{\theta}{2m} \epsilon_{ij} P_j$$

(26)

where $K_i$ and $P_i$ belong to the standard Galilean algebra with one central charge $m$:

$$[K_i, K_j] = [P_i, P_j] = 0, \quad [K_i, P_j] = i\delta_{ij} m.$$ 

(27)

$^3$Remaining generators $P_i, H, J$, stay unchanged.
Indeed, the generators \((K'_i, P'_i = P_i)\) in \(D = 2 + 1\) lead to the appearance of an ‘exotic’ central charge

\[
[K'_i, K'_j] = i\epsilon_{ij}\theta, \quad [P'_i, P'_j] = 0, \quad [K'_i, P'_j] = i\delta_{ij}m, \quad (28)
\]

which was recently reinterpreted as generating noncommutativity of the \(d = 2\) space coordinates \([8],[14]\).

### 3.2 Casimirs

\(\hat{g}_\theta\) has two Casimirs:

\[
C_H = H - \frac{1}{\theta} \epsilon_{ij} K_i P_j \quad (29)
\]

and

\[
C_J = J - \frac{1}{2\theta}(F_i P_i - K_i^2). \quad (30)
\]

The Casimirs for the case of three central charges \(\theta, \theta'\) and \(c\) can be easily obtained from (29) by the transformation (25) with the result

\[
C'_H = H - \frac{1}{\theta} \epsilon_{ij} K'_i P_j + \frac{c}{2\theta'} P_i^2 \quad (30)
\]

and

\[
C'_J = J - \frac{1}{2\theta}(F'_i P_i - K_i'^2) - \frac{c}{\theta'} H - \frac{\theta'}{8\theta} P_i^2. \quad (30)
\]

### 3.3 Enlargement by an \(O(2,1)\) algebra

We may add to \(\hat{g}_\theta\) two further generators

\[
J_\pm = \frac{1}{4\theta} (K_\pm^2 - F_\pm P_\pm), \quad (31)
\]

where \(K_\pm = K_1 \pm iK_2\) etc. For \(C_J = 0\) they form together with \(J\) an \(O(2,1)\) algebra

\[
[J_3, J_\pm] = \mp iJ_\pm, \quad [J_+, J_-] = 2iJ_3, \quad (32)
\]

where \(J_3 = \frac{\theta}{2}\). The remaining nonvanishing commutators of \(J_\pm\) describe the \(O(2,1)\) covariance of any two-vector \(A_i \in (P_i, K_i, F_i)\)

\[
[J_+, A_-] = -iA_+, \quad [J_-, A_+] = iA_- \quad (33)
\]

In the Lagrangian model discussed in \([1]\) we had \(C_J = 0\) and so this model provided a realization of this \(O(2,1)\) algebra.
3.4 Extension to $D = 2 + 1$ conformal Galilean algebra and central charges

To obtain the Galilean conformal algebra \[1\] one has to add to the acceleration-extended Galilean algebra \((2\cdot 4)\) two further generators: dilatation $D$ and expansion $K$, which together with the Hamiltonian, form an $O(2, 1)$ subalgebra

\[
[D, H] = -2H, \quad [D, K] = 2K, \quad [H, K] = D. \quad (34)
\]

The generators $D$ and $K$ are scalars\(^4\)

\[
[D, J] = [K, J] = 0 \quad (35)
\]

and, in addition, they satisfy

\[
[D, P_i] = -P_i, \quad [D, K_i] = 0, \quad [D, F_i] = F_i,
\]
\[
[K, P_i] = -2K_i, \quad [K, K_i] = -F_i, \quad [K, F_i] = 0. \quad (36)
\]

It is easy to argue that the Galilean conformal algebra does not permit the central extensions with parameters $c$ and $\theta'$. Indeed, if we observe that from \((34-36)\) we get the following mass dimensions of the generators

\[
[P_i] = M^1, \quad [K_i] = M^0, \quad [F_i] = M^{-1}
\]
\[
[H] = M^1, \quad [D] = M^0, \quad [K] = M^{-1} \quad [J] = M^0 \quad (37)
\]

we obtain from \((35)\), \((7)\) and \((8)\) that

\[
[\theta] = M^0, \quad [c] = M^{-1}, \quad [\theta'] = M^{-2}. \quad (38)
\]

We see from \((38)\) that the constants $c$ and $\theta'$ break the scale and conformal invariance. In fact, if we supplement the transformation \((25)\) by the relations $D' = D$ and $K' = K$ we get from the generators of the acceleration-extended Galilean algebra $\hat{g}_0$ with one nonvanishing central charge the following deformation of the conformal Galilean algebra\(^5\)

\[
[K', P'_i] = -2K'_i + \frac{\epsilon}{\theta} \epsilon_{ij} P'_j
\]
\[
[K', F'_i] = -\frac{\epsilon}{\theta} \epsilon_{ij} F'_j - \frac{3}{2} \frac{\epsilon}{\theta} \epsilon_{ij} K'_i + \frac{1}{\theta^2} \left( \frac{\epsilon^2}{\theta} - \frac{\theta'}{2} \right) \epsilon_{ij} P'_j
\]
\[
[K', K'_i] = -F'_i + \frac{1}{\theta^2} \left( \frac{\epsilon^2}{\theta} - \theta' \right) P'_i
\]
\[
[D', K'_i] = -\frac{\epsilon}{\theta^2} \epsilon_{ij} P'_j
\]
\[
[D', F'_i] = F'_i - \frac{\epsilon}{\theta} \epsilon_{ij} K'_j - \frac{1}{\theta} \left( \frac{\epsilon^2}{\theta} - \frac{\theta'}{2} \right) P'_i. \quad (39)
\]

\(^4\)We recall that in $d = 2$ $J_{ij} = \epsilon_{ij} J$.

\(^5\)We only list the modified relations.
The parameters $c$ and $\theta'$ cease to be central charges and take the role of deformation parameters (see eq [15]). The deformed conformal Galilean algebra with the generators $(P'_i, K'_i, F'_i, J', H', D')$ and the three parameters ($\theta$, $c$ and $\theta'$) can be treated as a particular choice of basis in the enveloping conformal Galilean algebra $U(\hat{g}_\theta)$ with one central charge.

4 Dynamical planar models realizing the acceleration extended Galilean symmetry with three central charges

In this section we introduce and discuss $D = 2 + 1$ nonrelativistic bilinear Lagrangian models which possess the acceleration-extended Galilean symmetry with three central charges ($\theta$, $\theta'$ and $c$).

4.1 Higher-order derivative model

The most general $D = 2 + 1$ Lagrangian which is bilinear in the time derivatives of the coordinates $x_i$ and is quasi-invariant under transformations (1) has the form:

$$L = -\frac{\theta}{2} \epsilon_{ij} \dot{x}_i \dot{x}_j + \frac{c}{2} \dot{x}_j^2 - \frac{\theta'}{8} \epsilon_{ij} \dot{x}_i \dot{x}_j.$$

(40)

The model consisting only of the first term in (40) (i.e. with $\theta \neq 0$, $\theta' = c = 0$) was considered in detail in [1].

The extended Galilean transformations (1) provide the following variation of the Lagrangian (40):

$$\delta L = \frac{d}{dt} \left( -\frac{\theta}{2} \epsilon_{ij} \dot{x}_j \right) \delta v_i + \frac{d}{dt} \left[ \theta \epsilon_{ij} (t \dot{x}_j - 2x_j) - 2c \dot{x}_i + \frac{\theta'}{4} \epsilon_{ij} \dot{x}_j \right] \delta b_i.$$

(41)

The invariance only modulo total derivative (quasi-invariance) implies the existence of central charges defined by the relations (5) and (7-9).

In the first order formalism the Lagrangian (41) provide the following variation of the Lagrangian (41):

$$L = p_i (\dot{x}_i - y_i) + q_i (\dot{y}_i - u_i) - \frac{\theta'}{8} \epsilon_{ij} u_i \dot{u}_j - \frac{\theta}{2} \epsilon_{ij} y_i \dot{y}_j + \frac{c}{2} u^2.$$

(42)

The Euler-Lagrange equations (EOM) which follow from the Lagrangian (42) are

$$y_i - \dot{x}_i = 0, \quad \dot{p}_i = 0, \quad u_i - \dot{y}_i = 0, \quad 4.$$  

$$p_i + \dot{q}_i + \theta \epsilon_{ij} \dot{y}_j = 0, \quad q_i + \frac{\theta'}{4} \epsilon_{ij} \dot{u}_j - cu_i = 0.$$
Using the Faddeev-Jackiw procedure [9]–[10] we obtain the following non-vanishing Poisson brackets:

\[
\{x_i, p_j\} = \delta_{ij}, \quad \{y_i, q_j\} = \delta_{ij}, \quad \{q_i, q_j\} = -\theta \epsilon_{ij}, \quad \{u_i, u_j\} = 4 \frac{\epsilon_{ij}}{\theta}.
\] (44)

To calculate from the Lagrangian (42) the Noether charges we consider the following acceleration extended Galilean transformations for a fixed but arbitrary value of the time parameter \(t\),

\[
\begin{align*}
\delta x_i &= -\delta a_i - \delta v_i t - \delta b_i t^2 + \delta \alpha \epsilon_{ij} x_j, \\
\delta y_i &= -\delta v_i - 2t \delta b_i + \delta \alpha \epsilon_{ij} x_j, \\
\delta u_i &= -2\delta b_i + \delta \alpha \epsilon_{ij} x_j, \\
\delta p_i &= 2\theta \epsilon_{ij} \delta b_j + \delta \alpha \epsilon_{ij} p_j, \\
\delta q_i &= -2c \delta b_i + \delta \alpha \epsilon_{ij} q_j.
\end{align*}
\] (45)

These transformations leave the EOM (43) invariant. The generators which provide the transformations (45) are obtained by the Noether theorem and are given by

\[
\begin{align*}
P_i &= p_i, \\
K_i &= p_i t + q_i + \theta \epsilon_{ij} y_j, \\
F_i &= -p_i t^2 + 2K_i t - 2\theta \epsilon_{ij} x_j - 2cy_i + \frac{\theta}{2} \epsilon_{ij} u_j, \\
J &= \epsilon_{ij} x_i p_j + \epsilon_{ij} y_i q_j - \frac{\theta}{2} y^2 - \frac{\theta}{8} u^2.
\end{align*}
\] (46)

4.2 Transmutation into a planar model with electromagnetic interaction

Note that if we consider the velocities \(y_i\) as the particle coordinates \(X_i\) then the model introduced in Section 4.1 would describe the interaction of noncommutative planar exotic particles with dynamical electric fields and a constant magnetic field (see [17], [18]).

Let us perform the substitution:

\[
\begin{align*}
p_i &\to -E_i, \quad x_i \to \pi_i, \quad y_i \to X_i, \quad q_i \to P_i, \quad u_i \to Y_i, \\
\theta &\to -B, \quad c \to m, \quad \text{and} \quad \theta' \to -4\theta m^2.
\end{align*}
\] (47)

Applying the substitution (47) into the Lagrangian (42) we obtain

\[
L = E_i X_i + \dot{E}_i \pi_i + P_i(\dot{X}_i - Y_i) + \frac{\theta m^2}{2} \epsilon_{ij} Y_i \dot{Y}_j + \frac{B}{2} \epsilon_{ij} X_i \dot{X}_j + \frac{m}{2} Y^2.
\] (48)
This is the Lagrangian of the higher-order derivative model introduced in \[8\] supplemented by the standard electromagnetic interaction with dynamical electric field \(E_i\) and a constant magnetic field \(B\) and a kinetic term \(\dot{E}_i\pi_i\) added for the additional phase space variables \(E_i\) and \(\pi_i\). In \[17\], \[18\] it was shown that such electromagnetic couplings of planar particle models lead to the electromagnetic enlargement of the Galilei symmetry with additional generators given by the electric field \(E_i\) and with three central charges \((m, \theta \text{ and } B)\). It can be checked that the Lagrangian \([18]\) is quasi-invariant under the electromagnetically enlarged \(D = (2+1)\) Galilei group with the symmetry algebra which is isomorphic to the acceleration-extended Galilei algebra with three central charges \((\dot{\theta}, \theta', c)\).

Concluding we see that our massless planar higher order Lagrangian model \((40)\) is classically equivalent to the Lagrangian for a massive exotic planar point particle interacting with dynamical electric and constant magnetic fields.

\section{Final Remarks}

\begin{itemize}
\item We would like to add that the extension of the equivalent coordinate frames to the ones with various constant accelerations may have possibly a physical application based on recent developments in astrophysics. If in accordance with recent measurements \(\text{see e.g.} [18]-[20]\) the universe expands with increasing velocity and approximately constant acceleration the coordinate frames extended by constant acceleration motions may be useful in its description.
\item Note that the Lagrangians invariant under acceleration-extended Galilean symmetries contain higher derivatives. Although such Lagrangians have been discussed in literature \(\text{see e.g.} [21], [22]\) their physical meaning is still unclear. We would like, however, to point out that following our discussion in Section 4.2 these models might have a meaning in the framework of classical mechanics for exotic particles in \(D = (2 + 1)\) dimensions.
\item Analogously to the model introduced by us in \[8\] \(\text{cp. also [23]}\) one can introduce for our model \([40]\) the phase space decomposition into “external” and “internal” dynamics. This decomposition can be understood as a generalization of the Galilean-invariant decomposition of phase space into the center of mass and relative motion in standard classical mechanics. We plan to treat this method in a general framework.
\end{itemize}

\footnote{In formulae (56-57) we assume that the electromagnetic coupling constant \(e = 1\).}
and discuss the solutions of the present model and the interpretation of “internal“ dynamics for the model with higher time derivatives in a forthcoming publication.

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