Maximal entanglement distillation with single-photon subtraction

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Entanglement is an indispensable quantum resource for quantum information technology. In continuous-variable quantum optics, photon subtraction can increase the entanglement between Gaussian states of light, but for mixed states the extent of this entanglement increase is poorly understood. In this work, we use an entanglement measure based on the Rényi-2 entropy to prove that single-photon subtraction increases bipartite entanglement by no more than log 2. This value coincides with the maximal amount of bipartite entanglement that can be achieved with one photon.

Quantum entanglement [1] is commonly seen as an important resource for quantum metrology [2], quantum communication [3] and quantum computation [4]. The increase of entanglement is an important task, since it can be used to strengthen the precision of quantum metrology [5], the fidelity of quantum information processing [6, 7] and the efficiency of quantum computing [8, 9]. However, in many platforms entanglement is fragile and rapidly vanishes due to environment-induced decoherence. In this regard, light is a particularly interesting platform due to its intrinsic resilience against its environment.

Continuous-variable (CV) quantum optics [10] allows for the deterministic generation of Gaussian entanglement between modes of light. In particular, it was used to produce large-scale entangled states [11–18]. However, because the entangled modes are Gaussian, they do not offer a genuine quantum advantage. To implement protocols that cannot be efficiently simulated by classical computers non-Gaussian states are required [19–21].

A common method to produce non-Gaussian modes of light is adding [22] or subtracting [23–26] a photon. Furthermore, this operation can also increase the entanglement between optical modes. For example, by adding/subtracting a photon to/from a two-mode squeezed vacuum state, entanglement of the resulting states is greater than that of the original squeezed state [27]. The entanglement of two-mode squeezed vacuum state was shown to be progressively enhanced with the number of photons added (subtracted) when acting on one mode only [28]. Similar results are obtained for single-photon addition and subtraction for pure multimode Gaussian systems [29–31].

Entanglement distillation is much harder to study for mixed states. Typical entanglement measures for multimode mixed states such as entanglement of formation require convex roof constructions [32–34] and the evaluation of the von Neumann entropy. Both are problematic for mixed non-Gaussian states. Here, we focus on the entanglement measure based on Rényi-2 entropy as a viable alternative, following [35, 36]. Because the Rényi-2 entropy is related to the purity of the state, it can be obtained directly from the Wigner function. As such, we will quantify the increase of entanglement due to photon addition or subtraction.

In this work, we will restrict ourselves to discussing bipartite entanglement. As shown in Fig. 1(a), for an arbitrary global pure state $\rho = |\psi\rangle\langle\psi|$, the entanglement between two subsystems $\mathcal{A}$ and $\mathcal{B}$ can be measured through the Rényi-2 entropy as

$$ E_{\rho}(|\psi\rangle) = -\log \mu_{\mathcal{A}}, $$

where $\mu_{\mathcal{A}} = \text{tr}_{\mathcal{B}}\rho$ is the reduced density operator for subsystem $\mathcal{A}$. The measure is based on the purity $\mu_{\mathcal{A}} = \text{tr}(\rho_{\mathcal{A}}^2)$, which can directly be calculated from the Wigner function by

$$ \mu_{\mathcal{A}} = (4\pi)^{m_{\mathcal{A}}} \int_{\mathbb{R}^{2m_{\mathcal{A}}}} |W_{\mathcal{A}}(\beta)|^2 d\beta, $$

where $m_{\mathcal{A}}$ is the total number of modes in subsystem $\mathcal{A}$. Importantly, this method works both for Gaussian and non-Gaussian states. Its extension to mixed states can be obtained by the following convex construction:

$$ E_{\rho}(\hat{\rho}) = \inf_{\{p(\lambda)\mid p(\lambda)\geq 0\}} \int d\lambda p(\lambda) E_{\rho}(|\psi_{\lambda}\rangle), $$

where we minimize over all decompositions $\hat{\rho} = \int d\lambda p(\lambda) |\psi_{\lambda}\rangle\langle\psi_{\lambda}|$. We use this measure to focus on the upper limit of the entanglement increase. We will show that for any Gaussian state –mixed or pure– the maximal increase of bipartite “Rényi-2 entanglement” through single-photon addition/subtraction is log 2.

In this work, we focus on entanglement increase via photon subtraction. This operation is theoretically described by the action of an annihilation operator on the (in our case Gaussian) state. We assume that this operation is implemented on subsystem $\mathcal{A}$, as shown in Fig. 1. To study the increase of entanglement, we start by analysing the purity of the photon...
subtracted state in subsystem $\mathcal{A}$:
\[
\hat{\rho}_\mathcal{A} = \frac{\hat{a}_g \hat{\rho}_\mathcal{A} \hat{a}_g^\dagger}{\text{tr}(\hat{a}_g^2 \hat{\rho}_\mathcal{A})}.
\]
(4)
The label $g$ indicates the mode in which the photon is subtracted. Our main tool to evaluate the purity of the state (4) are Gaussian transformations [37]. As shown in Fig. 1(b), we can write a thermal decomposition form for an arbitrary subsystem $\mathcal{A}$ as follows
\[
\hat{\rho}_\mathcal{A} = \hat{D}\hat{U} \otimes \hat{\rho}_\mathcal{A} \hat{U}^\dagger \hat{D}^\dagger,
\]
(5)
where $\hat{U}$ is a Bogoliubov transformation and $\hat{D} = \prod_{i=1}^{m_\mathcal{A}} \hat{D}(\alpha_i)$ is a displacement operator. Furthermore, single-mode thermal states $\hat{\rho}$ can be fully characterised by a covariance matrix $V_i = \text{diag}(n_i, n_i)$, where $n_i$ is the fraction of thermal noise compared to shot noise. Since the Gaussian unitary $\hat{D}\hat{U}$ does not affect the purity, the purity of subsystem $\hat{\rho}_\mathcal{A}$ can be directly expressed as $\mu_{\mathcal{A}} = 1/\prod_{i=1}^{m_\mathcal{A}} n_i$.

As shown by the green circles in Fig. 1(b) and Fig. 1(c), if we subtract a photon from the mode $g$ of subsystem $\mathcal{A}$, we can use Bogoliubov transform to convert the photon-subtraction operation at mode $g$ into a combined operator $\hat{b}$, which adds and subtracts a photon on each thermal mode
\[
\hat{\rho}_\mathcal{A} = \frac{\hat{D}\hat{U} \hat{b}_g \hat{\rho}_\mathcal{A} \hat{b}_g^\dagger \hat{U}^\dagger \hat{D}^\dagger}{\text{tr}(\hat{b}_g^2 \hat{\rho}_\mathcal{A})}, \quad \text{with} \quad \hat{b} = \hat{U}^\dagger \hat{D}^\dagger \hat{a}_g \hat{D}\hat{U}.
\]
(6)
When $m_\mathcal{A}$ is the number of modes in subsystem $\mathcal{A}$, we define $\hat{a}_g^\dagger = (\hat{a}_1^\dagger, \cdots, \hat{a}_{m_\mathcal{A}}^\dagger)^T$ and $\hat{a} = (\hat{a}_1, \cdots, \hat{a}_{m_\mathcal{A}})^T$. From the properties of Gaussian unitary transformations, we find that
\[
\hat{U}^\dagger \hat{D}^\dagger \hat{a} \hat{D}\hat{U} = K \hat{a}^\dagger + L \hat{a} + \hat{\alpha},
\]
(7)
with $L^T K - K^T L = \mathbf{1}$ and $K^T L = -L^T K$. We can thus explicitly write that $\hat{b} = \hat{k}^\dagger - \hat{a}^\dagger + \hat{l}^\dagger \hat{a} + \hat{\alpha}_g$, where $\hat{k} = (k_1, \cdots, k_{m_\mathcal{A}})$ and $\hat{l} = (l_1, \cdots, l_{m_\mathcal{A}})$ are $m_\mathcal{A}$-dimensional complex vectors, and $\alpha_g$ is a complex number.

Based on both the Williamson decomposition (5) and Bogoliubov transformation (6), we can derive (see Supplementary Material) a general expression of relative purity
\[
\frac{\mu_{\mathcal{A}}}{\mu_{\mathcal{A}}} = 1 + \left[ \frac{1}{2} \sum_{i=1}^{m_\mathcal{A}} \frac{\bar{N}_i}{n_i} \right]^2 + \frac{1}{2} |\alpha_g|^4 + \sum_{i=1}^{m_\mathcal{A}} k_i l_i \left( n_i^2 - 1 \right) \left( \frac{n_i^2 - 1}{2n_i} \right) \left( \frac{n_i^2 - 1}{2n_i} \right) \sum_{i=1}^{m_\mathcal{A}} k_i l_i \left( n_i^2 - 1 \right) \left( \frac{n_i^2 - 1}{2n_i} \right) \sum_{i=1}^{m_\mathcal{A}} k_i l_i \left( n_i^2 - 1 \right) \left( \frac{n_i^2 - 1}{2n_i} \right)
\]
\[
+ |\alpha_g|^2 \sum_{i=1}^{m_\mathcal{A}} N_i \left( \sum_{i=1}^{m_\mathcal{A}} N_i + |\alpha_g|^2 \right)^2,
\]
(8)
where $N_i = |k_i|^2 |l_i|^2 + |k_i|^2 |l_i|^2 - |k_i|^2 |l_i|^2$. By minimising this expression, we find that
\[
\frac{\mu_{\mathcal{A}}}{\mu_{\mathcal{A}}} \geq 1/2.
\]
(9)

We thus show that photon subtraction decreases the purity of an arbitrary state by at most a factor two. In the examples at the end of the manuscript, we show that this lower bound is tight in the sense that there are states for which $\mu_{\mathcal{A}} = \mu_{\mathcal{A}}/2$. Furthermore, when there is no entanglement in the initial Gaussian state, i.e. when $\mu_{\mathcal{A}} = 1$, we find that $\mu_{\mathcal{A}} = 1$.

When the state for the global system is pure, we can immediately combine the definition (1) with (9) to obtain that the entanglement increase is governed by
\[
\Delta E_R = \log \mu_{\mathcal{A}} - \log \mu_{\mathcal{A}} \leq \log 2.
\]
(10)

For mixed states, on the other hand, we use the definition of (3), which implies that we must consider all possible decompositions of the state. For each state in the decomposition we can use (2) to calculate $E_R(\hat{\rho}(\lambda))$ based on the Wigner function. This construction naturally implies that $E_R(\hat{\rho}(\lambda)) \leq \int d\lambda p(\lambda) E_R(\hat{\rho}(\lambda))$, for any possible decomposition of $\hat{\rho}$ in pure states. For Gaussian states this idea was explicitly used to define Gaussian entanglement of formation [38] as an upper bound to the entanglement of formation. At least for some classes of Gaussian states it was later proven that Gaussian entanglement of formation is the same as the general entanglement of formation [39, 40].

The Wigner function [37] of a Gaussian state $\hat{\rho}^G$ can be written as
\[
W^G(\beta) = \frac{\exp\left[-\frac{1}{2}(\beta - \bar{\alpha}_0)^T V^{-1} (\beta - \bar{\alpha}_0) \right]}{(2\pi)^m |\text{det}(V)|},
\]
(11)
where $\beta = (x_1, \cdots, x_m, p_1, \cdots, p_m) \in \mathbb{R}^{2m}$ is a set of ampli-
Rényi-2 entanglement of the mixed state $\hat{\rho}^G$ can be decomposed [38] as

$$\hat{\rho}^G = \int_{\mathbb{R}^{2n}} d^2\bar{\alpha} \mathcal{D}(\bar{\alpha}) \hat{\rho}_p^G \mathcal{D}^\dagger(\bar{\alpha}) p^G(\bar{\alpha}),$$  \hspace{1cm} (12)

where $\hat{\rho}_p$ is a pure squeezed vacuum state with covariance matrix $V_p$. We also introduce the probability distribution

$$p^G(\bar{\alpha}) = \frac{e^{-\frac{1}{2}(|\bar{\alpha}|^2 - \bar{\alpha}_p^2)^T V_p^{-1} (|\bar{\alpha}|^2 - \bar{\alpha}_p^2)}}{(2\pi)^n \sqrt{\det(V_p)}},$$  \hspace{1cm} (13)

with $\bar{\alpha}_p$ the mean field of $\hat{\rho}_p^G$. This gives us an explicit way to decompose any Gaussian state as a mixture of squeezed vacuum states. This decomposition of the Gaussian state can be used to decompose the associated photon-subtracted state as

$$\hat{\rho}^- = \int_{\mathbb{R}^{2n}} d^2\bar{\alpha} \hat{\rho}^-_{\bar{\alpha}} p^-_{\bar{\alpha}},$$  \hspace{1cm} (14)

where $\hat{\rho}^-_{\bar{\alpha}}$ is the state that is obtained by subtracting a single photon from $\mathcal{D}(\bar{\alpha}) \hat{\rho}_p \mathcal{D}^\dagger(\bar{\alpha})$, and $p^-_{\bar{\alpha}}(\alpha)$ a probability distribution for which the explicit expression is found in [31].

The decomposition (14) can be used to derive an upper bound for the amount of distilled entanglement, optimised over all possible decompositions of $V$. First, we find that

$$\mathcal{E}_R(\hat{\rho}^-) \leq \inf_{\mathcal{N} \in \mathcal{V}} \int_{\mathbb{R}^{2n}} d^2\bar{\alpha} p^-_{\bar{\alpha}}(\bar{\alpha}) \mathcal{E}_R(\hat{\rho}^-_{\bar{\alpha}}).$$  \hspace{1cm} (15)

Because $\hat{\rho}^-_{\bar{\alpha}}$ is a single-photon subtracted pure state, we can use (10) to obtain that $\mathcal{E}_R(\hat{\rho}^-_{\bar{\alpha}}) \leq \log 2 + \mathcal{E}_R(\hat{\rho}_p \mathcal{D}(\bar{\alpha})^\dagger)$. Furthermore, because entanglement is unchanged under local unitary transformations, we find that $\mathcal{E}_R(\hat{\rho}_p \mathcal{D}(\bar{\alpha})^\dagger) = \mathcal{E}_R(\hat{\rho}_p)$. Inserting all these elements in (15), we find that

$$\mathcal{E}_R(\hat{\rho}^-) \leq \log 2 + \inf_{\mathcal{N} \in \mathcal{V}} \mathcal{E}_R(\hat{\rho}_p).$$  \hspace{1cm} (16)

The second term on the right side of the inequality is defined to be the Gaussian Rényi-2 entanglement of the mixed state $\hat{\rho}^G$ [36]. This entanglement measure is defined by minimising over all possible Gaussian decompositions, but it is a priori not clear that there cannot be any decomposition in non-Gaussian states that yields a lower entanglement. In other words, we know that $\inf_{\mathcal{N} \in \mathcal{V}} \mathcal{E}_R(\hat{\rho}_p) \leq \mathcal{E}_R(\hat{\rho}^G)$, but it is not clear whether or not this is actually an equality.

Hence, we have shown that photon subtraction on any arbitrary Gaussian state $\hat{\rho}^G$ can increase the Rényi-2 entanglement by at most $\log 2$ as compared to its Gaussian Rényi-2 entanglement. A completely analogous calculation for photon-added state results in the same bound even though photon subtraction and photon addition tend to distill different amounts of entanglement depending on the state.

To get a better understanding of the quality of the bound and its physical interpretation, we will consider a range of examples. Because there is no exact method to calculate the entanglement for non-Gaussian mixed states, we restrict ourselves to pure states where we can use (10) to calculate the entanglement increase. We consider the following two typical entangled pure states as examples:

1. We subtract a photon from a linear entangled network, which is composed of several two-mode squeezer gates, as shown in Fig. 2(a). All the modes are initially in vacuum states $\hat{\rho}_i$, then the output state can be expressed as $\hat{\rho} = \hat{D} \hat{U} \hat{\rho}_1 \hat{U}^\dagger \hat{D}^\dagger$, where $\hat{U} = \hat{S}_{m-1} \cdots \hat{S}_1$ with $\hat{S}_i = \exp[r(\hat{a}_i \hat{a}_i - \hat{a}_i^\dagger \hat{a}_i^\dagger)/2]$. To test the effect of a mean field, we add the operator $\hat{D}$ that adds a displacement $(\alpha_p)$ to the mode-$g$. Thus the covariance matrix of the state $\hat{\rho}$ is $V_{\text{network}} = C^{-1} C$, where $C$ is a symplectic matrix that implements the multimode squeezing operation on the quadratures $\hat{U}^\dagger \hat{\rho} \hat{U} = C \beta$.

2. We subtract a photon from a graph state, as shown in Fig. 2(b), which is the backbone of measurement-based CV quantum computation. We start out from a set of independent squeezed modes, with a joint covariance matrix given by $V_0 = \text{diag}(s, \cdots, s, s^{-1}, \cdots, s^{-1})$. Every edge of the graph corresponds to a $C_Z$ gate, which turn those non-correlated modes into a graph state. On the level of covariance matrices, the action of the set of $C_Z$
gates is represented by a symplectic transformation $C$, applied to $V_0$ [29]: $V_{\text{graph}} = C' V_0 C$, with $C = \left( \begin{array}{cc} \mathbb{1} & \vec{A} \\ 0 & \mathbb{1} \end{array} \right)$, where $\vec{A}$ is $m \times m$ adjacency matrix.

We can use methods purely based on phase space representations [30, 31] to describe the Wigner function of a reduced photon subtracted state for any subsystem $\mathcal{A}$, which leads to the general expression

$$W_{\mathcal{A}}(\vec{\beta}_{\mathcal{A}}) = \frac{\|X V^{-1}_A (\vec{\beta}_{\mathcal{A}} - \vec{a}_{\mathcal{A}}) - \vec{a}_{\mathcal{A}}\|^2 + \text{tr}(V_g - X V^{-1}_A X') - 2}{\|\vec{a}_{\mathcal{A}}\|^2 + \text{tr}(V_g) - 2} \times W_{\mathcal{A}}^G(\vec{\beta}_{\mathcal{A}}),$$

(17)

where $\vec{\beta}_{\mathcal{A}} \in \mathbb{R}^{2m_a}$ and $X = G'(V - \mathbb{1}) A$. Here, $G$ is a $2m \times 2m$ matrix where the columns are the basis vectors $\vec{g}^{(x)}$ and $\vec{g}^{(p)}$ associated with the two phase space axes of mode $g$:

$$G = \left( \begin{array}{c|c} \vec{g}^{(x)} & \vec{g}^{(p)} \end{array} \right).$$

(18)

Analogously, $A$ is a $2m \times 2m$ matrix where the columns are the symplectic basis vectors $\vec{a}_1, \ldots, \vec{a}_{m_a}, \vec{d}_1, \ldots, \vec{d}_{m_a}$ that generate the phase space of subsystem $\mathcal{A}$:

$$A = \left( \begin{array}{cccc} \vec{a}_1 & \ldots & \vec{a}_{m_a} & \vec{d}_1 & \ldots & \vec{d}_{m_a} \end{array} \right).$$

(19)

In (17), we then obtain the matrices $V_A = A' VA$ and $V_g = G' VG$, and the displacement vectors $\vec{a}_{\mathcal{A}} = A' \vec{a}$ and $\vec{a}_g = G' \vec{a}_g$. We can now use standard techniques of Gaussian integrals to evaluate the purity $(2)$ from $W_{\mathcal{A}}(\vec{\beta}_{\mathcal{A}})$ for an arbitrary bipartition $\mathcal{A}$. This in turn allows to directly evaluate the entanglement increase $(10)$.

In Fig. 2, each circle represents a mode in a multimode quantum state, and the color code inside the circle indicate the increase in entanglement between the mode (subsystem $\mathcal{A}$) and the rest system (subsystem $\mathcal{B}$). When a photon is subtracted from the mode $g$ (marked by the green arrow) in the pure multimode systems, non-Gaussian features will spread from the mode $g$ to other modes, whose entanglement can also increase. In Fig. 3(a) and Fig. 3(b), we respectively show the increase of entanglement $(\Delta \mathcal{E}_R)$ related to mode $g$ and mode $g'$ (representing the neighboring mode of $g$ and marked by a green circle in Fig. 2). As shown by the blue and pink curves, $\Delta \mathcal{E}_R$ increases as the degree of squeezing increases until it approaches a maximum. The maximum values are usually different for different initial Gaussian states, but they are all less than log 2. In Fig. 3(c), each dots represents a different bi-partition for the example of Fig. 2(a), which we group depending on the number of modes in subsystem $\mathcal{A}$ (similar data for the example of Fig. 2(b) are shown in Supplementary Material). The entanglement increase varies depending on the modes that are comprised in each subsystem, but it is never greater than log 2.

Finally, we aim to provide an intuition for the value log 2 of entanglement increase. To this goal, we restrict to a two-mode system prepared in a non-displaced pure state $|\psi\rangle = \hat{U}(|0\rangle \otimes |0\rangle)$, where $\hat{U}$ is a Bogoliubov transformation. When we subtract a photon in one of the two modes, we can use (6) to write the photon-subtracted state $|\psi^-\rangle = \hat{U}b(|0\rangle \otimes |0\rangle)|N$ (with $N$ a normalisation factor). We recall that $\hat{b} = \hat{b}^\dagger$ and $\hat{a} = \hat{a}^\dagger$, which implies that

$$|\psi^-\rangle = \frac{1}{N} \hat{U} (k_1 |1\rangle \otimes |0\rangle + k_2 |0\rangle \otimes |1\rangle).$$

(20)

In particular in the limit where the squeezing induced by $\hat{U}$ is small, we find that $|\psi\rangle$ resembles the vacuum state with negligible entanglement. On the other hand, $|\psi^-\rangle$ is close to a single photon that passed through a beam splitter. The maximal entanglement in this low squeezing limit is achieved for the Bell state $|\psi^-\rangle \approx (|1\rangle \otimes |0\rangle \pm |0\rangle \otimes |1\rangle)/\sqrt{2}$, which has a Rényi-2 entanglement $\mathcal{E}_R(|\psi\rangle) = \log 2$. This corresponds exactly to the maximal entanglement increase and thus we can interpret our findings as a generalisation of this phenomenon to mixed states of arbitrary many modes with arbitrary amounts of squeezing.

In conclusion, we strictly prove that single-photon subtraction can reduce the purity of any Gaussian state by at most a
factor 1/2. This means that for any Gaussian state the amount of Rényi-2 entanglement that can be distilled by single-photon subtraction is at most log 2. This result holds for both pure and mixed states. A fully equivalent treatment of single-photon addition yields the same result (see Supplementary Material).

We evaluate the actual Rényi-2 entanglement of two examples based on the Wigner function and show that the bound is tight. The results of this work mark the limits of the non-Gaussian features induced by photon subtraction (or addition), which are in many ways complementary to other studies on induced Wigner negativity [41, 42].

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SUPPLEMENTARY MATERIAL

A. DECOMPOSITION OF SINGLE-PHOTON SUBTRACTION OPERATION

In CV quantum optics, for an arbitrary $m$-mode Gaussian state $\hat{\rho}_0$, we can write a thermal decomposition as

$$\hat{\rho}_0 = \hat{D}\hat{U}\hat{\rho}_M\hat{U}^\dagger\hat{D}^\dagger,$$  

(21)

where $\hat{U}$ is an arbitrary canonical unitary and $\hat{D} = \prod_{i=1}^{m} \hat{D}(\alpha_i)$ is displacement operator, which displace the vacuum state to generate coherent states $|\alpha_i\rangle = \hat{D}(\alpha_i)|0\rangle$. Furthermore, $\hat{\rho}_M = \bigotimes_{i=1}^{m} \hat{\rho}_i$, where single-mode thermal states $\hat{\rho}_i = (0, V_i)$ can be fully characterised by a covariance matrix $V_i = \text{diag}(n_i, n_i)$, where $n_i$ is the fraction of thermal noise compared to shot noise. Then the purity of $\hat{\rho}_0(\vec{x}, V)$ can be expressed as $\mu_0 = \prod_{i=1}^{m} \text{tr}(\hat{\rho}_i^2) = \prod_{n_i} \frac{1}{n_i}$.

When subtracting a photon to the mode-$g$, it corresponds to a unitary Bogoliubov transformation

$$\hat{U}^\dagger\hat{D}^\dagger \alpha_g \hat{D} \hat{U} = \hat{b} + \alpha^*_g, \text{ with } \hat{b} = \vec{k} \cdot \vec{a}^\dagger + \vec{l} \cdot \vec{a},$$

(22)

where $\vec{a}^\dagger = (\hat{a}_1^\dagger, \ldots, \hat{a}_m^\dagger)\vec{T}$ and $\vec{a} = (\hat{a}_1, \ldots, \hat{a}_m)\vec{T}$. A set of $1 \times m$ complex matrices $\vec{k} = (k_1, \ldots, k_m)$ and $\vec{l} = (l_1, \ldots, l_m)$ corresponds an canonical unitary $\hat{U}$. Thus, we convert the single-photon subtraction operation on the mode-$g$ into the addition and subtraction operation on each thermal mode. Thus the single-photon subtracted state can then be written as

$$\frac{\alpha_g\hat{\rho}_0\alpha_g^\dagger}{\text{tr}(\alpha_g\hat{\rho}_0\alpha_g^\dagger)} = \frac{\hat{D}\hat{U}(\hat{b} + \alpha_g\hat{\rho}_M(\hat{b}^\dagger + \alpha_g^*\hat{\rho}_M))}{N},$$

(23)

where the normalization factor $N = \text{tr}(\hat{b}\hat{\rho}_M\hat{b}^\dagger) + |\alpha_g|^2$. Due to $\hat{D}\hat{U}$ will not change the purity, so the purity of an arbitrary single-photon subtracted state can be expressed as

$$\mu N^2 = \text{tr}((\hat{b}\hat{\rho}_M\hat{b}^\dagger + |\alpha_g|^2\hat{\rho}_M + \alpha_g\hat{\rho}_M\hat{b}^\dagger + \alpha_g^*\hat{\rho}_M\hat{b}^\dagger)^2)$$

$$= \text{tr}((\hat{b}\hat{\rho}_M\hat{b}^\dagger + |\alpha_g|^2\hat{\rho}_M)^2 + (\alpha_g\hat{\rho}_M\hat{b}^\dagger + \alpha_g^*\hat{\rho}_M\hat{b}^\dagger)^2)$$

$$= \text{tr}(\hat{b}\hat{\rho}_M\hat{b}^\dagger\hat{b}\hat{\rho}_M\hat{b}^\dagger) + |\alpha_g|^2\text{tr}(\hat{\rho}_M\hat{b}\hat{\rho}_M\hat{b}^\dagger) + \alpha_g^2\text{tr}(\hat{\rho}_M\hat{b}\hat{\rho}_M\hat{b}) + |\alpha_g|^4\text{tr}(\hat{\rho}_M\hat{b}\hat{\rho}_M\hat{b}^\dagger\hat{b}\hat{\rho}_M\hat{b}^\dagger) + \alpha_g^2|\alpha_g|^2\text{tr}(\hat{\rho}_M\hat{b}\hat{\rho}_M\hat{b}^\dagger).$$

(24)

where we have used that $\text{tr}(\hat{b}\hat{\rho}_M\hat{b}^\dagger\hat{b}\hat{\rho}_M) = \text{tr}(\hat{b}\hat{\rho}_M\hat{b}^\dagger\hat{b}\hat{\rho}_M) = \text{tr}(\hat{\rho}_M\hat{b}\hat{\rho}_M\hat{b}) = 0$, since $\hat{\rho}_i$ is a diagonal matrix, and $\hat{a}_i^\dagger (\hat{a}_i)$ is off-diagonal matrices. Similarly, for single-photon addition, the calculation is equivalent the same expression just by swapping the values of $\vec{k}$ and $\vec{l}$ and the values of $\alpha_g^*$ with $\alpha_g$.

For the same Gaussian state, different form Eq.(21), we can also take the following decomposition

$$\hat{\rho}_0 = \hat{U}\hat{D}(\vec{x})\hat{\rho}_M\hat{D}(\vec{x})^\dagger\hat{U}^\dagger.$$  

(25)

In this case, we just need to to replace $\alpha_g^*$ with $\sum_{i=1}^{m} k_i a^* + l_i a$ in Eq.(24), which is still applicable, regardless of the decomposition form.

B. GENERALIZED EXPRESSION OF RELATIVE PURITY

1. When $\alpha_g = 0$

When $\alpha_g = 0$, only the first term of (24) is retained.

Let us take $m = 1$, so that (24) becomes

$$\mu_1 N^2 = \text{tr}(\hat{b}\hat{\rho}_M\hat{b}^\dagger\hat{b}\hat{\rho}_M\hat{b})$$

$$= |k_1|^4\text{tr}(\hat{a}_1^\dagger\hat{a}_1\hat{a}_1^\dagger\hat{a}_1) + |l_1|^4\text{tr}(\hat{a}_1\hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_1) + |k_1|^2|l_1|^2(\text{tr}(\hat{a}_1^\dagger\hat{a}_1\hat{a}_1^\dagger\hat{a}_1^\dagger) + \text{tr}(\hat{a}_1\hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_1)) + |k_1|^2|l_1|^2\text{tr}(\hat{a}_1^\dagger\hat{a}_1\hat{a}_1\hat{a}_1^\dagger) + |l_1|^2k_1^2\text{tr}(\hat{a}_1\hat{a}_1\hat{a}_1^\dagger\hat{a}_1)$$

$$+ |k_1|^2l_1^2\text{tr}(\hat{a}_1\hat{a}_1^\dagger\hat{a}_1\hat{a}_1) + |l_1|^2k_1^2\text{tr}(\hat{a}_1^\dagger\hat{a}_1\hat{a}_1^\dagger\hat{a}_1) + |l_1|^2k_1^2\text{tr}(\hat{a}_1\hat{a}_1\hat{a}_1^\dagger\hat{a}_1^\dagger) + |l_1|^2k_1^2\text{tr}(\hat{a}_1\hat{a}_1^\dagger\hat{a}_1\hat{a}_1^\dagger) + \text{tr}(\hat{a}_1\hat{a}_1\hat{a}_1\hat{a}_1^\dagger) + \text{tr}(\hat{a}_1\hat{a}_1\hat{a}_1^\dagger\hat{a}_1) + \text{tr}(\hat{a}_1\hat{a}_1\hat{a}_1^\dagger\hat{a}_1^\dagger) + \text{tr}(\hat{a}_1\hat{a}_1^\dagger\hat{a}_1\hat{a}_1^\dagger)).$$

(26)
Remove the zero traces and replace the subscript 1 with \( i \), we get any single-mode purity

\[
\mu_i N_i^2 = |k_i|^4 \text{tr}(\hat{A}^\dagger_i \hat{\rho_i} \hat{A}^\dagger_i \hat{\rho_i}) + |l_i|^4 \text{tr}(\hat{A}^\dagger \hat{\rho} \hat{A}^\dagger \hat{\rho}) + 2 |k_i|^2 |l_i|^2 \text{tr}(\hat{A} \hat{\rho}\hat{A}^\dagger \hat{\rho}) + 2 |k_i|^2 |l_i|^2 \text{tr}(\hat{A}^\dagger \hat{\rho}_{\hat{A}} \hat{A} \hat{\rho}_{\hat{A}}). 
\]

(28)

When we generalise this to a state with an arbitrary number of modes \( m \), we find that the purity can be expressed as

\[
\mu_m \left( \sum_{i=1}^{m} N_i \right)^2 = \sum_{i=1}^{m} \mu_i N_i^2 \cdot n_i \cdot \left( \frac{1}{n_i} + 4 \cdot \left( \sum_{i=1}^{m} \left[ k_i \text{tr}(\hat{A} \hat{\rho} \hat{A}^\dagger \hat{\rho}) \cdot n_i \right]^2 - \sum_{i=1}^{m} \left[ k_i \text{tr}(\hat{A} \hat{\rho} \hat{A}^\dagger \hat{\rho}) \cdot n_i \right]^2 \right) \cdot \frac{1}{n_i} + \left( \sum_{i=1}^{m} \left( k_i^2 \text{tr}(\hat{A} \hat{\rho} \hat{A}^\dagger \hat{\rho}) \cdot n_i \right)^2 \right) \cdot \frac{1}{n_i} + \left( \sum_{i=1}^{m} \left( k_i^2 + |l_i|^2 \text{tr}(\hat{A} \hat{\rho} \hat{A}^\dagger \hat{\rho}) \cdot n_i \right)^2 \right) \cdot \frac{1}{n_i} \right), 
\]

(29)

where \( N_i = |k_i|^2 \text{tr}(\hat{A}^\dagger \hat{\rho} \hat{A}) + |l_i|^2 \text{tr}(\hat{A} \hat{\rho} \hat{A}^\dagger) \). Therefore, the traces in (29) are related to the single-mode photon-added and -subtracted thermal states, which are the key to deriving the expression of final purity. Fortunately, Wigner function provides us with an easy way to calculate these traces. For a single-mode Gaussian thermal state \( \hat{\rho} \), the Wigner function is

\[
w_i(q_i, p_i) = \frac{1}{2\pi n_i} \exp \left[ \frac{q_i^2 + p_i^2}{2n_i} \right],
\]

(30)

where \( q_i \) and \( p_i \) are amplitude and phase quadrature. Then the Wigner functions due to photon addition or subtraction are

\[
w_{(i,+)}(q_i, p_i) = \left( \frac{n_i + 1}{2n_i} \right) w_i(q_i, p_i), \quad \text{and} \quad w_{(i,-)}(q_i, p_i) = \left( \frac{n_i - 1}{2n_i} \right) w_i(q_i, p_i).
\]

(31)

Thus, we can calculate the traces by the above Wigner functions:

\[
\text{tr}(\hat{A} \hat{\rho}^\dagger) = \int \frac{q_i^2 + p_i^2}{4} w_i dq_i dp_i - 1 = \frac{n_i - 1}{2}, \quad \text{and} \quad \text{tr}(\hat{A}^\dagger \hat{\rho}^\dagger) = \int \frac{q_i^2 + p_i^2}{4} w_i dq_i dp_i - 1 + 1 = \frac{n_i + 1}{2},
\]

(32)

which are the quantities related to the average-photon in the thermal state. Similarly, we can get that

\[
\text{tr}(\hat{A} \hat{\rho}^\dagger \hat{A}_i \hat{\rho} \hat{A}^\dagger) = \frac{\text{tr}(\hat{A} \hat{\rho}^\dagger \hat{A}_i \hat{\rho} \hat{A}^\dagger)}{\text{tr}(\hat{A} \hat{\rho} \hat{A}^\dagger)} = \frac{1 + n_i}{2n_i}, \quad \text{and} \quad \text{tr}(\hat{A} \hat{\rho} \hat{A}^\dagger \hat{A}_i \hat{\rho} \hat{A}^\dagger) = \frac{1 + n_i}{2n_i} \left( n_i - 1 \right)^2,
\]

(33)

which are quantities related to the purity of the photon-added and -subtracted states.

\[
\text{tr}(\hat{A} \hat{\rho}^\dagger \hat{A} \hat{\rho} \hat{A}^\dagger \hat{\rho} \hat{A}^\dagger) = \text{tr}(\hat{A} \hat{\rho} \hat{A}^\dagger \hat{A} \hat{\rho}^\dagger) = \frac{(n_i - 1)^2}{8n_i^3},
\]

(34)

which can be obtained by photon-added and -subtracted Wigner functions through the following integration

\[
\text{tr}(\hat{A}^\dagger \hat{\rho} \hat{A} \hat{\rho} \hat{A}^\dagger) = (4\pi) \int w_{(i,+)} \cdot w_{(i,-)} dq_i dp_i \cdot \left( \int \frac{q_i^2 + p_i^2}{4} w_i dq_i dp_i - 1 \right) \cdot \left( \int \frac{q_i^2 + p_i^2}{4} w_i dq_i dp_i - 1 \right) = \frac{(n_i - 1)^2}{8n_i^3}.
\]

(35)

With the help of Wigner function of single-photon added or subtracted and Gaussian thermal state Wigner function, we can get

\[
\text{tr}(\hat{A} \hat{\rho}^\dagger \hat{A}_i \hat{\rho} \hat{A}^\dagger) = (4\pi) \int w_i \cdot w_{(i,-)} dq_i dp_i \cdot \left( \int \frac{q_i^2 + p_i^2}{4} w_i dq_i dp_i - 1 \right) = \frac{n_i + 1}{2n_i} \left( n_i - 1 \right).
\]

Due to the noncommutative nature of creation and annihilation operators, we have

\[
\text{tr}(\hat{A} \hat{\rho} \hat{A}^\dagger \hat{A} \hat{\rho} \hat{A}^\dagger) = \text{tr}(\hat{A}^\dagger \hat{\rho} \hat{A} \hat{\rho}^\dagger) + \text{tr}(\hat{A}^\dagger \hat{\rho} \hat{A} \hat{\rho}^\dagger) - \text{tr}(\hat{A} \hat{\rho} \hat{A}^\dagger \hat{\rho} \hat{A}^\dagger) - \text{tr}(\hat{A} \hat{\rho} \hat{A}^\dagger \hat{\rho} \hat{A}^\dagger),
\]

then we can deduce the following traces

\[
\text{tr}(\hat{A}^\dagger \hat{\rho}^\dagger) = \frac{(n_i - 1)^2}{4n_i^3}, \quad \text{tr}(\hat{A} \hat{\rho} \hat{A}^\dagger) = \frac{(n_i + 1)^2}{4n_i^3}, \quad \text{and} \quad \text{tr}(\hat{A} \hat{\rho} \hat{A}^\dagger) = \frac{(n_i - 1)^2}{8n_i^3}.
\]

(36)

By bringing the results of these traces into (29), we get

\[
\frac{\mu_m}{\mu_0} = \frac{1}{2} \sum_{i=1}^{m} \frac{N_i}{n_i} + \frac{1}{2} \sum_{i=1}^{m} \frac{N_i}{n_i} \left( \sum_{i=1}^{m} k_i |k_i|^2 \frac{n_i - 1}{2n_i} - \frac{1}{(\sum_{i=1}^{m} N_i)^2} \right) \left( \sum_{i=1}^{m} k_i |k_i|^2 \frac{n_i - 1}{2n_i} - \frac{1}{(\sum_{i=1}^{m} N_i)^2} \right)^2.
\]

where \( N_i = |k_i|^2 + \frac{n_i - 1}{2} \) and \( \tilde{N}_i = |k_i|^2 - \frac{n_i - 1}{2} \). Hence, we can immediately know the relative purity

\[
\frac{\mu_m}{\mu_0} = \frac{1}{2} + \frac{1}{2} \sum_{i=1}^{m} \frac{N_i}{n_i} + \frac{1}{2} \sum_{i=1}^{m} k_i |k_i|^2 \frac{n_i - 1}{2n_i} \left( \sum_{i=1}^{m} N_i \right)^2 \geq 1/2.
\]
2. When $\alpha_g \neq 0$

According to similar approach above, we can derive the following traces

$$|\alpha_g|^2 |\text{tr}(\hat{p}_M^2)| = |\alpha_g|^2 |\mu_0|, \quad (37)$$

$$\alpha_g^2 |\text{tr}(\hat{\rho}_M \hat{b}^\dagger \hat{\rho}_M \hat{b})| = 2\alpha_g^2 \left( \sum_{i=1}^m k_i^2 |\text{tr}(\hat{\alpha}_i \hat{\rho}_M \hat{\alpha}_i^\dagger \hat{\rho}_M) n_i| \right) n_0 = \alpha_g^2 \left( \sum_{i=1}^m \frac{k_i^2 (n_i^2 - 1)}{2n_i} \right) |\mu_0|, \quad (38)$$

$$\alpha_g^2 |\text{tr}(\hat{b} \hat{\rho}_M \hat{b} \hat{\rho}_M) = 2\alpha_g^2 \left( \sum_{i=1}^m k_i n_i (n_i^2 - 1) \right) |\mu_0|, \quad (39)$$

and

$$|\alpha_g|^2 |\text{tr}(\hat{b} \hat{\rho}_M \hat{b} \hat{\rho}_M \hat{b})^\dagger| + |\text{tr}(\hat{\rho}_M \hat{b} \hat{b}^\dagger \hat{\rho}_M) + |\text{tr}(\hat{\rho}_M \hat{b} \hat{b}^\dagger \hat{\rho}_M) + |\text{tr}(\hat{\rho}_M \hat{b} \hat{b}^\dagger \hat{\rho}_M)\right| = 2|\alpha_g|^2 \left( \sum_{i=1}^m |k_i|^2 + |l_i|^2 |\text{tr}(\hat{\alpha}_i \hat{\rho}_M \hat{\alpha}_i^\dagger \hat{\rho}_M) n_i| + \sum_{i=1}^m |l_i|^2 |\text{tr}(\hat{\rho}_M \hat{\alpha}_i \hat{\alpha}_i^\dagger \hat{\rho}_M) n_i| \right) n_0$$

$$= 2|\alpha_g|^2 \sum_{i=1}^m N_i n_0. \quad (40)$$

Thus the relative purity is

$$\frac{\mu_\alpha}{\mu_0} = \frac{1}{2} \left( \sum_{i=1}^m N_i \right)^2 + \frac{1}{2} \left( \sum_{i=1}^m \frac{N_i}{n_i} \right)^2 + 2|\alpha_g|^2 \sum_{i=1}^m N_i + \left| \sum_{i=1}^m k_i \frac{n_i^2 - 1}{2n_i} \right|^2 + |\alpha_g|^4 +$$

$$\alpha_g^2 \left( \sum_{i=1}^m \frac{k_i^2 (n_i^2 - 1)}{2n_i} \right) + \alpha_g^2 \left( \sum_{i=1}^m \frac{k_i n_i (n_i^2 - 1)}{2n_i} \right)/\left( \sum_{i=1}^m N_i + |\alpha_g|^2 \right)^2 \quad (41)$$

Since $(\alpha_g + \alpha_g^2) \geq 0$ and $(\alpha_g - \alpha_g^2) \leq 0$, thus we have $2|\alpha_g| \leq \alpha_g^2 + \alpha_g^2 \leq 2|\alpha_g|^2$. Then

$$B^2 \left( \sum_{i=1}^m \frac{k_i^2 (n_i^2 - 1)}{2n_i} \right) + B^2 \left( \sum_{i=1}^m \frac{k_i n_i (n_i^2 - 1)}{2n_i} \right) \geq -2|B|^2 \sum_{i=1}^m \frac{k_i^2 (n_i^2 - 1)}{2n_i} \sum_{i=1}^m \frac{k_i n_i (n_i^2 - 1)}{2n_i} \quad (42)$$

where $B = \alpha_g$. Furthermore, The left side of the equation has

$$\left| \sum_{i=1}^m k_i n_i (n_i^2 - 1) \right|^2 = \sum_{i=1}^m \frac{|k_i|^2 |l_i|^2 (n_i^2 - 1)^2}{(2n_i)^2} + \sum_{i=1}^m \sum_{j=1}^m (k_i k_j^* l_j^* + k_i^* k_j l_j) \frac{(n_i^2 - 1)(n_j^2 - 1)}{2n_j} \quad (43)$$

So that

$$B^2 \left( \sum_{i=1}^m \frac{k_i^2 (n_i^2 - 1)}{2n_i} \right) + B^2 \left( \sum_{i=1}^m \frac{k_i n_i (n_i^2 - 1)}{2n_i} \right) \geq -2|B|^2 \sum_{i=1}^m \frac{|k_i|^2 |l_i|^2 (n_i^2 - 1)}{n_i} \quad (44)$$

and we add $|B|^2 \sum_{i=1}^m N_i$ to both sides of the equation

$$|B|^2 \left( \sum_{i=1}^m N_i + B^2 \left( \sum_{i=1}^m \frac{k_i^2 (n_i^2 - 1)}{2n_i} \right) + B^2 \left( \sum_{i=1}^m \frac{k_i n_i (n_i^2 - 1)}{2n_i} \right) \right) \geq |B|^2 \left( \sum_{i=1}^m N_i - \sum_{i=1}^m \sqrt{|k_i|^2 |l_i|^2} \frac{(n_i^2 - 1)}{n_i} \right). \quad (45)$$
Due to $\frac{\mu_0}{2} \geq 0$, thus
\[
\sum_{i=1}^{m} N_i - \sum_{i=1}^{m} \sqrt{\kappa_i^2 \lambda_i^2} \left( \frac{n_i^2 - 1}{n_i} \right) = \sum_{i=1}^{m} \left[ \sqrt{\kappa_i^2 \lambda_i^2} \left( \frac{n_i + 1}{2} \right) - \sqrt{\kappa_i^2 \lambda_i^2} \left( \frac{n_i - 1}{2} \right)^2 \right] + \sum_{i=1}^{m} \left[ \sqrt{\kappa_i^2 \lambda_i^2} \left( \frac{n_i^2 - 1}{n_i} \right) \right] \geq 0,
\]
so that
\[
|B|^2 \sum_{i=1}^{m} N_i + B^2 \left( \sum_{i=1}^{m} \kappa_i \lambda_i \left( \frac{n_i^2 - 1}{2n_i} \right) \right) + B^2 \left( \sum_{i=1}^{m} k_i l_i \left( \frac{n_i^2 - 1}{2n_i} \right) \right) \geq 0.
\]
Therefore, the relative purity is always $1/2 + a$ positive number, so that
\[
\frac{\mu_m}{\mu_0} \geq 1/2.
\]
It is a proof that for any Gaussian state, single-photon subtraction (or addition) can reduce the purity with at most a factor $1/2$.

The case of photon addition yields the same expressions with $\alpha_g$ and $\alpha_g^*$ interchanged. In practice, the case of photon addition will also lead to different values for $k_i$ and $l_i$. However, this does not change any of the proofs and as a consequence, we also find $\mu_m^+/\mu_0 \geq 1/2$ for photon addition.

C. ENTANGLEMENT INCREASE IN ALL POSSIBLE PARTITIONS FOR GRAPH STATES

In analogy to Fig. 3(c) of the main text, we present in Fig. 4 the entanglement increase for all the possible bi-partitions in the graph state example of Fig. 2(b) in the main text. Every dot represents the entanglement increase of a different bi-partition and the bi-partitions are grouped in terms of the number of modes $m_A$ in the subsystem $A$ where the photon is subtracted.