Rigidity and non local connectivity of Julia sets of some quadratic polynomials

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Abstract

For an infinitely renormalizable quadratic map \( f_c : z \mapsto z^2 + c \) with the sequence of renormalization periods \( \{n_m\} \) and the rotation numbers \( \{t_m = p_m/q_m\} \), we prove that if \( \limsup n_m^{-1} \log |p_m| > 0 \), then the Mandelbrot set is locally connected at \( c \). We prove also that if \( \limsup |t_{m+1}|^{1/q_m} < 1 \) and \( q_m \to \infty \), then the Julia set of \( f_c \) is not locally connected provided \( c \) is the limit of corresponding cascade of successive bifurcations. This quantifies a construction of A. Douady and J. Hubbard, and strengthens a condition proposed by J. Milnor.

1 Introduction

Theorem 1 Suppose that for some quadratic polynomial \( f(z) = z^2 + c \) there is an increasing sequence \( n_m \to \infty \) of integers, such that \( f^{n_m} \) is simply renormalizable, and

\[
\lim_{m \to \infty} \frac{\log |p_m|}{n_m} > 0,
\]

where \( p_m/q_m \in (-1/2, 1/2] \) denotes the rotation number of the separating fixed point of the renormalization \( f^{n_m} \).

Then \( c \) lies in the boundary of the Mandelbrot set \( M \) and \( M \) is locally connected at \( c \).

The following problems are central in holomorphic dynamics: MLC conjecture: “The Mandelbrot set is locally connected”, and its dynamical counterpart: “For which \( c \) is the Julia set \( J_c \) of \( f_c \) locally connected?” . The MLC conjecture is equivalent to the following rigidity conjecture: if two quadratic polynomials with connected Julia sets and all periodic points repelling are combinatorially equivalent, then they are conformally conjugate. The MLC implies that hyperbolic dynamics is dense in the space of complex quadratic polynomials [6] (see
also [39]). Yoccoz (see [14]) solved the above problems for finitely renormalizable quadratic polynomials as follows: for such a non hyperbolic map \( f_c \), the Julia set \( J_c \) of \( f_c \) is locally connected (provided \( f_c \) has no neutral periodic points), and the Mandelbrot set \( M \) is locally connected at \( c \). (For the (sub)hyperbolic maps and maps with a parabolic point the problem about the local connectivity of the Julia set had been settled before in the works by Fatou, Douady and Hubbard and others, see e.g. [3].) At the same time, infinitely renormalizable dynamical systems have been studied intensively, see e.g. [27], [28], [11] and references therein. In his work on renormalization conjectures, Sullivan [41] (see also [30]) introduced and proved so-called “complex bounds” for real infinitely renormalizable maps with bounded combinatorics. Roughly speaking, complex bounds mean that a sequence of renormalizations is precompact. This property became in the focus of research. It has played a basic role in recent breakthroughs in the problems of local connectivity of the Julia set and rigidity for real and complex polynomials, see [12], [22], [23], [38], [16], [17], [21], [1], [18].

On the other hand, there are maps without complex bounds. Indeed, Douady and Hubbard showed the existence of an infinitely renormalizable \( f_c \), such that its small Julia sets do not shrink (and thus \( f_c \) has no complex bounds) but still such that the Mandelbrot set is locally connected at this \( c \), see [31], [40].

Theorem 1 provides a first class of combinatorics of infinitely renormalizable maps \( f_c \) without (in general) complex bounds, for which the Mandelbrot set is locally connected at \( c \). Obviously, previous methods in proving the local connectivity of \( M \) do not work in this case. Our method is based on an extension result for the multiplier of a periodic orbit beyond the domain where it is attracting, see next Section.

That the maps with the combinatorics described in Theorem 1 in general do not have complex bounds and can have non locally connected Julia sets follow from the second result, Theorem 2 which makes explicit Douady-Hubbard’s construction. It is based on the phenomenon known as cascade of successive bifurcations, and is the following. Let \( W \) be a hyperbolic component of the Mandelbrot set, so that \( f_c \) has an attracting periodic orbit of period \( n \) for \( c \in W \). Given a sequence of rational numbers \( t_m = p_m/q_m \neq 0 \) in \((-1/2, 1/2]\), choose a sequence of hyperbolic components \( W^m \) as follows: \( W_0 = W \), and, for \( m \geq 0 \), the hyperbolic component \( W^{m+1} \) touches the hyperbolic component \( W^m \) at the point \( c_m \) with the internal argument \( t_m = p_m/q_m \) (see next Sect.). When the parameter \( c \) crosses \( c_m \) from \( W^m \) to \( W^{m+1} \), the periodic orbit of period \( n_m = n_{q_0}...q_{m-1} \) which is attracting for \( c \in W^m \) “gives rise” another periodic orbit of period \( n_{m+1} = n_m q_m \) which becomes attracting for \( c \in W^{m+1} \). Thus when the parameter \( c \) moves to a limit parameter through the hyperbolic components \( W^m \), the dynamics undergoes a sequence of bifurcations precisely at the parameters \( c_m, m = 0, 1, ... \).

In the case when \( n = 1 \) and \( t_m = 1/2 \) for all \( m \geq 0 \), the parameters \( c_m \) are real, and we get the famous period-doubling cascade on the real line known since
1960’s. The corresponding limit parameter \( \lim c_m = c_F = -1.4 \ldots \). The Julia set \( J_{c_F} \) is locally connected [15], [23].

Douady-Hubbard’s construction shows that if the sequence \( \{ t_m \} \) tends to zero fast enough, then periodic orbits generated by the cascade of successive bifurcations at the parameters \( c_m, m > 0 \), stay away from the origin. This implies that the Julia sets of the renormalizations of \( f_{c_m} \) for \( c_s = \lim c_m \) do not shrink, and \( J_{c_s} \) is not locally connected at zero. Their construction is by continuity, and it does not give any particular sequence of \( t_m \).

Milnor suggests in [32], p. 21, that the convergence of the series:

\[
\sum_{m=1}^{\infty} |t_m|^{1/q_m - 1} < \infty
\]

could be a criterion that the periodic orbits generated by the cascade after all bifurcations stay away from the origin. Theorem 2 shows that the condition (1) is indeed sufficient for this, but not the optimal one (cf. [24]). We give a weaker sufficient condition and thus prove:

**Theorem 2** For every \( a \in (0,1) \) there exists \( Q \) as follows. Let \( t_0, t_1, \ldots, t_m, \ldots \) be a sequence of rational numbers \( t_m = p_m/q_m \in (-1/2, 1/2) \setminus \{0\} \), such that

\[
\limsup_{m \to \infty} q_m > Q
\]

and

\[
\limsup_{m \to \infty} |t_m|^{1/q_m - 1} < a.
\]

Then:

1. The sequence \( \{c_m\}_{m=0}^{\infty} \) converges to a limit parameter \( c_s \).
2. The map \( f_{c_s} \) is infinitely renormalizable with non locally connected Julia set.

If, for instance, \( |t_{m+1}|^{1/q_m} \leq 1/2 \) for big indexes \( m \), then \( q_{m+1} \geq 2^{4m} \to \infty \), and Theorem 2 applies.

Let us make some further comments.

As it is mentioned above, we prove (2) by showing that the conditions of Theorem 2 imply that the cascade of bifurcated periodic orbits of \( f_{c_s} \) stay away from the origin.

By Theorem 1, if, for the sequence \( t_m = p_m/q_m \),

\[
\limsup_{m \to \infty} \frac{\log |p_m|}{q_0 \cdots q_{m-1}} > 0,
\]

then the components \( W^m \) tend to a single point, and \( M \) is locally connected at this point.
The conclusion (2) of Theorem 2 breaks down if not all its conditions are valid. Indeed, if \( t_m = 1/2 \) for all \( m \), then \( |t_{m+1}|^{1/q_m} = 1/2^{1/2} < 1 \) At the same time, as it is mentioned above, the limit Julia set \( J_{c_F} \) is locally connected.

Theorem 2 is a consequence of a more general Theorem 5, see Section 4.

Note in conclusion that there are similarities between Theorems 2, 5 and the celebrated Bruno-Yoccoz criterion of the (non-)linearizability of quadratic map near its irrational fixed point.

Throughout the paper, \( B(a, r) = \{ z : |z - a| < r \} \).

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2 Multipliers

2.1 Hyperbolic components

A component \( W \) of the interior of \( M \) is called an \( n \)-hyperbolic if \( f_c, c \in W \), has an attracting periodic orbit \( O(c) \) of period \( n \). Denote by \( \rho_W(c) \) the multiplier of \( O(c) \). By the Douady-Hubbard-Sullivan theorem [5], [29], \( \rho_W \) is a analytic isomorphism of \( W \) onto the unit disk, and it extends homeomorphically to the boundaries. Given a number \( t \in (-1/2, 1/2] \), denote by \( c(W, t) \) the unique point in \( \partial W \) with the \textit{internal argument} \( t \), i.e. \( \rho_W \) at this points is equal to \( \exp(2\pi t) \). Root of \( W \) is the point \( c_W = c(W, 0) \) with the internal argument zero.

If \( t = p/q \) is a rational number, we will always assume that \( p,q \) are co-primes. For any rational \( t \neq 0 \), denote by \( L(W, t) \) the connected component of \( M \setminus \{ c(W, t) \} \) which is disjoint with \( W \). It is called the \( t \)-limb of \( W \). Denote also by \( W(t) \) a \( nq \)-hyperbolic component with the root point \( c(W, t) \); it touches \( W \) at this point. The limb \( L(W, t) \) contains \( W(t) \). Root of \( W(t) \) is the point \( c_W = c(W, 0) \) with the internal argument zero. The root \( c_W \) of \( W \) is the landing point of precisely two parameter rays, i.e. external rays of the Mandelbrot set [6]. In what follows, it will be important the notion of the wake of a hyperbolic component \( W \) [14]: it is the only component \( W^* \) of the plane cut by two external rays to the root of \( W \) that contains \( W \). The points of periodic orbit \( O(c) \) as well as its multiplier \( \rho_W \) extend as analytic functions to the wake \( W^* \). Moreover, \( |\rho_W| > 1 \) in \( W^* \setminus \overline{W} \).
2.2 Analytic extension of the multiplier

Given $C > 1$, consider an open set $\Omega$ of points in the punctured $\rho$-plane defined by the inequality

$$|\rho - 1| > C \log |\rho|$$

(3)

It obviously contains the set $D_* = \{ \rho : 0 < |\rho| \leq 1, \rho \neq 1 \}$ and is disjoint with an interval $1 < \rho < 1 + \epsilon$. Denote by $\Omega(C)$ the connected component of $\Omega$ which contains the set $D_*$ completed by $0$. Denote also by $\Omega^{\log}(C)$ the set of points $L = \log \rho = x + iy$, $\rho \in \Omega(C)$, $|y| \leq \pi$. $\Omega(C)$ is simply-connected. More precisely, the intersection of $\Omega^{\log}(C)$ with any vertical line with $x = x_0 > 0$ is either empty or equal to two (mirror symmetric) intervals. If $C > 4$, then $x < 2/(C - 2)$ for all $L = x + iy$ in $\Omega^{\log}(C)$. If $C$ is large enough, $\Omega^{\log}(C)$ contains two (mirror symmetric) domains bounded by the lines $y = \pm(C/2)x$ $(x > 0)$ and $y = \pm \pi$.

Let $W$ be an $n$-hyperbolic component of $M$. The map $\rho_W$ from $W$ onto the unit disk $c \mapsto \rho_W(c)$ has an inverse, which we denote by $c = \psi_W(\rho)$. It is defined so far in the unit disk. In [24] we prove the following.

Theorem 3 (a) There exists $B_0$ as follows. Let $O(c)$ be a repelling periodic orbit of $f_c$ of exact period $n$, and the multiplier of $O(c)$ is equal to $\rho$. Assume that $|\rho| < e$. Then

$$|\rho - 1| \leq B_0 \frac{n}{4^n} \{ \log |\rho(c)| + \frac{|\rho'(c)|}{|\rho(c)|} (1 + o(1)) \}$$

(4)

as $n \to \infty$.

(b) Denote

$$\Omega_n = \Omega(n^{-1}4^n B_0)$$

and

$$\Omega_n^{\log} = \Omega^{\log}(n^{-1}4^n B_0) = \{ L = x + iy : \exp(L) \in \Omega_n, |y| \leq \pi \}$$

Then the function $\psi = \psi_W$ extends to a holomorphic function in the domain $\Omega_n$. Moreover, $\psi$ is univalent in a subset $\tilde{\Omega}_n$ of $\Omega_n$ defined by its log-projection $\tilde{\Omega}_n^{\log} = \{ \log \rho : \rho \in \tilde{\Omega}_n \}$ as follows:

$$\tilde{\Omega}_n^{\log} = \Omega_n^{\log} \setminus \{ L : |L - R_n| < R_n \}$$

where $R_n$ depends on $n$ only and has an asymptotics

$$R_n = (2 + O(2^{-n}))n \log 2$$

as $n \to \infty$.

Finally, the image of $\tilde{\Omega}_n$ by $\psi$ is contained in the wake $W^*$.

Next inequalities can be checked easily.
There exists \( K > 0 \), such that, for every \( n > 0 \) and \( t \in (-1/2, 1/2) \), the Euclidean distance between the point \( 2\pi it \) and the boundary of \( \tilde{\Omega}^\log_n \),

\[
K^{-1}P(t, n) < \text{dist}(2\pi it, \partial \tilde{\Omega}^\log_n) < KP(t, n)
\]

where

\[
P(t, n) = \min\left\{ \frac{n|t|}{4^n}, \frac{t^2}{n} \right\}.
\]

### 2.3 Limbs

Let \( W \) be an \( n \)-hyperbolic component. For every \( t = p/q \neq 0 \), consider the limb \( L(W, t) \). Then, for every \( c \in L(W, t) \), a branch of \( \log \rho_W(c) \) is contained in the following round disk (Yoccoz’s circle):

\[
Y_n(t) = \{ L : |L - (2\pi it + \frac{n \log 2}{q})| < \frac{n \log 2}{q} \},
\]

see [14], [37] and references therein.

In [24] we derive from Theorem 3 and the latter result the following bound.

**Theorem 4** There exists \( A > 0 \), such that, for every \( n \)-hyperbolic component \( W \) and every \( t = p/q \in [-1/2, 1/2] \), the diameter of the limb \( L(W, t) \) is bounded by:

\[
diamL(W, t) \leq A \frac{4^n}{p}
\]

### 3 Rigidity

#### 3.1 Simple renormalization

We follow some terminology as in [27]. Let \( f \) be a quadratic polynomial. The map \( f^n \) is called renormalizable if there are open disks \( U \) and \( V \) such that \( f^n : U \to V \) is a polynomial-like map with a single critical point at 0 and with connected Julia set \( J_n \). The map \( f^n : J_n \to J_n \) has two fixed points: \( \beta \) (non-separating) and \( \alpha \). Denote them by \( \beta_n \) and \( \alpha_n \). The renormalization is simple if any two small Julia sets \( f^i(J_n), i = 0, 1, \ldots, n-1 \) cannot cross each other, i.e. they can meet only at some iterate of \( \beta_n \).

The fixed point \( \alpha_n \) assuming it is repelling separates the Julia set of the renormalization \( f^n : U \to V \). Moreover, it has a well-defined nonzero rational rotation number \( p/q \in (-1/2, 1/2] \), which can be defined as follows: \( f^n : U \to V \) is hybrid equivalent to a quadratic polynomial which lies in the \( p/q \)-limb of the main cardioid.
3.2 Demonstration of Theorem 1

We split the proof into few steps.

A. Let \( f^n : U \to V \) be a simple renormalization of \( f \), \( \beta_n \) its \( \beta \)-fixed point, and \( O_n = \{ f^i(\beta_n) \}_{i=0}^{n-1} \) the periodic orbit containing \( \beta_n \). We use some notions and results from [6], [33]. The characteristic arc \( I(O_n) = (\tau_- (O_n), \tau_+ (O_n)) \) of \( O_n \) is the shortest arc (measured in \( S^1 \)) between the external arguments of the rays landing at the points of \( O_n \). Then \( \tau_\pm (O_n) \) are the arguments of two dynamical rays that land at the point \( \beta'_n = f(\beta_n) \) of \( O_n \), and \( c \) lies in the sector bounded by these rays and disjoint with 0. Furthermore, two parameter rays of the same arguments \( \tau_\pm (O_n) \) land at a single parameter \( c(O_n) \). The point \( c(O_n) \) is the root of a hyperbolic component denoted by \( W(O_n) \), and the above parameter rays completed by \( c(O_n) \) bound the wake \( W(O_n)^* \) of this component. Denote by \( L(O_n) = W(O_n)^* \cap M \) the corresponding limb. Thus,

(A1) \( c_0 \in L(O_n) \),

(A2) moreover, the rotation number of the \( \alpha \)-fixed point \( \alpha_n \) of the renormalization is \( p/q \) if and only if \( c_0 \in L(W(O_n), p/q) \) - the limb of \( W(O_n) \) which is attached to the point of \( \partial W(O_n) \) with the internal argument \( p/q \).

B. Let \( m < m' \). By [27], since all renormalizations \( f^{nm} \) are simple, \( n_m \) divides \( n_{m'} \). Also, the small Julia set \( J_{n_{m'}} \) is a proper subset of the small Julia set \( J_{n_m} \). Therefore,

(B1) \( \tau_- (O_{n_m}) < \tau_- (O_{n_{m'}}) < \tau_+ (O_{n_{m'}}) < \tau_+ (O_{n_m}) \), thus the sets \( \overline{T}(O_{n_m}) \) form a decreasing sequence of compact sets containing \( c_0 \). Therefore, for the local connectivity of \( M \) at \( c_0 \) it is enough to prove that the set

\[
S = \bigcap_m \overline{T}(O_{n_m})
\]

consists of a single point.

(B2). It is easy to see that \( \tau_+ (O_{n_m}) - \tau_- (O_{n_m}) \to 0 \). This follows from a well known inequality \( \tau_+ (O_{n_m}) - \tau_- (O_{n_m}) < (2^{n_m} - 1)^2 / (2^{n_m}q_m - 1) \) (see [2], [26]), and \( q_m > 2 \). Thus, there exists

\[
\lim \tau_\pm (O_{n_m}) = \tau_0.
\]  

Note that \( \tau_0 \) is not periodic under the uniformization map \( p(t) = 2\tau (\text{mod} 1) \).

C. Let \( c \) be any point from \( S \).

(C1) All periodic points of \( f_c \) are repelling. Indeed, obviously, \( f_c \) cannot have an attracting cycle. If \( f_c \) has an irrational neutral periodic orbit then \( c \) lies in the boundary of a hyperbolic component contained in \( S \), a contradiction with [5]. If \( f_c \) has a neutral parabolic periodic orbit, then \( c \) is the landing point of precisely two parameter rays with periodic arguments, again the same contradiction. Thus, all cycles are repelling. Consider the so-called real lamination \( \lambda(c) \) of \( f_c \) [19]. It is a minimal closed equivalence relation on \( S^1 \) that identifies two points whenever their prime end impressions intersect. For every \( m \), \( \tau_\pm (O_{n_m}) \subset \lambda(c) \). Since \( \lambda(c) \)
is closed, \( \tau_0 \in \lambda(c) \). Moreover, for every \( m \), there is a pair \( \{ \tau_m^-, \tau_m^+ \} \), such that 
\[
p(\tau_m^+) \subset \tau_0(O_{nm}),\quad \{ \tau_m^-, \tau_m^+ \}
\] is contained in the class of \( \lambda(c) \) corresponding to the point \( \beta_{nm} \). Passing to the limit, we get that the pair \( \{ \tau_0/2, \tau_0/2 + 1/2 \} \) is 
the critical class of \( \lambda(c) \). The following statements are known after Thurston and 
Douady and Hubbard and proved in a much more general form in [20], Proposition 4.10: if the critical class \( \{ \tau_0/2, \tau_0/2 + 1/2 \} \) is contained in a class of \( \lambda(c) \), then 
\( \lambda(c) \) is determined by (the itinerary of) \( \tau_0 \). Since \( \tau_0 \) is the same for all \( c \in S \), we 
conclude that the real laminations of all \( f_c \), \( c \in S \), coincide. In particular, for every 
\( c \in S \), \( f_c^{m} \) is simply renormalizable because by [27] the simple renormalizations 
are detected by the lamination. The renormalization \( f_c^{m} \) is hybrid equivalent 
to some \( f_{T(c,m)} \) [7]. Denote \( \hat{n}_k = n_{m+k}/n_m \), \( k > 0 \). Then \( f_{T(c,m)}^{\hat{n}_k} \) is simply 
renormalizable, and the rotation number of its \( \alpha \)-fixed point is \( p_{m+k}/q_{m+k} \) because 
it is a topological invariant.

(C2) By (A2), \( T(c,m) \) is contained in the intersection of a decreasing sequence 
of limbs \( L_{m,k} \), \( k = 1, 2, \ldots \), such that \( L_{m,k} \) is attached to a hyperbolic component 
of period \( \hat{n}_k \) at the internal argument \( p_{m+k}/q_{m+k} \). By Theorem 4, the diameter of \( L_{m,k} \),
\[
diam L_{m,k} \leq A4^{\frac{n_{m+k}}{p_{m+k}}} = A4^{\frac{n_{m+k}}{nm}}.
\]

Since \( \lim(\log p_m)/n_m > 0 \), one can find and fix \( m \) in such a way, that
\[
\lim_{k \to \infty} 4^{\frac{n_{m+k}}{n_m}} = 0.
\]

It means, that, for the chosen \( m \), the limbs \( L_{m,k} (k \to \infty) \) shrink to a point 
\( \hat{c} = T(c,m) \), so that \( \hat{c} \) depends on \( m \) but is independent of \( c \in S \). Thus \( f_c^{m} \) is 
quasi-conformally conjugate to \( f_{\hat{c}} \), for all \( c \in S \).

D. Assume that the compact \( S \) has at least two different points. Then \( S \) 
contains a point \( c_1 \in \partial M \) other than \( c_0 \). By (C1)-(C2), the renormalizations \( f_{c_0}^{m} \) 
of \( f_{c_0} \) and \( f_{c_1}^{m} \) of \( f_{c_1} \) are quasi-conformally conjugate, all periodic points of \( f_{c_0} \), 
\( f_{c_1} \) are repelling, and \( \lambda(c_0) = \lambda(c_1) \). We are in a position to apply Sullivan’s 
pullback argument (see [30]). Using a quasi-conformal conjugacy near small Julia 
sets (rather than on the postcritical set) and an appropriate puzzle structure, we 
arrive at a quasi-conformal conjugacy between \( f_{c_0}, f_{c_1} \). Since \( c_1 \in \partial M \), then 
according to [6], \( c_0 = c_1 \).
4 Non locally connected Julia sets

4.1 Statement

Theorem 5 Let \( t_0, t_1, \ldots, t_m, \ldots \) be a sequence of rational numbers \( t_m = p_m/q_m \in (-1/2, 1/2] \). Assume that the following conditions are satisfied.

\((E0)\) \[ \liminf_{m \to \infty} \frac{\log |t_{m+1}|}{q_0 \ldots q_{m-1}} > 0. \]  

\((Y0)\) \[ \sup_{m \geq 0} |t_m| q_0 \ldots q_{m-1} < \infty. \]

\((S0)\) for some \( k \geq 0, \)
\[ \sum_{m=k}^{\infty} \frac{u_{k,m}}{q_m (1 - u_{k,m})} H(u_{k,m+1}) < \infty, \]

where the smooth strictly increasing function \( H : [0, 1) \to [0, \infty) \) is defined below, and
\[ u_{k,m} = \theta_{k,m}^{1/q_{m}}; \quad \theta_{k,m} = C |t_{m+1}| \max\{q_k \ldots q_{m}, 4q_k \ldots q_{m-1}\}, \]

where \( C > 0 \) is an absolute constant.

Let now \( W \) be a hyperbolic component of some period \( n \geq 1 \), and the sequence \( W^m \) of hyperbolic component is built as above, in other words, the hyperbolic component \( W^{m+1} \) touches the hyperbolic component \( W^m \) at a point \( c_m \) with internal argument \( t_m \). Then:

1. the sequence of parameters \( c_m, m=0,1,2,\ldots, \) converges to a limit parameter \( c_* \),
2. the map \( f_{c_*} \) is infinitely renormalizable with non locally connected Julia set.

The function \( H \) used above is equal to
\[ H(u) = 16u \Pi_{k=1}^{\infty} \frac{(1 + u^{2k})^8}{(1 - u^{2k-1})^8}, \]

and is bounded by
\[ H(u) \leq \frac{1}{16} \exp (-\pi^2 / \log u). \]

Theorem 2 announced in the Introduction is a simple corollary of Theorem 5. Indeed, assume there exists \( a \in (0, 1) \), such that
\[ |t_m|^{1/q_{m-1}} < a \]
for all large $m$. If now some $q_k$ is big enough, then $q_m \to \infty$ rapidly enough, and it is not difficult to check that the conditions (E0), (Y0), and (S0) are satisfied.

The rest of the Section occupies the proof of Theorem [5]

### 4.2 Bifurcations

Let $W$ be a $n$-hyperbolic component, and let $c_0 \in \partial W$ have an internal argument $t_0 = p/q \neq 0$. Consider a periodic orbit $O(c) = \{b_j(c)\}_{j=1}^n$ of $f_c$ which is attracting when $c \in W$. Then all $b_j(c)$ as well as the multiplier $\rho(c)$ of $O(c)$ are holomorphic in $W$ and extend to holomorphic functions in $c$ in the whole wake $W^*$ of $W$. As we know, the multiplier $\rho(c)$ is injective near $c_0$. Denote the inverse function by $\psi$. It is well defined in the domain $\tilde{\Omega}_n$, which includes the unit disk and a neighborhood of the point $\rho_0 = \exp(2\pi it_0)$, so that $\psi(\rho_0) = c_0$. Remind that $W(t_0)$ denotes a hyperbolic component touching $W$ at the point $c_0$.

**Lemma 4.1** (cf. [22], [4]) There exists $Q > 0$ as follows. Given $n$, $q$, denote

$$r(t_0, n) = 1/2 \min \{1/(2nq^3), \text{dist}(2\pi it_0, \partial \tilde{\Omega}_n^{log})\},$$

Let $q > Q$, $n > 0$.

1. There exist $q$ function $F_k$ as follows. For each $k = 1, \ldots, q$, $F_k$ is a holomorphic function in the disk

$$S = \{|s| < r(t_0, n)^{1/q}\},$$

such that $F_k(0) = 0$, $F_k'(0) \neq 0$; for every $s \in S$ and corresponding $\rho = \rho_0 + s^q$, the points

$$b_j(c_0) + F_k(s \exp(2\pi j/q)), k = 1, \ldots, n, j = 0, \ldots, q - 1,$$

form a periodic orbit $O^q(c)$ of $f_c$ of period $nq$, where $c = \psi(\rho)$.

2. The disk

$$B_{t_0} = B(\rho_0, r(t_0, n))$$

is contained in $\tilde{\Omega}_n$, so that $\psi$ is univalent in $B_{t_0}$. The multiplier $\rho_{W(t_0)}$ of the $nq$-periodic orbit $O^q$ has a well-defined analytic extension from the hyperbolic component $W(t_0)$ to the union of the wake $W^*$ and the domain $\psi(B_{t_0})$. The function

$$\lambda = \rho_{W(t_0)} \circ \psi$$

is defined and holomorphic in $B_{t_0}$. For every real $T \in (-1/2, 1/2]$, the equation $\lambda(\rho) = \exp(2\pi iT)$ has at most one solution $\rho$ in $B_{t_0}$, and for such a solution the corresponding $c = \psi(\rho)$ lies on the boundary of the $nq$-hyperbolic component $W(t_0)$. A covering property holds: for every $r \leq r(t_0, n)$, the image of the disk $B(\rho_0, r)$ under the map $\lambda$ covers the disk $B(1, q^2r/16)$. 

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Proof. It is known (see e.g. [6]) that the periodic orbit $O^q(c)$ exists locally, for $c$ close enough to $c_0$, and this is the only periodic orbit other than $O(c)$ in its neighborhood. Therefore, by the Implicit Function theorem, (1) holds locally, in a neighborhood of the point $s = 0$.

Let us show that the functions $F_k$ don’t have singularities in the disk $S$. Indeed, if the periodic orbit $O^q$ of the period $nq$ has a singularity at some $c$, then $c$ lies in a limb $L(W, p'/q')$ other than $L(W, p/q)$. By a calculation in the external arguments [24] (cf. [1]), then $q \geq q' - 1$. On the other hand, a simple estimate shows that for all $n, q$, the ball $B(2\pi it_0, 1/(2nd^3))$ is disjoint with every Yoccoz’s circle that touches the vertical line at a point $2\pi ip'/q'$ with some $q' \leq q + 1$, $p'/q' \neq p/q$. Moreover, for $q$ large enough, and every $n \geq 1$, the log-projection of the latter ball covers $B_{t_0}$. Therefore, $\psi$ is univalent in $B_{t_0}$ and, furthermore, $V = \psi(B_{t_0})$ is disjoint with any limb $L(W, p'/q')$, with $q' \leq q + 1$, $p'/q' \neq p/q$. Now (1) follows from the Monodromy Theorem. We have shown also that $\lambda$ is holomorphic in $B_{t_0}$.

Let $\lambda(\rho_1) = \exp(2\pi iT_1)$ for some real $T_1$. Then, for $c_1 = \psi(\rho_1)$, the map $f_{c_1}$ has $nq$-periodic orbit $O^q(c_1)$ with the multiplier $\rho_1 = \exp(2\pi iT_1)$. Now, $V$ is disjoint with any $p'/q'$-limb of $W$ other than $L(W, t_0)$, where $q' \leq q + 1$, therefore, $c_1 \in L(W, t_0)$ and, moreover, lies in the boundary of an $nq$-hyperbolic component. This hyperbolic component belongs to some limb $L(W, t')$, $t' = p'/q'$, where, by the above, $q' > q + 1$. If $t' \neq t_0$, then $L(W, t')$ contains a parameter $\tilde{c}$ on the boundary of this component, such that $f_{\tilde{c}}$ has a $nq$-periodic orbit with the multiplier 1. But then $q \geq q' - 1$, in contradiction with the previous condition $q' > q + 1$ or $t' = t_0$. Thus $c_1$ is in the limb $L(W, t_0)$. Hence, it lies in the boundary of some $nq$-hyperbolic component belonging this time to $L(W, t_0)$. On the other hand, the multiplier of the periodic orbit $O^q(c)$ is bigger on modulus than 1 off the closure of the component $W(t_0)$. Thus the hyperbolic component containing $c_1$ in its boundary is just $W(t_0)$. Since the corresponding multiplier is injective on the boundary, it means that $\rho_1$ is the only solution of the equation $\lambda(\rho) = \exp(2\pi iT_1)$.

To prove the covering property, we start with the formula

$$
\frac{d\lambda}{dp}(\rho_0) = -\frac{q^2}{\rho_0}
$$

(see [11]). Given $r \leq r(t_0, n)$, consider a function $m(w) = (q^2r)^{-1}(\lambda(\rho_0 + rw) - 1)$. It is holomorphic in the unit disk, $m(0) = 0$, $|m'(0)| = 1$ and, by the proved part of the statement, $m(w) = 0$ if and only if $w = 0$. Therefore, by a classical result (Caratheodory-Fekete, see e.g. [9] or [33]), the disk $B(0, 1/16)$ is covered by the image of the unit disk under the map $m$. This is equivalent to the covering property.

□
4.3 The function $H$

**Definition 4.1** There exists a real strictly increasing smooth function

$$H : [0, 1) \to [0, \infty), \quad H(0) = 0$$

as follows. Let $G$ be the set of all holomorphic functions $g : B(0, 1) \to \mathbb{C} \setminus \{1\}$, such that $g(w) = 0$ if and only if $w = 0$.

Then $H(u)$ is defined by:

$$H(u) = \sup \{|g(w)| : |w| \leq u, g \in G\}.$$

There is an explicit expression for $H$. It is obtained as follows. Let $J(w)$ be a holomorphic function in $B(0, 1)$, such that $J(0) = 0$, $J'(0) > 0$, and

$$J : B(0, 1) \setminus \{0\} \to \mathbb{C} \setminus \{0, 1\}$$

is an infinite unbranched cover. Such function is investigated in [35], see also [8], [13], [36]. By the Schwarz lemma,

$$H(u) = \max_{|w|=u} |J(w)|.$$

On the other hand, by [35],

$$J(w) = 16w\prod_{k=1}^{\infty} \frac{(1 + w^{2k})^8}{(1 + u^{2k-1})^8}.$$

(Apparently, $J$ is equal to the square of the so-called elliptic modulus, see e.g. [36].) Thus,

$$H(u) = -J(-u) = 16u\prod_{k=1}^{\infty} \frac{(1 + u^{2k})^8}{(1 - u^{2k-1})^8}.$$

As it is shown in [35]:

$$H(u) \leq \frac{1}{16} \exp^{-\pi^2/\log u}.$$

4.4 Main Lemma

**Lemma 4.2** There exist $Q > 0$ and $L > 0$ as follows. Let $t_0, t_1, \ldots, t_m, \ldots$ be a sequence of rational numbers $t_m = p_m/q_m \in (-1/2, 1/2)$. Denote

$$n_0 = 1, \quad n_m = n_0 q_1 \ldots q_{m-1}, \quad m > 0.$$

Denote also

$$\tilde{d}_m = r(t_m, n_m) = 1/2 \min \left\{ \frac{1}{2n_m q_m^2}, \text{dist}(2\pi it_m, \partial \Omega_{n_m}^{\log}) \right\},$$
and, finally,
\[ d_m = \frac{320}{\beta q_m} |t_{m+1}|, \]

where \( \beta = 1/32000 \). Assume that, for every \( m \geq 0 \), the following conditions are satisfied.

(0) \( q_m > Q \),
(Y) \( d_m < \frac{d_m}{2} \).
(S)

\[ |t_0| H(u_0) + \sum_{m=1}^{\infty} \frac{u_{m-1}}{q_m(1 - u_m)} H(u_m) < L, \]  \( (13) \)

where \( u_m = \theta_m^{1/q_m} \) with

\[ \theta_m = 200 \beta d_m \tilde{d}_m, \quad m \geq 0. \]  \( (14) \)

Let now \( W_0 \) be the main cardioid. Denote \( W^m = W^{m-1}(t_{m-1}), \ m = 1, 2, \ldots \), in other words, the hyperbolic component \( W^m \) touches the hyperbolic component \( W^{m-1} \) at the point \( c_{m-1} := c(W^{m-1}, t_{m-1}) \) with the internal argument \( t_{m-1} \).

Then:

1. the sequence of parameters \( c_m, m=0,1,2,\ldots \), converges to a limit parameter \( c^* \),
2. the map \( f_{c^*} \) is infinitely renormalizable with non locally connected Julia set.

**Proof.** By the definition of \( \tilde{d}_m \), for every \( m = 0, 1, 2, \ldots \), the disk

\[ B_{t_m} := B(\exp(2\pi i t_m), \tilde{d}_m) \]

lies in \( \tilde{\Omega}_{n_m} \), and the conclusions of Lemma 4.1 holds.

Introduce more notations. Let \( \psi_{W^m} \) be a function, which is inverse to the multiplier function \( \rho_{W^m} \), and

\[ \psi_m(w) = \psi_{W^m}(\exp(w)), \quad m = 0, 1, \ldots \]

\( \psi_m \) is holomorphic in \( \Omega_{n_m}^\log \) and univalent in its subset \( \tilde{\Omega}_{n_m}^\log \). In particular, the function \( \psi_m \) extends in a univalent fashion to the disk \( B(2\pi it_m, d_m) \)

Note that \( c_m = \psi_m(2\pi it_m) \). Denote further

\[ R_m = \psi_m(B(2\pi it_m, d_m)), \quad D_m = \psi_m(B(2\pi it_m, \beta d_m)), \quad D'_m = \psi_m(B(2\pi it_m, 100\beta d_m)) \]

We use classical distortion bounds for univalent maps, see e.g. [9]: if \( g \) is univalent in a disk \( B(0, R) \) and \( r < R \), then

\[ B(g(0), \alpha(r/R)^{-1} r|g'(0)|) \subset g(B(0, r)) \subset B(g(0), \alpha(r/R)r|g'(0)|) \]  \( (15) \)
where \( \alpha(x) = (1 - x)^{-2} \).

In particular, if \( r \leq \tilde{d}_m/2 \), then \( \psi_m(B(2\pi it_m, r)) \) is “roughly” a disk:

\[
B(c_m, 4^{-1}r|\psi'_m(2\pi it_m)|) \subset \psi_m(B((2\pi it_m, r)) \subset B(c_m, 4r|\psi'_m(2\pi it_m)|)
\]

Hence, \( R_m, D_m \) and \( D'_m \) are “roughly” disks (around \( c_m \)), and

\[
D_m \subset D'_m \subset R_m, \quad m = 0, 1, \ldots,
\]

and, moreover, the diameter of each previous set is at least twice smaller than the diameter of the next one.

**Fact.** For \( m = 0, 1, 2, \ldots \), the following holds.

(i) \( c_{m+1} \in D_m \),

(ii) \( R_{m+1} \subset D'_m \).

For the proof, we use Lemma 4.1. Consider the function \( \lambda = \rho_{W^{m+1}} \circ \psi_{W^{m}} \). Then \( \lambda \) is holomorphic in \( B_{m} \). If \( d_m \) is small enough, then

\[
B(\exp(2\pi it_m), \beta d_m/2) \subset \exp(B(2\pi it_m, \beta d_m)).
\]

By the covering property of Lemma 4.1 \( \lambda(B(\exp(2\pi it_m), \beta d_m/2)) \) covers the disk \( B(1, q_m^2 \beta d_m/32) \). But

\[
\exp(2\pi it_{m+1}) \in B(1, q_m^2 \beta d_m/32),
\]

because \( |t_{m+1}| = q_m^2 \beta d_m/320 < q_m^2 \beta d_m/(64\pi) \). Therefore, by the part (2) of Lemma 4.1 the point

\[
c_{m+1} = \psi_m(2\pi it_{m+1}) \in \psi_m(B(2\pi it_m, \beta d_m)) = D_m.
\]

To prove that \( R_{m+1} \) is contained in \( D'_m \), let us notice that by part (b) of Theorem 3 the root \( c_m \) of \( W^{m+1} \) is outside of \( R_{m+1} \), and \( R_{m+1} \) is “roughly” a disk around \( c_{m+1} \). Hence, \( R_{m+1} \subset B(c_{m+1}, 4|c_{m+1} - c_m|) \) and then \( R_{m+1} \subset B(c_m, 17|c_{m+1} - c_m|) \).

Let us estimate \( |c_{m+1} - c_m| \). Since \( c_{m+1} \in D_m \), then \( |c_{m+1} - c_m| < \beta \alpha(\beta)r_m \) where \( r_m = \psi'_m(2\pi it_m)|d_m \) and \( \alpha(x) = (1-x)^{-2} \). Thus \( R_{m+1} \subset B(c_m, 17 \beta \alpha(\beta)r_m) \).

On the other hand, \( D'_m = \psi_m(B(2\pi it_m, 100\beta d_m) \) contains the disk \( B(c_m, 100\beta(\alpha(100\beta))^{-1}r_m) \). By the choice of \( \beta \), we check that \( 17 \alpha(\beta) < 100/\alpha(\beta) \), i.e. indeed

\[
R_{m+1} \subset B(c_m, 17 \beta \alpha(\beta)r_m) \subset B(c_m, 100\beta(\alpha(100\beta))^{-1}r_m) \subset D'_m.
\]

The Fact is proved.

It implies that for all \( m \), \( c_{m+1} \in D'_m \) and

\[
D'_{m+1} \subset R_{m+1} \subset D'_m.
\]
Since the diameters of $D'_m$ tend to zero, the sequence $c_m$ tends to the point

$$\{c_*\} = \cap_{m=0}^\infty R_m = \cap_{m=0}^\infty D'_m.$$  

It proves the conclusion (1).

Now we pass to the proof of the conclusion (2). Let us denote by $O_k(c)$ the $n_k$-periodic orbit of $f_c$, which is attracting if $c \in W_k$, $k = 0, 1, 2, \ldots$. As we know, $O_k$ extends holomorphically for $c$ in the wake $(W^k)^* \subset W^k$. Given $m \geq 1$, consider any point $b(c)$ of a periodic orbit $O_m(c)$, for $c \in (W^m)^*$. Lemma 4.1 tells us that $b_m$ extends also to a neighborhood of $c_{m-1}$ in the following sense. There is a holomorphic function $Z$ of a local parameter $s$ in the disc

$$S_{m-1} := \{|s| < v_{m-1}\}, \quad v_{m-1} = \bar{d}_{m-1}^{1/q_{m-1}},$$

such that it matches $b(c)$, i.e., $b(c) = Z(s)$ for $c = c_{m-1}(s) := \psi_{W_{m-1}^*}(\exp(2\pi it_{m-1}) + s^{q_{m-1}})$. Moreover, the points $Z(se_{q_{m-1}}^j)$, where $j = 1, \ldots, n_{m-1} - 1$ and $e_q$ denotes a primitive $q$-root of unity, also belong to $O_m$. We denote by $b^+$ the point $Z(se_{q_{m-1}}^1)$ of $O_m$, which is uniquely defined for $s \in S_{m-1}$. Let us estimate the distance between $b = Z(s)$ and $b^+ = Z(se_{q_{m-1}}^1)$. Since $|Z(s)| < 3$ for $s \in S_{m-1}$, we have:

$$|Z(se_{q_{m-1}}^1) - Z(s)| < 3 \frac{|s|}{q_{m-1} - |s|^2 v_{m-1}^{-1}}. \quad (16)$$

Let us detect $s$ for $O_m$, which corresponds to the limit parameter $c_*$. Since $c_* \in D'_{m-1} = \psi_m(B(2\pi it_{m-1}, 100/\bar{d}_{m-1}))$, then there is $s_{m-1} \in S_{m-1}$, such that $c_* = c_{m-1}(s_{m-1})$ and

$$|s_{m-1}| < (200/\bar{d}_{m-1})^{1/q_{m-1}}.$$

Now we consider two consecutive periodic orbits $O_m$ and $O_{m+1}$. Consider any point $b_{m+1}(c)$ of $O_{m+1}$ for $c \in (W^{m+1})^*$. By the above, there is a holomorphic function $Z_{m+1}$ of a local parameter $s \in S_m$, such that $b_{m+1}(c) = Z_{m+1}(s)$ where $c = c_m(s)$ provided $c(s) \in (W^{m+1})^*$. Moreover, $c_s \in (W^{m})^* \subset W^m$. Therefore, the points of $O_m$ can be represented as holomorphic functions of the local parameter $s \in S_m$ of $O_{m+1}$, too, because they are holomorphic functions of the parameter $c_m(s)$. Let $Z_m(s)$ be the point of $O_m$, such that $Z_m(0) = Z_{m+1}(0)$. For the point $b = Z_m(s)$ of $O_m$, there is the corresponding point $b^+ = Z_m^+(s)$ of $O_m$ determined above. Consider the functions $Z_{m+1}, Z_m, Z_m^+$ of $s \in S_m$. Observe that $Z_{m+1}(s) = Z_m(s)$ if and only if $s = 0$, and $Z_m^+(s) \neq Z_m(s)$ in $S_m$. Introduce now a new function

$$\zeta_m(s) = \frac{Z_{m+1}(s) - Z_m(s)}{Z_m^+(s) - Z_m(s)}.$$

By the above, $\zeta_m(s)$ obeys the following properties:

(i) it is holomorphic in the disc $S_m$,  

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(ii) \( \zeta_m(s) \neq 1 \) in \( S_m \),
(iii) \( \zeta_m(s) = 0 \) iff \( s = 0 \), and 0 is a simple zero.
(The latter fact holds because, by Lemma 4.1, \( Z_{m+1}(s) = B + Fs + O(s^2) \) with \( F \neq 0 \) while \( Z_m(s) = B + O(s^m) \).)

We conclude that
\[
|\zeta_m(s)| \leq H\left(\frac{|s|}{v_m}\right), \quad (17)
\]
where the function \( H \) is defined in Definition 4.1.

As for the limit parameter \( s = s_m \), we get
\[
|Z_{m+1}(s_m) - Z_m(s_m)| \leq H\left(\frac{|s_m|}{v_m}\right)|Z_m^+(s_m) - Z_m(s_m)|, \quad (18)
\]
In turn, by (16),
\[
|Z_m^+(s_m) - Z_m(s_m)| = |Z(s_{m-1}e_{q_m-1}) - Z_{m-1}(s_{m-1})| < 3\frac{2\pi}{q_{m-1}} \frac{|s_{m-1}|}{v_{m-1} - |s_{m-1}|^2}. \quad (19)
\]

Here \( Z_{m+1}^* := Z_{m+1}(s_m) \) is any point of the periodic orbit \( O_{m+1} \) of the limit map \( f_c \), while \( Z_m^* := Z_m(s_m) \) is a point of the periodic orbits \( O_m \) of the same map \( f_c \), which is determined by the \( Z_{m+1}^* \). Because \( H \) and \( t/(1 - t^2) \) are increasing functions, we conclude from (18) and (19):
\[
|Z_{m+1}^* - Z_m^*| \leq 3\frac{2\pi}{q_{m-1}} \frac{u_{m-1}}{1 - u_{m-1}} H(u_m) \quad (20)
\]
where we denote, for \( k = 0, 1, ... \),
\[
\bar{u}_k = \theta_k^{1/q_k}, \quad \theta_k = \frac{200\beta d_k}{d_k}.
\]

In turn, the point \( Z_m^* \) determines the point \( Z_{m-1}^* \) of the periodic orbit \( O_{m-1} \), and so on until the point \( Z_0^* \) of \( O_0 \). The inequality (20) holds for every \( m \geq 1 \).

It remains to estimate \( |Z_1^* - Z_0^*| \). Since \( Z_0^* \) is a fixed point of \( f_c \), we compare \( |Z_1^* - Z_0^*| \) with \( |\beta^* - Z_0^*| \), where \( \beta^* \) is the second fixed point of \( f_c \). By similar considerations, we can write
\[
|Z_1^* - Z_0^*| \leq H(\bar{u}_0)|\beta^* - Z_0^*|.
\]
On the other hand,
\[
|\beta^* - Z_0^*| = |1 - \rho^*|
\]
where \( \rho^* \) is the multiplier of \( Z_0^* \). Hence,
\[
|\beta^* - Z_0^*| \leq |1 - \exp(2\pi it_0)| + |\exp(2\pi it_0) - \rho^*| \leq 2\pi|t_0| + K_0/\bar{q}_0 \leq K_1|t_0|,
\]

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for some absolute constants $K_0, K_1$. We have finally, for $m = 1, 2, \ldots$,

$$|Z_{m+1}^* - Z_0^*| < K_2 \{|t_0|H(u_0) + \sum_{m=1}^{\infty} \frac{1}{q_{m-1}} \frac{u_{m-1}}{1 - u_{m-1}} H(u_m)\},$$

(21)

where $K_2$ is an absolute constant. Here $Z_{m+1}^*$ is any point of the periodic orbit $O_{m+1}$ of $f_{c*}$, for any $m \geq 0$. On the other hand, the fixed point $Z_0^*$ is away from a fixed neighborhood $B(0, \delta)$ of zero. Thus, under the condition (13), where we set $L = \delta/(2K_2)$, for every $m \geq 1$, the periodic orbit $O_m$ of $f_{c^*}$ lies outside of the fixed neighborhood $B(0, \delta/2)$. It is well known that this implies the non local connectivity of $J_{c^*}$ (see [40] for a detailed proof of this fact).

\[ \square \]

4.5 Proof of Theorem 5

We finally prove Theorem 5.

Let’s remind its conditions (rephrasing them a bit):

(E1) there is $\alpha > 0$, such that, for all $m$ large enough,

$$|t_{m+1}| < \exp(-\alpha q_0 \ldots q_{m-1}).$$

(22)

(Y1) There is $M > 0$, such that, for all $m$ large enough,

$$|t_m| q_0 \ldots q_{m-1} < M.$$  

(23)

(S1) for some $k \geq 0$,

$$\sum_{m=k}^{\infty} \frac{u_{k,m}}{q_m (1 - u_{k,m})} H(u_{k,m+1}) < \infty,$$

(24)

where

$$u_{k,m} = \theta_{k,m}^{1/q_m}, \quad \theta_{k,m} = C |t_{m+1}| \max\{q_k \ldots q_m, \frac{4^{q_k \ldots q_{m-1}}}{q_k \ldots q_{m-1} q_m}\},$$

(25)

for some universal $C > 0$.

It is enough to find a tail $\{t_{k_0}, t_{k_0+1}, \ldots\}$ of the sequence $\{t_m\}_{m=0}^{\infty}$, which satisfies all the conditions of Lemma 4.2 with $n = 1$. Indeed, assume this is the case. It means the following. Let’s start with the 1-hyperbolic component $W_6$ (the main cardioid) and the tail $\{t_{k_0}, t_{k_0+1}, \ldots\}$ in place of $W$ and $\{t_0, t_1, \ldots\}$. Then we get a sequence of hyperbolic components $W_{0,k_0,0}^{k_0,m}, m \geq k_0$, where $W_{0,k_0,0}^{k_0,0} = W_0$ and, for $m > k_0$, the component $W_{0,k_0,0}^{k_0,m}$ touches $W_{0,k_0,0}^{k_0,m-1}$ at the point $c_{k_0,m-1}$ with internal argument $t_{m-1}$. Then by Lemma 4.2 the sequence of parameters $c_{k_0,m}$ converges, as $m \to \infty$, to some $c_{k_0,*}$, and the Julia set $J^{c_{k_0,*}}$ is not locally connected. By a
well known straightening procedure, see [7] and the proof of Theorem 4, it implies the conclusions (1)-(2) of Theorem 5.

Now we find \( k_0 \). By (Y1), \( t_m \to 0 \), in particular, \( q_m \to \infty \). Given \( k \geq 0 \), denote \( n_{k,k} = 1 \), \( n_{k,m} = q_k \ldots q_{m-1} \) for \( m > k \), the relative period.

Denote, for \( m \geq k \),

\[
\tilde{d}_{k,m} = r(t_m, n_{k,m}) = 1/2 \min \left\{ \frac{1}{2n_m q_m^2}, \text{dist}(2\pi it_m, \partial \Omega_{n_{k,m}}) \right\},
\]

and, as before,

\[
d_m = \frac{320}{\beta q_m^2} |t_{m+1}|.
\]

Using Proposition 1 with the constant \( K \), we check that for all \( k \) large enough and \( m \geq k \), \( \theta_{k,m} = \frac{200\beta d_m}{\tilde{d}_{k,m}} \leq \theta_{k,m} \), where

\[
\theta_{k,m} = C|t_{m+1}| \max \{q_k \ldots q_m, \frac{4q_k \ldots q_{m-1}}{q_k \ldots q_{m-1} q_m} \},
\]

and

\[
C = 6400 \max \{2, K\}
\]
is the absolute constant.

By (E1)-(Y1), \( \theta_{k,m} \to 0 \) as \( k \to \infty \) uniformly in \( m \geq k \). In particular, conditions (0) and (Y) of Lemma 4,2 are satisfied, for the tail \( \{t_k, \ldots, t_m, \ldots\} \) for every \( k \) large enough.

Assume that the condition (S1) also holds, for some fixed \( k \) large enough. Then we show that this implies that the last condition (S) of Lemma 4,2 is satisfied, too, for the tail \( \{t_{k_0}, \ldots, t_m, \ldots\} \), for some \( k_0 > k \).

Indeed, \( u_{k,m} = \theta_{k,m}^{1/q_m} \) is decreasing in \( k \). Therefore, by (S1), for every \( k_0 \) large enough,

\[
\sum_{m=k_0}^{\infty} \frac{u_{k_0,m}}{q_m (1 - u_{k_0,m})} H(u_{k_0,m+1}) < L/2.
\]

Thus to satisfy the condition (S) of Theorem 4,2 it is enough to check that \( |t_{k_0}| H(u_{k_0,k_0}) \) tends to zero as \( k_0 \) tends to \( \infty \). Note that \( u_{k,m} \) decreases as \( k \) increases. Hence, it follows from (S1), that

\[
\frac{u_{k_0-1,k_0-1}}{q_{k_0-1} (1 - u_{k_0-1,k_0-1})} H(u_{k_0,k_0}) \leq \frac{u_{k_0,k_0-1}}{q_{k_0-1} (1 - u_{k_0,k_0-1})} H(u_{k_0,k_0}) \to 0
\]
as \( k_0 \to \infty \). On the other hand, using (Y1) it is easy to see that

\[
\frac{|t_{k_0}|}{q_{k_0-1} (1 - u_{k_0-1,k_0-1})} \leq \frac{|t_{k_0}|}{u_{k_0-1,k_0-1} (1 - u_{k_0-1,k_0-1})} < \frac{|t_{k_0}|}{u_{k_0-1,k_0-1} (1 - u_{k_0-1,k_0-1})}
\]
is bounded as \( k_0 \to \infty \). This completes the proof of Theorem 5.
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