ETERNAL SOLUTIONS TO THE RICCI FLOW ON $\mathbb{R}^2$

P. DASKALOPOULOS* AND N. SESUM

Abstract. We provide the classification of eternal (or ancient) solutions of the two-dimensional Ricci flow, which is equivalent to the fast diffusion equation $\frac{\partial u}{\partial t} = \Delta \log u$ on $\mathbb{R}^2 \times \mathbb{R}$. We show that, under the necessary assumption that for every $t \in \mathbb{R}$, the solution $u(\cdot, t)$ defines a complete metric of bounded curvature and bounded width, $u$ is a gradient soliton of the form $U(x, t) = \frac{2}{\beta (|x - x_0|^2 + \delta t^2)}$, for some $x_0 \in \mathbb{R}^2$ and some constants $\beta > 0$ and $\delta > 0$.

1. Introduction

We consider eternal solutions of the logarithmic fast diffusion equation

$$(1.1) \quad \frac{\partial u}{\partial t} = \Delta \log u \quad \text{on } \mathbb{R}^2 \times \mathbb{R}.$$ 

This equation represents the evolution of the conformally flat metric $g_{ij} = u I_{ij}$ under the Ricci Flow

$$\frac{\partial g_{ij}}{\partial t} = -2 R_{ij}.$$ 

The equivalence follows from the observation that the metric $g_{ij} = u I_{ij}$ has scalar curvature $R = -(\Delta \log u)/u$ and in two dimensions $R_{ij} = \frac{1}{4} R g_{ij}$.

Equation (1.1) arises also in physical applications, as a model for long Van-der-Wals interactions in thin films of a fluid spreading on a solid surface, if certain nonlinear fourth order effects are neglected, see [11, 2, 3].

Our goal in this paper is to provide the classification of eternal solutions of equation (1.1) under the assumption that for every $t \in \mathbb{R}$, the solution $u(\cdot, t)$ defines a complete metric of bounded curvature and bounded width. This result is essential in establishing the type II collapsing of complete (maximal) solutions of the Ricci flow (1.1) on $\mathbb{R}^2 \times [0, T)$ with finite area $\int_{\mathbb{R}^2} u(x, t) dx < \infty$ (c.f. in [9] for the result in the radially symmetric case). For an extensive list of results on the Cauchy problem $u_t = \Delta \log u$ on $\mathbb{R}^2 \times [0, T)$, $u(x, 0) = f(x)$ we refer the reader to [8], [12], [19] and [22].

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In [10] we introduced the width $w$ of the metric $g = u I_{ij}$. Let $F : \mathbb{R}^2 \to [0, \infty)$ denote a proper function $F$, such that $F^{-1}(a)$ is compact for every $a \in [0, \infty)$. The width of $F$ is defined to be the supremum of the lengths of the level curves of $F$, namely $w(F) = \sup_c L\{F = c\}$. The width $w$ of the metric $g$ is defined to be the infimum

$$w(g) = \inf_F w(F).$$

We will assume, throughout this paper, that $u$ is smooth, strictly positive and satisfies the following conditions:

The width of the metric $g(t) = u(\cdot, t) I_{ij}$ is finite, namely

\begin{equation}
(1.2) \quad w(g(t)) < \infty, \quad \forall t \in \mathbb{R}.
\end{equation}

The scalar curvature $R$ satisfies the $L^\infty$-bound

\begin{equation}
(1.3) \quad \|R(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} < \infty, \quad \forall t \in \mathbb{R}.
\end{equation}

Our goal is to prove the following classification result.

**Theorem 1.1.** Assume that $u$ is a positive smooth eternal solution of equation (1.1) which defines a complete metric and satisfies conditions (1.2) - (1.3). Then, $u$ is a gradient soliton of the form

\begin{equation}
U(x, t) = \frac{2}{\beta (|x - x_0|^2 + \delta e^{2\beta t})}
\end{equation}

for some $x_0 \in \mathbb{R}^2$ and some constants $\beta > 0$ and $\delta > 0$.

Under the additional assumptions that the scalar curvature $R$ is globally bounded on $\mathbb{R}^2 \times \mathbb{R}$ and assumes its maximum at an interior point $(x_0, t_0)$, with $-\infty < t_0 < +\infty$, i.e., $R(x_0, t_0) = \max_{(x,t) \in \mathbb{R}^2 \times \mathbb{R}} R(x, t)$, Theorem 1.1 follows from the result of R. Hamilton on eternal solutions of the Ricci Flow in [15]. However, since in general $\partial R/\partial t \geq 0$, without this rather restrictive assumption on the maximum curvature, Hamilton’s result does not apply.

Before we begin with the proof of Theorem 1.1, let us give a few remarks.

**Remarks:**

(i) The bounded width assumption (1.2) is necessary. If this condition is not satisfied, then (1.1) admits other solutions, in particular the flat (constant) solutions.
(ii) It is shown in [10] that maximal solutions \( u \) of the initial value problem \( u_t = \Delta \log u \) on \( \mathbb{R}^2 \times [0, T) \), \( u(x, 0) = f(x) \) which vanish at time \( T < \infty \) satisfy the width bound \( c(T-t) \leq w(g(t)) \leq C(T-t) \) and the maximum curvature bound \( c(T-t)^{-2} \leq R_{\text{max}}(t) \leq C(T-t)^{-2} \) for some constants \( c > 0 \) and \( C < \infty \), independent of \( t \). Hence, one may rescale \( u \) near \( t \to T \) and pass to the limit to obtain an eternal solution of equation (1.1) which satisfies the bounds (1.2) and (1.3) (c.f. in [9] for the radially symmetric case). Theorem 1.1 provides then a classification of the limiting solutions.

(iii) Since \( u \) is strictly positive at all \( t < \infty \), it follows that \( u(\cdot, t) \) must have infinite area, i.e.,

\[
\int_{\mathbb{R}^2} u(x, t) \, dx = +\infty, \quad \forall t \in \mathbb{R}.
\]

Otherwise, if \( \int_{\mathbb{R}^2} u(x, t) \, dx < \infty \), for some \( t < \infty \), then by the results in [8] the solution \( u \) must vanish at time \( t + T \), with \( T = (1/4\pi) \int_{\mathbb{R}^2} u(x, t) \, dx \), or before.

(iv) The proof of Theorem 1.1 only uses that actually \( u \) is an ancient solution of equation \( u_t = \Delta \log u \) on \( \mathbb{R}^2 \times (-\infty, T) \), for some \( T < \infty \), such that \( \int_{\mathbb{R}^2} u(x, t) \, dx = \infty \).

(v) Any eternal solution of equation (1.1) satisfies \( u_t \geq 0 \). This is an immediate consequence of the Aronson-Bénilan inequality (or the maximum principle on \( R = -u_t/u \)), which in the case of a solution on \( \mathbb{R}^2 \times [\tau, t) \) states as \( u_t \leq u/(t-\tau) \). Letting, \( \tau \to -\infty \), we obtain for an eternal solution the bound \( u_t \leq 0 \).

(vi) Any eternal solution of equation (1.1) satisfies \( R > 0 \). By the previous remark, \( R = -u_t/u \geq 0 \). Since, \( R \) evolves by \( R_t = \Delta_g R + R^2 \) the strong maximum principle guarantees that \( R > 0 \) or \( R \equiv 0 \) at all times. Solutions with \( R \equiv 0 \) (flat) violate condition (1.2). Hence, \( R > 0 \) on \( \mathbb{R}^2 \times \mathbb{R} \).

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2. A PRIORI ESTIMATES

We will establish in this section the asymptotic behavior, as \(|x| \to \infty\), for any eternal solution of equation (1.1) which satisfies the conditions (1.2)-(1.3). We will show that there exists constants \(c(t) > 0\) and \(C(t) < \infty\) such that

\[
(2.1) \quad c(t) |x|^{-2} \leq u(x, t) \leq C(t) |x|^{-2}, \quad t \in \mathbb{R}.
\]

This bound is crucial in the proof of Theorem 1.1.

We begin with the following lower bound which is a consequence of the results in [22].

**Proposition 2.1.** Assume that \(u\) is a positive smooth eternal solution of equation (1.1) which defines a complete metric and satisfies condition (1.3). Then,

\[
(2.2) \quad \frac{1}{u(x, t)} \leq O \left( r^2 \log^2 r \right), \quad \text{as } r = |x| \to \infty, \; \forall t \in \mathbb{R}.
\]

**Proof.** For \(t_0 \in \mathbb{R}\) fixed, let \(\bar{u}\) denote the maximal solution of the Cauchy problem

\[
(2.3) \quad \begin{cases}
\bar{u}_t = \Delta \log \bar{u}, & \text{on } \mathbb{R}^2 \times (0, \infty) \\
\bar{u}(x, 0) = u(x, t_0) & x \in \mathbb{R}^2.
\end{cases}
\]

It follows by the results of Rodriguez, Vazquez and Esteban in [12] that \(u\) satisfies the growth condition

\[
(2.4) \quad \frac{1}{\bar{u}(x, t)} \leq O \left( \frac{r^2 \log^2 r}{2t} \right), \quad \text{as } r = |x| \to \infty, \; \forall t > 0.
\]

In particular, \(\bar{u}\) defines a complete metric. Let us denote by \(\bar{R}\) the curvature of the metric \(\bar{g}(t) = \bar{u} I_{ij}\).

**Claim 2.2.** There exists \(\tau > 0\) for which \(\sup_{\mathbb{R}^2} |\bar{R}(\cdot, t)| \leq C(\tau)\) for \(t \in [0, \tau]\).

> From the claim the proof of the Proposition readily follows by the uniqueness result of Chen and Zhu ([6]; see also [22]). Indeed, since both \(u\) and \(\bar{u}\) define complete metrics with bounded curvature, the uniqueness result in [6] implies that \(\bar{u}(x, t) = u(x, t + t_0)\), for \(t \in [0, \tau]\). Hence, \(u\) satisfies (2.4) which readily implies (2.2), since \(u\) is decreasing in time.

**Proof of Claim 2.2.** Since \(\bar{g}(0) = g(t_0)\) and

\[
0 < \bar{R}(\cdot, 0) = R(\cdot, t_0) \leq C_0
\]
the classical result of Klingenberg (see [14]) implies the injectivity radius bound
\[ r_0 = \text{injrad}(\mathbb{R}^2, \bar{g}(0)) \geq \frac{\pi}{C_0}. \]
Moreover,
\begin{equation}
(2.5) \quad \text{Vol}_{\bar{g}(0)}B_{\bar{g}(0)}(x, r) \geq V_{C_0}(r),
\end{equation}
where \( V_{C_0}(r) \) is the volume of a ball of radius \( r \) in a space form of constant sectional curvature \( C_0 \) (see [20] for (2.5)). We will prove the desired curvature bound using (2.5) and Perelman’s pseudolocality theorem (Theorem 10.3 in [21]). Let \( \epsilon, \delta > 0 \) be as in the pseudolocality theorem. Choose \( r_1 < r_0 \) such that for all \( r \leq r_1 \), we have
\[ V_{C_0}(r) \geq (1-\delta)r^2. \]
Since also \( \sup_{\mathbb{R}^2} |\bar{R}(\cdot, 0)| \leq C_0 \), the pseudolocality theorem implies the bound
\[ |\bar{R}(x, t)| \leq (\epsilon r_1)^{-2} \quad \text{whenever} \quad 0 \leq t \leq (\epsilon r_1)^2, \quad \text{dist}_t(x, x_0) < \epsilon r_1. \]
Since the previous estimate does not depend on \( x \), we obtain the uniform bound
\[ \sup_{\mathbb{R}^2} |\bar{R}(\cdot, t)| \leq (\epsilon r_1)^{-2} \quad \forall t \in [0, (\epsilon r_1)^2] \]
finishing the proof of the claim. \( \square \)

We will next perform the cylindrical change of coordinates, setting
\begin{equation}
(2.6) \quad v(s, \theta, t) = r^2 u(r, \theta, t), \quad s = \log r
\end{equation}
where \( (r, \theta) \) denote polar coordinates. It is then easy to see that the function \( v \) satisfies the equation
\begin{equation}
(2.7) \quad v_t = (\log v)_{ss} + (\log v)_{\theta \theta}, \quad \text{for} \quad (s, \theta, t) \in C_\infty \times \mathbb{R}
\end{equation}
with \( C_\infty \) denoting the infinite cylinder \( C_\infty = \mathbb{R} \times [0, 2\pi] \). Notice that the nonnegative curvature condition \( R \geq 0 \) implies that
\begin{equation}
(2.8) \quad \Delta_s \log v := (\log v)_{ss} + (\log v)_{\theta \theta} \leq 0
\end{equation}
namely that \( \log v \) is superharmonic in the cylindrical \( (s, \theta) \) coordinates. Estimate (2.1) is equivalent to:

**Lemma 2.3.** For every \( t \in \mathbb{R} \), there exist constants \( c(t) > 0 \) and \( C(t) < \infty \) such that
\begin{equation}
(2.9) \quad c(t) \leq v(s, \theta, t) \leq C(t), \quad (s, \theta) \in C_\infty.
\end{equation}

The proof of Lemma 2.3 will be done in several steps. We will first establish the bound from below which only uses that the curvature \( R \geq 0 \) and that the metric is complete.
Proposition 2.4. If \( u(x,t) \) is a maximal solution of (1.1) that defines a metric of positive curvature, then for every \( t \in \mathbb{R} \), there exists a constant \( c(t) > 0 \) such that

\[
(2.10) \quad v(s, \theta, t) \geq c(t), \quad (s, \theta) \in C. 
\]

Proof. Fix a \( t \in \mathbb{R} \). We will show that \( (\log v)^- = \max(-\log v, 0) \) is bounded above. We begin by observing that

\[
\Delta_c (\log v) = (\log v)_{ss} + (\log v)_{\theta\theta} \geq 0
\]
since \( \Delta_c \log v = -Rv \leq 0 \). Hence, setting

\[
V^-(s, t) = \int_0^{2\pi} (\log v)^-(s, \theta, t) d\theta
\]
the function \( V^- \) satisfies \( V^-_{ss} \geq 0 \), i.e., \( V^- \) is increasing in \( s \). It follows that the limit \( \gamma = \lim_{s \to \infty} V^-_s(s, t) \in [0, \infty] \) exists. If \( \gamma > 0 \), then \( V^-_s(s, t) > \gamma_1 s \), with \( \gamma_1 = \gamma/2 \) for \( s \geq s_0 \) sufficiently large, which contradicts the pointwise bound (2.2), which when expressed in terms of \( v \) gives \( v(s, t) \leq c(t)/s^2 \), for \( s \geq s_0 \) sufficiently large. We conclude that \( \gamma = 0 \). Since \( V^- \) is increasing, this implies that \( V^- \leq 0 \), for all \( s \geq s_0 \), i.e. \( V^- \) is decreasing in \( s \) and therefore bounded above.

We will now derive a pointwise bound on \( (\log v)^- \). Fix a point \((\bar{s}, \bar{\theta})\) in cylindrical coordinates, with \( \bar{s} \geq s_0 + 2\pi \) and let \( B_{2\pi} = \{ (s, \theta) : |s - \bar{s}|^2 + |\theta - \bar{\theta}|^2 \leq (2\pi)^2 \} \).

By the mean value inequality for sub-harmonic functions, we obtain

\[
(\log v)^-(\bar{s}, \bar{\theta}, t) \leq \frac{1}{|B_{2\pi}|} \int_{B_{2\pi}} (\log v)^-(s, \theta, t) ds d\theta.
\]

Since \( B_{2\pi} \subset Q = \{ |s - \bar{s}| \leq 2\pi, \theta \in [-4\pi, 4\pi] \} \), we conclude the bound

\[
(\log v)^-(\bar{s}, \bar{\theta}, t) \leq C \int_Q (\log v)^-(s, \theta, t) ds d\theta \leq 4C \int_{s-2\pi}^{s+2\pi} V^-(s, t) ds \leq \hat{C}
\]
finishing the proof. \(\square\)

The estimate from above on \( v \) will be based on the following integral bound.

Proposition 2.5. Under the assumptions of Theorem 1.1, for every \( t \in \mathbb{R} \), we have

\[
(2.11) \quad \sup_{s \geq 0} \int_0^{2\pi} (\log v(s, \theta, t))^+ d\theta < \infty.
\]

Proof. Fix \( t \in \mathbb{R} \) and define

\[
V(s, t) = \int_0^{2\pi} \log v(s, \theta, t) d\theta.
\]
Since $\Delta_c \log v \leq 0$, $V$ satisfies $V_{ss} \leq 0$, i.e. $V_s$ decreases in $s$. Set $\gamma = \lim_{s \to \infty} V_s \in [-\infty, \infty)$.

**Claim 2.6.** We have $\gamma = 0$, i.e. $V$ is increasing in $s$, and $\lim_{s \to \infty} V(s, t) < \infty$.

**Proof of Claim:** We consider the following two cases:

**Case 1:** $\gamma \geq 0$. Then since $V_s$ decreases in $s$, we have $V_s \geq 0$ for all $s$, which implies that there is $\beta = \lim_{s \to \infty} V(s, t) \in (-\infty, \infty]$. If $\gamma > 0$, there is $s_0$ such that for $s \geq s_0$,

$$
\int_0^{2\pi} \log v(s, \theta, t) d\theta \geq \gamma_1 s
$$

where $\gamma_1 = \gamma / 2$.

(1a) If $\beta < \infty$, then for $s >> 1$,

$$
V(s, t_0) \leq \beta,
$$

which contradicts (2.12) for big $s$, unless $\gamma = 0$.

(1b) If $\beta = \infty$ and $\gamma > 0$ we will derive a contradiction using the boundness of the width. The function $\log v$ satisfies $\Delta_c \log v = -R v \leq 0$ that is, $\log v$ is the superharmonic function. Fix a point $(\bar{s}, \bar{\theta})$ in cylindrical coordinates, with $\bar{s} \geq s_0$ and let $B_{2\pi} = \{(s, \theta) : |s - \bar{s}|^2 + |\theta - \bar{\theta}|^2 \leq (2\pi)^2\}$. By the mean value inequality for superharmonic functions, we obtain

$$
\log v(\bar{s}, \bar{\theta}, t) \geq \frac{1}{|B_{2\pi}|} \int_{B_{2\pi}} \log v(s, \theta, t) ds d\theta.
$$

Expressing $\log v = (\log v)^+ - (\log v)^-$, we observe that

$$(\log v)^-(s, \theta, t) \leq C(t) + 2 \log s, \quad \text{on } B_{2\pi}$$

from the bound (2.12). Since $Q = \{|s - \bar{s}| \leq 1, |\theta - \bar{\theta}| \leq \pi\}$ is contained in $B_{2\pi}$, we obtain

$$
\int_{B_{2\pi}} (\log v)^+ \geq \int_{s-1}^{\bar{s}+1} \int_{\theta-\pi}^{\theta+\pi} (\log v)^+ = \int_{s-1}^{\bar{s}+1} \int_0^{2\pi} (\log v)^+ \\
\geq V(\bar{s}, t) - C(t) + 2 \log s.
$$

Combining the above with (2.12) yields the bound

$$
(\log v)^+(\bar{s}, \bar{\theta}, t) \geq \gamma_1 s - 2C(t) - 4 \log s \geq \bar{\gamma} s
$$

for $\bar{s} >> 1$ and $\bar{\gamma} < \gamma_1$. We conclude that

$$
v(\bar{s}, \bar{\theta}, t) \geq e^{\bar{\gamma} \bar{s}} \to \infty, \quad \text{as } \bar{s} \to \infty.
$$
We will next show that \( \bar{w}(\bar{g}(t_0)) < \infty \). Indeed, let \( F : \mathbb{R}^2 \to [0, \infty) \) be a proper function on the plane. Denoting by \( L_g \{ F = c \} \) the length of the \( c \)-level curve \( \sigma_c \) of \( F \), measured with respect to metric \( g(t) = u(t, t) I_{ij} \), and using the bound \( u(x, t) |x|^2 = v(\log |x|, \theta, t_0) \geq e^{\bar{\gamma} \log |x|} \), we obtain

\[
L_g(\sigma_c) = \int_{\sigma_c} \sqrt{u} d\sigma_c \\
\geq e^{\min_{x \in \sigma_c} (\bar{\gamma}/2 - 1) \log |x|} L_{\text{eucl}}(\sigma_c).
\]

If \( \bar{\gamma} \geq 2 \), then denoting by \( |x_c| = \min_{x \in \sigma_c} |x| \) we have

\[
L_g(\sigma_c) \geq e^{(\bar{\gamma}/2 - 1) \log |x_c|} L_{\text{eucl}}(\sigma_c) \geq e^{(\bar{\gamma}/2 - 1) \log |x_c|} 2\pi |x_c|
\]

where we have used the fact that the euclidean circle centred at the origin of radius \( |x_c| \) is contained in the region bounded by the curve \( \sigma_c \).

If \( 0 < \bar{\gamma} < 2 \), then \( \alpha = 1 - \bar{\gamma}/2 > 0 \) and since we may assume that the origin is contained in the interior of the region bounded by the level curve \( \sigma_c \), denoting by \( |x_c| = \max_{x \in \sigma_c} |x| \), we obtain

\[
L_g(\sigma_c) \geq e^{-\alpha \log |x_c|} L_{\text{eucl}}(\sigma_c) \\
\geq \frac{1}{|x_c|^{\alpha}} |x_c| \\
= |x_c|^{1-\alpha} \to \infty \quad \text{as} \quad c \to \infty.
\]

Here we have used the fact that

\[
L_{\text{eucl}}(\sigma_c) \geq \text{diam}_{\text{eucl}}(\text{Int} \sigma_c) \geq \max_{x \in \sigma_c} |x| = |x_c|.
\]

We conclude that

\[
\sup_c L_g \{ F = c \} \geq M
\]

for any proper function \( F \), which implies that \( w(g(t)) \geq M \), contradicting our width bound \( (1.2) \).

(1c) If \( \beta = \infty \) and \( \gamma = 0 \), we can argue similarly as in (1b), with the only difference that now \( V(s, t) \) increases in \( s \) and \( V(s, t) \gg M \) for \( s \gg 1 \), where \( M \) is an arbitrarily large constant. The mean value property applied to \( \log v \) now shows that \( \log v \) can be made arbitrarily large for \( s \gg 1 \). The rest of the argument is the same as in (1b).
So far we have proven that if $\gamma \geq 0$, the only possibility is that $\gamma = 0$ and $\beta < \infty$.

**Case 2:** $\gamma < 0$. Then, there is $s_0$ such that for $s \geq s_0$,

\[(2.15) \quad \int_0^{2\pi} \log v(s, \theta, t) d\theta < -|\gamma|s.\]

On the other hand, by (2.2) we have

\[V(s, \theta, t) \geq C(t) s^2, \quad \text{for } s \gg 1\]

which implies

\[-|\gamma|s > \int_0^{2\pi} \log v d\theta \geq 2\pi \log(C(t)) - 4\pi \log s = \tilde{C}(t) - 4\pi \log s,\]

which is not possible for large $s$.

We have shown that the only possibility is that $\gamma = 0$ (which means $V(s, t)$ is increasing in $s$) and $\beta < \infty$. This finishes the proof of the Claim.

We can now finish the proof of the Proposition. Expressing

\[\int_0^{2\pi} (\log v)^+(s, \theta, t) d\theta = V(s, t) + \int_0^{2\pi} (\log v)^-(s, \theta, t) d\theta\]

and using that $V(s, t) \leq \beta$ (as shown in Claim 2.6) together with Proposition 2.4, we readily obtain

\[\int_0^{2\pi} (\log v)^+(s, \theta, t) d\theta \leq \beta + 2\pi \delta(t) < \infty\]

as desired. \hfill $\square$

To prove Lemma 2.3, it only remains to show that $\|v(\cdot, t)\|_{L^\infty} < \infty$. The proof of this bound will use the ideas of Brezis and Merle (4), including the following result which we state for the reader’s convenience.

**Theorem 2.7** (Brezis-Merle). Assume $\Omega \subset \mathbb{R}^2$ is a bounded domain and let $u$ be a solution of

\[(2.16) \quad \begin{cases} -\Delta u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}\]

with $f \in L^1(\Omega)$. Then for every $\delta \in (0, 4\pi)$ we have

\[\int_{\Omega} e^{-\frac{(|x-y|^2+|u(x)|)}{4\pi\delta}} dx \leq \frac{4\pi^2}{\delta} (\text{diam} \Omega)^2.\]
Proof of Lemma 2.3. The proof follows the ideas of Brezis and Merle in [4]. Set $w = \log v$ so that

$$\Delta w = -R e^w$$

with $R$ denoting the scalar curvature.

Fix $\epsilon \in (0, 1)$. Since $\int_{\mathbb{R}^2} R e^w < \infty$, there is $s_0$ such that for all $s \geq s_0$

$$\int_{s-2\pi}^{s+2\pi} R e^w \, d\theta \, d\rho < \epsilon.$$

Fix $\bar{s} > s_0$ and set $B_{2\pi}(\bar{s}, \bar{\theta}) = \{(s, \theta) \mid (s-\bar{s})^2 + (\theta-\bar{\theta})^2 \leq (2\pi)^2 \} \subset \{(s, \theta) \mid |s-\bar{s}| \leq 2\pi, |\theta-\bar{\theta}| \leq 2\pi \} = Q(\bar{s}, \bar{\theta})$ so that

$$\int_{B_{2\pi}(\bar{s}, \bar{\theta})} R e^w < \epsilon.$$

We shall denote by $B_{2\pi}$ the ball $B_{2\pi}(\bar{s}, \bar{\theta})$ and by $Q$ the cube $Q(\bar{s}, \bar{\theta})$. Let $w_1$ solve problem (2.16) on $\Omega = B_{2\pi}$ with $f = R e^w \in L^1(B_{2\pi})$ and $\|f\|_{L^1(B_{2\pi})} < \epsilon$. By Theorem 2.7 there exists a constant $C > 0$ for which

$$\int_{B_{2\pi}} e^{\frac{4\pi - \delta}{\|f\|_{L^1}}} \leq \frac{C}{\delta}.$$

Taking $\delta = 4\pi - 1$ we obtain

(2.17) \hspace{1cm} \int_{B_{2\pi}} e^{\frac{4\pi - \delta}{\|f\|_{L^1}}} \leq C.

Combining (2.17) and Jensen’s inequality gives the estimate

(2.18) \hspace{1cm} \||w_1\|_{L^1(B_{2\pi})} \leq \tilde{C}(\epsilon).

The difference $w_2 = w - w_1$ satisfies $\Delta w_2 = 0$ on $B_{2\pi}$. Hence by the mean value inequality

(2.19) \hspace{1cm} \||w_2^+\|_{L^\infty(B_{2\pi})} \leq C \||w_2^+\|_{L^1(B_{2\pi})}.$

Since $w_2^+ \leq w^+ + |w_1|$ combining (2.19) and (2.18) yields

$$\||w_2^+\|_{L^1(B_{2\pi})} \leq C.$$

Expressing $Re^w = R e^{w_2 e^{w_1}}$ and observing that

- $R$ is bounded and (2.19) holds on $B_{2\pi}$
- $e^{w_1} \in L^2(B_{2\pi})$, where $\epsilon$ can be chosen small, so that $\frac{1}{\epsilon} > 1$, and
- $-\Delta w = Re^w \leq C e^{w_2}$
by standard elliptic estimates we obtain the bound
\begin{equation}
\|w^+\|_{L^\infty(B_{\pi/2})} \leq C \|w^+\|_{L^1(B_\pi)} + C \|e^{w_1}\|_{L^p(B_\pi)} \leq \tilde{C},
\end{equation}
where $p > 1$, finishing the proof of the Lemma. \qed

Fix $\tau \in \mathbb{R}$ and define the infinite cylinder
\[ Q_+(\tau) = \{(s, \theta, t) : s \geq 0, \theta \in [0, 2\pi], \tau - 1 \leq t \leq \tau \}. \]

By Lemma 2.9
\begin{equation}
0 < c(\tau) \leq v(s, \theta, t) \leq C(\tau) < \infty, \quad \text{on } Q_+(\tau).
\end{equation}

It follows that equation (2.21) in uniformly parabolic in $Q_+(\tau)$. Hence, the bounds (2.21) combined with classical derivative estimates for uniformly parabolic equations imply the following:

**Lemma 2.8.** Under the assumptions of Theorem 1.1 for every $\tau \in \mathbb{R}$, there exists a constant $C$ such that
\[ |v_s| \leq \frac{C}{s}, \quad |v_{ss}| \leq \frac{C}{s^2}, \quad |v_\theta| \leq C, \quad |v_{\theta s}| \leq \frac{C}{s}, \quad \text{on } Q_+(\tau). \]

We will next show that the curvature $R(x, t) = -(\Delta \log u(x, t))/u(x, t)$ tends to zero, as $|x| \to \infty$. We begin by reviewing the Harnack inequality satisfied by the curvature $R$, shown by R. Hamilton [18] and [16]. In the case of eternal solutions $u$ of (1.1) which define a complete metric, it states as
\begin{equation}
\frac{\partial \log R}{\partial t} \geq |D_\gamma \log R|^2.
\end{equation}

Since, $D_\gamma R = u^{-1} DR$, equivalently, this gives the inequality
\begin{equation}
\frac{\partial R}{\partial t} \geq \frac{|DR|^2}{Ru}.
\end{equation}

Let $(x_1, t_1), (x_2, t_2)$ be any two points in $\mathbb{R}^2 \times \mathbb{R}$, with $t_2 > t_1$. Integrating (2.22) along the path $x(t) = x_1 + \frac{t-t_1}{t_2-t_1} x_2$, also using the bound $u(x, t) \leq C(t_1)/|x|^2$, holding for all $t \in [t_1, t_2]$ by (1.2) and the fact that $u_t \leq 0$, we find the more standard parabolic inequality
\begin{equation}
R(x_2, t_2) \geq R(x_1, t_1) e^{-C \left(\frac{|x_2-x_1|^2}{|x_1|^2(t_2-t_1)}\right)}.
\end{equation}

One may now combine (2.24) with Lemmas 2.3 and 2.8 to conclude the following:

**Lemma 2.9.** Under the assumptions of Theorem 1.1 we have
\[ \lim_{|x| \to \infty} R(x, t) = 0, \quad \forall t \in \mathbb{R}. \]
Proof. For any \( t \in \mathbb{R} \) and \( r > 0 \), we denote by \( \bar{R}(r,t) \) the spherical average of \( R \), namely

\[
\bar{R}(r,t) = \frac{1}{|\partial B_r|} \int_{\partial B_r} R(x,t) \, d\sigma.
\]

We first claim that

\[
(2.25) \quad \lim_{r \to \infty} \bar{R}(r,t) = 0, \quad \forall t.
\]

Indeed, using the cylindrical coordinates, introduced previously, this claim is equivalent to showing that

\[
\lim_{s \to \infty} \int_0^{2\pi} - \left( \frac{\log v}{v} \right)_{ss}(s, \theta, t) + (\log v)_{\theta\theta}(s, \theta, t) \, d\theta = 0
\]

which is equivalent, using Lemma 2.3, to showing that

\[
\lim_{s \to \infty} \int_0^{2\pi} -(\log v)_{ss}(s, \theta, t) \, d\theta = 0.
\]

But this readily follows from Lemmas 2.3 and 2.8.

Fix \( t \in \mathbb{R} \). To prove that \( \lim_{|x| \to \infty} R(x,t) = 0 \), we use the Harnack inequality (2.24) to show that

\[
R(x,t) \leq C \inf_{y \in \partial B_r} R(y, t+1) \leq C \bar{R}(r, t+1), \quad \forall x \in \partial B_r, \forall t \in \mathbb{R}
\]

and use (2.25). \( \square \)

Combining the above with classical derivative estimates for linear strictly parabolic equations, gives the following.

**Lemma 2.10.** Under the assumptions of Theorem 1.1 the radial derivative \( R_r \) of the curvature satisfies

\[
\lim_{|x| \to \infty} |x| R_r(x,t) = 0, \quad \forall t \in \mathbb{R}.
\]

**Proof.** For any \( \rho > 1 \) we set \( \tilde{R}(x,t) = R(\rho x,t) \) and we compute from the evolution equation \( R_t = u^{-1} \Delta R + R^2 \) of \( R \), that

\[
\tilde{R}_t = (\rho^2 u)^{-1} \Delta \tilde{R} + \tilde{R}^2.
\]

For \( \tau < T \) consider the cylinder \( Q = \{(r,t) : 1/2 \leq |x| \leq 4, \tau - 1 \leq t \leq \tau \} \). From (2.1) we have \( 0 < c(\tau) \leq \rho^2 u(x,t) \leq C(\tau) < \infty \), for all \( x \in Q \), hence \( \tilde{R} \) satisfies a uniformly parabolic equation in \( Q \). Classical derivative estimates then imply that

\[
|(\tilde{R})_r(x,t)| \leq C \|\tilde{R}\|_{L^\infty(Q)}
\]
for all \( 1 \leq |x| \leq 2 \), \( \tau - 1/2 \leq t \leq \tau \), implying in particular that
\[
\rho |R(x, \tau)| \leq C \|R\|_{L^\infty(Q_\rho)}
\]
for all \( \rho \leq |x| \leq 2 \rho \), where \( Q_\rho = \{(x, t) : \rho/2 \leq |x| \leq 4\rho, \tau - 1 \leq t \leq \tau\} \). The proof now follows from Lemma 2.9. \( \square \)

3. Proof of Theorem 1.1

Most of the computations here are known in the case that \( u(dx_1^2 + dx_2^2) \) defines a metric on a compact surface (see for example in [7]). However, in the non-compact case an exact account of the boundary terms at infinity should be made.

We begin by integrating the Harnack inequality \( R_t \geq |DR|^2/Ru \) with respect to the measure \( d\mu = u dx \). Since the measure \( d\mu \) has infinite area, we will integrate over a fixed ball \( B_\rho \). At the end of the proof we will let \( \rho \to \infty \). Using also that \( R_t = u^{-1} \Delta R + R^2 \) we find
\[
\int_{B_\rho} \Delta R \, dx + \int_{B_\rho} R^2 u \, dx \geq \int_{B_\rho} \frac{|DR|^2}{R} \, dx
\]
and by Green’s Theorem we conclude
\[
(3.1) \quad \int_{B_\rho} \frac{|DR|^2}{R} \, dx - \int_{B_\rho} R^2 u \, dx \leq \int_{\partial B_\rho} \frac{\partial R}{\partial \nu} \, d\sigma.
\]

Next, following Chow ([7]), we consider the vector \( X = \nabla R + R \nabla f \), where \( f = -\log u \) is the potential function (defined up to a constant) of the scalar curvature, since it satisfies \( \Delta_g f = R \), with \( \Delta_g f = u^{-1} \Delta f \) denoting the Laplacian with respect to the conformal metric \( g = u(dx_1^2 + dy^2) \). As it was observed in [7] \( X \equiv 0 \) on Ricci solitons, i.e., Ricci solitons are gradient solitons in the direction of \( \nabla_g f \). A direct computation shows
\[
\int_{B_\rho} \frac{|X|^2}{R} \, dx = \int_{B_\rho} \frac{|DR|^2}{R} \, dx + 2 \int_{B_\rho} \nabla R \cdot \nabla f \, dx + \int_{B_\rho} R |Df|^2 \, dx.
\]
Integration by parts implies
\[
\int_{B_\rho} \nabla R \cdot \nabla f \, dx = -\int_{B_\rho} R \Delta f \, dx + \int_{\partial B_\rho} R \frac{\partial f}{\partial \nu} \, d\sigma = -\int_{B_\rho} R^2 u \, dx + \int_{\partial B_\rho} R \frac{\partial f}{\partial \nu} \, d\sigma
\]

since \( \Delta f = Ru \). Hence
\[
(3.2) \quad \int_{B_\rho} \frac{|X|^2}{R} \, dx = \int_{B_\rho} \frac{|DR|^2}{R} \, dx - 2 \int_{B_\rho} R^2 u \, dx
\]
\[
+ \int_{B_\rho} R |Df|^2 \, dx + 2 \int_{\partial B_\rho} R \frac{\partial f}{\partial \nu} \, d\sigma.
\]
Combining (3.1) and (3.2) we find that

\[ (3.3) \quad \int_{B_\rho} \frac{|X|^2}{R} \, dx \leq - \left( \int_{B_\rho} R^2 u \, dx - \int_{B_\rho} R |Df|^2 \, dx \right) + I_\rho = -M + I_\rho \]

where

\[ I_\rho = \int_{\partial B_\rho} \frac{\partial R}{\partial n} \, d\sigma + 2 \int_{\partial B_\rho} R \frac{\partial f}{\partial n} \, d\sigma. \]

Lemmas 2.8 - 2.10 readily imply that

\[ (3.4) \quad \lim_{\rho \to \infty} I_\rho = 0. \]

As in [7], we will show next that \( M \geq 0 \) and indeed a complete square which vanishes exactly on Ricci solitons. To this end, we define the matrix

\[ M_{ij} = D_{ij} f + D_i f D_j f - \frac{1}{2} |Df|^2 + Ru \]

with \( I_{ij} \) denoting the identity matrix. A direct computation shows that \( M_{ij} = \nabla_i \nabla_j f - \frac{1}{2} \Delta_g f \, g_{ij} \), with \( \nabla_i \) denoting covariant derivatives. It is well known that the Ricci solitons are characterized by the condition \( M_{ij} = 0 \), (see in [10]).

Claim:

\[ (3.5) \quad M := \int_{B_\rho} R^2 u \, dx - \int_{B_\rho} R |Df|^2 \, dx = 2 \int_{B_\rho} |M_{ij}|^2 \frac{1}{u} \, dx + J_\rho \]

where

\[ \lim_{\rho \to \infty} J_\rho = 0. \]

To prove the claim we first observe that since \( \Delta f = Ru \)

\[ \int_{B_\rho} R^2 u = \int_{B_\rho} \frac{(\Delta f)^2}{u} \, dx = \int_{B_\rho} D_{ij} f D_{ij} f \frac{1}{u} \, dx. \]

Integrating by parts and using again that \( \Delta f = Ru \), we find

\[ \int_{B_\rho} D_{ij} f D_{ij} f \frac{1}{u} \, dx = - \int_{B_\rho} D_{ij} f D_{ij} f \frac{1}{u} \, dx + \int_{B_\rho} \Delta f D_{ij} f \frac{1}{u} \, dx + \int_{\partial B_\rho} R \frac{\partial f}{\partial n} \, d\sigma. \]

Integrating by parts once more we find

\[ \int_{B_\rho} D_{ij} f D_{ij} f \frac{1}{u} \, dx = - \int_{B_\rho} |D_{ij} f|^2 \frac{1}{u} \, dx \]

\[ + \int_{B_\rho} D_{ij} f D_{ij} f \frac{Di u}{u^2} \, dx + \frac{1}{2} \int_{\partial B_\rho} \frac{\partial (|Df|^2)}{\partial n} \frac{1}{u} \, d\sigma \]

since

\[ \int_{\partial B_\rho} D_{ij} f D_{ij} f n_i \frac{1}{u} \, d\sigma = \frac{1}{2} \int_{\partial B_\rho} \frac{\partial (|Df|^2)}{\partial n} \frac{1}{u} \, d\sigma. \]
Combining the above and using that $Df = -u^{-1}Du$ and $\Delta f = Ru$ we conclude

$$\int_{B_{\rho}} R^2 u \, dx = \int_{B_{\rho}} |D_{ij} f|^2 \frac{1}{u} \, dx + \int_{B_{\rho}} D_{ij} f \, D_i f \, D_j f \frac{1}{u} \, dx - \int_{B_{\rho}} R |Df|^2 \, dx + J^1_{\rho}$$

where

$$J^1_{\rho} = \int_{\partial B_{\rho}} \frac{\partial f}{\partial n} \, d\sigma - \frac{1}{2} \int_{\partial B_{\rho}} \frac{\partial (|Df|^2)}{\partial n} \frac{1}{u} \, d\sigma.$$

Hence

$$\int_{B_{\rho}} R^2 u \, dx = \int_{B_{\rho}} |D_{ij} f|^2 \frac{1}{u} \, dx + \int_{B_{\rho}} D_{ij} f \, D_i f \, D_j f \frac{1}{u} \, dx - \int_{B_{\rho}} R |Df|^2 \, dx + J^1_{\rho}.$$

We will now integrate $|M_{ij}|^2$. A direct computation and $\Delta f = Ru$ imply

$$\int_{B_{\rho}} |M_{ij}|^2 \frac{1}{u} \, dx = \int_{B_{\rho}} |D_{ij} f|^2 \frac{1}{u} \, dx + 2 \int_{B_{\rho}} D_{ij} f \, D_i f \, D_j f \frac{1}{u} \, dx - \int_{B_{\rho}} R |Df|^2 \, dx + J^2_{\rho}.$$

Combining (3.6) and (3.8) we then find

$$M - 2 \int_{B_{\rho}} |M_{ij}|^2 \frac{1}{u} \, dx = - \int_{B_{\rho}} |D_{ij} f|^2 \frac{1}{u} \, dx - 3 \int_{B_{\rho}} D_{ij} f \, D_i f \, D_j f \frac{1}{u} \, dx$$

$$- \int_{B_{\rho}} |Df|^4 \frac{1}{u} \, dx + \int_{B_{\rho}} R^2 u \, dx + J^1_{\rho}.$$

Using (3.6) we then conclude that

$$M - 2 \int_{B_{\rho}} |M_{ij}|^2 \frac{1}{u} \, dx = - \int_{B_{\rho}} |D_{ij} f|^2 \frac{1}{u} \, dx$$

$$- \int_{B_{\rho}} |Df|^4 \frac{1}{u} \, dx - \int_{B_{\rho}} R |Df|^2 \, dx + J^2_{\rho}.$$

where

$$J^2_{\rho} = \int_{\partial B_{\rho}} \frac{\partial f}{\partial n} \, d\sigma - \int_{\partial B_{\rho}} \frac{\partial (|Df|^2)}{\partial n} \frac{1}{u} \, d\sigma.$$

We next observe that

$$2 \int_{B_{\rho}} D_{ij} f \, D_i f \, D_j f \frac{1}{u} \, dx = \int_{B_{\rho}} D_i (|Df|^2) \, D_i f \frac{1}{u}$$

and integrate by parts using once more that $\Delta f = Ru$ and that $D_i f = -u^{-1}D_i f$, to find

$$2 \int_{B_{\rho}} D_{ij} f \, D_i f \, D_j f \frac{1}{u} \, dx = - \int_{B_{\rho}} R |Df|^2 \, dx - \int_{B_{\rho}} |Df|^4 \frac{1}{u} \, dx + J^3_{\rho}$$

where

$$J^3_{\rho} = \lim_{\rho \to \infty} \int_{\partial B_{\rho}} |Df|^2 \frac{\partial f}{\partial n} \, d\sigma.$$
Combining the above we conclude that

$$M - 2 \int_{B_\rho} |M_{ij}|^2 \frac{1}{u} \, dx = J_\rho$$

with

$$J_\rho = \int_{\partial B_\rho} R \frac{\partial f}{\partial n} \, d\sigma - \int_{\partial B_\rho} \left( \frac{\partial(|Df|^2)}{\partial n} + |Df|^2 \frac{\partial f}{\partial n} \right) \frac{1}{u} \, d\sigma.$$

We will now show that \( \lim_{\rho \to \infty} J_\rho = 0 \). Clearly the first term tends to zero, because \( \int_{\partial B_\rho} |\partial f/\partial n| \, d\sigma \) is bounded Lemma 2.8, and \( R(x,t) \to 0 \), as \( |x| \to \infty \), by Lemma 2.9. It remains to show that

$$\lim_{\rho \to \infty} \int_{\partial B_\rho} \left( \frac{\partial(|Df|^2)}{\partial n} + |Df|^2 \frac{\partial f}{\partial n} \right) \frac{1}{u} \, d\sigma = 0.$$

We first observe that since \( f = -\log u \), we have

$$\left( \frac{\partial(|Df|^2)}{\partial n} + |Df|^2 \frac{\partial f}{\partial n} \right) \frac{1}{u} = \frac{\partial}{\partial n} \left( \frac{|D\log u|^2}{u} \right).$$

Expressing the last term in cylindrical coordinates, setting \( v(s, \theta, t) = r^2 u(r, \theta, t) \), with \( s = \log r \), we find

$$\int_{\partial B_\rho} \frac{\partial}{\partial n} \left( \frac{|D\log u|^2}{u} \right) \frac{1}{u} \, d\sigma = \int_0^{2\pi} \frac{\partial}{\partial s} \left( \frac{[2 - (\log v)_s]^2 + [(\log v)_t]^2}{v} \right) \, d\theta.$$

Further computation shows that

$$\frac{\partial}{\partial s} \left( \frac{[2 - (\log v)_s]^2 + [(\log v)_t]^2}{v} \right) = -2 \frac{2 - (\log v)_s}{v} (\log v)_s$$

$$+ 2 (\log v)_t (\log v)_t + \left( [2 - (\log v)_s]^2 + [(\log v)_t]^2 \right) (\log v)_s.$$

By Lemma 2.8, \( (\log v)_t \) is bounded as \( s \to \infty \), while \( (\log v)_s \), \( (\log v)_t \), and \( (\log v)_ss \) tend to zero, as \( s \to \infty \). Using also that \( v \) is bounded away from zero as \( s \to \infty \), we finally conclude that

$$\lim_{s \to \infty} \int_0^{2\pi} \frac{\partial}{\partial s} \left( \frac{[2 - (\log v)_s]^2 + [(\log v)_t]^2}{v} \right) \, d\theta = 0$$

implying (3.10) therefore finishing the proof of the claim (3.5).

We will now conclude the proof of the Theorem. From (3.3) and (3.5) it follows that

$$\int_{B_\rho} \frac{|X|^2}{R} \, dx + 2 \int_{B_\rho} |M_{ij}|^2 \frac{1}{u} \, dx \leq I_\rho + J_\rho$$

where both

$$\lim_{\rho \to \infty} I_\rho + J_\rho = 0.$$

This immediately gives that \( X \equiv 0 \) and \( M_{ij} \equiv 0 \) for all \( t \) showing that \( U \) is a gradient soliton. It has been shown by L.F. Wu [23] that there are only two types
of complete gradient solitons on $\mathbb{R}^2$, the standard flat metric ($R \equiv 0$) which is stationary, and the cigar solitons 1.1. The flat solitons violate condition 1.2. Hence, $u$ must be of the form 1.4, finishing the proof of the Theorem. □

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Department of Mathematics, Columbia University, New York, USA
E-mail address: pdaskalo@math.columbia.edu

Department of Mathematics, Columbia University, New York, USA
E-mail address: natasas@math.columbia.edu