On constant $Q$-curvature metrics with isolated singularities

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Abstract

In this paper we derive a refined asymptotic expansion, near an isolated singularity, for conformally flat metrics with constant positive $Q$-curvature and positive scalar curvature. The condition that the metric has constant $Q$-curvature forces the conformal factor to satisfy a fourth order nonlinear partial differential equation with critical Sobolev growth, whose leading term is the bilaplacian. We model our results on a similar asymptotic expansion for conformally flat, constant scalar curvature metrics proven by Korevaar, Mazzeo, Pacard, and Schoen. Along the way we analyze the linearization of the $Q$-curvature equation about the Delaunay metrics recently discovered by Frank and König, which may be of independent interest.

1 Introduction

Let $(M,g)$ be a Riemannian manifold of dimension $n \geq 5$. In this paper we investigate the behavior of a conformal metric $\tilde{g} = u^{\frac{4}{n-4}}g$ near an isolated singularity, subject to curvature conditions, namely constant and positive $Q$-curvature and positive scalar curvature.

To begin we fix some notation. Let $R_g$ and $\text{Ric}_g$ denote the scalar and Ricci curvature of $g$, and let $\Delta_g$ denote the Laplace-Beltrami operator. The (fourth-order) $Q$-curvature of $g$ is

\begin{equation}
Q_g = -\frac{1}{2(n-1)} \Delta_g R_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R_g^2 - \frac{2}{(n-2)^2} |\text{Ric}_g|^2.
\end{equation}

We can simplify this expression using the Schouten tensor

\begin{equation}
A_g = \frac{1}{n-2} \left( \text{Ric}_g - \frac{R_g g}{2(n-1)} \right), \quad J_g = \text{tr}_g(A_g) = \frac{R_g}{2(n-1)},
\end{equation}

so that (1) becomes

\begin{equation}
Q_g = -\Delta_g J_g - 2|A_g|^2 + \frac{n}{2} J_g^2.
\end{equation}

Associated to $Q_g$ we find the fourth order differential operator

\begin{equation}
P_g(u) = (-\Delta_g)^2 u + \text{div} \left( 4A_g(\nabla u, \cdot) - (n-2) J_g \nabla u \right) + \frac{n-4}{2} Q_g,
\end{equation}

which enjoys the transformation rule

\begin{equation}
\tilde{g} = u^{\frac{4}{n-4}}g \Rightarrow P_g(v) = u^{-\frac{n-4}{2}} P_g(\tilde{v}).
\end{equation}

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Substituting $v = 1$ into (5) we find
\[ \tilde{g} = u^{\frac{4}{n-2}} \Rightarrow Q_{\tilde{g}} = \frac{2}{n-4} u^{-\frac{n+4}{n-4}} P_g(u). \] (6)

S. Paneitz [22] first introduced the operator (4) and studied its transformation properties. Later T. Branson [3, 4] extended this operator to differential forms and studied a related sixth-order operator, as well as the $Q$-curvature. The reader can find summaries of the current understanding of $Q$-curvature in the survey articles [3], [8], and [13].

1.1 An aside on scalar curvature and the Yamabe problem

At this point we pause to discuss the Yamabe problem, which provides a guide for much of the investigation of the properties of $Q$-curvature. One can define the conformal Laplacian
\[ L_g(u) = -\Delta_g(u) + \frac{4(n-1)}{n-2} R_g u, \] (7)
which enjoys the transformation rule
\[ \tilde{g} = u^{\frac{4}{n-2}} g \Rightarrow L_{\tilde{g}}(v) = u^{-\frac{n+2}{n-2}} L_g(u v). \] (8)
Substituting $v = 1$ into (8) we obtain
\[ \tilde{g} = u^{\frac{4}{n-2}} g \Rightarrow R_{\tilde{g}} = \frac{n-2}{4(n-1)} u^{-\frac{n+2}{n-2}} L_g(u). \] (9)

This last transformation rule allows us to define the conformal invariant
\[ \mathcal{Y}(g) = \inf \left\{ \frac{\int_M R_{\tilde{g}} d\mu_{\tilde{g}}}{(\text{Vol}_{\tilde{g}}(M))^{\frac{n+2}{n}}} : \tilde{g} = u^{\frac{4}{n-2}} g \in [g] \right\} \] (10)
\[ = \inf \left\{ \frac{n-2}{4(n-1)} \frac{\int_M u L_g(u) d\mu_g}{\left( \int_M u^{\frac{2n}{n-2}} d\mu_g \right)^{\frac{n+2}{n}}} : u \in C^\infty(M), u > 0 \right\} \]

A straightforward computation shows that critical points of the functional
\[ \tilde{g} \in [g] \mapsto \frac{\int_M R_{\tilde{g}} d\mu_{\tilde{g}}}{(\text{Vol}_{\tilde{g}}(M))^{\frac{n+2}{n}}} \]
are precisely the constant scalar curvature metrics in the conformal class $[g]$.

In [27] Yamabe proposed finding a constant scalar curvature metric in a given conformal class $[g]$ through the family of related variational problems
\[ S_p([g]) = \inf \left\{ \frac{n-2}{4(n-1)} \frac{\int_M u^{p} L_g(u) d\mu_g}{\left( \int_M u^{\frac{2n}{n-2}} d\mu_g \right)^{\frac{n+2}{n}}} : u \in C^\infty(M), u > 0 \right\} \] (11)
for $p$ such that $1 < p < \frac{2n}{n-2}$. By Rellich’s compactness theorem the infimum $S_p([g])$ is a minimum, and realized by a smooth metric. Yamabe’s strategy was to find a constant scalar curvature metric as a limit of minimizers for $S_p([g])$ as $p \to \frac{2n}{n-2}$.

Trudinger [25] first pointed out the difficulties in extracting this limit, primarily that Rellich’s compactness fails exactly for $p = \frac{2n}{n-2}$. Later Aubin [1] resolved these difficulties in many cases, and Schoen [23] completed Yamabe’s original program. Schoen’s resolution of the Yamabe problem culminated the work of many people over 25 years, and continues to inspire new research.
One can see the lack of compactness directly in the case of the round metric $g_0$ on the sphere $S^n$. Using stereographic projection we can write the round metric as

$$g_0 = \frac{4}{(1 + |x|^2)^{2}} \delta = \left( \frac{1 + |x|^2}{2} \right)^{\frac{2-n}{2}} \delta = u^{\frac{4}{n+2}} \delta$$

(12)

where $\delta$ is the flat metric. Dilating the Euclidean coordinates by a factor of $\lambda > 0$ one finds the conformal factor

$$u_\lambda = \left( \frac{1 + \lambda^2 |x|^2}{2 \lambda} \right)^{\frac{2-n}{2}}$$

(13)

However, as $\lambda \to \infty$ we see that $u_\lambda \to 0$ outside of any fixed neighborhood of 0, while $u_\lambda(0) \to \infty$. Geometrically, this blow-up behavior concentrates the entirety of the sphere in a small neighborhood of the south pole, which corresponds to the origin in Euclidean coordinates, shrinking the complement of this neighborhood to be vanishingly small.

The blow-up described above motivates one to understand the asymptotics of constant scalar curvature metrics with isolated singularities. In the special case that the background metric $g$ is locally conformally flat one can (locally) write a constant scalar curvature metric in the conformal class $[g]$ as $u^{\frac{4}{n+2}} \delta$. A theorem of Schoen and Yau [24] implies the value of the scalar curvature must be a positive constant, which we normalize to be $n(n-1)$. In this case case (9) becomes

$$-\Delta u = \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}}$$

(14)

A computation shows that $u_\lambda$, defined in (13) solves (14), and indicates that one should study the possible asymptotic behavior of positive solutions of (14). In [6] Caffarelli, Gidas and Spruck proved that any positive solution of (14) in the punctured ball $B_1(0) \setminus \{0\}$ satisfies

$$u(x) = \bar{u}(|x|) (1 + O(|x|)), \quad \bar{u}(r) = \frac{1}{r^{n-1}|S^{n-1}|} \int_{|x|=r} u(x) d\sigma(x).$$

Later Korevaar, Mazzeo, Pacard and Schoen [15] refined this asymptotic expansion, obtaining the next term in the expansion and giving the terms in this expansion a more geometric interpretation.

### 1.2 Motivation for studying constant $Q$-curvature metrics with isolated singularities

Returning to our discussion of $Q$-curvature, we can follow the well-established model of scalar curvature and define the conformal invariant

$$\mathcal{Y}_Q^+(|g|) = \inf \left\{ \frac{\int_M Q g d\mu_g}{(\text{Vol}_g(M))^{\frac{2-n}{n}}} : \bar{g} = u^{\frac{4}{n+2}} g \in [g] \right\}$$

(15)

$$= \inf \left\{ \frac{2}{n-4} \left( \int_M u P_g(u) d\mu_g \right)^{\frac{2-n}{2}} : u \in C^\infty(M), u > 0 \right\}.$$ 

Once again, a straight-forward computation implies critical points of the functional

$$u \mapsto \frac{\int_M u P_g(u) d\mu_g}{\left( \int_M u^{\frac{2n}{n-2}} d\mu_g \right)^{\frac{n-2}{2}}}$$
give exactly the constant $Q$-curvature metrics within the conformal class $[g]$. In the case that this constant is positive, we normalize it to be $\frac{n(n^2-4)}{8}$, which is the value attained by the round metric on the sphere.

In contrast to the situation with scalar curvature, the existence and properties of minimizers and higher order critical points is poorly understood. However, the conformal class of the round metric on the sphere still provides an illustrative example. We once again write the round metric as a conformally flat metric using stereographic projection, so that

$$g_0 = \frac{4}{(1+|x|^2)^2} \delta \left( \left( \frac{1+|x|^2}{2} \right)^{\frac{4-n}{2}} \delta = U_1(x) \frac{4}{n-4} \delta. \right. \quad (16)$$

Once again we may apply a conformal dilation, which gives the solution

$$U_\lambda = \left( \frac{1+\lambda^2 |x|^2}{2\lambda} \right)^{\frac{n-4}{2}}$$

for any $\lambda > 0$.

We can seek metrics $g = u^{\frac{4}{n-4}} g_0$ in the conformal class of the round metric $g_0$ with constant $Q$-curvature $\frac{n(n^2-4)}{8}$. In stereographic coordinates (6) reduces to

$$\Delta^n u = \frac{n(n-4)(n^2-4)}{16} u^{\frac{n+4}{2}}. \quad (18)$$

By the above discussion $U_\lambda$ solves (18) for each $\lambda > 0$. In fact, a theorem of C. S. Lin [16] states that any positive solution of (18) must have the form $U_\lambda(x-x_0)$ for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$.

As before, we examine the behavior of $U_{\lambda,x_0} = U_\lambda(\cdot + x_0)$ as $\lambda \to \infty$, finding that $U_{\lambda,x_0} \to 0$ outside any fixed neighborhood of $x_0$, while it blows up at $x_0$. This behavior illustrates the natural lack of compactness in the infimum (13), and explains why we should study solutions of (18) with isolated singularities.

### 1.3 Main results

We concentrate on solutions of (18) in the unit ball $B_1(0)$ with an isolated singularity at the origin.

Recently Jin and Xiong [14] proved that if $u \in C^\infty(B_1(0) \setminus \{0\})$ is a positive solution of (18) which also satisfies $-\Delta u > 0$ then

$$u(x) = \bar{u}(|x|)(1 + O(|x|)), \quad \bar{u}(r) = \frac{1}{r^{n-1} |S^{n-1}|} \int_{|x|=r} u(x) d\sigma(x). \quad (19)$$

To understand our results we introduce some special solutions of (18). We change to cylindrical coordinates, letting $t = -\log |x|$, $\theta = x/|x|$, and

$$v : (0, \infty) \times S^{n-1} \to (0, \infty), \quad v(t, \theta) = e^{(\frac{n+4}{2})t} u(e^{-t} \theta). \quad (20)$$

Under this change of coordinates (18) becomes

$$\frac{n(n-4)(n^2-4)}{16} v^{\frac{n+4}{n-4}} = \frac{\partial^4 v}{\partial t^4} - \left( \frac{n(n-4)+8}{2} \right) \frac{\partial^2 v}{\partial t^2} + \frac{n^2(n-4)^2}{16} v + \Delta_\theta v + 2\Delta v \frac{\partial^2 v}{\partial t^2} - \frac{n(n-4)}{2} \Delta_\theta v, \quad (21)$$

where $\Delta_\theta$ is the Laplace-Beltrami operator on the round sphere $S^{n-1}$, and the condition $-\Delta u > 0$ becomes

$$-\ddot{v} + 2\dot{v} + \frac{n(n-4)}{4} v - \Delta_\theta v > 0. \quad (22)$$
In [16], Lin also proved that any global solution of (21) must be a function of $t$ alone, and so it must satisfy the ordinary differential equation (ODE)

$$\frac{n(n - 4)(n^2 - 4)}{16} v_{\epsilon}'' = \cdots - \left( \frac{n(n - 4) + 8}{2} \right) \dot{v} + \frac{n^2(n - 4)^2}{16} v. \quad (23)$$

One immediately sees two solutions of (23): the constant solution

$$v_{cyl} = \left( \frac{n(n - 4)}{n^2 - 4} \right)^{\frac{1}{8}}$$

and the spherical solution

$$u(x) = U_1(x) = \left( \frac{1 + |x|^2}{2} \right)^{\frac{n-4}{2}} \Leftrightarrow v_{sph} = (\cosh(\frac{1}{8}t))^\frac{1}{8}. \quad (24)$$

Frank and König [10] classified all global, positive solutions of (23). First they demonstrate the existence of a unique periodic solution $v_{\epsilon}$ of (23) attaining its minimum value of $\epsilon$ at $t = 0$ for each $\epsilon \in (0, v_{cyl})$. We call $v_{\epsilon}$ the Delaunay solution with necksize $\epsilon$. In the same paper they prove that each global, positive solution of (23) must either have the form $(\cosh(\cdot + T))^\frac{1}{8}$ or have the form $v_{\epsilon}(\cdot + T)$ for some $\epsilon \in (0, v_{cyl})$ and $T \in \mathbb{R}$.

We are now ready to state our main theorem.

**Theorem 1.** Let $v \in C^\infty((0, \infty) \times S^{n-1})$ be a positive solution of (21) which also satisfies (22). Then either $\limsup_{t \to \infty} v(t, \theta) = 0$ or there exists parameters $\epsilon \in (0, v_{cyl})$, $T \in \mathbb{R}$ and $a \in \mathbb{R}^n$, and positive constants $C$ and $\beta > 1$ such that

$$\left| v(t, \theta) - v_{\epsilon}(t + T) - e^{-\epsilon t}(\theta, a) \left( -\dot{v}_{\epsilon}(t + T) + \frac{n - 4}{2} v_{\epsilon}(t + T) \right) \right| \leq C e^{-\beta t}. \quad (26)$$

**Remark 1.** We will see below in (64) and (65) that one obtains

$$v_{\epsilon} + e^{-\epsilon}(\theta, a) \left( -\dot{v}_{\epsilon} + \frac{n - 4}{2} v_{\epsilon} \right)$$

by applying a certain translation to the Delaunay solution $v_{\epsilon}$.

**Remark 2.** We speculate that one can adapt our proof below to prove a further refinement of our estimate in Theorem 1, following the estimate of Han, Li and Li [22]. In (61) we define the increasing set of indicial roots

$$\Gamma_\epsilon = \{ \ldots, -\gamma_{\epsilon,2}, -\gamma_{\epsilon,1} = -1, 0, \gamma_{\epsilon,1} = 1, \gamma_{\epsilon,2}, \ldots \}, \quad \gamma_{\epsilon,j} \to \infty.$$

One should be able to prove an estimate of the form

$$\left| v(t, \theta) - v_{\epsilon}(t + T) - \sum_{i=1}^{m} \sum_{j=0}^{m-1} c_{ij}(t, \theta) t^j e^{-\gamma_{\epsilon,j} t} \right| \leq C t^m e^{-\gamma_{\epsilon,m+1}}$$

where the coefficient functions $c_{ij}(t, \theta)$ are bounded.

We recast Theorem 1 in geometric terms.

**Corollary 2.** Let $n \geq 5$ and let $g = u^{-\frac{n-4}{2}} \delta$ be a conformally flat metric on $B_1(0) \setminus \{0\}$ with positive scalar curvature and $Q_g = \frac{n(n^2 - 4)}{8}$. Then either $g$ extends to a smooth metric on $B_1(0)$ or there exist parameters $\epsilon \in (0, v_{cyl})$, $T \in \mathbb{R}$ and $a \in \mathbb{R}$ and $\beta > 1$ such that

$$u(x) = |x|^{-\frac{4}{n-4}} \left( v_{\epsilon}(-\log |x| + T) + \langle x, a \rangle \left( -\dot{v}_{\epsilon}(-\log |x| + T) + \frac{n - 4}{2} v_{\epsilon}(-\log |x| + T) \right) + O(|x|^\beta) \right)$$

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Proof. We have already seen that \( Q_g = \frac{n(n^2-4)}{8} \) is equivalent to (18). By (19)

\[
R_g = \frac{n-2}{4(n-1)} u^{-\left(\frac{n+4}{n-4}\right)} \left(-\Delta u^{\frac{n-4}{4}}\right),
\]

so that

\[
R_g > 0 \iff -\Delta u^{\frac{n-4}{4}} > 0 \iff -\Delta u > \frac{2}{n-4} |\nabla u|^2 u.
\]

In particular, \( R_g > 0 \) implies \(-\Delta u > 0\). Changing to cylindrical coordinates we obtain a function

\[
v \in C^\infty((0, \infty) \times S^{n-1}), \quad v(t, \theta) = e^{(\frac{1}{4n})t} u(e^{-t}\theta)
\]

which satisfies (21) and (22).

We constrain our methods with that of Jin and Xiong [13]. They use Green’s identity to transform (18) into an integral formula

\[
u(x) = \int_{B_1(0)} \frac{(u(y))^{\frac{n+4}{n-4}}}{|x-y|^{n-4}} \, d\mu(y) + h(x), \quad u \in L^{\frac{n+4}{n-4}}(B_1(0)) \cap C^1(B_1(0)) \cap \{0\}
\]

where \( h \in C^1(B_1(0)) \). In this setting the condition \(-\Delta u > 0\) implies \( h > 0 \), which allows Jin and Xiong to apply the method of moving spheres/planes. They first prove a priori upper and lower bounds for \( u \), and then prove their asymptotic estimate using moving spheres.

Our proof below follows the techniques in [13], extracting a limit from a slide-back sequence. We start with \( v : (0, \infty) \times S^{n-1} \to (0, \infty) \) which satisfies (21) and a sequence \( \tau_k \to \infty \), and let \( v_k(t, \theta) = v(t + \tau_k, \theta) \). The a priori estimates of Jin and Xiong allow us to extract a convergent subsequence, which is defined on all of \( \mathbb{R} \times S^{n-1} \). By the uniqueness theorem of Frank and König this limit must have the form \( v_\epsilon(t + T) \) for some \( \epsilon \in (0, v_{sup}] \) and \( T \in \mathbb{R} \). The main task in proving simple asymptotics involves proving \( \epsilon \) and \( T \) do not depend on the choice of the sequence \( \tau_k \to \infty \) or the choice of the convergent subsequence. Our proof that the parameters \( \epsilon \) and \( T \) are independent of all choices relies heavily on a careful analysis of the linearization of the PDE (18) about a Delaunay solution, including an asymptotic expansion of solutions of the linearized equation. Once we prove the simple asymptotics, we obtain the refined asymptotics from the asymptotic expansion of the linearized operator.

The two techniques are complementary. The proof of Jin and Xiong is more general, and applies to any solution of the integral equation

\[
u(x) = \int_{B_1(0)} \frac{(u(y))^{\frac{n+4}{n-4}}}{|x-y|^{n-4}} \, d\mu(y) + h(x), \quad h > 0, \quad 0 < \sigma < \frac{n}{2}
\]

whereas at this time one can apply our technique only in the cases \( \sigma = 1 \) and \( \sigma = 2 \). On the other hand, their technique does not give the refined asymptotic expansion.

Previously González [11] proved a similar asymptotics theorem for \( \sigma_k \)-curvature, which is a fully-nonlinear generalization of scalar curvature whose associated PDE is second order and fully nonlinear. More recently, Caffarelli, Jin, Sire, and Xiong [7] prove an asymptotic result for positive solutions of the nonlocal equation

\[
(-\Delta)^s u = u^{\frac{n+4}{n-4}}
\]

for any \( s \in (0, 1) \). Also Baraket and Rebhi [2] and Y.-J. Lin [17] construct many examples of constant \( Q \)-curvature metrics.

The rest of this paper proceeds as follows. Section 2 we collect some preliminary computations, most of which exist already in the literature. Within this section we include a detailed description
of the Delaunay solutions. Our analysis begins in earnest in Section 3 where we study the linearization of the PDE (21). Most importantly we prove the linear stability of the Delaunay solutions using the Fourier-Laplace transform as developed by Mazzeo, Pollack and Uhlenbeck [20]. In this section we also define the indicial roots mentioned above in Remark 2, which give the exponential growth rates of the solutions of the linearization of (21) when linearized about a Delaunay solution. In Section 4 we present an alternative proof of the simple asymptotics Jin and Xiong prove in [14], and in Section 5 we derive the refined asymptotics.

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2 Preliminaries

We collect some preliminary computations related to solutions of (18) and constant $Q$-curvature metrics.

2.1 Symmetries of the PDE

In this section we recall some of the symmetries of (18), which essentially arise from the transformation rule (15). One can of course translate solutions to obtain a new solution, but two more interesting symmetries reflect the scale invariance and conformal invariance outlined above.

We first discuss scale invariance. Let $u > 0$ solve (18) and let $\lambda > 0$. We seek $a > 0$ such that $u_\lambda(x) = \lambda^a u(\lambda x)$ is also a solution. Evaluating, we find

$$(-\Delta)^2 u_\lambda = \lambda^{a+4}(-\Delta)^2 u(x), \quad \lambda^{\frac{n-4}{4}} \frac{n(n-4)}{16} u_\lambda = \lambda^{\frac{n-4}{4}} \frac{n(n-4)}{16} u(\lambda x),$$

which coincide precisely when $a = \frac{n-4}{2}$. In other words we see

$$u \text{ solves (18)} \Rightarrow u_\lambda(x) = \lambda^{\frac{n-4}{2}} u(\lambda x) \text{ solves (18) for all } \lambda > 0. \quad (29)$$

This symmetry is much simpler in the cylindrical coordinates. Letting $T = -\log \lambda$ we see

$$v_\lambda(t, \theta) = e^{\left(\frac{4-4n}{2}\right)} u_\lambda(e^{-t} \theta) = \lambda^{\frac{n-4}{2}} e^{\left(\frac{4-4n}{2}\right)} u(\lambda e^{-t} \theta) = e^{\left(\frac{4-4n}{2}(t+T)\right)} u(e^{-(t+T)} \theta),$$

so that

$$v \text{ solves (21)} \Rightarrow v_T(t, \theta) = v(t+T, \theta) \text{ solves (21) for all } T \in \mathbb{R},$$

which is readily apparent directly from the PDE.

The second symmetry reflects the invariance under reflections through spheres, and one can write it explicitly by defining the Kelvin transforms

$$K_{x_0}(u)(x) = |x|^{-n} u \left( \frac{x}{|x|^2 + x_0} + x_0 \right), \quad \hat{K}_{x_0}(u)(x) = |x|^{4-n} u \left( \frac{x}{|x|^2 + x_0} \right). \quad (30)$$

Observe that the functions $K_{x_0}(u)$ and $\hat{K}_{x_0}(u)$ are now defined on different domains. For instance, if $u \in C^\infty(\mathbb{B}_1(0) \setminus \{0\})$ then $K_{x_0}(u), \hat{K}_{x_0} \in C^\infty(\mathbb{R}^n \setminus \mathbb{B}_1(x_0))$.

One can find the usual Kelvin transformation law

$$\Delta K_{x_0}(u)(x) = |x|^{-4} K_0(\Delta u)(x) \quad (31)$$

in many textbooks, and the transformation law

$$\Delta^2 \hat{K}_{x_0}(u)(x) = |x|^{-8} \hat{K}_{x_0}(\Delta^2 u)(x), \quad (32)$$

appears (for instance) in Lemma 3.6 of [26].
Remark 3. In the case $x_0 = 0$ this transformation looks particularly simple in cylindrical coordinates, namely $v(t, \theta)$ gets transformed to $\tilde{v}_0(v)(t, \theta) = v(-t, \theta)$. On the other hand, the transformation is much more complicated in cylindrical coordinates when the center is not 0.

2.2 Delaunay solutions

We have already introduced the Delaunay solutions, and in this section we give a more detailed description.

The family of Delaunay solutions account for all of the positive solutions of (21) on the whole cylinder $\mathbb{R} \times S^{n-1}$. Recall the cylindrical (constant) and spherical solutions

$$v_{cyl} = \left( \frac{n(n-4)}{n^2-4} \right)^{\frac{n-4}{n-4}} \quad \text{and} \quad v_{sph} = (\cosh t)^{\frac{4}{n}}. $$

It is convenient to observe that, because $n > 4$ we have

$$n^2-4-n(n-4) = 4n-4 > 0 \Rightarrow 0 < v_{cyl} < 1,$$

as well as

$$v_{sph}(0) = 1, \quad \dot{v}_{sph}(t) > 0 \text{ for } t < 0, \quad \dot{v}_{sph}(t) < 0 \text{ for } t > 0, \quad \lim_{t \to \pm \infty} v_{sph}(t) = 0.$$

Frank and König proved that for each $\epsilon \in (0, v_{cyl})$ there exists a unique positive solution $v_\epsilon$ of (23) attaining its minimum value of $\epsilon$ at $t = 0$. Furthermore they prove that any positive, global solution of (15) must be either $v_{sph}(\cdot + T)$ for some $T \in \mathbb{R}$ or $v_{cyl}(\cdot + T)$ for some $T \in \mathbb{R}$ and $\epsilon \in (0, v_{cyl})$.

Each $v_\epsilon$ is periodic with period $T_\epsilon$, has local minima at exactly $kT_\epsilon$ for each $k \in \mathbb{Z}$, and local maxima at exactly $(2k+1)T_\epsilon$ for each $T_\epsilon$, and no other critical points. For our purposes we take the period of the cylindrical solution to be $T_{cyl}$, given in (66). Please see the text surrounding (66) for our reasoning. One can also show each $v_\epsilon$ is symmetric about each of its critical point and that $T_\epsilon$ is a decreasing function of $\epsilon$ with $\lim_{\epsilon \searrow 0} T_\epsilon = \infty$. In this context one should think of $v_{sph}$ as the limit of $v_{\epsilon}(\cdot + T_\epsilon/2)$ as $\epsilon \searrow 0$.

It appears now that the Delaunay metrics occur in a two-parameter family, with the necksize $\epsilon$ and translation parameter $T$ as parameters. However, it is useful to enlarge this family to include a 2n-dimension family of ambient translations, which we first describe in Euclidean coordinates and then transform to cylindrical coordinates. In preparation we transform the Delaunay solution $v_\epsilon$ to Euclidean coordinates, obtaining

$$u_\epsilon(x) = |x|^\frac{4-n}{2} v_\epsilon(-\log |x|).$$

The first translation is

$$\tilde{u}_{\epsilon,a}(x) = u_\epsilon(x-a) = |x-a|^\frac{4-n}{2} v_\epsilon(-\log |x-a|)$$

$$= |x|^\frac{4-n}{2} \left( \frac{x-a}{|x|} \right)^\frac{4-n}{2} v_\epsilon \left( -\log |x| - \log \left| \frac{x-a}{|x|} \right| \right),$$

where $a \in \mathbb{R}^n$. Transforming this back to cylindrical coordinates we then obtain

$$\tilde{v}_{\epsilon,a} = |\theta - e^t a|^\frac{4-n}{2} v_\epsilon(t - \log |\theta - e^t a|).$$

The function $\tilde{v}_{\epsilon,a}$ is of course not a smooth global solution, and it has a singular point when

$$\theta = e^t a \Leftrightarrow t = -\log |a|, \quad \theta = \frac{a}{|a|}.$$
We obtain the remaining translations using the Kelvin transform defined in \([30]\). Given \(a \in \mathbb{R}^n\) we define

\[
u_{\varepsilon,a}(x) = \hat{K}_0(\hat{K}_0(u_\varepsilon(\cdot - a)))(x) = \hat{K}_0 \left( |\cdot - a|^{4-n} u_\varepsilon \left( \frac{\cdot - a}{|\cdot - a|^2} \right) \right)(x)
\]

\[
= |x|^{4-n} \left( \frac{x}{|x|^2} - a \right)^{4-n} u_\varepsilon \left( \frac{\hat{x}}{|\hat{x}|^2} - \hat{a} \right) = |x| - |x|a \left( |x|^{4-n} u_\varepsilon \left( \frac{\hat{x}}{|\hat{x}|^2} - \hat{a} \right) \right) \nabla \left( \log \left( \frac{|x|}{|x| - |x|a} \right) \right)
\]

which in turn gives us

\[
u_{\varepsilon,a}(t, \theta) = |\theta - e^{-t} a|^{\frac{4-n}{2}} \nu_{\varepsilon}(t + \log |\theta - e^{-t} a|).
\] (34)

This function has a singular point when \(t = \log |a|\) and \(\theta = a/|a|\).

Using the Taylor expansions

\[
\left| \frac{x}{|x|} - |x|a \right|^\frac{4-n}{2} = 1 + \frac{(n-4)}{2} \langle x, a \rangle + O(|x|^2)
\]

and

\[
\log \left| \frac{x}{|x|} - a|x| \right| = -\langle a, x \rangle + O(|x|^2)
\]

we expand \(u_{\varepsilon,a}(x)\) as

\[
\begin{align*}
u_{\varepsilon,a}(x) &= |x|^{\frac{4-n}{2}} \left( 1 + \frac{n-4}{2} \langle x, a \rangle + O(|x|^2) \right) \nu_{\varepsilon}(-\log |x|) - \langle a, x \rangle \nu_{\varepsilon}(-\log |x|) + O(|x|^2) \\
&= |x|^{\frac{4-n}{2}} \left( \nu_{\varepsilon}(-\log |x|) + \langle x, a \rangle \left( -\nu_{\varepsilon}(-\log |x|) + \frac{n-4}{2} \nu_{\varepsilon}(-\log |x|) \right) + O(|x|^2) \right) \\
&= u_{\varepsilon}(x) + |x|^{\frac{4-n}{2}} \langle x, a \rangle \left( -\nu_{\varepsilon} + \frac{n-4}{2} \nu_{\varepsilon} \right) + O(|x|^2)
\end{align*}
\]

as \(|x| \to 0\). Rewriting this in cylindrical coordinates gives

\[
\nu_{\varepsilon,a}(t, \theta) = \nu_{\varepsilon}(t) + e^{-t} \langle \theta, a \rangle \left( -\nu_{\varepsilon}(t) + \frac{n-4}{2} \nu_{\varepsilon}(t) \right) + O(e^{-2t})
\] (35)

as \(t \to \infty\). Unfortunately, the same expansion for \(\nu_{\varepsilon,a}\) reveals

\[
\nu_{\varepsilon,a}(t, \theta) = e^t \left( \nu_{\varepsilon}(t) + \frac{n-4}{2} \nu_{\varepsilon}(t) \right) + O(1).
\]

In fact, one expects this behavior, as the motion generating the family \(\nu_{\varepsilon,a}\) translates the origin in Euclidean coordinates, which translates the end in cylindrical coordinates in which \(t \to \infty\).

It will be convenient for our later computations to observe that

\[
\begin{align*}
(a, \theta) > 0 &\Rightarrow \nu_{\varepsilon,a}(t, \theta) > \nu_{\varepsilon}(t), & (a, \theta) < 0 &\Rightarrow \nu_{\varepsilon,a}(t, \theta) < \nu_{\varepsilon}(t) \\
\langle a, \theta \rangle > 0 &\Rightarrow \nu_{\varepsilon,a}(t, \theta) > \nu_{\varepsilon}(t), & \langle a, \theta \rangle < 0 &\Rightarrow \nu_{\varepsilon,a}(t, \theta) < \nu_{\varepsilon}(t).
\end{align*}
\] (36)
One can find a first integral for the ODE (23). Indeed, differentiating once shows
\[ \mathcal{H}_e = -\dot{v}_r \ddot{v} + \frac{1}{2} v_r^2 + \left( \frac{n(n-4)+8}{4} \right) \dot{v}_r^2 - \frac{n^2(n-4)^2}{32} v_r^2 + \frac{(n-4)^2(n-4)}{32} v_r^2 \frac{\partial Q}{\partial \theta} \] (37)
is constant function for each Delaunay solution \( v_r \). Evaluating this energy on the cylindrical solution gives
\[ \mathcal{H}_{cyl} = -\frac{(n-4)(n^2-4)}{8} \left( \frac{n(n-4)}{n^2-4} \right)^\frac{3}{2} < 0 \] (38)
and evaluating on the spherical solution gives
\[ \mathcal{H}_{sph} = \frac{(n-4)^2}{4} (\cosh t)^{-n} \left( \frac{n(n-2)}{4} \sinh^4 t + \left( \frac{4-n}{2} + 2 - n \right) \cosh^2 t \sinh^2 t - \frac{1}{2} \cosh^4 t \right) \] (39)
Finally, it is useful to rewrite this last linearized equation in cylindrical coordinates, obtaining
\[ \Delta^2 w + \epsilon \Delta^2 \bar{w} = \frac{n(n-4)(n^2-4)}{16} w + \epsilon \frac{n(n-2)(n^2-4)}{16} w + \frac{\partial Q}{\partial \theta} + Q(\epsilon w), \] (40)
where
\[ Q(\epsilon w) = \frac{n(n-4)(n^2-4)}{16} \left[ (u+\epsilon w)^{\frac{n+4}{2}} - u^{\frac{n+4}{2}} - \epsilon \frac{n+4}{n-4} w^{\frac{n}{2}} \right] = O(\epsilon^2 \|w\|^2). \] (41)
Combining (18) with (10) and (41) we have
\[ \Delta^2 u + \epsilon \Delta^2 \bar{w} = \frac{n(n-4)(n^2-4)}{16} u^{\frac{n-4}{2}} + \epsilon \frac{n(n-2)(n^2-4)}{16} w + \frac{\partial Q}{\partial \theta} + Q(\epsilon w), \] (42)
where \( Q(\epsilon w) \) is of order \( \epsilon^2 \|w\|^2 \). Selecting the terms of order \( \epsilon \) from the last equation we obtain the linearization
\[ \Delta^2 w = \frac{n(n-4)(n^2-4)}{16} w^{\frac{n-4}{2}}. \] (43)
We refer to a solution \( w \) of (42) as a Jacobi field associated to the solution \( u \).

Finally, it is useful to rewrite this last linearized equation in cylindrical coordinates, obtaining
\[ \frac{\partial^4 w}{\partial \theta^4} - \left( \frac{n(n-4)+8}{2} \right) \frac{\partial^2 w}{\partial \theta^2} + \frac{n(n-4)^2}{16} w + 2 \Delta^2 u + \Delta^2 w = \frac{n(n+4)(n^2-4)}{16} w \] (44)
which we can rearrange to read
\[ 0 = \frac{\partial^4 w}{\partial \theta^4} - \left( \frac{n(n-4)+8}{2} \right) \frac{\partial^2 w}{\partial \theta^2} + \left( \frac{n(n-4)^2}{16} - \frac{n(n+4)(n^2-4)}{16} \right) w \] (45)
Again, we refer to a solution \( w \) of (45) as a Jacobi field associated to the solution \( v \).
2.4 Integral identities

The following is essentially a special case of Proposition 4.2 in [14], and also a special case of Proposition A.2 of [14].

**Proposition 3.** Let $u \in C^\infty((0, \infty) \times S^{n-1})$ solve (21) and let $0 < T_1 < T_2$. Then

\[
\begin{align*}
\int_{(T_1) \times S^{n-1}} & \left( \frac{\partial v}{\partial t} \right)^2 \, dt - \frac{1}{2} \left( \frac{\partial^2 v}{\partial t^2} \right)^2 - \frac{(n(n-4)-8)}{4} \left( \frac{\partial v}{\partial t} \right)^2 + \frac{n^2(n-4)^2}{32} v^2 \\
& - \frac{(n-4)^2(n^2-4)}{32} v \frac{\partial v}{\partial t} + \frac{1}{2} \left( \frac{\partial^2 v}{\partial t^2} \right)^2 - \frac{(n(n-4)+8)}{4} \left( \frac{\partial v}{\partial t} \right)^2 + \frac{n^2(n-4)^2}{32} v^2 \\
& - \frac{(n-4)^2(n^2-4)}{32} v \frac{\partial v}{\partial t} + \frac{1}{2} \left( \frac{\partial^2 v}{\partial t^2} \right)^2 + \frac{n(n-4)}{4} |\nabla \theta v|^2 - \left| \frac{\partial v}{\partial t} \right|^2 \, d\sigma
\end{align*}
\]

(Eq. 44)

**Proof.** Multiplying (21) by $\frac{\partial v}{\partial t}$ and integrating over the sphere $\{t\} \times S^{n-1}$ we obtain

\[
\begin{align*}
0 &= \int_{(t) \times S^{n-1}} \frac{\partial v}{\partial t} \frac{\partial^2 v}{\partial t^2} \, dt - \frac{(n(n-4)+8)}{4} \left( \frac{\partial v}{\partial t} \right)^2 + \frac{n^2(n-4)^2}{32} v^2 \\
& + \frac{\partial v}{\partial t} \frac{\partial^2 v}{\partial t^2} - \frac{(n-4)^2(n^2-4)}{32} v \frac{\partial v}{\partial t} - \frac{1}{2} \left( \frac{\partial^2 v}{\partial t^2} \right)^2 - \frac{(n(n-4)+8)}{4} \left( \frac{\partial v}{\partial t} \right)^2 + \frac{n^2(n-4)^2}{32} v^2 \\
& - \frac{(n-4)^2(n^2-4)}{32} v \frac{\partial v}{\partial t} - \frac{1}{2} \left( \frac{\partial^2 v}{\partial t^2} \right)^2 + \frac{n(n-4)}{4} |\nabla \theta v|^2 \\
& - \frac{1}{2} \left( \frac{\partial^2 v}{\partial t^2} \right)^2 \left( \frac{\partial v}{\partial t} \right)^2 + \frac{n(n-4)}{4} |\nabla \theta v|^2 - \left| \frac{\partial v}{\partial t} \right|^2 \, d\sigma
\end{align*}
\]

(Eq. 45)

Following this integral identity, we define

\[
H_{\text{rad}}(v) = \int_{(t) \times S^{n-1}} \frac{\partial v}{\partial t} \frac{\partial^2 v}{\partial t^2} + \frac{1}{2} \left( \frac{\partial^2 v}{\partial t^2} \right)^2 + \frac{(n(n-4)+8)}{4} \left( \frac{\partial v}{\partial t} \right)^2
\]

\[
- \frac{n^2(n-4)^2}{32} v^2 + \frac{(n-4)^2(n^2-4)}{32} v \frac{\partial v}{\partial t} - \frac{1}{2} \left( \frac{\partial^2 v}{\partial t^2} \right)^2 - \frac{n(n-4)}{4} |\nabla \theta v|^2 - \left| \frac{\partial v}{\partial t} \right|^2 \, d\sigma,
\]

which does not depend on $t$ by (44).

**Corollary 4.** If $v_\epsilon$ is the Delaunay solution of (23) described above then

\[
H_{\text{rad}}(v_\epsilon) = n \omega_n \left( \frac{1}{2} v_\epsilon^2(0) + \frac{(n-4)^2}{32} \epsilon^2 |n^2-4| e^{-\frac{\epsilon^2}{2(n-4)^2}} - n^2 \right) = n \omega_n H_\epsilon < 0.
\]

(Eq. 46)

where $\omega_n$ is the volume of a unit ball in $\mathbb{R}^n$. 

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2.5 A priori estimates

Jin and Xiong [14] prove the following.

**Theorem 5.** Let \( u \in C^\infty(B_1(0) \setminus \{0\}) \) be a positive solution of (18) such that \(- \Delta u > 0\). Then either \( u \) admits a continuous extension to \( B_1(0) \) or there exist positive constants \( C_1 < C_2 \) such that

\[
C_1|x|^{\frac{n}{n-4}} \leq u(x) \leq C_2|x|^{\frac{n}{n-4}}.
\]

**Remark 4.** The example of the Delaunay solutions demonstrate that the constant \( C_1 \) in (14) must depend on the choice of the solution \( u \). However, the constant \( C_2 \) in the upper bound is universal.

**Lemma 6.** Let \( v \in C^\infty((0, \infty) \times S^{n-1}) \) be a positive solution of (21) which also satisfies (22). Then either \( \limsup_{t \to \infty} v(t, \theta) = 0 \) or there exist positive constants \( C_1 < C_2 \) such that

\[
C_1 < v(t, \theta) < C_2.
\]

Moreover, in this case \( \mathcal{H}_{\text{rad}}(v) \neq 0 \).

**Proof.** The bounds (17) imply \( C_1 < v(t, \theta) < C_2 \). Choose a sequence \( \tau_i \searrow \infty \) and define \( v_i(t, \theta) = v(t + \tau_i, \theta) \). This sequence is uniformly bounded, so we may extract a subsequence, still denoted by \( \{v_i\} \), which converges uniformly on compact subsets of \( \mathbb{R} \times S^{n-1} \) to a solution \( \bar{v} \) of (21). However, the limit \( \bar{v} \) must be a Delaunay solution \( v_e \), and so (using Proposition 3) we have

\[
\mathcal{H}_{\text{rad}}(v) = \int_{\{1\} \times S^{n-1}} - \frac{\partial v}{\partial t} \frac{\partial^3 v}{\partial t \partial r^2} + \frac{1}{2} \left( \frac{\partial^2 v}{\partial t^2} \right)^2 + \left( \frac{n(n-4)+8}{4} \right) \left( \frac{\partial v}{\partial t} \right)^2 + \frac{n^2(n-4)^2}{32} \frac{\Delta r}{r^2} + \frac{(n-4)^2(n-4)}{32} \frac{2}{r^{n+2}} - \frac{1}{2} (\Delta_\theta v)^2 - \frac{n(n-4)}{4} \left| \nabla_\theta \frac{\partial v}{\partial t} \right|^2 d\sigma
\]

\[
= \int_{\{\tau_i+1\} \times S^{n-1}} - \frac{\partial v}{\partial t} \frac{\partial^3 v}{\partial t \partial r^2} + \frac{1}{2} \left( \frac{\partial^2 v}{\partial t^2} \right)^2 + \left( \frac{n(n-4)+8}{4} \right) \left( \frac{\partial v}{\partial t} \right)^2 + \frac{n^2(n-4)^2}{32} \frac{\Delta r}{r^2} + \frac{(n-4)^2(n-4)}{32} \frac{2}{r^{n+2}} - \frac{1}{2} (\Delta_\theta v)^2 - \frac{n(n-4)}{4} \left| \nabla_\theta \frac{\partial v}{\partial t} \right|^2 d\sigma
\]

\[
= \mathcal{H}_{\text{rad}}(v_i) \to \mathcal{H}_{\text{rad}}(v_e) = n \omega_n \mathcal{H}_e 
eq 0.
\]

\[\square\]

3 Linear analysis

In this section we study the mapping properties of the linear operator (43), concentrating on the linearization about a Delaunay metric.

3.1 Definitions

Linearizing (21) about a Delaunay solution \( v_e \), we obtain the operator

\[
L_e = \frac{\partial^4}{\partial t^4} + \Delta_\theta^2 + 2\Delta_\theta \frac{\partial^2}{\partial t^2} - \frac{n(n-4)}{2} \Delta_\theta - \left( \frac{n(n-4)+8}{2} \right) \frac{\partial^2}{\partial t^2} + \left( \frac{n^2(n-4)^2}{16} - \frac{n(n+4)(n^2-4)}{16} \right) \frac{\partial^2}{\partial r^2}.
\]
We refer to solutions of the equation $L_t(w)$ as **Jacobi fields** associated to the Delaunay solution $u_t$. More generally, if $v$ satisfies \(21\) and $w$ satisfies \(43\) then we call $w$ a Jacobi field associated to the solution $v$. We are interested in the mapping properties of $L_t$, for instance as the map

$$L_t : W^{4,2}((0, \infty) \times S^{n-1}) \to L^2((0, \infty) \times S^{n-1}).$$

It turns out the operator written above does not have closed range (see pg. 21, pg. 216, and Theorem 5.40 of \[21\]), so we will define certain weighted function spaces to accommodate $L_t$.

**Definition 1.** Let $\gamma \in \mathbb{R}$ and when $u \in L^2((0, \infty) \times S^{n-1})_{\text{loc}}$ define

$$\|u\|_{L^2_{\gamma}}^2 = \int_0^\infty \int_{S^{n-1}} e^{-2\gamma t} |u(t, \theta)|^2 \, d\theta \, dt.$$  

The space $L^2_{\gamma}((0, \infty) \times S^{n-1})$ is the space of functions with finite norm, as defined above. One can similarly define the Sobolev spaces $W^{k,2}_{\gamma}$ of functions with $k$ weak derivatives in $L^2$ having finite weighted norms.

One can also define weighted Hölder spaces.

**Definition 2.** Let $\gamma \in \mathbb{R}$ and $\alpha \in (0, 1)$. For $u \in C^{0, \alpha}_{\text{loc}}((0, \infty) \times S^{n-1})$ define

$$\|u\|_{C^{0, \alpha}_{\gamma}} = \sup_{t_0 > 1} \sup_{(t_1, \theta_1), (t_2, \theta_2)} \left\{ \frac{e^{-\gamma t_1} u(t_1, \theta_1) - e^{-\gamma t_2} u(t_2, \theta_2)}{d((t_1, \theta_1), (t_2, \theta_2))^\alpha} : (t_1, \theta_1), (t_2, \theta_2) \in (t_0 - 1, t_0 + 1) \times S^{n-1} \right\}.$$  

One can similarly define weighted Hölder spaces with more derivatives.

Heuristically, a function in a weighted function space with weight $\gamma$ is bounded from above by a multiple of $e^\gamma t$ as $t \to \infty$. Observe then that when $\gamma_1 < \gamma_2$ we have the inclusions

$$W^{k,2}_{\gamma_1}((0, \infty) \times S^{n-1}) \subset W^{k,2}_{\gamma_2}((0, \infty) \times S^{n-1}), \quad C^{k, \alpha}_{\gamma_1}((0, \infty) \times S^{n-1}) \subset C^{k, \alpha}_{\gamma_2}((0, \infty) \times S^{n-1}).$$

The fact that $L_t$ the leading order terms of $L_t$ is $\frac{\partial^2}{\partial t^2} + \Delta_\theta^2$ tells us the following.

**Lemma 7.** For any $\gamma \in \mathbb{R}$ the operators

$$L_t : W^{k+4,2}_{\gamma}((0, \infty) \times S^{n-1}) \to W^{k,2}_{\gamma}((0, \infty) \times S^{n-1})$$

and

$$L_t : C^{k+4, \alpha}_{\gamma}((0, \infty) \times S^{n-1}) \to C^{k, \alpha}_{\gamma}((0, \infty) \times S^{n-1})$$

are bounded, linear, and elliptic.

We complete our understanding of $L_t$ by identifying the weights for which

$$L_t : W^{k+4,2}_{\gamma}((0, \infty) \times S^{n-1}) \to W^{k,2}_{\gamma}((0, \infty) \times S^{n-1})$$

is Fredholm, injective, and/or surjective. Indeed, this is a nontrivial task.

We make our analysis easier by decomposing $w$ in spherical harmonics, writing

$$w(t, \theta) = \sum_{j \in \mathbb{Z}} w_j(t) \phi_j,$$

where $\phi_j$ is a normalized eigenfunction of $\Delta_\theta$ on $S^{n-1}$, i.e.

$$\Delta_\theta \phi_j = -\lambda_j \phi_j, \quad \int_{S^{n-1}} \phi_j \phi_k \, d\theta = \delta_{jk}. \quad (50)$$
The eigenvalues $\lambda_j$ of the $(n - 1)$-dimensional sphere have the form $\lambda_j = k(n - 2 + k)$ for some $k = 0, 1, 2, 3, \ldots$

Under this decomposition the Fourier coefficient $w_j$ solves the ODE

$$0 = L_{\epsilon,j} w_j = \ddot{w}_j - \left( \frac{n(n - 4) + 8 + 4\lambda_j}{2} \right) \dot{w}_j + \left( \frac{n^2(n - 4)^2}{16} - \frac{n(n + 4)(n^2 - 4)}{16} \right) \ddot{v}_\epsilon - \frac{n(n - 4)}{2} \lambda_j + \frac{n^2}{8} \lambda_j^2 w_j. \quad (51)$$

It immediately follows that

$$\text{spec}(L_\epsilon) = \bigcup_{j=0}^{\infty} \text{spec}(L_{\epsilon,j}). \quad (52)$$

### 3.2 Low Fourier modes

We can explicitly identify some of the ODE solutions when $|j|$ is small. For instance, we have $\lambda_0 = 0$ and so (51) becomes

$$\ddot{w}_0 - \left( \frac{n(n - 4) + 8}{2} \right) \dot{w}_0 + \left( \frac{n^2(n - 4)^2}{16} - \frac{n(n + 4)(n^2 - 4)}{16} \right) \ddot{v}_\epsilon w_0 = 0, \quad (53)$$

which is the derivative of (53). Thus

$$w_0^+ (t) = \dot{v}_\epsilon(t), \quad w_0^- = \frac{d}{d\epsilon} v_\epsilon(t) \quad (54)$$

both solve (53).

**Lemma 8.** The function $w_0^+$ is periodic with period $T_\epsilon$ while the function $w_0^-$ grows linearly.

**Proof.** Differentiating the equation $v_\epsilon(t) = v_\epsilon(t + T_\epsilon)$ with respect to $t$ gives $w_0^+ (t + T_\epsilon) = w_0(t)$. Differentiating $v_\epsilon(t + T_\epsilon) = v_\epsilon(t)$ with respect to $\epsilon$ gives

$$w_0^- (t + T_\epsilon) \frac{d T_\epsilon}{d\epsilon} = w_0^- (t),$$

and so $w_0^-$ grows linearly. \( \square \)

We can also explicitly identify the Fourier modes when $j = 1, 2, \ldots, n$ and $\lambda_j = n - 1$. To do this we first let $e_j$ be the standard basis of $\mathbb{R}^n$ and observe $\phi_j = \langle e_j, \theta \rangle$ is the eigenfunction associated to $\lambda_j = n - 1$. Substituting $a = \tau e_j$ into (54) and (55) we find

$$v_{\epsilon,\tau e_j} = |\theta - e^{-t \tau e_j}| \frac{d^2}{dt^2} v_\epsilon (t + \log |\theta - e^{-t \tau e_j}|) = v_\epsilon (t + \tau e^{-t} (\theta, e_j) \left( -\dot{v}_\epsilon (t) + \frac{n - 4}{2} v_\epsilon (t) \right) + O(e^{-2t}) = v_\epsilon (t + \tau e^{-t} \phi_j \left( -\dot{v}_\epsilon (t) + \frac{n - 4}{2} v_\epsilon (t) \right) + O(e^{-2t}).$$

Differentiating with respect to $\tau$ we obtain

$$w_j^- (t) \phi_j (\theta) = \frac{d}{d\tau} \bigg|_{\tau=0} v_{\epsilon,\tau e_j} = e^{-t} \left( -\dot{v}_\epsilon (t) + \frac{n - 4}{2} v_\epsilon (t) \right) \phi_j (\theta) + O(e^{-2t}),$$

or

$$w_j^- = e^{-t} \left( -w_0^+ + \frac{n - 4}{2} v_\epsilon \right) + O(e^{-2t}) = e^{-t} \left( -\dot{v}_\epsilon + \frac{n - 4}{2} v_\epsilon \right) + O(e^{-2t}). \quad (55)$$
However, \( v_{\tau, \epsilon} \) satisfies (21) for each \( \tau \). Differentiating this relation and using \(-\Delta_\theta \phi_j = (n - 1) \phi_j = \lambda_j \phi_j\) we find

\[
\frac{n(n + 4)(n^2 - 4)}{16} \dot{w}_j - \frac{n - 4}{2} \epsilon, \tau \phi_j = \left( \frac{\partial^4}{\partial t^4} - \frac{n(n - 4) + 8}{2} \frac{\partial^2}{\partial t^2} + \frac{n^2(n - 4)^2}{16} \right) \dot{w}_j - \frac{n(n - 4) + 8 + 4 \lambda_j}{2} \dot{w}_j - \left( \frac{n^2(n - 4)^2}{16} + \lambda_j^2 + \frac{n(n - 4)}{2} \lambda_j \right) \dot{w}_j \phi_j,
\]

which we can rearrange to give \( \tilde{L}_{\epsilon, j}(\dot{w}_j) = 0 \).

A similar calculation, starting from

\[
\dot{w}_j^+(t)\phi_j(\theta) = \frac{d}{d\tau} \bigg|_{\tau=0} \tilde{v}_{\epsilon, \tau} \phi_j,
\]

gives the expansion

\[
w_j^+ = e^t \left( w_0^+ + \frac{n - 4}{2} \epsilon \right) + O(1) = e^t \left( \dot{\epsilon} + \frac{n - 4}{2} \epsilon \right) + O(1). \tag{56}
\]

Observe that (36) implies

\[
w_j^+ > 0, \quad w_j^- > 0. \tag{57}
\]

It is not surprising that \( w_j^+(t) \) all agree for \( j = 1, 2, \ldots, n \), as each translation is geometrically the same. It is also not surprising that \( w_j^+ \) grows exponentially while \( w_j^- \) decays exponentially.

The ambient motion generating \( w_j^+ \) translates the origin in Euclidean coordinates, which in cylindrical coordinates moves the end corresponding to \( t \to \infty \), whereas the ambient motion generating \( w_j^- \) moves the end corresponding to \( t \to -\infty \).

### 3.3 Indicial roots

We further analyze the ODE (51) for general values of \( j \), that is

\[
0 = L_{\epsilon, j} w
= \cdots - \left( \frac{n(n - 4) + 8 + 4 \lambda_j}{2} \right) \ddot{w} + \left( \frac{n^2(n - 4)^2}{16} - \frac{n(n + 4)(n^2 - 4)}{16} \epsilon, \tau \phi_j \right) \ddot{w} + \lambda_j^2 + \frac{n(n - 4)}{2} \lambda_j \dot{w}.
\]

The coefficients of this ODE are all periodic with period \( T_\epsilon \), so there is a (constant) \( 4 \times 4 \) matrix \( A_{\epsilon, j} \) such that

\[
\begin{pmatrix}
w(t + T_\epsilon) \\
\dot{w}(t + T_\epsilon) \\
\ddot{w}(t + T_\epsilon) \\
\dddot{w}(t + T_\epsilon)
\end{pmatrix} = A_{\epsilon, j}
\begin{pmatrix}
w(t) \\
\dot{w}(t) \\
\ddot{w}(t) \\
\dddot{w}(t)
\end{pmatrix}.
\tag{58}
\]
Now let \( w_1, w_2, w_3, w_4 \) be four solutions of (51), and let

\[
W(t) = \det \begin{pmatrix}
w_1 & w_2 & w_3 & w_4 \\
\dot{w}_1 & \dot{w}_2 & \dot{w}_3 & \dot{w}_4 \\
\ddot{w}_1 & \ddot{w}_2 & \ddot{w}_3 & \ddot{w}_4 \\
\dddot{w}_1 & \dddot{w}_2 & \dddot{w}_3 & \dddot{w}_4
\end{pmatrix}(t)
\]  

(59)

be the associated Wronskian determinant. By Abel’s identity, \( \frac{dW}{dt} = 0 \) and so \( W \) is constant. Combining (58) and (59) we see that \( \det A_{\epsilon,j} = 1 \). Moreover, the matrix \( A_{\epsilon,j} \) has real coefficients, so its eigenvalues occur in conjugate pairs. Suppressing the dependence on \( \epsilon \) for the moment, we denote these eigenvalues as

\[
\mu_j^\pm = e^{\pm i \xi_j}, \quad \tilde{\mu}_j^\pm = e^{\pm i \tilde{\xi}_j}.
\]

(60)

Here we define the **indicical roots** of the operator \( L_{\epsilon,j} \) as all the real numbers \( \gamma \) such that \( \gamma = \Im(\xi) \) where \( \mu = e^{i \xi} \) is an eigenvalue of \( A_{\epsilon,j} \). For convenience later on, we collect these numbers as

\[
\Gamma_{\epsilon,j} = \{ \gamma \in \mathbb{R} : \gamma = \Im(\xi) \text{ and } \mu = e^{i \xi} \text{ is an eigenvalue of } A_{\epsilon,j} \}, \quad \Gamma_{\epsilon} = \bigcup_{j=0}^{\infty} \Gamma_{\epsilon,j}.
\]

(61)

Observe that \( \Gamma_{\epsilon,j} \) has at most four elements, so in particular \( \Gamma_{\epsilon} \) is countable. Moreover, by construction \( \Gamma_{\epsilon,j} \) is even for each \( j \), i.e. \( \gamma \in \Gamma_{\epsilon,j} \) if and only if \( -\gamma \in \Gamma_{\epsilon,j} \).

The fact that the eigenvalues of \( A_{\epsilon,j} \) occur in conjugate pairs implies we can always write the eigenvalues as \( \mu^\pm = e^{\pm i \xi} \) and \( \tilde{\mu}^\pm = e^{\pm i \tilde{\xi}} \). Writing \( \xi = \eta + i \nu \) and \( \tilde{\xi} = \tilde{\eta} + i \tilde{\nu} \) we see that \( |\mu^\pm| = e^{\pm \nu} \) and \( |	ilde{\mu}^\pm| = e^{\pm \tilde{\nu}} \). In other words, the indicial roots precisely determine the rates of exponential growth of the solutions of (51).

**Lemma 9.** For each \( \epsilon \in (0, v_{cyl}) \) we have \( 0 \in \Gamma_{\epsilon,0} \) (with multiplicity 2) and \( \{-1,1\} \subset \Gamma_{\epsilon,j} \) for \( j = 1, 2, \ldots, n \).

**Proof.** Lemma 8 implies \( 0 \in \Gamma_{\epsilon,0} \) while (55) and (56) together imply \( \{\pm 1\} \subset \Gamma_{\epsilon,j} \) for \( 1 \leq j \leq n \). \( \Box \)

Now we explicitly compute the Jacobi fields and the indicial roots for the special case of the cylindrical metric. We let \( \epsilon_n \) denote the Delaunay parameter of the cylindrical solution, that is

\[
\epsilon_n = \left( \frac{n(n - 4)}{n^2 - 4} \right)^{\frac{1}{4}},
\]

(62)

so that

\[
L_{\epsilon_n,j} = \frac{d^4}{dt^4} - \left( \frac{n(n - 4) + 8 + 4\lambda_j}{2} \right) \frac{d^2}{dt^2} + \left( -\frac{n^2(n - 4)}{2} + \frac{n(n - 4)}{2} \lambda_j + \lambda_j^2 \right).
\]

(63)

Fortunately we can explicitly solve the ODEs \( L_{\epsilon_n,j}w = 0 \). Substituting

\[
w(t) = c_+ e^{\nu_j t} + c_- e^{-\nu_j t} + \tilde{c}_+ e^{\tilde{\nu}_j t} + \tilde{c}_- e^{-\tilde{\nu}_j t}
\]

and using \( \lambda_j = k(n - 2 + k) \) for some nonnegative integer \( k \) we find

\[
\mu_j^2 = \frac{1}{2} \left( \frac{n(n - 4) + 8 + 4\lambda_j}{2} + \sqrt{\frac{n^4}{4} - 16(n - 1 - \lambda_j)} \right)
\]

\[
= \frac{1}{2} \left( \frac{(n + 2(k - 1))^2}{2} + \sqrt{\frac{n^4}{4} + 16(k - 1)(n + k - 1)} \right)
\]

(64)
\[ \tilde{\mu}_j^2 = \frac{1}{2} \left( \frac{n(n - 4) + 8 + 4\lambda_j}{2} - \sqrt{\frac{n^4}{4} - 16(n - 1 - \lambda_j)} \right) \]  
\[ = \frac{1}{2} \left( \frac{(n + 2(k - 1))^2 + 4}{2} - \sqrt{\frac{n^4}{4} + 16(k - 1)(n + k - 1)} \right), \]  
which then gives all solutions after taking square roots. We remark on some properties of \( \mu_j \) and \( \tilde{\mu}_j \). First observe that, because \( n > 4 \),

\[
\frac{n^4}{4} - 16n + 16 + 16\lambda_j \geq n^4 - 16n + 16 > 16 \left( \frac{n^2}{4} - n + 1 \right) > 0,
\]
so that, in particular, \( \mu_j^2 \) and \( \tilde{\mu}_j^2 \) are real numbers. Next we observe that \( \mu_j^2 > 0 \) for each integer \( j \). After taking a positive and a negative square root gives us one Jacobi field which grows exponentially and one exponentially decaying Jacobi field, implying each \( \Gamma_{\epsilon,n,j} \) contains one positive and one negative index. On the other hand, \( \tilde{\mu}_0^2 < 0 \) while \( \tilde{\mu}_j^2 > 0 \) for \( j > 0 \). Furthermore, we can explicitly compute these indices when \( j = 1, 2, \ldots, n \), in which case \( k = 1 \) and \( \lambda_j = n - 1 \) and

\[
\mu_j^2 = \frac{n^2 + 2}{2}, \quad \tilde{\mu}_j^2 = 1.
\]
Moreover, the fundamental period of the Jacobi field associated to \( \tilde{\mu}_0 \)

\[
T_{cyl} = T_{\epsilon,n} = \frac{2\pi}{\mu_0}, \quad \tilde{\mu}_0 = \frac{1}{2} \sqrt{\frac{n^4}{4} - 64n + 64 - n(n - 4) + 8}. \]  

To summarize, we have proved the following lemma.

**Lemma 10.** We have

\[
\Gamma_{\epsilon,n,0} = \left\{ 0, \pm \frac{1}{2} \sqrt{n(n - 4) + 8 + n^4 - 64n + 64} \right\}, \]  

and for \( j > 0 \)

\[
\Gamma_{\epsilon,n,j} = \left\{ \pm \frac{1}{2} \sqrt{(n + 2(k - 1))^2 + 4 - n^4 + 64(k - 1)(n + k - 1)}, \right\} \]  
\[
\pm \frac{1}{2} \sqrt{(n + 2(k - 1))^2 + 4 + n^4 + 64(k - 1)(n + k - 1)} \right\}, \]

where \( k \) is the positive integer corresponding to \( \lambda_j = k(n - 2 + k) \). In particular,

\[
\Gamma_{\epsilon,n,1} = \cdots = \Gamma_{\epsilon,n,n} = \left\{ \pm 1, \pm \sqrt{\frac{n^2 + 2}{2}} \right\}. \]  

**Remark 5.** It follows from (64) and (65) that as \( k \to \infty \) we have \( \gamma_{\epsilon,n,j} \simeq \sqrt{2}(k - 1) + 2\sqrt{k - 1} \) and \( \tilde{\gamma}_{\epsilon,n,j} \simeq \sqrt{2}(k - 1) - 2\sqrt{k - 1} \). However, we will not need this information later.

For future calculations we will write the set of indicial roots as

\[
\Gamma_\epsilon = \{ \ldots, -\gamma_\epsilon,2, -\gamma_\epsilon,1 = -1, 0, \gamma_\epsilon,1 = 1, \gamma_\epsilon,2, \ldots \}
\]
where \( \gamma_{\epsilon,j} < \gamma_{\epsilon,j+1} \to \infty \). We will justify later the fact that \( \Gamma_\epsilon \) has no accumulation points.
3.4 The Fourier-Laplace transform

The following transform, defined in [20], plays a key role in our understanding of the mapping properties of $L_{\epsilon}$ and $\{L_{\epsilon,i}\}$.

**Definition 3.** Let $\gamma \in \mathbb{R}$ and let $w \in W^k(2)((0, \infty) \times \mathbb{S}^{n-1})$. Extend $w$ to be $0$ in the half-space $\{t < 0\}$ and define

$$\mathcal{F}_\epsilon(w)(t, \xi, \theta) = \hat{w}(t, \xi, \theta) = \sum_{k=-\infty}^{\infty} e^{-i\epsilon} w(t + kT_\epsilon, \theta). \quad (70)$$

Here $\xi \in \{\eta + i\nu \in \mathbb{C}: \nu < -\gamma T_\epsilon\}$.

**Lemma 11.** The sum in $(70)$ converges uniformly and absolutely when $w \in W^k(2)((0, \infty) \times \mathbb{S}^{n-1})$ and $\nu = \Im(\xi) < -\gamma T_\epsilon$. Equivalently,

$$w \in W^k(2)((0, \infty) \times \mathbb{S}^{n-1}) \Rightarrow \mathcal{F}_\epsilon(w) \in \mathcal{C}^\omega((\Im(\xi) < -\gamma T_\epsilon), W^k(2)((0, \infty) \times \mathbb{S}^{n-1})).$$

**Proof.** We have seen that $w \in W^k(2)((0, \infty) \times \mathbb{S}^{n-1})$ implies $|w(t, \theta)| = O(e^{\gamma t})$. Writing $\xi = \eta + i\nu$, with $\eta, \nu \in \mathbb{R}$, we have

$$|\mathcal{F}_\epsilon(w)(t, \xi, \theta)| \leq \sum_{k=-\infty}^{\infty} |e^{-i(\eta + i\nu)k} w(t + kT_\epsilon, \theta)| = \sum_{k=-\infty}^{\infty} e^{k\nu} |w(t + kT_\epsilon, \theta)| \leq Ce^{\gamma t} \sum_{k=-\infty}^{\infty} e^{(\nu + \gamma T_\epsilon)k}.$$

First observe that, since we have extended $w$ to be $0$ in the region $\{t < 0\}$, each choice of $\xi = \eta + i\nu$ only gives finitely many nonzero terms with $k < 0$, and so we only must resolve the convergence when $k \to \infty$. In this case, all exponents are negative precisely when $\nu < -\gamma T_\epsilon$. \qed

Heuristically, the parameter $\xi$ (more specifically $\nu = \Im(\xi)$) allows us to move the weight as a parameter in the function space to one in the operator.

One can invert this transform, but (as expected) one must choose a branch in of the inversion.

**Lemma 12.** Let $w \in W^k(2)((0, \infty) \times \mathbb{S}^{n-1})$ and let $\nu < -\gamma T_\epsilon$. For each $t$ choose $l \in \mathbb{Z}$ and $\tilde{t} \in [0, T_\epsilon)$ so that $t = \tilde{t} + lT_\epsilon$. Then

$$w(t, \theta) = \frac{1}{2\pi} \int_{\eta=0}^{2\pi} e^{i\epsilon T_\epsilon(\eta + i\nu)} \hat{w}(\tilde{t}, \eta + i\nu, \theta) d\eta. \quad (71)$$

**Proof.** Writing $\xi = \eta + i\nu$ we have

$$\frac{1}{2\pi} \int_{\eta=0}^{2\pi} e^{i\epsilon T_\epsilon(\eta + i\nu)} d\eta = \frac{1}{2\pi} \int_{\eta=0}^{2\pi} e^{i\epsilon T_\epsilon(\eta + i\nu)} \sum_{k=-\infty}^{\infty} e^{-i\epsilon k} w(\tilde{t} + kT_\epsilon, \theta) d\eta = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_{\eta=0}^{2\pi} e^{i(\eta + i\nu)(\tilde{t} - k)} w(\tilde{t} + kT_\epsilon, \theta) d\eta = w(\tilde{t} + lT_\epsilon, \theta) \int_{\eta=0}^{2\pi} d\eta = w(t, \theta).$$

Here we have used the fact that $\nu < -\gamma T_\epsilon$ to allow us to interchange the sum and the integral. \qed
In fact, we can treat $\nu$ as a parameter in this inversion, and we see that changing $\nu$ alters
the weight of the transformed function. We make this explicit with a version of the Parseval-
Plancherel identity.

**Lemma 13.** For each $\theta \in \mathbf{S}^{n-1}$ and $\nu \in \mathbf{R}$ we have
\[
\|\hat{w}(\cdot, \cdot + i\nu, \theta)\|_{L^2([0,T] \times [0,2\pi])}^2 \simeq 2\pi \|w(\cdot, \theta)\|_{L^2_{\nu,T\nu}((0,\mathbf{R}))},
\]
where $a \simeq b$ means both $a = \mathcal{O}(b)$ and $b = \mathcal{O}(a)$.

**Proof.** We compute
\[
\int_0^{T_0} \int_0^{2\pi} |\hat{w}(t, \eta + i\nu, \theta)|^2 dt d\eta = \int_0^{T_0} \int_0^{2\pi} \left( \sum_{k=-\infty}^{\infty} e^{-ik\eta} e^{ik\nu} w(t + kT\nu, \theta) \right) \left( \sum_{l=-\infty}^{\infty} e^{il\eta} e^{i\nu} w(t + lT\nu, \theta) \right) dt d\eta
\]
\[
= \int_0^{T_0} \int_0^{2\pi} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \left( \frac{k}{l} \right) e^{i(l-k)\eta} e^{i(k+l)\nu} w(t + kT\nu, \theta) w(t + lT\nu, \theta) dt d\eta
\]
\[
= \int_0^{T_0} \int_0^{2\pi} \sum_{k=-\infty}^{\infty} e^{2\nu k} (w(t + kT\nu, \theta))^2 dt d\eta
\]
\[
\simeq 2\pi \int_{\mathbf{R}} (e^{\nu/T\nu} w(t, \theta))^2 dt.
\]
Observe that the integrals of all the cross-terms in the sum all vanish because $\int_0^{2\pi} e^{ikt} dt = 0$ for
each $k \in \mathbf{Z}\setminus\{0\}$. \hfill \Box

Evaluating the computation above with the choice $\nu = 0$ we find

**Corollary 14.** For each $\theta \in \mathbf{S}^{n-1}$ we have
\[
\|\hat{w}(\cdot, \cdot, \theta)\|_{L^2([0,T] \times [0,2\pi])}^2 = 2\pi \|w(\cdot, \theta)\|_{L^2_{\nu,T\nu}((0,\mathbf{R}))}^2.
\]

**Proof.** Take $\nu = 0$ in (73). \hfill \Box

Furthermore one can reindex the sum in (73) to obtain
\[
\hat{w}(t + T\nu, \xi, \theta) = \sum_{k=-\infty}^{\infty} e^{-ik\xi} w(t + kT\nu + kT\theta, \theta) \quad (74)
\]
\[
= \sum_{k=-\infty}^{\infty} e^{-ik\xi} w(t + (k + 1)T\theta, \theta)
\]
\[
= \sum_{l=-\infty}^{\infty} e^{-i(l-1)\xi} w(t + lT\theta, \theta)
\]
\[
e^{i\xi \nu} \hat{w}(t, \xi, \theta),
\]
which we can write either as $\hat{w}(t, \xi, \theta) = e^{-i\xi} \hat{w}(t + T\xi, \xi, \theta)$ or as $w(t + T\nu, \theta) = \mathcal{F}_{T\nu}^{-1}(e^{i\xi} \mathcal{F}_{T\nu}(w))(t, \theta)$. In more geometric/invariant language, this last formula states $\hat{w}$ is a section of the flat bundle $\mathbf{S}^1 \times \mathbf{S}^{n-1}$ with holonomy $\xi$ around the $\mathbf{S}^1$ loop.

**Corollary 15.** The Fourier-Laplace transform gives a direct integral decomposition
\[
L^2(\mathbf{R} \times \mathbf{S}^{n-1}) = \int_{\eta \in [0,2\pi]} L^2_{\eta}([0,T\nu] \times \mathbf{S}^{n-1}) d\eta,
\]
where $L^2_{\eta}([0,T\nu] \times \mathbf{S}^{n-1})$ is the $L^2$-completion of
\[
\left\{ w \in \mathcal{C}^0([0,T\nu] \times \mathbf{S}^{n-1}) : w(T\nu, \theta) = e^{iT\nu} w(0, \theta) \right\}.
\]

**Proof.** Combine (73) with (72) and (71). \hfill \Box
3.5 Spectral bands of the Jacobi operator of the Delaunay metrics

At this point we use the decomposition (76) to prove a spectral gap result for the Jacobi operator of a Delaunay metric. Much of this discussion borrows from [19].

We restrict attention in this section to the space of quasi-periodic functions. For each \( \eta \in \mathbb{R} \) and nonnegative integer \( k \) define \( W^{k,2}_\eta([0, T_\epsilon]) \) to be the \( W^{k,2} \)-closure of the space of smooth functions on \([0, T_\epsilon]\) subject to the boundary conditions

\[
\frac{d^k w}{dt^k}(T_\epsilon) = e^{iT_\epsilon \nu} \frac{d^k w}{dt^k}(0), \quad l = 0, 1, \ldots, k - 1.
\] (76)

and denote by \( L_{\epsilon,j,\eta} \) the restriction

\[
L_{\epsilon,j,\eta} = L_{\epsilon,j} : W^{4,2}_\eta([0, T_\epsilon]) \to L^2_\eta([0, T_\epsilon]).
\]

In order to use the decomposition (76) we define the following twisted operator. To begin we define \( \tilde{L}_\epsilon(\xi) \) by \( \tilde{L}_\epsilon(\xi)(\tilde{v}) = L_{\epsilon}(v) \), or \( \tilde{L}_\epsilon(\xi) = F_\nu \circ L_\epsilon \circ F_\nu^{-1} \). Using (74) we see

\[
\tilde{L}_\epsilon(\xi)(e^{i\xi \tilde{w}})(t, \xi, \theta) = \tilde{L}_\epsilon(\xi)(\tilde{w})(t + T_\epsilon, \xi, \theta) = \tilde{L}_\epsilon(\xi)(\tilde{w})(t, \xi, \theta) = e^{i\xi \tilde{L}_\epsilon(\xi)(\tilde{v})(t, \xi, \theta)},
\]

which we can rearrange to read

\[
e^{-i\xi \tilde{L}_\epsilon(\xi)(e^{i\xi \tilde{v}}) = \tilde{L}_\epsilon(\xi)(\tilde{v}).
\]

This last transformation rule allows us to define the twisted operator

\[
\tilde{L}_\epsilon(\xi)(\tilde{v}) = e^{i\xi \tilde{L}_\epsilon v} \circ L_\epsilon \circ F_\nu^{-1}(e^{-i\xi \tilde{v}}),
\] (77)

which is now a well-defined operator

\[
\tilde{L}_\epsilon(\xi) : W^{k+4,2}(S^1 \times S^{n-1}) \to W^{k,2}(S^1 \times S^{n-1}),
\]

for each value of the parameter \( \xi \in \mathbb{C} \). Here we identify \( S^1 = \mathbb{R}/T \). Our key point here is that \( \tilde{L}_\epsilon(\xi) \) act on the same function space for each value of \( \xi \).

Observe that \( \tilde{L}_\epsilon \) has the same coordinate expression as \( L_\epsilon \). We can again decompose \( \tilde{L}_\epsilon \) and \( \tilde{L}_\epsilon \) into Fourier components, obtaining \( \tilde{L}_{\epsilon,j} \) and \( \tilde{L}_{\epsilon,j} \). In particular, by (74) the restriction of \( \tilde{L}_{\epsilon,j,\eta} \) to the interval \([0, T_\epsilon]\) is exactly the operator \( L_{\epsilon,j,\eta} \) defined above.

For each \( \epsilon, j, \) and \( \eta \) the operator \( L_{\epsilon,j,\eta} \) is a fourth order ordinary differential operator and we denote its eigenvalues by \( \sigma_k(\epsilon, j, \eta) \) for \( k = 0, 1, 2, \ldots \). Furthermore \( L_{\epsilon,j,0} = L_{\epsilon,j,2\pi} \) for each \( \epsilon \) and \( j \), so we may think of

\[
\sigma_k(\epsilon, j, \cdot) : S^1 \to \mathbb{R}.
\]

We denote the image of this eigenvalue map by

\[
B_k(\epsilon, j) = \{ \sigma \in \mathbb{R} : \sigma = \sigma_k(\epsilon, j, \eta) \text{ for some } \eta \in [0, 2\pi/T_\epsilon]\}
\] (78)

as the \textit{jth spectral band} of \( L_{\epsilon,j} \).

Lemma 16. Each band \( B_k(\epsilon, j) \) is a nondegenerate interval.

Proof. Each \( L_{\epsilon,j} \) is a fourth order ordinary differential operator, and so the ODE \( L_{\epsilon,j} v = \sigma v \) has a four-dimensional solution space. If the function \( \sigma_k(\epsilon, j, \cdot) \) is constant on the interval \([0, 2\pi]\) then \( L_{\epsilon,j} v = \sigma v \) must have an infinite dimensional solution space for \( \sigma \in B_k(\epsilon, j) \), which is impossible. We conclude that no band \( B_k(\epsilon, j) \) may collapse to a single point. \( \square \)
The eigenfunction \( w \) corresponding to the eigenvalue \( \sigma_k(\epsilon, j, \eta) \) satisfies \( w(t + 2\pi/T) = e^{i\eta t}w(t) = e^{i(2\pi-\eta)t}w(t) \) and so \( \tilde{w}(t+2\pi) = e^{-i\eta t}\tilde{w}(t) \). However, the coefficients of the ordinary differential operator \( L_{\epsilon,j} \) are real, so

\[
\sigma_k(\epsilon, j, 2\pi - \eta) = \sigma_k(\epsilon, j, \eta)
\]

and we may as well restrict \( \sigma \) to the half-circle corresponding to \( 0 \leq \eta \leq \pi \).

It follows from Floquet theory [18] that the band functions \( B_{2k} \) are nonincreasing for each \( k \in \mathbb{Z} \) while \( B_{2k+1} \) are all nondecreasing, so that for each \( \epsilon, j \) and \( k \) we have

\[
\sigma_0(\epsilon, j, 0) \leq \sigma_0(\epsilon, j, \pi) \leq \sigma_1(\epsilon, j, \pi) \leq \sigma_1(\epsilon, j, 0) \leq \ldots. \tag{79}
\]

This in turn implies the bands all have the structure

\[
\begin{align*}
B_{2k}(\epsilon, j)(\epsilon, j) &= [\sigma_{2k}(\epsilon, j, 0), \sigma_{2k}(\epsilon, j, \pi)], \\
B_{2j+1}(\epsilon, j) &= [\sigma_{2j+1}(\epsilon, j, \pi), \sigma_{2j+1}(\epsilon, j, 0)].
\end{align*} \tag{80}
\]

We can related to bands \( B_k(\epsilon, 0) \) to the bands \( B_k(\epsilon, j) \) using the identity

\[
L_{\epsilon,j} = L_{\epsilon,0} - 2\lambda_j \frac{d^2}{dt^2} + \frac{n(n-4)}{2} \lambda_j + \lambda_j^2. \tag{81}
\]

Let \( w \) be an eigenvalue of \( L_{\epsilon,j,\eta} \), so that \( \eqref{81} \) implies

\[
\sigma_k(\epsilon, j, \eta)w = L_{\epsilon,j}w = L_{\epsilon,0}w - 2\lambda_j \tilde{w} + \frac{n(n-4)}{2} \lambda_j w + \lambda_j^2 w. \tag{82}
\]

Writing \( w = \sum_{l=0}^{\infty} \alpha_l w_l \) where \( L_{\epsilon,0}w_l = \sigma_l(\epsilon, 0, \eta)w_l \) we rewrite \( \eqref{82} \) as

\[
\sum_l \alpha_l \sigma_k(\epsilon, j, \eta)w_l = \sum_l \alpha_l \left( \sigma_l(\epsilon, 0, \eta)w_l - 2\lambda_j \tilde{w}_l + \frac{n(n-4)}{2} \lambda_j w_l + \lambda_j^2 w_l \right),
\]

which in turn gives us

\[
2\lambda_j \tilde{w}_l = -\left( \sigma_k(\epsilon, j, \eta) - \sigma_l(\epsilon, 0, \eta) - \frac{n(n-4)}{2} \lambda_j + \lambda_j^2 \right) w_l. \tag{83}
\]

This last eigenvalue equation admits quasi-periodic solutions only if

\[
\sigma_k(\epsilon, j, \eta) > \sigma_l(\epsilon, 0, \eta) + \frac{n(n-4)}{2} \lambda_j + \lambda_j^2. \tag{84}
\]

We have just proved the following lemma.

**Lemma 17.** For any positive integer \( j \) we have the lower bound

\[
\sigma_k(\epsilon, j, 0) > \sigma_0(\epsilon, 0, \eta) + \frac{n(n-4)}{2} \lambda_j + \lambda_j^2 \geq \sigma_0(\epsilon, 0, \eta) + \frac{(n-1)}{2}(n^2 - 2n - 2). \tag{85}
\]

Our main characterization of the spectral bands is the following Proposition.

**Proposition 18.** For each \( \epsilon \in (0, \epsilon_n) \) we have

\[
-\frac{n(n^2-4)}{2} \left( \frac{1}{T \epsilon} \int_0^{T \epsilon} \psi \left( \frac{2n\epsilon}{T \epsilon} \right) dt \right)^{4/n} \leq \sigma_0(\epsilon, 0, 0) < 0 \tag{86}
\]

and

\[
either \sigma_1(\epsilon, 0, 0) = 0 or \sigma_2(\epsilon, 0, 0) = 0. \tag{87}
\]
Proof. Observe that \( \dot{\psi} \) is a periodic solution of the ODE \( L_{e,0}(\dot{v}_e) = 0 \), so it must be an eigenfunction with associated eigenvalue 0, subject to periodic boundary conditions, i.e. \( \eta = 0 \). This eigenfunction has precisely two modal domains within the interval \([0, T]\), so it must correspond either to \( \sigma_1(\epsilon, 0) \) or to \( \sigma_2(\epsilon, 0) \). We don’t have enough information at this point to distinguish these two cases.

The function \( v_e \) is also \( T\)-periodic, and so is an appropriate test function for \( \sigma_{0}(\epsilon, 0, 0) \). We have

\[
L_{e,0}(v_e) = \dddot{v}_e - \left( \frac{n(n-4)+8}{2} \right) \ddot{v}_e + \frac{n^2(n-4)^2}{16} v_e - \frac{n(n+4)(n^2-4)}{16} v_\epsilon^{n+4}
\]

\[
= \dddot{v}_e + \left( \frac{n(n-4)+8}{2} \right) \ddot{v}_e + \frac{n^2(n-4)^2}{16} v_e - \frac{n(n+4)(n^2-4)}{16} v_\epsilon^{n+4}
\]

\[
= - \frac{n(n-4)}{2} v_\epsilon^{n+4} < 0,
\]

and so

\[
\sigma_0(\epsilon, 0, 0) = \frac{1}{T_0} \int_0^{T_0} v_L L_{e,0}(v_e) \, dt = \frac{n(n-4)}{2} \frac{\int_0^{T_0} v_\epsilon^{2n} \, dt}{\int_0^{T_0} v_e^{\frac{2n}{n-4}} \, dt} < 0,
\]

which gives the upper bound in (88).

On the other hand, combining the uniqueness theorem of [7] and the variational characterization of the Delaunay solution in Section 5 of [14] we see that (up to translations) \( v_e \) is the unique minimizer of the functional

\[
W_0^{4,2}([0, T]) \ni v \mapsto \frac{\int_0^{T} \dot{v}^2 + \left( \frac{n(n-4)+8}{2} \right) \ddot{v}^2 + \frac{n^2(n-4)^2}{16} v^2 \, dt}{\left( \int_0^{T_0} v_e^{\frac{2n}{n-4}} \, dt \right)^{\frac{n-4}{n}}}.
\]

By (88) we then have

\[
\frac{\int_0^{T_0} \dot{v}^2 + \left( \frac{n(n-4)+8}{2} \right) \ddot{v}^2 + \frac{n^2(n-4)^2}{16} v^2 \, dt}{\left( \int_0^{T_0} v_e^{\frac{2n}{n-4}} \, dt \right)^{\frac{n-4}{n}}} \geq \frac{n(n-4)}{16} \left( \int_0^{T_0} v_e^{\frac{2n}{n-4}} \, dt \right)^{4/n},
\]

for each \( v \in W_0^{4,2}([0, T]) \), which then gives

\[
\int_0^{T_0} v L_{e,0}(v) \, dt = \int_0^{T_0} \dot{v}^2 + \left( \frac{n(n-4)+8}{2} \right) \ddot{v}^2 + \frac{n^2(n-4)^2}{16} v^2 - \frac{n(n+4)(n^2-4)}{16} v_\epsilon^{2n} \, dt
\]

\[
\geq \frac{n(n-4)}{16} \left( \int_0^{T_0} v_e^{\frac{2n}{n-4}} \, dt \right)^{4/n} \left( \int_0^{T_0} v_\epsilon^{\frac{2n}{n-4}} \, dt \right)^{\frac{n-4}{n}} - \frac{n(n+4)(n^2-4)}{16} \left( \int_0^{T_0} v_e^{\frac{2n}{n-4}} \, dt \right)^{4/n}.
\]

Hölder’s inequality with exponents \( \frac{n}{n-4} \) and \( n/4 \) implies

\[
\int_0^{T_0} v^2 \, dt \leq T_\epsilon^{4/n} \left( \int_0^{T_0} v_\epsilon^{\frac{2n}{n-4}} \, dt \right)^{\frac{n-4}{n}},
\]

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which we combine with (89) to see
\[ \int_0^T vL_{\epsilon,0}(v) dt \geq \frac{n(n - 4)(n^2 - 4)}{16} T_{\epsilon}^{-4/n} \int_0^T v^2 dt \left( \int_0^T \frac{2n}{v^\epsilon} \right)^{4/n} \]
\[ = -\frac{n(n^2 - 4)}{2} T_{\epsilon}^{-4/n} \left( \int_0^T v^2 dt \right) \left( \int_0^T \frac{2n}{v^\epsilon} \right)^{4/n}, \]
for each \( v \in W_0^{4,2}([0, T_{\epsilon}]) \). Finally, we are free to choose a scale for our test function \( v \), and we normalize so that \( \int_0^T v^{1,4} = \int_0^T \frac{2n}{v^\epsilon} dt \). Using this normalization (90) becomes
\[ \int_0^T vL_{\epsilon,0}(v) dt \geq \frac{n(n - 4)(n^2 - 4)}{16} T_{\epsilon}^{-4/n} \int_0^T v^2 dt \left( \int_0^T \frac{2n}{v^\epsilon} \right)^{4/n} \]
\[ = -\frac{n(n^2 - 4)}{2} T_{\epsilon}^{-4/n} \left( \int_0^T v^2 dt \right) \left( \int_0^T \frac{2n}{v^\epsilon} \right)^{4/n}, \]
which gives the lower bound in (86).

**Corollary 19.** We have \( B_k(\epsilon, 0) \subset (0, \infty) \) for each \( k \geq 3 \) and \( B_k(\epsilon, 0) \subset [0, \infty) \) for each \( k \geq 2 \).

**Proof.** This follows from the previous proposition and (80).

**Corollary 20.** For each positive integer \( j \) we have \( B_k(\epsilon, j) \subset (0, \infty) \).

**Proof.** When \( j > n \) we have \( \lambda_j \geq 2n \), and in this case (85) gives
\[ \sigma_k(\epsilon, j, 0) > \sigma_0(\epsilon, 0, 0) + n^3 \]
for all \( k \). On the other hand, \( 0 < v_\epsilon < 1 \), so the lower bound in (86) gives us
\[ \sigma_0(\epsilon, 0, 0) \geq -\frac{n(n^2 - 4)}{2} \left( \frac{1}{T_{\epsilon}} \int_0^T \frac{2n}{v^\epsilon} dt \right)^{4/n} \geq -\frac{n(n^2 - 4)}{2}, \]
which implies
\[ \sigma_k(\epsilon, j, 0) > \sigma_0(\epsilon, 0, 0) + n^3 \geq n^3 - \frac{n(n^2 - 4)}{2} > 0. \]
and the corollary follows.

The case of \( 1 \leq j \leq n \) requires some more attention. Luckily, we know from (55) and (56) that
\[ L_{\epsilon,j}(w_{j}^\pm) = 0, \quad w_{j}^\pm = e^{\pm t} \left( \pm \frac{n - 4}{2} v_{\epsilon} \right) + R_{\pm}, \]
where \( R_+ = O(1) \) and \( R_- = O(e^{-2t}) \). By (57) these are positive, periodic solutions, and so must correspond to the bottom of the spectrum of \( L_{\epsilon,j} \).

The following lemma relates the spectral bands \( B_k(\epsilon, j) \) and the set of indicial roots \( \Gamma_{\epsilon,j} \) of the operator \( L_{\epsilon,j} \).

**Lemma 21.** We have \( 0 \in B_k(\epsilon, j) \) for some \( k \) if and only if the ODE \( L_{\epsilon,j}w = 0 \) admits a quasi-periodic solution.
Proof. Let \( \hat{\nu} \) satisfy
\[
0 = L_{\epsilon,j,\eta} \hat{\nu} = e^{i\eta t} \mathcal{F}_{\epsilon}(L_{\epsilon,j}(\mathcal{F}_{\epsilon}^{-1}(e^{-i\eta t} \hat{\nu}))) = L_{\epsilon,j}(\mathcal{F}_{\epsilon}^{-1}(e^{-i\eta t} \hat{\nu})) = 0.
\]
In particular, \( v = \mathcal{F}_{\epsilon}^{-1}(e^{-i\eta t} \hat{\nu}) \) satisfies \( L_{\epsilon,j}(v) = 0 \). In addition, (74) implies \( v(t + T_\epsilon, \theta) = e^{i\eta t} v(t, \theta) \), and so \( v \) must be quasi-periodic. \( \square \)

We summarize the important conclusions of this section with the following Corollary.

**Corollary 22.** The indicial root \( 0 \) is isolated, i.e. there exists \( \delta > 0 \) such that no other indicial roots lie in the interval \( (-\delta, \delta) \). Moreover, any Jacobi field with sub-exponential growth (i.e. tempered) must be a linear combination of \( w_0^+ \) and \( w_0^- \), the Jacobi fields generated by translations along the axis and changes of the Delaunay parameter.

### 3.6 Mapping properties of the linearized operator

We have already seen that for each \( \delta \in \mathbb{R} \) and \( k \in \mathbb{N} \) the mapping
\[
L_\epsilon : W^{k+4,2}_\delta((0, \infty) \times S^{n-1}) \to W^{k,2}_\delta((0, \infty) \times S^{n-1})
\]
is a linear, elliptic operator and it has bounded coefficients.

**Proposition 23.** The operator
\[
L_\epsilon : W^{k+4,2}_\delta((0, \infty) \times S^{n-1}) \to W^{k,2}_\delta((0, \infty) \times S^{n-1})
\]
is Fredholm provided \( \delta \notin \Gamma_\epsilon \), where \( \Gamma_\epsilon \) is given in (61).

Our proof follows that of Proposition 4.8 of [21].

**Proof.** We recall the twisted operator define in (77) depending on a parameter \( \xi \in \mathbb{C} \), namely
\[
\tilde{L}_\epsilon(\xi) : W^{k+4,2}(S^1 \times S^{n-1}) \to W^{k,2}(S^1 \times S^{n-1}), \quad \tilde{L}_\epsilon(\xi)(\tilde{v}) = e^{i\xi t} \mathcal{F}_{\epsilon} \circ L_\epsilon \circ \mathcal{F}_{\epsilon}^{-1}(e^{-i\xi t} \tilde{v}).
\]
We then use the analytic Fredholm theorem to prove the twisted operator is Fredholm away from a discrete set of poles in \( \mathbb{C} \). Unwinding definition, we then translate the Fredholm property of the twisted operator to corresponding properties of \( L_\epsilon \) on a weighted function space.

The operator \( \tilde{L}_\epsilon(\xi) \) is linear, bounded, and elliptic for each choice of \( \xi \), and depends on \( \xi \) holomorphically. Thus, by the analytic Fredholm theorem (see Section 5.3 of [21]) \( \tilde{L}_\epsilon(\xi) \) is either never Fredholm for any value \( \xi \) or it is Fredholm for \( \xi \) not in a certain discrete set. We choose \( \xi = \eta \in (0, 2\pi) \) and suppose there exists \( \tilde{\nu} \) such that \( \tilde{L}_\epsilon(\eta)(\tilde{\nu}) = 0 \), then \( \tilde{L}_\epsilon(\eta) = 0 \) where \( \nu = \mathcal{F}_{\epsilon}^{-1}(e^{-i\eta t} \tilde{\nu}) \). Then \( \nu \) is quasi-periodic, in that \( \nu(t + T_\epsilon, \theta) = e^{i\eta t} \nu(t, \theta) \), and so in particular \( \nu \) is bounded. However, by Corollary [22] any bounded Jacobi field must be a multiple of \( w_0^+ \), which is not quasi-periodic. Thus \( \tilde{L}_\epsilon(\eta) \) is injective. However, this operator is formally self-adjoint, so it is also surjective, and in particular \( \tilde{L}_\epsilon(\eta) \) is Fredholm.

Therefore we may safely apply the analytic Fredholm theorem to conclude there exists a discrete set \( \mathcal{P} \subset \mathbb{C} \) and a meromorphic operator
\[
\hat{G}_\epsilon(\xi) : W^{k,2}(S^1 \times S^{n-1}) \to W^{k+4,2}(S^1 \times S^{n-1})
\]
such that \( \tilde{v} = \hat{G}_\epsilon(\xi) \circ \tilde{L}_\epsilon(\xi)(\tilde{v}) \) so long as \( \xi \notin \mathcal{P} \). Moreover, by our construction
\[
\Gamma_\epsilon = \{ \nu \in \mathbb{R} : \nu = \Im(\xi) \text{ for some } \xi \in \mathcal{P} \}.
\]
We unravel this relation to find the Greens operator $G_\epsilon$. Again we let $v = F_\epsilon^{-1}(e^{-i\xi t}\tilde{\nu})$, so that
\[
e^{i\xi t}F_\epsilon(v) = \tilde{\nu} = \tilde{G}_\epsilon(L_\epsilon(v)) = G_\epsilon(e^{i\xi t}F_\epsilon(L_\epsilon(e^{-i\xi t}e^{i\xi t}F_\epsilon(v)))) = G_\epsilon(e^{i\xi t}F_\epsilon(L_\epsilon(v))).\]

We thus conclude
\[
G_\epsilon(\phi) = F_\epsilon^{-1}(e^{-i\xi T}\epsilon(G_\epsilon(e^{i\xi T}\epsilon(\phi))))).
\]

By our construction we have $v = G_\epsilon(\phi) \in W_{k+4,2}^{k}(0,\infty) \times S^{n-1}$, so that
\[
\delta = \Im(\xi) \not\in \Gamma_\epsilon \Rightarrow \text{there exist a Greens operator } G_\epsilon : W_{\delta}^{k+4,2}((0,\infty) \times S^{n-1}) \rightarrow W_{\delta}^{k}((0,\infty) \times S^{n-1}),
\]
and by the Fredholm alternative we have completed our proof.

**Lemma 24.** The set of indicial roots $\Gamma_\epsilon$ does not have any accumulation points.

**Proof.** Recall that each $\gamma \in \Gamma_\epsilon$ is the imaginary part of a pole of $\tilde{G}_\epsilon$, which forms a discrete set in $\mathbb{C}$. Furthermore, the operator $L_\epsilon(\xi)$ is unitarily equivalent to $\tilde{L}_\epsilon(\xi + 2\pi k)$ for each $k \in \mathbb{Z}$, and so $\xi$ is a pole of $\tilde{G}_\epsilon$ if and only if $\xi + 2\pi k$ is as well for each $k \in \mathbb{Z}$. Thus $\tilde{G}_\epsilon$ can only have finitely many poles in each horizontal strip.

**Corollary 25.** For each $\delta \in (0,1)$ the operator
\[
L_\epsilon : W_{-\delta}^{k+4,2}((0,\infty) \times S^{n-1}) \rightarrow W_{-\delta}^{k+4,2}((0,\infty) \times S^{n-1})
\]
is injective and
\[
L_\epsilon : W_{\delta}^{k+4,2}((0,\infty) \times S^{n-1}) \rightarrow W_{\delta}^{k+4,2}((0,\infty) \times S^{n-1})
\]
is surjective.

**Proof.** In the proof of Proposition 24 we showed that $\tilde{L}_\epsilon(\xi)$ is injective for each $\xi$ with $-1 < \Im(\xi) < 0$, which in turn implies $L_\epsilon : W_{-\delta}^{k+4,2}((0,\infty) \times S^{n-1}) \rightarrow W_{-\delta}^{k+4,2}((0,\infty) \times S^{n-1})$ is injective. The surjectivity statement now follows from the injectivity statement and duality because $L_\epsilon : W_{0}^{k+4,2}((0,\infty) \times S^{n-1}) \rightarrow W_{0}^{k+4,2}((0,\infty) \times S^{n-1})$ is formally self-adjoint.

We can extract more information from the Fourier-Laplace transform by examining how it behaves when the contour we are using to define its inverse crosses a pole of $\tilde{G}_\epsilon$. We use Proposition 4.14 of [20] as a model for the following.

**Proposition 26.** Let $\phi \in C_{0}^{\infty}((0,\infty) \times S^{n-1})$ and let $\delta \in (0,1)$ and $v \in W_{-\delta}^{4,2}((0,\infty) \times S^{n-1})$ satisfy $L_\epsilon(v) = \phi$. Then $v$ has an asymptotic expansion $v = \sum_{k=1}^{\infty}v_{j}$ with $L_\epsilon(v_{j}) = 0$ and $\nu_{j} \in W_{-\nu}^{4,2}$ for any $\nu < \gamma_{\epsilon,j}$.

**Proof.** We begin by transforming the equation. Choose $\xi \in \mathbb{C}$ with $\Im(\xi) > \delta > 0$ and let
\[
\bar{v} = e^{i\xi t}\tilde{\nu}, \quad \bar{\phi} = e^{i\xi t}\tilde{\phi},
\]
so that $\tilde{L}_\epsilon(\xi)\bar{v} = \bar{\phi}$. Applying the Greens operator $\tilde{G}_\epsilon(\xi)$ we have $\bar{v} = \tilde{G}_\epsilon(\xi)(\bar{\phi})$. That $\phi \in C_{0}^{\infty}((0,\infty) \times S^{n-1})$ implies $\tilde{\phi}$ is entire in the $\xi$ variable and smooth in the $(t,\theta)$ variables. On the other hand, in the half-plane $\Im(\xi) > \delta$ the poles of $\tilde{G}_\epsilon$ occur at the same points as the zeroes of $\bar{\phi}$ (with the same degrees), so $\bar{v}$ is analytic in this half-plane. In fact, because $\gamma_{\epsilon,1} = 1$, for any $\gamma' \in (\delta,1)$ the operator $\tilde{G}_\epsilon$ has no poles in the strip $\delta' < \Im(\xi) < \delta$, we can shift the contour integral
defining \( F^{-1} \) up to \( \Im(\xi) = \delta' \) for any \( \delta' < 1 \), and so we may take \( v \in W^{k+4,2}_{-\nu}(\mathbb{R} \times S^{n-1}) \) for any \( \nu < 1 \).

We complete the proof by shifting the contour integral across a pole of \( \tilde{G}_\epsilon \). We sketch this contour in Figure 1. Choose \( \delta'' \in (1, \gamma \epsilon, 2) \) and \( \xi'' \) such that \( \Im(\xi'') = \delta'' \), and let \( \tilde{\nu}'' = \tilde{G}_\epsilon(\xi'') \).

In fact the difference \( \tilde{v} - \tilde{v}'' = (\tilde{G}_\epsilon(\xi) - \tilde{G}_\epsilon(\xi''))(\phi) \) is the residue of a meromorphic function with a pole at height \( \Im(\xi) = -1 \) about a rectangular contour of width \( 2\pi \) and height \( \delta'' - \delta \). We do not see the result of the contour integral along the vertical sides of the rectangle because all our transformed functions are periodic in the real direction with period \( 2\pi \).

The following is a special case of Proposition 26, corresponding to moving the contour integral across only the first pole.

Corollary 27. Let \( \delta \in (0, 1) \), let \( \phi \in C^\infty((0, \infty) \times S^{n-1}) \) and \( L_{\epsilon}(v) = \phi \). Then there exist \( w \in W^{k+2}_{-\delta}(0, \infty) \times S^{n-1} \) and \( \psi \in \text{Span}\{w_0^+, w_0^-\} \) such that \( v = w + \psi \).

4 Simple convergence to a radial solution

In this section we prove an asymptotic estimate.

Theorem 28. Let \( v \in C^\infty((0, \infty) \times S^{n-1}) \) be a positive solution of (21) which also satisfies (22). Then either \( \limsup_{t \to \infty} v(t, \theta) = 0 \) or there exists a Delaunay parameter \( \epsilon \in (0, v_{cyl}) \) and translation parameter \( T \in [0, T_c) \) and positive constants \( C \) and \( \alpha \) such that

\[
|v(t, \theta) - v_\epsilon(t + T)| \leq Ce^{-\alpha t}. \tag{92}
\]

Remark 6. Strictly speaking, the proof below is not entirely necessary, as one should be able to show the estimate (92) is equivalent to the simple asymptotics Jin and Xiong derive in [14]. However, we believe the proof is different and interesting enough to include.
Lemma 29. Let \( \varepsilon \) \( v \) satisfy (21). We let \( \tau_k \to \infty \) and define
\[
v_k : (-\tau_k, \infty) \times S^{n-1} \to \infty, \quad v_k(t, \theta) = v(t + \tau_k, \theta).
\]
By the bounds (47) there are constants \( 0 < c_1 < c_2 \) (which depend on the solutions \( v \)) such that \( c_1 < v_k(t, \theta) < c_2 \) for all \( k \). Moreover, elliptic estimates imply the sequence \( \{\nabla v_k\} \) is also uniformly bounded, so in fact the family \( \{v_k\} \) is also equicontinuous. Thus a subsequence converges uniformly on compact sets to a solution \( \bar{v} = \lim_{\tau \to \infty} v_k \to \bar{v} \). Since \( \tau_k \to \infty \) we now have a global solution \( \bar{v} : \mathbb{R} \times S^{n-1} \to (0, \infty) \) of the PDE (21). By the classification theorem in [10] we must have \( \bar{v}(t, \theta) = v_k(t + T) \) for some \( \epsilon \in (0, \epsilon_n] \) and \( t \in [0, T_c) \).

It remains to show that \( \epsilon \) and \( T \) do not depend on the choice of sequence \( \tau_k \to \infty \) or the choice of subsequence \( v_k \).

By (14) for each \( k \) have
\[
\omega_n H_{rad}(v_k) = \lim_{k \to \infty} H_{rad}(v_k)
\]
\[
= \lim_{k \to \infty} \int_{\{1\} \times S^{n-1}} -\frac{\partial v_k}{\partial t} \frac{\partial^2 v_k}{\partial t^2} + \frac{1}{2} \left( \frac{\partial^2 v_k}{\partial t^2} \right)^2 + \frac{n}{2} (\Delta \theta v_k)^2 - \frac{n(n - 4)}{4} \left| \frac{\nabla v_k}{\partial t} \right|^2 d\sigma
\]
\[
= \int_{\{1\} \times S^{n-1}} -\frac{\partial v}{\partial t} \frac{\partial^3 v}{\partial t \partial^3 \theta} + \frac{1}{2} \left( \frac{\partial^2 v}{\partial t^2} \right)^2 + \frac{n}{2} (\Delta \theta v)^2 - \frac{n(n - 4)}{4} \left| \frac{\nabla v}{\partial t} \right|^2 d\sigma
\]
so \( \epsilon \) does not depend on any choices.

We complete the proof with the help of the following lemmas.

Lemma 29. Let \( \theta \in S^{n-1} \) and let \( v : (0, \infty) \times S^{n-1} \to (0, \infty) \) solve (21). Then
\[
\lim_{t \to \infty} \frac{\partial v}{\partial \theta}(t, \cdot) = 0
\]
uniformly.

Proof. If this were not the case then there exist \( (\tau_j, \theta_j) \in (0, \infty) \times S^{n-1} \) and \( C > 0 \) such that \( \tau_j \to \infty \) and
\[
\left| \frac{\partial v}{\partial \theta}(\tau_j, \theta_j) \right| \geq C > 0.
\]
Now translate by \(-\tau_j\) and rotate by \(-\theta_j\) to obtain a new sequence of solutions \( v_j \), for which
\[
\left| \frac{\partial v_j}{\partial \theta}(0, \theta_0) \right| \geq C > 0.
\]
However, by the reasoning above the sequence \( v_j \) must converge to a Delaunay solution, which does not depend on \( \theta \) at all, contradicting the lower bound displayed above. \( \square \)
Lemma 30. Let \( \theta \in S^{n-1} \) and let \( v \) solve (21). The Jacobi field \( \phi = \partial_\theta v \) decays exponentially in \( t \).

Proof. Let \( \tau_j \to \infty \) and let

\[
v_j(t, \theta) = v(t + \tau_j, \theta), \quad A_j = \sup \left\{ \left| \frac{\partial v_j}{\partial \theta} \right| : t > 0 \right\},
\]

Further suppose there exist \((s_j, \theta_j)\) such that \( |\partial_\theta v_j(s_j, \theta_j)| = |\partial_\theta v(s_j + \tau_j, \theta_j)| = A_j \). If the sequence \( \{s_j\} \) is unbounded then we can translate further to obtain \( \phi_j(t, \theta) = \partial_\theta v_j(t + s_j, \theta) = \partial_\theta v(t + s_j + \tau_j, \theta) \). Letting \( j \to \infty \) we obtain \( \phi : \mathbb{R} \times S^{n-1} \to \mathbb{R} \), which solves the linearized equation (43) about \( v_s(\cdot + T) \).

By our construction \( \phi \) is not identically zero and bounded. However, by our construction \( \phi \notin \text{Span}\{\hat{v}_0^+, \hat{v}_0^- = \frac{d}{dt} v_\epsilon\} \), so by Corollary 22 \( \phi \) must grow exponentially either as \( t \to \infty \) or \( t \to -\infty \). We have already shown \( \phi(t, \theta) \) must decay at some rate as \( t \to \infty \), so there must exist \( c > 0 \) such that \( |\phi(t, \theta)| \leq ce^{-t} \).

A consequence of Lemma 30 is that any choice of \( s_j > 0 \) such that

\[
v_j(s_j, \theta_j) = A_j = \sup \left\{ \left| \frac{\partial v_j}{\partial \theta} \right| : t > 0 \right\}
\]

remains bounded. In particular, there exists a positive integer \( N \) such that \( NT_\epsilon > \sup_{j \in \mathbb{N}} s_j \), where \( T_\epsilon \) is the period of \( v_\epsilon \). We now define the intervals \( I_N = [0, NT_\epsilon] \) and \( J_N = [NT_\epsilon, 2NT_\epsilon] \), and observe that the limit Jacobi field \( \phi \) we have just constructed is bounded for \( t > 0 \) and attains its supremum in \( I_N \).

Lemma 31. Let \( \phi \) be the Jacobi field constructed in Lemma 30. Then there exists \( c \) independent of all choices such that \( |\phi(t, \theta)| \leq ce^{-t} \).

Proof. Expand the Jacobi field \( \phi \) we have just constructed in Fourier modes. Let \( \{\hat{w}_j^+, \hat{w}_j^-\} \) span the solutions space of (61) and let

\[
E' = \text{Span}\{w_j^+, \hat{w}_j^- : j = 0, 1, \ldots, n\}, \quad E'' = \text{Span}\{w_j^+, \hat{w}_j^- : j \geq n + 1\},
\]

and write \( \phi = \phi' + \phi'' \), \( \phi' \in E', \quad \phi'' \in E'' \).

We claim that there exists \( c > 0 \) independent of any choices we have made such that \( |\phi(t, \theta)| \leq ce^{-t} \). We write

\[
\phi' = \sum_{j=0}^{n} (c_j^+ w_j^+ + \tilde{c}_j^- \hat{w}_j^-), \quad \phi'' = \sum_{j=n+1}^{\infty} (c_j^+ w_j^+ + \tilde{c}_j^- \hat{w}_j^-).
\]

By construction \( \phi \) decays at some rate as \( t \to \infty \), so \( c_0^+ = 0 \) and \( \tilde{c}_0^- = 0 \). Also by construction \( |\phi(t, \theta)| \leq 1 \) for \( t \geq 0 \), which implies \( |c_j^+| \) and \( |\tilde{c}_j^-| \) are all bounded independent of all choices for \( j = 1, \ldots, n \). We conclude \( |\phi'(t, \theta)| \leq c' e^{-t} \) for some \( c' > 0 \) which is independent of all choices. We complete the proof of the claim by showing \( |\phi''| \leq c'' e^{-t} \) for some \( c'' > 0 \) not depending on any choices. Suppose otherwise, then there would exist a sequence \( \phi''_i \in E'' \) such that

\[
A_i = \sup_{t \geq 0} e^t |\phi''_i(t, \theta)| \to \infty, \quad \phi''_i(t_i, \theta_i) = e^{-t_i} A_i.
\]

Define

\[
\tilde{\phi}_i : [-t_i, \infty) \times S^{n-1} \to \mathbb{R}, \quad \tilde{\phi}_i(t, \theta) = \frac{e^{t_i}}{A_i} \phi''_i(t + t_i, \theta) \in E''.
\]
The PDE \([13]\) is uniformly elliptic on any bounded cylinder, so the sequence \(\{t_i\}\) cannot be bounded. Thus \(t_i \to \infty\) and (after passing to a subsequence) we obtain a limit \(\tilde{\phi} \in E''\) on the whole cylinder \(\mathbb{R} \times \mathbb{S}^{n-1}\). Moreover, for \(t \geq -t_i\) we have

\[
\tilde{\phi}_i(t, \theta) = e^{\frac{t_i}{A_i}} \phi_i''(t + t_i, \theta) \leq e^{\frac{t_i}{A_i}} e^{-(t+t_i)} A_i = e^{-t},
\]

so that in the limit we obtain a global solution \(\tilde{\phi} \in E''\) on the whole cylinder \(\mathbb{R} \times \mathbb{S}^{n-1}\) which satisfies the bound \(|\tilde{\phi}(t, \theta)| \leq e^{-t}\). However, this contradicts the definition of \(E''\), as all functions in this space must grow as fast as \(e^{\gamma_0 t}\) either as \(t \to \infty\) or \(t \to -\infty\).

**Lemma 32.** Let \(v : (0, \infty) \times \mathbb{S}^{n-1} \to (0, \infty)\) solve (24) and let \(\tau_j\) be a sequence such that \(\tau_j \to \infty\). Define \(v_j(t, \theta) = v(t + \tau_j, \theta)\) and

\[
w_j(t, \theta) = v(t + \tau_j, \theta) - v_i(t + T), \quad \alpha_j = \sup_{0 \leq t \leq N T_\epsilon} |w_j(t, \theta)|, \quad \phi_j = \frac{1}{\alpha_j} w_j.
\]

Here \(v_i(\cdot + T)\) is the limit extracted from the sequence \(\{v_j\}\) using a priori upper and lower bounds. The sequence \(\{\phi_j\}\) converges to a Jacobi field \(\phi\) for \(v_i(\cdot + T)\). Moreover, \(\phi\) is bounded for \(t \geq 0\).

**Proof.** Let \(\Delta^2_{\text{cyl}}\) denote \(\Delta^2\) written in cylindrical coordinates, i.e.

\[
\Delta^2_{\text{cyl}} = \frac{\partial^4}{\partial t^4} - \left(\frac{n(n-4)+8}{2}\right) \frac{\partial^2}{\partial t^2} + \frac{n^2(n-4)^2}{16} \Delta^2_{\theta} + 2 \Delta^2_{\theta} \frac{\partial^2}{\partial t^2} - \frac{n(n-4)}{2} \Delta_{\theta}.
\]  

Then

\[
\Delta^2_{\text{cyl}} w_j = \frac{n(n-4)(n^2-4)}{16} \left(\psi_j - \frac{n+4}{n-4} \psi_j \right) = \frac{n(n-4)(n^2-4)}{16} \left(\frac{n+4}{n-4} \right) v_i \psi_j + O(\|v_j - v_i\|^2)
\]

which proves the sequence \(\{w_j\}\) converges to a Jacobi field. It then follows that \(\{\phi_j\}\) also converges to a Jacobi field \(\phi\).

Next we prove \(\phi\) is bounded for \(t \geq 0\). Decompose \(\phi\) into Fourier modes, writing

\[
\phi = \phi' + \phi'' = a_+ w_+ - a_- w_- + \phi'',
\]

where \(\phi''\) is the sum of all the nonzero Fourier modes. We may further decompose \(\phi'' = \phi''' + \phi''''\), where \(\phi'''\) is the sum of all the Fourier modes which grow exponentially as \(t \to +\infty\) and \(\phi''''\) is the sum of Fourier modes which decay exponentially as \(t \to +\infty\). Observe that \(\phi''\) is bounded if and only if \(\phi'''' = 0\). Now fix \(\theta \in \mathbb{S}^{n-1}\) and observe \(\partial_\theta \phi''\) grows exponentially, at least as fast as \(e^{\gamma_0 t}\), while \(\partial_\theta \phi''''\) decays exponentially, at least as fast as \(e^{-\epsilon t}\). Thus \(\phi''\) is bounded in the half-cylinder \(t \geq 0\) if and only if \(\partial_\theta \phi''''\) is bounded.

We can write a similar Fourier decomposition for \(\phi_j = \phi'_j + \phi''_j\) for each \(j\). Observe that each \(\phi'_j\) does not depend at all on \(\theta\), so that

\[
\partial_\theta \phi'_j = \partial_\theta \phi_j = \frac{1}{\alpha_j} \partial_\theta w_j = \frac{1}{\alpha_j} \partial_\theta v_j.
\]

We have just shown in Lemma [31] that \(\partial_\theta v_j\) converges to a Jacobi field \(\psi\) such that \(|\psi(t, \theta)| \leq c e^{-\epsilon t}\) for \(t > 0\), where \(c\) does not depend on any choices. Now let

\[
A_j = \sup\{ |\partial_\theta v_j(t, \theta)| = |\partial_\theta v(t + \tau_j, \theta)| : 0 \leq t \leq N T_\epsilon \}.
\]
If there exist $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 A_j \leq \alpha_j \leq C_2 A_j,$$

i.e. if $A_j$ and $\alpha_j$ are commensurate, then $\partial_\nu \phi'' = \partial_\nu \phi$ is commensurate with the Jacobi field $\psi$, and hence decays exponentially. If no uniform upper bound $C_2$ exists, then the sequence

$$\left\{ \frac{1}{\alpha_j} \partial_\nu v_j \right\}
$$

is unbounded and cannot converge, contradicting the convergence $\phi_j \to \phi$. Additionally, if no lower bound $C_1$ exists then $\phi_j \to 0$ uniformly on $[0, NT_\epsilon] \times S^{n-1}$, which is also a contradiction. Thus $\{A_j\}$ and $\{\alpha_j\}$ must be commensurate and so $\phi''$ must decay exponentially.

Lastly we show $a_- = 0$, completing the proof that $\phi$ is bounded for $t \geq 0$. We have

$$v_j(t, \theta) = v_\epsilon(t + T) + \alpha_j \phi + o(\alpha_j),$$

which implies

$$\mathcal{H}_{\text{rad}}(v_j) = \mathcal{H}_{\text{rad}}(v_\epsilon) + a_- \frac{d}{d\epsilon} \mathcal{H}_{\text{rad}}(v_\epsilon) + o(\alpha_j). \tag{95}$$

However, we have already shown $\mathcal{H}_{\text{rad}}(v_j) = \mathcal{H}_{\text{rad}}(v_\epsilon)$ for each $j$, so (95) is only possible if $a_- = 0$.

We have now shown $\phi = a_+ w_0^+ + \phi''$, where $\phi''$ decays exponentially. However $w_0^+$ is periodic and hence bounded, and so $\phi$ is bounded as well.

**Lemma 33.** Let $v : (0, \infty) \times S^{n-1} \to (0, \infty)$ satisfy (21) and (22), let $\tau \geq 0$, and let $B > 0$. Let

$$w_\tau(t, \theta) = v_\tau(t, \theta) - v_\epsilon(t + T) = v(t + \tau, \theta) - v_\epsilon(t + T)$$

and for $N \in \mathbb{N}$ let

$$\eta(\tau) = \sup_{0 \leq t \leq NT_\epsilon} |w_\tau(t, \theta)|.$$

If $\tau$ is sufficiently large and $\eta(\tau)$ is sufficiently small then there exists $s$ such that $|s| \leq B \eta(\tau)$ such that

$$\eta(\tau + NT_\epsilon + s) \leq \frac{1}{2} \eta(\tau). \tag{96}$$

**Proof.** Fix $B > 0$ and $N \in \mathbb{N}$. If (96) does not hold then there must exist $\tau_j \to \infty$ such that $\eta_j = \eta(\tau_j) \to 0$ but for any $s$ such that $|s| \leq B \eta_j$ it holds $\eta(\tau_j + NT_\epsilon + s) \geq \frac{1}{2} \eta_j$. Let $\phi_j = \frac{1}{\eta_j} \cdot w_{\tau_j}$. By Lemma 32 the sequence $\{\phi_j\}$ converges uniformly on compact subsets to a Jacobi field $\phi$. Moreover, $\phi$ is bounded for $t \geq 0$. By hypothesis

$$0 \leq t \leq NT_\epsilon \Rightarrow |\phi(t, \theta)| \geq \frac{1}{2}$$

so $\phi$ is not identically zero. In the half-cylinder $(0, \infty) \times S^{n-1}$ we expand $\phi = aw_0^+ + \phi$ where $\phi$ is a sum of Fourier modes all of which decay at least like $e^{-\gamma \cdot t \epsilon}$. The coefficients of the other Fourier modes must all be zero because $\phi$ is bounded for $t \geq 0$.

Next we adjust $v_\epsilon(t + T)$ by the translation parameter $s_j = -\eta_j a$. Without loss of generality $B > |a|$, so that $|s_j| \leq B \eta_j$. Thus

$$w_{\tau_j + s_j}(t, \theta) = v(t + \tau_j - \eta_j a, \theta) - v_\epsilon(t + T) = w_{\tau_j}(t, \theta) - a \eta_j w_0^+(t) + o(\eta_j),$$

which implies

$$w_{\tau_j + s_j} = \eta_j \phi + o(\eta_j).$$
The function \( \tilde{\phi} \) decays at least like \( e^{-t} \) and \( \sup_{0 \leq t \leq NT_c} |\tilde{\phi}(t, \theta)| \leq 1 \), so we may choose \( N \) sufficiently large such that
\[
\sup_{0 \leq t \leq NT_c} |w_{r_j + s_j + NT_c}(t, \theta)| = \sup_{NT_c \leq t \leq 2NT_c} |w_{r_j + s_j}(t, \theta)|
\]
\[
= \eta_j \sup_{NT_c \leq t \leq 2NT_c} |\tilde{\phi}(t, \theta) + o(\eta_j)|
\]
\[
\leq \frac{1}{4} \eta_j.
\]
However, this contradicts the hypothesis \( \eta(\tau_j + NT_c + s) > \frac{1}{4} \eta(\tau_j) \), completing the proof.  

We complete our proof of (92) by showing we can choose \( \sigma \) such that \( w_\sigma \to 0 \). Again we choose \( B > 0 \) and \( N \in \mathbb{N} \). Letting \( t > 0 \) be sufficiently large we may assume \( B\eta(0) \leq \frac{1}{4} NT_c \). Let \( \tau_0 = 0 \) and choose \( s_0 \) such that (96) holds.

Next we choose the sequences \( \tau_j, s_j, \) and \( \sigma_j \) by
\[
\tau_j = \tau_{j-1} + s_{j-1} + NT_c, \quad \sigma_j = \sum_{i=0}^{j-1} s_i,
\]
where \( s_j \) satisfies (96) with the choice \( \tau = \tau_j \). Iterating we see
\[
\eta(\tau_j) \leq 2^{-j} \eta(0) \Rightarrow |s_j| \leq 2^{-j-1} NT_c \Rightarrow \sigma = \lim_{j \to \infty} \sigma_j = \sum_{i=0}^{\infty} s_j \leq NT_c.
\]
Finally we must show that \( \sigma \) is indeed the correct translation parameter. For any \( t > 0 \) write \( t = jNT_c + [t] \) where \( 0 \leq [t] < NT_c \). Then
\[
w_\sigma(t, \theta) = v(t + \sigma, \theta) - v(t + T)
\]
\[
= v(t + \sigma_j, \theta) - v(t + T) + v(t + \sigma, \theta) - v(t + \sigma_j, \theta)
\]
\[
= w_{r_j}([t], \theta) + O(2^{-j}).
\]
Our bound on \( \eta(\tau_j) \) then implies
\[
v(t + \sigma, \theta) - v(t + T) = w_\sigma(t, \theta) \leq C2^{-j},
\]
which is exactly the exponential decay we claimed.  

5 \hspace{1em} \textbf{Refined asymptotics}

We finally derive a refined asymptotic expansion of solutions of (18) with an isolated singularity, essentially writing out the next term in the Taylor expansion.

**Theorem 34.** Let \( v : (0, \infty) \times S^{n-1} \to (0, \infty) \) be a smooth solution of (21) such that the associated metric \( g_{ij} = v \frac{1}{n} (dt^2 + d\theta^2) \) has positive scalar curvature. Then either \( \limsup_{t \to \infty} v(t, \theta) = 0 \), in which case \( v \) is a spherical solution, or there exist \( \epsilon \in [\epsilon_n, 1], T \in [0, T_c], a \in \mathbb{R}^n, C > 0 \) and \( \beta > 1 \) such that
\[
|v(t, \theta) - v_{\epsilon, a}(t + T, \theta)| \leq Ce^{-\beta t}.
\]

**Proof.** By (92) there exist \( \epsilon, T, C_1, \) and \( \alpha \) such that
\[
|v(t, \theta) - v_{\epsilon}(t + T)| \leq C_1 e^{-\alpha t}.
\]
In other words
\[
w(t, \theta) = v(t, \theta) - v_{\epsilon}(t + T) \in C^\infty_{\alpha}((0, \infty) \times S^{n-1}).
\]
However,

\[
L_{c}(w) = \Delta^{2}_{\text{cyl}}(v - v_{c}(\cdot + T)) - \frac{n(n + 4)(n^{2} - 4)}{16} v_{c}^{\star} w
\]

\[
= \left( \frac{n(n - 4)(n^{2} - 4)}{16} \left( \frac{n+n+4}{v^{n-4}} - \frac{n+n+4}{v_{c}^{n-4}} \right) - \frac{n(n + 4)(n^{2} - 4)}{16} \right) v_{c}^{\star} w
\]

\[
= Q_{\text{cyl}}(w) \in C^{\infty}_{2\alpha}((0, \infty) \times S^{n-1})
\]

where $\Delta^{2}_{\text{cyl}}$ is given in (94). We combine this with Proposition 26 to see $w \in C^{\infty}_{2\alpha}((0, \infty) \times S^{n-1})$. We iterate this finitely many times to obtain $w \in C^{\infty}_{2\alpha}((0, \infty) \times S^{n-1})$ for some $\delta \in (1/2, 1)$, and so we can apply Proposition 26 once more to see

\[
w \in C^{\infty}_{\beta}((0, \infty) \times S^{n-1}) \oplus \text{Span}\{w_{j}^{\pm}, j = 0, 1, 2, \ldots, n\}, \quad \beta = \min\{2\delta, \gamma_{c, 2}\}.
\]

However, combining (35) and (54) we have

\[
v_{c,a}(t + T, \theta) = v_{c}(t + T) + e^{-t-t} \langle \theta, a \rangle w_{1}^{\ast}(t + T) + O(e^{-2t}),
\]

which completes the proof. \qed
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