THE $gl(M|N)$ SUPER YANGIAN AND ITS FINITE DIMENSIONAL REPRESENTATIONS

R. B. ZHANG
Department of Pure Mathematics
University of Adelaide
Adelaide, S. A., Australia

November 21, 2018

Abstract

Methods are developed for systematically constructing the finite dimensional irreducible representations of the super Yangian $Y(gl(M|N))$ associated with the Lie superalgebra $gl(M|N)$. It is also shown that every finite dimensional irreducible representation of $Y(gl(M|N))$ is of highest weight type, and is uniquely characterized by a highest weight. The necessary and sufficient conditions for an irrep to be finite dimensional are given.

Mathematics subject classifications(1991): 17B37, 81R50, 17A70
Running title: Representations of $Y(gl(M|N))$

1 Introduction

This is the second of a series of papers developing the representation theory of the super Yangians. In an earlier publication[1], the finite dimensional irreducible representations of the super Yangian $Y(gl(1|1))$ were classified, and explicit bases for such representations were constructed. It is the aim of this paper to carry out a similar program for the super Yangian associated with the Lie superalgebra $gl(M|N)$ for all $M$ and $N$. In particular, we will prove that every finite dimensional irrep of $Y(gl(M|N))$ is of highest weight type, and uniquely characterized by a highest weight, and give the necessary and sufficient conditions for an irrep to be finite dimensional. Methods will also be developed for systematically constructing the finite dimensional irreps.

The structures and representations of the Yangians associated with ordinary Lie algebras were studied extensively[2][3][4][5][6][7][8]. The central result in the representation theory is Drinfeld’s Theorem[2], which provides a characterization of finite
dimensional irreps in terms of highest weight polynomials, and also gives the necessary and sufficient conditions for an irrep to be finite dimensional. However, more detailed structural information on Yangian irreps, such as their dimensions and their decompositions with respect to the underlying Lie algebras, is difficult to obtain, primarily due to the existence of finite dimensional indecomposable but non-irreducible representations. At present, the understanding of the structures of irreps of Yangians is still incomplete.

Super Yangians and their quantum analogues arose naturally from the algebraic description of integrable lattice models with Lie superalgebra symmetries. Their representation theory plays a central role in the study of such models, e.g., by applying the algebraic Bethe Ansatz to diagonalize the transfer matrices. Super (quantum) Yangians are also algebraic systems of considerable mathematical interest; their structure and representations merit a thorough investigation in their own right.

Some structural features of super (quantum) Yangians, in particular, their connections with the Lie superalgebras and the related quantum supergroups, have already been studied by Nazarov [9] and also in [10]. The finite dimensional irreps of the simplest super Yangian $Y(gl(1|1))$ have been classified, and explicit bases for such irreps have been constructed [9]. It is the aim of this letter to systematically develop the representation theory of the super Yangian $Y(gl(M|N))$.

In section 2, we define the super Yangian $Y(gl(M|N))$, then study its structure. In particular, we will prove a BPW theorem, which will be our starting point for developing the representation theory. Section 3 contains the main results. We prove that every finite dimensional irrep of $Y(gl(M|N))$ is of highest weight type, and is uniquely characterized by the highest weight. A general method is developed to construct highest weight irreps; the necessary and sufficient conditions for an irrep to be finite dimensional are also given. In section 4 we outline another construction of the finite dimensional irreps of $Y(gl(M|N))$, which is similar to Kac' induced module construction [11] for Lie superalgebras. This construction should be useful for investigating detailed structures of irreps.

This letter relies heavily on [9]. The reasoning and techniques employed in proving some of the results of subsection 3.1 are adopted from that paper. To make this letter reasonably self-contained, we spell out these proofs in some detail.

## 2 Structure of $Y(gl(M|N))$

### 2.1 Definition

The super Yangian $Y(gl(M|N))$ was first defined by Nazarov [9] following the Faddeev - Reshetikhin - Takhtajian formalism of quantization of algebraic structures. We present the definition here to fix our notation.

We will work on the complex number field $\mathbb{C}$. The underlying vector space of the Lie superalgebra $gl(M|N)$ is $\mathbb{Z}_2$ graded, with a homogeneous basis $\{E_b^a | a, b = 1, 2, ..., M + N\}$. Introduce the gradation index $[ ] : \{1, 2, ..., M + N\} \rightarrow \mathbb{Z}_2$ such that $[a] = \begin{cases} 0, & a \leq M, \\ 1, & a > M. \end{cases}$ Let $gl(M|N)_\theta$, $\theta \in \mathbb{Z}_2$, be the vector space over $\mathbb{C}$ spanned by the $E_b^a$ with $[a] + [b] \equiv \theta \pmod{2}$. Then $gl(M|N)_0$ and $gl(M|N)_1$ are
the even and odd subspaces of $gl(M|N)$ respectively. We will abuse the notation a bit and define $[\ ] : gl(M|N)_0 \cup gl(M|N)_1 \rightarrow \mathbb{Z}_2$, $[x] = \begin{cases} 0, & x \in gl(M|N)_0, \\ 1, & x \in gl(M|N)_1. \end{cases}$ The Lie superalgebra $gl(M|N)$ is this $\mathbb{Z}_2$ graded vector space endowed with the bilinear graded bracket $[\ ,\ ] : gl(M|N) \otimes gl(M|N) \rightarrow gl(M|N),$

$$[E^a_b, E^c_d] = \delta^c_b E^a_d - (-1)^{([a]+[b])([c]+[d])}E^c_b \delta^a_d. \tag{1}$$

For convenience, we can regard $gl(M|N)$ as embedded in its universal enveloping algebra. Thus the graded bracket $[\ ,\ ]$ can be interpreted as the graded commutator

$$[x, y] = xy - (-1)^{[x][y]}yx. \tag{2}$$

The vector module of $gl(M|N)$ is an $(M + N)$-dimensional $\mathbb{Z}_2$-graded vector space $V$, spanned by the homogeneous elements $\{v^a|a = 1, 2, ..., M + N\}$ where $v^a$ is even if $[a] = 0$ and odd otherwise. The action of $gl(M|N)$ on $V$ is defined by $E^a_b v^c = \delta^c_b v^a$. We denote the associated vector representation of $gl(M|N)$ by $\pi$. Then in this basis $\pi(E^a_b) = e^a_b$, where $e^a_b \in End(V)$ are the standard matrix units.

Define the permutation operator $P : V \otimes V \rightarrow V \otimes V$ by

$$P(v^a \otimes v^b) = (-1)^{[a][b]}v^b \otimes v^a.$$

Then explicitly, we have

$$P = \sum_{a,b=1}^{M+N} e^a_b \otimes e^b_a (-1)^{[b]}.$$

It is well known that the following $R$ matrix

$$R(u) = 1 + \frac{P}{u}, \quad u \in \mathbb{C}, \tag{3}$$

satisfies the graded Yang - Baxter equation.

Let us introduce

$$L(u) = \sum_{a,b=1}^{M+N} (-1)^{[b]} t^a_b(u) \otimes e^b_a,$$

$$t^a_b(u) = (-1)^{[b]} \delta^a_b + \sum_{n=1}^{\infty} t^a_b[n] u^{-n},$$

where $u$ is an indeterminate, and the $t^a_b[n], 0 < n \in \mathbb{Z}_+$, are homogeneous elements of $Y(gl(M|N))$ such that $t^a_b[n]$ is even if $[a] + [b] \equiv 0 (mod 2)$ and odd otherwise. The $L(u)$ belongs to the $\mathbb{Z}_2$ graded vector space $Y(gl(M|N)) \otimes End(V)$ and is even. Now $Y(gl(M|N))$ is the $\mathbb{Z}_2$ graded associative algebra generated by the $t^a_b[n], 0 < n \in \mathbb{Z}_+$, with the following defining relations

$$L_1(u)L_2(v)R_{12}(v-u) = R_{12}(v-u)L_2(v)L_1(u). \tag{4}$$

Note that equation (4) lives in $Y(gl(M|N)) \otimes End(V) \otimes End(V)$. The multiplication of the factors on both sides are defined with respect to the grading of this triple tensor
product. To gain some concrete feel about this algebra, we put (4) into a more explicit form:

\[ [t^{a_1}_{b_1}(u), t^{a_2}_{b_2}(v)] = \frac{(-1)^{\eta(a_1,b_1;a_2,b_2)}}{u - v} \left[ t^{a_2}_{b_1}(u)t^{a_1}_{b_2}(v) - t^{a_2}_{b_2}(v)t^{a_1}_{b_1}(u) \right], \]

or equivalently,

\[ \eta(a_1,b_1; a_2, b_2) = [a_1][a_2] + [b_1]([a_1] + [a_2]) \text{(mod 2)}; \]

or equivalently,

\[
[t^{a_1}_{b_1}[m], t^{a_2}_{b_2}[n]] = \delta^{a_2 a_1}_{b_2 b_1} [m + n - 1] - (-1)^{([a_1] + [b_1])([a_2] + [b_2])} t^{a_2}_{b_1}[m + n - 1] \delta^{a_1}_{b_2} \\
+ \sum_{r=1}^{\min(m,n)-1} \left\{ t^{a_2}_{b_1}[r] t^{a_1}_{b_2}[m + n - 1 - r] - t^{a_2}_{b_2}[m + n - 1 - r] t^{a_1}_{b_1}[r] \right\}. \tag{5}
\]

\( Y(gl(M|N)) \) admits co-algebraic structures compatible with the associative multiplication. We have the co-unit \( \epsilon : Y(gl(M|N)) \rightarrow \mathbb{C} \), \( t^b_0[k] \mapsto \delta^k_0 \delta^b_0 (-1)^{[k]} \), the co-multiplication \( \Delta : Y(gl(M|N)) \rightarrow Y(gl(M|N)) \otimes Y(gl(M|N)) \), \( L(u) \mapsto L(u) \otimes L(u) \), and also the antipode \( S : Y(gl(M|N)) \rightarrow Y(gl(M|N)) \), \( L(u) \mapsto L^{-1}(u) \). Thus \( Y(gl(M|N)) \) is indeed a \( \mathbb{Z}_2 \) graded Hopf algebra.

Note that \( Y(gl(M|N)) \) also admits the following generalized tensor product structure:

\[
\Delta^{(k-1)}_\alpha : Y(gl(M|N)) \rightarrow Y(gl(M|N))^{\otimes k}, \quad L(u) \mapsto L(u + \alpha_1) \otimes L(u + \alpha_2) \otimes \ldots \otimes L(u + \alpha_k), \tag{6}
\]

where \( \alpha_1 = 0 \), and \( \alpha_i, i = 2, 3, \ldots, k \), are a set of arbitrary complex parameters. Explicitly, we have

\[
\Delta^{(k-1)}_\alpha(t^a_b(u)) = \sum_{a_1, \ldots, a_{k-1}} (-1)^{\sum_{i=1}^{k-1} \{ [a_i] + [a_0] + [a_i] + [a_{i+1}] \}} \times t^{a_1}_{b_1}(u) \otimes t^{a_2}_{b_2}(u + \alpha_2) \otimes \ldots \otimes t^{a}_{b_{k-1}}(u + \alpha_k)
\]

where \( a_0 = b \), and \( a_k = a \).

Another useful fact is the existence of an automorphism \( \phi_f : Y(gl(M|N)) \rightarrow Y(gl(M|N)) \) associated with each power series \( f(x) = 1 + f_1 x^{-1} + f_2 x^{-2} + \ldots \), which is defined by

\[
t^a_b(x) \mapsto \tilde{t}^a_b(x) = f(x)t^a_b(x). \tag{7}
\]

As can be easily seen, the \( \tilde{t}^a_b \) satisfy exactly the same relations as the \( t^a_b \) themselves.

Some further simple properties of the super Yangian are worth observing. Note that \( Y(gl(M|N)) \) as a Hopf algebra is a deformation of the universal enveloping algebra of the infinite dimensional Lie superalgebra \( \hat{gl}(M|N)^{(+)} = gl(M|N) \otimes \mathbb{C}[\![t]\!] \), where \( \mathbb{C}[\![t]\!] \) denotes the ring of polynomials in the indeterminate \( t \). Set \( E^a_b[k] = E^a_b \otimes t^k, k \in \mathbb{Z}_+ \). Then the graded bracket for \( \hat{gl}(M|N)^{(+)} \) reads

\[
[E^a_b[k], E^c_d[t]] = \delta^a_c E^a_d[k + l] - (-1)^{([a] + [b])([c] + [d])} E^a_b[k + l] \delta^a_d. \tag{8}
\]

The universal enveloping algebra \( U(\hat{gl}(M|N)^{(+)}) \) of this Lie superalgebra is a \( \mathbb{Z}_2 \)-graded associative algebra, which may be thought as generated by \( E^a_b[k], k \in \mathbb{Z}_+ \), subject to
the relations (8) but with the left hand side interpreted as the graded commutator defined in (2).

This algebra in fact has the structure of a $\mathbb{Z}_2$-graded Hopf algebra. In particular, its co-multiplication is given by

$$\delta : U(\hat{gl}(M|N)^{(+)}) \to U(\hat{gl}(M|N)^{(+)} \otimes U(\hat{gl}(M|N)^{+)})$$

$$E^a_k \mapsto E^a[k] \otimes 1 + 1 \otimes E^a[k].$$

In order to see that $Y(gl(M|N))$ is indeed a deformation of the Hopf superalgebra $U(\hat{gl}(M|N)^{(+)}),$ we set $t^a_b[m + 1] = \kappa^{-m} \delta^a_b[m], m \in \mathbb{Z}_+,$ where $\kappa$ is an indeterminate. Then $Y(gl(M|N))$ is isomorphic to the algebra $\hat{U}$ generated by $E^a_b[m], m \in \mathbb{Z}_+,$ subject to the relations

$$[E^{a_1}_{b_1}[m], E^{a_2}_{b_2}[n]] = \delta^{a_2}_{b_2} E^{a_1}_{b_2}[m + n] - (-1)^{([a_1] + [b_1])([a_2] + [b_2])} E^{a_2}_{b_1}[m + n] \delta^{a_1}_{b_2}$$

$$+ \kappa (-1)^{([a_1] + [b_1] + [a_2])} \sum_{r=1}^{\text{Min}(m,n)} \left\{ E^{a_2}_{b_1}[r - 1] E^{a_1}_{b_2}[m + n - r] - E^{a_2}_{b_1}[m + n - r] E^{a_1}_{b_2}[r - 1] \right\}.$$ 

Regard $\hat{U}$ as an algebra defined on the polynomial ring $\mathbb{C}[[\kappa]].$ Then it is clear from the above equation that $U(\hat{gl}(M|N)^{(+)} = \hat{U}/\kappa \hat{U}.$ Also, the co-multiplication $\Delta$ of $Y(gl(M|N))$ induces a co-associative co-multiplication $\Delta : \hat{U} \to \hat{U},$ which is clearly the deformation of the co-multiplication $\delta$ of $U(\hat{gl}(M|N)^{(+)}.$

Important structural and representation theoretical properties of $Y(gl(M|N))$ can be obtained by investigating this Hopf superalgebra within the framework of deformation theory$^{[12]}.$ Results will be reported in a future publication.

The super Yangian $Y(gl(M|N))$ contains several subalgebras, which will be useful for developing the representation theory. We can easily see from the defining relations (8) that the generators $t^a_b[1]$ form the Lie superalgebra $gl(M|N).$ For each $n > 1,$ the generators $t^a_b[n]$ transform as the components of an adjoint tensor operator of this $gl(M|N).$ Define a map $t^a_b[n] \mapsto \delta_{1n} E^a_b,$ where $E^a_b$ are the standard generators of $gl(M|N).$ Then it extends to an algebra homomorphism $Y(gl(M|N)) \to U(gl(M|N)).$

There also exist various Hopf (super) subalgebras of $Y(gl(M|N)).$ In particular, the following will be used in the remainder of the letter:

$$Y(gl(M)) \text{ generated by } \{ t^a_b[n] | a, b = 1, 2, ..., M, 0 < n \in \mathbb{Z}_+ \};$$

$$Y(gl(N)) \text{ generated by } \{ t^a_b[n] | a, b = M + 1, M + 2, ..., M + N, 0 < n \in \mathbb{Z}_+ \};$$

$$Y(gl(1|1)) \text{ generated by } \{ t^M_{M+1}[n], t^{M+1}_M[n], t^M_M[n], t^{M+1}_{M+1}[n] | 0 < n \in \mathbb{Z}_+ \}.$$ (9)

Note that although both $Y(gl(M))$ and $Y(gl(N))$ are even subalgebras, together they do not form a subalgebra of $Y(gl(M|N)).$ This leads to certain complications in the development of the representation theory.

### 2.2 BPW theorem

We now prove a version of the BPW theorem for the super Yangian $Y(gl(M|N)),$ which will be of crucial importance for developing the representation theory. Let us introduce a filtration on $Y(gl(M|N)).$ Define the degree of a generator $t^a_b[n]$ by $\text{deg}(t^a_b[n]) = n,$
and require that the degree of a monomial $t_{p_1}^{a_1}[n_1]t_{p_2}^{a_2}[n_2]...t_{p_k}^{a_k}[n_k]$ is $\sum_{r=1}^{k} n_r$. Let $Y_p$ be the vector space over $C$ spanned by monomials of degree not greater than $p$. Then

$$... \supset Y_p \supset Y_{p-1} \supset ... \supset Y_1 \supset Y_0 = C,$$

$$Y_pY_q \subset Y_{p+q}.$$

Let $z_1$, $z_2$, ..., $z_k$ be some $t^a_b[n]$'s. Consider the product $Z = z_1z_2...z_k$, which is assumed to have $deg(Z) = p$. It directly follows from the defining relations (5) of the super Yangian $Y(gl(M|N))$ that for any permutation $\sigma$ of $(1, 2, ..., k),$

$$z_1z_2...z_k - \epsilon(\sigma)z_{\sigma(1)}z_{\sigma(2)}...z_{\sigma(k)}$$

belongs to $Y_{p-1}$, where $\epsilon(\sigma)$ is $-1$ if $\sigma$ permutes the odd elements in $z_1$, $z_2$, ..., $z_k$ an odd number of times, and $+1$ otherwise. In particular, if $t^a_b[n]$ is odd, then $(t^a_b[n])^2 \in Y_{2n-1}$. Therefore, given any ordering of the generators $t^a_b[n]$, $0 < n \in Z_+$, $a, b \in \{1, 2, ..., M + N\}$, their ordered products of degrees less or equal to $p$ span $Y_p$, where the products do not contain factors $(t^a_b[n])^2$ if $[a] + [b] \equiv 1(\text{mod } 2)$. It immediately follows that the ordered products of all degrees span the underlying vector space of $Y(gl(M|N))$. As we will show presently, the ordered products are also linearly independent, thus form a basis for $Y(gl(M|N))$.

Define $U_p = Y_p/Y_{p-1}$. Then the multiplication of $Y(gl(M|N))$ defines a bilinear map $U_p \otimes U_q \to U_{p+q}$, which extends to a multiplication $U \otimes U \to U$ for the space $U = \oplus_{p=0}^\infty U_p$, turning $U$ into an associative algebra. This algebra is isomorphic to the algebra of polynomials $G[X]$ in the variables $X^a_b[n]$, $a, b \in \{1, 2, ..., M + N\}$, $n = 1, 2, ...$, with the isomorphism $U \cong G[X]$ defined by $t^a_b[n] \mapsto X^a_b[n]$, $\forall a, b, n$, where $X^a_b[n]$ is an ordinary indeterminate if $[a] + [b] \equiv 0(\text{mod } 2)$, and is a Grassmannian variable if $[a] + [b] \equiv 1(\text{mod } 2)$. (Note that for any Grassmannian variables $\zeta_1$ and $\zeta_2$ we have $\zeta_i\zeta_j = -\zeta_j\zeta_i$, $i, j = 1, 2$. ) Since monomials in $X^a_b[n]$ are linearly independent as elements of $G[X]$, we conclude that ordered products of the $t^a_b[n]$ (not allowing powers of order higher than 1 of the odd $t^a_b[n]$) are linearly independent. To summarize, we have

**Theorem 1**: For any given ordering of the generators $t^a_b[n]$, $a, b \in \{1, 2, ..., M + N\}$, $0 < n \in Z_+$, the ordered products of the $t^a_b[n]$ containing no second and higher order powers of the odd generators form a basis of $Y(gl(M|N))$.

It is useful to construct an explicit basis for $Y(gl(M|N))$. To do that, we need to fix some notations. Consider the pairs $(a, b)$, $a, b \in \{1, 2, ..., M + N\}$. Let

$$\Phi_+ = \{(a, b)|a < b\},$$

$$\Phi_- = \{(a, b)|a > b\},$$

$$\Phi_0 = \{((a, a)\},$$

$$\Phi_\pm = \{(a, b) \in \Phi_\pm|[a] + [b] \equiv \theta(\text{mod } 2)\}.$$ 

Given any $p = (a, b)$, we denote $\bar{p} = (b, a)$. We introduce a total ordering $\succ (=)$ of all the pairs in the following way: for any $p_+ \in \Phi_+$, $p_0 \in \Phi_0$, $p_- \in \Phi_-$, we define
unique (up to scalar multiples) maximal vector $v$.

Every finite dimensional irreducible representation $Y$ of $gl(M|N)$ contains a maximal vector which positions $Q^{(k)}_p\{n_p\}$ on the right of $Q^{(k_p')}_{p'}\{n_{p'}\}$ if $p \succ p'$. Now

**Lemma 1** The following elements

$$
\prod_{p \in \Phi_+}^{\succ} Q^{(k_p)}_p\{n_p\} \prod_{q \in \Phi_0} Q^{(k_q)}_q\{n_q\} \prod_{r \in \Phi_0}^{\succ} Q^{(k_r)}_r\{n_r\} \prod_{s \in \Phi_-} Q^{(k_s)}_s\{n_s\} \prod_{t \in \Phi_-} Q^{(k_t)}_t\{n_t\}
$$

form a basis of $Y(gl(M|N))$.

## 3 Finite Dimensional Irreps

We study structures of the finite dimensional irreps of $Y(gl(M|N))$ in this section. Some of the results reported here are generalizations of those on the representations of $Y(gl(1|1))$ to the present case, and the proofs of these results are also adopted from

### 3.1 Highest weight irreps

Let $V$ be an irreducible $Y(gl(M|N))$-module. A nonzero element $v^A_+ \in V$ is called maximal if

$$
t^n_a[n]v^A_+ = 0, \quad \forall (a, b) \in \Phi_+, \quad n > 0,
$$

$$
t^n_a[n]v^A_+ = \lambda_a[n]v^A_+, \quad a = 1, 2, ..., M + N, \quad n > 0,
$$

where $\lambda_a[n] \in \mathbb{C}$. An irreducible module is called a highest weight module if it admits a maximal vector. We define

$$
\Lambda(x) = (\lambda_1(x), \lambda_2(x), ..., \lambda_{M+N}(x)), \quad \lambda_a(x) = (-1)^{[a]} + \sum_{k>0} \lambda_a[k]x^{-k},
$$

and call $\Lambda(x)$ a highest weight of $V$.

Note that commutators amongst the $t^n_a[n]$ do not close on these generators themselves; the usual practice of using Lie’s Theorem to prove the existence of a common eigenvector for them does not work here. But nevertheless, we have

**Theorem 2** Every finite dimensional irreducible $Y(gl(M|N))$-module $V$ contains a unique (up to scalar multiples) maximal vector $v^A_+$. 

Lemma 2 Define $V_0 = \{ v \in V \mid t^a_0[n]v = 0, \forall (a, b) \in \Phi_+, n > 0 \}$. Then

1. $V_0 \neq 0$;
2. the generators $t^a_0[n]$ stabilize $V_0$;
3. for all $v \in V_0$,

$$[t^a_0[m], t^b_0[n]]v = 0, \forall a, b, m, n.$$ 

Proof: 1). Let $E$ be an $(M+N)$-dimensional vector space over $C$. We define a map $\alpha : \Phi_+ \cup \Phi_- \to E$ by setting $\alpha(a, b)$ to the vector with the $a-th$ component being $+1$ and the $b-th$ component being $-1$, e.g., $\alpha(1, 3) = (1, 0, -1, 0, ..., 0)$. We can introduce a partial ordering of all the elements of $E$ by requiring that $\mu \succ \nu$ (i.e., $\nu < \mu$) if $\mu = \nu + \sum_{p \in \Phi_+} l_p \alpha(p)$, $l_p \in Z_+$, where at least for one $p \in \Phi_+, l_p > 0$.

If the nonvanishing element $v \in V$ is a common eigenvector of the $t^a_0[1]$, then

$$t^a_0[1]v = \mu_a v, \mu_a \in C.$$ 

In this way, every such $v$ is associated with a unique vector $\mu = (\mu_1, \mu_2, ..., \mu_{M+N}) \in E$, which we call the $gl(M|N)$ weight of $v$. It is obvious that if a set of common eigenvectors all have different $gl(M|N)$ weights, then they must be linearly independent. Since \{ $t^a_0[1]|a = 1, 2, ..., M + N$ \} forms an abelian Lie algebra, it follows Lie’s Theorem that there exists at least one nonvanishing element of $V$, which is their common eigenvector.

Let $v \in V$ be a common eigenvector of the $t^a_0[1]$ with a $gl(M|N)$ weight $\mu$. If $v \in V_0$, we have proved the first part of the Lemma. If $v \not\in V_0$, by applying $t^b_0[n], (a, b) \in \Phi_+$, to $v$ we will arrive at other common eigenvectors of the $t^a_0[1]$, which have $gl(M|N)$ weights $\succ \mu$. Since $V$ is finite dimensional, there can only exist a finite number of vectors with $gl(M|N)$ weights $\succ \mu$. Hence repeated applications of the generators $t^b_0[n], (a, b) \in \Phi_+$, to $v$ will lead to a nonvanishing $v_0 \in V$ such that

$$t^b_0[n]v_0 = 0, \forall (a, b) \in \Phi_+, n > 0,$$

$$t^a_0[1]v_0 = \lambda_a[1]v_0, \forall a.$$ 

This proves that $V_0$ contains at least one nonzero element.

2). Let $v$ be a vector of $V_0$. We want to prove that all $t^a_{0k}[n_k]t^a_{0k-1}[n_{k-1}]...t^a_{01}[n_1]v$, $a_i = 1, 2, ..., M + N, n_i > 0, k \geq 0$, are annihilated by $t^b_0[m], (a, b) \in \Phi_+, m > 0$. The $k = 0$ case requires no proof. Assume that all the vectors $v_l = t^a_{01}[n_1]...t^a_{0l}[n_l]v, l < k$, are in $V_0$. Then

i). $(a, b) \in \Phi_+, b > c$,

$$t^a_0[m]t^c_0[n_k]v_{k-1} = -[t^c_0[n_k], t^a_0[m]]v_{k-1}$$

$$= (-1)^{|c| + 1} \sum_{r=0}^{\min(m, n_k)-1} (t^a_0[r]t^c_0[m + n_k - 1 - r] - t^c_0[m + n_k - 1 - r]t^a_0[r]) v_{k-1}$$

$$= 0,$$

ii). $(a, b) \in \Phi_+, b \leq c$,

$$t^a_0[m]t^c_0[n_k]v_{k-1} = [t^a_0[m], t^c_0[n_k]]v_{k-1}$$

$$= \sum_{r=0}^{\min(m, n_k)-1} (t^c_0[r]t^a_0[m + n_k - 1 - r] - t^a_0[m + n_k - 1 - r]t^c_0[r]) v_{k-1}$$

$$\times (-1)^{|a|+|b|(|a|+|c|)} = 0.$$
3). The following defining relations of $Y(gl(M|N))$

$$
[t^a_n, t^b_n] = 0, \quad a = 1, 2, ..., M + N,
$$

$$
[t^a_n, t^c_n] = (-1)^{|b|} \sum_{r=0}^{\min(m,n)-1} \left( t^b_r t^c_{m+n-1-r} - t^c_r t^b_{m+n-1-r} \right),
$$

and part 2) of the Lemma directly lead to part 3).

**Proof of theorem** By part 3) of Lemma 2, the action of the $t^a_n$ on $V_0$ coincides with an abelian subalgebra of $gl(V_0)$. Therefore, Lie’s Theorem can be applied, and we conclude that there exists at least one common eigenvector of all the $t^a_n$ in $V_0$. This proves the existence of the highest weight vector. Assume that $v_+$ and $v'_+$ are two highest weight vectors of $V$, which are not proportional to each other. Applying $Y(gl(M|N))$ to them generates two nonzero submodules of $V$, which are not equal. This contradicts the irreducibility of $V$.

We now turn to the construction of highest weight irreps of $Y(gl(M|N))$. Let $N^+, N^-$ and $Y^0$ be the vector spaces spanned by the ordered products (as defined in (14)) of the elements $t^a_n$, with $(a, b) \in \Phi^+$, $(a, b) \in \Phi^-$ and $(a, b) \in \Phi^0$ respectively. Set $Y^+ = Y^0 N^+$, and $Y^- = N^- Y^0$. We emphasize that these vector spaces do not form subalgebras of $Y(gl(M|N))$.

Consider a one dimensional vector space $C v^A_+$. We define a linear action of $Y^+$ on it by

$$
t^a_n v^A_+ = 0, \quad (a, b) \in \Phi_+,
$$

$$
t^a_n v^A_+ = \lambda_a [n] v^A_+,
$$

$$
t^a_n y_0 v^A_+ = \lambda_a [n] y_0 v^A_+, \quad \forall y_0 \in Y^0. \quad (13)
$$

From the proof of Lemma 2 we can see that the definition (13) is consistent with the commutation relations of $Y(gl(M|N))$. Now we define the following vector space

$$
\tilde{V}(\Lambda) = Y(gl(M|N)) \otimes_{Y^+} v^A_+.
$$

Then $\tilde{V}(\Lambda)$ is a $Y(gl(M|N))$ module, which is obviously isomorphic to $N^- \otimes v^A_+$.

The action of $Y(gl(M|N))$ on this module is defined in the following way. Every vector of $\tilde{V}(\Lambda)$ can be expressed as $y \otimes v^A_+$ for some $y \in N^-$. For simplicity, we write it as $y v^A_+$. Given any $u \in Y(gl(M|N))$, $uy$ can be expressed as a linear sum of the basis elements (14). We write

$$
uy = \sum a_{\alpha, \beta} y^{(\alpha)} y^{(\beta)} y^A_+, \quad y^{(\alpha)} y^{(\beta)} \in N^-, \quad y^0, y^\omega \in Y^0, \quad 1 \neq y^\sigma \in N^+.
$$

Then

$$
u(v^A_+) = \sum a_{\alpha, \beta} y^{(\alpha)} y^{(\beta)} v^A_+.$$
The $Y(gl(M|N))$ module $\bar{V}(\Lambda)$ is infinite dimensional. Standard arguments show that it is indecomposable, and contains a unique maximal proper submodule $M(\Lambda)$. Construct

$$V(\Lambda) = \bar{V}(\Lambda)/M(\Lambda).$$

Then $V(\Lambda)$ is an irreducible highest weight $Y(gl(M|N))$ module.

Let $V_1(\Lambda)$ and $V_2(\Lambda)$ be two irreducible $Y(gl(M|N))$ modules with the same highest weight $\Lambda(x)$. Denote by $v_{1,+}^\Lambda$ and $v_{2,+}^\Lambda$ their maximal vectors respectively. Set $W = V_1(\Lambda) \oplus V_2(\Lambda)$. Then $v_{+}^\Lambda = (v_{1,+}^\Lambda, v_{2,+}^\Lambda)$ is maximal, and repeated applications of $Y(gl(M|N))$ to $v_{+}^\Lambda$ generate an $Y(gl(M|N))$ submodule $V(\Lambda)$ of $W$. Define the module homomorphisms $P_i : V(\Lambda) \to V_i(\Lambda)$ by

$$P_1(v_1, v_2) = (v_1, 0),$$
$$P_2(v_1, v_2) = (0, v_2), \quad v_1 \in V_1(\Lambda), \ v_2 \in V_2(\Lambda).$$

Since

$$P_1(v_{1,+}^\Lambda, v_{2,+}^\Lambda) = (v_{1,+}^\Lambda, 0),$$
$$P_2(v_{1,+}^\Lambda, v_{2,+}^\Lambda) = (0, v_{2,+}^\Lambda),$$

it follows the irreducibility of $V_1(\Lambda)$ and $V_2(\Lambda)$ that $\text{Im}P_i = V_i(\Lambda)$. Now $\text{Ker}P_1$ is a submodule of $V_2(\Lambda)$. The irreducibility of $V_2(\Lambda)$ forces either $\text{Ker}P_1 = 0$ or $\text{Ker}P_1 = V_2(\Lambda)$. But the latter case is not possible, as $(0, v_{2,+}^\Lambda) \notin W$. Similarly we can show that $\text{Ker}P_2 = 0$. Hence, $P_i$ are $Y(gl(M|N))$ module isomorphisms.

To summarize the preceding discussions, we have

**Theorem 3** Corresponding to each $\Lambda(x)$ of the form $(\nu \Lambda)$, there exists a unique irreducible highest weight $Y(gl(M|N))$ module $V(\Lambda)$ with highest weight $\Lambda(x)$.

Before closing this subsection, we consider some useful facts about tensor products of highest weight irreps of $Y(gl(M|N))$. Let $W(\mu)$ and $W(\nu)$ be finite dimensional irreducible $Y(gl(M|N))$ modules with highest weights $\mu(x) = (\mu_1(x), \mu_2(x), ..., \mu_{M+N}(x))$ and $\nu(x) = (\nu_1(x), \nu_2(x), ..., \nu_{M+N}(x))$ respectively. (The existence of such modules will be proven in the next subsection.) Let $w_+^\mu$ and $w_+^\nu$ be the maximal vectors of these modules. Then $v_+ = w_+^\mu \otimes w_+^\nu \in W(\mu) \otimes W(\nu)$ is a maximal vector such that

$$t_+^a(x)v_+ = (-1)^{[a]} \mu_a(x) \nu_a(x)v_+, \ \forall a.$$

Set $\lambda_a(x) = (-1)^{[a]} \mu_a(x) \nu_a(x)$, and define the $\star$ - product of the highest weight $\mu(x)$ and $\nu(x)$ by

$$\mu(x) \star \nu(x) = (\lambda_1(x), \lambda_2(x), ..., \lambda_{M+N}(x)).$$

Applying $Y(gl(M|N))$ to $v_+$ generates an indecomposable $Y(gl(M|N))$ module $\bar{V}(\mu \star \nu)$, which is contained in $W(\mu) \otimes W(\nu)$. The quotient module of $\bar{V}(\mu \star \nu)$ by its unique maximal invariant submodule yields an irreducible $Y(gl(M|N))$ module $V(\mu \star \nu)$ with highest weight $\mu(x) \star \nu(x)$, which is necessarily finite dimensional. (The maximal invariant submodule of $\bar{V}(\mu \star \nu)$ can of course be trivial. In that case, $\bar{V}(\mu \star \nu) = V(\mu \star \nu)$.) Clearly this discussion can be generalized to tensor products of more than two irreps.
3.2 Finite dimensionality conditions

Let $V(\Lambda)$ be a finite dimensional irreducible $Y(gl(M|N))$ module with highest weight $\Lambda(x)$. Denote its maximal vector by $v^\Lambda_\mu$. We now consider the actions of the subalgebras of $Y(gl(M|N))$ defined by (3) on the maximal vector of $V(\Lambda)$. The following $V(\Lambda)$ subspaces $Y(gl(M))v^\Lambda_\mu$, $Y(gl(N))v^\Lambda_\mu$ and $Y(gl(1|1))v^\Lambda_\mu$ furnish indecomposable modules over the subalgebras $Y(gl(M))$, $Y(gl(N))$, and $Y(gl(1|1))$ respectively. It is obvious but very important to note that these modules, being subspaces of $V(\Lambda)$, are finite dimensional. Thus, the necessity part of the following theorem immediately follows from Drinfeld’s Theorem[2] (also see[8]) and a result of [1]:

**Theorem 4** The irreducible highest weight $Y(gl(M|N))$ - module $V(\Lambda)$ is finite dimensional if and only if its highest weight $\Lambda(x)$ satisfies the following conditions

$$\frac{\lambda_a(x)}{\lambda_{a+1}(x)} = \frac{P_a(x + (-1)^{|a|})}{P_a(x)}, \quad 1 \leq a < N + M, \ a \neq M,$$

$$\frac{\lambda_M(x)}{\lambda_{M+1}(x)} = \frac{\tilde{Q}_M(x)}{Q_M(x)},$$

where

$$P_a(x) = \prod_{i=1}^{K_a}(x + p_a^{(i)}), \quad 1 \leq a < N + M, \ a \neq M,$$

$$\tilde{Q}_M(x) = \prod_{i=1}^{K_M}\left(1 + \frac{r_2^{(i)}}{x}\right),$$

$$Q_M(x) = -\prod_{i=1}^{K_M}\left(1 + \frac{r_2^{(i)}}{x}\right), \quad \tilde{Q}_M(x), \ Q_M(x) \text{ co - prime.}$$

**Proof:** We only need to prove sufficiency. Let us consider the special case where the highest weight $\mu(x) = (\mu_1(x), \mu_2(x),...,\mu_{M+N}(x))$ of a $Y(gl(M|N))$ irrep is of the form

$$\mu_a(x) = (-1)^{|a|} + \mu_a x^{-1}, \ \forall a.$$  \hfill (15)

We denote the irreducible $Y(gl(M|N))$ module with highest weight $\mu(x)$ by $W(\mu)$, and the associated irrep by $\pi_\mu$. This irreducible representation can be explicitly constructed using the irreducible $gl(M|N)$ representation $\gamma_\mu$ with highest weight $\mu = (\mu_1, \mu_2,...,\mu_{M+N})$. We have

$$\pi_\mu(\epsilon^\mu_{a,b}(u)) = (-1)^{|b|}\delta^a_b + \gamma_\mu(E^a_b)u^{-1}, \ \forall a, b,$$

where $E^a_b$ are the standard $gl(M|N)$ generators.

In this case, the given conditions of the theorem are equivalent to the following constraints on $\mu$:

$$\mu_a - \mu_{a+1} \in \mathbb{Z}_+ , \quad 1 \leq a < N + M, \ a \neq M.$$  \hfill (16)

This is nothing else but the necessary and sufficient conditions in order for the $gl(M|N)$ irrep $\gamma_\mu$ to be finite dimensional. Therefore, the $Y(gl(M|N))$ irrep $\pi_\mu$ is also finite.
dimensional, and we have proved the sufficiency in this case. This also proves the fact that finite dimensional irreps of $Y(gl(M|N))$ indeed exist.

The next step in the proof involves showing that up to an automorphism $\phi_f$ of $Y(gl(M|N))$ defined by (7), every finite dimensional irreducible $Y(gl(M|N))$ module $V(\Lambda)$ with highest weight $\Lambda(x)$ sits inside the tensor product of some irreducible $Y(gl(M|N))$ modules $W(\mu)$, where the highest weights of these modules are of the form (15). To do this, we note that an automorphism $\phi_f$ transforms the highest weight according to $\Lambda(x) \mapsto f(x)\Lambda(x)$. Thus, it leaves the $P_a$, $Q_M$, and $\widetilde{Q}_M$ intact, but an appropriate choice of $f(x)$ will change the components of $\Lambda(x)$ into polynomials in $x^{-1}$ defined by

$$\lambda_a(x) = \prod_{c=1}^{a-1} Q_c(x) \prod_{d=a}^{M+N-1} \widetilde{Q}_d(x), \quad \forall a,$$

where we have used the following notation

$$Q_a(x) = (-1)^{(K_a+1)[a]} \prod_{i=1}^{K_a} q_a^{(i)}(x), \quad q_a^{(i)}(x) = (-1)^{[a]} \left(1 + \frac{1}{x^{q_a^{(i)}}}\right),$$

$$\widetilde{Q}_a(x) = (-1)^{(K_a+1)[a]} \prod_{i=1}^{K_a} \widetilde{q}_a^{(i)}(x), \quad \widetilde{q}_a^{(i)}(x) = (-1)^{[a]} \left(1 + \frac{1}{x^{(1-[a]})}\right), \quad a \neq M,$$

$$\widetilde{q}_M^{(i)}(x) = \left(1 + \frac{r_1^{(i)}}{x}\right), \quad q_M^{(i)}(x) = - \left(1 + \frac{1}{x^{q_M^{(i)}}}\right).$$

Define

$$\mu^{(t,i)}(x) = (\mu_1^{(t,i)}(x), \mu_2^{(t,i)}(x), \ldots, \mu_{M+N}^{(t,i)}(x)),$$

$$\mu_a^{(t,i)}(x) = \left\{ \begin{array}{ll}
 q_t^{(i)}(x), & t < a, \\
 \widetilde{q}_t^{(i)}(x), & t \geq a,
 \end{array} \right.$$

$$i = 1, 2, \ldots, K_t, \quad t = 1, 2, \ldots, M + N - 1.$$

Then $\Lambda(x)$ can be expressed as the $\star$- product of all the $\mu^{(t,i)}(x)$

$$\Lambda(x) = \star_{t=1}^{M+N-1} \star_{i=1}^{K_t} \mu^{(t,i)}(x).$$

The $\mu^{(t,i)}(x)$ are of the form (15) and satisfy the conditions (16). Therefore, corresponding to each $\mu^{(t,i)}(x)$, there exists a finite dimensional irreducible $Y(gl(M|N))$ module $W(\mu^{(t,i)})$ with highest weight $\mu^{(t,i)}(x)$. It follows the discussions at the end of the last subsection that the tensor product module $\otimes_{t=1}^{M+N-1} \otimes_{i=1}^{K_t} W(\mu^{(t,i)})$ (The order of the tensor product is not important for us.) contains as a submodule an indecomposable $Y(gl(M|N))$ module $\bar{V}(\Lambda)$. A quotient module of $\bar{V}(\Lambda)$ gives rise to an irreducible $Y(gl(M|N))$ module which is isomorphic to $V(\Lambda)$. Being a submodule of a finite dimensional module, $\bar{V}(\Lambda)$ is finite dimensional, and so is also $V(\Lambda)$. This proves the sufficiency of the given conditions of the theorem.

4 Another Construction of Irreps

Experiences with the representation theories of the Lie superalgebras and quantum supergroups urge us to ask whether a method similar to Kac’ induced module
construction\cite{11} can be developed to build irreps of $Y(gl(M|N))$. The answer to this question is affirmative. We now outline the construction.

Introduce an auxiliary algebra $Y(gl(M))\hat{+}Y(gl(N))$, which is the product of the two Yangians $Y(gl(M))$ and $Y(gl(N))$. In more explicit terms, $Y(gl(M))\hat{+}Y(gl(N))$ is generated by $\{\phi_{ij}^*[n], \psi_{\mu\nu}^*[n] | i, j = 1, 2, ..., M, \mu, \nu = M + 1, M + 2, ..., M + N, 0 < n \in \mathbb{Z}_+\}$, where the $\phi_{ij}^*[n]$ satisfy the standard defining relations of a $gl(M)$ Yangian algebra $Y(gl(M))$, and the $\psi_{\mu\nu}^*[n]$ satisfy relations of $Y(gl(N))$, while

$$[\phi_{ij}^*[m], \psi_{\mu\nu}^*[n]] = 0.$$  

Given a finite dimensional irreducible $Y(gl(M))\hat{+}Y(gl(N))$ module $V_0$, we define the action of the following $Y(gl(M|N))$ generators $\{t_{ij}^*[n], t_{\mu\nu}^*[n], t_{i\mu}^*[n] | i, j = 1, 2, ..., M, \mu, \nu = M + 1, M + 2, ..., M + N, 0 < n \in \mathbb{Z}_+\}$ on $V_0$ by

$$t_{\mu\nu}^*[n]v = 0,$$

$$t_{ij}^*[n]v = \phi_{ij}^*[n]v,$$

$$t_{i\mu}^*[n]v = \psi_{\mu\nu}^*[n]v, \forall v \in V_0.$$  

It follows from the first equation and the $Y(gl(M|N))$ defining relations (3) that $[t_{ij}^*[m], t_{\mu\nu}^*[n]]v = 0, \forall v \in V_0$. Thus this definition is self consistent.

We further define the vector space $\bar{V}$ spanned by

$$\left\{ \prod_{\{k_p\} \in \Phi \cap \{1\}} Q_p^{(k_p)}[\{n_p\}] \otimes v | \forall \{k_p\}, \{n_p\}; v \in V_0 \right\},$$

which furnishes an indecomposable $Y(gl(M|N))$ module, with the module action defined in the obvious way. Then the quotient module of $\bar{V}$ by its unique maximal invariant submodule is the irreducible $Y(gl(M|N))$ module which we intend to construct. From Theorems (3) and (4) we deduce that this construction yields all the finite dimensional irreps of $Y(gl(M|N))$.

Kac’ construction\cite{11} proves to be useful for studying the representation theory of Lie superalgebras. A generalization of the method also enabled us to develop a relatively satisfactory representation theory for the type I quantum supergroups\cite{13}. We hope that the construction presented here will also provides a practicable method for investigating detailed structures of the finite dimensional irreps of $Y(gl(M|N))$.

**Acknowledgements:** Part of this work was done while I visited the Institutes of Applied Mathematics and Theoretical Physics, Chinese Academy of Sciences, Beijing. I wish to thank Professors S. K. Wang and K. Wu for their hospitality. Financial support from the Australian Research Council is gratefully acknowledged.
References

[1] R. B. Zhang, Representations of super Yangian, J. Math. Phys., in press.

[2] V. G. Drinfeld, Soviet Math. Dokl. 36 (1988) 212.

[3] V. O. Tarasov, Theor. Math. Phys. 61 (1984) 163; 63 (1985) 175.

[4] I. V. Cherednik, Duke Math. J. 54 (1987) 563.

[5] V. Chari and A. Pressley, J. Reine Angew. Math. 417 (1991) 87.

[6] A. N. Kirillov and N. Yu Reshetikhin, Lett. Math. Phys. 12 (1986) 199.

[7] N. Yu Reshetikhin, Theor. Math. Phys. 63 (1985) 347.

[8] A. Molev, Lett. Math. Phys. 30 (1994) 53.

[9] M. L. Nazarov, Lett. Math. Phys. 21 (1991) 123.

[10] R. B. Zhang, Lett. Math. Phys. 33 (1995) 263.

[11] V. G. Kac, Adv. Math. 26 (1977) 8;
    M. Scheunert, Lecture Notes in Math., 716, Berlin: Springer (1979).

[12] M. Gerstenhaber, Ann. Math. 78 (1963) 267.

[13] R. B. Zhang, J. Math. Phys., 34 (1993) 1236; J. Phys. A26 (1993) 7041.