The cohomology of the projective unitary groups

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Abstract
The projective unitary group $PU(n)$ is the quotient group of the unitary group $U(n)$ by its center $S^1$. We determine the integral cohomology ring $H^*(PU(n))$ using explicitly constructed generators.

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1 Introduction
The projective unitary group $PU(n)$ is the quotient group $U(n)/S^1$ of the unitary group $U(n)$ by its center $S^1 = \{e^{i\theta}I_n; \theta \in [0, 2\pi]\}$, where $I_n$ denotes the identity, $n \geq 2$. In the cohomology theory of Lie groups these groups are of particular interest, because any integer dividing $n$ is the order of some torsion elements of its integral cohomology $H^*(PU(n))$, while for the other types of simple Lie groups $G$ (see [21, p.674]) the only orders of the torsion elements in their integral cohomologies $H^*(G)$ are

2 for $G$ with local types $SO(n), Sp(n), G_2, F_4, E_6, E_7, E_8$;
3 for $G$ with local types $F_4, E_6, E_7, E_8$;
4 for $G$ isomorphic to $SO(4n)/\mathbb{Z}_2$, and
5 for $G$ isomorphic to $E_8$.

In this paper we determine the integral cohomology ring $H^*(PU(n))$ using explicitly constructed generators.

Earlier in 1954 Borel [5] investigated the cohomologies of the quotients of the classical Lie groups $SO(n), SU(n)$ and $Sp(n)$ by their center, and obtained in particular a presentation of the mod $p$ cohomology $H^*(PU(n); \mathbb{F}_p)$ with $p$ a prime. Baum and Browder [6] recovered the result in their study of the Hopf algebra structure on $H^*(PU(n); \mathbb{F}_p)$. In [20] Ruiz stated a presentation of the cohomology of the projective complex Stiefel manifolds $Y_{n,n-m}$, that contains the group $PU(n)$ as the special case $Y_{n,n}$, which is obviously wrong, see Example 1.8. Another result relevant to ours is due to Petrie [16, 17], who showed that, if $n = p^r$, then the topological $K$-theory of $PU(n)$ is additively isomorphic to

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the cohomology $H^*(PU(n))$. Recently, inspired by questions arising from the twisted $K$-theory and divisibility of characteristic classes \cite{1 \ 2 \ 3}, the problem of determining the cohomology of the classifying space of $PU(n)$ has appeared on the agenda \cite{11}. Our result can of course be useful to the topic.

The binomial coefficients $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ appear naturally in our calculation and construction. It would be convenient for us to begin with an arithmetic result eliminating those numbers. Given an integer $1 \leq r \leq n$ we set

$$b_{n,r} := \text{g.c.d.}\left\{\binom{n}{1}, \ldots, \binom{n}{r}\right\}.$$  

Because $b_{n,r}$ divides $b_{n,r-1}$ one obtains the sequence $\{c_2, \ldots, c_n\}$ of integers with $c_r := \frac{b_{n,r-1}}{b_{n,r}}$. The following result, that expresses this sequence in term of the integer $n$, has been shown in \cite{10} Lemma 3.3. By the prime factorization of an integer $n \geq 2$ we mean the unique expression $n = p_1^{r_1} \cdots p_t^{r_t}$ with $1 < p_1 < \cdots < p_t$ the set of all primes factors of $n$.

**Lemma 1.1.** If $n$ is an integer with the prime factorization $n = p_1^{r_1} \cdots p_t^{r_t}$, then

i) $c_r = p_i$ or $1$ in accordance to $r = p_i^s$ with $1 \leq i \leq t$ and $1 \leq s \leq r_i$, or otherwise;

ii) $b_{n,1} = n$, $b_{n,r} = c_{r+1} \cdot c_{r+2} \cdots c_n$. □

Turning to the group $PU(n)$ we shall regard the quotient homomorphism

$$(1.1) \quad c : U(n) \to PU(n) = U(n)/S^1$$

as an oriented circle bundle on $PU(n)$, and let $\omega \in H^2(PSU(n))$ be its Euler class. The Gysin sequence \cite{15} p.149 for spherical fibration provides an exact sequence relating the cohomologies of the groups $U(n)$ and $PU(n)$

$$(1.2) \quad \cdots \to H^r(PU(n)) \xrightarrow{c_*} H^r(U(n)) \xrightarrow{\partial} H^{r-1}(PU(n)) \xrightarrow{\omega} H^{r+1}(PU(n)) \xrightarrow{c_*} \cdots$$

In addition to the routine property of the homomorphism $\theta$

$$(1.3) \quad \theta(c^*(y) \cup x) = y \cup \theta(x) \text{ for } y \in H^r(PU(n)), \ x \in H^r(U(n))$$

(where $\cup$ denotes cup product on cohomologies) it was shown by Borel that

$$(1.4) \quad H^*(U(n)) = \Lambda(\xi_1, \ldots, \xi_{2n-1}) \text{ with } \deg \xi_k = k.$$  

Our initial idea is to solve the ring $H^*(PU(n))$ from the exact sequence (1.2). To this end we need to specify a set of generators of the ring $H^*(PU(n))$ so that the homomorphism $\theta$ in (1.2) can be explicitly expressed. The following result, whose proof appears in Section 3.4, serves this purpose.

**Basic Lemma 1.2.** There exist cohomology classes $\rho_{2r-1} \in H^{2r-1}(PU(n))$, $2 \leq r \leq n$, that satisfy the following constraints:

i) $\rho_{2r-1}^2 = 0$; ii) $c^*(\rho_{2r-1}) = c_r \cdot \xi_{2r-1}$.

Moreover, the homomorphism $\theta$ in (1.2) satisfies that

a) $\theta(\xi_{2r-1}) = \binom{r}{r} \omega^{r-1},$
and, for a multi–index $I = \{i_1, \cdots, i_k\} \subseteq \{1, \cdots, n\}$ with $l(I) = k \geq 2$, that

b) $\theta(\xi_I) = -\theta(\xi_{I'}) \cup \rho_{2r-1}$ if $c_r = 1$;

c) $\theta(\xi_I) = -\frac{1}{p}\theta(\xi_{I'}) \cup \rho_{2r-1} + \frac{1}{p}p^{\rho_k - \rho_{k-1}} \cup \theta(\xi_{I''} \cup \xi_{2r-1}^{-1})$ if $c_r = p > 1$,

where $r = i_k$, $I^* = \{i_1, \cdots, i_{k-1}, \hat{i}_k\}$, $\xi_I = \bigcup_{s \in I} \xi_{2s-1}$, and where in the formula c) $p$ divides $r$ by Lemma 1.1.

In Lemma 1.2 the recurrences relations a), b) and c) imply a close formula expressing $\theta(\xi_I)$ in term of the $\omega$ and $\rho_{2r-1}$'s, by which the $p$-divisibility of the classes $\theta(\xi_{I'})$ and $\theta(\xi_{I''} \cdot \xi_{2r-1}^{-1})$ in c) becomes evident, too. As example we deduce below such a formula for a special case relevant to our main result.

Assume that the integer $n$ has the prime factorization $p_1^{i_1} \cdots p_r^{i_r}$. For each pair $(p, r) = (p_i, r_i)$, $1 \leq i \leq t$, define the ordered sequence

$$Q_p(n) := \{1, p, p^2, \cdots, p^r\}.$$ 

For a multi–index $I = \{p_1^{i_1}, \cdots, p_k^{i_k}\} \subseteq Q_p(n)$ (with $i_1 < \cdots < i_k$) we shall put

$$I^* := \{p_1^{i_1}, \cdots, p_r^{i_r-1}\} \text{ and } I^0 := \{p_1^{i_1}, \cdots, p_k^{i_k-1}, p_{k+1}^{i_k-1}\}.$$ 

With these notation the formula c) in Lemma 1.2 turns to be

1) $\theta(\xi_I) = -\frac{1}{p}\theta(\xi_{I'}) \cup \rho_{2r-1} + \frac{1}{p}p^{\rho_k - \rho_{k-1}} \cup \theta(\xi_{I''})$.

Repeatedly apply 1) to reduce the factor $\theta(\xi_{I''})$ on its right hand side to get

2) $\theta(\xi_I) = -\frac{1}{p}\theta(\xi_{I'}) \cup \left( \sum_{0 \leq t \leq i_k - i_{k-1} - 1} \frac{1}{p}p^{\rho_k - \rho_{k-1} - t} \cup \rho_{2r-1} \right)$,

where the sum must end at $t = i_k - i_{k-1} - 1$, because the relation $\xi_{2r-1} = 0$ on $H^*(U(n))$ implies that $\xi_{I''} = 0$ when $i_k = i_{k-1} + 1$.

To make use formula 2) we introduce for the $I = \{p_1^{i_1}, \cdots, p_k^{i_k}\}$ a set $S(I)$ of certain subsequences of $Q_p(n)$, as well as the functions $\varepsilon_I$, $\kappa_I : S(I) \to \mathbb{Z}$, by

$$S(I) := \{J = \{p_1^{i_2}, \cdots, p_h^{i_h}\} \subseteq Q_p(n); i_s \geq j_s > i_{s-1}\},$$

$$\varepsilon_I(J) := (i_k - j_k) + \cdots + (i_2 - j_2) \text{ and }$$

$$\kappa_I(J) := (p_1^{i_2} - p_1^{i_1}) + \cdots + (p_i^{i_k} - p_i^{i_{k-1}}) + p_i^{i_1} - 1, J \in S(I),$$

respectively. Then, taking 2) as a recurrence to reduce the length $k$ of $I$, and applying a) of Lemma 1.2 to evaluate $\theta(\xi_{2^{r-1}-1})$ in the final step, yields that

**Corollary 1.3.** For each $I = \{p_1^{i_1}, \cdots, p_k^{i_k}\} \subseteq Q_p(n)$ we have

$$\theta(\xi_I) = (-1)^{k-1} \frac{1}{p^{\rho_{i_1}}(p_1^{i_1})^n} \sum_{J \in S(I)} (-1)^{i_2 - i_1} \sum_{s \in I} \rho_s^{-i_1} - \rho_i^{-i_1} \cup \rho_{k+1} \cup \rho_{2r-1}, I \subseteq Q_p(n),$$

where $p^{i_1 - i_1}$ divides $\binom{n}{p_1^{i_1}}$ by ii) of Lemma 1.1, and where $\rho_J = \bigcup_{r \in J} \rho_{2r-1} \cdot \Box$

Granted with Lemma 1.2 the exact sequence (1.2) is solvable as to yield the ring $H^*(PU(n))$. To state the result so obtained recall that the integral cohomology of a finite CW–complex $X$ admits a decomposition of the form
(1.6) \( H^*(X) = F(X) \oplus \sigma_p(X) \),

where \( F(X) := H^*(X)/\text{Tor}H^*(X) \) is the free part of the cohomology \( H^*(X) \), the sum is over all prime \( p \geq 2 \), and where the summand \( \sigma_p(X) \) is the \( p \)-primary component of the cohomology \( H^*(X) \) defined by

\[
\sigma_p(X) := \{ x \in H^*(X) \mid p^r x = 0, \text{ for some } r \geq 1 \}.
\]

In addition, if \( \{ \alpha_1, \ldots, \alpha_r \} \) is a finite subset of a ring \( A \) denote by \( \langle \alpha_1, \ldots, \alpha_r \rangle \subset A \) the ideal generated by \( \alpha_1, \ldots, \alpha_r \). In particular, for a prime power \( p^r \) we introduce the graded ring

\[
(1.7) \quad J_{p,r}(\omega) = \frac{\mathbb{Z}[\omega]^+}{(p^r\omega, p^{r-1}\omega, \ldots, \omega^r)}, \quad \deg \omega = 2,
\]

where \( \mathbb{Z}[\omega]^+ \) is the ring of integral polynomials in \( \omega \) without constant terms. With these convention our main result is

**Theorem 1.4.** If the integer \( n \) has the prime factorization \( p_1^{r_1} \cdots p_t^{r_t} \), then

\[
H^*(PU(n)) = \Lambda(\rho_3, \ldots, \rho_{2n-1}) \oplus \bigoplus_{1 \leq i \leq t} \sigma_{p_i}(PU(n))
\]

where for each pair \((p, r) = (p_i, r_i)\) with \( 1 \leq i \leq t \),

\[
(1.8) \quad \sigma_{p_i}(PU(n)) = \frac{J_{p_i,r_i}(\omega) \otimes \Lambda(\rho_3, \ldots, \rho_{2n-1})}{(R_i, I \subseteq \mathcal{Q}_p(n), I(I \geq 2))}
\]

in which \( R_i = \sum_{J \in S(I)} p^r \epsilon_{J} \cdot \omega \kappa_{J}^{(J) + 1} \otimes \rho_J \) (compare with (1.5)).

Assume in Theorem 1.4 that \( r_1 = \cdots = r_t = 1 \). Then for each \( p \in \{ p_1, \ldots, p_t \} \) we have, in addition to

\[
J_{p,1}(\omega) = \mathbb{Z}_p[\omega]^+ / \langle \omega^p \rangle \text{ and } Q_p(n) = \{ 1, p \},
\]

that \( R_{(1,p)} = \omega \otimes \mathbb{Z}_{p^2}^{-1} \). Theorem 1.4 implies that

**Corollary 1.5.** If the integer \( n \) has the prime factorization \( p_1 \cdots p_t \), then

\[
H^*(PU(n)) = \Lambda(\rho_3, \ldots, \rho_{2n-1}) \oplus \bigoplus_{1 \leq i \leq t} \frac{\mathbb{Z}_p[\omega]^+}{\langle \omega^p \rangle} \otimes \Lambda(\rho_3, \ldots, \hat{\rho}_{2p_i-1}, \cdots \rho_{2n-1}) \\Box
\]

Assume next that \( n = p^n n' \) with \( p \) a prime, \( (n', p) = 1 \). Since \( Q_p(n) = Q_p(p') \), and for each \( I \subseteq Q_p(n) \) the set \( S(I) \), as well as the functions \( \epsilon_I, \kappa_I \), relies only on \( p' \) (i.e. is independent of \( n' \)), we get from (1.8) the isomorphism:

\[
\sigma_{p_i}(PU(n)) = \sigma_{p_i}(PU(p')) \otimes \Lambda(\rho_{2p_i+1}, \cdots, \rho_{2n-1})
\]

Theorem 1.4 yields that

**Corollary 1.6.** If the integer \( n \) has the prime factorization \( p_1^{r_1} \cdots p_t^{r_t} \), then

\[
H^*(PU(n)) = \Lambda(\rho_3, \ldots, \rho_{2n-1}) \oplus \bigoplus_{1 \leq i \leq t} \sigma_{p_i}(PU(p_i^{r_i})) \otimes \Lambda(\rho_{2p_i+1}, \cdots, \rho_{2n-1}) \\Box
\]
Example 1.7. If \( n = 2^3 \) we have \( J_{2,3}(\omega) = \frac{2[\omega]}{\langle 8\omega, 4\omega^2, 2\omega^4, \omega^8 \rangle} \) and
\[
H^*(PU(8)) = \Lambda(\rho_3, \cdots, \rho_{17}) \oplus \frac{J_{2,3}(\omega)\otimes \Lambda(\rho_3, \cdots, \rho_{17})}{\langle R_I \mid \imath(I) \geq 2 \rangle},
\]
where the generators \( R_I \) of the ideal are recorded in the table below:

| \( I \) | \( R_I \) |
|---|---|
| \{1, 2\} | \( 4\omega \otimes \rho_3 \) |
| \{1, 4\} | \( 4\omega \otimes \rho_7 + 2\omega^2 \otimes \rho_3 \) |
| \{1, 8\} | \( 4\omega \otimes \rho_{15} + 2\omega^6 \otimes \rho_7 + 8\omega^4 \otimes \rho_3 \) |
| \{2, 4\} | \( 8\omega^4 \otimes \rho_7 \) |
| \{2, 8\} | \( 8\omega^4 \otimes \rho_{15} + \omega^8 \otimes \rho_7 \) |
| \{4, 8\} | \( \omega^4 \otimes \rho_{15} \) |
| \{1, 2, 4\} | \( 2\omega \otimes \rho_3 \rho_7 \) |
| \{1, 2, 8\} | \( 2\omega \otimes \rho_3 \rho_{15} + \omega^6 \otimes \rho_3 \rho_7 \) |
| \{1, 4, 8\} | \( 2\omega \otimes \rho_7 \rho_{15} + \omega^4 \otimes \rho_3 \rho_{15} \) |
| \{2, 4, 8\} | \( \omega^4 \otimes \rho_7 \rho_{15} \) |
| \{1, 2, 4, 8\} | \( \omega \otimes \rho_3 \rho_7 \rho_{15} \) |

In view of these we note that the relations \( R_I \), together with the obvious ones
\[
\rho_{2r-1}^2 = 0, \quad 2 \leq r \leq 8, \quad \text{and} \quad 8\omega = 4\omega^2 = 2\omega^4 = \omega^8 = 0,
\]
provide us with a system of \( \kappa \)-invariants useful to build up the Postnikov tower of the group \( PU(8) \) from the classifying map
\[
PU(8) \rightarrow K(\mathbb{Z}, 2) \times \prod_{r \in \{2, 3, \cdots, 8\}} K(\mathbb{Z}, 2r - 1)
\]
of the set \( \{ \omega, \rho_3, \cdots, \rho_{15} \} \subset H^*(PU(8)) \) of generators. \( \square \)

Example 1.8. In [20] Theorem A] Ruiz claimed that
\[
H^*(PU(n)) = \frac{2[\omega]}{\langle v_2, \cdots, v_n \rangle} \otimes \Lambda(v_2, \cdots, v_n), \quad I = \langle b_n, r y^r : 1 \leq r \leq n \rangle,
\]
where \( \text{deg}(y) = 2, \text{deg}(v_k) = 2k - 1 \). It implies that the order of the product
\[
y \cup v_2 \cup \cdots \cup v_n \in H^{m+2}(PU(n)), \quad m = \text{dim} \ PU(n),
\]
is \( b_{n,1} = n \), contradicting to the geometric fact that \( H^{m+2}(PU(n)) = 0 \). \( \square \)

For an account on the arrangement of this paper let \( G \) is a compact connected Lie group with a maximal torus \( T \), and consider the associated torus fibration
\[
(1.9) \quad \pi : G \rightarrow G/T.
\]

In Section 2 we recall and develop general properties of the Leray–Serre spectral sequence \( \{ E_r^{*,*}(G), d_r \} \) of the fibration \( \pi \), where the terms \( E_2^{*,*}(G) \) with \( G = U(n), \ PU(n) \) are made explicit. The central part of this paper is Section 3, where we deduce a Gysin type exact sequence (3.4) relating \( E_3^{*,*}(PU(n)) \) and \( E_3^{*,*}(U(n)) \), construct the preliminary forms \( \rho_{3, \cdots, \rho_{2n-1}} \in E_3^{*,*}(PU(n)) \) of the classes \( \rho_{3, \cdots, \rho_{2n-1}} \) promised by Lemma 1.2, and demonstrated their
properties relevant to Lemma 1.2. Section §4 deals with the extension problem from $E_3^{*,*}(PU(n))$ to $H^*(PU(n))$, as well as the verification of Lemma 1.2. With these preparations Theorem 1.4 is established in Section §5.

An appendix to this paper is set to complete the calculation of Baum and Bredon on the Bockstein $\beta_p : H^*(PU(n); \mathbb{F}_p) \to H^*(PU(n))$, where the result for the case $p = 2$ is requested by showing the relation $\rho_{2^r-1}^2 = 0$ in Lemma 1.2. Finally, in this paper the cohomologies and spectral sequences are over the ring $\mathbb{Z}$ of integers, unless otherwise stated.

2 The Koszul complex of the fibration $\pi$

For a compact connected Lie group $G$ with a maximal torus $T$ the induced map $\pi^* : H^2(G/T) \to H^2(G, T)$ is always an isomorphism. The Borel transgression in the fibration $\pi$ is the composition ([9], [14, p.185])

$$\tau = (\pi^*)^{-1} \circ \delta : H^1(T) \to H^2(G/T),$$

where $\delta : H^1(T) \to H^2(G, T)$ is the connecting homomorphism in the cohomological exact sequence of the pair $(G, T)$. By the Leray–Serre theorem [14, p.135] we have that

**Lemma 2.1.** The $E_2^{*,*}$-term of the Leray–Serre spectral sequence of $\pi$ is the Koszul complex $E_2^{*,*}(G) = H^*(G/T) \otimes H^*(T)$ (see [14, p.259]) on which

i) if $x, x' \in H^*(G/T)$ and $t, t' \in H^*(T)$ then

$$(x \otimes t) \cdot (x' \otimes t') = (x \cup x') \otimes (t \cup t');$$

ii) the differential $d_2$ is characterized by

$$d_2(x \otimes 1) = 0, d_2(1 \otimes t) = \tau(t) \otimes 1, t \in H^1(T),$$

and by the Leibniz rule

$$d_2(z \cdot z') = d_2(z) \cdot z' + (-1)^{\deg z} z \cdot d_2(z'), z, z' \in E_2^{*,*}(G).$$

We shall furnish the higher terms $E_r^{*,*}(G)$, $r \geq 3$, with the products inherited from that on $E_2^{*,*}(G)$ ([21, P.668]), and denoted by $\cdot$ both of the products on $E_2^{*,*}(G)$ and on $E_r^{*,*}(G)$. In particular, $E_3^{*,0}(G)$ is a subring of $E_3^{*,*}(G)$, while $E_3^{*,k}(G)$ is module over the ring $E_3^{*,0}(G)$.

**Example 2.2.** The Koszul complex $\{E_2^{*,*}(U(n)), d_2\}$ has a beautiful description due to Borel [4]. Take diagonal subgroup

$$T = \{\text{diag}\{e^{i\theta_1}, \cdots, e^{i\theta_n}\} \in U(n); \theta_i \in [0, 2\pi]\}$$

as the fixed maximal torus on $U(n)$. Let $y_i \in H^1(T)$ be the Kronecker dual of the oriented circle subgroup $\theta_k = 0, k \neq i$, on $T$, and put

$$x_i = \tau(y_i) \in H^2(U(n)/T), 1 \leq i \leq n.$$

Then, in addition to $H^*(T) = \Lambda(y_1, \cdots, y_n)$, Borel showed that

$$H^*(U(n)/T) = \frac{\mathbb{Z}[x_1, \cdots, x_n]}{(\epsilon_1, \cdots, \epsilon_n)},$$

(2.1)
where $e_r$ is the $r^{th}$ elementary symmetric function in $x_1, \cdots, x_n$. That is

$$E_2^{\ast} (U(n)) = \frac{\mathbb{Z}[x_1, \cdots, x_n]}{(x_1, \cdots, x_n)} \otimes \Lambda(y_1, \cdots, y_n), \quad d_2(1 \otimes y_i) = x_i \otimes 1.$$

Turning to the group $G = PU(n)$ we shall take $T' := c(T) \subset PU(n)$ as the preferable maximal torus on $PU(n)$. Let $\pi$ and $\pi'$ be the corresponding torus fibrations on $U(n)$ and $PU(n)$, respectively. Then the circle bundle $c$ over $PU(n)$ can be viewed as a bundle map from $\pi$ to $\pi'$

$$\begin{array}{ccc}
S^1 & \xrightarrow{c} & T \\
\| & \downarrow & \| \\
S^1 & \xrightarrow{\pi} & U(n) \quad \xrightarrow{\pi'} & PU(n) \quad \xrightarrow{\pi'} & PU(n)/T' \\
\| & \downarrow & \| \\
\| & \downarrow & \| \\
U(n) & \xrightarrow{\pi} & U(n)/T \quad \xrightarrow{\pi'} & PU(n)/T' \\
\end{array}$$

(2.2)

where $c$ induces a diffeomorphism on the base manifolds because $\ker c = S^1 \subset T$. To get a presentation of the ring $E_2^{\ast \ast} (PU(n))$ so that the induced map $c^\ast$ becomes transparent, we change the basis $\{y_1, \cdots, y_n\}$ of $H^1(T)$ by setting

$$t_0 := y_1, \quad t_i := y_{i+1} - y_i, \quad 1 \leq i \leq n - 1.$$

**Lemma 2.3.** One has the presentations

$$E_2^{\ast \ast} (U(n)) = H^\ast (U(n)/T) \otimes \Lambda(t_0, t_1, \cdots, t_{n-1})$$

and

$$E_2^{\ast \ast} (PU(n)) = H^\ast (U(n)/T) \otimes \Lambda(t_1, \cdots, t_{n-1})$$

on which the following relations hold for $1 \leq i \leq n - 1$:

i) the map $c^\ast : E_2^{\ast \ast} (PU(n)) \to E_2^{\ast \ast} (U(n))$ is the inclusion $c^\ast (z \otimes t_i) = z \otimes t_i, \quad z \in H^\ast (U(n)/T)$;

ii) the differential $d_2$ on $E_2^{\ast \ast} (PU(n))$ is given by $d_2 (1 \otimes t_i) = d_2 (1 \otimes t_i) = (x_{i+1} - x_i) \otimes 1$.

**Proof.** The restriction $c' = c | T$ yields the short exact sequence

$$0 \to S^1 \to T \to T' \to 0$$

of the torus groups in which $\ker c' = \{e^{i\theta} I_n \mid \theta \in [0, 2\pi]\}$. It follows that the torus $T'$ has the factorization $S^1 \times \cdots \times S^1$ ($n - 1$ factors) so that

$$c'(\text{diag} [e^{i\theta_1}, \cdots, e^{i\theta_n}]) = (e^{i(\theta_2 - \theta_1)}, \cdots, e^{i(\theta_n - \theta_{n-1})})$$

That is, the map $c^\ast$ identifies $H^\ast (T')$ with the subring $\Lambda(t_1, \cdots, t_{n-1})$ of $H^\ast (T)$, showing the presentation of $E_2^{\ast \ast} (PU(n))$.

Properties i) and ii) are evident as $c^\ast$ is a map of Koszul complexes. □

If $z \in E_2^{\ast \ast} (G)$ is a $d_2$–closed cocycle (i.e. $d_2 (z) = 0$) we write $[z] \in E_3^{\ast \ast} (G)$ to denote its cohomology class. In particular we shall set

$$\varpi := [x_1 \otimes 1] \in E_3^{2, 0} (PU(n)).$$
As an application of Lemma 2.3 we show that

**Corollary 2.4.**  $E_3^{s,0}(PU(n)) = \frac{\mathbb{Z}[x]}{(b_{n,r}x^r, 1 \leq r \leq n)}$.

In particular, for each $1 \leq r \leq n$ there exists a class $\delta_r \in E_2^{2(r-1),1}(PU(n))$ so that $d_2(\delta_r) = b_{n,r}x^r$.

**Proof.**  Let $\langle \text{Im } \tau \rangle \subset H^*(G/T)$ be the ideal generated by the image of the transgression $\tau$. Then by Lemma 2.1

$$E_1^{s,0}(G) = H^*(G/T)/\langle \text{Im } \tau \rangle.$$  

From $e_r \mod \langle \text{Im } \tau \rangle = \binom{n}{r}x^n$ by ii) of Lemma 2.3 we obtain that

$$E_1^{s,0}(PU(n)) = \frac{\mathbb{Z}[x_1, \ldots, x_n]}{\langle \binom{n}{r}x^n | 1 \leq r \leq n \rangle}.$$

It implies that the order of $\omega^r = [x_1 \otimes 1]^r$ is precisely $b_{n,r} = \text{g.c.d.} \{(\binom{n}{r}) \mid 1 \leq r \leq n\}$.

We conclude this section with three properties of the spectral sequence \{ $E_r^{s,t}(G), d_r$ \} useful to solve the extension problem from $E_3^{s,t}(G)$ to $H^*(G)$.

Let $F^s$ be the filtration on $H^*(G)$ defined by (1.9). That is

$$0 = F^{r+1}(H^*(G)) \subseteq F^r(H^*(G)) \subseteq \cdots \subseteq F^0(H^*(G)) = H^*(G)$$

with $E_{\infty}^{s,t}(G) = F^s(H^{s+t}(G))/F^{s+1}(H^{s+t}(G))$.

The relation $d_r(E_r^{s,0}(G)) = 0$ for $r \geq 2$ yields the sequence of quotients

$$H^*(G/T) \to E_2^{s,0} \to E_3^{s,0} \to \cdots \to E_\infty^{s,0} = F^s(H^*(G)) \subset H^*(G)$$

whose composition agrees with the induced ring map $\pi^*$ (see [14, P.147]). For this reason we shall reserve $\pi^*$ also for the composition

(2.3)  $\pi^*: E_3^{s,0}(G) \to \cdots \to E_\infty^{s,0}(G) = F^s(H^*(G)) \subset H^*(G)$.

The fact $H^{2k+1}(G/T) = 0$ due to Bott and Samelson [7] implies that

a)  $E_2^{2k+1,*} = 0$ for $r \geq 0$,  

b)  $E_3^{4s,2} = E_4^{4s,2} = \cdots = E_{\infty}^{4s,2}$.

From $F^{2s+1}(H^{2s+1}(G)) = F^{2s+2}(H^{2s+1}(G)) = 0$ by a) one finds that

$$E_\infty^{2s,1}(G) = F^{2s}(H^{2s+1}(G)) \subset H^{2s+1}(G).$$

Combining this with $d_r(E_r^{s,1}) = 0$ for $r \geq 3$ yields the composition

(2.4)  $\kappa: E_3^{s,1}(G) \to E_4^{s,1}(G) \to \cdots \to E_\infty^{s,1}(G) \subset H^*(G)$

which interprets elements of $E_3^{s,1}$ directly as cohomology classes of $G$.

**Lemma 2.5.**  For any $\rho \in \text{Im } \kappa$ we have $\rho^2 \in \text{Im } \pi^*$.

**Proof.**  For an element $x \in E_3^{2s,1}$ the obvious relation $x^2 = 0$ in
\[ E_3^{4s,2} = E_\infty^{4s,2} = F^{4s}H^{4s+2}/F^{4s+1}H^{4s+2} \] (by b))

implies that \( \kappa(x)^2 \in F^{4s+1}H^{4s+2} \). From

\[ F^{4s+1}H^{4s+2}/F^{4s+2}H^{4s+2} = E_\infty^{4s+1,1} = 0 \] (by a))

one concludes further that \( \kappa(x)^2 \in F^{4s+2}H^{4s+2} \). The proof is completed by

\[ F^{4s+2}H^{4s+2} = E_\infty^{4s+2,0} \subset \text{Im} \pi^* \]

Assume next that \( \dim G/T = m \) and \( \dim T = n \). Since

\[ E_2^{m,n}(G) = \mathbb{Z}, \] and \( E_2^{s,t}(G) = 0 \) if either \( s > m \) or \( t > n \),

any differential \( d_x \) that acts or lands on the group \( E_2^{m,n}(G) \) must be trivial. This implies the following isomorphisms

\[ \text{(2.5)} \quad E_2^{m,n}(G) = H^m(G) \otimes H^n(T) = E_2^{m,n}(G) = H^{\dim G}(G) = \mathbb{Z}, \ r \geq 2. \]

**Lemma 2.6.** Let \( \eta_1, \cdots, \eta_n \in E_3^{*,1}(G) \) be \( n \) elements so that their product \( \eta_1 \cdots \eta_n \) generates the group \( E_3^{m,n}(G) = \mathbb{Z} \). Then

i) the free part of \( E_3^{*,*}(G) \) has the basis \( \Psi = \{1, \eta_I; I \subseteq \{1, \cdots, n\}\}; \)

ii) the free part of \( H^*(G) \) has the basis \( \Phi = \{1, \kappa(\eta_I); I \subseteq \{1, \cdots, n\}\}, \)

where \( \eta_I = \prod_{i \in I} \eta_i, \ k(\eta_I) = \bigcup_{i \in I} k(\eta_i) \).

**Proof.** According to Leray \[13\] the algebra \( E_3^{*,*}(G) \otimes \mathbb{Q} \) is generated multiplicatively by \( E_3^{*,1}(G) \otimes \mathbb{Q} \), and the map \( \kappa \) in (2.4) induces an isomorphism

\[ \text{(2.6)} \quad \kappa : E_3^{*,*}(G) \otimes \mathbb{Q} \cong H^*(G) \otimes \mathbb{Q} \quad \text{(e.g. \[19\] (6.2) Theorem)].} \]

of algebras. In particular, \( \text{rank} E_3^{*,*}(G) = \text{rank} H^*(G) = 2^n. \)

As the product \( \eta_1 \cdots \eta_n \) generates \( E_3^{m,n}(G) = \mathbb{Z} \) the set \( \Psi \) of \( 2^n \) monomials is linearly independent in \( E_3^{*,*}(G) \). With \( \text{rank} E_3^{*,*}(G) = 2^n \) for assertion i) it suffices to show that the set \( \Psi \) spans a direct summand of \( E_3^{*,*}(G) \). Assume on the contrary that there exist an \( \eta_I \in \Psi \), a class \( c \in E_3^{*,*}(G) \), as well as some integer \( a > 1 \), so that a relation of the form \( \eta_I = a \cdot c \) holds in \( E_3^{*,*}(G) \).

Multiplying both sides by \( \eta_T \) with \( T \) the complement of \( I \subseteq \{1, \cdots, n\} \) yields

\[ \eta_1 \cdots \eta_n = (-1)^r a \cdot \left( c \cup \eta_T \right) \quad \text{(for some } r \in \{0, 1\}\).} \]

This contradiction to the identification shows i).

In view of the identification (2.5) the isomorphism \( \kappa \) in (2.6) transforms the generator \( \eta_1 \cdots \eta_n \) of \( E_3^{m,n}(G) \) to the generator \( \kappa(\eta_1) \cup \cdots \cup \kappa(\eta_n) \) of \( H^{\dim G}(G) = \mathbb{Z} \). Assertion ii) follows from the same calculation. □

Since the ring \( E_2^{*,*}(G) \) is torsion free one has for a prime \( p \) the short exact sequence of Koszul complexes

\[ 0 \rightarrow E_2^{*,*}(G) \xrightarrow{p} E_2^{*,*}(G) \xrightarrow{r} E_2^{*,*}(G; \mathbb{F}_p) \rightarrow 0, \]
where \( \mathbb{F}_p \) is the field of characteristic \( p \). With respect to the maps \( \kappa \) and \( \pi^* \) the connecting homomorphism \( \tilde{\beta}_p \) in the associated cohomology exact sequence clearly satisfies the commutative diagram

\[
\begin{array}{ccc}
E_3^{n,0}(G; \mathbb{F}_p) & \xrightarrow{\tilde{\beta}_p} & E_3^{n,0}(G) \\
\kappa & \\nH^*(G; \mathbb{F}_p) & \xrightarrow{\tilde{\beta}_p} & H^*(G)
\end{array}
\]

(2.7)

where \( \tilde{\beta}_p \) is the Bockstein homomorphism on the cohomologies. In addition, in view of the proof of Corollary 2.4 the quotient map \( H^*(G/T) \to E_3^{n,0}(G) \) is \( x \to x \mod(\text{Im} \tau), x \in H^*(G/T) \). Letting \( z \in E_3^{n,0}(G; \mathbb{F}_p) \) be a \( d_2 \)-cocycle with an integral lift \( \tilde{z} \in E_3^{n,1}(G) \), the diagram chasing in the short exact sequence

\[
\begin{array}{ccc}
\mathbb{F}_p & \xrightarrow{d_2} & \mathbb{F}_p \\
\xrightarrow{\frac{1}{p}d_2(\tilde{z})} & \\
\xrightarrow{p} & \mathbb{F}_p
\end{array}
\]

shows the following formula that evaluates \( \beta_p \) on \( \text{Im} \kappa \).

**Lemma 2.7.** \( \beta_p(\kappa[z]) = \pi^* \left( \frac{1}{p}d_2(\tilde{z}) \mod(\text{Im} \tau) \right) \). \( \square \)

### 3 A refinement of the Gysin sequence (1.2)

To study the cohomology of a Lie group \( G \) the primary task is to specify a set of such generators by which the ring \( H^*(G) \) admits a concise formulation. In this section we boil down the constructions and computations in \( H^*(PU(n)) \) to \( E_2^{*,*}(PU(n)) \), whose structure has been made explicit by Lemma 2.3.

In view of the decomposition

\[ E_1^{*,*}(U(n)) = E_2^{*,*}(PU(n)) \oplus E_2^{*,*}(PU(n)) \oplus t_0 \]

by Lemma 2.3 we have the short exact sequence of Koszul complexes

\[ (1.1) \quad 0 \to E_2^{*,*}(PU(n)) \xrightarrow{c^*} E_2^{*,*}(U(n)) \xrightarrow{\theta} E_2^{*,*}(PU(n)) \to 0 \]

where, if \( z, z' \in E_2^{*,*}(PU(n)) \), then

\[ c^*(z) = z \oplus 0, \quad \theta(z \oplus z' \otimes t_0) = z'. \]

As a result, every element \( x \in E_2^{*,*}(U(n)) \) can be uniquely expressed as

\[ (3.2) \quad x = z_1 + \theta(x) \otimes t_0 \text{ with } z_1 \in E_2^{*,*}(PU(n)). \]

With \( (1 \otimes t_0)^2 = 0 \) we get from (3.2) the multiplicative rule of the map \( \theta \)

\[ (3.3) \quad \theta(x \cdot x') = (-1)^{\deg z_1} \theta(x) \cdot z_1' + z_1 \cdot \theta(x'). \]

**Lemma 3.1.** The map \( c^* \) in (3.1) induces the exact sequence

\[ (3.4) \quad \cdots \to E_3^{r,*)}(PU(n)) \xrightarrow{c} E_3^{r,*)}(U(n)) \xrightarrow{\theta} E_3^{r,*)}(PU(n)) \to \cdots \]
in which, if \( z \in E^*_3(\text{PU}(n)) \) and \( x \in E^*_3(U(n)) \), then

\begin{enumerate}
  \item \( \overline{\partial}(e^*(z) \cdot x) = z \cdot \overline{\partial}(x) \),
  \item \( \overline{w}(z) = \overline{w} \cdot z \) (recall that \( \overline{w} := [x_1 \otimes 1] \)).
\end{enumerate}

\textbf{Proof.} The sequence (3.4) is the exact sequence associated to the short exact sequence (3.1) of Koszul complexes. Formula i) is an instance of (3.3), while ii) comes directly from the definition of the connecting homomorphism in the exact sequence associated to a short exact sequence of cochain complexes. □

In view of the quotient ring map \( f : \mathbb{Z}[x_1, \cdots, x_n] \to H^*(U(n)/T) \) by (2.1) introduce the maps \( F \) and \( \overline{F} \) of Koszul complexes by

\[
F := f \otimes \text{id} : K^{*,*} := \mathbb{Z}[x_1, \cdots, x_n] \otimes H^*(T) \to E^*_2(U(n)),
\]

\[
\overline{F} := f \otimes \text{id} : \overline{K}^{*,*} := \mathbb{Z}[x_1, \cdots, x_n] \otimes H^*(T^\prime) \to E^*_2(\text{PU}(n)),
\]

where the differentials \( d \) on \( K^{*,*} \) and \( \overline{d} \) on \( \overline{K}^{*,*} \) are defined in accordance to \( d_2 \) and \( d_2' \), respectively. For a geometric interpretation on the map \( F \) (also on \( \overline{F} \)) consider the fibration induced by the inclusion \( T \subset U(n) \)

\[ U(n)/T \overset{i}{\to} BT \to BU(n), \]

where \( BH \) is the classifying space of the group \( H \). Evidently, \( K^{*,*} \) is the Koszul complex associated to the universal \( T \) bundle on \( BT \), \( f \) is the induced ring map of the fiber inclusion \( i \), while \( F \) is induced by the bundle map over \( i \).

The maps \( F \) and \( \overline{F} \) restrict to the commutative diagrams

\[
\begin{array}{ccc}
K^{*,1} & \xrightarrow{F} & E^*_2(U(n)) \\
\downarrow d & & \downarrow d_2 \\
\mathbb{Z}[x_1, \cdots, x_n] & \xrightarrow{f} & \frac{\mathbb{Z}[x_1, \cdots, x_n]}{I_{x_1, \cdots, x_n}} \\
\downarrow d & & \downarrow \overline{d} \\
\overline{K}^{*,1} & \xrightarrow{\overline{F}} & E^*_2(\text{PU}(n)) \\
\end{array}
\]

in which we notice the following relations.

\textbf{Lemma 3.2.} If \( g \in \mathbb{Z}[x_1, \cdots, x_n] \) and \( z, z' \in K^{*,1} \) (resp. \( z, z' \in \overline{K}^{*,1} \)), then

\begin{enumerate}
  \item \( g \in \text{Im } d \) (resp. \( g \in \text{Im } \overline{d} \)) if and only if
    \[ g \text{ mod(Im } \tau) \equiv 0 \] (resp. \( g \mid \text{mod(Im } \tau') \equiv 0 \));
  \item \( d_2(F(z)) = 0 \) (resp. \( d_2'(\overline{F}(z)) = 0 \)) if and only if
    \[ d(z) \in \ker f \] (resp. \( \overline{d}(z) \in \ker f \));
  \item \( d(z) = d(z') \) (resp. \( \overline{d}(z) = \overline{d}(z') \)) implies that
    \[ F(z) - F(z') \in \text{Im } d_2 \] (resp. \( \overline{F}(z) - \overline{F}(z') \in \text{Im } d_2' \)).
\end{enumerate}

\textbf{Proof.} Property i) comes from the definition of \( d \) (resp. \( \overline{d} \)), while ii) follows from the commutativity of the diagrams (3.5). To show iii) we take the case \( G = U(n) \) as an illustrative example. Suppose that

\[ z = p_1 \otimes t_1 + \cdots + p_n \otimes t_n \text{ and } z' = p'_1 \otimes t_1 + \cdots + p'_n \otimes t_n \in K. \]

If \( d(z) = d(z') \) then
\((p_1 - p'_1) \cdot \tau(t_1) + \cdots + (p_n - p'_n) \cdot \tau(t_n) = 0,\)

where we can assume that \(p_i - p'_i \neq 0.\) Since the subset \(\{\tau(t_1), \ldots, \tau(t_n)\}\) is algebraically independent in \(\mathbb{Z}[x_1, \ldots, x_n]\) (by ii) of Lemma 2.3), it implies that all the differences \(p_i - p'_i\) with \(i \neq 1\) are divisible by \(\tau(t_1)\). That is

\[p_i - p'_i = q_i \cdot \tau(t_1)\]

for some \(q_i \in \mathbb{Z}[x_1, \ldots, x_n],\) \(2 \leq i \leq n.\)

Assertion iii) is now verified by

\[d(q_2 \otimes t_1 t_2 + \cdots + q_n \otimes t_1 t_n) = z - z'.\]

Lemma 3.2 enables us to construct elements in \(E^1_\ast(G)\) from polynomials in \(\ker f.\) As applications we construct below explicit elements

\[\zeta_1, \ldots, \zeta_{2n-1} \in E^1_3(U(n))\]

and \(\rho_3', \ldots, \rho_{2n-1}' \in E^1_3(PU(n))\)

which will be shown to generate the rings \(E^{\ast\ast}_3(U(n))\) and \(E^{\ast\ast}_3(PU(n))\), respectively. The following identity is crucial in the construction of \(\rho_3', \ldots, \rho_{2n-1}'.\)

**Lemma 3.3.** Assume that \(n = p^k n', (p, n') = 1.\) For each \(1 \leq s \leq k\) one has

\[(3.6) \quad \binom{n}{p^s} = \frac{n!}{p^s} \sum_{1 \leq t \leq p^{s-1}} (-1)^{t-1} \binom{n}{p^s-t} + (-1)^{p^s-p^{s-1}+1} \frac{1}{p} \binom{n}{p^s-1}.\]

**Proof.** Let \(g_r = x_1^r + \cdots + x_n^r\) be the \(r\)th power sum. In the Newton’s formula

\[re_r = \sum_{1 \leq t \leq r} (-1)^{t-1} g_{e_{r-t}}, \quad 1 \leq r \leq n,\]

setting \(x_1 = \cdots = x_n = 1\) we get from \(g_t = n\) and \(e_{r-t} = \binom{n}{r-t}\) that

\[(3.6)_a \quad \binom{n}{p^s} = \frac{n}{p^s} \sum_{1 \leq t \leq r} (-1)^{t-1} \binom{n}{r-t}.\]

Taking \(r = p^s\) and separating the sum at \(t = p^s - p^{s-1}\) yields (3.6), because of

\[\frac{n}{p^s} \sum_{p^s - p^{s-1}+1 \leq t \leq p^s} (-1)^{t-1} \binom{n}{p^s-t} = (-1)^{p^s-p^{s-1}+1} \frac{1}{p} \binom{n}{p^s-1}\]

(again by (3.6)_a).

**Construction A.** Assume that \(G = U(n)\) and \(1 \leq r \leq n.\)

By i) of Lemma 3.2 the symmetric functions \(e_r\) belongs to \(\text{Im} d,\) hence \(d(\tilde{e}_r) = e_r\) for some \(\tilde{e}_r \in K^{n-1}\). Further, with \(e_r \in \ker f\) by (2.1) the element \(F(\tilde{e}_r) \in E^{\ast\ast}_2(U(n))\) is \(d_2\)-closed by ii) of Lemma 3.2, hence yields the cohomology class

\[(3.7) \quad \zeta_{2r-1} := [F(\tilde{e}_r)] \in E^{2r-2,1}_2(U(n)), \quad 1 \leq r \leq n.\]

We emphasis this point that, by the assertion iii) of Lemma 3.2, the class \(\zeta_{2r-1}\) relies only on the polynomial \(e_r,\) and is independent of the choices of \(\tilde{e}_r.\) For this reason we do not need to entail a particular formula of \(\tilde{e}_r.\) The same observation applies also to the next construction.

**Construction B.** Assume that \(G = PU(n).\) Let \(\delta_r \in E^{2(r-1),1}_2\) be a class specified in Corollary 2.4, \(2 \leq r \leq n.\)

By i) of Lemma 3.2 the polynomial \(a_r = e_r - \binom{n}{r} x_1^n\) belongs to \(\text{Im} d,\) hence
\[ \hat{a}_r := \hat{c}_r - \binom{n}{r} x_1^{r-1} \otimes t_0 \in K^{r-1} \text{ with } d'_{2}(\hat{a}_r) = a_r. \]

In accordance to the constants \( c_r \) decided by Lemma 1.1 introduce the elements
\[ \hat{h}_r \in E_{2}^{2(r-1),1}(PU(n)) \]
by the formulas
\begin{equation}
(3.8) \quad \hat{h}_r := F(\hat{a}_r) + \binom{n}{r} \cdot x_1 \cdot \delta_{r-1} \text{ if } c_r = 1;
\end{equation}
\[ \hat{h}_r := p \cdot F(\hat{a}_r) - \frac{n}{p^{r-1}} \sum_{1 \leq t \leq p^{r-1}} (-1)^{t-1} x_1^t \cdot F(\hat{a}_r) \]
\[ -(-1)^{p^{r-1}} x_1^{p^{r-1}} \cdot F(\hat{a}_r) \text{ if } c_r = p > 1 \text{ and } r = p^s, \]
where we note by Lemma 1.1 that \( b_{n,r} \) divides \( \binom{n}{r} \), and that \( r = p^s \) divides \( n \) when \( c_r = p \). Using the relations
\[ d'_{2}(\delta_{r-1}) = b_{n,r-1} \cdot x_1^{r-1}, \quad d'_{2}(F(\hat{a}_r)) = -\binom{n}{r} \cdot x_1^r \]
in \( H^*(U(n)/T) \), the identities (3.6), as well as the relations \( b_{n,r} = b_{n,r-1} \) when \( c_r = 1 \), one verifies easily that \( d'_{2}(\hat{h}_r) = 0 \), hence obtains the cohomology class
\begin{equation}
(3.9) \quad \rho_{2r-1} := [\hat{h}_r] \in E_{2}^{2(r-1),1}(PU(n)), \quad 2 \leq r \leq n. \quad \square
\end{equation}

Subject to the exact sequence (3.4) the classes \( \xi'_{2r-1} \) and \( \rho_{2r-1} \) just constructed possess the following properties that are compatible with Lemma 1.2.

**Lemma 3.4.** We have \( E_{3}^{*,*}(U(n)) = \Lambda(\xi'_1, \ldots, \xi'_{2n-1}) \). Moreover,

i) the map \( \kappa \) in (2.5) induces the isomorphism (see (1.4))
\[ L : E_{3}^{*,*}(U(n)) \to H^*(U(n)) \]
with \( L(\xi'_{2r-1}) = \xi_{2r-1} \);

ii) \( c^*(\rho_{2r-1}) = c_r \cdot \xi'_{2r-1}, \quad 2 \leq r \leq n \).

**Lemma 3.5.** The homomorphism \( \overline{\theta} \) in (3.4) satisfies that

a) \( \overline{\theta}(\xi'_{2r-1}) = \binom{r}{r} \cdot \omega^{r-1} \),

and, for a multi-index \( I = \{i_1, \ldots, i_k\} \) with length \( k \geq 2 \), that

b) \( \overline{\theta}(\xi'_I) = -\overline{\theta}(\xi'_{I_r}) \cdot \rho_{2r-1} \text{ if } c_r = 1; \)

c) \( \overline{\theta}(\xi'_I) = -\frac{1}{p} \overline{\theta}(\xi'_{I_r}) \cdot \rho_{2r-1} + \frac{1}{p} \omega^{r-1} \cdot \overline{\theta}(\xi'_{I_r} \cdot \xi_{2r-1}) \text{ if } c_r = p > 1, \)

where \( I_r = \{i_1, \ldots, i_{k-1}, \hat{i}_k\}, \quad r = i_k, \) and where \( p \) divides \( r \) if \( c_r = p > 1 \).

**Proof of Lemma 3.4.** For i) it can be shown that
\[ F(\hat{c}_1) \cdots F(\hat{c}_n) = \pm x_2 x_3^2 \cdots x_n^{n-1} \otimes t_1 \cdots t_n \in E_{2}^{m,n}(U(n)) = Z, \]
in which the monomial \( x_2 x_3^2 \cdots x_n^{n-1} \) generates the group \( H^m(U(n)/T) = \mathbb{Z} \), \( m = \text{dim } U(n)/T \) (e.g. [13] (4.1.3)). As a result the product \( \xi'_1 \cdots \xi'_{2n-1} \) generates the group \( E_{3}^{m,n}(U(n)) = \mathbb{Z} \) by (2.5). Since the ring \( H^*(U(n)) \) is torsion free with \( \text{rank } H^*(U(n)) = 2^n \), assertion i) follows from Lemma 2.6.

In accordance to \( c_r = 1 \) or \( > 1 \) the relation ii) is verified by the following calculations in \( E_{2}^{*,*}(U(n)) \). If \( c_r = 1 \) then
\[ e^*(\hat{h}_r) - F(\hat{e}_r) = \binom{n}{n_r} \cdot x_1 \cdot \delta_{r-1} - \binom{n}{r} x_1^{r-1} \otimes t_0 \]
\[ = \binom{n}{n_r} \cdot \delta_{r-1} (\delta_{r-1} \otimes t_0) \text{ (recall that } b_{n,r} \text{ divides } \binom{n}{r} ) \]

and if \( c_r = p > 1 \) (hence \( r = p^k \) divides \( n \) by Lemma 1.1) then

\[ e^*(\hat{h}_r) - p \cdot F(\hat{e}_r) = -\binom{n}{p^k} \sum (-1)^{t-1} x_1^t \cdot F(\hat{a}_{p^k-t}) \]
\[ = \binom{n}{p^k} \sum (-1)^{t-1} x_1^t \cdot d_2(F(\hat{a}_{p^k-t}) \otimes t_0) \]
\[ = \binom{n}{p^k} \sum (-1)^{t-1} x_1^t \cdot d_2(F(\hat{a}_{p^k-t}) \otimes t_0) \text{ (by (3.6))}, \]

where the sums \( \sum \) are over \( 1 \leq t \leq p^k - p^k - 1 \), \( e^* \circ F = F \).

The proof of Lemma 3.5 requests two additional formulas we are about to deduce. For a multi-index \( \mathcal{I} = \{i_1, \cdots, i_k\} \) with \( k \geq 2 \) we set

\[ \hat{\mathcal{I}} := \prod_{i \in \mathcal{I}} F(\hat{a}_i) \in E_{2^{*}}^{*}(U(n)); \]

\[ \hat{\mathcal{I}}_i := \prod_{i \in \mathcal{I}} \hat{a}_i, \quad \hat{\mathcal{I}} := \prod_{i \in \mathcal{I}} F(\hat{a}_i) \in E_{2^{*}}^{*}(PU(n)), \]

\[ \xi_{i} := \prod_{i \in \mathcal{I}} \xi_{i} \in E_{3}^{*}(U(n)), \quad \rho_{i} := \prod_{i \in \mathcal{I}} \rho_{i} \in E_{3}^{*}(PU(n)). \]

With these convention we have in term of (3.2) that

\[ \hat{\mathcal{I}} = \hat{\mathcal{I}}_i + \theta(\hat{\mathcal{I}}) \otimes t_0. \]

Applying the relation \( d_2^* \circ \theta = \theta \circ d_2 \) to the product

\[ \hat{\mathcal{I}}_i \cdot 1 \otimes t_0 = \hat{\mathcal{I}}_i \otimes t_0 \text{ (by } (1 \otimes t_0)^2 = 0) \]

gives rise to the identity on \( E_{2^{*}}^{*}(PU(n)) \)

(3.10) \((-1)^{1/2} \theta(\hat{\mathcal{I}}) \cdot x_1 = d_2^*(\hat{\mathcal{I}}).

In addition, setting \( r = i_k \) and applying \( \theta \) to the product

\[ \hat{\mathcal{I}} = \hat{\mathcal{I}} \cdot (\delta_{r} + \binom{n}{r} x_1^{r-1} \otimes t_0) \in E_{2}^{*}(U(n)) \]

we obtain from (3.3) the identity

(3.11) \( \theta(\hat{\mathcal{I}}) = -\theta(\hat{\mathcal{I}}) \cdot \delta_{r} + \binom{n}{r} x_1^{r-1} \cdot \delta_{r}. \)

**Proof of Lemma 3.5.** If \( \mathcal{I} = \{r\} \) is a singleton one gets formula a) directly from (3.11). Assume next that \( \mathcal{I} = \{i_1, \cdots, i_k\} \) with \( k \geq 2 \), and put \( r = i_k \).

If \( c_r = 1 \) we get formula b) from

\[ \overline{\theta}(\xi_{i_k}) = \overline{\theta}(\xi_{i_k} \cdot \xi_{2r-1}) = \overline{\theta}(\xi_{i_k} \cdot c^* \rho_{2r-1}) \text{ (by ii) of Lemma 3.4)} \]
\[ = -\overline{\theta}(\xi_{i_k}) \cdot \rho_{2r-1} \text{ (by i) of Lemma 3.1)}. \]

If \( c_r = p > 1 \) we have \( r = p^k \) by Lemma 1.1. In (3.11) substituting
Lemma 4.2. Combining this with Lemma 4.1 we proceed to show that

\[ \sum_\omega E \]  

which equality on \( H\) resulting equality yield

Characterization of the classes

This section is devoted to show Lemma 1.2. We begin with the following precise characterization of the classes \( \rho_{2r-1} \) promised in Lemma 1.2:

(4.1) \( \rho_{2r-1} := \kappa(\rho_{2r-1}) \in H^*(PU(n)) \), \( 2 \leq r \leq n \) (see (2.4) for the map \( \kappa \)).

The first half of Lemma 1.2 is shown by the following result.

Lemma 4.1. For each \( 2 \leq r \leq n \) we have

i) \( \beta_{2r-1}^2 = 0 \); ii) \( c^\ast(\rho_{2r-1}) = cr \cdot \xi_{2r-1} \).

Proof. For i) assume that \( n = 2^k(2b + 1) \), and let \( \beta_2 \) be the mod 2 Bockstein homomorphism. Then we have \( \rho_{2r-1}^2 \in \text{Im} \pi^\ast \) (by Lemma 2.5) and \( \rho_{2r-1}^2 \in \text{Im} \beta_2 \) (by \( 2\rho_{2r-1} = 0 \)). With \( \deg \rho_{2r-1} = 4r - 2 \) and \( \deg \omega = 2 \), relation i) follows from the following result by formula (6.2) of the Appendix:

\[ \text{Im} \pi^\ast \cap \text{Im} \beta_2 = \{ 0, 2^{k-t-1} \omega^{2t} \mid 0 \leq t \leq k - 1 \} \].

By the naturality of the transformation \( \kappa \) in (2.4) with respect to bundle maps, we have the commutative diagram

\[
\begin{array}{ccc}
E_3^*(PU(n)) & \xrightarrow{\kappa} & E_3^*(U(n)) \\
\downarrow & & \downarrow \\
H^*(PU(n)) & \xrightarrow{\kappa} & H^*(U(n))
\end{array}
\]

by which assertion ii) follows from ii) of Lemma 3.4.

Let \( \langle \omega \rangle \subset H^* (PU(n)) \) and \( \langle \varpi \rangle \subset E_3^*(PU(n)) \) be the ideals generated by the Euler class \( \omega \) and \( \varpi \), respectively. Then, by the exactness of (1.2) and (3.4)

(4.2) \( \langle \omega \rangle = \ker c^\ast, \langle \varpi \rangle = \ker c^\ast \).

Combining this with Lemma 4.1 we proceed to show that

Lemma 4.2. One has the following decompositions
(4.3) \( H^*(PU(n)) = \Lambda(\rho_3, \cdots, \rho_{2n-1}) \oplus \langle \omega \rangle; \)
\[ E_3^*(PU(n)) = \Lambda(\rho'_3, \cdots, \rho'_{2n-1}) \oplus \langle \varpi \rangle, \]

where the first and the second summands are respectively the free part and torsion ideal the corresponding rings.

**Proof.** Let \( i : SU(n) \to U(n) \) be the inclusion of the special unitary group. The map \( l : U(n) \to SU(n) \) by \( l(g) = \text{diag}\{\det(g)^{-1}, 1, \cdots, 1\} \cdot g, g \in U(n) \), is clearly a retraction. As a result \( l^* \) induces the decomposition

\[ H^*(U(n)) = H^*(SU(n)) \oplus \xi_1 \cdot H^*(SU(n)) \]
on which \( i^* \) annihilates the second summand, and carries the first summand isomorphically onto the subring (see (1.4))

\[ H^*(SU(n)) = \Lambda(\xi_3, \cdots, \xi_{2n-1}) \subset H^*(U(n)). \]

In addition, we note that the composition \( c \circ i : SU(n) \to PU(n) \) is the universal cover of the group \( PU(n) \) with \( \deg(c \circ i) = n \).

Set \( n = \dim U(n)/T. \) Since the product \( \xi_3 \cup \cdots \cup \xi_{2n-1} \) generates the group \( H^{m+n-1}(SU(n)) = \mathbb{Z} \), and since \( c_2 \cdots c_n = n \) by Lemma 1.1, the calculation

\[ i^* \circ c^* (\rho_3 \cup \cdots \cup \rho_{2n-1}) = c_2 \cdots c_n \cdot \xi_3 \cup \cdots \cup \xi_{2n-1} \]
by ii) of Lemma 4.1 implies that \( \rho_3 \cup \cdots \cup \rho_{2n-1} \) generates \( H^{m+n-1}(PU(n)) = \mathbb{Z} \). With \( \rho^2_{2n-1} = 0 \) we conclude by ii) of Lemma 2.6 that the free part of the ring \( H^*(PU(n)) \) is isomorphic to \( \Lambda(\rho_3, \cdots, \rho_{2n-1}) \).

Let \( z \in H^*(PU(n)) \) be a torsion element. Since \( H^*(U(n)) \) is torsion free we must have \( c^*(z) = 0 \). That is \( z \in \ker c^* = \langle \omega \rangle \) by (4.2), showing the decomposition (4.3) of \( H^*(PU(n)) \).

For the case \( E_3^*(PU(n)) \) we note by (4.1) that the identification (2.5) transforms the product \( \rho'_3 \cup \cdots \cup \rho'_{2n-1} \) to \( \rho_3 \cup \cdots \cup \rho_{2n-1} \), hence \( \rho'_3 \cdots \rho'_{2n-1} \) generates the group \( E_3^{m+n-1}(PU(n)) = \mathbb{Z} \). Applying i) of Lemma 2.6 one obtains the decomposition (4.3) of \( E_3^*(PU(n)) \) by the same calculation. \( \Box \)

By Lemma 4.2 every \( z \in H^*(PU(n)) \) (resp. \( z \in E_3^*(PU(n)) \)) can be expressed as a sum \( z = a_0 + \omega \cup z_1 \) with \( a_0 \in \Lambda(\rho_3, \cdots, \rho_{2n-1}), z_1 \in H^*(PU(n)) \).

By repetition and for the degree reason, we get an expansion

\[ z = a_0 + \omega \cup a_1 + \omega^2 \cup a_2 + \cdots + \omega^k \cup a_k \]
with \( a_i \in \Lambda(\rho_3, \cdots, \rho_{2n-1}) \). It shows that

**Corollary 4.3.** The ring \( H^*(PU(n)) \) is generated by \( \rho_3, \cdots, \rho_{2n-1} \) and \( \omega \).

The ring \( E_3^*(PU(n)) \) is generated by \( \rho'_3, \cdots, \rho'_{2n-1} \) and \( \varpi \). \( \square \)

With \( \varpi \in E_3^{0,0} \) and \( \rho'_{2n-1} \in E_3^{1,1} \) one gets by Corollary 4.3 that

\[ d_r(E_3^*(PU(n))) = 0, r \geq 3, \]
hence obtains the additive isomorphisms
(4.4) $E_3^{*,*}(PU(n)) = E_2^{*,*}(PU(n)) \cong H^*(PU(n))$.

To make this isomorphism transparent let $J(\omega) \subset E_3^{*,*}(PU(n))$ (resp. $J(\omega) \subset H^*(PU(n))$) be the subring generated by $\omega$ (resp. by $\omega$) with unit 1, and regard $E_3^{*,*}(PU(n))$ (resp. $H^*(PU(n))$) as a module over the ring $J(\omega)$ (resp. $J(\omega)$).

**Corollary 4.4.** The maps $\pi^*$ and $\kappa$ (in (2.3) and (2.4)) are injective, and induce a module isomorphism $K : E_3^{*,*}(PU(n)) \cong H^*(PU(n))$.

**Proof.** According to Corollary 2.4 we have

$$J(\omega) = E_3^{*,0}(PU(n)) = \frac{\mathbb{Z}[\omega]}{(6n, \omega, 1 \leq r \leq n)}.$$  

Since $K \mid E_3^{*,0}(PU(n)) = \pi^*$ by (2.3), and since $\omega \in E_3^{*,*}(PU(n))$ and $\omega \in H^*(PU(n))$ are the only generators in degree 2, the isomorphism (4.4) forces the relation $\pi^*(\omega) = \omega$. In particular, $K$ maps $J(\omega)$ isomorphically onto $J(\omega)$. □

We are ready to complete the proof of Lemma 1.2.

**Proof of Lemma 1.2.** Granted with Lemma 4.1 it remains to verify the formulas a), b), c) in the second part of Lemma 1.2. This will be done by comparing the two homomorphisms from the exact sequences (3.4) and (1.2)

$$\overline{\theta} : E_3^{*,*}(U(n)) \rightarrow E_3^{*,r-1}(PU(n)) \text{ and } \theta : H^*(U(n)) \rightarrow H^*(PU(n)).$$

Regarding $H^*(U(n))$ as a $\mathbb{Z}$–module let $H^*(U(n))^{(k)}$, $1 \leq k \leq n$, be the submodule spanned additively by the monomials $x_I$ with length $l(I) = k$, $I \subseteq \{1, \cdots, n\}$. Similarly, considering $H^*(PU(n))$ as a module over its subring $J(\omega)$ let $H^*(PU(n))^{(k)}$ be the submodule spanned by the monomials $\rho_I$ with $l(I) = k$, $I \subseteq \{2, \cdots, n\}$. Then the modules isomorphisms

$$L : E_3^{*,*}(U(n)) \rightarrow H^*(U(n)) \text{ and } K : E_3^{*,*}(PU(n)) \rightarrow H^*(PU(n))$$

restrict to the isomorphisms $L_k$ and $K_k$ that fit into the commutative diagram:

$$
\begin{array}{ccc}
E_3^{*,k}(U(n)) & \xrightarrow{\overline{\theta}} & E_3^{*,k-1}(PU(n)) \\
L_k \downarrow \cong & & K_{k-1} \downarrow \cong \\
H^*(U(n))^{(k)} & \xrightarrow{\theta} & H^*(PU(n))^{(k-1)},
\end{array}
$$

where $\theta_k$ denotes the restriction of $\theta$ to the submodule $H^*(U(n))^{(k)}$. Using this diagram one transforms the formulas a), b) and c) of Lemma 3.5 into their correspondences in Lemma 1.2. □

**Remark 4.5.** The proof of Lemma 1.2 clarifies the relationship between the two exact sequences (1.2) and (3.4), both of them are associated with the circle fibration $c : U(n) \rightarrow PU(n)$. Precisely, in terms of the bi–gradation of the “base degrees” and “fiber degrees” imposed on $H^*(U(n))$ and $H^*(PU(n))$, respectively by the isomorphisms $L$ and $K$, the exact sequence (3.4) provides us with the following additional information on its counterpart (1.2):

a) the map $c^*$ preserves both the base degrees and fiber degrees;

b) the map $\theta$ preserves the base degrees, but reduces the fiber degrees by 1;

c) the map $\omega$ increases the base degrees by 2, and preserves the fiber degrees.

Indeed, the bi–gradation on $E_3^{*,*}(G)$ has been an essential tool of our calculation and construction. These justify the title of Section §3. □
5 Proof of Theorem 1.4

Assume in this section that the prime factorization of the integer $n$ is $p_1^{i_1} \cdots p_t^{i_t}$.
Let $J(\omega) \subset H^*(PU(n))$ be the subring generated by the Euler class $\omega$. According to Corollary 4.3 the map

$$ h : J(\omega) \otimes \Lambda(\rho_3, \cdots, \rho_{2m-1}) \to H^*(PU(n)) \text{ by } h(\omega^r \otimes a) = \omega^r \cup a $$

is a surjective ring map. In [20, Theorem A] Ruiz claimed that $h$ is an isomorphism (see Example 1.8). In the course of showing Theorem 1.4 we correct this error by specifying all the nontrivial generators of the ideal $\ker h$.

Firstly, from the exactness of the sequence (1.2) and a) of Lemma 1.2 we get the presentation

$$ J(\omega) = \mathbb{Z}[\omega] / \langle \xi_i \rangle = \mathbb{Z}[\omega] / \langle (t \omega, 1 \leq t \leq n) \rangle $$(alternatively, see Corollary 4.4). It implies by ii) of Lemma 1.1 that

**Lemma 5.1.** The ring $J(\omega)$ admits the following decomposition into its free part and $p$–primary components

$$ J(\omega) = \bigoplus_{p \in \{p_1, \cdots, p_t\}} J_{p, r}(\omega) \text{ (see (1.7) for } J_{p, r}(\omega)). \blacksquare $$

As in Corollary 1.3 for each $1 \leq i \leq t$ we set $Q_{p_i}(n) = \{1, p_i, \cdots, p_i^i\}$. In the Gysin sequence (1.2) note that $\text{Im } \theta = \ker \omega$ is an ideal.

**Lemma 5.2.** We have that

(5.1) $\text{Im } \theta = \langle \theta(\xi_I); I \subseteq Q_{p_i}(n), 1 \leq i \leq t \rangle$,
and, for each $I = \{p_1^{i_1}, \cdots, p_t^{i_t}\} \subseteq Q_p(n)$ with $i_1 > 0$, that

(5.2) $p^k \cdot \theta(\xi_I) = 0; \quad p^k \cdot (\theta(\xi_I) \cup \xi_I) - p^k \cdot \rho_I = 0$. 

**Proof.** Since $\text{Im } \theta$ is spanned additively by $\theta(\xi_I)$ ($I \subseteq \{1, \cdots, n\}$) formula (5.1) amounts to that, for any multi–index $I \subseteq \{2, \cdots, n\}$ one has

(5.3) $\theta(\xi_I), \theta(\xi_I \cup \xi_I) \in \langle \theta(\xi_J); J \subseteq Q_{p_i}(n), 1 \leq i \leq t \rangle$.

For (5.3) we regard the ordered sequence $I$ as a set to form the partition

$$ I = I_0 \bigcup_{1 \leq i \leq t} I_i, \text{ where } I_i = I \cap Q_{p_i}(n), I_0 = I - \bigcup_{1 \leq i \leq t} I_i. $$

Denote by $|I_i|$ the cardinality of the set $I_i$ and put

$$ b_I(i) := p_1^{i_1} \cdots p_t^{i_t} \in \mathbb{Z}, 1 \leq i \leq t, $$
where $\hat{\cdot}$ denotes omission. Since the sequence $\{b_I(1), \cdots, b_I(t)\}$ so obtained is co–prime, there exists a sequence $\{q_I(1), \cdots, q_I(t)\}$ of integers satisfying

$$ \sum q_I(i) \cdot b_I(i) = 1. $$
By the additivity of $\theta$ and by the formula (1.3) we get

\[ (5.4) \quad \theta(\xi_I) = \sum q_I(i) \cdot \theta(b_I(i) \cdot \xi_I) = \sum (-1)^{|h|} q_I(i) \cdot \theta(\xi_I) \cup \rho_I; \]

\[ \theta(\xi_1 \cup \xi_I) = \sum q_I(i) \cdot \theta(b_I(i) \cdot \xi_1 \cup \xi_I) = \sum (-1)^{|h|} q_I(i) \cdot \theta(\xi_1 \cup \xi_1) \cup \rho_I, \]

where $J_i = I - I_i$, $h_i$ is the reverse order of the re–arrangement $I_i \cup J_i$ of the ordered sequence $I$. With $I_i, 1 \cup I_i \subseteq Q_p(n)$ the relation (5.3) is shown by (5.4).

Finally, if $J = \{p^1, \cdots, p^m\} \subseteq Q_p(n)$ with $i_1 > 0$ we have

\[ c^*(\rho_{2r-1}) = p \cdot \xi_{2r-1}, \quad r \in J \]

by Lemma 1.2. It follows that

\[ p^k \cdot \theta(\xi_I) = \theta(c^*(\rho_J)) = 0 \quad \text{(by (1.3) and } \theta \circ c^* = 0); \]

\[ p^k \cdot \theta(\xi_1 \cup \xi_I) = \theta(\xi_1 \cup c^*(\rho_J)) = n \cdot \rho_J \quad \text{(by (1.3) and } \theta(\xi_1) = n). \]

These verify (5.2) and therefore, complete the proof of Lemma 5.2. \[ \square \]

To relate the ideal $\ker h$ with $\text{Im} \theta$ we apply the second part of Lemma 1.2 to formulate the additive map

\[ \Theta : H^*(U(n)) \rightarrow J(\omega) \otimes \Lambda(\rho_3, \cdots, \rho_{2n-1}) \]

of degree $-1$ by two steps:

i) for each $I \subseteq Q_p(n)$ use the formula (1.5) to define

\[ \Theta(\xi_I) := (-1)^{k-1} \varepsilon_{\xi_I} \sum_{J \subseteq \xi_I} \pi_J \cdot \xi_{1 \cup J} \cdot \rho_J; \]

ii) use (5.4) to extend $\Theta$ to the basis $\{\xi_I; I \subseteq \{1, \cdots, n\}\}$ of $H^*(U(n))$.

Evidently, the map $\Theta$ so defined provides us a lift of $\theta$ with respect to $h$

\[ H^*(PU(n)) \xrightarrow{\Theta} H^*(U(n)) \xrightarrow{\theta} H^*(PU(n)) \]

where the bottom row is exact by (1.2). We use this to show that

**Lemma 5.3.** $\ker h = \langle \omega \otimes \Theta(\xi_I); I \subseteq Q_p(n), 1 \leq i \leq l \rangle$.

**Proof.** By the commutative triangle above, as well as the exactness of the bottom sequence, we already have

\[ \omega \otimes \Theta(\xi_I) \in \ker h, \quad I \subseteq Q_p(n). \]

So it suffices for us to show the converse

\[ \ker h \subseteq \langle \omega \otimes \Theta(\xi_I), I \subseteq \{1, \cdots, n\} \rangle \quad \text{(by (5.1))}. \]

This will be done by induction on the degrees of elements of $\ker h$.

The elements of the ring $J(\omega) \otimes \Lambda(\rho_3, \cdots, \rho_{2n-1})$ with lowest non–zero degree are those $m \omega \otimes 1$ with $m \in \mathbb{Z}$. Since $\theta(\xi_1) = n$ by Lemma 1.2 the statement $m \omega \otimes 1 \in \ker h$ is equivalent that $n$ divides $m$ by (1.2). That is
showing (5.5) in degree 2.

Consider next an element \( y \in \ker h \) with \( \deg(y) > 2 \) and with
\[
y = a_0 + \omega \otimes a_1 + \cdots + \omega^r \otimes a_r, \quad a_i \in \Lambda(p_3, \cdots, p_{2n-1}).
\]

One infers from \( h(y) = 0 \) that \( a_0 = 0 \), hence
\[
y = \omega \otimes x \quad \text{with} \quad x = 1 \otimes a_1 + \cdots + \omega^{r-1} \otimes a_r.
\]

By the exactness of (1.2) one gets from \( h(y) = 0 \) that \( h(x) \in \Im \theta \), hence the expansion
\[
h(x) = \Sigma b_I \cdot \theta(\xi_I), \quad I \subseteq \{1, \cdots, n\}, \quad b_I \in \mathbb{Z}.
\]

With \( \deg x = \deg y - 2 \) one obtains from \( h(x - \Sigma b_I \cdot \theta(\xi_I)) = 0 \), as well as the inductive hypothesis, that
\[
x - \Sigma b_I \cdot \theta(\xi_I) \in \langle \omega \otimes \Theta(\xi_I), I \subseteq \{1, \cdots, n\} \rangle,
\]
showing \( y = \omega \otimes x \in \langle \omega \otimes \Theta(\xi_I), I \subseteq \{1, \cdots, n\} \rangle \). This completes the proof.\( \Box \)

Combining results of Lemmas 5.1, 5.2 and 5.3 we establish our main result.

**Proof of Theorem 1.4.** Abbreviate the exterior ring \( \Lambda(p_3, \cdots, p_{2n-1}) \) by \( \Lambda \). In view of the decompositions of the rings \( J(\omega) \otimes \Lambda \) and \( H^*(PU(n)) \) by their free parts and primary components
\[
J(\omega) \otimes \Lambda = \bigoplus_{p \in \{p_1, \cdots, p_t\}} J_{p,r}(\omega) \otimes \Lambda \quad (\text{by Lemma 5.1})
\]
\[
H^*(PU(n)) = \Lambda \oplus \sigma_p(PU(n)) \quad (\text{by (1.6)})
\]
we examine the surjective ring map \( h : J(\omega) \otimes \Lambda \to H^*(PU(n)) \). Since \( h \) must preserve the \( p \)-primary components we have, in addition to

i) \( \sigma_p(PU(n)) = 0 \) for \( p \notin \{p_1, \cdots, p_t\} \),

ii) \( h(J_{p,r}(\omega) \otimes \Lambda) = \sigma_p(PU(n)), \quad p \in \{p_1, \cdots, p_t\} \)

that (by (5.2) and Lemma 5.3)

iii) \( J_{p,r}(\omega) \otimes \Lambda \cap \ker h = \langle \omega \otimes \Theta(\xi_I); I \subseteq Q_p(n) \rangle \)

It follows from ii) and iii) that the map \( h \) induces the ring isomorphism
\[
(5.6) \quad \frac{f_{p,r}(\omega) \otimes \Lambda(p_3, \cdots, p_{2n-1})}{\omega \otimes \Theta(\xi_I); I \subseteq Q_p(n), l(I) \geq 2} \cong \sigma_p(PU(n)), \quad p \in \{p_1, \cdots, p_t\},
\]
where we have added the constraint \( l(I) \geq 2 \) because the relations \( \omega \otimes \Theta(\xi_I) \) with \( l(I) = 1 \) has already been used in presenting \( J(\omega) \).

On the other hand comparing the definitions of \( \Theta(\xi_I) \) with \( R_I \) we find that
\[
\omega \otimes \Theta(\xi_I) = (-1)^{k-1} \frac{1}{p^\prime-1} \binom{n}{p^\prime} \cdot R_I, \quad I \subseteq Q_p(n),
\]
in which the multiplier \( (-1)^{k-1} \frac{1}{p^\prime-1} \binom{n}{p^\prime} \) is co-prime to \( p \) by ii) of Lemma 1.1. This implies that the relation \( \omega \otimes \Theta(\xi_I) = 0 \) on \( \sigma_p(PU(n)) \) is equivalent to \( R_I = 0 \), hence formula (5.6) is identical to (1.8). This completes the proof.\( \Box \)
6 Appendix: the Bockstein homomorphism \( \beta_p : H^*(PU(n); \mathbb{F}_p) \rightarrow H^*(PU(n)) \)

Assume in this section that \( n = p^kn' \) with \( p \) a prime, \( (p,n') = 1 \). In [5] Borel obtained the following presentation of the algebra \( H^*(PU(n); \mathbb{F}_p) \)

\[
(6.1) \quad H^*(PU(n); \mathbb{F}_p) = \Delta(\eta_1) \otimes \Lambda(\eta_3, \cdots, \eta_{2p^k-1}, \cdots \eta_{2n}) \otimes \mathbb{F}_p[\omega]/\langle \omega^{p^s} \rangle,
\]

where \( \eta_1 = \omega \) or 0 in accordance to \( p = 2 \) and \( k = 1 \), or otherwise. In [5] Baum and Browder studied the Bockstein homomorphism \( \beta_p : H^*(PU(n); \mathbb{F}_p) \rightarrow H^*(PU(n)) \) and obtained the formula (see [5] Lemma 6.7)

\[
\beta_p(\eta_{2p-1}) = p^{k-2}\omega^p \text{ (for the case } n' = 1). \]

In this appendix we complete the calculation of Baum and Browder by showing the following formula of \( \beta_p \).

**Theorem 6.1.** With respect to (6.1) the Bockstein \( \beta_p \) satisfies that

\[
(6.2) \quad \beta_p(\eta_{2r-1}) = \begin{cases} 
p^{k-t-1}\omega^{p^t} & \text{if } r = p^t \text{ with } 0 \leq t < k \\
0 & \text{otherwise.} \end{cases}
\]

We begin with explicit constructions of the generators \( \eta_{2r-1} \) in (6.1). Since the Koszul complexes \( \{ E_2^{*,*}(G), d_2 \} \) is torsion free one can deduces the \( \mathbb{F}_p \) analogous of Sections §2 and §3 by tensoring the graded groups and the homomorphisms there with \( \mathbb{F}_p \). In particular we can infer from (3.5) the following commutative diagram

\[
\begin{array}{ccc}
K^{t+1} \otimes \mathbb{F}_p & \xrightarrow{\mathcal{F}} & E_2^{t+1}(PU(n); \mathbb{F}_p) \\
\mathcal{D} & \downarrow & d_2' \\
[\mathbb{F}_p[x_1,\cdots,x_n]] & \xrightarrow{f} & [\mathbb{F}_p[x_1,\cdots,x_n]/(e_1,\cdots,e_s)]
\end{array}
\]

by which we have (compare with Lemma 3.2)

**Lemma 6.2.** If \( g \in \mathbb{F}_p[x_1,\cdots,x_n] \) and \( z, z' \in K^{t+1} \otimes \mathbb{F}_p \), then

i) \( g \in \text{Im } \mathcal{D} \) if and only if \( g \equiv 0 \mod (\text{Im } \mathcal{D}) \);

ii) \( d_2'(\mathcal{F}(z)) \equiv 0 \) if and only if \( \mathcal{D}(z) \in \text{ker } f \);

iii) \( \mathcal{D}(z) \equiv \mathcal{D}(z') \) implies that \( \mathcal{F}(z) - \mathcal{F}(z') \in \text{Im } d_2, \square \)

Consider the polynomials \( g_r \in \mathbb{F}_p[x_1,\cdots,x_n], 1 \leq r \leq n, r \neq p^k \), defined by

\[
g_r \equiv \begin{cases} 
e_r(p) & \text{if } 1 \leq r \leq p^k - 1, \\
e_r(p) - t_{n,r}e_{p^k}(p) \cdot x_1^{r-p^k} & \text{if } p^k < r \leq n, \end{cases}
\]

where \( e_r(p) \equiv e_r \mod p \), and where \( t_{n,r} \) is an integer satisfying the congruence

\[
t_{n,r} \left( \binom{n}{p^k} \right) \equiv \left( \binom{n}{r} \right) \mod p \text{ (note that } \binom{n}{p^k} \text{ is prime to } p). \]
Lemma 6.4. \(\text{of \cite[Lemma 3.3]{10}}\) implies that \(\text{Lemma 6.3. Theorem 4.5}}\)

Proof of Theorem 6.1. Applying Lemma 2.7 yields that \(\eta_r\) mod \((\text{Im }\tau')\) for \(r < p^k\) (by Lemma 1.1).

one finds that \(g_r\) mod \((\text{Im }\tau')\) \(\equiv 0\) mod \(p\). By i) of Lemma 6.2 there exist \(\hat{g}_r \in \mathfrak{K}^{1,1} \otimes \mathbb{F}_p\) so that \(g_r = \mathfrak{d}(\hat{g}_r)\). With

\(g_r \in \ker f \subset \mathbb{F}_p[x_1, \ldots, x_n]\)

the elements \(\mathfrak{F}(\hat{g}_r) \in E_2^{2,1}(PU(n); \mathbb{F}_p)\) are \(d_2\)-closed by ii) of Lemma 6.2, hence define the cohomology classes

\(\eta_{2r-1} = [\mathfrak{F}(\hat{g}_r)] \in E_3^{3,1}(PU(n); \mathbb{F}_p)\)

which are independent of the choices of \(\hat{g}_r\) by iii) of Lemma 6.2. Let us set

\(\eta_{2r-1} := \kappa(\eta'_{2r-1}) \in H^{2r-1}(PU(n); \mathbb{F}_p), 1 \leq r \leq n, r \neq p^k,\)

where \(\kappa\) is the \(\mathbb{F}_p\) analogous of the map \(\kappa\) in (2.4). It has been shown in \[8\] Theorem 4.5] that

Lemma 6.3. With respect to the classes \(\eta_{2r-1}\) defined by (6.3) the algebra \(H^*(PU(n); \mathbb{F}_p)\) has the presentation (6.1). □

For an integer \(m \geq 2\) the biggest integer \(a\) so that \(m\) is divisible by the power \(p^a\) is called the \(p\)-prime order of \(m\), and will be denoted by \(\text{ord}_p m\). The proof of \[10\] Lemma 3.3] implies that

Lemma 6.4. If \(p^t \leq r < p^{t+1}\) with \(t + 1 < k\), then

\(\text{(6.4) } \text{ord}_p(n) \geq k - t, \text{ where the equality holds if and only if } r = p^t.\) □

Proof of Theorem 6.1. According to Lemma 2.7 we have

\(\beta_p(\eta_{2r-1}) \in \text{Im }\pi^* = \mathcal{J}(\omega).\)

With \(p \cdot \beta_p(\eta_{2r-1}) = 0\) we have further that

\(\beta_p(\eta_{2r-1}) \in \mathcal{J}_{p,k}(\omega) = \mathbb{Z}[\omega]^+/(\mathbb{Z}[\omega]^{p^k} \mathbb{Z}[\omega^{p^{k-1}}, \ldots, \omega^{p^t}]).\)

With \(\omega^{p^k} = 0\) in \(\mathcal{J}_{p,k}(\omega)\) we obtain for the degree reason that

a) \(\beta_p(\eta_{2r-1}) = 0\) if \(r > p^k.\)

Assume below that \(p^t \leq r < p^{t+1}\) with \(t + 1 < k\). Then the \(d_2\)-cocycles \(\mathfrak{F}(\hat{g}_r) \in E_2^{2,1}(PU(n); \mathbb{F}_p)\) have the obvious integral lifts

\(\mathfrak{F}(\hat{c}_r) \in E_2^{2,1}(PU(n))\) (in the notation of Construction B).

Applying Lemma 2.7 yields that

b) \(\beta_p(\eta_{2r-1}) = \tfrac{1}{p} \pi^* (\chi_r \mod (\text{Im }\tau')) = \tfrac{1}{p} (\begin{pmatrix} n \\ r \end{pmatrix}) \omega^r \in \mathcal{J}_{p,k}(\omega).\)

Granted with Lemma 6.4 the proof of (6.2) is completed by a) and b). □

Remark 6.5. The study of the prime orders \(\text{ord}_p(n)\) of the binomial coefficients \(\binom{n}{r}\) has a long history in number theory, see the survey article \[12\] by Granville. □
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