The energy-carrying velocity and rolling of tachyons of unstable D-branes

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Abstract

It is shown that the tachyons which originate from unstable D-branes carry energy and momentum at the velocity $\beta = c^2/v$, where $v$ is the phase velocity which is greater than $c$. For an observer who moves with the velocity $\beta$, tachyon is observed to be moving from one of the ground states of the tachyon potential to the potential hill. It is found that tachyon either passes over the hill or bounces back to the original ground state. Another possible solution is the case which is marginal to these, that is tachyon reaches to the top of the potential hill and stays there forever.

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1 Introduction

Tachyons, which abound in non-BPS D-branes of type IIA/IIB superstring theories or in bosonic strings, can be studied by using relatively simple effective action\[1\]. The tachyon potential $V(T)$, which is an even function of the tachyon field $T$, has a run-away form: it is maximum at $T = 0$, and minimum at $T \to \pm\infty$. This means that if the initial string state is the tachyonic state around the unstable false vacuum $T \approx 0$, all the energy, because of the fact that the full string states contain various field modes, will eventually be distributed among various bulk modes.

This process of rolling of tachyons and tachyon condensations are discussed by various authors\[2\]. Recently it is pointed out that fluctuations around a static kink solution of $D_p$-brane produces the Dirac-Born-Infelt action of a BPS $D_{p-1}$-brane\[3\]. It is also found\[4\] that the only static finite energy solution of tachyon field which depends only on one of the spatial coordinate $x$ is $T(x) = Ax$, where $A \to \infty$.

In this paper, we investigate the traveling wave solutions of the tachyon field which propagate along $x$. It is found that the phase velocity $v$ of tachyon wave may have any value except the speed of light $c$. As usual, the phase velocity is quite different from the mechanical velocity\[1\] at which the energy-momentum is carried. If $v < c$, they are the same when one assumes that the wave amplitude $A$ becomes very large. On the other hand, if $v > c$, the mechanical velocity is $\beta = c^2/v$. Furthermore, the invariant mass $m = \sqrt{E^2 - p^2c^2/c^2}$ is always real. This important fact is pointed out in\[5\] where boundary state idea is used under different perspective. It means that tachyonic states of unstable D-branes can be interpreted as usual matter states moving at the mechanical velocity $\beta$. This energy distribution in space may be related to the final bulk states of

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\[1\]The usual terminology for the velocity at which waves carry the energy-momentum is the group velocity. But in this tachyonic case whose highly non-linear equation of motion does not admit the simple rule of linear superposition, it is more adequate to use the term mechanical velocity instead of group velocity.
strings.

For an observer who moves with the velocity $\beta$, tachyon is observed to be rolling from one of the ground states of the tachyon potential to the potential hill. It is found that there are three different types of motion. The first is that tachyon passes over the hill and rolls to another ground state. The second one is the case that it can not cross over the hill and bounces back to the original ground state. The last possibility is marginal to these, and tachyon reaches to the top of the potential hill and stays there forever.

In section 2, a propagating wave solution for the tachyon is obtained. It has either oscillating form or kink form. The oscillating solution is studied in section 3, and the kink solution is investigated in section 4. Physical interpretation of the solution is given if section 5. The conclusion is given in section 6.

## 2 Propagating wave solution for the tachyon field

Consider the following effective action of the tachyon field $T$ on a non-BPS D$_p$-brane,

$$S = - \int d^{p+1}x \sqrt{- \det(\eta_{\mu\nu} + \partial_\mu T \partial_\nu T)} V(T).$$  \hspace{1cm} (1)

Here, $\eta_{\mu\nu} = \text{diag}(-1, 1, ..., 1)$, and $\hbar = c = \alpha' = 1$. For the potential of the tachyon field, we assume the following form\cite{6, 7}

$$V(T) = \frac{\tau_p}{\cosh\left(\frac{T}{T_0}\right)}.$$  \hspace{1cm} (2)

where $\tau_p$ is the brane tension, and $T_0$ is a constant which is $\sqrt{2}$ for non-BPS D-branes in superstrings, and 2 for bosonic strings.

To solve the Lagrangian equation of motion, we introduce a new variable $\mathcal{T}$ defined by\cite{7, 8}

$$\mathcal{T} = T_0 \sinh\left(\frac{T}{T_0}\right).$$  \hspace{1cm} (3)
The Lagrangian in terms of $\mathcal{T}$, which is given by
\[ L = -\frac{\tau_p}{1 + \frac{T^2}{T_0^2}} \sqrt{1 + \frac{T^2}{T_0^2} + \partial_\mu \partial^\mu \mathcal{T}}, \] (4)
does not contain any transcendental function. The corresponding equation of motion is given by the following polynomial form of $\mathcal{T}$, $\partial_\mu \mathcal{T}$ and $\partial_\mu \partial_\nu \mathcal{T}$:
\[ (1 + \frac{T^2}{T_0^2})(\partial_\mu \partial^\mu \mathcal{T} + \frac{T}{T_0} \partial_\mu \partial_\nu \mathcal{T} - \partial_\mu \partial_\nu \mathcal{T}) = 0. \] (5)

We observe the following facts that in each term of this equation, when expanded, $\partial_\mu$ appears even times while $\mathcal{T}$ odd times. This means that for a given solution, one may obtain another solution by changing either the sign of a particular $x^\mu$ or/and $\mathcal{T}$.

It is known\[3\] that for the extreme static kink solution such as $T \rightarrow \pm \infty$ depending on the sign of $x^\rho$, the resulting configuration is a stable $D_{p-1}$-brane at $x = 0$. In this paper we assume that $\mathcal{T}$, or equivalently $T$, depends on both $x = x^p$ and $t$. In this case, (5) can be written as
\[ (1 + \frac{T^2}{T_0^2})(\partial_\mu \partial^\mu \mathcal{T} + \frac{T}{T_0} \partial_\mu \partial_\nu \mathcal{T} - \partial_\mu \partial_\nu \mathcal{T}) - \partial_\mu \partial_\nu \mathcal{T} \partial^\mu \partial^\nu \mathcal{T} = 0. \] (6)

It is easy to show that it has the following traveling wave solution,
\[ \mathcal{T} = \frac{T_0}{2} \left\{ A e^{ik(x-\nu t)} + B e^{-ik(x-\nu t)} \right\}, \] (7)
where $A$, $B$ and $\nu$ are all integration constants, and $k$ is given by
\[ k = \frac{1}{\sqrt{1 - \nu^2 T_0^2}}. \] (8)

If $|\nu| < 1$, $k$ is real and the solution is oscillatory. This is discussed in section 3. On the other hand, if $|\nu| > 1$, one may replace $ik$ by $\kappa$ to obtain a kink solution. It is discussed in section 4.
3 Traveling sinusoidal solution

Consider the case $|v| < 1$. For the convenience of computation we rewrite (7) in the following form

$$\mathcal{T} = T_0 A \cos k(x - vt). \tag{9}$$

To understand the physical meaning of the phase velocity $v$, we calculate the energy and momentum densities of the field configuration. The relevant energy-momentum tensor $T^{\mu\nu}$ is

$$T^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial \partial_{\mu} T} \partial_{\nu} T + \delta^{\mu\nu} \mathcal{L}, \tag{10}$$

where the Lagrangian expressed in terms of $T(x, t)$ is given by

$$\mathcal{L} = -V(T) \sqrt{1 + (\partial_x T)^2 - (\partial_0 T)^2}. \tag{11}$$

From this one can read off the following energy density $\rho = T^{00}$ and momentum density $P = T^{\nu 0}$,

$$\rho = \frac{V(T)}{\sqrt{1 + (\partial_x T)^2 - (\partial_0 T)^2}} \left\{ 1 + (\partial_x T)^2 \right\}, \tag{12}$$

$$P = \frac{V(T)}{\sqrt{1 + (\partial_x T)^2 - (\partial_0 T)^2}} \partial_x T \partial_0 T. \tag{13}$$

After non-trivial but straightforward calculations it can be shown that

$$\frac{V(T)}{\sqrt{1 + (\partial_x T)^2 - (\partial_0 T)^2}} = \frac{\tau_p}{\sqrt{1 + A^2}}. \tag{14}$$

On the other hand, by using the defining equation (3) and the solution (9), one may prove that

$$(\partial_x T)^2 = \frac{(\partial_x T)^2}{1 + \frac{T^2}{T_0^2}} \tag{15}$$

$$= k^2 T_0^2 \frac{A^2 \sin^2 k(x - vt)}{1 + A^2 \cos^2 k(x - vt)}. \tag{16}$$
That is, $\rho$ and $P$, for this traveling sinusoidal solution, are

\[
\rho = \frac{\tau_p}{\sqrt{1 + A^2}} \left\{ 1 + \frac{1}{1 - v^2} \frac{A^2 \sin^2 k(x - vt)}{1 + A^2 \cos^2 k(x - vt)} \right\},
\]
(17)

\[
P = \frac{\tau_p v}{\sqrt{1 + A^2} \left( 1 - \frac{v^2}{1 + A^2} \right)} \frac{A^2 \sin^2 k(x - vt)}{1 + A^2 \cos^2 k(x - vt)}.
\]
(18)

When $A = 0$, one has the trivial unstable solution, $T = 0$. In this case, $\rho = \tau_p$ and $P = 0$ as expected.

For general $A$, the energy density is oscillatory at the string length scale $T_0$. The average behaviour of it can be obtained by using the relation,

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{a \sin^2 \theta}{1 + a \cos^2 \theta} d\theta = \sqrt{1 + a} - 1.
\]
(19)

That is, the spatial average of $\rho$ is

\[
\bar{\rho} = \frac{\tau_p}{1 - v^2} \left( 1 - \frac{v^2}{\sqrt{1 + A^2}} \right).
\]
(20)

For large $A$, it becomes

\[
\bar{\rho} \simeq \frac{\tau_p}{1 - v^2}.
\]
(21)

It can be interpreted in the following way. The energy stored in length $L$ is $E_L = \bar{\rho}L$. Here, one should be careful that it is not $L$ but $L_0 = \gamma L$ which is Lorentz invariant, where $\gamma = 1/\sqrt{1 - v^2}$. It means that it is more appropriate to express $E_L$ in terms of $L_0$,

\[
E_L = \gamma \tau_p L_0.
\]
(22)

This shows that the (rest) mass density is $\rho_m = \tau_p$.

Similarly, the average momentum density is

\[
\bar{P} = \frac{\tau_p v}{1 - v^2} \left( 1 - \frac{1}{\sqrt{1 + A^2}} \right).
\]
(23)

This means that, for large $A$, the momentum in length $L$ is

\[
P_L = \bar{P}L = \gamma \tau_p v L_0.
\]
(25)
Combining this with (22) one gets \( P_L = vE_L \). It shows that \( v \) is in fact the velocity at which the energy is propagated. Similarly we have \( E_L^2 - P_L^2 = m_L^2 \), where \( m_L = \tau_p L_0 \) is the mass stored in the invariant length \( L_0 \).

4 Traveling kink solution

Consider the case \(|v| > 1\). In this case, by replacing \( ik \) of (7) with \( \kappa \) we obtain the following solution,

\[
\mathcal{T} = \frac{T_0}{2} \left\{ A e^{\kappa(x-\nu t)} + B e^{-\kappa(x-\nu t)} \right\}.
\]

(26)

Here,

\[
\kappa = \frac{1}{\sqrt{v^2 - 1}} T_0.
\]

(27)

The corresponding energy and momentum densities which can be determined from (12) and (13) are

\[
\rho = \frac{\tau_p}{\sqrt{1 + AB}} \left\{ 1 + (\partial_x T)^2 \right\},
\]

(28)

\[
P = \frac{\tau_p}{\sqrt{1 + AB}} (\partial_x T)^2 \nu.
\]

(29)

By using the same technique employed in section 3, one may prove that

\[
(\partial_x T)^2 = \kappa^2 T_0^2 \frac{A^2 e^{2\kappa(x-\nu t)} + B^2 e^{-2\kappa(x-\nu t)} - 2AB}{A^2 e^{2\kappa(x-\nu t)} + B^2 e^{-2\kappa(x-\nu t)} + 2AB + 4}.
\]

(30)

That is,

\[
\rho = \frac{\tau_p}{\sqrt{1 + AB}} \left\{ 1 + \kappa^2 T_0^2 \frac{A^2 e^{2\kappa(x-\nu t)} + B^2 e^{-2\kappa(x-\nu t)} - 2AB}{A^2 e^{2\kappa(x-\nu t)} + B^2 e^{-2\kappa(x-\nu t)} + 2AB + 4} \right\},
\]

(31)

\[
P = \frac{\tau_p}{\sqrt{1 + AB}} \kappa^2 T_0^2 \frac{A^2 e^{2\kappa(x-\nu t)} + B^2 e^{-2\kappa(x-\nu t)} - 2AB}{A^2 e^{2\kappa(x-\nu t)} + B^2 e^{-2\kappa(x-\nu t)} + 2AB + 4} \nu.
\]

(32)

To have real \( \rho \) and \( P \), it is required that \( AB > -1 \). We consider \( AB > 0 \), and \( 0 > AB > -1 \), \( AB = 0 \) cases separately.

Case 1) \( AB > 0 \).
In this case, one may assume that \( A = B \). It is always possible by properly choosing the origin of \( x \). The solution has the following form

\[
\mathcal{T} = T_0 A \cosh \kappa (x - vt).
\]  

(33)

The energy density \( \rho \) which can be read off from (33) is

\[
\rho = \frac{\tau_p}{\sqrt{1 + A^2}} \left\{ 1 + \frac{\kappa^2}{v^2 - A^2} \cosh^2 \kappa (x - vt) \right\}.
\]

(34)

This \( \rho \) is almost constant except at the region of a small dimple of size \( T_0 \).

By using the relation

\[
\int \frac{a \sinh^2 x}{1 + a \cosh^2 x} dx = x - \sqrt{1 + a} \tanh^{-1} \left( \frac{\tanh x}{\sqrt{1 + a}} \right) + \text{const.},
\]

(35)

one can show that the stored energy in the region \( -\frac{L}{2} < x < \frac{L}{2} \) of length \( L \) is the following,

\[
E_L = \frac{v^2}{v^2 - 1} \frac{\tau_p}{\sqrt{1 + A^2}} \left[ 1 - \frac{2 \tau_p}{v^2 - 1} \tanh^{-1} \left( \frac{\tanh \frac{L}{2}}{\sqrt{1 + A^2}} \right) \right].
\]

(36)

That is, the energy density \( \bar{\rho} = \lim_{L \to \infty} E_L/L \) effectively becomes

\[
\bar{\rho} = \frac{\tau_p}{\sqrt{1 + A^2}} \frac{1}{1 - \beta^2},
\]

(37)

where \( \beta = 1/v \).

Similarly, we have

\[
\bar{P} = \frac{\tau_p}{\sqrt{1 + A^2}} \frac{\beta}{1 - \beta^2}.
\]

(38)

It is interesting that the momentum density \( \bar{P} \) vanishes if \( \beta \to 0 \), or equivalently if \( v \to \infty \). This gives us a hint that \( v \) is not the mechanical velocity. In fact the relation \( P = \beta \bar{\rho} \) shows that it is not \( v \) but \( \beta \) which carries energy. This means that tachyon can be interpreted as a usual matter which moves at the mechanical velocity \( \beta \). Similar analysis done in section 3 may be employed here to show that the mass density is

\[
\rho_m = \frac{\tau_p}{\sqrt{1 + A^2}}.
\]

(39)
Case 2) $0 > AB > -1$.

By a suitable choice of the origin of $x$ one may assume that $B = -A$, where $0 < A^2 < 1$. Then, the solution has the following form,

$$
T = T_0 A \sinh \kappa (x - vt). \quad (40)
$$

The corresponding energy density $\rho$ is

$$
\rho = \frac{\tau_p}{\sqrt{1 - A^2}} \left\{ 1 + \frac{1}{v^2 - 1} \frac{A^2 \cosh^2 \kappa (x - vt)}{A^2 \sinh^2 \kappa (x - vt) + 1} \right\} \quad (41)
$$

As the case 1), $\rho$ has a small dimple of size $T_0$. By using the relation

$$
\int \frac{a \cosh^2 x}{1 + a \sinh^2 x} \, dx = x - \sqrt{1 - a} \tanh^{-1} \left\{ \sqrt{1 - a} \tanh x \right\} + \text{const}, \quad (42)
$$

it can be shown that the total energy and momentum stored in the region $-\frac{L}{2} < x < \frac{L}{2}$ of length $L$ are

$$
E_L = \frac{1}{1 - \beta^2} \frac{\tau_p}{\sqrt{1 - A^2}} L - \frac{2 \tau_p \beta^2}{1 - \beta^2} \tanh^{-1} \left( \sqrt{1 - A^2} \tanh \frac{L}{2} \right), \quad (43)
$$

$$
P_L = \frac{\beta}{1 - \beta^2} \frac{\tau_p}{\sqrt{1 - A^2}} L - \frac{2 \tau_p \beta}{1 - \beta^2} \tanh^{-1} \left( \sqrt{1 - A^2} \tanh \frac{L}{2} \right). \quad (44)
$$

That is, the energy and momentum densities, when $L \to \infty$, effectively become

$$
\bar{\rho} = \frac{1}{1 - \beta^2} \frac{\tau_p}{\sqrt{1 - A^2}}, \quad (45)
$$

$$
\bar{P} = \frac{\beta}{1 - \beta^2} \frac{\tau_p}{\sqrt{1 - A^2}}. \quad (46)
$$

These show that the tachyon can in fact be interpreted as a usual matter of mass density

$$
\rho_m = \frac{\tau_p}{\sqrt{1 - A^2}} \quad (47)
$$

moving at the velocity $\beta = 1/v$.

Case 3) $AB = 0$. 

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8
We assume that \( B = 0 \). The other case, that is \( A = 0 \), can be obtained from the results of this case by replacing \( \kappa \rightarrow -\kappa \). The corresponding solution is given by

\[
\mathcal{T} = A e^{\kappa (x - vt)}.
\]  

(48)

The energy and momentum densities are

\[
\rho = \tau_p \left\{ 1 + \frac{1}{v^2 - 1} \frac{A^2 e^{2\kappa (x - vt)}}{A^2 e^{2\kappa (x - vt)} + 4} \right\},
\]

(49)

\[
P = \tau_p \left\{ \frac{1}{v^2 - 1} \frac{A^2 e^{2\kappa (x - vt)}}{A^2 e^{2\kappa (x - vt)} + 4} \right\} v.
\]

(50)

These mean that the energy and momentum densities under the \( A \rightarrow \infty \) limits become

\[
\rho = \frac{\tau_p}{1 - \beta^2},
\]

(52)

\[
P = \frac{\tau_p}{1 - \beta^2} \beta.
\]

(53)

This means that it also describes the usual matter of the mass density \( \rho_m = \tau_p \) moving at the velocity \( \beta = 1/v \).

5 Interpretation of the solutions

Consider the traveling sinusoidal solution (9) first. For an observer with coordinates \( (x', t') \) who moves at the velocity \( v \) with respect to the frame \( (x, t) \), the solution can be written as

\[
\mathcal{T} = T_0 A \cos \frac{x'}{T_0}.
\]

(54)

This represents an infinite lattice of brain-antibrane system[9].

For the traveling kink solution, consider (26). For an observer with coordinates \( (x', t') \) who moves with velocity \( \beta \), phase \( \kappa(x - vt) \) can be written as \( -\frac{t'}{T_0} \), where we used

\[
\frac{x - vt}{\sqrt{v^2 - 1}} = -\frac{t'}{T_0}.
\]

(55)
This means that for kink solution it can be written as

\[
\mathcal{T} = \begin{cases} 
T_0 A \cosh\left(\frac{-t'}{T_0}\right) & \text{for Case 1}, \\
T_0 A \sinh\left(\frac{-t'}{T_0}\right) & \text{for Case 2}, \\
T_0 A e^{-\frac{t'}{T_0}} & \text{for Case 3}.
\end{cases}
\] (56)

It is clear that Case 2 represents motion of tachyon which moves from one ground state of the potential at \(t' \to -\infty\) to the other ground state at \(t' \to \infty\), thus crossing over the potential hill. For the Case 1, tachyon does not cross over the potential hill, and begins to bounce back to the original vacuum at \(t' = 0\). The Case 3 is marginal to these, and tachyon approaches to the maximum of the potential as \(t' \to \infty\).

6 Conclusion

The traveling wave solution of the Sen’s tachyon action, either in the oscillating form or kink form, represents usual matter that moves at the velocity \(\beta\) which is less than \(c\). If the phase velocity \(v\) is less than \(c\), the tachyon field oscillates in space. In this case, \(\beta = v\) and the mass density is \(\tau_p\).

If, on the other hand, the phase velocity of the tachyon field \(v\) is greater than the speed of light \(c\), tachyon field forms a kink. In this case, the velocity at which energy is carried is \(\beta = c^2/v\). This corresponds to the rolling of tachyons. The corresponding mass density is \(\tau_p/\sqrt{1+A^2}\), \(\tau_p/\sqrt{1-A^2}\) or \(\tau_p\) depending on the precise form of the solution. This energy-momentum may be distributed among various modes of strings.

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