A SIMPLE NEARLY-OPTIMAL RESTART SCHEME
FOR SPEEDING-UP FIRST ORDER METHODS

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Abstract. We present a simple scheme for restarting first-order methods for convex optimization problems. Restarts are made based only on achieving specified decreases in objective values, the specified amounts being the same for all optimization problems. Unlike existing restart schemes, the scheme makes no attempt to learn parameter values characterizing the structure of an optimization problem, nor does it require any special information that would not be available in practice (unless the first-order method chosen to be employed in the scheme itself requires special information). As immediate corollaries to the main theorems, we show that when some well-known first-order methods are employed in the scheme, the resulting complexity bounds are nearly optimal for particular – yet quite general – classes of problems.

1. Introduction

A “restart scheme” is, for us, a procedure that aims to improve the performance of an existing algorithm when the algorithm is applied to optimization problems possessing particular structure, where the procedure occasionally stops the algorithm, and then starts it again with new input. (In particular, a restart scheme does not modify the algorithm, just the input.)

An understanding that restarting can speed-up first-order methods goes back decades, to work of Nemirovski and Nesterov [13], although their focus was on the abstract setting where a scheme somehow knows nearly-ideal times at which to restart an algorithm it employs.

In the last ten years, the notion of “adaptivity” has come to play a foremost role in research on first-order methods. Here, an emerging line of work focuses on adaptive restart schemes, these being meta-algorithms capable of deciding when best to restart algorithms they employ. Before noting some highlights of the existing literature, we provide a summary of our contributions.

1.1. Overview of Our Contributions. We present a simple scheme for restarting first-order methods applied to optimization problems

\[ \min_{x \in Q} f(x) \]

where \( f \) is a convex function, and \( Q \) is a closed convex set contained in the (effective) domain of \( f \) (i.e., where \( f \) is finite). We assume the set of optimal solutions, \( X^* \), is nonempty. Let \( f^* \) denote the optimal value.

The scheme is simple to the extent of almost seeming naive. Indeed, restarts are made only on the basis of having obtained specified decreases in objective value, the same decreases applying to all optimization problems, and to all first-order methods employed in the scheme.

The scheme relies on multiple copies of a first-order method (whichever first-order method is chosen). For the best time bounds, the copies are run in parallel.

We present a synchronous scheme, and especially for first-order methods in which the amount of work can vary significantly among iterations, an asynchronous scheme. (Even for methods that do require the same amount of work in every iteration, the asynchronous scheme can have definite advantages.)
The minor requirements for a first-order method to be employed in the schemes are discussed in §2, where also some motivation for the design of the schemes is given. The schemes are presented in §3.

To theoretically demonstrate effectiveness of the schemes, we consider objective functions \( f \) possessing “Hölderian growth,” in that there exist constants \( \mu > 0 \) and \( d \geq 1 \) for which

\[
x \in Q \text{ and } f(x) \leq f(x_0) \implies f(x) - f^* \geq \mu \text{dist}(x, X^*)^d,
\]

where \( x_0 \) is the initial iterate, \( \text{dist}(x, X^*) := \min\{\|x - x^*\| \mid x^* \in X^*\} \), and \( \| \cdot \| \) is the Euclidean norm.\(^1\) We refer to \( d \) as the “degree of growth.” (The parameters \( \mu \) and \( d \) are not assumed to be known, nor is any attempt made to learn approximations to their values. Indeed, our main theorems do not assume \( f \) possesses Hölderian growth.)

As a simple corollary to the theorem for the synchronous scheme, we show that if \( f \) is Lipschitz continuous on an open neighborhood of \( Q \), and a standard subgradient method is employed in the scheme, the time sufficient to compute an \( \epsilon \)-approximate optimal solution (feasible \( x \) satisfying \( f(x) - f^* \leq \epsilon \)) is no greater than a multiple of \( \log(1/\epsilon) \) if \( d = 1 \), and no greater than a multiple of \( 1/\epsilon^2(1-1/d) \) if \( d > 1 \). These time bounds apply when the scheme is run in parallel, relying on max\( \{1, 2 + [\log_2(1/\epsilon)]\} \) copies of the subgradient method.

As there are \( O(\log(1/\epsilon)) \) copies of the subgradient method being run simultaneously, the total amount of work (total number of subgradient evaluations) is of order \( \log(1/\epsilon)^2 \) when \( d = 1 \), and of order \( \log(1/\epsilon)/\epsilon^{2(1-1/d)} \) when \( d > 1 \). These values also bound the amount of time when the scheme is run sequentially rather than in parallel.

Another simple corollary of the theorem pertains to the setting where \( Q = \mathbb{R}^n \) and \( f \) is \( L \)-smooth, that is, \( \| \nabla f(x) - \nabla f(y) \| \leq L\|x - y\| \) for all \( x, y \in \mathbb{R}^n \). (For smooth functions, the growth degree necessarily satisfies \( d \geq 2 \).) Here we show that if Nesterov’s original accelerated method is employed in the synchronous scheme, the time required to compute an \( \epsilon \)-approximate optimal solution is no greater than a multiple of \( \log(1/\epsilon) \) if \( d = 2 \), and no greater than a multiple of \( 1/\epsilon^{4 - 1/d} \) if \( d > 2 \). For the total amount of work – or for a time bound when the scheme is run sequentially rather than in parallel – the bounds are multiplied by \( \log(1/\epsilon) \). These bounds qualify as being “nearly optimal,” off by a factor proportional to \( \log(1/\epsilon) \) from the known lower bound (\([13\text{, page } 6]\)).

Each iteration of the subgradient method requires, approximately, the same amount of time (proportional to the time needed for evaluating a subgradient for the current iterate). Likewise for Nesterov’s original accelerated method. By contrast, the work required by Nesterov’s universal fast gradient method (\([17]\)) can depend significantly on the particular iteration, making the method inappropriate for use in a synchronous parallel scheme. Nonetheless, the universal method fits well in our asynchronous scheme. As a straightforward corollary to our theorem on the asynchronous scheme, we deduce nearly-optimal complexity bounds for the general class of problems in which \( f \) both has Hölderian growth and has “Hölder continuous gradient with exponent \( 0 < \nu \leq 1 \).”

1.2. Positioning of Our Contributions. As noted above, an understanding that restarting can speed-up first-order methods goes back to work of Nemirovski and Nesterov \([13]\), although their focus was on the abstract setting where a scheme somehow knows nearly-ideal times at which to make restarts. In other words, the schemes were not adaptive – were not meta-algorithms capable of deciding when to restart the algorithms they employ.

\(^1\) Elsewhere the property is referred to as a Hölderian error bound \([25]\), as sharpness \([23]\), as the Łojasiewicz property \([10]\), and the (generalized) Łojasiewicz inequality \([1]\).
In the last ten years, adaptivity has been a primary focus in research on optimization algorithms, but far more in the context of adaptive accelerated methods than in the context of adaptive schemes (meta-algorithms).

Impetus for the development of adaptive accelerated methods was provided by Nesterov in [16], where he designed and analyzed an accelerated method in the context of $f$ being $L$-smooth, the new method possessing the same convergence rate as his original (optimal) accelerated method (specifically, $O(1/\sqrt{\epsilon})$), but without needing $L$ as input. (Instead, input meant to approximate $L$ is needed, and during iterations, the method modifies the input value until reaching a value which approximates $L$ to within appropriate accuracy.) Secondary consideration, in §5.3 of the paper, was given specifically to strongly convex functions, with a procedure for approximating to within appropriate accuracy the so-called strong convexity parameter, leading overall to an adaptive accelerated method possessing an $O(\log(1/\epsilon))$ time bound for the class of $L$-smooth and strongly-convex functions, demonstrating how designing an adaptive method aimed at a narrower class of functions can result in a dramatically improved convergence rate (i.e., $O(\log(1/\epsilon))$ vs. $O(1/\sqrt{\epsilon})$).

A range of significant papers on adaptive accelerated methods followed, in the setting of $f$ being smooth and either strongly convex or uniformly convex (c.f., [8], [11]), but also relaxing the uniform convexity requirement to assuming Hölderian growth with degree $d = 2$ (c.f., [12]).

Far fewer have been the number of papers regarding adaptive schemes, but still, important contributions have been made, in earlier papers which focused largely on heuristics for the setting of $f$ being $L$-smooth and strongly convex (c.f., [18, 6, 3]), but more recently, papers analyzing schemes with proven guarantees (c.f., [3]), including for the setting when the smooth function $f$ is not necessarily strongly convex but instead satisfies the weaker condition of having Hölderian growth with $d = 2$ (c.f., [4]). Each of these schemes, however, is designed to employ a narrow family of first-order methods under assumptions on $f$ which are not especially general.

Going beyond the setting of $f$ being smooth and $d = 2$, and going beyond being designed to apply to a narrow family of first-order methods, the foremost contributions on restart schemes are due to Roulet and d’Aspremont [23], who consider all $f$ possessing Hölderian growth (which they call “sharpness”), and having Hölder continuous gradient. Their work aims, in part, to make algorithmically-concrete the earlier abstract analysis of Nemirovski and Nesterov [13]. The restart schemes of Roulet and d’Aspremont result in optimal time bounds when particular algorithms are employed in the schemes, assuming scheme parameters are set to appropriate values – values that generally would be unknown in practice. For smooth $f$, however, they develop an adaptive procedure within the scheme, to accurately approximate the required values, leading to overall time bounds that differ from the optimal bounds only by a logarithmic factor. Moreover, for general $f$ for which the optimal value $f^*$ is known, they show that when an especially-simple restart scheme employs Nesterov’s universal fast gradient method [17], optimal time bounds result.

The restart schemes with established time bounds either rely on values that would not be known in practice (e.g., $f^*$), or assume the function $f$ is of a particular form (e.g., smooth and having Hölderian growth), and adaptively approximate values characterizing the function (e.g., $\mu$ and $d$) in order to set scheme parameters to appropriate values. Our simple scheme entirely avoids depending on values that would be unknown in practice (unless the first-order method that is employed in the scheme itself depends on such values). Likewise, our scheme avoids assuming $f$ has a particular form, and certainly avoids adaptively approximating values characterizing $f$ (unless the employed first-order method does so itself).

In the setting when $f$ is smooth and possesses Hölderian growth, none of the consequences of our scheme are new – indeed, essentially the same results were established earlier by Roulet and d’Aspremont for their adaptive scheme [23, §3.2]. However, when $f$ has Hölder continuous
gradient “with parameter 0 < ν < 1,” the consequences are new, in that our scheme results in nearly-optimal time bounds without requiring any special knowledge (if, say, the first-order method employed in the scheme is Nesterov’s universal fast gradient method, the situation considered in §6).

Employing the scheme even with a simple and standard subgradient method leads to new and consequential results when \( f \) is a nonsmooth convex function that is Lipschitz-continuous on an open neighborhood of \( Q \). In this regard, we note that for such functions which in addition have growth degree \( d = 1 \), there was considerable interest in obtaining linear convergence even in the early years of research on subgradient methods (see [19, 20, 7, 24] for discussion and references). Linear convergence was established, however, only under substantial assumptions.

Interest in the setting continues today. In recent years, various algorithms have been developed with complexity bounds depending only logarithmically on \( \epsilon \) (c.f., [2, 5, 9, 21, 26]). However, each of the algorithms for which a logarithmic bound has been established depends on exact knowledge of values characterizing an optimization problem (e.g., \( f^* \)), or on accurate estimates of values (e.g., a Lipschitz constant for \( f \), the distance of the initial iterate from optimality, or a “growth constant”), that generally would be unknown in practice. None of the algorithms entirely avoids the need for special information, although a few of the algorithms are capable of learning appropriately-accurate approximations to nearly all of the values they rely upon. (Most notable for us, in part due to its generality, is Johnstone and Moulin [9].)

By contrast, given that some basic subgradient methods do not rely on values characterizing problem structure, the same is true for our scheme when such a subgradient method is employed. It is thus consequential that for a particularly-simple subgradient method, the resulting time bound, presented in Corollary 5.1, is proportional to \( \log(1/\epsilon) \). (However, the total amount of work is proportional to \( \log(1/\epsilon)^2 \).)

2. First-Order Methods Appropriate for the Scheme

2.1. Requirements and Notation. Let \( \text{fom} \) denote a first-order method capable of solving some class of convex optimization problems of the form (1.1), and for which the number of iterations required to be successful can be bounded by a function \( K_{\text{fom}} \) of the distance of the initial point \( x_0 \) to \( X^* \), and the desired accuracy \( \epsilon \):

\[
\text{dist}(x_0, X^*) \leq \delta \quad \Rightarrow \quad \min\{f(x_k) \mid 0 \leq k \leq K_{\text{fom}}(\delta, \epsilon)\} \leq \epsilon .
\] (2.1)

For example, consider the class of problems for which all subgradients of \( f \) on \( Q \) satisfy \( \|g\| \leq M \) for fixed \( M \) (we slightly abuse terminology and say that \( f \) is “\( M \)-Lipschitz continuous on \( Q \)”). An appropriate algorithm is the projected subgradient method,

\[
x_{k+1} = P_Q\left(x_k - (\epsilon/\|g_k\|^2) g_k\right),
\] (2.2)

where \( g_k \) is any subgradient of \( f \) at \( x_k \), and where \( P_Q \) denotes orthogonal projection onto \( Q \). Then it is well known that \( K_{\text{fom}} = K_{\text{subgrad}} \) given by

\[
K_{\text{subgrad}}(\delta, \epsilon) := \lceil(M\delta/\epsilon)^2 \rceil
\] (2.3)
satisfies (2.1). While the iteration bound depends on $M$, the algorithm itself does not require $M$ as input, although it does require $\epsilon$.

As another example, for problems in which $f$ is $L$-smooth and $Q = \mathbb{R}^n$, the original accelerated method can be applied, for which Nesterov ([14], [15, Thm 2.2.2]) proved (2.1) holds with $K_{\text{fom}} = K_{\text{accel}}$ given by

$$K_{\text{accel}}(\delta, \epsilon) := [2\delta \sqrt{L/\epsilon}] .$$

Here, the algorithm is independent of $\epsilon$, but dependent on $L$.

For both the subgradient method and the original accelerated method, as well as for numerous other first-order methods, each iteration requires (approximately) the same number of function/(sub)gradient evaluations – that is, the same number of “oracle calls” – making the value $K(\delta, \epsilon)$ be a reasonable proxy for the time sufficient to compute an $\epsilon$-approximate optimal solution.

For some first-order methods, however, the number of oracle calls can vary significantly among iterations, as is the case for Nesterov’s universal fast gradient method ([17, §4]), which applies to the general class of problems of the form (1.1) for which there exists a scalar $0 \leq \nu \leq 1$ satisfying

$$M_{\nu} := \sup \left\{ \frac{\|\nabla f(x) - \nabla f(y)\|}{\|x - y\|^{\nu}} : x, y \in Q, x \neq y \right\} < \infty$$

(i.e., is finite)

($f$ has “Hölder continuous gradient with exponent $\nu$”).

It is for methods involving significant variability in work per iteration that instead of giving focus to the value $K(\delta, \epsilon)$ bounding the number of iterations, we rely on a function $T(\delta, \epsilon)$ assumed to provide an upper bound on the amount of time (e.g., number of oracle calls) sufficient for $\text{fom}$ to to compute an $\epsilon$-approximate optimal solution. However, so as not to detract from the simplicity of the ideas underlying our restart scheme, we defer presenting these details to §6.

2.2. Motivation for the Scheme. Our approach relies on multiple copies of $\text{fom}$ (whichever first-order method $\text{fom}$ is chosen, perhaps one of the methods mentioned above, or perhaps a different method). We denote the copies $\text{fom}_n$ for $n = -1, 0, \ldots, N$, where $N$ is user-specified. Think of $\text{fom}_n$ as being a version of the first-order method aimed at achieving optimality within precision $2^n \epsilon$. Specifically for the examples above:

- If $\text{fom} = \text{subgrad}$, the subgradient method, then $\text{fom}_n (= \text{subgrad}_n)$ has iterates defined by (2.2) but with $\epsilon$ replaced by $2^n \epsilon$.
- If $\text{fom} = \text{accel}$, Nesterov’s original accelerated method, then $\text{fom}_n (= \text{accel}_n)$ is simply the accelerated method with no changes. (Here, $\text{fom}_n$ does not depend explicitly on

Indeed, letting $x^*$ be the closest point in $X^*$ to $x_k$, and defining $\alpha_k = \epsilon/\|g_k\|^2$, we have

$$\text{dist}(x_{k+1}, X^*)^2 \leq ||x_{k+1} - x^*||^2 = ||P(x_k - \alpha_k g_k) - x^*||^2 \leq ||(x_k - \alpha_k g_k) - x^*||^2$$

$$= ||x_k - x^*||^2 - 2\alpha_k \langle g_k, x_k - x^* \rangle + \alpha_k^2 \|g_k\|^2$$

$$\leq ||x_k - x^*||^2 - 2\alpha_k (f(x_k) - f^*) + \alpha_k^2 \|g_k\|^2 \quad \text{(by convexity)}$$

$$\leq \text{dist}(x_k, X^*)^2 - (2(f(x_k) - f^*) - \epsilon) \epsilon/M^2 ,$$

the final inequality due to $\alpha_k = \epsilon/\|g_k\|^2$, $\|g_k\| \leq M$ and the definition of $x^*$. Thus,

$$f(x_k) - f^* > \epsilon \Rightarrow \text{dist}(x_{k+1}, X^*)^2 < \text{dist}(x_k, X^*)^2 - (\epsilon/M)^2 .$$

Consequently, by induction, if $x_{k+1}$ is the first iterate satisfying $f(x_{k+1}) - f^* \leq \epsilon$, then

$$0 \leq \text{dist}(x_{k+1}, X^*)^2 < \text{dist}(x_0, X^*)^2 - (k+1)(\epsilon/M)^2 ,$$

implying the function $K_{\text{fom}} = K_{\text{subgrad}}$ defined by (2.3) satisfies (2.1).
n. Regardless of the choice of \( \text{fom} \), however, the algorithm \( \text{fom}_n \) is tied to the value \( n \) through specific tasks that \( \text{fom}_n \) is assigned to achieve (discussed below.)

- If \( \text{fom} = \text{univ} \), Nesterov’s universal fast gradient method, then \( \text{fom}_n = \text{univ}_n \) is the universal method with desired accuracy set equal to \( 2^n \epsilon \). (Among the inputs to the method, the desired accuracy is required, as is clarified in §6.)

Ideally, as will become apparent, \( N \) is chosen to be slightly larger than \( \lceil \log_2 (1/\epsilon) \rceil \), where \( x_0 \in Q \) is the initial point for the scheme. However, as we want not to assume \( f^* \) is known (not even approximately), our general results do not depend on a specific choice for \( N \). Nonetheless, the significance of the results will be evident for even the easily-computed choice \( N = \max\{\lceil \log_2 (1/\epsilon) \rceil, -1\} \).

To motivate the scheme, it is instructive to consider the subgradient method. Among the algorithms \( \text{subgrad}_n \) \( (n = -1, 0, \ldots, N) \), larger values of \( n \) correspond to longer steps, which typically are beneficial when far from optimality, in that great strides are made towards optimality. When a neighborhood of optimality is reached, however, it becomes a fluke for a long step to result in a point with improved objective value. Instead, shorter steps are warranted, that is, \( \text{subgrad}_n \) for a smaller value of \( n \), should take the lead role.

At issue, of course, is that we generally have no idea of the proximity to optimality, so we are not in position to know when to rely on \( \text{subgrad}_n \) for a larger value of \( n \), and when to rely on a smaller value of \( n \). To get around this conundrum, we simultaneously rely on \( \text{subgrad}_n \) for all \( n = -1, 0, \ldots, N \), running these algorithms in parallel.

More generally, we simultaneously rely on \( \text{fom}_n \) for all \( n \), running these algorithms in parallel.

We first explain a version of the scheme in which the various copies \( \text{fom}_n \) each performs one iteration in one time period. This synchronous scheme is appropriate if all iterations of \( \text{fom} \) require approximately the same amount of time, as certainly is the case for \( \text{subgrad} \) and \( \text{accel} \), as well as many other first-order methods.

The universal fast gradient method is far different. The proofs in [17] leave open the possibility for \( \text{univ} \) that the number of oracle calls made only in the \( k \)th iteration might exceed \( k \). Likewise, Bregman methods are inappropriate for the synchronous scheme, because of having to solve a subproblem at every iteration, some subproblems possibly being harder than others. While we perhaps could shoehorn these algorithms into the synchronous setting by redefining what constitutes an iteration, doing so would be unnatural, motivating us to develop an asynchronous scheme, in which an iteration by \( \text{fom}_n \) can be aborted early, at a cost. Additional motivation for developing the asynchronous scheme is the situation in which communication transmission can become congested, an extreme case being when all \( \text{fom}_n \) \( (n = -1, 0, \ldots, N) \) are required to send communication through a single server.

The asynchronous scheme requires an additional layer of detail, but the key ideas are the same as for the synchronous scheme.

3. The Parallel First-Order Method (\( \|\text{FOM} \) )

Now we present the general scheme, dubbed “the parallel first-order method,” and denoted \( \|\text{FOM} \). We begin with a synchronous version of the scheme, appropriate for methods which make (approximately) the same number of oracle calls at each iteration, methods such as \( \text{subgrad} \) and \( \text{accel} \). The synchronous scheme can easily be made sequential at the cost of requiring greater time.

For both the synchronous scheme and the asynchronous scheme, we assume each \( \text{fom}_n \) has its own oracle, in the sense that given \( x \), \( \text{fom}_n \) is capable of computing – or can readily access – both the value \( f(x) \) and a (sub)gradient at \( x \). (In other words, we assume \( \text{fom}_n \) can obtain the
function value and a (sub)gradient without having to wait in a queue with other copies $fom_m$ ($m \neq n$) in need of a function value and (sub)gradient.)

3.1. Synchronous Scheme ($\text{Sync}||\text{FOM}$). Here we speak of “time periods,” each having the same length, one unit of time.

Each of the algorithms $fom_n$ ($n = -1, 0, \ldots, N$) is initiated at the same feasible point, $x_0 \in Q$. During a time period, $fom_n$ performs one iteration, resulting in an iterate which at the beginning of the following time period will be its “current iterate.”

The algorithm $fom_N$ differs from the others in that it is never restarted. It also differs in that it has no “inbox” for receiving messages. For $n < N$, there is an inbox in which $fom_n$ can receive messages (only) from $fom_{n+1}$.

At any time, each algorithm has a “task.” Restarts occur only when tasks are accomplished. When a task is accomplished, the task is updated.

For all $n$, the initial task of $fom_n$ is to obtain a point $y_n$ satisfying $f(y_n) \leq f(x_0) - 2^n \epsilon$. Generally for $n < N$, at any time, $fom_n$ is pursuing the task of obtaining $y_n$ satisfying $f(y_n) \leq f(x_n, 0) - 2^n \epsilon$, where $x_n, 0$ is the most recent (re)start point for $fom_n$.

In a time period, the following steps are made by $fom_N$, the algorithm that never restarts: If the current iterate fails to accomplish the current task for $fom_N$, then $fom_N$ makes one iteration, and does nothing else in the time period. On the other hand, if the current iterate – call it $y_{N,i}$ – does fulfill the task, then (1) $y_{N,i}$ is sent to the inbox of $fom_{N-1}$ to be available for $fom_{N-1}$ at the beginning of the next time period, (2) the current task for $fom_N$ is replaced by the task of finding $y_{N,i+1}$ satisfying $f(y_{N,i+1}) \leq f(y_{N,i}) - 2^N \epsilon$, and (3) $fom_N$ finishes its work in the time period by making one iteration (without having restarted).

The steps made by $fom_n$ ($n < N$) are as follows: First the objective value for the current iterate of $fom_n$ is examined, and so is the objective value of the point in the inbox (if the inbox is nonempty). If the smallest of these function values – either one or two values, depending on whether the inbox is empty – does fulfill the current task for $fom_n$, then $fom_n$ completes its work in the time period by simply clearing its inbox (if it is not already empty), and making one iteration, without restarting. On the other hand, if the smallest of the function values fulfills the current task for $fom_n$, then: (1) $fom_n$ is restarted at the point with the smallest function value (this point becomes the new $x_n, 0$ in the updated task for $fom_n$ of obtaining $y_n$ satisfying $f(y_n) \leq f(x_n, 0) - 2^n \epsilon$), (2) the point is sent to the inbox of $fom_{n-1}$ (assuming $n > -1$) to be available for $fom_{n-1}$ at the beginning of the next time period, (3) the inbox of $fom_n$ is cleared, and (4) $fom_n$ finishes its work in the time period by completing the first iteration after restarting.

This concludes the description of $\text{Sync}||\text{FOM}$.

Remarks:

- Ideally for each $n > -1$, there is apparatus dedicated solely to the sending of messages from $fom_n$ to $fom_{n-1}$. If messages from all $fom_n$ are handled by a single server, then the length of periods might be increased to more than one unit of time, to reflect possible congestion.
- In light of $fom_n$ sending messages only to $fom_{n-1}$, the scheme can naturally be made sequential, with $fom_n$ performing an iteration, followed by $fom_{n-1}$ (and with $fom_N$ following $fom_{-1}$).

Analysis of $\text{Sync}||\text{FOM}$ is presented in §4, and corollaries for $\text{subgrad}$ and $\text{accel}$ are given in §5.

Regarding communication, the analysis reveals there potentially is much to be gained from an asynchronous scheme, even for methods which, like $\text{subgrad}$ and $\text{accel}$, require the same
number of oracle calls at every iteration. Indeed, it is shown that \( \text{fom}_n \) plays a critical role only in the last few messages (at most three, possibly none) that it sends to \( \text{fom}_{n-1} \). During the time that \( \text{fom}_n \) is performing iterations to arrive at the content of a critical message, \( \text{fom}_m \), for \( m \ll n \), can be sending scads of messages to \( \text{fom}_{m-1} \). It is thus advantageous if \( \text{fom}_n \) is disengaged, to the extent possible, from the timeline being followed by \( \text{fom}_m \), especially if all messages go through a single server.

3.2. Asynchronous Scheme (Async/FOM). Here we refer to “epochs” rather than time periods. Each \( \text{fom}_n \) has its own epochs.

The first epoch for every \( \text{fom}_n \) begins at time zero, when \( \text{fom}_n \) is started at \( x_{n,0} = x_0 \in Q \), the same point for every \( \text{fom}_n \).

As for the synchronous scheme, \( \text{fom}_N \) never restarts, nor receives messages. Its only epoch is of infinite length. During the epoch, \( \text{fom}_N \) proceeds as before, focused on accomplishing a sequence of tasks, each of the form of finding \( y_{N,i+1} \) satisfying \( f(y_{N,i+1}) \leq f(y_{N,i}) - 2^N \epsilon \). The only difference in the description of \( \text{fom}_N \) is that when \( \text{fom}_N \) sends the point \( y_{N,i+1} \) to the inbox of \( \text{fom}_{N-1} \), it arrives in the inbox no more than \( t_{\text{transit}} \) units of time after being sent, where \( t_{\text{transit}} > 0 \). The same transit time holds for messages sent by any \( \text{fom}_m \) to \( \text{fom}_{m-1} \). (After fully specifying the scheme, we explain that the assumption of a uniform time bound on message delivery is not overly restrictive.)

For \( n < N \), the beginning of a new epoch coincides with \( \text{fom}_n \) obtaining a point \( y_n \) satisfying its current task, that is, \( f(y_n) \leq f(x_{n,0}) - 2^n \epsilon \).

As for the synchronous scheme, for \( n < N \), \( \text{fom}_n \) can receive messages (only) from \( \text{fom}_{n+1} \). Now, however, it is assumed that when a message arrives in the inbox, a “pause” for \( \text{fom}_n \) begins, lasting a positive amount of time, but no more than \( t_{\text{pause}} \) time units. The pause is meant to capture various delays that can happen upon receipt of a new message, including time for \( \text{fom}_n \) to come to a good stopping place in its calculations before actually going to the inbox.

If a message from \( \text{fom}_{n+1} \) arrives in the inbox during a pause, then immediately the old message is overwritten by the new one, the old pause is cancelled, and a new pause begins.

If in an epoch for \( \text{fom}_n \), a message arrives after \( \text{fom}_n \) has restarted (i.e., after \( \text{fom}_n \) has begun making iterations), then during the pause, \( \text{fom}_n \) determines whether \( y_{n+1} \) (the point sent by \( \text{fom}_{n+1} \)) accomplishes its current task, that is, determines whether \( y_{n+1} \) satisfies \( f(y_{n+1}) \leq f(x_{n,0}) - 2^n \epsilon \), where \( x_{n,0} \) is the point at which \( \text{fom}_n \) (re)started at the beginning of the epoch. If during the pause, another message arrives, overwriting \( y_{n+1} \) with \( \tilde{y}_{n+1} \), then in the new pause, \( \text{fom}_n \) turns attention to determining if \( \tilde{y}_{n+1} \) satisfies the task (indeed, it is a better candidate, because \( f(\tilde{y}_{n+1}) \leq f(y_{n+1}) - 2^{n+1} \epsilon \). And so on, until the sequence of contiguous pauses ends.\(^3\) Let \( \tilde{y}_{n+1} \) be the final point.

If \( \tilde{y}_{n+1} \) satisfies the current task for \( \text{fom}_n \), then immediately at the end of the (contiguous) pause, \( \tilde{y}_{n+1} \) is relabeled as \( y_n \) and a new epoch begins. On the other hand, if \( \tilde{y}_{n+1} \) fails to satisfy the current task, then the current epoch continues, with \( \text{fom}_n \) returning to computing iterations.

The other manner an epoch for \( \text{fom}_n \) can end is by \( \text{fom}_n \) computing an iterate \( y_n \) satisfying \( f(y_n) \leq f(x_{n,0}) - 2^n \epsilon \). At the instant that \( \text{fom}_n \) computes such an iterate \( y_n \), a new epoch begins.

It remains to describe the beginning of an epoch for \( \text{fom}_n \).

As already mentioned, at time zero – the beginning of the first epoch for all \( \text{fom}_n \) – every \( \text{fom}_n \) is started at \( x_{n,0} = x_0 \). No messages are sent at time zero.

\(^3\)The sequence of contiguous pauses is finite, because (1) \( \text{fom}_{n+1} \) sends a message only when it has obtained \( y_{n+1} \) satisfying \( f(y_{n+1}) \leq f(x_{n+1,0}) - 2^{n+1} \epsilon \), and (2) we assume \( f^* \) is finite.
For \( n < N \), as explained above, the beginning of a new epoch occurs precisely when \( \text{fom}_n \) has a point \( y_n \) satisfying the task for \( \text{fom}_n \).

If no message from \( \text{fom}_{n+1} \) arrives simultaneously with the beginning of a new epoch for \( \text{fom}_n \), then \( \text{fom}_n \) instantly\(^4\) (1) relabels \( y_n \) as \( x_{n,0} \), (2) updates its task by substituting this point for the previous (re)start point, (3) restarts at \( x_{n,0} \), and (4) sends a copy of \( x_{n,0} \) to the inbox of \( \text{fom}_{n-1} \) (assuming \( n > -1 \)), where it will arrive no later than time \( t + t_{\text{transit}} \), with \( t \) being the time the new epoch for \( \text{fom}_n \) has begun.

On the other hand, if a message from \( \text{fom}_{n+1} \) arrives in the inbox exactly at the beginning of the new epoch, then \( \text{fom}_n \) immediately pauses. Here, \( \text{fom}_n \) makes different use of the pause than what occurs for pauses happening after \( \text{fom}_n \) has restarted. Specifically, letting \( y_{n+1} \) be the point in the message, then during the pause, \( \text{fom}_n \) determines whether \( f(y_{n+1}) < f(y_n) \) — if so, \( y_{n+1} \) is preferred to \( y_n \) as the restart point. If during the pause, another message arrives, overwriting \( y_{n+1} \) with \( \bar{y}_{n+1} \), then during the new pause, \( \text{fom}_n \) determines whether \( \bar{y}_{n+1} \) is preferred to \( y_n \) (\( \bar{y}_{n+1} \) is definitely preferred to \( y_{n+1} \) — indeed, \( f(\bar{y}_{n+1}) \leq f(y_{n+1}) - 2^{n+1} \)). And so on, until the sequence of contiguous pauses ends, at which time \( \text{fom}_n \) instantly (1) labels the most preferred point as \( x_{n,0} \), (2) updates its task by substituting this point for the previous (re)start point, (3) restarts at \( x_{n,0} \), and (4) sends a copy of \( x_{n,0} \) to the inbox of \( \text{fom}_{n-1} \) (assuming \( n > -1 \)).

The description of \( \text{Async} | \text{FOM} \) is now complete.

**Remarks:**

- Regarding the transit time, a larger value for \( t_{\text{transit}} \) might be chosen to reflect the possibility of greater congestion in message passing, as could occur if all messages go through a single server. We emphasize, however, that a long queue of unsent messages can be avoided, because (1) if two messages from \( \text{fom}_n \) are queued, the later message contains a point that has better objective value than the earlier message, and (2) as the proofs show, only the last few messages (at most three messages, perhaps none) sent by \( \text{fom}_n \) play a significant role. Thus, to avoid congestion, when a second message from \( \text{fom}_n \) arrives in the queue, delete the earlier message. So long as this policy of deletion is also applied to all of the other copies \( \text{fom}_m \), the fact that only the last few messages from \( \text{fom}_n \) are significant then implies there is not even a need to move the second message from \( \text{fom}_n \) forward into the place that had been occupied by the deleted message — any truly critical message from \( \text{fom}_n \) will naturally move from the back to the front of the queue within time proportional to \( N + 1 \) after its arrival. Managing the queue in this manner — deleting a message from \( \text{fom}_n \) as soon as another message from \( \text{fom}_n \) arrives in the queue — ensures that \( t_{\text{transit}} \) can be chosen proportional to \( N + 1 \) in the worst case of there being only a single server devoted to message passing.

- As for the synchronous scheme, the ideal arrangement is, of course, for each \( n > -1 \), to have apparatus dedicated solely to the sending of messages from \( \text{fom}_n \) to \( \text{fom}_{n-1} \). Then \( t_{\text{transit}} \) can be chosen as a constant independent of the size of \( N \).

- For first-order methods which require approximately the same number of oracle calls per iteration, it is natural to choose \( t_{\text{pause}} \) to be the maximum possible number of oracle calls in an iteration, in which case a pause for \( \text{fom}_n \) can be interpreted as the time needed for \( \text{fom}_n \) to complete its current iteration before checking whether any messages are in its inbox.

The synchronous scheme can be viewed as a special case of the asynchronous scheme by choosing \( t_{\text{transit}} = 1 \) and \( t_{\text{pause}} = 0 \) (rather, can be viewed as a limiting case, because for

\(^4\)We assume the listed chores are accomplished instantly by \( \text{fom}_n \). Assuming positive time is required would have negligible effect on the complexity results, but would make notation more cumbersome.
Async∥FOM, the length of a pause is assumed to be positive). Indeed, choices in designing the asynchronous scheme were made with this in mind, the intent being that the main result for the synchronous scheme would be a corollary of a theorem for the asynchronous scheme. However, as the key ideas for both proofs are the same, and since a proof for the synchronous scheme happens to have much greater transparency, in the end we chose to first present the theorem (and its proof) for the synchronous scheme.

The theorem for Async∥FOM is presented in §6, along with a corollary for Nesterov’s universal fast gradient method. The proof of the theorem is deferred to Appendix A, due to its similarities with the analysis of Sync∥FOM.

4. Analysis of Sync∥FOM

Here we present the main theorem regarding Sync∥FOM, and provide the elementary proof.

For scalars \( f \geq f^* \), define

\[
D(\hat{f}) = \sup \{ \text{dist}(x, X^*) \mid x \in Q \text{ and } f(x) \leq \hat{f} \}.
\]

For the following theorem to be meaningful, the values \( D(\hat{f}) \) must be finite, but \( f \) is not required to have Hölderian growth.

**Theorem 4.1.** Assume time periods are of unit length, with each \( \text{fom}_n \) making one iteration in a time period. Assume \( f(x_0) - f^* > \epsilon \) (else an \( \epsilon \)-approximate optimal solution already is in hand).

If \( f(x_0) - f^* < 5 \cdot 2^N \epsilon \), then Sync∥FOM computes an \( \epsilon \)-approximate optimal solution within time

\[
\bar{N} + 1 + 3 \sum_{n=1}^{\bar{N}} K_{\text{fom}}(D_n, 2^n \epsilon)
\]

with \( D_n := \min \{ D(f^* + 5 \cdot 2^n \epsilon), D(f(x_0)) \} \), \( \bar{N} \) is the smallest integer satisfying both \( f(x_0) - f^* < 5 \cdot 2^\bar{N} \epsilon \) and \( \bar{N} \geq -1 \).

In any case, Sync∥FOM computes an \( \epsilon \)-approximate optimal solution within time

\[
T_N + K_{\text{fom}}(\text{dist}(x_0, X^*), 2^N \epsilon),
\]

where \( T_N \) is the quantity obtained by substituting \( N \) for \( \bar{N} \) in (4.1).

As there are \( N+2 \) copies of \( \text{fom} \) being run in parallel, the total amount of work is proportional to the time bound multiplied by \( N+2 \). Of course the same bound on the total amount of work applies if the scheme is performed sequentially rather than in parallel (i.e., \( \text{fom}_n \) performs an iteration, followed by \( \text{fom}_{n-1} \), with \( \text{fom}_N \) following \( \text{fom}_{-1} \)). In the sequential case, trivially, the time is proportional to the total amount of work.

The theorem is obtained through a sequence of results in which we speak of \( \text{fom}_n \) “updating” at a point \( x \). The starting point \( x_0 \) is considered to be the first update point for every \( \text{fom}_n \). After Sync∥FOM has started, then for \( n < N \), updating at \( x \) means the same as restarting at \( x \). For \( \text{fom}_N \), updating at \( x \) is the same as having computed \( x \) satisfying the current task of \( \text{fom}_N \) (in which case the point is sent to \( \text{fom}_{N-1} \), even though \( \text{fom}_N \) never restarts).

**Lemma 4.2.** Assume that at time \( t \), \( \text{fom}_n \) updates at \( x \) satisfying \( f(x) - f^* \geq 2 \cdot 2^n \epsilon \). Then no later than time \( t + K_{\text{fom}}(D(f(x)), 2^n \epsilon) \), \( \text{fom}_n \) updates again.

**Proof:** Indeed, if \( \text{fom}_n \) has not updated by the specified time, then at that time, it has computed a point \( y \) satisfying \( f(y) - f^* \leq 2^n \epsilon \), simply by definition of the function \( K_{\text{fom}} \), the assumption
that $f_{om_{\alpha}}$ performs one iteration in each time period, and the assumption that time periods are of unit length. Since
\[ f(y) = f(x) + (f(y) - f^*) + (f^* - f(x)) \leq f(x) + 2^n \epsilon - 2 \cdot 2^n \epsilon = f(x) - 2^n \epsilon, \]
in that case an update happens precisely at the specified time.

**Proposition 4.3.** Assume that at time $t$, $f_{om_{\alpha}}$ updates at $x$ satisfying $f(x) - f^* \geq 2 \cdot 2^n \epsilon$. Let $j := \lceil \frac{f(x) - f^*}{2^n \epsilon} \rceil - 2$. Then $f_{om_{\alpha}}$ updates at a point $\tilde{x}$ satisfying $f(\tilde{x}) - f^* < 2 \cdot 2^n \epsilon$ no later than time
\[ t + \sum_{j=0}^{\tilde{J}} K_{om}(D(f(x) - j \cdot 2^n \epsilon), 2^n \epsilon). \tag{4.3} \]

**Proof:** Note that $j$ is the integer satisfying
\[ f(x) - (j + 1) \cdot 2^n \epsilon < f^* + 2 \cdot 2^n \epsilon \leq f(x) - j \cdot 2^n \epsilon. \tag{4.4} \]
Lemma 4.2 implies $f_{om_{\alpha}}$ has a sequence of update points $x_0, x_1, \ldots, x_J, x_{J+1}$, where $x_0 = x$,
\[ f(x_{j+1}) \leq f(x_j) - 2^n \epsilon \quad \text{for all } j = 0, \ldots, J, \tag{4.5} \]
and where $x_{J+1}$ is obtained no later than time
\[ t + \sum_{j=0}^{J} K_{om}(D(f(x_j)), 2^n \epsilon). \tag{4.7} \]

Observe that (4.4), (4.5) and (4.6) imply $J \leq j$, and $f(x_j) \leq f(x) - j \cdot 2^n \epsilon$. Consequently, since $\hat{f} \mapsto K_{om}(D(\hat{f}), 2^n \epsilon)$ is an increasing function, the quantity (4.7) is bounded above by (4.3), completing the proof.

**Corollary 4.4.** Assume that at time $t$, $f_{om_{\alpha}}$ updates at $x$ satisfying $f(x) - f^* < i \cdot 2^n \epsilon$, where $i$ is an integer and $i \geq 3$. Then $f_{om_{\alpha}}$ updates at a point $\tilde{x}$ satisfying $f(\tilde{x}) - f^* < 2 \cdot 2^n \epsilon$ no later than time
\[ t + \sum_{i=3}^{i} K_{om}(D_{n,i}, 2^n \epsilon) \tag{4.8} \]
where $D_{n,i} = \min\{D(f^* + i \cdot 2^n \epsilon), D(f(x_0))\}. \tag{4.9}$

**Proof:** Clearly, we may assume $f(x) \geq f^* + 2 \cdot 2^n \epsilon$. Let $j$ be as in Proposition 4.3, and hence the time bound (4.3) applies for updating at some $\tilde{x}$ satisfying $f(\tilde{x}) - f^* < 2 \cdot 2^n \epsilon$.

Note that for all non-negative integers $j$,
\[ f(x) - j \cdot 2^n \epsilon < f^* + (i - j) \cdot 2^n \epsilon. \tag{4.10} \]
Consequently, since $\hat{f} \mapsto K_{om}(D(\hat{f}), 2^n \epsilon)$ is an increasing function, and since $f(x) \leq f(x_0)$ (due to $x$ being an update point), the quantity (4.3) is bounded above by
\[ t + \sum_{i=3}^{i} K_{om}(D_{n,i}, 2^n \epsilon). \tag{4.11} \]
Finally, as $j$ is the integer satisfying the inequalities (4.4), the rightmost of those inequalities, and (4.10) for $j = j$, imply $i - j \geq 3$, and thus (4.11) is bounded from above by (4.8).
Corollary 4.5. Let \( n > -1 \). Assume that at time \( t \), \( \text{fom}_n \) updates at \( x \) satisfying \( f(x) - f^* < 5 \cdot 2^n \epsilon \). Then no later than time

\[
t + 1 + \sum_{i=3}^{5} K_{\text{fom}}(D_{n,i}, 2^n \epsilon)
\]

(where \( D_{n,i} \) is given by (4.9)), \( \text{fom}_{n-1} \) updates at \( x' \) satisfying \( f(x') - f^* < 5 \cdot 2^{n-1} \epsilon \).

**Proof:** By Corollary (4.4), no later than time

\[
t + \sum_{i=3}^{5} K_{\text{fom}}(D_{n,i}, 2^n \epsilon)
\]

\( \text{fom}_n \) updates at \( \bar{x} \) satisfying \( f(\bar{x}) < f^* + 2 \cdot 2^n \epsilon \). When \( \text{fom}_n \) updates at \( \bar{x} \), the point is sent to the inbox of \( \text{fom}_{n-1} \), where it is available to \( \text{fom}_{n-1} \) at the beginning of the next period.

Either \( \text{fom}_{n-1} \) restarts at \( \bar{x} \), or its most recent restart point \( \hat{x} \) satisfies \( f(\hat{x}) > f(\bar{x}) - 2^{n-1} \epsilon \). In the former case, \( \text{fom}_{n-1} \) restarts at a point \( x' = \hat{x} \) satisfying \( f(x') < f^* + 2 \cdot 2^n \epsilon = f^* + 4 \cdot 2^{n-1} \epsilon \), whereas in the latter case it has already restarted at a point \( x' = \hat{x} \) satisfying \( f(x') < f(\hat{x}) + 2^{n-1} \epsilon < f^* + 5 \cdot 2^{n-1} \epsilon \).

\[
\text{Corollary 4.6. For any } \bar{N} \in \{-1, \ldots, N\}, \text{ assume that at time } t, \text{fom}_N \text{ updates at } x \text{ satisfying } f(x) - f^* < 5 \cdot 2^N \epsilon. \text{ Then no later than time }
\]

\[
t + \bar{N} + 1 + \sum_{n=\bar{N}+1}^{N} \sum_{i=3}^{5} K_{\text{fom}}(D_{n,i}, 2^n \epsilon)
\]

\( \text{Sync} \parallel \text{FOM} \) has computed an \( \epsilon \)-approximate optimal solution.

**Proof:** If \( \bar{N} = -1 \), Corollary 4.4 implies the additional time required by \( \text{fom}_{-1} \) to compute a \( 2 \cdot 2^{-1} \epsilon \)-approximate optimal solution \( \bar{x} \) is bounded from above by

\[
\sum_{i=3}^{5} K_{\text{fom}}(D_{-1,i}, 2^{-1} \epsilon).
\]

(4.12)

The present corollary thus is established for the case \( \bar{N} = -1 \).

On the other hand, if \( \bar{N} > -1 \), induction using Corollary 4.5 shows that no later than time

\[
t + \bar{N} + 1 + \sum_{n=0}^{\bar{N}} \sum_{i=3}^{5} K_{\text{fom}}(D_{n,i}, 2^n \epsilon)
\]

\( \text{fom}_{-1} \) has restarted at \( x \) satisfying \( f(x) - f^* < 5 \cdot 2^{-1} \epsilon \). Then, as above, the additional time required by \( \text{fom}_{-1} \) to compute an \( \epsilon \)-approximate optimal solution does not exceed (4.12).

**Proof of Theorem 4.1:** For the case that \( f(x_0) - f^* < 2^N \epsilon \), the time bound (4.1) is immediate from Corollary 4.6 and the fact that \( K_{\text{fom}}(D_{n,i}, 2^n \epsilon) \leq K_{\text{fom}}(D_{n,i}, 2^n \epsilon) \) for \( i = 3, 4, 5 \).

In any case, because \( \text{fom}_N \) never restarts, within time

\[
K_{\text{fom}}(\text{dist}(x_0, X^*), 2^N \epsilon),
\]

(4.13)

\( \text{fom}_N \) computes a point \( x \) satisfying \( f(x) - f^* \leq 2^N \epsilon \). If \( x \) is not an update point for \( \text{fom}_N \), then the most recent update point \( \hat{x} \) satisfies \( f(\hat{x}) < f(x) + 2^N \epsilon \). Hence, irrespective of whether \( x \) is an update point, by the time \( \text{fom}_N \) has computed \( x \), it has obtained an update point \( x' \) satisfying \( f(x') - f^* < 2 \cdot 2^N \epsilon < 5 \cdot 2^N \epsilon \). Consequently, the time required for \( \text{Sync} \parallel \text{FOM} \) to compute an \( \epsilon \)-optimal approximate optimal solution does not exceed the value (4.13) plus the value (4.1), where in (4.1), \( N \) is substituted for \( \bar{N} \).
5. Corollaries for Sync∥FOM

Throughout the section, we assume $f$ has Hölderian growth, in that there exist constants $\mu > 0$ and $d \geq 1$ for which
\[ x \in Q \text{ and } f(x) \leq f(x_0) \implies f(x) - f^* \geq \mu \text{dist}(x, X^*)^d. \]
Consequently, the upper bound (4.1) in Theorem 4.1 is no greater than
\[ \bar{N} + 1 + 3 \sum_{n=-1}^{\bar{N}} K_{\text{fom}} \left( \left(5 \cdot 2^n \epsilon/\mu \right)^{1/d}, 2^n \epsilon \right). \tag{5.1} \]

(Deserving of mention is that for all corollaries in the paper, analogous results can be obtained under the weaker assumption that Hölderian growth holds on a smaller set $\{x \in Q \mid \text{dist}(x, X^*) \leq r\}$ for some $r > 0$. The analogous results are heavier in notation, but in some circumstances are preferable (specifically, when otherwise would necessarily cause $\mu$ to be a tiny value). However, as our goal is to present key ideas most clearly, we avoid the messier proofs that would result.)

The first corollary regards the projected subgradient method, when $f$ is $M$-Lipschitz continuous on $Q$. Recall
\[ K_{\text{subgrad}}(\delta, \epsilon) = \lceil (M\delta/\epsilon)^2 \rceil \]
(an upper bound on the number of iterations required to compute a $\epsilon$-approximate optimal solution when the subgradient method is initiated at an arbitrary point $x_0 \in Q$ satisfying $\text{dist}(x_0, X^*) \leq \delta$).

**Corollary 5.1.** Assume $f(x_0) - f^* > \epsilon$, and assume $f$ is $M$-Lipschitz continuous on $Q$. Consider Sync∥FOM with fom = subgrad.

If $f(x_0) - f^* < 5 \cdot 2^\bar{N} \epsilon$, then Sync∥FOM computes an $\epsilon$-approximate optimal solution in time $T$ for which
\[ d = 1 \implies T \leq \bar{N} + 1 + 3(\bar{N} + 2)(5M/\mu)^2, \tag{5.2} \]
\[ d > 1 \implies T \leq \bar{N} + 1 + 3 \left( \frac{5^{1/d} M}{\mu^{1/d} \epsilon^{1-1/d}} \right)^2 \min \left\{ \frac{16^{1-1/d}}{4^{1-1/d} - 1}, \bar{N} + 5 \right\}, \tag{5.3} \]
where $\bar{N}$ is the smallest integer satisfying both $f(x_0) - f^* < 5 \cdot 2^\bar{N} \epsilon$ and $\bar{N} \geq -1$.

On the other hand, if $f(x_0) - f^* \geq 5 \cdot 2^\bar{N} \epsilon$, then time bounds are obtained by substituting $N$ for $\bar{N}$ above, and adding
\[ \left( \frac{M \text{dist}(x_0, X^*)}{2^\bar{N} \epsilon} \right)^2. \tag{5.4} \]

**Remarks:** For $d = 1$, the easily computable choice $N = \max\{-1, \lceil \log_2(1/\epsilon) \rceil\}$ always results in a time bound that grows only like $\log(1/\epsilon)$ as $\epsilon \to 0$ (using that the term (5.4) is then upper bounded by a constant which is independent of $\epsilon$). However, per the discussion immediately following the statement of Theorem 4.1, the total amount of work – and the time required when the scheme is applied sequentially rather than in parallel – grows like $\log(1/\epsilon)^2$ as $\epsilon \to 0$.

---

\textsuperscript{5}The key is that due to convexity of $f$, for $\hat{f} \geq f^*$ we would then have
\[ D(\hat{f}) \leq \begin{cases} \left( (\hat{f} - f^*)/\mu \right)^{1/p} & \text{if } \hat{f} - f^* \leq \mu r^p \\ \left( \hat{f} - f^* \right)/(\mu r^{p-1}) & \text{if } \hat{f} - f^* \geq \mu r^p. \end{cases} \]
Proof of Corollary 5.1: Observe

\[ K_{\text{subgrad}} \left( (5 \cdot 2^n \epsilon / \mu)^{1/d}, 2^n \epsilon \right) \leq \left( \frac{M(5 \cdot 2^n \epsilon / \mu)^{1/d}}{2^n \epsilon} \right)^2 = \left( \frac{5^{1/d} M}{\mu^{1/d} \epsilon^{1-1/d}} \right)^2 \left( \frac{1}{4^{1-1/d}} \right)^n, \]

and hence (5.1) is bounded above by

\[ \tilde{N} + 1 + 3 \left( \frac{5^{1/d} M}{\mu^{1/d} \epsilon^{1-1/d}} \right)^2 \sum_{n=-1}^{\tilde{N}} \left( \frac{1}{4^{1-1/d}} \right)^n. \]

Since the upper bound (4.1) of Theorem 4.1 is no greater than (5.1), the implication (5.2) for \( d = 1 \) is immediate. On the other hand, since for \( d > 1 \),

\[ \sum_{n=-1}^{\tilde{N}} \left( \frac{1}{4^{1-1/d}} \right)^n \leq \min \left\{ \sum_{n=-1}^{\infty} \left( \frac{1}{4^{1-1/d}} \right)^n, \tilde{N} + 5 \right\} = \min \left\{ \frac{16^{1-1/d}}{4^{1-1/d} - 1}, \tilde{N} + 5 \right\}, \]

the implication (5.3) is immediate, too.

To obtain upper bounds when \( f(x_0) - f^* \geq 5 \cdot 2^N \epsilon \), then according to Theorem 4.1, simply substitute \( N \) for \( \tilde{N} \) in (5.2) and (5.3), and then add (5.4), completing the proof.

The next corollary regards accel, Nesterov’s original accelerated method. Recall \( K_{\text{accel}}(\delta, \epsilon) = [2\delta \sqrt{L/\epsilon}] \).

If \( f \) is smooth and has Hölderian growth, then necessarily the growth degree satisfies \( d \geq 2 \).

Corollary 5.2. Assume \( f(x_0) - f^* > \epsilon \), and assume \( f \) is \( L \)-smooth on \( \mathbb{R}^n \). Consider \( \text{Sync}\|\text{FOM} \) with \( \text{fom} = \text{accel} \).

If \( f(x_0) - f^* < 5 \cdot 2^N \epsilon \), then \( \text{Sync}\|\text{FOM} \) computes an \( \epsilon \)-approximate optimal solution in time \( T \) for which

\[ d = 2 \Rightarrow T \leq \tilde{N} + 1 + 6(\tilde{N} + 2)\sqrt{5L/\mu}, \]

\[ d > 2 \Rightarrow T \leq \tilde{N} + 1 + \frac{6(5/\mu)^{1/d} \sqrt{L}}{\epsilon^{1-1/d}} \min \left\{ \frac{4^\frac{1}{d} - \frac{3}{d}}{2^\frac{1}{d} - \frac{3}{d} - 1}, \tilde{N} + 3 \right\}, \]

where \( \tilde{N} \) is the smallest integer satisfying both \( f(x_0) - f^* < 5 \cdot 2^\tilde{N} \epsilon \) and \( \tilde{N} \geq -1 \).

If \( f(x_0) - f^* \geq 5 \cdot 2^N \epsilon \), time bounds are obtained by substituting \( N \) for \( \tilde{N} \) above, and adding

\[ 2 \text{dist}(x_0, X^*) \sqrt{L/(2^N \epsilon)}. \]

Remarks: Similar to the discussion following Corollary 5.1, but now for \( d = 2 \), the easily computable choice \( N = \max \{-1, \lfloor \log_2(1/\epsilon) \rfloor \} \) always results in a time bound that grows only like \( \log(1/\epsilon) \) as \( \epsilon \to 0 \), although the total amount of work – and the time bound for the scheme implemented sequentially rather than in parallel – grows like \( \log(1/\epsilon)^2 \). This upper bound on the total amount of work is within a factor of \( \log(1/\epsilon) \) of the known lower bound ([13, page 6]).

Proof of Corollary 5.2: Observe

\[ K_{\text{accel}} \left( (5 \cdot 2^n \epsilon / \mu)^{1/d}, 2^n \epsilon \right) \leq \frac{2\sqrt{L}(5 \cdot 2^n \epsilon / \mu)^{1/d}}{\sqrt{2^n \epsilon}} = \frac{2\sqrt{L}(5/\mu)^{1/d}}{\epsilon^{\frac{1}{d} - \frac{3}{d}}} \left( \frac{1}{2^\frac{1}{d} - \frac{3}{d}} \right)^n, \]

and hence (5.1) is bounded above by

\[ \tilde{N} + 1 + \frac{6\sqrt{L}(5/\mu)^{1/d}}{\epsilon^{\frac{1}{d} - \frac{3}{d}}} \sum_{n=-1}^{\tilde{N}} \left( \frac{1}{2^\frac{1}{d} - \frac{3}{d}} \right)^n. \]
Since the upper bound (4.1) of Theorem 4.1 is no greater than (5.1), the implication (5.5) for \( d = 2 \) is immediate. On the other hand, since for \( d > 2 \),

\[
\sum_{n=-1}^{\bar{N}} \left( \frac{1}{2^{\frac{d}{2}} - \frac{1}{d}} \right)^n < \min \left\{ \sum_{n=-1}^{\infty} \left( \frac{1}{2^{\frac{d}{2}} - \frac{1}{d}} \right)^n, \bar{N} + 3 \right\} = \min \left\{ \frac{4^{\frac{1}{2}} - \frac{1}{d}}{2^{\frac{1}{2}} - \frac{1}{d}} - 1, \bar{N} + 3 \right\},
\]

the implication (5.6) is immediate, too.

To obtain time bounds when \( f(x_0) - f^* \geq 5 \cdot 2^N \epsilon \), then according to Theorem 4.1, substitute \( \bar{N} \) for \( \bar{N} \) in (5.5) and (5.6), and add the quantity (5.7).

\[\square\]

6. The Theorem for Async||FOM, and a Corollary

So as to account for the possibility of variability in the amount of work among iterations, we now rely on a function \( T_{\text{FOM}}(\delta, \epsilon) \) assumed to provide an upper bound on the amount of time (e.g., number of oracle calls) required by \( \text{FOM} \) to to compute an \( \epsilon \)-approximate optimal solution when \( \text{FOM} \) is initiated at an arbitrary point \( x_0 \in Q \) satisfying \( \text{dist}(x_0, X^*) \leq \delta \).

Recall that for scalars \( \hat{f} \geq f^* \),

\[D(\hat{f}) := \sup\{\text{dist}(x, X^*) \mid x \in Q \text{ and } f(x) \leq \hat{f}\}.\]

**Theorem 6.1.** Assume \( f(x_0) - f^* > \epsilon \). If \( f(x_0) - f^* < 5 \cdot 2^N \epsilon \), then Async||FOM computes an \( \epsilon \)-approximate optimal solution within time

\[ (\bar{N} + 1)t_{\text{transit}} + 2(\bar{N} + 2)t_{\text{pause}} + 3 \sum_{n=-1}^{\bar{N}} T_{\text{FOM}}(D_n, 2^n \epsilon) \] \hspace{1cm} (6.1)

with \( D_n := \min\{D(f^* + 5 \cdot 2^n \epsilon), D(f(x_0))\} \),

where \( \bar{N} \) is the smallest integer satisfying both \( f(x_0) - f^* < 5 \cdot 2^N \epsilon \) and \( \bar{N} \geq -1 \).

In any case, Async||FOM computes an \( \epsilon \)-approximate optimal solution within time

\[ T_N + T_{\text{FOM}}(\text{dist}(x_0, X^*), 2^N \epsilon), \]

where \( T_N \) is the quantity obtained by substituting \( N \) for \( \bar{N} \) in (6.1).

**Remarks:**

- As remarked upon in §3.2, a larger value for \( t_{\text{transit}} \) might be chosen to reflect the possibility of greater congestion in message passing, although by appropriately managing communication, in the worst case (in which all communication goes through a single server), a message from \( \text{FOM}_n \) would reach \( \text{FOM}_{n-1} \) within time proportional to \( N + 1 \). For \( t_{\text{transit}} \) being proportional to \( N + 1 \), we see from (6.1) that the impact of congestion on Async||FOM is modest, adding an amount of time proportional to \( (N + 1)^2 \). Compare this with the synchronous scheme, where to incorporate congestion in message passing, every time period would be lengthened, in the worst case to length \( 1 + \kappa \cdot (N + 1) \) for some positive constant \( \kappa \). The time bound (4.2) would then be multiplied by \( 1 + \kappa \cdot (N + 1) \), quite different than adding only a single term proportional to \( (N + 1)^2 \).

- Unlike the synchronous scheme, however, for the asynchronous scheme there is no sequential analogue that in general is truly natural.

The proof is deferred to Appendix A, due to its similarities with the proof of Theorem 4.1.

To provide a representative application of Theorem 6.1, we consider Nesterov’s universal fast gradient method – denoted univ – which applies whenever \( f \) has Hölder continuous gradient with
exponent \( \nu (0 \leq \nu \leq 1) \), meaning
\[
M_\nu := \sup \left\{ \frac{\| \nabla f(x) - \nabla f(y) \|}{\| x - y \|^{\nu}} \mid x, y \in Q, x \neq y \right\} < \infty \quad \text{(i.e., is finite)}.
\]
(If \( \nu = 0 \) then \( f \) is \( M_0 \)-Lipschitz continuous on \( Q \), whereas if \( \nu = 1 \), \( f \) is \( M_1 \)-smooth on \( Q \).

The input to \( \text{univ} \) consists of the desired accuracy \( \epsilon \), an initial point \( x_0 \in Q \), and a value \( L_0 > 0 \) meant, roughly speaking, as a guess of \( M_\nu \). (The algorithm relies on the value \( \nu \) only through the assumption \( 0 \leq \nu \leq 1 \).) Nesterov \cite{B. Nesterov, 2004: 4} showed the function
\[
K_{\text{univ}}(\delta, \epsilon) = \left[ \frac{3 + 5 \nu}{2 + 3 \nu} \left( M_\nu \delta^{1+\nu}/\epsilon \right)^{4/(1+3\nu)} \right]
\]
provides an upper bound on the number of iterations sufficient to compute an \( \bar{\epsilon} \)-approximate optimal solution if \( \text{dist}(x_0, X^*) \leq \delta \).

The number of oracle calls in some iterations, however, can significantly exceed the number in other iterations. Indeed, the proofs in \cite{B. Nesterov, 2004: 4} leave open the possibility that the number of oracle calls made only in the \( k \text{th} \) iteration might exceed \( k \). Thus, the scheme \( \text{Async}\|\text{FOM} \) is highly preferred to \( \text{Sync}\|\text{FOM} \) when \( \text{fom} \) is chosen to be \( \text{univ} \).

For the universal fast gradient method, the upper bound established in \cite{B. Nesterov, 2004: 4} on the number of oracle calls in the first \( k \) iterations is
\[
4(k + 1) + \log_2 \left( \delta^{(1+\nu)/(1+3\nu)} (1/\epsilon)^{2/(1+3\nu)} M_\nu^{4/(1+3\nu)} \right) - 2 \log_2 L_0,
\]
assuming \( \text{dist}(x_0, X^*) \leq \delta \) (and assuming the prox function in \cite{B. Nesterov, 2004: 4} is chosen as \( d(x) = 1/2 \| x - x_0 \|^2 \)). This upper bound depends on an assumption such as
\[
L_0 \leq \left( \frac{1 - \nu}{1 + \nu} \right) \frac{1 - \nu}{1 + \nu} M_\nu \frac{2}{1 + \nu}
\]
(\( L_0 \leq M_1 \) when \( \nu = 1 \)). Presumably the assumption can be removed by a slight extension of the analysis, resulting in a time bound that differs insignificantly, just as the assumption can readily be removed for the first universal method introduced by Nesterov in \cite{B. Nesterov, 2004: 3} (essentially the same algorithm as \text{univ} when \( \nu = 1 \)).\footnote{For that setting, see \cite[Appendix A]{B. Nesterov, 2004: A} for a slight extension to Nesterov’s arguments that suffice to remove the assumption.} However, as the focus of the present paper is on the simplicity of \( \|\text{FOM} \) and not on \text{univ} per se, we do not digress to attempt removing the assumption, but instead make assumptions that ensure the results from \cite{B. Nesterov, 2004: 4} apply.

In particular, we assume \( L_0 \) is a positive constant satisfying (6.4) for \( \bar{\epsilon} = 2^{-1} \epsilon \), and thus satisfies (6.4) when \( \bar{\epsilon} = 2^n \epsilon \) for any \( n \geq -1 \). Moreover, we assume that whenever \( \text{univ} \) (\( n = -1, 0, \ldots, N \)) is (re)started, the input consists of \( \bar{\epsilon} = 2^n \epsilon \), \( x_{n,0} \) (the (re)start point), and \( L_0 \) (the same value at every (re)start).

Relying on the same value \( L_0 \) at every (re)start likely results in complexity bounds that are slightly-more suboptimal in some cases (specifically, when \( d = 1 + \nu \) and \( 0 \leq \nu < 1 \)), but not much more suboptimal.

In view of (6.2) and (6.3), and assuming (6.4) holds, it is natural to define
\[
T_{\text{univ}}(\delta, \epsilon) = 4K_{\text{univ}}(\delta, \epsilon) + 1 + \log_2 \left( \delta^{(1+\nu)/(1+3\nu)} (1/\epsilon)^{2/(1+3\nu)} M_\nu^{4/(1+3\nu)} \right) - 2 \log_2 L_0,
\]
an upper bound on the time (number of oracle calls) sufficient for \( \text{univ} \) to compute an \( \bar{\epsilon} \)-approximate optimal solution when initiated at an arbitrary point \( x_0 \in Q \) satisfying \( \text{dist}(x_0, X^*) \leq \delta \). In order to reduce notation, for \( \epsilon > 0 \) let
\[
C(\delta, \epsilon) := 4 + \log_2 \left( \delta^{(1+\nu)/(1+3\nu)} (2/\epsilon)^{2/(1+3\nu)} M_\nu^{4/(1+3\nu)} \right) - 2 \log_2 L_0,
\]
in which case for $n \geq -1$, from (6.2) and (6.5) follows
\[ T_{\text{univ}}(\delta, 2^n \epsilon) \leq 4K_{\text{univ}}(\delta, 2^n \epsilon) + C(\delta, \epsilon) \]
\[ \leq 4 \cdot 2^{\frac{3+5\nu}{1+3 \nu}} \left( M_\nu \delta^{1+\nu} / (2^n \epsilon) \right) \frac{2}{1+3 \nu} + C(\delta, \epsilon) . \quad (6.6) \]
Note that $C(\delta, \epsilon)$ is a constant independent of $\epsilon$ if $\nu = 1$, and otherwise grows like $\log(1/\epsilon)$ as $\epsilon \to 0$.

We assume $f$ has Hölderian growth, that is, there exist constants $\mu > 0$ and $d \geq 1$ for which
\[ x \in Q \text{ and } f(x) \leq f(x_0) \Rightarrow f(x) - f^* \geq \mu \text{dist}(x, X^*)^d . \]
Consequently, the values $D_n$ in Theorem 6.1 satisfy
\[ D_n \leq \min\{(5 \cdot 2^n \epsilon / \mu)^{1/d}, D(f(x_0))\} . \quad (6.7) \]
For a function $f$ which has Hölderian growth and has Hölder continuous gradient, necessarily the values $d$ and $\nu$ satisfy $d \geq 1 + \nu$ (see [23, §2.2]).

**Corollary 6.2.** Make the above assumptions, and also assume $f(x_0) - f^* > \epsilon$. Consider Async\|FOM with fom = univ.
If $f(x_0) - f^* < 5 \cdot 2^N \epsilon$, then Async\|FOM computes an $\epsilon$-approximate optimal solution in time $T$ for which
\[ d = 1 + \nu \Rightarrow T \leq (\bar{N} + 1) t_{\text{transit}} + 2(\bar{N} + 2) t_{\text{pause}} + 3(\bar{N} + 2) C(D(f(x_0)), \epsilon) \]
\[ + 12 (\bar{N} + 2) 2^{\frac{1+3 \nu}{1+3 \nu}} (5M_\nu / \mu)^{\frac{2}{1+3 \nu}} , \quad (6.8) \]
\[ d > 1 + \nu \Rightarrow T \leq (\bar{N} + 1) t_{\text{transit}} + 2(\bar{N} + 2) t_{\text{pause}} + 3(\bar{N} + 2) C(D(f(x_0)), \epsilon) \]
\[ + 12 \cdot 2^{\frac{3+5\nu}{1+3 \nu}} \left( M_\nu (5/\mu) \frac{1+\nu}{\nu} \right) \frac{2}{1+3 \nu} \min \left\{ \frac{4^{(1-1/\nu)} \frac{2}{1+3 \nu}}{2^{(1-1/\nu)} \frac{2}{1+3 \nu} - 1}, \bar{N} + 5 \right\} , \quad (6.9) \]
where $\bar{N}$ is the smallest integer satisfying both $f(x_0) < f^* + 5 \cdot 2^\bar{N} \epsilon$ and $\bar{N} \geq -1$.
If $f(x_0) - f^* \geq 5 \cdot 2^N \epsilon$, then time bounds are obtained by substituting $N$ for $\bar{N}$ above, and adding
\[ 4 \cdot 2^{\frac{3+5\nu}{1+3 \nu}} \left( M_\nu \text{dist}(x_0, X^*)^{1+\nu} / (2^N \epsilon) \right) \frac{2}{1+3 \nu} + C(\text{dist}(x_0, X^*), 2^N \epsilon) . \quad (6.10) \]

**Remarks:** According to Nemirovski and Nesterov [13, page 6], who provide lower bounds when $0 < \nu \leq 1$, for the easily computable choice $N = \max\{-1, \lceil \log_2(1/\epsilon) \rceil\}$, the time bounds of the corollary would be optimal – with regards to $\epsilon$ – for a sequential algorithm, except in the cases when both $d = 1 + \nu$ and $0 < \nu < 1$, where due to the term “$3(\bar{N} + 2) C(D(f(x_0)), \epsilon)$,” the bound (6.8) grows like $\log(1/\epsilon)^2$ as $\epsilon \to 0$, rather than growing like $\log(1/\epsilon)$. Consequently, in those cases, the total amount of work is within a multiple of $\log(1/\epsilon)^2$ of being optimal, whereas in the other cases for which they provide lower bounds, the total amount of work is within a multiple of $\log(1/\epsilon)$ of being optimal.

**Proof of Corollary 6.2:** Assume $f(x_0) - f^* < 5 \cdot 2^N \epsilon$. By (6.6) and (6.7), for $n \geq -1$ we have
\[ T_{\text{univ}}(D_n, 2^n \epsilon) \leq C + 4 \cdot 2^{\frac{3+5\nu}{1+3 \nu}} \left( M_\nu (5 \cdot 2^n \epsilon / \mu)^{\frac{1+\nu}{\nu}} \right) \frac{2}{1+3 \nu} \]
\[ = C + 4 \cdot 2^{\frac{3+5\nu}{1+3 \nu}} \left( M_\nu (5/\mu)^{\frac{1+\nu}{\nu}} \right) \frac{2}{1+3 \nu} \left( \frac{1}{2^{(1-1/\nu)} \frac{2}{1+3 \nu}} \right)^n . \quad (6.11) \]
Substituting this into the bound (6.1) of Theorem 6.1 establishes the implication (6.8)

For $d > 1 + \nu$, observe

$$
\sum_{n=-1}^{N} \left( \frac{1}{2 \left( 1 - \frac{x}{d} \right)^{1 + 3\nu}} \right)^n < \min \left\{ \sum_{n=-1}^{\infty} \left( \frac{1}{2 \left( 1 - \frac{x}{d} \right)^{1 + 3\nu}} \right)^n, \tilde{N} + 5 \right\}
$$

$$
= \min \left\{ \frac{4 \left( 1 - \frac{x}{d} \right)^{1 + 3\nu}}{2 \left( 1 - \frac{x}{d} \right)^{1 + 3\nu} - 1}, \tilde{N} + 5 \right\}, \tag{6.12}
$$

where for the inequality we have used $1 \leq 2 \left( 1 - \frac{x}{d} \right)^{1 + 3\nu} \leq 4$. Substituting (6.11) into the bound (6.1) of Theorem 6.1, and then substituting (6.12) for the resulting summation, establishes the implication (6.9), concluding the proof in the case that $f(x_0) - f^* < 5 \cdot 2^\nu \epsilon$.

To obtain time bounds when $f(x_0) - f^* \geq 5 \cdot 2^\nu \epsilon$, then according to Theorem 6.1, simply substitute $N$ for $\tilde{N}$ in the bounds above, and add

$$
T_{univ}(\text{dist}(x_0, X^*), 2^\nu \epsilon),
$$

which due to (6.6), is bounded above by (6.10).

\[ \Box \]

\section*{Appendix A. Proof of Theorem 6.1}

The proof proceeds through a sequence of results analogous to the sequence in the proof of Theorem 4.1, although bookkeeping is now more pronounced. As for the proof of Theorem 4.1, we refer to fom \textsubscript{n} “updating” at a point $x$. The starting point $x_0$ is considered to be the first update point for every fom \textsubscript{n}. After Async\|FOM has started, then for $n < N$, updating at $x$ means the same as restarting at $x$, and for fom\textsubscript{N}, updating at $x$ is the same as having computed $x$ satisfying the current task of fom\textsubscript{N} (in which case the point is sent to fom\textsubscript{N-1}, even though fom\textsubscript{N} does not restart).

Keep in mind that when a new epoch for fom \textsubscript{n} occurs, if a message from fom\textsubscript{n+1} arrives in the inbox exactly when the epoch begins, then the restart point $x$ will not be decided immediately, due to fom \textsubscript{n} being made to pause, possibly contiguously. In any case, the restart point is decided after only a finite amount of time, due to fom\textsubscript{n+1} sending at most finitely many messages in total.\footnote{The number of messages sent by fom\textsubscript{n+1} is finite because (1) fom\textsubscript{n+1} sends a message only when it has obtained $y_{n+1}$ satisfying $f(y_{n+1}) \leq f(x_{n+1,0}) - 2^{n+1} \epsilon$, where $x_{n+1,0}$ is the most recent (re)start point, and (2) we assume $f^*$ is finite.}

\begin{proposition}
Assume that at time $t$, fom \textsubscript{n} updates at $x$ satisfying $f(x) - f^* \geq 2 \cdot 2^n \epsilon$. Then fom \textsubscript{n} updates again, and does so no later than time

$$
t + m t_{\text{pause}} + T_{\text{fom}}(D(f(x)), 2^n \epsilon), \tag{A.1}
$$

where $m$ is the number of messages received by fom \textsubscript{n} between the two updates.

\end{proposition}

\textbf{Proof:} Let $\bar{m}$ be the total number of messages received by fom \textsubscript{n} from time zero onward. We know $\bar{m}$ is finite. Consequently, at time

$$
t + \bar{m} t_{\text{pause}} + T_{\text{fom}}(D(f(x)), 2^n \epsilon), \tag{A.2}
$$

fom \textsubscript{n} will have devoted at least $T_{\text{fom}}(D(f(x)), 2^n \epsilon)$ units of time to computing iterations. Thus, if fom \textsubscript{n} has not updated before time (A.2), then at that time, fom \textsubscript{n} will not be in a pause, and will compute $\bar{x}$ satisfying

$$
f(\bar{x}) - f^* \leq 2^n \epsilon \quad (\text{thus, } f(\bar{x}) \leq f(x) - 2^n \epsilon). \tag{A.3}
$$
Consequently, if $f_{om_n}$ has not updated by time (A.2), it will update at that time, at $\bar{x}$.

Having proven that after updating at $x$, $f_{om_n}$ will update again, it remains to prove the next update will occur no later than time (A.1). Of course the next update occurs at (or soon after) the start of a new epoch. Let $m_1$ be the number of messages received by $f_{om_n}$ after updating at $x$ and before the new epoch begins, and let $m_2$ be the number of messages received in the new epoch and before $f_{om_n}$ updates (i.e., before the restart point is decided). (Thus, from the beginning of the new epoch until the update, $f_{om_n}$ is contiguously paused for an amount of time not exceeding $m_2 t_{pause}$.) Clearly, $m_1 + m_2 = m$.

In view of the preceding observations, to establish the time bound (A.1) it suffices to show the new epoch begins no later than time

$$ t + m_1 t_{pause} + T_{om}(D(f(x)), 2^n\epsilon) . $$

(A.4)

However, if the new epoch has not begun prior to time (A.4) then at that time, $f_{om_n}$ is not in a pause, and has spent enough time computing iterations so as to have a point $\bar{x}$ satisfying (A.3), causing a new epoch to begin instantly.

**Proposition A.2.** Assume that at time $t$, $f_{om_n}$ updates at $x$ satisfying $f(x) - f^* \geq 2 \cdot 2^n\epsilon$. Let $j := \lceil (f(x) - f^*) \cdot 2^n \epsilon \rceil - 2$. Then $f_{om_n}$ updates at a point $\bar{x}$ satisfying $f(\bar{x}) - f^* < 2 \cdot 2^n\epsilon$ no later than time

$$ t + m t_{pause} + \sum_{j=0}^{j} T_{om}(D(f(x) - j \cdot 2^n\epsilon), 2^n\epsilon) , $$

where $m$ is the total number of messages received by $f_{om_n}$ between the updates at $x$ and $\bar{x}$.

**Proof:** The proof is essentially identical to the proof of Proposition 4.3, but relying on Proposition A.1 rather than on Lemma 4.2. \qed

**Corollary A.3.** Assume that at time $t$, $f_{om_n}$ updates at $x$ satisfying $f(x) - f^* < i \cdot 2^n\epsilon$, where $i$ is an integer and $i \geq 3$. Then $f_{om_n}$ updates at a point $\bar{x}$ satisfying $f(\bar{x}) - f^* < 2 \cdot 2^n\epsilon$ no later than time

$$ t + m t_{pause} + \sum_{i=3}^{i} T_{om}(D_{n,i}, 2^n\epsilon) $$

$$ \text{where } D_{n,i} = \min\{D(f^* + i \cdot 2^n\epsilon), D(f(x))\} , $$

(A.5)

and where $m$ is the number of messages received by $f_{om_n}$ between the updates at $x$ and $\bar{x}$.

Moreover, if $n > -1$, then after sending the message to $f_{om_{n-1}}$ containing the point $\bar{x}$, $f_{om_n}$ will send at most one further message.

**Proof:** Except for the final assertion, the proof is essentially identical to the proof of Corollary 4.4, but relying on Proposition A.2 rather than on Proposition 4.3.

For the final assertion, note that if $f_{om_n}$ sent two points after sending $\bar{x}$ - say, first $x'$ and then $x''$ - we would have

$$ f(x'') \leq f(x') - 2^n\epsilon \leq (f(\bar{x}) - 2^n\epsilon) - 2^n\epsilon < f^* , $$

a contradiction. \qed

**Corollary A.4.** Let $n > -1$. Assume that at time $t$, $f_{om_n}$ updates at $x$ satisfying $f(x) - f^* < 5 \cdot 2^n\epsilon$, and assume from time $t$ onward, $f_{om_n}$ receives at most $\hat{m}$ messages. Then for either $\hat{m}' = 0$ or $\hat{m}' = 1$, $f_{om_{n-1}}$ restarts at $x'$ satisfying $f(x') - f^* < 5 \cdot 2^{n-1}\epsilon$ no later than time

$$ t + t_{transit} + (\hat{m} + 2 - \hat{m}') t_{pause} + \sum_{i=3}^{5} T_{om}(D_{n,i}, 2^n\epsilon) $$
(where $D_{n,i}$ is given by (A.5)), and from that time onward, $\text{fom}_{n-1}$ receives at most $\tilde{m}'$ messages.

**Proof:** By Corollary A.3, no later than time

$$t + \tilde{m}t_{\text{pause}} + \sum_{i=3}^{5} T_{\text{fom}}(D_{n,i}, 2^n \epsilon),$$

$\text{fom}_n$ updates at $\tilde{x}$ satisfying $f(\tilde{x}) - f^* \leq 2 \cdot 2^n \epsilon$. When $\text{fom}_n$ updates at $\tilde{x}$, the point is sent to the inbox of $\text{fom}_{n-1}$, where it arrives no more than $t_{\text{transit}}$ time units later, causing a pause of $\text{fom}_{n-1}$ to begin immediately. Moreover, following the arrival of $\tilde{x}$, at most one additional message will be received by $\text{fom}_{n-1}$ (per the last assertion of Corollary A.3).

If during the pause, $\text{fom}_{n-1}$ does not receive an additional message, then at the end of the pause, $\text{fom}_{n-1}$ either updates at $\tilde{x}$ or continues its current epoch. The decision on whether to update at $\tilde{x}$ is then enacted no later than time

$$t + t_{\text{transit}} + (\tilde{m} + 1)t_{\text{pause}} + \sum_{i=3}^{5} T_{\text{fom}}(D_{n,i}, 2^n \epsilon),$$

after which $\text{fom}_{n-1}$ receives at most one message.

On the other hand, if during the pause, an additional message is received, then a second pause immediately begins, and $\tilde{x}$ is overwritten by a point $\hat{x}$ satisfying $f(\hat{x}) \leq f(\tilde{x}) - 2^n \epsilon < f^* + 2^n \epsilon$. Here, the decision on whether to update at $\hat{x}$ is enacted no later than time

$$t + t_{\text{transit}} + (\hat{m} + 2)t_{\text{pause}} + \sum_{i=3}^{5} T_{\text{fom}}(D_{n,i}, 2^n \epsilon),$$

(A.6)

after which $\text{fom}_{n-1}$ receives no messages.

If $\text{fom}_{n-1}$ chooses to update at $\hat{x}$ (or $\tilde{x}$), the corollary follows immediately from (A.6) and (A.7), as the value of $f$ at the update point is then less than $f^* + 4 \cdot 2^{n-1} \epsilon$. On the other hand, if $\text{fom}_{n-1}$ chooses not to update, it is only because its most recent update point satisfies $f(x_{n-1,0}) < f(\hat{x}) - 2^{n-1} \epsilon$ (resp., $f(x_{n-1,0}) < f(\hat{x}) - 2^{n-1} \epsilon$), and hence $f(x_{n-1,0}) < f^* + 5 \cdot 2^{n-1} \epsilon$. Thus, here as well, the corollary is established. \hfill $\square$

**Corollary A.5.** For any $\tilde{N} \in \{-1, \ldots, N\}$, assume that at time $t$, $\text{fom}_N$ updates at $x$ satisfying $f(x) - f^* < 5 \cdot 2^{\tilde{N}} \epsilon$, and assume from time $t$ onward, $\text{fom}_N$ receives at most $\hat{m}$ messages. Then no later than time

$$t + (\tilde{N} + 1)t_{\text{transit}} + (\hat{m} + 2\tilde{N} + 2)t_{\text{pause}} + \sum_{n=-1}^{\tilde{N}} \sum_{i=3}^{5} T_{\text{fom}}(D_{n,i}, 2^n \epsilon),$$

Async-FOM has computed an $\epsilon$-approximate optimal solution.

**Proof:** If $\tilde{N} = -1$, Corollary A.3 implies that after updating at $x$, the additional time required by $\text{fom}_1$ to compute a $(2 \cdot 2^{-1} \epsilon)$-approximate optimal solution $\tilde{x}$ is bounded from above by

$$\hat{m} t_{\text{pause}} + \sum_{i=3}^{5} T_{\text{fom}}(D_{-1,i}, 2^{-1} \epsilon).$$

(A.8)

The present corollary is thus immediate for the case $\tilde{N} = -1$.

On the other hand, if $\tilde{N} > -1$, induction using Corollary A.4 shows that for either $\hat{m}' = 0$ or $\hat{m}' = 1$, no later than time

$$t + (\tilde{N} + 1)t_{\text{transit}} + (\hat{m} + 2\tilde{N} + 2 - \hat{m}')t_{\text{pause}} + \sum_{n=0}^{\tilde{N}} \sum_{i=3}^{5} K_{\text{fom}}(D_{n,i}, 2^n \epsilon),$$
A.3

6.1

A.5

A.8

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