Disordered free fermions and the Cardy Ostlund fixed line at low temperature

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Using functional RG, we reexamine the glass phase of the 2D random-field Sine Gordon model. It is described by a line of fixed points (FP) with a super-roughening amplitude $\langle u(0) - u(r) \rangle^2 \sim A(T) \ln^2 r$ as temperature $T$ is varied. A speculation is that this line is identical to the one found in disordered free-fermion models via exact results from “nearly conformal” field theory. This however predicts $A(T = 0) = 0$, contradicting numerics. We point out that this result may be related to failure of dimensional reduction, and that a functional RG method incorporating higher harmonics and non-analytic operators predicts a non-zero $A(T = 0)$ which compares reasonably with numerics.

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I. INTRODUCTION

A. Overview

Few analytical methods can handle systems with quenched disorder beyond mean field and phenomenological arguments, and each seems adapted only to a specific class of models. One example is the functional RG (FRG) used to study elastic objects in a random potential or continuous (e.g. XY) spin models in a random field. When pure, these systems possess internal order and one studies the case where the (weak) quenched disorder produces only elastic deformations (no topological defects) parameterized by a displacement field $u(x)$ away from perfect order $u = 0$. They exhibit glass physics and pinning, i.e. thermal (and quantum) fluctuations are small and energy minimization yields multiple metastable states. The FRG handles this by summing an infinite set of operators, all relevant for internal dimension $d < d_{ac}$, usually $d_{ac} = 4$. The coupling constant is a function, $R(u)$, which represents the correlator of the coarse-grained disorder. FRG fixed points $R^*(u)$ were obtained in a $d = d_{ac} - \epsilon$ expansion for various disorder classes and recently it was shown how to directly measure them in numerics.

Disordered systems in $d = 2$ can sometimes be studied using methods inspired by conformal field theory (CFT). Exact results were obtained for free fermions with disorder of various symmetries using bosonization and supersymmetry. The physics there is localization, and in recently much studied symmetry classes, de-localization near a point where the density of state (DOS) is singular (e.g. bipartite lattices leading to Dirac fermions). There, sample (and local DOS) fluctuations can become broad leading to freezing transitions in the DOS dynamical exponent $z$, $\rho(E) \sim E^{-1 + 2/z}$, as found for Dirac fermions with random vector potential. Being equivalent to a (dilute) limit of a random gauge XY spin model it exhibits an infinite set of relevant operators which can be handled using another functional RG method based on the KPP-Fisher equation (hence noted here KPP-FRG). Freezing transitions are expected to impact a wider class of models, e.g. the time-reversal, particle-hole symmetric class, where the $\rho(E) \sim E^{-1} e^{-c(lnE)^\mu}$ Gade-Wegner singularity ($\mu = 1/2$) was argued to become $\mu = 2/3$. This was first found using KPP-FRG on the dual spin version (the Cardy-Ostlund model with dilute vortices) and, recently, within the fermion model and the sigma model. These emerging 2d field theories still need testing on whether they capture all (nonperturbative) freezing (and glassy) physics, for which the spin models can provide insight.

B. Model

In this paper we focus on the simpler 2D random field XY (Cardy-Ostlund) model without vortices. Being the bosonized form of a fermion model, exact results were claimed for this model using CFT inspired methods. Since it is also a (periodic) pinning problem, we can also use FRG and test the strong disorder regime, expected to be correctly described already to one loop. The model, also called random phase Sine Gordon model (referred to here simply as CO model) also describes a periodic object (such as an array of flux lines) at equilibrium on a 2d disordered substrate at temperature $T$, and of partition function $Z = \int D\epsilon e^{-\mathcal{S}_{CO}[\epsilon]}$ with $\mathcal{S}_{CO}[\epsilon] = H_{CO}[\epsilon]/T$ and:

$$H_{CO}[\epsilon] = \int d^2r \left( \frac{\kappa}{2} (\nabla_r \epsilon)^2 - h \cdot \nabla_r \epsilon - \text{Re}(\xi e^{i\epsilon}) \right)$$

where $\epsilon(r) \in (-\infty, +\infty)$ is an XY phase (excluded vortices), $\kappa$ the elastic coefficient and $\xi(r)$ a (complex) quenched random field with random phase $\xi(r) \xi^\ast (r') = $
2g_0\delta^2(r-r'). The additional random field \( h(r) \) is generated under coarse-graining with \( h'(r)h'(r') = \sigma \delta_{ij} \delta^2(r-r') \). Disorder being short ranged, the model [1] together with some of its discrete versions, exhibits the STS symmetry [19,20], i.e. statistical invariance of the nonlinear part under \( u(r) \rightarrow u(r) + v(r) \). The model [1] exhibits a super-rough phase for \( T < T_g = 4\pi\xi \) with anomalous growth of the 2-point correlation [16,18,19]:

\[
B(r) = \langle (u(r)-u(0))^2 \rangle = A(T)(\ln r)^2 + \mathcal{O}(\ln r)
\]

where \( \langle \ldots \rangle \) denotes thermal averages. This is in contrast with the usual logarithmic roughness for \( T > T_g \), disorder being irrelevant, i.e. \( A(T > T_g) = 0 \). The STS symmetry implies that \( T/T_g \) is uncorrected by disorder and does not flow under RG. This "marginal glass" below \( T_g \) is thus described at large scale by a line of fixed points (FP), indexed by \( T/T_g \) which predicts [21]:

\[
A(T) = 2\tau^2 + \mathcal{O}(\tau^3)
\]

for small \( \tau = (T_g - T)/T_g \ll 1 \) in agreement with precise numerics [22]. At low temperature, only numerical results are available at present: it was concluded from exact ground state determinations [23,24] for a closely related solid on solid model that super-roughening holds at \( T = 0 \), i.e. \( A(T = 0) \approx .5 > 0 \) (see below).

II. RELATIONS TO DISORDERED FERMIONS

A. General framework and bosonization

A recent study [25] of a free fermion model opened the possibility that \( A(T) \) could be obtained exactly. It has been known that weak disorder on the CO model at any \( T \) can be equivalently studied by adding quenched (dynamical) disorder to a (euclidean) \( d = 1 + 1 \) interacting fermion model [26]. Starting with the CO model without disorder, i.e. \( h, \xi = 0 \) in [1], one defines \( r = (x, \tau) \) where \( \tau \in [0, \beta h] \) is imaginary time, and \( u(r) = 2\phi(r) \). Then the bosonic action \( S_0[\phi] = S_{CO}[u = 2\phi] \), where one substitutes \( \frac{\partial}{\partial \phi} \rightarrow \frac{1}{\pi \xi} \left( \frac{\partial}{\partial \phi} + \nu \frac{\partial}{\partial \tilde{\phi}} \right) \) in [1], is the euclidean action (with \( Z = \int D\phi e^{-S_0[\phi]} \) the bosonized form of the fermion model (with \( Z = \int D\psi^\dagger D\psi e^{-S_f[\psi, \psi^\dagger]} \) of euclidean action \( S_f = S_0 + S_{int} \) with:

\[
S_0 = \int_{\tau\tau} [\psi_1^\dagger \partial_\tau \psi_R + \psi_R^\dagger \partial_\tau \psi_L] + \frac{1}{h} \int_{\tau\tau} H_0
\]

\[
H_0 = \int_x \nu F \psi^\dagger (i\partial_x - k_F) \psi_L - \psi_R^\dagger (i\partial_x + k_F) \psi_R:
\]

\[
S_{int} = \frac{1}{h} \int_{x\tau} \frac{g_1}{2} (\rho_R \rho_R + \rho_L \rho_L) + g_2 \rho_R \rho_L
\]

where \( \int_{\tau\tau} = \int dx \int_0^{\beta h} dr, \psi = \psi_L + \psi_R, \rho = \psi^\dagger \psi \) is the fermion density and \( g_{2,4} \) are (spinless) fermion interaction parameters with \( \rho_s = \psi_s^\dagger \psi_s, s = L, R \). Using bosonization:

\[
\psi_s = \frac{U_s}{\sqrt{2\pi\alpha}} e^{i k_F x} e^{i(\theta - s\phi)}
\]

with \( s = 1/ - 1 \) for \( R/L \) movers, the density reads:

\[
\rho = \rho_0 - \frac{\partial_x \phi}{\pi} + \frac{2}{2\pi\alpha} \Re(e^{2ik_F x - 2\phi})
\]

where \( \rho_0 \) is the average density and \( \alpha \) a UV cutoff. The (Tomonaga Luttinger) model \( S_0 + S_{int} \) is equivalent, using \( 2\pi\rho_0 = -\partial_x \phi + s\partial_x \theta \), to the quadratic action \( S_0[\phi] \) and to the CO model without disorder, with:

\[
K = T/T_g
\]

hence any temperature can be reached by varying the Luttinger parameter \( K \), i.e. the interactions, with:

\[
K^2 = (1 + y_4 - y_2)/(1 + y_4 + y_2)
\]

\[
y_i = g_i/(2\pi v_F)
\]

and the sound velocity

\[
v = v_F \sqrt{(1 + y_4)^2 - y_2^2}
\]

which can be set to one by a rescaling of space-time. Next, the (formal) perturbation of the fermion model \( S_f \rightarrow S_f + S_{dis} \) with

\[
S_{dis} = -\frac{1}{\hbar} \int_{x\tau} W(r)
\]

\[
W(r) = \mu(r) + \xi(r)e^{2ik_F x} + \tilde{\xi}(r)e^{-2ik_F x}
\]

by a small (dynamic) random potential \( W(r) \) (neglecting for now higher harmonics) is equivalent via \( S_0[\phi] = S_{CO}[u = 2\phi] \), to the disorder perturbation in the CO model, i.e \( h, \xi \) in [1]. The correspondence reads \( h^* = -\mu/(2\pi h), h^* = 0 \), \( \tilde{\xi}(r) = \xi(r)/\pi\alpha h \). An isotropic \( h \) distribution is obtained by coupling a random field to \( \partial_\tau \phi \) but it makes no difference at large scale. For \( T < T_g \) small disorder \( g_0 \) is relevant and grows under RG to finite value on the fixed line [27]. At a naive level (see below) the relation [28] still holds since being unrenormalized (for exact STS).

B. Predictions from "nearly conformal" field theory

The recent study [25] addresses a free fermion model in presence of (isotropic) disorder \( S_{ff} = S_0 + S_{dis} \). We will follow the notations of Ref [29] which uses disorder parameters \( g_A = \pi\sigma/T^2 \) and \( g_M \propto g_0 \), shifted fermion fields \( \psi_s = e^{i(\theta - s\phi)} \) and the same bosonic fields \( \phi_L/R = \phi - s\theta \).
One has chosen $\hbar = 1 = v_F$, $h_x = A_x/\pi$, $h_y = -A_y/\pi$. This model was also found to be described by a line of FP with the exact result (denoting $\zeta = 1/(1+g_M/\pi)$ and $\phi = u/2$ the same bosonic phase as above):

$$\begin{align*}
\partial_0 g_{M} &= 0 \quad , \quad \partial_0 g_{A} = \frac{(g_{A})^2}{2\pi^2} \\
\langle \phi_0(z, \bar{z})\phi_0(0) \rangle &= -\frac{g_3^2}{(2\pi)^2} \ln^2 z\bar{z} - \zeta(\delta_{ab} + \frac{g_{A}\zeta}{\pi}) \ln z\bar{z} 
\end{align*}$$

for real copies $a, b$ of the model (no replica). The vanishing beta function for $g_M$ strongly suggests that the free fermion model is exactly on the CO fixed line. From the above it means that the value of $K$ is shifted from $K = 1$ (no disorder) down to $K = 1/(1+g_M/\pi)$ (in presence of disorder), to all orders in $g_M$ according to Ref.\textsuperscript{27}. To lowest order in $g_M$ one easily finds such a shift on model $S_0 + S_{\text{dis}}$, e.g. considering the replicated action

$$S_0[\psi_a] = \frac{1}{2\pi^2} \int_{x\tau} (g_{MO_M} + g_{AO_A})$$

While bosonization of the terms $a \neq b$ yields the $\cos(2(\psi_a - \psi_b))$ disorder terms of the replicated CO action (see\textsuperscript{23} below), the $a = b$ term yields an interaction\textsuperscript{28} as in Eq. (14) with $g_2 = 2g_M$, compatible to lowest order with $K = 1 - g_2/\pi$. To understand higher orders one notes that the interaction used in\textsuperscript{27} is fully isotropic, and does not correct $v$ (while $g_2$ corrects $v$ to order $g_2^2$), hence involves also a $2k_F$ coupling to current. Note that density only couplings preserve $vK$ to all orders. In bosonized form it reads

$$\begin{align*}
\frac{g_M}{2\pi^2} \int_{x\tau} J_{ab}\bar{J}_{aa} &= \frac{g_M}{2\pi^2} \int_{x\tau} [(\partial_\tau \phi)^2 + (\partial_x \phi)^2] 
\end{align*}$$

where $J_{ab} = \psi_{La}^\dagger \psi_{La} \psi_{Lb}^\dagger \psi_{Lb}$. The bosonised theory of Ref.\textsuperscript{28} hence exhibits super-renormalization, and is "nearby conformal" as the $h$ random field (the $g_A$ sector) is a (Larkin type) random force trivially eliminated by a shift using STS. The $\delta_{ab}$ term in (13) gives thermal connected correlations hence one identifies $\zeta = T/T_g$. It implies the prediction (with $\tau = 1 - T/T_g$):

$$A_{ff}(T) = 2\tau^2(1 - \tau)^2$$

consistent with one loop at small $\tau$, exhibiting a maximum at $\tau = 1/2$ where $A_{ff} = 1/8$ (see Fig. 1 and vanishing at zero temperature. Clearly $A_{ff}(T = 0) = 0$ is incompatible with numerical results and equivalence of the models. The reentrant shape of the $A_{ff}(T)$ is surprising as one expects stronger effect of disorder at low $T$. This is reminiscent of reentrant phase boundaries found in the random gauge XY model, due to neglecting operators (moments of vortex fugacity) which become relevant at low $T$\textsuperscript{29,11}. Here higher harmonics of the $\exp(i\phi)$ field are known to be generated as an interplay of interactions and disorder. This can be seen in the Haldane representation\textsuperscript{28}.

$$\rho(r) = (\rho_0 - \frac{1}{\pi} \partial_\tau \phi(r)) \sum_p e^{2i p \rho(x)} e^{-2i p \rho(r)}$$

which yields higher harmonics $\cos(2(p\phi_a - \phi_b))$ in\textsuperscript{28}. Note that the isotropic disorder $\int_{x\tau} \rho W + j V$ where $j$ is the exact current, $\partial_\tau \rho + \partial_x j = 0$, produces a seemingly irrelevant coupling $\partial_\tau \phi(r) Re(e^{-2i \rho(r)} \xi(r))$. Upon replicating its $aa$ contribution generates a shift in $K$, and none in $v$. Whether the higher harmonics are generated in a free fermion model (e.g. as in\textsuperscript{28}) properly taking microscopic cutoff into account deserves further investigations\textsuperscript{29}. Weak disorder counting shows them to be irrelevant for $1/2 < T/T_g < 1$ but these dimensions could change along the fixed line well before that. We now study the effect of higher harmonics using the FRG.

### III. FUNCTIONAL RG

#### A. Effective action and general structure of the RG flow

We start by noting that $A_{ff}(T = 0) = 0$ is reminiscent of a dimensional reduction result, i.e. an artefact of considering an analytic disorder $R(u)$. This suggests a reexamination of the model at low $T$ using FRG. We now indeed show that the cusp in $R'(u)$ is necessary to get $A(T = 0) > 0$. Using replicas $u^a(r) = u^a_x$, $a = 1,..,n$ and averaging over $\zeta, h$ we obtain the bare replicated action

![FIG. 1](image-url)

Right $y$-axis: $A_{ff}(T)$ given by formula (15) obtained, as explained in the text, from the theory of Ref.\textsuperscript{28} and yielding $A_{ff}(T = 0) = 0$. Left $y$-axis: the exact ground state determination of disordered solid-on-solid models yield $A(T = 0) = a_2 = 0.57$ from Ref.\textsuperscript{24} and $A(T = 0) = 2(2\pi)^2 B = 0.51$ from Ref.\textsuperscript{27}, shown together with the value $A^{(2)}$ obtained by the present FRG method.
(with $\overline{Z_n} = \int D\mu e^{-S_0[\mu]}$):

$$S_0[\mu] = \frac{H_{\text{rep}}[\mu]}{T} = \frac{\kappa}{2T} \sum_a \int \langle u_0^a u_{-q}^a (G_0^q)^{-1} \rangle$$

$$- \frac{1}{2T^2} \sum_{ab} \int \left[ R_0(u_{r}^{ab}) - \frac{1}{2} \nabla u_{r}^{ab} \nabla u_{r}^{ab} G_0(u_{r}^{ab}) \right]$$

with $u_{r}^{ab} = u_0^a - u_0^b$, $\int = \int d^d r$, $\int_q = \int d^d q/(2\pi)^d$. The bare propagator is $G_0^q = q^{-2}c(q^2/2\Lambda_r^2, q^2/2\Lambda_l^2)$ where $\Lambda_0$ and $\Lambda_l = \Lambda_0 e^{-t}$ are UV and IR cutoffs and $c(z,s)$ is a smooth cut-off function. We denote $c(z,\infty) = 1 - c(z)$ with $c(0) = 1$ and $c(\infty) = 0$. At the bare level, $R_0(u) = g_0 \cos(u)$ and $G_0(u) = \sigma$ in $\mathbb{R}^d$. The effective action $\Gamma[u]$ (i.e. the Legendre tranform of the free energy in presence of sources) admits a double expansion in number of replica (cumulants) and gradients, which starts as

$$\Gamma[u] = \frac{\kappa}{2T} \sum_a \int \langle u_0^a u_{-q}^a (G_0^q)^{-1} \rangle$$

$$- \frac{1}{2T^2} \sum_{ab} \int \left[ R_t(u_{r}^{ab}) - \frac{1}{2} \nabla u_{r}^{ab} \nabla u_{r}^{ab} G_t(u_{r}^{ab}) \right]$$

as in $\mathbb{R}^d$ with (periodic) renormalized function $R_t(u)$ and $G_t(u)$. Note that the STS symmetry constrains the form of the one replica term to be identical to the bare one. In $d = 2$ a $p$-periodic $R_t$ becomes relevant below $T = T^{(p)}_t = T_0 2\pi/p$ and at $T = 0$ is relevant below $d = 4$. Higher cumulants are irrelevant compared to $R_t$ near $d = 4$. In $d = 2$ a 1-loop approximation keeping only $R_t$ yields the exact $T^{(p)}_t$ and reasonable properties at low $T$ (see [20,23]). A specificity of $d = 2$, not systematically studied with the FRG (see however [31,32]) is that the term $G_t$ must be retained. Indeed at $T = 0$ and $d = 2$ the whole function $G_t(u)$ is dimensionless at the bare level, and acquires anomalous dimensions due to temperature and disorder. The 2-point correlation function reads:

$$\langle u_q u_{-q} \rangle = \frac{T}{\kappa} G_t^q + \frac{G_t^q}{\kappa^2} (-R_t'(0) + G_t(0)q^2)$$

Near $T_y$, it was shown that $G_t(0) \sim l$ which yields, setting $l = \log(1/q)$, the $\ln^2 |r|$ behavior in $B(r)$ (2). We now investigate how a similar behavior, $G_t(0) \propto l$ can arise also at low $T$ where, as for $R_t(u)$, all harmonics in $G_t(u)$ must be kept. We first set $\kappa = 1$ in Eq. (20). Defining:

$$\tilde{T} = S_d A_t^{-d-2} T$$

$$\tilde{R}_t(u) = S_d A_t^{-d-4} R_t(u)$$

$$\tilde{G}(u) = S_d A_t^{-d-2} G_t(u)$$

with $S_d = (2\pi)^{-1}$, the coupled RG equations for $\tilde{R}_t$ and $\tilde{G}$ have the structure

$$\partial_t \tilde{T} = (\epsilon - 2) \tilde{T} , \quad \partial_t \tilde{R}_t(u) = \beta_{\tilde{R}}(u)$$

$$\partial_t \tilde{G}(u) = (\epsilon - 2) \tilde{G}(u) + \tilde{\beta}_{\tilde{G}}(u)$$

with $\epsilon = 4 - d, \beta_{\tilde{R}}(u) \equiv \beta_{\tilde{R},\tilde{G}}[\tilde{R}_t, \tilde{G}'', \tilde{G}, \dot{\tilde{G}}']$ where $\tilde{R}_t(u) = \tilde{R}_t''(u) = 0$ and $\tilde{G}(u) = \tilde{G}(u) - \delta_G(0)$. Under the following assumptions: i) $\tilde{G}(u)$ is continuous in $u = 0$, ii) $\tilde{\beta}_{\tilde{G}}(u)$ has a good limit when $u \to 0^+$:

$$\tilde{\beta}_{\tilde{G}}(0^+) = A_d$$

and iii) $\tilde{G}(u)$ has a good fixed point for all $d$ one obtains that $\tilde{G}(0) = \tilde{A}_d/(2 - \epsilon)$, where $\tilde{A}_d$ admits an $\epsilon$-expansion around $d = 4$. Thus $\tilde{G}(0)$ is diverging for $\epsilon \to 2$, yielding the announced behavior in $d = 2, \tilde{G}(0) = \tilde{A}_2$. Note that this implies:

$$\langle u_q u_{-q} \rangle \sim \frac{G(0)}{q^2} \sim \frac{A_2}{S_2} \ln(1/q)$$

hence, going back to real space $A(T = 0) = A_2$.

B. Lowest order calculation

We first compute, at $T = 0$, $\beta_{\tilde{R}}(u)$ to $\tilde{G}_0(u)$ (see Appendix [A]). From one loop 1PI graphs one finds $\Gamma[u] = S_0[u] + \delta\Gamma[u]$, $\delta\Gamma[u] = -\frac{1}{4T^2} \sum_{ab} \int \langle G_t^{-(r)} \rangle^2 R_0(u)^2 R_0(u)^2$ excluding terms proportional to $T$ ( tadpoles and some 3-replica terms). Differentiating the local component of $\delta\Gamma[u]$ in $\epsilon$ w.r.t. the IR cutoff $\Lambda_t$ at fixed $R_0$, and inverting $R_0 = \tilde{R}_t + O(R_t^2)$, one obtains the standard $T = 0$ 1-loop FRG equation for $\tilde{R}_t$ in $d = 4 - \epsilon$:

$$\partial_t \tilde{R}_t(u) = \epsilon \tilde{R}_t(u) + (1/2)(\tilde{R}_t''(u))^2 + O(\tilde{R}_t^3, \tilde{G}^3)$$

with $\tilde{R}_t(u) = 2I_{00} \tilde{R}_t(u)$ where

$$I_{nm} = S_d^{-1} \Lambda_t^{-d+4+n-m} \int_y y^n \partial y \Gamma_t^g y^m \Gamma_t^g y^r$$

for $n, m$ even (see also Appendix [B]). Eq. (31) admits a non-analytic, $p$-periodic fixed point solution given by

$$\tilde{R}^*(pu) = (ep^4/(72))(1/36 - u^2(1 - u)^2)$$

with $u \in [0, 1]$, yielding the amplitude of the cusp:

$$\tilde{R}'''(0^+) = -\tilde{R}''''(0^-) = pe/6$$

Next, the bilocal component of $\delta\Gamma[u]$ in $\epsilon$ when expanded in gradients generates a term $\nabla u_x^a \nabla u_x^b G_t(u_x^a)$. Proceeding as above one obtains (see Appendix [A]) up to $O(\tilde{R}_t^3, \tilde{G}^3)$:

$$\partial_t \tilde{G}_t(u) = (\epsilon - 2) \tilde{G}_t(u) + (I_{20}/d) \tilde{R}_t''''(u)^2$$

Eq. (33) yields, in $d = 2$, $\tilde{G}_t(0) \sim (I_{20}/2)(\tilde{R}_t''''(0))^2/2!$ which has no non ambiguous value. It also leads to:

$$\tilde{G}_t(u) = (I_{20}/d)((\tilde{R}_t''''(u))^2 - (\tilde{R}_t''''(0))^2)/(2 - \epsilon)$$
thus satisfying the requirement iii) above for $\epsilon < 2$. As shown below, the divergence for $\epsilon = 2$ is cured by considering higher order terms in Eq. (38). From (23), (38) a first estimate $A^{(1)}$ of $A(T = 0)$ in (2) up to order $O(R^2)$ is

$$A^{(1)} = \frac{I_{20}}{2} (\tilde{R}''(0^+))^2 = \frac{p^2 e^2 I_{20}}{288} \frac{I_{20}^2}{I_{00}}$$

(37)

As announced it is non zero at $T = 0$ only because the fixed point is non-analytic. The $p^2$ dependence in Eq. (37) yields that $A^{(1)}$ is independent of $\kappa$. Indeed one has $B_{\kappa,p}(r) = \kappa^{-1} B_{1,p}/\sqrt{T}(r)$ if $B_{\kappa,p}(r)$ denotes the correlation in (2) for model (1) with p-periodic potential (and $h = 0$). While $A_{d=2}$ is expected to be universal, our truncation (37) depends on the shape of the IR regulator $c(z)$, with local sensitivity minimum around the exponential, $c(x) = e^{-x}$ which yields $A^{(1)} = 0.881$.

C. Higher order calculation

We now push the $T = 0$ analysis further by computing $\beta_R, \beta_G$ up to order $O(RG, G^2)$, which are of order $O(1)$ in $d = 2$. From $\beta_R[u]$ to one loop, one finds:

$$\beta^2_R = \epsilon \bar{R} + I_{00} R \rho'' + 2 I_{00} \tilde{R}'' \tilde{G} + I_{00} \tilde{G}^2$$

(38)

$$\beta^2_G = (I_{20}/d) \tilde{R}'' \rho + \gamma_1 \tilde{R}'' \tilde{G} + 2 I_{00} \tilde{R}'' \tilde{G}$$

(39)

$$+ 2 I_{02} \tilde{G}'' + \gamma_2 \tilde{G}''$$

where (see Appendix [4] for more details):

$$\gamma_1 = 4 (I_{00} + K_0/d + I_{22}/2d)$$

(40)

$$\gamma_2 = 6 I_{02} + I_{24}/d + K_2/d$$

(41)

$$S^{-1} \tilde{G}'' \rho = -\frac{1}{\bar{G}^4} \tilde{G}'' \rho$$

(42)

Remarkably, (38), (39) admit a simple solution:

$$\tilde{R}''(pu) = p^2 a_d u(1 - u), \quad \tilde{G}(pu) = p^2 a_d u(1 - u)$$

(43)

with:

$$a_d = \frac{\epsilon}{12 (I_{00} + 2\alpha I_{02} + \alpha^2 I_{04})}$$

(44)

and $\epsilon = 4 - d$. Eqs. (39), (43) satisfy all above requirements i), ii), iii). It yields a value for $A_d$, up to order $O(RG, \tilde{G}^2)$ at one loop, for a $p$-periodic potential:

$$A_d = \frac{p^2 e^2 I_{20}/d + \alpha \gamma_1 + \alpha^2 \gamma_2}{444 (I_{00} + 2\alpha I_{02} + \alpha^2 I_{04})^2}$$

(45)

yielding back (37) if $\alpha$ is (formally) set to zero. Here $\alpha$ is the solution of a cubic equation for $\epsilon < 2$ (with $\alpha = O(\epsilon)$ near $d = 4$), and of a quadratic equation in $d = 2$

$$\alpha^2 (I_{02} + \gamma_2) + \alpha (I_{00} + \gamma_1) + \frac{I_{20}}{2} = 0$$

(46)

Among the two solutions of Eq. (46), only one is physical, and yields $\tilde{G}(0) \propto A_d$ and an estimate $A^{(2)}$ of $A(T = 0)$ up to order $O(RG, \tilde{G}^2)$ at one loop. Using relations valid in $d = 2$ it can also be written as:

$$A^{(2)} = -\frac{p^2 e^2 \alpha}{144} \frac{I_{00} + 2\alpha I_{02} + \alpha^2 I_{04})^2}$$

(47)

$A^{(2)}$ is independent of $\kappa$ and rather stable as the IR cutoff $c(z)$ is varied. A Gaussian $c'(z) \propto e^{-(x-1/2)^2/(2\sigma^2)}$ reaches a local maximum with $\sigma = 1/4$ and $A^{(2)} = 0.29$. Thus the numerical results, $A(T = 0) = a_d = 0.57$ of Ref. 22 (notice that $a_d$ refers here to the notation used in Ref. 22 and is different from $a_d = 2$ as in our Eq. (14)) and $A(T = 0) = 2(2\pi)^2 B = 0.51$ of Ref. 22 seem to lie in between our estimates $A^{(1)}, A^{(2)}$ (see Fig. 1).

D. Extension to low temperature

To be convincing the scenario should be stable to a small temperature $T$. To one loop (27) become:

$$\dot{\tilde{R}}(u) = \tilde{T} \tilde{R}''(u) - J_2 \tilde{T} \tilde{G}(u) + \beta^T_R (u)$$

(48)

$$\dot{\tilde{G}}(u) = (\epsilon - 2) \tilde{G}(u) + \beta^T_G (u)$$

(49)

where $J_2 = (S_d)^{-1} \Lambda_d^{-4} \tilde{G}'' \tilde{G}'^2$. At low $T$, and dimension $d = 2$, we look for a solution of Eq. (49) where for $u \gg T$ the functions $\tilde{R}$ and $\tilde{G}$ are given by the $T = 0$ solution (43), while for small $u \sim T$ there is a thermal boundary layer (TBL) solution of the form

$$\tilde{R}''(u) = T \rho''(u/T), \quad \tilde{G}(u) = A_d + c + T g(u/T)$$

(50)

where, with no loss of generality, $g(0) = r''(0) = 0$. The functions $r''(z)$ and $g(z)$ satisfy the coupled equations (for $p = 1$)

$$0 = -a_d^2 z^2 + r'' - J_2 g + I_{00} r'' + 2 I_{02} r'' g + I_{04} g^2$$

(51)

$$A_2 = g'' + \frac{I_{20}}{d} r''' + \gamma_1 r'' g' + 2 I_{00} r'' g'' + 2 I_{02} g''$$

and $\epsilon = 4 - d$. Eqs. (39), (43) satisfy all above requirements i), ii), iii). It yields a value for $A_d$, up to order $O(RG, \tilde{G}^2)$ at one loop, for a $p$-periodic potential:

yielding back (37) if $\alpha$ is (formally) set to zero. Here $\alpha$ is the solution of a cubic equation for $\epsilon < 2$ (with $\alpha = O(\epsilon)$ near $d = 4$), and of a quadratic equation in $d = 2$

$$\alpha^2 (I_{02} + \gamma_2) + \alpha (I_{00} + \gamma_1) + \frac{I_{20}}{2} = 0$$

(46)

which match correctly the cusp (43) outside the TBL. The constants are $r_0 = (J_2 I_{00} + I_{04})/(2 I_{02} - 2 I_{04} I_{00})$ and $g_0 = -(1 + 2r_0 I_{00})/(2 I_{02})$.

We have solved numerically these RG equations (41) and checked that it admits a well defined solution: this is depicted in Fig. 2 where we plot $r''(z)/a_d z$ (respectively $g(z)/a_d z$) as a function of $z$ in Fig. 2a (respectively
To conclude we applied the Functional RG to the random phase Sine-Gordon model, when gradient expansion is important. We showed that the super-roughening amplitude \( A(T) \) does not vanish at \( T = 0 \) thanks to the non-analyticity of the fixed point function \( R(u) \), which avoids dimensional reduction. The one loop estimates give an order of magnitude consistent with numerics. Our study suggests that either the mapping between free fermions model and the CO model fails below some temperature or that new operators become relevant, e.g. higher harmonics, as is the case for interacting fermions models. Such a scenario was demonstrated in random field spin models. Whether the path integral transformations of Ref. hold at low \( T \) needs to be clarified. A procedure exists to measure the FRG functions described here, in numerics and check the scenario. We hope that it allows progress in issues common to fermion models, localization and periodic pinned systems.

**IV. CONCLUSION**

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**APPENDIX A: GRADIENT EXPANSION**

The calculation of the effective action up to order \( R_0^2 \) yields:

\[
\delta \Gamma[u] = -\frac{1}{2 T^2} \sum_{ab} \int_x \left\langle R_0(u_{ab}^x + \delta u_{ab}^x) \right\rangle + \frac{1}{4 T^2} \sum_{abcd} \int_{xy} \left\langle [R_0(u_{ab}^x + \delta u_{ab}^x)R_0(u_{ac}^y + \delta u_{ac}^y)] \right\rangle 
\]

where \( R''_0(u) = R''_0(u) - R''_0(0) \). The last term is a third cumulant proportional to temperature, and here we ignore it. We now expand the second term in gradients.

\[
\int_{xy} F(u_x, u_y, y - x) \quad (A3)
\]

Consider a general bilocal functional:
We can either use \( z = (x + y)/2, t = y - x \) and expand everything to second order in \( t \):

\[
\int z \ F(u_z + \frac{1}{2}(t \cdot \nabla)u_z + \frac{1}{8}(t \cdot \nabla)^2 u_z, u_z - \frac{1}{2}(t \cdot \nabla)u_z + \frac{1}{8}(t \cdot \nabla)^2 u_z, t)
\]

\[
\int z \frac{1}{2}(t \cdot \nabla)u_z(t \cdot \nabla)u_z(\partial_1 - \partial_2)^2 F(u_z, u_z, t) + \frac{1}{8}(t \cdot \nabla)(t \cdot \nabla)u_z(\partial_1 + \partial_2) F(u_z, u_z, t)
\]

Using also that \( \int_z F(u, u, t) = 0 \) (we assume that the local part has already been subtracted) and performing integration by parts to get rid of the \( \nabla \nabla u F(u, u, t) \) term, we obtain a term containing \( (\partial_1 - \partial_2)^2 - (\partial_1 + \partial_2)^2 = -4\partial_1 \partial_2 \) and thus finally one obtains the gradient expansion as:

\[
\int x \ y \ F(u_x, u_y, y - x)
\]

\[
= -\frac{1}{2}(t \cdot \nabla)u_z(t \cdot \nabla)u_z(\partial_1 \partial_2) F(u_z, u_z, t) \quad (A6)
\]

Alternatively one can also perform the expansion as

\[
\int x \ y \ F(u_x, u_y, y - x)
\]

\[
= \int x y \ F(u_x, u_x + (t \cdot \nabla)u_x + \frac{1}{2}(t \cdot \nabla)^2 u_x, t)
\]

\[
= \int \frac{1}{2}[(t \cdot \nabla)u_z(t \cdot \nabla)u_z(\partial_2)^2
\]

\[
+ (t \cdot \nabla)(t \cdot \nabla)u_z(\partial_2) F(u_z, u_z, t)
\]

and integration by parts then yields the same result as above. Thus one finds that the part of interest of the effective action is:

\[
\delta \Gamma[u] = \frac{1}{8T^2b} \sum_{ab} \int_x R^{ab}(u_x) R^{ab}(u_x)
\]

\[
\times (G^2 - \delta(t) \int \nabla^2)u_x^a(t \cdot \nabla)u_x^b \quad (A9)
\]

yielding the result given in the text.

**APPENDIX B: CALCULATION OF INTEGRALS**

In this appendix, we give the detailed calculation of the coefficients \( I_{m,n}, K_n \) entering the RG equations given in the text. One introduces the notation \( g^y = \Lambda_{x^2}^{d-2} \gamma^y \). Focusing on the large \( l \) limit, one defines \( g^y = g^y \) and \( \partial_y \) as:

\[
g^y = \int \frac{e^{iyq}}{q^2} \left[ 1 - c(q^2/2) \right] \quad , \quad \partial_y = -\int \frac{c'(q^2/2)e^{iyq}}{q^2}
\]

Let us start with \( I_{00} \):

\[
I_{00} = \int \frac{1}{S_d} \int y \partial_y g^y = -\frac{1}{S_d} \int \int \frac{c'(q^2/2)}{q^2}(1 - c(q^2/2))
\]

\[
= -\frac{1}{2} \int \int \frac{c'(u)}{u}(1 - c(u)) \quad \text{in} \ d = 2 \quad (B2)
\]

In a similar way, one computes the coefficients (in \( d = 2 \))

\[
I_{02} = -\frac{1}{2} \quad (B3)
\]

\[
I_{04} = -2 \int_0^\infty du \int c'(u)(1 - c(u)) \quad (B4)
\]

Next, we compute \( I_{20} \)

\[
I_{20} = \frac{1}{S_d} \int y^2 \partial_y g^y g^p
\]

\[
= \frac{2}{S_d} \int \int \frac{c''(q^2/2)(1 - c(q^2/2))}{q^2} + \frac{1}{S_d} \int c''(q^2/2)(1 - c(q^2/2))
\]

\[
= \int_0^\infty du(1 - c(u))(c''(u)/u + c'''(u)) \quad \text{in} \ d = 2 \quad (B5)
\]

Similarly, one has (in \( d = 2 \))

\[
I_{22} = -2 \int_0^\infty du(1 - c(u))(c''(u) + uc'''(u)) \quad (B6)
\]

\[
I_{24} = 4 \int_0^\infty du(1 - c(u))(uc''(u) + u^2 c'''(u)) \quad (B7)
\]

Finally one has

\[
K_0 = \frac{1}{S_d} \int y \partial_y g^y (\nabla^3 g)^y g^p
\]

\[
= -\frac{1}{S_d} \int c''(q^2/2)(1 - c(q^2/2)) \quad (B8)
\]

\[
= \int_0^\infty du c''(u)(1 - c(u)) \quad \text{in} \ d = 2 \quad (B9)
\]

and similarly, in \( d = 2 \)

\[
K_2 = 8 \int_0^\infty du c''(u)(1 - c(u)) \quad (B10)
\]
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