BURGHELEA-FRIEDLANDER-KAPPELER’S GLUING FORMULA AND THE ADIABATIC DECOMPOSITION OF THE ZETA-DETERMINANT OF A DIRAC LAPLACIAN.

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Abstract. In this paper we first establish the relation between the zeta-determinant of a Dirac Laplacian with the Dirichlet boundary condition and the APS boundary condition on a cylinder. Using this result and the gluing formula of the zeta-determinant given by Burghelea, Friedlander and Kappeler with some assumptions, we prove the adiabatic decomposition theorem of the zeta-determinant of a Dirac Laplacian. This result was originally proved by J. Park and K. Wojciechowski in [11] but our method is completely different from the one they presented.

§1 Introduction

Let $M$ be a compact oriented $m$-dimensional Riemannian manifold and $E \to M$ be a Clifford module bundle. Suppose that $\mathcal{D}_M$ is a Dirac operator acting on smooth sections of $E$. Then $\mathcal{D}_M^2$ is a non-negative self-adjoint elliptic differential operator, which is called a Dirac Laplacian. From the standard elliptic theory it is well-known that the spectrum of $\mathcal{D}_M^2$ is discrete and tends to infinity. We define the zeta function associated to $\mathcal{D}_M^2$ by

$$\zeta_{\mathcal{D}_M^2}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr} e^{-t \mathcal{D}_M^2} dt,$$

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which is holomorphic for $Re(s) > \frac{m}{2}$. $\zeta_{D_M^2}(s)$ admits a meromorphic continuation to the whole complex plane having a regular value at $s = 0$. We define the zeta-determinant by

$$DetD_M^2 = e^{-\zeta_{D_M^2}'(0)}.$$  

We suppose that $Y$ is a hypersurface of $M$ such that $M - Y$ has two components and $N$ is a collar neighborhood of $Y$ which is diffeomorphic to $[-1,1] \times Y$. Choose a metric $g$ on $M$ which is a product metric on $N$. We now assume the product structures of the bundle $E$ and the Dirac operator $D_M$ in the following sense. We first assume that $E|_N = p^*E|_Y$, where $p : [-1,1] \times Y \rightarrow Y$ is the canonical projection. We also assume that $D_M$ is of the form on $N$

$$D_M = G(\partial_u + B),$$  

where $G : E|_Y \rightarrow E|_Y$ is a bundle automorphism, $\partial_u$ is the normal derivative to $Y$ in the usual direction and $B$ is a Dirac operator on $Y$. Here, we assume that $G$ and $B$ do not depend on the normal coordinate $u$ and satisfy

$$G^* = -G, \quad G^2 = -Id, \quad B^* = B \quad \text{and} \quad GB = -BG.$$  

Then we have, on $N$,

$$D_M^2 = -\partial_u^2 + B^2.$$  

We denote by $M_1, M_2$ the closure of each component of $M - Y$ so that $M_1$ ($M_2$) contains the part $[-1,0] \times Y$ ($[0,1] \times Y$). We also denote by $D_{M_1}, D_{M_2}$ the restriction of $D_M$ to $M_1, M_2$, respectively. In this paper we discuss the decomposition of $DetD_M^2$ into contributions coming from $M_1, M_2$ and the hypersurface $Y$ in the adiabatic sense. To do this we consider the following adiabatic setting.

We denote by $M_r$ the compact manifold without boundary obtained by attaching $N_{r+1} = [-r - 1, r + 1] \times Y$ on $M - (-\frac{1}{2}, \frac{1}{2}) \times Y$ by identifying $[-1, -\frac{1}{2}) \times Y$ with $[-r - 1, -r - \frac{1}{2}) \times Y$ and $[\frac{1}{2}, 1] \times Y$ with $[r + \frac{1}{2}, r + 1] \times Y$. We also denote by $M_{1_r}, M_{2_r}$ the manifolds with boundary which are obtained by attaching $[-r, 0] \times Y, [0, r] \times Y$ on $M_1, M_2$ by identifying $\partial M_1$ with $Y_{-r} := \{-r\} \times Y$ and $\partial M_2$ with $Y_r := \{r\} \times Y$, respectively. Then the bundle $E \rightarrow M$ and the Dirac operator $D_M$ on $M$ can be extended naturally to the bundle $E_r \rightarrow M_r$ and the Dirac operator $D_{M_r}$ on $M_r$. We also denote by $D_{M_{1_r}}, D_{M_{2_r}}$ ($E_{1_r}, E_{2_r}$) the restriction of $D_{M_r}$ ($E_r$) to $M_{1_r}, M_{2_r}$, respectively and by $D_{M_{1_r}, D_0}$ the Dirac Laplacian $D_{M_{1_r}}^2$ on $M_i$ with the Dirichlet boundary condition on $Y_0 := \{0\} \times Y$, i.e., $\text{Dom}(D_{M_{i_r}, D_0}) = \{\phi \in C^\infty(M_r) \mid \phi|_{Y_0} = 0\}$.

We define the operators $Q_i : C^\infty(Y) \rightarrow C^\infty(Y)$ ($i = 1, 2$) as follows. For $f \in C^\infty(Y)$, choose $\phi_i \in C^\infty(M_i)$ satisfying $D_{M_i}^2 \phi_i = 0$ and $\phi_i|_Y = f$. Then we define

$$Q_1(f) = (\partial_u \phi_1)|_Y, \quad Q_2(f) = (-\partial_u \phi_2)|_Y.$$  

(1.1)

We show in Proposition 4.5 that if both $Q_1 + \sqrt{B^2}$ and $Q_2 + \sqrt{B^2}$ are invertible, then $B$ is invertible for each $r > 0$ and $D_{M_r}$ is invertible for $r$ large enough. The following is the main result of [9] given by the author.
Theorem 1.1. Let $M$ be a compact oriented Riemannian manifold having the product structures near a hypersurface $Y$. We assume that both $Q_1 + \sqrt{B_2}$ and $Q_2 + \sqrt{B_2}$ are invertible operators. Then :

$$\lim_{r \to \infty} \left\{ \log Det(D^2_{M,r}) - \log Det(D^2_{M_1,r,D_0}) - \log Det(D^2_{M_2,r,D_0}) \right\} = \frac{1}{2} \log Det(B^2).$$

Remark : We denote $M_{1,\infty} = M_1 \cup_{\partial M_1} [0, \infty) \times Y$, $M_{2,\infty} = M_2 \cup_{\partial M_2} (-\infty, 0] \times Y$ and by $D_{M_{i,\infty}}$ ($i = 1, 2$) the natural extension of $D_{M_{i,r}}$ to $M_{i,\infty}$. Then we prove in Proposition 4.5 that the invertibility of both $Q_1 + \sqrt{B_2}$ and $Q_2 + \sqrt{B_2}$ is equivalent to the non-existence of extended $L^2$-solutions (see Definition 4.4) of $D_{M_{1,\infty}}$ and $D_{M_{2,\infty}}$ on $M_{1,\infty}$ and $M_{2,\infty}$.

The purpose of this paper is to establish a same type of formul a as Theorem 1.1 with the Atiyah-Patodi-Singer boundary condition (APS condition) instead of the Dirichlet boundary condition.

Recall that $B$ is a Dirac operator on $Y$ and its spectrum is distributed from negative infinity to positive infinity. Denote by $P_\prec (P_\succ)$, $P_\preceq (P_\succeq)$ the projections from $C^\infty(Y)$ to negative (positive) and non-negative (non-positive) eigensections of $B$, respectively. Then $D_{M_1,r}, P_\prec$ and $D_{M_2,r}, P_\preceq$ are defined by the same operators $D_{M_1,r}, D_{M_2,r}$ with

$$Dom(D_{M_1,r}, P_\prec) = \{ \phi \in C^\infty(M_1,r) \mid P_\prec(\phi|_Y) = 0 \},$$

$$Dom(D_{M_2,r}, P_\preceq) = \{ \phi \in C^\infty(M_2,r) \mid P_\preceq(\phi|_Y) = 0 \}.$$ 

Similarly, $D^2_{M_1,r}, P_\prec$ := $(D_{M_1,r}, P_\prec)(D_{M_1,r}, P_\preceq)$ and $D^2_{M_2,r}, P_\preceq$ := $(D_{M_2,r}, P_\preceq)(D_{M_2,r}, P_\preceq)$ are defined by the same operators $D^2_{M_1,r}, D^2_{M_2,r}$ with

$$Dom(D^2_{M_1,r}, P_\prec) = \{ \phi \in C^\infty(M_1,r) \mid P_\prec(\phi|_Y) = 0, P_\preceq((\partial_u \phi + B \phi)|_Y) = 0 \},$$

$$Dom(D^2_{M_2,r}, P_\preceq) = \{ \phi \in C^\infty(M_2,r) \mid P_\preceq(\phi|_Y) = 0, P_\succ((\partial_u \phi + B \phi)|_Y) = 0 \}.$$ 

$D_{M_{i,r}, P_\prec}$, $D_{M_{i,r}, P_\preceq}$, $D^2_{M_{i,r}, P_\prec}$ and $D^2_{M_{i,r}, P_\preceq}$ are defined similarly.

Put $N_{-r,0} = [-r, 0] \times Y$ and $N_{0,r} = [0, r] \times Y$. Then from the decomposition

$$M_{1,r} = M_1 \cup_{Y_{-r}} N_{-r,0}, \quad M_{2,r} = M_2 \cup_{Y_{r}} N_{0,r},$$

we have the following theorem, which we call Burghela-Friedlander-Kappeler’s gluing formula and refer to [9] for the proof (see also [4], [8]).
Theorem 1.2. Suppose that $k = \text{dim} \text{Ker} B$. Then:

1. $\log \text{Det} \mathcal{D}^2_{M_1,r,D_0} = \log \text{Det} \mathcal{D}^2_{M_1,D_{-r}} + \log \text{Det}(-\partial_u^2 + B^2)_{N_{-r,0},D_{-r},D_0} - \log 2 \cdot (\zeta_{B^2}(0) + k) + \log \text{Det} R_{M_1,r,D_{-r}}.$

2. $\log \text{Det} \mathcal{D}^2_{M_2,r,D_0} = \log \text{Det} \mathcal{D}^2_{M_2,D_r} + \log \text{Det}(-\partial_u^2 + B^2)_{N_0,r,D_0,D_r} - \log 2 \cdot (\zeta_{B^2}(0) + k) + \log \text{Det} R_{M_2,r,D_r}.$

Here the Dirichlet-to-Neumann operators $R_{M_1,r,D_{-r}} : C^\infty(Y_{-r}) \to C^\infty(Y_{-r})$ and $R_{M_2,r,D_r} : C^\infty(Y_r) \to C^\infty(Y_r)$ are defined as follows. For $f \in C^\infty(Y_{-r})$ and $\tilde{f} \in C^\infty(Y_r)$, choose $\phi \in C^\infty(M_1)$, $\psi \in C^\infty(N_{-r,0})$, $\tilde{\phi} \in C^\infty(M_2)$, $\tilde{\psi} \in C^\infty(N_{0,r})$ so that

\begin{align*}
\mathcal{D}^2_{M_1} \phi &= 0, \\ (-\partial_u^2 + B^2) \psi &= 0, \\ \phi|_{Y_{-r}} &= \psi|_{Y_{-r}} = f, \\ \psi|_{Y_0} &= 0,
\end{align*}

\begin{align*}
\mathcal{D}^2_{M_2} \tilde{\phi} &= 0, \\ (-\partial_u^2 + B^2) \tilde{\psi} &= 0, \\ \tilde{\phi}|_{Y_r} &= \tilde{\psi}|_{Y_r} = \tilde{f}, \\ \tilde{\psi}|_{Y_0} &= 0.
\end{align*}

Then we define:

$$R_{M_1,r,D_{-r}}(f) := (\partial_u \phi)|_{Y_{-r}} - (\partial_u \psi)|_{Y_{-r}} = Q_1(f) - (\partial_u \psi)|_{Y_{-r}},$$

$$R_{M_2,r,D_r}(\tilde{f}) := -(\partial_u \tilde{\phi})|_{Y_r} + (\partial_u \tilde{\psi})|_{Y_r} = Q_2(\tilde{f}) + (\partial_u \tilde{\psi})|_{Y_r}.$$ 

The operator $(-\partial_u^2 + B^2)_{N_{-r,0},D_{-r},D_0}$ is the Laplacian $(-\partial_u^2 + B^2)$ on $N_{-r,0}$ with the Dirichlet condition on $Y_{-r}$, $Y_0$ and $(-\partial_u^2 + B^2)_{N_0,r,D_0,D_r}$ is defined similarly.

By replacing the Dirichlet condition on $Y_0$ by the APS condition, we obtain the following theorem, which can be proved by the exactly same way as Theorem 1.2 without any modification.

Theorem 1.3. Suppose that $\mathcal{D}^2_{M_1,r,P_<}$ and $\mathcal{D}^2_{M_2,r,P_\geq}$ are invertible operators and $k = \text{dim} \text{Ker} B$. Then:

1. $\log \text{Det} \mathcal{D}^2_{M_1,r,P_<} = \log \text{Det} \mathcal{D}^2_{M_1,D_{-r}} + \log \text{Det}(-\partial_u^2 + B^2)_{N_{-r,0},D_{-r},P_<} - \log 2 \cdot (\zeta_{B^2}(0) + k) + \log \text{Det} R_{M_1,r,P_<}.$

2. $\log \text{Det} \mathcal{D}^2_{M_2,r,P_\geq} = \log \text{Det} \mathcal{D}^2_{M_2,D_r} + \log \text{Det}(-\partial_u^2 + B^2)_{N_0,r,P_\geq,D_r} - \log 2 \cdot (\zeta_{B^2}(0) + k) + \log \text{Det} R_{M_2,r,P_\geq}$.
Here \((-\partial^2_u + B^2)_{N_r,0,D_r,P_\prec}\) is the operator \(-\partial^2_u + B^2\) on \(N_{-r,0}\) with the Dirichlet boundary condition on \(Y_{-r}\) and the APS condition \(P_\prec\) on \(Y_0\), i.e.,

\[
\text{Dom}((-\partial^2_u + B^2)_{N_r,0,D_r,P_\prec}) = \\
\{ \phi \in C^\infty(N_{-r,0}) | \phi|_{Y_{-r}} = 0, P_\prec(\phi|_{Y_0}) = 0, P_\succ((\partial u \phi + B \phi)|_{Y_0}) = 0 \},
\]

and \((-\partial^2_u + B^2)_{N_0,r,P_\prec,D_r}\) is defined similarly. The operators \(R_{M_1,r,P_\prec} : C^\infty(Y_{-r}) \to C^\infty(Y_{-r})\) and \(R_{M_2,r,P_\succ} : C^\infty(Y_r) \to C^\infty(Y_r)\), which are the Dirichlet-to-Neumann operators with the APS condition on \(Y_0\), are defined as follows. For \(f \in C^\infty(Y_{-r})\) and \(\tilde{f} \in C^\infty(Y_r)\), choose \(\phi \in C^\infty(M_1), \psi \in C^\infty(N_{-r,0}), \tilde{\phi} \in C^\infty(M_2), \tilde{\psi} \in C^\infty(N_{0,r})\) satisfying

\[
\mathcal{D}^2_{M_1}\phi = 0, \quad (-\partial^2_u + B^2)\psi = 0, \quad \phi|_{Y_{-r}} = \psi|_{Y_{-r}} = f, \quad P_\prec(\psi|_{Y_0}) = P_\succ((\partial u \psi + B \psi)|_{Y_0}) = 0,
\]

\[
\mathcal{D}^2_{M_2}\tilde{\phi} = 0, \quad (-\partial^2_u + B^2)\tilde{\psi} = 0, \quad \tilde{\phi}|_{Y_r} = \tilde{\psi}|_{Y_r} = \tilde{f}, \quad P_\succ(\tilde{\psi}|_{Y_0}) = P_\prec((\partial u \tilde{\psi} + B \tilde{\psi})|_{Y_0}) = 0.
\]

Then we define

\[
R_{M_1,r,P_\prec}(f) := (\partial u \phi)|_{Y_{-r}} - (\partial u \psi)|_{Y_{-r}} = Q_1(f) - (\partial u \psi)|_{Y_{-r}},
\]

\[
R_{M_2,r,P_\succ}(\tilde{f}) := -(\partial u \tilde{\phi})|_{Y_r} + (\partial u \tilde{\psi})|_{Y_r} = Q_2(\tilde{f}) + (\partial u \tilde{\psi})|_{Y_r}.
\]

Now we discuss a relation on the cylinder part between the zeta-determinant with the Dirichlet condition and the APS condition. With the help of Theorem 1.2 and Theorem 1.3, it is enough to consider the terms coming from the cylinder parts \([-r,0] \times Y\) and \([0,r] \times Y\). To describe the main result of this paper, we define the operator \(Q_{(\partial u + |B|),r} : C^\infty(Y_0) \to C^\infty(Y_0)\) in the following way, which is suggested in [6]. For \(f \in C^\infty(Y_0)\), choose \(\phi \in C^\infty(N_{-r,0})\) satisfying

\[
(-\partial^2_u + B^2)\phi = 0, \quad \phi|_{Y_{-r}} = 0, \quad \phi|_{Y_0} = f.
\]

Then we define

\[
Q_{(\partial u + |B|),r}(f) := (\partial u \phi + |B| \phi)|_{Y_0}.
\]

We can check easily by direct computation that \(Q_{(\partial u + |B|),r}\) is a positive operator (Lemma 4.1, (5)) and have the following theorem, which is proved in Section 2 and 3 by modifying the proof of Theorem 1.2.
Theorem 1.4.

$$\log \text{Det}(-\partial_u^2 + B^2)_{N-r,0,D-r,P<} + \log \text{Det}(-\partial_u^2 + B^2)_{N_0,r,P_>,D_r} - \log \text{Det}(-\partial_u^2 + B^2)_{N_0,r,D_0,D_r} = \log \text{Det}Q(\partial_u + |B|),r.$$

Now we are ready to describe the main result of this paper. We prove in Proposition 4.5 that if each \(Q_i + \sqrt{B^2} \ (i = 1, 2)\) is invertible, then \(B, \mathfrak{D}_{M_1,r,P<}^2\) and \(\mathfrak{D}_{M_2,r,P>}^2\) are invertible for each \(r > 0\) and \(\mathfrak{D}_{M_r}^2\) is invertible for \(r\) large enough. Combining Theorem 1.1, 1.2, 1.3 and 1.4 with the Remark after Theorem 1.1, we have the following theorem, which is the main result of this paper.

**Theorem 1.5.** Let \(M\) be a compact oriented Riemannian manifold having the product structures near a hypersurface \(Y\). We assume that there are no extended \(L^2\)-solutions of \(\mathfrak{D}_{M_i,\infty}\) on \(M_i,\infty\) for \(i = 1, 2\). Then:

$$\lim_{r \to \infty} \{\log \text{Det} \mathfrak{D}_{M_r}^2 - \log \text{Det} \mathfrak{D}_{M_1,r,P<}^2 - \log \text{Det} \mathfrak{D}_{M_2,r,P>}^2\} = -\log 2 \cdot \zeta_{B^2}(0).$$

Theorem 1.5 was proved originally in [11] by Park and Wojciechowski on an odd dimensional compact oriented Riemannian manifold under the same assumption. They used the fact that \(\zeta_{\mathfrak{D}_{M_1,P>}^2}(0) = 0\) when \(\text{Ker} B = 0\) and \(\dim M\) is odd and then decomposed \(\text{Tr} e^{-t \mathfrak{D}_{M_r}^2}\) into contributions coming from \(M_1,r, M_2,r\) and a cylinder part plus some error terms. They proved that the error terms tend to zero as \(r \to \infty\) and finally computed the contribution coming from the cylinder part as \(r \to \infty\). Recently they improved their result in [12] by deleting the assumption of the non-existence of \(L^2\)-solutions on \(M_i,\infty\) \((i = 1, 2)\). For this work they strongly used the scattering theory developed in [10].

In this paper we, however, take different approach for the proof of Theorem 1.5. We are going to use Burghelea-Friedlander-Kappeler’s gluing formula established in case of the Dirichlet boundary condition. The original form of this formula contains a constant which can be expressed in terms of zero coefficients of some asymptotic expansions ([4], [8]). Under the assumption of the product structures near \(Y\), it is shown by the author in [9] that this constant is \(-\log 2 \cdot (\zeta_{B^2}(0) + \dim \text{Ker} B)\) and hence Theorem 1.2 and 1.3 are obtained. We are going to use this result intensively. Finally we use Theorem 1.4 to compare the case of the Dirichlet boundary condition with the APS boundary condition and use Theorem 1.1 to
compute the adiabatic limit as \( r \to \infty \). One of the advantages for this approach is that this method works in not only odd but also even dimensional manifolds.

\[ \text{§2 Proof of Theorem 1.4} \]

In this section we are going to prove Theorem 1.4. Throughout this section every computation will be done on the cylinders \([-r, 0] \times Y\) and \([0, r] \times Y\).

First of all, one can check by direct computation that the spectra of

\( (-\partial_u^2 + B^2)_{N_{-r,0},D_{-r},D_0} \), \( (-\partial_u^2 + B^2)_{N_{0,r},D_0,D_r} \), \( (-\partial_u^2 + B^2)_{N_{-r,0},D_{-r},P_<} \) and \( (-\partial_u^2 + B^2)_{N_{0,r},P_{q},D_r} \) are as follows.

1. \( \text{Spec} \left( (-\partial_u^2 + B^2)_{N_{-r,0},D_{-r},D_0} \right) = \text{Spec} \left( (-\partial_u^2 + B^2)_{N_{0,r},D_0,D_r} \right) = \)
   \[ \{ \lambda^2 + \left( \frac{k\pi}{r} \right)^2 \mid \lambda \in \text{Spec}(B), k = 1, 2, 3, \ldots \} \].

2. \( \text{Spec} \left( (-\partial_u^2 + B^2)_{N_{-r,0},D_{-r},P_<} \right) = \{ \mu_{\lambda,l} \mid \lambda \in \text{Spec}(B), \lambda \geq 0, \mu_{\lambda,l} > \lambda^2 \} \cup \)
   \[ \{ \lambda^2 + \left( \frac{k\pi}{r} \right)^2 \mid \lambda \in \text{Spec}(B), \lambda < 0, k = 1, 2, 3, \ldots \} \].

3. \( \text{Spec} \left( (-\partial_u^2 + B^2)_{N_{0,r},P_{<},D_r} \right) = \{ \mu_{\lambda,l} \mid \lambda \in \text{Spec}(B), \lambda > 0, \mu_{\lambda,l} > \lambda^2 \} \cup \)
   \[ \{ \lambda^2 + \left( \frac{k\pi}{r} \right)^2 \mid \lambda \in \text{Spec}(B), \lambda \geq 0, k = 1, 2, 3, \ldots \} \].

In (2) and (3) \( \mu_{\lambda,l} \)'s are the solutions of the equation

\[ \sqrt{\mu - \lambda^2} \cos(\sqrt{\mu - \lambda^2} r) + \lambda \sin(\sqrt{\mu - \lambda^2} r) = 0. \]

We next introduce the boundary condition \( \partial_u + |B| \) on \( Y_0 \) and consider the Laplacian \( (-\partial_u^2 + B^2)_{N_{-r,0},D_{-r},\partial_u + |B|} \) with

\[ \text{Dom} \left( (-\partial_u^2 + B^2)_{N_{-r,0},D_{-r},\partial_u + |B|} \right) = \]
\[ \{ \phi \in C^\infty(N_{-r,0}) \mid \phi|_{Y_{-r}} = 0, (\partial_u \phi + |B| \phi)|_{Y_0} = 0 \}. \]

Then the spectrum of this operator is:

4. \( \text{Spec} \left( (-\partial_u^2 + B^2)_{N_{-r,0},D_{-r},\partial_u + |B|} \right) = \{ \mu_{\lambda,l} \mid \lambda \in \text{Spec}(|B|), \mu_{\lambda,l} > \lambda^2 \} \),

where \( \mu_{\lambda,l} \)'s are the solutions of the equation

\[ \sqrt{\mu - \lambda^2} \cos(\sqrt{\mu - \lambda^2} r) + \lambda \sin(\sqrt{\mu - \lambda^2} r) = 0. \]
Hence from (1), (2), (3), (4), we have

\[
\zeta(-\partial_u^2 + B^2)_{N_{-r,0}, D_{-r}, p}(s) + \zeta(-\partial_u^2 + B^2)_{N_{0,r}, p, D_r}(s) \\
- \zeta(-\partial_u^2 + B^2)_{N_{-r,0}, D_{-r}, D_0}(s) - \zeta(-\partial_u^2 + B^2)_{N_{0,r}, D_0, D_r}(s) \\
= \zeta(-\partial_u^2 + B^2)_{N_{-r,0}, D_{-r}, (\partial_u + |B|)}(s) - \zeta(-\partial_u^2 + B^2)_{N_{-r,0}, D_{-r}, D_0}(s).
\]

Now we are going to use the method in [6] and [8] to analyze

\[
\log \text{Det}((-\partial_u^2 + B^2)_{N_{-r,0}, D_{-r}, (\partial_u + |B|)}) - \log \text{Det}((-\partial_u^2 + B^2)_{N_{-r,0}, D_{-r}, D_0}).
\]

From now on, we simply denote \((-\partial_u^2 + B^2), \partial_u + |B|\) by \(\Delta, C\) and the bundle \(E|_{N_{-r,0}}\) by \(E\). We first consider \(\Delta^l + t^l\) for any positive integer \(l > \left[\frac{m}{2}\right]\) and \(t > 0\) rather than \(\Delta\) itself because under some proper boundary condition \((\Delta^l + t^l)^{-1}\) is a trace class operator and in this case we are able to use the well-known formula about the derivative of \(\log \text{Det}(\Delta^l + t^l)\) with respect to \(t\) (c.f. (2.3)). For simplicity we put \(l = m\). Note that

\[
\Delta^m + t^m = \begin{cases} 
\prod_{k=-\left[\frac{m-1}{2}\right]}^{\left[\frac{m-1}{2}\right]}(\Delta + e^{\frac{2k\pi}{m}}t), & \text{if } m \text{ is odd} \\
\prod_{k=-\left[\frac{m-1}{2}\right]}^{\left[\frac{m-1}{2}\right]}(\Delta + e^{\frac{(2k+1)\pi}{m}}t), & \text{if } m \text{ is even} 
\end{cases}
\]

For \(-\left[\frac{m}{2}\right] \leq k \leq \left[\frac{m-1}{2}\right]\), denote

\[
\alpha_0 = \begin{cases} 
e^{-\frac{2\pi k}{m}}, & \text{if } m \text{ is odd} \\
e^{-\frac{(m-1)\pi}{m}}, & \text{if } m \text{ is even} 
\end{cases}
\]

and \(\alpha_k = \alpha_0 e^{\frac{2k\pi}{m}} (k = 0, 1, 2, \cdots, m - 1)\).

Suppose that \(\gamma_{-r}\) and \(\gamma_0\) are the restriction operators from \(C^\infty(E)\) to \(C^\infty(Y_{-r})\) and \(C^\infty(Y_0)\), respectively. Then the Dirichlet boundary conditions \(D_{-r}\) and \(D_0\) on \(Y_{-r}\) and \(Y_0\) are defined by the operators \(\gamma_{-r}\) and \(\gamma_0\). We can check easily that for each \(\alpha_k\) and \(t > 0\),

\[
(\Delta + \alpha_k t)_{D_{-r}, D_0} : \{\phi \in C^\infty(E) \mid \gamma_{-r}\phi = \gamma_0\phi = 0\} \to C^\infty(E)
\]

is an invertible operator and we can define the Poisson operator \(P_{D_0}(\alpha_k t) : C^\infty(E|_{Y_0}) \to C^\infty(E)\), which is characterized as follows.

\[
\gamma_{-r} P_{D_0}(\alpha_k t) = 0, \quad \gamma_0 P_{D_0}(\alpha_k t) = Id_{Y_0}, \quad (\Delta + \alpha_k t)P_{D_0}(\alpha_k t) = 0.
\]
Now we are going to define boundary conditions corresponding to the operator \( \Delta^m + t^m \). Define \( D_{-r,m}(t) \), \( D_{0,m}(t) \) and \( C_m(t) : C^\infty(E) \to \oplus_m C^\infty(E|_{Y_0}) \) as follows.

\[
D_{-r,m}(t) =
(\gamma_r, \gamma_r(\Delta + \alpha_0 t), \gamma_r(\Delta + \alpha_1 t)(\Delta + \alpha_0 t), \ldots, \gamma_r(\Delta + \alpha_{m-2} t)(\Delta + \alpha_0 t)),
\]

\[
D_{0,m}(t) =
(\gamma_0, \gamma_0(\Delta + \alpha_0 t), \gamma_0(\Delta + \alpha_1 t)(\Delta + \alpha_0 t), \ldots, \gamma_0(\Delta + \alpha_{m-2} t)(\Delta + \alpha_0 t)),
\]

\[
C_m(t) =
(\gamma_0 C, \gamma_0 C(\Delta + \alpha_0 t), \gamma_0 C(\Delta + \alpha_1 t)(\Delta + \alpha_0 t), \ldots, \gamma_0 C(\Delta + \alpha_{m-2} t)(\Delta + \alpha_0 t)).
\]

Then the Poisson operator \( \tilde{P}_{D_{0,m}(t)}(t) : \oplus_m C^\infty(E|_{Y_0}) \to C^\infty(E) \) associated to \((\Delta^m + t^m, D_{0,m}(t))\) is given as follows (c.f. [4], [8]).

\[
\tilde{P}_{D_{0,m}(t)}(t)(f_0, \ldots, f_{m-1}) = P_{D_0}(\alpha_0 t)f_0 + (\Delta + \alpha_0 t)^{-1}_{D_{-r}, D_0} P_{D_0}(\alpha_1 t)f_1 + \ldots + (\Delta + \alpha_0 t)^{-1}_{D_{-r}, D_0} (\Delta + \alpha_1 t)^{-1}_{D_{-r}, D_0} \cdots (\Delta + \alpha_{m-2} t)^{-1}_{D_{-r}, D_0} P_{D_0}(\alpha_{m-1} t)f_{m-1}.
\]

I.e. \( \tilde{P}_{D_{0,m}(t)}(t) \) defined as above satisfies the following properties.

\[
D_{-r,m}(t) \tilde{P}_{D_{0,m}(t)}(t) = 0, \quad D_{0,m}(t) \tilde{P}_{D_{0,m}(t)}(t) = Id_{\oplus_m C^\infty(E|_{Y_0})},
\]

and \((\Delta^m + t^m) \tilde{P}_{D_{0,m}(t)}(t) = 0\).

Note that for \( i = -r, 0, \)

\[
D_{i,m}(0) = (\gamma_i, \gamma_i \Delta, \ldots, \gamma_i \Delta^{m-1}), \quad C_m(0) = (\gamma_0 C, \gamma_0 C \Delta, \ldots, \gamma_0 C \Delta^{m-1}).
\]

Put

\[
\Omega(t) =
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 \\
-\alpha_0 t & 1 & 0 & \ldots & 0 & 0 \\
\alpha_0^2 t^2 & -t(\alpha_0 + \alpha_1) & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(-1)^m \alpha_0^{m-1} t^{m-1} & (-1)^m \alpha_0^{m-2} t^{m-2} \sum_{k=0}^{m-2} \alpha_0^{-k} \alpha_1^{k} & \ldots & \ldots & -t \sum_{i=0}^{m-2} \alpha_i & 1
\end{pmatrix}
\]
Then one can check by direct computation that

$$D_{-r,m}(0) = \Omega(t) D_{-r,m}(t), \quad D_{0,m}(0) = \Omega(t) D_{0,m}(t) \quad \text{and} \quad C_m(0) = \Omega(t) C_m(t).$$

Define $P_{D_{0,m}(0)}(t) = \tilde{P}_{D_{0,m}(t)}(t)\Omega(t)^{-1}$. Then $P_{D_{0,m}(0)}(t)$ is the Poisson operator associated to $(\Delta^m + t^m, D_{0,m}(0))$, which is characterized as follows.

$$D_{-r,m}(0) P_{D_{0,m}(0)}(t) = 0, \quad D_{0,m}(0) P_{D_{0,m}(0)}(t) = Id_{\mathcal{D}_m C^\infty(Y_0)},$$

and $(\Delta^m + t^m) P_{D_{0,m}(0)}(t) = 0$.

Next we consider

$$(\Delta^m + t^m)_{D_{-r,m}(0), D_{0,m}(0)} : \{ \phi \in C^\infty(E) \mid D_{-r,m}(0)\phi = D_{0,m}(0)\phi = 0 \} \to C^\infty(E)$$

and

$$(\Delta^m + t^m)_{D_{-r,m}(0), C_m(0)} : \{ \phi \in C^\infty(E) \mid D_{-r,m}(0)\phi = C_m(0)\phi = 0 \} \to C^\infty(E),$$

both of which are invertible operators and $(\Delta^m + t^m)^{-1}_{D_{-r,m}(0), D_{0,m}(0)}$ and $(\Delta^m + t^m)^{-1}_{D_{-r,m}(0), C_m(0)}$ are trace class operators. From the following identities

$$(\Delta^m + t^m)((\Delta^m + t^m)^{-1}_{D_{-r,m}(0), C_m(0)} - (\Delta^m + t^m)^{-1}_{D_{-r,m}(0), D_{0,m}(0)}) = 0,$$

$$D_{-r,m}(0)((\Delta^m + t^m)^{-1}_{D_{-r,m}(0), C_m(0)} - (\Delta^m + t^m)^{-1}_{D_{-r,m}(0), D_{0,m}(0)}) = 0,$$

$$D_{0,m}(0)((\Delta^m + t^m)^{-1}_{D_{-r,m}(0), C_m(0)} - (\Delta^m + t^m)^{-1}_{D_{-r,m}(0), D_{0,m}(0)})$$

$$= D_{0,m}(0)(\Delta^m + t^m)^{-1}_{D_{-r,m}(0), C_m(0)},$$

we have

$$(\Delta^m + t^m)_{D_{-r,m}(0), C_m(0)} - (\Delta^m + t^m)_{D_{-r,m}(0), D_{0,m}(0)} =$$

$$P_{D_{0,m}(0)}(t) D_{0,m}(0)(\Delta^m + t^m)^{-1}_{D_{-r,m}(0), C_m(0)}, \quad (2.2)$$
Hence, we have

\[
\frac{d}{dt}\{\log |\det (\Delta^m + t^m)_{D_{-r,m}(0),C_m(0)} - \log |\det (\Delta^m + t^m)_{D_{-r,m}(0),D_{0,m}(0)}|\}
\]

\[
= Tr\{m(t)^{m-1}\left((\Delta^m + t^m)^{-1}_{D_{-r,m}(0),C_m(0)} - (\Delta^m + t^m)^{-1}_{D_{-r,m}(0),D_{0,m}(0)}\right)\}
\]

\[
= m(t)^{m-1}Tr\{P_{D_{0,m}(0)}(t)D_{0,m}(0)(\Delta^m + t^m)^{-1}_{D_{-r,m}(0),C_m(0)}\}
\]

\[
= m(t)^{m-1}Tr\{D_{0,m}(0)(\Delta^m + t^m)^{-1}_{D_{-r,m}(0),C_m(0)}P_{D_{0,m}(0)}(t)\}
\]

(2.3)

We now define \(Q_{(\partial_u + |B|),r} : C^\infty(Y_0) \to C^\infty(Y_0)\) as in Section 1 by

\[
Q_{(\partial_u + |B|),r} = \gamma_0(\partial_u + |B|)P_{D_0}(0).
\]

and for simplicity we denote \(Q_{(\partial_u + |B|),r}\) by \(Q\). We also define \(Q_m(t), \tilde{Q}_m(t) : \oplus_mC^\infty(Y_0) \to \oplus_mC^\infty(Y_0)\) as follows.

\[
Q_m(t) = C_m(0)P_{D_{0,m}(0)}(t), \quad \tilde{Q}_m(t) = C_m(t)\tilde{P}_{D_{0,m}(t)}(t).
\]

Then

\[
\tilde{Q}_m(t) = \Omega(t)^{-1}C_m(0)P_{D_{0,m}(0)}(t)\Omega(t)
\]

\[
= \Omega(t)^{-1}Q_m(t)\Omega(t),
\]

(2.4)

and hence \(Q_m(t)\) and \(\tilde{Q}_m(t)\) are isospectral and have the same determinants.

Now we are going to describe \(\frac{d}{dt}Q_m(t) = C_m(0)\frac{d}{dt}P_{D_{0,m}(0)}(t)\).
Lemma 2.1.

\[
\frac{d}{dt} P_{D_{0,m}(0)}(t) = -mt^{m-1}(\Delta^m + t^m)_D^{-1} D_{-r,m(0),D_{0,m}(0)} P_{D_{0,m}(0)}(t).
\]

Proof Differentiating \((\Delta^m + t^m)P_{D_{0,m}(0)}(t) = 0\), we have

\[
(\Delta^m + t^m) \frac{d}{dt} P_{D_{0,m}(0)}(t) = -mt^{m-1} P_{D_{0,m}(0)}(t).
\] (2.5)

From the following identities

\[
D_{-r,m(0)} P_{D_{0,m}(0)}(t) = 0, \quad D_{0,m(0)} P_{D_{0,m}(0)}(t) = Id_{\oplus_m C^\infty(Y_0)},
\]

we have

\[
D_{-r,m(0)} \frac{d}{dt} P_{D_{0,m}(0)}(t) = 0, \quad D_{0,m(0)} \frac{d}{dt} P_{D_{0,m}(0)}(t) = 0.
\] (2.6)

From (2.5) and (2.6), the result follows. □

From Lemma 2.1 and (2.2),

\[
\frac{d}{dt} Q_m(t) = -mt^{m-1} C_m(0)(\Delta^m + t^m)_D^{-1} D_{-r,m(0),D_{0,m}(0)} P_{D_{0,m}(0)}(t)
\]

\[
= mt^{m-1} C_m(0) \left( (\Delta^m + t^m)_D^{-1} D_{-r,m(0),C_m(0)} - (\Delta^m + t^m)_D^{-1} D_{-r,m(0),D_{0,m}(0)} \right) P_{D_{0,m}(0)}(t)
\]

\[
= mt^{m-1} C_m(0) P_{D_{0,m}(0)}(t) D_{0,m(0)}(\Delta^m + t^m)_D^{-1} D_{-r,m(0),C_m(0)} P_{D_{0,m}(0)}(t)
\]

\[
= mt^{m-1} Q_m(t) D_{0,m(0)}(\Delta^m + t^m)_D^{-1} D_{-r,m(0),C_m(0)} P_{D_{0,m}(0)}(t).
\]

As a consequence, we have

\[
\frac{d}{dt} \log \text{Det} Q_m(t) = Tr \left( Q_m(t)^{-1} \frac{d}{dt} Q_m(t) \right)
\]

\[
= mt^{m-1} Tr \left( D_{0,m}(0)(\Delta^m + t^m)_D^{-1} D_{-r,m(0),C_m(0)} P_{D_{0,m}(0)}(t) \right). \quad (2.7)
\]

From (2.3), (2.4) and (2.7), we have

\[
\frac{d}{dt} \log \text{Det} \tilde{Q}_m(t) =
\]

\[
\frac{d}{dt} \left\{ \log \text{Det}(\Delta^m + t^m)_{D_{-r,m(0),C_m(0)}} - \log \text{Det}(\Delta^m + t^m)_{D_{-r,m(0),D_{0,m}(0)}} \right\}. \quad (2.8)
\]
Setting $Q(\partial_{u}+|B|),r(\alpha_{k}t) = \gamma_{0}CP_{D_{0}}(\alpha_{k}t) : C^{\infty}(E|Y_{0}) \rightarrow C^{\infty}(E|Y_{0})$ (briefly $Q(\alpha_{k}t)$), $\tilde{Q}_{m}(t)$ is of the following upper triangular matrix,

$$
\begin{pmatrix}
Q(\alpha_{0}t), \gamma_{0}C(\Delta+\alpha_{0}t)^{-1}_{D_{+},D_{0}}P_{D_{0}}(\alpha_{1}t), \ldots, \gamma_{0}C(\Delta+\alpha_{0}t)^{-1}_{D_{+},D_{0}} \cdots (\Delta+\alpha_{m-2}t)^{-1}_{D_{+},D_{0}}P_{D_{0}}(\alpha_{m-1}t) \\
0, Q(\alpha_{1}t), \ldots, \gamma_{0}C(\Delta+\alpha_{1}t)^{-1}_{D_{+},D_{0}} \cdots (\Delta+\alpha_{m-2}t)^{-1}_{D_{+},D_{0}}P_{D_{0}}(\alpha_{m-1}t) \\
\vdots \quad \vdots \quad \ddots \quad \vdots \\
0, 0, \ldots, Q(\alpha_{m-1}t)
\end{pmatrix}
$$

and hence we have

$$
\frac{d}{dt} \left( \log \text{Det} \tilde{Q}_{m}(t) \right) = \frac{d}{dt} \left( \sum_{k=0}^{m-1} \log \text{Det} Q(\alpha_{k}t) \right). 
$$

Finally by (2.8), (2.9), we have

$$
\log \text{Det} (\Delta^{m} + t^{m})_{D_{-r,m}(0),C_{m}(0)} - \log \text{Det} (\Delta^{m} + t^{m})_{D_{-r,m}(0),D_{0,m}(0)} = \tilde{c} + \sum_{k=0}^{m-1} \log \text{Det} Q(\alpha_{k}t),
$$

where $\tilde{c}$ does not depend on $t$. It is known in [4] that $\log \text{Det} (\Delta^{m} + t^{m})_{D_{-r,m}(0),C_{m}(0)}$, $\log \text{Det} (\Delta^{m} + t^{m})_{D_{-r,m}(0),D_{0,m}(0)}$ and $\log \text{Det} Q(\alpha_{k}t)$ have asymptotic expansions as $t \to \infty$ and the zero coefficients in the asymptotic expansions of $\log \text{Det} (\Delta^{m} + t^{m})_{D_{-r,m}(0),C_{m}(0)}$ and $\log \text{Det} (\Delta^{m} + t^{m})_{D_{-r,m}(0),D_{0,m}(0)}$ are zeros ([8], [13]). Denoting by $c_{k}$ the zero coefficient in the asymptotic expansion of $\log \text{Det} Q(\alpha_{k}t)$,

$$
\tilde{c} = - \sum_{k=0}^{m-1} c_{k}.
$$

Setting $t = 0$, we have the following theorem.

**Theorem 2.2.**

$$
\log \text{Det} \Delta_{D_{-r},C} - \log \text{Det} \Delta_{D_{-r},D_{0}} = -c + \log \text{Det} Q(\partial_{u}+|B|),r,
$$

where $c = \frac{1}{m} \sum_{k=0}^{m-1} c_{k}$.

**Proof** Setting $t = 0$ in (2.10), we have

$$
\log \text{Det} (\Delta^{m})_{D_{-r,m}(0),C_{m}(0)} - \log \text{Det} (\Delta^{m})_{D_{-r,m}(0),D_{0,m}(0)} = \tilde{c} + m \log \text{Det} Q(\partial_{u}+|B|),r.
$$

From the fact that $\lambda$ is an eigenvalue of $\Delta_{D_{-r},D_{0}} (\Delta_{D_{-r},C})$ if and only if $\lambda^{m}$ is an eigenvalue of $(\Delta^{m})_{D_{-r,m}(0),D_{0,m}(0)} ((\Delta^{m})_{D_{-r,m}(0),C_{m}(0)})$, the result follows. 

From (2.1) and Theorem 2.2, we have the following result.
\textbf{Theorem 2.3.}

\[
\log \text{Det}(-\partial_u^2 + B^2)_{N_{r,0}, D_{-r}, D_r} + \log \text{Det}(-\partial_u^2 + B^2)_{N_{0,r}, D_{-r}, D_r} - \\
\log \text{Det}(-\partial_u^2 + B^2)_{N_{-r,0}, D_{-r}, D_0} - \log \text{Det}(-\partial_u^2 + B^2)_{N_{0,r}, D_{0}, D_r}
\]
\[
= -c + \log \text{Det}Q(\partial_u + |B|), r.
\]

In the next section, we are going to show that this constant \(c = 0\) by using the method in [13], which completes the proof of Theorem 1.4

\section{3 Computation of the constant term in Theorem 2.3}

Recall that \(c = \frac{1}{m} \sum_{k=0}^{m-1} c_k\), where \(c_k\) is the zero coefficient in the asymptotic expansion of \(\log \text{Det}Q(\partial_u + |B|), r(\alpha_k t)\) as \(t \to \infty\). Note that for \(f \in C^\infty(Y)\) with \(Bf = \lambda f\),

\[
Q(\partial_u + |B|), r(\alpha_k t) f = \left( \sqrt{\lambda^2 + \alpha_k t} + |\lambda| + \frac{2 \sqrt{\lambda^2 + \alpha_k t} e^{-\sqrt{\lambda^2 + \alpha_k t} r}}{e^{\sqrt{\lambda^2 + \alpha_k t} r} - e^{-\sqrt{\lambda^2 + \alpha_k t} r}} \right) f.
\]

Here we take the negative real axis as a branch cut for square root of \(\alpha_k\). Since the asymptotic expansion of \(\log \text{Det}Q(\partial_u + |B|), r(\alpha_k t)\) as \(t \to \infty\) is completely determined up to smoothing operators (c.f. [4]), \(\log \text{Det}Q(\partial_u + |B|), r(\alpha_k t)\) and \(\log \text{Det}(\sqrt{B^2 + \alpha_k t} + |B|)\) have the same asymptotic expansions. Hence it’s enough to consider the asymptotic expansion of \(\log \text{Det}(\sqrt{B^2 + \alpha_k t} + |B|)\).

Since \(Re(\alpha_k)\) can be negative, to avoid this difficulty we choose an angle \(\phi_k\) so that \(0 \leq |\phi_k| < \frac{\pi}{2}\) and \(Re(e^{i(\theta_k - \phi_k)}) > 0\), where \(\alpha_k = e^{i\theta_k}\). Then,

\[
\log \text{Det}(\sqrt{B^2 + \alpha_k t} + |B|)
\]
\[
= \log \text{Det}\{e^{i\frac{\phi_k}{2}}(e^{-i\phi_k} B^2 + e^{-i\phi_k} B^2 + e^{i(\theta_k - \phi_k)t})\}
\]
\[
= -\frac{d}{ds}|_{s=0}\{e^{-i\frac{\phi_k}{2}s} \zeta(e^{-i\phi_k} B^2 + e^{-i\phi_k} B^2 + e^{i(\theta_k - \phi_k)t}) (s)\}
\]
\[
= \frac{i\phi_k}{2} \zeta(e^{-i\phi_k} B^2 + e^{-i\phi_k} B^2 + e^{i(\theta_k - \phi_k)t}) (0) + \\
\log \text{Det} \left( e^{-i\phi_k} B^2 + e^{-i\phi_k} B^2 + e^{i(\theta_k - \phi_k)t} \right). \quad (3.1)
\]
Now put $\tilde{\theta}_k = \theta_k - \phi_k$ and denote $\zeta(\sqrt{e^{-i\phi_k}B^2 + \sqrt{e^{-i\phi_k}B^2 + e^{i(\theta_k - \phi_k)}}})$ simply by $\zeta(s)$. Then,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty r^{s-1} \text{Re} e^{-r\left(\sqrt{e^{-i\phi_k}B^2 + \sqrt{e^{-i\phi_k}B^2 + e^{i\theta_k}}}\right)} dr$$

$$= \sum_{q=0}^\infty \frac{(-1)^q}{q!} \frac{1}{\Gamma(s)} \int_0^\infty r^{s+q-1} \text{Re} \{e^{-r\sqrt{e^{-i\phi_k}B^2 + e^{i\theta_k}}}\} dr.$$

Setting

$$\zeta_q(s) = \frac{1}{\Gamma(s)} \int_0^\infty r^{s+q-1} \text{Re} \{e^{-r\sqrt{e^{-i\phi_k}B^2 + e^{i\theta_k}}}\} dr,$$

we have

$$\zeta_q(s) = \frac{1}{\Gamma(s)} \sum_{\lambda_j \in \text{Spec}(B)} \frac{\sqrt{e^{-i\phi_k} \lambda_j^2}}{r^{s+q}} \int_0^\infty (r z_j)^{s+q-1} e^{-rz_j} z_j dr,$$

where $z_j = \sqrt{e^{-i\phi_k} \lambda_j^2 + e^{i\theta_k}}$. Consider the contour integral $\int_C z^{s+q-1} e^{-z} dz$ for $\text{Res} > -q$, where for $\arg(z_j) = \rho_j,$

$$C = \{re^{i\rho_j} | \epsilon \leq r \leq R\} \cup \{e^{i\theta} | 0 \leq \theta \leq \rho_j\} \cup \{r | \epsilon \leq r \leq R\} \cup \{e^{i\theta} | 0 \leq \theta \leq \rho_j\}$$

and oriented counterclockwise. Then one can check that

$$\int_0^\infty (r z_j)^{s+q-1} e^{-rz_j} z_j dr = \Gamma(s+q),$$

and hence we have

$$\zeta_q(s) = \frac{\Gamma(s+q)}{\Gamma(s)} \sum_{\lambda_j \in \text{Spec}(B)} \frac{\sqrt{e^{-i\phi_k} \lambda_j^2}}{r^{s+q}}.$$

Using the equation (3.3) again, we obtain that

$$\zeta_q(s) = \frac{\Gamma(s+q)}{\Gamma(s)} \frac{1}{\Gamma(s+q/2)} \int_0^\infty r^{s+q-1} \text{Re} \{e^{-r\sqrt{e^{-i\phi_k}B^2 + e^{i\theta_k}}}\} dr.$$

Putting $rt = u,$

$$\zeta_q(s) = \frac{\Gamma(s+q)}{\Gamma(s+q/2) \Gamma(s)} u^{s+q/2-1} e^{-ue^{i\theta_k}} \int_0^\infty u^{s+q/2-1} e^{-ue^{i\theta_k}} du.$$
It is known in [7] (see also [3]) that as $r \to 0$

\[
Tr \{ \sqrt{e^{-i\phi_k} B^2} e^{-re^{-i\phi_k} B^2} \} \sim \sum_{j=0}^{\infty} a_j r^{\frac{j-(m-1)-q}{2}} + \sum_{j=0}^{\infty} (b_j \log r + c_j) r^j. \tag{3.5}
\]

Hence, as $t \to \infty$,

\[
\zeta_q(s) \sim \frac{\Gamma(s+q)}{\Gamma(\frac{s+q}{2})\Gamma(s)} \left\{ \sum_{j=0}^{\infty} a_j \int_0^\infty u^{\frac{s+q-j-(m-1)-q}{2}} e^{-ue^{i\hat{\theta}_k}} du \cdot t^{\frac{(m-1)-j-1}{2}} + \sum_{j=0}^{\infty} b_j \int_0^\infty u^{\frac{s+q-j-1}{2}} e^{-ue^{i\hat{\theta}_k}} \log u \cdot t^{\frac{s+q-j}{2}} \right. \\
- \sum_{j=0}^{\infty} b_j \int_0^\infty u^{\frac{s+q-j-1}{2}} e^{-ue^{i\hat{\theta}_k}} du \cdot t^{\frac{s+q-j}{2}} \log t \\
+ \sum_{j=0}^{\infty} c_j \int_0^\infty u^{\frac{s+q-j-1}{2}} e^{-ue^{i\hat{\theta}_k}} du \cdot t^{\frac{s+q-j}{2}} \right\}. \tag{3.6}
\]

For $q \geq 1$, the zero coefficients in the asymptotic expansions of $\zeta_q(0)$ and $-\zeta_q'(0)$ as $t \to \infty$ can be obtained only from the term

\[
\sum_{j=0}^{\infty} a_{m-1} \int_0^\infty u^{\frac{s}{2}-1} e^{-ue^{i\hat{\theta}_k}} du \cdot t^{-\frac{s}{2}}. \tag{3.6}
\]

One can check by using (3.3) that

\[
\int_0^\infty (ue^{i\hat{\theta}_k})^{\frac{s}{2}-1} e^{-ue^{i\hat{\theta}_k}} (e^{i\hat{\theta}_k}) du = \int_0^\infty r^{\frac{s}{2}-1} e^{-r} dr = \Gamma\left(\frac{s}{2}\right). 
\]

Hence, the equation (3.6) can be simplified as

\[
\frac{\Gamma(s+q)}{\Gamma(\frac{s+q}{2})\Gamma(s)} e^{-i\hat{\theta}_k s} a_{m-1} t^{-\frac{s}{2}}. \tag{3.7}
\]

**Lemma 3.1.** For each $q \geq 1$, $a_{m-1} = 0$. Therefore, the zero coefficients in the asymptotic expansions of $\zeta_q(0)$ and $-\zeta_q'(0)$ as $t \to \infty$ are zero.

**Proof** We denote by $\beta(s)$

\[
\beta(s) = \frac{1}{\Gamma(s)} \int_0^\infty r^{\frac{s+q}{2}-1} Tr \{ \sqrt{e^{-i\phi_k} B^2} e^{-re^{-i\phi_k} B^2} \} dr.
\]
Then from (3.5),

\[
\beta(s) = \frac{1}{\Gamma(s)} \int_0^1 r^{\frac{s+q}{2}-1} Tr \{ \sqrt{e^{-i\phi k} B^2} e^{-re^{-i\phi k} B^2} \} dr
+ \frac{1}{\Gamma(s)} \int_1^\infty r^{\frac{s+q}{2}-1} Tr \{ \sqrt{e^{-i\phi k} B^2} e^{-re^{-i\phi k} B^2} \} dr
\]

\[
= \sum_{j=0}^{N} a_j \frac{1}{\Gamma(s)} \int_0^1 r^{\frac{s+q}{2}-1} r^{\frac{j-(m-1)-q}{2}} dr + \frac{1}{\Gamma(s)} \Psi_N(s)
\]

\[
= \sum_{j=0}^{N} a_j \frac{1}{\Gamma(s)} \frac{2}{s+j-(m-1)} + \frac{1}{\Gamma(s)} \Psi_N(s),
\]

where \( \Psi_N(s) \) is holomorphic for \( \text{Res} > -q \). Hence,

\[
\beta(0) = 2a_{m-1}.
\] (3.8)

On the other hand,

\[
\beta(s) = \sum_{\lambda_j \in \text{Spec}(|B|)} \frac{1}{\Gamma(s)} \int_0^\infty r^{\frac{s+q}{2}-1} \sqrt{e^{-i\phi k \lambda_j^2}} e^{-re^{-i\phi k \lambda_j^2}} dr.
\]

One can check by contour integration (c.f. (3.3)) that

\[
\beta(s) = \frac{\Gamma(\frac{s+q}{2})}{\Gamma(s)} \zeta_{\text{e}^{-i\phi k \lambda_j^2} |B|}(s) = e^{i\phi k \lambda_j^2} \frac{\Gamma(\frac{s+q}{2})}{\Gamma(s)} \zeta_{|B|}(s).
\]

Since \( \zeta_{|B|}(s) \) is regular at \( s = 0 \), \( \beta(0) = 0 \) and hence from (3.8) \( a_{m-1} = 0 \). □

From Lemma 3.1, it’s enough to consider \( \zeta_0(s) \) to compute the zero coefficients in the asymptotic expansions of \( \zeta(0) \) and \( -\zeta'(0) \) as \( t \to \infty \). Recall that from (3.4)

\[
\zeta_0(s) = \frac{1}{\Gamma(\frac{s}{2})} t^{-\frac{s}{2}} \int_0^\infty u^{\frac{s}{2}-1} e^{-ue^{i\theta k} Tr \{ e^{-\frac{u}{t} e^{-i\phi k} B^2} \}} du
\]

Then, as \( t \to \infty \),

\[
\zeta_0(s) \sim \sum_{j=0}^{\infty} d_j \frac{1}{\Gamma(\frac{s}{2})} t^{-\frac{s}{2}} \int_0^\infty u^{\frac{s}{2}-1} e^{-ue^{i\theta k} \left( \frac{u}{t} \right)^{(m-1)}} du
\]
\[
= \sum_{j=0}^{\infty} d_j \frac{1}{\Gamma(m/2)} \int_0^\infty u^{s+j-(m-1)} e^{-ue^{i\theta_k}} du
= \sum_{j=0}^{\infty} d_j \left(e^{-i\tilde{\theta}_k}\right)^{s+j-(m-1)} \frac{\Gamma(s+j-(m-1))}{\Gamma(m/2)} \frac{t^{-\frac{s+j-(m-1)}{2}}}{t^{-\frac{s+j-(m-1)}{2}}}.
\]  

(3.9)

Hence, the zero coefficients \( \pi_0(\zeta_0(0)) \), \( \pi_0(-\zeta'_0(0)) \) in the asymptotic expansions of \( \zeta_0(0) \) and \( -\zeta'_0(0) \) as \( t \to \infty \) come from only the term

\[
d_{m-1} \left(e^{-i\tilde{\theta}_k}\right)^{\frac{\sigma}{2}} t^{-\frac{m}{2}},
\]

and hence,

\[
\pi_0(\zeta_0(0)) = d_{m-1}. \quad \text{(3.11)}
\]

\[
\pi_0(-\zeta'_0(0)) = \frac{i}{2} \tilde{\theta}_kd_{m-1} = \frac{i}{2}(\theta_k - \phi_k)d_{m-1}. \quad \text{(3.12)}
\]

Therefore, from (3.1), (3.11) and (3.12) the zero coefficient \( c_k \) in the asymptotic expansion of \( \log \text{Det}(\sqrt{B^2 +\alpha_k t} + |B|) \) is as follows.

\[
c_k = \frac{i\phi_k}{2}d_{m-1} + \frac{i}{2}(\theta_k - \phi_k)d_{m-1} = \frac{i}{2}\theta_kd_{m-1}.
\]

Since

\[
d_{m-1} = \zeta_{e^{-i\phi_k}B^2}(0) + \dim\text{Ker}(e^{-i\phi_k}B^2) = \zeta_{B^2}(0) + \dim\text{Ker}B^2,
\]

\( d_{m-1} \) does not depend on \( e^{-i\phi_k} \) and hence

\[
c = \frac{1}{m} \sum_k c_k = \frac{i}{2m}d_{m-1} \sum_k \theta_k = 0,
\]

which completes the proof of Theorem 1.4.

§4 Proof of Theorem 1.5

In this section we are going to prove Theorem 1.5. We first assume that both \( Q_i + \sqrt{B^2} \) \( (i = 1, 2) \) are invertible operators. Then this implies that \( \text{Ker}B = 0 \) (Proposition 4.5). From Theorem 1.2, 1.3 and 1.4 we have:

\[
\log \text{Det}\mathfrak{S}^2_{M_r} - \log \text{Det}\mathfrak{S}^2_{M_{1,r},P_<} - \log \text{Det}\mathfrak{S}^2_{M_{2,r},P_>}.
\]
\[
\log \text{Det} \mathcal{D}_{M_r}^2 - \log \text{Det} \mathcal{D}_{M_1,r,D_0}^2 - \log \text{Det} \mathcal{D}_{M_2,r,D_0}^2 + \\
\log \text{Det} \mathcal{D}_{M_1,r,D_0}^2 + \log \text{Det} \mathcal{D}_{M_2,r,D_0}^2 - \log \text{Det} \mathcal{D}_{M_1,r,P_<}^2 - \log \text{Det} \mathcal{D}_{M_2,r,P_>^2}
\]

\[
= \log \text{Det} \mathcal{D}_{M_r}^2 - \log \text{Det} \mathcal{D}_{M_1,r,D_0}^2 - \log \text{Det} \mathcal{D}_{M_2,r,D_0}^2 + \\
\log \text{Det} \mathcal{D}_{M_1,r,D_0}^2 + \log \text{Det} \mathcal{D}_{M_2,r,D_0}^2 - \log \text{Det} \mathcal{D}_{M_1,r,P_<}^2 - \log \text{Det} \mathcal{D}_{M_2,r,P_>^2} - \log \text{Det} \mathcal{D}_{M_1,r,D_r}^2 - \log \text{Det} \mathcal{D}_{M_2,r,D_r}^2 - \log \text{Det} \mathcal{D}_{M_1,r,P_<}^2 - \log \text{Det} \mathcal{D}_{M_2,r,P_>^2}
\]

From Theorem 1.1, we have:

\[
\lim_{r \to \infty} \{ \log \text{Det} \mathcal{D}_{M_r}^2 - \log \text{Det} \mathcal{D}_{M_1,r,P_<}^2 - \log \text{Det} \mathcal{D}_{M_2,r,P_>^2} \}
\]

\[
= \frac{1}{2} \log \text{Det}(B^2) + \lim_{r \to \infty} \{- \log \text{Det} Q(\partial_u + |B|), r + \log \text{Det} R_{M_1,r,D_r}^2 - \log \text{Det} R_{M_1,r,P_<}^2 - \log \text{Det} R_{M_2,r,P_>^2} \}
\]

Now we describe the operators \(Q(\partial_u + |B|), r\), \(R_{M_1,r,D_r}\), \(R_{M_2,r,D_r}\), \(R_{M_1,r,P_<}\) and \(R_{M_2,r,P_>}\) in terms of \(Q_i\) and \(B\). One can check the following lemma by direct computation.
Lemma 4.1. Suppose that $f \in C^\infty(Y)$ with $Bf = \lambda f$ and $\text{Ker}B = 0$. Then:

1. $R_{M_1,D_r}(f) = Q_1(f) + \left( |\lambda| + \frac{2|\lambda|e^{-|\lambda|r}}{e^{\lambda|r} - e^{-|\lambda|r}} \right) f.$

2. $R_{M_2,D_r}(f) = Q_2(f) + \left( |\lambda| + \frac{2|\lambda|e^{-|\lambda|r}}{e^{\lambda|r} - e^{-|\lambda|r}} \right) f.$

3. $R_{M_1,P_{<}}(f) = \begin{cases} Q_1(f) + \left( |\lambda| + \frac{2|\lambda|e^{-|\lambda|r}}{e^{\lambda|r} - e^{-|\lambda|r}} \right) f & \text{for } \lambda < 0 \\ Q_1(f) + |\lambda|f & \text{for } \lambda > 0. \end{cases}$

4. $R_{M_2,P_{>}}(f) = \begin{cases} Q_2(f) + |\lambda|f & \text{for } \lambda < 0 \\ Q_2(f) + \left( |\lambda| + \frac{2|\lambda|e^{-|\lambda|r}}{e^{\lambda|r} - e^{-|\lambda|r}} \right) f & \text{for } \lambda > 0. \end{cases}$

5. $Q_{(\partial_u + |B|),r}(f) = \left( 2|\lambda| + \frac{2|\lambda|e^{-|\lambda|r}}{e^{\lambda|r} - e^{-|\lambda|r}} \right) f.$

The following lemma can be also checked easily.

Lemma 4.2. Let $A$ be an invertible elliptic operator of order $> 0$ and $K_r$ be a one-parameter family of trace class operators such that $\lim_{r \to \infty} \text{Tr}(K_r) = 0$. Then:

\[
\lim_{r \to \infty} \log \text{Det}(A + K_r) = \log \text{Det} A.
\]

Applying Lemma 4.2, we have:

\[
\lim_{r \to \infty} \log \text{Det} R_{M_1,D_r} = \lim_{r \to \infty} \log \text{Det} R_{M_1,P_{<}} = \log \text{Det}(Q_1 + |B|). \tag{4.2}
\]

\[
\lim_{r \to \infty} \log \text{Det} R_{M_2,D_r} = \lim_{r \to \infty} \log \text{Det} R_{M_2,P_{>}} = \log \text{Det}(Q_2 + |B|). \tag{4.3}
\]

Note that

\[
\zeta_{Q_{(\partial_u + |B|),r}}(s) = \sum_{\lambda \in \text{Spec}(B)} \left( 2|\lambda| + \frac{2|\lambda|e^{-|\lambda|r}}{e^{\lambda|r} - e^{-|\lambda|r}} \right)^{-s}.
\]

From Lemma 4.2 again, we have:

\[
\lim_{r \to \infty} \{ \log \text{Det} Q_{(\partial_u + |B|),r} \} = \log 2 \cdot \zeta_{B^2}(0) + \frac{1}{2} \log \text{Det} B^2. \tag{4.4}
\]

Therefore from (4.1), (4.2), (4.3) and (4.4), we have

\[
\lim_{r \to \infty} \{ \log \text{Det} D_{M_1}^2 - \log \text{Det} D_{M_2}^2 \} = -\log 2 \cdot \zeta_{B^2}(0).
\]

This completes the proof of Theorem 1.5.

Finally, we are going to discuss the invertibility conditions of both $Q_1 + \sqrt{B^2}$ and $Q_2 + \sqrt{B^2}$. For this purpose we need Green’s formula of the following form (c.f. Lemma 3.1 in [5]).
Lemma 4.3. Let $\phi$ and $\psi$ be smooth sections on $M_j$ ($j = 1, 2$). Then,

$$
\langle \mathcal{D}_{M_j} \phi, \psi \rangle_{M_j} - \langle \phi, \mathcal{D}_{M_j} \psi \rangle_{M_j} = \epsilon_j \langle \phi|_Y, G(\psi|_Y) \rangle_Y,
$$

where $\epsilon_j = 1$ for $j = 2$ and $\epsilon_j = -1$ for $j = 1$.

Suppose that for $f \in C^\infty(Y)$, $\phi_j$ is the solution of $\mathcal{D}_{M_j}$ with $\phi_j|_Y = f$. Then by Lemma 4.3

$$
\langle (Q_1 + |B|) f, f \rangle_Y = \langle \mathcal{D}_{M_1} \phi_1, \mathcal{D}_{M_1} \phi_1 \rangle_{M_1} + \langle |B| - B, f, f \rangle_Y, \quad (4.5)
$$

$$
\langle (Q_2 + |B|) f, f \rangle_Y = \langle \mathcal{D}_{M_2} \phi_2, \mathcal{D}_{M_2} \phi_2 \rangle_{M_2} + \langle |B| + B, f, f \rangle_Y. \quad (4.6)
$$

Hence we have

$$
f \in \text{Ker}(Q_1 + |B|) \quad \text{if and only if} \quad \mathcal{D}_{M_1} \phi_1 = 0 \text{ and } f \in \text{Im} P_\geq. \quad (4.7)
$$

$$
f \in \text{Ker}(Q_2 + |B|) \quad \text{if and only if} \quad \mathcal{D}_{M_2} \phi_2 = 0 \text{ and } f \in \text{Im} P_\leq. \quad (4.8)
$$

$\phi_1$ satisfying (4.7) can be expressed on the cylinder part by

$$
\phi_1 = \sum_{j=1}^k a_j g_j + \sum_{\lambda_j > 0} b_j e^{-\lambda_j u} h_j, \quad (4.9)
$$

where $Bg_j = 0$, $Bh_j = \lambda_j h_j$ and $k = \dim \ker B$. $\phi_2$ satisfying (4.8) can be expressed in the similar way.

Definition 4.4. We denote $M_{1,\infty} := M_1 \cup \partial M_1 \times [0, \infty)$ and by $\mathcal{D}_{M_{1,\infty}}, \ E_{1,\infty}$ the natural extensions of $\mathcal{D}_{M_{1,r}}, \ E_{1,r}$ to $M_{1,\infty}$. A section $\psi$ of $E_{1,\infty}$ is called an extended $L^2$-solution of $\mathcal{D}_{M_{1,\infty}}$ if $\psi$ is a solution of $\mathcal{D}_{M_{1,\infty}}$ which takes the form (4.9) on the cylinder part $[0, \infty) \times Y$. In this case $\sum_{j=1}^k a_j g_j$ is called the limiting value of $\psi$. We can define the same notions for $\mathcal{D}_{M_{2,\infty}}$ on $M_{2,\infty}$.

It is a well-known fact that if $L$ is the set of all limiting values of extended $L^2$-solutions of $\mathcal{D}_{M_{1,\infty}}$, $L$ is a Lagrangian subspace of $\ker B$ and in particular $\dim L = \frac{1}{2} \dim \ker B$ (c.f. [1], [2], [5], [10]). From Definition 4.4, $\phi_1$ satisfying (4.7) is the restriction of an extended $L^2$-solution of $\mathcal{D}_{M_{1,\infty}}$ on $M_{1,\infty}$. We can say similar assertion for $\phi_2$ and as a consequence we have the following proposition.
Proposition 4.5. The invertibility of $Q_1 + \sqrt{B^2}$ and $Q_2 + \sqrt{B^2}$ is equivalent to the non-existence of extended $L^2$-solutions of $\mathcal{D}_{M_1,\infty}$ and $\mathcal{D}_{M_2,\infty}$ on $M_{1,\infty}$ and $M_{2,\infty}$. Furthermore, this condition implies the invertibility of $B$, $\mathcal{D}^2_{M_1,r,P_2}$, $\mathcal{D}^2_{M_2,r,P_2}$ for each $r > 0$ and the invertibility of $\mathcal{D}^2_{M_2}$ for $r$ large enough.

Proof. We need to prove the second assertion. $\text{Ker}B = 0$ implies the invertibility of $B$. Suppose that $\psi \in \text{Ker} \mathcal{D}^2_{M_1,r,P_2}$. By Lemma 4.3 and (1.2) we have

$$
0 = \langle \mathcal{D}^2_{M_1,r,\psi}, \mathcal{D}^2_{M_1,r,\psi} \rangle_{M_1,r} = \langle \mathcal{D}_{M_1,r,\psi}, \mathcal{D}_{M_1,r,\psi} \rangle_{M_1,r} - \langle (\partial u + B)\psi|_{Y_0}, \psi|_{Y_0} \rangle_{Y_0}
$$

$$
= \langle \mathcal{D}_{M_1,r,\psi}, \mathcal{D}_{M_1,r,\psi} \rangle_{M_1,r}.
$$

Hence, $\mathcal{D}_{M_1,r}\psi = 0$ and by (4.7) $\psi|_{Y_0}$ belongs to $\text{Ker}(Q_1 + |B|)$, which implies that $\psi = 0$ and hence $\text{Ker} \mathcal{D}_{M_1,r,P_2} = 0$. Since $\mathcal{D}_{M_1,r,P_2}$ is a self-adjoint Fredholm operator (c.f. Proposition 2.4 in [5]), $\mathcal{D}_{M_1,r,P_2}$ is an invertible operator and so is $\mathcal{D}^2_{M_1,r,P_2}$. We can use the same argument for $\mathcal{D}^2_{M_2,r,P_2}$.

Now let us consider $\mathcal{D}^2_{M_1}$. Since $\mathcal{D}_M$ is a self-adjoint Fredholm operator, it’s enough to show that $\text{Ker} \mathcal{D}_{M_1} = 0$. From the decomposition $M_r = (M_1 \cup M_2) \cup (\partial M_1 \cup \partial M_2) N_{r,r}$, we define the Dirichlet-to-Neumann operator

$$
R_{r,r} : C^\infty(Y_{-r}) \oplus C^\infty(Y_r) \to C^\infty(Y_{-r}) \oplus C^\infty(Y_r)
$$

as follows. For $(f,g) \in C^\infty(Y_{-r}) \oplus C^\infty(Y_r)$, choose $\phi_i \in C^\infty(M_i)$ ($i = 1, 2$), $\psi \in C^\infty(N_{-r,r})$ such that

$$
\mathcal{D}^2_{M_i} \phi_i = 0, \quad (-\partial^{2}u + B^2)\psi = 0,
$$

$$
\phi_1|_{\partial M_1} = \psi|_{Y_{-r}} = f, \quad \phi_2|_{\partial M_2} = \psi|_{Y_r} = g.
$$

Then we define

$$
R_{r,r}(f,g) = \left((\partial u\phi_1)|_{Y_{-r}} - (\partial u\psi)|_{Y_{-r}}, -(\partial u\phi_2)|_{Y_r} + (\partial u\psi)|_{Y_r}\right) = \left(Q_1 f - (\partial u\psi)|_{Y_{-r}}, Q_2 g + (\partial u\psi)|_{Y_r}\right).
$$

If $(f,g) \in \text{Ker} R_{r,r}$, then $\phi_1 \cup_{Y_{-r}} \psi \cup_{Y_r} \phi_2$ is a smooth section which belongs to $\text{Ker} \mathcal{D}^2_{M_2}$ and vice versa. Hence $\text{Ker} R_{r,r} = 0$ if and only if $\text{Ker} \mathcal{D}^2_{M_2} = \text{Ker} \mathcal{D}_{M_2} = 0$. By direct computation (c.f. [9]), one can check that
\[
R_{-r,r} \left( \begin{array}{c} f \\ g \end{array} \right) = \left( \begin{array}{cc}
Q_1 + |B| & 0 \\
0 & Q_2 + |B| \end{array} \right) \left( \begin{array}{c} f \\ g \end{array} \right) + Ar \left( \begin{array}{cc}
e^{-2r|B|} & -1 \\
-1 & e^{-2r|B|} \end{array} \right) \left( \begin{array}{c} f \\ g \end{array} \right),
\]
where \( Ar = \frac{2|B|}{e^{2r|B|} - e^{-2r|B|}} \). Then we have

\[
\left\langle R_{-r,r} \left( \begin{array}{c} f \\ g \end{array} \right), \left( \begin{array}{c} f \\ g \end{array} \right) \right\rangle_{L^2(Y)} = \langle (Q_1 + |B|)f, f \rangle + \langle (Q_2 + |B|)g, g \rangle + \langle Ar e^{-2r|B|}f, f \rangle + \langle Ar e^{-2r|B|}g, g \rangle - \langle Ar g, f \rangle - \langle Ar f, g \rangle.
\]

Note that each \( Q_i + |B| \) is a non-negative operator by (4.5), (4.6). Let \( \lambda_0 \) be the minimum of the eigenvalues of \( Q_1 + \sqrt{\Delta_Y} \) and \( Q_2 + \sqrt{\Delta_Y} \). Since \( \lim_{r \to \infty} ||Ar||_{L^2} = 0 \), one can choose \( r_0 \) so that for \( r \geq r_0 \), \( ||Ar||_{L^2} < \lambda_0 \). Then for \( r \geq r_0 \), \( R_{-r,r} \) is injective and this completes the proof of Proposition 4.5. \( \square \)

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