Optimal Program Synthesis Over Noisy Data

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ABSTRACT

We explore and formalize the task of synthesizing programs over noisy data, i.e., data that may contain corrupted input-output examples. By formalizing the concept of a Noise Source, an Input Source, and a prior distribution over programs, we formalize the probabilistic process which constructs a noisy dataset. This formalism allows us to define the correctness of a synthesis algorithm, in terms of its ability to synthesize the hidden underlying program. The probability of a synthesis algorithm being correct depends upon the match between the Noise Source and the Loss Function used in the synthesis algorithm’s optimization process. We formalize the concept of an optimal Loss Function given prior information about the Noise Source. We provide a technique to design optimal Loss Functions given perfect and imperfect information about the Noise Sources. We also formalize the concept and conditions required for convergence, i.e., conditions under which the probability that the synthesis algorithm produces a correct program increases as the size of the noisy data set increases. This paper presents the first formalization of the concept of optimal Loss Functions, the first closed form definition of optimal Loss Functions, and the first conditions that ensure that a noisy synthesis algorithm will have convergence guarantees.

1 INTRODUCTION

Program synthesis has been successfully used to synthesize programs from examples, for domains such as string transformations [12, 21], data wrangling [7], data completion [23], and data structure manipulation [8, 16, 25]. In recent years, there has been interest in synthesizing programs from input-output examples in presence of noise/corruptions [13, 17, 19]. The motivation behind this line of work is to extract information left intact in noisy/corrupted data to synthesize the correct program. These techniques have empirically shown that synthesizing the correct program, even in presence of substantial noise, is possible.

1.1 Noisy Program Synthesis Framework

Previous work [13, 17] formulates the noisy program synthesis problem as an optimization problem over the program space and the noisy dataset. Given a Loss Function, these techniques return the program which best fits the noisy dataset. A Loss Function, within this context, measures the cost of the input-output examples on which the program $p$ produces a different output than the output in the noisy dataset.

However, no previous research explores the connection between the best-fit program and the hidden underlying program which produced (via the Noise Source) the noisy data set. No previous research characterizes when the noisy program synthesis algorithm, working with a given Loss Function, will synthesize a program equivalent to the hidden underlying program. Nor does it specify how to pick an appropriate Loss Function to maximize the synthesis algorithm’s probability of synthesizing the correct program.

We formulate the correctness of Loss Function based noisy program synthesis techniques in terms of their ability to find the underlying hidden program. We achieve this by formalizing and specifying the underlying hidden probabilistic process which selects the hidden program, generates inputs, and corrupts the program’s outputs to produce the noisy data set provided to the synthesis algorithm. Our formalism uses the following concepts:

- **Program Space**: A set of programs $p$ defined by a grammar $G$.
- **Prior Distribution over Programs**: $p_p$ from which a hidden underlying program $p_h$ is sampled from.
- **Input Source**: A probability distribution $p_{\vec{x}}$ which generates $n$ input examples $\vec{x} = (x_1, \ldots, x_n)$.
- **Hidden Outputs**: The correct outputs computed by the hidden program $p_h$ over the input examples $\vec{x}$.
- **Noise Source**: A probability distribution $p_{\vec{\epsilon}}$, which corrupts the hidden outputs to construct a set of noisy outputs $\vec{\theta} = (y_1, \ldots, y_n)$.
- **Noisy Dataset**: The collection of inputs and noisy outputs $\mathcal{D} = (\vec{x}, \vec{\theta})$ which are visible to the synthesis algorithm to synthesize the hidden underlying program $p_h$.

Working with this formalism, we present the following two results:

- **Optimal Loss Functions**: Given information about the Noise Source, we formally define the optimal Loss Function and provide a technique to design the optimal Loss Function given perfect and imperfect information about the Noise Source.
- **Convergence**: We formalize the concept of convergence and the conditions required for the program synthesis technique to converge to the correct program as the size of the noisy data set increases.

1.2 Optimal Loss Functions

Different Loss Functions can be appropriate for different Noise Sources and synthesis domains. For example, the 0/1 Loss Function, which counts the number of input-output examples where the noisy dataset and synthesized program $p$ disagree, is a general Loss Function that assumes that the Noise Source can corrupt the output arbitrarily. The Damerau-Levenshtein (DL) Loss Function [5] measures the edit difference under character insertions, deletions, substitutions, and/or transpositions. It can extract information present in partially corrupted outputs and thus can be appropriate for measuring loss when text strings are corrupted by a discrete Noise Source.

If the Noise Source and a Loss Function are a good match, the noisy synthesis is often able to extract enough information left
This paper makes the following contributions:

- An algorithm to guarantee convergence.
- Select an appropriate Loss Function, which will allow our synthesis algorithm to produce the correct underlying program, given an Input Source and a Noise Source. These conditions allow us to formalize the concept of optimal Loss Function and derive a closed-form expression for optimal Loss Functions (Section 3).

We also derive optimal Loss Functions for Noise Sources previously studied in the noisy synthesis literature [13] (Section 5). These case studies provide a theoretical explanation for the empirical results provided in [13].

1.3 Convergence

Convergence is an important property which is well studied in the statistics and machine learning literature [10, 15]. Convergence properties connect the size of the noisy dataset to the probability of synthesizing a correct underlying program. Given a synthesis setting (i.e., a set of programs, an Input Source, a Noise Source, a prior distribution, and a Loss Function), we define a convergence property that allows us to guarantee that the probability of the synthesis algorithm producing the correct underlying program increases as the size of the noisy dataset increases.

This is the first paper to formulate the conditions required for the synthesis algorithm to have convergence guarantees. Given an Input Source and a Noise Source, these conditions allow us to select an appropriate Loss Function, which will allow our synthesis algorithm to guarantee convergence.

1.4 Contributions

This paper makes the following contributions:

- **Formalism**: It presents and formalizes a framework for noisy program synthesis. The framework includes the concepts of a prior probability distribution over programs, an Input Source, and a Noise Source. It then formalizes how these structured distributions interact to form a probabilistic process which creates the Noisy Dataset. Given a noisy data set, this formalism allows us to define a correct solution for the noisy synthesis problem over that data set.

- **Optimal Loss Function**: It presents a framework to calculate the expected reward associated with predicting a program, given a noisy dataset. Based on this framework, it presents a technique to design Loss Functions which are optimal, i.e., have the highest probability of returning the correct solution to the synthesis problem.

- **Convergence**: It formalizes the concept of convergence, i.e., for any probability tolerance \( \delta \), there exists a threshold dataset size \( k \), such that, given a random noisy dataset of size \( n \geq k \) generated by a hidden program \( p \), the synthesis algorithm will synthesize a program equivalent to \( p \) with probability greater than \( \delta \). Based on this definition, this paper formulates conditions on the Input Source, the Loss Function, and the Noise Source which ensure convergence.

- **Case Studies**: It presents multiple case studies highlighting Input Sources, Noise Sources, and Loss Functions which break these conditions and thus make convergence impossible. It also proves that these conditions hold for some Noise Sources and Loss Functions studied in prior work, thus providing a theoretical explanation for the reported empirical results.

2 SYNTHESES OVER NOISY DATA

Within this section, we formalize a conceptual framework to view noisy program synthesis. We also introduce a modified noisy synthesis algorithm to accommodate the concept of prior distribution of programs.

2.1 Noisy Program Synthesis Framework

We first define the programs we consider, how inputs to the program are specified, and the program semantics. Without loss of generality, we assume programs \( p \) are specified as parse trees in a domain-specific language (DSL) grammar \( G \). Internal nodes represent function invocations; leaves are constants or 0-arity symbols in the DSL. A program \( p \) executes in an input \( x \). \( [p]|x \) denotes the output of \( p \) on input \( x \) (if defined in Figure 1).

All valid programs (which can be executed) are defined by a DSL grammar \( G = (T, N, P, s_0) \) where:

- \( T \) is a set of terminal symbols. These may include constants and symbols which may change value depending on the input \( x \).
- \( N \) is the set of nonterminals that represent subexpressions in our DSL.
- \( P \) is the set of production rules of the form \( s \rightarrow f(s_1, \ldots, s_n) \), where \( f \) is a built-in function in the DSL and \( s, s_1, \ldots, s_n \) are non-terminals in the grammar.
- \( s_0 \in N \) is the start non-terminal in the grammar.

We assume that we are given a black box implementation of each built-in function \( f \) in the DSL. In general, all techniques explored within this paper can be generalized to any DSL which can be specified within the above framework. This is a standard way of specifying DSLs in program synthesis literature [13, 22]

**Example 1.** The following DSL defines expressions over input \( x \), constants 2 and 3, and addition and multiplication:

\[
\begin{align*}
\text{n} & : = x \mid n + t \mid n \times t; \\
\text{t} & : = 2 \mid 3;
\end{align*}
\]

**Notation:** We will use the notation \( p(x) \) to denote \([p]|x\) within the rest of the paper. Given a vector of input values \( \vec{x} = (x_1, x_2, \ldots, x_n) \), we use the notation \( p(\vec{x}) \) to denote vector \((p(x_1), p(x_2), \ldots, p(x_n))\). We use the notation \( G \subseteq G \) to denote a finite subset of programs accepted by \( G \). Given an input vector \( \vec{x} \), we use the notation \( G[\vec{x}] \) to denote the set of outputs produced by programs in \( G \). Formally,

\[
G[\vec{x}] = \{ p(\vec{x}) \mid p \in G \}
\]

Given a set of programs \( G \), an input vector \( \vec{x} \), and an output vector \( \vec{y} \), we use the notation \( G[\vec{x}] = \vec{y} \) to denote the set of programs in \( G \),
which given input vector \( \mathbf{x} \) produce the output \( \mathbf{z} \). Formally,

\[
G_{\mathbf{x}} \mathbf{z} = \{ p \in G \mid p[\mathbf{x}] = \mathbf{z} \}
\]

Given two programs \( p_1, p_2 \in G \), we use the notion \( p_1 \approx p_2 \) to imply program \( p_1 \) is equivalent to \( p_2 \), i.e., for all \( \mathbf{x} \in X \), \( p_1(\mathbf{x}) = p_2(\mathbf{x}) \). Given an input vector \( \mathbf{x} \), we use the notion \( p_1 =_{\mathbf{x}} p_2 \) to imply that program \( p_1 \) and \( p_2 \) have the same outputs on input vector \( \mathbf{x} \), i.e., \( p_1[\mathbf{x}] = p_2[\mathbf{x}] \).

Given a set of programs \( G \), we use the notation \( G_{p_h} \) to denote the set of programs in \( G \) which are equivalent to program \( p_h \). Formally,

\[
G_{p_h} = \{ p \in G \mid p \approx p_h \}
\]

We use the notation \( G_{p_h}^c \) to denote the set of programs in \( G \) which are not equivalent to program \( p_h \). Formally,

\[
G_{p_h}^c = \{ p \in G \mid p \neq p_h \}
\]

**Prior Distribution over Programs:** Given a set of programs \( G \), let \( \rho_p \) be a prior distribution over programs in \( G \). We assume the hidden underlying program \( p_h \) is sampled from the prior distribution with probability \( \rho_p(p_h) \). Prior distributions over models are a standard way to introduce prior knowledge without systems which learn from data [10]. The prior distribution allows us to direct the synthesis procedure by introducing information about the underlying process. The synthesis algorithm uses this distribution to trade off performance over the noisy data set and synthesizing the most likely program.

Within this paper, we assume that \( \rho_p \) is expressed via a set \( G \subseteq G \) and a weight function \( w_G \) over the DSL \( G \). We assume, we are given a weight function \( w_G \), which assigns a weight to each terminal \( t \in T \), and each production \( s \leftarrow f(s_1, \ldots, s_k) \) is \( P \).

Given a program \( p \in G \), \( \rho_p(p) \) is defined as

\[
\rho_p(p) = \frac{w(s_0, p)}{\sum_{p \in G} w(s_0, p)}
\]

where \( w(p) \) is computed via the following recursive definition:

\[
w(t, t) := w_G(t, t)
\]

\[
w(s, f(e_1, \ldots, e_n)) := \sum_{p \in D} w_G(p) \times \prod_{i=1}^{n} w(s_i, e_i)
\]

where \( pD \) are productions of the form \( s \leftarrow f(s_1, \ldots, s_n) \).

**Input Source:** An Input Source is a probabilistic process which generates the inputs provided to the hidden underlying program. Formally, an Input Source is a probability distribution \( \rho_n \), from which \( n \in \mathbb{N} \) inputs \( \mathbf{x} = (x_1, \ldots, x_n) \) are sampled with probability \( \rho_n((x_1, \ldots, x_n) \mid p_h, n) \). Where \( p_h \) is hidden underlying program.

**Note:** Within this paper, we assume that inputs are independent of the hidden program \( p_h \). We leave the exploration of idea that the inputs maybe selected to provide information about the hidden program for future work.

**Noise Source:** A Noise Source \( N \) is a probabilistic process which corrupts the correct outputs returned by the hidden program to create the noisy outputs. Formally, a Noise Source \( N \) is attached with a probability distribution \( \rho_N \). Given a hidden program \( p_h \) and a set of outputs \( (z_1, \ldots, z_n) \), the noisy outputs \( (y_1, \ldots, y_n) \) are sampled from the probability distribution \( \rho_N \), with probability

\[
\rho_N((y_1, \ldots, y_n) \mid (z_1, \ldots, z_n))
\]

**Note:** Within this paper, we will use the notation \( N \) and \( \rho_N \) for a Noise Source interchangeably.

**Noisy Dataset:** A noisy dataset \( D \) is a set of input values, denoted by \( \mathbf{x} \) and noisy output values, denoted by \( \mathbf{y} \). Within this paper, we assume the dataset \( D = (\mathbf{x}, \mathbf{y}) \) of size \( n \) is constructed by the following process:

- A hidden program \( p_h \) is sampled from \( G \) with probability \( \rho_p(p_h) \).
- \( n \) inputs \( \mathbf{x} = (x_1, \ldots, x_n) \) are sampled from probability distribution \( \rho_p((x_1, \ldots, x_n) \mid n) \).
- We compute outputs \( \mathbf{y} = (y_1, \ldots, y_n) \), where \( y_i = p_h(x_i) \).
- The Noise Source introduces noise by corrupting outputs \( z_1, \ldots, z_n \) to \( \mathbf{y} = (y_1, \ldots, y_n) \) with probability

\[
\rho_N((y_1, \ldots, y_n) \mid (z_1, \ldots, z_n))
\]

**Correctness:** The goal of the synthesis procedure is to find a program \( p \) such that \( p \) is equivalent to \( p_h \), i.e., \( p \approx p_h \). But even in absence of noise, it maybe impossible to synthesize a program equivalent to \( p_h \). Therefore, similar to Noise-Free programming-by-example systems [18, 22], we relax the synthesis requirements to find a program \( p \) such that \( p \) and \( p_h \) have the same outputs on the input vector \( \mathbf{x} \) (\( p \approx_{\mathbf{x}} p_h \)), given input vector \( \mathbf{x} = (x_1, \ldots, x_n) \). Note that, even in this relaxed setting, the noise introduced by the Noise Source may make it impossible to acquire enough information to synthesize the correct program.

In Section 4, we will tackle the harder problem of convergence, i.e., probability of synthesizing a program \( p \), equivalent to the hidden program \( p_h (p \approx p_h) \) will improve as we increase the size of the noisy dataset.

**2.2 Loss Function**

Given a dataset \( D = (\mathbf{x}, \mathbf{y}) \) (inputs \( \mathbf{x} = (x_1, \ldots, x_n) \) and outputs \( \mathbf{y} = (y_1, \ldots, y_n) \)), and a program \( p \), a Loss Function \( \mathcal{L}(p, D) \) calculates how incorrect the program is with respect to the given dataset. A Loss Function, only depends on outputs \( \mathbf{y} \), and the outputs of the program \( p \) over inputs \( \mathbf{x} \) (i.e., \( \mathbf{z} = p[\mathbf{x}] \)). Given programs \( p_1, p_2 \), such that for all \( \mathbf{x} \in \mathbf{X} \), \( p_1(\mathbf{x}) = p_2(\mathbf{x}) \), then \( \mathcal{L}(p_1, D) = \mathcal{L}(p_2, D) \). We can rewrite the Loss Function as

\[
\mathcal{L}(p, D) = \mathcal{L}(p[\mathbf{x}], \mathbf{y})
\]

**Definition 1. Piecewise Loss Function:** A Piecewise Loss Function \( \mathcal{L}(p, D) \) can be expressed in the following form

\[
\mathcal{L}(p, D) = \sum_{i=1}^{n} L(p(x_i), y_i)
\]

where \( D = (\mathbf{x}, \mathbf{y}), \mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n), \) and \( L(z, y) \) is the per-example Loss Function.

**Definition 2. 0/1 Loss Function:** The 0/1 Loss Function \( \mathcal{L}_{0/1}(p, D) \) counts the number of input-output examples where \( p \) does not agree with the data set \( D \):

\[
\mathcal{L}_{0/1}(p, D) = \sum_{i=1}^{n} 1 \text{ if } (y_i \neq p(x_i)) \text{ else } 0
\]
We modify the synthesis algorithm presented in [13] to include the existence of state \( \vec{q}_1 \) (terminal or non-terminal) in our language or input symbol for constructing a FTA that accepts all programs in grammar \( \mathcal{A} \).

### 2.3 Complexity Measure

As previously defined in literature [13], given a program \( p \), a Complexity Measure \( C(p) \) ranks programs independent of the input-output examples in the dataset \( D \). This measure is used to synthesize a simple program out of the current correct candidate programs. Formally, a complexity measure is a function \( C(p) \) that maps each program \( p \) expressible in \( G \) to a real number. The following \( Cost(p) \) complexity measure computes the complexity of given program \( p \) presented as a parse tree recursively as follows:

\[
\text{Cost}(t) = \text{cost}(t) \\
\text{Cost}(f(e_1, e_2, \ldots, e_k)) = \text{cost}(f) + \sum_{i=1}^{k} \text{Cost}(e_i)
\]

where \( t \) and \( f \) are terminals and built-in functions in our DSL \( \mathcal{G} \) respectively. Setting \( \text{Cost}(t) = \text{cost}(f) = 1 \) delivers a complexity measure \( \text{Size}(p) \) that computes the size of \( p \).

### 2.4 Program Synthesis over Noisy Data

We modify the synthesis algorithm presented in [13] to include the concept of prior distributions over programs. The synthesis algorithm builds upon Finite Tree Automata.

**Definition 3 (FTA).** A finite tree automaton (FTA) over alphabet \( F \) is a tuple \( \mathcal{A} = (Q, F, Q_f, \Lambda) \) where \( Q \) is a set of states, \( F \subseteq Q \) is the set of accepting states, and \( \Lambda \) is a set of transitions of the form \( f(q_1, \ldots, q_k) \rightarrow q \) where \( q, q_1, \ldots, q_k \) are states, \( f \in F \).

Given an input vector \( \vec{x} = (x_1, \ldots, x_n) \), Figure 2 presents rules for constructing a FTA that accepts all programs in grammar \( \mathcal{G} \). The alphabet of the FTA contains built-in functions within the DSL. The states in the FTA are of the form \( q_s \), where \( s \) is a symbol (terminal or non-terminal) in \( \mathcal{G} \) and \( \vec{c} \) is a vector of values. The existence of state \( q_s \) implies that there exists a partial program which can map \( \vec{x} \) to values \( \vec{c} \). Similarly, the existence of transition \( f(q^{e_1}_s, q^{e_2}_s, \ldots, q^{e_k}_s) \rightarrow q^e_s \) implies \( \forall j \in [1, n], f(\vec{c}_{1,j}, \vec{c}_{2,j}, \ldots, \vec{c}_{k,j}) = \vec{c}_{j} \).

The **Term** rule states that if we have a terminal \( t \) (either a constant in our language or input symbol), execute it with the input \( x_i \) and construct a state \( q^e_t \) (where \( \vec{c}_i = [t]x_i \)). The **Prod** rule states that, if we have a production rule \( f(s_1, s_2, \ldots, s_k) \rightarrow s \in \Lambda \), and there exists states \( q_{s_1}^*, q_{s_2}^*, \ldots, q_{s_k}^* \in Q \), then we also have state \( q^e_s \) in the FTA and a transition \( f(q_{s_1}^*, q_{s_2}^*, \ldots, q_{s_k}^*) \rightarrow q^e_s \).

The FTA Final rule (Figure 2) marks all states \( q^e_{s_0} \) with start symbol \( s_0 \) as accepting states regardless of the values \( \vec{c} \) attached to the state.

The FTA divides the set of programs in the DSL into subsets. Given an input set \( \vec{x} \), all programs in a subset produce the same outputs (based on the accepting state), i.e., if a program \( p \) is accepted by the accepting state \( q^e_{s_0} \), then \( p(\vec{x}) = \vec{c} \).

In general, the rules in Figure 2 may result in a FTA which has infinitely many states. To control the size of the resulting FTA, we do not add a new state within the constructed FTA if the smallest tree it will accept is larger than a given threshold \( d \). This results in a FTA which accepts all programs which are consistent with the input-output example but are smaller than the given threshold (it may accept some programs which are larger than the given threshold but it will never accept a program which is inconsistent with the input-output example). This is a standard practice in the synthesis literature [18, 22].

We denote the FTA constructed from DSL \( \mathcal{G} \), given input vector \( \vec{x} \) and threshold \( b \) as \( \mathcal{A}_{\mathcal{G}}^d(\vec{x}) \). We omit the subscript grammar \( \mathcal{G} \) and threshold \( b \) wherever it is obvious from context.

**Example 2.** Consider the DSL presented in Example 1. Given input \( x = 1 \), Figure 3 presents the FTA which represents all programs of height less than 3.

For readability, we omit the states for terminals 2 and 3.

Algorithm 1 presents a modified version of the synthesis algorithm presented by [13] to synthesize programs within the noisy synthesis settings. The algorithm first constructs a FTA over input vector \( \vec{x} \). It then finds the state \( q^e_{s_0} \) which optimizes \( L(\vec{c}, \vec{g}) - \log \pi(q^e_{s_0}, \mathcal{A}, d) \), where \( \pi(q^e_{s_0}, \mathcal{A}, d) \) is defined in Figure 4.
w(q′ₐ, m) denotes the sum of weights of partial programs of size ≤ m, accepted by the state q′ₐ. π(q′ₐ, G, A, d) denotes the sum of probabilities of all programs accepted by state q′ₐ, of size ≤ d. Note that, π(q′ₐ, G, A, d) = ρₜ(Gₜ), where G is the subset of programs in G of height ≤ d.

Note that c minimizes L(c, y) − log ρₜ(Gₜ), where Gₜ is the set of programs in G which map input x to output c.

Given the optimal state q*, we find the program accepted by q* which minimizes our complexity metric C(p). The following equations hold true:

\[ c = \arg \min_{c \in G} L(c, y) - \log \rhoₜ(Gₜ) \]

Algorithm 1: Synthesis Algorithm

**Input:** DSL G, prior distribution π, threshold d, data set \( D = (x, y) \), Loss function L, complexity measure C

**Result:** Synthesized program \( p^* \)

\[ A_D(x) = (Q, F, Q_f, \Lambda) \]

q* \( \leftarrow \) argmin \( q \in Q \) \( L(c, y) = -\log \rhoₜ(c) \in G \) \( p^* \in G \)

Given a FTA \( A \), we can use dynamic programming to find the minimum complexity parse tree (under the certain Cost(p) like measures) accepted by \( A \) [9]. In general, given a FTA \( A \), we assume we are provided with a method to find the program \( p \) accepted by \( A \) which minimizes the complexity measure.

### 3.2 OPTIMAL LOSS FUNCTION

Within this section, we will formalize the connection between the Noise Source and the Loss Function. We then derive a closed form expression for the optimal Loss Function in case of perfect and imperfect information about the Noise Source.

#### 3.1 Optimal Loss Function given a Noise Source

Given dataset \( D = (x, y) \), let us assume that a synthesis procedure predicts \( p \) to be the underlying hidden program. The Expected Reward is the probability that \( p \) generated dataset \( D \). Formally, given \( D = (x, y) \),

\[ E(p|x, y) = \sum_{p \in G} I(p = \tau) p_h(p_h|x, y) \]

= \[ \frac{1}{\rho(y | x)} \sum_{p_h \in G} I(p = \tau) p_h(p_h|x, y) \rho_N(y|p_h) \]

\[ = \frac{1}{\rho(y | x) \rho_N(y|p_h)} \sum_{p_h \in G} I(p = \tau) p_h(p_h|y) \rho_N(y|p_h) \]

\[ E(p | x, y) = \frac{1}{\rho(y | x)} \rho_N(y|p_h) \rho_N(y|p_h) \]

Since the above reward only depends on the output of program \( p \) on input set \( x \), we can rewrite the above reward as

\[ E(c | x, y) = \frac{1}{\rho(y | x)} \rho_N(y|p_h) \rho_N(y|p_h) \]

where \( c = p[x] \).

Therefore, given dataset \( D \), the probability \( D \) was generated by the synthesized program \( p \) is \( E(p[x] | x, y) \).

Given a Loss Function \( L \) and prior distribution \( \rho_N \), our synthesis algorithm is correct with probability \( E(p_1[x] | x, y) \), where \( p_1 \) is:

\[ c = \arg \min_{c \in G} L(c, y) - \log \rho_N(c) \]

\[ \rho_N \] will be the ideal prediction if it maximizes the expected reward.

Let \( E_1(p[x] | x, y) = -\log E(p[x] | x, y) \).

Therefore, given a set of programs \( G \), dataset \( D \), prior distribution \( \rho_N \), and Noise Source \( N \), and no other information about the hidden program \( p_h \), the synthesis algorithm will always return the program which maximizes the expected reward if the Loss Function \( L(c, y) \) is equal to \( -\log \rho_N(c) + C \), for any constant \( C \). Hence, within this setting, \( -\log \rho_N(c) + C \) is the optimal Loss Function.

Note that, in expectation, no other Loss Function will outperform the above optimal version.

### 3.2 Optimal Loss Function given imperfect information

Let us now consider a scenario where we are presented with some imperfect information about the Noise Source, i.e., the Noise Source corrupting the output takes belongs to the set \( N \) and we are presented with a prior probability distribution \( \rho_N \) over possible Noise Sources in \( N \). The probability that \( N \) is the underlying Noise Source corrupting the correct dataset is \( \rho_N(N) \).

Given dataset \( D = (x, y) \), we assume it was constructed by the following underlying process:

- A Noise Source \( N \) is sampled from the prior distribution \( \rho_N \).
- A hidden program \( p_h \) is sampled from the set \( G \) with probability distribution \( \rho_p \).
- \( n \) inputs \( x = (x_1, \ldots, x_n) \) are sampled from the set \( G \) with probability distribution \( \rho_p \) with probability \( \rho_p(n) \).
- The process then computes outputs \( z = (z_1, \ldots, z_n) \), where \( z_i = p_h(x_i) \).
- The sampled Noise Source \( N \) introduces noise by corrupting outputs \( z_1, \ldots, z_n \) to \( y \) with probability \( \rho_C(y | z_1, \ldots, z_n) \) with probability

\[ \rho_C(y | z_1, \ldots, z_n) \]

which is equal to

\[ \rho_N(y | z_1, \ldots, z_n) \]

**Expected Reward**: Given dataset \( D \), let us assume that a synthesis procedure predicts \( p \) to be the underlying hidden program. The Expected Reward is the probability that \( p \) is the hidden underlying program. Formally, given \( D = (x, y) \),

\[ E(p[x] | x, y) = \sum_{p_h \in G} I(p = \tau) p_h(p_h|x, y) \]

\[ \propto \sum_{N \in N} \sum_{p_h \in G} I(p = \tau) p_h(p_h|x, y) \rho_C(y | p_h(x), N) \rho_N(N) \]

\[ = \sum_{N \in N} \sum_{p_h \in G} I(p = \tau) p_h(p_h|x, y) \rho_C(y | p_h(x), N) \rho_N(N) \]
Within this section, we explore the conditions under which the synthesizing program equivalent of a dataset will return an output dataset which will differentiate any two programs. For any underlying program and the synthesis algorithm, even in the absence of noise, cannot differentiate between datasets produced assuming an Input Source is the hidden program. This issue comes up in traditional Noise-Free programming-by-example synthesis as well. Noise-Free synthesis frameworks assume that the Input Source will eventually provide input-output examples to differentiate the underlying program from all other possible candidate programs to guarantee convergence. We take a similar approach and constrain the Input Source to provide convergence guarantees.

Let \( d \) be some distance metric which measures distance between two equally sized output vectors. For any underlying program and a probability tolerance, increasing the dataset size should eventually allow us to differentiate between this program and any other program in \( G \).

\[
\pi(Q_{i_0}, G, A, d) = \frac{w(q^{i_0}_{i_0}, d)}{\sum_{q^{i_0}_{i_0} \in Q_f} w(q^{i_0}_{i_0}, d)}
\]

**Figure 4: Rules for calculating \( \pi(q^{i_0}_{i_0}, G, A, d) \)**

\[
\rho_p(G_{\bar{x}, p}[\bar{y}]) = \rho_p(G_{\bar{x}, p}[\bar{y}]) \left( \sum_{N \in \mathbb{N}} \rho_C(\bar{y} | p[\bar{x}], N) \rho_N(N) \right)
\]

Let \( E_L(p[\bar{x}] | \bar{x}, \bar{y}) = -\log E(p[\bar{x}] | \bar{x}, \bar{y}) \)

\[
= -\log \rho_p(G_{\bar{x}, p}[\bar{z}]) + ( -\log (\sum_{N \in \mathbb{N}} \rho_N(N) \rho_C(\bar{y} | p[\bar{x}])) )
\]

Therefore, given a set of programs \( G \), dataset \( D \), prior distribution \( \rho_p \), and no other information about the hidden program \( p_h \) or the hidden Noise Source, the synthesis algorithm will always return the program which maximizes the expected reward if the Loss Function \( L(\bar{x}, \bar{y}) \) is equal to

\[
-\left[\log \sum_{N \in \mathbb{N}} \rho_C(\bar{y} | \bar{x}, N) \rho_N(N) \right] + C = -\log E[\rho_N(\bar{y} | \bar{x})] + C
\]

for any constant \( C \). Hence, the optimal Loss function, in presence of imperfect information of the Noise Source, is the negative log of the expected probability of output \( \bar{z} \) being corrupted to noisy output \( \bar{y} \).

## 4 CONVERGENCE

Within this section, we explore the conditions under which the synthesis algorithm will have convergence guarantees.

Given a synthesis setting (i.e., a finite set of programs \( G \), an Input Source \( \rho_i \), a Noise Source \( \rho_N \), a prior probability \( \rho_p \), and a Loss Function \( L(\bar{x}, \bar{y}) \), convergence property allows us to guarantee synthesis algorithm’s output will be the correct underlying program with high probability if we are providing the algorithm with a large enough dataset.

Given a Noise Source \( N \), a Loss Function \( L(\bar{x}, \bar{y}) \), prior probability \( \rho_p \), a positive probability hidden program \( p_h \) (i.e., \( \rho_p(p_h) > 0 \)), and a dataset size \( n \), let \( Pr[p^n_h \approx p_h | p_h, N] \) denote the probability on synthesizing program equivalent \( p_h \) on a random data set \( (\bar{x}, \bar{y}) \) of size \( n \), constructed assuming \( p_h \) as the hidden program. Formally, \( Pr[p^n_h \approx p_h | p_h, N] \) is the probability of the following process returning true:

- Sample \( n \) inputs \( \bar{x} \) with probability \( \rho_i(\bar{x} | n) \).
- \( \bar{z}_h \approx p_h[\bar{x}], \bar{y} \) is sampled from the distribution \( \rho_N(\bar{y} | \bar{z}_h) \).
- Return true, if for all programs \( p \in G_p^C \):
  \[
  L(p_h[\bar{x}], \bar{y}) - \log \rho_p(G_{\bar{z}, p_h}[\bar{z}]) > L(p[\bar{x}], \bar{y}) - \log \rho_p(G_{\bar{z}, p}[\bar{z}])
  \]
  and there exists a program \( p^n_h \in G_p^C \), such that for all \( p \in G_p^C \), where \( p = \bar{z}_h, C(p^n_h) < C(p) \).

Note that if the procedure returns true then the following is true:

\[
p_h[\bar{x}] = \arg \min_{\bar{z} \in G[\bar{x}]} L(\bar{z}, \bar{y}) - \log \rho_p(G_{\bar{z}}), \ p^n_h = \arg \min_{p \in G_{\bar{z}, p_h}[\bar{z}]} C(p)
\]

i.e., \( p^n_h \approx p_h \) is synthesized.

**Principle**: If the synthesis algorithm will have convergence guarantees.

**Definition 4. Convergence**: Given a finite set of programs \( G \), a Loss Function \( L \), a Noise Source \( N \), an Input Source \( \rho_i \), and a probability distribution over programs \( \rho_p \), the synthesis will converge, if for all \( \delta > 0 \), there exists a natural number \( k \), such that

\[
Pr[p^n_h \approx p_h | p_h, N] \geq (1 - \delta)
\]

i.e., for all probability tolerance \( \delta > 0 \), we can find a minimum dataset size \( k \), such that for all hidden programs and dataset sizes \( \geq k \), the probability the synthesized program will be equivalent to the hidden program is greater than \((1 - \delta)\).

If the convergence property is true for a finite set of programs \( G \), a Loss Function \( L \), a Noise Source \( N \), an Input Source \( \rho_i \), and a probability distribution over programs \( \rho_p \), then for all \( \delta > 0 \), there exists a natural number \( k \), such that for all positive probability programs \( p_h \) (i.e., there is positive probability that \( p_h \) will be sampled), for any dataset of size greater than \( k \), with probability \((1 - \delta)\), the algorithm will synthesize a program equivalent to \( p_h \).

## 4.1 Differentiating Input Distributions

Even in absence of noise, the Input Source may hinder the ability of the synthesis algorithm to converge to the hidden program. For example, consider an Input Source which only generates vectors \( \bar{x} \), such that, there exist two programs \( p_1 \) and \( p_2 \) which have the same outputs on input \( \bar{x} \) (i.e., \( p_1[\bar{x}] = p_2[\bar{x}] \)). For such an Input Source, the synthesis algorithm, even in the absence of noise, cannot differentiate between datasets produced assuming \( p_1 \) is the underlying program and datasets produced assuming \( p_2 \) is the underlying program. This issue comes up in traditional Noise-Free programming-by-example synthesis as well. Noise-Free synthesis frameworks assume that the Input Source will eventually provide input-output examples to differentiate the underlying program from all other possible candidate programs to guarantee convergence. We take a similar approach and constrain the Input Source to provide convergence guarantees.

Let \( d \) be some distance metric which measures distance between two equally sized output vectors. For any underlying program and a probability tolerance, increasing the dataset size should eventually allow us to differentiate between this program and any other program in \( G \).

**Definition 5. Differentiating Input Source**: Given a set of programs \( G \) and a distance metric \( d \), an Input Source \( \rho_i \) is differentiating, if for a large enough dataset size, the distribution will return an input dataset which will differentiate any two programs within \( G \), with a high probability.
An Input Source is differentiating if for all \( \delta > 0 \) and \( \epsilon > 0 \), there exists a minimum dataset size \( k \), such that for dataset sizes \( n \geq k \) and all programs \( p_h \in G \), the following process returns true with probability greater than \( (1 - \delta) \):

- Sample \( \tilde{x} \) of size \( n \) from the distribution \( \rho_l(\tilde{x}) \).
- Return true if \( \forall p \in \bar{G}_{p_h}, d(p[\tilde{x}], p_h[\tilde{x}]) > \epsilon \).

Formally, given a set of programs \( G \) and a distance metric \( d \), an Input Distribution \( \rho_l \) is differentiating, if for all \( \delta > 0 \), for all distance \( \epsilon > 0 \), there exists a natural number \( k \), such that for all natural numbers \( n \geq k \), and for all programs \( p_h \in G \), the following statement is true:

\[
\int_{\tilde{x} \in \chi^n} 1(\forall p \in \bar{G}_{p_h}, d(p[\tilde{x}], p_h[\tilde{x}]) > \epsilon) \rho_l(d\tilde{x}) \geq (1 - \delta)
\]

i.e., When sampling input vectors \( \tilde{x} \) of length \( n \), with probability greater than \((1 - \delta)\), the distance between \( p_h[\tilde{x}] \) (output of program \( p_h \) on input \( \tilde{x} \)) and \( p[\tilde{x}] \), for all programs \( p \) not equivalent to \( p_h \), is greater than \( \epsilon \).

Having a differentiating Input Source ensures that as we increase the size of the dataset, a random dataset will contain inputs which will allow us to differentiate a program with other programs with high probability.

### 4.2 Differentiating Noise Sources

Even if we are given an Input Source which allows us to differentiate between the hidden underlying program and other programs in \( G \) in the absence of noise, the Noise Source can, in theory, make convergence impossible. For example, consider a Noise Source which corrupts all outputs \( z \) to a single noisy output value \( y^* \). A dataset corrupted by this Noise Source contains no information about the underlying correct outputs. A synthesis process cannot extract any information about the hidden underlying program from such a dataset. Therefore, no synthesis algorithm will be able to differentiate between different programs in the input program space. Restrictions have to be placed on the types of Noise Sources a synthesis algorithm can handle in order to provide convergence guarantees.

**Definition 6. Differentiating Noise Source**: Given a finite set of programs \( G \), a distance metric \( d \), and a Loss Function \( L \), a Noise Source \( \rho_N \) is differentiating, if for all \( \delta > 0 \) and \( \gamma > 0 \), there exists a natural number \( k \), and \( \epsilon \in \mathbb{R}^+ \), such that for all \( n \geq k \), for all vectors \( \tilde{z}_h \) of length \( n \), the following is true:

\[
\rho_N(\forall z \in Z^n \cdot L(\tilde{z}, \tilde{y}) - L(\tilde{z}_h, \tilde{y}) \leq \gamma \implies d(\tilde{z}, \tilde{z}_h) < \epsilon \ | \ z_h ] \geq (1 - \delta)
\]

If we are using the optimal Loss Function for the given Noise Source (Subsection 3.1), the above condition reduces to:

\[
\rho_N(\forall z \in Z^n \cdot \rho_N(\tilde{y} | \tilde{x}_h) \leq \gamma \implies d(\tilde{z}, \tilde{z}_h) < \epsilon \ | \ z_h ] \geq (1 - \delta)
\]

**Convergence**: Convergence is guaranteed in presence of a differentiating Input Source and a differentiating Noise Source.

**Theorem 1.** Given a finite set of programs \( G \), distribution \( \rho_N \) from which the programs are sampled, a Loss Function \( L \), a differentiating Input Source \( \rho_l \), and a differentiating Noise Source \( \rho_N \), then our synthesis algorithm will guarantee convergence.

We present the proof of this theorem in the Appendix A.1.

### 5 CASE STUDIES

In the previous section, we proved how having a differentiating Input Source and a differentiating Noise Source are sufficient for the synthesis algorithm to have convergence guarantees. Within this section, we will prove that the Noise Sources and Loss Functions studied in [13] fulfill these requirements, and therefore, allow the synthesis algorithm to provide convergence guarantees. We will then show how breaking these requirements makes convergence impossible. We will also show the importance of picking an appropriate distance metric \( d \) which connects the Input Source to the Noise Source.

#### 5.1 Differentiating Input Distributions

In the special case, where each element of the input vector \( \tilde{x} \) are i.i.d., the Differentiating Input Distribution condition can be simplified. For any vector \( \tilde{x} = (x_1, x_2, \ldots, x_n) \),

\[
\rho_l(\tilde{x}) = \prod_{j=1}^{n} \rho_l(x_j)
\]

Given any two equal length vectors \( \tilde{z} = (z_1, \ldots, z_n) \) and \( \tilde{z}_h = (z'_1, \ldots, z'_n) \), let \( d_i \) be a distance metric such that

\[
d(\tilde{z}, \tilde{z}_h) = \sum_{j=1}^{n} d_i(z_j, z'_j)
\]

We say the input distribution \( \rho_l \) is differentiating, if for all program \( p_h \in G \), and \( p \in \bar{G}_{p_h} \) there exists an input \( x_p \), such that \( \rho_l(x_p) > 0 \), and

\[
d_i(p(x_p), p_h(x_p)) > 0
\]

**Theorem 2.** If \( \rho_l \) is differentiating then \( \rho_l \) is differentiating.

We present the proof of this theorem in the Appendix (Theorem A.2).

#### 5.2 Differentiating Noise Distributions and optimal Loss Function

**Case 1**: We first consider the case where the noise distribution never introduces any corruptions in the correct output. Formally, for all \( n \in \mathbb{N} \), for all \( \tilde{z} \in Z^n \), \( \rho_N(\tilde{z} \mid \tilde{z}) = 1 \). Consider the Loss Function \( L_{0/\infty} \) and the counting distance metric \( d_c \):

**Definition 7. 0/∞ Loss Function**: The 0/∞ Loss Function \( L_{0/\infty}(\tilde{z}, \tilde{y}) = 0 \) if \( \tilde{z} = \tilde{y} \) and \( \infty \) otherwise.

**Definition 8. Counting Distance** The counting distance metric \( d_c \) counts the number of positions between two equal length vectors disagree on, i.e.,

\[
d_c((z_1, \ldots, z_n), (z'_1, \ldots, z'_n)) = \sum_{i=1}^{n} 1(z_i \neq z'_i)
\]

Note that for all \( \gamma > 0 \), for all \( n \geq 1 \), for all \( \epsilon \geq 1 \), and for all \( \tilde{z}_h \in Z^n \), the following is true:

\[
\int_{\tilde{y} \in Y^n} 1(\forall z \in Z^n \cdot L(\tilde{z}, \tilde{y}) - L(\tilde{z}_h, \tilde{y}) \leq \gamma \implies d(\tilde{z}, \tilde{z}_h) < \epsilon \ | \ z_h ] \geq (1 - \delta)
\]

\[
\int_{\tilde{y} \in Y^n} 1(\forall z \in Z^n \cdot L(\tilde{z}, \tilde{y}) - L(\tilde{z}_h, \tilde{y}) \leq \gamma \implies d(\tilde{z}, \tilde{z}_h) < \epsilon ) \rho_N(\tilde{y} | \tilde{z}_h)\]

\[
= \int_{\tilde{y} \in Y^n} 1(\forall z \in Z^n \cdot L(\tilde{z}, \tilde{y}) - L(\tilde{z}_h, \tilde{y}) \leq \gamma \implies d(\tilde{z}, \tilde{z}_h) < \epsilon ) = 1
\]
Therefore, in this case $\rho_N$ is differentiating.

Case 2: Consider the following $n$-Substitution Noise Source and $n$-Substitution Loss Function studied in previous work [13].

**Definition 9. $n$-Substitution Noise Source:**
The $n$-Substitution Noise Source $N_{NS}$, given an output vector $(z_1, \ldots, z_n)$ corrupts each string $z_i$ independently. For each string $z = c_1 \cdots c_k$, it replaces character $c_j$ with a random character not equal to $c_j$ with probability $\delta_{NS}$.

**Definition 10. $n$-Substitution Loss Function:**
The $n$-Substitution Loss Function $L_{NS}(\tilde{z}, \tilde{y})$ uses per-example Loss Function $L_{NS}$ that captures a weighted sum of positions where the noisy output string agrees and disagrees with the output from the synthesized program. If the synthesized program produces an output that is longer or shorter than the output in the noisy data set, the Loss Function is $\infty$:

$$L_{NS}(z_1, \ldots, z_n, y_1, \ldots, y_n) = \sum_{i=1}^{n} L_{NS}(z_i, y_i),$$

where $L_{NS}(z, y) = \begin{cases} \infty & |z| \neq |y| \\ \sum_{i=1}^{\infty} \log \delta_i & \text{if } z[i] \neq y[i] \\ 0 & \text{else} \end{cases}$

Note that this Loss Function is a linear transformation of the $n$-Substitution Loss Function proposed in [13].

**Definition 11. Length Distance Metric**
Given two equal length vectors of strings, the length distance metric $d_l$ counts the number of positions, which have unequal length strings in the two vectors, i.e.,

$$d_l((z_1, \ldots, z_n), (z'_1, \ldots, z'_n)) = \sum_{i=1}^{n} \mathbb{I}(|z_i| \neq |z'_i|)$$

**Theorem 3. $n$-Substitution Loss Function $L_{NS}$ is the optimal Loss Function for the $n$-Substitution Noise Source $N_{NS}$. Also, given Length Distance Metric $d_l$ and the $n$-Substitution Loss Function $L_{NS}$, $n$-Substitution Noise Source $N_{NS}$ is differentiating.**

We present the proof of this theorem in the Appendix (Theorem A.3).

Therefore, given a differentiating Input Source, the synthesis algorithm will guarantee convergence in presence of $n$-Substitution Noise Source and $n$-Substitution Loss Function.

Case 3: Consider the 1-Delete Loss Function $L_{1D}$ and the 1-Delete Noise Source $N_{1D}$ studied in previous work [13].

**Definition 12. 1-Delete Noise Source:** The 1-Delete Noise source $N_{1D}$ given an output vector $(z_1, z_2, \ldots, z_n)$ independently corrupts each output $z_i$ by deleting a random character, with probability $0 < \delta_{1D} < 1$. Formally,

$$\rho_{1D}(y_1, \ldots, y_n) \mid (z_1, \ldots, z_n) = \prod_{i=1}^{n} \rho_{1D}(y_i \mid z_i)$$

where

$$\rho_{1D}(z \mid z) = 1 - \delta_{1D}$$

$$\rho_{1D}(a \cdot b \mid a \cdot x \cdot c) = \frac{1}{\text{len}(a \cdot c \cdot b)} \delta_{1D}$$

where $c$ is a character.

**Definition 13. 1-Delete Loss Function:** The 1-Delete Loss Function $L_{1D}(z_1, \ldots, z_n, y_1, \ldots, y_n)$ assigns loss $-\log(1 - \delta_i)$ if the output $z_i$ from the synthesized program and the data set $y_i$ match exactly, $-\log \delta_i$ if a single deletion enables the output from the synthesized program to match the output from the data set, and $\infty$ otherwise (for $0 < \delta_i < 1$):

$$L_{1D}(z_1, \ldots, z_n, y_1, \ldots, y_n) = \sum_{i=1}^{n} L_{1D}(z_i, y_i),$$

where

$$L_{1D}(z, y) = \begin{cases} -\log(1 - \delta_i) & z = y \\ -\log \delta_i & a \cdot x \cdot b = z \land a \cdot b = y \land |x| = 1 \\ \infty & \text{otherwise} \end{cases}$$

Note that this Loss Function is a linear transformation of the 1-Delete Loss Function proposed in [13].

**Definition 14. DL-k Distance**
The DL-k Distance metric $d_{DLk}$ given two vectors of strings, returns the count of string pairs with Damerau-Levenshtein distance greater than equal to $k$ between them.

Formally,

$$d_{DL}(z_1, \ldots, z_n, z'_1, \ldots, z'_n) = \sum_{i=1}^{n} \mathbb{I}(L_{DL}(z_i, z'_i) \geq k)$$

where $L_{DL}(z, z')$ is the Damerau-Levenshtein metric [5].

**Theorem 4. 1-Delete Loss Function $L_{1D}$ is the optimal Loss Function for the 1-Delete Noise Source $N_{1D}$. Also, given DL-2 Distance Metric $d_{DL2}$ [5] and the 1-Delete Loss Function $L_{1D}$, 1-Delete Noise Source $N_{1D}$ is differentiating.**

We present the proof of this theorem in the Appendix (Theorem A.4).

Therefore, given a differentiating Input Source, the synthesis algorithm will guarantee convergence in presence of 1-Delete Noise Source and 1-Delete Loss Function.

**Case 4: Consider the Damerau-Levenshtein Loss Function $L_{DL}$ and the 1-Delete Noise Source $N_{1D}$ studied in previous work [13].**

**Definition 15. Damerau-Levenshtein (DL) Loss Function:** The DL Loss Function $L_{DL}(p, D)$ uses the Damerau-Levenshtein metric [5], to measure the distance between the output from the synthesized program and the corresponding output in the noisy data set:

$$L_{DL}(p, D) = \sum_{(x, y) \in D} L_{DL}(p, y),$$

where $L_{DL}(p, y)$ is the Damerau-Levenshtein metric [5].

This metric counts the number of single character deletions, insertions, substitutions, or transpositions required to convert one text string into another. Because more than 80% of all human misspellings are reported to be captured by a single one of these four operations [5], the DL Loss Function may be appropriate for computations that work with human-provided text input-output examples.

**Theorem 5. Given DL-2 Distance Metric $d_{DL2}$ and the Damerau-Levenshtein Loss Function $L_{DL}$, the 1-Delete Noise Source $N_{1D}$ is differentiating.**
We present the proof of this theorem in the Appendix (Theorem A.5).

Therefore, given a differentiating Input Source, the synthesis algorithm will guarantee convergence in presence of 1-Delete Noise Source and Damerau-Levenshtein Loss Function.

**Case 5:** Consider the Damerau-Levenshtein Loss Function $L_{DL}$ and the $n$-Substitution Noise Source $N_{NS}$ studied in previous work [13].

**Theorem 6.** Given Length Distance Metric $d_{l}$ and the Damerau-Levenshtein Loss Function $L_{DL}$, the $n$-Substitution Noise Source $N_{NS}$ is differentiating.

We present the proof of this theorem in the Appendix (Theorem A.6).

Therefore, given a differentiating Input Source, the synthesis algorithm will guarantee convergence in presence of $n$-Substitution Noise Source and Damerau-Levenshtein Loss Function.

**Connections to Empirical Results:** Handa et al. within their paper [13], empirically evaluated Damerau-Levenshtein and 1-Delete Loss Functions in presence of 1-Delete style noise. Both 1-Delete Loss Function and Damerau-Levenshtein Loss Function are able to synthesize the correct program in presence of some noise. This is in line with our convergence results. 1-Delete Loss Function is also able to tolerate datasets with more noise, compared to Damerau-Levenshtein Loss Function. Even when all input-output examples were corrupted, 1-Delete Loss Function was able to synthesize the correct program. Note that 1-Delete Loss Function is the optimal Loss Function in presence of 1-Delete Noise Source, therefore it has a higher probability of synthesizing the correct program.

They also evaluated $n$-Substitution Loss Function in presence of the $n$-Substitution Noise Source. In line with our theoretical results, their technique was able to synthesize the correct answer over datasets corrupted by $n$-Substitution Noise Source.

### 5.3 Non-Differentiating Input Distributions

Let $G$ be a set of two programs $p_1$ and $p_2$ and let $x^*$ be the only input on which $p_1$ and $p_2$ disagree on, i.e., $p_1(x^*) \neq p_2(x^*)$ and $\forall x \in X, x \neq x^* \implies p_1(x) = p_2(x)$. Let $\rho_1$ be a probability distribution over programs in $G$. Without loss of generality, we assume $C(p_2) \geq C(p_1)$. Let $\rho_i$ be the probability distribution over input vectors.

**Case 1:** The input process never returns a differentiating input, i.e., for all $n \in \mathbb{N}$ and for all $\bar{x} \in X^n$, $x^* \in \bar{x} \implies \rho_1(\bar{x}) = 0$.

For all $n \in \mathbb{N}$ and for all $\epsilon > 0$, the following statement is true for both $p_h = p_1$ and $p_2$:

$$\int_{\bar{x} \in X^n} 1(\forall p \in C_{p_h}, d(p[\bar{x}], p_h[\bar{x}]) > \epsilon) \rho_1(d\bar{x})$$

$$\leq \int_{\bar{x} \in X^n} 1(p_1[\bar{x}] \neq p_2[\bar{x}]) \rho_1(d\bar{x}) = 0$$

Therefore, $\rho_1$ is non-differentiating.

**Theorem 7.** In this case, $Pr[p_2^n \approx p_1 | p_1, N] = 0$

We present the proof of this theorem in the Appendix (Theorem A.7).

Since $p_2[\bar{x}] = p_1[\bar{x}]$ for all $\bar{x}$, the best move for any algorithm is to always predict the simplest program (which in this case is $p_2$). Hence, if the hidden program is not the simplest program, even in the absence of noise, the synthesis algorithm will not converge to the correct hidden program.

**Case 2:** The input process never returns a differentiating input with sufficient probability, i.e., there exists a $\delta_i$, such that for all $n \in \mathbb{N}$,

$$\int_{\bar{x} \in X^n} 1(\forall x^* \in \bar{x}) \rho_1(d\bar{x}) < \delta_i$$

For all $n \in \mathbb{N}$ and for all $\epsilon > 0$, the following statement is true for both $p_h = p_1$ or $p_2$:

$$\int_{\bar{x} \in X^n} 1(\forall p \in \bar{C}_{p_h}, d(p[\bar{x}], p_h[\bar{x}]) > \epsilon) \rho_1(d\bar{x})$$

$$\leq \int_{\bar{x} \in X^n} 1(p_1[\bar{x}] \neq p_2[\bar{x}]) \rho_1(d\bar{x}) < \delta_i$$

Therefore, $\rho_1$ is non-differentiating.

**Theorem 8.** In this case, $Pr[p_2^n \approx p_1 | p_1, N] < \delta_i$.

We present the proof of this theorem in the Appendix (Theorem A.8).

Hence, no matter how much we increase $n$ (i.e., the size of the dataset sampled), the probability that $p_1$ will be synthesized (in presence of any Loss Function) is less than $\delta_i$.

### 5.4 Non-Differentiating Noise Distributions

**Case 1:** We first consider the case where the Noise Source corrupts all information identifying the hidden program. Let $G$ be a set containing two programs, $p_a$ and $p_b$. $p_a$ takes an input string $x$ and appends character “a” in front of it. $p_b$, similarly, takes an input string $x$ and appends character “b” in front of it.

$$p_a := \text{append}(\text{”a”, } x)$$

$$p_b := \text{append}(\text{”b”, } x)$$

Let $N$ be a Noise Source which deletes the first character of the output string with probability $1$. Note that in presence of this Noise Source, no synthesis algorithm can infer which of the given programs $p_a$ or $p_b$ is the hidden program.

**Theorem 9.** $Pr[p_2^n \approx p_1 | p_a, N] + Pr[p_2^n \approx p_b | p_b, N] \leq 1$

We present the proof of this theorem in the Appendix (Theorem A.9).

Therefore, any gains in improving convergence of the synthesis algorithm by increasing $n$, assuming $p_a$ is the hidden program, will be on the cost of convergence when $p_b$ is the hidden program. Formally,

$$Pr[p_2^n \approx p_a | p_a, N] \geq 1 - \delta \implies Pr[p_2^n \approx p_b | p_b, N] \leq \delta$$

**Case 2:** The choice of Loss Function affects whether the Noise Source is differentiating or not. For example, consider a Noise Source which reveals information identifying the hidden program with high probability, but the Loss Function does not capture this information. 1-Delete Noise Source $N_{1D}$, given the 1-Delete Loss function $L_{1D}$ (and DL-2 distance metric $d_{DL2}$), is differentiating.

Since, $n$-Substitution Loss Function $L_{NS}$ penalizes a deletion with infinite loss, the following is true:

**Theorem 10.** Given $n$-Substitution Loss Function $L_{NS}$, 1-Delete Noise Source $N_{1D}$ is non-differentiating. In this case, the synthesis algorithm will never converge.
We present the proof of this theorem in the Appendix (Theorem A.10).
Handa et al. empirically showed within their paper [13] that the n-Substitution Noise Source requires all input-output examples to be correct (i.e., it cannot tolerate any noise) when the specific noise is introduced by the 1-Delete Noise Source. This theorem is in line with their experimental results.

### 5.5 Necessity of the distance metric $d$

The distance metric serves as an important link between the Input Source and the Noise Source. A simple distance metric (like the counting distance) makes it easy to construct a differentiating Input Source but a simple distance metric lacks the restrictions required to prove a Noise Source differentiating. Similarly, a restricted distance metric may enable us to easily prove the Noise Source differentiating but make it hard to construct a differentiating Input Source.

For example, consider the following case. Let $G$ be a set containing two programs $p_a$ and $p_b$. $p_a$ takes an input tuple (string and a boolean) $(x,b)$ and appends character “$a$” in front of $x$ if $b$ is true, else it appends “$aa$” in front of $x$. $p_b$ similarly, takes an input tuple (string and a boolean) $(x,b)$ and appends character “$b$” in front of $x$ if $b$ is true, else it appends “$bb$” in front of $x$. Formally,

$$
p_a := \text{append}(\text{if } b \text{ then } \text{“}a\text{” } else \text{ “}aa\text{”}, x)\\
p_b := \text{append}(\text{if } b \text{ then } \text{“}b\text{” } else \text{ “}bb\text{”}, x)
$$

Consider a Noise Source which deletes the first character with probability 1. For counting distance metric $d_c$, an Input Source $\rho_i$ which only returns inputs with $b$ set to true, is differentiating, i.e., the outputs produced by $p_a$ and $p_b$ will be of the from $c \cdot x$, where $c$ is “$a$” and “$b$” respectively.

**Theorem 11.** There exists no Loss Functions for which the above Noise Source is differentiating. Note that for $\rho_i$ described above, the synthesis algorithm will not converge as well.

We present the proof of this theorem in the Appendix (Theorem A.11).

But for DL-2 Distance metric $d_{DL2}$ (which only counts the number of disagreements have at least 2 Damerau-Levenshtein between them), then an Input Source $\rho_i$ has to return inputs with $b$ set to true, with high probability, to be differentiating.

**Theorem 12.** Consider a Loss Function $L_{ab}$ which checks if the first character appended to a string is either “$a$” or “$b$”. Given DL-2 Distance metric $d_{DL2}$ and the Loss Function $L_{ab}$, the Noise Source described above is differentiating.

In this case, if we pick a differentiating Input Source, our synthesis algorithm will have convergence guarantees.

We present the proof of this theorem in the Appendix (Theorem A.12).

But note that for space of programs considered in Subsection 5.4 (case 1), even through the Noise Source is differentiating, the complex $d_{DL2}$ distance metric will not work as a differentiating Input Source is impossible to construct for $d_{DL2}$ distance.

### 6 RELATED WORK

**Noise-Free Programming-by-examples:** The problem of learning programs from a set of correct input-output examples has been studied extensively [12, 20, 21]. Even though these techniques provide correctness guarantees (and convergence guarantees if all inputs which allow us to differentiate it from other programs are provided), these techniques cannot handle noisy datasets.

**Noisy Program Synthesis:** There has been recent interest in synthesizing programs over noisy datasets [13, 17]. These systems do not formalize the requirements for their algorithms to converge to the correct underlying program. Handa et al’s work uses Loss Functions as a parameter in their synthesis procedure to tailor it to different noise sources. It does not provide any process to either pick or design loss functions, given information about the noise source.

**Neural Program Synthesis:** There is extensive work that uses machine learning/deep neural networks to synthesize programs [3, 6, 19]. These techniques require a training phase, a differentiable loss function, and provide no formal correctness or convergence guarantees. There also has been a lack of work on designing an appropriate loss function, given information about the noise source.

**Learning Theory:** Learning theory captures the formal aspects of learning models over noisy data [4, 14, 15]. Our work takes concepts from learning theory and applies them to the specific context of synthesizing programs over noisy data. To the best of our knowledge, the special case of noisy program synthesis has never been explored in learning theory.

There has considerable work done on designing loss functions for training neural networks [11, 24]. To the best of our knowledge, these works do not theoretically prove the optimality of their loss functions with respect to a given noise source.

### 7 CONCLUSION

Learning models from noisy data with guarantees is an important problem. There has been recent work on synthesizing programs over noisy datasets using loss functions. Even though these systems have delivered effective program synthesis over noisy datasets, they do not provide any guidance for constructing loss functions given prior information about the noise source, nor do they comment on the convergence of their algorithms to the correct underlying program.

We are the first paper to formalize the hidden process which samples a hidden program and constructs a noisy dataset, thus formally specifying the correct solution to a given noisy synthesis problem. We are the first paper to formalize the concept of an optimal loss function for noisy synthesis and provide a closed form expression for such a loss function, in presence of perfect and imperfect information about the noise source. We are the first paper of formalize the constraints on the input source, the noise source, and the loss function which allow our synthesis algorithm to eventually converge to the correct underlying program. The case studies highlight why these constraints are necessary. We also provide proofs of convergence for the noisy program synthesis problems studied in literature.

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A APPENDIX

A.1 Convergence

**Theorem A.1.** Given a finite set of programs $G$, a prior distribution $\rho_p$, a Loss Function $L$, a differentiating Input Source $\rho_i$, and a differentiating Noise Source $\rho_N$, the synthesis algorithm I will have convergence guarantees.

**Proof.** Given a $\delta > 0$, let $\delta_i > 0$ and $\delta_N > 0$ be two real numbers such that $\delta = \delta_i \delta_N$.

For all positive probability programs $p_h \in G$:

$$\Pr[p^n_i \approx p_h \mid p_h, N] \geq \int_{x^n, y^n} \mathbf{1}(\forall p \in \mathcal{G}_{p_h} L(p[x], y) - \log \pi(G_{x,p[x]}) > L(p_h[x], y) - \log \pi(G_{x,p_h[x]})) d\rho_i(d\tilde{x}) \rho_N(d\tilde{y} \mid p_h[x])$$

Let $\gamma = \arg \min_{p \in G} \rho_p(G_p)$. Note that $\log \pi(G_{x,p[x]}) - \log \pi(G_{x,p_h[x]}) > \gamma$

$$\Pr[p^n_i \approx p_h \mid p_h, N] \geq \int_{x^n, y^n} \mathbf{1}(\forall p \in \mathcal{G}_{p_h} L(p[x], y) - L(p_h[x], y) > \gamma) \rho_N(d\tilde{y} \mid p_h[x])$$

$$\geq \int_{x^n, y^n} \mathbf{1}(\forall p \in G d(p_h[x], p[x]) \geq \epsilon) \rho_i(d\tilde{x})$$

$$\geq \int_{x^n, y^n} \mathbf{1}(\forall z \in Z^n d(p_h[x], z) \geq \epsilon \implies L(z, y) - L(p_h[z], y) > \gamma) \rho_N(d\tilde{y} \mid p_h[x])$$

If $n \geq n_N, \epsilon \geq \epsilon_N, \delta_N > 0$, the following statement is true, for any $z_h$ and $\gamma = \log \pi(\mathcal{G}_{p_h}) - \log \pi(G_{p_h})$:

$$\rho_N(\forall z \in Z^n. d(z, z_h) \geq \epsilon \implies L(z, y) - L(z_h, y) > \gamma \mid z_h \geq (1 - \delta_N)$$

And for $n \geq n_i$, and $\delta_i > 0$, the following is true:

$$\int_{x^n} \mathbf{1}(\forall p \in \mathcal{G}_{p_h} d(p_h[x], p[x]) > \epsilon) \rho_i(d\tilde{x}) \geq (1 - \delta_i)$$

Then for $n \geq \max(n_N, n_i)$, the following is true:

$$\Pr[p^n_i \approx p_h \mid p_h, N] \geq (1 - \delta_N)(1 - \delta_i) \geq (1 - \delta)$$

\[\square\]

A.2 Supplementary Material: Case Studies

**Differentiating Input Distributions**

**Theorem A.2.** If $\rho_I$ is differentiating then $\rho_i$ is differentiating.

**Proof.** Let for all $p \in G_{p_h}$, $x_p$ be an input such that $\rho_I(x_p) > 0$

$$d_i(p(x_p), p_h(x_p)) > 0$$

Let $\delta_i$ and $\epsilon_i$ be the largest rational numbers such that, for all $p$, $\rho_I(x_p) \geq \delta_i$ and $d_i(p(x_p), p_h(x_p)) \geq \epsilon_i$.

Let $\epsilon$ and $\delta$ be any rational number greater than 0. Let $m$ and $n_0$ be natural numbers such that $m \geq \frac{1}{\epsilon}$, for $n \geq \lceil G \rceil n_0$,

$$\sum_{j=0}^{m} a^n_j \delta^j (1 - \delta_i)^{n_0 - j} \leq \frac{\delta}{n}$$

$$\int_{x^n} \mathbf{1}(\forall p \in \mathcal{G}_{p_h} d(p_h[x], p[x]) > \epsilon) \rho_i(d\tilde{x})$$

$$\geq \int_{x^n} \mathbf{1}(\forall p \in \mathcal{G}_{p_h} d(p_h[x], p[x]) > m\epsilon_i) \rho_i(d\tilde{x})$$
where:

\[ \delta \geq \int_{\mathbf{x} \in \mathcal{X}} \mathbb{1}(\forall \mathbf{p} \in C_{\mathbf{p}}^m \text{ at least } m \text{ } x_p \text{ occur in } \mathbf{x}) \rho_i(d\mathbf{x}) \]

\[ = (1 - \sum_{j=0}^{m} c_{ij}^m \delta^j (1 - \delta)^{m-j})^{\mathbb{E}} \geq (1 - \delta) \]

Differentiating Noise Distributions and optimal Loss Function

Case 2:

**Theorem A.3.** \( n \)-Substitution Loss Function \( L_{NS} \) is the optimal Loss Function for the \( n \)-Substitution Noise Source \( N_{NS} \). Also, given Length Distance Metric \( d_i \) and the \( n \)-Substitution Loss Function \( L_{NS} \), \( n \)-Substitution Noise Source \( N_{NS} \) is differentiating.

**Proof.**

\[-\log \rho_{NS}(\{y_1, \ldots, y_n\} | \{z_1, \ldots, z_n\}) = \sum_{i=1}^{n} -\log \rho_{NS}(y_i | z_i) \]

where:

\[-\log \rho_{NS}(s'_1 \cdots s'_k | s_1 \cdots s_k) = \sum_{i=1}^{n} (-\mathbb{1}(s'_i = s_i) \log(1 - \delta_i)) + (-\mathbb{1}(s'_i \neq s_i) \log \delta_i) \]

Note that \( L_{NS} = -\log \rho_{NS} \). Hence \( n \)-Substitution Loss Function \( L_{NS} \) is the optimal Loss Function for \( n \)-Substitution Noise Source.

If \( d_i(\tilde{z}, \tilde{z}_h) \geq 1 \), then for all samples \( \tilde{y} \), \( L_{NS}(\tilde{z}, \tilde{y}) = \infty \). Therefore for all \( y > 0, \delta > 0, \)

\[ \rho_N(\forall \tilde{z} \in \mathbb{Z}^n d_i(\tilde{z}, \tilde{z}_h) \geq 1 \implies L(\tilde{z}, \tilde{y}) - L(\tilde{z}_h, \tilde{y}) = \infty > y \mid \tilde{z}_h) = 1 \geq (1 - \delta) \]

Case 3:

**Theorem A.4.** 1-Delete Loss Function \( L_{1D} \) is the optimal Loss Function for the 1-Delete Noise Source \( N_{1D} \). Also, given DL-2 Distance Metric \( d_{DL2} \) and the 1-Delete Loss Function \( L_{1D} \), 1-Delete Noise Source \( N_{1D} \) is differentiating.

**Proof.**

\[-\log \rho_{1D}(\{y_1, \ldots, y_n\} | \{z_1, \ldots, z_n\}) = \sum_{i=1}^{n} -\log \rho_{1D}(y_i | z_i) \]

where \( \rho_{1D}(z | z) = (1 - \delta_i) \) and \( \rho_{1D}(y | z) = \delta_i \) if \( y \) has exactly one character deleted with respect to \( z \).

Note that \( L_{1D} = -\log \rho_{1D} \). Hence 1-Delete Loss Function \( L_{1D} \) is the optimal Loss Function for 1-Delete Noise Source.

If \( d_{DL2}(\tilde{z}, \tilde{z}_h) \geq 1 \) then \( L_{1D}(\tilde{z}, \tilde{y}) = \infty \). Therefore for all \( y > 0, \delta > 0, \)

\[ \rho_N(\forall \tilde{z} \in \mathbb{Z}^n d_{DL2}(\tilde{z}, \tilde{z}_h) \geq 1 \implies L(\tilde{z}, \tilde{y}) - L(\tilde{z}_h, \tilde{y}) = \infty > y \mid \tilde{z}_h) = 1 \geq (1 - \delta) \]

Case 4:

**Theorem A.5.** Given DL-2 Distance Metric \( d_{DL2} \) and the Damerau-Levenshtein Loss Function \( L_{DL} \), the 1-Delete Noise Source \( N_{1D} \) is differentiating.

**Proof.** If \( d_{DL2}(\tilde{z}, \tilde{z}_h) \geq m \), then \( L_{DL}(\tilde{z}, \tilde{y}) - L_{DL}(\tilde{z}_h, \tilde{y}) \geq m \). Therefore for all \( y > 0, \delta > 0, \)

\[ \rho_N(\forall \tilde{z} \in \mathbb{Z}^n d_{DL2}(\tilde{z}, \tilde{z}_h) \geq y \implies L(\tilde{z}, \tilde{y}) - L(\tilde{z}_h, \tilde{y}) > y \mid \tilde{z}_h) = 1 \geq (1 - \delta) \]

Case 5:

**Theorem A.6.** Given Length Distance Metric \( d_i \) and the Damerau-Levenshtein Loss Function \( L_{DL} \), the \( n \)-Substitution Noise Source \( N_{NS} \) is differentiating.

**Proof.** If \( d_i(\tilde{z}, \tilde{z}_h) \geq m \), then \( L_{DL}(\tilde{z}, \tilde{y}) - L_{DL}(\tilde{z}_h, \tilde{y}) \geq m \) (Assuming replacements take place from a very large set). Therefore for all \( y > 0, \delta > 0, \)

\[ \rho_N(\forall \tilde{z} \in \mathbb{Z}^n d_i(\tilde{z}, \tilde{z}_h) \geq y \implies L(\tilde{z}, \tilde{y}) - L(\tilde{z}_h, \tilde{y}) > y \mid \tilde{z}_h) = 1 \geq (1 - \delta) \]
Non-Differentiating Input Distributions

Case 1:

**Theorem A.7.** In this case, \( Pr[p^* = p_1 \mid p_1, N] = 0 \)

**Proof.**
\[
Pr[p^* = p_1 \mid p_1, N] \leq \int \mathbb{1}(\mathcal{L}(p_2[\bar{x}], \bar{y}) - \\
\log \rho_p(G_{\bar{x},p_2[\bar{x}]}) > \mathcal{L}(p_1[\bar{x}], \bar{y}) - \log \rho_p(G_{\bar{x},p_1[\bar{x}]}) \rho_N(d\bar{y} \mid p_1[\bar{x}]) \rho_1(d\bar{x})) \\
= \int \mathbb{1}(\mathcal{L}(p_2[\bar{x}], p_1[\bar{x}]) - \log \rho_p(G_{\bar{x},p_2[\bar{x}]}) > \mathcal{L}(p_1[\bar{x}], p_1[\bar{x}]) - \log \rho_p(G_{\bar{x},p_1[\bar{x}]}) \rho_1(d\bar{x}) \\
= 0
\]

\[\square\]

Case 2:

**Theorem A.8.** In this case, \( Pr[p^* = p_1 \mid p_1, N] < \delta_i \).

**Proof.**
\[
Pr[p^* = p_1 \mid p_1, N] \leq \int \mathbb{1}(\mathcal{L}(p_2[\bar{x}], \bar{y}) - (\log \rho_p(G_{\bar{x},p_2[\bar{x}]}) > \mathcal{L}(p_1[\bar{x}], \bar{y}) - \log \rho_p(G_{\bar{x},p_1[\bar{x}]}) \rho_N(d\bar{y} \mid p_1[\bar{x}]) \rho_1(d\bar{x})) \\
= \int \mathbb{1}(\mathcal{L}(p_2[\bar{x}], p_1[\bar{x}]) - (\log \rho_p(G_{\bar{x},p_2[\bar{x}]}) > \mathcal{L}(p_1[\bar{x}], p_1[\bar{x}]) - \log \rho_p(G_{\bar{x},p_1[\bar{x}]}) \rho_1(d\bar{x}) \\
\leq \int \mathbb{1}(p_1[\bar{x}] \neq p_2[\bar{x}]) \rho_1(d\bar{x}) < \delta_i
\]

\[\square\]

Non-Differentiating Noise Distributions

Case 1:

**Theorem A.9.** \( Pr[p^* = p_a \mid p_a, N] + Pr[p^* = p_b \mid p_b, N] \leq 1 \)

**Proof.** For distance metric \( d_n \), an Input Source \( p_1 \) which only returns \((x, \text{true})\) is differentiating, as for all \( x \), \( p_a((x, \text{true})) \neq p_b((x, \text{true})) \). Note that for all Loss Functions \( \mathcal{L} \),
\[
\rho_N[\mathcal{L}((x_1, \ldots, x_n), (b \cdot x_1, \ldots, b \cdot x_n)) - \mathcal{L}((x_1, \ldots, x_n), (a \cdot x_1, \ldots, a \cdot x_n))] > y \mid (a \cdot x_1, \ldots, a \cdot x_n)] \\
+ \rho_N[\mathcal{L}((x_1, \ldots, x_n), (a \cdot x_1, \ldots, a \cdot x_n)) - \mathcal{L}((x_1, \ldots, x_n), (b \cdot x_1, \ldots, b \cdot x_n))] > y \mid (b \cdot x_1, \ldots, b \cdot x_n)] \\
= \rho_N[\mathcal{L}((x_1, \ldots, x_n), (b \cdot x_1, \ldots, b \cdot x_n)) - \mathcal{L}((x_1, \ldots, x_n), (a \cdot x_1, \ldots, a \cdot x_n))] > y \mid (b \cdot x_1, \ldots, b \cdot x_n)] \\
+ \rho_N[\mathcal{L}((x_1, \ldots, x_n), (a \cdot x_1, \ldots, a \cdot x_n)) - \mathcal{L}((x_1, \ldots, x_n), (b \cdot x_1, \ldots, b \cdot x_n))] > y \mid (b \cdot x_1, \ldots, b \cdot x_n)] \\
\leq 1
\]

\[\square\]

Case 2:

**Theorem A.10.** Given \( n \)-Substitution Loss Function \( \mathcal{L}_{nS} \), 1-Delete Noise Source \( N_{1D} \) is non-differentiating. In this case, the synthesis algorithm will never converge.

**Proof.** If even a single deletions happen in \( \bar{y} \), then \( \mathcal{L}_{nS}(\bar{x}_n, \bar{y}) = \infty \). The prob of no deletions is equal to \((1 - \delta_i)^n \) which decreases with \( n \), therefore in this case 1-Delete Noise Source is non-differentiating.

\[\square\]

Necessity of the distance metric \( d \)

**Theorem A.11.** There exists no Loss Functions for which the above Noise Source is differentiating. Note that for \( p_1 \) described above, the synthesis algorithm will not converge as well.
Therefore, if for any $\rho_1^N$ which only returns $(x, \text{true})$ is differentiating, as for all $x$, $p_a((x, \text{true})) \neq p_b((x, \text{true}))$. Note that for all Loss Functions $\mathcal{L}$,  
\[
\rho_N[\mathcal{L}(\langle x_1, \ldots, x_n \rangle, \langle b \cdot x_1, \ldots, b \cdot x_n \rangle) - \mathcal{L}(\langle x_1, \ldots, x_n \rangle, \langle a \cdot x_1, \ldots, a \cdot x_n \rangle)] > \gamma \mid \langle a \cdot x_1, \ldots, a \cdot x_n \rangle 
\]  
\[
+ \rho_N[\mathcal{L}(\langle x_1, \ldots, x_n \rangle, \langle a \cdot x_1, \ldots, a \cdot x_n \rangle) - \mathcal{L}(\langle x_1, \ldots, x_n \rangle, \langle b \cdot x_1, \ldots, b \cdot x_n \rangle)] > \gamma \mid \langle b \cdot x_1, \ldots, b \cdot x_n \rangle 
\]  
\[
\leq 1
\]  
Therefore $\rho_N$ is not differentiating.

Similarly,  
\[
Pr[p_a^n \approx p_a \mid p_a] + Pr[p_b^n \approx p_b \mid p_b] \leq 1
\]

Therefore, if for any $n$, $Pr[p_a^n \approx p_a \mid p_a] \geq (1 - \delta)$ then $Pr[p_a^n \approx p_b \mid p_b] \leq \delta$  

\[ \square \]

**Theorem A.12.** Consider a Loss Function which checks if the first character appended to a string is either “a” or “b”. Given the above Loss Function and DL-2 Distance metric $d_{DL2}$, the Noise Source described above is differentiating.

In this case, if we pick a differentiating Input Source, our synthesis algorithm will have convergence guarantees.

**Proof.** If $\mathcal{L}(y_1, \ldots, y_n), (\langle a^a \cdot x_1, \ldots, a^a \cdot x_1 \rangle) = \infty$ if for some $i, y_i = “b^n \cdot x_i$. Similarly, we can define this in the case of Therefore if for two vectors $\bar{z}_1, \bar{z}_2, d_{DL2}(\bar{z}_1, \bar{z}_2) \geq 1$, then $\mathcal{L}(\bar{z}_2, \bar{y}) - \mathcal{L}(\bar{z}_1, \bar{y}) = \infty$ if $\rho_N(\bar{y} \mid \bar{z}) > 0$.  
\[
\rho_N[\forall \bar{z} \in \mathbb{Z}^n, d_{DL2}(\bar{z}, \bar{z}_h) \geq 1 \implies \mathcal{L}(\bar{z}, \bar{y}) - \mathcal{L}(\bar{z}_h, \bar{y}) = \infty \mid \bar{z}_h] = 1 = 1
\]  
Note that if the Input Source is differentiating then the following is true.  
\[
\mathbb{1}((?, \text{true}) \in \bar{x}) \rho_1(d\bar{x}) \geq (1 - \delta)
\]

Note that  
\[
Pr[p_a^n \approx p_a \mid p_a, N] = \mathbb{1}((?, \text{true}) \in \bar{x}) \rho_1(d\bar{x}) \geq (1 - \delta)
\]

Similarly,  
\[
Pr[p_b^n \approx p_b \mid p_b, N] = \mathbb{1}((?, \text{true}) \in \bar{x}) \rho_1(d\bar{x}) \geq (1 - \delta)
\]

\[ \square \]