A CHARACTERIZATION OF MUMFORD CURVES WITH GOOD REDUCTION

JIE XIA

ABSTRACT. Mumford defines a certain type of Shimura curves of Hodge type, parameterizing polarized complex abelian fourfolds. In this paper, we study the good reduction of such a curve in positive characteristic and give a characterization in the generically ordinary case.

1. INTRODUCTION

1.1. Background. This paper aims to characterize certain Shimura varieties of Hodge type with good reduction. This description will serve as the main example in our work on defining Shimura varieties in positive characteristic.

Let $A$ be an abelian variety over $\mathbb{C}$. The elements in $H^{2r}(A, \mathbb{Q}) \cap H^{r,r}(A)$ are called Hodge classes of $A$. The Hodge group of an abelian variety $A$ is the largest $\mathbb{Q}$-subgroup of $GL(H^1(A, \mathbb{Q}))$ which leaves all Hodge classes invariant. Mumford defines in [15], a Shimura variety of Hodge type as a moduli scheme of abelian varieties (with a suitable level structure) whose Hodge group is contained in a prescribed Mumford-Tate group, arising from a Hermittain symmetric pair $\gamma$.

Furthermore, Mumford exemplifies Shimura curves of Hodge type in [14]. He constructs a simple algebraic group $Q$ over $\mathbb{Q}$ which is the $\mathbb{Q}$-form of the real algebraic group $SU(2) \times SU(2) \times SL(2)$, a cocharacter $h$ of $Q_\mathbb{R}$:

$$h : S_m(\mathbb{R}) \longrightarrow Q(\mathbb{R})$$

$$e^{i\theta} \mapsto I_4 \otimes \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

and an eight dimensional absolute irreducible rational representation $V$ of $Q$. The pair $(Q, h)$ is a Shimura datum and with the representation $V$, it defines Shimura curves of Hodge type, parameterizing four dimensional polarized abelian varieties over $\mathbb{C}$.

Generalizing the construction, one is able to define Shimura curves of Hodge type parameterizing $2^m$ dimensional polarized abelian varieties (see Section 2). We call such Shimura curves (with its universal family) Shimura curves of Mumford type, or for simplicity Mumford curves, denoted as $M$ and let $A \longrightarrow M$ be a universal family of abelian varieties over $M$.

Mumford curves play a significant role among smooth Shimura curves of Hodge type. Specifically, Theorem 0.8 in [13] shows the universal family over a Shimura curve has a strictly maximal Higgs field. Theorem 0.5 in [20] shows that up to powers and isogenies, the only smooth families of abelian varieties curves with maximal Higgs field are Mumford curves.

Rutger Noot and Kang Zuo et al. have studied the reduction of a fiber of a Mumford curve([11], [17]). They especially classify the possible Newton polygons of such a good reduction, which we will use in this paper. Their approaches both have the flavor of $p$-adic Hodge theory while we mainly use crystalline cohomology and deformation theory.
1.2. Main Result.

**Definition 1.3.** For any prime number $p$ and integer $m$, we say the pair $(\tilde{X} \to \tilde{C}, k)$ satisfies $(s_{p,m})$ if it satisfies the following properties:

1. $k$ is an algebraically closed field of characteristic $p$,
2. $\tilde{C}$ is a proper smooth curve over $W(k)$ and $\tilde{X} \to \tilde{C}$ is a family of abelian varieties of dimension $2^m$ over $\tilde{C}$,
3. there exists a versally deformed height 2 Barsotti-Tate (BT) group $\tilde{G}$ and a height $2^m$ etale BT group $\tilde{H}$ over $\tilde{C}$ such that $\tilde{X}[p^\infty] \cong \tilde{G} \otimes \tilde{H} := \text{colim}_n(\tilde{G}[p^n] \otimes \tilde{H}[p^n])$,
4. the reduction of $\tilde{X} \to \tilde{C}$ at $k$ is generically ordinary.

As in [21], a height 2 BT group $\tilde{G}$ over $\tilde{C}$ is versally deformed if the Kodaira-Spencer map $T_{\tilde{C}} \to t_{\tilde{G}} \otimes t_{\tilde{G}^*}$ is an isomorphism, or equivalently the Higgs field $\theta_{\tilde{G}}$ (see Section 9) is an isomorphism.

**Theorem 1.4.** Let $A \to M$ be a Mumford curve, parameterizing principally polarized abelian varieties of dimension $2^m$. For infinitely many primes $p$, there exists a pair $(\tilde{X} \to \tilde{C}, k)$ satisfying $(s_{p,m})$ (see Definition 1.3) and

$$(\tilde{X} \to \tilde{C}) \otimes_{W(k)} C = (A \to M).$$

**Remark 1.5.** When choosing $p$, it suffices to require that

1. $p > 2$, see [7,7]
2. $M$ admits a good reduction at the place $p$, see 2.5
3. the reflex field of $M$ is splitting over $p$, see 6.6

Since the reduction $X \to C$ of $\tilde{X} \to \tilde{C}$ at $k$ also admits the decomposition $X[p^\infty] \cong G \otimes H$, [1,4] provides examples for Theorem 7.4 in [21].

1.6. Structure of the paper. The goal of the paper is to prove 1.4. In Section 2 we introduce the basic definitions. By Lefschetz principle 2.5, a Mumford curve with the universal family can descend to a Witt ring whose special fiber $X/C$ is smooth. The definition of Mumford curves implies the Dieudonne crystal $D(X/C)$ of the abelian scheme $X$ is a tensor product of $m + 1$ rank 2 crystals:

$$D(X) \cong V_1 \otimes V_2 \otimes V_3 \cdots \otimes V_{m+1}.$$

In Sections 3 and 4 we set up some notation and basic facts of Tannakian categories. In Section 5 we prove two important lemmas (5.2, 5.3) in context of abstract Tannakian categories. They are key ingredients in determining the Tannakian groups of the rank 2 isocrystals and their Frobenius pullback.

In Section 6 we describe the structure of $V_i$ in the terminology of Tannakian categories. It is shown in 6.2 that the Tannakian group of each isocrystals $V_i$ is $SL(2)$. Furthermore, we investigate the tensor decomposition of the Frobenius morphism $F$. Firstly, it is shown in 6.3 that $F$ can be decomposed to the tensor product of $\phi_i$ which are morphisms between rank 2 crystals. Secondly, imposing the generically ordinary assumption permits a refinement, i.e. the permutation $s$ in 6.3 fixes an index, say 1. Lastly, we adjust $\phi_1$ to be an actual Frobenius morphism of a rank 2 crystal. That requires proving that $\sigma^* - \text{Id}$ on $\text{Pic}(C/W(k)_{\text{cris}})$ is surjective. This step is in Section 7.

Summarizing the above results in Section 8 we construct a rank 2 Dieudonne crystal $V$ and a rank $2^m$ unit root crystal $T$, such that $D(\tilde{X}/C) \cong V \otimes T$. We conclude the proof of 1.4 in Section 9 by studying the BT groups corresponding to $V$ and $T$.

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2. Mumford curves and their reduction

We review the generalization of Mumford’s construction, following [20].

2.1. Mumford curves over \( \mathbb{C} \). Let \( K \) be a totally real field of degree \( m + 1 \) and \( D \) be a quaternion division algebra over \( K \) which splits only at one real place and \( \text{Cor}_{K/Q}(D) = M_{2m+1}(Q) \). In this case \( D \otimes_{Q} \mathbb{R} \cong \mathbb{H} \times \cdots \times \mathbb{H} \times M_2(\mathbb{R}) \) and \( m \) is even.

Let \( \Gamma \) be the standard involution of \( D \), and let
\[
Q = \{ x \in D^* | x\bar{x} = 1 \}.
\]

Then \( Q \) is a simple algebraic group over \( \mathbb{Q} \) which is the \( \mathbb{Q} \)-form of the real algebraic group
\[
SU(2)^{\times m} \times SL(2, \mathbb{R}).
\]

Since \( \text{Cor}_{K/Q}(D) = M_{2m+1}(Q) \), \( Q \) admits a natural \( 2^{m+1} \) dimensional rational representation \( V \) whose real form is
\[
\rho : SU(2)^{\times m} \times SL(2) \to SO(2^m) \times SL(2) \text{ acting on } \mathbb{R}^{2^{m+1}}.
\]

Note \( Q_\mathbb{C} = SL(2, \mathbb{C})^{\times m+1} \). Then \( V_\mathbb{C} \) is the tensor of \( m + 1 \) copies of standard representation \( \mathbb{C}^2 \) of \( SL(2, \mathbb{C}) \):
\[
(1) \quad V_\mathbb{C} = \mathbb{C}^2 \otimes \mathbb{C}^2 \cdots \otimes \mathbb{C}^2 \quad (m + 1 \text{ factors})
\]

Let
\[
h : \mathbb{S}_m(\mathbb{R}) \to Q(\mathbb{R})
\]
\[
e^{i\theta} \mapsto I_{2^m} \otimes \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.
\]

Then \((Q, h)\) defines a Shimura datum. Generically \( \rho(Q) \) is the Hodge group of \( V \).

Let \( \text{stab}(h) \subset Q_\mathbb{R} \) be the stabilizer of \( h \). Then \( \text{stab}(h) \) is a maximal compact subgroup of \( Q_\mathbb{R} \) and hence conjugate to \( SO(2) \times SU(2)^{\times m} \). So \( Q_\mathbb{R}/\text{stab}(h) \cong Sp(1, \mathbb{R})/SO(2, \mathbb{R}) \cong \mathfrak{h} \) the upper half plane. Since \( \ker \rho \subset \text{stab}(h) \), we have
\[
\rho(Q)/\text{stab}(\rho \circ h) = Q_\mathbb{R}/\text{stab}(h) = \mathfrak{h}.
\]

Let \( \Gamma \subset Q_\mathbb{R} \) be an arithmetic subgroup such that \( \Gamma \) acts freely and properly discontinuous on \( \mathfrak{h} \). Note \( \ker \rho \subset Z(Q) \) and then it fixes \( h \), \( \Gamma \leftarrow \rho(Q(\mathbb{R})) \).

The (one-dimensional) Shimura varieties defined by \((Q, h)\) are called Mumford curves. With a small enough level structure \( \Gamma \), a Mumford curve is proper and smooth. It is a Shimura curve of Hodge type, parameterizing a family \( A \) of polarized abelian varieties of dimension \( 2^m \). In particular, we can view the space \( V \) as a \( \mathbb{Q} \)-local system over \( M \).

As \[1, 4\] indicates, we study the good reduction of \( A \to M \) in this paper.

2.2. Monodromy. Since \( \mathfrak{h} \) is simply connected, \( \pi_1(M) = \Gamma \). The local system \( V \) induces a monodromy \( \Gamma \to \text{Aut}(V_\mathbb{C}) \). Further, the tensor components \( \mathbb{C}^2 \) of \( V_\mathbb{C} \) also admit representations of \( \Gamma \) and hence also monodromy. Since \( \Gamma_\mathbb{C} \subset Q_\mathbb{C} \cong SL(2)^{\times 3} \wedge^2 \mathbb{C}^2 \) is a trivial representation of \( \Gamma \).

Definition 2.3. For any monodromy \( \Gamma \to GL(n) \), the algebraic monodromy group is defined to be the Zariski closure of the image of the monodromy. The connected algebraic monodromy group is the connected component of the identity of the algebraic monodromy.

Proposition 2.4. The algebraic monodromy group induced by \( \mathbb{C}^2 \) in \([1]\) is \( SL(2, \mathbb{C}) \) and that of \( V_\mathbb{C} \) is the image of \( SL(2, \mathbb{C})^{\times m+1} \) in \( \text{Aut}(V_\mathbb{C}) \).
Proof. From above, the monodromy induced by the representation of \( Q \) is tensor of \( m + 1 \) copies of monodromies \( \mathbb{C}^2 \). Let \( K_i, 1 \leq i \leq n + 1 \) be the corresponding algebraic monodromy groups. Since \( \wedge^2 \mathbb{C}^2 \) is a trivial representation of \( \Gamma_C, K_i \subset SL(2, \mathbb{C}) \).

By [1], the connected algebraic monodromy on \( V_Q \) is a normal subgroup in the Hodge group \( \rho(Q) \). Since \( Q \) is simple, \( \rho(Q) \) is also simple over \( Q \). Thus the connected algebraic monodromy is \( \rho(Q) \). Since \( \rho(Q)_C = \text{im}(SL(2, \mathbb{C})^{x_m+1} \rightarrow \text{Aut}(V_C)) \) is connected, the connected complex algebraic monodromy of \( V_C \) is \( \text{im}(SL(2, \mathbb{C})^{x_m+1} \rightarrow \text{Aut}(V_C)) \).

Note the complex algebraic monodromy of \( V_C \) is \( \text{im}(\prod K_i \rightarrow \text{Aut}(V_C)) \). Therefore

\[
\text{im}(\prod K_i \rightarrow \text{Aut}(V_C))^o = \text{im}(SL(2, \mathbb{C})^{x_m+1} \rightarrow \text{Aut}(V_C)).
\]

Then necessarily, \( K_i = SL(2, \mathbb{C}) \) for each \( i \).

\[\square\]

2.5. **Lefschetz Principle.** By Lefschetz Principle (see [10]), we mean the process that all the co-efficients of polynomials, defining a variety of finite type over a field, generate a subring \( R \) of finite type over \( \mathbb{Z} \), such that the variety can be defined over \( R \). Note this process can be easily generalized to morphisms of finite type or vector bundles of finite rank.

Apply Lefschetz Principle to \( A \rightarrow M \) and the flat vector bundles induced by \( \mathbb{C}^2 \). We obtain these data can descend from \( K \) to a ring \( R \) finite type over \( \mathbb{Z} \). Throwing away finite places, we can assume \( R \) is smooth over \( \mathbb{Z} \). Let \( k \) be a residue field of \( R \) with characteristic \( > 2 \) such that \( M \) admits a good reduction over \( k \). By smoothness of \( R \), we have the lifting from \( \text{Spec} W_n(k) \) to \( \text{Spec} R \):

\[
\begin{array}{ccc}
\text{Spec} k & \longrightarrow & \text{Spec} R \\
\downarrow & & \downarrow \\
\text{Spec} W_n(k) & \longrightarrow & \text{Spec} \mathbb{Z}
\end{array}
\]

Therefore we find a morphism \( \text{Spec} W(k) \rightarrow \text{Spec} R \).

Let \( \tilde{X} \overset{\tilde{\pi}}{\rightarrow} \tilde{C} \) (resp. \( \mathcal{V}_i \)) be the base change \( A \rightarrow M \) (resp. the flat vector bundles) from \( \text{Spec} R \) to \( \text{Spec} W(k) \). Let \( X, C \) be the special fiber of \( \tilde{X}, \tilde{C} \). Let \( \mathcal{E} \) be the Hodge bundle \( \mathcal{E} = R^1\tilde{\pi}_*(\Omega^2_{\tilde{X}/\tilde{C}}) \) and \( \mathcal{E} \) admits the Gauss-Manin connection. By [3 Theorem 6.6], the category of crystals on \( C \) is equivalent to the category of modules with an integrable connection (MIC). In particular, the Hodge bundle \( \mathcal{E} \) corresponds to the Dieudonne crystal \( R^1\pi_{*, \text{cris}}(\mathcal{O}_X) \). Let us denote the crystal still as \( \mathcal{E} \). The vector bundles \( \mathcal{V}_i \) also correspond to crystals and denote the corresponding crystals as \( \mathcal{V}_i \) as well. Then as crystals

\[
\mathcal{E} \cong \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1}.
\]

3. **Tannakian Category**

In this section, we review some basic constructions and facts regarding Tannakian categories that we will need later.

**Definition 3.1.** Let \( L \) be a field of characteristic 0. A Tannakian category \( T \) (over \( L \)) is a \( L \)-linear neutral rigid tensor abelian category with an exact fiber functor \( \omega : T \rightarrow \text{Vect}_L \).

**Theorem 3.2.** ( [7 Theorem 2.11]) For any Tannakian category \( T \), there exists an \( L \)-algebraic group \( G \) such that \( T \) is equivalent to \( \text{Rep}_L(G) \) as tensor categories.

We mainly use the following two special Tannkian categories.
Example 3.3. Choose a point $c \in M$, the category of all MIC on $M$ with fiber functor $\mathcal{F} \rightarrow \mathcal{F}_c$ form a Tannakian category. By 3.2 it corresponds to $\text{Rep}\mathcal{C}(G_{\text{univ}})$.

By Riemann-Hilbert correspondence, the category of MIC on $\tilde{C}$ is equivalent to $\text{Rep}(\pi_1(M))$. The algebraic group $G_{\text{univ}}$ can be constructed from $\pi_1(M)$ by the following:

$$G_{\text{univ}} = \lim H$$

where $H$ lists the Zariski closure of image of $\pi_1(M)$ in $GL(W)$ for all complex representations $W$. Note the system of $H$ is projective. So the image of $G_{\text{univ}} \rightarrow Aut(W)$ is exactly the Zariski closure of the image of $\pi_1(M)$ in $GL(W)$.

Let $B(k)$ be the fraction field of $W(k)$.

Example 3.4. (VI 3.1.1, 3.2.1) Inverting $\rho$ in the category $\text{Cris}(C/W(k))$, we obtain the category of isocrystals $\text{Isocris}(C/W(k))$. Similar to 3.3 the category $\text{Isocris}(C/W(k))$ forms a Tannakian category over $B(k)$, with fiber functor associated to a $k$-point of $C$. So there exists a $B(k)$-affine group scheme $P_{\text{univ}}$ such that the following two categories are equivalent.

$$\{\text{finite locally free isocrystals on } C/W(k)\} \leftrightarrow \text{Rep}_{B(k)}(P_{\text{univ}}).$$

An object $\mathcal{F}'$ in $\text{Isocris}(C/W(k))$ is called effective if it is from an object $\mathcal{F}$ in $\text{Cris}(C/W(k))$, i.e. $\mathcal{F}' = \mathcal{F} \otimes B(k)$. For any morphism $f : \mathcal{F} \otimes B(k) \rightarrow \mathcal{G} \otimes B(k)$ between effective objects in $\text{Isocris}(C/W(k))$, there exists $m \in Z$ such that $p^m f : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism in $\text{Cris}(C/W(k))$.

Note different from [19 VI 3.1.1, 3.2.1], $\text{Isocris}(C/W(k))$ denotes just the isocrystals, not the $F$-isocrystals. So $P_{\text{univ}}$ is an affine group scheme over $B(k)$, note over $\mathbb{Q}_p$.

We conclude this section by a simple result.

Proposition 3.5. For any Tannakian category $T$ and $W, V \in T$, let $< W >$ denote the Tannakian subcategory generated by $W$, with Tannakian group $G_W$. Similarly, $< V > = \text{Rep}_k(G_V), < W, V > = \text{Rep}_k(K)$. Then there exists a natural injection $K \hookrightarrow G_W \times G_V$.

Proof. Since $W, V \in \text{Rep}(K)$, by (7 2.21), $K$ admits surjections onto $G_W$ and $G_V$. Then $K$ admits a map $K \rightarrow G_W \times G_V$. The induced morphism $\text{Rep}(G_W \times G_V) \rightarrow \text{Rep}(K)$ satisfies (7 2.21(2)). So the map is injective. \hfill \square

4. Notation

We summarize the notation and fix them till the end.

- By (iso)crystals over $C$, we always mean (iso)crystals in vector bundles over the crystalline site $\text{cris}(C/\mathbb{Z}_p)$.

- We use subscript $C$ to denote the reduction of an object or a morphism from $\tilde{C}$ to $C$. For instance, $\mathcal{E}_C$ naturally means the associated vector bundle over $\tilde{C}$ from the crystal $\mathcal{E}$ and $F_C$ is just the restriction of the morphism $F : \mathcal{E} \rightarrow \mathcal{E}$ to $C$.

$B(k)$ the fractional field of $W(k)$.

$Q$ the reductive group defining the Mumford curves.

$\tilde{X} \rightarrow \tilde{C}$ the descent of the Mumford curve with the family of abelian varieties $A/M$ to $\text{Spec} W(k)$.

$\sigma$ the absolute Frobenius on $C$.

$\mathcal{E}, V_1^\sigma, V_i \mathcal{E} = R^1\pi_{\text{cris},*}(O_X), \mathcal{E} \cong V_1 \otimes V_2 \otimes V_3 \cdots \otimes V_{m+1}$ and $\mathcal{E}^\sigma \cong V_1^\sigma \otimes V_2^\sigma \otimes V_3^\sigma \cdots \otimes V_{m+1}^\sigma$.

$P_{\text{univ}}$ the Tannakian group of the category of finitely locally free isocrystals on $\tilde{C}/W(k)$.

$E, W_i, V_i$ $B(k)$-representations of $P_{\text{univ}}$ corresponding to $(\mathcal{E}, V_i^\sigma, V_i)$, respectively.

$P$ the Tannakian group of the subcategory generated by $\mathcal{E}$, i.e. $\text{im}(P_{\text{univ}} \rightarrow Aut(E))$. 


\( P_i \) the Tannakian group of the subcategory generated by \( V_i \), i.e. \( \text{im } (P_{\text{univ}} \rightarrow \text{Aut}(V_i)) \).

\( Q_i \) the Tannakian group of the subcategory generated by \( V'_i \), i.e. \( \text{im } (P_{\text{univ}} \rightarrow \text{Aut}(W_i)) \).

\( Q'_0 \) the connected component of the identity in \( Q_i \).

\( P' \) the Tannakian group of the subcategory generated by \( \{E^\sigma\} \).

\( Q' = \text{im } (P_{\text{univ}} \rightarrow \prod Q_i) \).

\( K_{12} \) the Tannakian group of the subcategory generated by \( \{ V'_1, V_2 \} \).

5. Some Important Lemmas on Tannakian Categories

In this section, we prove some lemmas about general Tannakian categories. These lemmas will be applied to proving 1.4 in the next section.

**Lemma 5.1.** For any \( g \in GL(2) \), the centralizer \( Z(g) \) of \( g \) has dimension \( \geq 2 \) as a variety.

**Proof.** Let \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). The centralizer of \( g \)

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

implies

\[ bz = cy, (a - d)y = b(x - w). \]

Note \( \dim GL(2) = 4 \). As a subvariety of \( GL(2) \), \( Z(g) \) has dimension at least 2. □

The Tannakian category of isocrystals on \( C/W(k) \) is equivalent to \( \text{Rep}(P_{\text{univ}}) \).

**Lemma 5.2.** Let \( W_i, V_i \in \text{Rep}(P_{\text{univ}}) \) be representations over \( B(k) \), \( 1 \leq i \leq n \). Let \( E \cong \bigotimes V_i, P_i = \text{im } (P_{\text{univ}} \rightarrow \text{Aut}(V_i)) \) and \( Q_i = \text{im } (P_{\text{univ}} \rightarrow \text{Aut}(W_i)) \). Suppose we have that

\[ F : W_1 \otimes \cdots \otimes W_{m+1} \rightarrow V_1 \otimes \cdots \otimes V_{m+1} \]

is an isomorphism between representations and \( P_i = \text{SL}(2) \) for each \( i \). Then

\( Q_i = \text{GL}(2) \) or \( \text{SL}(2) \times \mu_k \) for some \( k \).

Let \( Q' \) be the image of \( P_{\text{univ}} \rightarrow \prod Q_i \), and then the projections \( Q' \rightarrow Q_i \) are surjective for each \( i \).

**Proof.** Let \( P' \) be \( \text{im } (Q' \rightarrow \text{Aut}(E) \cong \text{GL}(2^{m+1})) \). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
Q_1 \times Q_2 \times Q_3 \cdots \times Q_{m+1} & \xleftarrow{\text{twisted by } F} & Q' \\
\downarrow & & \downarrow \\
\text{GL}(2)^{m+1} & \xrightarrow{\text{twisted by } F} & \text{GL}(2^{m+1})
\end{array}
\]

Note \( \text{SL}(2)^{m+1} \rightarrow \text{GL}(2^{m+1}) \) is twisted by \( F \). The right triangle can be specified as

\[
\begin{array}{ccc}
Q' & \xrightarrow{\text{twisted by } F} & \text{SL}(2)^{m+1} \\
\downarrow & & \downarrow \\
P' & \xrightarrow{\text{twisted by } F} & \text{GL}(2^{m+1})
\end{array}
\]
where $P'$ is the common image.

Since $SL(2)$ is semisimple, so is $P'/Z(P')$. Since $\ker(Q' \rightarrow GL(2^{m+1})) \subset \ker(GL(2)^{x+m+1} \rightarrow GL(2^{m+1}))$, the kernel of $Q' \rightarrow P'$ consists of just central elements. The group $Q'$ is an extension of central elements and a semisimple group. Therefore $Q'$ is reductive and $P'$ is the adjoint group of $Q'$. Further, $Q' \rightarrow P'$ induces a morphism from the derived group $[Q', Q']$ to $P'$ which further induces a surjection to $P'/Z(P')$.

$$[Q', Q'] \rightarrow Q' \rightarrow P' \rightarrow P'/Z(P').$$

If the projection of $[Q', Q']$ to some factor $GL(2)$ has dimension less than 3, then one of the projections must have dimension 4 because of $\dim P' = 3 \times 2^m$. So one of the projections would be $GL(2)$. Since the kernel of $Q' \rightarrow P'$ is finite, $P'$, as the image of $SL(2)^{m+1} \rightarrow GL(2^{m+1})$ would have infinitely many centers, contradiction.

Now we have the projections of $[Q', Q']$ to each factor have precisely dimension 3. Therefore each projection has the form $SL(2) \times \mu_k$. By comparing the dimensions, $SL(2)^{x+m+1} \subset \text{im} (Q' \rightarrow GL(2^{x+m+1}))$. Then we have a lifting

$$SL(2)^{x+m+1} \rightarrow [Q', Q'] \subset Q'$$

such that the right triangle is commutative

\[
\begin{array}{ccc}
Q_1 \times Q_2 \times Q_3 \cdots \times Q_{m+1} & \xleftarrow{} & Q' \leftarrow \cdots \leftarrow SL(2)^{x+m+1} \\
GL(2)^{x+m+1} & \xrightarrow{\text{twisted by } F} & GL(2^{m+1})
\end{array}
\]

Now we classify the elements with finite kernel in $\text{Hom}(SL(2)^{x+m+1}, GL(2)^{x+m+1})$.

First, recall that all automorphisms of $SL(2)$ are inner and hence $\text{Hom}(SL(2), GL(2))$ consists of the trivial morphism and the conjugation by some element in $GL(2)$. For any morphism $f \in \text{Hom}(SL(2)^{x+m+1}, GL(2)^{x+m+1})$, restricting to each factor of $SL(2)$ gives $m + 1$ inclusions $SL(2) \hookrightarrow GL(2)$. Explicitly,

\[
\begin{align*}
(g_1, 1, 1, \cdots) & \mapsto (\psi_{11}(g_1), \psi_{12}(g_1), \psi_{13}(g_1), \cdots) \\
(1, g_2, 1, \cdots) & \mapsto (\psi_{21}(g_2), \psi_{22}(g_2), \psi_{23}(g_2), \cdots) \\
(1, 1, g_3, \cdots) & \mapsto (\psi_{31}(g_3), \psi_{32}(g_3), \psi_{33}(g_3), \cdots).
\end{align*}
\]

Then $\psi_{11}(g_1)$ and $\psi_{21}(g_2)$ commute for any $g_i \in SL(2)$. Note all the automorphisms of $SL(2)$ are inner. So if neither of $\psi_{11}$ and $\psi_{12}$ is an identity, then there exists $h, k \in GL(2)$ such that $\psi_{11}, \psi_{21}$ are conjugation by $h$ and $k$, respectively. Then for any $g_i \in SL(2),

(2) \quad h g_1 h^{-1} k g_2 k^{-1} = k g_2 k^{-1} h g_1 h^{-1}

(3) \quad k^{-1} h g_1 h^{-1} k g_2 k^{-1} h = g_2 k^{-1} h g_1.

Since $Z(k^{-1}g)$ in $GL(2)$ has dimension at least 2 by [5.1] we can choose $g_2 \in SL(2)$ such that $g_2 \neq \pm I$ and $g_2 \in Z(k^{-1}g)$. But then from (2), $g_2$ has to commute with $g_1$, i.e. $g_2 \in Z(SL(2)) = \pm I$, contradiction. Therefore at least one of $\psi_{11}$ and $\psi_{21}$ is identity. Further, each column $\psi_{ki}$ has at least $m$ identities. So each factor $SL(2)$ is embedded into exactly one of the $m + 1$ copies of $GL(2)$ and trivially to the others.

Then $\dim Q_i = 3$ or 4. Since $Q_i \subset GL(2)$, $Q_i = GL(2)$ or $SL(2) \times \mu_k$ for some integer $k$. \hfill \square

**Lemma 5.3.** Assumptions as [5.2] there exist a permutation $s \in S_{m+1}$, dimension 1 representation $L_i$ with $\otimes_i L_i$ trivial and isomorphisms

$$\phi_i : W_i \rightarrow V_{s(i)} \otimes L_i$$
such that

\[ F = \bigotimes_i \phi_i. \]

5.4. **Proof of 5.3.** From the proof of 5.2, for each \( i \), there exists a unique \( P_j = SL(2) \) such that \( P_j \hookrightarrow Q_i \). This inclusion is an isomorphism between \( P_j \) and \( Q_i^0 \), which is a conjugation by some \( l \in GL(2) \). Without loss of generality, assume \( i = 1 \) and \( j = 2 \).

Note we have the following diagram:

\[
\begin{array}{ccc}
Q_1 & \xrightarrow{f_1} & P_2 \\
\downarrow & & \downarrow \\
\text{PGL}(2) & \xrightarrow{f_2} & \text{PGL}(2)
\end{array}
\]

The morphism \( f_1 \) is the usual quotient by the center \( Q_i \hookrightarrow \text{PGL}(2) \). The morphism \( f_2 \) is twisted by the conjugation by \( l \).

**Claim 1.** this diagram is commutative.

**Proof.** We have

\[
\begin{array}{ccc}
P_{\text{univ}} & \xrightarrow{\Pi P_i} & Q' \\
\downarrow & & \downarrow \\
\text{GL}(2)^{\times m+1} & \xrightarrow{\text{twisted by } F} & \text{GL}(2^{m+1})
\end{array}
\]

For any \( h \in P_{\text{univ}} \), let \( (h_1, h_2, h_3, \ldots, h_{m+1}) \) be the image of \( h \) in \( \prod P_i \) and \( (g_1, g_2, g_3, \ldots, g_{m+1}) \) image in \( Q' \). Then \( \prod P_i \twoheadrightarrow Q' \) permutes the factors and sends \( (h_1, h_2, h_3, \ldots) \) to \( (lh_2l^{-1}, \ldots) \). Then \( (lh_2l^{-1}, \ldots) \) and \( (g_1, g_2, g_3, \ldots) \) have the same image under \( \text{GL}(2)^{\times m+1} \twoheadrightarrow \text{GL}(2^{m+1}) \). Therefore \( C(l)(h_2) = tg_1 \) for some scalar \( t \in B(k) \) where \( C_l \) is the adjoint action by \( l \). In particular, \( f_2(h_2) = f_1(g_1) \). The claim is true. \( \square \)

Then \( P_{\text{univ}} \twoheadrightarrow Q_1 \times P_2 \) factors through the limit of

\[
\begin{array}{ccc}
Q_1 & \xrightarrow{f_1} & P_2 \\
\downarrow & & \downarrow \\
\text{PGL}(2) & \xrightarrow{f_2} & \text{PGL}(2)
\end{array}
\]

**Claim 2.** the limit of the above diagram is \( P_2 \times Z(Q_1) = SL(2) \times \mu_n \) or \( SL(2) \times \mathbb{G}_m \) with

\[
P_2 \times Z(Q_1) \twoheadrightarrow P_2 \\
(h, k) \mapsto h
\]

\[
P_2 \times Z(Q_1) \twoheadrightarrow Q_1 \\
(h, k) \mapsto (khl^{-1}).
\]
Proof. We can prove it directly: for any $K'$ fitting in the diagram

$$
\begin{array}{ccc}
K' & \xrightarrow{s_1} & Q_1 \\
\downarrow f_1 & & \downarrow f_1 \\
 & \xleftarrow{s_2} & P_2 \\
& \downarrow s_2 & \\
PGL(2) & & ,
\end{array}
$$

we construct the map

$$K' \rightarrow Z(Q_1) \times SL(2)
\quad k \mapsto (s_1(k)C_1(s_2(k))^{-1}, s_2(k)).$$

Since the lower triangle is commutative, the map is well defined and obviously it is unique. \qed

Consider the Tannakian category generated by $\{W_1, V_2\}$. Then it is isomorphic to $\text{Rep}(K_{12})$ for some algebraic group $K_{12}$. By [3.5],

$$K_{12} = \text{im} \left( P_{\text{univ}} \rightarrow \text{Aut}(W_1) \times \text{Aut}(V_2) \right) \subset Q_1 \times P_2.$$ 

Therefore by Claim [2] $K_{12} \subset P_2 \times Z(Q_1) = SL(2) \times Z(Q_1)$.

- If $Q_1 = GL(2)$, then $\dim K_{12} = 4$ and by $GL(2)$ connected, $K_{12} = SL(2) \times \mathbb{G}_m$.
- If $Q_1 = SL(2)$ and $Z(Q_1) = \mu_n$, then $\dim K_{12} = 3$ and hence $K_{12}^0 = SL(2)$. It suffices to determine the number of the connected components of $K_{12}$. Let $\zeta$ be a generator of $\mu_n$. Then $\zeta$ and $-\zeta$ are in the same component of $Q_1$.

1. If $n \equiv 0 \pmod{4}$, then $-\zeta$ is also a generator of $\mu_n$. Therefore $K_{12}$ has to be $SL(2) \times \mu_n$ to cover the whole $Q_1$.
2. If $n \equiv 2 \pmod{4}$, then $\mu_n = \pm I \times \mu_{n/2}$ and hence $Q_1 \cong SL(2) \times \mu_{n/2}$. So besides $SL(2) \times \mu_n$, $K_{12}$ also can be $Q_1$.

In summary, $K_{12} = SL(2) \times \mathbb{G}_m$ or $SL(2) \times \mu_k$ for some $k$.

Therefore as an irreducible $K$–representation, $W_i$ is tensor of a $SL(2)$–representation and an irreducible $\mu_k$ or $\mathbb{G}_m$ representation, i.e. $W_i = V_{\sigma(i)} \otimes L_i$.

This is the end of the proof of [5.3]

6. Tensor decomposition the Frobenius

Now we come back to the context of [1.4] The Dieudonne crystal $E = R^1\pi_{\text{cris}*}(O_X)$ admits the Frobenius map:

$$E^\sigma \xrightarrow{F} E.$$ 

Then we have

$$F : V_1^\sigma \otimes V_2^\sigma \otimes V_3^\sigma \cdots \otimes V_{m+1}^\sigma \otimes B(k) \xrightarrow{\text{univ}} V_1 \otimes V_2 \otimes V_3 \cdots \otimes V_{m+1} \otimes B(k),$$

where $B(k)$ is the fractional field of $W(k)$. By [3.4] the category of isocrystals over $C$ is Tannakian.

Proposition 6.1. For each $i$, $P_i \cong SL(2, B(k))$ and $P_{\text{univ}} \rightarrow \prod P_i$ is surjective.

Proof. Since by [3] Theorem 6.6 the crystals on $C/W(k)_{\text{cris}}$ are exactly vector bundles with a connection over $\tilde{C}$, $\text{Rep}_C(P_{\text{univ}} \otimes \mathbb{C})$ is a Tannakian subcategory of $\text{Rep}_C(G_{\text{univ}})$. By functorality,
There exist a permutation 

\[ \text{Proposition 6.3.} \]

\[ P \otimes \mathbb{C} = \text{im} (G_{\text{univ}} \rightarrow \text{Aut}(V)) = \text{im} ((SL(2, \mathbb{C}))^{\times m+1} \rightarrow \text{Aut}(\mathbb{C}^{\otimes m+1})) \]

\[ P_i \otimes \mathbb{C} = \text{im} (G_{\text{univ}} \rightarrow \text{Aut}(\mathbb{C}^2)) = SL(2, \mathbb{C}). \]

The group \( P_i \) is a \( B(k) \)-form of \( SL(2) \) and admits a faithful two dimensional representation. Therefore \( P_i \cong SL(2, B(k)) \).

Therefore \( P = \text{im} (P_{\text{univ}} \rightarrow \prod_i P_i \rightarrow \text{Aut}(E)) \) is the same as \( \prod_i P_i \rightarrow \text{Aut}(E) \), after tensoring with \( \mathbb{C} \). Since it is faithfully flat, it is also true over \( B(k) \) and \( P = \text{im} (\prod_i P_i \overset{\otimes}{\rightarrow} \text{Aut}(E)) \). Further, since the kernel of \( (\prod P_i \rightarrow \text{Aut}(E)) \) is finite, \( \text{im} (P_{\text{univ}} \rightarrow \prod_i P_i) \) is an algebraic subgroup of \( \prod_i P_i \) with the same dimension. Since \( \prod_i P_i = SL(2, B(k))^{\times m+1} \) are connected,

\[ \text{im} (P_{\text{univ}} \rightarrow \prod_i P_i) = \prod_i P_i, \]

i.e. \( P_{\text{univ}} \rightarrow \prod_i P_i \) is surjective. \( \square \)

Now we can interpret isomorphism \( \mathfrak{4} \) as follows. We already have a rank \( 2^{m+1} \) isocrystal admitting a tensor decomposition to \( m+1 \) rank 2 isocrystals, each corresponding to a standard representation of \( SL(2) \). Then for another tensor decomposition to \( m+1 \) rank 2 isocrystals, just as left hand side of \( \mathfrak{4} \), we expect that each component also corresponds to a \( SL(2) \)-representation which is a corollary of \( \mathfrak{5.2} \).

**Proposition 6.2.** For each \( i, Q_i \cong SL(2, B(k)) \).

**Proof.** By \( \mathfrak{6.1} \) \( V_i, V_i'^{\sigma} \) and the isomorphism \( \mathfrak{4} \) satisfy the conditions of \( \mathfrak{5.2} \). Therefore the Tannakian group \( Q_i \) corresponds to \( V_i'^{\sigma} \) is either \( GL(2) \) or \( SL(2) \times \mu \).

Furthermore, note \( V_i \) comes from \( \mathbb{C}^2 \) in \( \mathfrak{1} \). Since the local system \( \mathbb{C}^2 \) on \( M \) has a trivial determinant, each isocrystal \( V_i \) has \( \wedge^2 V_i = O_{\tilde{\mathcal{C}}} \). So correspondingly \( \det Q_i = 1 \) and thus \( Q_i = SL(2) \). \( \square \)

Apply \( \mathfrak{5.3} \) and note that \( W_1 \) and \( V_2 \) are the corresponding objects of \( V_i'^{\sigma} \otimes B(k) \) and \( V_2 \otimes B(k) \) in \( \text{Rep}(P_{\text{univ}}) \), respectively. We have that there exist a permutation \( s \in S_{m+1}, \) rank 1 crystals \( L_i \) with \( \otimes_i L_i \cong O_{\tilde{\mathcal{C}}} \) and isomorphisms

\[ \phi_i : V_i'^{\sigma} \otimes B(k) \rightarrow V_{s(i)} \otimes L_i \otimes B(k) \]

such that

\[ F = \otimes_i \phi_i. \]

In fact, we can refine \( \phi_i \) to be a morphism between crystals.

**Proposition 6.3.** There exist a permutation \( s \in S_{m+1}, \) rank 1 crystals \( L_i \) with \( \otimes_i L_i \cong O_{\tilde{\mathcal{C}}} \) and isomorphisms

\[ \phi_i : V_i'^{\sigma} \rightarrow V_{s(i)} \otimes L_i \]

such that

\[ F = \otimes_i \phi_i. \]

**Proof.** Since \( \mathcal{E} \) is an \( F \)-crystal, we still have \( F : \otimes V_i'^{\sigma} \rightarrow \otimes V_i \). Since each \( \phi_i \) is a morphism between effective isocrystals, by \( \mathfrak{3.4} \) there exists an integer \( k_i \) such that \( p^{k_i} \phi_i \) is a morphism in \( \text{Cris}(C) \). We can assume \( p^{k_i} \phi_i \neq 0 \mod p \) at the generic point. Then \( p^{-k_1-k_2-\cdots-k_m} \phi_{m+1} \) is also a morphism in \( \text{Cris}(C) \). In fact, for any \( U \subset C \) and \( a_{m+1} \in V_{m+1}^{\sigma}(U) \), we can find \( a_1 \in V_{1}^{\sigma}(U), a_2 \in V_{2}^{\sigma}(U), \cdots \)

such that

\[ p^{k_i} \phi_i(a_i) \neq 0 \mod p \]

for \( 1 \leq i \leq m \). Then \( p^{-k_1-k_2-\cdots-k_m} \phi_{m+1}(a_{m+1}) \in V_{m+1}^{\sigma}(U) \). Otherwise,

\[ F(a_1 \otimes a_2 \cdots \otimes a_{n+1}) = p^{k_1} \phi_1(a_1) \otimes B(k) p^{k_2} \phi_2(a_2) \cdots \otimes B(k) p^{-k_1-k_2-\cdots-k_m} \phi_{m+1}(a_{m+1}) \]
is not in $\mathcal{V}_1 \otimes \mathcal{V}_2 \cdots \otimes \mathcal{V}_{m+1}(U)$.

A straightforward corollary of 6.3 is that

**Corollary 6.4.** Viewed as a morphism between crystals, $F$ still preserves pure tensors.

Let $\eta$ be the generic point of $C$ and $\mathcal{V}_{i,\eta}$ denote the restriction of $\mathcal{V}_i$ to the crystalline site $\text{cris}(\eta/W(k))$.

Since $C$ parametrizes a family of polarized abelian varieties (with a level structure), it admits a map to the moduli scheme $A_{2m,d,n} \otimes k$. If the image intersects with the ordinary locus in $A_{2m,d,n} \otimes k$, we say “$C$ intersects the ordinary locus” for simplicity. Note since the ordinary locus is open in $A_{2m,d,n} \otimes k$, the statement is equivalent to the universal family over $C$ is generically ordinary. Let

$$0 \to \omega \to \mathcal{E} \to \alpha \to 0$$

be the weight 1 Hodge filtration associated to $\tilde{X}/\tilde{C}$. Then from the definition of Mumford curves, especially the action of Hodge group $Q$ on $V$, we know $\omega$ is constructed from a line bundle $\mathcal{L}$ in $\mathcal{V}_i$ for some $i$, say $i = 1$, then

$$\omega \cong \mathcal{L} \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1}.$$  

Correspondingly $\alpha \cong \mathcal{V}_1/\mathcal{L} \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1}$ and the Hodge filtration of $\mathcal{E}$ comes from a filtration $\mathcal{L} \subset \mathcal{V}_1$:

$$\mathcal{L} \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1} \subset \mathcal{E} = \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1}.$$  

Base change from $W(k)$ to $k$. Denote the reduction of $\tilde{C}$ over $k$ as $C$ and the reduction of $\mathcal{E}$ as $\mathcal{E}_C$. Then the Frobenius $\mathcal{E}_C^{(p)} \xrightarrow[]{F} \mathcal{E}_C$ factors through $\alpha_C^{(p)}$ and then we have the conjugate spectral sequence:

$$0 \to \alpha_C^{(p)} \to \mathcal{E}_C \to \omega_C^{(p)} \to 0.$$  

**Proposition 6.5.** If $C$ intersects the ordinary locus, then $s(1) = 1$.

**Proof.** Let $c$ be a closed point in the intersection of ordinary locus and $C$. Then restricted to $c$, consider the composition $F' : \mathcal{E}_C^{(p)} \to \alpha_C$ in the following diagram

$$\begin{array}{ccc}
\mathcal{E}_C^{(p)} & \xrightarrow[]{F_C} & \mathcal{E}_C \\
\downarrow F' & & \downarrow \pi \\
\mathcal{E}_C & \xrightarrow[]{\alpha_C} & \mathcal{E}_C \\
\end{array}$$

Since $X_c$ is ordinary, $F'_C$ is surjective.

If $s(1) \neq 1$, Without loss of generality, suppose $s(1) = 2$. Note by 6.3

$$F'(\mathcal{E}_C^{(p)}) = \pi \circ F_C(\mathcal{V}_1^{(p)} \otimes \mathcal{V}_2^{(p)} \otimes \mathcal{V}_3^{(p)} \cdots \otimes \mathcal{V}_{m+1}^{(p)})_C$$

$$= \pi \circ \otimes \phi_1(\mathcal{V}_1^{(p)} \otimes \mathcal{V}_2^{(p)} \otimes \mathcal{V}_3^{(p)} \cdots \otimes \mathcal{V}_{m+1}^{(p)})_C$$

From the conjugate spectral sequence, $F_C$ factors through $\alpha_C^{(p)}$ and $\alpha = (\mathcal{V}_1/\mathcal{L}) \otimes \mathcal{V}_2 \otimes \mathcal{V}_3 \cdots \otimes \mathcal{V}_{m+1}$, thus $\phi_1 : \mathcal{V}_1^{(p)} \to \mathcal{V}_2 \otimes \mathcal{L}_1$ factors through $\mathcal{V}_1^{(p)} \otimes \mathcal{L}^{(p)}$, and the image of $\phi_1$ has rank 1. But $\dim_k \mathcal{V}_2|_c = 2$. So $F'$ can not be surjective. Contradiction. □
Therefore we have

\[(7) \quad \phi_1 : \mathcal{V}_1^\sigma \longrightarrow \mathcal{V}_1 \otimes L_1 \]

**Remark 6.6.** The Mumford curve $M$ is defined over the reflex field $K$, and let $p$ be the prime of $K$ over $p$.

Let $r = [K_p : \mathbb{Q}_p]$. Then by [11, Theorem 1.2], there are two Newton polynomials in $C/k$, it is either $\{2^{m+1+\epsilon(D)} \times \frac{1}{2}\}$ or $\{2^{m+1-r+\epsilon(D)} \times 0, 2^{m+1-r+\epsilon(D)} \times \frac{1}{2}, \ldots, 2^{m+1-r+\epsilon(D)} \times 1\}$. So $C$ intersects with ordinary locus if and only if $r = 1$.

So there are infinitely many prime $p$ over which the reduction of Mumford curve at $p$ is generically ordinary.

7. The Surjectivity of $\sigma^* - \text{Id}$ on the Picard Group

Our purpose is to construct a rank 2 Dieudonné crystal in the tensor decomposition of $\mathcal{E}$. We already have

\[ \phi_1 : \mathcal{V}_1 \longrightarrow \mathcal{V}_1 \otimes L_1. \]

So it only remains to "eliminate" $L_1$. We can achieve this goal in next section and the key ingredient is [7, 1] which we will prove in this section.

Let $\sigma$ be the absolute Frobenius of $C/k$ and $\text{Pic}(C/W(k)_{\text{cris}})$ denote the group of the rank 1 crystals on $C$. The following general principle guarantees $L_1 = (\sigma^* - \text{Id})(\mathcal{Z}^0)$.

**Proposition 7.1.** The group endomorphism $\sigma^* - \text{Id}$ of $\text{Pic}(C/W(k)_{\text{cris}})$ is surjective.

7.2. The proof.

**Lemma 7.3.** $\sigma^* - \text{Id}$ acts on $W(k)$ surjectively.

**Proof.** Since $k$ is algebraically closed, $(\sigma^* - \text{Id})$ acts on $k$ surjectively. Then for any $b \in W(k)$, we can find

\[ (\sigma^* - \text{Id})(a_0) = b + pb_1 \]

and there exists $a_1, a_2, a_3, \ldots$ such that

\[
(\sigma^* - \text{Id})(a_1) = b_1 + pb_2, \\
(\sigma^* - \text{Id})(a_2) = b_2 + pb_3, \\
(\sigma^* - \text{Id})(a_3) = b_3 + pb_4 \ldots
\]

Then $(\sigma^* - \text{Id})(\sum p^i a_i) = b$. In fact, since $W(k)$ is $p$-adically complete, $\sum p^i a_i \in W(k)$ and $(\sigma^* - \text{Id})(\sum p^i a_i) - b$ is contained in $p^n W(k)$ for any $n$. Therefore $(\sigma^* - \text{Id})(a + \sum p^i a_i) = b = 0$. \(\square\)

Now we recall the definition of Atiyah class. For a more detailed explanation, we refer the reader to [8, 10.1].

Let $\mathcal{J}$ be the ideal sheaf of the diagonal set of $\tilde{X} \times \tilde{X}$ and $\mathcal{O}_{\tilde{X} \times \tilde{X}} = \mathcal{O}_{\tilde{X} \times \tilde{X}} / \mathcal{J}^2$.

**Definition 7.4.** For any smooth proper variety $\tilde{X}$ and vector bundle $V$ over $\tilde{X}$, the Atiyah class is the extension class of

\[ 0 \longrightarrow V \otimes \Omega^1_{\tilde{X}} \longrightarrow p_{1*}(p_2^* V \otimes \mathcal{O}_{\tilde{X} \times \tilde{X}}) \longrightarrow V \longrightarrow 0. \]

Atiyah class is the unique obstruction to the existence of a connection on $V$.

By [13, Remark 3.7], the Atiyah class of any line bundle coincides with its first Chern class. So line bundles with a connection over a curve are exactly those of degree 0.

**Lemma 7.5.** The restriction of $\sigma^* - \text{Id}$ to $\text{Pic}^0(C/k_{\text{cris}})$ is surjective.
Proof. Note the rank 1 crystal on the site $C/k_{\text{cris}}$ is equivalent to a line bundle on $C/k$ with connection. For any $\mathcal{L} \in \text{Pic}(C/k_{\text{cris}})$, $\sigma^*(\mathcal{L}) = \mathcal{L}^p$. So it suffices to show that for any degree 0 line bundle with connection $(\mathcal{L}, \nabla)$, there exists a line bundle with connection $(L, \nabla_L)$ such that

$$(L, \nabla_L)^{p-1} \cong (\mathcal{L}, \nabla).$$

Since $k$ is algebraically closed, the Jacobian $\text{Jac}(C/k)$ is a divisible group. Therefore we always can find a line bundle $L \in \text{Jac}(C/k)$ such that $LP^{-1} \cong \mathcal{L}$.

Note the set of connections of $\mathcal{L}$ is a torsor under $\text{Hom}(\mathcal{L}, \mathcal{L} \otimes \omega_C) = \Gamma(\omega_C) = k^\theta$.

The same for $L$. For any two connections $\nabla_L, \nabla'_L$ on $L$, let $\nabla_L - \nabla'_L = h \in \Gamma(\omega_C)$.

Thus to find the connection $\nabla_L$, it suffices to show the $(p-1)$-th power is an injection from the connections on $L$ to the connections on $\mathcal{L}$. Then for any local section $\otimes_i s_i$,

$$((\nabla_L + h)^{p-1} - \nabla_L^{p-1})(\otimes_{i=1}^{p-1} s_i)$$

$$= \sum_{i=1}^{p-1} \cdots \otimes h.s_i \otimes \cdots$$

$$= (p-1)(\prod_i s_i)h(1 \otimes 1 \otimes \cdots \otimes 1).$$

Therefore for any connection $\nabla_{\mathcal{L}}$, if $g = \nabla_{\mathcal{L}} - \nabla_L^{p-1}$, then $(\nabla_L + g)^{p-1} = \nabla_{\mathcal{L}}$. So $(\sigma^* - \text{Id})$ acts on $\text{Pic}^0(C/k_{\text{cris}})$ surjectively. □

Lemma 7.6. $\sigma^* - \text{Id}$ maps $H^1(C/W(k)_{\text{cris}}, \mathcal{O}_{\tilde{C}})$ to itself surjectively.

Proof. By comparison theorem,

$$H^1(C/W(k)_{\text{cris}}, \mathcal{O}_{\tilde{C}}) \cong H_{\text{cris}}^1(\tilde{C}, \Omega_C) \cong W(k)^{2g}.$$ 

Let $N$ denote the free $W(k)$-module with $\sigma^*$ action. Then $V := N/pN$ is a $k$-vector space with $p$-linear action. By a result in [16 Page 143],

$$V = V_s \oplus V_n$$

where $V_s$ is the semisimple part and $V_n$ the nilpotent part. On $V_n$, since $\sigma^*$ acts nilpotently, $(\sigma^* - \text{Id})$ is invertible and hence surjective. On $V_s$, by [7,3] we can find $\lambda$ such that $(\sigma^* - \text{Id})(\lambda) = 1$. Then for each $k$, $(\sigma^* - \text{Id})(\lambda x_k) = x_k$. Therefore $(\sigma^* - \text{Id})$ acts on $V$ surjectively.

Back to $N$, for any $b \in N$, we can choose $a_0$ such that

$$(\sigma^* - \text{Id})(a_0) = b + pb_1.$$

Then choose $a_1$ such that

$$(\sigma^* - \text{Id})(a_1) = b_1 + pb_2.$$ 

Following this way, we can find $a_2, a_3, \cdots$. Similar to the proof of [7,3] we have

$$(\sigma^* - \text{Id})(a_0 + pa_1 + p^2a_2 + \cdots + p^n a_n + \cdots) = b.$$ 

□

Now we can prove [7,1]

Proof. Note $\text{Pic}(C/W(k)_{\text{cris}}) \cong H^1(C/W(k)_{\text{cris}}, \mathcal{O}_C^*)$. We have the sequence

$$0 \rightarrow (1 + p\mathcal{O}_C)^* \rightarrow \mathcal{O}_C^* \rightarrow (\mathcal{O}_C/p)^* \rightarrow 0$$
and \((\sigma^* - \text{Id})\) acts on the long exact sequence. Since \(\text{char } k > 2\), the exponential and logarithm maps converge and thus give an isomorphism between abelian groups

\[ \mathcal{O}_C \cong (1 + p\mathcal{O}_C)^*. \]

So the cohomology groups are isomorphic:

\[ H^1(C/W(k)_{\text{cris}}, \mathcal{O}_C) \cong H^1(C/W(k)_{\text{cris}}, (1 + p\mathcal{O}_C)^*). \]

We have the long exact sequence

\[ H^1(C/W(k)_{\text{cris}}, \mathcal{O}_C) \rightarrow H^1(C/W(k)_{\text{cris}}, \mathcal{O}_C^e) \xrightarrow{\partial} H^1(C/W(k)_{\text{cris}}, (\mathcal{O}_C/p)^*) \cong \text{Pic}(C/k)_{\text{cris}} \rightarrow H^2(C/W(k)_{\text{cris}}, \mathcal{O}_C). \]

By [3, Theorem 6.6], the category of crystals on \(C\) is equivalent to the category of vector bundles with a connection on \(\hat{C}\). Therefore \(\text{Pic}(C/W(k)_{\text{cris}})\) is isomorphic to the group of line bundles with a connection on \(\hat{C}\) and \(g\) is the pull back of such line bundle from \(\hat{C}\) to \(C\). Therefore \(\text{im } g \subset \text{Pic}^0(C/k_{\text{cris}})\).

Since the obstruction to deform the line bundle from \(C\) to \(\hat{C}\) vanishes and the deformation preserves the degree, \(\text{Pic}^0(\hat{C}) \rightarrow \text{Pic}^0(C)\) is surjective. In fact, for any degree 0 line bundle \(\mathcal{L}\) on \(\hat{C}\), it corresponds to a divisor \(\sum_i n_ip_i\) with each \(p_i\) a \(k\)-point. Then by Hensel’s lemma, each \(p_i\) lifts to a \(W(k)\)-point \(\hat{p}_i\) (though not uniquely). Let \(\sum_in_i\hat{p}_i = \hat{L} \in \text{Pic}^0(\hat{C})\) and then \(\hat{L}\) reduces to \(\mathcal{L}\).

For the connection, for any \((\mathcal{L}, \nabla) \in \text{Pic}^0(C/k_{\text{cris}})\), choose a lifting \(\hat{\mathcal{L}} \in \text{Pic}^0(\hat{C})\) of \(\mathcal{L}\) and a connection \(\hat{\nabla}\) on \(\hat{\mathcal{L}}\). Let \(\nabla'\) be the reduction of \(\hat{\nabla}\), then \(\nabla' - \nabla = f \in \Gamma(\omega_{\hat{C}})\). Choose \(\hat{f} \in \Gamma(\omega_{\hat{C}})\) such that \(\hat{f}\) reduces to \(f\). Then \(\hat{\nabla} - \hat{f}\) reduces to \(\nabla\). And \(g(\hat{\mathcal{L}}, \hat{\nabla} - \hat{f}) = (\mathcal{L}, \nabla)\).

Therefore \(\text{im } g = \text{Pic}^0(C/k_{\text{cris}})\). So we have the following sequence:

\[ H^1(C/W(k)_{\text{cris}}, \mathcal{O}_C) \rightarrow H^1(C/W(k)_{\text{cris}}, \mathcal{O}_C^e) \rightarrow \text{Pic}^0(C/k_{\text{cris}}) \rightarrow 0 \]

By [7,5] and [7,6], \(\sigma^* - \text{Id}\) induces surjective endomorphisms on \(H^1(C/W(k)_{\text{cris}}, \mathcal{O}_C)\) and \(\text{Pic}^0(C/k_{\text{cris}})\). Therefore \((\sigma^* - \text{Id})\) maps \(H^1(C/W(k)_{\text{cris}}, \mathcal{O}_C^e)\) surjectively on itself.

**Remark 7.7.** In the proof of [7,1], we use the convergence of exponential and logarithm, which are true if and only if the characteristic \(p > 2\).

8. **The Dieudonné Crystal \(\mathcal{V}\) and the Unit Crystal \(\mathcal{T}\)**

Now by [7,4] we can choose \(\mathcal{L}' \in \text{Pic}(C/W(k)_{\text{cris}})\) such that \((\sigma^* - \text{Id})(\mathcal{L}') = \mathcal{L}^{-1}_1\) (for \(\mathcal{L}_1\) see [7]). Then \(\phi_1\) induces an isomorphism

\[ (8) \quad \gamma : \mathcal{V}_1 \otimes \mathcal{L}^\sigma \otimes B(k) \rightarrow \mathcal{V}_1 \otimes \mathcal{L}' \otimes B(k). \]

Let \(\mathcal{V} = \mathcal{V}_1 \otimes \mathcal{L}\). Similarly, we have the isomorphism

\[ \beta : \mathcal{V}_2^\sigma \otimes \mathcal{V}_3^\sigma \otimes \cdots \otimes \mathcal{V}_{m+1}^\sigma \otimes (\mathcal{L}'^{-1})^\sigma \otimes B(k) \rightarrow \mathcal{V}_2 \otimes \mathcal{V}_3 \otimes \cdots \otimes \mathcal{V}_{m+1} \otimes \mathcal{L}'^{-1} \otimes B(k). \]

Denote \(\mathcal{V}_2 \otimes \mathcal{V}_3 \otimes \cdots \otimes \mathcal{V}_{m+1} \otimes \mathcal{L}'^{-1}\) as \(\mathcal{T}\). Therefore as crystals,

\[ \mathcal{E} \cong \mathcal{V} \otimes \mathcal{T} \]

and as a morphism between crystals \(F = \gamma \otimes \beta\). Then \(V = pF^{-1} = p\gamma^{-1} \otimes \beta^{-1}\).

**Lemma 8.1.** The morphism \(\beta : \mathcal{T}^\sigma \rightarrow \mathcal{T}\) is an isomorphism between crystals.
Proof. We have known that $\gamma \neq 0 \pmod{p}$. Over $C$, \[6\] shows the Frobenius

$$F_C = \gamma_C \otimes \beta_C : \mathcal{E}_C^\sigma \to \mathcal{E}_C$$

induces an injection

$$\alpha_C^{(p)} = (V_1/L \otimes \bar{\mathcal{L}}_C^{(p)}) \otimes T_C^{(p)} \to \mathcal{E}_C \cong \mathcal{V}_C \otimes \mathcal{T}_C.$$

Therefore $\beta_C$ is an isomorphism between $T_C$ and $\mathcal{T}_C$.

Note the fact that for any $W(k)$-algebra $R$ and any $r \in R$, if the image $\bar{r} \in \bar{R}$ over $k$ is a unit, then $r$ is a unit in $R$. So $\beta$ is an isomorphism between crystals $\mathcal{T}^\sigma$ and $\mathcal{T}$. \[ \square \]

Then $\beta^{-1}$ is also a morphism between crystals. Since $V = pF^{-1} = p\gamma^{-1} \otimes \beta^{-1}$, so is $p\gamma^{-1}$. Therefore $F_V := \gamma$ and $V_V := \gamma^{-1}$ can serve as Frobenius and Verschiebung of $V$, which makes $V$ a Dieudonne crystal. The fact that $p^{-k} \beta^{-1}$ and $p^{-k} \beta$ are isomorphisms implies $\mathcal{T}$ is a unit root crystal. We have the following summary.

Corollary 8.2.

$$(V, F_V = p^k \gamma, V_V = p^{k'} \gamma^{-1})$$

is a Dieudonne crystal,

$$(\mathcal{T}, F_\mathcal{T} = p^{-1} \beta)$$

is a unit root crystal and

$$\mathcal{(E, F)} \cong (V, F_V) \otimes (\mathcal{T}, F_\mathcal{T}).$$

The Hodge filtration of $\mathcal{E}$ comes from a sub line bundle $L \otimes L'$ of $\mathcal{V}$.

Let the filtration $\text{Fil}_\mathcal{T}$ be $L \otimes L' \subset \mathcal{V}$ and $\text{Fil}_\mathcal{T}$ be the trivial filtration.

Now we switch to BT groups.

9. THE BT GROUPS CORRESPONDING TO $\mathcal{V}$, $\mathcal{T}$ AND $\mathcal{E}$

From \[6\ Main Theorem 1\], we know that over a smooth curve $C/k$, the category of finite locally free Dieudonne crystals on $\text{cris}(C/W(k))$ is equivalent to the category of BT groups on $C$. Obviously $(\mathcal{E}, F, V)$ corresponds to $X[p^{\infty}]$. Let $G$ be the BT group over $C$ corresponding to $(\mathcal{V}, F_V, V_V)$.

From \[6\,remark 2.5.5\], we know the BT group $G$ induces a filtration of $\mathbb{D}(G)_C = \mathcal{V}_C$:

\[9\]

$$0 \to \omega_G \to \mathcal{V}_C \to t_G^* \to 0.$$ 

Lemma 9.1. The above filtration \[9\] coincides with the filtration $\text{Fil}_\mathcal{T}$ (mod $p$).

Proof. From \[8.2\] $\ker F_Y_C = (L \otimes L')^{(p)}_C$. By \[6\,Theorem 2.5.2 and Remark 2.5.5\], the subbundle of $\mathcal{V}$ satisfying this condition is unique and $\omega_G \cong L \otimes L'_C$.

Then the filtration $\text{Fil}_\mathcal{T}$ is just

$$0 \to \omega_G \to \mathcal{V}_C \to t_G^* \to 0.$$ 

Note $\mathcal{V}_C$ admits a connection $\nabla : \mathcal{V}_C \to \mathcal{V}_C \otimes \Omega_C$. The connection and the filtration induce the Higgs field: $\theta_G : \omega_G \to t_G \otimes \Omega_C$.

$$0 \to \omega_G \to \mathcal{V}_C \to t_G^* \otimes \Omega_C^1 \to 0.$$ 

Since $\mathcal{T}$ is a unit root crystal, by \[2\,2.4.10\], $\mathcal{T}$ comes from an etale BT group $H$ over $C$. In particular, $\mathbb{D}(H[p^n]) \cong \mathcal{T}/p^n$ and each truncated $\mathcal{T}/p^n$ comes from a local system \[4\,Theorem 2.2\] $\rho_n : \pi_1(C, c) \to GL(4, \mathbb{Z}/p^n)$. 

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Then there exists a finite etale covering \( f_n : C' \rightarrow C \) such that \( \pi_1(C', c) \cong \ker \rho_n \). Therefore we have \( f_n^*(\mathcal{T}/p^n) \cong \mathcal{O}_{C'/W_n} \) as unit root \( F_\cdot \)-crystals. By [2 2.4.1], over smooth curve \( C \), the category of finite locally free etale group schemes is equivalent to the category of \( p \)-torsion unit crystals. Thus

\[
f_n^*(H[p^n]) \cong (\mathbb{Z}/p^n)^{\oplus m}.
\]

**Definition 9.2.** Define the binary operation between two BT groups:

\[
G \otimes H := \text{colim}_n(G[p^n] \otimes_{\mathbb{Z}} H[p^n]).
\]

**Remark 9.3.** The inductive system \( (G[p^n] \otimes_{\mathbb{Z}} H[p^n]) \) is explained in [21]. Note in general \( G \otimes H \) is just an abelian sheaf rather than a group scheme. But in our case, \( H \) is etale and \( G \otimes H \) is indeed a BT group and \( (G \otimes H)[p^n] = G[p^n] \otimes_{\mathbb{Z}} H[p^n] \).

**Proposition 9.4.**

\[
X[p^n] \cong G[p^n] \otimes_{\mathbb{Z}} H[p^n].
\]

**Proof.** We will show that \( \mathbb{D}(G[p^n] \otimes_{\mathbb{Z}} H[p^n]) = \mathcal{V} \otimes T/p^n = \mathcal{E}/p^n(8.2) \) as Dieudonne crystals. Over \( C' \),

\[
\mathbb{D}_{C'}(f_n^*(G[p^n] \otimes_{\mathbb{Z}} H[p^n])) \cong f_n^*(\mathcal{V}/p^n)^{\oplus m} \cong f_n^*(\mathcal{V}/p^n) \otimes_{\mathcal{O}_C} f^*T/p^n
\]

as Dieudonne crystals. Both sides have effective descent datum with respect to \( C' \rightarrow C \). For any \( g \in \text{Aut}(C'/C) \), \( g^* \) acts on both of \( f^*(G[p^n] \otimes_{\mathbb{Z}} H[p^n]) \) and \( f_n^*(\mathcal{V}/p^n) \otimes_{\mathcal{O}_C} f^*T/p^n \) which is compatible with the functor \( \mathbb{D}_{C'} \):

\[
\begin{align*}
\mathbb{D}_{C'}(f_n^*(G[p^n] \otimes_{\mathbb{Z}} H[p^n])) & \xrightarrow{g^*} f^*(G[p^n] \otimes_{\mathbb{Z}} H[p^n]) \\
\xrightarrow{\mathbb{D}_{C'}} f^*(\mathcal{V}_p) & \xrightarrow{g^*} f^*(\mathcal{V}/p^n)
\end{align*}
\]

is commutative (we leave the details leave to the reader). Therefore the isomorphism (*) between effective descent datum also descends to \( C \).

Then we have

\[
\mathbb{D}_C(G[p^n] \otimes_{\mathbb{Z}} H[p^n]) = (\mathcal{V} \otimes T/p^n, F_{\mathcal{V}} \otimes F_T, V_{\mathcal{V}} \otimes F_{\mathcal{T}})^{-1}).
\]

Since \( C \) is smooth over an algebraically closed field \( k \), it has locally \( p \)-basis. Therefore we can apply ( [2], 4.1.1), the Dieudonne functor is fully faithful, so

\[
G[p^n] \otimes_{\mathbb{Z}} H[p^n] \cong X[p^n].
\]

\[\square\]

**Corollary 9.5.** \( G \otimes H = X[p^\infty] \).

To complete the proof of [1,4] it remains to show the isomorphism in [2,4] lifts to \( \tilde{C} \).

**Proposition 9.6.** The curve \( C \) is a versal deformation of the BT group \( G \).

**Proof.** From ( [13 Theorem 0.9]), any Shimura curve of Hodge type admits the maximal Higgs field. So \( \tilde{C} \) and thus \( C \) has maximal Higgs field:

\[
0 \longrightarrow \omega_{\tilde{C}} \longrightarrow \mathcal{E}_{\tilde{C}} \longrightarrow \mathcal{E}_{\tilde{C}} \otimes \Omega^1_{\tilde{C}} \longrightarrow \alpha_{\tilde{C}} \otimes \Omega^1_{\tilde{C}} \longrightarrow 0
\]

the induce map \( \theta : \omega_{\tilde{C}} \rightarrow \alpha_{\tilde{C}} \otimes \Omega^1_{\tilde{C}} \) is an isomorphism.
By [8.2], the filtration of $E_C$ comes from that of $V_C$. So $\theta = \theta_G \otimes \text{Id}_T$. Therefore the Higgs field of $\mathcal{V}_C / C$ is maximal and combining with [9.1] it implies $\omega_G \cong t_G^* \otimes \Omega^1_C$. By ([9.2.3.6]), $C$ is a versal deformation of the BT group $G$.

From ([21, Theorem 1.1]), such a curve $C$ admits a lifting to $W(k)$ over which $G$ admits a lifting as a BT group.

**Proposition 9.7.** The lifting of $C$ coincides with $\tilde{C}$.

**Proof.** From ([12, V, Theorem 1.6]), it is known that the lifting of the BT group $G$ is equivalent to lifting the filtration $\omega_G \rightarrow V_C$. By [9.1], it is equivalent to lifting $L \otimes \mathcal{L}' \rightarrow V_C$ and hence the curve $\tilde{C} / W(k)$ admits a lifting of $G$. From [21, Theorem 1.1], we know the lifting of $G$ is unique. $\square$

Let $\tilde{G}$ be the lifting of $G$ on $\tilde{C}$ and $\tilde{H}$ be the lifting of $H$ on $\tilde{C}$. Since $H$ is etale, $\tilde{H}$ is etale and unique up to isomorphism. By Proposition A.3 in [21], $\tilde{G} \otimes \tilde{H}$ is a BT group.

**Proposition 9.8.**

$$X[p^\infty] \cong \tilde{G} \otimes \tilde{H}.$$

**Proof.** In [9.4] we have shown that $X[p^\infty] \cong G \otimes H$ as BT groups over $C$. Both sides are liftable to $\tilde{C}$([9.7]), induced by the same filtration $\omega_{\tilde{G}} \otimes T \hookrightarrow E$([9.1]). Again by [12, V Theorem 1.6], $X[p^\infty] \cong \tilde{G} \otimes \tilde{H}$. $\square$

Now the proof of [1.4] is complete.

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