Comparison of optimum reconstruction filters for discrete and continuous-discrete linear observation models

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Abstract. In this paper, the expressions derivation for the frequency response of a reconstructing linear shift-invariant system and restoration error is performed for a continuously-discrete linear observation model. We analyzed three cases – two special ones and the general one. The first special case assumes the absence of dynamic degradations; the second one assumes the absence of additive noise. We presented results of a comparison of the restoration error values of the optimal filters with the same signal parameters, dynamic distortions and additive noise for the selected observation models.

1. Introduction
Traditionally, a discrete linear observation model (DLOM) [1 - 10] is used to describe degradations of digital signals, which operates only discrete signals. However, the original (physical) signal in many cases is a continuous function and is subject to distortion even before sampling, in a continuous domain [6, 7]. These, for example, are degradations of the optical signal in the imaging system [6, 8]. Signal distortions in the continuous domain can be taken into account by means of a continuous-discrete observation model (CDLOM) [4 - 6, 11]. It is interesting to find out will the results of digital processing be different, in particular, the signal reconstruction using these two models.

To propose our results in easier way, we operate one-dimensional signals in this paper, but all results and conclusions can be generalized to the two-dimensional case (images).

In this paper, we derive expressions for the frequency response of the optimal reconstruction filter using a continuous-discrete linear observation model. We also derive expressions for the minimally achievable signal restoration error.

Also, the results of calculations and comparison of signal restoration error s by the optimal restoring filters for the considered observation models are presented in the article.

The work is structured as follows. First, expressions are derived for the optimal restoring filter for CDLOM. Further, for the sake of completeness, the well-known expressions for the optimal restoring filter for DLOM are given. After that, the results of calculation of the restoration error for both
observation models using the same signal parameters, additive noise and dynamic distortions are given.

2. Optimum filter for continuous-discrete observation model

CDLOM assumes that the original signal undergoes dynamic degradations in a continuous time domain, then the degraded continuous signal is sampled, then an additional noise is added. Only a discrete output signal is available to observation. It is assumed that the distorting system is linear and invariant to a shift (LIS system):

\[
\begin{align*}
    y_n(t) &= \int_{-\infty}^{\infty} h_n(\tau) x_n(t - \tau) d\tau, \\
    y(n) &= y_n(t)|_{t = nT} + v(n),
\end{align*}
\]

where \(y_n(t)\) – is the degraded signal in a continuous time domain; \(h_n(t)\) – is the impulse response of the degradation system in a continuous time domain; \(x_n(t)\) – is the original continuous signal, \(y(n)\) – is the observed degraded discrete signal, \(v(n)\) – is the additive noise, which is uncorrelated with the signal, \(n\) – is the integer argument of the resulted sequences, given on the whole numerical axis, \(T\) – is the period of continuous signal sampling.

If we combine expressions (1) and (2) into one, a continuous-discrete model of observation can be written as follows:

\[
y(n) = \left[ \int_{-\infty}^{\infty} h_n(\tau) x_n(nT - \tau) d\tau \right]|_{t = nT} + v(n).
\]

The expression for the discrete values of the continuous source signal is as follows:

\[
x(n) = x_n(t)|_{t = nT}.
\]

The task of reconstruction is to obtain an estimate of the original signal:

\[
\hat{x}(n) = g(n) * y(n),
\]

where \(\hat{x}(n)\) – is the reconstructed signal, the estimate of the original signal; \(g(n)\) – is the impulse response of the restoring filter.

The optimal reconstruction filter is based on the root-mean-square error criterion minimization:

\[
\epsilon^2 = M[\{\hat{x}(n) - x(n)\}]^2 = M[\{\sum_{k=-\infty}^{\infty} g(k) y(n - k) - x(n)\}]^2 \rightarrow \min.
\]

Minimization of the criterion is achieved by varying the samples of the impulse response of the reconstruction filter. At the minimum point, all partial derivatives will be zero:

\[
\frac{\partial \epsilon^2}{\partial g(m)} = \frac{\partial}{\partial g(m)} M[\{\sum_{k=-\infty}^{\infty} g(k) y(n - k) - x(n)\}]^2 \rightarrow 0, \quad \forall m.
\]

From (6) the Wiener-Hopf equation is acquired [4, 5]:

\[
\sum_{k=-\infty}^{\infty} g(k) B_y(m - k) = B_{xy}(-m), \quad \forall m,
\]

where \(B_y(k)\) – is the autocorrelation function of the observed signal (ACF); \(B_{xy}(k)\) – is the cross-correlation function (CCF) of the original and observed signals.

Using the z-transformation from expression (8), we obtain the transfer function of the reconstruction filter:

\[
G(z) = \frac{\Phi_{xy}(z^{-1})}{\Phi_y(z)},
\]

where \(G(z)\) – is the transfer function of the optimal reconstruction filter; \(\Phi_{xy}(z)\) – is the z- transformation of the CCF of observed and original signals; \(\Phi_y(z)\) – is the z- transformation of the observed signal’s ACF.

Using these expressions, we can obtain the following expression:

\[
\Phi_\epsilon(z) = \Phi_x(z) - \frac{\Phi_{xy}(z)\Phi_{xy}(z^{-1})}{\Phi_y(z)}.
\]

While using this model of observation, it becomes necessary to connect the discrete and continuous time. Therefore, we should move from the z-transformation to the Fourier transform and use the mensural frequency scale:
In this case, the expressions (9) and (10) take the following form:

\[ G(\exp(i\Omega T)) = \frac{\phi_x(\exp(-i\Omega T))}{\phi_y(\exp(i\Omega T))}, \]  
\[ \Phi_y(\exp(i\Omega T)) = \Phi_x(\exp(i\Omega T)) - \frac{|\phi_x(\exp(i\Omega T))|^2}{\phi_y(\exp(i\Omega T))}. \]  

The restoration error can be calculated using the following expression:

\[ \varepsilon^2 = \frac{T}{2\pi} \int_{-\pi}^{\pi} \Phi_e(\exp(i\Omega T)) \, d\Omega. \]  

### 2.1. General case

Let us define the components entering into formulas (12) and (13) for the general case. The energy spectrum of the observed signal can be written using the known connection formula of the spectra of continuous and discrete signals [4]:

\[ \Phi_y(\exp(i\Omega T)) = \sum_{k=0}^{\infty} |H_u(\Omega + \frac{2\pi}{T} k)|^2 \Phi_{x_u}(\Omega + \frac{2\pi}{T} k) + \Phi_y(\exp(i\Omega T)). \]  

In order to determine the mutual energy spectrum of the original and observed signals, let us turn to the time domain and their CCF:

\[ B_{xy}(m) = M[x(n) \cdot y(n + m)]. \]  
\[ B_{xy}(m) = B_{x_u y_u}(\theta)|\theta = mT. \]  
\[ B_{x_u y_u}(\theta) = M[x_u(t - \theta) \cdot y_u(t)] = M \left[ h_{u}(t) \cdot x_u(t - \tau) \right] \int_{-\infty}^{\infty} h_u(\tau) \cdot x_u(\tau - \tau) \, d\tau. \]  

Using the Fourier transform from (18), we obtain:

\[ \Phi_{x y}(\exp(i\Omega T)) = \sum_{k=-\infty}^{\infty} H_u(\Omega + \frac{2\pi}{T} k) \Phi_{x_u}(\Omega + \frac{2\pi}{T} k). \]  

Taking into account (15) and (19), the expressions for the transfer function of the optimal reconstruction filter (12) and the energy spectrum of the signal restoration error (13) take the following form:

\[ G(\exp(i\Omega T)) = \frac{\sum_{k=-\infty}^{\infty} H_u(\Omega + \frac{2\pi}{T} k) \Phi_{x_u}(\Omega + \frac{2\pi}{T} k)}{\sum_{k=-\infty}^{\infty} |H_u(\Omega + \frac{2\pi}{T} k)|^2 \Phi_{x_u}(\Omega + \frac{2\pi}{T} k) + \Phi_y(\exp(i\Omega T))}, \]  
\[ \Phi_{\text{CDLMD}}(\exp(i\Omega T)) = \Phi_e(\exp(i\Omega T)) - \frac{\left( \sum_{k=-\infty}^{\infty} H_u(\Omega + \frac{2\pi}{T} k) \Phi_{x_u}(\Omega + \frac{2\pi}{T} k) \right)^2}{\sum_{k=-\infty}^{\infty} |H_u(\Omega + \frac{2\pi}{T} k)|^2 \Phi_{x_u}(\Omega + \frac{2\pi}{T} k) + \Phi_y(\exp(i\Omega T))}. \]  

The restoration error can be calculated using the following expression:

\[ \varepsilon^2_{\text{CDLMD}} = \frac{T}{2\pi} \int_{-\pi}^{\pi} \Phi_{\text{CDLMD}}(\exp(i\Omega T)) \, d\Omega. \]  

### 2.2. Absence of dynamic degradations

Let there be no effect of dynamic degradations on the original signal. In this case, expression (3) will look like:

\[ y(n) = x_u(t)|_{t = nT + \nu(n)}. \]  

Let us apply the same method as in expression (17) and consider the ACF of the discrete signal as discrete ACF values of the continuous signal, so the expression for the energy spectrum of the signal will look like:

\[ B_x(m) = B_{x_u}(\tau)|_{\tau = mT}. \]
In this case, the expressions for the frequency response of the optimal filter (20) and the restoration error spectrum (21) will look like this:

\[
G(\exp(i\Omega T)) = \frac{\sum_{k=-\infty}^{\infty} \Phi_{xu}(\Omega + \frac{2\pi}{T} k)}{\sum_{k=-\infty}^{\infty} \Phi_{xu}(\Omega + \frac{2\pi}{T} k) + \tau \Phi_{x}(\exp(i\Omega T))},
\]

\[
\Phi_{ECDLOM}(\exp(i\Omega T)) = \Phi_{x}(\exp(i\Omega T)) - \frac{\sum_{k=-\infty}^{\infty} \Phi_{x}(\Omega + \frac{2\pi}{T} k)^2}{\sum_{k=-\infty}^{\infty} \Phi_{xu}(\Omega + \frac{2\pi}{T} k) + \tau \Phi_{x}(\exp(i\Omega T))}.
\]

Taking into account expression (25), the previous pair of expressions will take the following form:

\[
G(\exp(i\Omega T)) = \frac{\Phi_{x}(\exp(i\Omega T))}{\Phi_{x}(\exp(i\Omega T)) + \Phi_{x}(\exp(i\Omega T))},
\]

\[
\Phi_{ECDLOM}(\exp(i\Omega T)) = \Phi_{x}(\exp(i\Omega T)) - \frac{\sum_{k=-\infty}^{\infty} \Phi_{x}(\Omega + \frac{2\pi}{T} k)^2}{\sum_{k=-\infty}^{\infty} \Phi_{x}(\Omega + \frac{2\pi}{T} k) + \tau \Phi_{x}(\exp(i\Omega T))}.
\]

### 2.3. Absence of additive noise

Suppose that the input signal is not affected by additive noise, then formula (3) takes the form:

\[
y(n) = \int_{-\infty}^{\infty} h_u(t) x_u(nT - t) \, dt = nT.
\]

In this particular case, the expressions (20) and (21) are as follows:

\[
G(\exp(i\Omega T)) = \frac{\sum_{k=-\infty}^{\infty} H_{ux}(\Omega + \frac{2\pi}{T} k) \Phi_{xu}(\Omega + \frac{2\pi}{T} k)}{\sum_{k=-\infty}^{\infty} H_{ux}(\Omega + \frac{2\pi}{T} k)^2 \Phi_{xu}(\Omega + \frac{2\pi}{T} k)},
\]

\[
\Phi_{ECDLOM}(\exp(i\Omega T)) = \Phi_{x}(\exp(i\Omega T)) - \frac{\sum_{k=-\infty}^{\infty} H_{ux}(\Omega + \frac{2\pi}{T} k)^2 \Phi_{xu}(\Omega + \frac{2\pi}{T} k)}{\sum_{k=-\infty}^{\infty} H_{ux}(\Omega + \frac{2\pi}{T} k)^2 \Phi_{xu}(\Omega + \frac{2\pi}{T} k)}.
\]

As can be seen from the formulas, the obtained optimal filter is not inverse for a given observation model.

### 3. Optimum filter for discrete observation model

As it is known, DLOM is given as follows [4-8]:

\[
y(n) = x(n) * h(n) + v(n),
\]

where \(h(n)\) – is the known impulse response of a degradation LIS system; 
\(\ast\) – is the discrete convolution. 

The task of recovery is to obtain an estimate of the original signal:

\[
\hat{x}(n) = g(n) * y(n),
\]

### 3.1. General case

Using the z-transformation from the expressions for the partial derivatives of the criterion, an expression for the transfer function of the restoring system follows:

\[
G(\exp(i\Omega T)) = \frac{H(\exp(i\Omega T))}{|H(\exp(i\Omega T))|^2 \Phi_{x}(\exp(i\Omega T)) + \Phi_{x}(\exp(i\Omega T))}.
\]

The expression for the energy spectrum of the restoration error will look like this:

\[
\Phi_{ECDLOM}(\exp(i\Omega T)) = \frac{\Phi_{x}(\exp(i\Omega T))}{|H(\exp(i\Omega T))|^2 \Phi_{x}(\exp(i\Omega T)) + \Phi_{x}(\exp(i\Omega T))}.
\]

The error is calculated as follows:

\[
e^2 = \frac{T}{2\pi} \int_{-\pi}^{\pi} \Phi_{ECDLOM}(\exp(i\Omega T)) \, d\Omega.
\]

### 3.2. Absence of dynamic degradations

Suppose that the original signal is not affected by dynamic degradations. The observation model is written as follows:

\[
y(n) = x(n) + v(n).
\]
The expression for the transfer function of the reconstruction filter (35) will have the following form:

\[ G(\exp (i\Omega T)) = \frac{\Phi_x(\exp (i\Omega T))}{\Phi_x(\exp (i\Omega T)) + \Phi_e(\exp (i\Omega T))}. \]  

(39)

Taking into account (39), the energy spectrum of the error and its value will look like this:

\[ \Phi_{\epsilon \text{DLOM}}(\exp (i\Omega T)) = \frac{\Phi_e(\exp (i\Omega T))}{\Phi_x(\exp (i\Omega T)) + \Phi_e(\exp (i\Omega T))}. \]  

(40)

We note that expressions (39) and (40) completely coincide with expressions (28) and (29) obtained for CDLOM.

### 3.3. Absence of additive noise

Suppose that the input signal is not affected by additive noise. In this case, the observation model is written as follows:

\[ y(n) = x(n) \ast h(n). \]  

(41)

In this case, the frequency response of the reconstruction filter will have the following form:

\[ G(\exp (i\Omega T)) = \frac{1}{H(\exp (i\Omega T))}. \]  

(42)

The resulting filter is an inverse one. Often the reverse filter is unstable, since for its stability it is required that all values of the frequency characteristic of the distorting LIS system be nonzero.

### 4. Experimental comparison of restoration error s for observation models

Let us compare the restoration error s of the optimal filters for the considered models with the same signal, noise and degradation system parameters.

The source signal is an exponentially correlated signal:

\[ B_{x_0}(t) = \sigma_x^2 \cdot \exp (-\beta |t|), \]  

(43)

\[ \Phi_{x_0} (\Omega) = \frac{2\beta \sigma_x^2}{\beta^2 + \Omega^2}, \]  

(44)

where \( \sigma_x^2 \) – is the original signal variance,

\( \beta \) – is the correlation coefficient,

It is assumed that the additive noise is white. Its energy spectrum:

\[ \Phi_e(\exp (i\Omega T)) = \sigma_e^2, \]  

(45)

where \( \sigma_e^2 \) – is the noise variance.

Two cases of distorting LIS systems were considered. The first one’s impulse response is the Gauss function, which is the traditional model of linear distortion systems for real optical systems [4]:

\[ h_u(t) = \frac{1}{\sqrt{2\pi} \sigma_u} \cdot \exp \left(-\frac{t^2}{2\sigma_u^2}\right), \]  

(46)

\[ H_u(\Omega) = \exp \left(-\frac{\sigma_u^2 \Omega^2}{2}\right), \]  

(47)

where \( \sigma_u^2 \) – is the degradation LIS system variance.

Another one’s impulse response is the rectangular impulse:

\[ h_u(t) = \begin{cases} \frac{1}{T}, & t \in [0, T], \\ 0, & \text{other}, \end{cases} \]  

(48)

\[ H_u(\Omega) = \text{sinc}(\Omega), \]  

(49)

For CDLOM, the error of restoring the optimal filter was calculated from expression (22), for DLOM using expression (37). Dependencies normalized to the signal variance of the restoration error on the noise variance were calculated. Some of the obtained dependencies are presented in Figures 1-8.

As can be seen from the presented dependences, the results for the selected observation models differ significantly. Use to recover a signal that has been degraded, DLOM, which does not take this fact into account, gives too optimistic results. With similar signal and distortion parameters, the CDLOM, which takes into account that the signal has undergone distortion in the continuous domain, reconstructs the signal with a larger error.
Figure 1. Normalized to the signal variance dependencies of the restoration error on the noise variance, Gauss function, $\sigma_R^2 = 2, \beta = 0.8$.

Figure 2. Normalized to the signal variance dependencies of the restoration error on the noise variance, Gauss function $\sigma_R^2 = 0.5, \beta = 0.95$.

Figure 3. Normalized to the signal variance dependencies of the restoration error on the noise variance, Gauss function $\sigma_R^2 = 1, \beta = 0.7$.

Figure 4. Normalized to the signal variance dependencies of the restoration error on the noise variance, Gauss function $\sigma_R^2 = 1, \beta = 0.9$.

Figure 5. Normalized to the signal variance dependencies of the restoration error on the noise variance, rectangular function, $T = 1, \beta = 0.9$.

Figure 6. Normalized to the signal variance dependencies of the restoration error on the noise variance, rectangular function, $T = 1, \beta = 0.7$. 
It was also examined what would be the restoration error of the optimal filter for DLOM in the case of processing a signal that has been degraded in the continuous domain, in other words, the signal with which CDLOM works. The obtained dependences are presented in Figures 9-12:

**Figure 7.** Normalized to the signal variance dependencies of the restoration error on the noise variance, rectangular function, $T = 2, \beta = 0.8$.

**Figure 8.** Normalized to the signal variance dependencies of the restoration error on the noise variance, rectangular function, $T = 0.5, \beta = 0.95$.

**Figure 9.** Normalized to the signal variance dependencies of the restoration error on the noise variance, Gauss function, $\sigma^2_N = 1, \beta = 0.9$, degraded in continuous domain signal.

**Figure 10.** Normalized to the signal variance dependencies of the restoration error on the noise variance, Gauss function, $\sigma^2_N = 2, \beta = 0.8$ degraded in continuous domain signal.

**Figure 11.** Normalized to the signal variance dependencies of the restoration error on the noise variance, Gauss function, $\sigma^2_N = 0.5, \beta = 0.95$, degraded in continuous domain signal.

**Figure 12.** Normalized to the signal variance dependencies of the restoration error on the noise variance, Gauss function, $\sigma^2_N = 0.5, \beta = 0.7$, degraded in continuous domain signal.
As can be seen from the resulting dependencies, in this case the restoration error of the optimal DLOM filter is significantly higher than the error of the optimal CDLOM filter.

5. Conclusion
This article presents the derivation of expressions for the frequency response and restoration error of the optimal reconstruction filter for CDLOM for the general case, the case of no dynamic distortions and the case of the absence of additive noise.

Normalized to the signal variance dependencies of the restoration error on the noise variance for DLOM and CDLOM are obtained. It was found that the use of DLOM gives too optimistic results, since it does not take into account the fact that the original signal has undergone distortions in the continuous region. When recovering a signal distorted in a continuous domain, the restoration error of the optimal DLOM filter is significantly higher than CDLOM filter.

6. References
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