ON FAKE ES-IRREDUCIBILE COMPONENTS OF CERTAIN STRATA OF SMOOTH PLANE SEXTICS

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Abstract. We construct the first examples of what we call fake $ES$-irreducible components; Definition 2.8. In our way to do so, we classify the automorphism groups of smooth plane sextics that only have automorphisms of order $\leq 3$; Theorems 2.1, 2.4 and 2.5, Corollaries 2.9 and 2.11.

1. Introduction

Let $\mathcal{M}_{g}^{3}$ be the set of $K$-isomorphism classes of smooth plane curves $C$ of a fixed degree $d \geq 4$. Here $K$ is an algebraically closed field of characteristic $p = 0$ or $p > 2g + 1$, where $g = (d - 1)(d - 2)/2 \geq 3$ is the geometric genus of $C$.

We can associate to any $[C] \in \mathcal{M}_{g}^{3}$ infinitely many non-singular plane models, each of them is given by a homogeneous polynomial equation $C : F(X,Y,Z) = 0$ of degree $d$ in $\mathbb{P}^{2}(K)$. Moreover, two such plane models for $C$ are $K$-isomorphic and their automorphism groups are $\text{PGL}_{3}(K)$-conjugated via a projective change of variables $\phi \in \text{PGL}_{3}(K)$.

Now, suppose that $G$ is a finite non-trivial group that can be embedded into $\text{PGL}_{3}(K)$. We write $[C] \in \mathcal{M}_{g}^{3}(G)$ when there exists an injective representation $\phi : G \hookrightarrow \text{PGL}_{3}(K)$ such that $\phi(G)$ is a subgroup of $\text{Aut}(C)$; the automorphism group of $C : F(X,Y,Z) = 0$ inside $\text{PGL}_{3}(K)$. More precisely, we say that $[C]$ belongs to the component $\mathcal{M}_{g}^{3}(\phi(G))$ of $\mathcal{M}_{g}^{3}(G)$. Similarly, we write $[C] \in \mathcal{M}_{g}^{3}(G)$ when $\phi(G) = \text{Aut}(C)$ for some $\phi$, and again we say that $[C]$ belongs to the component $\mathcal{M}_{g}^{3}(\phi(G))$ of $\mathcal{M}_{g}^{3}(G)$.

Clearly, if $\phi_i : G \hookrightarrow \text{PGL}_{3}(K)$, for $i = 1, 2$, are $\text{PGL}_{3}(K)$-conjugated, then $\mathcal{M}_{g}^{3}(\phi_1(G)) = \mathcal{M}_{g}^{3}(\phi_2(G))$ and $\mathcal{M}_{g}^{3}(\phi_1(G)) = \mathcal{M}_{g}^{3}(\phi_2(G))$. Accordingly,

$$\mathcal{M}_{g}^{3}(G) = \bigsqcup_{[\phi] \in R_{G}} \mathcal{M}_{g}^{3}(\phi(G)) \quad \text{and} \quad \mathcal{M}_{g}^{3}(G) = \bigsqcup_{[\phi] \in R_{G}} \mathcal{M}_{g}^{3}(\phi(G)).$$

Here $R_{G} := \{ [\phi] : G \hookrightarrow \text{PGL}_{3}(K) \} / \sim$, where $\phi_1 \sim \phi_2$ if and only if $\phi_1(G)$ and $\phi_2(G)$ are $\text{PGL}_{3}(K)$-conjugated.

Definition 1.1 (ES-irreducibility [3]). Each $[\phi] \in R_{G}$ such that $\mathcal{M}_{g}^{3}(\phi(G)) \neq \emptyset$ is called an ES-irreducible component for $\mathcal{M}_{g}^{3}(G)$. We call $\mathcal{M}_{g}^{3}(G)$ ES-irreducible if it has exactly one ES-irreducible component.

Clearly, if a non-empty $\mathcal{M}_{g}^{3}(G)$ is not ES-irreducible, then it is not irreducible and the number of its ES-irreducible components is a lower bound for the number of its irreducible components inside the coarse moduli space $\mathcal{M}_{g}$ of $K$-isomorphism classes of smooth curves of genus $g$.

Now, in the language of ES-irreducibility, one can interpret the results of Henn [10] and Komiya-Kuribayashi [11] for smooth plane quartic curves, which are genus
g = 3 curves, as follows: the strata \( \widetilde{M}^3_{13}(G) \) are either empty or ES-irreducible. Thus each non-empty \( \widetilde{M}^3_{13}(G) \) is described by a single normal form: a homogeneous polynomial equation \( F(X, Y, Z) = 0 \) in \( \mathbb{P}^2(K) \) equipped with parameters as its coefficients such that any \([C]\) in \( \widetilde{M}^3_{13}(G) \) can be described by a smooth plane model through a specialization of those parameters.

**Notation.** Throughout the paper, \( L_i, B \) denotes the generic homogeneous polynomial of degree \( i \) in the variables \( \{X, Y, Z\} \) - \( \{B\} \).

By \( \zeta_n \) we mean a fixed primitive \( n \)-th root of unity in \( K \).

A projective linear transformation \( A = (a_{i,j}) \in \mathrm{PGL}_3(K) \) is sometimes written as

\[
[a_{1,1}X + a_{1,2}Y + a_{1,3}Z : a_{2,1}X + a_{2,2}Y + a_{2,3}Z : a_{3,1}X + a_{3,2}Y + a_{3,3}Z].
\]

For example, \([X : Z : Y]\) represents the projective change of variables \( X \mapsto X, Y \mapsto Z, Z \mapsto Y \), and \( \text{diag}(1, a, b) \) represents \( X \mapsto X, Y \mapsto aY, Z \mapsto bZ \) with \( a, b \in K^* \).

We use the formal GAP library notations “\( \text{GAP}(n, m) \)” to refer the finite group of order \( n \) that appears in the \( m \)-th position of the atlas for small finite groups [8]. See also GroupNames.

Fix the following subgroups in \( \mathrm{PGL}_3(K) \):

- \( \varphi_1(\mathbb{Z}/3\mathbb{Z}) := \langle (\text{diag}(1, 1, -1)) \rangle \) and \( \varphi_1(\mathbb{Z}/2\mathbb{Z}) := \langle \varphi_1(\mathbb{Z}/3\mathbb{Z}), \text{diag}(1, -1, 1) \rangle \),
- \( \varphi_1(\mathbb{Z}/3\mathbb{Z}) := \langle (\text{diag}(1, 1, \zeta_3)) \rangle \),
- \( \varphi_2(\mathbb{Z}/3\mathbb{Z}) := \langle (\text{diag}(1, \zeta_3, \zeta_3^{-1})) \rangle \) and \( \varphi_2(\mathbb{Z}/3\mathbb{Z}) := \langle \varphi_2(\mathbb{Z}/3\mathbb{Z}), [Y : Z : X] \rangle \),
- \( \varphi_1(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) := \langle \varphi_1(\mathbb{Z}/3\mathbb{Z}), \varphi_2(\mathbb{Z}/3\mathbb{Z}) \rangle \),
- \( \varphi_1(\mathbb{Z}/2\mathbb{Z}) := \langle \varphi_1(\mathbb{Z}/2\mathbb{Z}), [Y : Z : X] \rangle \) and \( \varphi_2(\mathbb{Z}/2\mathbb{Z}) := \langle \varphi_1(\mathbb{Z}/2\mathbb{Z}), [\zeta_3^{-1}Y : Z : X] \rangle \).

**Remark 1.2.** P. Henn observed that \( \widetilde{M}^3_{13}(\mathbb{Z}/2\mathbb{Z}) \) admits two ES-components. One component corresponds to \( \varphi_1(\mathbb{Z}/3\mathbb{Z}) \) where any \([C]\) in \( \widetilde{M}^3_{13}(\varphi_1(\mathbb{Z}/3\mathbb{Z})) \) is given by an equation of the form \( Z^2Y + L_{1,3}Z = 0 \). The second component corresponds to \( \varphi_2(\mathbb{Z}/3\mathbb{Z}) \) such that any \([C]\) in \( \widetilde{M}^3_{13}(\varphi_2(\mathbb{Z}/3\mathbb{Z})) \) is given by an equation of the form \( X^4 + X(Y^3 + 3Z) + \alpha_{2,1}X^2YZ + \alpha_{1,1}X(YZ)^2 = 0 \) for some \( \alpha_{2,1}, \alpha_{1,1} \in K \). In particular, \( C' \) has \([X : Z : Y] \) as an extra involution, thus \( C' \) always has the symmetry group \( S_3 \) as a subgroup of automorphisms. Therefore, \( \widetilde{M}^3_{13}(\varphi_2(\mathbb{Z}/3\mathbb{Z})) = \emptyset \) and \( \widetilde{M}^3_{13}(\varphi_2(\mathbb{Z}/3\mathbb{Z})) \subseteq \widetilde{M}^3_{13}(S_3) \).

Concerning smooth plane quintic curves, which are genus \( g = 6 \) curves, Badr-Bars [1] showed that all the strata \( \widetilde{M}^5_{13}(G) \) are either empty or ES-irreducible except when \( G = \mathbb{Z}/4\mathbb{Z} \). In this case, \( \widetilde{M}^5_{13}(\mathbb{Z}/4\mathbb{Z}) \) has exactly two ES-irreducible components. Moreover, we generalized this result in [3] for any odd degree \( d \geq 5 \). More precisely, we proved that \( \widetilde{M}^5_{13}(\mathbb{Z}/(d - 1)\mathbb{Z}) \) has at least two ES-irreducible components for any \( g = (d - 1)(d - 2)/2 \) with \( d \geq 5 \) odd. However, each of the strata \( \widetilde{M}^5_{13}(\varphi(G)) \) is described again by a single normal form.

Accordingly, we were wondering if this is the situation in general. That is to say, there always exists a single normal form describing the elements of \( \widetilde{M}^5_{13}(\varphi(G)) \) for each \( \varphi \in R_G \). In this article, we will show that this impression is not true at least for smooth plane sextic curves, which are genus \( g = 10 \) curves. We establish three counter examples corresponding to \( G = \mathbb{Z}/3\mathbb{Z} \) and \( A_4 \) respectively.

On the other hand, classifying automorphism groups of smooth curves is a long standing problem that receives interest by many people. In the case of hyperelliptic
curve, the structure of the automorphism group is quite explicit, see [6, 7, 16, 17]. For non-hyperelliptic curves, we still have a lack of knowledge about the structure, except for some special cases. For example, the cases of low genus and also Hurwitz curves, see [5, 10, 12, 13, 14]. This lack motivates us to do more investigation in this direction, especially for the case of smooth plane curves of degree \( d \geq 4 \). In this paper, we classify the automorphism groups of smooth plane curves \( C \) of degree 6 such that 2 and 3 are the only divisors of \(| \text{Aut}(C) | \). A more detailed treatment of automorphisms of non-singular plane sextic curves is intended in [4].

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2. Main Results

Theorem 2.1. Let \( C \) be a smooth plane sextic curve that admits an automorphism of maximal order 2. Up to \( K \)-isomorphism, \( C \) is defined by an equation of the form:

\[ C : Z^6 + Z^4 L_{2, Z} + Z^2 L_{1, Z} + L_{6, Z} = 0 \]

such that \( L_{6, Z} \) is of degree \( \geq 5 \) in both \( X \) and \( Y \), and at least one of the binary forms \( L_{2, Z} \) and \( L_{4, Z} \) is non-zero. Moreover, \( \text{Aut}(C) = \mathbb{Z}/2\mathbb{Z} \) unless \( L_{2, Z}, L_{4, Z} \) and \( L_{6, Z} \) belong to the ring \( K[X^2, Y^2] \). In the latter case, \( \text{Aut}(C) = \mathbb{Z}/2\mathbb{Z}^4 \).

Corollary 2.2. The strata \( \mathcal{M}^P_{10}(\mathbb{Z}/2\mathbb{Z}) \) and \( \mathcal{M}^P_{10}(\mathbb{Z}/2\mathbb{Z})^2 \) are ES-irreducible.

Definition 2.3 ([15]). An homomorphism of period \( n \) is a projective linear transformation of the plane \( \mathbb{P}^2(K) \), which is \( \text{PGL}_3(K) \)-conjugate to \( \text{diag}(1, 1, \zeta_n) \). Such a transformation fixes pointwise a line \( \mathcal{L} \) (its axis) and a point \( P \) off this line (its center). In its canonical form, \( \mathcal{L} : Z = 0 \) and center \( P = (0 : 0 : 1) \).

Otherwise, it is called a non-homology.

Theorem 2.4. Let \( C \) be a smooth plane sextic curve that admits an homology of period 3 as an automorphism of maximal order. Up to \( K \)-isomorphism, \( C \) is defined by an equation of the form \( Z^6 + Z^3 L_{3, Z} + L_{6, Z} = 0 \) where neither \( L_{3, Z} \) nor \( L_{6, Z} \) equals 0. Moreover, \( \text{Aut}(C) \) is always \( \mathbb{Z}/3\mathbb{Z} \) except when \( C \) is \( K \)-isomorphic to \( C' \) of the form \( \mathcal{C}' : X^6 + Y^6 + Z^6 + Z^3 \left( \alpha_{3,0} X^3 + \alpha_{3,0} Y^3 \right) + \alpha_{3,3} X^3 Y^3 Z^3 = 0 \), such that \( \alpha_{3,3}, \alpha_{3,0} \), \( \alpha_{3,3} \) are pair-wise distinct modulo \( \{\pm 1\} \). In this case, \( \text{Aut}(C') = \mathbb{Z}/2\mathbb{Z}^3 \).

Theorem 2.5. Let \( C \) be a smooth plane sextic curve that admits a non-homology of period 3 as an automorphism of maximal order. Up to \( K \)-isomorphism, \( C \) is a member of one of the following families:

\[ \mathcal{C}_1 : \quad X^6 + Y^6 + Z^6 + XYZ \left( \alpha_{4,1} X^3 + \alpha_{4,1} Y^3 + \alpha_{1,1} Z^3 \right) + \alpha_{2,2} X^2 Y^2 Z^2 \\
\quad + \alpha_{3,3} X^3 Y^3 + \alpha_{3,0} X^3 Z^3 + \alpha_{0,3} Y^3 Z^3 = 0 \]

\[ \mathcal{C}_2 : \quad X^3 Y + Y^3 Z + X^3 Z + XYZ \left( \alpha_{3,2} X^2 Y + \alpha_{1,3} Y^2 Z + \alpha_{2,1} X Z^2 \right) \\
\quad + \alpha_{2,2} X^2 Y^4 + \alpha_{0,3} Y^2 Z^4 + \alpha_{4,0} X^4 Z^2 = 0. \]

In either way, \( \sigma = \text{diag}(1, \zeta_3, \zeta_3^{-1}) \) is an automorphism of maximal order 3.

(1) The automorphism group \( \text{Aut}(\mathcal{C}_1) = \mathbb{Z}/3\mathbb{Z} \) except when one of the following conditions holds.

(i) If \( \alpha_{4,1} = \alpha_{1,1} = \alpha_{2,2} = 0 \) such that \( \alpha_{3,3} \neq \alpha_{3,0} \), then \( \mathcal{C}_1 \) reduces to \( X^6 + Y^6 + Z^6 + X^3 \left( \alpha_{3,3} Y^3 + \alpha_{3,0} Z^3 \right) + \alpha_{0,3} Y^3 Z^3 = 0 \),
where $\text{Aut}(C_1) = \varrho_1(\mathbb{Z}/3\mathbb{Z})^2$.

(ii) If (a) $\alpha_{1,1} = \pm \alpha_{1,1}$ and $\alpha_{3,0} = \pm \alpha_{3,0}$, (b) $\alpha_{1,4} = \pm \alpha_{1,1}$ and $\alpha_{3,3} = \pm \alpha_{3,0}$, or (c) $\alpha_{1,1} = \pm \alpha_{1,1}$ and $\alpha_{3,3} = \pm \alpha_{3,0}$, then $C_1$ is $K$-isomorphic to

\[ C'_1 : X^6 + Y^6 + Z^6 + \alpha_{1,1}'X^4YZ + \alpha_{3,3}'X(Y^3 + Z^3) + \alpha_{2,2}'X^2Y^2Z^2 \]
\[ + \alpha_{1,2}'XYZ(Y^3 + Z^3) + \alpha_{3,3}'Y^3Z^3 = 0, \]

where $\text{Aut}(C'_1) = \varrho_2(S_3)$ if $\alpha_{1,1}' \neq \alpha_{1,1}$ or $\alpha_{3,3}' \neq \alpha_{3,3}$, and $\text{Aut}(C'_1) = \varrho_1(\mathbb{Z}/3\mathbb{Z} \ltimes S_3)$ otherwise.

Remark 2.6. $(\alpha_{3,3}', \alpha_{1,2}') \neq (0, 0)$ or diag$(1, \zeta_6, \zeta_6^{-1})$ will be an automorphism of order $6 > 3$.

(iii) If $\alpha_{1,1} = \zeta_6^{2\ell} \alpha_{1,1}$, $\alpha_{1,4} = \pm \zeta_3^{-\ell} \alpha_{1,1}$, $\alpha_{3,3} = \pm (-1)^\ell \alpha_{3,0}$, $\alpha_{3,0} = \pm \alpha_{3,0}$ for some $\ell \neq 0$ or $3 \bmod 6$, then $C_1$ is $K$-isomorphic to

\[ C''_1 : X^6 + \zeta_6^{2\ell} Y^6 + \zeta_6^{-2\ell} Z^6 + \alpha_{1,1}'XYZ(X^3 + \zeta_6^{2\ell} Y^3 + \zeta_6^{-2\ell} Z^3) + \alpha_{3,3}'X(Y^3 + \zeta_6^{-2\ell} X^3 Z^3 + \zeta_6^{2\ell} Y^3 Z^3) = 0, \]

where $\text{Aut}(C''_1) = \varrho_2(\mathbb{Z}/3\mathbb{Z})^2$.

(iv) If (a) $(\alpha_{1,1}, \alpha_{1,1}, \alpha_{1,1}, \alpha_{1,4})$, $(\alpha_{1,1}, \alpha_{1,4}, \alpha_{1,1}, \alpha_{4,1})$ or $(\alpha_{1,1}, \alpha_{1,4}, \alpha_{4,1})$ equals

\[
\frac{2(29 - 54\lambda^6 - 54\mu^6)}{27\lambda\mu}, \quad \frac{2(27\mu^6 - 54\lambda^6 - 52)}{27\lambda^4}, \quad \frac{2(27\lambda^6 - 54\mu^6 - 52)}{27\lambda^4},
\]

(b) $(\alpha_{3,0}, \alpha_{3,3}, \alpha_{3,0})$, $(\alpha_{3,3}, \alpha_{3,0}, \alpha_{3,3})$ or $(\alpha_{3,0}, \alpha_{3,0}, \alpha_{3,3})$ equals

\[
\frac{2(81\lambda^6 - 27\mu^6 - 26)}{27\lambda^3}, \quad \frac{2(81\mu^6 - 27\lambda^6 - 26)}{27\lambda^3}, \quad \frac{2(82 - 27\lambda^6 - 27\mu^6)}{27\lambda^3},
\]

and (c) $\alpha_{2,2} = \frac{9\lambda^6 + 9\mu^6 + 10}{3\lambda^2\mu^2}$ for some $\lambda, \mu \in K^*$, then $C_1$ is $K$-isomorphic to

\[ C_{1,\lambda,\mu} : X^6 + Y^6 + Z^6 + f_1(\lambda, \mu)X^2Y^2Z^2 + f_2(\lambda, \mu)(X^4Y^2 + X^2Z^4 + Y^4Z^2) + f_2(\mu, \lambda)(X^4Z^2 + X^2Y^4 + Y^2Z^4) = 0, \]

where

\[
f_1(\lambda, \mu) := 3(80 + 81\lambda^6 + 81\mu^6), \quad f_2(\lambda, \mu) := 81 \left(1 + \zeta_3\lambda^6 + \zeta_3^{-1}\mu^6\right).
\]

In this case, $\text{Aut}(C_{1,\lambda,\mu}) = \varrho_1(A_4)$.

(2) The automorphism group $\text{Aut}(C_2) = \langle \sigma \rangle = \varrho_2(\mathbb{Z}/3\mathbb{Z})$ except when one of the following conditions holds.

(i) If $\alpha_{2,4} = \zeta_{21}^{12r} \alpha_{4,0}$, $\alpha_{2,4} = \zeta_{21}^{r} \alpha_{4,0}$, $\alpha_{1,3} = \zeta_{21}^{-r} \alpha_{3,2}$, $\alpha_{2,1} = \alpha_{2,1}$, then $C_2$ is $K$-isomorphic to

\[ C'_2 : X^5Y + Y^5Z + XZ^5 + \alpha_{4,0}\zeta_{21}^{2r}(X^4Z^2 + X^2Y^4 + Y^2Z^4) + \alpha_{3,2}\zeta_{21}^{-r}XYZ(X^2Y + XZ^2 + Y^2Z) = 0, \]

where $\text{Aut}(C'_2) = \varrho_2(\mathbb{Z}/3\mathbb{Z})^2$.

Remark 2.7. $(\alpha_{2,4}, \alpha_{1,3}) \neq (0, 0)$ or diag$(1, \zeta_{21}, \zeta_{21}^{-1})$ will be an automorphism of order $21 > 3$. 

Theorem 2.8. (a) If $\alpha_{2,4} = \zeta_{21}^{12r} \alpha_{4,0}$, $\alpha_{2,4} = \zeta_{21}^{r} \alpha_{4,0}$, $\alpha_{1,3} = \zeta_{21}^{-r} \alpha_{3,2}$, $\alpha_{2,1} = \alpha_{2,1}$, then $C_2$ is $K$-isomorphic to
(ii) If (a) \((a_2, a_4, a_0, a_0, 2)\), \((a_0, 2, a_2, a_4, a_0)\) or \((a_4, 0, a_0, 2, a_2)\) equals
\[
\begin{pmatrix}
\frac{\lambda^3 \mu + 4 \mu^5}{2 \lambda^4}, & \frac{\lambda + 4 \lambda^5 \mu + 4 \lambda + \mu^5}{2 \mu^2}, & \frac{2 (2 \lambda^5 \mu + \lambda + 2 \mu^5)}{2 \lambda^2 \mu^4}
\end{pmatrix}
\]
and (b) \((a_1, a_3, a_2, a_2, 1)\), \((a_2, a_1, a_3, a_2)\) or \((a_3, a_2, a_2, 1, a_1)\) equals
\[
\begin{pmatrix}
\frac{2 (2 \lambda^5 \mu + 2 \lambda + \mu^5)}{\lambda^3 \mu^2}, & \frac{2 \lambda^5 \mu + 4 \lambda + 4 \mu^5}{\lambda^2 \mu}, & \frac{2 (2 \lambda^5 \mu + \lambda + 2 \mu^5)}{\lambda \mu^3}
\end{pmatrix}
\]
then \(C_2\) is \(K\)-isomorphic to
\[
\begin{align*}
C_{2, \lambda, \mu} : X^6 + Y^6 + Z^6 &= g_1(\lambda, \mu) (\zeta_3^{-1} X^4 Y^2 + X^2 Z^4 + Y^4 Z^2) \\
&+ g_2(\lambda, \mu) (X^4 Z^2 + \zeta_3 X^2 Y^4 + Y^2 Z^4) = 0,
\end{align*}
\]
where
\[
\begin{align*}
g_1(\lambda, \mu) &:= \sqrt[3]{9} \left( \frac{\zeta_4 \lambda \mu + \zeta_2 \lambda + \zeta_3 \mu^5}{\lambda^6 \mu + \lambda + \mu^5} \right), \\
g_2(\lambda, \mu) &:= \sqrt[3]{9} \left( \frac{\zeta_2 \lambda \mu + \zeta_1 \lambda + \zeta_4 \mu^5}{\lambda^6 \mu + \lambda + \mu^5} \right).
\end{align*}
\]
In this case, \(\text{Aut}(C_{2, \lambda, \mu}) = g_2(A_4)\).

We now introduce the notion of fake ES-irreducible components.

**Definition 2.8.** An ES-irreducible component \(\hat{M}_g^P(\varrho(G))\) is fake if it is not defined by a single normal form.

As a consequence of Theorems 2.4 and 2.5:

**Corollary 2.9.** The strata \(\hat{M}_1^P(\mathbb{Z}/3\mathbb{Z})\) and \(\hat{M}_1^P(\mathbb{Z}/3\mathbb{Z})^2\) are not ES-irreducible and each of them has exactly two ES-irreducible components namely, \(\hat{M}_1^P(\varrho_1(\mathbb{Z}/3\mathbb{Z}))\) and \(\hat{M}_1^P(\varrho_1((\mathbb{Z}/3\mathbb{Z})^2))\) respectively with \(i = 1\) and \(2\).

On the other hand, the components \(\hat{M}_1^P(\varrho_2(\mathbb{Z}/3\mathbb{Z}))\) and \(\hat{M}_1^P(\varrho_2((\mathbb{Z}/3\mathbb{Z})^2))\) are the first examples of fake ES-irreducible components. Any \([C]\) in the family \(C_2\) that belongs to \(\hat{M}_1^P(\varrho_2(\mathbb{Z}/3\mathbb{Z}))\) or \(\hat{M}_1^P(\varrho_2((\mathbb{Z}/3\mathbb{Z})^2))\) has the property that its automorphism group fixes the triangle \(\Delta\) whose vertices \(P_1 = (1 : 0 : 0), P_2 = (0 : 1 : 0)\) and \(P_2 = (0 : 0 : 1)\) lie on \(C\). This does not hold if \([C]\) is in the family \(C_1\), in the sense that it is not necessarily true that \(\text{Aut}(C) = \varrho_2(\mathbb{Z}/3\mathbb{Z})\) or \(\varrho_2((\mathbb{Z}/3\mathbb{Z})^2)\) fixes a triangle whose vertices lie on \(C\). For example, take \([C]\) as in \(C'_i\) with \(1 + \alpha_{i,1} + \alpha_{i,0} 
eq 0\).

**Corollary 2.10.** The strata \(\hat{M}_1^P(S_3)\) and \(\hat{M}_1^P(\mathbb{Z}/3\mathbb{Z} \times S_3)\) are ES-irreducible. More precisely, \(\hat{M}_1^P(S_3) = \hat{M}_1^P(\varrho_2(S_3))\) and \(\hat{M}_1^P(\mathbb{Z}/3\mathbb{Z} \times S_3) = \hat{M}_1^P(\varrho_1(\mathbb{Z}/3\mathbb{Z} \times S_3))\).

**Corollary 2.11.** The stratum \(\hat{M}_1^P(A_4)\) is ES-irreducible determined by \(\hat{M}_1^P(\varrho_1(A_4))\).

It represents the second example of fake ES-irreducible components. Indeed, \(C_{2, \lambda, \mu}\) is \(K\)-isomorphic, via a change of variables \(\phi = \text{diag}(1, s, t)\) such that \(s = t^2\) and \(t^3 = \zeta_6\), to \(\phi C_{2, \lambda, \mu} : X^6 + \zeta_3^{-1} Y^6 + \zeta_6 Z^6 + \text{lower order terms}\), where \(\text{Aut}(\phi C_{2, \lambda, \mu}) = \varrho_1(A_4)\). Moreover, any \([C] \in \hat{M}_1^P(\varrho_1(A_4))\) in the family \(C_{1, \lambda, \mu}\) is a descendant of the Fermat curve \(F_6\) in the sense of Theorem 3.1 via a change of variables in the normalizer of \(\varrho_1(A_4)\) in \(\text{PGL}_3(K)\). This does not hold if \([C]\) is in the family \(\phi C_{2, \lambda, \mu}\).
3. Preliminaries about automorphism groups

Based entirely on geometrical methods, H. Mitchell [15, §1-10] proved that if $G$ is a finite subgroup of $\text{PGL}_3(K)$, then it fixes a point, a line or a triangle unless it is primitive and conjugate to some group in a specific list. However, as a consequence of Maschke’s theorem in group representation theory, the first two cases are equivalent, in the sense that if $G$ fixes a point (respectively a line), then it also fixes a line not passing through the point (respectively a point not lying the line).

**Notation.** For a non-zero monomial $cX^{i_1}Y^{i_2}Z^{i_3}$ with $c \in K^*$, its exponent is defined to be $\max\{i_1, i_2, i_3\}$. For a homogenous polynomial $F(X,Y,Z)$, the core of it is defined to be the sum of all terms of $F$ with the greatest exponent. Now, let $C_0$ be a non-singular plane curve over $K$, a pair $(C,G)$ with $G \leq \text{Aut}(C_0)$ is said to be a descendant of $C_0$ if $C$ is defined by a homogenous polynomial whose core is a defining polynomial of $C_0$ and $G$ acts on $C_0$ under a suitable change of the coordinates system, i.e. $G$ is $\text{PGL}_3(K)$-conjugate to a subgroup of $\text{Aut}(C_0)$.

An element of $\text{PGL}_3(K)$ is called *intransitive* if it has the matrix shape
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & * & * \\
0 & * & *
\end{pmatrix}
\]
The subgroup of $\text{PGL}_3(K)$ of all intransitive elements is denoted by $\text{PBD}(2,1)$. Obviously, there is a natural map $\Lambda : \text{PBD}(2,1) \to \text{PGL}_2(K)$ given by
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & * & * \\
0 & * & *
\end{pmatrix} \in \text{PBD}(2,1) \mapsto \begin{pmatrix} * & * \\ * & * \end{pmatrix} \in \text{PGL}_2(K).
\]

**Theorem 3.1.** Let $C$ be a non-singular plane curve of degree $d \geq 4$ defined over an algebraically closed field $K$ of characteristic 0. Then, one of the following situations holds:
1. $\text{Aut}(C)$ fixes a point on $C$ and then it is cyclic.
2. $\text{Aut}(C)$ fixes a point not lying on $C$ where we can think about $\text{Aut}(C)$ in the following commutative diagram, with exact rows and vertical injective morphisms:

    \[
    \begin{array}{ccc}
    1 & \longrightarrow & K^* \\
    & & \downarrow \Lambda \\
    & \text{PBD}(2,1) & \longrightarrow \text{PGL}_2(K) & \longrightarrow & 1
    \end{array}
    \]

    \[
    \begin{array}{ccc}
    1 & \longrightarrow & N \\
    & & \downarrow \text{Aut}(C) \\
    & \longrightarrow & G' & \longrightarrow & 1
    \end{array}
    \]

Here, $N$ is a cyclic group of order dividing the degree $d$ and $G'$ is a subgroup of $\text{PGL}_2(K)$, which is conjugate to a cyclic group $\mathbb{Z}/m\mathbb{Z}$ of order $m$ with $m \leq d - 1$, a Dihedral group $D_{2m}$ of order $2m$ with $|N| = 1$ or $m|(d - 2)$, one of the alternating groups $A_4$, $A_5$, or the symmetry group $S_4$.

**Remark 3.2.** We note that $N$ is viewed as the part of $\text{Aut}(C)$ acting on the variable $B \in \{X,Y,Z\}$ and fixing the other two variables, while $G'$ is the part acting on $\{X,Y,Z\} - \{B\}$ and fixing $B$. For example, if $B = X$, then every automorphism in $N$ has the shape $\text{diag}(\zeta_n, 1, 1)$ for some $n$th root of unity $\zeta_n$. 
3. Aut($C$) is conjugate to a subgroup $G$ of Aut($\mathcal{F}_d$), where $\mathcal{F}_d$ is the Fermat curve $X^d + Y^d + Z^d = 0$. In particular, $|G|$ divides $|\text{Aut}(\mathcal{F}_d)| = 6d^2$, and $(C, G)$ is a descendant of $\mathcal{F}_d$.

4. Aut($C$) is conjugate to a subgroup $G$ of Aut($\mathcal{K}_d$), where $\mathcal{K}_d$ is the Klein curve curve $X^{d-1}Y + Y^{d-1}Z + XZ^{d-1} = 0$. In this case, $|\text{Aut}(C)|$ divides $|\text{Aut}(\mathcal{K}_d)| = 3(d^2 - 3d + 3)$, and $(C, G)$ is a descendant of $\mathcal{K}_d$.

5. Aut($C$) is conjugate to one of the finite primitive subgroup of $\text{PGL}_3(K)$ namely, the Klein group $\text{PSL}(2,7)$, the icosahedral group $A_5$, the alternating group $A_6$, or to one of the Hessian groups Hess, with $* \in \{36, 72, 216\}$.

Finally, we have:

**Proposition 3.3.** The automorphism groups of the Fermat sextic curve $\mathcal{F}_6$ generated by $[X : Z : Y]$, $[Y : Z : X]$, diag($\zeta_6$, 1, 1) and diag(1, $\zeta_6$, 1) of orders 2, 3, 6 and 6 respectively is isomorphic to GAP(216, 92) = ($\mathbb{Z}/6\mathbb{Z}$)$^2 \rtimes S_3$. On the other hand, the automorphism group of the Klein sextic curve $\mathcal{K}_6$ generated by diag(1, $\zeta_{21}$, $\zeta_{21}^4$) and $[Y : Z : X]$ of orders 21 and 3 respectively is isomorphic to GAP(63, 3) = $\mathbb{Z}/21\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$.

**Proof.** Regarding the generators of Aut($\mathcal{F}_6$) and Aut($\mathcal{K}_6$), we refer the reader to [9, Propositions 3.3, 3.5]. Now, for the Fermat curve $\mathcal{F}_6$, take $a = [X : Z : Y]$, $b = [Y : Z : X]$, $c = \text{diag}(\zeta_6, 1, 1)$ and $d = \text{diag}(1, \zeta_6, 1)$. One verifies that

\[(ab)^2 = (ac)(ca)^{-1} = (cd)(dc)^{-1} = ada(cd)^{-5} = bcb^{-1}(cd)^{-5} = 1.\]

These relations give us the 4th semidirect product of ($\mathbb{Z}/6\mathbb{Z}$)$^2$ and $S_3$ acting faithfully, see semidirect products of ($\mathbb{Z}/6\mathbb{Z}$)$^2$ and $S_3$ for more details.

For the Klein curve $\mathcal{K}_6$, the two generators $a = \text{diag}(1, \zeta_{21}, \zeta_{21}^4)$ and $b = [Y : Z : X]$ of orders 21 and 3 respectively produce GAP(63, 3) = $\mathbb{Z}/21\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ as $ba = (ab)^{-5}$. \hfill \square

4. **Proof of Theorem 2.4**

In this case, $C : F(X, Y, Z) = 0$ has an homology $\sigma$ of period 3 in its automorphism group. The results in [2] allows us to assume that $\sigma$ acts as

\[(X : Y : Z) \mapsto (X : Y : \zeta_3 Z)\]

up to $K$-isomorphism, where $\zeta_3$ is a fixed primitive 3rd root of unity in $K$. In particular, $C$ is defined over $K$ by a non-singular plane equation of the form:

\[C : Z^6 + 2L_{3,Y} + L_{6,Z} = 0,\]

where $\sigma = \text{diag}(1, 1, \zeta_3)$ is an automorphism of maximal order 3. By non-singularity, $L_{6,Z}$ should be of degree at least 5 in both variables $X$ and $Y$. Also, $L_{3,Y} \neq 0$ or diag(1, 1, $\zeta_3$) would be an automorphism of order $6 > 3$.

In the sense of Theorem 3.1, we have the following:

- First, Aut($C$) is not conjugate to any of the finite primitive subgroups of $\text{PGL}_3(K)$ since each of them contains elements of order $> 3$. Also, $C$ is not a descendant of the Klein sextic curve $\mathcal{K}_6$ because Aut($\mathcal{K}_6$) by Proposition 3.3 equals $\mathbb{Z}/21\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ and it does not contains homologies of order 3 similar to $\sigma$.

- Secondly, suppose that $C$ is a descendant of the Fermat curve $\mathcal{F}_6$. So there is a $\phi \in \text{PGL}_3(K)$ such that $\phi^{-1} \text{Aut}(C)\phi \leq \text{Aut}(\mathcal{F}_6)$ and the transformed equation $\phi C$ is $X^6 + Y^6 + Z^6 + \text{lower order terms in } X, Y, Z = 0$. There is no loss of generality to impose $\phi^{-1}(\sigma)\phi = (\sigma)$ since homologies of period
3 inside $\text{Aut}(F_6)$ form two conjugacy classes represented by $\sigma$ and $\sigma^{-1}$. Hence $\phi C$ reduces to

$$\phi C : X^6 + Y^6 + Z^6 + Z^3L_{3,2} + \text{lower order terms in } X, Y = 0$$

Furthermore, by assumption, the automorphisms of $C$ have orders $\leq 3$, then the group structure of $\text{Aut}(F_6) = (\mathbb{Z}/6\mathbb{Z})^2 \times S_3$ assures that $\text{Aut}(\phi C)$ would be one of the following groups inside $\text{Aut}(F_6)$:

$$\mathbb{Z}/3\mathbb{Z}, (\mathbb{Z}/3\mathbb{Z})^2, S_3, A_4, \mathbb{Z}/3\mathbb{Z} \times S_3, \text{He}_3.$$  

For more details, check the subgroups lattice of $\text{Aut}(F_6)$.

Now we tackle each of the above situations.

- Any copy of $S_3$ (respectively $A_4$) inside $\text{Aut}(F_6)$ is $\text{Aut}(F_6)$-conjugate to either $\phi_1(S_3)$ (respectively $\phi_1(A_4)$) with $i = 1$ or 2. But none of these subgroups has homologies of period 3 similar to $\sigma$. So $\text{Aut}(\phi C)$ can not be an $S_3$ or $A_4$ inside $\text{Aut}(F_6)$.

- If $\text{Aut}(\phi C)$ equals a $(\mathbb{Z}/3\mathbb{Z})^2, \mathbb{Z}/3\mathbb{Z} \rtimes S_3$ or $\text{He}_3$ in $\text{Aut}(F_6)$, then there must be $\sigma' \in \text{Aut}(F_6) \cap \text{Aut}(\phi C)$ of order 3 that commutes with $\sigma$ as in any of these groups $\mathbb{Z}/3\mathbb{Z}$ is always contained in a $(\mathbb{Z}/3\mathbb{Z})^2$. By Proposition 3.3, the elements of order 3 in $\text{Aut}(F_6)$ are $\text{diag}(1, s, t)$ with $s^3 = t^3 = 1$, $[sY : tZ : X]$ and $[tZ : X : sY]$ with $s^6 = t^6 = 1$. One easily verifies that only the diagonal shapes satisfies the description, equivalently, $\sigma' \in \langle \sigma, \text{diag}(1, \zeta_3, 1) \rangle$. In any case, we can reduce $C$ up to $K$-isomorphism to

$$\phi C : X^6 + Y^6 + Z^6 + Z^3(\alpha_{3,0}X^3 + \alpha_{0,3}Y^3) + \alpha_{3,3}X^3Y^3 = 0,$$

where $\phi_1((\mathbb{Z}/3\mathbb{Z})^2) \leq \text{Aut}(\phi C)$.

**Remark 4.1.** In this scenario, the parameters $\alpha_{3,0}, \alpha_{0,3}, \alpha_{3,3}$ must be pairwise distinct modulo $\{ \pm 1 \}$ or $\phi C$ will admit automorphisms of order $> 3$.

For example, $[\zeta_3Y : X : Z] \in \text{Aut}(\phi C)$ has order 6 if $\alpha_{3,0} = \alpha_{0,3}$ and $[\zeta_3Y : X : -Z] \in \text{Aut}(\phi C)$ has order 6 if $\alpha_{3,0} = -\alpha_{0,3}$.

A similar discussion shows that any $\sigma'' \in \text{Aut}(F_6)$ that commutes with $\sigma$ or $\sigma'$ belongs to $\langle \sigma, \sigma' \rangle$. Therefore, $\text{Aut}(\phi C)$ can not be the Heisenberg group $\text{He}_3$ because this requires another automorphism $\sigma'' \notin \langle \sigma, \sigma' \rangle$ that commutes with either $\sigma$ or $\sigma'$.

Finally, for $\text{Aut}(\phi C)$ to be $\mathbb{Z}/3\mathbb{Z} \rtimes S_3$, it is necessary that $\text{Aut}(F_6) \cap \text{Aut}(\phi C)$ has involutions in it. Proposition 3.3 tells us that the involutions of $F_6$ are $\text{diag}(-1, 1, 1), \text{diag}(1, -1, 1), \text{diag}(1, 1, -1), [X : sZ : s^{-1}Y], [s^{-1}Y : sX : Z]$ and $[sZ : Y : s^{-1}X]$ with $s^6 = 1$. If any of these involutions lies in $\text{Aut}(\phi C)$, then two of the parameters are equal modulo $\{ \pm 1 \}$, which is absurd by Remark 4.1. For example, $\text{diag}(-1, 1, 1) \in \text{Aut}(\phi C)$ only if $\alpha_{3,0} = \alpha_{3,3} = 0$, $[sY : s^{-1}X : Z] \in \text{Aut}(\phi C)$ only if $\alpha_{3,0} = \pm \alpha_{0,3}$, and so on.

- Third, if $\text{Aut}(C)$ fixes a line $L$ and a point $P$ not lying on $L$, then by Theorem 3.1 we can think about $\text{Aut}(C)$ in a short exact sequence

$$1 \to N = \langle \sigma \rangle \to \text{Aut}(C) \to \Lambda(\text{Aut}(C)) \to 1,$$

where $\Lambda(\text{Aut}(C)) \simeq \mathbb{Z}/3\mathbb{Z}, D_4$ or $A_4$.

- Any group of order 36 (respectively 12) that has a normal subgroup isomorphic to $\mathbb{Z}/3\mathbb{Z}$ contains elements of order $6 > 3$, see Groups of order 12 and Groups of order 36 for more details. This allows us to exclude that $\Lambda(\text{Aut}(C))$ equals $A_4$ or $D_4$.

- On the other hand, if $\Lambda(\text{Aut}(C))$ equals $\mathbb{Z}/3\mathbb{Z}$ in $\text{PGL}_2(K)$, then $\text{Aut}(C)$ equals $(\mathbb{Z}/3\mathbb{Z})^2$ in $\text{PBD}(2, 1)$. In particular, $C : \mathbb{Z}^6 + \mathbb{Z}^3L_{3,2} + \text{lower order terms in } X, Y, Z = 0$.
Let $C$ be a smooth plane sextic curve such that $L_{6,Z} = 0$. If $C$ admits an automorphism $\sigma' \in \text{PBD}(2,1) - \langle \sigma \rangle$ of order 3 that commutes with $\sigma$, then $C$ is conjugate to $C'$ up to $K$-isomorphism. Depending on whether $\sigma'$ is an homology or a non-homology, it is conjugate via a change of variables $\phi \in \text{PBD}(2,1)$, the normalizer of $\langle \sigma \rangle$, to diag$(1, \zeta_3, 1)$ or diag$(1, \zeta_3, \zeta_3^{-1})$ respectively. In either way, $\text{Aut}(\sigma C) = \varphi_1 \left( (\mathbb{Z}/3\mathbb{Z})^2 \right)$, which appeared earlier.

Summing up, we deduce that $\text{Aut}(C)$ is always cyclic of order 3 generated by $\sigma$ except when $C$ is projectively equivalent to $C'$ of the form

$$C' : X^6 + Y^6 + Z^6 + Z^3(\alpha_{3,0}X^3 + \alpha_{0,3}Y^3) + \alpha_{3,3}X^3Y^3 = 0,$$

such that $\alpha_{3,0}, \alpha_{0,3}, \alpha_{3,3}$ are pair-wise distinct modulo $\{ \pm 1 \}$. In this case, $\text{Aut}(C)$ is conjugate to $(\mathbb{Z}/3\mathbb{Z})^2$ generated by diag$(1, \zeta_3, 1)$ and diag$(1, \zeta_3, 1)$.

This proves Theorem 2.4.

### 5. Proof of Theorem 2.1

In this case, $C : F(X, Y, Z) = 0$ has an homology $\sigma$ of period 2 in its automorphism group. By [2], there is no loss of generality to assume that $\sigma$ acts as

$$(X : Y : Z) \mapsto (X : Y : -Z)$$

up to $K$-isomorphism. In particular, $C$ is defined over $K$ by a non-singular plane equation of the form:

$$C : Z^6 + Z^4L_{2,Z} + Z^2L_{4,Z} + L_{6,Z} = 0$$

where $\sigma = \text{diag}(1, 1, -1)$ is an automorphism of maximal order 2. Again $L_{6,Z}$ is of degree $\geq 5$ in $X$ and $Y$ by non-singularity. Also, $L_{2,Z}$ or $L_{4,Z}$ does not vanish or diag$(1, 1, \zeta_6)$ will be an automorphism of order $6 > 3$ otherwise.

- Obviously, $\text{Aut}(C)$ is not conjugate to any of the finite primitive subgroups of $\text{PGL}_3(K)$ as each of them contains elements of order $> 2$. Also, $C$ can not be a descendant of the Klein sextic curve $\mathcal{K}_6$ since $2 | | \text{Aut}(\mathcal{K}_6)|$, recall that $| \text{Aut}(\mathcal{K}_6)| = 63$ by Proposition 3.3.

- Secondly, if $\text{Aut}(C)$ fixes a line $\mathcal{L}$ and a point $P$ off $\mathcal{L}$, then, by Theorem 3.1, $\text{Aut}(C)$ is inside $\text{PBD}(2,1)$ and satisfies a short exact sequence

$$1 \rightarrow N = \langle \sigma \rangle \rightarrow \text{Aut}(C) \rightarrow \Lambda(\text{Aut}(C)) \rightarrow 1.$$

Our assumptions that any automorphism of $C$ has order $\leq 2$ implies that $\Lambda(\text{Aut}(C))$ is either $\mathbb{Z}/2\mathbb{Z}$ or $D_4$ inside $\text{PGL}_2(K)$, so $\text{Aut}(C)$ is conjugate to either $(\mathbb{Z}/2\mathbb{Z})^2$ or $(\mathbb{Z}/2\mathbb{Z})^3$. In both situations $\text{Aut}(C)$ has another involution $\sigma'$ that commutes with $\sigma$. Up to projective equivalence via a change of variables $\phi \in \text{PBD}(2,1)$, the normalizer of $\langle \sigma \rangle$ in $\text{PGL}_3(K)$, we can assume that $\sigma' = \text{diag}(1, -1, 1)$.

Consequently, $C$ is $K$-isomorphic to $C' : Z^6 + Z^4L_{2,Z} + Z^2L_{4,Z} + L_{6,Z} = 0$ for some $L_{i,Z} \in K[X^2, Y^2]$. Moreover, $\text{Aut}(C)$ equals $(\mathbb{Z}/2\mathbb{Z})^3$ only if there is an involution $\sigma'' \notin \text{PBD}(2,1) - \langle \sigma, \sigma' \rangle$ that commutes with both $\sigma$ and $\sigma'$. It is straightforward to check that such $\sigma''$ does not exist, hence $\text{Aut}(C)$ is not $(\mathbb{Z}/2\mathbb{Z})^3$ in this case.

- If $C$ is a descendant of the Fermat curve $\mathcal{F}_6$ via a change of variables $\phi \in \text{PGL}_3(K)$ with bigger automorphism group than $\langle \sigma \rangle$, then $\text{Aut}(\sigma C)$ is a copy of $(\mathbb{Z}/2\mathbb{Z})^2$ inside $\text{Aut}(\mathcal{F}_6)$. Indeed any other subgroup of $\text{Aut}(\mathcal{F}_6)$ has elements of order $> 2$, see subgroups lattice of $\text{Aut}(\mathcal{F}_6)$.

Up to $\text{Aut}(\mathcal{F}_6)$-conjugation, there are two copies of $(\mathbb{Z}/2\mathbb{Z})^2$ inside $\text{Aut}(\mathcal{F}_6)$ namely, $\langle \sigma, \sigma' \rangle$ and $\langle \sigma, \tau \rangle$ with $\sigma' = \text{diag}(1, -1, 1)$ and $\tau = [X : Y : Z]$. However, both groups are $\text{PGL}_3(K)$-conjugated via a transformation in $\text{PBD}(2,1)$, the normalizer of $\langle \sigma \rangle$ in $\text{PGL}_3(K)$. Thus there is no loss
of generality to assume that $\text{Aut}(C)$ is conjugate to $\varrho_1((\mathbb{Z}/2\mathbb{Z})^2)$, which was treated earlier.

Summing up, we deduce that $\text{Aut}(C)$ is always cyclic of order 2 generated by $\sigma$ except when $L_i, Z \in K[X^2, Y^2]$ for $i = 2, 4, 6$. In the latter case, $\text{Aut}(C)$ equals $\varrho_1((\mathbb{Z}/2\mathbb{Z})^2)$, which shows Theorem 2.1.

6. Proof of Theorem 2.5

In this case, $C : F(X, Y, Z) = 0$ has a non-homology $\sigma$ of period 3 in its automorphism group. By [2], one can assume that $\sigma$ acts as

$$(X : Y : Z) \mapsto (X : \zeta_3 Y : \zeta_3^{-1} Z)$$

up to $K$-isomorphism, where $\zeta_3$ is a fixed primitive 3rd root of unity in $K$. In particular, $C$ is a $K$-isomorphic to a non-singular plane model in one of the following families:

$$C_1 : X^6 + Y^6 + Z^6 + XYZ \left(\alpha_{4,1} X^3 + \alpha_{1,4} Y^3 + \alpha_{1,1} Z^3\right) + \alpha_{2,2} X^2 Y^2 Z^2 + \alpha_{3,3} X^3 Y^3 + \alpha_{3,0} X^3 Z^3 + \alpha_{0,3} Y^3 Z^3 = 0$$

$$C_2 : X^5 Y + Y^5 Z + X Z^5 + XYZ \left(\alpha_{3,2} X^2 Y + \alpha_{1,3} Y^2 Z + \alpha_{2,1} X Z^2\right) + \alpha_{2,4} X^2 Y^4 + \alpha_{0,2} Y^2 Z^4 + \alpha_{4,0} X^4 Z^2 = 0.$$ 

where $\sigma := \text{diag}(1, \zeta_3, \zeta_3^{-1})$ is an automorphism of maximal order 3.

- Again $\text{Aut}(C_i)$ for $i = 1$ and 2 is not conjugate to any of the finite primitive subgroups of $\text{PGL}_3(K)$.
- Suppose that $\text{Aut}(C_i)$ fixes a line $L$ and a point $P$ not lying on this line. Since $\sigma$ is a non-homology inside $\text{Aut}(C_i)$ in its canonical form, $L$ must be one of the reference lines; $B = 0$ with $B = X, Y$ or $Z$ and $P$ is the reference point $(1 : 0 : 0), (0 : 1 : 0)$ or $(0 : 0 : 1)$ respectively.
- For $C_2$, the point $P$ belongs to $C : F(X, Y, Z) = 0$. Hence $\text{Aut}(C_2)$ is cyclic, generated by $\langle \sigma \rangle$.
- For $C_1$, we can further impose $L : X = 0$ and $P = (1 : 0 : 0)$ (in the worst case scenario, one just needs to permute two of the variables and to fix the third one, which preserves the property that $\sigma$ remains an automorphism). In particular, by Theorem 3.1, $\text{Aut}(C_1) \subseteq \text{PBD}(2, 1)$ and lives in a short exact sequence: $1 \rightarrow N \rightarrow \text{Aut}(C_1) \rightarrow \Lambda(\text{Aut}(C_1)) \rightarrow 1$, where $N = \langle \tau \rangle$ has order 1, 2 or 3 and $\Lambda(\text{Aut}(C))$ is either $\mathbb{Z}/3\mathbb{Z}$, $S_3$ with $|N| = 1$ or $A_4$ in $\text{PGL}_2(K)$. First, we easily exclude the case when $\tau$ has order 2 because $\sigma \tau$ would be an automorphism of order $6 > 3$, a contradiction.

Secondly, we handle each of the remaining cases:

(i) If $\Lambda(\text{Aut}(C_1)) = \mathbb{Z}/3\mathbb{Z}$ and $N = 1$, then $\text{Aut}(C_1) = \mathbb{Z}/3\mathbb{Z}$ generated by $\sigma$.

(ii) If $\Lambda(\text{Aut}(C_1)) = \mathbb{Z}/3\mathbb{Z}$ and $N = \mathbb{Z}/3\mathbb{Z}$, then $\text{Aut}(C_1) = \varrho_1((\mathbb{Z}/3\mathbb{Z})^2)$ generated by $\sigma$ and $\tau = \text{diag}(\zeta_3, 1, 1)$. In particular, $\alpha_{4,1} = \alpha_{2,2} = \alpha_{1,1} = \alpha_{1,4} = 0$, and $C_1$ reduces to

$$X^6 + Y^6 + Z^6 + Z^3 \left(\alpha_{3,0} X^3 + \alpha_{0,3} Y^3\right) + \alpha_{3,3} X^3 Y^3 = 0,$$

which happened before in Theorem 2.4. We also remark that $\alpha_{3,0} \neq \alpha_{0,3}$ or $[Y : X : Z]$ will be an extra involution for $C_1$.

This clarifies part of Theorem 2.5, (1)-(i).

(iii) If $\Lambda(\text{Aut}(C_1)) = S_3$ and $N = 1$, then $C$ should have an involution $\tau$ such that $\tau \sigma \tau = \sigma^{-1}$. So $\tau = [X : s Z : s^{-1} Y], [s Y : s^{-1} X : Z]$ or $[s Z : Y : s^{-1} X]$ with $s^6 = 1$. This holds if we are in one of the situations: $\alpha_{3,3} = \pm \alpha_{3,0}$ and $\alpha_{1,1} = \pm \alpha_{1,4}, \alpha_{0,3} = \pm \alpha_{3,0}$ and
\[ C_1' : X^6 + Y^6 + \alpha_{4,1} X^4 Y Z + \alpha_{3,3} X^3 (Y^3 + Z^3) + \alpha_{2,2} X^2 Y^2 Z^2 + \alpha_{1,2} X Y Z (Y^3 + Z^3) + \alpha_{0,3} Y^3 Z^3 = 0. \]

Here \( \text{Aut}(C_1') = \langle \sigma, \tau \rangle \). In particular, we should impose \( \alpha'_{4,1} \neq \alpha_{1,2} \) or \( \alpha'_{3,3} \neq \alpha_{0,3} \) to avoid having \( \langle Y : Z : X \rangle \) as an extra automorphism. Also, \( (\alpha'_{3,3}, \alpha_{1,2}) \neq (0, 0) \) to avoid having \( \langle 1, \zeta_6, \zeta_6^{-1} \rangle \) as an extra automorphism of order 6 > 3.

This shows part of Theorem 2.5, (1)-(ii).

(iv) If \( \Lambda(\text{Aut}(C)) = A_4 \), then the Group Structure of \( A_4 \) assures that \( \Lambda(\text{Aut}(C)) \) contains \( \Lambda(\tau) \) and \( \Lambda(\tau') \) both of order 2 such that

\[ \Lambda(\sigma) \Lambda(\tau) \Lambda(\sigma)^{-1} = \Lambda(\tau') \]  
\[ \Lambda(\sigma) \Lambda(\tau') \Lambda(\sigma)^{-1} = \Lambda(\tau) \Lambda(\tau') \]

We aim to show that such \( \tau \) and \( \tau' \) do not exist. Write \( \Lambda(\tau) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), then being of order 2 yields \( (a + d)b = (a + d)c = 0 \) and \( a = \pm d \). So \( \Lambda(\tau) = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \) or \( \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \).

- If \( \Lambda(\tau) = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \), then

\[ \Lambda(\tau') = \Lambda(\sigma) \Lambda(\tau) \Lambda(\sigma)^{-1} = \begin{pmatrix} 0 & \zeta_3^{-1}b \\ \zeta_3 c & 0 \end{pmatrix} \]

which implies that \( \Lambda(\tau') \Lambda(\tau) \neq \Lambda(\sigma) \Lambda(\tau') \) a contradiction.

- If \( \Lambda(\tau) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \), then \( \Lambda(\tau') = \Lambda(\sigma) \Lambda(\tau) \Lambda(\sigma)^{-1} = \begin{pmatrix} a & \zeta_3^{-1}b \\ \zeta_3 c & -a \end{pmatrix} \)

such that \( \Lambda(\tau') \Lambda(\tau) = \Lambda(\tau') \Lambda(\tau) \). That is,

\[ \begin{pmatrix} a^2 + \zeta_3 bc & (\zeta_3^{-1} - 1)ab \\ (1 - \zeta_3) ac & a^2 + \zeta_3^{-1} bc \end{pmatrix} = \begin{pmatrix} a^2 + \zeta_3^{-1} bc & -(\zeta_3^{-1} - 1)ab \\ -(1 - \zeta_3) ac & a^2 + \zeta_3 bc \end{pmatrix} \]

in \( \text{PGL}_2(K) \).

For this to be true, either \( ab = ac = 0 \) or \( a^2 + \zeta_3 bc = -(a^2 + \zeta_3^{-1} bc) \). Assuming \( ab = ac = 0 \) yields \( \Lambda(\tau') = \begin{pmatrix} 0 & \zeta_3^{-1}b \\ \zeta_3 c & 0 \end{pmatrix} \).

\[ \Lambda(\tau) \] in \( \text{PGL}_2(K) \) or \( \Lambda(\tau') = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} = \Lambda(\tau) \) in \( \text{PGL}_2(K) \), which is again a contradiction. Assuming \( a^2 + \zeta_3 bc = -(a^2 + \zeta_3^{-1} bc) \) yields \( c = 2a^2/b \) with \( ab \neq 0 \). Moreover, \( \Lambda(\sigma) \Lambda(\tau') \Lambda(\sigma)^{-1} = \Lambda(\tau) \Lambda(\tau') \), hence

\[ \begin{pmatrix} a & \zeta_3 b \\ 2a^2/b & -a \end{pmatrix} \]

in \( \text{PGL}_2(K) \).

This is valid only if \( (\zeta_3 - \zeta_3^{-1}) \zeta_3 = (\zeta_3^{-1} - 1) \) and \( (\zeta_3 - \zeta_3^{-1}) = (1 - \zeta_3) \), however, the second equation is never valid. This means that \( \Lambda(\text{Aut}(C)) \neq A_4 \).

Finally, assume that \( C_1 \) is a descendant of the Klein sextic curve \( K_6 \).

**Claim 1.** For \( C_1 \), a descendant of \( K_6 \), \( \text{Aut}(C_1) = \varphi_2(\mathbb{Z}/3\mathbb{Z}) \).

**Proof.** (of Claim 1) If \( C_1 \) is a descendant of \( K_6 \) with bigger automorphism group than \( \langle \sigma \rangle \), then, from the Group Structure of \( \mathbb{Z}/21\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \) and since the automorphisms of \( C \) have orders \( \leq 3 \), \( \text{Aut}(C_1) \) should be
conjugate to a \((\mathbb{Z}/3\mathbb{Z})^2\) in \(\text{Aut}(K_6)\). Thus \(C_1\) has another automorphism \(\sigma' \notin \langle \sigma \rangle\) of order 3 that commutes with \(\sigma\). Direct calculations show that we can take \(\sigma' = \text{diag}(1, s, t)\) with \(s^3 = t^3 = 1\) or \([sY : tZ : X]\) with \(s, t \in K^*\).

In the first case, \(\sigma'\) reduces to an homology as \(\sigma' \notin \langle \sigma \rangle\). This is absurd because \(\text{Aut}(K_6)\) does not contain any homologies of period 3. Regarding the second case, any descendant \(C'\) of the Klein curve \(C' : X^6Y + Y^6Z + Z^6X + \text{lower terms in } X, Y, Z\) satisfies the property that its automorphism group fixes the triangle \(\Delta\) whose vertices are the three reference points \((1 : 0 : 0), (0 : 1 : 0)\) and \((0 : 0 : 1)\), moreover, those points all lie on \(C'\). Because \(\Delta\) is the only triangle fixed by \(\langle \sigma, [sY : tZ : X] \rangle\) for any \(s, t\) and because none of its vertices lies on \(C_1\), we conclude that \(\text{Aut}(C_1)\) can not equal \(\langle \sigma, [sY : tZ : X] \rangle\). This proves the claim for \(C_1\).

**Claim 2.** For \(C_2\) a descendant of \(K_6\), \(\text{Aut}(C_2)\) is either conjugate to \(\varphi_2(\mathbb{Z}/3\mathbb{Z})^2\) or \(\varphi_2((\mathbb{Z}/3\mathbb{Z})^2)\).

**Proof.** (of Claim 2) If \(C_2\) is a descendant of \(K_6\) with bigger automorphism group than \(\langle \sigma \rangle\), then \(\text{Aut}(C_2) = \langle \sigma, [sY : tZ : X] \rangle\) for some \(s, t \in K^*\).

For \(\sigma' \in \text{Aut}(C_2), s = \zeta_{21}^{17}, t = \zeta_{21}^{24r} \alpha_0, \alpha_2 = \zeta_{21}^{24r} \alpha_4, \alpha_3 = \zeta_{21}^{24r} \alpha_3,\) \(C_2\) reduces to
\[
X^6Y + Y^6Z + Z^6X + \text{lower terms in } X, Y, Z = 0.
\]

In any situation, there exists a change of variables \(\phi = \text{diag}(1, \zeta_{21}', \zeta_{21}^{17r}') \in \text{Aut}(K_6)\) such that \(21 | 18r' + r, 12r' - 4r\) for some \(r' \in \{0, 1, \ldots, 20\}\) that transforms \(C_2\) up to \(K\)-isomorphism to
\[
C'_2 : X^6Y + Y^6Z + Z^6X + \zeta_{21}^{24r}X^2Y^2 + \zeta_{21}^{24r}Z^2 = 0,
\]

where \(\text{Aut}(C'_2) = \varphi_2((\mathbb{Z}/3\mathbb{Z})^2) = \langle \sigma, [Y : Z : X] \rangle\). In particular, we must have \((\alpha_2, \alpha_3) \neq (0, 0)\) or \(\text{diag}(1, \zeta_{21}, \zeta_{21}^{17}) \in \text{Aut}(C'_2)\) of order 21 > 3.

This completes the proof, which in turns shows Theorem 2.5, (2)-(i). □

- Now, assume that \(C_1\) is a descendant of \(F_6\). From the Group structure of Aut(\(F_0\)), one sees that if \(C_1\) is a descendant of \(F_6\) with bigger automorphism group than \(\langle \sigma \rangle\), then \(\text{Aut}(C_1)\) is conjugate to one of the following groups inside \(\text{Aut}(F_0)\):

\[
(\mathbb{Z}/3\mathbb{Z})^2, S_3, A_4, \mathbb{Z}/3\mathbb{Z} \rtimes S_3, \text{He}_3.
\]

In what follows, we treat each of these cases for \(C_1\) and \(C_2\) respectively, more precisely, Claim 3 and Claim 4 below.

**Claim 3.** For \(C_1\) a descendant of \(F_6\), \(\text{Aut}(C_1)\) is conjugate to \(\varphi_2(\mathbb{Z}/3\mathbb{Z}), \varphi_2(S_3), \varphi_1(\mathbb{Z}/3\mathbb{Z} \rtimes S_3), \varphi_1((\mathbb{Z}/3\mathbb{Z})^2), \varphi_2((\mathbb{Z}/3\mathbb{Z})^2)\) or \(\varphi_1(A_4)\).

**Claim 4.** For \(C_2\) a descendant of \(F_6\), \(\text{Aut}(C_2)\) is conjugate to \(\varphi_2(\mathbb{Z}/3\mathbb{Z}), \varphi_2((\mathbb{Z}/3\mathbb{Z})^2)\) or \(\varphi_2(A_4)\).

**Proof.** (of Claim 3) - If \(\text{Aut}(C_1)\) is conjugate to \(S_3\) or \(\mathbb{Z}/3\mathbb{Z} \rtimes S_3\) inside \(\text{Aut}(F_0)\), then \(C_1\) has an involution \(\tau\) such that \(\tau \sigma = \sigma^{-1}\). Similarly as before, this holds only if \(\alpha_3 = \pm \alpha_3, \alpha_4 = \pm \alpha_4, \alpha_0 = \pm \alpha_0, \alpha_1 = \pm \alpha_1\). In this scenario, \(C_1\) is \(K\)-isomorphic to
\[
C'_1 : X^6 + Y^6 + Z^6 + \alpha'_4X^4YZ + \alpha'_3X^3(Y^3 + Z^3) + \alpha'_2X^2Y^2Z^2 + \alpha'_1XZ^3 + \alpha'_0Y^3Z = 0,
\]

\[sY : tZ : X] \]
where \( g_2(S_3) \) generated by \( \sigma = \text{diag}(1, \zeta_3, \zeta_3^{-1}) \) and \( \tau = [X : Z : Y] \) is a subgroup of \( \text{Aut}(C'_1) \). Furthermore, if \( \text{Aut}(C'_1) \) equals \( \mathbb{Z}/3\mathbb{Z} \rtimes S_3 \), then it must contain another automorphism \( \sigma' \notin \langle \sigma, \tau \rangle \) of order 3 that commutes with \( \sigma \) and satisfies \( \sigma' \cdot \tau = \sigma'^{-1} \cdot \tau \).

Thus \( \sigma' = [s'Y : s'Z : X] \) and the invariance of the defining equation for \( C_1 \) under the action of \( \sigma' \) yields \( s'^3 = 1, \alpha'_{1,1} = \alpha'_{1,2} \) and \( \alpha'_{6,3} = \alpha'_{6,3} \). Hence \( C_1' \) becomes

\[
\begin{align*}
X^6 &+ Y^6 + Z^6 + \alpha'_{1,2} XYZ(X^3 + Y^3 + Z^3) + \alpha'_{3,3}(X^3Y^3 + Y^3Z^3 + Z^3X^3) \\
&+ \alpha'_{2,2} X^2Y^2Z^2 = 0
\end{align*}
\]

with \( \text{Aut}(C'_1) = g_1(\mathbb{Z}/3\mathbb{Z} \rtimes S_3) \). This shows the rest of Theorem 2.5, (1)-(ii).

- If \( \text{Aut}(C_1) \) is conjugate to \( (\mathbb{Z}/3\mathbb{Z})^2 \) or \( \text{He}_3 \) inside \( \text{Aut}(F_0) \), then \( C_1 \) would have an automorphism \( \sigma'' \notin \langle \sigma, \sigma' \rangle \) of order 3 that commutes with \( \sigma \) and satisfies \( \sigma'' \cdot \sigma' \cdot \sigma''^{-1} = \sigma' \cdot \sigma^{-1} \). This gives us \( \sigma'' = [s''Y : t''Z : X] \) for some \( s'', t'' \in K^* \). Hence \( s''^6 = t''^6 = 1, \alpha_{3,3} = s''^3 \alpha_{3,3}, \alpha_{0,3} = t''^3 \alpha_{0,3} \), and \( C_1 \) becomes of the form:

\[
\begin{align*}
X^6 &+ Y^6 + Z^6 + \alpha_{3,3} X^3Y^3 + \alpha_{3,0} X^3Z^3 + \alpha_{0,3} Y^3Z^3 = 0,
\end{align*}
\]

In particular, \( [Y : X : t''Z] \) is an automorphism for \( C_1 \) of order divisible by 2. This is a contradiction as \( 2 \mid |\text{He}_3| (= 27) \).

- Suppose that \( \sigma'' = [s''Y : t''Z : X] \in \text{Aut}(C_1) \). For this to be true, we should have \( s''^6 = t''^6 = 1, \alpha_{4,1} = s''t'' \alpha_{1,1}, \alpha_{1,4} = s''t''^2 \alpha_{1,1}, \alpha_{3,3} = s''^3 \alpha_{3,3}, \alpha_{0,3} = t''^3 \alpha_{0,3} \), and \( C_1 \) is defined by

\[
\begin{align*}
X^6 &+ Y^6 + Z^6 + \alpha_{1,1} XYZ(s''t''X^3 \pm \frac{1}{s''t''} Y^3 + Z^3) + \alpha_{2,2} X^2Y^2Z^2 \\
&+ \alpha_{3,0}(s''^3 X^3Y^3 + X^3Z^3 \pm Y^3Z^3) = 0,
\end{align*}
\]

such that \( (s''t'')^2 = 1 \). Consequently, it must be the case that \( \alpha_{2,2} = 0 \) and \( \alpha_{1,1} \neq 0 \) or \( [t'Y : t''^{-1}X : Z] \) would be an extra involution, which violates the fact that \( |\text{Aut}(C_1)| = 9 \) or 27. That is, \( s''t'' = \zeta_6^\ell \) for some \( \ell \neq 0 \) or 3 mod 6, and \( C_1 \) becomes

\[
\begin{align*}
X^6 &+ Y^6 + Z^6 + \alpha_{1,1} XYZ(\zeta_6^\ell X^3 \pm \zeta_6^{-\ell} Y^3 + Z^3) \\
&+ \alpha_{3,0}(\pm(-1)^\ell X^3Y^3 + X^3Z^3 \pm Y^3Z^3) = 0,
\end{align*}
\]

for some \( \alpha_{1,1}, \alpha_{3,0} \in K^* \). Applying the projective change of variables \( \phi = \text{diag}(1, \frac{\sqrt{27}}{s''t''}, \frac{1}{s''t''}) \) we get

\[
\begin{align*}
C'_1 : \quad X^6 + \zeta_6^\ell Y^6 + \zeta_6^{-\ell} Z^6 &+ \alpha'_{1,1} XYZ(X^3 + \zeta_6^\ell Y^3 + \zeta_6^{-\ell} Z^3) \\
&+ \alpha'_{3,0}(X^3Y^3 + \zeta_6^\ell X^3Z^3 + \zeta_6^{-\ell} Y^3Z^3) = 0.
\end{align*}
\]

Now with \( \sigma \) and \( \sigma' = [Y : Z : X] \) as automorphisms for \( C'_1 \), we have that \( \langle \sigma, \sigma' \rangle = g_1(\mathbb{Z}/3\mathbb{Z}) \subseteq \text{Aut}(C'_1) \). Again it is impossible that we can enlarge \( \text{Aut}(C'_1) \) to \( \text{He}_3 \), since this requires \( \text{diag}(1, \zeta_3, \zeta_3^{-1}) \) to be in \( \text{Aut}(C'_1) \).

This cannot be as \( \alpha'_{1,1} = \frac{3\sqrt{27}}{s''t''} \neq 0 \).
- If $\text{Aut}(C_1)$ is conjugate to an $A_4$ inside $\text{Aut}(F_6)$, then it should be $\varphi_1(A_4)$ with $i = 1$ or 2.

(i) First, suppose that $\varphi^{-1} \text{Aut}(C_1) \varphi = \varphi_1(A_4)$. As all subgroups of $A_4$ of order 3 are $A_4$-conjugated, there is no loss of generality to take $\varphi^{-1} \sigma \varphi = [Y : Z : X]$ or $[Z : X : Y]$. In particular, $\varphi$ has one of the following shapes:

$$
\phi_1 := \begin{pmatrix} 1 & 1 & 1 \\ \lambda & \zeta_3^{-1} \lambda & \zeta_3 \lambda \\ \mu & \zeta_3 \mu & \zeta_3^{-1} \mu \end{pmatrix}, \quad \phi_2 := \begin{pmatrix} \mu & \zeta_3 \mu & \zeta_3^{-1} \mu \\ 1 & 1 & 1 \\ \lambda & \zeta_3^{-1} \lambda & \zeta_3 \lambda \end{pmatrix}, \quad \phi_3 := \begin{pmatrix} \lambda & \zeta_3^{-1} \lambda & \zeta_3 \lambda \\ \mu & \zeta_3 \mu & \zeta_3^{-1} \mu \\ 1 & 1 & 1 \end{pmatrix},
$$

$$
\phi_4 := \begin{pmatrix} 1 & 1 & 1 \\ \lambda & \zeta_3 \lambda & \zeta_3^{-1} \lambda \\ \mu & \zeta_3^{-1} \mu & \zeta_3 \mu \end{pmatrix}, \quad \phi_5 := \begin{pmatrix} \mu & \zeta_3^{-1} \mu & \zeta_3 \mu \\ 1 & 1 & 1 \\ \lambda & \zeta_3 \lambda & \zeta_3^{-1} \lambda \end{pmatrix}, \quad \phi_6 := \begin{pmatrix} \lambda & \zeta_3 \lambda & \zeta_3^{-1} \lambda \\ \mu & \zeta_3^{-1} \mu & \zeta_3 \mu \\ 1 & 1 & 1 \end{pmatrix},
$$

for some $\lambda, \mu \in K^*$.

Now, we handle each of these situations to determine the restrictions on the defining equation of $C_1$ for which this holds.

- For $\varphi_1 \text{diag}(1, 1, -1)\varphi^{-1}_1$ (respectively $\varphi_3 \text{diag}(1, 1, -1)\varphi^{-1}_1$) to be in $\text{Aut}(C_1)$, we must eliminate the coefficients of $X^5 Z, X^3 Y, Y^5 Z, X Z^5, Y^2 Z, X^4 Y^2, X^4 Z^2$ from the transformed equation $\varphi_1 \text{diag}(1, 1, -1)\varphi^{-1}_1 C_1 = C_1$ with $i = 1$ and 4 respectively. In this way, we obtain:

$$
\alpha_{4,1} = \frac{2 \left( 29 - 54 \lambda^6 - 54 \mu^6 \right)}{27 \lambda \mu}, \quad \alpha_{3,3} = \frac{2 \left( 81 \lambda^6 - 27 \lambda^6 - 26 \right)}{27 \lambda^3},
$$

$$
\alpha_{3,0} = \frac{2 \left( 81 \lambda^6 - 27 \mu^6 - 26 \right)}{27 \mu^4}, \quad \alpha_{1,4} = \frac{2 \left( 27 \lambda^6 - 54 \mu^6 - 52 \right)}{27 \lambda^4 \mu},
$$

$$
\alpha_{1,1} = \frac{2 \left( 27 \mu^6 - 54 \lambda^6 - 52 \right)}{27 \lambda^3 \mu^4}, \quad \alpha_{0,3} = \frac{2 \left( 82 - 27 \lambda^6 - 27 \mu^6 \right)}{27 \lambda^4 \mu^3},
$$

$$
\alpha_{2,2} = \frac{9 \lambda^6 + 9 \mu^6 + 10}{3 \lambda^2 \mu^2}.
$$

In particular, $C_1$ is $K$-isomorphic via $\varphi_1$ (respectively $\varphi_4$ followed by $Y' \leftrightarrow Z$) to $C_{1, \lambda, \mu}$ described in Theorem 2.5, (1)-(iii).

- For $\varphi_2 \text{diag}(1, 1, -1)\varphi^{-1}_2$ (respectively $\varphi_5 \text{diag}(1, 1, -1)\varphi^{-1}_5$) to be in $\text{Aut}(C_1)$, one notices that $\varphi_2 = [Z : X : Y] \varphi_1 = \varphi_1 \circ [Z : X : Y]$ (respectively $\varphi_5 = [Z : X : Y] \varphi_4 = \varphi_4 \circ [Z : X : Y]$). This means that we get the same conclusion as above up to a permutation of the parameters, more precisely, after

$$
(\alpha_{4,1}, \alpha_{1,1}, 1, 1) \mapsto (\alpha_{1,1,1}, \alpha_{1,4}, 4, 1),
$$

$$(\alpha_{0,3}, \alpha_{3,3}, \alpha_{3,0}) \mapsto (\alpha_{3,3}, \alpha_{3,0}, \alpha_{0,3}).$$

In other words, we have $\varphi_i \text{diag}(1, 1, -1)\varphi^{-1}_i$ with $i = 2$ or 5 inside $\text{Aut}(C_1)$ only if

$$
\alpha_{1,4} = \frac{2 \left( 29 - 54 \lambda^6 - 54 \mu^6 \right)}{27 \lambda \mu}, \quad \alpha_{0,3} = \frac{2 \left( 81 \lambda^6 - 27 \lambda^6 - 26 \right)}{27 \lambda^3},
$$

$$
\alpha_{3,3} = \frac{2 \left( 81 \lambda^6 - 27 \mu^6 - 26 \right)}{27 \lambda^4 \mu}, \quad \alpha_{1,1} = \frac{2 \left( 27 \lambda^6 - 54 \mu^6 - 52 \right)}{27 \lambda^4 \mu},
$$

$$
\alpha_{4,1} = \frac{2 \left( 27 \mu^6 - 54 \lambda^6 - 52 \right)}{27 \lambda^3 \mu^4}, \quad \alpha_{0,3} = \frac{2 \left( 82 - 27 \lambda^6 - 27 \mu^6 \right)}{27 \lambda^4 \mu^3},
$$

$$
\alpha_{2,2} = \frac{9 \lambda^6 + 9 \mu^6 + 10}{3 \lambda^2 \mu^2}.
$$

Once more $C_1$ reduces to $C_{1, \lambda, \mu}$ described in Theorem 2.5, (1)-(iii).
Similarly, \( \phi_3 = \phi_1 \circ [Y : Z : X] \) and \( \phi_6 = \phi_4 \circ [Y : Z : X] \). So \( \phi_i \) diag(1, 1, \(-1\)) with \( i = 3 \) or \( 6 \) is an automorphism for \( C_1 \) only if

\[
\begin{align*}
\alpha_{1,1} &= \frac{2 \left( 29 - 54\lambda^6 - 54\phi^6 \right)}{27\lambda\mu}, \\
\alpha_{0,3} &= \frac{2 \left( 81\lambda^6 - 27\phi^6 \right)}{27\lambda^3}, \\
\alpha_{1,4} &= \frac{2 \left( 2\phi^6 - 54\lambda^6 - 52 \right)}{27\lambda^4\mu}, \\
\alpha_{2,2} &= \frac{9\lambda^6 + 9\phi^6 + 10}{3\lambda^2\mu^2},
\end{align*}
\]

where \( C_1 \) becomes \( K \)-isomorphism to \( C_{1,\lambda,\mu} \).

This shows Theorem 2.5, (1)-(iii).

(ii) Second, suppose that \( \psi^{-1}\text{Aut}(C_1)\psi = \varphi_2(A_4) \). Again, we can impose \( \psi^{-1}\sigma\psi = [\zeta_6 : Y : Z : X] \) or \( [Z : \zeta_6 X : Y] \), in particular, \( \psi \) has the shape of \( \psi_1 \) below.

\[
\psi_1 := \begin{pmatrix}
1 & \zeta_{18}^{-2} & \zeta_{18}^{-1} \\
\lambda & \zeta_{18} & \zeta_{18}^{-4} \\
\mu & \zeta_{18}^{-8} & \zeta_{18}^{2}
\end{pmatrix}, \quad \psi_2 := \begin{pmatrix}
\mu & \zeta_{18}^{-4} & \zeta_{18}^{-7} \\
\lambda & \zeta_{18}^{-2} & \zeta_{18}^{5} \\
\lambda & \zeta_{18}^{-8} & \zeta_{18}^{4}
\end{pmatrix}, \quad \psi_3 := \begin{pmatrix}
\lambda & \zeta_{18}^{-8} & \zeta_{18}^{4} \\
\mu & \zeta_{18}^{-2} & \zeta_{18}^{4} \\
\mu & \zeta_{18}^{-8} & \zeta_{18}^{2}
\end{pmatrix},
\]

for some \( \lambda, \mu \in K^* \). However, it is straightforward to check that none of these transformation transforms \( C_1 \) to \( C' \) whose core is \( X^6 + Y^6 + Z^6 \). Consequently, \( C_1 \) is never a descendant of the Fermat curve \( F_6 \) with \( \text{Aut}(C_1) \) conjugate to \( \varphi_2(A_4) \).

This proves Claim 3.

\[ \square \]

It remains to prove Claim 4 for \( C_2 \) that is a descendant of the Fermat curve \( F_6 \).

\textbf{Proof.} (of Claim 4) - We easily discard the cases when \( \text{Aut}(C_2) \) equals an \( S_3 \) or \( \mathbb{Z}/3\mathbb{Z} \times S_3 \) inside \( \text{Aut}(F_6) \) as none of the involutions \([X : sZ : s^{-1}Y], [sY : s^{-1}X : Z]\) and \([sZ : Y : s^{-1}X]\) preserves the core \( X^5 Y + Y^5 Z + Z^5 X \) of \( C_2 \).

- On the other hand, if \( \text{Aut}(C_2) \) equals \( \mathbb{Z}/3\mathbb{Z}^2 \) or \( \text{He}_3 \), then the discussion we had to show Claim 2 applies to conclude that \( C_2 \) is \( K \)-isomorphic to

\[
\begin{align*}
C' : X^5 Y + Y^5 Z + XZ^5 + \alpha_{4,0} \zeta_{21}^4 (X^4 Z^2 + X^2 Y^4 + Y^2 Z^4) + \\
\alpha_{3,2} \zeta_{21}^2 XYZ (X^2 Y + XZ^2 + Y^2 Z) = 0,
\end{align*}
\]

where \( \varphi_2(\mathbb{Z}/3\mathbb{Z}^2) \subseteq \text{Aut}(C') \). Next, if \( \text{Aut}(C') \) is \( \text{He}_3 \), then there must be another automorphism \( \sigma' \notin \varphi_2(\mathbb{Z}/3\mathbb{Z}^2) \) of order 3 that commutes with \( \sigma \) such that \( \sigma' [Y : Z : X] \sigma'^{-1} = [Y : Z : X] \sigma^{-1} \). Straightforward calculations show that \( \sigma' = [sY : t'Z : X] \) or \([s'Z : tX : Y] \) with \( s't' = \zeta_3 \) and \( s^2 t'^{-1} = \zeta_3^{-1} \). So \( \sigma' \) belongs to \( \varphi_1(\mathbb{Z}/3\mathbb{Z}^2) \) modulo \([Y : Z : X]\). Obviously, none of these transformations leaves invariant the core of \( C' \). Therefore, \( \text{Aut}(C_2) \) is never conjugate to \( \text{He}_3 \) inside \( F_6 \).

- Thirdly, following the notations of Claim 3, a change of variables of the form \( \phi = \phi_i \) for \( i = 1, 2, \ldots, 6 \) does not transform \( C_2 \) to \( C_2' : X^6 + Y^6 + Z^6 \) of lower order terms in \( X, Y, Z \). Thus \( C_2 \) is not a descendant of \( F_6 \) such that \( \phi^{-1} \text{Aut}(C_2) \phi = \text{Aut}(C_2') \).
\( \varphi_1(A_4) \). On the other hand, \( \psi_1 \, \text{diag}(1,1,-1) \psi_1^{-1} \in \text{Aut}(C_2) \) with \( i = 1 \) or \( 4 \) only if
\[
\alpha_{2,4} = \frac{\lambda^5 \mu + 4 \mu^5}{2 \lambda^2}, \quad \alpha_{4,0} = \frac{\lambda + 4 \lambda^5 \mu}{2 \mu^2}, \quad \alpha_{0,2} = \frac{4 \lambda + \mu^5}{2 \lambda^2 \mu^4} \quad \alpha_{1,3} = \frac{2 \left(2 \lambda^5 \mu + 2 \lambda + \mu^5\right)}{\lambda^5 \mu^2}, \quad \alpha_{3,2} = \frac{2 \lambda^5 \mu + 4 \lambda + 4 \mu^5}{\lambda^2 \mu}, \quad \alpha_{2,1} = \frac{2 \left(2 \lambda^5 \mu + \lambda + 2 \mu^5\right)}{\lambda \mu^3}.
\]

The above restrictions are consequences of eliminating the coefficients of \( X^6, Y^6, Z^6, X^3Z, Y^4Z^2, X^4Y^2, X^4Z^2 \) from the transformed equation \( \psi_1 \, \text{diag}(1,1,-1) \psi_1^{-1} C_2 = C_2 \). Moreover, \( C_2 \) is \( K \)-isomorphic via \( \psi_1 \) (respectively \( \psi_4 \) followed by \( Y \leftrightarrow Z \)) to \( C_{2,\lambda,\mu} \) described in Theorem 2.5, (2)-(ii). The rest is obvious by noticing that \( \psi_2 = \psi_1 \circ [Z : X : Y], \psi_5 = \psi_4 \circ [Z : X : Y], \psi_3 = \psi_1 \circ [Y : Z : X] \) and \( \psi_6 = \psi_4 \circ [Y : Z : X] \).

This proves Claim 4. \( \square \)

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