Relativistic Kinetic Equations for Electromagnetic, Scalar and Pseudoscalar Interactions

Pengfei Zhuang

Gesellschaft für Schwerionenforschung, Theory Group,
P.O.Box 110552, D-64220 Darmstadt, Germany

Ulrich Heinz

Institut für Theoretische Physik, Universität Regensburg,
D-93040 Regensburg, Germany

(February 1, 2008)

Abstract

We derive the kinetic equations for both the covariant and equal-time Wigner functions of Dirac particles with electromagnetic, scalar and pseudoscalar interactions. We emphasize the constraint equations for the spinor components in the equal-time formulation.

PACS: 03.65.Bz, 05.60.+w, 52.60.+h.
One of the recent progresses in transport theory \cite{1,2} is the establishment of transport equations for spinor \cite{3} and scalar \cite{4} equal-time Wigner functions with abelian gauge interaction. The main advantage of the equal-time formulation lies in the fact that the initial value of the equal-time Wigner function can be directly obtained from the corresponding initial field operators, since there is only one time scale in the equal-time formulation. Therefore some quantum problems such as pair production \cite{5,6} in strong electric fields have so far only be solved in the equal-time formulation \cite{3,7}. The advantages of the covariant formulation, on the other hand, are explicit Lorentz invariance and the feature that the kinetic equations naturally split into a transport equation of Vlasov-Boltzmann type and a generalized mass-shell condition \cite{1,2} which makes explicit the off-shell effects generated by the collisions in the system. In Ref. \cite{8} we discussed the relationship between the covariant and equal-time kinetic equations. By taking the energy average of the covariant equations we obtained both transport and constraint equations for the equal-time Wigner functions in scalar and spinor electrodynamics.

Recently, Shin and Rafelski \cite{9} discussed a system which, in addition to the electromagnetic vector interaction, included also scalar and pseudoscalar (strong) interactions. For the system defined by the Lagrangian

\begin{equation}
\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m + g_\sigma \sigma + ig_\pi \pi \gamma_5)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial^\mu \sigma \partial_\mu \sigma + \partial^\mu \pi \partial_\mu \pi) - \mathcal{U}_M(\sigma, \pi),
\end{equation}

they derived a generalized group of transport equations for the spinor components of the equal-time Wigner function in the mean field approximation. Here $A_\mu, \sigma$ and $\pi$ are electromagnetic, scalar and pseudoscalar fields, respectively, $g_\sigma$ and $g_\pi$ are the scalar and pseudoscalar coupling constants, and $\mathcal{U}_M(\sigma, \pi)$ is the mesonic self-interaction potential. In this short paper, we first study the full covariant kinetic equations for such a system and then derive from them transport and constraint equations also for the equal-time Wigner function, using our recently introduced energy averaging method. We thereby supplement the equal-time transport equations derived in Ref. \cite{9} by a set of non-trivial constraint equa-
tions. The importance of the latter is emphasized both for the semiclassical limit and for the general quantum dynamics.

In the following we restrict ourselves to the kinetic equations for the spinor Wigner function in the mean field approximation, by replacing the field operators $A_{\mu}, \sigma$ and $\pi$ by their mean values. Such a Hartree approximation can be justified for sufficiently strong fields and has so far been used in most applications of quantum transport theory. Since the meson potential $U_M$ does not explicitly appear in the kinetic equations for the spinor Wigner function (it only enters the selfconsistency equations for the scalar and pseudoscalar mean fields), the discussions below holds universally for all potentials $U_M(\sigma, \pi)$.

The covariant spinor Wigner function $W_4(x,p)$ is the ensemble average of the Wigner operator $\hat{W}_4(x,p)$ which is the 4-dimensional Wigner transformation of the covariant density operator $\hat{\Phi}_4(x,y)$:

$$W_4(x,p) = \int d^4y e^{ipy} \langle \hat{\Phi}_4(x,y) \rangle$$

$$= \int d^4y e^{ipy} \langle \psi(x + \frac{y}{2}) \exp \left[ ie \int_{1/2}^{y/2} ds A(x + sy) \cdot y \right] \bar{\psi}(x - \frac{y}{2}) \rangle. \quad (2)$$

Calculating the first-order derivatives of the covariant density matrix $\hat{\Phi}_4(x,y)$ with respect to $x$ and $y$ and using the Dirac equation

$$i\gamma_\mu \left( \partial_\mu + ieA_\mu(x) \right) \psi(x) = m\psi(x) - g_\sigma(x)\psi(x) - ig_\pi(x)\gamma^5 \psi(x) \quad (3)$$

one obtains an evolution equation for $\hat{\Phi}_4(x,y)$. After Fourier transforming with respect to the 4 components of the relative coordinate $y$ one obtains for arbitrary external fields $A_\mu, \sigma$ and $\pi$ the generalized VGE \cite{10} equation

$$[\gamma^\mu K_\mu + \gamma^5 K_5 - M] \hat{W}_4 = 0, \quad (4)$$

where now

$$K_\mu = \Pi_\mu + \frac{i\hbar}{2} D_\mu,$$

$$K_5 = \Pi_5 + iD_5,$$

$$M = M_1 + iM_2,$$
\[ \Pi_\mu(x, p) = p_\mu - i\hbar \int_{-\frac{1}{2}}^{\frac{1}{2}} ds F_{\mu\nu}(x - i\hbar s\partial_p) \partial^\nu, \]
\[ D_\mu(x, p) = \partial_\mu - e \int_{-\frac{1}{2}}^{\frac{1}{2}} ds F_{\mu\nu}(x - i\hbar s\partial_p) \partial^\nu, \]
\[ \Pi_5(x, p) = g_\pi \sin \left( \frac{\hbar}{2} \Delta \right) \pi(x), \]
\[ D_5(x, p) = g_\pi \cos \left( \frac{\hbar}{2} \Delta \right) \pi(x), \]
\[ M_1(x, p) = m - g_\sigma \cos \left( \frac{\hbar}{2} \Delta \right) \sigma(x), \]
\[ M_2(x, p) = g_\sigma \sin \left( \frac{\hbar}{2} \Delta \right) \sigma(x), \]
\[ \Delta = \partial_x \cdot \partial_p. \] (5)

For the discussion of the semiclassical expansion below we have explicitly given all factors of \( \hbar \). If one neglects in these equations the abelian gauge field tensor and all terms of second or higher order in \( \hbar \) one recovers the results recently obtained by Florkowski et al. \[11\].

Inserting the spinor decomposition \[10\]
\[ W_4(x, p) = \frac{1}{4} \left[ F(x, p) + i\gamma_5 P(x, p) + \gamma_\mu V^\mu(x, p) + \gamma_\mu \gamma_5 A^\mu(x, p) + \frac{1}{2} \sigma_{\mu\nu} S^\mu{}^\nu(x, p) \right], \] (6)
we split the complex equation (4) into 10 real equations for the spinor components:
\[ \Pi^\mu V_\mu - D_5 P = M_1 F, \]
\[ \hbar D^\mu A_\mu + 2D_5 F = 2M_1 P, \]
\[ \Pi_\mu F + \frac{\hbar}{2} D^\nu S_{\mu\nu} - \Pi_5 A_\mu = M_1 V_\mu, \]
\[ -\hbar D_\mu P + \epsilon_{\mu\nu\sigma\rho} \Pi^\nu S^{\sigma\rho} - 2\Pi_5 V_\mu = 2M_1 A_\mu, \]
\[ \frac{\hbar}{2} (D_\mu V_\nu - D_\nu V_\mu) + \epsilon_{\mu\nu\sigma\rho} \Pi^\sigma A^{\rho} - \frac{1}{2} \epsilon_{\mu\nu\sigma\rho} D_5 S^{\sigma\rho} = M_1 S_{\mu\nu}, \]
\[ \hbar D^\mu V_\mu + 2\Pi_5 P = 2M_2 F, \]
\[ \Pi^\mu A_\mu + \Pi_5 F = -M_2 P, \]
\[ \frac{\hbar}{2} D_\mu F - \Pi^\nu S_{\nu\mu} - D_5 A_\mu = M_2 V_\mu, \]
\[ \Pi_\mu P - D_5 V_\mu + \frac{\hbar}{4} \epsilon_{\mu\nu\sigma\rho} D^\nu S^{\sigma\rho} = M_2 A_\mu, \]
\[ \Pi_\mu V_\nu - \Pi_\nu V_\mu - \frac{\hbar}{2} \epsilon_{\mu\nu\sigma\rho} D^\sigma A^{\rho} - \frac{1}{2} \epsilon_{\mu\nu\sigma\rho} \Pi_5 S^{\sigma\rho} = M_2 S_{\mu\nu}. \] (7)
Again the corresponding results up to first order in $\hbar$ can be found in Ref. [11] if we do not consider the abelian gauge interaction.

Following the treatment of Ref. [8], we write the equal-time Wigner function $W_3(x, p)$ as the energy average of the covariant Wigner function $W_4(x, p)$ and decompose it in a similar way [3] into its spinor components:

$$W_3(x, p) = \int d^3y e^{-ip\cdot y} \left\langle \psi \left( x + \frac{y}{2}, t \right) \exp \left[ ie \int_{-\frac{y}{2}}^{\frac{y}{2}} ds A(x + sy, t) \cdot y \right] \psi^\dagger \left( x - \frac{y}{2}, t \right) \right\rangle = \int dp_0 W_4(x, p) \gamma_0 \left[ \frac{1}{4} \left[ f_0(x, p) + \gamma_5 f_1(x, p) - i\gamma_0 \gamma_5 f_2(x, p) + \gamma_0 f_3(x, p) \right. \right. $$

$$\left. \left. + \gamma_5 \gamma_0 \gamma_1 \gamma_0 \gamma \gamma_1 (x, p) + \gamma_0 \gamma_1 \gamma_3 (x, p) - i\gamma_1 \gamma_2 (x, p) - \gamma_5 \gamma_1 \gamma_3 (x, p) \right] \right].$$

The second line in (8) suggests that the equal-time kinetic equations may be obtained by taking the energy average of the corresponding covariant equations. As pointed out in Ref. [8] and seen below, this procedure yields additional information on the equal-time Wigner function, in the form of additional constraint equations which are lost in the procedure used in Ref. [9] which is based on a 3-dimensional Wigner transformation of the equations of motion for the equal-time density matrix.

The energy average of the equations (7) gives rise to kinetic equations for the spinor components $f_i$ and $g_i$ ($i = 0, 1, 2, 3$) of the equal-time Wigner function. In direct analogy to Ref. [8] one obtains a set of transport equations which are identical to the ones derived in [9],

$$\hbar(D_t f_0 + D \cdot g_1) = -2g_1 \sigma_0 f_3 + 2g_2 \pi_0 f_2,$$

$$\hbar(D_t f_1 + D \cdot g_0) = 2g_2 \sigma_0 f_2 + 2g_2 \pi_0 f_3 - 2mf_2,$$

$$\hbar(D_t f_2 + 2\Pi \cdot g_3) = -2g_3 \sigma_0 f_1 + 2g_3 \pi_0 f_0 + 2mf_1,$$

$$\hbar(D_t f_3 - 2\Pi \cdot g_2) = -2g_2 \sigma_0 f_0 - 2g_3 \pi_0 f_1,$$

$$\hbar(D_t g_0 + D \cdot f_1) - 2\Pi \times g_1 = -2g_3 \sigma_0 g_3 + 2g_2 \pi_0 g_2,$$

$$\hbar(D_t g_1 + D \cdot f_0) - 2\Pi \times g_0 = 2g_2 \sigma_0 g_2 + 2g_3 \pi_0 g_3 - 2mg_2,$$

$$\hbar(D_t g_2 + D \times g_3) + 2\Pi f_3 = -2g_3 \sigma_0 g_1 + 2g_3 \pi_0 g_0 + 2mg_1.$$
\[ h(D_3 g_3 - D \times g_2) - 2 \Pi f_2 = -2 g_\sigma \sigma_o g_0 - 2 g_\pi \pi_e g_1, \]  

\hspace{1cm} (9)

as well as a set of additional constraint equations:

\[
\begin{align*}
\int dp_0 p_0 F - \frac{1}{2} \hbar D \cdot g_2 + \tilde{\Pi}_0 f_3 - g_\pi \pi_o f_1 &= (m - g_\sigma \sigma_e) f_0, \\
\int dp_0 p_0 P + \frac{1}{2} \hbar D \cdot g_3 + \tilde{\Pi}_0 f_2 - g_\pi \pi_e f_0 - g_\sigma \sigma_o f_1 &= 0, \\
\int dp_0 p_0 V_0 - \Pi_0 \cdot g_1 + \tilde{\Pi}_0 f_0 - g_\pi \pi_e f_2 &= (m - g_\sigma \sigma_e) f_3, \\
\int dp_0 p_0 V - \frac{1}{2} \hbar D \cdot g_0 - \Pi f_0 + \tilde{\Pi}_0 g_1 + g_\pi \pi_o g_3 + g_\sigma \sigma_o g_2 &= 0, \\
\int dp_0 p_0 A_0 + \Pi_0 \cdot g_0 - \tilde{\Pi}_0 f_1 - g_\pi \pi_o f_3 - g_\sigma \sigma_o f_2 &= 0, \\
\int dp_0 p_0 A + \frac{1}{2} \hbar D \times g_1 + \Pi f_1 - \tilde{\Pi}_0 g_0 + g_\pi \pi_e g_2 &= -(m - g_\sigma \sigma_e) g_3, \\
\int dp_0 p_0 S_{0i} e_i - \frac{1}{2} \hbar D f_3 + \Pi \times g_3 - \tilde{\Pi}_0 g_2 + g_\pi \pi_e g_0 + g_\sigma \sigma_o g_1 &= 0, \\
\int dp_0 p_0 S_{ijk} e^{ijk} e_i - \hbar D f_2 + 2 \Pi \times g_2 + 2 \tilde{\Pi}_0 g_3 - 2 g_\pi \pi_o g_1 &= 2 (m - g_\sigma \sigma_e) g_0. \\
\end{align*}
\]

(10)

Here we defined the following 3-dimensional operators:

\[ D_t(x, p) = \partial_t + e \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \mathbf{E}(x + is \hbar \nabla_p, t) \cdot \nabla_p, \]

\[ D(x, p) = \nabla + e \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \mathbf{B}(x + is \hbar \nabla_p, t) \times \nabla_p, \]

\[ \Pi(x, p) = p - i e \hbar \int_{-\frac{1}{2}}^{\frac{1}{2}} ds s \mathbf{B}(x + is \hbar \nabla_p, t) \times \nabla_p, \]

\[ \tilde{\Pi}_0(x, p) = i e \hbar \int_{-\frac{1}{2}}^{\frac{1}{2}} ds s \mathbf{E}(x + is \hbar \nabla_p, t) \cdot \nabla_p, \]

\[ \sigma_e(x, p) = \cos \left( \frac{\hbar}{2} \nabla \cdot \nabla_p \right) \sigma(x), \]

\[ \sigma_o(x, p) = \sin \left( \frac{\hbar}{2} \nabla \cdot \nabla_p \right) \sigma(x), \]

\[ \pi_e(x, p) = \cos \left( \frac{\hbar}{2} \nabla \cdot \nabla_p \right) \pi(x), \]

\[ \pi_o(x, p) = \sin \left( \frac{\hbar}{2} \nabla \cdot \nabla_p \right) \pi(x), \]

\hspace{1cm} (11)

which correspond to the covariant operators [3]. \( \mathbf{E} \) and \( \mathbf{B} \) are the electric and magnetic fields, respectively.

We now discuss the relevance of the additional constraint equations (10), both in the classical limit and for genuine quantum situations. In the classical limit, the covariant kinetic
equations have solutions of the form $W^\pm_1(x, p) = \tilde{W}^\pm_1(x, \mathbf{p}) \delta(p_0 \mp E_p)$. Here $E_p$ is still an arbitrary function of $p$; its concrete form will be seen momentarily as a result of the constraint equations. To zeroth order in $\hbar$, the kinetic equations (9) and (10) lead to the following constraints for the classical spinor components and the quasiparticle energy $E_p$:

$$
\begin{align*}
  f^\pm_1 &= \pm \frac{\mathbf{p} \cdot \mathbf{g}_0^\pm}{E_p}, \\
  f^\pm_2 &= \pm \frac{V_\pi f_0^\pm}{E_p}, \\
  f^\pm_3 &= \pm \frac{(m - V_\sigma) f_0^\pm}{E_p}, \\
  g^\pm_1 &= \pm \frac{\mathbf{p} f_0^\pm}{E_p}, \\
  g^\pm_2 &= \frac{\mathbf{p} \times \mathbf{g}_0^\pm + V_\pi \mathbf{g}_3^\pm}{m - V_\sigma}, \\
  g^\pm_3 &= \pm \frac{E^2_p(m - V_\sigma)g_0^\pm - (m - V_\sigma)\mathbf{p} \cdot \mathbf{g}_0^\pm + E_p V_\pi \mathbf{p} \times \mathbf{g}_0^\pm}{E_p (m^*)^2}, \\
  E^2_p &= (m^*)^2 + \mathbf{p}^2.
\end{align*}
$$

(12)

Here we defined

$$
V_\sigma = g_\sigma \sigma, \quad V_\pi = g_\pi \pi, \quad (m^*)^2 = (m - V_\sigma)^2 + V_\pi^2.
$$

(13)

The classical transport equations for the charge and spin densities are seen to originate from the equation (9) at first order in $\hbar$:

$$
\begin{align*}
  d_t f^\pm_0 + \mathbf{d} \cdot \mathbf{g}_1^\pm &= (\mathbf{F}_\sigma \cdot \nabla_p) f^\pm_3 - (\mathbf{F}_\pi \cdot \nabla_p) f^\pm_2, \\
  d_t g^\pm_3 - \mathbf{d} \times \mathbf{g}_2^\pm &= (\mathbf{F}_\sigma \cdot \nabla_p) g^\pm_0 - \frac{V_\pi}{m - V_\sigma} (\mathbf{F}_\pi \cdot \nabla_p) g^\pm_0 \\
  &\quad - \frac{\mathbf{p}}{m - V_\sigma} (d_t f^\pm_1 + \mathbf{d} \cdot \mathbf{g}_0^\pm) - \frac{V_\pi}{m - V_\sigma} (d_t g^\pm_2 + \mathbf{d} \times g^\pm_3),
\end{align*}
$$

(14)

where

$$
\begin{align*}
  d_t &= \partial_t + e \mathbf{E} \cdot \nabla_p, \\
  \mathbf{d} &= \nabla + e \mathbf{B} \times \nabla_p, \\
  \mathbf{F}_\sigma &= -g_\sigma \nabla \sigma, \\
  \mathbf{F}_\pi &= -g_\pi \nabla \pi.
\end{align*}
$$

(15)
Introducing the particle and antiparticle charge densities $f$ and $\bar{f}$ and spin densities $g$ and $\bar{g}$ through

$$f(x, p) = f_0^+(x, p), \quad \bar{f}(x, p) = f_0^-(x, -p),$$

$$g(x, p) = g_0^+(x, p), \quad \bar{g}(x, p) = g_0^-(x, -p),$$

and using the constraints (12), one derives from (14) a Vlasov-type transport equation for $f$,

$$\partial_t f + (v \cdot \nabla) f + \left( eE + ev \times B - \frac{(m - V_\sigma) F_\sigma - V_\pi F_\pi}{E_p} \right) \cdot \nabla_p f = 0,$$

and a similar equation for the spin density $g$:

$$\partial_t g + (v \cdot \nabla) g + \left( eE + ev \times B - \frac{(m - V_\sigma) F_\sigma - V_\pi F_\pi}{E_p} \right) \cdot \nabla_p g$$

$$= \frac{e}{E_p^2} \left( (p \cdot g) E - (E \cdot p) g \right) - \frac{e}{E_p} B \times g$$

$$+ \frac{(m - V_\sigma) F_\sigma + V_\pi F_\sigma}{(m^*)^2} \times \left( g - \frac{(p \cdot g)}{E_p^2} p \right) + \frac{(m - V_\sigma) F_\sigma - V_\pi F_\pi}{E_p(m^*)^2} \times (p \times g)$$

$$+ \frac{(m - V_\sigma) \partial_t V_\pi + V_\pi \partial_t V_\sigma}{(m - V_\sigma)^2(m^*)^2} \frac{V_\pi}{E_p} \left( E_p g - \frac{(p \cdot g)}{E_p} p - \frac{V_\pi}{m - V_\sigma} p \times g \right)$$

$$- \frac{m^* \partial_t m^*}{E_p^4} \left( (p \cdot g) p + (m^*)^2 g \right) + \frac{\partial_t V_\sigma}{(m - V_\sigma) E_p} \left( \frac{V_\pi}{m - V_\sigma} p \times g - \frac{(m^*)^2}{E_p^2} g \right).$$

Here $v = \frac{p}{E_p}$ is the velocity of the classical transport flow. The corresponding equations for the antiparticle densities $\bar{f}$ and $\bar{g}$ are obtained by changing the sign of the electric charge, $e \to -e$. The spin evolution equation (18) is the 3-dimensional phase-space version of a generalized Bargmann-Michel-Telegdi (BMT) equation [12,10] which describes spin precession in external electromagnetic, scalar and pseudoscalar fields. The first two terms of the r.h.s. which couple the spin density to the electromagnetic field have already been given in [8]. The remaining terms describe spin-precession effects caused by the coupling to the scalar and pseudoscalar fields. They are the 3-dimensional version of the much more compact covariant expressions derived in [11].
It is well known \[10,11\] that the covariant spin equation can be greatly simplified by introducing the covariant “spin up” and “spin down” distribution functions

\[
F_{\pm s}(x,p) = \frac{F(x,p)}{m-V_\sigma} \pm \frac{S_\mu(x,p)A^\mu(x,p)}{m^*(x)},
\]

where \(S_\mu = A_\mu/(-A\cdot A)^{1/2}\) is the covariant and normalized spin phase-space density.

The kinetic equations for the analogous equal-time distribution functions \(f_{\pm s}(x,p) = E_p \int dp_0 F_{\pm s}(x,p) \delta(p_0 - E_p)\) and \(\bar{f}_{\pm s}(x,p) = E_p \int dp_0 F_{\pm s}(x,-p) \delta(p_0 + E_p)\) decouple into scalar equations of the form (17) also obeyed by the charge density:

\[
\partial_t f_{\pm s} + (\mathbf{v} \cdot \nabla) f_{\pm s} + \left( e\mathbf{E} + e\mathbf{v} \times \mathbf{B} - \frac{(m-V_\sigma)F_\sigma - V_\pi F_\pi}{E_p} \right) \cdot \nabla_p f_{\pm s} = 0,
\]

and a similar one for the antiparticle distribution \(\bar{f}_{\pm s}\) obtained through \(e \rightarrow -e\).

Although the energy averaging method and the procedure of Ref. \[9\] lead to the same Vlasov transport equations for \(f\) and \(\bar{f}\), we would like to point out an essential difference: Since the equal-time constraint equations (10) do not disappear even in the classical limit, they reduce (together with the transport equations (9) to order \(\hbar^0\)) the number of independent spinor components from 8 in \[3\] (namely \(f_0, f_3, g_0\) and \(g_3\)) to 4 here (namely the charge density \(f_0\) and the spin density \(g_0\)). Furthermore, the form of the quasiparticle dispersion relation \(E_p\) need not be guessed as in \[3\], but follows from the constraints. In the treatment of Ref. \[3\], the equations for the charge density \(f_0\) and the mass density \(f_3\) remained coupled, and the Vlasov-type transport equation (17) was only obtained after a suitable redefinition of the classical distribution functions by hand (see equations (57) and (58) in \[3\]). Our constraint equations (14) turn these redefinition equations into identities.

For genuine quantum problems like pair production, one must consider the non-perturpative (in \(\hbar\)) form of the constraint equations (14). To get a closed set of equations for the spinor components of the equal-time Wigner function, one eliminates the first \(p_0\)-moments \(\int dp_0 p_0 W_4(x,p)\) from (14) by combining them with the first \(p_0\)-moments of those covariant equations in (7) which gave rise to the equal-time transport equations (9). The procedure is entirely analogous to the one detailed in Ref. \[8\], and one obtains the constraints

\[
L f_0 + \mathbf{M} \cdot \mathbf{g}_1 - \mathbf{F}_{\pi_0} \cdot \mathbf{g}_2 - \mathbf{F}_{\pi_0} \cdot \mathbf{g}_3 = 0,
\]
\[ Lf_1 + M \cdot g_0 - (F_{\sigma e} + F_\pi) \cdot g_2 + (F_{\pi e} + F_\sigma) \cdot g_3 = 0, \]
\[ Lf_2 + 2N \cdot g_3 - (F_{\pi e} - F_\pi) \cdot g_1 - F_{\sigma o} \cdot g_0 = 0, \]
\[ Lf_3 - 2N \cdot g_2 + (F_{\sigma e} - F_\sigma) \cdot g_1 - F_{\pi o} \cdot g_0 = 0, \]
\[ Lg_0 - Mf_1 - 2N \times g_1 - F_{\sigma o} f_2 - F_{\pi o} f_3 - (F_{\pi e} - F_\pi) \times g_3 - (F_{\sigma e} - F_\sigma) \times g_2 = 0, \]
\[ Lg_1 - Mf_0 - 2N \times g_0 - (F_{\pi e} - F_\pi) f_2 + (F_{\sigma e} - F_\sigma) f_3 + F_{\sigma o} \times g_3 - F_{\pi o} \times g_2 = 0, \]
\[ Lg_2 + M \times g_3 - 2N f_3 + F_{\sigma o} f_0 + F_{\pi e} f_1 - (F_{\sigma e} - F_\sigma) \times g_0 + F_{\pi o} \times g_1 = 0, \]
\[ Lg_3 - M \times g_2 + 2N f_2 + F_{\pi o} f_0 - (F_{\sigma e} - F_\sigma) f_1 - (F_{\pi e} - F_\pi) \times g_0 - F_{\sigma o} \times g_1 = 0, \]

with
\[
L(x, p) = ie \int_{-\frac{1}{2}}^{\frac{1}{2}} ds s \left( \nabla \times B(x + is\nabla_p, t) \right) \cdot \nabla_p, \\
M(x, p) = ie \int_{-\frac{1}{2}}^{\frac{1}{2}} ds s \nabla \left( E(x + is\nabla_p, t) \cdot \nabla_p \right) + e \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \left( E(x + is\nabla_p, t) - E(x) \right), \\
N(x, p) = \frac{1}{4} e \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \left( \nabla_p \cdot \nabla \right) E(x + is\nabla_p, t) \\
+ e \int_{-\frac{1}{2}}^{\frac{1}{2}} ds s^2 \left( \partial_t B(x + is\nabla_p, t) \right) \times \nabla_p + ie \int_{-\frac{1}{2}}^{\frac{1}{2}} dss E(x + is\nabla_p, t), \\
F_{\sigma o} = -g_\sigma \nabla \sigma_o, \\
F_{\sigma e} = -g_\sigma \nabla \sigma_e, \\
F_{\pi o} = -g_\pi \nabla \pi_o, \\
F_{\pi e} = -g_\pi \nabla \pi_e. \] 

These constraints, which are closely associated with the quantum corrections to the classical mass-shell condition, hold for arbitrary external fields. Only for homogeneous fields, the spatial derivatives of the electric, magnetic, scalar and pseudoscalar fields vanish, and therefore the constraint equations (21) disappear identically.

These results extend our work in [8] where we studied only electromagnetic interactions. They also complement the recent studies of Refs. [11] and [9] by generalizing the former to all orders in \( \hbar \) and supplementing the latter with the necessary additional constraints. The generalization to scalar and pseudoscalar interactions is relevant for the development
of a chiral kinetic theory with possible applications to the non-equilibrium dynamics in relativistic heavy ion collisions. Eq. (1) has the generic form of an effective chiral Lagrangian for low-energy hadronic physics; suitable choices for the meson potential $U_M$ correspond to different popular effective chiral models. For example, replacing the second line in Eq. (1) by $C(\sigma^2 + \pi^2)$ yields the semi-bosonized version of the Nambu–Jona-Lasinio model \[13\]. On the other hand, the meson potential $U_M \sim (\sigma^2 + \pi^2 - v^2)^2$, with $v$ as a parameter, represents the original linear $\sigma$ model \[14\]. These two models are widely used to describe low energy hadronic physics and the chiral phase transition at high temperatures and densities. Therefore, the study of the kinetic equations in both covariant and equal-time formulations for the Lagrangian (1) is helpful for an understanding of the dynamical consequences of the chiral properties of QCD.

We should not close without repeating from Ref. \[8\] an important cautionary remark. The energy averaging procedure elucidates that the kinetic equations for the equal-time Wigner function $W_3(x, p) = \int dp_0 W_4(x, p)$ form only the lowest level of a coupled hierarchy of equations for the energy moments of the covariant Wigner function, $\int dp_0 p_0^n W_4(x, p), \ n = 0, 1, 2, \ldots$. The full hierarchy can be obtained by taking the energy average of the appropriate $p_0$-moments of the generalized VGE equation (4). Its properties are presently under study in the context of the simpler case of scalar QED \[15\]. The connection between certain truncation schemes for this hierarchy of equal-time moment equations and the well-known gradient expansion in the covariant approach \[1, 2, 11\] is an interesting subject for further research.

**ACKNOWLEDGMENTS**

P.Z. wishes to thank the GSI for a fellowship. This work was supported in part by BMBF and DFG.
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