Pre-modular fusion categories of global dimensions $p^2$

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Abstract

Let $p \geq 5$ be a prime, we show that a non-pointed modular fusion category $C$ is Grothendieck equivalent to $C(sl_2, 2(p-1))^A_0$ if and only if $\dim(C) = p \cdot u$, where $u$ is a certain totally positive algebraic unit and $A$ is the regular algebra of the Tannakian subcategory $\text{Rep}(\mathbb{Z}_2) \subseteq C(sl_2, 2(p-1))$. As a direct corollary, we classify non-simple modular fusion categories of global dimensions $p^2$.

Keywords: Global dimension; pre-modular fusion category; modular fusion category

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1 Introduction

A fusion category $C$ over the complex field $\mathbb{C}$ is a semisimple finite tensor category, fusion categories are widely studied with restrictions on their Frobenius-Perron dimensions [10, 11], ranks (i.e., the number of the isomorphism classes of simple objects) [4, 21] etc. The global dimension of a fusion category is an important concept that deserved more attentions. Unlike the classification of fusion categories by Frobenius-Perron dimensions, however, we even don’t know the structures of fusion categories with a prime global dimension.

Recently, some progresses are made in topics related to the classifications of spherical fusion categories by their global dimensions. In [22], V. Ostrik gave lower bounds of the dimensions (more generally, the formal codegrees) of fusion categories, as a direct result, he classified spherical fusion categories of integer global dimensions less than 6. Braided spherical (or, pre-modular) fusion categories of global dimension less than or equal to 10 were completely classified by the author in [27], and spherical fusion categories of dimension 6 were also shown to be weakly integral. It was conjectured in [27] that pre-modular fusion categories of prime dimension $p$ are always pointed if $p \neq 5$, this is answered affirmatively in [25] lately. Moreover, by using a classical Siegel’s trace theorem about totally positive algebraic integers [26], spherical fusion categories of prime dimension $p$ are proved to be pointed if $(p - 1)/2$ is also an odd prime.

As a special class of fusion categories, modular fusion categories connect deeply with conformal field theory [2], vertex operator algebras [6], quantum groups at root of unity [2, 10]. The $S$-matrix and $T$-matrix of modular fusion categories (see section 2), which reflect many important properties of modular fusion categories, also enjoy interesting algebraic and arithmetic properties [3, 4, 6]. Therefore, modular fusion categories are inseparable with algebraic number...
theory and representations of the modular group \( SL(2, \mathbb{Z}) \) \[3, 4, 10\], in particular, one can peer into their properties by considering the number of Galois orbits of the simple objects \[16, 23\] and the representation type of \( SL(2, \mathbb{Z}) \) associated to a modular category \[17\], for example.

Let \( p \) be a prime, \( C \) a modular fusion category of global dimension \( p^2 \). Then \( C \) is tensor equivalent to an Ising category \( C(\mathfrak{sl}_2, 2) \) or \( C \) is pointed \[22\] if \( p = 2 \). Modular fusion categories of global dimension 9 are either pointed or braided tensor equivalent to a Galois conjugate of the non-pointed simple modular fusion category \( C(\mathfrak{so}_5, \frac{3}{2})_{ad} \) \[27, \text{Theorem 4.8}\]. Let \( C(\mathfrak{sl}_2, 3)_{ad} \) denote the Yang-Lee (or, Fibonacci) fusion category, which is a rank 2 transitive modular fusion category \[15\], and its Galois conjugate \( C(\mathfrak{sl}_2, 3)_{ad}^\sigma \) has global dimension \( \frac{25-\sqrt{25}}{2} \), where \( \sigma \in \text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) \) such that \( \sigma(\zeta_5) = \zeta_5^2 \). So modular fusion categories

\[
C(\mathfrak{sl}_2, 3)_{ad} \boxtimes C(\mathfrak{sl}_2, 3)_{ad}^\sigma \boxtimes C(\mathfrak{sl}_2, 3)_{ad} \boxtimes C(\mathfrak{sl}_2, 3)_{ad}^\sigma, \quad C(\mathfrak{sl}_2, 3)_{ad} \boxtimes C(\mathfrak{sl}_2, 3)_{ad}^\sigma \boxtimes C(\mathfrak{sl}_2, 3)_{ad} \boxtimes C(\mathfrak{sl}_2, 3)_{ad}^\sigma \boxtimes C(\mathbb{Z}_5, \eta)
\]

both have global dimension 25, where \( C(\mathbb{Z}_5, \eta) \) is a pointed modular fusion category of dimension 5. When \( p = 7 \), modular fusion category \( C(\mathfrak{sl}_2, 5)_{ad} \boxtimes C(\mathfrak{sl}_2, 5)_{ad}^\tau \boxtimes C(\mathfrak{sl}_2, 5)_{ad}^\tau \) and its Galois conjugates have global dimension 49, where \( \tau \in \text{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q}) \) such that \( \tau(\zeta_7) = \zeta_7^2 \), \( C(\mathfrak{sl}_2, 5)_{ad} \) is a transitive modular fusion category of rank 3 \[21, 16\]. It was asked in \[27, \text{Question 4.9}\] whether the modular fusion categories (and their Galois conjugates) mentioned above are all non-pointed modular fusion categories of global dimension \( p^2 \).

In this paper, from both the views of algebraic and arithmetic properties of modular fusion categories, we continue to investigate the structure of (pre-)modular fusion categories of global dimensions \( p^2 \) with \( p \geq 5 \). Let \( C \) be such a modular fusion category, if \( C \) does not contains a non-trivial fusion subcategory of integer dimension, we show that \( C \) always contains a Galois conjugate of the transitive modular fusion category \( C(\mathfrak{sl}_2, p-2)_{ad} \) \( \text{Theorem 4.3} \), then we give a complete classification of non-simple modular fusion category of global dimension \( p^2 \) \( \text{Theorem 4.15} \). Moreover, we find that there exists another non-pointed modular fusion category of global dimension \( 11^2 \), which then gives a negative answer to \[27, \text{Question 4.9}\].

### Table 1: Some Notations

| Notation       | Meaning                                                                 |
|----------------|-------------------------------------------------------------------------|
| \( \zeta_n \)  | the \( n \)-th primitive root of unity \( e^{\frac{2\pi i}{n}} \)       |
| \( N(f) \)     | the norm of \( f \), i.e., \( N(f) = \prod_{\sigma \in \text{Gal}(\mathbb{Q}(f)/\mathbb{Q})} \sigma(f) \) |
| \( O_X(C) \)   | the Galois orbit of simple object \( X \) of a modular fusion category \( C \) |
| \( C(g, k) \)  | the modular fusion category obtained from representation category \( \text{Rep}(U_q(g)) \) of quantum group \( U_q(g) \) at root of unity |

The paper is organized as follows. In section \[2\] we recall some basic notions and notations of (modular) fusion categories, such as global dimensions, formal codegrees and \( d \)-numbers, modular data, and the congruence representations of the modular group \( SL(2, \mathbb{Z}) \). In section \[3\] we consider modular fusion categories whose norm of global dimensions are powers of a prime \( p \), in particular, if \( p \geq 5 \), we show that a non-pointed modular fusion category \( C \) is Grothendieck equivalent to \( C(\mathfrak{sl}_2, 2(p-1))_{ad} \) if and only if \( \dim(C) = p \cdot u \) in Theorem \[3, 12\] and Corollary \[3, 13\] where \( u \) is a certain algebraic unit. In section \[4\] we first show that a non-simple modular
fusion category $C$ of global dimension $p^2$ contains a Galois conjugate of a transitive modular fusion subcategory if $C$ does not contain a non-trivial fusion subcategory with an integer global dimension in Theorem 4.3, then we give a complete classification of non-simple modular fusion categories of dimension $p^2$ in Theorem 4.15.

2 Preliminaries

In this section, we recall we will recall some important definitions and properties about fusion categories and modular fusion categories, we refer the readers to [2, 4, 7, 10, 11, 15, 20].

2.1 Fusion categories and dimensions

Let $C$ be a fusion category, $O(C)$ the set of isomorphism classes of simple objects of a fusion category $C$. Then the Frobenius-Perron homomorphism $FPdim(-) : Gr(C) \rightarrow C$ is the unique ring homomorphism such that $FPdim(X) \geq 1$ is an algebraic integer, $\forall X \in O(C)$, $FPdim(X)$ is called the Frobenius-Perron dimension of the object $X$, and the sum $FPdim(C) := \sum_{X \in O(C)} FPdim(X)^2$ is defined as the Frobenius-Perron dimension of $C$.

A fusion category $C$ is weakly integral if $FPdim(C) \in \mathbb{Z}$, $C$ is integral if $FPdim(X) \in \mathbb{Z}$ for all $X \in O(C)$. We use $C_{int}$ to denote the maximal integral fusion subcategory of $C$. The adjoint subcategory $C_{ad}$ of a weakly integral fusion category is always integral [10]. A simple object of $C$ is called invertible if $X \otimes X^* = I$, the unit object, equivalently, $FPdim(X) = 1$. A fusion category $C$ is pointed if $O(C)$ is a finite group, where the group multiplication is induced by the tensor product $\otimes$. In the following, we use $C_{pt}$ to denote the maximal pointed subcategory of $C$, that is, the fusion subcategory generated by invertible simple objects of $C$. And we say a fusion category $C$ is non-pointed if $C_{pt} \neq C$. In addition, two fusion categories $C$ and $D$ are Grothendieck equivalent if $Gr(C) \cong Gr(D)$ as fusion rings.

Let $C$ be a spherical fusion category $C$ with spherical structure $j$, which is a natural isomorphism $j = \{j_X : X \rightarrow X^{**}, X \in C\}$ such that $dim_j(X) = dim_j(X^*)$, $dim_j(X)$ is called the quantum (or, categorical) dimension of $X$ determined by $j$, where $dim_j(X)$ is defined as the (categorical) trace of $id_X$, that is,

$$dim_j(X) = Tr(id_X) := ev_X \circ (j_X \circ id_X) \otimes id_X^* \circ coev_X,$$

where $(X^*, ev_X, coev_X)$ is the dual object of $X$ and we suppress the associativity and unit constraints of $C$. We define the global (or, categorical) dimension of the fusion category $C$ as

$$dim(C) := \sum_{X \in O(C)} dim_j(X)^2,$$

the global dimension $dim(C)$ is independent of the choice of the spherical structure of $C$ and $dim_j(-)$ induces a homomorphism from $Gr(C)$ to $\mathbb{C}$ [10 Proposition 4.7.12].

Given an arbitrary spherical fusion category $C$, we can consider the twist (or Galois conjugate) $C^\sigma$ of $C$, where $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$. More precisely, $C^\sigma$ is a
fusion category with the same monoidal functor $\otimes$ as $C$ and the associator of $C''$ is obtained by composing the one of $C$ with automorphism $\sigma$. Moreover, $\dim(C'') = \sigma(\dim(C))$. A fusion category $C$ is said to be pseudo-unitary if $\dim(C) = \FPdim(C)$. For more properties of global dimension, we refer the readers to references [10][11][22]. In this paper, we will fix a spherical structure $j$ and denote $\dim_j(-)$ by $\dim(-)$.

2.2 Formal codegrees of fusion categories

Let $C$ be a fusion category, then the complexified ring $\Gr(C) \otimes \mathbb{C}$ is semisimple. Given an irreducible representation $\chi$ of $\Gr(C) \otimes \mathbb{C}$, the element
$$\alpha_\chi = \sum_{X \in \mathcal{O}(C)} \text{tr}_\chi(X)X^*$$
is central, where $\text{tr}_\chi$ is the ordinary trace function on the representation $\chi$, moreover $\chi'(\alpha_\chi) = 0$ if $\chi \not\cong \chi'$ and $f_\chi := \chi(\alpha_\chi)$ is a positive algebraic integer [14], $f_\chi$ is called a formal codegree of fusion category $C$ [20]. For example, $\FPdim(C)$ and $\dim(C)$ are formal codegrees of $C$.

It was showed that $\dim(C)f_\chi$ is always a totally positive algebraic integer [20, Corollary 2.14] and the set of formal codegrees of $C$ satisfy the following equation [20, Proposition 2.10]
$$\sum \chi(1) f_\chi = 1.$$
If $C \not\cong \text{Vec}$, then $f_\chi > \sqrt{2\rank(C)} + 1 \geq \sqrt{\frac{2}{3}}$ for all irreducible representations $\chi$ [22, Theorem 4.2.1]. Moreover, $\sigma(\dim(C)) > \sqrt{2}$ if the non-trivial fusion category $C$ is not a Galois conjugate of $C(sl_2,3)_{\text{ad}}$ [22, Proposition A.1.1], where $\sigma \in \text{Gal}(\mathbb{Q}/\mathbb{Q})$.

Let $\alpha$ be an algebraic integer with the minimal polynomial $g(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, then $\alpha$ is called a $d$-number if $(a_i)^j$ divides $(a_j)^n$ for all $1 \leq j \leq n$ [20, Definition 1.1], see [20, Lemma 2.7] for more equivalent conditions about $d$-numbers. The formal codegrees of a fusion category are $d$-numbers [20, Theorem 1.2], for example. In addition, the Frobenius-Perron dimensions and quantum dimensions of objects, and formal codegrees of a fusion category are cyclotomic algebraic integers [11, Corollary 8.53]. Hence, in order to determine whether a totally positive algebraic integer $\alpha$ is a formal codegree of a fusion category $C$, we can use the Program GAP to test whether $\alpha$ is a $d$-number and the Galois group of the minimal polynomial of $\alpha$ is abelian, this is called the $d$-number tests and cyclotomic test [20].

2.3 Modular fusion categories and representations of $\text{SL}(2,\mathbb{Z})$

Let $C$ be a braided fusion category with braiding $c$ and $D$ a fusion subcategory of $C$. Then the centralizer of $D$ in $C$ is the following fusion subcategory
$$D_C' = \{ X \in C | c_{Y,X}c_{X,Y} = \text{id}_{X \otimes Y}, \forall Y \in D \}.$$
We call $C' := C_C'$ the M"uger center of $C$ [15].
Let $\mathcal{C}$ be a braided spherical (i.e., pre-modular) fusion category with ribbon structure $\theta$, the matrices $S = (S_{X,Y})$ and $T = (d_{X,Y} \theta_X)$ are called the $S$-matrix and $T$-matrix of $\mathcal{C}$, respectively, where $S_{X,Y} = \text{Tr}(c_{YX}c_{X,Y})$, $X, Y \in \mathcal{O}(\mathcal{C})$. A modular fusion category is a pre-modular fusion category such that the $S$-matrix $S$ is non-degenerate, equivalently $\mathcal{C}' = \text{Vec}[7,10][15]$. Moreover, the $S$-matrix a modular fusion category $\mathcal{C}$ determines the multiplication of the Grothendieck ring $\text{Gr}(\mathcal{C})$ by the famous Verlinde formula [10], i.e., for $X,Y,Z \in \mathcal{O}(\mathcal{C})$,

\[
N_{X,Y}^Z := \dim_C(\text{Hom}(X \otimes Y, Z)) = \sum_{W \in \mathcal{O}(\mathcal{C})} \frac{S_{X,W}S_{Y,W}S_{Z,W}}{\dim(W)}.
\]

Moreover, given a modular fusion category $\mathcal{C}$, the Verlinde formula (2) also implies that the set of ring homomorphism from $\text{Gr}(\mathcal{C})$ to $\mathbb{C}$ is in bijective correspondence with $\mathcal{O}(\mathcal{C})$ [10]. Explicitly, let $X \in \mathcal{O}(\mathcal{C})$, then $h_X(Y) := \frac{S_{X,Y}}{\dim(X)}$ defines a ring homomorphism from $\text{Gr}(\mathcal{C})$ to $\mathbb{C}$, $\forall Y \in \mathcal{O}(\mathcal{C})$; notice that the set of formal coodegrees of $\mathcal{C}$ is $\{\frac{\dim(C)}{\dim(X)} | X \in \mathcal{O}(\mathcal{C})\}$ due to the Verlinde formula (2). Since $\sigma \circ h_X(-)$ is also a ring homomorphism of $\text{Gr}(\mathcal{C})$, where $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, there is a unique simple object $Y$ such that $\sigma \circ h_X(-) = h_Y(-)$. Hence, there is a unique permutation $\hat{\sigma}$ of $\mathcal{O}(\mathcal{C})$ such that $\hat{\sigma}(X) = Y$ and $\sigma \circ h_X(-) = h_{\hat{\sigma}(X)}(-)$. We call the subset

$\mathcal{O}_X(\mathcal{C}) := \{ Y \in \mathcal{O}(\mathcal{C}) | Y = \hat{\sigma}(X) \text{ for some } \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \}$

the Galois orbit of the simple object $X$. When $\mathcal{O}_I(\mathcal{C}) = \mathcal{O}(\mathcal{C})$, then $\mathcal{C}$ is said to be transitive; transitive modular fusion categories are classified explicitly [16] Theorem III.

The modular data $(S, T)$ of a modular fusion category $\mathcal{C}$ is also connected closely with the congruence representations of the modular group $\text{SL}(2, \mathbb{Z})$ [3][4][6], which is generated by $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ with relations $s^4 = 1$ and $(st)^3 = s^2$. Explicitly, the morphism $s \mapsto \frac{1}{\sqrt{\dim(C)}} S, t \mapsto T$ defines a projective representation of $\text{SL}(2, \mathbb{Z})$ [10] Theorem 8.16.1, where $\sqrt{\dim(C)}$ is the positive square root of $\dim(C)$.

Let $\tau(S) := \sum_{X \in \mathcal{O}(\mathcal{C})} \dim(X)^2 \theta_X^S$ be the Gauss sums of $\mathcal{C}$ [7][10], then $\xi(S) := \frac{\tau(S)}{\sqrt{\dim(C)}}$ is called the multiplicative central charge of $\mathcal{C}$. It follows from [4][6] that there always exists a third root $\xi$ of $\xi(S)$ such that $s := \rho_C(s) = \sqrt{\dim(C)} S$ and $t := \rho_C(t) = \sqrt{\dim(C)} T$ defines a finite-dimensional congruence representation $\rho_C$ of $\text{SL}(2, \mathbb{Z})$ of level $N$, that is, $\ker(\rho_C)$ is a congruence subgroup of level $N = \text{ord}(t)$. Moreover, $\mathbb{Q}(S) \subseteq Q(T) \subseteq \mathbb{Q}(t)$ [6] Theorem II].

Let $t_X := \theta_X/\xi$, the normalized ribbon structure of the simple object $X$, $\forall X \in \mathcal{O}(\mathcal{C})$, then we have the Galois symmetry [6] Theorem II], that is, $\sigma^2(t_X) = t_{\hat{\sigma}(X)}, \forall \sigma \in \text{Gal}(\mathbb{Q}(t)/\mathbb{Q})$.

Let $\rho$ be a finite-dimensional congruence representation of $\text{SL}(2, \mathbb{Z})$ of level $n$, where $n$ is a positive integer. Then $\rho$ factors through the finite groups

$\text{SL}(2, \mathbb{Z}_n) \cong \text{SL}(2, \mathbb{Z}_{p_1^{n_1}}) \times \cdots \times \text{SL}(2, \mathbb{Z}_{p_r^{n_r}})$

and $\rho \cong \otimes_{j=1}^r \rho_{p_j}$ by the Chinese Reminder Theorem, where $n = \prod_{j=1}^r p_j^{n_j}$ and $p_j$ are distinct primes, $\rho_{p_j}$ are finite-dimensional representations of subgroups $\text{SL}(2, \mathbb{Z}_{p_j^{n_j}})$. Moreover, given an arbitrary prime $p$, all finite-dimensional irreducible representations of the group $\text{SL}(2, \mathbb{Z}_{p^n})$ are completely classified and constructed explicitly in [18][19]. A finite-dimensional congruence representation $\rho$ of the modular group $\text{SL}(2, \mathbb{Z})$ is said to be non-degenerate if the eigenvalues
of \( \rho(t) \) are distinct; non-degenerate finite-dimensional congruence representations are irreducible [9] Lemma 1. In addition, the set of eigenvalues of \( \rho(t) \) is called the t-spectrum of \( \rho \) following [4] [17] [23]: we note that the t-spectrum of any finite-dimensional irreducible representation of \( \text{SL}(2, \mathbb{Z}_{p^m}) \) is produced in [23] Appendix.

The following remark is known to experts, we list it here for the reader’s convenience, and it will play a key role in the arguments of this paper.

**Remark 2.1.** Let \( C \) be a non-trivial modular fusion category with \( N(\dim(C)) = p^N \), where \( p \) is an odd prime. Then there exists a simple object \( X \) such that \( p \) divides \( \text{ord}(t_X) \).

Indeed, if \( t_Y = 1 \) for all \( Y \in \mathcal{O}(C) \), then \( \xi = 1 \) as \( t_Y = 1/\xi = 1 \), so \( \theta_Y = 1 \) (\( \forall Y \in \mathcal{O}(C) \)). Notice that the balancing equation \([10]\) then implies that \( s_{X,Y} = \dim(X) \dim(Y) \), particularly the \( S \)-matrix of \( C \) can’t be non-degenerate, it is a contradiction.

Moreover, let \( X \) be the simple object of \( C \) such that \( \text{ord}(t_X) = p^n \) is maximal. Then

\[
\text{rank}(C) \geq \frac{p^{n-1}(p-1)}{2}. 
\]

By the Galois symmetry of modular fusion categories [6], we have \( \sigma^2(t_X) = t_{\sigma(t_X)} \), then

\[ \text{Gal}(\mathbb{Q}(t_X)/\mathbb{Q}) \cdot t_X = \{ t_{\sigma(t_X)} = \sigma^2(t_X) | \sigma \in \text{Gal}(\mathbb{Q}(t_X)/\mathbb{Q}) \}, \]

hence the number of Galois conjugates of \( X \) is greater or equal to the order of the following subgroup

\[ \text{Gal}(\mathbb{Q}(t_X)/\mathbb{Q})^2 := \{ \sigma^2 | \sigma \in \text{Gal}(\mathbb{Q}(t_X)/\mathbb{Q}) \} \]

of \( \text{Gal}(\mathbb{Q}(t_X)/\mathbb{Q}) \). It follows immediately that the order of \( \text{Gal}(\mathbb{Q}(t_X)/\mathbb{Q})^2 \) is exactly \( \frac{p^{n-1}(p-1)}{2} \), since \( \text{Gal}(\mathbb{Q}(t_X)/\mathbb{Q}) \) is a cyclic group of order \( p^{n-1}(p-1) \).

Throughout this paper, we always use \( \rho_0 \) to denote the trivial representation of \( \text{SL}(2, \mathbb{Z}) \).

**Example 2.2.** Let \( p \geq 5 \) be a prime, and let \( C := C(\mathfrak{sl}_2, 2(p-1)) \). Then \( C \) contains a unique non-trivial connected étale algebra \( A \) [12] Theorem 6.5], i.e., the regular algebra of Tannakian subcategory \( \text{Rep}(\mathbb{Z}_2) \subseteq C \), such that \( C_A^0 \) is a modular fusion category and

\[ \dim(C_A^0) = \frac{\dim(C)}{\dim(A)^2} = \frac{p}{4 \cos^2(\frac{d}{p})} \]

by [12] Theorem 4.5], where \( d = \frac{p+1}{2} \). Moreover, the simple objects of \( C_A^0 \) are also characterized explicitly in [12] Theorem 7.1], i.e., \( \mathcal{O}(C_A^0) = \{ V_0, V_2, \cdots, V_{p-3}, V_{p-1}, V_{p-1}^- \} \), and

\[ \dim(V_j) = \frac{\zeta_{4p}^{j+1} - \zeta_{4p}^{-j+1}}{\zeta_{4p} - \zeta_{4p}^{-1}} = \frac{\cos(\frac{(d-1)(j+1)}{p})}{\cos(\frac{d-1}{p})}, 0 \leq j \leq p-3, \dim(V_{p-3}^-) = \frac{1}{2 \cos(\frac{d-1}{p})} \]

Hence, the formal codegrees of \( C(\mathfrak{sl}_2, 2(p-1))^0_\Lambda \) are

\[ p(\text{twice}), \sigma \left( \frac{p}{4 \cos^2(\frac{d}{p})} \right) \text{ where } \sigma \in \text{Gal}(\mathbb{Q}(\zeta_p)^+/\mathbb{Q}). \]

In particular, if \( p = 5 \), then \( C(\mathfrak{sl}_2, 8)^0_\Lambda \simeq C(\mathfrak{sl}_2, 3)_{\text{ad}} \boxtimes C(\mathfrak{sl}_2, 3)_{\text{ad}} \) as modular fusion category.
By [12] Theorem 1.17, $\theta_{V_{j}} = e^{2\pi i \frac{d_{j}+2j}{dp-1}}$ and $\theta_{V_{j}} = \zeta_{p}^{j}$. Notice that the multiplicative central charge $\xi(C_{A}) = \xi(C) = e^{2\pi i \frac{dp-1}{dp-1}}$, let $\xi$ be a third root of $\xi(C_{A})$, so $\xi$ is an $24$-th root of $\zeta_{p}^{3(p-1)}$, while $\zeta_{p}^{p-1} = \zeta_{p}^{(p-1)+p(p-1)} = \zeta_{p}^{2(p-1)}$, hence we can choose $\xi = \zeta_{p}^{\frac{p-1}{2}}$ to be the third root of $\xi(C_{A})$, then the t-spectrum of the normalized t-matrix is

$$\left\{ \sigma^{2}(\zeta_{p}^{\frac{p-1}{2}}) \mid \sigma \in \text{Gal}(Q(\zeta_{p})/Q) \right\} \cup \{\text{twice}\}.$$ 

Therefore, the associated modular representation $\rho_{\xi}$ is $\rho_{1} \oplus \rho_{0}$, where $\rho_{1}$ is a $d$-dimensional irreducible representation of $\text{SL}(2, \mathbb{Z})$ of level $p$ and $\rho_{0}$ is the trivial representation of $\text{SL}(2, \mathbb{Z})$.

### 3 Modular fusion categories with $N(\dim(C)) = p^{m}$

In this section, we always assume $p$ is a prime, and we study the structures of modular fusion categories $C$ such that $N(\dim(C))$ is a power of $p$. Notice that if $N(\dim(C)) = p$ and $\dim(C) \notin \mathbb{Z}$, then $C$ is braided tensor equivalent to a Galois conjugate of $C(sl_{2}, 3)_{ad}$ [25].

#### 3.1 Modular fusion category $C$ with $N(\dim(C)) = p^{2}$

Let $C$ be a modular fusion category with $N(\dim(C)) = p^{2}$. Then [25] Proposition 4.11] says that $\frac{1}{2}d(\dim(C)) \leq 2$, where $d(\dim(C)) = [Q(\dim(C)) : Q]$, that is, $\dim(C) = 1, 2, 3, 4$. Note that $Q(\dim(C)) \subseteq Q(T_{C})$ and the Cauchy’s Theorem [3] Theorem 3.9] shows that $\text{ord}(T_{C}) = p^{m}$ for some positive integer $m$. Since $Q(\dim(C))$ is a real subfield of $Q(T_{C})$, $Q(\dim(C)) \subseteq Q(\zeta_{p^{n}})^{+}$, the maximal real subfield of $Q(\zeta_{p^{n}})$, and the Galois group $\text{Gal}(Q(\zeta_{p^{n}})/Q)$ is

$$\text{Gal}(Q(\zeta_{p^{n}})/Q) \cong (\mathbb{Z}_{p^{n}})^{*} = \left\{ \begin{array}{ll}
\mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{p-1}, & p > 2; \\
\mathbb{Z}_{2}, & p = 2 \text{ and } n = 2; \\
\mathbb{Z}_{2} \times \mathbb{Z}_{2^{n-2}}, & p = 2 \text{ and } n \geq 3.
\end{array} \right.$$ 

and $[Q(\zeta_{p^{n}}) : Q(\zeta_{p^{n}})^{+}] = 2$. Moreover, for any odd prime $p$, it is well-known that the cyclotomic field $Q(\zeta_{p^{n}})$ contains a unique quadratic subfield $Q(\sqrt[p^{n}]{p})$, where $p^{n} := (-1)^{\frac{p-1}{2}} p$.

We first prove the following lemma, which is a direct corollary of [25] Theorem 4.4].

**Lemma 3.1.** Let $C$ be a modular fusion category such that $N(\dim(C)) = p^{m}$, where $p$ is a prime. If $d(\dim(C)) > m$, then $m = \frac{p-1}{2}$ and $C$ is braided equivalent to a Galois conjugate of the transitive modular fusion category $C(sl_{2}, p-2)_{ad}$.

**Proof.** Since $N(\dim(C)) = p^{m}$ and $m < d(\dim(C))$, $p$ does not divide $\dim(C)$ by [25] Proposition 4.2]. Thus, for any integer $q \in \mathbb{Z}_{\geq 2}$, $q$ does not divides $\dim(C)$. Hence, all formal codegrees of $C$ are Galois conjugates of $\dim(C)$ [25] Theorem 4.4]. Note that $C$ is a modular fusion category, formal codegrees of $C$ all have the form $\frac{\dim(C)}{\dim(X)^{2}}$ for simple objects $X$ of $C$, so for any object $X \in \mathcal{O}(C)$, there exists a $\sigma \in \text{Gal}(Q(\dim(C))/Q)$ such that

$$\frac{\dim(C)}{\dim(X)^{2}} = \sigma(\dim(C)) = \sigma \left( \frac{\dim(C)}{\dim(I)^{2}} \right) = \frac{\dim(C)}{\dim(\sigma(I))^{2}}.$$ 


Corollary 3.2. Let \( \sigma \) be a modular fusion category such that \( N_{\mathbb{Q}}(\dim(C)) = p^2 \). If \( d_{\dim(C)} > 2 \), then \( p = 7 \) and \( C \cong C(sl_2, 5)_{ad} \), where \( \tau \in \text{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q}) \).

Let \( C \) be a modular fusion category, \( D := \bigoplus_{\sigma \in \text{Gal}(\mathbb{Q}(\dim(C))/\mathbb{Q})} C^\sigma \), then

\[
\dim(D) = N(\dim(C)) = \prod_{\sigma \in \text{Gal}(\mathbb{Q}(\dim(C))/\mathbb{Q})} \dim(C^\sigma) = \prod_{\sigma \in \text{Gal}(\mathbb{Q}(\dim(C))/\mathbb{Q})} \sigma(\dim(C)).
\]

Hence, [22] Lemma 4.2.2, Remark 4.2.3] say that

\[
(3) \quad \text{rank}(D) = \text{rank}(C)^{d_{\dim(C)}} \leq \dim(D),
\]

and \( \text{rank}(D) = \dim(D) \) if and only if \( D \) is pointed. In particular, if \( N(\dim(C)) = p^2 \), then \( \text{rank}(C) \leq p^2 \). When \( p = 2 \), it is easy to see \( \dim(C) = 4 \), then \( C \cong \mathcal{I} \) is an Ising category or \( C \) is pointed [22] Example 5.1.2]; when \( p = 3 \), \( d_{\dim(C)} = 1 \), that is, \( \dim(C) = 9 \), then \( C \) is either pointed or \( C \) is braided equivalent to a Galois conjugate of \( C(sl_5, 9, \zeta_{18})_{ad} \) by [27] Theorem, where \( \zeta_{18} \) is a primitive 18-root of unity.

Lemma 3.3. Let \( C \) be a modular fusion category such that \( N(\dim(C)) = p^2 \), where \( p \) is a prime. If \( d_{\dim(C)} = 2 \), then \( p = 5 \), and as a modular fusion category

\[
C \cong C(sl_2, 3)_{ad} \boxtimes C(sl_2, 3)_{ad} \text{ or } C \cong C(sl_2, 3)_{ad} \boxtimes C(sl_2, 3)_{ad}.
\]

where \( \sigma \in \text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) \) such that \( \sigma(\sqrt{5}) = -\sqrt{5} \).

Proof. By using the Cauchy theorem [3] Theorem 3.9] and [25] Proposition 4.2], we know that \( \dim(C) \in \mathbb{Q}(\zeta_p^s) \) for some positive integer \( s \). If \( d_{\dim(C)} = 2 \), then \( \dim(C)/p \) is a totally positive algebraic unit in the unique quadratic subfield \( \mathbb{Q}(\sqrt{p}) \) of \( \mathbb{Q}(\zeta_p^s) \), in particular, \( p \equiv 1 \pmod{4} \) and \( N\left(\frac{\dim(C)}{p}\right) = 1 \). Let \( \sigma \in \text{Gal}(\mathbb{Q}(\sqrt{p})/\mathbb{Q}) \) be the unique non-trivial element, then \( \sigma(\epsilon_p)\epsilon_p = -1 \) since \( N_{\mathbb{Q}(\sqrt{p})}(\epsilon_p) = -1 \), where \( \epsilon_p \) is the fundamental unit in the quadratic field \( \mathbb{Q}(\sqrt{p}) \). As \( \dim(C) \) is a totally positive algebraic integer, we have \( \sigma(\dim(C)) > 0 \), so \( \frac{\dim(C)}{p} = \epsilon_p^n \) for a nonzero even integer \( n \) by the Fundamental unit theorem [1] Theorem 11.5.1]. Besides, for the Galois conjugate \( \sigma(\dim(C)) \) of \( \dim(C) \), we have \( \sigma(\dim(C)) > \frac{4\sqrt{3}}{5} \) and \( \dim(C) > \frac{4\sqrt{3}}{5} \) by [22] Proposition A.1.1]. Hence without loss of generality, we can assume \( n \geq 2 \) below, then

\[
\sigma(\dim(C)) = p\epsilon_p^{-n} > \frac{4\sqrt{3}}{5},
\]

also \( \epsilon_p^{-n} \leq \epsilon_p^{-2} \) as \( n \geq 2 \), thus

\[
pe_p^{-2} \geq pe_p^{-n} = \sigma(\dim(C)) > \frac{4\sqrt{3}}{5}.
\]
which then implies $\epsilon_p < \sqrt{\frac{5}{4\sqrt{3}}}$. Let 

$$\epsilon_p = \frac{a_p + b_p \sqrt{3}}{2}$$ 

for positive integers $a_p, b_p$. If $b_p \geq 2$, then $\epsilon_p > \sqrt{\frac{5}{4\sqrt{3}}}$ if $p_b = 1$, then 

$$a_p^2 - pb_p^2 = -4,$$ 

i.e., $a_p = \sqrt{p} - 4$, we also have $\epsilon_p > \sqrt{\frac{5}{4\sqrt{3}}}$ if $p > 5$. In fact, let $p > 5$ and

$$p \equiv 1 \pmod{4}, \epsilon_p > \sqrt{\frac{5}{4\sqrt{3}}}$$ 

and only if $\frac{\sqrt{5} - 1}{2} > \left(\sqrt{\frac{5}{4\sqrt{3}}} - \frac{1}{p}\right) \sqrt{p}$, which is equivalent to

$$\sqrt{\frac{5}{4\sqrt{3}} - \frac{5}{4\sqrt{3}}} \cdot \sqrt{p} > \frac{1}{p},$$

while we have inequalities

$$\sqrt{\frac{5}{4\sqrt{3}}} - \frac{5}{4\sqrt{3}} > 0.12 > \frac{1}{13} \geq \frac{1}{p} \quad \text{when} \quad p \geq 13,$$

which is a contradiction for $p \geq 13$. Then $p = 5$.

If $p = 5$ and $d_{\text{dim}(C)} = 2$, then $\mathbb{Q}(\text{dim}(C)) = \mathbb{Q}(\sqrt{5})$. Since $C$ is not a transitive modular fusion category, $5 | \text{FPdim}(C)$ by [25, Theorem 4.4]. Meanwhile, [25, Proposition 4.2] implies $N(\text{FPdim}(C)) = 25$. Therefore, $\text{FPdim}(C) = 5\epsilon_5^n$, where $n$ is a positive even integer. Let $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ such that $\sigma(\sqrt{5}) = -\sqrt{5}$, notice that $\text{FPdim}(C)$ and $\sigma(\text{FPdim}(C))$ are formal codegrees of $C$, so $\sigma(\text{FPdim}(C)) > \sqrt{\frac{5}{4\sqrt{3}}}$ by [22, Theorem 4.2.1], which implies $n = 2$ and $\text{FPdim}(C) = \frac{15 + 2\sqrt{5}}{2}$. Since $\text{rank}(C) \leq 4$ by equation (3), by using the argument of [22] Example 5.1.2(v), we obtain that $\mathcal{C} \cong \mathcal{C}(\mathfrak{sl}_2, 3)_{\text{ad}} \boxtimes \mathcal{C}(\mathfrak{sl}_2, 3)_{\text{ad}}$ or $\mathcal{C} \cong \mathcal{C}(\mathfrak{sl}_2, 5)_{\eta} \boxtimes \mathcal{C}(\mathfrak{sl}_2, 3)_{\text{ad}}$ as modular fusion category.

**Remark 3.4.** It follows from the proof of Lemma 3.3 and [22] Example 5.1.2] that a non-pointed modular fusion category $C$ is Grothendieck equivalent to $\mathcal{C}(\mathfrak{sl}_2, 8)_{\eta}$ if and only if $\text{dim}(C) = 5\epsilon_5^n$, where $m = 0, \pm 2$.

In summary, Corollary 3.2 and Lemma 5.3 imply the following theorem:

**Theorem 3.5.** Let $C$ be a modular fusion category with $N(\text{dim}(C)) = p^2$, where $p$ is a prime. Then either $\text{dim}(C) = p^2$, or $p = 5, 7$. Moreover, if $\text{dim}(C) \neq p^2$, then

$$\mathcal{C} \cong \begin{cases} 
\mathcal{C}(\mathfrak{sl}_2, 3)_{\text{ad}} \boxtimes \mathcal{C}(\mathfrak{sl}_2, 3)_{\text{ad}} & \text{if } p = 5; \\
\mathcal{C}(\mathfrak{sl}_2, 5)_{\eta} & \text{if } p = 7.
\end{cases}$$

as modular fusion category, where $\sigma \in \text{Gal}(\mathbb{Q}(\eta_5)/\mathbb{Q})$ and $\tau \in \text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$.

Let $C$ be a spherical fusion category of global dimension $p$, where $p$ is a prime. Then for any formal codegree $f$ of $C$, it follows from [25] Lemma 5.1 that $f = p \cdot u_f$, where $u_f$ is an algebraic unit in the field $\mathbb{Q}(\zeta_p)^+$. By using the argument of [25] Proposition 5.2, thus $d_f := [\mathbb{Q}(f) : \mathbb{Q}]$ divides $\frac{p - 1}{2}$.

**Proposition 3.6.** Assume that $C$ is not pointed. Then $2 < d_{\text{FPdim}(C)} < \frac{p - 1}{2}$ if $p > 5$.

**Proof.** Spherical fusion categories of integer dimensions less or equal to 5 are classified in [22] Example 5.2.2], so we assume $p > 5$ below. If $d_{\text{FPdim}(C)} = 2$, then $u_{d_{\text{FPdim}(C)}} = \epsilon_5^n$ [11, Theorem 11.5.1], where $n \neq 0$ is an even integer, as $C$ is not pointed and $u_{d_{\text{FPdim}(C)}}$ is a totally positive algebraic unit. Thus $\text{FPdim}(C) = p\epsilon_5^n$, while $\text{FPdim}(C) > \sqrt{\frac{5}{3}}$ and $\sigma(\text{FPdim}(C)) > \sqrt{\frac{5}{3}}$ [22, Theorem 4.2.1], where $\langle \sigma \rangle = \mathbb{Q}(\epsilon_p)/\mathbb{Q}$. By using the same argument of Lemma 5.3 we see $p \leq 11$. However, spherical fusion categories of global dimensions 7, 11 are pointed [25, Theorem 5.5], it is a contradiction. Hence $d_{\text{FPdim}(C)} > 2$ if $p > 5$. 

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If \( d_{\FPdim(C)} = \frac{p-1}{2} \), then \([21]\) Corollary 2.15 says
\[
\mathbb{Q}(\FPdim(C)) = \mathbb{Q}(\zeta_p)^+ = \mathbb{Q}(\dim(X), X \in \mathcal{O}(C)).
\]
That is, each of the homomorphisms \( \dim(-) \) and \( \FPdim(-) \) has \( \frac{p-1}{2} \) Galois conjugates. Since \( \mathcal{C} \) is not a pointed fusion category, \( \rank(C) < p \) by \([22]\) Remark 4.2.3, so \( \rank(C) = p - 1 \) and the Grothendieck ring \( \Gr(C) \) is commutative. Note that
\[
\sum_{\sigma \in \Gal(\mathbb{Q}(\zeta_p)^+)/\mathbb{Q}} \frac{1}{\sigma(\dim(C))} + \frac{1}{\sigma(\FPdim(C))} = 1,
\]
by \([20]\) Proposition 2.10, then we have
\[
\sum_{\sigma \in \Gal(\mathbb{Q}(\zeta_p)^+)/\mathbb{Q}} \frac{1}{\sigma(\FPdim(C))} = \frac{p+1}{2}.
\]
While \( \frac{p+1}{2} \neq 1 \) is a totally positive algebraic unit and \( \mathbb{Q}(\zeta_p)^+ \neq \mathbb{Q}(\sqrt{5}) \), the Siegel’s trace theorem \([26]\) Theorem III shows
\[
\sum_{\sigma \in \Gal(\mathbb{Q}(\zeta_p)^+)/\mathbb{Q}} \frac{1}{\sigma(\FPdim(C))} = \text{tr}(\frac{1}{\FPdim(C)}) \geq \frac{p-1}{2} - \frac{3}{2}.
\]
Thus we obtain \( \frac{p+1}{2} > \frac{p-1}{2} + \frac{3}{2} \), which then implies \( p < 5 \), it is impossible.
\( \square \)

**Remark 3.7.** In fact, let \( f_X \) be an arbitrary formal codegree of \( \mathcal{C} \), where \( \mathcal{C} \) is a spherical fusion category of global dimension \( p \) with \( p \neq 5 \). If \( f_X \notin \mathbb{Z} \), then the method of the above proposition says that \( 2 < [\mathbb{Q}(f_X) : \mathbb{Q}] < \frac{p-1}{2} \).

### 3.2 Modular fusion category \( \mathcal{C} \) with \( N(\dim(C)) = p^3 \)

Let \( \mathcal{C} \) be a modular fusion category such that \( N(\dim(C)) = p^3 \) and \( \dim(C) \notin \mathbb{Z} \). Then \( d_{\dim(C)} \leq 6 \) by \([25]\) Proposition 4.11, moreover Lemma 3.11 says \( d_{\dim(C)} = 2, 3 \). Hence, we know either 4 or 6 divides \( p - 1 \). In particular, there does not exist modular fusion categories such that \( N(\dim(C)) = p^3 \) and \( d_{\dim(C)} > 1 \) when \( (p - 1, 6) = 2 \). The inequality (3) implies \( \rank(C) \leq p - 1 \), and we know there does not exist such modular fusion category when \( p = 2, 3 \) \([21]\), so we assume \( p \geq 5 \) below. Moreover,

**Lemma 3.8.** Let \( \mathcal{C} \) be a modular fusion category such that \( N(\dim(C)) = p^3 \) and \( d_{\dim(C)} > 1 \). Then \( \mathcal{C} \) is simple when \( p > 5 \).

**Proof.** Assume \( \mathcal{C} \cong \mathcal{D} \cong \mathcal{D}' \) where \( \mathcal{D} \) is a modular fusion subcategory of \( \mathcal{C} \), then the proof of Lemma 3.11 shows that either \( \mathcal{D} \) or \( \mathcal{D}' \) is transitive, as \( p \) can not divide both \( \dim(D) \) and \( \dim(D') \). Without loss of generality, assume that \( \mathcal{D} \) is braided equivalent to a Galois conjugate of \( \mathcal{C}(\mathfrak{s}_2, p^2)_{\text{ad}} \), then \( N(\dim(D)) = \frac{p^3}{2} \) and
\[
N_{\mathbb{Q}(\zeta_p)^+}(\dim(C)) = \begin{cases} \frac{3(p-1)}{2}, & \text{if } d_{\dim(C)} = 2; \\ \frac{p(p-1)}{2}, & \text{if } d_{\dim(C)} = 3. \end{cases}
\]
thus we obtain \( N_{\mathbb{Q}(\zeta_p)^+}(\dim(D)) = \frac{p(p+1)}{2} \) if \( d_{\dim(C)} = 2 \), or \( p \) if \( d_{\dim(C)} = 3 \). Obviously, in each case \( p \) can’t divide \( \dim(D') \) by \([25]\) Proposition 4.2, so \( \mathcal{D}' \) is also transitive by \([25]\) Theorem 4.4. However, there is a contradiction when \( p > 5 \). \( \square \)
Lemma 3.9. Let $C$ be a modular fusion category such that $N(\dim(C)) = 5^2$ and $d_{\dim(C)} \neq 1$. Then $\dim(C) = \frac{25 + 5\sqrt{5}}{2}$ or $25 \pm 10\sqrt{5}$, and $FPdim(C) = 25 + 10\sqrt{5}$ or $\frac{25 + 3\sqrt{5}}{2}$.

Proof. It is easy to see $\mathbb{Q}(\sqrt{5}) = \mathbb{Q}(\dim(C))$. Since $d_{\dim(C)} = 2$ and $N\left(\frac{\dim(C)}{5\sqrt{5}}\right) = -1$, $\dim(C) = 5\sqrt{5}e_5^n$ with an odd integer $m$, where $e_5 = \frac{\sqrt{5} + 1}{2}$ is the fundamental algebraic unit of $\mathbb{Q}(\sqrt{5})$. By [22] Proposition A.1.1, $\sigma(\dim(C)) > \frac{4\sqrt{5}}{5}$ and $\dim(C) > \frac{4\sqrt{5}}{5}$, where $\sigma(\sqrt{5}) = -\sqrt{5}$. Hence, if $m \leq -1$, then $\dim(C) = 5\sqrt{5}\left(\frac{\sqrt{5} - 1}{2}\right)^{-m} > \frac{4\sqrt{5}}{5}$, i.e., $\left(\frac{5\sqrt{5} - 1}{2}\right)^{-m} > \frac{4\sqrt{5} - 5}{25\sqrt{5}}$, thus $m = -1, -3$; if $m > 0$, then $\sigma(\dim(C)) = \frac{5\sqrt{5}}{5\sqrt{5}e_5^{-m}} > \frac{4\sqrt{5}}{5}$, so $m = 1, 3$. In summary, $m = \pm 1, \pm 3$.

Assume that $\dim(\mathcal{O})$ takes value in a subfield $\mathcal{F}$ of the totally real $\mathbb{Q}(\zeta_{2n})^+$ for some positive integer $n$ with $n$ being minimal. If $n > 1$, then $[\mathcal{F} : \mathbb{Q}] \geq 10$ as $\mathbb{Q}(\sqrt{5}) \subseteq \mathcal{F}$, for any $\sigma \in \text{Gal}(\mathcal{F}/\mathbb{Q})$, there exists a unique simple object $\sigma(I)$ such that $\sigma(\dim(C)) = \frac{\dim(C)}{\dim(I)^2}$, while $\sigma(\dim(C)) \in \left\{\frac{25 + 5\sqrt{5}}{2}\right\}$ or $\left\{25 \pm 10\sqrt{5}\right\}$, then $C$ has at least 5 formal codegrees equal to $\frac{25 + 5\sqrt{5}}{2}$ and $\frac{25 - 5\sqrt{5}}{2}$ if $\dim(C) = \frac{25 + 5\sqrt{5}}{2}$; however $5(1/\frac{25 + 5\sqrt{5}}{2}) + 1/\frac{25 - 5\sqrt{5}}{2} = 1$, so these are all formal codegrees of $C$, it is a contradiction as $n = 1$ in this case; if $\dim(C) = \frac{25 + 5\sqrt{5}}{2}$, then $5(1/(25 + 10\sqrt{5}) + 1/(25 - 10\sqrt{5})) = 2$, impossible. Thus, $n = 1$. Since $\frac{\dim(C)}{\dim(I)^2}$ is a totally positive algebraic integer that is less than or equal to $1$, we see $N(FPdim(C)) = 25, 125$ by Lemma [5] and [25] Proposition 4.2.

If $N(FPdim(C)) = 25$, then $FPdim(C) = 5\epsilon_5^2 = \frac{25 + 5\sqrt{5}}{2}$ and $C$ is braided equivalent to a Deligne product of two Fibonacci fusion categories by [22] Example 5.1.2; however, in this case $N(\dim(C)) \neq 125$. Therefore, $N(FPdim(C)) = 125$, then $FPdim(C) = \frac{25 + 5\sqrt{5}}{2}$ or $25 + 10\sqrt{5}$, as $FPdim(C)$ is the largest among the set of formal codegrees of $C$.

Proposition 3.10. Let $C$ be a modular fusion category such that $N(\dim(C)) = 5^2$ and $d_{\dim(C)} \neq 1$. Then as modular fusion category,

$$C \cong C(sl_2, 3)_{ad} \boxtimes C(sl_2, 3)_{ad} \boxtimes C(sl_2, 3)_{ad} \text{ or } C \cong C(Z_5, \eta) \boxtimes C(sl_2, 3)_{ad},$$

where $\sigma_i \in \text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ for all $1 \leq i \leq 4$.

Proof. By Lemma [5] we know $FPdim(C) = 25 + 10\sqrt{5}$ or $\frac{25 + 3\sqrt{5}}{2}$.

If $FPdim(C) = \frac{25 + 5\sqrt{5}}{2}$, then $\dim(C) = \frac{25 + 5\sqrt{5}}{2}$, so $C$ is conjugated to a quasi-unitary fusion category. Notice that $C$ contains a self-dual simple object $X = \hat{\sigma}(I)$ of FP-dimension $\frac{1 + \sqrt{5}}{2}$. $X \otimes X = I \oplus Z$ where $Z$ is also a self-dual simple object of FP-dimension $\frac{1 + \sqrt{5}}{2}$. If $Z = X$, then $C$ contains $C(sl_2, 3)_{ad}$ as a fusion subcategory, hence $C \cong C(Z_5, \eta) \boxtimes C(sl_2, 3)_{ad}$; if not, $X \otimes Z = X \oplus g$, where $g$ is a non-trivial invertible object, so $C_{fp}$ is non-trivial. While $FPdim(C_{fp})$ divides $FPdim(C)$ by [11] Proposition 8.15, which implies $FPdim(C_{fp}) = 5$ and $C_{fp} \cong C(Z_5, \eta)$. By [7] Theorem 3.13, we have

$$C \cong C(Z_5, \eta) \boxtimes C(sl_2, 3)_{ad} \text{ if } \dim(C) = \frac{25 + 5\sqrt{5}}{2},$$

$$C \cong C(Z_5, \eta) \boxtimes C(sl_2, 3)_{ad} \text{ if } \dim(C) = \frac{25 + 3\sqrt{5}}{2}.$$
If $\text{FPdim}(C) = 25 + 10\sqrt{5}$, then $C_{pt} = \text{Vec}$. In fact, if $C_{pt} \neq \text{Vec}$, then $C_{pt} \cong C(\mathbb{Z}_5, \eta)$ and $C \cong C(\mathbb{Z}_5, \eta) \boxtimes D$ with $\dim(D) = 5 + 2\sqrt{5}$. However, note that $5 - 2\sqrt{5}$ is a formal codegree of $D$, which contradicts to \[\begin{array}{l}
\text{Theorem 1.1.2}\end{array}\]. If $\dim(C) = 25 + 10\sqrt{5}$, by using the same argument as above, we know $C$ contains a Galois conjugate of $C(sl_2, 3)_{ad}$ as a modular fusion subcategory. Consequently, $C \cong C(sl_2, 3)_{ad} \boxtimes D$ with $D$ being a modular fusion subcategory and $\sqrt{N(\dim(D))} = 5^2$, where $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$, and Lemma \[\begin{array}{l}
3.3\end{array}\] implies
\[
C \cong C(sl_2, 3)_{ad} \boxtimes C(sl_2, 3)_{ad} \boxtimes C(sl_2, 3)_{ad} \text{ if } \dim(C) = 25 + 10\sqrt{5},
C \cong C(sl_2, 3)_{ad} \boxtimes C(sl_2, 3)_{ad} \otimes C(sl_2, 3)_{ad} \text{ if } \dim(C) = 25 - 10\sqrt{5}.
\]

Notice that if $\dim(C) = \frac{25 + 5\sqrt{5}}{2}$, then $\dim(C') = \frac{25 + 5\sqrt{5}}{2}$, hence it suffices to consider when $\dim(C) = \frac{25 + 5\sqrt{5}}{2}$ below. We know that $C$ contains a self-dual simple object $X = \hat{s}(I)$ with $\dim(X)^2 = \frac{25 + 5\sqrt{5}}{2}$, simple objects $Y$ and $Z = s(Y)$ such that $\dim(Y)^2 = \frac{\dim(C)}{\text{FPdim}(C)} = \frac{25 + 5\sqrt{5}}{2}$ and $\dim(Z)^2 = \frac{\dim(C)}{\text{FPdim}(C)} = \frac{25 + 5\sqrt{5}}{2}$. Let $f$ be another formal codegree of $C$, then $N(f) = 25$ or $125$. If $N(f) = 25$, then $f$ is a root of $x^2 - ax + 25 = 0$ with $a^2 \geq 100$, we obtain $f = 5$ by equation \[\begin{array}{l}
1\end{array}\], thus $C$ contains a simple object $V$ with $\dim(V)^2 = \frac{25 + 5\sqrt{5}}{2}$, which is impossible as $\dim(V) \notin \mathbb{Q}(\zeta_5)$. Thus, $N(f) = 125$ for all other formal codegrees of $C$, a direct computation shows $\mathcal{O}(C) = \{I, X, Y, Z, V_1, V_2, W_1, W_2\}$ with $\dim(V_1)^2 = \dim(V_2)^2 = 1$ and $\dim(W_1)^2 = \dim(W_2)^2 = \frac{25 + 5\sqrt{5}}{2}$.

By decomposing $\text{FPdim}(C)$ into the sum of squares of Frobenius-Perron dimensions of eight simple objects over field $\mathbb{Q}(\sqrt{5})$, we find that $C$ always contains a simple object of Frobenius-Perron dimension $\frac{25 + 5\sqrt{5}}{2}$, therefore, $C \cong C(sl_2, 3)_{ad} \boxtimes D$ as modular fusion subcategory by \[\begin{array}{l}
7\end{array}\] Theorem 3.13]. Thus, again by Lemma \[\begin{array}{l}
3.3\end{array}\]
\[
C \cong C(sl_2, 3)_{ad} \boxtimes C(sl_2, 3)_{ad} \boxtimes C(sl_2, 3)_{ad} \text{ if } \dim(C) = \frac{25 + 5\sqrt{5}}{2},
C \cong C(sl_2, 3)_{ad} \boxtimes C(sl_2, 3)_{ad} \otimes C(sl_2, 3)_{ad} \text{ if } \dim(C) = \frac{25 - 5\sqrt{5}}{2},
\]
$\sigma \in \text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ such that $\sigma(\sqrt{5}) = -\sqrt{5}$, this completes the proof of the proposition.

Let $C$ be a modular fusion category. Let $p \geq 5$, and let $\rho_1$ be a $\frac{p+1}{2}$-dimensional irreducible representation of $\text{SL}(2, \mathbb{Z})$ with level equal to $p$, it is proved in \[\begin{array}{l}
17\end{array}\] Proposition 3.22] that if the associated modular representation $\rho_C$ is equivalent to $m\rho_0 \oplus \rho_1$, then $m = 1$.

We strengthen the above conclusion in the following proposition and theorem.

**Proposition 3.11.** Let $p \geq 7$ be a prime, and let $C$ be a modular fusion category such that $\rho_C \cong \rho_0 \oplus \rho_1$, where $\rho_1$ is the $\frac{p+1}{2}$-dimensional irreducible representation of $\text{SL}(2, \mathbb{Z})$ of level $p$. Then $C$ is a Galois conjugate of a pseudo-unitary fusion category.

**Proof.** Let $\mathcal{O}(C) = \{X_0, X_1, \cdots, X_d\}$ and $S = (S_{ij})$ be the un-normalized $S$-matrix of $C$. Let $d := \frac{p+1}{2}, \ p^* = \left(\frac{-1}{p}\right) p$ and $\beta := \left(\frac{a}{p}\right) / \sqrt{p}$, where $a$ is an integer coprime to $p$. By \[\begin{array}{l}
17\end{array}\] Proposition 3.22] there exists an real orthogonal matrix $U = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ such that
\[
\rho_C(s) = \frac{1}{\sqrt{\dim(C)}} S = U \begin{pmatrix} 1 & 0 \\ 0 & \rho_1(s) \end{pmatrix} U^T, \rho_C(t) = \text{diag}(1, \zeta_p^a, \cdots, \zeta_p^{(d-1)a}),
\]
Corollary 3.5.8. However, in this case
Notice that
\[ \rho_1(g) = \beta \begin{pmatrix}
1 & \sqrt{2} & \cdots & \sqrt{2} \\
\sqrt{2} & 2 \cos \left( \frac{4\pi a jk}{p} \right) & \cdots & \sqrt{2} \\
\vdots & \vdots & \ddots & \vdots \\
\sqrt{2} & \sqrt{2} & \cdots & \sqrt{2}
\end{pmatrix}, \]
for \(1 \leq j, k, l \leq d - 1\), \(A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\),
\[ A_2 = \text{diag}(\lambda_1, \ldots, \lambda_{d-1}), \] where \(\lambda_i \in \{\pm 1\}\). Moreover, \(\sqrt{2}a_{12}, \sqrt{2}a_{22} \in \mathbb{Q} \), \(0 < a_{12}^2 < 1\) and \(0 < a_{22}^2 < 1\).

Up to isomorphism, there are exactly two \(\frac{p+1}{4}\)-dimensional irreducible representations of level \(p\) [8, 19] depending on the value \(\left( \frac{2}{p} \right)\). We assume \(a = 1\) below, the other case is the same.

A direct computation shows
\[ \rho_C(g) = \begin{pmatrix}
1 + a_{12}^2(\beta - 1) & a_{12}a_{22}(\beta - 1) & \sqrt{2}\beta a_{12} \lambda_1 & \cdots & \sqrt{2}\beta a_{12} \lambda_{d-1} \\
\sqrt{2} \beta a_{12} \lambda_1 & 1 + a_{22}^2(\beta - 1) & \sqrt{2} \beta a_{22} \lambda_1 & \cdots & \sqrt{2} \beta a_{22} \lambda_{d-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\sqrt{2} \beta a_{12} \lambda_{d-1} & \sqrt{2} \beta a_{22} \lambda_{d-1} & \cdots & 2 \beta \lambda_j \lambda_k \cos \left( \frac{4\pi a jk}{p} \right)
\end{pmatrix}, \]
for \(1 \leq j, k \leq d - 1\). We show that either \(X_0\) or \(X_1\) don’t represent the unit object \(I\).

On the contrary, without loss of generality, let \(X_0 = I\), then \(\frac{1}{\sqrt{\dim(C)}} = |1 + (\beta - 1) a_{12}^2|\).
Notice that
\[ \beta = \left( \frac{1}{p} \right) / \sqrt{p} = \left\{ \begin{array}{ll}
\frac{1}{p}, & p \equiv 1 \mod 4; \\
\frac{-1}{p}, & p \equiv 3 \mod 4.
\end{array} \right. \]
then \(|1 + (\beta - 1) a_{12}^2|^2 = \left\{ \begin{array}{ll}
1 - 2a_{12}^2 + \frac{p+1}{p} a_{12}^4 + 2(a_{12}^2 - a_{12}^4) \sqrt{p}, & p \equiv 1 \mod 4; \\
1 - 2a_{12}^2 + \frac{p+1}{p} a_{12}^4, & p \equiv 3 \mod 4.
\end{array} \right. \]
and
\[ |\beta - 1|^2 = \left\{ \begin{array}{ll}
\frac{p+1-2\sqrt{p}}{p}, & p \equiv 1 \mod 4; \\
\frac{p+1}{p}, & p \equiv 3 \mod 4.
\end{array} \right. \]
When \(p \equiv 3 \mod 4\), the set of formal codegrees of \(C\) is
\[ \frac{\dim(C)}{(\sqrt{\dim(C)}|S_{1,1}|)^2} \in \left\{ \dim(C), \frac{1}{a_{12}^2 a_{22}^2 |\beta - 1|^2}, \frac{p}{\sqrt{2} a_{12} \lambda_j} \right\}, 1 \leq j \leq d, \]
so the formal codegrees of \(C\) are all rational, which implies they are integers. In particular, \(\text{FPdim}(C) = p^m\) by Cauchy’s theorem [3, Theorem 3.9], hence \(C\) must be integral as \(p\) is odd [10 Corollary 3.5.8]. However, in this case \(C^p\) must be non-trivial, so \(\text{rank}(C) \geq p\), it is impossible.

When \(p \equiv 1 \mod 4\), \(\frac{1}{\sqrt{\dim(C)}|S_{1,1}|^2} = 1 - 2a_{12}^2 + \frac{p+1}{p} a_{12}^4 + 2(a_{12}^2 - a_{12}^4) \sqrt{p} \notin \mathbb{Z}\), again the Cauchy’s theorem [3, Theorem 3.9] implies that \(N(\dim(C)) = p^m\) for \(m \geq 2\). Let \(\sigma \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})\) be a generator, then \(\sigma\) has no invariant simple objects [17, Proposition 3.22]. Hence
\[ \sigma(\dim(C)) = \frac{\dim(C)}{\dim(X_2)^2} = \frac{1}{a_{12}^2 a_{22}^2 |\beta - 1|^2} = \frac{p}{a_{12}^2 a_{22}^2} \left( \frac{p+1+2\sqrt{p}}{(p+1)^2} \right). \]
Therefore, \(N(\dim(C)) = \dim(C) \sigma(\dim(C)) = \frac{p^2}{(p-1)^2 a_{12}^2 a_{22}^2}\), that is, \(a_{12}^2 a_{22}^2 (p-1)^2 = p^m\) with \(2 \mid m\) and \(m \geq 0\), since \(a_{12}^2 a_{22}^2 \in \mathbb{Q}\). Assume \(m = 2l\), so \(a_{12}^2 a_{22}^2 = \frac{1}{p(p-1)^2}; \) meanwhile
\[
a_1^2 + a_2^2 = 1, \text{ so } a_1^2 \text{ and } a_2^2 \text{ are roots of equation } x^2 - x + \frac{1}{p^3(p-1)} = 0, \text{ consequently}
\]
\[
a_{12}^2, a_{22}^2 = \frac{\sqrt{(p-1)p^l} \pm \sqrt{(p-1)p^l - 4}}{2\sqrt{(p-1)p^l}}.
\]
Since \( p \geq 7 \) and \( p \equiv 1 \mod 4 \), \((p-1)p^l - 4, (p-1)p^l) = 4\); as \( a_{12}^2, a_{22}^2 \) are rational, we have \((p-1)p^l - 4 = q_1^2 \) and \((p-1)p^l = q_2^2 \) for nonnegative integers \( q_1, q_2 \). Note that \((q_2 - q_1)(q_2 + q_1) = 4\), then \( l = 0 \) and \( p = 5 \), it is a contradiction.

Therefore, the unit object \( I \in \{X_2, \ldots, X_d\} \). Meanwhile, the Galois symmetry \([6] \text{ Theorem II} \) implies that the unit object \( I \) has exactly \( \frac{p-1}{2} \) Galois conjugates, so \( \mathcal{O}_I(C) = \{X_2, \ldots, X_d\} \).

If the simple object \( X_1 \) or \( X_2 \) determines the homomorphism \( \text{FPdim}(-) \), then \( \text{FPdim}(C) = \frac{\dim(C)}{\dim(X_1)} \text{ or } \text{FPdim}(C) = \frac{\dim(C)}{\dim(X_2)} \). Note that \( \text{FPdim}(C) \in \mathbb{Q} \) in both cases, again \( C \) must be an integral fusion category, so \( \text{rank}(C) \geq p \), which is absurd. Therefore, the simple object which determines the homomorphism \( \text{FPdim}(-) \) belongs to \( \mathcal{O}_I(C) \), so \( C \) is Galois conjugate to a pseudo-unitary fusion category.

\[\Box\]

**Theorem 3.12.** Let \( p \geq 5 \) be a prime and \( C \) a modular fusion category such that \( \rho_C \cong \rho_0 \oplus \rho_1 \), where \( \rho_1 \) is the \( d \)-dimensional irreducible representation of \( SL(2, \mathbb{Z}) \) of level \( p \). Then \( C \) is Grothendieck equivalent to \( C(sl_2, (2p-1))_1^0 \). Moreover, \( \dim(C) = p \cdot \text{u} \) where \( \text{u} \) is a Galois conjugate of algebraic unit \( 4 \cos^2(\frac{d\pi}{p}) \).

**Proof.** By Proposition 3.11 we know that \( C \) is a Galois conjugate of a pseudo-unitary fusion category if \( p \geq 7 \). Without loss of generality, we assume that \( C \) is pseudo-unitary. As we have proved in Proposition 3.11 that \( X_0, X_1 \not\in \mathcal{O}_I(C) \), and \( \rho(s) = \frac{1}{\sqrt{\dim(C)}} S \).

\[
\dim(C) = \frac{1}{\vert S_{X_0, X_0} \vert^2} \text{ for some } 2 \leq k \leq d.
\]
Meanwhile, \( \text{FPdim}(C) \) is the largest among its Galois conjugates, we see \( \dim(C) = \frac{p}{4 \cos^2(\frac{d\pi}{p})} \).

In addition, if \( p = 5 \), then the argument of Proposition 3.11 also shows that \( \dim(C) = 5 \epsilon_2^2 \) if the unit object \( I \in \{X_0, X_1\} \) and that \( C \) is a Galois conjugate of a pseudo-unitary fusion category if \( I \in \{X_2, X_3\} \). Hence, in both cases, we see that \( C \) is braided tensor equivalent to a Galois conjugate of \( C(sl_2, 8)_1^0 \) by Lemma 3.3.

We assume \( p \geq 7 \) below, then \( C \) is a simple modular fusion category. Indeed, if \( C \) is not simple, then fusion subcategories of \( C \) are modular by [27] Theorem 3.1. Let \( D \) be an arbitrary non-trivial modular fusion subcategory of \( C \), then \( C \cong D \boxtimes D' \) by [17] Theorem 3.13. As \( p \) can’t divide both \( \dim(D) \) and \( \dim(D') \), therefore \( D \) and \( D' \) are transitive modular fusion categories. By [16], we know \( D \cong C(sl_2, p - 2)_{\sigma_1}^{\sigma_2} \) and \( D' \cong C(sl_2, p - 2)_{\sigma_2}^{\sigma_1} \) for \( \sigma_1, \sigma_2 \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \).

However, \( \dim(C(sl_2, p - 2)_{\sigma_1}) = \frac{6}{\sqrt{p^3(p-1)}} \) and
\[
\dim(C) = \dim(D) \text{ dim}(D') = \sigma_1(\dim(C(sl_2, p - 2)_{\sigma_1})) \sigma_1(\dim(C(sl_2, p - 2)_{\sigma_1})),
\]
so \( \frac{\nu_{ad}}{p^{\nu_{ad}}} = N(\dim(C)) = N(\dim(D)) N(\dim(D')) = p^{\nu_{ad} - 3} \), i.e., \( p = 5 \), it is impossible.

A direct computation shows that the set of formal codimensions of \( C \) is
\[
\left\{ \frac{\dim(C)}{\sqrt{\dim(C) S_{X_0, X_0}^2}}, \frac{\dim(C)}{\sqrt{\dim(C) S_{X_0, X_1}^2}} \sigma(\dim(C)), \forall \sigma \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \right\},
\]

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while norms of \( \frac{\dim(C)}{(\sqrt{\dim(C)})} \) and \( \frac{\dim(C)}{(\sqrt{\dim(C)})} \) divide that of \( \dim(C) \), which is power of \( p \), consequently \( (\sqrt{2a_{12}})^2 = \frac{1}{p^{m_1}} \) and \( (\sqrt{2a_{22}})^2 = \frac{1}{p^{m_2}} \) for nonnegative integers \( m_1, m_2 \), while \( (\sqrt{2a_{12}})^2 + (\sqrt{2a_{22}})^2 = 2 \), so \( a_{12} = \frac{m_1}{2} \) and \( a_{22} = \frac{m_2}{2} \). Hence, we obtain that

\[
    \rho_C(s) = \begin{pmatrix}
        \frac{\beta_{1\lambda_1}}{2} & \frac{\beta_{1\lambda_{d-1}}}{2} & \cdots & \frac{\beta_1\lambda_1}{2} & \cdots & \beta_{1\lambda_{d-1}} \\
        \beta_{1\lambda_1} & \beta_{2\lambda_1} & \cdots & \beta_{1\lambda_1} & \cdots & \beta_{2\lambda_{d-1}} \\
        \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
        \frac{\beta_{1\lambda_{d-1}}}{2} & \frac{\beta_{2\lambda_{d-1}}}{2} & \cdots & \beta_{2\lambda_1} & \cdots & \beta_{2\lambda_{d-1}} \\
    \end{pmatrix}, 1 \leq j, k \leq d - 1.
\]

We choose \(-a = \frac{1}{2} \mod p \), when \(-a = j = k = \frac{1}{2} \),

\[
    |2\beta_{l_j}\lambda_k \cos\left(\frac{4\pi a_k}{p}\right)|^2 = \frac{4 \cos^2\left(\frac{(d-1)x}{p}\right)}{\dim(C)},
\]

therefore \( X_d \) represents the isomorphism class of the unit object.

Consequently, the set of the Frobenius-Perron dimensions of simple objects of \( C \) is:

\[
    \left\{ \frac{\mu_1\lambda_{d-1}}{2 \cos\left(\frac{(d-1)x}{p}\right)} - \frac{\mu_2\lambda_{d-1}}{2 \cos\left(\frac{(d-1)x}{p}\right)} \right\}, 1 \leq j \leq d - 1,
\]

since the Frobenius-Perron dimensions of simple objects are positive, \( \mu_1 = \mu_2 = \lambda_j \) for all \( 1 \leq j \leq d - 1 \). By comparing the Frobenius-Perron dimensions of simple objects of \( C \) and \( C(s_{12}, 2(p - 1))_{\lambda} \) (see Example 2.2), let \( \phi : \text{Gr}(C(s_{12}, 2(p - 1))_{\lambda}) \rightarrow \text{Gr}(C) \) be a morphism such that \( \phi(V_{p-1}^\pm) = X_0, X_1 \) and \( \phi(V_{2(d-j)}^\pm) = X_j, 1 \leq j \leq d \), respectively. Notice that \( C \) and \( \text{Gr}(C) \) share the same modular data and \( \phi \) preserves the Frobenius-Perron dimensions of simple objects, hence \( \phi \) is an isomorphism of fusion ring by the Verlinde formula.

\[\Box\]

**Corollary 3.13.** Let \( C \) be a modular fusion category such that \( \dim(C) = p \cdot u \), where \( u \neq 1 \) and \( u \in \mathbb{Q}(\zeta_n)^+ \) is a totally positive algebraic unit. Then \( C \) is Grothendieck equivalent to \( C(s_{12}, 2(p - 1))_{\lambda} \).

**Proof.** Let \( D := \mathbb{E}_{\sigma \in \text{Gal}(\mathbb{Q}(u)/\mathbb{Q})}[^\sigma C] \). Since \( C \) is not pointed, the inequality \( \boxed{1} \) and \( \boxed{2} \) Remark 4.2.3 imply \( \text{rank}(C) \leq p - 1 \), so we can assume \( p \geq 5 \). If \( \mathbb{Q}(u) : \mathbb{Q} = 2 \), then \( N(\dim(C)) = p^2 \), \( p = 5 \) and \( \dim(C) = \frac{5^{[3x_2+5]}_2}{2} \) by Lemma \( \boxed{4} \) Assume \( \mathbb{Q}(u) : \mathbb{Q} > 2 \) and \( p > 5 \) below. By Remark \( \boxed{2} \) we know \( \text{ord}(t_{\mathbb{G}}) = p \) and at least one of the Galois orbits of simple objects of \( C \) contains \( \frac{u-1}{2} \) simple objects. Meanwhile \( C \) can not be a transitive modular fusion category \( \boxed{16} \) Theorem 1, therefore, \( \text{rank}(C) > \frac{p-1}{2} \).

Assume \( \rho_C \) is the modular representation associated to \( C \), then \( \rho_C \) is a direct sum of level \( p \) irreducible representations of \( \text{SL}(2, \mathbb{Z}) \). However, \( \rho_C \) can’t be decomposed as direct sum of irreducible representation of disjoint t-spectrum \( \boxed{4} \) Lemma 3.18\] and \( \rho_C \) also can’t be decomposed into direct sum of one-dimensional representations \( \boxed{9} \) Lemma 4. Therefore, by comparing the dimensions of level \( p \) irreducible representations of \( \text{SL}(2, \mathbb{Z}) \), we obtain that either \( \rho_C \) is an irreducible representation of dimension \( p - 1 \) or \( \frac{p+1}{2}, \) or \( \rho \) is a direct sum of two irreducible representations.
of dimension $\frac{p-1}{2}$, or $\rho$ is a direct sum of one-dimensional representations and an irreducible representation of dimension $\frac{p+1}{2}$.

If $\rho_C \cong \rho_1 \oplus \rho_2$ with $\dim(\rho_1) = \dim(\rho_2) = \frac{p-1}{2}$, since the $t$-spectrums of $\rho_1$ and $\rho_2$ intersect non-trivially [4 Lemma 3.18], $\rho_1 = \rho_2$, which is impossible by [23 Lemma 5.2.2]. And the irreducible representations of dimension $\frac{p+1}{2}$ can’t be realized as representations of modular fusion categories [8]. Therefore, $\rho_C$ is an irreducible representation of dimension $p - 1$ or $\rho_C = \rho_0 \oplus \rho_1$ by [17 Lemma 3.20, Proposition 3.22], where $\rho_0$ is the trivial representation and $\rho_1$ is the irreducible representation of dimension $\frac{p+1}{2}$.

If $\rho_C$ is irreducible, then $\text{rank}(C) = p - 1$. In particular, the Galois symmetry [6 Theorem II] implies that $\mathcal{O}(C)$ splits into two orbits and each Galois orbit have exactly $\frac{p-1}{2}$ simple objects. Notice that all formal codegrees of $C$ are divided by $p$. Indeed, if not, then $C$ is a transitive modular fusion category [25 Theorem 4.4] and

$$C \cong C(sl_2, p - 2)^u$$

for some $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$, however, $\text{rank}(C(sl_2, p - 2)) = \frac{p-1}{2}$, it is a contradiction. Meanwhile, $p^2 \nmid f$ for any formal codegree $f$ of $C$, otherwise, $p^2 \mid \dim(C)$, which implies $p \mid u$, it is impossible by [25 Proposition 4.2]. Thus, for any formal codegree $f$ of $C$, we have $f = p \cdot u_f$, where $u_f \in \mathbb{Q}(\dim(X), X \in \mathcal{O}(C))$ is a totally positive algebraic integer as $f \mid \dim(C)$. Since $f = \frac{\dim(C)}{\dim(X)}$, for some simple object $X \in \mathcal{O}(C)$, we see $u_X := u_f = \frac{u}{\dim(X)}$.

By definition,

$$p \cdot u = \dim(C) = \sum_{X \in \mathcal{O}(C)} \dim(X)^2 = \sum_{X \in \mathcal{O}(C)} \frac{u}{u_X},$$

and for any $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_p)^+/\mathbb{Q})$, $\sigma \left( \frac{\dim(C)}{\dim(X)^2} = \frac{\dim(C)}{\dim(\sigma(X))^2} \right)$, therefore,

$$\dim(\sigma(X))^2 = \frac{u \sigma(\dim(X))^2}{\sigma(u)} = \frac{u}{\sigma(u_X)}$$

and $\sigma(u_X) = \frac{u}{\dim(\sigma(X))^2} = u_{\sigma}(X)$.

Thus, we have the following equation

$$\sum_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta_p)^+/\mathbb{Q})} \dim(\sigma(X))^2 = \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta_p)^+/\mathbb{Q})} \frac{u}{\sigma(u_X)}.$$

Hence, let $\mathcal{O}(C) = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 := \{ \sigma(X_i) | \sigma \in \text{Gal}(\mathbb{Q}(u_{X_i})/\mathbb{Q}) \}$, then

$$p = \sum_{i=1}^{2} \sum_{\sigma \in \text{Gal}(\mathbb{Q}(u_{X_i})/\mathbb{Q})} \frac{1}{\sigma(u_{X_i})}.$$

Since $u_f = u \neq 1$, the Siegel’ trace theorem [26] says that

$$\sum_{\sigma \in \text{Gal}(\mathbb{Q}(u_{X_i})/\mathbb{Q})} \frac{1}{\sigma(u_{X_i})} > 1.79 \cdot \frac{p - 1}{2},$$

except for roots of the equation $x^3 - 5x^2 + 6x - 1 = 0$ when $p = 7$ [13 Theorem 1.1]. If $u_{X_1} = 1$, then $\sum_{\sigma \in \text{Gal}(\mathbb{Q}(u_{X_1})/\mathbb{Q})} \frac{1}{\sigma(u_{X_1})} = \frac{p+1}{2}$. Hence,

$$p \geq \frac{5}{3} \cdot \frac{p - 1}{2} + \frac{p - 1}{2},$$

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that is, \( p \leq 3 \), it is impossible.

When \( pC = \rho_0 \oplus \rho_1 \), we deduce from Theorem 4.12 that \( C \) is Grothendieck equivalent to the modular fusion category \( \mathcal{C}(\mathfrak{sl}_2, (2p - 1))_\Lambda \). This finishes the proof of the corollary.

**Corollary 3.14.** Let \( C \) be a modular fusion category such that \( N(\dim(C)) = p^3 \) and \( d_{\dim(C)} = 3 \). Then \( p = 7 \) and \( C \) is Grothendieck equivalent to \( \mathcal{C}(\mathfrak{sl}_2, 12)_\Lambda \).

## 4 Modular fusion category of global dimension \( p^2 \)

In this section, we always assume \( p \) is a prime and we study the structure of modular fusion categories \( C \) of global dimension \( p^2 \).

**Proposition 4.1.** Let \( C \) be a pre-modular fusion category of global dimension \( p^2 \). If \( C' \not\cong \text{Vec} \), then either \( C \) is pointed or \( C \cong \mathcal{C}(\mathfrak{sl}_2, 3)_{ad} \boxtimes \mathcal{C}(\mathfrak{sl}_2, 3)_{ad}'' \boxtimes \text{Rep}(\mathbb{Z}_3) \), where \( \sigma \in \text{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q}) \) such that \( \sigma(\zeta_3) = \zeta_3^2 \).

**Proof.** Since \( C' \not\cong \text{Vec} \), \( \dim(C') = p, p^2 \) by [22] Theorem 3.1. If \( \dim(C') = p^2 \), then \( C \) is symmetric, so it is pointed [11] Proposition 8.32. If \( \dim(C') = p \), then \( C' = \text{Rep}(\mathbb{Z}_p) \) is Tannakian, or \( C' \equiv \text{Vec} \) and \( p = 2 \). When \( p = 2 \), it is obviously \( C \) is pointed [22] Example 5.1.2. If \( C' = \text{Rep}(\mathbb{Z}_p) \), then \( C_{zp} \) is a modular fusion category of dimension \( p \), hence \( C_{zp} \) is pointed or \( C_{zp} \cong C(\mathfrak{sl}_2, 3)_{ad} \boxtimes C(\mathfrak{sl}_2, 3)_{ad}'' \) by [25] Theorem 5.12. Therefore, \( C \) is a pointed fusion category or \( C \cong C(\mathfrak{sl}_2, 3)_{ad} \boxtimes C(\mathfrak{sl}_2, 3)_{ad}'' \boxtimes \text{Rep}(\mathbb{Z}_3) \).

**Proposition 4.2.** Let \( D \) be a non-trivial fusion subcategory of \( C \). If \( \dim(D) \in \mathbb{Z} \), then either \( C \) is pointed, or \( C \cong C(\mathfrak{sl}_2, 2) \), or \( C \cong C(\mathfrak{sl}_2, 3)_{ad} \boxtimes C(\mathfrak{sl}_2, 3)_{ad}'' \boxtimes C(\mathfrak{sl}_2, 3)_{ad}'' \boxtimes C(\mathfrak{sl}_2, 3)_{ad}'' \boxtimes C(\mathfrak{sl}_2, 3)_{ad}'' \boxtimes \text{Vec}, \text{ad} \), or \( C \cong C(\mathbb{Z}_p, \eta) \boxtimes C(\mathfrak{sl}_2, 3)_{ad} \boxtimes C(\mathfrak{sl}_2, 3)_{ad}'' \), where \( \sigma \in \text{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q}) \) such that \( \sigma(\zeta_3) = \zeta_3^2 \).

**Proof.** Since \( D \) is a non-trivial fusion subcategory of \( C \) with integral global dimension, by [22] Theorem 3.1 \( \dim(D) = p \). Then we consider the global dimension of the Miyler center \( D' \) of \( D \). [22] Theorem 3.1 again implies that either \( D = D' \) is symmetric or \( D \) is a modular fusion category. If \( D \) is modular, then \( D \cong C(\mathfrak{sl}_2, 3)_{ad} \boxtimes C(\mathfrak{sl}_2, 3)_{ad}'' \) or \( D \) is pointed by [25] Theorem 5.12. Notice that \( C \cong D \boxtimes D' \) as modular fusion category [7] Theorem 3.13, so \( D' \) is also a modular fusion category of dimension \( p^2 \), thus the structure of \( D \) is known. If \( D \) is symmetric, then either it is a Tannakian fusion category or \( D \equiv \text{Vec} \). In the first case, \( D \) must be a Lagrange fusion category as \( D \subseteq D' \) and \( \dim(D) \dim(D'_C) = p^2 \) [7] Theorem 3.10], which implies \( C \cong Z(\text{Vec}_{Zp}) \) by [7] Theorem 4.64], where \( \omega \in Z^1(\mathbb{Z}_p, C^*) \) is a 3-cocycle; in the second case \( \dim(C) = 4 \), then \( C \cong C(\mathfrak{sl}_2, 2) \) is an Ising category or \( C \) is pointed [22] Example 5.1.2.

Let \( C \) be a modular fusion category of global dimension \( p^2 \), then \( \text{ord}(T_C) \) divides \( p^5 \) [10] Corollary 8.18.2]. Since the structures of modular fusion categories of global dimension 4 and 9 are known [22][27], we always assume \( p \geq 5 \) below. Let \( \text{ord}(t_C) = p^t \) where \( t \) is the normalized \( T \)-matrix of \( C \), since \( \text{rank}(C) \leq p^2 - 1 \) [22] Lemma 4.2.2], \( n \leq 2 \) by Remark [22]. In particular, the number of simple objects in each Galois orbit is less than or equal to \( |\text{Gal}(\mathbb{Q}(\zeta_{p^2})/\mathbb{Q})|^2 | = p^2 - 2 \).

Assume that \( C \) does not contain a non-trivial fusion subcategory of integer global dimension below, and let \( D \) be a non-trivial modular fusion subcategory of \( C \). Note that we can assume that
$\mathcal{D}$ to be a simple modular fusion category and that $\mathcal{C}$ can be decomposed as Deligne product of simple modular fusion subcategories. Indeed, since $\mathcal{D}'$ is an integral fusion subcategory, $\mathcal{D}' = \text{Vec}$, hence fusion subcategories of $\mathcal{C}$ are modular. Assume $\dim(\mathcal{D})$ be the largest among its Galois conjugates, otherwise we can replace $\mathcal{D}$ with one of its Galois conjugates $\mathcal{D}''$.

**Theorem 4.3.** Let $\mathcal{C}$ be a non-simple modular fusion category of dimension $p^2$, where $p > 3$. If $\mathcal{C}$ does not contain a non-trivial fusion subcategory with integer global dimension, then $\mathcal{C}$ contains a modular fusion subcategory that is braided equivalent to a Galois conjugate of $\mathcal{C}(sl_2, p-2)_{\text{ad}}$.

**Proof.** Let $\mathcal{D} \subseteq \mathcal{C}$ be a simple modular fusion subcategory with $\dim(\mathcal{D}) \notin \mathbb{Z}$, then $\mathcal{C} \cong \mathcal{D} \boxtimes \mathcal{D}'$ and $\mathcal{D}'$ is also a modular fusion category by [17] Theorem 3.13. By Lemma 4.4 it suffices to show that $\mathcal{C}$ contains a non-trivial fusion subcategory whose global dimension is not divided by $p$ in sense of algebraic integers. If $p \nmid \dim(\mathcal{D})$, then we are done; assume $\dim(\mathcal{D}) = pa \mathbb{Z}$ for some totally positive algebraic integer $a_\mathbb{D}$ below. If $\mathcal{D}'$ is not simple, let $\mathcal{D}' \cong \mathcal{A} \boxtimes \mathcal{B}$, where $\mathcal{A}, \mathcal{B}$ are non-trivial modular fusion subcategories, obviously $p$ can’t divide both $\dim(A)$ and $\dim(B)$, thus the argument of Lemma 4.4 says that either $\mathcal{A}$ or $\mathcal{B}$ is a transitive modular fusion category, so $\mathcal{C}$ contains a Galois conjugate of $\mathcal{C}(sl_2, p-2)_{\text{ad}}$ [16] Theorem 1.1].

Assume that $\mathcal{D}'$ is a simple modular fusion category such that $\dim(\mathcal{D}'^\prime) = pb_\mathcal{D}$, $a_\mathcal{D}$ and $b_\mathcal{D}$ are totally positive algebraic units with $1 = a_\mathcal{D}b_\mathcal{D}$. Notice that $a_\mathcal{D} \in \mathbb{Q}((\zeta_5))^+$, if not, each Galois orbit of the unit objects $I_\mathcal{D}$ and $I_{\mathcal{D}'}$ has at least $p$ simple objects, then $\text{rank}(\mathcal{C}) \geq p^2$, it is impossible. By Corollary 8.13 $\mathcal{D}$ and $\mathcal{D}'$ are Grothendieck equivalent to modular fusion category $\mathcal{C}(sl_2, 2(p-1))_{\text{ad}}$. Up to Galois conjugates, we can assume $\dim(\mathcal{D}) = \frac{p}{4 \cos^2(\frac{\pi}{p})}$ and $\dim(\mathcal{D}'^\prime) = 4p \cos^2(\frac{\pi}{p})$ with $d = \frac{p+1}{2}$.

If there exists a $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ such that

$$\sigma\left(\frac{p}{4 \cos^2(\frac{\pi}{p})}\right) = \dim(\mathcal{D}'^\prime) = 4p \cos^2(\frac{d\pi}{p}),$$

then $p = 5$; in addition, when $p = 5$ we have a braided tensor equivalence

$$\mathcal{D} \cong \mathcal{C}(sl_2, 3)_{\text{ad}} \boxtimes \mathcal{C}(sl_2, 3)_{\text{ad}} \text{ or } \mathcal{D} \cong \mathcal{C}(sl_2, 3)_{\text{ad}}^\sigma \boxtimes \mathcal{C}(sl_2, 3)_{\text{ad}}^\sigma$$

by Theorem 3.5 where $\sigma(\zeta_5) = \zeta_5^2$. However, $\mathcal{C}$ must contain a fusion subcategory of dimension 5 by Proposition 3.10 which contradicts to the assumption. Therefore, $\mathcal{C}$ contains a modular fusion subcategory that is braided equivalent to a Galois conjugate of $\mathcal{C}(sl_2, p-2)_{\text{ad}}$. \hfill \Box

**Lemma 4.4.** Let $f(x) = \frac{4 \pi \sin^2(x)}{x}$, where $x \in (0, \frac{\pi}{2}]$. Then $f(x)$ is an increasing function.

**Proof.** We have $f'(x) = \frac{4 \pi \sin(x)[2x \cos(x) - \sin(x)]}{x^2}$; since $x \in (0, \frac{\pi}{2}]$, $\frac{4 \pi \sin(x)}{x^2} > 0$. Let

$$g(x) := 2x \cos(x) - \sin(x), \quad x \in [0, \frac{\pi}{2}],$$

then $g'(x) = \cos(x) - 2x \sin(x)$, $g''(x) = -\sin(x) - 2x \cos(x)$, hence $g''(x) < 0$ for all $x \in [0, \frac{\pi}{2}]$, so $g'(x) \geq g'(\frac{\pi}{2}) > 0$. Thus $g(x) \geq g(0) = 0$ for $x \in [0, \frac{\pi}{2}]$, which shows $f'(x) > 0$ when $x \in (0, \frac{\pi}{2}]$. Consequently, $f(x)$ is a strictly increasing function. \hfill \Box
Theorem 4.5. Let $C$ be a modular fusion category of global dimension $p^2$, where $p > 3$ is a prime. If $C$ contains a modular fusion category that is a Galois conjugate of $C(s_{12}, p - 2)_{ad}$, then $p \leq 23$ and $Q(Sc) \subseteq Q(T_c) = Q(\zeta_p)$.

Proof. Let $D := C(s_{12}, p - 2)_{ad}$, then $\dim(D) = \frac{p}{\sin^2(\pi/p)}$ and $\dim(D_C) = 4p\sin^2(\pi/p)$. By [22, Proposition A.1.1], we obtain that $\dim(D_C) > \frac{2p}{\sqrt{3}}$, since $\dim(D_C) = f(\pi/p)$, where $f(x) = \frac{4x\sin^2(x)}{\sqrt{3}}$, and $f(\pi/29) < 1.38 < \frac{2p}{\sqrt{3}}$. Lemma 4.4 implies $p \leq 23$.

Since $C$ is not pointed, $\text{rank}(C) \leq p^2 - 1$ by [22, Remark 4.2.3]; while $\text{rank}(D) = \frac{2p}{\sqrt{3}}$, so $\text{rank}(D_C) \leq 2(p + 1)$. Assume that $Q(t_c) = Q(\zeta_p^m)$ with $m$ being minimal. When $p > 5$, if $n > 1$, then $\text{rank}(C) > p^2$ by Remark 2.1 which is a contradiction. When $p = 5$, the structure of $C$ is known by Proposition 3.10, so $n = 1$.

Lemma 4.6. Let $C$ be a modular fusion category of global dimension $p\beta_p$, where $p$ is a prime and $\beta_p := 2 - (\zeta_p + \zeta_p^{-1})$. Then $\dim(X)$ is an algebraic unit for all objects $X \in O(C)$.

Proof. Let $f = \frac{\dim(C)}{\dim(X)^2}$ be an arbitrary formal codegree of $C$ corresponding to a simple object $X$. Since $C$ is not transitive, $p | f$ by [25, Theorem 4.4], hence $\frac{\beta_p}{\dim(X)^2}$ is an algebraic integer, i.e., $N_{Q(\zeta_p^m)}(\dim(X)^2) | N_{Q(\zeta_p^m)}(\beta_p)$ by [23, Proposition 4.2]. While $N_{Q(\zeta_p^m)}(\dim(X)^2) = N_{Q(\zeta_p^m)}(\dim(X)^2)$ and $N_{Q(\zeta_p^m)}(\beta_p) = p$, so $N_{Q(\zeta_p^m)}(\dim(X)^2) = 1$, which means that $\dim(X)$ must be an algebraic unit.

Corollary 4.7. Let $D$ be a modular fusion category of global dimension $p\beta_p$, where $p > 3$ is a prime. Then $C$ does not contain simple object that is fixed by $\text{Gal}(Q(\zeta_p^m)/Q)$.

Proof. Let $X$ be a simple object that is fixed by the Galois group $\text{Gal}(Q(\zeta_p^m)/Q)$, then for any $\sigma \in \text{Gal}(Q(\zeta_p^m)/Q)$, we have

$$\frac{\dim(D)}{\dim(X)^2} = \sigma \left( \frac{\dim(D)}{\dim(X)^2} \right) = \frac{\dim(D)}{\dim(\sigma(X))^2} \in \mathbb{Z},$$

which implies $\dim(X)^2 = \beta_p$, it contradicts to Lemma 4.6.

Proposition 4.8. Let $C$ be a modular fusion category of global dimension $p\beta_p$, where $p \geq 7$. Then $\text{rank}(C) = p - 1$ or $\frac{3(p - 1)}{2}$.

Proof. Since $Q(\dim(C)) = Q(\zeta_p^m)$ and $\text{ord}(t) = p$ by Theorem 4.5, the orbit of $I$ has exactly $\frac{p}{\beta_p}$ simple objects, it is easy to see that $C$ can’t be a transitive modular fusion category, so $\text{rank}(C) > \frac{p}{\beta_p}$. By Corollary 4.5, $p \leq 23$; Corollary 4.7 says that each Galois orbit of simple objects of $C$ has exactly $\frac{p}{\beta_p}$ simple objects when $p = 7, 11, 23$.

When $p = 13, 17, 19$, for any formal codegree $f$ of $C$, and let $g(x) = x^n - a_1x^{n-1} + \cdots + (-1)^na_n$ be the minimal polynomial of $f$, where $a_j$ are positive integers and $n$ is a divisor of $\frac{p}{\beta_p}$. Notice that $a_0 = N(f)$ is a power of $p$ and that

$$\sum_{\sigma \in \text{Gal}(Q(\zeta_p^m)/Q)} \frac{1}{\sigma(\dim(C))} = \frac{a_{n-1}}{a_n} = \begin{cases} \frac{1}{p}, & \text{if } p = 13; \\ \frac{1}{7}, & \text{if } p = 17; \\ \frac{1}{17}, & \text{if } p = 19. \end{cases}$$
equation \(1\) implies \(\sum_{\sigma \in \text{Gal}(\mathbb{Q}(f))/\mathbb{Q}} \frac{d}{d(f)} \leq \frac{\alpha\cdot\omega_{\mathbb{Q}}}{\omega_{\mathbb{Q}}}\). By using the program GAP, we can show that the \(d\)-number test or cyclotomic test fail when \([\mathbb{Q}(f) : \mathbb{Q}] < \frac{p-1}{2}\) for any formal codegree \(f\) of \(C\), hence each Galois orbit of simple object of \(C\) has \(\frac{d}{d} \) simple objects. Since \(N(\dim(C)) = p^{\frac{d}{d}+\frac{1}{2}}\), the inequality \(3\) shows that \(\text{rank}(C) \leq \left[\frac{p+1}{2}\right] = \left[p \cdot \frac{p-2}{2}\right]\), where \([\alpha]\) is the integer part of a positive algebraic integer \(\alpha\). Therefore, \(\text{rank}(C) \leq 12\) if \(p = 7\), \(\text{rank}(C) = p-1 = \frac{3(p-1)}{2}\) when \(p = 11, 13, 17\) and \(\text{rank}(C) = p-1 = 1\) when \(p = 19, 23\).

When \(p = 7\), for any formal codegree \(f\) of \(C\), let \(g(x) = x^3 - ax^2 + bx - 7^4\) be the minimal polynomial of \(f\), where \(a, b\) are positive integers. Then the \(d\)-number test and cyclotomic test show that \(g(x) = x^3 - 49x^2 + 686x - 7^4\), which is the minimal polynomial of \(7\beta_7\), or \(g(x) = x^3 - 98x^2 + 1029x - 7^4\), the minimal polynomial of the totally positive algebraic integer \(\frac{4\alpha}{\beta_7^2}\). Then the set of formal codegrees of \(C\) are exactly the Galois conjugates of \(\dim(C)\) (with multiplicity equals two) and \(\text{FPdim}(C) = \frac{4\alpha}{\beta_7}\) by equation \(1\).

Lemma 4.9. Let \(C\) be a modular fusion category of global dimension \(p\beta_p\) where \(p = 11, 13, 23, \ldots\). Then \(\text{rank}(C) \neq \frac{3(p-1)}{2}\).

Proof. Assume \(\text{rank}(C) = \frac{3(p-1)}{2}\). By Proposition \(4.3\) each Galois orbits of the simple objects has exactly \(\frac{d}{d} \) simple objects. Let \(X_1 = I, X_2, X_3\) be the representatives of each Galois orbits. Let \(d := \frac{p-1}{2}\), let \(g_2(x) = x^d + \sum_{j=1}^{d} (-1)^ja_jx^{d-j}\) and \(g_1(x) = x^d + \sum_{j=1}^{d} (-1)^jb_jx^{d-j}\) be the minimal polynomials of the formal codegrees \(\frac{\dim(C)}{\text{dim}(X_i)}\) of \(C\), respectively, where \(a_j, b_j\) are positive integers for \(1 \leq j \leq d\). Since \(\dim(X_i)^2\) are algebraic units by Lemma \(4.9\) we obtain \(a_d = b_d = \frac{N(\dim(C))}{p^{\frac{d}{2}}+1}\). The \(d\)-number condition \(20\) says \(a_{d-1}^d \mid a_{d-1}\) and \(b_{d-1}^d \mid b_{d-1}\), that is, \(p^d\) divides both \(a_{d-1}\) and \(b_{d-1}\).

Assume \(b_{d-1} \geq a_{d-1}\) and \(a_{d-1} = mp^d\) with \(m\) being a positive integer. As \(\sigma\left(\frac{\dim(C)}{\dim(X_i)^2}\right) = \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\beta_p)^+)} \frac{\dim(C)}{\text{dim}(X_i)^2}\), then

\[
\dim(C) = \sum_{i=1}^{3} \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\beta_p)^+)} \dim(\sigma(X_i))^2 \leq \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\beta_p)^+)} \frac{\dim(C)}{\sigma(\dim(C))} + \sum_{i=2}^{3} \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\beta_p)^+)} \frac{\dim(C)}{\sigma(\dim(C))}\sigma(\dim(X_i))^2,
\]

\[
\sum_{\sigma \in \text{Gal}(\mathbb{Q}(\beta_p)^+)} \frac{\dim(C)}{\sigma(\dim(C))} = \begin{cases} 5\beta_{11}, & \text{if } p = 11; \\ 7\beta_{13}, & \text{if } p = 13, \end{cases}
\]

We assume \(p = 11\) below, the other case is same. Let \(f_1 \leq \cdots \leq f_5\) be the Galois conjugates of \(\frac{\dim(C)}{\dim(X_2)}\). If \(m = 1\), then \(f_j \geq j-1\) for all \(1 \leq j \leq 5\), consequently \(\alpha_5 \geq 11^5 \cdot 5! > 11^6\), it is impossible. If \(m = 2\), the \(d\)-number test shows \(11^2|a_1, 11^3|a_2\) and \(11^4 \mid a_4\); also note \(f_j \geq \frac{11}{2} (1 \leq j \leq 5)\), by using a similar restriction of \(27\) Theorem \(4.2\), we see \(27a_1 \leq a_2, 1, a_2 \leq a_3\) and \(4a_3 \leq a_4 = 2 \cdot 1, 11^5\); the cyclotomic test fails for all possible cases, however. If \(m = 3\), then \(f_j \geq \frac{11}{2} (1 \leq j \leq 5)\), \(18a_1 \leq a_2, 7a_2 \leq a_3\) and \(7a_3 \leq 2a_4 = 6 \cdot 11^5\), again the cyclotomic test fails for all possible cases. Therefore, \(\text{rank}(C) \neq \frac{3(p-1)}{2}\).

Lemma 4.10. Let \(C\) be a modular fusion category of global dimension \(17\beta_{17}\), then \(\text{rank}(C) \neq 24\).
Proposition 4.12. Let \( C \) be a modular fusion category of global dimension \( \text{FPdim}(C) = 24 \). Let \( X_1 = I, X_2, X_3 \) be the representatives of each Galois orbits. Therefore, same as Lemma 4.9

\[
\dim(C) = \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta_7)^+)/\mathbb{Q})} \frac{\dim(C)}{\sigma(\dim(C))} + \sum_{i=2}^{3} \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta_7)^+)/\mathbb{Q})} \frac{\dim(C)}{\sigma(\dim(C))} \sigma(\dim(X_i)^2)
\]

\[
> \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta_7)^+)/\mathbb{Q})} \frac{\dim(C)}{\sigma(\dim(C))} + \sum_{i=2}^{3} \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta_7)^+)/\mathbb{Q})} \frac{\dim(C)}{M} \sigma(\dim(X_i)^2),
\]

where \( M \) is the maximal Galois conjugate of \( \dim(C) \). Note that

\[
\sum_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta_7)^+)/\mathbb{Q})} \frac{\dim(C)}{\sigma(\dim(C))} = 12 \beta_{17},
\]

we have \( 5 \beta_{17} > \frac{\dim(C)}{\pi} \cdot 2 \cdot 8 \cdot 1.79 \) by the Siegel’s trace theorem [13], which then implies \( \sin^2 \frac{\pi}{5} > 1 \), it is a contradiction.

Lemma 4.11. Let \( C \) be a modular fusion category of global dimension \( p \beta_p \), where \( p \geq 7 \). If \( C \) is not simple, then \( p = 7 \) and \( C \cong C(\mathfrak{sl}_2, 5)^{ad}_1 \boxtimes C(\mathfrak{sl}_2, 5)^{ad}_2 \), where \( \sigma \in \text{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q}) \) such that \( \sigma(\zeta_7) = \zeta_7^2 \).

Proof. Indeed, if \( C \cong B_1 \boxtimes B_2 \) with \( B_1, B_2 \) being non-trivial modular fusion subcategories, then it is easy to see that \( p \) cannot divide both \( \dim(B_1) \) and \( \dim(B_2) \), so \( B_1 \) and \( B_2 \) are prime transitive modular fusion categories [25 Theorem 4.4]. Note that

\[
N(\dim(C)) = p^{\frac{\beta_p}{4}} = N(\dim(B_1))N(\dim(B_2)) = p^{\beta_p - 3},
\]

that is, \( p = 7 \), and \( C \cong C(\mathfrak{sl}_2, 5)^{ad}_1 \boxtimes C(\mathfrak{sl}_2, 5)^{ad}_2 \) as modular fusion category.

Proposition 4.12. Let \( C \) be a modular fusion category of global dimension \( 7 \beta_7 \), then as a modular fusion category \( C \cong C(\mathfrak{sl}_2, 5)^{ad}_1 \boxtimes C(\mathfrak{sl}_2, 5)^{ad}_2 \).

Proof. By Proposition 4.8, we know \( \text{rank}(C) = 9 \) and \( \text{FPdim}(C) = \frac{20}{3} \). Let \( X_1 = I \), and \( \mathcal{O}_X(C) = \{ X_1, X_2, X_3 \} \), that is, \( \sigma(\dim(C)) = \frac{\dim(C)}{\dim(X_1)^2}, \sigma^2(\dim(C)) = \frac{\dim(C)}{\dim(X_2)^2}, \sigma \in \text{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q}) \) such that \( \sigma(\zeta_7) = \zeta_7^2 \); \( \mathcal{O}_X(C) = \{ X_4, X_5, X_6 \} \), where \( \dim(X_4)^2 = 1 \),

\[
\sigma(\dim(C)) = \frac{\dim(C)}{\dim(X_5)^2}, \sigma^2(\dim(C)) = \frac{\dim(C)}{\dim(X_6)^2};
\]

and \( \mathcal{O}_X(C) = \{ X_7, X_8, X_9 \} \), where \( \dim(X_7)^2 = \frac{\dim(C)}{\text{FPdim}(C)} = \frac{3}{2} \).

\[
\frac{\dim(C)}{\dim(X_8)^2} = \sigma \left( \frac{\dim(C)}{\dim(X_7)^2} \right), \quad \frac{\dim(C)}{\dim(X_9)^2} = \sigma^2 \left( \frac{\dim(C)}{\dim(X_7)^2} \right).
\]
Then we obtain that
\[ \epsilon_2 \dim(X_2) = \epsilon_5 \dim(X_5) = \frac{1}{d_Y}, \epsilon_3 \dim(X_3) = \epsilon_6 \dim(X_6) = \frac{1}{d_Y}, \dim(X_4) = \epsilon_4, \]
\[ \dim(X_7) = \frac{\epsilon_7}{d_Y}, \dim(X_8) = \frac{\epsilon_8 d_Y}{d_Y}, \dim(X_9) = \frac{\epsilon_9 d_Y}{d_Y}, \epsilon_j \in \{\pm 1\}, 2 \leq j \leq 9. \]

Since \( \sigma(X_j) = X_{j+1} \) and \( \sigma^2(X_j) = X_{j+2} \) where \( j \in \{1, 4, 7\} \), for \( 1 \leq k \leq 9 \),
\[ \sigma \left( \frac{S_{X_j, x_k}}{\dim(X_j)} \right) = \frac{S_{X_{j+1}, x_k}}{\dim(X_{j+1})}, \quad \sigma^2 \left( \frac{S_{X_j, x_k}}{\dim(X_j)} \right) = \frac{S_{X_{j+2}, x_k}}{\dim(X_{j+2})}. \]

Meanwhile \( \text{FPdim}(C) = \frac{\dim(C)}{\dim(X_7)} \), so \( \text{FPdim}(X_j) = \frac{S_{X_j, X_7}}{\dim(X_7)} \) for all \( 1 \leq j \leq 9 \). Note that \( \sigma(d_X) = \frac{1}{d_Y} \) and \( \sigma(d_Y) = -\frac{d_Y}{d_Y} \), hence \( \text{FPdim}(X_2) = \frac{\dim(X_2) \sigma(d_X)}{\dim(X_7)} = -\epsilon_2 d_Y \) and
\[ \text{FPdim}(X_3) = \frac{\dim(X_3) \sigma^2(d_X)}{\dim(X_7)} = -\epsilon_3 d_Y, \] we see \( \epsilon_2 = \epsilon_3 = -1. \)

Notice that the \( S \)-matrix of \( C \) are presented by \( \epsilon_j \) \( (4 \leq j \leq 9) \), \( d_X, d_Y, \text{FPdim}(X_4), \text{FPdim}(X_7), S_{X_4, X_4} \) and their Galois conjugates. In particular,
\[ \text{FPdim}(X_5) = -\epsilon_4 \epsilon_5 d_Y^2 \sigma(\text{FPdim}(X_4)), \text{FPdim}(X_6) = -\epsilon_4 \epsilon_5 d_Y^2 \sigma^2(\text{FPdim}(X_4)), \]
\[ \text{FPdim}(X_8) = \epsilon_7 \epsilon_8 d_Y^2 \sigma(\text{FPdim}(X_7)), \text{FPdim}(X_9) = \epsilon_7 \epsilon_8 d_Y^2 \sigma^2(\text{FPdim}(X_7)). \]

The Verlinde formula \( [2] \) implies \( X_2 \otimes X_2 = I \oplus X_3 \oplus A \), where \( A \) is an object with \( \text{FPdim}(A) = 2d_X \), so either \( A \) is a simple object or \( A \) is a direct sum of two simple objects by \( [10] \) Corollary 3.1.6] and \( [5] \) Theorem 1.0.1, note that \( A \) contains a simple object of Frobenius-Perron dimension \( d_X \) or \( d_Y \) as a direct summand in the latter case \( [5] \) Theorem 1.0.1. We claim that \( A \) is a direct sum of two non-isomorphic simple objects of Frobenius-Perron dimension \( d_X \). Indeed, if \( A \) is simple or \( A = V \oplus W \) with \( \text{FPdim}(V) = d_Y \) and \( \text{FPdim}(W) = 2d_X - d_Y \), then a direct computation shows that \( \text{FPdim}(C) > \frac{2d_X - d_Y}{2d_X} \); if \( A = 2M \) with \( \text{FPdim}(M) = d_X \), then \( M \otimes X_2 = 2X_2 \oplus N \), however \( \text{FPdim}(N) = d_Y \), it is a contradiction.

Since the Frobenius-Perron dimensions of simple objects in the Galois orbits of \( X_4 \) and \( X_7 \) are distinct, it is easy to show that they are \( d_X, d_Y, d_X d_Y \), respectively. Assume \( A = W_1 \oplus W_2 \) and let \( V_1, V_2 \) be simple objects of Frobenius-Perron dimension \( d_Y \). Then \( V_1 \otimes V_2 \) and \( W_1 \otimes W_2 \) must be simple, as \( C \) does not contain non-trivial invertible simple objects. Assume \( V_1 \otimes V_1 = I \oplus W_1 \), if \( V_2 \otimes V_2 = I \oplus W_2 \), then \( 2I \leq (V_1 \otimes V_2) \otimes (V_1 \otimes V_2) \), it is impossible, so \( V_2 \otimes V_2 = 2I \otimes W_2 \); note that if \( V_1 \otimes W_1 = V_1 \otimes W_2 \), then \( 2I \leq W_2 \otimes (V_1 \otimes W_1) \equiv V_1 \otimes (W_1 \otimes W_2) \), it is impossible. Therefore, \( V_1 \otimes W_1 = V_1 \otimes W_1 \) and \( W_1 \otimes W_1 = I \oplus V_1 \otimes W_1 \), so \( C \) contains a fusion subcategory that is Grothendieck equivalent to \( C(sl_2, 5)_{ad} \). Consequently, \( C \cong C(sl_2, 5)_{ad} \otimes C(sl_2, 5)_{ad} \) as modular fusion category by Lemma 4.11. \[ \square \]

**Proposition 4.13.** Let \( C \) be a modular fusion category of global dimension \( p \tilde{d}_p \), where \( 11 \leq p \leq 23 \). If \( \text{rank}(C) = p - 1 \), then \( p = 11 \) and \( C \) is braided tensor equivalent to a Galois conjugate of modular fusion category \( C(sl_2, 5)_{ad} \).

**Proof.** Let \( \rho_C \) be the associated modular representation of \( C \). We know that \( \rho_C \) can’t be decomposed as direct sum of sub-representations with disjoint t-spectra [4] Lemma 3.18] and also that \( \rho_C \) can’t be decomposed as direct sum of non-degenerate sub-representations of same type [23] Lemma 5.2.2]. Since \( \text{rank}(C) = p - 1 \) and \( \text{ord}(t) = p \), by comparing the dimensions of level
Moreover, when $pC$ is irreducible and rank$(C)$ = 0, we obtain either $\rho C$ is irreducible or $\rho C = \rho_1 \oplus (\oplus_m \rho_0)$, where $\rho_1$ is an irreducible representation of rank $\frac{p+1}{2}$ and $m = \frac{p-3}{2}$. However, [17, Proposition 3.22] states $m = 1$ and then $p = 5$, which is impossible. Hence, $\rho C$ is irreducible.

As $\rho C$ is non-degenerate and rank$(C)$ \leq 23, [9, Main Theorem 4] implies that $p = 11, 17, 23$. Moreover, when $p = 17, 23$, the $(p - 1)$-dimensional non-degenerate representations are realized as modular representations of modular fusion categories $C(g_2, \frac{1}{2})$ and $C(\tau_7, 5)_{ad}$, respectively. However, neither of the Galois conjugates of $\dim(C(g_2, \frac{1}{2}))$ equal to $17\beta_7\tau$, and the norm of $\dim(C(\tau_7, 5)_{ad})$ is $23^{14}$, it is also impossible. When $p = 11$, the 10-dimensional non-degenerate representations can only be realized as modular representation of $C(s_{05}, \frac{1}{2})_{ad}$. Therefore, $C$ is braided tensor equivalent to a Galois conjugate of $C(s_{05}, \frac{1}{2})_{ad}$.

**Theorem 4.14.** Let $C$ be a modular fusion category of global dimension $p\beta_p$, where $p > 3$ is a prime. If $C_{p} = Vec$, then $C$ is braided equivalent to one of the following modular fusion categories

$$
\begin{cases}
C(sl_2, 3)_{ad} \boxtimes C(sl_2, 3)_{ad} \boxtimes C(sl_2, 3)_{ad}, & \text{if } p = 5; \\
C(sl_2, 5)_{ad} \boxtimes C(sl_2, 5)_{ad} \boxtimes C(sl_2, 5)_{ad}, & \text{if } p = 7; \\
C(sl_2, 2)_{ad}, & \text{if } p = 11.
\end{cases}
$$

where $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ such that $\sigma(\zeta_5) = \zeta_5^2$, $\tau \in \text{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})$ such that $\tau(\zeta_7) = \zeta_7^2$ and $\nu \in \text{Gal}(\mathbb{Q}(\zeta_{11})/\mathbb{Q})$ such that $\nu(\zeta_{11}) = \zeta_{11}^7$.

**Proof.** We know $p \leq 23$ by Theorem 4.5. When $p = 5$, this is the conclusion of Proposition 3.10 and Proposition 4.8 shows rank$(C) = p - 1$ or $\frac{3p-15}{2}$ if $p > 5$. If rank$(C) = \frac{3p-15}{2}$, then there exists such a modular fusion category $C$ only for $p = 7$ by Lemma 4.9, Lemma 4.10 and Proposition 4.12 if rank$(C) = p - 1$, we have $p = 11$ by Proposition 4.13.

Recall that modular fusion categories of global dimension 4 and 9 are either pointed, or braided equivalent to a Galois conjugate of Ising category $C(sl_2, 2)$ and $C(s_{05}, \frac{1}{2})_{ad}$ [27], respectively. Hence, combining with the conclusions of [27, Proposition 4.4], Theorem 4.3 and Theorem 4.14 together imply the following theorem:

**Theorem 4.15.** Let $C$ be a modular fusion category of global dimension $p^2$, where $p > 5$ is a prime. If $C$ contains a non-trivial fusion subcategory, then either $C$ is pointed, or $C$ is braided tensor equivalent to a Galois conjugate of one of the following modular fusion categories

$$
\begin{cases}
C(sl_2, 3)_{ad} \boxtimes C(sl_2, 3)_{ad} \boxtimes C(Z_5, \eta), & \text{if } p = 5; \\
C(sl_2, 5)_{ad} \boxtimes C(sl_2, 5)_{ad} \boxtimes C(sl_2, 3)_{ad}, & \text{if } p = 7; \\
C(sl_2, 9)_{ad} \boxtimes C(sl_2, 9)_{ad}, & \text{if } p = 11.
\end{cases}
$$

where $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ such that $\sigma(\zeta_5) = \zeta_5^2$, $\tau \in \text{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})$ such that $\tau(\zeta_7) = \zeta_7^2$ and $\nu \in \text{Gal}(\mathbb{Q}(\zeta_{11})/\mathbb{Q})$ such that $\nu(\zeta_{11}) = \zeta_{11}^7$.

This completes the classification of non-simple modular fusion categories of global dimension $p^2$. In addition, it is worth to note that $C(s_{05}, \frac{1}{2})_{ad}$ and its Galois conjugates are simple modular fusion categories of global dimension 9.
**Question 4.16.** Let \( p > 3 \) be a prime. Is there a simple modular fusion category \( C \) of global dimension \( p^2 \)?

Moreover, let \( D \) be a spherical fusion category of global dimension \( p \), then its Drinfeld center \( Z(D) \) is a modular fusion category of global dimension \( p^2 \). Therefore, a negative answer to Question 4.16 will also result in a complete classification of spherical fusion categories of prime global dimensions.

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