Abstract. The constituent counting rules, i.e., the scaling behavior of amplitudes (in terms of the number of fundamental constituents) for exclusive processes when high energy scales are present, have been known for decades, and have been borne out in a number of experiments. Such scaling would be sensitive, in particular, to possible exotic multiquark content. Here we examine how one may use the rules to test for pentaquarks in electroproduction, or for tetraquarks in $e^+e^-$ annihilation. An interesting new type of scaling (separate Mandelstam $s$ and $t$ behavior) arises in the forward scattering direction. The correct scaling arises naturally in AdS/QCD, in which the amplitudes can be computed explicitly.

Keywords. QCD scaling rules · Exotic hadrons

1 Introduction: Multiquark Hadrons

After more than 50 years of the quark model and more than 40 years of QCD, we find ourselves at last in the era of multiquark hadrons. The first such unambiguous candidate, the $X(3872)$, was discovered by BELLE in 2003 [1], and has been established as a primarily $c\bar{c}q\bar{q}$ state ($q$ a light quark). Its existence (with a well-defined mass and width) has been verified at multiple experiments (BaBar, CDF, D0, BESIII, LHCb, CMS, COMPASS), but the unfortunate decline [2] of the ostensible pentaquark state $\Theta^+(1540)$ seen around the same...
time led the physics community as a whole to be less confident in the acceptance of $X(3872)$ as a true exotic state. Numerous charmoniumlike (and bottomoniumlike) states—some neutral and some charged—were discovered in subsequent years. However, not until 2014 with the LHCb discovery of the charged $Z_{-}^{c}(4430)$ $c\bar{c}d\bar{u}$ state $^{3}$, in which was observed the looping Argand diagram production amplitude behavior characteristic of a resonant state, were most of the remaining multiquark-state skeptics persuaded.

As of the time of this writing, 35 heavy-quark exotics have been observed: 30 in the charmoniumlike sector, 4 in the bottomoniumlike sector, and 1 [$X(5568)$] believed to be $b\bar{d}d\bar{u}$ $^{4}$. Most are tetraquarks, but 2 discovered by LHCb $^{5}$ (called $P_{cc}^{+}$) have pentaquark quantum numbers. The discovery history and status of these heavy-quark exotics is summarized in the recent review $^{6}$, and various proposals for future research directions are discussed in $^{7}$. Several other reviews have recently appeared $^{8,9,10,11,12,13}$.

It is especially interesting that, after decades of searches, the first clear signals for multiquark hadrons appeared in the heavy-quark sector. In states of this sector, quantum-mechanical potential models characterize -onium states very well and heavy-light states reasonably well, exposing exotic states as ones that are superfluous in the spectrum or have anomalous properties. One may then hope that signals of exotic states might appear in the $s$-quark sector, such as the possibility of $\gamma p \rightarrow \phi p$ $^{14,15}$ revealing hints of $s\bar{s}uud$ pentaquarks $^{16}$.

2 Constituent Counting Rules

This paper focuses upon the use of one of the earliest diagnostics of QCD, the constituent counting rules $^{17,18}$ developed in its first decade, and specifically upon their use to reveal the multiquark nature of heavy-quark states, as recently discussed in Refs. $^{19,20,21}$. The counting rules are an expression of the approximate conformality and scale independence of QCD at high energies; formally, they give the leading-order scaling behavior of scattering amplitudes in terms of the twist dimension of the operator responsible for the process amplitude. Obtaining the relevant scaling power is straightforward when one uses the leading Feynman diagram for the process: Namely, every propagator and vertex through which flows a large momentum transfer [so that the overall process involves scattering through a finite center-of-momentum (c.m.) frame angle $\theta_{\text{CM}}$, i.e., fixed $t/s$] contributes to the leading-twist operator. Supposing that this large scale is uniformly Mandelstam $s$—the total c.m. energy for the process—then the invariant amplitude $\mathcal{M}$ and differential cross section $d\sigma/dt$ scale very simply in terms of the total number (initial plus final) of fundamental constituents (quarks, leptons, gauge bosons) $n$:

\[
\mathcal{M} \sim s^{\frac{1}{2}+2}, \quad \frac{d\sigma}{dt} \sim \frac{1}{s^{n-2}}. \tag{1}
\]

These scaling rules are genuinely rigorous and survive, up to $\ln s$ corrections, such varied effects as $\alpha_s$ running and renormalization-group effects, Su-
3 Constituent Counting Rules and Exotics

The main thrust of Ref. [23], however, was to test the oft-touted possibility (e.g., in [10]) that \( \Lambda(1405) \) is a (5-quark) \( KN \) molecule because it is so light even when compared to the nonstrange \( N^* \)'s such as \( N(1535) \). Applying Eq. (1) to such a state yields

\[
\frac{d\sigma}{dt}[\pi^- p \to K^0 \Lambda(1405)] \sim s^{-10},
\]

(2)

\[
\frac{d\sigma}{dt}[\gamma p \to K^0 \Lambda(1405)] \sim s^{-9}.
\]

(3)

The latter process, especially, is custom-made for measurement at Jefferson Lab.

If such unique signatures are possible for light exotic candidate states such as \( \Lambda(1405) \), then, as Ref. [10] reasoned, similar results should hold for the heavy-quark exotics as well. In particular, if \( Z_c^\pm \) is a tetraquark state, then

\[
\frac{d\sigma}{dt}[e^+ e^- \to Z_c^\pm \pi^\mp] \sim s^{-6},
\]

(4)

\[
\frac{d\sigma}{dt}[e^+ e^- \to Z_c^+ Z_c^-] \sim s^{-8}.
\]

(5)

On the other hand, if the quarks in \( Z_c^\pm (cc \bar{u}d) \) (or its charge conjugate \( Z_c^- \)) are arranged into two diquarks (\( cu \)) and (\( \bar{c}\bar{d} \)) that are tightly bound and therefore scatter as intact units, then

\[
\frac{d\sigma}{dt}[e^+ e^- \to Z_c^\pm \pi^\mp] \sim s^{-4},
\]

(6)

\[
\frac{d\sigma}{dt}[e^+ e^- \to Z_c^+ Z_c^-] \sim s^{-4}.
\]

(7)

These scalings provide rather distinctive experimental signals. However, one may object that they only apply at energies \( \sim \sqrt{s} \) rather high above threshold.

In order to remove systematic differences due to the presence of nontrivial resonant regions, one may consider ratios, e.g.,

\[
\frac{\sigma(e^+ e^- \to Z_c^\pm \pi^\mp)}{\sigma(e^+ e^- \to \mu^+ \mu^-)} = |F_{Z_c}(s)|^2 \propto \frac{1}{s^4},
\]

(6)
or $\propto s^{-2}$ if $Z_c$ is formed of tightly bound diquarks. Here, $F_{Z_c}$ is the electromagnetic form factor of the $Z_c^\pm$.

One may also consider ratios such as

$$\frac{\sigma (e^+ e^- \rightarrow Z_c^\pm \pi^-)}{\sigma (e^+ e^- \rightarrow A_1^\pm A_1^-)} ,$$

(7)
in which both final states contain the same number (6) of fundamental constituents and the same heavy quarks ($c\bar{c}$), their effects cancelling in the ratio. In this case, the value of the ratio is a direct measure of how the components of the $Z_c$ are assembled; one may expect a smaller [but still $O(1)$] value if the $Z_c$ consists of two loosely bound mesons. The same types of counting considerations apply to the electroproduction of hidden-charm pentaquarks, $e^- p \rightarrow e^- P_c^\pm (c\bar{c}uud)$, a process of great interest to JLab.

Another interesting exception to the scalings given above was noted in Ref. [24]: The naive scalings require every constituent of the process to participate in the scattering through a high momentum transfer. If some of them are produced softly with respect to any of the energetic constituents—for example, in $Z_c$ production, the $c\bar{c}$ pair might be produced from the vacuum through gluons emitted from one of the light quarks $q$ that are soft in the $q$ rest frame—then the $1/s$ scaling occurs at a lower power than advertised. In this particular case, the $c\bar{c}$ production amplitude is suppressed by $1/m_c$ due to the charm-quark propagator in $gg$ fusion, while $c$ and $\bar{c}$ are created with virtuality $m_c^2$ and therefore must exchange gluons of $q^2 \sim m_c^2$ in order to be promoted to becoming nearly on-shell $Z_c$ components. One can check that this mechanism is not competitive with ones in which the $c\bar{c}$ pair is created from the original hard subprocess until $\sqrt{s} = O(20 \text{ GeV})$ [25].

4 New Constituent Counting Rules in the Forward Region

Interesting novel constituent counting rules arise when any of the constituents undergoing hard momentum transfer are constrained to lie along the forward direction of scattering [20]. As noted above, traditional constituent counting rules require scattering through a finite angle $\theta_{CM}$. Neglecting masses, one has

$$t, u = -\frac{s}{2} (1 \mp \cos \theta_{CM}) ,$$

(8)

so that $s, t, u$ are all of the same order for finite $\theta_{CM}$. However, in the extreme forward (backward) direction, for which $\theta_{CM} = \varepsilon (\theta_{CM} = \pi - \varepsilon)$ with $\varepsilon \ll 1$, then:

$$-t = s \frac{\varepsilon^2}{4} \ll s \approx -u \left( -u = s \frac{\varepsilon^2}{4} \ll s \approx -t \right) .$$

(9)
The existence of a new scale (specifically, the hierarchy here reads $A_{QCD}^2 \ll -t \ll s$) implies the existence of new scaling laws for cross sections.

Consider the textbook example [26] of $e^+ e^- \rightarrow \gamma^* \rightarrow q\bar{q} \rightarrow \text{hadrons}$, whose usual scale-invariant cross section goes as $\alpha_{EM}^2 / s$. The $1/s$ factor arises through
a combination of the photon propagator, fermion traces, and phase space. But one can also show for the process $e^+e^- \rightarrow \gamma\gamma$, which has $t$ (forward) and $u$ (backward) singularities due to the lepton propagator (see Fig. 1), that in fact its cross section also scales as $1/s$ specifically, as $\alpha_{EM}^2/s|t|$ in the forward direction.

Fig. 1 Leading-order diagrams ($t$- and $u$-channel, respectively) for $e^+e^- \rightarrow \gamma\gamma$.

It was further argued in Ref. [20] that additional suppressions by $\alpha_{EM}\kappa^2/|t|$ or $\alpha_{EM}^2\kappa^4/|t|^2$, respectively [where $\kappa = O(\Lambda_{QCD})$], arise when one or both of the photons convert to neutral vector mesons $V^0$, are the cost of constraining the $qq$ components of $V^0$ both to propagate in the forward direction. However, this result must be reassessed in light of the objection noted above, that obtaining the full naive scaling behavior requires all relevant lines in the diagram to carry large momentum transfers; if not, then the twist of the corresponding operator is smaller [21]. Such is the case (anticipated in [24]) when the photon couples solely to a single $V^0$ of fixed squared mass $m_{V^0}^2 = s_1$; then the photon—which generically would have $O(|t|)$ virtuality—carries a precisely fixed virtuality, $s_1$. Since the photon propagator is one of the contributors to the total counting rule, the scaling law is modified: The $\kappa^2/|t|$ suppressions are replaced by $O(1)$ factors of $\kappa^2/s_1$. In this exceptional configuration, one recovers the expectation of vector meson dominance (VMD): The $V^0$ behaves as a single fundamental constituent.

Away from this special point, the original expectations of [20] remain valid [21]. In particular, the inclusive forward cross section $\sigma(e^+e^- \rightarrow \gamma\gamma^* \rightarrow \gamma q\bar{q})$ scales as $1/|t|^2$, while $V^0$ created by hadronic processes (e.g., in $pp$ scattering) no longer behave as though composed of a single constituent.

5 Explicit Checks in AdS/QCD

One may observe the explicit manifestation of the constituent counting rules in AdS/QCD, which is a model for QCD that treats strongly coupled confining theories as dual to weakly coupled gravity theories in 5 spacetime dimensions, in such a way that confinement is represented by the familiar 4-dimensional
fields only being able to penetrate a finite distance \( z \) into the 5th bulk dimension. In the particular variant known as the soft-wall AdS/QCD light-front model [28,29], each coupling is weighted by the Gaussian factor \( e^{-\kappa^2 z^2} \), and then all nonperturbative overlap integrals, such as form factors, can be computed in closed form as confluent hypergeometric (Kummer) functions or Bessel functions.

In particular, not only can one show that the square of the \( \gamma^+ V^0 \) transition form factor \( |G_V(s_1)|^2 \) scales as \( \kappa^2/s_1 \) as discussed above, but in soft-wall AdS/QCD in the large-\( s_1 \) limit one calculates exactly that:

\[
\frac{|G_V(s_1)|^2}{|G_V(0)|^2} = \frac{9\pi^3}{256} \cdot \frac{\kappa^2}{s_1}.
\]

Returning to the issue of counting constituents, let us consider the process \( e^+ e^- \rightarrow H_{n_1} H_{n_2} \) where hadrons \( H \) contain \( n_1 \) and \( n_2 \) constituents, respectively. Then the form factors \( F \) in

\[
\frac{\sigma(e^+ e^- \rightarrow H_{n_1} H_{n_2})}{\sigma(e^+ e^- \rightarrow \mu^+ \mu^-)} = |F_{H_{n_1} H_{n_2}}(s)|^2,
\]

are computed in AdS/QCD as the overlap integrals:

\[
F_{H_{n_1} H_{n_2}}(Q^2) = \int_0^\infty dz V(Q,z) \phi_{n_1}(z) \phi_{n_2}(z),
\]

where \( V(Q,z) = \Gamma(1+a)/U(a,0,\kappa^2 z^2) \) is the vector-field bulk-to-boundary propagator, \( a \equiv Q^2/4\kappa^2 \), \( U \) is a Kummer function, and the bulk profiles of the hadrons are given by

\[
\phi_n(z) = \sqrt{\frac{2}{\Gamma(n-1)}} \kappa^{-1} z^{n-3/2} e^{-\kappa^2 z^2/2}.
\]

One obtains the closed-form result

\[
F_{H_{n_1} H_{n_2}}(Q^2) = \frac{\Gamma \left( \frac{n_1+n_2}{2} \right) \Gamma \left( \frac{n_1+n_2}{2} - 1 \right)}{\sqrt{\Gamma(n_1-1)} \sqrt{\Gamma(n_2-1)}} \frac{\Gamma(a+1)}{\Gamma(a+1 + \frac{n_1+n_2}{2} - 1)}.
\]

Using the large-\( a \) (\( Q^2 \to s \)) expansion of \( \Gamma \) functions, one immediately has

\[
F_{Z_c^+ \pi^-} \sim s^{-2} \quad \Rightarrow \quad \frac{d\sigma}{dt} [e^+ e^- \to Z_c^+ \pi^-] \sim s^{-6},
\]

\[
F_{Z_c^+ Z_c^-} \sim s^{-3} \quad \Rightarrow \quad \frac{d\sigma}{dt} [e^+ e^- \to Z_c^+ Z_c^-] \sim s^{-8},
\]

exactly as predicted by Eqs. (11). If the components of \( Z_c \) are treated as fundamental diquarks, then Eqs. (15) immediately follow.
6 Conclusions

Exotic hadrons, at least in the form of tetraquarks and pentaquarks, are here to stay: Over 30 such states ($X,Y,Z,P_c$) have thus far been observed. Producing and characterizing them and their decay modes will be one of the major thrusts of experimental facilities: not just the existing LHCb, BESIII, and Belle(II) programs, but at JLab, COMPASS, PANDA at FAIR, and elsewhere.

The old and well-known constituent counting rules provide simple and straightforward tests of exoticity: Scalings of both cross sections at high energies and ratios of cross sections at energies all the way down to threshold provide important handles on exotic versus nonexotic structure. New forward-scattering ($s \gg |t| \gg A_{QCD}^2$) constituent counting rules provide yet more useful experimental tests of hadronic substructure. One must, however, in all processes confirm the expected scaling behavior based upon the leading twist dimension/number of propagators and vertices through which flows large momentum transfers.

One may also use AdS/QCD, in which the nonperturbative amplitudes can be computed explicitly, to obtain expressions in which the expectations of the constituent counting rules are explicitly manifested.

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References

1. S.-K. Choi et al. [Belle Collaboration], Observation of a Narrow Charmonium-Like State in Exclusive $B^\pm \rightarrow K^\pm \pi^+\pi^-J/\psi$ Decays, Phys. Rev. Lett. 91, 262001 (2003) [hep-ex/0309032].
2. R.A. Schumacher, The Rise and Fall of Pentaquarks in Experiments, AIP Conf. Proc. 842, 409 (2006) [nucl-ex/0512042].
3. R. Aaij et al. [LHCb Collaboration], Observation of the Resonant Character of the $Z(4430)^-$ State, Phys. Rev. Lett. 112, 222002 (2014) [arXiv:1404.1903 [hep-ex]].
4. V.M. Abazov et al. [D0 Collaboration], Evidence for a $B_s^0\pi^\pm$ State, Phys. Rev. Lett. 117, 022003 (2016) [arXiv:1602.07588 [hep-ex]].
5. R. Aaij et al. [LHCb Collaboration], Observation of $J/\psi p$ Resonances Consistent with Pentaquark States in $A_0^0 \rightarrow J/\psi K^-p$ Decays, Phys. Rev. Lett. 115, 072001 (2015) [arXiv:1507.03414 [hep-ex]].
6. R.F. Lebed, R.E. Mitchell, and E.S. Swanson, Heavy-Quark QCD Exotica, Prog. Part. Nucl. Phys. 93, 143 (2017) [arXiv:1610.04528 [hep-ph]].
7. R.A. Briceno et al., Issues and Opportunities in Exotic Hadrons, Chin. Phys. C 40, 042001 (2016) [arXiv:1511.06779 [hep-ph]].
8. H.-X. Chen, W. Chen, X. Liu, and S.-L. Zhu, The Hidden-Charmonium Pentaquark and Tetraquark States, Phys. Rept. 639, 1 (2016) [arXiv:1601.02092 [hep-ph]].
9. A. Esposito, A. Pilloni, and A.D. Polosa, Multiquark Resonances, Phys. Rept. 668, 1 (2016) [arXiv:1611.07920 [hep-ph]].
10. F.-K. Guo, C. Hanhart, U.-G. Meißner, Q. Wang, Q. Zhao, and B.-S. Zou, Hadronic Molecules, Rev. Mod. Phys. 90, 015004 (2018) [arXiv:1705.00141 [hep-ph]].
11. A. Ali, J.S. Lange, and S. Stone, Exotics: Heavy Pentaquarks and Tetraquarks, Prog. Part. Nucl. Phys. 97, 123 (2017) [arXiv:1706.00619 [hep-ph]].
12. S.L. Olsen, T. Skwarnicki, and D. Zieminska, Non-Standard Heavy Mesons and Baryons, an Experimental Review, Rev. Mod. Phys. 90, 015003 (2018) [arXiv:1708.04012 [hep-ph]].
13. M. Karliner, J.L. Rosner, and T. Skwarnicki, Multiquark States, [arXiv:1711.10626 [hep-ph]].
14. B. Dey et al. [CLAS Collaboration], Data Analysis Techniques, Differential Cross Sections, and Spin Density Matrix Elements for the Reaction $\gamma p \rightarrow \phi p$, Phys. Rev. C 89, 055208 (2014); addendum: [Phys. Rev. C 90, 019901 (2014)] [arXiv:1403.2110 [nucl-ex]].
15. B. Dey, Phenomenology of $\phi$ Photoproduction from Recent CLAS Data at Jefferson Lab, [arXiv:1405.3770 [hep-ex]].
16. R.F. Lebed, Diquark Substructure in $\phi$ Photoproduction, Phys. Rev. D 92, 114006 (2015) [arXiv:1510.01412 [hep-ph]].
17. S.J. Brodsky and G.R. Farrar, Scaling Laws at Large Transverse Momentum, Phys. Rev. Lett. 31, 1153 (1973).
18. V.A. Matveev, R.M. Muradian, and A.N. Tavkhelidze, Automodellism in the Large-Angle Elastic Scattering and Structure of Hadrons, Lett. Nuovo Cim. 7, 719 (1973).
19. S.J. Brodsky and R.F. Lebed, QCD Dynamics of Tetraquark Production, Phys. Rev. D 91, 114025 (2015) [arXiv:1505.00803 [hep-ph]].
20. S.J. Brodsky, R.F. Lebed, and V.E. Lyubovitskij, QCD Compositeness As Revealed in Exclusive Vector Boson Reactions through Double-Photon Annihilation: $e^+e^- \rightarrow \gamma\gamma^* \rightarrow V^0$ and $e^+e^- \rightarrow \gamma\gamma^* \rightarrow V^0V^0$, Phys. Lett. B 764, 174 (2017) [arXiv:1609.06639 [hep-ph]].
21. S.J. Brodsky, R.F. Lebed, and V.E. Lyubovitskij, QCD Constituent Counting Rules for Neutral Vector Mesons, Phys. Rev. D 97, 034009 (2018) [arXiv:1712.08853 [hep-ph]].
22. C. White et al., Comparison of 20 Exclusive Reactions at Large $t$, Phys. Rev. D 49, 58 (1994).
23. H. Kawanura, S. Kumano, and T. Sekihara, Determination of Exotic Hadron Structure by Constituent-Counting Rule for Hard Exclusive Processes, Phys. Rev. D 88, 034010 (2013) [arXiv:1307.0369 [hep-ph]].
24. F.-K. Guo, U.-G. Meißen, and W. Wang, On the Constituent Counting Rule for Hard Exclusive Processes Involving Multi-Quark States, Chin. Phys. C 41, 053108 (2017) [arXiv:1607.04020 [hep-ph]].
25. S.J. Brodsky and V.E. Lyubovitskij, private communication.
26. M.E. Peskin and D.V. Schroeder, An Introduction to Quantum Field Theory, Westview Press, New York (1995).
27. M. Davier, M.E. Peskin, and A. Snyder, Two-Photon Exchange Model for Production of Neutral Meson Pairs in $e^+e^-$ Annihilation, [hep-ph/0606155].
28. S.J. Brodsky and G.F. de Teramond, Light-Front Dynamics and AdS/QCD Correspondence: The Pion Form Factor in the Space- and Time-Like Regions, Phys. Rev. D 77, 056007 (2008) [arXiv:0707.3850 [hep-ph]].
29. T. Branz, T. Guttridge, V.E. Lyubovitskij, I. Schmidt, and A. Vega, Light and Heavy Mesons in a Soft-Wall Holographic Approach, Phys. Rev. D 82, 074022 (2010) [arXiv:1008:0268 [hep-ph]].