Non-Abelian anyonic interferometry with a multi-photon spin lattice simulator

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\textbf{Abstract.} Recently, a pair of experiments (Lu \textit{et al} 2009 \textit{Phys. Rev. Lett.} \textbf{102} 030502; Pachos \textit{et al} 2009 \textit{New J. Phys.} \textbf{11} 083010) demonstrated the simulation of Abelian anyons in a spin network of single photons. The experiments were based on the Abelian discrete gauge theory spin lattice model of Kitaev (Kitaev 2003 \textit{Ann. Phys., NY} \textbf{303} 2). Here, we propose an experiment using linear optics and single photons to simulate non-Abelian anyons. The scheme makes use of joint qutrit–qubit encoding of the spins, and the resources required are three pairs of parametric downconverted photons and 14 beam splitters.

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1. Introduction

In three-dimensional space, particles are restricted to bosons, which give no phase change when exchanged, or fermions, which give a sign change when exchanged. In contrast, in two dimensions, more general particle statistics are possible [4]–[6]. Abelian anyons can accumulate an arbitrary phase under exchange. In the non-Abelian case, there is a multitude of distinguishable states corresponding to the outcomes of fusing particles, and the exchange of particles is a unitary transformation on this space [7]. Non-Abelian anyons are a promising candidate for performing quantum computation, because the quantum information is topologically protected, potentially leading to very low error rates [3, 8].

Anyons are not elementary particles in three dimensions but can arise as quasiparticles: particle-like excitations in many-body systems. This is manifested in two-dimensional electron gases in the quantum Hall effect, where at fractional filling of the particle orbitals, the low-lying excitations above the ground state are particle-like and predicted to behave like anyons [9]–[11]. To date, there has been some experimental evidence in support of the existence of Abelian anyons in fractional quantum Hall fluids [12, 13] although there are caveats to interpretation of the data [14], and a vigorous experimental effort is in place to demonstrate interferometry of non-Abelian anyons. Recently, two experiments have reported observation of e/4 electron charge in the ν = 5/2 filled state [15, 16], which is predicted to have non-Abelian quasiparticles. In [17] a theoretical interpretation of the experimental results from [16] was given and a suggestion was provided for how to witness non-Abelian statistics in that experiment. There are several proposals for performing non-Abelian interferometry in the ν = 5/2 filled quantum Hall state [8, 18, 19], and for the general theory of interferometry of non-Abelian anyons, see [20]–[22].
Another promising alternative platform to observe anyons is in spin lattices. Here, the strategy is to take a two-dimensional array of spins (i.e. a group of spins whose coupling graph defines a surface) and prepare a highly entangled ground state with respect to some physical theory. Early work by Bais [23] showed that discrete gauge theories in two dimensions could support non-Abelian particles, and the full mechanics of braiding in those models was given in [24]. Kitaev [3] suggested a spin lattice Hamiltonian that is a sum of vertex and face operators, of the form

\[ H = -\sum_v A(v) - \sum_f B(f), \]

where spins are placed on the edges of a lattice. Here interactions among edges meeting at a vertex \( v \) (a neighborhood denoted \( \text{star}(v) \)) are given by the vertex operators \( A(v) = \prod_{e \in \text{star}(v)} \sigma^x_e \), while the interactions on the edges on the boundary of a face \( f \) (a neighborhood denoted by \( \partial f \)) are given by the face operators \( B(f) = \prod_{e \in \partial f} \sigma^z_e \). This Hamiltonian is known as the surface code Hamiltonian and corresponds to a discrete gauge theory with the simplest Abelian group \( G = \mathbb{Z}_2 \). The ground state has +1 outcome for the measurement of either a face or vertex operator and we call it the vacuum state. A localized particle residing at a vertex or face is by definition the state corresponding to a −1 outcome of measuring \( A(v) \) or \( B(f) \) at that vertex or face. Particle states are eigenstates of the Hamiltonian, and because the vertex and face operators mutually commute, these particles are localized. We will also see that these particles have well-defined statistics.

Several proposals have been made for engineering such a Hamiltonian in physical systems and to demonstrate interferometry of Abelian anyons (for a review, see [25] and references therein). These proposals suggest realizing the Hamiltonian in the perturbative regime corresponding to the gapped phase of a two-body Hamiltonian on a honeycomb lattice that is realistic [26], and can be designed in the laboratory. With an additional magnetic field perturbation, the honeycomb model also supports a phase with non-Abelian anyons, and there are proposals for observing the statistics in these models [27, 28]. Such simulations require sufficiently large lattices to overcome finite size effects [29].

Another approach, and the one we adopt here, is to not try to build the Hamiltonian at all but rather just simulate the kinematics of the anyonic states. This involves creating the highly entangled ground state and performing operations on the spins for creating, braiding and annihilating excitations. The operations are the same regardless of whether the background Hamiltonian is present. Such a digital simulation of the model is not as robust against error, because there is no energy gap for creating excitations, i.e. processes where anyons are created out of the vacuum and cause errors. However, there is still topological order, which means that the outcomes of anyonic interferometry are invariant under deformations of the braiding path.

A promising platform for this kind of digital simulation is to use entangled states of single photons. This brings the advantage of easy single qudit manipulation with passive optical elements, low decoherence and well-characterized loss channels. Entangling gates between qudits is more difficult, but there are well-studied pathways for nondeterministically creating such operations. The key limitations for optics are poor single photon sources, limiting experiments to about six photons, and the difficulty of building circuits with large depth. Large depth amounts to complex nested interferometers, which require tuning and stabilization to a fraction of the wavelength. Recent advances in integrated quantum photonics [30, 31] hold the key to increasing the depth of circuits that can be created, and we envisage using just such a platform for simulating non-Abelian anyons.
Figure 1. A simulation of non-Abelian anyonic interferometry using entangled photons (based on a generic protocol in [33, 34]). (a) Begin with a surface cellulation where multilevel spins reside on all the links. The spins are prepared in a highly entangled state corresponding to the ground, i.e. vacuum, state of $H_{TO}$. An implementation with non-interacting photons would have $H_{TO} = 0$; yet provided that the entangled vacuum state is prepared and the same operations are done as if $H_{TO}$ were present, one obtains the same measurement outcomes. (b) A single plaquette of the lattice with physical spins indicated as dots, and red dots indicate where an operation is performed. Here beginning with the ground state of $H_{TO}$, a single-spin operation creates localized anyonic excitations. The electric charge/anti-charge excitations are drawn as diamonds and the flux excitations as squares. (c) Braiding of the flux around one charge. For a single plaquette a complete braiding path includes only the spins located at the two red dots. (d) Fusion of the electric charges realized with an operation on a single spin located at the red dot. An incomplete fusion into the vacuum state indicates the effect of non-Abelian statistics.

It is challenging to build up many-body entanglement using single photons and linear optics and so prototype experiments will be limited to small systems. Yet probing the physics of small scale systems is rewarding. In fact, it is the very virtue of topologically ordered systems that the physics is scale invariant. As shown in figure 1 a single plaquette of a spin lattice is enough to demonstrate braiding statistics of non-Abelian anyons. Of course, for such a small system it would be easy to perform a classical simulation to predict such behavior. Yet the point would be to exhibit the emergent behavior on small prototypes, using tools of coherent control, that could be extended to scalable systems. Two recent experiments have demonstrated a simulation of Abelian statistics in a single plaquette of a spin lattice [1, 2] using six and four photons, respectively. A simulation using nuclear magnetic resonance (NMR) control of four carbon atoms in a molecule of crotonic acid was also realized [32].

A simulation of non-Abelian statistics would mark significant progress for two reasons: firstly, the effect of non-Abelian statistics is more dramatic and easier to discriminate from geometric or dynamical phases, and secondly, several novel phenomena occur such as the existence of topological entropy, which could, in principle, be measured in small-scale systems. Another advantage of non-Abelian anyons is that they can be used to achieve general quantum computation [8]. The trade-off is an increased complexity in the physical operations. Yet for our scheme, only three type-I parametric downconversion crystals are needed to prepare the initial state, which is the same number used in the Lu et al [1] experiment to measure Abelian statistics.

The paper is organized in three parts: the generic lattice model; mapping to a single-plaquette system, and mapping to a physical system. Readers just wishing to see the details of the steps involved in the experimental proposal can skip to section 4. In section 2, we introduce the simplest spin lattice model of the discrete gauge theories of Kitaev that has
non-Abelian anyonic excitations. There are several different types of non-Abelian particles arising from this model, and in section 3 we describe one of several types of braiding experiments that could be done to reveal the non-Abelian statistics. The operations are done using six-level particles, which can be encoded in a qutrit–qubit pair. A physical realization of these ideas using linear optical elements and single photons is given in section 4. Finally, we conclude in section 5 with a discussion of the viability of this technique and possible further avenues for experimental inquiry.

2. Spin lattice model for non-Abelian anyons

2.1. Non-Abelian discrete gauge theory

Kitaev’s generalization of the surface code Hamiltonian is also a sum of vertex and face operators and has localized particle-like excitations, but now the ground states are invariant under gauge transformations generated by some finite group \( G = \{g_j\} \) of choice. We consider the smallest non-Abelian group \( G = S_3 \), which is the permutation group on three objects. To specify the spin lattice, we consider a cellulation of a two-dimensional surface with the vertex set \( V = \{v_i\} \), edge set \( E = \{e_j\} \), and face set \( F = \{f_j\} \). Qudits with \( d = |G| = 6 \) levels are placed on the edges and physical states are elements of a Hilbert space \( \mathcal{H} = \mathcal{H}(6)^{\otimes |E|} \) where \( \mathcal{H}(6) = \mathbb{C}|0\rangle + \cdots + \mathbb{C}|5\rangle \). Particles on edges that meet at a vertex \( v \) all interact via a vertex operator \( A(v) \). Similarly, all particles on edges that are on the boundary of a face \( f \) interact via \( B(f) \). We pick an orientation for each edge with \( e = [v_j, v_k] \) denoting an edge with an arrow pointing from vertex \( v_j \) to \( v_k \). The choice of edge orientations is not important as long as a consistent convention is used. The Hamiltonian is a sum of operators chosen such that the ground states of \( H_{10} \) are invariant under local gauge transformations

\[
T_g(v) = \prod_{e_j \in [v, v]} L_g(e_j) \prod_{e_j \in [v, v]} R_{g^{-1}}(e_j).
\]

Here \( L_g(e_j), R_g(e_j) \in U(6) \) are the permutation representations of the left and right action of multiplication by the group element \( g \in S_3 \) on the system particle located at edge \( e_j \). For the particle states we make the identification \( |j\rangle \equiv |g_j\rangle \), where by convention \( |0\rangle \equiv |g_0\rangle \equiv |e\rangle \), with \( e \) being the identity element. The action of left and right group multiplication on the basis states is then \( L_h|j\rangle = |h g_j\rangle \), and \( R_h|j\rangle = |g_j h\rangle \).

The spin lattice model is

\[
H_{10} = - \sum_v A(v) - \sum_f B(f),
\]

where

\[
A(v) = \frac{1}{6} \sum_{g \in S_3} T_g(v),
\]

\[
B(f) = \sum_{|h_k| \prod_{e_j \in f} h_2 = e} \bigotimes_{e_k} \left\{ h_k^{-o_f(e_j)} \right\}.
\]

In the definition of \( B(f) \), the sum is taken over all products of group elements \( h_k \) acting on a counterclockwise cycle of edges on the boundary of \( f \) such that the accumulated left action is the identity element \( e \in S_3 \) (i.e. \( h_1 h_{-1} \cdots h_2 h_1 = e \) for the counterclockwise cycle starting at edge \( e_1 \) and ending at edge \( e_5 \)). The function \( o_f(e_j) = \pm 1 \) according to whether

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Figure 2. The triangle of nodes on which to demonstrate the non-Abelian topological action. The white (black) diamonds indicate the vertices where the charges (anti-charges) reside, and the arrows indicate the directions of the edges. The dotted ellipses indicate the six-level qudits on the edges. Each qudit consists of a qutrit, labeled $b$ and indicated by the solid circle, and a qubit, labeled $a$ and indicated by the empty circle. The dashed lines indicate the entangled state that needs to be created between the three qutrits, and the dotted line indicates the two qubits that must also be entangled.

the orientation of the edge is the same as (or opposite to) the face orientation. By construction $[A(v), A(v')] = [B(f), B(f')] = [A(v), B(f)] = 0$. Furthermore, it is straightforward to verify that since $A(v)$ is a symmetrized gauge transformation it is a projection, as is $B(f)$. The ground states of $H_{TO}$ are then manifestly gauge invariant states. For a planar surface with boundary, the ground state $|GS\rangle$ of $H_{TO}$ is unique. Excited states can be described by localized particles in the sense that the expectation value $\langle A(v) \rangle = 0$ corresponds to an excited state with an electric charge located at vertex $v$ while $\langle B(f) \rangle = 0$ indicates a magnetic flux particle at face $f$. Note that if one were to begin in the ground state of $H_{TO}$, applying an operation to a single edge spin will produce a state where either $A(v) = 0$ on both vertices that are boundaries of the edge or $B(f) = 0$ on both faces that share that edge as a boundary, or both. These particles have anyonic statistics and we refer the reader to [33, 34] for a detailed description of the particle spectrum of this model and the operations used to create, manipulate and fuse the anyons.

The specific interferometry experiment we consider is the creation of an electric charge pair, i.e. a state that has $A(v_1) = A(v_2) = 0$, but $B(f) = 1$ for all faces and $A(v) = 1$ for all other vertices, followed by the braiding of a magnetic flux around one charge, and the subsequent fusion of the charges. This entire process can be simulated on a single lattice plaquette as summarized in figure 1. Pure electric charges are labeled by irreps $R$ of the group $S_3 = \{e, c_+, c_-, t_0, t_1, t_2\}$, where $e$ is the identity, $t_j$ are transpositions, and $c_\pm$ are three-cycles. There are three irreps of $S_3$. Two are one dimensional, namely $R^+_1$ (the identity irrep) and $R^-_1$ (the signed irrep), and both correspond to Abelian anyons. The two dimensional irrep $R_2$ corresponds to a non-Abelian anyon. For future reference, the two dimensional irrep of $S_3$ is

$$R_2(e) = 1_2, \quad R_2(t_k) = \sigma^x e^{i(2\pi/3)k} \sigma^z, \quad R_2(c_\pm) = e^{\pm i(2\pi/3)} \sigma^z. \quad (5)$$

Consider the lattice with a single face and three edges indicated in figure 2. The edges are oriented $e_1 = [v_1, v_2]$, $e_2 = [v_1, v_3]$, $e_3 = [v_2, v_3]$. For simplicity of notation, we label the edges
by the numbers 1, 2 and 3. The ground state is quite easy to write down:

\[ |GS⟩ = \frac{1}{6} \sum_{g,k} |k⟩_1 \otimes |g⟩_2 \otimes |k^{-1}g⟩_3 = \frac{1}{6} \sum_{g,k} |g⟩_2 |k, k^{-1}g⟩_{1,3}. \]  

(6)

The state of a generic electric charge anti-charge pair located at vertices \((v_1, v_3)\) in the irrep \(R\) (with dimension \(|R|\)) is given by

\[ |M_R; (v_1, v_3)⟩ \equiv \frac{1}{6} \sum_{g, k} \text{tr}[M_R R^\dagger(g)] |g⟩_2 \sum_{k} |k, k^{-1}g⟩_{1,3}. \]  

(7)

with \(M_R\) being an \(|R| \times |R|\) matrix normalized so that \(\sum_{a,b=1}^{|R|} |(M_R)_{a,b}|^2 = |R|\). Under a local gauge transformation \(T_h\) at vertex \(v_1\), the basis states undergo the mapping:

\[ T_h(v_1) |M_R; (v_1, v_3)⟩ = \frac{1}{6} \sum_{g, k} \text{tr}[M_R R^\dagger(g)] |hg⟩_2 |hk, k^{-1}g⟩_{1,3}, \]

and located at face \(f\) is braided in a counterclockwise sense around a charge at location \(v\), the action is

\[ \mathcal{R}^2_{j,v} |h⟩ |M_R; (v, v′)⟩ = |h⟩ |R(h)M_R; (v, v′)⟩, \]  

(9)

where \(\mathcal{R}^2_{i,j}\) is the square of the monodromy operator (it represents the braiding particle at location \(i\) around the particle at location \(j\) in a counterclockwise sense). If a flux with value \(h\) is braided around both charges, then the action on the state is conjugation:

\[ \mathcal{R}^2_{j,v} \mathcal{R}^2_{j,v′} |h⟩ |M_R; (v, v′)⟩ = |h⟩ |R(h)M_R R(h^{-1}); (v, v′)⟩. \]  

(10)

There is one state that is invariant under conjugation, the fluxless charge: \(|1_R; (v, v′)⟩\).

In fact, there is a simple interpretation of this state when the vertices are nearest neighbors, i.e. there is an edge \(e = [v, v′]\). In that case, the pair of charges is created out of the vacuum by acting on the single spin living on the edge \(e\) with

\[ W_R(e) |GS⟩ = |1_R; (v, v′)⟩, \]  

(11)

where

\[ W_R(e) = \sum_{g \in S_3} |g⟩_e ⟨g| \chi_R^*(g) \]  

(12)
with $\chi_R$ being the character of the group element in the irrep $R$. This operator is in general neither unitary nor Hermitian, but for the group $S_3$ it is Hermitian for all irreps.

Finally, the amplitude for the process of fusion of the electric charge pair into the vacuum is given by

$$F(R(h) \to vac) \equiv \langle 1_R; (v, v') | R(h); (v, v') \rangle = \frac{\text{tr}\{R(h)\}}{|R|}. \quad (13)$$

For non-Abelian electric charges, the fusion probability, i.e. the squared modulus of the fusion amplitude, will fail to be unity for some group element $h$. This is because for non-Abelian anyons there can be multiple outcomes when fusing particles. In contrast, Abelian charges are one-dimensional representations in which case equation (13) shows that the fusion probability into the vacuum is unity.

The fusion amplitudes can be measured by using an ancillary qubit to perform conditional gauge transformations and then measuring the ancilla. For example, say we prepare an ancilla in $|\pm_x\rangle_{anc}$, then apply the controlled operation

$$|0\rangle_{anc} \otimes 1 + |1\rangle_{anc} \otimes T_h(v), \quad (14)$$

followed by measurement of the ancilla in the basis $|\pm_x\rangle$ with outcome $m = \pm 1$. The outcome distribution satisfies

$$P(m = 1) - P(m = -1) = \Re[\langle 1_R; (v, v') | R(h); (v, v') \rangle], \quad (15)$$

which is the real part of the fusion amplitude for $R(h) \to 1$. Similarly, measuring the ancilla in the basis $|\pm_y\rangle$ yields the imaginary part of the fusion amplitude for $R(h) \to \text{vacuum}$.

Let us now calculate the expected outcomes for non-Abelian anyonic interferometry. Using equations (5) and (13), the fusion amplitudes to the vacuum are

$$F(R_2(h) \to vac) = \begin{cases} 1 & h = e \\ -\frac{1}{2} & h = c_+ \\ 0 & h = t_j \forall j \end{cases}. \quad (16)$$

For the measurement proposed here, it is also instructive to see how these arise explicitly in the state overlap. The projection operator onto $R_2$ charge is

$$W_{R_2} = 2|e\rangle\langle e| - |c_+\rangle\langle c_+| - |c_-\rangle\langle c_-|. \quad (17)$$

Say we have a fluxless charge pair on nearest-neighbor vertices bounding the edge $e = [v, v']$. Then the overlap is

$$\langle 1_{R_2}; (v, v') | R_2(h); (v, v') \rangle = \langle 1_{R_2}; (v, v') | T_h | 1_{R_2}; (v, v') \rangle = \langle GS | W_{R_2}(e) T_h W_{R_2}(e) | GS \rangle = \langle GS | W_{R_2}(e) T_h W_{R_2}(e) T_h^\dagger | GS \rangle = \langle GS | W_{R_2}(e) W_{R_2}^h(e) | GS \rangle = \frac{1}{6} \text{tr} \{ W_{R_2}(e) W_{R_2}^h(e) \}, \quad (18)$$

where

$$W_{R_2}^h(e) = 2|h\rangle\langle h| - |hc_+\rangle\langle hc_+| - |hc_-\rangle\langle hc_-|. \quad (19)$$
In the third line, we have used the fact that the group state is invariant under gauge transformations $T_h$ and in the last line we have used the fact that the reduced state of any edge qudit in the group state is maximally mixed. Evaluating the trace we recover the fusion amplitudes in equation (16) by considering the state overlap explicitly. For the given initial state $|1_R; (v, v') \rangle$ it is also possible to infer a subset of the fusion rules for the theory without using an ancilla at all as we show in section 4.3.

3. Single plaquette scheme

3.1. Initial state preparation

An algorithm to prepare the ground state of $H^{TO}$ in a spin lattice is given in [33]. However, since we only want to demonstrate the action of braiding a flux around an electric charge pair, it is sufficient to begin with the state $|1_R; (v, v') \rangle$. Specifically, we want to create the state where the charges are on vertices $v_1$ and $v_3$ as in figure 2, i.e.

$$|1_R; (v_1, v_3) \rangle \equiv \frac{1}{6}[2|e \rangle_1|0 \rangle_2|0 \rangle_3 + |t_0 \rangle_0|t_1 \rangle_1|t_2 \rangle_2 + |c_+ \rangle_4|c_- \rangle_5 + |c_- \rangle_4|c_+ \rangle_5]$$

$$-|c_+ \rangle_4|c_- \rangle_5 + |t_0 \rangle_0|t_1 \rangle_1|t_2 \rangle_2 + |c_+ \rangle_4|c_- \rangle_5 + |c_- \rangle_4|c_+ \rangle_5]$$

$$-|c_- \rangle_4|c_+ \rangle_5 + |t_0 \rangle_0|t_1 \rangle_1|t_2 \rangle_2 + |c_- \rangle_4|c_+ \rangle_5 + |c_+ \rangle_4|c_- \rangle_5].$$

(20)

Here the subscripts indicate the qudit that this is the state for. In figure 2, the qudits correspond to the qutrit/qubit pairs, and the numbers are shown next to the pair. The qutrits and qubits in the pairs are further labeled ‘$a$’ for the qubits and ‘$b$’ for the qutrits.

There are a number of options on how the six-level qudit can be encoded in the qutrit/qubit pair. It is convenient to use the encoding that respects the semidirect product structure of the group $S_3$ (see appendix B of [34]). We pick for qudits 1 and 2 the encoding:

$$|e \rangle \equiv |1 \rangle_a|0 \rangle_b, \quad |t_2 \rangle \equiv |0 \rangle_a|2 \rangle_b, \quad |t_0 \rangle \equiv |0 \rangle_a|0 \rangle_b,$$

$$|c_+ \rangle \equiv |1 \rangle_a|2 \rangle_b, \quad |t_1 \rangle \equiv |0 \rangle_a|1 \rangle_b, \quad |c_- \rangle \equiv |1 \rangle_a|1 \rangle_b,$$

and for qudit 3 the encoding:

$$|e \rangle \equiv |1 \rangle_a|0 \rangle_b, \quad |t_2 \rangle \equiv |0 \rangle_a|2 \rangle_b, \quad |t_0 \rangle \equiv |0 \rangle_a|0 \rangle_b,$$

$$|c_+ \rangle \equiv |1 \rangle_a|1 \rangle_b, \quad |t_1 \rangle \equiv |0 \rangle_a|1 \rangle_b, \quad |c_- \rangle \equiv |1 \rangle_a|2 \rangle_b.$$  

(21)

(22)

Using this encoding, the initial state becomes

$$|1_R; (v_1, v_3) \rangle \equiv \frac{1}{6}[|1 \rangle_2|0 \rangle_1|0 \rangle_3|0 \rangle_2 + |1 \rangle_2|0 \rangle_1|0 \rangle_3|0 \rangle_2 + |2 \rangle_2|0 \rangle_1|0 \rangle_3|0 \rangle_2 + |1 \rangle_2|1 \rangle_1|1 \rangle_3|0 \rangle_2 + |2 \rangle_2|1 \rangle_1|1 \rangle_3|0 \rangle_2 + |1 \rangle_2|1 \rangle_1|1 \rangle_3|0 \rangle_2$$

$$+ |2 \rangle_2|1 \rangle_1|1 \rangle_3|0 \rangle_2 + |1 \rangle_2|1 \rangle_1|1 \rangle_3|0 \rangle_2 + |2 \rangle_2|1 \rangle_1|1 \rangle_3|0 \rangle_2 + |1 \rangle_2|1 \rangle_1|1 \rangle_3|0 \rangle_2]$$

(23)

This factorizes into an entangled pair of qubits and an entangled triple of qutrits, as indicated in figure 2.

Given the initial state preparation, it remains to be shown how to perform controlled $T_{c\pm}$ and $T_{t_i}$ operations.
3.2. The controlled \( T_{c_+}(v_1) \) operations

From equation (2), the operation \( T_{c_+}(v_1) \) involves operations on the adjacent edges, i.e. with qudits 1 and 2. Therefore, the effect of this operation is to exchange the basis states of qudits 1 and 2 as

\[
|e\rangle \rightarrow |c_+\rangle, \quad |t_0\rangle \rightarrow |t_2\rangle, \quad |c_+\rangle \rightarrow |c_-\rangle, \quad |t_1\rangle \rightarrow |t_0\rangle, \quad |c_-\rangle \rightarrow |e\rangle, \quad |t_2\rangle \rightarrow |t_1\rangle. \tag{24}
\]

With encoding (21), this corresponds to cyclically permuting the qutrit states as

\[
|0\rangle \rightarrow |2\rangle, \quad |1\rangle \rightarrow |0\rangle, \quad |2\rangle \rightarrow |1\rangle. \tag{25}
\]

This means that in order to measure the fusion amplitudes and perform the controlled \( T_{c_+}(v_1) \) operations of equation (14), we need to do two controlled permutations—one on 1b and one on 2b.

The operation \( T_{c_-}(v_1) \) is the inverse, and gives the opposite permutation of the qutrit states

\[
|0\rangle \rightarrow |1\rangle, \quad |1\rangle \rightarrow |2\rangle, \quad |2\rangle \rightarrow |0\rangle. \tag{26}
\]

In this case, the controlled operation is just the same, except that it uses the opposite permutations.

3.3. The controlled \( T_{c_0}(v_1) \) operation

If, instead, we want to perform the operation \( T_{c_0}(v_1) \), then we need to exchange the states for qudits 1 and 2 as

\[
|e\rangle \leftrightarrow |t_0\rangle, \quad |t_1\rangle \leftrightarrow |c_+\rangle, \quad |t_2\rangle \leftrightarrow |c_-\rangle. \tag{27}
\]

This operation can be achieved by employing a modified \( \text{CNOT} \) between the qubit and qutrit that only acts on two states of the qutrit. A \( \text{CNOT} \) usually performs a \( \text{NOT} \) operation on a target qubit if the control qubit is in the state \( |1\rangle \), and has been demonstrated using linear optics on dual-rail qubits [35]. The modified \( \text{CNOT} \) has a qutrit as the target, but only performs the \( \text{NOT} \) operation on a two-dimensional subspace of this qutrit. This can be achieved simply by applying a linear-optical \( \text{CNOT} \) on two modes of the target qutrit. These types of modified qubit gates have demonstrated an advantage in constructing multiqubit gates [36, 37]. First apply such a \( \text{CNOT} \) with the qubit as control and on states 1 and 2 of the qutrit, followed by a \( \text{NOT} \) gate on the qubit, followed by another \( \text{CNOT} \) between the qubit and qutrit. Explicitly this sequence of operations gives

\[
|e\rangle \equiv |1\rangle_a|0\rangle_b \rightarrow |1\rangle_a|0\rangle_b \rightarrow |0\rangle_a|0\rangle_b \equiv |t_0\rangle,
|t_0\rangle \equiv |0\rangle_a|0\rangle_b \rightarrow |0\rangle_a|0\rangle_b \rightarrow |1\rangle_a|0\rangle_b \equiv |e\rangle,
|t_1\rangle \equiv |0\rangle_a|1\rangle_b \rightarrow |0\rangle_a|1\rangle_b \rightarrow |1\rangle_a|1\rangle_b \equiv |c_+\rangle,
|t_2\rangle \equiv |0\rangle_a|2\rangle_b \rightarrow |0\rangle_a|2\rangle_b \rightarrow |1\rangle_a|2\rangle_b \equiv |c_-\rangle,
|c_+\rangle \equiv |1\rangle_a|2\rangle_b \rightarrow |1\rangle_a|2\rangle_b \rightarrow |0\rangle_a|1\rangle_b \equiv |t_1\rangle,
|c_-\rangle \equiv |1\rangle_a|1\rangle_b \rightarrow |1\rangle_a|1\rangle_b \rightarrow |0\rangle_a|2\rangle_b \equiv |t_2\rangle. \tag{28}
\]

Here the first and third arrows correspond to \( \text{CNOTs} \) and the second corresponds to the \( \text{NOT} \) gate on the qubit.
To achieve a controlled $T_{t_0}(v_1)$, we can take advantage of the fact that the CNOT applied twice is the identity. Therefore, we can apply the CNOT, then apply a CNOT with the ancilla as control and the qubit in the qubit/qudit pair as target, then apply another CNOT between the qubit and qudit. This needs to be done twice—one for qubit/qudit pair 1 and again for pair 2. A further simplification may be obtained by noting that the measurement of the ancilla can take place immediately after the CNOTs with the ancilla as the control. The final CNOTs within the qubit–qudit pairs can be omitted, as they do not affect the probabilities of the measurement results.

3.4. The controlled $T_{t_1}(v_1)$ operation

Next, the operation $T_{t_1}(v_1)$ exchanges the states for qudits 1 and 2 as

$$|e\rangle \leftrightarrow |t_1\rangle, \quad |r_2\rangle \leftrightarrow |c_+\rangle, \quad |t_0\rangle \leftrightarrow |c_-\rangle.$$  \hspace{1cm} (29)

This operation can be achieved in a similar way to $T_{t_0}(v_1)$, but with CNOTs acting on states 0 and 1 for the qudit. Explicitly, the sequence of operations gives

$$|e\rangle \equiv |1\rangle_a|0\rangle_b \rightarrow |1\rangle_a|1\rangle_b \rightarrow |0\rangle_a|1\rangle_b \rightarrow |0\rangle_a|1\rangle_b \equiv |t_1\rangle,$$

$$|t_0\rangle \equiv |0\rangle_a|0\rangle_b \rightarrow |0\rangle_a|0\rangle_b \rightarrow |1\rangle_a|0\rangle_b \rightarrow |1\rangle_a|1\rangle_b \equiv |c_-\rangle,$$

$$|t_1\rangle \equiv |0\rangle_a|1\rangle_b \rightarrow |0\rangle_a|1\rangle_b \rightarrow |1\rangle_a|1\rangle_b \rightarrow |1\rangle_a|0\rangle_b \equiv |e\rangle,$$

$$|t_2\rangle \equiv |0\rangle_a|2\rangle_b \rightarrow |0\rangle_a|2\rangle_b \rightarrow |1\rangle_a|2\rangle_b \rightarrow |1\rangle_a|2\rangle_b \equiv |t_2\rangle,$$

$$|c_+\rangle \equiv |1\rangle_a|2\rangle_b \rightarrow |1\rangle_a|2\rangle_b \rightarrow |0\rangle_a|2\rangle_b \rightarrow |0\rangle_a|2\rangle_b \equiv |t_2\rangle,$$

$$|c_-\rangle \equiv |1\rangle_a|1\rangle_b \rightarrow |1\rangle_a|0\rangle_b \rightarrow |0\rangle_a|0\rangle_b \rightarrow |0\rangle_a|0\rangle_b \equiv |t_0\rangle.$$ \hspace{1cm} (30)

3.5. The controlled $T_{t_2}(v_1)$ operation

Similarly, the operation $T_{t_2}(v_1)$ exchanges the states for qudits 1 and 2 as

$$|e\rangle \leftrightarrow |t_2\rangle, \quad |t_0\rangle \leftrightarrow |c_+\rangle, \quad |t_1\rangle \leftrightarrow |c_-\rangle.$$  \hspace{1cm} (31)

Again, the operation follows the above procedure, but with the CNOTs acting on states 0 and 2 for the qudit. Explicitly the sequence of operations gives

$$|e\rangle \equiv |1\rangle_a|0\rangle_b \rightarrow |1\rangle_a|2\rangle_b \rightarrow |0\rangle_a|2\rangle_b \rightarrow |0\rangle_a|2\rangle_b \equiv |t_2\rangle,$$

$$|t_0\rangle \equiv |0\rangle_a|0\rangle_b \rightarrow |0\rangle_a|0\rangle_b \rightarrow |1\rangle_a|0\rangle_b \rightarrow |1\rangle_a|2\rangle_b \equiv |c_+\rangle,$$

$$|t_1\rangle \equiv |0\rangle_a|1\rangle_b \rightarrow |0\rangle_a|1\rangle_b \rightarrow |1\rangle_a|1\rangle_b \rightarrow |1\rangle_a|1\rangle_b \equiv |c_-\rangle,$$

$$|t_2\rangle \equiv |0\rangle_a|2\rangle_b \rightarrow |0\rangle_a|2\rangle_b \rightarrow |1\rangle_a|2\rangle_b \rightarrow |1\rangle_a|2\rangle_b \equiv |t_2\rangle,$$

$$|c_+\rangle \equiv |1\rangle_a|2\rangle_b \rightarrow |1\rangle_a|0\rangle_b \rightarrow |0\rangle_a|0\rangle_b \rightarrow |0\rangle_a|0\rangle_b \equiv |t_0\rangle,$$

$$|c_-\rangle \equiv |1\rangle_a|1\rangle_b \rightarrow |1\rangle_a|1\rangle_b \rightarrow |0\rangle_a|1\rangle_b \rightarrow |0\rangle_a|1\rangle_b \equiv |t_1\rangle.$$ \hspace{1cm} (32)

Other configurations of the charges turn out to be equivalent to the states above. In the case where both charges are on the lower nodes, the encoding on edges 1 and 2 is the same.
as before, as are the operations that need to be performed on them. The only difference is in
the encoding of the third edge, which simply corresponds to a different interpretation of the
experiment. The state obtained if both charges are on the right is again equivalent, although this
time with different operations. For details see the appendix.

4. Linear optics implementation

Next, we consider how this scheme can be achieved using a linear optics implementation. We
first describe, in section 4.1, how to prepare the initial state via an interferometric circuit and a
tri-rail encoding of the qutrits. The advantage of the tri-rail encoding is that arbitrary unitaries
can be performed on these qutrits using linear optics [38]. This entangled state encodes
the three-node plaquette depicted in figure 2. Then, in section 4.2, we show how to implement
controlled $T_j$ operations, using additional ancillary modes and photons, to measure the fusion
amplitudes given by equation (13). Remarkably, in section 4.3, we find that for a subset of the
fusion rules, we do not need to introduce any ancillary modes or use controlled operations,
which substantially simplifies the experiment. It is this simplified method that we propose for
the experiment, as it introduces very little additional complexity. The experimental signature to
be detected is then the expectation value of the operator $W_{R_2}$, which can be determined simply
by photon counting in the output modes.

4.1. Initial state preparation

One possible way of creating the three-qutrit entangled state required for $|1_{R_2}: (v_1, v_3)\rangle$,
\[
\{2|0\rangle_{2b}(|0\rangle_{1b}|0\rangle_{3b} + |1\rangle_{1b}|1\rangle_{3b} + |2\rangle_{1b}|2\rangle_{3b}) - |2\rangle_{2b}(|0\rangle_{1b}|1\rangle_{3b} + |1\rangle_{1b}|2\rangle_{3b} + |2\rangle_{1b}|0\rangle_{3b}) \\
- |1\rangle_{2b}(|0\rangle_{1b}|2\rangle_{3b} + |1\rangle_{1b}|0\rangle_{3b} + |2\rangle_{1b}|1\rangle_{3b})\}/(3\sqrt{2}),
\]
(33)
is to prepare the initial state $|0\rangle_{2b}|\psi_1\rangle_{1b,3b}$, where
\[
|\psi_1\rangle_{1b,3b} = (|0\rangle_{1b}|0\rangle_{3b} + |1\rangle_{1b}|1\rangle_{3b} + |2\rangle_{1b}|2\rangle_{3b})/\sqrt{3}
\]
(34)
is a maximally entangled two-qutrit state, and to propagate this state through the circuit in
figure 3. The phase angles $\phi$ and $\theta$ in the circuit are given by
\[
\theta = \arcsin\left[\frac{10}{\sqrt{247}}\right], \quad \phi = \arcsin\left[\frac{7 + \sqrt{3}}{2\sqrt{26}}\right] - \frac{\pi}{4}.
\]
(35)
Each of the beam splitters depicted acts in a symmetric way, transforming two optical
modes described by the boson creation operators $a^\dagger$ and $b^\dagger$, as $a^\dagger \rightarrow i\sqrt{R} a^\dagger + \sqrt{1 - R} b^\dagger$ and $b^\dagger \rightarrow i\sqrt{R} b^\dagger + \sqrt{1 - R} a^\dagger$, where $R$ is the reflectivity indicated. The circuit relies on a
final postselection to ensure only one photon is present in each qutrit, which happens with a
probability of $9/55$. The remaining time, invalid qutrit states are produced such as having two
photons in three modes and these can be postselected out.

Although the circuit in figure 3 looks daunting, each cnot gate of the type in [35] takes
five beam splitter transformations to implement, so the circuit is the same order in complexity
as three such gates. In fact, the transformation implemented by this circuit is equivalent to a
single qutrit transformation on qutrit 2 followed by a ternary adder gate between qutrits 2 and
Figure 3. The scheme used to produce the desired three-qutrit state from a pair of entangled qutrits and an additional photon (the state $|0\rangle_{2b}$ corresponds to a single photon in mode $2b_0$). The horizontal lines indicate the different optical modes, and the vertical lines indicate symmetric beam splitters connecting modes with reflectivities shown. The open circles indicate a relative phase shift in that mode. Note that the beam splitter connecting mode $v_0$, which is initially in a vacuum, and mode $1b_1$ is to introduce loss to balance the circuit. The phase angles $\phi$ and $\theta$, and the beam splitter convention, are given in the text.

3 ($|x, y\rangle \rightarrow |x, x \oplus y\rangle$). The single qubit version of this transformation would take at least two beam splitters and the ternary version can be implemented by four qubit \textsc{cnot} gates acting between pairs of qutrit levels. That is, the circuit that can be used to perform a two-qubit \textsc{cnot} is instead applied to two modes out of the three encoding each qutrit. If such a naive application of qubit gates were used to synthesize the transformation, the circuit would already require 22 beam splitters. The circuit in figure 3 has been optimized in comparison but it may be possible to optimize it further.

There are potentially a number of ways of preparing the entangled state (34). A fairly direct way, which can make use of the same final postselection as the circuit, is to use the output of three type-I spontaneous parametric downconversion (SPDC) crystals in a similar way to (1). Each crystal produces an infinite-dimensional entangled state between two spatial modes,

$$|\text{SPDC}\rangle = \sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} \lambda^n |nn\rangle.$$  

Here $\lambda$ characterizes the strength of squeezing in the SPDC process, and $n$ denotes the photon number. The two occurrences of $n$ in the state indicate identical photon numbers in two spatial modes. For quantum information applications, the state is usually postselected on a single photon being obtained in each spatial mode, thus producing two photons entangled in energy
and momentum. For three independent crystals with equal amplitudes for pair production, the state is of the form

$$|\text{SPDC}\rangle^{\otimes 3} = (1 - \lambda^2)^{3/2} \sum_{n_1, n_2, n_3 = 0}^{\infty} \lambda^{n_1+n_2+n_3} |n_1 n_1, n_2 n_2, n_3 n_3\rangle. \quad (37)$$

Here the $n_j$ are the photon numbers for the modes produced by crystal $j$. If we postselect on obtaining a single pair of photons, then since the pair may have arisen from any of the three crystals, we obtain the state (34) with probability $\lambda^2(1 - \lambda^2)^3$. So, by using three type-I SPDC sources and an additional photon and counting three photons in the output, one for each qutrit, we can ensure that the only component in the input that contributes to the output corresponds to the entangled state (34), and ensure the correct operation of the circuit in figure 3.

An alternative method of producing entangled qutrits has been proposed recently [39]. That method has an advantage in that it is heralded, but it produces qutrits encoded in the photon-number basis, which makes it more difficult to perform subsequent operations.

The generation of the three-qutrit entangled state uses two pairs of downconverted photons: one for the state $|\psi_3\rangle$, and a second to produce the single photon input to qutrit $2b$ (by postselection on detection of the other photon in the pair). The state $|1_{R_2}; (v_1, v_3)\rangle$ also requires a two-qubit entangled state, which may also be produced by parametric downconversion. Except for cases where these photons must be interfered with those forming the three-photon entangled state, this does not significantly increase the complexity of the scheme, because these photons may be produced independently.

4.2. Implementing the controlled $T_g$ operations

Given our initial state of a pair of electric charge anyons, we can measure the effect of braiding a flux around one member of the pair by performing controlled gauge transformations $T_g$ at one vertex (here vertex $v_1$) as described in section 2. There are six operators $T_g$, one for each group element, and recall that they involve a product of operations on spins that reside on edges that meet at vertex $v_1$. The operator $T_e$ is just the identity and is therefore trivial. First consider how to perform the operators $T_c \pm$ controlled on the state of a qubit ancilla. For these two operators, it is necessary to perform controlled swaps on two qutrits in the entangled state, with a qubit as the control. These operations can be performed using repeated CNOTs.

Here we consider the linear optical CNOT of the type demonstrated in [35]. For this gate to function correctly it is necessary to ensure that each pair of output modes contains a single photon. This can make it problematic to chain these gates together, as it may be necessary to ensure that a single photon is present in an intermediate step where we cannot detect the photon directly. However, it is still possible provided that it can be inferred that a single photon was present. For example, that is the approach used in [36]. In the circuits to follow, we make use of CNOT gates that act on pairs of qubits and also between one qubit and two modes of a qutrit. Both types of gates have been demonstrated in the laboratory [36].

One method to achieve the repeated linear-optical CNOTS is to replace the single control qubit with two control qubits, as shown in figure 4. The CNOTS controlled by the lower rails simply give the required permutation on the first qutrit. The CNOTS with the upper rail as control give the inverse permutation. This is followed by the desired permutation without controls, with the effect that the desired permutation is performed if the qubit photons are in the lower rails.
Figure 4. The sequence of linear optical cnots to perform the controlled $T_{c_2}$ operation in the case where there is initially a single photon for each qubit and qutrit. The pairs of modes 4 and 5 are the two ancillas, and are initially in an entangled state.

Provided that there is a single photon in each qubit or qutrit in the input and we postselect on single photons at the output, the possibility of photon transfer is eliminated and the controlled operation is performed correctly. For example, if a photon were transferred from the control (qubit 5) to the target (qutrit 1) at the first cnot, then a photon must be transferred from the target (qutrit 1) to the control (qubit 4) at the second one to ensure that qutrit 1 has no more than one photon. However, this would imply that qubit 4 will have more than one photon.

For the initial state produced by the scheme in section 4.1, it is necessary to postselect on a single photon in each of the output qutrits. The above scheme with two ancilla qubits cannot be applied, because it is not possible to infer that there was only one photon in the qutrits between preparation of the state and application of the cnots. The cnots can be achieved with four ancillas in an entangled state and each cnot applied with a different ancilla as the control. Because no ancilla is reused, any photon transfer could be detected. Although this works with the states produced by the state preparation scheme, it would be very challenging to perform experimentally due to the requirement of generating four entangled photons.

Next, to achieve the controlled $T_{t_1}$ operation, we can use the following approach.

(i) Generate the three-qutrit and two-qubit entangled states.
(ii) Apply cnots between the qubits and qutrits in pairs 1 and 2.
(iii) Apply nots on the qubits in pairs 1 and 2.
(iv) Apply cnots with an ancilla qubit as the control and the qubits in pairs 1 and 2 as targets.
(v) Measure the ancilla qubit in the $\pm$ basis.
(vi) Perform cnots between the qubits and qutrits in pairs 1 and 2 again.
The sequence of linear optical CNOTs to perform the controlled $T_{t_0}$ operation. The pair of modes 4 is the ancilla and is initially in a superposition state. The qubit 2a is initially in the state $|1\rangle$, so the CNOT just simplifies to a NOT operation.

The final step (vi) can be omitted, because it has no effect on the probabilities of the measurement results. In addition, the NOT between 2a and 2b just simplifies to a NOT, and can be performed deterministically. Steps (iii) and (iv) can be combined by making the NOTs controlled by the 0 state (of the ancilla), rather than 1. The scheme can then be achieved as shown in figure 5. This example is for $T_{t_0}$; the cases of $T_{t_1}$ and $T_{t_2}$ can be addressed by applying the CNOT and NOT to different pairs of modes in the qutrits.

There are only three CNOTs required in each case. There is one CNOT between 1a and 1b. Then there is a CNOT between the ancilla qubits 4 and 1a and another between 4 and 2a. These three CNOTs require that there is no photon transfer. In this case, photon transfer can be ruled out because the ancilla qubits have single photons input, and we postselect on single photons output. We also postselect on single photons at the outputs of the qutrits. This ensures that there can be no photon transfer, and the CNOTs must have worked. This case is simpler than that for $T_{c_0}$, because only one ancilla photon is required for the control, rather than four in an entangled state, and one less CNOT is required.

In contrast to the case of $T_{c_0}$, there is now interaction with two of the qubits in the initial state. However, qubit 2a is not entangled with any other subsystem, and qubit 1a is entangled with qubit 3a, which is discarded. Qubit 3a can be discarded before the operations in figure 5 without altering the results. Therefore, for the controlled $T_i$ operations, we require a total of three ancilla photons, and they need not be entangled.

The anyonic fusion rules given above require that the probabilities for measuring the ancilla qubit in the states + and − are equal. It can immediately be seen that this is what will be obtained,

**Figure 5.** The sequence of linear optical CNOTs to perform the controlled $T_{t_0}$ operation. The pair of modes 4 is the ancilla and is initially in a superposition state. The qubit 2a is initially in the state $|1\rangle$, so the CNOT just simplifies to a NOT operation.
because the \textsc{cnot} between 4 and 2\textsubscript{a} results in these qubits being in an entangled state. As 2\textsubscript{a} is then discarded without further interaction, this is equivalent to just decoherence of the ancilla qubit 4. Then the probabilities of measuring both + and — are equal to 1/2. The difference in probabilities for measuring + or — in the case of \( T_{e\alpha} \) is because the state is not orthogonal to the initial state. Therefore, the controlled operation does not result in the ancilla being completely entangled with the system.

In summary, performing the controlled \( T_{e\alpha} \) operations requires four entangled ancilla photons and four \textsc{cnot} operations. If the required three-qutrit entangled state did not require postselection of the number of photons at its output, the number of ancilla photons required could be reduced to two. Performing the controlled \( T_{i} \) operations requires three unentangled ancilla photons and three \textsc{cnots} and is therefore simpler.

4.3. Measuring fusion data without controlled operations

An alternative, simpler, approach to extract information about the non-Abelian statistics is to bypass controlled operations altogether and directly measure operators \( W_{R} \) at edge \( e \), in electric charge states \( |M_{R}\rangle \), where \( R \) and \( R' \) are possibly different irreducible representations. These measurements give nontrivial information about the anyon model, namely information about a subset of the fusion rules of the theory. In particular, \( \langle M_{R}|W_{R}|M_{R}\rangle \) is always zero if \( R' \times R \) does not contain \( R' \); and by varying \( M \), one can probe the matrices \( Q^{[RR' \times R]} \) (to be defined below) implementing the fusion rules as projectors made of 3\textit{j} symbols.

Let us consider first the case where \( R = R' = R_{2} \), the two-dimensional irrep of \( S_{3} \). The outcome of the measurement of \( W_{R_{2}} \) in the gauge transformed state \( T_{h}(v)|1_{R_{2}}; (v, v')\rangle = |R_{2}(h); (v, v')\rangle \) is

\[
\langle R_{2}(h); (v, v')|W_{R_{2}}(e)|R_{2}(h); (v, v')\rangle = \langle 1_{R_{2}}; (v, v')|T_{h}^{\dagger}W_{R_{2}}(e)T_{h}(v)|1_{R_{2}}; (v, v')\rangle \\
= \langle GS|W_{R_{2}}(e)W_{R_{2}}^{h^{-1}}(e)W_{R_{2}}(e)|GS\rangle \\
= \frac{1}{5}\text{tr}[(W_{R_{2}}(e))^{2}W_{R_{2}}^{h^{-1}}(e)].
\]

We find

\[
\langle R_{2}(h); (v, v')|W_{R_{2}}(e)|R_{2}(h); (v, v')\rangle = \begin{cases} 
1 & h = e, \\
-1 & h = c_{\pm}, \\
0 & h = t_{j} \forall j.
\end{cases}
\]

This reproduces the fusion amplitudes computed before, but this is just an accident. In fact, such an expectation value probes the fusion rules of the theory:

\[
\langle R(h)|W_{R}^{\dagger}|R(h)\rangle = \sum_{a, b, \alpha, \beta, \alpha', \beta'} |R|^{\alpha} |R'|^{\beta} \sum_{c, e=1} Q^{[RR' \times R]}_{abcd, bce} R_{ab}^{\ast}(h)R_{de}(h),
\]

where \( Q^{[R^{(1)}R^{(2)} \times R^{(3)}]} \) are the projectors onto the vacuum fusion channel for three irreducible representations, that is,

\[
Q^{[R^{(1)}R^{(2)}R^{(3)}]}_{ace, bcf} = \frac{1}{|G|} \sum_{g} R_{ab}^{(1)}(g)R_{cd}^{(2)}(g)R_{ef}^{(3)}(g) \\
= \sum_{\alpha} q^{[R^{(1)}R^{(2)} \times R^{(3)}]}_{ace, \alpha} q^{[R^{(3)}R^{(2)} \times R^{(1)}]}_{bcf, \ast \alpha},
\]

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where the $q^{(R^{(1)} R^{(2)}) a}$ form an orthonormal basis for the $+1$ eigenspace of $Q^{(R^* R^* R)}$; they are more familiar as $3j$ symbols in the case of angular momentum.

Measurements for $(R_2(h); (v, v')|W_{R_2}(e)|R_2(h); (v, v'))$ can easily be performed deterministically. $T_{c_\pm}$ is just a permutation of the three modes on the qutrit. The operation $T_{t_0}$ gives the change in the encoded states

$$|1\rangle_a|0\rangle_b \leftrightarrow |0\rangle_a|1\rangle_b, \quad |0\rangle_a|1\rangle_b \leftrightarrow |1\rangle_a|2\rangle_b, \quad |0\rangle_a|2\rangle_b \leftrightarrow |1\rangle_a|1\rangle_b. \quad (42)$$

This can be achieved by a NOT gate on the qubit, and swapping states 1 and 2 for the qutrit. Then operation $T_{t_1}$ is the same, except for swapping states 0 and 1 for the qutrit, and $T_{t_2}$ is the same except for swapping states 0 and 2. These are all operations that can be performed deterministically. Then the operation $W_{R_2}$ can just be measured by measuring the qubits and qutrits in their computational basis.

In particular, with the operation $T_{c_1}$ the unnormalized state becomes

$$|1\rangle_{2a}|0\rangle_{1a}|0\rangle_{3a} + |1\rangle_{1a}|1\rangle_{3a}|2\rangle_{2b}|2\rangle_{1b}|0\rangle_{3b} + |0\rangle_{1b}|1\rangle_{3b}|1\rangle_{2b}|2\rangle_{1b}|1\rangle_{3b} - |1\rangle_{2b}|(2\rangle_{1b}|1\rangle_{3b} + |0\rangle_{1b}|2\rangle_{2b}|1\rangle_{3b} + |1\rangle_{1b}|0\rangle_{3b} + |1\rangle_{1b}|1\rangle_{3b}]. \quad (43)$$

Then the probabilities of measuring $|e\rangle$, $|c_+\rangle$ and $|c_-\rangle$ on edge 2 are 1/6, 2/3 and 1/6 respectively, which gives $\langle W_{R_2} \rangle = -1/2$.

With the operation $T_{t_0}$ the unnormalized state becomes

$$|0\rangle_{2a}|1\rangle_{1a}|0\rangle_{3a} + |0\rangle_{1a}|1\rangle_{3a}|2\rangle_{2b}|(0\rangle_{1b}|0\rangle_{3b} + |2\rangle_{1b}|1\rangle_{3b} + |1\rangle_{1b}|2\rangle_{3b} - |1\rangle_{2b}|(0\rangle_{1b}|1\rangle_{3b} + |2\rangle_{1b}|2\rangle_{3b} + |1\rangle_{1b}|0\rangle_{3b} + |1\rangle_{1b}|1\rangle_{3b}]. \quad (44)$$

In this case, the state $|0\rangle_{2a}$ means that the probabilities of measuring $|e\rangle$, $|c_+\rangle$ and $|c_-\rangle$ on edge 2 are all zero, so $\langle W_{R_2} \rangle = 0$.

On the other hand, for the original state the probabilities of $e$, $c_+$ and $c_-$ are 2/3, 1/6 and 1/6, respectively, giving $\langle W_{R_2} \rangle = 1$. This approach gives us a method to test the system deterministically once the correct initial entangled state is prepared, without increasing the complexity. We can simply measure the operator $W_{R_2}$ using photon counting, which should yield the expectation values given by equation (39). No interactions with the two-qubit entangled state (qubits 1a and 3a) are required. That state can be discarded and therefore does not add to the complexity of the protocol.

4.4. Tolerance to errors

There are two main potential sources of error in this simulation: detector inefficiencies with higher order photon number terms, and gate errors. With regard to the former, even though we use three independent SPDC crystals, only the correct set of four photons is postselected out. Beyond this, the leading source of error from our source is likely to be higher order terms in photon number. These terms, which may be mistaken for valid four-photon events due to detector inefficiencies, naturally occur with smaller probability as they arise from a higher power of $\lambda$, and thus are a small contribution to error.

Regarding gate errors, there are several measures to quantify how close a noisy output state $\sigma$ is from an ideal state $\rho$. Among them are the fidelity $F$, the Bures distance $B$ [40] and the

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trace distance $D$ defined as follows:

$$F(\rho, \sigma) = \text{tr} \left\{ \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}} \right\}^2, \quad B(\rho, \sigma) = \sqrt{2 - 2F(\rho, \sigma)}, \quad D(\rho, \sigma) = \frac{1}{2} \text{tr} |\rho - \sigma|_{tr},$$

(45)

where the trace norm of an operator $O$ is $|O|_{tr} = \sqrt{O^\dagger O}$. The trace distance $D(\rho, \sigma)$ measures the maximum difference in expectation value taken with respect to the two states $\rho$ and $\sigma$ of any positive operator $O$ satisfying $O \leq \mathbf{1}$. Since we can scale and shift our observable for statistics to be positive and $\leq \mathbf{1}$, e.g. by mapping $W_{R_2} \rightarrow (1 + W_{R_2})/3$, trace distance upper bounds the error in the measurement of anyonic statistics. Furthermore, the trace distance is upper bounded by the Bures distance $D(\rho, \sigma) \leq B(\rho, \sigma)$ [41]. We have found that the circuit in figure 3 is not overly sensitive to any of the parameters used. Numerically it was found that, when a parameter is perturbed, the Bures distance between the perturbed state and the ideal state is approximately proportional to the size of the perturbation. To evaluate the sensitivity, this proportionality constant was determined for small perturbations in each of the reflectivities and phase shifts in the interferometer. It was found that the proportionality constant is less than 1 for 8 of the 14 beam splitter reflectivities. The largest proportionality constant is for the $6/7$ reflectivity beam splitters and is approximately 1.87. The system is less sensitive to phase shifts, with the proportionality constant less than $1/2$ for errors in most phases in the circuit. The largest proportionality constant is 0.63, for phase errors adjacent to the $27/76$ reflectivity beam splitter.

5. Conclusions

In summary, we have proposed a physical implementation of non-Abelian interferometry using entangled multi-photon states. This implementation requires 14 beam splitters and three pairs of downconverted photons. One of these pairs is required for the full representation of the state, but is not otherwise used in our proposal. The anyon statistics may be probed in a simple way by measuring the projector onto the fluxless anyon pair $W_{R_2}$, which simply requires photon counting at the output. This approach does not add any additional complexity and the overall expected success probability for each measurement is $9/55$. Alternatively, the fusion amplitudes $F(R_2(h) \rightarrow \text{vac})$ may be measured directly given additional ancilla photons and linear-optical CNOTs at the cost of reduced success probability.

Our proposal corresponds to a minimal construction of a non-Abelian discrete gauge theory in a single plaquette of a spin lattice. The creation and manipulation of the anyons correspond to transformations of entangled states, which are topologically ordered eigenstates of a spin lattice Hamiltonian, but we do not need the presence of this Hamiltonian. Our construction uses three parametric downconversion crystals to generate entangled photon pairs, which are then processed using beam splitters and phase shifters to simulate the braiding and fusion of non-Abelian charges. By choosing an initial state that is already an excited state of the model with a pair of vacuum electric charge pairs, and by judicious choice of witness to the fusion data, we have found a substantially simplified protocol for the interferometry.

Given recent advances in integrated photonic devices, our protocol should be within experimental reach in the relatively near future. Possible extensions of this model include the simulation of thermal states of topologically ordered media by introducing mixedness using ancillary degrees of freedom (see e.g. [42]). Such measurements could then be used to explore phenomena like thermal fragility of topological entanglement [43].

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An interesting topic for future research is the scalability to larger systems. It is nontrivial to scale this experiment, due to the use of postselection. Nevertheless, it is possible to scale experiments with postselection by using cluster states for measurement-based quantum computing [44]. It is known that Abelian anyons can be supported on a cluster state [45] and therefore can be scaled in this way. This provides a promising approach to the scaling of experiments simulating non-Abelian anyons with photonic networks.

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Appendix. Other charge configurations

Alternative configurations of the charges turn out to be equivalent. Consider the case when the charges are on the lower two nodes. The state in this case is

\[
|1_{R_2}; (v_1, v_2)\rangle = \frac{1}{\sqrt{6}} [2|e\rangle_1 |(e, e)_{23} + |t_0, t_0\rangle_{23} + |t_1, t_1\rangle_{23} + |t_1, t_2\rangle_{23} + |c_+, c_+\rangle_{23} + |c_-, c_-\rangle_{23})
- |c_+\rangle_1 |(e, e)_{23} + |t_0, t_1\rangle_{23} + |t_1, t_2\rangle_{23} + |t_2, t_0\rangle_{23} + |c_+, e\rangle_{23} + |c_-, c_+\rangle_{23})
- |c_-\rangle_1 |(e, e)_{23} + |t_0, t_2\rangle_{23} + |t_1, t_0\rangle_{23} + |t_2, t_1\rangle_{23} + |c_-, c_-\rangle_{23} + |c_-, e\rangle_{23})].
\]

(A.1)

In this case, we can use the encoding (21) on all three qudits, and the state becomes the same as (23), except with labels 1 and 2 interchanged.

Another case is where both charges are on the right, and the state becomes

\[
|1_{R_2}; (v_2, v_3)\rangle = \frac{1}{\sqrt{6}} [2|e\rangle_3 |(e, e)_{12} + |t_0, t_0\rangle_{12} + |t_1, t_1\rangle_{12} + |t_2, t_2\rangle_{12} + |c_+, c_+\rangle_{12} + |c_-, c_-\rangle_{12})
- |c_+\rangle_3 |(e, e)_{12} + |t_0, t_1\rangle_{12} + |t_1, t_2\rangle_{12} + |t_2, t_0\rangle_{12} + |c_-, e\rangle_{12} + |c_+, c_-\rangle_{12})
- |c_-\rangle_3 |(e, e)_{12} + |t_0, t_2\rangle_{12} + |t_1, t_0\rangle_{12} + |t_2, t_1\rangle_{12} + |c_-, c_+\rangle_{12} + |c_+, e\rangle_{12})].
\]

(A.2)

Here, we would use the encoding (21) on qudit 3 and the encoding (22) on qudits 1 and 2. Then we would again get the state (23), except that this time with labels 2 and 3 interchanged.

The encodings are different on the nodes we wish to perform the operations on, but the operations are different, and the scheme is again isomorphic. To achieve the operations $T_\sigma(v_2)$ we need to apply operations to qudits 1 and 3. The encoding on 3 now is the same as the encoding on 1 and 2 previously, and we are again performing $L$ operations on this qudit, so the analysis is identical to that above. For qudit 1, we now need to perform $R$ operations, because this edge is directed inwards towards node 2. The effect of the operation $T_{c_-(v_2)}$ is to exchange the basis states of qudit 1 as

\[
|e\rangle \rightarrow |c_-\rangle, \quad |t_0\rangle \rightarrow |t_2\rangle, \quad |c_-\rangle \rightarrow |c_+\rangle, \quad |t_1\rangle \rightarrow |t_0\rangle, \quad |c_+\rangle \rightarrow |e\rangle, \quad |t_2\rangle \rightarrow |t_1\rangle.
\]

(A.3)

In this case, because we are now using the encoding (22) on qudit 1, this corresponds to permuting the basis states as

\[
|0\rangle \rightarrow |2\rangle, \quad |1\rangle \rightarrow |0\rangle, \quad |2\rangle \rightarrow |1\rangle.
\]

(A.4)
This is identical to that obtained previously. The requirement to perform the $R$ operation, together with the different encodings, means that the end result is identical.

The results are similar for the other cases. $T_{c -} (v_2)$ is just the inverse, and the operations $T_{t_i} (v_1)$ are also the same as before. For example, to perform the operation $T_{t_0} (v_1)$, we need to exchange the states for qudit 1 as

$$|e\rangle \leftrightarrow |t_0\rangle, \quad |t_1\rangle \leftrightarrow |c -\rangle, \quad |t_2\rangle \leftrightarrow |c +\rangle.$$  \hspace{1cm} (A.5)

Again, the roles of $c -$ and $c +$ are exchanged, but the encoding also exchanges the roles of $c -$ and $c +$, so the operations required on the encoded states are identical.

Recall also that in this case, the roles of qudits 2 and 3 are exchanged for the state, so operations on qudits 1 and 3 here are equivalent to operations on qudits 1 and 2 previously. As a result of this, in each of the three possibilities for the charges, both the encoded states and the operations on these encoded states are the same. Hence, any experiment can be interpreted in each of three ways: as with the charges on the nodes shown in figure 2, as with the charges on the bottom two nodes, or with the charges on the right two nodes.

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