Contest Design with Threshold Objectives

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Abstract. We study contests where the designer’s objective is an extension of the widely studied objective of maximizing the total output: The designer gets zero marginal utility from a player’s output if the output of the player is very low or very high. We consider two variants of this setting, which correspond to two objective functions: binary threshold, where a player’s contribution to the designer’s utility is 1 if her output is above a certain threshold, and 0 otherwise; and linear threshold, where a player’s contribution is linear in her output if the output is between a lower and an upper threshold, and becomes constant below the lower and above the upper threshold. For both of these objectives, we study (1) rank-order allocation contests, which assign prizes based on players’ rankings only, and (2) general contests, which may use the numerical values of the players’ outputs to assign prizes. We characterize the contests that maximize the designer’s objective and indicate techniques to efficiently compute them. We also prove that for the linear threshold objective, a contest that distributes the prize equally among a fixed number of top-ranked players offers a factor-2 approximation to the optimal rank-order allocation contest.

Keywords: contest theory · mechanism design · all-pay auctions

1 Introduction

Contests are games in which (1) players invest effort and produce outputs toward winning one or more prizes, (2) those investments of effort are costly and irreversible, and (3) the prizes are allocated based on the values of outputs. They are prevalent in many areas, including sports, rent-seeking, patent races, innovation inducement, labor markets, college admissions, scientific projects, crowdsourcing and other online services.

We study contests as incomplete information games. We assume that the players are self-interested and exert costly effort in order to win valuable prizes. Each player is associated with a private ability (or quality), and their cost, as a function of their output, is linear with a slope equal to the inverse of their ability. The players know the prize allocation scheme, their own ability, and the prior distributions of other players’ abilities, and play strategically, reaching a

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1 See the book by Vojnovic [15] for an introduction to and applications of contest theory.
Bayes–Nash equilibrium. On the other hand, the contest designer knows the prior distributions of the players’ abilities, and can therefore compute the equilibrium behavior of the players. She wants to design the prize allocation scheme to elicit equilibrium behavior that optimizes her own objective.

The most widely studied designer’s objective is the total output, i.e., the sum of the outputs generated by the players. Under the total output objective, the designer values equally the marginal output by a player producing a low level of output and the marginal output by a player already producing a high level of output. However, in several practical scenarios, the designer may want to focus on the output generated by a section of players producing low/middle/high level of output or to elicit an adequate output from several players instead of very high output from a few players.

Consider, for instance, a crowdsourcing task, such as a survey. It may be more valuable to get many contributors to give adequate responses than to get a few people to submit perfect responses. Similarly, a health insurance company promoting a fitness/exercise app aims to encourage many subscribers to start exercising regularly rather than to get a few fitness enthusiasts to log many hours each day. As another example, consider an instructor who is preparing the students for a standardized test that measures the school performance, but has a limited impact on the students’ educational trajectories. The instructor/school may want to incentivize her students to perform better by awarding prizes. If a student is far below the pass/fail threshold, they are likely to fail even if they improve their performance a bit; similarly, there is no need to push students who are sure to excel to work even harder. The crucial students are the ones in between, and the instructor wants to award the prizes to elicit additional output from these students.

We focus on two types of objective functions. Under the binary threshold objective, there is a fixed threshold $B$, and a player contributes 1 to the designer’s objective if her output is at least $B$, and contributes 0 otherwise. In contrast, under the linear threshold objective, there is a lower threshold $B_L$ and an upper threshold $B_H$. All players who produce an output below $B_L$ make the same contribution to the designer’s objective; similarly, all players who produce an output above $B_H$ contribute the same amount. However, between these thresholds, a player’s output contributes linearly to the designer’s objective, just like in case of the total output objective.

Depending upon the situation, a contest designer might be restricted to use only the relative value of the players’ outputs to award prizes, which motivates the study of contests with a rank-order allocation of prizes (see, e.g., [12]). On the other hand, the contest designer might be allowed to use the numerical values of the players’ outputs and design the optimal general contest. In the latter case,

\footnote{In the appendix, for rank-order allocation of prizes, we also study a model where the designer’s objective is the sum of a concave (or convex) transformation of players’ outputs. A concave (convex) transformation captures decreasing (increasing) marginal utility for the designer as a player’s output increases.}
the contest design problem is similar to that of an all-pay auction design (see, e.g., [6]).

In this paper, we assume that the prizes are non-negative, and we normalize them in two ways: unit-sum and unit-range. The unit-sum constraint is a budget constraint, which requires that the total prize money does not exceed 1. The unit-range constraint restricts the individual prizes awarded to the players to be between 0 and 1. Such a constraint is suitable when the designer is not restricted by a budget, or when the monetary value of the prizes is much less important to her compared to the players' outputs, but there are limits on the prize range. The unit-range constraint for the general contest allows us to optimize for an individual player independently of other players.

1.1 Our Results

Contests with a rank-order allocation of prizes. For the binary threshold objective, the optimal contest equally distributes the prize among the top \( k \) players, where \( k \) depends upon the distribution \( F \) of the players’ abilities (Theorem 4). For the linear threshold objective, the optimal contest has up to three levels of prizes: the top \( k \) players get the first-level prize, the next \( \ell \) players get the second-level prize, the next \( m \) players get the third level prize, and the last \( n - k - \ell - m \) players do not get anything; here, \( n \) is the number of players, and the values of \( k, \ell, \) and \( m \) depend upon \( F \) (Theorem 6). We also prove that a simple contest that gives an equal prize to the first few players and nothing to others (recall that this format is optimal for the binary threshold) has an approximation ratio of 2 for the linear threshold (Theorem 7). Both for the binary and for the linear threshold, the results apply to both unit-sum and unit-range constraints on the prizes, although the numbers of players at the different prize levels depend upon the type of constraint.

General contests. For the binary threshold objective, the optimal contest equally distributes the prize to the players who produce an output above a reserve output level, and this reserve is equal to the threshold (Theorem 8). This is true for both unit-sum and unit-range constraints. For the linear threshold objective, the optimal contest is an extension of the revenue-maximizing all-pay auction with a reserve. For the unit-range constraint, if the distribution of the players’ abilities, \( F \), is regular\(^3\), then there is a reserve output between the lower and the upper threshold, and any player with an output above the reserve gets a prize of 1 (Theorem 9). For the unit-sum constraint and regular \( F \), the contest has a reserve output and a saturation output. In this case, the prize allocation depends on whether the player with the highest output is: (i) below the reserve, (ii) between the reserve and the saturation level, or (iii) above the saturation level. In case (i) no one gets a prize; in case (ii) the player with the highest output gets the entire prize; and in case (iii) the prize is distributed equally

\(^3\) See Definition 3. This is a weaker assumption than the monotone hazard rate condition.
among the players with outputs above the saturation level (Theorem 9). The reserve and the saturation levels depend upon $F$. For irregular $F$, following techniques from optimal auction design, we iron the virtual ability function to get an optimal contest that is a generalization of the optimal contest for the regular case (Theorem 10).

1.2 Related Work

Rank-order allocation of prizes is the dominant paradigm in contest theory. Our work is closely related to prior work on contest design with incomplete information model and unit-sum constraints [9,12,6]. Glazer and Hassin [9] show that for linear cost functions and players’ abilities sampled i.i.d. from a uniform distribution, the contest that maximizes the total output awards the entire prize to the top-ranked player. Moldovanu and Sela [12] give the symmetric Bayes–Nash equilibrium (Theorem 1) that we use in our analysis. They also generalize the result of Glazer and Hassin [9] and show that awarding the entire prize to the top-ranked player is optimal when the players have (weakly) concave cost functions with the abilities sampled i.i.d. from any distribution with continuous density function; however, with convex cost functions, the optimal mechanism can have multiple prizes. Chawla et al. [6] optimize maximum individual output instead of total output, in a similar incomplete information setup with linear cost functions.

Our study of the optimal general contest design rests on the framework established in the seminal work by Myerson [14] on revenue optimal auction design. Di-Palantino and Vojnovic [7] and Chawla et al. [6] connect crowdsourcing contests with all-pay auctions. For general contests with linear cost functions, the optimal contest for total output has been studied by Vojnovic [16] and the optimal contest for maximum individual output has been considered by Chawla et al. [6].

To the best of our knowledge, there has not been any work on maximizing the threshold objectives studied in this paper. In addition to studying total output (e.g., [9,12,13,11]) and maximum individual output (e.g., [6,12,15]), other objectives that have been investigated include maximizing the cumulative output from the top $k$ agents (e.g., [18]).

On the technical side, our work on rank-order allocation builds upon the equilibrium characterization of Moldovanu and Sela [12]. As in the prior work,

\footnote{Specific cases of our problem (such as the linear threshold objective with only an upper threshold) are related, although not equivalent, to the problem of maximizing total output when players have non-linear cost functions. A non-linear cost function affects the players’ equilibrium behavior, but a threshold objective is associated with the designer and does not directly affect the equilibrium behavior. For example, if every player has a convex cost function that is linear up to a certain threshold and then goes to infinity (this setting is similar in spirit to the linear threshold objective with an upper threshold), then no player would produce an output above the threshold, but we will see that the optimal rank-order allocation contest with a linear threshold objective may have players that produce an output above the upper threshold.}
the single-crossing property (Definition 2) and properties of order statistics are useful for the characterization of the optimal rank-order allocation contest. For general contests, we build upon the work on revenue optimal auction design by Myerson [14] and on the implementability of auctions by Matthews [10]. Matthews [10] characterizes which expected allocation functions can be implemented by some allocation function, and therefore allows us to focus on expected allocation functions instead of allocation functions. Previous works in general contest design, such as the work of Chawla et al. [6], did not require the result of Matthews [10] because, unlike in our model, their objective functions were linear in the expected allocation.

There have also been several studies in the complete information settings (e.g., [3, 2]). We point the readers to the book by Vojnovic [16], particularly Chapter 3, for a survey on contest theory.

2 Notation and Preliminaries

There are \( n \) players. Let \( v = (v_1, v_2, \ldots, v_n) \) be the ability profile of the players, where the values \( v_i \) are drawn independently from a continuous and differentiable distribution \( F \) with support \([0, 1]\). Let \( f \) be the probability density function (PDF) of \( F \). The \( n \) players simultaneously produce outputs \( b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n_+ \). For a function \( g(x) \) that is not one-to-one, let \( g^{-1}(y) \) denote the minimum value of \( x \) such that \( g(x) \geq y \), for \( y \) in the range of \( g \).

2.1 Contests with a Rank-Order Allocation of Prizes

The contest has \( n \) prizes \( w = (w_1, w_2, \ldots, w_n) \), where \( 0 \leq w_{j+1} \leq w_j \leq 1 \) for \( j \in [n-1] \). For the unit-sum model, we additionally require that \( \sum_j w_j \leq 1 \). The prize \( w_1 \) is awarded to the highest-performing player, \( w_2 \) to the second highest, and so on; a player receives one of the prizes based on the rank of their outputs, with ties broken uniformly. Fix an output vector \( b = (b_i)_{i \in [n]} \) and suppose that \( b_{i_1} \geq b_{i_2} \geq \ldots \geq b_{i_n} \). Then the utility of player \( i \) (scaled up by \( v_i \) for convenience) is given by:

\[
u(v_i, b) = v_i \sum_{j \in [n]} w_j \frac{1\{b_i = b_j\}}{\{k \mid b_k = b_{i_j}\}} - b_i,
\]

where \( 1 \) is the indicator random variable. To interpret this formula, observe that \( \frac{1\{b_i = b_j\}}{\{k \mid b_k = b_{i_j}\}} \) is the probability that player \( i \) receives the \( j \)-th prize, whereas \( \frac{b_i}{v_i} \) is the cost of producing output \( b_i \) for player \( i \).

Let \( p_j(v) \) be the probability that a value \( v \in [0, 1] \) is the \( j \)-th highest among \( n \) i.i.d. samples from \( F \), given by the expression:

\[
p_j(v) = \binom{n-1}{j-1} F(v)^{n-j}(1 - F(v))^{j-1}.
\]
Let $f_{n,j}$ be the PDF of the $j$-th highest order statistic out of $n$ i.i.d. samples from $F$, given by the expression:

$$f_{n,j}(v) = \frac{n!}{(j-1)!(n-j)!} F(v)^{n-j}(1 - F(v))^{j-1} f(v).$$

(3)

A key role in the symmetric Bayes–Nash equilibrium of the contest is played by the order statistics of the abilities of the players.

Moldovanu and Sela [12] characterize the unique symmetric Bayes–Nash equilibrium in rank-order allocation contests. Chawla and Hartline [5] prove the uniqueness of this equilibrium in general (see also [6] for more details).

**Theorem 1.** ([12,5]) Consider the game that models the rank-order allocation contest with the values of placement prizes $w_1 \geq w_2 \geq \ldots \geq w_n \geq 0$. The unique Bayes–Nash equilibrium is given by

$$\beta(v) = \sum_{j \in [n]} w_j \int_0^v t p_j'(t) dt,$$

(4)

where $\beta(v)$ is the output generated by a player with ability $v$.

Note that $\beta$ depends on the prize vector $w$, but we are suppressing it to keep the notation cleaner.

**Definition 1 (Simple Contest).** A rank-order allocation contest is called simple if there exists a $j \in [n]$ such that it gives a positive prize of equal value to the first $j$ players and 0 to the other $n - j$ players.

**Definition 2 (Single-Crossing).** ([15], [16] chapter 3) A function $f : [a,b] \to \mathbb{R}$ is single-crossing with respect to a function $g : [a,b] \to \mathbb{R}$ if there exists a point $x^* \in [a,b]$ such that $f(x) \leq g(x)$ for all $x \leq x^*$ and $f(x) > g(x)$ for all $x > x^*$; when this is the case, we will also say that $f$ is single-crossing with respect to $g$ at $x^*$.

### 2.2 General Contests

We utilize the connection between contests and all-pay auctions made by [7,6]. Leveraging the revelation principle [14], for most of the analysis of general contests, we shall restrict our attention to direct revelation mechanisms and optimize over allocation rules $x(v) = (x_i(v))_{i \in [n]}$ that determine the allocation based on the abilities of the players $v = (v_i)_{i \in [n]}$. We interpret $x_i(v)$ as the expected value of the prize obtained by player $i$ given the ability profile $v$, where the expectation is taken over the choices of the tie-breaking mechanism. We recognize that, when running a contest, we do not have direct access to players’ abilities; rather, the players must produce outputs, and the allocation function must be based upon the observed outputs. After deriving the optimal allocation function based on the players’ abilities, we shall convert it to an allocation function based on the players’ outputs.
For both unit-range and unit-sum settings, we have the restriction $0 \leq x_i(v) \leq 1$ for all $i \in [n]$. For unit-sum, we additionally have $\sum_i x_i(v) \leq 1$. The unit-range case is comparatively easier, because $x_i(v)$ can be optimized independently for every player $i$, whereas for unit-sum, for two players $j$ and $\ell$, $x_j(v)$ and $x_\ell(v)$ are not independent as we have to satisfy the $\sum_i x_i(v) \leq 1$ constraint. While studying unit-sum contests, we shall by default assume that $n \geq 2$.

We assume that the allocation rule $x(v)$ is symmetric with respect to the players. As $x(v)$ is symmetric, we have $\mathbb{E}[x_i(v) \mid v_i = v] = \mathbb{E}[x_j(v) \mid v_j = v]$ for any $i, j \in [n]$. Let the expected allocation function be $\xi(v) = \mathbb{E}[x_i(v) \mid v_i = v]$. Using Myerson’s [14] characterization of allocation rules that allow a Bayes–Nash equilibrium, we conclude that $\xi(v)$ should be non-negative and non-decreasing in $v$, and in the equilibrium, the output of a player as a function of her ability is given by:

$$\beta(v) = v\xi(v) - \int_0^v \xi(t)dt. \quad (5)$$

We slightly abuse the notation by representing the output function as $\beta$ for both rank-order allocation contests and general contests.

Let us make a few crucial observations about $x(v)$ and $\xi$. Both our objective functions, binary threshold (4) and linear threshold (5), depend upon the output function $\beta(v)$, which further depends upon the expected allocation function $\xi(v)$, but not directly on the allocation function $x(v)$. So, any allocation function $x(v)$ that leads to the same expected allocation $\xi(v)$ leads to the same objective value.

Observe that for any expected allocation function $\xi$ that is induced by an allocation function with unit-sum constraints, the following condition holds (check [10] for more details):

$$\int_V^1 \xi(v)f(v)dv \leq \frac{1 - F(V)^n}{n}. \quad (6)$$

In plain words, inequality (6) says that the probability that any player with an ability above $V$ gets a prize is at most the probability that any player has an ability above $V$. Now, we state a result by Matthews [10] that we shall use in the remainder of the paper.

**Theorem 2 ([10]).** Any non-decreasing expected allocation function $\xi$ that satisfies inequality (6) is implementable by some allocation function $x$ that satisfies unit-sum constraints.

Given this result, we can focus on finding a $\xi$ that is non-decreasing and satisfies inequality (6) without worrying about the unit-sum constraint $\sum_i x_i(v) \leq 1$, because there will be some $x$ that implements $\xi$. Further, for the optimal $\xi$, as we shall see later, the optimal $x$ will be easy to derive.

In our study of optimal general contests, we shall give special attention to regular distributions, defined below. This property of the distribution of the players’ abilities allows for simpler and efficiently computable optimal contests (see [14] for their use in optimal auction design).
Definition 3 (Regular Distributions). A distribution $F$ is regular if $v(1 - F(v))$ is concave with respect to $(1 - F(v))$, or, equivalently, if the virtual ability $\psi(v) = v - \frac{1 - F(v)}{f(v)}$ is non-decreasing in $v$.

2.3 Objective Functions

We formally define the binary threshold and linear threshold objective functions. They apply to both rank-order allocation contests and general contests. We use the same notation for the output function for both cases, $\beta$.

Definition 4 (Binary Threshold Objective). Under the binary threshold objective, the contest designer gets a utility of 1 from a player $i$ if $i$’s output is at least a specified threshold $B$, and 0 otherwise. The expected utility of the designer is given by:

$$BT = \mathbb{E}_v[\sum_{i \in [n]} 1\{\beta(v_i) \geq B\}] = \sum_{i \in [n]} \mathbb{E}_v[1\{\beta(v_i) \geq B\}] = n \mathbb{E}_v[1\{\beta(v) \geq B\}]$$

$$= \mathbb{E}_v[1\{\beta(v) \geq B\}] = \int_0^1 1\{\beta(v) \geq B\} f(v) dv. \quad (7)$$

Definition 5 (Linear Threshold Objective). Under the linear threshold objective, the contest designer’s utility increases linearly with a player’s output if the player’s output is between a lower threshold of $B_L$ and an upper threshold of $B_H$. Formally, we define it as:

$$LT = \mathbb{E}_v[\sum_{i \in [n]} \max(0, \min(B_H, \beta(v_i)) - B_L)]$$

$$= n \mathbb{E}_v[\max(0, \min(B_H, \beta(v)) - B_L)] = n \mathbb{E}_v[\max(B_L, \min(B_H, \beta(v)))] - nB_L$$

$$= \mathbb{E}_v[\max(B_L, \min(B_H, \beta(v)))] = \int_0^1 \max(B_L, \min(B_H, \beta(v))) f(v) dv. \quad (8)$$

3 Rank-order allocation of prizes

In this section, we study contests that allocate prizes based on the players’ ranks. We first present some useful properties of the equilibrium output function $\beta$ given in Theorem 1. Then we study the two objective functions based on these properties.

From Theorem 1 we have

$$\beta(v) = \sum_{j \in [n]} w_j \int_0^v t p_j'(t) dt.$$

Writing $p_j'(t)$ using order statistics:

$$p_j'(t) = \begin{cases} f_{n-1,1}(t), & \text{if } j = 1 \\ f_{n-1,j}(t) - f_{n-1,j-1}(t), & \text{if } 1 < j < n \\ -f_{n-1,n-1}(t), & \text{if } j = n \end{cases}$$
Substituting \( p'_j(t) \) into the formula for \( \beta(v) \), we get
\[
\beta(v) = \sum_{j \in [n-1]} (w_j - w_{j+1}) \int_0^v t f_{n-1,j}(t) dt.
\] (9)

From equation (9) we can observe that decreasing \( w_n \) to 0 does not decrease \( \beta(v) \) for any \( v \). Changing \( \beta(v) \) so that it becomes greater or higher for every \( v \) leads to an equal or higher utility for the designer for both binary and linear threshold objectives. So, from now on, we shall assume that \( w_n = 0 \).

Depending upon whether we are looking at the unit-range or the unit-sum constraint on prizes, we have different constraints on \( w \). We now transform \( \beta(v) \) further to make it more convenient to work with.

**Unit-Sum.** We have the constraints \( \sum_j w_j \leq 1 \) and \( w_j \geq w_{j+1} \geq 0 \). Let \( \alpha_j = j(w_j - w_{j+1}) \). The set of constraints on \( \alpha = (\alpha_j)_{j \in [n-1]} \) that are equivalent to the constraints on \( w \) are: \( \alpha_j \geq 0 \) for all \( j \in [n-1] \) and \( \sum_{j \in [n-1]} \alpha_j \leq 1 \). Let \( \beta_S \) denote the output function \( \beta \) with unit-sum constraints. We can rewrite equation (9) as
\[
\beta_S(v) = \sum_{j \in [n-1]} \alpha_j \frac{1}{j} \int_0^v t f_{n-1,j}(t) dt = \sum_{j \in [n-1]} \alpha_j \beta_{Sj}(v),
\] (10)

where \( \beta_{Sj}(v) := \frac{1}{j} \int_0^v t f_{n-1,j}(t) dt \). Observe that the simple contest that awards a prize of \( 1/j \) to the first \( j \) players and 0 to others has \( \alpha_j = 1 \) and \( \alpha_k = 0 \) for \( k \neq j \). Moreover, this contest induces an output of \( \beta_{Sj}(v) \) from a player with ability \( v \). Thus, any rank-order prize structure can be written as a convex combination of these \( (n-1) \) simple contests where the first \( j \) players get awarded \( 1/j \), for \( j \in [n-1] \).

**Unit-Range.** We have the constraints \( 1 \geq w_j \geq w_{j+1} \geq 0 \) for all \( j \in [n-1] \). In this case, let \( \alpha_j = w_j - w_{j+1} \). We have the same set of constraints on \( \alpha \) as with unit-sum: \( \alpha_j \geq 0 \) for all \( j \) and \( \sum_{j \in [n-1]} \alpha_j \leq 1 \). However, we have a slightly different formula for \( \beta \), which we denote by \( \beta_R \):
\[
\beta_R(v) = \sum_{j \in [n-1]} \alpha_j \int_0^v t f_{n-1,j}(t) dt = \sum_{j \in [n-1]} \alpha_j \beta_{Rj}(v),
\] (11)

where \( \beta_{Rj}(v) := \int_0^v t f_{n-1,j}(t) dt \). Thus, similarly to the unit-sum case, any unit-range contest and the respective \( \beta_R(v) \) can be written as a convex combination of \( (n-1) \) simple unit-range contests \( \beta_{Rj} \), \( j \in [n-1] \). However, the unit-range contest that induces \( \beta_{Rj} \) awards a prize of \( 1 \) to the first \( j \) players and 0 to others, whereas the unit-sum contest \( \beta_{Sj} \) awards a prize of \( 1/j \) to the first \( j \) players.

Using the characterization of \( \beta \) in equation (10) for unit-sum and (11) for unit-range, we can easily prove that the total output objective is maximized by \( \beta_{S1} \) for unit-sum contests (proved by [112]) and by a simple contest for unit-range contests (the proof is provided in the full version of the paper).
Most of our analysis in this section applies to both unit-range and unit-sum settings; we shall use $\beta$ to denote either $\beta_S$ or $\beta_R$, and $\beta_j$ to denote either $\beta_{Sj}$ or $\beta_{Rj}$. Also, we shall assume without loss of generality that $\sum_j \alpha_j = 1$, because increasing $\alpha_i$ for some $i \in [n-1]$ while keeping $\alpha_j$ constant for all $j \in [n-1]\{i\}$ does not decrease $\beta(v)$ for any $v \in [0, 1]$, and therefore does not decrease either of our two objective functions.

**Theorem 3.** Fix an $\alpha$ and the corresponding output function $\beta$. Consider a pair of indices $j, k$ s.t. $1 \leq j < k \leq n - 1$, and $\epsilon > 0$. Suppose both the vector $\alpha$ and the vector $\alpha'$ given by $\alpha'_j = \alpha_j + \epsilon$, $\alpha'_k = \alpha_k - \epsilon$, $\alpha'_\ell = \alpha_\ell$ for $\ell \notin \{j, k\}$ satisfy the required constraints. Let $\beta'$ be the output function that corresponds to $\alpha'$. Then, $\beta'$ is single-crossing w.r.t. $\beta$, and $\beta^{-1}$ is single-crossing w.r.t. $\beta'^{-1}$.

The proof of Theorem 3 for unit-sum is available in [16] (chapter 3); in the full version of the paper, for completeness, we provide a similar proof for both unit-sum and unit-range settings.

### 3.1 Binary Threshold Objective

We first focus on the binary threshold objective (Definition 4): $E_{v}[1\{\beta(v) \geq B\}]$.

**Theorem 4.** The rank-order allocation contest that optimizes the binary threshold objective is simple, and the output function for the optimal contest is $\beta^*_j$, where $j^*$ is selected from the set $\arg \min_j \beta_j^{-1}(B)$.

Given Theorem 4, we can design the optimal contest by first finding the root of the equation $\beta_j(v) - B = 0$ for each $j \in [n-1]$. We can do this efficiently using a root-finding algorithm such as the bisection method because $\beta_j$ is continuous and monotone. Then, we select a $j$ with the smallest $\beta_j^{-1}(B)$.

### 3.2 Linear Threshold Objective

Next, we consider the linear threshold objective: $E[\max(B_L, \min(B_H, \beta(v)))]$ (Definition 5). For the binary threshold objective, the optimal contest was simple, but for the linear threshold this is not true in general. The following example illustrates this:

**Example 1.** Consider a contest with: unit-sum prizes; three players, $n = 3$; uniform distribution, $F(v) = v$, $f(v) = 1$; lower threshold $B_L = 0$; upper threshold $B_H = \frac{1}{4} \times \left(\frac{2}{3}\right)^2 + \frac{1}{6} \times \left(\frac{2}{3}\right)^3 \approx 0.1605$. The output function is $\beta(v) = \alpha_1 \beta_1(v) + \alpha_2 \beta_2(v)$ where $\alpha_1 + \alpha_2 = 1$ and $\beta_1(v) = \int_0^v tf_{2,1}(t)dt = \int_0^v 2t^2 dt = \frac{2}{3} v^3$ and $\beta_2(v) = \frac{1}{2} \int_0^v tf_{2,2}(t)dt = \int_0^v t(1-t)dt = \frac{v^3}{2} - \frac{v^2}{3}$. We consider three contests: the two simple contests and a mixed one.
– Simple Contest 1: $\alpha_1 = 1$ and $\beta(v) = \beta_1(v)$. We have $\beta_1^{-1}(B_H) \approx 0.6221$ and the objective value is $\int_0^{\beta_1^{-1}(B_H)} \beta_1(v)dv + B_H(1 - \beta_1^{-1}(B_H)) \approx 0.8577$.
– Simple Contest 2: $\alpha_2 = 1$ and $\beta(v) = \beta_2(v)$. The objective value is $\int_0^1 \min(B_H, \beta_2(v))dv \leq \int_0^1 \beta_2(v)dv = 1/12 \approx 0.0833$.
– Mixed Contest: $\alpha_1 = \alpha_2 = 1/2$ and $\beta(v) = \beta_1(v)/2 + \beta_2(v)/2$. We have $\beta^{-1}(B_H) = 2/3$ and the objective value is $\int_0^{2/3} \left(\frac{v^2}{4} + \frac{v^3}{6}\right)dv + B_H(1 - \frac{2}{3}) \approx 0.0864$.

We observe that the given mixed contest outperforms the two simple contests.

For the case where there is only an upper threshold, i.e., $B_L = 0$, there is an optimal contest that is a convex combination of only two simple contests.

**Theorem 5.** For a linear threshold objective with upper threshold only, i.e., with $B_L = 0$, there is an optimal $\alpha$ with at most two positive entries $\alpha_i$ and $\alpha_j$, i.e., with $\alpha_k = 0$ for $k \in [n - 1] \setminus \{i, j\}$. For this $\alpha$, $i$ and $j$, we also have:

$$
\int_0^{V_H} \beta_i(v)f(v)dv = \int_0^{V_H} \beta_j(v)f(v)dv = \int_0^{V_H} \beta(v)f(v)dv,
$$

where $V_H = \beta^{-1}(B_H)$ and $\beta$ is the output function induced by $\alpha$.

Theorem 5 suggests an algorithm for finding the optimal $\alpha$, sketched below:

1. For every $i, j \in [n - 1]$, $i < j$, find a $V_{ij} > 0$ (if any) such that $\int_0^{V_{ij}} \beta_i(v)f(v)dv = \int_0^{V_{ij}} \beta_j(v)f(v)dv$. Note that there might be multiple such values for $V_{ij}$, but these values form an interval of $[0, 1]$ because $\int_0^1 \beta_i(v)f(v)dv$ is single-crossing w.r.t. $\int_0^v \beta_j(v)f(v)dv$. Select any one of those values as $V_{ij}$.
2. If $V_{ij}$ exists and $\beta_i(V_{ij}) \geq B \geq \beta_j(V_{ij})$ then this pair $i, j$ is a candidate for being the optimal.
3. The objective value for this pair is $\int_0^{V_{ij}} \beta_i(v)f(v)dv + B(1 - F(V_{ij}))$.
4. Comparing $O(n^2)$ such pairs along with the $O(n)$ simple contests, we find the optimal contest.
5. $\alpha$ corresponding to pair $i, j$ can be calculated as $\alpha_i = \frac{B - \beta_i(V_{ij})}{\alpha_j(V_{ij}) - \beta_i(V_{ij})}$, $\alpha_j = 1 - \alpha_i$, and $\alpha_k = 0$ for $k \in [n - 1] \setminus \{i, j\}$. (Check the proof of Theorem 5 for more details about this step.)

We can prove a result analogous to Theorem 5 if we only have a lower threshold and no upper threshold, i.e., $B_H = 1$. Now, we give a result that applies for arbitrary thresholds.

**Theorem 6.** For a linear threshold objective, there is an optimal $\alpha$ with at most three positive entries $\alpha_i$, $\alpha_j$, and $\alpha_k$, i.e., with $\alpha_{\ell} = 0$ for $\ell \in [n - 1] \setminus \{i, j, k\}$. For this $\alpha$ and $i, j, k$, we also have:

$$
\int_{V_L}^{V_H} \beta_i(v)f(v)dv = \int_{V_L}^{V_H} \beta_j(v)f(v)dv = \int_{V_L}^{V_H} \beta_k(v)f(v)dv = \int_{V_L}^{V_H} \beta(v)f(v)dv,
$$

where $V_L = \beta^{-1}(B_L)$, $V_H = \beta^{-1}(B_H)$, and $\beta$ is the output function induced by $\alpha$. 
The main ingredients used in the proof of Theorem 6 (and Theorem 5) are: first-order optimality condition of the objective w.r.t. $\alpha$; single-crossing property of $\beta_i$ w.r.t. $\beta_j$ for $i < j$; and the fact that every linear programming problem has an optimal solution that lies at a corner of the feasible region.

Recall that for the case $B_L = 0$, we used Theorem 5 to obtain an algorithm that compares at most $O(n^2)$ contests to find an optimal one. In a similar spirit, for general $B_L$ and $B_H$, we can use Theorem 6 to obtain an algorithm that finds an optimal contest by comparing at most $O(n^3)$ contests.

**Simple vs Optimal.** In Theorem 6, we proved that a convex combination of at most three simple contests is optimal. We now compare this optimal contest with the best simple contest.

**Theorem 7.** For the linear threshold objective, the objective value of the optimal contest is at most $2$ times that of the best simple contest.

In the proof of Theorem 7, we use the expression given in Theorem 6 and the single-crossing property of $\beta_i$ w.r.t. $\beta_j$ for any $i < j$ to show that the objective value of the optimal contest is at most the sum of the objective values of two simple contests.

**4 General Optimal Contests.**

In the previous section, we restricted our focus to contests that awarded prizes based on players’ ranks only. In this section, we relax this restriction and consider contests that may use the numerical values of the players’ outputs to award prizes.

As we discussed in the preliminaries (Section 2), for the unit-range constraint, the allocation function must satisfy $0 \leq x_i(v) \leq 1$. Equivalently, the expected allocation function $\xi$ must satisfy $0 \leq \xi(v) \leq 1$. For the unit-sum constraint, in addition to the constraints for the unit-range case, the allocation function must also satisfy $\sum x_i(v) \leq 1$. Equivalently (by Theorem 2), the expected allocation function $\xi$ must satisfy inequality (6).

The following allocation rules award the entire prize to players with abilities above $V$, if there are such players in a given ability profile. Therefore, they maximize $\int_V^1 \xi(v) f(v) dv$, and the inequality (6) is satisfied with an equality.

1. Give the prize to the player with the highest ability. Then the expected allocation function is $\xi(v) = F(v)^n$, and $\int_V^1 \xi(v) f(v) dv = \int_V^1 F(v)^n f(v) dv = \int_{F(V)}^1 y^{n-1} dy = 1 - F(V)^{n}$. We can also observe that for this allocation rule, inequality (6) is tight for every $V \in [0, 1]$.

2. Uniformly distribute the prize among the players with abilities above $V$. Then the expected allocation function is $\xi(v) = \frac{1 - F(V)^n}{n(1 - F(V))}$, and $\int_V^1 \xi(v) f(v) dv = \int_V^1 \frac{1 - F(V)^n}{n(1 - F(V))} f(v) dv = \frac{1 - F(V)^n}{n}$. 
4.1 Binary Threshold Objective

For the binary threshold objective, we have:

$$\max E\{\mathbf{1}\{\beta(v) \geq B\}\} = \max \int_0^1 1\{\beta(v) \geq B\} f(v) dv = \min(\beta^{-1}(B)).$$

Thus, we want to find a $\xi$ that minimizes $\beta^{-1}(B)$.

**Theorem 8.** The optimal contest with the binary threshold objective gives a prize of 1 to all players who produce an output above the threshold $B$ in the unit-range model and equally distributes the total prize of 1 to all players who produce an output above the threshold $B$ in the unit-sum model.

4.2 Linear Threshold Objective

The linear threshold objective is $E[\max(B_L, \min(B_H, \beta(v)))]$ (Definition 5).

**Regular Distributions.** Let us first focus on the case when $F$ is regular. We shall see that the optimal contest resembles a highest bidder wins all-pay auction with a reserve bid (if every player bids below this, no one gets the prize) and a saturation bid (all bids above this level are considered equal), and the ties are broken uniformly. We shall also give an efficient way to find the reserve and the saturation bids.

We can, w.l.o.g., make the following assumptions on the optimal expected allocation function:

**Lemma 1.** If $\xi$ is optimal, we can assume w.l.o.g. that $\xi(v) = 0$ for $v < \beta^{-1}(B_L)$.

**Lemma 2.** If $\xi$ is optimal, we can assume w.l.o.g. that $\xi(v) = \xi(V_H)$ for $v \geq \beta^{-1}(B_H)$, i.e., $\xi(v)$ is constant for $v \geq \beta^{-1}(B_H)$.

From now on, we shall assume that $\xi$ satisfies the assumptions formulated in these two lemmas.

We can write our linear threshold objective, using the notation $V_L = \beta^{-1}(B_L)$ and $V_H = \beta^{-1}(B_H)$, as:

$$B_L F(V_L) + \int_{V_L}^{V_H} \beta(v) f(v) dv + B_H (1 - F(V_H)).$$

(12)

Intuitively, the following lemma says that we should push the area under the curve $\xi$ to the right, as much as possible. We prove this result for unit-sum; for unit-range it holds, as a corollary, if we set $n = 1$.

**Lemma 3.** Let $F$ be a regular distribution and suppose that the allocation function has to satisfy unit-sum constraints. Then there is an optimal $\xi$ such that $\int_0^1 \xi(t) f(t) dt = \frac{1}{n}(1 - F(v)^n)$ for all $v \in [0, 1]$ where $\beta(v) < B_H$ and $\xi(v) > 0$.
The previous lemma effectively says that for \( v \in [V_L, V_H) \) we have
\[
\int_0^1 \xi(t) f(t) dt = \frac{1 - F(v)^n}{n}.
\]
As both sides of the above equation are continuous, taking the limit \( v \to V_H \), we have:
\[
\int_{V_H}^1 \xi(t) f(t) dt = \frac{1 - F(V_H)^n}{n}.
\]
We already know that \( \xi \) is 0 for \( v < V_L \) and constant for \( v \geq V_H \). So, we get
\[
\xi(v) = \begin{cases} 
0, & \text{if } v < V_L \\
F(v)^{n-1}, & \text{if } V_L \leq v < V_H \\
\frac{1 - F(V_H)^n}{n(1 - F(V_H))}, & \text{if } v \geq V_H
\end{cases}
\] (13)

We now prove the following lemma, which says that for an optimal allocation rule, the output generated by a player never goes strictly above the upper threshold \( B_H \) almost surely in \( v \in [0, 1] \).

**Lemma 4.** If \( \xi \) is an optimal expected allocation function, then the induced output function \( \beta \) almost surely satisfies \( \beta(v) \leq B_H \) for \( v \in [0, 1] \). This result holds whether or not the distribution \( F \) is regular, and both for unit-range and for unit sum constraints.

Combining the optimal expected allocation rule given in (13) with Lemma 4, we get
\[
\beta(V_H) = \frac{V_H(1 - F(V_H)^n)}{n(1 - F(V_H))} - \int_{V_L}^{V_H} F(v)^{n-1} dv = B.
\]
Note that the above formula is applicable only if there exists a \( V_H \). It may be possible that \( \beta(v) \) never reaches the threshold \( B_H \), i.e., \( \beta(v) < B_H \) for \( v \in [0, 1] \).

The above formula also tells us that given \( V_L \) (or \( V_H \)), \( V_H \) (or \( V_L \)) can be calculated efficiently because, keeping \( V_L \) (or \( V_H \)) fixed, \( \beta(v) \) is continuous and monotone in \( V_H \) (or \( V_L \)).

We now discuss how to efficiently compute the reserve \( V_L \) and the saturation \( V_H \). We shall use the following notation in the next theorem: \( \eta(z) = \frac{1 - x^z}{n(1 - x^z)} \), \( \psi_u(v) = v - F(v)^{n-1} \), \( V_{low} \) is the solution of \( V_{low} F(V_{low})^{n-1} = B_L \), \( V_{mid} \) is the solution of \( \int_{V_{mid}}^{V_H} F(v)^{n-1} dv = 1 - B_H \), and \( V_{up} \) is the solution of \( B_H = V_{up} \eta(F(V_{up})) \).

**Theorem 9.** The contest that optimizes the linear threshold objective for regular distributions has the following allocation and expected allocation functions:
1. For unit-range, the expected allocation function \( \xi \) and the allocation function \( x(v) \) are given as:
\[
\xi(v) = \begin{cases} 
0, & \text{if } v < V \\
1, & \text{if } v \geq V
\end{cases}, \quad x_i(v) = \begin{cases} 
0, & \text{if } v_i < V \\
1, & \text{if } v_i \geq V
\end{cases}
\]
where \( V = \max(B_L, \min(B_H, \psi^{-1}(B_L))) \).
2. For unit-sum, the optimal solution is given by one of the two cases below:

(a) The expected allocation function $\xi$ is:

\[
\xi(v) = \begin{cases} 
0, & \text{if } v < V_L \\
F(v)^{n-1}, & \text{if } v \geq V_L 
\end{cases}
\]  

and the allocation function $x(v)$ is:

\[
x_i(v) = \begin{cases} 
0, & \text{if } \max_j(v_j) < V_L \text{ or } i \notin W \\
1/|W|, & \text{if } i \in W \text{ and } \max_j(v_j) \geq V_L 
\end{cases}
\]  

where $W = \{k \mid v_k = \max_j(v_j)\}$ and $V_L = \min(V_{\text{mid}}, \max(V_{\text{low}}, V_L))$; where $V_L$ is the solution of $\frac{B_L}{F(V_L)^{n-1}} - \psi(V_L) = 0$. 

(b) The expected allocation function $\xi$ is:

\[
\xi(v) = \begin{cases} 
0, & \text{if } v < V_L \\
F(v)^{n-1}, & \text{if } V_L \leq v < V_H \\
\eta(F(V_H)), & \text{if } v \geq V_H 
\end{cases}
\]  

and the allocation function $x(v)$ is:

\[
x_i(v) = \begin{cases} 
0, & \text{if } \max_j(v_j) < V_L \text{ or } i \notin W \\
1/|\tilde{W}|, & \text{if } i \in \tilde{W} \text{ and } \max_j(v_j) \geq V_H 
\end{cases}
\]  

where $W = \{k \mid v_k = \max_j(v_j)\}$, $\tilde{W} = \{k \mid v_k \geq V_H\}$, $V_L = \min(V_{\text{up}}, \max(V_{\text{mid}}, V_L))$ and $V_H = \min(1, \max(V_{\text{up}}, V_H))$; where $V_L$ and $V_H$ are the solutions of equations $\frac{B_L}{F(V_L)^{n-1}} - \psi(V_L) = 0$ and $\frac{V_H}{V_H} = \eta(F(V_H)) - \int_{V_L}^{V_H} F(v)^{n-1} dv = \frac{B_H}{V_H}$. 

The values of $V_L$ and $V_H$ derived in the theorem above can be efficiently computed if the distribution $F$ is known, see the proof for more details. Also note that, although the contest may seem complicated, it is reasonably simple from a player’s perspective. The optimal contest that maximizes the total output has a reserve \[16\], here we have a saturation value in addition to the reserve. A player need not know how these reserve and saturation values are computed.

Implementing this mechanism in practice, i.e., finding the allocation as a function of the outputs of the players, is not difficult. Given a player’s output $b$, map it to $g(b)$, where:

\[
g(b) = \begin{cases} 
0, & \text{if } b < F(V_L)^{n-1} \\
b, & \text{if } F(V_L)^{n-1} \leq b \leq F(V_H)^{n-1} \\
F(V_H)^{n-1}, & \text{if } F(V_H)^{n-1} < b < \frac{1-F(V_H)^{n}}{n(1-F(V_H))} \\
\frac{1-F(V_H)^{n}}{n(1-F(V_H))}, & \text{if } b \geq \frac{1-F(V_H)^{n}}{n(1-F(V_H))} 
\end{cases}
\]  

One can then distribute the prize equally among the players who have the maximum positive $g(b)$.
Irregular Distributions. In the study of the optimal linear threshold contest for regular distributions, we used the regularity condition at two places: first, in Lemma \[3\] to pack the area under \(\xi\) to the right; second, in Theorem \[4\] to prove that we obtain the optimal values for \(V_L\) and \(V_H\) by solving for the roots of the specific equations given in the theorem statement using an efficient root-finding method. For irregular \(F\), first, we give Lemma \[5\] analogous to Lemma \[3\]; second, we find an approximate solution by discretizing the feasible space of \(V_L\), \(V_H\), and \(\xi(V_H)\) \[5\].

We first introduce some additional notation. Consider the function \(\psi_{U,V}(v) = v - \frac{F(V) - F(v)}{f(v)}\), defined on the interval \([U, V]\). We now define \(\overline{\psi}_{U,V}\), which is the ironed version of \(\psi_{U,V}\), also defined on \([U, V]\). Our definition proceeds in several steps.

1. Let \(h_{U,V}(y) = \psi_{U,V}(F^{-1}(y))\) and \(H_{U,V}(y) = \int_{F^{-1}(U)}^y h_{U,V}(y)dy\);
2. Let \(\overline{H}_{U,V}(y)\) be the point-wise maximum convex function less that or equal to \(H_{U,V}(y)\). Note that at the boundary \(\overline{H}_{U,V}(F^{-1}(V)) = H_{U,V}(F^{-1}(V))\) and \(\overline{H}_{U,V}(F^{-1}(V)) = H_{U,V}(F^{-1}(V))\);
3. Let \(\overline{h}_{U,V}(y) = \overline{H}_{U,V}(y)\) and \(\overline{\psi}_{U,V}(v) = \overline{h}_{U,V}(F(v))\).

Let \(l_{U,V}(v) = \min_{u \in [U,V]} \overline{\psi}_{U,V}(u) = \overline{\psi}_{U,V}(v)\) \(u\) and \(r_{U,V}(v) = \max_{u \in [U,V]} \overline{\psi}_{U,V}(u) = \overline{\psi}_{U,V}(v)\) \(u\), and let \(l(u) = l_{0,1}(u)\) and \(r(u) = r_{0,1}(u)\).

**Lemma 5.** For an irregular distribution \(F\), in the unit-sum setting there is an optimal \(\xi\) such that \(\int_1^v \xi(t)f(t)dt = \frac{1}{n}(1 - F(v)^n)\) for all \(v\) where \(\beta(v) < B_H\), \(\xi(v) > 0\), and \(\overline{H}(F^{-1}(v)) = H(F^{-1}(v))\).

Note that Lemma \[5\] does not apply to points \(v \in [0, 1]\) where \(\overline{H}(F^{-1}(v)) < H(F^{-1}(v))\). So, unlike the case for regular distributions, where we had \(\int_{V_H}^1 \xi(t)f(t)dt = \frac{1}{n}(1 - F(V_H)^n)\) by applying Lemma \[3\] for \(v \rightarrow V_H\), we may not have a similar result for irregular distributions.

For regular \(F\), the allocation function \(x\) has three cases depending upon the highest ability, as given in \[15\]. On the other hand, for irregular \(F\), the allocation function \(x\) has five cases depending upon the highest ability, as given in the theorem below:

---

\[5\] We would like to note here that the algorithm we provide finds an approximate solution in time polynomial in the reciprocal of the parameter used to discretize \(V_L\), \(V_H\), and \(\xi(V_H)\). One could have discretized the entire optimization problem (the functions are evaluated at discrete values, the integrals are written as finite summations, etc.) and directly found an approximately optimal discretized \(\xi\) and \(\beta\) using linear programming, also in polynomial time. The advantage of our analysis is that it characterizes the optimal solution and gives a more intuitive algorithm.
Theorem 10. The contest that optimizes the linear threshold objective for irregular distributions and unit-sum constraints has the following allocation function:

\[
x_i(v) = \begin{cases} 
0, & \text{if } \max_j(v_j) < V_L \text{ or } i \notin W \\
\frac{1}{|W|}, & \text{if } r_{V_L,V_H}(V_L) \leq \max_j(v_j) < l_{V_L,V_H}(V_H) \text{ and } i \in W \\
\frac{n(V_L)(1-F(V_H))}{W(r_{V_L,V_H}(V_L)-(F(V_H))}, & \text{if } V_L \leq \max_j(v_j) < r_{V_L,V_H}(V_L) \text{ and } i \in W_L \\
\frac{n(V_H)(1-F(V_H))}{W}, & \text{if } V_H \leq \max_j(v_j) \text{ and } i \in \hat{W} \\
\frac{n(V_H)(F(V_H))}{W}, & \text{if } l_{V_L,V_H}(V_L) \leq \max_j(v_j) < V_H \text{ and } i \in W_H \end{cases}
\]

where \( W = \{ k \mid r_{V_L,V_H}(V_L) \leq v_k < l_{V_L,V_H}(V_H), \psi_{V_L,V_H}(v_k) = \max_j(\psi_{V_L,V_H}(v_j)) \}, \) \( W_L = \{ k \mid v_k \in [V_L, r_{V_L,V_H}(V_L)] \}, \) \( W_H = \{ k \mid v_k \in [l_{V_L,V_H}(V_H), V_H] \}, \) and \( \hat{W} = \{ k \mid v_k \geq V_H \}. \)

In the appendix, we have sketched an approximation algorithm to find the parameters used in the theorem, like \( V_L, V_H, \xi(V_L), \xi(V_H) \), etc.

We can transform the allocation function \( x(v) \) given in the theorem to an allocation function based on outputs in a manner analogous to [18]. The optimal contest for unit-range can be derived by combining ideas from the solution for unit-range with regular \( F \) and the solution for unit-sum with irregular \( F \).

5 Conclusion

In this paper, we looked at two natural and practically useful objectives for a contest designer, and we described optimal contests for both of them. An interesting open problem is to find how well can a contest without a reserve and/or a saturation output, or a rank-order allocation contest, approximate the optimal general contest, possibly with an additional player; we may expect to get a result along the lines of [3]. Another extension of this work would be to study other practically relevant objective functions for the designer, monotone transformations other than the threshold transformations we studied in this paper. Combining the objectives for the designer studied in this paper with non-linear utility and cost functions for the players is also an interesting research direction.

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A Appendix for Section 2

A.1 Expected allocation

We use the expression given below in Section 4. For completeness, we provide its derivation.

Consider a player \( i \) with ability \( v \), and fix \( a \) and \( b \) so that \( 0 \leq a \leq v \leq b \leq 1 \). For each \( k \in [n-1] \), let \( p(v, a, b, k) \) be the probability that all players other than player \( i \) have ability at most \( b \), with \( k \) of them having ability in \([a, b]\). We have:

\[
p(v, a, b, k) = \binom{n-1}{k} F(a)^{n-1-k} (F(b) - F(a))^k.
\]
Lemma 6. For two functions $f, g : [a, b] \to \mathbb{R}$, if $f$ is single-crossing with respect to $g$, then $F(x) = \int_a^x f(x)\,dx$ is single-crossing with respect to $G(x) = \int_a^x g(x)\,dx$.

Proof. Let $y^* \in [a, b]$ be the maximum value such that $F(y) \leq G(y)$ for all $y \leq y^*$. Such a value exists because the set $\{y : F(y) \leq G(y)\}$ is closed and $F(a) = G(a) = 0$.

Suppose that $f$ is single-crossing with respect to $g$ at $x^*$. Then we have $y^* \geq x^*$, because for every $x \leq x^*$ we have $g(x) - f(x) \geq 0$, and therefore for every $y \leq x^*$ we have $G(y) - F(y) = \int_y^y (g(x) - f(x))\,dx \geq 0$. Now, for $y > x^*$, we know that $f(y) > g(y)$ so $F(y) - G(y)$ is strictly increasing. So, if $F(x)$ goes above $G(x)$ at some point in $[a, b]$, then it will remain that way. □

Lemma 7. For two continuous and strictly increasing functions $f, g : [a, b] \to \mathbb{R}$ with $f(a) = g(a) = 0$, if $f$ is single-crossing with respect to $g$ at $x^*$, then $g^{-1}$ is single-crossing with respect to $f^{-1}$ (in the intersection of the domains of $f^{-1}$ and $g^{-1}$) at $y^* = f(x^*) = g(x^*)$.

Proof. Consider a point $y$ that belongs to the domains of $f^{-1}$ and $g^{-1}$. Suppose $y \leq y^*$. Since $f$ and $g$ are continuous and strictly increasing, there exists a unique point $x$ with $y = f(x)$ and a unique point $z$ with $y = g(z)$. Further, $y \leq y^*$ implies $x \leq x^*$ and hence $f(x) \leq g(x)$. It follows that $z \leq x$, i.e., $g^{-1}(y) \leq f^{-1}(y)$. In a similar manner, we can prove that $g^{-1}(y) > f^{-1}(y)$ for each point $y > y^*$ in the intersection of the domains of $f^{-1}$ and $g^{-1}$. □
B Appendix for Section 3

B.1 Total Output

If $B_L = 0$ and $B_H = 1$ in the linear threshold objective (Definition 5), then the objective becomes simply $\mathbb{E}[\beta(v)]$, i.e., the well-studied objective of maximizing the total output. For the unit-sum case, it has been shown by [9] that the optimal allocation of prizes is to award the entire prize to the top-ranked player, i.e., set $\alpha_1 = 1$ and $\alpha_j = 0$ for $j > 1$. This can also be observed from the following analysis:

$$\int_0^1 \beta_S(v) f(v) dv = \int_0^1 \left( \sum_{j \in [n-1]} \frac{\alpha_j}{j} \int_0^v x f_{n-1,j}(x) dx \right) f(v) dv$$

$$= \sum_{j \in [n-1]} \frac{\alpha_j}{j} \int_0^1 \left( \int_0^v f_{n-1,j}(x) dx \right) f(v) dv$$

$$= \sum_{j \in [n-1]} \alpha_j \int_0^1 \left( \int_0^v x f_{n-1,j}(x) dx \right) f(v) dv$$

$$= \sum_{j \in [n-1]} \alpha_j \int_0^1 \frac{1}{j} \int_0^v x f_{n-1,j}(x) dx$$

$$= \frac{1}{n} \sum_{j \in [n-1]} \alpha_j \int_0^v x f_{n,j+1}(x) dx.$$ 

As $\int_0^1 x f_{n,j+1}(x) dx$ is the expectation of the $(j + 1)$-st highest value out of $n$ i.i.d. samples from $F$, we have

$$\int_0^1 x f_{n,j+1}(x) dx \geq \int_0^1 x f_{n,k+1}(x) dx$$ for $j \leq k$.

Therefore, the optimal output function is $\beta_{S1}$.

For the unit-range objective, following similar steps, we get:

$$\int_0^1 \beta_R(v) f(v) dv = \frac{1}{n} \sum_{j \in [n-1]} \alpha_j \cdot \int_0^1 x f_{n,j+1}(x) dx.$$ 

While for unit-sum, $\int_0^1 x f_{n,j+1}(x) dx$ decreases as $j$ increases, for unit-range, $j \int_0^1 x f_{n,j+1}(x) dx$ does not necessarily decrease as $j$ increases. Nevertheless, we can see that the optimal contest is a simple contest and the output function in the optimal contest is $\beta_{Rj}$, where:

$$j^* \in \arg \max_{j \in [n-1]} \left( j \int_0^1 x f_{n,j+1}(x) dx \right).$$
B.2 Concave and Convex Transformations

Let \( h : [0, 1] \rightarrow [0, 1] \) be a monotone non-decreasing differentiable function, either concave or convex. The contest designer’s objective function is:

\[
E_v \left[ \sum_{i \in [n]} h(\beta(v_i)) \right] = \mathbb{E}_v[h(\beta(v))] = \int_0^1 h(\beta(v)) f(v) dv.
\]

(20)

In Section 3, for both unit-sum and unit-range, we derived that \( \beta(v) = \sum_{j \in [n-1]} \alpha_j \beta_j(v) \) where \( \alpha = (\alpha_j)_{j \in [n-1]} \) s.t. \( \sum_{j \in [n-1]} \alpha_j = 1 \) and \( \alpha_j \geq 0 \) for \( j \in [n-1] \). For unit-sum, \( \beta_j(v) = \beta S_j(v) = \frac{1}{f} \int_0^v tf_{n-1,j}(t) dt \) (equation (10)), and for unit-range, \( \beta_j(v) = \beta R_j(v) = \int_0^v tf_{n-1,j}(t) dt \) (equation (11)). We shall use these results in this section.

Lemma 8. If \( h \) is convex (concave), then the objective function of the contest designer is also convex (concave) w.r.t. \( \alpha \).

Proof. Take \( \gamma \in [0, 1] \) and two vectors \( \alpha^{(1)} \) and \( \alpha^{(2)} \), and let \( \alpha = \gamma \alpha^{(1)} + (1 - \gamma) \alpha^{(2)} \). Let \( OBJ(\alpha) \) denote the objective value corresponding to \( \alpha \), similarly, \( OBJ(\alpha^{(1)}) \) and \( OBJ(\alpha^{(2)}) \).

\[
OBJ(\alpha) = \int_0^1 h(\sum_{j \in [n-1]} \alpha_j \beta_j(v)) f(v) dv
\]

\[
= \int_0^1 h(\gamma \sum_{j \in [n-1]} \alpha_j^{(1)} \beta_j(v) + (1 - \gamma) \sum_{j \in [n-1]} \alpha_j^{(2)} \beta_j(v)) f(v) dv.
\]

If \( h \) is convex, then \( h(\gamma x + (1 - \gamma) y) \geq \gamma h(x) + (1 - \gamma) h(y) \) for any \( x \) and \( y \), and therefore, we get \( OBJ(\alpha) \geq \gamma OBJ(\alpha^{(1)}) + (1 - \gamma) OBJ(\alpha^{(2)}) \), so \( OBJ \) is convex. Similarly, we can prove that if \( h \) is concave, then \( OBJ \) is concave.

For convex \( h \), the results are similar to the results for the total output objective (Section B.1). For both unit-sum and unit-range, there is a simple contest that is also optimal. Moreover, for unit-sum, the simple contest that awards the entire prize to the first ranked player is also optimal. These results are similar to the results for a model where the players have concave cost functions, as studied by Moldovanu and Sela [12]. There is a similar correspondence for concave \( h \) and convex cost functions.

Theorem 11. If \( h \) is convex, then there is a simple and optimal rank-order allocation contest.

Proof. From Lemma 8, we know that if \( h \) is convex then the objective function is also convex w.r.t. \( \alpha \). Any \( \alpha \) can be written as a convex combination of the corner points, where a corner point has \( \alpha_j = 1 \) for some \( j \in [n-1] \) and \( \alpha_k = 0 \) for all \( k \neq j \). So, the optimal value of the objective is achieved at some of the corner points (it may be achieved at other points also).
Theorem 12. If \( h \) is convex, and the prizes are unit-sum, then the simple contest that awards the entire prize to the first-ranked player is optimal.

Proof. From Theorem 11, we know that there is an optimal simple contest. Let this simple contest have \( \alpha_j = 1 \) for some \( j \in \{n-1\} \) and \( \alpha_k = 0 \) for all \( k \neq j \), so \( \beta(v) = \beta_{S_j}(v) \). Comparing the objective value of the contest where \( \alpha_1 = 1 \) with this contest, we have:

\[
\int_0^1 h(\beta_{S_1}(v)) f(v) dv - \int_0^1 h(\beta_{S_j}(v)) f(v) dv = \int_0^1 (h(\beta_{S_1}(v)) - h(\beta_{S_j}(v))) f(v) dv
\]

\[
\geq \int_0^1 h'(\beta_{S_j}(v))(\beta_{S_1}(v) - \beta_{S_j}(v)) f(v) dv, \quad \text{as} \ h \text{ is convex.}
\]

Using (i) \( h'(\beta_{S_j}(v)) \) is monotonically increasing and non-negative because \( \beta_{S_j}(v) \) is monotonically increasing and \( h \) is convex, (ii) \( \beta_{S_1}(v) \) is single-crossing w.r.t. \( \beta_{S_j}(v) \), and (iii) \( \int_0^1 (\beta_{S_1}(v) - \beta_{S_j}(v)) f(v) dv \geq 0 \), as proved while studying total output (Section B.1), we have our required result. \( \square \)

Theorem 13. If \( h \) is concave, then the optimal rank-order allocation contest need not be simple, but can be efficiently found by solving a concave maximization problem.

Proof. The linear threshold objective with only an upper threshold is an example of a concave \( h \) (we smooth the non-differentiable point), and for this objective, we already gave an example that shows that a simple contest may not be optimal (Section 3.2). From Lemma 8, we know that the objective is concave if \( h \) is concave, so the optimal contest can be solved efficiently by solving a concave maximization problem (equivalently a convex minimization problem). \( \square \)

B.3 Omitted Proofs

Proof (Theorem 3). First, let us prove that \( tf_{n-1,j}(t) \) is single-crossing w.r.t. \( tf_{n-1,k}(t) \) for \( j < k \). We will argue that the following inequality is true for sufficiently large values of \( t \):

\[
\frac{(n-1)!}{(j-1)!(n-j-1)!} F(t)^{n-j-1}(1 - F(t))^{j-1} f(t) > \frac{(n-1)!}{(k-1)!(n-k-1)!} F(t)^{n-k-1}(1 - F(t))^{k-1} f(t)
\]

\[
\iff \frac{(k-1)!(n-k-1)!}{(j-1)!(n-j-1)!} \left( \frac{1 - F(t)}{F(t)} \right)^{k-j}.
\]

Indeed, for fixed values of \( j, k, \) and \( n \), the left-hand side of the above inequality is a positive constant. As \( t \) increases, the right-hand side monotonically decreases
from $\infty$ to 0. Thus, the above inequality is true for all $t$ above some $t^*$, and we have proved that $tf_{n-1,j}(t)$ is single-crossing w.r.t. $tf_{n-1,k}(t)$.

Following similar steps, we can prove that $t_{f_{n-1,j}}(t)$ is single-crossing w.r.t. $t_{f_{n-1,k}}(t)$ for $j < k$. Instead of the inequality $(k-1)!(n-k-1)! \geq (k-1)! \geq (n-k)! = (n-j)!$ above, we have the inequality $k(n-k-1)! \geq (k-1)! \leq (n-j)!$, but the subsequent argument applies.

Now, let us prove that $\beta$ is single-crossing w.r.t. $\beta'$. Let us assume that the following inequality is true:

$$\beta'(v) > \beta(v).$$

For unit-range, we have:

$$\beta'_R(v) > \beta_R(v) \quad \iff \quad \sum_{t \in [n-1]} \alpha_t \int_0^v x f_{n-1,t}(x) dx > \sum_{t \in [n-1]} \alpha_t \int_0^v x f_{n-1,t}(x) dx \quad \iff \quad \int_0^v x f_{n-1,j}(x) dx > \int_0^v x f_{n-1,k}(x) dx.$$

Using Lemma 6 and the fact that $tf_{n-1,j}(t)$ is single-crossing w.r.t. $tf_{n-1,k}(t)$ for $j < k$, we obtain that $\beta'_R$ is single-crossing w.r.t. $\beta_R$ for unit-range. Similarly, for unit-sum, we use Lemma 6 together with the result that $t_{f_{n-1,j}}(t)$ is single-crossing w.r.t. $t_{f_{n-1,k}}(t)$ for $j < k$ to prove that $\beta'_S$ is single-crossing w.r.t. $\beta_S$.

As $\beta'$ is single-crossing w.r.t. $\beta$, using Lemma 7, we get that $\beta'^{-1}$ is single-crossing w.r.t. $\beta'^{-1}$. 

**Proof (Theorem 4).** We need to prove that for any threshold value $B$, the optimal $\alpha$ has $\alpha_j = 1$ for some $j$ and $\alpha_k = 0$ for $k \neq j$. In other words, $\beta = \beta_j$ for some $j \in [n-1]$.

From Definition 3 and equations 10 and 11, we have:

$$\int_0^1 1\{\beta(v) \geq B\} f(v) dv = \int_0^1 1\{v \geq \beta'^{-1}(B)\} f(v) dv = \int_{\beta'^{-1}(B)}^1 f(v) dv = 1 - F(\beta'^{-1}(B)).$$

Thus, to maximize the binary threshold objective, we need to minimize $F(\beta'^{-1}(B))$, and as $F$ is non-decreasing, we need to minimize $\beta'^{-1}(B)$. For any $\alpha$ we have $\alpha_j \geq 0$ and $\sum_j \alpha_j = 1$, and for every $j$, the $\beta_j$ functions are monotone. Therefore $\min_j \beta'^{-1}(B) \leq \beta^{-1}(B)$. Thus, the optimal contest is simple, and the output function for the optimal contest is $\beta'^\star$, where $j^\star$ is:

$$j^\star = \arg \min_j \beta'^{-1}(B).$$

\[\square\]
Proof (Theorem 5). Let us assume that \( \alpha \) is optimal and \( \beta \) is the induced output function. Let \( V_H = \beta^{-1}(B_H) \). Let \( I = \{ i \in [n-1] \mid \alpha_i > 0 \} \). If \( |I| < 2 \), we are trivially done so we can assume w.l.o.g. that \( |I| \geq 2 \). Select arbitrary \( i, j \in I, i \neq j \). Let \( \alpha_{ij} = \alpha_i + \alpha_j \) and \( \gamma = \alpha_i / \alpha_{ij} \). Observe that \( \alpha_i = \gamma \alpha_{ij} \) and \( \alpha_j = (1 - \gamma) \alpha_{ij} \). Also, as \( \alpha_i, \alpha_j > 0 \), we have \( 0 < \gamma < 1 \).

Now, let us fix \( \alpha_{ij} \) and \( \alpha_k \ (k \notin \{i, j\}) \) and focus on \( \gamma \). As \( \alpha \) is optimal and \( \gamma \) is strictly between 0 and 1, we have \( \partial \gamma = 0 \):

\[
\frac{\partial \gamma}{\partial \alpha} \frac{\partial \gamma}{\partial \beta} = 0
\]

Also, it is easy to verify that the objective value does not change:

\[
\frac{\partial \gamma}{\partial \alpha} \frac{\partial \gamma}{\partial \beta} = 0
\]

\[
\Rightarrow \int_0^{V_H} (\beta_i(v) - \beta_j(v)) f(v) dv = 0
\]

\[
\Rightarrow \int_0^{V_H} \beta_i(v) f(v) dv = \int_0^{V_H} \beta_j(v) f(v) dv.
\]

(21)

For any \( i, j \in I \), we get \( \int_0^{V_H} \beta_i(v) f(v) dv = \int_0^{V_H} \beta_j(v) f(v) dv = \int_0^{V_H} \beta(v) f(v) dv \).

Now, we will construct an \( \alpha' \) with at most two of its components strictly greater than 0. As \( \beta \) is a convex combination of the \( \beta_i \) output functions, we must either have (i) \( \beta_i(V_H) = \beta(V_H) \) for some \( i \in I \), or (ii) \( \beta_i(V_H) > \beta(V_H) \) and \( \beta_j(V_H) < \beta(V_H) \) for some \( i, j \in I \). For (i) set \( \alpha'_i = 1 \) and \( \alpha'_k = 0 \) for \( k \notin [n-1] \setminus \{ i \} \), and for (ii) set \( \alpha'_i = \frac{\beta - \beta_i(V_H)}{\beta_i(V_H) - \beta_j(V_H)} \) and \( \alpha'_j = (1 - \alpha'_i) \) and \( \alpha'_k = 0 \) for \( k \notin [n-1] \setminus \{ i, j \} \).

We can check that \( \sum_{k \in [n-1]} \alpha_k \beta_k(V_H) = \sum_{k \in [n-1]} \alpha'_k \beta_k(V_H) = B \). So, we get \( \beta^{-1}(B_H) = \beta^{-1}(B) = V_H \), where \( \beta' \) is the output function induced by \( \alpha' \).

Also, it is easy to verify that the objective value does not change:

\[
\int_0^1 \min(B, \sum_{j \in [n-1]} \alpha_j \beta_j(v)) f(v) dv = \int_0^1 \min(B, \sum_{j \in I} \alpha_j \beta_j(v)) f(v) dv
\]

\[
= \int_0^{V_H} \sum_{j \in I} \alpha_j \beta_j(v) f(v) dv + B(1 - F(V_H))
\]

\[
= \sum_{j \in I} \alpha_j \int_0^{V_H} \beta_j(v) f(v) dv + B(1 - F(V_H))
\]

\[
= \sum_{j \in I} \alpha'_j \int_0^{V_H} \beta_j(v) f(v) dv + B(1 - F(V_H)) \quad \text{(using equation (21))}
\]

\[
= \int_0^1 \min(B, \sum_{j \in [n-1]} \alpha'_j \beta_j(v)) f(v) dv.
\]

\( \square \)
Proof (Theorem 6). Let us assume that $α$ is optimal and $β$ is the induced output function. Let $V_L = β^{-1}(B_L)$ and $V_H = β^{-1}(B_H)$. Let $V_H = β^{-1}(B_H)$. Let $I = \{i \in [n-1] \mid α_i > 0\}$; if $|I| < 2$, we are trivially done, so assume w.l.o.g. that $|I| \geq 2$. Select arbitrary $i, j \in I, i \neq j$. Let $α_{ij} = α_i + α_j$ and $γ = α_i/α_{ij}$. Observe that $α_i = γα_{ij}$ and $α_j = (1 - γ)α_{ij}$. Also, as $α_i, α_j > 0$, we have $0 < γ < 1$.

Let us fix $α_{ij}$ and $α_l (l \notin \{i, j\})$ and focus on $γ$. As $α$ is optimal and $γ$ is strictly between 0 and 1, we have $\frac{∂LT}{∂γ} = 0$:

\[
\frac{∂LT}{∂γ} = \frac{∂}{∂γ} \int_0^1 \max(B_L, \min(B_H, \sum_{i \in [n-1]} α_i β_i(v))) f(v) dv = 0
\]

\[
⇒ \frac{∂}{∂γ} \int_0^1 \max(B_L, \min(B_H, \sum_{k \in [n-1] \setminus \{i, j\}} α_k β_k(v) + α_{ij}(γ β_i(v) + (1 - γ) β_j(v)))) f(v) dv = 0
\]

\[
⇒ \int_{V_L}^{V_H} (β_i(v) - β_j(v)) f(v) dv = 0
\]

\[
⇒ \int_{V_L}^{V_H} β_i(v) f(v) dv = \int_{V_L}^{V_H} β_j(v) f(v) dv.
\]

Thus, for every $i ∈ I$, we have $\int_{V_L}^{V_H} β_i(v) f(v) dv = \int_{V_L}^{V_H} β_i(v) f(v) dv$.

Now, let us look at the constraints that $α$ satisfies:

1. $β(V_L) = \sum_{i ∈ I} α_i β_i(V_L) = B_L$;
2. $β(V_H) = \sum_{i ∈ I} α_i β_i(V_H) = B_H$;
3. $\sum_{i ∈ I} α_i = 1$;
4. $α_i ≥ 0$ for $i ∈ I$;
5. $α_i = 0$ for $i \notin I$.

Observe that any other $α'$ that satisfies the $3 + |I| + (n-1-|I|) = n+2$ constraints given above will also be optimal (because $\int_{V_L}^{V_H} β_i(v) f(v) dv = \int_{V_L}^{V_H} β_i(v) f(v) dv$ for all $i ∈ I$ and any $α$ that satisfies the $(n+2)$ constraints will have the same value for the objective). The feasible region of the $(n+2)$ constraints is bounded on all sides and therefore has corner points. At a corner point, at least $(n-1)$ of the constraints must be satisfied with an equality, so at least $(n-4)$ of them are of the type $α_i = 0$ for some $i ∈ [n-1]$. Hence, selecting such a corner point, we will get a solution with at most 3 of the coordinates of $α$ strictly greater than 0.

Proof (Theorem 7). Let us take an optimal solution $α$ with the minimum number of non-zero entries, i.e., the minimum number of indices $i$ with $α_i > 0$.

1. If there is only one such index, then we have an approximation ratio of 1 and we are done.
2. Now suppose there are three such indices. Let $i, j, k$ be the three indices for which $α_i, α_j, α_k > 0$ in the optimal solution. W.l.o.g. let $β_i(V_L) ≤
\[ \beta_j(V_L) \leq \beta_k(V_L) \]. We claim that \[ \beta_i(V_H) \geq \beta_j(V_H) \geq \beta_k(V_H) \]. If this were not true, then for a pair, say \( i, j \), we would have had \[ \beta_i(V_L) \leq \beta_j(V_L) \] and \[ \beta_i(V_H) \leq \beta_j(V_H) \], with at least one of the two inequalities strict. As the output functions are single-crossing, we would have \[ \beta_i(v) \leq \beta_j(v) \] for all \( v \in [V_L, V_H] \), and we could increase \( \alpha_j \) by \( \alpha_i \) and decrease \( \alpha_i \) to 0 to get a better solution.

As \( \beta_i(V_L) \leq \beta_j(V_L) \leq \beta_k(V_L) \) and \( \beta_i(V_H) \geq \beta_j(V_H) \geq \beta_k(V_H) \), their convex combination, \( \beta \), has: \( \beta_i(V_L) \leq \beta(V_L) \leq \beta_k(V_L) \) and \( \beta_i(V_H) \geq \beta(V_H) \geq \beta_k(V_H) \).

3. Finally, if there are only two positive entries, say \( \alpha_i \) and \( \alpha_k \), then also we can prove a similar condition: \( \beta_i(V_L) \leq \beta(V_L) \leq \beta_k(V_L) \) and \( \beta_i(V_H) \geq \beta(V_H) \geq \beta_k(V_H) \).

Look at a generic plot of \( \beta_i, \beta_k, \beta \) given in Figure 1. With reference to the figure, we have:

\[ - \int_0^1 \max(B_L, \min(B_H, \beta_i(v))) f(v) dv = C + D + E + B_L; \]
\[ - \int_0^1 \max(B_L, \min(B_H, \beta_k(v))) f(v) dv = A + B + E + B_L; \]
- \int_0^1 \max(B_L, \min(B_H, \beta(v))) f(v) dv = B_L F(V_L) + \int_{V_L}^{V_H} \beta(v) f(v) dv + B_H (1 - F(V_H)), which is, by Theorem 6, equal to: B_L F(V_L) + \int_{V_L}^{V_H} \beta(v) f(v) dv + B_H (1 - F(V_H)) = B + D + E + B_L.

The approximation ratio is at most

\[
\frac{B + D + E + B_L}{\max(A + B + E + B_L, C + D + E + B_L)} = \frac{E + B_L + B + D}{B + D} \leq \frac{B + D}{\max(A + B, C + D)} \leq 2.
\]

\[\Box\]

C Appendix for Section 4

C.1 Linear Threshold Objective: Irregular Distributions (continued)

Here, we provide an algorithm to find the (approximately) optimal contest. We perform an approximate search on the three parameters \(V_H, \xi(V_H), \) and \(V_L,\) to maximize the linear threshold objective, using an algorithm sketched below:

1. Select values for \(V_H\) and \(\xi(V_H)\).
2. Assume that \(\xi\) is constant in the interval \([l(V_H), V_H]\). Applying Lemma 5 to the point \(l(V_H)\), compute \(\xi(l(V_H)):\)

\[
\int_{l(V_H)}^1 \xi(t) f(t) dt = \xi(l(V_H))(F(V_H) - F(l(V_H))) + \xi(V_H)(1 - F(V_H))
\]

\[
= \frac{1 - F(l(V_H))^n}{n(1 - F(l(V_H)))} = \frac{\xi(V_H)(1 - F(V_H))}{F(V_H) - F(l(V_H))}. \tag{1}
\]

3. Compute \(V_L\) by solving \(\beta(V_H) = V_H \xi(V_H) - \int_{V_L}^{V_H} \xi(v) dv\) by the bisection method, where the value of \(\xi(v)\) for \(V_L \leq v < l(V_H)\) is given as

\[
\xi(v) = \begin{cases} 
F(v)^{n-1}, & \text{if } H(F^{-1}(v)) = H(F^{-1}(v)) \text{ or } v \in [l(V_L), r(V_L)] \\
\frac{F(r(v))^{n-1} - F(l(v))^{n-1}}{F(r(v)) - F(l(v))}, & \text{if } H(F^{-1}(v)) < H(F^{-1}(v)) \text{ and } v > r(V_L)
\end{cases}
\]

4. If \(H(F^{-1}(V_L)) = H(F^{-1}(V_L))\), then set \(V_L = V_L\), otherwise search for the \(V_L \in [l(V_L), r(V_L)]\) (and automatically for \(\xi(V_L) \geq B_L\)) that satisfies

\[
F_L^{V_L} F(v)^{n-1} dv = \int_{V_L}^{V_H} \xi(v) F_L^{V_L} dv + \int_{r(V_L)}^{r(V_L)} F(v)^{n-1} dv
\]

maximizes \(B_L F(V_L) + \int_{V_L}^{V_H} \beta(v) f(v) dv\). This step selects \(V_L\) to optimally redistribute the area under \(\xi\) in the interval \([V_L, r(V_L)]\).

We skipped a corner case: if \(l(V_H) \xi(l(V_H)) < B_L\), then \(V_L\) must be greater than \(l(V_H)\). Find the \(V_L\) in \([l(V_H), V_H]\) that optimizes the objective following a procedure similar to step (4). Also, we need to redistribute \(\xi\) in \([l(V_H), V_H]\) according to \(\xi(V_H)\) to find \(\xi(l(V_L), V_H)\) in a manner similar to step (2).
C.2 Omitted Proofs

Proof (Theorem 8). For unit-range allocations, we optimize $\xi$ subject to the constraints: $0 \leq \xi(v) \leq 1$. We have

$$\beta(v) = v\xi(v) - \int_0^v \xi(t)dt \leq v\xi(v) \leq v,$$

where the first inequality holds because $\xi(t) \leq 1$ for all $t \in [0, 1]$. We have $\beta(v) \leq v \implies B \leq \beta^{-1}(B)$. Set $\xi(v) = 0$ for $v < B$ and $\xi(v) = 1$ for $v \geq B$. We have $\xi(B) = 1$ and $\int_0^B \xi(t)dt = 0$, so we get $\beta(B) = B\xi(B) - \int_0^B \xi(t)dt = B \implies \beta^{-1}(B) \leq B$. As we have already seen that $\beta^{-1}(B) \geq B$, this is optimal.

For unit-sum, we have an additional constraint on $\xi$, inequality (6): $\int_V^1 \xi(v)f(v)dv \leq \frac{1 - F(V)^n}{n}$ for every $V$.

Lemma 9. If $\xi$ is optimal, we can assume w.l.o.g. that $\xi(v) = 0$ for $v < \beta^{-1}(B)$.

Proof. We know that $\beta(V) = V\xi(V) - \int_0^V \xi(x)dx \leq V\xi(V)$. Hence, by setting $\xi(v) = 0$ for $v < V$ we still have $\beta^{-1}(B) = V$.

Lemma 10. If $\xi$ is optimal, we can assume w.l.o.g. that $\xi(v) = \xi(V)$ for $v \geq \beta^{-1}(B)$, i.e., $\xi(v)$ is constant for $v \geq \beta^{-1}(B)$.

Proof. We define a transformed expected allocation function $\tilde{\xi}$, where $\tilde{\xi}(v) = \xi(v)$ for $v < V$, and $\tilde{\xi}(v) = \frac{1}{1 - F(V)} \int_V^1 \xi(t)f(t)dt$ for $v \geq V$. Let $\tilde{\beta}$ be the output function for $\tilde{\xi}$. As $\xi$ is monotone, $\tilde{\xi}(V) \geq \xi(V)$, and therefore $\tilde{\beta}(V) \geq \beta(V) \geq B$. So, we still have $\beta^{-1}(B) = V$.

From the previous two lemmas, we know that the allocation function equally distributes the prize among the players who have an ability above some value $V$, where $\beta(v) \geq B$ for $v \geq V$ and $\beta(v) < B$ otherwise. As $\xi$ satisfies $\int_V^1 \xi(v)f(v)dv \leq \frac{1 - F(V)^n}{n}$, we have:

$$\xi(V) \leq \frac{1 - F(V)^n}{n(1 - F(V))} \implies B \leq \beta(V) \leq \frac{V(1 - F(V)^n)}{n(1 - F(V))}.$$

As the expression $\frac{V(1 - F(V)^n)}{n(1 - F(V))}$ increases with $V$, and we want to minimize $V$, it is optimal to satisfy the above inequality with an equality. This gives us $B = \frac{V(1 - F(V)^n)}{n(1 - F(V))}$.

Also, as $\frac{V(1 - F(V)^n)}{n(1 - F(V))}$ is continuous and non-decreasing in $[0, 1]$, we can efficiently find $V$. However, we do not need to explicitly compute $V$ because the contest that equally distributes the prize to all players who generate an output above $B$, if there are any such players in a given ability profile, automatically induces the required contest.
Proof (Lemma 3). We will show that pushing the area under ξ to the right does not decrease the objective. For the initial portion of the proof, let us disregard these two constraints: ξ is monotone and ∫₀¹ ξ(t)f(t)dt ⩽ \frac{1}{n}(1 - F(v)^n). At the end, we will briefly explain how to incorporate these constraints into the proof.

Take two points u and v such that u < v, ξ(u) > 0 and β(v) < B. For very small δ and ε greater than 0, let us decrease the area under ξ in a small neighborhood [u − ε, u + ε] of u by δ/f(u) and increase the area under ξ in a small neighborhood [v − ε, v + ε] of v by δ/f(v). Let \tilde{ξ} and \tilde{β} be the transformed ξ and β after the update. Select ε and δ small enough to maintain ξ(u − ε) ≥ 0 and \tilde{β}(v + ε) ≤ B_H.

Let us compute the change in the linear threshold objective value. To do that, first, let us look at \tilde{β}(y) − β(y) for y ∈ [0, 1]:

\[
\tilde{β}(y) - β(y) = \begin{cases} 
0, & \text{if } y \in [0, u - ε) \\
\frac{-uδ}{2f(u)}, & \text{if } y \in [u - ε, u + ε) \\
\frac{δ}{f(u)}, & \text{if } y \in [u + ε, v - ε) \\
\frac{2f(v)δ}{v} + \frac{δ}{f(v)}, & \text{if } y \in [v - ε, v + ε) \\
\frac{δ}{f(v)} - \frac{2f(v)δ}{v}, & \text{if } y \in [v + ε, 1]
\end{cases}
\]

As we are moving area from left to right, i.e., u < v, it is easy to check that for the lower threshold B_L, V_L = \tilde{β}^{-1}(B_L) does not decrease, so neither does B_L F(V_L). For the remaining portion of the linear threshold objective (12), we have the following:

\[
\int_{y \geq V_L} \min(B_H, \tilde{β}(y)) - \min(B_H, β(y))f(y)dy
\]

\[
= \int_{y = u - ε}^{v + ε} \frac{-uδ}{2f(u)}f(y)dy + \int_{y = u + ε}^{v - ε} \frac{δ}{f(u)}f(y)dy + \int_{y = v - ε}^{v + ε} \frac{vδ}{2f(v)} + \frac{δ}{f(v)}f(y)dy
\]

\[
+ \int_{y = v + ε}^{1} \min(B_H, \tilde{β}(y)) - \min(B_H, β(y))f(y)dy, \quad \text{because } \tilde{β}(v + ε) \leq B_H
\]

\[
= -uδ + \frac{δ}{f(u)}(F(v) - F(u)) + vδ
\]

\[
+ \int_{y = v}^{1} \min(B_H, \tilde{β}(y)) - \min(B_H, β(y))f(y)dy, \quad \text{as } ε \to 0.
\]

We will now consider two cases: (1) f(u) ≤ f(v) and (2) f(u) > f(v).

Case (1): f(u) ≤ f(v) \implies δ/f(u) ≥ δ/f(v). For y ≥ v + ε, \tilde{β}(y) − β(y) = \frac{δ}{f(u)} − \frac{δ}{f(v)} ≥ 0, so min(B_H, \tilde{β}(y)) − min(B_H, β(y)) ≥ 0. The total change in the objective is:

\[
-uδ + \frac{δ}{f(u)}(F(v) - F(u)) + vδ + \int_{y = v}^{1} \min(B_H, \tilde{β}(y)) - \min(B_H, β(y))f(y)dy
\]

\[
≥ 0, \quad \text{as } v > u, F(v) \geq F(u), \text{ and } min(B_H, \tilde{β}(y)) \geq min(B_H, β(y)).
\]
We proved that transforming $\xi$ to $\delta/f$ change in the objective is: the beginning: non-decreasing property of $\xi$

the constraint: select $v$ to be close to the rightmost point with $a$ in this manner does not decrease the objective

For the rest of the proof, let $a(y) = \int_0^1 \xi(t)f(t)dt$ and $b(y) = (1 - F(y)^n)/n$.

Now, we explain how to incorporate the two constraints we disregarded in the beginning: non-decreasing property of $\xi$ and $a(y) \leq b(y)$. For the $a(y) \leq b(y)$ constraint: select $v$ to be close to the rightmost point for which $a(v) < b(v)$ and has $\beta(v) < B_H$, select $u$ to be close to the rightmost point less that $v$ for which $a(u) = b(u)$ and $\beta(u) > 0$, if there is no such point $u$ less than $v$ with $a(u) = b(u)$, then select any point with $\beta(u) > 0$. If we select a small enough area to move from $u$ to $v$ (parameterized by $\delta$ in the first part of the proof) and select the neighborhood of $u$ and $v$ suitably (parameterized by $\epsilon$ earlier) we can satisfy the constraint. For the monotonically non-decreasing property, selecting $v$ to be close to the rightmost point with $a(v) < b(v)$ and $\beta(v) < B_H$, and increasing $\xi$ in very small increments, and repeating until convergence, maintains the non-decreasing property of $\xi$ in the aggregate.

Starting from an arbitrary $\xi$, one can reach a $\xi$ that satisfies the condition given in the statement of the lemma by transformations to $\xi$ as given above. This completes the proof.

\[ \square \]

Proof (Theorem 4). Let us assume that the lemma is false, which gives us $\int_{V_H}^1 \beta(v)f(v)dv > B_H(1 - F(V_H))$.

We average out $\xi$ in the interval $v \in [V - \delta, 1]$, for some $\delta > 0$, while maintaining $\beta(v) \geq B$ for $v \geq V - \delta$. This will improve our objective. We can check that the transformed $\xi$ satisfies $\xi(v) = \frac{1}{1 - F(V - \delta)} \int_{V - \delta}^1 \xi(v)f(v)dv$ for $v \geq V - \delta$.

\footnote{If the set is open, we select $v$ in limit. Same for $u$.}
We can find the \( \delta \) by solving the following equation:
\[
B = \beta(V - \delta) = (V - \delta) \frac{1}{1 - F(V - \delta)} \int_{V - \delta}^{1} \xi(v)f(v)dv - \int_{0}^{V - \delta} \xi(v)dv.
\]

Observe that the right-hand side is a non-increasing continuous function of \( \delta \). As \( \delta \) goes from 0 to \( V \), the right-hand side goes from strictly above \( B \) to 0, so we obtain the required solution for \( \delta \).

\( \square \)

**Proof (Theorem 4).** We provide separate proofs for unit-range and unit-sum settings.

**Unit-Range.** Note that Lemmas 1, 2, and 4 are applicable for unit-range, and Lemma 3 is applicable with slight modification. Together, they imply that \( \xi(v) = 0 \) for \( v < V \) and \( \xi(v) = 1 \) for \( v \geq V \), for some \( V \in [0,1] \). From Lemma 1 we also have \( V_L = V \geq B_H \), and from Lemma 4, \( V \leq V_H \leq B_H \) if there exists a \( V_H \), or \( V < B_H \) if not. The objective value can be written as:
\[
OBJ = B_L F(V) + V(1 - F(V)).
\]

Differentiating w.r.t. \( V \) and equating to 0, we obtain
\[
B_L f(V) - V f(V) + (1 - F(V)) = f(V)(B_L - (V - 1 - F(V))) = f(V)(B_L - \psi(V)) = 0 \implies \psi(V) = B_L \implies V = \psi^{-1}(B_L).
\]

We can also observe that the solution to the above equation is a global maximum because the derivative of the objective is greater than 0 for \( V < \psi^{-1}(B_L) \) and less than 0 afterwards. Plugging in the constraints on \( V \), \( B_L \leq V \leq B_H \), the optimal solution is \( V = \max(B_L, \min(B_H, \psi^{-1}(B_L))) \).

**Unit-Sum.** We divide the analysis into two cases, depending on whether \( \beta \) hits the upper threshold \( B_H \).

1. \( \beta(v) < B_H \) for \( v \in [0,1] \). We do not have a \( V_H \), and for \( V_L \) we have the following inequality:
\[
\beta(1) < B_H \implies 1\xi(1) - \int_{V_L}^{1} \xi(v)dv < B_H \implies \int_{V_L}^{1} F(v)^{n-1}dv > 1 - B_H \implies V_L < V_{\text{mid}},
\]

where \( V_{\text{mid}} \) is the solution of the equation \( \int_{V_{\text{mid}}}^{1} F(v)^{n-1}dv = 1 - B_H \). We also know that
\[
\beta(V_L) \geq B_L \implies V_L \xi(V_L) = V_L F(V_L)^{n-1} \geq B_L \implies V_L \geq V_{\text{low}},
\]

where \( V_{\text{low}} \) is the solution to \( V_{\text{low}} F(V_{\text{low}})^{n-1} = B_L \). Thus, a value of \( V_L \) in \( [V_{\text{low}}, V_{\text{mid}}] \) maximizes the objective:
\[
OBJ = B_L F(V_L) + \int_{V_L}^{1} \beta(v)f(v)dv.
\]
Now, $\beta(v) = vF(v)^{n-1} - \int_{V_L}^{V} F(t)^{n-1} dt \implies \frac{d\beta(v)}{dV} = F(V_L)^{n-1}$. Differentiating OBJ w.r.t. $V_H$ we get:

$$\frac{dOBJ}{dV_H} = B_L f(V_L) - \beta(V_L) f(V_L) + \int_{V_L}^{1} \frac{d\beta(v)}{dV} f(v) dt$$

$$= B_L f(V_L) - V_L F(V_L)^{n-1} f(V_L) + F(V_L)^{n-1}(1 - F(V_L))$$

$$= f(V_L) F(V_L)^{n-1} \left( \frac{B_L}{F(V_L)^{n-1}} - V_L + \frac{1 - F(V_L)}{f(V_L)} \right)$$

$$= f(V_L) F(V_L)^{n-1} \left( \frac{B_L}{F(V_L)^{n-1}} - \psi(V_L) \right).$$

As $\frac{B_L}{F(V_L)^{n-1}}$ decreases with $V_L$, $\psi(V_L)$ increases with $V_L$, and $f(V_L) F(V_L)^{n-1}$ is non-negative, the root of $\frac{B_L}{F(V_L)^{n-1}} - \psi(V_L) = 0$ is the global maximum.

Also, as the function is continuous, we can efficiently find a solution using a root finding algorithm such as the bisection method; let the solution be $V_L$.

The optimal $V_L$ for this case will be $V_L = \max(V_{low}, \min(V_{mid}, V_H))$.

2. $\beta(v) \geq B$ for $v \geq V_H \in [0, 1]$. We have the following equality:

$$\beta(V_H) = V_H \xi(V_H) - \int_{V_L}^{V_H} \xi(v) dv = V_H \eta(F(V_H)) - \int_{V_L}^{V_H} F(v)^{n-1} dv = B_H,$$

where $\eta(x) = \frac{1-x^n}{n(1-x)}$. Note that $\eta'(x) = \frac{1-x^n}{n(1-x)^2} - \frac{n x^{n-1}}{1-x} = \frac{1}{1-x} (\eta(x) - x^{n-1})$ and also that $\eta(x) \geq x^{n-1}$ and $\eta'(x) \geq 0$ for $x \in [0, 1]$.

For $u \geq v$, $\psi_u(v) = v - \frac{F(u) - F(v)}{f(v)}$. Observe that $\psi_u(v)$ is non-decreasing in $v$ because $\frac{\psi_u(v)}{dv} = 2 + \frac{(F(u) - F(v)) f'(v)}{f(v)^2}$ is obviously non-negative if $f'(v) \geq 0$, and if $f'(v) < 0$, then $\frac{\psi_u(v)}{dv} = 2 + \frac{(F(u) - F(v)) f'(v)}{f(v)^2} \geq 2 + \frac{(1-F(v)) f'(v)}{f(v)^2} = \frac{\psi(v)}{dv} \geq 0$.

As $\beta(V_H) = B_H$, we get $V_H \in [V_{up}, 1]$ and $V_L \in [V_{mid}, V_{up}]$ where $V_{up}$ is the solution of the equation $B_H = V_{up} \eta(F(V_{up}))$. Differentiating $\beta(V_H) = B_H$ w.r.t. $V_H$ we get:

$$\frac{d\beta(V_H)}{dV_L} = \frac{dB_H}{dV_L} = 0$$

$$\implies (V_H \eta'(F(V_H)) f(V_H) + \eta(F(V_H)) - F(V_H)^{n-1}) \frac{dV_H}{dV_L} + F(V_L)^{n-1} = 0$$

$$\implies (V_H \eta'(F(V_H)) f(V_H) + (1 - F(V_H)) \eta'(F(V_H)) \frac{dV_H}{dV_L} + F(V_L)^{n-1} = 0$$

$$\implies \frac{dV_H}{dV_L} = \frac{-F(V_L)^{n-1}}{\eta'(F(V_H))(V_H f(V_H) + (1 - F(V_H)))} \leq 0.$$
Differentiating $OBJ$ w.r.t. $V_L$ we get:
\[
\frac{dOBJ}{dV_L} = B_L f(V_L) + \beta(V_H) f(V_H) \frac{dV_H}{dV_L} - \beta(V_L) f(V_L) + \int_{V_L}^{V_H} F(V_L)^{n-1} f(v) dv - B_H f(V_H) \frac{dV_H}{dV_L} = B_L f(V_L) - V_L F(V_L)^{n-1} f(V_L) + \int_{V_L}^{V_H} F(V_L)^{n-1} f(v) dv + (\beta(V_H) - B_H) f(V_H) \frac{dV_H}{dV_L} = B_L f(V_L) - V_L F(V_L)^{n-1} f(V_L) + \int_{V_L}^{V_H} F(V_L)^{n-1} f(v) dv = F(V_L)^{n-1} f(V_L) \left( \frac{B_L}{F(V_L)^{n-1}} - \frac{V_L F(V_L)^{n-1}}{f(V_L)} \right) = F(V_L)^{n-1} f(V_L) \left( \frac{B_L}{F(V_L)^{n-1}} - \psi_{V_H}(V_L) \right).
\]

From the equation above, to find the solution of $\frac{dOBJ}{dV_L} = 0$, we need to solve for the values of $V_L$ and $V_H$ that satisfy $\frac{B_L}{F(V_L)^{n-1}} - \psi_{V_H}(V_L) = 0$ (and $\beta(V_H) = B_H$). As the $F$ and $\psi_{V_H}$ are continuous, we can efficiently find a solution using a root finding algorithm. Moreover, the pair of values for $V_L$ and $V_H$ that satisfies $\frac{B_L}{F(V_L)^{n-1}} - \psi_{V_H}(V_L) = 0$ is optimal because:
- The first term, $\frac{B_L}{F(V_L)^{n-1}}$, decreases with $V_L$.
- The second term, $\psi_{V_H}(V_L)$, has a derivative: $\frac{d\psi_{V_H}(V_L)}{dV_L} = \frac{\partial \psi_{V_H}(V_L)}{\partial V_L} \frac{dV_H}{dV_L} + \frac{\partial \psi_{V_H}(V_L)}{\partial V_H} \frac{dV_H}{dV_L}$. As $\frac{\partial \psi_{V_H}(V_L)}{\partial V_H} = -\frac{f(V_H)}{f(V_L)} \leq 0$ and $\frac{dV_H}{dV_L} \leq 0$, we get $\frac{\partial \psi_{V_H}(V_L)}{\partial V_L} \frac{dV_H}{dV_L} \geq 0$. Also, $\frac{\partial \psi_{V_H}(V_L)}{\partial V_L} \geq 0$ as shown earlier. So, $\psi_{V_H}(V_L)$ is a non-decreasing function of $V_L$.

Let the values of $V_L$ and $V_H$ that satisfy $\frac{B_L}{F(V_L)^{n-1}} - \psi_{V_H}(V_L) = 0$ be $\overline{V}_L$ and $\overline{V}_H$, respectively. Overall, we have the optimal $V_L = \min(V_{up}, \max(V_{mid}, \overline{V}_L))$ and the optimal $V_H = \min(1, \max(V_{up}, \overline{V}_H))$.

One of the two cases, either $\beta(v)$ touches the upper threshold $B_H$ or it does not, will give us the overall optimal solution.

Given the optimal expected allocation function $\xi(v)$, we can easily derive the optimal allocation function $x(v)$, given in the theorem statement.

\[\text{Proof (Proof Sketch of Lemma 3)}\] The proof is very similar to Lemma 3. The main modifications are: first, we can check that in $[l(v), r(v)]$ if we flatten $\xi$, i.e., we set $\xi(y) = \frac{\int_{l(v)}^{r(v)} \xi(t)f(t)dt}{F(r(v)) - F(l(v))}$ for $y \in [l(v), r(v)]$, then we do not decrease the objective; second, we account for the change in the objective for transferring the first term for all points except for the points in $[l(V_L), r(V_L)]$ if $l(V_L) < V_L$, otherwise it might change $\beta^{-1}(B_L)$. Note that the statement of the lemma accommodates for this.
area under $\xi$ from $[l(u), r(u)]$ to $[l(v), r(v)]$, in aggregate, rather than from $u$ to $v$ as we did in Lemma 3.