EXTREMES OF REFLECTING GAUSSIAN PROCESSES ON DISCRETE GRID

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Abstract: For \( \{X(t), t \in G_\delta\} \) a centered Gaussian process with stationary increments on a discrete grid \( G_\delta = \{0, \delta, 2\delta, \ldots\} \), where \( \delta > 0 \), we investigate the stationary reflected process

\[
Q_{\delta,X}(t) = \sup_{s \in [t, \infty) \cap G_\delta} (X(s) - X(t) - c(s-t)), \quad t \in G_\delta
\]

with \( c > 0 \). We derive the exact asymptotics of \( \mathbb{P} \left( \sup_{t \in [0, T] \cap G_\delta} Q_{\delta,X}(t) > u \right) \) and \( \mathbb{P} \left( \inf_{t \in [0, T] \cap G_\delta} Q_{\delta,X}(t) > u \right) \), as \( u \to \infty \), with \( T > 0 \). It appears that \( \varphi = \lim_{u \to \infty} \sigma^2(u)/u \) determines the asymptotics, leading to three qualitatively different scenarios: \( \varphi = 0 \), \( \varphi \in (0, \infty) \) and \( \varphi = \infty \).

Key Words: Gaussian process; storage process; exact asymptotics; Piterbarg property; fractional Brownian motion; discrete grid.

AMS Classification: Primary 60G15; secondary 60G70, 60K25

1. Introduction

For \( X(t), t \geq 0 \) a centered Gaussian process with a.s. continuous sample paths, stationary increments and variance function \( \sigma^2(t) := \text{Var} (X(t)) \) such that \( \sigma^2(0) = 0 \), consider the reflected (at 0) process

\[
\hat{Q}_X(t) = X(t) - ct + \max \left( \hat{Q}_X(0), - \inf_{s \in [0,t]} (X(s) - cs) \right), \quad t \geq 0,
\]

where \( c > 0 \).

Due to its relation with the solution of the Skorokhod problem that describes the dynamics of the buffer content process in a fluid queue fed by \( X \) and emptied at rate \( c \), properties of \( \hat{Q}_X \) have been investigated for a wide class of processes \( X \), see e.g. [1–4] and references therein. Within this framework, the process \( \hat{Q}_X(t) \) is called in the literature as the storage process [3, 5].

Distributional properties of the unique stationary solution of (1), which has the following representation

\[
Q_X(t) = \sup_{t \leq s} (X(s) - X(t) - c(s-t)),
\]

were intensively analyzed. In particular, the exact asymptotics of \( \mathbb{P} (Q_X(0) > u) \), as \( u \to \infty \), was derived for a wide class of Gaussian processes \( X \); see, e.g., [2, 4, 6, 7] and references therein. An important direction of research on extremal behaviour of the process \( Q_X(t) \), \( t \geq 0 \) was initiated by Piterbarg [3], where asymptotics of \( \sup_{t \in [0,T]} Q_{B_\alpha}(t) \) was derived, with \( B_\alpha(t) \) a fractional Brownian motion with Hurst parameter \( \alpha \in (0,1) \). Notably, the following asymptotic equivalence

\[
\mathbb{P} (Q_{B_\alpha}(0) > u) \sim \mathbb{P} \left( \sup_{t \in [0,T]} Q_{B_\alpha}(t) > u \right)
\]
for \( \alpha > 1/2 \), called in [5] the Piterbarg property, took particular interest and was further observed for more broad class of processes \( X \) [4, 5, 8]. Complementary, in [4] it was shown that if \( \alpha > 1/2 \), then the process \( Q_{B_\alpha} \) also possesses strong Piterbarg property, i.e. for all \( T > 0 \)

\[
\mathbb{P}(Q_{B_\alpha}(0) > u) \sim \mathbb{P}\left( \inf_{t \in [0,T]} Q_{B_\alpha}(t) > u \right)
\]
as \( u \to \infty \).

From the point of view of the stochastic modelling or simulation techniques, discrete-time models frequently appear to be more natural. However, despite of its relevance in modelling of, e.g., queueing systems, much less is known on distributional properties of the discrete counterpart of (2), i.e.,

\[
Q_{\delta,X}(t) = \sup_{s \in [t,\infty) \cap G_\delta} (X(s) - X(t) - c(s - t)), \quad t \in G_\delta,
\]
where \( G_\delta = \{0, \delta, 2\delta, \ldots, \} \). A notable exception are recent works [9, 10], where the exact asymptotics of \( \mathbb{P}(Q_{\delta,B_\alpha}(0) > u) \), as \( u \to \infty \), was derived.

In this contribution we extend the findings of [9–11] to a more general class of Gaussian processes with stationary increments and derive the exact asymptotics of

\[
\psi_{T,\delta}^{\sup}(u) := \mathbb{P}\left( \sup_{t \in [0,T] \cap G_\delta} Q_{\delta,X}(t) > u \right), \quad \psi_{T,\delta}^{\inf}(u) := \mathbb{P}\left( \inf_{t \in [0,T] \cap G_\delta} Q_{\delta,X}(t) > u \right),
\]
as \( u \to \infty \), for \( T > 0 \) and \( \delta > 0 \), complementing results for continuous time given in [3, 4].

It appears that the influence of the grid size \( \delta \) on the asymptotics of (4) strongly depends on the value of

\[
\varphi := \lim_{u \to \infty} \frac{\sigma^2(u)}{u} \in [0, \infty],
\]
leading to three scenarios: \( \varphi = 0, \varphi \in (0, \infty) \) and \( \varphi = \infty \). The case \( \varphi = \infty \) leads to the same asymptotics as its continuous-time counterpart, which reflects the long-range dependance property of \( X \) when its variance \( \sigma^2 \) is superlinear. On the other hand, the asymptotics for the case \( \varphi = 0 \) strongly depends on the grid size and is asymptotically negligible with respect to its continuous counterpart. The third case \( \varphi \in (0, \infty) \) needs particularly precise analysis and leads to the asymptotics which differs with the continuous-time case only by a constant (depending on \( \delta \)).

Outline of the paper. We introduce notation and assumptions on the process \( X \) in Section 2. Then, in Theorem 2.1 we derive exact asymptotics of \( \mathbb{P}(Q_{\delta,X}(0) > u) \) as \( u \to \infty \). Main results of this contribution are presented in Section 3. In Section 4 we illustrate the main findings of this paper by analysis of two important classes of Gaussian processes, i.e. fractional Brownian motions and Gaussian integrated processes. Section 5 consists of the proofs of the results derived in this paper.

2. Notation and preliminary results

Let \( X(t), t \in \mathbb{R} \) be a centered Gaussian process with stationary increments, as introduced in Section 1. Suppose that

**A:** \( \sigma^2 \) is regularly varying at \( \infty \) with index \( 2\alpha \in (0,2) \) and \( \sigma^2(t) \) is twice continuously differentiable for any \( t \in (0, \infty) \). Further, the first and second derivatives of \( \sigma^2 \) are ultimately monotone.

We note that it follows straightforwardly from **A** that \( \varphi = 0 \) if \( \alpha < 1/2 \) while for \( \alpha > 1/2 \) we have \( \varphi = \infty \). It appears that case \( \alpha = 1/2 \), i.e. when \( \sigma^2 \) is asymptotically close to a linear function, needs particularly precise analysis, for which the following condition is a tractable assumption:
B: If \( \sigma^2 \) satisfies A with \( \alpha = \frac{1}{2} \), then \( \varphi > 0 \).

Condition B excludes the cases when \( \alpha = 1/2 \) but \( \varphi = 0 \). For this scenario we were able to give only partial results. We refer to Remark 3.2 for the discussion of the extension of the results derived in Section 3 to the case \( \alpha = 1/2, \varphi = 0 \) under an additional constrain on the variance function \( \sigma^2 \) of \( X \).

Conditions A and B are satisfied for a wide class of Gaussian processes with stationary increments, including family of fractional Brownian motions and integrated stationary Gaussian processes; see Section 4 for details. We note that quantity \( \varphi \) already appeared in [7], where it was observed that the form of the asymptotic behavior, as \( u \to \infty \), of \( P(Q_X(0) > u) \) introduced in (2) is determined by the value of \( \varphi \). Let

\[
\mathcal{H}_\xi(M) = \mathbb{E} \left( \sup_{t \in M} e^{\sqrt{2}\xi(t)} \right) \in (0, \infty), \quad G_\xi(M) = \mathbb{E} \left( \inf_{t \in M} e^{\sqrt{2}\xi(t)} \right) \in (0, \infty),
\]

where \( M \) is a compact subset of \( \mathbb{R} \) and \( \xi(t), t \in \mathbb{R} \) is a Gaussian process with stationary increments and a.s. continuous sample paths. Then, we define Pickands constant by

\[
\mathcal{H}_\xi^\delta = \lim_{S \to \infty} \frac{\mathcal{H}_\xi([0, S] \cap \delta \mathbb{Z})}{S}, \quad \delta \geq 0,
\]

where we set \( \delta \mathbb{Z} = \mathbb{R}^+ \) if \( \delta = 0 \). We refer to [12] for properties of \( \mathcal{H}_\xi^0 \). In Lemma 5.3 we prove that for \( \delta > 0 \) it is sufficient to suppose that \( \xi \) satisfies A to claim that \( \mathcal{H}_\xi^\delta \in (0, \infty) \). Later on for \( \delta = 0 \) we simply write \( \mathcal{H}_\xi \) instead of \( \mathcal{H}_\xi^0 \).

Next, let us recall the findings of [7][Proposition 2] (see also [8] [Theorems 3.1-3.3]) for the asymptotics of \( P(Q_X(0) > u) \), as \( u \to \infty \), which will be a useful benchmark for the results derived in the next section. Let \( \overline{\sigma}(t), t \geq 0 \) stand for the asymptotic inverse function of \( \sigma \), i.e., \( \overline{\sigma}(x) = \inf\{y \in [0, \infty) : f(y) > x\} \) (for details and properties of the asymptotic inverse functions see, e.g., [13]) and let

\[
t_\ast = \frac{\alpha}{c(1-\alpha)}, \quad m(u) = \inf_{t > 0} \frac{u(1 + ct)}{\sigma(ut)}, \quad \Delta(u) = \begin{cases} \overline{\sigma} \left( \sqrt{x^2(ut_\ast)} \right), & \varphi \notin (0, \infty), \\ 1, & \varphi \in (0, \infty), \end{cases}
\]

Let for \( X \) such that \( \varphi \in (0, \infty) \),

\[
\eta(t) = \frac{c\sqrt{\varphi}}{\varphi} X(t), \quad t \geq 0.
\]

As shown in [7][Proposition 2], if \( \sigma^2 \) is regularly varying at 0 with index \( 2\alpha_0 \in (0, 2) \) and A is satisfied, then for \( \hat{Q}_X \) defined in (1), we have

\[
P\left(\hat{Q}_X(0) > u\right) \sim f(u)\Psi(m(u)) \times \begin{cases} \mathcal{H}_{B_\alpha}, & \varphi = \infty, \\ \mathcal{H}_\eta, & \varphi \in (0, \infty), \quad u \to \infty, \\ \mathcal{H}_{B_{\alpha_0}}, & \varphi = 0 \end{cases}
\]

where \( \Psi \) is the survival function of a standard Gaussian random variable and

\[
f(u) = \sqrt{\frac{2\pi A}{B}} \frac{u}{m(u)\Delta(u)}, \quad A = \frac{1}{(1-\alpha)t_\ast^\alpha}, \quad B = \frac{\alpha}{t_\ast^{\alpha+2}}.
\]

The following result establishes the asymptotics of \( P(Q_{\delta,X}(0) > u) \) for \( \delta > 0 \) as \( u \to \infty \). It generalizes the findings of [9, 10], where the special case of \( X \) being a fractional Brownian motion was analyzed.
Theorem 2.1. Let $X(t), t \geq 0$ be a centered Gaussian process with continuous trajectories and stationary increments satisfying $A$, $B$. Then, for $\delta > 0$, as $u \to \infty$, it holds that

$$
\mathbb{P}(Q^{\delta,X}(0) > u) \sim \Psi(m(u)) \times \begin{cases}
    \frac{\sqrt{2\pi}u}{\delta(1-\alpha)^{3/2}m(u)}, & \varphi = 0 \\
    \mathcal{H}_n^\delta f(u), & \varphi \in (0, \infty) \\
    \mathcal{H}_{B_n} f(u), & \varphi = \infty.
\end{cases}
$$

(11)

Remark 2.2. Providing that both the asymptotics (9) and (11) hold, we have that

$$
\lim_{u \to \infty} \mathbb{P}(Q^{\delta,X}(0) > u | Q_X(0) > u) = \begin{cases}
    0, & \varphi = 0 \\
    \frac{4e^\delta}{\pi n}, & \varphi \in (0, \infty) \\
    1, & \varphi = \infty.
\end{cases}
$$

3. Main Results

In this section we derive the exact asymptotics of

$$
\psi^{\text{sup}}_{T,\delta}(u) := \mathbb{P}\left(\sup_{t \in [0,T]} Q^{\delta,X}(t) > u\right) \quad \text{and} \quad \psi^{\text{inf}}_{T,\delta}(u) := \mathbb{P}\left(\inf_{t \in [0,T]} Q^{\delta,X}(t) > u\right),
$$

(12)
as $u \to \infty$, for $T > 0$ and $\delta > 0$, where for any real $a < b$ and positive $\delta$

$$
[a, b]_\delta = [a, b] \cap \delta \mathbb{Z}.
$$

We begin with the asymptotics of $\psi^{\text{sup}}_{T,\delta}(u)$ as $u \to \infty$. Let in the following $[\cdot]$ stand for the integer part.

Theorem 3.1. Let $X(t), t \geq 0$ be a centered Gaussian process with continuous trajectories and stationary increments satisfying $A-B$. Then for $\delta > 0$ as $u \to \infty$ it holds that

$$
\psi^{\text{sup}}_{T,\delta}(u) \sim \Psi(m(u)) \times \begin{cases}
    (1 + \left[\frac{T}{\delta}\right]) \frac{\sqrt{2\pi}u}{\delta(1-\alpha)^{3/2}m(u)}, & \varphi = 0 \\
    \mathcal{H}_n([0,T]_\delta) \mathcal{H}_n^\delta f(u), & \varphi \in (0, \infty) \\
    \mathcal{H}_{B_n} f(u), & \varphi = \infty.
\end{cases}
$$

Remark 3.2. It follows straightforwardly from the proof of Theorem 2.1 and 3.1 that condition $B$ can be relaxed a bit. Namely, if $\varphi = 0$ and $\alpha = \frac{1}{2}$ and for $\kappa = \sqrt{c \inf_{t \in \{\delta, 2\delta, \ldots\}} \sigma(t) - \varepsilon}$, with sufficiently small $\varepsilon > 0$,

$$
\sigma(u) \leq \kappa \frac{\sqrt{u}}{\ln^{1/4} u}, \quad u \to \infty
$$

(13)

then both Theorem 2.1 and 3.1 hold. If $\varphi = 0$ and $\alpha = 1/2$ in the Theorem 3.1 and (13) does not hold, then it follows from its proof that (11) reduces to the upper bound.

Next we analyze the asymptotical behaviour of $\psi^{\text{inf}}_{T,\delta}(u)$, as $u \to \infty$.

Theorem 3.3. Let $X(t), t \geq 0$ be a centered Gaussian process with continuous trajectories and stationary increments satisfying $A-B$. Then for $\delta > 0$ as $u \to \infty$ it holds that

$$
\psi^{\text{inf}}_{T,\delta}(u) \sim f(u)\Psi(m(u)) \times \begin{cases}
    G_n([0,T]_\delta) \mathcal{H}_n^\delta, & \varphi \in (0, \infty) \\
    \mathcal{H}_{B_n}, & \varphi = \infty.
\end{cases}
$$
If \( \varphi = 0 \), then for the non-degenerated scenario (when set \([0, T]_\delta \) consists of more than 1 element, i.e, for \( T \geq \delta \)) it seems difficult to derive even logarithmic asymptotics of \( \psi_{T, \delta}^{\inf}(u) \). One can argue that \( \psi_{T, \delta}^{\inf}(u) \) is exponentially smaller than \( P(Q_{\delta, X}(0) > u) \) in this case, as \( u \to \infty \). We have the following proposition giving an upper bound for \( \psi_{T, \delta}^{\inf}(u) \).

**Proposition 3.4.** If \( \varphi = 0 \) and for some \( \varepsilon > 0 \)

\[
\sigma(u) \leq \frac{\sqrt{u}}{\ln^{1/4+\varepsilon} u}, \quad u \to \infty,
\]

then for \( T \geq \delta \) with any \( C < \frac{1+ct}{2\pi u^2} \) \( \sup_{t \in [0,T]_\delta} \sigma(t) \) it holds that

\[
\psi_{T, \delta}^{\inf}(u) \leq \Psi(m(u)) \Psi\left(\frac{u}{\sigma^2(u)^2}\right), \quad u \to \infty.
\]

**Remark 3.5.** Comparing the asymptotics in Theorems 2.1, 3.1 and 3.3 for case \( \alpha > 1/2 \) we observe that the process \( Q_{\delta, X} \) possesses the so-called strong Piterbarg property, that is

\[
\psi_{T, \delta}^{\sup}(u) \sim P(Q_{\delta, X}(0) > u) \sim \psi_{T, \delta}^{\inf}(u), \quad T \geq 0, \quad u \to \infty.
\]

The analogous property was observed for the continuous-times analog of \( Q_{\delta, X} \); see [4].

4. **Examples**

We illustrate the findings of this contribution by application of Theorems 3.1 and 3.3 to the family of fractional Brownian motions and Gaussian integrated processes.

4.1. Fractional Brownian motion. Let

\[
C_H = \frac{c_H}{H^{1-H}}, \quad D_H = \frac{\sqrt{2\pi} H^{H+1/2}}{c_{H+1}(1-H)^{H+1/2}}, \quad E_H = \frac{2^{1-3H} \sqrt{\pi}}{H^{1/2}(1-H)^{1/2}}.
\]

Applying Theorems 2.1, 3.1 and 3.3 for \( X(t) = B_H(t) \) being a standard fractional Brownian motion with Hurst parameter \( H \in (0,1) \), we obtain the following results.

**Corollary 4.1.** As \( u \to \infty \) it holds that

\[
P(Q_{\delta, B_H}(0) > u) \sim \begin{cases}
\frac{\partial u^H}{\partial \delta} \Psi(C_H u^{1-H}), & H < 1/2 \\
\mathcal{H}_{B_1/2}^{2c^2} e^{-2cu}, & H = 1/2 \\
\mathcal{H}_{B_1/2} E_H (C_H u^{1-H})^{1/H-1} \Psi(C_H u^{1-H}), & H > 1/2.
\end{cases}
\]

**Corollary 4.2.** For \( T, \delta > 0 \) as \( u \to \infty \) it holds that

\[
P\left(\sup_{t \in [0,T]_\delta} Q_{\delta, B_H}(t) > u\right) \sim \begin{cases}
(1 + [\frac{T}{\delta}]) \frac{\partial u^H}{\partial \delta} \Psi(C_H u^{1-H}), & H < 1/2 \\
\mathcal{H}_{B_1/2}([0,2c^2T]_{2^\varepsilon \delta}) \mathcal{H}_{B_1/2}^{2c^2} e^{-2cu}, & H = 1/2 \\
\mathcal{H}_{B_1/2} E_H (C_H u^{1-H})^{1/H-1} \Psi(C_H u^{1-H}), & H > 1/2.
\end{cases}
\]

and

\[
P\left(\inf_{t \in [0,T]_\delta} Q_{\delta, B_H}(t) > u\right) \sim \begin{cases}
\mathcal{G}_{B_1/2}([0,2c^2T]_{2^\varepsilon \delta}) \mathcal{H}_{B_1/2}^{2c^2} e^{-2cu}, & H = 1/2 \\
\mathcal{H}_{B_1/2} E_H (C_H u^{1-H})^{1/H-1} \Psi(C_H u^{1-H}), & H > 1/2.
\end{cases}
\]
Note that Corollary 4.1 intersects with the results in [9–11] while Corollary 4.2 provides a discrete counterpart of Theorems 5-7 in [3] and Theorem 1 in [4], respectively.

4.2. Gaussian integrated processes. For a stationary centered Gaussian process with a.s. continuous sample paths \( \zeta(s) \), \( s \geq 0 \) define the integrated process by

\[
Z(t) = \int_0^t \zeta(s)ds, \quad t \geq 0.
\]

(15)

This process is also Gaussian, has a.s. continuous sample paths and stationary increments. In what follows we consider two classes of processes \( Z \), which differ by property of the correlation function \( R(t) := \mathbb{E}(\zeta(0)\zeta(t)) \) of \( \zeta \) as \( t \to \infty \).

**SRD case.** Following, e.g., [14] (see also [15]), we impose the following conditions on the correlation of \( \zeta \):

\[ S1: R(t) \in C([0, \infty)), \quad \lim_{t \to \infty} tR(t) = 0; \]

\[ S2: \int_0^t R(s)ds > 0 \quad \forall t \in (0, \infty); \]

\[ S3: \int_0^\infty t^2|R(t)|dt < \infty. \]

The above assertions imply the existence of the first and second derivatives of \( \sigma_Z^2(t) = \text{Var}(Z(t)) \) and establish the asymptotic behavior of \( \sigma_Z^2(t) \) at \( \infty \) (see e.g., Remark 6.1 in [14]):

\[
\sigma_Z^2(t) = \frac{2}{G}t - 2D + o(t^{-1}), \quad t \to \infty,
\]

where \( G = 1/\int_0^\infty R(t)dt \) and \( D = \int_0^\infty tR(t)dt \). Thus, \( \sigma_Z^2 \) satisfies A-B with \( \alpha = 1/2 \) and \( \varphi = \frac{2}{G} > 0 \). Hence applying Theorems 2.1, 3.1 and 3.3, the following corollary holds.

**Corollary 4.3.** Suppose that \( \zeta \) satisfies S1-S3. Then for \( T \geq 0 \) and \( \delta > 0 \) as \( u \to \infty \)

\[
P \left( Q_{\delta,Z}(0) > u \right) \sim \mathcal{A}H^\delta\xi e^{-cGu},
\]

\[
P \left( \sup_{t \in [0,T]_\delta} Q_{\delta,Z}(t) > u \right) \sim \mathcal{A}H\xi([0,T]_\delta)H^\delta\xi e^{-cGu},
\]

\[
P \left( \inf_{t \in [0,T]_\delta} Q_{\delta,Z}(t) > u \right) \sim \mathcal{A}G\xi([0,T]_\delta)H^\delta\xi e^{-cGu},
\]

where \( \mathcal{A} = \frac{1}{c^2Ge^2G^2D} \) and \( \xi(t) = cGZ(t)/\sqrt{t} \).

Note that the first asymptotics in Corollary 4.3 differs from its continuous-time analog (Theorem 5.1 in [14]) only by the corresponding Pickands constants.

**LRD case.** Following, e.g., [6, 7] we characterize LRD case by the following assumptions on the covariance function \( R(t) \) of the process \( \zeta \):

\[ L1: R(t) \text{ is a continuous strictly positive function for } t \geq 0; \]

\[ L2: R(t) \text{ is regularly varying at } \infty \text{ with index } 2\alpha - 2, \quad \alpha \in (1/2, 1). \]

Under the above assumptions, by Karamata’s theorem, \( \sigma_Z^2 \) is regularly varying at \( \infty \) with index \( 2\alpha \). Since \( 2\alpha > 1 \) we are in \( \varphi = \infty \) scenario. Hence, applying Theorems 2.1, 3.1 and 3.3 we obtain the following result, which shows that \( Q_{\delta,Z}(t) \) possesses the strong Piterbarg property.
Corollary 4.4. Suppose that \( \zeta \) satisfies L1-L2. Then as \( u \to \infty \) it holds that
\[
\mathbb{P} \left( \inf_{t \in [0,T]} Q_{\delta,Z}(t) > u \right) \sim \mathbb{P} (Q_{\delta,Z}(0) > u) \sim \mathbb{P} \left( \sup_{t \in [0,T]} Q_{\delta,Z}(t) > u \right) \sim \mathcal{H}_{B,u} f(u) \Psi(m(u)),
\]
where \( m(u) \) and \( f(u) \) are defined in (7) and (10), respectively.

5. Proofs

In this section we give proofs of all the results presented in this contribution. Hereafter, denote by \( C, C_i, i = 1, 2, 3, \ldots \) positive constants that may differ from line to line and \( X := \frac{X}{\sqrt{Var(X)}} \) for any nontrivial random variable \( X \). For any \( u > 0 \) we have
\[
\mathbb{P} (Q_{\delta,X}(0) > u) = \mathbb{P} \left( \sup_{t \in G_\delta} (X(t) - ct) > u \right) = \mathbb{P} \left( \sup_{t \in G_{\delta/u}} X_u(t) > m(u) \right),
\]
where \( m(u) \) is defined in (7) and
\[
X_u(t) = \frac{X(ut)}{u(1 + ct)} m(u).
\]
Denote by \( \sigma_{X_u}^2 \) the variance function of \( X_u(t), t \geq 0 \). In the next lemma we focus on asymptotic properties of the variance and correlation functions of \( X_u(t) \); we refer to, e.g., [8] for the proof.

Lemma 5.1. Suppose that A is satisfied. For \( u \) large enough the maximizer \( t_u \) of \( \sigma_{X_u} \) is unique and \( t_u \to t^* = \frac{\alpha}{c(1-\alpha)} \) as \( u \to \infty \). Moreover, for \( \delta_u > 0 \) satisfying \( \lim_{u \to \infty} \delta_u = 0 \) (A, B are defined in (10))
\[
\lim_{u \to \infty} \sup_{t \in (t_u - \delta_u, t_u + \delta_u)\setminus \{t_u\}} \left| \frac{1 - \sigma_{X_u}(t)}{\frac{\sigma^2(ut)}{2\sigma^2(ut)}} - 1 \right| = 0
\]
and (recall, \( \sigma^2 \) is the variance of \( X \))
\[
\lim_{u \to \infty} \sup_{u \neq s, t, s, t \in (t_u - \delta_u, t_u + \delta_u)} \left| \frac{1 - Cor (X(us), X(ut))}{\frac{\sigma^2(u(s-t))}{2\sigma^2(ut)}} - 1 \right| = 0.
\]
By Lemma 5.1 we have that \( t_u \) is the unique minimizer of \( \frac{u(1 + ct_u)}{\sigma(ut_u)} \) for large \( u \) and hence by Potter’s theorem (Theorem 1.5.6 in [16]) we obtain useful in the following proofs asymptotics of \( m(u) \)
\[
m(u) = \frac{u(1 + ct_u)}{\sigma(ut_u)} \sim \frac{u(1 + ct_\tau)}{\sigma(ut_\tau)} \sim \frac{u(1 + ct_\tau)}{t_\tau^2 \sigma(u)}, \quad u \to \infty.
\]
Observe that
\[
\psi_{T,\delta}^{\sup}(u) = \mathbb{P} \left( \sup_{t \in [0,T]u, t \leq s \in G_{\delta/u}} Z_u(t,s) > m(u) \right),
\]
where
\[
Z_u(t,s) = \frac{X(us) - X(ut)}{u(1 + c(s-t))} m(u).
\]
Notice that for the variance \( \sigma_{Z_u}^2 \) of \( Z_u \) it holds, that \( \sigma_{Z_u}^2(s,t) = \sigma_{X_u}^2(s - t) \) and for correlation \( r_{Z_u} \) we have
\[
\lim_{u \to \infty} \sup_{|t-t_1| < \delta_u, s-t, s_1 - t_1 \in (-\delta_u, t_u + \delta_u), (s,t) \neq (s_1,t_1)} \left| \frac{1 - r_{Z_u}(s,t,s_1,t_1)}{\frac{\sigma^2(u(s-t_1) + \sigma^2(u(t-t_1))}{2\sigma^2(ut)} - 1} \right| = 0.
\]
To the rest of the paper we suppose that

\[
\delta_u = \begin{cases} 
  u^{-1/2} \ln u, & \varphi < \infty \\
  u^{-1} \ln(u) \sigma(u), & \varphi = \infty
\end{cases}
\]

and set

\[
I(t_u) = G_{\frac{\delta_u}{u}} \left( -\delta_u + t_u, t_u + \delta_u \right)
\]

for \( u > 0 \). The following lemma allow us to extract the main area contributing in the asymptotics of \( \psi_{\sup}^{T,\varphi}(u), \psi_{\inf}^{T,\varphi}(u) \) and \( \mathbb{P}(Q_{\delta,X}(0) > u) \) as \( u \to \infty \).

**Lemma 5.2.** For any \( T \geq 0 \) it holds that, as \( u \to \infty \),

\[
\psi_{\sup}^{T,\varphi}(u) \sim \mathbb{P} \left( \sup_{t \in [0,T/u] \delta_u / u, s \in I(t_u)} Z_{u}(t,s) > m(u) \right)
\]

\[
\psi_{\inf}^{T,\varphi}(u) \sim \mathbb{P} \left( \inf_{t \in [0,T/u] \delta_u / u, s \in I(t_u)} Z_{u}(t,s) > m(u) \right).
\]

In the next lemma we prove that the discrete Pickands constant appearing in Theorems 2.1, 3.1 and 3.3 is well defined, positive and finite.

**Lemma 5.3.** For any \( \delta \geq 0 \) and \( \eta \) a centered Gaussian process with stationary increments, a.s. continuous sample paths and variance satisfying \( A \) it holds, that

\[
\lim_{S \to \infty} \frac{\mathcal{H}_\eta([0,S]^{\delta})}{S} = \mathcal{H}_\eta^{\delta} \in (0,\infty).
\]

The lemma below allows us to give upper bounds for the double-sum terms appearing in case \( \varphi = 0 \) in Theorems 2.1 and 3.3.

**Lemma 5.4.** Assume that \( \varphi = 0 \). Then uniformly for \( t \neq s \in I(t_u) \) and all large \( u \) with some \( \varepsilon > 0 \) it holds that

\[
\mathbb{P} (X_u(t) > m(u), X_u(s) > m(u)) \leq u^{-1/2-\varepsilon} \Psi(m(u)).
\]

The proofs of Lemmas 5.2, 5.3 and 5.4 are given in the Appendix.

5.1. **Proof of Theorem 2.1**. Taking \( T = 0 \) in Lemma 5.2 we obtain that

\[
\mathbb{P}(Q_{\delta,X}(0) > u) \sim \mathbb{P} \left( \sup_{t \in I(t_u)} X_u(t) > m(u) \right), \quad u \to \infty,
\]

where \( I(t_u) \) is defined in (19). Next we consider 3 cases: \( \varphi = 0, \varphi \in (0,\infty) \) and \( \varphi = \infty \).

*Case \( \varphi = 0 \).* We have by Bonferroni inequality

\[
\sum_{t \in I(t_u)} \mathbb{P} (X_u(t) > m(u)) \geq \mathbb{P} \left( \sup_{t \in I(t_u)} X_u(t) > m(u) \right)
\]
\[ (22) \quad \geq \sum_{t \in I(t_u)} \mathbb{P}(X_u(t) > m(u)) - \sum_{t \neq s \in I(t_u)} \mathbb{P}(X_u(t) > m(u), X_u(s) > m(u)). \]

There are less then \( C u \ln^2 u \) summands in the double-sum above, hence by Lemma 5.4 we have
\[ (23) \quad \sum_{t \neq s \in I(t_u)} \mathbb{P}(X_u(t) > m(u), X_u(s) > m(u)) \leq C \ln^2(u) u^{1/2 - \varepsilon} \Psi(m(u)), \quad u \to \infty. \]

Next we focus on calculation of the single sum in (22). Since by Lemma 5.1 \( \sup_{t \in I(t_u)} |\sigma_X(t) - 1| \to 0 \) as \( u \to \infty \) the following inequality (see, e.g., Lemma 2.1 in [17])
\[ (1 - \frac{1}{x^2}) \frac{1}{\sqrt{2\pi x}} e^{-x^2/2} \leq \Psi(x) \leq \frac{1}{\sqrt{2\pi x}} e^{-x^2/2}, \quad x > 0, \]
implies as \( u \to \infty \)
\[ \sum_{t \in I(t_u)} \mathbb{P}(X_u(t) > m(u)) = \sum_{t \in I(t_u)} \Psi\left(\frac{m(u)}{\sigma_X(t)}\right) \sim \sum_{t \in I(t_u)} \frac{\sigma_X(t)}{\sqrt{2\pi m(u)}} e^{-\frac{m^2(u)}{2 \sigma_X(t)}} \sim \frac{e^{-\frac{m^2(u)}{2}}}{\sqrt{2\pi m(u)}} \sum_{t \in I(t_u)} e^{-\frac{m^2(u)}{2 \sigma_X(t)}}. \]
\[ (24) \]
By Lemma 5.1 we have that as \( u \to \infty \) the last sum above is asymptotically equivalent to
\[ \sum_{t \in I(t_u)} e^{-m^2(u) \frac{\partial}{\partial t} (t-t_u)^2} = \sum_{t \in (-\frac{\ln n}{\sqrt{u}}, \frac{\ln n}{\sqrt{u}}) / u} e^{-m^2(u) \frac{\partial}{\partial t} t^2} = \frac{u}{\delta m(u)} \left( \frac{\delta m(u)}{u} \sum_{t \in (-\frac{m(u) \ln u}{\sqrt{u}}, \frac{m(u) \ln u}{\sqrt{u}}) / \delta m(u) / u} \right) \sim \frac{u}{\delta m(u)} \int_{\mathbb{R}} e^{-\frac{\partial}{\partial t} t^2} dt = \frac{u}{\delta m(u)} \sqrt{2\pi A} = \frac{u}{\delta m(u)} \frac{\sqrt{2\pi \alpha}}{B} = \frac{u}{\delta m(u)} c(1 - \alpha)^{3/2}, \]
where the asymptotic equivalence in (25) holds, since by (16), \( \frac{\delta m(u)}{u} \to 0 \) and \( \frac{m(u) \ln u}{\sqrt{u}} \to \infty \) as \( u \to \infty \). Thus,
\[ (26) \quad \sum_{t \in I(t_u)} \mathbb{P}(X_u(t) > m(u)) \sim \frac{\sqrt{2\pi \alpha u \Psi(m(u))}}{\delta c(1 - \alpha)^{3/2} m(u)}, \quad u \to \infty \]
and hence by (16), (22) and (23) we have that
\[ \mathbb{P}\left( \sup_{t \in I(t_u)} X_u(t) > m(u) \right) \sim \frac{\sqrt{2\pi \alpha u \Psi(m(u))}}{\delta c(1 - \alpha)^{3/2} m(u)}, \quad u \to \infty \]
and the claim follows by (21).
For any fixed \( u > 0 \) and \( S \in \{0, \delta, 2\delta, \ldots\} \) denote
\[
N_u = \lceil \frac{u n\delta_u}{S\Delta(u)} \rceil, \quad t_j = \frac{\Delta(u)jS}{u}, \quad \Delta_{j,S,u} = \lfloor t_u + t_j, t_u + t_{j+1} \rfloor j / u, \quad j \in [-N_u - 1, N_u],
\]
where \( \lfloor \cdot \rfloor \) is the ceiling function. We have by Bonferroni inequality that
\[
(27) \sum_{-N_u \leq j \leq N_u - 1} p_{j,S,u} - \sum_{-N_u - 1 \leq i \neq j \leq N_u} p_{i,j,S,u} \leq \mathbb{P} \left( \sup_{t \in I(t_u)} X_u(t) > m(u) \right) \leq \sum_{-N_u - 1 \leq j \leq N_u} p_{j,S,u},
\]
where
\[
p_{j,S,u} = \mathbb{P} \left( \sup_{t \in \Delta_{j,S,u}} X_u(t) > m(u) \right) \quad \text{and} \quad p_{i,j,S,u} = \mathbb{P} \left( \sup_{t \in \Delta_{i,j,S,u}} X_u(t) > m(u), \sup_{t \in \Delta_{j,S,u}} X_u(t) > m(u) \right).
\]
By [8] (proof of lower bound of \( \pi_T(u) \), p. 288) we have as \( u \to \infty \) and then \( S \to \infty \)
\[
\sum_{-N_u - 1 \leq j \leq N_u - 1} p_{j,S,u} = o \left( \frac{u}{m(u)\Delta(u)\Psi(m(u))} \right).
\]
Hence from the asymptotics of \( \sum_{-N_u \leq j \leq N_u - 1} p_{j,S,u} \) given in (28) and (29) we obtain that as \( u \to \infty \) and then \( S \to \infty \)
\[
\mathbb{P} \left( \sup_{t \in I(t_u)} X_u(t) > m(u) \right) \sim \sum_{-N_u \leq j \leq N_u} p_{j,S,u}
\]
and we need to calculate the asymptotics of the sum above. That can be done via uniform approximation of \( p_{j,S,u} \) for all \( -N_u - 1 \leq j \leq N_u \) separately for cases \( \varphi \in (0, \infty) \) and \( \varphi = \infty \).

**Case \( \varphi \in (0, \infty) \)**. Let \( \Delta(u) = 1 \), \( N_u = \lceil \frac{u n\delta_u}{S} \rceil \), \( t_j = \frac{jS}{u} \) and \( \Delta_{j,S,u} = \lfloor t_u + t_j, t_u + t_{j+1} \rfloor j / u \). We have for any \( \varepsilon > 0 \), \( 0 \leq j \leq N_u \) for all \( u \) large enough with \( m_j^-(u) = \frac{m(u)}{1 - (1 - \varepsilon)\frac{\sqrt{2\pi}}{2\sqrt{t_u}}(\frac{\Delta(u)}{u})^2} \)
\[
p_{j,S,u} = \mathbb{P} \left( \exists t \in \Delta_{j,S,u} : X_u(t) > \frac{m(u)}{\sigma_u(t)} \right) \leq \mathbb{P} \left( \exists t \in \Delta_{j,S,u} : X_u(t) > \frac{m(u)}{1 - (1 - \varepsilon)^2(t - t_u)^2} \right)
\]
\[
\leq \mathbb{P} \left( \sup_{t \in \Delta_{j,S,u}} X_u(t) > m_j^-(u) \right) = \mathbb{P} \left( \sup_{t \in [0,S]} X_u(t + ut_u + ut_j) > m_j(u) \right).\]
By Lemma 5.1 we have that \( 1 - r_{X_u}(t,s) \sim \frac{\sigma^2(t-s)}{2\sigma^2(t_u)} \), \( u \to \infty \), \( t, s \in [0, S] \). Thus, by Lemma 1 in [4] the last probability above is asymptotically equal to \( \mathcal{H}_\eta([0, S])\Psi(m_j^-(u)) \), as \( u \to \infty \), where \( \eta' \) is a centered Gaussian process with stationary increments, a.s. continuous sample paths and variance (asymptotics of \( m(u) \) is given in (16))
\[
\sigma^2_{\eta'}(t) = \lim_{u \to \infty} \frac{m^2(u)}{2\sigma^2(t_u)}\sigma^2(t) = \lim_{u \to \infty} \frac{u^2(1 + ct_u)^2}{2\sigma^2(t_u)}\sigma^2(t) = \frac{2c^2}{\varphi^2}\sigma^2(t).
\]
Note that \( \eta' \) and \( \eta \) defined in (8) have the same distributions. Thus,
\[
\sum_{0 \leq j \leq N_u} p_{j,S,u} \leq \mathcal{H}_\eta([0, S]) \sum_{0 \leq j \leq N_u} \Psi(m_j^-(u)), \quad u \to \infty.
\]
Next, as \( u \to \infty \), (set \( C_- = \frac{(1-\varepsilon)B}{2A} \)) using the same reasoning as in case \( \varphi = 0 \), we have

\[
\sum_{0 \leq j \leq N_u} \frac{\Psi(m_j(u))}{\Psi(m(u))} \sim \sum_{0 \leq j \leq N_u} e^{-\frac{m_j^2(u)}{2} \left( \frac{1}{1-C_- \left( \frac{m_j(u)}{u} \right)^2} \right)}
\]

\[
\sim \sum_{0 \leq j \leq N_u} e^{-\frac{m_j^2(u)}{2} 2C_- \left( \frac{tS(j)}{u} \right)^2}
\]

\[
= \frac{S(m(u))}{u} \sum_{t \in \left[ 0, \frac{m(u)\ln u}{u} \right]} e^{-C_- \left( \frac{tS(m(u))}{u} \right)^2}
\]

\[
= \frac{u}{S(m(u))} \left( \frac{S(m(u))}{u} \sum_{t \in \left[ 0, \frac{m(u)\ln u}{u} \right]} e^{-C_- t^2} \right).
\]

Due to (16) we have \( \frac{u}{S(m(u))} \to 0 \) and \( \frac{m(u)\ln u}{u} \to \infty \), as \( u \to \infty \). Thus, the sum above converges to \( \int_0^\infty e^{-C_- t^2} dt = \frac{\sqrt{\pi}}{2\sqrt{C_-}} \) as \( u \to \infty \). Similar calculation can be done for \( j < 0 \). Hence, we have as \( u \to \infty \) and then \( S \to \infty \)

\[
\sum_{-N_u \leq j \leq N_u} p_{j,S,u} \leq \mathcal{H}_{\eta,j}([0,S]_{\delta}) \frac{u\Psi(m(u))}{S(m(u))} \frac{\sqrt{\pi}}{\sqrt{C_-}}.
\]

By Lemma 5.3 we know that \( \mathcal{H}_{\eta,j}([0,S]_{\delta}) \to \mathcal{H}_{\eta,j} \in (0, \infty) \) as \( S \to \infty \). Hence, letting \( S \to \infty \), we have

\[
\sum_{-N_u \leq j \leq N_u} p_{j,S,u} \leq \mathcal{H}_{\eta,j} \frac{u\Psi(m(u))}{m(u)} \frac{\sqrt{\pi}}{\sqrt{C_-}} (1 + o(1)), \quad u \to \infty.
\]

By the same arguments we have the asymptotic lower bound

\[
\sum_{-N_u \leq j \leq N_u} p_{j,S,u} \geq \mathcal{H}_{\eta,j} \frac{u\Psi(m(u))}{m(u)} \frac{\sqrt{\pi}}{\sqrt{C_+}} (1 + o(1)), \quad u \to \infty,
\]

with \( C_+ = \frac{(1+\varepsilon)B}{2A} \). Hence letting \( \varepsilon \to 0 \) we have that, as \( S \to \infty \) and then \( u \to \infty \),

\[
\sum_{-N_u \leq j \leq N_u} p_{j,S,u} \sim \mathcal{H}_{\eta,j} \frac{u}{m(u)} \sqrt{\frac{2\pi A}{B}} \Psi(m(u)),
\]

which completes the proof of the case \( \varphi \in (0, \infty) \).

Case \( \varphi = \infty \). In this case we have that \( \Delta(u), N_u \to \infty \) and \( \Delta(u)/u \to 0 \) as \( u \to \infty \). Moreover, for \( m_k(u) = \frac{m(u)}{1-C_+ \left( \frac{k}{u} \right)^2} \), \(-N_u \leq k \leq N_u\) for large \( S,u \)

\[
p_{k,S,u} \geq \mathbb{P} \left( \sup_{t \in [k,k+1]_{\delta_u}/u} X_u(t) > m_k(u) \right)
\]

\[
\geq \mathbb{P} \left( \sup_{t \in [k,k+1]_{\delta_u/\Delta(u)}} X_u(t) > m_k(u) \right)
\]

\[
\geq \mathbb{P} \left( \sup_{t \in [0,S(1-\delta_u)]_{\delta_u/\Delta(u)}} X_u \left( t \frac{\Delta(u)}{u} + t_k \right) > m_k(u) \right)
\]

\[
\geq \mathbb{P} \left( \sup_{t \in [0,S(1-\delta_u)]_{\delta_u/\Delta(u)}} X_u \left( t \frac{\Delta(u)\delta_u}{u} + t_k \right) > m_k(u) \right)
\]
Next by Lemma 5.1 in [8] using the notation introduced therein with index set $K$ consisting of 1 element and

$$g(u) = m_k(u), \quad \theta(u,s,t) = |t - s|^{2\alpha}, \quad V = B_{\alpha}, \quad \sigma_{V}(t) = |t|^{2\alpha}$$

we have uniformly for $-N_u \leq k \leq N_u$

$$P \left( \sup_{t \in [0,(1-\varepsilon_2)S[\delta_{x_1}]} Y_u(t) > m_k(u) \right) \sim \mathcal{H}_{B_{\alpha}}([0,S(1-\varepsilon_2)]_{\delta_{x_1}}) \Psi(m_k(u)), \ u \rightarrow \infty.$$

Thus, for large $S$, as $u \rightarrow \infty$,

$$\sum_{-N_u \leq k \leq N_u} p_{k,S,u} \geq \mathcal{H}_{B_{\alpha}}([0,(1-\varepsilon_2)S[\delta_{x_1}]) \sum_{-N_u \leq k \leq N_u} \Psi(m_k(u))(1 + o(1)).$$

Next we calculate the sum above. Similarly to the previous cases we have as $u \rightarrow \infty$ with $\hat{C}_+ = \frac{C_1(1+ct_*)^2}{(t_*)^{2\alpha}}$ and $l_u = \frac{S\Delta(u)}{\sigma(u)} \rightarrow 0$, that

$$\frac{\sum_{k=-N_u}^{N_u} \Psi(m_k(u))}{\Psi(m(u))} \sim \sum_{k=-N_u}^{N_u} e^{-\frac{m^2(u)}{2}(1-C_+ t^2 u)} = \sum_{k=-N_u}^{N_u} e^{-C_+ m^2(u) t^2 u} \sim \sum_{k=-N_u}^{N_u} e^{-\frac{\sigma(u)\ln u}{S\Delta(u)} kS \Delta(u)} e^{-\frac{C_+ (1+ct_*)^2}{t^2 u} (kS \Delta(u))^2} = \sum_{kl_u \in (-\ln u,\ln u)_u} e^{-\hat{C}_+(k^2 u)^2} = \frac{1}{l_u} \left( \sum_{t \in (-\ln u,\ln u)_u} e^{-\hat{C}_+(t^2 u)} \right).$$

Since $l_u \rightarrow 0$ as $u \rightarrow \infty$, the expression in the parentheses above converges as $u \rightarrow \infty$ to

$$\int_{\mathbb{R}} e^{-\hat{C}_+(t^2 u)} dt = \sqrt{\pi} \hat{C}_+ = \sqrt{\frac{2A\pi}{B}} \frac{t^\alpha}{1 + ct_*} \sqrt{\frac{1}{1 + \varepsilon}}.$$

Thus, summarizing the calculations above we have as $u \rightarrow \infty$ for large $S$

$$\sum_{-N_u \leq k \leq N_u} p_{k,S,u} \geq \frac{1}{\sqrt{1 + \varepsilon}} \mathcal{H}_{B_{\alpha}}([0,(1-\varepsilon_2)S[\delta_{x_1}]) \Psi(m(u)) \frac{\sigma(u)}{\Delta(u)} \sqrt{\frac{2A\pi}{B}} \frac{t^\alpha}{1 + ct_*} (1 + o(1)).$$

Letting $S \rightarrow \infty$ then $\varepsilon_2 \rightarrow 0$ and then $\varepsilon_1 \rightarrow 0$, in view of Lemma 12.2.7 ii) and Remark 12.2.10 in [13] we obtain $\frac{1}{S} \mathcal{H}_{B_{\alpha}}([0,(1-\varepsilon_2)S[\delta_{x_1}]) \rightarrow \mathcal{H}_{B_{\alpha}}$. Letting then $\varepsilon \rightarrow 0$ we get the asymptotic lower bound

$$\sum_{-N_u \leq k \leq N_u} p_{k,S,u} \geq \mathcal{H}_{B_{\alpha}} \Psi(m(u)) \frac{\sigma(u)}{\Delta(u)} \sqrt{\frac{2A\pi}{B}} \frac{t^\alpha}{1 + ct_*} (1 + o(1)), \ u \rightarrow \infty.$$

Similarly, we have for $\varepsilon > 0$ with $m_{\tilde{C}}(u) = \frac{m(u)}{1-C_-|k|^2}$ (recall, $C_- = \frac{B(1-\varepsilon)}{2A}$) as $S \rightarrow \infty$

$$\sum_{-N_u \leq k \leq N_u} p_{k,S,u} \leq \frac{1}{\sqrt{1 - \varepsilon}} \mathcal{H}_{B_{\alpha}} \Psi(m(u)) \frac{\sigma(u)}{\Delta(u)} \sqrt{\frac{2A\pi}{B}} \frac{t^\alpha}{1 + ct_*} (1 + o(1)), \ u \rightarrow \infty.$$
Letting $\varepsilon \to 0$ we obtain the right side in (29), thus, the claim is established. \hfill $\square$

5.2. **Proof of Theorem 3.1.** By Lemma 5.2 we have

$$
\psi(u) = \max_{t \in [0, T]} \max_{s \in I(u)} Z_u(t, s) > m(u), \quad u \to \infty.
$$

As in the proof of Theorem 2.1 next we consider three cases: $\varphi = 0$, $\varphi \in (0, \infty)$ and $\varphi = \infty$.

**Case $\varphi = 0$.** We have by Bonferroni inequality

$$
\sum_{t \in [0, T]} \sum_{s \in I(u)} \mathbb{P}(Z_u(t, s) > m(u)) \geq \mathbb{P}\left( \max_{t \in [0, T]} \max_{s \in I(u)} Z_u(t, s) > m(u) \right)
$$

where the last inequality above follows from (23). Next we have

$$
\sum_{t \in [0, T]} \sum_{s \in I(u)} \mathbb{P}(Z_u(t, s), Z_u(t, s') > m(u)) \leq (1 + \left[ \frac{T}{\delta} \right]) \sup_{t \in [0, T]} \sum_{s \in I(u)} \mathbb{P}(X_u(s) > m(u))
$$

where the last inequality above follows from (23). Next we have

$$
\sum_{t \in [0, T]} \sum_{s \in I(u)} \mathbb{P}(Z_u(t, s) > m(u)) = \sum_{t \in [0, T]} \mathbb{P}(X_u(s) > m(u))
$$

where the last inequality above follows from (23). Next we have

$$
\sum_{t \in [0, T]} \sum_{s \in I(u)} \mathbb{P}(Z_u(t, s) > m(u)) \sim (1 + \left[ \frac{T}{\delta} \right]) \frac{2\pi \alpha \Psi(m(u))}{\delta c(1 - \alpha)^{3/2} m(u)}, \quad u \to \infty.
$$

By the line above combined with (31) we obtain

$$
\mathbb{P}\left( \max_{t \in [0, T]} \max_{s \in I(u)} Z_u(t, s) > m(u) \right) \sim (1 + \left[ \frac{T}{\delta} \right]) \frac{2\pi \alpha \Psi(m(u))}{\delta c(1 - \alpha)^{3/2} m(u)}, \quad u \to \infty.
$$

and the claim follows by (30).

**Case $\varphi \in (0, \infty)$.** With the notation of Theorem 2.1 we have by Bonferroni inequality for $u > 0$

$$
\sum_{j=-N_u}^{N_u} q_{j, S, u} \geq \mathbb{P}\left( \max_{t \in [0, T]} \max_{s \in I(u)} Z_u(t, s) > m(u) \right) \geq \sum_{j=-N_u}^{N_u} q_{j, S, u} - \sum_{-N_u \leq i < j \leq N_u} q_{i, j, S, u},
$$

where the last inequality above follows from (23). Next we have

$$
\sum_{t \in [0, T]} \sum_{s \in I(u)} \mathbb{P}(Z_u(t, s) > m(u)) \leq C(1 + \left[ \frac{T}{\delta} \right]) \ln^2(u) u^{1/2 - \varphi} \Psi(m(u)),
$$

where the last inequality above follows from (23). Next we have

$$
\sum_{t \in [0, T]} \sum_{s \in I(u)} \mathbb{P}(Z_u(t, s) > m(u)) = \sum_{t \in [0, T]} \sum_{s \in I(u)} \mathbb{P}(X_u(s) > m(u))
$$

where the last inequality above follows from (23). Next we have

$$
\sum_{t \in [0, T]} \sum_{s \in I(u)} \mathbb{P}(Z_u(t, s) > m(u)) \sim (1 + \left[ \frac{T}{\delta} \right]) \frac{2\pi \alpha \Psi(m(u))}{\delta c(1 - \alpha)^{3/2} m(u)}, \quad u \to \infty.
$$

By the line above combined with (31) we obtain

$$
\mathbb{P}\left( \max_{t \in [0, T]} \max_{s \in I(u)} Z_u(t, s) > m(u) \right) \sim (1 + \left[ \frac{T}{\delta} \right]) \frac{2\pi \alpha \Psi(m(u))}{\delta c(1 - \alpha)^{3/2} m(u)}, \quad u \to \infty.
$$

and the claim follows by (30).
where

\[ q_{j,S,u} = \mathbb{P}\left( \sup_{t \in [0,T_u]} Z_u(t, s) > m(u) \right), \quad q_{i,j,S,u} = \mathbb{P}\left( \exists t \in [0,T_u] : \sup_{s \in \Delta_{j,S,u}} Z_u(t, s) > m(u), \sup_{s \in \Delta_{i,j,S,u}} Z_u(t, s) > m(u) \right) \]

By [8] (proof of lower bound of \( \pi_{TL}(u) \), p. 288) we have that as \( u \to \infty \) and then \( S \to \infty \)

\[ \sum_{-N_u \leq i < j \leq N_u} q_{i,j,S,u} = o\left( \frac{u}{m(u)\Delta(u)} \Psi(m(u)) \right). \]

Since

\[ \mathbb{P}\left( \sup_{t \in [0,T_u]} Z_u(t, s) > m(u) \right) \geq \mathbb{P}\left( \sup_{t \in I(t_u)} X_u(t) > m(u) \right) \geq C \frac{u}{m(u)\Delta(u)} \Psi(m(u)), \quad u \to \infty, \]

we have that as \( u \to \infty \) and then \( S \to \infty \)

\[ \mathbb{P}\left( \sup_{t \in [0,T_u]} Z_u(t, s) > m(u) \right) \sim \sum_{j=-N_u}^{N_u} q_{j,S,u} \]

and we need to calculate the asymptotics of the sum above. Next we uniformly approximate each summand in the sum above. For \( \varepsilon > 0, j \geq 1, S > T \) and \( u \) large enough we have (recall, \( m_{j-1}(u) = \frac{m(u)}{1-(1-\varepsilon) \frac{\beta}{2A}(\frac{1-S}{u})^{2}} \))

\[ q_{j,S,u} = \mathbb{P}\left( \exists (t, s) \in [0,T_u] : \sup_{\Delta_{j,S,u}} Z_u(t, s) > \frac{m(u)}{\sigma_u(s-t)} \right) \]
\[ \leq \mathbb{P}\left( \exists (t, s) \in [0,T_u] : \sup_{\Delta_{j,S,u}} Z_u(t, s) > \frac{m(u)}{1-(1-\varepsilon) \frac{\beta}{2A}(s-t)^2} \right) \]
\[ = \mathbb{P}\left( \sup_{t \in [0,T], s \in [0,S]} Z_u(t,u,t_u + s/u) > \frac{m(u)}{1 - C_{-} (\frac{1-S}{u})^{2}} \right) \]
\[ =: \mathbb{P}\left( \sup_{t \in [0,T], s \in [0,S]} Z_u(t,s) > m^{-}_{j-1}(u) \right). \]

By (18) correlation of \( Z_u \) satisfies

\[ 1 - r_{Z_u}(t, s, t_1, s_1) \sim \frac{\sigma(|s-s_1|)^2 + \sigma^2(|t-t_1|)}{2\sigma^2(ut_u)}, \quad (t, s) \in [0,T] \times [0,S], \quad u \to \infty. \]

Applying Lemma 5.1 in [8] with

\[ \Phi := \sup, \quad \theta(u, s, t) := \frac{2u^2}{\varphi^2}(\sigma^2(|s-s_1|) + \sigma^2(|t-t_1|)), \quad V(t, s) := \frac{\sqrt{2c}}{\varphi} (X^{(1)}(t) + X^{(2)}(s)), \]

where \( X^{(1)} \) and \( X^{(2)} \) are independent copies of \( X \) we have

\[ \mathbb{P}\left( \sup_{t \in [0,T], s \in [0,S]} Z_u(t,u,s) > m^{-}_{j-1}(u) \right) \sim \mathcal{H}_{\frac{\varphi}{\varphi}} X([0,T]_{\delta}) \mathcal{H}_{\frac{\varphi}{\varphi}} X([0,S]_{\delta}) \Psi(m^{-}_{j-1}(u)). \]

Finally, for \( \varepsilon > 0, j \geq 1, S > T \) and \( u \) large we have

\[ \mathbb{P}\left( \sup_{t \in [0,T], s \in \Delta_{j,S,u}} Z_u(t,s) > m(u) \right) \leq \mathcal{H}_{\frac{\varphi}{\varphi}} X([0,T]_{\delta}) \mathcal{H}_{\frac{\varphi}{\varphi}} X([0,S]_{\delta}) \Psi(m^{-}_{j-1}(u))(1 + o(1)). \]
The rest of the proof is the same as in Theorem 2.1 case \( \varphi \in (0, \infty) \), thus, the claim is established.

**Case \( \varphi = \infty \).** By Theorem 2.1 we have
\[
\psi_{T, \delta}^{\sup}(u) \geq \mathbb{P}(M_{\delta} > u) \sim \mathcal{H}_{B_u} f(u) \Psi(m(u)), \quad u \to \infty.
\]

By (30) we have
\[
\psi_{T, \delta}^{\sup}(u) \leq \mathbb{P} \left( \sup_{t \in [0,T/\delta], s \in \Delta_j} Z_u(t, s) > m(u) \right) (1 + o(1)), \quad u \to \infty.
\]

From the proof of Theorem 3.1 in [8] it follows that the last probability above does not exceed \((1 + o(1))\mathcal{H}_{B_u} f(u) \Psi(m(u)), \quad u \to \infty.\) Combining both bounds above we obtain the claim.

\(\square\)

**Proof of Theorem 3.3.** *Case \( \varphi \in (0, \infty) \).* By Lemma 5.2 we have
\[
\psi_{T, \delta}^{\inf}(u) \sim \mathbb{P} \left( \inf_{t \in [0,T/\delta], s \in \Delta_j} Z_u(t, s) > m(u) \right), \quad u \to \infty.
\]

With notation of Theorem 2.1 repeating the proof of Theorem 3.1 we have as \( u \to \infty \) and then \( S \to \infty \)
\[
\mathbb{P} \left( \inf_{t \in [0,T/\delta], s \in \Delta_j} Z_u(t, s) > m(u) \right) \sim \sum_{j=-N_u}^{N_u} \mathbb{P} \left( \inf_{t \in [0,T/\delta], s \in \Delta_j} Z_u(t, s) > m(u) \right).
\]

Next we uniformly approximate each summand in the sum above. For \( \varepsilon > 0, j \geq 1, S > T \) and \( u \) large enough similarly to the proof of Theorem 3.1 we obtain (recall, \( m_{j-1}(u) = \frac{m(u)}{1 - (1 - \varepsilon) \frac{2(1 - \frac{1}{u^2})}{2}} \))
\[
\mathbb{P} \left( \inf_{t \in [0,T/\delta], s \in [0,S]_\delta} Z_u(t, s) > m(u) \right) \leq \mathbb{P} \left( \inf_{t \in [0,T/\delta], s \in [0,S]_\delta} Z_u(t, s) > m_{j-1}(u) \right),
\]

where \( Z_u(t, s) \) is a centered Gaussian process with unit variance and correlation satisfying (32). Applying Lemma 5.1 in [8] with (with \( X^{(1)}, X^{(2)} \) being independent copies of \( X \))
\[
\Phi := \inf_{u, s, t} \theta(u, s, t) := \frac{2c^2}{\varphi^2} (\sigma^2(|s - s_i|) + \sigma^2(|t - t_i|)), \quad V(t, s) := \frac{\sqrt{2c}}{\varphi} (X^{(1)}(t) + X^{(2)}(s))
\]

we have as \( u \to \infty \)
\[
\mathbb{P} \left( \inf_{t \in [0,T/\delta], s \in [0,S]_\delta} Z_u(t, s) > m_{j-1}(u) \right) \sim \mathbb{E} \left( \inf_{t \in [0,T/\delta], s \in [0,S]_\delta} e^{\sqrt{2V(t,s)} - \text{Var}(V(t,s))} \right) \Psi(m_{j-1}(u)).
\]

Since \( X^{(1)}(t) \) and \( X^{(2)}(s) \) are independent we have
\[
\mathbb{E} \left( \inf_{t \in [0,T/\delta], s \in [0,S]_\delta} e^{\sqrt{2V(t,s)} - \text{Var}(V(t,s))} \right) = \mathcal{G}_{\sqrt{2} \varphi} X([0,T]_\delta \mathcal{H}_{\sqrt{2} \varphi} X([0,S]_\delta).
\]

Thus, as \( u \to \infty \)
\[
\mathbb{P} \left( \inf_{t \in [0,T/\delta], s \in [0,S]_\delta} Z_u(t, s) > m(u) \right) \leq \mathcal{H}_{\sqrt{2} \varphi} X([0,T]_\delta \mathcal{H}_{\sqrt{2} \varphi} X([0,S]_\delta) \Psi(m_{j-1}(u)).
\]

The rest of the proof is the same as in the proof of Theorem 2.1. Thus, the claim is established.
Case $\varphi = \infty$. Let $R(s,t) = X(s) - X(t) - c(s-t)$, $t, s \geq 0$. Using the idea from [4], (the next equation after (2)) we write

$$\frac{\psi_{\inf,T}(u)}{\psi_{\sup,T,\delta}(u)} = \mathbb{P}\left( \inf_{t \in [0,T]} \sup_{s \leq m_\delta} R(s,t) > u \mid \sup_{t \in [0,T]} \sup_{s \geq m_\delta} R(s,t) > u \right)$$

$$\geq 1 - \sum_{t \in [0,T]} \left( 1 - \mathbb{P}\left( \sup_{s \leq m_\delta} R(s,t) > u \right) \right).$$

By Theorems 2.1 and 3.1 we have that the right part of the expression above tends to 1 as $u \to \infty$. Thus, we have $\psi_{\inf,T,\delta}(u) \sim \psi_{\sup,T,\delta}(u) \sim \mathcal{H}_{B_u}(f(u)\Psi(m(u)))$, $u \to \infty$ and the claim follows.

**Proof of Proposition 3.4.** Since $T \geq \delta$ with any $K \in [\delta,T]_\delta$, $a \in (0,t_s)$, $b > 0$ and $J(t_u) = [-a + t_u, t_u + b]$ we have

$$\psi_{\inf,T}(u) \leq \psi_{\inf,K}(u) \leq \mathbb{P}\left( \exists s_1, s_2 \in J(t_u) \cap G_{\frac{K}{u}} : Z_u(0,s_1) > m(u), Z_u(\frac{K}{u},s_2) > m(u) \right)$$

$$+ \mathbb{P}(\exists s \notin J(t_u) : Z_u(0,s) > m(u)) =: p_1(u) + p_2(u).$$

**Estimation of $p_1(u).** Fix some $s_1, s_2 \in J(t_u) \cap G_{\frac{K}{u}}$ and let $W_1, W_2 = (Z_u(0,s_1), Z_u(\frac{K}{u},s_2))$. We have that $(W_1,W_2)$ is a centered Gaussian vector with $Var(W_1), Var(W_2) \leq 1$ and correlation $r_{W_u}(s_1,s_2)$ satisfying (see (18))

$$1 - r_{W_u}(s_1,s_2) \geq \frac{\sigma^2(u|s_1-s_2|)}{2\sigma^2(ut_\alpha)} + \frac{\sigma^2(K)}{2(1+o(1))\sigma^2(u)}, \quad s_1, s_2 \in J(t_u).$$

Thus, by Lemma 2.3 in [17]

$$\Psi(m(u))^{-1}\mathbb{P}(W_1 > m(u), W_2(u) > m(u)) \leq 3\Psi(m(u)\sqrt{\frac{1 - r_{W_u}(s_1,s_2)}{2}})$$

$$= 3\Psi\left(1 + o(1)\frac{\sigma(K)m(u)}{2\sigma^2(u)}\right) = 3\Psi\left(1 + o(1)\frac{\sigma(K)(1 + ct_\alpha)u}{2\sigma^2(\alpha)}\right).$$

Note that $\varphi = 0$ implies $\frac{u}{\sigma(u)} \to \infty$ as $u \to \infty$. Thus, since there are less than $Cu^2\ln^2 u$ points in $(J(t_u) \cap G_{\frac{K}{u}}) \times (J(t_u) \cap G_{\frac{K}{u}})$ we have with any $C_K < \frac{\sigma(K)(1 + ct_\alpha)}{2\sigma^2(\alpha)}$ by (14) as $u \to \infty$

$$p_1(u) \leq Cu^2(\ln^2 u) \cdot 3\Psi(m(u))\Psi\left(1 + o(1)\frac{\sigma(K)(1 + ct_\alpha)u}{2\sigma^2(\alpha)}\right) \leq \Psi(m(u))\Psi\left(C_K \frac{u}{\sigma^2(u)}\right).$$

**Estimation of $p_2(u).** Since $Z_u(0,s) \overset{d}{=} X_u(s)$, $s \geq 0$, it follows from the estimation of $R_1(u)$ and $R_2(u)$ in the proof of Lemma 5.2 below (see (35) and (36), respectively) that for appropriately chosen $a \in (0,t_s), b > 0$ and small $\varepsilon > 0$

$$p_2(u) \leq \mathbb{P}\left(\exists \varepsilon \in [0,a] : X_u(s) > m(u)\right) + \mathbb{P}(\exists \varepsilon \in [t_s + b, \infty) : X_u(s) > m(u)) \leq \Psi(m(u))Ce^{-\frac{m^2(u)}{u^{\alpha}}}. $$

Combining this inequality with the upper bound of $p_1(u)$ we obtain that for any $K \in [\delta,T]_\delta$ it holds that

$$\psi_{\inf,T}(u) \leq \Psi(m(u))\Psi\left(C_K \frac{u}{\sigma^2(u)}\right), \quad u \to \infty.$$
and taking the supremum with respect to \( K \) we obtain the claim. \( \square \)

**Proof of Corollary 4.3.** We start with the proof of the first statement. Since \( Z(t) \) is a Gaussian process with stationary increments, a.s. sample paths and variance satisfying \( A \) applying Theorem 2.1 with parameters

\[
\varphi = \lim_{u \to \infty} \frac{\sigma^2(u)}{u} = \frac{2}{G} \in (0, \infty), \quad \alpha = 1/2, \quad t_* = 1/c, \quad \Delta(u) = 1, \quad A = 2\sqrt{C}, \quad D = c^2 \sqrt{C}/2,
\]

\[
m(u) = \sqrt{2G}c\sqrt{u} + \frac{c^3/2G^{3/2}D}{\sqrt{2}}u^{-1/2} + o(u^{-1/2}), \quad f(u) = \frac{2\sqrt{\pi}}{c\sqrt{C}}\sqrt{u} + O(u^{-1}), \quad \eta(t) = \frac{cG}{\sqrt{2}}Z(t)
\]

we have \( \Psi(m(u)) = e^{-ucG-c^2G^2D/2}\sqrt{u} \to \infty \) implying

\[
\mathbb{P}(\exists t \in [0, \infty) : Z(t) - ct > u) \sim \mathcal{H}_{tGZ(t)/c^2G}^{\delta/u} e^{-ucG-c^2G^2D}, \quad u \to \infty
\]

and the first claim follows. Applying Theorems 3.1 and 3.3 with the same parameters we obtain the second and third claims, respectively. \( \square \)

**Proof of Lemma 5.2.** We start with the first claim. We have

\[
\mathbb{P} \left( \sup_{t \in [0, T/u]} \sup_{s \in I(t_u)} Z_u(t, s) > m(u) \right) \leq \mathcal{V}^{\sup}_{\delta, T}(u)
\]

(33)

\[
\leq \mathbb{P} \left( \sup_{t \in [0, T/u]} \sup_{s \in I(t_u)} Z_u(t, s) > m(u) \right) + \mathbb{P} \left( \sup_{t \in [0, T/u]} \sup_{s \in (G_{\delta/u} \setminus I(t_u))} Z_u(t, s) > m(u) \right).
\]

Our first aim is to show that

(34)

\[
\mathbb{P} \left( \sup_{t \in [0, T/u]} \sup_{s \in (G_{\delta/u} \setminus I(t_u))} Z_u(t, s) > m(u) \right) = o(\Psi(m(u))), \quad u \to \infty.
\]

Since for any fixed \( 0 \leq t \leq s \) random variables \( Z_u(t, s) \) and \( X_u(s - t) \) have the same distributions we have with \( I'(t_u) = (-\frac{\delta u}{2} + t, \frac{\delta u}{2} + t) \cap G_{\delta/u} \)

\[
\mathbb{P} \left( \sup_{t \in [0, T/u]_{\delta/u}} \sup_{s \in (G_{\delta/u} \setminus I(t_u))} Z_u(t, s) > m(u) \right) \leq \sum_{t \in [0, T/u]_{\delta/u}} \mathbb{P} \left( \sup_{s \in (G_{\delta/u} \setminus I(t_u))} Z_u(t, s) > m(u) \right)
\]

\[
= \sum_{t \in [0, T/u]_{\delta/u}} \mathbb{P} \left( \sup_{s \in (G_{\delta/u} \setminus I(t_u))} X_u(s - t) > m(u) \right)
\]

\[
\leq (1 + \frac{T}{\delta}) \mathbb{P} \left( \sup_{s \in (G_{\delta/u} \setminus I'(t_u))} X_u(s) > m(u) \right).
\]

We have that for any chosen small \( \varepsilon \) and large \( M \) the last probability above does not exceed

\[
\sum_{t \in (G_{\delta/u} \setminus I'(t_u))} \mathbb{P} \left( X_u(t) > m(u) \right) = \sum_{t \in (G_{\delta/u} \setminus I'(t_u))} \Psi \left( \frac{m(u)}{\sigma X_u(t)} \right)
\]

\[
\leq 2\Psi(m(u)) \left( \sum_{t \in [\varepsilon, M]_{\delta/u}} e^{-m^2(u)/(\sigma X_u(t)^2)} + \sum_{t \in [M, \infty)_{\delta/u}} e^{-m^2(u)/(\sigma X_u(t)^2)} + \sum_{t \in [\varepsilon, M]_{\delta/u} \setminus I'(t_u)} e^{-m^2(u)/(\sigma X_u(t)^2)} \right)
\]
\[ = 2\Psi(m(u))(R_1(u) + R_2(u) + R_3(u)). \]

Thus, to establish (34) we need to prove that \( R_1(u) + R_2(u) + R_3(u) \to 0 \) as \( u \to \infty \). By Lemma 5.1, \( t_u \) is unique for large \( u \) and we have

\[ \sigma_{X_u}(t) = \frac{\sigma(ut)}{u(1+ct)} m(u) = \frac{\sigma(ut)}{\sigma(ut_u)} \frac{1+ct}{1+ct_u}. \]

*Estimation of \( R_1(u) \).* We have for all large \( u \) and \( t \in [0, \varepsilon]_{\delta/u} \)

\[ \sigma_{X_u}(t) \leq C \frac{\sigma(ut)}{\sigma(ut_u)}. \]

i) Assume that \( ut \geq \ln u \). Then with \( h \) being a slowly varying function at \( \infty \) and \( 0 < \varepsilon < \alpha \) by Potter’s theorem (Theorem 1.5.6 in [16]) we have

\[ \frac{\sigma(ut)}{\sigma(ut_u)} = \left( \frac{t}{t_u} \right)^{\alpha} \frac{h(ut)}{h(ut_u)} \leq C t^{\alpha} \frac{t_u}{t} \leq C t^{\alpha - \varepsilon}. \]

ii) Assume that \( ut < \ln u \). Since \( t \in [0, \varepsilon]_{\delta/u} \) we have \( ut \geq \delta \) for \( t \neq 0 \). Then for \( \varepsilon \in (0, \alpha) \) and large \( u \)

\[ \frac{\sigma(ut)}{\sigma(ut_u)} \leq u^{-(\alpha - \varepsilon)} \sup_{t \in [0, \ln u]} \sigma(t) \leq u^{-(\alpha - \varepsilon) \ln u}. \]

Combining the above inequalities we have that for sufficiently small \( \varepsilon \) and for all \( t \in [0, \varepsilon]_{\delta/u} \) uniformly for large \( u \) it holds that \( \frac{1}{\sigma_{X_u}(t)} - 1 \geq 2 \). Thus, for small enough \( \varepsilon > 0 \)

\[ (35) \quad R_1(u) \leq C e^{-m^2(u)} \to 0, \quad u \to \infty. \]

*Estimation of \( R_2(u) \).* By Potter’s theorem we have for \( M \) large enough and \( 0 < \varepsilon \) \( \in (1 - \alpha, 0) \)

\[ \frac{\sigma(ut)}{\sigma(ut_u)} \leq C \left( \frac{t}{t_u} \right)^{1+\varepsilon} \]

Since \( t_u \to t^* \) as \( u \to \infty \) we have for some small \( \varepsilon > 0 \)

\[ \sigma_{X_u}(t) \leq C \frac{t^{\alpha+\varepsilon}}{1+ct} \leq t^{-\varepsilon}, \]

hence for all \( t > M \) uniformly for \( u \) large it holds that \( \frac{1}{\sigma_{X_u}(t)} - 1 \geq 2t^{\varepsilon} \). Choosing \( M \) large enough and \( \varepsilon \) sufficiently small we have as \( u \to \infty \)

\[ (36) \quad R_2(u) \leq \sum_{t \in [M, \infty)_{\delta/u}} e^{-t^{\varepsilon} m^2(u)} = \sum_{t \in [M, \infty)_{\delta}} e^{-t^{\varepsilon} m^2(u)} = \leq C e^{-m^2(u)} \to 0. \]

*Estimation of \( R_3(u) \).* We have by Lemma 5.1 that with some \( C > 0 \)

\[ 1 - \sigma_{X_u}(t) \geq C(t-t_u)^2, \quad t \in [\varepsilon, M] \]

and hence by (16) for \( t \in [\varepsilon, M]_{\delta} \setminus I(t_u) \) it holds that

\[ m^2(u) \left( \frac{1}{\sigma_{X_u}(t)} - 1 \right) \geq m^2(u)(1 - \sigma_{X_u}(t)) \geq C(t-t_u)^2 m^2(u) \geq C \ln^2 u. \]

Thus, for \( t \in [\varepsilon, M]_{\delta} \setminus I(t_u) \) it holds that \( e^{-m^2(u) \left( \frac{1}{\sigma_{X_u}(t)} - 1 \right)} \leq u^{-C \ln u}, \) and we obtain

\[ R_3(u) \leq C_1 u^{-C_2 \ln u} \to 0, \quad u \to \infty. \]

Combining the estimate above with (35) and (36) obtain (34). It follows from the calculations in Theorem 3.1 that

\[ \mathbb{P} \left( \sup_{t \in [0, \frac{1}{\alpha_0^2} \delta]} \sup_{s \in I(t_u)} Z_u(t, s) > m(u) \right) \geq \Psi(m(u)), \quad u \to \infty \]
and the first claim follows by (33) and (34).

Next we show the second statement of the lemma. Again, by Bonferroni inequality we have

$$
\mathbb{P} \left( \inf_{t \in [0, \frac{1}{u})} \sup_{s \in I(t_u)} Z_u(t, s) > m(u) \right) \leq \nu_{(1, T)}^{\inf}(u)
$$

$$
\leq \mathbb{P} \left( \inf_{t \in [0, \frac{1}{u})} \sup_{s \in I(t_u)} Z_u(t, s) > m(u) \right) + \mathbb{P} \left( \inf_{t \in [0, \frac{1}{u})} \sup_{s \in (G_u \setminus I(t_u))} Z_u(t, s) > m(u) \right).
$$

It follows from the calculations in Theorem 3.1 that

$$
\mathbb{P} \left( \inf_{t \in [0, \frac{1}{u})} \sup_{s \in I(t_u)} Z_u(t, s) > m(u) \right) \geq \Psi(m(u)), \quad u \to \infty
$$

and by (34) we obtain that

$$
\mathbb{P} \left( \inf_{t \in [0, \frac{1}{u})} \sup_{s \in (G_u \setminus I(t_u))} Z_u(t, s) > m(u) \right) = o(\Psi(m(u))), \quad u \to \infty.
$$

Combining both statements above we obtain the second claim of the lemma.

**Proof of Lemma 5.4.** Fix some \( t \neq s \in I(t_u) \). Since \( \sigma_{X_u}(t), \sigma_{X_u}(s) \leq 1 \) we have by Lemma 2.3 in [17], with \( r_u(t, s) = \text{Corr}(X_u(t), X_u(s)) \),

$$
\mathbb{P}(X_u(t) > m(u), X_u(s) > m(u)) \leq \mathbb{P}(X_u(t) > m(u), X_u(s) > m(u)) \leq \Psi(m(u))\Psi(m(u)\sqrt{1 - r_u(t, s)})/2.
$$

If \( \alpha < 1/2 \), then by Lemma 5.1 as \( u \to \infty \) for some \( \epsilon > 0 \) it holds that \( m(u)\sqrt{1 - r_u(t, s)} / 2 \) \( \geq u^\epsilon \) hence uniformly for \( t \neq s \in I(t_u) \)

$$
(37) \quad \Psi(m(u)\sqrt{1 - r_u(t, s)})/2 \leq e^{-\frac{1}{4}u^2}, \quad u \to \infty.
$$

If \( \alpha = 1/2 \), then \( t_* = 1/c \) and by Lemma 5.1 and (13) we have that for some \( \epsilon' > 0 \) as \( u \to \infty \)

$$
\frac{m^2(u) - 1 - r_u(t, s)}{2} / 2 \sim \frac{u^2(1 + ct_u^2)\sigma^2(u|s - t|)}{8\sigma^4(u)} \sim \frac{u^2\sigma^2(u|s - t|)}{2\sigma^4(u)} \geq (1/2 + \epsilon')\ln u
$$

implying as \( u \to \infty \)

$$
\Psi(m(u)\sqrt{1 - r_u(t, s)})/2 \leq u^{-1/2 - \epsilon'}.
$$

Combining the line above with (37) we obtain the claim.

**Proof of Lemma 5.3.** Let \( a = K\delta \), where \( K \) is a large natural number that we shall choose later on. By the proofs of Theorem 15 and Lemma 16 in [18] we have with \( \sigma_{\eta}^2 \) being the variance of \( \eta \) that

$$
\lim_{S \to \infty} \mathcal{H}_{\eta}([0, S]_{[a])} \geq \frac{1}{a} \left( 1 - 2 \int_0^\infty e^{-\frac{\sigma_{\eta}^2(t)}{4} dt} \right).
$$
We have that for all \( u \) large enough \( \sigma^2(t) \geq \mathbb{C}t \) implying that \( \int_0^\infty e^{-\sigma^2(t)/4} dt < \infty \). Choosing sufficiently large \( K \) we have
\[
\liminf_{S \to \infty} \frac{\mathcal{H}_\eta([0,S]\delta)}{S} \geq \liminf_{S \to \infty} \frac{\mathcal{H}_\eta([0,S]\alpha)}{S} \geq \frac{1}{a} \cdot \frac{1}{2} > 0.
\]
Next we prove that for \( I(S) := \frac{\mathcal{H}_\eta([0,S]\delta)}{S} \) it holds that for large \( S \in G_\delta \)
\[
(38) \quad I(S) \geq I(S + \delta).
\]
We have
\[
(S + \delta)I(S + \delta) \leq \mathbb{E}\left( \sup_{t \in \{0,\delta,\ldots,S\}} e^{\sqrt{2}\eta(t)} - \sigma^2_\eta(t)\right) + \mathbb{E}\left( e^{\sqrt{2}\eta(S+\delta)} - \sigma^2_\eta(S+\delta)\right) F(S + \delta) = SI(S) + F(S + \delta),
\]
where \( F(M) = \mathbb{P}\left( \argmax_{t \in \{0,\delta,\ldots,M\}} (\sqrt{2}\eta(t) - \sigma^2_\eta(t)) = M \right) \) for \( M \in G_\delta \). Thus, to claim (38) we need to show that for large \( S \)
\[
(39) \quad \delta I(S) \geq F(S + \delta).
\]
Since \( \liminf_{S \to \infty} I(S) > 0 \), we have that \( \delta I(S) > \varepsilon \) for all \( S \) and some positive \( \varepsilon \), but on the other hand as \( S \to \infty \) it holds that
\[
F(S + \delta) \leq \mathbb{P}\left( \sqrt{2}\eta(S + \delta) - \sigma^2_\eta(S + \delta) \geq \sqrt{2}\eta(0) - \sigma^2_\eta(0) \right) = \mathbb{P}\left( \sqrt{2}\eta(S + \delta) - \sigma^2_\eta(S + \delta) \geq 0 \right) \to 0,
\]
consequently (39) holds and hence \( I(S) \) is non-increasing for large \( S \). Thus, \( \lim_{S \to \infty} I(S) \in (0, \infty) \) and the claim holds.

Acknowledgement: K. Dębicki was partially supported by NCN Grant No 2018/31/B/ST1/00370 (2019-2022). G. Jasnovidov was supported by the Ministry of Science and Higher Education of the Russian Federation, agreement 075-15-2019-1620 date 08/11/2019 and 075-15-2022-289 date 06/04/2022. The Authors would like to thank Professor Enkelejd Hashorva for fruitful discussions that significantly improved the paper.

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