How many roots of a system of random Laurent polynomials are real?

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Abstract. We say that a zero of a Laurent polynomial that lies on the unit circle with centre $0 \in \mathbb{C}$ is real. We also say that a Laurent polynomial that is real on this circle is real. In contrast with ordinary polynomials, it is known that for random real Laurent polynomials of increasing degree the average proportion of real roots tends to $1/\sqrt{3}$ rather than to 0. We show that this phenomenon of the asymptotically nonvanishing proportion of real roots also holds for systems of Laurent polynomials of several variables. The corresponding asymptotic formula is obtained in terms of the mixed volumes of certain convex compact sets determining the growth of the system of polynomials.

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§ 1. Introduction

1.1. It is known (see [1]) that the expected value of the proportion of real zeros of a real polynomial of increasing degree $m$ is asymptotically equal to $(2/\pi) \ln m / m \approx 0$. It is assumed in this evaluation that the polynomial has independent normally distributed coefficients with zero mean and unit variance. For more information about the distribution of the number of real solutions of systems of random polynomial equations, see [2] and the bibliography there.

The transition from ordinary polynomials to Laurent ones leads to an unexpected result. The restriction of a real Laurent polynomial of degree $m$

$$P(z) = a_0 + \sum_{1 \leq k \leq m} a_k z^k + \overline{a}_k z^{-k}$$

(see Definition 1 below) to the circle $z = e^{i\theta}$ is a trigonometric polynomial:

$$P(e^{i\theta}) = a_0 + \sum_{1 \leq k \leq m} \alpha_k \cos(k\theta) + \beta_k \sin(k\theta),$$

where $\alpha_k = (a_k + \overline{a}_k)/2$ and $\beta_k = (a_k - \overline{a}_k)/(2i)$. The restriction map is an isomorphism between the space of real Laurent polynomials of degree $m$ and the space of real trigonometric polynomials with spectrum $(-m, \ldots, -1, 0, 1, \ldots, m)$. Here ‘random’ means that the quantities $a_0/\sqrt{2\pi}, \alpha_1/\sqrt{\pi}, \beta_1/\sqrt{\pi}, \ldots, \alpha_m/\sqrt{\pi}, \beta_m/\sqrt{\pi}$ are

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independent and normally distributed, with zero mean and unit variance. It turns out that the average proportion of the zeros of a random real Laurent polynomial that lie on the unit circle tends to \(1/\sqrt{3}\) rather than to 0 with growth of the degree. This is a consequence of well-known results on the distribution of the zeros of random trigonometric polynomials of one variable; for instance, see [3] and the bibliography there. We present the proof of this result in §1.3 below (see Example 2). Its multivariate analogue is stated in §1.2.

1.2. Here we present a theorem that is a multidimensional analogue of the result on the average proportion of real zeros of a Laurent polynomial. To do this we need the following concepts.

For \(\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n\) set \(z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n}\). Let \(\Lambda\) be a finite subset of \(\mathbb{Z}^n\). Then a function \(\sum_{\lambda \in \Lambda} a_{\lambda} z^\lambda\) on the complex torus \((\mathbb{C} \setminus \{0\})^n\) is called a Laurent polynomial with support \(\Lambda\).

**Definition 1.** A Laurent polynomial that is real on the compact subtorus \(T^n = \{z \in (\mathbb{C} \setminus \{0\})^n : z = (e^{i\theta_1}, \ldots, e^{i\theta_n})\}\) of \((\mathbb{C} \setminus \{0\})^n\) is called a real Laurent polynomial. If a Laurent polynomial vanishes at a point \(z \in T^n\), then \(z\) is called a real zero of the polynomial.

The following are direct consequences of Definition 1.

(A) A Laurent polynomial \(P(z) = \sum_{\lambda \in \Lambda} a_{\lambda} z^\lambda\) is real if and only if 1) its support \(\Lambda\) has central symmetry and 2) \(a_{-\lambda} = \overline{a_\lambda}\) for each \(\lambda \in \Lambda\).

(B) The set of zeros of a real Laurent polynomial is invariant under the map \((z_1, \ldots, z_n) \mapsto (\overline{z_1}, \ldots, \overline{z_n})\).

In what follows we look at Laurent polynomials of \(n\) variables and use the terms system of random real Laurent polynomials with support \(\Lambda\) and average proportion real \(n(\Lambda)\) of real zeros of such a random system. For \(n = 1\) these notions are defined in §1.1, and for arbitrary \(n\) we give the corresponding definitions in §2. Theorem 1 is a multivariate version of the result in §1.1 on the average proportion of real zeros of a random real Laurent polynomial.

**Theorem 1.** Let \(B_m\) be a ball with centre \(O\) and radius \(m\) in the space \(\mathbb{R}^n\). Viewing \(\mathbb{Z}^n\) as the integer lattice in \(\mathbb{R}^n\), set \(\Lambda_m = B_m \cap \mathbb{Z}^n\). Then

\[
\lim_{m \to \infty} \text{real}_n(\Lambda_m) = \left(\frac{\sigma_{n-1}}{\sigma_n} \beta_n\right)^{n/2},
\]

where \(\beta_n = \int_{-1}^1 x^2(1-x^2)^{(n-1)/2} dx\), and \(\sigma_k\) is the volume of a unit \(k\)-dimensional ball.

In the table below we present the quantities \(\beta_n\) for \(1 \leq n \leq 20\). Note that \(\sqrt{(\sigma_0/\sigma_1)\beta_1} = 1/\sqrt{3}\); cf. §1.1 and Example 2.

**Remark 1.** The expression \(x^2(1-x^2)^{(n-1)/2} dx\) is a so-called Chebyshev differential binomial. Chebyshev [4] showed that the binomial \(x^m(a + bx^n)^p dx\) is not integrated by elementary functions apart from the three cases of integrability discovered by Euler. For odd \(n\) the above expression falls into the first case, and for even \(n\) it belongs to the third case.
1.3. Recall that the Newton polytope of a Laurent polynomial is the convex hull $\text{conv}(\Lambda)$ of the support $\Lambda$ of the polynomial. We also assign to $\Lambda$ an ellipsoid $\text{ell}(\Lambda) \subset \text{conv}(\Lambda)$, which is called its Newton ellipsoid (see Definition 3), and we show that

$$\text{real}_n(\Lambda) = \frac{\text{vol}(\text{ell}(\Lambda))}{\text{vol}(\text{conv}(\Lambda))}.$$ \hspace{1cm} (1.1)

It follows from Definition 3 that if $n = 1$, then the Newton ellipsoid is the interval with endpoints $\pm \sqrt{(1/N) \sum_{\lambda \in \Lambda} \lambda^2}$, where $N = \#\Lambda$.

**Example 1.** In the one-dimensional case the intervals $\text{ell}(\Lambda)$ and $\text{conv}(\Lambda)$ coincide only when the support has the form $\Lambda = \{\lambda, -\lambda\}$. Then it follows from (1.1) that $\text{real}_1(\Lambda) = 1$. This means that all zeros of a polynomial of the form $az^\lambda + \bar{a}z^{-\lambda}$ lie on the unit circle: this is indeed true because these zeros are the $2\lambda$th roots of $-\bar{a}/a$.

**Example 2.** Let $\Lambda_m = \{-m, \ldots, -1, 0, 1, \ldots, m\}$. Then

$$\sqrt{\frac{1}{\#\Lambda_m} \sum_{k \in \Lambda_m} k^2} = \sqrt{\frac{2(1^2 + \cdots + m^2)}{2m + 1}} = \sqrt{\frac{m(m + 1)}{3}}.$$ 

Hence $\text{real}_1(\Lambda_m) = \sqrt{(m + 1)/(3m)}$ by (1.1), so that $\lim_{m \to \infty} \text{real}_1(\Lambda_m) = 1/\sqrt{3}$.

We also find the proportion of real zeros $\text{real}_n(\Lambda_1, \ldots, \Lambda_n)$ of a system of Laurent polynomials with supports $\Lambda_1, \ldots, \Lambda_n$. Then the volumes in the numerator and denominator of (1.1) must be replaced by the appropriate mixed volumes of suitable ellipsoids and Newton polytopes; see Theorem 3. Using the geometry underlying (1.1) we find the asymptotic behaviour of $\text{real}_n(\Lambda_1, \ldots, \Lambda_n)$ as the supports $\Lambda_1, \ldots, \Lambda_n$ increase (Theorem 4), and we apply these calculations to the proof of Theorem 1.

In our proofs we use two results on the number of roots of a system of equations: a theorem on the average number of roots of a system of smooth equations on a differentiable manifold (see [5] and also [6] and [7]) and the Bernstein-Kushnirenko theorem [8], also called the BKK formula (after Bernstein, Kushnirenko and Khovanskii).
§ 2. Average number of roots, ellipsoids and Newton polytopes

2.1. The average number of zeros of a system of trigonometric polynomials. Let \( \Lambda \) be a finite centrally symmetric subset of \( \mathbb{Z}^n \) and let

\[
T^n = \{ z \in (\mathbb{C} \setminus 0)^n : z = (e^{i\theta_1}, \ldots, e^{i\theta_n}) \}.
\]

Then we call functions on \( T^n \) of the form

\[
a_0 + \sum_{0 \neq \lambda \in \Lambda, \alpha_\lambda \in \mathbb{R}, \beta_\lambda \in \mathbb{R}} \alpha_\lambda \cos \langle \theta, \lambda \rangle + \beta_\lambda \sin \langle \theta, \lambda \rangle,
\]

where \( \theta = (\theta_1, \ldots, \theta_n) \), trigonometric polynomials with support \( \Lambda \). For \( 0 \notin \Lambda \) we assume that \( a_0 = 0 \). We denote the set of trigonometric polynomials with support \( \Lambda \) by \( \text{Trig}(\Lambda) \). In what follows we always regard \( \text{Trig}(\Lambda) \) as a subspace of the Hilbert space \( L^2(T^n, d\chi) \), where \( d\chi \) is the associated invariant Haar measure on \( T^n \). We can construct orthogonal bases in \( \text{Trig}(\Lambda) \) as follows. We treat \( \mathbb{Z}^n \) as the integer lattice in \( \mathbb{R}^n \). Let \( l \) be a linear functional in \( \mathbb{R}^n \) that is distinct from zero at the nonzero points of \( \Lambda \). Let \( \Lambda_+ \) denote the intersection of \( \Lambda \) with the half-space \( l \geq 0 \). For \( 0 \neq \lambda \in \Lambda_+ \) set

\[
\tau_\lambda(\theta) = \sqrt{2} \cos \langle \theta, \lambda \rangle \quad \text{and} \quad \tau_{-\lambda}(\theta) = \sqrt{2} \sin \langle \theta, \lambda \rangle.
\]

(2.1)

If \( 0 \in \Lambda \), then we set \( \tau_0(\theta) = 1 \). The functions \( \{ \tau_\lambda : \lambda \in \Lambda \} \) form an orthonormal basis of \( \text{Trig}(\Lambda) \).

Let \( \Lambda_1, \ldots, \Lambda_n \) be finite centrally symmetric subsets of \( \mathbb{Z}^n \). Regarding the \( f_i \in \text{Trig}(\Lambda_i) \) as functions on \( T^n \), we let \( N(f_1, \ldots, f_n) \) denote the set of isolated points of the set of their common zeros.

Definition 2. By the average number of zeros of a system of trigonometric polynomials with supports \( \Lambda_1, \ldots, \Lambda_n \) we mean

\[
\mathcal{M}(\Lambda_1, \ldots, \Lambda_n) = \int_{S_1 \times \cdots \times S_n} N(s_1, \ldots, s_n) \, ds_1 \cdots ds_n,
\]

where \( S_i \) is the unit sphere in \( \text{Trig}(\Lambda_i) \) and \( ds_i \) is the normalized orthogonally-invariant measure on \( S_i \).

2.2. Newton ellipsoids. Using the scalar product \( \langle *, * \rangle \) in \( \text{Trig}(\Lambda) \) we consider the map \( \Theta : T^n \to \text{Trig}(\Lambda) \) such that

\[
\forall f \in \text{Trig}(\Lambda) : \quad \langle \Theta(\theta), f \rangle = \frac{1}{\sqrt{N}} f(\theta),
\]

(2.2)

where \( N = \dim \text{Trig}(\Lambda) = \#\Lambda \).

Lemma 1. The embedding \( \Theta(T^n) \subset S \) holds, where \( S \) is the unit sphere in \( \text{Trig}(\Lambda) \).

Proof. Using the orthonormal basis \( \{ \tau_\lambda : \lambda \in \Lambda \} \) of \( \text{Trig}(\Lambda) \) defined in (2.1) we can express \( \Theta \) as

\[
\Theta(\theta) = \frac{1}{\sqrt{N}} \sum_{\lambda \in \Lambda} \tau_\lambda(\theta) \tau_\lambda.
\]

(2.3)

Now the required result follows from the identity \( \cos^2 + \sin^2 = 1 \). The proof is complete.
Let $t$ and $t^*$ be the tangent and cotangent spaces at the point 0 in $T^n$. Let $\mathbb{Z}^n$ denote the character lattice of $T^n$ in $t^*$, and let $\mathbb{Z}^n$ be the dual lattice in $t$. In what follows we assume that $\Lambda \subset \mathbb{Z}^n$. Let $F_\Lambda$ denote the quadratic form in $t$ that is the pullback by $\Theta$ of the quadratic form of the Riemannian metric on $S$ at $\Theta(0)$.

**Lemma 2.** The equality $F_\Lambda(\xi) = (1/N) \sum_{\lambda \in \Lambda} \lambda^2(\xi)$ holds.

The proof follows from (2.3).

Recall that the support function of a compact convex set $A \subset \mathbb{R}^n$ is a function $h(x): \mathbb{R}^n \to \mathbb{R}$ such that $h(x) = \max_{a \in A} a(x)$.

**Lemma 3.** The function $h_\Lambda(\xi) = \sqrt{F_\Lambda(\xi)} = \sqrt{1/N} \sum_{\lambda \in \Lambda} \lambda^2(\xi)$ is the support function of a centrally symmetric ellipsoid $\text{ell}(\Lambda)$ in $t^*$. If $\Lambda$ does not lie in a proper subspace of $t^*$, then $\dim \text{ell}(\Lambda) = n$.

**Proof.** For any nonnegative square form $g$ in $\mathbb{R}^n$, $\sqrt{g}$ is the support function of a centrally symmetric ellipsoid in the dual space $\mathbb{R}^n$. If $g$ is nondegenerate, then this ellipsoid has full dimension, which completes the proof.

**Definition 3.** The ellipsoid $\text{ell}(\Lambda)$ with support function $h_\Lambda$ is called the Newton ellipsoid of $\Lambda$.

**Theorem 2.** The equality $M(\Lambda_1, \ldots, \Lambda_n) = n! \text{vol}(\text{ell}(\Lambda_1), \ldots, \text{ell}(\Lambda_n))$ holds, where in the measurements of the mixed volume $\text{vol}(\text{ell}(\Lambda_1), \ldots, \text{ell}(\Lambda_n))$ of the Newton ellipsoids $\text{ell}(\Lambda_i)$ the volume of the fundamental cube of the character lattice $\mathbb{Z}^n$ of $T^n$ is set to be equal to 1.

**Proof.** Recall that a Banach body in a smooth manifold $X$ is a family of centrally symmetric compact convex sets $\mathcal{B} = \{\mathcal{B}(x)\}$ in fibres $T_x^*X$ of the cotangent bundle $T^*X$ of $X$; see [5] and [7]. By definition, the volume $V(\mathcal{B})$ of a Banach body $\mathcal{B}$ is the volume of the domain $\bigcup_{x \in X} \mathcal{B}(x) \subset T^*X$ which is measured with respect to the standard symplectic structure on the total space of the cotangent bundle. More precisely, if $\omega$ is the standard symplectic form on $T^*X$ (see [9]), then we use the form $\omega^n/n!$, where $n = \dim X$, to measure the volume.

Using Minkowski addition we can consider linear combinations of convex bodies with nonnegative coefficients. A linear combination of Banach bodies is defined by $(\sum_{i} \lambda_i \mathcal{B}_i)(x) = \sum_{i} \lambda_i \mathcal{B}_i(x)$. The volume of $\lambda_1 \mathcal{B}_1 + \cdots + \lambda_k \mathcal{B}_k$ is a homogeneous polynomial of degree $n$ in the variables $\lambda_1, \ldots, \lambda_k$. If $k = \dim X = n$, then the coefficient of $\lambda_1 \cdots \lambda_n$ in this polynomial, divided by $n!$, is called the symplectic mixed volume $V(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ of the Banach bodies.
Let $V \subset C^\infty(X)$ be a finite-dimensional space. If

$$\forall x \in X \ \exists f \in V: \ f(x) \neq 0,$$

then to any scalar product in $V$, [5] assigns a Banach ellipsoid $B_V$ in $X$. For $\dim X = n$, given any finite-dimensional Euclidean spaces $V_1, \ldots, V_n \subset C^\infty(X)$, the average number of common zeros $M(V_1, \ldots, V_n)$ of a system of random functions $f_1 \in V_1, \ldots, f_n \in V_n$ was defined in [5], where it was also proved ([5], Theorem 1) that

$$M(V_1, \ldots, V_n) = \frac{n!}{(2\pi)^n} \mathcal{V}(B_{V_1}, \ldots, B_{V_n}). \quad (2.4)$$

In the case when $X = T^n$, $V_1 = \text{Trig}(\Lambda_1)$, $\ldots$, $V_n = \text{Trig}(\Lambda_n)$, it is easy to verify that 1) the Banach ellipsoids $B_{V_i}$ are invariant under the action of $T^n$, and 2) if $e$ is the identity element of the group $T^n$, then $B_{V_i}(e) = \text{ell}(\Lambda_i)$. Hence, using (2.4) we obtain

$$M(\Lambda_1, \ldots, \Lambda_n) = \frac{n!}{(2\pi)^n} \mathcal{V}(B_{\Lambda_1}, \ldots, B_{\Lambda_n}) = n! \, \text{vol}(\text{ell}(\Lambda_1), \ldots, \text{ell}(\Lambda_n)),$$

where we measure the mixed volume $\text{vol}(\text{ell}(\Lambda_1), \ldots, \text{ell}(\Lambda_n))$ under the assumption that the fundamental cube of the character lattice of $T^n$ has measure 1. The proof is complete.

Note that Theorem 2 yields the following inequalities for the average numbers of zeros.

**Corollary 1.** For any supports $\Lambda_1, \ldots, \Lambda_n$

$$M^2(\Lambda_1, \ldots, \Lambda_n) \geq M(\Lambda_1, \ldots, \Lambda_{n-1}, \Lambda_{n-1})M(\Lambda_1, \ldots, \Lambda_n),$$

$$M^n(\Lambda_1, \ldots, \Lambda_n) \geq M(\Lambda_1) \cdots M(\Lambda_n),$$

where $M(\Lambda) = M(\Lambda, \ldots, \Lambda)$ by definition.

This follows from the Aleksandrov-Fenchel inequalities (see [10]) for the mixed volumes of the ellipsoids $\text{ell}(\Lambda_i)$.

**Remark 2.** The inequalities for the numbers of zeros in Corollary 1 are analogues of Hodge’s inequalities for the intersection indices of hypersurfaces in projective algebraic varieties. This connection between the Hodge and Aleksandrov-Fenchel inequalities has independently been discovered by Khovanskii and Teissier; for instance, see [11].

**2.3. Newton polytopes.** The restriction of Laurent polynomials to $T^n \subset (C \setminus 0)^n$ establishes an isomorphism between the real vector space of real Laurent polynomials with support $\Lambda$ and the space $\text{Trig}(\Lambda)$. Hence, by Theorem 2 the average number of zeros of a system of real Laurent polynomials with supports $\Lambda_i$ is $n! \, \text{vol}(\text{ell}(\Lambda_1), \ldots, \text{ell}(\Lambda_n))$. To calculate the number of complex zeros of a system of polynomials we use the Kushnirenko-Bernstein theorem (see [8]).
Recall that we assign two compact convex sets to any centrally symmetric set \( \Lambda \subset \mathbb{Z}^n \subset \mathfrak{t}^* \), namely, the Newton polytope \( \text{conv}(\Lambda) \) of a polynomial with support \( \Lambda \) and the ellipsoid \( \text{ell}(\Lambda) \) with support function

\[
h_\Lambda(\xi) = \sqrt{\frac{1}{N} \sum_{\lambda \in \Lambda} \lambda^2(\xi)},
\]

where \( N \) is the cardinality of \( \Lambda \).

**Proposition 1.** For almost all tuples \((P_1, \ldots, P_n)\) of real Laurent polynomials with supports \( \Lambda_1, \ldots, \Lambda_n \) the system \( P_1 = \ldots = P_n = 0 \) has \( n! \text{vol}(\text{conv}(\Lambda_1), \ldots, \text{conv}(\Lambda_n)) \) solutions in the complex torus \((\mathbb{C} \setminus 0)^n\).

**Proof.** Let \( \mathfrak{A}_i \) be the space of Laurent polynomials with support in \( \Lambda_i \). Let \( \mathfrak{Q} \subset (\mathfrak{A}_1 \setminus 0) \times \cdots \times (\mathfrak{A}_n \setminus 0) \) denote the set of tuples of nontrivial Laurent polynomials such that the number of their common zeros is distinct from \( n! \text{vol}(\text{conv}(\Lambda_1), \ldots, \text{conv}(\Lambda_n)) \). Let \( \mathbb{P}_i \) be the projectivization of \( \mathfrak{A}_i \). Then the Kushnirenko-Bernstein theorem states that \( \mathfrak{Q} \) lies in the preimage of a certain complex algebraic hypersurface \( \mathfrak{P} \) in the product \( \mathbb{P}_1 \times \cdots \times \mathbb{P}_n \) under the projection \( (\mathfrak{A}_1 \setminus 0) \times \cdots \times (\mathfrak{A}_n \setminus 0) \to \mathbb{P}_1 \times \cdots \times \mathbb{P}_n \).

The real Laurent polynomials in \( \mathfrak{A}_i \) form a real vector subspace \( \text{Trig}(\Lambda_i) \), and we have \( \dim_{\mathbb{R}} \text{Trig}(\Lambda_i) = \dim_{\mathbb{C}} \mathfrak{A}_i \). Consider the projection \( \Pi: S_1 \times \cdots \times S_n \to \mathbb{P}_1 \times \cdots \times \mathbb{P}_n \) (see Definition 2). Then \( \Pi(S_1 \times \cdots \times S_n) \) is a dense subset of the algebraic variety \( \mathbb{P}_1 \times \cdots \times \mathbb{P}_n \) in the Zariski topology. Therefore, \( \Pi^{-1}\mathfrak{P} \) lies in a closed real hypersurface in \( S_1 \times \cdots \times S_n \). This yields the required result and completes the proof.

**Corollary 2.** The inequality \( \text{vol}(\text{ell}(\Lambda)) \leq \text{vol}(\text{conv}(\Lambda)) \) holds.

**Proof.** By Proposition 1 the right-hand side of this inequality is equal to the average number of real roots of some system of equations, while by Theorem 2 the left-hand side is the average number of complex (that is, all) roots of the same system.

The next result refines Corollary 2.

**Proposition 2.** For any centrally-symmetric set \( \Lambda \), \( \text{ell}(\Lambda) \subset \text{conv}(\Lambda) \).

**Proof.** If \( h_A \) and \( h_B \) are the support functions of compact convex sets \( A \) and \( B \), respectively, then the inclusion \( A \subset B \) is equivalent to \( h_A \leq h_B \). The support function \( h_{\text{conv}(\Lambda)} \) of \( \text{conv}(\Lambda) \) at \( \xi \) is \( \max_{\lambda \in \Lambda} \lambda(\xi) \). If \( \Lambda \) is centrally symmetric, then this function is also equal to \( \max_{\lambda \in \Lambda} |\lambda(\xi)| \). The support function \( h_{\Lambda} \) of the ellipsoid \( \text{ell}(\Lambda) \) is equal to the root mean square of the quantities \( \{|\lambda(\xi)|: \lambda \in \Lambda\} \) by Definition (2.5). The root mean square of a finite system of nonnegative numbers does not exceed the maximum of these numbers. The proof is complete.

We call the ratio of the number of real zeros of a system of Laurent polynomials and the total number of its zeros the proportion of real zeros of this system.

**Theorem 3.** Let \( \text{real}(\Lambda_1, \ldots, \Lambda_n) \) be the average proportion of real zeros of systems of real Laurent polynomials with supports \( \Lambda_i \). Then

\[
\text{real}(\Lambda_1, \ldots, \Lambda_n) = \frac{\text{vol}(\text{ell}(\Lambda_1), \ldots, \text{ell}(\Lambda_n))}{\text{vol}(\text{conv}(\Lambda_1), \ldots, \text{conv}(\Lambda_n))}.
\]

This follows from Theorem 2 and Proposition 1.
§ 3. Asymptotic behaviour of the average proportion of real roots

Let $\Delta \subset t^*$ be a centrally symmetric convex compact set and let $\Lambda = \Delta \cap \mathbb{Z}^n$. In what follows we assume that

(*) if $\dim \Delta = k < n$, then $\Delta$ lies in a $k$-dimensional subspace $V_\Delta \subset t^*$ spanned by vectors in $\mathbb{Z}^n$.

For an integer $m > 0$ set $\Delta_m = m\Delta$, $\Lambda_m = \Delta_m \cap \mathbb{Z}^n$ and $N_{\Lambda,m} = \# \Lambda_m$. Recall that to $\Lambda$ we have assigned the ellipsoid $\text{ell}(\Lambda) \subset t^*$ with support function $h_\Lambda = \sqrt{F_\Lambda}$, where

$$F_\Lambda(\xi) = \frac{1}{\#\Lambda} \sum_{\lambda \in \Lambda} \lambda^2(\xi).$$

Set

$$F_{\Lambda,m}(\xi) = \frac{1}{N_{\Lambda,m}} \sum_{\lambda \in \Lambda_m} \lambda^2(\xi).$$

Then $h_{\Lambda,m} = \sqrt{F_{\Lambda,m}}$ is the support function of an ellipsoid $\text{ell}(\Lambda_m) \subset t^*$.

It follows from (*) that $\mathbb{Z}^n \cap V_\Delta$ is a full-rank lattice in the subspace $V_\Delta$. Consider the volume form in $V_\Delta$ such that the unit cube of the lattice $\mathbb{Z}^n \cap V_\Delta$ has volume 1. For $\xi \in t$ set

$$F_\Delta(\xi) = \frac{1}{\text{vol}_k(\Delta)} \int_\Delta \langle x, \xi \rangle^2 dx.$$ 

The function $F_\Delta : t \to \mathbb{R}$ is a nonnegative quadratic form. We set $h_\Delta = \sqrt{F_\Delta}$ and let $\text{ell}(\Delta)$ denote the ellipsoid with support function $h_\Delta$ in $t^*$.

**Lemma 4.** As $m \to \infty$, the sequence of functions $(1/m^2)F_{\Lambda,m}$ converges to $F_\Delta$ locally uniformly.

**Proof.** Let $\dim \Delta = k$. Then

$$1/m^2 F_{\Lambda,m}(\xi) = \frac{1}{m^2 N_{\Lambda,m}} \sum_{\lambda \in \Lambda_m} \langle \xi, \lambda \rangle^2 = \frac{m^k}{N_{\Lambda,m}} \sum_{\alpha \in \Lambda_m/m} \langle \xi, \alpha \rangle^2 \frac{1}{m^k}.$$ 

Now note that $N_{\Lambda,m} \approx m^k \text{vol}_k(\Delta)$, and $\sum_{\alpha \in \Lambda_m/m} \langle \xi, \alpha \rangle^2 1/m^k$ is an integral sum for the integral $\int_\Delta \langle x, \xi \rangle^2 dx$ which corresponds to the partition of $\Delta$ with nodes $\Lambda_m/m$. Hence $(1/m^2)F_{\Lambda,m}(\xi) \to F_\Delta(\xi)$. The proof is complete.

**Corollary 3.** As $m \to \infty$,

(1) the sequence of functions $(1/m)h_{\Lambda,m}$ converges locally uniformly to the support function $h_\Delta$ of $\text{ell}(\Delta)$;

(2) the sequence of ellipsoids $(1/m)\text{ell}(\Lambda_m)$ converges to $\text{ell}(\Delta)$ in the Hausdorff topology.

By definition, $h_\Delta = \sqrt{F_\Delta}$, so that both assertions are direct consequences of Lemma 4.

**Lemma 5.** If condition (*) holds for the convex body $\Delta$, then as $m \to \infty$, the sequence of convex polytopes $(1/m)\text{conv}(\Lambda_m)$ converges to $\Delta$ in the Hausdorff topology.

This is a consequence of the definition of the set $\Lambda_m$. 
Theorem 4. Let $\Delta_1, \ldots, \Delta_n$ be convex bodies in $t^*$ which satisfy $(\ast)$ and let $\Lambda_i = \Delta_i \cap \mathbb{Z}^n$. Then
\[
\lim_{\inf(m_1, \ldots, m_n) \to \infty} \text{real}_n((\Lambda_1)_{m_1}, \ldots, (\Lambda_n)_{m_n}) = \frac{\text{vol}(\ell(\Delta_1), \ldots, \ell(\Delta_n))}{\text{vol}(\Delta_1, \ldots, \Delta_n)}.
\] (3.1)

Proof. Let $m_k = ((m_k)_1, \ldots, (m_k)_i, \ldots)$, where $1 \leq k \leq n$, be an $n$-tuple of increasing sequences of positive numbers. Then applying Theorem 3 to the sequence of tuples of supports $(\Lambda_1)_{m_1}, \ldots, (\Lambda_n)_{m_n}$ and making the limiting procedure on the basis of Corollary 3, (2) and Lemma 5, we arrive at the required result and complete the proof.

Corollary 4. Let $\alpha_1, \ldots, \alpha_n$ be positive numbers. Then the asymptotic behaviour of the quantities $\text{real}_n(\Lambda_{m_1}, \ldots, \Lambda_{m_n})$ is preserved after the change of variables $\Delta_i \to \alpha_i \Delta_i$ (this change preserves condition $(\ast)$).

It fact, it follows from Lemma 4 that $h_{\alpha \Delta_i} = \alpha h_{\Delta_i}$. Hence both the numerator and denominator on the right-hand side of (3.1) are multiplied by $\alpha_1 \cdots \alpha_n$.

§ 4. The proof of Theorem 1

In the proof of Theorem 1 we use the notation and results from § 3. We identify $t^*$ and $\mathbb{R}^n$ so that the character lattice of $T^n$ coincides with the standard integer lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$. Let $B_m \subset \mathbb{R}^n$ be the ball of radius $m$ with centre 0. Then by Theorem 4
\[
\lim_{m \to \infty} \text{real}_n(B_m \cap \mathbb{Z}^n) = \frac{\text{vol}(\ell(B_1))}{\sigma_n}.
\]
(Recall that $\sigma_k$ is the volume of the $k$-dimensional ball of radius 1.) Hence Theorem 1 reduces to the following.

Proposition 3. The equality
\[
\ell(B_1) = \sqrt{\frac{\sigma_{n-1}}{\sigma_n}} \beta_n B_1
\]
holds.

Proof. By definition, the support function $h_{B_1}$ of $\ell(B_1)$ is equal to $\sqrt{F_{B_1}}$, where $F_{B_1}(\xi) = \frac{1}{\sigma_n} \int_{B_1} (x_1 \xi_1 + \cdots + x_n \xi_n)^2 \, dx_1 \cdots dx_n$. Since the quantity $F_{B_1}(\xi)$ only depends on $|\xi|$, $\ell(B_1)$ is in fact a ball of radius $\sqrt{F_{B_1}(\xi_0)}$, where $\xi_0 = (1, 0, \ldots, 0)$. Going over to repeated integration in the definition of $F_{B_1}$ we obtain
\[
F_{B_1}(\xi_0) = \frac{\sigma_{n-1}}{\sigma_n} \int_{-1}^{1} x_1^2 (1 - x_1^2)^{(n-1)/2} \, dx_1 = \frac{\sigma_{n-1}}{\sigma_n} \beta.
\]
Hence the radius of $\ell(B_1)$ is $\sqrt{\sigma_{n-1}/\sigma_n} \beta_n$.

Corollary 5. The limit relation
\[
\lim_{\inf(m_1, \ldots, m_n) \to \infty} \text{real}_n(B_{m_1} \cap \mathbb{Z}^n, \ldots, B_{m_n} \cap \mathbb{Z}^n) = ((\sigma_{n-1}/\sigma_n) \beta_n)^{n/2}
\]
holds.
Proof. The mixed volume of \( n \) balls of radii \( r_1, \ldots, r_n \) is \( r_1 \cdots r_n \cdot \sigma_n \). Hence the required result follows from Theorem 4 and Proposition 3.

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