Abstract. We consider minimizing harmonic maps from $\Omega \subset \mathbb{R}^3$ into the unit sphere $S^2$ and extend Almgren and Lieb’s linear law on the bound of the singular set as well as Hardt and Lin’s stability theorem for singularities. Both results are shown to hold with weaker hypotheses, i.e., only assuming that the trace of our map lies in the fractional space $W^{s,p}$ with $s \in (\frac{1}{2}, 1]$ and $p \in [2, \infty)$ satisfying $sp \geq 2$.

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1. Introduction

A minimizing harmonic map from an $n$-dimensional domain $\Omega \subseteq \mathbb{R}^n$ into $\mathcal{N}$ is a map $u \in W^{1,2}(\Omega, \mathcal{N})$ that minimizes the Dirichlet energy

$$E(u) := \int_{\Omega} |\nabla u|^2 \, dx$$

among all maps in $W^{1,2}(\Omega, \mathcal{N})$ with the same boundary data $\varphi : \partial \Omega \to \mathcal{N}$. Here, the target manifold $\mathcal{N}$ is a smooth, closed (i.e., compact and without boundary) Riemannian manifold. For convenience, assume it is isometrically embedded in $\mathbb{R}^N$. The Sobolev space $W^{1,2}(\Omega, \mathcal{N})$ is defined as a restriction space, namely

$$W^{1,2}(\Omega, \mathcal{N}) := \{ u \in W^{1,2}(\Omega, \mathbb{R}^N) : u(x) \in \mathcal{N} \text{ almost everywhere} \}.$$

In such a geometrical setup, one might suspect that these minimizers are smooth maps. However, this holds only in the case of geodesics, i.e., for $n = 1$, and in the conformal case $n = 2$ (where the energy is conformally invariant), see Morrey’s classic [18].

However, in dimensions $n \geq 3$ even continuity cannot be guaranteed anymore. Minimizers (and more generally critical points) of the Dirichlet energy satisfy the Euler–Lagrange system of equations

$$-\Delta u = A(u)(\nabla u, \nabla u) \quad \text{in } \Omega,$$

where $A$ is the second fundamental form of the isometric embedding $\mathcal{N} \subset \mathbb{R}^N$. In the special case when $\mathcal{N} = S^{n-1}$, this system takes the form

$$-\Delta u = |\nabla u|^2 u \quad \text{in } \Omega.$$

For $n \geq 3$ solutions to these equations might be everywhere discontinuous, see Rivière’s seminal [21].

Minimizers enjoy better regularity on a large set, but there might be discontinuities. The simplest example are topological obstructions: the “hedgehog” map $u(x) := \frac{x}{|x|}$ has finite energy for $n \geq 3$ and is a minimizing harmonic map from $B^n$ into $\mathcal{N} := S^{n-1}$ — and certainly $\frac{x}{|x|}$ is not smooth (the minimality of this map was proved by Brezis–Coron–Lieb [4, Theorem 7.1] in dimension $n = 3$ and for $n \geq 3$ by Lin [14]). Actually, the map $u : B^3 \to S^2$ is the unique minimizer for identity boundary data $id : \partial B^3 \to S^2$. By Hopf–Brouwer theorem we know that since the topological degree of identity map $id : \partial B^3 \to S^2$ is not zero, there is no continuous extension to $B^3$.

But even the continuity of minimizers may also fail without such a topological obstruction. Hardt and Lin in [9] constructed a boundary data $\varphi \in C^\infty(\partial B^3, S^2)$ with $\deg \varphi = 0$ for which all minimizers must have singularities.

Consequently, in dimensions $n \geq 3$, the analysis of singularities of minimizing harmonic maps is an intriguing theory. Here, the singular set $\text{sing } u$ of a mapping $u \in W^{1,2}(\Omega, \mathcal{N})$ is
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defined as the complement of its regular set

\[ \text{sing} \, u = \{ x \in \Omega : u \text{ is smooth on some neighborhood of } x \}^c. \]

For harmonic maps this analysis started with the work of Schoen and Uhlenbeck [22]. They showed that one can estimate the Hausdorff dimension of the singular set. Namely,

\[ \dim_H(\text{sing} \, u) \leq n - 3 \]

for any minimizing harmonic map from an \( n \)-dimensional domain \( \Omega \subseteq \mathbb{R}^n \) into an arbitrary closed Riemannian manifold \( \mathcal{N} \).

For certain target manifolds \( \mathcal{N} \) this estimate is optimal, take for example the “hedgehog” example for \( \mathcal{N} = S^2 \) in dimension \( n = 3 \). Even in this dimension, there is still more that can be said about possible singularities — e.g., in the case \( \mathcal{N} = S^2 \) the singularities are classified and the behavior of the map around those points is well understood, see [4].

Some more quantitative results were also obtained in the golden age of minimizing harmonic maps (late 80’s) — for \( n = 3 \) and \( \mathcal{N} = S^2 \), Almgren and Lieb [1] showed that one can estimate the numbers of singularities in terms of their trace maps, which became to be known as Almgren and Lieb’s linear law:

\[ \#\{ \text{singularities of } u \} \leq C(\Omega) \int_{\partial \Omega} |\nabla_T u|^2 \, d\mathcal{H}^2. \]

Moreover, in [11] Hardt and Lin showed that for a unique harmonic minimizer \( v \in W^{1,2}(\Omega, S^2) \) the number of singularities remains the same for all minimizers whose trace is close in the Lipschitz-norm to the trace of \( v \). This is known as Hardt and Lin’s stability theorem.

We extend these two theorems to the case when the trace map is controlled in a weaker norm, the fractional Sobolev spaces \( W^{s,p} \). For \( s \in (0,1) \) the fractional Sobolev space on the boundary of a smooth enough set \( \Omega \) is defined as

\[ W^{s,p}(\partial \Omega) := \left\{ f \in L^p(\partial \Omega) : [f]_{W^{s,p}(\partial \Omega)} < \infty \right\}, \]

where the Gagliardo seminorm \([f]_{W^{s,p}(\partial \Omega)}\) is given by

\[ [f]_{W^{s,p}(\partial \Omega)} := \left( \int_{\partial \Omega} \int_{\partial \Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}}. \]

In this work we focus on the case \( n = 3 \) and \( \mathcal{N} = S^2 \) and we extend Almgren and Lieb’s linear law (see Theorem 8.1) and Hardt and Lin’s stability theorem (see Theorem 7.1) to trace spaces \( W^{s,p} \) with \( s \in (\frac{1}{2}, 1] \) and \( p \in [2, \infty) \), where \( sp = 2 \). These trace space estimates are sharp in the sense of scaling: For \( sp < 2 \) the results fail, see Lemma 8.6, Example 8.8. For Hardt and Lin’s stability theorem this follows already from an argument by Strzelecki and the first-named author, [17].
Using the techniques recently developed by Naber and Valtorta [19] we are actually able to generalize both stability theorem and the linear law to higher dimensions \( n \geq 4 \). This will be the subject of our future work [16].

For an excellent discussion on the topic concerning singularities of harmonic maps we refer the reader to the survey [12]. Streamlined proofs for the regularity theory for minimizing harmonic maps into spheres were obtained in [7].

During the lengthy preparation of this manuscript and our future work [16] we learned that Hardt and Lin’s stability theorem was extended by Li [13] from Lipschitz trace assumptions to \( W^{1,2} \)-trace assumptions. This interesting paper is concerned not only with stability of singularities, but also with stability of maps via diffeomorphisms of the domain. However, it is worth noting that our methods are different — we apply the same techniques to both problems (linear bound and stability) — and enable us to work with even weaker trace assumptions.

**Notation.** We denote by \( B_r(x) \) balls centered in \( x \) with radius \( r \). For \( x = 0 \), we simply write \( B_r = B_r(0) \) and for \( r = 1 \), \( B = B_1 \). By \( \mathbb{R}^3_+ = \mathbb{R}^3 \cap \{ x \in \mathbb{R}^3 : x_3 > 0 \} \) we denote the upper half-space. \( B^+_r \) is given by \( B^+_r = B_r \cap \mathbb{R}^3_+ \). For any \( \rho > 0 \) we write \( T_{\rho} = B_{\rho} \cap \{ x \in \mathbb{R}^3 : x_3 = 0 \} \) for the flat part and \( S^+_{\rho} = \partial B_{\rho} \cap \mathbb{R}^3_+ \) for the curved part of the boundary of the half ball \( B^+_\rho \).

For simplicity we use Greek letters \( \psi, \varphi, \) etc. for boundary maps and \( u, v, \) etc. for interior maps. The letters \( r, R, \rho \) will be usually reserved for the radii. We use \( \nabla_T u \) for the tangential gradient of \( u \), i.e., the gradient of its restriction \( u|_{\partial \Omega} \). We also use \( (u)_{\Omega} \) to denote the mean value of \( u \) on \( \Omega \), i.e., \( (u)_{\Omega} := \frac{1}{|\Omega|} \int_{\Omega} u \). As usual, the constant \( C \) will denote a generic constant that may vary from line to line.

Throughout the paper the term **minimizer** or **energy minimizer** will refer to an \( S^2 \)-valued map minimizing the Dirichlet energy among \( W^{1,2}(\Omega, S^2) \) maps with same boundary data, unless otherwise stated.

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2. General facts about harmonic maps

The regularity theory of harmonic maps is based on the $\varepsilon$-regularity theorem from the seminal work of Schoen–Uhlenbeck, see [22, Theorem 3.1].

**Theorem 2.1** ($\varepsilon$-regularity of minimizers). Let $\Omega \subset \mathbb{R}^3$ be a domain with smooth boundary. Then there exists $\varepsilon > 0$ such that the following holds: if $u : \Omega \to S^2$ is a minimizing harmonic map, and if for a ball $B_R(x_0) \subset \Omega$ we have

$$R^{-1} \int_{B_R(x_0)} |\nabla u|^2 \, dx < \varepsilon,$$

then $u$ is analytic in $B_{\frac{R}{2}}(x_0)$.

The following is a result of [22, Theorem II]

**Theorem 2.2** (Partial regularity of minimizers). Let $\Omega \subset \mathbb{R}^3$ and $u \in W^{1,2}(\Omega, S^2)$ be a minimizing harmonic map, then $u$ is analytic in $\Omega \setminus \text{sing } u$ and $\text{sing } u$ consists of isolated points.

2.1. Monotonicity formulas. The first published version of a monotonicity formula for minimizing harmonic maps was in Schoen and Uhlenbeck’s [22, Proposition 2.4]. We remark here that the result was also generalized for stationary harmonic maps (and Yang–Mills fields) by Price in [20]. For a nice presentation see [24, Section 2.4].

**Theorem 2.3** (Interior Monotonicity formula). Let $\Omega \subset \mathbb{R}^3$ and let $u \in W^{1,2}(\Omega, S^2)$ be a minimizing harmonic map. Then for any $0 < r < R < \text{dist } (y, \partial \Omega)$

$$R^{-1} \int_{B_R(y)} |\nabla u|^2 \, dx - r^{-1} \int_{B_r(y)} |\nabla u|^2 \, dx = 2 \int_{B_R(y) \setminus B_r(y)} |x - y|^{-1} \left| \frac{\partial u}{\partial \nu} \right|^2 \, dx,$$

where $\frac{\partial u}{\partial \nu}$ is the directional derivative in the radial direction $\frac{x-y}{|x-y|}$.

As an immediate corollary of the monotonicity formula, note that there exists the limit

$$\Theta_u(y) := \lim_{r \to 0} r^{-1} \int_{B_r(y)} |\nabla u|^2 \, dx,$$

often referred to as the energy density. It plays a crucial role in the study of singularities.

We will also employ a boundary monotonicity formula of [23, Lemma 1.3]. We will use it only for boundary data constant on a portion of the boundary.

**Theorem 2.4** (Boundary Monotonicity formula). Let $B_1^+ \subset \mathbb{R}^3$ and let $u \in W^{1,2}(B_1^+, S^2)$ be a minimizing harmonic map with $u \bigg|_{\partial \Omega} \in W^{1,\infty}(\partial B_1^+, S^2)$. Then for $y \in T_1$

$$R^{-1} \int_{B_R^+(y)} |\nabla u|^2 \, dx - r^{-1} \int_{B_r^+(y)} |\nabla u|^2 \, dx \geq 2 \int_{B_R^+(y) \setminus B_r^+(y)} |x - y|^{-1} \left| \frac{\partial u}{\partial \nu} \right|^2 \, dx.$$
for all $0 < r < R < 1 - |y|$.

2.2. **Tangent maps.** Let $u \in W^{1,2}(\Omega, \mathbb{S}^2)$ be a minimizing harmonic map, $y \in \Omega$ and $\lambda > 0$. We define the rescaled maps $u_{y,\lambda} \in W^{1,2}(\frac{1}{\lambda}(\Omega - y), \mathbb{S}^2)$ by

$$u_{y,\lambda}(x) := u(y + \lambda x).$$

We say that map $v \in W^{1,2}(\mathbb{R}^n, \mathbb{S}^2)$ is a tangent map to $u$ at point $y$ if there exists a sequence $\{\lambda_i\}_{i}, \lambda_i \searrow 0$ such that $u_{y,\lambda_i} \to v$ strongly in $W^{1,2}$ as $i \to \infty$.

As a consequence of the monotonicity formula one has the following

**Lemma 2.5** (tangent maps, [22, Lemma 2.5]). For any $y \in \Omega$ and any sequence $\{\lambda_i\}_{i}, \lambda_i \searrow 0$ there exists a minimizing tangent map $u_0 \in W^{1,2}(\mathbb{R}^n, \mathbb{S}^2)$ such that, up to a subsequence, $u_{y,\lambda_i} \to u_0$ strongly in $W^{1,2}$ as $i \to \infty$. Moreover, $u_0$ is homogeneous of degree 0.

**Proof.** Let $B_R(y) \subset \subset \Omega$. As a consequence of the monotonicity formula (2.1) we obtain the bound

$$\|\nabla u_{y,\lambda}\|_{L^2(B_R)}^2 = \int_{B_R} |\nabla u_{y,\lambda}|^2 \, dx \leq \lambda^{-1} \int_{B_{\lambda R}(y)} |\nabla u|^2 \, dx \leq \int_{B_R(y)} |\nabla u|^2 \, dx.$$

Thus

$$\sup_{\lambda > 0} \|u_{\lambda, y}\|_{W^{1,2}(B_R)} < \infty,$$

which implies that for a $u_0 \in W^{1,2}(\Omega, \mathbb{S}^2)$ up to a subsequence, denoted the same $u_{\lambda, y} \to u_0$ weakly in $W^{1,2}_{loc}$.

By the compactness result — Theorem 4.6 — we get that the convergence is in fact strong and that the limiting map $u_0$ is minimizing.

Recall the energy density (2.2)

$$\Theta_u(y) = \lim_{\lambda \to 0} \lambda^{-1} \int_{B_{\lambda}(y)} |\nabla u|^2 \, dx = \lim_{\lambda \to 0} \int_{B_1} |\nabla u_\sigma|^2 \, dx.$$

After passing to the limit in (2.1) with the smaller radius, we obtain

$$\int_{B_{\lambda}(y)} |x - y|^{-1} |\frac{\partial u}{\partial \nu}|^2 \, dx = \lambda^{-1} \int_{B_{\lambda}(y)} |\nabla u|^2 \, dx - \Theta_u(y).$$

(2.3)

After a change of variable the left-hand side becomes

$$\int_{B_{\lambda}(y)} |x - y|^{-1} |\frac{\partial u_{y,\lambda}}{\partial \nu}|^2 \, dx = \int_{B_1} |x|^{-1} |\frac{\partial u_{\lambda,y}}{\partial \nu}|^2 \, dx.$$
Inserting the last equality into (2.3) and passing with \( \lambda \to 0 \) we infer
\[
\int_{B_1} |x|^{-1} \left| \frac{\partial u_{\lambda, y}}{\partial \nu} \right|^2 \, dx \leq 0,
\]
which implies that
\[
\frac{\partial u_0}{\partial \nu} = 0.
\]
\[\square\]

The following result, due to [23], states that there exist no nonconstant boundary tangent maps. This is the main reason why at the boundary we have full regularity for certain boundary data, c.f., Section 5.

Lemma 2.6 (Tangent maps (boundary)). Assume \( \Omega = B_2^+ \) and assume that the map \( u \) is continuous at some \( y \in T_1 \). Then for any \( \lambda \) and any sequence \( \lambda_i \to 0 \), there exists a minimizing tangent map \( u_0 \in W^{1,2}(B_2^+, \mathbb{S}^2) \) such that, up to a subsequence, \( u_{y, \lambda_i} \to \text{const} \) strongly in \( W^{1,2}(B_1^+) \).

Proof. By Theorem 4.7 we obtain that up to a subsequence \( u_{y, \lambda_i} \) converges strongly in \( W^{1,2}(B_1^+) \) to a \( \tau \in W^{1,2}(B_2^+, \mathbb{S}^2) \) and \( \tau \) is minimizing harmonic. Moreover, by continuity \( \tau \) is constant at \( T_1 \). Then since the limit of the rescaled maps is homogeneous of degree 0 one has by [10, Theorem 5.7] that a minimizing 0-homogeneous map from a half-ball, which is constant at \( T_1 \) must be constant itself. \( \square \)

3. Uniform boundedness of Minimizers

Here we collect the results concerning local uniform boundedness of minimizers into spheres. We will often use this result to infer that from any sequence of minimizing harmonic maps one can choose a subsequence that converges locally weakly, in fact strongly, see Theorem 4.6. An underlying tool of the arguments in [1] is the following \( W^{1,2} \)-extension property of the \( W^{1/2,2} \) maps into \( \mathbb{S}^2 \). This result was first proved in Hardt, Kinderlehrer, and Lin (see [7, p.556]) and then generalized by Hardt and Lin [10, Theorem 6.2] for \( W^{1,p} \)-maps into manifolds with sufficiently simple topology.

Theorem 3.1 (Extension Property). Let \( \Omega \) be a bounded domain and let \( v \in W^{1,2}(\Omega, \mathbb{R}^3) \) with \( v(x) \in \mathbb{S}^2 \) for a.e. \( x \in \partial \Omega \). Then there exists a map \( u \in W^{1,2}(\Omega, \mathbb{S}^2) \),
\[
\left. u \right|_{\partial \Omega} = \left. v \right|_{\partial \Omega}
\]
with the estimate
\[
\|\nabla v\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}
\]
for a uniform constant \( C \).

For reader’s convenience we recall the proof from [8, Lemma A.1].
Proof. Let $a \in \mathbb{R}^3$, $|a| < 1$, consider the maps

$$u_a(x) = \frac{v(x) - a}{|v(x) - a|}, \quad x \in \Omega.$$  

Then, 

$$\nabla u_a(x) = \frac{\nabla v(x)}{|v(x) - a|} - \frac{(v(x) - a) \otimes (v(x) - a)}{|v(x) - a|^3} \nabla v(x), \quad \text{for } x \in \Omega.$$  

Thus,

$$|\nabla u_a(x)| \leq \sqrt{2} \frac{|\nabla v(x)|}{|v(x) - a|}.$$  

Hence,

$$\int_{B_\frac{1}{2}} |\nabla u_a|^2 \, da \leq 2 \int_{B_\frac{1}{2}} \frac{|\nabla v|^2}{|v - a|^2} \, da \leq 8\pi |\nabla v|^2;$$  

because for any $p \in \mathbb{R}^3$ we have, for a $q \in B_\frac{1}{2}$, that

$$\int_{B_\frac{1}{2}} \frac{1}{|p - a|^2} \, da \leq \int_{B_1} \frac{1}{|q - a|^2} \, da \leq \int_{B_\frac{1}{2}(q)} \frac{1}{|y|^2} \, dy \leq \int_{B_1} \frac{1}{|y|^2} \, dy = 4\pi.$$  

Integrating the above inequality over $\Omega$ we get by Fubini’s theorem

$$\int_{B_\frac{1}{2}} \int_{\Omega} |\nabla u_a|^2 \, dx \, da = \int_{\Omega} \int_{B_\frac{1}{2}} |\nabla u_a|^2 \, dx \, da \leq 8\pi \int_{\Omega} |\nabla v|^2 \, dx.$$  

Thus, again by Fubini’s theorem, we infer the existence of $a_0 \in B_\frac{1}{2}$ for which

$$\int_{\Omega} |\nabla u_{a_0}|^2 \, dx \leq 8\pi \left|B_\frac{1}{2}\right|^{-1} \int_{\Omega} |\nabla v|^2 \, dx = 48 \int_{\Omega} |\nabla v|^2 \, dx.$$  

For all $a \in B_\frac{1}{2}$ let

$$\Pi_a(\xi) = \frac{\xi - a}{|\xi - a|} \quad \text{for } \xi \in \mathbb{S}^2.$$  

This is a $C^1$ homeomorphism of $\mathbb{S}^2$ onto itself. Indeed, the inverse map is given by

$$\Pi_a^{-1}(\xi) = a + [(a \cdot \xi)^2 + (1 - |a|^2)]^{1/2}\xi.$$  

Thus, after simple computations

$$|\nabla \Pi_a^{-1}(\xi)| \leq 2, \quad \text{for all } |a| \leq \frac{1}{2}.$$  

We now observe that since $|v| \equiv 1$ almost everywhere on $\partial \Omega$ we have

$$u_a = \Pi_a \circ v \quad \text{on } \partial \Omega.$$  

Finally, we define $u := \Pi_a^{-1} \circ u_{a_0}$. By (3.3) we have $u|_{\partial \Omega} = v|_{\partial \Omega}$ and combining (3.2) with (3.1) we conclude

$$\int_{\Omega} |\nabla u|^2 \, dx \leq \left(\text{Lip}(\Pi_a^{-1})\right)^2 \int_{\Omega} |\nabla u_{a_0}|^2 \, dx \leq 192 \int_{\Omega} |\nabla v|^2 \, dx.$$
One of the most important applications of the extension property above is that we can compare the energy of harmonic maps mapping into the sphere with harmonic maps into the surrounding Euclidean space, i.e., if \( u \in W^{1,2}(\Omega, S^2) \) is a minimizing harmonic map into the sphere on the domain \( \Omega \subset \mathbb{R}^3 \) and \( u^h \in W^{1,2}(\Omega, S^2) \) denotes the harmonic extension

\[
\begin{cases}
\Delta u^h = 0 & \text{in } \Omega \\
u^h = u & \text{on } \partial \Omega,
\end{cases}
\]

then we have as a consequence of Theorem 3.1 and trace estimates,

\[
\| \nabla u^h \|_{L^2(\Omega)} \lesssim \| \nabla u^h \|_{L^2(\Omega)} \lesssim [u]_{W^{1/2,2}(\partial \Omega)}.
\]

Now, by Gagliardo-Nirenberg inequality, see for example [5],

\[
[u]_{W^{1/2,2}(\partial \Omega)} \lesssim \sqrt{\| \nabla_T u \|_{L^2(\partial \Omega)}} \sqrt{\| u \|_{L^\infty(\partial \Omega)}}.
\]

That is, using trace theorems as in Theorem A.4, Sobolev embedding, and Gagliardo-Nirenberg inequalities, we obtain as a corollary of Theorem 3.1 the following.

**Corollary 3.2.** If \( u : B_r \to S^2 \) is a minimizing harmonic map, then the following estimates hold

\[
(3.4) \quad \| \nabla u \|_{L^2(B_r)} \lesssim \sqrt{r \| \nabla_T u \|_{L^2(S^2)}}.
\]

If \( u : B_r^+ \to S^2 \) is a minimizing harmonic map with \( u = \varphi \) on the flat part of the boundary \( T_r \). Then the following estimates hold for any \( s \in (1/2, 1] \), and \( sp > 1 \)

\[
(3.5) \quad \| \nabla u \|_{L^2(B_r^+)} \lesssim \sqrt{r \| \nabla_T u \|_{L^2(S^2_+)} + r^{sp-2} [\varphi]_{W^{s,p}(T_r)}}.
\]

In particular, in Theorem 3.5 we obtain an extension to more general trace spaces of Almgren and Lieb’s uniform boundedness with \( W^{1,2} \)-boundary data, see [1, Theorem 2.3 (2)]. We first state the interior result.

**Theorem 3.3 (Uniform Boundedness of Minimizers (interior)).** Let \( u \in W^{1,2}(B_R, S^2) \) be a minimizing harmonic map. Then for any \( r < R \),

\[
\int_{B_r} |\nabla u|^2 \, dx \leq C \frac{rR}{R-r},
\]

where \( C \) is an absolute constant.

**Remark 3.4.** Theorem 3.3 does not hold for general target manifolds. A counterexample is due to Hardt–Kinderlehrer–Lin [8, p.22]: the energy minimizers \( u_j \in W^{1,2}(B^n, S^1) \), \( u_j(x) = (\cos jx_1, \sin jx_1) \) have unbounded energies on each subdomain.
Proof of Theorem 3.3. By minimality of \( u \) we may apply (3.4) and we get

\[
D(\rho) := \int_{B^+} |\nabla u|^2 \, dx \leq C \rho \left( \int_{\partial B^\rho} |\nabla_T u|^2 \, d\mathcal{H}^2 \right)^{\frac{1}{2}},
\]

for a uniform constant \( C > 0 \). This means

\[
D(\rho) \leq \rho C \sqrt{D'(\rho)}
\]

and taking the square

\[
\frac{1}{C^2} \rho^{-2} \leq \frac{D'(\rho)}{D(\rho)^2}.
\]

Integrating the last inequality from \( r \) to \( R \) we obtain

\[
\left( \frac{1}{r} - \frac{1}{R} \right) \frac{1}{C^2} \leq \frac{1}{D(r)} - \frac{1}{D(R)}
\]

In particular,

\[
D(r) \leq C^2 \frac{rR}{R - r}.
\]

\[\square\]

Theorem 3.5 (Uniform boundedness (boundary)). Let \( u \in W^{1,2}(B^r_{2r}, S^2) \) be a minimizing harmonic map. Then, whenever \( s \in \left( \frac{1}{2}, 1 \right], \, p \in [2, \infty), \)

\[
\int_{B^r_{2r}} |\nabla u|^2 \, dx \leq C \max \left\{ 1, \rho^{sp-2}[u]_{W^{s,p}(T_2)}^p \right\}.
\]

where \( C \) is an absolute constant.

Proof. It suffices to consider the case \( r = 1 \), the claim then follows from rescaling.

For any \( \rho \in [1/2, 1] \). In view of Corollary 3.2 we have

\[
D(\rho) := \int_{B^+_{\rho}} |\nabla u|^2 \, dx \lesssim \rho \left( \int_{S^+_{\rho}} |\nabla_T u|^2 \, d\mathcal{H}^2 \right)^{\frac{1}{2}} + [u]_{W^{s,p}(T_1)}^p.
\]

Now, since \( D'(\rho) = \int_{S^+_{\rho}} |\nabla u|^2 \, d\mathcal{H}^2 \) and \( |\nabla u|^2 = |u_\nu|^2 + |\nabla_T u|^2 \)

\[
\int_{S^+_{\rho}} |\nabla_T u|^2 \, d\mathcal{H}^2 \leq D'(\rho).
\]

Thus, we have found

\[
D(\rho) \leq C \left( (D'(\rho))^{\frac{1}{2}} + [u]_{W^{s,p}(T_1)}^p \right)
\]

Thus, for any \( \rho \in (1/2, 1), \)

\[
(3.6) \quad D'(\rho) \geq \left( \frac{1}{C} D(\rho) - [u]_{W^{s,p}(T_1)}^p \right)^2.
\]
Assume now that
\begin{equation}
D(1/2) > (2C^2 + 2C) \max \left\{ 1, [u]^p_{W^{s,p}(T_1)} \right\}.
\end{equation}
Then, since $D(\rho) \geq D(1/2)$ for $\rho \in (1/2, 1)$,
\begin{align*}
\left( \frac{1}{C} D(\rho) - [u]^2_{W^{1/2}(T_1)} \right)^2 & \geq \frac{1}{C^2} D^2(\rho) - \frac{2}{C} D(\rho) \cdot [u]^p_{W^{s,p}(T_1)} \\
& > \frac{1}{C^2} D^2(\rho) - \frac{2}{C(2C^2 + 2C)} D^2(\rho) \\
& = \frac{2}{2C^2 + 2C} D^2(\rho).
\end{align*}
Consequently we find from (3.6)
\begin{equation*}
D'(\rho) \geq \frac{2}{2C^2 + 2C} D^2(\rho).
\end{equation*}
Similarly, as in the proof of the previous theorem, integrating on the interval $(1/2, 2)$ we would get
\begin{equation*}
-D(1)^{-1} + D(1/2)^{-1} \geq \frac{1}{2C^2 + 2C}.
\end{equation*}
But this would imply
\begin{equation*}
D(1/2) \leq 2C^2 + 2C \leq (2C^2 + 2C) \max \left\{ 1, [u]^p_{W^{s,p}(T_1)} \right\}.
\end{equation*}
A contradiction to (3.7).

3.1. Classification of singularities for harmonic maps into the sphere. In the case of the target space being a sphere, one can actually classify singularities explicitly. Maps near singular points are in the form $g(\frac{x}{|x|})$, where $g : S^2 \to S^2$, then [4] classification of such minimizing harmonic maps gives the following.

**Theorem 3.6 (Classification of tangent maps, [4, Corollary 7.2]).** Let $u \in W^{1,2}(\Omega, S^2)$ be a minimizing harmonic map. Then each singular point has degree $\pm 1$. Moreover, for every singularity $y \in \Omega$ we have
\begin{equation*}
\lim_{\varepsilon \to 0} u(\varepsilon(x - y)) = \pm \mathcal{R} \left( \frac{x - y}{|x - y|} \right),
\end{equation*}
where $\mathcal{R}$ is a rotation.

**Corollary 3.7.** Let $u \in W^{1,2}(\Omega, S^2)$ be a minimizing harmonic map and $\Omega \subset \mathbb{R}^3$. If $y \in \Omega$ is a singular point of $u$, then
\begin{equation*}
r^{-1} \int_{B_r(y)} |\nabla u|^2 \, dx \geq 8\pi \quad \text{for each } B_r(y) \subset \Omega.
\end{equation*}
Moreover, for each \( y \in \Omega \),

\[
(3.8) \quad \lim_{r \to 0} r^{-1} \int_{B_r(y)} |\nabla u|^2 \, dx \leq 8\pi.
\]

**Proof.** By (2.1) we have

\[
\begin{align*}
\quad r^{-1} \int_{B_r(y)} |\nabla u|^2 \, dx & \geq \lim_{\rho \to 0} \rho^{-1} \int_{B_\rho(y)} |\nabla u|^2 \, dx \\
& = \lim_{\rho \to 0} \rho^{-1} \int_{B_\rho(y)} \left| \nabla \left( \pm R \left( \frac{x - y}{|x - y|} \right) \right) \right|^2 \, dx = 8\pi,
\end{align*}
\]

where the first equality is a consequence of the classification of singular points in Theorem 3.6.

\[ \square \]

## 4. Strong convergence of minimizers

Compactness of minimizers has been historically a huge challenge, partial results were obtained gradually by Schoen–Uhlenbeck [22, Lemma 4.3] (tangent maps), Hardt–Lin [10, Theorem 6.4] (target manifolds with \( \pi_1(N) = 0 \)), and finally the general case was solved with the help of the celebrated Luckhaus Lemma, [15, Lemma 1]. As observed in [7], in our special case \( N = S^2 \) the extension property (Theorem 3.1) simplifies the situation.

### 4.1. Caccioppoli inequality and higher local integrability.

In this section we derive the Caccioppoli inequality for minimizing harmonic maps. The \( W^{1,2} \)-extension property (Theorem 3.1) will play here a crucial role as it provides us a tool to compare energies of maps that agree on the boundary but do not have to take values in the manifold. We will use a variant of an iteration lemma just a

**Lemma 4.1** (Iteration lemma). *Let \( 0 \leq a < b < \infty \) and \( f[a, b] : \rightarrow [0, \infty) \) be a measurable function. Suppose that there are \( \theta \in (0, 1) \), \( A, \Lambda > 0 \), and \( \alpha > 0 \) such that for \( a \leq s < t \leq b \) we have

\[
(4.1) \quad f(s) \leq \theta f(t) + \frac{A}{(t-s)^\alpha} + \Lambda.
\]

Then for all \( a \leq r < R \leq b \) we obtain

\[
(4.1) \quad f(r) \leq C \frac{A}{(R-r)^\alpha} + \frac{\Lambda}{1-\theta}.
\]
Proof. We fix $a \leq r < R \leq b$ and define a sequence $\{r_i\}$ by $r_0 = r$, $r_{i+1} - r_i = (1-\tau)r^i(R-r)$, for a $\tau \in (0,1)$ to be chosen later. Then by iterating $n$-times the inequality (4.1) we obtain
\[
f(r) \leq \theta f(r_1) + \frac{A}{(r_1 - r_0)\alpha} + \Lambda = \theta f(r_1) + \frac{A}{(1-\tau)^\alpha(R-r)^\alpha} + \Lambda
\]
\[
\leq \theta \left( \theta f(r_2) + \frac{A}{(1-\tau)^\alpha(R-r)^\alpha} + \Lambda \right) + \frac{A}{(1-\tau)^\alpha(R-r)^\alpha} + \Lambda \leq \ldots
\]
\[
\leq \theta^n f(r_n) + \frac{A}{(1-\tau)^\alpha(R-r)^\alpha} \sum_{i=0}^{n-1} \theta^i \tau^{-i\alpha} + \Lambda \sum_{i=0}^{n-1} \theta^i.
\]
Now we choose $\tau$ in such a way that $\tau^{-\alpha} \theta < 1$ and let $n \to \infty$ in the above inequality. □

The following two propositions are Caccioppoli’s interior and boundary inequalities. The proofs are similar and thus we will only include the proof in the boundary case. For the proof of the interior case we refer the interested reader, e.g., to [7, Lemma 2.3].

**Proposition 4.2** (Caccioppoli inequality (interior)). Let $u \in W^{1,2}(\Omega, S^2)$ be a minimizing harmonic map. Suppose that $B_R(y) \subset \subset \Omega$ for some $y \in \Omega$ and $R > 0$. Then there exists a constant $C$, such that
\[
\int_{B_R/2(y)} |\nabla u|^2 \, dx \leq CR^{-2} \int_{B_R(y)} |u - (u)_{B_R(y)}|^2 \, dx.
\]

**Proposition 4.3** (Caccioppoli inequality (boundary)). Let $u \in W^{1,2}(B^+, S^2)$ be a minimizing harmonic map and let $u = \varphi$ on $T_1$ for a $\varphi \in W^{1,2}(T_1, S^2)$. Then for all $r < 1$ we have
\[
\int_{B^+_r} |\nabla u|^2 \, dx \leq C \int_{B^+_2r} |u(x) - \varphi^h(x)|^2 \, dx + C \int_{B^+_2r} |\nabla \varphi^h(x)|^2 \, dx,
\]
where $\varphi^h \in W^{1,2}(\mathbb{R}^3_+, \mathbb{R}^3)$ is any harmonic function with $\varphi^h = \varphi$ on $T_1$.

**Proof.** By Theorem 3.1 we have
\[
\int_{B^+} |\nabla u|^2 \, dx \leq C \int_{B^+} |\nabla v|^2 \, dx
\]
for any $v \in W^{1,2}(B^+, \mathbb{R}^3)$ which coincides with $u$ on $\partial B^+$.

Let $v(x) = \varphi^h(x) + (1-\eta(x))(u(x) - \varphi^h(x))$, where $\eta \in C^\infty_c(B, [0,1])$ is a cutoff function such that $\eta \equiv 1$ in $B_\rho$, $\eta \equiv 0$ outside $B_R$ and $|\nabla \eta| \leq \frac{C}{R-\rho}$ for $r \leq \rho < R \leq 1$. One can readily verify that
\[
v(x) = \varphi(x) \text{ on } T_1 \text{ and } v(x) = u(x) \text{ on } S^+.
\]
We compute
\[
\nabla v(x) = \eta(x)\nabla u(x) + \nabla \eta(x)(u(x) - \varphi^h(x)) + (1-\eta(x))\nabla \varphi^h(x).
\]
Thus, by (4.3)
\[
\int_{B_R^+} |\nabla u|^2 \, dx \leq C \int_{B_R^+ \setminus B_R^\rho} |\nabla u|^2 \, dx + \frac{C}{(R-\rho)^2} \int_{B_R^\rho} |u(x) - \phi^h(x)|^2 \, dx \\
+ C \int_{B_R^+} |\nabla \phi^h(x)|^2 \, dx.
\]
Therefore, we get that there exists a $0 < \theta < 1$ such that
\[
\int_{B_R^+} |\nabla u|^2 \, dx \leq \theta \int_{B_R^+} |\nabla u|^2 \, dx + \frac{C}{(R-\rho)^2} \int_{B_R^\rho} |u(x) - \phi^h(x)|^2 \, dx \\
+ C \int_{B_R^+} |\nabla \phi^h(x)|^2 \, dx
\]
for all $r \leq \rho < R \leq 1$.

Thus, by Lemma 4.1 we obtain
\[
\int_{B_R^+} |\nabla u|^2 \, dx \leq \frac{C}{(R-\rho)^2} \int_{B_R^\rho} |u(x) - \phi^h(x)|^2 \, dx + C \int_{B_R^+} |\nabla \phi^h(x)|^2 \, dx.
\]
Taking $R = 2r$ we conclude. \qed

As consequences of Poincaré inequality, Sobolev embedding and Gehring Lemma we readily obtain

**Corollary 4.4** (Higher integrability in the interior). Let $u$ and $R$ be as in Proposition 4.3. There exists a $p > 2$ such that
\[
\left( \int_{B_{R/2}} |\nabla u|^p \, dx \right)^{1/p} \leq C \left( \int_{B_R} |\nabla u|^2 \, dx \right)^{1/2}.
\]
Observing that for $s \in \left( \frac{1}{2}, 1 \right]$ if $\phi \in W^{s,p}(T_{2R})$ then, by Sobolev embedding, its harmonic extension $\phi^h$ belongs to the homogeneous Sobolev space $\dot{W}^{1,q}(\mathbb{R}^3_+)$ if
\[
s - \frac{2}{p} > 1 - \frac{1}{q} - \frac{2}{q}.
\]
In particular, $\phi^h \in \dot{W}^{1,q}(\mathbb{R}^3_+)$ for $q > 2$ if
\[
s - \frac{2}{p} > \frac{1}{2}.
\]
This is in particular satisfies if $p \geq 2$, so we get the following corollary.
Corollary 4.5 (Higher integrability up to the boundary). Let \( u \) and \( R \) be as in Proposition 4.3. For any \( s \in \left( \frac{1}{2}, 1 \right] \) and \( p \in [2, \infty) \) there exists a \( q > 2 \) such that

\[
\left( \int_{B^+_R/2} |\nabla u|^q \, dx \right)^{1/q} \leq C \left( \int_{B^+_R} |\nabla u|^2 \, dx \right)^{1/2} + [u]_{W^{s,p}(T_R)}.
\]

Theorem 4.6 (strong convergence of minimizers (interior)). The limit \( u \in W^{1,2}(\Omega, S^2) \) of any weakly convergent sequence \( u_i \) of minimizers in \( W^{1,2}(\Omega, S^2) \) is locally a minimizer and the convergence is locally strong.

Proof. The following proof is due to Hardt and Lin [10, Theorem 6.4].

For simplicity of notation we assume that \( \Omega = B_2 \) and show that \( u \) is a minimizing harmonic map in \( B_1 \), and that the convergence \( u_i \to u \) is strong in \( W^{1,2}(B_1) \).

For any small \( \delta > 0 \) take a cutoff function \( \eta \equiv 1 \) in \( B_{1+\delta} \) and \( \eta \equiv 0 \) outside of \( B_{1+2\delta} \), with \( [\eta]_{Lip} \lesssim \frac{1}{\delta} \). We use this cutoff to interpolate between \( u_i \) on \( \partial B_{1+2\delta} \) and \( u \) on \( \partial B_{1+\delta} \).

\[
v_i := \eta \delta u_i + (1 - \eta \delta) u.
\]

We claim that

\[
\lim_{\delta \to 0} \lim_{i \to \infty} \int_{B_{1+2\delta} \setminus B_1} |\nabla v_i|^2 \, dx = 0.
\]

Indeed, we have

\[
\int_{B_{1+2\delta} \setminus B_1} |\nabla v_i|^2 \, dx \lesssim \delta^{-2} \int_{B_{1+2\delta} \setminus B_1} |u_i - u|^2 \, dx + \int_{B_{1+2\delta} \setminus B_1} |\nabla u|^2 \, dx + \int_{B_{1+2\delta} \setminus B_1} |\nabla u_i|^2 \, dx.
\]

Since \( u_i \) converges weakly to \( u \) in \( W^{1,2}(B_2) \), up to a subsequence, it converges strongly in \( L^2(B_2) \). In particular,

\[
\lim_{i \to \infty} \delta^{-2} \int_{B_{1+2\delta} \setminus B_1} |u_i - u|^2 \, dx = 0.
\]

Moreover, by absolute continuity of the integral,

\[
\lim_{\delta \to 0} \int_{B_{1+2\delta} \setminus B_1} |\nabla u|^2 \, dx = 0.
\]

Finally, we would like to use this argument also for the last term, but it depends on \( i \). Thus we use Corollary 4.4 which by Hölder’s inequality implies for some \( p > 2 \),

\[
\int_{B_{1+2\delta} \setminus B_1} |\nabla u_i|^2 \, dx \lesssim |B_{1+2\delta} \setminus B_1|^\frac{p-2}{p} \int_{B_2} |\nabla u_i|^2 \, dx.
\]

It suffices to prove the statement for a subsequence of \( u_i \), since then every sequence has a strongly converging subsubsequence converging to \( u \), which implies that \( u_i \) strongly converges to \( u \).
Since $u_i$ is weakly convergent in $W^{1,2}(B_2)$, the sequence $u_i$ is in particular bounded in $W^{1,2}(B_2)$ and we find
\[
\lim \lim_{\delta \to 0} \int_{B_{1+2\delta} \setminus B_1} |\nabla u_i|^2 \, dx \approx \lim_{\delta \to 0} \sup_{B_{1+2\delta}} \int_{B_2} |\nabla u_i|^2 \, dx = 0.
\]
Thus (4.4) is established.

Now we use the $W^{1,2}$-extension property of $S^2$ (Theorem 3.1) and thus from $v_i$ construct map $w_i \in W^{1,2}(B_{1+2\delta} \setminus B_1, S^2)$ such that (extended by $u$ to the interior of the ball $B_{1-\delta}$)
\[
\begin{cases}
  w_i \equiv u & \text{in } B_{1+\delta} \\
  w_i = u_i & \text{on } \partial B_{1+2\delta}
\end{cases}
\]
such that in view of (4.4)
\[
\lim_{\delta \to 0} \lim_{i \to \infty} \int_{B_{1+2\delta}} |\nabla w_i|^2 \, dx = \int_{B_1} |\nabla u|^2 \, dx.
\]
This implies, by minimality of $u_i$,
\[
\lim_{\delta \to 0} \sup_{i \to \infty} \int_{B_{1+2\delta}} |\nabla u_i|^2 \, dx \leq \lim_{\delta \to 0} \lim_{i \to \infty} \int_{B_{1+2\delta}} |\nabla w_i|^2 \, dx = \int_{B_1} |\nabla u|^2 \, dx.
\]
That is,
\[
\lim_{i \to \infty} \int_{B_1} |\nabla u_i|^2 \, dx \leq \lim_{\delta \to 0} \sup_{i \to \infty} \int_{B_{1+2\delta}} |\nabla u_i|^2 \, dx \leq \int_{B_1} |\nabla u|^2 \, dx
\]
By lower semicontinuity of the $W^{1,2}$-norm with respect to the weak convergence we obtain also the other direction and find that
\[
\lim_{i \to \infty} \int_{B_1} |\nabla u_i|^2 \, dx = \int_{B_1} |\nabla u|^2 \, dx
\]
From this, the weak convergence and the following identity for $L^2$ we obtain strong convergence:
\[
(4.5) \quad \|\nabla (u_i - u)\|_{L^2}^2 = \|\nabla u\|_{L^2}^2 - \|\nabla u_i\|_{L^2}^2 + 2\langle \nabla (u_k - u), \nabla u \rangle_{L^2}.
\]
The minimizing property follows along the same lines, if in the argument above we choose $w_i$ so that
\[
\begin{cases}
  w_i \equiv v & \text{in } B_1 \\
  w_i \equiv u & \text{in } B_{1+\delta} \setminus B_1 \\
  w_i = u_i & \text{on } \partial B_{1+2\delta}
\end{cases}
\]
for any competitor $v$ with $v = u$ on $\partial B_1$. Then we obtain
\[
\int_{B_1} |\nabla u|^2 \, dx = \lim_{i \to \infty} \int_{B_1} |\nabla u_i|^2 \, dx \leq \int_{B_1} |\nabla v|^2 \, dx,
\]
which is the desired minimizing property. \qed
Theorem 4.7 (strong convergence of minimizers (boundary)). Let \( \{u_i\}_{i \in \mathbb{N}} \in W^{1,2}(B^+, S^2) \) be a sequence of minimizing harmonic maps such that \( u_i = \varphi_i \) on \( T_1 \) for each \( i \in \mathbb{N} \). If we have strong convergence at the flat boundary, namely \( \varphi_i \to \varphi \) in \( W^{s,p}(T_1, S^2) \) for some \( s \in (\frac{1}{2}, 1] \) and \( p \geq 2 \), then \( u_i \to u \) strongly in \( W^{1,2}(B^+, S^2) \) for any \( r < 1 \) and \( u \in W^{1,2}(B^+, S^2) \) is a minimizing harmonic map.

Proof. It suffices to show that \( u \big|_{B^+} \) is minimizing for any \( r < 1 \). For clarity, without loss of generality, we will prove the assertion for \( r = \frac{1}{2} \).

We note that by Theorem 3.5,

\[
\sup_{i \in \mathbb{N}} [u_i]_{W^{1,2}(B^+, S^2)} \leq C \sup_{i \in \mathbb{N}} [\varphi_i]_{W^{s,p}(T_1, S^2)} < \infty.
\]

Thus, combing this with Corollary 4.5, we get for some \( 2 \leq p > 2 \)

\[
(4.6) \quad \sup_{i \in \mathbb{N}} \|u_i\|_{W^{1,p}(B^+_{\frac{1}{2}/2}, S^2)} < \infty.
\]

Let \( v \in W^{1,2}(B^+, S^2) \) be such that \( v = \varphi \) on \( T_1 \) and \( v \big|_{B^+\setminus B_{\frac{1}{2}/2}} = u \big|_{B^+\setminus B_{\frac{1}{2}/2}} \). We choose a small positive \( \delta > 0 \) and a cutoff function \( \eta_\delta \in C_c^\infty(B_{1/2+3\delta}; [0,1]) \) satisfying \( \eta_\delta \equiv 1 \) in \( B_{1/2+\delta} \) with \( |\nabla \eta_\delta| \leq \frac{C}{\delta} \).

For \( A_\delta^+ \equiv B_{\frac{1}{2}+\delta}^+ \setminus B_{\frac{1}{2}+\delta} \) we define maps \( v_i \in W^{1,2}(A_\delta^+; \mathbb{R}^3) \) by

\[
v_i(x) = \eta_\delta(x)v(x) + (1 - \eta_\delta(x))u_i(x).
\]

We note that

\[
\begin{align*}
v_i(x) &= u_i(x) &\text{on } &S_{\frac{1}{2}+\delta}^+ \\
v_i(x) &= u(x) &\text{on } &S_{\frac{1}{2}+\delta}^+ \\
v_i(x) &= \eta_\delta(x)\varphi(x) + (1 - \eta_\delta(x))\varphi_i(x) &\text{on } &T_{\frac{1}{2}+\delta}.
\end{align*}
\]

Now let us define \( \tilde{v}_i(x) \in W^{1,2}(B_{1/2+3\delta}^+; \mathbb{R}^3) \) by

\[
\tilde{v}_i(x) = \begin{cases} v_i(x) &\text{for } x \in A_\delta^+ \\ v(x) &\text{for } x \in B_{\frac{1}{2}+\delta}^+ \end{cases}
\]

and observe that on the boundary

\[
\begin{align*}
\tilde{v}_i(x) &= u_i &\text{on } &S_{\frac{1}{2}+\delta}^+ \\
\tilde{v}_i(x) &= \varphi_i(x) + \eta_\delta(x)(\varphi(x) - \varphi_i(x)) &\text{on } &T_{\frac{1}{2}+\delta}.
\end{align*}
\]

Let us define \( \psi_i(x) := \eta_\delta(x)(\varphi(x) - \varphi_i(x)) \) for \( x \in T_{\frac{1}{2}+\delta} \). By the definition of \( \eta_\delta \) we have \( \psi_i = 0 \) on \( T_{\frac{1}{2}+\delta} \setminus T_{\frac{1}{2}+3\delta} \). Now we take \( \psi^h_i \in W^{1,2}(B_{1/2+3\delta}^+; \mathbb{R}^3) \) a harmonic extension of \( \psi_i \) such that \( \psi^h_i = 0 \) on \( S_{\frac{1}{2}+\delta}^+ \) and set

\[
w_i(x) := \tilde{v}_i(x) - \psi^h_i, \quad w_i \in W^{1,2}(B_{1/2+3\delta}^+; \mathbb{R}^3).
\]
It is easy to see that for such defined map we have \( w_i = u_i \) on \( \partial B^{+}_{\frac{1}{2} + 4\delta} \).

By Theorem 3.1 we obtain maps \( \bar{w}_i \in W^{1,2}_{u_i}(B^{+}_{1/2+4\delta}, \mathbb{S}^2) \) for which

\[
\int_{B^{+}_{\frac{1}{2} + 4\delta}} |\nabla \bar{w}_i|^2 \, dx \leq C \int_{B^{+}_{\frac{1}{2} + 4\delta}} |\nabla w_i|^2 \, dx.
\]

Combining this with the minimality of \( u_i \) we have

\[
\int_{B^{+}_{\frac{1}{2} + 4\delta}} |\nabla u_i|^2 \, dx \leq \int_{B^{+}_{\frac{1}{2} + 4\delta}} |\nabla \bar{w}_i|^2 \, dx \leq \int_{B^{+}_{\frac{1}{2} + 4\delta}} |\nabla w_i|^2 \, dx \leq \int_{B^{+}_{\frac{1}{2} + 4\delta}} |\nabla \bar{w}_i|^2 \, dx + \int_{B^{+}_{\frac{1}{2} + 4\delta}} |\nabla \psi_i^h|^2 \, dx.
\]

First, we observe that

\[
\lim_{i \to \infty} \int_{B^{+}_{\frac{1}{2} + 4\delta}} |\nabla \psi_i^h|^2 \, dx \leq C \lim_{i \to \infty} \| \psi_i \|^2_{W^{s,p}(T_{1/2+4\delta}, \mathbb{S}^2)} = 0,
\]

where in the last equality we used that \( (\varphi_i - \varphi) \to 0 \) in \( W^{s,p}(T_{1/2+4\delta}, \mathbb{S}^2) \) as \( i \to \infty \).

We compute

\[
\nabla v_i(x) = \nabla \eta_{\delta}(x) (v(x) - u_i(x)) + (1 - \eta_{\delta}(x)) \nabla u_i(x) + \eta_{\delta}(x) \nabla v(x).
\]

Thus, since \( \psi = u \) on \( A_{\delta}^+ \)

\[
\int_{A_{\delta}^+} |\nabla v_i|^2 \, dx \leq \frac{C}{\delta^2} \int_{A_{\delta}^+} |u(x) - u_i(x)|^2 \, dx + \int_{A_{\delta}^+} |\nabla u_i|^2 \, dx + \int_{A_{\delta}^+} |\nabla v|^2 \, dx.
\]

Now passing with \( i \) to the limit we get by the strong convergence of \( u_i \) in \( L^2 \) that

\[
\lim_{i \to \infty} \int_{A_{\delta}^+} |\nabla v_i|^2 \, dx \leq \lim_{i \to \infty} \int_{A_{\delta}^+} |\nabla u_i|^2 \, dx + \int_{A_{\delta}^+} |\nabla v(x)|^2 \, dx
\]

\[
\leq \lim_{i \to \infty} \left( \int_{A_{\delta}^+} |\nabla u_i|^p \, dx \right)^{\frac{2}{p}} \int_{A_{\delta}^+} |\nabla v(x)|^2 \, dx,
\]

where in the last estimate we used Hölder’s inequality.

By (4.6) the term \( \int_{A_{\delta}^+} |\nabla u_i|^p \, dx \) is uniformly bounded, clearly \( |A_{\delta}^+| \to 0 \) as \( \delta \to 0 \), and by the absolute continuity of the Lebesgue integral \( \int_{A_{\delta}^+} |\nabla v(x)|^2 \, dx \to 0 \) as \( \delta \to 0 \). Thus, the right-hand side of the inequality (4.9) converges to 0 as \( \delta \to 0 \).
Combining (4.7), the definition of \( \mathcal{v}_i \) and (4.9) we get for any \( v \) with \( v = \varphi \) on \( T_1 \) and \( v|_{B^+ \setminus B_{1/2}} = u|_{B^+ \setminus B_{1/2}} \)

\[
\lim_{i \to \infty} \int_{B^+_{\frac{3}{2} + 4\delta}} |
abla u_i|^2 \, dx \leq \int_{B^+_{\frac{3}{2} + \delta}} |
abla v|^2 \, dx + \mathcal{O}(1), \quad \text{as } \delta \to 0.
\]

Now, to obtain the strong convergence of \( \nabla u_i \) in \( L^2 \), we choose in inequality (4.10) \( v = u \) in \( B^+_{\frac{3}{2}} \). By lower semicontinuity we obtain

\[
\lim_{i \to \infty} \int_{B^+_{\frac{3}{2}}} |
abla u_i|^2 \, dx = \int_{B^+_{\frac{3}{2}}} |
abla u|^2 \, dx.
\]

This implies the strong convergence, just as in the interior case by (4.5).

Moreover,

\[
\int_{B^+_{\frac{3}{2} + 4\delta}} |
abla u|^2 \, dx \leq \int_{B^+_{\frac{3}{2} + \delta}} |
abla v|^2 \, dx + \mathcal{O}(1), \quad \text{as } \delta \to 0.
\]

for any \( v \) such that \( v = u \) on \( \partial B^+_{\frac{3}{2}} \). This finishes the proof of the minimality of \( u \). \( \square \)

4.2. **Singular points converge to singular points and have uniform distance to each other.** In general, a limit of singular points has to be singular. This is basically a consequence of the \( \varepsilon \)-regularity theorem and upper-semicontinuity of \( \Theta_u(y) \) (see (2.2)) with respect to both \( u \) and \( y \).

An inverse statement can be shown in the special case when the target manifold is \( S^2 \), fundamentally because then the singularities are known to have non-trivial homotopy type.

**Theorem 4.8** (Singular points converge to singular points, [1, Thm 1.8]). Assume \( u_i \in W^{1,2}(\Omega, S^2) \) is a sequence of minimizing maps in \( \Omega \subset \mathbb{R}^3 \), which converges strongly in \( W^{1,2}_{loc} \) to \( u \). Then

1. If \( y_i \) is a singular point of \( u_i \), such that \( y_i \to y \in \Omega \), then \( y \) is a singular point of \( u \).
2. If \( y \in \Omega \) is a singular point of \( u \), then for all sufficiently large \( i \), \( u_i \) has a singularity at a point \( y_i \) such that \( y_i \to y \).

**Proof.** (1). Choose \( \varepsilon \) according to the \( \varepsilon \)-regularity theorem (Corollary 2.1) and fix small \( r > 0 \), then for each \( i \) we have

\[
\frac{1}{r} \int_{B_r(y_i)} |
abla u_i|^2 \, dx \geq \varepsilon.
\]

Convergence \( y_i \to y \) and strong convergence \( u_i \to u \) now imply that

\[
\frac{1}{r} \int_{B_{2r}(y)} |
abla u|^2 \, dx \geq \varepsilon.
\]
for every small enough radius \( r > 0 \). On the other hand, if we suppose on the contrary that \( y \) is a regular point of \( u \), then the integral \( \int_{B_{2r}(y)} |\nabla u|^2 \, dx \) decays like \( r^3 \), thus yielding a contradiction.

(2). By classification of tangent maps, Theorem 3.6, if \( y \) is a singular point of \( u \) we know that on balls of small radius \( B_r(y) \)

\[
u \sim \pm \mathcal{R} \left( \frac{x - y}{|x - y|} \right),
\]

for an orthogonal rotation \( \mathcal{R} \) of \( \mathbb{R}^3 \). Since, the map \( \frac{x}{|x|} \in W^{1,2}(B, \mathbb{S}^2) \) cannot be approximated by \( C^\infty(B, \mathbb{S}^2) \) maps, see [23, Section 4] or [2], we infer that for \( i \geq i_0(r) \), \( u_i \) must have a singular point \( y_i \in B_r(y) \).

Applying this reasoning for a sequence \( r_i \searrow 0 \), we obtain a sequence of singularities \( y_i \in \text{sing } u_i \) converging to \( y \).

The following two results exploit the classification of singularities into \( \mathbb{S}^2 \) (Theorem 3.6) even further – here it will be important that at each singular point \( y \in \text{sing } u \) the energy density \( \Theta_u(y) \) (2.2) has the same value

\[
\Theta := \int_{B_1} \left| \nabla \frac{x}{|x|} \right|^2 \, dx = 8\pi,
\]

as noted in Corollary 3.7.

**Lemma 4.9** (Liouville theorem, [1, Thm 2.2]). Let \( u : \mathbb{R}^3 \to \mathbb{S}^2 \) be locally energy minimizing in all of \( \mathbb{R}^3 \). Then, up to a translation, \( u \) is a tangent map, i.e., \( u \) is either constant or has the form \( u(x) = \pm \mathcal{R}(\frac{x - y}{|x - y|}) \) for some \( y \in \mathbb{R}^3 \) and some rotation \( \mathcal{R} \).

**Proof.** Let us first consider the singular case, so that without loss of generality we may assume \( 0 \in \text{sing } u \). By Theorem 3.3 (uniform boundedness), the monotone quantity from Theorem 2.3 is bounded, so it has a finite limit

\[
\Theta' := \lim_{r \to \infty} r^{-1} \int_{B_r} |\nabla u|^2 \, dx.
\]

Moreover, the sequence of rescaled maps \( u_r(x) = u(rx) \) is bounded in \( W^{1,2}_{\text{loc}}(\mathbb{R}^3, \mathbb{S}^2) \), and after choosing a subsequence, we can obtain in the limit an energy minimizing limit map \( \varphi : \mathbb{R}^3 \to \mathbb{S}^2 \) (a tangent map at infinity). Note that for each \( s > 0 \) we have

\[
s^{-1} \int_{B_s} |\nabla \varphi|^2 = \lim_{r \to \infty} s^{-1} \int_{B_s} |\nabla u_r|^2 = \lim_{r \to \infty} (rs)^{-1} \int_{B_{rs}} |\nabla u|^2 = \Theta',
\]

in particular \( \varphi \) has energy density \( \Theta_{\varphi}(0) = \Theta' \). As there is only one possible energy density, we infer that \( \Theta' \) is equal to \( \Theta = \Theta_u(0) \). Monotonicity formula (Theorem 2.3) now implies that \( u \) is 0-homogeneous (i.e., a tangent map), since the monotone quantity has the same limit for \( r \to 0 \) and \( r \to \infty \).
If $u$ is smooth, most of the above discussion still applies. If $\Theta' = 0$, then in particular $\int_{B_r} |\nabla u|^2$ tends to zero as $r \to \infty$, so $u$ is constant. If $\Theta' > 0$, then the obtained map $\varphi$ has a singularity at the origin, and by Theorem 4.8 the rescaled maps $u_r$ are also singular for large $r$. Since $u$ is smooth, this yields a contradiction.

**Theorem 4.10** (Uniform distance between singular points, [1, Thm 2.1]). There is a constant $c > 0$ such that the following holds. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, and $u \in W^{1,2}(\Omega, S^2)$ a minimizing harmonic map with a singularity at $y \in \Omega$. Then there is no other singularity within distance $cD$ of $y$, where $D := \text{dist}(y, \partial \Omega)$ is its distance to the boundary.

**Proof.** Assume the claim is false. Then we can find a sequence $u_i \in W^{1,2}(\Omega_i, S^2)$ with two distinct singularities $x_i, y_i \in B_{D_i/i}(y_i)$, where $D_i = \text{dist}(y_i, \partial \Omega_i)$. For each $i$ consider the rescaled map

$$\tilde{u}_i(z) := u_i \left( y_i + \frac{x_i - y_i}{|x_i - y_i|} z \right),$$

which is a minimizing harmonic map in a large ball $B_i(0)$. This map has two singularities at $0$ and $\frac{x_i - y_i}{|x_i - y_i|}$. Using Theorem 3.3 (uniform boundedness), compactness of minimizers and of $S^2$, and a diagonal argument, we obtain an energy minimizing limit map $u: \mathbb{R}^3 \to S^2$, which is singular at least in two points $0$ and $x$ with $|x| = 1$. However, the possibility of two singularities is excluded by the Liouville theorem (Lemma 4.9).

**Corollary 4.11** (Uniform bound for singularities in the interior). Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, and $u \in W^{1,2}(\Omega, S^2)$ a minimizing harmonic map. Then for any $\sigma > 0$, the number of singularities with distance to the boundary at least $\sigma$ is bounded by a constant depending only on $\Omega$ and $\sigma$:

$$\#\{x \in \text{sing } u : \text{dist}(x, \partial \Omega) \geq \sigma\} \leq C(\Omega, \sigma).$$

5. **Boundary regularity for smooth and singular boundary data**

It is a classical result by Schoen and Uhlenbeck that minimizing harmonic maps with smooth boundary data are smooth close to the boundary [23]. In the case when $N = S^2$, this result can be strengthened to uniform boundary regularity.

One of the quite surprising results in [1] is that even possibly singular boundary data (they consider $W^{1,2}(\partial B^3, S^2)$) prevents singularities from reaching the boundary.

In this section we extend this result to larger trace spaces. In [16] we show that this holds true in a suitable sense also for dimension $n \geq 4$.

5.1. **Uniform boundary regularity for constant boundary data.** The first step is uniform boundary regularity for constant boundary data, see [1, Theorem 1.10].
Theorem 5.1 (Boundary regularity). There exists a uniform constant $\lambda > 0$ such that the following holds: Let $u \in W^{1,2}(B^3_+, S^2)$ be a minimizer and assume that $\varphi = u\big|_{T^1_1}$ is constant. Then $u$ is real analytic in the region $[0, \lambda] \times B^{2}_2$.

The main ingredient in Theorem 5.1 is the following.

Lemma 5.2. For any $\varepsilon > 0$ there is a uniform constant $R_0(\varepsilon) \in (0, \frac{1}{2})$ so that the following holds: Let $u \in W^{1,2}(B^3_+, S^2)$ be a minimizer and assume that $\varphi = u\big|_{T^1_1}$ is a constant. Then for any $x_0 \in T^1_\frac{1}{2}$

$$\sup_{r < R_0(\varepsilon)} r^{-1} \int_{B_r(x_0)} |\nabla u|^2 \, dx < \varepsilon.$$ 

Proof. Assume that the claim is false for some $\varepsilon > 0$, then we find a sequence $R_i \to 0$, $x_i \in T^1_\frac{1}{2}$, and a sequence of minimizers $u_i$ with constant boundary data $\varphi_i \in S^2$, such that

$$R_i^{-1} \int_{B^{1+}_{R_i}(x_i)} |\nabla u_i|^2 \, dx \geq \varepsilon.$$ 

By the boundary monotonicity formula, Theorem 2.4,

$$\inf_{r \geq R_i} r^{-1} \int_{B^{1+}_{r}(x_i)} |\nabla u_i|^2 \, dx \geq \varepsilon.$$ 

By Theorem 3.5 we know that maps $u_i$ are uniformly bounded and thus by strong convergence of minimizing harmonic maps, Theorem 4.7, up to taking a subsequence we find a limit minimizing harmonic map $u \in W^{1,2}(B^1_+, S^2)$ and a cluster point of $\{x_i\}$, say $x_0 \in T^1_\frac{1}{2}$ such that

$$\inf_{r \geq 0} r^{-1} \int_{B^{1+}_{r}(x_0)} |\nabla u|^2 \, dx \geq \frac{1}{2} \varepsilon. \quad (5.1)$$ 

Now take any sequence $r_i \to 0$ such that

$$u(x_0 + r_ix) \xrightarrow{i \to \infty} \tau(x) \quad \text{in } W^{1,2}(B^1_+).$$ 

By Lemma 2.6 we find that $\tau$ is constant. Thus by the strong convergence in $W^{1,2}$ to a constant we find $r > 0$ such that

$$\int_{B^{1+}_r} |\nabla (u(x_0 + rx))|^2 \, dx = r^{-1} \int_{B_r(x_0)} |\nabla u|^2 \, dx < \frac{1}{2} \varepsilon.$$ 

Contradicting (5.1).
Proof of Theorem 5.1. For any given $\epsilon > 0$ let $R_0(\epsilon)$ be the radius from Lemma 5.2 and set $\lambda := R_0(\epsilon)/2$. For $x_0 \in [0, \lambda] \times B^2_\frac{\epsilon}{2}$ denote by $x_1 \in \{0\} \times B^2_\frac{\epsilon}{2}$ the projection of $x_0$ onto $T_\frac{\epsilon}{2}$. Then for $\rho := |x_0 - x_1| < \lambda$ we have
\[ \rho^{-1} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx \lesssim (2\rho)^{-1} \int_{B_{2\rho}(x_1)} |\nabla u|^2 \, dx \leq \epsilon. \]
The last estimate is due to Lemma 5.2. This implies
\[ \lim_{r \to 0} r^{-1} \int_{B_r(x_0)} |\nabla u|^2 \, dx < C \epsilon \]
holds for every $x_0 \in (0, \lambda) \times B^2_\frac{\epsilon}{2}$.

The interior regularity now follows from $\epsilon$-regularity, namely Corollary 2.1. Unique analytic continuation allows to extend this result up to the boundary. □

5.2. Uniform boundary regularity for singular boundary data. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded set. It is a classical fact that for any map $\varphi \in W^{1,2}(\partial \Omega, \mathbb{R}^N)$ its harmonic extension
\[ \begin{cases} \Delta u = 0 \quad \text{in } \Omega \\ u = \varphi \quad \text{on } \partial \Omega. \end{cases} \]
i.e., any minimizer of the Dirichlet energy $E(u)$ has no singularities (it is analytic) in $\Omega$, even if $\varphi$ is non-smooth on the boundary. This remains true, when $\mathbb{R}^N$ is replaced by a negatively curved manifold, but it fails for $n \geq 3$ when $\mathbb{R}^N$ is replaced by a positively curved manifold such as the sphere $\mathbb{S}^{N-1}$, $x/|x|$ being the classical counterexample. However, the singularities do not approach the boundary, they come from the interior: That is, for boundary data in certain trace spaces one has a regularity theory close to the boundary, even if those trace spaces are not subsets of the continuous functions. This was established for $C^{2,\alpha}$ boundary data by Schoen and Uhlenbeck [23]. In the special case $n = 3$, $\mathcal{N} = \mathbb{S}^2$ a stronger result of uniform boundary regularity was obtained by Hardt and Lin for $W^{1,\infty}$ boundary data [11, Lemma 2.1] and finally by Almgren and Lieb [1, Corollary 2.5] for $W^{1,2}$-boundary data (which in general are not continuous).

Theorem 5.3 (Uniform boundary regularity for singular boundary data). Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. For $s \in (\frac{1}{2}, 1)$, $p \in [2, \infty)$ and $sp > 1$ there are constants $R = R(\Omega)$ and $\epsilon = \epsilon(\Omega)$ such that the following holds. Take any minimizing harmonic map $u : \Omega \to \mathbb{S}^2$ and denote the trace of $u$ on $\partial \Omega$ by $\varphi \in W^{1,2}(\partial \Omega, \mathbb{S}^2)$.

If for some $x_0 \in \partial \Omega$ and some $\rho_0 < R$ we have the estimate
\[ \Lambda := \sup_{B_\rho(y) \subset B_{\rho_0}(x_0)} \rho_0^{sp-2} \| \varphi \|_{W^{s,p}(T_1 \cap B_\rho(y))} \leq \epsilon \]
then $u$ is smooth in $B_{\rho/2}(x_0) \cap \Omega$. 
Theorem 5.3 is a generalization of the results of Almgren and Lieb \cite[Corollary 2.5]{1}, who assumed an $W^{1,2}$-estimate on $\varphi$, and by Hardt and Lin \cite[Lemma 2.1]{11} who proved an analog for Lipschitz-maps $\varphi$.

For simplicity we assume that $\Omega$ is the half-ball $B_1^+$. Then Theorem 5.3 is the consequence of the following Proposition.

**Proposition 5.4.** For $s \in (\frac{1}{2}, 1]$, $p \in [2, \infty)$ there exist uniform constants $R_0$ and $\varepsilon$ such that the following holds. Take any minimizing harmonic map $u : B_1^+ \to S^2$ and denote the trace of $u$ on $T_1$ by $\varphi = u \big|_{T_1}$.

If for some $\rho_0 < R_0$ we have the estimate

$$\rho_0^{sp-2} [\varphi]_{W^{s,p}(T_1 \cap B_{\rho_0})} \leq \varepsilon,$$

then $u$ is smooth in

$$B_{\lambda \rho_0} \cap \{x_3 > \lambda \rho_0 / 2\}$$

where $\lambda$ is from Theorem 5.1.

**Proof.** Assume the claim is false. Then we find a sequence $\rho_k \to 0$, a sequence of minimizing harmonic maps $u_k \in W^{1,2}(B_1^+, S^2)$ with trace $\varphi_k = u_k \big|_{T_1}$ satisfying

$$\rho_k^{sp-2} [\varphi_k]_{W^{s,p}(T_1 \cap B_{\rho_k})} \leq \frac{1}{k},$$

however there is a singularity

$$y_k \in B_{\rho_k \lambda} \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > \rho_k \lambda / 4\}.$$

We rescale, setting $v_k(x) := u_k(\rho_k x)$, $\psi_k(x) := \varphi_k(\rho_k x)$, and find $v_k \in W^{1,2}(B_1^+, S^2)$ is a minimizing harmonic map with trace $\psi_k$ on $T_1$ satisfying

$$[\psi_k]_{W^{s,p}(T_1)} \leq \frac{1}{k}.$$

Moreover, $v_k$ has a singularity

$$z_k = \frac{1}{\rho_k} y_k \in B_{\lambda} \cap \{x \in \mathbb{R}^3 : x_3 \geq \frac{\lambda}{4}\}.$$

By strong convergence of minimizers, Theorem 4.7, and convergence of singularities, Theorem 4.8, up to taking a subsequence, we find in the limit a minimizing harmonic map $v \in W^{1,2}(B_1^+, S^2)$ which in view of (5.2) is constant on $T_1$, but has a singularity

$$z = \lim_{k \to \infty} \frac{1}{\rho_k} y_k \in B_{\lambda} \cap \{x \in \mathbb{R}^3 : x_3 \geq \frac{\lambda}{2}\}.$$

This contradicts Theorem 5.1. \qed

By a covering argument, we obtain in particular the following regularity up to the boundary (but of course not including the boundary).
Corollary 5.5. There exist uniform constants $R_0$ and $\varepsilon$ such that the following holds. Take any minimizing harmonic map $u : B^+_2 \to S^2$ and denote the trace of $u$ on $T_2$ by $\varphi \in W^{2,2}(T_2, S^2)$.

If for some $x_0 \in T_1$ and some $\rho_0 < R_0$ we have the estimate

$$\sup_{B_{\rho}(y) \subset B_{\rho_0}(x_0)} \rho^{sp-2}[\varphi]^p_{W^{s,p}(T_1 \cap B_\rho(y))} \leq \varepsilon$$

then $u$ is smooth in $B_{\frac{\lambda}{2}\rho_0}(x_0) \cap B^+_2$, where $\lambda$ is as in Theorem 5.1.

The proof of Theorem 5.3 follows now from Corollary 5.5 by a blowup argument.

As a corollary from Theorem 5.3 we also obtain that there are only finitely many singularities of a minimizing harmonic map even if the boundary is nonsmooth.

Corollary 5.6. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Let $s \in (\frac{1}{2}, 1]$, $p \in [2, \infty)$. Then there exist constants $R = R(\Omega)$ and $\varepsilon = \varepsilon(\Omega)$ such that the following holds.

Take any minimizing harmonic map $u : \Omega \to S^2$ and denote the trace of $u$ on $\partial\Omega$ by $\varphi \in W^{2,2}(\partial\Omega, S^2)$.

If there exists $\rho_0 < R$ so that for all $x_0 \in \partial\Omega$ we have the estimate

$$\sup_{B_{\rho}(y) \subset B_{\rho_0}(x_0)} \rho^{sp-2}[\varphi]^p_{W^{s,p}(T_1 \cap B_\rho(y))}$$

then $u$ has only finitely many singularities in $\Omega$.

Proof. By Theorem 5.3 there are no singular points close to the boundary. By Theorem 4.10 between each two singularities there is a distance proportional to the distance to the boundary. Since $\Omega$ is bounded this implies that there are at most finitely many singularities.

Later, in Theorem 8.1 we obtain a more precise bound on the number of singularities in terms of the boundary data.

6. Refined boundary energy estimates and “hot spots”

The uniform boundedness of minimizing harmonic maps in the interior and at the boundary, Theorem 3.3 and Theorem 3.5, together with a covering argument imply the following theorem

Theorem 6.1. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. There exists a number $r_0 = r_0(\Omega) > 0$, with the following property. Let $N > 0$ be a natural number. Suppose $K$ is a collection of $N$ points on the boundary $\partial\Omega$ and $A(r, s) := \{x \in \mathbb{R}^3 : r <$
We obtain (6.1) \( r^{-1} \int_{\Omega \cap A(r/2)} |\nabla u|^2 \, dx \leq CN + Cr^{sp-2} \int_{A(\frac{r}{2}, \frac{r}{2}) \cap \partial \Omega} \int_{A(\frac{r}{2}, \frac{r}{2}) \cap \partial \Omega} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{2+sp}} \, dx \, dy \),

where \( C = C(s, p) \) is a constant independent of \( \Omega, K, u, \) and \( \varphi \).

**Remark 6.2.** This theorem extends [1, Theorem 2.3 (v)]. There, instead of the term

\[ C r^{sp-2} \int_{A(\frac{r}{2}, \frac{r}{2}) \cap \partial \Omega} \int_{A(\frac{r}{2}, \frac{r}{2}) \cap \partial \Omega} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{2+sp}} \, dx \, dy \]

they obtained an estimate in terms of

\[ Cr \int_{A(\frac{r}{2}, \frac{r}{2}) \cap \partial \Omega} |\nabla T \varphi|^2 \, d\mathcal{H}^2. \]

which follows from Theorem 6.1 by Poincaré inequality.

**Proof.** Since \( \Omega \) is bounded and smooth, we may choose \( r_0 \) so small, such that the boundary \( \partial \Omega \cap B_{r_0}(x) \) is almost uniformly flat for all \( x \in \partial \Omega \). Then, there is a uniform combinatorial number \( M_1 \) such that the for every \( r \in (0, r_0) \) we find \( (x_i)_{i=1}^{M_1} \) with the following properties

\[
\bigcup_{i=1}^{M_1} B_{r/4}(x_i) \cap \partial \Omega \supset A(r, 2r) \cap \partial \Omega.
\]

and

\[
dist \left( \partial \Omega, A(r, 2r) \backslash \bigcup_{i=1}^{M_1} B_{r/4}(x_i) \right) \geq \frac{r}{8}.
\]

Also, there exists a uniform number \( M_2 \) such that we always find \( (y_j)_{j=1}^{M_2} \) so that \( B_{r/8}(y_j) \subset \Omega \) and

\[
\Omega \supset \bigcup_{j=1}^{M_2} B_{r/8}(y_j) \supset \Omega \cap A(r, 2r) \backslash B_{r/8}(\partial \Omega).
\]

One checks this first for \( N = 1 \), and then observes that for \( N \geq 2 \) the worst-case situation of \( A(r, 2r) \) are disjoint annuli.

From Theorem 3.5 we obtain

\[
\int_{B_{r/4}(x_i) \cap \Omega} |\nabla u|^2 \, dx \lesssim r + r^{sp-1} \int_{B_{r/2}(x_i) \cap \partial \Omega} \int_{B_{r/2}(x_i) \cap \partial \Omega} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{2+sp}} \, dx \, dy
\]

Summing over all the \( x_i \) we obtain

(6.2) \[
\int_{A(r, 2r) \cap \Omega \cap B_{r/8}(\partial \Omega)} |\nabla u|^2 \, dx \lesssim M_1 r + M_1 r^{sp-1} \int_{A(\frac{r}{2}, \frac{r}{2}) \cap \partial \Omega} \int_{A(\frac{r}{2}, \frac{r}{2}) \cap \partial \Omega} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{2+sp}} \, dx \, dy.
\]
From Theorem 3.3 we obtain
\[
\int_{B_{r/8}(y_j)} |\nabla u|^2 \, dx \lesssim \frac{r}{r - \frac{r}{6}} \approx r.
\]
and thus summing over \(j\),
\[
(6.3) \quad \int_{A(r,2r) \cap \Omega \setminus B_{r/8}(\partial \Omega)} |\nabla u|^2 \, dx \lesssim M_2 r.
\]
Together (6.2) and (6.3) give the claim. \(\square\)

**Theorem 6.3** (regularity away from “hot spots”). Let \(s > \frac{1}{2}\), \(p \in [2, \infty)\), \(sp > 1\). Let \(N > 0\) be a natural number. Then there exists an \(\varepsilon > 0\) with the following property: Suppose \(W := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1 \text{ and } 0 \leq z \leq 1\}\) and let \(V\) be the base of \(W\)
\[V := W \cap \{(x,y,z) \in \mathbb{R}^3 : z = 0\}.\]
Suppose \(u \in W^{1,2}(W,S^2)\) is a minimizer with a boundary map \(\varphi\), about which we know a priori only that it satisfies the following bound on the energy: There are balls \(B_1, \ldots, B_N\) of radius \(\varepsilon_N\) such that
\[
\int_{V \setminus (B_1 \cup \ldots \cup B_N)} \int_{V \setminus (B_1 \cup \ldots \cup B_N)} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{2+sp}} \, dx \, dy < \varepsilon_N.
\]
Then \(u\) is real analytic in the smaller cylinder separated from the base \(V\)
\[\widetilde{W} := \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq \left(\frac{1}{2}\right)^2 \text{ and } \delta \leq z \leq 4\delta\right\}.
\]
Here \(\delta = \frac{\varepsilon}{4}\), for \(\varepsilon\) from Theorem 5.1.

**Proof.** For simplicity suppose that \(N = 1\), \(\varepsilon_N = \varepsilon_1 = \varepsilon\). Thus we have only one ball \(B_1 = B_\varepsilon(p)\), for a point \(p \in V\). The case of general \(N\) easily follows.

Again, we argue by contradiction. Assume that \(u_i : W \to S^2\) is a sequence of minimizers with boundary maps \(\varphi_i\) such that
\[
\left(\frac{1}{i}\right)^{sp-2} \int_{V \setminus B_{\varepsilon+i}(p_i)} \int_{V \setminus B_{\varepsilon+i}(p_i)} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{2+sp}} \, dx \, dy < \frac{1}{i},
\]
however assume that, contrary to the claim, \(u_i\) have a point singularity \(y_i \in \widetilde{W}\) for all \(i \in \mathbb{N}\).

By Theorem 6.1, for large enough \(i\) and for any \(r \geq \frac{2}{i}\)
\[
r^{-1} \int_{W \cap A_i(r,2r)} |\nabla u_i|^2 \, dx \leq C,
\]
where $A_i$ denotes the annular region around $p_i$. Summing up over a geometric sequence of radii, we obtain
\[
\int_{W \cap A_{i(2/3,1/2)}} |\nabla u_i|^2 \, dx \leq C.
\]
By a diagonal argument together with Theorem 4.7, we can assume that $p_i \to p_0$ and $u_i \to u$ in $W_{1,2}^{1,2}(W \cap B_{1/2}(p_0) \setminus B_{\varepsilon}(p_0))$ for any $\varepsilon > 0$. Moreover,
\[
\int_{W \cap B_{1/2}(p_0)} |\nabla u|^2 \, dx \leq C.
\]
We have that $u|_V$ is constant, and $u$ is a (locally) minimizing harmonic map away from $p_0$. By Theorem 4.8, the sequence $y_i$ can be assumed to converge to a singular point $y \in \tilde{W}$ of $u$.

On the other hand, in view of Lemma 6.4 below, the singularity $p_0$ is removable, and so $u$ is a minimizing harmonic map in $W \cap B_{r_0/2}(p_0)$. Since $u$ is constant on $T$, Theorem 5.1 contradicts the possibility of a singular point. $\square$

To complete the proof of Theorem 6.3, we need the following removability lemma.

**Lemma 6.4** (Removability of points for minimizing harmonic maps). Let $u \in W^{1,2}(B^+_1, S^2)$ be a minimizer away from the origin, i.e., assume that for any $\delta > 0$ and any $v \in W_{1,2}^{1,2}(B^+_1, S^2)$ satisfying $v = u$ on $\partial B^+_1$ and $v \equiv u$ on $B^+_\delta$, we have
\[
(6.4) \quad \int_{B^+_1 \setminus B^+_\delta} |\nabla u|^2 \, dx \leq \int_{B^+_1 \setminus B^+_\delta} |\nabla v|^2 \, dx.
\]
Then, $u$ is a minimizing harmonic map in all of $B^+_1$.

**Proof.** Let $w \in W^{1,2}(B^+_1, S^2)$ with $u \equiv w$ on $\partial B^+_1$. We need to show that
\[
(6.5) \quad \int_{B^+_1} |\nabla u|^2 \, dx \leq \int_{B^+_1} |\nabla w|^2 \, dx.
\]
For $\delta > 0$ let $\eta_\delta \in C_0^\infty(B_{2\delta})$ be the typical cutoff function, $\eta_\delta \equiv 1$ in $B_\delta$ and $|\nabla \eta_\delta| \lesssim \delta$.

We set $\tilde{w}_\delta \in W^{1,2}(B^+_1, \mathbb{R}^3)$ by
\[
\tilde{w}_\delta := (1 - \eta_\delta)w + \eta_\delta u,
\]
which satisfies $\tilde{w}_\delta = u$ on $\partial B^+_1$, $\tilde{w}_\delta \equiv u$ in $B^+_\delta$ and $\tilde{w}_\delta \equiv w$ in $B^+_1 \setminus B_{2\delta}$. By the extension property, Theorem 3.1, applied in $B_{2\delta}^+ \setminus B_\delta$ we find $w_\delta \in W^{1,2}(B^+_1, S^2)$ such that
\[
w_\delta = \begin{cases} 
  u & \text{in } B^+_\delta \\
  w & \text{in } B^+_1 \setminus B_{2\delta} \\
  u & \text{on } \partial B_1
\end{cases}
\]
and
\[ \int_{B_{1/2}^+ \setminus B} |\nabla w_\delta|^2 \, dx \lesssim \int_{B_{1/2}^+ \setminus B} |\nabla \tilde{w}_\delta|^2 \, dx. \]

In particular, \( \tilde{w}_\delta \) is a competitor in the sense of (6.4), and we have
\[
\int_{B_{1/2}^+ \setminus B} |\nabla u|^2 \, dx \leq \int_{B_{1/2}^+ \setminus B} |\nabla w_\delta|^2 \, dx
= \int_{B_{1/2}^+ \setminus B} |\nabla w_\delta|^2 \, dx + \int_{B_{1/2}^+ \setminus B} |\nabla \tilde{w}_\delta|^2 \, dx
\leq \int_{B_{1/2}^+ \setminus B} |\nabla w|^2 \, dx + C \int_{B_{1/2}^+ \setminus B} |\nabla \tilde{w}_\delta|^2 \, dx.
\]

Since \( u, w \in W^{1,2}(B_1^+) \) using the absolute continuity of the integral we find that
\[
(6.6) \quad \int_{B_{1/2}^+} |\nabla u|^2 \, dx \leq \int_{B_{1/2}^+} |\nabla w|^2 \, dx + C \liminf_{\delta \to 0} \int_{B_{1/2}^+} |\nabla \tilde{w}_\delta|^2 \, dx.
\]

Now
\[
\int_{B_{1/2}^+} |\nabla \tilde{w}_\delta|^2 \, dx \lesssim \frac{1}{\delta^2} \int_{B_{1/2}^+} |u - v|^2 \, dx + \int_{B_{1/2}^+} |\nabla u|^2 \, dx + \int_{B_{1/2}^+} |\nabla v|^2 \, dx.
\]

Observe that since we are in dimension \( n = 3 \) and \( S^2 \) is compact,
\[
\frac{1}{\delta^2} \int_{B_{1/2}^+} |u - v|^2 \, dx \lesssim \delta.
\]

Thus, using again the absolute continuity of the integral and that \( u, w \in W^{1,2} \) we find
\[
\lim_{\delta \to 0} \int_{B_{1/2}^+} |\nabla \tilde{w}_\delta|^2 \, dx = 0.
\]

Plugging this into (6.6) we conclude. \( \square \)

We note the following corollary of Theorem 6.3, which is essentially just a useful reformulation. It extends [1, Cor. 2.7] to our larger class of trace spaces. To transfer between flat boundary case and generic boundary, one rescales small balls at the boundary to the unit size and employs an additional contradiction argument.

**Corollary 6.5.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded open set with smooth boundary, \( s \in (\frac{1}{2}, 1] \) and \( p \in [2, \infty) \). There exists a \( \sigma = \sigma(\Omega) > 0 \), a small constant \( \varepsilon > 0 \), and two scales \( \Lambda_1, \Lambda_2 > 0 \) so that the following holds.

If \( u \in W^{1,2}(\Omega, S^2) \) is a minimizing harmonic map with trace \( \varphi := u \big|_{\partial \Omega} \), a singular point \( p \in \text{sing} \, u \) with \( r := \text{dist} (p, \partial \Omega) < \sigma \) and \( B \subset \mathbb{R}^3 \) is an arbitrary ball with radius \( \Lambda_2 r \), then
\[
r^{sp-2} \int_{\partial \Omega \cap (B_{\Lambda_1 r}(p) \setminus B)} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{2 + sp}} \, dx \, dy \geq \varepsilon.
\]
7. Hardt and Lin’s stability theorem

By an adaption of the arguments in Hardt and Lin’s [11] one can show that for a unique harmonic minimizer $v \in W^{1,2}(\Omega, S^2)$ the number of singularities stays the same for all minimizers whose trace is close in the right $W^{s,p}$-norm, for $s \in (\frac{1}{2}, 1]$ and $p \in [2, \infty)$ with $sp \geq 2$.

The counter-example by Strzelecki and the first-named author showed in [17] implies that this does not hold for $W^{s,p}$ with $sp < 2$. In this sense the following Theorem 7.1 is the sharp limit case.

Let us remark that recently and independently Li [13] obtained a similar theorem for $W^{1,2}$-boundary data (i.e. our $s = 1$, $p = 2$-case). So while the theorem below is more general than [13] in terms of the trace spaces, let us stress that his theorem obtains much better estimates than (7.3), which is easy and follows from strong convergence, see Proposition 7.2. He obtains $C^\alpha$-estimates of the distance $u - v$ away from the boundary (modulo bi-Lipschitz homeomorphisms).

Let us also remark that one can extend Theorem 7.1 suitably to dimensions $n \geq 4$, and this will be part of [16].

**Theorem 7.1.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. For $s \in (\frac{1}{2}, 1]$, $p \in [2, \infty)$ there exists $R = R(\Omega)$ and $\gamma = \gamma(\Omega) > 0$ such that the following holds.

Assume that $v \in W^{1,2}(\Omega, S^2)$ is the unique minimizing harmonic map with boundary $v|_{\partial \Omega} \psi$. Then for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, \Omega, \psi) > 0$ such that whenever $u$ is a minimizing harmonic map $u \in W^{1,2}(\Omega, S^2)$ such that its trace $\varphi := u|_{\partial \Omega}$ satisfies

$$\sup_{B_\rho(y) \subset B_{\rho_0}(x_0)} \rho^{sp-2} [v|_{W^{s,p}(\partial \Omega \cap B_\rho(y))}]^p < \gamma$$

and $\varphi$ is close to $\psi$,

$$[\psi - \varphi]_{W^{s,p}(\partial \Omega)} \leq \delta$$

then $u$ has the same number of singularities as $v$. Moreover

$$\|u - v\|_{W^{1,2}(\Omega)} \leq \varepsilon.$$  

The statement (7.3) actually follows from weaker assumptions already,

**Proposition 7.2.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, and $s \in (\frac{1}{2}, 1]$, $p \in [2, \infty)$. Assume that $v \in W^{1,2}(\Omega, S^2)$ is the unique minimizing harmonic map with boundary $v|_{\partial \Omega} \psi$. Then for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, \Omega, \psi) > 0$ such that whenever $u$ is a minimizing harmonic map $u \in W^{1,2}(\Omega, S^2)$ such that its trace $\varphi := u|_{\partial \Omega}$ satisfies

$$[\psi - \varphi]_{W^{s,p}(\partial \Omega)} \leq \delta$$
then
\[ \|u - v\|_{W^{1,2}(\Omega)} \leq \varepsilon. \]

**Proof.** Assume the claim is false for a given unique minimizer \( v \) and for some \( \varepsilon > 0 \). Then we find a sequence of minimizers \( u_i \) with trace \( \varphi_i \) which satisfy
\[ [u_i - \varphi]_{W^{s,p}(\partial \Omega)} \leq \frac{1}{i} \]
but
\[ \|u_i - v\|_{W^{1,2}(\Omega)} > \varepsilon. \]
Now we obtain a contradiction since by strong convergence of minimizers, Theorem 4.7, converge to the unique minimizer of \( v \) in \( W^{1,2} \). In particular the last displayed inequality cannot be true for all \( i \in \mathbb{N} \). \( \square \)

**Proof of Theorem 7.1.** Statement (7.3) follows from Proposition 7.2. Regarding the singularities, take \( R \) and \( \sigma = \varepsilon \) from Corollary 5.6. Assume the theorem is false for a unique minimizer \( v \) with \( N < \infty \) singularities \( x_1, \ldots, x_N \).

Then we find a sequence \( u_i \in W^{1,2}(\Omega, \mathbb{S}^2) \) of minimizing harmonic maps with traces \( \varphi_i := u_i|_{\partial \Omega} \) such that

- \( \varphi_i \) satisfy (7.1),
- either all \( u_i \) have \( M < N \) singularities \( (y_i,k)_{k=1}^M \) or all \( u_i \) have at least \( N + 1 \) singularities \( (y_i,k)_{k=1}^{N+1} \)
- By (7.1) we can assume in particular
  \[ [\varphi_i - \psi]^p_{W^{s,p}(\partial \Omega \cap B_\rho(y))} < \frac{1}{i} \]
- We may assume w.l.o.g. from the strong convergence of minimizing harmonic maps, Theorem 4.7 and Theorem 4.6, and the uniqueness of \( v \), that
  \[ u_i \to v \quad \text{in } W^{1,2}(\Omega, \mathbb{S}^2). \]
If \( u_i \) had \( M < N \) singularities we find a contradiction to Theorem 4.8 (2), since all the singularities of \( v \) have to come as limits of singularities of \( u_i \).

So we may assume that each \( u_i \) has at least \( M > N \) singularities. Since singularities of \( u_i \) which do not approach the boundary \( \partial \Omega \) converge to singularities of \( v \) Theorem 4.8, and two different singularities of \( u_i \) cannot converge to the same singularity of \( v \) by uniform distance (proportional to the distance of the boundary) of singularities, Theorem 4.10, the only way this is possible is if singularities of \( u_i \) approach the boundary \( \partial \Omega \).

However, we can rule this out simply by the assumption (7.1) and uniform boundary regularity, Theorem 5.3 since \( \rho_0 \) in (7.1) is fixed. \( \square \)
Remark 7.3. We end this section with two remarks. First, the uniqueness assumption in Theorem 7.1 was necessary. It is known that there are “unstable” boundary maps $\varphi : S^2 \to S^2$, i.e., having at least two minimizers with different number of singularities (see [11, Section 5], see also [17, Section 4]). Second, we would like to note that boundary maps with unique minimizers are dense in $W^{1,2}(\partial \Omega)$, as observed by Almgren and Lieb in [1, Theorem 4.1 (1)].

8. Almgren and Lieb’s linear law: size of the singular set

Finally we obtain as one of our main results, a sharper (in the sense of trace spaces) version of Almgren and Lieb’s linear law for $W^{1,2}$-traces, [1, Theorem 2.12].

**Theorem 8.1** (Almgren and Lieb’s linear law for trace spaces). Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, $s \in (\frac{1}{2}, 1]$, $p \in (1, \infty)$ and $sp = 2$. Then there exist constants $\sigma = \sigma(\Omega) > 0$ and $C = C(\Omega, s, p) > 0$ such that if $u \in W^{1,2}(\Omega, S^2)$ is a minimizing harmonic map with trace $\varphi := u_{\mid \partial \Omega}$, then

\[(8.1) \quad \# \{\text{singularities of } u \} \leq C \int_{\Omega} \int_{\Omega} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{2+sp}} \, dx \, dy.\]

**Remark 8.2.** By the Gagliardo-Nirenberg inequality we have for $sp = 2$ and $s < 1$

\[\int_{\Omega} \int_{\Omega} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{2+sp}} \, dx \, dy \leq \|\varphi\|_{L^p(\partial \Omega)}^{p-2} \|\nabla \varphi\|_{L^2(\partial \Omega)}^2,\]

thus we obtain in particular Almgren and Lieb’s linear law from [1, Theorem 2.12].

Also, in view of our arguments it seems possible to improve Theorem 8.1 to Morrey-trace spaces and $1 < sp \leq 2$. We will make no effort to do this here, and leave this to the interested reader.

The proof of Theorem 8.1 consists of two parts: estimates close to the boundary, where the hot spots come into play, and away from the boundary, where the uniform distance between the singularities is enough to conclude, see Corollary 4.11. Cf. [1, Theorem 2.11]

**Theorem 8.3** (Singularities at the boundary). Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, $s \in (\frac{1}{2}, 1]$, $p \in (1, \infty)$ and $sp = 2$. Then there exist constants $\sigma = \sigma(\Omega) > 0$ and $C = C(\Omega, s, p) > 0$ such that if $u \in W^{1,2}(\Omega, S^2)$ is a minimizing harmonic map with trace $\varphi := u_{\mid \partial \Omega}$, and $N$ denotes the number of singularities of $u$ in $\Omega$ within $\sigma$ distance of the boundary $\partial \Omega$, then

\[N \leq C \int_{\partial \Omega} \int_{\partial \Omega} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{2+sp}} \, dx \, dy.\]
A crucial technical ingredient to the above linear law is a suitable covering lemma, which we here cite from [1, Theorem 2.8, 2.9].

**Theorem 8.4** (Covering lemma). Let $\mathcal{B}$ be a family of closed balls in $\mathbb{R}^n$, $\mu$ be a Borel measure over $\mathbb{R}^n$, and let $\tau, \omega \in (0,1)$. Moreover, assume that the following two hypotheses hold:

1. For any two different $B_r(p), B_s(q) \in \mathcal{B}$ we have
   \[ |p - q| \geq \omega \min(r, s). \]
2. Suppose that $B_r(p) \in \mathcal{B}$ and $q \in \mathbb{R}^n$ is an arbitrary point, then
   \[ \mu(B_r(p) \setminus B_{\tau r}(q)) \geq 1. \]

Then
\[ \# \text{balls in } \mathcal{B} \leq C(\mu(\mathbb{R}^n)), \]
for a constant $C(\omega, \tau, n) > 0$.

**Proof of Theorem 8.3.** Fix $\Omega, u$ and $\varphi$ as in the theorem. As in [1, Proof of Theorem 2.11] we aim to apply the covering lemma (Theorem 8.4).

We define the Borel measure $\mu$ as follows. For open $U \subset \mathbb{R}^3$ we define for $s \in (0, 1)$ and $p \in (2, \infty)$ such that $sp = 2$ and for some $\nu > 0$ chosen below
\[ \mu(U) := \nu \int_{U \cap \partial \Omega} \int_{\partial \Omega} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{2+sp}} \, dx \, dy. \]

Obviously, $\mu$ is a measure, and we have
\[ \mu(U) \geq \nu \int_{U \cap \partial \Omega} \int_{U \cap \partial \Omega} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{2+sp}} \, dx \, dy. \]

By $\mathcal{B}$ we denote the collection of all balls $B_{\Lambda_1 D(x_0)}(x_0)$ centered at a singularity $x_0 \in \Omega$ of $u$ with $D(x_0) := \text{dist}(x_0, \partial \Omega) < \sigma$, where $\sigma$ and $\Lambda_1$ will be chosen below.

The statement follows once we confirm Properties (1) and (2) from Theorem 8.4.

Property (1) follows from the uniform distance between singularities, Theorem 4.10: given any two singularities $x_0$ and $x_1$ of $u$ we have by Theorem 4.10
\[ |x_0 - x_1| \geq C \min\{D(x_0), D(y_0)\}. \]

Regarding Property (2) we use Corollary 6.5: let $x_0 \in \Omega$ be a singularity of $u$ with $\text{dist}(x_0, \partial \Omega) < \sigma$. Then if $\Lambda_1$ is chosen as in Corollary 6.5, and $\tau := \frac{\Lambda_2}{\Lambda_1}$, where $\Lambda_2$ is from Corollary 6.5, then we have
\[ \mu\left( B_{\Lambda_1 D(x_0)}(x_0) \setminus B_{\tau \Lambda_1 D(x_0)}(q) \right) \geq \nu \varepsilon, \]
where $\varepsilon$ is also taken from Corollary 6.5. Choosing $\nu := \frac{1}{\varepsilon}$ Property (2) is established. \qed
To complete the proof of Theorem 8.1 we need one more ingredient: that small boundary data in the right norm on all of \( \partial \Omega \) means that no singularities can appear. It follows from strong convergence of minimizing harmonic maps.

**Lemma 8.5.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary, \( s \in (\frac{1}{2}, 1] \) and \( p \in [2, \infty) \) such that \( sp = 2 \). There exists \( \varepsilon = \varepsilon(\Omega, s, p) > 0 \) such that any minimizing harmonic map \( u \in W^{1,2}(\Omega, \mathbb{S}^2) \) with trace \( \varphi := u \big|_{\partial \Omega} \) is smooth in \( \Omega \) if only
\[
[\varphi]_{W^{s,p}(\partial \Omega)} \leq \varepsilon.
\]

**Proof.** Assume that the claim was false, then we find minimizing harmonic maps \( u_i \) with singularities \( \Omega \ni x_i \xrightarrow{i \to \infty} x_0 \in \overline{\Omega} \) so that the respective traces \( \varphi_i := u_i \big|_{\partial \Omega} \) satisfy
\[
[\varphi_i]_{W^{s,p}(\partial \Omega)} \xrightarrow{i \to \infty} 0.
\]
This and the strong convergence of minimizers, Theorem 4.7, implies that up to a subsequence \( u_i \to u \) in \( W^{1,2}(\Omega) \) where \( u \) is a minimizing harmonic map with constant trace, thus constant and in particular smooth.

Thus, in view of Theorem 4.8, the singularities \( x_i \) converge to \( x_0 \in \partial \Omega \). But this contradicts the uniform boundary regularity for singular boundary data, Theorem 5.3 which by the assumptions on \( \varphi_i \) implies that for all large enough \( i \),
\[
\text{dist} (x_i, \partial \Omega) > \rho_0
\]
for some \( \rho_0 = \rho_0(\Omega) \). \( \square \)

**Proof of Theorem 8.1.** Let \( u \) be a minimizing harmonic map in \( \Omega \) with trace \( \varphi := u \big|_{\partial \Omega} \). We have two cases. If
\[
\int_{\partial \Omega} \int_{\partial \Omega} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{2+sp}} \, dx \, dy \leq \varepsilon
\]
for \( \varepsilon = \varepsilon(\Omega) \) from Lemma 8.5, then \( u \) has no singularities, so the claim is proven.

If on the other hand,
\[
\int_{\partial \Omega} \int_{\partial \Omega} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{2+sp}} \, dx \, dy > \varepsilon
\]
then combining Theorem 8.3 and Corollary 4.11 we obtain that the number \( N \) of singularities of \( u \) can be bounded by
\[
N \leq C(\Omega) + C(\Omega) \int_{\partial \Omega} \int_{\partial \Omega} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{2+sp}} \, dx \, dy
\]
\[\leq \left( \frac{C(\Omega)}{\varepsilon} + C(\Omega) \right) \int_{\partial \Omega} \int_{\partial \Omega} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{2+sp}} \, dx \, dy. \]
This proves Theorem 8.1. \( \square \)
8.1. Sharpness of the linear law. We present an example that shows that the linear law we obtain is sharp. The argument is based on the fact that a weaker norm is not scaling invariant. In this example we show that there is no continuity (at 0) for boundary energies $[\varphi]_{W^{s,p}(\partial \Omega)}$ if $sp < 2$. In particular there cannot be any linear law (or a similar result, e.g., a power law) for these spaces.

**Lemma 8.6.** Let $u \in W^{1,2}(B_1, S^2)$ be a minimizer with trace $\varphi$. Assume that the following holds for $s > 0$, $p \geq 1$: For every $\varepsilon > 0$ there exist a $\delta$ such that

$$(8.2) \quad [\varphi]_{W^{s,p}(\partial B_1)} \leq \delta$$

implies

$$(8.3) \quad \# \{ \text{singularities of } u \} \leq \varepsilon.$$  

Then $sp \geq 2$.

This lemma follows from the example below.

**Example 8.7.** Let $\varphi_0 : S^2 \to S^2$ be the identity map. The corresponding minimizer $u_0$ has one singularity.

Now we rescale. For $\lambda \in (0, 1)$ define $\varphi_\lambda$ as follows. Identify $S^2$ with $\mathbb{R}^2$ via the stereographic projection, and for $\tilde{\varphi}$ the map on $\mathbb{R}^2$ set $\tilde{\varphi}_\lambda := \tilde{\varphi}(\lambda^{-1} \cdot)$. Now call $\varphi_\lambda$ the corresponding map from $S^2 \to S^2$ (which essentially concentrates at the northpole as $\lambda \to 0$). Scaling gives

$$[\varphi_\lambda]_{W^{s,p}(S^2)} \lesssim \lambda^{2-sp}[\varphi]_{W^{s,p}(S^2)} \xrightarrow{\lambda \to 0} 0.$$  

However, $\varphi_\lambda$ is topologically equivalent to $\varphi$, hence we keep having one singularity. That is, (8.2) is satisfied for small enough $\lambda$, but (8.3) cannot be satisfied.

It is also possible to construct infinitely many singularities (and suitable generalization for $n \geq 4$) with small $W^{s,p}$-energy at the boundary if $sp < 2$, see [16].

**Example 8.8.** Let $\varepsilon > 0$ be any positive number. There is a boundary map $\varphi \in W^{1,2-\varepsilon}(\partial B, S^2)$ $u : B \to S^2$ with $u|_{\partial B} = \varphi$ at the boundary, which has countably infinitely many singularities near the boundary.

### Appendix A. Trace theorems

**A.1. A trace theorem.** In this section we review the trace theorems used throughout the paper. Here we present the results for domains in $\mathbb{R}^n$ for $n \geq 3$.

**Lemma A.1.** Let $B_1^+ \subset \mathbb{R}^n$, $n \geq 3$, and $u \in W^{1,2}(B_1^+)$ be a solution to

\[
\begin{cases}
\Delta u = 0 & \text{in } B_1^+,
\quad u = 0 & \text{on } S^+,
\quad u = \psi & \text{on } T_1.
\end{cases}
\]
Then for any \( s > \frac{1}{2} \),
\[
\| \nabla u \|_{L^2(B_1^+)} \lesssim [\psi]_{W^{s,2}(T_1)}.
\]

Proof. By trace theorem and Poincaré inequality we have
\[
\int_{B_1^+} |\nabla u|^2 \, dx \lesssim [u]_{W^{s,2}(\partial B_1^+)}^2 \lesssim [u]_{W^{s,2}(\partial B_1^+)}^2.
\]
Moreover, since \( u = 0 \) on \( S^+ \) we have
\[
[u]_{W^{s,2}(\partial B_1^+)}^2 \lesssim [\varphi]_{W^{s,2}(\partial T_1)}^2 + \int_{T_1} \int_{S^+} |\varphi(x)|^2 \, dy \, dx \lesssim \int_{T_1} \int_{\partial T_1} \frac{|\varphi(x)|^2}{\text{dist}(x, \partial T_1)^s} \, dx.
\]
Now, since \( 2s > 1 \) we can apply Hardy’s inequality in convex sets, [6]. Observe that since \( \chi_{T_1} \varphi \) is the trace of a \( W^{1,2} \)-function, it can be approximated in \( C^\infty_c \) (simply by scaling the support inside and convolution). Thus
\[
\int_{T_1} \frac{|\varphi(x)|^2}{\text{dist}(x, \partial T_1)^s} \, dx \lesssim \int_{T_1} \int_{T_1} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{n+2s}} \, dx \, dy.
\]
This proves the claim. \( \square \)

As a consequence we can split the influence from \( S^+ \) and \( T_1 \).

Lemma A.2. Let \( B_1^+ \subset \mathbb{R}^n, \, n \geq 3 \), and \( u \in W^{1,2}(B_1^+) \) be a solution to
\[
\begin{aligned}
\Delta u &= 0 \quad \text{in } B_1^+ \\
u &= \varphi \quad \text{on } S^+ \\
u &= \psi \quad \text{on } T_1.
\end{aligned}
\]
Then for any \( s > \frac{1}{2}, \, sp > 1 \) we have
\[
\| \nabla u \|_{L^2(B_1^+)} \lesssim [\psi]_{W^{s,2}(T_1)} + [\varphi]_{W^{s,2}(S^+)}.
\]

Proof. By an even reflection we can extend \( \varphi \) to all of \( \partial B_1 \) with
\[
[\varphi]_{W^{s,2}(\partial B_1^+)} \lesssim [\varphi]_{W^{s,2}(S^+)}.
\]
Now we solve the equation
\[
\begin{aligned}
\Delta v &= 0 \quad \text{in } B_1^+ \\
v &= \varphi \quad \text{on } \partial B_1^+.
\end{aligned}
\]
Then we have
\[
(A.1) \quad \| \nabla v \|_{L^2(\partial B_1^+)} \lesssim [\varphi]_{W^{s,2}(S^+)}. 
\]
On the other hand
\[
\begin{aligned}
\Delta (u - v) &= 0 \quad \text{in } B_1^+ \\
u - v &= 0 \quad \text{on } S^+ \\
u - v &= \psi - \varphi \quad \text{on } T_1.
\end{aligned}
\]
By Lemma A.1,
\[
\| \nabla (u - v) \|_{L^2(B_1^+)} \lesssim [\psi - \varphi]_{W^{s,2}(T_1)} \lesssim [\psi]_{W^{s,2}(T_1)} + [\varphi]_{W^{s,2}(\partial B_1^+)} \\
\lesssim [\psi]_{W^{s,2}(T_1)} + [\varphi]_{W^{s,2}(S^+)}.
\]
(A.2)

Together, (A.1) and (A.1) imply the claim.

We also need the following

**Lemma A.3.** For every \( \varphi \in W^{1,2} \cap L^\infty(\mathbb{S}^{n-1}) \) the following interpolation inequality holds for a constant independent of \( \varphi \):
\[
\int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{n+1}} \, dx \, dy \lesssim \| \varphi \|_{L^\infty(\mathbb{S}^{n-1})} \| \nabla \varphi \|_{L^2(\mathbb{S}^{n-1})}.
\]
(A.3)

Also, for every \( \varphi \in W^{1,2} \cap L^\infty(\mathbb{S}^+_{n-1}) \), where \( \mathbb{S}^+_{n-1} = \mathbb{S}^{n-1} \cap \mathbb{R}^n_+ \), the following interpolation inequality holds for a constant independent of \( \varphi \):
\[
\int_{\mathbb{S}^+_{n-1}} \int_{\mathbb{S}^+_{n-1}} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{n+1}} \, dx \, dy \lesssim \| \varphi \|_{L^\infty(\mathbb{S}^+_{n-1})} \| \nabla \varphi \|_{L^2(\mathbb{S}^+_{n-1})}.
\]
(A.4)

**Proof.** Firstly by interpolation we get
\[
[f]_{W^{1,2}(\mathbb{R}^{n-1})} \lesssim \| f \|_{L^2(\mathbb{R}^{n-1})} + \| \nabla f \|_{L^2(\mathbb{R}^{n-1})}.
\]
Scaling with \( f_\lambda := f(\lambda \cdot) \) and choosing a good \( \lambda \) this becomes
\[
[f]_{W^{1/2,2}(\mathbb{R}^{n-1})} \lesssim \sqrt{\| f \|_{L^2(\mathbb{R}^{n-1})} \| \nabla f \|_{L^2(\mathbb{R}^{n-1})}}.
\]

Let \( B \subset \mathbb{R}^{n-1} \) be the unit ball, then for \( f \in W^{1,2}(B) \) we define
\[
\tilde{f}(x) := \eta_{B(2)}(x) \left\{ \begin{array}{ll}
 f(x) & \text{for } |x| < 1 \\
 f(x/|x|^2) & \text{for } |x| \geq 1.
\end{array} \right.
\]

Here \( \eta_{2B} \in C^\infty_c(2B) \) is a cutoff function so that \( \eta_{2B} \equiv 1 \) in \( B \). Then we obtain
\[
[f]_{W^{1/2,2}(B)} \lesssim \sqrt{\| f \|_{L^2(B)} \| \nabla f \|_{L^2(B)} + \| f \|_{L^2(B)}}.
\]

Apply this to \( f - (f)_B \), and since \( \| f - (f)_B \|_{L^2(B)} \lesssim \| f \|_{L^2(B)} \) we obtain by Poincaré inequality
\[
[f]_{W^{1/2,2}(B)} \lesssim \sqrt{\| f \|_{L^2(B)} \| \nabla f \|_{L^2(B)} \| \nabla f \|_{L^2(B)}} \lesssim \sqrt{\| f \|_{L^\infty(B)} \| \nabla f \|_{L^2(B)}}.
\]

Using a diffeomorphism \( \tau : \mathbb{S}^{n-1} \to B \) and setting \( f := \varphi \circ \tau^{-1} \) we obtain the second claim (A.4). The first claim follows from a covering argument (A.4), namely by covering \( \mathbb{S}^{n-1} \).
by many half-spheres from (A.3) we can obtain
\[
\int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{n+1}} \, dx \, dy \\
\lesssim \|\varphi\|_{L^\infty(\mathbb{S}^{n-1})} \|\nabla\varphi\|_{L^2(\mathbb{S}^{n-1})} + \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |\varphi(x) - \varphi(y)|^2 \, dx \, dy.
\]
Now,
\[
\int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |\varphi(x) - \varphi(y)|^2 \, dx \, dy \\
\lesssim \|\varphi\|_{L^2(\mathbb{S}^{n-1})} \|\varphi - \varphi(\mathbb{S}^{n-1})\|_{L^2(\mathbb{S}^{n-1})} \lesssim \|\varphi\|_{L^2(\mathbb{S}^{n-1})} \|\nabla\varphi\|_{L^2(\mathbb{S}^{n-1})}.
\]

From the above lemmata we obtain the following trace estimate.

Theorem A.4 (Trace Theorem). Let \( B_r \subset \mathbb{R}^n, \ n \geq 3, \) be a ball of radius \( r > 0. \) If \( \varphi^h : B_r \to \mathbb{R}^N \) denotes the harmonic extension of \( \varphi : \partial B_r \to \mathbb{R}^N, \) then
\[
\int_{B_r} |\nabla\varphi^h|^2 \lesssim \int_{\partial B_r} \int_{\partial B_r} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{n+1}} \, dx \, dy,
\]
and
\[
\int_{B_r} |\nabla\varphi^h|^2 \lesssim r^{\frac{n-1}{2}} \|\varphi\|_{L^\infty(\partial B_r)} \|\nabla\varphi\|_{L^2(\partial B_r)}.
\]

If \( \varphi^h : B_r^+ \to \mathbb{R} \) denotes the harmonic extension of \( \varphi : \partial B_r^+ \to \mathbb{R}^N, \) then
\[
\int_{B_r^+} |\nabla\varphi^h|^2 \lesssim \int_{\partial B_r^+} \int_{\partial B_r^+} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{n+1}} \, dx \, dy,
\]
and moreover for \( s > \frac{1}{2}, \ p \in (1, \infty) \) so that \( sp > 1, \) then
\[
\int_{B_r^+} |\nabla\varphi^h|^2 \lesssim r^{\frac{n-2}{2}} \|\varphi\|_{L^\infty(B_r^+)} \|\nabla\varphi\|_{L^2(B_r^+)} + r^{sp-1} |\varphi|_{W^{s,p}(I(r))}^p.
\]

A.2. A Counterexample. The above trace theorem does not hold with \( W^{\frac{1}{2}, 2}. \) Indeed, this follows essentially from a counterexample to Hardy-Sobolev inequality on bounded domains for \( W^{\frac{1}{2}, 2} \) by Dyda [6] (attributed to an idea by Bogdan). For an overview on available Hardy-Sobolev inequalities see also [3].

Lemma A.5. There does not exist a constant \( C > 0 \) such that the following holds.

Let \( u \in W^{1,2}(B_1^+) \) be harmonic in \( B_1^+ \subset \mathbb{R}^n, \ n \geq 3, \ \varphi \in W^{\frac{1}{2}, 2}(T_2), \ \psi \in W^{1,2}(S_1^+) \) so that in the trace sense \( u \equiv \varphi \) on \( T_1 \) and \( u \equiv \psi \) on \( S_1^+. \) Then
\[
\|\nabla u\|_{L^2(B_1^+)}^2 \leq C \left( \int_{T_2} \int_{T_2} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{n+1}} \, dx \, dy + \int_{S_1^+} |\nabla \tau \psi|^2 \right).
\]
Proof. In [6] Dyda constructs an example $\tilde{\varphi}_i^1 \in C_c^\infty(T_1)$ such that $\tilde{\varphi}_i^1 \equiv 1$ on $T_{1-\frac{1}{i}}$,  
\[ \int_{T_1} \int_{T_1} \frac{|\tilde{\varphi}_i^1(x) - \tilde{\varphi}_i^1(y)|^2}{|x - y|^{n+1}} \, dx \, dy \xrightarrow{i \to \infty} 0, \]
but
\[ \int_{T_2} \int_{T_2} \frac{|\chi_{T_1}(x)\tilde{\varphi}_i^1(x) - \chi_{T_2}(y)\tilde{\varphi}_i^1(y)|^2}{|x - y|^{n+1}} \, dx \, dy \xrightarrow{i \to \infty} \infty. \]
By the same arguments one can find also $\tilde{\varphi}_i^2 \in C_c^\infty(T_2 \setminus T_1)$ so that $\tilde{\varphi}_i^2 \equiv 1$ on $T_{2-\frac{1}{i}} \setminus T_{1+\frac{1}{i}}$, and
\[ \int_{T_2 \setminus T_1} \int_{T_2 \setminus T_1} \frac{|\tilde{\varphi}_i^2(x) - \tilde{\varphi}_i^2(y)|^2}{|x - y|^{n+1}} \, dx \, dy \xrightarrow{i \to \infty} 0. \]
Since along $\partial T_2 \setminus T_1$ this can be done by a even reflection, one can even require that for some fixed bump function $\eta \in C_c^\infty(T_2)$, $\eta \equiv 1$ in $T_{\frac{1}{2}}$ we have
\[ \varphi_i := \begin{cases} \tilde{\varphi}_i^1 & \text{in } T_1 \\ \eta \tilde{\varphi}_i^2 & \text{in } T_2 \setminus T_1 \end{cases} \]
satisfies
\[ \int_{T_2} \int_{T_2} \frac{|\varphi_i(x) - \varphi_i(y)|^2}{|x - y|^{n+1}} \, dx \, dy \xrightarrow{i \to \infty} \int_{T_2} \int_{T_2} \frac{|\eta(x) - \eta(y)|^2}{|x - y|^{n+1}} \, dx \, dy < \infty. \]
Setting now $\psi := 0$, we find that $\psi_i \chi_{S^+} + \varphi_i \chi_{T_1} \in W^{2,2}(\partial B_1^+)$, and thus we can solve
\[ \begin{cases} \Delta u^i = 0 & \text{in } B_1^+ \\ u^i = \psi_i & \text{in } S_1^+ \\ u^i = \varphi_i & \text{in } T_1. \end{cases} \]
Using a Lipschitz diffeomorphism we find that
\[ \int_{T_2} \int_{T_2} \frac{|\chi_{T_1}(x)\varphi_i(x) - \chi_{T_2}(y)\varphi_i(y)|^2}{|x - y|^{n+1}} \, dx \, dy \approx \int_{\partial B_1^+} \int_{\partial B_1^+} \frac{|\chi_{T_1}(x)\varphi_i(x) - \chi_{T_2}(y)\varphi_i(y)|^2}{|x - y|^{n+1}} \, dx \, dy \approx [u_i]^2_{W^{2,4}(\partial B_1^+)}. \]
If (A.9) were to hold, by the trace theorem we would thus find
\[ \int_{T_2} \int_{T_2} \frac{|\chi_{T_1}(x)\varphi_i(x) - \chi_{T_2}(y)\varphi_i(y)|^2}{|x - y|^{n+1}} \, dx \, dy \lesssim \int_{T_2} \int_{T_2} \frac{|\varphi_i(x) - \varphi_i(y)|^2}{|x - y|^{n+1}} \, dx \, dy. \]
But now by construction, the right-hand side stays bounded, while the left-hand side blows up as $i \to \infty$. Thus (A.9) was false. \hfill \Box
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