ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A FREE BOUNDARY PROBLEM MODELLING THE GROWTH OF TUMORS WITH STOKES EQUATIONS

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Abstract. We study a free boundary problem modelling the growth of non-necrotic tumors with fluid-like tissues. The fluid velocity satisfies Stokes equations with a source determined by the proliferation rate of tumor cells which depends on the concentration of nutrients, subject to a boundary condition with stress tensor effected by surface tension. It is easy to prove that this problem has a unique radially symmetric stationary solution. By using a functional approach, we prove that there exists a threshold value $\gamma^* > 0$ for the surface tension coefficient $\gamma$, such that in the case $\gamma > \gamma^*$ this radially symmetric stationary solution is asymptotically stable under small non-radial perturbations, whereas in the opposite case it is unstable.

1. Introduction. In this paper we study the following free boundary problem modelling the growth of non-necrotic tumors with fluid-like tissues:

\[ \Delta \sigma = f(\sigma) \quad \text{in} \quad \Omega(t), \quad t > 0, \]  
\[ \nabla \cdot v = g(\sigma) \quad \text{in} \quad \Omega(t), \quad t > 0, \]  
\[ -\nu \Delta v + \nabla p - \frac{\nu}{3} \nabla (\nabla \cdot v) = 0 \quad \text{in} \quad \Omega(t), \quad t > 0, \]  
\[ \sigma = \bar{\sigma} \quad \text{on} \quad \partial \Omega(t), \quad t > 0, \]  
\[ T(v,p)n = -\gamma \kappa n \quad \text{on} \quad \partial \Omega(t), \quad t > 0, \]  
\[ V_n = v \cdot n \quad \text{on} \quad \partial \Omega(t), \quad t > 0, \]  
\[ \int_{\Omega(t)} v \, dx = 0, \quad t > 0, \]  
\[ \int_{\Omega(t)} v \times x \, dx = 0, \quad t > 0, \]  
\[ \Omega(0) = \Omega_0, \]

where $\sigma = \sigma(x,t)$, $v = v(x,t)$ ($= (v_1(x,t), v_2(x,t), v_3(x,t))$) and $p = p(x,t)$ are unknown functions representing the concentration of nutrient, the velocity of the fluid and the internal pressure, respectively, $f$ and $g$ are given functions representing...

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the nutrient consumption rate and tumor cell proliferation rate, respectively, which typically have the following forms respectively:

\[ f(\sigma) = \lambda \sigma, \quad g(\sigma) = \mu (\sigma - \sigma_c), \]

(10)

where \( \lambda, \mu \) and \( \sigma_c \) are positive constants, \( \sigma_c < \bar{\sigma} \), and \( \Omega(t) \) is an a priori unknown bounded domain in \( \mathbb{R}^3 \) representing the region occupied by the tumor at time \( t \). Besides, \( \nu, \bar{\sigma} \) and \( \gamma \) are positive constants, among which \( \nu \) is the viscosity coefficient of the fluid, \( \gamma \) is the surface tension coefficient of the tumor surface, and \( \bar{\sigma} \) is the concentration of nutrient in tumor’s host tissues, \( \kappa, V_n, n \) denote the mean curvature, the normal velocity and the unit outward normal, respectively, of the tumor surface \( \partial \Omega(t) \), and \( T(v, p) \) represents the stress tensor, i.e.,

\[ T(v, p) = \nu \left[ \nabla \otimes v + \left( \nabla \otimes v \right)^T \right] - (p + \frac{2\nu}{3} \nabla \cdot v)I, \]

(11)

where \( I \) denotes the unit tensor. We note that the sign of the mean curvature \( \kappa \) is defined such that it is nonnegative for convex hyper-surfaces. Without loss of generality, later on we assume that

\[ \nu = 1 \quad \text{and} \quad \bar{\sigma} = 1. \]

Note that the general situation can be easily reduced into this special situation by using the rescaling \( \sigma \to \sigma/\bar{\sigma}, \quad p \to p/\nu \) and \( \gamma \to \gamma/\nu \).

Tumor growth modelling and analysis has attracted considerable attention during the past more than ten years. Most tumor models assume that the tumor tissue has the structure of a porous medium for which Darcy’s law applies (see, e.g., [2], [3] and the references cited therein). For such tumor models, many interesting results of rigorous analysis have been obtained, for which we refer the interested reader to see [4–8], [17]–[21], [26], [27] and the references cited therein. The tumor whose tissue does not have the structure of a porous medium but instead is more like a fluid was recently considered by Franks et al in the literatures [11]–[14], where some new models were proposed to mimic the early stages of the growth of ductal carcinoma in the breast. A basic feature of a ductal carcinoma in the breast in early stages is that it is confined to the duct of a mammary gland, which consists of epithelial cells, a meshwork of proteins, and extracellular fluid. In modelling, this leads to the replacement of the Darcy’s law used in porous medium structured tumor models by the Stokes equations. See [11]–[14] for details. The models of Franks et al [11]–[14] have been concisely reformulated by Friedman in [15] (see also [16]).

The problem (1)–(9) above is a simplification of the tumor model proposed in [15]. The simplifications are made in two aspects. First, the model in [15] contains a system of nonlinear hyperbolic conservation laws with source terms describing the movements and interchanges of three different species of cells: proliferating cells, quiescent cells and necrotic cells. In this paper we only consider one species of proliferating cells, so that no hyperbolic conservation laws appear in (1)–(9). Second, in [15] the equation for \( \sigma \) is of the following evolutionary type:

\[ \partial_t \sigma = \Delta \sigma - f(\sigma) \quad \text{in} \quad \Omega(t), \quad t > 0, \]

where \( f \) is as given in (10), but in this paper the stationary form (1) is considered. All these simplifications are made for the purpose to make the analysis simpler. If either one of the above two aspects of simplifications are not made, then the model will be much more complicated to analyze, and new mathematical techniques have to be employed. We leave it for future work.
In [15] Friedman established local wellposedness in Hölder spaces of his model. Meanwhile, he proved that in the special case that the tumor contains only one species of cells (i.e., the tumor contains only proliferating cells), there exists a unique radially symmetric stationary solution. Based on these results, a number of interesting questions are raised in [15] (see also [16]), one of which is as follows: Is this radially symmetric stationary solution asymptotically stable under non-radial perturbations? A heuristic result toward an answer to this question was obtained by Friedman and Hu in [18], where they proved that this radially symmetric stationary solution is linearly stable for small \( \mu/\gamma \), i.e., there exists a threshold value \( \mu/\gamma^* \) such that if we denote by \((\sigma_s, v_s, p_s, \Omega_s)\) this stationary solution, then in the case \( \mu/\gamma < (\mu/\gamma)^* \) the trivial solution of the linearization at \((\sigma_s, v_s, p_s, \Omega_s)\) of the original problem is stable. Moreover, they also proved that in the case \( \mu/\gamma > (\mu/\gamma)^* \) the radially symmetric stationary solution is unstable. We also refer the interested reader to see [17] for the study of existence of non-radial stationary solutions.

The purpose of this paper is to prove that, at least for the simplified model \((1)\)–\((9)\), the answer to the above question is yes for large \( \gamma \) but no for small \( \gamma \). To this end, we shall use a functional approach inherited from the references [7], [10], [26] and [27] to study the problem \((1)\)–\((9)\), namely, we shall first reduce the problem \((1)\)–\((9)\) into an evolution equation containing merely the function \( \rho \) describing the free boundary \( \partial \Omega(t) \), which can be considered as a differential equation in certain Banach space. We shall prove that this differential equation is of the parabolic type. Next we use the geometric theory for parabolic differential equations in Banach spaces (see [1] and [24]) to study the stability of the stationary solution. Since our discussion does not depend on the specific linear forms of the equations \((1)\) and \((2)\), throughout this paper we shall not consider the specific forms of \( f \) and \( g \) given by \((10)\), but instead assume that they are general smooth functions satisfying the following assumptions:

\[
\begin{align*}
(A1) \quad & f \in C^\infty[0, \infty), f'(\sigma) > 0 \text{ for } \sigma \geq 0 \text{ and } f(0) = 0. \\
(A2) \quad & g \in C^\infty[0, \infty), g'(\sigma) > 0 \text{ for } \sigma \geq 0 \text{ and } g(\sigma_c) = 0 \text{ for some } \sigma_c > 0. \\
(A3) \quad & \sigma_c < 1.
\end{align*}
\]

To give a precise statement of our main result, let us first introduce some notations.

Given a bounded domain \( \Omega \subseteq \mathbb{R}^3 \) and two numbers \( m \in \mathbb{N} \) and \( \theta \in (0, 1) \), we denote by \( h^{m+\theta}(\Omega) \) the so-called little Hölder space on \( \Omega \) of index \( m + \theta \), which is, by definition, the closure of \( C^\infty(\Omega) \) in the usual Hölder space \( C^{m+\theta}(\Omega) \). Similarly, given a smooth hypersurface \( \Gamma \) in \( \mathbb{R}^3 \), we denote by \( h^{m+\theta}(\Gamma) \) the closure of \( C^\infty(\Gamma) \) in \( C^{m+\theta}(\Gamma) \).

It can be easily shown (see Theorem A.1 in Appendix A) that under assumptions \((A1)\)–\((A3)\), the problem \((1)\)–\((8)\) has a unique radially symmetric stationary solution. Later on we use the same notation \((\sigma_s, v_s, p_s, \Omega_s)\) as before to denote this radially symmetric stationary solution of the problem \((1)\)–\((8)\). Note that this means that there exists \( R_s > 0 \) such that \( \Omega_s = \{ x \in \mathbb{R}^3 : |x| < R_s \} \) and

\[
\sigma_s(x) = \sigma_s(r), \quad v_s(x) = v_s(r) \frac{x}{r}, \quad p_s(x) = p_s(r) \quad \text{for } x \in \Omega_s,
\]

where \( r = |x| \) and \( v_s \) represents a scalar function. Clearly, a coordinate translation of a solution of \((1)\)–\((8)\) is still a solution of it. Thus, for any \( x_0 \in \mathbb{R}^3 \), we denote by \((\sigma_{[x_0]}, v_{[x_0]}, p_{[x_0]}, \Omega_{[x_0]})\) the stationary solution obtained by the coordinate translation \( x \rightarrow x + x_0 \) of the stationary solution \((\sigma_s, v_s, p_s, \Omega_s)\). Given \( \rho \in C^1(\partial \Omega_s) \) with \( ||\rho||_{C^1(\partial \Omega_s)} \) sufficiently small, we denote by \( \Omega_\rho \) the domain enclosed by the
hypersurface $r = R_s + \rho(\omega)$, where $\omega \in \partial \Omega_s$. It is obvious that for $x_0 \in \mathbb{R}^3$ such that $|x_0|$ is sufficiently small, there exists a smooth function $\rho(x_0)$ on $\partial \Omega_s$ such that $\Omega_{[x_0]} = \Omega_{\rho(x_0)}$. Since we shall only be concerned with small perturbations of the stationary solution $(\sigma_s, v_s, p_s, \Omega_s)$, it is natural to assume that the domains $\Omega(t)$ and $\Omega_0$ in (1.1)–(9) are small perturbations of $\Omega_s$. It follows that there exist functions $\rho(t) (= \rho(\omega, t))$ and $\rho_0 (= \rho_0(\omega))$ on $\partial \Omega_s$ such that $\Omega(t) = \Omega_0(\rho(t))$ and $\Omega_0 = \Omega_0(\rho_0)$. Using these notations, the initial condition (9) can be rewritten as follows:

$$\rho(\omega, 0) = \rho_0(\omega) \quad \text{for} \quad \omega \in \partial \Omega_s. \quad (13)$$

The solution $(\sigma, v, p, \Omega)$ of the problem (1.1)–(9) will be correspondingly rewritten as $(\sigma, v, p, \rho)$, and the radially symmetric stationary solution $(\sigma_s, v_s, p_s, \Omega_s)$ will be re-denoted as $(\sigma_s, v_s, p_s, 0)$.

The main result of this paper is the following theorem:

**Theorem 1.1.** Assume that assumptions (A1)–(A3) hold. For given $m \in \mathbb{N}$, $m \geq 3$, and $0 < \theta < 1$, we have the following assertion: There exists a positive threshold value $\gamma_*$ such that for any $\gamma > \gamma_*$, the radially symmetric stationary solution $(\sigma_s, v_s, p_s, 0)$ is asymptotically stable in the following sense: There exists constant $\varepsilon > 0$ such that for any $\rho_0 \in h^{m+\theta}(\partial \Omega_s)$ satisfying $\|\rho_0\|_{C^{m+\theta}(\partial \Omega_s)} < \varepsilon$, the problem (1.1)–(1.9) has a unique solution $(\sigma, v, p, \rho)$ for all $t \geq 0$, and there are positive constants $\omega, K$ independent of the initial data and a point $x_0 \in \mathbb{R}^3$ uniquely determined by the initial data, such that the following holds for all $t \geq 0$:

$$
\begin{align*}
\|\sigma(\cdot, t) - \sigma_{[x_0]}\|_{C^m(\Omega(t))} &+ \|v(\cdot, t) - v_{[x_0]}\|_{C^{m+\theta}(\Omega(t))} \\
+\|p(\cdot, t) - p_{[x_0]}\|_{C^{m+\theta}(\partial \Omega_s)} &\leq Ke^{-\omega t}.
\end{align*}
$$

For $\gamma < \gamma_*$ the stationary solution $(\sigma_s, v_s, p_s, 0)$ is unstable.

It is interesting to compare this result with the corresponding result for the porous medium structured tumor model obtained by Cui and Escher in [8], where it is proved that, for the porous medium structured tumor model, there exists a threshold value for the surface tension coefficient $\gamma$, which we denote as $\tilde{\gamma}_*$, such that the unique radially symmetric stationary solution is asymptotically stable if $\gamma > \tilde{\gamma}_*$, but unstable if $\gamma < \tilde{\gamma}_*$. We shall show that $\gamma_*>\tilde{\gamma}_*$. This implies that radially symmetric stationary solution is more stable for a tumor whose tissue has a porous medium structure than a tumor whose tissue is more like a fluid. See Lemma 3.5 for the proof of the assertion that $\gamma_* > \tilde{\gamma}_*$.

The structure of the rest part is as follows. In Section 2 we first convert the problem into an equivalent initial-boundary value problem on a fixed domain by using the so-called Hanzawa transformation, and next further reduce it into a scalar equation containing the single function $\rho$, which can be regarded as a differential equation in the Banach space $h^{m-1+\theta}(S^2)$. We shall also prove that this equation is of the parabolic type. In Section 3 we study the linearization of (1.1)–(8) at the radially symmetric stationary solution, and study the spectrum of the linearized operator. In the last section we give the proof of Theorem 1.1.

2. Reduction of the problem. In this section we reduce the problem (1.1)–(9) into a differential equation in a Banach space. For simplicity of the notation, later on we always assume that $R_s = 1$. Note that this assumption is reasonable because the case $R_s \neq 1$ can be easily reduced into this case after a rescaling. It follows that

$$
\Omega_s = \mathbb{B}^3 = \{x \in \mathbb{R}^3 : |x| < 1\} \quad \text{and} \quad \partial \Omega_s = S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}.
$$
Let $m$ and $\theta$ be as in Theorem 1.1. For $u \in h^{m+\theta}(\mathbb{B}^3)$, we denote by $\text{tr}_{S^2}(u)$ the trace of $u$ on $S^2$, i.e., $\text{tr}_{S^2}(u) = u|_{S^2}$. We know that $\text{tr}_{S^2}(u) \in h^{m+\theta}(S^2)$ and the operator $\text{tr}_{S^2} : u \rightarrow \text{tr}_{S^2}(u)$ from $h^{m+\theta}(\mathbb{B}^3)$ to $h^{m+\theta}(S^2)$ is linear, bounded and surjective. For any given $\rho \in h^{m+\theta}(S^2)$, by the well-known theory for elliptic problems we see that there exists a unique solution $u \in h^{m+\theta}(\mathbb{B}^3)$ of the following boundary value problem

$$\Delta u = 0 \quad \text{in} \quad \mathbb{B}^3, \quad u = \rho \quad \text{on} \quad S^2.$$ 

We denote the solution mapping by $u = \Pi(\rho)$. It is clear that $\Pi \in L(h^{m+\theta}(S^2), h^{m+\theta}(\mathbb{B}^3))$, and $\Pi$ is a right inverse of the trace operator, i.e., $\Pi \in L(h^{m+\theta}(\mathbb{B}^3), h^{m+\theta}(S^2))$ and $\text{tr}_{S^2}(\Pi(\rho)) = \rho$ for any $\rho \in h^{m+\theta}(S^2)$. Let $E \in L(C^{m+\theta}(\mathbb{B}^3))$, $BUC^{m+\theta}(\mathbb{R}^3)$ be an extension operator, i.e., $E$ has the property that $E(u)(x) = u(x)$ for any $u \in C^{m+\theta}(\mathbb{B}^3)$ and $x \in \mathbb{B}^3$. Here $BUC^{m+\theta}(\mathbb{R}^3)$ denotes the space of all $C^m$ functions $u$ on $\mathbb{R}^3$ such that $u$ itself and all its partial derivatives of order $\leq m$ are bounded and uniformly $\theta$-th order Hölder continuous in $\mathbb{R}^3$. We denote $\Pi_1 = E \circ \Pi$. Then clearly $\Pi_1 \in L(h^{m+\theta}(S^2), h^{m+\theta}(\mathbb{R}^3))$, where $h^{m+\theta}(\mathbb{R}^3)$ represents the closure of $BUC^{\infty}(\mathbb{R}^3)$ in $BUC^{m+\theta}(\mathbb{R}^3)$. Hence there exists a constant $C_0 > 0$ such that

$$||\Pi_1(\rho)||_{BUC^{m+\theta}(\mathbb{R}^3)} \leq C_0||\rho||_{C^{m+\theta}(S^2)} \quad \text{for} \quad \rho \in h^{m+\theta}(S^2). \quad (15)$$

Take a constant $0 < \delta < \min\{1/6, 1/(3C_0)\}$ and fix it, where $C_0$ is the constant in (15). We choose a cut-off function $\chi \in C^\infty([-1, \infty))$ such that

$$0 \leq \chi \leq 1, \quad \chi(\tau) = \begin{cases} 1, & \text{for } |\tau| \leq \delta, \\ 0, & \text{for } |\tau| \geq 3\delta, \end{cases} \quad \text{and} \quad |\chi'(\tau)| \leq \frac{2}{3\delta}. \quad (16)$$

We denote

$$O^{m+\theta}_\delta(S^2) = \{ \rho \in h^{m+\theta}(S^2) : ||\rho||_{C^{m+\theta}(S^2)} < \delta^2 \}. \quad \text{Given } \rho \in O^{m+\theta}_\delta(S^2), \text{ we define the Hanzawa transformation } \Phi_\rho : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \text{ as follows:}$$

$$\Phi_\rho(x) = x + \chi(r - 1)\Pi_1(\rho)(x) \frac{x}{r} \quad \text{for} \quad x \in \mathbb{R}^3. \quad (17)$$

Using (15) and (16) we can easily verify that the function $|r| \rightarrow r + \chi(r-1)\Pi_1(\rho)(r\omega)$ is strictly monotone increasing, where $\omega \in S^2$, so that $\Phi_\rho$ is an $1$-$1$ correspondence. Moreover, $\Phi_\rho$ is a $h^{m+\theta}$ diffeomorphism from $\mathbb{R}^3$ onto itself, i.e., $\Phi_\rho \in \text{Diff}^{m+\theta}(\mathbb{R}^3, \mathbb{R}^3)$, and each component of $\Phi_\rho$ and $\Phi_\rho^{-1}$ belongs to $h^{m+\theta}(\mathbb{R}^3)$. Later on we write $\Phi_\rho \in \text{Diff}_{h^{m+\theta}}(\mathbb{R}^3, \mathbb{R}^3)$ to indicate this fact. We define $\omega_\rho = \Phi_\rho|_{S^2}$, and denote

$$\Omega_\rho = \Phi_\rho(\mathbb{B}^3), \quad \Gamma_\rho = \partial \Omega_\rho = \Phi_\rho(S^2) = \text{Im}(\omega_\rho).$$

Clearly,

$$\omega_\rho(\omega) = [1 + \rho(\omega)]\omega \quad \text{for} \quad \omega \in S^2.$$ 

This implies that $x \in \Gamma_\rho$ if and only if there exists $\omega \in S^2$ such that $x = [1 + \rho(\omega)]\omega$. Thus, in the polar coordinates $(r, \omega)$ of $\mathbb{R}^3$, where $r = |x|$ and $\omega = x/|x|$, the hypersurface $\Gamma_\rho$ has the following equation: $r = 1 + \rho(\omega)$.

Next, given $\rho \in C([0, T], O^{m+\theta}_\delta(S^2))$, for each $t \in [0, T]$ we denote

$$\Gamma_\rho(t) = \Gamma_{\rho(t)} \quad \text{and} \quad \Omega_\rho(t) = \Omega_{\rho(t)}.$$ 

Since our purpose is to study asymptotical stability of the radially symmetric stationary solution, later on we always assume that the initial domain $\Omega_0$ is contained in a small neighborhood of $\Omega_\omega = \mathbb{B}^3$. It follows that there exists $\rho_0 \in O^{m+\theta}_\delta(S^2)$ such that $\Gamma_0 \equiv \partial \Omega_0 = \Gamma_{\rho_0}$. 


Let \( \rho \) be as above, and let \( \Phi^i_\rho \) be the \( i \)-th component of \( \Phi_\rho \), \( i = 1, 2, 3 \). We denote

\[
[D\Phi^i_\rho]_{ij} := \partial_j \Phi^i_\rho = \frac{\partial \Phi^i_\rho}{\partial x_j}, \quad a^i_{\rho j}(x) = [D\Phi^i_\rho(x)]^{-1} \quad (i, j = 1, 2, 3),
\]

\[
G_\rho(x) = \det(D\Phi_\rho(x)) \quad \text{for} \ x \in \mathbb{R}^3,
\]

\[
H_\rho(\omega) = |\phi_\rho|^2 \sqrt{1 + |\nabla_\omega \phi_\rho|^2} \quad \text{for} \ \omega \in \mathbb{S}^2,
\]

where \( \nabla_\omega \) represents the orthogonal projection of the gradient \( \nabla_x \) onto the tangent space \( T_x(S^2)^1 \). Here and hereafter, for a matrix \( A \) we use the notation \( A_{ij} \) to denote the element of \( A \) in the \((i, j)\)-th position. Since \( \Phi_\rho \in \text{Diff}^{m+\theta}(\mathbb{R}^3, \mathbb{R}^3) \), we have \( a^i_{\rho j} \in h^{m-1+\theta}(\mathbb{R}^3), \ i, j = 1, 2, 3, \ G_\rho \in h^{m-1+\theta}(\mathbb{R}^3), \) and \( H_\rho \in h^{m-1+\theta}(S^2) \).

We now introduce four partial differential operators \( A(\rho), \bar{B}(\rho), \tilde{B}(\rho), \) and \( \bar{B}(\rho) \otimes \) on \( \mathbb{R}^3 \) as follows:

\[
A(\rho)u(x) = a^i_{\rho j}(x) \partial_j(a^j_{\rho k}(x) \partial_k u(x)) \quad \text{for scalar function} \ u, \quad (18)
\]

\[
\bar{B}(\rho)u(x) = (a^i_{\rho j}(x) \partial_j u(x), a^j_{\rho k}(x) \partial_k u(x), a^i_{\rho j}(x) \partial_j u(x)) \quad \text{for vector function} \ v = (v_1, v_2, v_3), \quad (19)
\]

\[
\tilde{B}(\rho) \otimes v(x) = (a^i_{\rho j}(x) \partial_j v_j(x)) \quad \text{for vector function} \ v = (v_1, v_2, v_3), \quad (20)
\]

\[
\bar{B}(\rho) \otimes v(x) = (\bar{B}(\rho) \otimes v(x), \tilde{B}(\rho) \otimes v(x)) \quad \text{for vector function} \ v = (v_1, v_2, v_3). \quad (21)
\]

Here and hereafter we use the convention that repeated indices represent summations with respect to these indices, and \( \partial_j = \partial/\partial x_j, \ j = 1, 2, 3 \). Obviously,

\[
A(\rho) \in L(h^{m+\theta}(\mathbb{R}^3), h^{m-2+\theta}(\mathbb{R}^3)), \quad \bar{B}(\rho) \in L(h^{m+\theta}(\mathbb{R}^3), (h^{m-1+\theta}(\mathbb{R}^3))^3), \quad \tilde{B}(\rho) \in L((h^{m+\theta}(\mathbb{R}^3))^3, (h^{m-1+\theta}(\mathbb{R}^3))^3), \quad \bar{B}(\rho) \otimes \in L((h^{m+\theta}(\mathbb{R}^3))^3, (h^{m-1+\theta}(\mathbb{R}^3))^3 \times 3) \times 3).
\]

The definitions \((18)-\(21\)) can be respectively briefly rewritten as follows:

\[
A(\rho)u = (\Delta(u \circ \Phi^{-1}_\rho)) \circ \Phi_\rho, \quad \bar{B}(\rho)u = (\nabla(u \circ \Phi^{-1}_\rho)) \circ \Phi_\rho,
\]

\[
\tilde{B}(\rho) \cdot v = (\nabla \cdot (v \circ \Phi^{-1}_\rho)) \circ \Phi_\rho, \quad \bar{B}(\rho) \otimes v = (\nabla \otimes (v \circ \Phi^{-1}_\rho)) \circ \Phi_\rho.
\]

As in \[15\] we introduce the following vector functions:

\[
w_1(x) = (0, x_3, -x_2), \quad w_2(x) = (-x_3, 0, x_1), \quad w_3(x) = (x_2, -x_1, 0).
\]

Then clearly \( v \times x = (v \cdot w_1), v \cdot w_2, v \cdot w_3 \).

Let \( \mathbf{n}_\rho \) and \( \kappa_\rho \) be respectively the unit outward normal and the mean curvature of \( \Gamma_\rho \) (see \(5\)). We denote

\[
\bar{n}_\rho(x) = \mathbf{n}_\rho(\varphi_\rho(x)) \quad \text{and} \quad \bar{\kappa}_\rho(x) = \kappa_\rho(\varphi_\rho(x)) \quad \text{for} \ x \in \mathbb{S}^2.
\]

A direct computation shows that

\[
\bar{n}_\rho(x) = \frac{x \cdot [(D\Phi_\rho(x))]^{-1}}{|x \cdot [(D\Phi_\rho(x))]^{-1}|} e_1 = \frac{a^i_{\rho j}(x) x_j e_1}{|a^i_{\rho j}(x) x_j e_1|} = (\bar{n}_\rho^1(x), \bar{n}_\rho^2(x), \bar{n}_\rho^3(x)), \quad (22)
\]

where

\[
e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1),
\]

\[1\] In the coordinate \( \omega = (\vartheta, \varphi) = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta) \quad (0 \leq \vartheta \leq \pi, \ 0 \leq \varphi \leq 2\pi) \) of the sphere we have

\[
\nabla_\omega f(\omega) = (\cos \vartheta \cos \varphi, \cos \vartheta \sin \varphi, -\sin \vartheta) \partial_\vartheta f(\omega(\vartheta, \varphi)) + \frac{1}{\sin \vartheta} (-\sin \vartheta, \cos \varphi, 0) \partial_\varphi f(\omega(\vartheta, \varphi)).
\]

Note that \( \nabla f = \frac{\partial f}{\partial \varphi} \varphi + \frac{1}{r} \nabla_\omega f. \)
Thus we have
\[ \tilde{\kappa}_\rho(x) = \frac{1}{2} a_{ij}^\rho(x) \partial_j \tilde{n}_i(x). \] (23)

We remind the reader to notice that, obviously,
\[ [\rho \to \Phi_\rho] \in C^\infty(O_\delta^{m+\theta}(S^2), \text{Diff}^{m+\theta}_h(R^3, R^3)) \).

Thus we have
\[
\begin{align*}
[\rho \to a_{ij}^\rho] &\in C^\infty(O_\delta^{m+\theta}(S^2), h^{m-1+\theta}(R^3)), \quad i, j = 1, 2, 3, \\
[\rho \to G_\rho] &\in C^\infty(O_\delta^{m+\theta}(S^2), h^{m-1+\theta}(R^3)), \\
[\rho \to H_\rho] &\in C^\infty(O_\delta^{m+\theta}(S^2), h^{m-1+\theta}(S^2)), \\
[\rho \to \tilde{\kappa}_\rho] &\in C^\infty(O_\delta^{m+\theta}(S^2), h^{m-2+\theta}(S^2)), \\
[\rho \to \tilde{n}_\rho] &\in C^\infty(O_\delta^{m+\theta}(S^2), (h^{m-1+\theta}(S^2))^3). \tag{24}
\end{align*}
\]

Finally, for \( \sigma, \mathbf{v} \) and \( p \) as in (1)–(8), we denote
\[ \tilde{\sigma} = \sigma \circ \Phi_\rho, \quad \tilde{\mathbf{v}} = \mathbf{v} \circ \Phi_\rho, \quad \tilde{p} = p \circ \Phi_\rho. \]

We also denote \( \tilde{\omega}^\rho_j = \mathbf{w}_j \circ \Phi_\rho, \quad j = 1, 2, 3. \)

Using these notations, we see easily that the Hanzawa transformation transforms the equations (1)–(5), (7) and (8) into the following equations, respectively:
\[
\begin{align*}
\mathcal{A}(\rho) \tilde{\sigma} &= f(\tilde{\sigma}) \quad \text{in } \mathbb{B}^3, \quad t > 0, \\
\tilde{\mathbf{B}}(\rho) \cdot \tilde{\mathbf{v}} &= g(\tilde{\sigma}) \quad \text{in } \mathbb{B}^3, \quad t > 0, \\
- \mathcal{A}(\rho) \tilde{\mathbf{v}} + \tilde{\mathbf{B}}(\rho) \tilde{p} - \frac{1}{3} \nabla (\tilde{\mathbf{B}}(\rho) \cdot \tilde{\mathbf{v}}) &= 0 \quad \text{in } \mathbb{B}^3, \quad t > 0, \\
\tilde{\sigma} &= 1 \quad \text{on } S^2, \quad t > 0, \\
\tilde{T}_p(\tilde{\mathbf{v}}, \tilde{p}) \tilde{n}_\rho &= -\gamma \tilde{\kappa}_\rho \tilde{n}_\rho \quad \text{on } S^2, \quad t > 0, \\
\int_{|x| < 1} \tilde{\mathbf{v}}(x) G_\rho(x) dx &= 0, \quad t > 0, \\
\int_{|x| < 1} \tilde{\mathbf{v}}(x) \cdot \tilde{\omega}^\rho_j(x) G_\rho(x) dx &= 0, \quad j = 1, 2, 3, \quad t > 0. \tag{31}
\end{align*}
\]

Here \( \tilde{T}_p(\tilde{\mathbf{v}}, \tilde{p}) = [\tilde{\mathbf{B}}(\rho) \otimes \tilde{\mathbf{v}} + (\tilde{\mathbf{B}}(\rho) \otimes \tilde{\mathbf{v}})^T] - [\tilde{p} + (2/3) \tilde{\mathbf{B}}(\rho) \cdot \tilde{\mathbf{v}}] I. \)

In what follows we rewrite (6) into an explicit equation expressed with the function \( \rho = \rho(\omega, t) \). Let \( \psi_\rho(x, t) = r - 1 - \rho(\omega, t) \), where \( r = |x| \) and \( \omega = x/|x| \). Then \( x \in \Gamma_\rho(t) \) if and only if \( \psi_\rho(x, t) = 0 \). It follows that the normal velocity of \( \Gamma_\rho(t) \) is as follows (see [10]):
\[ V_n(x, t) = \frac{\partial_\nu \rho(\omega, t)}{\sqrt{\nabla \psi_\rho(x, t)}} \quad \text{for } x \in \Gamma_\rho(t), \quad t > 0. \]

Moreover, \( n(x, t) = \nabla \psi_\rho(x, t)/|\nabla \psi_\rho(x, t)| \). Hence (6) can be rewritten as follows:
\[ \partial_t \rho(\omega, t) = \mathbf{v}(x, t) \cdot \nabla \psi_\rho(x, t) \quad \text{for } x \in \Gamma_\rho(t), \quad t > 0, \]
where \( \omega = x/|x| \). Since \( \nabla \psi_\rho = \frac{\partial \psi_\rho}{\partial r} \omega + r^{-1} \nabla_\omega \psi_\rho = \omega - r^{-1} \nabla_\omega \rho \), we see that after the Hanzawa transformation, this equation has the following form:
\[ \partial_t \rho(\omega, t) = \tilde{\mathbf{v}}(\omega, t) \cdot \left[ \omega - \frac{\nabla_\omega \rho(\omega, t)}{1 + \rho(\omega, t)} \right] \quad \text{for } \omega \in S^2, \quad t > 0. \tag{32} \]
Finally, we rewrite (13) as follows:

$$\rho(\omega, 0) = \rho_0(\omega) \quad \text{for } \omega \in S^2. \quad (33)$$

In summary, we have the following preliminary result:

**Lemma 2.1.** If \((\sigma, \mathbf{v}, p, \rho)\) is a solution of the problem (1)–(9), then by letting \(\tilde{\sigma} = \sigma \circ \Phi_{\rho}, \tilde{\mathbf{v}} = \mathbf{v} \circ \Phi_{\rho}\) and \(\tilde{p} = p \circ \Phi_{\rho}\), we have that \((\tilde{\sigma}, \tilde{\mathbf{v}}, \tilde{p}, \rho)\) is a solution of the problem (25)–(33). Conversely, If \((\tilde{\sigma}, \tilde{\mathbf{v}}, \tilde{p}, \rho)\) is a solution of the problem (25)–(33), then by letting \(\sigma = \tilde{\sigma} \circ \Phi_{\rho}^{-1}, \mathbf{v} = \tilde{\mathbf{v}} \circ \Phi_{\rho}^{-1}\) and \(p = \tilde{p} \circ \Phi_{\rho}^{-1}\), we have that \((\sigma, \mathbf{v}, p, \rho)\) is a solution of the problem (1)–(9).

In the sequel we further reduce the problem (25)–(33) into a scalar equation containing the single unknown \(\rho\). The idea is to first solve the system of equations (25)–(31) to get \(\tilde{\sigma}, \tilde{\mathbf{v}}\) and \(\tilde{p}\) as functionals of \(\rho\), and next substitute \(\tilde{\mathbf{v}}\) obtained in this way into the equation (32).

We first consider (25) and (28). We have:

**Lemma 2.2.** Let \(\delta\) be sufficiently small. Then, given \(\rho \in O^{m+\theta}(S^2)\), the boundary value problem

$$A(\rho)\tilde{\sigma} = f(\tilde{\sigma}) \quad \text{in } \mathbb{B}^3, \quad \tilde{\sigma} = 1 \quad \text{on } S^2 \quad (34)$$

has a unique solution \(\tilde{\sigma} = R(\rho) \in h^{m+\theta}(\mathbb{B}^3)\) which satisfies \(0 < \tilde{\sigma} \leq 1\). Moreover, we have

$$R \in C^\infty(O^{m+\theta}(S^2), h^{m+\theta}(\mathbb{B}^3)). \quad (35)$$

**Proof.** See Lemma 3.1 of [7]. \(\Box\)

Next, for given \(\rho \in O^{m+\theta}(S^2)\) we consider the following boundary value problem:

$$\bar{B}(\rho) \cdot \bar{\mathbf{v}} = \varphi \quad \text{in } \mathbb{B}^3, \quad (36)$$

$$-A(\rho)\mathbf{v} + \bar{B}(\rho)\mathbf{p} = \mathbf{g} \quad \text{in } \mathbb{B}^3, \quad (37)$$

$$\bar{T}_\rho(\mathbf{v}, \mathbf{p})\bar{n}_\rho = \mathbf{h} \quad \text{on } S^2, \quad (38)$$

$$\int_{|x|<1} \bar{\mathbf{v}}(x)G_\rho(x)dx = 0, \quad (39)$$

$$\int_{|x|<1} \bar{\mathbf{v}}(x) \cdot \bar{\mathbf{v}}^j(x)G_\rho(x)dx = 0, \quad j = 1, 2, 3, \quad (40)$$

where \(\varphi \in h^{m-k-1+\theta}(\mathbb{B}^3)\), \(\mathbf{g} \in (h^{m-k-2+\theta}(\mathbb{B}^3))^3\) and \(\mathbf{h} \in (h^{m-k-1+\theta}(S^2))^3\) for some \(0 \leq k \leq m-2\).

**Lemma 2.3.** Let \(\delta\) be sufficiently small and let \(\rho \in O^{m+\theta}(S^2)\) be given. A necessary and sufficient condition for (36)–(40) to have a solution is that \(\varphi, \mathbf{g}\) and \(\mathbf{h}\) satisfy the following relations:

$$\int_{|x|<1} (\mathbf{g}(x) - \frac{1}{3}\bar{B}(\rho)\varphi(x)) \cdot \bar{\mathbf{v}}^j(x)G_\rho(x)dx$$

$$+ \int_{|x|=1} \mathbf{h}(x) \cdot \bar{\mathbf{v}}^j(x)H_\rho(x)dS_x = 0, \quad j = 1, 2, 3, \quad (41)$$

$$\int_{|x|<1} (\mathbf{g}(x) - \frac{1}{3}\bar{B}(\rho)\varphi(x)) \cdot \mathbf{e}_jG_\rho(x)dx$$

$$+ \int_{|x|=1} \mathbf{h}(x) \cdot \mathbf{e}_jH_\rho(x)dS_x = 0, \quad j = 1, 2, 3. \quad (42)$$
If these conditions are satisfied, then (26)–(40) has a unique solution \((\tilde{v}, \tilde{p}) \in \left( H^{m-\kappa+\theta}(\mathbb{R}^3) \right)^3 \times H^{m-\kappa+\theta}(\mathbb{R}^3) \).

**Proof.** Integrating by parts and employing the divergence theorem we see that for any \(v, w \in (C^2(\Omega)^3)\) and \(p \in C^1(\Omega)^3\) there holds the following integral identity (cf. [22]):

\[
\frac{1}{2} \int_{\Omega} \left[ \nabla \otimes v + (\nabla \otimes v)^T \right] \cdot \left[ \nabla \otimes w + (\nabla \otimes w)^T \right] dx - \int_{\Omega} \left[ p + \frac{2}{3}(\nabla \cdot v) \right] \nabla \cdot w dx = \int_{\Omega} \left[ -\Delta v + \nabla p - \frac{1}{3}(\nabla \cdot v) \right] \cdot w dx + \int_{\Gamma_p} T(v, p) n \cdot w dS_x. \tag{43}
\]

Here, for two matrix \(A\) and \(B\) we use the notation \(A \cdot B\) to denote the inner product of \(A\) and \(B\), i.e., \(A \cdot B = A_{ij}B_{ij}\). By making the Hanzawa transformation, we see that for any \(\tilde{v}, \tilde{w} \in (C^2(\mathbb{R}^3))^3\) and \(\tilde{p} \in C^1(\mathbb{R}^3)\) there holds

\[
\frac{1}{2} \int_{|x|<1} \left[ \tilde{B}(\rho) \otimes \tilde{v} + (\tilde{B}(\rho) \otimes \tilde{v})^T \right] \cdot \left[ \tilde{B}(\rho) \otimes \tilde{w} + (\tilde{B}(\rho) \otimes \tilde{w})^T \right] G_\rho dx - \int_{|x|<1} \left[ \tilde{p} + \frac{2}{3}\tilde{B}(\rho) \cdot \tilde{v} \right] \tilde{B}(\rho) \cdot \tilde{w} G_\rho dx = \int_{|x|<1} \left[ -A(\rho) \tilde{v} + \tilde{B}(\rho) \tilde{p} - \frac{1}{3}\tilde{B}(\rho) (\tilde{B}(\rho) \cdot \tilde{v}) \right] \cdot \tilde{w} G_\rho dx + \int_{|x|=1} \tilde{T}_\rho(\tilde{v}, \tilde{p}) n \cdot \tilde{w} H_\rho dS_x. \tag{44}
\]

Besides, clearly \(\nabla \otimes w_j + (\nabla \otimes w_j)^T = 0, \nabla \cdot w_j = 0, j = 1, 2, 3\), which yield, after the Hanzawa transformation, that

\[
\tilde{B}(\rho) \otimes \tilde{w}_j^\rho + (\tilde{B}(\rho) \otimes \tilde{w}_j^\rho)^T = 0, \quad \tilde{B}(\rho) \cdot \tilde{w}_j^\rho = 0, \quad j = 1, 2, 3.
\]

Hence, if \((\tilde{v}, \tilde{p})\) is a solution of (36)–(40), then by taking \(\tilde{w} = \tilde{w}_j^\rho (j = 1, 2, 3)\) in (44), we see that (41) holds. Similarly we have (42). This proves the necessity of (41) and (42).

Next we assume that the conditions (41) and (42) are satisfied, and proceed to prove that there exists a unique solution to the problem (36)–(40). We first prove uniqueness of the solution. Let \((\tilde{v}_1, \tilde{p}_1)\) and \((\tilde{v}_2, \tilde{p}_2)\) be two solutions of (36)–(40). Then \(\tilde{v} = \tilde{v}_1 - \tilde{v}_2\) and \(\tilde{p} = \tilde{p}_1 - \tilde{p}_2\) satisfy the corresponding homogeneous equations. Thus, by letting \(\tilde{w} = \tilde{v}\) in (44), we get

\[
\int_{|x|<1} \left[ \tilde{B}(\rho) \otimes \tilde{v} + (\tilde{B}(\rho) \otimes \tilde{v})^T \right] \cdot \left[ \tilde{B}(\rho) \otimes \tilde{v} + (\tilde{B}(\rho) \otimes \tilde{v})^T \right] G_\rho(x) dx = 0,
\]

so that

\[
\tilde{B}(\rho) \otimes \tilde{v} + (\tilde{B}(\rho) \otimes \tilde{v})^T = 0 \quad \text{in} \quad \mathbb{R}^3.
\]

This combined with (39) and (40) yields, by the Korn inequality

\[
\|u\|^2_{H^1(\Omega))^3} \leq C_1\|S(u)\|^2_{L^2(\Omega))^{3 \times 3}} + C_2 \left( \left\| \int_{\Omega} u dx \right\|^2 + \left\| \int_{\Omega} u \times x dx \right\|^2 \right), \tag{45}
\]

where \(S(u) = \nabla \otimes u + (\nabla \otimes u)^T\) (cf. Proposition 8.1 of [25]), that \(\tilde{v} = 0\). From this it follows immediately that also \(\tilde{p} = 0\). Hence the solution is unique if it exists.
To prove existence we denote, for a given $0 \leq k \leq m-2$,
\[
\begin{align*}
\mathbb{X} &= (h^{m-k+\theta}(\mathbb{R}^3))^3 \times h^{m-k-1+\theta}(\mathbb{R}^3) \times \mathbb{R}^6, \\
\mathbb{Y} &= h^{m-k-1+\theta}(\mathbb{R}^3) \times (h^{m-k-2+\theta}(\mathbb{R}^3))^3 \times (h^{m-k-1+\theta}(\mathbb{S}^2))^3 \times \mathbb{R}^3 \times \mathbb{R}^3,
\end{align*}
\]
and regard $O^{m+\theta}(\mathbb{S}^2)$ as an open subset of the Banach space $h^{m+\theta}(\mathbb{S}^2)$. For every $\rho \in O^{m+\theta}(\mathbb{S}^2)$, we define a linear operator $L(\rho) : \mathbb{X} \to \mathbb{Y}$ as follows:

\[
L(\rho)U = \begin{bmatrix}
\bar{B}(\rho) \cdot \bar{v} \\
-\mathcal{A}(\rho)\bar{v} + \tilde{B}(\rho)\tilde{p} + l_\rho(\zeta) \\
T(\bar{v}, \tilde{p})\bar{\eta}_\rho \\
\int_{|x|<1} \bar{v}(x)G_\rho(x)dx \\
\int_{|x|<1} \bar{v}(x) \cdot \tilde{w}_j(x)G_\rho(x)dx
\end{bmatrix}_T
\quad \text{for } U = (\bar{v}, \tilde{p}, \zeta) \in \mathbb{X},
\]
where $l_\rho$ is the linear operator from $\mathbb{R}^6$ to $(h^{m-k-2+\theta}(\mathbb{R}^3))^3$ defined by $l_\rho(\zeta) = a + b_1\tilde{w}_1 + b_2\tilde{w}_2 + b_3\tilde{w}_3$ for $\zeta = (a, b_1, b_2, b_3, \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$. By (24) it is clear that

\[
\mathcal{L} \in C^\infty(O^{m+\theta}(\mathbb{S}^2), L(\mathbb{X}, \mathbb{Y})),
\]
and

\[
L(0)U = \begin{bmatrix}
\nabla \cdot \nu \\
-\Delta \nu + \nabla p + l_0(\zeta) \\
T(\nu, p)\eta \\
\int_{|x|<1} \nu(x)dx \\
\int_{|x|<1} \nu(x) \times xdx
\end{bmatrix}_T
\quad \text{for } U = (\nu, p, \zeta) \in \mathbb{X},
\]
where $l_0$ is the linear operator from $\mathbb{R}^6$ to $(h^{m-k-2+\theta}(\mathbb{R}^3))^3$ defined by $l_0(\zeta) = a + b_1w_1 + b_2w_2 + b_3w_3$ for $\zeta = (a, b_1, b_2, b_3) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$. From the proof of Theorem 2.1 of [15] we see that $L(0)$ is an isomorphism from $\mathbb{X}$ to $\mathbb{Y}$ (cf. also Lemma A.1 of [9]) since all isomorphisms from $\mathbb{X}$ to $\mathbb{Y}$ forms an open set in $L(\mathbb{X}, \mathbb{Y})$, we conclude that for $\delta$ sufficiently small, $L(\rho)$ is also an isomorphism from $\mathbb{X}$ to $\mathbb{Y}$ for any $\rho \in O^{m+\theta}(\mathbb{S}^2)$. This particularly implies that given $\varphi \in h^{m-k-1+\theta}(\mathbb{R}^3)$, $\mathbf{g} \in (h^{m-k-2+\theta}(\mathbb{R}^3))^3$ and $\mathbf{h} \in (h^{m-k-1+\theta}(\mathbb{R}^3))^3$, there exist unique $\mathbf{v} \in (h^{m-k+\theta}(\mathbb{R}^3))^3$, $\mathbf{\phi} \in h^{m-k-1+\theta}(\mathbb{R}^3)$ and $\zeta \in \mathbb{R}^6$ such that they satisfy (36), (38)–(40) and

\[
-\mathcal{A}(\rho)\bar{v} + \tilde{B}(\rho)\tilde{p} + l_\rho(\zeta) = \mathbf{g} \quad \text{in } \mathbb{R}^3. \tag{46}
\]
We claim that $\zeta = 0$. Indeed, taking $\tilde{w} = \tilde{w}_j, \bar{e}_j$ in (44) and using (36), (38), (41), (42) and (46), we get

\[
\int_{|x|<1} l_\rho(\zeta) \cdot \tilde{w}_j G_\rho dx = 0, \quad \int_{|x|<1} l_\rho(\zeta) \cdot \bar{e}_j G_\rho dx = 0, \quad j = 1, 2, 3.
\]
From these relations we can easily show that if $\rho = 0$ then $\zeta = 0$. By continuity (a small perturbation of a nonsingular matrix is still nonsingular), this implies that if $\delta$ is sufficiently small then for any $\rho \in O^{m+\theta}(\mathbb{S}^2)$ we also have $\zeta = 0$. Hence our
claim is true. It follows that \((\bar{\nu}, \bar{\rho})\) is a solution of (36)–(40). This completes the proof of Lemma 2.3. \(\square\)

Lemma 2.4. For the solution of the problem (36)–(40), we have \(\bar{\nu} = \bar{\mathcal{P}}(\rho)\phi + Q(\rho)g + R(\rho)h\), where

\[
\begin{align*}
\bar{\mathcal{P}} &\in \bigcap_{k=0}^{m-2} \mathcal{C}^\infty(O_3^{m+\theta}(\mathbb{S}^2), L(h^{m-k-1+\theta}(\mathbb{B}^3)), \langle h^{m-k+\theta}(\mathbb{B}^3) \rangle^3), \\
Q &\in \bigcap_{k=0}^{m-2} \mathcal{C}^\infty(O_3^{m+\theta}(\mathbb{S}^2), L(h^{m-k-2+\theta}(\mathbb{B}^3))^3, \langle h^{m-k+\theta}(\mathbb{B}^3) \rangle^3), \\
R &\in \bigcap_{k=0}^{m-2} \mathcal{C}^\infty(O_3^{m+\theta}(\mathbb{S}^2), L(h^{m-k-1+\theta}(\mathbb{S}^2))^3, \langle h^{m-k+\theta}(\mathbb{B}^3) \rangle^3).
\end{align*}
\]

Proof. Let notations be as in the proof of Lemma 2.3. We denote by \(\mathcal{I}_1, \mathcal{I}_2\) and \(\mathcal{I}_3\) the natural embedding operators from \(h^{m-k-1+\theta}(\mathbb{B}^3)\), \((h^{m-k-2+\theta}(\mathbb{B}^3))^3\) and \((h^{m-k-1+\theta}(\mathbb{S}^2))^3\), respectively, into \(\mathbb{Y}\), and by \(\mathcal{J}\) the projection operator from \(\mathbb{X}\) onto \((h^{m-k+\theta}(\mathbb{B}^3))^3\). Then by letting

\[
\bar{\mathcal{P}}(\rho) = \mathcal{J} \circ \mathcal{L}(\rho)^{-1} \circ \mathcal{I}_1, \quad Q(\rho) = \mathcal{J} \circ \mathcal{L}(\rho)^{-1} \circ \mathcal{I}_2, \quad R(\rho) = \mathcal{J} \circ \mathcal{L}(\rho)^{-1} \circ \mathcal{I}_3,
\]

we immediately see that the desired assertion follows. \(\square\)

The system of equations (26), (27) and (29)–(31) can be rewritten in the form of (36)–(40), with

\[
\phi = g(\bar{\sigma}), \quad g = \frac{1}{3} \mathcal{B}(\rho)g(\bar{\sigma}), \quad h = -\gamma \bar{\kappa}_\rho \bar{\mathbf{n}}_\rho.
\]

We assert that (41) and (42) are satisfied by these functions. Indeed, since \(g = \frac{1}{3} \mathcal{B}(\rho)\phi\), this assertion follows if we show that

\[
\int_{|x|=1} h(x) \cdot \bar{w}^j H_\rho(x) dS_x = 0, \quad \int_{|x|=1} h(x) \cdot e_j H_\rho(x) dS_x = 0, \quad j = 1, 2, 3.
\]

Let \(\Delta_\rho\) be the Laplace-Beltrami operator on \(\Gamma_\rho\). Then we have \(\kappa(x)\mathbf{n}(x) = -\Delta_\rho x\) for \(x \in \Gamma_\rho\) (cf. [9], [25]). Since \(\Delta_\rho\) is a symmetric operator in \(L^2(\Gamma_\rho, dS_x)\), we see that

\[
\int_{\Gamma_\rho} (\Delta_\rho x_i \cdot x_j - \Delta_\rho x_j x_i) dS_x = 0, \quad i, j = 1, 2, 3.
\]

Thus

\[
\int_{|x|=1} \bar{\kappa}_\rho \bar{\mathbf{n}}_\rho \cdot \bar{w}^j H_\rho dS_x = \int_{\Gamma_\rho} \kappa_\rho \mathbf{n}_\rho \cdot \mathbf{w}_j dS_x = - \int_{\Gamma_\rho} \Delta_\rho x \cdot \mathbf{w}_j dS_x = 0, \quad j = 1, 2, 3.
\]

Similarly we have

\[
\int_{|x|=1} \bar{\kappa}_\rho \bar{\mathbf{n}}_\rho \cdot e_j H_\rho dS_x = \int_{\Gamma_\rho} \kappa_\rho \mathbf{n}_\rho \cdot e_j dS_x = - \int_{\Gamma_\rho} \Delta_\rho x \cdot e_j dS_x = 0, \quad j = 1, 2, 3.
\]

This verifies (48).

Now, given \(\rho \in O_3^{m+\theta}(\mathbb{S}^2)\), we first use Lemma 2.2 to solve the equations (25) and (28). This gives \(\bar{\sigma} = \mathcal{R}(\rho) \in h^{m+\theta}(\mathbb{B}^3)\). Next we use Lemma 2.3 to solve equations (26), (27) and (29)–(31). Note that with \(\phi, g\) and \(h\) given in (47), we have \(\phi \in h^{m+\theta}(\mathbb{B}^3) \subseteq h^{m-2+\theta}(\mathbb{B}^3)\), \(g \in (h^{m-1+\theta}(\mathbb{B}^3))^3 \subseteq (h^{m-3+\theta}(\mathbb{B}^3))^3\), and, by (44), \(h \in (h^{m-2+\theta}(\mathbb{S}^2))^3\). Hence, by Lemma 2.3 (with \(k = 1\)) it follows that these
equations have a unique solution \((\vec{v}, \vec{p}) \in (h^{m-1+\theta}(\mathbb{R}^3))^3 \times h^{m-2+\theta}(\mathbb{R}^3)\). Moreover, since \(\bar{\sigma} = \mathcal{R}(\rho)\), by Lemma 2.4 we have
\[
\bar{v} = \bar{P}(\rho)g(\mathcal{R}(\rho)) + \frac{1}{3}Q(\rho)\mathcal{E}(\rho)\bar{g}(\mathcal{R}(\rho)) - \gamma \mathcal{R}(\rho)(\bar{\kappa}(\rho)\bar{\n}(\rho)).
\] (50)
where \(\mathcal{K}(\rho) = \tilde{\kappa}_\rho\) and \(\bar{\n}(\rho) = \bar{\n}_\rho\). We note that
\[
\mathcal{K} \in C^\infty(O_{\delta}^{m+\theta}(\mathbb{S}^2), h^{m-2+\theta}(\mathbb{S}^2)), \quad \bar{\n} \in C^\infty(O_{\delta}^{m+\theta}(\mathbb{S}^2), (h^{m-1+\theta}(\mathbb{S}^2))^3).
\] (51)
Substituting the expression of \(\bar{v}\) in (50) into (32), and introducing the operator \(Q: O_{\delta}^{m+\theta}(\mathbb{S}^2) \rightarrow h^{m-1+\theta}(\mathbb{S}^2)\) by
\[
Q(\rho) = \text{tr}_{S^2} \left[ \bar{P}(\rho)g(\mathcal{R}(\rho)) + \frac{1}{3}Q(\rho)\mathcal{E}(\rho)\bar{g}(\mathcal{R}(\rho)) - \gamma \mathcal{R}(\rho)(\bar{\kappa}(\rho)\bar{\n}(\rho)) \right] \left[ \omega - \frac{\nabla_\omega \rho}{1 + \rho} \right].
\] (52)
(for \(\rho \in O_{\delta}^{m+\theta}(\mathbb{S}^2)\)), where as before \(\omega\) represents the variable in \(\mathbb{S}^2\), we see that the problem (25)–(33) is reduced into the following initial value problem for a differential equation in the Banach space \(h^{m-1+\theta}(\mathbb{S}^2)\):
\[
\begin{align*}
\partial_t \rho &= Q(\rho), & t > 0, \\
\rho|_{t=0} &= \rho_0.
\end{align*}
\] (53)
Here \(Q\) is regarded as an unbounded operator in \(h^{m-1+\theta}(\mathbb{S}^2)\) with domain \(O_{\delta}^{m+\theta}(\mathbb{S}^2)\).

We summarize:

**Lemma 2.5.** Let \((\bar{\sigma}, \bar{v}, \bar{p}, \rho)\) be a solution of the problem (25)–(33). Then \(\rho\) is a solution of the initial value problem (53). Conversely, if \(\rho\) is a solution of the initial value problem (53), then by letting \(\bar{\sigma} = \mathcal{R}(\rho)\) and \((\bar{v}, \bar{p}) = P\mathcal{L}(\rho)^{-1}(g(\bar{\sigma}), \frac{1}{3}\mathcal{E}(\rho)g(\bar{\sigma}), -\gamma \mathcal{R}(\rho)(\bar{\kappa}(\rho)\bar{n}(\rho), 0, 0)\), where \(P\) denotes projection from \(X = (h^{m-1+\theta}(\mathbb{R}^3))^3 \times h^{m-2+\theta}(\mathbb{R}^3)\) onto \((h^{m-1+\theta}(\mathbb{S}^2))^3 \times h^{m-2+\theta}(\mathbb{S}^2)\), we have that \((\bar{\sigma}, \bar{v}, \bar{p}, \rho)\) is a solution of (25)–(33).

From (24), (35), (51) and Lemma 2.4 we see that
\[
Q \in C^\infty(O_{\delta}^{m+\theta}(\mathbb{S}^2), h^{m-1+\theta}(\mathbb{S}^2)).
\] (54)
In the sequel we prove that if \(\delta\) is sufficiently small then for any \(\rho \in O_{\delta}^{m+\theta}(\mathbb{S}^2)\), \(DQ(\rho)\) is a infinitesimal generator of an analytic semigroup in \(h^{m-1+\theta}(\mathbb{S}^2)\) with domain \(h^{m+\theta}(\mathbb{S}^2)\), so that the differential equation in (53) is of parabolic type. Here and in what follows, the notation \(D\) represents Fréchet derivatives of smooth operators from \(h^{m+\theta}(\mathbb{S}^2)\) to \(h^{m-1+\theta}(\mathbb{S}^2)\).

We first note that the mean curvature operator \(\mathcal{K}\) has the following expression
\[
\mathcal{K}(\rho) = \mathcal{K}_1(\rho)\rho + \mathcal{K}_0(\rho),
\] (55)
where, for each \(\rho\), \(\mathcal{K}_1(\rho)\) is a second-order linear elliptic partial differential operator on \(\mathbb{S}^2\) with coefficients being functions of \(\rho\) and its first-order derivatives, and \(\mathcal{K}_0\) is a first-order nonlinear partial differential operator on \(\mathbb{S}^2\), so that
\[
\begin{align*}
\mathcal{K}_1 &\in C^{\infty}(O_{\delta}^{m+\theta}(\mathbb{S}^2), L(h^{m+\theta}(\mathbb{S}^2), h^{m-2+\theta}(\mathbb{S}^2))), \\
\mathcal{K}_0 &\in C^{\infty}(O_{\delta}^{m+\theta}(\mathbb{S}^2), h^{m-1+\theta}(\mathbb{S}^2)).
\end{align*}
\] (56)
(see [7] and [10]). Substituting (55) into (52) we see that
\[
Q(\rho) = Q_2(\rho)[\rho, \rho] + Q_1(\rho)\rho + Q_0(\rho),
\] (57)
where, for each \( \rho \), \( Q_2(\rho) \) is a bilinear operator, \( Q_1(\rho) \) is a linear operator, and \( Q_0 \) is a nonlinear operator; they are respectively defined as follows:

\[
Q_2(\rho)[\eta_1, \eta_2] = \gamma \text{tr}_{S^2} \left\{ R(\rho) [K_1(\rho) \eta_1 \mathcal{N}(\rho)] \right\} \frac{\nabla_\omega \eta_2}{1 + \rho},
\]
\[
Q_1(\rho) \eta = -\text{tr}_{S^2} \left\{ \tilde{F}(\rho) g(\mathcal{R}(\rho)) + \frac{1}{3} Q(\rho) \tilde{B}(\rho) g(\mathcal{R}(\rho)) - \gamma R(\rho) [K_0(\rho) \mathcal{N}(\rho)] \right\} \frac{\nabla_\omega \eta}{1 + \rho} - \gamma \text{tr}_{S^2} \left\{ R(\rho) [K_1(\rho) \eta \mathcal{N}(\rho)] \right\} \cdot \omega,
\]
\[
Q_0(\rho) = \text{tr}_{S^2} \left\{ \tilde{F}(\rho) g(\mathcal{R}(\rho)) + \frac{1}{3} Q(\rho) \tilde{B}(\rho) g(\mathcal{R}(\rho)) - \gamma R(\rho) [K_0(\rho) \mathcal{N}(\rho)] \right\} \cdot \omega.
\]

We note that, by (35), (51), (56) and Lemma 2.4 we have

\[
Q_2 \in C^\infty(\mathcal{O}_\delta^{m+\theta}(S^2), BL(h^{m+\theta}(S^2) \times h^{m+\theta}(S^2), h^{m-1+\theta}(S^2))),
\]
\[
Q_1 \in C^\infty(\mathcal{O}_\delta^{m+\theta}(S^2), L(h^{m+\theta}(S^2), h^{m-1+\theta}(S^2))),
\]
\[
Q_0 \in C^\infty(\mathcal{O}_\delta^{m+\theta}(S^2), h^{m+\theta}(S^2)),
\]

where \( BL(\cdot \times \cdot, \cdot) \) denotes the Banach space of all bilinear operators with respect to the corresponding spaces.

Given two Banach spaces \( E_0 \) and \( E_1 \) such that \( E_1 \) is continuously and densely embedded into \( E_0 \), we denote by \( \mathcal{H}(E_1, E_0) \) the subset of all linear operators \( A \in L(E_1, E_0) \) such that \( -A \) generates a strongly continuous analytic semigroup on \( E_0 \).

**Lemma 2.6.** \( -DQ(0) \in \mathcal{H}(h^{m+\theta}(S^2), h^{m-1+\theta}(S^2)) \).

**Proof.** For any \( \rho \in \mathcal{O}_\delta^{m+\theta}(S^2) \) and \( \eta \in h^{m+\theta}(S^2) \) we have

\[
DQ(\rho) \eta = Q_2(\rho)[\rho, \eta] + Q_2(\rho)[\rho, \eta] + [DQ_2(\rho) \eta][\rho, \rho] + Q_1(\rho) \eta + [DQ_1(\rho) \eta] \rho + DQ_0(\rho) \eta.
\]

In particular,

\[
DQ(0) \eta = \lambda(0) \eta + DQ(0) \eta \quad \text{for} \quad \eta \in h^{m+\theta}(S^2), \tag{58}
\]

i.e., \( DQ(0) = \lambda(0) + DQ(0) \). We note that \( \lambda(0) \in L(h^{m+\theta}(S^2), h^{m-1+\theta}(S^2)) \) and \( DQ(0) \in L(h^{m+\theta}(S^2), h^{m+\theta}(S^2)) \). Thus, by a standard perturbation result for infinitesimal generators of continuous analytic semigroups (see [1] and [24]), the desired assertion follows if we prove that \( -\lambda(0) \in \mathcal{H}(h^{m+\theta}(S^2), h^{m-1+\theta}(S^2)) \).

Since (cf. [20])

\[
\mathcal{K}(\varepsilon) = 1 - \varepsilon[\eta(\omega) + \frac{1}{2} \Delta_\omega \eta(\omega)] + o(\varepsilon),
\]

where \( \Delta_\omega \) is the Laplace-Beltrami operator on the sphere \( S^2 \), we have \( \lambda(0) = \mathcal{K}(0) = 1 \) and \( \lambda(0) \eta = -\frac{1}{4} \Delta_\omega \eta \). Hence, from the definition of \( \lambda \) we see that

\[
\lambda(0) \eta = -v_{|S^2} \cdot \nabla_\omega \eta - \gamma u_{|S^2} \cdot \omega, \tag{59}
\]

where \( v \) is the solution of the following boundary value problem:

\[
\Delta \sigma = f(\sigma) \quad \text{in} \quad |x| < 1,
\]
\[
\nabla \cdot v = g(\sigma) \quad \text{in} \quad |x| < 1,
\]
\[
-\Delta v + \nabla p - \frac{1}{3} \nabla (\nabla \cdot v) = 0 \quad \text{in} \quad |x| < 1,
\]
σ = 1 on |x| = 1,
\begin{align*}
\mathbf{T}(\mathbf{v}, p) \mathbf{n} &= -\gamma \mathbf{n} \quad \text{on } |x| = 1, \\
\int_{|x|<1} \mathbf{v} \, dx &= 0, \\
\int_{|x|<1} \mathbf{v} \times dx &= 0,
\end{align*}
and \( \mathbf{u}_\eta = -\frac{1}{2} \mathbf{R}(0)(\Delta_S \eta \cdot \mathbf{n}) \). Clearly, \( \mathbf{v} = \mathbf{v}_s \) — the radially symmetric stationary solution of the problem (1)–(8) (see Appendix A), so that \( \mathbf{v}_{|S^2} = 0 \). It follows that
\begin{equation}
Q_1(0) \eta = -\gamma \mathbf{u}_\eta|_{S^2} \cdot \omega = -\gamma \mathbf{u}_\eta|_{S^2} \cdot \mathbf{n} = \frac{\gamma}{2} \mathbf{n} \cdot \mathbf{R}(0)[\mathbf{n} \cdot \Delta_S \eta] \equiv -\frac{\gamma}{2} A_1 \eta. \tag{60}
\end{equation}
Define \( A_0 = \partial_\eta(\Delta, tr_{S^2})^{-1}(0, \cdot) \), i.e., \( A_0 \eta = \partial_\eta \psi_\eta \) for \( \eta \in h^{m+\theta}(S^2) \), where \( \psi_\eta \) is the solution of the following boundary value problem:
\[ \Delta \psi_\eta = 0 \quad \text{in } |x| < 1, \quad \psi_\eta = \eta \quad \text{on } |x| = 1. \tag{61} \]
It is well-known that (cf. [9]),
\begin{equation}
A_0 \in C^\infty(h^{m+\theta}(S^2), h^{m+1+\theta}(S^2)) \cap H(h^{m+\theta}(S^2), h^{m+1+\theta}(S^2)). \tag{62}
\end{equation}
We rewrite
\[ Q_1(0) = -\frac{\gamma}{4} A_0 - \frac{\gamma}{4}(2A_1 - A_0). \]
In what follows we prove that
\[ 2A_1 - A_0 \in L(h^{m+\theta}(S^2), h^{m+\theta}(S^2)). \tag{63} \]
Note that if this assertion is proved, then the desired assertion follows.
Since \( A_0 \eta = \partial_\eta \psi_\eta \) and \( A_1 \eta = 2\mathbf{u}_\eta|_{S^2} \cdot \mathbf{n} \), we have \( (2A_1 - A_0) \eta = 4\mathbf{u}_\eta|_{S^2} \cdot \mathbf{n} - \partial_\eta \psi_\eta \).
Since \( \mathbf{u}_\eta = -\frac{1}{2} \mathbf{R}(0)(\Delta_S \eta \cdot \mathbf{n}) \), by definition of the operator \( \mathbf{R}(0) \) we see that there exist \( q_\eta \in h^{m+1+\theta}(S^2) \) and \( \zeta_\eta \in \mathbb{R}^6 \) such that
\[ \mathcal{L}(0)(\mathbf{u}_\eta, q_\eta, \zeta_\eta) = (0, 0, -\frac{1}{2} \Delta_S \eta \cdot \mathbf{n}, 0, 0). \]
Besides, a simple computation shows that
\[ \mathcal{L}(0)(\nabla \psi_\eta, 0, 0) = (0, 0, \mathbf{T}(\nabla \psi_\eta, 0, \mathbf{n}), \int_{|x|<1} \nabla \psi_\eta \, dx, \int_{|x|<1} \nabla \psi_\eta \times dx) \]
\[ = (0, 0, 2\partial_\eta \nabla \psi_\eta, \int_{|x|=1} \eta \cdot \mathbf{n} \, dS_x, 0). \]
Since \( 0 = \Delta \psi_\eta |_{r=1} = \left( \frac{1}{r^2} \frac{\partial^2 \psi_\eta}{\partial r^2} + \frac{2}{r} \frac{\partial \psi_\eta}{\partial r} + \frac{1}{r^2} \Delta_S \psi_\eta \right) \bigg|_{r=1} = \frac{\partial^2 \psi_\eta}{\partial r^2} \bigg|_{r=1} + 2\partial_r \psi_\eta + \Delta_S \psi_\eta, \)
we have
\[ \partial_\eta \nabla \psi_\eta = \frac{\partial}{\partial r} \nabla \psi_\eta \bigg|_{r=1} = \frac{\partial^2 \psi_\eta}{\partial r^2} \bigg|_{r=1} \mathbf{n} + \nabla_\omega \left( \frac{\partial \psi_\eta}{\partial r} \bigg|_{r=1} \right) \]
\[ = -\Delta_S \psi_\eta \cdot \mathbf{n} - 2\partial_\eta \psi_\eta \cdot \mathbf{n} + \nabla_\omega (\partial_\eta \psi_\eta). \]
Hence
\[ \mathcal{L}(0)(4\mathbf{u}_\eta - \nabla \psi_\eta, 4q_\eta, 4\zeta_\eta) = (0, 0, 4\partial_\eta \psi_\eta \cdot \mathbf{n} - 2\nabla_\omega (\partial_\eta \psi_\eta), \int_{|x|=1} \eta \cdot \mathbf{n} \, dS_x, 0). \]
It follows that
\[(2A_1 - A_0)\eta = 4u_\eta |_{S^2} \cdot n - \partial_n \psi_\eta = (4u_\eta - \nabla \psi_\eta)|_{S^2} \cdot n \]
\[= \text{tr}_{S^2} \left\{ \mathcal{J} \mathcal{L}(0)^{-1} \left( 0, 0, 4\partial_n \psi_\eta \cdot n - 2\nabla_\omega (\partial_n \psi_\eta), \int_{|x|=1} \eta \cdot ndS_x, 0 \right) \right\} \cdot n \]
\[= -2R(0)(\nabla_\omega (\partial_n \psi_\eta))|_{S^2} \cdot n \]
\[+ \text{tr}_{S^2} \left\{ \mathcal{J} \mathcal{L}(0)^{-1} \left( 0, 0, 4\partial_n \psi_\eta \cdot n, \int_{|x|=1} \eta \cdot ndS_x, 0 \right) \right\} \cdot n \]
\[\equiv B_1 \eta + B_0 \eta.\]

It can be easily seen that \(B_0 \in L(h^{m+\theta}(S^2), h^{m+\theta}(S^2))\). Furthermore, minor changes to the proof of Lemma A.2 in [9] show that also \(B_1 \in L(h^{m+\theta}(S^2), h^{m+\theta}(S^2))\) (see Lemma B.1 and Corollary 2 in Appendix B for details). Hence (63) follows. This completes the proof of Lemma 2.6.

Since \(H(h^{m+\theta}(S^2), h^{m-1+\theta}(S^2))\) is open in \(L(h^{m+\theta}(S^2), h^{m-1+\theta}(S^2))\), from Lemma 2.6 we immediately get

**Corollary 1.** For sufficiently small \(\delta\) we have
\[- DQ(\rho) \in H(h^{m+\theta}(S^2), h^{m-1+\theta}(S^2))\]
\[\text{for } \rho \in O^{m+\theta}(S^2).\]  

By this corollary we see that, at least in a small neighborhood of the origin, the differential equation (53) is of the parabolic type in the sense of Amann [1] and Lunardi [24], so that the geometric theory for parabolic differential equations in Banach spaces presented in these literatures can be applied to (53). In the following sections we shall use this theory to prove Theorem 1.1.

3. Linearization. In this section we compute the spectrum of the operator \(DQ(0)\). Note that since \(h^{m+\theta}(S^2)\) is compactly embedded in \(h^{m-1+\theta}(S^2)\), by Lemma 2.6 we see that the spectrum of the operator \(DQ(0)\) consists of all eigenvalues.

To compute the eigenvalues of \(DQ(0)\) we first derive a useful expression of this operator. Consider a perturbation of the radially symmetric stationary solution \((\sigma_s, v_s, p_s, 0)\) (see (12)):
\[
s(x, t) = \sigma_s(r) + \varepsilon \phi(r,\omega,t), \]
\[
v(x, t) = v_s(x) + \varepsilon \psi(r,\omega,t), \]
\[
p(x, t) = p_s(r) + \varepsilon \psi(r,\omega,t), \]
\[
\Omega(t) = \{ x \in \mathbb{R}^3 : r < 1 + \varepsilon \eta(\omega,t) \} \quad (r = |x|, \omega \in S^2), \]
where \(\varepsilon\) is a small parameter, and \(\phi, \psi = (v_1, v_2, v_3), \psi \) and \(\eta\) are new unknown functions. From [17] and [18] we see that the linearizations of equations (1)–(8) are respectively as follows:
\[
\Delta \phi = f'(\sigma_s)\phi \quad \text{in } \mathbb{B}^3, \quad \text{(65)}
\]
\[
\nabla \cdot \psi = g'(\sigma_s)\phi \quad \text{in } \mathbb{B}^3, \quad \text{(66)}
\]
\[
- \Delta \psi + \nabla \psi \cdot \frac{1}{3} \nabla(\nabla \cdot \psi) = 0 \quad \text{in } \mathbb{B}^3, \quad \text{(67)}
\]
\[
\phi = -\sigma_s'(1)\eta \quad \text{on } S^2, \quad \text{(68)}
\]
\[
T(\tilde{\psi}, \psi)n = -2g(1)\nabla_\omega \eta + \gamma(\eta + \frac{1}{2}\Delta_\omega \eta)n + 4g(1)\eta n \quad \text{on } S^2, \quad \text{(69)}
\]
\[
\partial_\nu \eta = \tilde{\psi} \big|_{S^2} \cdot n + g(1)\eta \quad \text{on } S^2, \quad \text{(70)}
\]
\[
\int_{|x|<1} \vec{v} \, dx = 0, \quad (71)
\]
\[
\int_{|x|<1} \vec{v} \times x \, dx = 0. \quad (72)
\]

Similarly as before, the system (65)–(72) can be reduced into a scalar equation in the unknown function \( \eta \) only. Indeed, given \( \eta \in C([0, \infty), H^{m+\theta}(S^2)) \), we first solve the second-order elliptic equation (65) subject to the boundary condition (68) to get \( \phi(\cdot, t) \in H^{m+\theta}(\mathbb{R}^3) \) as a functional of \( \eta \), and next substitute this solution \( \phi \) into (66). It can be easily checked that (41) and (42) are satisfied by equations (66), (67), (69), (71) and (72). Thus by using Lemma 2.3 we get a unique solution \( (\vec{v}(\cdot, t), \psi(\cdot, t)) \in H^{m-1+\theta} \times H^{m-2+\theta}(\mathbb{R}^3) \) as a functional of \( \eta \). Substituting \( \vec{v} = \vec{v}(r, \omega, t) \) obtained in this way into (70) and denoting
\[
\mathcal{B}_\gamma \eta = \left. \vec{v} \right|_{S^2} \cdot \mathbf{n} + g(1) \eta, \quad (73)
\]
we see that the system of equations (65)–(72) reduces into the scalar equation
\[
\partial_t \eta = \mathcal{B}_\gamma \eta. \quad (74)
\]

Now, since the problem (1)–(8) is equivalent to the equation (53) with \( Q(\rho) \) given by (52), its linearization should correspondingly be equivalent to the linearization of (53) which reads as follows:
\[
\partial_t \eta = D Q(0) \eta. \quad (75)
\]

Comparing (74) with (75), we get the following result:

**Lemma 3.1.** \( D Q(0) = \mathcal{B}_\gamma. \)

In the sequel we deduce the expression of \( \mathcal{B}_\gamma \) in terms of Fourier expansions of functions on the sphere \( S^2 \).

For each \( l \in \mathbb{N} \cup \{0\} \), let \( Y_{lm}(\omega) \) \((m = -l, -l + 1, \ldots, l - 1, l)\) be a normalized orthogonal basis of the space of all spherical harmonics of degree \( l \). Then \( \{Y_{lm}(\omega) : l = 0, 1, 2, \ldots ; m = -l, -l + 1, \ldots, l - 1, l\} \) is a normalized orthogonal basis of the scalar \( L^2 \)-space on \( S^2 \). As in [23], let \( \vec{V}_{lm}(\omega), \vec{X}_{lm}(\omega) \) and \( \vec{W}_{lm}(\omega) \), where \( l = 0, 1, 2, \ldots \) and \( m = -l, -l + 1, \ldots, l - 1, l \), be the corresponding vector spherical harmonics. From [23] we know that all these vector spherical harmonics form a normalized orthogonal basis of the vector \( L^2 \)-space on \( S^2 \) (see also Appendix A of [17] and Section 2 of [21] for this assertion). Besides, for every \( l \in \mathbb{N} \cup \{0\} \) we denote
\[
L_l = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l^2 + l - 2}{r^2}.
\]

For simplicity of the notation, we shall not write out the whole expansions of \( \phi \), \( \vec{v} \), \( \psi \) and \( \eta \), but instead merely consider each monomials in the expansions of these functions. This is reasonable because of the special forms of the operators appearing in the (65)–(72). Thus we put
\[
\eta = Y_{lm}(\omega).
\]

Then it can be easily verified that the corresponding solution of (65) and (68) is as follows:
\[
\phi(r, \omega) = F_l(r) Y_{lm}(\omega), \quad (76)
\]
where \( F_l(r) \) is the unique solution of the following problem:
\[
L_l F_l(r) = f'(\sigma(r)) F_l(r) \quad \text{for} \quad 0 < r < 1, \quad F_l(0) = 0, \quad F_l(1) = -\sigma'(1). \quad (77)
\]
Observe that by (66) and (67) we have
\[ \Delta (\psi - \frac{4}{3} g'(\sigma_s)\phi) = 0 \quad \text{in} \quad \mathbb{B}^3, \] (78)
\[ \Delta \vec{\psi} - \nabla (g'(\sigma_s)\phi) = \nabla (\psi - \frac{4}{3} g'(\sigma_s)\phi) \quad \text{in} \quad \mathbb{B}^3. \] (79)
Thus the solution of (66), (67), (69), (71) and (72) has the following expressions:
\[ \psi(r, \omega) = \frac{4}{3} g'(\sigma_s(r))\phi(r, \omega) + P_{lm}(r)Y_{lm}(\omega) \]
(80)
\[ = \frac{4}{3} g'(\sigma_s(r))F_l(r)Y_{lm}(\omega) + P_{lm}(r)Y_{lm}(\omega), \]
\[ \vec{\psi} = \vec{a} + \vec{b} \times x + v_{lm}\vec{V}_{lm} + x_{lm}\vec{X}_{lm} + w_{lm}\vec{W}_{lm}, \]
(81)
where \( \vec{a}, \vec{b} \) are unknown constant vectors, and \( P_{lm}(r), v_{lm}(r), x_{lm}(r) \) and \( w_{lm}(r) \) are unknown functions defined on \([0, 1] \) such that \( v_{lm}(0) = x_{lm}(0) = w_{lm}(0) = 0 \).

Using some well-known formulas for vector spherical harmonics (see [23] or [17], [21]), we have
\[ \nabla \phi = \left[ -F'_l(r) + \frac{l}{r}F_l(r) \right] \sqrt{\frac{l+1}{2l+1}} \vec{V}_{lm}(\omega) \]
(82)
\[ + \left[ F'_l(r) + \frac{l+1}{r}F_l(r) \right] \sqrt{\frac{l}{2l+1}} \vec{W}_{lm}(\omega), \]
\[ \nabla (g'(\sigma_s)\phi) = \left[ -\frac{\partial}{\partial r}[g'(\sigma_s(r))F_l(r)] + \frac{l}{r}g'(\sigma_s(r))F_l(r) \right] \sqrt{\frac{l+1}{2l+1}} \vec{V}_{lm}(\omega) \]
(83)
\[ + \left[ \frac{\partial}{\partial r}[g'(\sigma_s(r))F_l(r)] + \frac{l+1}{r}g'(\sigma_s(r))F_l(r) \right] \sqrt{\frac{l}{2l+1}} \vec{W}_{lm}(\omega) \]
\[ \equiv F'_l(r)\vec{V}_{lm}(\omega) + F^2_l(r)\vec{W}_{lm}(\omega), \]
\[ \nabla (\psi - \frac{4}{3} g'(\sigma_s)\phi) = \sqrt{\frac{l+1}{2l+1}} \left[ -\frac{\partial}{\partial r}P_{lm} + \frac{l}{r}P_{lm} \right] \vec{V}_{lm} \]
(84)
\[ + \sqrt{\frac{l}{2l+1}} \left[ \frac{\partial}{\partial r}P_{lm} + \frac{l+1}{r}P_{lm} \right] \vec{W}_{lm}, \]
\[ \nabla \cdot \vec{\psi} = -\left[ v_{lm}'(r) + \frac{l+2}{r}v_{lm}(r) \right] \sqrt{\frac{l+1}{2l+1}} Y_{lm}(\omega) \]
(85)
\[ + \left[ w_{lm}'(r) - \frac{l-1}{r}w_{lm}(r) \right] \sqrt{\frac{l}{2l+1}} Y_{lm}(\omega), \]
and
\[ \Delta \vec{\psi} = L_{l+1}(v_{lm}(r))\vec{V}_{lm}(\omega) + L_l(x_{lm}(r))\vec{X}_{lm}(\omega) + L_{l-1}(w_{lm}(r))\vec{W}_{lm}(\omega). \]
(86)
By (79), (83), (84) and (86), we have
\[ \sqrt{\frac{l+1}{2l+1}} \left[ -P'_{lm}(r) + \frac{l}{r}P_{lm}(r) \right] = L_{l+1}(v_{lm}(r)) - F'_l(r), \]
(87)
\[ L_l(x_{lm}(r)) = 0. \]
(88)
\[ \sqrt{\frac{l}{2l+1}} \left[ P'_{lm}(r) + \frac{l+1}{r}P_{lm}(r) \right] = L_{l-1}(w_{lm}(r)) - F^2_l(r). \]
(89)
By (78) we have
\[ L_l(P_{lm}(r)) = 0. \]
(90)
By (66) and (85) we have
\[
\sqrt{\frac{l}{2l+1}} \left[ w'_{lm}(r) - \frac{l-1}{r} w_{lm}(r) \right] - \sqrt{\frac{l+1}{2l+1}} \left[ v'_{lm}(r) + \frac{l+2}{r} v_{lm}(r) \right] = g'(\sigma_s(r)) F_l(r).
\] (91)

Solving the ODE problem (87)–(91), we get
\[
P_{lm}(r) = 2(2l+3)A_1 r^l, \quad x_{lm}(r) = B_1 r^l, \tag{92}
\]
\[
v_{lm}(r) = \sqrt{\frac{l+1}{2l+1} \frac{2l}{l+1}} A_1 r^{l+1}
- \frac{r}{(2l+3)} \sqrt{\frac{l+1}{2l+1}} g'(\sigma_s(r)) F_l(r) - \frac{r^{-l-2}}{2l+3} \int_0^r s^{l+3} F_l^1(s) \, ds \tag{93}
\]
\[
= \sqrt{\frac{l+1}{2l+1} \frac{2l}{l+1}} A_1 r^{l+1} - \tilde{v}(r),
\]
and
\[
w_{lm}(r) = C_1 r^{l-1} + \sqrt{\frac{l}{2l+1} \frac{2l}{l+1} (2l+3) A_1 r^{l+1}}
- \frac{r}{(2l-1)} \sqrt{\frac{l}{2l+1}} g'(\sigma_s(r)) F_l(r) - \frac{r^{l-1}}{2l-1} \int_r^{R_1} s^{-l+2} F_l^1(s) \, ds \tag{94}
\]
\[
= C_1 r^{l-1} + \sqrt{\frac{l}{2l+1} \frac{2l}{l+1} (2l+3) A_1 r^{l+1}} - \tilde{w}(r),
\]
where $A_1$, $B_1$ and $C_1$ are constants.

Next we consider the boundary condition (69). Again by using some well-known properties of vector spherical harmonics (see [23] or [17], [21]), we can rewrite $\tilde{v}$ in (81) as follows
\[
\tilde{v}(r, \omega) = a + b \times x + H_{l1}(r) Y_{lm}(\omega) \omega + H_{l2}(r) \nabla_\omega Y_{lm}(\omega), \tag{95}
\]
where
\[
H_{l1}(r) = - \sqrt{\frac{l+1}{2l+1}} v_{lm}(r) + \sqrt{\frac{l}{2l+1}} w_{lm}(r),
H_{l2}(r) = \frac{v_{lm}(r)}{\sqrt{(l+1)(2l+1)}} + \frac{w_{lm}(r)}{\sqrt{l(2l+1)}}.
\]

Thus
\[
T(\tilde{v}, \psi) \mathbf{n} = \frac{2}{3} g'(1) \sigma'_s(1) Y_{lm}(\omega) \omega + \left[ 2H_{l1}'(1) Y_{lm}(\omega) - \psi(1, \omega) \right] \omega
+ \left[ H_{l1}(1) - H_{l2}(1) + H_{l2}'(1) \right] \nabla_\omega Y_{lm}(\omega). \tag{96}
\]

Note that by (80) and (92) we have
\[
\psi(1, \omega) = Y_{lm}(\omega) \left[ - \frac{4}{3} g'(1) \sigma'_s(1) + 2(2l+3)A_1 \right].
\]

Substituting this expression into (96) we get
\[
T(\tilde{v}, \psi) \mathbf{n} = \left[ 2g'(1) \sigma'_s(1) + 2H_{l1}'(1) - 2(2l+3)A_1 \right] Y_{lm}(\omega) \omega
+ \left[ H_{l1}(1) - H_{l2}(1) + H_{l2}'(1) \right] \nabla_\omega Y_{lm}(\omega). \tag{97}
\]
On the other hand, putting $\eta = Y_{lm}(\omega)$ in (69) and using the well-known relation $\Delta_\omega Y_{lm}(\omega) = -l(l+1)Y_{lm}(\omega)$, we get

$$
\mathbf{T}(\vec{v}, \psi)\mathbf{n} = -2g(1)\nabla_\omega Y_{lm}(\omega) + \left[\gamma(1 - \frac{l(l+1)}{2}) + 4g(1)\right]Y_{lm}(\omega)\omega. \quad (98)
$$

Since $\nabla_\omega Y_{lm}(\omega)$ and $Y_{lm}(\omega)\omega$ are mutually orthogonal, by comparing their coefficients in (97) and (98) and using the relations

$$
H_{11}(1) = -\sqrt{\frac{l+1}{2l+1}}v_{lm}(1) + \sqrt{\frac{l}{2l+1}}w_{lm}(1),
$$

$$
H_{12}(1) = \frac{\sqrt{l(l+1)(2l+1)}}{\sqrt{l(l+1)(2l+1)}} + \frac{(l+1)}{\sqrt{l(l+1)(2l+1)}}w_{lm}(1),
$$

$$
H_{11}'(1) = -\sqrt{\frac{l+1}{2l+1}}v_{lm}'(1) + \sqrt{\frac{l}{2l+1}}w_{lm}'(1),
$$

$$
H_{12}'(1) = \frac{\sqrt{l(l+1)(2l+1)}}{\sqrt{l(l+1)(2l+1)}} + \frac{(l+1)}{\sqrt{l(l+1)(2l+1)}}w_{lm}'(1),
$$

we obtain

$$
-\sqrt{\frac{l+1}{2l+1}}v_{lm}'(1) + \sqrt{\frac{l}{2l+1}}w_{lm}'(1)
= \frac{\gamma}{4}(2 - l^2 - l) + 2g(1) - g'(1)\sigma_s'(1) + (2l + 3)A_1,
$$

and

$$
\frac{1}{\sqrt{2l+1}}\left[ - \frac{l+2}{\sqrt{l+1}}v_{lm}(1) + \frac{l-1}{\sqrt{l}}w_{lm}(1) \right]
+ \frac{v_{lm}'(1)}{\sqrt{l(l+1)(2l+1)}} + \frac{w_{lm}'(1)}{\sqrt{l(l+1)(2l+1)}} = -2g(1). \quad (100)
$$

We now proceed to consider the equation (70). By (73) and (93)–(95) we have

$$
\mathcal{E}_s Y_{lm}(\omega) = \vec{v} |_{s^2} \cdot \mathbf{n} + g(1)Y_{lm}(\omega) = \mathbf{a} \cdot \omega + [H_{11}(1) + g(1)]Y_{lm}(\omega)
= \mathbf{a} \cdot \omega + Y_{lm}(\omega)\left[ g(1) - \sqrt{\frac{l+1}{2l+1}}v_{lm}(1) + \sqrt{\frac{l}{2l+1}}w_{lm}(1) \right]
= \mathbf{a} \cdot \omega + Y_{lm}(\omega)\left[ g(1) + l(A_1 + \tilde{C}_1) + \sqrt{\frac{l+1}{2l+1}}\tilde{v}_l(1) \right]
- \sqrt{\frac{l}{2l+1}}\tilde{v}_l(1),
$$

where $\tilde{C}_1 = C_1/\sqrt{l(2l+1)}$. Thanks to the constrain condition (71) we see that

$$
a = -\frac{3}{4\pi} \int_{|x|<1} \{H_{11}(r)Y_{lm}(\omega) + H_{12}(r)\nabla_\omega Y_{lm}(\omega)\} \, dx
= 0 \quad \text{for } l \in \mathbb{N}, \; l \neq 1 \quad (102)
$$

(cf. (5.8) in [17]). To compute $A_1 + \tilde{C}_1$ we substitute (93) and (94) into (99) and (100), which gives

$$
\sqrt{\frac{l}{2l+1}}\tilde{v}_l'(1) - \sqrt{\frac{l+1}{2l+1}}\tilde{v}_l'(1) = -\frac{\gamma}{4}(2 - l^2 - l) - 2g(1) + g'(1)\sigma_s'(1) + \tilde{C}_1(l^2 - l) + (l^2 - l - 3)A_1,
$$

$$
(103)
$$
and
\[
\sqrt{\frac{l}{2l+1}}\left(\frac{l-1}{l}\tilde{v}_l(1) + \frac{1}{7}\tilde{v}'_l(1)\right) - \sqrt{\frac{l+1}{2l+1}}\left(\frac{l+2}{l+1}\tilde{v}_l(1) - \frac{1}{l+1}\tilde{v}'_l(1)\right) = 2g(1) + 2\tilde{C}_l(l-1) + \frac{2l^2 + 4l}{l+1}A_1.
\]
(104) yields:
\[
A_1 + \tilde{C}_l = \frac{1}{2(l-1)(2l^2 + 4l + 3)}\left\{\frac{2}{2}(1-l)(2l^2 + 5l + 2) - (4l + 2)g'(1)\sigma'_*(1) + (2l - 2)g(1) + \frac{4l^2 + 5l + 3}{\sqrt{l(2l+1)}}\tilde{w}_l(1) - \sqrt{\frac{l+1}{2l+1}}(4l - 1)\tilde{v}_l(1) + \frac{3(l^2 - 1)}{\sqrt{l(2l+1)}}\tilde{w}_l(1) - 3(l + 2)\sqrt{\frac{l+1}{2l+1}}\tilde{v}_l(1)\right\}.
\]
By the definitions of \(F_]^i\) (\(i = 1, 2\)), \(\tilde{v}_l\) and \(\tilde{w}_l\) respectively in (83), (93) and (94), and by straightforward calculation we easily have
\[
\tilde{v}_l(1) = \sqrt{\frac{l+1}{2l+1}}\int_0^1 g'(\sigma_s(r))F_i(r)r^{l+2} dr,
\]
\[
\tilde{w}_l(1) = -\sqrt{\frac{l}{2l+1}}\frac{g'(1)}{2l-1}\sigma'_*(1),
\]
\[
\tilde{v}'_l(1) = -\sqrt{\frac{l+1}{2l+1}}g'(1)\sigma'_*(1) - (l + 2)\sqrt{\frac{l+1}{2l+1}}\int_0^1 g'(\sigma_s(r))F_i(r)r^{l+2} dr,
\]
\[
\tilde{w}'_l(1) = \sqrt{\frac{l}{2l+1}}\frac{g'(1)}{2l-1}\sigma'_*(1).
\]
Substituting (102) and (105) into (101) and using (106)–(109), we see that, for \(l \neq 1\),
\[
\mathcal{B}_sY_{lm}(\omega) = \frac{1}{2l^2 + 4l + 3}\left\{g(1)(2l + 3)(l + 1) - \frac{5}{4}(2l + 1)(l + 2) + (2l + 3)(l + 1)\int_0^1 g'(\sigma_s(r))F_i(r)r^{l+2} dr\right\}Y_{lm}(\omega).
\]
We define, for \(l \geq 2\),
\[
\gamma_l = \frac{4(2l + 3)(l + 1)}{l(l+2)(2l+1)}\left[g(1) + \int_0^1 g'(\sigma_s(r))F_i(r)r^{l+2} dr\right],
\]
\[
\alpha_l(\gamma) = -\frac{l(l+2)(2l+1)}{4(2l^2 + 4l + 3)}(\gamma - \gamma_l).
\]
Then in case \(l \geq 2\) (110) can be rewritten as follows:
\[
\mathcal{B}_sY_{lm}(\omega) = \alpha_l(\gamma)Y_{lm}(\omega).
\]
In the case \(l = 0\) we have, directly from (110), the last equation also holds with
\[
\alpha_0 \equiv \alpha_0(\gamma) = g(1) + \int_0^1 g'(\sigma_s(r))F_0(r)r^2 dr.
\]
(113)
Finally, we consider the case \( l = 1 \). Since the problem (1)–(8) is translation invariant, by some similar argument as those in [6] and [17] we see that \( B, \gamma \) is 0 for any spherical harmonics of degree 1. In particular, we have

\[ B, Y_{1m}(\omega) = 0, \quad m = -1, 0, 1. \]

In summary, we have proved the following result:

**Lemma 3.2.** DQ(0) = \( B, \gamma \) is a Fourier multiplication operator having the following expression: For any given \( \eta \in C^\infty(\mathbb{S}^2) \) with Fourier expansion \( \eta(\omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{lm} Y_{lm}(\omega) \), we have

\[ B, \gamma \eta(\omega) = \alpha_0 c_{00} Y_{00} + \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \alpha_l(\gamma) c_{lm} Y_{lm}(\omega), \quad (114) \]

where \( \alpha_0 \) and \( \alpha_l(\gamma) \) defined in (113) and (112), respectively.

As usual, for a given closed linear operator \( B \) in a Banach space \( X \), we denote by \( \rho(B) \) and \( \sigma(B) \) the resolvent set and the spectrum of \( B \), respectively. The set of all eigenvalues of \( B \) is denoted by \( \sigma_p(B) \). As mentioned in the beginning of this section, we have \( \sigma(DQ(0)) = \sigma_p(DQ(0)) \). Hence, from Lemma 3.2 we immediately obtain the following result:

**Lemma 3.3.** The spectrum of DQ(0) = \( B, \gamma \) is given by

\[ \sigma(B, \gamma) = \{ \alpha_0, 0 \} \cup \{ \alpha_l(\gamma) : l = 2, 3, 4, \ldots \}. \]

Moreover, the multiplicity of the eigenvalue \( 0 \) is 3.

The next result shows some useful properties of \( \alpha_0 \) and \( \gamma_l \) (\( l \geq 2 \)):

**Lemma 3.4.** We have the following assertions:

(i) \( \alpha_0 < 0 \).

(ii) \( \gamma_l > 0 \) for all \( l \geq 2 \), and \( \lim_{l \to \infty} \gamma_l = 0 \).

(iii) There exists an integer \( l^* \geq 2 \) such that \( \gamma_{l+1} < \gamma_l \) for all \( l \geq l^* \).

**Proof.** By assumption (A1) we have \( f' > 0 \). Thus, by the maximum principle we see that \( F_0(r) \leq 0 \). Furthermore, since \( u(r) = -\sigma'_s(r) \) is a solution of the problem

\[ \begin{cases} u''(r) + \frac{2}{r}u'(r) = f'(\sigma_s(r))u(r) + \frac{2}{r^2} u \quad \text{for} \quad 0 < r < 1, \\ u(0) = 0, \quad u(1) = -\sigma'_s(1), \end{cases} \quad (115) \]

by comparison we easily get \( F_0(r) \leq -\sigma'_s(r) \). Thus, since \( g' > 0 \) (by assumption (A2)), we have

\[ \int_0^1 g'(\sigma_s(r))F_0(r)r^2 dr \leq - \int_0^1 g'(\sigma_s(r))\sigma'_s(r)r^2 dr = -g(1) + 2 \int_0^1 g(\sigma_s(r))r dr. \]

Hence,

\[ \alpha_0 = g(1) + \int_0^1 g'(\sigma_s(r))F_0(r)r^2 dr \leq 2 \int_0^1 g(\sigma_s(r))r dr < 0. \]

The last inequality follows from the facts that \( g' > 0 \) and \( \int_0^1 g(\sigma_s(r))r^2 dr = 0 \) (by (129) and (130) in Appendix A). This proves (i).

Next, from (77) we have, for any \( l \geq 2 \), that

\[ F_l'(r) + \frac{2}{r}F_l'(r) - \frac{2}{r^2}F_l(r) - f'(\sigma_s(r))F_l(r) = \frac{l^2 + l - 2}{r^2} F_l(r) \leq 0. \]
Since $f' > 0$ and $u(r) = -\sigma'_s(r)$ satisfies (115), by comparison we get that $-\sigma'_s(r) < F_i(r) < 0$. Hence

$$g(1) + \int_0^1 g'(\sigma_s(r))F_i(r)r^{l+2} dr > g(1) - \int_0^1 g'(\sigma_s(r))\sigma'_s(r)r^{l+2} dr$$

$$> g(1) - \int_0^1 g'(\sigma_s(r))\sigma'_s(r)r^3 dr$$

$$= 3 \int_0^1 g(\sigma_s(r))v^2 dr = 0,$$

so that $\gamma_l > 0$ for all $l \geq 2$. Moreover, since $|F_i(r)| \leq \sigma'_s(1)$ and $0 < g'(\sigma_s(r)) \leq g'(1)$, we have

$$\gamma_l = \frac{4(2l + 3)(l + 1)}{l(l + 2)(2l + 1)} [g(1) + \int_0^1 g'(\sigma_s(r))F_i(r)r^{l+2} dr]$$

$$\leq 8l^{-1} [g(1) + \frac{g'(1)\sigma'_s(1)}{l + 3}].$$

Hence $\lim_{l \to \infty} \gamma_l = 0$. This proves (ii).

Finally, by direct computation we have

$$\gamma_{l+1} - \gamma_l = -4g(1)(1 + o(1))l^{-2} \quad \text{as} \quad l \to \infty.$$ 

From this fact the assertion (iii) immediately follows. The proof is complete. \qed

By virtue of the the assertion (ii) of the above lemma, we introduce

$$\gamma_* = \max_{l \geq 2} \gamma_l. \quad (117)$$

Note that Lemma 3.4 ensures that $0 < \gamma_* < \infty$.

It is interesting to compare the threshold number $\gamma_*$ defined above with the corresponding threshold number for the porous medium structured tumor model obtained by Cui and Escher [7, 8], which we denote by $\tilde{\gamma}_*$. Recall that $\tilde{\gamma}_* = \max_{l \geq 2} \tilde{\gamma}_l$, where, for the case $R_s = 1$ and $\tilde{\sigma} = 1$,

$$\tilde{\gamma}_l = \frac{2}{l(l-1)(2l+1)} \left[g(1) + \int_0^1 g'(1)F_1(r)r^{l+2} dr\right] \quad \text{for} \quad l \geq 2, \ l \in \mathbb{N}. \quad (118)$$

From Lemma 3.2 of [8] we know that $\{\tilde{\gamma}_l\}_{l \geq 2}$ has the same properties as $\{\gamma_l\}_{l \geq 2}$ presented in Lemma 3.4.

**Lemma 3.5.** $\tilde{\gamma}_l < \gamma_l$ for all $l \geq 2$, $l \in \mathbb{N}$, so that $\tilde{\gamma}_* < \gamma_*$. 

**Proof.** For any $l \geq 2$ we have, by (111), (116) and (118), that

$$\tilde{\gamma}_l - \gamma_l = \left[\frac{2}{l(l-1)(2l+1)} - \frac{4(2l + 3)(l + 1)}{l(l + 2)(2l + 1)}\right] \left[g(1) + \int_0^1 g'(1)F_1(r)r^{l+2} dr\right]$$

$$= \frac{2}{l(l + 2)} \left[\frac{1}{l} - \frac{2(2l + 3)(l + 1)}{2l + 1}\right] \left[g(1) + \int_0^1 g'(1)F_1(r)r^{l+2} dr\right]$$

$$< \frac{2}{l(l + 2)} (1 - 1) \left[g(1) + \int_0^1 g'(1)F_1(r)r^{l+2} dr\right] = 0.$$ 

This completes the proof. \qed
4. The proof of Theorem 1.1. In this section we give the proof of Theorem 1.1. Note that since $0 \in \sigma(DQ(0))$, the standard linearized stability theorem for parabolic differential equations in Banach spaces cannot be applied to treat (53), and we have to employ the method of center manifold analysis. We shall construct a locally invariant center manifold, which consists only of equilibria, and show that this manifold attracts nearby transient solutions at an exponential rate. Similar method was applied in [7], [10] and [26].

Proof of Theorem 1.1. We fulfill this proof in four steps.

(i) By the definition of $\gamma$, we see that for any $\gamma > \gamma$, $\alpha_l(\gamma) < 0$ for all $l \geq 2$. Besides, by (114) we see that 0 is an eigenvalue of geometric multiplicity 3, and the kernel of $DQ(0) = B_\gamma$ is the space $X_\gamma := \text{span}\{Y_{1m}; m = -1, 0, 1\}$. Let $X_\gamma^\perp$ be the orthogonal complement of $X_\gamma$ in $L^2(S^2)$, and for fixed $m \geq 3$ we denote

$$h^{m+\theta}(S^2) = h^{m+\theta}(S^2) \cap X_\gamma^\perp.$$  

Then we have

$$h^{m+\theta}(S^2) = h^{m+\theta}(S^2) \oplus X_\gamma.$$  

This decomposition induces two projection operators $\pi^c$ and $\pi^s$ which map $h^{m+\theta}(S^2)$ onto $X_\gamma$ and $h^{m+\theta}(S^2)$, respectively. From Lemma 3.2 we know that $B_\gamma$ commutes with them.

(ii) Let $M(\eta) = Q(\eta) - B_\gamma \eta$. Then the equation (53) can be rewritten as follows:

$$\partial_t \eta = B_\gamma \eta + M(\eta) \quad \text{for} \quad t > 0, \quad \text{and} \quad \eta(0) = \eta_0.$$  

(119)

The little Hölder spaces have the following well-known interpolation property

$$(h^{\sigma_0}(S^2), h^{\sigma_1}(S^2))_\theta = h^{(1-\theta)\sigma_0 + \theta \sigma_1}(S^2), \quad \text{if} \quad (1-\theta)\sigma_0 + \theta \sigma_1 \notin \mathbb{Z},$$  

where $0 < \theta < 1$ and $(\cdot, \cdot)_\theta$ denotes the continuous interpolation of Da Prato and Grisvard (see [24]). By Lemma 3.1 we know that $B_\gamma = DQ(0)$ generates a strongly continuous analytic semigroup on $h^{m-1+\theta}(S^2)$ with domain $h^{m+\theta}(S^2)$. Thus by Propositions 6.2, 6.4 and Theorem 6.5 in [10] we conclude that there exists an open neighborhood $\mathcal{O}$ of the origin in $X_\gamma$ and a mapping

$$\mathcal{C} \in C^m(\mathcal{O}, h^{m+\theta}(S^2)) \quad \text{with} \quad \mathcal{C}(0) = 0, \quad \partial \mathcal{C}(0) = 0,$$  

such that the 3-dimensional submanifold $\mathcal{M}_c := \text{graph}(\mathcal{C})$ of $h^{m+\theta}(S^2)$ is a locally invariant and stable manifold for the evolution equation (119). Note that $\mathcal{M}_c$ consists only of radial equilibria, i.e. $\mathcal{M}_c$ is the set of all spheres of radius 1 with centers sufficiently close to 0. Furthermore, by the above-mentioned results of [10] we know that $\mathcal{M}_c$ attracts at an exponential rate all small global solutions of (119) in $h^{m+\theta}(S^2)$. More precisely, there exists $\varepsilon > 0$ such that the solution to (119) exists globally for any $\eta_0$ with $\|\eta_0\|_{h^{m+\theta}(S^2)} \leq \varepsilon$, and, moreover, there exist $c > 0$, $K > 0$ and a unique $z_0 = z_0(\eta_0) \in \mathcal{O}$ such that for any $t \geq 0$ there holds

$$\left\| (\pi^c \eta(t), \pi^s \eta(t)) - (z_0, \mathcal{C}(z_0)) \right\|_{h^{m+\theta}(S^2)} \leq K \exp(-ct) \|\pi^c \eta_0 - \mathcal{C}(\pi^c \eta_0)\|_{h^{m+\theta}(S^2)}.$$  

(120)

(iii) Now let $\eta_0 \in h^{m+\theta}(S^2)$ be given and $\|\eta_0\|_{C^{m+\theta}(S^2)} \leq \varepsilon$. Then the solution of the equation (119) $\eta \in C([0, \infty), h^{m+\theta}(S^2)) \cap C^1((0, \infty), h^{m+\theta-1}(S^2))$. By Lemma 2.1 and Lemma 2.5, it follows that the problem (1)–(9) has a global-in-time solution $\sigma(\cdot, t), v(\cdot, t), p(\cdot, t), \Omega(t)$, where $\Omega(t) = \{x \in \mathbb{R}^3 : x = r\omega, 0 \leq r < 1 + \eta(\omega, t), \omega \in S^2\}$. Since $\mathcal{M}_c$ is the set of equilibrium solutions which are sufficiently close to $S^2$,
there exists a \( x_0 \in \mathbb{R}^3 \) such that \((z_0, C(z_0)) = \eta_{[x_0]}\), where \( \eta_{[x_0]} \) is the distance function on \( \mathbb{S}^2 \) introduced in Section 1. Then (120) implies that

\[
\|\eta'(t) - \eta_{[x_0]}\|_{H^{m+\sigma}(\mathbb{S}^2)} \leq K \exp(-ct) \quad \text{for any} \ t \geq 0. \tag{121}
\]

By Lemmas 2.2–2.4 we have

\[
\sigma(\cdot, t) = \mathcal{R}(\eta(t)) \circ \Phi_{\eta(t)}^{-1}, \quad \nu(\cdot, t) = \tilde{\nu}(\eta(t)) \circ \Phi_{\eta(t)}^{-1}, \quad p(\cdot, t) = \tilde{p}(\eta(t)) \circ \Phi_{\eta(t)}^{-1}. \tag{122}
\]

Recalling the definition of \((\sigma_{[x_0]}, \nu_{[x_0]}, p_{[x_0]}, \Omega_{[x_0]})\), we have

\[
\sigma_{[x_0]} = \mathcal{R}(\eta_{[x_0]}) \circ \Phi_{\eta_{[x_0]}}^{-1}, \quad \nu_{[x_0]} = \tilde{\nu}(\eta_{[x_0]}) \circ \Phi_{\eta_{[x_0]}}^{-1}, \quad p_{[x_0]} = \tilde{p}(\eta_{[x_0]}) \circ \Phi_{\eta_{[x_0]}}^{-1}. \tag{123}
\]

The explicit construction of \( \Phi_{\eta} \), (35) and the mean value theorem immediately imply that there is a positive constant \( C \) such that for any \( t \geq 0 \)

\[
\|\mathcal{R}(\eta(t)) - \mathcal{R}(\eta_{[x_0]})\|_{C^{m+\sigma}(\mathbb{B}^3)} \leq C\|\eta(t) - \eta_{[x_0]}\|_{C^{m+\sigma}(\mathbb{B}^3)}. 
\]

Then using (122) and (123) we have

\[
\|\sigma(\cdot, t) - \sigma_{[x_0]}\|_{C^{m+\sigma}(\bar{\Omega}(t))} \leq C\|\eta(t) - \eta_{[x_0]}\|_{C^{m+\sigma}(\bar{\Omega}(t))}, \tag{124}
\]

for any \( t \geq 0 \). Similarly, by Lemma 2.3 and Lemma 2.4 we also have

\[
\|\nu(\cdot, t) - \nu_{[x_0]}\|_{C^{m-1+\sigma}(\bar{\Omega}(t))} \leq C\|\eta(t) - \eta_{[x_0]}\|_{C^{m-1+\sigma}(\bar{\Omega}(t))}, \tag{125}
\]

\[
\|p(\cdot, t) - p_{[x_0]}\|_{C^{m-2+\sigma}(\bar{\Omega}(t))} \leq C\|\eta(t) - \eta_{[x_0]}\|_{C^{m-2+\sigma}(\bar{\Omega}(t))}, \tag{126}
\]

for any \( t \geq 0 \). Combining (120), (121) and (124)–(126), we see that (14) holds.

(iv) Finally, if \( 0 < \gamma < \gamma_s \) then by Lemma 3.3 we see that \( \sigma(\mathcal{B}_s) \cap \{ \lambda \in \mathbb{C} : \Re \lambda > 0 \} \) is not empty. It follows from Theorem 9.1.3 in [24] that the zero equilibrium of (119) is unstable. This completes the proof of Theorem 1.1. \( \square \)

Appendix A. Radially symmetric stationary solution.

**Theorem A.1.** Under assumptions (A1)–(A3), the problem (1)–(8) has a unique radially symmetric stationary solution \((\sigma_s, \nu_s, p_s, \Omega_s)\) with components having expressions in (12).

**Proof.** It suffices to consider the following problem:

\[
\sigma''_s(r) + \frac{2}{r} \sigma'_s(r) = f(\sigma_s(r)) \quad \text{for} \quad 0 < r \leq R_s, \tag{127}
\]

\[
\sigma'_s(0) = 0, \quad \sigma_s(R_s) = 1, \tag{128}
\]

\[
v'_s(r) + \frac{2}{r} v_s(r) = g(\sigma_s(r)) \quad \text{for} \quad 0 < r \leq R_s, \tag{129}
\]

\[
v_s(0) = 0, \quad v_s(R_s) = 0. \tag{130}
\]

\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right] \left[ p_s(r) - \frac{4}{3} g(\sigma_s(r)) \right] = 0 \quad \text{for} \quad 0 < r \leq R_s, \tag{131}
\]

\[
p'_s(0) = 0, \quad p_s(R_s) = \frac{\gamma}{R_s} + \frac{4}{3} g(1). \tag{132}
\]

Indeed, it is straightforward to verify that for a radially symmetric stationary solution the equations (1) and (2) have respectively the forms (127) and (129), the
boundary condition (4) has the form of the second equation in (128), and (6) becomes the second equation in (130). Next, taking divergence in both sides of (3) and using (2), we see that (131) holds, by which we have
\[
T(v_s,p_s)n|_{r=R_s} = \left[2v'_s(R_s) - p_s(R_s) - \frac{2}{3}g(1)\right]n = \left[2g(1) - p_s(R_s) - \frac{2}{3}g(1)\right]n = \frac{4}{3}g(1) - p_s(R_s)n, \quad (\text{by (129), (130)})
\]
where \(n(x) = x/|x|\) for \(x \in \mathbb{R}^3\setminus\{0\}\). Hence, the second equation in (132) follows from the boundary condition (5). Finally, the first equations in (128), (130) and (132) are imposed to rule out possible solutions possessing singularities at \(r = 0\) for the problem without these equations, which are not meaningful solutions of (1)–(8).

From [4] we know that under assumptions (A1)–(A3), the problem (127)–(130) has a unique solution \((\sigma_s(r), v_s(r), R_s)\). Besides, using (131) we immediately see that the function
\[
p_s(r) = \gamma \frac{1}{R_s} + \frac{4}{3}g(\sigma_s(r))
\]
solves (131) and (132). Since the solution of (127)–(132) is obviously unique, we see that the desired assertion follows. \(\square\)

Appendix B. Boundedness of the operator \(B_1\).

**Lemma B.1.** Let \(m \in \mathbb{N}, m \geq 3,\) and \(0 < \theta < 1\). Let \(s \in (h_m^{−2+\theta}(\mathbb{S}^2))^3\) satisfy \(s \cdot n = 0\) and let \(v = R(0)s\). Then \(\text{tr}_{\mathbb{S}^2}(v) \cdot n \in h_m^{\theta}(\mathbb{S}^2)\), and there exists a constant \(C\) independent of \(s\) such that
\[
||\text{tr}_{\mathbb{S}^2}(v) \cdot n||_{h_m^{\theta}(\mathbb{S}^2)} \leq C||s||_{(h_m^{−2+\theta}(\mathbb{S}^2))^3}.
\]

**Proof.** Fix a function \(d \in BUC^\infty(\mathbb{R}^3)\) such that
\[
\text{tr}_{\mathbb{S}^2}(d) = 0, \quad \text{tr}_{\mathbb{S}^2}(\nabla d) = n, \quad \partial_n(\nabla d) = 0. \quad (133)
\]

Extend the normal vector field on \(\mathbb{S}^2\) into \(\mathbb{R}^3\) by setting \(n = \nabla d\). From the proof of Lemma 2.3 we know that there exists a unique \((v, p, \zeta) \in (h_m^{−1+\theta}(\mathbb{S}^2))^3 \times h_m^{−2+\theta}(\mathbb{S}^2) \times \mathbb{R}^6\), where \(v = (v_1, v_2, v_3)\), such that
\[
\mathcal{L}(0)(v, p, \zeta) = (0, 0, h, 0, 0). \quad (134)
\]

We have
\[
\Delta p = \nabla \cdot (\Delta v - l_0(\zeta)) = 0 \quad \text{in } \mathbb{R}^3. \quad (135)
\]

Since, on one hand,
\[
T(v, p)n \cdot n = 2n_n v_j \partial_n v_j - p = 2n_n \partial_n v_j - p = 2\partial_n(v \cdot n) - 2v \cdot \partial_n(\nabla d) - p = 2\partial_n(v \cdot n) - p \quad \text{on } \mathbb{S}^2,
\]
and on the other hand, by (134),
\[
T(v, p)n \cdot n = h \cdot n = 0 \quad \text{on } \mathbb{S}^2.
\]

Hence,
\[
2\partial_n(v \cdot n) - p = 0 \quad \text{on } \mathbb{S}^2. \quad (136)
\]
Define $\Psi = v \cdot n - \frac{1}{2} p d$. Then we have $\operatorname{tr} g^2(v) \cdot n = \operatorname{tr} g^2(\Psi)$ and
\[
\Delta \Psi = \Delta (v \cdot n - \frac{1}{2} p d)
= \Delta v \cdot n + 2 \partial_i v_j \partial_i n_j + v \cdot \Delta n - \frac{1}{2} p \Delta d - \nabla d \cdot \nabla p - \frac{1}{2} d \Delta p
= (\nabla p + l_0(\zeta)) \cdot n + 2 \partial_i v_j \partial_i n_j + v \cdot \Delta n - \frac{1}{2} p \Delta d - \nabla p \cdot n
= l_0(\zeta) \cdot n + 2 \partial_i v_j \partial_i n_j + v \cdot \Delta n - \frac{1}{2} p \Delta d \quad \text{in } \mathbb{B}^3.
\]
Furthermore, by (133) and (136) we have
\[
\partial_n \Psi = \partial_n (v \cdot n) - \frac{d}{2} \partial_n p - \frac{p}{2} n \cdot (\nabla d) = \partial_n (v \cdot n) - \frac{1}{2} p \quad \text{on } \mathbb{S}^2.
\]
Hence $\Psi$ is the solution of the problem
\[
\begin{cases}
\Delta \Psi = l_0(\zeta) \cdot n + 2 \partial_i v_j \partial_i n_j + v \cdot \Delta n - \frac{1}{2} p \Delta d & \text{in } \mathbb{B}^3, \\
\partial_n \Psi = 0 & \text{on } \mathbb{S}^2.
\end{cases}
\]
(137)
It follows by classical Hölder estimates for second-order partial differential equations of the elliptic type that
\[
\|\operatorname{tr} g^2(v) \cdot n\|_{h^{m+\theta}(\mathbb{S}^2)} = \|\Psi\|_{h^{m+\theta}(\mathbb{S}^2)} \leq C\|\Psi\|_{h^{m+\theta}(\mathbb{S}^2)}
\leq C(\|\Delta \Psi\|_{h^{m-2+\theta}(\mathbb{T}^2)} + \|\partial_n \Psi\|_{h^{m-1+\theta}(\mathbb{S}^2)} + \|\Psi\|_{h^{m-2+\theta}(\mathbb{T}^2)})
\leq C(\|v\|_{h^{m-1+\theta}(\mathbb{S}^2)} + \|p\|_{h^{m-2+\theta}(\mathbb{T}^2)} + |\zeta|)
\leq C\|h\|_{h^{m-2+\theta}(\mathbb{T}^2)}^3.
\]
The proof is complete. \hfill \Box

Since $\nabla_\omega(\partial_n \psi) \cdot n = \omega \cdot \nabla_\omega(\partial_n \psi) = 0$, by Lemma B.1 we immediately obtain

Corollary 2. Let $B_1 \eta = -2 \operatorname{Re} (0)(\nabla_\omega(\partial_n \psi))|_{\mathbb{S}^2} \cdot n$ for $\eta \in h^{m+\theta}(\mathbb{S}^2)$. Then we have $B_1 \in L(h^{m+\theta}(\mathbb{S}^2), h^{m+\theta}(\mathbb{S}^2))$.

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REFERENCES

[1] H. Amann, “Linear and Quasilinear Parabolic Problems, Vol. I,” Birkhäuser, Basel, 1995.
[2] H. M. Byrne, A weakly nonlinear analysis of a model of avascular solid tumor growth, J. Math. Biol., 39 (1999), 59–89.
[3] H. M. Byrne and M. A. J. Chaplain, Growth of nonnecrotic tumors in the presence and absence of inhibitors, Math. Biosci., 130 (1995), 151–181.
[4] S. Cui, Analysis of a free boundary problem modeling tumor growth, Acta Mathematica Sinica (English Series), 21 (2005), 1071–1092.
[5] S. Cui, Well-posedness of a multidimensional free boundary problem modelling the growth of nonnecrotic tumors, J. Func. Anal., 245 (2007), 1–18.
[6] S. Cui, Lie group action and stability analysis of stationary solutions for a free boundary problem modelling tumor growth, J. Differential Equations, 246 (2009), 1845–1882.
[7] S. Cui and J. Escher, Asymptotic behavior of solutions of multidimensional moving boundary problem modeling tumor growth, Comm. Partial Differential Equations., 33 (2008), 636–655.
[8] S. Cui and J. Escher, Bifurcation analysis of an elliptic free boundary problem modelling the growth of avascular tumors, SIAM J. Math. Anal., 39 (2007), 210–235.
[9] J. Escher and G. Prokert, Analytic of solutions to nonlinear parabolic equations on manifolds and an application to Stokes flow, J. Math. Fluid Mech., 8 (2006), 1–35.
[10] J. Escher and G. Simonett, A center manifold analysis for the Mullins-Sekerka model, J. Differential Equations, 143 (1998), 267–292.

[11] S. J. H. Franks, H. M. Byrne, J. P. King, J. C. E. Underwood and C. E. Lewis, Modelling the early growth of ductal carcinoma in situ of the breast, J. Math. Biol., 47 (2003), 424–452.

[12] S. J. H. Franks, H. M. Byrne, J. P. King, J. C. E. Underwood and C. E. Lewis, Modelling the growth of comedo ductal carcinoma in situ, Math. Med. Biol., 20 (2003), 277–308.

[13] S. J. H. Franks, H. M. Byrne, J. C. E. Underwood and C. E. Lewis, Biological inferences from a mathematical model of comedo ductal carcinoma in situ of the breast, J. Theor. Biol., 232 (2005), 523–543.

[14] S. J. H. Franks and J. P. King, Interactions between a uniformly proliferating tumour and its surroundings: Uniform material properties, Math. Med. Biol., 20 (2003), 47–89.

[15] A. Friedman, A free boundary problem for a coupled system of elliptic, hyperbolic, and Stokes equations modeling tumor growth, Interfaces and Free Boundaries, 8 (2006), 247–261.

[16] A. Friedman, Mathematical analysis and challenges arising from models of tumor growth, Math. Models Methods Appl. Sci., suppl., 17 (2007), 1751–1772.

[17] A. Friedman and B. Hu, Bifurcation for a free boundary problem modeling tumor growth by Stokes equation, SIAM J. Math. Anal., 39 (2007), 174–194.

[18] A. Friedman and B. Hu, Bifurcation from stability to instability for a free boundary problem modeling tumor growth by Stokes equation, J. Math. Anal. Appl., 327 (2007), 643–664.

[19] A. Friedman and F. Reitich, Analysis of a mathematical model for growth of tumors, J. Math. Biol., 38 (1999), 262–284.

[20] A. Friedman and F. Reitich, Symmetry-breaking bifurcation of analytic solutions to free boundary problems, Trans. Amer. Math. Soc., 353 (2001), 1587–1634.

[21] A. Friedman and F. Reitich, Quasi-static motion of capillary drop II: The three dimensional case, J. Differential Equations, 186 (2002), 509–557.

[22] M. Günther and G. Prokert, Existence results for the quasistationary motion of a free capillary liquid drop, Z. Anal. Anwendungen, 16 (1997), 311–348.

[23] E. L. Hill, The theory of vector spherical harmonics, Amer. J. Phys., 222 (1954), 211–214.

[24] A. Lunardi, “Analytic Semigroups and Optimal Regularity in Parabolic Problems,” Birkhäuser, Basel, 1995.

[25] V. A. Solonnikov, “Lectures on Evolution Free Boundary Problems: Classical Solutions,” Mathematical Aspects of Evolving Interfaces, J. M. Morel, F. Takens and B. Teissier eds., Lecture Notes in Math., 1812, Springer, Berlin, 2003, 123–175.

[26] J. Wu and S. Cui, Asymptotic behavior of solutions of a free boundary problem modeling the growth of tumors in the presence of inhibitors, Nonlinearity, 20 (2007), 2389–2408.

[27] F. Zhou and S. Cui, Well-posedness and stability of a multidimensional moving boundary problem modeling the growth of tumor cord, Discrete Contin. Dyn. Syst., 21 (2008), 929–943.

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