Higher Dimensional Systems of Differential Equations Obtained by Iterative Use of Complex Methods

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Abstract.

Systems of two ordinary and partial differential equations (ODEs and PDEs) had been obtained from a scalar complex ODE by splitting it into its real and imaginary parts. The procedure was also carried out to obtain a four dimensional system by splitting a complex system of two ODEs into its real and imaginary parts. Systems of three ODEs had not been accessible by these methods. In this paper the complex splitting is used iteratively to obtain three and four dimensional systems of ODEs and four dimensional systems of PDEs for four functions of two and four variables. The new systems of four ODEs are distinct from the class obtained by the single split of a two dimensional system. Illustrative examples are provided.

1 Introduction

Lie [1] developed his study of the symmetry of differential equations for complex functions. Of course, to be differentiable they have to be analytic. Though this was assumed, analyticity was not used in the sense of treating the complex function as the real and imaginary parts connected to each other by the Cauchy-Riemann (CR) conditions. Thus, while the complex nature of the function was very important for the topological properties of the Lie groups that arose, it was not used directly for the differential equations. More recently [2] the splitting of a scalar ODE, into its real and imaginary parts, was exploited to obtain methods to deal with systems of ODEs and PDEs. This was called complex symmetry analysis (CSA). Of particular interest was its application to dealing with the variational principle for systems of ODEs [3, 4] and to linearization (conversion of the equation to linear form by transformation of the dependent and independent variables) of ODEs [5, 6, 7]. The latter was of special interest because of the connection between geometry and the symmetries of systems of ODEs [8, 9, 10, 11]. This enabled one to use geometric methods to linearize
systems (including the scalar case) of ODEs [12, 13]. This method allowed one to not only write down the linearizing transformation but also to directly provide the solution. The procedure also led to some new insights regarding standard systems of equations and to methods for solving systems that were not amenable to solution by standard symmetry analysis [4, 14].

One might have expected that a system of four ODEs or PDEs could be obtained by using quaternions. This turns out to be impossible. The reason is that while algebra works for quaternions calculus does not. Consider \( q = w + ix + jy + kz \), subject to the usual quaternion rules that \( i^2 = j^2 = k^2 = -1 \), \( ij = k = -ji \), \( jk = i = -kj \), \( ki = j = -ik \). For \( dq/dq = 1 \) the derivative operator is

\[
\frac{d}{dq} = a \frac{\partial}{\partial w} + ib \frac{\partial}{\partial x} + jc \frac{\partial}{\partial y} + kd \frac{\partial}{\partial z},
\]

subject to the condition that \( a - b - c - d = 1 \). Now ask that \( dq^2/dq = 2q \). This requires the above condition along with the additional conditions, \( a - b = a - c = a - d = 1 \), which are inconsistent with the earlier requirement. One has to obtain the four dimensional system by other means, such as using a complex system of two ODEs and splitting it into a system of four ODEs.

Though one can split a system of two complex ODEs into one of four ODEs by this procedure, it is not possible to use it to obtain a system of three ODEs from either a scalar or a vector equation. In fact, all odd dimensional systems are inaccessible by the usual CSA splitting procedure. However, one would like to be able to use the power of CSA for three (and other odd dimensional systems) dimensional systems as well. In this paper the complex splitting is used iteratively to be able to generate three and four dimensional systems of ODEs and to obtain systems of four PDEs for four functions of two or four variables. The procedure could be used with more than two iterations to generate higher dimensional systems as well, but we will not follow that up here.

The method used to obtain the system of ODEs is to start with a scalar ODE and regard the dependent variable as a complex function of a real variable, as was done for CSA, and split it into a system of two ODEs. Now regard the two dependent variables as themselves complex functions of a real variable. This provides a system of four ODEs. Of course, if both steps had been merged into one, only a system of two equations would again have been obtained. (This would be the reduced system obtainable from the new system of four equations.) Instead, we first close our eyes to the fact that we are going to treat the two dependent variables as complex, and only after obtaining the system of two equations do we treat each of the dependent variables as themselves complex. The result of repeating the CSA procedure is different from treating the dependent variable of the original scalar equation as a function of two complex functions of a real variable as the symmetry structure of the systems is different.
In the CSA procedure it was found that one may be able to linearize the base scalar equation even though the corresponding real system does not have enough symmetries to allow linearization, or even to permit of solution by usual symmetry methods. The system of four ODEs could then have much fewer symmetries while the base system is linearizable, or even have a base system with too few symmetries while the scalar equation is linearizable [14]. It is to be expected that here we will not only get the same sort of cases, but that there would be some even stranger cases arising than for CSA.

One has another option. For the system obtained by the first split, one can treat one of the two dependent variables as real and the other as complex. This provides the system of three ODEs. In fact, it provides two systems of three ODEs, as we can choose either of the dependent variables to be real and the other complex. The two systems obtained are dual to each other in some sense. That sense will be clarified by some examples that we will provide.

Of course, one could take the CSA system of PDEs and treat each of the dependent variables as complex functions of the two real variables. This would yield a system of four PDEs for four functions of two variables. Alternatively, we could have started with the system of two ODEs and now treated both the dependent and independent variables as complex. This split also gives a system of four PDEs for four functions of two variables. The two systems obtained are dual to each other in a fairly obvious way. Splitting the system of two PDEs by treating the dependent and independent variables as complex gives a system of four PDEs for four functions of four variables.

The role of the CR-equations was fairly obvious in the original CSA and had not been spelled out explicitly. A geometric description of these equations was given later [14] but the detailed requirements for a general scalar ODE were not. On the double use of the splitting procedure the conditions become more thoroughly coupled and need to be spelled out explicitly. It turns out that even for the original CSA the conditions for the derivatives of the functions involved on the right side of the semilinear equations with respect to the derivatives of the dependent variables are more complicated than was envisaged. These have been stated explicitly here.

We have limited the discussion to second order equations only. One can, of course, go to higher order equations but that complicates the expressions without providing any further understanding of the procedures being developed. Also, there is no direct equivalent of the geometric connection and procedures for the higher order equations. (There is an indirect connection provided by differentiating the second order ODE and requiring that the original equation hold [15, 16], but that will not be followed up here.) On the other hand, one could have limited oneself to first order equations but that is the degenerate case and results for it will not hold more generally.
The basic view adopted here is to use the connection of a second order system of differential equations with a scalar second order ODE to be able to solve, or otherwise deal with, the given system. For this purpose, one ideally needs clear criteria to be able to determine whether the given system is, or is not, related to some scalar ODE. The criteria should be such that a computer code could be written to check the given system for the relationship, and if it is related to construct the required ODE, which could then be appropriately dealt with. This had not been done even for the original CSA. Here we state the explicit criteria as theorems for the original CSA and then for the double split systems.

The plan of the paper is as follows. In the next section we present the basics of the CSA splitting procedure and briefly mention symmetries of differential equations. We also present two characterization theorems there that had not been provided earlier. In the subsequent section we give the split into a system of three ODEs. In section four we give the split into a system of four ODEs. The section after that deals with the split into a system of four PDEs for four functions of two variables. In section six we present the split into a system of four PDEs for four functions of four variables. For each of the systems we provide some examples in the same sections. In the concluding section we give a brief summary and discussion of the results.

## 2 Complex Splitting and Symmetries

Consider a general second order ODE

\[ u''(r) = f(r; u, u') \].

(2)

We can now take either \( u \) to be a complex function of the real variable \( r \) or take both \( u \) and \( r \) to be complex. Let \( u = p + iq \) in the former case and put

\[ f(r; u, u') = f^r(r; p, q; p', q') + if^i(r; p, q; p', q') \].

(3)

The resulting system of ODEs is

\[ p''(r) = f^r(r; p, q; p', q') , q''(r) = f^i(r; p, q; p', q') \].

(4)

The CR-equations for this system are

\[ f^r_p = f^r_q , f^r_q = -f^i_p ; f^r_{p'} = f^i_q ; f^r_{q'} = -f^i_{p'} ; \]

(5)

with no conditions on \( p \) and \( q \) other than second differentiability.

For the latter case let \( r = s + it \) as well. Then

\[ f(r; u, u') = f^r(s, t; p, q; p_s, q_s, p_t, q_t) + if^i(s, t; p, q; p_s, q_s, p_t, q_t) \].

(6)
Splitting the equation into its real and imaginary parts then gives the system of two PDEs
\[ p_{ss} - p_{tt} + 2q_{st} = 4f^r, \quad q_{ss} - q_{tt} - 2p_{st} = 4f^i. \]  \tag{7}

The CR-equations for this system include conditions for \( p \) and \( q \), apart from the previous ones (which are now more complicated),
\[ 
\begin{align*}
p_s &= q_t, \quad p_t = -q_s; \\
\frac{f_r}{s} &= \frac{f}{i}, \quad f_i^t = -f_s^i; \\
\frac{f_r}{p} &= \frac{f}{q}, \quad f^i_q = -f^i_p.
\end{align*}
\]  \tag{8}

The derivative of the functions with respect to the derivatives is more complicated to state. The problem is that
\[ 2u' \to \left( \frac{\partial}{\partial s} - \iota \frac{\partial}{\partial t} \right) (p + \iota q) = (p_s + q_t) - \iota (p_t - q_s). \]  \tag{9}

Thus, for \( f(r; u, u') \) to be analytic, \( f^r \) and \( f^i \) cannot depend arbitrarily on \( p_s, q_s, p_t \) and \( q_t \) but must depend on \( p_s + q_t \) and \( p_t - q_s \). If we call these variables \( \phi \) and \( \psi \), respectively, then the last conditions are
\[ 
\begin{align*}
f_r^\phi &= f_i^\psi, \\
f_r^\psi &= -f_i^\phi.
\end{align*}
\]  \tag{10}

Though the CR-equations were taken as obvious, they are needed to characterize systems of (real) ODEs and PDEs that can arise by splitting a scalar ODE by CSA methods. To complete the CSA procedure we state the following two theorems.

**Theorem 1:** A system of two second order ODEs \( (4) \) corresponds to a scalar second order ODE \( (2) \) if and only if it satisfies the system of CR-conditions \( (5) \).

**Theorem 2:** A system of two second order PDEs \( (7) \) corresponds to a scalar second order ODE \( (2) \) if and only if it satisfies the system of CR-conditions \( (8) \) and \( (10) \) where \( \phi = p_s + q_t \) and \( \psi = p_t - q_s \).

Symmetry analysis deals with the infinitesimal generators that leave the differential equation invariant under point transformations \( (r, u) \to (R, U) \) say,
\[ X = \xi(r, u) \frac{\partial}{\partial r} + \eta(r, u) \frac{\partial}{\partial u}. \]  \tag{11}

To be able to apply the operators to differential equations these generators have to be *prolonged* or *extended* to include the higher derivatives. For the first derivative
\[ X^{[1]} = \xi \frac{\partial}{\partial r} + \eta \frac{\partial}{\partial u} + \eta^{[1]}(r; u, u') \frac{\partial}{\partial u'}, \]  \tag{12}

where
\[ \eta^{[n]} = \frac{d\eta^{[n-1]}}{dr} - u^n(r) \frac{d\eta}{dr}, \]  \tag{13}
\( (\eta[0] = \eta) \), \( d/dr \) stands for the total derivative,

\[
\frac{d}{dr} = \frac{\partial}{\partial r} + u' \frac{\partial}{\partial u} + u'' \frac{\partial}{\partial u'}
\]

and \( u^n(r) \) stands for the \( n^{th} \) derivative. The prolonged generators for higher order differential equations can be similarly obtained using (13).

We can extend the analysis to systems of ODEs by replacing the scalar \( u \) by a vector \( \mathbf{u} \) and the corresponding partial derivative by \( \nabla \mathbf{u} \). Consequently we must replace the scalar \( \eta \) and \( \eta'[1] \) by the vectors \( \mathbf{\eta} \) and \( \mathbf{\eta}'[1] \). The extension to the PDEs is more complicated. The scalar \( r \) has to now also be replaced by a vector \( \mathbf{s} \) and along with it the scalar \( \xi \) by the vector \( \mathbf{\xi} \), but now the derivative of \( \mathbf{u} \) becomes \( \nabla_{\mathbf{s}} \mathbf{u} \) and the derivative with respect to this vector of partial derivatives becomes too messy to read. As such, we write \( \nabla_{\mathbf{s}} \mathbf{u} = \mathbf{u}_1 \). Of course, we need to also bear in mind that the variables that the functions depend on will not be the derivatives given but linear combinations as we saw for the CR-conditions. The second derivative can then be written as \( \mathbf{u}_2 \) and so on. For the complex case the \( \mathbf{X} \) was written as \( \mathbf{Z} \) and similar changes were made for the coefficients but it will be more convenient to use the same notation throughout here.

### 3 System of Three ODEs by Double Splitting

Consider (2) as the base scalar equation with the split of the function given by (3) leading to the system (4) subject to the CR-equations (5). Now regard \( p \) as a real variable \( x \) and \( q \) as the complex variable \( y + i z \). We run into a problem here. There are three second order ODEs but the number of functions to be obtained from \( f \) must be even. To avoid this problem we put

\[
f(r; u, u') = g(r; p, p') + i \ G(r; p, q; p, q') \ .
\]

The choice of what part to put into \( g \) and what part to put into \( G \) is clearly arbitrary. For definiteness, we define \( g \) to consist of all those terms that do not involve \( q \) or \( q' \). Then \( G \) consists of all those terms that do. Now we proceed with the second split by putting

\[
G(r; p, q; p, q') = k(r; x, y, z; x', y', z') + i \ l(r; x, y, z; x', y', z') \ ,
\]

yielding the system of three ODEs

\[
x'' = h(r; x, x') \ , \quad y'' = k(r; x, x') \ , \quad z'' = l(r; x, x') \ ,
\]

subject to the CR-equations

\[
k_y = l_z \ , \quad k_z = -l_y \ ; \quad k_{y'} = l_{z'} \ , \quad k_{z'} = -l_{y'} \ .
\]
The first prolonged symmetry generator is:

\[
X^{[1]} = \xi(r, x) \frac{\partial}{\partial r} + \eta^y(r, x) \frac{\partial}{\partial x} + \eta^z(r, x) \frac{\partial}{\partial y} + \eta^z(r, x) \frac{\partial}{\partial z} + \eta^y(r, x, \nabla x') \frac{\partial}{\partial x'} + \eta^z(r, x, \nabla x') \frac{\partial}{\partial y'} + \eta^z(r, x, \nabla x') \frac{\partial}{\partial z'} .
\] (19)

Instead of this procedure, at the second split we could have taken \( p(r) = x(r) + \iota y(r) \) and set \( q(r) = z(r) \). This will give a dual system in some sense. We would now have to set

\[
f(r; u, u') = g(r; x, y, z; x', y', z') + \iota h(r; x, y, z; x', y', z') + \iota k(r; z, z') ,
\] (20)

and the slightly modified CR-equations

\[
g_y = h_z , \quad g_z = -h_y ; \quad g_{y'} = h_{z'} , \quad g_{z'} = -h_{y'} .
\] (21)

The prolonged symmetry generator remains unaltered in form except that now the coefficient of the last term is pure imaginary, to account for the last imaginary term in (20). The sense of the duality will be clarified by the examples.

For completeness we state a theorem for the characterization of systems of three ODEs that correspond to a scalar ODE by double splitting.

**Theorem 3:** The system of three ODEs (17) corresponds to the scalar ODE (2), for any consistent identification of the function given by (15) and (16), provided the CR-conditions (18) hold.

**Remark:** The dual procedure gives the same system but now we require that the split given by (20) and (21) holds.

**Example 1:** Consider the free-particle scalar ODE, \( u'' = 0 \). It clearly yields the system \( x'' = y'' = z'' = 0 \). The splitting of the functions is obviously trivial. However, the infinitesimal symmetry generators of the system are not trivially related to the generators of the scalar ODE. Even for the original CSA, it had been noted that the symmetries for the system could not be a simple doubling of the symmetries of the original ODE, as the maximal Lie algebra for the system is \( sl(4, \mathbb{R}) \), which has 15 generators, while the algebra for the scalar free particle equation is \( sl(3, \mathbb{R}) \), which has 8. Doubling gives one extra generator. It is also clear that it cannot be a simple matter of leaving one generator out, as the complex generators occur in pairs. What happened there was that we lost two generators and got one new one. The system must clearly have 24 generators, as the algebra is \( sl(5, \mathbb{R}) \). However, the double splitting gives only 23 operators of which 15 are symmetries and 8 are Lie-like [2], five of the nine dilations coming from the dependent variable and the four local projective symmetries are missing. We have to use the actual symmetries
obtained here and require closure of the algebra to generate the full 24. The dual system also gives the same symmetry structure, as expected. However, due to the fact that the $k$ in (20) has a coefficient with iota, even though $k$ itself is zero, the operator for the symmetry carries an imaginary. If we do not put in the iota for the operator, we lose some of the symmetry generators.

**Example 2:** Consider the scalar equation [17]

$$u'' = s^{-5}u^2 .$$  \hfill (22)

It splits into the following system of ODEs:

$$x'' = -2s^{-5}yz ;$$
$$y'' = -2s^{-5}zx ;$$
$$z'' = -2s^{-5}xy ;$$  \hfill (23)

subject to the further algebraic constraint

$$x^2 + y^2 = z^2 .$$  \hfill (24)

There are no symmetries among the 8 Lie-like operators obtained from the splitting of the complex generators

$$X_1 = s \frac{\partial}{\partial s} + 3u \frac{\partial}{\partial u} ; \quad X_2 = s^2 \frac{\partial}{\partial s} + su \frac{\partial}{\partial u} .$$  \hfill (25)

However, the system admits the two symmetry generators:

$$Y_1 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} ;$$
$$Y_2 = s^2 \frac{\partial}{\partial s} + sx \frac{\partial}{\partial x} + sy \frac{\partial}{\partial y} + sz \frac{\partial}{\partial z} .$$  \hfill (26)

**Example 3:** Consider the scalar (Emden-Fowler) equation [18]

$$u'' + 5s^{-1}u' + u^2 = 0 .$$  \hfill (27)

It splits into the following system of ODEs:

$$x'' + 5s^{-1}x' - 2yz = 0 ;$$
$$y'' + 5s^{-1}y' - 2zx = 0 ;$$
$$z'' + 5s^{-1}z' - 2xy = 0 ;$$  \hfill (28)

subject to the further algebraic constraint

$$x^2 + y^2 = z^2 .$$  \hfill (29)
The Emden-Fowler equation has only the one scaling symmetry

\[ X = s \frac{\partial}{\partial s} - 2u \frac{\partial}{\partial u}. \]  

(30)

The split system has 4 Lie-like operators, none of which are symmetries of the system. However, the system admits the scaling

\[ Y = s \frac{\partial}{\partial s} - 2x \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial y} - 2z \frac{\partial}{\partial z}. \]  

(31)

4 System of Four ODEs by Double Splitting

For the four dimensional system, after the first split of (2) given by (3), (4) and (5) we could set

\[ p(r) = w(r) + \iota x(r) \text{ and } q(r) = y(r) + \iota z(r) \] 

and

\[ f^r(r; p, q; p', q') = g(r; w, w') + \iota h(r; w, w'), \]

\[ f^i(r; p, q; p', q') = k(r; w, w') + \iota l(r; w, w'), \]  

(32)

yielding the system of four ODEs

\[ w''(r) = g(r; w, w'), \quad x''(r) = h(r; w, w'), \]

\[ y''(r) = k(r; w, w'), \quad z''(r) = l(r; w, w'), \]  

(33)

subject to the CR-conditions

\[ g_w + h_x = k_y + l_z, \quad g_x - h_w = k_z - l_y, \]

\[ g_y + h_z = -k_w - l_x, \quad g_z - h_y = -k_x + l_w, \]

\[ g_{w'} + h_{x'} = k_{y'} + l_{z'}, \quad g_{x'} - h_{w'} = k_{z'} - l_{y'}, \]

\[ g_{y'} + h_{z'} = -k_{w'} - l_{x'}, \quad g_{z'} - h_{y'} = -k_{x'} + l_{w'}, \]  

(34)

where \( w = (w, x, y, z) \). The prolonged symmetry generator can now be written as

\[ X = \xi(r, w) \frac{\partial}{\partial r} + \eta(r, w).\nabla_w + \eta^{[1]}(r; w, w').\nabla_{w'}. \]  

(35)

Writing this equation out in detail makes it too unwieldy to convey much wisdom.

There is no dual system to this as the splitting is direct. We could have obtained a system of four ODEs by a three stage splitting, setting one of the dependent variables in the systems of three ODEs as a complex pair. For each of the three dimensional systems obtained, one splitting would give the above system and one would give a new system. We are, here, limiting our discussion to a two-step splitting only.

The characterization theorem is:

**Theorem 4:** The system of four second order ODEs (33) corresponds to the scalar
ODE (2) by double complex splitting if and only if the CR-conditions (34) hold with the splitting (32).

The examples will illustrate our procedure further.

**Example 4:** The free particle scalar equation obviously yields the free-particle system of four equations, \( w'' = x'' = y'' = z'' = 0 \). The symmetry algebra must be \( sl(6, \mathbb{R}) \), which has 35 generators. There are a total of 18 symmetries and 8 Lie-like operators. Again, the missing ones come from the dilations involving the dependent variables and local projective symmetries. Again, the closure of the algebra starting with the derived symmetries generates the full \( sl(6, \mathbb{R}) \).

**Example 5:** Consider (22) and now split into the system of four ODEs:

\[
egin{align*}
  w'' &= s^{-5}(w^2 - x^2 - y^2 + z^2) \\
  x'' &= s^{-5}(2wx - 2yz) \\
  y'' &= s^{-5}(2wy - 2xz) \\
  z'' &= s^{-5}(2wz + 2xy).
\end{align*}
\]  

(36)

This has 8 Lie-like operators of which none are symmetries of the system. However, the system does admit the two symmetries

\[
\begin{align*}
  Y_1 &= s \frac{\partial}{\partial s} + 3w \frac{\partial}{\partial w} + 3x \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y} + 3z \frac{\partial}{\partial z} \\
  Y_2 &= s^2 \frac{\partial}{\partial s} + sw \frac{\partial}{\partial w} + sx \frac{\partial}{\partial x} + sy \frac{\partial}{\partial y} + sz \frac{\partial}{\partial z}.
\end{align*}
\]  

(37)

**Example 6:** Again consider the scalar (Emden-Fowler) equation (27). It splits into the following system of ODEs:

\[
\begin{align*}
  w'' + 5s^{-1}w' + w^2 - x^2 - y^2 + z^2 &= 0 \\
  x'' + 5s^{-1}x' + 2wx - 2yz &= 0 \\
  y'' + 5s^{-1}y' + 2wy - 2xz &= 0 \\
  z'' + 5s^{-1}z' + 2wz + 2xy &= 0.
\end{align*}
\]  

(38)

The split system again has 4 Lie-like operators, none of which are symmetries of the system. However, the system admits the scaling

\[
Y = s \frac{\partial}{\partial s} - 2w \frac{\partial}{\partial w} - 2x \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial y} - 2z \frac{\partial}{\partial z}.
\]  

(39)
5 System of Four PDEs for Four Functions of Two Variables by Double Splitting

For the PDEs, we need to treat the dependent variable as complex. However, there are two ways of doing so if we are to obtain functions of only two variables. We could retain the independent variable as real in the first step and then make it complex in the second step, or first treat it as complex and then retain the same independent variables in the second step. We follow the former procedure first. Starting with (2), with the first step splitting given by (3), (4) and (5), we put \( r = s + it \) and proceed with treating the dependent variables as complex exactly as in the case for the system of four ODEs.

The system of equations is now

\[
\begin{align*}
   w_{ss} - w_{tt} + 2x_{st} &= 4g(s; w; w_s, w_t), \\
   x_{ss} - x_{tt} - 2w_{st} &= 4h(s; w; w_s, w_t), \\
   y_{ss} - y_{tt} + 2z_{st} &= 4k(s; w; w_s, w_t), \\
   z_{ss} - z_{tt} - 2y_{st} &= 4l(s; w; w_s, w_t), \\
\end{align*}
\]  

subject to the CR-equations

\[
\begin{align*}
   w_s &= x_t, & w_t &= -x_s, & y_s &= z_t, & y_t &= -z_s; \\
   g_w &= h_x, & g_x &= -h_w, & g_y &= h_z, & g_z &= -h_y; \\
   k_w &= l_x, & k_x &= -l_w, & k_y &= l_z, & k_z &= -l_w. \\
\end{align*}
\]  

Now, as for (10), defining

\[
\phi = w_s + x_t, \quad \psi = w_t - s_x, \quad \kappa = y_s + z_t, \quad \lambda = y_t - z_s,
\]

the conditions for the derivatives with respect to the derivatives can be written as

\[
\begin{align*}
   g_\phi &= h_\psi, & g_\psi &= -h_\phi; & k_\kappa &= l_\lambda, & k_\lambda &= -l_\kappa.
\end{align*}
\]

Writing \((s,t) = \xi\) for the infinitesimal symmetry generator we will now have two components for \(\xi\), namely \((\xi^s, \xi^t)\) and \(\eta\) will have four components as for the system of four ODEs. The additional feature is that the prolonged derivatives will be for \(\nabla_{w_s}\) and the coefficients will be \(\eta^{[1]}\). Thus the prolonged generator can be written as

\[
X = \xi(s,w) \cdot \nabla_s + \eta(s,w) \cdot \nabla_w + \eta^{[1]}(s,w) \cdot \nabla_{\nabla_{w_s}},
\]

where \(\eta^{[1]}(s,w)\) is the generalization of \(\eta^{[1]}\) for the case of PDEs.

The characterization theorem here is:

**Theorem 5:** The system of four PDEs for four functions of two variables (40) corresponds to the scalar ODE (2) by double complex splitting, if and only if the
CR-conditions (41) - (43) hold, provided that \( g, h, k, l \) depend on the derivatives only in the combinations given by (43).

Instead, if we had put \( r = s + t \) in the first step and then proceeded to split the dependent variables twice, we would have got the dual system

\[
\begin{align*}
w_{ss} - w_{tt} + 2y_{st} &= 4g(s; w; w_x, w_t), \\
x_{ss} - x_{tt} + 2z_{st} &= 4h(s; w; w_x, w_t), \\
y_{ss} - y_{tt} - 2w_{st} &= 4k(s; w; w_x, w_t), \\
z_{ss} - z_{tt} - 2x_{st} &= 4l(s; w; w_x, w_t).
\end{align*}
\]

(45)

The CR-conditions are considerably more involved. The simple ones are

\[
\begin{align*}
w_s &= y_t, \\
w_t &= -y_s, \\
x_s &= z_t, \\
x_t &= -z_s; \\
g_w + h_x &= k_y + l_z, \\
g_x - h_w &= k_z - l_y; \\
g_y + h_z &= -k_w - l_x, \\
g_z - h_y &= -k_x + l_w.
\end{align*}
\]

(46)

For the derivatives with respect to derivatives we have to define the new variables

\[
\alpha = w_s + y_t, \\
\beta = x_s + z_t, \\
\gamma = w_t - y_s, \\
\delta = x_t - z_s
\]

(47)

to get

\[
\begin{align*}
g_\alpha + h_\beta &= k_\gamma + l_\delta, \\
g_\beta - h_\alpha &= k_\delta - l_\gamma, \\
g_\gamma + h_\delta &= -k_\alpha - l_\beta, \\
g_\delta - h_\gamma &= -k_\beta + l_\alpha.
\end{align*}
\]

(48)

The form of the symmetry generator remains unchanged. Once again, we rely on the examples to illustrate the procedure.

Here we need a separate theorem because the systems are apparently different, though they are dual to each other in some sense.

**Theorem 6:** The (dual) system of four PDEs for four functions of two variables (45) corresponds to the scalar ODE (2) by double complex splitting, if and only if the CR-conditions (46) - (48) hold, provided that \( g, h, k, l \) only depend on the derivatives in the combinations given by (48).

**Example 7:** Consider the free particle equation split into the system of four PDEs for two independent variables

\[
\begin{align*}
w_{ss} - w_{tt} + 2x_{st} &= 0, \\
x_{ss} - x_{tt} - 2w_{st} &= 0, \\
y_{ss} - y_{tt} + 2z_{st} &= 0, \\
z_{ss} - z_{tt} - 2y_{st} &= 0.
\end{align*}
\]

(49)

The symmetry generators split into 28 Lie-like operators of which 20 are symmetries. However, the system admits infinitely many symmetries.
The dual system is very similar. More precisely, it is the the system (45) with the right side set equal to zero. Since the CR-conditions are trivial there is no significant difference between the original and the dual system.

Example 8: Again consider the scalar (Emden-Fowler) equation (27). It splits into the following system of four PDEs of two independent variables:

\[
\begin{align*}
    w_{ss} - w_{tt} + 2x_{st} + 10\frac{s}{s^2 + t^2}(w_s + x_t) + 10\frac{t}{s^2 + t^2}(x_s - w_t) & + 4(w^2 - x^2 - y^2 + z^2) = 0; \\
x_{ss} - x_{tt} - 2w_{st} + 10\frac{s}{s^2 + t^2}(x_s - w_t) - 10\frac{t}{s^2 + t^2}(w_s + x_t) + 8(wx - yz) & = 0; \\
y_{ss} - y_{tt} + 2z_{st} + 10\frac{s}{s^2 + t^2}(y_s + z_t) + 10\frac{t}{s^2 + t^2}(z_s - y_t) + 8(wy - xz) & = 0; \\
z_{ss} - z_{tt} - 2y_{st} + 10\frac{s}{s^2 + t^2}(z_s - y_t) - 10\frac{t}{s^2 + t^2}(y_s + z_t) + 8(wz + xy) & = 0.
\end{align*}
\]

This split system again has 4 Lie-like operators, none of which are symmetries of the system. However, it admits the scaling

\[
\mathbf{Y} = s \frac{\partial}{\partial s} + t \frac{\partial}{\partial t} - 2w \frac{\partial}{\partial w} - 2x \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial y} - 2z \frac{\partial}{\partial z}.
\]

6 System of Four PDEs for Four Functions of Four Variables by Double Splitting

This is the most straight forward (and the most complicated) of the various possibilities considered. At the first step we regard both the independent and the dependent variables as given by (6) to (8). For the second step we run short of symbols for the variables. As such, we now write the first variable (previously written as \( s \)), as the complex variable \( s + \iota t \) and the second variable (previously written as \( t \)), as the complex variable \( u + \iota v \) and write \( s \) for \((s, t, u, v)\). Further, we put

\[
\begin{align*}
p(s, t) & \to w(s) + \iota x(s), \\
q(s, t) & \to y(s) + \iota z(s).
\end{align*}
\]

\[
\begin{align*}
f^s(s, t; p, q; p_s, q_s, p_t, q_t) & = g(s; w, \nabla_s w) + \iota h(s; w, \nabla_s w); \\
f^i(s, t; p, q; p_s, q_s, p_t, q_t) & = k(s; w, \nabla_s w) + \iota l(s; w, \nabla_s w).
\end{align*}
\]

The system of equations is

\[
\begin{align*}
w_{ss} - w_{tt} + 2x_{st} - w_{uu} + w_{vv} - 2x_{uv} + 2y_{su} - 2y_{tv} & + 2z_{sv} + 2z_{tv} = 4g(s; w, \nabla_s w);
\end{align*}
\]
\[ x_{ss} - x_{tt} - 2w_{st} - x_{uu} + x_{vv} + 2w_{uv} + 2z_{su} - 2z_{tv} 
- 2y_{uw} - 2y_{tv} = 4h(s; w, \nabla_s w); \]
\[ y_{ss} - y_{tt} + 2z_{st} - y_{uu} + y_{vv} - 2z_{uv} + 2w_{su} - 2w_{tv} + 2x_{sv} + 2x_{tv} = 4k(s; w, \nabla_s w); \]
\[ z_{ss} - z_{tt} - 2y_{st} - z_{uu} + z_{vv} + 2y_{uv} + 2x_{su} - 2x_{tv} - 2w_{sv} - 2w_{tv} = 4l(s; w, \nabla_s w); \]

subject to the CR-conditions

\[ w_s + x_t = y_u + z_v, \quad w_t - x_s = y_v - z_u, \]
\[ w_u + x_v = -y_s - z_t, \quad w_v - x_u = -y_v + z_u; \]
\[ g_s + h_t = k_u + l_v, \quad g_t - h_s = k_v - l_u, \]
\[ g_u + h_v = -k_s - l_t, \quad g_v - h_u = -k_t + l_s; \]
\[ g_w + h_x = k_y + l_z, \quad g_x - h_w = k_z - l_y, \]
\[ g_y + h_z = -k_w - l_x, \quad g_z - h_y = -k_x + l_w. \]  \tag{55}

The derivatives with respect to the derivatives require the variables

\[ \alpha = w_s + x_t + y_u + z_v, \quad \beta = w_t - x_s + y_v - z_u; \]
\[ \gamma = w_u + x_v - y_s - z_t, \quad \delta = w_v - x_u - y_t + z_s. \]  \tag{56}

Then the rest of the CR-conditions are

\[ g_\alpha - h_\beta = k_\gamma - l_\delta, \quad g_\beta + h_\alpha = k_\delta + l_\gamma; \]
\[ g_\gamma - h_\delta = -k_\alpha + l_\beta, \quad g_\delta + h_\gamma = -k_\beta - l_\alpha. \]  \tag{57}

The characterization theorem in this case is:

**Theorem 7:** The system of four second order PDEs for four functions of four variables (54) corresponds to the scalar second order ODE (2) by double complex splitting provided the functions \( g, h, k, l \) depend on the derivatives only in the combinations given by (57) and the CR-conditions (55) and (57) hold.

The prolonged symmetry generator for the system is

\[ X^{[1]} = \xi(s, g) \nabla_s + \eta(s, g) \nabla_g + \eta^{[1]}(s, g, \nabla_s g) \nabla_{\nabla_s g}. \]  \tag{58}

The derivatives with respect to derivatives are to be taken bearing in mind the discussion for the CR-equations. However, even if we ignore it in taking the derivatives, no error will ensue.

We again rely on the examples to illustrate our systems.

**Example 9:** The free-particle system of equations is given by (54), with the right side set equal to zero. The CR-conditions are trivial. There are now 32 Lie-like
operators, of which only 24 are symmetry generators. As before the local projective symmetries are lost. Here there are 8 such. However, the dilations are not lost here. The system, itself, has an infinite number of symmetry generators.

**Example 10:** Consider the double splitting of the Emden-Fowler equation (27) into a system of four PDEs for four functions of four variables. It is given by (54) with

\[
\begin{align*}
g &= C[(sA + tB)\alpha + (tA - sB)\beta + (uA + vB)\gamma + (vA - uB)\delta] \quad -w^2 + x^2 + y^2 + z^2; \\
h &= C[(sA + tB)\beta - (tA - sB)\alpha + (uA + vB)\delta - (vA - uB)\gamma] \quad -2wx + 2yz; \\
k &= C[(sA + tB)\gamma + (tA - sB)\delta - (uA + vB)\alpha - (vA - uB)\beta] \quad -2wy + 2xz; \\
l &= C[-(sA + tB)\delta + (tA - sB)\gamma + (uA + vB)\beta - (vA - uB)\alpha] \quad -2wz - 2xy;
\end{align*}
\]

where

\[
C = -\frac{5}{A^2 + B^2}, \quad A = s^2 - t^2 + u^2 - v^2, \quad B = 2st + 2uv.
\]

There are 4 Lie-like operators none of which are symmetries. The system, like the scalar equation, has a scaling symmetry.

### 7 Summary and Discussion

In this paper we have considered systems of three and four second order ODEs, and systems of four second order PDEs for four functions of two or four variables, that correspond to a scalar equation, that we shall call a **base equation** by a specific procedure, that we call **double complex splitting**. We have also provided characterization criteria for such systems to correspond to the base equation and a clear procedure to be able to construct the base equation. Thus, in principle, we could write a computer code that could take any such system given and check if it corresponds to a base equation. It could then construct the base equation.

What is the advantage of having such base equations and constructing them? The point is that it is much easier to deal with the base equation than with the system. Thus, for example, if the base system has two infinitesimal symmetry generators it could be solved by symmetry methods. In fact it could have eight symmetry generators and thus be linearizable. In that case we could write down the solution directly. Following the double splitting procedure by which the system corresponds, we could then write down the solution for the system of ODEs or PDEs. Note that in this procedure the system need not have the required symmetry for being directly solvable by symmetry methods.
The procedure adopted for CSA could only give an even dimensional system as it simply split $n$ (complex) equations into $2n$ (real) equations. Also, the number of independent variables in the PDE equals the number of dependent variables. In the complex double split procedure we are being considerably more adventurous. Having obtained the system of two “real” equations we conveniently forget that they arose from a scalar complex equation, treat it as complex and promptly split the equations again. Now we have the earlier freedom of choosing the independent variable to be either real or complex, while treating the dependent variables to be complex, but we have the additional freedom to choose to treat one of the dependent variables as real and the other as complex. This provides the possibility of obtaining an odd dimensional system. Further, to obtain the PDEs, we could choose to treat the independent variables as real first and then complex, complex first and then real or complex both times. Thus we also get the system of four PDEs for four functions of two variables.

In the cases of full double complex splitting, where either the dependent variables were fully split or both the dependent and independent variables were double split, giving the system of four ODEs or four PDEs for four functions of four variables, there were no complications of additional dual systems arising. However, in the case of the system of three ODEs or the system of four PDEs for four functions of two variables, we got dual systems arising. The duality in the former case was very obvious but in the latter it was considerably more involved on account of the CR-conditions. Even in the case of the system of three ODEs the symmetry generators had to carry an iota to make sense. The “duality” of these systems needs to be better understood. Note that $\alpha$ in (56) is simply $\nabla_s w$. It would be interesting to find out what the operators for the other variables are. Presumably, they would be “dual” divergence operators in some sense. This may shed some light on the structure of the double split systems.

For the system of three ODEs obtained by double splitting, it would be of interest to consider the ambiguity due to the choice of the function $g$ in (15), or $k$ in (20) for the dual system. In some sense all choices must be “equivalent”. The question is whether one gets an equivalence class. Further, would they be equivalent under point transformations or possibly some more general transformations like contact or higher order transformations [19].

The algebraic constraint that arises in the system of three ODEs was not apparent in setting up the system but was found in the examples. It is interesting to note that it geometrically amounts to the solution lying on a cone. This seems to be generic for the three dimensional system. It also shows that it can be written as a system of two ODEs. However, that system is much more complicated.

One would have hoped that for the system of PDEs corresponding to a base ODE one could use the symmetries of the ODE to obtain a “core” set of symmetries for
the system of PDEs. Even if the system has infinitely many symmetries, the base equation can only have a finite number. However, the examples show that we can lose all the Lie symmetry generators and be left only with Lie-like ones. This applies even to the free particle equation. Further for the Emden-Fowler equation, we are left with no Lie symmetries from the Lie-like operators, though the equation has a scaling symmetry and so does the double-split system. In general, we obtain Lie-like operators and not Lie-symmetry generators that would form an algebra. The Lie-like operators somehow encode the symmetries of the base equation. It would be most important to learn how they do so. It may be that the CR-conditions will enable us to re-construct the Lie from the Lie-like symmetry.

It is of interest to note that not only for the PDEs but also for the systems of ODEs, we get Lie-like operators arising and lose some Lie-symmetry generators. It would be worth while to see this encoding of symmetry as distinct from the PDE case.

We hope that in future the use of these systems for the variational principle and with linearization will be followed up. It should lead to interesting and useful results.

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