Probabilistic cellular automata for interacting fermionic quantum field theories

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A particular version of the Thirring model with imaginary coupling can be represented as a cellular automaton. This interacting fermionic quantum field theory obeys a unitary time evolution and shows all properties of quantum mechanics. Cellular automata with probabilistic initial conditions can take the form of quantum theories. Our model exhibits interesting features as spontaneous symmetry breaking or solitons. The same model can be formulated as a generalized Ising model. This euclidean lattice model can be investigated by standard techniques of statistical physics as Monte Carlo simulations. Our model is an example how quantum mechanics emerges from classical statistics.

1. Introduction

Cellular automata are used in a wide range of physics and more general science [1]. They are the basis of t’Hooft’s deterministic proposal for quantum mechanics [2], see also [3]. Probabilistic cellular automata are based on a probability distribution of initial condition. They constitute simple examples for information transport in classical statistics [4]. Probabilistic cellular automata correspond to unique jump step evolution operators and assure a “unitary evolution” of the classical wave function. In a probabilistic view of cellular automata, and more generally of classical probabilistic systems, the concepts of wave functions and density matrices arise in a natural way [4, 5]. Many features of the quantum formalism are present in classical probabilistic theories [6]. Cellular automata induce a time evolution for which the norm of the wave function is preserved. This property singles out quantum mechanics from more general probabilistic evolution laws. Approximate cellular automata constitute interesting static memory materials [7]. Free fermionic quantum field theories have been realized as generalised Ising models [8–10]. The associated evolution of the quantum wave function corresponds to simple probabilistic cellular automata. There exists a general “fermion-bit map” [8, 9] that maps the weight distribution for generalized Ising models to the weight distribution of fermionic models defined by a Grassmann functional integral. This map is based on the construction of the transfer matrix [11, 12] or the step evolution operator in both formulations. The question arises which unitary quantum field theories for fermions can be mapped to a positive weight distribution for Ising spins and vice versa. A positive weight distribution is a classical probability distribution for which many highly developed methods of statistical physics, including efficient numerical simulations, can be employed. Cellular automata are prime examples for a map between probability distributions for Ising spins and unitary fermionic quantum field theories.

So far, generalized Ising models and cellular automata have been constructed only for free fermionic quantum field theories. The present paper provides a first example for an interacting fermionic quantum field theory. We construct explicitly a rather simple cellular automaton for a particular version of the Thirring model [13]. The Thirring model is a rather simple two-dimensional model that allows for exact solutions [13–19]. Nevertheless, it admits rich features as spontaneous symmetry breaking and solitons, which have been used by Coleman [20] for a map to a bosonic model.

The cellular automaton constructed in the present paper corresponds to a particular model in this class, characterized by a special value of an imaginary coupling. Despite the coupling being imaginary, the quantum field theory is unitary, as implied directly by its equivalence to a cellular automaton. Our model again can describe spontaneous symmetry breaking and solitons. It can be investigated either by a numerical or analytic solution for the cellular automaton, or by methods of fermionic quantum field theories. We also construct the associated generalized Ising model. This allows for the use of Monte-Carlo simulations or similar numerical techniques. It may provide for a new direction how fermionic quantum field theories could become accessible to numerical simulations.

2. Probabilistic cellular automata

Cellular automata can be described by a series of discrete time steps $t$, $t + \varepsilon$, $t + 2\varepsilon$ etc., starting at some initial time $t_{\text{in}}$. Every state $\rho$ at $t$ is transformed to a state $\overline{\rho}(\rho)$ at $t + \varepsilon$, and so forth for increasing time. To be specific, we consider at each $t$ a number $M$ of Ising spins $s_{\gamma}(t)$, $\gamma = 1 \ldots M$, $s_{\gamma}^{2}(t) = 1$. The $N = 2^{M}$ states $\rho$ are the possible configurations of Ising spins. A given cellular automaton is characterized by a sequence of maps $\rho \rightarrow \overline{\rho}(\rho)$ for a sequence of time steps. These maps can be represented by $N \times N$ - matrices. We consider here invertible cellular automata for which at every $t$ the inverse map $\tau \rightarrow \overline{\rho}(\tau)$ exists.

Step evolution operator and wave function for cellular automata

A convenient formalism uses for each time step the step evolution operator $\hat{S}_{\tau,\rho}(t)$, given by

$$\hat{S}_{\tau,\rho}(t) = \delta_{\tau,\overline{\rho}(\rho)} = \delta_{\overline{\rho}(\tau),\rho}.$$ \hspace{1cm} (1)
It is a matrix with precisely one element equal to one in each row and column, and zeros otherwise. The state of the system at $t$ can be described by a vector or “wave function” $q(t)$ with $N$ components $q_{\rho}(t)$. The evolution law for a cellular automaton reads

$$q_{\tau}(t + \varepsilon) = \hat{S}_{\tau \rho}(t) q_{\rho}(t) .$$  \hspace{1cm} (2)

For a fixed state $\sigma$ at $t$ one takes $q_{\rho}(t) = \delta_{\rho, \sigma}$. This implies

$$q_{\tau}(t + \varepsilon) = \delta_{\tau \rho}(t) \delta_{\rho, \sigma} = \delta_{\tau \rho}(t) \delta_{\sigma \sigma}(t) ,$$  \hspace{1cm} (3)

and zero otherwise. The state $\sigma$ at $t$ is indeed transported to $\tau(\sigma)$ at $t + \varepsilon$. The sequence of time steps corresponds to an ordered matrix multiplication of step evolution operators.

$$q_{\tau}(t + n \varepsilon) = \left[ \hat{S}(t + (n - 1) \varepsilon) \ldots \hat{S}(t + \varepsilon) \hat{S}(t) \right] \tau \rho q_{\rho}(t) .$$  \hspace{1cm} (4)

Probabilistic cellular automata are cellular automata for which the initial condition is given by a probability distribution \{pr(tin)\}, $p_{\tau} > 0$, $\sum_{\tau} p_{\tau} = 1$. It is convenient to associate at every $t$ the probability distribution to a wave function [5, 8, 21]

$$p_{\tau}(t) = q_{\tau}^{2}(t) .$$  \hspace{1cm} (5)

Probabilistic initial conditions are then encoded in a more general form of the initial wave function $q(t_{in})$, given by a unit vector with

$$p_{\tau}(t_{in}) = q_{\tau}^{2}(t_{in}), \quad \sum_{\tau} q_{\tau}^{2}(t_{in}) = 1 .$$  \hspace{1cm} (6)

The probability distribution at $t > t_{in}$ is obtained from eqs. (4), (5). The relation $p_{\tau}(t) = q_{\tau}^{2}(t)$ guaranties the positivity and normalization of the probability distribution for every $t > t_{in}$, since $S$ is an orthogonal matrix which preserves the norm of $q$. Expectation values of local observables at every $t$ can be extracted from the probability distribution \{pr(t)\} by the standard rules of classical statistics.

### Alternating switch gates

A simple example is the "switch operator" for two bits, $M = 2$. We label the four states $\tau = 1 \ldots 4$ by the configurations of Ising spins $(1,1), (1,-1), (-1,1), (-1,-1)$. For the analogy with fermions we actually will rather use occupation numbers $n_{\gamma} = (s_{\gamma} + 1)/2$, $n_{\gamma} = 1, 0$, and label the states by $(1,1), (1,0), (0,1), (0,0)$. The switch operation maps

$$S : (1,0) \leftrightarrow (0,1),$$

$$\quad \quad \quad \quad (0,0), (1,1) \ \text{invariant} .$$  \hspace{1cm} (7)

The corresponding step evolution operator reads

$$\hat{S}_{S} = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) , \quad \hat{S}_{S}^{2} = 1 .$$  \hspace{1cm} (8)

This is a rather trivial automaton for which two particles of different type are interchanged at every time step.

Cellular automata describing Thirring type models can be based on an alternating sequence of generalized switch operations or switch gates. We start with the simplest type of switch gates. At a given $t$ the spins are on a one-dimensional chain with positions $x$ separated by $\varepsilon$, e.g. $s_{x}(t) = s(t, x)$. Correspondingly, at every $t$ the states are distributions of particles, given by occupation numbers $n(t, x)$. We define even and odd $t$ or $x$ by $t = m_{t} \varepsilon$, $x = m_{x} \varepsilon$, with even or odd integers $m_{t}, m_{x}$. At even $t$ we consider pairs of neighboring positions $x$ and $x + \varepsilon$, with $x$ even. They define blocks with two sites. The cellular automaton moves a single particle at $x$ to $x + \varepsilon$, and vice versa. Configurations with two particles or zero particles in the block remain invariant. If we associate the two species of particles with particles at $x$ and $x + \varepsilon$, this amounts to the switch (7), (8). The switch operation acts independently on all pairs or in all blocks.

At $t + \varepsilon$ we change the grouping into pairs of neighbors at $x$ and $x - \varepsilon$, for even $x$. The switch operates now between the particles in the new blocks. As a result, $\hat{S}(t)$ and $\hat{S}(t + \varepsilon)$ do not commute. In the following, we alternate the pairing, $\hat{S}(t + 2 \varepsilon) = \hat{S}(t)$. Repetition of a sequence of two switches does no longer produce unity as in eq. (8), since

$$\left( \hat{S}(t + \varepsilon) \hat{S}(t) \right)^{2} = 1 + \hat{S}(t + \varepsilon) \hat{S}(t) \left[ \hat{S}(t + \varepsilon), \hat{S}(t) \right] .$$  \hspace{1cm} (9)

As a result, this rather simple automaton allows already for a non-trivial evolution. For only a few particles present, and most positions empty, the evolution can be viewed as the propagation of the particles on light cones with occasional scattering, as depicted in Fig. 1. The totally empty and the totally occupied states are invariant. For the regular half-filled state with alternating occupied and empty sites one finds the same state at $t + 2 \varepsilon$ and $t$. For this state all particles are either freely right moving, and empty sites or holes left moving, or all particles are left moving with right moving holes. These features show qualitatively already many analogies to fermions.

### 3. Step evolution operator for fermions

We next develop the language of a fermionic quantum field theory for this and similar cellular automata. It is based on fermion-bit map [8, 9] between Ising spins and fermions in the occupation number basis.

#### Grassmann functional integral

In order to establish an isomorphic map between some discretized version of a type of Thirring model for fermions and the particular cellular automaton described above we extract the step evolution operator from the weight function for fermions in a formulation with Grassmann variables [8, 9]. We assume that the weight function $w_{0}[\psi]$ can be written as a product of local factors $\tilde{K}(t)$

$$w_{0}[\psi] = \exp \left\{ - S[\psi] \right\} = \prod_{t} \tilde{K}(t) ,$$  \hspace{1cm} (10)
where $S[\psi]$ is the action and $\hat{K}(t)$ depends on Grassmann variables on neighboring $t$-layers $\psi_\gamma(t)$ and $\psi_\gamma(t+\varepsilon)$. Each local factor $\hat{K}(t)$ can be written as a polynomial in the Grassmann variables involved. We assume that all terms involve an even number of Grassmann variables, such that local factors at different $t$ commute.

An element of a local Grassmann algebra is a function of the local Grassmann variables $\psi_\gamma(t)$, where locality refers to $t$. It is a linear combination of basis elements $g_\tau$,

$$\hat{q}(t) = g_\tau(t) g_\tau[\psi(\tau)] .$$

Each basis element is a polynomial of factors $\psi_\gamma(t)$, where we take some ordering with the smaller $\gamma$ to the left, with a conveniently chosen sign. Since at each "place" $\gamma$ in the polynomial there can be either a factor $\psi_\gamma$ or one (i.e. no $\psi_\gamma$), there are $N = 2^M$ possibilities, similar to the configurations of $M$ occupation numbers. We identify a factor $\psi_\gamma$ with an empty site, and a factor one with an occupied site. An example with $M = 2$ is

$$(1,1) : \quad g_1 = g'_1 = \bar{\eta}_4 = - \bar{\eta}_4 = 1 ,$$

$$(1,0) : \quad g_2 = g'_2 = - \bar{\eta}_3 = - \bar{\eta}_3 = \psi_2 ,$$

$$(0,1) : \quad g_3 = g'_3 = \bar{\eta}_2 = \bar{\eta}_2 = \psi_1 ,$$

$$(0,0) : \quad g_4 = - g'_4 = \bar{\eta}_1 = - \bar{\eta}_1 = \psi_1 \psi_2 .$$

We will later identify the coefficients $q_\tau(t)$ for $t$ even with the wave function of the probabilistic cellular automaton, with local probabilities $p_\tau(t) = q_\tau(t)$.

We also introduce conjugate basis elements $\bar{g}_\tau$ by the relation

$$\int \mathcal{D} \psi \bar{g}_\tau[\psi] \eta_\rho[\psi] = \delta_{\tau\rho} .$$

Here all elements are local at $t$, and the local Grassmann integral is given by $\int \psi_\gamma(t) \ldots \psi_\gamma(t)$ . In order to organize the signs arising from the anticommutation of Grassmann variables in an efficient way, we also employ

$$g'_\tau = (-1)^{m_\tau(m_\tau-1)} g_\tau , \quad \bar{g}'_\tau = (-1)^{m_\tau(m_\tau-1)} g_\tau ,$$

with $m_\tau$ the number of $\psi$ factors in $g_\tau$, and $m_\tau$ the number of $\psi$-factors in $g_\tau$ . The different sets of basis functions obey

$$\exp(\psi_\gamma \varphi_\gamma) = g_\tau(\psi) g'_\tau(\varphi) = \bar{g}'_\tau(\psi) \bar{g}_\tau(\varphi) ,$$

and

$$\int \mathcal{D} \psi g'_\tau(\psi) \bar{g}_\rho(\psi) = \eta_\rho \delta_{\tau\rho} , \quad \eta_\rho = (-1)^{\frac{M(M-1)}{2}} .$$

**Step evolution operator from Grassmann functional integral**

A Grassmann functional integral defines a sequence of step evolution operators or normalized transfer matrices. For this purpose we expand for even $t$ the local factor as

$$\hat{K}(t) = \bar{g}'_\tau(t+\varepsilon) \hat{S}_{\tau\rho}(t) \bar{g}_\rho(t) .$$

The coefficients $\hat{S}_{\tau\rho}(t)$ will be identified with the matrix elements of the step evolution operator. With eq. (13) one finds

$$\int \mathcal{D} \psi(t) \hat{K}(t) \hat{q}(t) = \hat{S}_{\tau\rho}(t) q_\rho(t) \bar{g}_\rho(t+\varepsilon) ,$$

where we expand for odd $t$

$$\hat{q}(t) = g_\tau(t) \bar{g}_\tau[\psi(\tau)] ,$$

instead of eq. (11) for even $t$. According to the modulo two properties in time we employ for odd $t$ the expansion

$$\hat{K}(t) = g_\tau(t+\varepsilon) \hat{S}_{\tau\rho}(t) g'_\rho(t) .$$

Integrating out the variables at $t+\varepsilon$ the product of odd local factors results in matrix multiplication ($t$ even)

$$\int \mathcal{D} \psi(t+\varepsilon) \hat{K}(t+\varepsilon) \hat{K}(t) = \eta_\rho g_\tau(t+2\varepsilon) \hat{S}(t+\varepsilon) \hat{S}(t) \bar{g}_\rho(t) .$$

Considering longer chains of neighboring local factors and integrating out intermediate Grassmann variables one arrives at eq. (4), multiplied on both sides with appropriate Grassmann basis functions. (A possible factor $(-1)$ from powers of $\eta_\rho$ will be omitted - if necessary it can be absorbed by a slight redefinition of expansions.) In general, the matrices $\hat{S}$ defined by eqs. (17), (20) are the transfer matrices. We normalize $\hat{K}(t)$ by multiplication with a constant such that the largest eigenvalue of $\hat{S}$ obeys $|\lambda| = 1$. With this normalization the transfer matrix becomes the step evolution operator.
The "fermion-bit map" discussed here allows us to describe the evolution of the local probability distribution for cellular automata by a fermion model that produces the same step evolution operator. The fermion-bit map can be extended to maps of observables, not necessarily involving variables only at a given time [9]. Our aim will be to find fermionic models which represent a given cellular automaton, as the one described above.

**Simple fermion model**

A first Thirring type model is defined by a Grassmann functional integral

$$Z = \int D\psi \exp\{-S[\psi]\} = \int D\psi \exp\{-iS_M[\psi]\},$$  \hspace{1cm} (22)

with euclidean action $S$ related to the action $S_M$ for Minkowski signature by $S = iS_M$. The action is local and invariant under Lorentz-transformations, $S = \int \mathcal{L}(t)$, with

$$\mathcal{L}(t) = -\int_x \left\{ \frac{\gamma^\mu \partial_\mu \psi - \frac{g}{2} (\gamma^\mu \psi)(\gamma^\mu \psi)\} \right\}.$$  \hspace{1cm} (23)

With two-dimensional Dirac matrices given by the Pauli matrices as $\gamma^0 = -i\sigma_2$, $\gamma^1 = \tau_1$, and $\psi^T = (\psi_1, \psi_2)$, $\overline{\psi} = (\overline{\psi}_1, \overline{\psi}_2)$, this reads

$$\mathcal{L}(t) = -\int_x \left\{ \overline{\psi}_2 (\partial_t + \partial_x)\psi_1 - \overline{\psi}_1 (\partial_t - \partial_x)\psi_2 - 2g\overline{\psi}_2 \psi_1 \psi_1 \right\}.$$  \hspace{1cm} (24)

We emphasize that the Minkowski action $S_M = -iS$ is not obtained by analytic continuation from the euclidean action. The Grassmann functional is not changed, the formulation with $S_M$ is just a different notation. In $S_M$ the interaction term is multiplied by $-ig/2$. This corresponds to an imaginary coupling in the Thirring model. Nevertheless we will see that the time evolution of this model is unitary for $g = 1$. The Grassmann variables $\psi_\gamma$ and $\overline{\psi}_\gamma$ are independent.

For a well defined discrete formulation we choose for $t$ and $x$ a quadratic lattice with lattice distance $\varepsilon$. For a discretized version of our model we first consider the action

$$S = \sum_{t,\text{even}} [\mathcal{L}_{\text{kin}}(t) + \mathcal{L}_{\text{int}}(t)],$$  \hspace{1cm} (25)

with

$$\mathcal{L}_{\text{kin}}(t) = -\sum_{x,\text{even}} \left[ \varphi(t+\varepsilon, x+\varepsilon)\varphi(t, x) + \varphi(t, x)\varphi(t+\varepsilon, x) + \varphi(t+2\varepsilon, x+\varepsilon)\varphi(t+\varepsilon, x-\varepsilon) + \varphi(t+2\varepsilon, x-\varepsilon)\varphi(t+\varepsilon, x+\varepsilon)\right],$$  \hspace{1cm} (26)

and

$$\mathcal{L}_{\text{int}}(t) = g \sum_{x,\text{even}} \left[ \varphi(t+\varepsilon, x+\varepsilon)\varphi(t+\varepsilon, x)\varphi(t, x+\varepsilon)\varphi(t, x) + \varphi(t+2\varepsilon, x+\varepsilon)\varphi(t+\varepsilon, x-\varepsilon)\varphi(t+\varepsilon, x)\right].$$  \hspace{1cm} (27)

The action is an element of a real Grassmann algebra with a single Grassmann variable $\varphi(t, x)$ on each point of the lattice. The sum runs only over a coarse grained lattice with even $t$ and even $x$. $m_t$, $m_x$ even integers. We take periodic boundary conditions in $x$, with an even total number of $x$ points.

**Continuum limit**

We introduce lattice derivatives by

$$\begin{align*}
(\partial_t + \partial_x)\varphi(t + \varepsilon, x + \varepsilon) &= \frac{1}{2\varepsilon} [\varphi(t + 2\varepsilon, x + 2\varepsilon) - \varphi(t, x)] , \\
(\partial_t - \partial_x)\varphi(t + \varepsilon, x) &= \frac{1}{2\varepsilon} [\varphi(t + 2\varepsilon, x - \varepsilon) - \varphi(t, x + \varepsilon)].
\end{align*}$$  \hspace{1cm} (28)

The kinetic term,

$$\mathcal{L}_{\text{kin}}(t) = 2g \sum_{x,\text{even}} \left\{ \varphi(t+\varepsilon, x+\varepsilon)(\partial_t + \partial_x)\varphi(t+\varepsilon, x+\varepsilon) + \varphi(t, x)\varphi(t+\varepsilon, x)\right\},$$  \hspace{1cm} (29)

describes right movers on the sublattice with $m_t + m_x$ even, and left movers on the sublattice with $m_t + m_x$ odd. For $t$, $x$ both even we choose the naming conventions and normalization

$$\begin{align*}
\varphi(t, x) &= \sqrt{2\varepsilon} \psi_1(t, x), \\
\varphi(t, x + \varepsilon) &= \sqrt{2\varepsilon} \psi_2(t, x + \varepsilon), \\
\varphi(t + \varepsilon, x) &= \sqrt{2\varepsilon} \overline{\psi}_1(t + \varepsilon, x), \\
\varphi(t + \varepsilon, x + \varepsilon) &= -\sqrt{2\varepsilon} \overline{\psi}_2(t + \varepsilon, x + \varepsilon).
\end{align*}$$  \hspace{1cm} (30)

such that

$$\mathcal{L}_{\text{kin}}(t) = -4\varepsilon g \sum_{x,\text{even}} \left\{ \overline{\psi}_2(t + \varepsilon, x + \varepsilon)(\partial_t + \partial_x)\psi_1(t + \varepsilon, x + \varepsilon) - \overline{\psi}_1(t + \varepsilon, x)(\partial_t - \partial_x)\psi_2(t + \varepsilon, x) \right\}.$$  \hspace{1cm} (31)

In the continuum limit the lattice derivatives become partial derivatives and $\sum_{t,\text{even}} = (2\varepsilon)^{-1} \int dt$, $\sum_{x,\text{even}} = (2\varepsilon)^{-1} \int dx$. One recovers the derivative term in the action (24). In terms of $\psi$, $\overline{\psi}$ the interaction term (27) reads

$$\begin{align*}
\mathcal{L}_{\text{int}}(t) &= -4\varepsilon g \sum_{x,\text{even}} \left\{ \overline{\psi}_2(t + \varepsilon, x + \varepsilon) \times \overline{\psi}_1(t + \varepsilon, x)\psi_1(t, x) + \overline{\psi}_2(t + \varepsilon, x - \varepsilon)\overline{\psi}_1(t + \varepsilon, x) \times \psi_2(t + 2\varepsilon, x - \varepsilon)\psi_1(t + 2\varepsilon, x) \right\}.
\end{align*}$$  \hspace{1cm} (32)

We find the interaction term of eq. (24) in the continuum limit.
Local factors

For the action (25)-(27) the weight function \( \exp \{-S[\varphi]\} \) can be written as a product of local factors \( \mathcal{K}(t) \) using

\[
\exp\{-S\} = \prod_t \exp\{-L(t)\} = \prod_t \tilde{\mathcal{K}}(t),
\]
where we take for \( t \) even

\[
\exp\{-L(t)\} = \tilde{\mathcal{K}}(t+\varepsilon)\tilde{\mathcal{K}}(t).
\]

These local factors take the form

\[
\tilde{\mathcal{K}}(t) = \prod_{x,\text{even}} \left\{ 1 + \varphi(t+\varepsilon, x+\varepsilon)\varphi(t, x) + \varphi(t, x+\varepsilon)\varphi(t+\varepsilon, x) + (1-\varepsilon)\varphi(t+\varepsilon, x+\varepsilon)\varphi(t, x+\varepsilon, x)\varphi(t, x) \right\}
\]

and

\[
\tilde{\mathcal{K}}(t+\varepsilon) = \prod_{x,\text{even}} \left\{ 1 + \varphi(t+2\varepsilon, x-\varepsilon)\varphi(t+\varepsilon, x) + \varphi(t+\varepsilon, x-\varepsilon)\varphi(t+2\varepsilon, x) + (1-\varepsilon)\varphi(t+\varepsilon, x-\varepsilon)\varphi(t+\varepsilon, x) \right\}.
\]

Each factor for a given \( x \) in \( \tilde{\mathcal{K}}(t) \) for \( t \) even involves only four Grassmann variables in a block of four lattice sites. The quadratic term multiplies variables at opposite ends of the diagonals in the block, and the interaction term involves all four variables in the block. For odd \( t \) the structure is the same, only the blocks are shifted by one place in \( x \) and \( t \). As a consequence, every block in eq. (35) has precisely one common variable with the two diagonal neighboring blocks from eq. (36).

The block structure makes no difference between space and time. We may view the model (25)-(27) as a euclidean fermion model on a two-dimensional square lattice. The choice of the time-direction is arbitrary at this stage. Time is singled out only by the direction in which we study the evolution. The Lorentz symmetry of the action (23), (24) with the characteristic difference in signature for space and time is not yet directly visible in the model (25)-(27), even though the diagonal structure may be seen already as a hint for light cones.

4. Cellular automata for fermions

Map to cellular automaton

From eqs. (33)-(36) we can extract the step evolution operators for the different blocks defined by appropriate pairs of positions. For each block it is a \( 4 \times 4 \) matrix. For even \( t \) the expansion (17) yields with eq. (12)

\[
\tilde{S}(t, x) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & g-1
\end{pmatrix},
\]
while the expansion (20) implies

\[
\tilde{S}(t+\varepsilon, x) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & g-1
\end{pmatrix}.
\]

For \( g = 2 \) one finds that \( \tilde{S}(t+\varepsilon, x) \) in eq. (38) is indeed the switch operator (8) employed for the cellular automaton discussed above. It involves pairs at \( x \) and \( x+\varepsilon \) (for even \( x \), even \( t \), and therefore odd \( t+\varepsilon \)), as appropriate. For \( t \) even the operator \( \tilde{S}(t, x) \) in eq. (37) acts on pairs at \( x \) and \( x+\varepsilon \), again in accordance with the cellular automaton. The only difference concerns the minus signs in the off-diagonal elements. These minus signs have no physical significance. They can be absorbed by a different definition of Grassmann variables. Indeed, a variable transformation \( \varphi(t, x) \rightarrow -\varphi(t, x) \) for \( t = m_1 \varepsilon \), \( m_k = 1, 2 \mod 4 \), changes the sign of the off-diagonal elements in eq. (37).

For \( g = 0 \) one recovers freely propagating fermions. The operators (37) (38) have eigenvalues \( \pm 1 \). The product of these operators over all \( x \) produces a simple step evolution operator. All particles on the sub-lattice with even \( m_1 + m_x \) move one place to the right if \( m_1 \) increases by one unit, while all particles on the other sublattice with odd \( m_1 + m_x \) move one place to the left. Thus \( \psi_1, \psi_1^- \) describe "right movers", and \( \psi_2, \psi_2^- \) "left-movers".

For identical particles the model with \( g = 2 \) is actually the same as the free model for \( g = 0 \). These two values can be mapped onto each other by a change of sign of Grassmann basis functions. This can also be seen from Fig. 1. Instead of moving the states (1, 1) and (0, 0) in the blocks by one position in \( t \), corresponding to vertical black or red lines, we can also exchange the positions at \( t+\varepsilon \), producing instead black or red crosses. As a result, all filled or empty positions move on diagonals, being right - or left - movers according to the appropriate sublattice. For identical particles Fig. 1 does not describe a non-trivial scattering. We could equally draw lines for unperturbed left and right movers. This can also be seen in the continuum limit. We recall that \( \overline{\varphi}(t+\varepsilon, x) \) is related to \( \overline{\varphi}(t+\varepsilon, x) \) by eq. (30), while \( \psi(t, x) \) corresponds to \( \varphi(t, x) \). For sufficiently smooth wave functions \( \varphi(t, x) \), for which a continuum limit applies, one would like to identify \( \overline{\varphi}(t+\varepsilon, x) \) with variables \( \pm \psi(t, x) \). In this limit the interaction term vanishes since only two distinct Grassmann variables are available at every point \( (t, x) \).

Dirac spinors

A fermion model with non trivial interaction in the continuum limit can be obtained if the fermions carry different colors. If for Fig. 1 one particle is red, the other is green, they can be distinguished and the figure describes scattering due to interaction. We therefore consider next on each site of the lattice two different Grassmann variables. We may denote the two "colors" by \( \varphi_R(t, x) \) and \( \varphi_L(t, x) \). The kinetic term (26) becomes a sum of two kinetic terms, one for \( \varphi_R \) and the other for \( \varphi_L \).
We can introduce a complex structure by defining the complex Grassmann variable
\[ \varphi(t, x) = \varphi_\nu(t, x) + i\varphi_\mu(t, x) . \] (39)
Correspondingly, on the level of \( \psi, \overline{\psi} \) in eq. (30) we define complex Grassmann variables by
\[ \psi_\nu(t, x) = \psi_\nu \nu(t, x) + i\psi_\nu \mu(t, x) , \]
\[ \overline{\psi}_\nu(t, x) = \overline{\psi}_\nu \nu(t, x) - \overline{\psi}_\nu \mu(t, x) . \] (40)

The kinetic term (31) retains its form, except that \( \psi_\nu \) and \( \overline{\psi}_\nu \) are now complex Grassmann variables and the real part of \( \mathcal{L}_{\text{kin}}(t) \) is taken. The continuum limit yields the kinetic term in eq. (23), now with complex Grassmann variables \( \psi, \overline{\psi} \) as appropriate for Dirac spinors. An interaction term becomes possible in the continuum limit. If it is given by eq. (23) one realizes the Thirring model with imaginary coupling.

**Discrete model for interacting fermions**

For a discrete formulation of Thirring type models we need to add an interaction term whose continuum limit yields the interaction term in eq. (23), with \( \psi, \overline{\psi} \) now complex Grassmann variables. We propose local factors \( \tilde{K}(t, x) \) for the blocks of four lattice sites at even \( t, x \), with \( \alpha = R, I \),
\[ \tilde{K}(t, x) = \exp \{ -\hat{L}_{\text{kin}}(t, x) \} + \tilde{K}_{\text{int}}(t, x) , \] (41)
with
\[ \hat{L}_{\text{kin}}(t, x) = -\left\{ \sum_\alpha \left[ \varphi_\alpha(t + \varepsilon, x + \varepsilon) \varphi_\alpha(t, x) + \varphi_\alpha(t + \varepsilon, x) \varphi_\alpha(t, x + \varepsilon) \right] \right\} . \] (42)
The interaction part reads
\[ \tilde{K}_{\text{int}}(t, x) = (\zeta_1 \zeta_4 - \zeta_2 \zeta_3)(\zeta_1 \zeta_4 - \zeta_2 \zeta_3) - (\zeta_1 \zeta_4 + \zeta_2 \zeta_3)(\zeta_1 \zeta_4 + \zeta_2 \zeta_3) , \] (43)
where the Grassmann variables on the four sites of the block are given by
\[ \zeta_1 = \varphi_R(t, x) = \sqrt{2\varepsilon} \psi_{1R}(t, x) , \]
\[ \zeta_2 = \varphi_I(t, x) = \sqrt{2\varepsilon} \psi_{1I}(t, x) , \]
\[ \zeta_3 = \varphi_R(t + \varepsilon, x + \varepsilon) = \sqrt{2\varepsilon} \psi_{2R}(t, x + \varepsilon) , \]
\[ \zeta_4 = \varphi_I(t + \varepsilon, x + \varepsilon) = \sqrt{2\varepsilon} \psi_{2I}(t, x + \varepsilon) , \] (44)
and
\[ \zeta_1 = \varphi_R(t + \varepsilon, x) = \sqrt{2\varepsilon} \psi_{1R}(t + \varepsilon, x) , \]
\[ \zeta_2 = \varphi_I(t + \varepsilon, x) = \sqrt{2\varepsilon} \psi_{1I}(t + \varepsilon, x) , \]
\[ \zeta_3 = \varphi_R(t + \varepsilon, x + \varepsilon) = \sqrt{2\varepsilon} \psi_{2R}(t + \varepsilon, x + \varepsilon) , \]
\[ \zeta_4 = \varphi_I(t + \varepsilon, x + \varepsilon) = \sqrt{2\varepsilon} \psi_{2I}(t + \varepsilon, x + \varepsilon) . \] (45)

Omitting the differences between the different positions one finds in terms of the complex doublets \( \psi \) and \( \overline{\psi} \)
\[ \tilde{K}_{\text{int}}(t, x) = \xi^2 (\overline{\psi} \gamma^\mu \psi)^* (\overline{\psi} \gamma^\mu \psi) . \] (46)
For odd \( t + \varepsilon \) we use for \( \tilde{K}(t + \varepsilon) \) the same from (41), with \( t \) replaced by \( t + \varepsilon \) and \( x \) replaced by \( x + \varepsilon \). Equivalently, we could also make the shift \( x \to x - \varepsilon \).

We next establish that the local factors (41) correspond to the discretization of a particular Thirring model. Writing the block local factor in exponential form
\[ \tilde{K}(t, x) = \exp \{ -\hat{L}(t, x) \} , \] (47)
one has
\[ \hat{L}(t, x) = \tilde{K}_{\text{kin}}(t, x) - \tilde{K}_{\text{int}} - \Delta , \] (48)
where the "correction term" \( \Delta \) obeys
\[ \Delta = \tilde{K}_{\text{int}}(\hat{L}_{\text{kin}} + \frac{1}{2} \hat{L}_{\text{kin}}^2) - \frac{1}{2} \tilde{K}_{\text{int}}^2 . \] (49)

Here we use that \( \tilde{K}_{\text{int}} \) involves four Grassmann variables, and \( \Delta \) at least six Grassmann variables, such that \( \tilde{K}_{\text{int}}^3 = 0 \), \( \tilde{K}_{\text{int}} \Delta = 0 \), \( \Delta^2 = 0 \).

In the continuum limit the first two terms in eq. (48) yield
\[ S = \int_{t,x} \{ -\text{Re}(\overline{\psi} \gamma^\mu \partial_\mu \psi) - \frac{1}{2} (\overline{\psi} \gamma^\mu \psi)^* (\overline{\psi} \gamma^\mu \psi) \} . \] (50)
The term \( \Delta \) vanishes in the continuum limit \( \varepsilon \to 0 \), with
\[ \Delta \sim \varepsilon^3 (\overline{\psi} \psi)^3 , \quad \sum_{t,x} \Delta \sim \varepsilon \int_{t,x} (\overline{\psi} \psi)^3 . \] (51)

Also the differences between the different positions vanish in the transition from eq. (44) to eq. (46). Only the part (50) remains. In the continuum limit one also wants to express \( \overline{\psi}(t, x) \) in terms of \( \psi(t, x) \) since it originates from \( \psi(t + \varepsilon, x) \). The identification reads
\[ \overline{\psi}(t, x) = \psi^*(t, x) \gamma^0 . \] (52)

With this identification \( \int \overline{\psi} \gamma^\mu \partial_\mu \psi \) is real by use of partial integration, and \( (\overline{\psi} \gamma^\mu \psi)^* = -\overline{\psi} \gamma^\mu \psi \). The continuum limit yields indeed the Thirring model (23) with \( g = 1 \), now with complex Grassmann variables \( \psi_1, \psi_2 \).

**Cellular automaton for interacting fermions**

For the particular Thirring type model with \( g = 1 \) we can extract the step evolution operator from the local factor (41). For each block it is a \( 16 \times 16 \) matrix. The kinetic part alone (first term in eq. (41)) moves all four configurations of particle numbers from the lower left corner \( (t, x) \) of the block to the upper right corner \( (t + \varepsilon, x + \varepsilon) \). Similarly, it transports all four configurations in the lower right corner \( (t + \varepsilon, x) \) to the upper left corner \( (t + \varepsilon, x) \). This would be a generalized switch operator. In the presence of the interaction term, however, this behavior is modified if both
lower corners of the block are occupied by a single particle. This concerns the products of variables $\zeta_1 \zeta_3$, $\zeta_1 \zeta_4$, $\zeta_2 \zeta_3$ and $\zeta_2 \zeta_4$. In this sector the kinetic term contributes a factor

$$K_{\text{kin}} = (\zeta_1^* \zeta_3 + \zeta_2^* \zeta_4 + \zeta_1^* \zeta_4 + \zeta_2^* \zeta_3)^2$$

as obtained from the expansion of the exponential and omitting contributions $\sim \zeta_1 \zeta_2$ or $\sim \zeta_3 \zeta_4$. The interaction part contributes

$$K_{\text{int}} = \zeta_1^* \zeta_4 \zeta_1 \zeta_4 - \zeta_1^* \zeta_3 \zeta_2^* \zeta_3 - \zeta_1^* \zeta_3 \zeta_2^* \zeta_3 - \zeta_1^* \zeta_4 \zeta_1 \zeta_4 - \zeta_2^* \zeta_4 \zeta_2 \zeta_4 .$$

(53)

(54)

The last two terms cancel $K_{\text{kin}}$, such that only the first two terms remain in this sector.

With a suitable choice of sign for the Grassmann basis functions the step evolution operator obtained from eqs. (17) (20) is a unique jump operator with one element equal to one in each line and column, and zeros otherwise. It describes a cellular automaton with the following rules for the blocks

1. If the lower left corner is occupied by a single red particle (R) and the lower right corner by a single green particle (I), the red particle moves to the upper left corner and the green particle to the upper right corner. They jump one unit in $t$ without changing position $x$. The same rule holds if red and green colors are interchanged.

2. If the lower left corner and the lower right corner are both occupied by a single red particle, the upper left corner and upper right corner are occupied by green particles. Both particles move on as $t$ progresses, but change color. The same holds if the red and green colors are interchanged. The processes 1.) and 2.) arise from $K_{\text{int}} + K_{\text{kin}}$ in eqs (53), (54).

3. All other configurations except the ones of 1.) and 2.) switch site as $t$ increases, from lower left to upper right and lower right to upper left.

4. The blocks alternate, comprising at $t$ the sites $x$ and $x + \varepsilon$, and at $t + \varepsilon$ the sites $x$ and $x - \varepsilon$, where $t$ and $x$ are taken even.

We have depicted the evolution of a configuration with two particles in Fig. 2. Single particles move on diagonal straight lines. They change color whenever they meet another single particle. In addition, there are two particle lines (not shown in Fig. 2) where a red and a green particle move together on the same diagonal trajectory without scattering. Fig. 2 also demonstrates that the cellular automaton is invariant under $\pi/2$-rotations in the $(t,x)$-plane. Zooming on the two scatterings in the third row from below we observe scattering events as the one shown in Fig. 1, now for two particles with different color which cannot be identified. Note that the drawing of colored lines inside the scattering boxes is a particular picture. We also could draw the lines on the vertical boundaries of the box, as in Fig. 1. In Fig. 2 we have not indicated lines for empty sites (or holes). For two holes on each site we could draw a cross, or vertical lines as in Fig. 1.

For initial conditions with a unique sharp configuration at $t_{\text{in}}$ the dynamics is solved by registering the crossings of single particle lines and accounting for the change of colors. It is obvious that all non-trivial dynamics is related to color and cannot be detected by color-blind observables. A fixed configuration of occupation number of fermions with sharp values of one or zero is a possible initial state for the particular Thirring-type model. It is an allowed quantum state since the occupation number operators at different sites $x$ commute. For this particular initial condition the quantum wave function evolves according to a deterministic cellular automaton.

5. Symmetries, ground states and dynamics

Symmetries

Many symmetries of the continuous Thirring-type model (23) are already present in the discrete fermion model (41)-(43) and can be seen directly for the cellular automaton. Chiral transformations rotate the two colors into each other, separately for right movers and left movers. Infinitesimal chiral transformations act on the complex fields $\psi_1$ and $\psi_2$ as separate phase transformations. On the Grassmann variables (44) infinitesimal chiral transformations act as

$$\delta \zeta_1 = \xi_+ \zeta_2 , \quad \delta \zeta_2 = -\xi_+ \zeta_1 ,$$

$$\delta \zeta_3 = \xi_- \zeta_4 , \quad \delta \zeta_4 = -\xi_- \zeta_3 ,$$

$$\delta \zeta_1' = \xi_- \zeta_2' , \quad \delta \zeta_2' = -\xi_- \zeta_1' ,$$

$$\delta \zeta_3' = \xi_+ \zeta_4' , \quad \delta \zeta_4' = -\xi_+ \zeta_3' .$$

(55)
Both $\tilde{\mathcal{L}}_{\text{kin}}$ and $\tilde{\mathcal{K}}_{\text{int}}$ in eqs. (42), (43) are invariant. The chiral symmetries imply the separate conservation of the numbers of right movers and left movers, which is clearly a property of the cellular automaton.

Discrete symmetries comprise parity (reflection of $x$) and time reversal (reflection of $t$). Charge conjugation corresponds to complex conjugation of $\psi$, or a change of sign of the Grassmann variables $\varphi_{n,t}$. For the cellular automaton this means that the number of green particles can only change in units of two. The exchange of red and green particles in the cellular automaton corresponds to invariance under the discrete symmetry $\psi \rightarrow i\psi$, $\psi \rightarrow -i\psi$.

Ground states

We define a ground state as a time-invariant state. For a ground state configuration at $t$ the configuration at $t + \epsilon$ is the same as for $t$. Because of the modulo two properties of our model for even and odd $t$ we take a coarse grained approach and weaken the ground state condition. Only the configuration at $t + 2\epsilon$ has to be the same as at $t$. This guarantees that the state for all even $t + 2n\epsilon$ is the same as at $t$.

There are several possible ground states according to this definition. First of all, we have the totally empty state where all occupation numbers vanish or all Ising spins take negative values. A single one-particle excitation in this ground state takes at $t_{\text{in}}$ a single occupation number different from zero. One has $n_\alpha(t_{\text{in}}, x) = 1$ for one particular $x$ and one particular $\alpha$, and keeps all other occupation numbers at zero. The single particle will propagate for increasing $t$ as a right mover or a left mover without scattering, depending on $x$ being even or odd. The ground state is stable in the sense that for later $t$ it remains the same as at $t_{\text{in}}$ for all locations $x$ except for the position of the particle. A second ground state is the totally filled state, with all $n_\alpha(t, x) = 1$, or all Ising spins positive. The single particle excitation is now a hole, corresponding to $n_\alpha(t_{\text{in}}, x) = 0$ for a single $x$ and $\alpha$. The hole propagates in the same way as a particle in the first ground state. Particle-hole symmetry maps the two ground states and the two one-particle states onto each other. These two ground states preserve the chiral symmetry of the action.

Another type of ground states are the half-filled states. They have precisely one particle at each location $x$

$$n_\alpha(t, x) + n_\bar{\alpha}(t, x) = 1. \quad (56)$$

The colors do change, however, from $t$ to $t + \epsilon$. They are again the same at $t + 2\epsilon$. For a first ground state of this type we take at $t_{\text{in}}$ only red particles, $n_R(t_{\text{in}}, x) = 1$, $n_G(t_{\text{in}}, x) = 0$. The evolution is shown in Fig. 3. At $t + \epsilon$ all particles are green, and at $t + 2\epsilon$ red again. A second possible ground state of this type has only green particles at $t_{\text{in}}$. Chiral symmetry transformations change red into green particles. The two half-filled ground states correspond to spontaneous chiral symmetry breaking???.

The symmetry of the ground state is reduced as compared to the symmetry of the action. We call this structure of the half-filled ground state ”type A”.

The condition (56) allows for two more different ground states of ”type B”. At $t_{\text{in}}$ red and green particles may fill the positions in an alternating way. We depict this ground state in Fig. 4. The other ground state of this type interchanges the red and green particles. As compared to the type A in Fig. 3, the ground state of type B in Fig. 4 obtains by interchanging $x$ and $t$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3}
\caption{Half filled ground state (A). Green and red lines link sites occupied by a single particle of the corresponding color.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig4}
\caption{Half filled ground state (B). It can be obtained from Fig. 3 by a $\pi/2$ - rotation.}
\end{figure}

Solitons

All the different ground states break some of the discrete symmetries of the action spontaneously. We may therefore have solutions where different regions in spacetime are characterized by different ”vacua” or ground states. At the boundaries of these regions there will be ”domain walls” or ”kinks” or ”solitons”. There exist various solitons of this type, as determined by different vacua in the different regions. We only briefly discuss one type here.

Let us add a particle to the half filled vacuum of type A. If the initial state of the vacuum at $t_{\text{in}}$ is green, with all sites occupied by a single green particle, the additional particle can only be red. At the position of the red particle we therefore have a two particle state which propagates freely to the left or to the right, according to its position.
We have depicted this two particle state in Fig. 5 by a black line. The two particle state disturbs the vacuum, as visible in Fig. 5. While to the right of the black line one finds the vacuum A, we observe to its left the vacuum B. The vacuum B is found within the whole forward light cone of the initial position of the two-particle state. Its left boundary is indicated in Fig. 5 by a dashed blue line. (The dashed blue line is not a particle line). To the left of the dashed blue line one finds again vacuum A. The same picture obtains if instead of adding a red particle at the initial position where the black and blue dashed line meet, we take a green particle away. This is a hole excitation of the half filled vacuum A. The black line is now a line of empty positions.

![Fig. 5. Solitons. The black line is either doubly occupied by a red and a green particle, or empty with no particle. It separates at region of vacuum (A) to its right from a region of vacuum (B) on its left. The other boundary of the vacuum (B) is the blue dashed line. To the left of the blue dashed line one finds again vacuum (A). The vacuum (B) forms within the light cone of the initial position of the black two-particle or zero-particle line.](image)

The black and dashed blue lines can be viewed as solitons separating different types of vacuum. They propagate with light velocity, similar to the fermions. They are topologically stable since matching two different types of vacuum needs a defect. Solitonic excitations are the basis of Coleman’s bosonization of the Thirring model [20]. One can continue Fig. 5 to the past. The crossing of the black and blue dashed line appears as “soliton-soliton scattering”.

**Probabilistic initial conditions and wave function**

Arbitrary probabilistic initial conditions for the cellular automata are given by the wave function \( q(t_{in}) = \{ q_r(t_{in}) \} \). The evolution of the wave function for larger \( t > t_{in} \) obeys eq. (2), and for all \( t \) the probabilities for finding a configuration of local occupation numbers \( \tau \) is given by eq. (5). The expectation values of observables constructed from occupation numbers at \( t \) can be obtained from the local probabilities \( \{ p_r(t) \} = \{ q_r^2(t) \} \) in the standard way. This extends to time ordered products of local observables, for a discussion see ref [4].

The same wave function \( q(t) \) also describes initial conditions for the fermionic quantum theory as well as their evolution to later times. The initial wave function \( q(t_{in}) \) can be implemented in the fermionic quantum field theory by adding an appropriate boundary term [9] in the functional integral (22). The evolution of \( q(t) \) is the same as for the cellular automaton, since the step evolution operator \( \hat{S}(t) \) in eq. (2) is the same. Quantum observables constructed from local occupation numbers are represented by diagonal operators \( \hat{A}_{\tau \rho} = A_\tau \delta_{\tau \rho} \), with \( A_\tau \) the value of the observable in the state \( \tau \) [4]. Indeed, if \( A \) is a function of the occupation numbers \( n_\tau \), it obtains by inserting for \( n_\tau \) the value \( \langle n_\tau \rangle_\tau \) for the configuration \( \tau \). The expectation value obeys the quantum rule

\[
\langle A(t) \rangle = \langle q | \hat{A} | q \rangle = q_\tau \hat{A}_{\tau \rho} q_\rho = A_\tau p_\tau .
\]

One can employ all the usual formalism of quantum mechanics, with extension to correlations for time ordered observables and additional observables represented by off-diagonal operators [6].

**Schrödinger equation**

For a large number of sites \( t \) and \( x \), and wave functions depending sufficiently smoothly on \( x \), we can employ the continuum limit. In this limit \( q(t) \) is considered as a differentiable function of time - and space - coordinates, such that lattice derivatives become partial derivatives. We can classify the states according to the conserved particle number. For one-particle states the wave function depends on the position \( x \) of the particle, \( q_1(t) = q(t,x) \). For two-particle states two positions have to be specified, \( q_2(t) = q_2(t,x,y) \), and so on. We define for \( t \) even the time derivative

\[
\partial_t q(t) = \frac{1}{4\varepsilon} (q(t+2\varepsilon) - q(t-2\varepsilon)) = W(t)q(t) ,
\]

where eq. (2) implies for the matrix \( W \)

\[
W(t) = \frac{1}{4\varepsilon} (\hat{S}(t + \varepsilon)\hat{S}(t) - \hat{S}^{-1}(t-2\varepsilon)\hat{S}^{-1}(t-\varepsilon)) .
\]

This definition takes into account that the step evolution operators may differ between even and odd \( t \). For \( \hat{S}(t+2\varepsilon) = \hat{S}(t) \), \( \hat{S}(t+\varepsilon) = \hat{S}(t-\varepsilon) \) and \( \hat{S}(t) = \hat{S}^T(t) \) for all \( t \), the matrix \( W(t) \) is antisymmetric

\[
W(t) = \frac{1}{4\varepsilon} \left\{ \hat{S}(t + \varepsilon)\hat{S}(t) - (\hat{S}(t + \varepsilon)\hat{S}(t))^T \right\} .
\]

Eq. (58) is the Schrödinger equation in a real formulation. (Every complex Schrödinger equation can be written as a real Schrödinger equation with twice the number of components of the wave function). For antisymmetric \( W \) eq. (58) describes a rotation of the wave function that preserves its norm. Cellular automata always produce a ”unitary time evolution” in this sense.

In the presence of a complex structure the Schrödinger equation can be written as a complex differential equation, and \( W \) is transformed to a hermitian Hamilton operator \( H \) [4, 5]. The evolution is unitary. For the example of an one-particle state the real wave function has four components, \( q(t,x) = \{ q_1(x), q_{1\nu}(x), q_2(x), q_{2\nu}(x) \} \), according to the four types of particles - two colors for right movers
and two colors for left movers. Combining them into a complex doublet wave function
\[ \psi(t, x) = \begin{pmatrix} q_{1\rho}(t, x) + i q_{1i}(t, x) \\ q_{2\rho}(t, x) + i q_{2i}(t, x) \end{pmatrix}, \]
the Schrödinger equation involves the momentum operator \( \hat{P} \) according to
\[ i \partial_t \psi = H \psi, \quad H = \hat{P} \tau_3, \quad \hat{P} = -i \partial_x. \]
This is the free motion of massless particles - scattering plays no role for one-particle states. The interaction term will matter for the evolution of two-particle states, or states with more than two particles. We observe that the one-particle state is defined here for the totally empty vacuum. One particle excitations of other vacua will involve a more complex Schrödinger equation.

### 6. Generalized Ising model

One more way to represent cellular automata are generalized Ising models [4]. For generalized Ising models the weight function
\[ w[s] = \exp \{ -S[s] \}, \]
involves an action that depends on Ising spins \( s, n \), \( s^2 = 1 \), or associated occupation numbers \( n(t) = (s(t) + 1)/2 \), not that takes values one or zero. Similarly to eqs (33), (34) for the fermionic model, we assume that \( w[s] \) can be written as a product of local factors, with \( K(t) \) replaced by \( K(t) \). The local factors depend on the occupation numbers \( n(t), n(t + \varepsilon) \) for two neighboring time layers.

We can define a wave function
\[ q(t) = q_\tau(t) h_\tau[n(t)] = q_\tau h_\tau(t), \]
with basis functions \( h_\tau(t) \) that are functions of the local occupation numbers \( n_\tau(t) \). These basis functions are constructed in close analogy to the Grassmann basis functions as a product of factors \( a_\gamma \) for each \( \gamma \). For a particle at \( \gamma \) \( (n_\gamma = 1) \) one has \( a_\gamma = n_\gamma \), while in the absence of a particle \( (n_\gamma = 0) \) the factor is \( (1 - n_\gamma) \). In comparison to the fermionic model, \( \psi \) corresponds to \( 1 - n_\gamma \), and a factor 1 at the place \( \gamma \) corresponds to \( n_\gamma \). There is no issue of signs since the integers \( n_\gamma \) commute, and we choose the overall sign positive, e.g. \( h_\tau \geq 0 \).

For the local factors we use a double expansion,
\[ K(t) = h_\tau(t + \varepsilon) \tilde{S}_{\tau \rho}(t) h_\rho(t). \]
"Integrating" the occupation numbers at \( t \) one has [4]
\[ \int Dn(t) K(t) q(t) = q_\tau(t + \varepsilon) h_\tau(t + \varepsilon), \]
with \( \int Dn(t) = \prod_\gamma \sum_{n_\gamma(t) = 0, 1} \), and
\[ q_\tau(t + \varepsilon) = \tilde{S}_{\tau \rho}(t) q_\rho(t). \]
One also finds the product rule
\[ \int Dn(t + \varepsilon) K(t + \varepsilon) K(t) = h_\tau(t + 2\varepsilon) \tilde{S}_{\tau \rho}(t + \varepsilon) \tilde{S}_{\rho \sigma}(t) h_\sigma(t), \]
such that the integrated multiplication of local factors results in matrix multiplication of the step evolution operator.

For an orthogonal step evolution operator \( \tilde{S}_{\tau \rho}(t) \) identical to a given fermionic model or cellular automaton, the evolution of the wave function in the generalized Ising model is the same as for the other two formulations. Also the meaning of the wave function is the same, with local probabilities \( p_\tau(t) = q_\tau^2(t) \). For positive matrix elements \( \tilde{S}_{\tau \rho} \geq 0 \) we can write \( K(t) \) in an exponential form
\[ K(t) = \exp \{ -L(t) \} = \exp \{ -h_\tau(t + \varepsilon)L_{\tau \rho}(t) h_\rho(t) \}. \]
The properties of the basis function imply [4] for all elements of \( \tilde{S} \) the relation
\[ \tilde{S}_{\tau \rho}(t) = \exp \{ -L_{\tau \rho}(t) \}. \]
The weight function can therefore be written in the standard form of a generalized Ising model
\[ w = \exp(-S) = \exp \{ -\sum L(t) \}. \]
According to eq. (69), \( L(t) \) contains interactions between occupation numbers or Ising spins on two neighboring layers.

The generalized Ising models are constrained Ising models in the sense that many combinations of neighboring configurations \( \{n_\gamma(t)\} \) and \( \{n_\gamma(t + \varepsilon)\} \) are forbidden. This concerns the zero elements in the step evolution operator \( \tilde{S}_{\tau \rho} \). For \( \tilde{S}_{\tau \rho} = 0 \) one takes \( L_{\tau \rho} \to \infty \), such that the probability of a configuration \( \tau \) at \( t + \varepsilon \) coexisting with a configuration \( \rho \) at \( t \) vanishes. For the allowed sequences of the cellular automaton one has \( \tilde{S}_{\tau \rho} = 1 \), realized by \( L_{\tau \rho} = 0 \). In the absence of interactions the generalized Ising model for free Dirac spins [9] is given by
\[ L_{\text{free}}(t) = \beta \sum_\alpha \left\{ \begin{array}{c} 4 - n_{1\alpha}(t + \varepsilon, x + \varepsilon)n_{1\alpha}(t, x) \\
+ n_{2\alpha}(t + \varepsilon, x - \varepsilon)n_{2\alpha}(t, x) \\
+ (1 - n_{1\alpha}(t + \varepsilon, x + \varepsilon))(1 - n_{1\alpha}(t, x)) \\
+ (1 - n_{2\alpha}(t + \varepsilon, x - \varepsilon))(1 - n_{2\alpha}(t, x)) \end{array} \right\} \]
(72)
taking the limit \( \beta \to \infty \). Only for the allowed neighboring spin configurations \( L(t) \) equals zero, otherwise it diverges.

### Generalized Ising model for the Thirring type model

For the interaction in our model one needs to add an interaction term \( L_{\text{int}}(t) \) that vanishes for the configurations of allowed color changes, and diverges for the propagation on light cones of states with single neighboring
right and left movers. The latter would be allowed for free Dirac fermions, but is forbidden for our particular Thirring model. For the construction of $\mathcal{L}_{\text{int}}(t)$ we use projectors on single particle states

$$P_k = \frac{1}{2} (1 - s_{kr}s_{kl}) = n_{kr}(1 - n_{kl}) + n_{kl}(1 - n_{kr}).$$  

(73)

This projector only equals one if the Ising spins $s_{k,a}$ of two different colors $R$ and $I$ are opposite, which happens if either a red or a green particle is present. It vanishes if no particle or two particles of type $k$ are present. We write

$$\mathcal{L}_{\text{int}} = 8\beta P' P'' L^{(2)} P_1 P_2,$$

(74)

where $P$ refers to right movers or left movers at $t$, and $P'$ correspondingly at $t + \varepsilon$. For $L^{(2)}$ we take

$$L^{(2)} = (n_{1R}^{(t)} n_{2R}^{(t)} - n_{1I}^{(t)} n_{2I}^{(t)})(n_{1R} n_{2I} - n_{1I} n_{2R}) + (n_{1R}^{(t)} n_{2R}^{(t)} - n_{1I}^{(t)} n_{2I}^{(t)})(n_{1R} n_{2R} - n_{1I} n_{1I}),$$  

(75)

where we denote occupation numbers at $t + \varepsilon$ by $n'_{k,a}$. In the presence of the projectors only one out of the four combinations $n_{1R} n_{2R}$, $n_{1I} n_{2I}$, $n_{1R} n_{1I}$, $n_{1I} n_{2I}$ can be equal to one at $t$, and similarly at $t + \varepsilon$. If at $t + \varepsilon$ colors are exchanged, $L^{(2)}$ equals $-8\beta$. On the other hand, if the colors at $t + \varepsilon$ are the same as at $t$ the value $L^{(2)} = 8\beta$ suppresses now these configuration pairs that were allowed by $\mathcal{L}_{\text{free}}$. For the color exchange configurations $\mathcal{L}_{\text{free}}$ takes the value $8\beta$. This is cancelled by $\mathcal{L}_{\text{int}}$ such that these configuration pairs are now allowed. The projections in $\mathcal{L}_{\text{int}}$ replace effectively in $L^{(2)}$ each factor $n_{IR}$ by $n_{IR}(1 - n_{II})$ etc., such that for $n_{IR} = 1$ only $n_{II} = 0$ contributes.

Combining $\mathcal{L}(t) = \mathcal{L}_{\text{free}}(t) + \mathcal{L}_{\text{int}}(t)$ defines the generalized Ising model (71). The action involves Ising spins at two neighboring $t$ and two neighboring $x$. It can again be written as a sum over blocks. For even $t$ the blocks involve the four sites $(t, x)$, $(t, x + \varepsilon)$, $(t + \varepsilon, x)$, $(t + \varepsilon, t + \varepsilon)$ with $s_{1\alpha} = s_{0}(t, x)$, $s_{2\alpha} = s_{0}(t + \varepsilon, x)$, $s_{1'\alpha} = s_{0}(t + \varepsilon, x + \varepsilon)$, $s_{2'\alpha} = s_{0}(t + \varepsilon, t + \varepsilon)$, and similar for the occupation numbers. For odd $t + \varepsilon$ the blocks comprise the four sites $(t + \varepsilon, x)$, $(t + \varepsilon, x - \varepsilon)$, $(t + 2\varepsilon, x)$, $(t + 2\varepsilon, x - \varepsilon)$, with the identifications $s_{1\alpha} = s_{0}(t + \varepsilon, x - \varepsilon)$, $s_{2\alpha} = s_{0}(t + \varepsilon, x)$, $s_{1'\alpha} = s(t + 2\varepsilon, x)$. We can define a coarse grained lattice with even $t$ and $x$, and combine two blocks at $(t - \varepsilon)$ and $t$ which comprise the seven sites $(t - \varepsilon, x - \varepsilon)$, $(t - \varepsilon, x)$, $(t - \varepsilon, t)$, $(t, x)$, $(t, x + \varepsilon)$, $(t + \varepsilon, x)$, $(t + \varepsilon, x + \varepsilon)$. On this level a distinction between even and odd $t$-layers is no longer needed. Two neighboring blocks share precisely one common site. For the example of blocks at $(t, x)$ and $(t, x + 2\varepsilon)$ the spins $s_{0}(t, x + \varepsilon)$ appear in both blocks. The equivalence between the ”functional integral” for Ising spins (71) and the Grassmann functional integral (10), (41) is a first example of a fermion-bit map for a model with interactions.

The generalized Ising model is a well defined euclidean statistical model. The factor $e^{-S}$ defines a probability distribution. Many powerful methods of statistical mechanics and condensed matter physics can be applied in this setting, as block spinning or (functional) renormalization. In particular, one can perform Monte Carlo simulations, for example for large finite $\beta$ and taking the limit $\beta \to \infty$.

7. Discussion

We have proposed a simple two-dimensional model for which three equivalent formulations exist:

1. a probabilistic cellular automaton,
2. a Grassmann functional for fermions,
3. a generalized Ising model.

This allows for a simultaneous use of methods from all three fields. We emphasize that the generalized Ising model is a standard euclidean functional integral formulated on a square lattice. On the other hand, the Grassmann functional exhibits Lorentz symmetry in the continuum limit, with its characteristic Minkowski signature differentiating between time and space. Also the cellular automaton singles out time as the direction in which the steps of the automaton are performed. Our example shows that there is no contradiction between these concepts.

Our model of interacting fermions is an example how quantum mechanics emerges from classical statistics [22, 23]. A quantum field theory is quantum mechanics for many particles. The quantum mechanics of a single particle is a special case. The evolution of the one-particle wave function obeys the Schrödinger equation. All features of quantum mechanics as non-commuting operators and the quantum laws for the computation of expectation values emerge in a natural way [4, 5, 8]. There are no contradictions to no-go theorems [22, 23]. Even though our model is simple, it exhibits non-trivial features of interacting quantum field theories as spontaneous symmetry breaking or solitons. The one-particle Schrödinger equation depends on the ground state.

Our construction of a cellular automaton may appear somewhat special. For the same cellular automaton one can use a different construction of alternating step evolution operators for the propagation and the interaction, that will be published elsewhere. We believe that a wide range of fermionic quantum field theories can be constructed as cellular automata in this way.
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