Supersymmetry and shape invariance of the effective screened potential

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Abstract. This paper discusses the supersymmetry and shape invariance (SI) of the effective screened potential. It is shown that the effective screened potential (ESP) has SI and belongs to the first class of shape-invariant potentials, thence the energy levels of this potential are obtained. Furthermore, by using the method of point canonical transformation we find that the ESP belongs to the same subclass as Pöschl–Teller potential ¹; the bound-state spectra and the eigenfunctions of this potential are obtained. The results obtained can readily be applied to the special case of the ESP, the famous Hulthén potential.

1. Introduction

Many exactly solvable potentials are exponential functions of the spatial coordinate. These exponential potentials are widely used in many branches of physics. The effective screened potential (ESP) [1] belongs to this group of exactly solvable exponential potentials. Though the solutions of the ESP have already been given by some authors [1], its principal features, such as supersymmetry (SUSY) and shape invariance (SI) [2, 3], have not been systematically discussed in the recent literature. Since the ESP is of considerable importance in nuclear physics and elementary particle physics, the famous Hulthén potential is just a special case of the ESP and also it is closely related to the famous Yukawa potential, we feel that it is necessary to systematically discuss the SUSY and SI of the ESP, such that one can easily grasp the principal features of this very important potential. First we shall discuss the SUSY and SI of the ESP from the Hamiltonian formulation of supersymmetric quantum mechanics (SUSYQM) [4, 5].
is well known that SUSY relates bosonic and fermionic degrees of freedom; it is a necessary ingredient in any unifying approach. The algebra involved in SUSY is a graded Lie algebra, which closes under a combination of commutation and anti-commutation relations. SUSYQM is the simplest case. Once people started studying various aspects of SUSYQM, it was soon clear that this field was interesting in its own right, not just as a model for testing field theory methods. Gradually a whole technology was evolved based on SUSY to understand the solvable potential problem. Gendenstein [6] in 1983 pointed out that all analytically solvable potentials in quantum mechanics have the property of SI. Recently it has been shown that almost all of the second-order differential equations in mathematical physics have the convenient property of SUSY and SI [7].

Nine years ago it was found that there are two classes of shape-invariant potentials (SIPs) [8, 9]: for the first class of SIPs (SIP1), the parameters \( a_1 \) and \( a_2 \) of the two supersymmetric partners are related to each other by translation \( a_2 = a_1 + \alpha \); for the second class of SIPs (SIP2), the parameters \( a_1 \) and \( a_2 \) of the two supersymmetric partners are related to each other by scaling \( a_2 = qa_1 \). For SIP1 and SIP2, we have shown that these two classes are interrelated with each other [10]; in general SIP2 with \( 0 < q < 1 \) can be regarded as the multi-parameter deformation of SIP1 with \( q \) acting as the deformation parameter. In section 2 of this article we shall show that the ESP has SI and belongs to SIP1, and thence we obtain energy levels of the ESP. At present we know that all SIPs in SIP1 can be grouped into two subclasses in the sense that the potentials in any subclass can be mapped to a single potential of that class through point canonical transformation (PCT) [11]. In order to find out which subclass ESP belongs to, in section 3 we shall use the method of PCT to map ESP into the Pöschl–Teller-I potential (PT-I), thus we show that the ESP belongs to the same subclass of SIP1 as PT-I and its eigenfunctions correspond to hypergeometric functions; meanwhile we shall use this mapping to find the bound-state spectra and eigenfunctions of the ESP. In principle we can obtain completely identical bound-state eigenfunctions from the well known method suggested by SUSYQM [2, 3].

2. Shape invariance of the ESP

The ESP is of the form

\[
V(r) = -\frac{\lambda - \mu}{e^{r/a} - 1} + \frac{\mu}{(e^{r/a} - 1)^2}, \quad \mu = \frac{\hbar^2 l(l+1)}{2ma^2}.
\]

The famous Hulthén potential is the particular case when \( \mu = 0 \). It should be pointed out that in SUSYQM we only consider the bound-state spectra and the corresponding bound state wavefunctions. The configuration space for the ESP is obviously the positive half-line \( r > 0 \). Since we only consider the bound-state problem, in that case the bound-state energy \( E < 0 \). In order to find the physically acceptable solutions for the bound-state problem, we have imposed a boundary condition concerning the energy eigenfunctions: they tend to zero when \( r \to \infty \). As for the boundary condition at \( r = 0 \), it has already been discussed in detail in the book Quantum Mechanics written by E Merzbacher for all central force problems, i.e. the wavefunction tends to zero when \( r \to 0 \). When we introduce the quantities

\[
E_0 = -\frac{\hbar^2}{2ma^2} \left[ \frac{b^2 - (l+1)^2}{2(l+1)} \right], \quad b^2 = \frac{2ma^2\lambda}{\hbar^2},
\]

we can define a superpotential \( W(r) \) of the ESP:

\[
W(r) = \sqrt{-E_0 - \frac{\hbar^2}{a\sqrt{2m}e^{r/a} - 1}}.
\]
From (3) we obtain the supersymmetric partner potentials:

\[
V_+ = W^2 + \frac{\hbar}{\sqrt{2m}} W' = -E_0 - \frac{\lambda}{e^{r/a} - 1} + \frac{\hbar^2 l(l+1)}{2ma^2} \left[ \frac{1}{e^{r/a} - 1} + \frac{1}{(e^{r/a} - 1)^2} \right] 
\]

(4)

\[
V_- = W^2 + \frac{\hbar}{\sqrt{2m}} W' = -E_0 - \frac{\lambda}{e^{r/a} - 1} + \frac{\hbar^2 l(l+1)(l+2)}{2ma^2} \left[ \frac{1}{e^{r/a} - 1} + \frac{1}{(e^{r/a} - 1)^2} \right].
\]

(5)

Obviously

\[
V(r) = V_+ + E_0.
\]

If we take \(a_0 = l, a_1 = l + 1\), then from (4) and (5) we readily see

\[
V_+(r, a_0) = V_-(r, a_1) + R(a_1)
\]

where

\[
R(a_1) = -E_0 + E_1, \quad E_1 = -\frac{\hbar^2}{2ma^2} \left[ \frac{b^2 - (l+2)^2}{2(l+2)} \right].
\]

(8)

Equation (7) is just the mathematical expression of SI, hence the ESP has the required property of SI; furthermore, from the relation \(a_1 = l + 1 = a_0 + 1\), we know that the ESP belongs to SIP1. The series of Hamiltonians \(H^{(s)}(s = 0, 1, 2, \ldots)\) is defined as

\[
H^{(s)} = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V_-(r, a_s) + \sum_{k=1}^{s} R(a_k)
\]

(9)

where \(H^{(0)} \equiv H_-, H^{(1)} = H_+\).

\[
a_s = f^s(a_0) = a_0 + s = l + s
\]

(10)

hence

\[
V_-(r, a_s) = -E_s - \frac{\lambda}{e^{r/a} - 1} + \frac{\hbar^2 (l+s)(l+s+1)}{2ma^2} \left[ \frac{1}{e^{r/a} - 1} + \frac{1}{(e^{r/a} - 1)^2} \right].
\]

(11)

In view of (7) and (9) we have

\[
H^{(s+1)} = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V_-(r; a_{s+1}) + \sum_{k=1}^{s+1} R(a_k) = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V_+(r, a_s) + \sum_{k=1}^{s} R(a_k).
\]

(12)

Hence the complete energy spectrum of \(H_+\) is given by

\[
E_n^{(-)} = \sum_{k=1}^{n} R(a_k) = E_n - E_0, \quad (n \geq 1), \quad E_0^{(-)} = 0
\]

(13)

hence we obtain the energy levels of the ESP:

\[
E_n^{(-)} + E_0 = E_n = -\frac{\hbar^2}{2ma^2} \left[ \frac{b^2 - (n+l+1)^2}{2(n+l+1)} \right]^2 = -\frac{\hbar^2 p_{n+1}^2}{2ma^2}, \quad n = 0, 1, 2, \ldots
\]

\[
p_{n+1} = \frac{b^2 - (n+l+1)^2}{2(n+l+1)}.
\]

(14)

From equation (10), we already know the ESP belongs to class ESP1; in order to reveal the relationship between the ESP and other SIPs in class SIP1, in the next section we shall introduce the method of mapping of SIPs under the PCT. Incidentally, we can easily obtain the corresponding energy eigenfunctions, hence we can also use the standard method given by SUSYQM to obtain the bound-state wavefunctions [2, 3], but we think there is no need to perform such exercises in this paper; this method has already been systematically discussed in [3].
3. Mapping of the ESP into Pöschl–Teller potential 1 (PT-1)

First we briefly review the method of mapping of an SIP under PCT. For a given potential \( V(\alpha_i; x) \) the Schrödinger equation is

\[
\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_-(\alpha_i; x) - E(\alpha_i) \right] \psi(\alpha_i; x) = 0
\]  

(15)

where \([\alpha_i]\) represents set of parameters of the given potential. We invoke a transformation of both the independent and dependent variables of the form

\[ x = f(z), \quad \psi(\alpha_i; x) = v(z)\tilde{\psi}(\tilde{\alpha}_i; z) \]  

(16)

then equation (15) becomes

\[
-\frac{\hbar^2}{2m} \frac{d^2\tilde{\psi}}{dz^2} - \frac{\hbar^2}{m} \frac{d\tilde{\psi}}{dz} \left( \frac{v'}{v} - \frac{f''}{2f'} \right) + \left[ f'^2 \{ V(\alpha_i; f(z)) - E(\alpha_i) \} + \frac{\hbar^2}{2m} \left( \frac{f''}{f'} \right)^2 - \frac{f''}{f'} \right] \tilde{\psi} = 0.
\]  

(17)

We require (17) in the form

\[
\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + \tilde{V}(\tilde{\alpha}_i; z) - \tilde{E}(\tilde{\alpha}_i) \right] \tilde{\psi}(\tilde{\alpha}_i; z) = 0
\]  

(18)

for which \( \tilde{\psi}_n(\tilde{\alpha}_i; z) \) and \( \tilde{E}_n(\tilde{\alpha}_i) \) are known for the SIP \( \tilde{V}(\tilde{\alpha}_i; z) \) for each state labelled by the quantum number \( n = 0, 1, 2, \ldots \). Here \([\tilde{\alpha}_i]\) represents set of parameters of the transformed potential. To remove the first-derivative term from (17) one requires

\[ v(z) = C\sqrt{f'(z)}. \]  

(19)

Using (19) and comparing (17) and (18) we obtain

\[
\tilde{V}(\tilde{\alpha}_i; z) - \tilde{E}(\tilde{\alpha}_i) = f'^2 \{ V(\alpha_i; f(z)) - E(\alpha_i) \} + \frac{\hbar^2}{4m} \left( \frac{3}{2} \left( \frac{f''}{f'} \right)^2 - \frac{f''}{f'} \right). \]  

(20)

By using the known values of \( \tilde{E}(\tilde{\alpha}_i) \) and known functions \( \tilde{\psi}_n(\tilde{\alpha}_i; z) \), from (16), (19) and (20) we can easily find the bound-state spectra \( E_n(\alpha_i) \) and eigenfunctions \( \psi_n(\alpha_i; x) \) of the given potential \( V(\alpha_i; x) \).

Now we turn to map the ESP into PT-1. Using PCT

\[ r = f(z) = -2a \ln[\cos(\alpha z)] \]  

(21)

equation (1) becomes

\[ V(r) = -\left( \lambda - \mu \right) \cot^2(\alpha z) + \mu \cot^4(\alpha z) = -\lambda \cot^2(\alpha z) + \mu \csc^2(\alpha z) \cot^2(\alpha z). \]  

(22)

Substituting (21) and (22) into (20) we obtain

\[
\tilde{V} - \tilde{E} = -4a^2 \alpha^2 (\lambda - E) - \left( 4a^2 \alpha^2 E + \frac{\alpha^2 \hbar^2}{8m} \right) \sec^2(\alpha z) + \left( 4a^2 \alpha^2 \mu + \frac{3a^2 \hbar^2}{8m} \right) \csc^2(\alpha z).
\]  

(23)

For PT-1 we have

\[
\tilde{V} = -(A + B)^2 + A \left( A - \frac{\alpha \hbar}{\sqrt{2m}} \right) \sec^2(\alpha z) + B \left( B - \frac{\alpha \hbar}{\sqrt{2m}} \right) \csc^2(\alpha z)
\]  

(24)
\[ \hat{E} = \left( A + B + \frac{2n\alpha\hbar}{\sqrt{2m}} \right)^2 - (A + B)^2 \quad (25) \]

\[ \check{\psi} = (1 - y)^{\lambda/2} (1 + y)^{s/2} P_n^{(\lambda-1/2, s-1/2)}(y), \]

\[ (y = 1 - 2\sin^2(\alpha z), \quad s = \frac{\sqrt{2mA}}{\alpha\hbar}, \quad \lambda = \frac{\sqrt{2mB}}{\alpha\hbar}) \quad (26) \]

where \( P_n^{(\alpha, \beta)}(x) \) is the Jacobi polynomial. Substituting (24) and (25) into the LHS of (23) and comparing the coefficients of the corresponding terms of the two sides, we obtain

\[ A \left( A - \frac{\alpha\hbar}{\sqrt{2m}} \right) = - \left( 4a^2\alpha^2 E + \frac{\alpha^2\hbar^2}{8m} \right) \quad (27) \]

\[ B \left( B - \frac{\alpha\hbar}{\sqrt{2m}} \right) = 4a^2\alpha^2 \mu + \frac{3\alpha^2\hbar^2}{8m} \quad (28) \]

\[ (A + B + \frac{2n\alpha\hbar}{\sqrt{2m}})^2 = 4a^2\alpha^2(\lambda - E). \quad (29) \]

Solving for (27) and (28) we obtain

\[ A = 2a\alpha \sqrt{-E} + \frac{\alpha\hbar}{2\sqrt{2m}}, \quad B = \frac{3\alpha\hbar}{2\sqrt{2m}} + \frac{2\lambda\hbar}{\sqrt{2m}}. \quad (30) \]

Substituting (30) into (29) we obtain once again equation (14). Now we start to find the eigenfunctions of the ESP from (16) and (26).

\[ y = 1 - 2\sin^2(\alpha z) = 2e^{-r/a} - 1, \quad s = \frac{\sqrt{2mA}}{\alpha\hbar} = \frac{1}{2} + 2p_{n+1}, \quad \lambda = \frac{\sqrt{2mB}}{\alpha\hbar} = \frac{3}{2} + 2l \quad (31) \]

hence from (26) we have

\[ \check{\psi} = [2(1 - e^{-r/a})]^{(3/4+l)}[2e^{-r/a}]^{(p_{n+1}+1/4)} P_n^{(1+2l, 2p_{n+1})}(2e^{-r/a} - 1). \quad (32) \]

From (21) and (31) we obtain

\[ f' = 2a\alpha \tan(\alpha z) = 2aa \left( \frac{1 - y}{1 + y} \right)^{1/2} = 2aa \left( \frac{1 - e^{-r/a}}{e^{-r/a}} \right)^{1/2}. \quad (33) \]

From (16), (19), (32) and (33) we obtain

\[ \psi = e^{\sqrt{f't}\check{\psi}} = \text{const}(1 - e^{-r/a})^{(l+1)}e^{-p_{n+1}r/a} P_n^{(1+2l, 2p_{n+1})}(2e^{-r/a} - 1). \quad (34) \]

Using the formula

\[ P_n^{(\alpha, \beta)}(x) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(n + 1)} F \left( -n, \alpha + 1 + \beta + n, \alpha + 1; \frac{1 - x}{2} \right) \quad (35) \]

\[ F(\alpha, \beta, \gamma; z) = F(\beta, \alpha, \gamma; z) \quad (36) \]

from (34) we obtain

\[ \psi = C_n(e^{-r/a})^{p_{n+1}}(1 - e^{-r/a})^{(l+1)} F(-n, 2l + 2 + 2p_{n+1} + n, 2l + 2; 1 - e^{-r/a}) \quad (37) \]

which coincides completely with (27) of [1].
For the famous Hulthén potential, which is just the special case of the ESP when \( l = 0 \), i.e. \( \mu = 0 \),

\[
V(r) = -V_0 \frac{e^{-r/a}}{1 - e^{-r/a}}.
\]

Hence we obtain

\[
E_n = -\frac{\hbar^2}{2ma^2} \left[ \frac{b^2 - (n + 1)^2}{2(n+1)} \right]^2 = -\frac{\hbar^2 p_{n+1}^2}{2ma^2}, \quad n = 0, 1, 2, \tag{39}
\]

\[
\psi = C_n (e^{-r/a})^{p_{n+1}} (1 - e^{-r/a}) F(-n, 2 + 2p_{n+1} + n, 2; 1 - e^{-r/a}) \tag{40}
\]

which coincides completely with the results of [12].

Hence the ESP belongs to the same subclass of SIP1 as PT-I; this class contains some very famous SIPs, such as the Rosen–Morse potential, the Eckart potential, the Scarf potential and the Pöschl–Teller potential. Their eigenfunctions all correspond to hypergeometric functions and can be mapped to each other by PCT.

### 4. Conclusion

The ESP is of considerable importance in nuclear physics and elementary particle physics. In this paper we have discussed the SUSY and SI of the ESP; we have found the energy levels and energy eigenfunctions of this potential. First we showed that the ESP has SI and belongs to SIP1, thence the energy levels of this potential are obtained from SUSYQM; of course we can also find the energy eigenfunctions by this method. Furthermore, by using the method of PCT, we reveal the relationship between the ESP and other SIPs in class SIP1: we find that the ESP belongs to the same subclass of SIP1 as PT-I; meanwhile we find the energy eigenfunctions of this potential. The results obtained can readily be applied to the special case of the ESP, the famous Hulthén potential.

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