Liveness of Heap Data for Functional Programs

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Abstract

Functional programming languages use garbage collection for heap memory management. Ideally, garbage collectors should reclaim all objects that are dead at the time of garbage collection. An object is dead at an execution instant if it is not used in future. Garbage collectors collect only those dead objects that are not reachable from any program variable. This is because they are not able to distinguish between reachable objects that are dead and reachable objects that are live. In this paper, we describe a static analysis to discover reachable dead objects in programs written in first-order, eager functional programming languages. The results of this technique can be used to make reachable dead objects unreachable, thereby allowing garbage collectors to reclaim more dead objects.

Keywords: Compilers, Liveness, Garbage Collection, Memory Management, Data Flow Analysis, Context Free Grammars

1 Introduction

Garbage collection is an attractive alternative to manual memory management because it frees the programmer from the responsibility of keeping track of object lifetimes. This makes programs easier to implement, understand and maintain. Ideally, garbage collectors should reclaim all objects that are dead at the time of garbage collection. An object is dead at an execution instant if it is not used in future. Since garbage collectors are not able to distinguish between reachable objects that are live and reachable objects that are dead, they conservatively approximate the liveness of an object by its reachability from a set of locations called root set (stack locations and registers containing program variables) [14]. As a consequence, many dead objects are left uncollected. This has been confirmed by empirical studies for Haskell [19], Scheme [16] and Java [22,23,24].

Compile time analysis can help in distinguishing reachable objects that are live from reachable objects that are dead. This is done by detecting unused references to objects. If an object is dead at a program point, none of its references are used by the program beyond that program point. If every unused reference is nullified, then the dead objects may become unreachable and may be claimed by garbage collector.

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Example 1.1 Figure 1(a) shows an example program. The label \( \pi \) of an expression \( e \) denotes the program point just before the evaluation of \( e \). At a given program point, the heap memory can be viewed as a (possibly unconnected) directed acyclic graph called memory graph. The locations in the root set form the entry nodes for the memory graph. Figure 1(b) shows the memory graph at \( \pi_{14} \). Each \( \text{cons} \) cell is an intermediate node in the graph. Elements of basic data types and the 0-ary constructor \( \text{Nil} \) form leaf nodes of the graph. They are assumed to be boxed, i.e. stored in separate heap cells and are accessed through references. The edges in the graph are called links.

If we consider the execution of the program starting from \( \pi_{14} \), the links in the memory graph that are traversed are shown by thick arrows. These links are live at \( \pi_{14} \). Links that are not live can be nullified by the compiler by inserting suitable statements. If an object becomes unreachable due to nullification, it can be collected by the garbage collector.

In the figure, the links that can be nullified are shown with a \( \times \). Note that a link need not be nullified if nullifying some other link makes it unreachable from the root set. If a node becomes unreachable from the root set as a consequence of nullifying the links, it will be collected during the next invocation of garbage collector.

In this example, starting at \( \pi_{14} \), there is only one execution path. In general, there could be multiple execution paths starting from a program point \( \pi \). The liveness information at \( \pi \) is a combination of liveness information along every execution path starting at \( \pi \).

In this paper, we describe a static analysis for programs written in first-order, eager functional programming languages. The analysis discovers live references at every program point, i.e. the references that may be used beyond the program point in any execution of the program. We use context free grammars as a bounded representation for the set of live references. The result of the analysis can be used by the compiler to decide whether a given reference can be nullified at a given program point. Our analysis is context-sensitive yet modular in that a function is analyzed only once.

The rest of the paper is organized as follows: Section 2 describes the language used to explain our analysis along with the basic concepts and notations. The analysis in Section 3 captures the liveness information of a program as a set of equations. The method to solve
these equations is given in Section 4. Section 5 describes how the result of the analysis can be used to nullify unused references. Finally, we compare our approach with related work in Section 6 and conclude in Section 7.

2 Language, Concepts and Notations

The syntax of our language is described in Figure 2. The language has call-by-value semantics. The argument expressions are evaluated from left to right. We assume that variables in the program are renamed so that the same name is not defined in two different scopes.

For notational convenience, the left link (corresponding to the \texttt{car}) of a \texttt{cons} cell is denoted by 0 and the right link (corresponding to the \texttt{cdr}) is denoted by 1. We use \texttt{e.0} to denote the link corresponding to \texttt{(car e)} for an expression \texttt{e} (assuming \texttt{e} evaluates to a list) and \texttt{e.1} to denote the link corresponding to \texttt{(cdr e)}. A composition of several \texttt{car}s and \texttt{cdr}s is represented by a string \(\alpha \in \{0, 1\}^*\). If an expression \texttt{e} evaluates to a \texttt{cons} cell then \texttt{e.0} corresponds to the reference to the \texttt{cons} cell.

For an expression \texttt{e}, let \([e]\) denote the location in the root set holding the value of \texttt{e}. Given a memory graph, the string \texttt{e.\alpha} describes a path in the memory graph that starts at \([e]\). We call the string \texttt{e.\alpha} an access expression, the string \texttt{\alpha} an access pattern, and the path traced in the memory graph an access path. In Figure 1, the access expression \texttt{w.100} represents the access path from \texttt{w} to the node containing the value 4. Most often, the memory graph being referred to is clear from the context, and therefore we shall use access expressions to refer to access paths. When we use an access path to refer to a link in the memory graph, it denotes the last link in the access path. Thus, \texttt{w.100} denotes the link incident on the node containing the value 4. If \(\sigma\) denotes a set of access patterns, then \texttt{e.\sigma} is the set of access paths rooted at \([e]\) and corresponding to \(\sigma\), i.e.

\[
e.\sigma = \{e.\alpha \mid \alpha \in \sigma\}
\]

A link in a memory graph is live at a program point \(\pi\) if it is used in some path from \(\pi\)
to the program exit. An access path is defined to be live if its last link is live. In Example 1, the set of live access paths at \( \pi_{14} \) is \( \{ w.\varepsilon, w.1, w.10, w.100, z.0, z.00 \} \). Note that the access paths \( z.0 \) and \( z.00 \) are live at \( \pi_{14} \) due to sharing. We do not discover the liveness of such access paths directly. Instead, we assume that an optimizer using our analysis will use alias analysis to discover liveness due to sharing.

The end result of our analysis is the annotation of every expression in the program with a set of access paths rooted at program variables. We call this liveness environment, denoted \( L \). This information can be used to insert nullifying statements before expressions.

The symbols \( \bar{0} \) and \( \bar{1} \) extend the access patterns of a structure to describe the access patterns of a larger structure. In some situations, we need to create access patterns of a substructure from the access patterns of a larger structure. For this purpose, we extend our alphabet of access patterns to include symbols \( \bar{0} \) and \( \bar{1} \). The following example motivates the need for these symbols.

**Example 2.1** Consider the expression at program point \( \pi_{1} \) in

\[
\pi_{1} : (\text{let } w \leftarrow \pi_{2} : (\text{cons } x \ y) \text{ in } \pi_{3} : \cdots)
\]

Assuming \( L_{\pi_{1}} = \{ w.\alpha \} \), we would like to find out which reference of the list \( x \) and \( y \) are live at \( \pi_{1} \). Let \( x.\alpha' \) be live at \( \pi_{1} \). Then, the two possible cases are:

- If \( \alpha = 1\beta \) or \( \alpha = \varepsilon \), no link in the structure rooted at \( x \) is used. We use \( \perp \) to denote the access pattern describing such a situation. Thus, \( \alpha' = \perp \).
- If \( \alpha = 0\beta \) then the link represented by \( w.\alpha \) that is \( x \) rooted and live at \( \pi_{1} \) can be represented by \( x.\beta \). Thus, \( \alpha' = \beta \).

This relation between \( \alpha \) and \( \alpha' \) is expressed by \( \alpha' = \bar{0}\alpha \). \( \bar{1} \) can be interpreted similarly.

With the inclusion of \( \bar{0} \), \( \bar{1} \) and \( \perp \) in the alphabet for access patterns, an access pattern does not directly describe a path in the memory graph. Hence we define a Canonical Access Pattern as a string restricted to the alphabet \( \{ 0, 1 \} \). As a special case, \( \perp \) is also considered as a canonical access pattern.

We define rules to reduce access patterns to their canonical forms. For access patterns \( \alpha_{1} \) and \( \alpha_{2} \):

\[
(1) \quad \alpha_{1}\bar{0}\alpha_{2} \rightarrow \begin{cases} \alpha_{1}\alpha'_{2} & \text{if } \alpha_{2} \equiv 0\alpha'_{2} \\ \perp & \text{if } \alpha_{2} \equiv 1\alpha'_{2} \text{ or } \alpha_{2} \equiv \varepsilon \end{cases}
\]

\[
(2) \quad \alpha_{1}\bar{1}\alpha_{2} \rightarrow \begin{cases} \alpha_{1}\alpha'_{2} & \text{if } \alpha_{2} \equiv 1\alpha'_{2} \\ \perp & \text{if } \alpha_{2} \equiv 0\alpha'_{2} \text{ or } \alpha_{2} \equiv \varepsilon \end{cases}
\]

\[
(3) \quad \alpha_{1}\perp\alpha_{2} \rightarrow \perp
\]

\( \alpha \rightarrow^{k} \alpha' \) denotes the reduction of \( \alpha \) to \( \alpha' \) in \( k \) steps, and \( \rightarrow^{*} \) denotes the reflexive and transitive closure of \( \rightarrow \). The concatenation \( (\cdot) \) of a set of access patterns \( \sigma_{1} \) with \( \sigma_{2} \) is defined as a set containing concatenation of each element in \( \sigma_{1} \) with each element in \( \sigma_{2} \), i.e.

\[
\sigma_{1} \cdot \sigma_{2} = \{ \alpha_{1}\alpha_{2} \mid \alpha_{1} \in \sigma_{1}, \alpha_{2} \in \sigma_{2} \}
\]

### 3 Computing Liveness Environments

Let \( \sigma \) be the set of access patterns specifying the liveness of the result of evaluating \( e \). Let \( L \) be the liveness environment after the evaluation of \( e \). Then the liveness environment
before the computation of \( e \) is discovered by propagating \( \sigma \) backwards through the body of \( e \). This is achieved by defining an environment transformer for \( e \), denoted \( XE \).

Since \( e \) may contain applications of primitive operations and user defined functions, we also need transfer functions that propagate \( \sigma \) from the result of the application to the arguments. These functions are denoted by \( XP \) and \( Xf \). While \( XP \) is given directly based on the semantics of the primitive, \( Xf \) is inferred from the body of a function.

3.1 Computing \( XE \)

For an expression \( e \) at program point \( \pi \), \( XE(e, \sigma, L) \) computes liveness environment at \( \pi \) where \( \sigma \) is the set of access patterns specifying the liveness of the result of evaluating \( e \) and \( L \) is the liveness environment after the evaluation of \( e \). Additionally, as a side effect, the program point \( \pi \) is annotated with the value computed. However, we do not show this explicitly to avoid clutter. The computation of \( XE(e, \sigma, L) \) is as follows.

\[
(4) \quad XE(k, \sigma, L) = L
\]
\[
(5) \quad XE(v, \sigma, L) = L \cup v.\sigma
\]
\[
(6) \quad XE\left(\left(\left(P e_1 e_2\right), \sigma, L\right) = \text{let } L' \leftarrow XE\left(e_2, XP^1_{P}\left(\sigma, L\right)\right) \text{ in } XE(e_1, XP^1_{P}\left(\sigma, L'\right)\right)
\]
\[
\text{where } P \text{ is one of } \text{ cons, } +
\]
\[
(7) \quad XE\left(\left(\left(P e_1\right), \sigma, L\right) = XE(e_1, XP^1_{P}\left(\sigma, L\right)
\]
\[
\text{where } P \text{ is one of } \text{ car, cdr, null?, } \text{ pair?}
\]
\[
(8) \quad XE\left(\left(\left(if e_1 e_2 e_3\right), \sigma, L\right) = \text{let } L' \leftarrow XE\left(e_3, \sigma, L\right) \text{ in } \text{let } L'' \leftarrow XE\left(e_2, \sigma, L\right) \text{ in } XE\left(e_1, \{e_1\}, L' \cup L''\right)
\]
\[
(9) \quad XE\left(\left(\left(let v_1 \leftarrow e_1 in e_2\right), \sigma, L\right) = \text{let } L' \leftarrow XE\left(e_2, \sigma, L\right) \text{ in } XE\left(e_1, \sigma', L' - v_1.\sigma\right)
\]
\[
\text{where } \sigma' = \{\alpha \mid v_1.\alpha \in L'\}
\]
\[
(10) \quad XE\left(\left(\left(f e_1 \dots e_n\right), \sigma, L\right) = \text{let } L_1 \leftarrow XE\left(e_n, Xf^n_{f}(\sigma), L\right) \text{ in } \text{let } L_{n-1} \leftarrow XE\left(e_2, Xf^n_{f}(\sigma), L_{n-2}\right) \text{ in } XE\left(e_1, Xf^n_{f}(\sigma), L_{n-1}\right)
\]

We explain the definition of \( XE \) for the \( if \) expression. Since the value of the conditional expression \( e_1 \) is boolean and this value is used, the liveness access pattern with respect to which \( e_1 \) is computed is \( \{e\} \). Further, since it is not possible to statically determine whether \( e_2 \) or \( e_3 \) will be executed, the liveness environment with respect to which \( e_1 \) is computed is the union of the liveness environments arising out of \( e_2 \) and \( e_3 \).

3.2 Computing \( XP \) and \( Xf \)

If \( \sigma \) is the set of access patterns specifying the liveness of the result of evaluating \( (P e_1 \dots e_n) \), where \( P \) is a primitive, then \( XP^1_{P}(\sigma) \) gives the set of access patterns specifying the liveness of \( e_i \). We describe the transfer functions for the primitives in our language: \text{ car, cdr, cons, null?, } \text{ pair? and } +. \text{ The 0-ary constructor } \text{ Nil does not accept any argument and is ignored.}
Assume that the live access pattern for the result of the expression (\texttt{car }e) is \(\alpha\). Then, the link that is denoted by the path labeled \(\alpha\) starting from location \([[\texttt{car } e]]\) can also be denoted by a path \(0\alpha\) starting from location \([e]\). We can extend the same reasoning for set of access patterns (\(\sigma\)) of result, i.e. every pattern in the set is prefixed by \(0\) to give live access pattern of \(e\). Also, since the cell corresponding to \(e\) is used to find the value of \texttt{car}, we need to add \(\epsilon\) to the live access patterns of \(e\). Reasoning about (\texttt{cdr }e) similarly, we have

\[
(11) \quad \mathcal{XP}_{\text{car}}(\sigma) = \{\epsilon\} \cup \{0\} \cdot \sigma, \quad \mathcal{XP}_{\text{cdr}}(\sigma) = \{\epsilon\} \cup \{1\} \cdot \sigma
\]

As seen in Example 2.1, an access pattern of \(\alpha\) for result of \texttt{cons} translates to an access pattern of \(0\alpha\) for its first argument, and \(1\alpha\) for its second argument. Since \texttt{cons} does not read its arguments, the access patterns of the arguments do not contain \(\epsilon\).

\[
(12) \quad \mathcal{XP}_{\text{cons}}(\sigma) = \{0\} \cdot \sigma, \quad \mathcal{XP}_{\text{cons}}(\sigma) = \{1\} \cdot \sigma
\]

Since the remaining primitives read only the value of the arguments, the set of live access patterns of the arguments is \{\(\epsilon\}\}.

\[
(13) \quad \mathcal{XP}_{\text{null}}(\sigma) = \{\epsilon\}, \quad \mathcal{XP}_{\text{pair}}(\sigma) = \{\epsilon\}, \quad \mathcal{XP}_{\text{1}}(\sigma) = \{\epsilon\}, \quad \mathcal{XP}_{\text{2}}(\sigma) = \{\epsilon\}
\]

We now consider the transfer function for a user defined function \(f\). If \(\sigma\) is the set of access patterns specifying the liveness of the result of evaluating \((f \ e_1 \ldots e_n)\), then \(\mathcal{XF}_f(\sigma)\) gives the set of access patterns specifying the liveness of \(e_i\). Let \(f\) be defined as:

\[
\text{(define } (f \ v_1 \ldots v_n) \ \pi : e )
\]

Assume that \(\sigma\) is the live access pattern for the result of \(f\). Then, \(\sigma\) is also the live access pattern for \(e\). \(\mathcal{XE}(e, \sigma, \emptyset)\) computes live access patterns for \(v_i\) \((1 \leq i \leq n)\) at \(\pi\). Thus, the transfer function for the \(i^{th}\) argument of \(f\) is given by:

\[
(14) \quad \mathcal{XF}_f(\sigma) = \{\alpha \mid v_i, \alpha \in \mathcal{XE}(e, \sigma, \emptyset)\} \quad 1 \leq i \leq n
\]

The following example illustrates our analysis.

**Example 3.1** Consider the program in Figure 1. To compute the transfer functions for \texttt{app}, we compute the environment transformer \(\mathcal{XE}(e, \sigma, \emptyset)\) in terms of a variable \(\sigma\). Here \(e\) is the body of \texttt{app}. The value of the liveness environment at each point in the body of \texttt{app} is shown in Appendix A. From the liveness information at \(\pi_1\) we get:

\[
\begin{align*}
\mathcal{XF}_{\text{app}}(\sigma) &= \{\epsilon\} \cup \{0\} \cdot \sigma \cup \{1\} \cdot \mathcal{XF}_{\text{app}}(\{1\} \cdot \sigma) \\
\mathcal{XF}_{\text{app}}(\sigma) &= \sigma \cup \mathcal{XF}_{\text{app}}(\{1\} \cdot \sigma)
\end{align*}
\]

Let \(e_{\text{pgm}}\) represent the entire program being analyzed and \(\sigma_{\text{pgm}}\) be the set of access patterns describing the liveness of the result. Then, the liveness environment at various points in the \(e_{\text{pgm}}\) can be computed as \(\mathcal{XE}(e_{\text{pgm}}, \sigma_{\text{pgm}}, \emptyset)\). The liveness environments at \(\pi_{14}\) and \(\pi_{12}\) are as follows:

\[
\begin{align*}
\mathcal{XE}_{\pi_{14}} &= \{ \langle \text{w}, (\{\epsilon, 1, 10\} \cup \{100\} \cdot \sigma_{\text{pgm}}) \rangle \} \\
\mathcal{XE}_{\pi_{12}} &= \left\{ \langle y, \mathcal{XF}_{\text{app}}(\{\epsilon, 1, 10\} \cup \{100\} \cdot \sigma_{\text{pgm}}), \rangle, \langle z, \mathcal{XF}_{\text{app}}(\{\epsilon, 1, 10\} \cup \{100\} \cdot \sigma_{\text{pgm}}) \rangle \right\}
\end{align*}
\]

We assume that the entire result of the program is needed, i.e., \(\sigma_{\text{pgm}}\) is \(\{0, 1\}^*\).
4 Solving the Equations for $X\mathcal{F}$

In general, the equations defining the transfer functions $X\mathcal{F}$ will be recursive. To solve such equations, we start by guessing that the solution will be of the form:

$$(15) \quad X\mathcal{F}^j_i(\sigma) = I^j_i \cup D^j_i \cdot \sigma,$$

where $I^j_i$ and $D^j_i$ are sets of strings over the alphabet $\{0, 1, \bar{0}, \bar{1}\}$. The intuition behind this form of solution is as follows: The function $f$ can use its argument locally and/or copy a part of it to the return value being computed. $I^j_i$ is the live access pattern of $f^j_i$ argument due to local use in $f$. $D^j_i$ is a sort of selector that selects the liveness pattern corresponding to the $f^j_i$ argument of $f$ from $\sigma$, the liveness pattern of the return value.

If we substitute the guessed form of $X\mathcal{F}^j_i$ in the equations describing it and equate the terms containing $\sigma$ and the terms without $\sigma$, we get the equations for $I^j_i$ and $D^j_i$. This is illustrated in the following example.

**Example 4.1** Consider the equation for $X\mathcal{F}^1_{app}(\sigma)$ from Example 3.1:

$$X\mathcal{F}^1_{app}(\sigma) = \{\varepsilon\} \cup \{\bar{0}\} \cdot \sigma \cup \{1\} \cdot X\mathcal{F}^1_{app}(\bar{1}) \cdot \sigma$$

Decomposing both sides of the equation, and rearranging the RHS gives:

$$I^1_{app} \cup D^1_{app} \cdot \sigma = \{\varepsilon\} \cup \{\bar{0}\} \cdot \sigma \cup \{1\} \cdot (I^1_{app} \cup D^1_{app} \cdot \bar{1}) \cdot \sigma$$

$$= \{\varepsilon\} \cup \{1\} \cdot I^1_{app} \cup \{0\} \cdot \sigma \cup \{1\} \cdot D^1_{app} \cdot \bar{1} \cdot \sigma$$

Separating the parts that are $\sigma$ dependent and the parts that are $\sigma$ independent, and equating them separately, we get:

$$I^1_{app} = \{\varepsilon\} \cup \{1\} \cdot I^1_{app}$$

$$D^1_{app} \cdot \sigma = \{0\} \cdot \sigma \cup \{1\} \cdot D^1_{app} \cdot \bar{1} \cdot \sigma$$

As the equations hold for any general $\sigma$, we can simplify them to:

$$I^1_{app} = \{\varepsilon\} \cup \{1\} \cdot I^1_{app} \quad \text{and} \quad D^1_{app} = \{0\} \cup \{1\} \cdot D^1_{app} \cdot \bar{1}$$

Similarly, from the equation describing $X\mathcal{F}^2_{app}(\sigma)$, we get:

$$I^2_{app} = I^1_{app} \quad \text{and} \quad D^2_{app} = \{\varepsilon\} \cup D^2_{app} \cdot \bar{1}$$

The liveness environment at $\pi_{12}$ and $\pi_{14}$ in terms $I_{app}$ and $D_{app}$ are:

$$L_{\pi_{14}} = \{w. \{100\} \cdot \sigma_{\text{pgm}}\}$$

$$L_{\pi_{12}} = \{y.(I^1_{app} \cup D^1_{app} \cdot \{\varepsilon, 1, 10\} \cup \{100\} \cdot \sigma_{\text{pgm}})), z.(I^2_{app} \cup D^2_{app} \cdot \{\varepsilon, 1, 10\} \cup \{100\} \cdot \sigma_{\text{pgm}})\}$$

Solving for $I_{app}$ and $D_{app}$ gives us the desired liveness environments at these program points.

4.1 Representing Liveness by Context Free Grammars

The values of $I$ and $D$ variables of a transfer function are sets of strings over the alphabet $\{0, 1, \bar{0}, \bar{1}\}$. We use context free grammars (CFG) to describe these sets. The set of terminal symbols of the CFG is $\{0, 1, \bar{0}, \bar{1}\}$. Non-terminals and associated rules are constructed as illustrated in Examples 4.2 and 4.3.
Example 4.2 Consider the following constraint from Example 4.1:

\[ I_{app}^1 = \{ \varepsilon \} \cup \{ 1 \} \cdot I_{app}^1 \]

We add non-terminal \( \langle I_{app}^1 \rangle \) and the productions with right hand sides directly derived from the constraints:

\( \langle I_{app}^1 \rangle \rightarrow \varepsilon \mid 1 \langle I_{app}^1 \rangle \)

The productions generated from other constraints of Example 4.1 are:

\( \langle D_{app}^1 \rangle \rightarrow 0 \bar{0} \mid 1 \langle D_{app}^1 \rangle \bar{1} \)

\( \langle I_{app}^2 \rangle \rightarrow \langle I_{app}^2 \rangle \)

\( \langle D_{app}^2 \rangle \rightarrow \varepsilon \mid \langle D_{app}^2 \rangle \bar{1} \)

These productions describe the transfer functions of \( app \).

The liveness environment at each program point can be represented as a CFG with a start symbol for every variable. To do so, the analysis starts with \( \langle S_{pgm} \rangle \), the non-terminal describing the liveness pattern of the result of the program, \( \sigma_{pgm} \). The productions for \( \langle S_{pgm} \rangle \) are:

\( \langle S_{pgm} \rangle \rightarrow \varepsilon \mid 0 \langle S_{pgm} \rangle \mid 1 \langle S_{pgm} \rangle \)

Example 4.3 Let \( S_v^\pi \) denote the non-terminal generating liveness access patterns associated with a variable \( v \) at program point \( \pi \). For the program of Figure 1:

\( \langle S_{pgm} \rangle \rightarrow \varepsilon \mid 1 \mid 10 \mid 100 \langle S_{pgm} \rangle \)

\( \langle S_{pgm} \rangle \rightarrow \langle I_{app}^1 \rangle \mid \langle D_{app}^1 \rangle \mid \langle D_{app}^1 \rangle 1 \mid \langle D_{app}^1 \rangle 10 \mid \langle D_{app}^1 \rangle 100 \langle S_{pgm} \rangle \)

\( \langle S_{pgm} \rangle \rightarrow \langle I_{app}^1 \rangle \mid \langle D_{app}^1 \rangle \mid \langle D_{app}^1 \rangle 1 \mid \langle D_{app}^1 \rangle 10 \mid \langle D_{app}^1 \rangle 100 \langle S_{pgm} \rangle \)

The access patterns in the access paths used for nullification are in canonical form but the access patterns described by the CFGs resulting out of our analysis are not. It is not obvious how to check the membership of a canonical access pattern in such CFGs. To solve this problem, we need equivalent CFGs such that if \( \alpha \) belongs to an original CFG and \( \alpha \rightarrow \beta \), where \( \beta \) is in canonical form, then \( \beta \) belongs to the corresponding new CFG. Directly converting the reduction rules (Equations (1, 2, 3)) into productions and adding it to the grammar results in unrestricted grammar [11]. To simplify the problem, we approximate original CFGs by non-deterministic finite automata (NFAs) and eliminate \( \bar{0} \) and \( \bar{1} \) from the NFAs.

4.2 Approximating CFGs using NFAs

The conversion of a CFG \( G \) to an approximate NFA \( N \) should be safe in that the language accepted by \( N \) should be a superset of the language accepted by \( G \). We use the algorithm described by Mohri and Nederhof [18]. The algorithm transforms a CFG to a restricted form called strongly regular CFG which can be converted easily to a finite automaton.

Example 4.4 We show the approximate NFAs for each of the non-terminals in Example 4.2 and Example 4.3.
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Note that there is no automaton for $\langle I^2_{\text{app}} \rangle$. This is because the least solution of the equation $\langle I^2_{\text{app}} \rangle \rightarrow \langle I^2_{\text{app}} \rangle$ is $\emptyset$. Also, the language accepted by the automaton for $D^1_{\text{app}}$ is approximate as it does not ensure that there is an equal number of $1$ and $\bar{1}$ in the strings generated by rules for $\langle D^1_{\text{app}} \rangle$.

4.3 Eliminating $\bar{0}$ and $\bar{1}$ from NFA

We now describe how to convert an NFA with transitions on symbols $\bar{0}$ and $\bar{1}$ to an equivalent NFA without any transitions on these symbols.

**Input:** An NFA $\mathcal{N}$ with underlying alphabet $\{0, 1, \bar{0}, \bar{1}\}$ accepting a set of access patterns

**Output:** An NFA $\mathcal{N}$ with underlying alphabet $\{0, 1\}$ accepting the equivalent set of canonical access patterns

**Steps:**

1. $i \leftarrow 0$
2. $\mathcal{N}_0 \leftarrow$ Equivalent NFA of $\mathcal{N}$ without $\varepsilon$-moves [11]
3. do
   1. $\mathcal{N}'_{i+1} \leftarrow \mathcal{N}_i$
   2. foreach state $q$ in $\mathcal{N}_i$ such that $q$ has an incoming edge from $q'$ with label $0$ and outgoing edge to $q''$ with label $0$
      1. /* bypass $00$ using $\varepsilon$ */
      2. add an edge in $\mathcal{N}'_{i+1}$ from $q'$ to $q''$ with label $\varepsilon$.
   3. foreach state $q$ in $\mathcal{N}_i$ such that $q$ has an incoming edge from $q'$ with label $\bar{1}$ and outgoing edge to $q''$ with label $1$
      1. /* bypass $\bar{1}1$ using $\varepsilon$ */
      2. add an edge in $\mathcal{N}'_{i+1}$ from $q'$ to $q''$ with label $\varepsilon$.
4. $\mathcal{N}_{i+1} \leftarrow$ Equivalent NFA of $\mathcal{N}'_{i+1}$ without $\varepsilon$-moves
5. $i \leftarrow i + 1$
6. while $(\mathcal{N}_i \neq \mathcal{N}_{i-1})$
7. $\mathcal{N} \leftarrow \mathcal{N}_i$
8. delete all edges with label $\bar{0}$ or $\bar{1}$ in $\mathcal{N}$.

The algorithm repeatedly introduces $\varepsilon$ edges to bypass a pair of consecutive edges labeled $00$ or $\bar{1}1$. The process is continued till a fixed point is reached. When the fixed point is reached, the resulting NFA contains the canonical access patterns corresponding to all the
access patterns in the original NFA. The access patterns not in canonical form are deleted by removing edges labeled $\bar{0}$ and $\bar{1}$. Note that by our reduction rules if $\alpha$ is accepted by $\overline{N}$ and $\alpha \xrightarrow{\epsilon} \perp$, then $\perp$ should be accepted by $N$. However, $N$ returned by our algorithm does not accept $\perp$. This is not a problem because the access patterns which are tested for membership against $N$ do not include $\perp$ as well.

**Example 4.5** We show the elimination of $\bar{0}$ and $\bar{1}$ for the automata for $\langle S_y^{\pi_{12}} \rangle$ and $\langle S_z^{\pi_{12}} \rangle$. The automaton for $\langle S_w^{\pi_{14}} \rangle$ remains unchanged as it does not contain transitions on $\bar{0}$ and $\bar{1}$.

The automata at the termination of the loop in the algorithm are:

\[
\langle S_y^{\pi_{12}} \rangle: \quad \begin{array}{c}
\text{start} \\
0 \\
1 \\
0 \\
\bar{0} \\
\bar{1} \\
0 \\
\end{array}
\]

\[
\langle S_z^{\pi_{12}} \rangle: \quad \begin{array}{c}
\text{start} \\
\bar{1} \\
1 \\
0 \\
0 \\
\bar{0} \\
0 \\
\end{array}
\]

Eliminating the edges labeled $\bar{0}$ and $\bar{1}$, and removing the dead states gives:

\[
\langle S_y^{\pi_{12}} \rangle: \quad \begin{array}{c}
\text{start} \\
0 \\
0 \\
\bar{0} \\
\bar{1} \\
0 \\
\end{array}
\]

\[
\langle S_z^{\pi_{12}} \rangle: \quad \begin{array}{c}
\text{start} \\
1 \\
0 \\
0 \\
\bar{0} \\
0 \\
\end{array}
\]

The language accepted by these automata represent the live access paths corresponding to $y$ and $z$ at $\pi_{12}$. $\square$

We now prove the termination and correctness of our algorithm.

**Termination**

Termination of the algorithm follows from the fact that every iteration of the do-while loop adds new edges to the NFA, while old edges are not deleted. Since no new states are added to NFA, only a fixed number of edges can be added before we reach a fix point.

**Correctness**

The sequence of obtaining $N$ from $\overline{N}$ can be viewed as follows, with $N_m$ denoting the NFA at the termination of while loop:

\[
\overline{N} \xrightarrow{\text{deletion of } \epsilon\text{-edges}} N_0 \xrightarrow{\text{addition of } \epsilon\text{-edges}} N'_0 \xrightarrow{\text{deletion of } \epsilon\text{-edges}} N_1 \xrightarrow{\text{addition of } \epsilon\text{-edges}} N'_1 \cdots \xrightarrow{\text{addition of } \epsilon\text{-edges}} N_i \xrightarrow{\text{deletion of } \epsilon\text{-edges}} N'_i \cdots \xrightarrow{\text{deletion of } \epsilon\text{-edges}} N_m
\]

Then, the languages accepted by these NFAs have the following relation:

\[
L(\overline{N}) = L(N_0) \subseteq L(N'_1) = L(N_1) \subseteq \cdots \subseteq L(N'_i) = L(N_i) \subseteq \cdots = L(N_m)
\]

We first prove that the addition of $\epsilon$-edges in the while loop does not add any new information, i.e. any access pattern accepted by the NFA after the addition of $\epsilon$-edges is a reduced version of some access pattern existing in the NFA before the addition of $\epsilon$-edges.
Lemma 4.6 for \( i > 0 \), if \( \alpha \in L(N_i) \) then there exists \( \alpha' \in L(N_{i-1}) \) such that \( \alpha' \xrightarrow{\epsilon} \alpha \).

**Proof.** As \( L(N_i) = L(N'_i) \), we have \( \alpha \in L(N'_i) \). Only difference between \( N'_i \) and \( N_{i-1} \) is that \( N'_i \) contains some extra \( \epsilon \)-edges. Thus, any \( \epsilon \)-edge free path in \( N'_i \) is also in \( N_{i-1} \). Consider a path \( p \) in \( N'_i \) that accepts \( \alpha \). Assume the number of \( \epsilon \) edges in \( p \) is \( k \). The proof is by induction on \( k \).

(BASE) \( k = 0 \), i.e. \( p \) does not contain any \( \epsilon \)-edge: As the path \( p \) is \( \epsilon \)-edge free, it must be present in \( N_{i-1} \). Thus, \( N_{i-1} \) also accepts \( \alpha \). \( \alpha \xrightarrow{\epsilon} \alpha \).

(HYPOTHESIS) For any \( \alpha \in L(N_i) \) with accepting path \( p \) having less than \( k \) \( \epsilon \)-edges there exists \( \alpha' \in L(N_{i-1}) \) such that \( \alpha' \xrightarrow{\epsilon} \alpha \).

(INDUCTION) \( p \) contains \( k \) \( \epsilon \)-edges \( e_1, \ldots, e_k \). Assume \( e_1 \) connects states \( q' \) and \( q'' \) in \( N'_i \). By construction, there exists a state \( q \) in \( N'_i \) such that there is an edge \( e'_1 \) from \( q' \) to \( q \) with label \( \overline{0(1)} \) and an edge \( e''_1 \) from \( q \) to \( q'' \) with label \( 0(1) \) in \( N'_i \). Replace \( e_1 \) by \( e'_1 e''_1 \) in \( p \) to get a new path \( p'' \) in \( N'_i \). Let \( \alpha'' \) be the access pattern accepted by \( p'' \). Clearly, \( \alpha'' \xrightarrow{\epsilon} \alpha \). Since \( p'' \) has \( k-1 \) \( \epsilon \)-edges, \( \alpha'' \) is accepted by \( N'_i \) along a path \( (p'') \) that has less than \( k \) \( \epsilon \)-edges. By induction hypothesis, we have \( \alpha' \in L(N_{i-1}) \) such that \( \alpha' \xrightarrow{\epsilon} \alpha'' \). This along with \( \alpha'' \xrightarrow{\epsilon} \alpha \) gives \( \alpha' \xrightarrow{\epsilon} \alpha \). \( \square \)

**Corollary 4.7** for each \( \alpha \in L(N_m) \), there exists \( \alpha' \in L(\overline{N}) \) such that \( \alpha' \xrightarrow{\epsilon} \alpha \).

**Proof.** The proof is by induction on \( m \), and using Lemma 4.6. \( \square \)

The following lemma shows that the the language accepted by \( N_m \) is closed with respect to reduction of access patterns.

**Lemma 4.8** For \( \alpha \in L(N_m) \), if \( \alpha \xrightarrow{\epsilon} \alpha' \) and \( \alpha' \neq \bot \), then \( \alpha' \in L(N_m) \).

**Proof.** Assume \( \alpha \xrightarrow{k} \alpha' \). The Proof is by induction on \( k \), number of steps in reduction.

(BASE) case \( k = 0 \) is trivial as \( \alpha \xrightarrow{0} \alpha \).

(HYPOTHESIS) Assume that for \( \alpha \in L(N_m) \), if \( \alpha \xrightarrow{k-1} \alpha' \), then \( \alpha' \in L(N_m) \).

(INDUCTION) \( \alpha \in L(N_m) \), \( \alpha \xrightarrow{k} \alpha' \). There exists \( \alpha'' \) such that: \( \alpha \xrightarrow{k-1} \alpha'' \xrightarrow{1} \alpha' \). By induction hypothesis, we have \( \alpha'' \in L(N_m) \).

For \( \alpha'' \xrightarrow{1} \alpha' \) to hold we must have \( \alpha'' = \alpha_1 \overline{00} \alpha_2 \) and \( \alpha' = \alpha_1 \alpha_2 \), or \( \alpha'' = \alpha_1 \overline{11} \alpha_2 \) and \( \alpha' = \alpha_1 \alpha_2 \). Consider the case when \( \alpha'' = \alpha_1 \overline{00} \alpha_2 \). Any path in \( N_m \) accepting \( \alpha'' \) must have the following structure (The states shown separately may not necessarily be different):

![Diagram](https://via.placeholder.com/150)

As \( N_m \) is the fixed point NFA for the iteration process described in the algorithm, adding an \( \epsilon \)-edge between states \( q' \) and \( q'' \) will not change the language accepted by \( N_m \). But, the access pattern accepted after adding an \( \epsilon \)-edge is \( \alpha_1 \alpha_2 = \alpha' \). Thus, \( \alpha' \in L(N_m) \). The case when \( \alpha'' = \alpha_1 \overline{11} \alpha_2 \) is identical. \( \square \)

**Corollary 4.9** For \( \alpha \in L(\overline{N}) \), if \( \alpha \xrightarrow{\epsilon} \alpha' \) and \( \alpha' \neq \bot \), then \( \alpha' \in L(N_m) \).

**Proof.** \( L(\overline{N}) \subseteq L(N_m) \Rightarrow \alpha \in L(N_m) \). The proof follows from Lemma 4.8. \( \square \)

The following theorem asserts the equivalence of \( \overline{N} \) and \( N \) with respect to the equivalence of access patterns, i.e. every access pattern in \( \overline{N} \) has an equivalent canonical access
pattern in $N$, and for every canonical access pattern in $N$, there exists an equivalent access pattern in $\bar{N}$.

**Theorem 4.10** Let $\bar{N}$ be an NFA with underlying alphabet $\{0, 1, \bar{0}, \bar{1}\}$. Let NFA $N$ be the NFA with underlying alphabet $\{0, 1\}$ returned by the algorithm. Then,

(i) if $\alpha \in L(N)$, $\beta$ is a canonical access pattern such that $\alpha \rightarrow \beta$ and $\beta \neq \bot$, then $\beta \in L(N)$.  

(ii) if $\beta \in L(N)$ then there exists an access pattern $\alpha \in L(N)$ such that $\alpha \rightarrow \beta$.

**Proof.**

(i) From Corollary 4.9: $\alpha \in L(\bar{N}), \alpha \rightarrow \beta$ and $\beta \neq \bot \Rightarrow \beta \in L(N_m)$. As $\beta$ is in canonical form, the path accepting $\beta$ in $N_m$ consists of edges labeled $0$ and $1$ only. The same path exists in $N$. Thus $N$ also accepts $\beta \Rightarrow \beta \in L(N)$.

(ii) $L(N) \subseteq L(N_m) \Rightarrow \beta \in L(N_m)$. Using Corollary 4.7, there exists $\alpha \in L(\bar{N})$ such that $\alpha \rightarrow \beta$.

\[\square\]

5 An Application of Liveness Analysis

The result of liveness analysis can be used to decide whether a given access path $v\alpha$ can be nullified at a given program point $\pi$. Let the link corresponding to $v\alpha$ in the memory graph be $l$. A naive approach is to nullify $v\alpha$ if it does not belong to the liveness environment at $\pi$. However, the approach is not safe because of two reasons: (a) The link $l$ may be used beyond $\pi$ through an alias, and may therefore be live. (b) a link $l'$ in the access path from the root variable $v$ to $l$ may have been created along one execution path but not along another. Since the nullification of $v\alpha$ requires the link $l'$ to be dereferenced, a run time exception may occur.

To solve the first problem, we need an alias analysis phase to detect sharing of links among access paths. A link in the memory graph can be nullified at $\pi$ if none of the access paths sharing it are live at $\pi$. To solve the second problem, we need an availability analysis phase. It detects whether all links in the access path have been created along all execution paths reaching $\pi$. The results of these analysis are used to filter out those access paths whose nullification may be unsafe. We do not address the descriptions of these analyses in this paper.

6 Related Work

In this paper, we have described a static analysis for inferring dead references in first order functional programs. We employ a context free grammar based abstraction for the heap. This is in the spirit of the work by Jones and Muchnick [13] for functional programs. The existing literature related to improving memory efficiency of programs can be categorized as follows:

*Compile time reuse.* The method by Barth [2] detects memory cells with zero reference count and reallocates them for further use in the program. Jones and Le Metayer [15] describe a sharing analysis based garbage collection for reusing of cells. Their analysis incorporates liveness information: A cell is collected even when it is shared provided expressions sharing it do not need it for their evaluation.
Explicit reclamation. Shaham et al. [25] use an automaton called heap safety automaton to model safety of inserting a free statement at a given program point. The analysis is based on shape analysis [20,21] and is very precise. The disadvantage of the analysis is that it is very inefficient and takes large time even for toy programs. Free-Me [7] combines a lightweight pointer analysis with liveness information that detects when short-lived objects die and insert statements to free such objects. The analysis is simpler and cheaper as the scope is limited. The analysis described by Inoue et al. [12] detects the scope (function) out of which a cell becomes unreachable, and claims the cell using an explicit reclaim procedure whenever the execution goes out of that scope. Like our method, the result of their analysis is also represented using CFGs. The main difference between their work and ours is that we detect and nullify dead links at any point of the program, while they detect and collect objects that are unreachable at function boundaries. Cherem and Rugina [5] use a shape analysis framework [8] to analyze a single heap cell at a time for deallocation. However, multiple iterations of the analysis and the optimization steps are required, since freeing a cell might result in opportunities for more deallocations.

Making dead objects unreachable. The most popular approach to make dead objects unreachable is to identify live variables in the program to reduce the root set to only the live reference variables [1]. The major drawback of this approach is that all heap objects reachable from the live root variables are considered live, even if some of them may not be used by the program. Escape analysis [3,4,6] based approaches discover objects escaping a procedure (an escaping object being an object whose lifetimes outlives the procedure that created it). All non-escaping objects are allocated on stack, thus becoming unreachable whenever the creating procedure exits. In Region based garbage collection [9], a static analysis called region inference [26] is used to identify regions that are storage for objects. Normal memory blocks can be allocated at any point in time; they are always allocated in a particular region and are deallocated at the end of that region’s lifetime. Approaches based on escape analysis and region inference detect garbage only at the boundaries of certain predefined areas of the program. In our previous work [17], we have used bounded abstractions of access paths called access graphs to describe the liveness of memory links in imperative programs and have used this information to nullify dead links.

A related work due to Heine and Lam [10] attempts to find potential memory leaks in C/C++ programs by detecting the earliest point in a program when an object becomes unreachable.

7 Conclusions

In this paper we presented a technique to compute liveness of heap data in functional programs. This information could be used to nullify links in heap memory to improve garbage collection. We have abstracted the liveness information in the form of a CFG, which is then converted to NFAs. This conversion implies some imprecision. We present a novel way to simplify the NFAs so they directly describe paths in the heap. Unlike the method described by Inoue et al. [12], our simplification does not cause any imprecision.

In future, we intend to take this method to its logical conclusion by addressing the issue of nullification. This would require us to perform alias analysis which we feel can be done in a similar fashion. We also feel that with minor modification our method can be used for dead code elimination and intend to extend our analysis to higher order languages.
References

[1] Agesen, O., D. Detlefs and J. E. Moss, Garbage collection and local variable type-precision and liveness in Java virtual machines, in: PLDI ’98: Proceedings of the ACM SIGPLAN 1998 conference on Programming language design and implementation (1998), pp. 269–279.

[2] Barth, J. M., Shifting garbage collection overhead to compile time, Commun. ACM 20 (1977), pp. 513–518.

[3] Blanchet, B., Escape analysis for object-oriented languages: application to Java, in: OOPSLA ’99: Proceedings of the 14th ACM SIGPLAN conference on Object-oriented programming, systems, languages, and applications (1999), pp. 20–34.

[4] Blanchet, B., Escape analysis for JavaTM: Theory and practice, ACM Transactions on Programming Languages and Systems 25 (2003), pp. 713–753.

[5] Cherem, S. and R. Rugina, Compile-time deallocation of individual objects, in: ISMM ’06: Proceedings of the 2006 international symposium on Memory management (2006), pp. 138–149.

[6] Choi, J.-D., M. Gupta, M. Serrano, V. C. Sreedhar and S. Midkiff, Escape analysis for Java, in: OOPSLA ’99: Proceedings of the 14th ACM SIGPLAN conference on Object-oriented programming, systems, languages, and applications (1999), pp. 1–19.

[7] Guyer, S. Z., K. S. McKinley and D. Frampton, Free-me: a static analysis for automatic individual object reclamation, in: PLDI ’06: Proceedings of the 2006 ACM SIGPLAN conference on Programming language design and implementation (2006), pp. 364–375.

[8] Hackett, B. and R. Rugina, Region-based shape analysis with tracked locations, in: POPL ’05: Proceedings of the 32nd ACM SIGPLAN-SIGACT symposium on Principles of programming languages (2005), pp. 310–323.

[9] Hallenberg, N., M. Elsman and M. Tofte, Combining region inference and garbage collection, in: PLDI ’02: Proceedings of the ACM SIGPLAN 2002 Conference on Programming language design and implementation (2002), pp. 141–152.

[10] Heine, D. L. and M. S. Lam, A practical flow-sensitive and context-sensitive c and c++ memory leak detector, in: PLDI ’03: Proceedings of the ACM SIGPLAN 2003 conference on Programming language design and implementation (2003), pp. 168–181.

[11] Hopcroft, J. E. and J. D. Ullman, “Introduction To Automata Theory, Languages, And Computation,” Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA, 1990.

[12] Inoue, K., H. Seki and H. Yagi, Analysis of functional programs to detect run-time garbage cells, ACM Trans. Program. Lang. Syst. 10 (1988), pp. 555–578.

[13] Jones, N. D. and S. S. Muchnick, Flow analysis and optimization of list-like structures, in: POPL ’79: Proceedings of the 6th ACM SIGACT-SIGACT symposium on Principles of programming languages (1979), pp. 244–256.

[14] Jones, R. and R. Lins, “Garbage collection: algorithms for automatic dynamic memory management,” John Wiley & Sons, Inc., New York, NY, USA, 1996.

[15] Jones, S. B. and D. L. Metayer, Compile-time garbage collection by sharing analysis, in: FPCA ’89: Proceedings of the fourth international conference on Functional programming languages and computer architecture (1989), pp. 54–74.

[16] Karkare, A., A. Sanyal and U. Khedker, Effectiveness of garbage collection in MIT/GNU scheme, http://arxiv.org/abs/cs/0611093 (2006).

[17] Khedker, U., A. Sanyal and A. Karkare, Heap reference analysis using access graphs, Submitted to ACM Transactions on Programming Languages and Systems, copy available at http://arxiv.org/abs/cs.PL/0608104 (2006).

[18] Mohri, M. and M.-J. Nederhof, Regular approximation of context-free grammars through transformation, in: J.-C. Junqua and G. van Noord, editors, Robustness in Language and Speech Technology, Kluwer Academic Publishers, Dordrecht, 2000 pp. 251–261.

[19] Röjemo, N. and C. Runciman, Lag, drag, void and use—heap profiling and space-efficient compilation revisited, in: ICFP ’96: Proceedings of the first ACM SIGPLAN international conference on Functional programming (1996), pp. 34–41.

[20] Sagiv, M., T. Reps and R. Wilhelm, Parametric shape analysis via 3-valued logic, in: POPL ’99: Proceedings of the 26th ACM SIGPLAN-SIGACT symposium on Principles of programming languages (1999), pp. 105–118.

[21] Sagiv, M., T. Reps and R. Wilhelm, Parametric shape analysis via 3-valued logic, ACM Transactions on Programming Languages and Systems 24 (2002), pp. 217–298.

[22] Shaham, R., E. K. Kolodner and M. Sagiv, On the effectiveness of gc in java, in: ISMM ’00: Proceedings of the 2nd international symposium on Memory management (2000), pp. 12–17.

[23] Shaham, R., E. K. Kolodner and M. Sagiv, Heap profiling for space-efficient java, in: PLDI ’01: Proceedings of the ACM SIGPLAN 2001 conference on Programming language design and implementation (2001), pp. 104–113.

[24] Shaham, R., E. K. Kolodner and M. Sagiv, Estimating the impact of heap liveness information on space consumption in Java, in: ISMM ’02: Proceedings of the 3rd international symposium on Memory management (2002), pp. 64–75.

[25] Shaham, R., E. Yahav, E. K. Kolodner and M. Sagiv, Establishing local temporal heap safety properties with applications to compile-time memory management, Sci. Comput. Program. 58 (2005), pp. 264–289.

[26] Tofte, M. and L. Birikzal, A region inference algorithm, ACM Transactions on Programming Languages and Systems 20 (1998), pp. 724–767.
| Program Point (π) | Live Access Patterns for e at π (σ) | Liveness Environment after e (ζ) | Liveness Environment at π (XE(e, σ, ζ)) |
|------------------|-------------------------------------|----------------------------------|----------------------------------------|
| π₁               | σ                                   | ∅                               | list₁.(σ) ∪ {00} · σ ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) |
| π₂               | {ε}                                | list₁.(σ) ∪ {ε} ∪ {1} · X \_\_\_app(ε) ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) | list₁.(σ) ∪ {00} · σ ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) |
| π₃               | {ε}                                | list₁.(σ) ∪ {ε} ∪ {1} · X \_\_\_app(ε) ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) | list₁.(σ) ∪ {00} · σ ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) |
| π₄               | σ                                   | list₁.(σ) ∪ {00} · σ ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) | list₁.(σ) ∪ {00} · σ ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) |
| π₅               | σ                                   | ∅                               | list₁.(σ) ∪ {00} · σ ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) |
| π₆               | {0} · σ                             | list₁.(σ) ∪ {0} · σ ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) | list₁.(σ) ∪ {00} · σ ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) |
| π₇               | {ε} ∪ {00} · σ                       | list₁.(σ) ∪ {ε} ∪ {1} · X \_\_\_app(ε) ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) | list₁.(σ) ∪ {00} · σ ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) |
| π₈               | {1} · σ                             | list₁.(σ) ∪ {0} · σ ∪ {1} · X \_\_\_app(ε) ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) | list₁.(σ) ∪ {00} · σ ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) |
| π₉               | X \_\_\_app(1) · σ                  | list₁.(σ) ∪ {0} · σ ∪ {1} · X \_\_\_app(ε) ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) | list₁.(σ) ∪ {00} · σ ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) |
| π₁₀              | {1} · X \_\_\_app(1) · σ            | list₁.(σ) ∪ {0} · σ ∪ {1} · X \_\_\_app(ε) ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) | list₁.(σ) ∪ {00} · σ ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) |
| π₁₁              | X \_\_\_app(1) · σ                  | list₁.(σ) ∪ {0} · σ ∪ {1} · X \_\_\_app(ε) ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) | list₁.(σ) ∪ {00} · σ ∪ {1} · X \_\_\_app({1} · σ), list₂.(σ ∪ X \_\_\_app({1} · σ)) |