DECONVOLUTION OF BAND LIMITED FUNCTIONS ON NON-COMPACT SYMMETRIC SPACES

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Abstract. It is shown that a band limited function on a non-compact symmetric space can be reconstructed in a stable way from some countable sets of values of its convolution with certain distributions of compact support. A reconstruction method in terms of frames is given which is a generalization of the classical result of Duffin-Schaeffer about exponential frames on intervals. The second reconstruction method is given in terms of polyharmonic average splines.

1. Introduction

One of the most interesting properties of the so called band limited functions, i.e. functions whose Fourier transform has compact support, is that they are uniquely determined by their values on some countable sets of points and can be reconstructed from such values in a stable way. The sampling problem for band limited functions had attracted attention of many mathematicians [3], [4], [5], [14].

The mathematical theory of reconstruction of band limited functions from discrete sets of samples was introduced to the world of signal analysis and information theory by Shannon [21]. After Shannon the concept of band limitedness and the Sampling Theorem became the theoretical foundation of the signal analysis. In the classical signal analysis it is assumed that a signal (≡ a function) propagates in a Euclidean space. The most common method to receive, to store and to submit signals is through sampling using not the entire signal but rather a discrete set of its amplitudes measured on a sufficiently dense set of points.

In the present paper we consider non-compact symmetric spaces which include hyperbolic spaces. A sampling Theory on Riemannian manifolds and in Hilbert spaces was started by the author in [16]- [20]. It was shown in [16] that for properly defined band limited functions on manifolds the following sampling property holds true: they can be recovered from countable sets of their amplitudes measured on a sufficiently dense set of points.

Recently A. Kempf [12], [13] explored our results [16], [18] to developed a theory which approaches quantization of space-time and information through the sampling ideas on manifolds.

By using Harmonic Analysis on symmetric spaces we give further development of the theory of band limited functions on manifolds. It is shown that under certain

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assumptions a band limited function is uniquely determined and can be reconstructed in a stable way from some countable sets of values of its convolution with a distribution.

The problem of deconvolution is very natural in the context of Signal Analysis since convolution is used for denoising a signal with subsequent reconstruction of the band limited component. On the other hand, since it is impossible to perform an exact measurement of a signal at a point, convolution with a distribution can represent a measuring device.

As an illustration of the main result it is shown in Section 7 that every band limited function is uniquely determined and can be reconstructed in a stable way from a set of averages of its "derivatives" $\Delta^nf, n \in \mathbb{N} \cup \{0\}$, over a set of small spheres of a fixed radius whose centers form a sufficiently dense set, here $\Delta$ is the Laplace-Beltrami operator of $X$. As a particular case we obtain that every band limited function on a non-compact symmetric space of rank one is uniquely determined and can be reconstructed in a stable way from a set of samples of its "derivatives" $\Delta^nf(x_i), n \in \mathbb{N} \cup \{0\}$, for a countable and sufficiently dense set of points $\{x_i\}$. Another particular case is, that a band limited function $f$ on $X$ is uniquely defined and can be reconstructed in a stable way from a set of its average values over a set of small spheres whose centers form a sufficiently dense set.

The main result of Section 7 is the Theorem 7.3 which shows that for all these results the density of centers of the spheres depends just of the band width $\omega$ and independent of the radius of the spheres $\tau$ and of the smoothness index $n$. The radius of the spheres $\tau$ depends on $\omega$ and smoothness $n$ but it is independent of the distance $\tau$ between centers and can be relatively large compare to $r$. In other words, a stable reconstruction can still take place even if spheres intersect each other.

Reconstruction of band limited functions in $L^2(\mathbb{R}^d)$ from their average values over sets of full measure were initiated in the paper [6]. Our approach to this problem is very different from the approach of [6] and most of our results are new even in the one-dimensional case. It seems that reconstruction from averages over sets of measure zero (other then points) was never considered for band limited functions on $\mathbb{R}^d$.

Let $G$ be a semi-simple Lie group with a finite center and $K$ its maximal compact subgroup. We consider a non-compact symmetric space $X = G/K$. We say that a function $f$ from $L^2(X)$ is $\omega$-band limited if its Helgason-Fourier transform $\hat{f}(\lambda,b)$ is zero for $< \lambda, \lambda >^{1/2} = \|\lambda\| > \omega$, where $< . . >$ is the Killing form. We discuss some properties of such functions in the Section 3. It is shown in particular, that for any $\omega > 0$ and any ball $U \subset X$ restrictions of all $\omega$-band limited functions to the ball $U$ are dense in $L^2(U,dx)$.

In Section 4 we prove uniqueness and stability theorems. In Section 5 a reconstruction algorithm in terms of frames is presented. This result is a generalization of the results of Duffin-Schaeffer [5] about exponential frames on intervals. Let us recall, that the main result of [5] states, that for so called uniformly dense sequences of scalars $\{x_j\}$ the exponentials $\{e^{ix_j\xi}\}$ form a frame in $L^2(I)$ on a certain interval $I$ of the real line. In other words, if the support of the Fourier transform of a function $f \in L^2(I)$ belongs to $I$, then $f$ is uniquely determined and can be reconstructed in a stable way from its values on a countable set of points $\{x_j\}$.

In Section 6 we consider another way of reconstruction by using spline-like functions on $X$. It is a far going generalization of some results of Schoenberg [22].
2. Harmonic Analysis on symmetric spaces

A Riemannian symmetric space $X$ is defined as $G/K$, where $G$ is a connected non-compact semi-simple group Lie with Lie algebra with finite center and $K$ its maximal compact subgroup. Their Lie algebras will be denoted respectively as $\mathfrak{g}$ and $\mathfrak{k}$. The group $G$ acts on $X$ by left translations and it has the "origin" $o = eK$, where $e$ is the identity in $G$. Every such $G$ admits Iwasawa decomposition $G = NAK$, where nilpotent Lie group $N$ and abelian group $A$ have Lie algebras $\mathfrak{n}$ and $\mathfrak{a}$ respectively. The dimension of $\mathfrak{a}$ is known as the rank of $X$. Letter $M$ is usually used to denote the centralizer of $A$ in $K$ and letter $B$ is commonly used for the factor $B = K/M$.

Let $\mathfrak{a}^*$ be the real dual of $\mathfrak{a}$ and $W$ be the Weyl’s group. The $\Sigma$ will be the set of all bounded roots, and $\Sigma^+$ will be the set of all positive bounded roots. The notation $\mathfrak{a}^+$ has the following meaning
\[
\mathfrak{a}^+ = \{ h \in \mathfrak{a} | \alpha(h) > 0, \alpha \in \Sigma^+ \}
\]
and is known as positive Weyl’s chamber. Let $\rho \in \mathfrak{a}^*$ is defined in a way that $2\rho$ is the sum of all positive bounded roots. The Killing form $\langle , \rangle$ on $\mathfrak{g}$ defines a metric on $\mathfrak{a}$. By duality it defines a scalar product on $\mathfrak{a}^*$. The $\mathfrak{a}_+^*$ is the set of $\lambda \in \mathfrak{a}^*$, whose dual belongs to $\mathfrak{a}^+$. According to Iwasawa decomposition for every $g \in G$ there exists a unique $A(g) \in \mathfrak{a}$ such that
\[
g = n \exp A(g)k, \quad k \in K, \quad n \in N,
\]
where $\exp : \mathfrak{a} \to A$ is the exponential map of the Lie algebra $\mathfrak{a}$ to Lie group $A$. On the direct product $X \times B$ we introduce function with values in $\mathfrak{a}$ using the formula
\[
A(x, b) = A(u^{-1}g)
\]
where $x = gK, \ g \in G, \ b = uM, \ u \in K$.

For every $f \in C_0^\infty(X)$ the Helgason-Fourier transform is defined by the formula
\[
\hat{f}(\lambda, b) = \int_X f(x) e^{-i\lambda x + \rho A(x, b)} \, dx,
\]
where $\lambda \in \mathfrak{a}^*, \ b \in B = K/M$, and $dx$ is a $G$-invariant measure on $X$. This integral can also be expressed as an integral over group $G$. Namely, if $b = uM, \ u \in K$, then
\[
\hat{f}(\lambda, b) = \int_G f(x) e^{-i\lambda x + \rho A(u^{-1}g)} \, dg.
\]
The following inversion formula holds true
\[
f(x) = w^{-1} \int_{\mathfrak{a}_+^* \times B} \hat{f}(\lambda, b) e^{i\lambda x + \rho (A(x, b))} |c(\lambda)|^{-2} \, d\lambda db,
\]
where $w$ is the order of the Weyl’s group and $c(\lambda)$ is the Harish-Chandra’s function, $d\lambda$ is the Euclidean measure on $\mathfrak{a}^*$ and $d\lambda db$ is the normalized $K$-invariant measure on $B$. This transform can be extended to an isomorphism between spaces $L_2(X, dx)$ and $L_2(\mathfrak{a}_+^* \times B, |c(\lambda)|^{-2} \, d\lambda db)$ and the Plancherel formula holds true
\[
\|f\| = \left( \int_{\mathfrak{a}_+^* \times B} |\hat{f}(\lambda, b)|^2 |c(\lambda)|^{-2} \, d\lambda db \right)^{1/2}.
\]

An analog of the Paley-Wiener Theorem is known which says in particular that a Helgason-Fourier transform of a compactly supported distribution is a function which is analytic in $\lambda$. 

To introduce convolutions on $X$ we will need the notion of the spherical Fourier transform on the group $G$.

For a $\lambda \in a^*$ a zonal spherical function $\varphi_\lambda$ on the group $G$ is introduced by the Harish-Chandra’s formula

$$\varphi_\lambda(g) = \int_K e^{i(\lambda + \rho)(A(kg))} dk.$$  

(2.3)

If $f$ is a smooth bi-invariant function on $G$ with compact support its spherical Fourier transform is a function on $a^*$ which is defined by the formula

$$\hat{f}(\lambda) = \int_G f(g) \varphi_{-\lambda}(g) dg.$$  

The inversion formula is

$$f(g) = w^{-1} \int_{a^*} \hat{f}(\lambda) \varphi_\lambda(g) dg.$$  

The corresponding Plancherel formula has the form

$$\left( \int_G |f(g)|^2 dg \right)^{1/2} = \left( \int_{a^*} |\hat{f}(\lambda)|^2 d\lambda \right)^{1/2}.$$  

If $f$ is a function on the symmetric space $X$ and $\psi$ is a $K$-bi-invariant function on $G$ their convolution is a function on $X$ which is defined by the formula

$$f \ast \psi(g \cdot o) = \int_G f(gh^{-1} \cdot o) \psi(h) dh, \quad g \in G.$$  

By using duality arguments the last definition can be extended from functions to distributions. It is known, that

$$\hat{f} \ast \hat{\psi}(\lambda, b) = \hat{f}(\lambda, b) \hat{\psi}(\lambda).$$

(2.4)

Denote by $T_x(M)$ the tangent space of $M$ at a point $x \in M$ and let $exp_x : T_x(M) \rightarrow M$ be the exponential geodesic map i. e. $exp_x(u) = \gamma(1), u \in T_x(M)$ where $\gamma(t)$ is the geodesic starting at $x$ with the initial vector $u : \gamma(0) = x, \frac{d\gamma(0)}{dt} = u$. In what follows we assume that local coordinates are defined by $exp$.

By using a uniformly bounded partition of unity $\{\varphi_\nu\}$ subordinate to a cover of $X$ of finite multiplicity

$$X = \bigcup_\nu B(x_\nu, r),$$  

where $B(x_\nu, r)$ is a metric ball at $x_\nu \in X$ of radius $r$ we introduce Sobolev space $H^\sigma(X), \sigma > 0$, as the completion of $C^\infty_0(X)$ with respect to the norm

$$\|f\|_{H^\sigma(X)} = \left( \sum_\nu \|\varphi_\nu f\|_{H^\sigma(B(x_\nu, r))}^2 \right)^{1/2}.$$  

(2.5)

The usual embedding Theorems for the spaces $H^\sigma(X)$ hold true.

The Killing form on $G$ induces an inner product on tangent spaces of $X$. Using this inner product it is possible to construct $G$-invariant Riemannian structure on $X$. The Laplace-Beltrami operator of this Riemannian structure is denoted as $\Delta$.

If the space $X$ has rank one, then in the polar geodesic coordinate system $(r, \theta_1, ..., \theta_{d-1})$ on $X$ at every point $x \in X$ it has the form [9]

$$\Delta = \partial_r^2 + \frac{1}{S(r)} \frac{dS(r)}{dr} \partial_r + \Delta_S,$$  

where $S(r)$ is the area element of the sphere of radius $r$. 

The usual embedding Theorems for the spaces $H^\sigma(X)$ hold true.
where $\Delta_S$ is the Laplace-Beltrami operator on the sphere $S(x,r)$ of the induced Riemannian structure on $S(x,r)$ and $S(r)$ is the surface area of a sphere of radius $r$ which depends just on $r$ and is given by the formula

$$S(r) = \Omega_d 2^{-q} e^{-p-\eta} s h^p(c r) s h^q(2 c r),$$

where $d = \dim X = p + q + 1$, $c = (2 p + 8 q)^{-1/2}$, $p$ and $q$ depend on $X$ and

$$\Omega_d = \frac{2 \pi^{d/2}}{\Gamma(d/2)}$$

is the surface area of the unit sphere in $d$-dimensional Euclidean space.

In particular, if a function $f$ is zonal, i.e. depends just on distance $r$ from the origin $o$, we have

$$\Delta f(r) = \frac{1}{S(r)} \frac{d}{dr} \left( S(r) \frac{df(r)}{dr} \right).$$

Since for a smooth function $f$ with compact support the function $\Delta f$ can be expressed as a convolution of $f$ with the distribution $\Delta \delta_e$, where the Dirac measure $\delta_e$ is supported at the identity $e$ of $G$, the formula (2.4) implies

$$\hat{\Delta} f(\lambda, b) = - (\|\lambda\|^2 + \|\rho\|^2) \hat{f}(\lambda, b), f \in C_0^\infty(X),$$

where $\|\lambda\|^2 = <\lambda, \lambda>$, $\|\rho\|^2 = <\rho, \rho>$, $<,>$ is the Killing form.

It is known that for a general $X$ the operator $(-\Delta)$ is a self-adjoint positive definite operator in the corresponding space $L_2(X, dx)$, where $dx$ is the $G$-invariant measure. The regularity Theorem for the Laplace-Beltrami operator $\Delta$ states that domains of the powers $(-\Delta)^{\sigma/2}$ coincide with the Sobolev spaces $H^\sigma(X)$ and the norm (2.5) is equivalent to the graph norm $\|f\| + \|(-\Delta)^{\sigma/2} f\|$.

We consider a ball $B(o, r/4)$ in the invariant metric on $X$. Now we choose such elements $g_\nu \in G$ that the family of balls $B(x_\nu, r/4), x_\nu = g_\nu \cdot o$, has the following maximal property: there is no ball in $X$ of radius $r/4$ which would have empty intersection with every ball from this family. Then the balls of double radius $B(x_\nu, r/2)$ would form a cover of $X$. Of course, the balls $B(x_\nu, r)$ will also form a cover of $X$. Let us estimate the multiplicity of this cover.

Note, that the Riemannian volume $B(\rho)$ of a ball of radius $\rho$ in $X$ is independent of its center and is given by the formula

$$B(\rho) = \int_0^\rho S(t)dt,$$

where the surface area $S(t)$ of any sphere of radius $t$ is given by the formula (2.4). Every ball from the family $\{B(x_\nu, r)\}$, that has non-empty intersection with a particular ball $B(x_j, r)$ is contained in the ball $B(x_j, 3r)$. Since any two balls from the family $\{B(x_\nu, r/4)\}$ are disjoint, it gives the following estimate for the index of multiplicity of the cover $\{B(x_\nu, r)\}$:

$$\frac{B(3r)}{B(r/4)} = \frac{\int_0^{3r} S(t)dt}{\int_0^{r/4} S(t)dt}.$$
It is clear that for all sufficiently small $r > 0$ this fraction is bounded. In what follows we we will use the notation

$$N = \sup_{0 < r < 1} \frac{B(3r)}{B(r/4)}.$$

So, we proved the following Lemma.

**Lemma 2.1.** For any sufficiently small $r > 0$ there exists a set of points $\{x_\mu\}$ from $X$ such that

1) balls $B(x_\mu, r/4)$ are disjoint,
2) balls $B(x_\mu, r/2)$ form a cover of $X$,
3) multiplicity of the cover by balls $B(x_\mu, r)$ is not greater $N$.

We will use notation $Z(\{x_\mu\}, r, N)$ for any set of points $\{x_\mu\} \in X$ which satisfies the properties 1)- 3) from the last Lemma and we will call such set a metric $(r, N)$-lattice of $X$.

The following results can be found in [16] for any homogeneous manifold $X$.

**Theorem 2.2.** For any $k > d/2$ there exist constants $C = C(X, k, N) > 0, r_0(X, k, N)$, such that for any $0 < r < r_0$ and any $(r, N)$-lattice $Z(\{x_\mu\}, r, N)$ the following inequality holds true

$$(2.10) \quad \|f\| \leq C \left\{ r^{d/2} \left( \sum_{x_j \in Z} |f(x_j)|^2 \right)^{1/2} + r^k \|\Delta^{k/2} f\| \right\}, k > d/2.$$ 

and there exists a constant $C_1 = C_1(X, k, N)$ such that

$$\left( \sum_{x_j \in Z} |f(x_j)|^2 \right)^{1/2} \leq C_1 \|f\|_{H^k(X)}, f \in H^k(X).$$

3. Band limited functions

**Definition 1.** We will say that $f \in L_2(X, dx)$ belongs to the class $B_\omega(X)$ if its Helgason-Fourier transform has compact support in the sense that $\hat{f}(\lambda, b) = 0$ for $\|\lambda\| > \omega$. Such functions will be called $\omega$-band limited.

We have the following important result, which is a specification of some results in [15]- [18].

**Theorem 3.1.** A function $f$ belongs to $B_\omega(X)$ if and only if the following Bernstein inequality holds true for all $\sigma > 0$,

$$(3.1) \quad \|\Delta^\sigma f\| \leq (\omega^2 + \|\rho\|^2)^\sigma \|f\|.$$ 

**Proof.** By using the Plancherel formula and (2.8) we obtain that for every $\omega$-band limited function

$$\|\Delta^\sigma f\|^2 = \int_{|\lambda| < \omega} \int_B (|\lambda|^2 + \|\rho\|^2)^\sigma |\hat{f}(\lambda, b)|^2 |c(\lambda)|^{-2} d\lambda db \leq$$

$$(\omega^2 + \|\rho\|^2)^\sigma \int_{a^*} \int_B |\hat{f}(\lambda, b)|^2 |c(\lambda)|^{-2} d\lambda db = (\omega^2 + \|\rho\|^2)^\sigma \|f\|^2.$$
Conversely, if \( f \) satisfies (3.1), then for any \( \varepsilon > 0 \) and any \( \sigma > 0 \) we have
\[
\int_{\|\lambda\| \geq \omega + \varepsilon} \left| \hat{f}(\lambda, b) \right|^2 |c(\lambda)|^{-2} d\lambda dB \leq
\int_{\|\lambda\| \geq \omega + \varepsilon} \left( \omega^2 + \|\rho\|^2 \right) d\lambda dB \leq \frac{2^{2\sigma}}{\left( \omega + \varepsilon \right)^2 + \|\rho\|^2} \| f \|^2.
\]
(3.2)
It means, that for any \( \varepsilon > 0 \) the function \( \hat{f}(\lambda, b) \) is zero on \( \{ \lambda : \|\lambda\| \geq \omega + \varepsilon \} \times B \).

The statement is proved. \( \square \)

Now we are going to prove the following "density" result.

**Theorem 3.2.** For every \( \omega > 0 \) and every ball \( U \subset X \) restrictions to \( U \) of all functions from \( B_\omega(X) \) are dense in the space \( L_2(U, dx) \).

**Proof.** Indeed, assume that \( \psi \in L_2(U, dx) \) is a function which is orthogonal to all restrictions to \( U \) of all functions from \( B_\omega(X) \). We extend \( \psi \) by zero outside of \( U \). By the Paley-Wiener Theorem the Helgason-Fourier transform \( \hat{\psi}(\lambda, b) \) is holomorphic in \( \lambda \) and at the same time should be orthogonal to all functions from \( L_2(B(0, \omega) \times B; |c(\lambda)|^{-2} d\lambda dB) \), where
\[
\mathcal{B}(0, \omega) = \{ \lambda \in \mathfrak{a}^* : \|\lambda\| \leq \omega \}.
\]
It implies that \( \hat{\psi} \) is zero. Consequently, the function \( \psi \) is zero. It proves the Theorem. \( \square \)

### 4. Uniqueness and Stability

We say that set of points \( M = \{ x_j \} \) is a uniqueness set for \( B_\omega(X) \), if every \( f \in B_\omega(X) \) is uniquely determined by its values on \( M \).

**Theorem 4.1.** If a set \( M = \{ x_j \} \) is a uniqueness set for the space \( B_\omega(X) \), then for any bi-invariant distribution of compact support \( \phi \) every function \( f \in B_\omega(X) \) is uniquely determined by the set of values \( f \ast \phi(x_j) \).

**Proof.** If \( f \in B_\omega(X) \) then because
\[
\hat{f} \ast \phi = \hat{f} \hat{\phi}
\]
the function \( f \ast \phi \) also belongs to \( B_\omega(X) \) and by assumption is uniquely determined by its values \( f \ast \phi(x_j) \). But by the Paley-Wiener Theorem the zero set of the function \( \hat{\phi} \) has measure zero. Thus, the equality
\[
\hat{f}_1 \ast \phi - \hat{f}_2 \ast \phi = \hat{\phi}(\hat{f}_1 - \hat{f}_2) \equiv 0,
\]
implies that \( f_1 = f_2 \) as \( L_2 \)-functions. The statement is proved. \( \square \)

For any uniqueness set \( M \) and any \( \omega > 0 \) the notation \( l_2^\omega(M) \) will be used for a linear subspace of all sequences \( \{ v_j \} \) in \( l_2 \) for which there exists a function \( f \) in \( B_\omega(X) \) such that
\[
f \ast \phi(x_j) = v_j, x_j \in M.
\]
In general \( l_2^\omega(M) \neq l_2 \).
Definition 2. A linear reconstruction method $R$ from a uniqueness set $M$ is a linear operator

$$ R : l_2^2(M) \to B_{\omega}(X) $$

such that

$$ R : \{ f \ast \phi(x_j) \} \to f. $$

The reconstruction method is said to be stable, if it is continuous in topologies induced respectively by $l_2$ and $L_2(X)$.

Theorem 4.1 does not guarantee stability but the next one does.

Theorem 4.2. There exists a constant $c = c(X,N) > 0$ such that for any $\omega > 0$, any $(r,N)$-lattice $Z(\{x_{\mu}\},r,N)$ with

$$ 0 < r < c(\omega^2 + \| \rho \|^2)^{-1/2}, $$

and every bi-invariant distribution $\phi$ with compact support, whose Helgason-Fourier transform $\hat{\phi}$ does not have zeros on $B(0,\omega)$, every function $f \in B_{\omega}(X)$ is uniquely determined and reconstruction method from the set of samples $f \ast \phi(x_j)$ is stable.

Proof. The formulas (2.10) and (3.1) imply, that there exist constants $C, c$ such that for any $\omega > 0$, any $Z = Z(\{x_{\mu}\},r,N)$ with

$$ 0 < r < c(\omega^2 + \| \rho \|^2)^{-1/2}, $$

and every $f \in B_{\omega}(X)$

$$ \| f \| \leq C r^{-d/2} \left( \sum_{x_j \in Z} |f(x_j)|^2 \right)^{1/2}. \quad (4.1) $$

Indeed, by the Theorem 2.2 and the Theorem 3.1 we have

$$ \| f \| \leq C \left\{ r^{d/2} \left( \sum_{x_j \in Z} |f(x_j)|^2 \right)^{1/2} + r^k (\omega^2 + \| \rho \|^2)^{k/2} \| f \| \right\}, \quad (4.2) $$

where $k > d/2$. If we choose

$$ 0 < r < c(\omega^2 + \| \rho \|^2)^{-1/2}, c = C^{-1}, $$

then the inequality (4.2) gives the inequality (4.1) which implies uniqueness. Next, by applying the Plancherel Theorem and the inequality (4.1) to the function $f \ast \phi$ we obtain the inequality

$$ \| f \| = \| \hat{f} \| = \| \hat{\phi}^{-1} \hat{f} \hat{\phi} \| \leq C_1 \| \hat{f} \ast \phi \| \leq $$

$$ Cr^{-d/2} \left( \sum_{x_j \in Z(x_j,r,N)} |f \ast \phi(x_j)|^2 \right)^{1/2}, \quad (4.3) $$

where

$$ C_1 = \left( \inf_{\lambda \in B(0,\omega)} \hat{\phi}(\lambda) \right)^{-1}. $$

This inequality (4.3) implies stability of the reconstruction method from the samples $f \ast \phi(x_j)$. The Theorem is proved. \hfill \Box
5. Reconstruction in terms of frames

In this section we are going to present a way of reconstruction of a function $f \in B_\omega(X)$ from a set of samples of its convolution $f \ast \phi$ in terms of frames in Hilbert spaces [1], [5].

If $\delta_{x_j}$ is a Dirac distribution at a point $x_j \in X$ then according to the inversion formula for the Helgason-Fourier transform we have

$$\langle \delta_{x_j}, \hat{f} \rangle = w^{-1} \int_{a^* \times B} \hat{f}(\lambda, b) e^{i(\lambda + \rho)(A(x_j, b))} |c(\lambda)|^{-2} d\lambda db.$$  

It implies that if $f \in L^2(X)$ then the action on $\hat{f}(\lambda, b)$ of the Helgason-Fourier transform $\hat{\delta}_{x_j}$ of $\delta_{x_j}$ is given by the formula

$$\hat{\delta}_{x_j}(\lambda, b) \rightarrow \langle \delta_{x_j}, \hat{f} \rangle = w^{-1} \int_{a^* \times B} e^{i(\lambda + \rho)(A(x_j, b))} \hat{f}(\lambda, b) |c(\lambda)|^{-2} d\lambda db.$$  

**Theorem 5.1.** There exists a constant $c = c(X, N)$ such that for any given $\omega > 0$, for every $(r, N)$-lattice $Z = Z(\{x_\mu\}; r, N)$ with

$$0 < r < c(\omega^2 + \|\rho\|^2)^{-1/2},$$  

the following statements hold true.

1) The set of functions $\{\hat{\delta}_{x_j}\}$ forms a frame in the space $L^2(\mathcal{B}(0, \omega) \times B; |c(\lambda)|^{-2} d\lambda db)$

and there exists a frame $\{\Theta_j\}$ in the space $B_\omega(X)$ such that every $\omega$-band limited function $f \in B_\omega(X)$ can be reconstructed from a set of samples $\{\delta_{x_j}(f)\}$ by using the formula

$$f = \sum_{x_j \in Z} \delta_{x_j}(f) \Theta_j.$$  

2) If the Helgason-Fourier transform of a compactly supported distribution $\phi$ does not have zeros on the ball $\mathcal{B}(0, \omega)$, then the Helgason-Fourier transforms of the distributions

$$f \rightarrow f \ast \phi(x_j)$$  

form a frame in the space $L^2(\mathcal{B}(0, \omega) \times B; |c(\lambda)|^{-2} d\lambda db)$ and there exists a frame $\{\Phi_j\}$ in the space $B_\omega(X)$ such that every $f \in B_\omega(X)$ can be reconstructed from the samples of the convolution $f \ast \phi$ by using the formula

$$f = \sum_{x_j \in Z} \delta_{x_j}(f \ast \phi) \Phi_j.$$  

**Proof.** An application of the Plancherel formula along with the inequality (4.3) and the second inequality in the Theorem 2.2 give that for any $f \in B_\omega(X)$

$$C_1 \|\hat{f}\|_{\Lambda_2} \leq \left( \sum_{x_j \in Z} |\langle \delta_{x_j}, \hat{f} \rangle|^2 \right)^{1/2} \leq C_2 \|\hat{f}\|_{\Lambda_2},$$  

where $\langle \cdot, \cdot \rangle$ is the scalar product in the space $\Lambda_2 = L^2(\mathcal{B}(0, \omega) \times B; |c(\lambda)|^{-2} d\lambda db)$. The first statement of the Theorem is just another interpretation of the inequalities (5.5).
We consider the so called frame operator
\[ F(\hat{f}) = \sum_j <\delta_{x_j}, \hat{f}> \delta_{x_j}. \]
It is known that the operator \( F \) is invertible and the formula
(5.6) \( \hat{\Theta}_j = F^{-1} \delta_{x_j} \)
gives a dual frame \( \hat{\Theta}_j \) in \( L_2(\mathcal{B}(0,\omega) \times B; |c(\lambda)|^{-2}d\lambda db) \). A reconstruction formula of a function \( f \) can be written in terms of the dual frame as
(5.7) \( \hat{f} = \sum_j <\hat{\Theta}_j, \hat{f}> \delta_{x_j} = \sum_j <\delta_{x_j}, \hat{f}> \hat{\Theta}_j, \)
where inner product is taken in the space \( L_2(\mathcal{B}(0,\omega) \times B; |c(\lambda)|^{-2}d\lambda db) \).

Taking the Helgason-Fourier transform of both sides of the last formula we obtain the formula (5.3) of our Theorem.

The second statement of the Theorem is a consequence of the first one, of the Plancherel formula and of our assumption that \( \hat{\phi} \) does not have zeros in \( \mathcal{B}(0,\omega) \). It is clear, that in the formula (5.4) every \( \Phi_j \) is the inverse Helgason-Fourier transform of the function \( \hat{\Theta}_j/\hat{\phi} \). ⊓⊔

In the classical case when \( X \) is the one-dimensional Euclidean space we have
(5.8) \( \delta_{x_j}(\lambda) = e^{ix_j \lambda}. \)
In this situation the first statement of the Theorem means that the complex exponentials \( e^{ix_j \lambda}, x_j \in Z(\{x_{\mu}\}, r, N) \) form a frame in the space \( L_2([-\omega, \omega]). \)

Note that in the case of a uniform point-wise sampling in the space \( L_2(R) \) this result gives the classical sampling formula
\[ f(t) = \sum f(\gamma n \Omega) \frac{\sin(\omega(t - \gamma n \Omega))}{\omega(t - \gamma n \Omega)}, \Omega = \pi/\omega, \gamma < 1, \]
with a certain oversampling.

6. Reconstructions by using polyharmonic splines on \( X \)

To present our second method of reconstruction we consider the following optimization problem. Although the results of this section are similar to the corresponding results from our paper [17], the presence of the Helgason-Fourier transform allows to make all our constructions much more explicit.

For a given \((r,N)\)-lattice \( Z = Z(\{x_{\mu}\}, r, N) \) find a function \( L^k_\nu \) in the Sobolev space \( H^{2k}(X) \) for which \( L^k_\nu(x_{\mu}) = \delta_{\nu\mu}, x_{\mu} \in Z \), and which minimizes the functional \( u \to \|\Delta^k u\| \). Here the \( \delta_{\nu\mu} \) is the Kronecker delta.

**Theorem 6.1.** For a given \((r,N)\)-lattice \( Z(\{x_{\mu}\}, r, N) \) the following statements are true.
1) The above optimization problem does have a unique solution.
2) For every \( L^k_\nu \) there exists a sequence \( \alpha = \{\alpha_{j,\nu}\} \in l_2 \) such that
(6.1) \( \hat{L}^k_\nu(\lambda,b) = \sum_{x_j \in Z} \alpha_{j,\nu} \hat{\delta}_j(\lambda,b), \)
where $\hat{\delta}_j$ is the Helgason-Fourier transform of the distribution $\delta_j$ and is given by the formula

$$\hat{\delta}_j = e^{(-i\lambda + \rho)(A(x_j, b))}, x_j \in \mathbb{Z}.$$  

Proof. The Theorem 2.2 implies that for a fixed set of values $\{v_j\} \in l_2$ the minimum of the functional

$$u \rightarrow \|\Delta^k u\|$$

is the same as the minimum of the functional

$$u \rightarrow \|\Delta^k u\| + \left( \sum_{x_j \in \mathbb{Z}} |v_j|^2 \right)^{1/2},$$

with a set of constrains $u(x_j) = v_j, x_j \in \mathbb{Z}(\{x_\mu\}, r, N)$.

Since the last functional is equivalent to the Sobolev norm it allows to perform the following procedure.

For the given sequence $v_j = \delta_{j, \nu}$ where $\nu$ is fixed natural number consider a function $f$ from $H^{2k}(X)$ such that $f(x_j) = v_j$. Let $Pf$ denote the orthogonal projection of this function $f$ (in the Hilbert space $H^{2k}(X)$ with natural inner product) on the subspace of functions vanishing on $Z(\{x_\mu\}, r, N)$. Then the function $L^k_{\nu} = f - Pf$ will be the unique solution of the above minimization problem.

The condition that $L^k_{\nu} \in H^{2k}(X)$ is a solution to the minimization problem implies, that $\Delta^k L^k_{\nu}$ should be orthogonal to all functions of the form $\Delta^k h$, where $h \in H^{2k}(X)$ and has the property $h(x_\mu) = 0$ for all $\mu$. This leads to a differential equation

$$\int_X (\Delta^k L^k_{\nu}) \overline{\Delta^k h} dx = \sum_j \alpha_{j, \nu} \overline{h}(x_j),$$

for an $l_2$- sequence $\{\alpha_{j, \nu}\}$. Thus, in the sense of distributions

$$\Delta^{2k} L^k_{\nu} = \sum_j \alpha_{j, \nu} \delta_{x_j}.$$  

(6.3)

Taking the Helgason-Fourier transform of both sides of (6.3) in the sense of distributions we obtain the equation (6.1).

The Theorem is proved. \qed

We will need the following Lemma.

**Lemma 6.2.** If for some $f \in H^{2\sigma}(X), a, \sigma > 0$,

$$\|f\| \leq a \|\Delta^{\sigma} f\|,$$  

then for the same $f, a, \sigma$ and all $s \geq 0, m = 2^l, l = 0, 1, \ldots$,

$$\|\Delta^{s} f\| \leq a^m \|\Delta^{m\sigma+s} f\|,$$  

if $f \in H^{2(m\sigma+s)}(X)$.

Proof. In what follows we use the notation $d\mu(\lambda)$ for the measure $|c(\lambda)|^{-2} d\lambda$. The Plancherel Theorem allows to write our assumption (6.4) in the form

$$\|f\|^2 = \int_{a+ \times B} |\hat{f}(\lambda, b)|^2 d\mu(\lambda)db \leq$$

...
\[
\int_{\mathbb{R}^2} (|\lambda|^2 + \|\rho\|^2)^{2\sigma} \hat{f}(\lambda, b)^2 d\mu(\lambda)\|
\]
where
\[
C_{\sigma}\|
\]
for any \(s \geq 0\). Moreover, if \(s > d/2 + k\) then
\[
\int_{\mathbb{R}^2} (|\lambda|^2 + \|\rho\|^2)^{2\sigma} \hat{f}(\lambda, b)^2 d\mu(\lambda)\|
\]

**Lemma 6.3.** There exists a constant \(C = C(X, N)\), and for any \(\omega > 0\) there exists a \(r_0(X, N, \omega) > 0\) such that for any \(0 < r < r_0(X, N, \omega)\), any \((r, N)\)-lattice \(Z = Z(\{x_i\}, r, N)\), the following inequality holds true

\[
\int \sum_j f(x_j) L_{\omega}^{2d+s} - f\|_{H^s(\mathbb{R})} \leq (Cr^2(\omega^2 + \|\rho\|^2))^{2d} (\omega^2 + \|\rho\|)^s\|f\|,
\]

for any \(s \geq 0\), \(l = 0, 1, \ldots\), and any \(f \in B_{\omega}(X)\). Moreover, if \(s > d/2 + k\) then

\[
\int \sum_j f(x_j) L_{\omega}^{2d+s} - f\|_{C^k_b(X)} \leq (Cr^2(\omega^2 + \|\rho\|^2))^{2d} (\omega^2 + \|\rho\|)^s\|f\|, l = 0, 1, \ldots
\]

where \(C^k_b(X)\) the space of \(k\) continuously differentiable bounded functions on \(X\).
where \( C \) gives from its values \( f \) imply the Lemma. \( \square \)

**Theorem 6.4.** There exists a constant \( c = c(X, N) \), and \( r_0(X, N) > 0 \) such that for any \( 0 < r < r_0(X, N) \), any \( (r, N) \)-lattice \( Z = Z(\{ x_\mu \}, r, N) \), the following inequality holds true

\[
(6.8) \quad \| \sum_j f(x_j)L_j^{2^d+s} - f \|_{H^r(X)} \leq 2(Cr^{2d})^2 \| \Delta^{2^d+s} f \|.
\]

for any \( s \geq 0, l = 0, 1, \ldots \), and any \( f \in H^s(X) \).

Indeed, by the Theorem 2.2, since for \( k = d \) the function \( \sum_j f(x_j)L_j^{2^d+s} \) interpolates \( f \) we have

\[
\| \sum_j f(x_j)L_j^{2^d+s} - f \| \leq Cr^{2d} \| \Delta^{2^d+s}(\sum_j f(x_j)L_j^{2^d+s} - f) \|.
\]

By the Lemma 6.2 we obtain

\[
\| \Delta^s(\sum_j f(x_j)L_j^{2^d+s} - f) \| \leq (Cr^{2d})^2 \| \Delta^{2^d+s}(\sum_j f(x_j)L_j^{2^d+s} - f) \|.
\]

Using the minimization property we obtain (6.7).

If \( s > d/2 + k \) then an application of (6.7) and the Sobolev embedding Theorem gives

\[
(6.9) \quad \| \sum_j f(x_j)L_j^{2^d+s} - f \|_{C^k(X)} \leq (Cr^{2d})^2 \| \Delta^{2^d+s} f \|, l = 0, 1, \ldots,
\]

where \( C^k(X) \) the space of \( k \) continuously differentiable bounded functions on \( X \).

The inequalities (6.7) and (6.8) along with the inequality

\[
\| \Delta^s f \| \leq (\omega^2 + \| \rho \|^2)^s \| f \|, f \in B_\omega(X),
\]

imply the Lemma. \( \square \)

Inequalities (6.6) and (6.7) show that a function \( f \in B_\omega(X) \) can be reconstructed from its values \( f(x_j) \) as a limit of interpolating splines

\[
\sum_j f * \phi(x_j)L_j^{2^d+s}
\]

in Sobolev or uniform norms when \( l \) goes to infinity. By using the Plancherel formula for the Helgason-Fourier transform we obtain the following reconstruction algorithm in which we assume that a reciprocal of the Helgason-Fourier transform of a distribution \( \phi \) is defined almost everywhere and bounded.

**Theorem 6.4.** There exists a constant \( c = c(X, N) \), so that for any \( \omega > 0 \), any \( (r, N) \)-lattice \( Z = Z(\{ x_\mu \}, r, N) \) with

\[
r < (\omega^2 + \| \rho \|^2)^{-1/2},
\]
every function \( f \in B_\omega(X) \) is uniquely determined by the values of \( f * \phi \) at the points \( \{ x_j \} \) and can be reconstructed as a limit of

\[
\sum_j f * \phi(x_j)\Lambda_j^{2^d+s}, l \to \infty,
\]
in Sobolev and uniform norms, where
\[ \Lambda_j^{2d+s} = \frac{L_j^{2d+s}}{\phi}. \]

7. Example: Convolution with spherical average distribution

In this section we assume that the symmetric space \( X = G/H \) has rank one, which means that the algebra Lie \( \mathfrak{a} \) of the abelian component \( A \) in the Iwasawa decomposition \( G = NAK \) has dimension one. For a point \( y \in X \), the \( S(y, \tau), \tau > 0 \), will denote the sphere with center \( y \) and of radius \( \tau \). For a smooth function \( f \) with compact support we define the spherical average
\[ (M_\tau^* f)(y) = S(\tau)^{-1} \int_{S(y, \tau)} f(z) ds(z), \tau > 0, \]
where \( ds \) is the measure on \( S(y, \tau) \) and \( S(\tau) \) is its volume, i.e. the surface integral of \( ds \) over \( S(\tau) \) which is independent on the center.

It is clear, that \( M_\tau^* f \) is a convolution of \( f \) with the distribution
\[ m_\tau^*(f) = S^{-1}(\tau) \int_{S(o, \tau)} f(x) ds(x), \tau > 0, f \in C_0^\infty(X). \]

Since \( X \) is a symmetric space of rank one, the group \( K \) is transitive on every \( S(o, \tau) \) and this fact allows to reduce integration over the sphere \( S(o, \tau) \) to an integral over group \( K \). Because the invariant measure \( dk \) on \( K \) is normalized, it results in the following change of variables formula
\[ (M_\tau^* f)(g \cdot o) = \int_K f(gkz_\tau) dk, \]
for any \( z_\tau \in S(o, \tau), g \in G \).

Next, by using the following formula
\[ \int_X f(x) dx = \int_G f(g \cdot o) dg, f \in C_0^\infty(X) \]
along with Minkowski inequality and the formula (7.1) we obtain
\[ \|M_\tau^*(f)\| \leq \left\{ \int_G \int_K |f(gkz_\tau)|^2 dk dg \right\}^{1/2} \leq \]
\[ \left( \int_K \left\{ \int_G |f(go)|^2 dg \right\}^{1/2} dk = \|f\|, f \in C_0^\infty(X), \right. \]
where we have used the fact that the invariant measure \( dk \) is normalized on \( K \).

Since the set \( C_0^\infty(X) \) is dense in \( L_2(X) \) it implies the following.

**Lemma 7.1.** Operator \( f \to M_\tau^*(f) \) has continuous extension from \( C_0^\infty(X) \) to \( L_2(X) \) which will be also denoted as \( M_\tau^*(f) \) and for this extension
\[ \|M_\tau^*(f)\| \leq \|f\| \]
for all \( f \in L_2(X), \tau > 0. \)
Since the operator $M^\tau$ is a convolution with the distribution $m^\tau$, we have

$$M^\tau(f) = \hat{m}^\tau \hat{f}.$$  

Our goal is to consider even more general operator. We choose an $n \in \mathbb{N} \cup \{0\}$, and consider the operator

$$f \rightarrow M^\tau((-\Delta)^n f).$$  

It is clear, that this operator is a convolution with the distribution $(-\Delta)^n m^\tau$.

The nearest goal is to find the Fourier transform $\hat{(-\Delta)^n m^\tau}$ of the distribution $(-\Delta)^n m^\tau$. Namely, we prove the following.

**Lemma 7.2.** The Helgason-Fourier transform of the distribution $\hat{(-\Delta)^n m^\tau}$ is given by the formula

$$\hat{(-\Delta)^n m^\tau} = \left(\|\lambda\|^2 + \|\rho\|^2\right)^n \hat{m}^\tau,$$

where $\hat{m}^\tau(\lambda, b)$ is the zonal spherical function i.e.

$$\varphi_\lambda(g) = \int_K e^{(i\lambda + \rho)(k g)} dk, g \in G.$$  

**Proof.** Since the operator $(-\Delta)^n$, $n \in \mathbb{N} \cup \{0\}$, can be represented as a convolution with the distribution $\Delta^n \delta_e$, where $\delta_e$ has support at the identity $e \in G$, it is clear, that

$$(-\Delta)^n m^\tau = \left(\|\lambda\|^2 + \|\rho\|^2\right)^n \hat{m}^\tau.$$  

But

$$\hat{m}^\tau(\lambda, b) = m^\tau(e^{(i\lambda + \rho)A(x, b)}).$$  

By the formula (7.1) and by the Harish-Chandra’s formula (2.3) the last integral is

$$S^{-1}(\tau) \int_{S(o, \tau)} e^{(i\lambda + \rho)A(x, b)} ds(x), \tau > 0.$$  

The Lemma is proved.  

The following result is a specification of the Theorem 5.1.

**Theorem 7.3.** There exists a constant $c = c(X, N)$ such that for any given $\omega > 0$, $n \in \mathbb{N} \cup \{0\}$, for every $(r, N)$-lattice $Z(\{x_{\mu}\}, r, N)$ with

$$0 < r < c(\omega^2 + \|\rho\|^2)^{-1/2}$$  

and any

$$\tau < (\omega^2 + \|\rho\|^2)^{-(n+1)/2},$$

the following hold true.
1) There exists a frame $\Phi_j \in B_\omega(X)$ such that for any $f \in B_\omega(X)$ the following reconstruction formula holds true

$$f = \sum_j \delta_{x_j}((-\Delta)^m \ast f) \Phi_j,$$

which means that $f$ can be reconstructed from averages of $(-\Delta)^n f$ over spheres of radius $\tau$ with centers at $x_j \in Z(x_\mu, r, N)$.

2) Every $f \in B_\omega(X)$ is a limit (in Sobolev and uniform norms) of the functions

$$\sum_j f \ast \phi(x_j) \Lambda_j^{2d+s},$$

when $l$ goes to infinity and

$$\Lambda_j^{2d+s} = \frac{L_j^{2d+s}}{\langle \|\lambda\|^2 + \|\rho\|^2 \rangle^n \varphi_\lambda(\exp\tau V)},$$

where $L_j^{2d+s}$ is given by the formula (6.1).

**Proof.** First we are going to show that the following estimate holds true

$$|\langle \|\lambda\|^2 + \|\rho\|^2 \rangle^n \varphi_\lambda(\exp\tau V) - 1| \leq \min \{2(\|\lambda\|^2 + \|\rho\|^2)^n; \tau^2(\|\lambda\|^2 + \|\rho\|^2)^{n+1}\}.$$

Indeed, since $\varphi_\lambda$ is a spherical function, the function

$$\langle \|\lambda\|^2 + \|\rho\|^2 \rangle^n \varphi_\lambda(\exp\tau V) = \Phi_\lambda(\tau)$$

is zonal and $\Delta \Phi_\lambda(\tau)$ can be calculated according to the formula (2.5). By using this fact along with the fact that the spherical function $\Phi_\lambda$ is an eigenfunction the Laplace-Beltrami operator $(-\Delta)$ with the eigenvalue $(\|\lambda\|^2 + \|\rho\|^2)^{n+1}$ we obtain

$$1 - \Phi_\lambda(\tau) =$$

$$-\int_0^\tau (S(\sigma))^{-1} \left( \int_0^\sigma S(\gamma) \left(S^{-1}(\gamma) \frac{d}{d\gamma} \left(S(\gamma) \frac{d\Phi_\lambda(\gamma)}{d\gamma}\right)\right)d\gamma \right)d\sigma =$$

$$\langle \|\lambda\|^2 + \|\rho\|^2 \rangle^{n+1} \int_0^\tau (S(\sigma))^{-1} \left( \int_0^\sigma S(\gamma) \Phi_\lambda(\gamma)d\gamma \right)d\sigma.$$

By using the following change of variables: $\sigma = s\tau, \gamma = s\tau$, and the inequality $|\varphi_\lambda(g)| \leq 1$, we obtain the estimate (7.7).

According to inequalities (7.5) and (7.7) the Helgason-Fourier transform of the distribution $(-\Delta)^m \ast f$ does not have zeros on the interval $[-\omega, \omega]$. Consequently, the first statement of the Theorem is a consequence of the Theorem 5.1. Note, that since the Helgason-Fourier transform of the distribution $(-\Delta)^m m_\tau$ is given by the formula (7.3) the Theorem 5.1 gives that every function $\Phi_j$ is the inverse Helgason-Fourier transform of the function

$$\widehat{\Theta}_j = \frac{1}{\langle \|\lambda\|^2 + \|\rho\|^2 \rangle^n \varphi_\lambda(\exp\tau V)},$$

where functions $\widehat{\Theta}_j$ form a frame which is dual to the frame $\delta_{x_j}$ in the space $L_2([-\omega, \omega] \times B; |c(\lambda)|^{-2}d\lambda d\theta)$.

The second part of the Theorem is a consequence of the Theorem 6.4. □
As a particular case with $\tau = 0$ we have the so called "derivative" sampling, which means that a function can be reconstructed from the values of $\Delta^nf$ as

$$f = \sum_j \delta_{x_j}((-\Delta)^nf)\Phi_j, n \in \mathbb{N} \cup \{0\},$$

for corresponding frame $\Phi_j$.

Another particular case we obtain when $n = 0$. In this situation the formula (6.6) tells that a function can be reconstructed from its averages over spheres of radius $\tau$ with centers at $x_j$.

The case $\tau = 0, n = 0$, corresponds to a point-wise sampling.

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