Hausdorff dimension of affine random covering sets in torus

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Joint work with E. Järvenpää, H. Koivusalo, B. Li and V. Suomala
Introduction

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Introduction: the case of full measure

Dvoretzky covering problem (1956)

What conditions on \((l_n)\) guarantee that \(E = S^1\) almost surely?

Theorem (Shepp 1972)

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E = S^1\text{ almost surely if and only if } \sum_{n=1}^{\infty} \frac{1}{n^2} \exp \left( l_1 + \cdots + l_n \right) = \infty.
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What is the dimension of $E$?

$\cdot$ Fan and Wu (2004): almost surely $\dim H E = \frac{1}{\alpha}$ in the case $\ln l = a/n\alpha$ for some $a > 0$ and $\alpha > 1$.

$\cdot$ Durand (2010): $\dim H E = \inf \{0 < s < 1 \mid \sum_{n=1}^{\infty} l s^n < \infty\}$ almost surely.

$\cdot$ Li, Shieh and Xiao Maarit Järvenpää Hausdorff dimension of affine random covering sets in torus AFRT, Hong Kong 2012
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Higher dimensional case: uniformly ball like sets

For a ball $B = B(x, r) \subset \mathbb{R}^d$ and $0 < s < d$ write $B^s = B(x, r^{\frac{s}{d}})$. 
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Then for any ball $B \subset \mathbb{R}^d$

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- Assume that \(B(x_n, r_n) \subset g_n\) and there is \(C < \infty\) such that \(\rho_n \leq Cr_n\) for all \(n\).

Proposition

Almost surely \(\dim H = \min\{s_0, d\}\), where \(s_0 = \inf\{s \geq 0 : \sum_{n=1}^{\infty} \rho_n^s < \infty\}\).
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Proof: the upper bound

For $s > s_0$ we obtain

$$\mathcal{H}^s(E) \leq \liminf_{N \to \infty} \sum_{n=N}^{\infty} \rho_n^s = 0,$$
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Obviously, $E \supset \limsup_{n \to \infty} B_n$ where $B_n = B(x_n, r_n)$. Consider $s < \min\{s_0, d\}$. Then

$$\sum_{n=1}^{\infty} \mathcal{L}(B_n^s) = K \sum_{n=1}^{\infty} r_n^s \geq KC^{-1} \sum_{n=1}^{\infty} \rho_n^s = \infty.$$
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Hence $\mathcal{L}(\limsup_{n \to \infty} B_n^s) = 1$, implying

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for any ball $B \subset \mathbb{T}^d$. 

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The mass transference principle gives \( \mathcal{H}^s(\limsup_{n \to \infty} B_n) = \infty \), which leads to \( \dim_H E \geq \min\{s_0, d\} \).
Higher dimensional case: main theorem

- Given a contractive linear injection $L : \mathbb{R}^d \rightarrow \mathbb{R}^d$, let $0 < \alpha_d(L) \leq \cdots \leq \alpha_1(L) < 1$ be the singular values of $L$. 
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• Given a contractive linear injection $L : \mathbb{R}^d \to \mathbb{R}^d$, let $0 < \alpha_d(L) \leq \cdots \leq \alpha_1(L) < 1$ be the singular values of $L$.

• For $0 < s \leq d$, define the singular value function by

$$\Phi^s(L) = \alpha_1(L) \cdots \alpha_{m-1}(L) \alpha_m(L)^{s-m+1},$$

where $m$ is the integer such that $m - 1 < s \leq m$. 
Higher dimensional case: main theorem

- Assume that $g_n = \Pi(L_n(R))$ where $R \subset [0, 1]^d$ has non-empty interior and and $\Pi : \mathbb{R}^d \to \mathbb{T}^d$ is the natural covering map.
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Theorem

Almost surely $\dim H = s_0$. 

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with the interpretation \( s_0 = d \) if \( \sum_{n=1}^{\infty} \Phi^d(L_n) = \infty \).
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**Theorem**

Almost surely \( \dim_H E = s_0 \).
Outline of the proof: the upper bound

• Enough to consider the case where $g_n$ is a rectangular parallelepiped.
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- Consider $m - 1 < s < s_0(g_n) \leq m$ where $m$ is an integer.
- Construct an event $\Omega(\infty) \subset \Omega$, having positive probability, and a random Cantor set $C^\omega$ such that $C^\omega \subset E^\omega$ for all $\omega \in \Omega(\infty)$. 

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- This gives \( P(\dim_H E^\omega \geq s) > 0 \).
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- Using potential theoretical methods, verify that $\dim_H C^\omega \geq s$ almost surely conditioned on $\Omega(\infty)$.
- This gives $P(\dim_H E^\omega \geq s) > 0$.
- The Kolmogorov zero-one law implies that $P(\dim_H E \geq s) = 1$. 