Time-Varying Dispersion Integer-Valued GARCH Models

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Abstract

We propose a general class of INteger-valued Generalized AutoRegressive Conditionally Heteroscedastic (INGARCH) processes by allowing time-varying mean and dispersion parameters, which we call time-varying dispersion INGARCH (tv-DINGARCH) models. More specifically, we consider mixed Poisson INGARCH models and allow for dynamic modeling of the dispersion parameter (as well as the mean), similar to the spirit of the ordinary GARCH models. We derive conditions to obtain first and second-order stationarity, and ergodicity as well. Estimation of the parameters is addressed and their associated asymptotic properties are established as well. A restricted bootstrap procedure is proposed for testing constant dispersion against time-varying dispersion. Monte Carlo simulation studies are presented for checking point estimation, standard errors, and the performance of the restricted bootstrap approach. We apply the tv-DINGARCH process to model the weekly number of reported measles infections in North...
Rhine-Westphalia, Germany, from January 2001 to May 2013, and compare its performance to the ordinary INGARCH approach.

**Keywords:** Autocorrelation; Count time series; Overdispersion; Time-varying dispersion parameter; Volatility.

1 Introduction

Modeling count time series data is a challenging and very exciting research topic with applications in many different areas such as epidemiology, sociology, economics, and health science. A well-established methodology, for dealing with count time series data, has been developed under the framework of INteger-valued Generalized AutoRegressive Conditional Heteroscedastic (INGARCH) models, which have been initially studied and explored by Heinen (2003), Ferland et al. (2006), Fokianos et al. (2009), and Fokianos (2011). The INGARCH nomenclature emerges from the fact that for a Poisson distribution (this assumption is imposed in the aforementioned papers), the mean equals its variance, therefore modeling of mean implies modeling of variance like the classic continuous GARCH models introduced by Bollerslev (1986). Consequently, such models are considered in the literature as integer counterpart of the GARCH models although this terminology should be used cautiously because ordinary GARCH processes do not consider dynamics for the conditional mean; for instance, see Fokianos et al. (2009).

A Poisson INGARCH $(p,q)$ (with $p,q \in \mathbb{N}_0$) model $\{Y_t\}_{t \in \mathbb{N}}$ is defined by

$$Y_t | F_{t-1} \sim \text{Poisson}(\lambda_t), \quad \lambda_t \equiv E(Y_t | F_{t-1}) = \beta_0 + \sum_{i=1}^{p} \beta_i Y_{t-i} + \sum_{j=1}^{q} \alpha_j \lambda_{t-j}, \quad (1)$$

where $F_{t-1} = \sigma\{Y_{t-1}, \ldots, Y_0, \lambda_0\}$, $\beta_0 > 0$, $\beta_i \geq 0$ and $\alpha_j \geq 0$, for all $i = 1, \ldots, p$ and $j = 1, \ldots, q$. Since $E(Y_t | F_{t-1}) = Var(Y_t | F_{t-1}) = \lambda_t$, the model specifies dynamic processes for both the conditional mean and variance, as previously stated.

The Poisson INGARCH model has been extensively considered in the literature. However, due to limitations for fitting adequately real count time series data (e.g. not capturing all the
sources of overdispersion although the model itself is overdispersed, inflation or deflation of zeros), other variants have been proposed based on such as the negative binomial INGARCH process by Zhu (2011) and Christou and Fokianos (2014) or the more general mixed Poisson INGARCH models by Christou and Fokianos (2015), Silva and Barreto-Souza (2019), among others. Zero-inflated versions of the Poisson and negative binomial INGARCH models were proposed by Zhu (2012a), while processes dealing with both overdispersion and underdispersion were proposed by Zhu (2012b,c) and Xu et al. (2012).

An alternative process to the linear model for the conditional mean, as in Equation (1), was introduced by Fokianos and Tjøstheim (2011) through the log-linear INGARCH processes. These models cope with both negative and positive autocorrelation function and allow inclusion of co-variates in a straightforward way. Another different approach was recently proposed by Weiß et al. (2022), where a softplus link function is assumed instead of a logarithmic one.

Based on the discussion so far, the Poisson and Negative Binomial distributions are frequently used in applications. In several cases, the Negative Binomial model fits more adequate due to the fact that this distribution depends on a dispersion parameter which facilitates flexible modeling. The dispersion parameter is usually assumed fixed and is estimated by using the Pearson residuals; see Christou and Fokianos (2014) for more. The main goal of this paper is to propose a novel and general class of INGARCH processes based on the mixed Poisson distributions (the Negative Binomial distribution falls in this class) by allowing time-varying mean and dispersion parameters. We call these models time-varying dispersion INGARCH (tv-DINGARCH) models. Hence, we develop methodology and study new models for integer-valued time series, where the assumption of constant dispersion might not be hold. The advantage of the tv-DINGARCH processes over the ordinary INGARCH models will be illustrated in Section 5, where we show that such processes fit better real data whose stylistic facts cannot be dealt with the existing INGARCH models.

For example, in a count regression context, Barreto-Souza and Simas (2016) demonstrated through a data application on the attendance behavior of high school juniors that a constant dispersion assumption can be violated. Generalized linear models allowing regression structures for
both mean and dispersion/precision parameters have been discussed previously by Efron (1986) and Smyth (1989), among others. In this work, we develop further this line of research by studying in detail joint modeling of mean and dispersion parameters in the context of dependent data. We put special emphasis in the case of INGARCH models for count time series and demonstrate the usefulness of such methodological development. Besides exploring the traditional INGARCH models, the proposed class contains the case that admits constant mean and time-dependent variance, similar to the spirit of the ordinary GARCH models. Another important feature is volatility modeling, which is a well-explored topic for the case of continuous valued time series but has been neglected in the context of integer-valued time series. Although the traditional Poisson INGARCH processes consider a time-dependent conditional variance, this in turn is driven by the dynamics of the mean process. The main disadvantage of this approach is that imposes severe restrictions as illustrated in the real data application in this paper. The tv-DINGARCH models relax such an assumption by considering a time-dependent dispersion parameter, therefore also controlling the conditional variance and allowing for additional source of volatility.

In the context of time-varying models for INGARCH processes, Roy and Karmakar (2021) introduced a Poisson INGARCH model with time-varying coefficients and Ratnayake and Samaranayake (2023) study a time-varying INGARCH model based on the zero-inflated Poisson distribution. These authors consider dynamics for the inflation parameter which is assumed to depend on exogenous variables. Furthermore, Doukhan et al. (2022) study in detail first order non-stationary INGARCH models and prove mixing conditions under natural assumptions. Our approach is different in several aspects. First, we consider mixed Poisson distributions (rather than Poisson and zero-inflated Poisson models) which cover a broad class of count distributions that include the Poisson and Negative Binomial. In addition, we consider dynamics that enter through the dispersion parameter. One advantage of the latter feature is that we are able to establish desirable stability properties for the models we consider, such as stationarity and ergodicity, based on e-chain theory.

The paper is organized as follows. In Section 2, we define the class of tv-DINGARCH models and then derive its stochastic properties. We establish conditions ensuring stationarity and ergodicity
of the count processes. Section 3 is devoted to the estimation of the parameters and the derivation of their associated asymptotic properties as well. Aiming at testing constant dispersion in practice, a restricted bootstrap procedure is proposed in Section 4. Monte Carlo simulation studies are presented for checking point estimation, standard errors, and the performance of the restricted bootstrap approach. In Section 5, we apply the tv-DINGARCH process to model the weekly number of reported measles infections in North Rhine-Westphalia, Germany, from January 2001 to May 2013, and compare its performance to the ordinary INGARCH approach. Some technical results and proofs are provided in the Appendix. This paper contains Supplementary Material, which is available upon request.

**Notation:** We say that a random variable $Y$ follows a mixed Poisson (MP) distribution if satisfies the stochastic representation $Y|Z=z \sim \text{Poisson}(\lambda z)$, with $Z$ following some non-negative distribution with $E(Z) = 1$ (standardization) and $\text{Var}(Z) = \phi^{-1}$, for $\lambda, \phi > 0$. We denote $Y \sim \text{MP}(\lambda, \phi)$. In this case, $E(Y) = \lambda$ and $\text{Var}(Y) = \lambda + \lambda^2/\phi$. A random variable $Z$ following a Gamma distribution, with respective shape and scale parameters $a > 0$ and $b > 0$, is denoted by $Z \sim \text{Gamma}(a,b)$, where $E(Z) = a/b$ and $\text{Var}(Z) = a/b^2$. If the mixing variable $Z \sim \text{Gamma}(\phi,\phi)$, then we obtain that $Y$ follows a negative binomial (NB) distribution, i.e. $Y \sim \text{NB}(\lambda, \phi)$.

For an $l$-dimensional vector $x = (x_1, \ldots, x_l)^\top$, let $\|x\|_p = (\sum_{i=1}^l |x_i|^p)^{1/p}$, for $p \in [1, \infty)$, and $\|x\|_\infty = \max_{1 \leq i \leq l} |x_i|$ for $p = \infty$. The induced $p$-norm for a $m \times l$ matrix $C$ is then defined by $\|C\|_p = \max_{x \neq 0} \{\|Cx\|_p/\|x\|_p : x \in \mathbb{R}^l\}$, for $p \in [1, \infty]$.

### 2 The tv-DINGARCH Models

In this section we propose a class of time-varying dispersion INGARCH models by allowing both the mean and the dispersion parameter to depend on lagged valued of the observed series and their past values as follows.

**Definition 2.1.** (tv-DINGARCH processes) A tv-DINGARCH$(p_1,p_2,q_1,q_2)$ process $\{Y_t\}_{t \in \mathbb{N}}$ is de-
fined by $Y_t | \mathcal{F}_{t-1} \sim MP(\lambda_t, \phi_t)$, with

$$
\lambda_t = f(Y_{t-1}, \ldots, Y_{t-p_1}, \lambda_{t-1}, \ldots, \lambda_{t-q_1}), \quad \phi_t = g(Y_{t-1}, \ldots, Y_{t-p_2}, \phi_{t-1}, \ldots, \phi_{t-q_2}),
$$

where $\mathcal{F}_{t-1} = \sigma\{Y_{t-1}, \ldots, Y_0, \phi_0, \lambda_0\}$, and $(\lambda_0, \phi_0)$ denoting some starting value, $f : \mathbb{N}^{p_1} \times (0, \infty)^{q_1} \to (0, \infty)$, and $g : \mathbb{N}^{p_2} \times (0, \infty)^{q_2} \to (0, \infty)$.

Of particular interest to our study and to diverse applications is the negative binomial (NB) model when it is imposed in Definition 2.1. Then, the conditional probability function of $Y_t$ given $\mathcal{F}_{t-1}$ is given by

$$
P(Y_t = y | \mathcal{F}_{t-1}) = \frac{\Gamma(y + \phi_t)}{y! \Gamma(\phi_t)} \left( \frac{\lambda_t}{\lambda_t + \phi_t} \right)^y \left( \frac{\phi_t}{\lambda_t + \phi_t} \right)^{\phi_t}, \quad y \in \mathbb{N}_0.
$$

The negative binomial model will be assumed to prove finiteness of moments and to derive the asymptotic theory for conditional maximum likelihood estimators in what follows. Some of results hold in general for mixed Poisson models, see Theorem 2.1, which states stationarity and ergodicity of time-varying dispersion INGARCH models.

Definition 2.1 is general and some additional assumptions on the function $f$ and $g$ are necessary to study in detail these processes. We consider the first order model ($p_1 = p_2 = q_1 = q_2 = 1$) linear tv-DINGARCH processes, which will be defined in what follows. This particular linear parametric form is a common choice considered in the literature but also enables a thorough study of stability properties of processes defined by Definition 2.1.

**Definition 2.2.** (Linear tv-DINGARCH processes) A linear tv-DINGARCH(1, 1, 1, 1) process $\{Y_t\}_{t \in \mathbb{N}}$ is given as in Definition 2.1 with $f(\cdot)$ and $g(\cdot)$ being linear parametric functions of the forms

$$
\lambda_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 \lambda_{t-1}, \quad \phi_t = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 \phi_{t-1},
$$

where $\beta_0, \alpha_0 > 0$ and $\beta_1, \beta_2, \alpha_1, \alpha_2 \geq 0$.

**Remark 2.1.** Note that we are using $Y_{t-1}$ to model $\phi_t$ in eq. (3), and more generally in Def. 2.1, instead of terms such as $Y_{t-1}^2$ and $|Y_{t-1}|$ considered for the continuous GARCH models and their modification. This is so because $Y_{t-1}$ is non-negative in our case (count time series).
Remark 2.2. Some particular cases are obtained from the linear tv-DINGARCH class defined above. For $\alpha_1 = \alpha_2 = 0$, we obtain the ordinary linear mixed Poisson INGARCH (Christou and Fokianos, 2015; Silva and Barreto-Souza, 2019) models as particular cases. Additionally, by taking $\alpha_0 \to 0^+$, we also obtain the Poisson INGARCH model as a limiting member of our proposed class. Another interesting and novel model arises when $\beta_1 = \beta_2 = 0$. Under this setting, the mean of the INGARCH process is constant and the variance is time-dependent as in the case of ordinary GARCH models (Bollerslev, 1986). Such property does not hold for the standard Poisson INGARCH model (1). In this case, we refer to (3) as the Pure INGARCH (P-INGARCH) process. We use the term “pure” to connect the fact that our model mimics the traditional continuous GARCH models (constant mean and time-varying variance). Hence, our approach is general and encompasses many different models studied earlier in the literature.

Some simulated trajectories of the linear tv-DINGARCH models for some parameter settings are shown in the Supplementary Material.

2.1 Stationarity and Ergodicity

We now explore the stochastic properties of the tv-DINGARCH(1,1,1,1) models. Linearity is a common assumption in the literature as discussed before, which is justified due to successful empirical applications.

Conditions for the existence and stationarity of the process (3) can be established, for example, by following the strategy by Christou and Fokianos (2014), which relies on establishing weak dependence by Doukhan and Wintenberger (2008). We here obtain such desirable properties based on another approach, the e-chains theory (Meyn and Tweedie, 1993), as follows.

The dynamic latent processes $\{\lambda_t\}_{t \geq 1}$ and $\{\phi_t\}_{t \geq 1}$ given in (3) can be rewritten in a matrix form. By defining $\xi_t = (\lambda_t, \phi_t)^\top$, for $t \geq 1$, $\tau = (\beta_0, \alpha_0)^\top$ $A = \begin{pmatrix} \beta_2 & 0 \\ 0 & \alpha_2 \end{pmatrix}$, and $B = \begin{pmatrix} \beta_1 & 0 \\ 0 & \alpha_1 \end{pmatrix}$, we have that $\xi_t = \tau + B(Y_{t-1}, Y_{t-1})^\top + A\xi_{t-1}$. Under this matrix representation, we can adapt the strategy in Liu (2012) to establish the existence and stationarity of our process. In that work, the
author provided stochastic properties of a bivariate Poisson INGARCH model. The key point is to show that \( \{ \xi_t \}_{t \in \mathbb{N}} \) is an e-chain Meyn and Tweedie (1993, Ch.18). The following theorem holds true for any mixed Poisson model including the Negative Binomial model.

**Theorem 2.1.** Let \( \{ Y_t \}_{t \in \mathbb{N}} \) be a tv-DINGARCH process as in (3). If there exists \( p \in [1, \infty) \) such that \( \| A \|_p + 2^{1-1/p} \| B \|_p < 1 \), then the trivariate process \( \{(Y_t, \lambda_t, \phi_t)\}_{t \in \mathbb{N}} \) has a unique stationary and ergodic solution.

The proof is given in the Appendix. The importance of Theorem 2.1 for data analysis and modeling count time series is that these conditions are sufficient to have consistency and asymptotic normality of the conditional maximum likelihood estimators, as will be addressed in the next section. In addition, Theorem 2.1 can be extended to the higher-order linear tv-DINGARCH \((p_1, p_2, q_1, q_2)\) processes, that is consider Def. 2.1 with \( f = g \) the identity function, i.e.

\[
\lambda_t = \beta_0 + \sum_{i=1}^{p_1} \beta_{1i} Y_{t-i} + \sum_{j=1}^{p_2} \beta_{2j} \lambda_{t-j}, \quad \phi_t = \alpha_0 + \sum_{i=1}^{q_1} \alpha_{1i} Y_{t-i} + \sum_{j=1}^{q_2} \alpha_{2j} \phi_{t-j}.
\]

(4)

Then by considering \( \xi_t = (\lambda_t, \phi_t)^\top = \tau + \sum_{j=1}^{q} B_j (Y_{t-j}, Y_{t-j})^\top + \sum_{i=1}^{m} A_i \xi_{t-i} \), where \( \{ A_i \}_{i=1}^{m} \) and \( \{ B_j \}_{j=1}^{q} \) are

\[
A_j = \begin{pmatrix}
\beta_{2j} & 0 \\
0 & \alpha_{2j}
\end{pmatrix}
\quad \text{and} \quad
B_j = \begin{pmatrix}
\beta_{1j} & 0 \\
0 & \alpha_{1j}
\end{pmatrix},
\]

\( m = \max(p_1, p_2) \), and \( q = \max(q_1, q_2) \). Then, following the same steps as in the proof of Theorem 2.1 the following result can be established.

**Corollary 2.1.** Consider the linear linear tv-DINGARCH \((p_1, p_2, q_1, q_2)\) processes (4). Put \( m = \max(p_1, p_2) \), and \( q = \max(q_1, q_2) \). Then, with the same notation as in Thm. 2.1, \( \{(Y_t, \lambda_t, \phi_t)\}_{t \in \mathbb{N}} \) has a unique stationary and ergodic solution if there exists \( p \in [1, \infty) \) such that \( \sum_{i=1}^{m} \| A_i \|_p + 2^{1-1/p} \sum_{j=1}^{q} \| B_j \|_p < 1 \).

We close this section by proving finiteness of the moments (see Appendix for a proof) of the first-order negative binomial INGARCH process, which will be used in the next section to establish the asymptotic normality of the conditional maximum likelihood estimators.
Theorem 2.2. For $\|A\|_1 + \|B\|_1 < 1$, the first-order negative binomial \(\text{INGARCH}\) process \(\{Y_t\}_{t \in \mathbb{N}}\) given by (3) has finite moments of order \(k \in \mathbb{N}\), for an arbitrary \(k\).

3 Estimation and Asymptotic Theory

In this section we study conditional maximum likelihood estimation and provide some numerical experiments for the linear tv-DINGARCH(1, 1, 1, 1) process in Definition 2.2 under the assumption of negative binomial conditional distributions, which is the focus of the present paper. Furthermore, asymptotic results will be established for this model. These results can be generalized to the case of general order linear models but we restrict our attention to the first-order case (3) for ease of presentation.

3.1 Estimators and Asymptotics

Let \(\theta = (\theta_1, \ldots, \theta_6)^\top = (\beta_0, \beta_1, \alpha_0, \alpha_1, \alpha_2)^\top\) be the parameter vector, and \(y_1, \ldots, y_n\) be a realization of a NB tv-DINGARCH process \(\{Y_t\}_{t=1}^n\). The conditional log-likelihood function of \(Y_2, \ldots, Y_n\) given \(Y_1 = y_1\) is given by \(\ell(\theta) = \sum_{t=2}^n \ell_t(\theta)\), where

\[
\ell_t(\theta) = y_t [\log \lambda_t - \log(\lambda_t + \phi_t)] + \phi_t [\log \phi_t - \log(\lambda_t + \phi_t)] + \log \Gamma(y_t + \phi_t) - \log \Gamma(\phi_t) - \log y_t!,
\]

for \(t = 2, \ldots, n\), where we have omitted the dependence of \(\lambda_t\) and \(\phi_t\) on \(\theta\) for simplicity of notation. In practice, it is necessary to set some initial values for \(\lambda_1\) and \(\phi_1\), which are fixed in this paper. More specifically, we get such initial values based on the two first empirical moments of the count time series. This strategy has worked well as demonstrated in the simulated results to be discussed in the sequence. The conditional maximum likelihood estimator (CMLE) of \(\theta\), say \(\hat{\theta}\), is given by \(\hat{\theta} = \arg\max_{\theta \in \Theta} \ell(\theta)\), where \(\Theta = (0, \infty) \times [0, \infty)^2 \times (0, \infty) \times [0, \infty)^2\) denotes the parameter space.

The score function associated to the conditional log-likelihood function is \(U(\theta) = \partial \ell(\theta)/\partial \theta = \sum_{t=2}^n U_t(\theta)\), with

\[
U_t(\theta) = \left( S_{1t} \frac{\partial \lambda_t}{\partial \beta_0}, S_{1t} \frac{\partial \lambda_t}{\partial \beta_1}, S_{1t} \frac{\partial \lambda_t}{\partial \beta_2}, S_{2t} \frac{\partial \phi_t}{\partial \alpha_0}, S_{2t} \frac{\partial \phi_t}{\partial \alpha_1}, S_{2t} \frac{\partial \phi_t}{\partial \alpha_2} \right)^\top,
\]
For Proposition 3.1. To establish the asymptotic normality of the CMLEs, the following proposition is necessary.

\[ S_{1t} = \frac{\phi_t (y_t - \lambda_t)}{\lambda_t (\lambda_t + \phi_t)}, \quad S_{2t} = -y_t - \lambda_t + \log \left( \frac{\phi_t}{\lambda_t + \phi_t} \right) + \Psi(y_t + \phi_t) - \Psi(\phi_t), \]

for \( t = 2, \ldots, n \), where \( \Psi(x) = d\log \Gamma(x)/dx \), for \( x > 0 \), is the digamma function. Explicit expressions for the derivatives involving \( \lambda_t \) and \( \phi_t \) are presented in the Supplementary Material.

To establish the asymptotic normality of the CMLEs, the following proposition is necessary.

**Proposition 3.1.** Let \( U_t(\theta) \) be the \( t \)-th term of the score function, for \( t \geq 2 \). For \( \|A\|_1 + \|B\|_1 < 1 \), \( E(U_t(\theta)) = 0 \) for all \( t \geq 2 \), where the expectation is taken regarding the model with true parameter vector \( \theta \).

**Proof.** Since \( E(Y_t|\mathcal{F}_{t-1}) = \lambda_t \) by definition, we immediately obtain that \( E(S_{1t}(\theta)|\mathcal{F}_{t-1}) = 0 \). We now compute the conditional expectation of \( S_{2t} \) given \( \mathcal{F}_{t-1} \), which involves \( E(\Psi(Y_t + \phi_t)|\mathcal{F}_{t-1}) \).

For \( a > 0 \) and \( |c| < 1 \), we have that

\[
\sum_{y=0}^{\infty} \frac{\Gamma'(y+a)}{y!} c^y = \frac{d}{da} \sum_{y=0}^{\infty} \frac{\Gamma(y+a)}{y!} c^y = \frac{d}{da} \frac{\Gamma(a)}{(1-c)^a} = \frac{\Gamma(a)}{(1-c)^a} \{ \Psi(a) - \log(1 - c) \},
\]

where \( \Gamma'(a) = d\Gamma(a)/da \). Using the above result, it follows that

\[
E(\Psi(Y_t + \phi_t)|\mathcal{F}_{t-1}) = \frac{(\lambda_t \phi_t^{-1} + 1)^{1-\phi_t}}{\Gamma(\phi_t)} \sum_{y=0}^{\infty} \frac{\Gamma'(y+\phi_t)}{y!} \left( \frac{\lambda_t}{\lambda_t + \phi_t} \right)^y = \Psi(\phi_t) - \log \left( \frac{\phi_t}{\lambda_t + \phi_t} \right),
\]

and then we obtain that \( E(S_{2t}(\theta)|\mathcal{F}_{t-1}) = 0 \). Hence, for all \( t \geq 1 \), \( E(U_t(\theta)|\mathcal{F}_{t-1}) = 0 \).

We will now show that \( U_t(\theta) \) is integrable. We have that \( |S_{1t}| \leq |y_t/\lambda_t - 1| \leq y_t/\beta_0 + 1 \). Furthermore, \( \partial \lambda_t / \partial \beta_i \) is a linear combination of \( y_{t-1}, \ldots, y_1 \) and \( \lambda_1 \), for \( i = 0, 1, 2 \). Using these results and under condition \( \|A\|_1 + \|B\|_1 < 1 \), Theorem 2.2 gives that the score function associated with \( \beta_0, \beta_1 \) and \( \beta_2 \) is integrable. We now study the score function associated with \( \alpha_0, \alpha_1 \) and \( \alpha_2 \). Note that

\[
\left| \frac{y_t - \lambda_t}{\lambda_t + \phi_t} \right| \leq \frac{y_t}{\lambda_t + \phi_t} + 1 \leq \frac{y_t}{2} \left( \frac{1}{\beta_0} + \frac{1}{\alpha_0} \right) + 1,
\]

where we have used that fact that \( \frac{1}{a+b} \leq \frac{1}{2} \left( \frac{1}{a} + \frac{1}{b} \right) \) for \( a, b > 0 \). Also, by using that \( \frac{1}{2x} < \log x - \Psi(x) < \frac{1}{x} \) for \( x > 0 \) (Alzer, 1997), we obtain that \( 0 < \log \phi_t - \Psi(\phi_t) - \phi_t^{-1} \leq \alpha_0^{-1} \). Similarly, it follows that \( \mid \Psi(\lambda_t + \phi_t) - \log(\lambda_t + \phi_t) \mid < (\lambda_t + \phi_t)^{-1} \leq \frac{1}{2} \left( \frac{1}{\beta_0} + \frac{1}{\alpha_0} \right) \). Therefore, \( |S_{2t}| \) is bounded above by a linear combination of \( y_t \).

Moreover, \( \partial \phi_t / \partial \alpha_j \) is a linear combination of \( y_{t-1}, \ldots, y_1 \) and \( \phi_1 \), for \( j = 0, 1, 2 \). Again, an application of Theorem 2.2 gives that the score function associated with \( \alpha_0, \alpha_1 \) and \( \alpha_2 \) is also integrable. \( \square \)
Let us now discuss the asymptotic distribution of the CMLE for the NB tv-DINGARCH model. A more detailed technical result will be established in Theorem 3.1. By using Proposition 3.1 and the square integrability of $\mathbf{U}(\theta)$ (this can be demonstrated using very similar arguments to the proof of Proposition 3.1), we can apply the Central Limit Theorem for martingale difference (for instance, see Corollary 3.1 from Hall and Heyde (1980)) to obtain that

$$n^{-1/2} \mathbf{U}(\theta) \xrightarrow{d} N(0, \Omega_1(\theta)), \quad \Omega_1(\theta) \equiv \text{plim}_{n \to \infty} \mathbf{J}_1(\theta),$$

as $n \to \infty$, with

$$\mathbf{J}_1(\theta) = \frac{1}{n} \sum_{t=2}^{n} \text{Var}(\mathbf{U}_t(\theta)|\mathcal{F}_{t-1}) = \frac{1}{n} \sum_{t=2}^{n} \begin{pmatrix} b_t & 0 \\ l_t & d_t \end{pmatrix},$$

$$\theta_* = (\theta_1, \theta_2, \theta_3)^\top; \quad \theta^* = (\theta_4, \theta_5, \theta_6)^\top; \quad \Psi'(x) = d\Psi(x)/dx,$$

$$b_t = \frac{\phi_t}{\lambda_t(\lambda_t + \phi_t)}, \quad l_t = E \left( \Psi^2(Y_t + \phi_t)|\mathcal{F}_{t-1} \right) - \frac{\lambda_t}{\phi_t(\lambda_t + \phi_t)} - \left( \psi(\phi_t) - \log \left( \frac{\phi_t}{\lambda_t + \phi_t} \right) \right)^2,$$

for $t = 2, \ldots, n$, where we have used that $E(Y_t\Psi(Y_t + \phi_t)|\mathcal{F}_{t-1}) = \lambda_t \left\{ \Psi(\phi_t + 1) - \log \left( \frac{\phi_t}{\lambda_t + \phi_t} \right) \right\}$ and that $\Psi(x + 1) = \Psi(x) + x^{-1}$, for $x > 0$.

Then, we apply the Law of Large Numbers for stationary and ergodic sequences to obtain that

$$n^{-1} \frac{\partial \mathbf{U}(\theta)}{\partial \theta} \xrightarrow{p} \Omega_2(\theta), \quad \Omega_2(\theta) \equiv \text{plim}_{n \to \infty} \mathbf{J}_2(\theta),$$

as $n \to \infty$, where

$$\mathbf{J}_2(\theta) = \frac{1}{n} \sum_{t=2}^{n} \sum_{l=2}^{n} E(-\nabla \mathbf{U}_t(\theta)|\mathcal{F}_{t-1}) = \frac{1}{n} \sum_{t=2}^{n} \begin{pmatrix} b_t & 0 \\ l_t & d_t \end{pmatrix},$$

with

$$d_t = \Psi'(\phi_t) - E \left( \Psi'(Y_t + \phi_t)|\mathcal{F}_{t-1} \right) - \frac{\lambda_t}{\phi_t(\lambda_t + \phi_t)}, \quad t = 2, \ldots, n,$$

where $\Psi'(x) = d\Psi(x)/dx$ is the trigamma function. By using the facts that $\Psi'(x) = \frac{\Gamma''(x)}{\Gamma(x)} - \Psi^2(x)$ and

$$\sum_{y=0}^{\infty} \frac{\Gamma''(y+a)}{y!} c^y = \frac{\Gamma(a)}{(1-c)^a} \left\{ \Psi'(a) + [\Psi(a) - \log(1-c)]^2 \right\},$$

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for \( a > 0 \) and \( |c| < 1 \), we obtain that \( l_t = d_t \) for all \( t \). Therefore, \( J_1(\theta) = J_2(\theta) \) and \( \Omega_1(\theta) = \Omega_2(\theta) \).

By combining the above results and using Taylor’s expansion to obtain that
\[
\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \Sigma(\theta)) \quad \text{as} \quad n \to \infty,
\] (7)
with \( \Sigma(\theta) = \Omega_2^{-1}(\theta)\Omega_1(\theta)\Omega_2^{-1}(\theta) = \Omega_1^{-1}(\theta) \). We have that \( \hat{\Sigma} = J_1^{-1}(\hat{\theta}) \) is a consistent estimator for \( \Sigma \). Another two consistent estimators for \( \Sigma \) are \( S_1^{-1}(\hat{\theta}) \) and \( S_2^{-1}(\hat{\theta}) \), where \( S_1(\theta) = n^{-1} \sum_{t=2}^{n} U_t(\theta)U_t(\theta)^\top \) and \( S_2(\theta) = -n^{-1} \sum_{t=2}^{n} \nabla U_t(\theta) \).

There are required additional technical conditions to ensure the consistency and asymptotic normality of the CML estimators, which are provided in the next theorem, where its proof can be found in the Appendix. Before doing that, we have to consider lower and upper bounds for the possible values that the parameters can assume and then we define
\[
\mathcal{B} = \{ \theta : \beta_{i,\text{low}} < \beta_i < \beta_{i,\text{up}}, \alpha_{j,\text{low}} < \alpha_j < \alpha_{j,\text{up}}, \ i, j = 0, 1, 2 \}. \] (8)

**Theorem 3.1.** Let \( \{Y_t\}_{t=1}^{n} \) follow the linear negative binomial tv-DINGARCH(1,1,1,1) process. Assume that \( \|A\|_1 + \|B\|_1 < 1 \) and that the true value of \( \theta \) is an interior point of \( \mathcal{B} \). Then, there exists an open set \( A \subset \mathcal{B} \) such that the conditional log-likelihood function \( \ell(\cdot) \) has a global maximum point on \( A \), say \( \hat{\theta} \), with probability tending to 1 as \( n \to \infty \). Furthermore, \( \hat{\theta} \) is consistent for \( \theta \) and satisfy the asymptotic normality (7).

### 3.2 Monte Carlo Simulation

The finite-sample behaviour of the NB tv-DINGARCH(1,1,1,1) model CMLEs is investigated in the subsection. To that end, we conduct a Monte Carlo study with 1000 replications of trajectories with length \( n = 500, 1000 \) and four parameter configurations. Two configurations are reported in what follows, and additional results are given in the Supplementary Material. The true values of \( \theta \) in our first two configurations (Settings I and II) are \( \theta = (\beta_0, \alpha_0, \beta_1, \beta_2, \alpha_1, \alpha_2)^\top = (15, 0.5, 0.2, 0.25, 0.1, 0.3)^\top \) and \( \theta = (\beta_0, \alpha_0, \beta_1, \beta_2, \alpha_1, \alpha_2)^\top = (3, 0.1, 0.3, 0.15, 0.2, 0.3)^\top \), respectively. Stationarity and uniqueness of the simulated tv-DINGARCH processes is guaranteed by
ensuring that Theorem 2.1 holds when $p = 1$, that is, $\|A\|_1 + \|B\|_1 = \beta_2 + \alpha_2 + \beta_1 + \alpha_1 < 1$.

All results are obtained by employing restricted optimization such that stationarity conditions are satisfied.

Table 1 summarises the results obtained for settings I and II. Empirical means and standard deviation (SD) of the CMLEs are provided by sample size. Results show adequate performance of the estimation method, with empirical Monte Carlo means close to the values and standard deviation decreasing with the increase in sample size, as expected.

| Setting | $\beta_0$ | $\alpha_0$ | $\beta_1$ | $\beta_2$ | $\alpha_1$ | $\alpha_2$ |
|---------|-----------|------------|-----------|-----------|------------|------------|
|         | Mean      | SD         | Mean      | SD         | Mean       | SD         |
| I       |           | $n = 500$  |           | $n = 1000$ |            |            |
|         | Mean      | SD         | Mean      | SD         | Mean       | SD         |
| II      |           | $n = 500$  |           | $n = 1000$ |            |            |

Table 1: Empirical mean and standard deviation (SD) of the Monte Carlo estimates for the NB tv-DINGARCH model parameters under Settings I and II. Results are based on 1000 Monte Carlo replications.

Figure 1 illustrates the asymptotic normality of the CMLEs for setting I. Non-parametric density estimator plots of the standardized parameter estimates are displayed alongside the standard Gaussian density (solid line). Dashed and dotted curves indicate densities estimated from the experiments carried out with $n = 500$ and $n = 1000$, respectively. The density curves of the pa-
Figure 1: Non-parametric density plots of standardized parameter estimates due to Setting I under sample sizes $n = 500$ and $n = 1000$. The solid line corresponds to the standard Gaussian density function.

Parameter estimates are mostly overlapping, but some improvement can be seen with the increase of the sample size. Additional density plots for the other parameter configurations can be found in the Supplementary Material.

4 Testing Constant Dispersion

Testing constant value of the dispersion parameter, i.e. $\phi_t = \alpha_0$ for all $t$ under model (3), is equivalent to testing $H_0 : \alpha_1 = \alpha_2 = 0$ against $H_1 : \alpha_1 \neq 0$ or $\alpha_2 \neq 0$. Note that the null hypothesis belongs to the boundary of the parameter space, which is a non-standard testing problem; for instance, see Self and Liang (1987), Andrews (2001), and Crainiceanu and Ruppert (2004). We develop and compare heuristically two parametric bootstrap methods; the classical or unrestricted, and the restricted bootstrap recently developed by Cavaliere et al. (2016). The first method considers the usual parametric bootstrap (Efron and Tibshirani, 1993) replications based on the unrestricted CMLEs, while the latter uses the CMLEs under $H_0$. Algorithm 1 describes the
estimation of the test’s p-value with $B$ replications of the restricted or unrestricted bootstrap.

**Algorithm 1:** Bootstrap likelihood ratio test of constant ($H_0 : \alpha_1 = \alpha_2 = 0$) versus time-varying dispersion ($H_1 : \alpha_1 \neq 0$ or $\alpha_2 \neq 0$) for a tv NB-INGARCH model. Alternatives to step 3 yield restricted (3A) or unrestricted bootstrap (3B) estimators of the test’s p-value, $p_B$, where $B$ is the number of replications.

**Input:** $Y$ observed count time series data

- $B$ bootstrap replications
- $\alpha$ significance level

1. Obtain $\hat{\theta}_{H_0}$ and $\hat{\theta}$, the model CMLEs under $H_0$ and $H_1$;
2. Compute the observed likelihood ratio $LR = -2(\ell(\hat{\theta}_{H_0}) - \ell(\hat{\theta}))$;
3. For $b \leftarrow 1 : B$ do
   - 3A) $Y_b \sim$ tv NB-INGARCH($\hat{\theta}_{H_0}$); // if restricted bootstrap
   - 3B) $Y_b \sim$ tv NB-INGARCH($\hat{\theta}$); // if unrestricted bootstrap
4. Obtain $\hat{\theta}_{H_0}^b$ and $\hat{\theta}^b$ fitting tv NB-INGARCH models to $Y_b$ under the null and alternative hypothesis;
5. Let $LR^b = -2(\ell(\hat{\theta}_{H_0}^b) - \ell(\hat{\theta}^b))$, the replicated LR statistic;
6. If $p_B = \sum_{b=1}^{B} I\{LR^b > LR\}/B < \alpha$ reject $H_0$;

We use a Monte Carlo simulation study to investigate how these methods achieve the desirable significance levels. Time series from the NB tv-DINGARCH process with 200 time observations are simulated under the null hypothesis using four different (varying) settings of the parameter vector $(\beta_0, \beta_1, \beta_2, \alpha_0)\top$ as follows: (C1) $\beta_0 = 2, \alpha_0 = 1, \beta_1 = 0.4, \beta_2 = 0.3$, (C2) $\beta_0 = 2, \alpha_0 = 1, \beta_1 = 0, \beta_2 = 0$, (C3) $\beta_0 = 3, \alpha_0 = 0.5, \beta_1 = 0.3, \beta_2 = 0.4$, and (C4) $\beta_0 = 3, \alpha_0 = 0.5, \beta_1 = 0, \beta_2 = 0$.

For each configuration, 500 Monte Carlo replications are used to calculate the empirical significance levels by employing the competing methodologies. The number of replications used to estimate bootstrap $p$-values is $B = 500$. Table 2 displays the proportion of times that the restricted and
unrestricted bootstrap procedures rejected the null hypothesis. Parameter configurations are set in a way that C2 and C4 are variations of C1 and C3 that do not include effects on the mean. Additional evidence is provided in the Supplementary Material.

| Configuration | Significance level | Restricted Bootstrap | Unrestricted Bootstrap |
|---------------|--------------------|-----------------------|------------------------|
| C1: $\beta_0 = 2, \alpha_0 = 1$ | 0.05 | 0.046 | 0.000 |
| $\beta_1 = 0.4, \beta_2 = 0.3$ | 0.10 | 0.088 | 0.002 |
| C2: $\beta_0 = 2, \alpha_0 = 1$ | 0.05 | 0.052 | 0.012 |
| $\beta_1 = 0, \beta_2 = 0$ | 0.10 | 0.098 | 0.036 |
| C3: $\beta_0 = 3, \alpha_0 = 0.5$ | 0.05 | 0.064 | 0.002 |
| $\beta_1 = 0.3, \beta_2 = 0.4$ | 0.10 | 0.104 | 0.002 |
| C4: $\beta_0 = 3, \alpha_0 = 0.5$ | 0.05 | 0.042 | 0.022 |
| $\beta_1 = 0, \beta_2 = 0$ | 0.10 | 0.094 | 0.058 |

Table 2: Nominal significance levels produced by the restricted and unrestricted bootstrap hypothesis tests for $H_0: \alpha_1 = \alpha_2 = 0$ for various parameter configurations.

Notably, the restricted parametric bootstrap agrees with the set significance levels, something that occurs under the four parameter configurations investigated in this study, while the unrestricted parametric bootstrap does not provide satisfactory results by underestimating the significance levels. These results show the importance of considering a restricted bootstrap for testing constant dispersion in the tv-DINGARCH models.

5 Measles Data Analysis

The linear negative binomial tv-DINGARCH model is now applied to modeling the weekly number of reported measles infections in North Rhine-Westphalia, Germany. The series is observed between January 2001 and May 2013 (646 observations), publicly available in the R package
Figure 2 displays \( \{Y_t\}_{t=1}^{646} \) on the left, and the series autocorrelation function on the right. Figure 2: On the left, weekly cases of measles reported in North Rhine-Westphalia, Germany, between January 2001 and May 2013. The autocorrelation function of the series is shown on the right.

Fitting of the NB tv-DINGARCH model will be compared to the fit of ordinary NBINGARCH model with an identical mean time series structure. The purpose of this exercise is to assess how a time-varying dispersion changes the model adequacy to this real data. To this end, we will evaluate goodness-of-fit and predictions for each model. \( \mathcal{M}_{tv} \) will denote the NB tv-DINGARCH model while \( \mathcal{M}_{ord} \) denotes the ordinary case (which assumes a constant dispersion).

Summary fits for \( \mathcal{M}_{tv} \) and \( \mathcal{M}_{ord} \) models are reported in Table 3. Conditional maximum likelihood estimates of the parameters, their standard errors (SE) in parenthesis and associated approximate 95% confidence intervals are provided. Uncertainty quantification relies on 500 replications of parametric bootstrap for both \( \mathcal{M}_{tv} \) and \( \mathcal{M}_{ord} \). Inspection of Table 3 shows that \( \mathcal{M}_{ord} \) and \( \mathcal{M}_{tv} \) are in close agreement concerning the INGARCH mean structure as \( \beta_0, \beta_1 \) and \( \beta_2 \) are in resemblance.

To compare between models, we compute model information criteria and perform the constant dispersion test developed in Section 4. The values of AIC and BIC are respectively 2670.568 and 2697.393 for \( \mathcal{M}_{tv} \) and 2797.216 and 2815.099 for \( \mathcal{M}_{ord} \), both supporting time-varying dispersion. The likelihood ratio test for \( H_0 : \alpha_1 = \alpha_2 = 0 \) versus \( H_1 : \alpha_1 \neq 0 \) or \( \alpha_2 \neq 0 \) is in line with the
### Table 3: Conditional maximum likelihood estimates, standard errors (se), and 95% confidence intervals for log-linear tv-DINGARCH and ordinary log-linear INGARCH models applied to weekly counts of measles in Germany.

| Parameter | tv-DINGARCH | INGARCH |
|-----------|-------------|---------|
|           | Estimate (se) | 95% CI | Estimate (se) | 95% CI |
| $\beta_0$ | 0.259 (0.048) | (0.164, 0.353) | 0.194 (0.127) | (0.000, 0.444) |
| $\beta_1$ | 0.579 (0.034) | (0.512, 0.645) | 0.583 (0.090) | (0.407, 0.759) |
| $\beta_2$ | 0.342 (0.039) | (0.265, 0.419) | 0.390 (0.094) | (0.205, 0.574) |
| $\alpha_0$ | 0.775 (0.057) | (0.663, 0.886) | 0.736 (0.110) | (0.521, 0.952) |
| $\alpha_1$ | 0.079 (0.007) | (0.065, 0.093) | $-$ | $-$ |
| $\alpha_2$ | 0.000 (0.010) | (0.000, 0.020) | $-$ | $-$ |

In addition, we consider Probability Integral Transform (PIT) plots (Czado et al., 2009), an approach that enables the comparison of count data models via their predictive distributions. A model providing a good fit to the data in this aspect will render a PIT plot resembling a uniform distribution, where major deviations typically indicate problems of overdispersion or underdispersion of the model’s predictive distribution. These are reported in Figure 3 for both models, and while that of $M_{tv}$ is near the uniform as desired, the upside-down U shaped PIT of $M_{ord}$ points to a non adequate fit.

It is often of interest to practitioners to select between models according to their forecasting power. To that end, we consider a forecasting exercise that explores one-step-ahead (OSA) prediction from $M_{tv}$ and $M_{ord}$. This is done in a recursive manner in order to provide multiple OSA predictions from each model. We start by defining the initial training data from the beginning of the study until October 2004 (week 4) which contains 200 observations. Both models are fitted to the training set and used to predict the next week’s counts (week 1 of November 2004) via the...
conditional median and mode of the distributions. The conditional (or predictive) distributions are Negative-Binomial($\hat{\mu}_{t+1}, \hat{\phi}_{t+1}$), or Negative-Binomial($\hat{\mu}_{t+1}, \hat{\phi}$) respectively for $M_{tv}$ and $M_{ord}$ and $t = 200$ at this step. Once the prediction is obtained, we add week 1 of November 2004 to the training set, refit both models and gather the new OSA predictions. Proceeding until the end of the study period gives a total of 446 predictions from each model. Pseudocode describing the steps to this prediction exercise is given in Algorithm 2.

Algorithm 2 is presented in terms of the $M_{tv}$ model but works similarly for $M_{ord}$. In this case, $M_{ord}$ is fitted in step two and the predictions in 5A) and 5B) take in the (fixed) dispersion parameter estimated in step 2. By employing the Algorithm with $M_{tv}$ and $M_{ord}$, their predictions are summarized via the root mean forecasting error (RMSFE). Let $n_0$ denote the time point chosen to start the prediction exercise; in our case $n_0 = 200$ the RMSFE of the forecasting step $t$ is

$$\text{RMSFE}_t = \sqrt{\frac{1}{t-n_0} \sum_{s=n_0+1}^{t} (Y_s - \hat{Y}_s)^2},$$

where $Y_s$ and $\hat{Y}_s$, are the observed and predicted counts, respectively. Calculation of RMSFE, from
Algorithm 2: Recursive algorithm for obtaining $\hat{Y}$, the one-step-ahead (OSA) predicted values of $Y_{[n_0+1:n]}$. The training data at iteration $s$, $Y(s)$, is incremented and the model is refitted to obtain the OSA forecast $\hat{Y}_{s+1}$. Steps 5A) and 5B) provide alternative prediction methods via the mean or median.

**Input:** $Y$ observed count trajectory of length $n$

$n_0$ starting point of prediction exercise

0. $s \leftarrow n_0$;

while $s < n$ do

1. $Y(s) \leftarrow Y_{[1:s]}$: // train data

2. Fit the tv-NBINGARCH model to $Y(s)$;

3. From 2, gather the CMLEs $\hat{\theta}(s)$ and the fitted $\hat{\lambda}(s)$ and $\hat{\phi}(s)$ of step $s$;

4. Obtain the OSA mean $\hat{\mu}_{s+1} = \beta_0 + \beta_1 Y_s + \beta_2 \hat{\mu}_s$ and $\phi_{s+1} = \alpha_0 + \alpha_1 Y_s + \alpha_2 \phi_s$;

5A) $\hat{Y}_{s+1} = \hat{\mu}_{s+1}$ // prediction via mean

5B) $\hat{Y}_{s+1} = \text{qbinom}(0.5, \text{size} = \phi_{s+1}, \text{mu} = \hat{\mu}_{s+1})$ // prediction via median

$s = s + 1$; // increment step

end

**Output:** $\hat{Y}$ vector of $(n - n_0)$ OSA predictions.

the 446 total predictions from $M_{tv}$ and $M_{ord}$, yields Figure 4. On the left, $\hat{Y}_s$ is the mode of the conditional distribution and on the right it is taken as the median. Prediction by the median yields a smaller prediction error in comparison to the mode, and in both cases the tv-DINGARCH model produces the smallest RMSFE values.

This empirical illustration on the weekly number of measles cases in North Rhine-Westphalia, Germany, demonstrated that the time-varying dispersion is a promising and important extension of the ordinary INGARCH processes that achieved improvement in terms of goodness-of-fit and
Figure 4: RMSFE of predictions obtained with the fit of ordinary (dashed lines) and time-varying dispersion (solid lines) INGARCH models to the weekly count of measles in North Rhine-Westphalia, Germany. On the left, predicted values are the mode of the predictive distributions, whereas the median is taken as the point on the right.

6 Conclusions

We proposed a class of time-varying dispersion INGARCH (tv-DINGARCH) models and explored stochastic properties such as stationarity and ergodicity. Estimation of parameters was addressed through conditional maximum likelihood estimation (CMLE) and its associated asymptotic theory was established. Monte Carlo simulations were conducted to evaluate the performance of the CMLE. Moreover, we developed bootstrap methodologies to test for constant dispersion and showed via simulated studies that the restricted bootstrap is preferred over the unrestricted parametric one. We analyzed the weekly number of reported measles cases in north Rhine-Westphalia, Germany, from January 2001 to May 2013, and found that the tv-DINGARCH approach delivers much better forecasting.
results regarding goodness-of-fit and prediction when compared to the ordinary INGARCH model.

A log-linear version of our model allowing for the inclusion of covariates deserves future re-
search. A first-order log-linear tv-DINGARCH process \( \{Y_t\}_{t \geq 1} \) allowing for covariates/exogenous
time series is given as in Definition 2.1 with \( \lambda_t \equiv \exp(\mu_t) \) and \( \phi_t \equiv \exp(\nu_t) \), where

\[
\mu_t = \beta_0 + \beta_1 \log(Y_{t-1} + 1) + \beta_2 \mu_{t-1} + \delta^T X_t, \quad \nu_t = \alpha_0 + \alpha_1 \log(Y_{t-1} + 1) + \alpha_2 \nu_{t-1} + \gamma^T W_t,
\]

\( \beta_i, \alpha_i \in \mathbb{R} \) for \( i = 0, 1, 2 \), with \( X_t \) and \( W_t \) covariates with associated real-valued coefficients \( \delta \) and \( \gamma \). The ordinary log-linear INGARCH model (Fokianos and Tjøstheim, 2011) is obtained as a
particular case from the log-linear tv-DINGARCH process by taking \( \alpha_1 = \alpha_2 = 0 \) and \( \gamma = 0 \). This
topic will be investigated in a future communication. Finally, it is worth mentioning that other
mixed Poisson distributions rather than negative binomial can be used for our tv-DINGARCH
formulation such as Poisson-inverse Gaussian distribution.

Appendix

A-1 Proof of Theorem 2.1

The key ingredient to establish the desired result is to prove that \( \{\xi_t\}_{t \in \mathbb{N}} \) is an e-chain, that is, for
any continuous function \( w \) with compact support on \((0, \infty)^2\) and for every \( \epsilon > 0 \), there exists \( \eta > 0 \)
such that, for \( x, z \in (0, \infty)^2 \), \( \|x - z\| < \eta \) implies that \( |E(w(\xi_k)|\xi_0 = x) - E(w(\xi_k)|\xi_0 = z)| < \epsilon \)
\( \forall k \geq 1 \), where \( \| \cdot \| \) is some norm.

Let \( w \) be a continuous function with compact support on \((0, \infty)^2\) and assumed to be bounded
\( |w| < 1 \) without loss of generality. Consider \( k = 1, x, z \in (0, \infty)^2 \), and \( \epsilon > 0 \) arbitrary. Denote by
\( f_{\xi^{mp}}(\cdot) \) the probability function of a mixed Poisson distribution with mean \( \lambda \) and variance \( \lambda + \lambda^2 / \phi \),
where \( \xi = (\lambda, \phi)^T \). It follows that

\[
|E(w(\xi_1)|\xi_0 = x) - E(w(\xi_1)|\xi_0 = z)| \leq \\sum_{y=0}^{\infty} |w(\tau + B(y, y)^T + A\mathbf{z})f^\text{mp}_x(y) - w(\tau + B(y, y)^T + A\mathbf{z})f^\text{mp}_z(y)| \\leq \\sum_{y=0}^{\infty} f^\text{mp}_x(y)|w(\tau + B(y, y)^T + A\mathbf{z}) - w(\tau + B(y, y)^T + A\mathbf{z})| + (A-1) \\sum_{y=0}^{\infty} |w(\tau + B(y, y)^T + A\mathbf{z})| f^\text{mp}_x(y) - f^\text{mp}_z(y)|. (A-2)
\]

We now find an upper bound for the term \( (A-2) \). Denote by \( f^\text{pois}_\lambda(\cdot) \) the probability function of a Poisson distribution with mean \( \lambda \). From the mixed Poisson stochastic representation, we have that \( f^\text{mp}_x(y) = E(f^\text{pois}_{\lambda_1}(y)) \), where \( Z_1 \) is the associated latent random variables with distribution depending on \( x_2 \). Similar representation holds for \( f^\text{mp}_z(y) \) in terms of an associated latent factor \( Z_2 \) with distribution depending on \( z_2 \). Using this and the recalling the fact that \( |g| < 1 \), we obtain that

\[
\sum_{y=0}^{\infty} |w(\tau + B(y, y)^T + A\mathbf{z})| f^\text{mp}_x(y) - f^\text{mp}_z(y)| = \\
\sum_{y=0}^{\infty} |E(f^\text{pois}_{\lambda_1}(y)) - E(f^\text{pois}_{\lambda_2}(y))| \leq E \left( \sum_{y=0}^{\infty} |f^\text{pois}_{\lambda_1}(y) - f^\text{pois}_{\lambda_2}(y)| \right). (A-3)
\]

By using inequality (A.1) from Wang et al. (2014) (see also Liu (2012)), we have that \( \sum_{y=0}^{\infty} |f^\text{pois}_{\lambda_1}(y) - f^\text{pois}_{\lambda_2}(y)| \leq 2 \left( 1 - \exp \{-|x_1 Z_1 - x_2 Z_2|\} \right) \). Hence, (A-3) is bounded above by

\[
E \left( \sum_{y=0}^{\infty} |f^\text{pois}_{\lambda_1}(y) - f^\text{pois}_{\lambda_2}(y)| \right) \leq 2E \left( 1 - e^{-|x_1 Z_1 - x_2 Z_2|} \right) \leq \\
2E \left( 1 - e^{-|x_1 - x_2|(Z_1 + Z_2)} \right) \leq 2E \left(|x_1 - x_2||(Z_1 + Z_2)| = 2|x_1 - x_2| \leq 2\|x - z\|_p, \right.
\]

for \( p \in [1, \infty] \), where we have used in the second inequality the fact that \( |ab - cd| \leq |a - c|(b + d) \) for \( a, b, c, d > 0 \). In the third inequality, we used that \( 1 - e^{-x} \leq x \) for all \( x > 0 \). Finally, the fourth inequality follows, for instance, by (Liu, 2012, pp. 108).

The upper bound for the term given in (A-1) follows exactly as discussed in Liu (2012): choose \( \epsilon' > 0 \) and \( \eta > 0 \) small enough such that \( \epsilon' + \frac{8\eta}{1 - \|A\|_p} < \epsilon \) and \( \|x - z\|_p < \eta \) implying \( |w(x) - w(z)| < \)
\( \epsilon' \), with \( p \in [1, \infty] \). In this part, we are using the fact that \( \|A\|_p < 1 \) for some \( p \in [1, \infty] \), which follows from the assumption that there exists \( p \) such that \( \|A\|_p + 2^{1/p}\|B\|_p < 1 \).

By combining the above results, we obtain that
\[
|E(w(\xi_1)|\xi_0 = x) - E(w(\xi_1)|\xi_0 = z)| < \epsilon' + 2\|x - z\|_p.
\]
For general \( k \geq 2 \), the result follows by using mathematical induction, exactly as done in Chapter 4 from Liu (2012), and therefore it is omitted. With the e-chain property established for the bivariate process \( \{\xi_t\} \) and under the assumption of existence of \( p \in [1, \infty] \) such that \( \|A\|_p + 2^{1/p}\|B\|_p < 1 \), following the same steps that proofs of Proposition 4.2.1(b) and Proposition 4.3.1 from Liu (2012), we obtain the desired result.

\section*{A-2 Proof of Theorem 2.2}

Let \( \{Y_t\}_{t \in \mathbb{N}} \) be a first-order negative binomial INGARCH process. Under condition \( \|A\|_1 + \|B\|_1 < 1 \), Theorem 2.1 holds (for \( p = 1 \)) and gives us that \( \{Y_t\}_{t \in \mathbb{N}} \) has a unique stationary and ergodic solution. Moreover, this process is a Markov chain. Therefore, we will now show that Tweedie’s criterion is satisfied for a test function to show the finiteness of arbitrary moments stated in the lemma (Meyn and Tweedie, 1993).

Consider the test function \( V(x, z) = 1 + x^k + z^k \), for \( x, z > 0 \) and arbitrary \( k \in \mathbb{N} \). It follows that
\[
E(V(\lambda_t, \phi_t)|\lambda_{t-1} = \lambda, \phi_{t-1} = \phi) = 1 + E(\lambda^{k}_{t-1} = \lambda, \phi_{t-1} = \phi) + E(\phi^{k}_{t-1} = \lambda, \phi_{t-1} = \phi)
\]
\[
= 1 + E((\beta_0 + \beta_1 Y + \beta_2 \lambda)^k) + E((\alpha_0 + \alpha_1 Y + \alpha_2 \phi)^k), \quad (A-4)
\]
where \( Y \sim NB(\lambda, \phi) \). From proof of Lemma (A.1) from Christou and Fokianos (2014), we have that the \( d \)-th moment of \( Y \sim NB(\lambda, \phi) \), for \( d \geq 2 \), satisfies the recursive equation
\[
E(Y^d) = \lambda \left\{ 1 + \sum_{j=1}^{d-1} \left[ \binom{d-1}{j} \frac{1}{\phi^2} \right] E(Y^j) \right\}, \quad (A-5)
\]
with \( E(Y) = \lambda \).

From (A-5), we immediately obtain that \( E(Y^d) = \lambda^d(1 + 1/\phi) + O(\lambda^{d-1}) \) as \( \lambda \rightarrow \infty \). Using this
result, it follows that
\[
E \left( (\beta_0 + \beta_1 Y + \beta_2 \lambda)^k \right) = \sum_{j=0}^{k} \binom{k}{j} (\beta_0 + \beta_2 \lambda)^{k-j} \beta_1^j E(Y^j) = \sum_{j=0}^{k} \binom{k}{j} (\beta_2 \lambda)^{k-j} \beta_1^j \lambda^j + O(\lambda^{k-1})
\]
\[= \lambda^k (\beta_1 + \beta_2)^k + O(\lambda^{k-1}).\]

Similarly, we obtain that
\[
E \left( (\alpha_1 + \alpha_1 Y + \alpha_2 \phi)^k \right) = (\alpha_1 \lambda + \alpha_2 \phi)^k + O(\lambda^{k-1}).
\]

Hence, we can express the conditional expectation (A-4) as
\[
E \left( V(\lambda_t, \phi_t) | \lambda_{t-1} = \lambda, \phi_{t-1} = \phi \right) = V(\lambda, \phi) \frac{1 + \lambda^k (\beta_1 + \beta_2)^k + (\alpha_1 \lambda + \alpha_2 \phi)^k + O(\lambda^{k-1})}{1 + \lambda^k + \phi^k}. \quad (A-6)
\]

We will now analyze the behavior of the terms multiplying \( V(\lambda, \phi) \) in (A-6) for large values of \( \lambda \) and \( \phi \). First, note that \( \lambda^* = \frac{-\beta_0}{1 - \beta_2} \) and \( \phi^* = \frac{-\alpha_0}{1 - \alpha_2} \) are the smallest reachable points of \( \lambda_t \) and \( \phi_t \), respectively, for all \( t \). Hence, we consider \( \lambda \in [\lambda^*, \infty) \) and \( \phi \in [\phi^*, \infty) \), both bounded away from zero. Consider the polar coordinates transformation \( \lambda = r \cos \delta \) and \( \phi = r \sin \delta \), for \( r > 0 \) and \( \delta \in (0, \pi/2) \) such that \( \lambda \geq \lambda^* \) and \( \phi \geq \phi^* \). In terms of polar coordinates, we have that
\[
1 + \frac{\lambda^k (\beta_1 + \beta_2)^k + (\alpha_1 \lambda + \alpha_2 \phi)^k}{1 + \lambda^k + \phi^k} = 1 + r^k \frac{[ (\beta_1 + \beta_2)^k \cos \delta + (\alpha_1 \cos \delta + \alpha_2 \sin \delta)^k ]}{1 + r^k (\cos \delta + \sin \delta)^k}
\]
\[\rightarrow \frac{(\beta_1 + \beta_2)^k \cos \delta + (\alpha_1 \cos \delta + \alpha_2 \sin \delta)^k}{\cos \delta + \sin \delta} \leq \frac{(\beta_1 + \beta_2)^k \cos \delta + \max\{\cos \delta, \sin \delta\} (\alpha_1 + \alpha_2)^k}{\cos \delta + \sin \delta}
\]
\[\leq (\beta_1 + \beta_2)^k + (\alpha_1 + \alpha_2)^k \leq \beta_1 + \beta_2 + \alpha_1 + \alpha_2
\]
\[= \|A\|_1 + \|B\|_1 < 1,
\]
as \( r \to \infty \), where the last inequality follows from lemma’s assumption implying that \( \beta_1 + \beta_2 < 1 \) and \( \alpha_1 + \alpha_2 < 1 \). Further, the remaining term converges to 0 as \( r \to \infty \):
\[
O(\lambda^{k-1}) \frac{1}{1 + \lambda^k + \phi^k} = O(r^{k-1})/O(r^k) = O(r^{-1}).
\]

Therefore, there exist real constants \( \kappa_1 \in (0, 1), \kappa_2 > 0 \), and \( L > 0 \) such that (A-4) is bounded above as follows:
\[
E \left( V(\lambda_t, \phi_t) | \lambda_{t-1} = \lambda, \phi_{t-1} = \phi \right) \leq (1 - \kappa_1) V(\lambda, \phi) + \kappa_2 I \{ (\lambda, \phi) \in G \},
\]

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where \( G = \{ \lambda \geq \lambda^*, \phi \geq \phi^* : \lambda = r \cos \delta, \phi = r \sin \delta, 0 < r < L, \delta \in (0, \pi/2) \} \).

In other words, Tweedie’s criterion is satisfied and we conclude that the \( k \)-th moment of \( \lambda_t \) and \( \phi_t \) are finite for arbitrary \( k \in \mathbb{N} \). Now, using the fact that \( Y_t | \mathcal{F}_{t-1} \sim \text{NB}(\lambda_t, \phi_t) \) and Eq. (A-5), we have that the \( k \)-th conditional moment of \( Y_t \) given \( \mathcal{F}_{t-1} \) is a polynomial on \( \lambda_t \) and \( 1/\phi_t \); note that \( 1/\phi_t \leq 1/\alpha_0 \) for all \( t \). Since the \( l \)-th moment of \( \lambda_t \) is finite for arbitrary \( l \in \mathbb{N} \), we obtain that the unconditional \( k \)-th moment of \( Y_t \) is finite for all \( k \geq 1 \) as well.

**Lemma A-1.** Let \( \mathcal{B} \) as defined in (8). Under the linear negative binomial tv-DINGARCH(1,1,1,1) model with \( \| \mathbf{A} \|_1 + \| \mathbf{B} \|_1 < 1 \), there exists a sequence of random variables \( \{ M_n \}_{n \in \mathbb{N}} \) such that

\[
\max_{i,j,k=1,2,...,6} \sup_{\theta \in \mathcal{B}} \left| \frac{1}{n} \frac{\partial^3 \ell_t(\theta)}{\partial \theta_i \partial \theta_j \partial \theta_k} \right| \leq M_n,
\]

where \( M_n \xrightarrow{p} m \) as \( n \to \infty \), with \( m < \infty \) and \( \theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) = (\beta_0, \beta_1, \beta_2, \alpha_0, \alpha_1, \alpha_2)^\top \).

**Proof.** We will show that the result holds for the third derivative of \( \ell(\cdot) \) with respect to \( \beta_1 \) and also \( \alpha_1 \). The remaining cases follow the very same steps as these two and therefore are assumed.

The third derivative of \( \ell_t(\theta) \) with respect to \( \beta_1 \) is given by

\[
\frac{\partial^3 \ell_t(\theta)}{\partial \beta_1^3} = \frac{\partial^2 S_{1t}}{\partial \phi_t^2} \left( \frac{\partial \lambda_t}{\partial \beta_1} \right)^3,
\]

where

\[
\frac{\partial^2 S_{1t}}{\partial \phi_t^2} = \frac{2\phi_t}{\lambda_t^2(\lambda_t + \phi_t)} \left[ \frac{y_t(2\lambda_t + \phi_t) - \lambda_t^2(2\lambda_t + \phi_t)}{\lambda_t(\lambda_t + \phi_t)} - (y_t - \lambda_t) \right]
\]

and

\[
\frac{\partial \lambda_t}{\partial \beta_1} = \sum_{j=1}^{t-1} \beta_2^{-1} y_{t-j}.
\]

It is straightforward to obtain that there are constants \( c_1, c_2 > 0 \) that do not depend on \( \theta \) such that \( \left| \frac{\partial^3 \ell_t(\theta)}{\partial \beta_1^3} \right| \leq c_2 y_t + c_1 \). Also, \( \left| \frac{\partial \lambda_t}{\partial \beta_1} \right| \leq \sum_{j=1}^{t-1} \lambda_t^{j-1} y_{t-j} \equiv \mu_{1t} \). Therefore, \( \left| \frac{\partial^3 \ell_t(\theta)}{\partial \beta_1^3} \right| \leq c_2 y_t \mu_{1t}^3 + c_1 \mu_{1t}^3 = M_{1t} \), where \( M_{1t} \) does not depend on \( \theta \). Under the assumption \( \| \mathbf{A} \|_1 + \| \mathbf{B} \|_1 < 1 \) and using Theorem 2.2, the Law of Large Numbers for stationary and ergodic sequences can be applied to obtain that \( n^{-1} \sum_{t=2}^{n} M_{1t} \xrightarrow{p} m_1 \) as \( n \to \infty \), where \( m_1 \) is a finite constant, as similarly argued by Fokianos et al. (2009) (proof of Lemma 3.4 of their Supplementary Material).

The case for \( \alpha_1 \) is more involving. We have that

\[
\frac{\partial^3 \ell_t(\theta)}{\partial \alpha_1^3} = \frac{\partial^2 S_{2t}}{\partial \phi_t^2} \left( \frac{\partial \phi_t}{\partial \alpha_1} \right)^3,
\]

where

\[
\frac{\partial^2 S_{2t}}{\partial \phi_t^2} = -2 \frac{y_t - \lambda_t}{(\lambda_t + \phi_t)^3} + \frac{\lambda_t^2 + 2\lambda_t \phi_t}{\phi_t^2(\lambda_t + \phi_t)^2} + \Psi''(y_t + \phi_t) - \Psi''(\phi_t) \tag{A-7}
\]
and
\[ \frac{\partial \phi_t}{\partial \alpha_1} = \sum_{j=1}^{t-1} \alpha_2^{j-1} y_{t-j}. \]

Inequality (15) from Guo and Qi (2013) gives that
\[ |\Psi''(x)| < \frac{1}{x} + \frac{2}{x^3}, \text{ for } x > 0. \]
Hence, it follows that
\[ |\Psi''(y_t + \phi_t)| < \frac{1}{(y_t + \phi_t)^2} + \frac{2}{(y_t + \phi_t)^3} \leq \frac{1}{\phi_t^2} + \frac{2}{\phi_t^3} \leq \frac{1}{\alpha_0^2} + \frac{2}{\alpha_0^3} \leq \frac{1}{\alpha_{0,inf}^2} + \frac{2}{\alpha_{0,inf}^3}. \]
Similarly, \[ |\Psi''(\phi_t)| < \frac{1}{\alpha_{2,inf}^2} + \frac{2}{\alpha_{3,inf}^3}. \] The modulus of the remaining terms in (A-7) are bounded above (as done for the third derivative involving $\beta_1$) by a linear function of $y_t$ with positive coefficients do not depend on $\theta$. Moreover,
\[ \left| \frac{\partial \phi_t}{\partial \alpha_1} \right| \leq \sum_{j=1}^{t-1} \alpha_2^{j-1} y_{t-j} = \mu_{2t}. \]

From the above results, we conclude that there exist constants $c_3 > 0$ and $c_4 > 0$ that do not depend on $\theta$ such that
\[ \left| \frac{\partial^3 \ell_t(\theta)}{\partial \alpha_1^3} \right| \leq c_3 y_t \mu_{2t}^3 + c_4 \mu_{2t}^3 \equiv M_{2t}. \]
Now, by arguing exactly as done for the case $\beta_1$ above, we obtain that $n^{-1} \sum_{t=2}^{n} M_{2t} \xrightarrow{p} m_2$ when $n \to \infty$, where $m_2$ is a finite constant.

\section*{A-3 Proof of Theorem 3.1}

We will show that the Conditions (A1), (A2), and (A3) from Lemma 3.1 by Jensen and Rahbek (2004) are satisfied, which give us the desired results stated in the theorem. As argued before for the general case, we use Proposition 3.1 and the Central Limit Theorem for martingale difference to establish the weak convergence involving the score function in (5). The existence and finiteness of the matrix $\Omega_1(\theta)$ follow from Theorem 2.2 that provides the finiteness of the moments of all orders for the linear NB tv-DINARCH process under the assumption that $\|A\|_1 + \|B\|_1 < 1$, which is in force. Therefore, Condition (A1) holds.

Moreover, under the assumption $\|A\|_1 + \|B\|_1 < 1$, we obtain from Theorem 2.1 that $\{Y_t\}_{t \in \mathbb{N}}$ is a stationary and ergodic sequence. Then, the Law of Large Numbers for stationary and ergodic sequences gives us that (6) is valid, with the existence and finiteness of the matrix $\Omega_2(\theta)$ being ensured again by Theorem 2.2. That is, Condition (A2) from Lemma 3.1 by Jensen and Rahbek (2004) is satisfied. Finally, Condition (A3) has been established in Lemma A-1.
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