On complexity of structure and substructure connectivity, component connectivity and restricted connectivity of graphs

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Abstract

The connectivity of a graph is an important parameter to measure its reliability. k-restricted connectivity of a graph G is the minimum cardinality of a set of vertices in G, if exists, whose deletion disconnects G and makes each remaining component of G more than k vertices. In contrast, k-component connectivity of G is the minimum cardinality of a set of vertices in G, if exists, whose deletion makes the resulting graph have at least k components. When determining the structure (substructure) connectivity, to make the graph disconnected, the vertices deleted induce a specific subgraph of G. As generalizations of the concept of connectivity, structure (substructure) connectivity, component connectivity and restricted connectivity have been extensively studied from the combinatorial point of view. Very little result is known about

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their complexity other than the recently obtained computational complexity of \(k\)-restricted edge-connectivity. In this paper, we zero in on characterizing the complexity of structure and substructure connectivity, component connectivity and \(k\)-restricted connectivity of graphs, showing that they are all NP-complete.

**Key words:** Structure connectivity; Component connectivity; \(k\)-restricted connectivity; Reliability; NP-complete

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1 Introduction

The graph we considered throughout this paper is simple and undirected. Let \(G = (V, E)\) be a graph, where \(V\) is the vertex-set of \(G\) and \(E\) is the edge-set of \(G\). The degree of a vertex \(v\) is the number of incident edges, written \(d_G(v)\) or \(d(v)\) when the context is clear. The minimum degree of \(G\) is \(\delta(G)\) and the maximum degree is \(\Delta(G)\). For any subset \(X \subset V\), the closed neighborhood of \(X\) is defined to be all neighbors of any vertex \(x \in X\) together with \(X\), denoted by \(N[X]\), while the open neighborhood of \(X\) is \(N[X] \setminus X\), denoted by \(N(X)\). If \(X = \{x\}\), then we write \(N[x]\) and \(N(x)\), respectively. The subgraph induced by \(X\) is denoted by \(G[X]\). A matching of \(G\) is a set of independent edges of \(G\). For other standard graph-theoretical terminologies and notations not defined here, we follow Bondy [1].

The connectivity and edge-connectivity of a graph play important roles in measuring network reliability and have been widely studied in the literature from a combinatorial point of view. Generally, the higher the connectivity, the more reliable the network is. With the rapid development of multiprocessor systems, it is meaningful and necessary to accurately evaluate their reliability. However, the classical connectivity and edge-connectivity always underestimate the resilience of large networks as these two parameters tacitly assume that all vertices adjacent to, or all edges incident to, the same vertex can potentially fail simultaneously. This is practically impossible in large networks.

To address the deficiencies of connectivity and edge-connectivity, several generalizations of them were introduced by graph theorists and computer scientists. In the seminal paper [15], Harary introduced the concept of conditional connectivity to evaluate the reliability of the graph that each component of the graph has a given property \(P\) when the faulty graph is disconnected. As a kind of conditional connec-
tivity, the component connectivity was introduced in [4] and [29] independently. The $k$-component connectivity (resp. $k$-component edge-connectivity) of a non-complete graph $G$ is the minimum number of vertices (resp. edges) whose deletion results in the graph with at least $k$ components. The $k$-component edge-connectivity problem, also referred as minimum $k$-way cut problem, has important applications in VLSI design, task allocation in distributed computing systems and graph strength, see [17] and the references therein. In particular, Goldschmidt and Hochbaum [13] showed that $k$-component edge-connectivity problem is NP-hard if $k$ is an input parameter but admits a polynomial time algorithm if $k$ is a constant.

In 1988, Esfahanian and Hakimi [8] proposed the definition of restricted edge-connectivity. An edge-cut $S \subseteq E(G)$ is called a restricted edge-cut if there exists no isolated vertices in $G - S$. The restricted edge-connectivity is the minimum cardinality over all restricted edge-cuts $S$. Motivated by this definition, Fàbrega and Fiol [10, 11] introduced the following definitions to assess connectivity of a graph, which have different flavours with component connectivity. Given a graph $G$ and a non-negative integer $k$, the $k$-restricted connectivity (resp. $k$-restricted edge-connectivity) of $G$, denoted by $\kappa_k(G)$ (resp. $\lambda_k(G)$), is the minimum cardinality of a set of vertices (resp. edges) in $G$, if exists, whose deletion disconnects $G$ and makes each remaining component of $G$ more than $k$ vertices. In particular, a connected graph $G$ is called $\lambda_k$-connected if $\lambda_k(G)$ exists.

Recently, as another variant of traditional connectivity, Lin et al. [20] introduced structure and substructure connectivity to evaluate the fault tolerance of a network from the perspective of a single vertex, as well as some special structures of the network. Let $\mathcal{F} = \{F_1, F_2, \cdots, F_t\}$ be a set of pairwise disjoint connected subgraphs of $G$ and let $V(\mathcal{F}) = \bigcup_{i=1}^{t} V(F_i)$. Then $\mathcal{F}$ is a subgraph-cut of $G$ provided that $G - V(\mathcal{F})$ is disconnected or trivial. Let $H$ be a connected subgraph of $G$, then $\mathcal{F}$ is an $H$-structure-cut if $\mathcal{F}$ is a subgraph-cut, and each element in $F$ is isomorphic to $H$. The $H$-structure connectivity of $G$, written $\kappa(G; H)$, is the minimum cardinality over all $H$-structure-cuts of $G$. Similarly, if $\mathcal{F}$ is a subgraph-cut and each element of $\mathcal{F}$ is isomorphic to a connected subgraph of $H$, then $\mathcal{F}$ is called an $H$-substructure-cut. The $H$-substructure connectivity of $G$, written $\kappa^*(G; H)$, is the minimum cardinality over all $H$-substructure-cuts of $G$.

From definitions of component connectivity and restricted connectivity, it can be seen that the common point is that vertex deletion results in the remaining graph no longer connected, and the difference is the restriction on components of
the remaining graph. Whereas structure (substructure) connectivity requires that vertices deleted induce a specific subgraph of $G$. These variations of connectivity have been widely studied for some famous interconnection networks, for example, structure and substructure connectivity [18,20,21,25,28], component connectivity [2,16,19,21,22], $k$-restricted connectivity [3,19,23]. In particular, Montejano and Sau [26] showed that it is NP-hard to determine the exact value of $\lambda_k(G)$ even for $\lambda_k$-connected graphs, implying that determining $k$-restricted edge-connectivity of graphs is NP-hard. Interestingly, the computational complexity of $k$-restricted connectivity is still open. Thus, a natural question arise: what are computational complexities of structure and substructure connectivity, component connectivity and $k$-restricted connectivity in general graphs? In this paper, we study these problems.

The rest of the paper is organized as follows. In Sections 2 and 3, we prove NP-completeness of the structure connectivity and substructure connectivity problems respectively. In Sections 4 and 5, we show that the component connectivity problem is NP-complete and $k$-restricted connectivity problem is NP-complete for all integers $k \geq 3$, respectively. Finally, we conclude this paper in Section 6.

2 Structure connectivity

3-dimensional matching, 3DM for short, is one of the most standard NP-complete problems to prove NP-complete results. An instance of 3DM consists of three disjoint sets $R$, $B$ and $Y$ with equal cardinality $q$, and a set of triples $T \subseteq R \times B \times Y$. For convenience, let $W = R \cup B \cup Y$. The question is to decide whether there is a subset $T_1 \subseteq T$ covering $W$, that is, $|T_1| = q$ and each element of $W$ occurs in exactly one triple of $T_1$. This instance can be associated with a bipartite graph $G_b$ as follows. Each element of $W$ and each triple of $T$ is represented by a vertex of $G_b$. There is an edge $wt$ between an element $w \in W$ and a triple $t \in T$ if and only if the element is a member of the triple.

It has been proved in [6,7] that 3DM is NP-complete when each element of $W$ appears in only two or three triples of $T$, i.e., each vertex in the partite set $W$ of $G_b$ has degree two or three only. We shall show that the decision problem of structure connectivity is NP-complete by reducing from 3DM stated previously.

To this end, we state the following decision problem.

**Problem:** The $H$-structure connectivity of a graph.

**Instance:** Given a nonempty graph $G = (V, E)$, a subgraph $H$ of $G$ and a positive
integer $q < |V|$.

**Question:** Is $\kappa(G; H) \leq q$?

Now we are ready to prove the following theorem.

**Theorem 2.1.** The $H$-structure connectivity is NP-complete when $H = K_{1,M}$ for any integer $M \geq 5$.

**Proof.** Obviously, the structure connectivity problem is in NP, because we can check in polynomial time whether a set of disjoint $K_{1,M}$s is a structure cut or not. It remains to show that the structure connectivity is NP-hard when $H = K_{1,M}$ for any integer $M \geq 5$. We prove this argument by reducing 3DM to it.

Let $G_b$ be an instance of 3DM defined previously. For convenience, let $T = \{t_k|1 \leq k \leq |T|\}$ and $W = \{w_l|1 \leq l \leq 3q\}$. We make a further assumption that each vertex in the partite set $W$ of $G_b$ has degree two or three only.

Now we construct a graph $G = (V, E)$ from $G_b$ as follows (see Fig. 1).

![Fig. 1. The graph $G$ constructed for proving NP-completeness of the structure connectivity.](image)

Set

$$V_j = \{v_{ij}^j|1 \leq i \leq |T|\} \text{ for each } j = 1, 2, \cdots, M - 3,$$
\[ \hat{V} = \bigcup_{j=1}^{M-3} V_j, \]
\[ V' = \{v_1, v_2, \ldots, v_{(M+1)(M-3)|T|}\}, \]
\[ U = \{u_1, u_2, \ldots, u_{3qM}\}, \quad \text{and} \]
\[ U' = \{u_1', u_2', \ldots, u_{3qM}'\}. \]

The vertex set of \( G \) is \( V = V(G_b) \cup \hat{V} \cup V' \cup U \cup U' \). The subgraph induced by \( V \) is clearly the graph \( G_b \). The subgraph induced by \( V_j \) is an independent set of size \( |T| \) for each \( j = 1, 2, \ldots, M - 3 \), and the subgraph induced by \( V' \) is \( K_{(M+1)(M-3)|T|} \). Similarly, the subgraph induced by \( U \) is \( 3q \) vertex disjoint \( K_M \) and the subgraph induced by \( U' \) is \( K_{3qM} \).

The edge set of \( G \) is
\[ E = E(G_b) \cup 3qE(K_M) \cup E(K_{3qM}) \cup E(K_{(M+1)(M-3)|T|}) \cup E_t \cup E_u \cup E_z \cup E_p, \]
where \( E_t = \{\{t_i, t_j\} | 1 \leq i \leq |T|, 1 \leq j \leq M - 3\} \), \( E_u = \{\{u_l, u_{l+M}\} | 1 \leq l \leq 3q\} \), \( E_z = \{\{u_k, u_k'\} | 1 \leq k \leq 3qM\} \) and \( E_p \) contains all edges between \( V' \) and \( \hat{V} \). In precise, each vertex \( v_i', 1 \leq i \leq |T|, 1 \leq j \leq M - 3 \), is joined to exactly \( M + 1 \) distinct vertices in \( V' \).

We show that \( G_b \) has a 3DM \( T_1 \subseteq T \) covering \( W \) if and only if \( \kappa(G; K_{1,M}) \leq q \). First suppose that \( G_b \) has a subset \( T_1 \subseteq T \) covering \( W \), that is, \( |T_1| = q \) and each element of \( W \) occurs in exactly one triple of \( T_1 \). We show that \( \kappa(G; K_{1,M}) \leq q \). Clearly, \( d_G(t_i) = M \) for each vertex \( t_i \in T_1 \) and hence the subgraph of \( G \) induced by \( N[t_i] \) is isomorphic to \( K_{1,M} \). Thus, \( \mathcal{F} = \{G[N[t_i]] | t_i \in T_1\} \) is a structure-cut of \( G \) with \( |\mathcal{F}| \leq q \).

Next suppose that \( \mathcal{F} \) is a structure-cut of \( G \) with \( |\mathcal{F}| \leq q \). We shall show that \( G_b \) has a subset \( T' \) of \( T \) covering \( W \) with \( |T'| \leq q \). Recall that each vertex in the partite set \( W \) of \( G_b \) has degree two or three only, we may assume that the number of vertices with degree two (resp. three) in \( W \) of \( G_b \) is \( m \) and (resp. \( n \), and consequently, \( 2m + 3n = 3|T| \), which implies that \( 3q = |W| \geq |T| > q \).

Since each element of \( \mathcal{F} \) is a graph isomorphic to \( K_{1,M} \), we focus on the center vertex of \( K_{1,M} \in \mathcal{F} \). Let \( S \) be the set of center vertices of all \( K_{1,M} \in \mathcal{F} \), and let \( S \cap V(G_b) = S' \) and \( S'' = S \cap (\hat{V} \cup V' \cup U \cup U') \). Since each vertex in \( W \) has degree less than \( M \), any vertex in \( W \) can not be center vertices of \( K_{1,M} \in \mathcal{F} \), that is, \( S \cap W = \emptyset \). We claim that \( \mathcal{F} \) covers \( W \). Suppose on the contrary that \( \mathcal{F} \) does not cover \( W \), we shall show that \( G - V(\mathcal{F}) \) is connected. Thus, two cases arise.
Case 1. \( S'' = \emptyset \). So \( S = S' \). Then there exists an edge \( w_x t_y \in E(G - V(\mathcal{F})) \) such that \( w_x \in W \) and \( t_y \in T \). It is not hard to see that \( G - V(\mathcal{F}) \) is connected, contradicting that \( \mathcal{F} \) is a structure-cut of \( G \). Hence, all vertices in \( W \) are covered by \( V(\mathcal{F}) \). Clearly, components in \( \mathcal{F} \) restricted on \( G_b \) form a 3-dimensional matching of \( G_b \).

Case 2. \( S'' \neq \emptyset \). Note that \( d_G(t_i) = 3 \) for any vertex \( t_i \in T \) (\( 1 \leq i \leq |T| \)) and \( d_G(w_j) = 2 \) or 3 for any vertex \( w_j \in W \) (\( 1 \leq j \leq 3q \)). By the structure of \( G \), each vertex in \( S'' \) (as the center vertex of a \( K_{1,M} \in \mathcal{F} \)) can subvert at most one vertex in \( G_b \). Similarly, each vertex in \( S' \) can subvert precisely one vertex in \( T \) together with three vertices in \( W \).

Obviously, when a vertex in \( \hat{V} \) is subverted, it is no longer connected to any vertex in \( V' \) or disappears. Similarly, after subverting vertices in \( U \), each clique \( K_M \) in \( G[U] \) is either joined to \( K_{3qM} \) or disappears.

Since \( S'' \neq \emptyset \), we have \(|S'| < q < |T|\). This implies that there exists an edge \( w_x t_y \in E(G - V(\mathcal{F})) \) such that \( t_y \) is connected to a vertex in \( K_{(M+1)(M-3)|T|} \) via a vertex in \( \hat{V} \) and \( w_x \) is connected to \( K_{3qM} \) via a clique \( K_M \). Thus, \( G - V(\mathcal{F}) \) is connected.

This complete the proof. \( \square \)

3 Substructure connectivity

A dominating set of \( G \) is a subset \( D \subseteq V \) such that every vertex not in \( D \) is adjacent to one member of \( D \). The dominating number is the number of vertices in a smallest dominating set. The decision version of the dominating set problem is:

Given a graph \( G = (V, E) \) and a positive integer \( d \leq |V| \), is there a dominating set \( D \) of size not greater than \( d \) such that for each vertex \( u \in V - D \) there is a vertex \( v \in D \) with \( uv \in E \)? It is a classical NP-complete problem showed as problem GT2 in p. 190 [12].

We present the decision problem of the substructure connectivity as follows.

Problem: The substructure connectivity of a graph.

Instance: Given a nonempty graph \( G = (V, E) \) with \( \Delta(G) = M \), a subgraph \( H \) of \( G \) and a positive integer \( k < |V| \).

Question: Is \( \kappa^*(G, H) \leq k \)?

Theorem 3.1. The \( H \)-substructure connectivity is NP-complete when \( H = K_{1,M} \).
Proof. Obviously, the substructure connectivity problem is in NP, because we can check in polynomial time whether a set of disjoint subgraphs of $K_{1,M}$ is a substructure cut. It remains to show that the substructure connectivity is NP-hard when $H = K_{1,M}$. We prove this argument by reducing dominating set problem to it.

Given a graph $G = (V, E)$ with $\Delta(G) = M$, we construct a graph $G' = (V', E')$ from $G$ as follows (see Fig. 2).

Set

\[ X = \{x_{ij} | 1 \leq i \leq |V|, 1 \leq j \leq |V|\}, \]
\[ Y = \{y_{ij} | 1 \leq i \leq |V|, 1 \leq j \leq |V|\}, \]
\[ X' = \{x'_{ij} | 1 \leq i \leq |V|, 1 \leq j \leq |V|\}, \]
\[ Y' = \{y'_{ij} | 1 \leq i \leq |V|, 1 \leq j \leq |V|\}, \]
\[ V = \{v_1, v_2, \ldots, v_{|V|}\}. \]

The vertex set of $G'$ is $V' = V \cup X \cup Y \cup X' \cup Y'$. The subgraph $G'[X]$ (resp. $G'[Y]$) induced by $X$ (resp. $Y$) are $|V|$ disjoint complete graphs of order $|V|$, and the subgraph $G'[X']$ (resp. $G'[Y']$) induced by $X'$ (resp. $Y'$) is a complete graph of order $|V|^2$. The subgraph induced by $V$ is clearly the graph $G$. For each $i \in \{1, 2, \ldots, |V|\}$,
there exists a substructure cut $X$ and $Y$ with center vertex $x$. Thus, $G$ has a substructure-cut of size at most $k$. Let $F = \{x\}$ be the set of center vertices of all complete subgraphs of $K_{1,M}$, and let $S = F$.

**Case 1.** $S \subseteq V$. That is, $S = S'$. Note that each vertex of $V$ is contained in at most one of $F_i \in F$ for some $i$, $1 \leq i \leq k$. We claim that the set of all center vertices of $F_i \in F$ forming a dominating set $D$ of $G$ with $|D| \leq k$. Suppose on the contrary that $D$ is not a dominating set of $G$, then there is a vertex $u \in V \setminus D$ not dominated by any vertex in $D$. It implies that $u \notin V(F)$. By our construction, it can be seen that $u$ is connected to one of complete subgraphs induced by $X$ (resp. $Y$). Clearly, $G' - V(F)$ is connected, which is a contradiction. Thus, the claim holds.

**Case 2.** $S \not\subseteq V$. Then $S' \neq \emptyset$. If $S'$ is a dominating set of $G$, then $|S'| < k$, and we are done. So we assume that $S'$ is not a dominating set of $G$. Let $X \subseteq V$ be the set of vertices in $G$ not in $V(F)$, that is, $X \cap V(F) = \emptyset$.

Since there are $|V|$ disjoint complete graphs of size $|V|$ in $G'[X]$ (resp. $G'[Y]$), each $F_i$ ($1 \leq i \leq k$) with center vertex in $S \setminus S'$ could subvert at most one vertex in $G$, that is, $|S \setminus S'| \geq |X|$, otherwise, $G' - V(F)$ is connected. Deletion of some $F_i$'s in any $K_{|V|}$ in $G'[X]$ or $G'[Y]$ will either result in a complete subgraph or make it disappear. Since $V(F)$ covers all vertices in $V \setminus X$, it implies that $S'$ dominates all vertices in $V \setminus X$. Therefore, $S' \cup X$ is clearly a dominating set of $G$ with size not greater than $k$.

This completes the proof.

**Remark 1.** Subversion of a vertex means that the entire closed neighborhood of the
vertex is deleted from the graph. A set of vertices $B$ is called a *subversion strategy* if all the vertices in $N[B]$ are deleted from the graph $G$. If the resulting graph $G - N[B]$ is empty, complete or disconnected, then $B$ is called an *effective subversion strategy*. The *neighbor connectivity* of $G$ is the minimum size of an effective subversion strategy $B$, namely $|B|$. In [3], Doty et al. showed that the decision problem of neighbor connectivity is NP-complete. In substructure connectivity problem, as a special restriction, we consider $K_{1,M}$-substructure connectivity. At each step we can delete a vertex and, optionally, some neighbors (not necessarily all) from the remaining graph. Whereas, in neighbor connectivity problem, we must delete a vertex and all its neighbors recursively in the resulting graph. Thus, neighbor connectivity problem is a special case of $K_{1,M}$-substructure connectivity.

4 Component connectivity

To begin with this section, we state the decision problem of 2-partition connectivity as follows.

*Problem:* 2-partition connectivity of an arbitrary graph (2PCP).

*Instance:* Given a nonempty graph $G = (V, E)$ and a positive integer $k$.

*Question:* Is there a set $S$ of vertices (edges) and three unspecified vertices $u, v$ and $w$ not in $S$ such that $|S| \leq k$ and $G - S$ separates $u, v$ and $w$?

The complexity of 2PCP is characterized in the following theorem.

**Theorem 4.1** [27]. 2PCP is NP-complete.

Next we are ready to present the decision problem of component connectivity as follows.

*Problem:* Component connectivity (edge-connectivity) of an arbitrary graph.

*Instance:* Given a nonempty graph $G = (V, E)$ and a positive integer $k \geq 3$.

*Question:* Is there a set $S$ of vertices (or edges) such that $|S| \leq k$ and $G - S$ contains at least $k$ components?

Notice that when $k = 2$, the component connectivity problem states that after deletion of a set $S$ of vertices (or edges) from $G$, $G - S$ has at least two components, which is actually the same as the traditional connectivity (or edge-connectivity) problem in graphs. The connectivity [9] and edge-connectivity [30] problems have already been shown to be polynomial-time solvable. Thus, the component connectivity problem is, clearly, in P when $k = 2$. Therefore, we only consider its
computational complexity for \( k \geq 3 \). In [13], Goldschmidt and Hochbaum showed that component edge-connectivity problem (called the minimum \( k \)-way cut problem in the article) is NP-hard if \( k \geq 3 \) is an input parameter but admits a polynomial time algorithm if \( k \) is regarded as a constant.

By the definitions of 2PCP and component connectivity problem, we know that \( S \) is a set of vertices (edges) of a graph \( G \) with \( |S| \leq k \) such that there exist three unspecified vertices \( u, v \) and \( w \) not in \( S \) separating \( u, v \) and \( w \) if and only if \( S \) is a component cut of \( G \) leaving \( G - S \) with at least three components containing \( u, v \) and \( w \) respectively. Hence, it is straightforward to derive the complexity of the decision problem of component connectivity (edge-connectivity), where the result of the complexity of component edge-connectivity has already been proved in [13].

**Corollary 4.2.** Component connectivity (edge-connectivity) problem is NP-complete.

The above corollary implies that determining minimum number of vertex/edge deletions that break a network into three or more components is NP-complete.

## 5 \( k \)-restricted connectivity

In view of the fact that the NP-completeness of \( k \)-restricted edge-connectivity, we consider the complexity of \( k \)-restricted connectivity problems in this section. To characterize the complexity of \( k \)-restricted connectivity problem, we present the following decision problem.

**Problem:** \( k \)-restricted connectivity with three specified vertex sets (RCTSVS).

**Instance:** Given a nonempty graph \( G = (V, E) \), a positive integer \( k \) and three distinct vertex sets \( X, Y \) and \( Z \) with \( |X| = |Y| = |Z| = k - 1 \).

**Question:** Is there a set \( S \) of vertices \( (S \cap (X \cup Y \cup Z) = \emptyset) \) such that \( S \) separates \( X, Y \) and \( Z \) (i.e. \( X, Y \) and \( Z \) lie in different components of \( G - S \) and each component of \( G - S \) has order at least \( k \)?)

To characterize the computational complexity of RCTSVS, we state the following decision problem called NON-MONOTONE 2-3SAT.

**Problem:** NON-MONOTONE 2-3SAT.

**Instance:** Given a set \( U \) of variables, a collection \( C \) of clauses over \( U \) such that each clause \( c \in U \) contains two or three variables and for each clause with three variables, it contains at least one negated variable and one un-negated variable.

**Question:** Is there a satisfying truth assignment for \( C \)?
Clearly, NON-MONOTONE 2-3SAT is a variant of well-known 3SAT, and its complexity is characterized in the following theorem.

**Theorem 5.1** [27]. NON-MONOTONE 2-3SAT is NP-complete.

**Theorem 5.2.** RCTSVS is NP-complete for any integer \( k \geq 2 \).

*Proof.* Clearly, the problem is in NP, because we can check in polynomial time whether a set \( S \) of vertices such that \( G - S \) separates \( X, Y \) and \( Z \) and each of which is contained in a component of \( G - S \) of order at least \( k \). It remains to show that the RCTSVS is NP-hard for any specific integer \( k \geq 3 \). We prove this argument by transforming NON-MONOTONE 2-3SAT to it.

We first construct a graph \( G \) from an instance of NON-MONOTONE 2-3SAT. Let \( U \) be the set of variables and \( C \) the collection of clauses. We may assume that \(|U| = n\) and there are \( m \) clauses of three literals and \( p \) clauses of two literals in \( C \) (that is, \( C \) contains \( m + p \) clauses in total). For each variable \( u_i \in U \), we associate two literal vertices \( u_i, \overline{u}_i \) connecting by an edge. For each clause \( c_j \in C \), we associate a clause vertex to each literal in it, and label it with that literal, then connect clause vertices each other to form a complete graph of order \(|c_j|\). Furthermore, connect each literal vertex to the clause vertex with the same label.

Additionally, create three new independent sets \( X, Y, Z \) of size \( k - 1 \), respectively. Connect all literal vertices with labels of un-negated variables to all the vertices in \( X \) and all literal vertices with labels of negated variables to all the vertices in \( Y \). According to literals in each \( c_j \), we consider the following three possibilities:

(1) \( c_j \) has two un-negated variables. Connect one of the un-negated variables to all the vertices in \( Y \) and the other to all the vertices in \( Z \). Moreover, if \( c_j \) contains one more literal, it must be a negated variable, then connect it to all the vertices in \( X \).

(2) \( c_j \) has two negated variables. Connect one of the negated variables to all the vertices in \( X \) and the other to all the vertices in \( Z \). Moreover, if \( c_j \) contains one more literal, it must be an un-negated variable, then connect it to all the vertices in \( Y \).

(3) \( c_j \) has exactly one un-negated variable and exactly one negated variable. Connect the negated variable to all the vertices in \( X \) and the un-negated to all the vertices in \( Y \).

Until now, we have completed the construction of the graph \( G \).
We claim that if two distinct vertices \( u, v \notin X \cup Y \cup Z \) have the same neighbors in \( X, Y \) or \( Z \), then \( u \) and \( v \) are not adjacent in \( G \). Clearly, this is true if \( u \) and \( v \) are literal vertices. Similarly, the clause vertices in the same clause are connected to vertices of distinct independent sets \( X, Y \) and \( Z \) respectively, and the clause vertices in different clauses are non-adjacent. Moreover, two vertices, one literal and one clause, with the same literal label are connected to vertices in distinct independent sets \( X, Y \) or \( Z \), respectively. Thus, the claim holds.

For convenience, let \( s = n + 2m + p \). In the following, we shall show that there is a satisfying truth assignment for the instance of NON-MONOTONE 2-3SAT if and only if there is a set \( S \subseteq V(G) \) with \( |S| = s = n + 2m + p \) such that \( S \) separates \( X, Y \) and \( Z \) and each component of \( G - S \) is of order at least \( k \).

**Necessity.** Assume that there exists a satisfying truth assignment for the instance of NON-MONOTONE 2-3SAT. Clearly, for each clause \( c_j \), there exists a literal, say \( u_i \), with a true assignment in it. First choose the literal vertices with the same label as \( u_i \), and next choose the remaining two or one (noting that the size of \( c_j \) is 3 or 2) vertices in \( c_j \) whose labels are different from \( u_i \), which generates a set \( S \) of vertices of size \( s = n + 2m + p \). We shall show that the resulting graph \( G - S \) is disconnected, each component of \( G - S \) has at least \( k \) vertices, and \( X, Y, Z \) lie in different components of \( G - S \) respectively.

It is clear that there exists no edges between \( X, Y \) and \( Z \) in \( G \). It can be further deduced that there exists no edges between literal vertices and clause vertices in \( G - S \), so all the remaining edges in \( G - S \) have exactly one endpoint in \( X, Y \) or \( Z \). It follows that \( G - S \) is a bipartite graph. Therefore, \( S \) separates \( X, Y \) and \( Z \). When a literal vertex or a clause vertex is connected to one vertex in \( X \) (or \( Y, Z \)), by the construction of \( G \), it is connected to all vertices in \( X \), yielding a component of at least \( k \) vertices containing all vertices of \( X \).

**Sufficiency.** Suppose that there exists a set \( S \) of vertices with \( |S| = s = n + 2m + p \) such that \( S \) separates \( X, Y \) and \( Z \) and each component of \( G - S \) has order at least \( k \). By the construction of \( G \), each literal vertex (resp. clause vertex) is connected to all the vertices in exactly one of \( X, Y \) and \( Z \).

Observe that \( X, Y \) and \( Z \) are independent sets, to separate \( X, Y \) and \( Z \), it can be deduced from the construction of \( G \) that there exists no edges between literal vertices, no edges between literal and clause vertices, and no edges between clause vertices in \( G - S \). This implies that each edge of \( G - S \) has exactly one endpoints in \( X, Y \) or \( Z \). It follows that for each variable \( u_i \), at least one of the literal \( u_i \) or \( \overline{u}_i \) is
contained in $S$. Similarly, at least two vertices of the clause vertices are contained in $S$ for each 3-literal clause, and at least one vertex of the clause vertices is contained in $S$ for each 2-literal clauses. By this argument, it follows that $S$ contains at least $s$ vertices. That is, $S$ must in fact contain exactly one of the literal $u_i$ or $\overline{u}_i$, exactly two vertices of the clause vertices for each 3-literal clause, and exactly one vertex of the clause vertices for each 2-literal clauses.

If two (or one) clause vertices of a clause $c_j$ are contained in $S$, then the remaining clause vertex, say $u_i$ of $c_j$ is not in $S$. Correspondingly, the corresponding literal vertex with label $u_i$ must be in $S$. Thus, the true assignment to each literal vertex of $c_j$ not in $S$ is obvious a satisfying truth assignment.

This completes the proof.

To characterize computational complexity of $k$-restricted connectivity of a nonempty graph, we make a step further to state the following decision problem.

\textbf{Problem:} $k$-restricted connectivity ($k$-RC).

\textbf{Instance:} Given a nonempty graph $G = (V, E)$ and a positive integer $k \geq 2$.

\textbf{Question:} Is there a set $S$ of vertices such that $G - S$ is disconnected and each component of $G - S$ has order at least $k$?

\textbf{Theorem 5.3.} $k$-RC is NP-complete for any given integer $k \geq 2$.

\textbf{Proof.} Clearly, the problem is in NP, because we can check in polynomial time whether a set $S$ of vertices such that $G - S$ is disconnected and each component of $G - S$ is of order at least $k$. It remains to show that the $k$-RC is NP-hard for any specific integer $k \geq 2$. We prove this argument by reducing RCTSVS to it.

Consider an instance of RCTSVS problem. Given the positive integer $k$ and the graph $G$ with three specific vertex set $X, Y, Z$ of size $k - 1$ respectively. We construct a graph $G'$ as an instance of $k$-RC based on the instance of RCTSVS as follows. For each pair of vertices $u$ and $v$ with $u, v \not\in X \cup Y \cup Z$, associate a complete graph $K(u, v)$ of order $s + 1 = n + 2m + p + 1$. Then join $u$ and $v$ to each vertex of $K(u, v)$ respectively, that is, the edges added between $u, v$ and $K(u, v)$ induced a complete bipartite subgraph.

Observe that for any two vertices $u, v \not\in X \cup Y \cup Z$, to separate $u$ and $v$ in $G'$, at least $s + 1$ vertices of $G'$ need to be deleted, whereas to separate $X, Y$ and $Z$, by the previous argument, at most $s$ vertices are needed, which is the same as the proof in Theorem 5.2. It implies that there is a set $S \subseteq V(G)$ with $|S| = n + 2m + p$ such that $S$ separates $X, Y$ and $Z$, and each component of $G - S$ has order at least
if and only if there is a set $S' \subseteq V(G')$ with $|S'| = n + 2m + p$ such that $G' - S'$ is disconnected and each component of $G - S$ has order at least $k$. In fact, it is clear that $S' \cap V(G') \subset V(G)$.

This completes the proof. \qed

6 Conclusions

In this paper, we present several complexity results concerning structure connectivity, substructure connectivity, component connectivity and $k$-restricted connectivity. In characterizing the complexity of component connectivity, we show that it happens to be essentially the same as a variant of vertex connectivity ($2\text{PCP}$) investigated by Ramarao [27], leading to NP-completeness of component connectivity. In particular, the results showing that these network reliability measurements are all NP-complete, which indicates that it is unlikely that polynomially bounded algorithms exist to solve these problems. Consequently, it is meaningful to determine the general upper and lower bounds of these parameters for general networks and exact values for graphs with excellent network properties, as well as search for polynomially bounded algorithms for well-structured networks.

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