A NEW APPROACH TO THE INVERSE DISCRETE TRANSMISSION EIGENVALUE PROBLEM

NATALIA P. BONDARENKO*

Department of Applied Mathematics and Physics, Samara National Research University
Moskovskoye Shosse 34, Samara 443086, Russia
Department of Mechanics and Mathematics, Saratov State University
Astrakhanskaya 83, Saratov 410012, Russia

VJACHESLAV A. YURKO

Department of Mechanics and Mathematics, Saratov State University
Astrakhanskaya 83, Saratov 410012, Russia

(Communicated by Fioralba Cakoni)

Abstract. A discrete analog is considered for the inverse transmission eigenvalue problem, having applications in acoustics. We provide a well-posed inverse problem statement, develop a constructive procedure for solving this problem, prove uniqueness of solution, global solvability, local solvability, and stability. Our approach is based on the reduction of the discrete transmission eigenvalue problem to a linear system with polynomials of the spectral parameter in the boundary condition.

1. Introduction. The paper is concerned with the discrete transmission eigenvalue problem. The motivation of our study is related with the acoustic inverse scattering problem in an inhomogeneous medium (see [19]):

\[
\begin{aligned}
\Delta u + \lambda \rho(x) u &= 0, & x \in B, \\
\Delta v + \lambda v &= 0, & x \in B, \\
u(x) &= v(x), & \frac{\partial u(x)}{\partial n} = \frac{\partial v(x)}{\partial n}, & x \in \partial B,
\end{aligned}
\]

where \( B \) is the ball of radius \( b > 0 \) in \( \mathbb{R}^3 \), \( \partial B \) is its boundary, \( \rho(x) > 0 \) is the refraction index, \( \frac{\partial}{\partial n} \) is the normal derivative. The inverse transmission eigenvalue problem consists in reconstruction of the function \( \rho(x) \), which is related with the speed of sound, from the eigenvalues of the problem (1). The majority of the studies of the inverse transmission eigenvalue problem (see [19, 20, 11, 1, 12, 10, 7, 9, 13, 24, 8]) deal with the radially symmetric case, when the problem (1) is reduced to the one-dimensional form

\[
\begin{aligned}
u'' + \lambda \rho(x) u &= 0, & 0 < x < b, \\
v'' + \lambda v &= 0, & 0 < x < b, \\
u(0) &= v(0) = 0, & u(b) = v(b), & u'(b) = v'(b).
\end{aligned}
\]

2020 Mathematics Subject Classification. Primary: 15A29; Secondary: 15A18, 34A55.

Key words and phrases. Inverse problems, discrete transmission eigenvalue problem, Weyl function, global solvability, local solvability, stability.

The authors are supported by RFBR grants 20-31-70005, 19-01-00102.

*Corresponding author: Natalia P. Bondarenko.
There were some attempts to study the inverse discrete transmission eigenvalue problems in [21, 22, 2, 23]. In particular, the following discrete analogs of the continuous problem (2) have been considered:

\[
\begin{align*}
-\psi_{n+1} + 2\psi_{n} - \psi_{n-1} &= \lambda \rho_n \psi_n, \quad n = \overline{1,l}, \\
-\psi_{n+1}^* + 2\psi_{n}^* - \psi_{n-1}^* &= \lambda \psi_n^*, \quad n = \overline{1,l}, \\
\psi_0 &= \psi_0^* = 0, \quad \psi_l = \psi_l^*, \quad \psi_{l+1} = \psi_{l+1}^*,
\end{align*}
\]

and

\[
\begin{align*}
-\psi_{n+1} + V_n \psi_n - \psi_{n-1} &= \lambda \psi_n, \quad n = \overline{1,l}, \\
-\psi_{n+1}^* + 2\psi_{n}^* - \psi_{n-1}^* &= \lambda \psi_n^*, \quad n = \overline{1,l}, \\
\psi_0 &= \psi_0^* = 0, \quad \psi_l = \psi_l^*, \quad \psi_{l+1} = \psi_{l+1}^*.
\end{align*}
\]

The corresponding inverse transmission problems consist in recovering the coefficients \(\{\rho_n\}_{n=1}^l\) or \(\{V_n\}_{n=1}^l\) from the eigenvalues of (3) or (4), respectively. However, the important disadvantage of those problems is their ill-posedness, since they consist in determining \(l\) unknown numbers by \((2l - 1)\) or \((2l - 2)\) known eigenvalues. Therefore, a small perturbation of the transmission eigenvalues influences the existence of inverse problem solution. Consequently, the known results in this direction are limited to uniqueness theorems and constructive methods of solving inverse problems, but the solvability issues remain open.

In this paper, we suggest a new approach to the inverse discrete transmission eigenvalue problem. We provide a well-posed inverse problem statement (without overdetermination in the input data), develop a constructive procedure for solving this problem, prove uniqueness of solution, global solvability, local solvability, and stability.

Let us briefly describe our approach. Consider the generalized discrete transmission boundary value problem with respect to vectors \(\psi_{n=0}^{l+1}\) and \(\psi_{n=0}^{l+1}\):

\[
\begin{align*}
\alpha_n \psi_{n+1} + \beta_n \psi_n + \alpha_n \psi_{n-1} &= \lambda \psi_n, \quad n = \overline{1,l}, \\
\alpha_n^* \psi_{n+1}^* + \beta_n^* \psi_n^* + \alpha_n^* \psi_{n-1}^* &= \lambda \psi_n^*, \quad n = \overline{1,l}, \\
\psi_0 &= \psi_0^* = 0, \quad \psi_l = \psi_l^*, \quad \psi_{l+1} = \psi_{l+1}^*,
\end{align*}
\]

where \(\lambda\) is the spectral parameter, \(\alpha_n, \beta_n, \alpha_n^*, \beta_n^* \in \mathbb{C}, n = \overline{1,l}\).

Note that every linear system in the form

\[
an_{u_{n+1}} + b_n u_n + c_n u_{n-1} = \lambda \rho_n u_n, \quad n = \overline{1,l},
\]

where \(a_n, c_n, \rho_n \neq 0, n = \overline{1,l}\), can be reduced to the form (5) by the change of variables \(u_n = d_n \psi_n\) with some coefficients \(d_n \neq 0, n = \overline{1,l}\).

Assume that \(\alpha_n \neq 0, \alpha_n^* \neq 0, n = \overline{1,l}\), and \(\alpha_l \neq \alpha_l^*\), Then, the boundary value problem (5)-(7) has \((2l - 1)\) eigenvalues \(\{\lambda_j\}_{j=1}^{2l-1}\) (counting with multiplicities). Our inverse discrete transmission problem is stated as follows.

**Inverse Problem 1.1.** Suppose that \(\{\alpha_n^*\}_{n=1}^l, \{\beta_n^*\}_{n=1}^l\), and \(\alpha_l\) are known a priori. Given the eigenvalues \(\{\lambda_j\}_{j=1}^{2l-1}\), find \(\{\alpha_n\}_{n=1}^l\) and \(\{\beta_n\}_{n=1}^l\).

Our method of solution is based on the reduction of the problem (5)-(7) to the form

\[
\begin{align*}
a_n y_{n+1} + b_n y_n + y_{n-1} &= \lambda y_n, \quad n = \overline{1,l}, \\
R_0(\lambda) y_1 - R_1(\lambda) y_0 &= 0, \quad y_{l+1} = 0,
\end{align*}
\]
where \( a_n, b_n \in \mathbb{C}, a_n \neq 0, n = 1, l \), and \( R_0(\lambda), R_1(\lambda) \) are relatively prime polynomials (i.e. not having common roots). The latter polynomials are constructed by the coefficients \( \{\alpha_n^\bullet\}_{n=1}^l, \{\beta_n^\bullet\}_{n=1}^l, \alpha_l \) so that the eigenvalues of the problem (8)-(9) coincide with the eigenvalues of (5)-(7). Thus, Inverse Problem 1.1 is reduced to the following problem.

**Inverse Problem 1.2.** Given eigenvalues \( \{\lambda_j\}_{j=1}^{2l-1} \) of the problem (8)-(9), find \( \{a_n\}_{n=1}^{l-1} \) and \( \{b_n\}_{n=1}^{l} \).

Note that the coefficient \( a_l \) is multiplied by \( y_{l+1} = 0 \) in (8), so this coefficient cannot be recovered. Although we are primarily interested in Inverse Problem 1.2 in connection with Inverse Problem 1.1, our method for solving Inverse Problem 1.2 is developed for the case of arbitrary relatively prime polynomials \( R_0(\lambda) \) and \( R_1(\lambda) \) satisfying some additional conditions. We reduce Inverse Problem 1.2 to the reconstruction of \( \{a_n\}_{n=1}^{l-1} \) and \( \{b_n\}_{n=1}^{l} \) from the Weyl coefficients defined in Section 2.

The latter problem is equivalent to the classical inverse problem that consists in the reconstruction of the coefficients from the two spectra \( \{\mu_j\}_{j=1}^{l-1} \) and \( \{\nu_j\}_{j=1}^{l} \) of the eigenvalue problems for equations (8) with the boundary conditions \( y_0 = y_{l+1} = 0 \) and \( y_1 = y_{l+1} = 0 \), respectively. In order to deal with this classical inverse problem, we adapt the methods of Yurko [25, 26]. Note that in [25, 26] these methods are developed for arbitrary discrete systems of triangular structure corresponding to arbitrary-order differential operators.

It is worth mentioning that (8)-(9) is the discrete analog of the Sturm-Liouville problem

\[
\begin{align*}
-\ y'' + q(x)y = \lambda y, & \quad x \in (0, \pi), \\
y(0) = 0, & \quad y'(\pi) + f_1(\lambda)y(\pi) + f_2(\lambda)y(\pi) = 0,
\end{align*}
\]

with entire analytic functions \( f_1(\lambda) \) and \( f_2(\lambda) \) in the boundary condition. Bondarenko [3, 4, 5] has developed a unified approach, which is based on the reduction to the form (10), to a wide class of inverse problems for differential operators, including the Hochstadt-Lieberman half-inverse problem [16], the inverse transmission eigenvalue problem [19, 7], partial inverse problems on metric graphs [6, 5]. The approach of [3, 4, 5] allows not only to prove the uniqueness theorems, but also to develop constructive algorithms for solution and to obtain solvability conditions for various types of partial inverse problems. In this paper, the idea of such reduction is transferred to the discrete case.

The paper is organized as follows. In Section 2, the solution of the auxiliary inverse problem for equations (8) by the Weyl coefficients is described. In Section 3, we provide our main results concerning Inverse Problems 1.1-1.2. In Section 4, we discuss the connection of our results with the classical results of Hochstadt [14, 15]. In particular, we show that the inverse problem for a Jacobi matrix by mixed data [15] can be easily reduced to Inverse Problem 1.2 and solved by our method.

2. **Auxiliary inverse problem.** Denote by \( \{P_n\}_{n=0}^{l+1}, \{Q_n\}_{n=0}^{l+1}, \) and \( \{\Phi_n\}_{n=0}^{l+1} \) the solutions of the system (8) satisfying the initial conditions

\[ P_0(\lambda) = 0, \quad P_1(\lambda) = 1, \quad Q_0(\lambda) = 1, \quad Q_1(\lambda) = 0, \]

and the boundary conditions

\[ \Phi_0(\lambda) = 1, \quad \Phi_{l+1}(\lambda) = 0, \]
Inverse Problem 2.3. Problem 2.2 is equivalent to the following classical inverse problem by two spectra. Given the Weyl coefficients \( \Phi_n(\lambda) = Q_n(\lambda) + M(\lambda)P_n(\lambda), \quad n = \overline{1, l}, \)
where \( M(\lambda) \) is called the Weyl function of (8). Using (11), we obtain
\[
M(\lambda) = -\frac{Q_{l+1}(\lambda)}{P_{l+1}(\lambda)}. 
\]

Below the notation \( f(\lambda) \sim f_0 \lambda^k \) means that \( f(\lambda) \) is a polynomial of degree \( k \) with the leading coefficient \( f_0 \). By induction, we show that
\[
P_{n+1}(\lambda) \sim \left( \prod_{k=1}^{n} a_k^{-1} \right) \lambda^n, \quad Q_{n+1}(\lambda) \sim -\left( \prod_{k=1}^{n} a_k^{-1} \right) \lambda^{n-1}, \quad n = \overline{1, l}.
\]
Therefore, \( M(\lambda) \) is a rational function, so it can be represented in the form
\[
M(\lambda) = \sum_{k=1}^{\infty} M_k \lambda^{-k}, \quad M_1 = 1,
\]
where the series uniformly converges for sufficiently large \( r > 0, |\lambda| \geq r \). We call the numbers \( \{M_k\}_{k=1}^{2l} \) the Weyl coefficients.

**Lemma 2.1.** The Weyl coefficients \( \{M_k\}_{k=1}^{2l} \) uniquely specify \( M(\lambda) \).

**Proof.** Suppose that the Weyl functions \( M(\lambda) \) and \( \tilde{M}(\lambda) \) have the Weyl coefficients \( \{M_k\}_{k=1}^{2l} \) and \( \{\tilde{M}_k\}_{k=1}^{2l} \), respectively. Using (13), we obtain
\[
\frac{Q_{l+1}(\lambda)}{P_{l+1}(\lambda)} - \frac{\tilde{Q}_{l+1}(\lambda)}{\tilde{P}_{l+1}(\lambda)} = O \left( \lambda^{-(2l+1)} \right), \quad |\lambda| \to \infty.
\]
This implies
\[
Q_{l+1}(\lambda)\tilde{P}_{l+1}(\lambda) - P_{l+1}(\lambda)\tilde{Q}_{l+1}(\lambda) \equiv 0.
\]
Hence \( M(\lambda) \equiv \tilde{M}(\lambda). \) \( \square \)

**Inverse Problem 2.2.** Given the Weyl coefficients \( \{M_k\}_{k=2}^{2l} \), find \( \{a_n\}_{n=1}^{l-1} \) and \( \{b_n\}_{n=1}^{l-1} \).

The coefficient \( a_l \) cannot be recovered because of the condition \( \Phi_{l+1} = 0 \). Without loss of generality assume that \( a_1 = 1 \).

Denote by \( \{\mu_j\}_{j=1}^{l} \) and \( \{\nu_j\}_{j=1}^{l-1} \) the zeros of the polynomials \( P_{l+1}(\lambda) \) and \( Q_{l+1}(\lambda) \), respectively (counting with multiplicities). Clearly, \( \{\mu_j\}_{j=1}^{l} \) and \( \{\nu_j\}_{j=1}^{l-1} \) are the eigenvalues of the boundary value problems for the system (8) with the boundary conditions \( y_0 = y_{l+1} = 0 \) and \( y_1 = y_{l+1} = 0 \), respectively. For \( \{\nu_j\}_{j=1}^{l-1} \), it is supposed that \( n = \overline{2, l} \) in (8). In view of (13), (14), and Lemma 2.1, the two spectra \( \{\mu_j\}_{j=1}^{l} \) and \( \{\nu_j\}_{j=1}^{l-1} \) uniquely specify \( M(\lambda) \), and vice versa. Hence, Inverse Problem 2.2 is equivalent to the following classical inverse problem by two spectra.

**Inverse Problem 2.3.** Given \( \{\mu_j\}_{j=1}^{l} \) and \( \{\nu_j\}_{j=1}^{l-1} \), find \( \{a_n\}_{n=1}^{l-1} \) and \( \{b_n\}_{n=1}^{l-1} \).

Proceed with constructive solution of Inverse Problem 2.2. In view of (14), we have
\[
P_{n+1}(\lambda) = \sum_{i=0}^{n} c_i \lambda^i, \quad n = \overline{0, l},
\]
where \( c_{in} \in \mathbb{C}, c_{nn} \neq 0 \). Substituting (16) into (8), we obtain
\[
a_n c_{nn} = c_{n-1,n-1}, \quad a_n c_{n-1,n} + b_n c_{n-1,n-1} = c_{n-2,n-1}, \quad n = 1, l,
\]
where \( c_{-1,0} = 0 \). Hence
\[
a_n = \frac{c_{n-1,n-1}}{c_{nn}}, \quad b_n = \frac{c_{n-2,n-1} - a_n c_{n-1,n}}{c_{n-1,n-1}}, \quad n = 1, l.
\]

Denote by \( [v_i]_{i=0}^{l+1} \) the solution of equation (8) satisfying the initial conditions \( v_{l+1} = 0, v_1 = 1 \). One can easily show that
\[
(18) \quad \Phi_n(\lambda) = \frac{v_n(\lambda)}{v_0(\lambda)}, \quad n = 0, l + 1,
\]
and \( v_n(\lambda) \sim \lambda^{l-n} \). Consequently,
\[
(19) \quad \Phi_n(\lambda) = \lambda^{-n} + O\left(\lambda^{-\left(n+1\right)}\right), \quad |\lambda| \to \infty,
\]
that is, there are the zero coefficients at \( \lambda^{-k}, k = 1, n-1 \). Using (12), (15), (16), and (19), we obtain the relations
\[
(20) \quad \sum_{i=0}^{n} c_{in} M_{i+k+1} = \delta_{nk}, \quad k = 0, n, \quad n = 1, l,
\]
where \( \delta_{nk} \) is the Kronecker delta.

**Theorem 2.4.** For complex numbers \( \{M_k\}_{k=1}^{2l} \) \( (M_1 = 1) \) to be the Weyl coefficients of (8) with some \( \{a_n\}_{n=1}^{l} \) and \( \{b_n\}_{n=1}^{l} \), \( a_n \neq 0, n = 1, l-1, a_l = 1 \), the following condition is necessary and sufficient:
\[
(21) \quad \Delta_n := \det([M_{i+j-1}]_{i,j=1}^{n+1}) \neq 0, \quad n = 1, l-1.
\]
Under the condition (21), the solution of Inverse Problem 2.2 is unique.

**Proof. Necessity.** Let \( \{M_k\}_{k=1}^{2l} \) be the Weyl coefficients corresponding to some \( \{a_n\}_{n=1}^{l} \) and \( \{b_n\}_{n=1}^{l} \). By the above construction, there exist \( c_{in}, i = 0, n, n = 1, l, \) satisfying (20) and \( c_{nn} \neq 0 \). Obviously, \( \Delta_0 = 1 \). Suppose that \( \Delta_n \neq 0 \) and \( \Delta_{n+1} = 0 \) for some \( n \in \{0, \ldots, l-2\} \). Then the linear system (20) is inconsistent for this value of \( n \). This contradiction yields the claim.

**Sufficiency.** Suppose that \( \Delta_n \neq 0, n = 1, l-1 \). Then, for each fixed \( n = 1, l-1 \), the linear system (20) is uniquely solvable with respect to \( \{c_{in}\}_{i=0}^{n} \). In particular,
\[
c_{nn} = \frac{\Delta_{n-1}}{\Delta_n} \neq 0, \quad n = 1, l-1.
\]
Put \( c_{00} := 1, c_{ll} := c_{l-1,l-1}, \Delta_l := \Delta_{l-1} c_{ll}^{-1} \). Find \( \{c_{il}\}_{i=0}^{l-1} \) from the system (20) for \( n = l \) by Cramer’s rule, replacing the determinant of this system by \( \Delta_l \). (We cannot directly calculate \( \det([M_{i+j-1}]_{i,j=1}^{n+1}) \), since \( M_{2l+1} \) is unknown). Using (17), find \( \{a_n\}_{n=1}^{l} \) and \( \{b_n\}_{n=1}^{l} \). Clearly, \( a_n \neq 0, n = 1, l-1, \) and \( a_l = 1 \). It remains to prove that \( \{M_k\}_{k=1}^{2l} \) are the Weyl coefficients corresponding to \( \{a_n\}_{n=1}^{l} \) and \( \{b_n\}_{n=1}^{l} \). Suppose that equation (8) has the Weyl coefficients \( \{M_k\}_{k=1}^{2l}, M_l = 1 \). Then the relations (20) are valid for \( M_n \) replaced by \( M_k \). Using this fact together with \( c_{nn} \neq 0, n = 1, l \), one can easily show that \( M_k = M_{k+1} = 1, M_k = 1, l-2l \). The described algorithm of unique reconstruction of \( \{a_n\}_{n=1}^{l} \) and \( \{b_n\}_{n=1}^{l} \) by \( \{M_k\}_{k=1}^{2l} \) implies the uniqueness of the inverse problem solution. \( \square \)
Thus, we have arrived at the following constructive algorithm for solving Inverse Problem 2.2.

**Algorithm 2.5.** Let \( \{M_k\}_{k=1}^{2l} \), \( M_1 = 1 \) be given. We have to find \( \{a_n\}_{n=1}^{l-1} \) and \( \{b_n\}_{n=1}^{l} \).

1. For each \( n = 1, l-1 \), find \( \{c_{in}\}_{i=0}^{n} \) by solving the system (20).
2. Put \( c_{00} := 1 \), \( c_l := c_{l-l, l-1} \), \( \Delta_l := \Delta_{l-l}c_{ll}^{-1} \), and find \( \{c_{il}\}_{i=0}^{l-1} \) by solving the system (20) for \( n = l \), replacing its determinant by \( \Delta_l \).
3. Find \( \{a_n\}_{n=1}^{l-1} \) and \( \{b_n\}_{n=1}^{l} \) by (17).

Using Theorem 2.4 and Algorithm 2.5, we easily obtain the following theorem on local solvability and stability of Inverse Problem 2.2.

**Theorem 2.6.** Let \( \{M_k\}_{k=1}^{2l} \) be the Weyl coefficients corresponding to \( \{a_n\}_{n=1}^{l-1} \) and \( \{b_n\}_{n=1}^{l} \), \( a_n \neq 0 \), \( n = 1, l-1 \), \( a_l = 1 \). Then there exists \( \epsilon > 0 \) (depending on \( \{a_n\}_{n=1}^{l-1} \) and \( \{b_n\}_{n=1}^{l} \)) such that, for any complex numbers \( \{\tilde{M}_k\}_{k=2}^{2l} \) satisfying the estimate

\[
\delta := \max_{k \geq 2l} |M_k - \tilde{M}_k| \leq \epsilon
\]

there exist unique complex numbers \( \{\tilde{a}_n\}_{n=1}^{l-1} \) and \( \{\tilde{b}_n\}_{n=1}^{l} \) such that \( \tilde{a}_n \neq 0 \), \( n = 1, l-1 \), \( \tilde{a}_l = 1 \), and \( \{\tilde{M}_k\}_{k=2}^{2l} \) are the corresponding Weyl coefficients. Moreover, the following estimates hold:

\[
|a_n - \tilde{a}_n| \leq C\delta, \quad |b_n - \tilde{b}_n| \leq C\delta, \quad n = 1, l,
\]

where the constant \( C \) depends only on \( \{a_n\}_{n=1}^{l-1} \) and \( \{b_n\}_{n=1}^{l} \).

3. Main results. In this section, we investigate Inverse Problems 1.1 and 1.2. The uniqueness of solution, global solvability, local solvability, stability are proved, and also reconstruction algorithms are obtained for these two inverse problems.

Denote by \( [\varphi_n]_{n=0}^{l+1} \) and \( [\varphi^*_n]_{n=0}^{l+1} \) the solutions of equations (5) and (6), respectively, satisfying the initial conditions \( \varphi_0(\alpha) = \varphi_0^*(\alpha) = 0 \), \( \varphi_1(\alpha) = \varphi_1^*(\alpha) = 1 \). Clearly, the eigenvalues of the boundary value problem (5)-(7) coincide with the zeros of the characteristic function

\[
D(\lambda) = \varphi_1(\lambda)\varphi_{l+1}^*(\lambda) - \varphi_{l+1}\varphi_1^*(\lambda).
\]

By induction, we show that

\[
\varphi_n(\lambda) \sim \left( \prod_{k=1}^{n-1} \frac{1}{\alpha_k} \right) \lambda^{n-1}, \quad \varphi_n^*(\lambda) \sim \left( \prod_{k=1}^{n-1} \frac{1}{\alpha_k^*} \right) \lambda^{n-1}, \quad n = 1, l+1.
\]

Consequently,

\[
D(\lambda) \sim \left( \prod_{k=1}^{l} \frac{1}{\alpha_k \alpha_k^*} \right) (\alpha_l^* - \alpha_l)^{2l-1}.
\]

Recall that \( \alpha_k \neq 0 \), \( \alpha_k^* \neq 0 \), \( k = 1, l \), and \( \alpha_l \neq \alpha_l^* \). Therefore, the problem (5)-(7) has exactly \( (2l-1) \) eigenvalues \( \{\lambda_j\}_{j=1}^{2l-1} \) (counting with multiplicities).

Denote

\[
\begin{align*}
d_{l+1} := 1, & \quad d_n := a^{-1}_{n-1} d_{n+1}, \quad n = l, l-1, \ldots, 1, \\
a_{l+n} := d_n a_n d_{n-1}, & \quad n = 2l, \quad a_l := 1, \\
b_{l+n} := \beta_n, & \quad b_l := d_n \psi_n, \quad n = 1, l, \\
R_0(\lambda) := \alpha_l \varphi_{l+1}^*(\lambda), & \quad R_1(\lambda) = \varphi_1^*(\lambda).
\end{align*}
\]
This change of variables reduces the eigenvalue problem (5)-(7) to the equivalent form (8)-(9). On the other hand, if the coefficients \( \{a_n\}_{n=1}^l \), \( \{b_n\}_{n=1}^l \), and \( \alpha_i \) are known, one can find \( \{\alpha_n\}_{n=1}^{l-1} \) and \( \{\beta_n\}_{n=1}^l \) by using the formulas

\[
\begin{align*}
(24) & \quad d_{l+1} := 1, \quad d_l := \alpha^{-1}_l, \quad d_{n-1} := d_{n+1}a_{l+1-n}, \quad n = l-1, \ldots, 2, \\
(25) & \quad \alpha_n := d^{-1}_n d_{n+1}, \quad \beta_n := b_{l+1-n}, \quad n = 1, \ldots, l.
\end{align*}
\]

It follows from (22)-(23) and the condition \( \alpha_l \neq \alpha_i^* \) that

\[
(26) \quad R_0(\lambda) \sim r_0 \lambda^l, \quad R_1(\lambda) \sim r_1 \lambda^{l-1}, \quad r_0 \neq r_1, \quad r_j \neq 0, \quad j = 0, 1
\]

\[
(27) \quad r_0 = \frac{\alpha_l}{\alpha_i^* r_1}, \quad r_1 = \prod_{k=1}^{l-1} \frac{1}{\alpha_k^*}.
\]

Let us consider the problem (8)-(9) in the general form with arbitrary relatively prime polynomials \( R_0(\lambda) \) and \( R_1(\lambda) \) satisfying (26). The eigenvalues of (8)-(9) coincide with the zeros of the characteristic function

\[
(28) \quad E(\lambda) = R_0(\lambda)v_1(\lambda) - R_1(\lambda)v_0(\lambda).
\]

Since \( v_n \sim \lambda^{l-n} \), we have

\[
(29) \quad E(\lambda) \sim (r_0 - r_1)\lambda^{2l-1}.
\]

Hence, under our assumptions, the problem (8)-(9) has exactly \( 2l - 1 \) eigenvalues \( \{\lambda_j\}_{j=1}^{2l-1} \) (counting with multiplicities). If (23) holds, then \( \{\lambda_j\}_{j=1}^{2l-1} \) coincide with the eigenvalues of the problem (5)-(7). Clearly, the polynomial \( E(\lambda) \) can be constructed by its zeros as follows:

\[
(30) \quad E(\lambda) = (r_0 - r_1) \prod_{j=1}^{2l-1} (\lambda - \lambda_j).
\]

**Lemma 3.1.** Suppose that \( R_0(\lambda), R_1(\lambda), E(\lambda) \) are arbitrary polynomials satisfying (26), (29), and \( R_0(\lambda), R_1(\lambda) \) are relatively prime. Then, there exist unique polynomials \( v_j(\lambda) \sim \lambda^{l-j}, j = 0, 1 \), satisfying (28).

**Proof.** Denote by \( \{\xi_{n,j}\}_{n=1}^{l-j} \) the zeros of the polynomial \( R_j(\lambda) \), \( j = 0, 1 \) (counting with multiplicities). Introduce the notation

\[
S_j := \{1\} \cup \{n = 2, l-j: \xi_{n,j} \neq \xi_{n-1,j}\},
\]

\[
m_{n,j} := \#\{k = 1, l-j: \xi_{k,j} = \xi_{n,j}\}, \quad n \in S_j, \quad j = 0, 1,
\]

that is, \( S_j \) is the set of indices of all the distinct values among \( \{\xi_{n,j}\}_{n=1}^{l-j} \) and \( m_{n,j} \) is the multiplicity of \( \xi_{n,j} \). Denote

\[
f^{<\nu>}(\lambda) = \frac{1}{\nu!} \frac{d^{\nu} f(\lambda)}{d\lambda^{\nu}}, \quad \nu \geq 0.
\]

Using (28), we obtain

\[
E^{<\nu>}(\xi_{n,j}) = (-1)^{1-j} \sum_{s=0}^{\nu} \nabla^{<\nu-s>} R_{1-j}^{<\nu-s>}(\xi_{n,j}) y_{j}^{<s>}(\xi_{n,j}),
\]

\[
\forall n \in S_j, \quad \nu = 0, m_{n,j} - 1, \quad j = 0, 1.
\]
Hence

\[
\psi_{j}^{<\nu>} (\xi_{n,j}) \equiv R_{1-j}^{-1} (\xi_{n,j}) \left( (-1)^{1-j} s^{<\nu>} (\xi_{n,j}) - \sum_{s=0}^{\nu-1} R_{1-j}^{-s} (\xi_{n,j}) \psi_{s}^{<\nu>} (\xi_{n,j}) \right),
\]

\( n \in S_{j}, \quad \nu = 0, m_{n,j} - 1, \quad j = 0, 1. \)

Note that \( R_{1-j} (\xi_{n,j}) \neq 0, \) since \( R_{0} (\lambda) \) and \( R_{1} (\lambda) \) do not have common roots. Clearly, the polynomials \( \psi_{j} (\lambda) \sim \lambda^{l-j} \) can be uniquely constructed by the values \( \{ \psi_{j}^{<\nu>} (\xi_{n,j}) \}_{n \in S_{j}, \nu = 0, m_{n,j} - 1}, \) \( j = 0, 1, \) by using the Hermite interpolation.

Although the proof of Lemma 3.1 is constructive, it is more convenient to use another method for finding \( \psi_{0} (\lambda) \) and \( \psi_{1} (\lambda). \) Represent the polynomials appearing in (28) in the form

\[
E (\lambda) = \sum_{s=0}^{2l-1} e_{s} \lambda^{s}, \quad R_{j} (\lambda) = \sum_{n=0}^{l-j} r_{n,j} \lambda^{n}, \quad \psi_{j} (\lambda) = \sum_{k=0}^{l-j} v_{k,j} \lambda^{k}, \quad j = 0, 1.
\]

Note that

\[
e_{2l-1} = r_{0} - r_{1}, \quad r_{l-j,j} = r_{j}, \quad v_{l-j,j} = 1, \quad j = 0, 1.
\]

Substituting (32) into (28) and taking (33) into account, we obtain the following system of \((2l-1)\) linear equations

\[
\sum_{n+k=s} r_{n,0} v_{k,1} - \sum_{n+k=s} r_{n,1} v_{k,0} = e_{s}, \quad s = 0, 2l-2,
\]

with respect to the \((2l-1)\) unknown values \( \{ v_{k,j} \}_{k=0}^{l-j-1}, \) \( j = 0, 1. \) By virtue of Lemma 3.1, the system (34) is uniquely solvable, since its determinant is non-zero. Solving (34), one can find \( \{ v_{k,j} \}_{k=0}^{l-j-1}, \) \( j = 0, 1 \) and construct the polynomials \( \psi_{0} (\lambda), \psi_{1} (\lambda). \)

It follows from (12) and (18) that \( M (\lambda) = \frac{\psi_{1} (\lambda)}{\psi_{0} (\lambda)}. \) Therefore, using \( \psi_{j} (\lambda), \) \( j = 0, 1, \)

one can construct the Weyl function \( M (\lambda) \) and the Weyl coefficients \( \{ M_{k} \}_{k=1}^{2l} \), and then solve Inverse Problem 2.2 to determine \( \{ a_{n} \}_{n=1}^{l-1} \) and \( \{ b_{n} \}_{n=1}^{l-1} \). Thus, we have arrived at the following algorithm for solving Inverse Problem 1.2.

**Algorithm 3.2.** Suppose that the relatively prime polynomials \( R_{j} (\lambda), \) \( j = 0, 1, \)

satisfying (26) and the complex numbers \( \{ \lambda_{j} \}_{j=1}^{2l-1} \) are given. We have to find \( \{ a_{n} \}_{n=1}^{l-1} \) and \( \{ b_{n} \}_{n=1}^{l-1}. \)

1. Construct \( E (\lambda) \) by (30).
2. Put \( v_{l-j,j} = 1, \) \( j = 0, 1, \) find \( \{ v_{k,j} \}_{k=0}^{l-j-1}, \) \( j = 0, 1, \) by solving the linear system (34), and construct the polynomials \( \psi_{0} (\lambda), \psi_{1} (\lambda) \) by (32).
3. Construct the coefficients \( \{ M_{k} \}_{k=1}^{2l} \) in the expansion (15) of the function \( M (\lambda) = \frac{\psi_{1} (\lambda)}{\psi_{0} (\lambda)}. \)
4. Apply Algorithm 2.5 to find \( \{ a_{n} \}_{n=1}^{l-1} \) and \( \{ b_{n} \}_{n=1}^{l-1} \) by \( \{ M_{k} \}_{k=1}^{2l}. \)

**Remark 1.** For definiteness, we assume throughout the paper that \( a_{l} \neq a_{l}^{*}, r_{0} \neq r_{1}. \) However, our results are also valid for the opposite case with some technical modifications. Indeed, if \( a_{l} = a_{l}^{*} \) and \( r_{0} = r_{1}, \) then \( \deg (E) < 2l-1, \) and the number
Suppose that \( \lambda_j \) is a solution of \( \lambda^p \) is less than \((2l - 1)\). In this case, the polynomial \( E(\lambda) \) has the form

\[
E(\lambda) = c \prod_{j=1}^{p} (\lambda - \lambda_j)
\]

instead of \((30)\). The constant \( c \neq 0 \) should be added to the initial data of the inverse problem. Thus, given the eigenvalues \( \{\lambda_j\}_{j=1}^{p} \) and the constant \( c \), one can use \((35)\) to construct \( E(\lambda) \). It is easy to check that Lemma 3.1, Algorithm 3.2, and the subsequent results remain valid with necessary technical changes.

Using Theorem 2.4, we immediately obtain necessary and sufficient conditions of solvability for Inverse Problem 1.2 together with the uniqueness of solution.

**Theorem 3.3.** Suppose that \( R_0(\lambda) \) and \( R_1(\lambda) \) are relatively prime polynomials satisfying \((26)\). Then, for complex numbers \( \{\lambda_j\}_{j=1}^{2l-1} \) to be the eigenvalues of a problem \((8)-(9)\), it is necessary and sufficient that the numbers \( \{M_k\}_{k=1}^{2l-1} \) constructed by steps 1-3 of Algorithm 3.2 fulfill the condition \((21)\). Under the latter condition, the solution of Inverse Problem 1.2 is unique.

Using the reduction of the problem \((5)-(7)\) to \((8)-(9)\), we obtain the algorithm for solving the discrete inverse transmission eigenvalue problem (Inverse Problem 1.1).

**Algorithm 3.4.** Suppose that \( \{\alpha_n\}_{n=1}^{l-1}, \{\beta_n\}_{n=1}^{l-1}, \alpha_l, \) and \( \{\lambda_j\}_{j=1}^{2l-1} \) are given. We have to find \( \{\alpha_n\}_{n=1}^{l-1}, \{\beta_n\}_{n=1}^{l-1} \).

1. Find the solution \( \varphi_{n,l} \) of equation \((6)\) satisfying the initial conditions \( \varphi_0 = 0, \varphi_{l-1} = 1 \).
2. Construct the numbers \( r_0, r_1 \) by \((27)\) and the polynomials \( R_0(\lambda), R_1(\lambda) \) by \((23)\).
3. Using \( R_j(\lambda), j = 0, 1, \) and \( \{\lambda_j\}_{j=1}^{2l-1} \), construct \( \{a_n\}_{n=1}^{l-1}, \{b_n\}_{n=1}^{l-1} \) by Algorithm 3.2.
4. Find \( \{\alpha_n\}_{n=1}^{l-1}, \{\beta_n\}_{n=1}^{l-1} \) by \((24)-(25)\).

Relying on Algorithm 3.4 and Theorems 2.4 and 3.3, we obtain necessary and sufficient conditions of solvability and the uniqueness of solution for Inverse Problem 1.1.

**Theorem 3.5.** Suppose that \( \{\alpha_n\}_{n=1}^{l-1}, \{\beta_n\}_{n=1}^{l-1}, \alpha_l \) are fixed complex numbers, \( \alpha_n \neq 0, n = 1, l, \alpha_l \neq 0, \alpha_l \neq \alpha_0 \). Then, for complex numbers \( \{\lambda_j\}_{j=1}^{2l-1} \) to be the eigenvalues of a problem \((5)-(7)\), it is necessary and sufficient that the numbers \( \{M_k\}_{k=1}^{2l-1} \) constructed by steps 1-3 of Algorithm 3.4 by steps 1-3 of Algorithm 3.2 fulfill the condition \((21)\). Under the latter condition, the solution of Inverse Problem 1.1 is unique.

**Remark 2.** Theorems 3.3 and 3.5 contain the implicit condition \((21)\). Observe that analogous implicit conditions arise in investigation of the half-inverse problem for the Sturm-Liouville differential equation (see [17, 18]). It seems that such kind of conditions is unavoidable for these problems.

Since each step of Algorithms 3.2 and 3.4 is stable, we obtain the following theorems on local solvability and stability of Inverse Problems 1.1 and 1.2.

**Theorem 3.6.** Let \( R_0(\lambda) \) and \( R_1(\lambda) \) be fixed relatively prime polynomials satisfying \((26)\), let \( \{a_n\}_{n=1}^{l}, \{b_n\}_{n=1}^{l} \) be fixed complex numbers, \( a_n \neq 0, n = 1, l-1, \)
Let \( a_1 = 1 \), and let \( \{\lambda_j\}_{j=1}^{2l-1} \) be the eigenvalues of the corresponding problem (8)-(9). Then, there exists \( \alpha > 0 \) (depending on \( R_0, R_1, \{a_n\} \), and \( \{b_n\} \)) such that, for any complex numbers \( \{\tilde{\lambda}_j\}_{j=1}^{2l-1} \) and for any polynomials

\[
\tilde{R}_j(\lambda) = \sum_{n=0}^{l-j} \tilde{r}_{n,j} \lambda^n, \quad j = 0, 1,
\]

satisfying the estimate

\[
\delta := \max_{j=1,2l-1} |\lambda_j - \tilde{\lambda}_j| + \max_{n=0,1} |r_{n,j} - \tilde{r}_{n,j}| \leq \epsilon
\]

there exist unique complex numbers \( \{\tilde{a}_n\}_{n=1}^l \) and \( \{\tilde{b}_n\}_{n=1}^l \), \( \tilde{a}_l = 1 \), such that \( \{\tilde{\lambda}_j\}_{j=1}^{2l-1} \) are the eigenvalues of the problem (8)-(9) with the coefficients \( \{a_n\}, \{b_n\}, R_0, R_1 \) replaced by \( \{\tilde{a}_n\}, \{\tilde{b}_n\}, \tilde{R}_0, \tilde{R}_1 \), respectively. Moreover, the following estimates hold:

\[
|a_n - \tilde{a}_n| \leq C\delta, \quad |b_n - \tilde{b}_n| \leq C\delta, \quad n = 1, \ldots, l,
\]

where the constant \( C \) depends only on \( R_0, R_1, \{a_n\}, \{b_n\} \).

**Theorem 3.7.** Let \( \{a_n\}_{n=1}^l, \{\beta_n\}_{n=1}^l, \{a_n^\star\}_{n=1}^l, \{\beta_n^\star\}_{n=1}^l \) be fixed complex numbers, \( a_n \neq 0, \beta_n \neq 0, \alpha \neq \alpha^\star, \lambda \neq \lambda^\star \), and let \( \{\lambda_j\}_{j=1}^{2l-1} \) be the eigenvalues of the corresponding problem (5)-(7). Then, there exists \( \alpha > 0 \) (depending on \( \{a_n\}, \{\beta_n\}, \{a_n^\star\}, \{\beta_n^\star\} \)) such that, for any complex numbers \( \{\tilde{\lambda}_j\}_{j=1}^{2l-1}, \{\tilde{a}_n\}_{n=1}^l, \{\tilde{\beta}_n\}_{n=1}^l, \tilde{a}_l \) satisfying the estimate

\[
\delta := \max_{j=1,2l-1} |\lambda_j - \tilde{\lambda}_j| + \max_{n=1,l} (|a_n^\star - \tilde{a}_n^\star| + |\beta_n - \tilde{\beta}_n|) + |a_l - \tilde{a}_l| \leq \epsilon
\]

there exist unique complex numbers \( \{\tilde{a}_n\}_{n=1}^{l-1} \) and \( \{\tilde{\beta}_n\}_{n=1}^l \) such that \( \{\tilde{\lambda}_j\}_{j=1}^{2l-1} \) are the eigenvalues of the problem (5)-(7) with the coefficients \( \{a_n\}, \{\beta_n\}, \{a_n^\star\}, \{\beta_n^\star\} \) replaced by \( \{\tilde{a}_n\}, \{\tilde{\beta}_n\}, \{a_n^\star\}, \{\beta_n^\star\} \), respectively. Moreover, the following estimates hold:

\[
|a_n - \tilde{a}_n| \leq C\delta, \quad |\beta_n - \tilde{\beta}_n| \leq C\delta, \quad n = 1, \ldots, l,
\]

where the constant \( C \) depends only on \( \{a_n\}, \{\beta_n\}, \{a_n^\star\}, \{\beta_n^\star\} \).

Note that Theorems 3.6-3.7 do not contain implicit conditions mentioned in Remark 2.

**Remark 3.** Our approach can be generalized to the case when (6) contains \( \tilde{l} \) equations, \( \tilde{l} > l \), and \( \deg(R_j) > j - l, \) \( j = 0, 1 \). In this case, a part of the spectrum is sufficient for the recovery of the unknown coefficients. In the case \( \tilde{l} < l \), on the contrary, one has to use additional spectral data together with the eigenvalues for solving the inverse problem.

4. **Connection with Hochstadt’s results.** In this section, we discuss the relation of our results with the classical results of Hochstadt [14, 15].

In [14], the reconstruction of the tridiagonal symmetric Jacobi matrix has been studied, by using its spectrum and the spectrum of the truncated matrix obtained by deleting the first row and the first column. In our notations, this problem can be formulated as follows. Let \( \{\mu_j\}_{j=1}^{2l-1} \) and \( \{\nu_j\}_{j=1}^{2l-1} \) be the eigenvalues (counting with their multiplicities) of the boundary value problems for the system

\[
A_{n-1}u_{n-1} + B_nu_n + A_nu_{n+1} = \lambda u_n, \quad n = 1, \ldots, l,
\]
with the boundary conditions \( u_0 = u_{t+1} = 0 \) and \( u_1 = u_{t+1} = 0 \), respectively. For the second boundary value problem, we suppose that \( n = \frac{2}{l} \) in (36). Suppose that \( A_n \neq 0 \), \( n = 0, l \). The coefficients \( A_0 \) and \( A_l \) are unimportant, so we assume for definiteness that \( A_0 = A_l = 1 \).

**Inverse Problem 4.1.** Given \( \{ \mu_j \}_{j=1}^l \) and \( \{ \nu_j \}_{j=1}^{l-1} \), find \( \{ A_n \}_{n=1}^{l-1} \) and \( \{ B_n \}_{n=1}^l \).

This inverse problem, studied in [14], is closely related with Inverse Problems 2.2 and 2.3. Indeed, the system (36) is reduced to the form (8) by the change of variables:

\[
(37) \quad u_n = d_n y_n, \quad n = 0, l+1, \quad a_n = A_n^2, \quad b_n = B_n, \quad n = 1, l,
\]

where the coefficients \( \{ d_n \}_{n=0}^{l+1} \) can be chosen uniquely up to a multiplicative constant so that \( d_{n+1} = A_n d_n, n = 0, l \). Consequently, the spectra \( \{ \mu_j \}_{j=1}^l \) and \( \{ \nu_j \}_{j=1}^{l-1} \) defined in this section coincide with the ones defined in Section 2. Hochstadt [14] considered the case of real \( A_n > 0 \). In this case, the correspondence (37) between the coefficients \( \{ a_n \}_{n=1}^{l-1} \), \( \{ b_n \}_{n=1}^l \) and \( \{ A_n \}_{n=1}^{l-1} \), \( \{ B_n \}_{n=1}^l \) is one-to-one, so Inverse Problems 2.2, 2.3, and 4.1 are equivalent. However, in the general case of complex coefficients \( A_n \neq 0 \), they are determined by \( a_n \) uniquely up to the sign.

In [15], the Jacobi matrix has been constructed by using the known spectrum and a half of the matrix coefficients. For definiteness, consider the odd \( l = 2m - 1 \). Then the inverse problem from [15] can be formulated as follows.

**Inverse Problem 4.2.** Given \( \{ A_n \}_{n=1}^{m-1} \), \( \{ B_n \}_{n=1}^{m-1} \), and \( \{ \mu_j \}_{j=1}^l \), find \( \{ A_n \}_{n=m}^l \) and \( \{ B_n \}_{n=m}^l \).

Let us show that Inverse Problem 4.2 can be reduced to Inverse Problem 1.2. By using \( \{ A_n \}_{n=1}^{m-1} \) and \( \{ B_n \}_{n=1}^{m-1} \), one can find \( \{ a_n \}_{n=1}^{m-1} \) and \( \{ b_n \}_{n=1}^{m-1} \) via (37), and then construct the polynomials \( P_n(\lambda) \) defined in Section 2. It can be easily shown that the eigenvalue problem for (36) with the boundary conditions \( u_0 = u_{t+1} = 0 \) is equivalent to the following eigenvalue problem:

\[
\begin{align*}
a_n y_{n+1} + b_n y_n + y_{n-1} = \lambda y_n, & \quad n = m, l, \\
P_{m-1}(\lambda) y_m - P_m(\lambda) y_{m-1} = 0, & \quad y_{l+1} = 0,
\end{align*}
\]

where \( P_{m-1}(\lambda) \) and \( P_m(\lambda) \) are relatively prime polynomials of degrees \( (m - 2) \) and \( (m - 1) \), respectively. The latter problem is similar to (8)-(9), so the results of Section 3 (Algorithm 3.2, Theorems 3.3 and 3.6) can be applied to this problem. After recovering \( \{ a_n \}_{n=m}^{l-1} \) and \( \{ b_n \}_{n=m}^l \) from the spectrum \( \{ \mu_j \}_{j=1}^l \), one can uniquely find \( \{ A_n \}_{n=m}^{l-1} \) and \( \{ B_n \}_{n=m}^l \) from (37) if \( A_n > 0 \). Thus, our approach based on Inverse Problem 1.2 with polynomials in the boundary condition can be applied to the Hochstadt inverse problem by mixed data.

**Acknowledgments.** The work of the author N.P. Bondarenko was supported by Grant 20-31-70005 of the Russian Foundation for Basic Research. The work of the author V.A. Yurko was supported by Grant 19-01-00102 of the Russian Foundation for Basic Research.

**Authors’ contributions.** Sections 1, 3-4 belong to N.P. Bondarenko, Section 2 belongs to V.A. Yurko.
REFERENCES

[1] T. Aktosun, D. Gintides and V. G. Papanicolaou, The uniqueness in the inverse problem for transmission eigenvalues for the spherically symmetric variable-speed wave equation, *Inverse Problems*, 27 (2011), 115004.

[2] T. Aktosun and V. G. Papanicolaou, Inverse problem with transmission eigenvalues for the discrete Schrödinger equation, *J. Math. Phys.*, 56 (2015), 082101.

[3] N. P. Bondarenko, Inverse Sturm-Liouville problem with analytical functions in the boundary condition, *Open Math.*, 18 (2020), 512–528.

[4] N. P. Bondarenko, Solvability and stability of the inverse Sturm-Liouville problem with analytical functions in the boundary condition, *Math. Meth. Appl. Sci.*, 43 (2020), 7009–7021.

[5] N. P. Bondarenko, A partial inverse Sturm-Liouville problem on an arbitrary graph, *Math. Meth. Appl. Sci.*, 44 (2021), 6896–6910.

[6] N. P. Bondarenko, A partial inverse problem for the Sturm-Liouville operator on a star-shaped graph, *Anal. Math. Phys.*, 8 (2018), 155–168.

[7] N. P. Bondarenko and S. Buterin, On a local solvability and stability of the inverse transmission eigenvalue problem, *Inverse Problems*, 33 (2017), 115010.

[8] S. A. Buterin, A. E. Choque-Rivero and M. A. Kuznetsova, On a regularization approach to the inverse transmission eigenvalue problem, *Inverse Problems*, 36 (2020), 105002.

[9] S. A. Buterin and C.-F. Yang, On an inverse transmission problem from complex eigenvalues, *Results Math.*, 71 (2017), 859–866.

[10] S. A. Buterin, C.-F. Yang and V. A. Yurko, On an open question in the inverse transmission eigenvalue problem, *Inverse Problems*, 31 (2015), 045003.

[11] F. Cakoni, D. Colton and P. Monk, On the use of transmission eigenvalues to estimate the index of refraction from far field data, *Inverse Problems*, 23 (2007), 507–522.

[12] D. Colton and Y.-J. Leung, Complex eigenvalues and the inverse spectral problem for transmission eigenvalues, *Inverse Problems*, 29 (2013), 104008.

[13] D. Gintides and N. Pallikarakis, The inverse transmission eigenvalue problem for a discontinuous refractive index, *Inverse Problems*, 33 (2017), 055006.

[14] H. Hochstadt, On the construction of a Jacobi matrix from spectral data, *Linear Algebra Appl.*, 8 (1974), 435–446.

[15] H. Hochstadt, On the construction of a Jacobi matrix from mixed given data, *Lin. Alg. Appl.*, 28 (1979), 113–115.

[16] H. Hochstadt and B. Lieberman, An inverse Sturm-Liouville problem with mixed given data, *SIAM J. Appl. Math.*, 34 (1978), 676–680.

[17] R. O. Hryniv and Y. V. Mykytyuk, Half-inverse spectral problems for Sturm-Liouville operators with singular potentials, *Inverse Problems*, 20 (2004), 1423–1444.

[18] O. Martinyuk and V. Pivovarchik, On the Hochstadt–Lieberman theorem, *Inverse Problems*, 26 (2010), 035011.

[19] J. R. McLaughlin and P. L. Polyakov, On the uniqueness of a spherically symmetric speed of sound from transmission eigenvalues, *J. Diff. Equas.*, 107 (1994), 351–382.

[20] J. R. McLaughlin, P. L. Polyakov and P. E. Sacks, Reconstruction of a spherically symmetric speed of sound, *SIAM J. Appl. Math.*, 54 (1994), 1203–1223.

[21] V. G. Papanicolaou and A. V. Doumas, On the discrete one-dimensional inverse transmission eigenvalue problem, *Inverse Problems*, 27 (2011), 015004.

[22] G. Wei, The uniqueness for inverse discrete transmission eigenvalue problems, *Linear Algebra Appl.*, 439 (2013), 3699–3712.

[23] Z. Wei and G. Wei, The inverse discrete transmission eigenvalue problem for absorbing media, *Inverse Probl. Sci. Eng.*, 26 (2018), 83–99.

[24] X.-C. Xu and C.-F. Yang, On the inverse spectral stability for the transmission eigenvalue problem with finite data, *Inverse Problems*, 36 (2020), 085006.

[25] V. A. Yurko, An inverse problem for operators of a triangular structure, *Results Math.*, 30 (1996), 346–373.
[26] V. A. Yurko, *Inverse Spectral Problems for Differential Operators and their Applications*, Analytical Methods and Special Functions, Gordon and Breach Science Publishers, Amsterdam, 2000.

Received June 2021; revised September 2021; early access December 2021.

E-mail address: bondarenkop@email.sgu.ru
E-mail address: yurkova@email.sgu.ru