STABILITY CONDITIONS AND EXTREMAL
CONTRACTIONS

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Dedicated to Professor Yujiro Kawamata on the occasion of his 60-th birthday

Abstract. We show that any extremal contraction from a smooth
projective variety with dimension less than or equal to three ap-
ppears as a moduli space of (semi)stable objects in the derived cat-
egory of coherent sheaves.

Contents

1. Introduction 1
2. Background on stability conditions 7
3. Extremal contractions of projective surfaces 10
4. Extremal contractions of projective 3-folds 23
5. Conjectural Bridgeland stability conditions 37

References 50

1. Introduction

1.1. Motivation. Let \( X \) be a smooth projective variety over \( \mathbb{C} \). Re
call that a Minimal Model Program (MMP) is a sequence of divisorial
contractions or flips

\[
X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_N = X_{\text{min}}
\]

such that \( X_{\text{min}} \) is either a minimal model (i.e. \( K_{X_{\text{min}}} \) is nef) or has
a Mori fiber space structure. In two dimensional case, the MMP is
just contracting \((-1)\)-curves. In three dimensional case, the MMP is
completed in 1980’s by allowing some mild singularities on each \( X_i \).
(cf. [22].) In a higher dimensional case, the MMP is known to exist
when Kodaira dimension of \( X \) is equal to the dimension of \( X \). (cf. [7].)

On the other hand, a surprising relationship between MMP and de-
duced categories of coherent sheaves was found by Bondal and Orlov [9].
They observed a phenomena that the derived category gets smaller by
MMP, at least each \( X_i \) is smooth and birational map \( X_i \dashrightarrow X_{i+1} \) is a
standard one. This result was generalized by Bridgeland [10] for arbi-
trary three dimensional flops, and by Kawamata [20] for toroidal cases.
It is now an interesting research subject to see an interaction between
MMP and derived category.
In this paper, we study a relationship between MMP and derived category from the viewpoint of stability conditions and moduli spaces. A relationship between MMP and stability conditions was first pointed out in [30], in which flips appeared as a variation of GIT stability. This relationship was also applied for the study of wall-crossing of moduli spaces of sheaves [26]. The derived category appeared in this context in [10], in which three dimensional flops were constructed as moduli spaces of objects in the derived category. This result should be interpreted that three dimensional flops are obtained as variations of stability conditions in the derived category, and we expect that any birational map which appears in MMP is always realized in this way.

Now there is a notion of stability conditions on derived categories by Bridgeland [11], which provides a mathematical formulation of Douglas’s II-stability [15]. We address the following question in this paper:

**Question 1.1.** Is each \( X_i \) a moduli space of Bridgeland (semi)stable objects in the derived category of \( X \), and MMP is interpreted as wall-crossing under a variation of Bridgeland stability conditions?

If the answer to the above question is true, then we are able to analyze the geometry of each \( X_i \), especially the minimal model \( X_{\min} \), from a categorical data of \( D^b\text{Coh}(X) \). The purpose of this paper is to answer the above question for the first step of MMP when \( \dim X \leq 3 \), that is an extremal contraction. This is a birational morphism

\[
f: X \to Y
\]

such that \( Y \) is a normal projective variety, \( -K_X \) is \( f \)-ample and the relative Picard number of \( f \) is equal to one. When \( \dim X = 2 \), then \( f \) is just a contraction of a \((-1)\)-curve. When \( \dim X = 3 \), \( f \) contracts a divisor \( D \) to a curve or point in \( Y \), and it is classified by Mori [29].

**1.2. Result for surfaces.** We first study the case of \( \dim X = 2 \). Let us fix the notation: for a Bridgeland stability condition \( \sigma \) on \( D^b\text{Coh}(X) \), we denote by \( M^\sigma([O_x]) \) the set of isomorphism classes of \( \sigma \)-semistable objects \( E \) with phase one, satisfying \( \text{ch}(E) = \text{ch}(O_x) \) for \( x \in X \). The following is the result for the surface case.

**Theorem 1.2. (Theorem 3.16)** Let \( X \) be a smooth projective surface and \( f: X \to Y \) an extremal contraction. Then there is a one parameter family of Bridgeland stability conditions \( \{\sigma_t\}_{t \in (-1,1)} \) on \( D^b\text{Coh}(X) \) satisfying the following:

- If \( t < 0 \), then \( X \) is the fine moduli space of \( \sigma_t \)-stable objects in \( M^{\sigma_t}([O_x]) \).
- If \( t = 0 \), then \( Y \) is the coarse moduli space of \( S \)-equivalence classes\(^1\) of objects in \( M^{\sigma_t}([O_x]) \).

\(^1\) The notion of ‘\( S \)-equivalence’ is a direct analogue used in the study of moduli of sheaves. See Definition 2.2.
If \( t > 0 \), then \( Y \) is the fine moduli space of \( \sigma_t \)-stable objects in \( M^{\sigma_t}(\mathcal{O}_x) \).

The above result is based on the following expectation on the relationship between the space of stability conditions and the nef cone of \( X \): let us consider the subset of the space of Bridgeland stability conditions \((\mathcal{Z}, \mathcal{A})\) so that all the skyscraper sheaves \( \mathcal{O}_x \) are stable of phase one. (We call it a geometric chamber.) Then \( \text{Im} \mathcal{Z} \) restricted to the curve classes determines an ample class on \( X \), by the argument of \[12, \text{Lemma 10.1}\]. So the geometric chamber is closely related to the ample cone of \( X \). Now if we consider the boundary of the nef cone given by the pull-back of the ample cone of \( Y \), we expect that there is a corresponding boundary of the geometric chamber in the space of stability conditions. If this is true, it would be interesting to investigate the wall-crossing phenomena of the moduli spaces of (semi)stable objects after crossing the above boundary (wall).

Indeed, the stability conditions \( \sigma_t \) for \( t < 0 \) are contained in the geometric chamber constructed by Arcara and Bertram \[1\], and \( \sigma_0 \) lies at its boundary corresponding to the pull-back of the ample cone of \( Y \) to \( X \). The stability conditions \( \sigma_t \) for \( t > 0 \) are obtained by crossing the boundary at \( \sigma_0 \), although it is not clear how to describe them explicitly. Now Theorem 1.3 answers Question 1.1 for the contraction \( f : X \to Y \): it is realized by crossing the wall given by the boundary of the geometric chamber.

More precisely, the one parameter family \( \{\sigma_t\}_{t \in (-1,1)} \) is constructed as follows: we first construct \( \sigma_0 \) to be a pair \( \sigma_0 = (Z_{f^*\omega}, B_{f^*\omega}) \) for an ample divisor \( \omega \) on \( Y \). The central charge \( Z_{f^*\omega} \) is given by

\[
Z_{f^*\omega}(E) = -\int_X e^{-f^*\omega} \text{ch}(E),
\]

and the heart \( B_{f^*\omega} \) is a tilting of the perverse heart \( \text{Per}(X/Y) \) in the sense of \[10\]. The tilting here is similar to the one constructed in \[12\], \[1\] applied for \( \text{Coh}(X) \). Then we deform \( \sigma_0 \) to a one parameter family \( \{\sigma_t\}_{t \in (-1,1)} \), so that any object \( \mathcal{O}_x \) with \( x \in X \) becomes \( \sigma_t \)-stable for \( t < 0 \), and any object \( Lf^*\mathcal{O}_y \) with \( y \in Y \) becomes \( \sigma_t \)-stable for \( t > 0 \).

The most serious technical issue is to show that \( \sigma_0 \) satisfies the condition which guarantees the existence of the deformation, formulated as a support property. The key ingredient to prove the support property is an analogue of Bogomolov-Gieseker (BG) inequality for certain semistable objects in the perverse heart \( \text{Per}(X/Y) \). By combining it with the techniques developed in \[3\], \[6\], we establish the BG inequality for \( \sigma_0 \)-semistable objects in \( B_{f^*\omega} \). By using the above BG inequality, we are able to evaluate Chern characters of \( \sigma_0 \)-semistable objects, and prove the support property for \( \sigma_0 \).
1.3. Result for 3-folds I: Derived equivalence. In the three-dimensional case, the extremal contraction is classified into the following five types [29]:

**Type I:** $Y$ is smooth and $f$ is a blow up at a smooth curve.

**Type II:** $Y$ is smooth and $f$ is a blow-up at a point.

**Type III:** $Y$ has an ordinary double point, and $f$ is a blow-up at the singular point.

**Type IV:** $Y$ has an orbifold singularity $\mathbb{C}^3/\mathbb{Z}/2\mathbb{Z}$ and $f$ is a blow-up at the singular point.

**Type V:** $Y$ has a $cA_2$-singularity and $f$ is a blow-up at the singular point.

We begin the study of three dimensional case with the construction of the perverse heart and relating it to a sheaf of non-commutative algebras on $Y$. When $f(D)$ is a curve, Van den Bergh [14] shows that $X$ is derived equivalent to a certain coherent $\mathcal{O}_Y$-algebras $\mathcal{A}$, and the perverse heart in [10] corresponds to the category of right $\mathcal{A}$-modules. When $f(D)$ is a point, we show that the previous result [34] on the local construction of the perverse heart and a derived equivalence with non-commutative algebras can be applied to our situation. By improving the argument of [34], we show the following:

**Theorem 1.3.** (Theorem 4.5) Let $X$ be a smooth projective 3-fold and $f : X \to Y$ an extremal contraction. Then there is a vector bundle $E$ on $X$ and a derived equivalence

$$Rf_*R\text{Hom}(E, *) : D^b\text{Coh}(X) \xrightarrow{\sim} D^b\text{Coh}(\mathcal{A}).$$

Here $\mathcal{A} := f_*\text{End}(E)$ and $\text{Coh}(\mathcal{A})$ is the category of coherent right $\mathcal{A}$-modules on $Y$.

The construction of $E$ in Theorem 1.3 is more concrete than the one in [34], and we can explicitly study it. We note that it is not difficult to construct $E$ in types I and II: in type I, the result is contained in [14]. In type II, $E$ is constructed by taking the direct sum of line bundles associated to the exceptional divisor $D \cong \mathbb{P}^2$, which restrict to a full exceptional collection on $D$. However it is not obvious to construct $E$ in other cases:

- In types III and IV, there is a full exceptional collection of line bundles on $D$, and their direct sum extends to a formal neighborhood of $D$. However it does not necessary extend to the whole space $X$.

- In type V, $D$ is a singular quadric surface in $\mathbb{P}^3$, and there is no full exceptional collection on it. In this case, the construction of $E$ is new even in a formal neighborhood of $D$.

Similarly to the surface case, there is a perverse heart $\text{Per}(X/Y)$ in $D^b\text{Coh}(X)$ which corresponds to $\text{Coh}(\mathcal{A})$ under the derived equivalence (3). The information of simple objects in $\text{Per}(X/Y)$ is required
for the study of stability conditions, and the above result enables us to describe them explicitly via the equivalence \(3\). In Proposition \(1.14\) and Proposition \(1.21\) we will give a complete description of these simple objects in \(\text{Per}(X/Y)\).

1.4. Result for 3-folds II: Conjectural Bridgeland stability conditions. In three dimensional case, we wish to realize a story similar to Theorem \(1.2\) starting from \(\text{Per}(X/Y)\) which corresponds to \(\text{Coh}(\mathcal{O})\) under the equivalence \(3\). However, there is a serious issue in studying Bridgeland stability conditions on 3-folds: we don’t even know whether there is a Bridgeland stability condition on any projective 3-fold or not. In [6], Bayer, Macri and the author constructed the heart of a bounded t-structure on any projective 3-fold as a double tilting of \(\text{Coh}(X)\) which, together with a suitable central charge, conjecturally gives a point in the geometric chamber of the space of Bridgeland stability conditions. As an analogy of the work [6], we construct a pair of the form

\[
\sigma_{B,f^*\omega} = (Z_{B,f^*\omega}, A_{B,f^*\omega}).
\]

(4)

Here the central charge \(Z_{B,f^*\omega}\) is of the form \(2\), replacing \(\text{ch}(E)\) by \(e^{-B}\text{ch}(E)\) where \(B\) is a \(\mathbb{Q}\)-multiple of \(D\). The heart \(A_{B,f^*\omega}\) is constructed as a double tilting of \(\text{Per}(X/Y)\), similarly to the construction in [6]. We conjecture that the pair (4) gives a Bridgeland stability condition. In this case, the subcategory

\[
\mathcal{P}_{B,f^*\omega}(1) := \{E \in A_{B,f^*\omega} : \text{Im} Z_{B,f^*\omega}(E) = 0\}
\]

should be the category of \(\sigma_{B,f^*\omega}\)-semistable objects of phase one. By denoting \(M^\sigma_{B,f^*\omega}([\mathcal{O}_x])\) the set of isomorphism classes of \(E \in \mathcal{P}_{B,f^*\omega}(1)\) with \(\text{ch}(E) = \text{ch}(\mathcal{O}_x)\) for \(x \in X\), the result for 3-folds is formulated as follows:

**Theorem 1.4. (Theorem [5.21])** Let \(X\) be a smooth projective 3-fold and \(f : X \to Y\) an extremal contraction. Then there is a conjectural Bridgeland stability condition (4) with \(\mathcal{P}_{B,f^*\omega}(1)\) finite length\(^2\) abelian category, such that

- If \(f(D)\) is a curve, then \(Y\) is one of the irreducible components of the coarse moduli space of \(S\)-equivalence classes of objects in \(M^\sigma_{B,f^*\omega}([\mathcal{O}_x])\).
- If \(f(D)\) is a point, then \(Y\) is the coarse moduli space of \(S\)-equivalence classes of objects in \(M^\sigma_{B,f^*\omega}([\mathcal{O}_x])\).

The finite length property of \(\mathcal{P}_{B,f^*\omega}(1)\) is a necessary condition for the pair (4) to give a Bridgeland stability condition, and required to define \(S\)-equivalence classes of objects in \(\mathcal{P}_{B,f^*\omega}(1)\). Also when \(f(D)\) is a curve, the coarse moduli space contains another component \(f(D) \times f(D)\), so \(Y\) appears only as an irreducible component. (cf. Remark 5.22)

\(^2\)This means that \(\mathcal{P}_{B,f^*\omega}(1)\) is a noetherian and artinian abelian category.
The pair (4) can be shown to give a Bridgeland stability condition if a conjectural BG type inequality evaluating \( \text{ch}_3 \), similar to the one conjectured in [6], holds. (cf. Conjecture 5.9.) If this is true, then Theorem 1.4 realizes \( Y \) as a moduli space of Bridgeland semistable objects. The conjectural stability condition (4) should be in a boundary of the conjectural geometric chamber constructed in [6]. So the pair (4) is an analogue of \( \sigma_0 \) in Theorem 1.2.

In types I and II, similarly to the surface case, the variety \( Y \) is likely to be a fine moduli space, if we are able to deform \( \sigma_{B,f^*\omega} \). However in other cases, the variety \( Y \) may not be regarded as a fine moduli space, even if we deform \( \sigma_{B,f^*\omega} \). This is due to the fact that the object \( L_{f^*\mathcal{O}_y} \) is not an object in \( D^b_{\text{Coh}}(X) \) for \( y \in \text{Sing}(Y) \).

In order to show that the pair (4) satisfies the desired property, we use a description of simple objects in \( \text{Per}(X/Y) \), and investigate the \( S \)-equivalence classes of objects in \( \mathcal{P}_{B,f^*\omega}(1) \). The most difficult and interesting case is the type V case, and the arguments will be focused in this case.

1.5. Relation to existing works. There are several recent works relating Bridgeland stability conditions and MMP. In [2], the MMP of the Hilbert scheme of points in \( \mathbb{P}^2 \) is realized as a variation of Bridgeland stability conditions on \( \mathbb{P}^2 \). In [4], the moduli spaces of generic Bridgeland stability conditions on K3 surfaces are shown to be projective varieties, and they are related by flops under variations of Bridgeland stability conditions. The motivation of this paper is similar to these works, but we concentrate on moduli spaces of objects whose numerical class is equal to that of a skyscraper sheaf.

The Bridgeland stability conditions on arbitrary surfaces are constructed in [1]. In the works [27], [28], [35], [24], [25], the structure of walls and wall-crossing phenomena with respect to these stability conditions are studied. Our construction of \( \sigma_0 \) is a generalization of the construction in [1], and similar to the one in [27] for the perverse heart on K3 surfaces. Its deformation \( \sigma_t \) for \( t > 0 \) provides a new example of Bridgeland stability conditions on arbitrary non-minimal surfaces. It would be interesting to study the moduli spaces of \( \sigma_t \)-semistable objects with arbitrary numerical classes, and investigate their behavior under wall-crossing.

In the 3-fold case, the construction of \( \mathcal{E} \) in Theorem 1.4 seems to have an application of the minimal saturated triangulated category associated to the singular variety. In type V contraction, the direct summand of \( \mathcal{E} \) tensored by a line bundle provides a local tilting generator of the saturated triangulated category \( \mathcal{D}_Y \) constructed in [21]. (cf. Remark 4.10) It would be interesting to study \( \mathcal{D}_Y \) in terms of our vector bundle \( \mathcal{E} \).
1.6. Plan of the paper. In Section 2, we recall some background on Bridgeland stability conditions. In Section 3, we prove Theorem 1.2. In Section 4, we prove Theorem 1.3 and classify simple objects in the perverse heart. In Section 5.1, we construct a conjectural Bridgeland stability condition and prove Theorem 1.4.

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1.8. Notation and convention. In this paper, all the varieties are defined over \( \mathbb{C} \). For a triangulated (or abelian) category \( \mathcal{D} \) and a set of objects \( \mathcal{S} \subset \mathcal{D} \), we denote by \( \langle \mathcal{S} \rangle \) the smallest extension closed subcategory of \( \mathcal{D} \) which contains objects in \( \mathcal{S} \). For a sheaf of (not necessarily commutative) \( \mathcal{O}_Y \)-algebras \( \mathcal{A} \) on a variety \( Y \), we denote by \( \text{Coh} (\mathcal{A}) \) the abelian category of coherent right \( \mathcal{A} \)-modules on \( Y \). The subcategory \( \text{Coh}_{\leq i}(\mathcal{A}) \subset \text{Coh}(\mathcal{A}) \) is the category of \( E \in \text{Coh}(\mathcal{A}) \) with \( \dim \text{Supp}(E) \leq 1 \) as \( \mathcal{O}_Y \)-module. For \( E \in D^b \text{Coh}(X) \), we denoted by \( \mathcal{H}^i(E) \) the \( i \)-th cohomology sheaf of the complex \( E \).

2. Background on stability conditions

In this section, we review the theory of stability conditions by Bridgeland [11].

2.1. Definitions. Let \( X \) be a smooth projective variety and \( N(X) \) the numerical Grothendieck group of \( X \). This is the quotient of the usual Grothendieck group \( K(X) \) by the subgroup of \( E \in K(X) \) with \( \chi(E,F) = 0 \) for any \( F \in K(X) \), where \( \chi(E,F) \) is the Euler pairing

\[
\chi(E,F) := \sum_{i \in \mathbb{Z}} (-1)^i \dim \text{Ext}^i(E,F).
\]

**Definition 2.1.** ([11]) A stability condition\(^3\) on \( X \) is a pair

\[
(Z, \mathcal{A}), \quad \mathcal{A} \subset D^b \text{Coh}(X),
\]

where \( Z : N(X) \to \mathbb{C} \) is a group homomorphism and \( \mathcal{A} \) is the heart of a bounded t-structure, such that the following conditions hold:

- For any non-zero \( E \in \mathcal{A} \), we have

\[
Z(E) \in \{ r \exp(i\pi\phi) : r > 0, \phi \in (0,1] \}.
\]

\(^3\)A stability condition in Definition 2.1 was called numerical stability condition in [11]. We omit ‘numerical’ since we do not deal with non-numerical stability conditions.
(Harder-Narasimhan property) For any $E \in \mathcal{A}$, there is a filtration in $\mathcal{A}$

$$0 = E_0 \subset E_1 \subset \cdots \subset E_N$$

such that each subquotient $F_i = E_i/E_{i-1}$ is $Z$-semistable with $\arg Z(F_i) > \arg Z(F_{i+1})$.

Here an object $E \in \mathcal{A}$ is $Z$-(semi)stable if for any subobject $0 \neq F \subset E$ we have

$$\arg Z(F) < (\leq) \arg Z(E).$$

The group homomorphism $Z$ is called a central charge. In this paper, the central charge is always of the form $Z = Z_{B,\omega}$:

$$Z_{B,\omega}(E) = -\int_X e^{-i\omega} \text{ch}^B(E).$$

Here $B, \omega$ are elements of $H^2(X, \mathbb{R})$ and $\text{ch}^B(E) := e^{-B} \text{ch}(E) \in H^*(X, \mathbb{R})$ is the twisted Chern character. If $B = 0$, we just write $Z_{\omega} := Z_{0,\omega}$. By setting $d = \dim X$, $Z_{B,\omega}(E)$ is written as

$$\sum_{j \geq 0} \frac{(-1)^j}{(2j)!} \omega^{2j} \text{ch}_d^{B}(E) + \sqrt{-1} \left( \sum_{j \geq 0} \frac{(-1)^j}{(2j + 1)!} \omega^{2j+1} \text{ch}_{d-2j-1}^{B}(E) \right).$$

2.2. Slicing. Given a stability condition $\sigma = (Z, \mathcal{A})$, we can construct subcategories $\mathcal{P}(\phi) \subset D^b \text{Coh}(X)$ for $\phi \in \mathbb{R}$ as follows: if $0 < \phi \leq 1$, then $\mathcal{P}(\phi)$ is defined by

$$\mathcal{P}(\phi) := \left\{ E \in \mathcal{A} : E \text{ is } Z\text{-semistable with } Z(E) \in \mathbb{R}_{>0} \exp(i\pi \phi) \right\} \cup \{0\}.$$

Other $\mathcal{P}(\phi)$ are defined by the rule

$$\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1].$$

A family of subcategories $\{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}$ forms a slicing in the sense of [11, Definition 3.3]. As proved in [11, Proposition 5.3], giving a stability condition is equivalent to giving a pair

$$(Z, \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}),$$

where $\{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}$ is a slicing, $Z : N(X) \to \mathbb{C}$ is a group homomorphism satisfying a certain axiom. (cf. [11, Definition 1.1].)

A non-zero object $E \in \mathcal{P}(\phi)$ is called $\sigma$-semistable of phase $\phi$. Each subcategory $\mathcal{P}(\phi) \subset D^b \text{Coh}(X)$ is shown to be an abelian category, and a simple object in $\mathcal{P}(\phi)$ is called $\sigma$-stable. For an interval $I \subset \mathbb{R}$, we set

$$\mathcal{P}(I) := (\mathcal{P}(\phi) : \phi \in I).$$
A stability condition (8) is called **locally finite** if for any \( \phi \in \mathbb{R} \), there is \( \epsilon > 0 \) so that \( \mathcal{P}((\phi - \epsilon, \phi + \epsilon)) \) is of finite length, i.e. it is a noetherian and artinian with respect to strict epimorphisms and strict monomorphisms. (cf. [11, Definition 5.7].) In particular each \( \mathcal{P}(\phi) \) is a finite length abelian category.

In general if \( \mathcal{P} \) is a finite length abelian category, any object \( F \in \mathcal{P} \) admits a Jordan-Hölder filtration in \( \mathcal{P} \)

\[
0 = F_0 \subset F_1 \subset \cdots \subset F_N = F
\]

so that each subquotient \( F_i/F_{i-1} \) is simple. We set

\[
\text{gr}(F) := \bigoplus_{i=1}^{N} F_i/F_{i-1} \in \mathcal{P}.
\]

Although the filtration (9) is not unique, the object \( \text{gr}(F) \) is uniquely determined.

**Definition 2.2.** Let \( \mathcal{P} \) be a finite length abelian category. Then \( F, F' \in \mathcal{P} \) is called \( S \)-equivalent if there is an isomorphism \( \text{gr}(F) \cong \text{gr}(F') \).

### 2.3. The space of stability conditions.

Let \( \|*\| \) be a fixed norm on \( N(X)_R \). It is useful (but often difficult to check) to put the following additional condition on the pair (8):

**Definition 2.3.** A stability condition (8) satisfies the **support property** if there is a positive constant \( C > 0 \) such that the following inequality holds for any non-zero \( E \in \bigcup_{\phi \in \mathbb{R}} \mathcal{P}(\phi) \):

\[
\frac{\|E\|}{|Z(E)|} \leq C.
\]

The space of stability conditions is defined as follows:

**Definition 2.4.** (i) We define \( \text{Stab}^\dagger(X) \) to be the set of locally finite stability conditions on \( D^b \text{Coh}(X) \).

(ii) We define \( \text{Stab}(X) \) to be the set of stability conditions on \( D^b \text{Coh}(X) \) satisfying the support property.

It is easy to see that the local finiteness of the stability condition follows from the support property, i.e. we have the inclusion

\[
\text{Stab}(X) \subset \text{Stab}^\dagger(X).
\]

As we will mention in Remark 2.6 the set \( \text{Stab}(X) \) has the better property than \( \text{Stab}^\dagger(X) \). The following is Bridgeland’s main theorem.

**Theorem 2.5.** ([11, Theorem 1.2]) There is a natural topology on \( \text{Stab}(X) \) such that the forgetting map

\[
\text{Stab}(X) \to N(X)^{\vee}_C
\]

sending \((Z, A)\) to \(Z\) is a local homeomorphism. In particular, each connected component of \( \text{Stab}(X) \) is a complex manifold.
Remark 2.6. The set $\text{Stab}^\dagger(X)$ was considered in the original paper [11], and there is also a natural topology on $\text{Stab}^\dagger(X)$. However the forgetting map $\text{Stab}^\dagger(X) \to N(X)_C^\dagger$ may not be a local homeomorphism, but so on a certain linear subspace of $N(X)_C^\dagger$. The subspace $\text{Stab}(X) \subset \text{Stab}^\dagger(X)$ is shown to be the union of connected components on which the forgetting map is a local homeomorphism.

2.4. Bogomolov-Gieseker inequality. We use the following (generalized) Bogomolov-Gieseker inequality to show the axiom (5) or support property.

Theorem 2.7. ([8], [16], [23]) Let $X$ be a smooth projective variety of $\dim X = d \geq 2$, and $H_1, \cdots, H_{d-1}$ nef divisors on $X$ such that the 1-cycle $H_1 \cdots H_{d-1}$ is numerically nontrivial. Then for any torsion free $(H_1, \cdots, H_{d-1})$-slope semistable sheaf $E$ on $X$, we have the inequality

$$(\text{ch}_1(E)^2 - 2 \text{ch}_0(E) \text{ch}_2(E))H_2 \cdots H_{d-1} \geq 0.$$ 

3. Extremal contractions of projective surfaces

The goal of this section is to prove Theorem 1.2. Throughout this section, $X$ is a smooth projective surface and $f$ is a birational morphism $f: X \to Y$ which contacts a single $(-1)$-curve $C \subset X$ to a point in $Y$.

3.1. Perverse t-structure on $D^b\text{Coh}(X)$. We recall the construction of the hearts of perverse t-structures

$$^p\text{Per}(X/Y) \subset D^b\text{Coh}(X)$$

for $p \in \mathbb{Z}$, studied in [10], [14]. Let $\mathcal{C}$ be the triangulated subcategory of $D^b\text{Coh}(X)$, defined by

$$\mathcal{C} := \{ E \in D^b\text{Coh}(X) : Rf_! E = 0 \}.$$ 

Since the dimensions of the fibers of $f$ are at most one dimensional, the standard t-structure on $D^b\text{Coh}(X)$ induces a t-structure on $\mathcal{C}$,

$$(\mathcal{C}^\leq 0, \mathcal{C}^\geq 0)$$

with heart $\mathcal{C}^0 = \mathcal{C} \cap \text{Coh}(X)$. (cf. [10, Lemma 3.1].)

Definition 3.1. ([10], [14]) We define $^p\text{Per}(X/Y)$ to be

$$^p\text{Per}(X/Y) := \left\{ E \in D^b\text{Coh}(X) : \begin{array}{l} Rf_! E \in \text{Coh}(Y), \\ \text{Hom}(\mathcal{C}^{<p}, E) = \text{Hom}(E, \mathcal{C}^{>p}) = 0 \end{array} \right\}.$$

In what follows, we mainly use the case $p = -1$, and denote

$$\text{Per}(X/Y) := ^{-1}\text{Per}(X/Y).$$

Also we denote by $H^i_\text{Per}(\ast)$ the $i$-th cohomology functor with respect to the t-structure with heart $\text{Per}(X/Y)$. 

As proved in [11, 14], the subcategory $\text{Per}(X/Y)$ is the heart of a bounded t-structure on $D^b\text{Coh}(X)$. Indeed, Van den Bergh [14] shows that the t-structure $\text{Per}(X/Y)$ is related to a sheaf of non-commutative algebras on $Y$. Let us set

$$ E := \mathcal{O}_X \oplus \mathcal{O}_X(-C), \quad \mathcal{A} := f_*\text{End}(\mathcal{E}). $$

Note that $\mathcal{A}$ is a sheaf of non-commutative algebras on $Y$.

**Theorem 3.2.** ([14]) We have the derived equivalence

$$ \Phi := Rf_*R\text{Hom}(\mathcal{E}, \ast) : D^b\text{Coh}(X) \sim \rightarrow D^b\text{Coh}(\mathcal{A}), $$

which restricts to an equivalence between $\text{Per}(X/Y)$ and $\text{Coh}(\mathcal{A})$.

We define the following subcategory of $\text{Per}(X/Y)$:

$$ \text{Per}_{\leq i}(X/Y) := \{ E \in \text{Per}(X/Y) : \Phi(E) \in \text{Coh}_{\leq i}(\mathcal{A}) \}, $$

and write $\text{Per}_0(X/Y) := \text{Per}_{\leq 0}(X/Y)$.

**Remark 3.3.** By the construction of $\mathcal{E}$, an object $E \in \text{Per}(X/Y)$ is an object in $\text{Per}_{\leq 1}(X/Y)$ iff $\text{Supp}(E)$ is zero dimensional outside $C$, and an object in $\text{Per}_{\leq 1}(X/Y)$ iff $\text{Supp}(E)$ is at most one dimensional.

We collect some known results used in this paper.

**Proposition 3.4.** (i) The category $\text{Per}_0(X/Y)$ is a finite length abelian category with simple objects given by

$$ \mathcal{O}_C, \mathcal{O}_C(-1)[1], \mathcal{O}_x, \ x \in X \setminus C. $$

(ii) An object $E \in D^b\text{Coh}(X)$ is an object in $\text{Per}(X/Y)$ if and only if $\mathcal{H}^i(E) = 0$ for $i \neq 0, -1$, $\text{Hom}(\mathcal{H}^0(E), \mathcal{O}_C(-1)) = 0$ and

$$ R^1f_*\mathcal{H}^0(E) = f_*\mathcal{H}^{-1}(E) = 0. $$

**Proof.** The result of (i) follows from [14, Proposition 3.5.8]. The result of (ii) follows from [10, Lemma 3.2]. \(\square\)

### 3.2. Slope stability on $\text{Per}(X/Y)$

Let $\omega$ be an ample divisor on $Y$. Similarly to the usual slope function on coherent sheaves, we define the slope function $\mu_{f^*\omega}$ on $\text{Per}(X/Y)$ as follows: if $E \in \text{Per}(X/Y)$ satisfies $\text{ch}_0(E) > 0$, we set

$$ \mu_{f^*\omega}(E) := \frac{\text{ch}_1(E) \cdot f^*\omega}{\text{ch}_0(E)}. $$

Otherwise we set $\mu_{f^*\omega}(E) = \infty$. Similarly to the usual slope stability, the slope $\mu_{f^*\omega}$ satisfies the weak seesaw property, and defines the weak stability condition on $\text{Per}(X/Y)$ as follows:

**Definition 3.5.** An object $E \in \text{Per}(X/Y)$ is $\mu_{f^*\omega}$-(semi)stable if for any exact sequence $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ in $\text{Per}(X/Y)$, we have the inequality

$$ \mu_{f^*\omega}(F) < (\leq) \mu_{f^*\omega}(G). $$
Similarly to the usual slope stability, we have the following:

**Lemma 3.6.** Any object in $\text{Per}(X/Y)$ admits a Harder-Narasimhan filtration with respect to $\mu_{f^*\omega}$-stability.

*Proof.* By Theorem 3.2, the category $\text{Per}(X/Y)$ is noetherian. Therefore it is enough to check that there is no infinite sequence

$$E_1 \supset E_2 \supset \cdots \supset E_i \supset \cdots$$

such that $\mu_{f^*\omega}(E_{i+1}) > \mu_{f^*\omega}(E_i/E_{i+1})$ for all $i$. (cf. [33, Proposition 2.12].) Suppose that there is such a sequence. Since $\text{ch}_0(*)$ is non-negative on $\text{Per}(X/Y)$, we may assume $\text{ch}_0(E_i)$ is constant. But then $\mu_{f^*\omega}(E_i/E_{i+1}) = \infty$, which is a contradiction. $\square$

### 3.3. Bogomolov-Gieseker inequality for perverse coherent sheaves.

We are going to construct a Bridgeland stability condition on $D^b\text{Coh}(X)$ by using the tilting of $\text{Per}(X/Y)$. For this purpose, we need to evaluate Chern classes of $\mu_{f^*\omega}$-semistable perverse coherent sheaves. First we show the following lemma:

**Lemma 3.7.** Let $A \subset D^b\text{Coh}(X)$ be the heart of a bounded t-structure with $\mathcal{O}_C, \mathcal{O}_C(-1)[1] \in A$. For $E \in A$, suppose that $\text{Hom}(\mathcal{O}_C, E) = 0$. Then we have $C \cdot \text{ch}_1(E) \geq 0$.

*Proof.* By the Serre duality, we have

$$\text{Ext}^i(\mathcal{O}_C, E) \cong \text{Hom}^{1-i}(E, \mathcal{O}_C(-1)[1])^\vee$$

$$\cong 0,$$

for $i \geq 2$. Here the last isomorphism follows from that both of $E$ and $\mathcal{O}_C(-1)[1]$ are objects in the heart $A$. Also by the assumption, we have

$$\text{Hom}^{\leq 0}(\mathcal{O}_C, E) = 0.$$

By the above vanishing and the Riemann-Roch theorem, we have

$$-C \cdot \text{ch}_1(E) = \chi(\mathcal{O}_C, E)$$

$$= -\dim \text{Ext}^1(\mathcal{O}_C, E).$$

Therefore $C \cdot \text{ch}_1(E) \geq 0$ holds. $\square$

Another lemma we use is the following:

**Lemma 3.8.** For $T \in \text{Per}_0(X/Y)$, we have the inequality

$$\text{ch}_2(T) \geq \frac{1}{2} C \cdot \text{ch}_1(T).$$

*Proof.* By the definition of $\text{Per}_0(X/Y)$, the object $Rf_*T$ is a zero dimensional sheaf on $Y$. Therefore we have

$$\chi(\mathcal{O}_X, T) = \chi(\mathcal{O}_Y, Rf_*T) \geq 0.$$
On the other hand, the Riemann-Roch theorem implies
\[ \chi(O_X, T) = \text{ch}_2(T) - \frac{1}{2}C \cdot \text{ch}_1(T). \]
Therefore the desired inequality holds. \qed

The following is the Bogomolov-Gieseker inequality for perverse coherent sheaves.

\textbf{Proposition 3.9.} For any \( \mu_{f^*, \omega} \)-semistable \( E \in \text{Per}(X/Y) \) with \( \text{ch}_0(E) > 0 \), we have the inequality
\[ \text{ch}_1(E)^2 - 2 \text{ch}_0(E) \text{ch}_2(E) \geq 0. \] (12)

\textit{Proof.} Since \( E \in \text{Per}(X/Y) \) is \( \mu_{f^*, \omega} \)-semistable and \( \mathcal{H}^{-1}(E)[1] \) is a subobject of \( E \) in \( \text{Per}(X/Y) \) supported on \( C \) by Proposition 3.4 (ii), we must have \( \mathcal{H}^{-1}(E) = 0 \). Hence \( E \) is a sheaf on \( X \) but it may have a torsion. Let us take an exact sequence of sheaves
\[ 0 \to T \to E \to F \to 0, \] (13)
where \( T \) is a torsion sheaf and \( F \) is a torsion free sheaf. The above exact sequence and Proposition 3.4 (ii) easily imply that \( F \in \text{Per}(X/Y) \). We have the exact sequence in \( \text{Per}(X/Y) \)
\[ 0 \to \mathcal{H}^0_p(T) \to E \to F \to \mathcal{H}^1_p(T) \to 0, \]
and \( \mathcal{H}^i_p(T) = 0 \) for \( i \neq 0, 1 \). By the \( \mu_{f^*, \omega} \)-stability of \( E \), we have \( \mathcal{H}^0_p(T) = 0 \) hence \( T[1] \in \text{Per}(X/Y) \) follows. This also implies that \( f_*T = 0 \), hence \( T \) is supported on \( C \).

Since sheaves supported on \( C \) do not affect \( \mu_{f^*, \omega}(\ast) \), it follows that \( F \) is a \( \mu_{f^*, \omega} \)-semistable sheaf in the category of coherent sheaves. Therefore Theorem 2.7 implies
\[ \text{ch}_1(F)^2 - 2 \text{ch}_0(F) \text{ch}_2(F) \geq 0. \] (14)

Let us write
\[ \text{ch}(T) = (0, aC, \text{ch}_2(T)) \in H^0(X) \oplus H^2(X) \oplus H^4(X) \]
for \( a \in \mathbb{Z}_{\geq 0} \). By Lemma 3.3, we have \( \text{ch}_2(T) \leq -a/2 \). Also Lemma 3.7 implies
\[ C \cdot \text{ch}_1(E) = C \cdot \text{ch}_1(F) - a \geq 0. \] (15)

By combining these inequalities, we have
\[ \text{ch}_1(E)^2 - 2 \text{ch}_0(E) \text{ch}_2(E) \]
\[ = (\text{ch}_1(F) + aC)^2 - 2 \text{ch}_0(F)(\text{ch}_2(F) + \text{ch}_2(T)) \]
\[ = \text{ch}_1(F)^2 - 2 \text{ch}_0(F) \text{ch}_2(F) + 2aC \cdot \text{ch}_1(F) - a^2 - 2 \text{ch}_0(F) \text{ch}_2(T) \]
\[ \geq a (\text{ch}_0(F) + 2C \cdot \text{ch}_1(F) - a) \]
\[ \geq 0. \]
Here we have used (14), (15) and \( \text{ch}_2(T) \leq -a/2 \) in the third inequality. Hence the inequality (12) is proved. \( \square \)

**Remark 3.10.** By the Hodge index theorem, the inequality (12) also implies

\[
(\text{ch}_1(E) \cdot f^*\omega)^2 \geq 2\omega^2 \text{ch}_0(E) \text{ch}_2(E).
\]

However the weaker inequality (16) is much easier to prove: it immediately follows from the exact sequence (13) and the inequality (14), together with \( \text{ch}_2(T) \leq -a/2 \leq 0 \). The weaker inequality (16) will be used in the construction of a stability condition, while the stronger one (12) will be required to show the support property.

3.4. **Construction of \( \sigma_0 \).** In this subsection, we construct a stability condition \( \sigma_0 \) on \( D^b \text{Coh}(X) \) via tilting of \( \text{Per}(X/Y) \). Let \((\mathcal{T}_{f^*\omega}, \mathcal{F}_{f^*\omega})\) be the pair of subcategories of \( \text{Per}(X/Y) \) defined by

\[
\mathcal{T}_{f^*\omega} := \{ E : E \text{ is } \mu_{f^*\omega}-\text{semistable with } \mu_{f^*\omega}(E) > 0 \},
\]

\[
\mathcal{F}_{f^*\omega} := \{ E : E \text{ is } \mu_{f^*\omega}-\text{semistable with } \mu_{f^*\omega}(E) \leq 0 \}.
\]

The pair of subcategories \((\mathcal{T}_{f^*\omega}, \mathcal{F}_{f^*\omega})\) forms a torsion pair [17] on \( \text{Per}(X/Y) \) by Lemma 3.6. The associate tilting is defined in the following way:

**Definition 3.11.** We define \( \mathcal{B}_{f^*\omega} \subset D^b \text{Coh}(X) \) to be

\[
\mathcal{B}_{f^*\omega} := \langle \mathcal{F}_{f^*\omega}[1], \mathcal{T}_{f^*\omega} \rangle.
\]

As for the central charge, we take \( Z_{f^*\omega} \) in the notation of (7). It is written as

\[
Z_{f^*\omega}(E) = \left( -\text{ch}_2(E) + \frac{\omega^2}{2} \text{ch}_0(E) \right) + \sqrt{-1} \text{ch}_1(E) \cdot f^*\omega.
\]

We show the following lemma:

**Lemma 3.12.** The pair \((Z_{f^*\omega}, \mathcal{B}_{f^*\omega})\) determines a stability condition on \( D^b \text{Coh}(X) \).

**Proof.** We first check the property (6). Let us take a non-zero object \( E \in \mathcal{B}_{f^*\omega} \). By the construction of \( \mathcal{B}_{f^*\omega} \), we have \( \text{Im} Z_{f^*\omega}(E) \geq 0 \). Suppose that \( \text{Im} Z_{f^*\omega}(E) = 0 \). Then we have

\[
E \in \left\langle F[1], T : F \in \text{Per}(X/Y) \text{ is } \mu_{f^*\omega}-\text{semistable with } \mu_{f^*\omega}(F) = 0, T \in \text{Per}_0(X/Y). \right\rangle.
\]

It is enough to check (6) for \( E = F[1] \) where \( F \) is a \( \mu_{f^*\omega}-\text{semistable object in } \text{Per}(X/Y) \) or \( E \in \text{Per}_0(X/Y) \). In the former case, the inequality in Remark 3.10 implies \( \text{ch}_2(F) \leq 0 \). Combined with \( \text{ch}_0(F) > 0 \), we have \( Z_{f^*\omega}(F[1]) \in \mathbb{R}_{<0} \). In the latter case, noting Proposition 3.4 (i), the property (6) follows from

\[
Z_{f^*\omega}(\mathcal{O}_C) = Z_{f^*\omega}(\mathcal{O}_C(-1)[1]) = -\frac{1}{2},
\]
and $Z_{f^*\omega}(\mathcal{O}_x) = -1$ for $x \in X \setminus C$.

As for the Harder-Narasimhan property, we will give a 3-fold version of a similar statement in Lemma 5.7. The same proof is applied. □

Below we fix an ample divisor $\omega$ on $Y$ and set

$\sigma_0 := (Z_{f^*\omega}, \mathcal{B}_{f^*\omega}).$

(18)

Note that, at this moment, $\sigma_0$ is just an element of $\text{Stab}^1(X)^4$ in the notation of Definition 2.4. The following result will be required in order to deform $\sigma_0$:

**Proposition 3.13.** We have $\sigma_0 \in \text{Stab}(X)$, i.e. $\sigma_0$ satisfies the support property.

The proof of Proposition 3.13 is postponed until Subsection 3.7.

### 3.5. One parameter family of stability conditions.

By Proposition 3.13 and Theorem 2.5, a small deformation of $Z_{f^*\omega}$ uniquely lifts to a small deformation of $\sigma_0$. Therefore for $0 < \varepsilon \ll 1$, there is a unique continuous map

$\sigma : (-1, 1) \to \text{Stab}(X)$

such that $\sigma_t := \sigma(t)$ satisfies the following: $\sigma_0$ coincides with the stability condition (18), and other $\sigma_t$ is written as

$\sigma_t = (Z_{f^*\omega + tC}, \mathcal{P}_t),$

where $\mathcal{P}_t = \{\mathcal{P}_t(\phi)\}_{\phi \in \mathbb{R}}$ is a slicing on $D^b \text{Coh}(X)$. Let $M^\sigma([\mathcal{O}_x])$ be the set of isomorphism classes of objects $E \in \mathcal{P}_t(1)$ with $\text{ch}(E) = \text{ch}(\mathcal{O}_x)$ for $x \in X$. We have the following proposition:

**Proposition 3.14.** By taking $\varepsilon$ smaller if necessary, we have the following:

**(i)** If $t < 0$, we have $M^\sigma([\mathcal{O}_x]) = \{\mathcal{O}_x : x \in X\}.$

**(ii)** If $t > 0$, we have $M^\sigma([\mathcal{O}_x]) = \{Lf^*\mathcal{O}_y : y \in Y\}.$

**(iii)** If $t = 0$, we have

(19) $M^\sigma([\mathcal{O}_x]) = \{\mathcal{O}_x, Lf^*\mathcal{O}_0, \mathcal{O}_C \oplus \mathcal{O}_C(-1)[1] : x \in X\}.$

Here $0 = f(C) \in Y.$

**Proof.** We first show (iii). Let us take $E \in \mathcal{B}_{f^*\omega}$ with $\text{ch}(E) = \text{ch}(\mathcal{O}_x)$. Since $\text{Im} Z_{f^*\omega}(E) = 0$, $E$ is contained in the RHS of (17), hence the condition $\text{ch}(E) = \text{ch}(\mathcal{O}_x)$ implies $E \in \text{Per}_0(X/Y)$. By Proposition 3.4 (i), we have the following possibilities: $E \cong \mathcal{O}_x$ for $x \in X \setminus C$ and it is $\sigma_0$-stable, or $E$ is not $\sigma_0$-stable with

$\text{gr}(E) \cong \mathcal{O}_C \oplus \mathcal{O}_C(-1)[1].$

\footnote{The locally finiteness of $\sigma_0$ is obvious since the image of $Z_{f^*\omega}$ is a discrete subgroup.}
Here $\text{gr}(E)$ is defined by (10). In the latter case, if $E$ is not isomorphic to $\mathcal{O}_C \oplus \mathcal{O}_C(-1)[1]$, $E$ fits into one of the following non-split exact sequences in $\text{Per}_0(X/Y)$:

\begin{align*}
(20) & 
0 \to \mathcal{O}_C \to E \to \mathcal{O}_C(-1)[1] \to 0, \\
(21) & 
0 \to \mathcal{O}_C(-1)[1] \to E \to \mathcal{O}_C \to 0.
\end{align*}

In the case of (20), we have $E \cong \mathcal{O}_x$ for $x \in C$. In the case of (21), we have $E \cong \mathcal{L} f^* \mathcal{O}_0$. Indeed, it is easy to check that there is a distinguished triangle

\begin{equation}
\mathcal{O}_C(-1)[1] \to \mathcal{L} f^* \mathcal{O}_0 \to \mathcal{O}_C,
\end{equation}

hence $E \cong \mathcal{L} f^* \mathcal{O}_0$ follows since $\text{Ext}^1_X(\mathcal{O}_C, \mathcal{O}_C(-1)[1]) = \mathbb{C}$.

Next we show (i) and (ii). The support property of $\sigma_0$ ensures the existence of a wall and chamber structure on $\text{Stab}(X)$ near $\sigma_0$, i.e. the set of semistable objects $E$ with $\text{ch}(E) = \text{ch}(\mathcal{O}_x)$ is constant on a chamber. Therefore, by choosing $\varepsilon$ smaller if necessary, we may assume that $M^s([\mathcal{O}_x])$ is constant for $t \in (-1, 0)$ and $t \in (0, 1)$. Since the set of points in $\text{Stab}(X)$ in which a given object is semistable is closed, we have the inclusion $M^s([\mathcal{O}_x]) \subset M^s([\mathcal{O}_x])$. It is enough to check $\sigma_t$-stability for the objects in the RHS of (19).

Since $\mathcal{O}_x$ for $x \in X \setminus C$, $\mathcal{O}_C$ and $\mathcal{O}_C(-1)[1]$ are stable in $\sigma_0$, they are also stable in $\sigma_t$ for any $t \in (-1, 1)$. Let us take real numbers $\phi_t, \psi_t$ so that the following holds:

\[
\mathcal{O}_C \in \mathcal{P}_t(\phi_t), \quad \mathcal{O}_C(-1)[1] \in \mathcal{P}_t(\psi_t).
\]

We have $\phi_t < \psi_t$ for $t < 0$ and $\phi_t > \psi_t$ for $t > 0$. Therefore, by looking at the exact sequence (20) for $E = \mathcal{O}_x$ with $x \in C$, we see that $E$ is $\sigma_t$-stable for $t < 0$ but not $\sigma_t$-semistable for $t > 0$. Similarly by the sequence (22), for $E = \mathcal{L} f^* \mathcal{O}_0$, we see that $E$ is $\sigma_t$-stable for $t > 0$ but not $\sigma_t$-semistable for $t < 0$. Finally, it is obvious that $\mathcal{O}_C \oplus \mathcal{O}_C(-1)[1]$ is not $\sigma_t$-semistable for $t \neq 0$. Therefore (i) and (ii) are proved. \qed

**Remark 3.15.** The proof of the above proposition shows that $M^s([\mathcal{O}_x])$ consists of $\sigma_t$-stable objects when $t \neq 0$.

We now give a proof of Theorem 1.2.

**Theorem 3.16.** We have the following:

(i) If $t < 0$, then $X$ is the fine moduli space of $\sigma_t$-stable objects in $M^s([\mathcal{O}_x])$.

(ii) If $t = 0$, then $Y$ is the coarse moduli space of $S$-equivalence classes of objects in $M^s([\mathcal{O}_x])$.

(iii) If $t > 0$, then $Y$ is the fine moduli space of $\sigma_t$-stable objects in $M^s([\mathcal{O}_x])$. 

We now give a proof of Theorem 1.2.
Proof. Let $\mathcal{M}$ be the algebraic space which parameterizes objects $E \in D^b\text{Coh}(X)$ satisfying \[ \text{Ext}^{<0}(E, E) = 0, \quad \text{Hom}(E, E) = \mathbb{C}, \]
constructed by Inaba [19]. We have the open sub algebraic space $\mathcal{M}^{\sigma_t}(\mathcal{O}_X) \subset \mathcal{M}$ whose closed points correspond to $\sigma_t$-stable objects in $M^{\sigma_t}(\mathcal{O}_X)$. If $t > 0$, Proposition 3.14 (ii) implies that there is a morphism
\[ Y \to \mathcal{M}^{\sigma_t}(\mathcal{O}_X) \]
sending $y \in Y$ to the point corresponding to $Lf^*\mathcal{O}_y$, which is bijective on $\mathbb{C}$-valued points. Also since we have the fully faithful functor $Lf^*: D^b\text{Coh}(Y) \to D^b\text{Coh}(X)$ the morphism (23) induces an isomorphism on the tangent space. Therefore (23) is an isomorphism, and (iii) is proved. The proof of (i) is similar by considering the morphism
\[ X \to \mathcal{M}^{\sigma_t}(\mathcal{O}_x) \]
for $t < 0$, sending $x \in X$ to $\mathcal{O}_x$.

In order to prove (ii), we need to show that $Y$ corepresents the functor of families of objects in $M^{\sigma_0}(\mathcal{O}_x)$. In Theorem 5.21 we will discuss a similar statement for 3-folds. Noting that $\mathcal{O}_x$ for $x \in C$ and $Lf^*\mathcal{O}_0$ are $S$-equivalent to $\mathcal{O}_C \oplus \mathcal{O}_C(-1)[1]$, the same argument as in Theorem 5.21 is applied. □

Remark 3.17. If $t < 0$, then any skyscraper sheaf $\mathcal{O}_x$ for $x \in X$ is $\sigma_t$-stable of phase one. By using the proof of [12, Lemma 10.1], we can show that $\sigma_t$ for $t < 0$ coincides with a stability condition constructed in [1].

3.6. Bogomolov-Gieseker inequality for $\sigma_0$-semistable objects. The rest of this section is devoted to proving Proposition 3.13. The key ingredient is to prove Bogomolov-Gieseker type inequality for $\sigma_0$-semistable objects in $B_{f^*\omega}$. The desired inequality is proved by a somewhat tricky argument: we first prove a version of BG inequality, evaluating $(\text{ch}_1(*) \cdot f^*\omega)^2$. It is not enough to show the support property, but ensures the ‘support property in one direction’. By this property, we are able to apply wall-crossing argument with respect to the central charges $Z_{s f^*\omega}$ for $s \in \mathbb{R}_{>0}$. Then BG inequality evaluating $(\text{ch}_1(*))^2$ is proved by the induction on $\text{ch}_1(E) \cdot f^*\omega$ as in [6, Theorem 7.3.1].

The following is the BG type inequality evaluating $(\text{ch}_1(*) \cdot f^*\omega)^2$.

**Proposition 3.18.** For any $\sigma_0$-semistable object $E \in B_{f^*\omega}$, we have the inequality
\[ (\text{ch}_1(E) \cdot f^*\omega)^2 \geq 2\omega^2 \text{ch}_0(E) \text{ch}_2(E). \]
Proof. The argument is borrowed from the proof of the support property for the local projective plane in [5, Lemma 4.5], although a similar inequality is not stated explicitly there.

Let \( E \in \mathcal{B}_{f^*\omega} \) be a \( \sigma_0 \)-semistable object. Since we may assume \( \text{ch}_0(E) \text{ch}_2(E) > 0 \), we have the two possibilities:

\[ \text{ch}_0(E), \text{ch}_2(E) > 0 \quad \text{or} \quad \text{ch}_0(E), \text{ch}_2(E) < 0. \]

We first prove the inequality \((24)\) in the case \( \text{ch}_0(E), \text{ch}_2(E) > 0 \). There is an exact sequence in \( \mathcal{B}_{f^*\omega} \)

\[ 0 \to \mathcal{H}_p^{-1}(E)[1] \to E \to \mathcal{H}_p^0(E) \to 0. \]

By the assumption, we have the two inequalities

\[ 0 < \text{ch}_0(E) \leq \text{ch}_0(\mathcal{H}_p^0(E)), \]

\[ 0 \leq \text{Im} \mathcal{Z}_{f^*\omega}(\mathcal{H}_p^0(E)) \leq \text{Im} \mathcal{Z}_{f^*\omega}(E). \]

The above inequalities immediately imply \( \mu_{f^*\omega}(E) \geq \mu_{f^*\omega}(\mathcal{H}_p^0(E)). \) On the other hand, there is a surjection \( \mathcal{H}_p^0(E) \to F \) in \( \text{Per}(X/Y) \) such that \( F \) is \( \mu_{f^*\omega} \)-semistable with \( \text{ch}_0(F) > 0 \) and

\[ 0 < \mu_{f^*\omega}(F) \leq \mu_{f^*\omega}(\mathcal{H}_p^0(E)) \leq \mu_{f^*\omega}(E). \]

We have the composition

\[ E \to \mathcal{H}_p^0(E) \to F, \]

which is surjective in \( \mathcal{B}_{f^*\omega} \). Therefore the \( \sigma_0 \)-stability of \( E \) yields \( \arg \mathcal{Z}_{f^*\omega}(E) \leq \arg \mathcal{Z}_{f^*\omega}(F) \), or equivalently

\[ \frac{-\text{ch}_2(E) + \omega^2 \text{ch}_0(E)/2}{\text{ch}_1(E) \cdot f^*\omega} \geq \frac{-\text{ch}_2(F) + \omega^2 \text{ch}_0(F)/2}{\text{ch}_1(F) \cdot f^*\omega}. \]

Combined with \((25)\) and the assumption \( \text{ch}_2(E) > 0 \), we obtain the inequality

\[ 0 < \frac{\text{ch}_2(E)}{\text{ch}_1(E) \cdot f^*\omega} \leq \frac{\text{ch}_2(F)}{\text{ch}_1(F) \cdot f^*\omega}. \]

Then the inequality \((24)\) follows from

\[ \frac{1}{2\omega^2} \geq \frac{\text{ch}_0(F)}{\text{ch}_1(F) \cdot f^*\omega} \cdot \frac{\text{ch}_2(F)}{\text{ch}_1(F) \cdot f^*\omega} \]

\[ \geq \frac{\text{ch}_0(E)}{\text{ch}_1(E) \cdot f^*\omega} \cdot \frac{\text{ch}_2(E)}{\text{ch}_1(E) \cdot f^*\omega}. \]

Here the first inequality follows from Remark 3.10 and the second inequality follows from \((25)\) and \((26)\).

The proof for the case of \( \text{ch}_0(E), \text{ch}_2(E) < 0 \) is similar, by replacing the quotient \( \mathcal{H}_p^0(E) \to F \) by a \( \mu_{f^*\omega} \)-semistable subobject \( G \subset \mathcal{H}_p^{-1}(E) \) with \( \mu_{f^*\omega}(G) \geq \mu_{f^*\omega}(\mathcal{H}_p^{-1}(E)). \) \( \square \)
For \( s \in \mathbb{R}_{>0} \), note that \( B_{sf^*\omega} = B_{f^*\omega} \). Let us consider the element
\[
\gamma_s := (Z_{sf^*\omega}, B_{f^*\omega}) \in \text{Stab}^1(X).
\]
Using the inequality (24), we show the following proposition.

**Proposition 3.19.** The map \( \gamma : s \mapsto \gamma_s \) is continuous. Moreover for each \( E \in D^b\text{Coh}(X) \), there is a wall and chamber structure on the image of \( \gamma \) so that the Harder-Narasimhan filtration of \( E \) is constant on each chamber.

**Proof.** As for the first statement, by [11, Proposition 6.3, Theorem 7.1], it is enough to show the following: for any \( s \in \mathbb{R}_{>0} \), there is a positive constant \( C \) such that
\[
\frac{|Z_{sf^*\omega}(E) - Z_{f^*\omega}(E)|}{|Z_{f^*\omega}(E)|} \leq C
\]
for any \( \sigma_0(= \gamma_1) \)-semistable object \( E \in B_{f^*\omega} \). If \( \text{ch}_0(E) = 0 \), then the LHS of (27) is less than or equal to \(|s - 1|\). Hence we may assume that \( \text{ch}_0(E) \neq 0 \), and set
\[
x = \frac{\text{ch}_1(E) \cdot f^*\omega}{\text{ch}_0(E)}, \quad y = \frac{\text{ch}_2(E)}{\text{ch}_0(E)}.
\]
Then the LHS of (27) is written as
\[
|s - 1| \sqrt{\frac{(s + 1)^2/4 + x^2}{(-y + \omega^2/2)^2 + x^2}}.
\]
By the inequality (24), we have \( x^2 \geq 2\omega^2 y \). Hence the bound (27) is obtained by the following elementary fact:
\[
\inf \{(-y + \omega^2/2)^2 + x^2 : (x, y) \in \mathbb{R}^2, x^2 \geq 2\omega^2 y \} > 0.
\]
The latter statement follows from the same argument describing the wall in [12, Proposition 9.3], after replacing [12, Lemma 5.1] by Proposition 3.18.

The following lemma is required to show another version of BG type inequality.

**Lemma 3.20.** There is a constant \( C_\omega > 0 \), which depends only on the class \( [\omega] \in \mathbb{P}(H^2(Y)_{\mathbb{R}}) \), such that for any \( E \in \text{Per}_{<1}(X/Y) \) with \( \text{Hom}(\text{Per}_0(X/Y), E) = 0 \), we have the inequality
\[
\text{ch}_1(E)^2 \omega^2 + C_\omega (\text{ch}_1(E) \cdot f^*\omega)^2 \geq 0.
\]

**Proof.** Let us write
\[
\text{ch}_1(E) = f^*\alpha + k[C],
\]
for some \( k \in \mathbb{Z} \) and \( \alpha \in H^2(Y) \). Since \( \text{ch}_1(E) \cdot C \geq 0 \) by Lemma 3.7, we have \( k \leq 0 \). Next let us take \( a_\omega > 0 \) so that \( f^*\omega - a_\omega \sqrt{\omega^2 C} \) is ample. Note that we can take \( a_\omega \) so that it depends only on the class...
$[\omega] \in \mathbb{P}(H^2(Y)_{\mathbb{R}})$. By Proposition 5.3 (ii) and the assumption, we must have $\mathcal{H}^{-1}(E) = 0$. Therefore $E$ is a one dimensional sheaf, hence $\text{ch}_1(E) \cdot (f^*\omega - a_\omega \sqrt{\omega^2 C}) \geq 0$. This implies

$$-\frac{\alpha \cdot \omega}{a_\omega \sqrt{\omega^2}} \leq k \leq 0.$$  

On the other hand, since $\alpha$ is an effective class on $Y$, there is $C'_\omega > 0$ which only depends on the class $[\omega] \in \mathbb{P}(H^2(Y)_{\mathbb{R}})$ such that (cf. the proof of [6, Corollary 7.3.1])

$$\alpha^2 \omega^2 + (\alpha \cdot \omega)^2 C'_\omega \geq 0.$$  

By combining the above inequalities, we obtain

$$\text{ch}_1(E)^2 = \alpha^2 - k^2 $$

$$\geq - \left( C'_\omega + \frac{1}{a_\omega^2} \right) \frac{(\alpha \cdot \omega)^2}{\omega^2}.$$  

Hence $C_\omega := C'_\omega + 1/a_\omega^2$ satisfies the desired property. \(\Box\)

Let $D$ be the set of isomorphism classes of objects $E \in \mathcal{B}_{f^*\omega}$ satisfying one of the following:

- $E \in \text{Per}(X/Y)$, $\text{ch}_0(E) > 0$ and it is $\mu_{f^*\omega}$-semistable.
- $E \in \text{Per}_{\leq 1}(X/Y)$ and it satisfies $\text{Hom}(\text{Per}_0(X/Y), E) = 0$.
- $E$ fits into an exact sequence in $\mathcal{B}_{f^*\omega}$,

$$0 \to F[1] \to E \to T \to 0 \quad (28)$$

where $F \in \text{Per}(X/Y)$ is $\mu_{f^*\omega}$-semistable, $T \in \text{Per}_0(X/Y)$, and $E$ satisfies $\text{Hom}(\text{Per}_0(X/Y), E) = 0$.

**Lemma 3.21.** Suppose that an object $E \in \mathcal{B}_{f^*\omega}$ satisfies $\text{Im} Z_{f^*\omega}(E) > 0$ and $Z_{s f^*\omega}$-semistable for $s \gg 0$. Then we have $E \in D$.

**Proof.** The proof is given by the same argument as in [12, Proposition 14.2], [3, Lemma 4.2], after replacing $\text{Coh}(X)$ by $\text{Per}(X/Y)$. We omit the detail. \(\Box\)

Let $c \in \mathbb{Z}_{>0}$ be

$$c := \min \{ \text{ch}_1(E) \cdot f^*\omega > 0 : E \in \mathcal{B}_{f^*\omega} \}.$$  

**Lemma 3.22.** For an object $E \in \mathcal{B}_{f^*\omega}$, suppose that $\text{ch}_1(E) \cdot f^*\omega = c$. Then $E \in D$.

**Proof.** The same argument of [6, Lemma 7.2.2] is applied, after replacing $\text{Coh}(X)$ by $\text{Per}(X/Y)$. Again we omit the detail. \(\Box\)

Combining the above results, we prove the BG type inequality evaluating $(\text{ch}_1(*))^2$. 

Theorem 3.23. There is a constant $C_{\omega} > 0$, which depends only on the class $[\omega] \in \mathbb{P}(H^2(Y)_{\mathbb{R}})$, such that the following holds: for any $\sigma_0$-semistable object $E \in \mathcal{B}_{f^*\omega}$ with $\text{ch}_1(E) \cdot f^*\omega > 0$, we have the inequality

$$
\text{ch}_1(E)^2 - 2\left(\text{ch}_0(E) \cdot \text{ch}_2(E) + C_{\omega}\frac{(\text{ch}_1(E) \cdot f^*\omega)^2}{\omega^2}\right) \geq 0.
$$

Proof. The proof proceeds as in the proof of [6, Theorem 7.3.1], so we just give an outline of the proof. We show that the inequality (29) holds for any $Z_{s'}f^*\omega$-semistable object $E \in \mathcal{B}_{f^*\omega}$ with $\text{ch}_1(E) \cdot f^*\omega > 0$ and $s' \in \mathbb{R}_{>0}$, by the induction on $\text{ch}_1(E) \cdot f^*\omega$.

The first step is the case that $\text{ch}_1(E) \cdot f^*\omega = c$. In this case, we have $E \in \mathcal{D}$ by Lemma 3.22. Hence the inequality (29) follows from Proposition 3.9, Lemma 3.20, except the case that $E$ fits into an exact sequence (28). In the last case, the inequality (29) can be proved along with the same argument of Proposition 3.9. Indeed it is enough to replace the sequence (13) by (28) and the same computation works.

Suppose that (29) holds for any $Z_{s'}f^*\omega$-semistable object $F \in \mathcal{B}_{f^*\omega}$ with $s' > 0$ and $0 < \text{ch}_1(F) \cdot f^*\omega < \text{ch}_1(E) \cdot f^*\omega$. We consider the following set:

$$
\mathcal{S} := \{s'' \geq s : E \text{ is } Z_{s''}f^*\omega \text{-semistable}\},
$$

and set $s_0 := \sup\{s'' \in \mathcal{S}\}$. If $s_0 = \infty$, we have $E \in \mathcal{D}$ by Lemma 3.22, so the inequality (29) holds. Otherwise, by the result of Proposition 3.19, the set $\mathcal{S}$ is a closed subset of $\mathbb{R}_{>0}$. By the existence of wall and chamber structure in Proposition 3.19, we have the following: there is $0 < \epsilon \ll 1$ and an exact sequence in $\mathcal{B}_{f^*\omega}$

$$
0 \to E_1 \to E \to E_2 \to 0
$$

such that

$$
\text{arg } Z_{s_0}f^*\omega(E_1) = \text{arg } Z_{s_0}f^*\omega(E_2),
$$

$$
\text{arg } Z_{(s_0+\epsilon)}f^*\omega(E_1) > \text{arg } Z_{(s_0+\epsilon)}f^*\omega(E_2).
$$

Note that $E_i$ are $Z_{s_0}f^*\omega$-semistable with $0 < \text{ch}_1(E_i) \cdot f^*\omega < \text{ch}_1(E) \cdot f^*\omega$. Hence by the assumption of the induction, the objects $E_i$ satisfy the inequality (29). Together with the equality (30) and the inequality (31), we can apply the exactly same computation in [6, Theorem 7.3.1], and show that $E$ satisfies (29).

Corollary 3.24. Let $C_{\omega} > 0$ be as in Theorem 3.23. Then for any $\sigma_0$-stable object $E \in \mathcal{B}_{f^*\omega}$, we have the inequality

$$
\text{ch}_1(E)^2 - 2\left(\text{ch}_0(E) \cdot \text{ch}_2(E) + C_{\omega}\frac{(\text{ch}_1(E) \cdot f^*\omega)^2}{\omega^2}\right) \geq -1.
$$

Proof. By Theorem 3.23, we may assume $\text{ch}_1(E) \cdot f^*\omega = 0$. In this case, $E$ is contained in the RHS of (17). Since $E$ is $\sigma_0$-stable, either $E = F[1]$ for a $\mu_{f^*\omega}$-semistable sheaf $F$ with $\mu_{f^*\omega}(F) = 0$ or $E$ is a simple object in $\text{Per}_0(X/Y)$. In the former case, the inequality (32)
follows from Proposition 3.9. In the latter case, the inequality (32) follows since it is satisfied for simple objects in $\text{Per}_0(X/Y)$ described in Proposition 3.4 (i).

\[ \square \]

Remark 3.25. The equality is achieved in (32) if and only if $E$ is isomorphic to $O_C$ or $O_C(-1)[1]$.  

3.7. Proof of Proposition 3.13.

Proof. We first fix the norm $\|\cdot\|$ on $N(X)_\mathbb{R}$. Let us embed $N(X)_\mathbb{R}$ into $H^*(X, \mathbb{R})$ via the Chern character map. For an element $(r, \beta, n) \in H^0(X) \oplus H^2(X) \oplus H^4(X)$, we can write

$$
\beta = \beta_+ + \beta_- , \quad \beta_+ \in \mathbb{R}[f^*\omega], \quad \beta_- \in (f^*\omega)^\perp.
$$

Note that $\beta_-^2 \leq 0$ by the Hodge index theorem. We set

$$
\| (r, \beta, n) \| := \max \{ |r|, |n|, |\beta_+ \cdot f^*\omega|, \sqrt{-\beta_-^2} \}.
$$

Let us bound $\| E \|/|Z_{f^*\omega}(E)|$ for $\sigma_0$-semistable objects $E \in B_{f^*\omega}$. By the triangle inequality, it is enough to bound it for $\sigma_0$-stable objects. If we write $\text{ch}(E) = (r, \beta_+ + \beta_- , n)$ as above, then

$$
\| E \|/|Z_{f^*\omega}(E)|
$$

coincides with

$$
\frac{\max \{ |r|, |n|, |\beta_+ \cdot f^*\omega|, \sqrt{-\beta_-^2} \}}{\sqrt{(-n + r\omega^2/2)^2 + (\beta_+ \cdot f^*\omega)^2}}.
$$

By the same argument of Proposition 3.19, the values

$$
\frac{|r|}{|Z_{f^*\omega}(E)|}, \quad \frac{|n|}{|Z_{f^*\omega}(E)|}, \quad \frac{|\beta_+ \cdot f^*\omega|}{|Z_{f^*\omega}(E)|}
$$

can be shown to be bounded. It remains to show the boundedness of $|\sqrt{-\beta_-^2}|/|Z_{f^*\omega}(E)|$.

In the above notation, the inequality (32) is written as

$$
\beta_+^2 + \beta_-^2 - 2rn + \frac{C_\omega(\beta_+ \cdot f^*\omega)^2}{\omega^2} \geq -1.
$$

Noting $\beta_+^2 \omega^2 = (\beta_+ \cdot f^*\omega)^2$, we have

$$
\frac{-\beta_-^2}{|Z_{f^*\omega}(E)|^2} \leq \frac{1 + C_\omega + 1) \beta_-^2 - 2rn}{(-n + r\omega^2/2)^2 + \beta_+^2 \omega^2}.
$$

In order to obtain the upper bound of the RHS, we may assume $rn < 0$. By setting $u = -n/r > 0$, the RHS of (33) is shown to be bounded above by the fact

$$
\sup \left\{ \frac{u}{(u + \omega^2/2)^2} : u > 0 \right\} < \infty.
$$

\[ \square \]
4. Extremal contractions of projective 3-folds

In what follows, \( X \) is a smooth projective 3-fold and \( f \) is a birational extremal contraction

\[
f : X \to Y,
\]

i.e. \( Y \) is a normal projective variety, \(-K_X\) is \( f \)-ample and the relative Picard number of \( f \) is equal to one. In this section, we construct the perverse heart associated to \( f \), and establish a derived equivalence between \( X \) and a certain sheaf of non-commutative algebras on \( Y \).

4.1. Classification. By [29], the extremal contraction (34) is classified into five types:

- **Type I**: \( Y \) is smooth and \( f \) is a blow up at a smooth curve.
- **Type II**: \( Y \) is smooth and \( f \) is a blow-up at a point.
- **Type III**: \( Y \) has an ordinary double point, and \( f \) is a blow-up at the singular point.
- **Type IV**: \( Y \) has an orbifold singularity \( \mathbb{C}^3/(\mathbb{Z}/2\mathbb{Z}) \) and \( f \) is a blow-up at the singular point.
- **Type V**: \( Y \) has a cA\(_2\)-singularity and \( f \) is a blow-up at the singular point.

We denote by \( D \subset X \) the exceptional divisor. When \( f(D) \) is a point \( 0 \in Y \), (i.e. \( f \) is not type I,) we denote by \( \hat{O}_{Y,0} \) the completion of \( O_{Y,0} \) at 0. In types III, IV and V, we have

\[
\hat{O}_{Y,0} \cong \begin{cases} 
\mathbb{C}[[x, y, z, w]]/(xy + zw) & \text{in type III} \\
\mathbb{C}[[x, y, z]]^{(2)} & \text{in type IV} \\
\mathbb{C}[[x, y, z, w]]/(xy + z^2 + w^3) & \text{in type V}
\end{cases}
\]

and

\[
(D, O_D(D)) \cong \begin{cases} 
(\mathbb{P}^1 \times \mathbb{P}^1, O_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)) & \text{in type III} \\
(\mathbb{P}^2, O_{\mathbb{P}^2}(-2)) & \text{in type IV} \\
(D \subset \mathbb{P}^3, O_D(-1)) & \text{in type V}
\end{cases}
\]

Here \( \mathbb{C}[[x, y, z]]^{(2)} \) is the subring of invariants of \( \mathbb{C}[[x, y, z]] \) for the involution \((x, y, z) \mapsto (-x, -y, -z)\), and \( D \subset \mathbb{P}^3 \) in type V is the normal singular quadric surface.

4.2. Perverse t-structure for extremal contractions. In this subsection, we construct a perverse t-structure following [34]. We set

\[
E_0 := O_X \oplus O_X(D).
\]

First we note the following lemma:

**Lemma 4.1.** (i) The line bundle \( O_X(-D) \) is \( f \)-ample and \( f \)-globally generated.
(ii) The vector bundles $E_0, E_0|_D$ satisfy the following for $i > 0$:

\[ H^i(Rf_*\text{Hom}(E_0, E_0)) \cong 0, \]

\[ H^i(Rf|_D*\text{Hom}(E_0|_D, E_0|_D)) \cong 0. \]

Proof. Both of (i) and (ii) are easily proved by the classification in Subsection 4.1, or using vanishing theorem. □

The result of (ii) in particular implies

\[ R^2 f_*\mathcal{O}_X(D) = R^2 f|_D*\mathcal{O}_D(D) = 0. \]

Hence we are in the situation where the result of [34] is applied for $f$ and $f|_D$. In particular, we can define the perverse t-structures on $D^b\text{Coh}(X)$ and $D^b\text{Coh}(D)$. Below we recall their constructions.

Let $A_0 := f_*\text{End}(E_0)$ be the sheaf of non-commutative algebras on $Y$, and $\Phi_0$ be the functor

\[ \Phi_0 := Rf_*R\text{Hom}(E_0, *) : D^b\text{Coh}(X) \to D^b\text{Coh}(A_0). \]

The above functor is an equivalence when $f$ is type I by [14], since the dimension of the fiber of $f$ is at most one dimensional in this case. However $\Phi_0$ is not an equivalence in other cases. Indeed, the following subcategory is non-trivial:

\[ C := \{ E \in D^b\text{Coh}(X) : \Phi_0(E) \cong 0 \}. \]

The key result of [34, Theorem 6.1] is that the standard t-structure on $D^b\text{Coh}(X)$ induces a t-structure on $C$. Let

\[ (C^{\leq 0}, C^{\geq 0}) \]

be the induced t-structure with heart $C^0 := C \cap \text{Coh}(X)$. The heart $^p\text{Per}(X/Y)$ is defined as follows:

**Definition 4.2.** We define $^p\text{Per}(X/Y)$ to be

\[ ^p\text{Per}(X/Y) := \left\{ E \in D^b\text{Coh}(X) : \Phi_0(E) \in \text{Coh}(A_0), \text{Hom}(C^{<p}, E) = \text{Hom}(E, C^{-p}) = 0 \right\}. \]

As in [34], the category $^p\text{Per}(X/Y)$ is the heart of a bounded t-structure on $D^b\text{Coh}(X)$. In the 3-fold case, we use the perversity $p = 0$. By abuse of notation, we set

\[ \text{Per}(X/Y) := 0\text{Per}(X/Y). \]

Again we denote by $H^i_p(\cdot)$ the $i$-th cohomology functor with respect to the t-structure with heart $\text{Per}(X/Y)$.

By replacing $E_0, A_0$ by $E_0|_D, A_{D,0} := f|_D*\text{End}(E|_D)$ respectively, we can define the similar functor

\[ \Phi_{D,0} = Rf|_D*R\text{Hom}(E|_D, *) : D^b\text{Coh}(D) \to D^b\text{Coh}(A_{D,0}) \]

\[ ^5\text{This was denoted by } ^p\text{Per}(X/A_0) \text{ in [34].} \]

\[ ^6\text{Here we use the different perversity from the surface case.} \]
and a similar subcategory
$$C_D := \{ E \in D^b \text{Coh}(D) : \Phi_{D,0}(E) \cong 0 \},$$
such that $$C_D^0 := C_D \cap \text{Coh}(D)$$ is the heart of a bounded t-structure on $$C_D$$. The perverse heart is also defined similarly,
$$\text{Per}(D) := \left\{ E \in D^b \text{Coh}(D) : \Phi_{D,0}(E) \in \text{Coh}(A_D,0), \ \text{Hom}(C_{D,0}^0, E) = \text{Hom}(E, C_D^0) = 0 \right\}.$$

**Remark 4.3.** When $$f(D)$$ is a point, then $$A_D,0$$ is the path algebra of a quiver with two vertices and $$\dim H^0(D, O_D(-D))$$-arrows between them in one direction.

### 4.3. Derived equivalence

The purpose here is to give an analogue of Theorem 3.2 for our threefold situation. Suppose that $$f(D)$$ is a point $$0 = f(D) \in Y$$. If we denote by $$m_0 \subset O_Y$$ the sheaf of defining ideal of $$0 \in Y$$, then the sheaf of algebras $$A_0$$ is written as a matrix

$$A_0 = \begin{pmatrix} O_Y & m_0 \\ O_Y & O_Y \end{pmatrix},$$

since $$f_* O_X(D) \cong O_Y$$ and $$f_* O_X(-D) = m_0$$. Let $$S \in \text{Coh}(A_0)$$ be the object, which is isomorphic to $$O_Y/m_0$$ as a $$O_Y$$-module, and the right $$A_0$$-action is given by

$$x \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = x \cdot a.$$

We have the following lemma:

**Lemma 4.4.** Suppose that $$f(D)$$ is a point. Then there is a distinguished triangle in $$D^b \text{Coh}(A_0),$$

$$\Phi_0(O_X(D)) \to \Phi_0(O_X(2D)) \to S^{\leq l}[-2].$$

Here $$l := \dim H^2(D, O_D(2D)).$$

**Proof.** By the exact sequence of sheaves

$$0 \to O_X(D) \to O_X(2D) \to O_D(2D) \to 0,$$

we have the distinguished triangle

$$\Phi_0(O_X(D)) \to \Phi_0(O_X(2D)) \to \Phi_0(O_D(2D)).$$

On the other hand, we have the isomorphism $$\Phi_0(O_D(2D)) \cong S^{\leq l}[-2].$$

If $$f$$ is either type III, IV or V, we have $$H^2(D, O_D(2D)) \neq 0$$. In this case, the above lemma implies that the vector bundle

$$O_X \oplus O_X(D) \oplus O_X(2D)$$

does not satisfy the property similar to Lemma 4.1 (ii). In order to give a derived equivalence as in Theorem 1.3, a modification of $$O_X(2D)$$ is required. By improving the argument of [34], we have the following result:
Theorem 4.5. Let $X$ be a smooth projective 3-fold, $f: X \to Y$ an extremal contraction and $D \subset X$ the exceptional divisor. Then there is a vector bundle $V$ on $X$ satisfying

- If $f(D)$ is a curve, then $V = 0$.
- If $f(D)$ is a point, then $V$ fits into an exact sequence of sheaves

$$0 \to O_X(2D) \to V \to O_X(D - 2f^*H)_{\text{et}} \to O_X(-f^*H)_{\text{et}} \to 0,$$

where $H$ is a sufficiently ample divisor on $Y$ and $e, l$ are given by

$$e = \dim H^0(X, O_X(f^*H - D)), \quad l = \dim H^2(D, O_D(2D)),$$

such that, setting $E = O_X \oplus O_X(D) \oplus V$ and $A = f_*\text{End}(E)$, we have the equivalence of derived categories

$$\Phi := Rf_*R\text{Hom}(E, \ast) : D^b\text{Coh}(X) \xrightarrow{\sim} D^b\text{Coh}(A),$$

which restricts to an equivalence between $\text{Per}(X/Y)$ and $\text{Coh}(A)$.

Proof. If $f(D)$ is a curve, the result follows from [14]. Below we assume $f(D)$ is a point $0 \in Y$.

Let $H$ be a sufficiently ample divisor on $Y$ so that $m_0(H)$ is globally generated, and satisfies

$$\text{Ext}^j_{\mathcal{A}_0}(\mathcal{A}_0, O_Y(iH) \otimes O_Y \mathcal{A}_0) \cong H^j(Y, O_Y(iH) \otimes O_Y \mathcal{A}_0) \cong 0$$

for any $i \geq 1$ and $j \geq 1$. Let us take a basis $x_1, \ldots, x_e$,

$$x_i \in \Gamma(X, m_0(H)) = \Gamma(X, O_X(f^*H - D)).$$

We have the right exact sequence in $\text{Coh}(\mathcal{A}_0)$

$$\mathcal{A}_0 \oplus (O_Y(-H) \otimes O_Y \mathcal{A}_0^{\text{et}}) \to \mathcal{A}_0 \to S \to 0.$$

Here the middle morphism takes $1 \in \mathcal{A}_0$ to $1 \in S$, and the left one is given by the $(e + 1)$-matrices:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_i \\ 0 & 0 \end{pmatrix}, \quad 1 \leq i \leq e,$$

in the expression [35]. Below if we write $(\mathcal{F} \to \mathcal{G})$, this always means the two term complex with $\mathcal{F}$ located in degree one. By taking the $l$-direct sum of the resolution [41] tensored by $O_Y(-H)$, and combining the distinguished triangle [35], we obtain the morphisms

$$O_Y(-H) \otimes O_Y (\mathcal{A}_0 \oplus (O_Y(-H) \otimes O_Y \mathcal{A}_0^{\text{et}})) \to \mathcal{A}_0^{\text{et}} \xrightarrow{\psi} S^{\oplus t}[-2] \to \Phi_0(O_X(D))[1].$$

Since $\Phi_0(O_X(D))$ is a direct summand of $\mathcal{A}_0$, the condition [39] implies that the composition of the above morphisms is zero. Hence $\psi$ lifts to
a morphism
\[ (42) \quad \mathcal{O}_Y(-H) \otimes_{\mathcal{O}_Y} (\mathcal{A}_0 \oplus (\mathcal{O}_Y(-H) \otimes_{\mathcal{O}_Y} \mathcal{A}_0^{\oplus e})) \rightarrow \mathcal{A}_0^{\oplus l} \rightarrow \Phi_0(\mathcal{O}_X(2D)). \]

Note that the functor \( \Psi_0: D^b \text{Coh}(\mathcal{A}_0) \rightarrow D^- \text{Coh}(X) \) given by
\[ \Psi_0(\mathcal{M}) := \mathcal{M} \otimes_{\mathcal{A}_0} \mathcal{E}_0 \]
is a left adjoint of \( \Phi_0 \). Taking the adjunction of \( (42) \), we obtain the morphism
\[ (43) \quad \left( \mathcal{E}_0 \oplus \mathcal{E}_0^{\oplus e}(-f^*H) \rightarrow \mathcal{E}_0 \right)^{\oplus l} \rightarrow \mathcal{O}_X(2D). \]

Since \( m_0(H) \) is globally generated and \(-D\) if \( f\)-globally generated, the morphism \( \phi \) is surjective and \( \text{Ker}(\phi) \) is given by
\[ \mathcal{O}_X \oplus \mathcal{O}_X(-f^*H)^{\oplus e} \oplus \text{Ker}(\mathcal{O}_X(D - f^*H)^{\oplus e} \rightarrow \mathcal{O}_X). \]

Here the morphism \( \psi' \) is given by the sections \( (40) \). Let \( \mathcal{V}' \) be the cone of the morphism \( (43) \). It fits into the exact sequence of sheaves
\[ 0 \rightarrow \mathcal{O}_X(2D) \rightarrow \mathcal{V}' \rightarrow \text{Ker}(\phi)^{\oplus l}(-f^*H) \rightarrow 0. \]

If we take \( H \) sufficiently ample, the vector bundle \( \mathcal{V}' \) splits into
\[ \mathcal{V}' \cong \mathcal{V} \oplus \mathcal{O}_X(-f^*H)^{\oplus l} \oplus \mathcal{O}_X(-2f^*H)^{\oplus d}, \]
where \( \mathcal{V} \) is a vector bundle on \( X \) which fits into a desired exact sequence \( (38) \).

We claim that \( \mathcal{E} = \mathcal{E}_0 \oplus \mathcal{V} \) gives the desired derived equivalence \( (38) \). Let \( 0 \in U \subset Y \) be an affine open neighborhood. Then by the construction and Lemma \( 4.4 \), we can take a free left \( \mathcal{A}_0 \mid_U \)-resolution \( \mathcal{P}^* \) of \( \Phi_0(\mathcal{O}_X(2D)) \mid_U \), so that its stupid filtration \( \sigma_{\geq 1} \mathcal{P}^* \) together with the canonical morphism
\[ \sigma_{\geq 1} \mathcal{P}^* \rightarrow \mathcal{P}^* \Rightarrow \Phi_0(\mathcal{O}_X(2D)) \mid_U \]
is identified with \( (42) \) restricted to \( U \). Hence the vector bundle \( \mathcal{V}' \mid_{f^{-1}(U)} \) fits into the distinguished triangle
\[ \Psi_0(\sigma_{\geq 1} \mathcal{P}^*) \rightarrow \mathcal{O}_X(2D) \mid_{f^{-1}(U)} \rightarrow \mathcal{V}' \mid_{f^{-1}(U)}. \]

By \( 34 \), Equation (13)], this implies that the vector bundle
\[ \mathcal{E}_0 \mid_{f^{-1}(U)} \oplus \mathcal{V}' \mid_{f^{-1}(U)} \]
on \( f^{-1}(U) \) is nothing but a projective generator of \( \text{Per}(f^{-1}(U)/U) \) constructed in \( 34 \), Subsection 4.4. Since \( \mathcal{V}' \mid_{f^{-1}(U)} \) is a direct sum of \( \mathcal{V} \mid_{f^{-1}(U)} \) and \( \mathcal{O}_{f^{-1}(U)}^{\oplus d} \), the vector bundle \( \mathcal{E} \mid_{f^{-1}(U)} \) is also a projective generator of \( \text{Per}(f^{-1}(U)/U) \). Noting that \( f \) is an isomorphism outside \( 0 \in Y \), this implies that \( \mathcal{E} \) is a local projective generator of \( \text{Per}(X/Y) \) in the sense of \( 14 \), Proposition 3.3.1]. Therefore the equivalence \( (38) \) follows from \( 14 \), Proposition 3.3.1. \( \square \)
Similarly to (11), we define the following subcategory
\[(44) \quad \text{Per}_{\leq i}(X/Y) := \{E \in \text{Per}(X/Y) : \Phi(E) \in \text{Coh}_{\leq i}(\mathcal{A})\},\]
and write \(\text{Per}_0(X/Y) := \text{Per}_{\leq 0}(X/Y)\).

**Remark 4.6.** By the construction of \(\mathcal{E}\), it is easy to see that \(E \in \text{Per}(X/Y)\) is an object in \(\text{Per}(X/Y)\) iff \(\text{Supp}(E)\) is contained in a finite number of fibers of \(f\), an object in \(\text{Per}_{\leq 1}(X/Y)\) iff \(\text{Supp}(E)\) is at most one dimensional outside \(D\), and an object in \(\text{Per}_{\leq 2}(X/Y)\) iff \(\text{Supp}(E)\) is at most two dimensional.

If \(f(D)\) is a point \(0 \in Y\), we set
\[
\hat{Y} := \text{Spec} \hat{O}_{Y,0}, \quad \hat{X} := X \times_Y \hat{Y}.
\]
The categories \(\text{Per}(\hat{X}/\hat{Y})\), \(\text{Per}_0(\hat{X}/\hat{Y})\) are similarly defined. The following local version is also proved in a similar way.

**Corollary 4.7.** Suppose that \(f(D)\) is a point \(0 \in Y\). Then there is a vector bundle \(\hat{V}\) on \(\hat{X}\) which fits into an exact sequence of sheaves
\[(45) \quad 0 \to \mathcal{O}_X(2D) \to \hat{V} \to \mathcal{O}_X(D)^{\oplus kl} \to \mathcal{O}_{\hat{X}}^{|D|} \to 0 \]
where \(k\) is given by \(k := \dim H^0(D, \mathcal{O}_D(-D))\) such that, setting \(\hat{\mathcal{E}} = \mathcal{O}_X \oplus \mathcal{O}_X(D) \oplus \hat{V}\) and \(\hat{A} = \text{End}(\hat{\mathcal{E}})\), we have the equivalence of derived categories
\[
\hat{\Phi} := \mathbf{R} \text{Hom}(\hat{\mathcal{E}}, *) : D^b \text{Coh}(\hat{X}) \xrightarrow{\sim} D^b \text{mod}(\hat{A})
\]
which restricts to an equivalence between \(\text{Per}(\hat{X}/\hat{Y})\) and \(\text{mod}(\hat{A})\).

**Proof.** We note that \(k = \dim m_0/m_0^2\). The proof is obtained by modifying the proof of Theorem 4.5 by replacing the sections (40) by the minimal generators of \(m_0 \subset \hat{O}_{Y,0}\). □

We also have the version for the exceptional locus.

**Corollary 4.8.** In the situation of Corollary 4.7, setting \(\mathcal{V}_D := \hat{V}|_D\), \(\mathcal{E}_D := \mathcal{O}_D \oplus \mathcal{O}_D(D) \oplus \mathcal{V}_D\) and \(A_D := \text{End}(\mathcal{E}_D)\), we have the derived equivalence
\[
\Phi_D := \mathbf{R} \text{Hom}(\mathcal{E}_D, *) : D^b \text{Coh}(D) \xrightarrow{\sim} D^b \text{mod}(A_D)
\]
which restricts to an equivalence between \(\text{Per}(D)\) and \(\text{mod}(A_D)\). Furthermore \(\mathcal{V}_D\) fits into the universal extension
\[(46) \quad 0 \to \mathcal{O}_D(2D) \to \mathcal{V}_D \to \Omega^{|D|}_{P^{k-1}}|_D \to 0.
\]
Here \(D\) is embedded into \(P^{k-1}\) by the linear system \(|\mathcal{O}_D(-D)|\).
Proof. By the construction of \( \hat{\mathcal{E}} \) in Corollary 4.7, the vector bundle \( \mathcal{E}_D \) coincides with the construction of the tilting generator on \( D^b \text{Coh}(D) \) given in [34]. The exact sequence (46) is obtained by restricting the sequence (45) to \( D \). Noting that 
\[
\text{Ext}_D^1(\Omega_{\mathbb{P}^{k-1}|D}, \mathcal{O}_D(2D)) \cong H^2(D, \mathcal{O}_D(2D)),
\]
the universality of (46) obviously follows from the construction of the exact sequence (45).

\[\square\]

Remark 4.9. The numbers \((k, l)\) are \((3, 0)\) in type II, \((4, 1)\) in types III, V and \((6, 3)\) in type IV.

Remark 4.10. The result of Theorem 4.5 may be applied to the study of minimal saturated triangulated category of singular varieties. Suppose that \( f \) is a type V extremal contraction. In [21], Kawamata proposed that the subcategory 
\[
\mathcal{D}_Y := \{ E \in D^b \text{Coh}(X) : R\text{Hom}(E, \mathcal{O}_D) = 0 \}
\]
is the minimal saturated triangulated subcategory of \( D^b \text{Coh}(X) \) which contains \( f^*\text{Perf}(Y) \). By our construction of the vector bundle \( \mathcal{V} \), we have 
\[
\mathcal{O}_X \oplus \mathcal{V}(-D) \in \mathcal{D}_Y,
\]
and it is a local tilting generator of \( \mathcal{D}_Y \) in the sense of [14, Proposition 3.3.1]. Hence we have the equivalence
\[
Rf_* R\text{Hom}(\mathcal{O}_X \oplus \mathcal{V}(-D), x) : \mathcal{D}_Y \rightarrow D^b \text{Coh}(\mathcal{B}),
\]
where \( \mathcal{B} := f_* \mathcal{E}nd(\mathcal{O}_X \oplus \mathcal{V}(-D)) \). It would be interesting to study the category \( \mathcal{D}_Y \) in terms of the sheaf of non-commutative algebras \( \mathcal{B} \).

4.4. Derived category of the exceptional locus. The rest of this section is devoted to giving a complete description of simple objects in \( \text{Per}_0(X/Y) \) via Theorem 4.5. The strategy is as follows: in this subsection, we describe simple objects in \( \text{Per}(D) \) using Corollary 4.8. Then in the next subsection, we show that simple objects in \( \text{Per}_0(X/Y) \) are either \( \mathcal{O}_x \) for \( x \notin D \) or a pushforward of some simple object in \( \text{Per}(D) \).

Below we use the notation in Corollary 4.8. First we note the following:

Lemma 4.11. We have \( \mathcal{V}_D \in \mathcal{C}_D^0 \), i.e.
\[
R\text{Hom}_D(\mathcal{O}_D, \mathcal{V}_D) = R\text{Hom}_D(\mathcal{O}_D(D), \mathcal{V}_D) = 0.
\]

Proof. The result obviously follows from the exact sequence (46).

Remark 4.12. A similar result is not true for the vector bundle \( \mathcal{V} \) on \( X \). Indeed we have \( \text{Hom}(\mathcal{O}_X, \mathcal{V}) \neq 0 \).

\[\text{7Kawamata informed the author that he later proved the minimality of } \mathcal{D}_Y.\]
Next we investigate the structure of $V_D$.

**Lemma 4.13.** Suppose that $f(D)$ is a point. Then the vector bundle $V_D$ on $D$ is described as follows:

- If $f$ is type II, then $V_D = O_{\mathbb{P}^2}(-2)$.
- If $f$ is type III, then we have
  \[ V_D \cong O_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -2)^{\oplus 2} \oplus O_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -1)^{\oplus 2}. \]
- If $f$ is type IV, then $V_D \cong \Omega_{\mathbb{P}^2}^{\oplus 8}(-1)$.
- If $f$ is type V, then $V_D \cong U(-1)^{\oplus 2}$, where $U$ is the non-split extension

\[ 0 \to \mathcal{O}_D(-C) \to U \to \mathcal{O}_D(-C) \to 0 \] (47)

and $C \subset D$ is a ruling.

**Proof.** The statement is obvious for type II. As for types III and IV, the statement easily follows from Lemma 4.11 and using exceptional collections on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. We focus on the case of type V, which is a less obvious case. It may be possible to construct an isomorphism $U(-1)^{\oplus 2} \cong V_D$ directly, but here we give an indirect proof using the abelian category $\mathcal{C}_D$ given in Subsection 4.2.

Let $D$ be the singular quadric defined by

\[ D = \{xy + z^2 = 0\} \subset \mathbb{P}^3 \]

where $[x : y : z : w]$ is the homogeneous coordinate of $\mathbb{P}^3$. Let $C \subset D$ be a ruling, defined by

\[ C = \{y = z = 0\} \subset D. \]

It is easy to check that

\[ \text{Ext}^i_D(\mathcal{O}_D(-C), \mathcal{O}_D(-C)) = \mathbb{C}, \quad i \geq 0, \] (48)

using the resolution

\[ \cdots \to \mathcal{O}_D(-2)^{\oplus 2} \to \mathcal{O}_D(-1)^{\oplus 2} \xrightarrow{(y,x)} \mathcal{O}_D(-C) \to 0. \]

Here the morphisms between rank two vector bundles are given by the matrix

\[ \begin{pmatrix} -z & x \\ y & z \end{pmatrix}. \]

It follows that there is a unique extension (47) up to isomorphism, and $U$ is a vector bundle of rank two on $D$. Moreover, using (47) and (48), we can check that

\[ \text{Ext}^i_D(\mathcal{O}_D(-C), \mathcal{U}) = 0, \quad i \geq 1. \] (49)
Next we consider the abelian category $\mathcal{C}_D^0$. We first note that any non-zero object $E \in \mathcal{C}_D^0$ is a torsion free sheaf. Indeed $E \in \mathcal{C}_D^0$ implies that
\[ H^0(D, E) = H^0(D, E|_H(H)) = 0, \]
where $H \in |\mathcal{O}_D(1)|$ is a general element. The above vanishing shows that $E$ is torsion free. By this fact, it immediately follows that the object
\[ \mathcal{O}_D(-3C) \in \mathcal{C}_D^0 \]
is a simple object in $\mathcal{C}_D^0$. Conversely it is easy to check that an object $E \in \mathcal{C}_D^0$ with rank$(E) = 1$ should be isomorphic to $\mathcal{O}_D(-3C)$.

By computing Hom-groups using exact sequences (46), (47), it is easy to see that
\[
\text{Hom}(U(−1), \mathcal{V}_D) \cong \mathbb{C}^4, \quad \text{Hom}(\mathcal{O}_D(-3C), \mathcal{V}_D) \cong \mathbb{C}^2.
\]
This implies that there is a non-trivial morphism
\[
(50) \quad U(−1) \to \mathcal{V}_D
\]
such that the composition
\[ \mathcal{O}_D(-3C) \to U(−1) \to \mathcal{V}_D \]
is non-zero. Also note that $\mathcal{V}_D \in \mathcal{C}_D^0$ by Lemma 4.11. Since $\mathcal{O}_D(-3C)$ is the unique simple objects of $\mathcal{C}_D^0$ with rank one, and the sequence (47) is non-split, the morphism (50) should be injective.

Let us take the exact sequence in $\mathcal{C}_D^0$
\[
(51) \quad 0 \to U(−1) \to \mathcal{V}_D \to F \to 0.
\]
By the above sequence and using (49), we see that
\[
\text{Hom}(U(−1), F) \cong \mathbb{C}^2, \quad \text{Hom}(\mathcal{O}_D(-3C), F) \cong \mathbb{C}.
\]
By the same argument as above, we have an injection $U(−1) \to F$, which must be an isomorphism since $\mathcal{C}_D^0$ does not contain non-zero torsion sheaves. Therefore $U(−1) \cong F$, and $\mathcal{V}_D \cong U(−1)^{\oplus 2}$ follows since (51) splits by (19).

Using the above lemma, we determine the simple objects in Per($D$).

**Proposition 4.14.** Suppose that $f(D)$ is a point. Then the abelian category Per($D$) is described by simple objects as follows:

- If $f$ is type II, we have
  \[ \text{Per}(D) = \langle \mathcal{O}_{p_2}(-3)[2], \Omega_{p_2}(-1)[1], \mathcal{O}_{p_2}(-2) \rangle. \]

- If $f$ is type III, we have
  \[ \text{Per}(D) = \langle \mathcal{O}(-2,-2)[2], S_3(-1,-1)[1], \mathcal{O}(-1,-2), \mathcal{O}(-2,-1) \rangle. \]

Here $S_3$ is the kernel of the universal morphism
\[ 0 \to S_3 \to \mathcal{O}(-1,0)^{\oplus 2} \oplus \mathcal{O}(0,-1)^{\oplus 2} \to \mathcal{O} \to 0. \]
• If \( f \) is type IV, we have
\[
\text{Per}(D) = \langle \mathcal{O}_{\mathbb{P}^2}(-3)[2], S_4(-1)[1], \Omega_{\mathbb{P}^2}(-1) \rangle.
\]
Here \( S_4 \) is the kernel of the universal morphism
\[
0 \to S_4 \to \Omega_{\mathbb{P}^2}^3 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to 0.
\]

• If \( f \) is type V, we have
\[
\text{Per}(D) = \langle \mathcal{O}_D(-2)[2], S_5(-1)[1], \mathcal{O}_D(-3C) \rangle.
\]
Here \( C \subset D \) is a ruling and \( S_5 \) is a rank three vector bundle on \( D \) which fits into an exact sequence
\[
0 \to S_5 \to U^{\oplus 2} \to \mathcal{O}_D \to 0,
\]
where \( U \) is defined by (47).

Proof. Let \( \mathcal{V}_D \) be
\[
\mathcal{V}_D = \begin{cases} 
\mathcal{O}_{\mathbb{P}^2}(-2) & \text{in Type II,} \\
\mathcal{O}(-1,-2) \oplus \mathcal{O}(-2,-1) & \text{in Type III,} \\
\Omega_{\mathbb{P}^2}(-1) & \text{in Type IV,} \\
U(-1) & \text{in Type V.}
\end{cases}
\]

By Corollary 4.8 and Lemma 4.13, setting \( \mathcal{E}_D = \mathcal{O}_D \oplus \mathcal{O}_D(-1) \oplus \mathcal{V}_D \) and \( A_D = \text{End}(\mathcal{E}_D) \), we have the derived equivalence
\[
\Phi_D^\dagger = \mathbf{R} \text{Hom}(\mathcal{E}_D, *) : D^b \text{Coh}(D) \xrightarrow{\sim} D^b \text{mod}(A_D^\dagger),
\]
which restricts to an equivalence between \( \text{Per}(D) \) and \( \text{mod}(A_D^\dagger) \). The algebra \( A_D^\dagger \) is a finite dimensional \( \mathbb{C} \)-algebra, and there is a finite number of simple objects in \( \text{mod}(A_D^\dagger) \). Therefore it is enough to find objects in \( \text{Per}(D) \) which are sent to simple objects in \( \text{mod}(A_D^\dagger) \) after applying \( \Phi_D^\dagger \).

We only discuss the type V case. The other cases are similarly discussed. By the sequence (47), we have an isomorphism of \( \mathbb{C} \)-algebras
\[
\text{End}(U(-1)) \cong \mathbb{C}[T]/T^2.
\]
Also noting Lemma 4.11, we see that the algebra \( A_D^\dagger \) is a path algebra of a quiver of the form
\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]
with relations \( T^2 = 0 \) and others which we omit. Therefore there are only three simple objects in \( \text{mod}(A_D^\dagger) \) corresponding to the three vertices.

We have
\[
\dim_{\mathbb{C}} \Phi_D^\dagger(\mathcal{O}_D(-2)[2]) = \dim_{\mathbb{C}} \Phi_D^\dagger(\mathcal{O}_D(-3C)) = 1.
\]
Hence $\mathcal{O}_D(-2)[2]$ and $\mathcal{O}_D(-3C)$ are simple objects in $\text{Per}(D)$, and they correspond to the left vertex, right vertex of the quiver respectively. In order to describe the simple object corresponding to the middle vertex, we note that

$$\mathbf{R} \text{Hom}_D(\mathcal{O}_D, \mathcal{O}_D(-1)) = 0,$$
$$\mathbf{R} \text{Hom}_D(\mathcal{O}_D(-1), \mathcal{O}_D(-1)) = \mathbb{C},$$
$$\mathbf{R} \text{Hom}_D(\mathcal{U}(-1), \mathcal{O}_D(-1)) = \mathbb{C}^4.$$ 

In particular, we have $\mathcal{O}_D(-1) \in \text{Per}(D)$.

The $T$-action on $\text{Hom}_D(\mathcal{U}(-1), \mathcal{O}_D(-1))$ is given by the composition

$$\text{Hom}_D(\mathcal{U}(-1), \mathcal{O}_D(-1)) \rightarrow \text{Hom}_D(\mathcal{O}_D(-3C), \mathcal{O}_D(-1)) \cong \mathbb{C}^{\oplus 2} \hookrightarrow \text{Hom}_D(\mathcal{U}(-1), \mathcal{O}_D(-1)),$$

induced by the exact sequence (47). Therefore we have an isomorphism as $\text{End}(\mathcal{U}(-1))$-modules

$$\text{Hom}_D(\mathcal{U}(-1), \mathcal{O}_D(-1)) \cong \left( \mathbb{C}[T]/T^2 \right)^{\oplus 2},$$

which is isomorphic to $\Phi^1_D(\mathcal{U}(-1)^{\oplus 2})$ as $A^1_D$-modules. If $S_5(-1)[1] \in \text{Per}(D)$ is the simple object corresponding to the middle vertex, the above argument implies the existence of an exact sequence in $\text{Per}(D)$

$$\mathcal{U}(-1)^{\oplus 2} \xrightarrow{\phi_1, \phi_2} \mathcal{O}_D(-1) \rightarrow S_5(-1)[1].$$

Here $\phi_1, \phi_2$ form a basis of $\text{Hom}_D(\mathcal{U}(-1), \mathcal{O}_D(-1))$ as $\mathbb{C}[T]/T^2$-module.

We show that $S_5$ must be a sheaf. Suppose that $S_5$ is not a sheaf, or equivalently $\mathcal{H}^0(S_5(-1)[1])$ is non-zero. By the construction of $\phi_i$, the composition

$$\mathcal{O}_D(-3C) \hookrightarrow \mathcal{U}(-1) \xrightarrow{\phi_i} \mathcal{O}_D(-1)$$

is non-zero. Therefore we have

$$\text{Cok}(\phi_i) \cong \text{Cok}(\phi'_i: \mathcal{O}_D(-3C) \rightarrow \mathcal{O}_C(-1))$$

for some non-zero morphism $\phi'_i$. This implies that $\text{Cok}(\phi_i)$ is zero dimensional, hence $\mathcal{H}^0(S_5(-1)[1])$ is also a zero dimensional sheaf. On the other hand, by the quiver description (53) of $\text{mod}(A^1_D)$, it is easy to see that there is no morphism from $\Phi^1_D(S_5(-1)[1])$ to $\Phi^1_D(\mathcal{O}_x)$ for any $x \in D$. This is a contradiction since the natural morphism

$$S_5(-1)[1] \rightarrow \mathcal{H}^0(S_5(-1)[1])$$

also becomes a zero map. Therefore $S_5$ is a sheaf (indeed a vector bundle) on $D$ which fits into an exact sequence (52).

**Remark 4.15.** The above analysis of the derived category of the singular quadric $D \subset \mathbb{P}^3$ shows that there is a semi orthogonal decomposition

$$D^b \text{Coh}(D) = \langle \mathcal{O}_D, \mathcal{O}_D(-1), D^b \text{mod} (\mathbb{C}[T]/T^2) \rangle.$$
This fact seems to be not seen in literatures.

4.5. **Simple objects in** \(\text{Per}_0(X/Y)\). In this subsection, we determine the simple objects in \(\text{Per}_0(X/Y)\), where \(\text{Per}_0(X/Y)\) is defined in \((44)\).

We first recall the result by Van den Bergh \([14]\).

**Proposition 4.16.** \((14)\) Suppose that \(f(D)\) is a curve. Then \(\text{Per}_0(X/Y)\) is described by simple objects as

\[
\text{Per}_0(X/Y) = \langle O_x, O_{L_y}(-2)[1], O_{L_y}(-1) : x \in X \setminus D, y \in f(D) \rangle.
\]

Here \(L_y := f^{-1}(y) \cong \mathbb{P}^1\).

Below we assume that \(f(D)\) is a point. We need the following property of \(\text{Per}(X/Y)\), which will be also used in the next section:

**Proposition 4.17.** Suppose that \(f(D)\) is a point. For \(E \in \text{Per}(X/Y)\), we have

(i) \(H^i(E) = 0\) for \(i \notin \{-2, -1, 0\}\).
(ii) \(H^{-1}(E), H^{-2}(E)\) are supported on \(D\).
(iii) \(H^0(E), H^{-2}(E)[2] \in \text{Per}(X/Y)\).

**Proof.** The statement is local on \(Y\), so we may assume that \(Y\) is affine. Let us take a general member \(H \in |O_X(-D)|\). We have the distinguished triangle

\[
Rf_*E \to Rf_* (E(-D)) \to Rf_* (E|_H(-D)).
\]

Since \(\Phi_0(E) \subset \text{Coh}(\omega_0)\), the left two objects are coherent sheaves on \(Y\). Therefore

\[
R^i f_* (E|_H(-D)) = 0, \quad i \neq -1, 0.
\]

On the other hand, we have the spectral sequence

\[
E_2^{p,q} = R^p f_* (H^q(E)|_H(-D)) \Rightarrow R^{p+q} f_* (E|_H(-D))
\]

which degenerates since the fibers of \(f|_H\) are at most one dimensional. (cf. \([11]\) Lemma 3.1.) Therefore we have

\[
R^p f_* (H^q(E)|_H(-D)) = 0
\]

if \(p + q \neq -1, 0\), or equivalently

\[
(p, q) \neq (0, 0), (0, -1), (1, -1), (1, -2).
\]

Hence for \(q \notin \{-2, -1, 0\}\), we have the isomorphism

\[
Rf_* H^q(E) \cong Rf_* (H^q(E)(-D))
\]

We show that \((56)\) vanishes for \(q \notin \{-2, -1, 0\}\). Let us consider the spectral sequence

\[
E_2^{p,q} = R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E.
\]
Since $Rf_*E \in \text{Coh}(Y)$, the morphism
\begin{equation}
(58) \quad f_*\mathcal{H}^q(E) \to R^2f_*\mathcal{H}^{q-1}(E)
\end{equation}
is an isomorphism for $q \notin \{-1, 0\}$, injective for $q = -1$ and surjective for $q = 0$. Also we have
\[ R^1f_*\mathcal{H}^q(E) = 0, \quad q \neq -1. \]
Hence if we have
\begin{equation}
(59) \quad R^2f_*\mathcal{H}^q(E) = 0, \quad q \leq -3, q \geq 0,
\end{equation}
then the vanishing of (56) for $q \notin \{-2, -1, 0\}$ follows. The condition (59) follows from the vanishing (55) for $p = 1$, $q \leq -3$, $q \geq 0$ and the duality argument given in the proof of [34, Theorem 6.1].

By the vanishing of (56) for $q \notin \{-2, -1, 0\}$, we have
\[ \mathcal{H}^q(E) \in \mathcal{C}^0, \quad q \notin \{-2, -1, 0\}. \]
Therefore we must have $\mathcal{H}^0(E) = 0$ by the definition of $\text{Per}(X/Y)$. So (i) is proved. The result of (ii) follows from that (58) is injective for $q = -2, -1$.

Now since (i) is proved, the spectral sequence (57) and a similar one for $R^0f_*\mathcal{H}^0(E)(-D)$ shows
\[ R^i f_*\mathcal{H}^0(E) = R^i f_* (\mathcal{H}^0(E)(-D)) = 0, \quad i \geq 1. \]
Hence we have $\Phi_0(\mathcal{H}^0(E)) \in \text{Coh}(\mathfrak{X}_0)$. Since we have
\[ \text{Hom}(\mathcal{C}^<0, \mathcal{H}^0(E)) = \text{Hom}(\mathcal{C}^>0, \mathcal{H}^0(E)) = 0, \]
it follows that $\mathcal{H}^0(E) \in \text{Per}(X/Y)$. A similar argument also shows $\mathcal{H}^{-2}(E)[2] \in \text{Per}(X/Y)$.

Using the notation of Corollary 4.7 and Corollary 4.8, we show the following:

**Lemma 4.18.** Suppose that $f(D)$ is a point. Then the restriction morphism of algebras
\begin{equation}
(60) \quad \text{End}(\hat{\mathcal{E}}, \hat{\mathcal{E}}) \to \text{End}(\mathcal{E}_D, \mathcal{E}_D)
\end{equation}
is surjective. In particular, the inclusion $i: D \hookrightarrow X$ induces a fully faithful functor
\begin{equation}
(61) \quad i_*: \text{Per}(D) \hookrightarrow \text{Per}_0(X/Y).
\end{equation}

**Proof.** As for the first statement, it is enough to show $\text{Ext}^1_X(\hat{\mathcal{E}}(-D), \hat{\mathcal{E}}) = 0$. By the formal function theorem, we have
\[ \text{Ext}^1_X(\hat{\mathcal{E}}(-D), \hat{\mathcal{E}}) \cong \lim_{\leftarrow} \text{Ext}^1_{D_n}(\hat{\mathcal{E}}(-D)|_{D_n}, \hat{\mathcal{E}}|_{D_n}). \]
Here $D_n$ is the divisor $nD$. Applying $\text{Hom}_X(\hat{\mathcal{E}}(-D), *)$ to the exact sequence
\[ 0 \to \mathcal{E}_D(-nD) \to \hat{\mathcal{E}}|_{D_{n+1}} \to \hat{\mathcal{E}}|_{D_n} \to 0, \]
it is enough to show
\[ \text{Ext}^1_D(\mathcal{E}_D, \mathcal{E}_D(-nD)) = 0, \quad n \geq 0. \]

The above vanishing holds if \( \mathcal{E}_D(-nD) \in \text{Per}(D) \), since \( \mathcal{E}_D \) is a projective object of \( \text{Per}(D) \). Using the exact sequence (46), it is easy to check that
\[ \text{R Hom}_D(\mathcal{O}_D \oplus \mathcal{O}_D(D), \mathcal{E}_D(-nD)) \in \text{mod}(A_D) \]
for \( n \geq 0 \). Noting \( \mathcal{E}_D(-nD) \in \text{Coh}(D) \), it follows that \( \mathcal{E}_D(-nD) \) is an object in \( \text{Per}(D) \).

As for the latter statement, the morphism (60) induces a functor
\[ \text{mod}(A_D) \to \text{mod}(\hat{A}) \]
which is fully faithful since (60) is surjective. Applying \( \Phi_D \) and \( \hat{\Phi} \), we see that \( i_* : D^b \text{Coh}(D) \to D^b \text{Coh}(\hat{X}) \) restricts to a fully faithful functor
\[ (62) \quad i_* : \text{Per}(D) \hookrightarrow \text{Per}_0(\hat{X}/\hat{Y}). \]

On the other hand, the morphism of schemes \( \hat{X} \to X \) obviously induces a fully faithful functor
\[ (63) \quad \text{Per}_0(\hat{X}/\hat{Y}) \hookrightarrow \text{Per}_0(X/Y). \]

By (62) and (63), it follows that \( i_* \) induces the fully faithful functor (61).

**Remark 4.19.** By the surjection (61), it follows that the subcategory
\[ (64) \quad \text{Per}(D) \subset \text{Per}_0(\hat{X}/\hat{Y}) \]
via (62) is closed under subobjects and quotients.

We also prepare another lemma.

**Lemma 4.20.** Let \( E \in \text{Per}_0(\hat{X}/\hat{Y}) \) be a simple object. Then \( \mathcal{H}_i(E) \) is \( \mathcal{O}_D \)-module for all \( i \in \mathbb{Z} \).

**Proof.** For \( x \in m_0 \subset \mathcal{O}_{\hat{Y}} \), we have the morphism
\[ E \xrightarrow{x} E. \]
Since \( E \) is simple and supported on \( D \), the above morphism must be a zero map. In particular, the induced morphism on \( \mathcal{H}_i(E) \) is also a zero map. Since \( \text{Cok}(f^*m_0 \to \mathcal{O}_{\hat{X}}) = \mathcal{O}_D \), this implies that each \( \mathcal{H}_i(E) \) is \( \mathcal{O}_D \)-module.

Using the above lemmas, we show the following:
Proposition 4.21. Suppose that \( f(D) \) is a point. Then \( E \in \text{Per}_0(X/Y) \) is a simple object if and only if \( E \cong \mathcal{O}_x \) for \( x \notin D \) or \( E \cong i_* F \) for a simple object \( F \in \text{Per}(D) \), where \( i_* \) is given in (61). In particular, we have

\[
\text{Per}_0(X/Y) = \langle \mathcal{O}_x, i_* \text{Per}(D) : x \in X \setminus D \rangle.
\]

Proof. It is obvious that an object \( \mathcal{O}_x \) for \( x \notin D \) is a simple object in \( \text{Per}_0(X/Y) \). Let \( E \in \text{Per}_0(X/Y) \) be a simple object which is not isomorphic to \( \mathcal{O}_x \) for any \( x \notin D \). Then \( E \) is supported on \( D \), hence \( E \in \text{Per}_0(\hat{X}/\hat{Y}) \). By Lemma 4.18 and Remark 4.19, it is enough to show that \( E \) is contained in the subcategory (64).

First suppose that \( \mathcal{H}^0(E) \) is non-zero. By Proposition 4.17, \( \mathcal{H}^0(E) \in \text{Per}_0(\hat{X}/\hat{Y}) \) and we have the non-zero morphism

\[
E \to \mathcal{H}^0(E).
\]

The above morphism must be injective in \( \text{Per}_0(\hat{X}/\hat{Y}) \), since \( E \) is a simple object in \( \text{Per}_0(\hat{X}/\hat{Y}) \). By Lemma 4.20, the sheaf \( \mathcal{H}^0(E) \) is \( \mathcal{O}_D \)-module, hence \( \mathcal{H}^0(E) \in \text{Per}(D) \). Then we have \( E \in \text{Per}(D) \) by Remark 4.19.

Next suppose that \( \mathcal{H}^0(E) = 0 \) and \( \mathcal{H}^{-2}(E) \neq 0 \). By the same argument as above, we have the non-zero morphism

\[
\mathcal{H}^{-2}(E)[2] \to E
\]

which must be surjective in \( \text{Per}_0(\hat{X}/\hat{Y}) \). Hence we have \( E \in \text{Per}(D) \) by Lemma 4.20 and Remark 4.19. Also if \( \mathcal{H}^0(E) = \mathcal{H}^{-2}(E) = 0 \), then \( E = \mathcal{H}^{-1}(E)[1] \) and \( E \in \text{Per}(D) \) follows from Lemma 4.20.

Finally \( \text{Per}_0(X/Y) \) is a finite length abelian category by Theorem 4.5, so it is generated by its simple objects. The last statement follows from this fact. \( \square \)

5. Conjectural Bridgeland stability conditions

In this section, we prove Theorem 1.4 using the results in the previous section. Throughout this section, \( X \) is a smooth projective 3-fold, \( f: X \to Y \) is an extremal contraction and \( D \subset X \) is the exceptional divisor.

5.1. First tilting of \( \text{Per}(X/Y) \). For an ample divisor \( \omega \) on \( Y \), we consider \( f^*\omega \)-slope function on \( \text{Per}(X/Y) \) and apply a similar construction as in Subsection 3.4.

For \( E \in \text{Per}(X/Y) \), the \( f^*\omega \)-slope is defined by

\[
\mu_{f^*\omega}(E) = \frac{\text{ch}_1(E) \cdot (f^*\omega)^2}{\text{ch}_0(E)}.
\]

Here \( \mu_{f^*\omega}(E) = \infty \) when \( \text{ch}_0(E) = 0 \). Similarly to Definition 3.5, we can define the \( \mu_{f^*\omega} \)-stability on \( \text{Per}(X/Y) \), which has the Harder-Narasimhan property:
Lemma 5.1. Any object in $\text{Per}(X/Y)$ admits a Harder-Narasimhan filtration with respect to $\mu_{f^\ast \omega}$-stability.

Proof. By Theorem 4.5, the abelian category $\text{Per}(X/Y)$ is noetherian, and the lemma is proved along with the same argument of Lemma 3.6.

By the above lemma, the torsion pair $(T_{f^\ast \omega}, F_{f^\ast \omega})$ is defined exactly as in the same way of Subsection 3.4. Its tilting is similarly defined, 

$$B_{f^\ast \omega} = \langle F_{f^\ast \omega}[1], T_{f^\ast \omega} \rangle.$$ 

We have the following lemma:

Lemma 5.2. The abelian category $B_{f^\ast \omega}$ is noetherian.

Proof. The statement follows from the same argument of [12, Proposition 7.1], with a minor modification. Suppose that there is an infinite sequence of surjections in $B_{f^\ast \omega}$,

$$E = E_1 \to E_2 \to \cdots \to E_i \to E_{i+1} \to \cdots.$$ 

Let us take the exact sequence in $B_{f^\ast \omega}$

$$0 \to L_i \to E \to E_i \to 0.$$

Since $\text{ch}_1(\ast) \cdot (f^\ast \omega)^2 \geq 0$ on $B_{f^\ast \omega}$, we may assume that $\text{ch}_1(E_i) \cdot (f^\ast \omega)^2$ is constant. Hence $\text{ch}_1(L_i) \cdot (f^\ast \omega)^2 = 0$, which implies that

$$L_i \in \left\langle F[1], T : F \in \text{Per}(X/Y) \text{ is } \mu_{f^\ast \omega}\text{-semistable with } \mu_{f^\ast \omega}(F) = 0, T \in \text{Per}_{\leq 1}(X/Y) \right\rangle.$$ 

Applying the same argument of [12, Proposition 7.1], replacing $\text{Coh}(X)$ by $\text{Per}(X/Y)$, we arrive at the exact sequences in $\text{Per}(X/Y)$

$$0 \to Q \to \mathcal{H}^{-1}_p(E_i) \to \mathcal{H}^0_p(L_i) \to 0,$$

where $Q \in F_{f^\ast \omega}$ is independent of $i$ and inclusions in $\text{Per}(X/Y)$

$$\mathcal{H}^{-1}_p(E_1) \subset \mathcal{H}^{-1}_p(E_2) \subset \cdots \subset \mathcal{H}^{-1}_p(E_i) \subset \mathcal{H}^{-1}_p(E_{i+1}) \subset \cdots.$$

We apply the equivalence $\Phi$ in Theorem 4.5 and forget the $\mathcal{A}$-module structures. We obtain the exact sequence in $\text{Coh}(Y)$

$$0 \to \Phi(Q) \to \Phi(\mathcal{H}^{-1}_p(E_i)) \to \Phi(\mathcal{H}^0_p(L_i)) \to 0.$$ 

Since $\mathcal{H}^0_p(L_i) \in \text{Per}_{\leq 1}(X/Y)$, the sheaf $\Phi(\mathcal{H}^0_p(L_i))$ is at most one dimensional. On the other hand, since $Q \in F_{f^\ast \omega}$ satisfies

$$\text{Hom}(\text{Per}_{\leq 2}(X/Y), Q) = 0,$$

the sheaf $\Phi(Q)$ is a torsion free sheaf on $Y$. Similarly $\Phi(\mathcal{H}^{-1}_p(E_i))$ is also torsion free, so we have the sequence in $\text{Coh}(Y)$

$$\Phi(Q) \subset \Phi(\mathcal{H}^{-1}_p(E_1)) \subset \Phi(\mathcal{H}^{-1}_p(E_2)) \subset \cdots \subset \Phi(Q)^{\vee \vee}.$$ 

The above sequence terminates since $\text{Coh}(Y)$ is noetherian. Therefore the sequence (65) also terminates. \qed
5.2. **Second tilting of** $\text{Per}(X/Y)$. In this subsection, we tilt $B_{f, \omega}$ again. This is an analogy of the double tilting of $\text{Coh}(X)$ in [6] for $\text{Per}(X/Y)$.

We first show the following lemma, whose proof relies on the results in the previous section.

**Lemma 5.3.** There is $b \in \mathbb{Q}$ such that we have

$$ \text{ch}^b (E) > 0 $$

for any $0 \neq E \in \text{Per}_0(X/Y)$.

**Proof.** Suppose that $f(D)$ is a curve. Then by calculating Chern characters of simple objects in the RHS of (54), the desired inequality (67) is checked to hold when

$$ \frac{1}{2} < b < \frac{3}{2}. $$

Next suppose that $f(D)$ is a point. By Proposition 4.21, it is enough to find $b \in \mathbb{Q}$ so that the inequality (67) holds for any object $E = i_* F$, where $F \in \text{Per}(D)$ is a simple object. Similarly to the case that $f(D)$ is a curve, such $b$ can be found by computing Chern characters of simple objects in $\text{Per}(D)$ appearing in Proposition 4.14.

We give some more detail on the computation for the type V case. We have

$$ \text{ch}(i_* O_D(-1)) = (0, D, -C, 1/3), $$

$$ \text{ch}(i_* S_5) = (0, 3D, -C, 1), $$

$$ \text{ch}(i_* O_D(-C)) = (0, D, 0, -1/6). $$

Therefore, setting $c = 1 - b$, we have

$$ \text{ch}_3^{bD}(i_* O_D(-2)[2]) = c^2 + c + 1/3, $$

$$ \text{ch}_3^{bD}(i_* S_5(-1)[1]) = -3c^2 - c + 1, $$

$$ \text{ch}_3^{bD}(i_* O_D(-3C)) = c^2 - 1/6. $$

We want to find $b$ so that all the above values are positive. The solution of the inequalities is

$$ 1 + \frac{\sqrt{6}}{6} < b < \frac{7}{6} + \frac{\sqrt{13}}{6} \quad \text{or} \quad \frac{7}{6} - \frac{\sqrt{13}}{6} < b < 1 - \frac{\sqrt{6}}{6}. $$

The other cases are similarly calculated. The result is as follows: in type III case, the result is the same as in type V case, and

$$ 2 - \frac{\sqrt{6}}{3} < b < 2 + \frac{\sqrt{6}}{3} \quad \text{in type II}, $$

$$ \frac{3}{4} + \frac{\sqrt{15}}{12} < b < \frac{4}{5} + \frac{\sqrt{6}}{6} \quad \text{in type IV}. $$

\qed
Let us take $b \in \mathbb{Q}$ as in Lemma 5.3 and set $B = bD$. Note that we have
\begin{equation}
ch_i^B(*) (f^* \omega)^{3-i} = ch_i(*) (f^* \omega)^{3-i}, \quad 0 \leq i \leq 1.
\end{equation}
Next we generalize Bogomolov-Gieseker inequality for perverse coherent sheaves to our 3-fold situation.

**Proposition 5.4.** For any $\mu_{f^* \omega}$-semistable object $E \in \text{Per}(X/Y)$ with $\text{ch}_0(E) > 0$, we have the inequality
\begin{equation}
\left\{ \text{ch}_1(E) \cdot (f^* \omega)^2 \right\}^2 \geq 2 \omega^3 \text{ch}_0(E)(ch^B_2(E) \cdot f^* \omega).
\end{equation}

**Proof.** First suppose that $f(D)$ is a point. Note that the equality (68) also holds for $i = 2$ in this case. By Proposition 4.17, $H^0(E)$ is torsion free outside $D$. Since sheaves supported on $D$ do not affect $\mu_{f^* \omega}$, the free part of $H^0_0(E)$ is a $\mu_{f^* \omega}$-semistable sheaf. Then the desired inequality (69) follows from the Bogomolov-Gieseker inequality for the free part of $H^0(E)$.

Next suppose that $f(D)$ is a curve. Similarly to the proof of Proposition 3.9, $E$ is a coherent sheaf. Also there is an exact sequence
\[ 0 \to T \to E \to F \to 0 \]
such that $F$ is a torsion free $\mu_{f^* \omega}$-semistable sheaf, $T$ is a torsion sheaf supported on $D$ with $T[1] \in \text{Per}(X/Y)$. By Theorem 2.7, the Hodge index theorem and noting (68), we have
\begin{align*}
\left\{ \text{ch}_1(F) \cdot (f^* \omega)^2 \right\}^2 / \omega^3 &\geq \text{ch}_1^B(F)^2 f^* \omega \\
&\geq 2 \text{ch}_0(F)(ch^B_2(F) \cdot f^* \omega).
\end{align*}
By the argument as in Remark 3.10 it is enough to show that $\text{ch}^B_2(T) \cdot f^* \omega \leq 0$.

By Theorem 4.5 the condition $T[1] \in \text{Per}(X/Y)$ is equivalent to $f_* T = f_* (T(-D)) = 0$.

Let $W \subset Y$ be a divisor which is linearly equivalent to some multiple of $\omega$, and $Z := f^{-1}(W)$. If we take $W$ to be smooth and general, then $f|_Z : Z \to W$ is a blow up at a finite number of points, and
\[ f|_Z(T|_Z) = f|_Z(T|_Z(-D)) = 0. \]
Then $T|_Z[1] \in \text{Per}_0(X/Y)$ by Theorem 4.5, so Lemma 5.3 implies
\[ \text{ch}^B_2(T[1]) \cdot Z = \text{ch}^B_3(T|_Z[1]) > 0. \]
Therefore $\text{ch}^B_2(T) \cdot f^* \omega \leq 0$ follows. \hfill \Box

We consider the central charge $Z_{B,f^* \omega}$ given by (7). Noting (68), $Z_{B,f^* \omega}(E)$ is written as
\begin{align*}
&\left( -\text{ch}^B_3(E) + \frac{(f^* \omega)^2}{2} \text{ch}_1(E) \right) + \sqrt{-1} \left( f^* \omega \text{ch}^B_2(E) - \frac{\omega^3}{6} \text{ch}_0(E) \right).
\end{align*}
As an analogy of [6] Lemma 3.2.1, we have the following lemma:

**Lemma 5.5.** For any non-zero object $E \in B_{f^*\omega}$, one of the following conditions hold:

(a) $\chi_1(E) \cdot (f^*\omega)^2 > 0$.

(b) $\chi_1(E) \cdot (f^*\omega)^2 = 0$ and $\text{Im} Z_{B,f^*\omega}(E) > 0$.

(c) $\chi_1(E) \cdot (f^*\omega)^2 = \text{Im} Z_{B,f^*\omega}(E) = 0$ and $\text{Re} Z_{B,f^*\omega}(E) < 0$.

**Proof.** By the construction of $B_{f^*\omega}$, we always have $\chi_1(E) \cdot (f^*\omega)^2 \geq 0$. Suppose that $\chi_1(E) \cdot (f^*\omega)^2 = 0$. Then $E$ is contained in the RHS of (66). If $F$ is a $\mu_{f^*\omega}$-semistable object in $\text{Per}(X/Y)$ with $\mu_{f^*\omega}(F) = 0$, then $\chi_2^B(F) \cdot f^*\omega \leq 0$ by Proposition 5.4. Hence $\text{Im} Z_{B,f^*\omega}(F[1]) > 0$ holds. Also for an object $T \in \text{Per}_{\leq 1}(X/Y)$, we have $\chi_2(T) \cdot f^*\omega \geq 0$, hence $\text{Im} Z_{B,f^*\omega}(T) \geq 0$. Then the inequality $\text{Im} Z_{B,f^*\omega}(E) \geq 0$ follows from these inequalities. Finally if $\chi_1(E) \cdot (f^*\omega)^2 = \text{Im} Z_{B,f^*\omega}(E) = 0$, the above argument shows $E \in \text{Per}_0(X/Y)$. Hence $\text{Re} Z_{B,f^*\omega}(E) < 0$ follows from our choice of $B = bD$ that the condition of Lemma 5.3 is satisfied.

We define the slope function $\nu_{B,f^*\omega}$ on $B_{f^*\omega}$ to be

$$\nu_{B,f^*\omega}(E) = \frac{\text{Im} Z_{B,f^*\omega}(E)}{\chi_1(E) \cdot (f^*\omega)^2}.$$ 

Here $\nu_{B,f^*\omega}(E) = \infty$ if $\chi_1(E) \cdot (f^*\omega)^2 = \infty$. By Lemma 5.5, the above slope function is an analogy of the slope function for torsion free sheaves on algebraic surfaces. Also a similar slope function was defined on a tilting of $\text{Coh}(X)$ in [6]. The following is the analogy of ‘tilt stability’ in [6] for the tilting of perverse t-structure.

**Definition 5.6.** A non-zero object $E \in B_{f^*\omega}$ is $\nu_{B,f^*\omega}$-(semi)stable if for any exact sequence $0 \to F \to E \to G \to 0$ in $B_{f^*\omega}$, we have the inequality

$$\nu_{B,f^*\omega}(F) \leq (\leq) \nu_{B,f^*\omega}(G).$$

**Lemma 5.7.** Any object in $B_{f^*\omega}$ admits a Harder-Narasimhan filtration with respect to $\nu_{B,f^*\omega}$-stability.

**Proof.** Since $B_{f^*\omega}$ is noetherian by Lemma 5.2, the same argument of [12] Proposition 7.1 is applied.

We consider the following subcategories of $B_{f^*\omega}$:

$$\mathcal{T}_{B,f^*\omega} := \{E : E \text{ is } \nu_{B,f^*\omega}\text{-semistable with } \nu_{B,f^*\omega}(E) > 0\},$$

$$\mathcal{F}_{B,f^*\omega} := \{E : E \text{ is } \nu_{B,f^*\omega}\text{-semistable with } \nu_{B,f^*\omega}(E) \leq 0\}.$$ 

By Lemma 5.7 the pair of subcategories $(\mathcal{T}_{B,f^*\omega}, \mathcal{F}_{B,f^*\omega})$ forms a torsion pair on $B_{f^*\omega}$. By tilting, we have the following heart of a t-structure:

**Definition 5.8.** We define $\mathcal{A}_{B,f^*\omega} \subset D^b \text{Coh}(X)$ to be

$$\mathcal{A}_{B,f^*\omega} := \langle \mathcal{F}_{B,f^*\omega}[1], \mathcal{T}_{B,f^*\omega} \rangle.$$
Note that, by the construction of $A_{B, f^*}\omega$, we have
\begin{equation}
Z_{B, f^*}\omega(A_{B, f^*}\omega \setminus \{0\}) \subset \{ z \in \mathbb{C} : \text{Im } z \geq 0 \}. \tag{70}
\end{equation}

5.3. Conjectures. Let us consider the pair
\[
\sigma_{B, f^*}\omega = (Z_{B, f^*}\omega, A_{B, f^*}\omega).
\]
Similarly to [6, Conjecture 3.2.6], we propose the following conjecture:

**Conjecture 5.9.** We have $\sigma_{B, f^*}\omega \in \text{Stab}(X)$.

**Remark 5.10.** For $0 < \varepsilon \ll 1$ so that $f^*\omega - \varepsilon D$ is ample, a conjectural Bridgeland stability condition
\begin{equation}
\sigma_{B, f^*}\omega - \varepsilon D = (Z_{B, f^*}\omega - \varepsilon D, A_{B, f^*}\omega - \varepsilon D) \tag{71}
\end{equation}
is constructed in [6]. If both of [6, Conjecture 3.2.6] and Conjecture 5.9 are true, then we conjecture that
\begin{equation}
\lim_{\varepsilon \to 0^+} \sigma_{B, f^*}\omega - \varepsilon D = \sigma_{B, f^*}\omega \tag{72}
\end{equation}
in $\text{Stab}(X)$. The above equality should follow from the support property of $\sigma_{B, f^*}\omega$. Proving this would require further evaluations of Chern classes of $\sigma_{B, f^*}\omega$-semistable objects, as we discussed in Section 3 for surfaces.

If Conjecture 5.9 is true, we are interested in the subcategory of semistable objects of phase one. Although we are not able to prove Conjecture 5.9 yet, such a subcategory is well-defined as follows:

**Definition 5.11.** We define $P_{B, f^*}\omega(1)$ to be
\[
P_{B, f^*}\omega(1) := \{ E \in A_{B, f^*}\omega : \text{Im } Z_{B, f^*}\omega(E) = 0 \}.
\]

Note that $P_{B, f^*}\omega(1)$ is an abelian subcategory of $A_{B, f^*}\omega$. The construction of $A_{B, f^*}\omega$ immediately implies
\begin{equation}
P_{B, f^*}\omega(1) = \left\{ F[1], T : F \in B_{f^*}\omega \text{ is } \nu_{B, f^*}\omega\text{-semistable with } \nu_{B, f^*}\omega(F) = 0, T \in \text{Per}_0(X/Y) \right\}. \tag{73}
\end{equation}

**Remark 5.12.** The abelian categories $A_{B, f^*}\omega$ and $P_{B, f^*}\omega(1)$ are independent of $B$ if $f(D)$ is a point.

In order to show Conjecture 5.9 we need to show the axiom (6). As in [6], this condition is equivalent to the following Bogomolov-Gieseker type inequality evaluating $\text{ch}_3$:

**Conjecture 5.13.** Take $b \in \mathbb{Q}$ as in Lemma 5.3 and set $B = bD$. Then for any $\nu_{B, f^*}\omega$-semistable object $E \in B_{f^*}\omega$ with $\nu_{B, f^*}\omega(E) = 0$, we have the inequality
\[
\text{ch}_3^B(E) < \frac{(f^*\omega)^2}{2} \text{ch}_1(E).
\]

Indeed we have the following:
Proposition 5.14. Suppose that Conjecture 5.13 is true. Then we have $(Z_{B,f*ω}, A_{B,f*ω}) \in \text{Stab} \, †(X)$.

Proof. First we check (6). By the condition (70), it is enough to check that any non-zero object $E \in \mathcal{P}_{B,f*ω}(1)$ satisfies $\text{Re} \, Z_{B,f*ω}(E) < 0$. Since $E$ is contained in the RHS of (73), the condition $\text{Re} \, Z_{B,f*ω}(E) < 0$ follows from Conjecture 5.13 and our choice of $B = bD$ so that the condition of Lemma 5.3 is satisfied.

In Proposition 5.15 below, we show that the abelian category $A_{B,f*ω}$ is noetherian. The Harder-Narasimhan property follows from this fact and the same argument of Lemma 3.6. The locally finiteness is obvious since the image of $Z_{B,f*ω}$ is a discrete subgroup in $\mathbb{C}$. □

5.4. Finite length property. This subsection is devoted to showing some technical results: noetherian property of $A_{B,f*ω}$ and finite length property of $\mathcal{P}_{B,f*ω}(1)$. Both of them are necessary conditions for the Conjecture 5.9 to hold.

Proposition 5.15. The abelian category $A_{B,f*ω}$ is noetherian.

Proof. We give a direct proof for this fact without passing to polynomial stability conditions as in [6, Proposition 5.2.2]. Suppose that there is an infinite sequence of surjections in $A_{B,f*ω}$

$$E = E_1 \to E_2 \to \cdots \to E_i \to E_{i+1} \to \cdots.$$ 

Let us take the exact sequence in $A_{B,f*ω}$

$$0 \to L_i \to E \to E_i \to 0.$$ 

Since $\text{Im} \, Z_{B,f*ω}(E_i) \geq 0$ on $A_{B,f*ω}$, we may assume that $\text{Im} \, Z_{B,f*ω}(E_i)$ is constant. Hence $\text{Im} \, Z_{B,f*ω}(L_i) = 0$, which implies that $L_i$ is contained in the RHS of (73). Similarly to the proof of Lemma 5.7 if we apply the same argument of [12, Proposition 7.1], we arrive at the exact sequences in $B_{f*ω}$

$$0 \to Q \to \mathcal{H}_{B}^{-1}(E_i) \to \mathcal{H}_{B}^{0}(L_i) \to 0$$

where $Q \in \mathcal{F}_{B,f*ω}$ is independent of $i$, and $\mathcal{H}_{B}^{i}(*)$ is the $i$-th cohomology functor with respect to the t-structure with heart $B_{f*ω}$. We also have the inclusions in $B_{f*ω}$

$$(74) \quad \mathcal{H}_{B}^{0}(L_1) \subset \mathcal{H}_{B}^{0}(L_2) \subset \cdots \subset \mathcal{H}_{B}^{0}(L_i) \subset \mathcal{H}_{B}^{0}(L_{i+1}) \subset \cdots.$$ 

Since $\mathcal{H}_{B}^{0}(L_i) \in \text{Per}_0(X/Y)$, the object $\Phi(\mathcal{H}_{B}^{0}(L_i))$ is a zero dimensional sheaf as $\mathcal{O}_Y$-module. Hence it is enough to bound the length of $\Phi(\mathcal{H}_{B}^{0}(L_i))$.

In order to reduce the notation, we set

$$V_i = \mathcal{H}_{B}^{-1}(E_i), \quad T_i = \mathcal{H}_{B}^{0}(L_i).$$
By Lemma 5.16 below, we have the exact sequences in $\mathcal{B}_{f^*\mathcal{O}}$

\[
0 \to Q^{(1)} \to Q \to Q^{(2)} \to 0
\]
\[
0 \to V_i^{(1)} \to V_i \to V_i^{(2)} \to 0
\]

which respect the torsion pair $(\mathcal{T}^\dagger, \mathcal{F}^\dagger)$. Using the snake lemma, it is easy to see that there are exact sequences in $\mathcal{B}_{f^*\mathcal{O}}$

\[
0 \to Q^{(j)} \to V_i^{(j)} \to T_i^{(j)} \to 0,
\]
\[
0 \to T_i^{(1)} \to T_i \to T_i^{(2)} \to 0,
\]

for some $T_i^{(j)} \in \text{Per}_0(X/Y)$ with $j = 1, 2$. Since $\Phi(T_i^{(j)})$ is a zero dimensional sheaf, it is enough to bound its length.

First we bound the length of $\Phi(T_i^{(1)})$. By applying $D\Phi$ given in Lemma 5.17 below to the sequence (75) for $j = 1$, we obtain the distinguished triangle in $D^b\text{Coh}(Y)$

\[
D\Phi(T_i^{(1)}) \to D\Phi(V_i^{(1)}) \to D\Phi(Q^{(1)}).
\]

Note that $Q^{(1)}, V_i^{(1)} \in \mathcal{T}^\dagger \cap \mathcal{F}_{B,f^*\mathcal{O}}'$. Therefore Lemma 5.17 implies that $H^2(D\Phi(Q^{(1)}))$ is zero dimensional, and we obtain the exact sequence in $\text{Coh}(Y)$

\[
H^2(D\Phi(V_i^{(1)})) \to H^2(D\Phi(Q^{(1)})) \to H^3(D\Phi(T_i^{(1)})) \to 0.
\]

Since $H^3(D\Phi(T_i^{(1)}))$ is a zero dimensional sheaf whose length is equal to the length of $\Phi(T_i^{(1)})$, we obtain the bound of the length of $\Phi(T_i^{(1)})$.

As for the bound of the length of $\Phi(T_i^{(2)})$, let $Q^{(3)}, V_i^{(3)}$ be the maximal subobjects of $Q^{(2)}, V_i^{(2)}$ in $\text{Per}(X/Y)$ contained in $\text{Per}_{\leq 2}(X/Y)$, and set

\[
Q^{(4)} = Q^{(2)}/Q^{(3)}, \quad V_i^{(4)} = V_i^{(2)}/V_i^{(3)}.
\]

Similarly as above, we have the exact sequences in $\text{Per}(X/Y)$

\[
0 \to Q^{(j)} \to V_i^{(j)} \to T_i^{(j)} \to 0
\]
\[
0 \to T_i^{(3)} \to T_i^{(2)} \to T_i^{(4)} \to 0,
\]

for some $T_i^{(j)} \in \text{Per}_0(X/Y)$ with $j = 3, 4$. Since $\Phi(Q^{(4)})$ and $\Phi(V_i^{(4)})$ are torsion free sheaves on $Y$, the bound of the length of $\Phi(T_i^{(4)})$ is obtained since

\[
\Phi(T_i^{(4)}) \subset \Phi(Q^{(4)})^\vee/\Phi(Q^{(4)}).
\]

Also since $\Phi(Q^{(3)})$ and $\Phi(V_i^{(3)})$ are pure two dimensional sheaves on $Y$, the bound of the length of $\Phi(T_i^{(3)})$ is obtained by taking the projection
$Y \xrightarrow{\phi} \mathbb{P}^2$, which is defined and finite over the support of $\Phi(Q^{(3)})$, and noting that
\[
\phi_* \Phi(T_i^{(3)}) \subseteq \phi_* \Phi(Q^{(3)})^{\vee} / \phi_* \Phi(Q^{(3)}).
\]

We have used the following lemmas:

**Lemma 5.16.** There is a torsion pair $(\mathcal{T}^\dagger, \mathcal{F}^\dagger)$ on $\mathcal{B}_{f, \omega}$ such that
- $E \in \mathcal{T}^\dagger$ if and only if $\mathcal{H}^{-1}_p(E) \in \mathcal{F}_{f, \omega}$ and $\mathcal{H}^0_p(E) \in \text{Per}_{\leq 1}(X/Y)$.
- $E \in \mathcal{F}^\dagger$ if and only if $\mathcal{H}^{-1}_p(E) = 0$, $\mathcal{H}^0_p(E) \in \mathcal{T}_{f, \omega}$ and
\[
\text{Hom}(\text{Per}_{\leq 1}(X/Y), \mathcal{H}^0_p(E)) = 0.
\]

**Proof.** If we define $(\mathcal{T}^\dagger, \mathcal{F}^\dagger)$ in a required way, then it is obvious that $\text{Hom}(\mathcal{T}^\dagger, \mathcal{F}^\dagger) = 0$. For any $E \in \mathcal{B}_{f, \omega}$, the decomposition of $E$ into objects in $\mathcal{T}^\dagger$ and $\mathcal{F}^\dagger$ is obtained by composing the exact sequence in $\mathcal{B}_{f, \omega}$
\[0 \to \mathcal{H}^{-1}_p(E)[1] \to E \to \mathcal{H}^0_p(E) \to 0\]
with the exact sequence
\[0 \to T_1 \to \mathcal{H}^0_p(E) \to T_2 \to 0\]
where $T_1$ is the maximal subobject of $\mathcal{H}^0_p(E)$ in $\text{Per}(X/Y)$ contained in $\text{Per}_{\leq 1}(X/Y)$. □

**Lemma 5.17.** For an object $E \in \mathcal{T}^\dagger \cap \mathcal{F}^\dagger_{B, f, \omega}$, the object
\[
\mathbb{D} \Phi(E) := \mathbb{R} \text{Hom}_{O_Y} (\Phi(E), O_Y) \in D^b \text{Coh}(Y)
\]
satisfies that $\mathcal{H}^1(\mathbb{D} \Phi(E))$ is a torsion free sheaf, $\mathcal{H}^2(\mathbb{D} \Phi(E))$ is zero dimensional and $\mathcal{H}^i(\mathbb{D} \Phi(E)) = 0$ for $i \neq 1, 2$.

**Proof.** We first show that
\[(76) \quad \text{Hom}_Y(\text{Coh}_{\leq 1}(Y), \Phi(E)) = 0.\]
Suppose that there is $F \in \text{Coh}_{\leq 1}(Y)$ and a non-trivial morphism $F \to \Phi(E)$. By taking the adjunction, we have
\[(77) \quad \text{Hom}_{\mathcal{A}} (F \otimes_{O_Y} \mathcal{A}, \Phi(E)) \neq 0.\]

Let us consider the object $\Phi^{-1}(F \otimes_{O_Y} \mathcal{A})$. We have $\mathcal{H}^i_p \Phi^{-1}(F \otimes_{O_Y} \mathcal{A}) = 0$ for $i \geq 1$ and it is an object in $\text{Per}_{\leq 1}(X/Y)$ for $i \leq 0$. On the other hand, since $E \in \mathcal{T}^\dagger \cap \mathcal{F}^\dagger_{B, f, \omega}$, we have
\[
\text{Hom}(\text{Per}_{\leq 1}(X/Y)[i], E) = 0, \quad i \geq 0.
\]
This contradicts to (77), so (76) holds.
By the assumption and (76), $\Phi(E)$ fits into the distinguished triangle in $D^b\text{Coh}(Y)$

$$U[1] \to \Phi(E) \to F$$

such that $U$ is a reflexive sheaf on $Y$ and $F \in \text{Coh}_{\leq 1}(Y)$. Then the required property for $D\Phi(E)$ is proved by dualizing the above sequence and using (76), the property of the derived dual of reflexive sheaves and that of one dimensional sheaves. The detail is found in the proof of [31, Lemma 3.8].

We also show the following:

**Proposition 5.18.** The abelian category $\mathcal{P}_{B,f^*\omega}(1)$ is of finite length.

**Proof.** By Proposition 5.15, $\mathcal{P}_{B,f^*\omega}(1)$ is noetherian, so it is enough to show that $\mathcal{P}_{B,f^*\omega}(1)$ is artinian. Suppose that there is an infinite sequence in $\mathcal{P}_{B,f^*\omega}(1)$

$$E = E_1 \supset E_2 \supset \cdots$$

(78)

We take the exact sequence in $\mathcal{P}_{B,f^*\omega}(1)$

$$0 \to E_i \to E \to F_i \to 0.$$  

Since $\chi_1(\ast)(f^*\omega)^2$ is non-positive on $\mathcal{P}_{B,f^*\omega}(1)$ by (73), we may assume that $\chi_1(E_i)(f^*\omega)^2$ is independent of $i$. Then $\chi_1(F_i)(f^*\omega)^2 = 0$, hence $F_i \in \text{Per}_0(X/Y)$ by (73). In the notation of the proof of Proposition 5.15 we have the surjection in $\text{Per}_0(X/Y)$:

$$\mathcal{H}_B^0(E) \twoheadrightarrow F_i.$$  

Applying $\Phi$ in Theorem 4.5, we see that $\Phi(F_i)$ is a zero dimensional sheaf whose length is bounded above by the length of $\Phi(\mathcal{H}_B^0(E))$. Hence the sequence (78) terminates. 

□

5.5. **Proof of Theorem 1.4.** Let $M^{\sigma_{B,f^*\omega}}([O_x])$ be the set of isomorphism classes of objects $E \in \mathcal{P}_{B,f^*\omega}(1)$ satisfying $\chi(E) = \chi(O_x)$ for $x \in X$. Note that by Proposition 5.18, we can define $S$-equivalence classes of objects in $M^{\sigma_{B,f^*\omega}}([O_x])$. Also by (73), it follows that

$$M^{\sigma_{B,f^*\omega}}([O_x]) = \{ E \in \text{Per}_0(X/Y) : \chi(E) = \chi(O_x) \}.$$  

In particular we have $O_x \in M^{\sigma_{B,f^*\omega}}([O_x])$ for any $x \in X$. We investigate $S$-equivalence classes of objects $O_x$ in the following proposition:

**Proposition 5.19.** (i) For $x, x' \in X$, the objects $O_x$ and $O_{x'}$ are $S$-equivalent in $M^{\sigma_{B,f^*\omega}}([O_x])$ if and only if $f(x) = f(x')$.

(ii) An object $E \in M^{\sigma_{B,f^*\omega}}([O_x])$ is isomorphic to $O_x$ for $x \notin D$ or supported on $D$. In the latter case, suppose moreover that $\text{Supp}(E)$ is connected when $f(D)$ is a curve. Then $E$ is $S$-equivalent to $O_x$ for $x \in D$. 


Proof. (i) Suppose that $\mathcal{O}_x$ and $\mathcal{O}_{x'}$ are $S$-equivalent. If $x \notin D$, then $\mathcal{O}_x$ is a simple object in $\text{Per}_0(X/Y)$. Hence $\mathcal{O}_{x'}$ should be isomorphic to $\mathcal{O}_x$, which implies $x = x'$. If $x \in D$, then $x' \in D$ by the above argument. This implies that $f(x) = f(x')$ when $f(D)$ is a point. When $f(D)$ is a curve and $x, x' \in D$, we have

$$(79) \quad \text{gr}(\mathcal{O}_x) \cong \mathcal{O}_{L_{f(x)}}(-2)[1] \oplus \mathcal{O}_{L_{f(x')}}(-1),$$

by Proposition 4.16 Therefore we must have $L_{f(x)} = L_{f(x')}$, which is equivalent to $f(x) = f(x')$.

Conversely suppose that $f(x) = f(x')$. If $x \notin D$, we have $x = x'$, so $\mathcal{O}_x$ and $\mathcal{O}_{x'}$ are isomorphic. Suppose that $x, x' \in D$. When $f(D)$ is a curve, the isomorphism (79) shows that $\mathcal{O}_x$ and $\mathcal{O}_{x'}$ are $S$-equivalent. When $f(D)$ is a point, then $\mathcal{O}_x, \mathcal{O}_{x'}$ are objects in $\text{Per}_0(\hat{X}/\hat{Y})$, which has only a finite number of simple objects. Since $D$ is connected, the argument of [1 Lemma 3.7] shows that $\mathcal{O}_x$ and $\mathcal{O}_{x'}$ are $S$-equivalent. This fact can be also checked by using simple objects in Proposition 4.14 directly. For instance if $f$ is type V, we have the resolutions

$$0 \to \mathcal{O}_D(-1) \to S_5 \to \mathcal{O}_D(-C)^{\oplus 2} \to \mathcal{O}_x \to 0, \quad x \in \text{Sing}(D),$$

$$0 \to \mathcal{O}_D(-1) \to S_5 \to U \to \mathcal{O}_x \to 0, \quad x \notin \text{Sing}(D).$$

Therefore for any $x \in D$, $\mathcal{O}_x$ is $S$-equivalent to

$$(80) \quad \mathcal{O}_D(-2)[2] \oplus S_5(-1)[1] \oplus \mathcal{O}_D(-3C)^{\oplus 2}.$$

(ii) Let us take an object $E \in \text{Per}_0(X/Y)$. Then $E$ is a direct sum of zero dimensional sheaves supported outside $D$ and an object in $\text{Per}_0(X/Y)$ supported on $D$. If we take $b \in \mathbb{Q}$ as in Lemma 5.3, then any direct summand of $E$ satisfies (67). Therefore if $E$ satisfies $\text{ch}(E) = \text{ch}(\mathcal{O}_x)$ for $x \in X$, then $E$ is isomorphic to $\mathcal{O}_x$ for $x \notin D$ or supported on $D$.

In the latter case, we first discuss the case that $f(D)$ is a curve. By Proposition 4.14 $E$ is $S$-equivalent to

$$(81) \quad E_{y,y'} := \mathcal{O}_{L_y}(-2)[1] \oplus \mathcal{O}_{L_{y'}}(-1)$$

for some $y, y' \in f(D)$. If $\text{Supp}(E)$ is connected, then $y = y'$, hence $E$ is $S$-equivalent to $\mathcal{O}_x$ for $x \in f^{-1}(y)$ by (79).

When $f(D)$ is a point, we only discuss the type V case. The other cases are similarly discussed. By Proposition 4.14 and Proposition 4.21 $E$ is $S$-equivalent to

$$\mathcal{O}_D(-2)[2]^{\oplus a} \oplus S_5(-1)[1]^{\oplus b} \oplus \mathcal{O}_D(-3C)^{\oplus c}$$

for some $a, b, c \in \mathbb{Z}_{\geq 0}$. By the Chern character computation in the proof of Lemma 5.3, the condition $\text{ch}(E) = \text{ch}(\mathcal{O}_x)$ becomes $a - 3b + c = 0$, $-a + b = 0$ and $a/3 + b - c/6 = 1$. The solution is $a = b = 1, c = 2$, hence $E$ is $S$-equivalent to $\mathcal{O}_x$ for $x \in D$ by (80).
Remark 5.20. In Proposition 5.19 (ii), suppose that \( f(D) \) is a curve and \( \text{Supp}(E) \) is not connected. Then the above proof shows that \( E \) is isomorphic to \( E_{y,y'} \) for \( y \neq y' \) given by (81).

The following is the main result in this section:

Theorem 5.21. We have the following:

- If \( f(D) \) is a curve, then \( Y \) is one of the irreducible components of the coarse moduli space of \( S \)-equivalence classes of objects in \( M_{\sigma, r^n - \infty}([O_x]) \).
- If \( f(D) \) is a point, then \( Y \) is the coarse moduli space of \( S \)-equivalence classes of objects in \( M_{\sigma, r^n - \infty}([O_x]) \).

Proof. We first discuss the case that \( f(D) \) is a point. The statement is equivalent to that \( Y \) corepresents the functor (cf. [18, Definition 2.2.1])

\[
\mathcal{M}_{\sigma, r^n - \infty}([O_x]): \text{Sch}/\mathbb{C} \rightarrow \text{Set}
\]

which assigns a \( \mathbb{C} \)-scheme \( S \) to isomorphism classes of objects

\[
Q \in D^b \text{Coh}(X \times S)
\]

such that for each \( s \in S \), its derived restriction \( Q_s \) to \( X \times \{s\} \) is an object in \( \text{Per}_0(X/Y) \) with \( \text{ch}(Q_s) = \text{ch}(O_x) \). Let us consider the object

\[
R(f \times \text{id}_S)_*Q \in \text{Coh}(Y \times S).
\]

The above object is a flat family of skyscraper sheaves of points in \( Y \) over \( S \). Hence it induces a morphism \( S \rightarrow Y \), giving a natural transformation

\[
F_Y: \mathcal{M}_{\sigma, r^n - \infty}([O_x]) \rightarrow \text{Hom}(\ast, Y).
\]

Suppose that there is another \( \mathbb{C} \)-scheme \( Z \) and a natural transformation

\[
F_Z: \mathcal{M}_{\sigma, r^n - \infty}([O_x]) \rightarrow \text{Hom}(\ast, Z).
\]

By applying \( F_Z \) to the family \( \{O_x\}_{x \in X} \), we obtain a morphism

\[
g: X \rightarrow Z.
\]

Note that any two \( S \)-equivalent objects in \( \text{Per}_0(X/Y) \) are mapped to the same point by \( F_Z(\text{Spec} \mathbb{C}) \). (cf. [18, Lemma 4.1.2].) Therefore by Proposition 5.19 (i), \( g(D) \) must be a point in \( Z \). Since \( f_*O_X = O_Y \), the morphism \( g \) descends to the morphism from \( Y \),

\[
h: Y \rightarrow Z.
\]

We need to show that

\[
h_* \circ F_Y = F_Z.
\]

The above relationship follows from Proposition 5.19 (ii). Hence \( Y \) corepresents the functor (82).

In the case that \( f(D) \) is a curve, we consider the scheme

\[
\tilde{Y} := Y \cup (f(D) \times f(D)),
\]
where $Y$ and $f(D) \times f(D)$ are glued along $f(D) \subset Y$ and the diagonal $f(D) \subset f(D) \times f(D)$. The scheme structure of $\tilde{Y}$ along the intersection is given by the fiber product

$$\mathcal{O}_{\tilde{Y}} = \mathcal{O}_Y \times_{\mathcal{O}_{f(D)}} \mathcal{O}_{f(D) \times f(D)}.$$ 

Note that for $p \in f(D)$, the ring $\hat{\mathcal{O}}_{\tilde{Y},p}$ is written as

$$\hat{\mathcal{O}}_{\tilde{Y},p} \cong \mathbb{C}[x, y, z, w]/(xz, yz).$$

We show that $\tilde{Y}$ corepresents the functor (82). Let $S_i, Q$ be as before, and $S_1, \ldots, S_N$ be the irreducible components of $S$. If $\text{Supp} \ Q_s$ is connected for general (hence for any) point $s \in S_i$, then we have the morphism $S_i \to Y$ induced by the flat family of skyscraper sheaves on $Y$ over $S_i$,

(85) \[ R(f \times \text{id}_{S_i})_* (Q|_{X \times S_i}) \in \text{Coh}(Y \times S_i). \]

If $\text{Supp} \ Q_s$ is not connected for general $s \in S_i$, we consider another object

(86) \[ R(f \times \text{id}_{S_i})_* (Q|_{X \times S_i}(-D \times S_i)) \in \text{Coh}(Y \times S_i). \]

The objects (85), (86) are flat families of skyscraper sheaves of points in $f(D)$ over $S_i$, hence induce a morphism

$$S_i \to f(D) \times f(D).$$

The above morphisms on irreducible components glue on the intersections, so induce a (unique) morphism $S \to \tilde{Y}$. In this way, we obtain a natural transformation

$$F_{\tilde{Y}} : \mathcal{M}^{a.b.\tau-*}([\mathcal{O}_x]) \to \text{Hom}(\ast, \tilde{Y}).$$

Suppose that there is another natural transformation $F_Z$ as in (84). Similarly to the case that $f(D)$ is point, $F_Z \{\mathcal{O}_x\}_{x \in X}$ induces the morphism $Y \to Z$. Also applying $F_Z$ to the family

$$\{E_{y,y'}\}_{(y,y') \in f(D) \times f(D)}$$

given by (81), we obtain a morphism

$$f(D) \times f(D) \to Z.$$ 

These morphisms glue along the intersection, giving a morphism $\tilde{h} : \tilde{Y} \to Z$. Similarly to the case that $f(D)$ is a point, the relationship

$$\tilde{h}_* \circ F_{\tilde{Y}} = F_Z$$

follows from Proposition 5.19 (ii) and Remark 5.20. Therefore $\tilde{Y}$ corepresents the functor (82), and $Y$ is the desired irreducible component. \[ \square \]
Remark 5.22. Since Proposition 5.19 (ii) is not true when \( f(D) \) is a curve and \( \text{Supp}(E) \) is not connected, the statement of Theorem 5.21 is weaker in this case. Indeed the objects \( E_{y,y'} \) in (81) provide another component \( f(D) \times f(D) \).

Remark 5.23. Let us consider a conjectural Bridgeland stability condition in (71), and a category \( \mathcal{P}_{B,f^+\omega-\varepsilon D}(1) \) defined similarly to Definition 5.11. Then \( X \) is shown to be the fine moduli space of objects \( E \in \mathcal{P}_{B,f^+\omega-\varepsilon D}(1) \) with \( \text{ch}(E) = \text{ch}(\mathcal{O}_x) \). If the relation (72) holds, then we can consider a one parameter family

\[
\sigma_t = (Z_{B,f^+\omega+\varepsilon tD}, \mathcal{P}_t), \quad t \in (-1, 1),
\]

similarly to the surface case. The behavior of moduli spaces of \( E \in \mathcal{P}_t(1) \) with \( \text{ch}(E) = \text{ch}(\mathcal{O}_x) \) under change of \( t \in (-1, 0) \) is similar to the surface case in Theorem 3.16 by Theorem 5.21. On the other hand, it would be interesting to study such a moduli space for \( t > 0 \). The moduli space itself may be studied by investigating the space of stability conditions on \( D^b(\text{Per}_0(X/Y)) \), which is much easier than \( \text{Stab}(X) \).

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