Sigma models in the presence of dynamical point-like defects

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Abstract

Point-like Liouville integrable dynamical defects are introduced in the context of the Landau-Lifshitz and Principal Chiral (Faddeev-Reshetikhin) models. Based primarily on the underlying quadratic algebra we identify the first local integrals of motion, the associated Lax pairs as well as the relevant sewing conditions around the defect point. The involution of the integrals of motion is shown taking into account the sewing conditions.
1 Introduction

The issue of integrable defects has been quite intriguing, and a considerable number of studies have been devoted to this particular problem, both at classical and quantum level [1]–[22]. From a physical point of view, the insertion of defects or impurities within a given physical system renders the latter more interesting and realistic. Applications of such studies are important for a number of different disciplines, including models in condensed matter theory (see e.g. [1, 23, 24]) and quantum information (see e.g. [25, 26]). It is desirable then to introduce defects in integrable theories, in such a way that integrability is preserved, so that the corresponding tools can be used in order to obtain precise information about the defect model.

A systematic algebraic formulation for describing Liouville integrable point-like defects was recently introduced in [21, 22]. The description was primarily based on the underlying quadratic algebra satisfied by the bulk monodromy matrices, which describe the left and right theories, as well as by the defect L-matrix. In fact, this is the main necessary requirement so that Liouville integrability may be by construction guaranteed. The main steps of the proposed algebraic methodology are provided in the subsequent sections, however for a more detailed description of the process we refer the interested reader to [21, 22].
An efficient description of defects becomes more intricate at the level of classical integrable field theories, where usually the defect is introduced as a discontinuity (jump) together with suitable sewing conditions [8, 10]. In the present scheme the sewing conditions naturally emerge as continuity conditions on the time components of the Lax pair around the defect point. It was also rigorously proven in [21] that the sewing conditions are compatible with the hierarchy of the Hamiltonians, a fact that ensures that the proposed formulation is well defined and consistent.

In the present work, we consider dynamical point-like defects in the context of sigma models, such as the Landau-Lifshitz (L-L) model and a variation of the familiar principal chiral model (PCM), the so called Faddeev-Reshetikhin (F-R) [27] model, in such a way that integrability is preserved. It is worth noting that this kind of systems, in addition to their own physical and mathematical value have also attracted considerable interest lately due to the fact that they typically arise within the AdS/CFT context (see e.g. [28, 29, 30] and references therein). Thus, it would be of great significance to investigate possible relevant physical implications in this particular frame. Recall also that the typical PCM possesses a non-ultra local algebra rendering its quantization a very intricate task, whereas its variation, the F-R model is associated to a more familiar ultra-local algebra, much easier to deal with. In any case, both PCM and F-R models share the same Lax pair, and consequently the same equations of motion, hence they are physically quite similar. Therefore, the findings presented for the F-R model are naturally relevant for the conventional PCM model.

Based on the formulation of [21, 22] we introduce the defect matrix, and the associated modified monodromy matrix. Using this as our starting point we extract the first couple of the local integrals of motion, and the corresponding time components of the Lax pairs for both models. Due to analyticity requirements imposed on the time components of the Lax pairs, certain sewing conditions emerge at the defect point. The involution of the charges is shown based on the underlying algebra, and the invariance of the sewing conditions under the Hamiltonian action is also explicitly checked.

2 The Landau-Lifshitz model with defect

We first consider integrable point-like dynamical defects in the context of the isotropic Landau-Lifshitz model. We assume periodic boundary conditions and restrict our attention to the \( su_2 \) classical algebra [31]. Results regarding higher rank algebras may also be considered [32].

The equations of motion associated with the isotropic Landau-Lifshitz model are given
as:
\[
\frac{\partial \vec{S}}{\partial t} = i\vec{S} \times \frac{\partial^2 \vec{S}}{\partial x^2}, \tag{2.1}
\]
where the vector-valued functions \(\vec{S}(x) = (S_1(x), S_2(x), S_3(x))\) which describe the physical quantities of the model take values on the unit 2-sphere, i.e. \(\vec{S} \cdot \vec{S} = 1\), and satisfy an \(\mathfrak{su}_2\) Poisson structure given by the following Poisson brackets
\[
\{S_a(x), S_b(y)\} = 2i\varepsilon_{abc} S_c(x) \delta(x - y), \tag{2.2}
\]
with \(\varepsilon_{abc}\) being the totally antisymmetric Levi-Civita tensor with value \(\varepsilon_{123} = 1\). We note that throughout the present section we shall also use the following linear combinations of the fields \(S_i(x)\):
\[
S^\pm(x) = \frac{1}{2}(S_1(x) \pm iS_2(x)). \tag{2.3}
\]
The Hamiltonian and the momentum of the model are given by the following expressions
\[
\mathcal{H} = -\frac{1}{4} \int_{-L}^{L} dx \left(\frac{\partial \vec{S}}{\partial x}\right)^2,
\]
\[
\mathcal{P} = \frac{i}{2} \int_{-L}^{L} dx \frac{S_1 S'_2 - S_1' S_2}{1 + S_3}, \tag{2.4}
\]
where the prime denotes derivative with respect to \(x\). Using the Hamiltonian, the equations of motion can also be expressed as
\[
\frac{\partial \vec{S}}{\partial t} = \{\mathcal{H}, \vec{S}\}. \tag{2.5}
\]

### 2.1 The Lax and defect operators

We may now introduce a single point-like defect of dynamical type in the L-L model at the point \(x_0\). The starting point for analyzing the system is the derivation of the modified monodromy matrix, which is built as a co-action [21]
\[
\mathcal{T}(L, -L, \lambda) = T^+(L, x_0, \lambda) \mathcal{D}(x_0, \lambda) T^-(x_0, -L, \lambda)
= \mathcal{P} \exp \left( \int_{x_0}^{L} dx \mathcal{U}^+(x) \right) \mathcal{D}(x_0, \lambda) \mathcal{P} \exp \left( \int_{-L}^{x_0} dx \mathcal{U}^-(x) \right). \tag{2.6}
\]
Recall the Lax pair \((\mathcal{U}, \mathcal{V})\) for the bulk L-L model
\[
\mathcal{U}(x) = \frac{1}{\lambda} \begin{pmatrix} S_2 \ - S_3 \\ S_3 \ \frac{S_1}{2} \end{pmatrix} = \frac{1}{2\lambda} S, \quad \mathcal{V}(x) = \frac{1}{2\lambda^2} S - \frac{1}{2\lambda} \frac{\partial S}{\partial x} S. \tag{2.7}
\]
As indicated in e.g. [21], the defect operator is required to satisfy the same quadratic algebra as the monodromy matrix, i.e.

\[ \{ \mathcal{D}(\lambda) \otimes \mathcal{D}(\mu) \} = \left[ r(\lambda - \mu), \mathcal{D}(\lambda) \otimes \mathcal{D}(\mu) \right]. \tag{2.8} \]

In the case of the L-L model, the classical \( r \)-matrix is the Yangian solution [33]

\[ r(\lambda) = -\frac{\mathcal{P}}{\lambda}, \tag{2.9} \]

with \( \mathcal{P} \) being the permutation operator: \( \mathcal{P}(\vec{a} \otimes \vec{b}) = \vec{b} \otimes \vec{a} \). It turns out that the generic defect operator satisfying the quadratic Poisson structure with the Yangian \( r \)-matrix has the following form

\[ \mathcal{D}(\lambda) = \lambda \mathbb{I} + \begin{pmatrix} \xi^z & \xi^- \\ \xi^+ & -\xi^z \end{pmatrix}, \tag{2.10} \]

due to (2.8) the elements satisfy the \( \mathfrak{sl}_2 \) algebraic relations

\[ \begin{align*}
\{ \xi^z, \xi^\pm \} &= \pm \xi^\pm \\
\{ \xi^+, \xi^- \} &= 2\xi^z. \tag{2.11} \end{align*} \]

Notice that the associated Casimir is given as: \( C = (\xi^z)^2 + \xi^+\xi^- \) hence, the algebra (2.11) may be parameterized by two free fields. Similarly for (2.7).

The continuum “bulk” monodromy matrices \( T^\pm \) satisfy the usual differential equation

\[ \frac{\partial T^\pm(x, y; \lambda)}{\partial x} = U^\pm(x, \lambda)T^\pm(x, y; \lambda), \tag{2.12} \]

and the zero curvature condition is then expressed as:

\[ \dot{U}^\pm(x, t) - \dot{V}^\pm(x, t) + \left[ U^\pm(x, t), V^\pm(x, t) \right] = 0, \quad x \neq x_0. \tag{2.13} \]

On the defect point in particular the zero curvature condition is formulated as (see e.g. [19, 21] for more details)

\[ \frac{d}{dt} \mathcal{D}(x_0) = \tilde{V}^+(x_0)\mathcal{D}(x_0) - \mathcal{D}(x_0)\tilde{V}^-(x_0), \tag{2.14} \]

and describes explicitly the jump occurring across the defect point. This will be a major consistency check of the prescription followed here. The time components \( \tilde{V}^\pm \) of the Lax pairs for the left and right bulk theories as well as the defect point will be explicitly derived subsequently.
2.2 Local integrals of motion

We first derive the tower of involutive local integrals of motion. Regarding the left and right bulk parts, we consider the usual ansatz for the respective monodromy matrices

\[ T^\pm(x, y; \lambda) = (1 + W^\pm(x)) \ e^{Z^\pm(x,y)} \ (1 + W^\pm(y))^{-1}, \tag{2.15} \]

with \( W^\pm \) and \( Z^\pm \) being purely off-diagonal and diagonal matrices respectively. We also assume that they admit an expansion in terms of the spectral parameter \( \lambda \) as

\[ W^\pm(x, \lambda) = \sum_{n=0}^{\infty} \lambda^n W^\pm_n(x), \quad Z^\pm(x, y, \lambda) = \sum_{n=-1}^{\infty} \lambda^n Z^\pm_n(x, y). \tag{2.16} \]

Substituting the ansatz (2.15) into the relation (2.12), and splitting the resulting equation into a diagonal and an off-diagonal part one obtains

\[ \frac{dW^\pm}{dx} + W^\pm U^\pm_d - U^\pm_d W^\pm + W^\pm U^\pm_a W^\pm - U^\pm_a = 0, \]

\[ \frac{\partial Z^\pm}{\partial x} = U^\pm_d + U^\pm_a W^\pm. \tag{2.17} \]

The off-diagonal part corresponds to a typical Riccati type differential equation. Solving this set of differential equations provides explicit expressions of the \( W^\pm, Z^\pm \) matrices. As for the local integrals of motion, recall that they are provided by the generating functional

\[ G(\lambda) = \ln \left( \text{tr} \ T(\lambda) \right). \tag{2.18} \]

Substituting the ansatz (2.15), the latter expression can be formulated as:

\[ G(\lambda) = \ln \text{tr} \left[ e^{Z^+(L,x_0)} (1 + W^+(x_0))^{-1} D(x_0) \ (1 + W^-(x_0)) e^{Z^-(x_0,-L)} \right], \tag{2.19} \]

where Schwartz boundary conditions at the endpoints \( x = \pm L \) have been implemented.

Solving (2.17) one determines \( W^\pm \) and \( Z^\pm \) order by order. The first three terms of the expansion suffice to compute the Hamiltonian and the momentum of the bulk theories, and are found to be

\[ O(1/\lambda) : \quad W^\pm_0 = \begin{pmatrix} 0 & -\bar{a} \\ a & 0 \end{pmatrix}, \quad a = \frac{1 - S_3}{2S^-} = \frac{2S^+}{1 + S_3}, \]

\[ O(\lambda^0) : \quad W^\pm_1 = \begin{pmatrix} 0 & -\bar{a}' \\ -a' & 0 \end{pmatrix}, \]

\[ O(\lambda) : \quad W^\pm_2 = \begin{pmatrix} 0 & -\bar{a}'' + (\bar{a}')^2 S^- \\ a'' - (a')^2 S^- & 0 \end{pmatrix}. \tag{2.20} \]
Note that \( \bar{a} \) is obtained by interchanging \( S^- \leftrightarrow S^+ \). Correspondingly, we find that \( Z^\pm \) is given by
\[
Z^\pm(x, y, \lambda) = \frac{1}{2\lambda}(x - y)\sigma^x + \sum_{n=1}^\infty \lambda^{n-1} \int_y^x dz \left( S^+\sigma^- + S^-\sigma^+ \right) W^\pm_n(z). \tag{2.21}
\]
At order \( O(1/\lambda) \) one finds
\[
Z^\pm_{-1} = \frac{1}{2}(x - y)\sigma^x. \tag{2.22}
\]
This term is important, since it provides the leading contribution of \( e^Z \) as \( \lambda \to 0 \), a fact that is used when expanding the generating function \( G \). The next two orders provide the first two physical integrals of motion for the right and left theories. More specifically, one concludes that
\[
(Z^\pm_0)_{11} = -P^\pm, \quad \text{and} \quad (Z^\pm_1)_{11} = H^\pm. \tag{2.23}
\]
It should be stressed out that the implementation of special boundary conditions at the defect point \( x_0 \) amounts to the emergence of certain boundary type terms in both charges, which have to be taken into account.

Expanding then the generating function \( G \) in powers of \( \lambda \) leads to the local integrals of motion of the defect L-L model. More precisely, the generating function of the local integrals of motion becomes
\[
G(\lambda) = Z^+_1(\lambda) + Z^-_1(\lambda) + \ln \left[ (1 + W^+(x_0))^{-1} \mathcal{D}(x_0) \left( 1 + W^-(x_0) \right) \right]_{11}, \tag{2.24}
\]
so that the terms \( Z^\pm_{11} \) provide the left and right bulk charges computed above, whereas the third term gives the defect contribution.

Gathering all the information given above we derive the explicit expressions for the momentum and Hamiltonian of the system:
\[
P = \frac{i}{2} \int_{-L}^{x_0} dx \left( \frac{S_1 S_2' - S_2 S_1'}{1 + S_3} \right) + \frac{i}{2} \int_{x_0}^{L} dx \left( \frac{\tilde{S}_1 \tilde{S}_2' - \tilde{S}_2 \tilde{S}_1'}{1 + \tilde{S}_3} \right) \\
+ \frac{1}{2} \ln(1 + \tilde{S}_3(x_0)) - \frac{1}{2} \ln(1 + S_3(x_0)) - \ln \mathcal{D}^{(0)}(x_0),
\]
\[
H = -\frac{1}{4} \int_{-L}^{x_0} \left( \frac{\partial \tilde{S}}{\partial x} \right)^2 - \frac{1}{4} \int_{x_0}^{L} \left( \frac{\partial S}{\partial x} \right)^2 + \mathcal{D}^{(1)}(x_0) \\
- \frac{1}{2} \left( \tilde{S}_1(x_0) - \tilde{S}_2(x_0) \right) a^+(x_0) + \frac{1}{2} \left( S_1(x_0) - S_2(x_0) \right) a^-(x_0), \tag{2.25}
\]
where we define
\[
a^+ = \frac{1 - \tilde{S}_3}{2\tilde{S}^-} = \frac{2\tilde{S}^+}{1 + \tilde{S}_3}, \quad a^- = \frac{1 - S_3}{2S^-} = \frac{2S^+}{1 + S_3}, \tag{2.26}
\]
and \( \bar{a}^\pm \) are defined analogously. Clearly the variables \( S_i \) describe the left bulk theory, whereas the variables \( \tilde{S}_i \) describe the right bulk theory. Note also the emergence of some non-trivial
boundary type terms at the defect point, in both charges. Regarding the defect contributions, we have found the following explicit expressions for the first two physical charges

\[ D^{(0)} = \frac{1}{3} \left[ a^- \xi^- + a^+ \xi^+ + (1 - \bar{a}^+ a^-) \xi_- \right] \]

\[ D^{(1)} = \frac{1}{3} \left[ \xi_+ \left(1 + a^- \bar{a}^+ \right) + \left(\bar{a}^+ + (\bar{a}^+)^2 a^+ \right) \xi^- + \left(\bar{a}^+ - \bar{a}^+ a^+ \right) a^- + \left(\bar{a}^+ a^+ \right) \xi_+ \right], \] (2.27)

with

\[ \hat{3} \equiv \xi_+ \left(1 - \bar{a}^+ a^- \right) = (\xi_+)^2 D^{(0)}, \quad \xi_+ \equiv 1 + a^+ \bar{a}^-. \] (2.28)

These two terms are the defect contributions in the momentum and Hamiltonian charges respectively. For the sake of clarity, we regard from now on \( a^\pm, \bar{a}^\pm \) as the fundamental variables, instead of the fields \( S_i, \bar{S}_i \). We should also note that these expressions simplify, when one considers the sewing conditions on the defect point, as will be transparent in the subsequent section.

### 2.3 The modified Lax pair

From the general theory of classical integrable systems, the time components \( \mathbb{V} \) of the Lax pair are given by [21, 31]

\[ \mathbb{V}(x, \lambda, \mu) = t^{-1}(\lambda) \text{tr}_a \left( T_a(L, x, \lambda) r_{ab}(\lambda, \mu) T_a(-L, -x) \right). \] (2.29)

In the case where defects are taken into account, one has to compute \( \mathbb{V} \) in the bulk, as well as at the defect point. If the \( r \)-matrix of the model is the Yangian, as it happens for the L-L model, the corresponding time components are simplified and for a single point-like defect are expressed as [21]

\[ \mathbb{V}^+(x, \lambda, \mu) = \frac{t^{-1}}{\lambda - \mu} T^+(x, x_0) D(x_0) T^-(x_0, -L) T^+(L, x) \]

\[ \mathbb{V}^-(x, \lambda, \mu) = \frac{t^{-1}}{\lambda - \mu} T^-(x, -L) T^+(L, x_0) D(x_0) T^-(x_0, x) \]

\[ \tilde{\mathbb{V}}^+(x_0, \lambda, \mu) = \frac{t^{-1}}{\lambda - \mu} D(x_0) T^-(x_0, -L) T^+(L, x_0) \]

\[ \tilde{\mathbb{V}}^-(x_0, \lambda, \mu) = \frac{t^{-1}}{\lambda - \mu} T^-(x_0, -L) T^+(L, x_0) D(x_0). \] (2.30)
We provide explicit expressions for the first three orders of the $V$ operators below. It should be clear that the following expressions hold for the left bulk part

$$
\mathcal{O}(\lambda^0) : \quad V^\perp_{(0)}(x, \mu) = -\frac{I + S}{2\mu} = -\frac{1}{2\mu} \begin{pmatrix} 1 + S_3 & 2S^- \\ 2S^+ & 1 - S_3 \end{pmatrix}
$$

(2.31)

$$
\mathcal{O}(\lambda) : \quad V^\perp_{(1)}(x, \mu) = \frac{1}{\mu} \begin{pmatrix} S^+S'^- - S^-S'^+ & S^-S_3^l - S_3S'^- \\ S_3S'^+ - S'^+S^+ & S^+S'^+ - S^+S'^- \end{pmatrix} + \frac{1}{\mu} V^\perp_{(0)}(x, \mu).
$$

Similar expressions hold for $V^\parallel_{(i)}$, but with $S^i \to \tilde{S}^i$. At order $\mathcal{O}(\lambda)$ one naturally finds the time component associated with the Hamiltonian. For the sake of clarity, as noted before, it is more useful to write down the operators in terms of the fields $a^\pm$, defined in (2.20)

$$
\mathcal{O}(\lambda^0) : \quad V^\pm_{(0)}(x, \mu) = -\frac{1}{\mu^2} \begin{pmatrix} \bar{a}^\pm \\ a^\pm \end{pmatrix} \left( \begin{array}{c} \bar{\alpha}^\pm \\ \alpha^\pm \end{array} \right)
$$

(2.32)

$$
\mathcal{O}(\lambda) : \quad V^\pm_{(1)}(x, \mu) = \frac{1}{\mu^2(\bar{\delta})^2} \begin{pmatrix} \bar{a}^\pm(a^\pm - \bar{a}^\pm) - \bar{\delta}^\pm & -a^\pm - \bar{a}^\pm(\bar{\delta}^\pm + \bar{a}^\pm a^\pm) \\ a^\pm(a^\pm \bar{a}^\pm - \bar{\delta}^\pm) + a^\pm & a^\pm(\bar{a}^\pm - a^\pm) - a^\pm a^\pm \bar{\delta}^\pm \end{pmatrix}.
$$

At the defect point, we have the following expressions at the first order

$$
\tilde{V}^+_0(x_0, \mu) = -\frac{1}{\mu^2} \begin{pmatrix} a^- \xi^- + \xi^z & a^+(a^- \xi^- + \xi^z) \\ \bar{a}^+(\bar{a}^- + \bar{a}^+) \end{pmatrix}
$$

$$
\tilde{V}^-_0(x_0, \mu) = -\frac{1}{\mu^2} \begin{pmatrix} \bar{a}^+ \xi^+ + \xi^z & \xi^- - \bar{a}^+ \xi^z \\ a^-(a^+ \xi^+ + \xi^z) & \xi^- - a^+ \xi^z \end{pmatrix}.
$$

(2.33)

Gluing the bulk and defect $V$ operators at the defect point leads to sewing conditions that relate the bulk fields at the defect point with the defect fields. In particular, setting

$$
V^\parallel_{(0)}(x \to x_0) = \tilde{V}^+_0(x_0) ,
$$

(2.34)

leads to a unique sewing condition for the four matrix entries

$$
\mathcal{C}^{(0)} : \quad \xi^+ = a^- a^+ \xi^- - (a^- + a^+) \xi^z = 0 .
$$

(2.35)

Solving this condition with respect to $\xi^-$ and substituting the solution back to the defect contribution of the first charge (2.27) simplifies the latter one significantly, which can be written in a very compact form as

$$
\mathcal{D}^{(0)} = a^- \xi^- + \xi^z .
$$

(2.36)

The sewing condition emerging when gluing the $V$ operators at the defect point from the left is related with eq. (2.35) by complex conjugation.
At the next order, the modified $\mathbb{V}$ operator at the defect point from the right is

$$
\mathbb{V}^+_{(1)} = \frac{1}{\mu^2} \left( \frac{\partial^+}{\partial^+} \right)^2 \left( \begin{array}{cc}
\Omega_{11} & \Omega_{12} \\
\Omega_{21} & \Omega_{22}
\end{array} \right),
$$

with

$$
\Omega_{11} = (a^- a'^+ + (a'^+ - a^- a'^+) \xi^+ \xi^- + a^- (a'^+ - a'^+ - 2) \xi^- \xi^+ \\
+ (a'^+ - a^+ \xi^+ + (a^- a'^+ - 1 + a'^+ a^-) (\xi^+)^2 - (a^- \xi^-)^2)
$$

$$
\Omega_{12} = a^+ (a'^+ - a^-) \xi^+ \xi^- + a^- (a'^+ (a^- a'^+ - 2) - 2 a'^+) \xi^- \xi^+ - (a^+)^2 \xi^+ \xi^+
+ (a^+ (-1 + a^- a'^+ - 1 + a'^+ a^-) (\xi^+)^2 - (a^+ + a'^+) (a^- \xi^-)^2)
$$

$$
\Omega_{21} = (a^-)^2 \xi^- \xi^+ + (2 a^+ a^- - 2 a'^+ a^-) \xi^+ \xi^- - (a^- + a'^-) \xi^+ \xi^-
+ (a^- - a^+ (a^-)^2 + (a^-)^2 a'^+ - a'^-) (\xi^+)^2 + (a'^+ - a^+ (a^-)^2)
$$

$$
\Omega_{22} = -(a^- a'^+ + a^+(a^- + a'^-)) \xi^+ \xi^- + (a^-)^2 (a^+ + a'^+) \xi^- \xi^+ - (a^+ \xi^+)^2
+ (a^+ a^- - (a^+ a^-)^2 + a^- a'^+ - a'^- (a^-)^2 + (a'^+ (1 - 2 a^+ a^-) - a^+ (a^-)^2)
\xi^+ \xi^+.
$$

Equating the first order Lax operators at the defect point, $\mathbb{V}^+_{(1)}(x \rightarrow x_0) = \mathbb{V}^+_{(1)}(x_0)$, yields a unique second sewing condition, $C^{(1)}$, for all four entries of the matrices, which is rather complicated and long. However, using the previously found sewing condition of the lower order, $C^{(0)}$, the new condition simplifies greatly and assumes the form

$$
\bar{a}^+(\partial^+)^2 (a^- \xi^- + \xi^+ (a^- + a^- a'^+ \xi^- + a^+(1 + a'^- \xi^-) + (a^- + a^+) \xi^+) = 0,
$$

or, in other words,

$$
C^{(1)}: \quad (a^- - a^+) + (a^- a'^+) \xi^- + (a^- + a^+) \xi^+ = 0,
$$

since the factor $\bar{a}^+(\partial^+)^2$ in front cannot be zero and the next term inside the parenthesis is just $D^{(0)}$. We observe that the sewing condition at this order involves not only the left and right fields, as well as the defect fields, but also the derivatives of the bulk fields (see also [21, 22]). Proceeding to higher orders, one builds an infinite tower of sewing conditions, which in principle contain higher order derivatives of the bulk and defect fields. In fact, roughly speaking the sewing conditions of each order are associated to the corresponding integral of motion. In the same spirit that there exists an infinite tower of independent charges in involution in integrable field theories, so that the system is integrable, an infinite tower of
independent sewing conditions also exists (see also relevant proof on the invariance of the sewing conditions under the generic Hamiltonian action in [21]). In any case, this is not really surprising, given for instance that the same scenario applies in integrable field theories with non trivial boundaries, where an infinite tower of suitable boundary conditions is also consistently extracted [34]. Moreover, given that the sewing conditions reflect the analyticity of the fields at the defect point, we expect that they should provide useful information in order to solve the equations of motion derived below.

It has been explicitly checked that the extracted sewing conditions for the first two orders, i.e. (2.35) and (2.38), are as expected compatible with the Hamiltonian hierarchy, confirming the relevant generic statement proven in [21] regarding the invariance of all sewing conditions under the Hamiltonian action. Note also that the commutativity of the momentum and Hamiltonian is valid as expected, but on-shell, i.e. the sewing conditions must be taken into account, as it happens in the case of the sine-Gordon model with defect (see also [22]),

\[ \{ \mathcal{H}, \mathcal{P} \} = 0. \] (2.39)

It is worth noting that taking into account the second sewing condition (2.38), the first order defect contribution simplifies significantly; it is eventually given by the following simple expression

\[ \mathcal{D}^{(1)} = \frac{a^{+} + a^{-}}{a^{+} - a^{-}}. \] (2.40)

It is interesting that on-shell the defect fields \( \xi_i \) cancel out completely from the defect Hamiltonian. This simplification could be a hint that a simpler formulation of implementing dynamical defects in the theory could exist. On the other hand, it is possible that this effect emerges as a bi-product from the existence of integrability of the model. Furthermore, one should keep in mind that this effect is present only in the on-shell Hamiltonian of the model, once the sewing conditions are taken into account, and is related to the on-shell versus off-shell integrability issue of the model. For a more detailed discussion on the on-shell vs off shell integrability we refer the interested reader to [22].

Finally, the equations of motion for the left and right bulk theories are given by (2.5), for the right theory the variables are simply \( \tilde{S}_i \). The time evolution on the defect point is expressed as:

\[
\begin{align*}
\dot{\xi}^- &= \frac{2}{3} \left( A - \mathcal{D}^{(1)} \tilde{a}^+ \tilde{d}^+ \right) \xi^z + \frac{1}{3} \left( \mathcal{D}^{(1)} \tilde{d}^+ (1 - a^- a^+) - C \right) \xi^-
- \frac{2}{3} \left( - B + \mathcal{D}^{(1)} a^- \tilde{d}^+ \right) \xi^z + \frac{1}{3} \left( - \mathcal{D}^{(1)} \tilde{d}^+ (1 - a^- \tilde{a}^+) + C \right) \xi^+
- \frac{1}{3} \left( A - \mathcal{D}^{(1)} \tilde{a}^+ \tilde{d}^+ \right) \xi^+ + \frac{1}{3} \left( B - \mathcal{D}^{(1)} a^- \tilde{d}^+ \right) \xi^-,
\end{align*}
\] (2.41)

where we define

\[ A = \tilde{a}^{+'} + (\tilde{a}^+)^2 a^{+'}, \quad B = (a^+ - a^{+'}) a^- \tilde{a}^{+'} + \tilde{d}^+ a^{+} \]
\[ C = \bar{a}^\dagger (a^+ + a^-) + \bar{a}^\dagger (\bar{a}^+ a^+ + a^- - a^+ - \bar{a}^+ a^-). \]  

The equations above, relevant to the defect point, emerge either from the Hamiltonian description through
\[ \dot{\xi}^i = \{ H, \xi^i \}, \]

or the zero curvature condition on the defect point (2.14). Recall that throughout this work we consider only dynamical type defects, so that the equations of motion above describe their time evolution. It is an interesting task to attempt to solve this set of equations, as well as determining the nature of the physical information obtained from the solutions. It should be stressed out that not only the sewing conditions presented before should have to be taken into account when solving the equations of motion, but it is expected that they will also simplify the somehow complicated expressions of the latter ones. We hope to address these interesting issues in a forthcoming publication, starting from the less complicated NLS defect model [21].

3 The Faddeev-Reshetikhin model with defect

The F-R model was introduced in [27], as a convenient modification of the familiar non-ultra local PCM model. Despite the different underlying algebras, it was shown that the equations of motion of these two models coincide, hence they may be considered as equivalent. Moreover, the F-R model is easier to quantize, due to the absence of non-ultra local terms in the algebra.

3.1 The model

The Lax operator of the model is the same with the familiar PCM one and has the form
\[ \mathbb{U}(x, \lambda) = \frac{\lambda J_0(x) + J_1(x)}{\lambda^2 - 1}, \]

with \( J_i \) being currents taking values on the Lie algebra \( \mathfrak{g} \), corresponding to a Lie group \( G \). In the present work, we restrict ourselves to the simple case of the \( SU(2) \) group. We choose the Pauli matrices \( (\sigma^+, \sigma^-, \frac{1}{2} \sigma^z) \) as the generators of the \( \mathfrak{su}_2 \) Lie algebra. We also define the currents
\[ S = J^+ = \frac{J_0 + J_1}{2}, \quad T = J^- = \frac{J_0 - J_1}{2}, \]

so that the Lax operator may also be written as
\[ \mathbb{U}(x, \lambda) = \frac{S}{\lambda - 1} + \frac{T}{\lambda + 1}. \]
Expanding in the Lie algebra generators, the currents are expressed in the following component form

\[ S = S^a t^a = S^z z^z + S^+ \sigma^+ + S^- \sigma^-, \]

(3.4)

and similarly for \( T \). The time component of the Lax pair is given by:

\[ \mathbb{V}(x, \lambda) = -\frac{S}{\lambda - 1} + \frac{T}{\lambda + 1}. \]

(3.5)

The Lax operator satisfies the fundamental Poisson bracket relation

\[ \{ \mathbb{U}(x, \lambda) \otimes \mathbb{U}(y, \mu) \} = [r(\lambda - \mu), \mathbb{U}(x, \lambda) \otimes I + I \otimes \mathbb{U}(x, \mu)] \delta(x - y), \]

(3.6)

which reflects the absence of non-ultra-local terms in the level of the algebra for \( S \) and \( T \). In particular, the latter ones satisfy the following algebraic relations

\[
\begin{align*}
\{ S^a(x), S^b(y) \} &= f^{ab}_{\ c} S^c(x) \delta(x - y) \\
\{ S^a(x), T^b(y) \} &= 0 \\
\{ T^a(x), T^b(y) \} &= f^{ab}_{\ c} T^c(x) \delta(x - y).
\end{align*}
\]

(3.7)

As mentioned above, we restrict our attention in the simplest case, i.e. the \( SU(2) \) group. The underlying algebra consists of two independent copies of the \( su_2 \) Lie algebra and the structure constants are proportional to the usual Levi-Civita tensor \( (f^{ab}_{\ c} = 2i \epsilon_{abc}) \). Moreover, \( S \) and \( T \) may be interpreted as three-vectors and their lengths can be fixed as: \( |S| = |T| = C = \text{constant} \); i.e. they are simply the \( su_2 \) Casimir operators. Note that the latter restriction \( (C = \text{constant}) \) is known as the Virasoro constraint in the string theory frame (see e.g. \[28\]), rendering essentially the familiar PCM model ultra-local.

### 3.2 Local integrals of motion

Let us first briefly review the bulk theory. From the Lax operator of the F-R model, it is manifest that there are two expansions that one may obtain, around two singular points. The local conserved charges, such as the momentum and the Hamiltonian are then linear combinations of the quantities derived. We present details for the \( \lambda = 1, \ (y = \lambda - 1 = 0) \) expansion, the results for the \( \lambda = -1 \) are similar and are omitted. In particular, for the bulk parts we have

\[ W = \sum_{n=0}^{\infty} y^n W_n = \sum_{n=0}^{\infty} y^n \begin{pmatrix} 0 & a_n \\ b_n & 0 \end{pmatrix}, \]

(3.8)

and

\[ Z = \sum_{n=-1}^{\infty} y^n Z_n = \sum_{n=-1}^{\infty} y^n \begin{pmatrix} \gamma_n & 0 \\ 0 & \delta_n \end{pmatrix}. \]

(3.9)
For the first order we have found for $W$

$$ a_0 = \frac{1}{2S^z}(S^z \mp C), \quad -\bar{a}_0 = b_0 = \frac{1}{2S^-}(-S^z \pm C), \quad (3.10) $$

with

$$ C \equiv \sqrt{(S^z)^2 + 4(S^-)(S^+)} = 2\text{tr}(S^z) = 2\text{tr}(T^2) = \sqrt{(T^z)^2 + 4(T^-)(T^+)} . \quad (3.11) $$

Setting the Casimir $C$ equal to one amounts to coinciding with the results of the L-L model.

In the next order

$$ \pm a_1 (2C) = -T^+ - a_0 T^z + a_0^2 T^+ + 2a'_0 $$
$$ \pm b_1 (2C) = T^- - b_0 T^z - b_0^2 T^- - 2b'_0 . \quad (3.12) $$

Observe also that

$$ b_1 = -\bar{a}_1 + \frac{2}{C} \bar{a}'_0 , \quad (3.13) $$

so that everything can be written in terms of $a_0$ and $a_1$, which may now be regarded as the fundamental fields. We carry on the computations for the $Z$ elements. The results we have found are the following:

$$ 2 \gamma'_{-1} = +S^z + 2b_0 \ S^- = \pm C $$
$$ 2 \delta'_{-1} = -S^z + 2a_0 \ S^+ = \mp C $$
$$ 4 \gamma'_{0} = +T^z + 2b_0 \ T^- + 4b_1 \ S^- $$
$$ 4 \delta'_{0} = -T^z + 2a_0 \ T^+ + 4a_1 \ S^+ . \quad (3.14) $$

It is manifest that the first integrals are trivial since the corresponding densities are constant

$$ \mathcal{I}_{-1} = \hat{\mathcal{I}}_{-1} = L C . \quad (3.15) $$

Henceforth we choose to consider $C = 1$, as well as the first of the two signs appearing in all the expressions above. Let us first derive the expression for the bulk theory with Schwartz type boundary conditions. Substituting the exact expressions for $a_i$’s and $b_i$’s we obtain the charges in terms of the fields. Bear in mind that some boundary terms vanish due to Schwartz boundary conditions, that will have to be taken into account when considering non-trivial boundary conditions at the defect point. Eventually, one is left with

$$ \mathcal{I}_0 = \frac{1}{2} \mathcal{P}_S - \frac{1}{2} \int_{-L}^{L} dx \text{tr}(S \ T) $$
$$ \hat{\mathcal{I}}_0 = \frac{1}{2} \mathcal{P}_T + \frac{1}{2} \int_{-L}^{L} dx \text{tr}(S \ T) , \quad (3.16) $$

and

$$ \mathcal{P}_S = i \int_{-L}^{L} dx \ S_1 S'_2 - S'_1 S_2 \frac{1}{1 + S^z} , \quad \mathcal{P}_T = i \int_{-L}^{L} dx \ T_1 T'_2 - T'_1 T_2 \frac{1}{1 + T^z} . \quad (3.17) $$
Taking their sum and their difference yield the Hamiltonian and momentum densities of the system respectively. More specifically,

\[
P = \frac{1}{2} P_S + \frac{1}{2} P_T
\]

\[
H = \frac{1}{2} P_S - \frac{1}{2} P_T - \int_{-L}^{L} dx \text{tr}(S \ T).
\] (3.18)

The above results hold for the right and left bulk parts of the F-R model. Recalling the generic expression for the generating function (2.24), we find the defect contributions at all orders.

Now consider the system in the presence of a point-like integrable defect at the position \(x_0\), described by the generic \(L\)-matrix (2.10). The expressions of the momentum and the Hamiltonian emerging from the expansion of \(\lambda\) around the values \(\pm 1\), are given as:

\[
\mathcal{I}_0 = \frac{1}{2} \int_{-L}^{x_0} dx \left( i \frac{S_1 S'_2 - S'_1 S_2}{1 + S_3} - \text{tr}(ST) \right) + \frac{1}{2} \int_{x_0}^{L} dx \left( i \frac{\tilde{S}_1 \tilde{S}'_2 - \tilde{S}'_1 \tilde{S}_2}{1 + \tilde{S}_3} - \text{tr}(\tilde{S}\tilde{T}) \right)
\]

\[
- \frac{1}{2} \ln \left( (1 + S_3)(1 + \tilde{S}_3) \right)(x_0) - \ln \mathfrak{W}(x_0),
\] (3.19)

\[
\hat{\mathcal{I}}_0 = \frac{1}{2} \int_{-L}^{x_0} dx \left( i \frac{T_1 T'_2 - T'_1 T_2}{1 + T_3} + \text{tr}(ST) \right) + \frac{1}{2} \int_{x_0}^{L} dx \left( i \frac{\tilde{T}_1 \tilde{T}'_2 - \tilde{T}'_1 \tilde{T}_2}{1 + \tilde{T}_3} + \text{tr}(\tilde{S}\tilde{T}) \right)
\]

\[
- \frac{1}{2} \ln \left( (1 + T_3)(1 + \tilde{T}_3) \right)(x_0) - \ln \hat{\mathfrak{W}}(x_0),
\] (3.20)

where we define

\[
\mathfrak{W} = \xi^z(1 + a_0^+ b_0^-) - a_0^+ \xi^+ + b_0^- \xi^- + 1 - a_0^+ b_0^-
\]

\[
\hat{\mathfrak{W}} = \xi^z(1 + \hat{a}_0^+ \hat{b}_0^-) - \hat{a}_0^+ \xi^+ + \hat{b}_0^- \xi^- + 1 + \hat{a}_0^+ \hat{b}_0^-,
\] (3.21)

where \(a_0^-, b_0^-\) are defined as in (3.10) with the \(S\) fields, \(a_0^+, b_0^+\) are defined with the \(\tilde{S}\) fields, and

\[
\hat{a}_0^- = \frac{T_3 - 1}{2T^+}, \quad \hat{b}_0^- = \frac{1 - T_3}{2T^-}, \quad \hat{a}_0^+ = \frac{\tilde{T}_3 - 1}{2\tilde{T}^+}, \quad \hat{b}_0^+ = \frac{1 - \tilde{T}_3}{2\tilde{T}^-}.
\] (3.22)

The fields \(S, T\) correspond to the left theory whereas the fields \(\tilde{S}, \tilde{T}\) correspond to the right theory. The complete momentum and Hamiltonian of the system are given by the following expressions:

\[
P = \mathcal{I}_0 + \hat{\mathcal{I}}_0
\]

\[
H = \mathcal{I}_0 - \hat{\mathcal{I}}_0.
\] (3.23)
3.3 The modified Lax pair

The next step is the derivation of the associated time components of the Lax pairs for the bulk theories and at the defect point, from left and right. Requiring continuity around the defect point the relevant sewing conditions are extracted. The bulk quantity associated to the Hamiltonian is given by (3.5). The \( \tilde{V} \) operator at the defect point from the right, from the expansion around \( \lambda = 1 \) has the following expression:

\[
\tilde{V}_0^+ = -\frac{1}{(\lambda - 1)\mathfrak{M}} \begin{pmatrix}
\xi^z + 1 + b_0^- \xi^- & -a_0^+ (\xi^z + 1) - a_0^+ b_0^- \xi^- \\
\xi^z + b_0^- (-\xi^z + 1) & -a_0^+ \xi^z - a_0^+ b_0^- (-\xi^z + 1)
\end{pmatrix},
\]

(3.24a)

while from the expansion around \( \lambda = -1 \) one obtains

\[
\tilde{V}_0^- = -\frac{1}{(\lambda + 1)\mathfrak{M}} \begin{pmatrix}
\xi^z - 1 + \hat{b}_0^- \xi^- & -\hat{a}_0^+ (\xi^z - 1) - \hat{a}_0^+ \hat{b}_0^- \xi^- \\
\xi^z + \hat{b}_0^- (-\xi^z - 1) & -\hat{a}_0^+ \xi^z - \hat{a}_0^+ \hat{b}_0^- (-\xi^z - 1)
\end{pmatrix},
\]

(3.24b)

where \( \mathfrak{M}, \hat{\mathfrak{M}} \) are defined in (3.21). Similar expressions hold for \( \tilde{V}^- \), in particular:

\[
\tilde{V}_0^- = -\frac{1}{(\lambda - 1)\mathfrak{M}} \begin{pmatrix}
\xi^z + 1 - a_0^+ \xi^+ & a_0^+ (\xi^z - 1) + \xi^- \\
-a_0^- b_0^+ \xi^+ + b_0^- (\xi^z + 1) & b_0^- \xi^- - a_0^+ b_0^- (-\xi^z + 1)
\end{pmatrix},
\]

(3.24c)

and

\[
\tilde{V}_0^- = -\frac{1}{(\lambda + 1)\mathfrak{M}} \begin{pmatrix}
\xi^z - 1 - \hat{a}_0^+ \xi^+ & \hat{a}_0^+ (\xi^z + 1) + \xi^- \\
-\hat{a}_0^- \hat{b}_0^+ \xi^+ + \hat{b}_0^- (\xi^z - 1) & \hat{b}_0^- \xi^- + \hat{a}_0^+ \hat{b}_0^- (\xi^z + 1)
\end{pmatrix}.
\]

(3.24d)

Due to the continuity requirements, the corresponding sewing conditions arise, from the expansion around \( \lambda = 1 \):

\[
\mathcal{C}^{(0)}: \quad \xi^+ - b_0^+ b_0^- \xi^- - (b_0^+ + b_0^-) \xi^z - (b_0^+ - b_0^-) = 0,
\]

(3.25)

and from the expansion around \( \lambda = -1 \):

\[
\hat{\mathcal{C}}^{(0)}: \quad \xi^+ - \hat{b}_0^+ \hat{b}_0^- \xi^- - (\hat{b}_0^+ + \hat{b}_0^-) \xi^z - (\hat{b}_0^+ - \hat{b}_0^-) = 0.
\]

(3.26)

Commutativity between the momentum and the Hamiltonian of the F-R system is shown via the algebraic relations of the defect degrees of freedom, and as expected:

\[
\{ \mathcal{H}, \mathcal{P} \} = 0.
\]

(3.27)

Furthermore, as in the case of the L-L model, the compatibility of the sewing constraints with the charges in involution is explicitly checked and confirmed.

Finally, by means of the underlying Poisson structure, or the zero curvature condition, one may derive the equations of motion for the bulk theories:

\[
\dot{S}(x, t) = -S'(x, t) - [S(x, t), T(x, t)]
\]
\[ \dot{T}(x, t) = T'(x, t) - \left[ T(x, t), S(x, t) \right], \quad x \neq x_0. \quad (3.28) \]

Similar equations of motion hold for the fields of the right theory \( \tilde{S}, \tilde{T} \), which are omitted here for brevity. The time evolution of the defect degrees of freedom reads as:

\[
\begin{align*}
\dot{\xi}^+ &= 2\left( \frac{b}{\mathcal{M}} - \frac{\hat{b}}{\mathcal{M}} \right) \xi^z - \left( \frac{c}{\mathcal{M}} - \frac{\hat{c}}{\mathcal{M}} \right) \xi^+
\dot{\xi}^- &= -2\left( \frac{a}{\mathcal{M}} - \frac{\hat{a}}{\mathcal{M}} \right) \xi^z + \left( \frac{c}{\mathcal{M}} - \frac{\hat{c}}{\mathcal{M}} \right) \xi^-
\dot{\xi}^z &= \left( \frac{a}{\mathcal{M}} - \frac{\hat{a}}{\mathcal{M}} \right) \xi^+ - \left( \frac{b}{\mathcal{M}} - \frac{\hat{b}}{\mathcal{M}} \right) \xi^-.
\end{align*}
\[
(3.29)
\]

where we define

\[
\begin{align*}
& a = -a_0^+, \quad b = b_0^-, \quad c = 1 + a_0^+ b_0^- \\
& \hat{a} = -\hat{a}_0^+, \quad \hat{b} = \hat{b}_0^-, \quad \hat{c} = 1 + \hat{a}_0^+ \hat{b}_0^-.
\end{align*}
\[
(3.30)
\]

Note that expressions (3.29) arise either from the Hamiltonian or from the zero curvature condition at the defect point. With this, we complete our analysis on sigma models in the presence of dynamical point-like defect. In the last section we summarize our results and discuss some interesting extensions that will be analyzed in future works.

### 4 Discussion

We have applied the methodology developed in [21] for the Landau-Lifshitz and the Faddeev-Reshetikhin models. Due to the existence of the classical \( r \)-matrix, we have been able to formulate a modified monodromy matrix, and hence identify the hierarchy of Poisson commuting Hamiltonian alongside the associated Lax pairs. Certain sewing conditions compatible with the Hamiltonian action have also been identified.

In the present and previous studies dynamical type defects have been considered, the so-called type-II defect according to the terminology of [8, 10]. This type of defects is associated to a Poisson structure strictly related to the point-like defect, hence time evolution of the defect degrees of freedom occurs. The next natural step would be the investigation of the type-I or non-dynamical defects. Study of non-dynamical defects may lead to Bäcklund transformations associated to the defect point as pointed out in [8, 10]. It would be quite interesting to check whether these Bäcklund transformations could naturally emerge through the proposed algebraic formulation. For instance if the defect is movable then possibly there exists some Bäcklund transformation associated, although this is not a priori clear. Nevertheless, still the question of the relevant Poisson structure is raised. Another key issue
raised is about dressing and its compatibility with the Poisson structure, in other words is the dressing/Bäcklund transformation group a Lie Poisson group? This is true usually in the bulk case, but not obvious in the defect case. Regarding the dynamical defects that we have studied here, despite our efforts, we have not been able so far to show the natural emergence of a similar geometric origin, which consequently remains an open issue even in the frame of other models studied in earlier works [21, 22].

We have restricted our attention so far to point-like defects, although in principle spatially extended impurities may also be investigated within the proposed scheme. Moreover, at the quantum level one may study the associated Bethe ansatz equations, and hence extract significant information regarding the interaction of particle like excitations displayed by such models with the defect. All the above are highly intricate issues and hopefully will be addressed in forthcoming publications.

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