Counting distinct states in physical dynamics

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A finite physical system has a finite entropy—a finite number of distinct possible states. Here we show that finite maximum counts of distinct (orthogonal) states also define other basic quantities of physics, including energy, momentum and Lagrangian action. A finite number of distinct states also limits the resolution of measurements and makes classical spacetime effectively discrete. Our analysis generalizes speed limits on time evolution: we count the distinct states possible in a finite-length of unitary transformation. As in Nyquist’s bound on distinct signal values in classical waves, widths of superpositions bound the distinct states per unit length. Maximally distinct transformations are effectively discrete, allowing us to simplify analysis and simulation of maximally distinct dynamics.

INTRODUCTION

We live in a quantum world in which distinctness is a finite resource and counting distinct states defines basic quantities such as entropy, energy and momentum.

Entropy was the first of these quantities to be recognized as a count (or conventionally, as the log of a count). This was discovered by Boltzmann, who coarse-grained classical mechanical state-space in order to count states and apply statistics to thermal systems [1, 2]. Planck applied Boltzmann’s statistical entropy to the interaction of light and matter, and found it matched experiment if the grain-size had a particular value $h$. Entropy was an absolute count! Graininess of state was reconciled with continuity of dynamics when energy and momentum were recognized as wave phenomena [3–7]: a minimum size for a distinct state is like the minimum product of spatial and frequency widths of a classical wavepacket [8, 9]. This idea has been formalized in uncertainty bounds [10–25], but these only roughly define counts of distinct states.

More general bounds on certainty, introduced here, directly count the maximum number of states in a quantum evolution that can be perfectly distinguished from each other: how many orthogonal states there can be. These distinctness bounds reveal, for example, that macroscopic energy and momentum are counts of distinct states—in time and space respectively—and explain stationary action as a counting principle. The prototype for the new bounds is Nyquist’s classical bound on communication with waves [26]. He showed that the critical resource that limits the number of distinct signal values that can be transmitted per unit time is bandwidth, the width of the range of frequencies that can appear in the wave’s Fourier decomposition. Intuitively, doubling bandwidth lets us double all frequencies, making everything happen twice as fast—including distinct values. In quantum evolutions, more general measures of frequency width similarly limit how fast things can change.

To briefly review, Nyquist’s classical bandwidth bound derives from the fact that a periodic wave has a discrete spectrum, with a finite number of frequencies in a finite range. For example, consider a complex-valued signal $s(t)$ periodic with period $T$. Only frequency components with an integer number $n$ of oscillations in time $T$ can appear in its Fourier series: $s(t) = \sum c_n e^{2\pi i nt/T}$. If the sum has just $N$ terms, all coefficients $c_n$ are determined by choosing the value of the signal at $N$ times. This defines maximum distinctness: $N$ chosen signal values with $N$ frequency components (Figure 1). Since allowed frequencies are $1/T$ apart, the minimum frequency range to have $N$ of them is

$$\nu_{\text{max}} - \nu_{\text{min}} \geq \frac{N - 1}{T}. \quad (1)$$

Thus the bandwidth $\nu_{\text{max}} - \nu_{\text{min}}$ of a long signal is the maximum average number of chosen values per unit time, $N/T$ [27]. Moreover, since finitely spaced signal values determine all coefficients in the Fourier sum, the continuous signal is completely determined by a discrete subset of its values [28]. A finite bandwidth wave in time (or similarly in space) is effectively discrete.

In the wave-like evolution of an isolated quantum system, energies are the frequencies that determine how fast the system changes: double all energies and the evolution goes twice as fast. Bandwidth is then the width of the range of energy frequencies in the superposition, $\nu_{\text{max}} - \nu_{\text{min}} = E_{\text{max}}/h - E_{\text{min}}/h$, and (1) limits the number $N$ of distinct states that can occur in an evolution with period $T$. The analysis is similar to Nyquist’s, since at least $N$ distinct energy eigenstates are needed to construct $N$ distinct states in time. Again finite bandwidth evolution is effectively discrete in time—and similarly for momentum and discreteness in space. Since energy bandwidth cannot be finite unless all momentum bandwidths are finite, evolution of a state with finite energy bandwidth is effectively discrete in both space and time—even though symmetries remain continuous [29, 30]. For example, the continuous state can be translated and rotated.
by any amount, and yet is always completely determined by its values at a discrete set of locations.

More generally, doubling any sensible measure of frequency width allows us to double all frequencies, just like bandwidth. We find that any well-behaved width provides a bound on distinct states in periodic evolution, equivalent to a bandwidth bound and giving the same effective discreteness when achieved. If the evolution has $N$ distinct states in period $T$, minimum bandwidth is achieved by uniformly weighting $N$ minimally-separated energies (Figure 2)—and so are all other minimum widths! Thus we can write down an achievable bound for any width simply by evaluating it at minimum bandwidth. For example, the width $E - E_0$ is minimized by a uniform rectangular distribution starting at $E_0$. With average energy $E$ in the middle, the rectangle is $2(E - E_0)$ wide. Dividing by $h$ gives the frequency bandwidth, which satisfies \[ \frac{2(E - E_0)}{h} \geq \frac{N - 1}{T}. \] (2)

The $E - E_0$ bound is particularly useful, since it identifies average energy as a conserved resource governing distinguishability \[14\] and hence measurability (cf. \[31, 42\]). It also shows a system’s ground state energy cannot cause distinct state change within the system (cf. \[43–45\]).

As with \[1\], the new periodic bounds all have limits for large $N$ that apply to a long portion of any evolution. For normal widths, the periodic bounds also govern any finite portion of an evolution. This is because periodic evolution is the optimal completion of a constrained portion (Figure 3). Given only that a portion of an evolution has average separation $\tau$ between $\Delta N$ distinct states, the complete evolution that allows the smallest energy widths is periodic with $\Delta N$ equally separated distinct states and period $\Delta N \tau$. For example, with $T \to \Delta N \tau$, \[2\] becomes

\[ \frac{2(E - E_0)}{h} \tau \geq \frac{\Delta N - 1}{\Delta N}. \] (3)

For any natural definition of energy width $h \Delta \nu$, we get a similar bound $\Delta \nu \tau \geq f$, where $f$ increases with $\Delta N$ but is always of order unity. The $\Delta N = 2$ cases have a single interval $\tau$ and the smallest $f$. These special cases include known uncertainty bounds that treat time as a classical parameter: quantum speed limits \[12, 19\].

Of course there is also a bandwidth bound like \[4\] on the number of distinct values that can be put in a classical wave periodic in space. In the quantum case, we count mutually orthogonal states traversed while shifting a quantum wavefunction a given distance in space, without any other evolution. If $\Delta \mu$ is an average width of the spatial frequency \( (\text{momentum}/h) \) distribution along the direction of shift and $\lambda$ the average distance between distinct states, $\Delta \mu \lambda \geq f$ (formally identical to the time bounds). The usual counting of distinct states in classical phase space \[46, 47\] is the maximum distinctness allowed by the momentum bandwidth bound.

There is no general spatial-shift analog of \[3\], since there is generally no smallest momentum $p_0$ analogous to $E_0$. Nevertheless, special relativity demands such a bound, but only on distinctness due to frame motion: extra distinct states seen in a moving inertial frame but not in the rest frame (Figure 4). Frame motion is a uniformly shifting classical coordinate system, and overall momentum is due entirely to this motion and moves with it, so $p_0 = 0$. Relativistic energy $E_0$ of the physical vacuum in flat spacetime is also zero in any inertial frame. Thus for large $\Delta N$, in units with $h = 2$, average energy bound \[3\] simplifies and the momentum bound has the same form:

\[ E_{\text{rel}} \geq \frac{1}{\tau}, \quad p \geq \frac{1}{\lambda_{\text{motion}}}. \] (4)

$E_{\text{rel}}$ is the maximum number of distinct events per unit time, $p$ the maximum due to motion of a unit distance. The bounds are achieved in ordinary macroscopic evolutions, making basic classical mechanical quantities counts of distinct states, and relativistic invariance a relationship between counts with and without overall motion. Maximum distinctness also makes classical evolution effectively discrete, and finite-state models realistic \[49\].

This paper is organized as follows: We first establish general energy-width bounds on number of distinct states that can occur in a given time. The arguments used are elementary and all results are verified numerically. The same analysis is then applied to other single parameter unitary transformations to establish bounds on shifts in space, rotations, relativistic events, and resolution of measurements. Finally, we discuss the classical character of finite distinctness and its relevance to classical mechanics, relativistic information, and the discreteness of spacetime. Some preliminary results appeared in \[49\].
COUNTING DISTINCT STATES IN TIME

A set of quantum states are perfectly distinct if each is orthogonal to all the others: the states can be distinguished from each other with certainty. Here we count the number of distinct states that can occur in a given length of time. This is a bound on certainty.

Only frequencies matter

Consider a finite-sized isolated system in flat spacetime. We can express the time evolution as a superposition of discrete energy (frequency) eigenstates:

$$|\psi(t)\rangle = \sum_n a_n e^{-2\pi i \nu_n t} |E_n\rangle ,$$

with $$\nu_n = E_n/h.$$ For a normalized state the $$|a_n|^2$$ add up to one, and play the role of probabilities for each $$\nu_n.$$ This allows us to define a width $$\Delta \nu$$ for the distribution of frequencies—for example, the standard deviation.

If the time evolution (5) passes through a sequence of mutually orthogonal states $$|\psi(t_k)\rangle$$ at times $$t_k,$$ then

$$\langle \psi(t_m)|\psi(t_k)\rangle = \sum_n |a_n|^2 e^{2\pi i \nu_n (t_m - t_k)} = \delta_{mk} .$$

Thus the frequencies $$\nu_n$$ are the only characteristic of the dynamical law that constrains orthogonal evolution.

Given a definition of frequency width $$\Delta \nu,$$ we seek the minimum value of $$\Delta \nu$$ possible in an evolution (5) with $$\Delta N$$ mutually orthogonal states in time $$\Delta t.$$ This also determines the maximum $$\Delta N$$ for given $$\Delta \nu$$ and $$\Delta t.$$

Defining frequency width

We define a well-behaved average frequency-width $$\Delta \nu$$ to be a non-negative function of a discrete set of frequencies and their assigned probabilities, with Properties:

1. **Scales with Frequency.** It is a homogeneous function of the frequencies, of degree one.
2. **Measures the Spread.** It is a function only of frequency differences, not absolute frequencies.
3. **Weights Frequencies.** It does not change if probability is redistributed among equal frequencies.
4. **Centered.** It does not decrease if probability mass is moved farther from some central frequency $$\alpha.$$

We call the width natural if it also has the Property:

5. **Natural.** It is of the same order of magnitude as bandwidth for a uniform probability distribution.

For example, if $$\alpha$$ is the average frequency or the smallest with non-zero weight, the generalized average deviation

$$\langle \nu - \alpha \rangle_M \equiv \left( \sum_n |a_n|^2 |\nu_n - \alpha|^M \right)^{1/M} ,$$

the $$M^{th}$$ root of the $$M^{th}$$ moment of absolute deviation, has Properties 1–4 for $$M > 0.$$ Twice average deviation from $$\alpha$$ is a natural width with Property 5 for $$M \geq 2/3.$$

The most general frequency spectrum

There is no loss of generality in assuming the evolution is periodic with a period $$T$$ that may be much longer than the interval $$\Delta t$$ of interest; the evolution of a finite-sized isolated system comes arbitrarily close to repeating [50]. There is also no loss in assuming the $$\nu_n$$ are distinct; moving probability between states with identical frequencies does not affect orthogonality [6] or widths (Property 3), so repeated frequencies can be consolidated.

Like in classical waves, evolution that is periodic (up to an overall phase, which does not affect distinctness) constrains the frequency spectrum. The spectrum is bounded from below [51], including at most all frequencies that cycle with period $$T,$$ up to an overall constant:

$$\nu_n = \nu_0 + n/T ,$$

with $$n$$ a non-negative integer. Most spectra will include just a subset of these frequencies, but by allowing them all we calculate minimum widths independent of the spectrum of any particular physical system: the least possible constraint gives the smallest possible minimum.

Spectrum (8) restricts the maximum period to be $$T,$$ but the first-recurrence time $$T_1$$ of the evolution may be much smaller. For $$T$$ sufficiently large (8) approaches a continuous spectrum, allowing us to minimize over the union of all possible spectra and first-recurrence times.

Bounds on entire periodic evolutions

Following Nyquist, we begin by analyzing entire periodic evolutions that traverse $$N$$ distinct states. This also provides tight bounds on any portion of evolution containing a large number of distinct states. We will later show that periodic bounds for any $$N,$$ expressed in terms of average separation $$\tau$$ between distinct states, normally apply also to a portion of evolution.

Here we show that the minimum of $$\Delta \nu \tau$$ is of order unity for any natural width $$\Delta \nu.$$ In all cases, the minimizing frequency distribution has $$N$$ consecutive equally-weighted $$\nu_n.$$ As examples, we evaluate bounds for all generalized deviations (7), and bounds for a frequency range $$\Delta \nu_q$$ holding a fraction $$q$$ of the probability.

**Proposition 1:** Given an isolated periodic quantum evolution that traverses $$N$$ distinct states in period $$T,$$ there is a minimum possible width for the range of energy eigenfrequencies $$\nu_n = E_n/h$$ of the state represented in the energy basis: a minimum bandwidth of $$(N - 1)/T.$$

*Proof:* At least $$N$$ distinct $$|E_n\rangle$$ must appear in the superposition to add up to $$N$$ distinct states $$|\psi(t_k)\rangle$$ at different times $$t_k.$$ Since the possible frequencies $$\nu_n$$ for the periodic evolution differ by at least $$1/T,$$ the minimum possible bandwidth with $$N$$ distinct $$\nu_n$$ is $$(N - 1)/T.$$ □

**Proposition 2:** Minimum bandwidth is only possible if the $$N$$ distinct states are evenly spread out in time.

*Proof:* Each different separation time $$0 \leq t_m - t_k < T$$ in (6) provides a linearly independent constraint. Evenly
spaced states give \(N\) constraints, allowing a solution with \(N\) consecutive non-zero \(|a_n|^2\) (minimum bandwidth). Uneven spacing gives more constraints. \(\square\)

**Proposition 3:** A minimum-bandwidth superposition contains \(N\) consecutive frequencies \(\nu_n\), equally weighted. 

*Proof:* With equal separations \(\tau = T/N\) between consecutive distinct states, \(t_m = m\tau\). From [8] and [6],

\[
\langle \psi(t_{k+m}) | \psi(t_k) \rangle = e^{2\pi i \nu t_m} \sum_{n=0}^{\infty} |a_n|^2 e^{2\pi i n m / N} .
\]  

(9)

There are only \(N\) different phase factors in the sum and so we can pick any \(N\) consecutive values of \(n\), and provide one non-zero coefficient for each distinct phase. Then, since \(\langle \psi(t_{k+m}) | \psi(t_k) \rangle = \delta_{m0} = e^{2\pi i \nu t m \delta_{m0}}\), the non-zero \(|a_n|^2\) are just the discrete Fourier transform of a Kronecker delta impulse, and so they all equal 1/\(N\). \(\square\)

**Proposition 4:** With equal spacing between \(N\) distinct states, any width \(\Delta \nu\) centered on \(\alpha\) is minimized by a minimum-bandwidth superposition centered on \(\alpha\).

*Proof:* As shown above, with equal spacing there are only \(N\) distinct phases and [6] can only be satisfied if they are equally weighted. This can be done by putting all weight on \(N\) consecutive \(\nu_n\) centered on \(\alpha\). Then, from Property 4 of \(\Delta \nu\), no rearrangement of weights that keeps the same net weight on each distinct phase can decrease \(\Delta \nu\). \(\square\)

**Proposition 5:** With equal spacing \(\tau\) between \(N\) distinct states, and \(\Delta \nu\) centered on \(\alpha\),

\[
\Delta \nu \tau \geq f ,
\]  

(10)

where \(f\) is a dimensionless constant that depends only on \(N\) and the definition of \(\Delta \nu\). The constant is determined by evaluating the dimensionless product \(\Delta \nu \tau\) for a minimum bandwidth superposition centered on \(\alpha\).

*Proof:* From [8] and Properties 1 and 2, \(\Delta \nu \propto 1/N\tau\), so \(\Delta \nu \tau\) depends only on \(N\) and the probabilities. The minimum \(\tau\) then follows from Proposition 4. \(\square\)

In fact, (10) also holds with \(\tau = T/N\) the average separation between consecutive distinct states, since uneven spacing requires a larger width for any \(\Delta \nu\)–not just for bandwidth. The larger the largest separation is, the faster the rest of the evolution traverses \(N\) distinct states, and the larger \(\Delta \nu\) must be (see discussion of portions). Figure 5 illustrates how extra constraints due to unequal spacing give larger minima (numerical methods are discussed later). The smallest minima (dashed line) are achieved as separations approach equality. Even a tiny departure from equality increases minima, since it gives uniform-spacing constraints plus additional ones.

**Proposition 6:** For any natural measure \(\Delta \nu\) of average frequency-width, \(f\) is of order one.

*Proof:* Since \(\Delta \nu\) is minimized by a uniform minimum-bandwidth distribution, from Property 5 its minimum has the same order of magnitude as \((N-1)/T\). Thus \(f\) has the same order of magnitude as \((N-1)/N\). \(\square\)

![Figure 5](image-url)  

**Examples of minimum-bandwidth bounds**

With width \(\Delta \nu = 2(\nu - \alpha)_M\), twice the average deviation (7) of \(\nu\) from \(\alpha\) for moment-\(M\), (10) becomes

\[
2(\nu - \alpha)_M \tau \geq f_\nu(M, N) \tag{11}
\]

for some \(f_\nu(M, N)\) defined by a minimum-bandwidth frequency distribution. For example, a width above a lowest frequency \(\alpha\) is centered (Property 4). For \(\alpha = \nu_0\), \(f\) is determined by equally weighting \(N\) consecutive frequencies starting with \(\nu_0\). Equality in (11) then gives

\[
f_\nu(\nu_0, N) = 2N^{-(1+\frac{1}{M})} \left( \sum_{n=0}^{N-1} n^M \right)^{\frac{1}{M}} , \tag{12}
\]

which ranges from 1/2 to 2 for \(M = 1, f_\nu(\nu, M, 2) = 2^{-\frac{1}{M}}\). Of course we get exactly the same bounds if some other \(\alpha = \nu_0\) is the lowest that has weight in the wavefunction.

Another interesting case is a width about a mean frequency: \(\alpha = \bar{\nu}\). The mean of a uniform minimum bandwidth distribution is its center, so equality in (11) then gives

\[
f_\nu(\bar{\nu}, M, N) = 2N^{-(1+\frac{1}{M})} \left( \sum_{n=0}^{N-1} n^M \right)^{\frac{1}{M}} \tag{13}
\]

ranging from 4/9 to 1 for \(M \geq 1, f_\nu(\bar{\nu}, M, 2) = 1/2\) for all \(M\). Because of the symmetry of sums about the mean, the smallest \(f_\nu(M, N)\) for any \(\alpha = f_\nu(M, N)\), if \(M \geq 1\). For \(N \to \infty, f_\nu\) is smallest for any \(M > 0\) and

\[
f_\nu(\bar{\nu}, \infty) = \frac{1}{2} f_\nu(\bar{\nu}, M, N) = \left( \frac{1}{1+M} \right)^{\frac{1}{M}} , \tag{14}
\]

which ranges from 1/e to 1 for \(M > 0\). For \(M \to \infty, f_\nu(M, N)\) is the bandwidth and

\[
f_\nu(\infty, N) = \frac{1}{2} f_\nu(\infty, N) = \frac{N-1}{N} , \tag{15}
\]

which ranges from 1/2 to 1. The \(N \to \infty\) limit (14) is not restricted to periodic systems; it applies to any long portion of an evolution. The finite-\(N\) bounds also normally
apply to a portion of an evolution, as we will discuss, allowing comparisons with known speed limit results.

We can bound other well-behaved widths similarly. For example, let $\Delta \nu_q$ be the width of a finite range of frequencies—starting at some frequency $\alpha$—that make up a fraction $q$ of the total probability of the wavefunction. This width has been studied using Fourier analysis in the context of uncertainty bounds [17–19]. From (10),

$$\Delta \nu_q \tau \geq f_{\text{prob}}(q, N)$$

for some $f_{\text{prob}}(q, N)$ defined by aligning the beginning of the minimum bandwidth distribution with the beginning of $\Delta \nu_q$. Since the minimum $\Delta \nu_q$ must encompass at least $\lceil qN \rceil$ of $N$ equally weighted frequencies $1/N\tau$ apart,

$$f_{\text{prob}}(q, N) = \frac{\lceil qN \rceil - 1}{N}.$$  

In Figure 6 this bound (blue lines) is tested numerically. For $q \leq 1/2$, $\Delta \nu_q$ is not a natural width, since then it can be zero for a discrete uniform distribution; for $q > 1/2$, $f$ ranges from 1/3 to 1. Asymptotically, $f_{\text{prob}}(q, \infty) = q$.

**Bounds on a portion of an evolution**

Given that an isolated system traverses $\Delta N$ distinct states in time $\Delta t$, there is a minimum value of any average width $\Delta \nu$. We find the minimum by making the least constraining assumptions about the full (recurrent) evolution $\Delta t$ is a part of: we place no constraint on the length of the evolution outside of $\Delta t$, and we do not require additional distinct states outside of $\Delta t$.

The periodic bounds [10] provide an example of what is possible. If $\tau$ is the average time between consecutive distinct states within $\Delta t$, we can imagine $\Delta t$ is part of an evolution with period $\Delta N \tau$. This allows the most symmetric, least constrained and smallest bandwidth evolution containing $\Delta t$. In fact, for every $\Delta \nu$ we have studied, the corresponding periodic bound is optimal—as long as it never decreases with $\Delta N$. Non-decreasing is necessary for the bound to be a minimum over all continuations of a portion, since adding a distinct state is a continuation.

![FIG. 6. Width of frequency range vs. probability within range. Each dot represents an evolution with $N \leq 10$ distinct states, with separations and width-of-range $\Delta \nu$ chosen randomly. Maximum probability $q$ in chosen $\Delta \nu$ is found numerically.](image)

![FIG. 7. Using two frequencies gives fastest orthogonality. The earliest $\langle \psi(t) | \psi(0) \rangle = 0$ is when fastest and slowest changing phases (black) cancel, and there are no other phases (gray). Equally weighting max and min frequencies then gives zero.](image)
frequencies at the minimum spacing for period $2\tau$.

**Proposition 9:** For $\Delta N = 2$, minimum bandwidth minimizes the width $\Delta \nu$ of deviation from $\bar{\nu}$ for $M \geq \pi/2$.

*Proof:* The narrowest distribution with $T \neq \Delta N \tau$ has a mean that is the middle of three consecutive $\nu_n$, allowing probability mass closer to the mean than with two $\nu_n$. To find $T$ that minimizes $\Delta \nu \tau$ we insert into deviation [7] from the mean the distribution $\{p, 1 - 2p, p\}$ given above; the derivative of $\Delta \nu \tau$ with respect to $T$ is zero if

$$\frac{2\pi T}{\tan \pi T/T} = M. \quad (18)$$

For $0 < M < \pi/2$ this has solutions with $2\tau < T < 4\tau$. In all cases, minimum $\Delta \nu \tau$ is smaller than for $T = 2\tau$. For $M = \pi/2$, maximum period $T = 4\tau$, actual period is $2\tau$, and we revert to the minimum bandwidth bound. \(\square\)

For $M = 1$, for example, [18] gives $T/\tau = 2.69535 \ldots$ and hence $2(\nu - \bar{\nu}), \tau \geq 0.439284 \ldots$, which is exactly what we find minimizing numerically (see Figure 5).

**Proposition 10:** For $\Delta N = 2$, minimum bandwidth minimizes the width $\Delta \nu$ of deviation from $\nu_0$ for $M > 0$.

*Proof:* If we insert the distribution $\{p, 1 - 2p, p\}$ into deviation [7] from $\nu_0$, $\Delta \nu \tau$ always attains its minimum at $T = 4\tau$, the maximum allowed. This gives the minimum bandwidth distribution $\{1/2, 0, 1/2\}$ and period $2\tau$. \(\square\)

For $\Delta N = 2$, the bounds above on deviations from $\nu_0$ are known for all $M$ [14,16], and on deviations from $\bar{\nu}$ for $M = 1$ and $M = 2$ [12,13] with extension to $M > 2$ obvious [55,56]. For $\Delta N > 2$, analysis similar to the above shows that using slightly more than minimum bandwidth does not allow smaller bounds as long as the min-bandwidth distribution is centered on one of the $\nu_n$. We also verify numerically no amount of extra bandwidth helps (Figure 9). As we saw for $\Delta N = 2$, the cases not so centered are widths about the mean with $\Delta N$ even, and again slightly wider bandwidth only helps if $M$ is small. We find numerically that [13] is an exact bound if $M \geq 2$, which is the threshold above which [13] increases with $\Delta N$, as it must to be a portion bound (Figure 10).

**Summary of energy distinctness bounds**

Given any well-behaved energy-width $\Delta E = \Delta \nu \tau$ of a wavefunction, we provide bounds on the average time $\tau$ between distinct states, hence on the number of distinct states in a given time. All bounds are tight, since they are achievable if the spectrum of the Hamiltonian includes appropriate eigenvalues. The situation is simplest for a periodic evolution that traverses $N$ distinct states. Then

$$\Delta \nu \tau \geq f(N), \quad (19)$$

with equality for a uniform superposition of energies with $N$ frequencies $1/N\tau$ apart. This minimum bandwidth superposition defines $f(N)$, which is of order one if $\Delta \nu$ is a natural width: same order of magnitude as bandwidth for a uniform superposition. As examples, we evaluate $f(N)$ for all generalized $M$th deviations from the mean or minimum (these widths are natural for $M \geq 2/3$), and for a range of energies with total probability weight $q$ (these widths are natural for $q > 1/2$). The limit $N \rightarrow \infty$ of $f(N)$ bounds any evolution with a long distinct portion.

In all cases studied, the periodic bounds also apply to any portion of an evolution whenever they possibly can. That is, given that an evolution includes a portion with
\[ \Delta \nu_q \tau \geq \frac{1}{\pi} \arccos \left( \frac{1}{q} - 1 \right) \text{ for } 1/2 \leq q \leq 1. \]
Bounds on shifts in space

The analysis of distinctness under shifts in space is a one-dimensional problem, identical in form to the problem already studied of translations in time. Consider an isolated scalar particle. Its state can be described as a sum of products of eigenstates of momentum for three orthogonal directions in space. If the particle is confined to a finite region by periodic boundary conditions, the momentum spectrum is discrete and the wavefunction is

\[ |\psi\rangle = \sum_{nlj} c_{nlj} |p_n| |p_l| |p_j\rangle. \]  

(22)

If \( |p_n\rangle \) is periodic in space along a direction with period \( L \), possible spatial frequencies are \( \mu = p_n / h = n / L \), with \( n \) an integer. Shifting the system a distance \( r \) along this direction, the wave function becomes

\[ |\psi(r)\rangle = \sum_{nlj} c_{nlj} e^{-(2\pi i / h) \mu r} |p_n| |p_l| |p_j\rangle. \]  

(23)

This follows from the form of the one-dimensional eigenstate \( |p_n\rangle \). Now if shifts of \( r_k \) and \( r_m \) give orthogonal states for \( m \neq k \), and \( |a_n|^2 = \sum_{lj} |c_{nlj}|^2 \), we get

\[ \langle \psi(r_m)|\psi(r_k)\rangle = \sum_n |a_n|^2 e^{2\pi i \mu_n (r_m - r_k)} = \delta_{mk}. \]  

(24)

This is identical in form to [6] and so yields the same one-dimensional minimization problem as before—with \( r \) and \( \mu \) playing the roles of \( t \) and \( \nu \). In general, the total momentum operator \( \vec{p} \) for any isolated quantum system is defined to be the generator of spatial shifts [58]: the unitary operator \( e^{-(2\pi i / h) \vec{p} \cdot \vec{r}} \) shifts the wavefunction a fixed distance \( \vec{r} \) in space with no other changes, as in (23).

The Hamiltonian operator \( H \) is similarly the generator of shifts in time: the unitary operator \( e^{-(2\pi i / h) H \cdot t} \) evolves the wavefunction by an amount \( t \) in time. We can simply substitute one Hermitian generator for another, and conclude the momentum and energy bounds are formally the same: if \( \vec{r} = r \hat{x} \) then \( H \rightarrow p_x \) and \( t \rightarrow r \).

We can gain some insight into the relationship between space and time bounds by formally adding time to the shift evolution. Imagine turning off the actual dynamics and replacing it with \( H = p_x v \), with \( v \) a constant speed. This dynamics shifts the state a distance \( \lambda = vt \) in time \( \tau \), since \( H \tau = p_x \lambda \). Periodicity in time becomes periodicity in space; distinct states in space become distinct states in space. If \( \tau \) is the average time between distinct shifts, then \( \lambda = v \tau \) is the average distance between them. Thus substituting \( p_x \lambda \) for \( H \tau \) in (20) gives bounds on \( \Delta p_x \lambda \) of

\[ 2 \langle |p_x - p_x \lambda|^2 \rangle^{1/2} \geq h f_\alpha (M, \Delta N) . \]  

(25)

The bounds \( 25 \) have the same dimensionless \( f_\alpha \) as the time bounds with \( \alpha \) defined relative to the momentum distribution. They also have exactly the same applicability to a portion of shift evolution, with the same additional bounds about \( \hat{\mu} \) given by [18]. If there is no lowest frequency for the spatial superposition, bounds about the mean \( p_x \hat{\mu} = \langle p_x \rangle \) apply but not ones about a minimum. Other energy bounds, such as [19], similarly apply.

Bounds \( 25 \) include the \( \Delta N \gg 1 \) bandwidth bound that counts distinct states in phase space [16, 21]. Yu’s \( \Delta N = 2 \) bound [21], \( \langle (p_x - \langle p_x \rangle)^2 \rangle^{1/2} \lambda \geq \hbar / 4 \), and Luo’s \( \Delta N = 2 \) bound [13], \( \langle |p_x| \rangle \lambda \geq \hbar / 4a \) with \( a = 1.1382 \ldots \), which is an exceptional bound given by [18], achievable only if the mean momentum is zero.

Bounds on rotations

For periodic unitary evolution, there is no acceptable Hermitian operator for the length of a portion of a finite period. This makes uncertainty bounds based on non-commuting operators problematic for evolution length [59]. The distinctness bounds discussed here avoid this issue by treating evolution length as a classical parameter, hence there is no impedance to applying these bounds to the separation between distinct rotations in space (or between distinct phases of an oscillation).

As we did with shifts, we can formally make spatial rotation a special case of time evolution by making evolution length depend on time. Now, the generator of rotations about an origin \( O \) in space is the operator \( \hat{J} \) representing total angular momentum about \( O \): \( e^{-(2\pi i / h) \hat{J} \cdot \vec{\phi}} \) rotates the wavefunction by an angle \( \phi \) about an axis \( \vec{\phi} \), where \( \omega = \theta / \tau \) is the ratio of average separation in angle and time between distinct states. To express this as an evolution \( e^{-(2\pi i / h) H \tau} \) we let \( H = H \; \vec{\phi} \), giving \( H = \vec{J} \cdot \vec{\phi} \). Letting \( J_z = \vec{J} \cdot \vec{\phi} \) (the component of \( \vec{J} \) along the rotation axis) this becomes \( H \tau = J_z \theta \). Making this substitution we can, for example, rewrite (20) in terms of \( \Delta J_z \theta \) as

\[ 2 \langle |J_z - J_z \theta|^2 \rangle^{1/2} \geq h f_\alpha (M, \Delta N) . \]  

(26)

The maximum angular period for mutually orthogonal rotations of any state is \( 2\pi \)—an overall phase factor does not affect orthogonality. This is apparent in the spectrum of \( J_z \), which has the same spacing \( h / 2 \pi \) between eigenvalues for both fermionic and bosonic systems. Distinctness depends on eigenvalue differences. For \( \Delta N \) distinct states in a full rotation, (26) requires \( J_z \max - J_z \min \geq (\Delta N - 1) h / 2\pi \); the superposition must contain at least \( \Delta N \) different eigenvalues and eigenstates. For \( \Delta N = 2 \) this agrees with a limit of [18]. For an entire periodic evolution, minimum bandwidth bounds are achievable for any \( \Delta J_z \) (e.g., [16]) is better than [21]). For a system in an eigenstate of total angular momentum \( J^2 \), bounds about a minimum or maximum \( J_z \) always apply.

Maximally distinct evolution

Distinctness bounds for periodic evolution are achieved by uniform superpositions with finite bandwidth. For
For finite period $T$, a superposition with $N$ frequencies $m/N$ centered on $b$ replaces $\sin c_b u$ in (28) and (29):

$$\sin c_{b,N} u = \sum_{m = bN - N/2}^{bN + N/2} e^{2\pi i u m/N}, \quad (30)$$

and $\tau = T/N$. The limit $\sin c_{b,\infty}$ is $\sin c_b$. To have periodicity in both magnitude and phase, all frequencies must have a whole number of oscillations per period: for example, $\nu_0 = 0$ in (8). Thus $m$ must be an integer in (30), making this a periodic delta function for integer $u$: $\sin c_{b,N} u = 1$ if $u \equiv 0 \mod N$, and 0 otherwise. Then, as before, we see from (28) that the $N$ discrete values at $t = k\tau$ suffice to determine the $N$ Fourier coefficients and hence $s(t)$, making it effectively discrete. Similarly from (29), $N$ states $\tau$ apart determine $|\Psi(t)|$; if these states are distinct, evolution is maximally distinct.

With a half-integer number $m$ oscillations per “period” (as in some rotations), the $|\Psi(n\tau)|$ for $0 \leq n < N - 1$ still determine all Fourier coefficients in (29), but we gain a factor of $-1$ for each successive interval $N\tau$. For example, $b = 0, N = 2$ gives $|\Psi(t)| = \cos \frac{\pi t}{2}\Psi(0) + \sin \frac{\pi t}{2}\Psi(\tau)$.

**Bounds on measurement**

Finite distinctness limits resolution in measurements. For example, suppose we want to measure the time between two states of a unitary evolution with average energy $E_0 + \Delta E$. For maximum resolution the evolution should have as many distinct states as possible to give the minimum spacing $\tau \approx h/2\Delta E$ between distinguishable moments. Distinct states then form a discrete basis for the time evolution, as in (29), so the time between any two states is only defined up to a resolution $\tau$. In general, a minimum spacing between distinct “shifts” of a state limits the resolution of a measured shift (cf. 18).

Since light is often used to probe systems of interest, the maximum density of distinct states in a periodic optical evolution is a key constraint on measurements. This is given by the energy bounds. Historically, resolution limits stem from a photon number bound on phase resolution in interferometry using monochromatic light [31–42]. With light of frequency $\nu$, average energy bound $\tau\Delta E \geq h/4$ implies an average photon number $\bar{n}$ bound, since $\Delta E = \bar{n}\nu$ and the period of time evolution $T \leq 1/\nu$. Thus $\bar{n} \theta = \bar{n}(2\pi|\tau|/T) \geq 2\pi\tau\bar{n}\nu \geq \pi/2$. In general, though, only energy bounds distinctness in time.

For an ideal interferometer, maximally distinct evolution allows the best resolution in time differences [38]. The unitary transformation performed by the interferometer may, however, allow less distinctness than the energy bound (e.g., for $\theta$ it is half). With monochromatic light and beam splitters, overall transformations are isomorphic to 3D rotations [32–34] and so all bounds [26] apply. In practice, highest resolution is currently achieved using squeezed states that are not maximally distinct [31–42]. Their resolution scales sub-linearly with energy, but they can be constructed with macroscopic amounts of energy.
DISTINCT EVENTS IN SPACETIME

We have studied distinct change due to shifts of a classical space or time coordinate. This symmetric treatment of space and time extends naturally to spacetime, with an inertial reference frame modeled as a uniformly shifting classical coordinate system. Evolution in a single coordinate direction has a positive spectrum, providing bounds about a minimum energy or momentum, and redefining four-momentum as a resource for distinct events.

Distinctness defines relativistic energy

We consider only bounds in flat spacetime (cf. [68 69]), where equivalence of relativistic quantum field theory to unitary evolution in Hilbert space—the case we have analyzed—is well accepted [70]. Equivalence is easily demonstrated for field theories restricted to a finite range of energies [68]; particles with arbitrarily small energies are undetectable, and our standard field theories are not effective descriptions at arbitrarily large energies. This equivalence is also supported by accurate predictions made using finite-lattice models of fields [71–73].

In flat spacetime, the relativistic ground state (the physical vacuum) must have energy zero; this is required by frame invariance [51 74 76]. Within any system, only energy above its ground state can cause distinct change, so relativistic energy \(E\) in any frame is the total energy that can cause distinct change. For evolution that traverses many distinct states, with \(E_0 = 0\), (3) becomes \(1/\tau \leq 2E/h\). In natural units with \(h = 2\), this is simply

\[
1/\tau \leq E .
\]

\[\text{(31)}\]

Total relativistic energy \(E\) is the maximum average rate of distinct state change in a frame, so \(E \Delta t\) is the maximum number of distinct events possible in time \(\Delta t\).

Kinetic energy vs. energy of distinct motion

We are free to divide total conserved relativistic energy \(E\) into different forms of energy, and correspondingly partition the total maximum rate of distinct state change into a sum of different kinds of change. For example, for a system with a rest frame energy \(E_r\), the energy \(E - E_r\) is the maximum rate of distinct change due to not being at rest. This is the conventional kinetic energy.

Kinetic energy does not, however, provide a natural relativistic division of distinct change into that seen in the rest frame, plus additional change seen only in a moving frame (Figure 3). The quantities \(E\) and \(E_r\) are maximum rates in two different frames, and so their difference is not a rate in either frame. The natural relativistic measure of extra distinctness possible in a moving frame is

\[
E - E_r/\gamma = v p ,
\]

\[\text{(32)}\]

the maximum total rate seen in the moving frame, minus events that are also seen in the rest frame. This equals \(v p\), which we identify as the average energy of \(H_{\text{shift}} = \vec{v} \cdot \vec{p}\).

Shift energy \(v p\) is a portion of the total energy \(E\) that allows additional distinct state change in a moving frame, shifting at speed \(v\), that is not visible in the rest frame.

Distinctness defines relativistic momentum

The appearance of shift energy \(v p\) in (32) reflects the fundamental division of relativistic dynamics into rest frame evolution and overall motion. This division is possible because overall momentum is conserved separately from energy: we can define a zero-average-momentum rest frame for any isolated system moving at speed \(v < c\). We will assume we can also define a wavefunction that describes just the overall motion seen in the lab frame, without including any rest frame dynamics. After all, our choice of frame from which to view the system should not affect the rest frame. Then, since overall momentum is due entirely to overall motion, it is associated entirely with the overall motion wavefunction. To ensure this wavefunction has no dynamics in its rest-frame (no change), it must contain only momenta with a single direction and phase-velocity. This makes a separate description of overall motion a one dimensional shift with a positive momentum spectrum—true also for motion with \(v = c\). Thus bounds (25) with a minimum \(p_0 = 0\) apply.

For a state well localized in space the number of effectively distinct shifts is large, so again using units with \(h = 2\), (25) gives an average momentum bound like (31),

\[
1/\lambda_{\text{motion}} \leq p .
\]

\[\text{(33)}\]

Momentum \(p\) is the maximum average spatial rate of distinct change due to motion, \(p \Delta x\) the maximum number of states distinct due to a motion of length \(\Delta x\), and \(v p\) the maximum per unit time—the energy of distinct motion.

Time-energy interval counts distinct events

The bounds (31) and (33), which derive from separate descriptions of shifts in time and space, together imply a relationship—for a long evolution of a well-localized system—between maximum counts of distinct states in two different frames:

\[
E \Delta t - p \Delta x = E_r \Delta t_r .
\]

\[\text{(34)}\]

This is the classical time-energy interval for a system moving between two events separated by time \(\Delta t\) and distance \(\Delta x = v \Delta t\) in a moving frame, with separation \(\Delta t_r\) in the rest frame. If the system is changing at the maximum rate then \(E \Delta t\) is the total number of distinct events seen in the moving frame, of which \(p \Delta x\) are distinct due to frame motion. The difference \(E_r \Delta t_r\) is the number distinct without frame motion. This partitioning of action into different resources for distinct events is
equivalent to partitioning a conserved energy into different resources for change: \( \text{\textbf{\textit{34}}} \) divided by \( \Delta t \) is \( \text{\textbf{\textit{32}}} \).

We argue below that we can treat ordinary macroscopic systems as changing at the maximum rate, so \( \text{\textbf{\textit{34}}} \) becomes a relationship between actual counts of distinct events underlying classical mechanics. Relativistic action \( L dt = (H - \sum p_i v_i) dt \) for a system of interacting particles similarly becomes a count of events not due to particle motion. Dynamics over short time intervals follows a path where action (defined with this sign) is greatest \( \text{\textbf{\textit{74}}} \). Here this is the path of most non-motion events (cf. \( \text{\textbf{\textit{78}}} \)) and, since energy is conserved, least distinct-motion.

FINITE DISTINCTNESS IS CLASSICAL

Finite distinctness is a classical property of continuous wave evolution with a finite average frequency-width. This challenges some common assumptions about both classical and quantum dynamics.

Classical systems are maximally distinct. We can regard classical mechanics as the infinitely distinct \( h \rightarrow 0 \) limit of a unitary quantum evolution in which all amplitude lies on a single path with stationary classical action \( \text{\textbf{\textit{79}}} \text{\textbf{\textit{84}}} \). With finite energy the limit approximates a unitary evolution with finite distinctness (cf. \( \text{\textbf{\textit{81}}} \text{\textbf{\textit{85}}} \)). To infer a possible finite-energy evolution from just the limit we assume maximum distinctness. This is a generic property of large evolutions \( \text{\textbf{\textit{83}}} \text{\textbf{\textit{86}}} \) (cf. \( \text{\textbf{\textit{87}}} \)).

Maximum distinctness defines how distinct states are counted in classical phase space \( \text{\textbf{\textit{46}}} \text{\textbf{\textit{47}}} \), since all width bounds are equivalent to bandwidths for large \( N \). Consider the number of distinct shifts allowed by the momentum bandwidth bound \( \text{\textbf{\textit{25}}} \) for a gas of conserved distinguishable particles: for each spatial degree of freedom, maximum distinctness allows one distinct state per area \( h \) of accessible momentum-range \( \times \) spatial-length. For identical particles, we divide by a permutation count.

Incompressible flow in phase space is the signature that if momentum bounds are achieved, energy bounds are also achieved. Consider, for example, the classical phase space of a simple harmonic oscillator, shown in Figure \( \text{\textbf{\textit{13}}} \). The evolution of a state (point) is a circle of constant energy. A distinct area \( h \) crosses a line joining energies \( E_1 \) and \( E_2 \) in time \( h/(E_2 - E_1) \): all points move from one distinct area to another in the minimum time allowed by \( \text{\textbf{\textit{16}}} \) for a distinct change in a long evolution with that range of energies. Any single degree of freedom system similarly achieves this bound \( \text{\textbf{\textit{88}}} \): Hamilton’s equations require an infinitesimal flow in the \( q \) direction crossing an energy gradient \( dE \) in time \( dt \) to obey \( dq \, dp = dE \, dt \), and integrating this for a time \( \Delta t \) along a line joining \( E_1 \) and \( E_2 \) gives \( h = (E_2 - E_1) \Delta t \). This extends to any number of degrees of freedom: in the simplest case flow is locally uniform, so all change as a distinct volume moves distinctly can be attributed to a single degree of freedom.

With a maximally-distinct unitary evolution underlying it, classical mechanics becomes effectively discrete. Different forms of classical energy and related mechanical quantities become counts of distinct states (finite energy-bandwidth constrains related bandwidths). Lagrangian action counts events not modeled as distinct motion.

Some classical models are fundamental. It is well known that classical lattice gases, such as the Ising model, can also be regarded as quantum models \( \text{\textbf{\textit{89}}} \). These are simple finite-state models of thermal systems that capture the finite distinctness that defines entropy. They also exhibit realistic phase change behavior, with the same critical exponents as real physical systems \( \text{\textbf{\textit{90}}} \). In view of the effective discreteness and finite distinctness of classical systems, simple finite-state models of classical mechanics acquire a similar status \( \text{\textbf{\textit{91}}} \). Maximum distinctness defines their energies and momenta \( \text{\textbf{\textit{18}}} \).

Consider, for example, the finite-state evolution depicted in Figure \( \text{\textbf{\textit{14}}} \). This classical-mechanical lattice gas discretely samples a continuous classical evolution \( \text{\textbf{\textit{22}}} \). Here two streams of elastically colliding balls are shown at one moment of time. The continuous dynamics is contrived so that, started from a suitably constrained initial state, the evolution is equivalent to a finite-state lattice computation at integer times. We can infer the energy of the lattice gas from the momentum and speed of the particles when they are moving freely. Assuming bound \( \text{\textbf{\textit{33}}} \) is always achieved classically, the momentum required to
have distinct particle positions \( \lambda \) apart is \( p = h/2\lambda \). If the distinct positions are \( \tau \) apart in time, \( v = \lambda/\tau \). Then relativistic energy is \( c^4 p/v \), equal to \( h/2\tau \) if \( v = c \): all change is motion for a massless particle [93]. Momentum is defined by the lattice spacing, so energy is defined by \( \tau \).

Other lattice gases can similarly be constructed by devising a continuous classical dynamics plus constraints on the initial state that produce a finite-state evolution at discrete times. The continuous dynamics can be somewhat stylized; it generally allows classical particles to sometimes pass through each other without interacting. Such models enable, for example, discrete molecular-dynamics simulations of hydrodynamics and complex fluids [94–98]. The microscopic evolution can have exact conservations of a continuously symmetric dynamics, as in Figure 14 but lattice constraints reduce the symmetry of the state. If the lattice has sufficient discrete symmetry, continuous symmetry is recovered in macroscopic evolutions. Treating free motion in such models as maximally distinct defines intrinsic momenta and energies.

**The most-classical states are not Gaussian.** A Gaussian wave packet has the minimum product of standard deviations \( \Delta x \Delta p \) allowed by the Heisenberg-Kennard bound [10]. For this reason it is often taken as the most classical state describing both location and momentum of a particle. Distinctness bounds suggest otherwise. Using any natural measure of momentum width, a wavepacket of width \( \Delta p \) has a minimum separation of at least about \( h/\Delta p \) between distinct shifts in space. This is the most distinctness possible, and minimum separation is achieved by a uniformly weighted superposition of momenta with the given \( \Delta p \)—not by a Gaussian.

This is illustrated in Figure 15 for a maximally distinct wavepacket (solid) in an unbounded space (\( \Delta N \to \infty \)). The separation \( \lambda \) between distinct shifts determines minimum bandwidth and hence the shape. We compare this with a Gaussian distribution of the same height (dashed). Using standard deviation to measure width in space, as the Heisenberg-Kennard bound does, the wavepacket with maximum spatial distinctness is assigned infinite width. Gaussian wavepackets provide only a fuzzy bound on distinctness, given a range of allowed momenta and positions in classical phase space; minimum bandwidth states achieve maximum distinctness (cf. 99). They are also maximally distinct in time, for free motion described relativistically as a uniformly shifting frame.

**Ground state energy is not uniquely quantum.** We define an isolated system by identifying a set of degrees of freedom that evolve independently of everything else—to some level of approximation. If the isolated dynamics (classical or quantum) is described as wavefunction evolution, the structure of the dynamics generally imposes constraints on its degrees of freedom, allowing less distinct change than energy (average frequency) would seem to permit. For example, in the infinite square well of Figure 16 part of the energy of the wavefunction is used to define the position of the well in space, rather than the position of the particle within the well. We can estimate the energy \( E_0 \) required to localize the well using distinctness bounds. Clearly repeated shifts of the wavefunction by the width \( \lambda \) of the well are distinct, and so for a wavefunction with mean momentum zero, [14] and [25] require that the average kinetic energy of a massless particle \( \langle |p| \rangle \geq h/4\lambda \), and for a slow massive particle \( \langle p^2 \rangle/2m \geq h^2/24m\lambda^2 \). Similarly, from (2) with \( N = 2 \), the minimum average energy above the physical vacuum for an isolated dynamics to have a distinct period \( T \) is that of a simple harmonic oscillator, \( h/2T \).

Energy required to define the structure of a system is not available to cause distinct change within it. This property may be relevant to the question of whether vacuum energy gravitates in general relativity [15–16].

**Energy bounds maximum classical information.** Momentum counts distinct states possible in space with a given set of degrees of freedom. This allows us to assign an entropy to an isolated system. Relativistic energy, on the other hand, counts all distinct states nature allows per unit time, without requiring any knowledge of the ultimate degrees of freedom. This allows us to place fundamental bounds on information in time and space.

For example (taking \( h = 2 \)), the maximum rate of distinct change \( E \) is also the maximum rate of classical information communication [100–102]: bit changes are distinct changes. This also bounds the amount of classical information that can be stored using energy \( E \) in a region of radius \( R \) [103–104]: intuitively, the fastest all
of the energy can leave the region is in time \( R/c \), and at most \( ER/c \) bits can change when this happens.

Since maximum rate of distinct change \( E \) bounds storage it seems that, at the limit, all bits must be dynamic. We can recover the storage bound from this assumption. Consider the minimum energy \( E_{\text{min}} \) needed to represent two distinct states in a region of radius \( R \) as distinct motion (Figure 17). Momentum is smallest if the bit is as large as possible, so \( p_{\text{min}} = 1/R \). Then (taking \( c = 1 \)), \( E_{\text{min}}^2 = p_{\text{min}}^2 + m_{\text{min}}^2 \) is smallest if \( m_{\text{min}} = 0 \). Thus the most binary-subsystems we can have is \( E/E_{\text{min}} = ER \). This bound holds as long as, for minimum energy, all bits stored in any form can ultimately be treated as motion.

**Classical spacetime is effectively discrete.** In our analysis of information bounds, maximum separation between pairs of distinct states, within a region of space or time, determines the least energy required for a two-state system. At the opposite extreme, the Planck length and Planck time estimate minimum separations between pairs of distinct states, regardless of how much energy is used \([105] \). Limits to proximity are expected because high energy is needed to strongly localize a state in space or time, but too much energy in a region creates a black hole, making the region inaccessible \([106] \). This suggests continuous evolution may be effectively discrete at the Planck scale \([99] \), with finer scales inaccessible.

If we model classical mechanics as maximally distinct, though, all evolution is effectively discrete. For large objects, discrete separations \( h/2E \) in time can be arbitrarily small, but this is just a property of aggregated change: if enough clocks tick, average time between ticks can be arbitrarily short. Given the energy density \( \rho \) in a region, we can identify the scale of individual “ticks” there. If we assume that, at the most microscopic scale, all motion is at the speed of light and all change is motion, we can estimate the distance \( \lambda \) between distinct positions as a function of \( \rho \). In natural units with \( c = 1 \) and \( \hbar = 2 \), \( \rho \lambda^3 = E = p = 1/\lambda \), since minimum distinct volume \( \lambda^3 \) must move a width \( \lambda \) to move distinctly, so \( \lambda = \rho^{-1/4} \). In the extreme case of \( \rho \) the Planck density, \( \lambda \) is the Planck length and \( \tau = 1/E = \lambda \) is the Planck time. Effective discreteness provides a fundamental link between classical spacetime and quantum dynamics, and makes discrete integration of Lagrangian action exact \([62] \) (cf. \([107]–[110] \)).

**Energy bounds rate of classical computation.** Since the state at each step of a classical reversible computation is distinct from all other steps, it seems obvious that average energy bounds the rate of computation for any quantum system performing a classical computation. This statement has, however, been challenged \([111] \). Suppose that a quantum computer can perform a classical computation. It can then also perform a uniform superposition of time-delayed versions of the same classical computation \([66]–[112]–[113] \). Such a superposition describes a computation in which the time-step it has reached is uncertain. With a sufficiently wide uncertainty in time, a single distinct change can encompass the entire computation. Does this mean distinctness does not give a lower bound on energy per step of classical computation?

This analysis ignores the relationship between energy and momentum. A computer capable of performing a classical computation rapidly one step at a time must have small elements that are close together in light-travel time. This means that the signals that constitute the computation must be small in three dimensional space, at a scale set by the speed of the distinct-step computation. This also sets localized energy scales, since distinctness in space is governed by momentum bounds. Superposing signals along the length of their paths does not reduce their localization in the other dimensions, so distinctness continues to give a lower bound on energy per step.

**CONCLUSIONS**

Distinctness is a finite resource in nature. This is often represented in physics indirectly, by quantities that constrain an otherwise continuous description. For example, the bandwidth of a classical wave’s Fourier decomposition bounds the rate at which distinct signal values can occur in the continuous wave: signaling-rate \( \leq \) bandwidth.

Quantum mechanics generalizes Fourier analysis to describe all dynamics as unitary evolution of linear superpositions. Widths of generators of such evolutions play the role of bandwidth in bounding the number of distinct states possible in the course of the evolution. Taking \( h \) as the unit of action, any natural width of the eigenvalue distribution, times the length of evolution, normally equals the maximum number of distinct states—up to a factor of order unity. For example, if \( \Delta E \) is any deviation-width of the energy distribution for moment one or greater, at most about \( \Delta E \Delta t \) distinct states can occur in time \( \Delta t \).

Action similarly counts distinct states possible in other single-parameter unitary transformations such as a shift in space \( \Delta p \Delta x \) or a rotation \( \Delta J \Delta \phi \), since momentum is the generator of shifts and angular momentum the generator of rotations. Maximum distinctness implies discreteness: continuous evolution becomes just interpolation between distinct states, and discrete integration of expectation values becomes exact. The number of distinct states possible in an evolution defines the resolution with which the length of the evolution can be measured.

The minimum amount of action needed to have at least
two distinct states is of order unity. This is an absolute bound on distinguishability. Such bounds are usually called uncertainty relations, but they are really constraints on the representation of finite distinctness using continuous waves, applicable even to describing classical computations as continuous unitary evolution. Ground state energy is similarly generic: in a wave description, some of a system’s energy is dedicated to localizing its state energy is similarly generic: in a wave description, continuous waves, applicable even to describing classical constraints on the representation of finite distinctness using an underlying quantum evolution. This is, in fact, how a finite entropy is assigned to a classical system. Treating classical mechanics as maximally distinct, basic mechanical quantities, such as different forms of energy, become counts of distinct states, and the principle of least action becomes least distinct-motion. Spacetime becomes effectively discrete, at a scale set by the local energy density, not by the maximum density that defines the Planck scale. Information represented and transformed by the maximally distinct evolution is classical and finite.

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R. F. Streater, A. S. Wightman, 100th anniversary of the Sackur–Tetrode Statistical Mechanics in a Nutshell
M. J. W. Hall, D. W. Berry, M. Zwierz, H. M. Wiseman, Integrals equal sums. If \( s(t) \) has bandwidth \( 1/\tau \) centered on \( b/\tau \), then \( \int_{-\infty}^{\infty} dt s(t) = \sum_{n=\infty}^{|t|} r' s(n \tau) \) \( \forall \tau' \in (0, \tau], |b| < \frac{1}{2} \).
Proof: For \( \tau' = \tau \), this is just term by term integration of \([28]\), since \( \int_{-\infty}^{\infty} dt \sin(\tau' t - n) \tau = \tau \) if \( |b| < \frac{1}{4} \) (Fourier transform of sinc\( (t/\tau - n) \) evaluated at \( \nu = 0 \)). Then, since \( \int_{-\infty}^{\infty} dt s(t) = \int_{-\infty}^{\infty} dt s(t + \delta) \), we get the same sum if we shift the origin, so \( \sum \tau s(n \tau) = \sum_{\nu} s(n \pm \nu) \), \( \kappa \geq 1 \) an integer. To generalize to \( \kappa \) real, let \( F(t) = \frac{1}{\kappa} \sin(\frac{\pi}{\kappa} \delta - \delta) \).
\( \sum F(n) = \sum_{\nu} F(n) = F(0) = 1 \), and use \([28]\) (cf. \([61]\)).
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From \([20]\) \( \int_{-\infty}^{\infty} dt \int \sin(c \tau t - b s) \sin(c \tau t + n) \) since \( \int_{-\infty}^{\infty} dt \sin\left(\frac{t}{\tau} - n\right) = \tau \). Space is similar. Sums using \( \tau' < \tau \) are similar to \([60]\).
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Periodic integrals equal periodic sums \([60]\) when the \( N \) frequencies are integer multiples of \( 1/T \) and include \( 0/T \).
Then \( \int_{T/2}^{T/2} dt \sin(c \tau t - n \tau) \), \( |b| \leq \frac{N-1}{N} \) (cf. \([60]\)).
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Let initial state \( \ket{\psi} = \sqrt{|0|} \phi_{0} + \sqrt{|n|} \phi_{n} \), an equal superposition of \( n \) photons in arm \( a \) of the interferometer with 0 in arm \( b \), and vice versa. The free Hamiltonian for each arm, is \( H_{a} = (n + \frac{1}{2}) \hbar \nu \), and if there is a time delay \( \delta t \) in arm \( a \) relative to arm \( b \), the generator of distinct evolution is \( H_{a} - H_{b} + \hbar \delta t \) since the rest of the dynamics contributes just an overall phase: \( H_{a} + H_{b} (\delta t - \delta t) = H_{a} + H_{b} (\delta t + \hbar / (4 \pi \nu)) \).
Since \( \ket{\psi} \) is an equal superposition of eigenvalues \( \pm \nu \nu / 2 \) of \( H_{a} \), it achieves \([4]\) and the two distinct states of \( \delta t \) are \( T/2 = 1/2 \nu \).a part.
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