HARMONIC GAUSS MAPS AND SELF-DUAL EQUATIONS
IN STRING THEORY

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Abstract

The string world sheet, regarded as Riemann surface, in background $\mathbb{R}^3$ and $\mathbb{R}^4$ is described by the generalised Gauss map. When the Gauss map is harmonic or equivalently for surfaces of constant mean scalar curvature, we obtain an Abelian self-dual system, using $SO(3)$ and $SO(4)$ gauge fields constructed in our earlier studies. This compliments our earlier result that $h\sqrt{g} = 1$ surfaces exhibit Virasaro symmetry. The self-dual system so obtained is compared with self-dual Chern-Simons system and a generalized Liouville equation involving extrinsic geometry is obtained.

The world sheet in background $\mathbb{R}^n$, $n > 4$ is described by the generalized Gauss map. It is first shown that when the Gauss map is harmonic, the scalar mean curvature is constant. $SO(n)$ gauge fields are constructed from the geometry of the surface and expressed in terms of the Gauss map. It is shown that the harmonic map satisfies a non-Abelian self-dual system of equations for the gauge group $SO(2) \times SO(n - 2)$. 
I. INTRODUCTION

The study of Yang-Mills connections on 2-dimensional Riemann surfaces is of importance in string theory. The space of self-dual connections provides a model for Teichmüller space [1]. Clearly, the string world sheet is a 2-dimensional surface immersed in $R^n$ (For convenience we consider both the world sheet and the target space as Euclidean). The immersion induces a metric (the first fundamental form) on the world sheet. The second fundamental form of the surface determines its extrinsic geometry. We [2] have developed a formalism to study the dynamics of the world sheet conformally immersed (by conformal immersion it is meant that the induced metric is in the conformal gauge) in $R^3$ and $R^4$ using the generalized Gauss map [3]. The string action can be written as a constrained Grassmannian $\sigma$-model action and the theory is asymptotically free.

Subsequently [4] we found a hidden Virasaro symmetry for surfaces of constant scalar mean curvature density ($h\sqrt{g} = 1$). An action exhibiting this symmetry has recently been constructed [5] and it is a WZNW action. The quantum theory of this action has also been studied in [5]. It would be of interest to know if surfaces of constant scalar mean curvature ($h=$constant) exhibit novel properties, so that we get a better understanding of the many facets of the string world sheet. For such surfaces, it is known that the Gauss map from the world sheet $M$ (regarded as a Riemann surface) into the Grassmannian $G_{2,n} \simeq SO(n)/(SO(2) \times SO(n - 2))$ is harmonic for $n = 3,4$. It is the purpose of this paper to show that there exists an Abelian self-dual system on such surfaces, and a non-Abelian self-dual system for $n > 4$.

In this paper, we first consider the Gauss map of a 2-dimensional surface into the Grassmannian $G_{2,n} \simeq SO(n)/(SO(2) \times SO(n - 2))$ for $n = 3,4$. We [4] have previously
constructed $SO(n)$ connections on the string world sheet. We project these onto $SO(2) \times SO(n - 2)$ and the coset. The projection onto $SO(2) \times SO(n - 2)$ is identified as the gauge field (see sec.III) and that on the coset as the Higgs field. Using the Euler-Lagrange equations (harmonic map equations) for the surface, we show that this system is self-dual. In this analysis, the harmonic map has a geometrical interpretation. For immersion in $R^3$ and $R^4$, the Gauss map is harmonic if the mean curvature scalar $h$ is constant [6].

We next take up immersion in $R^n$ for $n > 4$ and show explicitly that when the Gauss map is harmonic, the mean scalar curvature of the surface is constant. The $SO(n)$ gauge fields are constructed from the geometry of the surface. Using their projections into the subgroup $SO(2) \times SO(n - 2)$ and its compliment in $G_{2,n}$, a non-Abelian self-dual system is obtained when the Gauss map is harmonic. Thus our main result is:

**Theorem.1**

*Let $M$ be a 2-dimensional surface conformally immersed in $R^n$. Let $M_0$ be the Riemann surface obtained by the induced conformal structure on $M$. Let $G : M_0 \to G_{2,n}$ be the Gauss map. Let $A$ be the flat $SO(n)$ connection on $M_0$ defined by the adapted frame of tangents and $(n - 2)$ normals to $M_0$. The projections of $A$ onto $SO(2) \times SO(n - 2)$ and its orthogonal compliment in $G_{2,n}$ satisfy self-dual system when the Gauss map $G$ is harmonic.*

Recently, Dunne, Jackiw, Pi and Trugenberger [7] made a systematic analysis of the Yang-Mills non-linear Schrödinger equation and demonstrated self-dual Chern-Simons equations for the static configurations. Here the matter density $\rho$ is in the adjoint representation. By choosing the Chern-Simons gauge field in the commuting set of the Cartan subalgebra and $\Psi (\rho = -i[\Psi, \Psi^\dagger])$ in terms of the ladder operators with positive roots,
they [7] and Dunne [8] obtain Toda equations. We obtain similar results in the case of
string world sheet in background $R^n$, restricting to world sheets of constant mean scalar
curvature.

In this way, we find that the (extrinsic) geometry of the string world sheet described
by harmonic Gauss maps is closely related to the self-dual system of Hitchin [1] and to
the static configuration of (2+1) self-dual non-Abelian Chern-Simons theory [7,8].

II. PRELIMINARIES

Consider a 2-dimensional (Euclidean) string world sheet regarded as a Riemann surface
conformally immersed in $R^n$. The induced metric is $g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\mu$, with $X^\mu(\xi_1, \xi_2)$
as immersion coordinates ($\mu = 1,2,...,n$) and $\xi_1, \xi_2$ as local isothermal coordinates on the
surface. The Gauss-Codazzi equations introduce the second fundamental form $H^i_{\alpha\beta}$, $i = 1,2,...,(n-2)$. Locally on the surface, we have two tangents and (n-2) normals. The Gauss
map is

$$ G : M \to G_{2,n} \simeq SO(n)/(SO(2) \times SO(n-2)). $$

$G_{2,n}$ can be realized as a quadric $Q_{n-2}$ in $CP^{n-1}$ defined by $\sum_{i=1}^{n} Z_i^2 = 0$, where $Z_i$ are
the homogeneous coordinates on $CP^{n-1}$ [3]. A local tangent plane to $M_0$ is an element of
$G_{2,n}$, or equivalently a point in $Q_{n-2}$. Then [3] we have

$$ \partial_z X^\mu = \psi \Phi^\mu, $$

where $z = \xi_1 + i\xi_2$, $\bar{z} = \xi_1 - i\xi_2$, with $\xi_{1,2}$ as isothermal coordinates on $M_0$, $\Phi^\mu \in Q_{n-2}$,
$\Phi^\mu \Phi^\mu = 0$ and $\psi$ is a complex function determined in terms of the geometrical properties
of the surface. As not every element of $G_{2,n}$ is a tangent plane to $M_0$, the Gauss map (2)
has to satisfy (n-2) conditions of integrability [3]. These were explicitly derived in Ref.3
for immersion in $R^3$ and $R^4$ and by us [11] for $R^n$ ($n > 4$). We first consider immersion in $R^3$ and $R^4$.

For immersion in $R^3$, $\Phi^\mu$ is parametrized as

$$\Phi^\mu = [1 - f^2, i(1 + f^2), 2\bar{f}],$$

where $f \in CP^1$. The integrability condition $f$ is,

$$Im \left[ \frac{f_{zz}}{f_z} - \frac{2\bar{f}f_z}{1 + |f|^2} \right] = 0.$$ 

(4)

The scalar mean curvature $h(= N^\alpha H_{\alpha\alpha}^\mu)$ is given by

$$(\ell nh)_z = \frac{f_{zz}}{f_z} - \frac{2\bar{f}f_z}{1 + |f|^2},$$

which is known as the Kenmotsu equation [10]. The normal $N^\mu$ to the surface can be expressed in terms of $f$ as

$$N^\mu = \frac{1}{1 + |f|^2} [f + \bar{f}, -i(f - \bar{f}), |f|^2 - 1].$$

(6)

The energy integral of the surface is

$$S = \int |f_z|^2 + |f_{\bar{z}}|^2 d\bar{z}d\bar{z},$$

which is also the extrinsic curvature action $\int \sqrt{g} |H|^2$. The Euler-Lagrange equation from (7) is

$$L(f) \equiv f_{zz} - \frac{2\bar{f}f_zf_{\bar{z}}}{1 + |f|^2} = 0.$$ 

(8)

The Gauss map is said to be harmonic if $f$ satisfies the Euler-Lagrange equations (8) and it then follows from (5) that $h$ is constant [7].
For immersion in $R^4$, we have $G_{2,4} \simeq SO(4)/(SO(2) \times SO(2)) \simeq CP^1 \times CP^1$ and so $\Phi^\mu$ is parametrized in terms of the two $CP^1$ fields, $f_1$ and $f_2$ as

$$\Phi^\mu = [1 + f_1 f_2, i(1 - f_1 f_2), f_1 - f_2, -i(f_1 + f_2)].$$

(9)

The Gauss map integrability conditions are

$$Im \left[ \sum_{i=1}^{2} \frac{f_{iz} \bar{z} - 2 \bar{f}_i f_{iz}}{1 + |f_i|^2} \right] = 0,$$

$$|F_1| = |F_2|,$$

(10)

where $F_i = \frac{f_i}{1 + |f_i|^2}$. There are two normals $N_1^\mu, N_2^\mu$ to the surface which can be written in terms of $f_1$ and $f_2$ as

$$N_1^\mu = \frac{1}{2D} (A^\mu + \bar{A}^\mu),$$

$$N_2^\mu = \frac{1}{2iD} (A^\mu - \bar{A}^\mu),$$

(11)

where,

$$D = \left( (1 + |f_1|^2)(1 + |f_2|^2) \right)^{1/2},$$

$$A^\mu = \left[ f_2 - \bar{f}_1, -i(f_2 + \bar{f}_1), 1 + \bar{f}_1 f_2, -i(1 - \bar{f}_1 f_2) \right],$$

The projections of $H_\alpha^{\mu\alpha}$ along $N_1^\mu$ and $N_2^\mu$ are given by [2]

$$h_1 = \frac{F_1 - F_2}{2\psi D},$$

$$h_2 = \frac{i(F_1 + F_2)}{2\psi D},$$

(12)

and the scalar mean curvature $h^2 = (h_1^2 + h_2^2)$ satisfies the equation [2]

$$2(\ell n h)_z = \sum_{i=1}^{2} \left[ \frac{f_{iz} \bar{z} - 2 \bar{f}_i f_{iz}}{1 + |f_i|^2} \right].$$

(13)
The energy integral of the surface is

\[ S = \int \sum_{i=1}^{2} |F_i|^2 + |\hat{F}_i|^2, \quad (14) \]

where \( \hat{F}_i = \frac{f_i}{1+|f_i|^2} \). The Euler-Lagrange equations for (14) are

\[ L(f_1) = 0 ; \quad L(f_2) = 0, \quad (15) \]

where \( L(f) \) is defined in (8).

The Gauss map is harmonic if \( f_1 \) and \( f_2 \) satisfy (15) and from (13) it follows that the immersed surface has constant \( h \) for immersion in \( \mathbb{R}^4 \).

It is to be noted that \( f_1 \) and \( f_2 \) should also satisfy the second requirement in (10) to describe the Gauss map. In Ref.4, we have considered tangents to \( M \) as \( \hat{e}_1 = \frac{1}{\sqrt{2|\Phi|}}(\Phi^\mu + \bar{\Phi}^\mu) \) and \( \hat{e}_2 = \frac{1}{\sqrt{2|\Phi|}}(\Phi^\mu - \bar{\Phi}^\mu) \) along with the (n-2) normals. Then, the local orthonormal frame \( (\hat{e}_1, \hat{e}_2, N^\mu_i) \) satisfies

\[ \partial_z \hat{e}_i = (A_z)_{ij} \hat{e}_j; \quad i, j = 1 \text{ to } n, \quad (16) \]

where \( \hat{e}_i = N^\mu_i \), for \( i = 3 \) to \( n \). A similar equation for the \( \bar{z} \) derivative defines \( (A_{\bar{z}})_{ij} \). \( A_z \) and \( A_{\bar{z}} \) transform as \( SO(n, C) \) gauge fields under local \( SO(n) \) transformations of \( (\hat{e}_1, \hat{e}_2, N^\mu_i) \) which follows from (16) [4]. These non-Abelian gauge fields are constructed from the geometrical properties of the surface alone and so they are characteristics of the world sheet.

Using (3) and (6), it is easily verified that \( A_z \) for immersion in \( \mathbb{R}^3 \) is given by

\[ A_z = \frac{1}{1 + |f|^2} \begin{bmatrix} 0 & -i(f\bar{f}_z - \bar{f}f_z) & -(f_z + \bar{f}_z) \\ i(f\bar{f}_z - \bar{f}f_z) & 0 & i(f_z - \bar{f}_z) \\ f_z + \bar{f}_z & -i(f_z - \bar{f}_z) & 0 \end{bmatrix}, \quad (17) \]
Similarly, using (9),(11),(16) and, denoting 
\[ d_i = 1 + |f_i|^2, \quad m_i = f_i \bar{f}_{i2} - \bar{f}_i f_{i2}, \quad p_i = f_{i2} + \bar{f}_{i2}, \]
and \( q_i = f_{i2} - \bar{f}_{i2} \), \( A_z \) for immersion in \( R^4 \) is obtained as

\[
A_z = \frac{1}{2} \begin{bmatrix}
0 & -i\left(\frac{m_1}{d_1} + \frac{m_2}{d_2}\right) & \frac{p_1}{d_1} - \frac{p_2}{d_2} & i\left(\frac{q_1}{d_1} + \frac{q_2}{d_2}\right) \\
-\left(\frac{p_1}{d_1} - \frac{p_2}{d_2}\right) & 0 & -i\left(\frac{q_1}{d_1} - \frac{q_2}{d_2}\right) & \frac{p_1}{d_1} + \frac{p_2}{d_2} \\
i\left(\frac{m_1}{d_1} + \frac{m_2}{d_2}\right) & i\left(\frac{q_1}{d_1} - \frac{q_2}{d_2}\right) & 0 & \frac{p_1}{d_1} + \frac{p_2}{d_2} \\
i\left(\frac{q_1}{d_1} + \frac{q_2}{d_2}\right) & i\left(\frac{m_1}{d_1} - \frac{m_2}{d_2}\right) & -\left(\frac{p_1}{d_1} + \frac{p_2}{d_2}\right) & 0 
\end{bmatrix}
\] (18)

The gauge field \( A_{\bar{z}} \) can be obtained by replacing \( z \) derivatives by \( \bar{z} \) derivatives and it is seen that \( (A_z)^\dagger = -A_{\bar{z}} \). Further, from (16) it is easily verified that the gauge fields satisfy

\[
\partial_z A_z - \partial_{\bar{z}} A_{\bar{z}} + [A_z, A_{\bar{z}}] = 0. \quad (19)
\]

### III. HARMONIC GAUSS MAP AND SELF-DUAL SYSTEM

We now project the gauge fields constructed in the previous section onto \( SO(2) \times SO(n-2) \) and its orthogonal compliment in \( G_{2,n} \) for \( n = 3 \) and 4. The general procedure is briefly outlined here. (For details see Ref.12). Consider a \( G/H \) sigma model on a two dimensional Riemann surface. Denote the generators of the Lie algebra \( L_G \) of \( G \) by \( L(\tilde{\sigma}), \tilde{\sigma} = 1,2,...[G] \) and those of \( L_H \) of \( H \) by \( L(\bar{\sigma}); \bar{\sigma} = 1,2,...,[H]; [H]<[G]. \) The remaining generators of \( L_G \) will be denoted by \( L(\sigma) \). Consider a local gauge group associated with \( G \). We have

\[
M \ni (z, \bar{z}) \xrightarrow{g} g(z, \bar{z}) \in G, \quad (20)
\]

and introduce,

\[
\omega_\alpha(g) = g^\dagger \partial_\alpha g. \quad (21)
\]
The field strength associated with $\omega_\alpha(g)$ is zero. In fact, $\omega_\alpha(g)$ is same as $-A_z$ and $-A_{\bar{z}}$ and is equivalent to (16) with $g(z, \bar{z})$ as the $n \times n$ matrix formed by the two tangent vectors $\hat{e}_1$ and $\hat{e}_2$ and the $(n-2)$ normals $N^\mu_i$. Under a local gauge transformation generated by $u(z, \bar{z}) \in H$, we have

$$g(z, \bar{z}) \rightarrow g(z, \bar{z})u(z, \bar{z}),$$

$$\omega_\alpha(g) \rightarrow \omega_\alpha(gu) = u^\dagger \omega_\alpha(g)u + u^\dagger \partial_\alpha u.$$  \hfill (22)

Thus $-A_z$ and $-A_{\bar{z}}$ transform as gauge fields under $SO(2) \times SO(n-2)$ gauge transformation. The projection of $\omega_\alpha(g)$ onto $L_H$ and its orthogonal compliment are

$$a_\alpha(g) = L(\bar{\sigma})tr(L(\bar{\sigma})\omega_\alpha(g)),$$

$$b_\alpha(g) = L(\sigma)tr(L(\sigma)\omega_\alpha(g)),$$  \hfill (23)

and it is straightforward to verify that under (22),

$$a_\alpha(g) \rightarrow a_\alpha(gu) = u^\dagger a_\alpha u + u^\dagger \partial_\alpha u,$$

$$b_\alpha(g) \rightarrow b_\alpha(gu) = u^\dagger \partial_\alpha u.$$  \hfill (24)

So, $a_\alpha(g)$ transforms as a gauge field under local gauge transformations belonging to $H$ and $b_\alpha(g)$ transforms homogeneously.

Now we consider immersion in $R^3$. The $SO(3)$ gauge fields $A_z$ and $A_{\bar{z}}$ in (17) are projected onto $SO(2)$ and its orthogonal compliment in $G_{2,3}$. Denoting the anti-Hermitian generators of $SO(3)$ as $T_1, T_2, T_3$; $[T_1, T_2] = T_3$, (cyclic), we have

$$a_z = \frac{1}{2} T_3 tr(T_3 A_z),$$

$$b_z = \frac{1}{2} T_1 tr(T_1 A_z) + \frac{1}{2} T_2 tr(T_2 A_z).$$  \hfill (25)
The gauge group $H$ in (22) is $SO(2) \sim U(1)$. Similar projections for $A_z$ are made. It can be verified that $a_z + b_z = -A_z = g^\dagger \partial_z g$. Here we have a flat connection which is decomposed as $a_z$ and $b_z$ (similarly for $A_{z\bar{}}$). Then (19) leads to

$$
\begin{align*}
\partial_z a_z - \partial_{\bar{z}} a_z + [a_z, a_{\bar{z}}] + [b_z, b_{\bar{z}}] &= 0, \\
\partial_z b_z + [a_z, b_z] &= \partial_{\bar{z}} b_z + [a_{\bar{z}}, b_{\bar{z}}],
\end{align*}
$$

(26)

where we have made use of the group structure underlying (25), viz, the first equation in (26) is in the Cartan subalgebra while the second in $T_1$ and $T_2$ directions: hence both must separately vanish. The second equation in (26) gives the self-dual property if each side vanishes. This we shall prove by the equations of motion (15), namely for harmonic Gauss map. Explicitly, from (17), we find

$$
a_z = \frac{1}{1 + |f|^2} \begin{bmatrix} 0 & i(f \bar{f}_z - \bar{f}f_z) & 0 \\ -i(f \bar{f}_z - \bar{f}f_z) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
$$

(27)

and

$$
b_z = \frac{1}{1 + |f|^2} \begin{bmatrix} 0 & 0 & f_z + \bar{f}_z \\ 0 & 0 & -i(f_z - \bar{f}_z) \\ -(f_z + \bar{f}_z) & i(f_z - \bar{f}_z) & 0 \end{bmatrix}.
$$

(28)

$a_{\bar{z}} = -a_z^\dagger; b_{\bar{z}} = -b_z^\dagger$. Then we find when $L(f) = 0$

$$
\partial_{\bar{z}} b_z + [a_{\bar{z}}, b_{\bar{z}}] = 0,
$$

(29)

Thus for harmonic Gauss maps, we have the following self-dual system

$$
\begin{align*}
\partial_z a_z - \partial_{\bar{z}} a_z + [a_z, a_{\bar{z}}] + [b_z, b_{\bar{z}}] &= 0, \\
\partial_z b_z + [a_z, b_z] &= 0, \\
\partial_{\bar{z}} b_z + [a_{\bar{z}}, b_{\bar{z}}] &= 0.
\end{align*}
$$

(30)
Note that $a_z$ transforms as an $SO(2)$ gauge field while $b_z$ transform homogeneously and so, $b_z$ is identified with the Higg’s field in Ref.1. The self-dual system (30) is also equivalent to the static self-dual Chern-Simons system [7,8] if we identify the matter density $\rho$ as $-i [b_z, b_z^\dagger]$ which lies in the Cartan subalgebra of $SO(3)$.

Next consider surfaces in $R^4$. Here we have the Grassmannian, $G_{2,4} \simeq SO(4)/(SO(2) \times SO(2))$. We choose $T_1$ to $T_6$ as generators of $SO(4)$ such that $[T_1, T_2] = T_3$, cyclic; $[T_4, T_5] = T_6$, cyclic; and $[T_i, T_j] = 0$ for $i = 1, 2, 3; j = 4, 5, 6$. The explicit form of $A_z$ has been given in (18) and the projection of $A_z$ and $\bar{A}_z$ onto $SO(2) \times SO(2)$ and its compliment in $G_{2,4}$ are

$$a_z = T_3 \text{tr}(T_3 A_z) + T_6 \text{tr}(T_6 A_z),$$

$$b_z = T_1 \text{tr}(T_1 A_z) + T_2 \text{tr}(T_2 A_z) + T_4 \text{tr}(T_4 A_z) + T_5 \text{tr}(T_5 A_z).$$ (31)

Equations similar to (26) readily follow from (18). The explicit forms of $a_z$ and $b_z$ are not displayed as the procedure is straightforward. The self-dual property can be verified for harmonic maps by computing each side of the second equation in (26) for immersion in $R^4$. Introducing

$$\mathcal{L}(f_i) = \frac{(L(f_i) + \bar{L}(f_i))}{1 + |f_i|^2},$$

$$\mathcal{L}'(f_i) = \frac{(L(f_i) - \bar{L}(f_i))}{1 + |f_i|^2},$$

$$\mathcal{S} = \mathcal{L}(f_1) + \mathcal{L}(f_2),$$

$$\mathcal{D} = \mathcal{L}(f_1) - \mathcal{L}(f_2),$$

$$\mathcal{S}' = \mathcal{L}'(f_1) + \mathcal{L}'(f_2),$$

$$\mathcal{D}' = \mathcal{L}'(f_1) - \mathcal{L}'(f_2).$$ (32)
for \( i = 1, 2 \) and where \( L(f) \) is defined in (8), we find

\[
\partial \bar{z} b_z + [a_{\bar{z}}, b_z] = \frac{1}{2} \times \begin{bmatrix}
0 & 0 & -\mathcal{D} & -i\mathcal{S'} \\
0 & 0 & i\mathcal{D'} & -\mathcal{S} \\
\mathcal{D} & -i\mathcal{D'} & 0 & 0 \\
i\mathcal{S'} & \mathcal{S} & 0 & 0
\end{bmatrix}.
\] (33)

It can be seen that when the Gauss map is harmonic (the Euler-Lagrange equations of motion (15) are satisfied) it follows that \( \partial \bar{z} b_z + [a_{\bar{z}}, b_z] = 0 \), which is the self-dual equation. It is pertinent to note that \( b_z \) transforms homogeneously under the local \( SO(2) \times SO(2) \) gauge transformation. The field \( a_z \) which is in the Cartan subalgebra \( SO(2) \times SO(2) \) transforms as a gauge field. It is important to reiterate that \( a_z \) and \( b_z \) are embedded in \( SO(4) \). Explicit solutions to the self-dual equations for the gauge group studied here are given by (27) and (28) for \( R^3 \) and (18) and (31) for \( R^4 \), where the complex functions \( f_1 \) and \( f_2 \) satisfy the equations, \( L(f_1) = 0 \) and \( L(f_2) = 0 \). As harmonic Gauss map implies surfaces of constant scalar mean curvature, it follows that the self-dual system is on surfaces of constant \( h \).

Fujii [14] examined the relationship between Toda systems and the Grassmannian \( \sigma \)-models. We now examine the self-dual system on surfaces in \( R^3 \). In analogy with [7], the matter density \( \rho = \rho_3 T_3 \) is

\[
\rho_3 = \frac{2(f \bar{z} \bar{f}_z - f \bar{f}_z)}{(1 + |f|^2)^2}.
\] (34)

We recall the following relations for Gauss map in \( R^2 \) [2]. Writing \( \sqrt{g} = \exp(\phi) \), and \( \hat{F} = \frac{f_z}{1 + |f|^2} \), we have

\[
|F|^2 = \frac{h}{2} H_{z\bar{z}} = \frac{h^2}{2} \exp(\phi),
\]
where $H_{\alpha\beta}$ is the second fundamental form. It can be verified by using the equation of motion (8) and the Gauss map relation, $(\ell n\psi)_{z} = -\frac{2ff_{z}}{(1+|f|^{2})}$, that $\partial_{z}\partial_{\bar{z}}\ell n|H_{zz}|^{2} = 0$. The Gauss curvature $R = -\exp(-\phi)\partial_{z}\partial_{\bar{z}}\phi$. Writing the Gauss curvature in terms of $H_{\alpha\beta}$, we have a modified Liouville equation for extrinsic curvature as

$$\partial_{z}\partial_{\bar{z}}\phi = -2h^{2}\exp(\phi) + 2\exp(-\phi)\exp(\phi_{E}),$$

(36)

where $|H_{zz}|^{2} = \exp(\phi_{E})$ and $\partial_{z}\partial_{\bar{z}}\phi_{E} = 0$. When we consider $f$ to be anti-holomorphic, (36) reduces to the Liouville equation for $h = 1$

$$\partial_{z}\partial_{\bar{z}}\phi = -2\exp(\phi),$$

(37)

which is also the Toda equation for $SO(3)$.

For immersion in $R^{4}$, when we consider $f_{1}$ and $f_{2}$ both anti-holomorphic, we obtain

$$\rho = \frac{f_{1\bar{z}}\bar{f}_{1z}}{(1+|f_{1}|^{2})^{2}}T_{6} + \frac{f_{2\bar{z}}\bar{f}_{2z}}{(1+|f_{2}|^{2})^{2}}T_{3}.$$  

(38)

The Cartan matrix for $SO(4)$ is $K_{\alpha\beta} = -2\delta_{\alpha\beta}$. Then we obtain

$$\partial_{z}\partial_{\bar{z}}\ell n\rho_{6} = -2\rho_{6},$$

$$\partial_{z}\partial_{\bar{z}}\ell n\rho_{3} = -2\rho_{3},$$

(39)

where $\rho_{6}$ and $\rho_{3}$ are the coefficients of $T_{6}$ and $T_{3}$ in (38). (39) is identified as the $SO(4)$ Toda system. Thus, harmonic Gauss maps lead to $SO(3)$ and $SO(4)$ Toda system for immersion of the world sheet in background $R^{3}$ and $R^{4}$ respectively.
IV. HARMONIC GAUSS MAPS IN $R^n$ ($n > 4$)

We now consider immersion of 2-dimensional surfaces in $R^n$, $n > 4$. There are two reasons for this consideration. First of all, the gauge field $a_z$ in the two cases considered ($n=3$ and 4) is an Abelian embedding in $SO(3)$ and $SO(4)$. This is similar to the choice in Ref. 8 and 9. We would like to realize a non-Abelian self-dual system, which occurs when $n > 4$. Secondly, the result that harmonic Gauss map implies constant scalar mean curvature, has been proved for immersion in $R^3$ by Ruh and Vilms [7] and can be proved from our [2] results for $h$ and the Euler-Lagrange equations for immersion in $R^4$. For immersion in $R^n$, $n > 4$, such a result has not yet been explicitly obtained to the best of our knowledge. In this paper we prove this and use it to obtain the self-duality equations for harmonic Gauss maps in $R^n$, $n > 4$.

We recall the essential details of the Gauss map of surfaces in $R^n$ from our earlier paper [10] and from Hoffman and Osserman [11]. $\Phi^\mu$ in $Q_{n-2}$ in (2) is parametrized in the following manner. Let $(z_1, z_2, \ldots, z_n)$ be the homogeneous coordinates of $CP^{n-1}$. The quadric $Q_{n-2} \in CP^{n-1}$ is defined by

$$\sum_{k=1}^{n} z_k^2 = 0.$$  \hspace{1cm} (40)

Let $H$ be the hyperplane in $CP^{n-1}$ defined by $H : (z_1 - iz_2) = 0$. Then $Q_{n-2}^* = Q_{n-2} \setminus \{H\}$ is biholomorphic to $C^{n-2}$ under the correspondence [10,11]

$$(z_1, \ldots, z_n) = \frac{z_1 - iz_2}{2} \left[ 1 - \zeta_k^2, i(1 + \zeta_k^2), 2\zeta_1, \ldots, 2\zeta_{n-2} \right],$$  \hspace{1cm} (41)

where,

$$\zeta_j = \frac{z_j + 2}{z_1 - iz_2}. \hspace{1cm} (42)$$
for \( j = 1, 2, ..., n - 2 \). (In (41) and in what follows we use the summation convention that repeated indices are summed from 1 to \( n - 2 \), unless otherwise stated.) Conversely, given any \((\zeta_1, ..., \zeta_{n-2}) \in \mathbb{C}^{n-2}\),

\[
\Phi^\mu = \begin{bmatrix}
1 - \zeta_k^2, i(1 + \zeta_k^2), 2\zeta_1, ..., 2\zeta_{n-2}
\end{bmatrix},
\]

satisfies (40) and hence defines a point in the complex quadric \( Q_{n-2} \). The Fubini-Study metric on \( CP^{n-1} \) induces a metric on \( Q_{n-2} \) [10] which is computed as

\[
g_{ij} = \frac{4}{|\Phi|^2} \delta_{ij} + \frac{16}{|\Phi|^4} [\zeta_i \bar{\zeta}_j - \zeta_j \bar{\zeta}_i + 2\zeta_i \bar{\zeta}_j | \zeta_k |^2 - \zeta_i \zeta_j \bar{\zeta}_k - \bar{\zeta}_i \bar{\zeta}_j \zeta_k^2],
\]

where,

\[
|\Phi|^2 = 2 + 4\zeta_k \bar{\zeta}_k + 2\zeta_k^2 \zeta_m.
\]

We [10] found it convenient to introduce an n-vector

\[
A_k^\mu = -[\bar{\zeta}_k + \zeta_k \zeta_m] \Phi^\mu + \frac{|\Phi|^2}{2} v_k^\mu,
\]

for \( k = 1, 2, ..., (n - 2) \) and

\[
v_k^\mu = (-\zeta_k, i\zeta_k, 0, 0, 0, ..., 1_k, 0, ..),
\]

where \( 1_k \) stands for 1 in the \((k + 2)\)th position. The algebraic properties of \( A_k^\mu \) and \( a_k^\mu \) have been established in [10]. The \((n - 2)\) real normals to the surface have been derived as

\[
N_i^\mu = \frac{4}{|\Phi|^2} (O^T)_{ij} A_j^\mu,
\]

where the \((n - 2) \times (n - 2)\) matrix \( O \) has been defined in [10].

The \((n - 2)\) complex functions \( \zeta_i(z, \bar{z}) \) where \( z, \bar{z} \) are the isothermal coordinates on \( M_0 \) have been shown to satisfy \((n - 2)\) conditions so that they can represent the Gauss map.
The mean curvature scalar \( h \) of the surface has been shown to be related to Gauss map by

\[
(\ell nh)_z = \frac{\sum_{j=1}^{n-2} \zeta_j \bar{z} \zeta_j \bar{z}}{\sum_{j=1}^{n-2} (\zeta_j \bar{z})^2} - \frac{4}{|\Phi|^2} \sum_{j=1}^{n-2} \zeta_j \bar{z} \left( \zeta_j + \zeta_j \sum_{k=1}^{n-2} \bar{\zeta}_k \bar{z} \right),
\]

which is the generalization of the Kenmotsu equation to immersion in \( R^n \).

\textbf{Theorem 2}

Let \( M \) be a 2-dimensional surface defined by a conformal immersion \( X : M \to R^n \).

Then if the Gauss map \( G : M \to G_{2,n} \) is harmonic, the mean curvature scalar \( h \) of \( M \) is constant.

\textit{Proof}

The Gauss map is said to be harmonic if the \((n-2)\) complex functions \( \zeta_i \) satisfy the Euler-Lagrange equations of the energy integral \([10]\)

\[
\mathcal{E} = \int g_{ij} \zeta_i \bar{z} \zeta_j \bar{z} \ dz \wedge d\bar{z}.
\]

The above ‘energy integral’ is also the action for the extrinsic curvature of the surface \( M \), namely, \( \int \sqrt{g} |H|^2 \). The Euler-Lagrange equations that follow from the extremum of \( \mathcal{E} \) are obtained as

\[
\zeta_{k \bar{z} \bar{z}} = -\frac{4}{|\Phi|^2} \sum_{i=1}^{n-2} \zeta_i \zeta_i \bar{z} \left( \zeta_k + \zeta_k \sum_{m=1}^{n-2} \bar{\zeta}_m \right) + \frac{4}{|\Phi|^2} \sum_{i=1}^{n-2} (\bar{\zeta}_i + \zeta_i \sum_{m=1}^{n-2} \bar{\zeta}_m) [\zeta_k \zeta_i \bar{z} \zeta_i + \zeta_k \zeta \bar{z} \bar{z}].
\]

Upon using the above equations in the expression for \( (\ell nh)_z \) in (49) it follows that the mean curvature scalar \( h \) of the surface \( M_0 \) is constant. This completes the proof.
We now proceed to construct $SO(n)$ gauge fields on the surface $M_0$. The defining equation for them is (16) with $\Phi^\mu$ given by (43) and the $(n-2)$ normals by (48). The various components of $A_z$ are given below.

$$
\begin{bmatrix}
0 & (A_z)_{12} & (A_z)_{1i}(i = 3 \text{ to } n) \\
(A_z)_{21} & 0 & (A_z)_{2i}(i = 3 \text{ to } n) \\
(A_z)_{31} & (A_z)_{32} & \ldots \\
\vdots & \vdots & \vdots \\
(A_z)_{n1} & (A_z)_{n2} & \ldots
\end{bmatrix}
$$

(52)

where,

$$(A_z)_{12} = \frac{2}{i|\Phi|^2}[(\zeta_j + \bar{\zeta}_j \zeta_i^2)\bar{\zeta}_{jz} - (\bar{\zeta}_j + \zeta_{j2}\bar{\zeta}_i)\zeta_{jz}]$$

$$(A_z)_{1i} = \frac{2\sqrt{2}}{|\Phi|^5}(O^T)_{ij}[|\Phi|^2\zeta_{jz} - 4(\bar{\zeta}_j + \zeta_{j2}\bar{\zeta}_m)(\zeta_k + \bar{\zeta}_k\zeta_q^2)\bar{\zeta}_{kz}$$

$$+ |\Phi|^2(2\zeta_j\bar{\zeta}_k\bar{\zeta}_{kz} + \bar{\zeta}_{jz})]$$

$$(A_z)_{2i} = \frac{2\sqrt{2}}{|\Phi|^5}(O^T)_{ij}[|\Phi|^2\zeta_{jz} + 4(\bar{\zeta}_j + \zeta_{j2}\bar{\zeta}_m)(\zeta_k + \bar{\zeta}_k\zeta_q^2)\bar{\zeta}_{kz}$$

$$- |\Phi|^2(2\zeta_j\bar{\zeta}_k\bar{\zeta}_{kz} + \bar{\zeta}_{jz})]$$

$$(A_z)_{ij} = -\frac{1}{|\Phi|^7}\partial_z(|\Phi|^2)\delta_{ij} + \frac{4}{|\Phi|^7}(O^T)_{jk}\partial_zO_{ki}$$

$$+ \frac{16}{|\Phi|^7}(\bar{\zeta}_k + \zeta_k\zeta_q^2)\zeta_{mz}[(O^T)_{jk}(O^T)_{im} - (O^T)_{jm}(O^T)_{ik}],$$

(53)

$A_z$ can be obtained by replacing the $z$-derivatives by $\bar{z}$ derivatives. They together define the $SO(n)$ gauge fields on the surface $M_0$. We now project them on to $SO(2) \times SO(n-2)$ and its orthogonal compliment in $G_{2,n} \simeq SO(n)/(SO(2) \times SO(n-2))$. Denoting
these projections by $a_z$ and $b_z$ respectively (and similarly for $a_z$ and $b_z$), we have

$$a_z = - \begin{bmatrix} 0 & (A_z)_{12} & 0(13 \text{ to } 1n) \\ (A_z)_{21} & 0 & 0(23 \text{ to } 2n) \\ 0 & 0 & \vdots \\ \vdots & \vdots & (A_z)_{ij} \\ 0 & 0 & \end{bmatrix}$$

(54)

and

$$b_z = - \begin{bmatrix} 0 & 0 & (A_z)_{1i}(i = 3 \text{ to } n) \\ 0 & 0 & (A_z)_{2i}(i = 3 \text{ to } n) \\ (A_z)_{31} & (A_z)_{32} & \vdots \\ \vdots & \vdots & 0 \\ (A_z)_{n1} & (A_z)_{n2} & \end{bmatrix}$$

(55)

From the general considerations described in (24), it follows that $a_z$ transform as gauge fields under local $SO(2) \times SO(n-2)$ gauge transformation while $b_z$ transforms homogeneously. It can be verified that $a_z$ in (54) is indeed a non-Abelian gauge field when $n > 4$. The gauge connection $a_z$ contains contributions from both the tangent space and the normal frame to $M$ reflecting $SO(2) \times SO(n-2)$ group structure. The orthogonal compliment $b_z$ on the other hand receives contributions from interaction of tangents with the normals. As $b_z$ transforms homogeneously under the local $SO(2) \times SO(n-2)$ gauge transformations, it can be identified with the Higg’s field. Realizing that $A_z = -(a_z + b_z)$ and using (19) we immediately obtain (26) exploiting the group structure underlying (54) and (55). In order to prove that second equation in (26) gives the self-duality, namely the vanishing of both the sides, we make use of the Euler-Lagrange equation (51) or harmonic Gauss map. To prove the self-duality equation, namely, $\partial b_z + [a_z, b_z] = 0$, when
the Gauss map is harmonic, we proceed as below. We consider (1i) component of the self-duality equation for \( i \geq 3 \) which follows from (54) and (55).

\[
(\partial_z b_z + [a_z, b_z])_{1i} = \partial_z (b_z)_{1i} + (a_z)_{12} (b_z)_{2i} - (b_z)_{1j} (a_z)_{ji},
\]

\[
= -\partial_z (A_z)_{1i} + (A_z)_{12} (A_z)_{2i} - (A_z)_{1j} (A_z)_{ji}, \quad j \geq 3,
\]

(56)

where the structure of (54) and (55) have been used. Using the definition that \((A_z)_{1i} = N^\mu_i (\partial_z \hat{e}_1)\) and (16), we find,

\[
(\partial_z b_z + [a_z, b_z])_{1i} = (A_z)_{12} (A_z)_{2i} + (A_z)_{12} (A_z)_{2i} - N^\mu_i (\partial_z \partial_z \hat{e}_1).
\]

(57)

The expressions for \((A_z)_{12}\), \((A_z)_{2i}\), \((A_z)_{12}\) and \((A_z)_{2i}\) have been given in (53). The quantity \(N^\mu_i \partial_z \partial_z \hat{e}_1\) is calculated using (43) to be

\[
N^\mu_i \partial_z \partial_z \hat{e}_1 = \frac{4}{|\Phi|^4} (O_T)_{ik} [\partial_z (\frac{1}{\sqrt{2} |\Phi|})(A^\mu_k \partial_z \Phi^\mu + A^\mu_k \partial_z \Phi^\mu)]
\]

\[
+ \frac{1}{\sqrt{2} |\Phi|}(A^\mu_k \partial_z \Phi^\mu + A^\mu_k \partial_z \Phi^\mu)\]

\[
+ \frac{1}{\sqrt{2} |\Phi|}(A^\mu_k \partial_z \partial_z \hat{e}_1 + A^\mu_k \partial_z \partial_z \Phi^\mu)]
\]

(58)

Using the expressions for \(A^\mu_k\) in (46) and \(\Phi^\mu\) in (43), the above quantity has been evaluated using,

\[
A^\mu_k \partial_z \Phi^\mu = |\Phi|^2 \zeta_{kz},
\]

\[
A^\mu_k \partial_z \Phi^\mu = |\Phi|^2 \zeta_{kz},
\]

\[
A^\mu_k \partial_z \Phi^\mu = -4(\zeta_k + \zeta_k \zeta_k^2) (\zeta_j + \zeta_j \zeta_j^2) \zeta_{jz} + |\Phi|^2 (\zeta_{kz} + 2\zeta_k \zeta_m \zeta_{mz})
\]

\[
A^\mu_k \partial_z \Phi^\mu = -4(\zeta_k + \zeta_k \zeta_k^2) (\zeta_j + \zeta_j \zeta_j^2) \zeta_{jz} + |\Phi|^2 (\zeta_{kz} + 2\zeta_k \zeta_m \zeta_{mz})
\]

\[
A^\mu_k \partial_z \partial_z \Phi^\mu = 4(\zeta_k + \zeta_k \zeta_m) \zeta_{qz} \zeta_{qz} + |\Phi|^2 \zeta_{kz}
\]

\[
A^\mu_k \partial_z \partial_z \Phi^\mu = -4(\zeta_k \zeta_{mz}) [\zeta_{jz} \zeta_{jz} + \zeta_{jz} \zeta_{jz}]
\]

\[
+ |\Phi|^2 [2\zeta_k \zeta_{jz} \zeta_{jz} + 2\zeta_k \zeta_{jz} \zeta_{jz} + \zeta_{jz}]
\]

(59)
and the Euler-Lagrange equations of motion (51) (harmonic map requirement) for the $z\bar{z}$-derivatives of $\zeta$’s. We then find,

$$\left(\partial_{z} b_{z} + [a_{z}, b_{z}]\right)_{1i} = 0,$$

(60)

which is the required self-duality equation. Similarly the other components have been verified. This proves our main result that for immersed surfaces in $R^{n}$ for $n > 4$, the surface $M$ admits self-dual system when the Gauss map is harmonic. Explicit solutions to the self-dual equations for the gauge group $SO(2) \times SO(n - 2)$ are given by (54) and (55), where the complex functions $\zeta$’s satisfy the equation of motion (51).

VI. CONCLUSIONS

We have considered the string world sheet regarded as a Riemann surface immersed in $R^{n}$. The immersion is described by the Gauss map. For $n = 3$ and 4, when the Euler-Lagrange equations following from the extrinsic curvature action are satisfied, the Gauss map is harmonic and the mean curvature scalar of the immersed surface is constant. For such a class of surfaces, we have made use of the $SO(3)$ and $SO(4)$ two dimensional gauge fields constructed by us [4] and projected them onto the subgroup and its orthogonal complement in the Grassmannian $G_{2,3}$ and $G_{2,4}$. The projection into the subgroup transforms as a gauge field belonging to the subgroup $SO(2)$ and $SO(2) \times SO(2)$, while the complement transforms homogeneously. By identifying the complement with the complex Higgs field, we are able to prove the existence of solutions to Hitchin’s self-dual equation for the constant $h$ immersions in $R^{3}$ and $R^{4}$. This study compliments our earlier result that $h \sqrt{g} = 1$, surfaces exhibit Virasaro symmetry.

The self-dual system so obtained for harmonic maps is compared with the self-dual Chern-Simons system. A generalized Liouville equation involving extrinsic geometry is
obtained. As a particular case, when the map is anti-holomorphic the familiar Toda equations are obtained.

We have generalized the results to conformal immersion of 2-dimensional surfaces in $\mathbb{R}^n$, using the results of the generalized Gauss map. We prove that the surface has constant mean curvature when the Gauss map is harmonic. This harmonicity condition or Euler-Lagrange equation is used to show that for such surfaces, there exists Hitchin’s self-dual system. It is to be noted that the procedure to construct self-dual system for harmonic Gauss maps here is a variant of the one by Donaldson \[14\]. In particular, the gauge fields are constructed from the geometry of the surface itself (and so characteristic of the surface and not external) and the self-dual gauge fields $a_z$ and $a_{\bar{z}}$ belong to the gauge group $SO(2) \times SO(n-2)$ while the fields $b_z$ and $b_{\bar{z}}$ belong to the compliment in $G_{2,n}$, transform homogeneously under $SO(2) \times SO(n-2)$ gauge transformation.

The general action for the string theory will be a sum of the Nambu-Goto action and the action involving extrinsic geometry. When the theory is described in terms of the Gauss map, we have earlier \[2\] noted that both the actions can be expressed as Grassmannian sigma model. Explicitly,

$$S = S_{NG} + S_{Extrinsic}$$

$$= \sigma \int \sqrt{g} dz \wedge d\bar{z} + \frac{1}{\alpha_0} \int \sqrt{g}|H|^2 dz \wedge d\bar{z},$$

$$= \sigma \int \frac{1}{h^2} g_{ij} \zeta_i \zeta_j dz \wedge d\bar{z} + \frac{1}{\alpha_0} \int g_{ij} \zeta_i \zeta_j dz \wedge d\bar{z}. \quad (61)$$

In general the mean curvature scalar $h$ will be a (real) function of $(z, \bar{z})$ and so the study of the action $S$ will be complicated since we have a space-dependent coupling for $S_{NG}$ in this framework. When the Gauss map is harmonic (surfaces of constant mean scalar curvature), it is easy to see that the total action is just a Grassmannian sigma model with
one effective coupling constant. Since the classical equations of motion for this action are identical to (51), it is possible to study the quantization in the background field method.

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References

1. N.J.Hitchin, Proc.London.Math.Soc.(3)\textbf{55},59(1987).

2. K.S.Viswanathan,R.Parthasarathy and D.Kay,Ann.Phys.(N.Y)\textbf{206}237(1991).

3. D.A.Hoffman and R.Osserman,J.Diff.Geom.\textbf{18},733(1983);
Proc.London.Math.Soc.(3)\textbf{50},21(1985).

4. R.Parthasarathy and K.S.Viswanathan,Int.J.Mod.Phys.\textbf{A7},317(1992).

5. K.S.Viswanathan and R.Parthasarathy, Ann.Phys.(N.Y)(submitted).

6. E.A.Ruh and J.Vilms,Trans.Amer.Math.Soc.\textbf{149},569(1979).

7. G.V.Dunne,R.Jackiw,S-Y.Pi and C.A.Trugenberger, Phys.Rev.\textbf{D43},1332(1991).

8. G.V.Dunne, Comm.Math.Phys.\textbf{150},519(1992).
9. K.Kenmotsu, Math.Ann.245,89(1979).

10. R.Parthasarathy and K.S.Viswanathan, Int.J.Mod.Phys. A7,2819(1992).

11. D.A.Hoffman and R.Osserman, Mem.Amer.Math.Soc. No.236 (1980).

12. A.P.Balachandran, A.Stern and G.Trahern, Phys.Rev.D19,2416(1979).

13. K.Fujii, Lett.Math.Phys.25,203(1992).

14. S.K.Donaldson, Proc.London.Math.Soc.(3) 55, 127(1987).