Can quantum Rabi model with $A^2$-term avoid no-go theorem for spontaneous SUSY breaking?

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(Dated: October 21, 2022)

Abstract

The hierarchy problem asks why the mass of the Higgs particle is so much lighter than the Planck-scale mass. Considering the interaction of the Higgs particle and an elementary particle in the Planck-scale, to cope with that big difference, the conventional calculation needs the help of an arbitrary, excessive fine-tuning, that is, the huge cancellation between the bare mass term and the quantum correction, without obeying a physical principle such as symmetry. Thus, it is often said to be unnatural. On the other hand, the theory of supersymmetry (SUSY) is a strong candidate naturally to solve the hierarchy problem. However, any sign of SUSY even for the quantum mechanics (QM) version [1, 2] had not been firmly, directly observed in the physical reality until Cai et al. reported the observation of the $N = 2$ SUSY and its spontaneous breaking in a trapped ion quantum simulator [3] for the prototype model for SUSY QM [4]. In this discussion, I derive a no-go theorem for the spontaneous SUSY breaking in the strong coupling limit for the quantum Rabi model [5] with the $A^2$-term, and at the same time, I show another limit proposed in the scheme by Cai et al. [3] can avoid the no-go theorem and take that model from the $N = 2$ SUSY to its spontaneous breaking. I propose a theoretical method to observe how the effect of $A^2$-term appears in the spontaneous SUSY breaking.
In the discussion below, the annihilation and creation operators of a 1-mode boson are respectively denoted by \( a \) and \( a^\dagger \). The annihilation operator \( \sigma_- \) and the creation operator \( \sigma_+ \) of a 2-level atom (i.e., spin) are given by \( \sigma_\pm := \frac{1}{2}(\sigma_x \pm i\sigma_y) \). The standard notation \( \sigma_x \), \( \sigma_y \), and \( \sigma_z \) are used for the Pauli matrices. The Hamiltonian is defined by

\[
H(\omega_a, \omega_c, g, C) = \frac{\hbar \omega_a}{2} \sigma_z + \hbar \omega_c \left( a^\dagger a + \frac{1}{2} \right) + \hbar g \sigma_x (a + a^\dagger) + \hbar C g^2 (a + a^\dagger)^2 ,
\]

where the first and second terms of RHS of equation (1) are respectively the free energies of the 2-level atom and the 1-mode boson. The third term is the linear interaction between the atom and boson with the parameter \( g \) representing the coupling strength. The last term is the quadratic interaction \( \hbar C \{g\sigma_x(a + a^\dagger)\}^2 \) with the parameter \( C \) which controls the dimension and volume of the quadratic interaction energy. This is often called \( A^2 \)-term.

It is already proved that, tuning the parameters, \( \omega_a, \omega_c \), as \( \omega_a = \omega_c = \omega \) for a positive constant \( \omega \), the Hamiltonian \( H(\omega, \omega, 0, 0) \) without the \( A^2 \)-term has the \( N = 2 \) SUSY [4]. As the coupling strength \( g \) gets stronger enough, the \( A^2 \)-term may appear. Similarly to the case of the superradiant phase transition [6, 7], a no-go theorem caused by the \( A^2 \)-term [8] should be minded, and its avoidance should be argued [9] also for the prototype model for SUSY QM.

For every non-negative \( C \), there exists a unitary operator \( U_{A^2} \) such that

\[
U_{A^2} H(\omega_a, \omega_c, g, C) U_{A^2} = H(\omega_a, \omega(g), \tilde{g}, 0) \]

\[
= \frac{\hbar \omega_a}{2} \sigma_z + \hbar \omega(g) \left( a^\dagger a + \frac{1}{2} \right) + \hbar \tilde{g} \sigma_x (a + a^\dagger) ,
\]

where \( \omega(g) = \sqrt{\omega_a^2 + 4C\omega_c g^2} \) and \( \tilde{g} = g \sqrt{\omega_c / \omega(g)} \). This unitary operator \( U_{A^2} \) is obtained in the meson pair theory of nuclear physics [10, 11]. Equation (2) is often called the Hopfield-Bogoliubov transformation of \( H(\omega_a, \omega_c, g, C) \). The effect of the \( A^2 \)-term is stuffed into \( \omega(g) \) and \( \tilde{g} \). We note that, for \( C = 0 \), the parameters satisfy \( \omega(g) = \omega_c, \tilde{g} = g \), and then, the unitary operator is \( U_{A^2} = I \) (i.e., the identity operator).

For the displacement operator \( D(g/\omega_c) := \exp \left[ g(a^\dagger - a)/\omega_c \right] \), a unitary operator is defined by \( U(g/\omega_c) := \frac{1}{\sqrt{2}} \{(\sigma_- - 1)\sigma_+ D(g/\omega_c) + (\sigma_+ + 1)\sigma_- D(-g/\omega_c)\} \). Then, it makes the equation,

\[
U(g/\omega_c)^* \left\{ H(\omega_a, \omega_c, g, 0) + \frac{\hbar g^2}{\omega_c} \right\} U(g/\omega_c)
\]

\[
= \hbar \omega_c \left( a^\dagger a + \frac{1}{2} \right) - \frac{\hbar \omega_a}{2} \left\{ \sigma_+ D(g/\omega_c)^2 + \sigma_- D(-g/\omega_c)^2 \right\} .
\]
Since the arguments on the limit of Hamiltonians used below are already established as a mathematical method \[4, 10, 11\]. Thus, for simplicity, mathematically naive arguments are made in this discussion to explain the no-go theorem and its avoidance.

Now we consider the strong coupling limit for the quantum Rabi model with the \(A^2\)-term. For instance, the strong coupling limit is experimentally realized for the quantum Rabi model in circuit QED \[12\] as deep-strong coupling regime \[13\]. The parameters, \(\omega_a, \omega_c, g\), are set as \(\omega_a = \omega_c = \omega\) and \(g = g\) for a non-negative parameter \(g\). The Hamiltonian \(H(\omega, \omega, g, 0)\) is for the quantum Rabi model, and denoted by \(H_{\text{Rabi}}(g)\) for simplicity. In the renormalization for \(A^2\)-term, the quantities, defined by \(\omega(g) := \sqrt{\omega^2 + 4C\omega g^2}\) and \(\tilde{g} := g\sqrt{\omega^2/\omega(g)}\), are used.

In case \(C = 0\), according to the mathematical results \[4, 10\], the approximation,

\[
H_{\text{Rabi}}(g) + \hbar \frac{g^2}{\omega} \approx U(g/\omega) \left[ \hbar \omega \left( a^+a + \frac{1}{2} \right) \right] U(g/\omega)^*, \quad (4)
\]

is obtained as \(g \to \infty\). Since the mechanism for spontaneous SUSY breaking for the quantum Rabi model \[4\] works by the appearance of only the boson free energy in RHS of equation (4), the \(N = 2\) SUSY of \(H_{\text{Rabi}}(g)\) is spontaneously broken in the strong coupling limit \(g \to \infty\) (Fig.1a). Naively, how to obtain equation (4) is explained in the following. Due to the violent vibrations in the displacement operator \(D(\pm g/\omega)\) as \(g/\omega\) grows larger, the displacement operator decays. Thus, the second term of RHS of equation (3) disappears in the limit \(g \to \infty\).

In the case \(C > 0\), on the other hand, the mathematical results \[11\] says that

\[
H_{\text{Rabi}}(g) + \hbar C g^2 \left( a + a^+ \right)^2 + \hbar \tilde{g}^2 \frac{\omega}{\omega(g)} \\
\approx U_{A^2}U(\tilde{g}/\omega(g)) \left[ \hbar \omega(g) \left( a^+a + \frac{1}{2} \right) - \frac{\hbar \omega}{2} \sigma_x \right] U(\tilde{g}/\omega(g))^*U_{A^2}^* \quad (5)
\]

as \(g \to \infty\). The atomic term \(\hbar \omega \sigma_x/2\) appears in addition to the boson free energy in RHS of equation (5), and then, this appearance interferes with the conversion to the spontaneous SUSY breaking, and moreover, the divergence of \(\omega(g)\), together with the atomic term, rudely crushes that SUSY (Fig.1b). Thus, the above quantum Rabi model with the \(A^2\)-term cannot go to its spontaneous breaking as \(g\) changes from \(g = 0\) to \(g \approx \infty\). This is the no-go theorem for the spontaneous SUSY breaking in the strong coupling limit caused by the \(A^2\)-term. The reason why \(\hbar \omega \sigma_x/2\) appears in RHS of equation (5) is naively explained as follows: Since \(\lim_{g \to \infty} \tilde{g}/\omega(g) = 0\) for \(C > 0\) though \(\lim_{g \to \infty} \tilde{g}/\omega(g) = \lim_{g \to \infty} g/\omega = \infty\) for \(C = 0\), the
displacement operator \( D(\pm \tilde{g}/\omega(g)) \) in equation (3) remains as the identity operator \( I \) in the limit \( g \to \infty \).

The approximations given by equations (4) and (5) are mathematically established in the norm resolvent sense, and the limit is valid over the energy spectrum \([14]\). Thus, the limit energy spectrum is obtained by those approximations. Whether \( N = 2 \) SUSY of \( H(\omega, \omega, 0, 0) \) is taken to its spontaneous breaking is checked by the energy degeneracy between the bosonic and the fermionic states. The energy spectrum by the numerical analysis with QuTiP \([15]\) is obtained, for instance, as in Fig. 1.

(a) Energy Spectrum: \( C = 0 \)
(b) Energy Spectrum: \( C = 0.37699111843077515 \)

FIG. 1. Energy Spectrum of \( H_{\text{Rabi}}(g) + \hbar C g^2 (a + a^\dagger)^2 \) with \( \omega = 6.2832 \): a shows the energy spectrum for \( C = 0 \). The right graph is for \( C = 0.3770 \). The quantum Rabi model (without \( A^2 \)-term) has the transition from the \( N = 2 \) SUSY to its spontaneous breaking. On the other hand, b shows the loss of the spontaneous breaking. Here, it should be noted \( \lim_{g \to \infty} \hbar \omega(g) = \infty \) and \( \lim_{g \to \infty} \hbar \tilde{g}^2/\omega(g) = \hbar/(4C) \).

As stated above, the Rabi model with the \( A^2 \)-term is faced with the no-go theorem in the strong coupling limit, which is caused by the effect coming from the \( A^2 \)-term. Indeed the no-go theorem appears in the strong coupling limit, but there is another limit in the scheme which is proposed by Cai et al. \([3]\). From now on, it is proved that the limit has the advantage over the strong coupling limit in order that the Rabi model with the \( A^2 \)-term avoid the no-go theorem and has the spontaneous SUSY breaking.

Let \( \omega[r] \) be a continuous function of variable \( r \), \( 0 \leq r \leq 1 \), satisfying \( \omega[0] = \omega \) and \( \omega[1] = 0 \). The parameters \( \omega_a, \omega_a, g \) are given by \( \omega_a = \omega[r], \omega_c = \omega, g = rg_0 \) for a positive constant \( g_0 \). The Hamiltonian \( H(\omega[r], \omega, rg_0, 0) \) for the quantum Rabi model is denoted by \( H_{\text{Rabi}}[r] \) for simplicity. The quantities are given by \( \tilde{\omega}[r] := \sqrt{\omega^2 + 4C\omega r^2 g_0^2} \) and
\( \bar{g}[r] := r g_0 \sqrt{\omega/\bar{\omega}[r]} \). Then, the same argument as in \([4, 10, 11]\) gives

\[
H_{\text{Rabi}}[r] + \hbar C r^2 g_0^2 (a + a^\dagger)^2 + \hbar \frac{\bar{g}[r]^2}{\bar{\omega}[r]}
\]

\[
\rightarrow U_{A^2} U(\bar{g}[1]/\bar{\omega}[1]) \left[ \hbar \bar{\omega}[1] \left( a^\dagger a + \frac{1}{2} \right) \right] U(\bar{g}[1]/\bar{\omega}[1])^* U_{A^2}^*
\]

in the norm resolvent sense \([14]\) as \( r \rightarrow 1 \). The naive reason why this limit is obtained is because the limit \( \omega [r] \rightarrow \omega [1] = 0 \) eliminates the second term of RHS of equation (3).

Equation (6) says that the mechanism for spontaneous SUSY breaking for the quantum Rabi model \([4]\) works, and therefore, the Rabi model with \( A^2 \)-term, described by \( H_{\text{Rabi}}[r] + \hbar C r^2 g_0^2 (a + a^\dagger)^2 + \hbar \bar{g}[r]^2/\bar{\omega}[r] \), yields the spontaneous SUSY breaking in the limit \( r \rightarrow 1 \).

The Lagrangian for this Hamiltonian obtains the extra second-order field, \(- (m[r]^2/(2\hbar^2)) \phi^2\), from the \( A^2 \)-term, where \( m[r] \) is a mass and \( \phi = \sqrt{\hbar/(2\omega)}(a + a^\dagger) \) the 1-mode Bose field. We have \( m[r]^2 = 4\hbar^2 C \omega r^2 g_0^2 \) then. The limit in the norm resolvent sense guarantees the convergence of each energy level \([14]\). Thus, it is worthy to note that the energy gap is produced in the process from the \( N = 2 \) SUSY to its spontaneous breaking. The energy gap is governed by the parameter \( C \) of the \( A^2 \)-term. Whether \( N = 2 \) SUSY of \( H(\omega, \omega, 0, 0) \) goes to its spontaneous breaking can be shown by the energy degeneracy between the bosonic and the fermionic states. The energy spectrum is checked with QuTiP \([15]\), for instance, as in Fig.2. In particular, the comparison of Fig.2a and Fig.2b shows the energy gap caused by the \( A^2 \)-term.
FIG. 2. Energy Spectrum of $H_{\text{Rabi}}[r] + \hbar C r^2 g_0^2 (a + a^\dagger)^2$ with $\omega = 6.2832$ and $g_0 = 6.2832$: a shows the energy spectrum for $C = 0$. b is for $C = 0.3770$. The quantum Rabi models without $A^2$-term (i.e., $C = 0$) and with $A^2$-term (i.e., $C > 0$) have the transition from the $N = 2$ SUSY to its spontaneous breaking. In particular, the energy gap by the $A^2$-term appears in $\hbar \tilde{\omega}[1]$ of the right graph.

ACKNOWLEDGMENTS

The author acknowledges the support from JSPS Grant-in-Aid for Scientific Researchers (C) 20K03768. He would like to dedicate this study to Hiroshi Ezawa and Elliott H. Lieb on the occasions of their 90th birthdays.

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