On cross-diffusion systems for two populations subject to a common congestion effect

Maxime Laborde *

Abstract

In this paper, we investigate the existence of solutions for systems of Fokker-Planck equations coupled through a common nonlinear congestion. Two different kinds of congestion are considered: a porous media congestion or soft congestion and the hard congestion given by the constraint $\rho_1 + \rho_2 \leq 1$. We show that these systems can be seen as gradient flows in a Wasserstein product space and then we obtain a constructive method to prove the existence of solutions. Therefore it is natural to apply it for numerical purposes and some numerical simulations are included.

Keywords: Wasserstein gradient flows, Jordan-Kinderlehrer-Otto scheme, crowd motion, nonlinear cross-diffusion systems.

1 Introduction

The modelling of crowd behaviour has become a very active field of applied mathematics in recent years. These models permit to understand many phenomena such as cell migration, tumor growth, etc. Several models already exist to tackle this problem. The first one, microscopic, consists in seeing a population as a high number of individuals which satisfy ODEs, see for instance [35] and the second is macroscopic and consists in describing a population by a density $\rho$ satisfying one PDE, where $\rho(t,x)$ represents the number of individuals in $x$ at time $t$. In the latter framework, different methods to handle the congestion effect have been proposed. The first one consists in saying that the motion has to be slower when the density is very high, see for example [14, 13, 12] for a different approach with applications to crowd dynamics. Another way of modelling the congestion effect is to use a threshold: the density evolves as we would expect until it touches a maximal level and then the motion has to be adapted in these regions, see for example [32] for crowd motion model and [33] for application to dendritic growth. For a comparison between microscopic and macroscopic models, we refer to [34]. In [37], Mészáros and Santambrogio proposed a model in hard congestion where individuals are subject to a Brownian diffusion. This corresponds to modified Fokker-Planck equation with a constraint on the density.

Since in macroscopic models, we have mass conservation, the theory of optimal transportation is a very natural tool to attack it. In [32], the authors investigated a model of room evacuation. They showed that if the velocity field is given by a gradient, say $V = \nabla D$, where $D$ is the distance to a given target, then the problem has a gradient flow structure in the Wasserstein space and the velocity field has to be adapted by a pressure field to handle congestion effect. More recently in [37], a splitting scheme has been introduced to handle velocity fields which are -not necessarily potential- vector field. The scheme consists in combining steps where the density follows Fokker-Planck equation and Wasserstein projections over the set of densities which cannot exceed 1.

A natural variant of the model of [37], consists in considering two (or more) populations, each of whom having its own potential but coupled through the constraint that the total density cannot exceed 1 and then subject to a common pressure field. Note that variant problems with total density equal to 1 are treated in [15, 11, 5, 9] and for more general cross-diffusion systems, we refer,
estimates on the gradient of ingredient is the flow interchange argument (see [31, 20, 27] for example) which gives separated our main results. In section 3, we prove existence of weak solution for system (1.2). The key
We give numerical simulations implemented by the augmented Lagrangian scheme introduced in [6].

is to handle the cross diffusive term which needs to have strong compactness in and ordered drifts. In any dimension, they prove existence of very weak solutions. The difficulty
ual diffusions. They prove existence of weak solution in dimension
Laurençot and Matioc give a similar result in
Here, this difficulty is overcome by assuming that individuals of each populations are subject to

This paper is organized as follows. In section 2, we introduce our assumptions and we state our main results. In section 3, we prove existence of weak solution for system (1.2). The key ingredient is the flow interchange argument (see [31, 20, 27] for example) which gives separated estimates on the gradient of \( \rho_1 + \rho_2 \) and on the gradient of \( \rho_i \). Section 4 provides the proof of
existence of weak solution for system with hard congestion \[1.1\]. In this section we use again the flow interchange argument to obtain stronger estimates. In section \[4\] we focus on the particular case where \(\nabla V_1 = \nabla V_2\). In this case, we are able to show the convergence when \(m \to +\infty\) of a solution to \[1.2\] to a solution to \[1.1\] and we prove a \(L^1\)-contraction theorem. In the final section \[6\] numerical simulations are presented.

## 2 Preliminaries and main results

Throughout the paper, \(\Omega\) is a smooth convex bounded subset of \(\mathbb{R}^n\). We start to recall some results from the optimal transportation theory and then we will state our main results.

### 2.1 Wasserstein space

For a detailed exposition, we refer to reference textbooks \[11, 12, 3, 40\]. We denote \(\mathcal{M}^+(\Omega)\) the set of nonnegative finite Radon measures on \(\Omega\), \(\mathcal{P}(\Omega)\) the space of probability measures on \(\Omega\), and \(\mathcal{P}^{ac}(\Omega)\), the subset of \(\mathcal{P}(\Omega)\) of probability measures on \(\Omega\) absolutely continuous with respect to the Lebesgue measure.

For all \(\rho, \mu \in \mathcal{P}(\Omega)\), we denote \(\Pi(\rho, \mu)\), the set of probability measures on \(\Omega \times \Omega\) having \(\rho\) and \(\mu\) as first and second marginals, respectively, and an element of \(\Pi(\rho, \mu)\) is called a transport plan between \(\rho\) and \(\mu\). Then for all \(\rho, \mu \in \mathcal{P}(\Omega)\), we denote by \(W_2(\rho, \mu)\) the Wasserstein distance between \(\rho\) and \(\mu\),

\[
W_2^2(\rho, \mu) = \min \left\{ \int_{\Omega \times \Omega} |x - y|^2 \, d\gamma(x, y) : \gamma \in \Pi(\rho, \mu) \right\}.
\]

Since this optimal transportation problem is a linear problem under linear constraints, it admits a dual formulation given by

\[
W_2^2(\rho, \mu) = \sup \left\{ \int_{\Omega} \varphi(x) \, d\rho(x) + \int_{\Omega} \psi(y) \, d\mu(y) : \varphi(x) + \psi(y) \leq |x - y|^2 \right\}.
\]

Optimal solutions to the dual problem are called Kantorovich potentials between \(\rho\) and \(\mu\). If \(\rho \in \mathcal{P}^{ac}(\Omega)\), a well-known result proved by Brenier, \[7\], states that the optimal transport plan, \(\gamma\), is unique and is induced by an optimal transport map, \(T\), i.e \(\gamma\) is of the form \((\text{Id} \times T)\# \rho\), where \(T\# \rho = \mu\) and \(T\) is the gradient of a convex function. Moreover, the optimal transport map is given by \(T = \text{Id} - \nabla \varphi\) where \(\varphi\) is the Kantorovich potential between \(\rho\) and \(\mu\).

It is well known that \(\mathcal{P}(\Omega)\) endowed with the Wasserstein distance defines a metric space and \(W_2\) metrizes the narrow convergence of probability measures.

### 2.2 Assumptions and main results

For \(i \in \{1, 2\}\), we define \(V_i : \mathcal{P}(\Omega) \to \mathbb{R}\) the potential energy associated to \(V_i \in W^{1,\infty}(\Omega)\) as

\[
V_i(\rho) := \int_{\Omega} V_i(x) \, d\rho(x).
\]

We introduce the Entropy \(\mathcal{H}\) defined, for all probability measures \(\rho\), as

\[
\mathcal{H}(\rho) := \begin{cases} 
\int_{\Omega} H(\rho(x)) \, dx & \text{if } \rho \ll \mathcal{L}_{|\Omega}, \\
+\infty & \text{otherwise},
\end{cases} \quad H(z) := z \log(z) \text{ for all } z \in \mathbb{R}^+.
\]

Finally, for \(m \in [1, +\infty)\), we define \(F_m : \mathcal{M}^+(\Omega) \to \mathbb{R}\) as

\[
F_m(\rho) := \begin{cases} 
\int_{\Omega} F_m(\rho(x)) \, dx & \text{if } \rho \ll \mathcal{L}_{|\Omega}, \\
+\infty & \text{otherwise},
\end{cases} \quad F_m(z) := \begin{cases} 
\frac{z \log z}{m-1} & \text{if } m = 1, \\
\frac{z^m}{m-1} & \text{if } m > 1.
\end{cases}
\]

and, for \(m = +\infty\), \(F_\infty : \mathcal{M}^+(\Omega) \to \mathbb{R}\) is defined by

\[
F_\infty(\rho) := \begin{cases} 
0 & \text{if } \|\rho\|_{\infty} \leq 1, \\
+\infty & \text{otherwise}.
\end{cases}
\]
Definition 2.1.

1. We say that \((p_1, p_2) : [0, +\infty) \to \mathcal{P}_{ac}(\Omega)^2\) is a weak solution to (1.1) if for all \(i \in \{1, 2\}\) and for all \(T < +\infty\), \(p_i \in C^{0,1/2}([0, T], \mathcal{P}_{ac}(\Omega))\), \(p_i \in L^\infty([0, T], W^{1,1}(\Omega))\), \(\rho_i \partial_t F_m(p_1 + p_2) \in L^1((0, T) \times \Omega)\) and for all \(\phi \in C^\infty_0([0, +\infty) \times \mathbb{R}^n)\),

\[
\int_0^{+\infty} \int_\Omega [p_i \partial_t \phi - (\rho_i \nabla V_i + \rho_i \nabla F_m'(p_1 + p_2) + \nabla \rho_i) \cdot \nabla \phi] \, dx \, dt = -\int_\Omega \phi(0, x) \rho_{i,0}(x) \, dx.
\]

2. We say that \((p_1, p_2, p) : [0, +\infty) \to \mathcal{P}_{ac}(\Omega)^2 \times H^1(\Omega)\) is a weak solution to (1.2) if for all \(i \in \{1, 2\}\) and for all \(T < +\infty\), \(p_i \in C^{0,1/2}([0, T], \mathcal{P}_{ac}(\Omega))\), \(p_i \in L^\infty([0, T], W^{1,1}(\Omega))\), \(p \in L^2([0, T], H^1(\Omega))\) with \(p \geq 0\), \(p_1 + p_2 \leq 1\) and \(p(1 - p_1 - p_2) = 0\) a.e. In addition, for all \(\phi \in C^\infty_0([0, +\infty) \times \mathbb{R}^n)\),

\[
\int_0^{+\infty} \int_\Omega [p_i \partial_t \phi - (\rho_i \nabla V_i + \rho_i \nabla p + \nabla \rho_i) \cdot \nabla \phi] \, dx \, dt = -\int_\Omega \phi(0, x) \rho_{i,0}(x) \, dx.
\]

The main results of this paper are

**Theorem 2.2.** Assume that \(\rho_{1,0}, \rho_{2,0} \in \mathcal{P}_{ac}(\Omega)\) satisfy

\[
\mathcal{H}(\rho_{1,0}) + \mathcal{H}(\rho_{2,0}) + \mathcal{F}_m(\rho_{1,0} + \rho_{2,0}) < +\infty,
\]

then (1.2) admits at least one weak solution.

and

**Theorem 2.3.** Assume that \(|\Omega| > 2\). If \((\rho_{1,0}, \rho_{2,0}) \in \mathcal{K} := \{(\rho_1, \rho_2) \in \mathcal{P}_{ac}(\Omega)^2 : \rho_1 + \rho_2 \leq 1\}\) satisfies

\[
\mathcal{H}(\rho_{1,0}) + \mathcal{H}(\rho_{2,0}) < +\infty,
\]

then there exists at least one weak solution to (1.1).

The assumption on \(|\Omega|\) is to ensure that \(\mathcal{K}\) is not empty or trivial.

**Remarks on possible extensions:**

1. These models can be generalized to more than two species. Moreover, instead of assuming that individuals of different populations take the same space, we can generalize to densities evolving under the constraints on \(\alpha_1 \rho_1 + \alpha_2 \rho_2\). Then system (1.2) becomes

\[
\partial_t \rho_i = \text{div}(\rho_i \nabla V_i) + \Delta \rho_i + \alpha_i \text{div}(\rho_i \nabla F_m'(\alpha_1 \rho_1 + \alpha_2 \rho_2)), \quad i = 1, 2,
\]

and system with hard congestion becomes

\[
\begin{cases}
\partial_t \rho_1 - \Delta \rho_1 - \text{div}(\rho_1 (\nabla V_1 + \nabla \rho)) = 0, \\
\partial_t \rho_2 - \Delta \rho_2 - \text{div}(\rho_2 (\nabla V_2 + \nabla \rho)) = 0,
\end{cases}
\]

\[
p \geq 0, \alpha_1 \rho_1 + \alpha_2 \rho_2 \leq 1, \quad p(1 - \alpha_1 \rho_1 - \alpha_2 \rho_2) = 0.
\]

2. These results can be generalized to more general velocities. Indeed, using the semi-implicit scheme introduced by DiFrancesco and Fagioli in [19] and developed in [27], or the splitting method introduced in [10], we can treat vector fields depending on the densities and which come not necessarily from a potential. These extensions allow to treat nonlocal interactions between different species, of the form \(V_i[p_1, p_2] = K_{i,1} \ast \rho_1 + K_{i,2} \ast \rho_2\) where \(K_{i,j} \in W^{1,\infty}\), which are subject to a common congestion effect.
3 Coupling through common soft congestion

In this section, we prove theorem 2.2 using the implicit JKO scheme, firstly introduced by Jordan, Kinderlherer and Otto in [22]. Given a time step $h > 0$, we construct by induction two sequences $\rho_{1,h}^k$ and $\rho_{2,h}^k$ with the following scheme: $\rho_{1,h}^0 = \rho_{1,0}$ and for all $k \geq 0$,

$$(\rho_{1,h}^{k+1}, \rho_{2,h}^{k+1}) \in \arg\min_{(\rho_1, \rho_2) \in \mathcal{P}^ac(\Omega)^2} \left\{ \sum_{i=1}^2 \left( W_2^2(\rho_i, \rho_{i,h}^k) + 2h (\mathcal{H}(\rho_i) + V_i(\rho_i)) + 2h \mathcal{F}_m(\rho_1 + \rho_2) \right) \right\}. \quad (3.1)$$

These sequences are well-defined by compactness and l.s.c standard argument. Then we define the piecewise constant interpolations $\rho_{i,h}^k : \mathbb{R}^+ \to \mathcal{P}^{ac}(\Omega)$ by

$$\rho_{i,h}(t) := \rho_{i,h}^{k+1}, \quad \text{if } t \in (kh, (k+1)h].$$

In the first part of this section, we study the convergence of these sequences and then we give the proof of theorem 2.2.

3.1 Estimates and convergences

We start retrieving classical estimates coming from the JKO scheme, [22], and then, we develop stronger estimates using the flow interchange argument, [31] [20]. First, the minimization scheme gives

**Proposition 3.1.** For all $T < +\infty$ and for all $i \in \{1, 2\}$, there exists a constant $C < +\infty$ such that for all $k \in \mathbb{N}$ and for all $h$ with $kh \leq T$ and let $N = \lceil \frac{T}{h} \rceil$, we have

$$\mathcal{H}(\rho_{i,h}^k) \leq C, \quad (3.2)$$

$$\mathcal{F}_m(\rho_{1,h}^k + \rho_{2,h}^k) \leq C, \quad (3.3)$$

$$\sum_{k=0}^{N-1} W_2^2(\rho_{1,h}^k, \rho_{i,h}^{k+1}) \leq Ch. \quad (3.4)$$

**Proof.** These results are obtained easily taking $\rho_i = \rho_{i,h}^k$ as competitors in [3.1], see [22].

**Remark 3.2.** Notice that estimate (3.4) does not depend on $m$. This remark will be useful in section 5 to show that a solution to (1.2) converges to a solution to (1.1).

In the next proposition, stronger estimates are obtained in order to pass to the limit in the nonlinear diffusive term.

**Proposition 3.3.** For all $T > 0$, there exists a constant $C_T > 0$ such that,

$$\|\rho_{1,h}^{1/2}\|_{L^2((0,T),H^1(\Omega))} + \|\rho_{2,h}^{1/2}\|_{L^2((0,T),H^1(\Omega))} + \|((\rho_{1,h}^1 + \rho_{2,h}^1)^{m/2}\|_{L^2((0,T),H^1(\Omega))} \leq C_T. \quad (3.5)$$

**Proof.** We use the flow interchange argument, introduced in [31], to find a stronger estimate as in [20] [27]. In other words, we perturb $\rho_{1,h}^k$ and $\rho_{2,h}^k$ by the heat flow. Let $\eta_i$ be the solution to

$$\begin{cases}
\partial^+_t \eta_i = \Delta \eta_i & \text{in } (0,T) \times \Omega, \\
\nabla \eta_i \cdot \nu = 0 & \text{in } (0,T) \times \partial \Omega, \\
\eta_i|_{t=0} = \rho_{i,h}^k.
\end{cases} \quad (3.6)$$

Since the entropy is geodesically convex then the heat flow is a 0-flow of the Entropy $\mathcal{H}$, and satisfies the Evolution Variational Inequality, [22] [31] [16] [10],

$$\frac{1}{2} \frac{d^+}{ds}|_{s=s} W_2^2(\eta_i(s), \rho) \leq \mathcal{H}(\rho) - \mathcal{H}(\eta_i(s)), \quad (3.7)$$

for all $s > 0$ and $\rho \in \mathcal{P}_2^{ac}(\Omega)$, where

$$\frac{d^+}{dt} f(t) := \lim_{s \to 0^+} \frac{f(t + s) - f(t)}{s},$$

with

$$d^+ f(t) = \frac{f(t^+)}{t^+}.$$
Moreover, using the scheme $\text{(3.1)}$, we get
\[
\sum_{i=1}^{2} \frac{1}{2} \frac{d^+}{d\tau} |_{\sigma=s} W^2_{\rho}(\eta_i(s), \rho_{i,h}^{k-1}) \geq -2h \frac{d^+}{d\sigma} \left( \sum_{i=1}^{2} \frac{1}{2} (\mathcal{H}(\eta_i(s)) + \mathcal{V}_i(\eta_i(s))) + \mathcal{F}_m(\eta_1(s) + \eta_2(s)) \right) \tag{3.8}
\]
Since $\eta_i(s)$ is a smooth positive function for $s > 0$, the following computations are justified
\[
\partial_s \left( \sum_{i=1}^{2} \frac{1}{2} (\mathcal{H}(\eta_i(s)) + \mathcal{V}_i(\eta_i(s))) + \mathcal{F}_m(\eta_1(s) + \eta_2(s)) \right)
= \sum_{i=1}^{2} \left( \int_{\Omega} \Delta \eta_i(s)((1 + \log(\eta_i(s))) + V_i) + \int_{\Omega} \Delta(\eta_1(s) + \eta_2(s)) F_m'(\eta_1(s) + \eta_2(s)) \right) \tag{3.9}
= \sum_{i=1}^{2} \left( -\int_{\Omega} \frac{|\nabla \eta_i(s)|^2}{\eta_i(s)} + \int_{\Omega} \nabla V_i \cdot \nabla \eta_i(s) - \int_{\Omega} |\nabla(\eta_1(s) + \eta_2(s))|^2 F_m'(\eta_1(s) + \eta_2(s)) \right).
\]
In addition, Young’s inequality gives
\[
-\int_{\Omega} \nabla V_i(s) \cdot \nabla \eta_i \leq \int_{\Omega} |\nabla V_i||\nabla \eta_i(s)| \leq \frac{1}{2} \int_{\Omega} |\nabla V_i|^2 \eta_i(s) + \frac{1}{2} \int_{\Omega} |\nabla \eta_i(s)|^2
\]
Then, we have
\[
\partial_s \left( \sum_{i=1}^{2} \frac{1}{2} (\mathcal{H}(\eta_i(s)) + \mathcal{V}_i(\eta_i(s))) + \mathcal{F}_m(\eta_1(s) + \eta_2(s)) \right)
= \sum_{i=1}^{2} \left( \frac{1}{2} \int_{\Omega} |\nabla \eta_i(s)|^2 \eta_i(s) + \int_{\Omega} |\nabla V_i|^2 \eta_i(s) - \int_{\Omega} |\nabla(\eta_1(s) + \eta_2(s))|^2 F_m'(\eta_1(s) + \eta_2(s)) \right).
\]
By definition of $F_m$, for $m \geq 1$, $F_m''(z) = mz^{m-2}$ for all $z > 0$. And since $V_i \in W^{1,\infty}(\Omega),
\partial_s \left( \sum_{i=1}^{2} \frac{1}{2} (\mathcal{H}(\eta_i(s)) + \mathcal{V}_i(\eta_i(s))) + \mathcal{F}_m(\eta_1(s) + \eta_2(s)) \right)
\leq C - \frac{1}{2} \sum_{i=1}^{2} \int_{\Omega} |\nabla \eta_i(s)|^{1/2}|^2 - \frac{4}{m} \int_{\Omega} |\nabla(\eta_1(s) + \eta_2(s))|^{m/2}|^2.
\]
By $\text{(3.7)}$ and a lower semi-continuity argument,
\[
h \sum_{i=1}^{2} \int_{\Omega} |\nabla(\rho_{1,h}^k)|^{1/2}|^2 + \frac{4h}{m} \int_{\Omega} |\nabla(\rho_{1,h}^k + \rho_{2,h}^k)|^{m/2}|^2 \leq \sum_{i=1}^{2} \left( \mathcal{H}(\rho_{i,h}^{k-1}) - \mathcal{H}(\rho_{i,h}^k) \right) + C_h.
\]
Then summing over $k$, we obtain
\[
\|\rho_{1,h}^{1/2}\|_{L^2((0,T),H^1(\Omega))} + \|\rho_{2,h}^{1/2}\|_{L^2((0,T),H^1(\Omega))} + \|\rho_{1,h} + \rho_{2,h}\|^{m/2}_{L^2((0,T),H^1(\Omega))} \leq C_T.
\]

**Remark 3.4.** The bound on $\|\rho_{1,h}^{1/2}\|_{L^2((0,T),H^1(\Omega))}$ does not depend on $m$. However, if we multiply the Entropy $\mathcal{H}$ by a small parameter $\varepsilon > 0$ in the JKO scheme $\text{(3.1)}$, individual bounds blow up as $\varepsilon$ goes to 0.
Now we can deduce the following convergences.

**Proposition 3.5.** For all $T < +\infty$, there exist $\rho_1$ and $\rho_2$ in $C^{0,1/2}([0,T], \mathcal{P}^{ac}(\Omega))$ such that, up to a subsequence,

1. $\rho_{i,h}$ converges to $\rho_i$ in $L^\infty([0,T], \mathcal{P}^{ac}(\Omega))$,
2. $\rho_{i,h}$ converges strongly to $\rho_i$ in $L^1((0,T) \times \Omega)$,
3. $(\rho_{1,h} + \rho_{2,h})^{m/2}$ converges strongly to $(\rho_1 + \rho_2)^{m/2}$ and $\nabla(\rho_{1,h} + \rho_{2,h})^{m/2}$ converges weakly to $\nabla(\rho_1 + \rho_2)^{m/2}$ in $L^2((0,T) \times \Omega)$.

**Proof.**

1. The first convergence is classical. We use the refined version of Ascoli-Arzelà’s theorem, Proposition 3.3.1, and we immediately deduce that $\rho_{i,h}$ converges to $\rho_i \in C^{1/2}([0,T], \mathcal{P}^{ac}(\Omega))$ in $L^\infty([0,T], \mathcal{P}^{ac}(\Omega))$.

The next two strong convergence results are obtained applying an extension of the Aubin-Lions lemma proved by Rossi and Savaré in Theorem 2.

2. Let $\mathcal{G} : L^1(\Omega) \to (-\infty, +\infty]$ and $g : L^1(\Omega) \times L^1(\Omega) \to [0, +\infty]$ defined by

\[ G(\rho) := \begin{cases} \|\rho^{1/2}\|_{H^1(\Omega)} & \text{if } \rho \in \mathcal{P}^{ac}(\Omega) \text{ and } \rho^{1/2} \in H^1(\Omega) \\ +\infty & \text{otherwise}, \end{cases} \]

and

\[ g(\rho,\mu) := \begin{cases} W_2(\rho,\mu) & \text{if } \rho,\mu \in \mathcal{P}(\Omega) \\ +\infty & \text{otherwise}, \end{cases} \]

$\mathcal{G}$ is l.s.c and its sublevels are relatively compact in $L^1(\Omega)$ (see [20, 27]) and $g$ is a pseudo-distance. According to (3.4) and (3.5), we have

\[ \sup_{h \leq 1} \int_0^T G(\rho_{i,h}(t)) \, dt < +\infty, \text{ and } \limsup_{\tau \nearrow 0} \sup_{h \leq 1} \int_0^{T-\tau} g(\rho_{i,h}(t+\tau),\rho_{i,h}(t)) \, dt = 0, \]

then applying Rossi-Savaré’s theorem, there exists a subsequence, not-relabeled, such that for $i = 1, 2$, $\rho_{i,h}$ converges in measure with respect to $t$ in $L^1(\Omega)$ to $\rho_i$. Moreover by Lebesgue’s dominated convergence theorem, $\rho_{i,h}$ converges to $\rho_i$ strongly in $L^1((0,T) \times \Omega)$.

3. With the same argument, we get a strong convergence on a nonlinear quantity of $\rho_{1,h} + \rho_{2,h}$. Let $\mathcal{G}$ define by

\[ G(\rho) := \begin{cases} \|\rho^{m/2}\|_{H^1(\Omega)} & \text{if } \rho \in \mathcal{P}^{ac}(\Omega) \text{ and } \rho^{m/2} \in H^1(\Omega) \\ +\infty & \text{otherwise}, \end{cases} \]

and $g$ defined as before. We want to apply theorem 2 of [38] in $L^m(\Omega)$ over the sequence $\rho_{1,h} + \rho_{2,h}$. By (3.5), we obtain

\[ \sup_{h \leq 1} \int_0^T G\left(\frac{\rho_{1,h}(t) + \rho_{2,h}(t)}{2}\right) \, dt < +\infty. \]

Since, it is well-known that for all $\rho_1, \rho_2, \mu_1, \mu_2 \in \mathcal{P}^{ac}(\Omega),

\[ W_2^2\left(\frac{\rho_1 + \rho_2}{2}, \frac{\mu_1 + \mu_2}{2}\right) \leq \frac{1}{2} W_2^2(\rho_1, \mu_1) + \frac{1}{2} W_2^2(\rho_2, \mu_2), \]

by (3.4), we obtain

\[ \limsup_{\tau \nearrow 0} \sup_{h \leq 1} \int_0^{T-\tau} g\left(\frac{\rho_{1,h} + \rho_{2,h}}{2}(t+\tau), \frac{\rho_{1,h} + \rho_{2,h}}{2}(t)\right) \, dt = 0. \]
Theorem 2 in [38] and Lebesgue’s dominated convergence theorem imply that \( \rho_{1,h} + \rho_{2,h} \) converges strongly to \( \rho_1 + \rho_2 \) in \( L^m((0,T) \times \Omega) \). In addition, Krasnoselskii’s theorem, [17], Chapter 2, implies that \((\rho_{1,h} + \rho_{2,h})^{m/2}\) converges to \((\rho_1 + \rho_2)^{m/2}\) in \( L^2((0,T) \times \Omega) \). To conclude, \( \nabla((\rho_{1,h} + \rho_{2,h})^{m/2}) \) is bounded in \( L^2((0,T) \times \Omega) \), thanks to (3.5), then \( \nabla((\rho_{1,h} + \rho_{2,h})^{m/2}) \) weakly converges to \( \nabla((\rho_1 + \rho_2)^{m/2}) \) in \( L^2((0,T) \times \Omega) \).

Remark 3.6. Observe that it is possible to show that \( F_m(\rho_{1,h} + \rho_{2,h}) \) strongly converges to \( F_m(\rho_1 + \rho_2) \) in \( L^1((0,T) \times \Omega) \). Since \( \rho_{1,h} + \rho_{2,h} \) strongly converges in \( L^m((0,T) \times \Omega) \), then up to a subsequence, it converges a.e. Moreover, \((F_m(\rho_{1,h} + \rho_{2,h})(t,y)\) is uniformly integrable because \((F_m(\rho_{1,h} + \rho_{2,h}))_{h} \) is bounded in \( L^{m/(m-1)}((0,T) \times \Omega) \) and De La Vallée Poussin’s theorem. Then Vitali’s convergence theorem implies the result.

Remark 3.7. Notice that we can drop one individual diffusion. Assume that we drop the individual entropy in the JKO scheme (3.1) for one of the two densities, for instance \( \rho_2 \). The difficulty is to obtain a strong convergence for the sequence \((\rho_{2,h})_{h}\). Proposition 3.3 gives the strong convergence of \( \rho_{1,h} \) and \( \rho_{1,h} + \rho_{2,h} \) in \( L^1((0,T) \times \Omega) \) and \( L^m((0,T) \times \Omega) \) respectively, and then pointwise on \((0,T) \times \Omega\). Consequently, \( \rho_{2,h} = (\rho_{1,h} + \rho_{2,h}) - \rho_{1,h} \) converges pointwise on \((0,T) \times \Omega\). Moreover,

\[
\int_0^T \int_{\Omega} \rho_{2,h}(t,x)^m \, dx \, dt \leq \int_0^T \int_{\Omega} (\rho_{1,h}(t,x) + \rho_{2,h}(t,x))^m \, dx \, dt \leq C_T.
\]

Then Vitali’s convergence theorem implies that \( \rho_{2,h} \) strongly converges to \( \rho_2 \) in \( L^1((0,T) \times \Omega) \).

3.2 Existence of weak solutions of (1.2)

In this section, we start by giving the optimality conditions of (3.1). Instead of using horizontal perturbations, \( \rho_{i,\varepsilon} = \Phi_\varepsilon(\rho_{i,h}^{k+1}) \), as introduced in [22] by Jordan, Kinderlherer and Otto, we will perturb \( \rho_{i,h}^{k+1} \) with vertical perturbations introduced in [8, 11, 39, 40] which consist in taking \( \rho_{i,\varepsilon} = (1 - \varepsilon)\rho_{i,h}^{k+1} + \varepsilon \tilde{\rho}_i \), for all \( \tilde{\rho}_i \in L^\infty(\Omega) \). Before giving the optimality conditions of (3.1), we state the following lemma.

Lemma 3.8. For all \( k \geq 1 \), \( \rho_{i,h}^k > 0 \) a.e. and \( \log(\rho_{i,h}^k) \in L^1(\Omega) \).

Proof. The proof is the same as [40] Lemma 8.6. Let \( \tilde{\rho} = c = \frac{1}{|\Omega|} \) the uniform density on \( \Omega \). We define \( \rho_{i,\varepsilon} \) as the vertical perturbation of \( \rho_{i,h}^{k+1} \) by \( \tilde{\rho} \),

\[
\rho_{i,\varepsilon} := (1 - \varepsilon)\rho_{i,h}^{k+1} + \varepsilon \tilde{\rho}.
\]

for \( 0 < \varepsilon < 1 \). Using \((\rho_{i,\varepsilon}, \rho_{2,\varepsilon})\) as a competitor in (3.1), we obtain

\[
\mathcal{H}(\rho_{1,h}^{k+1}) - \mathcal{H}(\rho_{1,h}) + \mathcal{H}(\rho_{2,h}^{k+1}) - \mathcal{H}(\rho_{2,h}) \\
= \sum_{i=1}^2 \left( \int_{\Omega} V_i(\rho_{i,\varepsilon} - \rho_{i,h}^{k+1}) + \frac{1}{2h} W_2(\rho_{i,\varepsilon}, \rho_{i,h}^k) - \frac{1}{2h} W_2(\rho_{i,h}^{k+1}, \rho_{i,h}^k) \right) \\
+ \mathcal{F}_m(\rho_{1,\varepsilon} + \rho_{2,\varepsilon}) - \mathcal{F}_m(\rho_{1,h}^{k+1} + \rho_{2,h}^{k+1}).
\]

We remark that

\[
\int_{\Omega} V_i(\rho_{i,\varepsilon} - \rho_{i,h}^{k+1}) \leq C\varepsilon,
\]

and using the convexity of \( \frac{1}{2h} W_2(\cdot, \cdot) \) we obtain

\[
\frac{1}{2h} W_2(\rho_{i,\varepsilon}, \rho_{i,h}^k) - \frac{1}{2h} W_2(\rho_{i,h}^{k+1}, \rho_{i,h}^k) \leq \frac{1}{2h} W_2(\tilde{\rho}, \rho_{i,h}^k) - \frac{1}{2h} W_2(\rho_{i,h}^{k+1}, \rho_{i,h}^k) \leq C\varepsilon.
\]

If \( m > 1 \), then since \( \mathcal{F}_m \) is convex and \( \rho_{1,\varepsilon} + \rho_{2,\varepsilon} \in L^m(\Omega) \cap L^1(\Omega) \),

\[
\mathcal{F}_m(\rho_{1,\varepsilon} + \rho_{2,\varepsilon}) - \mathcal{F}_m(\rho_{1,h}^{k+1} + \rho_{2,h}^{k+1}) \leq C\varepsilon.
\]
Now we denote $A_i$ and $B_i$ the sets defined by
$$A_i := \{ \rho_{i,h}^{k+1} > 0 \} \text{ and } B_i := \{ \rho_{i,h}^{k+1} = 0 \}.$$  
On $A_i$, since $H(x) = x \log(x)$ is convex, $H(\rho_{i,h}^{k+1}) - H(\rho_{i,c}) \geq (\rho_{i,h}^{k+1} - \rho_{i,c})H'(\rho_{i,c}) \geq \varepsilon(\rho_{i,h}^{k+1} - c)(1 + \log(c))$. And on $B_i$, $H(\rho_{i,h}^{k+1}) - H(\rho_{i,c}) = -\varepsilon c \log(\varepsilon c)$. This implies
$$- c \log(\varepsilon c)(|B_1| + |B_2|) + \sum_{i=1}^{2} \int_{A_i} (\rho_{i,h}^{k+1} - c)(1 + \log(\rho_{i,c})) \leq C. \quad (3.10)$$

Since $-c \log(\varepsilon c) \to +\infty$, when $\varepsilon \searrow 0$, we conclude that $|B_1| = |B_2| = 0$. If $m = 1$, the proof is the same as before introducing additionally
$$A := \{ \rho_{1,h}^{k+1} + \rho_{2,h}^{k+1} > 0 \} \text{ and } B := \{ \rho_{1,h}^{k+1} + \rho_{2,h}^{k+1} = 0 \}.$$ Then in both case we obtain that $\rho_{i,h}^{k+1} > 0$ a.e.

Now using (3.10), the fact that $(\rho_{i,h}^{k+1} - c)(1 + \log(\rho_{i,c}))$ is bounded from below by $(\rho_{i,h}^{k+1} - c)(1 + \log(c)) \in L^1$ and applying Fatou’s lemma, we obtain
$$\int_{\Omega} (\rho_{i,h}^{k+1} - c)(1 + \log(\rho_{i,h}^{k+1})) \leq C.$$
This implies that $(\rho_{i,h}^{k+1} - c)(1 + \log(\rho_{i,h}^{k+1}))$ is $L^1(\Omega)$ and since $\rho_{i,h}^{k+1}$ and $\rho_{i,h}^{k+1} \log(\rho_{i,h}^{k+1})$ are in $L^1(\Omega)$, we conclude that $\log(\rho_{i,h}^{k+1})$ is in $L^1(\Omega)$. 

This lemma ensures the uniqueness (up to a constant) of the Kantorovich potential in the transport from $\rho_{i,h}^{k+1}$ to $\rho_{i,h}^k$ and then, we can easily compute the first variation of $W_2(\cdot, \rho_{i,h}^k)$ according to [40] Proposition 7.17.

**Proposition 3.9.** For $i \in \{1, 2\}$, $\rho_{i,h}^{k+1}$ satisfies
$$\nabla V_i(\rho_{i,h}^{k+1}) + \nabla \rho_{i,h}^{k+1} + \nabla F'(\rho_{i,h}^{k+1} + \rho_{i,h}^{k+1})\rho_{i,h}^{k+1} + \nabla \varphi_{i,h}^{k+1} = 0 \quad \text{a.e.}, \quad (3.11)$$
where $\varphi_{i,h}^{k+1}$ is the (unique) Kantorovich potential from $\rho_{i,h}^{k+1}$ to $\rho_{i,h}^k$.

**Proof.** We prove the result for $i = 1$ and the other case is analogous. Define $\rho_{1,c} = \rho_{1,h}^{k+1} + \varepsilon(\bar{\rho} - \rho_{1,h}^{k+1})$, for $\bar{\rho} \in L^\infty(\Omega)$. Using optimality of $\rho_{1,h}^{k+1}$ in (3.1), we obtain
$$\frac{1}{\varepsilon} \left( \frac{1}{2} W_2^2(\rho_{1,c}, \rho_{1,h}^{k+1}) - \frac{1}{2} W_2^2(\rho_{1,h}^{k+1}, \rho_{1,h}^{k+1}) + \frac{1}{2} W_2^2(\rho_{1,h}^{k+1}, \rho_{1,h}^{k+1}) \right) + \frac{1}{\varepsilon} \left( V_i(\rho_{1,c}) - V_i(\rho_{1,h}^{k+1}) + \mathcal{H}(\rho_{1,c}) - \mathcal{H}(\rho_{1,h}^{k+1}) + \mathcal{F}_m(\rho_{1,c}) - \mathcal{F}_m(\rho_{1,h}^{k+1}) \right) \geq 0. \quad (3.12)$$

Lemma 3.8 ensures uniqueness (up to a constant) of the Kantorovich potential between $\rho_{1,h}^{k+1}$ and $\rho_{1,h}^k$. [40] Proposition 7.18. Then applying proposition 7.17 in [40], the first variation of the Wasserstein distance exists and
$$\limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left( \frac{1}{2} W_2^2(\rho_{1,c}, \rho_{1,h}^{k+1}) - \frac{1}{2} W_2^2(\rho_{1,h}^{k+1}, \rho_{1,h}^{k+1}) \right) \leq \int_{\Omega} \varphi_{1,h}^{k+1} d(\bar{\rho} - \rho_{1,h}^{k+1}), \quad (3.13)$$
where $\varphi_{1,h}^{k+1}$ is the (unique) Kantorovich potential from $\rho_{1,h}^{k+1}$ to $\rho_{1,h}^k$. It is clear that
$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left( V_i(\rho_{1,c}) - V_i(\rho_{1,h}^{k+1}) \right) = \int_{\Omega} V_i d(\bar{\rho} - \rho_{1,h}^{k+1}). \quad (3.14)$$
Arguing as in [8] Lemma 3.1, since $H : x \mapsto x \log(x)$ is convex, the monotonicity of the incremental ratio gives for $\varepsilon < 1$,

$$\frac{|H(\rho) - H(\rho^{k+1})|}{\varepsilon} \leq |H(\rho^{k+1}) - H(\bar{\rho})|.$$  

Since $H(\rho^{k+1}) - H(\bar{\rho}) \in L^{1}(\Omega)$, Lebesgue’s dominated convergence theorem implies

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left( H(\rho) - H(\rho^{k+1}) \right) = \int_{\Omega} (1 + \log(\rho^{k+1})) \, d(\bar{\rho} - \rho^{k+1}). \quad (3.15)$$

Analogously, since $F_m$ is convex,

$$\frac{|F_m(\rho) + \rho^{k+1} - F_m(\rho^{k+1})|}{\varepsilon} \leq |F_m(\rho^{k+1}) - F_m(\bar{\rho} + \rho^{k+1})|.$$  

Then we have

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left( F_m(\rho) + \rho^{k+1} - F_m(\rho^{k+1}) \right) = \int_{\Omega} F_m'(\rho^{k+1}) (\bar{\rho} - \rho^{k+1}) \, d\rho. \quad (3.16)$$

Combining (3.12), (3.13), (3.14), (3.15), (3.16), we obtain, for all $\rho \in L^{1}(\Omega)$,

$$\int_{\Omega} \left( \varphi^{k+1} + hV_1 + h(1 + \log(\rho^{k+1})) + hF_m'(\rho^{k+1}) + hF_m(\rho^{k+1} + \rho^{k+1}) \right) \, d\rho \geq \int_{\Omega} \left( \varphi^{k+1} + hV_1 + h(1 + \log(\rho^{k+1})) + hF_m'(\rho^{k+1} + \rho^{k+1}) \right) \, d\rho^{k+1}.$$  

Applying proposition 7.20 in [40], there exists a constant $C_1$ such that

$$\varphi^{k+1} + hV_1 + h(1 + \log(\rho^{k+1})) + hF_m'(\rho^{k+1} + \rho^{k+1}) = C_1, \quad \rho^{k+1} - a.e. \quad (3.17)$$

Since $\varphi^{k+1} + hV_1$ is a Lipschitz function, it is differentiable a.e by using Rademacher’s theorem. Moreover $\rho^{k+1}$ is absolutely continuous with respect to Lebesgue measure then $\varphi^{k+1}$ is differentiable $\rho^{k+1}$-a.e as well as $h(1 + \log(\rho^{k+1})) + hF_m'(\rho^{k+1} + \rho^{k+1})$ because of (3.17). Then, we conclude

$$\nabla \varphi^{k+1} + h\nabla V_1 + h\nabla \log(\rho^{k+1}) + h\nabla F_m'(\rho^{k+1} + \rho^{k+1}) = 0, \quad \rho^{k+1} - a.e.$$  

A classical consequence of the previous proposition is that $\rho_{1,h}$ and $\rho_{2,h}$ are solutions to a discrete approximation of system $1,2$.

**Proposition 3.10.** Let $h > 0$, for all $T > 0$, let $N$ such that $N = \lfloor \frac{T}{h} \rfloor$. Then for all $(\phi_1, \phi_2) \in C_{c}^{\infty}([0, T] \times \mathbb{R}^{n})$ and for all $i \in \{1, 2\}$,

$$\int_{0}^{T} \int_{\Omega} \rho_{i,h}(t, x) \partial_{t}\phi_{i}(t, x) \, dx \, dt + \int_{\Omega} \rho_{i,0}(x) \phi_{i}(0, x) \, dx$$

$$= h \sum_{k=0}^{N-1} \int_{\Omega} \nabla V_1 \cdot \nabla \phi_{i}(t, x) \rho^{k+1}_{i,h}(x) \, dx + h \sum_{k=0}^{N-1} \int_{\Omega} \nabla \rho^{k+1}_{i,h}(x) \cdot \nabla \phi_{i}(t, x) \, dx$$

$$+ h \sum_{k=0}^{N-1} \int_{\Omega} \nabla F_{m}'(\rho^{k+1}_{i,h} + \rho^{k+1}_{2,h}) \cdot \nabla \phi_{i}(t, x) \rho^{k+1}_{i,h}(x) \, dx + \sum_{k=0}^{N-1} \int_{\Omega \times \Omega} \mathcal{R}[\phi_{i}(t, \cdot)](x, y) \, d\gamma^{k}_{i,h}(x, y)$$

where $t_k = kh$ ($t_N := T$) and $\gamma^{k}_{i,h}$ is the optimal transport plan in $W_2(\rho_{1,h}, \rho_{i,h}^{k+1})$. Moreover, $\mathcal{R}$ is defined such that, for all $\phi \in C_{c}^{\infty}([0, T] \times \mathbb{R}^{n})$,

$$|\mathcal{R}[\phi](x, y)| \leq \frac{1}{2} \|D^2\phi\|_{L^{\infty}([0, T] \times \mathbb{R}^{n})} |x - y|^2.$$
Proof. We take the scalar product between \( (3.11) \) and \( \nabla \phi_i \), for all \( \phi_i \in C_c^\infty([0, T) \times \mathbb{R}^n) \) and the proof is the same as in [11][27], for example.

Now we are able to prove theorem 2.2.

Proof of theorem 2.2. We have to pass to the limit in all terms in proposition 3.10 as \( h \to 0 \). The remainder term converges to 0 using the total square distance estimate (3.4) and the linear term converges to

\[
\int_0^T \int_\Omega \rho_i \partial_t \phi_i - \int_0^T \int_\Omega \nabla V_i \cdot \nabla \phi_i \rho_i,
\]
when \( h \) goes to 0 thanks to proposition 3.5.

Furthermore, since \( \nabla \rho_{i,h} = 2 \rho_{1,h}^{1/2} \nabla \rho_{1,h}^{1/2} \), then \( \nabla \rho_{i,h} \) is bounded in \( \mathcal{M}^n((0, T) \times \Omega) \) because proposition 3.5. We conclude that \( \nabla \rho_{i,h} \) narrowly converges to \( \nabla \rho_i \) because \( \rho_{i,h} \) strongly converges to \( \rho_i \) in \( L^1((0, T) \times \Omega) \). On the other hand, since \( \rho_{i,h} \) and \( \nabla \rho_{i,h} \) weakly converge to \( \rho_i \) and \( \nabla \rho_i \) in \( L^2((0, T) \times \Omega) \), \( \nabla \rho_i \in L^1((0, T), W^{1,1}(\Omega)) \). This implies that the individual diffusive term converges to

\[
\int_0^T \int_\Omega \nabla \phi_i \cdot \nabla \rho_i dx dt.
\]

It remains to study the convergence of the nonlinear cross diffusive term. First, we remark that \( \nabla F_m(\rho_{1,h}^{k+1} + \rho_{2,h}^{k+1}) \) can be rewritten as

\[
\nabla F_m'(\rho_{1,h}^{k+1} + \rho_{2,h}^{k+1}) = \frac{2 (\rho_{1,h}^{k+1} + \rho_{2,h}^{k+1})^{m/2}}{\rho_{1,h}^{k+1} + \rho_{2,h}^{k+1}} \nabla (\rho_{1,h}^{k+1} + \rho_{2,h}^{k+1})^{m/2}.
\]

Then

\[
\nabla F_m(\rho_{1,h}^{k+1} + \rho_{2,h}^{k+1}) = 2G_{1-m/2}(\rho_{1,h}^{k+1} + \rho_{2,h}^{k+1}) \nabla (\rho_{1,h}^{k+1} + \rho_{2,h}^{k+1})^{m/2},
\]

with \( G_\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \) is the continuous function (for \( \alpha < 1 \)) defined by

\[
G_\alpha(x, y) := \begin{cases} \frac{x}{(x+y)^\alpha} & \text{if } x > 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases}
\]

As \( m \geq 1, 1 - \frac{m}{2} < 1 \) so \( G_{1-m/2} \) is continuous and since, up to a subsequence, \( \rho_{i,h} \) converges to \( \rho_i \) a.e., we obtain that \( G_{1-m/2}(\rho_{1,h} + \rho_{2,h}) \) converges to \( G_{1-m/2}(\rho_1 + \rho_2) \) a.e. in \( (0, T) \times \Omega \). In addition,

\[
|G_{1-m/2}(\rho_{1,h} + \rho_{2,h})| = \left| (\rho_1 + \rho_2)^{m/2} \frac{\rho_{1,h}}{\rho_{1,h} + \rho_{2,h}} \right| \leq (\rho_1 + \rho_2)^{m/2}.
\]

Up to a subsequence, \( \rho_{i,h} \) and \( \rho_{1,h} + \rho_{2,h} \) converge a.e. in \( (0, T) \times \Omega \), and, since \( (\rho_{1,h} + \rho_{2,h})^{m/2} \) converges to \( (\rho_1 + \rho_2)^{m/2} \) in \( L^2((0, T) \times \Omega) \), there exists a function \( g \in L^2((0, T) \times \Omega) \) such that,

\[
|G_{1-m/2}(\rho_{1,h} + \rho_{2,h})| \leq \frac{g}{(\rho_1 + \rho_2)^{m/2}}
\]

Then Lebesgue’s dominated convergence theorem implies that \( G_{1-m/2}(\rho_{1,h} + \rho_{2,h}) \) converges strongly in \( L^2((0, T) \times \Omega) \) to \( G_{1-m/2}(\rho_1 + \rho_2) \). Moreover, \( \nabla (\rho_{1,h}^{k+1} + \rho_{2,h}^{k+1})^{m/2} \) converges weakly in \( L^2((0, T) \times \Omega) \), by proposition 3.5 then \( \nabla F_m'(\rho_{1,h} + \rho_{2,h})\rho_{1,h} \) converges weakly in \( L^1((0, T) \times \Omega) \) to \( \nabla F_m'(\rho_1 + \rho_2)\rho_1 \) and

\[
h \sum_{k=0}^{N-1} \int_\Omega \nabla F_m'(\rho_{1,h}^{k+1} + \rho_{2,h}^{k+1}) \cdot \nabla \phi_i(t_k, x) \rho_{1,h}^{k+1}(x) dx \to \int_0^T \int_\Omega \nabla F_m'(\rho_1 + \rho_2) \cdot \nabla \phi_i \rho_1 dx dt.
\]
4 Coupling by hard congestion

In this section we will prove an existence result for system of Fokker-Planck equations coupled by hard congestion on the sum. In other word we prove the existence to weak solution to (1.1). This system can be seen as gradient flow in a Wasserstein product space. Using the Jordan-Kinderlherer-Otto scheme, we construct two sequences defined in the following way: let \( h > 0 \) be a time step, we construct a sequence \((\rho_{1,h}^k, \rho_{2,h}^k)\) with \((\rho_{1,0,h}^0, \rho_{2,0,h}^0)\) = \((\rho_{1,0}, \rho_{2,0})\) and \((\rho_{1,h}^{k+1}, \rho_{2,h}^{k+1})\) is a solution to
\[
\inf_{(\rho_1,\rho_2) \in \mathcal{K}} \sum_{i=1}^2 \left[ \frac{1}{2h} W_2^2(\rho_i, \rho_{i,h}^k) + \mathcal{H}(\rho_i) + \mathcal{V}_i(\rho_i) \right],
\]
where \( \mathcal{K} := \{(\rho_1, \rho_2) \in \mathcal{P}^{ac}(\Omega)^2 : \rho_1 + \rho_2 \leq 1 \} \) and \(|\Omega| > 2\). The direct method shows that these sequences are well-defined. As before, we define the piecewise constant interpolations \( \rho_{i,h} : \mathbb{R}^+ \to \mathcal{P}^{ac}(\Omega) \) by
\[
\rho_{i,h}(t) := \rho_{i,h}^{k+1}, \quad \text{if } t \in (kh, (k+1)h].
\]

4.1 Estimates and convergences

In the following proposition, we list the classical estimates coming from the Wasserstein gradient flow theory.

**Proposition 4.1.** Let \( T > 0 \), for \( i \in \{1,2\} \) and for all \( k \geq 0 \) such that \( k \leq N := \lceil \frac{T}{h} \rceil \),
\[
\rho_{1,h}^k + \rho_{2,h}^k \leq 1, \quad \mathcal{H}(\rho_{i,h}^k) \leq C, \quad \sum_{k=0}^{N-1} W_2^2(\rho_{i,h}^k, \rho_{i,h}^{k+1}) \leq C h.
\]

As in the previous section, we need stronger estimates in order to handle the very degenerate cross diffusive term, \( \text{div}(\rho_{i} \nabla p) \).

**Proposition 4.2.** For all \( T > 0 \), there exists a constant \( C_T > 0 \) such that
\[
\|\rho_{1,h}^{1/2}\|_{L^2((0,T), H^1(\Omega))} + \|\rho_{2,h}^{1/2}\|_{L^2((0,T), H^1(\Omega))} \leq C_T.
\]

**Proof.** We apply the flow interchange technique as previously, proposition 3.3. Keeping the same notations as in the previous section, we note \( \eta_i \) the heat flow with initial condition \( \rho_{i,h}^0 \). Since the heat flow decreases the \( L^\infty \)-norm, (\( \eta_1(s), \eta_2(s) \)), defined in (3.6), is admissible for the minimization problem (4.1), for all \( s \geq 0 \). Then the same computations as in proposition 3.3 give the result. \( \square \)

Consequently, we deduce the following convergences.

**Proposition 4.3.** For all \( T > 0 \), the exists \( \rho_1 \) and \( \rho_2 \) in \( C^{0,1/2}([0,T], \mathcal{P}^{ac}(\Omega)) \) such that, up to a subsequence,
1. \( \rho_{i,h} \) converges to \( \rho_i \) in \( L^\infty([0,T], \mathcal{P}^{ac}(\Omega)) \),
2. \( \rho_{i,h} \) converges strongly in \( L^1((0,T) \times \Omega) \) and \( \nabla\rho_{i,h} \) converges narrowly to \( \nabla\rho_i \).

**Proof.** The total square distance estimate (4.1) and the refined version of Ascoli-Arzelà’s theorem, \([3] \) Proposition 3.3.1], implies that \( \rho_{i,h} \) converges to \( \rho_i \in C^{1/2}([0,T], \mathcal{P}^{ac}(\Omega)) \) in \( L^\infty([0,T], \mathcal{P}^{ac}(\Omega)) \). The second part of the proposition is obtained as in proposition 3.3 applying [35] Theorem 2. To conclude, we remark that \( \nabla\rho_{i,h} = 2\rho_{i,h}^{1/2} \nabla\rho_{i,h}^{1/2} \), \( \rho_{i,h}^{1/2} \) strongly converges to \( \rho_i^{1/2} \) in \( L^2((0,T) \times \Omega) \) and \( \nabla\rho_{i,h}^{1/2} \) weakly converges to \( \nabla\rho_i^{1/2} \) in \( L^2((0,T) \times \Omega) \).

We end this section by a lemma implying the uniqueness of the pair of Kantorovich potentials from \( \rho_{i,h}^{k+1} \) to \( \rho_{i,h}^k \) and then the existence of the first variation of \( W_2^2(\cdot, \rho_{i,h}^k) \) (propositions 7.18 and 7.17 from [40]).

**Lemma 4.4.** Minimizers of (4.1) satisfy \( \rho_{i,h}^k > 0 \) a.e. and \( \log(\rho_{i,h}^k) \in L^1(\Omega) \).

**Proof.** The proof is the same as in [40] Lemma 8.5] (see also lemma 3.8). Indeed we can use a constant perturbation \( \tilde{\rho} \) because \((\tilde{\rho}, \rho)\) is admissible in (4.1) \((\tilde{\rho} + \rho = 2/|\Omega| \leq 1) \). \( \square \)
4.2 Pressure field associated to the constraint

In this section, we introduce a discrete pressure associated to the constraint $\rho^{k+1}_{1,h} + \rho^{k+1}_{2,h} \leq 1$. This common pressure is obtained arguing as in [31] in the case of one population.

**Lemma 4.5.** Let $(\rho^{k+1}_{1,h}, \rho^{k+1}_{2,h})$ be the unique solution to (4.1). Then for all $(\rho_1, \rho_2) \in \mathcal{K}$,

$$
\int_{\Omega} \psi^{k+1}_{1,h}(\rho_1 - \rho^{k+1}_{1,h}) + \int_{\Omega} \psi^{k+1}_{2,h}(\rho_2 - \rho^{k+1}_{2,h}) \geq 0,
$$

where $\psi^{k+1}_{1,h} = \frac{\phi^{k+1}_{1,h}}{h} + V_i + \log(\rho^{k+1}_{1,h})$ and $\phi^{k+1}_{1,h}$ is the optimal (up to a constant) Kantorovich potential in $W_2(\rho_{1,h}^{k+1}, \rho_{2,h}^{k+1})$.

**Proof.** The proof of this result is the same as lemma 3.1 in [32]. \qed

**Remark 4.6.** Notice that (4.4) can be rewritten as

$$
\int_{\Omega} \psi^{k+1}_{1,h} f_1 + \int_{\Omega} \psi^{k+1}_{2,h} f_2 \geq 0,
$$

for all functions $f_1, f_2$ such that

$$
f_1 + f_2 \leq \frac{1 - \rho^{k}_{1,h} - \rho^{k}_{2,h}}{\varepsilon}, \quad f_i \geq \frac{-\rho^{k}_{i,h}}{\varepsilon} \quad \text{and} \int_{\Omega} f_i = 0, \quad (4.5)
$$

for all $0 < \varepsilon \ll 1$.

In the next proposition, we introduce the common discrete pressure belonging to the subdifferential $-\partial F_\infty(\rho^{k}_{1,h} + \rho^{k}_{2,h})$.

**Proposition 4.7.** There exists $p^k_h \geq 0$ such that for all, $k \geq 1$,

$$
p^k_h(1 - \rho^k_{1,h} - \rho^k_{2,h}) = 0 \quad \text{a.e.}
$$

In addition, $p^k_h$ satisfies

$$
\nabla p^k_h = \frac{\nabla \phi^k_{1,h}}{h} - \nabla V_i - \nabla \log(\rho^k_{i,h}) \quad \text{a.e.,} \quad (4.6)
$$

for $i = 1, 2$.

**Proof.** Let $S := \{\rho^k_{1,h} + \rho^k_{2,h} = 1\}$ be the set where the constraint is saturated. Firstly, we choose $f_2 = 0$ on $\Omega$ and $f_1 = 0$ on $S$ in remark 4.6. Then we have

$$
\int_{S^c} \psi^k_{1,h} f_1 \geq 0,
$$

for all $f_1 \in L^\infty(\Omega)$. This implies that there exists a constant $C_1$ such that $\psi^k_{1,h} = C_1$ a.e. on $S^c$. Applying the same argument with $f_1 = 0$ on $\Omega$ and $f_2 = 0$ on $S$, we find a constant $C_2$ such that $\psi^k_{2,h} = C_2$ a.e. on $S^c$. And since $f_1$ and $f_2$ satisfy (4.5), we have

$$
\int_{\Omega} (\psi^k_{1,h} - C_1)f_1 + \int_{\Omega} (\psi^k_{2,h} - C_2)f_2 \geq 0.
$$

Now, choosing $f_1 = f$ and $f_2 = -f$ on $S$ and by symmetry ($f_1 = -f$ and $f_2 = f$), we find

$$
\int_{S} ((\psi^k_{1,h} - C_1) - (\psi^k_{2,h} - C_2))f = 0,
$$

for all $f \in L^\infty(\Omega)$. We conclude that $(\psi^k_{1,h} - C_1) = (\psi^k_{2,h} - C_2) =: \psi^k_h$ a.e. on $S$ and consequently

$$
\int_{S} \psi^k_h(f_1 + f_2) \geq 0.
$$
On the other hand, since \( f_1 + f_2 \leq 0 \) on \( S \), \( \psi^k \leq 0 \) a.e. on \( S \), then we define \( p^k_h \) by
\[
p^k_h := (C_1 - \psi^k_{1,h})_+ = (C_2 - \psi^k_{2,h})_+.
\]
By definition, we have \( p^k_h (1 - \rho^k_{1,h} - \rho^k_{2,h}) = 0 \) a.e. and since \( \psi^k_{i,h} \) is differentiable a.e., the proof is completed. \( \square \)

Now, we define the piecewise interpolation \( p_h : \mathbb{R}^+ \to L^1(\Omega) \) by
\[
p_h(t) := p^{k+1}_h, \quad \text{if } t \in (kh, (k+1)h).
\]
Notice that \( p_h(t) \geq 0 \) and for all \( t \geq 0 \), \( p_h(t)(1 - \rho_{1,h}(t) - \rho_{2,h}(t)) = 0 \) a.e. Therefore, we immediately deduce the following estimate on the pressure.

**Proposition 4.8.** For all \( T > 0 \), \( p_h \) is bounded in \( L^2((0,T), H^1(\Omega)) \).

**Proof.** First, we prove that \( \nabla p_h \) is bounded in \( L^2((0,T) \times \Omega) \) and then we will conclude using Poincaré’s inequality. By definition of \( p^{k+1}_h \), we have
\[
\int_\Omega |\nabla p^{k+1}_h|^2 (\rho^k_{1,h} + \rho^k_{2,h}) = \sum_{i=1}^2 \int_\Omega |\nabla \psi^k_{i,h} + \rho^k_{i,h}|^2 \\
\leq C \sum_{i=1}^2 \left( \int_\Omega |\nabla \psi^k_{i,h}|^2 \rho^k_{i,h} + \int_\Omega |\nabla V_i|^2 \rho^k_{i,h} + \int_\Omega |\nabla \psi^k_{i,h} + \rho^k_{i,h}|^2 \right) \\
\leq C \sum_{i=1}^2 \left( \frac{1}{h^2} W_2^2(\rho_{i,h}, \rho^k_{i,h}) + C + \|\rho^k_{i,h} \|_{H^1(\Omega)} \right),
\]
where the last line is obtained using the fact that \( \nabla V_i \in L^\infty(\Omega) \). Summing the previous inequalities over \( k \) and by \([4,1] \) and \([4,3] \), we obtain that
\[
\int_0^T \int_\Omega |\nabla p_h(t)|^2 (\rho_{1,h}(t) + \rho_{2,h}(t)) \leq C.
\]
Since \( p_h(t) = 0 \) a.e. on \( \{ \rho_{1,h}(t) + \rho_{2,h}(t) < 1 \} \), we deduce
\[
\int_0^T \int_\Omega |\nabla p_h(t)|^2 = \int_0^T \int_\Omega |\nabla p_h(t)|^2 (\rho_{1,h}(t) + \rho_{2,h}(t)) \leq C.
\]
We conclude with the same argument as \([37] \). Using Poincaré’s inequality, since \( |\{ p_h(t) = 0 \} | \geq |\{ \rho_{1,h}(t) + \rho_{2,h}(t) < 1 \}| \geq |\Omega| - 2 > 0 \), we obtain that \( p_h \) is bounded in \( L^2((0,T), H^1(\Omega)) \). \( \square \)

To analyse the pressure field \( p_h \), we recall the following lemma, \([32, 37] \).

**Lemma 4.9.** \([32] \) Let \( (p_h)_{h>0} \) be a bounded sequence in \( L^2([0,T], H^1(\Omega)) \) and \( (\rho_h)_{h>0} \) a sequence of piecewise constant curves valued in \( \mathcal{P}(\Omega) \) which satisfy \( W_2(\rho_h(t), \rho_h(s)) \leq C \sqrt{t - s} - \hat{h} \) for all \( s < t \in [0,T] \) and \( \rho_h \leq C \) for a fixed constant \( C \). Suppose that
\[
p_h \geq 0, \quad \rho_h(1 - \rho_h) = 0, \quad \rho_h \leq 1,
\]
and that
\[
p_h \to p \text{ weakly in } L^2([0,T], H^1(\Omega)) \text{ and } p_h \to p \text{ uniformly in } \mathcal{P}(\Omega).
\]
Then \( p(1 - \rho) = 0 \).

Consequently, one has

**Proposition 4.10.** There exists \( p \in L^2([0,T], H^1(\Omega)) \) such that \( p_h \) converges weakly in \( L^2([0,T], H^1(\Omega)) \) to \( p \), where \( p \) satisfies
\[
p \geq 0, \quad p(1 - \rho_1 - \rho_2) = 0, \quad \rho_1 + \rho_2 \leq 1 \text{ a.e. in } [0,T] \times \Omega.
\]
In addition, \( \rho_{i,h} \nabla p_h \) narrowly converges to \( \rho_i \nabla p \).

14
Proof. We apply lemma 4.9 to $\rho_h := \rho_{1,h} + \rho_{2,h}$ and $p_h$. According to proposition 4.8, $p_h$ weakly converges in $L^2((0,T) \times \Omega)$ to $p$ such that

$$p \geq 0, \quad p(1 - p_h - p_2) = 0, \quad p_1 + p_2 \leq 1.$$ \hspace{1cm} (4.7)

Moreover, using the estimate on $p_h$, we know that $\nabla p_h$ weakly converges to $\nabla p$ in $L^2((0,T) \times \Omega)$. Then since $\rho_{i,h}$ strongly converges to $\rho_i$ in $L^1((0,T) \times \Omega)$ with $\rho_{i,h}, \rho_i \leq 1$, $\rho_{i,h} \nabla p_h$ narrowly converges to $\rho_i \nabla p$.

$$\Box$$

4.3 Existence of weak solutions of (1.1)

Arguing as in proposition 3.10, $(\rho_{1,h}, \rho_{2,h})$ is solution to a discrete approximation of system (1.1).

Proposition 4.11. Let $h > 0$, for all $T > 0$, let $N$ such that $N = \lceil \frac{T}{h} \rceil$. Then for all $(\phi_1, \phi_2) \in C^\infty_c([0,T] \times \mathbb{R}^n)$ and for all $i \in \{1, 2\}$,

$$\int_0^T \int_\Omega \rho_{i,h}(t,x) \partial_t \phi_i(t,x) \, dt \, dx + \int_\Omega \rho_{i,0}(x) \phi_i(0,x) \, dx = h \sum_{k=0}^{N-1} \int_\Omega \nabla V_1(t) \cdot \nabla \phi_i(t_k,x) \rho_{i,h}^{k+1}(x) \, dx + h \sum_{k=0}^{N-1} \int_\Omega \nabla p_{i,h}^{k+1}(x) \cdot \nabla \phi_i(t_k,x) \, dx$$

$$+ h \sum_{k=0}^{N-1} \int_\Omega \nabla p_{i,h}^{k+1} \cdot \nabla \phi_i(t_k,x) \rho_{i,h}^{k+1}(x) \, dx + \int_\Omega \sum_{k=0}^{N-1} \mathcal{R}[\phi_i(t_k,\cdot)](x,y) \, dx \, dy$$

where $t_k = kh$ (for $N := T$) and $\gamma_i^{k+1}$ is the optimal transport plan in $W_2(\rho_{i,h}^k, \rho_{i,h}^{k+1})$. Moreover, $\mathcal{R}$ is defined such that, for all $\phi \in C^\infty_c([0,T] \times \mathbb{R}^n)$,

$$|\mathcal{R}[\phi](x,y)| \leq \frac{1}{2} \|D^2\phi\|_{L^\infty([0,T] \times \mathbb{R}^n)} |x - y|^2.$$

Combining propositions 4.1, 4.3, 4.10 and 4.11, the rest of the proof of theorem 2.3 is identical to the previous section and we omit the details.

Remark 4.12. We can show $\nabla \rho_i \in L^2((0,T) \times \Omega)$. Indeed, if we use again 4.6 combined with $\rho_{i,h}^{k+1} \leq 1$, we obtain that

$$|\nabla \rho_{i,h}^{k+1}|^2 \leq C \left( \frac{|\nabla V_1|^2}{h^2} \rho_{i,h}^{k+1} + |\nabla V_2|^2 \rho_{i,h}^{k+1} + |\nabla p_{i,h}^{k+1}|^2 \right) \quad \text{a.e.}$$

Since $\nabla p_h$ is bounded in $L^2((0,T) \times \Omega)$ and

$$h \sum_{k=0}^{N-1} \int_\Omega \frac{|\nabla \rho_{i,h}^{k+1}|^2}{h^2} \rho_{i,h}^{k+1} \leq C,$$

because of (4.1), then

$$\|\nabla \rho_{i,h}\|_{L^2((0,T) \times \Omega)} \leq C,$$

and $\nabla \rho_{i,h}$ converges weakly to $\nabla \rho_i$ in $L^2((0,T) \times \Omega)$.

5 Systems with a common drift

In this section, we focus on the special case where $\nabla V_1 = \nabla V_2 = \nabla V \in L^\infty(\Omega)$. We will prove that a solution to (1.2) converges to a solution to (1.1), when $m$ goes to $+\infty$. Moreover, under some regularity we give a $L^1$-contraction result for both systems (1.2) and (1.1).
Remark 5.1. It is well-known in the Wasserstein gradient flow theory that the \(\lambda\)-geodesic convexity of the functional implies a \(W_2\)-contraction of the flow. Unfortunately, in general, \((\rho_1, \rho_2) \in \mathcal{P}^{\infty}(\Omega)^2 \mapsto \mathcal{F}_m(\rho_1 + \rho_2)\) is not displacement convex. Indeed, for \(m = 2\), we can rewrite the functional as

\[
\mathcal{F}_2(\rho_1 + \rho_2) = \mathcal{F}_2(\rho_1) + \mathcal{F}_2(\rho_2) + 2 \int_{\Omega} \rho_1 \rho_2.
\]

Let \(\rho_2\) be a fixed density, we study the displacement convexity of \(\rho \mapsto \mathcal{F}_2(\rho) + 2 \int_{\Omega} \rho_2 \rho\). We know, see [36], that \(\rho \in \mathcal{P}^{\infty}(\Omega) \mapsto \mathcal{F}_2(\rho)\) is displacement convex but \(\rho \mapsto \int_{\Omega} \rho_2 \rho\) is displacement convex if \(\rho_2\) is \(\lambda\)-convex.

To overcome this lack of convexity, we need to obtain a stronger estimate, independent on \(m\), on \(\nabla \mathcal{F}_m'(\rho_1 + \rho_2, \rho_m)\), where \((\rho_{1,m}, \rho_{2,m})\) is a solution to \((1.2)\). In the case of a commun drift, this estimate can be found observing that \(\rho_{1,m} + \rho_{2,m}\) is the Wasserstein gradient flow of \(\mathcal{E}\) + \(\mathcal{V}\) + \(\mathcal{F}_m\) and using the flow interchange argument.

Proposition 5.2. Let \((\rho_{1,m}, \rho_{2,m})\) be a solution to \((1.1)\) with \(\nabla \mathcal{V}_1 = \nabla \mathcal{V}_2 =: \nabla \mathcal{V} \in L^\infty(\Omega)\). Then \(\rho_{1,m} + \rho_{2,m}\) is unique and \(\mathcal{F}_m(\rho_m)\) is bounded independently of \(m\) in \(L^2((0,T), H^1(\Omega))\), for all \(T < +\infty\).

Proof. We start remarking that \(\rho_m\) is solution to

\[
\partial_t \mu - \Delta \mu - \text{div}(\mu \nabla \mathcal{V}) - \text{div}(\mu \nabla \mathcal{F}_m'(\mu)) = 0,
\]

with initial condition \(\mu_{|t=0} = \rho_{1,0} + \rho_{2,0}\). By geodesic convexity of \(\mathcal{E}\) and \(\mathcal{F}_m\), we know that solution to \((5.1)\) is unique, \([3]\). To conclude, we reason as in [21] Lemma 5.6]. The proof is based on the flow interchange technique with the (smooth) solution to

\[
\begin{cases}
\partial_t \eta = \Delta \eta \mu - \epsilon \Delta \eta & \text{in } (0,T) \times \Omega, \\
(\nabla \eta \mu - \epsilon \nabla \eta) \cdot \nu = 0 & \text{in } (0,T) \times \partial \Omega,
\end{cases}
\]

where \(\rho_{h,m}^k\) is constructed using the JKO scheme. We obtain, when \(\epsilon\) goes to 0 and using a lower semi-continuity argument, \(\|\nabla \mathcal{F}_m'(\rho_m)\|_{L^1((0,T), H^1(\Omega))} \leq C_T\), for all \(T > 0\), where \(C_T\) is a constant independent on \(m\). The \(L^1\)-estimate of \(\mathcal{F}'(\rho_m)\) and the Poincaré-Wirtinger inequality conclude the proof.

Now, we show that \((\rho_{1,m}, \rho_{2,m})\) converges to a solution to \((2.3)\), \((\rho_{1,\infty}, \rho_{2,\infty})\), as \(m \to +\infty\).

Theorem 5.3. Assume that the initial data satisfy \(\rho_{1,0} + \rho_{2,0} \leq 1\). Up to a subsequence, as \(m \to +\infty\), a solution to \((2.2)\), \((\rho_{1,m}, \rho_{2,m})\), converges strongly in \(L^2((0,T) \times \Omega)\) to \((\rho_{1,\infty}, \rho_{2,\infty})\) and \(p_m := \mathcal{F}_m', (\rho_{1,m} + \rho_{2,m})\) converges weakly in \(L^2((0,T), H^1(\Omega))\) to \(p_\infty\), where \((\rho_{1,\infty}, \rho_{2,\infty}, p_\infty)\) is a solution to \((2.3)\).

Proof. First we prove the convergence of \(\rho_{i,m}\). We start noticing that the estimate \((5.5)\) does not depend on \(m\) and then by remark 5.2 we have

\[
\|\rho_{i,m}^{1/2}\|_{L^2((0,T), H^1(\Omega))} \leq C_T \text{ and } W_2(\rho_{i,m}(t), \rho_{i,m}(s)) \leq C_T|t - s|^{1/2},
\]

for all \(t, s \leq T\) and where \(C_T\) is a constant independent on \(m\). Then using the Rossi-Savaré theorem we obtain that \(\rho_{i,m}\) converges to \(\rho_{i,\infty}\) in \(L^1((0,T) \times \Omega)\). In fact, \(\rho_{i,m}\) converges strongly to \(\rho_{i,\infty}\) in \(L^2((0,T) \times \Omega)\). Indeed, for \(m \gg 2\), \(\|\rho_{i,m}\|_{L^2((0,T) \times \Omega)}\) is uniformly bounded in \(m\) so \((\rho_{i,m})_{m}\) is uniformly integrable. Then, \(\rho_{i,m}\) converges weakly in \(L^2((0,T) \times \Omega)\) to \(\rho_{i,\infty}\) and Vitali’s convergence theorem implies that

\[
\|\rho_{i,m}\|_{L^2((0,T) \times \Omega)} = \|\rho_{i,m}^{1/2}\|_{L^1((0,T) \times \Omega)} \to \|\rho_{i,\infty}^{1/2}\|_{L^1((0,T) \times \Omega)} = \|\rho_{i,\infty}\|_{L^2((0,T) \times \Omega)}.
\]

Furthermore, \(p_m\) converges weakly in \(L^2((0,T), H^1(\Omega))\) to \(p_\infty\), proposition 5.2, and obviously \(p_\infty \geq 0\). Consequently, we can pass to the limit in the weak formulation of the system \((1.2)\) to obtain the weak formulation of system \((1.1)\).
To conclude the proof, it remains to prove that
\[ \rho_{1,\infty} + \rho_{2,\infty} \leq 1 \] and \[ \rho_\infty(1 - \rho_{1,\infty} - \rho_{2,\infty}) = 0 \text{ a.e.} \]

We start to show that \( \rho_{1,\infty} + \rho_{2,\infty} \leq 1 \). The argument is the same as in [2] Lemma 4.4. The estimate (5.3) does not depend on \( m \) so we have
\[ \int_0^T \int_\Omega (\rho_{1,m} + \rho_{2,m} - 1)^2 \, dx dt \leq \frac{2C}{m} \to 0, \] (5.2)
when \( m \to +\infty \), which implies that \( \rho_{1,\infty} + \rho_{2,\infty} \leq 1 \).

To obtain the second part of the claim, we start proving
\[ \int_0^T \int_\Omega \rho_m(1 - \rho_{1,m} - \rho_{2,m}) \varphi \, dx dt \to \int_0^T \int_\Omega \rho_\infty(1 - \rho_{1,\infty} - \rho_{2,\infty}) \varphi \, dx dt, \]
for all \( \varphi \in C_c^\infty((0, T) \times \Omega) \). With the same argument as before, \( \rho_{1,m} + \rho_{2,m} \to \rho_{1,\infty} + \rho_{2,\infty} \) strongly in \( L^2((0, T) \times \Omega) \) and \( \rho_m \to \rho_\infty \) weakly in \( L^2((0, T) \times \Omega) \), then by strong-weak convergence, we obtain the result. Now, we show that
\[ \int_0^T \int_\Omega \rho_m(1 - \rho_{1,m} - \rho_{2,m}) \varphi \, dx dt \to 0, \]
for all nonnegative \( \varphi \in C_c^\infty((0, T) \times \Omega) \). We start splitting the integral,
\[ \int_0^T \int_\Omega \rho_m(1 - \rho_{1,m} - \rho_{2,m}) \varphi \, dx dt = \int_{\{\rho_{1,m} + \rho_{2,m} \leq 1\}} \rho_m(1 - \rho_{1,m} - \rho_{2,m}) \varphi \, dx dt + \int_{\{\rho_{1,m} + \rho_{2,m} \geq 1\}} \rho_m(1 - \rho_{1,m} - \rho_{2,m}) \varphi \, dx dt. \]

We remark that, up to a subsequence, \( \rho_{1,m} + \rho_{2,m} \to \rho_{1,\infty} + \rho_{2,\infty} \) on \( (0, T) \times \Omega \). If \( \rho_{1,\infty}(t, x) + \rho_{2,\infty}(t, x) < 1 \), then \( \rho_{1,m}(t, x) - \rho_{2,m}(t, x) \leq (1 - \varepsilon) \), for large \( m \) and \( \rho_m(t, x) \leq \frac{m}{m-1} (1 - \varepsilon)^{m-1} \to 0 \), therefore \( \rho_m(t, x)(1 - \rho_{1,m}(t, x) - \rho_{2,m}(t, x)) \to 0 \). On the other hand, if \( \rho_{1,\infty}(t, x) + \rho_{2,\infty}(t, x) = 1 \) and, for large \( m \), \( \rho_{1,m}(t, x) + \rho_{2,m}(t, x) \leq 1 \), then \( 1 - \rho_{1,m}(t, x) - \rho_{2,m}(t, x) \to 0 \) and \( \rho_m(t, x) \leq \frac{m}{m-1} \) remains bounded. Thus, \( \rho_m(t, x)(1 - \rho_{1,m}(t, x) - \rho_{2,m}(t, x)) \to 0 \) and
\[ \int_{\{\rho_{1,m} + \rho_{2,m} \leq 1\}} \rho_m(1 - \rho_{1,m} - \rho_{2,m}) \varphi \, dx dt \to 0. \]
The convergence of the second term is obtained by (5.2) and proposition 5.2
\[ \left| \int_{\{\rho_{1,m} + \rho_{2,m} \geq 1\}} \rho_m(1 - \rho_{1,m} - \rho_{2,m}) \varphi \, dx dt \right| \leq \|\rho_m\|_{L^2((0,T)\times\Omega)} \frac{C}{m^{3/2}} \to 0, \]
when \( m \to +\infty \). Then, for all \( \varphi \in C_c^\infty((0, T) \times \Omega) \),
\[ \int_0^T \int_\Omega \rho_\infty(1 - \rho_{1,\infty} - \rho_{2,\infty}) \varphi \, dx dt = 0. \]
Since \( \rho_\infty(1 - \rho_{1,\infty} - \rho_{2,\infty}) \geq 0 \), we conclude that \( \rho_\infty(1 - \rho_{1,\infty} - \rho_{2,\infty}) = 0 \) a.e. in \( (0, T) \times \Omega \).

To end this section, we give a \( L^1 \)-contraction result for \( m \in [1, +\infty) \) under some regularity on solutions.

**Theorem 5.4.** Assume \( m \in [1, +\infty) \). Let \( (\rho_{1,m}^1, \rho_{2,m}^1) \) and \( (\rho_{1,m}^2, \rho_{2,m}^2) \) be two solutions to (2.2) \((2.3) \) if \( m = +\infty \). If \( \partial_t \rho_{1,m}^i, \partial_t \rho_{2,m}^i \in L^1((0, T) \times \Omega) \), then
\[ \|\rho_{1,m}^1(t, \cdot) - \rho_{1,m}^2(t, \cdot)\|_{L^1(\Omega)} \leq \|\rho_{1,m}^1(0, \cdot) - \rho_{1,m}^2(0, \cdot)\|_{L^1(\Omega)}. \]

17
Proof. First if \( m < +\infty \), since \( \rho_{1,m} + \rho_{2,m} \) solves (5.1), then it is unique and according to proposition 5.2. \( p_m := F'_m(\rho_{1,m} + \rho_{2,m}) \) is in \( L^2((0,T), H^1(\Omega)) \). Moreover, if \( m = +\infty \), we have already shown that the pressure \( p_\infty \) associated to the constraint \( \rho_{1,\infty} + \rho_{2,\infty} \leq 1 \) in theorem 2.3 is in \( L^2((0,T), H^1(\Omega)) \). According to [20], we know that \( (\rho_{1,\infty} + \rho_{2,\infty}, p_\infty) \) is unique.

Now, by the same argument as [1], we prove the \( L^1 \)-contraction. We remark that \( \rho_{1,m} \) solves

\[
\partial_t \rho_{1,m} - \Delta \rho_{1,m} - \text{div}(\rho_{1,m}(\nabla V + \nabla p_m)) = 0,
\]

for all \( 1 \leq m \leq +\infty \). We note\( \Omega_T := (0,T) \times \Omega \). For \( \delta > 0 \), define

\[
\zeta_\delta := \phi_\delta(\rho_{1,m} - \rho_{2,m}^2),
\]

where

\[
\phi_\delta(z) := \begin{cases} 
0 & \text{if } z \leq 0, \\
\frac{z}{\delta} & \text{if } 0 \leq z \leq \delta, \\
1 & \text{if } z \geq \delta.
\end{cases}
\]

Using \( \zeta_\delta \), or a smooth approximation of \( \zeta_\delta \) in the equation satisfies by \( \rho_{1,m} \) and \( \rho_{2,m} \), we obtain

\[
\iint_{\Omega_T} \partial_t (\rho_{1,m}^2 - \rho_{1,m}^2) \zeta_\delta = -\iint_{\Omega_T} ((\rho_{1,m}^2 - \rho_{1,m}^2)(\nabla V + \nabla p_m) \cdot \nabla \zeta_\delta + \nabla (\rho_{1,m}^2 - \rho_{2,m}^2) \cdot \nabla \zeta_\delta) \, dxdt.
\]

We introduce \( \Omega^\delta_T := \Omega_T \cap \{0 < \rho_{1,m}^2 - \rho_{2,m}^2 < \delta\} \). Then by definition of \( \zeta_\delta \) and using Young’s inequality

\[
\iint_{\Omega_T} \partial_t (\rho_{1,m}^2 - \rho_{2,m}^2) \zeta_\delta \\
= -\frac{1}{\delta} \iint_{\Omega^\delta_T} ((\rho_{1,m}^2 - \rho_{1,m}^2)(\nabla V + \nabla p_m) \cdot \nabla (\rho_{1,m}^2 - \rho_{1,m}^2) + |(\nabla (\rho_{1,m}^2 - \rho_{1,m}^2)|^2) \, dxdt \\
\leq \frac{1}{2\delta} \iint_{\Omega^\delta_T} (\rho_{1,m}^2 - \rho_{1,m}^2)^2 |\nabla V + \nabla p_m|^2 \, dxdt - \frac{1}{2\delta} \iint_{\Omega^\delta_T} |(\nabla (\rho_{1,m}^2 - \rho_{1,m}^2)|^2 \, dxdt \\
\leq \frac{1}{2} ||\nabla V + \nabla p_m||_{L^2(\Omega_T)}^2 \delta \to 0,
\]

when \( \delta \downarrow 0 \). Reversing the roles of \( \rho_{1,m}^2 \) and \( \rho_{2,m}^2 \), we have

\[
\iint_{\Omega_T} \partial_t (\rho_{1,m}^2 - \rho_{2,m}^2) \leq 0,
\]

which concludes the proof.

\[
\square
\]

6 Numerical simulations

To end this paper, we use the algorithm introduced in [6] to obtain numerical simulations. The first system we study is the transport equation with common porous media congestion, without individual diffusions,

\[
\partial_t \rho_i - \alpha_i \text{div}(\rho_i \nabla F'_m(\alpha_1 \rho_1 + \alpha_2 \rho_2)) - \text{div}(\rho_i \nabla V_i) = 0, \quad i = 1, 2,
\]

which, at least formally, is the gradient flow in Wasserstein space for the energy

\[
E(\rho_1, \rho_2) := \int_\Omega V_1 \rho_1 + \int_\Omega V_2 \rho_2 + \int_\Omega F_m(\alpha_1 \rho_1 + \alpha_2 \rho_2).
\]
Arguing as in [6], setting \( \phi = (\phi_1, \phi_2) \), \( (D \phi_1, D \phi_2) := (\partial_t \phi_1, \nabla \phi_1, \partial_t \phi_2, \nabla \phi_2) \), \( q = (q_1, q_2) = (a_1, b_1, c_1, a_2, b_2, c_2) \), \( \sigma = (\sigma_1, \sigma_2) = (\{\mu_1, m_1, \tilde{\mu}_1\}, \{\mu_2, m_2, \tilde{\mu}_2\}) \) and defining the convex set \( K := \{(a, b) \in \mathbb{R}^{m+1} : a + \frac{1}{2} |b|^2 \leq 0 \} \), one can rewrite one step of the JKO scheme, (3.1), with \( E \) as a saddle-point problem for the augmented Lagrangian

\[
L_r(\phi, q, \sigma) = \sum_{i=1}^{2} \int_{\Omega} \phi_i(0, x) \rho^E_{r,h}(x)dx + \sum_{i=1}^{2} \int_{0}^{1} \int_{\Omega} \chi_K(a_i(t, x), b_i(t, x))dxdt
\]

\[
+ \sum_{i=1}^{2} \int_{0}^{1} \int_{\Omega} \left( (\mu_i, m_i) \cdot (D \phi_i - (a_i, b_i)) + \frac{r}{2} |D \phi_i - (a_i, b_i)|^2 \right) dxdt
\]

\[
+ \sum_{i=1}^{2} \int_{0}^{1} \int_{\Omega} \left( \frac{r}{2} \phi_i(1, x) + c_i(x))^2dx - (\phi_i(1, x) + c_i(x))\tilde{\mu}_i(x) \right) dx
\]

\[
+ hE^* \left( \frac{c_1}{h}, \frac{c_2}{h} \right).
\]

where \( E^* \) is the Legendre transform of \( E \) extended by \(+\infty \) on \((-,0] \). A saddle point of \( L_r \) satisfies \( \rho_i^{k+1} = \tilde{\mu}_i \) and the solution to one JKO step is \( \rho_{i,h}^{k+1} = \tilde{\mu}_i \). Then, we use the augmented Lagrangian algorithm, ALG2-JKO, introduced in [6] to compute numerically \( (\rho_{1,h}^{k+1}, \rho_{2,h}^{k+1}) \) and we refer to [6] for a detailed exposition.

Figure 1 represents two populations crossing each other subject to porous media congestion with \( \alpha_1 = \alpha_2 = 1 \) and \( m = 50 \). We remark that the two populations have the same behaviour and when they cross each other, the density has to spread. In figure 2, we study the same behaviour but subject to the porous medium constraint on \( \rho_1 + 2\rho_2 \). We can see that the population where the constraint plays a higher role, \( \rho_2 \), has to deviate in order to let pass \( \rho_1 \) through.

In the two populations crowd motion model with linear diffusion, we saw that we can find a solution as the gradient flow of

\[
E(\rho_1, \rho_2) := \int_{\Omega} (V_1 + \log(\rho_1))\rho_1 + \int_{\Omega} (V_2 + \log(\rho_2))\rho_2 + \mathcal{F}_\infty(\alpha_1, \rho_1 + \alpha_2\rho_2).
\]

In figure 3, we see two populations which cross each other. When they start to cross each other at time \( t = 0.05 \), we remark that the density of \( \rho_1 \) and \( \rho_2 \) decrease and the sum is saturated. In this situation, individuals of both populations take the same space.

Now assume that an individual of the second population takes twice the space than an individual of the first population. Then if we study the one population model (without interaction), populations \( \rho_1 \) and \( \rho_2 \) are subject to constraints \( \rho_1(x) \leq 1 \) and \( \rho_2(x) \leq \frac{1}{2} \). In our case, where populations interact each other, \( \rho_1 \) and \( \rho_2 \) are subject to the common constraint \( \rho_1(x) + 2\rho_2(x) \leq 1 \). Notice that when \( \rho_1(x) = 0 \) or \( \rho_2(x) = 0 \), we recover the expected behaviour, \( \rho_2(x) \leq \frac{1}{2} \) and \( \rho_1(x) \leq 1 \). In Figure 4 we represents two populations crossing each other subject to this constraint. Immediately, the second population sprawls to saturate the constraints \( \rho_2(x) \leq \frac{1}{2} \) and then when they start crossing the density of \( \rho_1 \) and \( \rho_2 \) decrease and we have \( \rho_1(x) + 2\rho_2(x) = 1 \).

In figures 5 and 6, the same situations as in figures 3 and 4 are presented adding an obstacle in the middle of \( \Omega \). This can be done using a potential with very high value in this area.
Acknowledgements

The author gratefully thanks G. Carlier for suggesting this problem and for fruitful discussions about this work.

References

[1] Martial Agueh. Existence of solutions to degenerate parabolic equations via the Monge-Kantorovich theory. *Adv. Differential Equations*, 10(3):309–360, 2005.

[2] Damon Alexander, Inwon Kim, and Yao Yao. Quasi-static evolution and congested crowd transport. *Nonlinearity*, 27(4):823–858, 2014.

[3] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.
Figure 3: Evolution of two species crossing each other with density constraint. Top row: display of $\rho_1 + \rho_2$. Bottom row: display of $\rho_1$.

Figure 4: Evolution of two species crossing each other with weighted density constraint, $\rho_1 + 2\rho_2 \leq 1$. Top row: display of $\rho_1 + \rho_2$. Middle row: display of $\rho_1$. Bottom row: display of $\rho_2$.

[4] Athmane Bakhta and Virginie Ehrlacher. Cross-diffusion systems with non-zero flux and moving boundary conditions. November 2016. preprint.

[5] Jean-David Benamou, Yann Brenier, and Kevin Guittet. Numerical analysis of a multi-phasic mass transport problem. Contemporary Mathematics, 353:1–18, 2004.

[6] Benamou, Jean-David, Carlier, Guillaume, and Laborde, Maxime. An augmented lagrangian approach to wasserstein gradient flows and applications. ESAIM: ProcS, 54:1–17, 2016.

[7] Yann Brenier. Polar factorization and monotone rearrangement of vector-valued functions. Comm. Pure Appl. Math., 44(4):375–417, 1991.

[8] Giuseppe Buttazzo and Filippo Santambrogio. A model for the optimal planning of an urban area. SIAM J. Math. Anal., 37(2):514–530 (electronic), 2005.

[9] Clément Cancès, Thomas Gallouët, and Leonard Monsaingeon. Incompressible immiscible multiphase flows in porous media: a variational approach. preprint, 2016.
Figure 5: Evolution of two species crossing each other with density constraint and an obstacle. Top row: display of $\rho_1 + \rho_2$. Bottom row: display of $\rho_1$.

Figure 6: Evolution of two species crossing each other with weighted density constraint, $\rho_1 + 2\rho_2 \leq 1$, and an obstacle. Top row: display of $\rho_1 + \rho_2$. Middle row: display of $\rho_1$. Bottom row: display of $\rho_2$.

[10] G. Carlier and M. Laborde. A splitting method for nonlinear diffusions with nonlocal, nonpotential drifts. *Nonlinear Analysis: Theory, Methods & Applications*, 150:1 – 18, 2017.

[11] Guillaume Carlier and Filippo Santambrogio. A variational model for urban planning with traffic congestion. *ESAIM Control Optim. Calc. Var.*, 11(4):595–613 (electronic), 2005.

[12] Rinaldo M. Colombo, Mauro Garavello, and Magali Lécureux-Mercier. A class of nonlocal models for pedestrian traffic. *Math. Models Methods Appl. Sci.*, 22(4):1150023, 34, 2012.

[13] Rinaldo M. Colombo and Magali Lécureux-Mercier. Nonlocal crowd dynamics models for several populations. *Acta Math. Sci. Ser. B Eng. Ed.*, 32(1):177–196, 2012.

[14] Gianluca Crippa and Magali Lécureux-Mercier. Existence and uniqueness of measure solutions for a system of continuity equations with non-local flow. *NoDEA Nonlinear Differential Equations Appl.*, 20(3):523–537, 2013.
[15] Julien Dambrine, Nicolas Meunier, Bertrand Maury, and Aude Roudneff-Chupin. A congestion model for cell migration. *Commun. Pure Appl. Anal.*, 11(1):243–260, 2012.

[16] Sara Daneri and Giuseppe Savaré. Eulerian calculus for the displacement convexity in the Wasserstein distance. *SIAM J. Math. Anal.*, 40(3):1104–1122, 2008.

[17] Djaïro Guedes De Figueiredo. *Lectures on the Ekeland variational principle with applications and detours*. Springer Berlin, 1989.

[18] L. Desvillettes, T. Lepoutre, A. Moussa, and A. Trescases. On the entropic structure of reaction-cross diffusion systems. *Comm. Partial Differential Equations*, 40(9):1705–1747, 2015.

[19] Marco Di Francesco and Simone Fagioli. Measure solutions for non-local interaction PDEs with two species. *Nonlinearity*, 26(10):2777–2808, 2013.

[20] Marco Di Francesco and Daniel Matthes. Curves of steepest descent are entropy solutions for a class of degenerate convection-diffusion equations. *Calc. Var. Partial Differential Equations*, 50(1-2):199–230, 2014.

[21] Thomas Gallouët, Maxime Laborde, and Leonard Monsaingeon. An unbalanced optimal transport splitting scheme for general advection-reaction-diffusion problems. preprint, April 2017.

[22] Richard Jordan, David Kinderlehrer, and Felix Otto. The variational formulation of the Fokker-Planck equation. *SIAM J. Math. Anal.*, 29(1):1–17, 1998.

[23] Ansgar Jüngel. The boundedness-by-entropy method for cross-diffusion systems. *Nonlinearity*, 28(6):1963–2001, 2015.

[24] Ansgar Jüngel and Nicola Zamponi. A cross-diffusion system derived from a fokker-planck equation with partial averaging. *Zeitschrift für angewandte Mathematik und Physik*, 68(1):28, 2017.

[25] I. Kim and A. R. Mészáros. On nonlinear cross-diffusion systems: an optimal transport approach. preprint, May 2017.

[26] Stanislav Kondratyev, Léonard Monsaingeon, and Dmitry Vorotnikov. A fitness-driven cross-diffusion system from population dynamics as a gradient flow. *Journal of Differential Equations*, 261(5):2784–2808, 2016.

[27] M. Laborde. On some non linear evolution systems which are perturbations of Wasserstein gradient flows. to appear in *Radon Ser. Comput. Appl. Math.*, 2015.

[28] Philippe Laurençot and Bogdan-Vasile Matioc. A gradient flow approach to a thin film approximation of the Muskat problem. *Calc. Var. Partial Differential Equations*, 47(1-2):319–341, 2013.

[29] Thomas Lepoutre, Michel Pierre, and Guillaume Rolland. Global well-posedness of a conservative relaxed cross diffusion system. *SIAM J. Math. Anal.*, 44(3):1674–1693, 2012.

[30] Simone Di Marino and Alpar Richard Mészáros. Uniqueness issues for evolution equations with density constraints. *Math. Models Methods Appl. Sci.*, 2016.

[31] Daniel Matthes, Robert J. McCann, and Giuseppe Savaré. A family of nonlinear fourth order equations of gradient flow type. *Comm. Partial Differential Equations*, 34(10-12):1352–1397, 2009.

[32] Bertrand Maury, Aude Roudneff-Chupin, and Filippo Santambrogio. A macroscopic crowd motion model of gradient flow type. *Math. Models Methods Appl. Sci.*, 20(10):1787–1821, 2010.

[33] Bertrand Maury, Aude Roudneff-Chupin, and Filippo Santambrogio. Congestion-driven dendritic growth. *Discrete Contin. Dyn. Syst.*, 34(4):1575–1604, 2014.
[34] Bertrand Maury, Aude Roudneff-Chupin, Filippo Santambrogio, and Juliette Venel. Handling congestion in crowd motion modeling. *Netw. Heterog. Media*, 6(3):485–519, 2011.

[35] Bertrand Maury and Juliette Venel. *Handling of Contacts in Crowd Motion Simulations*, pages 171–180. Springer Berlin Heidelberg, 2009.

[36] Robert J. McCann. A convexity principle for interacting gases. *Adv. Math.*, 128(1):153–179, 1997.

[37] Alpar Richard Mészáros and Filippo Santambrogio. Advection-diffusion equations with density constraints. *Analysis and PDEs*, 2016.

[38] Riccarda Rossi and Giuseppe Savaré. Tightness, integral equicontinuity and compactness for evolution problems in Banach spaces. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 2, 2003.

[39] Filippo Santambrogio. Gradient flows in Wasserstein spaces and applications to crowd movement. In *Seminaire: Equations aux Dérivées Partielles. 2009–2010*, Sémin. Équ. Dériv. Partielles, pages Exp. No. XXVII, 16. École Polytech., Palaiseau, 2012.

[40] Filippo Santambrogio. *Optimal Transport for Applied Mathematicians*. Progress in Nonlinear Differential Equations and Their Applications 87. Birkasauser Verlag, Basel, 2015.

[41] Cédric Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[42] Cédric Villani. *Optimal transport*, volume 338 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2009. Old and new.