Photonic reservoir computer based on frequency multiplexing: supplemental document

1. DEPENDENCE OF PERFORMANCES ON PHASE MODULATION FREQUENCY

The optical cavity is a complex interferometer whose behaviour strongly depends on the frequencies of the circulating radiation. The two parameters defining these frequencies are the laser frequency $\omega$, which determines the position of the central comb line, and the RF phase modulation frequency $\Omega$, which is the spacing of the comb lines.

A rough characterisation of the optical cavity is its transfer function which we measure by recording optical power reflected by the cavity onto photodiode PD2 when the laser frequency $\omega$ is varied. In a typical transfer function measurement we scan $\omega$ in a 20 MHz range around its central value. 20 MHz is the experimentally accessible range closest to the cavity free spectral range, which is the period of the transfer function. The video reported in Visualization 1 shows how the cavity transfer function changes its shape when $\Omega$ is shifted: for each value of $\Omega$ we scan $\omega$ in a 20 MHz range.

The behaviour of the cavity determines the way in which comb lines interfere with each other and thus determines the set $W$ of internal connection between the neurons of the reservoir (see Eq. (1) in the main text). The Reservoir Computing scheme relies on random internal connections, but in experimental systems such as the present one, in which only a few parameters can be tuned, some parameters result in better performance than others. In particular certain sets of parameters may give rise to a bad operating condition. In Fig. S1 we plot the experimentally measured performances on both the tasks reported in the main text while sweeping $\Omega$ in the range between 16.970 GHz and 16.994 GHz. The figure also reports a compressed representation of the cavity transfer function where only minimum and maximum values are plotted. We do not see a clear correlation between the transfer function extension and the reservoir performances.

Fig. S2 reports some examples of transfer function measured both for good-performing $\Omega$ values and bad-performing ones. We do not identify a clear relation between transfer function shapes and performances, but it appears that bad performing configurations are characterized by transfer functions less complex than usual and almost sinusoidal, which could be related to the presence of a resonance in the cavity. Note that most of the $\Omega$ values giving bad performances are common to both tasks.

2. ANALYTICAL MODEL

A. Introduction

Here we describe the model behind the numerical simulation whose results are reported in the main text (Fig. 4 in the main text). The model runs in discrete time (a timestep is a roundtrip around the cavity) and accounts for noise in the photodiode measurements, for phase noise in the cavity and for refractive index dispersion (different wavelengths see different refractive indexes and accumulate different phases during the propagation). When set with realistic values for detector noise, phase noise and index dispersion, the simulation is in good agreement with experimental data. However in its current state it does not reproduce the detailed dependence of the performances on $\Omega$ (see experimental data reported in Fig. S1), most likely because we do not know the exact length of optical fiber between optical components.

We represent the complex field amplitude of each of the $N$ comb lines as an element of a $N \times 1$ vector, such that the central element of the vector is the amplitude of the central comb line. The state of the system is then defined in each timestep $n$ by two vectors: $x_{in}(n)$ and $x(n)$, representing respectively the amplitudes of the lines of the input comb and the amplitudes of the lines of the comb propagating inside the cavity. Both vectors are assumed to represent the field at the entrance of the cavity, at the coupler C1 (see Fig. 1 in the main text).
Fig. S1. (top and middle) Performance on nonlinear channel equalization task (top panel, SNR is set on 32 dB) and chaotic time series prediction (middle panel, time-shift is set on +1) varying $\Omega$. The red circle indicates the configuration in which we run the experiment reported in the main text. The laser wavelength, defining the position of the central comb line, is the same employed for the experiment reported in the main text ($\lambda = 1554.6$ nm). The vertical lines indicate the configurations which we analyzed further in Fig. S2. (bottom) Summary representation of the transfer function. Only minimum and maximum values are reported.
Fig. S2. Example of cavity transfer function measured for values of $\Omega$ giving good performances (a) and bad performances (b).

### B. Phase modulation

The effect of the first phase modulator, PM1, which is placed before the cavity, is represented by the $N \times 1$ complex vector $W_{\text{in}}$ such that $W_{\text{in}}^j$ represents how strongly the input radiation is coupled with the $j$-th comb line, i.e. the $j$-th neuron. The effect of the second phase modulator, PM2, which is placed inside the cavity, is represented by the $N \times N$ complex matrix $W_{\text{PM}}$ such that $W_{\text{PM}}^j, k$ represents how strongly the phase modulation couples the $j$-th comb line (i.e. the $j$-th neuron) with the $k$-th one.

$W_{\text{in}}$ and $W_{\text{PM}}$ can be derived based on the effect described by Eq. (5) in the main text:

$$W_{\text{in}}^j = i^{-N_0} j_{j-N_0}(m_1),$$  \hspace{1cm} (S1)

$$W_{\text{PM}}^j, k = i^{-k} j_{j-k}(m_2),$$  \hspace{1cm} (S2)

where $N_0$ is the index of the vector element representing the central comb line (if $N$ is odd, $N_0 = (N + 1)/2$), while $j_n$ is the Bessel function of the first kind and $m_1$ and $m_2$ are adimensional values representing the strength of the modulations generated by PM1 and PM2 respectively. Fig. S3 shows the shape of $W_{\text{in}}$ and $W_{\text{PM}}$.

### C. Input and propagation in the optical cavity

At the timestep $n$ the Mach Zehnder modulator is driven by the signal $u(n)$ ($u(n) \in [-1, +1]$).

We model the effect of the modulator on the light amplitude, $E_{\text{in}}(n)$, as

$$E_{\text{in}}(n) = E_0 \sin(\gamma u(n) + \frac{\pi}{4}),$$  \hspace{1cm} (S3)

where $\gamma$ represents the modulator driving strength. In the simulation we assume that the light amplitude input to the modulator is constant and unitary ($E_0 = 1$) and we neglect the effect of the propagation outside the cavity, hence $E_{\text{in}}(n)$ is real. The comb that at timestep $n$ enters the cavity is described by the vector

$$x_{\text{in}}(n) = \beta \cdot W_{\text{in}} \cdot E_{\text{in}}(n),$$  \hspace{1cm} (S4)

where $\beta < 1$ is a factor accounting for the coupling losses toward the cavity.
The effects of a roundtrip inside the cavity are summarized in the complex $N \times N$ matrix
\[
W = \alpha \cdot \Phi_{\text{noise}} \cdot \Phi^{(2)} \cdot W_{\text{PM}} \cdot \Phi^{(1)},
\]
where $\alpha$ is a real scalar value accounting for losses (including the amount of power leaving the cavity for readout) and gain (optical amplifier) affecting the radiation in one roundtrip, $\Phi_{\text{noise}}$ is a unit-modulus scalar value randomly extracted at each roundtrip to represent phase noise, while $\Phi^{(1)}$ and $\Phi^{(2)}$ are complex diagonal matrices whose elements account for the phase accumulated by each comb line during propagation respectively between the cavity entrance and PM2 and between PM2 and the cavity entrance. For $l \in [1, 2]$: 
\[
\Phi_{(l)}_{j,k} = \begin{cases} 
2^{\text{int}(\omega_j) \cdot L_l} & \text{if } j = k \\
0 & \text{otherwise} 
\end{cases},
\]
where $n(\omega_j)$ is the refractive index seen by wavelength $\omega_j$ (wavelength of the $j$-th comb line) and $L_1$ and $L_2$ represent respectively the length of the path between the cavity entrance and PM2 and the length of the path between PM2 and the cavity entrance.

For $n(\omega_j)$ we use the Sellmeier equation for single mode fiber. However, linearizing $n(\omega_j)$ as
\[
n(\omega_j) = \beta_0 + j\Omega\beta_1
\]
shows negligible differences, where $\beta_0$ is the propagation constant at frequency $\omega$ and $\beta_1 = v_g^{-1}$ the inverse of the group velocity.

For the propagation lengths we use $L_1 = pL$ and $L_2 = (1-p)L$ with $L = 10$ m, and $p \in [0, 1]$. Simulations show that $p$ can be chosen arbitrarily, except for a few values for which performance decreases.

At the timestep $n$ the comb inside the cavity is given by
\[
x(n) = W \cdot x(n-1) + x^{\text{in}}(n).
\]

**D. Absorbing propagation phases of the input**

Note that in Eq. (S8) we do not take into account the phases accumulated by the comb lines between PM1 and C1. However, we now show this is not a loss of generality as we can absorb this phase in a redefinition of $x(n)$. Indeed, denote the action of propagation between PM1 and C1 by the matrix $\Phi^{(3)}_{j,k}$ (similar to Eq. (S6)). Then Eq. (S8) should be replaced by
\[
x(n) = W \cdot x(n-1) + \beta \cdot \Phi^{(3)} \cdot x^{\text{in}}(n).
\]
If we define
\[
x(n) = \Phi^{(3)} \cdot x'(n)
\]
then
\[
x(n) = W \cdot x'(n-1) + x^{\text{in}}(n).
\]
then Eq. (S9) takes the form
\[ x'(n) = \alpha \cdot \Phi^{\text{noise}} \cdot \Phi^{(2)} \cdot W^{\text{PM}} \cdot \Phi^{(1)} \cdot x'(n - 1) + \beta \cdot W^{\text{in}} \cdot E^{\text{in}}(n). \] (S11)

where \( \Phi^{(2)} = \Phi^{(3)} - 1 \) and \( \Phi^{(1)} = \Phi^{(1)} \cdot \Phi^{(3)} \). This has exactly the same form as Eq. (S8), but with different matrices \( \Phi^{(1)} \) and \( \Phi^{(2)} \).

E. Readout

In the numerical model we assume that the output coupler C2 is placed just before coupler C1. Consequently, at timestep \( n \) the optical intensities of the output comb lines are described by the \( N \times 1 \) vector
\[ I^{\text{out}}(n) = |W \cdot x(n - 1)|^2, \] (S12)

where the \( | \cdot |^2 \) operation acts element-wise. The multiplication by \( W \) takes into account that the radiation is extracted just before the injection of the new input at C1, in other words, an input always propagates at least once round the cavity before being extracted at the output.

We define \( F \) the \( 1 \times N \) vector describing the attenuation that the spectral filter applies to each comb line, such that \( F_i \) is the attenuation applied to the \( i \)-th line. Hence, the power reaching PD1 is given by
\[ y(n) = F \cdot I^{\text{out}}(n) + y^{\text{noise}}, \] (S13)

where \( y^{\text{noise}} \) is a scalar value randomly extracted at each timestep to simulate the detector noise.

F. Analytical expression for output intensities

Eq. (S8) is a linear recurrence with source given by \( E^{\text{in}}(n) \). If we neglect the phase noise \( \Phi^{\text{noise}} \) in Eq. (S5), then we can resum the recurrence to obtain
\[ x(n) = \sum_{k=0}^{\infty} \beta \cdot W^k \cdot W^{\text{in}} \cdot E^{\text{in}}(n - k). \] (S14)

Then using Eq. (S12), we see that the optical intensities of the output comb lines take the form
\[ I^{\text{out}}(n) = |\sum_{k=0}^{\infty} \beta \cdot W^{k+1} \cdot W^{\text{in}} \cdot E^{\text{in}}(n - k)|^2 + y^{\text{noise}}, \] (S15)

which is a quadratic function of the previous inputs \( (E^{\text{in}}(n), E^{\text{in}}(n - 1), ...) \).

3. SIMULATION RESULTS

In simulations we take the parameters to be \( \alpha = 0.754, \beta = 0.43, \gamma = 0.33, L_1 = L_2 = 5 \text{ m}, \Omega = 16.983 \text{ GHz} \) and \( \omega = (2\pi c)/\lambda \) where \( \lambda = 1554.6 \text{ nm} \) and \( c \) is the speed of light.

The simulations are in good agreement with the experimental results, as demonstrated by Fig. 4 in the main text.

The numerical simulations allow to study the behaviour of the reservoir computer in conditions difficult or impossible to reach experimentally. In Fig. S4 we compare a simulation with realistic noise (which is in agreement with measurements) and a simulation without noise.

In Fig. S5 we study the dependence of the performances on the strength of the phase modulation, varying the parameters \( m_1 \) and \( m_2 \). Smaller (larger) values of \( m_1 \) and \( m_2 \) correspond to a reservoir computer with less (more) neurons (i.e. comb lines). Indeed when \( m_1 \) and \( m_2 \) are small, there will be more very small amplitude comb lines which will be masked by noise. In Fig. S5 we also indicate the approximate number of comb lines that can be used, i.e. the effective number of neurons in the reservoir computer.
Fig. S4. Numerical simulation of the reservoir computer performances in presence of noise and neglecting noise effects. Error bars represent the standard deviation of the score over 100 different random partitions of test and train datasets.

Fig. S5. Numerical simulation of the dependence of the reservoir computer performances on the strength of the phase modulation. Both $m_1$ (strength of the PM1 modulator) and $m_2$ (strength of the PM2 modulator) are scanned at the same time according to $m_1 = \rho \cdot 7.9$ and $m_2 = \rho \cdot 2.2$, with $\rho \in [0.1, 2]$, and $\rho = 1$ corresponding to modulation strengths similar to the experimental ones. Top horizontal axes report the number of usable neurons (i.e. number of comb lines encoding a signal above the noise floor). Error bars represent the standard deviation of the score over 100 partitions of test and train datasets. (a) Nonlinear channel equalization, SNR = 8 dB (b) Chaotic time series prediction, shift = −1.