Integers in the Open String

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We show that the $Y_{ab}^c$ of Pradisi-Sagnotti-Stanev are indeed integers, and we prove a conjecture of Borisov-Halpern-Schweigert. We indicate some of the special features which arise when the order of the modular matrix $T$ is odd. Our arguments are general, applying to arbitrary “parent” RCFT assuming only that $T$ has odd order.

1. Introduction

There has been considerable interest recently in analysing aspects of the open string in terms of the “parent” rational conformal field theory (RCFT) on 2-dimensional surfaces with boundaries (this strategy is called the method of open descendants) — see e.g. [1–6] and references therein for samples from this work.

Cardy’s influential work [7] investigated the boundary conditions on a cylinder (=annulus), for those RCFT whose torus partition function is given by the charge-conjugation matrix $C = S^2$:

$$Z(\tau) = \sum_a \chi_a \chi_a^*.$$  \hspace{1cm} (1)

In this case, the boundary conditions and primary fields are in one-to-one correspondence, and the annulus coefficients $A_{ab}^c$ are given by the chiral fusion coefficients $N_{ab}^c$. In other words, Cardy discovered a new interpretation for the coefficients $N_{ab}^c$: they also count the number of open string states in sector $c$ with boundary conditions $a$ and $b$.

Specifying the theory also requires knowing the Möbius strip and Klein bottle coefficients $M_a$ and $K_a$. These coefficients must satisfy certain consistency conditions, and Pradisi-Sagnotti-Stanev [8] expressed a solution in terms of an “integer-valued tensor” $Y_{ab}^c$ (see (9) below). These numbers $Y_{ab}^c$ also appear in the fusion coefficients of $Z_2$ permutation-orbifolds [9,10].

In this paper we find a novel interpretation for these $Y_{ab}^c$ (up to a sign they equal certain fusion coefficients!) which makes it manifest that they indeed are integers. A general proof of this seems to be lacking in the literature (see the last paragraph of this section). This also yields a positivity condition on the $K_a$. We also use this to prove a conjecture in [9].

Our results are very general: they apply to any RCFT, provided only that the order of the modular matrix $T = \text{diag}(\exp[2\pi i (h_a - \frac{c}{24})])$ is odd. In particular, we require that Verlinde’s formula for the “parent” RCFT yields nonnegative integers, and that equations (5) below hold. The odd-order hypothesis is vital for our interpretation, and for the positivity result on the $K_a$, but both the integrality of the $Y_{ab}^c$ and the conjecture of [9] should continue to hold when $T$ has even order.
In a recent paper [11], Bantay also relates the conjecture of [9] to that of [8]. In particular, he proves the integrality of \( Y \), assuming that Verlinde’s formula will yield nonnegative integers for an appropriate cyclic orbifold, and assuming results from the theory of permutation orbifolds [9]. On the one hand, his argument does not assume \( T \) has odd order. On the other hand, his argument is built on a somewhat less solid foundation than that of this paper: e.g. the formula for the modular matrix \( S \) for cyclic orbifolds is read off from expressions for the orbifold characters, but that technique works only when those characters are linearly independent, which for general RCFT is false. More precisely, the formula (2a) below uniquely specifies the coefficients \( S_{ab} \) only if the characters \( \chi_b(\tau) \) are linearly independent, and yet e.g. \( \chi_{a^{-1}}(\tau) = \chi_a(\tau) \). Reference [9] includes Cartan angles (corresponding to commuting spin one currents) in their characters, but there seems to be no reason why these should be sufficient to guarantee linear independence for arbitrary RCFT. It thus appears that the formulas [9,10] for the modular matrix \( S \) in permutation orbifolds is conjectural.

2. Background material

The characters \( \chi_a(\tau) \) of an RCFT carry a representation of \( \text{SL}_2(\mathbb{Z}) \) [12]:

\[
\chi_a(-1/\tau) = \sum_b S_{ab} \chi_b(\tau) \tag{2a}
\]

\[
\chi_a(\tau + 1) = \sum_b T_{ab} \chi_b(\tau) \tag{2b}
\]

The subscripts \( a, b \) label the (finitely many) primary fields. The matrices \( S \) and \( T \) are unitary and symmetric, and \( T \) is diagonal with entries \( \exp[2\pi i \left( h_a - \frac{c}{24}\right)] \) where \( h_a \) are the conformal weights and \( c \) is the central charge. \( T \) has finite order [13]: i.e. \( T^N = I \) for some \( N \). The matrix \( C = S^2 \) is the order-2 permutation matrix called charge-conjugation; we’ll write \( a \) for \( C a \).

There are two choices of \( 2 \times 2 \) matrices corresponding to the fractional linear transformation \( \tau \mapsto -1/\tau \) of (2a); we’ll adopt the more common choice in RCFT, namely

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

For this choice, \((ST)^3 = (TS)^3 = C\).

The fusion coefficients \( N_{ab}^c \) can be expressed using Verlinde’s formula:

\[
N_{ab}^c = \sum_d \frac{S_{ad} S_{bd} S_{ed}^*}{S_{0d}} \tag{3}
\]

where ‘0’ denotes the vacuum.

We will need two lesser known properties of RCFT. Both hold for any RCFT. The first is due to Bantay [14]. Define the numbers

\[
Z(a,b) := \exp[-\pi i h_b] \sum_{i,j} N_{ij}^a S_{bi}^* S_{0j} T_{jj}^2 T_{ii}^{-2} \tag{4}
\]
Then
\[ Z(a, b) \equiv N_{ab}^b \pmod{2} \quad (5a) \]
\[ |Z(a, b)| \leq N_{aa}^b \quad (5b) \]

\( Z(a, 0) \) is Bantay’s Frobenius-Schur indicator: \( Z(a, 0) = 0 \) if \( a \neq \overline{a} \), and \( Z(a, 0) = \pm 1 \) depending on whether \( a \) is real or pseudo-real.

The other property is the Galois symmetry [15]. The entries of \( S \) will lie in some cyclotomic extension of \( \mathbb{Q} \) — that just means each \( S_{ab} \) can be written as a polynomial \( p_{ab}(\xi_n) \), with rational coefficients, evaluated at some root of unity \( \xi_n := \exp[2\pi i/n] \). By the “Galois group” of this cyclotomic extension, we mean the integers mod \( n \) which are coprime to \( n \) — we write this \( \mathbb{Z} \times \mathbb{Z}_n \). For example, \( \mathbb{Z} \times \mathbb{Z}_{12} = \{1, 5, 7, 11\} \).

\[ \sigma_\ell S_{ab} = \epsilon_\ell(a) S_{\sigma_\ell a,b} = \epsilon_\ell(b) S_{a,\sigma_\ell b} \quad (6a) \]

These orthogonal matrices \( G_\ell \) in fact constitute a representation of the multiplicative group \( \mathbb{Z} \times \mathbb{Z}_n \), i.e. \( G_\ell G_m = G_{\ell m} = G_m G_\ell \).

This Galois symmetry should be regarded as a generalisation of charge-conjugation. In particular, \( \ell = -1 \) corresponds to the rather familiar Galois automorphism \( \sigma_{-1} = * \) (complex conjugation), and \( \epsilon_{-1}(a) = +1, \sigma_{-1} a = \overline{a} \), and \( G_{-1} = C \). The commuting of the \( G_\ell \) then implies \( \epsilon_\ell(\overline{a}) = \epsilon_\ell(a) \) and \( \sigma_\ell(\overline{a}) = \sigma_\ell(a) \).

Next, turn to the open string amplitudes. The case we will consider here is the best-understood one: where the torus partition function is given by (1).

There are two channels (transverse and direct) for the annulus, Klein bottle and M"obius strip, corresponding to the two different choices of time (“horizontal” and “vertical”). The two channels give identical amplitudes, provided the modular transformation relating the natural modular parameters in the two channels is taken into consideration. This modular transformation is \( S \), for both the annulus and Klein bottle, but is

\[ P := \sqrt{T} S T^2 S \sqrt{T} \quad (7) \]

for the M"obius strip. By ‘\( \sqrt{T} \)’ here we mean the diagonal matrix with entries \( \exp[\pi i (h_a - \overline{c})] \). Note that \( P \) is unitary and symmetric and \( P^2 = C \), hence \( P^* = CP = PC \).

The direct channel amplitude for the annulus is the open string partition function:

\[ A_{ab} = \frac{1}{2} \sum_c A_{ab}^c \chi_c \quad (8a) \]
where as already mentioned \( A_{ab}^c = N_{ab}^c \) here. The direct channel amplitudes for the Möbius strip and Klein bottle are

\[
M_a = \pm \frac{1}{2} \sum_b M_b^b \hat{\chi}_b \tag{8b}
\]

\[
K = \sum_a K^a \chi_a \tag{8c}
\]

where in (8b) the \( \hat{\chi}_a \) are a basis of “real characters” (see [1,2] for details).

There are a number of constraints on these coefficients [1,2], and the task is to find solutions to them. Cardy gives \( A_{ab}^c = N_{ab}^c \), and Pradisi-Sagnotti-Stanev give \( M_b^a = Y_{ab}^b \) and \( K^a = Y_{ab}^0 \), where \( Y_{ab}^c \) is given by

\[
Y_{ab}^c := \sum_d \frac{S_{ad} P_{bd} P_{cd}^*}{S_{0d}} \tag{9}
\]

Note that \( Y_{0b}^c = \delta_{b,c} \) and \( Y_{ab}^c = Y_{ba}^c = Y_{ab}^{c\pi} \), from the given properties of \( P \), \( Y_{ab}^c \) will automatically be real. Writing \( Y_a \) for the matrix with entries \( Y_{ab}^c \), we find that \( Y_a \) give a representation of the fusion ring: \( Y_a Y_b = \sum_c N_{ab}^c Y_c \).

It is conjectured in [8] that the \( Y_{ab}^c \) are integers (though not necessarily positive). There does not appear to be a proof of this, at least not in full generality (see the last paragraph of §1), although [16] show that \( Y_{a0}^b = Z(a,b) \) and hence those particular numbers are necessarily integers by (5a). In the next section we demonstrate the integrality of all \( Y_{ab}^c \), assuming only that the order \( N \) of \( T \) is odd.

There is no unique solution to the various constraints: e.g. [16] shows that provided simple-currents are present, there will be others. We will consider here only the “standard” Klein bottle of [8], given above.

3. The open string when \( N \) is odd

This section is the heart of the paper.

For the remainder of this paper, assume the order \( N \) of \( T \) is odd. This assumption can be rephrased as follows. Write \( t(r) \) for the exponent of 2 that appears in the prime decomposition of the rational number \( r \): e.g. \( t(3.7) = -1 \). Then \( N \) odd means that each \( t(h_a) \geq 0 \), as well as \( t(c) \geq 3 \).

Examples of odd \( N \) occur for instance in the WZW theories \( SU(M) \) level \( k \) when both \( M \) is odd and \( k \) is even. Another large class of examples are holomorphic orbifolds (with or without discrete torsion) by a finite group \( G \) with odd order.

The main result of this paper is the following new expression for \( Y_{ab}^c \):

\[
Y_{ab}^c = s(b) s(c) \varepsilon_{\frac{1}{2}}(b) \varepsilon_{\frac{1}{2}}(c) N_{a,\sigma b}^{\sigma c} \tag{10a}
\]

where ‘\( \sigma \)’ denotes the Galois permutation \( \sigma_{\frac{1}{2}} \), and where \( s(a) := +1 \) if \( t(h_a - \frac{a}{24}) > 0 \), otherwise \( s(a) := -1 \). The fraction ‘\( \frac{1}{2} \)’ here denotes the mod \( N \) inverse of 2, i.e. \( \frac{1}{2} = \)
\( \frac{N+1}{2} \in \mathbb{Z}_N \). So up to a sign, \( Y_{ab}^c \) is a fusion coefficient! Similarly, we obtain the curious expression

\[
P = s S G_2 s = \sigma_{\frac{1}{2}}(s S s) \tag{10b}
\]

where \( s = \text{diag}(s(a)) \) and the matrix \( G_2 \) is as in (6b). We will prove equations (10) in the final section. But first let’s explore some of their consequences.

Since the \( \epsilon \)'s and \( s \)'s in (10a) are signs, and fusion coefficients are nonnegative integers, the \( Y_{ab}^c \) will necessarily be integers.

Recall the observation [16] that \( Y_{a0}^b = Z(a, b) \). Then (5) become [17] the unobvious

\[
N_{a,\sigma_0}^{\sigma_b} \equiv N_{aa}^{\sigma_0} \pmod{2} \tag{11a}
\]

\[
N_{a,\sigma_0}^{\sigma_b} \leq N_{aa}^{\sigma_0}. \tag{11b}
\]

Putting \( b = 0 \) in (11) gives a curious fact [17]: the fusion product of \( \sigma_{\frac{1}{2}}(0) \) with itself consists of all self-conjugate fields, each appearing with multiplicity 1!

The Möbius strip and Klein bottle coefficients become

\[
M_a^b = \epsilon_{\frac{1}{2}}(b) \epsilon_{\frac{1}{2}}(0) N_{a,\sigma_0}^{\sigma_b} \tag{12a}
\]

\[
K^a = N_{\sigma_0,\sigma_0}^{a,\sigma_0} \tag{12b}
\]

(12b) is the interesting one: most curiously, it is never negative! In particular, it is now a consequence of (11) that

\[
K^a = \begin{cases} 
1 & \text{if } a = \pi \\
0 & \text{otherwise}
\end{cases} \tag{12c}
\]

This positivity is related to an observation made in [17]: when \( N \) is odd, there are no pseudo-real primary fields. The signs \( \epsilon_a \) of the coefficients \( K^a \) play a role in the open string theory [1,2], and when \( N \) is odd we see they all equal +1.

As a final result, let us prove a conjecture made in [9]. In their equation (4.38), they find that the combination

\[
\frac{1}{2} \sum_d S^2_{ad} S_{bd} S^\ast_{cd} \pm \frac{1}{2} Y_{ab}^c \tag{13}
\]

should equal a fusion coefficient in a \( \mathbb{Z}_2 \) cyclic-orbifold theory. They conjecture that for any RCFT, and any choice of signs, (13) should be a nonnegative integer.

First, note that the first term (dropping the factor \( \frac{1}{2} \)) is

\[
(N_a N_a)_{bc} = \sum_d N_{aa}^d N_{db}^c, \tag{14a}
\]

since the fusion matrices \( N_a \) (with entries \( (N_a)_{bc} := N_{bc}^a \)) form a representation of the fusion ring. Using (10a), the second term can be written as

\[
N_{a,\sigma_0}^{\sigma_c} = \sum_d \frac{\epsilon_{\frac{1}{2}}(d)}{\epsilon_{\frac{1}{2}}(b) \epsilon_{\frac{1}{2}}(c)} N_{bd}^c N_{a,\sigma_0}^{\sigma_d}, \tag{14b}
\]
where we use equation (18) of [15]. In order to prove the conjecture of [9], we must show both that (14a) and (14b) are congruent mod 2, and that the absolute value of (14b) is bounded above by (14a). Their congruence mod 2 follows immediately from (11a), and the desired inequality follows from (11b) and the triangle inequality applied to (14b).

Thus, provided the order of $T$ is odd, we have proven the conjecture of [9] that (13) is always a nonnegative integer.

The assumption that the order $N$ of $T$ is odd, is vital for the interpretation involving the Galois symmetry. For further consequences of $N$ being odd, see §§2.5,3.3 of [17].

It would be very interesting to find interpretations à la (10) for the other integral representations $Y_\ell^{(k)} := G_{\ell}^{-1}N_aG_{\ell}$ of the fusion ring. This should involve the “twisted dimensions” $D_0 \left( \frac{p}{0} \frac{q}{r} \frac{s}{t} \right)$ of [11] — see also §3.3 of [17].

4. Examples

In this section we work out the relevant quantities for two WZW examples. The WZW model for $SU(M)$ level $k$ has primaries most conveniently denoted by $M$-tuples of nonnegative integers which sum to $k$.

Consider first the WZW model for $SU(3)$ at level 2. There are 6 primaries: (2,0,0) (the vacuum ‘0’), (0,2,0) and (0,0,2) (the two nontrivial simple-currents), and (1,1,0), (1,0,1) and (0,1,1). The matrix $T$ is diag($\xi^2$, $\xi^8$, $\xi^8$, $\xi^2$, $\xi^2$, $\xi^7$) where $\xi = \exp[2\pi i/15]$, so its order is $N = 15$. The matrix $S$ here equals

$$
S = \frac{2 \sin(\frac{\pi}{5})}{\sqrt{15}} \left( \begin{array}{cccccc} 1 & 1 & 1 & 2c & 2c & 2c \\ 1 & \omega^2 & \omega & 2\omega c & 2\omega^2 c & 2c \\ 1 & \omega & \omega^2 & 2\omega^2 c & 2\omega c & 2c \\ 2c & 2\omega c & 2\omega^2 c & -\omega^2 & -\omega & -1 \\ 2c & 2\omega^2 c & 2\omega c & -\omega & -\omega^2 & -1 \\ 2c & 2c & 2c & -1 & -1 & -1 \end{array} \right)
$$

where $\omega = \exp[2\pi i/3]$ and $c = \cos(\frac{\pi}{5})$. The matrix $s$ used in equations (10) is $s = \text{diag}(+1,+1,+1,+1,+1,-1)$. Write $\sigma$ for $\sigma_{\frac{1}{2}}$: then $\sigma \sqrt{15} = -\sqrt{15}$, $\sigma \sin(\frac{\pi}{5}) = \sin(\frac{2\pi}{5})$, $\sigma\omega = \omega^2$, and $\sigma \cos(\frac{\pi}{5}) = \cos(\frac{2\pi}{5})$. The Galois matrix is thus

$$
G_{\frac{1}{2}} = G_{\frac{1}{2}}^t = \left( \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{array} \right)
$$

For our other example, consider $SU(3)$ at level 4. There are 15 primaries, and the vacuum is $0 = (4,0,0)$. The matrix $T$ has order $N = 21$. The sign $s(0)$ is +1, while $\epsilon_{\frac{1}{2}}(0) = +1$ and $\sigma_{\frac{1}{2}}(0) = (0,2,2)$. 
More generally when $k$ is even, the $T$ matrix for $SU(3)$ will have order $3(k+3)$. The vacuum is $0 = (k,0,0)$ and $\sigma_{\frac{k}{2}}(0) = (0,\frac{k}{2},\frac{k}{2})$. As mentioned in [17], (11) implies the following fusion product:

$$(0,\frac{k}{2},\frac{k}{2}) \boxtimes (0,\frac{k}{2},\frac{k}{2}) = (k,0,0), (k-2,1,1), \ldots, (0,\frac{k}{2},\frac{k}{2})$$

as can be verified explicitly by [18].

5. The derivation

We conclude this paper with the derivation of equations (10). This is similar to some calculations in [17].

Note that $T_{\frac{N+1}{2}} = s \sqrt{T}$, as can be seen by raising both sides to the $N$th power. Define $T_h := G_{\frac{1}{2}} T^4 G^{-1}_{\frac{1}{2}}$. We will eventually show that $T_h = T$. Hit the identity $C = (TS)^3$ with the Galois automorphism $\sigma_2$. The integral matrix $C$ is unchanged, and $T$, being a diagonal matrix with roots of 1 down the diagonal, gets sent to $T^2$. Using (6b), we get

$$C = T^2 G_2 S T^2 S G_2 T^2 G_2 S = G_2 T^{\frac{N+1}{2}} S T^2 S T^{\frac{N+1}{2}} S,$$

i.e. $G_{\frac{1}{2}} = G_2^{-1} = C T^{\frac{N+1}{2}} S T^2 S T^{\frac{N+1}{2}} S$.

Calling $D_a$ the diagonal matrix with entries $S_{ad}/S_{0d}$, equation (9) becomes the matrix equation

$$Y_a = (s T^{\frac{N+1}{2}} S T^2 S T^{\frac{N+1}{2}} s) D_a (s T^{\frac{N+1}{2}} S T^{\frac{N+1}{2}} s)$$

$$= C s T^{\frac{N+1}{2}} T^{\frac{N+1}{2}} G_{\frac{1}{2}} S D_a S G_2 T^{\frac{N+1}{2}} S T^{\frac{N+1}{2}} s$$

where we repeatedly commute diagonal matrices. Putting $\alpha = T_{00} T^*_h$, the integrality (5a) of $Z(a,b) = Y^b_{a0}$ (5a) tells us

$$\alpha^{\frac{N+1}{2}} N^\sigma_{a,0} T^{\frac{N+1}{2}}_{h bb} T^{\sigma N+1}_{bb} \in \mathbb{Z}$$

for all $a,b$, where we write $\sigma$ for $\sigma_{\frac{1}{2}}$. Choosing any $a$ in the fusion product of the primary field $\sigma 0$ with the primary $\sigma b$, we get then that $\alpha T_{h bb} = T^*_h bb \in \mathbb{Q}$ for all $b$. But it also must be an $N$th root of 1, and $N$ is odd, so in fact $\alpha T_{h bb} T^{*}_{bb h} = 1$, i.e. $T = \alpha T_h$. Substituting this into (15) gives us (10a), as well as $P = \alpha s G_{\frac{1}{2}} S s$. Using (6b), we find $P^2 = \alpha^2 C$, i.e. $\alpha^2 = 1$. Hence $\alpha$, which is an $N$th root of 1 for odd $N$, equals 1 and $P$ is given by (10b).

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