ON THE R-MATRIX IDENTITIES RELATED TO ELLIPTIC ANISOTROPIC SPIN RUIJSENAARS–MACDONALD OPERATORS

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We propose and prove a set of identities for the elliptic GLM R-matrix (in the fundamental representation).
In the scalar case (M = 1), these are elliptic function identities derived by Ruijsenaars as necessary and sufficient conditions for his kernel identity underlying the construction of integral solutions of quantum spinless Ruijsenaars–Schneider model. In this respect, our result can be regarded as a first step toward constructing solutions of the quantum eigenvalue problem for the anisotropic spin Ruijsenaars model.

Keywords: quantum integrable spin many-body system, spin Ruijsenaars–Schneider model, R-matrix identities, kernel identity

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1. Introduction

In [1] Ruijsenaars introduced a set of commuting elliptic difference operators

\[ D_k = \sum_{|I|=k} \prod_{i \in I, j \notin I} \phi(z_j - z_i) \prod_{i \in I} p_i, \quad k = 1, \ldots, N, \quad (1.1) \]

where \( \phi(z) = \phi(h, z) \) is the elliptic Kronecker function (see Eq. (A.2)). The sum in (1.1) is over all size-\( k \) subsets \( I \) of \( \{1, \ldots, N\} = \mathcal{N}_1 \). The shift operators \( p_i \) act on a function \( f(z_1, \ldots, z_N) \) of complex variables \( z = z_1, \ldots, z_N \) as

\[ (p_i f)(z_1, z_2, \ldots, z_N) = \exp\left(-\eta \frac{\partial}{\partial z_i}\right) f(z_1, \ldots, z_N) = f(z_1, \ldots, z_i - \eta, \ldots, z_N). \quad (1.2) \]

Operators (1.1) are called the Ruijsenaars–Macdonald operators because they reproduce the Macdonald operators in the trigonometric limit.

The mutual commutativity of \( D_k \) was shown in [1] to be equivalent to the set of functional equations

\[ \sum_{|I|=k} \left( \prod_{i \in I, j \notin I} \phi(z_j - z_i) \phi(z_i - z_j - \eta) - \prod_{i \in I, j \notin I} \phi(z_i - z_j) \phi(z_j - z_i - \eta) \right) = 0, \quad k = 1, \ldots, N, \quad (1.3) \]

and the function \( \phi(z) \) in (A.2) was shown to satisfy Eq. (1.3).

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Next, it was shown in [2] that the kernel identity $(D_k(x) - D_k(-y))\Psi(x,y) = 0$ holds for $\Psi(x,y)$ given by a certain product (and/or ratio) of elliptic gamma functions. This allows constructing integral solutions of the quantum eigenproblem for the operators $D_k$ [3], [4], [5]. This approach was later extended to a more general class of operators [6]. The kernel identity is proved by direct substitution, which then leads to a set of identities for $\phi$. Namely, it was proved in [2] that the kernel identity holds if and only if the following set of relations for $2N$ complex variables $(x = x_1, \ldots, x_N$ and $y = y_1, \ldots, y_N$) holds:

$$\sum_{I \subseteq \mathbb{N}} \left( \prod_{i \in I} \phi(x_i - x_j) \prod_{j \not\in I} \phi(y_j - x_i) - \prod_{i \in I} \phi(y_j - y_i) \prod_{j \not\in I} \phi(y_i - x_j) \right) = 0, \quad k = 1, \ldots, N. \quad (1.4)$$

Here, $I^c$ is the complement of a set $I$ in $\mathbb{N}$. It was shown in [2] that elliptic Kronecker function (A.2) satisfies (1.4). For example, for $k = 1$, Eq. (1.4) yields

$$\sum_{i=1}^N \left( \prod_{j \neq i} \phi(x_i - x_j) \prod_{j = 1}^N \phi(y_j - x_i) - \prod_{j \neq i} \phi(y_j - y_i) \prod_{j = 1}^N \phi(y_i - x_j) \right) = 0. \quad (1.5)$$

Under the substitution $x_i = z_i$, $y_i = z_i - \eta$, the last identity reproduces the first one (with $k = 1$) in (1.3). For $k > 1$, the substitution $x_i = z_i$, $y_i = z_i - \eta$ yields a particular case of (1.3) with $\hbar = 2\eta$.

We note that identities (1.4) arise in the context of Bäcklund transformations in the classical (elliptic) Ruijsenaars–Schneider model [3]. Also, Eq. (1.5) arises in the context of the so-called self-dual form of the Ruijsenaars–Schneider model, which follows from a multipole ansatz for the intermediate long-wave equation [7].

The purpose of the paper is to extend identities (1.4) to $R$-matrix identities understood as follows. The elliptic Baxter–Belavin [8], [9] $R$-matrix in Eq. (A.12), in the fundamental representation of the $GL_M$ Lie group, can be regarded as a matrix generalization of Kronecker function (A.2) because Eq. (A.2) is a particular case of (A.12) corresponding to $M = 1$. The $R$-matrices do not commute (in the general case). Our aim is to find a noncommutative analogue of (1.4) that reproduces (1.4) in the case $M = 1$.

In our previous paper [10] (see also [11] for applications to long-range spin chains), we introduced an anisotropic spin generalization of operators (1.1)). Similarly to the scalar case [1], we deduced and proved $R$-matrix generalizations of identities (1.3). These $R$-matrix identities are necessary and sufficient conditions for the commutativity of the spin Ruijsenaars–Macdonald operators. In this paper, we propose and prove $R$-matrix generalizations for identities (1.4) from [2]:

$$\sum_{1 \leq i_1, \ldots, i_k \leq N} \left( \prod_{i_k = i_k + 1}^{N} R_{k_{i_k - 1}i_{k - 1}}^{h} \prod_{i_{k - 1} = i_{k - 1} + 1}^{l_{k - 1}} R_{i_{k - 1} - 1i_{k - 1}}^{h} \cdots \prod_{l_1 = l_1 + 1}^{N} R_{i_{1}i_{1}}^{h} \right) \times \prod_{j_1 = N + 1}^{2N} R_{j_1i_{j_{1}}}^{h} \prod_{j_2 = N + 1}^{2N} R_{j_2i_{j_{2}}}^{h} \cdots \prod_{j_k = N + 1}^{2N} R_{j_ki_{j_{k}}}^{h} \times \prod_{m_{k - 1} = m_{k - 1} + 1}^{l_{k - 1}} \prod_{m_{k - 2} = m_{k - 2} + 1}^{l_{k - 2}} R_{i_{1}m_{1}}^{h} \left( \prod_{m_{k} \neq i_{1} \cdots i_{k - 1}} \prod_{m_{k - 1} \neq i_{1} \cdots i_{k - 2}} R_{i_{1}m_{1}}^{h} \cdots \prod_{m_{1} = 1}^{l_1} R_{i_{1}m_{1}}^{h} \right) - \sum_{N + 1 \leq j_1, \ldots, j_k \leq 2N} \left( \prod_{i_k = i_k + 1}^{N} R_{k_{j_i}j_{i}}^{h} \prod_{i_{k - 1} = i_{k - 1} + 1}^{l_{k - 1}} R_{i_{k - 1} - 1i_{k - 1}}^{h} \cdots \prod_{l_1 = l_1 + 1}^{N} R_{i_{1}j_{1}}^{h} \right) \times \prod_{j_1 = N + 1}^{2N} R_{j_1i_{j_{1}}}^{h} \prod_{j_2 = N + 1}^{2N} R_{j_2i_{j_{2}}}^{h} \cdots \prod_{j_k = N + 1}^{2N} R_{j_ki_{j_{k}}}^{h} \times \prod_{m_{k - 1} = m_{k - 1} + 1}^{l_{k - 1}} \prod_{m_{k - 2} = m_{k - 2} + 1}^{l_{k - 2}} R_{i_{1}m_{1}}^{h} \left( \prod_{m_{k} \neq i_{1} \cdots i_{k - 1}} \prod_{m_{k - 1} \neq i_{1} \cdots i_{k - 2}} R_{i_{1}m_{1}}^{h} \cdots \prod_{m_{1} = 1}^{l_1} R_{i_{1}m_{1}}^{h} \right)$$
increasing order, that is, for $j \leq i$, in the next section. This result is a first step toward constructing the kernel identity and solutions of the quantum eigenproblem for the (anisotropic) spin Ruijsenaars model. This is quite a nontrivial problem because it requires defining a matrix generalization of the elliptic gamma function. We hope to clarify these questions in our future publications.

2. Spin operators and $R$-matrix identities

We recall some notation and statements from [10]. Some additional notation is introduced for the purposes of this paper.

2.1. $R$-matrices and Yang–Baxter equations. Let $\mathcal{H}$ be a vector space $\mathcal{H} = (\mathbb{C}^M)^\otimes N$. The elliptic Baxter–Belavin $R$-matrix [8], [9] is the linear map in (A.12), $R_{ij}^h(z) \in \text{End}(\mathcal{H})$, acting nontrivially on the $i$th and $j$th tensor components of $\mathcal{H}$ only and satisfying the quantum Yang–Baxter equation

$$ R_{ij}^h(u)R_{jk}^h(u + v)R_{ik}^h(v) = R_{jk}^h(v)R_{ik}^h(u + v)R_{ij}^h(u) $$

(2.1)

for all pairwise distinct integers $1 \leq i, j, k \leq N$. Also, for any pairwise distinct integers $1 \leq i, j, k, l \leq N$,

$$ [R_{ij}^h(u), R_{kl}^h(v)] = 0. $$

(2.2)

We consider a pair $I, J$ of disjoint subsets in $\{1, \ldots, N\}$. The elements in $I$ and $J$ are arranged in increasing order, that is, for $J = \{j_1, j_2, \ldots, j_k\}$, we have $j_1 < j_2 < \cdots < j_k$. Similarly, for $I = \{i_1, i_2, \ldots, i_l\}$, we have $i_1 < i_2 < \cdots < i_l$. We use the notation $\prod_{j=1}^N R_{ij}$ and $\prod_{j=1}^N R_{ij}^\prime$, where the arrows denote ordering. For example, $\prod_{j=1}^N R_{ij} = R_{i1}R_{i2} \cdots R_{iN}$ and $\prod_{j=1}^N R_{ij}^\prime = R_{iN}R_{iN-1} \cdots R_{i1}$.

We introduce the notation

$$ R_{I,J} = \prod_{i_1 < i_2} R_{i_1,j_1}(z_{i_1} - z_{j_1}) \prod_{i_2 < i_3} R_{i_2,j_2}(z_{i_2} - z_{j_2}) \cdots \prod_{i_k < j_k} R_{i_k,j_k}(z_{i_k} - z_{j_k}). $$

(2.3)

Using property (2.2), we rewrite $R_{I,J}$ in the form

$$ R_{I,J} = \prod_{j_1 < j_2} R_{i_1,j_1}(z_{i_1} - z_{j_1}) \prod_{j_1 < j_2} R_{i_2,j_2}(z_{i_2} - z_{j_2}) \cdots \prod_{j_1 < j_2} R_{i_k,j_k}(z_{i_k} - z_{j_k}). $$

(2.4)

Similarly, we introduce

$$ R_{I,J}^\prime = \prod_{j_1 < j_2} R_{i_1,j_1}(z_{i_1} - z_{j_1}) \prod_{j_1 < j_2} R_{i_2,j_2}(z_{i_2} - z_{j_2}) \cdots \prod_{j_1 < j_2} R_{i_k,j_k}(z_{i_k} - z_{j_k}). $$

(2.5)
For any pairwise disjoint subsets $A$, $B$, $C$ of $\{1, 2, \ldots, N\}$, the following identities hold as a corollary of Yang–Baxter equation (2.1):

\[
R_{C,A} \circ R_{B,A} = R_{B,C} \circ R_{A,C},
\]
\[
R'_{A,B} \circ R'_{A,B,C} = R'_{B,C} \circ R'_{A,B,C}.
\]

The elliptic $R$-matrix satisfies the unitarity property

\[
R^h_{ij}(z)R^h_{ji}(-z) = \text{Id} \phi(h, z)\phi(h, -z).
\]

Below, we also use $R$-matrices in a slightly different normalization:

\[
R^h_{ij}(z) = \phi(h, z)R^h_{ij}(z).
\]

Then

\[
\overline{R}^h_{ij}(z)\overline{R}^h_{ji}(-z) = \text{Id}.
\]

For the products defined above, we have

\[
R_{I,J}R'_{J,I} = \text{Id} \prod_{i \in I, j \in J, i<j} \phi(h, z_i - z_j)\phi(h, z_j - z_i)
\]

and

\[
R'_{I,J}R_{J,I} = \text{Id} \prod_{i \in I, j \in J, i>j} \phi(h, z_i - z_j)\phi(h, z_j - z_i).
\]

Similarly,

\[
\overline{R}_{I,J}R'_{J,I} = \overline{R}'_{J,I}R_{I,J} = \text{Id}.
\]

Finally, elliptic $R$-matrix (A.12) is skew-symmetric:

\[
R^{h}_{12}(z) = -R^{-h}_{21}(-z).
\]

In what follows, we understand $R^{-h}_{12}(z)$ as $R^{h}_{21}(-z)$.

Our additional notation is as follows. For disjoint sets $I = \{i_1 < i_2 < \cdots < i_k\}$ and $J = \{j_1 < j_2 < \cdots < j_l\}$, we let $\mathcal{Y}^h_{I,J}$ denote the $R$-matrix product

\[
\mathcal{Y}^h_{I,J} = \prod_{j \in J} R^h_{i_1,j} \cdots \prod_{j \in J} R^h_{i_k,j}.
\]

By virtue of (2.2) it can be rewritten as

\[
\mathcal{Y}^h_{I,J} = \prod_{i \in I} R^h_{i,j_1} \cdots \prod_{i \in I} R^h_{i,j_l}.
\]

We also need an auxiliary statement based on Yang–Baxter equation (2.1), (2.2).
Lemma 1. Let $I = \{i_1 < \cdots < i_{m-1} < i_m = a < i_{m+1} < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_{n-1} < j_n = b < j_{n+1} < \cdots < j_l\}$ be disjoint sets. Then the following relations hold:

$$Y_{I,J}^h R_{b,j_n+1}^h \cdots R_{b,j_1}^h = R_{b,j_n+1}^h \cdots R_{b,j_1}^h Y_{I,J}^h$$

(2.17)

and

$$Y_{I,J}^h R_{i_m+1,a}^h \cdots R_{i_k,a}^h = R_{i_m+1,a}^h \cdots R_{i_k,a}^h Y_{I\setminus\{a\},J\setminus\{a\},J}^h$$

(2.18)

The proof is given in the Appendix.

Spin operators are defined as

$$D_k = \sum_{|I|=k} \prod_{i \in I^c, \ j \in I} \phi(z_i - z_j) \cdot \mathcal{R}_{I^c,I} \cdot \mathbf{p}_I \cdot \mathcal{R}_{I,I^c}, \quad \mathbf{p}_I = \prod_{i \in I} p_i,$$

(2.19)

where $p_i$ are shift operators (1.2) and the bars over $\mathcal{R}$ mean that the $R$-matrices normalized as in (2.9) and (2.10) are used in definitions (2.3) and (2.5). For $M = 1$, operators (2.19) reproduce the Ruijsenaars–Macdonald operators (1.1). The operators of type (2.19) were introduced in [12] in the trigonometric $M = 2$ with the help of the $U_q(gl_2)$ $R$-matrix.

We also note that using the so-called freezing trick, spin operators (2.19) provide an elliptic integrable long-range spin chain [11]. In classical mechanics, operators (1.1) are the Hamiltonians of the elliptic Ruijsenaars–Schneider model, while the (anisotropic) spin operators are quantum analogues of the Hamiltonians of a (relativistic) system of interacting tops [13], [14]. The description of relativistic tops can be found in [15], [16].

The commutativity of $D_k$ turns out to be equivalent to the following set of $R$-matrix identities:

$$\sum_{|I|=k} (\mathcal{R}_{I^c,I} \cdot \mathcal{R}_{I^c,I}^t \cdot \mathcal{R}_{I,I^c} \cdot \mathcal{R}_{I,I^c}^t - \mathcal{R}_{I,I^c} \cdot \mathcal{R}_{I,I^c}^t \cdot \mathcal{R}_{I,I^c} \cdot \mathcal{R}_{I,I^c}^t) = 0,$$

(2.20)

which hold for any $k = 1, \ldots, N$. Here, we use the notation $\mathcal{R}_{I,J} = \mathbf{p}_I \mathcal{R}_{I,J} \mathbf{p}_I^{-1}$. The derivation and proof of (2.20) as well as explicit examples can be found in [10]. For $M = 1$, relations (2.20) turn into identities (1.3).

3. New $R$-matrix identities

3.1. The main statement. It is convenient to define a set of $2N$ complex variables $z_1, \ldots, z_{2N}$ that unifies $x$ and $y$:

$$z_i = \begin{cases} x_i, & i \in N_1, \quad N_1 = \{1, \ldots, N\}, \\ y_i, & i \in N_2, \quad N_2 = \{N+1, \ldots, 2N\}. \end{cases}$$

(3.1)

The $R$-matrices act on the space $\mathcal{H} = (\mathbb{C}^M)^{\otimes 2N}$, and the following short notation is assumed:

$$R_{ij}^h(x_i - x_j), \quad i,j \in N_1,$$

$$R_{ij}^h(x_i - y_j), \quad i \in N_1, \quad j \in N_2,$$

$$R_{ij}^h(y_i - x_j), \quad i \in N_2, \quad j \in N_1,$$

$$R_{ij}^h(y_i - y_j), \quad i,j \in N_2.$$

(3.2)
Theorem 1. The following identities hold for any $k = 1, \ldots, N$:

$$
\sum_{I \subset N_1 \atop |I| = k} R^h_{I,I} \mathcal{Y}_{N_1}^h \mathcal{Y}_{N_2}^r R^h_{I,I^c} + (-1)^{k-1} \sum_{J \subset N_2 \atop |J| = k} R^{-h}_{J,J^c} \mathcal{Y}_{N_1}^h \mathcal{Y}_{N_2}^r R^{-h}_{J,J^c} = 0,
$$

(3.3)

where $R_{ij} = R^h_{ij}(z_i - z_j)$ is the elliptic $R$-matrix (A.12) acting on the $i$th and $j$th tensor components, where $i, j = 1, \ldots, 2N$ are chosen in accordance with (3.1) and (3.2). In the $M = 1$ case, $R$-matrix identities (3.3) turn into (1.4).

The idea of the proof is as follows. We let $\mathcal{F}[k,N] = \mathcal{F}_1[k,N] - \mathcal{F}_2[k,N]$ denote the left-hand side of (3.3), with $\mathcal{F}_1[k,N]$ and $\mathcal{F}_2[k,N]$ being the respective first and second sum in (3.3):

$$
\mathcal{F}_1[k,N] = \sum_{I \subset N_1 \atop |I| = k} R^h_{I,I} \mathcal{Y}_{N_1}^h \mathcal{Y}_{N_2}^r R^h_{I,I^c},
$$

(3.4)

$$
\mathcal{F}_2[k,N] = (-1)^k \sum_{J \subset N_2 \atop |J| = k} R^{-h}_{J,J^c} \mathcal{Y}_{N_1}^h \mathcal{Y}_{N_2}^r R^{-h}_{J,J^c}.
$$

(3.5)

More explicitly,

$$
\mathcal{F}_1[k,N] = \sum_{1 \leq i_1 < \cdots < i_k \leq N} \left( \prod_{l_k = i_k+1}^{N} R^h_{i_k l_k} \prod_{l_k-1 = i_{k-1}+1}^{l_k} R^h_{i_{k-1} l_{k-1}} \cdots \prod_{l_1 = i_1+1}^{1} R^h_{i_1 l_1} \times \prod_{j_1 = N+1}^{2N} R^h_{j_1 i_1} \prod_{j_2 = N+1}^{2N} R^h_{j_2 i_2} \cdots \prod_{j_k = N+1}^{2N} R^h_{j_k i_k} \times \prod_{m_k = 1}^{N} R^h_{j_k m_k} \prod_{m_{k-1} = 1}^{m_k} R^h_{i_{k-1} m_{k-1}} \cdots \prod_{m_1 = 1}^{m_2} R^h_{i_1 m_1} \right).
$$

(3.6)

We also rewrite expression (3.5) in terms of the $R$-matrices depending on $\hbar$ (but not on $-\hbar$) using the skew-symmetry property (2.14):

$$
\mathcal{F}_2[k,N] = \sum_{N+1 \leq j_k < \cdots < j_1 \leq 2N} \left( \prod_{l_k = j_k+1}^{N} R^h_{i_k j_k} \prod_{l_{k-1} = j_{k-1}+1}^{l_k} R^h_{i_{k-1} j_{k-1}} \cdots \prod_{l_1 = j_1+1}^{1} R^h_{i_1 j_1} \times \prod_{i_1 = 1}^{N} R^h_{i_1 j_1} \prod_{i_2 = 1}^{N} R^h_{i_2 j_2} \cdots \prod_{i_k = 1}^{N} R^h_{i_k j_k} \times \prod_{m_k = 1}^{N+1} R^h_{m_k j_k} \prod_{m_{k-1} = 1}^{m_k} R^h_{i_{k-1} m_{k-1}} \cdots \prod_{m_1 = 1}^{m_2} R^h_{i_1 m_1} \right).
$$

(3.7)

In this way, Eq. (3.3) takes form (1.6).

\footnote{In the first sum in (3.3), the set $I^c$ is the complement of $I$ in $N_1$. Similarly, in the second sum in (3.3), the set $J^c$ is the complement of $J$ in $N_2$.}

\footnote{We sometimes omit the indices $k$ and $N$ and write $\mathcal{F}$ instead of $\mathcal{F}[k,N]$.}

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We regard $\mathcal{F}$ as a (matrix-valued) function of $z_1 = x_1$. Our strategy is to use Lemma 4 from the Appendix similarly to the proof of (1.4) given in [2]. It can be shown that $\mathcal{F}$ is an entire function of $z_1$. The behavior of the $R$-matrices on the period lattice $\mathbb{Z} \oplus \tau \mathbb{Z}$ is nontrivial. It is given by (A.15). For this reason, we use the approach in [10]: we consider $\mathcal{F}(z_1)$ on the “large” torus with periods $M$ and $M\tau$. By (A.10), $Q^M = \Lambda^M = 1_M$, and the quasiperiodic behavior of the $R$-matrices (and of $\mathcal{F}$) on the large torus is simplified and we can use Lemma 4. We first prove the absence of poles (in $z_1$) in the fundamental parallelogram $0 \leq z_1 < 1$, $0 \leq z_1 < \tau$ and then extend the proof to the large torus.

The proof is done by induction on $k$ and $N$. For $k = 1$, identity (3.3) can be deduced from the higher-order $R$-matrix identities based on the associative Yang–Baxter equation.

### 3.2. Proof of Theorem 1 for $k = 1$. The proof of (1.5) can be performed straightforwardly using (A.7), which is a higher-order generalization of the addition formula (A.5). The details can be found in the Appendix in [7]. Here, we prove (3.3) for $k = 1$ in a similar way. For this, we recall that the elliptic $R$-matrix in the fundamental representation, Eq. (A.12), satisfies the so-called associative Yang–Baxter equation [17]

$$R_{12}^u R_{23}^{u'} = R_{13}^{u'} R_{12}^{u - u'} + R_{23}^{u - u} R_{13}^u, \quad R_{ab}^h = R_{ab}^h(z_a - z_b). \quad (3.8)$$

In [18], a higher-order analogue of (3.8) was derived, which we here a slightly different form suggested in [10],

$$\prod_{i=1}^n R_{a,i}(w_i) = \sum_{m=1}^n \prod_{j=m+1}^n R_{ai,j}(w_j - w_m) \cdot R_{b,m}(w_m) \cdot \prod_{j=1}^{m-1} R_{m,j}(w_j - w_m), \quad (3.9)$$

where $U = \sum_{k=1}^n u_k$ and $a \notin \{1, \ldots, n\}$, $n \in \mathbb{Z}_+$. We fix $a = 1$ and replace the set $\{1, \ldots, n\}$ with $\{2, \ldots, n+1\}$. Then (3.9) takes the form

$$\prod_{i=2}^{n+1} R_{1,i}(w_i) = \sum_{m=2}^{n+1} \prod_{j=m+1}^{n+1} R_{ai,j}(w_j - w_m) \cdot R_{1,m}(w_m) \cdot \prod_{j=2}^{m-1} R_{m,j}(w_j - w_m). \quad (3.10)$$

We set $n = 2N - 1$ and identify $w_i = z_1 - z_i$ for $i = 2, \ldots, 2N$ (see (3.1) and (3.2)). We also set $u_2 = \cdots = u_N = h$ and $u_{N+1} = \cdots = u_{2N} = -h$. Then $U = -h$. After all substitutions, the left-hand side of (3.10) becomes $R_{12}^h \cdots R_{1N}^h \cdots R_{12N}^h$. Transforming the right-hand side of (3.10) finally yields

$$\sum_{i=1}^N R_{i,i+1}^h \cdots R_{i,N}^h \cdot R_{N+1,N}^h \cdots R_{2N,i}^h \cdot R_{i,1}^h \cdots R_{i,i-1}^h - \sum_{i=1}^N R_{N+i+1,N+i}^h \cdots R_{N+i,N}^h \cdots R_{N+1,N+i}^h \cdots R_{N+i-1,N+i}^h = 0, \quad (3.11)$$

where we used the skew-symmetry of the $R$-matrices, Eq. (2.14). The obtained relation (3.11) is (3.3) for $k = 1$.

### 3.3. Auxiliary lemmas. Here we formulate two lemmas that are useful in proving Theorem 1. The proof of the Lemmas is given in Appendix B.

**Lemma 2.** As a function of $z_a = x_a$, $\mathcal{F} = \mathcal{F}[k, N]$ has no poles at $x_1, \ldots, x_{a-1}, x_{a+1}, \ldots, x_N$. 1549
Lemma 3. As a function of $z_a = x_a$, $\mathcal{F} = \mathcal{F}[k, N]$ has the following residues for any $b \in \mathbb{N}_2$:

$$ \text{Res}_{z_a = z_b} \mathcal{F}[k, N] = -G \cdot \mathcal{F}[k-1, N-1] \cdot H, \quad b = N + 1, \ldots, 2N, \quad (3.12) $$

where

$$ G = R_{b,a+1}^h \cdots R_{a,N}^h \cdot R_{b+1,b}^h \cdots R_{2N,b}^h, \quad (3.13) $$

$$ H = R_{N+1,a}^h \cdots R_{b-1,a}^h \cdot R_{1,b}^h \cdots R_{b,a-1}^h P_{ab}, \quad (3.14) $$

and by $\mathcal{F}[k-1, N-1]$ we mean the expression $\mathcal{F} = \mathcal{F}_1 - \mathcal{F}_2$ (see (3.4) and (3.5)) taken at $k-1$ and depending on the $2(N-1)$ variables $\{x_1, \ldots, x_N\} \setminus \{x_a\}$ and $\{y_1, \ldots, y_N\} \setminus \{y_b\}$.

The last statement is used in the proof of Theorem 1 with $a = 1$.

3.4. Proof of Theorem 1 for $k > 1$. We prove identities (3.3) by induction on $k$. For $k = 1$, the statement was proved in (3.11). We suppose that (3.3) holds for $k-1$. We then need to prove it for $k$.

The main idea is the same as in the scalar case. We use Lemma 4 given in Appendix A. For this, we consider the quasiperiodic properties of $\mathcal{F}$ as a function of $z_1$. By (A.15), any $R$-matrix of the form $R_{1k}^h = R_{1k}^h(z_1, z_k) = R_{1k}^h(z_1 - z_k), k = 2, \ldots, 2N$ transforms as

$$ R_{1k}^h(z_1 + 1, z_k) = Q_1^{-1} R_{1k}^h(z_1, z_k) Q_1, \quad (3.15) $$

where $Q_1 = Q \otimes 1_M \otimes \cdots \otimes 1_M = Q \otimes 1_M^{(2N-1)}$ and similarly for $\Lambda_1$. For $R$-matrices of the form $R_{k1}^h = R_{k1}^h(z_k, z_1) = R_{k1}^h(z_k - z_1)$, we have

$$ R_{k1}^h(z_k + 1, z_1) = R_{k1}^h(z_k - 1, z_1) = Q_k R_{k1}^h(z_1) Q_k^{-1} = Q_k^{-1} R_{12}^h(z_k, z_1) Q_k \quad (3.16) $$

and similarly,

$$ R_{k1}^h(z_k, z_1 + \tau) = e^{2\pi i h / M} \Lambda_1^{-1} R_{k1}^h(z_k, z_1) \Lambda_1. \quad (3.17) $$

Thus, any $R$-matrix that contains the index 1 transforms as (3.15) or (3.16), (3.17). The number of $R$-matrices of the form $R_{1k}^h$ or $R_{k1}^h$ is different in different terms. But the difference of these numbers is the same for all the terms: the number of $R$-matrices of the form $R_{1k}^h$ is less than the number of $R$-matrices of the form $R_{k1}^h$ by $k$. Therefore,

$$ \mathcal{F}(z_1 + 1) = Q_1^{-1} \mathcal{F}(z_1) Q_1, \quad (3.18) $$

$$ \mathcal{F}(z_1 + \tau) = e^{2\pi i k h / M} \Lambda_1^{-1} \mathcal{F}(z_1) \Lambda_1. \quad (3.19) $$

To use Lemma 4, we extend the torus to the large torus with the periods $M$ and $M \tau$. Because $Q^M = \Lambda^M = 1_M$, we then have

$$ \mathcal{F}(z_1 + M) = \mathcal{F}(z_1), $$

$$ \mathcal{F}(z_1 + M \tau) = e^{2\pi i k h} \mathcal{F}(z_1). $$

Next, we note that $\mathcal{F}(z_1)$ is an entire function of $z_1$ on the fundamental parallelogram ($0 \leq z_1 < 1$, $0 \leq z_1 < \tau$). Indeed, due to (A.14), all possible poles inside the fundamental parallelogram are at the points $x_2, \ldots, x_N$ and $y_1, \ldots, y_N$. By Lemma 2, the poles at the points $x_2, \ldots, x_N$ are absent. The absence of poles at $y_1, \ldots, y_N$ easily follows from Lemma 3 and the induction assumption (which implies that $\mathcal{F}[k-1, N-1] = 0$). Finally, by taking residue $\text{Res}_{z_1 = z_1}$, $k = 2, \ldots, 2N$ of both sides of (3.18), we conclude that the absence of poles in the fundamental parallelogram is extended to the absence of poles on the large torus. Then all conditions of Lemma 4 are satisfied and the theorem is proved.
3.5. Examples. We give several explicit examples of identities (1.6).

Example 1. \( N = 2, k = 1 \):

\[
R^{h}_{12}R^{h}_{31}R^{h}_{41} + R^{h}_{32}R^{h}_{42}R^{h}_{21} - R^{h}_{43}R^{h}_{31}R^{h}_{32} - R^{h}_{41}R^{h}_{42}R^{h}_{34} = 0.
\] (3.20)

Example 2. \( N = 3, k = 1 \):

\[
R^{h}_{12}R^{h}_{13}R^{h}_{41}R^{h}_{51}R^{h}_{61} + R^{h}_{23}R^{h}_{42}R^{h}_{52}R^{h}_{62}R^{h}_{21} + R^{h}_{34}R^{h}_{43}R^{h}_{53}R^{h}_{63}R^{h}_{32} - \\
- R^{h}_{54}R^{h}_{64}R^{h}_{44}R^{h}_{42}R^{h}_{43} - R^{h}_{65}R^{h}_{51}R^{h}_{61}R^{h}_{43}R^{h}_{53}R^{h}_{52}R^{h}_{63}R^{h}_{63}R^{h}_{45} - \\
- R^{h}_{61}R^{h}_{62}R^{h}_{63}R^{h}_{46}R^{h}_{56} = 0.
\] (3.21)

Example 3. \( N = 3, k = 2 \):

\[
R^{h}_{23}R^{h}_{13}R^{h}_{41}R^{h}_{51}R^{h}_{61} + R^{h}_{42}R^{h}_{41}R^{h}_{43}R^{h}_{51}R^{h}_{53}R^{h}_{52}R^{h}_{62} + \\
+ R^{h}_{42}R^{h}_{52}R^{h}_{62}R^{h}_{43}R^{h}_{53}R^{h}_{63}R^{h}_{31}R^{h}_{61}R^{h}_{43}R^{h}_{53}R^{h}_{63}R^{h}_{32} + \\
- R^{h}_{54}R^{h}_{64}R^{h}_{44}R^{h}_{42}R^{h}_{43}R^{h}_{46}R^{h}_{66} - R^{h}_{41}R^{h}_{51}R^{h}_{52}R^{h}_{53}R^{h}_{51}R^{h}_{61}R^{h}_{63}R^{h}_{46}R^{h}_{45} = 0.
\] (3.22)

We remark on the trigonometric and rational degenerations. All trigonometric and rational \( R \)-matrices considered in [10], [11] satisfy the obtained set of identities; the details will be given elsewhere.

Another remark is that in the trigonometric and rational cases, scalar identities (1.4) can be extended to the case where the number of variables \( x \) is different from the number of variables \( y \), such that \( |x| = N_1 \) and \( |y| = N_2 \) with \( N_1 \neq N_2 \). This can be explained as follows. In the trigonometric or rational case, we can consider the limit \( x_i \to \infty \), which decreases the number of variables \( x \) by one. Similar argument applies to the \( R \)-matrix identities if the limit \( \lim_{x_i \to \infty} R_{ij}(x_i - x_j) \) (or \( \lim_{x_i \to \infty} R_{i,N+j}(x_i - y_j) \)) is well defined. This is not true for all possible degenerations of the elliptic \( R \)-matrix. At the same time, this is true for a number of well-known cases. For example, it is obviously true for Yang’s rational \( R \)-matrix \( R^{h}_{12}(z) = 1_M \otimes 1_M h^{-1} + P_{12} z^{-1} \); then \( \lim_{z \to \infty} R^{h}_{12}(z) = 1_M \otimes 1_M h^{-1} \). Therefore, the limit \( x_i \to \infty \) applied to some identity provides another identity, where all \( R \)-matrices containing \( x_i \) are replaced with the scalar factor \( h^{-1} \). We will consider these cases in our future work.

Appendix A: Elliptic functions

Definitions and properties. We use the theta-function

\[
\vartheta(z) = \vartheta(z | \tau) = - \sum_{k \in \mathbb{Z}} \exp \left( \pi i \tau \left( k + \frac{1}{2} \right)^2 + 2\pi i \left( z + \frac{1}{2} \right) \left( k + \frac{1}{2} \right) \right), \quad \text{Im}(\tau) > 0.
\] (A.1)

It is odd, \( \vartheta(-z) = -\vartheta(z) \), and has a simple zero at \( z = 0 \). The Kronecker elliptic function is defined as

\[
\phi(z, u) = \frac{\vartheta'(0) \vartheta(z + u)}{\vartheta(z) \vartheta(u)} = \phi(u, z).
\] (A.2)

As a function of \( z \), it has a simple pole at \( z = 0 \) with \( \text{Res}_{z=0} \phi(z, u) = 1 \). The quasiperiodicity properties on the period lattice \( \Gamma = \mathbb{Z} \oplus \mathbb{Z} \tau \) (of the elliptic curve \( \mathbb{C}/\Gamma \)) are

\[
\vartheta(z + 1) = -\vartheta(z), \quad \vartheta(z + \tau) = -e^{-\pi i \tau - 2\pi i z} \vartheta(z),
\] (A.3)

whence

\[
\phi(z + 1, u) = \phi(z, u), \quad \phi(z + \tau, u) = e^{-2\pi i u} \phi(z, u).
\] (A.4)
Function (A.2) satisfies the addition formula
\[
\phi(z_1, u_1)\phi(z_2, u_2) = \phi(z_1, u_1 + u_2)\phi(z_2 - z_1, u_2) + \phi(z_2, u_1 + u_2)\phi(z_1 - z_2, u_1)
\]  
(A.5)
and the identity
\[
\phi(z, u)\phi(z, -u) = \wp(z) - \wp(u),
\]  
(A.6)
where \(\wp(x)\) is the Weierstrass \(\wp\)-function. Higher-order analogues of (A.5) are given by
\[
\prod_{i=1}^{n} \phi(w_i, u_i) = \prod_{i=1}^{n} \phi(w_i, \sum_{l=1}^{n} u_l) \prod_{j \neq i} \phi(w_j - w_i, u_j), \quad n \in \mathbb{Z}_+.
\]  
(A.7)

**R-matrix.** We define the set of \(M^2\) functions
\[
\varphi_a(z, \omega_a + h) = e^{2\pi i a_2 z/M} \phi(z, \omega_a + h), \quad \omega_a = \frac{a_1 + a_2 \tau}{M},
\]  
(A.8)
where \(M \in \mathbb{Z}_+\) is an integer and \(a = (a_1, a_2) \in \mathbb{Z}_M \times \mathbb{Z}_M\).

To construct the elliptic Baxter–Belavin \(R\)-matrix [8, 9] (also see [19]), we introduce a matrix basis in \(\text{Mat}(M, \mathbb{C})\),
\[
T_\alpha = e^{\alpha_1 \alpha_2 \pi i / M} Q^{\alpha_1} \Lambda^{\alpha_2}, \quad \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_M \times \mathbb{Z}_M.
\]  
(A.9)
where \(Q, \Lambda \in \text{Mat}(M, \mathbb{C})\) are given by
\[
Q_{kl} = \delta_{kl} e^{2\pi i k/M}, \quad \Lambda_{kl} = \delta_{k-l+1=0 \mod M}, \quad Q^M = \Lambda^M = 1_M.
\]  
(A.10)
For example, \(T_0 = T_{(0,0)} = 1_M\) is the \(M \times M\) identity matrix. The matrices \(Q\) and \(\Lambda\) satisfy the relations
\[
\Lambda^{a_2} Q^{\alpha_1} = e^{2\pi i a_1 a_2 / M} Q^{\alpha_1} \Lambda^{a_2}, \quad a_{1,2} \in \mathbb{Z}.
\]  
(A.11)
It then follows that
\[
T_\alpha T_\beta = \kappa_{\alpha,\beta} T_{\alpha+\beta}, \quad \kappa_{\alpha,\beta} = e^{\pi i (\alpha_2 \beta_1 - \alpha_1 \beta_2) / M},
\]
where \(\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)\).

Using (A.8) and (A.9), we introduce the Baxter–Belavin elliptic \((\mathbb{Z}_M\) symmetric) \(R\)-matrix
\[
R^h_{12}(x) = \frac{1}{M} \sum_{\alpha} \varphi_a \left( x, \frac{h}{M} + \omega_a \right) T_\alpha \otimes T_{-\alpha} \in \text{Mat}(M, \mathbb{C})^{\otimes 2}.
\]  
(A.12)
The \(\mathbb{Z}_M\) symmetry means that
\[
(Q \otimes Q) R^h_{12}(x) = R^h_{12}(x) (Q \otimes Q), \quad (\Lambda \otimes \Lambda) R^h_{12}(x) = R^h_{12}(x) (\Lambda \otimes \Lambda).
\]  
(A.13)
The \(R\)-matrix in (A.12) has simple poles in both variables \((h, z)\) with the residues
\[
\text{Res}_{z=0} R^h_{12}(z) = P_{12}, \quad \text{Res}_{h=0} R^h_{12}(z) = 1_M \otimes 1_M.
\]  
(A.14)
The quasiperiodic behavior on the period lattice is as follows:
\[
R^h_{12}(z + 1) = (Q^{-1} \otimes 1_M) R^h_{12}(z) (Q \otimes 1_M),
\]
\[
R^h_{12}(z + \tau) = e^{-2\pi i h / M} (\Lambda^{-1} \otimes 1_M) R^h_{12}(z) (\Lambda \otimes 1_M).
\]  
(A.15)
More properties can be found in the Appendix in [10].
Proof of identities. The main idea behind the proof of the identities in this paper is formulated in Lemma 4 below. The same lemma was used in [2]. Although it is widely known in the theory of elliptic functions, we briefly recall its proof for the convenience of the reader.

**Lemma 4.** Let \( f(z) \) be an entire function on the elliptic curve \( \Sigma = \mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z}) \) with the following quasiperiodicity properties on the period lattice \( \mathbb{Z} \oplus \tau \mathbb{Z} \):

\[
\begin{align*}
    f(z + 1) &= f(z), \\
    f(z + \tau) &= e^{2\pi i \alpha} f(z), \quad \alpha \in \mathbb{C}.
\end{align*}
\]

Then this function is equal to zero if \( \alpha \notin \mathbb{Z} \oplus \tau \mathbb{Z} \) (that is, if \( \alpha \neq n_1 + n_2 \tau, n_1, n_2 \in \mathbb{Z} \)).

**Proof.** We consider the function

\[
\tilde{f}(z) = e^{2\pi i \beta z} f(z), \quad \beta \in \mathbb{R}.
\]

(A.17)

\[
\begin{align*}
    \tilde{f}(z + 1) &= e^{2\pi i \beta} \tilde{f}(z), \\
    \tilde{f}(z + \tau) &= e^{2\pi i (\alpha + \beta \tau)} \tilde{f}(z).
\end{align*}
\]

(A.18)

By fixing the real-valued parameter \( \beta \) as

\[
\beta = \frac{\text{Im}(\alpha)}{\text{Im}(\tau)} = -\frac{\alpha - \bar{\alpha}}{\tau - \bar{\tau}},
\]

(A.19)

we obtain

\[
\begin{align*}
    \tilde{f}(z + 1) &= g_1 \tilde{f}(z), \quad g_1 = \exp\left(-2\pi i \frac{\alpha - \bar{\alpha}}{\tau - \bar{\tau}}\right), \\
    \tilde{f}(z + \tau) &= g_\tau \tilde{f}(z), \quad g_\tau = \exp(2\pi i (\text{Re}(\alpha) + \beta \text{Re}(\tau))) = \exp\left(2\pi i \frac{\tau \bar{\alpha} - \alpha \bar{\tau}}{\tau - \bar{\tau}}\right).
\end{align*}
\]

(A.20)

In the case where \( \alpha = n_1 + n_2 \tau \) for some \( n_1, n_2 \in \mathbb{Z} \), the function \( \tilde{f} \) becomes double periodic, i.e., \( g_1 = g_\tau = 1 \). Any entire double-periodic function on an elliptic curve is a constant (because an elliptic curve is compact). Therefore, due to (A.17) and (A.19), we obtain

\[
\tilde{f}(z) = \text{const} \cdot \exp\left(-2\pi i z \frac{\alpha - \bar{\alpha}}{\tau - \bar{\tau}}\right).
\]

If \( \alpha \notin \mathbb{Z} \oplus \tau \mathbb{Z} \), then \( g_1 \neq 1 \) and/or \( g_\tau \neq 1 \). At the same time, \( |g_1| = |g_\tau| = 1 \). Then due to the Liouville theorem (which says that every bounded entire holomorphic function on an elliptic curve is constant) we conclude that \( \tilde{f}(z) = \text{const} \). The constant is equal to zero because \( g_1 \neq 1 \) and/or \( g_\tau \neq 1 \).

**Appendix B: Proof of auxiliary lemmas**

**Proof of Lemma 1.** We consider the left-hand side of (2.17),

\[
Y_{l,j}^h R_{b,j_{n+1}}^h \cdots R_{b,j_1}^h = \prod_{j \in J} R_{b,j}^h \cdots \prod_{j \in J} R_{b,j_{n+1}}^h \cdot R_{b,j_{n+1}}^h \cdots R_{b,j_1}^h.
\]

(B.1)
and move $R^{h}_{b_{j_{n+1}}} \to$ to the left using the Yang–Baxter equation:

$$
\prod_{j \in J} R^{h}_{b_{j}} R^{h}_{b_{j_{n+1}}} = (R^{h}_{b_{j_{1}}} \cdots R^{h}_{b_{j_{n+1}}}) = R^{h}_{b_{j_{1}}} \cdots R^{h}_{b_{j_{n+1}}},
$$

In the second and in the last line we used (2.2), and in the third line, we used (2.1). In the same way, by moving the product $R^{h}_{b_{j_{n+1}}} \cdots R^{h}_{b_{j_{1}}}$ to the left through $\prod_{j \in J} R^{h}_{b_{j}}$, we obtain

$$
\prod_{j \in J} R^{h}_{b_{j}} R^{h}_{b_{j_{n-1}}} \cdots R^{h}_{b_{j_{1}}} = R^{h}_{b_{j_{n-1}}} \cdots R^{h}_{b_{j_{1}}},
$$

Applying (B.3) to each product $\prod_{j \in J} R^{h}_{b_{j}}$, we obtain

$$
\prod_{j \in J} R^{h}_{b_{j}} = R^{h}_{b_{j_{n}}} \cdots R^{h}_{b_{j_{1}}},
$$

Applying (B.3) to each product $\prod_{j \in J} R^{h}_{b_{j}}$ in (B.1) and moving each $R^{h}_{b_{j_{n}}}$ to the right, we arrive at (2.18).

**Proof of Lemma 2.** The $R$-matrix arguments of the form $x_{a} - x_{b}$ appear in $\mathcal{F}_{1}$, and there are no such arguments in $\mathcal{F}_{2}$. Therefore,

$$
\text{Res}_{x_{a}=x_{b}} \mathcal{F} = \text{Res}_{x_{a}=x_{b}} \mathcal{F}_{1}, \quad b = 1, \ldots, a - 1, a + 1, \ldots, N,
$$

and we need to prove that

$$
\text{Res}_{x_{a}=x_{b}} \mathcal{F}_{1} = 0. \quad (B.5)
$$

We consider the expression for $\mathcal{F}_{1}$ in (3.4). Due to the structure of poles in (A.14), a nonzero contribution to the left-hand side of (B.5) is made by the terms in $\mathcal{F}_{1}$ that contain $R^{h}_{ab}$ or $R^{h}_{ba}$. This can happen in two cases:

1. $a \in I$ and $b \in I^{c}$,
2. $b \in I$ and $a \in I^{c}$.

Let $a < b$. We fix some set $J$ of $(k-1)$ elements that do not contain $a$ and $b$ and set $J^{\bullet} = J^{c} \setminus \{a, b\}$. We choose $I = J \cup \{a\}$ and $I^{c} = J^{\bullet} \cup \{b\}$ in the first case and $I = J \cup \{b\}$ and $I^{c} = J^{\bullet} \cup \{a\}$ in the second case. We now show that the residues coming from the first case are exactly canceled by the residues coming from the second case:

$$
\text{Res}_{z_{a}=z_{b}} R^{h}_{J \cup \{a\}, J^{\bullet} \cup \{b\}} Y^{h}_{N_{2}, J \cup \{a\}} R^{h}_{J^{\bullet} \cup \{b\}} Y^{h}_{N_{2}, J \cup \{b\}} R^{h}_{J \cup \{b\}, J^{\bullet} \cup \{a\}} = - \text{Res}_{z_{a}=z_{b}} R^{h}_{J \cup \{b\}, J^{\bullet} \cup \{a\}} Y^{h}_{N_{2}, J \cup \{b\}} R^{h}_{J \cup \{b\}, J^{\bullet} \cup \{a\}}. \quad (B.6)
$$

To prove this, we use relations (2.6), (2.7) and (2.17) to separate those $R$-matrices in the left-hand side of (B.6) that contain the index $a$ or $b$:

$$
R^{h}_{J \cup \{a\}, J^{\bullet} \cup \{b\}} = (R^{h}_{J \cup \{a\}, J^{\bullet} \cup \{b\}} R^{h}_{J^{\bullet} \cup \{b\}, J \cup \{a\}})^{-1}, \\
Y^{h}_{N_{2}, J \cup \{a\}} = R^{h}_{a_{j_{n+1}}, \cdots a_{j_{1}}}, \quad Y^{h}_{N_{2}, J \cup \{b\}} = R^{h}_{a_{ik}, \cdots a_{jk}}, \\
R^{h}_{J \cup \{a\}, J^{\bullet} \cup \{b\}} = (R^{h}_{J \cup \{a\}, J^{\bullet} \cup \{b\}})^{-1}. \quad (B.7)
$$

We note that the structure of terms in $\mathcal{F}_{1}$ excludes the appearance of higher-order poles. This is why we are only interested in simple poles.
Then using (B.7)–(B.9), we simplify the left-hand side of (B.6) before taking the residue:

\[
\mathcal{R}^{h}_{j,a},J\cup\{b\} \mathcal{Y}^{h}_{N_{2},J},J\cup\{a\} \mathcal{R}^{h}_{j,a},J\cup\{b\} = \mathcal{R}^{h}_{a},J\cup\{a\} \mathcal{R}^{h}_{j,a},J\cup\{b\} \mathcal{R}^{h}_{j,a},J\cup\{b\} (\mathcal{R}^{h}_{j,a},J\cup\{b\})^{-1} \times \\
\times \mathcal{Y}^{h}_{N_{2},J} \mathcal{Y}^{h}_{N_{2},J} (\mathcal{Y}^{h}_{a},J^{-1})^{-1} (\mathcal{R}^{h}_{j,a},J\cup\{b\})^{-1} (\mathcal{R}^{h}_{j,a},J\cup\{b\})^{-1} (\mathcal{R}^{h}_{j,a},J\cup\{b\})^{-1}.
\]

(B.10)

Here, we canceled \((\mathcal{R}^{h}_{J,a},J\cup\{b\})^{-1}\) from (B.7) and \(\mathcal{R}^{h}_{a,j_{1n+1}} \cdots \mathcal{R}^{h}_{a,j_{k-1}}\) from (B.8) and also used the identity

\[
(\mathcal{R}^{h}_{a,j_{1}},J\cup\{b\})^{-1} (\mathcal{R}^{h}_{J,a},J^{-1}) = (\mathcal{Y}^{h}_{a},J^{-1}).
\]

Next, we can move \((\mathcal{R}^{h}_{j,a},J\cup\{b\})^{-1}\) to the right through \(\mathcal{Y}^{h}_{N_{2},J} (\mathcal{Y}^{h}_{a},J^{-1})^{-1}\) because these expressions contain pairwise distinct indices:

\[
\mathcal{R}^{h}_{J,a},J\cup\{b\} \mathcal{Y}^{h}_{N_{2},J\cup\{a\}} \mathcal{R}^{h}_{j,a},J\cup\{b\} = \mathcal{R}^{h}_{a},J\cup\{a\} \mathcal{R}^{h}_{j,a},J\cup\{b\} \mathcal{R}^{h}_{j,a},J\cup\{b\} \times \\
\times \mathcal{Y}^{h}_{N_{2},J\cup\{a\}} (\mathcal{Y}^{h}_{a},J^{-1})^{-1} (\mathcal{R}^{h}_{j,a},J\cup\{b\})^{-1} (\mathcal{R}^{h}_{j,a},J\cup\{b\})^{-1} \times \\
\times \mathcal{R}^{h}_{j,a},J\cup\{b\} \mathcal{R}^{h}_{J,a},J\cup\{b\} \mathcal{R}^{h}_{j,a},J\cup\{b\}.
\]

(B.11)

Calculating the residue at \(z_{a} = z_{b}\), replacing \(\mathcal{R}^{h}_{a,b}\) in \(\mathcal{R}^{h}_{a,b},J\cup\{a\}\) with the permutation operator \(P_{ab}\), and moving it to the right, we then obtain

\[
\text{Res}_{z_{a} = z_{b}} \mathcal{R}^{h}_{J,a},J\cup\{b\} \mathcal{Y}^{h}_{N_{2},J\cup\{a\}} \mathcal{R}^{h}_{j,a},J\cup\{b\} = \\
= \mathcal{R}^{h}_{a,a+1} \cdots \mathcal{R}^{h}_{a,b-1} \mathcal{R}^{h}_{a,N} \mathcal{R}^{h}_{b-1,b} \cdots \mathcal{R}^{h}_{a+1,b} \times \\
\times \mathcal{Y}^{h}_{N_{2},J\cup\{a\}} (\mathcal{Y}^{h}_{a},J^{-1})^{-1} (\mathcal{R}^{h}_{j,a},J\cup\{b\})^{-1} (\mathcal{R}^{h}_{j,a},J\cup\{b\})^{-1} \times \\
\times \mathcal{R}^{h}_{J,a},J\cup\{b\} \mathcal{R}^{h}_{j,a},J\cup\{b\} \mathcal{R}^{h}_{j,a},J\cup\{b\}.
\]

(B.12)

In the last equality, we used (2.8) to simplify the underbraced factors to \(\prod_{j=a+1}^{b-1} \phi(h, z_{a} - z_{j}) \phi(h, z_{j} - z_{a}) \mathcal{R}^{h}_{a},J\cup\{a\} \mathcal{R}^{h}_{j,a},J\cup\{a\} \mathcal{R}^{h}_{j,a},J\cup\{a\} \mathcal{Y}^{h}_{N_{2},J\cup\{a\}} \mathcal{Y}^{h}_{N_{2},J\cup\{a\}} \)

Also, following definitions (2.3) and (2.4), we performed the transformations \(\mathcal{R}^{h}_{b,b+1} \cdots \mathcal{R}^{h}_{b,N} = \mathcal{R}^{h}_{b},J\cup\{a\}\) and \(\mathcal{R}^{h}_{a-1,a} \cdots \mathcal{R}^{h}_{a} = \mathcal{R}^{h}_{J,a},J\cup\{a\}\).

We transform the right-hand side of (B.6) similarly,

\[
\text{Res}_{z_{a} = z_{b}} \mathcal{R}^{h}_{J,b},J\cup\{a\} \mathcal{Y}^{h}_{N_{2},J\cup\{b\}} \mathcal{R}^{h}_{j,b},J\cup\{a\} = \text{Res}_{z_{a} = z_{b}} \mathcal{R}^{h}_{b},J\cup\{a\} \mathcal{R}^{h}_{j,a},J\cup\{a\} \mathcal{R}^{h}_{J,a},J\cup\{a\} \times \\
\times \mathcal{Y}^{h}_{N_{2},J\cup\{b\}} (\mathcal{Y}^{h}_{b},J^{-1})^{-1} (\mathcal{R}^{h}_{j,a},J\cup\{a\})^{-1} (\mathcal{R}^{h}_{j,a},J\cup\{a\})^{-1} \times \\
\times \mathcal{R}^{h}_{j,a},J\cup\{a\} \mathcal{R}^{h}_{j,a},J\cup\{a\} \mathcal{R}^{h}_{j,a},J\cup\{a\}.
\]

(B.13)
and move the permutation $P_{ab}$ to the left:

$$
\text{Res}_{z_a = z_b} \mathcal{R}^h_{J \cup \{b\}, J' \cup \{a\}} y^h_{N_2, J \cup \{b\}, J' \cup \{a\}} = - \mathcal{R}^h_{\{b\}, J \cup \{a\}, J' \cup \{a\}} \mathcal{R}^h_{J, J', \{a\}} \mathcal{R}^h_{J, J'} \times \\
\times \phi(h, z_a - z_j) \phi(h, z_j - z_a).
$$

(B.14)

In the last equality, we used (2.8) to simplify the underbraced factors to $\prod_{j=a+1}^{b-1} \phi(h, z_a - z_j) \phi(h, z_j - z_a)$ and used definitions (2.5). The right-hand side of (B.14) equals minus the right-hand side of (B.12). This finishes the proof.

**Proof of Lemma 3.** The proof is done for $\mathcal{F}_1$ and $\mathcal{F}_2$ separately because relation (3.12) holds for $\mathcal{F}_1$ and for $\mathcal{F}_2$ separately. The poles at $z_a = z_b$ are contained in the middle products $\mathcal{Y}^h_{N_2, I}$ and $\mathcal{Y}^{-h}_{N_1, J}$ in (3.4) and (3.5) for $I$ and $J$ such that $a \in I$ and $b \in J$.

The calculation for $\mathcal{F}_1$ is straightforward. For any set $I$ that contains $a$, the product $\mathcal{Y}^h_{N_2, I}$ has only one term with $R^h_{ba}$ and therefore has a simple pole at $z_a = z_b$. Due to property (A.14), we must replace $R_{ba}$ with the permutation $-P_{ab}$ and set $z_a = z_b$. To transform the residue to form (3.12), we move the permutation operator to the right.

First, we use relations (2.6), (2.7) and (2.17), (2.18) to rewrite each term in $\mathcal{F}_1$ by separating the $R$-matrices that contain the indices $a$ and $b$. We fix

$$
I = \{i_1 < \cdots < i_m < i_{m+1} < \cdots < i_k\}
$$

and transform $\mathcal{R}^h_{I, I'}$ using (2.7):

$$
\mathcal{R}^h_{I, I'} = \mathcal{R}^h_{\{a\}, N_1 \setminus \{a\}} \mathcal{R}^h_{\{a\}, I'} (\mathcal{R}^h_{\{a\}, I'} \mathcal{R}^h_{\{a\}, I'})^{-1} = \\
= R^h_{a, a+1} \cdots R^h_{a, N_1 \setminus \{a\}} R^h_{I \setminus \{a\}, I'} (R^h_{a, i_{m+1}} \cdots R^h_{a, i_k})^{-1},
$$

(B.15)

Similar transformations are performed using (2.7):

$$
\mathcal{R}^h_{I, I'} = (\mathcal{R}^h_{\{a\}, I'} \mathcal{R}^h_{\{a\}, N_1 \setminus \{a\}})^{-1} R^h_{\{a\}, I'} R^h_{a, a+1} \cdots R^h_{a, a-1}.
$$

(B.16)

Consecutively applying (2.17) and (2.18) to $\mathcal{Y}^h_{N_2, I}$, we rewrite it as

$$
\mathcal{Y}^h_{N_2, I} = R^h_{b, i_1} \cdots R^h_{b, i_m+1} R^h_{b, b+1, b} \cdots R^h_{2N_2, b} y^h_{N_2, \{b\}, I \setminus \{a\}} (\mathcal{Y}^h_{N_2 \setminus \{b\}, I \setminus \{a\}} (\mathcal{Y}^h_{N_2 \setminus \{b\}, I \setminus \{a\}} y^h_{N_2 \setminus \{b\}, I \setminus \{a\}}) \times \\
\times R^h_{b, b+1, b} \cdots R^h_{2N_2, b} (R^h_{b, i_1} \cdots R^h_{b, i_m+1} R^h_{b, b+1, b} \cdots R^h_{2N_2, b})^{-1}.
$$

(B.17)

We calculate the residue of $\mathcal{F}_1$ at $z_a = z_b$. The nontrivial part is $\mathcal{Y}^h_{N_2, I}$, while the terms $\mathcal{R}^h_{I, I'}$ and $\mathcal{R}^h_{I, I'}$ have no poles. Substituting

$$
\mathcal{Y}^h_{\{b\}, I \setminus \{a\}} = R^h_{b, i_1} \cdots R^h_{b, i_m+1} R^h_{b, b+1, b} \cdots R^h_{b, i_k}
$$

we get

$$
\mathcal{Y}^h_{N_2, I} = R^h_{b, i_1} \cdots R^h_{b, i_m+1} R^h_{b, b+1, b} \cdots R^h_{b, i_k}.
$$

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and
\[ \mathcal{Y}^h_{N_2 \setminus \{b\}, \{a\}} = R^h_{N+1, a} \cdots R^h_{b-1, a} R^h_{b+1, a} \cdots R^h_{2N, a} \]
in (B.17), we obtain
\[
\text{Res}_{z=a} \mathcal{Y}^h_{N_2, I} = -R^h_{a, i_{m+1}} \cdots R^h_{a, i_k} R^h_{b+1, b} \cdots R^h_{2N, b} \mathcal{Y}^h_{N_2 \setminus \{b\}, \{a\}} \times
\]
\[
\times {R^h_{b, i_j} \cdots R^h_{b, i_{m-1}} R^h_{b+1, i_{m+1}} \cdots R^h_{b+1, i_k}} 
\times P_{ab}(R^h_{a, i_{m+1}} \cdots R^h_{a, i_k} R^h_{b+1, b} \cdots R^h_{2N, b})^{-1}.
\]
(B.18)

To simplify, we move the permutation operator \( P_{ab} \) to the right and then cancel the underlined factors:
\[
\text{Res}_{z=a} \mathcal{Y}^h_{N_2, I} = -R^h_{a, i_{m+1}} \cdots R^h_{a, i_k} R^h_{b+1, b} \cdots R^h_{2N, b} \mathcal{Y}^h_{N_2 \setminus \{b\}, \{a\}} \times
\]
\[
\times {R^h_{b, i_j} \cdots R^h_{b, i_{m-1}} R^h_{b+1, i_{m+1}} \cdots R^h_{b+1, i_k}} 
\times P_{ab}(R^h_{a, i_{m+1}} \cdots R^h_{a, i_k} R^h_{b+1, b} \cdots R^h_{2N, b})^{-1}.
\]
(B.19)

Indeed, \( R^h_{b, i_{m+1}} \cdots R^h_{b, i_k} \) commutes with \( R^h_{N+1, a} \cdots R^h_{b-1, a} \) due to (2.2) because each factor has pairwise different indices. The underlined factors therefore cancel.

We next calculate the residue at \( z = z_b \) in each summand containing the index \( a \in I \) for \( \mathcal{F}_1 \) given by (3.4). We use (B.15), (B.16), and (B.19) to obtain
\[
\text{Res}_{z=a} \mathcal{R}^h_{I, I', \mathcal{Y}^h_{N_2, I} \mathcal{R}^h_{I', I''}} = -R^h_{a, a+1} \cdots R^h_{a, N} R^h_{b+1, b} \cdots R^h_{2N, b} \mathcal{Y}^h_{N_2 \setminus \{b\}, \{a\}} \times
\]
\[
\times {R^h_{b, i_j} \cdots R^h_{b, i_{m-1}} R^h_{b+1, a} \cdots R^h_{b-1, a}} 
\times P_{ab}(R^h_{a, i_{m+1}} \cdots R^h_{a, i_k} R^h_{b+1, b} \cdots R^h_{2N, b})^{-1}.
\]
(B.20)

Next, we use (2.14) to move \( P_{ab} \) to the right and the underbraced factors to the left through \( \mathcal{R}^h_{I \setminus \{a\}, I''} \), which does not contain the indices \( b, b + 1, \ldots, 2N \):
\[
\text{Res}_{z=a} \mathcal{R}^h_{I, I', \mathcal{Y}^h_{N_2, I} \mathcal{R}^h_{I', I''}} = -R^h_{a, a+1} \cdots R^h_{a, N} R^h_{b+1, b} \cdots R^h_{2N, b} \mathcal{Y}^h_{N_2 \setminus \{b\}, \{a\}} \times
\]
\[
\times {R^h_{b, i_j} \cdots R^h_{b, i_{m-1}} R^h_{b+1, a} \cdots R^h_{b-1, a}} 
\times P_{ab}(R^h_{a, i_{m+1}} \cdots R^h_{a, i_k} R^h_{b+1, b} \cdots R^h_{2N, b})^{-1}.
\]
(B.21)

In the last equality, we combined the first factors into \( G \) using definition (3.13) and canceled the underlined factors. Then, moving \( R^h_{N+1, a} \cdots R^h_{b-1, a} \) through \( \mathcal{R}^h_{I \setminus \{a\}, I''} \) to the right and using definition (3.14), we obtain
\[
\text{Res}_{z=a} \mathcal{R}^h_{I, I', \mathcal{Y}^h_{N_2, I} \mathcal{R}^h_{I', I''}} = -G \mathcal{R}^h_{I \setminus \{a\}, I'} \mathcal{Y}^h_{N_2 \setminus \{b\}, \{a\}} \mathcal{R}^h_{I \setminus \{a\}, I''}.
\]
(B.22)

This yields
\[
\text{Res}_{z=a} \mathcal{F}_1[k, N] = -G \mathcal{F}_1[k - 1, N - 1] H.
\]
(B.23)
The same calculation can be performed for $\mathcal{F}_2$, which differs from $\mathcal{F}_1$ by only replacing $I$ with $J$, $N_2$ with $N_1$, $\hbar \to -\hbar$, and by the overall sign. Therefore, using the result in (B.22), we have

$$\text{Res}_{z_a = z_b} R_{J,J'}^{h} Y_{N_1}^{-h} R_{J,J'}^{-h} = -G' R_{J,J'}^{-h} Y_{N_1}^{-h} R_{J,J'}^{-h} H', \quad (B.24)$$

where

$$G' = R_{b,b+1}^{-h} \cdots R_{b,2N}^{-h} R_{a+1,a}^{-h} \cdots R_{N,a}^{-h},$$

$$H' = R_{1,b}^{-h} \cdots R_{a-1,b}^{-h} R_{a,N+1}^{-h} \cdots R_{a,b-1}^{-h} P_{ab}. \quad (B.25)$$

We used (2.14) to replace each $R_{i,j}^{-h}$ in (B.25) and (B.26) with $-R_{j,i}^{-h}$:

$$G' = (-1)^{3N-a-b} R_{b+1,b}^{h} \cdots R_{2N,b}^{h} R_{a,a+1}^{h} \cdots R_{N,a}^{h} = (-1)^{a+b-N} G, \quad (B.27)$$

$$H' = (-1)^{a-1+b-N-1} R_{b,b+1}^{h} \cdots R_{b,a-1}^{h} R_{a+1,a}^{h} \cdots R_{N-1,a}^{h} P_{ab} = (-1)^{a+b-N-1} H. \quad (B.28)$$

Due to (B.27) and (B.28), we can replace $G'$ in (B.24) with $G$ and $H'$ with $H$, whence

$$\text{Res}_{z_a = z_b} \mathcal{F}_2[k, N] = -G\mathcal{F}_2[k - 1, N - 1]H. \quad (B.29)$$

This finishes the proof.

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