Günter’s formalism ($k$-symplectic formalism) in classical field theory: Skinner–Rusk approach and the evolution operator

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Abstract

The first aim of this paper is to extend the Skinner-Rusk formalism on classical mechanics for first-order field theories. The second is to generalize the definition and properties of the evolution $K$-operator on classical mechanics for first-order field theories using in both cases Günter’s formalism ($k$-symplectic formalism).

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1 Introduction

The Skinner-Rusk formalism \cite{11} was developed in order to give a geometrical unified formalism for describing mechanical systems. It incorporates all the characteristics of Lagrangian and Hamiltonian descriptions of these systems (including dynamical equations and solutions, constraints, Legendre map, evolution operators, equivalence, etc.).

This formalism has been generalized to time-dependent mechanical systems \cite{7}, and also to the multisymplectic description of first-order field theories \cite{8, 25}.

The first aim of this paper is to extend this unified framework to Günther’s description of first-order classical field theories \cite{21}, and show how this description comprises the main features of the Lagrangian and Hamiltonian formalisms, both for the regular and singular cases.

Let us point out that Günther’s formalism should be also called $k$-symplectic formalism because the base of this formalism are the standard polysymplectic manifolds, introduced by Günther in \cite{21}, which coincide with the $k$-symplectic manifolds introduced by Awane in \cite{1, 2, 3}. Günther’s paper gives a geometric Hamiltonian formalism for field theories. The crucial device is the introduction of a vector-valued generalization of a symplectic form, called a polysymplectic form. One of the advantages of this formalism is that only the tangent and cotangent bundle of a manifold are required to develop it. In \cite{31} Günther’s formalism was revised and clarified. It was shown that the polysymplectic structures used by Günther to develop his formalism could be replaced by the $k$-symplectic structures defined by Awane \cite{1, 2, 3}. So this formalism could be called $k$-symplectic formalism.

The $k$-symplectic formalism is the generalization to field theories of the standard symplectic formalism in mechanics, which is the geometric framework for describing autonomous dynamical systems. In this sense, the $k$-symplectic formalism is used to give a geometric description of certain kind of field theories: in a local description, those whose Lagrangian does not depend on the coordinates in the basis (in many of them, the space-time coordinates); that is, it is only valid for Lagrangian $L(q^i, v^j_A)$ and Hamiltonian $H(q^i, p^A_i)$ that depends on the field coordinates $q^i$ and on the partial derivatives of the field $v^j_A$. A natural extension of this formalism is the so-called $k$-cosymplectic formalism, which is the generalization to field theories of the cosymplectic formalism which describes geometrically non-autonomous mechanical systems (this description can be found in \cite{28, 20}). It is devoted to describing field theories involving the independent parameters $(t^1, \ldots, t^k)$ on the Lagrangian $L(t^A, q^i, v^j_A)$ and on the Hamiltonian $H(t^A, q^i, p^A_i)$.

It is interesting to remark here that the polysymplectic formalism developed by G. Sardanashvily et al. \cite{13, 14, 40}, based on a vector valued form on some associated fiber bundle, is a different description of classical field theories of first order than the polysymplectic formalism proposed by Günther. (See also \cite{28} for more details on the polysymplectic formalism). In addition, we must remark that the soldering form on the linear frames bundles is a polysymplectic form, and its study and applications to field theory constitute the $n$-symplectic geometry developed by L. K. Norris in \cite{34, 35, 36, 37, 32}.

The so-called time-evolution $K$-operator in mechanics (also known by some authors as the relative Hamiltonian vector field \cite{38}) is a tool which has mainly been developed in order to study the Lagrangian and Hamiltonian formalisms for singular mechanical systems and their equivalence. This operator was introduced in \cite{4} and \cite{22}, and later it was defined geometrically in two different but equivalent ways \cite{5, 16} for autonomous dynamical systems. In \cite{16}, a further different
geometric construction is given, using a canonical map introduced by Tulczyjew [42]. The $K$-operator relates the sets of solutions of the Euler-Lagrange equations and the Hamilton equations; it also relates constraints on the Lagrangian and Hamiltonian sides, and allows us to obtain a complete classification of constraints [4]; as well as Lagrangian Noether infinitesimal symmetries from a Hamiltonian generator of symmetries [33 11 12 17]. It is also used for studying Lagrangian systems whose Legendre map has generic singularities [38 39].

The second aim of this paper is to generalize the definition and properties of this operator for first-order field theories in order to describe the relationship between the Lagrangian and Hamiltonian $k$-symplectic formalisms. In particular we extend the results in [16], showing how to obtain the solutions of Lagrangian and Hamiltonian field equations by means of this operator. The same idea has been developed in [2] but using the multisymplectic description of classical field theories.

The organization of the paper is as follows: Section 2-4 are devoted to reviewing the main features of Günther’s formalism or $k$-symplectic formalism [21 31] of Lagrangian and Hamiltonian field theories.

In particular, in Section 2 the field theoretic phase space is introduced as the Whitney sum $(T^1_k)^*Q = T^*Q \oplus \cdots \oplus T^*Q$ of $k$-copies of the cotangent bundle $T^*Q$ of a manifold $Q$. This space is the canonical example of a polysymplectic manifold. A particular case of polysymplectic manifolds are the $k$-symplectic manifolds (see Refs. [3 4 5 8 9]) which coincide with the standard polysymplectic manifolds.

The field theoretic state space is introduced as the Whitney sum $T^k_1Q = TQ \oplus \cdots \oplus TQ$ of $k$-copies of the tangent bundle $TQ$ of a manifold $Q$. This manifold has a canonical $k$-tangent structure defined by $k$ tensor fields of type $(1,1)$ satisfying certain algebraic properties. The $k$-tangent manifolds were introduced in de León et al. [26 27], and they generalize the tangent manifolds (see Refs. [6 10 19 20 24 27]).

Section 3 is devoted to giving a geometric interpretation of the second order partial differential equations. Here we show that these equations can be characterized by using the canonical $k$-tangent structure of $T^1_kQ$, which generalizes the case of Classical Mechanics.

The Hamiltonian and Lagrangian formalisms are developed in Section 4. Lagrangian formalism is developed using the canonical $k$-tangent structure of $T^1_kQ$, or the Legendre transformation as in Günther [21].

In section 5 we develop the unified formalism for field theories, which is based on the use of the Whitney sum $T^1_kQ \oplus (T^1_k)^*Q$ of $T^1_kQ$ and $(T^1_k)^*Q$. There are canonical presymplectic forms on it (the pull-back of the canonical symplectic form on each $T^*Q$) and a natural coupling function which is defined by the contraction between vectors and covectors. Then, given a Lagrangian $L \in C^\infty(T^1_kQ)$ we can state a field equation on $T^1_kQ \oplus (T^1_k)^*Q$. This equation has solution only on a submanifold $M_L$, which is the graph of the Legendre map. Then we prove that if $Z = (Z_1, \ldots, Z_k)$ is an integrable $k$-vector field, solution to this equation and tangent to $M_L$, then the projection onto the first factor $T^1_kQ$ of the integral sections of $Z$ are solutions of the Euler-Lagrange field equations. If $L$ is regular the converse also holds. Furthermore, we establish the relationship between $Z$ and the Hamiltonian and the Lagrangian $k$-vector fields of the $k$-symplectic formalism, $X_H$ and $X_L$.

In Section 6 we review the definition and the main properties of the evolution operator $K$ for autonomous mechanics. Next we define the field operators which, as a consequence of the field equations on the $k$-symplectic formalism, are given as a $k$-vector field along the Legendre
transformation FL, associated to the lagrangian \( L : T^1_0 Q \to \mathbb{R} \), satisfying certain properties. Finally we finish with similar results for field theories to those obtained in [16] and [9].

In a forthcoming paper we shall extend the results of this paper to the \( k \)-cosymplectic formalism [28, 29].

Manifolds are real, paracompact, connected and \( C^\infty \). Maps are \( C^\infty \). Sum over crossed repeated indices is understood.

## 2 Geometric framework: autonomous case

### 2.1 The cotangent bundle of \( k^1 \)-covelocities of a manifold

Let \( Q \) be a differentiable manifold of dimension \( n \) and \( \tau^* : T^* Q \to Q \) its cotangent bundle. Let us denote by \( (T^1_k)^* Q = T^* Q \oplus \cdots \oplus T^* Q \) the Whitney sum of \( k \) copies of \( T^* Q \), with projection map 
\[
\tau^* : (T^1_k)^* Q \to Q, \quad \tau^* (\alpha_1, \ldots, \alpha_k) = q, \quad \text{for every} \ (\alpha_1, \ldots, \alpha_k) \in (T^1_k)^* Q.
\]

\((T^1_k)^* Q\) can be canonically identified with the vector bundle \( J^1(Q, \mathbb{R}^k)_0 \) of \( k^1 \)-covelocities of the manifold \( Q \), that is the vector bundle of 1-jets of maps \( \sigma : Q \to \mathbb{R}^k \) with target at \( 0 \in \mathbb{R}^k \) and projection map \( \tau^*_Q : J^1(Q, \mathbb{R}^k)_0 \to Q, \quad \tau^*_Q (j^1_q \sigma) = q \), say,
\[
J^1(Q, \mathbb{R}^k)_0 \equiv \frac{T^* Q \oplus \cdots \oplus T^* Q}{(d\sigma^1(q), \ldots, d\sigma^k(q))}
\]

where \( \sigma^A = \pi^A \circ \sigma : Q \to \mathbb{R} \) is the \( A \)-th component of \( \sigma \), and \( \pi^A : \mathbb{R}^k \to \mathbb{R} \) is the canonical projection \( 1 \leq A \leq k \). For this reason to \( (T^1_k)^* Q \) is also called the bundle of \( k^1 \)-covelocities of the manifold \( Q \).

If \((q^i)\) are local coordinates on \( U \subseteq Q \), then the induced local coordinates \((q^i, p_i), 1 \leq i \leq n\), on \( T^* U = (\tau^*)^{-1}(U) \), are given by
\[
q^i(\alpha_q) = q^i(q), \quad p_i(\alpha_q) = \alpha_q \left( \frac{\partial}{\partial q^i} \bigg|_q \right), \quad \alpha_q \in T^* Q,
\]
and the induced local coordinates \((q^i, p^A_i), 1 \leq i \leq n, 1 \leq A \leq k\), on \( (T^1_k)^* U = (\tau^*_Q)^{-1}(U) \) are given by
\[
q^i(\alpha^1_q, \ldots, \alpha^k_q) = q^i(q), \quad p^A_i(\alpha^1_q, \ldots, \alpha^k_q) = \alpha^A_q \left( \frac{\partial}{\partial q^A} \bigg|_q \right).
\]

Let us denote by \( \{r_1, \ldots, r_k\} \) the canonical basis of \( \mathbb{R}^k \).

**Definition 2.1** (Günther [27]) A closed non-degenerate \( \mathbb{R}^k \)-valued 2-form
\[
\tilde{\omega} = \sum_{A=1}^k \omega_A \otimes r_A
\]
on a manifold \( M \) of dimension \( N \) is called a polysymplectic form. The pair \((M, \tilde{\omega})\) is a polysymplectic manifold.
The manifold \((T^1_k)^*Q\) is endowed with a \textit{canonical polysymplectic structure}. This canonical structure \(\bar{\omega} = \sum_{A=1}^{k} (\omega_0)_A \otimes r_A\), on \((T^1_k)^*Q\) is defined by
\[
(\omega_0)_A = (\tau_A^*)^*(\omega_0), \quad 1 \leq A \leq k,
\]
where \(\tau_A^* : (T^1_k)^*Q \to T^*Q\) is the projection on the \(A^{th}\)-copy \(T^*Q\) of \((T^1_k)^*Q\), and \(\omega_0 = -d\theta_0\) is the canonical symplectic structure of \(T^*Q\), \(\theta_0\) being the Liouville 1-form defined by
\[
\theta_0(\alpha_q)(\tilde{X}_{\alpha_q}) = \alpha_q((\tau^*)^*(\alpha_q)(\tilde{X}_{\alpha_q})), \quad \alpha_q \in T^*Q, \quad \tilde{X}_{\alpha_q} \in T_{\alpha_q}(T^*Q).
\]

One can also define the 2-forms \((\omega_0)_A\) by \((\omega_0)_A = -d(\theta_0)_A\) where \((\theta_0)_A = (\tau_A^*)^*\theta_0\).

Thus the Liouville 1-form and the canonical symplectic structure on \(T^*Q\) are locally given by
\[
\theta_0 = p_i dq^i, \quad \omega_0 = -d\theta_0 = dq^i \wedge dp_i,
\]
and the canonical polysymplectic structure \(((\omega_0)_1, \ldots, (\omega_0)_k)\) on \((T^1_k)^*Q\) is locally given by
\[
(\omega_0)_A = -d(\theta_0)_A = -d(p_i^A dq^i) = dq^i \wedge dp_i^A.
\]

\textbf{Definition 2.2} \textit{(Günter [21]) A polysymplectic form \(\bar{\omega}\) on a manifold \(M\) is called standard iff for every point of \(M\) there exists a local coordinate system such that \(\omega_A\) is written locally as in (1).}

So the canonical polysymplectic form \(\bar{\omega}\) on \((T^1_k)^*Q\) is standard.

\textbf{Remark 2.1} The \(k\)-symplectic manifolds were introduced in Awane [1, 2, 3] and they coincide with the \textit{standard polysymplectic} manifolds, as we now shall show.

\textbf{Definition 2.3} \textit{(Awane [4]) A \(k\)-symplectic structure on a manifold \(M\) of dimension \(N = n + kn\) is a family \((\omega_A, V; 1 \leq A \leq k)\), where each \(\omega_A\) is a closed 2-form and \(V\) is an integrable \(nk\)-dimensional distribution on \(M\) such that}
\[
(i) \quad \omega_A|_{V \times V} = 0, \quad (ii) \quad \cap_{A=1}^{k} \ker \omega_A = \{0\}.
\]

\textit{In this case \((M, \omega_A, V)\) is called a \(k\)-symplectic manifold.}

\textbf{Theorem 2.1} \textit{(Awane [4]) Let \((\omega_A, V; 1 \leq A \leq k)\) be a \(k\)-symplectic structure on \(M\). About every point of \(M\) we can find a local coordinate system \((q^i, p_i^A), 1 \leq i \leq n, 1 \leq A \leq k\), such that}
\[
\omega_A = dq^i \wedge dp_i^A, \quad 1 \leq A \leq k.
\]
The canonical model of $k$-symplectic manifolds is also $(T_k^1)^*Q$ and the canonical $k$-symplectic structure $(\omega_A, V; 1 \leq A \leq k)$, on $(T_k^1)^*Q$ is given by
\[ \omega_A = (\omega_0)_A = (\tau_A^*)^*(\omega_0), \quad V(j_{q,v}^1) = \ker(\tau_Q^*)(j_{q,v}^1). \]

Therefore, the 2-forms of the canonical polysymplectic structure and the canonical $k$-symplectic structure on $(T_k^1)^*Q$ coincide.

From [22] we know that the standard polysymplectic structures and the $k$-symplectic structures coincide. Indeed, if $\bar{\omega} = \sum_{A=1}^k \omega_A \otimes r_A$ is a standard polysymplectic structure on $M$, given a local adapted coordinate system $(q^i, p_A^i)$ we can define, locally, the distribution $V$, of dimension $nk$, by $dq^1 = \ldots = dq^n = 0$. Then, $(\omega_1, \ldots, \omega_k, V)$ is a $k$-symplectic structure on $M$.

Conversely if $(\omega_1, \ldots, \omega_k, V)$ is a $k$-symplectic structure on $M$ then $\bar{\omega} = \sum_{A=1}^k \omega_A \otimes r_A$ is a standard polysymplectic structure on $M$, because it is trivially standard and is non degenerate as a consequence of (ii) in Definition 2.3.

As we shall see later, in his Hamiltonian formalism, Günther uses a standard polysymplectic manifold because he needs to have local coordinates $(q^i, p_A^i)$ in the manifold $M$ where the Hamiltonian is defined, which is equivalent to considering a $k$-symplectic manifold. For this reason we will call the Günther’s formalism, called polysymplectic formalism, $k$-symplectic formalism.

### 2.2 The tangent bundle of $k^1$-velocities of a manifold

Let $\tau : TQ \to Q$ be the tangent bundle of $Q$. Let us denote by $T_k^1Q$ the Whitney sum $TQ \oplus . k \cdot TQ$ of $k$ copies of $TQ$, with projection $\tau_Q : T_k^1Q \to Q, \tau_Q(v_{1q}, \ldots, v_{kq}) = q$.

$T_k^1Q$ can be identified with the vector bundle $J_0^1(\mathbb{R}^k, Q)$ of the $k^1$-velocities of the manifold $Q$, that is, the vector bundle of 1-jets of maps $\sigma : \mathbb{R}^k \to Q$ with source at $0 \in \mathbb{R}^k$, and projection map $\tau_Q : T_k^1Q \to Q, \tau_Q(J_0^1(q, \sigma)) = \sigma(0) = q$, say
\[ J_k^1(\mathbb{R}^k, Q) \equiv TQ \oplus . k \cdot TQ \]
\[ J_0^1(q, \sigma) \equiv (v_{1q}, \ldots, v_{kq}) \]

where $q = \sigma(0)$, and $v_{Aq} = \sigma_*((\partial/\partial t^A)(0))$, $1 \leq A \leq k$. For this reason $T_k^1Q$ is called the tangent bundle of $k^1$-velocities of $Q$.

If $(q^i)$ are local coordinates on $U \subseteq Q$ then the induced local coordinates $(q^i, v^i)$, $1 \leq i \leq n$, on $TU = \tau^{-1}(U)$ are given by
\[ q^i(v_q) = q^i(q), \quad v^i(v_q) = v_q(q^i), \quad v_q \in TQ, \]
and the induced local coordinates $(q^i, v_A^i)$, $1 \leq i \leq n$, $1 \leq A \leq k$, on $T_k^1U = \tau_Q^{-1}(U)$ are given by
\[ q^i(v_{1q}, \ldots, v_{kq}) = q^i(q), \quad v_A^i(v_{1q}, \ldots, v_{kq}) = v_{Aq}(q^i). \]

We now introduce the canonical $k$-tangent structure on $T_k^1Q$. 

Definition 2.4 For a vector $X_q$ at $Q$, and for $1 \leq A \leq k$, we define its vertical $A$-lift $(X_q)^A$ as the vector on $T^1_k Q$ given by

$$(X_q)^A(v_1q, \ldots, v_kq) = \frac{d}{ds}(v_1q, \ldots, v_{A-1}q, v_Aq + sX_q, v_{A+1}q, \ldots, v_kq)|_{s=0}$$

for all points $(v_1q, \ldots, v_kq) \in T^1_k Q$.

In local coordinates we have

$$(X_q)^A = a^i \frac{\partial}{\partial v^i_A} |_q$$

for a vector $X_q = a^i (\partial/\partial q^i)(q)$.

The canonical $k$-tangent structure on $T^1_k Q$ is the set $(S^1, \ldots, S^k)$ of tensor fields of type $(1,1)$ defined by

$$S^A(v)(Z_v) = (\tau_Q)_*(v)(Z_v))^A, \text{ for all } Z_v \in T_v(T^1_k Q), v = (v_1q, \ldots, v_kq),$$

for each $1 \leq A \leq k$.

From (3) we have in local coordinates

$$S^A = \frac{\partial}{\partial v^i_A} \otimes dq^i$$

The tensors $S^A$ can be regarded as the $(0, \ldots, 0, 1, 0, \ldots, 0)$-lift of the identity tensor on $Q$ to $T^1_k Q$ defined by Morimoto [30].

Remark 2.2 The $k$-tangent manifolds were introduced as a generalization of the tangent manifolds by de León et al. [26, 27]. The canonical model of these manifolds is $T^1_k Q$ with the structure given by $(S^1, \ldots, S^k)$.

To develop later the Lagrangian formalism, we now construct a polysymplectic structure on $T^1_k Q$, for each regular Lagrangian $L: T^1_k Q \to \mathbb{R}$, using its canonical $k$–tangent structure.

Definition 2.5 A Lagrangian $L: T^1_k Q \to \mathbb{R}$ is called regular if and only if

$$\det \left( \frac{\partial^2 L}{\partial v^i_A \partial v^j_B} \right) \neq 0, \quad 1 \leq i, j \leq n, \quad 1 \leq A, B \leq k.$$
Proposition 2.1 $L : T^1_k Q \rightarrow \mathbb{R}$ is a regular Lagrangian if and only if $(\omega_L)_1, \ldots, (\omega_L)_k$ is a polysymplectic structure on $T^1_k Q$.

This polysymplectic structure, associated to $L$, was also introduced by Günther [21] using the Legendre transformation.

The Legendre map $F_L : T_1^k Q \rightarrow (T^1_k)^* Q$, was introduced by Günther [21], and we rewrite it as follows: if $(v_1, \ldots, v_k) \in (T^1_k)_q Q$

$$[F_L(v_1, \ldots, v_k)]^A(w_q) = \frac{d}{ds} L(v_1, \ldots, v_A + s w_q, \ldots, v_k)|_{s=0},$$

for each $1 \leq A \leq k$. We deduce that $F_L$ is locally given by

$$(q^i, v^i_A) \mapsto \left( q^i, \frac{\partial L}{\partial v^i_A} \right). \quad (6)$$

In fact, from (5) and (6), we easily obtain the following Lemma.

Lemma 2.1 For every $1 \leq A \leq k$, $(\omega_L)_A = (F_L)^*(\omega_0)_A$, where $(\omega_0)_1, \ldots, (\omega_0)_k$ are the 2-forms of the canonical polysymplectic structure or canonical $k$-symplectic structure of $(T^1_k)^* Q$.

Then, from (6) we get:

Proposition 2.2 Let $L$ be a Lagrangian. The following conditions are equivalent:

1) $L$ is regular. 2) $F_L$ is a local diffeomorphism. 3) $(\omega_L)_1, \ldots, (\omega_L)_k$ is a polysymplectic structure on $T^1_k Q$.

Remark 2.3 If $F_L$ is a global diffeomorphism, then $L$ is called a hyper-regular Lagrangian.

3 $k$-vector fields. Second order partial differential equations on $T^1_k Q$

3.1 $k$-vector fields

Let $M$ be an arbitrary manifold and $\tau_M : T^1_k M \rightarrow M$ its tangent bundle of $k^1$-velocities.

Definition 3.1 A section $X : M \rightarrow T^1_k M$ of the projection $\tau_M$ will be called a $k$-vector field on $M$. 

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Since $T_k^1 M$ is the Whitney sum $TM \oplus T^k. \oplus TM$ of $k$ copies of $TM$, we deduce that a $k$-vector field $X$ defines a family of $k$ vector fields $\{X_1, \ldots, X_k\}$ on $M$ by projecting $X$ onto every factor. For this reason we will denote a $k$-vector field $X$ by $(X_1, \ldots, X_k)$.

**Definition 3.2** An integral section of the $k$-vector field $X = (X_1, \ldots, X_k)$ passing through a point $x \in M$ is a map $\phi : U_0 \subset \mathbb{R}^k \to M$, defined on some neighborhood $U_0$ of $0 \in \mathbb{R}^k$, such that

$$\phi(0) = x, \quad \phi_*(t) \left( \frac{\partial}{\partial t^A} \right)_t = X_A(\phi(t)) \quad \text{for every} \quad t \in U_0, \quad 1 \leq A \leq k,$$

or equivalently, $\phi$ satisfies $X \circ \phi = \phi^{(1)}$, where $\phi^{(1)}$ is the first prolongation of $\phi$ defined by

$$\phi^{(1)} : U_0 \subset \mathbb{R}^k \to T_k^1 M \to \mathbb{R}^k \to (1) = \phi^{(1)}(t) = j_0^1 \phi_t, \quad \phi_*(t) = \phi(t + t),$$

for every $t, \bar{t} \in \mathbb{R}^k$ such that $\bar{t} + t \in U_0$.

In local coordinates:

$$\phi^{(1)}(t^1, \ldots, t^k) = \left( \phi^i(t^1, \ldots, t^k), \frac{\partial \phi^i}{\partial t^A}(t^1, \ldots, t^k) \right), \quad 1 \leq A \leq k, \quad 1 \leq i \leq n. \quad (7)$$

We say that a $k$-vector field $X = (X_1, \ldots, X_k)$ on $M$ is integrable if there is an integral section passing through each point of $M$.

We remark that a $k$-vector field $X$ is integrable if, and only if, $\{X_1, \ldots, X_k\}$ define an involutive distribution on $M$.

### 3.2 Second-order partial differential equations in $T_k^1 Q$

The aim of this subsection is to characterize the integrable $k$-vector fields on $T_k^1 Q$ such that their integral sections are canonical prolongations of maps from $\mathbb{R}^k$ to $Q$.

In general, if $F : M \to N$ is a differentiable map, then the induced map $T_k^1(F) : T_k^1 M \to T_k^1 N$ defined by $T_k^1(F)(j_0^1 g) = j_0^1(F \circ g)$ is given by

$$T_k^1(F)(v_{q,1}^1, \ldots, v_{q,k}^1) = (F_*(q)v_{1,q}, \ldots, F_*(q)v_{k,q}) \quad ,$$

where $v_{1,q}, \ldots, v_{k,q} \in T_q Q, q \in Q$, and $F_*(q) : T_q M \to T_{F(q)} N$.

**Definition 3.3** A $k$-vector field on $T_k^1 Q$, that is, a section $X : T_k^1 Q \to T_k^1(T_k^1 Q)$ of the projection $\pi_{T_k^1 Q} : T_k^1(T_k^1 Q) \to T_k^1 Q$, is a second order partial differential equation (SOPDE) if it is also a section of the vector bundle $T_k^1 \tau_Q : T_k^1(T_k^1 Q) \to T_k^1 Q$; that is,

$$T_k^1 \tau_Q \circ X = Id_{T_k^1 Q} \quad (8)$$

where $T_k^1(\tau)$ is defined by $T_k^1(\tau)(j_0^1 \gamma) = j_0^1(\tau \circ \gamma)$. 
Let \((q^i)\) be a coordinate system on \(Q\) and \((q^i, v^i_A)\) the induced coordinate system on \(T^1_k Q\). From a direct computation in local coordinates we obtain that the local expression of a SOPDE \((X_1, \ldots, X_k)\) is
\[
X_A(q^i, v^i_A) = v^i_A \frac{\partial}{\partial q^i} + (X_A)_B^i \frac{\partial}{\partial v^i_B}, \quad 1 \leq A \leq k.
\]
(9)

If \(\varphi : \mathbb{R}^k \to T^1_k Q\), is an integral section of \((X_1, \ldots, X_k)\) locally given by \(\varphi(t) = (\varphi_i^j(t), \varphi_j^i(t))\) then
\[
X_A(\varphi(t)) = \varphi_A^i(t) \frac{\partial}{\partial q^i} + (X_A)_B^i \frac{\partial}{\partial v^i_B}, \quad 1 \leq A \leq k.
\]
(10)

From (9) we obtain the following:

**Proposition 3.1** Let \(X = (X_1, \ldots, X_k)\) be an integrable SOPDE. If \(\varphi\) is an integral section then \(\varphi = \phi^{(1)}\) where \(\phi^{(1)}\) is the first prolongation of the map \(\phi = \tau \circ \varphi : \mathbb{R}^k \to T^1_k Q \to Q\), and satisfies
\[
\frac{\partial \phi^i_A}{\partial t^A(t)} = (X_A)_B^i(\phi^{(1)}(t)).
\]
(10)

Conversely, if \(\phi : \mathbb{R}^k \to Q\) is any map satisfying (10) then \(\phi^{(1)}\) is an integral section of \((X_1, \ldots, X_k)\).

**Definition 3.4** Let \((X_1, \ldots, X_k)\) be an integrable SOPDE. A map \(\phi : \mathbb{R}^k \to \mathbb{R}\) is said to be a solution to the SOPDE if the first prolongation \(\phi^{(1)}\) is an integral section of \((X_1, \ldots, X_k)\).

A \(k\)-vector field which is an integrable SOPDE is called a holonomic \(k\)-vector field, and its integral sections \(\varphi = \phi^{(1)}\) are called holonomic sections.

Now we show how to characterize the SOPDE’s using the canonical \(k\)-tangent structure of \(T^1_k Q\).

**Definition 3.5** The Liouville vector field \(C\) on \(T^1_k Q\) is the infinitesimal generator of the following flow
\[
\mathbb{R} \times T^1_k Q \quad \rightarrow \quad T^1_k Q \quad (s, (v_1, \ldots, v_k)) \quad \rightarrow \quad (e^s v_1, \ldots, e^s v_k),
\]
and in local coordinates has the form
\[
C = \sum_{i,B} v^i_B \frac{\partial}{\partial v^i_B}.
\]
(11)

We can write \(C = C_1 + \ldots + C_k\) where \(C_A, 1 \leq A \leq k\), are the canonical vector fields on \(T^1_k Q\) given by the following flows
\[
\mathbb{R} \times T^1_k Q \quad \rightarrow \quad T^1_k Q \quad (s, (v_1, \ldots, v_k)) \quad \rightarrow \quad (v_1, \ldots, v_{A-1q}, e^s v_{Aq}, v_{A+1q}, \ldots, v_k).
\]
In local coordinates

\[ C_A = \sum_i v_A^i \frac{\partial}{\partial v_A^i}. \] (12)

From (4), (9), (11) and (12) we deduce the following:

**Proposition 3.2** A \( k \)-vector field \( X = (X_1, \ldots, X_k) \) on \( T^1_k Q \) is a SOPDE if, and only if, \( S^A(X_A) = C_A \), for all \( 1 \leq A \leq k \), where \( (S^1, \ldots, S^k) \) is the canonical \( k \)-tangent structure on \( T^1_k Q \).

### 4 Hamiltonian and Lagrangian formalism [21, 31]

#### 4.1 Hamiltonian formalism

Let \((M, \omega_A, V)\) be a \( k \)-symplectic manifold, and \( H : M \to \mathbb{R} \) a Hamiltonian function. Let \( X = (X_1, \ldots, X_k) \) be a \( k \)-vector field on \( M \) that satisfies the equations

\[ \sum_{i=1}^k i_X A \omega = dH. \] (13)

If \( X_A \) is locally given by

\[ X_A = (X_A)^i \frac{\partial}{\partial q^i} + (X_A)^B \frac{\partial}{\partial v^B}, \]

in a local system of canonical coordinates \((q^i, p^A_i)\), (whose existence is ensured by the Theorem 2.1) then (13) is equivalent to the equations

\[ \frac{\partial H}{\partial q^i} = -\sum_{A=1}^k (X_A)_i^A, \quad \frac{\partial H}{\partial p^A_i} = (X_A)^i. \]

So if \((X_1, \ldots, X_k)\) is also integrable then its integral sections \( \varphi : \mathbb{R}^k \to M \), with \( \varphi(t) = (\varphi^i(t), \varphi^A_A(t)) \) are solutions to the Hamilton-De Donder-Weyl field equations

\[ \frac{\partial H}{\partial q^i} = -\sum_{A=1}^k \frac{\partial \varphi^A}{\partial t^A}, \quad \frac{\partial H}{\partial p^A_i} = \frac{\partial \varphi^i}{\partial t^A}, \quad 1 \leq A \leq k, \quad 1 \leq i \leq n. \] (14)

So, equation (13) is a geometric version of the Hamilton-De Donder-Weyl field equations.

#### 4.2 Lagrangian formalism

In this subsection, we recall the Lagrangian formalism developed by Günter [21].
In general, given a Lagrangian function of the form \( L = L(q^i, v^i_A) \), and using a variational principle, one obtains the Euler-Lagrange field equations for \( L \):

\[
\sum_{A=1}^{k} \frac{d}{dt^A} \left( \frac{\partial L}{\partial v^i_A} \right) - \frac{\partial L}{\partial q^i} = 0, \quad v^i_A = \frac{\partial q^i}{\partial t^A}. \tag{15}
\]

Then, let \( L : T^1_k Q \rightarrow \mathbb{R} \) be a Lagrangian, and let us consider the 2-forms \( (\omega^L_1, \ldots, \omega^L_k) \) on \( T^1_k Q \) defined by \( L \), and \( E_L = C(L) - L \), \( C \) being the Liouville vector field in \( T^1_k Q \). Now, let \( X = (X_1, \ldots, X_k) \) be a \( k \)-vector field in \( T^1_k Q \) (that is, a section \( X : T^1_k Q \rightarrow T^1_k(T^1_k Q) \)) of the projection \( \tau_{T^1_k Q} : T^1_k(T^1_k Q) \rightarrow T^1_k Q \). Then:

**Proposition 4.1** If \( X = (X_1, \ldots, X_k) \) is an integrable SOPDE, and \( \psi \equiv \phi^{(1)} : \mathbb{R}^k \rightarrow T^1_k Q \) is an integral section of \( X \), then \( X \) is a solution to the equation

\[
\sum_{A=1}^{k} t_{X_A}(\omega^L)_A = dE_L \tag{16}
\]

if, and only if, \( \phi : \mathbb{R}^k \rightarrow Q \) is a solution to the Euler-Lagrange equations (15).

**Proof:** If each \( X_A \) is locally given by

\[
X_A = (X^i_A) = \left( \frac{\partial}{\partial q^i} + (X^j_A)_B \frac{\partial}{\partial v^j_B} \right),
\]

then, from (5), (11) and (16) we deduce that \( (X_1, \ldots, X_k) \) is a solution to (16) if, and only if, \( (X^i_A) \) and \( (X^j_A)_B \) satisfy the system of equations

\[
\left( \frac{\partial^2 L}{\partial q^i \partial v^j_A} - \frac{\partial^2 L}{\partial q^j \partial v^i_A} \right) (X^j_A) - \frac{\partial^2 L}{\partial v^i_A \partial v^j_B} (X^j_A) = v^i_A \frac{\partial^2 L}{\partial q^i \partial v^j_A} - \frac{\partial L}{\partial q^i}, \tag{17}
\]

\[
\frac{\partial^2 L}{\partial v^j_B \partial v^i_A} (X^j_A) = \frac{\partial^2 L}{\partial v^j_B \partial v^i_A} v^i_A. \tag{18}
\]

But, as \( X \) is a SOPDE, we have

\[
(X^i_A) = v^i_A, \tag{19}
\]

then (18) holds identically, and (17) is equivalent to

\[
\frac{\partial^2 L}{\partial q^i \partial v^j_A} v^i_A + \frac{\partial^2 L}{\partial v^i_A \partial v^j_B} (X^j_A) = \frac{\partial L}{\partial q^i}. \tag{20}
\]

Now, if \( \psi(t) = \phi^{(1)} = (\phi^i(t), \phi^i_A(t)) \) is an integral section of \( X \), then

\[
(X^i_A)(\psi(t)) = \phi^i_A(t) = \frac{\partial \phi^i}{\partial t^A}, \tag{21}
\]

\[
(X^j_A)_B(\psi(t)) = \frac{\partial \phi^j_B}{\partial t^A} = \frac{\partial^2 \phi^i}{\partial t^A \partial t^B}, \tag{22}
\]
and going to (20) we obtain that
\[
\frac{\partial^2 L}{\partial q_i \partial v^i_A} (\phi(t)) \frac{\partial \phi^i}{\partial t^A} + \frac{\partial^2 L}{\partial v^i_B \partial v^i_A} (\phi(t)) \frac{\partial \phi^i}{\partial t^A} = \frac{\partial^2 L}{\partial q_i \partial v^i_A} (\phi(t)) \frac{\partial \phi^i}{\partial t^A} + \frac{\partial^2 L}{\partial v^i_B \partial v^i_A} (\phi(t)) \frac{\partial^2 \phi^i}{\partial t^A \partial B} = \frac{\partial L}{\partial q^i} (\phi(t))
\]
which are the Euler-Lagrange equations for the map \( \phi \).

Conversely, let \( X \) be an integrable SOPDE having \( \psi(t) = \phi^{(1)} = (\phi^i(t), \phi^A_A(t)) \) as integral sections, for every \( (\phi^i(t)) \) solution to the Euler-Lagrange equations. Therefore (21) and (22) hold since \( X \) is a SOPDE, and then (23), which holds because \( (\phi^i(t)) \) is a solution to the Euler-Lagrange equations, is equivalent to (20). Hence \( X \) is a solution to (16).

\[ \blacksquare \]

In this way, equation (16) can be considered as a geometric version of the Euler-Lagrange field equations.

Observe that, if the Lagrangian is regular, equation (18) leads to conclude that every solution to (16) is a SOPDE. In addition, equation (20) leads to defining local solutions to (16) in a neighborhood of each point of \( T_k^1 Q \) and, using a partition of unity, global solutions to (16).

Now let us suppose that the Lagrangian \( L : T^1_k Q \to \mathbb{R} \) is hyper-regular, that is, \( FL \) is a diffeomorphism. We consider the Hamiltonian \( H : (T^1_k)^* Q \to \mathbb{R} \) defined by \( H = E_L \circ FL^{-1} \) where \( FL^{-1} \) is the inverse map of \( FL \). Then:

**Theorem 4.1**

a) If \( X_L = ((X_L)_1, \ldots, (X_L)_k) \) is a solution to (16) then \( X_H = ((X_H)_1, \ldots, (X_H)_k) \), where \( (X_H)_A = FL_*((X_L)_A), 1 \leq A \leq k \), is a solution to (15) with \( \omega_A = (\omega_0)_A \) and \( H = E_L \circ FL^{-1} \).

b) If \( X_L = ((X_L)_1, \ldots, (X_L)_k) \) is integrable, \( \phi^{(1)} \) is an integral section and \( \phi = \tau \circ \phi^{(1)} \), then \( \varphi = FL \circ \phi^{(1)} \) is an integral section of \( X_H = ((X_H)_1, \ldots, (X_H)_k) \) and thus it is a solution to the Hamilton-De Donder Weyl equations (14) for \( H = E_L \circ FL^{-1} \).

**Proof:** a) It is an immediate consequence of (13) and (16) using that \( FL_* (\omega_0)_A = (\omega_L)_A \) and \( E_L = H \circ FL^{-1} \).

b) It is an immediate consequence of Definition 3.2 of integral section of a \( k \)-vector field.

\[ \blacksquare \]

**Definition 4.1** A singular Lagrangian system \( (T_k^1 Q, (\omega_L)_1, \ldots, (\omega_L)_k) \) is called almost-regular if \( P := FL(T_k^1 Q) \) is a closed submanifold of \( (T_k^1)^* Q \) (we will denote the natural imbedding by \( j_0 : P \hookrightarrow (T_k^1)^* Q, FL \) is a submersion onto its image, and the fibres \( FL^{-1}(FL(v)) \), for every \( v \in T_k^1 Q \), are connected submanifolds of \( T_k^1 Q \).
In this case there exists \( H_0 \in C^\infty(\mathcal{P}) \) such that \((FL_0)^*H_0 = E_L\), where \( FL_0 : T^1_k Q \to \mathcal{P} \) is defined by \( j_0 \circ FL_0 = FL \), and the Hamiltonian field equation analogous to \( (13) \) is

\[
\sum_{i=1}^k \iota_{(X_0)_A} \omega^0_A = dH_0
\]

(24)

where \( \omega^0_A = j_0(\omega)_A \), for every \( 1 \leq A \leq k \), and \( X_0 = ((X_0)_1, \ldots, (X_0)_k) \) (if it exists) is a \( k \)-vector field on \( \mathcal{P} \).

5  Skinner-Rusk formulation

5.1  The Skinner-Rusk formalism for \( k \)-symplectic field theories

Let us consider the Whitney sum \( T^1_k Q \oplus_Q (T^1_k)^*Q \), with coordinates \((q^i, v^i_A, \alpha^A_q)\). It has natural bundle structures over \( T^1_k Q \) and \((T^1_k)^*Q\). Let us denote by \( pr_1 : T^1_k Q \oplus_Q (T^1_k)^*Q \to T^1_k Q \) the projection into the first factor, \( pr_1(q^i, v^i_A, \alpha^A_q) = (q^i, v^i_A) \), and \( pr_2 : T^1_k Q \oplus_Q (T^1_k)^*Q \to (T^1_k)^*Q \) the projection into the second factor, \( pr_2(q^i, v^i_A, \alpha^A_q) = (q^i, \alpha^A_q) \).

In this bundle, we have some canonical structures. First, let \( ((\omega^0)_1, \ldots, (\omega^0)_k) \) be the canonical polysymplectic structure on \((T^1_k)^*Q\). We shall denote by \((\Omega_1, \ldots, \Omega_k)\) the pull-back by \( pr_2 \) of these 2-forms to \( T^1_k Q \oplus_Q (T^1_k)^*Q \), that is, \( \Omega_A = (pr_2)^*(\omega^0)_A \), \( 1 \leq A \leq k \).

Furthermore, the coupling function in \( T^1_k Q \oplus_Q (T^1_k)^*Q \), denoted by \( C \), is defined as follows:

\[
C : \quad T^1_k Q \oplus_Q (T^1_k)^*Q \quad \to \quad \mathbb{R}
\]

\[
(v_1, \ldots, v_k, \alpha^1_q, \ldots, \alpha^k_q) \quad \mapsto \quad \sum_{A=1}^k \alpha^A_q(v_A q)
\]

Given a Lagrangian \( L \in C^\infty(T^1_k Q) \), we can define the Hamiltonian function in \( T^1_k Q \oplus_Q (T^1_k)^*Q \), denoted by \( \mathcal{H} \in C^\infty(T^1_k Q \oplus_Q (T^1_k)^*Q) \), as

\[
\mathcal{H}(v_1, \ldots, v_k, \alpha^1_q, \ldots, \alpha^k_q) = C(v_1, \ldots, v_k, \alpha^1_q, \ldots, \alpha^k_q) - (pr_1^* L)(v_1, \ldots, v_k, \alpha^1_q, \ldots, \alpha^k_q)
\]

which, in local coordinates, is given by

\[
\mathcal{H} = \sum_{A=1}^k \sum_{i=1}^n p^A_i v^i_A - L(q^i, v^i_A) .
\]

(25)

Now, the problem consists in finding the integral sections \( \psi : \mathbb{R}^k \to T^1_k Q \oplus (T^1_k)^*Q \) of an integrable \( k \)-vector field \( Z = (Z_1, \ldots, Z_k) \) on \( T^1_k Q \oplus_Q (T^1_k)^*Q \), such that

\[
\sum_{A=1}^k \iota_{Z_A} \Omega_A = d\mathcal{H} .
\]

(26)

Equation (26) gives a different kind of information. In fact, writing locally each \( Z_A \) as

\[
Z_A = (Z_A)^i \frac{\partial}{\partial q^i} + (Z_A)^i_B \frac{\partial}{\partial v^i_B} + (Z_A)^i_B \frac{\partial}{\partial p^i_B} ,
\]

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then, from (1), (25) and (26) we obtain

$$ p_i^A = \frac{\partial L}{\partial v_i^A} \circ pr_1 $$

(27)

$$(Z_A)^i = v_i^A$$

(28)

$$ \sum_{A=1}^{k} (Z_A)^i_A = \frac{\partial L}{\partial q_i} \circ pr_1 $$

(29)

where $1 \leq A \leq k$, $1 \leq i \leq n$. Then from (28) we have that $Z_A$ is locally given by

$$ Z_A = v_i^A \frac{\partial}{\partial q_i} + (Z_A)^i_B \frac{\partial}{\partial v^B_i} + (Z_A)^B_i \frac{\partial}{\partial p_i^B} . $$

(30)

So, in particular, we have obtained information of three different classes:

1. The constraint equations (27), which are algebraic (not differential) equations defining a submanifold $M_L$ of $T^1_kQ \oplus_Q (T^1_k)^* Q$ where the equation (26) has solution. Let us observe that this submanifold is just the graph of the Legendre map $FL$ defined by the Lagrangian $L$.

   We denote by $j: M_L \to T^1_kQ \oplus_Q (T^1_k)^* Q$ the natural imbedding, and by $pr^0_1: M_L \to T^1_k Q$ and $pr^0_2: M_L \to (T^1_k)^* Q$ the restricted projections of $pr_1$ and $pr_2$.

2. Equations (28) which are a holonomy condition similar to (19) and, as we will see in the next subsection (see Theorem 5.1), they force the integral sections of the $k$-vector field $Z$ to be lifting of sections $\phi: \mathbb{R}^k \to Q$. This property is similar to the one in the unified formalism of Classical Mechanics, and it reflects the fact that the geometric condition in the unified formalism is stronger than the usual one in the Lagrangian formalism.

3. Equations (29) which, taking into account (27) and (28), are just the classical Euler-Lagrange equations (see Theorem 5.1).

If $Z = (Z_1, \ldots, Z_k)$ is a solution to (26), then each $Z_A$ is tangent to the submanifold $M_L$ if, and only if, the functions $Z_A \left( p_j^B - \frac{\partial L}{\partial v^B_j} \circ pr_1 \right)$ vanish at the points of $M_L$, for every $1 \leq A, B \leq k$, $1 \leq j \leq n$. Then from (30) we deduce that this is equivalent to the following equations

$$ (Z_A)^B_j = v_i^A \frac{\partial^2 L}{\partial q^i \partial v^j_B} + (Z_A)^i_C \frac{\partial^2 L}{\partial v^i_C \partial v^j_B} . $$

(31)

Thus the problem to be solved is the following:

**Statement 5.1** To find an integral section $\psi: \mathbb{R}^k \to M_L \subset T^1_kQ \oplus (T^1_k)^* Q$ of an integrable $k$-vector field $Z = (Z_1, \ldots, Z_k)$ on $T^1_kQ \oplus_Q (T^1_k)^* Q$ solution to (26) taking values on $M_L$. (This means that $Z$ is tangent to $M_L$).
Remark 5.1 1. Equations (26) have not, in general, a unique solution. The solutions to (26) are given by \((Z_1, \ldots, Z_k) + \ker \Omega^\sharp\), where \((Z_1, \ldots, Z_k)\) is a particular solution, and \(\Omega^\sharp : T_k^1 (T_k^1 Q \oplus Q) \to T^*(T_k^1 Q \oplus Q)\) is defined as \(\Omega^\sharp (Y_1, \ldots, Y_k) = \sum_{A=1}^k \iota_{Y_A} \Omega_A\).

2. If \(L\) is regular, then taking into account (28) and (29) we can define a local \(k\)-vector field \((Z_1, \ldots, Z_k)\) on a neighborhood of each point in \(M_L\) which is a solution to (26). Each \(Z_A\) is locally given by \((Z_A)^i = v_A^i\), \((Z_A)^B_i = \frac{1}{k} \frac{\partial L}{\partial q^i} \delta^B_A\), with \((Z_A)^B_i\) satisfying (31). Now, by using a partition of the unity, one can construct a global \(k\)-vector field which is a solution to (26).

When the Lagrangian function \(L\) is singular we cannot assure the existence of consistent solutions for equation (26). Then we must develop a constraint algorithm for obtaining a constraint submanifold (if it exists) where these solutions exist. Next, we outline this procedure (see also [25], where a similar algorithm is sketched in the multisymplectic formulation).

First, in order to assure the existence of a Hamiltonian counterpart for the singular Lagrangian system we assume, from now on, that the singular Lagrangians are almost-regular.

We begin with \(P_0 = M_L\). Then, let \(P_1\) be the subset of \(P_0\) made of those points where there exists a solution to (26), that is,

\[P_1 = \{z \in P_0 | \exists (Z_1, \ldots, Z_k) \in (T_k^1)_z P_0 \text{ solution to (26)}\}\]

If \(P_1\) is a submanifold of \(P_0\), then there exists a section of the canonical projection \(\tau_{P_0} : T_k^1 P_0 \to P_0\) defined on \(P_1\) which is a solution to (26), but that does not define, in general, a \(k\)-vector field on \(P_1\). To find solutions taking values into \(T^1_k P_1\), we define a new subset \(P_2\) of \(P_1\) as follows

\[P_2 = \{z \in P_1 | \exists (Z_1, \ldots, Z_k) \in (T_k^1)_z P_0 \text{ solution to (26)}\}\]

If \(P_2\) is a submanifold of \(P_1\), then there exists a section of the canonical projection \(\tau_{P_2} : T_k^1 P_1 \to P_1\) defined on \(P_2\) which is a solution to (26), but that does not define, in general, a \(k\)-vector field on \(P_2\).

Proceeding further, we get a family of constraint manifolds

\[\ldots \hookrightarrow P_2 \hookrightarrow P_1 \hookrightarrow P_0 = M_L \hookrightarrow (T_k^1)^* Q \oplus T_k^1 Q\]

If there exists a natural number \(f\) such that \(P_{f+1} = P_f\) and \(\dim P_f > k\) then we call \(P_f\) the final constraint submanifold over which we can find solutions to equation (26). Let us observe that the solutions will not be unique (even in the regular case) and, in general, will not be integrable. In order to find integrable solutions to equation (26), a constraint algorithm based on the same idea must be developed.
5.2 The field equations for sections

$M_L$ being the graph of $FL$, it is diffeomorphic to $T_k^1Q$ (so $pr^0_1$ is a diffeomorphism). Let $Z = (Z_1, \ldots, Z_k)$ be an integrable $k$-vector field solution to $26$. Every integral section $\psi: t \in \mathbb{R}^k \rightarrow (\psi^1(t), \psi^A_i(t), \psi^A_i(t)) \in T_k^1Q \oplus_Q (T_k^1)^*Q$ of $Z$ solution to $26$ is of the form $\psi = (\psi_L, \psi_H)$, with $\psi_L = pr_1 \circ \psi: \mathbb{R}^k \rightarrow T_k^1Q$, and if $\psi$ takes values in $M_L$ then $\psi_H = FL \circ \psi_L$; in fact, from $27$ we obtain

$$\psi_H(t) = (pr_2 \circ \psi)(t) = (\psi^1(t), \psi^A_i(t)) = (\psi^1(t), \frac{\partial L}{\partial v_A^i} |_{\psi_L(t)}) = (FL \circ \psi_L)(t).$$

In this way, every constraint, differential equation, etc. in the unified formalism can be translated to the Lagrangian or the Hamiltonian formalisms by restriction to the first or the second factors of the product bundle. In particular, conditions $26$ generate, by $pr_2$-projection, the primary constraints of the Hamiltonian formalism for singular Lagrangians (i.e., the image of the Legendre $(\psi)$ obtain $27$ being the graph of $L$ $17$), $i$ be an integrable $k$-vector field in $T_k^1Q \oplus_Q (T_k^1)^*Q$ solution to $26$ and let $\psi: \mathbb{R}^k \rightarrow M_L \subset T_k^1Q \oplus_Q (T_k^1)^*Q$ be an integral section of $Z = (Z_1, \ldots, Z_k)$, with $\psi = (\psi_L, \psi_H) = (\psi_L, FL \circ \psi_L)$. Then $\psi_L$ is the canonical lift $\phi^{(1)}$ of the projected section $\phi = \tau_Q \circ pr_1 \circ \psi: \mathbb{R}^k \rightarrow Q$, and $\phi$ is a solution to the Euler-Lagrange field equations $15$.

**Theorem 5.1** Let $Z = (Z_1, \ldots, Z_k)$ be an integrable $k$-vector field in $T_k^1Q \oplus_Q (T_k^1)^*Q$ solution to $26$ and let $\psi: \mathbb{R}^k \rightarrow M_L \subset T_k^1Q \oplus_Q (T_k^1)^*Q$ be an integral section of $Z = (Z_1, \ldots, Z_k)$, with $\psi = (\psi_L, \psi_H) = (\psi_L, FL \circ \psi_L)$. Then $\psi_L$ is the canonical lift $\phi^{(1)}$ of the projected section $\phi = \tau_Q \circ pr_1 \circ \psi: \mathbb{R}^k \rightarrow Q$, and $\phi$ is a solution to the Euler-Lagrange field equations $15$.

**Proof:** If $\psi(t) = (\psi^1(t), \psi^A_i(t), \psi^A_i(t))$ is an integral section of $Z = (Z_1, \ldots, Z_k)$, then

$$Z_A(\psi(t)) = \frac{\partial \psi^1}{\partial q_t^A}(t) \frac{\partial}{\partial q_t^1}|_{\psi(t)} + \frac{\partial \psi^B}{\partial q_t^A}(t) \frac{\partial}{\partial q_t^B}|_{\psi(t)} + \frac{\partial \psi^A_i}{\partial q_t^B}(t) \frac{\partial}{\partial v_t^B}|_{\psi(t)}$$

(32)

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From (27), (28) and (32) we obtain

\[
\psi^i_A(t) = v^i_A(\psi(t)) = (Z_A)^i(\psi(t)) = \frac{\partial \psi^i}{\partial t^A}(t) \tag{33}
\]

\[
\psi^A_i(t) = p^A_i(\psi(t)) = \left(\frac{\partial L}{\partial v^A_i} \circ \text{pr}_1\right)(\psi(t)) = \frac{\partial L}{\partial v^A_i}\big|_{\psi(t)} \tag{34}
\]

\[
\frac{\partial \psi^B_i}{\partial t^A}(t) = (Z_A)^B_i(\psi(t)), \tag{35}
\]

Therefore from (29), (34) and (35) we obtain

\[
\frac{\partial L}{\partial q^i}(\psi_L(t)) = \sum_{A=1}^k (Z_A)^A_i(\psi(t)) = \sum_{A=1}^k \frac{\partial \psi^A_i}{\partial t^A}(t) = \sum_{A=1}^k \frac{\partial}{\partial t^A} \left(\frac{\partial L}{\partial v^A_i}\big|_{\psi_L(t)}\right)
\]

and from (33)

\[
\psi^A_i(t) = \frac{\partial \psi^i}{\partial t^A}(t).
\]

The last two equations are the Euler-Lagrange field equations for the section \(\phi(t) = (\psi^i(t)) = (\tau \circ \text{pr}_1 \circ \psi)(t)\), and \(\psi_L = \phi^{(1)}\).

\[\blacksquare\]

In addition, for the regular case we can prove:

**Proposition 5.1** Under the hypothesis of Theorem 5.1, if \(L\) is regular then \(\psi_H = FL \circ \psi_L\) is a solution to the Hamilton-De Donder-Weyl field equations (14), where the Hamiltonian \(H\) is locally given by \(H = FL = E_L\).

**Proof:** Since \(L\) is regular, \(FL\) is a local diffeomorphism and thus we can choose for each point in \(T^1_kQ\) an open neighborhood \(U \subset T^1_kQ\) such that \(FL|_U : U \to FL(U)\) is a diffeomorphism. So we can define \(H_U : FL(U) \to \mathbb{R}\) as \(H_U = (E_L)|_U \circ (FL|_U)^{-1}\).

Denoting by \(H = H_U\), \(E_L = (E_L)|_U\) and \(FL = FL|_U\), we have \(E_L = H \circ FL\) which we provides the identities

\[
\frac{\partial H}{\partial p^A_i} \circ FL = v^A_i, \quad \frac{\partial H}{\partial q^i} \circ FL = -\frac{\partial L}{\partial q^i} \tag{36}
\]

Now considering the open subset \(V = \psi_L^{-1}(U) \subset \mathbb{R}^k\) we have \(\psi_V : V \subset \mathbb{R}^k \to U \oplus FL(U) \subset M_L\), where \((\psi_L)|_V : V \subset \mathbb{R}^k \to U \subset T^1_kQ\) and \((\psi_H)|_V = FL \circ (\psi_L)|_V : V \subset \mathbb{R}^k \to FL(U) \subset (T^1_k)^*Q\).

Therefore from (28), (33), (34) and (36), for every \(t \in V \subset \mathbb{R}^k\) we obtain

\[
\frac{\partial H}{\partial p^A_i}\big|_{\psi_H(t)} = \left(\frac{\partial H}{\partial p^A_i} \circ FL\right)(\psi_L(t)) = v^A_i(\psi_L(t)) = \frac{\partial \psi^i}{\partial t^A}(t)
\]
and

\[ \frac{\partial H}{\partial q^i} \bigg|_{\psi_H(t)} = \left( \frac{\partial L}{\partial q^i} \circ FL \right)(\psi_L(t)) = \frac{\partial L}{\partial q^i} \bigg|_{\psi_L(t)} = -\sum_{A=1}^{k} (Z_A)^A_i(\psi(t)) = -\sum_{A=1}^{k} \frac{\partial \psi^A_i}{\partial t^A}(t) \]

from which we deduce that \((\psi_H)|_V\) is a solution to the Hamilton-De Donder-Weyl field equations (14).

Conversely, we can state:

**Proposition 5.2** If \(L\) is regular and \(X = (X_1, \ldots, X_k)\) is a solution to (16) then:

1. The \(k\)-vector field \(Z = (Z_1, \ldots, Z_k)\) given by
   \[ Z_A = (Id_{T^1_kQ} \oplus FL)_*(X_A), \quad 1 \leq A \leq k \]
   is a solution to (26).
2. If \(\psi_L : \mathbb{R}^k \to T^1_kQ\) is an integral section of \(X = (X_1, \ldots, X_k)\) (and thus, from Proposition 4.1 a solution to the Euler-Lagrange field equations) then \(\psi = (\psi_L, FL \circ \psi_L) : \mathbb{R}^k \to M_L \subset T^1_kQ \oplus (T^1_k)^*Q\) is an integral section of \(Z = (Z_1, \ldots, Z_k)\).

**Proof:**

1. If \(L\) is regular and \(X = (X_1, \ldots, X_k)\) is a solution to (16), then from Proposition 4.1 we know that \(X_A\) is a SOPDE and thus \(X_A\) is locally given by
   \[ X_A = v^i_A \frac{\partial}{\partial q^i} + (X_A)^i_B \frac{\partial}{\partial v^j_B}, \quad (37) \]
   where \((X_A)^i_B\) satisfy (20). Since the map \(Id_{T^1_kQ} \oplus FL : T^1_kQ \to M_L \subset T^1_kQ \oplus (T^1_k)^*Q\), is locally given by
   \[ (q^1, v^1) \to \left( q^1, v^j_A, \frac{\partial L}{\partial v^j_A} \right), \quad (38) \]
   from (37) and (38) we obtain
   \[ Z_A = (Id_{T^1_kQ} \oplus FL)_*(X_A) = v^i_A \frac{\partial}{\partial q^i} + \left( v^j_A \frac{\partial^2 L}{\partial q^j \partial v^i_C} + (X_A)^i_B \frac{\partial^2 L}{\partial v^j_B \partial v^i_C} \right) \frac{\partial}{\partial p^j_C} + (X_A)^i_B \frac{\partial}{\partial v^j_B} \quad (39) \]

Then from (20) and (39) we have that

\[ \sum_{A=1}^{k} (Z_A)^A_j = v^j_A \frac{\partial^2 L}{\partial q^j \partial p^j_A} + (X_A)^i_B \frac{\partial^2 L}{\partial v^i_B \partial v^j_A} = \frac{\partial L}{\partial q^j}, \quad (Z_A)^i_j = v^j_A, \quad Z_A \left( p^B_k - \frac{\partial L}{\partial v^j_B} \right) = 0, \]

that is, the \(k\)-vector field \(Z = (Z_1, \ldots, Z_k)\) is a solution to (26) and each \(Z_A\) is tangent to \(M_L\) for \(A : 1, \ldots, k\).
2. It follows from Definition 3.2 taking into account that $\text{pr}_2 \circ \psi = FL \circ \psi_L$.

\section*{Remark 5.2} The last result really holds for regular and almost-regular Lagrangians. In the almost-regular case, the proof is the same, but the sections $\psi$, $\psi_L$ and $\psi_H$ take values not on $M_L$, $T^1_kQ$ and $(T^1_k)^*Q$, but in the final constraint submanifold $P_f$ and on the projection submanifolds $\text{pr}_1(P_f) \hookrightarrow T^1_kQ$ and $\text{pr}_2(P_f) \hookrightarrow (T^1_k)^*Q$, respectively.

\subsection*{5.3 The field equations for $k$-vector fields}

The aim of this subsection is to establish the relationship between $k$-vector fields that are solutions to (16) and $k$-vector fields that are solutions to (26). The main result is the following:

\section*{Theorem 5.2} Let $Z = (Z_1, \ldots, Z_k)$ be a $k$-vector field on $M_L$ solution to (26). Then the $k$-vector field $X_L = ((X_L)_1, \ldots, (X_L)_k)$ on $T^1_kQ$ defined by

$$X_L \circ \text{pr}_1^0 = T^1_k(\text{pr}_1^0) \circ Z$$

(40) is a $k$-vector field solution to (16) (where $T^1_k(\text{pr}_1^0) : T^1_k(M_L) \to T^1_k(T^1_kQ)$ is the natural extension of $(\text{pr}_1^0)_*$).

Conversely, every $k$-vector field $X_L$ solution to (16) can be recovered in this way from a $k$-vector field $Z$ in $M_L$ solution to (26).

Moreover, the $k$-vector field $Z$ is integrable iff the $k$-vector field $X_L$ is holonomic.

\section*{Proof:} Since $\text{pr}_1^0 : M_L \to T^1_kQ$ is a diffeomorphism, then the $k$-vector field $X_L$ on $T^1_kQ$ defined by (40) is given by

$$(X_L)_A = ((\text{pr}_1^0)^{-1})^* Z_A, \quad 1 \leq A \leq k.$$  

(41)

Now, for every $1 \leq A \leq k$ we have that

$$j^* \Omega_A = (\text{pr}_1^0)^* (\omega_L)_A,$$

(42)

which follows from Lemma 2.1

$$j^* \Omega_A = j^*(pr_2)^*(\omega_0)_A = (pr_2^0)^*(\omega_0)_A = (FL \circ pr_1^0)^*(\omega_0)_A = (pr_1^0)^* FL^*(\omega_0)_A = (pr_1^0)^* (\omega_L)_A.$$  

(43)

On the other hand we obtain that

$$j^* \mathcal{H} = (pr_1^0)^* E_L,$$  

(43)
from the following computation

\[ j^*\mathcal{H} = j^*(C - (pr_1)^*L) = j^*C - j^*(pr_1)^*L \]

\[ = (pr_1^0)^*CL - (pr_1^0)^*L = (pr_1^0)^*E_L. \]

From (41) and (42) we deduce that

\[ \sum_{A=1}^{k} \iota_{Z_A}j^*\Omega_A = \sum_{A=1}^{k} \iota_{(pr_1^0)^*(X_L)_A}(pr_1^0)^*(\omega_L)_A = (pr_1^0)^*\left( \sum_{A=1}^{k} \iota_{(X_L)_A}(\omega_L)_A \right), \tag{44} \]

and from (43) we deduce that

\[ d(j^*\mathcal{H}) = d((pr_1^0)^*E_L) = (pr_1^0)^*dE_L. \tag{45} \]

Since \( pr_1^0 \) is a diffeomorphism, from (44) and (45) we deduce that the \( k \)-vector field \( Z \) is a solution to (26) iff the \( k \)-vector field \( X_L \) is a solution to (16).

Let us suppose now that the \( k \)-vector field \( Z \) is integrable. As a consequence of Theorem 6.1 for every integral section \( \psi = (\psi_L, FL \circ \psi_L) \) of \( Z \), \( \psi_L = \phi^{(1)} \), for \( \phi = \tau \circ pr_1 \circ \psi \). Then

\[ (X_L)_A(pr_1^0(\psi(t))) = (pr_1^0)_*(\psi(t))(Z_A(\psi(t))) = (pr_1^0 \circ \psi)_*(t) \left( \frac{\partial}{\partial t_A} \bigg|_q \right) \]

So, \( \psi_L = \phi^{(1)} \) is an integral section of \( X_L \), and hence \( X_L \) is holonomic.

Conversely, if \( X_L \) is holonomic then for every integral section \( \psi_L = \phi^{(1)} \) with \( \phi : \mathbb{R}^k \to Q \), the map \( \psi = (\psi_L, FL \circ \psi_L) \) is an integral section of \( Z \). In fact, from (41), for every \( 1 \leq A \leq k \)

\[ Z_A(\psi(t)) = ((pr_1^0)^*(X_L)_A)(\psi(t)) = ((pr_1^0)^{-1})_* (\psi_L(t))(X_L)_A(\psi_L(t)) \]

\[ = ((pr_1^0)^{-1})_* (\psi_L(t)) \left( \psi_L(t) \left( \frac{\partial}{\partial t_A}(t) \right) \right) = ((pr_1^0)^{-1} \circ \psi_L)_*(t) \left( \frac{\partial}{\partial t_A} \bigg|_q \right) \]

\[ = \psi_L(t) \left( \frac{\partial}{\partial t_A} \bigg|_q \right). \]

If \( L \) is regular, in a neighborhood of each point of \( T_k^1Q \) there exists a local solution \( X_L = ((X_L)_1, \ldots, (X_L)_k) \) to (16). As \( L \) is regular, \( FL \) is a local diffeomorphism, so this open neighborhood can be chosen in such a way that \( FL \) is a diffeomorphism onto its image. Thus in a neighborhood of each point of \( FL(T_k^1Q) \) we can define

\[ (X_H)_A = [(FL)^{-1}]^*(X_L)_A, \quad 1 \leq A \leq k. \]

or equivalently, in terms of \( k \)-vector fields

\[ T_k^1(FL) \circ X_L = X_H. \]

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Proposition 5.3 1. The local $k$-vector field $X_H = ((X_H)_{1}, \ldots, (X_H)_{k})$ is a solution to (13), where the Hamiltonian $H$ is locally given by $H \circ FL = E_L$. (In other words, the local $k$-vector fields $X_L$ and $X_H$ solution to (16) and (13), respectively, are FL-related).

2. Every local integrable $k$-vector field solution to (13) can be recovered in this way from a local integrable $k$-vector field $Z$ in $T^1_kQ \oplus (T^1_k)^*Q$ solution to (26).

Proof:

1. This is the local version of Theorem 4.1 a).

2. On the other hand, if $X_H$ is a local integrable $k$-vector field solution to (13), then we can obtain the FL-related local integrable $k$-vector field $X_L$ solution to (16). By Theorem 5.2 we recover $X_L$ by a local integrable $k$-vector field $Z$ solution to (26).

6 Field operators

6.1 The evolution operator $K$ in mechanics

The so-called time-evolution $K$-operator in mechanics (also known by some authors as the relative Hamiltonian vector field [33]) is a tool which has mainly been developed in order to study the Lagrangian and Hamiltonian formalisms for singular mechanical systems and their equivalence. It was first introduced in a non-intrinsic way in [4] as an “evolution operator” to connect both formalisms.

In Classical Mechanics, the evolution operator $K$ associated with a Lagrangian $L : TQ \rightarrow \mathbb{R}$ is a map $K : TQ \rightarrow T(T^*Q)$ satisfying the following conditions (see [16]):

1. (Structural condition): $K$ is a vector field along $FL$, that is, $\tau_{T^*Q} \circ K = FL$, where $FL$ is the Legendre map defined by $L$ and $\tau_{T^*Q} : T(T^*Q) \rightarrow T^*Q$ is the natural projection.

2. (Dynamical condition): $(FL)^*(\omega \circ FL) = dE_L$, where $\omega$ is the canonical symplectic form on $T^*Q$ and $E_L = CL - L$, being $C$ the Liouville vector field on $TQ$.

3. (Second-order condition): $T(\tau^*) \circ K = Id_{TQ}$, where $\tau^* : T^*Q \rightarrow Q$ is the canonical projection.

The existence and uniqueness of this operator is studied in [16]. Its local expression is

$$K = v^i \left( \frac{\partial}{\partial q^i} \circ FL \right) + \frac{\partial L}{\partial q^i} \left( \frac{\partial}{\partial p_i} \circ FL \right).$$

By definition $\varphi : \mathbb{R} \rightarrow TQ$ is an integral curve of $K$ if

$$T(FL) \circ \varphi = K \circ \varphi ,$$

(46)
where \( \varphi : \mathbb{R} \to T(TQ) \) is the prolongation of \( \varphi \) to the tangent bundle \( T(TQ) \) of \( TQ \). So we have the diagram

\[
\begin{array}{c}
\mathbb{R} \\
\varphi \\
\downarrow \\
TQ \\
\varphi \\
\downarrow \\
T(TQ) \\
\downarrow T(FL) \\
\downarrow T(T^*Q) \\
\downarrow \tau_{TQ} \\
\downarrow K \\
\downarrow \tau_{T^*Q} \\
T^*Q \\
\end{array}
\]

Moreover, \( \varphi = \overline{\varphi} \), for \( \phi : \mathbb{R} \to Q \), that is, \( \varphi \) is holonomic.

The most relevant properties of this operator are the following:

- If there exists an Euler-Lagrange vector field \( X_L \) on \( TQ \), that is, a solution to the equation \( \iota_{X_L} \omega_L = dE_L \), then \( \varphi : \mathbb{R} \to TQ \) is an integral curve of \( X_L \) if, and only if, it is an integral curve of \( \mathcal{K} \); that is, relation (46) holds.

As a direct consequence of this fact, the relation between \( \mathcal{K} \) and \( X_L \) is

\[
T(FL) \circ X_L = \mathcal{K}.
\]  

In general, if the dynamical system is singular, the Euler-Lagrange vector fields exist only on a submanifold \( S \hookrightarrow TQ \).

- If there exists a Hamilton-Dirac vector field \( X_H \) on \( T^*Q \) associated with the the Lagrangian system \( (TQ, \omega_L, E_L) \) (that is, a vector field solution to the Hamilton-Dirac equations in the Hamiltonian formalism), then \( \psi : \mathbb{R} \to T^*Q \) is an integral curve of \( X_H \) if, and only if,

\[
\dot{\psi} = K \circ T(\tau^*_Q) \circ \dot{\psi}.
\]  

As a consequence, the relation between \( \mathcal{K} \) y \( X_H \) is

\[
X_H \circ FL = \mathcal{K}.
\]  

- If \( \xi \in C^\infty(T^*Q) \) is a Hamiltonian constraint, then \( \iota_K (d\xi \circ FL) \) is a Lagrangian constraint.

Relations (46), (47), (48) and (49) show how the Lagrangian and Hamiltonian descriptions can be unified by means of the operator \( \mathcal{K} \).

Some relevant results obtained using this operator are:

- The equivalence between the Lagrangian and Hamiltonian formalisms is proved by means of this operator in the following way: there is a bijection between the sets of solutions of Euler-Lagrange equations and Hamilton equations, even though the dimensions of the final constraint submanifold in both formalisms are not the same, in general [4, 15].
The complete classification of constraints is achieved. All the Lagrangian constraints can be obtained from the Hamiltonian ones using the $\mathcal{K}$-operator [4].

Noether’s theorem is proved and the relation between the generators of gauge and “rigid” symmetries in the Lagrangian and Hamiltonian formalisms is studied [11], [12], [17], [18].

This operator has been applied to studying Lagrangian systems whose Legendre map has generic singularities; that is, it degenerates on a hypersurface $\mathbb{R}^n$, [38], [39].

6.2 Field operators $\mathcal{K}$ in field theories

Next we generalize the definition, properties and some of the applications of the evolution operator for the $k$-symplectic formulation of field theories, in order to describe the relationship between the Lagrangian and Hamiltonian formalisms (the generalization for the multisymplectic formulation is given [9]). In particular, we will study how to obtain the solutions of Lagrangian and Hamiltonian field equations by means of this operator, and the relation between them.

**Definition 6.1** A field operator $\mathcal{K}$ associated with a Lagrangian $L : T^1_k Q \to \mathbb{R}$ is a map

$$\mathcal{K} : T^1_k Q \to T^1_k ((T^1_k)^* Q)$$

satisfying the following conditions:

1. **Structural condition**: $\mathcal{K}$ is a $k$-vector field along $FL$, that is

$$\tau(T^1_k)^* Q \circ \mathcal{K} = FL.$$  \hspace{1cm} (50)

2. **Field equation condition**:

$$\sum_{A=1}^k (FL)^* [\kappa_A (\omega_0)_A \circ FL)] = dE_L.$$ \hspace{1cm} (51)

Hence $\mathcal{K} = (\mathcal{K}_1, \ldots, \mathcal{K}_k)$, where each $\mathcal{K}_A$, $1 \leq A \leq k$, is a vector field along $FL$. 

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3. Second-order condition:

\[ T_k^1(\tau_Q^* Q) \circ K = Id_{T_k^1Q}. \]  

Now we are going to calculate the local expression of a field operator \( K \). If \( v = (v_1, \ldots, v_k) \in T_k^1Q \) then from (50) we have that

\[
K_A(v) = (K_A)^i(v) \frac{\partial}{\partial q^i} (FL)_*(v) \bigg|_{v} + (K_A)^B(v) \frac{\partial}{\partial p^i_B(v)} (FL)_*(v), \quad 1 \leq A \leq k.
\]

Taking into account (52) and that the map \( T_k^1(\tau_Q^*) : T_k^1((T_k^1)^*Q) \to T_k^1Q \) is locally given by

\[
T_k^1(\tau_Q^*)(q^i, p^A_i, (u_A)^i_B) = (q^i, (u_A)^i),
\]

we obtain that

\[
(K_A)^j = v_A^j. \tag{53}
\]

Then, writing in local coordinates the expression (51)

\[
\sum_{A=1}^{k} \omega_0^A (FL)_*(v) \left( (K_A)_j(v), (FL)_*(v) \left( \frac{\partial}{\partial q^i} \bigg|_{v} \right) \right) = dE_L \left( \frac{\partial}{\partial q^i} \bigg|_{v} \right),
\]

we obtain that

\[
\sum_{A=1}^{k} \left( v_A^k \frac{\partial^2 L}{\partial q^i \partial v_A^k}(v) - (K_A)^A_j(v) \right) = \sum_{A=1}^{k} v_A^k \frac{\partial^2 L}{\partial q^i \partial v_A^k}(v) - \frac{\partial L}{\partial q^i}(v).
\]

Therefore

\[
\sum_{A=1}^{k} (K_A)^A_j = (K_1)_1^1 + (K_2)_1^2 + \ldots + (K_k)_1^k = \frac{\partial L}{\partial q^i}, \tag{54}
\]

which means that every field operator \( K \) is locally given by

\[
K_A = v_A^j \left( \frac{\partial}{\partial q^i} \circ FL \right) + (K_A)^B \left( \frac{\partial}{\partial p^i_B} \circ FL \right), \quad 1 \leq A \leq k.
\]

where the components \((K_A)^B\) satisfy the identity (54).

Equations (53) and (54) lead us to define local solutions in a neighborhood of each point of \( T_k^1Q \) satisfying conditions 1, 2 and 3 in definition 6.1.

\[
K_A = v_A^j \left( \frac{\partial}{\partial q^i} \circ FL \right) + \frac{1}{k} \frac{\partial L}{\partial q^i} \left( \frac{\partial}{\partial q^i} \circ FL \right), \quad 1 \leq A \leq k,
\]

and, by using a partition of the unity, we obtain global solutions.

**Definition 6.2** \( \psi : \mathbb{R}^k \to T_k^1Q \) is an integral section of the field operator \( K \) if

\[
T_k^1(FL) \circ \psi(1) = K \circ \psi.
\]
Definition 6.2 means that, for every \( t \in \mathbb{R}^k \),
\[
\mathcal{K}_A(\psi(t)) = (FL)_*(\psi(t)) \left( \psi_*(t) \left( \frac{\partial}{\partial t^A} \bigg|_t \right) \right), \quad 1 \leq A \leq k,
\]
because
\[
(T^1_k(FL) \circ \psi(t))(t) = T^1_k(FL)(j^1_0 \psi_t) = j^1_0(FL \circ \psi_t),
\]
where \( \psi_t(t) = \psi(t + \bar{t}) \). Thus, the following diagram is commutative

\[
\begin{array}{c}
\psi^{(1)} \downarrow \\
T^1_k(T^1_k Q) \xrightarrow{T^1_k(FL)} T^1_k((T^1_k)^* Q)
\end{array}
\]

6.3 Properties of the field operators related to the Lagrangian formalism

In this section we study the properties of the field operator in relation to the Lagrangian field equations. In particular, we generalize the properties of the evolution operator in mechanics given in equation (47).

**Proposition 6.1** Let \( L : T^1_k Q \to \mathbb{R} \) be a Lagrangian. \( \psi : \mathbb{R}^k \to T^1_k Q \) is an integral section of \( \mathcal{K} \) if, and only if, \( \tau_\mathcal{Q} \circ \psi : \mathbb{R}^k \xrightarrow{\psi} T^1_k Q \xrightarrow{\tau_\mathcal{Q}} Q \) is a solution to the Euler-Lagrange equations (15).

**Proof:** If \( \psi : \mathbb{R}^k \to T^1_k Q \) is locally given by \( \psi(t) = (\psi^i(t), \psi_A^i(t)) \), then from (55) we obtain that
\[
(FL \circ \psi)_*(t) \left( \frac{\partial}{\partial t^A} \bigg|_t \right) = \left. \frac{\partial \psi^j}{\partial t^A}(t) \right|_{FL(\psi(t))} \frac{\partial}{\partial q^j} + \left. \frac{\partial^2 L}{\partial q^j \partial v^C}(\psi(t)) \frac{\partial \psi^j}{\partial t^A}(t) \right|_{FL(\psi(t))} \frac{\partial^2 L}{\partial v^j \partial v^C}(\psi(t)) \frac{\partial}{\partial p^C_j} \bigg|_{FL(\psi(t))}.
\]
On the other hand
\[
\mathcal{K}_A(\psi(t)) = v^j_A(\psi(t)) \left. \frac{\partial}{\partial q^j} \right|_{FL(\psi(t))} + (\mathcal{K}_A)^{\mathcal{C}}_j(\psi(t)) \left. \frac{\partial}{\partial p^C_j} \right|_{FL(\psi(t))}.
\]
So if \( \psi \) is a solution to \( \mathcal{K} \), then from (55) and (56) we obtain the equations
\[
\frac{\partial \psi^j}{\partial t^A}(t) = v^j_A(\psi(t)) = \psi^j_A(t),
\]

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and
\[
\frac{\partial}{\partial t^A} \left( \frac{\partial L}{\partial v_C}(\psi(t)) \right) = \frac{\partial^2 L}{\partial q \partial v_C}(\psi(t)) \frac{\partial \psi^i}{\partial A}(t) + \frac{\partial^2 \psi^i}{\partial v_B \partial v_C}(\psi(t)) \frac{\partial^2 L}{\partial v_B}(\psi(t)) = (K_A)^C_j(\psi(t)), \tag{58}
\]
for every \( A = 1, \ldots, k \). Therefore, from (54), (57) and (58) we obtain
\[
\sum_{A=1}^k \frac{\partial}{\partial t^A} \left( \frac{\partial L}{\partial v_A}(\psi(t)) \right) = \sum_{A=1}^k (K_A)^A_j(\psi(t)) = \frac{\partial L}{\partial q^i}(\psi(t)) = \psi^i_A(t) = \frac{\partial \psi^i}{\partial t^A}(t),
\]
that is \((\tau_Q \circ \psi)(t) = (\psi^i(t))\) is a solution to the Euler-Lagrange equations (13).

The proof of the converse follows the same pattern than in the proof of the converse statement of proposition 4.1.

\[\boxdot\]

**Theorem 6.1** Let \( L : T^1_kQ \to \mathbb{R} \) be a Lagrangian and let \( K \) be a \( k \) vector field along the Legendre map \( FL : T^1_kQ \to (T^1_k)^*Q \). If \( X_L : T^1_kQ \to T^1_k(T^1_kQ) \) is a \( k \)-vector field on \( T^1_kQ \) and \( j_S : S \hookrightarrow T^1_kQ \) is a submanifold of \( T^1_kQ \) such that
\[
T^1_k(FL) \circ X_L = K, \tag{59}
\]
then \( K \) is a field operator associated with the Lagrangian \( L \) if, and only if, \( X_L \) is a SOPDE solution to the equation (16).

**Proof:** We must prove that both the second-order condition, and the field equation condition hold for \( K \) if, and only if, they hold for \( X_L \). In this proof all the equalities hold on \( S \).

First, if \( K = (K_1, \ldots, K_k) \) and \( X_L = ((X_L)_1, \ldots, (X_L)_k) \), then equation (59) is equivalent to
\[
T(FL) \circ (X_L)_A = K_A, \quad 1 \leq A \leq k.
\]

On the other hand \((\omega_L)_A = (FL)^*(\omega_0)_A\) so one easily proves that
\[
\tau(X_L)_A (\omega_L)_A = (FL)^*(\tau_{K_A}(\omega_0)_A \circ FL),
\]
and for the field equation we obtain
\[
\sum_{A=1}^k [(FL)^*(\tau_{K_A}(\omega_A \circ FL)) - dE_L] = \sum_{A=1}^k [\tau_{X_A}(\omega_L)_A] - dE_L
\]

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hence the field equation condition holds for $K$ if, and only if, the Lagrangian field equation holds for $X_L$.

Furthermore, in relation to the second-order condition (see Definition 3.3) we have that

$$T_k^1(\tau_Q) \circ K = Id_{T^1_kQ} \iff T_k^1(\tau_Q) \circ T_k^1(FL) \circ X_L = Id_{T^1_kQ} \iff T_k^1(\tau_Q) \circ X_L = Id_{T^1_kQ}$$

because $FL$ is a fiber preserving map, that is $\tau_Q \circ FL = \tau_Q$, and hence $T_k^1(\tau_Q) \circ T_k^1(FL) = T_k^1(\tau_Q)$. Thus the last equality is equivalent to (3), and so the second order conditions for $K$ and $X_L$ are related.

Finally, as an immediate consequence of propositions 4.1 and 6.1, and theorem 6.1, we have:

**Corollary 6.1** Under the hypotheses of Theorem 6.1, $\psi: \mathbb{R}^k \rightarrow S \subset T^1_kQ$ is an integral section of the field operator $K$ if, and only if, it is an integral section of the SOPDE $X_L$. (This means that $K$ is integrable if, and only if, $X_L$ is integrable).

Moreover, every integral section $\psi: \mathbb{R}^k \rightarrow S \subset T^1_kQ$ is an holonomic section.

### 6.4 Properties of the field operators related to the Hamiltonian formalism

Next we analyze the properties of the field operator in relation to the Hamilton-de Donder-Weyl field equations, generalizing the properties of the evolution operator in mechanics given in Eqs. (48) and (49).

**Theorem 6.2** Let $L$ be an almost-regular Lagrangian function, and $K$ a field operator associated with $L$. If there exist a $k$-vector field $X_0: P \rightarrow T^1_kP$, and a submanifold $\mathcal{S}: S \hookrightarrow T^1_kQ$, such that

$$T_k^1j_0 \circ X_0 \circ FL_0 = K,$$

then $X_0$ is a solution to the equation (24) on $P = FL_0(S)$.

Conversely, if $X_0$ is a $k$-vector field solution to the equation (24), then the above relation defines a $k$-vector field $K$ along $FL$, which satisfy conditions 1 and 2 of Definition 6.1 on $S$, but not condition 3 (second-order condition) necessarily.

If $L$ is a hyper-regular Lagrangian function, then the same results hold (with $S = T^1_kQ$). But in addition, in the converse statements the $k$-vector field $K$ along $FL$ also satisfies the second-order condition 3 of Definition 6.1 and hence it is a field operator for $L$. 

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Proof: Equation (60) means that
\[(j_0)_*(FL_0(s))((X_0)_A(FL_0(s))) = K_A(s), \quad s \in S, \quad 1 \leq A \leq k.\] (61)

Then, since \(j_0 \circ FL_0 = FL\) and \((j_0)_*(\omega_0)_A = \omega_A^0\) we deduce, from (61), that
\[
(FL)^*(i_{K_A}((\omega_0)_A \circ FL)) = (FL_0)^*(i_{(X_0)_A} \omega_A^0)
\]
and since \((FL_0)^*H_0 = E_L\) we obtain
\[
\sum_{A=1}^{k} (FL)^*(i_{K_A}((\omega_0)_A \circ FL)) - dE_L = (FL_0)^* \left( \sum_{A=1}^{k} (i_{(X_0)_A} \omega_A^0) - dH_0 \right)
\]
where all the equalities hold on \(S\). But, as \(FL_0\) is a submersion, we obtain that
\[
\sum_{A=1}^{k} (FL)^*(i_{K_A}((\omega_0)_A \circ FL)) - dE_L = 0 \iff \sum_{A=1}^{k} (i_{(X_0)_A} \omega_A^0) - dH_0 = 0
\]
hence the field equation condition holds for \(K\) on \(S\) if, and only if, the Hamiltonian field equation holds for \(X_0\) on \(P = FL_0(S)\).

For hyper-regular systems, the proof of these properties is the same, but taking into account that now \(P = (T^1_k)^*Q\), and \(FL_0 = FL\). In addition, the \(k\)-vector field \(X_0 \equiv X\) is defined everywhere in \((T^1_k)^*Q\). Thus, the only addendum is to prove that, if \(X\) is a solution to the equation (24), then its associated \(k\)-vector field along \(FL, K\), satisfies the second-order condition. As \(X\) is a \(k\)-vector field in \((T^1_k)^*Q\), by definition it is a section of \(\tau(T^1_k)^*Q\), thus \(\tau(T^1_k)^*Q \circ X = Id(T^1_k)^*Q\). Then, taking into account that \(FL\) is a diffeomorphism, and that (60) reduces to \(X \circ FL = K\), we have that
\[
T^1_k(\tau_Q^0) \circ K = T^1_k(\tau_Q^0) \circ X \circ FL = FL^{-1} \circ \tau(T^1_k)^*Q \circ X \circ FL = IdT^1_kQ
\]
which is the second-order condition for \(K\).

Then assuming all these relations, we have:

**Theorem 6.3** \(K\) is integrable if, and only if, \(X_0\) is integrable. In particular:

1. Let \(FL_\Sigma: S \to P\) be the restriction of \(FL_0\) to \(S\) (that is, \(j_\Sigma \circ FL_\Sigma = FL_0 \circ j_\Sigma\)). If \(\psi: \mathbb{R}^k \overset{\psi_S}{\to} S \overset{\tau}{\to} T^1_kQ\) is an integral section of \(K\) on \(S\), then \(\psi_0: \mathbb{R}^k \overset{\psi_P}{\to} P \overset{\tau_P}{\to} P\) is an integral section of \(X_0\) on \(P\), where \(\psi_P := FL_\Sigma \circ \psi_S\).

2. Conversely, if \(\psi_0: \mathbb{R}^k \overset{\psi_P}{\to} P \overset{\tau_P}{\to} P\) is an integral section of \(X_0\) on \(P\), then the section \(\psi: \mathbb{R}^k \overset{\psi_S}{\to} S \overset{\tau}{\to} T^1_kQ\) is an integral section of \(K\) on \(S\), for every \(\psi_S: \mathbb{R}^k \to S \subseteq T^1_kQ\) such that \(\psi_P = FL_\Sigma \circ \psi_S\).

The section \(\psi_S\), and hence \(\psi := j_\Sigma \circ \varphi_S\), are holonomic if, and only if, \(K\) satisfies the second-order condition (and hence it is a field operator).
**Proof:** If the system is almost-regular, consider the diagram

\[
\begin{array}{c}
\text{T}_k^1(FL \circ \psi) \\
\text{T}_k^1(FL_0 \circ \psi) \\
\text{T}_k^1P \\
\text{T}_k^1(\text{T}_k^1)^*Q \\
\text{K} \\
\text{T}(\text{T}_k^1)^*Q \\
\text{X}_0 \\
\end{array}
\]

(62)

(where \(\text{X}_0\) denotes any extension of the \(k\)-vector field solution on \(P\) to \(P\)).

1. If \(\psi\) is an integral section of \(K\) then

\[
K_A(\psi(t)) = (FL \circ \psi)_*(t) \left( \frac{\partial}{\partial t^A} \big|_t \right), \quad 1 \leq A \leq k, \tag{63}
\]

but \(FL \circ \psi = j_0 \circ \psi_0\) because

\[
FL \circ \psi = FL \circ j_S \circ \psi_S = j_0 \circ j_P \circ FL_S \circ \psi_S = j_0 \circ j_P \circ \psi = j_0 \circ \psi_0,
\]

therefore (63) is equivalent to

\[
K_A(\psi(t)) = (j_0)_*(\psi_0(t)) \left( (\psi_0)_*(t) \left( \frac{\partial}{\partial t^A} \big|_t \right) \right), \quad 1 \leq A \leq k, \tag{64}
\]

Furthermore, from (61) and taking into account that \(FL_0 \circ \psi = \psi_0\), we have that

\[
K_A(\psi(t)) = (j_0)_*(FL_0(\psi(t)))(X_0)_A(FL_0(\psi(t))) = (j_0)_*(\psi_0(t))((X_0)_A(\psi_0(t))) \tag{65}
\]

then, from (64) and (65), taking into account that \(j_0\) is an imbedding, we deduce

\[
(\psi_0)_*(t) \left( \frac{\partial}{\partial t^A} \big|_t \right) = (X_0)_A(\psi_0(t)) \tag{66}, \quad 1 \leq A \leq k.
\]

Hence, \(\psi_0\) is integral section of \(X_0\).

2. The converse is proved by reversing the above reasoning. In addition, the sections \(\psi_S\) and \(\psi := j_S \circ \psi_S\) are holonomic if, and only if, they are integral sections of a second-order \(k\)-vector field along the Legendre map.

If the system is hyper-regular the proof is analogous, but taking \(P = (T_k^1)^*Q\) and \(FL_0 = FL\).
It is important to point out that, if the integrability condition holds only in a submanifold $\mathcal{I} \hookrightarrow S$, then Theorem 6.3 only holds on $\mathcal{I}$ and $FL(\mathcal{I})$ (which is assumed to be a submanifold of $P$).

Observe also that Theorem 6.3 together with Theorem 6.1 establish the equivalence between the Lagrangian and Hamiltonian formalisms.

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