A study on optimal consumption and portfolio with labour income under inflation

Fanhong Zhang\textsuperscript{a}, Weiyin Fei\textsuperscript{a,b}, Mingxuan Shen \textsuperscript{a,b} and Kui Jiang\textsuperscript{a}

\textsuperscript{a,b}School of Mathematics and Physics, Anhui Polytechnic University, Wuhu, People's Republic of China; \textsuperscript{b}Key Laboratory of Advanced Perception and Intelligent Control of High-end Equipment, Ministry of Education, Wuhu, People's Republic of China

ABSTRACT
This paper studies the problem of optimal consumption and portfolio with labour income under inflation. The purpose of investor is to maximize the expected utility of personal terminal real wealth. Firstly, the life cycle of the investor is divided into two phases of retirement and employment, and a stochastic optimal control model for the optimal consumption and portfolio problem with labour income is established under inflation. Secondly, with the method of stochastic control, the HJB equation of the optimal consumption and portfolio under inflation is built, and then an explicit expression of the valued function of the optimization problem in the case of hyperbolic absolute risk aversion (HARA) utility is obtained. Finally, through the numerical simulation under the condition of given parameters, the economic analysis of the results is provided.

ARTICLE HISTORY
Received 16 August 2019
Accepted 10 October 2019

KEYWORDS
Inflation; labour income; optimal consumption and portfolio; HARA utility; HJB equation

1. Introduction
In the field of finance, optimal consumption and portfolio has always been one of the fundamental issues. In the 1960s, Merton (1969) studied the optimal consumption and portfolio under a multi-period framework in continuous time by using the dynamic programming method, and he created a new theory of consumption and portfolio in continuous time. Afterwards, researchers made a lot of explorations based on Merton's seminal work. Kim and Omberg (1996) analysed the dynamic nonmyopic portfolio behaviour of the investors, and they examined the portfolio problem with the investor maximizing expected utility of HARA function for terminal wealth. And then Wachter (2002) gave an exact solution of the portfolio and consumption problem under mean-reverting returns and complete markets. In addition, Liu (2007) studied the portfolio selection problems in stochastic environments, in his paper when the asset returns are quadratic and the agent has a constant relative risk aversion (CRRA) coefficient, the explicit solutions of dynamic portfolio choice problems are derived. Based on the theory of differential game, Liu and Wu (2001) studied an optimal consumption and investment decision problem under the condition of considering transaction costs and the hypothesis that security returns had bounded uncertainty. Li and Yuan (2010) investigated a continuous-time mean–variance portfolio selection problem under parameter uncertainty and Bayesian learning.

In the real world, however, investors earn labour income in their wealth. Therefore, the optimal consumption and portfolio problem with labour income was worthy research direction. And the initial work is Merton (1971), which introduced the optimal policies on condition that an investor has a constant wage and his work was extended in several dimensions. Furthermore, Bodie, Merton, and Samuelson (1992) examined the effect of the labour-leisure choice on portfolio and consumption decisions over an individual's life cycle, and they employed a life cycle model that allowed continuous consumption decisions and traded in risky financial assets, then discussed the cases of non-stochastic wage and stochastic wage respectively, emphasizing the value of a flexible labour supply. Moreover, Moos (2011) studied the optimal consumption and portfolio problem under time-varying investment opportunities and dynamic non-financial (labour) income by using analytical methods. In Moos’s model, the life cycle of an investor was divided into two phases of retirement and employment, and the optimal consumption and portfolio strategy was obtained, where the risky asset with mean-reverting return and the labour income with a mean-reverting growth rate were assumed under a continuous time complete market. Koo (1998) explored the consumption and
portfolio selection problem of an agent who was liquidity constrained and had uninsurable income risk in a continuous time, and Koo drew a conclusion that for a given level of financial wealth and labour income, optimal consumption and risk taking of a liquidity-constrained agent are generally higher in a complete financial market. Besides, Viceira (2001) analysed optimal consumption-portfolio decisions of long-horizon investors with undiversifiable labour income risk and exogenous expected retirement and lifetime horizons, he not only examined the effect of increasing labour income risk on savings and portfolio choice, but also provided intuition about the optimal portfolio choice of young investors in the standard life-cycle model. Moreover, Dybvig and Liu (2010) studied an optimal consumption and portfolio problem with voluntary or mandatory retirement, and with or without a non-negative wealth constraint (which prevents borrowing against future wages), in the paper they considered three models which varied in the treatment of retirement and borrowing against future labour income.

Benzoni, Collin-Dufresne, and Goldstein (2007) discussed portfolio choice when labour income and dividends were cointegrated, and found that the young agents were likely to invest more risky asset with high labour income and their human capital effectively became stock-like because of cointegration. However, for older agents with shorter times-to-retirement, cointegration did not have enough time to act, and thus their human capital became more bond-like. In order to illustrate the role of labour income on the predictability of stock returns, Santos and Veronesi (2005) considered a general equilibrium model, where a representative agent receives income from two sources, financial and non-financial (human), and where the mix between the two sources of income varies over time, and showed that equilibrium expected returns change as the fraction of total income funded by labour income fluctuates over time. Moreover, Cocco, Gomes, and Maenhout (2005) investigated the problem of consumption and portfolio choice over the life cycle, and solved a realistically calibrated life cycle model of consumption and portfolio choice with non-tradable labour income and borrowing constraints, and they found that the presence of labour income increased the demand for stocks, especially early in life. Besides, Liu, Fei, Zhu, and Zheng (2014) studied an optimal consumption and portfolio problem with an investor’s heritage and insurance under Knightian uncertainty and three different borrowing constraints, the three different borrowing constraints associated with labour income. From what has been discussed above, we can recognize the importance of accounting for labour income and human capital in asset pricing tests.

Therefore, we consider a labour income process under the inflation environment.

There is much literature on the effect of inflation on prices by using historical data and empirical models. Vavra (2014) argued that greater volatility led to an increase in aggregate price flexibility so that nominal stimulus mostly generated inflation rather than output growth. And Shremirov (2015) found that if the transitory price changes were removed from data, the comovement of price dispersion and inflation was positive by standard models, but a wide variety of price-stickiness models that ignore sales cannot quantitatively match the comovement even for regular prices. Furthermore, Wulfsberg (2016) investigated monthly retail price data over three decades characterized by both high and low inflation in Norway, and added the empirical evidence on price adjustments in advanced economies with low and stable inflation. However, Nakamura and Steins-son (2018) found little evidence that the Great Inflation of the late 1970s and early 1980s in U.S. led to a substantial increase in price dispersion which the costs of inflation emphasized in standard New Keynesian models of the economy. Through the analysis of existing economic data, we find that inflation and price are closely linked. Hence, when considering the impact factors of asset price changes in the portfolio, inflation is worth noting. For example, Ding and Zou (2004) treated the problem of the optimal consumption and investment when there was a currency inflation or deflation in financial market and consumer goods market by means of martingale and stochastic analysis theory. Moreover, Wang and Wu (2009) were concerned with a kind of problem of maximizing the expected utility from the terminal wealth in the case of inflation, and they obtained by a direct method an agent’s explicit optimal portfolio strategy and the corresponding maximum expected utility for a class of CRRA utility. Furthermore, Fei and Li (2012) investigated the optimal consumption and portfolio choice problem of an investor who differentiates ambiguity and ambiguity attitude under the Knightian uncertainty (or ambiguity) and inflation, and they deduced the modified mutual fund theorem consisting of three funds by using the dynamic programming principle, and They also analysed the influence of ambiguity and inflation on optimal consumption and portfolio policies in the special case of the constant relative risk-aversion utility. Under inflation and Markovian switching, Fei (2013) explored optimal consumption and portfolio problem. Besides, Fei, Xia, and Liu (2014) concerned with the optimal portfolio choice of an investor under the inflation and rare events impact. Furthermore, Fei and Fei (2015) studied an optimal consumption and portfolio problem with the financial markets of Markovian switching and inflation.
In addition, Yao, Jiang, Ma, and Li (2013) studied a continuous time mean–variance portfolio selection problem under inflation. Based on Yao et al. (2013), Yao, Wu, and Zeng (2014) considered a portfolio selection problem under inflation when time-horizon was uncertain, but the financial market they considered consists of only risky assets. Moreover, Fei, Cai, and Xia (2015) studied the problem of dynamic asset allocation with inflation under jump-diffusion environment. Fei, Li, and Xia (2015) examined the factors such as inflation which exerted influence on optimal investment strategies of fund manager with incentive fees. And Fei, Chen, and Fei (2019) investigated an optimal consumption/portfolio problem of an agent with voluntary retirement and subsistence consumption constraints under inflation. Besides, Fei, Fei, Rui, and Yan (2019) developed an optimal intertemporal asset allocation strategy of a multinational corporation, which invested in foreign market under exchange rate risk. Moreover, Fei, Du, Fei, and Yan (2019) studied the entrepreneur’s investment-consumption and hedging under inflation risk, and showed that when hedging without investing in stocks, the entrepreneur’s optimal consumption depended on the expected appreciation rate and volatility of the project value, and the expected inflation and volatility of the expected inflation.

In addition, Sun, Hong, and Wang (2019) investigated the out-of-sample forecasts of China’s economic growth and inflation by using rolling weighted least squares, and found that asset prices were key variables for forecasting macroeconomic variables in most cases by using rolling weighted least squares, especially output growth rate. Inflation risk in markets also be considered in the study of contract theory, for example, Fei, Zhang, and Yang (2019) considered that an executive invests personal assets in the risk-free asset, the market portfolio, and holdings of own-company stocks in order to maximize his expected utility of the terminal real wealth under inflation risk.

In fact, the inflation risk from the market affects the real wealth of investor’s asset. The mainly contribution of this paper is that we extend the framework of Moos (2011) to inflation environment, which is more general and realistic. The innovation of this paper is as follow: we first get the real financial wealth of investor by using Itô formula. Next, we obtain the optimal consumption and investment strategies under inflation with or without labour income by using dynamic programming. Finally, the results are numerically simulated and quantitatively analysed in view of the effect of inflation risk, and give the explanation from the perspective of economics. The rest of the paper is organized as follows. In Section 2, we build the model under inflation including the financial market. In Section 3, the explicit solution for the model is obtained under HARA utility function. In Section 4, under the condition of the given parameters, we use the Matlab software to simulate and analyse the effect of inflation on the optimal consumption and portfolio of the investor with labour income. In Section 5, we conclude.

2. Model framework

We set a complete probability space \((\Omega, \mathcal{F}_T, P)\), where \(\Omega\) is the sample space, and \(P\) is the probability measure. \(\mathcal{F}_t = \sigma\{B_s(s), B_x(s), B_y(s), B_q(s); 0 \leq s \leq t\}\) is the natural filtration satisfying the usual condition for \(0 \leq t \leq T\), where \(B_s(t), B_x(t), B_y(t)\) and \(B_q(t)\) are one-dimensional standard Brownian motions defined on the above probability space. And \(B_s(t), B_x(t)\) and \(B_y(t)\) model the uncertainties in the risky asset, state variable and risky labour income, \(B_q(t)\) models the random nature of inflation. Assume they have their own correlated coefficients, i.e.

\[
\begin{align*}
    dB_s(t) dB_x(t) &= \rho_{sx} dt, \\
    dB_s(t) dB_y(t) &= \rho_{sy} dt, dB_x(t) dB_y(t) = \rho_{xy} dt, \\
    dB_s(t) dB_q(t) &= \rho_{sq} dt, \\
    dB_y(t) dB_q(t) &= \rho_{yq} dt, dB_x(t) dB_q(t) = \rho_{xq} dt.
\end{align*}
\]

We suppose that the financial market is composed of two kinds of assets: a risk-free asset and a single risky asset. The nominal price of the risk-free asset at time \(t\) is \(P(t)\), which evolves according to the following equation:

\[
dP(t) = r_0 P(t) dt, \quad 0 \leq t \leq T, \tag{1}
\]

where \(r_0\) is the nominal risk free interest rate. We assume that the risky asset’s expected returns is affine in a state variable and has constant volatility, and denote the nominal price of the risky asset at time \(t\) by \(S(t)\), whose evolution is described by

\[
dS(t) = S(t)[(\lambda_1 X(t) + r_0) dt + \sigma_x dB_s(t)], \quad 0 \leq t \leq T, \tag{2}
\]

where \(\lambda_1\) is the expected returns parameter of risky asset and \(\lambda_1 > 0\), \(\sigma_x\) is the diffusion term of risky asset and \(\sigma_x > 0\). In this framework, the market price of risk is \(\theta(t) \equiv (\lambda_1/\sigma_x) X(t)\).

The dynamics of the state variable \(X(t)\) are given by

\[
dX(t) = -\kappa_x(X(t) - \bar{X}) dt + \sigma_x dB_x(t), \quad 0 \leq t \leq T, \tag{3}
\]

where \(\kappa_x\) is the mean reversion parameter of state variable and \(\kappa_x \geq 0\), \(\bar{X}\) is the long-run mean of state variable and \(\bar{X} \geq 0\), \(\sigma_x\) is the diffusion term of state variable and \(\sigma_x > 0\).

We assume that the life cycle of the investor is divided into two phases of retirement and employment. For \(0 \leq t \leq T_r\), the investor is active in the labour market, he earns a dynamic labour income, consumes and invests in both
risk-free and risky asset. During the phase of retirement, i.e. \( T_r < t \leq T \), the investor has no non-financial (labour) income and has to ensure consumption from the accumulated financial wealth. Here \( T_r \) is the instant of the investor’s retirement, and \( T \) is the investor’s lifetime.

For \( 0 \leq t \leq T_r \), the investor gets a risky labour income from the labour market \( Y(t) \) at time \( t \), which is described by the following stochastic differential equation:

\[
dY(t) = Y(t)\left[ (y_0 + y_1X(t)) \, dt + \sigma_q \, dB_q(t) \right], \quad 0 \leq t \leq T_r,
\]

where \( y_0 = \tilde{y} - y_1 \tilde{X}(t) \), \( y_1 \) is the sensitivity of labour income growth on the state variable, and \( \tilde{y} \) is the labour income growth at \( X = \tilde{X} \), \( \sigma_q \) is the diffusion term of risky labour income and \( \sigma_q \geq 0 \).

The inflation rate is supposed to be random, the price level \( Q(t) \) at time \( t \) follows a stochastic differential equation given by

\[
dQ(t) = Q(t)\left[ \mu_q dt + \sigma_q dB_q(t) \right], \quad 0 \leq t \leq T,
\]

where \( \mu_q \) is the expected rate of inflation at time, \( \sigma_q \) describes the volatility of the inflation rate at time \( t \) and \( \sigma_q > 0 \). Thus we assume that \( P(t), S(t), Q(t) \) use the same monetary unit. Furthermore, considering the effect of inflation, the real price of risk-free asset at time \( t \) is \( \tilde{P}(t) = P(t)/Q(t) \), and the real price of risky asset is \( \tilde{S}(t) = S(t)/Q(t) \). At same time, the risky labour income of the investor is also affected by inflation, so the real labour income is \( \tilde{Y}(t) = Y(t)/Q(t) \). Hence, we obtain \( \tilde{P}(t), \tilde{S}(t) \) and \( \tilde{Y}(t) \) by using Itô formula:

\[
\begin{align*}
\tilde{P}(t) &= \tilde{P}(t)[(r_0 + \sigma_2^2 - \mu_q) \, dt - \sigma_q \, dB_q(t)], \\
\tilde{S}(t) &= \tilde{S}(t)[(\lambda_1X(t) + r_0 + \sigma_2^2 - \mu_q - \sigma_q\rho_y) \, dt + \sigma_q \, dB_q(t)], \\
\tilde{Y}(t) &= \tilde{Y}(t)[(y_0 + y_1X(t) + \sigma_2^2 - \mu_q - \sigma_q\rho_y) \, dt + \sigma_q \, dB_q(t)],
\end{align*}
\]

where \( r_0 + \sigma_2^2 - \mu_q \), \( \lambda_1 \), \( \rho_y \), \( y_0 \), \( y_1 \), \( \sigma_q \), \( \sigma_y \), \( \rho_y \), \( \rho_q \), \( \sigma_q \), \( \rho_q \) are all deterministic and uniformly bounded on \([0, T]\).

3. The explicit solutions for HARA utility

The conditional expected utility over the remaining lifetime for an investor at time \( t \) is

\[
E_t \left[ \int_t^T e^{-\delta s} \left( C(s) - \bar{c} \right)^{1-\gamma} \, ds \right], \quad \gamma > 1,
\]

where the consumption utility is determined by the real consumption level \( c(t) \) of the investor at time \( t \) and the subsistence level of real consumption \( \bar{c} \), and \( \bar{c} > 0 \). And the constant \( \delta \geq 0 \) is the subjective discount rate, \( \tau \equiv T - t \) is the fixed and certain time horizon, \( \gamma \) is the relative risk aversion coefficient.

Under the model framework of Section 2, the investor’s life cycle is divided into two phases of retirement and employment. Since the investor in employment just has a labour income, we focus on the analysis of the employment phase. For the phase of employment, \( 0 \leq t \leq T_r \), the investor has a risky labour income, we can get the financial real wealth \( W(t) \) dynamics from (6), (7) and (8)

\[
dW(t) = \left[ (W(t)\tau)(\lambda_1X(t) - \sigma_q\rho_y) + \tilde{W}(t)\tau(\sigma_q\rho_y - \sigma_q\rho_y) \right] \, dt + \tilde{W}(t)\tau(\sigma_q\rho_y - \sigma_q\rho_y) \, dB_q(t), \quad \gamma > 1,
\]

where \( \tau \) is the proportion of financial wealth invested in the risky asset. Since the consumption rate is not affected by inflation. Hence, our problem is

\[
J(t, X, W, \tilde{Y}) = \max_{(c_t, \pi_t)} \left[ \int_t^T e^{-\delta s} \left( C(s) - \bar{c} \right)^{1-\gamma} \, ds \right],
\]

and the optimal consumption and portfolio strategy under inflation is \((c^*_t, \pi^*_t)\).

Based on the principle of dynamic programming, we can obtain the HJB equation of the value function

\[
0 = J_t + \sup_{c_t} \left[ e^{-\delta t} \left( C_t - \bar{c} \right)^{1-\gamma} - J_W c^*_t \right] + \sup_{\pi_t} \left[ J_W W(t) \pi(t)(\lambda_1X(t) - \sigma_q\rho_y) + \tilde{W}(t)\pi(t)(\sigma_q\rho_y - \sigma_q\rho_y) \right] + \tilde{W}(t)\pi(t)(\sigma_q\rho_y - \sigma_q\rho_y)
\]

\[
+ \int_t^T e^{-\delta s} \left( C(s) - \bar{c} \right)^{1-\gamma} \, ds \right],
\]
where

\[
\rho_{sx} \in (-1, 1).
\]

(c.2) Assume the risky asset and the risky labour income are perfectly correlated or the labour income is locally risk-free, i.e.,

\[
\rho_{sy} \in (-1, 1) \Rightarrow \rho_{xy} = \rho_{sx}\rho_{sy} \in (-1, 1) \text{ or } \sigma_y = 0.
\]

(c.3) Assume the inflation is perfectly correlated with the risky asset and the risky labour income respectively, i.e.

\[
\rho_{sq} \in (-1, 1) \Rightarrow \rho_{qa} = \rho_{sx}\rho_{sq} \in (-1, 1) \text{ and } \rho_{qy} = \rho_{sy}\rho_{qy} \in (-1, 1).
\]

These assumptions might not match explicitly a reality, but they come with an advantage beside the interpretability of closed-form solutions.

Under these assumptions, \(C(s, X) = (1/\gamma)(c_0(s) + c_1(s)X + \frac{1}{2}c_2(s)X^2)\), where \(c_0(s), c_1(s)\) and \(c_2(s)\) are solved by the following three nonlinear ordinary differential equations with initial conditions \(c_0(s) = c_1(s) = c_2(s) = 0\)

\[
\frac{\partial c_2(s)}{\partial s} = k_0 + k_1 c_2(s) + k_2 c_2(s)^2,
\]

\[
\frac{\partial c_1(s)}{\partial s} = k_4 + k_5 c_2(s) + \frac{k_1}{2} c_1(s) + k_2 c_2(s) c_1(s),
\]

\[
\frac{\partial c_0(s)}{\partial s} = k_5 + k_3 c_1(s) + k_6 c_2(s) + \frac{k_2}{2} c_1(s)^2,
\]

where

\[
k_0 = \frac{1 - \gamma \lambda_1^2}{\gamma \sigma_x^2}, \\
k_1 = 2 \left(1 - \frac{\gamma \sigma_x^2}{\gamma \sigma_x^2 - k_x} \right), \\
k_2 = \frac{1}{\gamma} \sigma_x^2, \\
k_3 = \left(\frac{k_x}{\gamma} - 1 - \frac{\gamma}{\gamma} \sigma_q \rho_{sq} \sigma_x \rho_{sx}\right), \\
k_4 = -\frac{(1 - \gamma)^2}{\gamma} \sigma_q \rho_{sq} \lambda_1, \\
k_5 = -\delta + (1 - \gamma)(t_0 + \sigma_q^2 - \mu_q) + \frac{2\gamma^2}{2\gamma} - 3\gamma + \frac{1}{2} \sigma_q^2, \\
k_6 = \frac{1}{2} \sigma_x^2.
\]

Then the solution of the equation (15)–(17) is

\[
c_2(s) = \frac{2k_0(1 - \eta s)}{2\eta - (k_1 + \eta)(1 - \eta s)},
\]

\[
c_1(s) = \int_0^s k_4 + k_3 c_2(s) + \frac{k_1}{2} c_1(s) + k_2 c_2(s) c_1(s) \, ds,
\]

\[
c_0(s) = \frac{k_5}{\gamma} + k_6 c_1(s) + k_7 c_2(s) + \frac{k_2}{2} c_1(s)^2.
\]
where nary differential equation (See Appendix for the process of a derivation)

\[ c_0(s) = \int_0^s k_5 + k_2 c_1(s) + k_6 c_2(s) + \frac{k_2}{2} c_1(s)^2 \, ds, \quad (21) \]

where \( \eta = \sqrt{q}, \quad q = k_1^2 - 4k_0k_2 \geq 0. \) The formula \( k(t, X) = \int_0^t e^{d_0(s)+d_1(s)x} \) represents the value of one unit of income, \( d_0(s) \) and \( d_1(s) \) are solved by the following two ordinary differential equations with initial \( d_0(s) = d_1(s) = 0 \) (See Appendix for the process of a derivation):

\[ \frac{d d_1(s)}{ds} = l_0 + l_1 d_1(s), \quad (22) \]

\[ \frac{d d_0(s)}{ds} = l_2 + l_3 d_1(s) + l_4 d_1(s)^2, \quad (23) \]

where

\[
\begin{align*}
    l_0 & = y_1 - \frac{\sigma_x \rho_{xy}}{s} \lambda_1, \\
    l_1 & = -\kappa_x - \frac{\sigma_x \rho_{xy}}{s} \lambda_1, \\
    l_2 & = y_0 - r_0, \\
    l_3 & = \kappa_x X + \sigma_x \rho_{xy}, \\
    l_4 & = \frac{1}{2} \sigma_x^2.
\end{align*}
\]

Then the solution of the equation (22)–(23) is

\[
\begin{align*}
    d_1(s) & = \begin{cases} 
        \frac{l_0}{l_1} (e^{l_1s} - 1) & \text{if } l_1 \neq 0, \\
        l_0 s & \text{if } l_1 = 0,
    \end{cases} \\
    d_0(s) & = \begin{cases} 
        \left( l_2 - l_3 \frac{l_0}{l_1} + l_4 \frac{l_0^2}{l_1^2} \right) s \\
        + \left( l_2 \frac{l_0}{l_1} - 2l_3 \frac{l_0^2}{l_1^2} \right) (e^{l_1s} - 1) & \text{if } l_1 \neq 0, \\
        + \frac{1}{2} l_4 \frac{l_0^2}{l_1^2} (e^{2l_1s} - 1) \\
        l_2 s + \frac{1}{2} l_3 l_0 s^2 + \frac{1}{3} l_4 l_0 s^3 & \text{if } l_1 = 0.
    \end{cases}
\end{align*}
\]

Here \( R(\tau) \) can be obtained by the following linear ordinary differential equation (See Appendix for the process of a derivation)

\[
\frac{d R}{d \tau} = \ddot{c} - \left( r_0 - \mu_q + \frac{\sigma_q \rho_{aq}}{\sigma_s} \lambda_1 X \right) \dot{R}. \quad (27)
\]

When the initial condition is \( R(0) = 0 \), the solution of equation (27) reads

\[ R(\tau) = \frac{\ddot{c}}{m} (1 - e^{-m \tau}), \quad (28) \]

where \( m = r_0 - \mu_q + (\sigma_q \rho_{aq} / \sigma_s) \lambda_1 X. \)

In this part, we consider the optimal consumption and portfolio problem under inflation when the investor is in the employment phase. Under assumptions (c.1)–(c.3), the optimal total wealth follows a geometric Brownian motion with time-varying coefficients and will stay non-negative in all cases if the initial total wealth is positive. From the above model and the derivation process, the following proposition can be obtained.

**Proposition 3.1:** Under the assumptions (c.1)–(c.3), given initial total wealth \( \hat{W} \geq 0 \), the optimal consumption and risky investment of the investor with labour income are given by

\[ c_t^* = \frac{\hat{W}}{\int_0^t e^{C(s,X)} \, ds} + \ddot{c}, \quad (29) \]

\[
\begin{align*}
    W_{\pi_t^*} & = \frac{1}{\gamma} \hat{W} + \frac{1}{\gamma} \frac{\sigma_x \rho_{sx}}{\sigma_s} \left( \int_0^t d_1(s) e^{d_0(s)+d_1(s)X} \, ds \right) \hat{W} \\
    & - \frac{\sigma_x \rho_{xy}}{\sigma_s} \left( \int_0^t d_0(s) e^{d_0(s)+d_1(s)X} \, ds \right) \hat{c} + \frac{\sigma_q \rho_{aq}}{\sigma_s} W \\
    & - \frac{1}{\gamma} \frac{\sigma_q \rho_{aq}}{\sigma_s} \hat{W} + \frac{\sigma_q \rho_{aq}}{\sigma_s} \left( \int_0^t e^{d_0(s)+d_1(s)X} \, ds \right) \hat{c}, \quad (30)
\end{align*}
\]

where \( C(s, X) = (1/\gamma)(c_0(s) + c_1(s)X + \frac{1}{2} c_2(s)X^2), \ c_0(s), \ c_1(s) \) and \( c_2(s) \) satisfy (19)–(21), \( k_0, k_1, k_2, k_3, k_4, k_5, k_6 \) are given by (18) and \( \eta = \sqrt{q}, \quad q = k_1^2 - 4k_0k_2 \geq 0. \ d_0(s) \) and \( d_1(s) \) fulfill (25)–(26), \( l_0, l_1, l_2, l_3, l_4 \) are given by (24). \( R(\tau) \) is given by (29).

In the phase of retirement, for \( t \geq T_r \), it must be noticed that during retirement the real consumption has to exceed the subsistence level (i.e. \( c(t) \geq \ddot{c} \)). Therefore, we have to make a assumption (c.4) over the phase of retirement under inflation environment

\[ \hat{W}(T) = W(T) - \frac{\ddot{c}}{r_0 + \mu_q + \frac{\sigma_q \rho_{aq}}{\sigma_s} \lambda_1 X} \geq 0, \quad (c.4) \]

where \( \hat{W}(T) \) is the financial wealth that guarantees the future survival. Assumption (c.4) ensures that at the beginning of the phase of retirement financial wealth is sufficient to afford the future subsistence consumption.

In the phase of retirement, the investment environment of an investor is also affected by inflation, but the investor does not have labour income, the financial...
wealth dynamic is obtained from (6) and (7)
\[
dW(t) = [W(t) \pi(t)(\lambda_1 X(t) - \sigma_s \sigma_q \rho_q q) + W(t)(r_0 + \sigma_q^2 - \mu_q) - c(t)] \, dt \\
+ W(t) \pi(t) \sigma_d B_d(t) - W(t) \sigma_q dB_q(t).
\]

The value function of the following form
\[
J(t, X, W) = e^{-\delta(T-t)} \left[ \int_t^T e^{C(s,X)} \, ds \right]^{\gamma} (W - R(t))^{-1 - \gamma},
\]
where \( \tau \equiv T - t \), \( C(s, X) = (1/\gamma)(c_0(s) + c_1(s)X + \frac{1}{2}c_2(s)X^2) \), the solutions of \( c_0(s) \), \( c_1(s) \) and \( c_2(s) \) are identical to the phase of employment. From the analysis of the phase of employment, we know that \( R(t) \) represents the reserve for covering subsistence consumption and fulfills (28).

Similar to the analysis of the phase of employment, the investment environment of the investor is affected by inflation and he does not have labour income, and we can obtain the following proposition.

**Proposition 3.2:** Under the assumptions of (c.1), (c.4) and \( \rho_{sq} \in [-1, 1] \Rightarrow \rho_{sq} = \rho_{xq} \rho_{sq} \in [-1, 1] \), the optimal consumption and risky investment of the investor without the labour income are given by
\[
c_t^* = \frac{W - R(t)}{\int_0^T e^{C(s,X)} \, ds} + \bar{c},
\]
\[
W \pi_t^* = \frac{1}{\gamma} \frac{\lambda_1}{\sigma_s^2} (W - R(t)) + \frac{\sigma_q \rho_{sq}}{\sigma_s} W \\
- \frac{1}{\gamma} \frac{\sigma_q \rho_{sq}}{\gamma} (W - R(t)) + \frac{1}{\gamma} \frac{\sigma_x \rho_{sx}}{\sigma_s} \\
\times \int_0^T \left( (c_1(s) + c_2(s)X)e^{C(s,X)} \, ds \right) \left( W - R(t) \right),
\]
where \( C(s, X) = (1/\gamma)(c_0(s) + c_1(s)X + \frac{1}{2}c_2(s)X^2) \), \( c_0(s) \), \( c_1(s) \) and \( c_2(s) \) satisfy (19)-(21), \( k_0, k_1, k_2, k_3, k_4, k_5, k_6 \) are given by (18) and \( \eta = \sqrt{q} \), \( q = k_1^2 - 4k_0k_2 \geq 0 \). \( R(t) \) is given by (28).

4. **Numerical analysis**

Since the analysis of this paper is focused on the phase of employment, the numerical simulation and economic analysis are mainly aimed at the phase of employment. In order to simplify our analysis and focus on the effect of inflation risk on the optimal consumption and investment portfolio, namely the case of \( \sigma_y = 0 \), i.e. the real labour income process’s random noise is \( B_q(t) \). The corresponding parameters are given as follows:
\[
\begin{align*}
r_0 &= 0.0168, \quad \lambda_1 = 1, \quad \sigma_s = 0.1710, \quad \kappa_x = 0.2812, \\
\sigma_x &= 0.0348, \quad \rho_{sx} = -1, \quad \bar{x} = 0.0408, \\
X &= 0.0027, \quad \mu_q = 0.05, \quad \rho_{sq} = 1, \\
\gamma &= 5, \quad \delta = 0.04, \\
\vec{y} &= 0.0150, \quad \bar{c} = 10, \quad T_r = 45, \quad T = 70,
\end{align*}
\]

The numerical simulation by using the Matlab software, then the following conclusions are obtained.

Figure 1 shows the proportion of financial wealth in the risky asset \( \pi_t \) over inflation volatility. The investor is assumed to be 45 years old (\( T_r = 20 \)) with a financial wealth of 180 (thousand) and an income of 60 (thousand). The parameter values are given above. As shown in Figure 1, the proportion of the invested risky asset increases gradually with inflation volatility, because when inflation volatility begins to increase, an investor with the lower risk aversion thinks the market trend is expected to better this time, he will soon invest his wealth in the risky asset, and then with the increasing inflation volatility, the investor begins to increase the share of investment gradually, when inflation volatility is big enough, the investor may continue to invest by risk-free borrowing. We also find that the proportion of investment is also related to the labour income from Figure 1, so we display three cases with time-variation labour income growth \( (\gamma_1 = 0.2, y_1 = 0, y_1 = -0.2) \). It shows that the investment proportion of the investor is different with different labour income growth, the more labour income, the higher proportion of the risky asset.

In Figure 2 the investor is assumed to be 45 years old (\( T_r = 20 \)) with a financial wealth of 180 (thousand) and an income of 60 (thousand). According to the parameter values given above we obtain Figure 2. As shown in Figure 2, at the beginning increasing of the inflation volatility, the
investor increases the money supply in financial market, while he is investing, he will also stimulate consumption. So we can see that moderate inflation can stimulate consumption. But with the increasing of the larger inflation volatility, the investor invest more wealth (including labour income) to the risky asset, the level of consumption decreased, the final level of consumption tends to the subsistence level of consumption. From Figure 2, we also find that the consumption level of the investor has a relationship with the amount of labour income, we display three cases with time-variation labour income growth \((y_1 = 0.2, y_1 = 0, y_1 = -0.2)\), which shows that the consumption level of the investor is different with different labour income growth, the more labour income, the higher level of the consumption.

### 5. Conclusion

For various reasons the investors will face the different inflation volatilities at different times. Thus, investors should adjust the consumption and portfolio strategies to reduce risky exposure and get more benefits according to the changes of the volatility of inflation. Based on the framework of Moos (2011), This paper studies the consumption and portfolio problem of an investor with labour income under inflation. The innovation of this paper are as follows. First, we divide the investor’s cycle life into two phases of retirement and employment and focus on the phase of employment, then the mathematical model under inflation and real financial process are built. Next, we obtain the HJB equation of investor’s terminal utility maximization problem by using the method of stochastic control, and an explicit expression in the case of hyperbolic absolute risk aversion (HARA) utility is obtained. Finally, the range of inflation volatility is given, and according to the numerical simulation method we analysed the effects of inflation volatility for the optimal consumption and portfolio of the investor with labour income.

Compared with Moos (2011), this paper care about the effect of inflation risk on the optimal consumption and investment strategies, and we show that the investor with the lower risk aversion will increase the proportion of investment risky asset as inflation volatility gradually increases. Moreover, the more labour income, the greater proportion of investment, and the relatively higher consumption level. But with the more wealth investment into the risky asset, the consumption level gradually declines, and eventually reduces to the subsistence level of consumption. The model in this paper is more realistic, and of certain theoretical value, so it can provide some reference for the consumption and investment decisions of investors.

The results of the paper will provide practical guidance for investment behaviour in financial market, and scientific theoretical support for investment practice. Compared with the assumption that risk asset and inflation risk are driven by the geometric Brownian motion in this paper, if the risk source is driven by G-Brownian motion (see, e.g. Peng, 2010 for the related theory), how to solve the optimal consumption and investment strategies with the labour income can be further researched (see, e.g. Fei & Fei, 2013 for the related topics).

### Disclosure statement

No potential conflict of interest was reported by the authors.

### Funding

This paper is supported by the National Natural Science Foundation of China (71571001) and the Promoting Plan of Higher Education of Anhui Province (TSKJ2016B11).

### ORCID

Mingxuan Shen https://orcid.org/0000-0002-2197-7289

### References

Benzoni, L., Collin-Dufresne, P., & Goldstein, R. S. (2007). Portfolio choice over the life-cycle when the stock and labor markets are cointegrated. *Journal of Finance*, 62(5), 2123–2167.

Bodie, Z., Merton, R. C., & Samuelson, W. F. (1992). Labor supply flexibility and portfolio choice in a life cycle model. *Journal of Economic Dynamics and Control*, 16(3-4), 427–449.

Cocco, J. F., Gomes, F. J., & Maenhout, P. J. (2005). Consumption and portfolio choice over the life cycle. *The Review of Financial Studies*, 18(2), 491–533.

Ding, C. M., & Zou, J. Z. (2004). Optimal consumption and investment model considering effect of currency inflation. *Journal of Central South University (Natural Science)*, 35(1), 167–170 (in Chinese).
Dybvig, P. H., & Liu, H. (2010). Lifetime consumption and investment: Retirement and constrained borrowing. *Journal of Economic Theory*, 145(3), 885–907.

Fei, W. Y. (2013). Optimal consumption and portfolio under inflation and Markovian switching. *Stochastics*, 85(2), 272–285.

Fei, W. Y., Cai, Z. Q., & Xia, D. F. (2015). Dynamic asset allocation with inflation under jump diffusion environment. *Journal of Management Sciences in China*, 18(8), 83–94 (in Chinese).

Fei, W. Y., Chen, Y. H., & Fei, C. (2019). An optimal consumption, investment and voluntary retirement choice problem with subsistence consumption constraints under inflation. *Journal of Systems Engineering*. Accepted (in Chinese).

Fei, C., Du, H. J., Fei, W. Y., & Yan, L. T. (2019). Entrepreneur's investment-consumption and hedging under inflation risk. *Journal of Systems Engineering*, 34(3), 383–394 (in Chinese).

Fei, C., & Fei, W. Y. (2015). Optimal control of Markovian switching systems with applications to portfolio decisions under inflation. *Acta Mathematica Scientia*, 35(2), 439–458.

Fei, W. Y., & Fei, C. (2013). Optimal stochastic control and optimal consumption and portfolio with G-Brownian motion. see arXiv:1309.0209v1.

Fei, C., Fei, W. Y., Rui, Y. Y., & Yan, L. T. (2019). International investment with exchange rate risk. *Asia-Pacific Journal of Accounting and Economics*. doi:10.1080/16081625.2019.1569539.

Fei, W. Y., & Li, S. J. (2012). Study on optimal consumption and portfolio with inflation under Knightian uncertainty. *Chinese Journal of Engineering Mathematics*, 29(6), 799–806 (in Chinese).

Fei, W. Y., Li, Y. H., & Xia, D. F. (2015). Optimal investment strategies of hedge funds with incentive fees under inflationary environment. *Systems Engineering: Theory and Practice*, 35(11), 2740–2748 (in Chinese).

Fei, W. Y., Xia, D. F., & Liu, P. (2014). An investor’s optimal portfolio with rare events and model uncertainty under inflation. *Chinese Journal of Applied Probability and Statistics*, 30(3), 322–336 (in Chinese).

Fei, W. Y., Zhang, F. H., & Yang, X. G. (2019). The impact of inflation on executive’s equity incentive and work effort. *Chinese Journal of Management Science*. Accepted (in Chinese).

Kim, T. S., & Omberg, E. (1996). Dynamic nonmyopic portfolio behavior. *The Review of Financial Studies*, 9(1), 141–161.

Koo, H. K. (1998). Consumption and portfolio selection with labor income: A continuous time approach. *Mathematical Finance*, 8(1), 49–65.

Li, Z. F., & Yuan, Z. J. (2010). A dynamic mean–variance model of portfolio selection under parameter uncertainty. *Journal of Management Sciences in China*, 13(12), 1–9 (in Chinese).

Liu, J. (2007). Portfolio selection in stochastic environments. *The Review of Financial Studies*, 20(1), 1–39.

Liu, H. J., Fei, W. Y., Zhu, Y. W., & Zheng, A. M. (2014). Optimal consumption–portfolio and bequest with insurance and retirement under Knightian uncertainty. *Operations Research Transactions*, 18(3), 88–98 (in Chinese).

Liu, H. L., & Wu, C. F. (2001). Optimal consumption and investment strategy based on worst-case. *Journal of Management Sciences in China*, 4(6), 48–54 (in Chinese).

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous time case. *The Review of Economics and Statistics*, 51(3), 247–257.

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3(4), 373–413.

Moos, D. (2011). Portfolio and consumption decisions under mean-reverting returns and mean-reverting labor income growth. Preprint. Retrieved from https://www.alexandria.unisg.ch/Publikationen/72495/L-en.

Nakamura, E., & Steinsson, J. (2018). The elusive costs of inflation: Price dispersion during the US great inflation. *The Quarterly Journal of Economics*, 133(4), 1933–1980.

Peng, S. G. (2010). Nonlinear expectations and stochastic calculus under uncertainty. See arXiv:1002.4546v1.

Santos, T., & Veronesi, P. (2005). Labor income and predictable stock returns. *The Review of Financial Studies*, 19(1), 1–44.

Sheremirov, V. (2015). Price dispersion and inflation: New facts and theoretical implications. Working Paper, Federal Reserve Bank of Boston. Retrieved from http://hdl.handle.net/10419/130698.

Sun, Y. Y., Hong, Y. M., & Wang, S. Y. (2019). Out-of-sample forecasts of China’s economic growth and inflation using rolling weighted least squares. *Journal of Management Science and Engineering*, 4, 1–11.

Vavra, J. (2014). Inflation dynamics and time-varying uncertainty: New evidence and an SS interpretation. *The Quarterly Journal of Economics*, 129(1), 215–258.

Viceira, L. M. (2001). Optimal portfolio choice for long-horizon investors with nontradable labor income. *The Journal of Finance*, 56(2), 433–470.

Wachter, J. A. (2002). Portfolio and consumption decisions under mean-reverting returns: An exact solution for complete markets. *Journal of Financial and Quantitative Analysis*, 37(1), 63–91.

Wang, G. C., & Wu, Z. (2009). A kind of problem of maximizing the expected utility from the terminal wealth: The case of inflation. *Chinese Journal of Applied Probability and Statistics*, 23(4), 345–353 (in Chinese).

Wulfsberg, F. (2016). Inflation and price adjustment, evidence from Norwegian consumer price data 1975–2004. *American Economic Journal: Macroeconomics*, 8(3), 175–194.

Yao, H. X., Jiang, L. M., Ma, Q. H., & Li, Y. (2013). Continuous-time mean–variance portfolio selection under inflation. *Control and Decision*, 28(1), 43–48 (in Chinese).

Yao, H. X., Wu, H. L., & Zeng, Y. (2014). Optimal investment strategy for risky assets under uncertain time-horizon and inflation. *Systems Engineering – Theory & Practice*, 34(5), 1089–1098 (in Chinese).

**Appendix**

The derivation process of Equations (15)–(17), (22)–(23) and (27). According to (14), the relevant partial derivatives can be obtained as follows:

\[
J_x = e^{-\delta(T-t)} \left[ \frac{1}{1-\gamma} A^{1-\gamma} B^{-\gamma} \int_0^T \left( \frac{\partial C_0(s)}{\partial s} + \frac{\partial C_1(s)}{\partial s} \right) X \right. \\
+ \left. \frac{1}{2} \frac{\partial C_2(s)}{\partial s} \right] e^{(\delta \chi)} ds + \frac{\delta}{1-\gamma} A^{1-\gamma} B^{-\gamma} \frac{\partial k}{\partial \tau} - \frac{\partial R}{\partial \tau} \right], \\
J_W = e^{-\delta(T-t)} A^{1-\gamma} B^{-\gamma}, \quad J_{WW} = -\gamma e^{-\delta(T-t)} A^{1-\gamma} B^{-\gamma-1}, \\
J_p = e^{-\delta(T-t)} A^{1-\gamma} k, \quad J_{pp} = -\gamma e^{-\delta(T-t)} A^{1-\gamma} B^{-\gamma-1} k^2, \\
J_X = xe^{-\delta(T-t)} \left[ \frac{1}{\gamma} A^{1-\gamma} B^{-\gamma} \int_0^T \left( C_1(s) \right) \right]
\]
\[
J_{XX} = e^{-(t-T)} \left[ -\frac{1}{\gamma} A^{(t-T)} B^{-1} \gamma \left[ \int_0^\tau (c_1(s) + c_2(s)X) e^{C(s)} ds \right]^2 \right.
\]
\[
+ \frac{1}{1-\gamma} A^{(t-T)} B^{-1} \gamma \left[ \int_0^\tau \frac{1}{\gamma} (c_2(s) + c_2(s)^2) e^{C(s)} ds \right]
\]
\[
+ 2A^{(t-T)} B^{-1} \gamma \left[ \int_0^\tau (c_1(s) + c_2(s)X) e^{C(s)} ds \right]
\]
\[
+ \gamma A^{(t-T)} B^{-1} \gamma \left[ \int_0^\tau (c_1(s) + c_2(s)X) e^{C(s)} ds \right]
\]
\[
+ \gamma A^{(t-T)} B^{-1} \gamma \left[ \int_0^\tau (c_1(s) + c_2(s)X) e^{C(s)} ds \right]
\]
\[
+ 2 \gamma A^{(t-T)} B^{-1} \gamma \left[ \int_0^\tau (c_1(s) + c_2(s)X) e^{C(s)} ds \right]
\]
\[
\left. + \gamma A^{(t-T)} B^{-1} \gamma \left[ \int_0^\tau (c_1(s) + c_2(s)X) e^{C(s)} ds \right] \right] \]