Onsager and Kaufman’s calculation of the spontaneous magnetization of the Ising model: II

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Abstract
In 2011 I reviewed the calculation by Onsager and Kaufman of the spontaneous magnetization of the square-lattice Ising model, which Onsager announced in 1949 but never published. I have recently been alerted to further original papers that bear on the subject. It is quite clear that the draft paper on which I relied was indeed written by Onsager, who was working on the problem with Kaufman, and that they had two derivations of the result.

KEY WORDS: Statistical mechanics, Ising model, spontaneous magnetization, Toeplitz matrices.

Introduction
In my original paper of 2011 on Onsager and Kaufman’s work on the spontaneous magnetization $M_0$ of the Ising model[1], I presented a draft paper that I had been given some years earlier by John Stephenson, who had received it in about 1965 from Ren Potts. It bears the hand-written names of Onsager and Kaufman. I presented it both as a directly scanned version and (for clarity) a transcript: I shall refer to it herein as OK.

I was concerned with the puzzle of why Onsager had announced in 1949 that he and Kaufman had a proof of the result for spontaneous magnetization of the Ising model, but had never published that proof. I was aware of various writings that bore on the problem, including those in the Onsager archive in Trondheim, at http://www.ntnu.no/ub/spesialsamlingene/tekark/tek5/arkiv5.php

In particular, I was aware of the section “Selected research material and writings” and the sub-sections 9.94 – 10.104 headed “Ising Model”. I quoted sub-section 9.97, which contains material relevant to the calculation of the spontaneous magnetization.
Professor Percy Deift of the Courant Institute in New York has recently alerted me to another sub-section of the Onsager archive, namely 17.120 – 17.129, headed simply “Writings” and containing three documents. I was remiss not to have found this earlier as it is very relevant. Here I briefly review my previous comments in the light of these documents.

Onsager and Kaufman’s two methods

In [1] I quoted Onsager as saying that he had reduced the problem to one of calculating a particular $k$ by $k$ Toeplitz determinant $D_m$ in the limit $m \to \infty$, and that he solved the problem in two ways, the first by using generating functions and using a parametrization in terms of elliptic functions to reduce it to an integral equation problem with a kernel that was the sum of two parts, one a function of the difference of the two parameters, the other a sum. (2, p. 11), [1, eqn 2.7])

In the second way, Onsager found a formula for the $m \to \infty$ limit of $\Delta_r$ for a general class of Toeplitz determinants $\Delta_r$. It is contained in the draft paper OK and in a letter from Onsager to Kaufman. We outline the method below.

Summary of the second method

The draft paper OK is concerned with the evaluation of an $r$ by $r$ Toeplitz determinant $\Delta_r$, with entries $c_{j-i}$ in row $i$ and column $j$, for $1 \leq i, j \leq r$. Let $f(z)$ be the generating function with coefficients $c_j$:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n . \quad (1)$$

The paper shows algebraically that if, for a finite number of $\alpha_j, \beta_k$,

$$f(z) = \frac{\prod_j (1 - \alpha_j z)^{m_j}}{\prod_k (1 - \beta_k z^{-1})^{n_k}} , \quad (2)$$

then

$$\Delta_r = \prod_j \prod_k (1 - \alpha_j \beta_k)^{m_j n_k} \quad (3)$$

provided the $m_j, n_k$ are non-negative integers, $|\beta_k| < 1$ and $r \geq \sum_k n_k$.

Further, on pages 21-24 of sub-section 9.97 of the Onsager archive, there is a letter from Onsager to Kaufman. He defines $\Delta_r$ (or $\Delta_k$) as above and sets

$$e^{\eta_+} = \prod_j (1 - \alpha_j z)^{m_j} , \quad e^{-\eta_-} = \prod_k (1 - \beta_k z^{-1})^{n_k} . \quad (4)$$

He then quotes the result (3) (in this letter Onsager negates the $n_k$) and states that this implies

$$\log \Delta_\infty = \frac{i}{2\pi} \int_{\omega=0}^{2\pi} \eta_+ d\eta_-(\omega) \quad (5)$$

where Onsager takes $z$ above to be $e^{i\omega}$. 

If we define \( b_n \) so that
\[
\log f(z) = \sum_{n=-\infty}^{\infty} b_n z^n ,
\] (6)
then \( b_0 = 0 \),
\[
\eta_+ = \sum_{n=1}^{\infty} b_n z^n , \quad \eta_- = \sum_{n=1}^{\infty} b_{-n} z^{-n}
\] (7)
and we can write (5) as
\[
\log \Delta_\infty = \sum_{n=1}^{\infty} n b_n b_{-n} .
\] (8)

This reasoning depends on \( f(z) \) having the form (2), with integer \( m_j, n_k \). However, Onsager obviously realises that the result (5) - (8) must have greater validity. He begins his letter to Kaufman by saying “This is to let you know that I have found a general formula for the value of an infinite recurrent determinant”. He applies this formula to the determinant needed for the spontaneous magnetization and obtains the now well-known result \( M_0 = (1 - k^2)^{1/8} \).

He and Kaufman preferred this method, but were working on “how to to fill out the holes in the mathematics and show the epsilons and deltas and all of that”[3, p. xxiii], when the mathematicians Kakutani and Szegö became aware of their work and “got there first”[2, p. 12] In fact Szegö’s paper, in which he gives the result (8) for the case when \( b_{-n} = b_n^* \), did not appear until 1952[4] and did not refer to the spontaneous magnetization problem. The first publication of a proof of the formula for \( M_0 \) was by C.N. Yang, also in 1952.[5] The general formula (8) appears in a 1963 paper by Montroll, Potts and Ward.[6, eqn. 68]

Further material

Sub-section 17.121 of the Onsager archive (the one of which I was unaware in 2011) is entitled “Crystal Statistics. IV. Long-range order in a binary crystal” (with B. Kaufman). It contains three documents:

1. Pages 1 – 4 is the letter from Onsager to Kaufman mentioned above, dated April 12, 1950, giving a formula for the determinant of a general \( k \) by \( k \) Toeplitz determinant in the limit \( k \to \infty \). This is also on pages 21 - 24 of sub-section 9.97 and is the letter quoted above. I refer to it section 5 (sub-section 1) of my 2011 paper.

2. Pages 5 –12 contain the draft paper OK I presented in [1]. I was working from a photo-copy of a photo-copy of a carbon copy. The version on the archive is clearer and appears to be from the original type-script (or at least a better carbon). My copy is interesting in that it contains hand-written corrections and additions, probably by Onsager or Kaufman themselves. The fact that the paper is on the Onsager archive is a clear indication that it is indeed by Onsager, who was working in collaboration with Kaufman.

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1 If \( f(z) \) is analytic on the unit circle \( |z| = 1 \), then the sum in (8) converges and can be approximated to any accuracy by a finite truncation of the values of \( n \). Since (2) is sufficiently general to fit any finite number of the \( b_n \), this suggests that (8) should apply to any function \( f(z) \) analytic on the unit circle. One needs to worry about convergence.
3. Pages 13 - 22 contain the draft of a paper headed “Crystal Statistics. IV. Long-range order in a binary crystal” by Lars Onsager and Bruria Kaufman. This appears to be giving Onsager’s first method. It is unfinished, but is certainly leading towards an integral equation with a kernel with difference and sum properties. In Appendix A of [1] I give a calculation of $M_0$ that involves such a kernel. Onsager and Kaufman’s draft begins differently, but appears to be heading in a similar direction.

So Onsager and Kaufman did indeed have two ways of proving the formula for $M_0$. They turned their attention to other problems when the mathematicians became interested in their preferred method, so never published either way of solving the problem.

References

[1] R. J. Baxter, Onsager and Kaufman’s Calculation of the Spontaneous Magnetization of the Ising Model, J. Stat. Phys. 145 518–548 (2011)

[2] L. Onsager, The Ising model in two dimensions, in Critical Phenomena in Alloys, Magnets and Superconductors, eds. R.E. Mills, E. Ascher and R. I. Jaffee (New York: McGraw-Hill) pp. 3 – 12 (1971)

[3] L. Onsager, Autobiographical commentary of Lars Onsager, in Critical Phenomena in Alloys, Magnets and Superconductors, eds. Mills R E, Ascher E and Jaffee R I (New York: McGraw-Hill) pp. xix – xxiv (1971)

[4] G. Szegő, On certain hermitian forms associated with the Fourier series of a positive function, Communications du Seminaire mathematique de l’universite de Lund, tome supplementaire dedie a Marcel Riesz, 228 – 238 (1952)

[5] C. N. Yang, The spontaneous magnetization of a two-dimensional Ising model, Phys. Rev. 85 808 – 816 (1952)

[6] E. W. Montroll, R. B. Potts and J. C. Ward, Correlations and Spontaneous Magnetization of the Two-Dimensional Ising Model, J. Math. Phys. 4 308 – 322 (1963)