Abstract. We consider the Szegő kernel associated with domains $\Omega$ in $\mathbb{C}^2$ given by

$$\Omega = \{ (z, w) : \text{Im} w > b(\text{Re} z) \}$$

for $b$ a non-convex polynomial of even degree with positive leading coefficient. Such domains are not pseudoconvex. We give a precise description of a subset of $\partial \Omega \times \partial \Omega$ on which the kernel and all of its derivatives are finite. We show, in particular, that for such domains, the Szegő kernel has singularities off the diagonal of $\partial \Omega \times \partial \Omega$ as well as points on the diagonal at which it is finite.

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1. Introduction

Let $M$ be a CR submanifold of $\mathbb{C}^n$ and consider the Szegő projection operator $S$, which is the orthogonal projection of $L^2(M)$ onto the subspace of functions annihilated (in the sense of distributions) by all tangential Cauchy-Riemann operators. In the singular-integral approach to the study of this operator, one identifies a distribution $S$ on $M \times M$ so that for $f \in L^2(M)$,

$$S[f](z) = \int_M f(w)S(z, w)d\mu(w).$$

$S$ is called the Szegő kernel. Theorems concerning, for example, the mapping properties of the operator often follow from estimates on the kernel and its derivatives. The kernel will have singularities. A central idea in the modern theory of singular integral operators such as the above is that the nature of the singularities is closely tied to the geometry of the CR manifold; there should be associated with $M$ a certain natural metric, and when the estimates on the kernel are stated in terms of this metric, one may be able to recognize the integral operator as a representative of a larger class of operators whose properties are understood.

Such a program has been carried out for a large class of hypersurfaces bounding domains in $\mathbb{C}^2$ and for a more restricted class of boundaries of domains in higher dimensions. In particular, Nagel [Nag86] and Nagel, Rosay, Stein, and Wainger [NRSW89] consider the Bergman and Szegő projections for (weakly) pseudoconvex domains in $\mathbb{C}^2$ under a finite type hypothesis. For such boundaries, they identify a natural metric which can be defined in terms of geometric properties of the domain. Of particular importance is the fact that this metric is nonisotropic, meaning that different directions carry different weights. They use this non-isotropic metric to
define a class of integral operators called non-isotropic smoothing (NIS) operators and show that the Szegö projection for such hypersurfaces in \( \mathbb{C}^2 \) is an NIS operator.

From this point, several avenues of investigation present themselves. In one direction, one continues to investigate CR manifolds arising as boundaries of domains in \( \mathbb{C}^n \) but relaxes some of the hypotheses (allowing, for example, points of infinite type, as in [HNW10]). Moving in another direction, one could consider model CR submanifolds of \( \mathbb{C}^n \) with codimension larger than 1. See, for example, [NRS01]. Two quite simple model CR submanifolds are the tube model

\[
T_k = \{ (z_1, z_2, z_3, z_4) : \text{Re } z_j = |\text{Re } z_1|^j, j = 2, 3, 4 \},
\]

which carries a Lie group structure, or the related manifold

\[
\overline{M} = \{ (z_1, z_2, z_4) : \text{Re } z_j = |\text{Re } z_1|^j, j = 2, 4 \}.
\]

Both are CR manifolds of CR dimension 1. Neither is the boundary of a domain. For any tube manifold of CR dimension 1, one can follow the approach in [Nag86] to derive an explicit expression for the Szegö kernel. Consider

\[
M = \{ (z_1, \ldots, z_n) : \text{Im } z_j = b_j(\text{Re } z_1), 2 \leq j \leq n \}
\]

for smooth real-valued functions \( b_j \). Identify \( M \) with \( \mathbb{R}^{n+1} \) via the correspondence

\[
(x + iy, t_2 + ib_2(x), \ldots, t_n + ib_n(x)) \leftrightarrow (x, y, t_2, \ldots, t_n).
\]

Since \( M \) has CR dimension 1, the set of all CR operators on \( M \) is generated by a single element \( L \). Push this forward to \( \mathbb{R}^{n+1} \) and consider the orthogonal projection of \( L^2(\mathbb{R}^{n+1}) \) onto the null space of this operator. We still call this operator the Szegö projection operator associated with \( M \). The associated kernel is

\[
S((x, y, t), (r, s, u)) = c \int_{\tau_n > 0} \int_{-\infty}^{\infty} \frac{e^{i(\eta(\tau-s)+\tau (t-u))}e^{\eta(\tau+\tau)+\tau |b(x) + b(t)|}}{N(\eta, \tau)} \, d\eta \, d\tau,
\]

where \( b = (b_2, \ldots, b_n), t = (t_2, \ldots, t_n) \), etc., and

\[
N(\eta, \tau) = \int_{-\infty}^{\infty} e^{2[\eta \lambda - \tau \lambda]} \, d\lambda.
\]

Let us focus on the \( \overline{M} \) from (1.2). In this case,

\[
N(\eta, \tau_2, \tau_4) = \int_{-\infty}^{\infty} e^{-2[2\tau_4 \lambda^4 + \tau_2 \lambda^2 - \eta \lambda]} \, d\lambda, \quad \tau_4 > 0.
\]

\( N \) is a function of the three parameters \( \eta, \tau_2, \) and \( \tau_4 \), and the main difficulties in the analysis arise from the region of the parameter space in which the integrand is of the form \( e^{-P(\lambda; \eta, \tau_2, \tau_4)} \) for \( P \) non-convex.

Compare this to the situation for

\[
\partial \Omega = \{ (z_1, z_2) : \text{Im } z_2 = b(\text{Re } z_1) \}.
\]

Suppose \( b \) is an even-degree polynomial with positive leading coefficient. One verifies easily that \( \partial \Omega \) bounds a pseudoconvex domain if and only if \( b \) is convex. Thus for convex \( b \), the Szegö kernel is well-understood.

Suppose, then, that \( b \) fails to be convex on all of \( \mathbb{R} \). In the expression for the Szegö kernel, the corresponding denominator integral \( N \) is

\[
\int_{-\infty}^{\infty} e^{-2[\tau \lambda - \eta \lambda]} \, d\lambda.
\]
In this case, $N$ is a function of just two parameters $\eta$ and $\tau$, and the integrand is always of the form $e^{-P(\lambda, \eta, \tau)}$ for $P$ non-convex. We thus see that the study of certain non-pseudoconvex domains in $\mathbb{C}^2$ is an entirely natural step if one is interested in these operators for CR manifolds of codimension larger than one.

In contrast with the situation for pseudoconvex domains, comparatively little is known about the Szegő kernel for non-pseudoconvex domains. Some of the first results in the non-pseudoconvex context are due to Carracino ([Car05], [Car07]). She obtains detailed estimates for the Szegő kernel on the boundary of a model domain of the type (1.6) with $b$ a non-smooth, non-convex, piecewise quadratic function. She shows that the Szegő kernel has singularities off of $\Delta$ in this case. Then in [GHar], the current authors identify a subset of $\Omega \times \Omega$ on which the integrals defining the Szegő kernel and its derivatives are absolutely convergent for the case in which $b$ is a non-convex quartic polynomial. In particular, this work shows that there are points on the diagonal $\Delta$ at which the Szegő kernel is finite as well as points off the diagonal at which it is infinite.

In this paper, we explore this phenomenon in the much more general setting in which $b$ is a non-convex even-degree polynomial with positive leading coefficient. Without loss of generality, we may suppose

$$b(x) = \frac{1}{2n} x^{2n} + \sum_{j=2}^{2n-1} a_j x^j, \quad n \geq 2. \quad (1.7)$$

Although the statements of the theorems in this paper closely resemble those in [GHar], the technical challenges in proving the theorems are rather different. We will comment on these substantial differences in due course.

We close this introductory section with a comment on the rationale for studying non-pseudoconvex domains from the perspective of singular integral operators. Whereas the Szegő kernel for a pseudoconvex domain of finite type is an example of a non-isotropic smoothing operator, Carracino’s work shows that the structure of the singularities of the Szegő kernel can be very different in the non-pseudoconvex setting, but it is inconclusive on the question of whether these kernels are related to flag kernels [NRS01] or product singular integral operators [NS04].

2. Definitions, Notation, and Statement of Results

We begin with a more precise discussion of the Szegő projection operator and its associated integral kernel for domains in $\mathbb{C}^2$ having the form (1.7). We take $b$ smooth so that $\Omega \subset \mathbb{C}^2$ is smoothly-bounded. As above, let $\mathcal{O}(\Omega)$ denote the space of functions holomorphic on $\Omega$. Define

$$\mathcal{H}^2(\Omega) := \left\{ F \in \mathcal{O}(\Omega) : \sup_{\varepsilon > 0} \int_{\partial \Omega} |F(x + iy, t + ib(x) + i\varepsilon)|^2 \, dx \, dy \, dt < \infty \right\}.$$  

$\mathcal{H}^2(\Omega)$ can be identified with the set of all functions $f$ in $L^2(\partial \Omega)$ (which is itself identified with $L^2(\mathbb{R}^3)$) which are solutions in the sense of distributions to

$$\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} - ib'(x) \frac{\partial}{\partial t} \right) |f| \equiv 0. \quad (2.1)$$

We define the Szegő projection operator $S$ to be the orthogonal projection of $L^2(\partial \Omega)$ onto this (closed) subspace $\mathcal{H}^2(\Omega)$. 

One establishes the existence of a unique integral kernel associated with the operator. This is discussed, for example, in [Ste72], where the approach is as follows: Begin with an orthonormal basis \( \{ \phi_j \} \) for \( \mathcal{H}^2(\Omega) \) and form the sum
\[
S(z, w) = \sum_{j=1}^{\infty} \phi_j(z)\overline{\phi_j(w)}.
\]
One shows that this converges uniformly on compact subsets of \( \Omega \times \Omega \), that \( S(z, \cdot) \in \mathcal{H}^2(\Omega) \) for each \( z \in \Omega \), and that for \( g \in \mathcal{H}^2(\Omega) \),
\[
g(z) = \int_{\partial \Omega} S(z, w)g(w) \, d\sigma(w).
\]
S is then the Szegő kernel. From its construction it is clear that it will be smooth on \( \Omega \times \Omega \). It may extend to a smooth function on some larger subset of \( \Omega \times \Omega \).

For domains of the form \( \mathbb{C}^2 \), one can derive an explicit formula for the Szegő kernel. Let \( z = (z_1, z_2) \) and \( w = (w_1, w_2) \) be elements of \( \mathbb{C}^2 \). Set
\[
N(\eta, \tau) = \int_{-\infty}^{\infty} e^{2\tau|\eta\lambda - b(\lambda)|} \, d\lambda.
\]
Then
\[
S(z, w) = c \int_{\tau>0} \tau e^{\eta[z_1+w_1]+i\tau[z_2-w_2][N(\eta, \tau)]^{-1}} \, d\eta \, d\tau,
\]
where \( c \) is an absolute constant.

**Remark 2.1.** See [HNW10] for detailed discussions of \( \mathcal{H}^p \) spaces for unbounded domains, the derivations of such integral formulas, and the identification of \( \mathcal{H}^2(\Omega) \) with \( L^2(\partial \Omega) = L^2(\mathbb{R}^3) \) functions satisfying the differential equation (2.1).

**Remark 2.2.** Many authors only consider \( S \) as a distribution on \( \partial \Omega \times \partial \Omega \) since \( S \) is smooth on \( \Omega \times \Omega \). In this situation, one can identify the boundary with \( \mathbb{R}^3 \) and consider the integral kernel
\[
S(x, y, t, r, s, u) =
\]
\[
c \int_{\tau>0} \tau e^{\eta[t-u]+i\eta(y-s)-[b(x)+b(r)-\eta(x+r)]} [N(\eta, \tau)]^{-1} \, d\eta \, d\tau.
\]
This is done, for example, in the work of Nagel [Nag86], Haslinger [Has95], and Carracino [Car05], [Car07].

We may now state our results:

Let \( b \) be as in (1.7). For each real \( \eta \), set \( B_\eta(x) := -\eta x + b(x) \). The set of minimizers of this function is of vital importance in our analysis. Thus we define
\[
\Lambda_\eta = \{ \lambda : \inf_x B_\eta(x) = B_\eta(\lambda) \},
\]
with \( \Lambda := \bigcup_\eta \Lambda_\eta \) and \( C = \{ \eta : |\Lambda_\eta| > 1 \} \). Furthermore, set
\[
z = (z_1, z_2) = (x + iy, t + ib(x) + ih)
\]
\[
w = (w_1, w_2) = (r + is, u + ib(r) + ik),
\]
and define
\[
\Sigma = \{ (z, w) : x = r \text{ and } x \in \Lambda \} \cup \{ (z, w) : x, r \in \Lambda_c \text{ for } c \in C \}.
\]
Theorem 2.5. \[ G_{\text{Har}} \]

in which

\[ b \] \hspace{1cm} (2.9)

(2.10)

in which

Compare this with Theorem 3.2 in Remark 2.4.

The integral defining Theorem 2.3.

Finally, for a function \( \eta \) in (2.2). Observe that for fixed \( b \) if and only if

This is an open neighborhood of \( (\Omega \times \Omega) \setminus \Sigma \). More generally, if \( i_1, j_1, i_2, \) and \( j_2 \) are non-negative integers, then

(2.11)

\[ \partial_{z_{i_1}} \partial_{\bar{w}_{j_1}} \partial_{z_{i_2}} \partial_{\bar{w}_{j_2}} S(z, w) = e' \int_{\tau>0} e^{\eta [z_1 + \bar{w}_1 + i\tau z_2 - w_2]} \frac{\eta^{i_1} \tau^{j_1} + \eta^{i_2} \tau^{j_2}}{N(\eta, \tau)} d\eta d\tau \]

is absolutely convergent in the same region.

Remark 2.4. Compare this with Theorem 3.2 in [HNW10] and with Theorem 2.3 in [GHar].

Theorem 2.5. If \( [(x + iy, t + ib(x)), (r + iy, t + ib(r))] \in \Sigma, S[(x, y, t), (r, y, t)] \) is infinite. Also, if \( \delta = h + k > 0 \),

\[ \lim_{\delta \to 0^+} S[(x + iy, t + i(b(x) + h)), (r + iy, t + i(b(r) + k))] = \infty. \]

We will show that the set \( \Sigma \) is equal to the diagonal \( \Delta \) of \( \partial \Omega \times \partial \Omega \) precisely when the polynomial \( b \) is convex. For non-convex \( b \), there are both points \textit{off the diagonal} that are contained in \( \Sigma \) and points \textit{on the diagonal} that are not in \( \Sigma \). We summarize this important observation in a corollary.

Corollary 2.6. For tube domains (??) in \( \mathbb{C}^2 \) with \( b \) an even-degree polynomial with positive leading coefficient, the Szegő kernel extends smoothly to \( (\Omega \times \Omega) \setminus \Delta \) if and only if \( b \) is convex.

An analysis of the Szegő kernel begins with estimates of the integral \( N \) defined in (2.2). Observe that for fixed \( \eta \in \mathbb{R} \) and \( \tau > 0 \), \( \lim_{|\lambda| \to \infty} 2\pi [\eta \lambda - b(\lambda)] = -2\pi \lim_{|\lambda| \to \infty} B_{\eta}(\lambda) = -\infty \). The heuristic principle that guides the analysis of such integrals is that the main contribution comes from a neighborhood of the point(s) at which the exponent attains its global maximum. For our integral \( N \), let \( \lambda(\eta) \) denote the largest real number at which \( \inf_{\lambda} B_{\eta}(\lambda) \) is attained. Then

\[ N(\eta, \tau) = e^{-2\pi B_{\eta}(\lambda(\eta))} \int_{-\infty}^{\xi} e^{-2\pi [-\eta \lambda + b(\lambda) - B_{\eta}(\lambda(\eta))]} d\lambda \]

\[ = e^{2\pi b^*(\eta)} \int_{-\infty}^{\xi} e^{-2\pi p_{\eta}(\xi)} d\xi, \]

where

(2.12)

\[ p_{\eta}(\xi) := -\eta \xi + b(\xi + \lambda(\eta)) - b(\lambda(\eta)) \]

is a non-negative polynomial vanishing to even order at the origin. Furthermore, by our choice of \( \lambda(\eta) \), if for some \( \eta \), \( p_{\eta}(\xi) = 0 \) for non-zero \( \xi \), necessarily \( \xi < 0 \).

In Sections 3 and 4, we focus on understanding the main contribution to the integral \( N \) by exploring \( B_{\eta} \) and \( \lambda(\eta) \), while in Section 5, we focus on estimating the integral that remains once we have taken out this main contribution. The theorems are established in Section 6.
3. Global properties of $\lambda(\eta)$ and $B_\eta$

This section contains a number of technical lemmas on the long-term behavior of $\lambda(\eta)$ and $B_\eta(\lambda(\eta)) = -b^*(\eta)$. Most of these results follow rather easily from the fact that $B_\eta$ is a polynomial and $\lambda(\eta)$ is one of its critical points.

**Lemma 3.1.** \( \lim_{\eta \to -\infty} \lambda(\eta) = -\infty \) and \( \lim_{\eta \to \infty} \lambda(\eta) = \infty \). Furthermore, \( \lambda(\eta) \sim \eta^\frac{2}{2n-1} \) as \( |\eta| \to \infty \).

**Proof.** We consider the case \( \eta \to -\infty \). The case \( \eta \to \infty \) is established similarly.

Consider the equation \( b'(\omega) = \eta \). Since \( b \) has even degree and positive leading coefficient, there exists an interval \((-\infty, \beta)\) on which \( b \) is convex. Thus on this interval, \( b' \) is an increasing function with a well-defined inverse function \( \eta \mapsto \omega(\eta) \).

We claim that for any \( L > 0 \) with \(-L \leq \beta\) there exists \( m \) such that for \( \eta < m \), \( \omega(\eta) = \lambda(\eta) \). Indeed, since \( b' \) is an odd-degree polynomial with positive leading coefficient, the number \( m = \inf\{b'(\omega) : \omega \geq -L\} \) is finite. If \( \eta < m \), the only solution to \( b'(\omega) = \eta \) on \( \mathbb{R} \) must lie in \((-\infty, -L) \subseteq (-\infty, \beta)\). Since \( \lambda(\eta) \) is a solution, it lies in this interval. Thus for \( \eta < m \), \( \lambda(\eta) = \omega(\eta) \).

Note that \( b'(\omega) = \omega^{2n-1} + \sum_{j=2}^{2n-1} ja_j \omega^{j-1} \). Take \( L > 0 \) so that \( |\omega| \geq L \) implies

\[
\sum_{j=2}^{2n-1} j|a_j||\omega|^{2n+j} \leq \frac{1}{2}.
\]

Then for \( \omega \leq -L \),

\[
\frac{3}{2} \xi^{2n-1} \leq \omega^{2n-1} \left( 1 + \sum_{j=2}^{2n-1} j|a_j||\omega|^{2n+j} \right) = \omega^{2n-1} - \sum_{j=2}^{2n-1} j|a_j||\omega|^{j-1} \leq b'(\omega).
\]

Since for \( \eta < m \) the solution to \( b'(\omega) = \eta \) is \( \lambda(\eta) \), this shows that \( \lambda(\eta) \to -\infty \) as \( \eta \to -\infty \). Furthermore, if \( b'(\omega) = \eta \),

\[
(3.1) \quad \omega^{2n-1} = \eta - \sum_{j=2}^{2n-1} ja_j \omega^{j-1} \iff 1 = \frac{\eta}{\omega^{2n-1}} - \sum_{j=2}^{2n-1} ja_j \omega^{2n-j} = \frac{\eta}{\omega^{2n-1}} + o(1)
\]

as \( \eta \to -\infty \). Thus \( \lambda(\eta)^{2n-1} \sim \eta \) as \( \eta \to -\infty \), i.e., \( \lambda(\eta)^{2n-1} = \eta[1 + o(1)] \) as \( \eta \to -\infty \). It follows that \( \lambda(\eta) \sim \eta^\frac{2}{2n-1} \) as \( \eta \to -\infty \). \( \square \)

This allows us immediately to obtain size estimates for \( B_\eta(\lambda(\eta)) = -b^*(\eta) \) for large \( \eta \).

**Lemma 3.2.**

\[
b^*(\eta) \sim \left( \frac{2n-1}{2n} \right) \eta^{\frac{2}{2n-1}} \quad \text{as} \quad |\eta| \to \infty.
\]

**Proof.**

\[
B_\eta(\lambda(\eta)) = b(\lambda(\eta)) - \eta\lambda(\eta)
\]

\[
= \frac{1}{2n} \lambda(\eta)^{2n} + \sum_{j=2}^{2n-1} a_j \lambda(\eta)^j - \eta\lambda(\eta).
\]
By Lemma 3.1,

\[ B_\eta(\lambda(\eta)) = \frac{1}{2n} \eta^{2n} (1 + o(1))^{2n} + \sum_{j=2}^{2n-1} a_j \left( \eta^{\frac{j-1}{2n}} (1 + o(1)) \right)^j - \eta^{2n} (1 + o(1)) \]

\[ = \frac{1}{2n} \eta^{2n} (1 + o(1)) + \sum_{j=2}^{2n-1} a_j \eta^{\frac{j-1}{2n}} (1 + o(1)) \]

\[ = \left( \frac{1 - 2n}{2n} \right) \eta^{\frac{2n}{2n-1}} (1 + o(1)) + \sum_{j=2}^{2n-1} a_j \eta^{\frac{j-1}{2n}} (1 + o(1)) \]

\[ = \left( \frac{1 - 2n}{2n} \right) \eta^{\frac{2n}{2n-1}} (1 + o(1)) \]

as \(|\eta| \to \infty\), i.e.,

\[ B_\eta(\lambda(\eta)) \sim \left( \frac{1 - 2n}{2n} \right) \eta^{\frac{2n}{2n-1}} \]

as \(|\eta| \to \infty\). By our definition of \(b^*\), the result is established.

We will also need asymptotic estimates for \(b^{(j)}(\lambda(\eta))\):

**Lemma 3.3.** For \(j = 2, \ldots, 2n\),

\[ b^{(j)}(\lambda(\eta)) \sim \frac{(2n - 1)!}{(2n - j)!} \eta^{\frac{2n - j}{2n}} \] as \(|\eta| \to \infty\).

**Proof.** The proof is similar to that for Lemma 3.2 and is omitted.

We close this section with a proposition stating several properties of \(b^*\).

**Proposition 3.4.** For \(b\) as in (1.7), \(b^*(\eta) = \sup_x [\eta x - b(x)]\) is finite and convex on \(\mathbb{R}\). It is therefore continuous.

**Proof.** We merely sketch the proof since these are known properties of the Legendre transform. The finiteness of \(b^*\) comes from the fact that \(x \mapsto \eta x - b(x)\) is a non-constant polynomial with even degree and negative leading coefficient. The convexity comes from the fact that \(b^*\) is the supremum of a family \(\{ \eta \mapsto \eta x - b(x) : x \in \mathbb{R} \}\) of convex functions. Furthermore, \(b^*\) is continuous since every (finite) convex function is continuous.

\[ \square \]

4. Local properties of \(\lambda(\eta)\) and \(B_\eta\)

The main result of this section describes those points in \(\mathbb{R}\) that can be (global) minimizers of one of the members of the family of polynomials \(\{ B_\eta(\lambda) := -\lambda \eta + b(\lambda) : \eta \in \mathbb{R} \}\).

**Definition 4.1.** For each \(\eta \in \mathbb{R}\), define \(\Lambda_\eta\) to be the set of all points at which the polynomial \(B_\eta\) attains its global minimum. Let \(\sigma(\eta)\) be the smallest element of \(\Lambda_\eta\) and let \(\lambda(\eta)\) be the largest. Let \(\mathcal{C} = \{ \eta : |\Lambda_\eta| > 1 \}\). Finally, let \(\Lambda = \bigcup_\eta \Lambda_\eta\) and \(\lambda[\mathbb{R}] = \{ \lambda(\eta) : \eta \in \mathbb{R} \}\).
Theorem 4.2. $\lambda[\mathbb{R}] = \mathbb{R} \setminus \bigcup_{c \in C} [\sigma(c), \lambda(c)]$.

In the case of a convex polynomial $b$, $b'$ is one-to-one and hence $B_\eta$ has precisely one critical point for each $\eta$. Thus in the convex case, $C = \emptyset$ and $b'$ and $\lambda$ are inverses. These statements are not true in the non-convex case, though there are partial analogues.

Since all elements of $\Lambda_\eta$ are solutions to $\eta = b'(\lambda)$, the following is immediate.

Corollary 4.3. $\eta \mapsto \lambda(\eta)$ is injective.

We easily verify several other properties of $\lambda(\cdot)$ and $b'$.

Lemma 4.4. If $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 < \lambda_2$, then $b'(\lambda_1) < b'(\lambda_2)$.

Proof. Since $\lambda_1 \in \Lambda$, there exist $\eta_1 \neq \eta_2$ such that $\lambda_1 = \lambda(\eta_1)$ and $\lambda_2 = \lambda(\eta_2)$. Since $\eta_1 = b'(\lambda(\eta_1)) = b'(\lambda_1)$, we must show that $\eta_1 < \eta_2$.

Suppose, on the contrary, that $\eta_2 < \eta_1$. Since $\lambda(\eta_1)$ is a point at which $B_{\eta_1}(\lambda) = -\eta_1\lambda + b(\lambda)$ attains its global minimum, $B_{\eta_2}(\lambda_2) < B_{\eta_1}(\lambda_1)$. If $\eta_2 < \eta_1$,

$$B_{\eta_1}(\lambda_2) - B_{\eta_1}(\lambda_1) = -\eta_1\lambda_2 + b(\lambda_2) - (-\eta_1\lambda_1 + b(\lambda_1)) = (\lambda_1 - \lambda_2)[\eta_2 + (\eta_1 - \eta_2)] + b(\lambda_2) - b(\lambda_1) = (\lambda_1 - \lambda_2)(\eta_1 - \eta_2) + B_{\eta_2}(\lambda_2) - B_{\eta_1}(\lambda_1) < 0.$$  

This contradicts the fact that $B_{\eta_1}$ takes its global minimum at $\lambda_1$ and proves the result. \hfill \Box

Lemma 4.5. $\lambda : \mathbb{R} \to \lambda[\mathbb{R}]$ and $b' : \lambda[\mathbb{R}] \to \mathbb{R}$ are inverses.

Proof. We have already observed that $\eta = b'(\lambda(\eta))$ for all $\eta \in \mathbb{R}$.

Thus consider $\omega \in \lambda[\mathbb{R}]$. There exists a unique $\nu$ such that $\omega = \lambda(\nu)$. Since $\nu = b'(\lambda(\nu)) = b'(\omega)$, we have $\omega = \lambda(b'(\omega))$, as desired. \hfill \Box

Corollary 4.6. $\lambda : \mathbb{R} \to \lambda[\mathbb{R}]$ is increasing.

The proof of Theorem 4.2 requires a number of additional technical lemmas.

Lemma 4.7. Take $c \in C$. If $\omega \in (\sigma(c), \lambda(c)) \setminus \Lambda_c$, then there does not exist an $\eta$ for which $\omega \in \Lambda_\eta$.

Proof. Since $\omega$ is not a location of the global minimum of $B_c$, $-\omega + b(\omega) > -\sigma(c) + b(\sigma(c))$ and $-\omega + b(\omega) > -\lambda(c) + b(\lambda(c))$.

Since $\sigma(c) < \omega < \lambda(c)$, if $\eta > c$,

$$B_\eta(\omega) - B_\eta(\lambda(c)) = -\eta\omega + b(\omega) + \eta\lambda(c) - b(\lambda(c)) = (\omega + \lambda(c))[\eta + (\eta - c)] + b(\omega) - b(\lambda(c)) = B_c(\omega) - B_c(\lambda(c)) + [\lambda(c) - \omega][\eta - c] > 0.$$  

Similarly, for $\eta < c$, $B_\eta(\omega) - B_\eta(\sigma(c)) > 0$. We conclude that there is no $\eta \in \mathbb{R}$ for which $\omega$ is the location of the global minimum of $B_\eta$. \hfill \Box

Corollary 4.8. If $\eta_1 \neq \eta_2$, then $\Lambda_{\eta_1} \cap \Lambda_{\eta_2} = \emptyset$. Furthermore, if $c_1, c_2 \in C$ with $c_1 \neq c_2$, $[\sigma(c_1), \lambda(c_1)) \cap [\sigma(c_2), \lambda(c_2)] = \emptyset$.

Lemma 4.9. Let $\deg b = 2n$. Then $|C| \leq n - 1$. 

Proof. Let $c \in C$. Then $(\sigma(c), \lambda(c))$ is non-empty. Since $\lambda \mapsto -c\lambda + b(\lambda)$ takes the same value at $\sigma(c)$ and $\lambda(c)$, by Rolle’s theorem, there exists $\omega_0 \in (\sigma(c), \lambda(c))$ at which $-c + b'(\omega_0) = 0$, i.e., $\omega_0$ is another critical point of $B_c$. Since $-c + b'(\sigma(c)) > 0$ but $-c + b'(\lambda(c)) < 0$, we may take the point $\omega_0$ to be a local maximum of $B_c$. Thus $B_c'' = b''$ must change sign in each of $(\sigma(c), \omega_0)$ and $(\omega_0, \lambda(c))$. Since by Corollary 4.8 the intervals in the collection $\{(\sigma(c), \lambda(c)) : c \in C\}$ are disjoint, the total number $|C|$ can not exceed $\frac{1}{2} \deg b'' = n - 1$. \hfill \square

The next lemma is central, as it identifies subintervals of $\mathbb{R}$ in which $\lambda[\mathbb{R}]$ is dense.

**Lemma 4.10.** Let $\alpha, \eta_0 \in \mathbb{R}$.

1. If $a < \alpha(\eta_0)$, $(a, \alpha(\eta_0)) \cap \lambda[\mathbb{R}] \neq \emptyset$.
2. If $\alpha(\eta_0) < a$, $(\alpha(\eta_0), a) \cap \lambda[\mathbb{R}] \neq \emptyset$.

**Proof.** We note that $\eta_0$ need not be an element of $C$. If it is not, $\sigma(\eta_0) = \lambda(\eta_0)$.

We prove the first statement. The proof of the second is similar. If $\omega < \sigma(\eta_0)$, then

\[(4.1) \quad B_{\eta_0}(\omega) > B_{\eta_0}(\sigma(\eta_0)).\]

Fix $a < \sigma(\eta_0)$. Since $B_{\eta_0}$ is continuous, for any $L > 0$ satisfying $-L < a$, there exists $d$ (depending on $L$) such that for all $\omega \in [-L, a]$,

\[B_{\eta_0}(\omega) \geq d > B_{\eta_0}(\sigma(\eta_0)) \iff B_{\eta_0}(\omega) - B_{\eta_0}(\sigma(\eta_0)) > d - B_{\eta_0}(\sigma(\eta_0)) := \alpha > 0.\]

We choose $L$ as follows: Since, by Lemma 3.1, $\lambda(\eta) \to -\infty$ as $\eta \to -\infty$, there exists $\eta^* < \eta_0 - 1$ satisfying $\lambda(\eta^*) < -|a| - 1$. Set $-L := \lambda(\eta^*)$.

Set $\varepsilon = \min\{1, \frac{\alpha}{\sigma(\eta_0) + 2L}\}$. We claim that for all $\eta \in (\eta_0 - \varepsilon, \eta_0)$ and $\omega \in [-L, a]$,

\[(4.2) \quad B_{\eta}(\omega) > B_{\eta}(\sigma(\eta_0)).\]

Indeed, since $B_{\eta}(\omega) = B_{\eta_0}(\omega) - (\eta - \eta_0)\omega$ and $B_{\eta}(\sigma(\eta_0)) = B_{\eta_0}(\sigma(\eta_0)) - (\eta - \eta_0)\sigma(\eta_0)$,

\[
B_{\eta}(\omega) - B_{\eta}(\sigma(\eta_0)) = B_{\eta_0}(\omega) - B_{\eta_0}(\sigma(\eta_0)) + (\eta - \eta_0)(\sigma(\eta_0) - \omega) \\
\geq \alpha - \varepsilon(\sigma(\eta_0) - \omega) \\
\geq \alpha - \frac{\alpha}{2\sigma(\eta_0) + 2L} = \frac{\alpha}{2}.
\]

This proves (4.2).

Finally, we claim that if $\eta \in (\eta_0 - \varepsilon, \eta_0)$, then $\lambda(\eta) \in (a, \sigma(\eta_0))$. Since $\eta < \eta_0$ and $\lambda$ is an increasing function, $\lambda(\eta) < \lambda(\eta_0)$. By Lemma 4.7 this forces $\lambda(\eta) < \sigma(\eta_0)$. Since $B_{\eta}(\lambda(\eta)) < B_{\eta}(\sigma(\eta_0))$, by (4.2), $\lambda(\eta) \notin [-L, a]$. If $\lambda(\eta)$ were less than $-L = \lambda(\eta^*)$, then the fact that $\lambda$ is increasing would imply $\eta < \eta^* < \eta_0 - 1$, which is false. We conclude that $\lambda(\eta) \in (a, \sigma(\eta_0))$. \hfill \square

We are now ready to prove Theorem 4.2.

**Proof.** That $\lambda[\mathbb{R}] \subseteq \mathbb{R} \setminus \bigcup_{c \in C}[\sigma(c), \lambda(c))$ follows from Lemma 4.7 and Corollary 4.8.

Next we prove $\mathbb{R} \setminus \bigcup_{c \in C}[\sigma(c), \lambda(c)) \subseteq \lambda[\mathbb{R}]$. Suppose $C \neq \emptyset$. Since $|C|$ is finite, we may order the elements of $C$ so that $c_i < c_{i+1}$, $1 \leq i \leq k - 1$. The left-hand set is made up of three kinds of intervals: two semi-infinite intervals $(-\infty, \sigma(\eta_1))$ and $[\lambda(\eta_k), \infty)$, and (if $k \geq 2$) the intervals $[\sigma(\eta_i), \sigma(\eta_{i+1})]$. We must show that every $\omega$ in one of these intervals is in $\lambda[\mathbb{R}]$. 


By Lemma 4.10, \( U := (\lambda(c_i), \omega) \cap \lambda[\mathbb{R}], \) \( V := (\omega, \sigma(c_{i+1})) \cap \lambda[\mathbb{R}], \) \( \nu := \inf\{ b'(\lambda) : \lambda \in V \}. \)

By Lemma 4.10, \( V \neq \emptyset \) and hence \( \nu \) is defined. We claim \( \omega = \lambda(\nu) \).

If \( \lambda(\eta) \in V \), then \( \lambda(\eta) < \sigma(c_{i+1}) < \lambda(c_{i+1}) \), and the monotonicity of \( b' \) on \( \lambda[\mathbb{R}] \) implies

\[
\nu \leq b'(\lambda(\eta)) < b'(\lambda(c_{i+1})) = c_{i+1}.
\]

Furthermore, since \( U \) contains some \( \lambda(\eta_0) \), for \( \lambda(\eta) \in V \), \( \lambda(c_i) < \lambda(\eta_0) < \omega < \lambda(\eta) \), so that \( c_i < \nu \). It follows from the monotonicity of \( \lambda(\cdot) \) that \( \lambda(c_i) < \lambda(\nu) < \sigma(c_{i+1}) \).

Thus either \( \lambda(\nu) \in U \), \( \lambda(\nu) \in V \), or \( \lambda(\nu) = \omega \).

Suppose \( \lambda(\nu) \in U \). By Lemma 4.10, \( (\lambda(\nu), \omega) \cap \lambda[\mathbb{R}] \) is not empty. It thus contains \( \lambda(\eta_0) \) for some \( \eta_0 > \nu \). But then \( \eta_0 \) would be a lower bound for \( \{ b'(\lambda) : \lambda \in V \} \), contradicting the definition of \( \nu \). Thus \( \lambda(\nu) \notin U \).

Suppose \( \lambda(\nu) \in V \). Since \( \nu \notin C \), \( \lambda(\nu) = \sigma(\nu) \). Consider \( (\omega, \sigma(\nu)) \cap \lambda[\mathbb{R}] \). By Lemma 4.10, this is not empty, and thus there exists \( \lambda(\eta_0) \) in this set, hence in \( V \), with \( \eta_0 = b'(\lambda(\eta_0)) < b'(\lambda(\nu)) = \nu \). This contradicts the fact that \( \nu \) is a lower bound for \( \{ b'(\lambda) : \lambda \in V \} \). Thus \( \lambda(\nu) \notin V \). We conclude that \( \lambda(\nu) = \omega \).

The proof in the case of the semi-infinite intervals \( (-\infty, \sigma(c_1)) \) and \( (\lambda(c_k), \infty) \) is virtually identical. If \( |C| = 0 \), one can take \( \sigma(c_1) (= \lambda(c_1)) \) arbitrarily large since \( \lambda(\eta) \to \infty \) as \( \eta \to \infty \) to conclude that \( \lambda[\mathbb{R}] = \mathbb{R} \). \( \blacksquare \)

This theorem, together with Lemma 4.9, yields the following:

**Corollary 4.11.** \( b \) is convex on \( \mathbb{R} \setminus \bigcup_{c \in C} [\sigma(c), \lambda(c)] \).

### 5. Estimates for \( \int_{-\infty}^{\infty} e^{-2\tau p_\xi(\xi)} d\xi \)

Recall that \( p_\xi(\xi) = -\eta\xi + b(\xi + \lambda(\eta)) - b(\lambda(\eta)) \). In what follows, we will sometimes suppress the dependence of \( p \) and \( \lambda \) on \( \eta \). Within this section, we define

\[
I := \int_{-\infty}^{\infty} e^{-2\tau p_\xi(\xi)} d\xi.
\]

In the paper [GHar], we obtained sharp estimates on integrals of the form \( I \) that are uniform in the coefficients of \( p \) under the hypothesis that \( p \) has degree four. We showed there that those estimates do not generalize to polynomials of higher degree. The estimates obtained below are less precise but are nonetheless sufficient to prove our results on absolute convergence of the integral defining Szegő kernel.

#### 5.1. Estimates for \( I \) for convex \( p \)

As in the fourth-degree setting, our analysis makes use of known results on the integral of \( e^{-p} \) over intervals on which \( p \) is convex. Such results do not require \( p \) to have degree four. We recall the main result here:

**Lemma 5.1** (Lemma 4.9, [GHar]). Let \( n \) be a positive integer and define \( p_\xi(\xi) = \sum_{j=2}^{2n} \beta_j \xi^j \). Suppose \( p \) is convex on \( J \), where \( J \) is one of the intervals \( (-\infty, \infty), (0, \infty), \text{ or } (-\infty, 0) \). Then

\[
\int_J e^{-p_\xi(\xi)} d\xi = \left[ \sum_{j=2}^{2n} |\beta_j|^j \right]^{-1}.
\]
This lemma follows from work of Bruna, Nagel, and Wainger [BNW88]. A more detailed discussion, including variations and proofs, can be found in Section 4.2 of [GHar].

5.2. A lower bound for $I$ for non-convex $p$. By construction, $p_\eta$ vanishes to at least second order at the origin. Thus

$$p_\eta(\xi) = \sum_{j=2}^{2n} \frac{\eta^{(j)}(0)}{j!} \xi^j = \sum_{j=2}^{2n} \frac{\eta^{(j)}(\lambda(\eta))}{j!} \xi^j \leq \frac{1}{2} \sum_{j=2}^{2n} |\eta^{(j)}(\lambda(\eta))| |\xi|^j,$$

and (suppressing the dependence of $\lambda$ on $\eta$)

$$I \geq \int_{-\infty}^\infty e^{-\tau \sum_{j=2}^{2n} |\eta^{(j)}(\xi)|} d\xi$$

$$\geq \int_0^\infty e^{-\tau \sum_{j=2}^{2n} |\eta^{(j)}(\xi)|} d\xi$$

$$\approx \left[ \sum_{j=2}^{2n} \tau^{\frac{j}{2}} |\eta^{(j)}(\xi)| \right]^{-1},$$

where in the last line we have used Lemma 5.1 applied to $\xi \mapsto \sum_{j=2}^{2n} \tau |\eta^{(j)}(\xi)|$. This lemma follows from work of Bruna, Nagel, and Wainger [BNW88]. A more detailed discussion, including variations and proofs, can be found in Section 4.2 of [GHar].

5.3. An upper bound for $I$ for non-convex $p$. An upper bound for $I$ will give rise to a lower bound for the factor $[N(\eta, \tau)]^{-1}$ appearing in the integrand for the Szegö kernel. These estimates are therefore necessary for the results on the divergence of the integral $S[[x, y, t], (r, y, t)]$. We will see in Section 6 that for $M$ sufficiently large, the contribution to $S$ from $\{(\eta, \tau) : |\eta| > M, \tau > 0\}$ is finite for any $x$ and $r$. Thus when $S[[x, y, t], (r, y, t)]$ is divergent, it is because there exists some finite $\eta_0$ for which the contribution to $S$ from $\{(\eta, \tau) : \eta_0 < \eta < \eta_0 + \epsilon, \tau > 0\}$ is infinite. The following proposition is therefore sufficient to establish these results.

**Proposition 5.2.** Fix $\eta_0 \in \mathbb{R}$ and $\epsilon > 0$. Then there exists $c := c(\eta_0, \epsilon)$ such that for all $\eta \in (\eta_0, \eta_0 + \epsilon)$ and for all $\tau > 0$,

$$I \leq c \frac{1 + \tau^{\frac{1}{2}}}{\tau^{\frac{1}{4}}}.\tag{5.4}$$

We begin by factoring $p_\eta$. For fixed $\eta$, this is a non-negative polynomial vanishing to even order at the origin, its real roots are of even multiplicity, and its non-real roots occur in complex conjugate pairs. Its factorization over $\mathbb{C}$ may therefore be written

$$p_\eta(\xi) = \frac{1}{2^n} \xi^2 \prod_{j=2}^n (\xi - \alpha_j(\eta))[\xi - \overline{\alpha_j(\eta)}],\tag{5.5}$$

where the $\alpha_j$ may be real and need not be distinct. Furthermore, if $\alpha_j(\eta) = h_j(\eta) + ik_j(\eta)$, we order the roots so that $h_2(\eta) \leq h_3(\eta) \leq \ldots \leq h_n(\eta)$. The factorization of $p_\eta$ over $\mathbb{R}$ is thus

$$p_\eta(\xi) = \frac{1}{2^n} \xi^2 \prod_{j=2}^n (\xi - h_j(\eta))^2 + k_j^2(\eta).\tag{5.6}$$

In what follows, we denote the $j$-th quadratic factor in the above product by $q_j(\xi, \eta)$. 
Since the $h_j$ are functions of $\eta$ and we seek estimates for $I$ that are valid for all $\eta$ throughout an interval, we need a lemma on the local behavior of the $h_j$:

**Lemma 5.3.** Fix $\eta_0 \in \mathbb{R}$ and $\varepsilon > 0$. Then there exists $C > 1$ such that for all $\eta \in J := [\eta_0, \eta_0 + \varepsilon]$ and for all $j$, $|h_j(\eta)| \leq C - 1$.

**Proof.** This is a standard argument. Suppose the result fails, so that for some $j$, $h_j$ is unbounded on $J$. Assume without loss of generality that $j = 2$. Let $(\eta_q)$ be a sequence in $J$ for which $|h_2(\eta_q)| \to \infty$. By extracting subsequences if necessary, we may assume that $(\eta_q)$ converges to some $\eta' \in J$.

Recall that, by definition, $p_\eta(\xi) = -\eta(\xi + \lambda(\eta)) + b(\xi + \lambda(\eta)) - b^*(\eta)$. Since $b, b^*$ are continuous and $\lambda$ is bounded on $J$, for $\xi$ fixed, there exists $M_\xi > 0$ such that for all $\eta \in J$, $0 \leq p_\eta(\xi) \leq M_\xi$. Fix $\xi = 1$. Then for all $\ell$,

$$0 \leq \frac{1}{2n} \prod_{j=2}^n q_j(1, \eta) \leq M_1.$$

Since $\lim_{\ell \to -\infty} q_2(1, \eta) = \infty$, $\lim_{\ell \to -\infty} \prod_{j=3}^n q_j(1, \eta) = 0$. Thus there exists a factor $q_{j_1}$ and a subsequence $(\eta_{q_1}^{(1)})$ such that $\lim_{\ell \to -\infty} q_{j_1}(1, \eta_{q_1}^{(1)}) = 0$. This forces $\lim_{\ell \to -\infty} h_{j_1}(\eta_{q_1}^{(1)}) = 1$.

Now take $\xi = 2$. It is still the case that $\lim_{\ell \to \infty} q_2(2, \eta_{q_1}^{(1)}) = \infty$, but now for $\ell$ sufficiently large, $q_{j_2}(2, \eta_{q_1}^{(1)})$ is bounded away from zero. These facts, together with the boundedness of $p_\eta(2)$ on $J$, allow us to find a different factor $q_{j_2}$ and a subsequence $(\eta_{q_2}^{(2)})$ of $(\eta_{q_1}^{(1)})$ such that $q_{j_2}(2, \eta_{q_2}^{(2)}) \to 0$ and $h_{j_2}(\eta_{q_2}^{(2)}) \to 2$. Repeating this process at most $n - 2$ times leads to a subsequence $(\nu_{q_i})$ of the original such that $h_{j_i}(\nu_{q_i})$ tends to $i$.

Fix $\xi = n$. There exists $M_n$ such that $0 \leq p_\eta(n) \leq M_n$ for all $\eta \in J$. Furthermore, each $q_i(n, \nu_{q_i}), 2 \leq j \leq n$ is bounded away from zero for $\ell$ sufficiently large, but $q_1(n, \nu_{q_i})$ is unbounded. This is a contradiction, and the lemma is proved. \qed

We now prove Proposition 5.2. With notation as in the proof of Lemma 5.3, we find

$$I = \int_{-\infty}^{\infty} \exp \left( -\frac{\tau}{n} \xi^2 \prod_{j=2}^n (\xi - h_j)^2 + k_j^2 \right) d\xi$$

$$\leq \int_{-\infty}^{\infty} \exp \left( -\frac{\tau}{n} \xi^2 \prod_{j=2}^n (\xi - h_j)^2 \right) d\xi.$$

Write this last integral as the sum of integrals $I_1, I_2,$ and $I_3$, where $I_1$ is over the interval $(-\infty, C)$, $I_2$ is over $[-C, C]$, and $I_3$ is over $(C, \infty)$. 

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Since \(- (C - 1) \leq h_j(\eta) \leq C - 1\) for each \(j\) and for all \(\eta \in J\),

\[
I_1 \leq \int_{-\infty}^{C} \exp \left( -\frac{\tau}{n} \xi^2 \prod_{j=2}^{n} (C - h_j)^2 \right) d\xi
\]

\[
\leq \int_{-\infty}^{C} \exp \left( -\frac{\tau}{n} \xi^2 \prod_{j=2}^{n} (C - h_j)^2 \right) d\xi
\]

\[
\approx \left( \frac{\tau}{n} \prod_{j=2}^{n} (C + h_j)^2 \right)^{-\frac{1}{2}}
\]

since \(2C - 1 \geq C + h_j(\eta) \geq 1\) for all \(j\) and for all \(\eta \in J\).

We make the simplest possible estimate of \(I_2\); since the integrand is less than 1,

\[
I_2 \leq 2 C \approx 1.
\]

We estimate \(I_3\) in the same way as \(I_1\), using now the fact that for all \(j\), \(h_j(\eta) < C - 1\) for all \(\eta \in J\).

\[
I_3 \leq \int_{-C}^{\infty} \exp \left( -\frac{\tau}{n} \xi^2 \prod_{j=2}^{n} (C - h_j)^2 \right) d\xi
\]

\[
\leq \int_{-\infty}^{\infty} \exp \left( -\frac{\tau}{n} \xi^2 \prod_{j=2}^{n} (C - h_j)^2 \right) d\xi
\]

\[
\approx \left( \frac{\tau}{n} \prod_{j=2}^{n} (C - h_j)^2 \right)^{-\frac{1}{2}}
\]

Putting the three estimates together yields \(I \approx \max \{\tau^{-\frac{1}{2}}, 1\} \approx 1 + \frac{1 + \frac{3}{2}}{\tau^{\frac{1}{2}}},\) as claimed.

6. **Proofs of Theorems**

If we show that for all non-negative integers \(i_1, j_1, i_2\) and \(j_2\), each integral

\[
\int_{\tau > 0} e^{\tau \eta [z_1 + w_1] + i \tau [z_2 - w_2]} \eta^{i_1 + j_1 + i_2 + j_2 + 1} \frac{N(\eta, \tau)}{N(\eta, \tau)} d\eta d\tau
\]

is absolutely convergent in the region in which

\[
h + k + b(x) + b(r) - 2b^{**} \left( \frac{x + r}{2} \right) > 0,
\]

it will follow that this integral is in fact equal to \(\partial_{z_1}^{i_1} \partial_{w_1}^{j_1} \partial_{z_2}^{i_2} \partial_{w_2}^{j_2} S(z, w)\).

Set \(\delta = h + k, (z_1, z_2) = (x + iy, t + ib(x) + ih), (w_1, w_2) = (r + is, u + ib(r) + ik), s = i_1 + j_1,\) and \(m = i_1 + j_1 + i_2 + j_2\) (so that \(m \geq s\)). The integral becomes

\[
S^{s, m, \delta} := \int_{\tau > 0} e^{\tau \eta [x + r + (y - s)] + i \tau [t - u + b(x) + b(r) + \delta]} \eta^{s + m + 1} \frac{N(\eta, \tau)}{N(\eta, \tau)} d\eta d\tau,
\]
and it converges absolutely if and only if
\begin{equation}
\tilde{I}^{s,m,\delta} := \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-\tau(b(x)+b(r)-\eta(x+r))} \frac{|\eta|^{s+m+1}}{N(\eta, \tau)} d\tau d\eta < \infty.
\end{equation}

From (5.3),
\begin{align*}
\tilde{I}^{s,m,\delta} & = \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-\tau(b(x)+b(r)-\eta(x+r))} \frac{|\eta|^{s+m+1}}{e^{2b^*(\eta)}} d\tau d\eta \\
& \geq \sum_{j=2}^{2n} \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-\tau(b(x)+b(r)-\eta(x+r)+2b^*(\eta))} |\eta|^{s+m+1+\frac{1}{2} |\eta^j(\lambda(\eta))|} d\tau d\eta \\
& := \sum_{j=2}^{2n} I_j^{s,m,\delta}(\eta) d\eta \\
& := \sum_{j=2}^{2n} I_j^{s,m,\delta}.
\end{align*}

Further, set \(A(x, r, \eta) := b(x) + b(r) - \eta(x + r) + 2b^*(\eta)\). If \(\delta + A(x, r, \eta) > 0\), we can evaluate the \(\tau\) integral.
\begin{equation}
I_j^{s,m,\delta}(\eta) \approx \frac{|\eta|^{s+|\lambda^j(\lambda)|}}{[\delta + A(x, r, \eta)]^{m+2+\frac{1}{2}}}. 
\end{equation}

We see that there are two possible barriers to the convergence of the integrals \(I_j^{s,m,\delta}\):
\begin{enumerate}
\item insufficient decay of \(I_j^{s,m,\delta}(\eta)\) for fixed \(x, r\) as \(|\eta| \to \infty\);
\item vanishing of \(\delta + A(x, r, \eta)\) at some finite \(\eta\) for certain choices of \(x, r, \) and \(\delta\).
\end{enumerate}

We deal with these in turn.

6.1. Behavior of \(I_j^{s,m,\delta}(\eta)\) for large \(|\eta|\).

**Lemma 6.1.** Fix \(x, r \in \mathbb{R}, \delta > 0\). Then
\[\delta + A(x, r, \eta) \sim \left(\frac{2n-1}{n}\right) \eta^{\frac{2n}{m-1}}, \quad |\eta| \to \infty.\]

**Proof.** This follows immediately from Lemma 3.2, since
\begin{align*}
\delta + A(x, r, \eta) & = \delta + b(x) + b(r) - \eta(x + r) + 2b^*(\eta) \\
& = \left(\frac{2n-1}{n}\right) \eta^{\frac{2n}{m-1}} (1 + o(1))
\end{align*}
as \(|\eta| \to \infty\). \(\square\)

We may use Lemmas 3.3 and 6.1 to estimate the integrals for large \(|\eta|\):
\[|I_j^{s,m,\delta}(\eta)| \sim c |\eta|^s \frac{|\eta|^{\frac{2n-j}{m-1}}}{\eta^{\frac{2n}{m-1}} \eta^{\frac{|\eta^j(\lambda(\eta))|}}}.\]
as \(|\eta| \to \infty\). Since \(m \geq s \geq 0, -2 - (m - s) - \frac{m+3}{2n-1} < -2\). Thus for any fixed \(s, m, j, \) and \(\delta > 0\), \(I_j^{s,m,\delta}\) is convergent at infinity.
6.2. Vanishing of $\delta + A(x, r, \eta)$. In light of the previous section, we see that whether or not one of the integrals $I_j$ converges depends on whether or not the function $\eta \mapsto \delta + A(x, r, \eta)$ vanishes at a finite $\eta_0$ for some fixed $x$, $r$, $\delta$, and the behavior of this function near such zeros. In fact, we have proved

**Proposition 6.2.** If for some fixed $x$, $r$, and $\delta$

\begin{equation}
\inf_{\eta} \delta + A(x, r, \eta) > 0,
\end{equation}

then all the $I_j^{s, m, \delta}$ are finite.

Furthermore,

\[
\inf_{\eta} \delta + A(x, r, \eta) = \delta + b(x) + b(r) - 2\sup \left[ \eta \left( \frac{x + r}{2} \right) - b^*(\eta) \right]
\]

\[
= \delta + b(x) + b(r) - 2b^{**} \left( \frac{x + r}{2} \right),
\]

where the convexity of $b^*$ and its super-linear growth at infinity (Lemma 3.2) guarantee the finiteness of the supremum in the first line. It follows that the integrals defining the Szeg"o kernel and all of its derivatives converge absolutely in the region in which

\[
\delta + b(x) + b(r) - 2b^{**} \left( \frac{x + r}{2} \right) > 0.
\]

This is precisely the region defined in (2.10). To prove the remainder of Theorem 2.3, we must use the results of Section 4 to identify points $(z, w) = [(z_1, z_2), (w_1, w_2)] = [(x + iy, t + i(b(x) + h)), (r + is, u + i(b(r) + k))]$ satisfying (2.10).

If $(z, w) \in (\Omega \times \overline{\Omega}) \cup (\overline{\Omega} \times \Omega)$, $\delta > 0$. Since $A(x, r, \eta) \geq 0$, such $(z, w)$ are indeed in the region (2.10). We turn our attention, then, to points $(z, w) \in \partial \Omega \times \partial \Omega$, where $\delta = 0$.

Set

\begin{equation}
A_x(\eta) := b^*(\eta) - [\eta x - b(x)] \quad A_r(\eta) := b^*(\eta) - [\eta r - b(r)],
\end{equation}

so that

\begin{equation}
A(x, r, \eta) = A_x(\eta) + A_r(\eta).
\end{equation}

Fix $x$ and $r$, and recall the definition of $\Lambda_{\eta_0}$ from (2.5). $A(x, r, \eta_0) = 0$ if and only if $A_x(\eta_0) = A_r(\eta_0) = 0$. This, in turn, happens precisely when $x, r \in \Lambda_{\eta_0}$.

From Lemma 3.2 and Proposition 3.4, it follows that $A(x, r, \cdot)$ is a continuous function of $\eta$ which grows at infinity like $c|\eta|^{\frac{2}{2-n}}$. Thus if for some fixed $x$ and $r$ it does not vanish, it is bounded below by a positive constant. Together with Lemma 6.1 this shows that if $(z, w) \in (\partial \Omega \times \partial \Omega) \setminus \Sigma$, the integrals defining the Szeg"o kernel and all its derivatives are absolutely convergent.

Finally, we turn to the proof of Theorem 2.5. We must consider the integrals $S^{00, \delta}$ and $\bar{S}^{00, \delta}$ from (6.1) and (6.2). To simplify notation, we drop the additional superscripts. We must show

(i) $\bar{S}^0$ is divergent, and

(ii) $\lim_{\delta \to 0^+} \bar{S}^0 = \infty$,

whenever there exists $\eta_0$ such that $x, r \in \Lambda_{\eta_0}$. Clearly (i) implies (ii) since the integrand of $\bar{S}^0$ is non-negative and converges pointwise and monotonically to the integrand of $\bar{S}^0$ as $\delta \to 0^+$. We thus consider (i).
Fix \( \eta_0 \in \mathbb{R}, \varepsilon > 0, \) and \( x, r \in \Lambda_{\eta_0}. \) Applying Proposition 5.2,
\begin{equation}
\frac{\tau^2}{1 + \tau^2} \lesssim e^{2r\phi^*(\eta)} N(\eta, \tau)^{-1},
\end{equation}
for all \( \tau > 0 \) and \( \eta \in (\eta_0, \eta_0 + \varepsilon). \)
Substituting into (6.2) and recalling the definition of \( A(=A(x, r, \eta)) \) from (6.6) gives
\begin{align}
\tilde{S}_0 & > \int_{\eta_0}^{\eta_0 + \varepsilon} \int_0^\infty \tau^2 e^{-\tau} e^{2r\phi^*(\eta)} N(\eta, \tau)^{-1} \, d\tau \, d\eta \\
& \gtrsim \int_{\eta_0}^{\eta_0 + \varepsilon} \int_0^\infty \frac{\tau^2}{1 + \tau^2} e^{-\tau} \, d\tau \, d\eta \\
& = \int_{\eta_0}^{\eta_0 + \varepsilon} \int_0^\infty e^{-\tau^2} \frac{d\tau}{A^2[1 + \tau^2]} \, d\eta.
\end{align}

It is now clear that we need a lemma on the order of vanishing of \( A(x, r, \eta) \) at \( \eta_0. \)

**Lemma 6.3.** Take \( b \) as in (1.7), \( \eta_0 \in \mathbb{R}, \) and \( x \in \Lambda_{\eta_0}. \) Then
\[ A_x(\eta) = (\eta - \eta_0)F_x(\eta) \]
for all \( \eta \in (\eta_0, \infty), \) where \( F_x \) is bounded on each interval \((\eta_0, \eta_0 + \varepsilon).\)

**Proof.** Since \( \lambda(\eta_0) \) is the largest element of \( \Lambda_{\eta_0}, \lambda(\eta_0) \geq x. \) Since \( \lambda(\cdot) \) injective and increasing, for any \( \eta > \eta_0, \lambda(\eta) > x. \) Thus for \( \eta > \eta_0, \)
\begin{align}
A_x(\eta) &= b(x) - \eta x + b^*(\eta) \\
&= b(x) - \eta x + \eta \lambda(\eta) - b(\lambda(\eta)) \\
&= \eta(\lambda(\eta) - x) + b(x) - b(\lambda(\eta)) - \eta_0(\lambda(\eta) - x) + \eta_0(\lambda(\eta) - x) \\
&= (\eta - \eta_0)(\lambda(\eta) - x) - (\lambda(\eta) - x) \left[ \frac{b(\lambda(\eta)) - b(x) - \eta_0(\lambda(\eta) - x)}{\lambda(\eta) - x} \right] \\
&= (\eta - \eta_0)(\lambda(\eta) - x) - (\lambda(\eta) - x)\phi_x(\eta).
\end{align}
Observe,
\[ \phi_x(\eta) = \frac{b(\lambda(\eta)) - \eta_0\lambda(\eta) - [b(x) - \eta_0 x]}{\lambda(\eta) - x} = \frac{B_{\eta_0}(\lambda(\eta)) - B_{\eta_0}(x)}{\lambda(\eta) - x}. \]
Since \( x \in \Lambda_{\eta_0}, \) the minimality of \( B_{\eta_0}(x) \) yields \( B_{\eta_0}(\lambda(\eta)) \geq B_{\eta_0}(x) \) for all \( \eta \in \mathbb{R}. \)
Therefore \( \phi_x \) is non-negative on the interval \((\eta_0, \infty). \) It follows that for \( \eta > \eta_0, \)
\begin{equation}
A_x(\eta) = (\eta - \eta_0)(\lambda(\eta) - x) - (\lambda(\eta) - x)\phi_x(\eta) \geq 0 \iff 1 \geq \frac{\phi_x(\eta)}{\eta - \eta_0}.
\end{equation}
Hence, on \((\eta_0, \infty),\)
\begin{align}
A_x(\eta) &= (\eta - \eta_0)(\lambda(\eta) - x) - (\lambda(\eta) - x)\phi_x \\
&= (\eta - \eta_0)(\lambda(\eta) - x) \left[ 1 - \frac{\phi_x(\eta)}{\eta - \eta_0} \right] \\
&=: (\eta - \eta_0)F_x(\eta).
\end{align}
By inequality (6.9) and the local boundedness of \( \lambda(\eta), F_x \) is bounded on each interval \((\eta_0, \eta_0 + \varepsilon). \) This proves the lemma. \( \Box \)
We use this lemma to substitute for $A$ in (6.8)

$$\tilde{S}_0 \geq \int_{0}^{1} \frac{e^{-\tau} \left( \frac{2}{\sqrt{\eta - \eta_0}} \right)^2}{(\eta - \eta_0)^3 (F_x(\eta) + F_r(\eta))^{\frac{3}{2}} + 1} \, d\tau \, d\eta$$

$$= \left( \int_{0}^{1} \frac{\tau^2 e^{-\tau} \, d\tau}{(\eta - \eta_0)^3 (F_x(\eta) + F_r(\eta))^{\frac{3}{2}} + 1} \right) \int_{0}^{\eta_0 + \epsilon} \frac{1}{(\eta - \eta_0)^{\frac{5}{2}} (F_x(\eta) + F_r(\eta))^{\frac{3}{2}} + 1} \, d\eta$$

$$\approx \int_{0}^{\eta_0 + \epsilon} \frac{G(\eta)}{(\eta - \eta_0)^2 (F_x(\eta) + F_r(\eta))^2} \, d\eta,$$

where $G(\eta)$ is right-continuous and bounded away from zero. Since $F_x + F_r$ is also a locally-bounded positive function, the divergence of the integral follows. This completes the proof of Theorem 2.5.

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