$B_c \rightarrow B_{sJ}$ form factors and $B_c$ decays into $B_{sJ}$ in covariant light-front approach

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We suggest to study the $B_c$ and its excitations $B_{sJ}$ in the $B_c$ decays. We calculate the $B_c \rightarrow B_{sJ}$ and $B_c \rightarrow B_J$ form factors within the covariant light-front quark model, where the $B_{sJ}$ and $B_J$ denote an s-wave or p-wave $bs$ and $bd$ meson, respectively. The form factors at $q^2 = 0$ are directly computed while their $q^2$-distributions are obtained by the extrapolation. The derived form factors are then used to study semileptonic $B_c \rightarrow (B_{sJ}, B_J)ℓν$ decays, and nonleptonic $B_c \rightarrow B_{sJ}π$. Branching fractions and polarizations are predicted in the standard model. We find that the branching fractions are sizable and might be accessible at the LHC experiment and future high-energy $e^+e^−$ colliders with a high luminosity at the Z-pole. The future experimental measurements are helpful to study the nonperturbative QCD dynamics in the presence of a heavy spectator and also of great value for the spectroscopy study.

I. INTRODUCTION

In the past decades, there have been a lot of progresses in hadron spectroscopy, thanks to the well-operating experiments including the $e^+e^−$ colliders and hadron colliders. The immense interest in spectroscopy is not only due to the fact that one is able to find many missing hadrons to complete the quark model, but more importantly due to the observations of states that are unexpected in the simple quark model. The latter ones are generally called hadron exotics. A milestone in the exotics exploration is the discovery of $X(3872)$, firstly in $B$ decays by Belle Collaboration [1] and subsequently confirmed in many distinct processes in different experiments [2–4]. It was found the properties of this meson is peculiar. Since then the identification of multiquark hadrons becomes a hot topic in hadron physics. Inspired by the discovery of $X(3872)$, a number of new interesting structures were discovered in the mass region of heavy quarkonium. Refer to Refs. [5–8] for recent reviews.

On theoretical side, deciphering the underlying dynamics of these multiquark states is a formidable challenge, and is often based on explicit and distinct assumptions. In many assumptions, the quarkonium-like states are usually composed of a pair of heavy constituents, which makes it vital to study first the heavy-light hadron. In the system with one heavy charm quark, a series of important results start with the discoveries of the narrow states $D_s(2317)$ in the $D_s^+(π^0)$ final state and $D_s(2460)$ in the $D_s^0π^0$ and $D_sγ$ final state [9, 10]. Along this line, a few other new states, such as $D_{s1}(2536)$, $D_{s2}(2573), D_s(2710)$, have been observed at the B factory and other facilities [11].

Bottomed hadrons are related to charmed mesons by heavy quark symmetry. But compared to the charm sector, there are less progresses in the bottomed hadrons. In experiment, only a few bottom-strange mesons are observed, and most of which are believed to be filled in the quark model. In this paper, we propose to use the $B_c$ decays and study the spectrum of the $B_{sJ}$. It gains a few advantages. First, the large production rates of the $B_c$ is in expectation, in particular the LHCb will produce a number of $B_c$ events and thus the $B_c \rightarrow B_{sJ}$ decays will have a large potential to be observed. Secondly, the scale over the $m_W$ can be computed in the perturbation theory and the QCD evolution between the $m_W$ and the low energy scale $m_c$ is well organized by making use of the renormalization group improved perturbation theory. Consequently, the $B_c$ decays into $B_s$ and other excited states have received some theoretical attentions [12–30]. In the following we will be dedicated to investigate the production rates of $B_{sJ}$ meson (an s-wave or p-wave $bs$ hadron) in semileptonic and nonleptonic $B_c$ mesons decays under the framework of the covariant light-front quark model (LFQM) [31].

In the $B_c \rightarrow B_{sJ}$ decays, the quark level transition is the $c \rightarrow s$ in which the heavy bottom quark acts as a spectator. Since most of the momentum of the hadron is carried by the spectator, there is no large momentum transfer and the transition is dominated by the soft mechanism. A form factor can then be expressed as a overlap of the wave functions of the initial and final state hadrons. Treatments in quark models like the LFQM are of this type.

As pointed out in Ref. [32], the light-front approach owns some unique features which are suitable to handle a hadronic bound state. The LFQM [33–36] can provide a relativistic treatment of moving hadrons and give a fully

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treatment of hadron spins in terms of the Melosh rotation. Light-front wave functions, which characterize the hadron in terms of their fundamental quark and gluon degrees of freedom, are independent of hadron momentum and thus are Lorentz invariant. Moreover, in covariant LFQM, the spurious contribution which depends on the orientation of light-front is elegantly eliminated by including zero-mode contributions. This covariant model has been successfully extended to study the decay constants and form factors of various mesons. Through this study of $B_c \to B_{sJ}$ in LFQM, we believe that one will not only gain the information about the decay dynamics in the presence of a heavy spectator but will also provide a side-check for the classification of the heavy-light mesons. It is also helpful towards the establishment of a global picture of the heavy-light spectroscopy including the exotic spectrum.

The rest of this paper is organized as follows. In Sec. II, we will give a brief description of the parametrization of form factors, the framework of covariant LFQM, and the form factor calculation in this model. We present our numerical results for various transitions in Sec. III. In Sec. IV, we use the form factors to study semileptonic and nonleptonic $B_c$ decays. In this section, we will present our predictions for branching fractions and polarizations. The last section contains a brief summary.

II. TRANSITION FORM FACTORS IN THE COVARIANT LFQM

A. $B_c \to B_{sJ}$ form factors

The effective electroweak Hamiltonian for the $B_c \to B_{sJ} \bar{\nu} \nu$ reads

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{cs}^* \bar{s} \gamma_\mu (1 - \gamma_5) c [\bar{\nu} \gamma^\mu (1 - \gamma_5) \nu],$$

where the $G_F$ and $V_{cs}$ is Fermi constant and Cabibbo-Kobayashi-Maskawa matrix element, respectively. Leptonic parts can be computed in perturbation theory while hadronic contributions are parametrized in terms of form factors.

An $s$-wave meson corresponds to a pseudo-scalar meson or a vector meson, abbreviated as $P$ and $V$ respectively. For a $p$-wave meson, the involved state is a scalar $S$, an axial-vector $A$ or a tensor meson $T$. In the following we introduce the abbreviations $P = P' + P''$, $q = P' - P''$ and adopt the convention of $\epsilon_{0123} = 1$. The $B_c \to P, V$ form factors can be defined as follows:

$$\langle P(P'')|V_\mu|B_c(P') \rangle = \left( P_\mu - \frac{m_{B_c}^2 - m_{P'}^2}{q^2} q_\mu \right) F_1^{B_cP}(q^2) + \frac{m_{B_c}^2 - m_{P'}^2}{q^2} q_\mu F_0^{B_cP}(q^2),$$

$$\langle V(P'', \epsilon'')|V_\mu|B_c(P') \rangle = -\frac{1}{m_{B_c} + m_V} \epsilon_{\mu\nu\alpha\beta} \epsilon^{\epsilon''\nu\sigma} P^\alpha q^\beta V_{B_cV}(q^2),$$

$$\langle V(P'', \epsilon'')|A_\mu|B_c(P') \rangle = 2im_V \frac{\epsilon^{\epsilon''\nu} \cdot q}{q^2} q_\nu A_0^{B_cV}(q^2) + i(m_{B_c} + m_V) A_1^{B_cV}(q^2) \left[ \epsilon^{\epsilon''\nu} \cdot q - \frac{q^2}{q^2} q_\nu \right] \left[ P_\mu - \frac{m_{B_c}^2 - m_{P'}^2}{q^2} q_\mu \right].$$

In analogy with $B_c \to V$ form factors, we parametrize $B_c \to T$ form factors as

$$\langle T(P'', \epsilon'')|V_\mu|B_c(P') \rangle = -\frac{2V_{cT}(q^2)}{m_{B_c} + m_T} \epsilon^{\mu\nu\sigma} (\epsilon^*_T)_{\nu}(P'')_{\rho}(P''\sigma),$$

$$\langle T(P'', \epsilon'')|A_\mu|B_c(P') \rangle = 2im_T \frac{\epsilon^{T\nu} \cdot q}{q^2} q_\nu A_0^{B_cT}(q^2) + i(m_{B_c} + m_T) A_1^{B_cT}(q^2) \left[ \epsilon^{T\nu} \cdot q - \frac{q^2}{q^2} q_\nu \right] \left[ P_\mu - \frac{m_{B_c}^2 - m_T^2}{q^2} q_\mu \right].$$

with

$$\epsilon_T(h) = \frac{1}{m_{B_c}} \epsilon^*_T(h) P^{\rho\nu}.$$
The $B_c \to S, A$ form factors can be defined by exchanging the vector and axial-vector current:

$$\langle S(P')|A_\mu|B_c(P')\rangle = -i \left[ (P_\mu - \frac{m_{B_c}^2 - m_S^2}{q^2} q_\mu) f_1^{B_cS}(q^2) + \frac{m_{B_c}^2 - m_S^2}{q^2} q_\mu f_0^{B_cS}(q^2) \right],$$

$$\langle A(P', \varepsilon'')|V_\mu|B_c(P')\rangle = -2m_A \frac{\varepsilon'' \cdot q}{q^2} q_\mu V^{B_cA}_1(q^2) - (m_{B_c} + m_A) V^{B_cA}_1(q^2) \left[ \varepsilon''_\mu - \frac{\varepsilon'' \cdot q}{q^2} q_\mu \right]$$

$$+ \frac{\varepsilon''_\mu \cdot P}{m_{B_c} + m_A} V^{B_cA}_2(q^2) \left[ P_\mu - \frac{m_{B_c}^2 - m_A^2}{q^2} q_\mu \right],$$

$$\langle A(P', \varepsilon'')|A_\mu|B_c(P')\rangle = -i \frac{1}{m_{B_c} - m_A} \epsilon_{\mu\nu\alpha\beta} \varepsilon''^{\nu\gamma} P^{\rho} q^\beta A^{B_cA}(q^2).$$

(5)

The spin-2 polarization tensor can be constructed using the standard polarization vector $\varepsilon$:

$$\varepsilon''_{\mu\nu}(P'', \pm 2) = \varepsilon(\pm)\varepsilon(\pm), \quad \varepsilon''_{\mu\nu}(P'', \pm 1) = \frac{1}{\sqrt{2}} [\varepsilon(\pm)\varepsilon(0) + \varepsilon(0)\varepsilon(\pm)],$$

$$\varepsilon''_{\mu\nu}(P'', 0) = \frac{1}{\sqrt{6}} [\varepsilon(\pm)\varepsilon(-) + \varepsilon(+)\varepsilon(-)] + \sqrt{\frac{2}{3}} \varepsilon(0)\varepsilon(0).$$

(6)

It is symmetric and traceless, and $\varepsilon''_{\mu\nu}P^{\mu\nu} = 0$. If the recoiling meson is moving on the plus direction of the $z$ axis, their explicit structures are chosen as

$$\varepsilon(0) = \frac{1}{m_T}(p^0_T, 0, 0, -E_T), \quad \varepsilon(\pm) = \frac{1}{\sqrt{2}}(0, \pm 1, 0),$$

(7)

where $E_T$ and $|p_T|$ are the energy and the momentum magnitude of the tensor meson in the $B_c$ rest frame, respectively.

**B. Covariant light-front approach**

In the covariant LFQM, it is convenient to use the light-front decomposition of the momentum $P' = (P'^-, P'^+, P'_{\perp})$, with $P'^\pm = P'^0 \pm P'^3$, and thus $P'^2 = P'^0 (P'^- - P'^\perp)^2$. The incoming (outgoing) meson has the momentum $P' = p'_1 + p_2$ ($P'' = p''_1 + p_2$) and the mass $M'$ ($M''$). The quark and antiquark inside the incoming (outgoing) meson have the mass $m_1^{(\text{in})}$ and $m_2$, respectively. Their momenta are denoted as $p'^{(1)}_1$ and $p_2$ respectively. In particular these momenta can be written in terms of the internal variables $(x_1, p'^{(1)}_1)$ by

$$p'^{(1)}_{1,2} = x_{1,2} P'^+, \quad p'^{(1)}_{1,2,\perp} = x_{1,2} P'^{\perp} \pm p'^{\perp},$$

(8)

with the momentum fractions $x_1 + x_2 = 1$. With these internal variables, one can define some useful quantities for both incoming and outgoing mesons:

$$M'^2_0 = (e'_1 + e'_2)^2 = \frac{p'^2_1 + m_1^{(1)}_1}{x_1} + \frac{p'^2_2 + m_2^2}{x_2}, \quad \tilde{M}'_0 = \sqrt{M'^2_0 - (m'_1 - m_2^2)^2},$$

$$e_i^{(1)} = \sqrt{m_i^{(1)} + p'^2_i + p'^2_{1,\perp}}, \quad \tilde{p}'_i = \frac{x_2 M'_0}{2} - \frac{m_2^2 + p'^2_{1,\perp}}{2 x_2 M'_0}. \quad (9)$$

Feynman rules for meson-quark-antiquark vertices can be derived using the conventional light-front approach, whose forms for the s-wave and p-wave states are collected in Table 31 37. An extension to the d-wave vertices has been conducted in Ref. 50. In the following we will take the $B_c \to B_s$ transition as the example and illustrate the calculation. To do so, we will consider the matrix element

$$\langle P(P'')|V_\mu|P(P')\rangle \equiv B^{PP}_{\mu},$$

(10)

whose Feynman diagram is shown in Fig. 4. It is straightforward to obtain

$$B^{PP}_{\mu} = i^3 \frac{N_c}{(2\pi)^4} \int d^4p' \frac{H_P^{\mu} H^{\nu}_P}{N_1 N_1' N_2} S^{PP}_{\nu\mu},$$

(11)
type wave function can be adopted \([31, 37]\):

\[
\phi_{\gamma}(\mathbf{p}) = \frac{1}{\sqrt{p_{\perp}}^{\gamma}} (p'_{\perp} - p_{\perp})(p'_{\perp} - p_{\perp})_{\mu} \gamma_{\mu}.
\]

Here we consider the \(q_{\perp} = 0\) frame. The \(p'_{\perp}\) integration picks up the pole \(p_{2} = \hat{p}_{2} = [(p_{2 \perp}^{2} + m_{2}^{2})/p_{2}^{+}]^{1/2}, p_{2}^{+}, p_{2 \perp}\) and leads to

\[
N_{1}^{\prime(\mu)} = p_{1}^{\prime(\mu)} = m_{1}^{\prime(\mu)} + m_{1}^{\prime(\mu)}\gamma_{\mu}(p'_{\perp} - m_{2}(p_{2}^{2} - m_{2}^{2})),
\]

\[
H'_{P}^{\prime(\mu)} = \gamma_{\mu} = p_{1}^{\prime(\mu)} - m_{1}^{\prime(\mu)} + i\epsilon, \quad N_{2} = p_{2}^{2} - m_{2}^{2} + i\epsilon.
\]

where

\[
S_{P}^{PP} = \text{Tr} \gamma_{5}(p_{1}^{\prime(\mu)} + m_{1}^{\prime(\mu)}\gamma_{\mu}(p'_{\perp} + m_{1}^{\prime(\mu)}\gamma_{5}(p'_{\perp} - m_{2}^{2})),
\]

\[
N_{1}^{\prime(\mu)} = p_{1}^{\prime(\mu)} - m_{1}^{\prime(\mu)}\gamma_{\mu}(p'_{\perp} - m_{2}^{2} - m_{2}^{2} + i\epsilon).
\]

Here we consider the \(q_{\perp} = 0\) frame. The \(p'_{\perp}\) integration picks up the pole \(p_{2} = \hat{p}_{2} = [(p_{2 \perp}^{2} + m_{2}^{2})/p_{2}^{+}]^{1/2}, p_{2}^{+}, p_{2 \perp}\) and leads to

\[
N_{1}^{\prime(\mu)} = p_{1}^{\prime(\mu)} = m_{1}^{\prime(\mu)} + m_{1}^{\prime(\mu)}\gamma_{\mu}(p'_{\perp} - m_{2}(p_{2}^{2} - m_{2}^{2})),
\]

\[
H'_{P}^{\prime(\mu)} = \gamma_{\mu} = p_{1}^{\prime(\mu)} - m_{1}^{\prime(\mu)} + i\epsilon, \quad N_{2} = p_{2}^{2} - m_{2}^{2} + i\epsilon.
\]

where

\[
M_{0}^{\prime(\mu)} = p_{1}^{\prime(\mu)} + m_{1}^{\prime(\mu)} + m_{1}^{\prime(\mu)}\gamma_{\mu}(p'_{\perp} - m_{2}(p_{2}^{2} - m_{2}^{2})),
\]

\[
M_{0}^{\prime(\mu)} = p_{1}^{\prime(\mu)} + m_{1}^{\prime(\mu)} + m_{2}^{2} + m_{2}^{2},
\]

with \(p'_{\perp} = p'_{\perp} - x_{2}q_{\perp}\). The explicit form of \(h_{P}^{\prime(\mu)}\) has been derived in Ref. \([31, 37]\)

\[
h_{P}^{\prime(\mu)} = (M^{\prime(\mu)} - M_{0}^{\prime(\mu)})^{1/2} \frac{x_{1}x_{2}}{N_{c}} \frac{1}{\sqrt{2M_{0}^{\prime(\mu)}}} \varphi'
\]

where \(\varphi'\) is the light-front momentum distribution amplitude for \(s\)-wave meson. In practice, the following Gaussian-type wave function can be adopted \([31, 37]\):

\[
\varphi' = \varphi'(x_{2}, p'_{\perp}) = 4 \left( \frac{\pi}{\beta^{2}} \right)^{3/4} \sqrt{\frac{dp'_{\perp}}{dx_{2}}} \exp \left( -\frac{p'_{\perp}^{2} + p_{\perp}^{2}}{2\beta^{2}} \right).
\]

As shown in Ref. \([31, 37]\), the inclusion of the so-called zero mode contribution in the above matrix elements in practice amounts to the replacements

\[
\hat{p}_{1\mu} = P_{\mu}A_{1}^{(1)} + q_{\mu}A_{2}^{(1)}, \quad \hat{N}_{2} \rightarrow Z_{2}, \quad \hat{p}_{1\mu}^{\prime(\mu)} \hat{N}_{2} \rightarrow q_{\mu} \left[ A_{2}^{(1)}Z_{2} + \frac{q_{\mu}P_{\perp}}{q^{2}}A_{1}^{(2)} \right].
\]
where the symbol \( \hat{d} \) in the above equation reminds us that it is true only in the \( B^{P \bar{P}} \) integration. \( A^{(i)}_1 \) and \( Z_2 \), which are functions of \( x_{1,2}, P_{1,2}^2, P_{1,2} \cdot q_{2} \) and \( q^2 \), are listed in Appendix A. After the replacements, we arrive at

\[
\begin{align*}
    f_+(q^2) &= \frac{N_c}{16\pi^3} \int dx_2 dx_2^2 \frac{h_P h_P''}{x_2 N_i^1 N_i^1} \left[ x_1 (M_0^2 + M_0'^2) + x_2 q^2 - x_2 (m_1^2 - m_1')^2 - x_1 (m_1^2 - m_2^2) - x_1 (m_1''^2 - m_2^2) \right], \\
    f_-(q^2) &= \frac{N_c}{16\pi^3} \int dx_2 dx_2^2 \frac{2 h_P h_P''}{x_2 N_i^1 N_i^1} \left\{ -x_1 x_2 M'^2 - p_{1,2}^2 - m_1^2 + (m_1' - m_2) \right\} \\
    &\times (x_2 m_1' + x_1 m_2) + 2 \frac{q \cdot P}{q^2} \left( p_{1,2}^2 + 2 \frac{(p_{1,2} \cdot q_{1,2})^2}{q^2} + 2 \frac{(p_{1,2} \cdot q_{1,2})^2}{q^2} \right) \\
    &\times \frac{p_{1,2} \cdot q_{1,2}}{q^2} \left[ M'^2 - x_2 (q^2 + q \cdot P) - (x_2 - x_1) M'^2 + 2 x_1 M_0'^2 - 2 (m_1' - m_2) (m_1'' + m_1') \right].
\end{align*}
\]

Finally we get the form factors through the relations:

\[
F_1^{P \bar{P}}(q^2) = f_+(q^2), \quad F_0^{P \bar{P}}(q^2) = f_+(q^2) + \frac{q^2}{q \cdot P} f_-(q^2).
\]

Similarly, one can derive the other form factors, whose expressions are collected in Appendix A.

Before closing this section, it is worth mentioning that the axial-vector mesons may not be classified as \( ^3P_1 \) or \( ^1P_1 \) state. In the quark limit with \( m_Q \to \infty \), the QCD interaction is independent of the heavy quark spin and thus it will decouple with the light system. A consequence of this decoupling is that heavy mesons are classified into multiplets labeled by the total angular momentum of the light degrees of freedom. The s-wave pseudo-scalar and vector states are in the same multiplets denoted as \( s_l = 1/2 \). For the p-wave states, two kinds of axial-vector mesons \( P_1^{3/2} \) and \( P_1^{1/2} \) are mixtures of \( ^3P_1 \) or \( ^1P_1 \):

\[
|P_1^{3/2}\rangle = \sqrt{\frac{2}{3}} |^1P_1\rangle + \sqrt{\frac{1}{3}} |^3P_1\rangle, \quad |P_1^{1/2}\rangle = \sqrt{\frac{1}{3}} |^1P_1\rangle - \sqrt{\frac{2}{3}} |^3P_1\rangle.
\]

Since the form factors involving \( P_1^{3/2} \) and \( P_1^{1/2} \) can be straightforwardly obtained by the linear combination for those given above, we shall calculate the form factors using the \( ^{2S+1}L_J \) basis in the following analysis.

### III. NUMERICAL RESULTS FOR FORM FACTORS

#### A. Input parameters

In the covariant LFQM, the constituent quark masses are used as (in units of GeV):

\[
m_u = m_d = 0.25, \quad m_s = 0.37, \quad m_c = 1.4, \quad m_b = 4.8,
\]

which have been widely used in various \( B \) and \( B_c \) decays. The masses of the \( B_c \) and \( B_{s,J} \) are taken from the PDG (in units of GeV) \( ^{11} \):

\[
m_{B_c} = 6.276, \quad m_{B_s} = 5.367, \quad m_{B_c} = 5.415, \quad m_{B_{s,2}} = 5.840,
\]

while for the \( B_{s,0} \) and \( B_{s,1} \), we quote the results \( ^{51,52} \) (see also estimates in Refs. \( ^{53,55} \)):

\[
m_{B_{s,0}} = 5.782, \quad m_{B_{s,1}(P_1^{1/2})} = 5.843, \quad m_{B_{s,1}(P_1^{3/2})} = 5.833.
\]

Since masses of the \( P_1^{1/2} \) and \( P_1^{3/2} \) are close to the observed state \( B_{s,1}(5830) \) \( ^{11} \),

\[
m_{B_{s,1}(5830)} = 5.829 \text{GeV},
\]

we use the same value for both the \( ^3P_1 \) and \( ^1P_1 \) state.
The parameter $\beta$, characterizing the momentum distribution, is usually determined by fitting the meson decay constant. For instance, in this approach the pseudoscalar and vector meson’s decay constants read

$$
\begin{align*}
 f_P &= \frac{N_c}{16\pi^3} \int \frac{d^2p'_\perp}{x_1x_2(M'^2 - M_0'^2)} \frac{h'_P}{x_1x_2(M'^2 - M_0'^2)} A(m'_1x_2 + m_2x'_1), \\
 f_V &= \frac{N_c}{4\pi^3 M^4} \int \frac{d^2p'_\perp}{x_1x_2(M'^2 - M_0'^2)} \frac{h'_V}{x_1x_2(M'^2 - M_0'^2)} \left[ x_1M_0'^2 - m'_1(m'_1 - m_2) - p'^2 + \frac{m'_1 + m_2p'_2}{w'_V} \right].
\end{align*}
$$

For the $B_c$ meson, the decay constant can be in principle determined by leptonic and radiative-leptonic decays $^{56-59}$, both of which are lack of experimental data yet. Two loop contributions in the NRQCD framework have been calculated in Ref. $^{58}$ and the authors have found:

$$
 f_{B_c} = 398\text{MeV}.
$$

We will adopt this result, but it is necessary to note the above value is smaller than Lattice QCD result by approximately $2\sigma$: $f_{B_c} = (343 \pm 15)\text{MeV}$. We use the recent Lattice QCD result for the $B_s$ decay constant with $N_f = 2 + 1 + 1$ $^{60}$

$$
 f_{B_s} = (229 \pm 5)\text{MeV}.
$$

This is close to the previous Lattice QCD result $^{61, 62}$: $f_{B_s} = (224 \pm 5)\text{MeV}$. Using the decay constants, the shape parameters are fixed as

$$
\beta_{B_c} = 0.886\text{GeV}, \quad \beta_{B_s} = 0.623\text{GeV},
$$

and we assume that the values of $\beta$ for other $B_{s, J}$ mesons are approximately equal to that for the $B_s$, that is

$$
\beta_{B_s'} = \beta_{B_s0} = \beta_{B_s1} = \beta_{B_{s1}'} = \beta_{B_s A} = 0.623\text{GeV}.
$$

We will also calculate the $B_c \to B_J$ form factors, for which we use the masses $^{51, 52}$

$$
\begin{align*}
 m_B &= 5.279\text{GeV}, \quad m_{B^*} = 5.325\text{GeV}, \quad m_{B_s} = 5.749\text{GeV}, \quad m_{B_1} = m_{B'_1} = 5.731\text{GeV}, \quad m_{B_2} = 5.746\text{GeV},
\end{align*}
$$

and the shape parameter $\beta$ for the $B_J$ meson:

$$
\beta_{B_J} = 0.562\text{GeV}.
$$

The above result is derived from decay constant result $^{60}$:

$$
 f_B = (193 \pm 6)\text{MeV}.
$$

**B. Form factors and momentum transfer distribution**

With the inputs in the previous subsection, we can predict the $B_c \to B_s, B'^* , B_{s0}, B_{s1}, B'_{s1}$ and $B_{s2}$ form factors in the LFQM and we show our results in Table III. In order to access the $q^2$ distribution, one may adopt the fit formula:

$$
 F(q^2) = \frac{F(0)}{1 - \frac{q^2}{m_H^2} + \delta (\frac{q^2}{m_H^2})^2}.
$$

In the literature, the dipole form has been used to parametrize the $q^2$ distribution:

$$
 F(q^2) = \frac{F(0)}{1 - a \frac{q^2}{m_H^2} + b (\frac{q^2}{m_H^2})^2},
$$

with the $m_H = m_D$ for $D$ decays and $m_H = m_B$ for $B$ decays. This parametrization is inspired by the analyticity. Taking the $F_{1}^{B \to \pi}$ as an example, we consider the timelike matrix element:

$$
\langle 0|\bar{u}\gamma_{\mu}b|\pi(-p_\pi)\overline{B}(p_B)\rangle \sim \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m_X^2} \langle 0|\bar{u}\gamma_{\mu}b|X\rangle \langle X|\pi(-p_\pi)\overline{B}(p_B)\rangle,
$$

(34)
where the one-particle contribution has been singled out. The lowest resonance that can contribute is the vector $B^*$. This leads to the pole structure at large $q^2$:

$$F_{1}^{B \rightarrow \pi}(q^2) \sim \frac{F_1(0)}{\left(1 - q^2/m_{B^*}^2\right)}.$$  

(35)

Except the pole at $m_{B^*}$, there are residual dependences on $q^2$ which can be effectively incorporated into the $a, b$ of the dipole parametrization as shown in Eq. (33). However for the $B_c \rightarrow B_s \pi$ transition, one can not simply apply Eq. (33), since the contributing states are the $D_s$ resonances. Using the $m_H = m_{B_s}$ will not only disguise the genuine poles, but also lead to irrationally large results for parameters $a$ and $b$. So in order to avoid this problem, we have adopted the parametrization in Eq. (32). From the results in Tab. II one can see that the $m_{fit}$ for most form factors is between 1.5 GeV to 2.0 GeV, close to the mass of a $D_{s1}$ resonance. This has validated our parametrization.

For the $A_{2}^{B_s B_s^*}, F_{0}^{B_s B_s^*}, V_{0}^{B_s B_s^*}$ and $A_{2}^{B_s B_s^*}$, we found that the fitted values for the $m_{fit}^2$ are negative, and thus we use the following formula:

$$F(q^2) = \frac{F(0)}{1 + \frac{q^2}{m_{fit}^2} + \delta \left(\frac{q^2}{m_{fit}^2}\right)^2}.$$  

(36)

The $q^2$-dependent form factors of $B_c \rightarrow B_s$ are shown in Fig. 2. From this figure, we can see that except for the $B_c \rightarrow B_{s1}$ transition, most form factors are rather stable against the variation of $q^2$. This is partly because of the limited phase space. This will also lead to a reliable prediction for the branching fractions given in the next section.
IV. PHENOMENOLOGICAL APPLICATIONS

A. Semileptonic $B_c$ decays

Decay width for semileptonic decays of $B_c \to M\bar{l}\nu$, where $M = P, V, S, A, T$, can be derived by dividing the decay amplitude into hadronic part and leptonic part, both of which are Lorentz invariant so that can be readily evaluated. Then the differential decay widths for $B_c \to P\bar{l}\nu$ and $B_c \to V\bar{l}\nu$ turn out to be

$$
\frac{d\Gamma(B_c \to P\bar{l}\nu)}{dq^2} = \left(1 - \tilde{m}_l^2\right)^2 \frac{\sqrt{\lambda(m_{B_c}^2, m_P^2, q^2)} G_F^2 |V_{CKM}|^2}{384m_{B_c}^3 \pi^3} \left\{ (\tilde{m}_l^2 + 2) \lambda(m_{B_c}^2, m_P^2, q^2) F_1^2(q^2) \\
+ 3\tilde{m}_l^2 (m_{B_c}^2 - m_P^2)^2 F_0^2(q^2) \right\},
$$

(37)

$$
\frac{d\Gamma_L(B_c \to V\bar{l}\nu)}{dq^2} = \left(1 - \tilde{m}_l^2\right)^2 \frac{\sqrt{\lambda(m_{B_c}^2, m_V^2, q^2)} G_F^2 |V_{CKM}|^2}{384m_{B_c}^3 \pi^3} \left\{ 3\tilde{m}_l^2 \lambda(m_{B_c}^2, m_V^2, q^2) A_0^2(q^2) + (\tilde{m}_l^2 + 2) \\
\times \left| \frac{1}{2m_V} \left[ (m_{B_c}^2 - m_V^2 - q^2)(m_{B_c} + m_V) A_1(q^2) - \frac{\lambda(m_{B_c}^2, m_V^2, q^2)}{m_{B_c} + m_V} A_2(q^2) \right] \right|^2 \right\},
$$

(38)
\[
d\frac{d\Gamma^\pm(B_c \to V \ell\nu)}{dq^2} = (1 - \bar{m}_l^2)^2 \left\{ \frac{\sqrt{\lambda(m_{B_c}^2, m_i^2, q^2) G_F^2 |V_{CKM}|^2}}{384 m_{B_c} \pi^3} \right\} \left\{ (m_i^2 + 2q^2) \lambda(m_{B_c}^2, m_i^2, q^2) \right. \\
\times \left. \left| \frac{V(q^2)}{m_{B_c} + m_V} + \frac{(m_{B_c} + m_V) A_1(q^2)}{\sqrt{\lambda(m_{B_c}^2, m_i^2, q^2)}} \right|^2 \right\}, 
\]

where the superscript \(+(-)\) denotes the right-handed (left-handed) polarizations of vector mesons. \(\lambda(m_{B_c}^2, m_i^2, q^2) = (m_{B_c}^2 + m_i^2 - q^2)^2 - 4m_{B_c}^2 m_i^2 \) with \(i = P, V\). \(\bar{m}_l = m_l / \sqrt{q^2}\). The combined transverse and total differential decay widths are given by

\[
d\Gamma_T/dq^2 = d\Gamma_+/dq^2 + d\Gamma_-/dq^2, \quad d\Gamma_L/dq^2 = d\Gamma_{L1}/dq^2 + d\Gamma_{L2}/dq^2.
\]

The differential decay widths for \(B_c \to S \ell\nu\) and \(B_c \to A \ell\nu\) can be obtained by making the following replacements in the above expressions for \(B_c \to P \ell\nu\) and \(B_c \to V \ell\nu\)

\[
\begin{align*}
    m_P &\rightarrow m_S, \\
    F_i^{B_c P}(q^2) &\rightarrow F_i^{B_c S}(q^2), \quad i = 0, 1
\end{align*}
\]

and

\[
\begin{align*}
    m_{B_c} + m_V &\rightarrow m_{B_c} - m_A, \\
    V_i^{B_c V}(q^2) &\rightarrow A_i^{B_i A}(q^2), \quad i = 0, 1, 2
\end{align*}
\]

respectively. The \(d\Gamma_L/dq^2\) and \(d\Gamma^\pm/dq^2\) for \(B_c \to T \ell\nu\) is given by equation (38) multiplied \((\sqrt{\frac{\bar{m}_l}{3m_T}})^2\) and Eq. (39) multiplied \((\frac{1}{\sqrt{2m_T}})^2\), respectively. Here the \(\tilde{p}_T\) denotes the momentum of the tensor meson in the \(B_c\) rest frame and \(m_T\) is mass of the tensor meson.

For the \(B_{sJ}\) final state, the inputs are form factors given in table III and the masses of \(B_c\) and \(B_{sJ}\)s given in Eqs. (21). The other input parameters are given as follows [11]:

\[
\begin{align*}
    \tau_{B_c} &= (0.452 \times 10^{-12}) s, \\
    m_c &= 0.511 \text{MeV}, \quad m_\mu = 0.106 \text{GeV}, \quad m_\tau = 1.78 \text{GeV}, \\
    G_F &= 1.166 \times 10^{-5} \text{GeV}^{-2}, \quad |V_{cs}| = 0.973,
\end{align*}
\]

and our predictions for branching fractions are given in Table [X] It should be mentioned that in the above calculation we have considered \(B_{s1}(B_{s1}')\) and \(B_1(B_1')\) to be in \(^3 P_1(1 P_1)\) eigenstates.

For the \(B_j\) final state, we need to evaluate the form factors for \(B_c \to B_j\) by following the same method, and our results are given in Table [X]. The masses of \(B_c\) and \(B_j\)s are also given in Eqs. (21) and Eqs. (29). The other inputs are the same as Eq. (41) but with \(|V_{cs}| = 0.973\) replaced by \(|V_{cd}| = 0.225\). With these inputs, our results for branching fractions and ratios are given in Table [X].

From these tables, we can see that the branching fractions for \(B_c \to B_j \ell\nu\) and \(B_c \to B_j^* \ell\nu\) are at the percent level, while those for the \(B_c \to B^* \ell\nu\) and \(B_c \to B^* \ell\nu\) are suppressed by one order of magnitude. This is consistent with the results in the literature [12-24]. Branching fractions for channels with \(p\)-wave bottomed mesons in the final state range from \(10^{-4}\) to \(10^{-6}\). In decays with large phase space, the electron and muon masses can introduce about a few percents to branching ratios. While for those limited phase space like the \(B_c \to B_{s2} \ell\nu\), the effects due to the lepton mass difference can reach 30\%. We hope these predictions can be examined in future on the experimental side.

B. Nonleptonic \(B_c\) decays

Since our main purpose of this work is to investigate the production of \(B_{sJ}\), we will focus on the decay modes which can be controlled under the factorization approach. Such decay modes are usually dominated by tree operators with
effective Hamiltonian

\[ \mathcal{H}_{\text{eff}}(c \to sud) = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \left\{ C_1 [\bar{s}_\alpha \gamma^\mu (1 - \gamma_5) c_\beta] [\bar{u}_\beta \gamma_\mu (1 - \gamma_5) d_\alpha] \right. \\
\left. + C_2 [\bar{s}_\alpha \gamma^\mu (1 - \gamma_5) c_\beta] [\bar{u}_\beta \gamma_\mu (1 - \gamma_5) d_\beta] \right\}, \]

(45)

where \( C_1 \) and \( C_2 \) are the Wilson coefficients, \( \alpha \) and \( \beta \) denote the color indices.

With the definitions of decay constants,

\[ \langle \pi^+ (p) | \bar{u} \gamma_\mu \gamma_5 d | 0 \rangle = -i f_{\pi} p_\mu, \]

(46)

one can expect the factorization formula to have the following forms

\[ i\mathcal{M}(B_c^+ \to B_1^+ \pi^+) = N m_{B_c}^2 (1 - r_{B_1}^2) F_0^{B_c, B_1} (m_\pi^2), \]

(47)

\[ i\mathcal{M}(B_c^+ \to B_2^+ \pi^+) = ( -i ) N \sqrt{\lambda(m_{B_c}^2, m_{B_2}^2, m_\pi^2)} A_0^{B_c, B_2} (m_\pi^2), \]

(48)

\[ i\mathcal{M}(B_c^+ \to B_0^+ \pi^+) = ( -i ) N m_{B_c}^2 (1 - r_{B_0}^2) F_0^{B_c, B_0} (m_\pi^2), \]

(49)

\[ i\mathcal{M}(B_c^+ \to B_{31}^+ \pi^+) = ( -i ) N \sqrt{\lambda(m_{B_c}^2, m_{B_{31}}^2, m_\pi^2)} V_0^{B_c, B_{31}} (m_\pi^2), \]

(50)

\[ i\mathcal{M}(B_c^+ \to B_{32}^+ \pi^+) = ( -i ) N \sqrt{\lambda(m_{B_c}^2, m_{B_{32}}^2, m_\pi^2)} V_0^{B_c, B_{32}} (m_\pi^2), \]

(51)

\[ i\mathcal{M}(B_c^+ \to B_{22}^+ \pi^+) = ( -i ) \frac{1}{\sqrt{6}} N \lambda(m_{B_c}^2, m_{B_{22}}^2, m_\pi^2) A_0^{B_c, B_{22}} (m_\pi^2), \]

(52)

where \( N = G_F / \sqrt{2} V_{us}^* V_{ud} a_1 f_\pi \), with \( a_1 = C_2 + C_3 / N_c (N_c = 3) \).

The partial decay width for \( B_c \to B_{32} \pi \) is given as

\[ \Gamma = \frac{|\tilde{p}_1|}{8 \pi m_{B_c}^2} |\mathcal{M}|^2 \]

(53)
with $|\vec{p}_1|$ being the magnitude of three-momentum of $B_{sJ}$ or $\pi$ meson in the final state in the $B_c$ rest frame.

We use the transition form factors given in Table 11 and the masses of $B_c$ and $B_{sJ}$s given in Eqs. (21), (22) and (23) and the other inputs which are given as follows (11, 44):

$$\tau_{B_c} = (0.452 \times 10^{-12})s, \quad m_\pi = 0.140 \text{GeV},$$

$$|V_{cs}| = 0.973, \quad |V_{ud}| = 0.974,$$

$$f_\pi = 130.4 \text{MeV}, \quad a_1 = 1.07,$$  \hspace{1cm} (54)

where $f_\pi$ can be extracted from $\pi^- \to \ell^- \bar{\nu}$ data and $a_1$ is evaluated at the typical fatorization scale $\mu \sim m_c$. Then our theoretical results for $B_c \to B_{sJ} \pi$ branching ratios turn out to be as follows:

$$B(B_c^+ \to B_s \pi^+) = 4.1\%,$$

$$B(B_c^+ \to B_s^0 \pi^+) = 2.0\%,$$

$$B(B_c^+ \to B_{sJ} \pi^+) = 0.68\%,$$

$$B(B_c^+ \to B_{sJ}^0 \pi^+) = 0.0082\%,$$

$$B(B_c^+ \to B_{sJ}^* \pi^+) = 0.36\%,$$

$$B(B_c^+ \to B_{sJ}^{*0} \pi^+) = 0.023\%.$$  \hspace{1cm} (55)

Using the $1 \text{fb}^{-1}$ data of proton-proton collisions collected at the center-of-mass energy of 7 TeV and $2 \text{fb}^{-1}$ data accumulated at 8 TeV, the LHCb collaboration has observed the decay $B_c \to B_s \pi^+$ [64]:

$$\frac{\sigma(B_c^+)}{\sigma(B_s^0)} \times B(B_c^+ \to B_{sJ}^0 \pi^+) = (2.37 \pm 0.31 \pm 0.11^{+0.17}_{-0.13}) \times 10^{-3}. $$  \hspace{1cm} (56)

The first uncertainty is statistical, the second is systematic and the third arises from the uncertainty on the $B_c^+$ lifetime. The ratio of cross sections $\sigma(B_c^+)/\sigma(B_s^0)$ depends significantly on the kinematics, and a rough estimate has lead to the branching ratio for $B_c^+ \to B_{sJ}^0 \pi^+$ of about 10% [64]. The estimated branching fraction is somewhat larger than but still at the same magnitude with our result. Moreover, our results have indicated that the LHCb collaboration might be able to discover other channels with similar branching fractions like the $B_c \to B_{sJ}^* \pi$.

V. CONCLUSIONS

To understand the structure of the heavy-light mesons, especially the newly observed states, and to establish an overview of the spectroscopy, a lot of effort are requested on both experiment and theory sides. One particular remark is the classification of these states. In the heavy quark limit, the charm quark will decouple with the light degree of freedom and acts as a static color source. Strong interactions will be independent of the heavy flavor and spin. In this case, heavy mesons, the eigenstates of the QCD Lagrangian in the heavy quark limit, can be labeled according to the freedom and acts as a static color source. Strong interactions will be independent of the heavy flavor and spin. In this is the classification of these states. In the heavy quark limit, the charm quark will decouple with the light degree of freedom. The heavy mesons with the same angular momentum $s_J$ of the light degree of freedom. The heavy mesons with the same angular momentum $s_J$ but different orientations of the heavy quark spin degenerate. One consequence is that heavy mesons can be classified by the multiplets characterized by $s_J$ instead of the usual scheme using the $2S+1L_J$.

In this work, we have suggested to study the $B_s$ and its excitations $B_{sJ}$ in the $B_c$ decays. We have calculated the $B_c \to B_{sJ}$ and $B_c \to B_J$ form factors within the covariant light-front quark model, where the $B_{sJ}$ and $B_J$ denotes an s-wave or p-wave $\bar{b}s$ and $b\bar{d}$ meson, respectively. The form factors at $q^2 = 0$ are directly calculated while the $q^2$-distribution is obtained by the extrapolation. The derived form factors are then used to study semileptonic $B_c \to (B_{sJ}, B_J)\ell \nu$ decays, and nonleptonic $B_c \to B_{sJ} \pi$. Branching fractions and polarizations are predicted, through which we find that the predicted branching fractions are sizable, especially at the LHC experiment and future high-energy $e^+e^-$ colliders with a high luminosity at the $Z$-pole. The future experimental measurements are helpful to study the nonperturbative QCD dynamics in the presence of a heavy spectator and also of great value for the spectroscopy study.

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Appendix A: EXPLICIT EXPRESSIONS FOR FORM FACTORS

In the LFQM, it is more convenient to adopt a new set of parametrization of form factors, with the relations:

\[ V^{B,V}(q^2) = -(m_B + m_V)g(q^2), \quad A_1^{B,V}(q^2) = -\frac{f(q^2)}{m_B + m_V}, A_2^{B,V}(q^2) = (m_B + m_V)a_+(q^2), \]
\[ A_0^{B,V}(q^2) = \frac{m_B + m_V}{2m_V} A_1(q^2) - \frac{m_B - m_V}{2m_V} A_2(q^2) - \frac{q^2}{2m_V} a_-(q^2), \]
\[ F_1^{B,S}(q^2) = -u_+(q^2), \quad F_0^{B,S}(q^2) = -u_+(q^2) - \frac{q^2}{q\cdot P} u_-(q^2), \]
\[ A^{B,A}(q^2) = -(m_B - m_A)g(q^2), \quad A_1^{B,A}(q^2) = -\frac{f(q^2)}{m_B - m_A}, A_2^{B,A}(q^2) = (m_B - m_A)c_+(q^2), \]
\[ V_0^{B,A}(q^2) = \frac{m_B - m_A}{2m_A} V_1(q^2) - \frac{m_B + m_A}{2m_A} V_2(q^2) - \frac{q^2}{2m_A} c_-(q^2), \]
\[ V^{B,T}(q^2) = -m_B(m_B + m_T)h(q^2), \quad A_1^{B,T}(q^2) = -\frac{m_B}{m_B + m_T} k(q^2), A_2^{B,T}(q^2) = m_B(m_B + m_T)b_+(q^2), \]
\[ A_0^{B,T}(q^2) = \frac{m_B + m_T}{2m_T} A_1(q^2) - \frac{m_B - m_T}{2m_T} A_2(q^2) - \frac{m_B^2}{2m_T} b_-(q^2). \]

The analytic expressions for \( P \to P \) transition form factors in the covariant LFQM have been given in Eq. (7), while for the \( P \to V \) transition, they are given as follows [31, 35]:

\[ g(q^2) = -\frac{N_c}{16\pi^3} \int dx_2dx_2' \frac{2h_{i_i} h_{i_2}^*}{x_2 N_1' N_1} \left\{ x_2 m_1' + x_1 m_2 + (m_1' - m_1)^2 \frac{p_1^2 \cdot q_\perp}{q^2} + \frac{2}{w_{i_i}} \left[ p_1^2 + (p_1^2 \cdot q_\perp)^2 \right] \right\}, \]
\[ f(q^2) = \frac{N_c}{16\pi^3} \int dx_2dx_2' \frac{h_{i_i} h_{i_2}^*}{x_2 N_1' N_1} \left\{ 2x_1(m_2 - m_1')(M_0'^2 + M_0'^2) - 4x_1 m_1' M_0'^2 \\
+ 2x_2 m_1' q \cdot P + 2m_2 q^2 - 2x_1 m_2 (M_0'^2 + M_0'^2) + 2(m_1' - m_2) (m_1' + m_1'^2) \\
+ 8(m_1' - m_2) \left[ p_1^2 + (p_1^2 \cdot q_\perp)^2 \right] + 2(m_1' + m_1'^2) (q^2 + q \cdot P) \frac{p_1^2 \cdot q_\perp}{q^2} \\
- 4q^2 + (p_1^2 \cdot q_\perp)^2 \right\} \left[ 2x_1 (M_0'^2 + M_0'^2) - q^2 - q \cdot P - 2(q^2 + q \cdot P) \frac{p_1^2 \cdot q_\perp}{q^2} \\
- 2(m_1' - m_1'^2)(m_1' - m_2) \right\}, \]
\[ a_+(q^2) = \frac{N_c}{16\pi^3} \int dx_2dx_2' \frac{h_{i_i} h_{i_2}^*}{x_2 N_1' N_1} \left\{ (x_1 - x_2)(x_2 m_1' + x_1 m_2) - [2x_1 m_2 + m_1' + (x_2 - x_1) m_1'] \frac{p_1^2 \cdot q_\perp}{q^2} \\
- \frac{2}{x_2 q^2 + p_1^2 \cdot q_\perp} \left[ p_1^2 \cdot p_1^2 + (x_1 m_2 + x_2 m_1')(x_1 m_2 - x_2 m_1') \right] \right\}. \]
The explicit expressions for $a_\pm (q^2)$ can be obtained by making the following replacements \[37\]:

$$u_\pm (q^2) = -f_\pm (q^2)|m_i' \rightarrow m_i''', h_\psi' \rightarrow h_\psi''$$

\[\{[\ell^3 A, 1, A(q^2), q^3 A, 1, A(q^2), h_{\pm, 1, A}(q^2)] = \{f(q^2), g(q^2), \alpha_\pm (q^2)|m_i' \rightarrow m_i''', h_\psi' \rightarrow h_\psi''_{A, 1, A}, w_\psi' \rightarrow w_\psi''_{A, 1, A}\right\} \tag{A8}\]

where only the $1/W''$ terms in $P \rightarrow 1/A$ form factors are kept. It should be cautious that the replacement of $m_i' \rightarrow m_i''$ should not be applied to $m_i''$ in $w''$ and $h''$. The $P \rightarrow T$ transition form factors are calculated \[37\]:

$$h(q^2) = -g(q^2)|h_\psi' \rightarrow h_\psi'' + \frac{N_c}{16\pi^3} \int dx_2 d^2p'_1 \frac{2\hbar_p h_\psi''}{x_2 N_1'' N_1} \left\{ (m_i' - m_i'')(A_2^{(2)} + A_4^{(2)}) + (m_i'' + m_i' - 2m_2)(A_2^{(3)} + A_3^{(3)}) - m_i'(A_1^{(3)} + A_2^{(3)}) + \frac{2}{w''}(2A_1^{(3)} + 2A_3^{(1)} - A_1^{(2)}) \right\}, \tag{A9}\]

$$k(q^2) = -f(q^2)|h_\psi' \rightarrow h_\psi'' + \frac{N_c}{16\pi^3} \int dx_2 d^2p'_1 \frac{\hbar_p h_\psi''}{x_2 N_1'' N_1} \left\{ 2(A_1^{(1)} + A_2^{(1)}) \times [m_2(q^2 - \hat{N}_1' - \hat{N}_1'' - m_i''') - m_1'(M'' - \hat{N}_1'' - m_i'' - m_2')] - m_i'(M'' - \hat{N}_1' - m_i'' - m_2') - 2m_i'm_i'm_2] + 2(m_i' + m_i'') \left( A_2^{(1)} Z_2 + \frac{q \cdot P}{q^2}(A_2^{(2)}) \right) + 16(m_2 - m_i')(A_2^{(3)} + A_3^{(3)}) + 4(2m_i' - m_i'' - m_2)A_1^{(2)} + \frac{4}{w''}(2M'' - q^2 + 2(m_i' - m_2)(m_i' + m_2))(2A_1^{(3)} + 2A_3^{(1)} - A_1^{(2)}) + 4 \left( A_2^{(3)} Z_2 + \frac{q \cdot P}{3q^2}(A_1^{(2)})^2 + 2A_1^{(2)} Z_2 \right) \right\}, \tag{A10}\]

$$b_\pm (q^2) = -a_\pm (q^2)|h_\psi' \rightarrow h_\psi'' + \frac{N_c}{16\pi^3} \int dx_2 d^2p'_1 \frac{\hbar_p h_\psi''}{x_2 N_1'' N_1} \left\{ 8(m_2 - m_i')(A_3^{(3)} + 2A_4^{(3)} + A_5^{(3)}) - 2m_i'(A_1^{(3)} + A_2^{(3)})(A_2^{(2)} + A_4^{(2)}) + 2(m_i' + m_i'')(A_2^{(2)} + 2A_3^{(2)} + A_4^{(2)}) + \frac{2}{w''}(2M'' - q^2 + 2(m_i' - m_2)(m_i'' + m_2))(A_3^{(3)} + 2A_4^{(3)} + A_5^{(3)} - A_2^{(3)} - A_3^{(2)}) + [q^2 - \hat{N}_1' - \hat{N}_1'' - (m_i' + m_i'')][A_2^{(2)} + 2A_3^{(2)} + A_4^{(2)} - A_1^{(1)} - A_1^{(2)}] \right\}, \tag{A11}\]
The explicit forms of $h'_M$ and $w'_M$ are given by $[37]$:

$$h'_p = h'_v = (M^2 - M_0^2) \sqrt{\frac{x_1x_2}{N_c}} \frac{1}{\sqrt{2M_0}} \varphi',$$

$$h'_s = \sqrt{\frac{2}{3}} h'_{A} = (M^2 - M_0^2) \sqrt{\frac{x_1x_2}{N_c}} \frac{1}{\sqrt{2M_0}} \frac{M_0^2}{2\sqrt{3M_0}} \varphi'_p,$$

$$h'_{A} = h'_T = (M^2 - M_0^2) \sqrt{\frac{x_1x_2}{N_c}} \frac{1}{\sqrt{2M_0}} \varphi'_p,$$

$$w'_v = M'_0 + m'_1 + m'_2, \quad w'_{A} = -\frac{M_0^2}{m'_1 - m'_2}, \quad w'_{s} = 2,$$

(A14)

where $\varphi'$ and $\varphi'_p$ are the light-front momentum distribution amplitudes for s-wave and p-wave mesons, respectively $[37]$:

$$\varphi' = \varphi'(x_2, p'_\perp) = 4 \left( \frac{\pi}{\beta^2} \right)^{3/4} \sqrt{\frac{dp'_\perp}{dx_2}} \exp \left( -\frac{p'_\perp^2 + x_2^2}{2\beta^2} \right), \quad \varphi'_p = \varphi'_p(x_2, p'_\perp) = \sqrt{\frac{2}{\beta^2}} \varphi', \quad \frac{dp'_\perp}{dx_2} = \frac{c'_1c_2}{x_1x_2M_0}.$$  

(A15)
