A competition on blow-up of solutions to semilinear wave equations with scale-invariant damping and nonlinear memory term

Wenhui Chen*1 and Ahmad Z. Fino2

1School of Mathematical Sciences, Shanghai Jiao Tong University, 200240 Shanghai, China
2Department of Mathematics, Faculty of Sciences, Lebanese University, P.O. Box 826, Tripoli, Lebanon

Abstract

In this paper, we investigate blow-up of solutions to the Cauchy problem for semilinear wave equations with scale-invariant damping and nonlinear memory term, which can be represented by the Riemann-Liouville fractional integral of order $1 - \gamma$ with $\gamma \in (0, 1)$. Our main interest is to study mixed influence of various kinds from damping term and the nonlinear memory kernel on the blow-up condition for the power of nonlinearity by using test function method or generalized Kato’s type lemma. We find a new competition, particularly for the small value of $\gamma$, on the blow-up between the effective case and the non-effective case.

Keywords: Semilinear wave equation, scale-invariant damping, nonlinear memory term, Riemann-Liouville fractional integral, blow-up.

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1 Introduction

We study blow-up of solutions to the Cauchy problem for semilinear wave equations with scale-invariant damping of time-dependent type, namely,

\[
\begin{aligned}
&\left\{ \begin{array}{l}
u_{tt} - \Delta u + \frac{\mu}{1 + t} u_t = N_{\gamma,p}(u), \quad x \in \mathbb{R}^n, \quad t > 0, \\
&(u, u_t)(0, x) = (u_0, u_1)(x), \quad x \in \mathbb{R}^n, 
\end{array} \right.
\end{aligned}
\]  

(1.1)

where $\mu \in (0, \infty)$ and the nonlinear term on the right-hand side of the equation in (1.1) is the Riemann-Liouville fractional integral of order $1 - \gamma$ of the $p$ power of the solution, which can be represented by

\[
N_{\gamma,p}(u)(t, x) := c_\gamma \int_0^t (t - \tau)^{-\gamma} |u(\tau, x)|^p d\tau \quad \text{carrying} \quad c_\gamma := \frac{1}{\Gamma(1 - \gamma)},
\]

(1.2)

where $p > 1$, $\gamma \in (0, 1)$ and $\Gamma$ denotes the Euler integral of the second kind.

In the last two decades, the wave equations with scale-invariant damping have caught a lot of attention. Let us begin with the corresponding linear Cauchy problem with vanishing right-hand side to (1.1) as follows:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
u_{tt} - \Delta u + \frac{\mu}{1 + t} u_t = 0, \quad x \in \mathbb{R}^n, \quad t > 0, \\
&(u, u_t)(0, x) = (u_0, u_1)(x), \quad x \in \mathbb{R}^n, 
\end{array} \right.
\end{aligned}
\]  

(1.3)

*Corresponding author: Wenhui Chen (wenhui.chen.math@gmail.com)
where $\mu \in (0, \infty)$. According to the classification introduced by [30], thanks to the hyperbolic scaling $\tilde{u}(t, x) := u(\sigma(t + 1) - 1, \sigma x)$ with $\sigma \in (0, \infty)$, then the unknown function $\tilde{u} = \tilde{u}(t, x)$ satisfies the same wave equation with the damping term, which is the so-called scale-invariant. Indeed, the behavior of the solutions to (1.3) is mainly determined by the value of parameter $\mu$, which provides a threshold between the effective damping and the non-effective damping. Here, the effective damping stands for its solution somehow having the behavior of the corresponding parabolic equation, and the non-effective damping stands for its solution somehow having the behavior of the free wave equation.

We now turn to the semilinear Cauchy problem with the power source nonlinear term. Concerning the semilinear wave equations with scale-invariant damping

$$
\begin{aligned}
  u_{tt} - \Delta u + \frac{\mu}{1 + t} u_t &= |u|^p, & x \in \mathbb{R}^n, & t > 0, \\
  (u, u_t)(0, x) &= (u_0, u_1)(x), & x \in \mathbb{R}^n,
\end{aligned}
$$

we next introduce some results by distinguishing the value of $\mu$. First of all, by employing a test function method (see, for example, [32]), the author of [28] proved blow-up of solutions providing that $1 < p \leq p_{Fuj}(n + \mu - 1)$ if $\mu \in (0, 1]$, and $1 < p \leq p_{Fuj}(n)$ if $\mu \in (1, \infty)$, where $p_{Fuj}(n) := 1 + 2/n$ is the so-called Fujita exponent. We should underline that the Fujita exponent is the critical exponent for semilinear heat equations and semilinear classical damped wave equations. Later, [7] derived the global existence result providing that $p > p_{Fuj}(n)$, and $p \leq n/(n - 2)$ if $n \geq 3$, with the value of $\mu$ such that $\mu \geq 5/3$ for $n = 1$, $\mu \geq 3$ for $n = 2$, and $\mu \geq n + 2$ for $n \geq 3$. Therefore, these results show the critical exponent for (1.4) is the Fujita exponent $p_{Fuj}(n)$ with a large parameter $\mu$. Next, let us consider the “not large” value of $\mu$. In the special case $\mu = 2$, the authors of [8] proved that for $1 \leq n \leq 3$ the critical exponent is given by the competition $\max\{p_{Fuj}(n), p_{Str}(n + 2)\}$, where the so-called Strauss exponent $p_{Str}(n)$ is the critical exponent for semilinear wave equations. To be specific, the Strauss exponent is represented by

$$
p_{Str}(n) := \frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n - 1)}
$$

and it is a positive root of the quadratic equation $(n - 1)p^2 - (n + 1)p - 2 = 0$. Particularly, the blow-up result for (1.4) with $\mu = 2$ was proved by the application of the Liouville transform and the classical Kato’s lemma. In the same year, global existence results were extended for $p_{Str}(n) < p < 1 + 2/\max\{2, (n - 3)/2\}$ in odd dimensions $n \geq 5$ in [9]. The global existence results for general dimensional cases were derived in [22, 23]. Recently, [20] found a shift-Strauss exponent for the blow-up results, to be specific, $p_{Fuj}(n) \leq p < p_{Str}(n + 2\mu)$ when $\mu \in (0, \mu^*/2)$, where $\mu^* = (n^2 + n + 2)/(n + 2)$. By the aid of hypergeometric functions motivated by [33], the authors of [14] got the sharper blow-up results if $p_{Fuj}(n) \leq p \leq p_{Str}(n + \mu)$ when $\mu \in (0, \mu^*)$. It was also conjecture that the critical exponent could be $p_{Str}(n + \mu)$ for some value of $\mu$. Under $1 < p \leq p_{Str}(n + \mu)$ with $\mu \in (0, \infty)$, the lifespan estimates are improved in the papers [26, 27, 24] by applying iteration argument associated with modified Bessel functions. Concerning other studies on semilinear scale-invariant damped wave equations, we refer to [18, 16, 15, 19].

We now recall some studies for semilinear damped wave equations with nonlinear memory term.
Let us start by the case of constant coefficient as follows:

\[
\begin{cases}
  u_{tt} - \Delta u + u_t = N_{\gamma,p}(u), & x \in \mathbb{R}^n, \ t > 0, \\
  (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n.
\end{cases}
\] (1.5)

According to the recent studies of [12, 2, 6, 3], the critical exponent for (1.5) is given by the competition such that \(\max\{p_\gamma(n), 1/\gamma\}\), where

\[
p_\gamma(n) := 1 + \frac{2(2 - \gamma)}{\max\{n - 2(1 - \gamma), 0\}}.
\]

We remark that the proof of blow-up result is based on a test function method. Later, the authors of [10] investigated blow-up of solutions to

\[
\begin{cases}
  u_{tt} - \Delta u + a(x)b(t)u_t = N_{\gamma,p}(u), & x \in \mathbb{R}^n, \ t > 0, \\
  (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n,
\end{cases}
\] (1.6)

where the time-space-dependent coefficient in the damping term is defined by

\[
a(x)b(t) := a_0(1 + |x|^2)^{-\alpha/2}(1 + t)^{-\beta}
\] (1.7)

with \(a_0 > 0, \alpha, \beta \geq 0\) and \(\alpha + \beta < 1\). Roughly speaking, [10] just considered the effective case due to \(\alpha + \beta < 1\). Furthermore, new blow-up results for \(\alpha = 0\) and \(\beta \in (-1, 1)\) were obtained in the recent paper [13]. To the best of the authors’ knowledge, so far it is still unknown for the existence/nonexistence of global (in time) solutions for the scale-invariant case of time-dependent type, i.e. (1.7) with \(a_0 = \mu, \alpha = 0\) and \(\beta = 1\). In this paper, we will give a positive answer of blow-up of solutions. However, we should underline that the study of blow-up for (1.1) is not a trivial generalization of the study of the previous case. As we mentioned in the last part, the parameter \(\mu\) in the scale-invariant damping will play a significant role in the behavior of solutions. Thus, we may somehow observe a competition between the behavior for the effective case and the non-effective case influenced by the relaxation function.

Finally, we recall the recent result for the semilinear wave equation with nonlinear memory term, i.e. the Cauchy problem (1.1) formally carrying \(\mu = 0\), as follows:

\[
\begin{cases}
  u_{tt} - \Delta u = N_{\gamma,p}(u), & x \in \mathbb{R}^n, \ t > 0, \\
  (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n.
\end{cases}
\] (1.8)

The authors of [5] recently proved blow-up of energy solutions to (1.8) if \(p > 1\) for \(n = 1\) and \(1 < p \leq p_0(n, \gamma)\) for \(n \geq 2\), where \(p = p_0(n, \gamma)\) solves \((n - 1)p^2 - (n + 3 - 2\gamma)p - 2 = 0\), with \(\gamma \in (0, 1)\) and \(p > 1\). Here, for any \(n \geq 2\) we denote by \(p_0(n, \gamma)\) the positive root of the last equation by

\[
p_0(n, \gamma) := \frac{n + 3 - 2\gamma + \sqrt{n^2 + (14 - 4\gamma)n + (3 - 2\gamma)^2 - 8}}{2(n - 1)}.
\]

In the case when \(n = 1\) we set formally \(p_0(1, \gamma) = \infty\) for any \(\gamma \in (0, 1)\). This is a generalized Strauss exponent and satisfies \(\lim_{\gamma \to 1-} p_0(n, \gamma) = p_{Str}(n)\) for all \(n \geq 2\). Furthermore, the research concerning blow-up of solutions to (1.8) with general nonlinear memory terms has been done recently in [4].
The purpose of the present paper is to investigate blow-up of solutions for (1.1) with any $\mu \in (0, \infty)$ and $\gamma \in (0, 1)$. Especially, we are interested in the influence from various kinds of damping term (dominant by parameter $\mu$) and different order of nonlinear memory kernel (dominant by parameter $\gamma$) on blow-up conditions describing by the upper bounds of the exponent $p$. The main results and some discussions of the influence of $\mu$ and $\gamma$ will be showed in Section 2.

Actually, the paper is two-fold. For one thing, by using test function methods with suitable time-dependent weighted functions, we will prove blow-up of solutions to (1.1) in the case when $1 < p \leq p_1(n, \gamma)$ if $\mu \in (1, \infty)$, and $1 < p \leq p_2(n, \gamma, \mu)$ if $\mu \in (0, 1]$, where we denoted

$$p_1(n, \gamma) := \begin{cases} \min \left\{ \frac{1 + \frac{3 - \gamma}{n - 1 + \gamma} + \frac{2 - \gamma}{n - 2 + \gamma}}{1 + \frac{3 - \gamma}{n - 1 + \gamma}} \right\} & \text{if } \mu \in (1, 2) \cup (2, \infty) \text{ for } n \geq 2, \\ \min \left\{ \frac{1 + \frac{3 - \gamma}{n - 1 + \gamma} + \frac{2 - \gamma}{n - 2 + \gamma}}{1 + \frac{3 - \gamma}{n - 1 + \gamma}} \right\} & \text{if } \mu \in (1, 2) \cup (2, \infty) \text{ for } n = 1, \\ \text{or } \mu = 2 & \text{for } n \geq 1, \end{cases} \quad (1.9)$$

$$p_2(n, \gamma, \mu) := \min \left\{ \frac{1 + \frac{3 - \gamma}{n + \mu + \gamma - 2} + \frac{2 - \gamma}{n - 2 + \gamma}}{1 + \frac{3 - \gamma}{n + \mu + \gamma - 2} + \frac{2 - \gamma}{n - 2 + \gamma}} \right\} \quad (1.10)$$

For another, let us recall the following relation:

$$\lim_{\gamma \to 1} c_\gamma s_+^{-\gamma} = \delta_0(s) \text{ in the sense of distributions with } s_+^{-\gamma} := \begin{cases} s^{-\gamma} & \text{if } s > 0, \\ 0 & \text{if } s < 0. \end{cases} \quad (1.11)$$

It seems reasonable to derive in the blow-up results an upper bound $p_0(n + \mu, \gamma)$ for the exponent $p$ in (1.2) that fulfills

$$\lim_{\gamma \to 1} p_0(n + \mu, \gamma) = p_{\text{str}}(n + \mu),$$

where $p_{\text{str}}(n + \mu)$ is the critical exponent for (1.4) for some value of $\mu$. In Section 4, we first derive a generalized Kato’s type lemma of integral type. Then, by employing the new Kato’s type lemma associated with suitable test function from [26, 24], we will prove blow-up of solutions to (1.1) in the case when $1 < p < p_0(n + \mu, \gamma)$ for any $\mu \in (0, \infty)$ if initial data satisfies certain sign assumptions. Then, in Section 5, strongly motivated by [8, 29], the blow-up result for (1.1) with $\mu = 2$ in the case when $p = p_0(n + 2, \gamma)$ will be derived by using an iteration method with slicing procedure. Therefore, we may observe a new competition for the blow-up condition. Particularly, for the small value of $\gamma$, the model performs parabolic-like rather than hyperbolic-like even for the small value of $\mu$. This effect does not appear in the classical model. We will give more detail explanation on this competition later.

Notation: We give some notations to be used in this paper. $f \lesssim g$ means that there exists a positive constant $C$ such that $f \leq C g$. $B_R$ denotes the ball around the origin with radius $R$ in $\mathbb{R}^n$.

2 Main result

According to the recent paper [10], one may derive well-posedness for (1.1).
Lemma 2.1. Let $n \geq 1$ and $\gamma \in (0, 1)$. Let us assume $p \leq n/(n-2)$ if $n \geq 3$. Let us consider initial data $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$. Then, there is a unique maximal mild solution $u = u(t, x)$ to the Cauchy problem (1.1) such that

$$u \in \mathcal{C} \left( [0, T), H^1(\mathbb{R}^n) \right) \cap \mathcal{C}^1 \left( [0, T), L^2(\mathbb{R}^n) \right),$$

where $0 < T \leq \infty$. Particularly, we say $u$ is a global (in time) solution to (1.1) if $T = \infty$, while in the case $T = \infty$, we say $u$ blows up in finite time.

Theorem 2.1. Let us assume $p \leq n/(n-2)$ if $n \geq 3$ and

$$1 < p \leq \begin{cases} p_1(n, \gamma) & \text{if } \mu \in (1, \infty), \\ p_2(n, \gamma, \mu) & \text{if } \mu \in (0, 1], \end{cases}$$

for all $n \geq 1$ and $\gamma \in (0, 1)$. Let us consider initial data $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} (u_1(x) + (\mu - 1)u_0(x))dx > 0 \text{ if } \mu \in (1, \infty),$$

$$\int_{\mathbb{R}^n} u_1(x)dx > 0 \text{ if } \mu \in (0, 1].$$

Then, there is no global (in time) mild solution to the Cauchy problem (1.1).

Remark 2.1. Let us take $\gamma$ tending to $1^-$. We observe that

$$\lim_{\gamma \to 1^-} p_1(n, \gamma) = p_{\text{Fuji}}(n)$$

if $\mu \in (1, 2) \cup (2, \infty)$ for $n = 1, 2$ or $\mu = 2$ for $n \geq 1$. Moreover, it also holds that

$$\lim_{\gamma \to 1^-} p_2(n, \gamma, \mu) = p_{\text{Fuji}}(n + \mu - 1)$$

if $\mu \in (0, 1]$ for $n = 1$. By taking the consideration of (1.11), they provide some relations between our results and the blow-up result in [28].

Next, we will show the blow-up result when $1 < p < p_0(n + \mu, \gamma)$ for any $\mu \in (0, \infty)$ and $\gamma \in (0, 1)$. Let us introduce a suitable definition of energy solution to the Cauchy problem (1.1).

Definition 2.1. Let us assume $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$. We say that

$$u \in \mathcal{C} \left( [0, T), H^1(\mathbb{R}^n) \right) \cap \mathcal{C}^1 \left( [0, T), L^2(\mathbb{R}^n) \right) \text{ such that } N_{\gamma, p}(u) \in L^1_{\text{loc}}([0, T) \times \mathbb{R}^n)$$

is an energy solution of (1.1) on $[0, T)$ if $u$ fulfills $u(0, \cdot) = u_0$ in $H^1(\mathbb{R}^n)$ and the integral relation

$$\int_{\mathbb{R}^n} u_t(t, x)\psi(t, x)dx - \int_{\mathbb{R}^n} u_1(x)\psi(0, x)dx + \int_0^t \int_{\mathbb{R}^n} (\nabla u(s, x) \cdot \nabla \psi(s, x) - u_t(s, x)\psi_x(s, x))dxds$$

$$+ \int_0^t \int_{\mathbb{R}^n} \frac{\mu u_t(s, x)}{1 + s} \psi(s, x)dxds = c_\gamma \int_0^t \int_{\mathbb{R}^n} \psi(s, x) \int_0^s (s - \tau)^{-\gamma}|u(\tau, x)|^p d\tau dxds \quad (2.1)$$

for any $\psi \in \mathcal{C}^\infty_0 ([0, T) \times \mathbb{R}^n)$ and any $t \in [0, T)$. 
By using integration by parts, we can show that \( u = u(t, x) \) introduced in Definition 2.1 is a weak solution to (1.1) as \( t \to T \).

Let us begin with stating the second blow-up result to (1.1).

**Theorem 2.2.** Let us assume \( p \leq n/(n-2) \) if \( n \geq 3 \) and

\[
1 < p < p_0(n + \mu, \gamma)
\]

for all \( n \geq 1 \) and \( \gamma \in (0, 1) \). Let \( u_0 \in H^1(\mathbb{R}^n) \) and \( u_1 \in L^2(\mathbb{R}^n) \) be nonnegative, nontrivial and compactly supported functions with supports contained in \( B_R \) for some \( R \geq 1 \). Moreover, let \( u \) be an energy solution on \([0, T)\) to (1.1) with \( \mu > 0 \) according to Definition 2.1. Then, there is no global (in time) energy solution to the Cauchy problem (1.1).

**Remark 2.2.** The phenomenon of shift Strauss type exponent is exactly the same as those for the power nonlinearity (1.4). We refer the interested readers to [8, 14, 26, 24]. Precisely, concerning the subcritical case, the recent paper [5] proved blow-up of solutions to semilinear wave equation with memory nonlinearity (1.8) if \( 1 < p < p_0(n, \gamma) \) for \( \gamma \in (0, 1) \). In Theorem 2.2, the scale-invariant behavior can be expressed by a shift of Strauss type exponent \( p_0(n, \gamma) \) to \( p_0(n + \mu, \gamma) \).

Finally, let us turn to the critical case for the Cauchy problem (1.1) with \( \mu = 2 \). At this time, by defining a new variable \( v = v(t, x) \) such that

\[
v(t, x) := (1 + t)u(t, x),
\]

we may reduce the Cauchy problem (1.1) with \( \mu = 2 \) to the following semilinear wave equation with another nonlinear memory term:

\[
\begin{align*}

v_{tt} - \Delta v &= c_\gamma (1 + t) \int_0^t (t - \tau)^{-\gamma}(1 + \tau)^{-p} |v(\tau, x)|^p d\tau, \\
(v, v_t)(0, x) &= (v_0, v_1)(x),
\end{align*}
\]

where initial data is given by \( v_0(x) := u_0(x) \), \( v_1(x) := u_0(x) + u_1(x) \). Obviously, from the relation (2.3), we just need to establish the blow-up of solution \( v \), which implies the blow-up of \( u \) immediately. Before stating our main result on blow-up for (2.4), we introduce a definition of energy solutions to the Cauchy problem (2.4).

**Definition 2.2.** Let \( u_0 \in H^1(\mathbb{R}^n) \) and \( u_1 \in L^2(\mathbb{R}^n) \). We say that

\[
v \in C \left([0, T), H^1(\mathbb{R}^n)\right) \cap C^1 \left([0, T), L^2(\mathbb{R}^n)\right) \quad \text{such that} \quad \tilde{N}_{\gamma, p}(v) \in L^1_{\text{loc}}([0, T) \times \mathbb{R}^n)
\]

is an energy solution of (2.4) on \([0, T)\) if \( v \) fulfills \( v(0, \cdot) = v_0 \) in \( H^1(\mathbb{R}^n) \) and the integral relation

\[
\begin{align*}

\int_{\mathbb{R}^n} v_t(t, x) \psi(t, x) dx &= - \int_{\mathbb{R}^n} v_1(x) \psi(0, x) dx + \int_0^t \int_{\mathbb{R}^n} \nabla v(s, x) \cdot \nabla \psi(s, x) - v_t(s, x) \psi_s(s, x) dx ds \\
&= c_\gamma \int_0^t \int_{\mathbb{R}^n} \psi(s, x)(1 + s) \int_0^s (s - \tau)^{-\gamma}(1 + \tau)^{-p} |v(\tau, x)|^p d\tau ds \\
&= c_\gamma \int_0^t \int_{\mathbb{R}^n} \psi(s, x)(1 + s) \int_0^s (s - \tau)^{-\gamma}(1 + \tau)^{-p} |v(\tau, x)|^p d\tau ds
\end{align*}
\]

for any \( \psi \in C_0^\infty ([0, T) \times \mathbb{R}^n) \) and any \( t \in [0, T) \).

Then, an application of a further step of integration by parts, which shows that \( v = v(t, x) \) introduced in Definition 2.2 is also a weak solution to (2.4) as \( t \to T \).
Remark 2.3. Indeed, Definition 2.1 with \( \mu = 2 \) is equivalent to Definition 2.2, which can be proved by choosing the test function as \((1 + t)\psi(t, x)\) and using integration by parts.

Theorem 2.3. Let us assume \( p \leq n/(n - 2) \) if \( n \geq 3 \) and

\[
p = p_0(n + 2, \gamma)
\]

for all \( n \geq 1 \) and \( \gamma \in (0, 1) \). Let \( v_0 \in H^1(\mathbb{R}^n) \) and \( v_1 \in L^2(\mathbb{R}^n) \) be nonnegative, nontrivial and compactly supported functions with supports contained in \( B_R \) for some \( R > 0 \). Moreover, let \( v \) be an energy solution on \([0, T]\) to (2.4) according to Definition 2.2. Then, the energy solution \( v \) blows up. Furthermore, according to Remark 2.3 and the backward transform \( u(t, x) = (1 + t)^{-1}v(t, x) \), the solution to (1.1) with \( \mu = 2 \) blows up in finite time providing that the condition (2.6) hold for all \( n \geq 1 \) and \( \gamma \in (0, 1) \), moreover, \( u_0 \) and \( u_0 + u_1 \) are nonnegative, nontrivial and compactly supported functions with supports contained in \( B_R \) for some \( R > 0 \).

2.1 Some explanation for the competition

Let us summarize the derived results in Theorems 2.1, 2.2 and 2.3. Under the assumptions that \( p \leq n/(n - 2) \) if \( n \geq 3 \) and some conditions for initial data, we may derive blow-up of solutions to the Cauchy problem (1.1) if

\[
1 < p < \begin{cases} 
\max \{p_1(n, \gamma), p_0(n + \mu, \gamma)\} & \text{if } \mu \in (1, \infty), \\
\max \{p_2(n, \gamma, \mu), p_0(n + \mu, \gamma)\} & \text{if } \mu \in (0, 1],
\end{cases}
\]

where \( n \geq 1 \) and \( \gamma \in (0, 1) \). Furthermore, in the limit cases \( p = p_1(n, \gamma) \) if \( \mu \in (1, \infty) \), or \( p = p_2(n, \gamma, \mu) \) if \( \mu \in (0, 1] \), or \( p = p_0(n + 2, \gamma) \) if \( \mu = 2 \), the nontrivial local (in time) solutions to (1.1) also blow up in finite time. Honestly, the competition is different from those in semilinear wave equations with scale-invariant damping and power nonlinearity (1.4).

Let us focus on the subcritical case in Theorem 2.1 and Theorem 2.2. For the Cauchy problem (1.4), as shown in the introduction, the critical exponent is conjectured by \( p_{\text{Str}}(n + \mu) \) if \( \mu \) is not a large number, and \( p_{\text{Wid}}(n) \) if \( \mu \) is a large number. In other words, the wave equation with scale-invariant damping and power nonlinearity is explained by hyperbolic-like model if \( \mu \) is not large, and by parabolic-like model if \( \mu \) is large. Therefore, the competition between hyperbolic-like and parabolic-like comes, which is determined by the value of \( \mu \).

Nevertheless, our results in blow-up of solutions indicate another competition. This competition strongly relies on the value of \( \gamma \) also. A new phenomenon is that the model shows parabolic-like for the small value of \( \mu \) instead of hyperbolic-like if \( \gamma \) is a small number. To explain this effect clearly, we will concentrate on two dimensional case.

Let us take \( n = 2 \). Then, we conclude the following statements:

- Concerning \( 0 < \mu \leq 1 \), we find that there is a competition between \( p_0(2 + \mu, \gamma) \) and \( p_2(2, \gamma, \mu) \). Precisely, when \( \gamma \in (0, \gamma_0] \), the exponent \( p_2(2, \gamma, \mu) \) plays a dominant role. But when \( \gamma \in (\gamma_0, 1) \), the generalized shift-Strauss exponent \( p_0(2 + \mu, \gamma) \) has a stronger effect.

- Concerning \( 1 < \mu \leq 2 \), we observe the similar effect between \( p_0(2 + \mu, \gamma) \) and \( p_1(2, \gamma) \) to the first point. Furthermore, we denote the intersection of two curves in the \( \gamma - \mu \) plane by \( \gamma_1 \). We have to emphasize that when \( \mu = 2 \) and \( \gamma \to 1^- \), it holds \( p_0(4, \gamma) \to p_1(2, \gamma) \), which is exactly critical exponent for (1.4) in two dimensional space [8].
Concerning $\mu > 2$, the situation is completely changed that the exponent $p_1(2, \gamma)$ has a dominant influence comparing with $p_0(2+\mu, \gamma)$. We should underline again $\lim_{\gamma \to 1^-} p_1(2, \gamma) = p_{\text{Fuj}}(2)$ and $\lim_{\gamma \to 1^-} p_0(2+\mu, \gamma) = p_{\text{Str}}(2+\mu)$.

![Graphs showing blow-up range in the $\gamma - p$ plane](image)

Figure 1: Blow-up range in the $\gamma - p$ plane

### 3 Proof of Theorem 2.1: Blow-up via test function method

#### 3.1 Preliminaries

In this part, we will recall some definitions and derive some useful lemmas concerning the fractional integrals and fractional derivatives that will be used later.

According to Chapter 1 in [25], the Riemann-Liouville fractional integrals and their derivatives can be shown by the next definitions.

**Definition 3.1.** Let $f \in L^1((0, T))$ with $T > 0$. The Riemann-Liouville left- and right-sides fractional integrals of order $\alpha \in (0, 1)$ are defined by the following way:

\[
I_{0+}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{-(1-\alpha)} f(s) ds \quad \text{for} \quad t > 0,
\]

\[
I_{T-}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{-(1-\alpha)} f(s) ds \quad \text{for} \quad t < T.
\]
**Definition 3.2.** Let \( f \in \text{AC}([0,T]) \) with \( T > 0 \), i.e. \( f \) is an absolutely continuous functions. The Riemann-Liouville left- and right-sides fractional derivatives of order \( \alpha \in (0,1) \) are defined by the following way:

\[
D^\alpha_{0^+} f(t) := \frac{d}{dt} J^1_{0^+} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds \quad \text{for } t > 0,
\]

\[
D^\alpha_T f(t) := \frac{d}{dt} J^1_{t^-} f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T (s-t)^{-\alpha} f(s) ds \quad \text{for } t < T.
\]

Now, we will show some rules in the calculation of fractional derivatives, which were introduced in the books [25, 17].

**Proposition 3.1.** Let \( T > 0 \) and \( \alpha \in (0,1) \). The fractional integration by parts

\[
\int_0^T f(t) D^\alpha_{0^+} g(t) dt = \int_0^T g(t) D^\alpha_{0^+} f(t) dt
\]

holds for every \( f \in I^\alpha_{0^+} (L^p((0,T))), \ g \in I^\alpha_{0^+} (L^q((0,T))), \) where \( 1/p + 1/q \leq 1 + \alpha \) with \( p, q > 1 \) and

\[
I^\alpha_{0^+} (L^q((0,T))) := \{ f = I^\alpha_{0^+} h \ \text{for} \ h \in L^q((0,T)) \},
\]

\[
I^\alpha_{0^+} (L^p((0,T))) := \{ f = I^\alpha_{0^+} h \ \text{for} \ h \in L^p((0,T)) \}.
\]

**Proposition 3.2.** Let \( T > 0 \) and \( \alpha \in (0,1) \). The following identities hold:

\[
D^\alpha_{0^+} I^\alpha_{0^+} f(t) = f(t) \ a.e. \ t \in (0,T) \ \text{for all} \ f \in L^p((0,T)),
\]

and

\[
(-1)^k D^k D^\alpha_{0^+} f(t) = D^k f(t) \ \text{for all} \ f \in AC^{k+1}([0,T]),
\]

where \( 1 \leq p \leq \infty \) and \( k \in \mathbb{N} \).

Let us now define a time-dependent function \( w = w(t) \) for any \( T > 0 \) such that

\[
w(t) := (1 - t/T)^\sigma \ \text{for any} \ t \in [0,T],
\]

where \( \sigma \gg 1 \). According to Property 2.1 in [17], the function \( w(t) \) fulfills

\[
D^k w(t) = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma+1-k-\alpha)} T^{-(k+\alpha)}(1-t/T)^{\sigma-(\alpha+k)} \ \text{for any} \ t \in [0,T],
\]

where \( T > 0, \ \alpha \in (0,1) \) and \( k \geq 0 \).

By this way, we may introduce the useful lemma, which will be applied later in the proof.

**Lemma 3.1.** Let \( T \gg 1, \ \alpha \in (0,1), \ k \geq 0 \) and \( p > 1 \). The following estimates hold:

\[
\int_0^T (1+t)^{-\frac{1}{p-1}} \left| D^{k+\alpha}_{0^+} w(t) \right|^p dt \lesssim T^{\frac{k}{p-1}}
\]

and

\[
\int_0^T (1+t)^{-\frac{1}{p-1}} \left| D^{k+\alpha}_{0^+} w(t) \right|^p dt \lesssim T^{-\frac{k+\alpha}{p-1}} \times \begin{cases} T^{\frac{p-2}{p-1}} & \text{if } p > 2, \\ \ln(T) & \text{if } p = 2, \\ 1 & \text{if } p < 2. \end{cases}
\]
Proof. For one thing, with the aim of proving (3.3), we may use (3.2) to derive
\[
\int_0^T (1 + t)(w(t))^{-\frac{1}{p-1}} |D_t^{k+\alpha} w(t)|^{\frac{p}{p-1}} dt \lesssim T^{-\frac{(k+\alpha)p}{p-1}} \int_0^T (1 + t)(1 - t/T)^{\sigma - \frac{(k+\alpha)p}{p-1}} dt,
\]
where we used the definition of \( w(t) \). Then, it holds that
\[
\int_0^T (1 + t)(1 - t/T)^{\sigma - \frac{(k+\alpha)p}{p-1}} dt \lesssim \int_0^T (1 + t) dt \lesssim T^2
\]
for \( T \gg 1 \), which implies our desired estimates (3.3).

For another, we may apply the similar approach to prove (3.4). Precisely, by using (3.2), one observes
\[
\int_0^T (1 + t)^{-\frac{1}{p-1}} |D_t^{k+\alpha} w(t)|^{\frac{p}{p-1}} dt \lesssim T^{-\frac{(k+\alpha)p}{p-1}} \int_0^T (1 + t)^{-\frac{1}{p-1}} (1 - t/T)^{\sigma - \frac{(k+\alpha)p}{p-1}} dt
\]
\[
\lesssim T^{-\frac{(k+\alpha)p}{p-1}} \int_0^T (1 + t)^{-\frac{1}{p-1}} dt.
\]
Concerning the integrability of \((1 + t)^{-1/(p-1)}\) influenced by the parameter \( p \), we deduce
\[
\int_0^T (1 + t)^{-\frac{1}{p-1}} dt \lesssim \begin{cases} T \frac{p^2}{p-1} & \text{if } p > 2, \\ \ln(T) & \text{if } p = 2, \\ 1 & \text{if } p < 2. \end{cases}
\]
Thus, the proof is complete. \( \square \)

3.2 Blow-up result in the case when \( \mu \in (1, \infty) \)

In this case, we choose an auxiliary time-dependent function \( g = g(t) \) by
\[
g(t) := \frac{t + 1}{\mu - 1} \quad \text{for any } \mu > 1, \tag{3.5}
\]
so that from the equation in (1.1), we have
\[
gN_{\gamma,p}(u) = (gu)_t - \Delta (gu) - (g'u)_t + u_t.
\]

We assume, on the contrary, that \( u = u(t,x) \) is a global (in time) mild solution to (1.1), then as mild solutions being weak solutions (see [10]) and regarding \( g(t)\psi(t,x) \) as a test function, one can derive
\[
\int_0^T \int_{\mathbb{R}^n} N_{\gamma,p}(u)(t,x)g(t)\psi(t,x)dx dt - \int_{\mathbb{R}^n} u_0(x)g(0)\psi_t(0,x)dx
\]
\[
= - \int_{\mathbb{R}^n} (u_1(x)g(0) + u_0(x))\psi(0,x)dx + \int_0^T \int_{\mathbb{R}^n} u(t,x)g(t)\psi_t(t,x)dx dt
\]
\[
+ \int_0^T \int_{\mathbb{R}^n} u(t,x)(g'(t) - 1)\psi_t(t,x)dx dt - \int_0^T \int_{\mathbb{R}^n} u(t,x)g(t)\Delta \psi(t,x)dx dt \tag{3.6}
\]
for any \( T \gg 1 \) and compactly supported function \( \psi(t,x) \in C^2([0,T] \times \mathbb{R}^n) \) such that \( \psi(T,x) = \psi_t(T,x) = 0 \). Then, we define the test function by separating the variables fulfilling
\[
\psi(t,x) := D_{1,T}^{1-\gamma} (\tilde{\psi}(t,x)) := D_{1,T}^{1-\gamma} ((\tilde{\varphi}_T(x))^\prime w(t)),
\]
where \( w(t) \) is defined in (3.1) and \( \varphi_T(x) := \varphi(|x| R/T) \) with a positive parameter \( R \) to be determined later in each case. Here, \( \varphi = \varphi(x) \) is a radial test function with \( \varphi \in C^\infty(\mathbb{R}^n) \) such that

\[
\varphi(x) := \begin{cases} 
1 & \text{if } 0 \leq |x| \leq 1, \\
0 & \text{if } |x| > 2, 
\end{cases}
\]

moreover, \( \varphi(x) \in [0, 1] \) and \( |d_{|x|} \varphi(|x|)| \lesssim |x|^{-1} \). We remark that \( d_{|x|} \) stands for the derivative with respect to \( |x| \).

To begin with, let us define

\[
I_T := \int_0^T \int_{B_{2T/R}} |u(t, x)|^p g(t) \tilde{\psi}(t, x) dx dt.
\]

We notice that

\[
\int_0^T \int_{\mathbb{R}^n} N_{\gamma, p}(u(t, x)) g(t) \psi(t, x) dx dt = \int_0^T \int_{\mathbb{R}^n} I_{0|t|}^{1-\gamma}(|u|^p)(t, x) g(t) D_{t|T}^{1-\gamma} \left( \tilde{\psi}(t, x) \right) dx dt
\]

\[
\geq \int_0^T \int_{\mathbb{R}^n} D_{0|t|}^{1-\gamma} I_{0|t|}^{1-\gamma}(|u|^p)(t, x) \tilde{\psi}(t, x) dx dt = I_T
\]

by employing Proposition 3.1 and Proposition 3.2.

Then, because initial data fulfills

\[
\int_{\mathbb{R}^n} (u_1(x) + (\mu - 1) u_0(x)) dx > 0,
\]

which implying

\[
\int_{B_{2T/R}} (u_1(x) + (\mu - 1) u_0(x)) (\varphi_T(x))^\ell dx \geq 0,
\]

we make use of (3.6) and the derived estimates in the above to get

\[
I_T \lesssim -\frac{T^{-2+\gamma}}{\mu - 1} \int_{B_{2T/R}} u_0(x) (\varphi_T(x))^\ell dx - \frac{T^{-1+\gamma}}{\mu - 1} \int_{B_{2T/R}} (u_1(x) + (\mu - 1) u_0(x)) (\varphi_T(x))^\ell dx
\]

\[
+ \int_0^T \int_{B_{2T/R}} u(t, x) g(t) \psi_T(t, x) dx dt + \int_0^T \int_{B_{2T/R}} u(t, x) (g'(t) - 1) \psi_T(t, x) dx dt
\]

\[
- \int_0^T \int_{B_{2T/R}} u(t, x) g(t) \Delta \psi_T(t, x) dx dt
\]

\[
\lesssim -\frac{T^{-2+\gamma}}{\mu - 1} \int_{B_{2T/R}} u_0(x) (\varphi_T(x))^\ell dx + \int_0^T \int_{B_{2T/R}} |u(t, x)| g(t) (\varphi_T(x))^\ell \left| D_{t|T}^{3-\gamma} w(t) \right| dx dt
\]

\[
+ |\mu - 2| \int_0^T \int_{B_{2T/R}} |u(t, x)| (\varphi_T(x))^\ell \left| D_{t|T}^{2-\gamma} w(t) \right| dx dt
\]

\[
+ \int_0^T \int_{B_{2T}} |u(t, x)| g(t) \left| \Delta (\varphi_T(x)) \right| \left| D_{t|T}^{1-\gamma} w(t) \right| dx dt
\]

\[
= -\frac{T^{-2+\gamma}}{\mu - 1} \int_{B_{2T/R}} u_0(x) (\varphi_T(x))^\ell dx + J_{1, T} + J_{2, T} + J_{3, T},
\]
where we used Proposition 3.2 and (3.2).

Next, from Young’s inequality, we obtain

\[ J_1 \lesssim \frac{I_T}{6} + \int_0^T \int_{B_{2T/R}} g(t)(\varphi_T(x))^\ell(w(t))^{-\frac{1}{p-1}} |D_{4T/(1+\gamma/w)}^3 w(t)|^{p'} \, dx \, dt, \]

\[ J_2 \lesssim \frac{I_T}{6} + |\mu - 2|^{p'} \int_0^T \int_{B_{2T/R}} (g(t))^{-\frac{1}{p-1}} (\varphi_T(x))^\ell(w(t))^{-\frac{1}{p-1}} |D_{4T/(1+\gamma/w)}^2 w(t)|^{p'} \, dx \, dt, \]

\[ J_3 \lesssim \frac{I_T}{6} + \int_0^T \int_{B_{2T/R}} g(t)(\varphi_T(x))^{\ell-2p'} (w(t))^{-\frac{1}{p-1}} (|\Delta \varphi_T(x)|^{p'} + |\nabla \varphi_T(x)|^{2p'}) \, dx \, dt. \]

In the last estimate, we used

\[ \Delta (\varphi_T(x)) = \ell(\varphi_T(x)^{\ell-1} \Delta \varphi_T(x) + \ell(\ell - 1)(\varphi_T(x))^{\ell - 2} |\nabla \varphi_T(x)|^2. \]

Summarizing the derived estimates, we conclude

\[ I_T \lesssim -T^{-2+\gamma} \int_{B_{2T/R}} u_0(x)(\varphi_T(x))^\ell \, dx \int_0^T g(t)(w(t))^{-\frac{1}{p-1}} |D_{4T/(1+\gamma/w)}^3 w(t)|^{p'} \, dt \int_{B_{2T/R}} (\varphi_T(x))^\ell \, dx \]

\[ + |\mu - 2|^{p'} \int_0^T (g(t))^{-\frac{1}{p-1}} |D_{4T/(1+\gamma/w)}^2 w(t)|^{p'} \, dt \int_{B_{2T/R}} (\varphi_T(x))^\ell \, dx \]

\[ + \int_0^T g(t)(w(t))^{-\frac{1}{p-1}} |D_{4T/(1+\gamma/w)} w(t)|^{p'} \, dt \int_{B_{2T/R}} (\varphi_T(x))^{\ell-2p'} (|\Delta \varphi_T(x)|^{p'} + |\nabla \varphi_T(x)|^{2p'}) \, dx. \quad (3.7) \]

Obviously, the value of \( \mu \) influences on the last inequality due to the fact that when \( \mu = 2 \), the third term on the right-hand side of (3.7) will be vanishing.

We will divide the proof into two cases: \( 1 < p < p_1(n, \gamma) \) and \( p = p_1(n, \gamma) \).

Let us begin with the case when \( 1 < p < p_1(n, \gamma) \). In this case, we may consider \( R = 1 \) in the test function \( \varphi_T(x) \) so that \( \varphi_T(x) := \varphi(|x|/T) \). By applying Lemma 3.1 in (3.7), it yields

\[ I_T \lesssim -T^{-2+\gamma} \int_{B_{2T}} |u_0(x)|(\varphi_T(x))^\ell \, dx + T^{2-(3-\gamma)p'+n} \]

\[ + |\mu - 2|^{p'} T^{n-(2-\gamma)p'} \times \begin{cases} T^{\frac{p-2}{p-1}} & \text{if } p > 2, \\ \ln(T) & \text{if } p = 2, \\ 1 & \text{if } p < 2. \end{cases} \quad (3.8) \]

The condition \( 1 < p < 1 + \frac{3-\gamma}{n-1+\gamma} \) leads to

\[ 2 - (3-\gamma)p' + n < 0 \quad \text{and} \quad n - (2-\gamma)p' + \frac{p-2}{p-1} < 0. \]

What’s more, another condition \( 1 < p < 1 + \frac{2-\gamma}{n-2+\gamma} \) implies

\[ n - (2-\gamma)p' < 0. \]

We should remark that the last inequality is trivial for \( n = 1 \) due to \( p' > 1 \).

Finally, in the limit case \( p = 2 \), we may observe

\[ p = 2 < 1 + \frac{3-\gamma}{n-1+\gamma} \quad \text{iff} \quad n < 2(2-\gamma). \]
In other words, for the case \( p = 2 \), we just need to consider the dimension satisfying \( n < 2(2 - \gamma) \). Thus, we may use \( \ln(T) \leq T^{((2 - \gamma)p' - n)/2} = T^{2 - \gamma - n/2} \) if \( n < 2(2 - \gamma) \) to derive

\[
T^{n - (2 - \gamma)p'} \ln(T) \leq T^{2 - \gamma - n/2} = T^{n/2 - 2 + \gamma}.
\]

In conclusion,

- providing that \( \mu \neq 2 \), the assumption \( 1 < p < p_1(n, \gamma) \) shows the right-hand sides of (3.8) tends to 0 as \( T \to \infty \), and thanks to the Lebesgue dominated convergence theorem, it yields

\[
\int_0^\infty \int_{\mathbb{R}^n} |u(t, x)|^p g(t) dx dt = 0,
\]

which implies \( u(t, x) \equiv 0 \) for all \( t \) and almost everywhere \( x \). This contradicts to our assumption on initial data.

- providing that \( \mu = 2 \), the assumption \( 1 < p < p_1(n, \gamma) \) shows the right-hand sides of (3.8) tends to 0 as \( T \to \infty \), and similarly as the above discussion, it is a contradiction on our assumption.

To derive blow-up for the limit case when \( p = p_1(n, \gamma) \), we take \( 1 \ll R < T \) such that \( T \) and \( R \) do not tend simultaneously to infinity. By doing direct calculation, there exists a positive constant \( \tilde{C} \) independent of \( T \) such that

\[
\int_0^\infty \int_{\mathbb{R}^n} |u(t, x)|^p g(t) dx dt \leq \tilde{C},
\]

which implies that

\[
\int_0^T \int_{B_{2T/R} \setminus B_{T/R}} |u(t, x)|^p g(t) \tilde{\psi}(t, x) dx dt \to 0 \quad \text{as} \quad T \to \infty.
\]  

On the other hand, using Hölder’s inequality instead of Young’s inequality, we estimate \( J_{3,T} \) by

\[
J_{3,T} \lesssim \left( \int_0^T \int_{B_{2T/R} \setminus B_{T/R}} |u(t, x)|^p g(t) \tilde{\psi}(t, x) dx dt \right)^{1/p} \times \left( \int_0^T \int_{B_{2T/R}} g(t) (\varphi_T(x))^{t - 2p'} (w(t))^{-\frac{1}{p'}} \left( |\Delta \varphi_T(x)|^p' + |\nabla \varphi_T(x)|^{2p'} \right) D_{w(t)}^{1 - \gamma} w(t) |p'| dx dt \right)^{1/p'}.
\]

Similarly as the last cases, by considering \( p = p_1(n, \gamma) \), using (3.9), and letting \( T \to \infty \), we are able to claim that

\[
\int_0^\infty \int_{\mathbb{R}^n} |u(t, x)|^p g(t) dx dt \lesssim R^{-n},
\]

which implies a contradiction when \( R \) is suitably large. This completes the proof of the blow-up result in the case when \( \mu \in (1, \infty) \).

### 3.3 Blow-up result in the case when \( \mu \in (0, 1] \)

In this case, we will apply the similar idea as the last subsection so that we only sketch the proof. Let us replace the time-dependent function \( g = g(t) \) with \( g(0) = 1 \) in (3.5) by

\[
g(t) := (1 + t)^\mu,
\]
which results that
\[ gN_{\epsilon,p}(u) = (gu)_{t} - \Delta (gu) - (g'u)_{t}. \]

By choosing the same test function \( \psi = \psi(t,x) \) as the previous result, we may compute
\[
I_T \lesssim -T^{-2+\gamma} \int_{B_{2T/R}} u_0(x)(\varphi_T(x))^\ell\,dx - T^{-1+\gamma} \int_{B_{2T/R}} u_1(x)(\varphi_T(x))^\ell\,dx + \bar{J}_{1,T} + \bar{J}_{2,T} + \bar{J}_{3,T},
\]
where we used \( u_0 \equiv 0 \) and we defined
\[
\bar{J}_{1,T} := \int_0^T \int_{B_{2T/R}} |u(t,x)|g(t)|\psi_\alpha(t,x)|\,dxdt,
\]
\[
\bar{J}_{2,T} := \int_0^T \int_{B_{2T/R}} |u(t,x)|g'(t)|\psi_\alpha(t,x)|\,dxdt,
\]
\[
\bar{J}_{3,T} := \int_0^T \int_{B_{2T/R}} |u(t,x)|g(t)|\Delta \psi_\alpha(t,x)|\,dxdt.
\]

By using Hölder’s inequality, one may find
\[
I_T \lesssim -T^{-2+\gamma} \int_{B_{2T/R}} u_0(x)(\varphi_T(x))^\ell\,dx + \int_0^T \int_{B_{2T/R}} g(t)(\varphi_T(x))^\ell(w(t))^{-\frac{1}{\ell}} |D_t^{\frac{\gamma}{2}} w(t)|^{p'\prime} \,dxdt
\]
\[
+ \int_0^T \int_{B_{2T/R}} (g(t))^{-\frac{1}{\ell}} (g'(t))^{p'}(\varphi_T(x))^\ell(w(t))^{-\frac{1}{\ell}} |D_t^{\frac{\gamma}{2}} w(t)|^{p'\prime} \,dxdt
\]
\[
+ \int_0^T \int_{B_{2T/R}} g(t)(\varphi_T(x))^{\ell-2p'}(w(t))^{-\frac{1}{p'}} \left( |\Delta \varphi_T(x)|^{p'} + |\nabla \varphi_T(x)|^{2p'} \right) |D_t^{\frac{\gamma}{2}} w(t)|^{p'\prime} \,dxdt,
\]
where we considered our assumption
\[
\int_{\mathbb{R}^n} u_1(x)\,dx > 0 \quad \text{leading to} \quad \int_{B_{2T/R}} u_1(x)(\varphi_T(x))^\ell\,dx \geq 0.
\]

Similarly as Lemma 3.1, the next inequalities hold:
\[
\int_0^T (1+t)^\mu(w(t))^{-\frac{1}{p'-1}} |D_t^{k+\alpha} w(t)|^{\frac{\mu}{p'-1}} \,dt \lesssim T^{\mu+1-(k+\alpha)p/(p'-1)},
\]
and
\[
\int_0^T (1+t)^{-\frac{1}{p'-1}} w(t)^{\frac{p}{p'-1}} |D_t^{k+\alpha} w(t)|^{\frac{p}{p'-1}} \,dt \lesssim T^{-\frac{(k+\alpha)p}{(k+\alpha)p'-1} + \frac{\ln(T)}{p-1}} \times \begin{cases} T^{p-1} \quad &\text{if } p > 1 + \frac{1}{\mu}, \\
\ln(T) &\text{if } p = 1 + \frac{1}{\mu}, \\
1 &\text{if } p < 1 + \frac{1}{\mu}, \end{cases}
\]
where \( k \geq 0 \) and \( \alpha \in (0,1) \). For this reason, we proved
\[
I_T \lesssim -T^{-2+\gamma} \int_{B_{2T/R}} u_0(x)(\varphi_T(x))^\ell\,dx + T^{n+\mu+1-(3-\gamma)p'} + T^{n-(2-\gamma)p'} \times \begin{cases} T^{p-1} \quad &\text{if } p > 1 + \frac{1}{\mu}, \\
\ln(T) &\text{if } p = 1 + \frac{1}{\mu}, \\
1 &\text{if } p < 1 + \frac{1}{\mu}. \end{cases}
\]

By the same procedure as the last part, we can find some contradiction if \( 1 < p \leq p_2(n,\gamma,\mu) \). Therefore, our proof is complete.
4 Proof of Theorem 2.2: Blow-up via generalized Kato’s type lemma

4.1 Generalized Kato’s type lemma

In this subsection, we will derive generalized Kato’s type lemma of the integral type, whose proof is based on the iteration argument. This lemma is useful for us to prove Theorem 2.2.

**Lemma 4.1.** Let \( p > 1 \). Let us assume that the time-dependent functional \( \mathcal{F}(t) \in \mathcal{C}([0, T]) \) satisfies
\[
\mathcal{F}(t) \geq K_0(1 + t)^{-\alpha_0}(t - T_0)^{\beta_0}
\]
\[
\mathcal{F}(t) \geq \tilde{K}_0(1 + t)^{-\alpha_0} \int_{T_0}^{t} (1 + \eta)^{a_1} \int_{T_0}^{\eta} (1 + s)^{a_2} \int_{T_0}^{s} (1 + \tau)^{a_3} (\mathcal{F}(\tau))^p d\tau d\eta
\]
for any \( t \geq T_0 \geq 0 \), where \( \alpha_0, \beta_0, a_0, \ldots, a_3 \) are nonnegative constants, and \( K_0, \tilde{K}_0 \) are positive constants. If these constants fulfill
\[
(\beta_0 - \alpha_0)(p - 1) + a_1 + a_2 + a_3 + 3 - a_0 > 0,
\]
then the functional \( \mathcal{F}(t) \) blows up in finite time.

**Remark 4.1.** In the case when \( a_k < 0 \) for \( k = 1, 2, 3 \), we still can prove blow-up for the functional \( \mathcal{F}(t) \). For example, when \( a_3 < 0 \), from \( (1 + \tau)^{a_3} \geq (1 + t)^{a_3} \) for any \( \tau \leq t \), one has
\[
\mathcal{F}(t) \geq \tilde{K}_0(1 + t)^{-\alpha_0} \int_{T_0}^{t} (1 + \eta)^{a_1} \int_{T_0}^{\eta} (1 + s)^{a_2} \int_{T_0}^{s} (\mathcal{F}(\tau))^p d\tau d\eta.
\]
From Lemma 4.1, we can get blow-up of the functional \( \mathcal{F}(t) \) if
\[
(\beta_0 - \alpha_0)(p - 1) + a_1 + a_2 + 3 - (a_0 - a_3) > 0,
\]
which is exactly the same as (4.3).

**Remark 4.2.** Actually, the generalized Kato’s type lemma stated in Lemma 4.1 can be widely used in hyperbolic equation with nonlinear memory terms. Later, we will apply this lemma to prove blow-up for semilinear wave equation with scale-invariant damping and nonlinear memory term \( N_{\gamma, p}(u) \).

**Proof.** Motivated by (4.1), we will demonstrate the functional \( \mathcal{F}(t) \) having the following lower bound estimates:
\[
\mathcal{F}(t) \geq K_j(1 + t)^{-\alpha_j}(t - T_0)^{\beta_j},
\]
for any \( t \geq T_0 \), where the sequences \( \{K_j\}_{j \in \mathbb{N}}, \{\alpha_j\}_{j \in \mathbb{N}} \) and \( \{\beta_j\}_{j \in \mathbb{N}} \) consist of nonnegative real numbers to be determined later. Clearly from our observation, the initial case when \( j = 0 \) is given by (4.1). To prove (4.4) by deriving the sequences, we may use an iteration procedure. Precisely, we assume (4.4) hold for \( j \) and it still remains to do the inductive step, i.e. we will show (4.4) also holds for \( j + 1 \).
First of all, we combine (4.4) with (4.2) to get immediately
\[ \mathcal{F}(t) \geq K_j^p \tilde{K}_0 (1 + t)^{-a_0} \int_{T_0}^t (1 + \eta)^{a_1} \int_{T_0}^\eta (1 + s)^{a_2} \int_{T_0}^s (1 + \tau)^{a_1 - p\alpha_j} (\tau - T_0)^{p\beta_j} d\tau ds d\eta \]
\[ \geq K_j^p \tilde{K}_0 (1 + t)^{-a_0 - p\alpha_j} \int_{T_0}^t (1 + \eta)^{a_1} \int_{T_0}^\eta (s - T_0)^{a_2 + a_3 + 1 + p\beta_j} d\eta \]
\[ \geq \frac{K_j^p \tilde{K}_0}{a_3 + 1 + p\beta_j} (1 + t)^{-a_0 - p\alpha_j} \int_{T_0}^t (1 + \eta)^{a_1} \int_{T_0}^\eta (s - T_0)^{a_2 + a_3 + 1 + p\beta_j} d\eta \]
\[ \geq \frac{K_j^p \tilde{K}_0}{(a_1 + a_2 + a_3 + 3 + p\beta_j)^3} (1 + t)^{-a_0 - p\alpha_j} (t - T_0)^{a_1 + a_2 + a_3 + 3 + p\beta_j} \]
for all \( t \geq T_0 \) and we used nonnegativity of \( a_1, a_2, a_3 \). Therefore, the desired estimate (4.4) for \( j + 1 \) is concluded, provided that the recursive relations
\[ K_{j+1} := \frac{K_j^p \tilde{K}_0}{(a_1 + a_2 + a_3 + 3 + p\beta_j)^3}, \]
and \( \alpha_{j+1} := a_0 + p\alpha_j, \beta_{j+1} := a_1 + a_2 + a_3 + 3 + p\beta_j \) are fulfilled.

To determine the estimate for the multiplicative constant \( K_j \) from below, we should derive the explicit representation for \( \alpha_j \) and \( \beta_j \) in the first place. From the relations
\[ \alpha_j = a_0 + p\alpha_{j-1} \quad \text{and} \quad \beta_j = a_1 + a_2 + a_3 + 3 + p\beta_{j-1}, \quad (4.5) \]
we can deduce by iteration calculations
\[ \alpha_j = p^j \alpha_0 + a_0 \sum_{k=0}^{j-1} p^k = \left( \alpha_0 + \frac{a_0}{p-1} \right) p^j - \frac{a_0}{p-1}, \quad (4.6) \]
\[ \beta_j = p^j \beta_0 + (a_1 + a_2 + a_3 + 3) \sum_{k=0}^{j-1} p^k = \left( \beta_0 + \frac{a_1 + a_2 + a_3 + 3}{p-1} \right) p^j - \frac{a_1 + a_2 + a_3 + 3}{p-1}. \quad (4.7) \]
One may observe that
\[ (a_1 + a_2 + a_3 + 3 + p\beta_{j-1})^3 = \beta_j^3 \leq \left( \beta_0 + \frac{a_1 + a_2 + a_3 + 3}{p-1} \right)^3 p^{3j}, \]
where we used (4.5) and (4.7).

Then, it follows that
\[ K_j \geq \tilde{K}_0 \left( \beta_0 + \frac{a_1 + a_2 + a_3 + 3}{p-1} \right)^{-3} p^{-3j} K_{j-1}^p = D p^{-3j} K_{j-1}^p \quad (4.8) \]
for any \( j \in \mathbb{N} \), where we defined
\[ D := \tilde{K}_0 \left( \beta_0 + \frac{a_1 + a_2 + a_3 + 3}{p-1} \right)^{-3} > 0. \]
In order to achieve our aim, we employ the logarithmic function to both sides of (4.8) to get
\[
\log K_j \geq p^j \log K_0 - 3 \log p \sum_{k=0}^{j-1} (j-k)p^k + \log D \sum_{k=0}^{j-1} p^k
\]
\[
\geq p^j \left( \log K_0 - \frac{3p \log p}{(p-1)^2} + \frac{\log D}{p-1} \right) + \frac{3j \log p}{p-1} + \frac{3p \log p}{(p-1)^2} - \frac{\log D}{p-1}
\]
for any \(j \in \mathbb{N}\), where the next formula:
\[
\sum_{k=0}^{j-1} (j-k)p^k = \frac{1}{p-1} \left( \frac{p^{j+1} - p}{p-1} - j \right)
\]
was applied. Let us choose \(j_0 = j_0(p, a_1, a_2, a_3)\) to be the smallest positive integer fulfilling the relation
\[
j_0 \geq \frac{\log D}{3 \log p} - \frac{p}{p-1}.
\]
Taking into account \(j \geq j_0\) the inequality holds
\[
\log K_j \geq p^j \left( \log K_0 - \frac{3p \log p}{(p-1)^2} + \frac{\log D}{p-1} \right) = p^j \log E_0
\]
with a suitable constant \(E_0 = E_0(p, a_1, a_2, a_3) > 0\).

Finally, let us associate (4.4), (4.6), (4.7) with (4.10). By this way, it yields
\[
\mathcal{F}(t) \geq \exp \left( p^j \left( \log E_0 - \left( \alpha_0 + \frac{a_0}{p-1} \right) \log(1+t) + \left( \beta_0 + \frac{a_1 + a_2 + a_3 + 3}{p-1} \right) \log(t-T_0) \right) \right)
\]
\[
\times (1+t)^{\frac{a_0}{p-1}}(t-T_0)^{-\frac{a_1+a_2+a_3+3}{p-1}}
\]
for any \(j \geq j_0\) and \(t \geq T_0\). Let us assume \(t \geq \max\{1,2T_0\}\), which implies \(\log(1+t) \leq \log(2t)\) and \(\log(t-T_0) \geq \log(t/2)\). Therefore, from the above result, we may write
\[
\mathcal{F}(t) \geq \exp \left( p^j \log \left( E_0 2^{-a_0-\beta_0-\frac{a_0+a_1+a_2+a_3+3}{p-1} t \beta_0-\alpha_0+\frac{a_1+a_2+a_3+3}{p-1}} \right) \right)
\]
\[
\times (1+t)^{\frac{a_0}{p-1}}(t-T_0)^{-\frac{a_1+a_2+a_3+3}{p-1}}
\]
for any \(j \geq j_0\) and \(t \geq \max\{1,2T_0\}\). With our assumption on \(p\) such that (4.3) holds, we claim that the exponent for \(t\) in the exponential term of (4.11) is positive. Thus, we may find
\[
\log \left( E_0 2^{-a_0-\beta_0-\frac{a_0+a_1+a_2+a_3+3}{p-1} t \beta_0-\alpha_0+\frac{a_1+a_2+a_3+3}{p-1}} \right) > 0
\]
for suitably large \(t\). Letting \(j \to \infty\), we observe blow-up phenomenon of the functional \(\mathcal{F}(t)\). Thus, the proof is complete.

### 4.2 Blow-up result in the case when \(\mu \in (0, \infty)\)

Let us denote a time-dependent functional
\[
F(t) := \int_{\mathbb{R}^n} u(t,x)dx.
\]
We now take the test function \( \psi = \psi(t, x) \) in (2.1) satisfying \( \psi \equiv 1 \) over the set \( \{(s, x) \in [0, t] \times B_{R+s}\} \). It immediately results

\[
\int_{\mathbb{R}^n} u_t(t, x)dx - \int_{\mathbb{R}^n} u_1(x)dx + \int_0^t \frac{\mu}{1 + s} \int_{\mathbb{R}^n} u_t(s, x)dxds
= c_\gamma \int_0^t \int_{\mathbb{R}^n} \int_0^s (s - \tau)^{-\gamma} |u(\tau, x)|^p d\tau dxds,
\]

and differentiate (4.12) with respect to the time variable to conclude

\[
\int_{\mathbb{R}^n} u_{tt}(t, x)dx + \frac{\mu}{1 + t} \int_{\mathbb{R}^n} u_t(t, x)dx = c_\gamma \int_{\mathbb{R}^n} \int_0^t (t - \tau)^{-\gamma} |u(\tau, x)|^p d\tau dx.
\]

Clearly, the previous equality can be reformulated as

\[
(1 + t)^{-\mu} (F'(t)(1 + t)^\mu)' = F''(t) + \frac{\mu}{1 + t} F'(t) = c_\gamma \int_0^t (t - \tau)^{-\gamma} \int_{\mathbb{R}^n} |u(\tau, x)|^p d\tau d\tau.
\]

Then, multiplying (4.13) by \((1 + t)^\mu\) and integrating the resultant over \([0, t]\), we may see

\[
F(t) \geq c_\gamma \int_0^t (1 + \eta)^{-\mu} \int_0^\eta (1 + s)^\mu \int_0^s (s - \tau)^{-\gamma} \int_{\mathbb{R}^n} |u(\tau, x)|^p d\tau dxds \\geq C_0 (1 + \tau)^{-n(p-1)} (F(\tau))^p,
\]

where we used nonnegativities of \(u_0\) and \(u_1\). Furthermore, by using Hölder’s inequality and the support condition given by finite proposition speed, one has

\[
\int_{\mathbb{R}^n} |u(\tau, x)|^p dx = \int_{B_{R+t}} |u(\tau, x)|^p dx \geq C_0 (1 + \tau)^{-n(p-1)} (F(\tau))^p,
\]

with a positive constant \(C_0 = C_0(n, R, p)\). For this reason, the desired inequality (4.2) is constructed by plugging (4.15) into (4.14) so that

\[
F(t) \geq c_\gamma \int_0^t (1 + \eta)^{-\mu} \int_0^\eta (1 + s)^\mu \int_0^s (s - \tau)^{-\gamma} (1 + \tau)^{-n(p-1)} (F(\tau))^p d\tau dxds
\]

\[
\geq C_0 c_\gamma (1 + t)^{-\mu - \gamma - n(p-1)} \int_0^t \int_0^\eta (1 + s)^\mu \int_0^s (F(\tau))^p d\tau dxds \\geq C_0 c_\gamma (1 + t)^{-\mu - \gamma - n(p-1)} \int_0^t \int_0^\eta (1 + s)^\mu \int_0^s (F(\tau))^p d\tau dxds
\]

for any \(t \geq 0\).

The main approach of our proof is based on Lemma 4.1, which needs the lower bound estimate for the functional. Hence, our aim is to derive the lower bound estimate for the functional \(F(t)\). We are motivated by the paper [31], in other words, we introduce the function \(\Phi = \Phi(x)\) such that

\[
\Phi(x) := \begin{cases} e^x + e^{-x} & \text{if } n = 1, \\ \int_{S^{n-1}} e^{x \cdot \omega} d\sigma_\omega & \text{if } n \geq 2, \end{cases}
\]

(4.17)

where \(S^{n-1}\) is the \(n-1\) dimensional sphere. The above function is a positive smooth and fulfills the properties

\[
\Delta \Phi = \Phi, \quad \text{as well as } \Phi(x) \sim |x|^{-\frac{n-1}{2}} e^{|x|} \text{ as } |x| \to \infty.
\]
Moreover, according to [26, 24], we define the modified Bessel function of the second kind by
\[ \mathcal{K}_\nu(t) : = \int_0^t \exp (-t \cosh z) \cosh(\nu z) \, dz \]
for any \( \nu \in \mathbb{R} \), which solves the \( \nu \)-dependent second-order ordinary differential equation
\[ \left( t^2 \frac{d^2}{dt^2} + t \frac{d}{dt} - (t^2 + \nu^2) \right) \mathcal{K}_\nu(t) = 0 \quad \text{with} \quad \mathcal{K}_\nu(0) = 0. \]
Recalling [11], the asymptotic behavior of the modified Bessel function is showed for \( t \to \infty \) as
\[ \mathcal{K}_\nu(t) = \sqrt{\frac{\pi}{2t}} e^{-t} \left( 1 + \mathcal{O} \left( t^{-1} \right) \right). \]
Its derivative fulfills
\[ \frac{d}{dt} \mathcal{K}_\nu(t) = -\mathcal{K}_{\nu+1}(t) + \frac{\nu}{t} \mathcal{K}_\nu(t) = -\frac{1}{2} \left( \mathcal{K}_{\nu+1}(t) + \mathcal{K}_{\nu-1}(t) \right). \]
Setting the auxiliary function with respect to the time variable
\[ \lambda(t) : = (1 + t)^{-\frac{\mu+1}{2}} \mathcal{K}_{(\mu-1)/2}(1 + t), \]
we observe that it is the solutions to the following differential equation:
\[ \left( (1 + t)^2 \frac{d^2}{dt^2} - \mu(1 + t) \frac{d}{dt} + (\mu - (1 + t)^2) \right) \lambda(t) = 0 \quad \text{with} \quad \lambda(0) = \mathcal{K}_{(\mu-1)/2}(1), \quad \lambda(\infty) = 0. \]
We now introduce the test function \( \Psi = \Psi(t, x) \) with separate variables denoting by
\[ \Psi(t, x) : = \lambda(t) \Phi(x). \quad (4.18) \]
Indeed, we find that [26, Lemma 2.1] or [24, Lemma 2.1] is still valid for (1.1) due to the fact that the proof of such lemma is independent of nonnegative nonlinearity \( N_{\gamma,p}(u)(t, x) \geq 0 \) for any \( \gamma \in (0, 1) \) and \( p > 1 \). Consequently, it holds
\[ \int_{\mathbb{R}^n} |u(t, x)|^p \, dx \geq C_1(1 + t)^{n-1-\frac{n+\mu-1}{2}} \quad (4.19) \]
for any \( t \geq T_0 \), where \( T_0 \) is a large number independent of \( u_0, u_1 \) and \( C_1 = C_1(u_0, u_1, n, p, \mu, R, \Phi) \) is a positive constant. Eventually, combining (4.19) with (4.14) and using \( (1 + \tau) \geq (\tau - T_0) \) yield
\[
F(t) \geq C_1c_\gamma \int_{T_0}^t (1 + \eta)^{-\mu} \int_{T_0}^{\eta} (1 + s)^\mu \int_{T_0}^s (s - \tau)^{-\gamma}(1 + \tau)^{n-1-\frac{n+\mu-1}{2}} \, d\tau \, ds \, d\eta \\
\geq \frac{C_1c_\gamma}{n} \int_{T_0}^t (1 + \eta)^{-\mu} \int_{T_0}^{\eta} (1 + s)^{\mu-\frac{n+\mu-1}{2}} (s - T_0)^{-\gamma+n} \, ds \, d\eta \\
\geq \frac{C_1c_\gamma}{n(n + \mu + 1)} \int_{T_0}^t (1 + \eta)^{-\mu-\frac{n+\mu-1}{2}} (\eta - T_0)^{-\gamma+n+\mu+1} \, d\eta \\
\geq \frac{C_1c_\gamma}{n(n + \mu + 1)(n + \mu + 2)} (1 + t)^{-\mu-\frac{n+\mu-1}{2}} (t - T_0)^{-\gamma+n+\mu+2}
\]
for any \( t \geq T_0 \). In other words, we have already obtained the first estimate of the functional \( F(t) \) from below by

\[
F(t) \geq K_0 (1 + t)^{-\alpha_0} (t - T_0)^{\beta_0}
\]

for any \( t \geq T_0 \), where the multiplicative constant is defined by

\[
K_0 := \frac{C_1 c_\gamma}{n(n + \mu + 1)(n + \mu + 2)}
\]

and the exponents are given by \( \alpha_0 := \mu + (n + \mu - 1)p/2 \) and \( \beta_0 := -\gamma + n + \mu + 2 \).

Finally, from (4.16) and (4.20), we apply generalized Kato’s type lemma to get blow-up for the functional \( F(t) \) if

\[
-\frac{(n + \mu) - 1}{2} p^2 + \left( \frac{(n + \mu) + 1}{2} + 1 - \gamma \right) p + 1 > 0.
\]

It completes the proof of Theorem 2.2.

5 Proof of Theorem 2.3: Blow-up in the critical case when \( \mu = 2 \)

The purpose of this section devotes to the blow-up result for (1.1) with \( \mu = 2 \) in the case \( p = p_0(n + 2, \gamma) \). We will employ iteration method, which are strongly motivated by the recent paper [29]. For this reason, we just sketch the proof.

5.1 Auxiliary functions

To begin with this part, let us recall a pair of auxiliary functions introduced by [29]. These functions play a significant role in the procedure of iteration in the critical case \( p = p_0(n + 2, \gamma) \).

Let \( r > -1 \) be a real parameter. Let us introduce the following functions:

\[
\xi_r(t, x) := \int_0^{\lambda_0} e^{-\lambda(R+t)} \cosh(\lambda t) \Phi(\lambda x) \lambda^r d\lambda,
\]

\[
\eta_r(t, s, x) := \int_0^{\lambda_0} e^{-\lambda(R+t)} \sinh(\lambda(t-s)) \lambda(t-s)^{\gamma} \Phi(\lambda x) \lambda^r d\lambda,
\]

where \( \lambda_0 \) is a fixed positive parameter and \( \Phi \) is defined by (4.17).

Next, we will show some useful properties of these auxiliary functions \( \xi_r \) and \( \eta_r \). One may check the lemma in [29, Lemma 3.1]. Actually, the parameters shown in Lemma 3.1 in the paper [29] can be relaxed into \( r > -1 \) and \( n = 1 \).

Lemma 5.1. Let \( n \geq 1 \). There exist \( \lambda_0 > 0 \) such that

(i) if \( r > -1 \), \( |x| \leq R \) and \( t \geq 0 \), the estimates hold

\[
\xi_r(t, x) \geq A_0 \quad \text{and} \quad \eta_r(t, 0, x) \geq B_0(t)^{-1};
\]

(ii) if \( r > -1 \), \( |x| \leq s + R \) and \( t > s \geq 0 \), the estimate holds

\[
\eta_r(t, s, x) \geq B_1(t)^{-1}(s)^{-r};
\]
(iii) if \(r > (n-3)/2\), \(|x| \leq R + t\) and \(t > 0\), the estimate holds
\[
\eta_r(t, t, x) \leq B_2(t)^3 \frac{n-1}{r} (t - |x|)^{\frac{n-3}{r} - r}.
\]

Here, \(A_0\) and \(B_k\), with \(k = 0, 1, 2\), are positive constants depending on \(\lambda_0\), \(r\) and \(R\) only and we take the notation \(\langle y \rangle_3 := 3 + |y|\).

**Proposition 5.1.** Let \(r > -1\). Let us assume that \(v_0 \in H^1(\mathbb{R}^n)\) and \(v_1 \in L^2(\mathbb{R}^n)\) are nonnegative, nontrivial and compactly supported in \(B_R\) functions. Moreover, let \(v\) be an energy solution to (2.4) on \([0, T)\) according to Definition 2.2. Then, the next integral equation holds:
\[
\int_{\mathbb{R}^n} v(t, x) \eta_r(t, t, x) dx = \int_{\mathbb{R}^n} v_0(x) \xi_r(t, x) dx + t \int_{\mathbb{R}^n} v_1(x) \eta_r(t, 0, x) dx \\
+ c_\gamma \int_0^t (t - s)(1 + s)^{-\gamma} (1 + \tau)^{-p} \int_{\mathbb{R}^n} |v(\tau, x)|^p \eta_r(t, s, x) dx d\tau ds,
\]
for any \(t \in (0, T)\), where the functions \(\xi_r\) and \(\eta_r\) have been defined in (5.1) and (5.2).

**Proof.** Let us choose the test function \(\psi = \psi(s, x)\) in the definition of energy solutions by
\[
\psi(s, x) = \lambda^{-1} \sinh(\lambda(t - s)) \Phi(\lambda x),
\]
where the space-dependent test function \(\Phi\) is given in (4.17). Clearly, the function
\[
y(t, s; \lambda) := \lambda^{-1} \sinh(\lambda(t - s))
\]
is the solution to the \(\lambda\)-dependent ordinary differential equation \((\partial_s^2 - \lambda^2)y(t, s; \lambda) = 0\) associated with the conditions \(y(t, t; \lambda) = 0\) and \(\partial_s y(t, t; \lambda) = -1\).

Due to the fact that \(\Phi\) is an eigenfunction of the Laplace operator, from the relation \(\psi(s, x) = y(t, s; \lambda) \Phi(\lambda x)\), it tells us that the function \(\psi\) solves the wave equation \(\psi_{ss} - \Delta \psi = 0\). Therefore, applying the derived properties of \(\psi\) in (2.5), and multiplying the resultant equation by \(e^{-(\lambda(R + t))\lambda r}\), taking integration with respect to \(\lambda\) over \([0, \lambda_0]\) and using Tonelli’s theorem, we may complete the proof of the proposition. \(\Box\)

### 5.2 Iteration argument

Let us consider \(v\) be an energy solution of (2.4) on \([0, T)\). Then, we may construct a time-dependent functional
\[
G(t) := \int_{\mathbb{R}^n} v(t, x) \eta_r(t, t, x) dx,
\]
with the parameter \(r := (n-1)/2 - 1/p\). According to (5.3), we immediately conclude that \(G(t) \geq 0\) for any \(t \geq 0\), where we used the nonnegative assumption on initial data.

Our aim in the next step is to derive an integral inequalities involving \(G(t)\), i.e. the construction of the iteration frame. Basically, similarly as Proposition 4.2 in [29], by applying Hölder’s inequality and (ii) and (iii) in Lemma 5.1, we can conclude the next proposition.
Proposition 5.2. Let us assume that \( r = (n-1)/2 - 1/p \). Let \( G(t) \) be the time-dependent functional introduced in (5.4). Then, there exist positive constants \( C_2 \) depending on \( n, p, \gamma, \lambda_0, R \) such that the estimate

\[
G(t) \geq C_2(t)^{-1} \int_0^t (t-s)(1+s)^{\frac{3}{2} + \frac{1}{p}} \int_0^s (s-\tau)^{-\gamma} \langle \tau \rangle_3^{\frac{n+1}{2} p - np + n - 1} \frac{(G(\tau))^p}{(\log(\tau))^{p-1}} d\tau ds
\]

holds for any \( t \geq 0 \).

Proposition 5.3. Let us assume that \( r = (n-1)/2 - 1/p \) and \( p = p_0(n+2, \gamma) \). Let \( G(t) \) be the functional introduced in (5.4). Then, there exist positive constant \( C_7 \) depending on \( n, p, \gamma, \lambda_0, R, v_0, v_1 \) such that

\[
G(t) \geq C_7 \log(2t/3)
\]

holds for any \( t \geq 3/2 \).

Proof. Firstly, let us introduce the test function \( \tilde{\Psi} = \tilde{\Psi}(t,x) \) with separate variables denoting by \( \tilde{\Psi}(t,x) := e^{-\Phi(x)} \). One may observe that the function \( \tilde{\Psi} \) solves the wave equation and its adjoint equation \( \tilde{\Psi}_{tt} - \Delta \tilde{\Psi} = 0 \).

Let us denote the auxiliary functional by

\[
G_0(t) := \int_{\mathbb{R}^n} v(t,x) \tilde{\Psi}(t,x) dx.
\]

Then, the application of Hölder’s inequality yields

\[
\int_{\mathbb{R}^n} |v(\tau,x)|^p dx \geq |G_0(\tau)|^p \left( \int_{B_{R+\tau}} |\tilde{\Psi}(\tau,x)|^{\frac{p}{p-1}} dx \right)^{(p-1)}.
\]

Providing that we derive a lower bound estimate for the functional \( G_0(\tau) \), the last inequality will give us a lower bound for the left-hand side of (5.7). Again, according to [31, 21] the time-dependent functional \( G_0(t) \) fulfills the next inequality:

\[
G_0(t) \geq \frac{1}{2} \left( 1 - e^{-2t} \right) \int_{\mathbb{R}^n} (v_0(x) + v_1(x)) \Phi(x) dx + e^{-2t} \int_{\mathbb{R}^n} v_0(x) \Phi(x) dx \geq C_3
\]

for any \( t \geq 0 \) with a suitable positive constant \( C_3 = C_3(v_0, v_1) \), where we considered our assumption on initial data that \( v_0 \neq 0 \). By the asymptotic behavior of the test function \( \tilde{\Psi} \), the following estimate holds (cf. [31, Estimate (2.5)]):

\[
\int_{B_{R+\tau}} |\tilde{\Psi}(\tau,x)|^{\frac{p}{p-1}} dx \leq C_4(R + \tau)^{(n-1)(1-\frac{p}{2})}.
\]

So, from (5.7) we have

\[
\int_{\mathbb{R}^n} |v(\tau,x)|^p dx \geq C_5(R + \tau)^{n-1-\frac{n-1}{p}}
\]

for any \( \tau \geq 0 \), where \( C_5 = C_5^p C_4^{1-p} \).

Obviously, (5.8) may be rewritten by the form

\[
\int_{\mathbb{R}^n} |v(\tau,x)|^p dx \geq C_5(\tau)^{n-1-\frac{n-1}{p}}
\]

(5.9)
for any $\tau \geq 1$ up to a modification of the multiplicative constant. Let us apply (5.3), Lemma 5.1 (ii) and (5.9) to have

$$G(t) \geq c_\gamma \int_0^t (t-s)(1+s) \int_0^s (s-\tau)^{-\gamma}(1+\tau)^{-p} \int_{\mathbb{R}^n} |v(\tau, x)|^p \eta_r(t, s, x) dxd\tau ds$$

$$\geq B_1c_\gamma(t)_3^{-1} \int_0^t (t-s)(1+s)s_3^{n+1+p} \int_{s/2}^s (s-\tau)^{-\gamma}(1+\tau)^{-p} \int_{\mathbb{R}^n} |v(\tau, x)|^p dxd\tau ds$$

$$\geq B_1C_\gamma(t)_3^{-1} \int_1^t (t-s)(1+s)s_3^{n+1+p} \int_1^{s}(s-\tau)^{-\gamma}(1+\tau)^{-p} \tau_3^{n+1-n_p} d\tau ds.$$

For $t \geq 2$, by shrinking the domain of integration we derive

$$G(t) \geq B_1C_\gamma(t)_3^{-1} \int_2^t (t-s)(1+s)s_3^{n+1+p} \int_{s/2}^{s} (s-\tau)^{-\gamma}(1+\tau)^{-p} \tau_3^{n+1-n_p} d\tau ds$$

$$\geq C_6(t)_3^{-1} \int_2^t (t-s)(1+s)^{-p} s_3^{n+1-p} \int_{s/2}^{s} (s-\tau)^{-\gamma} s^{1-\gamma} ds$$

$$\geq C_7(t)_3^{-1} \int_2^t (t-s)s_3^{1} ds,$$

for some constants $C_6, C_7 > 0$, where we considered $p = p_0(n+2, \gamma)$ resulting

$$-\frac{n+1}{2}p + \frac{n+1}{2} + 1 - \gamma + \frac{1}{p} = -1. \quad (5.10)$$

For $t \geq 3$, by shrinking the domain of the integration from $[2, t]$ to $[2t/3, t]$ and using $(t)_3^{-1} \geq (3t)_3^{-1}$, one may immediately get

$$G(t) \geq CC_7(3t)_3^{-1} \int_{2t/3}^t \log s ds \geq C_8 \log (2t/3)$$

for some positive constants $C_8$. It completes the proof.

Finally, by following the similar procedure as [5], i.e. iteration argument by slicing procedure developed in [1], we may conclude

$$G(t) \geq \exp \left( p^j \log \left( 2^{-\frac{2p}{p-1}} L_0 \log (t)^{1/(p-1)} \right) \right) \log (t)_3 \log (t/2)^{-1/(p-1)} \quad (5.11)$$

for $t \geq 4$ with a positive constant $L_0 := L_0(n, p, \gamma)$. By concerning

$$t > \exp \left( 2^{2p-1} L_0^{-p} \right) \quad \text{so that} \quad 2^{-\frac{2p}{p-1}} L_0 \log (t)^{1/(p-1)} > 1,$$

and letting $j \to \infty$, we may observe that the lower bound for the functional $G(t)$ blows up. Namely, the energy solution $v$ to the Cauchy problem (2.4) does not globally (in time) existence.

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