COUNTING LATTICE POINTS IN THE MODULI SPACE OF CURVES.

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Abstract. We show how to define and count lattice points in the moduli space \( \mathcal{M}_{g,n} \) of genus \( g \) curves with \( n \) labeled points. This produces a polynomial with coefficients that include the Euler characteristic of the moduli space, and tautological intersection numbers on the compactified moduli space.

1. Introduction

Let \( \mathcal{M}_{g,n} \) be the moduli space of genus \( g \) curves with \( n \) labeled points. The decorated moduli space \( \mathcal{M}_{g,n} \times \mathbb{R}^n_+ \) equips the labeled points with positive numbers \( (b_1, ..., b_n) \). It has a cell decomposition due to Penner, Harer, Mumford and Thurston

\[
\mathcal{M}_{g,n} \times \mathbb{R}^n_+ \cong \bigcup_{\Gamma \in \text{Fat}_{g,n}} P_{\Gamma}
\]

where the indexing set \( \text{Fat}_{g,n} \) is the space of labeled fatgraphs of genus \( g \) and \( n \) boundary components. See Section 2 for definitions of a fatgraph \( \Gamma \), its automorphism group \( \text{Aut}\Gamma \) and the cell decomposition (1) realised as the space of labeled fatgraphs with metrics. Restricting this homeomorphism to a fixed \( n \)-tuple of positive numbers \( (b_1, ..., b_n) \) yields a space homeomorphic to \( \mathcal{M}_{g,n} \) decomposed into compact convex polytopes \( P_{\Gamma}(b_1, ..., b_n) \). When the \( b_i \) are positive integers the polytope \( P_{\Gamma}(b_1, ..., b_n) \) is an integral polytope and we define \( N_{\Gamma}(b_1, ..., b_n) \) to be its number of positive integer points. The weighted sum of \( N_{\Gamma} \) over all labeled fatgraphs of genus \( g \) and \( n \) boundary components is the lattice count polynomial:

\[
N_{g,n}(b_1, ..., b_n) = \sum_{\Gamma \in \text{Fat}_{g,n}} \frac{1}{|\text{Aut}\Gamma|} N_{\Gamma}(b_1, ..., b_n)
\]

Each integral point in the polytope \( P_{\Gamma}(b_1, ..., b_n) \) corresponds to a Dessin d’enfants defined by Grothendieck [2] which represents a curve in \( \mathcal{M}_{g,n} \) defined over \( \overline{\mathbb{Q}} \). Thus the lattice count polynomial \( N_{g,n}(b_1, ..., b_n) \) counts curves defined over \( \overline{\mathbb{Q}} \). This is described in Section 2 where the integral points in \( P_{\Gamma}(b_1, ..., b_n) \) represent metrics on labeled fatgraphs with integer edge lengths, or equivalently curves equipped with a canonical meromorphic quadratic (Strebel) differential with integral residues.

Quite generally the number of integer points in a convex polytope is a piecewise defined polynomial. Nevertheless the following theorem shows that a weighted sum of the piecewise defined polynomials \( N_{\Gamma}(b_1, ..., b_n) \) is a polynomial.

Theorem 1. The number of lattice points \( N_{g,n}(b_1, ..., b_n) \) is a degree \( 3g - 3 + n \) polynomial in the integers \( (b_1^2, ..., b_n^2) \) depending on the parity of the \( b_i \).

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The dependence on the parity means that \( N_{g,n}(b_1, \ldots, b_n) \) is represented by \( 2^n \) polynomials (by symmetry at most \( \frac{n(n+1)}{2} + 2 \) are different.) The polynomials are symmetric under permutations of \( b_i \) of the same parity. If the number of odd \( b_i \) is odd then \( N_{g,n}(b_1, \ldots, b_n) = 0. \) Otherwise, the top degree homogeneous part of \( N_{g,n}(b_1, \ldots, b_n) \) is independent of the parity. Table 1 shows the simplest polynomials.

Harer and Zagier [5] calculated the orbifold Euler characteristic \( \chi(M_{g,1}) \) and Penner [10] calculated \( \chi(M_{g,n}) \) for general \( n. \) This information is encoded in the lattice count polynomial for all even \( b_i. \)

**Theorem 2.** \( N_{g,n}(0, \ldots, 0) = \chi(M_{g,n}). \)

Kontsevich [6] defined the volume polynomial

\[
V_{g,n}(b_1, \ldots, b_n) = \sum_{\Gamma \in \mathcal{F}_{g,n}} \frac{1}{|Aut\Gamma|} Vol_{\Gamma}(b_1, \ldots, b_n)
\]

where \( Vol_{\Gamma}(b_1, \ldots, b_n) \) is the volume of the convex polytope \( P_{\Gamma}(b_1, \ldots, b_n). \) (The Laplace transform of \( V_{g,n} \) appears as \( I_g \) in [6].) He showed that the coefficients give intersection numbers of Chern classes of the tautological line bundles \( L_i \) over the compactified moduli space \( \overline{M}_{g,n}. \) By considering finer and finer meshes it follows that the homogeneous top degree part of the lattice point count polynomial is the volume polynomial.

**Theorem 3.** \( N_{g,n}(b_1, \ldots, b_n) = V_{g,n}(b_1, \ldots, b_n) + \text{lower order terms}. \)

**Corollary 1.** For \( |d| = \sum_i d_i = 3g - 3 + n \) and \( d! = \prod d_i! \) the coefficient \( c_d \) of \( b^{2d} = \prod b_i^{2d_i} \) in \( N_{g,n}(b_1, \ldots, b_n) \) is the intersection number

\[
c_d = \frac{1}{2^{6g-6+2n-3|d|}} \int_{\overline{M}_{g,n}} c_1(L_1)^{d_1} \cdots c_1(L_n)^{d_n}.
\]

Kontsevich proved that these tautological intersection numbers satisfy a recursion relation conjectured by Witten [12] that determine the intersection numbers. The lattice count polynomials satisfy a recursion relation that uniquely determine the polynomials and when restricted to the top degree terms imply Witten’s recursion.

**Table 1.** Lattice count polynomials for even \( b_i \)

| g | n | \( N_{g,n}(b_1, \ldots, b_n) \) |
|---|---|------------------|
| 0 | 3 | 1 |
| 1 | 1 | \( \frac{1}{18} (b_1^2 - 4) \) |
| 0 | 4 | \( \frac{1}{4} (b_1^2 + b_2^2 + b_3^2 + b_4^2 - 4) \) |
| 1 | 2 | \( \frac{1}{36} (b_1^2 + b_2^2 - 4) (b_2^2 + b_3^2 - 8) \) |
| 2 | 1 | \( \frac{1}{2^{10} \cdot 15} (b_1^2 - 4) (b_1^2 - 16) (b_1^2 - 36) (5b_1^2 - 32) \) |
Theorem 4. The lattice count polynomials satisfy the following recursion relation which determines the polynomials uniquely from $N_{0,3}$ and $N_{1,1}$.

\[
\left( \sum_{i=1}^{n} b_i \right) N_{g,n}(b_1, \ldots, b_n) = \sum_{i \neq j} \sum_{p+q=b_i+b_j} pq N_{g,n-1}(p, b_1, \ldots, \hat{b}_i, \ldots, b_n) + \sum_{i} \sum_{p+q+r=b_i} pqr \left[ N_{g-1,n+1}(p, q, b_1, \ldots, \hat{b}_i, \ldots, b_n) \right] + \sum_{g_1+g_2=g} \sum_{\text{I} \cup \text{J} = \{1, \ldots, n\}} \left[ N_{g_1,|\text{I}|}(p, b_I) \right] \left[ N_{g_2,|\text{J}|}(q, b_J) \right]
\]

The proof of Theorem 4 is elementary. The recursion relation (2) is used to prove Theorem 1. It resembles Mirzakhani’s recursion relation [7] between polynomials giving the Weil-Petersson volume of the moduli space. In fact the top homogeneous degree part of $N_{g,n}(b_1, \ldots, b_n)$ coincides with the top homogeneous degree part of Mirzakhani’s Weil-Petersson volume polynomial (after multiplying by an appropriate power of 2) since both of these coincide with Kontsevich’s volume. Mirzakhani [8] already showed the coefficients of the Weil-Petersson volume polynomial are the intersection numbers given in Corollary 1. Do and Safnuk [2] use fatgraphs to give a simpler proof of Mirzakhani’s recursion relation restricted to the top homogeneous degree part and show that it is a rescaled version of Mirzakhani’s proof.

Although Table 1 shows only even $b_i$, the recursion relation needs the odd cases too. We will fill in the cases of odd $b_i$ here. When $\sum b_i$ is odd, $N_{g,n}(b_1, \ldots, b_n) \equiv 0$. The polynomial $N_{0,4}(b_1, \ldots, b_4)$ is the same as in the table when $b_1, \ldots, b_4$ are all odd, and when exactly two of the $b_i$ are odd $N_{0,4}(b_1, \ldots, b_4) = \frac{1}{4} \left( b_1^2 + b_2^2 + b_3^2 + b_4^2 - 2 \right)$. For genus 1 when $b_1$ and $b_2$ are odd $N_{1,2}(b_1, b_2) = \frac{1}{432} \left( b_1^2 + b_2^2 - 2 \right) \left( b_1^2 + b_2^2 - 10 \right)$.

Section 2 contains preliminaries on fatgraphs and lattice point counting. Theorems 1 and 4 are proven in Section 2.2. Section 2.3 contains a simple vanishing result for $N_{g,n}(b_1, \ldots, b_2)$ which has powerful consequences. In Section 3 we prove Theorem 2 and treat the special case of $n = 1$ labeled points.

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2. Fatgraphs

A fatgraph is a graph $\Gamma$ with vertices of valency $> 2$ equipped with a cyclic ordering of edges at each vertex. In Figure 1 we use the projection to define the cyclic ordering to be anticlockwise at each vertex. The two pictured fatgraphs are

![Fatgraphs](image-url)
different, although the underlying graphs are the same. A fatgraph structure on
a graph is equivalent to an embedding of a graph into a surface \( \Gamma \to \Sigma \) such that
\( \Sigma - \Gamma \) is a union of disks. This gives a genus \( g \) and number of boundary components
\( n \) to \( \Gamma \). The examples in Figure 1 have genus 0 and 1 shown in Figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Graphs embedded in genus 0 and 1 surfaces}
\end{figure}

A labeled fatgraph is a fatgraph with boundary components labeled \( 1, \ldots, n \). The
set of all labeled fatgraphs of genus \( g \) and \( n \) boundary components is notated by
\( \mathcal{F}_{g,n} \).

It is useful to describe a fatgraph in the following equivalent way [6] which makes
the automorphisms transparent. Given a graph \( \Gamma \) with vertices of valency > 2, let
\( X \) be the set of oriented edges, so each edge of \( \Gamma \) appears in \( X \) twice. Define the
map \( \tau_1 : X \to X \) that flips the orientation of each edge. A fatgraph, or ribbon,
structure on \( \Gamma \) is a map \( \tau_0 : X \to X \) that permutes cyclically the oriented edges with
a common source vertex. Let \( X_0, X_1 \) and \( X_2 \) be the vertices, edges and boundary
components of the fatgraph \( \Gamma \). Then \( X_0 = X/\tau_0 \), \( X_1 = X/\tau_1 \) and \( X_2 = X/\tau_2 \) for
\( \tau_2 = \tau_0 \tau_1 \). An automorphism of the labeled fatgraph \( \Gamma \) is a permutation
\( \phi : X \to X \) that commutes with \( \tau_0 \) and \( \tau_1 \) and acts trivially on \( X_2 \). The examples in Figure 1
given any labeling have automorphism groups \( \{1\} \) and \( \mathbb{Z}_6 \).

A metric on a labeled fatgraph \( \Gamma \) assigns positive numbers—lengths—to each
edge of the fatgraph. If \( \Gamma \in \mathcal{F}_{g,n} \) then the valency > 2 conditions on the vertices
ensures that the number of edges \( e(\Gamma) \) of \( \Gamma \) is bounded \( e(\Gamma) \leq 6g - 6 + 3n \). Let \( P_\Gamma \)
be the \( 6g - 6 + 3n \) cell consisting of all metrics on \( \Gamma \). Construct the cell-complex
\[ M_{g,n}^{\text{combinatorial}} = \bigcup_{\Gamma \in \mathcal{F}_{g,n}} P_\Gamma \]
where we identify isometric metrics on fatgraphs, and when the length of an edge
\( l_E \to 0 \) we identify this with the metric on the fatgraph with the edge \( E \) contracted.
By the existence and uniqueness of meromorphic quadratic differentials with foli-
ations having compact leaves, known as Strebel differentials, the cell complex is
homeomorphic to the decorated moduli space \( M_{g,n}^{\text{combinatorial}} \cong M_{g,n} \times \mathbb{R}_+^n \) [4].

Denote by \( P_\Gamma(b_1, \ldots, b_n) \subset P_\Gamma \) the metrics on \( \Gamma \) with fixed boundary lengths
\( b = (b_1, \ldots, b_n) \in \mathbb{R}_+^n \) or equivalently with specified residues of the (square root of
the) associated Strebel differential. Then
\[ M_{g,n}^{\text{combinatorial}}(b_1, \ldots, b_n) = \bigcup_{\Gamma \in \mathcal{F}_{g,n}} P_\Gamma(b_1, \ldots, b_n) \cong M_{g,n}. \]

2.1. Counting lattice points in convex polytopes. A convex polytope \( P \subset \mathbb{R}^n \)
can be defined as the convex hull of a finite set of vertices in \( \mathbb{R}^n \). We will consider
integral polytopes \( P \) where the vertices lie in \( \mathbb{Z}^n \). Define the number of integral points
in \( P \) by \( N_P = \# \{ P \cap \mathbb{Z}^n \} \) and \( N_P(k) = \# \{ kP \cap \mathbb{Z}^n \} \) where \( kP \) rescales
Also, define $N_P^0(k)$ to be the number of integral points in the interior of $kP$.

**Theorem 2.1 (Ehrhart).** If $P \subset \mathbb{R}^n$ is an $n$-dimensional convex polytope then

$$N_P(k) = \text{Vol}(P)k^n + ...$$

is a degree $n$ polynomial in $k$ with top coefficient the volume of $P$. Furthermore,

$$N_P^0(k) = (-1)^n N_P(-k).$$

We can define a convex polytope with positive codimension as follows. Given a linear map $A : \mathbb{R}^N \to \mathbb{R}^n$ and $b \in \mathbb{R}^n$ define $P_A(b) = \{ x \in \mathbb{R}_+^N | Ax = b \}$. If $A$ and $b$ have integer entries (with respect to the standard bases) then $P_A(b)$ is integral and we define $N_{P_A}(b) = \# \{ P_A \cap \mathbb{Z}^N \}$. In this case $N_{P_A}(b)$ is a piecewise defined polynomial in $b$ - for example, $N_{P_A}(b)$ may be zero for some values of $b$.

The set $P_{\Gamma}(b)$ in (3) is a convex polytope defined by solutions $x \in \mathbb{R}^{e(\Gamma)}$ of

$$A_F x = b$$

where $A_F$ is the incidence matrix that maps the vector space generated by edges of $\Gamma$ to the vector space generated by boundary components of $\Gamma$—an edge maps to the sum of its two incident boundary components. In the examples in Figure 1 the incidence matrices are

$$A_F = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} , \quad A_{F'} = \begin{pmatrix} 2 & 2 & 2 \end{pmatrix}.$$ 

We define

$$N_{\Gamma}(b) = \# \{ P_A \cap \mathbb{Z}^N \}.$$ 

It is natural to allow non-negative solutions although we allow only positive integer solutions. This is justified by the fact that if some of the $x_i$ vanish then this will be counted using a fatgraph obtained by collapsing edges of $\Gamma$. (If the collapsing of edges of $\Gamma$ does not yield a fatgraph, for example collapsing a loop, then we do not want to count such solutions.)

Since each edge is incident to exactly two (not necessarily distinct) boundary components the columns of $A_{\Gamma}$ add to 2, or equivalently $(1, 1, ..., 1) \cdot A_{\Gamma} = (2, 2, ..., 2)$. Thus,

$$\sum b_i = (1, 1, ..., 1) \cdot b = (1, 1, ..., 1) \cdot A_{\Gamma} x = (2, 2, ..., 2) \cdot x \in 2\mathbb{Z}$$

so $N_{\Gamma}(b) = 0$ if $\sum b_i$ is odd. Hence the lattice count polynomial $N_{g,n}(b_1, ..., b_n)$ given in Definition 3 also vanishes when $\sum b_i$ is odd.

If we relax the condition on fatgraphs that the valency of each vertex must be $> 2$ then Grothendieck [3] showed that fatgraphs with all edge lengths 1 possess branched covers of $\mathbb{P}^1$ branched over 0, 1 and $\infty$. By a theorem of Belyi these correspond to curves defined over $\overline{\mathbb{Q}}$. When the length of each edge is a positive integer this is the same as a string of length 1 edges joined by valency 2 vertices. Thus, $N_{g,n}(b_1, ..., b_n)$ counts curves defined over $\overline{\mathbb{Q}}$ branched over of 0, 1, $\infty \in \mathbb{P}^1$ with all points over 1 of ramification 2, and all points over 0 of ramification $> 2$. 
For a convex polytope $P \subset \mathbb{R}^N$ and a polynomial $\phi$ on $\mathbb{R}^N$ define the following generalisation of counting lattice points.

$$N_P(\phi, k) = \sum_{x \in kP \cap \mathbb{Z}^N} \phi(x)$$

and $N_P^0(\phi, k)$ the sum over interior integer points of $kP$. Later when applying the recursion relation we will need to calculate sums with a parity restriction as in Lemma 1 because the polynomials $N_{g,n}$ vanish if the sum of the arguments is odd.

**Lemma 1.**

\begin{equation}
S_m(k) = \sum_{p+q=k, q \text{ even}} p^{2m+1} q, \quad R_{m,m'}(k) = \sum_{p+q+r=k, r \text{ even}} p^{2m+1} q r^{2m'+1}
\end{equation}

are odd polynomials in $k$ of degree $2m + 3$, respectively $2m + 2m' + 5$, depending on the parity of $k$.

**Proof.** The dependence on the parity means that there are two polynomials $S_m^{\text{even}}(k)$ and $S_m^{\text{odd}}(k)$ depending on whether $k$ is even or odd. The same is true for $R_{m,m'}(k)$. Notice that

$$S_m(k) = 2N_P(\phi_1, k)$$

for $P = \{(x, y) \in \mathbb{R}^2_+ | x + 2y = 1\}$ and $\phi_1 = x^{2m+1}y$ (substitute $q = 2Q$.) Similarly,

$$R_{m,m'}(k) = 2N_{P'}(\phi_2, k)$$

for $P' = \{(x, y, z) \in \mathbb{R}^3_+ | x + y + 2z = 1\}$ and $\phi_2 = x^{2m+1}y^{2m'+1}z$.

The polytopes $P$ and $P'$ are rational, not integral. They can be expressed in terms of the integral convex polytopes of higher dimension

$$P_1 = \{x \geq 0, y \geq 0, x + 2y \leq 2\}, \quad P_2 = \{x \geq 0, y \geq 0, z \geq 0, x + y + 2z \leq 2\}.$$

For $k$ even

$$S_m^{\text{even}}(k) = N_{P_1}(\phi_1, \frac{k}{2}) - N_{P_1}^0(\phi_1, \frac{k}{2}), \quad R_{m,m'}^{\text{even}}(k) = N_{P_2}(\phi_2, \frac{k}{2}) - N_{P_2}^0(\phi_2, \frac{k}{2}).$$

A generalisation of Ehrhart’s theorem states that for a dimension $n$ integral convex polytope $P \subset \mathbb{R}^n$, $N_P(\phi, k)$ is a degree $\deg \phi + n$ polynomial in $k$ and

$$N_P^0(\phi, k) = (-1)^{\deg \phi + n} N_P(\phi, -k).$$

For the cases at hand, $\deg \phi + n$ is even so the right hand side is $N_P(\phi, -k)$ and $S_m^{\text{even}}(k)$ and $R_{m,m'}^{\text{even}}(k)$ are odd polynomials in $k$ of degree $2m + 3$, respectively $2m + 2m' + 5$. For $k$ odd,

$$S_m^{\text{odd}}(k) = N_{P_1}(\phi_1, \frac{k+1}{2}) - N_{P_1}(\phi_1, \frac{k-1}{2})$$

and $R_{m,m'}^{\text{odd}}(k)$ is the same expression with $P_2$ in place of $P_1$. Once again $S_m^{\text{odd}}(k)$ and $R_{m,m'}^{\text{odd}}(k)$ are odd polynomials in $k$ of degree $2m + 3$, respectively $2m + 2m' + 5$. □
2.2. Recursion.

Proof of Theorem 4. The lattice count polynomial $N_{g,n}(b_1,\ldots,b_n)$ counts labeled fatgraphs with positive integer edge lengths which we call integer fatgraphs in $P_\Gamma(b_1,\ldots,b_n)$. We can produce an integer fatgraph in $P_\Gamma(b_1,\ldots,b_n)$ from simpler integer fatgraphs in the three ways shown in Figures 3, 4 and 5. Choose a graph in $P_\Gamma(p,b_3,\ldots,b_n)$ and add an edge of length $q/2$ inside the boundary of length $p$ as in Figure 3 so that $p+q = b_1 + b_2$. Similarly, attach an edge and a loop of total length $q/2$ inside the boundary of length $p$ as in Figure 4 so that $p+q = b_1 + b_2$. In both cases for each $\Gamma'$ there are $p$ possible ways to attach the edge so this construction contributes $pN_{g,n-1}(p,b_3,\ldots,b_n)$ to $N_{g,n}(b_1,\ldots,b_n)$. However we have overcounted, particularly when we repeat this construction for any pair $b_i$ and $b_j$, since each integer fatgraph in $P_\Gamma(b_1,\ldots,b_n)$ can be produced in many ways like this. To deal with this, we overcount even further by taking $pqN_{g,n-1}(p,b_3,\ldots,b_n)$, i.e. taking each constructed fatgraph $q$ times. But now we see that for each edge that we attach of length $q/2$ we have overcounted $q$ times. If we were to use all of the edges of $\Gamma$ in this way then we would have overcounted by

$$\sum_{E \in \Gamma} t(E) = \sum_{i=1}^n b_i.$$  

Indeed all of the edges of $\Gamma$ are used, exactly once, when we include one further construction of the integer fatgraph $\Gamma$. 

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**Figure 3.** $\Gamma$ is obtained from a simpler fatgraph by adding the broken line.

**Figure 4.** $\Gamma$ is obtained by adding a line and loop of total length $q/2$. 
Choose an integer fatgraph in $P_{\Gamma'}(p, q, b_2, ..., b_n)$ for $\Gamma' \in \text{Fat}_{g-1,n+1}$ or choose two integer fatgraphs in $P_{\Gamma_1}(p, b_2, ..., b_j)$ and $P_{\Gamma_2}(q, b_{j+1}, ..., b_n)$ for $\Gamma_1 \in \text{Fat}_{g_1,j}$ and $\Gamma_2 \in \text{Fat}_{g_2,n+1-j}$ where $g_1 + g_2 = g$ and attach an edge of length $r/2$ connecting these two boundary components as in Figure 5 so that $p + q + r = b_1$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{\(\Gamma\) is obtained from a single fatgraph or two disjoint fatgraphs by adding the broken line.}
\end{figure}

In the diagram, the two boundary components of lengths $p$ and $q$ are part of a fatgraph that may or may not be connected. There are $pq$ possible ways to attach the edge so this construction contributes $pqN_{g-1,n+1}(p, q, b_2, ..., b_n)$ and $pqN_{g_1,j}(p, b_2, ..., b_j)N_{g_2,n+1-j}(q, b_{j+1}, ..., b_n)$ to $N_{g,n}(b_1, ..., b_n)$ and again we have overcounted. We overcount further by a factor of $r$ to get $pqrN_{g-1,n+1}(p, q, b_2, ..., b_n)$ and $pqrN_{g_1,j}(p, b_2, ..., b_j)N_{g_2,n+1-j}(q, b_{j+1}, ..., b_n)$. We repeat this for each $g_1 + g_2 = g$ and $I \cup J = \{2, ..., n\}$ and then for each $b_j$ in place of $b_1$.

As previewed above, each edge of $\Gamma$ has been attached to construct $\Gamma$ and $N_{g,n}(b_1, ..., b_n)$ has been overcounted $\sum_{i=1}^{n} b_i$ times yielding (2).

Remark. The idea in the proof above to overcount by the length of each edge of the graph $\Gamma$ comes from the similar idea introduced by Mirzakhani [7] where she unfolds a function on Teichmüller space that sums to the analogue of $b_1$.

To apply the recursion we need to first calculate $N_{0,3}(b_1, b_2, b_3)$ and $N_{1,1}(b_1)$. There are seven labeled fatgraphs in $\text{Fat}_{0,3}$ coming from three unlabeled fatgraphs. It is easy to see that $N_{0,3}(b_1, b_2, b_3) = 1$ if $b_1 + b_2 + b_3$ is even (and 0 otherwise.) This is because for each $(b_1, b_2, b_3)$ there is exactly one of the seven labeled fatgraphs $\Gamma$ with a unique solution of $A_{\Gamma}\mathbf{x} = \mathbf{b}$ while the other six labeled fatgraphs yield no solutions. For example, if $b_1 > b_2 + b_3$ then only the fatgraph $\Gamma$ with $A_{\Gamma} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has a solution and that solution is unique.

To calculate $N_{1,1}(b_1)$, note that $A_{\Gamma} = [2 \ 2 \ 2]$ or $[2 \ 2]$ for the 2-vertex and 1-vertex fatgraphs. Hence

$$N_{1,1}(b_1) = a_1 \left( \frac{b_1}{2} - 1 \right) + a_2 \left( \frac{b_1}{2} - 1 \right)$$

where $a_1$ is the number of trivalent fatgraphs (weighted by automorphisms) and $a_2$ is the number of 1-vertex fatgraphs. The genus 1 graph $\Gamma$ from Figure 4 has $|\text{Aut}\Gamma| = 6$ so $a_1 = 1/6$, and $a_2$ uses the genus 1 figure 8 fatgraph which has
automorphism group \( \mathbb{Z}_4 \) hence \( a_2 = 1/4 \). Thus
\[
N_{1,1}(b_1) = \frac{1}{6} \left( \frac{b_1}{2} - 1 \right) + \frac{1}{4} \left( \frac{b_1}{1} - 1 \right) = \frac{1}{48} \left( b_1^2 - 4 \right).
\]
We can also calculate \( N_{1,1}(b_1) \) using a version of the recursion
\[
b_1 N_{1,1}(b_1) = \frac{1}{2} \sum_{p + q = b \text{ even}} p q.
\]
We will calculate \( N_{0,4}[b_1, b_2, b_3, b_4] \) to demonstrate the recursion relation and the parity issue.
\[
\left( \sum_{i=1}^{4} b_i \right) N_{0,4}(b_1, b_2, b_3, b_4) = \sum_{i \neq j} \sum_{q \text{ even}} \sum_{p + q = b_i + b_j} p q.
\]
If all \( b_i \) are even, or all \( b_i \) are odd, then \( b_i + b_j \) is always even so the sum is over \( p \) and \( q \) even. We have
\[
S_{0, \text{even}}(k) = \sum_{i \neq j} \sum_{p + q = k} p q = 4 \left( \frac{k}{3} + 1 \right)
\]
so
\[
\left( \sum_{i=1}^{4} b_i \right) N_{0,4}(b) = \sum_{i \neq j} 4 \left( \frac{b_i + b_j}{2} \right) + 1 = \left( \sum_{i=1}^{4} b_i \right) \frac{1}{4} \left( b_1^2 + b_2^2 + b_3^2 + b_4^2 - 4 \right)
\]
agreeing with Table 1. If \( b_1 \) and \( b_2 \) are odd and \( b_3 \) and \( b_4 \) are even then we need
\[
S_{0, \text{odd}}(k) = \sum_{i \neq j} \sum_{p + q = k} p q = \frac{1}{2} \left( \frac{k+1}{3} \right)
\]
so
\[
\left( \sum_{i=1}^{4} b_i \right) N_{0,4}(b) = \sum_{(i,j) = (1,2) \text{ or } (3,4)} 4 \left( \frac{b_i + b_j}{2} \right) + 1 + \sum_{(i,j) \neq (1,2) \text{ or } (3,4)} \frac{1}{2} \left( b_i + b_j + 1 \right)
\]
\[
= \left( \sum_{i=1}^{4} b_i \right) \frac{1}{4} \left( b_1^2 + b_2^2 + b_3^2 + b_4^2 - 2 \right)
\]
so we see that the polynomial representatives of \( N_{0,4}(b) \) agree up to a constant term.

**Proof of Theorem 7.** We can use the recursion (2) to prove that \( N_{g,n}(b_1, \ldots, b_n) \) is a polynomial of the right degree but to prove that it is a polynomial in \( b_i^2 \) we need a different recursion formula (5). For simplicity we use (5) to prove each part of
Theorem 1

\[ b_1 N_{g,n}(b_1, \ldots, b_n) = \sum_{j>1} \frac{1}{2} \left( \sum_{p+q=b_1+b_j} pq N_{g,n-1}(p, b_2, \ldots, \hat{b}_j, \ldots, b_n) \right. \]
\[ \left. + \sum_{p+q=b_1-b_j} pq N_{g,n-1}(p, b_2, \ldots, \hat{b}_j, \ldots, b_n) \right) \]
\[ + \sum_{p+q+r=b_1} pqr \left[ N_{g-1,n+1}(p, q, b_2, \ldots, b_n) \right. \]
\[ \left. + \sum_{g_1, g_2 : g_1+g_2=g, I \cup J = \{2, \ldots, n\}} N_{g_1, g_2, |I|, |J|}(p, q, b_I, b_J) \right] \]

This differs from the recursion formula (2) by breaking the symmetry around \( b_1 \).

The sum over the term \( p + q = b_1 - b_j \) needs to be interpreted as follows. If \( b_1 - b_j > 0 \) it is read as written, whereas if \( b_1 - b_j < 0 \) then replace \( b_1 - b_j \) by \( b_j - b_1 \) and negate the sum. (This is not the same as sending \( (p, q) \) to \( (-p, -q) \).)

We will prove the recursion (5) below. Before that we will prove that given \( N_{0,3} \) and \( N_{1,1} \) then (5) determines polynomials \( N'_{g,n}(b_1, \ldots, b_n) \) of degree \( 3g - 3 + n \) in \( b_1^2 \). By induction, the simpler polynomials are polynomials in \( b_1^2 \) so monomials on the right hand side of the recursion are of the form

\[ S_m(k) = \sum_{p+q=k, q \text{ even}} p^{2m+1} q, \quad R_{m, m'}(k) = \sum_{p+q+r=k, r \text{ even}} p^{2m+1} q^{2m'+1} r \]

as in (4). In Lemma 4 it is proven that \( S_m(k) \) and \( R_{m, m'}(k) \) are odd polynomials in \( k \). In particular, \( S_m(b_1 - b_j) = -S_m(b_j - b_1) \) explaining the interpretation of the sum over \( b_1 - b_j < 0 \).

The sums over \( p + q + r = b_1 \) yield terms which are odd in \( b_1 \) from \( R_{m, m'}(b_1) \) and even in \( b_i \) for \( i > 1 \) hence \( 1/b_1 \) times these terms is even in all \( b_i^2 \). The sums over \( p + q = b_1 + b_j \) and \( p + q = b_1 - b_j \) have the same summands so each monomial occurs with the same coefficient. Hence the terms involving \( b_1 \) are of the form \( S_m(b_1 + b_j) + S_m(b_1 - b_j) \) and since \( S_m \) is odd, this sum is odd in \( b_1 \) and even in \( b_j \), and even in all the other \( b_i \). Again \( 1/b_1 \) times these terms is even in all \( b_i^2 \). Thus by induction the polynomials generated by the recursion relation (5) from \( N_{0,3} \) and \( N_{1,1} \) are polynomials in \( b_1^2 \).

We will now calculate the degree in \( b_1^2 \). By induction \( \deg N_{g,n-1} = 3g - 3 + n - 1 \) and by Lemma 2 \( S_m(k) \) takes a term \( p^{2m+1} q \) and produces a degree \( 2m + 3 \) polynomial, i.e. it increases the degree by 1. In this case \( 3g - 3 + n - 1 + 1 = 3g - 3 + n \) as required. Similarly, by induction \( \deg N_{g-1,n+1} = 3g - 3 + n - 2 \) and \( \deg N_{g_1, g_2, |I|, |J|} = 3g - 3 + n - 2 \). By Lemma 4 \( R_{m, m'}(k) \) increases the degree of its summand by 2. Since \( 3g - 3 + n - 2 + 2 = 3g - 3 + n \) the result is proven by induction starting from the degrees of \( N_{0,3} \) and \( N_{1,1} \).

As above, write \( N'_{g,n} \) for the polynomials produced from the recursion (5). To prove the recursion (5) we use the fact that both (2) and (5) uniquely determine
$N_{g,n}$ and $N'_{g,n}$ respectively. It remains to show that $[1] \Rightarrow [2]$, hence $N_{g,n}$ and $N'_{g,n}$ necessarily coincide.

Apply (5) to each $b_i$ to calculate $b_i N'_{g,n}(b_1, \ldots, b_n)$ and add.

\[
\sum_{i=1}^{n} b_i \right) N'_{g,n}(b_1, \ldots, b_n) = \sum_{i \neq j} \sum_{p+q=b_i+b_j} pq N'_{g,n-1}(p, b_1, \ldots, \hat{b}_i, \ldots, \hat{b}_j, \ldots, b_n) \\
+ \sum_{p+q+r=b_i} pqr \left[ N'_{g-1,n+1}(p, q, b_1, \ldots, \hat{b}_i, \ldots, b_n) \\
+ \sum_{g_1+g_2=g} N'_{g_1,|I|}(p, b_I) N'_{g_2,|J|}(q, b_J) \right] \\
+ \Delta
\]

where

\[
\Delta = \sum_{i \neq j} \frac{1}{2} \left( \sum_{p+q=b_i-b_j} + \sum_{p+q=b_j-b_i} \right) pq N'_{g,n-1}(p, b_1, \ldots, \hat{b}_i, \ldots, \hat{b}_j, \ldots, b_n) = 0
\]

since the sums contain only canceling odd terms $S_m(b_i-b_j) + S_m(b_j-b_i) = 0$.

Thus $N_{g,n}$ and $N'_{g,n}$ satisfy the recursion relation (2) which uniquely determines them, hence

$N_{g,n} = N'_{g,n}$

so it follows that $N_{g,n}$ satisfies the recursion (5). \hfill \Box

Remark. The top degree term of recursion (5) is a discrete version of the integration recursion for volume given by Do and Safnuk [2]. They show their recursion is a rescaled version of Mirzakhani’s recursion relation [7] which give the Virasoro relations among tautological classes [8].

2.3. Vanishing.

**Lemma 2.** If $\sum_{i=1}^{n} b_i < 4g + 2n$ then $N_{g,n}(b_1, \ldots, b_n) = 0$.

**Proof.** A labeled fatgraph $\Gamma \in \mathcal{F}at_{g,n}$ has at least one vertex and hence at least $2g + n$ edges since $\chi(\Gamma) = 1 - 2g - n$. Since $N_{g,n}$ counts positive integers solutions of $A \Gamma x = b$, each $x_i \geq 1$, thus $\sum x_i \geq 2g + n$. Each edge contributes twice to the boundary of $\Gamma$ so

\[
\sum_{i=1}^{n} b_i = 2 \sum_{i=1}^{c(\Gamma)} x_i \geq 4g + 2n
\]

and the lemma follows. \hfill \Box

Lemma 2 can be used to get strong information about the lattice count polynomial. For example, $N_{1,1}(2) = 0$ and since it is a polynomial in $b_1^2$ of degree 1 we get $N_{1,1}(b_1) = c(b_1^2 - 4)$. The genus 2 case gives $N_{2,1}(2) = 0 = N_{2,1}(4) = N_{2,1}(6)$ hence

$N_{2,1}(b_1) = c_1(b_1^2 - 4)(b_1^2 - 16)(b_1^2 - 36)(b_1^2 + c_2)$. \hfill \Box
Although it is very difficult to calculate $N_{g,n}$ directly using fatgraphs, in the simplest cases it is calculable by extending the idea behind the vanishing Lemma 2. When $\sum_{i=1}^n b_i = 4g + 2n$ the argument in the proof of Lemma 2 shows that each $x_i = 1$ so $N_{g,n}$ counts 1-vertex fatgraphs.

3. Euler characteristic

Using the cell decomposition (1), the orbifold Euler characteristic of the moduli space can be calculated via a sum over labeled fatgraphs. Expressing the sum as a Feynman expansion Penner [10] calculated the following.

$$\chi(M_{g,n}) = \sum_{\Gamma \in \mathcal{F}_{at, g,n}} (-1)^{e(\Gamma)} |\text{Aut}\Gamma| = \left\{ \begin{array}{ll} (-1)^{n-1}(n-3)! & g = 0 \\
(-1)^{n-1} \frac{(2g+n-3)!}{(2g-2)!} \zeta(1-2g) & g > 0 \end{array} \right.$$ 

The exponent is the dimension of the cell since $\dim P_\Gamma = e(\Gamma) - n$.

The lattice count polynomial gives another way to calculate the Euler characteristic via $N_{g,n}(0, \ldots, 0) = \chi(M_{g,n})$. We will prove this here.

Proof of Theorem 2. Define

$$R_{g,n}(z) = \sum_{b \in \mathbb{Z}_+^n} N_{g,n}(b_1, \ldots, b_n) z^{b_1+\ldots+b_n}.$$ 

It has the following properties:

1. $R_{g,n}(z)$ is a meromorphic function, holomorphic on $\overline{\mathbb{C}} - \{\pm 1\}$.
2. $R_{g,n}(0) = 0$.
3. $R_{g,n}(\infty) = (-1)^n N_{g,n}(0, \ldots, 0)$.

Recall that $N_{g,n}(b)$ is represented by a collection of polynomials depending on the parity of $b_i$. By the symmetry of these polynomials we can set $R_{g,n}(z) = \sum_{k=0}^n \binom{n}{k} R_{g,n}^{(k)}(z)$ where $k =$ the number of odd $b_i$. The basic idea behind property (1) is that if $p(n) = \sum_{j=0}^k p_j n^j$ is a polynomial then

$$\sum_{n>0} p(n) z^n = \sum_{j=0}^k p_j \sum_{n>0} n^j z^n = \sum_{j=0}^k p_j \left( \frac{z}{d/dz} \right)^j \frac{z}{1-z},$$

which is a meromorphic function with pole at $z = 1$ and known behaviour at $z = 0$ and $z = \infty$. If we restrict the parity of $n$ then

$$\sum_{n \text{ even}} p(n) z^n = \sum_{j=0}^k p_j \left( \frac{z}{d/dz} \right)^j \frac{z^2}{1-z^2}, \quad \sum_{n \text{ odd}} p(n) z^n = \sum_{j=0}^k p_j \left( \frac{z}{d/dz} \right)^j \frac{z}{1-z^2},$$

which are both meromorphic functions with poles at $z = \pm 1$. Furthermore,

$$\left( \frac{z}{d/dz} \right)^j \frac{z^2}{1-z^2} \bigg|_{z=\infty} = \left\{ \begin{array}{ll} -1 & j = 0 \\
0 & j > 0 \end{array} \right., \quad \left( \frac{z}{d/dz} \right)^j \frac{z}{1-z^2} \bigg|_{z=\infty} = 0.$$
Each polynomial \(N_{g,n}(b_1, ..., b_n)\) is a sum of monomials of the form \(\prod_i b_i^{2m_i}\) so \(R^{(k)}_{g,n}(z)\) is a sum of finitely many series

\[
R^{(k)}_{g,n}(z) = \sum_{m} c_m \sum_{\substack{b \in \mathbb{Z}_+^n \\ b_i \text{ odd } i \leq k}} b_1^{2m_1} ... b_n^{2m_n} z^{b_1 + ... + b_n}
\]

which consists of terms of the form

\[
\sigma^{(k)}_{m}(z) = \sum_{\substack{b \in \mathbb{Z}_+^n \\ b_i \text{ odd } i \leq k}} b_1^{2m_1} ... b_n^{2m_n} z^{b_1 + ... + b_n} = \prod_{i=1}^k \sum_{b_i > 0 \ \ b_i \text{ odd}} b_i^{2m_i} z^{b_i} \cdot \prod_{i=k+1}^n \sum_{b_i > 0 \ \ b_i \text{ even}} b_i^{2m_i} z^{b_i}.
\]

This is a finite product of meromorphic functions each with poles only at \(z = \pm 1\) by (3). Furthermore, from the evaluation at \(\infty\) of such functions, \(\sigma^{(k)}_{m}(\infty) = (-1)^n\) if \(m = 0\) and \(k = 0\) and it vanishes otherwise. Thus, \(R^{(0)}_{g,n}(\infty) = R^{(0)}_{g,n}(\infty) = (-1)^n N_{g,n}(0, ..., 0)\) where we evaluate using the polynomial \(N_{g,n}\) that takes in all even \(b_i\).

We have proven (1) and (3). Property (2) follows from the strict positivity of the \(b_i\) and the convergence of the series which follows from the convergence of \(1 + z + z^2 + ...\) for \(|z| < 1\).

We can calculate \(R^{(0)}_{g,n}(\infty)\) in another way. For a vector \(v = (v_1, ..., v_n)\) with \(v_i \in \mathbb{Z}_+\) define (the semigroup homomorphism) \(|v| = \sum_{i=1}^n v_i\). Recall that the incident matrix \(A_{\Gamma} = \left[\alpha_1, ..., \alpha_{e(\Gamma)}\right]\) for \(\alpha_i \in \mathbb{R}^n\) of a labeled fatgraph \(\Gamma\) defines a convex polytope \(A_{\Gamma}x = b\) and \(N_{\Gamma}(b)\) counts integral points \(x \in \mathbb{Z}_+^e(\Gamma)\). Thus

\[
R_{\Gamma}(z) = \sum_{b \in \mathbb{Z}_+^e(\Gamma)} N_{\Gamma}(b_1, ..., b_n) z^{b_1 + ... + b_n} = \sum_{x \in \mathbb{Z}_+^e(\Gamma)} z^{e(\Gamma)} x |A_{\Gamma}x| = \sum_{x \in \mathbb{Z}_+^e(\Gamma)} z^{e(\Gamma)} x |A_{\Gamma}x| = \prod_{i=1}^{e(\Gamma)} \frac{z^{a_i}}{1 - z^{a_i}}
\]

so \(R_{\Gamma}(\infty) = (-1)^{e(\Gamma)}\) and

\[
R^{(0)}_{g,n}(\infty) = \sum_{\Gamma \in \mathcal{Fat}_{g,n}} \frac{(-1)^{e(\Gamma)}}{|\text{Aut}\Gamma|} = (-1)^n \chi(M_{g,n}).
\]

Combining this with property (3) yields the theorem

\[
N_{g,n}(0, ..., 0) = \chi(M_{g,n}).
\]
has \( \left( \frac{k}{e(\Gamma) - 1} \right) \) positive integral solutions. Hence

\[
N_{g,1}(b) = c_{6g-3}^{(g)} \left( \frac{b}{6g-4} - 1 \right) + c_{6g-4}^{(g)} \left( \frac{b}{6g-5} - 1 \right) + \ldots + c_k^{(g)} \left( \frac{b}{k-1} - 1 \right) + \ldots + c_{2g}^{(g)} \left( \frac{b}{2g-1} - 1 \right)
\]

where the coefficients are weighted counts of fatgraphs of genus \( g \) with \( n = 1 \) boundary component

\[
c_k^{(g)} = \sum_{\Gamma \in \mathcal{F}_{\text{Aut}g,1}} \frac{1}{|\text{Aut}\Gamma|}.
\]

The polynomial \( \left( \frac{b}{k} - 1 \right) \) evaluates at \( b = 0 \) to \((-1)^k\) which gives a direct proof that the Euler characteristic is given by evaluation at 0.

\[
N_{g,1}(0) = \sum_{\Gamma \in \mathcal{F}_{\text{Aut}g,1}} (-1)^{e(\Gamma) - 1} = \chi(M_{g,1}).
\]

When \( n = 1 \) the weighted number of trivalent fatgraphs and 1-vertex fatgraphs are known [10].

\[
c_{6g-3}^{(g)} = \frac{1}{12g}\frac{(6g-5)!}{g!(3g-3)!}, \quad c_k^{(g)} = \frac{(4g-1)!}{4^g(2g+1)!}
\]

We can calculate \( N_{2,1}(b) \) without using the recursion relation (except to deduce that \( N_{2,1}(b) \) is a polynomial of degree 4 in \( b^2 \)) by applying Lemma [2] to get \( N_{2,1}(b) = 0 \) for \( b = 2, 4 \) and 6. This leaves two unknown coefficients which can be calculated from any two of the three pieces of known information \( c_9^{(2)}, c_4^{(2)} \) and \( N_{2,1}(0) \).

\[
N_{2,1}(b) = \frac{1}{216315}(b^2 - 4)(b^2 - 16)(b^2 - 36)(5b^2 - 32)
\]

\[
= \frac{25}{12}(\frac{b}{8} - 1) + \frac{105}{4}(\frac{b}{7} - 1) + \frac{29}{2}(\frac{b}{6} - 1) + \frac{161}{4}(\frac{b}{5} - 1) + \frac{46}{3}(\frac{b}{4} - 1) + \frac{21}{2}(\frac{b}{3} - 1).
\]

The polynomial \( N_{2,1}(b) \) enables us to calculate the weighted counts of fatgraphs \( c_k^{(2)} \). We can similarly calculate \( N_{3,1} \) and hence deduce the weighted counts of fatgraphs.

\[
N_{3,1}(b) = \frac{1}{23456789}(b^2 - 4)(b^2 - 16)(b^2 - 36)(b^2 - 64)(b^2 - 100)(5b^2 - 188b^2 + 1152)
\]

\[
= \frac{5005}{3}(\frac{b}{14} - 1) + \frac{25025}{2}(\frac{b}{13} - 1) + \frac{41118}{12}(\frac{b}{12} - 1) + \frac{92929}{12}(\frac{b}{11} - 1) + \frac{18355}{2}(\frac{b}{10} - 1)
\]

\[
+ \frac{283767}{3}(\frac{b}{9} - 1) + \frac{31775}{9}(\frac{b}{8} - 1) + 10813(\frac{b}{7} - 1) + \frac{25441}{14}(\frac{b}{6} - 1) + \frac{405}{4}(\frac{b}{5} - 1).
\]

REFERENCES

[1] Brion, Michel and Vergne, Michèle *Lattice points in simple polytopes*. J. Amer. Math. Soc. **10** (1997), 371–392.

[2] Do, Norman and Safnuk, Brad *Hyperbolic rescaling and symplectic structures on moduli spaces of curves*. Preprint.

[3] Grothendieck, Alexandre *Esquisse d’un Programme*. London Math. Soc. Lecture Note Ser. **242**, Geometric Galois actions, 1, 3–48, Cambridge Univ. Press, Cambridge, 1997.

[4] Harer, John *The virtual cohomological dimension of the mapping class group of an orientable surface*. Invent. Math. **84** (1986), 157–176.

[5] Harer, John and Zagier, Don *The Euler characteristic of the moduli space of curves*. Invent. Math. **85** (1986), 457–485.
[6] Kontsevich, Maxim Intersection theory on the moduli space of curves and the matrix Airy function. Comm. Math. Phys. 147 (1992), 1-23.

[7] Mirzakhani, Maryam Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces. Invent. Math. 167 (2007), 179-222.

[8] Mirzakhani, Maryam Weil-Petersson volumes and intersection theory on the moduli space of curves. J. Amer. Math. Soc., 20 (2007), 1-23.

[9] Penner, R. C. The decorated Teichmüller space of punctured surfaces. Comm. Math. Phys. 113 (1987), 299-339.

[10] Penner, R. C. Perturbative series and the moduli space of Riemann surfaces. J. Diff. Geom. 27 (1988), 35-53.

[11] Walsh, T.R.S. and Lehman, A.B. Counting rooted maps by genus. J. Combin. Theory Ser. B 13 (1972), 192-218.

[12] Witten, Edward Two-dimensional gravity and intersection theory on moduli space. Surveys in differential geometry (Cambridge, MA, 1990), 243–310, Lehigh Univ., Bethlehem, PA, 1991.

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