The BFKL-Regge Expansion for the Proton Structure Function at Small $x$

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Abstract. We report an evaluation of subleading eigenvalues and eigenfunctions of the BFKL equation in the color dipole representation with a running gauge coupling. We present an expansion of the small-$x$ proton structure function $F_2(x, Q^2)$ in terms of the rightmost BFKL-singularities. The BFKL-Regge phenomenology of DIS structure functions is developed which is shown to provide remarkably good description of the data on $F_2(x, Q^2)$ from E665 to HERA.

Introduction

In this talk I address the issue of extrapolation of the proton structure functions to a domain of very small $x$ from the attainable kinematical region of $x$ and $Q^2$. So, this is an old problem of predicting the future from the known past. The standard predictor is the GLDAP evolution equation [1]. However, the art of predicting is difficult. The numerous pre-HERA GLDAP fits to the proton structure functions are equally good for large-$x$-data but seldom come close to the new data points at smaller $x$. The lack of predictive power is not surprising and we comment on this point below. It is well known that the GLDAP evolution breaks down at small $x$ and is superseded by the log($1/x$)-BFKL-evolution [2]. Contrary to the GLDAP approach, the BFKL evolution predicts uniquely the proton structure functions at an arbitrarily small $x$ from the input at a starting point $x = x_0$. DIS structure functions at asymptotically large $1/x$ are dominated by the rightmost Regge-singularity with the intercept $\Delta_0$ [3] and

$$F_{2p}(x, Q^2) = F_2^{(0)}(Q^2) \left( \frac{1}{x} \right)^{\Delta_0}.$$  

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At moderate $x$, however, the subleading contributions to $F_{2p}(x, Q^2)$ with smaller intercepts $\Delta_1, \Delta_2, ...$ cannot be neglected. The task of my talk is to present an evaluation of the intercepts $\Delta_n$, the pomeron trajectory slopes $\alpha'_n$ and the corresponding structure functions $F_2^n(Q^2)$, to arrive finally at the representation

$$F_{2p}(x, Q^2) = \sum_n A_n F_2^{(n)}(Q^2) \left( \frac{x_0}{x} \right)^{\Delta_n}. \quad (2)$$

I conclude with the expansion of the proton structure function $F_2(x, Q^2)$ in terms of the three rightmost BFKL singularities. Such a three pole approximation seems to exhaust the existing experimental data thus providing a reliable basis for the BFKL-Regge phenomenology of diffractive DIS.

**The BFKL Eigenvalue Problem in the Dipole Picture**

The virtual photo-absorption cross section $\sigma(r, x)$, where $r$ is the color dipole size, satisfies the BFKL equation [4] (hereafter $\xi = \log(1/x)$):

$$\frac{\partial \sigma(\xi, r)}{\partial \xi} = K \otimes \sigma(r, x) \quad (3)$$

with the kernel $K$ which involves the running gauge coupling and the infrared cutoff - the correlation radius $R_c$ of perturbative gluons. We look for the solution with the Regge behavior

$$\sigma_n(r, x) = \sigma_n(r) \left( \frac{1}{x} \right)^{\Delta_n}. \quad (4)$$

Then the eigenfunctions $\sigma_n(r)$ and the eigenvalues $\Delta_n$ are determined from

$$K \otimes \sigma_n(r) = \Delta_n \sigma_n(r). \quad (5)$$

The short-distance asymptotics of the eigenfunctions is known in the analytic form [5]

$$\sigma_n(r) = r^2 \left[ \frac{1}{\alpha_s(r)} \right]^{\gamma_n^{-1}}, \quad (6)$$

where $\gamma_n\Delta_n = 4/3$. At large distances, $r \gg R_c$,

$$\sigma_n(r) \equiv \bar{\sigma}_n = \text{const} \quad (7)$$

due to the finite $R_c$. Another useful clue is that the leading eigenfunction $\sigma_0(r)$ is node free and the $n$-th subleading solution must have $n$ nodes. Then a practical approach to the eigenvalue problem is a variational procedure [6] applied to a class of $n$-node polynomials $\mathcal{P}_n(z)$ in a variable $z \sim [1/\alpha_s(r)]^{\gamma}$. 
Results and Discussion

The BFKL equation with running coupling and infrared cutoff has a discrete spectrum. The eigenvalues $\Delta_n$ obtained by the variational method for $n = 0, 1, 2, 3, ...$ are as follows

$$ \Delta_n = 0.40, 0.220, 0.148, 0.111, 0.088, 0.073, 0.063, ... $$

To an accuracy better than 10% the above series follows the law

$$ \Delta_n = \frac{\Delta_0}{(n + 1)} $$

derived by Lipatov [3] from quasi-classical considerations.

The BFKL eigenfunctions are represented (Figure 1) in term of the quantity $\sigma_n(r)/r$ which to a crude approximation is similar to Lipatov’s quasi-classical eigenfunctions, which are $E_n(r) \sim \cos[\phi(r)]$ for $n \gg 1$ [3].

Once $\sigma_n(r)$ is known, the pomeron trajectory slope $\alpha'_n$ can readily be derived. In the notations of ref. [7]

$$ \alpha'_n = \frac{3}{32\pi R_c^2} \int d\rho^2 \rho^2 \alpha_S(\rho) K_1^2(\rho/R_c) [1 + \sigma_n(\rho)/\bar{\sigma}_n] , $$

where (see eq.7)

$$ \bar{\sigma}_n = \frac{3}{2\pi R_c^2} \int d\rho^2 \alpha_S(\rho) K_1^2(\rho/R_c) \sigma_n(\rho) . $$

FIGURE 1. BFKL eigenfunctions.
Numerically, $\alpha_0' = 0.072\, GeV^{-2}$, $\alpha_1' = 0.066\, GeV^{-2}$, $\alpha_2' = 0.062\, GeV^2$, $\alpha_3' = 0.060\, GeV^2$.

The color dipole factorization relates the dipole cross sections with the structure functions $F_2^{(n)}(Q^2)$ which are shown in Figure 2. At large $Q^2$, far beyond the nodal region,

$$F_2^n(x, Q^2) \propto \left(\frac{x_0}{x}\right)^{\Delta_n} \left[\frac{1}{\alpha_S(Q^2)}\right]^{4/3\Delta_n}.$$  \hspace{1cm} (12)

Since the relevant variable is a power of the inverse gauge coupling the nodes are spaced by 2-3 orders of magnitude in $Q^2$-scale and only the first two of them are in the accessible range of $Q^2$. The first nodes of $F_2^{(n)}(Q^2)$ are located at $Q^2 \sim 20 - 60\, GeV^2$. Hence, only the leading structure function contributes significantly in this region. This explains the precocious BFKL asymptotics found from the numerical solution of the BFKL equation [8].

In the accessible range of $x$ the subleading contributions are numerically large. In particular, the BFKL expansion of the Born approximation dipole cross section which is used as the boundary condition at $x = x_0 = 0.03$ suggests more than 60% contribution from $F_2^{(n)}$ with $n > 0$. This implies the subleading terms determine the $Q^2$ dependence of $F_2^{p}(Q^2)$ at $x = x_0$ and simultaneously the $x$ dependence of the structure functions. Notice that in the pre-nodal region of $Q^2 \lesssim 20\, GeV^2$ the leading and subleading structure functions are very similar in shape. This explains the failure of the early GLDAP fits: only the limited region of $Q^2 \lesssim 10\, GeV^2$ was accessible and they

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The modulus of structure function $F_2^{(n)}(Q^2)$ for $n = 0, 1, 2, 3$}
\end{figure}
could not catch and correctly describe a very different $x$-evolution of pomerons with different $n$.

At small $x$ only the region $Q^2 \lesssim 10^3 GeV^2$ is accessible. In this range the structure functions with $n \geq 3$ are hardly distinguishable. Besides, the splitting of the intercept with $n \geq 3$ is much smaller than for $n = 0, 1, 2$. Hence, the Regge expansion (2) can be truncated at $n = 2$ and $F_2^{(2)}(Q^2)$ comprises contributions from all poles with $n \geq 2$.

The BFKL equation allows one to determine the intercepts and structure functions $F_2^{(n)}(Q^2)$. The only adjustable parameters are ”the pole residues” $A_0, A_1, A_2$ in (2) which are fixed, in fact, by the boundary condition at $x = x_0$. With the proper account of the valence [9] and non-perturbative [8] corrections to (2) we arrive at the three-pole-approximation which appears to be very successful when confronted with the data [10]. in a wide kinematical range (Figure 3). The effective pomeron intercept

$$\Delta_{eff} = -\frac{\partial \log F_{2p}(x, Q^2)}{\partial \log x}$$

(13)

gives an idea of the role of the subleading singularities. The intercept $\Delta_{eff}$ calculated with the experimental kinematic constraints is much smaller than $\Delta_0 = 0.4$ which is expected to dominate asymptotically. The agreement of our numerical estimates with the $H1$ determination (Figure 4) is quite satisfactory.

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![Figure 3](image-url)  # FIGURE 3. Three-Pole Approximation vs. H1, ZEUS and E665 data.
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FIGURE 4. Effective intercept vs. H1 data.