1 The Non-Euclidean Plane

In case the work of Bolyai \cite{Bo} and Lobatschevsky \cite{Lo} left any doubts about the existence of non-Euclidean geometry these doubts were removed by the work \cite{Be} of Beltrami. With a modification made possible by hindsight one can state the following result.

**Theorem 1.1.** Given a simply connected region \( D \subset \mathbb{R}^2 (D \neq \mathbb{R}^2) \) there exists a Riemannian metric \( g \) on \( D \) which is invariant under all conformal transformations of \( D \). Also \( g \) is unique up to a constant factor.

Because of the Riemann mapping theorem we can assume \( D \) to be the unit disk. Given \( a \in D \) the mapping \( \varphi : z \to \frac{a - z}{1 - \bar{a}z} \) is conformal and \( \varphi(a) = 0 \). The invariance of \( g \) requires

\[
g_a(u, u) = g_0(d\varphi(u), d\varphi(u)) \tag{1.1}
\]

for each \( u \in D_a \) (the tangent space to \( D \) at \( a \)). Since \( g_0 \) is invariant under rotations around 0,

\[
g_0(z, z) = c|z|^2, \tag{1.2}
\]

where \( c \) is a constant. Here \( D_0 \) is identified with \( \mathbb{C} \). Let \( t \to z(t) \) be a curve with \( z(0) = a, z'(0) = u \in \mathbb{C} \). Then \( d\varphi(u) \) is the tangent vector

\[
\left\{ \frac{d}{dt} \varphi(z(t)) \right\}_{t=0} = \left( \frac{d\varphi}{dz} \right)_a \left( \frac{dz}{dt} \right)_0 = \left\{ \frac{1 - |a|^2}{(1 - \bar{a}z)^2} \right\}_{z=a} u
\]

so by (1.1), (1.2)

\[
g_a(u, u) = c \frac{1}{(1 - |a|^2)^2} |u|^2.
\]
Thus $g$ is the Riemannian structure

$$ds^2 = c \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

(1.3)

and the proof shows that it is indeed invariant.

We shall now take $D$ as the unit disk $|z| < 1$ with $g = ds^2$ given by (1.3) with $c = 1$. In our analysis on $D$ we are mainly interested in the geodesics in $D$ (the arcs orthogonal to the boundary $B = \{ z \in \mathbb{C} : |z| = 1 \}$) and the horocycles in $D$ which are the circles inside $D$ tangential to $B$. Note that a horocycle tangential to $B$ at $b$ is orthogonal to all the geodesics in $D$ which end at $b$.

2 The Non-Euclidean Fourier Transform

We first recall some of the principal results from Fourier analysis on $\mathbb{R}^n$. The Fourier transform $f \rightarrow \tilde{f}$ or $R^n$ is defined by

$$\tilde{f}(u) = \int_{R^n} f(x)e^{-i(x,u)} \, dx$$

(2.1)

where $(, )$ denotes the scalar product and $dx$ the Lebesgue measure. In polar coordinates $u = \lambda w \lambda \in R$, $w \in SS^{n-1}$ we can write

$$\tilde{f}(\lambda w) = \int_{R^n} f(x)e^{-i\lambda(x,w)} \, dx.$$  

(2.2)

It is then inverted by

$$f(x) = (2\pi)^{-n} \int_{R^+ \times SS^{n-1}} \tilde{f}(\lambda w)e^{i\lambda(x,w)}\lambda^{n-1} \, d\lambda \, dw$$

(2.3)

say for $f \in \mathcal{D}(R^n) = C^\infty_0(R^n)$, $dw$ denoting the surface element on the sphere $SS^{n-1}$. The Plancherel formula

$$\int_{R^n} |f(x)|^2 \, dx = (2\pi)^{-n} \int_{R^+ \times SS^{n-1}} |\tilde{f}(\lambda, w)|^2 \lambda^{n-1} \, d\lambda \, dw$$

(2.4)

expresses that $f \rightarrow \tilde{f}$ is an isometry of $L^2(R^2)$ onto $L^2(R^+ \times SS^{n-1}, (2\pi)^{-n}\lambda^{n-1} \, d\lambda \, dw)$.

The range of the mapping $f(x) \rightarrow \tilde{f}(\lambda w)$ as $f$ runs through $\mathcal{D}(R^n)$ is expressed in the following theorem [He7]. A vector $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ is said to be isotropic if $(a, a) = a_1^2 + \cdots + a_n^2 = 0$. 

2
Theorem 2.1. The Fourier transform $f(x) \rightarrow \hat{f}(\lambda w)$ maps $\mathcal{D}(\mathbb{R}^n)$ onto the set of functions $\hat{f}(\lambda w) = \varphi(\lambda, w) \in \mathcal{C}^\infty(\mathbb{R} \times SS^{n-1})$ satisfying:

(i) There exists a constant $A > 0$ such that for each $w \in SS^{n-1}$ the function $\lambda \rightarrow \varphi(\lambda, w)$ extends to a holomorphic function on $\mathbb{C}$ with the property

$$
\sup_{\lambda \in \mathbb{C}, w \in SS^{n-1}} |\varphi(\lambda, w)|(1 + |\lambda|)^N e^{-A|\text{Im} \lambda|} < \infty \quad (2.5)
$$

for each $N \in \mathbb{Z}$. ($\text{Im} \lambda = \text{imaginary part of} \lambda$).

(ii) For each $k \in \mathbb{Z}^+$ and each isotropic vector $a \in \mathbb{C}^n$ the function

$$
\lambda \rightarrow \lambda^{-k} \int_{SS^{n-1}} \varphi(\lambda, w)(a, w)^k \, dw \quad (2.6)
$$

is even and holomorphic in $\mathbb{C}$.

Condition (2.5) expresses that the function $\lambda \rightarrow \varphi(\lambda, w)$ is of uniform exponential type: The classical Paley–Wiener theorem states that $\mathcal{D}(\mathbb{R}^n)^\sim$ consists of entire functions of exponential type in $n$ variables whereas in the description above only $\lambda$ enters.

Formula (2.2) motivates a Fourier transform definition on $\mathcal{D}$. The inner product $(x, \omega)$ equals the (signed) distance from 0 to the hyperplane through $x$ with normal $\omega$. A horocycle in $\mathcal{D}$ through $b$ is perpendicular to the (parallel) family of geodesics ending at $b$ so is an analog of a hyperplane in $\mathbb{R}^n$. Thus if $z \in \mathcal{D}$, $b \in B$ we define $\langle z, b \rangle$ as the (signed) distance from 0 to the horocycle through $z$ and $b$. The Fourier transform $f \rightarrow \hat{f}$ on $\mathcal{D}$ is thus defined by

$$
\hat{f}(\lambda, b) = \int_{\mathcal{D}} f(z) e^{-i\lambda \langle z, b \rangle} \, dz \quad (2.7)
$$

for all $b \in B$ and $\lambda \in \mathbb{C}$ for which integral converges. Here $dz$ is the invariant surface element on $\mathcal{D}$

$$
dz = (1 - x^2 - y^2)^{-2} \, dx \, dy. \quad (2.8)
$$

The $+1$ term in (2.7) is included for later technical convenience.

The Fourier transform (2.7) is a special case of the Fourier transform on a symmetric space $X = G/K$ of the non-compact type, introduced in [He3]. Here $G$ is a semisimple connected Lie group with finite center.
and $K$ is a maximal compact subgroup. In discussing the properties of $f \rightarrow \tilde{f}$ below we stick to the case $X = D$ for notational simplicity but shall indicate (with references) the appropriate generalizations to arbitrary $X$. Some of the results require a rank restriction on $X$.

**Theorem 2.2.** The transform $f \rightarrow \tilde{f}$ in (2.7) is inverted by

$$f(z) = \frac{1}{4\pi} \int \int_{\mathbb{R} \times B} \tilde{f}(\lambda, b) e^{i(\lambda+1)(z,b)} \lambda \frac{\pi \lambda}{2} d\lambda db, \quad f \in D(D). \quad (2.9)$$

Also the map $f \rightarrow \tilde{f}$ extends to an isometry of $L^2(D)$ onto $L^2(\mathbb{R}^+ \times B, \mu)$ where $\mu$ is the measure

$$\mu = \frac{\lambda}{2\pi} \frac{\pi \lambda}{2} d\lambda db \quad (2.10)$$

and $db$ is normalized by $\int db = 1$.

This result is valid for arbitrary $X = G/K$ ([He3] [He4]), suitably formulated in terms of the fine structure of $G$. While this result resembles (2.3)—(2.4) closely the range theorem for $D$ takes a rather different form.

**Theorem 2.3.** The range $\mathcal{D}(D)^\sim$ consists of the functions $\varphi(\lambda, b)$ which (in $\lambda$) are holomorphic of uniform exponential type and satisfy the functional equation

$$\int_B \varphi(\lambda, b) e^{i(\lambda+1)(z,b)} db = \int_B \varphi(-\lambda, b) e^{-i(\lambda+1)(z,b)} db. \quad (2.11)$$

One can prove that condition (2.11) is equivalent to the following conditions (2.12) for the Fourier coefficients $\varphi_k(\lambda)$ of $\varphi$

$$\varphi_k(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\lambda, e^{i\theta}) e^{-ik\theta} d\theta$$

$$\varphi_k(-\lambda)p_k(-i\lambda) = \varphi_k(\lambda)p_k(i\lambda), \quad k \in \mathbb{Z}, \quad (2.12)$$

where $p_k(x)$ is the polynomial

$$p_k(x) = \frac{\Gamma\left(\frac{1}{2}(x+1) + |k|\right)}{\Gamma\left(\frac{1}{2}(x+1)\right)}.$$
The Paley–Wiener type theorems can be extended to the Schwartz spaces $S^p(D)$ ($0 < p \leq 2$). Roughly speaking, $f$ belongs to $S^p(D)$ if each invariant derivative $Df$ belongs to $L^p(D)$, more precisely, it is rapidly decreasing in the distance from 0 even after multiplication by the $p$th root of the volume element. Let $S_p$ denote the strip $|\text{Im } \lambda| < \frac{2}{p} - 1$ in $\mathbb{C}$ and $S(S_p \times B)$ the space of smooth functions on $S(S_p \times B)$ holomorphic (in $\lambda$) in $S_p$ and rapidly decreasing (uniformly for $b \in B$) on each line $\lambda = \xi + i\eta$ ($|\eta| < \frac{2}{p} - 1$). Then we have

**Theorem 2.4.** The Fourier transform $f \rightarrow \hat{f}$ on $D$ is a bijection of $S^p(D)$ onto the set of $\varphi \in S(S_p \times B)$ satisfying (2.11).

The theorem holds for all $X = G/K$ (Eguchi [Eg]). The proof is complicated. For the case of $K$-invariant functions (done for $p = 2$ by Harish–Chandra [H] and Trombi–Varadarajan [TV] for general $p$) a substantial simplification was done by Anker [A]. A further range theorem for the space of functions for which each invariant derivative has arbitrary exponential decay was proved by Oshima, Saburi and Wakayama [OSW]. See also Barker [Bar] (p. 27) for the operator Fourier transform of the intersection of all the Schwartz spaces on $SL(2\mathbb{R})$.

In classical Fourier analysis on $\mathbb{R}^n$ the Riemann–Lebesgue lemma states that for $f \in L^1(\mathbb{R})$, $\hat{f}$ tends to 0 at $\infty$. For $D$ the situation is a bit different.

**Theorem 2.5.** Let $f \in L^1(D)$. Then there exists a null set $N$ in $B$ such that if $b \in B - N$, $\lambda \rightarrow \hat{f}(\lambda, b)$ is holomorphic in the strip $|\text{Im } \lambda| < 1$ and

$$\lim_{\xi \rightarrow \infty} \hat{f}(\xi + i\eta, b) = 0$$

(2.13)

uniformly for $|\eta| \leq 1$.

The proof [HRSS] is valid even for symmetric spaces $X = G/K$ of arbitrary rank. Moreover

$$\|\hat{f}(\lambda, \cdot)\|_1 \rightarrow 0 \text{ as } \lambda \rightarrow \infty,$$

(2.14)

uniformly in the strip $|\text{Im } \lambda| \leq 1$, and this extends to $f \in L^p$ ($1 \leq p < 2$) this time in the strip $|\text{Im } \lambda| < \frac{2}{p} - 1$ ([SS]). In particular, if $f \in L^p(D)$ then there is a null set $N$ in $B$ such that $\hat{f}(\lambda, b)$ exists for $b \notin N$ and all $\lambda$ in the strip $|\text{Im } \lambda| < \frac{2}{p} - 1$.

The classical inversion formula for the Fourier transform on $\mathbb{R}^n$ now extends to $f \in L^p(D)$ ($1 \leq p < 2$) as follows.
Theorem 2.6. Let \( f \in L^p(D) \) and assume \( \tilde{f} \in L^1(\mathbb{R} \times B, \mu) \) (with \( \mu \) as in (2.10)). Then the inversion formula (2.9) holds for almost all \( z \in D \) (the Lebesgue set for \( f \)).

Again this holds for all \( X = G/K \). A result of this type was proved by Stanton and Thomas [ST] without invoking \( \tilde{f} \) explicitly (since the existence had not been established). The version in Theorem 2.6 is from [SS].

In Schwartz’s theory of mean–periodic functions [Sc] it is proved that any closed translation–invariant subspace of \( \mathcal{C}^\infty(\mathbb{R}) \) contains an exponential \( e^{ix} \). The analogous question here would be:

Does an arbitrary closed invariant subspace of \( \mathcal{C}^\infty(D) \) contain an exponential

\[
e^{\mu,b}(z) = e^{\mu(z,b)} \tag{2.15}
\]

for some \( \mu \in \mathbb{C} \) and some \( b \in B \)?

Here the topology of \( \mathcal{C}^\infty(D) \) is the usual Fréchet space topology and “invariant” refers to the action of the group \( G = SU(1,1) \) on \( D \). The answer is yes.

Theorem 2.7. Each closed invariant subspace \( E \) of \( \mathcal{C}^\infty(D) \) contains an exponential \( e^{\mu,b} \).

This was proved in [HS] for all symmetric \( X = G/K \) of rank one. Here is a sketch of the proof. By a result of Bagchi and Sitaram [BS] \( E \) contains a spherical function

\[
\varphi_\lambda(z) = \int_B e^{i\lambda(z,b)} \, db, \quad \varphi_\lambda = \varphi_{-\lambda}. \tag{2.16}
\]

For either \( \lambda \) or \( -\lambda \) it is true ([He9], Lemma 2.3, Ch. III) that the Poisson transform \( P_\lambda : F \to f \) where

\[
f(z) = \int_B e^{i\lambda+1(z,b)} F(b) \, db, \tag{2.17}
\]

maps \( L^2(B) \) into the closed invariant subspace of \( E \) generated by \( \varphi_\lambda \). On the other hand it is proved in [He9] (Ex. B1 in Ch. III) that \( e^{i\lambda+1,b} \) is a series of terms \( P_\lambda(F_n) \) where \( F_n \in L^2(B) \) and the series converges in the topology of \( \mathcal{C}^\infty(D) \). Thus \( e^{i\lambda+1,b} \in E \) as desired.

The following result for the Fourier transform on \( \mathbb{R}^n \) is closely related to the Wiener Tauberian theorem.
Let $f \in L^1(\mathbb{R}^n)$ be such that $\tilde{f}(u) \neq 0$ for all $u \in \mathbb{R}^n$. Then the translates of $f$ span a dense subspace of $L^1(\mathbb{R}^n)$.

There has been considerable activity in establishing analogs of this theorem for semisimple Lie groups and symmetric spaces. See e.g. [EM], [Sa], [Si1], [Si2]. The neatest version for $D$ seems to me to be the following result from [SS] which remains valid for $X = G/K$ of rank one.

Let $d(z, w)$ denote the distance in $D$ and if $\epsilon > 0$, let $L_\epsilon(D)$ denote the space of measurable functions $f$ on $D$ such that $\int_D |f(z)|e^{\epsilon d(0, z)} \, dz < \infty$. Let $T_\epsilon$ denote the strip $|\text{Im} \lambda| \leq 1 + \epsilon$.

**Theorem 2.8.** Let $f \in L_\epsilon(X)$ and assume $f$ is not almost everywhere equal to any real analytic function. Let

$$Z = \{\lambda \in T_\epsilon : \tilde{f}(\lambda, \cdot) \equiv 0\}.$$  

If $Z = \emptyset$ then the translates of $f$ span a dense subspace of $L^1(D)$.

A theorem of Hardy’s on Fourier transforms on $\mathbb{R}^n$ asserts in a precise fashion that $f$ and its Fourier transform cannot both vanish too fast at infinity. More precisely ([Ha]):

Assume

$$|f(x)| \leq Ae^{-\alpha|x|^2}, \quad |\tilde{f}(u)| \leq Be^{-\beta|u|^2},$$

where $A$, $B$, $\alpha$ and $\beta$ are positive constants and $\alpha \beta > \frac{1}{4}$. Then $f \equiv 0$.

Variations of this theorem for $L^p$ spaces have been proved by Morgan [M] and Cowling–Price [CP].

For the Fourier transform on $D$ the following result holds.

**Theorem 2.9.** Let $f$ be a measurable function on $D$ satisfying

$$|f(x)| \leq Ce^{-\alpha d(0, x)^2}, \quad |\tilde{f}(\lambda, b)| \leq Ce^{-\beta |\lambda|^2}$$

where $C$, $\alpha$, $\beta$ are positive constants. If $\alpha \beta > 16$ then $f \equiv 0$.

This is contained in Sitaram and Sundari [SiSu] §5 where an extension to certain symmetric spaces $X = G/K$ is also proved. The theorem for all such $X$ was obtained by Sengupta [Se], together with refinements in terms of $L^p(X)$.

Many such completions of Hardy’s theorem have been given, see [RS], [CSS], [NR], [Shi].
3 Eigenfunctions of the Laplacian

Consider first the plane $\mathbb{R}^2$ and the Laplacian

$$L^0 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}. $$

Given a unit vector $\omega \in \mathbb{R}^2$ and $\lambda \in \mathbb{C}$ the function $x \to e^{i\lambda(x,\omega)}$ is an eigenfunction

$$L_0 e^{i\lambda(x,\omega)} = -\lambda^2 e^{i\lambda(x,\omega)}. \quad (3.1)$$

Because of (2.3) one might expect all eigenfunctions of $L$ to be a “decomposition” into such eigenfunctions with fixed $\lambda$ but variable $\omega$.

Note that the function $\omega \to e^{i\lambda(x,\omega)}$ is the restriction to $SS^1$ of the holomorphic function

$$z \to \exp \left[ \frac{i}{2}(i\lambda)x_1(z + z^{-1}) + \frac{1}{2}i\lambda x_2(z - z^{-1}) \right] \quad z \in \mathbb{C} - (0),$$

which satisfies a condition

$$\sup_z \left( |f(z)|e^{-a|z|^{-b}|z|^{-1}} \right) < \infty, \quad (3.2)$$

with $a, b \geq 0$. Let $E_{a,b}$ denote the Banach space of holomorphic functions satisfying (3.2), the norm being the expression in (3.2). We let $E$ denote the union of the spaces $E_{a,b}$ and give it the induced topology. We identify the elements of $E$ with their restrictions to $SS^1$ and call the members of the dual space $E^\prime$ entire functionals.

**Theorem 3.1.** ([He6]) The eigenfunctions of $L^0$ on $\mathbb{R}^2$ are precisely the harmonic functions and the functions

$$f(x) = \int_{SS^1} e^{i\lambda(x,\omega)} dT(\omega) \quad (3.3)$$

where $\lambda \in \mathbb{C} - (0)$ and $T$ is an entire functional on $SS^1$.

For the non-Euclidean metric (1.3) (with $c = 1$) the Laplacian is given by

$$L = (1 - x^2 - y^2)^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \quad (3.4)$$

and the exponential function $e_{\mu,b}(z) = e^\mu(z,b)$ is an eigenfunction:

$$L z e^{i(\lambda+1)(z,b)} = -(\lambda^2 + 1)e^{i(\lambda+1)(z,b)}. \quad (3.5)$$
In particular, the function \( z \rightarrow e^{2(z,b)} \) is a harmonic function and in fact coincides with the classical Poisson kernel from potential theory:

\[
e^{2(z,b)} = \frac{1 - |z|^2}{|z - b|^2}.
\]  

(3.6)

Again the eigenfunctions of \( L \) are obtained from the functions \( e_{\mu,b} \) by superposition. To describe this precisely consider the space \( \mathcal{A}(B) \) of analytic functions on \( B \). Each \( F \in \mathcal{A}(B) \) extends to a holomorphic function on a belt \( B_\epsilon : 1 - \epsilon < |z| < 1 + \epsilon \) around \( B \). The space \( \mathcal{H}(B_\epsilon) \) of holomorphic functions on \( B_\epsilon \) is topologized by uniform convergence on compact subsets. We can view \( \mathcal{A}(B) \) as the union \( \bigcup_{n=1}^{\infty} \mathcal{H}(B_{1/n}) \) and give it the inductive limit topology. The dual space \( \mathcal{A}'(B) \) then consists of the analytic functionals on \( B \) (or hyperfunctions in \( B \)).

**Theorem 3.2.** ([He4], IV, §1). The eigenfunctions of \( L \) are precisely the functions

\[
u(z) = \int_B e^{\mu(z,b)} \, dT(b),
\]  

(3.7)

where \( \mu \in \mathbb{C} \) and \( T \in \mathcal{A}'(B) \).

Lewis in [L] has proved (under minor restriction on \( \mu \)) that \( T \) in (3.7) is a distribution if and only if \( u \) has at most an exponential growth (in \( d(0, z) \)). On the other hand, Ban and Schlichtkrull proved in [BaS] that

\[
T \in \mathcal{C}_c(B) \iff \text{all the invariant derivatives of } u \text{ have the same exponential growth.}
\]

We consider now the natural group representations on the eigenspaces. The group \( \mathbf{M}(2) \) of isometries of \( \mathbb{R}^2 \) acts transitively on \( \mathbb{R}^2 \) and leaves the Laplacian \( L^0 \) invariant: \( L^0(f \circ \tau) = (L^0f) \circ \tau \) for each \( \tau \in \mathbf{M}(2) \). If \( \lambda \in \mathbb{C} \) the eigenspace

\[
\mathcal{E}_\lambda = \{ f \in \mathcal{C}_c(\mathbb{R}^2) : L^0f = -\lambda^2f \}
\]

is invariant under the action \( f \rightarrow f \circ \tau^{-1} \) so we have a representation \( T_\lambda \) of \( \mathbf{M}(2) \) on \( \mathcal{E}_\lambda \) given by \( T_\lambda(\tau)(f) = f \circ \tau^{-1} \), the eigenspace representation.

**Theorem 3.3.** ([He6]) The representation \( T_\lambda \) is irreducible if and only if \( \lambda \neq 0 \).

Similarly the group \( G = \text{SU}(1,1) \) of conformal transformations

\[
z \rightarrow \frac{az + b}{bz + a} \quad (|a|^2 - |b|^2 = 1)
\]

9
leaves $\mathfrak{L}$ and the operator $L$ in $\mathfrak{R}$ invariant. Thus we get again an eigenspace representation $\tau_\lambda$ of $G$ on each eigenspace

$$E_\lambda = \{ f \in C^\infty(D) : Lf = -(\lambda^2 + 1)f \} .$$

**Theorem 3.4. ([He4])** The representation $\tau_\lambda$ is irreducible if and only if $i\lambda + 1 / 2 \notin \mathbb{Z}$.

Again all these results extend to Euclidean spaces of higher dimensions and suitably formulated, to all symmetric spaces $G/K$ of the noncompact type.

### 4 The Radon Transform

#### A. The Euclidean Case.

Let $d$ be a fixed integer, $0 < d < n$ and let $G(d, n)$ denote the space of $d$-dimensional planes in $\mathbb{R}^n$. To a function $f$ on $\mathbb{R}^n$ we associate a function $\hat{f}$ on $G(d, n)$ by

$$\hat{f}(\xi) = \int f(x) \, dm(x), \quad \xi \in G(d, n),$$

where $dm$ is the Euclidean measure on $\xi$. The transform $f \to \hat{f}$ is called the $d$-plane transform. For $d = 1, n = 2$ it is the classical Radon transform. The parity of $d$ turns out to be important.

The inversion of the transform $f \to \hat{f}$ is well known (case $d = n - 1$ in [R], [J], [GS], general $d$ in [F], [He1], [He2]). We shall give another group-theoretic method here, resulting in alternative inversion formulas.

The group $G = M(n)$ acts transitively both on $\mathbb{R}^n$ and on $G(d, n)$. In particular, $\mathbb{R}^n = G/K$ where $K = O(n)$. Let $p > 0$. Consider a pair $x \in \mathbb{R}^n, \xi \in G(d, n)$ at distance $p = d(x, \xi)$. Let $g \in G$ be such that $g \cdot 0 = x$. Then the family $kg^{-1} \cdot \xi$ constitutes the set of elements in $G(d, n)$ at distance $p$ from 0. Along with the transform $f \to \hat{f}$ we consider the dual transform $\varphi \to \check{\varphi}$ given by

$$\check{\varphi}(x) = \int_{\xi \ni x} \varphi(\xi) \, d\mu(\xi),$$

the average of $\varphi$ over the set of $d$-planes passing through $x$. More generally we put

$$\check{\varphi}_p(x) = \int_{d(\xi, x) = p} \varphi(\xi) \, d\mu(\xi),$$
the average of \( \varphi \) over the set of \( d \)-planes at distance \( p \) from \( x \). Since \( K \) acts transitively on the set of \( d \)-planes through 0 we see by the above that
\[
\hat{\varphi}_p(g \cdot 0) = \int_K \varphi(gkg^{-1} \cdot \xi) \, dk, \tag{4.4}
\]
d\( k \) being the normalized Haar measure on \( K \). Let \( (M^r f)(x) \) denote the mean-value of \( f \) over the sphere \( S_r(x) \) of radius \( r \) with center \( x \). If \( z \in \mathbb{R}^n \) has distance \( r \) from 0 we then have
\[
(M^r f)(g \cdot 0) = \int_K f(gk \cdot z) \, dk. \tag{4.5}
\]
We thus see that since \( d(0, g^{-1} \cdot y) = d(x, y) \),
\[
(\hat{f})_p^\vee(x) = \int_K \hat{f}(gkg^{-1} \cdot \xi) \, dk = \int_K \int_{\xi} f(gkg^{-1} \cdot y) \, dm(y) \, dk
= \int_{\xi} dm(y) \int_K f(gkg^{-1} \cdot y) \, dk = \int_{\xi} (M^{d(x, y)} f)(x) \, dm(y).
\]
Let \( x_0 \) be the point in \( \xi \) at minimum distance \( p \) from \( x \). The integrand \( (M^{d(x, y)} f)(x) \) is constant in \( y \) on each sphere in \( \xi \) with center \( x_0 \). It follows that
\[
(\hat{f})_p^\vee(x) = \Omega_d \int_0^{\infty} (M^q f)(x) r^{d-1} \, dr, \tag{4.6}
\]
where \( r = d(x_0, y), \, q = d(x, y), \, \Omega_d \) denoting the area of the unit sphere in \( \mathbb{R}^d \). We have \( q^2 = p^2 + r^2 \) so putting \( F(q) = (M^q f)(x), \hat{F}(p) = (\hat{f})_p^\vee(x) \) we have
\[
\hat{F}(p) = \Omega_d \int_{p}^{\infty} F(q)(q^2 - p^2)^{d/2 - 1} \, dq. \tag{4.7}
\]
This Abel-type integral equation has an inversion
\[
F(r) = c(d) \left( \frac{d}{d(r^2)} \right)^d \int_{r}^{\infty} p(p^2 - r^2)^{d/2 - 1} \hat{F}(p) \, dp, \tag{4.8}
\]
where \( c(d) \) is a constant, depending only on \( d \). Putting \( r = 0 \) we obtain the inversion formula
\[
(f(x) = c(d) \left[ \left( \frac{d}{d(r^2)} \right)^d \int_{r}^{\infty} p(p^2 - r^2)^{d/2 - 1} (\hat{f})_p^\vee(x) \, dp \right]_{r=0}. \tag{4.9}
\]
Note that in (4.8)
\[ p(p^2 - r^2)^{d/2-1} = \frac{d}{dp}(p^2 - r^2)^{(d/2)} \cdot \frac{1}{d} \]
so in (4.8) we can use integration by parts and the integral becomes
\[ C \int_{r}^{\infty} (p^2 - r^2)^{d/2} \hat{F}'(p) \, dp . \]

Applying \( \frac{d}{d(r^2)} = \frac{1}{2r} \frac{d}{dr} \) to this integral reduces the exponent \( d/2 \) by 1. For \( d \) odd we continue the differentiation \( \frac{d^{d+1}}{2} \) times until the exponent is \( -\frac{1}{2} \). For \( d \) even we continue until the exponent is 0 and then replace \( \int_{r}^{\infty} \hat{F}'(p) \, dp \) by \(-\hat{F}(r)\). This \( \hat{F}(r) \) is an even function so taking \( (d/d(r^2))^{d/2} \) at \( r = 0 \) amounts to taking a constant multiple of \( (d/dr)^d \) at \( r = 0 \). We thus get the following refinement of (4.9) where we recall that \( (\hat{f})_{\nu}^{-\nu}(x) \) is the average of the integrals of \( f \) over the \( d \)-planes tangent to \( S_r(x) \).

**Theorem 4.1.** The \( d \)-plane transform is inverted as follows:

(i) If \( d \) is even then
\[ f(x) = C_1 \left[ \frac{d}{dr} \right]^d (\hat{f})_{\nu}^{-\nu}(x) \bigg|_{r=0} . \tag{4.10} \]

(ii) If \( d \) is odd then
\[ f(x) = C_2 \left[ \frac{d}{d(r^2)} \right]^{(d-1)/2} \int_{r}^{\infty} (p^2 - r^2)^{-1/2} \frac{d}{dp} (\hat{f})_{\nu}^{-\nu}(x) \, dp \bigg|_{r=0} . \tag{4.11} \]

(iii) If \( d = 1 \) then
\[ f(x) = -\frac{1}{\pi} \int_{0}^{\infty} \frac{d}{dp} (\hat{f})_{\nu}^{-\nu}(x) \, dp . \tag{4.12} \]

For \( n = 2 \) formula (4.12) is proved in Radon’s original paper [R]. Note that the constant \(-1/\pi\) is the same for all \( n \). In the case \( d = n - 1 \) the formula in (i) coincides with formula (21) in Rouvière [Ro].
Another inversion formula ([He1], [He2]) valid for all \( d \) and \( n \) is
\[
f = c(-L)^{d/2}((\hat{f})^\vee)
\] (4.13)
where
\[
c = \frac{\Gamma \left( \frac{n-d}{2} \right)}{(4\pi)^{d/2} \Gamma \left( \frac{d}{2} \right)}.
\]
Here the fractional power of \( L \) is defined in the usual way by the Fourier transform. The parity of \( d \) shows up in the same way as in Theorem 4.1.

For range questions for the transform \( f \to \hat{f} \) see an account in [He10] and references there.

B. The Hyperbolic Case.

The hyperbolic space \( H^n \) is the higher-dimensional version of (1.3) and its Riemannian structure is given by
\[
ds^2 = 4 \frac{dx_1^2 + \cdots + dx_n^2}{(1 - x_1^2 - \cdots - x_n^2)^2}
\] (4.14)
in the unit ball \( |x| < 1 \). The constant 4 is chosen such that the curvature is now \(-1\). The \( d \)-dimensional totally geodesic submanifolds are spherical caps perpendicular to the boundary \( B : |x| = 1 \). They are natural analogs of the \( d \)-planes in \( R^n \). We have accordingly a Radon transform \( f \to \hat{f} \), where
\[
\hat{f}(\xi) = \int_\xi f(x) \, dm(x) \quad \xi \in \Xi,
\] (4.15)
where \( \Xi \) is the space of \( d \)-dimensional totally geodesic submanifolds of \( H^n \).

The group \( G \) of isometries of \( H^n \) acts transitively on \( \Xi \) as well. As in (4.2)—(4.3) we consider the dual transform
\[
\check{\varphi}(x) = \int_{\xi \ni x} \varphi(\xi) \, d\mu(\xi)
\] (4.16)
and more generally for \( p \geq 0 \),
\[
\check{\varphi}_p(x) = \int_{d(\xi,x)=p} \varphi(\xi) \, d\mu(\xi),
\] (4.17)
the mean value of $\varphi$ over the set of $\xi \in \Xi$ at distance $p$ from $x$. The formula

$$\left( \hat{f} \right)^\vee_p(x) = \int_\xi (M^{d(x,y)} f)(x) \, dm(y) \quad (4.18)$$

is proved just as before. Let $x_0$ be the point in $\xi$ at minimum distance $p$ from $x$ and put $r = d(x_0, y), q = d(x, y)$. Since the geodesic triangle $(xx_0y)$ is right angled at $x_0$ we have by the cosine rule

$$\cosh q = \cosh p \cosh r. \quad (4.19)$$

Also note that since $\xi$ is totally geodesic, distances between two points in $\xi$ are the same as in $\textbf{H}^n$. In particular $(M^{d(x,y)} f)(x)$ is constant as $y$ varies on a sphere in $\xi$ with center $x_0$. Therefore (4.18) implies

$$\left( \hat{f} \right)^\vee_p(x) = \Omega_d \int_0^\infty (M^q f)(x) \sinh^{d-1} r \, dr. \quad (4.20)$$

For $x$ fixed we put

$$F(\cosh q) = (M^q f)(x), \quad \hat{F}(\cosh p) = (\hat{f})^\vee_p(x),$$

substitute in (4.20) and use (4.19). Writing $t = \cosh p, s = \cosh r$ we obtain the integral equation

$$\hat{F}(t) = \Omega_d \int_1^\infty F(ts)(s^2 - 1)^{d/2 - 1} \, ds. \quad (4.21)$$

Putting here $u = ts, \, ds = t^{-1} du$ we get the Abel–type integral equation

$$t^{d-1} \hat{F}(t) = \Omega_d \int_t^\infty u^{-1} F(u)(u^2 - t^2)^{d/2 - 1} \, du,$$

which by (4.8) is inverted by

$$r^{-1} F(r) = c(d) \left( \frac{d}{d(r^2)} \right)^d \int_r^\infty \frac{t(t^2 - r^2)^{d/2 - 1} \, t^{d-1}}{r} \hat{F}(t) \, dt. \quad (4.22)$$

Here we put $r = 1$ and $s(p) = \cosh^{-1} p$. We then obtain the following variation of Theorem 3.12, Ch. I in [He9]:

14
Theorem 4.2. The transform \( f \to \tilde{f} \) is inverted by

\[
f(x) = C \left[ \frac{d}{d(r^2)} \int_r^\infty (t^2 - r^2)^{d/2-1} t^d (\hat{f})^{\vee}_{\nu(t)} (x) \, dt \right]_{r=1}. \tag{4.23}
\]

As in the proof of Theorem 4.1 we can obtain the following improvement.

Theorem 4.3. (i) If \( d \) is even the inversion can be written

\[
f(x) = C \left[ \frac{d}{d(r^2)} \right]^{d/2} (r^d-1) (\hat{f})^{\vee}_{\nu(r)} (x) \right]_{r=1}. \tag{4.24}
\]

(ii) If \( d = 1 \) then

\[
f(x) = -\frac{1}{2} \int_0^\infty \frac{d}{\sinh p \, dp} \left( (\hat{f})^{\vee}_{p} (x) \right) \, dp. \tag{4.25}
\]

Proof: Part (i) is proved as (4.10) except that we no longer can equate \( (d/d(r^2))^{d/2} \) with \((d/dr)^d\) at \( r = 1 \).

For (ii) we deduce from (4.22) since \( t(t^2 - r^2)^{-1/2} = \frac{d}{dt}(t^2 - r^2)^{1/2} \) that

\[
F(1) = -\frac{c(1)}{2} \left[ \frac{d}{dr} \int_r^\infty (t^2 - r^2)^{1/2} \frac{d}{dt} \hat{F}(t) \, dt \right]_{r=1}
= \frac{c(1)}{2} \int_1^\infty (t^2 - 1)^{-1/2} \frac{d}{dt} \hat{F}(t) \, dt.
\]

Putting again \( t = \cosh p, \, dt = \sinh p \, dp \) our expression becomes

\[
\frac{c(1)}{2} \int_0^\infty \frac{1}{\sinh p} \frac{d}{dp}((\hat{f})^{\vee}_{p})(x) \, dp.
\]

Remark For \( n = 2, \, d = 1 \) formula (4.25) is stated in Radon [R], Part C. The proof (which is only indicated) is very elegant but would not work for \( n > 2 \).
For $d$ even (4.24) can be written in a simpler form ([He1]) namely

$$f = c \, Q_d(L)((\hat{f})^\lor),$$

(4.26)

where $c = \frac{\Gamma\left(\frac{n-d}{2}\right)}{(-4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}$ and $Q_d$ is the polynomial

$$Q_d(x) = (x + (d-1)(n-d)) \cdots (x + (n+1)(n+2)).$$

The case $d = 1, n = 2$ is that of the X-ray transform on the non-Euclidean disk (4.15) for $n = 2$. Here are two further alternatives to the inversion formula (4.25). Let $S$ denote the integral operator

$$(Sf)(x) = \int_D (\coth d(x, y) - 1)f(y) \, dy.$$  

(4.27)

Then

$$LS(\hat{f})^\lor = -4\pi^2 f, \quad f \in \mathcal{D}(X).$$

(4.28)

This is proved by Berenstein–Casadio [BC]; see [He10] for a minor simplification. By invariance it suffices to prove (4.28) for $f$ radial and then it is verified by taking the spherical transform on both sides. Less explicit versions of (4.28) are obtained in [BC] for any dimension $n$ and $d$.

One more inversion formula was obtained by Lissiano and Ponomarev [LP] using (4.23) for $d = 1, n = 2$ as a starting point. By parameterizing the geodesics $\gamma$ by the two points of intersection of $\gamma$ with $B$ they prove a hyperbolic analog of the Euclidean formula:

$$f(x) = \int_{S^1} \left\{ \mathcal{H}_p \frac{d}{dp} \hat{f}(\omega, p) \right\}_{p=(\omega, x)} d\omega,$$

(4.29)

which is an alternative to (4.12). Here $\mathcal{H}_p$ is a normalized Hilbert transform in the variable $p$ and $\hat{f}(\omega, p)$ is $\hat{f}(\xi)$ for the line $(x, \omega) = p$, where $|\omega| = 1$.

In the theorems in this section we have not discussed smoothness and decay at infinity of the functions. Here we refer to [Je], [Ru1], [Ru2], [BeR1] and [BeR2] as examples.

Additional inversion formulas for the transform $f \to \hat{f}$ can be found in [Sm], [Ru3] and [K]. The range problem for the transform $f \to \hat{f}$ is treated in [BCK] and [I].
Added in Proof:

I have since this was written proved an inversion formula for the X-ray transform on a noncompact symmetric space of rank \( l > 1 \). It is similar to (4.12) except that in (4.17) one restricts the averaging to the set of geodesics each of which lies in a flat \( l \)-dimensional totally geodesic submanifold through \( x \) and at distance \( p \) from \( x \). On the other hand, Rouvière had proved earlier that the inversion formula (4.25) holds almost unchanged for the X-ray transform on a noncompact symmetric space of rank 1.

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