EMBEDDINGS OF HOMOLOGY EQUIVALENT MANIFOLDS WITH BOUNDARY

D. Gonçalves and A. Skopenkov

Abstract. We prove a theorem on equivariant maps implying the following two corollaries:

1. Let $N$ and $M$ be compact orientable $n$-manifolds with boundaries such that $M \subset N$, the inclusion $M \to N$ induces an isomorphism in integral cohomology, both $M$ and $N$ have $(n - d - 1)$-dimensional spines and $m \geq \max\{n + 3, \frac{3n+2-d}{2}\}$. Then the restriction-induced map $\text{Emb}^m(N) \to \text{Emb}^m(M)$ is bijective. Here $\text{Emb}^m(X)$ is the set of embeddings $X \to \mathbb{R}^m$ up to isotopy (in the PL or smooth category).

2. For a 3-manifold $N$ with boundary whose integral homology groups are trivial and such that $N \not\cong D^3$ (or for its special 2-sphere $N$) there exists an equivariant map $\tilde{N} \to S^2$, although $N$ does not embed into $\mathbb{R}^3$.

The second corollary completes the answer to the following question: for which pairs $(m,n)$ for each $n$-polyhedron $N$ the existence of an equivariant map $\tilde{N} \to S^{m-1}$ implies embeddability of $N$ into $\mathbb{R}^m$? An answer was known for each pair $(m,n)$ except $(3,3)$ and $(3,2)$.

This note is on the classical problem of classification of embeddings into Euclidean spaces. For recent surveys see [Sk08, MA]; whenever possible we refer to these surveys not to original papers. As a main tool we use the Haefliger-Wu invariant defined below.

We begin with the formulation of our main homotopy result. Let $\tilde{N} = \{(x,y) \in N \times N \mid x \neq y\}$ be the deleted product of $N$. Let $\mathbb{Z}_2$ act on $\tilde{N}$ and on $S^{m-1}$ by exchanging factors and antipodes. Denote by $\pi_{eq}^{m-1}(\tilde{N})$ be the set of equivariant maps $\tilde{N} \to S^{m-1}$ up to equivariant homotopy. The set $\pi_{eq}^{m-1}(\tilde{N})$ can be effectively calculated [CF60, beginning of §2, Ad93, 7.1, Sk02, §6, Sk08, §5]. Note that $\pi_{eq}^{m-1}(\tilde{N}) = 0$ for $m < n$ because $\tilde{N} \supset \tilde{D}^n \simeq_{eq} S^{n-1}$.

We omit $\mathbb{Z}$-coefficients from the notation.

Theorem. Let $N$ and $M$ be compact orientable connected $n$-manifolds with non-empty boundaries such that $M \subset N$ and the inclusion $M \to N$ induces an isomorphism in cohomology. Then the restriction-induced map $\pi_{eq}^{m-1}(\tilde{N}) \to \pi_{eq}^{m-1}(\tilde{M})$ is bijective.

This homotopy result is interesting because of the following topological corollaries. Denote $\text{CAT} = \text{DIFF}$ or $\text{PL}$. For a CAT manifold $N$ let $\text{Emb}^m_{\text{CAT}}(N)$ be the set of CAT embeddings $N \to \mathbb{R}^m$ up to CAT isotopy. A folklore general conjecture, supported by some known results (for a survey see e.g. [RS99]) is that $\text{Emb}^m_{\text{CAT}}(N)$ is not changed under homology equivalence of $N$ (i.e. under a map $f : M \to N$ between manifolds inducing an isomorphism in (co)homology), in the PL case for $m \geq n + 3$ and in the DIFF case for $m \geq \frac{3n}{2} + 2$.

1991 Mathematics Subject Classification. Primary: 57Q35, 57R40; Secondary: 55S15, 57Q30, 57Q60.

Key words and phrases. Embedding, deleted product, local coefficients, homology ball.

This is a new version of [GS06]. Gonçalves is supported in part by FAPESP ‘Projecto Temático Topologia Algébrica, Geométrica e Diferencial’ number 2008/57607-6. Skopenkov is supported in part by the Russian Foundation for Basic Research Grant No. 12-01-00748-a and by Simons-IUM Fellowship. We are grateful to S. Melikhov and A. Volovikov for useful discussions.
**Corollary.** Let $N$ and $M$ be compact orientable $n$-manifolds with non-empty boundaries such that $M \subset N$, the inclusion $M \to N$ induces an isomorphism in cohomology, both $M$ and $N$ have $(n - d - 1)$-dimensional spines and $m \geq \max\{n + 3, \frac{3n+2-d}{2}\}$. Then the restriction-induced map $\text{Emb}^m_{\text{CAT}}(N) \to \text{Emb}^m_{\text{CAT}}(M)$ is bijective.

Recall that
- a subpolyhedron $K$ of a manifold $N$ is called a spine of $N$ if $N$ is a regular neighborhood of $K$ in $N$ (or, equivalently, if $N$ collapses to $K$) [RS72].
- a closed manifold $N$ (or a pair $(N, \partial N)$) is called homologically $d$-connected, if $N$ is connected and $H_i(N) = 0$ for each $i = 1, \ldots, d$ (or $H_i(N, \partial N) = 0$ for each $i = 0, \ldots, d$).

In the DIFF category the restriction $m \geq \frac{3n+2-d}{2}$ can be relaxed to $m \geq \frac{3n+1-d}{2}$.

By the Corollary, any homology ball unknots in codimension at least 3, cf. [Sc77].

The Corollary follows from the Theorem and the bijectivity of $\alpha$-invariant [RS99, §4, Sk02, Theorems 1.1α∂ and 1.3α∂], which is defined as follows. For an embedding $f : N \to \mathbb{R}^m$ define a map

$$\widetilde{f} : \widetilde{N} \to S^{m-1} \quad \text{by} \quad \widetilde{f}(x, y) = \frac{fx - fy}{|fx - fy|}.$$ 

The equivariant homotopy class $\alpha(f)$ of the above-defined $\widetilde{f}$ in $\pi_{eq}^{m-1}(\widetilde{N})$ is clearly an isotopy invariant. Thus is defined the Haefliger-Wu (deleted product) invariant

$$\alpha = \alpha^m_{\text{CAT}}(N) : \text{Emb}^m_{\text{CAT}}(N) \to \pi_{eq}^{m-1}(\widetilde{N}).$$

**Remarks.**

(a) If in Theorem and in Corollary the inclusion-induced homomorphism $H^i(N) \to H^i(M)$ is an isomorphism only for $i \geq l > 0$, then the corresponding restriction-induced maps are bijective for $m \geq n + l$ and surjective for $m = n + l - 1$.

(b) The assumption that $(N, M)$ is a codimension 0 pair is essential in the Theorem and the Corollary. Indeed, take $N = D^p \times S^q$ and $M = S^q$. For $m \geq 3q/2 + 2$ we have $\# \text{Emb}^m(S^q) = 1$ while $\text{Emb}^m(D^p \times S^q) = \pi_q(V_{m-q,p})$ can contain more than one element (specific examples are particularly easy to find for $p = 1$, when $V_{m-q,p} \simeq S^{m-q-1}$).

(c) The assumption that $N$ has boundary is not essential in the Theorem and the Corollary. But these results are trivial for closed $N$: if $N$ is closed and $M \neq N$, then the assumptions are never fulfilled because $H^n(N) \not\cong H^n(M)$.

(d) The conclusion of the Theorem for closed manifolds is not always fulfilled, because there are closed manifolds non-embeddable in the same dimension as the corresponding punctured manifolds.

(e) The Theorem is clearly true for $m < n$ because both sets are empty. We conjecture that the Theorem holds for $m = n$ and for $m = n + 1$.

Now let us present motivation for the second corollary of the Theorem. From the construction of the map $\tilde{f}$ above it follows that

(*) if $N$ embeds into $\mathbb{R}^m$, then there exists an equivariant map $\widetilde{N} \to S^{m-1}$.

The existence of an equivariant map $\widetilde{N} \to S^{m-1}$ can be checked for many cases [CF60, beginning of §2, Ad93, 7.1, Sk08, §5]. Thus if a converse to (*) is true, the embedding problem is reduced to a manageable (although not trivial) algebraic problem. So in 1960s

---

1We remark that for a compact connected $n$-manifold $N$ with boundary, the property of having an $(n - d - 1)$-dimensional spine is close to $d$-connectedness of $(N, \partial N)$. Indeed, for a compact connected $n$-manifold $N$ with boundary and an $(n - d - 1)$-dimensional spine, the pair $(N, \partial N)$ is homologically $d$-connected. On the other hand, every compact connected $n$-manifold $N$ with boundary for which $(N, \partial N)$ is $d$-connected, $\pi_1(\partial N) = 0$, $d + 3 \leq n$ and $(n, d) \not\in \{(5, 2), (4, 1)\}$, has an $(n - d - 1)$-dimensional spine [Wa64, Theorem 5.5, Ho69, Lemma 5.1 and Remark 5.2].
there appeared a problem to find conditions under which the converse to (*) is true. The converse for (*) was known to be

true for an n-polyhedron $N$ and $2m \geq 3n + 3$ or $m = 2n = 2$ [RS99, §4, Sk08, §5], cf. [Sk98, Theorem 1.3];

false for each pair $(m, n)$ such that $\max\{4, n\} \leq m \leq \frac{3n}{2} + 1$ and some n-polyhedron $N$ [RS99, §4, Sk08, §5], cf. [Sk98, Example 1.4].

In the only remaining cases $m = 3$ and $n \in \{2, 3\}$ it was unknown if the converse to (*) is true. The counterexamples to the converse of (*) for $m = n \geq 4$ and $m = n + 1 \geq 4$ [MS67, Hu88] cannot be directly extended to $m = 3$ because they used $m$-dimensional contractible manifold distinct from the $m$-ball, which apparently does not exist for $m = 3$.

Recall that a homology $n$-ball is an $n$-manifold with boundary whose homology groups are the same as those of the $n$-ball. A special spine is defined e.g. in [Ca65].

**Proposition.** The converse to (*) is false in the cases $m = 3$ and $n \in \{2, 3\}$: if $N$ is either a non-trivial homology ball or a special spine of a non-trivial homology ball, then $N$ does not embed into $\mathbb{R}^3$ but there exists an equivariant map $\tilde{N} \to S^2$.

**Proof.** The non-embeddability follows because if a special spine of a homology ball $N$ embeds into $\mathbb{R}^3$, then the regular neighborhood in $\mathbb{R}^3$ of this spine is homeomorphic to $N$ [Ca65], which contradicts to the non-triviality of $N$.

It suffices to prove the existence of an equivariant map $\tilde{N} \to S^2$ for a homology 3-ball $N$.\(^2\)

Analogously to [Ad93, end of §7.1] (or by Lemma 2 below) it suffices to prove that $H^i(\tilde{N}) = 0$ for each $i \geq 3$. We prove this for $i = 3$; the proof for each $i \geq 4$ is analogous. Let $\Delta$ be the interior of a closed regular neighborhood in $N \times N$ of the diagonal. Then

$$H^3(\tilde{N}) \cong H^3(N \times N - \Delta) \cong H_3(N \times N - \Delta, \partial(N \times N - \Delta)) \cong$$

$$\cong H_3(N \times N, Cl \Delta \cup \partial(N \times N)) \cong H_2(Cl \Delta \cup \partial(N \times N)) = 0,$$

where

- the first isomorphism follows because $N \times N - \Delta$ is a deformation retract of $\tilde{N}$,
- the second one by Lefschetz duality (recall that $\tilde{N}$ is orientable if $N$ is a homology ball),
- the third one by excision,
- and the fourth one by exact sequence of pair.

Using the Mayer-Vietoris sequence for

$$\partial(N \times N) = N \times \partial N \cup \partial N \times N$$

and noting that $\partial N \cong S^2$, we prove that $H_2(\partial(N \times N)) = 0$. Using the Mayer-Vietoris sequence for $Cl \Delta \cup \partial(N \times N)$ and noting that $\Delta \cong N$ and $Cl \Delta \cap \partial(N \times N)$ is a regular neighborhood in $N \times N$ of the diagonal of $\partial N$, i.e. is homotopy equivalent to $\partial N \cong S^2$, we prove the last isomorphism. \(\square\)

Manifolds $\tilde{N}$ and $\tilde{M}$ are homotopy equivalent to $(2n - 1)$-dimensional CW complexes. Hence the Theorem follows by the cases $l = 0$ of Lemmas 1 and 2, see below. Remark (a) follows by (the general cases of) Lemma 1 and 2.

---

\(^2\)This existence follows from the (unproved) case $m = n = 3$ of the Theorem because an inclusion of the standard ball into $N$ induces isomorphisms in cohomology.

\(^3\)Another proof of the Proposition could possibly be obtained by using the fact that for the homology 3-ball $N$, which is a punctured boundary of the Mazur 4-manifold, there exists an equivariant map $\Sigma N \to S^3$ [MRS03]. The obstruction to equivariant desuspension of this map on $\tilde{P}$ (where $P$ is the special spine of $N$) apparently lies in $H^4(\tilde{P})$, which group is trivial because $P$ is acyclic [We68].
Lemma 1. Let $N$ and $M$ be compact orientable connected $n$-manifolds with non-empty boundaries such that $M \subset N$ and the inclusion induces an isomorphism $H^i(N) \to H^i(M)$ for $i \geq l$. Then $H^i(\tilde{N}, \tilde{M}) = 0$ for each $i \geq n + l$.

Lemma 2. [BG71, 3.2] Suppose $X, Y$ are finite connected CW-complexes with free involutions, $f : X \to Y$ is an equivariant map and $l$ is a non-negative integer. If $f^* : H^i(Y) \to H^i(X)$ is an isomorphism for each $i > l$ and is onto for $i = l$, then
\[(d_1) \quad f^2 : \pi^1_{eq}(Y) \to \pi^1_{eq}(X) \text{ is a 1-1 correspondence for } i > l \text{ and is onto for } i = l.\]

We give a proof of Lemma 2 (which was not presented in [BG71]) using standard argument and following [HH62, pp. 236-237], cf. [Me09, Proof of Lemma 8.1]. Lemma 2 was used in the previous version [GS06] of this paper; the proof was essentially presented there but contains mistakes which are corrected here.

Proof of Lemma 1 for $l = 0$. Let $N_0$ and $M_0$ be the interiors of $N$ and $M$, respectively. It suffices to prove Lemma 1 for $N$ and $M$ replaced by $N_0$ and $M_0$.

(Indeed, the collar theorem for the boundary of a manifold states that there is a neighborhood of $\partial M$ in $M$ which is homeomorphic to the product $\partial M \times [0, 1]$ so that $\partial M \times \{0\}$ is mapped homeomorphically to the boundary. Therefore there is an embedding $\phi : N \to N_0$ which is a homotopy inverse of the inclusion $N_0 \to N$. Analogously $\phi \times \phi : \tilde{N} \to \tilde{N}_0$ is a homotopy inverse of the inclusion $\tilde{N}_0 \to \tilde{N}$. Same observations hold for $N$ replaced by $M$. So it suffices to prove Lemma 1 for $N$ and $M$ replaced by $N_0$ and $M_0$.)

Let $x_0 \in M_0 \subset N_0$ be a base point for $M_0$ and $N_0$. Consider the following mapping of bundles (which are given by projections onto the first factor):
\[
\begin{array}{ccc}
M_0 - x_0 & \to & \tilde{M}_0 & \to & M_0 \\
\downarrow \subset & & \downarrow \subset & & \downarrow \subset \\
N_0 - x_0 & \to & \tilde{N}_0 & \to & N_0
\end{array}
\]

The action of $\pi_1(M_0)$ in the cohomology $H^i(M_0 - x_0)$ of the fiber is trivial for each $i$.

(Indeed, this follows for $i = n$ because $H^n(M_0 - x_0) = 0$ and for $i < n - 1$ because $H^i(M_0 - x_0) \cong H^i(M_0)$ and the bundle is the restriction of the trivial bundle $M_0 \times M_0 \to M_0$. For $i = n - 1$ we have $M_0 - x_0 \simeq M_0 \vee S^{n-1}$, so $H^{n-1}(M_0 - x_0) \cong H^{n-1}(M_0) \oplus \mathbb{Z}$. The action of an element $\alpha \in \pi_1(M_0)$ is given by the identity on the first summand and multiplication by the sign of the loop on $\mathbb{Z}$. Since $M_0$ is orientable, the action is identical.)

The same holds for the second bundle, where $M$ is replaced by $N$.

By excision the inclusion of the pairs $(M_0, M_0 - x_0) \to (N_0, N_0 - x_0)$ induces an isomorphism in cohomology.

Proof of Lemma 1: completion for $l = 0$. Applying 5-lemma for the inclusion-induced mapping of exact sequences of these pairs we obtain that the inclusion $M_0 - x_0 \to N_0 - x_0$ induces an isomorphism in cohomology. Hence using the triviality of the action and the Universal Coefficients Theorem we obtain that the restriction induces an isomorphism
\[r : H^p(N_0; H^q(N_0 - x_0)) \to H^p(M_0; H^q(M_0 - x_0)) \quad \text{for each } p, q.\]

This $r$ is a homomorphism of the $E_2$-terms of the Leray-Serre cohomology spectral sequences of the above bundles. By the Zeeman Comparison Theorem of spectral sequences [Ze57], the restriction $H^i(\tilde{N}_0) \to H^i(\tilde{M}_0)$ is an isomorphism for each $i$. This implies Lemma 1. $\square^4$

\[\text{4A statement on cohomology of compact manifolds should have a proof involving only cohomology of compact manifolds (recall that we may assume that } \tilde{N} = \tilde{N}_c \text{ is compact). The above proof has such an interpretation in terms of only compact spaces. Lemma 1 can also be proved analogously to proof of the Proposition above.}\]
Proof of Lemma 1: completion for the general case. Applying the 5-lemma for the inclusion-induced mapping of exact sequences of these pairs we obtain that the inclusion $M_0 - x_0 \to N_0 - x_0$ induces an isomorphism in $H^i$ for $i \geq l$. Hence using the triviality of the action and the Universal Coefficients Theorem we obtain that the restriction induces an isomorphism

$$r : H^p(N_0; H^q(N_0 - x_0)) \to H^p(M_0; H^q(M_0 - x_0)) \quad \text{for} \quad p + q \geq n + l - 1. $$

Hence $r$ is an isomorphism of for $p + q \geq n + l$ and an epimorphism for $p + q = n + l - 1$. This $r$ is a homomorphism of the $E_2$-terms of the Leray-Serre cohomology spectral sequences of the above bundles. Now using standard argument of homological algebra as in the Zeeman Comparison Theorem of spectral sequences [Ze57] we obtain that the restriction-induced homomorphism between $E^p_{r,q}$ terms is an isomorphism for $p + q \geq n + l$ and an epimorphism for $p + q = n + l - 1$. Since $E_{n-l} = E_{n-l+1} = ... = E_{\infty}$, the restriction induces on $E_{\infty}$ terms an isomorphism for $p + q \geq n + l$ and an epimorphism for $p + q = n + l - 1$. Hence the restriction $H^i(\tilde{N}_0) \to H^i(\tilde{M}_0)$ is an isomorphism for each $i \geq n + l$ and an epimorphism for $i = n + l - 1$. Therefore by the exact sequence of pair $H^i(\tilde{N}_0, \tilde{M}_0) = 0$ for each $i \geq n + l$.

Hence $H^i(\tilde{N}, \tilde{M}) = 0$ for each $i \geq n + l$. □

Proof of Lemma 2. We may assume that $f : X \to Y$ is an inclusion. Consider the following assertion:

(c1) $H^i(Y', X'; G_\varphi) = 0$ for each $i > l$, finitely-generated abelian group $G$, involution $\varphi : G \to G$ and local coefficient system $G_\varphi$ associated to $\varphi$ and double cover $(Y, X) \to (Y', X')$.

(Local coefficient system $G_\varphi$ is defined by the following action of $\pi_1(Y')$ on $G$. Take a representative $\alpha' : [0, 1] \to Y'$, $\alpha'(0) = \alpha'(1)$, of $[\alpha'] \in \pi_1(Y')$. Take a lift $\alpha : [0, 1] \to Y$ of $\alpha'$. If $\alpha(0) = \alpha(1)$, then $[\alpha']$ acts identically on $G$. If $\alpha(0) \neq \alpha(1)$, then $[\alpha']$ acts as $\varphi$. Clearly, this action is well-defined.)

Since $Y$ is finite-dimensional, 5 (c1) holds for large enough $l$. Consider the following part of the Smith-Richardson-Thom-Gysin sequences associated to the double cover $(Y, X) \to (Y', X')$ (see the Smith-Richardson-Thom-Gysin Sequence Theorem below):

$$0 = H^i(Y, X; G) \to H^i(Y', X'; G_\varphi) \to H^{i+1}(Y', X'; G_{-\varphi}).$$

By the hypothesis of Lemma 2 $H^i(Y, X) = 0$ for each $i > l$. So by the Universal Coefficients Formula $H^i(Y, X; G) = 0$ for each $i > l$. Then by downward induction on $l$ we obtain (c1).

Denote by $a$ the involution on $\pi_k(S^i)$ induced by the antipodal involution on $S^i$. The obstructions to extension to $Y$ of an equivariant map $X \to S^i$, and to homotopy uniqueness of such an extension, assume values in $H^{k+1}(Y', X'; \pi_k(S^i)_a)$ and $H^k(Y', X'; \pi_k(S^i)_a)$. These groups are trivial for $k < i$ because $\pi_k(S^i) = 0$, and for $k \geq i > l$ by (c1). So (d1) holds. □

For a reader’s convenience we present the following slight and possibly known extension of the Smith-Richardson-Thom-Gysin sequence. Cf. [Me09, arxiv v4, Remark 2.3 and p.9, lines 14-25].

5It would be interesting to know if Lemma 2 holds for infinite-dimensional complexes. Note that it does hold for infinite-dimensional complexes $S^{l-1} \to S^{\infty}$.

6Note that $a = id$ for $i$ odd and $a = -id$ for $i$ even and $k \leq 2i - 2$.

7This can be deduced either from obstruction theory for extension of maps with non-simply-connected range $\mathbb{R}P^\infty$ [HW60] or analogously to [CF60, beginning of §2, Ad93, 7.1] as follows. Denote by $t$ the involution on $Y$ and its restriction to $X$. Define a bundle $g : Y \times S^i \to (Y, X)$ by $g(x, s) = [x]$. Equivariant maps $Y \to S^i$ up to equivariant homotopy are in 1–1 correspondence with cross-sections of $g$ up to equivalence. So the required obstructions are obstructions to

(*) extendability of a section on $X'$ to a section on $Y'$ for each $i \geq l$, and to

(**) uniqueness of such an extension (up to equivalence) for $i > l$.

The action of $\pi_1(Y')$ on homotopy groups of the fiber $S^i$ gives rise to local coefficient system $\pi_k(S^i)_a$. 

8This can be deduced either from obstruction theory for extension of maps with non-simply-connected range $\mathbb{R}P^\infty$ [HW60] or analogously to [CF60, beginning of §2, Ad93, 7.1] as follows. Denote by $t$ the involution on $Y$ and its restriction to $X$. Define a bundle $g : Y \times S^i \to (Y, X)$ by $g(x, s) = [x]$. Equivariant maps $Y \to S^i$ up to equivariant homotopy are in 1–1 correspondence with cross-sections of $g$ up to equivalence. So the required obstructions are obstructions to

(*) extendability of a section on $X'$ to a section on $Y'$ for each $i \geq l$, and to

(**) uniqueness of such an extension (up to equivalence) for $i > l$.

The action of $\pi_1(Y')$ on homotopy groups of the fiber $S^i$ gives rise to local coefficient system $\pi_k(S^i)_a$. 

9This can be deduced either from obstruction theory for extension of maps with non-simply-connected range $\mathbb{R}P^\infty$ [HW60] or analogously to [CF60, beginning of §2, Ad93, 7.1] as follows. Denote by $t$ the involution on $Y$ and its restriction to $X$. Define a bundle $g : Y \times S^i \to (Y, X)$ by $g(x, s) = [x]$. Equivariant maps $Y \to S^i$ up to equivariant homotopy are in 1–1 correspondence with cross-sections of $g$ up to equivalence. So the required obstructions are obstructions to
Smith-Richardson-Thom-Gysin Sequence Theorem. Let $X'$ be a connected space, $X \to X'$ a double covering and $G$ a module with an involution $\varphi$. Consider the local coefficient system $G_\varphi$ on $X'$ associated to the double covering and $\varphi$. Then there is a long exact sequence

$$\cdots \to H^{p-1}(X'; G_\varphi) \to H^p(X'; G_{-\varphi}) \to H^p(X; G) \to H^p(X'; G_\varphi) \to H^{p+1}(X'; G_{-\varphi}) \to \cdots$$

If $2$ is invertible in $G$ (in particular, if either $G = \mathbb{Q}$ or $G = \mathbb{Z}_p$ for $p$ an odd prime), then we have splittable short exact sequence

$$0 \to H^p(X'; G_{-\varphi}) \to H^p(X; G) \to H^p(X'; G_\varphi) \to 0 \text{ so that } H^p(X; G) \cong H^p(X'; G_{-\varphi}) \oplus H^p(X'; G_\varphi).$$

If $G = \mathbb{Z}$ and $\varphi = \text{id}$, then we get long exact sequence

$$\cdots \to H^{p-1}(X') \to H^p(X'; \mathbb{Z}_{-\text{id}}) \to H^p(X) \to H^p(X') \to H^{p+1}(X'; \mathbb{Z}_{-\text{id}}) \to \cdots$$

If $G = \mathbb{Z}$ and $\varphi = -\text{id}$, then we get long exact sequence

$$\cdots \to H^{p-1}(X'; \mathbb{Z}_{-\text{id}}) \to H^p(X') \to H^p(X) \to H^p(X'; \mathbb{Z}_{-\text{id}}) \to H^{p+1}(X') \to \cdots$$

Proof. Consider the fibration $F \to X \to X'$ which is the double covering, where $F$ is a two-points set. For the spectral sequence with local coefficients [Si97, Theorem 2.9] we have $E_2^{p,q} = H^p(X', H^q(F; G)_\tau)$, where the coefficients are twisted according to double covering $X \to X'$ and the following involution $\tau$ of $H^q(F; G)$:

- $H^q(F; G) = 0$ and $\tau$ is trivial for $q > 0$, and
- $H^0(F; G) \cong G \oplus G$ and $\tau(a, b) := (\varphi(b), \varphi(a))$.

Then the spectral sequence contains at most one non-vanishing line. Hence

$$H^p(X; G) \cong E_\infty^{p,0} \cong E_2^{p,0} \cong H^p(X', (G \oplus G)_\tau).$$

Let $H = \{(m, -m) \in G \oplus G \mid m \in G\}$. We have $\tau(m, -m) = (\varphi(-m), \varphi(m)) = (-\varphi(m), \varphi(m))$. Hence $\tau(H) = H$ and $(H, \tau|_H) \cong (G, -\varphi)$. Then $(G \oplus G)/H$ has ‘the quotient’ involution $\tau/H$. Clearly, $((G \oplus G)/H, \tau/H) \cong (G, \varphi)$. Now the first part of the theorem follows from the cohomological long exact sequence associated with the short exact sequence of twisted coefficients $(H, \tau|_H) \to (G \oplus G, \tau) \to ((G \oplus G)/H, \tau/H)$.

The ‘further’ part where $2$ is invertible follows from the fact that the above short exact sequence splits: the homomorphism $s : (G \oplus G)/H \cong G \to G \oplus G$ defined by $s(m) = (m/2, m/2)$ respects involutions and is a splitting. The ‘further’ part where $G = \mathbb{Z}$ is clear. □

---

8The twisting of $H^q(F; G)$ is as required by [Si97, 2.7]. Note that [Si97, 2.8] is not required for the statement of [Si97, Theorem 2.9] (but is required for the proof). Note that the purpose of [Si97, Theorem 2.9] was to calculate cohomology of the total space of a fibration not with any non-twisted coefficient system but with the twisted coefficient system coming from a twisted coefficient system in the base.

The isomorphism $H^p(X; G) \cong H^p(X', (G \oplus G)_\tau)$ has two simpler proofs not involving spectral sequences. According to S. Melikhov, it follows easily from definitions, as explained in [Ha, Example 3.H.2] (the case of arbitrary $\varphi$ follows from the case $\varphi = \text{id}$ because the involution $(a, b) \mapsto (\varphi(b), \varphi(a))$ is obtained from the involution $(a, b) \mapsto (b, a)$ by an automorphism of $G \oplus G$ [Br82, Corollary III.5.7]), or, alternatively, is a special case of the Vietoris Mapping Theorem, [Br97, Theorem 11.1].
EMBEDDINGS OF HOMOLOGY EQUIVALENT MANIFOLDS WITH BOUNDARY

REFERENCES

[Ad93] M. Adachi, *Embeddings and Immersions*, Transl. of Math. Monographs 124, AMS, 1993.
[Br82] K.S. Brown, *Cohomology of Groups*, Graduate Texts in Mathematics 87, Springer, 1982.
[BG71] J. C. Becker and H. H. Glover, *Note on the Embedding of Manifolds in Euclidean Space*, Proc. of the Amer. Math. Soc. 27:2 (1971), 405–410; doi:10.2307/2036329.
[Br97] G. Bredon, *Sheaf Theory*, Graduate Texts in Mathematics 170, Springer, 1997.
[Ca65] B. G. Casler, *An embedding theorem for connected 3-manifolds with boundary*, Proc. Amer. Math. Soc. 16 (1965), 559–556.
[CF60] P. E. Conner and E. E. Floyd, *Fixed points free involutions and equivariant maps*, Bull. Amer. Math. Soc. 66 (1960), 416–441.
[CE56] H. Cartan and S. Eilenberg, *Homological Algebra*, Oxford University Press, Princeton University Press, Oxford-Princeton, 1956.
[Ha] A. Hatcher, *Algebraic Topology*, http://www.math.cornell.edu/~hatcher/AT/ATpage.html.
[Ho69] K. Horvatić, *Embedding manifolds with low-dimensional spine*, Glasnik Mat. 4(24):1 (1969), 101–116.
[HW60] P.J. Hilton and S. Wylie, *Homology Theory: An introduction to algebraic topology*, MR0115161 (22 #5963), Cambridge University Press, Cambridge, 1960.
[Hu88] L. S. Husch, *ε-maps and embeddings*, General Topological Relations to Modern Analysis and Algebra, VI, Heldermann, Berlin, 1988, pp. 273–280.
[MA] www.map.him.uni-bonn.de/index.php/High_codimension_embeddings_classification, Manifold Atlas Project; (unrefereed page: revision no. 2694).
[Mo09] S.A. Melikhov, *The van Kampen obstruction and its relatives*, Tr. Mat. Inst. Steklova 266 (2009), 149–183; English transl.: Proc. Steklov Inst. Math. 266 (2009), 142-176. arXiv: math.GT/0612082.
[MS67] S. Mardešić and J. Segal, *ε-mappings and generalized manifolds*, Michigan Math. J. 14 (1967), 171–182.
[MRS03] J. Malešić, D. Repovš and A. Skopenkov, *Embeddings into Rm and the deleted product obstruction*, Boletin de la Soc. Mat. Mexicana 9 (2003), 165–170.
[RS99] D. Repovš and A. Skopenkov, *New results on embeddings of polyhedra and manifolds into Euclidean spaces*, Uspekhi Mat. Nauk 54:6 (1999), 61–109 (in Russian); English transl., Russ. Math. Surv. 54:6 (1999), 1149–1196.
[RS72] C. P. Rourke and B. J. Sanderson, *Introduction to Piecewise-Linear Topology*, Ergeb. der Math. 69, Springer-Verlag, Berlin, 1972.
[Sc77] M. Scharlemann, *Isotopy and cobordism of homology spheres in spheres*, J. London Math. Soc., Ser. 2 16:3 (1977), 559–567.
[Si97] J. Siegel, *Higher order cohomology operations in local coefficient theory*, Amer. J. Math. 89 (1967), 909–931.
[Sk97] A. B. Skopenkov, *On the deleted product criterion for embeddability of manifolds in Rm*, Comment. Math. Helv. 72 (1997), 543–555.
[Sk98] A. B. Skopenkov, *On the deleted product criterion for embeddability in Rm*, Proc. Amer. Math. Soc. 126:8 (1998), 2467–2476.
[Sk02] A. B. Skopenkov, *On the Haefliger-Hirsch-Wu invariants for embeddings and immersions*, Comment. Math. Helv. 77 (2002), 78–124.
[Sk08] A. Skopenkov, *Embedding and knotting of manifolds in Euclidean spaces*, in: *Surveys in Contemporary Mathematics, Ed. N. Young and Y. Choi*, London Math. Soc. Lect. Notes 347 (2008), 248–342; arxiv:math/0604045.
[Wa64] C. T. C. Wall, *Differential topology, IV (theory of handle decompositions)*, Cambridge (1964), mimeographed notes.
[We68] C. Weber, *Deux remarques sur les plongements d’un AR dans un espace euclidien*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys 16 (1968), 851–855.
[Ze57] E. C. Zeeman, *A proof of the comparison theorem for spectral sequence*, Proc. Cambridge Philos. Soc. 53 (1957), 57–62.

Departamento de Matemática, IME, University of São Paulo, Caixa Postal 66281, Agência Cidade de São Paulo 05311-970, São Paulo, SP, Brasil. e-mail: dlgoncal@ime.usp.br

Independent University of Moscow, B. Vlasyevskiy, 11, 119002, Moscow, Russia. e-mail: skopenko@mccme.ru