GENERALIZED FILTRATIONS AND ITS APPLICATION TO BINOMIAL ASSET PRICING MODELS

TAKANORI ADACHI, KATSUSHI NAKAJIMA AND YOSHIHIRO NYU

Abstract. We introduce generalized filtration with which we can represent situations such as some agents forget information at some specific time. The filtration is defined as a functor to a category \textbf{Prob} whose objects are all probability spaces and whose arrows correspond to measurable functions satisfying an absolutely continuous requirement [Adachi and Ryu, 2019]. As an application of a generalized filtration, we develop a binomial asset pricing model, and investigate the valuations of financial claims along this type of non-standard filtrations.

1. Introduction

It is well known that in stochastic process theory and theories developed on it such as stochastic differential equation theory and stochastic control theory, the concept of filtration that expresses increasing information along time is important. The idea that the world’s information grows over time seems to be quite natural, but in a sense it is a divine perspective of omniscience and almighty, and it would be a little different if we say that the amount of information that an individual has always increases with time. People forget and misunderstand. The transition of such individuals’ information may therefore be reduced, and may be remembered as a different form of experience than objective information. The purpose of the first half of this paper is to propose a kind of subjective filtration that expresses the transition of such information.

In this way, we generalize the concept of filtration so that we can handle subjective situations, but the purpose of generalized filtration is not limited to that. For example, consider a situation in which Black Swan, who no one had imagined up to a certain point in time, was falling. The financial crisis that hit the world in 2008 and the COVID-19 pandemic in 2020 are typical examples. When Black Swan suddenly appeared, which was not included among the possible future world lines, we could not give a probability for that event and we were greatly upset. Of course, God could have a sufficiently large set of primitive events to take into account such possibilities, it would have been possible to give a probability to an event that ordinary people did not expect. But can such an idealized perspective really create a theory that averts the risk of Black Swan?

The generalized filtration formulated in this paper allows even the underlying set of probability space, which is the set of primitive events, to change over time. And it allows the sudden appearance of Black Swan to be incorporated into the theory in a natural way.

In the second half of this paper, we consider two types of filtration on the binomial asset price model as an application of generalized filtration. In particular, we show that there is a risk-neutral filtration associated with subjective filtration that a person who has lost memory for a certain period of time, and use it to price securities. This indicates that people with a lack of memory can price securities.

Finally, in summary, other applications of generalized filtration and future development directions are described.

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2. Generalized Filtrations

In this section, we define generalized filtration by gradually extending the classical filtration.

2.1. Time Domains. A filtration represents a set of information that increases with time. The set of times here is called a time domain and is represented by $\mathcal{T}$. Typical $\mathcal{T}$ has the following forms.

(1) $\mathcal{T} := \{0, 1, 2, \ldots, T\}$,
(2) $\mathcal{T} := \{0, 1, 2, \ldots\}$,
(3) $\mathcal{T} := [0, T]$,
(4) $\mathcal{T} := [0, +\infty)$,

where $T$ is a time horizon. In general time domain may be a totally ordered set having the minimum element 0.

2.2. Classical Filtrations. Let $\tilde{\Omega} := (\Omega, \mathcal{F}, P)$ be a probability space. Let $\{t_n\}$ be an increasing sequence in a time domain $\mathcal{T}$. Then, an increasing sequence of $\sigma$-fields $\mathcal{F}_{t_0} \subset \mathcal{F}_{t_1} \subset \cdots \subset \mathcal{F}_{t_n} \subset \mathcal{F}_{t_{n+1}} \subset \cdots$ with $\mathcal{F}_{t_n} \subset \mathcal{F}$ is called a classical filtration. In other words, a filtration is a family of set-inclusion relations like

$$\{\mathcal{F}_s \subset \mathcal{F}_t\}_{s \leq t}.$$ 

Now let $\tilde{\Omega}_t := (\Omega, \mathcal{F}_t, P)$ be probability spaces whose $\sigma$-fields are changing per time $t$. Then, for $s \leq t$ in $\mathcal{T}$, the condition that the function below defined as an identity function $i_{s,t}$ is measurable is equivalent to the condition $\mathcal{F}_s \subset \mathcal{F}_t$

$$\tilde{\Omega}_s \xrightarrow{i_{s,t}} \tilde{\Omega}_t.$$ 

In other words, the filtration can be identified with a family of measurable functions

$$\{\tilde{\Omega}_s \xrightarrow{i_{s,t}} \tilde{\Omega}_t\}_{s \leq t}.$$ 

Therefore, in the following, instead of using the $\sigma$-field $\mathcal{F}_t$, filtration will be considered as a family of measurable functions.

2.3. Generalization of Filtrations. As we see in the previous section, a filtration can be seen as a family of identity functions $i_{s,t}$ as measurable functions. Now what if we generalize them to arbitrary measurable functions like the following?

$$\{\tilde{\Omega}_s \xrightarrow{f_{s,t}} \tilde{\Omega}_t\}_{s \leq t}$$

satisfying

$$f_{t,t} = Id_{\tilde{\Omega}_t} \quad \text{and} \quad f_{s,t} \circ f_{t,u} = f_{s,u}$$

for any $s \leq t \leq u$ in $\mathcal{T}$, where $Id_{\tilde{\Omega}_t}$ is an identity function on $\tilde{\Omega}_t$. However, this definition is too general for a random variable $X : \tilde{\Omega}_t \to \mathbb{R}$ to define its conditional expectation $E^{f_{s,t}}(X) : \tilde{\Omega}_s \to \mathbb{R}$ satisfying

$$\int_A E^{f_{s,t}}(X) dP = \int_{f_{s,t}^{-1}(A)} X dP \quad (\forall A \in \mathcal{F}_s).$$
In order to make it possible, we need to add an extra condition to the measurable function $f_{s,t}$ called null-preserving, that is, for any $A \in \mathcal{F}_s$, $P(A) = 0$ implies $P(f_{s,t}^{-1}(A)) = 0$ [Adachi, 2014]. In fact, if $f_{s,t}$ is null-preserving, as we will see later, we can define a conditional expectation $E_{f_{s,t}}(X) : \bar{\Omega}_s \to \mathbb{R}$. Note that when the identity function is generalized to a null-preserving function, the corresponding sequence of the $\sigma$-fields is not necessarily monotonically increasing.

In order to give a further generalization, we consider that the probability space at each time fluctuates not only with the $\sigma$-fields but also with probability measures and underlying sets. In other words, the probability space $\bar{\Omega}_t$ at time $t$ is redefined as follows.

$$\bar{\Omega}_t := (\Omega_t, F_t, P_t).$$

Along with this, the definition of null-preserving functions is extended as follows.

**Definition 2.1.** Let $\bar{\Omega} = (\Omega, F, P)$ and $\bar{\Omega}' = (\Omega', F', P')$ be two probability spaces and $f : \bar{\Omega} \to \bar{\Omega}'$ be a measurable functions between them. Then $f$ is called null-preserving if $P \circ f^{-1} \ll P'$ (absolutely continuous).

**Definition 2.2.** A generalized filtration is a family of null-preserving functions

$$\{\bar{\Omega}_s \xrightarrow{f_{s,t}} \bar{\Omega}_t\}_{s \leq t}$$

satisfying

$$f_{t,t} = 1_{\Omega_t}$$

and $f_{s,t} \circ f_{t,u} = f_{s,u}$ for all triples $s \leq t \leq u$ in $\mathcal{T}$.

Then, we obtain a following theorem.

**Theorem 2.3.** ([Adachi and Ryu, 2019]) For any random variable $X$ on $\bar{\Omega}_t$ and any null-preserving function $f : \bar{\Omega}_t \to \bar{\Omega}_s$, there exists a random variable $Y$ on $\bar{\Omega}_s$ such that for every $A \in \mathcal{F}_s$,

$$\int_A Y dP_s = \int_{f^{-1}(A)} X dP_t. \tag{2.1}$$

We write $E_f(X)$ for the random variable $Y$, and call it a conditional expectation of $X$ along $f$.

**Proof.** Define a measure $X^*$ on $(\Omega_t, F_t)$ as in the following diagram.

$$\begin{array}{c}
D \\
\xrightarrow{f} \\
\xrightarrow{X^*} \\
\xrightarrow{P_t} \\
\xrightarrow{\mathbb{R}} \\
\end{array}
\begin{array}{c}
\mathcal{F}_s \\
\cap \\
\mathcal{F}_t \\
\cap \\
\mathbb{P}_s \\
\end{array}
\begin{array}{c}
\int_D X d\mathbb{P}_t \\
\end{array}
$$

Then, since $X^* \ll \mathbb{P}_t$ and $f$ is null-preserving, we have

$$X^* \circ f^{-1} \ll \mathbb{P}_t \circ f^{-1} \ll \mathbb{P}_s.$$ 

Therefore, we get a following Radon-Nikodym derivative.

$$Y := \partial(X^* \circ f^{-1})/\partial \mathbb{P}_s.$$

With this $Y$ we obtain for every $A \in \mathcal{F}_s$,

$$\int_A Y d\mathbb{P}_s = \int_A d(X^* \circ f^{-1}) = (X^* \circ f^{-1})(A) = X^*(f^{-1}(A)) = \int_{f^{-1}(A)} X d\mathbb{P}_t.$$
2.4. Filtration is a Functor. In this subsection, we will try to redefine the filtration introduced in Section 2.3 using Category Theory [MacLane, 1997].

**Definition 2.4.** [Two Categories Prob and T]

1. All probability spaces and all null-preserving functions between them form a category. This category is denoted by Prob. 
2. A time domain $T$ can be regarded as a category if we consider its elements as objects, and if two objects $s$ and $t$ have one and only one arrow from $t$ to $s$ when there is a relation $s \leq t$.

Then, the filtration introduced in Section 2.3 can be regarded as a functor $F : T \to \text{Prob}$ (Figure 2.1). We sometimes call $F$ a $T$-filtration in order for clarifying its time domain.

![Figure 2.1. Filtration $F : T \to \text{Prob}$](image)

Henceforth, generalized filtration will be referred to simply as filtration.

For further discussion about the category Prob, see [Adachi and Ryu, 2019].
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3. Filtrations over a Binomial Asset Pricing Model

In this section, as a concrete example of the filtration introduced in Section 2, we look at an unusual filtration on a binomial asset pricing model.

3.1. Filtration $B^N$. First, we define a general scheme of our model by introducing a filtration $B^N$ for an integer $N$.

**Definition 3.1** (Time Domain and Probability Space). Let $N \in \mathbb{N}$ and $s, t \in \mathbb{R}$ be non-negative real numbers.

1. *Discrete intervals.*
   
   $$[s, t]^N := \{ n2^{-N} | n \in \mathbb{Z} \text{ and } s \leq n2^{-N} \leq t \},$$
   $$[s, t)^N := \{ n2^{-N} | n \in \mathbb{Z} \text{ and } s \leq n2^{-N} < t \},$$
   $$(s, t]^N := \{ n2^{-N} | n \in \mathbb{Z} \text{ and } s < n2^{-N} \leq t \},$$
   $$(s, t)^N := \{ n2^{-N} | n \in \mathbb{Z} \text{ and } s < n2^{-N} < t \}.$$

2. Let $\mathcal{T}^N$ be a category whose objects are elements of $[0, \infty)^N$. For $s, t \in [0, \infty)^N$, $\mathcal{T}^N$ has the unique arrow $\iota^N_{s,t}$ from $t$ to $s$ if and only if $t \geq s$.

3. $B^N_1 := \{0, 1\}^{(0, t)^N}$. (function space)

4. $\mathcal{F}^N_t := 2^{B^N_t}$. (powerset)

5. Let $p^N_s \in [0, 1]$ for each $s \in (0, \infty)^N$. Then, a probability measure $\mathbb{P}^N_t : \mathcal{F}^N_t \rightarrow [0, 1]$ is defined for every $\omega \in B^N_t$ by
   
   $$\mathbb{P}^N_t(\{\omega\}) := \prod_{s \in [0, t]^N} (p^N_s)^{\omega(s)}(1 - p^N_s)^{1-\omega(s)}.$$

6. $\bar{B}^N_t := (B^N_1, \mathcal{F}^N_t, \mathbb{P}^N_t)$ (probability space)

**Definition 3.2.** A filtration $B^N$ is determined by defining arrows $f^N_{s,t}$ below:

$$\mathcal{T}^N \xrightarrow{B^N} \text{Prob}$$

$$\begin{array}{c|c}
\bar{B}^N_s & f^N_{s,t} := \iota^N_{s,t} \\
\hline
\bar{B}^N_t & \\
\end{array}$$

The filtration $B^N$ is called *non-trivial* if there exists $t \in (0, \infty)^N$ such that $0 < p_t < 1$.

Note that for a non-trivial filtration $B^N$, every function from $B^N_t$ to $B^N_s$ becomes a null-preserving function from $\bar{B}^N_t$ to $\bar{B}^N_s$.

As we introduced, the functor $B^N$ is a generalized filtration, representing a filtration over the classical binomial model developed, for example in [Shreve, 2005].

The classical version requires the terminal time horizon $\bar{T}$ for determining the underlying set $\Omega := \{0, 1\}^\bar{T}$ while our version does not require it since the time variant probability spaces can evolve without any limit. That is, our version allows unknown future elementary events, which, we believe, shows a big philosophical difference from the traditional Kolmogorov world.

**Proposition 3.3.** For a random variable $X$ on $\bar{B}^N_t$ and $\omega \in \bar{B}^N_s$, we have

$$E^{f^N_{s,t}}(X)(\omega)\mathbb{P}^N_s(\{\omega\}) = \sum_{\omega' \in (f^N_{s,t})^{-1}(\omega)} X(\omega')\mathbb{P}^N_t(\{\omega'\}).$$
Proof. Put $A := \{\omega\}$ and $f_{s,t} := f_{s,t}^N$ in \((\ref{e:2})\). Then the result is straightforward.

In order to see a variety of filtrations, we introduce two candidate $f_{s,t}^N$ introduced in Definition 3.2.

**Definition 3.4** (Two Candidates of $f_{s,t}^N$). Let $s, t$ be objects of $\mathcal{T}^N$ satisfying $s < t$.

1. **full**$_{s,t}^N$

\[
\begin{align*}
\mathcal{B}_s^N & \xrightarrow{\text{full}_{s,t}^N} \mathcal{B}_t^N \\
\omega |_{(0,s)^N} & \xrightarrow{\omega |_{(0,s)^N}} \omega
\end{align*}
\]

2. **drop**$_{s,t}^N$

\[
\begin{align*}
\mathcal{B}_s^N & \xrightarrow{\text{drop}_{s,t}^N} \mathcal{B}_t^N \\
\omega |_{(0,s)^N} & \xrightarrow{\omega |_{(0,s)^N}} \omega
\end{align*}
\]

The function **drop**$_{s,t}^N$ can be interpreted to forget what happens at time $s$.

We can easily show the following proposition.

**Proposition 3.5.** For $s < t < u$ in $[0, \infty]^N$,

1. **full**$_{s,t}^N \circ \text{full}^N_{t,u} = \text{full}^N_{s,u}$,
2. **full**$_{s,t}^N \circ \text{drop}^N_{t,u} = \text{full}^N_{s,u}$,
3. **drop**$_{s,t}^N \circ \text{full}^N_{t,u} = \text{drop}^N_{s,u}$,
4. **drop**$_{s,t}^N \circ \text{drop}^N_{t,u} = \text{drop}^N_{s,u}$.

**Definition 3.6** (Examples of (Subjective) Filtrations). Let $s, t$ be any objects of $\mathcal{T}^N$ such that $s < t$.

1. Classical filtration: **Full**$^N : \mathcal{T}^N \to \mathbf{Prob}$ is defined by

\[
\text{Full}^N(t_{s,t}^N) := \text{full}^N_{s,t}.
\]

2. Dropped filtration: **Drop**$^N_{\alpha, \beta} : \mathcal{T}^N \to \mathbf{Prob}$ where $\alpha, \beta \in \mathbb{R}$ are constants, is defined by

\[
\text{Drop}^N_{\alpha, \beta}(s_{s,t}^N) := \begin{cases} 
\text{drop}^N_{s,t} & \text{if } s \neq t \text{ and } s \in [\alpha, \beta]^N, \\
\text{full}^N_{s,t} & \text{otherwise.}
\end{cases}
\]

A person who has a subjective filtration **Drop**$^N_{\alpha, \beta}$ forgets the events happened during $[\alpha, \beta]$.

Note also that the dropped filtration is well-defined by Proposition 3.5.

**Definition 3.7.** ($\mathcal{B}$-Adapted Process $\xi_t^N$) Let $t \in [0, +\infty)^N$. a stochastic process $\xi_t^N : B_t^N \to \mathbb{R}$ is defined by

\[
\xi_t^N(\omega) := 2\omega(t) - 1 \quad (\forall \omega \in B_t^N).
\]

**Definition 3.8.** For $j = 0, 1$ and $\omega \in B_t^N$,

\[
I_t^N(j, \omega) := \{\omega' \in (f_{t,t+2}^N)^{-1}(\omega) \mid \omega'(t + 2^{-N}) = j\}.
\]

**Proposition 3.9.** For $\omega \in B_t^N$ with $\mathbb{P}_t^N(\omega) \neq 0$,

\[
E_{t,t+2}^N(\xi_{t+2}^N)(\omega) = \#((f_{t,t+2}^N)^{-1}(\omega))N_{t+2}^N - \#I_t^N(0, \omega),
\]

where $\#A$ denotes the cardinality of the set $A$. 

Proof. By Proposition 3.3,

\[ E^{I_N}_{t,1} (\xi_{t+2-N}) (\omega) = \sum_{\omega' \in \mathcal{F}_{t+2-N}^{-1} (\omega)} \frac{\xi_{t+2-N} (\omega') \mathbb{P}_{t+2-N} (\omega')}{\mathbb{P}_{t} (\omega)} \]

\[ = \sum_{\omega' \in \mathcal{F}_{t+2-N}^{-1} (\omega)} \frac{\mathbb{P}_{t+2-N} (\omega')}{\mathbb{P}_{t} (\omega)} - \sum_{\omega' \in \mathcal{F}_{t}^{-1} (0, \omega)} \frac{\mathbb{P}_{t+2-N} (\omega')}{\mathbb{P}_{t} (\omega)} \]

\[ = \sum_{\omega' \in \mathcal{F}_{t}^{-1} (1, \omega)} p_{t+2-N} (\omega') + \sum_{\omega' \in \mathcal{F}_{t}^{-1} (0, \omega)} (1 - p_{t+2-N} (\omega')) \]

\[ = \#((f_{t+2-N}^{-1} (\omega)) p_{t+2-N} (\omega') - \#I_{t}^{-1} (0, \omega). \]

\[ \square \]

### 3.2. Arbitrage Strategies

Now we define two instruments tradable in our market.

**Definition 3.10.** [Stock and Bond Processes] Let \( \mu, \sigma, r \in \mathbb{R} \) be constants such that \( \sigma > 0 \), \( \mu > \sigma - 1 \) and \( r > -1 \). We have the following \( \mathcal{B}^N \)-adapted processes which are two instruments tradable in our market. Let \( t \in [0, +\infty)^N \).

1. A **stock process** \( S^N_t : B^N_t \to \mathbb{R} \) over the filtration \( \mathcal{B}^N \) is defined by

   \[ S^N_0 (\ast) := s_0, \quad S^N_{t+2-N} := (S^N_t \circ f_{t+2-N} t + 2^{-N} \mu + 2^{-N} \sigma \xi_{t+2-N}^N) \]

   where \( \ast \in B^N_0 \) is the unique element.

2. A **bond process** \( B^N_t : B^N_t \to \mathbb{R} \) over the filtration \( \mathcal{B}^N \) is defined by

   \[ b^N_0 (\ast) := 1, \quad b^N_{t+2-N} := (b^N_t \circ f_{t+2-N} t + 2^{-N} r). \]

   The condition \( \mu > \sigma - 1 \) is necessary for keeping the stock price positive.

We sometimes call the triple \( (\mathcal{B}^N, S^N, B^N) \) a **market**. But, it does not mean that the market will not contain other instruments.

The following proposition is straightforward.

**Proposition 3.11.** Let \( 1_{B^N_t} \) be a random variable on \( B^N_t \) defined by \( 1_{B^N_t} (\omega) = 1 \) for every \( \omega \in B^N_t \). Then, we have for any \( \omega \in B^N_t \),

1. \( E^{I_N}_{t,1} (S^N_{t+2-N}) = S^N_t ((1 + 2^{-N} \mu)E^{I_N}_{t,1} (1_{B^N_{t+2-N}}) + 2^{-N} \sigma E^{I_N}_{t,1} (\xi_{t+2-N}^N) \)

2. \( E^{I_N}_{t,1} (B^N_{t+2-N}) (\omega) = \{ \mathbb{P}_{t+2-N} (\mathcal{F}_{t+2-N}^{-1} (\omega)) \} / \{ \mathbb{P}_{t} (\omega) \} \)

3. \( b^N_t (\omega) = (1 + 2^{-N} r) \omega_t \)

**Definition 3.12.** [Strategies] A **strategy** is a sequence \( (\phi, \psi) = \{(\phi_t, \psi_t)\}_{t \in (0, \infty)^N} \), where

\[ \phi_t : B_{t+2-N}^N \to \mathbb{R} \text{ and } \psi_t : B_{t+2-N}^N \to \mathbb{R}. \]

Each element of the strategy \( (\phi_t, \psi_t) \) is called a **portfolio**. For \( t \in [0, \infty)^N \), the **value** \( V_t \) of the portfolio at time \( t \) is determined by:

\[ V_t := \begin{cases} S^N_0 \phi_2-N + b^N_2-N \psi_2-N & \text{if } t = 0, \\ S^N_t (\phi_t \circ f_{t+2-N}^N) + b_t (\psi_t \circ f_{t+2-N}^N) & \text{if } t > 0. \end{cases} \]
Definition 3.13. [Gain Processes] A **gain process** of the strategy \((\phi, \psi)\) is the process 
\( \{G^{(\phi,\psi)}_t\}_{t \in [0,\infty]^N} \) defined by

\[
G^{(\phi,\psi)}_t := \begin{cases} 
-(S^N_0 \phi_{t-\omega} + b^N_0 \psi_{t-\omega}) & \text{if } t = 0, \\
(S^N_t(\phi_t \circ f^N_{t-\omega},N) + b^N_t(\psi_t \circ f^N_{t-\omega},N)) - (S^N_t \phi_{t+2^{-N}} + b^N_t \psi_{t+2^{-N}}) & \text{if } t > 0.
\end{cases}
\]

**Lemma 3.14.** Let \( t \in [0,\infty)^N \) with

\[
S^N_{t+2^{-N}}(\phi_{t+2^{-N}} \circ f^N_{t+2^{-N}}) + b_{n+1}(\psi_{t+2^{-N}} \circ f^N_{t+2^{-N}}) = (2^{-N} \mu + 2^{-NN} \sigma \xi_{t+2^{-N}} - 2^{-N}r)((S^N_t \phi_{t+2^{-N}}) \circ f^N_{t+2^{-N}}).
\]

**Proof.**

\[
LHS = (S^N_t \circ f^N_{t+2^{-N}})(1 + 2^{-N} \mu + 2^{-NN} \sigma \xi_{t+2^{-N}})(\phi_{t+2^{-N}} \circ f^N_{t+2^{-N}}) + (b^N_t \circ f^N_{t+2^{-N}})(1 + 2^{-N}r)(\psi_{t+2^{-N}} \circ f^N_{t+2^{-N}}) = (1 + 2^{-N} \mu + 2^{-NN} \sigma \xi_{t+2^{-N}})((S^N_t \phi_{t+2^{-N}}) \circ f^N_{t+2^{-N}}) + (1 + 2^{-N}r)((b^N_t \psi_{t+2^{-N}}) \circ f^N_{t+2^{-N}})
\]

\[
=RHS.
\]

Definition 3.15. [Arbitrage Strategies]

1. A strategy \((\phi, \psi)\) is called a **\(B^N\)-arbitrage strategy** if \(\mathbb{P}^N_t(G^{(\phi,\psi)}_t \geq 0) = 1\) for every \( t \in [0,\infty)^N \), and \(\mathbb{P}^N_{t_0}(G^{(\phi,\psi)}_t > 0) > 0\) for some \( t_0 \in [0,\infty)^N \).

2. The market is called **non-arbitrage** or NA if it does not allow \(B^N\)-arbitrage strategies.

**Proposition 3.16.** If the market \((B^N, S^N, b^N)\) with a non-trivial filtration \(B^N\) is non-arbitrage, then \(|\mu - r| < 2^{-N} \sigma\).

**Proof.** Assuming that \( r \leq \mu - 2^{-N} \sigma \) or \( r \geq \mu - 2^{-N} \sigma \), we will construct an arbitrage strategy \((\phi, \psi)\) by using the following algorithm.

```
for t = 0, 1, 2, ...:
    t := 2^{-(-N) n}
    observe S(t) and b(t)
    if r <= \mu - 2^{-N} \sigma
        phi(t+2^{-(-N)}) := \phi(t+2^{-(-N)}) # pick arbitrarily
    elif r >= \mu + 2^{-N} \sigma
        phi(t+2^{-(-N)}) := \phi(t+2^{-(-N)}) # pick arbitrarily
        psi(t+2^{-(-N)}) := -S(t) / b(t) \phi(t+2^{-(-N)})
        # then, observe S(t) \phi(t+2^{-(-N)}) + b(t) \psi(t+2^{-(-N)}) = 0,
        # which simplifies the computation of G(t) in the following.
        if t == 0:
            G(0) := 0
        else:
            # t > 1
            G(t) := S(t) (phi(t) * f(t-2^{-(-N)})) + b(t) (psi(t) * f(t-2^{-(-N)}))
```

In the above code, \('*'\) is the function composition operator. By Lemma 3.14 we have

\[
G^{(\phi,\psi)}_t = 2^{-N} (\mu + 2^{-NN} \sigma \xi_{t+2^{-N}} - r)((S^N_{t-2^{-N}} \phi_t) \circ f^N_{t-2^{-N}.t}).
\]
So we have $G_{t}^{(\phi, \psi)} \geq 0$ as long as $r \leq \mu - 2N\sigma$ or $r \geq \mu + 2N\sigma$.

By the way, since our filtration is non-trivial, there exists a number $t_{0}$ such that $0 < p_{t_{0}} < 1$. It is easy to check that

\[ \mathbb{P}^{N}_{t_{0}}(G_{t_{0}}^{(\phi, \psi)} > 0) > 0, \]

which concludes that $(\phi, \psi)$ is an arbitrage strategy. \qed

### 3.3. Risk-Neutral Filtrations.

In this subsection, we assume that $|\mu - r| < 2^{\frac{N}{T}}\sigma$.

Let us consider about the following discounted stock process

**Definition 3.17.** A **discount stock process** $(S_{t}^{N})': \mathcal{B}^{N} \rightarrow \mathbb{R}$ is defined by

\[ (S_{t}^{N})' := (b_{t}^{N})^{-1}S_{t}^{N}. \]

**Definition 3.18.** A **risk-neutral filtration** with respect to the filtration $\mathcal{B}^{N}$ is a filtration $\mathcal{C}^{N}$ such that

\[ U \circ C^{N} = U \circ \mathcal{B}^{N}, \]

where $U: \text{Prob} \rightarrow \text{Meas}$ is the forgetful functor to the category of measurable spaces,

\[ \mathcal{T}^{N} \xrightarrow{\mathcal{C}^{N}} \text{Prob} \xrightarrow{U} \text{Meas} \]

and with which $(S_{t}^{N})'$ becomes a $\mathcal{C}^{N}$-martingale, that is,

\[ E^{\mathcal{C}^{N}(\omega,t)}((S_{t}^{N})') = (S_{s}^{N})'. \]

In the remainder of this subsection, we will focus on proving the following theorem.

**Theorem 3.19.** There exists a risk-neutral filtration with respect to the filtration $\text{Drop}^{N}_{a, b}$.

First, we examine what form the probability measure $Q_{t}^{N}: \mathcal{F}_{t}^{N} \rightarrow [0, 1]$ takes when $\mathcal{C}^{N}(t) = (B_{t}^{N}, \mathcal{F}_{t}^{N}, Q_{t}^{N})$ for a risk-neutral filtration $\mathcal{C}^{N}$, in general.

**Theorem 3.20.** A stochastic process $(S_{t}^{N})'$ is a $\mathcal{C}^{N}$-martingale if and only if the following equation holds for every $t \in [0, \infty)^{N}$ and $\omega \in B_{t}^{N}$.

\[ Q_{t}^{N}(\{\omega\}) = c_{1}Q_{t+2-N}^{N}(I_{t}^{N}(1, \omega)) + c_{0}Q_{t+2-N}^{N}(I_{t}^{N}(0, \omega)) \]

where

\[ c_{1} := \frac{1 + 2^{-N}\mu + 2^{-\frac{N}{T}}\sigma}{1 + 2^{-N}r}, \quad c_{0} := \frac{1 + 2^{-N}\mu - 2^{-\frac{N}{T}}\sigma}{1 + 2^{-N}r}. \]

**Proof.** Let $\omega \in B_{t}^{N}$. Then, by Proposition 3.3 we have

\[ E^{\mathcal{C}^{N}(t_{t+2-N})}((S_{t+2-N}^{N})')(\omega)Q_{t}^{N}(\{\omega\}) \]

\[ = \sum_{\omega' \in (f_{t+2-N})^{N}} (S_{t+2-N}^{N})(\omega')Q_{t+2-N}^{N}(\{\omega'\}) \]

\[ = \sum_{\omega' \in (f_{t+2-N})^{N}} (b_{t+2-N}^{N})^{-1}(\omega')(S_{t}^{N}(\mathcal{F}_{t+2-N}^{N})(\omega')(1 + 2^{-N}\mu + 2^{-\frac{N}{T}}\sigma\xi_{t+2-N}^{N}(\omega'))Q_{t+2-N}^{N}(\{\omega'\}) \]

\[ = \sum_{\omega' \in (f_{t+2-N})^{N}} (1 + 2^{-N}r)^{-1}(\omega')S_{t}^{N}(\omega)(1 + 2^{-N}\mu + 2^{-\frac{N}{T}}\sigma\xi_{t+2-N}^{N}(\omega'))Q_{t+2-N}^{N}(\{\omega'\}) \]

\[ = (S_{t}^{N}')^{N}(\omega) \sum_{\omega' \in (f_{t+2-N})^{N}} \frac{1 + 2^{-N}\mu + 2^{-\frac{N}{T}}\sigma\xi_{t+2-N}^{N}(\omega')}{1 + 2^{-N}r}Q_{t+2-N}^{N}(\{\omega'\}). \]
Therefore, the condition \( (S_t^N)' = E^{G_N(t_{t+2}^N)}((S_t^{N-2})') \) is equivalent to
\[
Q_t^N(\{\omega\}) = \sum_{\omega \in I_t^N(1, \omega)} 1 + \frac{2^{-N} \mu + 2^{-\frac{N}{2}} \sigma}{1 + 2^{-N} r} Q_{t+2}^N(\{\omega^t\}) + \sum_{\omega \in I_t^N(0, \omega)} 1 + \frac{2^{-N} \mu - 2^{-\frac{N}{2}} \sigma}{1 + 2^{-N} r} Q_{t+2}^N(\{\omega^t\})
\]
\[
= c_1 Q_{t+2}^N(I_t^N(1, \omega)) + c_0 Q_{t+2}^N(I_t^N(0, \omega)).
\]
\(\square\)

**Definition 3.21.** For \( \omega \in B_t^N \) and \( d \in \{0, 1\} \), \( (\omega d) \in B_{t+2}^N \) is an element satisfying
\[
(\omega d)(s) :=\begin{cases} 
\omega(s) & (s \leq t) \\
1 & (s = t + 2^{-N})
\end{cases}
\]
for any \( s \in (0, t + 2^{-N}]^N \).

Unless there is confusion, we will omit the parentheses in \((\omega d_1) d_2\) and write \( \omega d_1 d_2 \).

In order to determine more detail of \( C \), we need the following condition for \( Q_t^N \).

**Proposition 3.22.** The following conditions for \( Q_t^N \) are equivalent.

1. For all \( t \in [0, \infty)^N \) and \( \omega \in B_t^N \),
\[
Q_t^N(I_t^N(1, \omega)) = Q_t^N(\{\omega\}).
\]
2. For all \( t \in [0, \infty)^N \), \( \text{full}_{t, t+2}^N \) is measure-preserving w.r.t. \( Q_t^N \), that is,
\[
Q_t^N = Q_{t+2}^N \circ (\text{full}_{t, t+2}^N)^{-1}.
\]
3. There exists a sequence of functions \( \{q_t : B_t^N \rightarrow [0, 1]\}_{t \in (0, \infty)^N} \) satisfying the following conditions for every \( t \in (0, \infty)^N \) and \( \omega \in B_t^N \),
   \ dated misspelled question
   (a) \( Q_t^N(\{\omega\}) = \prod_{s \in [0,1]^N} q_s(\omega | (0, s)^N) \),
   (b) \( q_{t+2}^N(\omega^0) + q_{t+2}^N(\omega^1) = 1 \).

In the following discussion, we assume the following assumption which is the condition (3) of Proposition 3.22

**Assumption 3.23.** Suppose that there exists a sequence of functions \( \{q_t : B_t^N \rightarrow [0, 1]\}_{t \in (0, \infty)^N} \) satisfying the following conditions for every \( t \in (0, \infty)^N \) and \( \omega \in B_t^N \),

1. \( Q_t^N(\{\omega\}) = \prod_{s \in (0, s)^N} q_s(\omega | (0, s)^N) \),
2. \( q_{t+2}^N(\omega^0) + q_{t+2}^N(\omega^1) = 1 \).

In the rest of this section, we assume Assumption 3.23 and then will determine the risk-neutral filtration \( C^N \) by calculating \( \{q_t\}_{t \in (0, \infty)^N} \).

**Lemma 3.24.** Let \( c_1 \) and \( c_0 \) are constants defined in Theorem 3.20. Then for any \( x \in \mathbb{R} \) we have
\[
1 = c_1 x + c_0 (1 - x) \iff x = \frac{1}{2} - \frac{2^\frac{N}{2} - 1}{\sigma} \mu - \frac{r}{\sigma} \quad \text{and} \quad 1 - x = \frac{1}{2} + \frac{2^\frac{N}{2} - 1}{\sigma} \mu - \frac{r}{\sigma}.
\]

**Proposition 3.25.** For \( t \in (0, \infty)^N \) if \( f_{t, t+2}^N = \text{full}_{t, t+2}^N \), then for \( \omega \in B_t^N \) such that \( Q_t^N(\{\omega\}) \neq 0 \), the following holds.
\[
q_{t+2}^N(\omega^1) = \frac{1}{2} - \frac{2^\frac{N}{2} - 1}{\sigma} \mu - \frac{r}{\sigma}, \quad q_{t+2}^N(\omega^0) = \frac{1}{2} + \frac{2^\frac{N}{2} - 1}{\sigma} \mu - \frac{r}{\sigma}.
\]
Proof. By observing Figure 3.1, we have
\[
(full_{t+2-N}^{-1})^{-1}(\omega) = \{\omega_0, \omega_1\}, \quad I_t^N(1, \omega) = \{\omega_1\}, \quad I_t^N(0, \omega) = \{\omega_0\}.
\]
Then, by Theorem 3.20,
\[
Q_{t+2-N}(\{\omega\}) = c_1 Q_{t+2-N}(I_t^N(1, \omega)) + c_0 Q_{t+2-N}(I_t^N(0, \omega)) = c_1 Q_{t+2-N}(\{\omega_1\}) + c_0 Q_{t+2-N}(\{\omega_0\}).
\]
Since
\[
Q_{t+2-N}(\{\omega_{d_{t+2-N}}\}) = Q_t(\{\omega\}) q_{t+2-N}(\omega_{d_{t+2-N}})
\]
by Assumption 3.23 and \(Q_t(\{\omega\}) \neq 0\), we have
\[
1 = c_1 q_{t+2-N}(\omega_1) + c_0 q_{t+2-N}(\omega_0).
\]
Hence, by Lemma 3.24, we obtain
\[
q_{t+2-N}(\omega_1) = \frac{1}{2} - 2^{\frac{N}{2}} - 1 \frac{\mu - r}{\sigma}, \quad q_{t+2-N}(\omega_0) = \frac{1}{2} + 2^{\frac{N}{2}} - 1 \frac{\mu - r}{\sigma}.
\]
□

Note that the probability obtained in Proposition 3.25 does not depend on either \(\omega\) or \(t\).

**Proposition 3.26.** For \(t \in (0, \infty)^N\), if \(I_t^N = drop_{t,t+2-N}\), then for \(\omega \in B_{t-2-N}\) such that \(Q_{t-2-N}(\{\omega\}) \neq 0\), the following holds.

\[
q_t(\omega_1) = 0, \quad q_t(\omega_0) = 1, \\
q_{t+2-N}(\omega_01) = \frac{1}{2} - 2^N - 1 \frac{\mu - r}{\sigma}, \\
q_{t+2-N}(\omega_00) = \frac{1}{2} + 2^N - 1 \frac{\mu - r}{\sigma}.
\]

Proof. By observing Figure 3.2, we have
\[
(drop_{t,t+2-N}^{-1})^{-1}(\omega_1) = \emptyset, \\
I_t^N(1, \omega_1) = I_t^N(0, \omega_1) = \emptyset, \\
(drop_{t,t+2-N}^{-1})^{-1}(\omega_0) = \{\omega_{00}, \omega_{01}, \omega_{10}, \omega_{11}\}, \\
I_t^N(1, \omega_0) = \{\omega_{01}, \omega_{11}\}, \\
I_t^N(0, \omega_0) = \{\omega_{00}, \omega_{10}\}.
\]
We have the following remarks for Figure 3.2.

**Remark 3.27.** We have the following remarks for Figure 3.2.

1. Since the agent evaluates stock and bond along the function $\text{drop}^N_{t,t+2-N}$, she can recognize only the nodes $\omega_0$, $\omega_1$ and $\omega_00$ and can not recognize the nodes $\omega_1$, $\omega_11$ and $\omega_{10}$. We interpret these nodes $\omega_1$, $\omega_{11}$ and $\omega_{10}$ as invisible.

2. The values $q_{t+2-N}(\omega_{11}) \in [0, 1]$ can be arbitrarily selected, and $q_{t+2-N}(\omega_{10})$ is computed by $1 - q_{t+2-N}(\omega_{10})$. That is, the probability measure $Q^N_{t+2-N}$ is not determined uniquely, so is not the risk-neutral filtration $C^N$. 

**Figure 3.2.** $\text{drop}^N_{t,t+2-N}$ followed by $\text{full}^N_{t+2-N,t}$

Then, by Theorem 3.20

$$Q^N_t(\{\omega_1\}) = c_1 Q^N_{t+2-N}(I^N_1(1, \omega_1)) + c_0 Q^N_{t+2-N}(I^N_1(0, \omega_1)) = 0.$$ 

Now, since $Q^N_t(\{\omega d_t\}) = Q^N_{t-2-N}(\{\omega\}) q_t(\omega d_t)$ by Assumption 3.23 and $Q^N_{t-2-N}(\{\omega\}) \neq 0$, we have

$$q_t(\omega_1) = 0, \quad q_t(\omega_0) = 1 - q_t(\omega_1) = 1.$$ 

Next, again by Theorem 3.20

$$Q^N_t(\{\omega_0\}) = c_1 Q^N_{t+2-N}(I^N_t(1, \omega_0)) + c_0 Q^N_{t+2-N}(I^N_t(0, \omega_0))$$

$$= c_1 (Q^N_{t+2-N}(\{\omega_01\}) + Q^N_{t+2-N}(\{\omega_{11}\}))$$

$$+ c_0 (Q^N_{t+2-N}(\{\omega_{00}\}) + Q^N_{t+2-N}(\{\omega_{10}\})).$$

By dividing both sides by $Q^N_{t-2-N}(\{\omega\}) \neq 0$, we obtain

$$q_t(\omega_0) = c_1 (q_t(\omega_0) q_{t+2-N}(\omega_{01}) + q_t(\omega_1) q_{t+2-N}(\omega_{11}))$$

$$+ c_0 (q_t(\omega_0) q_{t+2-N}(\omega_{00}) + q_t(\omega_1) q_{t+2-N}(\omega_{10})).$$

Hence, since $q_t(\omega_1) = 0$ and $q_t(\omega_0) = 1$, we get

$$1 = c_1 q_{t+2-N}(\omega_01) + c_0 q_{t+2-N}(\omega_{00}).$$

Therefore, by Lemma 3.24

$$q_{t+2-N}(\omega_{01}) = \frac{1}{2} - 2^{\frac{1}{2}} \frac{\mu - \gamma}{\sigma}, \quad q_{t+2-N}(\omega_{00}) = \frac{1}{2} + 2^{\frac{1}{2}} \frac{\mu - \gamma}{\sigma}.$$ 

$$\square$$

**Remark 3.27.** We have the following remarks for Figure 3.2.

1. Since the agent evaluates stock and bond along the function $\text{drop}^N_{t,t+2-N}$, she can recognize only the nodes $\omega_0$, $\omega_1$ and $\omega_00$ and can not recognize the nodes $\omega_1$, $\omega_{11}$ and $\omega_{10}$. We interpret these nodes $\omega_1$, $\omega_{11}$ and $\omega_{10}$ as invisible.

2. The values $q_{t+2-N}(\omega_{11}) \in [0, 1]$ can be arbitrarily selected, and $q_{t+2-N}(\omega_{10})$ is computed by $1 - q_{t+2-N}(\omega_{10})$. That is, the probability measure $Q^N_{t+2-N}$ is not determined uniquely, so is not the risk-neutral filtration $C^N$. 

**Figure 3.2.** $\text{drop}^N_{t,t+2-N}$ followed by $\text{full}^N_{t+2-N,t}$
(3) The probability measure $Q^N_t$ is not equivalent to the original measure $P^N_t$. Therefore, it is not an EMM.

**Proposition 3.28.** Both $\text{full}^N_{t,t+2-N}$ and $\text{drop}^N_{t,t+2-N}$ are null-preserving with respect to $Q^N_t$ and $Q^N_{t+2-N}$.

**Proof.** Let $\omega \in B^N_t$. Then, by Assumption 3.23

$$Q^N_{t+2-N} \circ (\text{full}^N_{t,t+2-N})^{-1})(\omega) = Q^N_{t+2-N}(\{\omega 1, \omega 0\}) = Q^N_t(\omega).$$

Hence, $\text{full}^N_{t,t+2-N}$ is null-preserving.

Next, consider the case when $\text{drop}^N_{t,t+2-N}$. Then for $\omega' \in B^N_{t-2-N}$, by Proposition 3.26 we have $Q^N_t(\omega' 1) = 0$. On the other hand, we get

$$(Q^N_{t+2-N} \circ (\text{drop}^N_{t,t+2-N})^{-1})(\omega' 1) = Q^N_{t+2-N}(\emptyset) = 0.$$ 

Therefore, $\text{drop}^N_{t,t+2-N}$ is also null-preserving.

\[\square\]

**Theorem 3.29.** There exists a risk-neutral filtration $C^N$ for the dropped filtration $\text{Drop}_{\alpha,\beta}$. In this case, the probability measure $Q^N_t$ of the probability space $C^N(t)$ is not equivalent to the probability measure $P_t$ of $\text{Drop}_{\alpha,\beta}(t)$. Therefore, it is not an EMM. In fact, the probability measure $Q^N_t$ is not uniquely determined. Similarly, the risk-neutral filtration $C^N$ is not uniquely determined.

**Proof.** Substituting the $q_t$ obtained by Propositions 3.25 and 3.26 into Assumption 3.23, we obtain the probability measure $Q^N_t$. On the other hand, from Proposition 3.28 the arrows $\text{full}^N_{t,t+2-N}$ and $\text{drop}^N_{t,t+2-N}$ are null-preserved under $Q^N_t$. Therefore, we can say that $C^N$ is a filtration. Moreover, $Q^N_t$ clearly satisfies the necessary and sufficient conditions of Theorem 3.20 from the way it is constructed. Therefore, the filtration $C^N$ is a risk-neutral filtration with respect to $\text{Drop}_{\alpha,\beta}$. By the way, in Proposition 3.26, $q_{t+2-N}(\omega 11) \in [0,1]$ can take any value. Then $q_{t+2-N}(\omega 10)$ can be computed by $1 - q_{t+2-N}(\omega 11)$. That is, the probability measure $Q^N_{t+2-N}$ is not uniquely determinable.

\[\square\]

3.4. **Valuation.** Let $C^N : T^N \to \text{Prob}$ be a risk-neutral filtration and $Y : B^N_t \to \mathbb{R}$ be a payoff at time $T$. Then, the price $Y_t$ of $Y$ at time $t$ is given by the equation

$$Y_t := E^{C^N}(c^N_t, T) ((b^N_T)^{-1} Y)$$

with the unique arrow $t^N_{T,T} : T \to t$.

That is to say, even those who have a dropped subjective filtration can price Securities $Y$. However, additional consideration is needed on how these prices affect the market equilibrium price.

For $\omega \in B^N_{t-2-N}$, you can see in Figure 3.4 that at time $t - 2^{-N}$ the value of $Y_t(\omega 1)$ is discarded and use only the value of $Y_t(\omega 0)$ for computing $Y_{t-2-N}(\omega)$.

3.4.1. **Replication Strategies.** Let us investigate the situation where a given strategy $(\phi, \psi)$ becomes a replication strategy of the payoff $Y$ at time $T$.

**Definition 3.30.** [Self-Financial Strategies] A self-financial strategy is a strategy $(\phi, \psi)$ satisfying

$$S^N_t \phi_{t+2-N} + b^N_t \psi_{t+2-N} = Y_t$$

for every $t \in (0,\infty)^N$. 

For a self-financial strategy \((\phi_t, \psi_t)_{t \in (0, \infty)^N}\), we have:

\[
V_{t+2-N} = S_{t+2-N}^N (\phi_{t+2-N} \circ f_{t,t+2-N}^N) + b_{t+2-N}^N (\psi_{t+2-N} \circ f_{t,t+2-N}^N)
\]

\[
= (S_t^N \circ f_{t,t+2-N}^N) (1 + 2^{-N} \mu + 2^{-N} \frac{\r}{\sigma \xi_{t+2-N}}) (\phi_{t+2-N} \circ f_{t,t+2-N}^N)
\]

\[
+ b_{t+2-N}^N ((b_t^N)^{-1} (V_t - S_t^N \phi_{t+2-N}) \circ f_{t,t+2-N}^N)
\]

\[
= (1 + 2^{-N} \mu + 2^{-N} \sigma \xi_{t+2-N}) ((S_t^N \phi_{t+2-N}) \circ f_{t,t+2-N}^N) + (1 + 2^{-N} r) ((V_t - S_t^N \phi_{t+2-N}) \circ f_{t,t+2-N}^N)
\]

\[
= (2^{-N} \mu - 2^{-N} r + 2^{-N} \sigma \xi_{t+2-N}) ((S_t^N \phi_{t+2-N}) \circ f_{t,t+2-N}^N)
\]

\[
+ (1 + 2^{-N} r) (V_t^N \circ f_{t,t+2-N}^N).
\]
Therefore, for $\omega \in B_t^N$ and $d_{t+2-N} \in \{0, 1\}$,

\[
V_{t+2-N}(\omega d_{t+2-N}) = (2^{-N}\mu - 2^{-N}r + 2^{-N}\sigma(2d_{t+2-N}-1))S_t^N(\omega t)\phi_{t+V}(\omega t) + (1 + 2^{-N}r)V_t(\omega t)
\]

where

\[
\omega_t := f_{t,t+2-N}^N(\omega d_{t+2-N}).
\]

Now let us assume that there exists a function $g_t : B_t^N \rightarrow B_t^N$ such that $f_{t,t+2-N}^N = g_t \circ \text{full}_{t,t+2-N}$.

Then $f_{t,t+2-N}^N(\omega d_{t+2-N}) = g_t(\omega)$ for every $\omega \in B_t^N$ and $d_{t+2-N} \in \{0, 1\}$. So the equation (3.11) becomes

\[
V_{t+2-N}(\omega d_{t+2-N}) = (2^{-N}\mu - 2^{-N}r + 2^{-N}\sigma(2d_{t+2-N}-1))S_t^N(g_t(\omega))\phi_{t+2-N}(g_t(\omega)) + (1 + 2^{-N}r)V_t(g_t(\omega)).
\]

Hence, we have:

\[
\phi_{t+2-N}(g_t(\omega)) = \frac{V_{t+2-N}(\omega 1) - V_{t+2-N}(\omega 0)}{2^{1-\frac{N}{2}}S_t^N(g_t(\omega))}
\]

\[
V_t(g_t(\omega)) = \frac{(2^{\frac{N}{2}}\sigma - \mu + r)V_{t+2-N}(\omega 1) + (2^{\frac{N}{2}}\sigma - \mu + r)V_{t+2-N}(\omega 0)}{2^{1+\frac{N}{2}}(1 + 2^{-N}r)}.
\]

Therefore, we can determine the appropriate strategy $(\phi_{t+2-N}, \psi_{t+2-N})$ on $g_t(B_t^N) \subset B_t^N$ by (3.14). We actually do not care the values of $(\phi_{t+2-N}, \psi_{t+2-N})$ on $B_t^N \setminus g_t(B_t^N)$.

For example, in the case of $f_{t,t+2-N}^N = \text{full}_{t,t+2-N}$, the function $g_t : B_t^N \rightarrow B_t^N$ satisfies

\[
g_t(\omega d_{t}) = $\omega 0$
\]

for all $\omega \in B_{t-2-N}$ and $d_t \in \{0, 1\}$. Looking at Figure 3.4, values in the region $B_t^N \setminus g_t(B_t^N)$ are not necessary for computing $V_{t-2-N}(\omega)$. Hence, determining the values of $(\phi_{t+2-N}, \psi_{t+2-N})$ in $g_t(B_t^N)$ is enough for making the practical valuation.

3.5. Experienced Paths. In this subsection, we introduce a concept of experienced paths that corresponds to a subjective recognition of a person’s experience.

**Definition 3.31.** Let $B^N : \mathcal{T}^N \rightarrow \text{Prob}$ be a filtration and $t \in [0, \infty]^N$.

1. Define a function $e^B_{t} : B_t^N \rightarrow B_t^N$ by

\[e^B_t(\omega)(s) := f_{s,t}^N(\omega)(s)\]

for $\omega \in B_t^N$, $s \in (0, t]^N$ and $f_{s,t}^N := B^N(t_{s,t}^N)$.

We call $e^B_t(\omega)$ an **experienced path** of $\omega$.

2. $\tilde{B}_t^N := \{e^B_t(\omega) \mid \omega \in B_t^N\}$.

3. $\tilde{\mathcal{F}}_t^N := 2^{\tilde{B}_t^N}$.
\( * \in B_0^2 \quad B_0^3 \ni * \quad 0 \)

\[ \text{full} \]

\( B_1^3 \ni d_1 \quad 1/8 \)

\[ \text{full} \]

\( B_2^3 \ni d_1d_2 \quad 2/8 \)

\[ \text{full} \]

\( B_3^3 \ni d_1d_20 \quad 3/8 = \alpha \)

\[ \text{drop} \]

\( B_4^3 \ni d_1d_2d_3d_40 \quad 5/8 = \beta \)

\[ \text{drop} \]

\( B_5^3 \ni d_1d_2d_3d_4d_5d_6 \quad 6/8 \)

\[ \text{full} \]

\( B_6^3 \ni d_1d_2d_3d_4d_5d_6d_7d_8 \quad 7/8 \)

\[ \text{full} \]

\( B_7^3 \ni d_1d_2d_3d_4d_5d_6d_7d_8 \quad 1 = t \)

Figure 3.5. Experienced Paths for \( B^N := \text{Drop}_{\frac{3}{8}}^{\frac{5}{8}} \)

(4) \( \tilde{\mathcal{P}}_t^N := \mathcal{P}_t^N \circ (e_{B,N}^N)^{-1}. \)

\[
\begin{align*}
B_t^N & \xrightarrow{e_{B,N}^N} \tilde{B}_t^N \\
[0,1] & \xrightarrow{\mathcal{P}_t^N} \mathcal{F}_t^N \xrightarrow{(e_{B,N}^N)^{-1}} \tilde{\mathcal{F}}_t^N
\end{align*}
\]

(5) \( \tilde{B}_t^N := (\tilde{B}_t^N, \tilde{\mathcal{F}}_t^N, \tilde{\mathcal{P}}_t^N). \)

(6) For \( s, t \in [0, \infty)^N (s \leq t), \)

\[
\tilde{f}_{s,t}^N : \tilde{B}_t^N \rightarrow \tilde{B}_s^N \text{ is a function defined by}
\]

\[
\tilde{f}_{s,t}^N := \text{full}_{s,t}^N |_{\tilde{B}_s^N}.
\]

Proposition 3.32. A correspondence \( \tilde{B}^N : \mathcal{T}^N \rightarrow \text{Prob} \) defined by

\[
\tilde{B}^N(t) := \tilde{B}_t^N \quad \text{and} \quad \tilde{B}^N(t_{s,t}) := \tilde{f}_{s,t}^N
\]

is a functor, that is, a \( \mathcal{T}^N \)-filtration.

Example 3.33. [Experienced Paths for \( B^N := \text{Drop}_{\frac{3}{8}}^{\frac{5}{8}} \)] Let \( B^N := \text{Drop}_{\frac{3}{8}}^{\frac{5}{8}} \) and \( d_i \in \{0, 1\} \) for \( i \in \mathbb{N} \). Then, as seen in Figure 3.5, we have

\[
e_{B,t}^0(d_1d_2d_3d_4) = d_10d_3d_4,
\]

\[
e_{B,t}^3(d_1d_2d_3d_4d_5d_6d_7d_8) = d_1d_2000d_6d_7d_8.
\]
**Theorem 3.34.** The correspondence $e^{\mathbb{B}^N} : \mathbb{B}^N \to \tilde{\mathbb{B}}^N$ is a natural transformation. That is, for $s,t \in [0,\infty)^N \ (s \leq t)$, the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{T}^N & \xrightarrow{B^N} & \mathbb{B}^N \\
\downarrow{s,t} & \swarrow{e^{\mathbb{B}^N}} & \downarrow{\tilde{\mathbb{B}}^N} \\
\tilde{\mathbb{B}}^N & \xrightarrow{\tilde{\mathbb{B}}^N} & \mathbb{B}^N \\
\end{array}
\]

Proof. For $\omega \in \mathbb{B}^N_t$ and $u \in (0,s]^N$, \(f^N_{s,t}(\omega)(u) = (\text{full}^N_{s,t} |_{\mathbb{B}^N})_{|_{\mathbb{B}^N}}(\omega)(u) = (e^N_{t}(\omega))(u) = (e^N_{t}(\omega))(u) = f^N_{u,t}(\omega)(u).\)

On the other hand, \(e^N_{s,t} (f^N_{s,t}(\omega))(u) = f^N_{u,t}(f^N_{s,t}(\omega))(u) = f^N_{u,t}(\omega)(u).\)

□

Here is an implication of Theorem 3.34: The person who dropped her memory believes that her memory is perfect (full), while others observe that she lost her memory.

Lastly, we mention the fact that in a case the given filtration is full, experienced paths coincide with objective paths.

**Proposition 3.35.** If $\mathbb{B}^N = \text{Full}^N$, then $\tilde{\mathbb{B}}^N = \mathbb{B}^N$.

### 4. Concluding Remarks

In this paper, we proposed the concept of generalized filtration. It is an extended filtration that goes beyond the conventional framework of monotonically increasing information sequences and allows the development of information to not only increase, but also to decrease or be twisted. It is an extended concept, just like the subjective probability measure attributed to an individual, of a subjective filtration as a history of personal information evolution. A natural interest is to see how far conventional theories of stochastic analysis and control can be developed under such generalized filtration.

In this paper, as an example of an application, in addition to conventional filtration (classical filtration) in a binomial asset price model, we introduce a dropped filtration with loss of memory for a certain period of time to see whether individuals with the latter as her subjective filtration can in any sense price securities. This resulted in the question of whether there is a risk-neutral filtration corresponding to this subjective filtration. We have shown the existence of such a filtration. However, the obtained risk-neutral filtration is not uniquely determined, unlike the classical risk-neutral probability measure observed in a complete market. This means that a market with such a generalized filtration is not complete (at least for individuals who have such a filtration as a subjective filtration). For other subjective filtrations not discussed in this paper, it is possible that there may be no risk-neutral probability measure. How equilibrium market prices are determined in such cases may be one of important themes for future research.

Needless to say, the application of generalized filtrations shown in this paper is only one example, and many other applications are possible. As mentioned above, generalized filtrations
can be used to develop conventional theories of stochastic control and stochastic differential equations. For example, it can be used to transform a problem that is not time-consistent under classical filtration into a time-consistent problem by twisting the filtration. The theory of filtration enlargement used for credit risk calculation and insider trading analysis in finance may be able to be considered in the framework of generalized filtration [Aksamit and Jeanblanc, 2017]. Furthermore, in order to study the relationship between a filtration and related risk-neutral filtration, or filtrations defined on several different time domains, it is necessary to consider the transformation and convergence of filtrations in a space of filtrations.

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Graduate School of Management, Tokyo Metropolitan University, 1-4-1 Marunouchi, Chiyoda-ku, Tokyo 100-0005, Japan
Email address: Takanori Adachi <tadachi@tmu.ac.jp>

College of International Management, Ritsumeikan Asia Pacific University, 1-1 Jumonjibaru, Beppu, Oita, 874-8577 Japan
Email address: Katsushi Nakajima <knakaji@apu.ac.jp>

Department of Mathematical Sciences, Ritsumeikan University, 1-1-1 Nojihigashi, Kusatsu, Shiga, 525-8577 Japan
Email address: Yoshihiro Ryu <iti2san@gmail.com>