Computational method for singularly perturbed parabolic differential equations with discontinuous coefficients and large delay

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ABSTRACT

This paper deals with the computational method for a class of second-order singularly perturbed parabolic differential equations with discontinuous coefficients involving large negative shift. The formulated method comprises the implicit Euler and the cubic-spline in compression methods for time and spatial dimensions, respectively. Intensive numerical experimentation has been done on some model examples and the results are tabulated. The results depict that the present method is more accurate than some methods existing in the literature. Further, the layer behavior of the solutions is presented using graphs and observed to agree with the existing theories. Finally, error analysis of the scheme is done and observed that the proposed method is parameter uniform convergent with the order of convergence \(O(\Delta t + h^2)\).

1. Introduction

Many mathematical models assume specified behavior or phenomena that depend on the present and the past states of a system [1, 2]. In other words, past events explicitly affect future results. For this reason, functional differential equations are more realistic and frequently appear in many applications such as medicine [3], mechanics [4, 5], biology [6], population dynamics [7], and economics [8], in which future behavior depends implicitly on the past. These problems provide the best, and sometimes the only realistic simulation of the phenomena observed [6]. The differential equations in which the highest order derivative is multiplied by a small positive number named singular perturbation parameter and containing a shifting parameter are termed singularly perturbed delay differential equations. The solutions to such types of problems have a boundary or interior layers. The boundary layer occurs when a term containing the highest order derivative is multiplied by a singular perturbation parameter, and the interior layer arises when there is a discontinuity in the given data. Moreover, the simultaneous presence of a delay parameter and discontinuous coefficient makes the problem stiff and the solution to the problem exhibits multi-scale character as \(\epsilon \to 0\). There exist narrow regions across the turning points where the solution varies exponentially and approaches a discontinuous limit [9, 10]. In these regions, the solution reveals sharp interior layers, and it is difficult to find the analytical solution to the problems.

Due to the presence of interior layers appearing in the solution, the classical numerical methods, when applied to these problems on a uniform mesh, are unable to provide an efficient numerical solution unless they are applied with very fine meshes inside the regions. Thus, to resolve this issue, the fitted operator and layer-adapted mesh methods are competitive computational techniques to overcome the limitations of the classical numerical methods.

In this paper, we introduce a method to solve the following singularly perturbed delay parabolic differential equation with discontinuous coefficients and source terms on the domain \(D = \mathbb{S}^- \cup \mathbb{S}^+ = (0, 1) \times (0, T] \cup (1, 2) \times (0, T]\), where \(\mathbb{S}^- = (0, 1) \times (0, T], \mathbb{S}^+ = (1, 2) \times (0, T]\), \(\partial D = \overline{\mathbb{T}} D\) and \(T\) is some fixed positive time:

\[
\begin{aligned}
L_\epsilon u(x, t) &= \epsilon \frac{\partial^2 u(x, t)}{\partial x^2} + v(x) \frac{\partial u(x, t)}{\partial x} - \varphi(x) u(x - 1, t) - m(x) u(x, t) - \frac{\partial u(x, t)}{\partial t} = 0(x, t), \\
\quad u(x, t) &= \Xi_0(x), x \in [0, 2], \\
\quad u(x, t) &= \Xi_1(x, t), x \in [-1, 0] \times [0, T], \\
\quad u(2, t) &= \Xi_2(2, t), t \in [0, T].
\end{aligned}
\]

where \(0 < \epsilon \ll 1\) is a perturbation parameter, \(\varphi(x)\) and \(m(x)\) are suf-}

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Lemma 2.1 (Minimum Principle). If $Y \in C^{0,0}(\overline{D}) \cap C^{1,1}(D) \cap C^{2,2}(\mathbb{S}^3 \cup \mathbb{S}^4)$ such that $Y(0,t) \geq 0, Y(x,0) \geq 0, Y(t,2) \geq 0$, and $Y(1^+,t) = Y(1^-,t) - Y(1^-,t) \leq 0$, and $L_\varepsilon Y(x,t) \leq 0, \forall (x,t) \in D$, then $Y(x,t) \geq 0, \forall (x,t) \in \overline{D}$.

Proof. Assume that $(x,\dot{x}) \in D$ such that $Y(x,\dot{x}) = \min_{(x,t) \in \overline{D}} Y(x,t)$ and suppose $Y(\dot{x}) < 0$, then it follows that $(x,\dot{x}) \not\in \partial D$. Consequently, we have
\[
\frac{\partial Y(\dot{x})}{\partial x} = \frac{\partial Y(\dot{x})}{\partial t} = 0, \quad \text{and} \quad \frac{\partial^2 Y(\dot{x})}{\partial x^2} > 0.
\]
To show $L_\varepsilon Y(x,\dot{x}) > 0$, we consider the following cases:

Case 1: If $(x,\dot{x}) \in \mathbb{S}^3$,
\[
L_\varepsilon Y(x,\dot{x}) = \frac{\partial^2 Y(x,\dot{x})}{\partial x^2} + v(x) \frac{\partial Y(x,\dot{x})}{\partial t} - \sigma(x) Y(x,\dot{x}) - \frac{\partial Y(x,\dot{x})}{\partial t} > 0,
\]
from Eqs. (2) and (4).

Case 2: If $(x,\dot{x}) \in \mathbb{S}^4$,
\[
L_\varepsilon Y(x,\dot{x}) = \frac{\partial^2 Y(x,\dot{x})}{\partial x^2} + v(x) \frac{\partial Y(x,\dot{x})}{\partial t} - \sigma(x) Y(x,\dot{x}) - \frac{\partial Y(x,\dot{x})}{\partial t} > 0.
\]
Combining the above two cases, we obtain $L_\varepsilon Y(x,\dot{x}) > 0$, that contradicts the assumption made above $L_\varepsilon Y(x,t) \leq 0, \forall (x,t) \in D$. If $x = 1$, we have $Y(1^+,t) \geq 0$, $Y(1^-,t) \leq 0$ implying $Y(1,t) \not\equiv 0$. This gives $Y(1^+,t) = Y(1^-,t) = Y(1^-,t) = 0$ to avoid contradiction to the assumption that $Y(1,t) \not\equiv 0$. But we have already proved that $Y(\dot{x}) \geq 0$ for $(x,\dot{x}) \in (\mathbb{S}^3 \cup \mathbb{S}^4)$. Therefore, $Y(x,t) \geq 0, \forall (x,t) \in \overline{D}$.

An immediate application of the minimum principle gives the following boundedness of the solution.

Lemma 2.2 (Uniform Stability Estimate). Let $u(x,t)$ be the solution of (1). Then
\[
|u(x,t)|_{\infty,\overline{D}} \leq \frac{\|\varphi\|_{\infty,\overline{D}}}{\zeta} + \max |u(x,t)|_{\infty,\partial D}.
\]
Proof. See [28].

3. Description of the numerical scheme

3.1. Temporal discretisation

On applying the implicit Euler method on the time variable of Eq. (1) with uniform step size $\Delta t$ such that $D_\varepsilon \Delta t = \{t_j = j\Delta t, \Delta t = T/M, 0 \leq j \leq M\}$ yields:
\[
\begin{align*}
L_\varepsilon^M U(x,t_{j+1}) = & \xi(x,t_{j+1}), \\
U(x,0) = & \Xi_0(x), x \in [0,2], \\
U^{(j+1)}(1) = & \Xi_1(1,t_{j+1}), x \in [-1,0] \times [0,T], \\
U^{(j+1)}(2) = & \Xi_2(2,t_{j+1}), t \in [0,2],
\end{align*}
\]
where
\[
L_\varepsilon^M U(x,t_{j+1}) = \begin{cases} \\
\frac{\partial^2 u(x,t_{j+1})}{\partial x^2} + v(x) \frac{\partial u(x,t_{j+1})}{\partial t} - \mu(x) U(x,t_{j+1}) \\
\frac{\partial^2 u(x,t_{j+1})}{\partial x^2} + v(x) \frac{\partial u(x,t_{j+1})}{\partial t} - \mu(x) U(x,t_{j+1}) \\
\frac{\partial^2 u(x,t_{j+1})}{\partial x^2} + v(x) \frac{\partial u(x,t_{j+1})}{\partial t} - \mu(x) U(x,t_{j+1}) \\
\end{cases}
\]
and $\xi(x,t_{j+1})$.
\[ \xi(x, t_j) = \begin{cases} \rho(x, t_j) + \phi(x)U(x, t - 1, t_j), & \text{if } x \in [0, 1], j = 0(1)M - 1, \\ 0, & \text{else} \end{cases} \]

where \( \mu(x) \equiv \sigma(x) + 1/\Delta t \). \( \rho(x, t_j) = \phi(x)U(x, t_j) - \frac{\partial U(x, t_j)}{\partial x} \).

The operator \( L^M \) satisfies the following semi-discrete minimum principle.

**Lemma 3.1** (Semi-discrete minimum principle). Let \( Y(x, t_{j+1}) \) be a smooth function satisfies \( Y(x, t_{j+1}) \geq 0 \) for \( x = 0, 2, \ldots, Y'(t_j) = Y'(1, t_j) \geq 0 \) and \( L^M Y(x, t_{j+1}) \leq 0 \), \( \forall x \in (0, 2) \). Then \( Y(x, t_{j+1}) \geq 0, \forall x \in (0, 2) \).

**Proof.** Suppose \( (\hat{x}, t_{j+1}) \in (x, t_{j+1}) : x \in [0, 2] \) be such that \( Y(\hat{x}, t_{j+1}) = \min_{x \in [0, 2]} Y(x, t_{j+1}) \) and assume that \( Y(\hat{x}, t_{j+1}) < 0 \). Consequently, we have

\[ \frac{\partial Y(\hat{x}, t_{j+1})}{\partial x} = 0, \quad \frac{\partial^2 Y(\hat{x}, t_{j+1})}{\partial x^2} > 0, \quad (6) \]

and \( (\hat{x}, t_{j+1}) \notin D \) since \( Y(x, t_{j+1}) \geq 0 \) for \( x = 0, 2 \).

To show \( L^M Y(\hat{x}, t_{j+1}) > 0 \), we consider the following cases:

**Case 1:** \( \hat{x} \notin D^* \),

\[ L^M Y(\hat{x}, t_{j+1}) = \frac{\partial^2 Y(\hat{x}, t_{j+1})}{\partial x^2} + \phi(\hat{x}) \frac{\partial Y(\hat{x}, t_{j+1})}{\partial x} - \mu(\hat{x}) Y(\hat{x}, t_{j+1}) \]

**Case 2:** \( \hat{x} \in D^* \),

\[ L^M Y(\hat{x}, t_{j+1}) = \frac{\partial^2 Y(\hat{x}, t_{j+1})}{\partial x^2} + \phi(\hat{x}) \frac{\partial Y(\hat{x}, t_{j+1})}{\partial x} - \mu(\hat{x}) Y(\hat{x}, t_{j+1}) \]

Combining the above two cases, we obtain \( L^M Y(\hat{x}, t_{j+1}) > 0 \), that contradicts the assumption made above \( L^M Y(x, t_{j+1}) \leq 0, \forall x \in D \).

**Lemma 3.2** (Semi-discrete Stability Estimate). Let \( U(x, t_{j+1}) \) be the solution to Eq. (5). Then

\[ |U(x, t_{j+1})| \leq \max \left\{ \left[ \xi^{i+1}(x) \right], \frac{\| U \|}{\xi^{i+1}(x)} \right\}, \quad \text{for } x \in [0, 2]. \]

**Proof.** Defining the barrier functions as

\[ \Theta^+(x, t_{j+1}) = \max \left\{ \left[ \xi^{i+1}(x) \right], \frac{\| U \|}{\xi^{i+1}(x)} \right\}, \quad \Theta^-(x, t_{j+1}) = 0 \]

Now, \( \Theta^+(x, t_{j+1}) \geq 0 \) and \( \Theta^-(x, t_{j+1}) \geq 0 \) and we have

**Case I:** \( 0 < x < 1 \),

\[ L^M \Theta^+(x, t_{j+1}) = -\mu(x) \max \left\{ \left[ \xi^{i+1}(x) \right], \frac{\| U \|}{\xi^{i+1}(x)} \right\} \leq L^M U(x, t_{j+1}) \leq 0, \quad (\text{since } \mu(x) \geq 0) \]

**Case II:** \( 1 < x < 2 \),

\[ L^M \Theta^-(x, t_{j+1}) = -\mu(x) \max \left\{ \left[ \xi^{i+1}(x) \right], \frac{\| U \|}{\xi^{i+1}(x)} \right\} \leq L^M U(x, t_{j+1}) \leq 0, \quad (\text{since } \mu(x) \geq 0) \]

Therefore, using Lemma 2.1, we obtain

\[ |U(x, t_{j+1})| \leq \max \left\{ \left[ \xi^{i+1}(x) \right], \frac{\| U \|}{\xi^{i+1}(x)} \right\}, \quad \text{for } x \in [0, 2]. \]

**Lemma 3.3.** Having \( \frac{\partial U(x, t_j)}{\partial x} \leq C, \forall (x, t) \in \overline{D}, k = 0, 1, 2 \) implies the local error estimate \( e_{i+1} = u(x, t_{j+1}) - U(x, t_{j+1}) \) at \((j+1)\) time step satisfies the following bounds

\[ \| e_{i+1} \| \leq C (\Delta t)^2, \quad \text{for some constant } C. \]

**Lemma 3.4.** Let \( E_j = u(x, t_j) - U(x, t_j) \) be the global error estimate in the time direction. Then the following bound holds

\[ \| E_j \| \leq C (\Delta t). \]

**Proof.** From Lemma 3.3 it follows that

\[ \| E_j \| = \sum_{i=0}^{\infty} \| e_i \|, \quad \| e_i \| \leq C (\Delta t) \]

3.2. Spatial semi-discretization

To approximate Eq. (5), we used the cubic spline in compression method described as follows. Let us discretize the domain \([0, 2]\) as \( x_i = ih, i = 0, 1, 2, \ldots, N \) with \( x_0 = 0, x_N = 2 \) and \( h \) is mesh length defined as \( h = 2/N \) where \( N \) number of subintervals. A function \( S_j(x_i) \) \( \in C^2[0, 2] \) which interpolates \( U_j(x_i) \) \( U_j(x_i) = U(x_i, t_{j+1}) \) at the mesh points \( x_i, i = 0, 1, \ldots, N \) depends on parameter \( t \) > 0 reduces to cubic spline in \([0, 2]\) as \( t = 0 \) is called parametric cubic spline function. In \([x_1, x_N]\), the parametric cubic spline function \( S_j(x, t) \) \( S_j(x, t) \) satisfies the differential equation

\[ \frac{d^2 S_j(x, t)}{dx^2} + \alpha S_j(x, t) = \frac{\alpha x - (x_i - x)}{h} \left[ \frac{d^2 S_j(x, t)}{dx^2} + \frac{\alpha S_j(x, t)}{h} \right] \]

where \( S_j(x_i) = U_j(x_i) \) and \( t > 0 \) is said to be the spline in compression. Following [12] Eq. (7) becomes

\[ \beta = h \sqrt{\alpha}, \quad \alpha_t = \beta^2 \left( \frac{x}{\alpha h} + 1 \right), \quad \beta_2 = \beta(t - 1 - \cot \beta), \]

\[ M_j = \frac{d^2 U_j(x_i)}{dx^2}, \quad \alpha_j = \beta_2 + 1, \quad \alpha = \alpha_2 / i, \quad t = 1)N - 1. \]

Let us rewrite Eq. (5) at \( x = x_k \), \( k = 1, \ldots, N \) as

\[ \xi(x_k, t_{j+1}) = 1 - \alpha \xi(x_k, t_{j+1}) \]

where \( \alpha \xi(x, t_{j+1}) = U_j(x_i) \) \( (x_i + 1) \) and we approximate \( \frac{d^2 U_j(x_i)}{dx^2} \) using:

\[ \frac{d^2 U_j(x_i)}{dx^2} = \frac{U_j(x_i + 1) - 2U_j(x_i) + U_j(x_i - 1)}{h^2} \]

Now substituting Eq. (9) into Eq. (8) using Eq. (10) we obtain

\[ \frac{d^2 U_j(x_i)}{dx^2} = \frac{U_j(x_i + 1) - 2U_j(x_i) + U_j(x_i - 1)}{h^2} + \frac{\alpha_j}{\alpha h} V_{x_i} - \frac{\alpha_j}{\alpha h} V_{x_i} + \alpha_j \mu_{x_i} \]

Now multiplying \( \alpha_1 \) term with constant \( \varepsilon \) in Eq. (11), we get

\[ \frac{d^2 U_j(x_i)}{dx^2} = \frac{U_j(x_i + 1) - 2U_j(x_i) + U_j(x_i - 1)}{h^2} + \frac{\alpha_j}{\alpha h} V_{x_i} - \frac{\alpha_j}{\alpha h} V_{x_i} + \alpha_j \mu_{x_i} \]
The fitting factor $\sigma(\rho)$ is evaluated in such a way that the solution of Eq. (12) converges uniformly to the solution of Eq. (1). Multiplying Eq. (12) by $h$ and taking the limit as $h \to 0$, we obtain

$$\frac{-\sigma(\rho)}{\rho} \left[ U^i_{i-1} - 2U^i_{i} + U^i_{i+1} \right] + (a_1 + a_2) y_0 \left[ U^i_{i-1} - U^i_{i+1} \right] = 0. \quad (13)$$

When the boundary layer is on the right side of the domain, from the theory of singular perturbation [30], the solution of Eq. (5) is of the form:

$$U^{i+1}(x) = U^i_{0+}(x) + \frac{v(2)}{v(1)} \left( \Xi^2_2 (2) - U^i_{0+}(2) \right) e^{-\left( \frac{i(x-x_0)}{h} \right)} + O(e). \quad (14)$$

where $U^i_{0+}(x)$ is the solution of the reduced problem of Eq. (5). Using Taylor’s series expansion for $v(x)$ about $x = 2$ and restricting to their first terms, Eq. (14) becomes

$$U^{i+1}(x) = U^i_{0+}(x) + \left( \Xi^2_{2+} (2) - U^i_{0+}(2) \right) e^{-\left( \frac{i(x-x_0)}{h} \right)} + O(e). \quad (15)$$

From Eq. (15), we have

$$\lim_{h \to 0} U^i_{0+}(i) U^i_{0+}(0) + \left( \Xi^2_{2+} (2) - U^i_{0+}(2) \right) e^{-\left( \frac{i(x-x_0)}{h} \right)} = \lim_{h \to 0} U^i_{0+}(i) U^i_{0+}(0) + \left( \Xi^2_{2-} (2) - U^i_{0+}(2) \right) e^{-\left( \frac{i(x-x_0)}{h} \right)} = \lim_{h \to 0} U^i_{0+}(i) U^i_{0+}(0) + \left( \Xi^2_{2-} (2) - U^i_{0+}(2) \right) e^{-\left( \frac{i(x-x_0)}{h} \right)}.$$

Using Eq. (16) into Eq. (13), we obtain the fitting parameter

$$\sigma(\rho) = (a_1 + a_2) \rho v(0) \coth \left( \frac{v(2)}{2} \right).$$

In general, a variable fitting parameter is given as

$$\sigma(\rho) = (a_1 + a_2) \rho v(0) \coth \left( \frac{v(2)}{2} \right). \quad (17)$$

Finally, using Eq. (12) and $\sigma(\rho)$ has given in Eq. (17), we obtain

$$\beta^{i-1} U^i_{i-1} + \beta^i U^i_{i} + \beta^{i+1} U^i_{i+1} = X_i, \quad (18)$$

where

$$\beta^{i-1} = \frac{-\sigma(\rho) v_0}{2N} + \frac{2a_1}{2N} v_0 vi + \frac{2a_2}{2N} v_0 v_0 + a_1 h_{i-1},$$

$$\beta^i = \frac{2\sigma(\rho) v_0}{2N} + \frac{2a_1}{2N} v_0 vi + \frac{2a_2}{2N} v_0 v_0 + a_1 h_{i-1},$$

$$\beta^{i+1} = \frac{-\sigma(\rho) v_0}{2N} + \frac{2a_1}{2N} v_0 vi + \frac{2a_2}{2N} v_0 v_0 + a_1 h_{i+1},$$

$$X_i = \left\{ \begin{array}{ll}
-\alpha_1 \left( v^i_{i+1} + \alpha h_i \right) + \frac{\Xi^2_{i+1} (i+2)}{2} - 2a_2 \left( v^i_{i+1} + \alpha h_i \right), & \text{for } i = (1)N/2,
-\alpha_1 \left( v^i_{i+1} + \alpha h_i \right) + \frac{\Xi^2_{i+1} (i+2)}{2} - 2a_2 \left( v^i_{i+1} + \alpha h_i \right), & \text{for } i = (N/2 + 1)(N - 1).
\end{array} \right.$$
Let \( \hat{U} = \left[ \hat{U}^{i+1}_1, \hat{U}^{i+1}_2, \ldots, \hat{U}^{i+1}_{N-1} \right] \) satisfy the following equation

\[
[H + G] \hat{U} + K = 0.
\]  

(22)

Subtracting Eq. (20) from Eq. (22), we have

\[
[H + G] E = TE(h).
\]  

(23)

where \( E = \left[ 0, 0, \ldots, 0 \right] \). Let \( |v_i| \leq C_1, |v| \leq C_2 \), so that if \( g_{ij} \) is the \((i, j)\)th element of the matrix \( G \), then

\[
\left| G_{i,j+1} \right| = |z_i| \leq \left( \frac{1}{1 + x} \right) C_1 + x_i h C_2, \quad i = 1(1)N - 2,
\]

\[
\left| G_{i,j-1} \right| = |a_i| \leq \left( \frac{1}{1 + x^2} \right) C_1 + a_i h C_2, \quad i = 2(1)N - 1.
\]

Thus, as \( h \to 0 \), we have

\[
\begin{cases}
\frac{d}{dh} |G_{i,j+1}| < 0, & i = 1(1)N - 2, \\
\frac{d}{dh} |G_{i,j-1}| < 0, & i = 2(1)N - 1.
\end{cases}
\]

Therefore, the matrix \((H + G)\) is irreducible.

Let \( L_i \) be the sum of the elements of the \( i \)th row of the matrix \((H + G)\), then

\[
L_i = \sum_{j=1}^{N-1} G_{i,j} + |z_i| - \left( \frac{1}{1 + x} \right) a_i h C_2, \quad i = 1(1)N - 2,
\]

\[
L_{N-1} = \sum_{j=1}^{N-2} G_{i,j} + |a_i| + |a_i| h C_2, \quad i = 1(1)N - 1.
\]

For sufficiently small \( h \), the matrix \((H + G)\) is monotone. So \((H + G)^{-1}\) exists and \((H + G)^{-1} \geq 0\). Hence, from Eq. (23), we have

\[
E = (H + G)^{-1} TE(h).
\]  

(24)

Equation (24) can be written as

\[
\|E\| \leq \|H + G\|^{-1} \|TE(h)\|.
\]  

(25)

Let \( g_{kj} \) be the \((k, l)\)th elements of \((H + G)^{-1}\). As \( g_{kj} \geq 0 \), from the operators of matrices we have

\[
\sum_{i=1}^{N-1} g_{kj} L_i = 1, \quad k, l = 1, 2, \ldots, N - 1.
\]  

(26)

Hence, Eq. (26) follows

\[
\sum_{i=1}^{N-1} (g_{kj}) L_i \leq \frac{1}{\min_{0 \leq i \leq N-1} L_i} \leq \frac{1}{h}, \quad k, l = 1, 2, \ldots, N - 1.
\]  

(27)

Thus, from Eqs. (19), (25) and (27), we obtain

\[
E_i = \sum_{i=1}^{N-1} g_{kj} T E(h) \leq \frac{O(h^2)}{|h|}, \quad i = 1, 2, \ldots, N - 1.
\]

Thus, \( \|E\| \leq O(h^2)\). This implies the semi-discretization process in the spatial direction of the presented method is of quadratic order of convergence.

**Lemma 4.3.** Let \( u(x, t) \) and \( U^{i+1} \) are solution of continuous problem (1) and discrete problem (18), respectively. Then, the error estimate for the fully discrete scheme is given by

\[
\left| u(x, t) - U^{i+1} \right| \leq C(\Delta t + h^2).
\]

**Proof.** Combining Lemma 3.4 and Lemma 4.2, we get the required estimate.

5. Numerical illustration

To validate the applicability of the proposed scheme three numerical examples are presented. In all cases, we performed numerical experiments by taking \( a_1 = 1 \epsilon - 0.2 \) and \( a_2 = 0.9 \epsilon - 0.3 \). As exact solutions of these problems are not available, the maximum point-wise absolute errors are calculated by using the double mesh principle [31]:

\[
E_{ij}^N = \max_{N-1 \geq i \geq j} \left| U_{ij}^N - U_{ij}^{2N^2} \right|,
\]

where \( U_{ij}^N \) and \( U_{ij}^{2N^2} \) are computed numerical solutions obtained on the mesh \( D_{N}^N = \Omega N \times \Omega M \) and \( D_{2N^2} = \Omega_{2N^2} \times \Omega^2 \) respectively.

The point-wise maximum absolute errors \((E_i^N)\), order of convergence \((R_{ij}^N)\) and point-wise order of convergence \((R_{ij}^N)\) are calculated using

\[
E_{ij}^N = \max \{ E_{ij}^N \}, \quad R_{ij}^N = \log_2 \left( \frac{E_{ij}^N}{E_{ij}^{2N^2}} \right)
\]

and

\[
R_{ij}^N = \log_2 \left( \frac{E_{ij}^N}{E_{ij}^{2N^2}} \right)
\]

respectively.

**Example 5.1.** Consider the following problem of the form in (1)

\[
\frac{\partial^2 u(x, t)}{\partial x^2} + \frac{\partial u(x, t)}{\partial t} + 2u(x - 1, t) - 5u(x, t) - 3u(x, t) = \theta(x, t),
\]

\[
\begin{cases}
\frac{\partial u(x, t)}{\partial t} = 0, & \text{if } (x, t) \in [0, 2] \times \{t = 0\}, \\
\frac{\partial u(x, t)}{\partial x} = 0, & \text{if } (x, t) \in \{-1, 0\} \times [0, 2],
\end{cases}
\]

\[
u(x) = \begin{cases}
-(4 + x^2), & \text{if } 0 \leq x \leq 1, \\
(8 - x^2), & \text{if } 1 < x \leq 2,
\end{cases}
\]

\[
\theta(x) = \begin{cases}
4x^2 \exp(-x), & \text{if } (x, t) \in [0, 1] \times [0, 2], \\
4(2 - x)^2 \exp(-x), & \text{if } (x, t) \in (1, 2) \times [0, 2].
\end{cases}
\]

**Example 5.2.** Consider the following problem of the form in (1)

\[
\frac{\partial^2 u(x, t)}{\partial x^2} + \frac{\partial u(x, t)}{\partial t} + u(x - 1, t) - 3u(x, t) - \frac{\partial u(x, t)}{\partial t} = \theta(x, t),
\]

\[
\begin{cases}
\frac{\partial u(x, t)}{\partial t} = 0, & \text{if } (x, t) \in [0, 2] \times \{t = 0\}, \\
\frac{\partial u(x, t)}{\partial x} = 0, & \text{if } (x, t) \in \{-1, 0\} \times [0, 2],
\end{cases}
\]

\[
u(x) = \begin{cases}
-(4 + x^2), & \text{if } 0 \leq x \leq 1, \\
(3 + x^2), & \text{if } 1 < x \leq 2,
\end{cases}
\]

\[
\theta(x) = \begin{cases}
-1, & \text{if } (x, t) \in [0, 1] \times [0, 2], \\
1, & \text{if } (x, t) \in (1, 2) \times [0, 2].
\end{cases}
\]

**Example 5.3.** Consider the following problem of the form in (1)

\[
\frac{\partial^2 u(x, t)}{\partial x^2} + \frac{\partial u(x, t)}{\partial t} - u(x - 1, t) - x(2 - x)u(x, t) - \frac{\partial u(x, t)}{\partial t} = \theta(x, t),
\]

\[
\begin{cases}
\frac{\partial u(x, t)}{\partial t} = \sin(x, t), & \text{if } (x, t) \in [0, 2] \times \{t = 0\}, \\
\frac{\partial u(x, t)}{\partial x} = 0, & \text{if } (x, t) \in \{-1, 0\} \times [0, 2],
\end{cases}
\]

\[
u(x) = \begin{cases}
-(2 + x(2 - x)), & \text{if } 0 \leq x \leq 1, \\
(2 + x(2 - x)), & \text{if } 1 < x \leq 2,
\end{cases}
\]

\[
\theta(x) = \begin{cases}
2(1 + x^2)^2, & \text{if } (x, t) \in [0, 1] \times [0, 2], \\
3(1 + x^2)^2, & \text{if } (x, t) \in (1, 2) \times [0, 2].
\end{cases}
\]
The calculated $E^{N,M}_{\text{EM}}, E^{N,M}_{\text{EN}}, R^{N,M}_{\text{EM}}$ and $R^{N,M}_{\text{EN}}$ by the scheme in (18) for Examples 5.1, 5.2, and 5.3 are given in Tables 1-4. Besides, the comparison of numerical results obtained by the proposed scheme and results in [28] is tabulated in Table 3. From this table, one can conclude that the proposed scheme gives better results than the scheme in [28]. As shown in Figs. 1-6, the numerical solutions of the considered examples possess singular behavior at the point $x = 0, 1, 2$. It is observed that singularity at point $x = 0$ on the spatial domain is propagated at the point $x = 1$ due to presence of the negative shift in the reaction term. As a result of this singularity, the numerical solution exhibits an interior layer at $x = 1$. The numerical solutions plotted in Figs. 4-6, depict that as $\varepsilon \to 0$ an interior layer appears at $x = 1$. Further, the log-log plot of the maximum absolute errors is plotted in Figs. 7-9. From these figures, one can see that the error decreases monotonically as $N$ increases.

6. Conclusions

In this work, a class of time-dependent singularly perturbed delay parabolic differential equations exhibits turning point behavior across discontinuities and weak boundary layers and is solved numerically. The developed scheme constitutes the implicit Euler in the time direction and the cubic spline in compression method in the space direction on uniform step size. The scheme has shown to be $\varepsilon$-uniformly con-
Fig. 1. Numerical solution profiles of Example 5.1 for $M = N = 128$.

(a) $\epsilon = 2^{-2}$  
(b) $\epsilon = 2^{-10}$

Fig. 2. Numerical solution profiles of Example 5.2 for $M = N = 128$.

(a) $\epsilon = 2^{-2}$  
(b) $\epsilon = 2^{-10}$

Fig. 3. Numerical solution profiles of Example 5.3 for $M = N = 128$.

(a) $\epsilon = 2^{-2}$  
(b) $\epsilon = 2^{-10}$
Fig. 4. Numerical solutions of Example 5.1 for different values of \( \varepsilon \) at \( M = N = 128 \).

Fig. 5. Numerical solutions of Example 5.2 for different values of \( \varepsilon \) at \( M = N = 128 \).

Fig. 6. Numerical solutions of Example 5.3 for different values of \( \varepsilon \) at \( M = N = 128 \).

Fig. 7. The Logplot of the maximum absolute error for different values of \( \varepsilon \) for Example 5.1.

Fig. 8. The Logplot of the maximum absolute error for different values of \( \varepsilon \) for Example 5.2.

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**Additional information**

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Fig. 9. The Loglog plot of the maximum absolute error for different values of $\epsilon$ for Example 5.3.

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