GLOBAL EXISTENCE AND SCATTERING OF EQUIVARIANT DEFOCUSING CHERN-SIMONS-SCHRODINGER SYSTEM

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Abstract. In this paper, we consider the following equivariant defocusing Chern-Simons-Schrödinger system,

\begin{align*}
  i\partial_t \phi + \Delta \phi &= \frac{2m}{r^2} A_\theta \phi + A_0 \phi + \frac{1}{r^2} A_\theta^2 \phi - \lambda |\phi|^{p-2} \phi, \\
  \partial_r A_0 &= \frac{1}{r} (m + A_\theta) |\phi|^2, \\
  \partial_t A_\theta &= \text{rIm}(\bar{\phi} \partial_r \phi), \\
  \partial_r A_\theta &= -\frac{1}{2} |\phi|^2 r, \\
  A_r &= 0.
\end{align*}

where \( \phi(t, x_1, x_2) : \mathbb{R}^{1+2} \to \mathbb{R} \) is a complex scalar field, \( A_\mu(t, x_1, x_2) : \mathbb{R}^{1+2} \to \mathbb{R} \) is the gauge field for \( \mu = 0, 1, 2 \), \( A_r = \frac{r}{r^2} A_1 + \frac{r}{r^2} A_2 \), \( A_0 = -x_2 A_1 + x_1 A_2 \), \( \lambda < 0 \) and \( p > 4 \).

When \( p > 4 \), the system is in the mass supercritical and energy subcritical range. By using the conservation law of the system and the concentration compactness method introduced in [17], we show that the solution of the system exists globally and scatters.

1. Introduction. We investigate the following Chern-Simons gauged nonlinear Schrödinger system

\begin{align*}
  iD_0 \phi + (D_1 D_1 + D_2 D_2) \phi + \lambda |\phi|^{p-2} \phi &= 0, \\
  \partial_0 A_1 - \partial_1 A_0 &= -\text{Im}(\bar{\phi} D_2 \phi), \\
  \partial_0 A_2 - \partial_2 A_0 &= \text{Im}(\bar{\phi} D_1 \phi), \\
  \partial_1 A_2 - \partial_2 A_1 &= -\frac{1}{2} |\phi|^2,
\end{align*}

where \( \phi : \mathbb{R}^{1+2} \to \mathbb{R} \) is a complex scalar field and \( A_\mu : \mathbb{R}^{1+2} \to \mathbb{R} \) is the gauge field, \( D_\mu = \partial_\mu + iA_\mu \) is the covariant derivative for \( \mu = 0, 1, 2 \), \( \lambda \in \mathbb{R} \) and \( p > 2 \) is a constant.

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The system (1.1)-(1.4) conserves the mass and the energy
\[ M(t) := \| \phi(t, \cdot) \|_{L^2(\mathbb{R}^2)}^2 = M(0), \] 
\[ E(t) := \frac{1}{2} \sum_{j=1}^{2} \int_{\mathbb{R}^2} |D_j \phi(t, \cdot)|^2 dx - \lambda \frac{1}{p} \int_{\mathbb{R}^2} |\phi(t, \cdot)|^p dx = E(0). \] 

The system (1.1)-(1.4) is invariant under the following gauge transformation
\[ \phi \rightarrow \phi e^{i\chi}, A_\mu \rightarrow A_\mu - \partial_\mu \chi, \] 
where \( \chi : \mathbb{R}^{1+2} \rightarrow \mathbb{R} \) is an arbitrary \( C^\infty \) function. To determine the gauge field \( A_\mu \), we impose the Coulomb gauge condition
\[ \sum_{j=1}^{2} \partial_j A_j = 0. \]

This model was proposed to study vortex solutions, which carry both electric and magnetic charges. It has applications in high-temperature superconductivity, we refer the reader to [14], [15], [6], for the physical background. For the mathematical study of this model, see e.g. [2], [12], [21], [20], [11], [3], [4], [25], [16], [27], [18].

For the pure monomial nonlinearity Schrödinger equation,
\[ iu_t + \Delta u + \mu |u|^{p-1} u = 0, \] 
where \( u(t, x) : \mathbb{R}^{1+d} \rightarrow \mathbb{R} \) is a complex valued function. In the literature, when \( \mu < 0 \), (1.9) is referred to be defocusing, when \( \mu > 0 \), (1.9) is referred to be focusing. Analogous to (1.9), we call (1.1)-(1.4) to be defocusing when \( \lambda < 0 \) and focusing when \( \lambda > 0 \).

In the mass supercritical and energy subcritical case, now the scattering for (1.9) is well being understood. In [9], [23], in the defocusing case, by using the Morawetz type inequality, the authors show that the finite energy solutions of (1.9) scatter to a free Schrödinger solution. In [13], [7], in the focusing case, by using the concentration compactness method developed in [17], the authors show that scattering also occurs for the finite energy solutions of (1.9) below a mass and energy constraint threshold when \( N = 3, p = 3 \). For the arbitrary space dimensions, see [8].

Expanding out (1.1) and using the coulomb gauge condition (1.8), we have
\[ i\partial_t \phi + \Delta \phi + 2iA_j \partial_j \phi - A_0 \phi - \sum_{j=1}^{2} A_j^2 \phi + \lambda |\phi|^{p-2} \phi = 0, \] 
\[ \Delta A_0 = Im(Q_{12}(\tilde{\phi}, \phi)) + \partial_1(A_2 |\phi|^2) - \partial_2(A_1 |\phi|^2), \] 
\[ \Delta A_1 = \partial_2\left( \frac{1}{2} |\phi|^2 \right), \Delta A_2 = -\partial_1\left( \frac{1}{2} |\phi|^2 \right), \]
where \( Q_{\alpha\beta}(\bar{\phi}, \phi) = \partial_\alpha \bar{\phi} \partial_\beta \phi - \partial_\beta \bar{\phi} \partial_\alpha \phi. \)

Now we consider the equivariant case \( \phi(t, x) = e^{im\theta} u(t, |x|) \) with \( m \in \mathbb{Z} \).

Following [20], we use the polar coordinates representation of (1.10)-(1.12). Motivated by the transformations
\[ \partial_r = \frac{x_1}{|x|} \partial_1 + \frac{x_2}{|x|} \partial_2, \partial_\theta = -x_2 \partial_1 + x_1 \partial_2, \]
and
\[
\partial_1 = (\cos \theta) \partial_r - \frac{1}{r} (\sin \theta) \partial \theta, \quad \partial_2 = (\sin \theta) \partial_r + \frac{1}{r} (\cos \theta) \partial \theta.
\]

In [20], the authors introduce
\[
A_r = \frac{x_1}{|x|} A_1 + \frac{x_2}{|x|} A_2, \quad A_\theta = -x_2 A_1 + x_1 A_2,
\]
which are easily seen to satisfy
\[
A_1 = A_r \cos \theta - \frac{1}{r} A_\theta \sin \theta, \quad A_2 = A_r \sin \theta + \frac{1}{r} A_\theta \cos \theta.
\]

Using these transformations, we can express \(A_1, A_2, \partial_1, \partial_2\) in terms of \(A_r, A_\theta, \partial_r, \partial_\theta\). In particular,
\[
A_j \partial_j = A_r \partial_r + \frac{1}{r^2} A_\theta \partial_\theta, \quad \partial_j A_j = \partial_r A_r + \frac{1}{r} A_r + \frac{1}{r^2} \partial_\theta A_\theta, \quad A_1^2 + A_2^2 = A_r^2 + \frac{1}{r^2} A_\theta^2.
\]

Now, the equation (1.10) has the following representation
\[
(i \partial_t + \Delta) \phi = -2i (A_r \partial_r + \frac{1}{r^2} A_\theta \partial_\theta) \phi - i (\partial_r A_r + \frac{1}{r} A_r + \frac{1}{r^2} \partial_\theta A_\theta) \phi + A_0 \phi + A^2 \phi + \frac{1}{r^2} A_\theta^2 \phi - \lambda |\phi|^{p-2} \phi,
\]
which can be written in more compact form
\[
D_t \phi = i(D_r^2 + \frac{1}{r} D_r + \frac{1}{r^2} D_\theta^2) \phi + i \lambda |\phi|^{p-2} \phi.
\]

Write the curvature \(F = dA\) in terms of variables \(t, r, \theta\), with
\[
F_{0r} = \partial_t A_r - \partial_r A_0, \quad F_{0\theta} = \partial_t A_\theta - \partial_\theta A_0, \quad F_{r\theta} = \partial_r A_\theta - \partial_\theta A_r.
\]

Similarly, we have
\[
F_{0r} = \partial_t A_r - \partial_r A_0 = \partial_t \left( \frac{x_1}{r} A_1 + \frac{x_2}{r} A_2 \right) - \left( \frac{x_1}{r} \partial_1 A_0 + \frac{x_2}{r} \partial_2 A_0 \right)
= \frac{x_1}{r} (\partial_t A_1 - \partial_1 A_0) + \frac{x_2}{r} (\partial_t A_2 - \partial_2 A_0)
= -\frac{x_1}{r} \text{Im}(\bar{\phi} D_2 \phi) + \frac{x_2}{r} \text{Im}(\bar{\phi} D_1 \phi)
= \frac{1}{r} \text{Im}(\bar{\phi} (x_2 D_1 \phi - x_1 D_2 \phi)) = -\text{Im}(\bar{\phi} D_\theta \phi).
\]
By calculating, we have $A_r = 0$ and $A_0$ is a radial function. By the equivariance of $\phi$, the system (1.1)-(1.4) transforms further to

$$i\partial_t \phi + \Delta \phi = \frac{2m}{r^d} A_\theta \phi + A_0 \phi + \frac{1}{r^2} A_0^2 \phi - \lambda |\phi|^{p-2} \phi,$$

(1.19)

$$\partial_r A_0 = \frac{1}{r} (m + A_\theta) |\phi|^2,$$

(1.20)

$$\partial_r A_\theta = r \text{Im}(\bar{\phi} \partial_r \phi),$$

(1.21)

$$\partial_r A_\theta = \frac{1}{2} |\phi|^2 r,$$

(1.22)

$$A_r = 0.$$  

(1.23)

In [20], the authors consider the $p = 4$ case, they prove that the solution of the system scatters when $\lambda \leq 0$ in the equivariant case, and also scatters in the focusing case with the mass below the threshold when $\lambda < 1$.

When $p > 4$, $\lambda > 0$, in [3], the authors show there are standing wave solutions of the system, so scattering does not always occur in the focusing case.

The $m = 0$ case corresponds to the radial case, when $\lambda < 0$, by using the concentration compactness method, in [28], we show that the solution of the system scatters. So we will focus on $m \neq 0$ case in this paper.

Let $\mathcal{V}_m = \{f(x)|f(x) = e^{i\theta} f(|x|), f(x) \in H^1(\mathbb{R}^2)\}$.

**Theorem 1.1.** Let $\lambda < 0$, $p > 4$, $m \in \mathbb{Z}\setminus\{0\}$ and $\phi_0 \in \mathcal{V}_m$, then the system (1.19)-(1.23) is globally wellposed in $\mathcal{V}_m$ and $\phi$ scatters, i.e., there exists $\phi_+ \in \mathcal{V}_m$, such that

$$\lim_{t \to \pm\infty} \|\phi(t, x) - e^{it\Delta} \phi_+\|_{H^1(\mathbb{R}^2)} = 0.$$  

(1.24)

If the term $\lambda |\phi|^{p-2} \phi$ does not exist in (1.13), the equation is invariant under the transform $\phi(t, x) \rightarrow \mu \phi(\mu^2 t, \mu x)$ for $\mu > 0$, and $\|\mu^2 \phi(t, x)\|_{L^2} = \|\phi(t, x)\|_{L^2}$, so it corresponds to the $L^2$-critical case, and $\lambda |\phi|^{p-2} \phi$ for $p > 4$ can be seen as a perturbation of the $L^2$-critical case. This is similar to the combined nonlinearities with power exponents studied in [5],

$$i\partial_t u + \Delta u = |u|^{q-1} u,$$

(1.25)

$$u(0) = u_0 \in H^1(\mathbb{R}^d)$$

where $u : \mathbb{R} \times \mathbb{R}^d \rightarrow C$ is a complex-valued function, $1 + \frac{4}{d} < q < 1 + \frac{4}{d-2}$, $d = 3, 4$ and $1 + \frac{2}{d} < q < \infty$, $d = 1, 2$. The term $|u|^{q-1} u$ can also seen as a perturbation of the $L^2$-critical equation. In [5], the authors use the concentration compactness method to prove the solution scatters below the ground state when $d \leq 4$ in the radial case. We also refer the reader to [5] for a review of combined nonlinearities.

In section 2, we will make some preparations which will be used in section 3. In this section, we will give the well-posedness result of (1.19)-(1.23), and use the conservation laws of the system to show the global existence of the solution. In section 3, we will establish the short-time and long-time perturbation theory. The perturbation terms need to be carefully analysed, here we put them in $L^1_{t,x} + L^1_t L^2_x$, see (3.6) and (3.42). In section 4, we will argue by contradiction to prove the scattering

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Footnote 1: Here $\lambda = 1$ is the borderline of the energy positivity of the system in the $p = 4$ case, when $\lambda < 1$, the energy of the system is positive, while when $\lambda \geq 1$ the energy of the system is not necessarily positive, see Proposition 6.1 and Section 7 in [20].
part of the Theorem 1.1. If Theorem 1.1 fails, firstly, by using the concentration-compactness method and inducting on $E(\phi) + M(\phi)$, we show that there exists a nonzero critical element with some infinite Lebesgue space-time norms which is compact in $H^1(\mathbb{R}^2)$ on its trajectories in $t$, then following [20], we establish the localized viral identity to give a contradiction.

**Notation.** Throughout the article we will use the letter $C$ to denote various constants whose exact value may change from line to line. We will use the notation $X \lesssim Y$ whenever there exists some positive constant $C$ so that $X \leq CY$.

2. Preliminaries. We assume that all spatial $L^p$ spaces are based on the 2-dimensional Lebesgue measure. Define the operators $[r^{-n} \partial_r]^{-1}$ and $[r \partial_r]^{-1}$ by

$$[r^{-n} \partial_r]^{-1} f(r) = \int_0^r f(s)s^n ds, \quad [r \partial_r]^{-1} f(r) = -\int_r^\infty \frac{1}{s} f(s)ds.$$ 

Then by the one dimensional Hardy’s inequality, we have

$$||[r \partial_r]^{-1} f||_{L^p} \lesssim_p ||f||_{L^p}, 1 \leq p < \infty. \quad (2.1)$$

$$||r^{-n-1}[r^{-n} \partial_r]^{-1} f||_{L^p} \lesssim_p ||f||_{L^p}, 1 < p \leq \infty. \quad (2.2)$$

By integrating (1.22) and (1.20), we have $A_0 = -\frac{1}{2} \int_0^t |\phi|^2 ds$, $A_0 = A_0^{(1)} + A_0^{(2)}$ with $A_0^{(1)} := -\int_0^\infty \frac{m}{s} |\phi|^2 ds$, $A_0^{(2)} := -\int_\infty^0 \frac{m}{s} |\phi|^2 ds$. By using (2.1) and (2.2), the authors in [20] prove the following Lemmas (see Lemma 2.1-2.3 in [20]).

**Lemma 2.1.** Let $I$ be a time interval and let $\phi$ be a solution to (1.19)-(1.23) on $I \times \mathbb{R}^2$, then

$$||A_0||_{L^\infty} \lesssim ||\phi||_{L^2_x}^2, \quad \frac{1}{r} \cdot A_0||_{L^\infty} \lesssim ||\phi||_{L^2_x}^2, \quad (2.3)$$

$$||\frac{1}{r^2} A_0||_{L^2_x} \lesssim ||\phi||_{L^2_x}^2, 1 < p \leq \infty, \quad (2.4)$$

$$||\frac{1}{r^2} A_0^2||_{L^2_{t,x}} \lesssim ||\phi||_{L^2_{t,x}}^4, ||\frac{1}{r^2} A_0^2||_{L^4_{t,x}} \lesssim ||\phi||_{L^4_{t,x}}^2, \quad (2.5)$$

$$||A_0^{(1)}||_{L^2_{t,x}} \lesssim ||\phi||_{L^2_{t,x}}^2, ||A_0^{(1)}||_{L^2_{t,x}} \lesssim ||\phi||_{L^2_{t,x}}^2, \quad (2.6)$$

$$||A_0^{(2)}||_{L^2_x} \lesssim ||m||_{L^4_x}^2. \quad (2.7)$$

Since $\phi(t, x) = e^{im\theta}u(t, |x|)$, $|\nabla \phi|^2 = u^2 + \frac{m^2}{r^2} u^2$. In particular, when $m \neq 0$,

$$\frac{\phi}{r} \lesssim |\nabla \phi|. \quad (2.8)$$

Let $\Lambda(\phi) = \frac{2m}{r^2} A_0 \phi + A_0 \phi + \frac{1}{r^2} A_0^2 \phi - \lambda |\phi|^{p-2} \phi$. Since for $i = 1, 2$,

$$\frac{2m}{r^2} A_0 \phi_i = -4m \frac{1}{r^3} x_i A_0 \phi - m |\phi|^2 \frac{x_i}{r} \phi + \frac{2m}{r^2} A_0 \phi_i, \quad (2.9)$$

$$A_0 \phi_i = \frac{A_0}{r} |\phi|^2 \frac{x_i}{r} \phi + m |\phi|^2 \frac{x_i}{r} \phi + A_0 \phi_i, \quad (2.10)$$

$$\frac{1}{r^2} A_0^2 \phi_i = -2 \frac{1}{r^2} A_0 \phi_i - \frac{A_0}{r} |\phi|^2 \frac{x_i}{r} \phi + \frac{1}{r^2} A_0^2 \phi_i, \quad (2.11)$$

$$(-\lambda |\phi|^{p-2} \phi)_i = -\frac{p\lambda}{2} |\phi|^{p-2} \phi_i - \frac{(p-2)\lambda}{2} |\phi|^{p-4} \phi^2 \phi_i, \quad (2.12)$$

\[\hline\]
so, by (2.8), we have

\[
|\Lambda(\phi)| \leq |A_0||\phi|^2|\nabla \phi| + \frac{|A_0|}{r^2}|\nabla \phi| + \frac{A_0^2}{r^2}|\nabla \phi| + |A_0||\nabla \phi| + |\phi|^3 + |\phi|^{p-2}|\nabla \phi|. \tag{2.13}
\]

\[
\Lambda(\phi_1) - \Lambda(\phi_2) = \frac{2m}{r^2}(A_0(\phi_1)\phi_1 - A_0(\phi_2)\phi_2) + (A_0(\phi_1)\phi_1 - A_0(\phi_2)\phi_2)
\]

\[
+ \frac{1}{r^2}(A_0^2(\phi_1)\phi_1 - A_0^2(\phi_2)\phi_2) - \lambda(|\phi_1|^{p-2}\phi_1 - |\phi_2|^{p-2}\phi_2). \tag{2.14}
\]

By using (2.1) and (2.2), similar to the proof of Lemma 2.1-2.3 in [20], we also have

**Lemma 2.2.** Let I be a time interval and let $\phi_1$ and $\phi_2$ be two solutions to (1.19)-(1.23) on $I \times \mathbb{R}^2$, then

\[
\|A_0(\phi_1) - A_0(\phi_2)\|_{L^\infty_t L^x_{1,\infty}} \lesssim \|\phi_1 - \phi_2\|_{L^2_x}(\|\phi_1\|_{L^4_x} + \|\phi_2\|_{L^4_x}), \tag{2.15}
\]

\[
\|A_0(\phi_1) - A_0(\phi_2)\|_{L^\infty_t L^x_{1,1}} \lesssim \|\phi_1 - \phi_2\|_{L^2_x}(\|\phi_1\|_{L^4_x} + \|\phi_2\|_{L^4_x}), \tag{2.16}
\]

\[
\|A_0(\phi_1) - A_0(\phi_2)\|_{L^\infty_t L^x_{1,4}} \lesssim \|\phi_1 - \phi_2\|_{L^2_x}(\|\phi_1\|_{L^8_x} + \|\phi_2\|_{L^8_x}), \tag{2.17}
\]

\[
\leq ((\|\phi_1\|_{L^4_{t,x}}^3 + \|\phi_2\|_{L^4_{t,x}}^3)\|\phi_1 - \phi_2\|_{L^4_{t,x}}, \tag{2.18}
\]

\[
\|A_0(\phi_1) - A_0(\phi_2)\|_{L^\infty_t L^x_{1,1}} \lesssim \|\phi_1 - \phi_2\|_{L^2_x}(\|\phi_1\|_{L^4_x} + \|\phi_2\|_{L^4_x}), \tag{2.19}
\]

\[
\|A_0(\phi_1) - A_0(\phi_2)\|_{L^\infty_t L^x_{1,1}} \lesssim ((\|\phi_1\|_{L^4_{t,x}}^3 + \|\phi_2\|_{L^4_{t,x}}^3)\|\phi_1 - \phi_2\|_{L^4_{t,x}}), \tag{2.20}
\]

\[
\|A_0(\phi_1) - A_0(\phi_2)\|_{L^\infty_t L^x_{1,1}} \lesssim ((\|\phi_1\|_{L^4_{t,x}}^3 + \|\phi_2\|_{L^4_{t,x}}^3)\|\phi_1 - \phi_2\|_{L^4_{t,x}}), \tag{2.21}
\]

\[
\|A_0(\phi_1) - A_0(\phi_2)\|_{L^\infty_t L^x_{1,1}} \lesssim ((\|\phi_1\|_{L^4_{t,x}}^3 + \|\phi_2\|_{L^4_{t,x}}^3)\|\phi_1 - \phi_2\|_{L^4_{t,x}}), \tag{2.22}
\]

By Lemma 2.2, we have

\[
\left|\frac{\Lambda(\phi_1) - \Lambda(\phi_2)}{r^2}\right|_{L^\infty_{t,x}} \lesssim \left|\bigg(\|\phi_1\|_{L^4_{t,x}}^3 + \|\phi_2\|_{L^4_{t,x}}^3\bigg)\|\phi_1 - \phi_2\|_{L^4_{t,x}}\right|
\]
Theorem 2.4. Proof. Let $t \in I$ and $u(t_0) = u_0$. Call a pair $(q, r)$ of exponents admissible if $2 \leq q, r \leq \infty, \frac{1}{q} + \frac{1}{r} = \frac{1}{2}$ and $(q, r) \neq (2, \infty)$. Let $(q, r)$ and $(\tilde{q}, \tilde{r})$ be admissible pairs of exponents. Then

$$
\|u\|_{L^{\infty}_{t \in I} L^{2}_{x \in \mathbb{R}^{2}}} + \|u\|_{L^{\tilde{q}}_{t \in I} L^{\tilde{r}}_{x \in \mathbb{R}^{2}}} \leq C\|u_0\|_{L^{2}_{x \in \mathbb{R}^{2}}} + \|f\|_{L^{q}_{t \in I} L^{r}_{x \in \mathbb{R}^{2}}},
$$

where the prime denotes the dual exponent, i.e., $\frac{1}{q'} = 1 - \frac{1}{q}$.

For $\phi$ defined on an interval $I$, we define

$$
\|\phi\|_{S_{I}} := \|\phi\|_{L^{\infty}_{t \in I} L^{2}_{x}} + \|\phi\|_{L^{\tilde{q}}_{t \in I} L^{\tilde{r}}_{x}},
$$

$$
\|\phi\|_{S_{I}'} := \|\nabla \phi\|_{L^{\infty}_{t \in I} L^{2}_{x}} + \|\nabla \phi\|_{L^{\tilde{q}}_{t \in I} L^{\tilde{r}}_{x}},
$$

$$
\|\phi\|_{S_{I}''} := \|\phi\|_{S_{I}} + \|\phi\|_{S_{I}'}.
$$

Lemma 2.3. (Strichartz estimates). Let $i\partial_{t}u + \Delta u = f$ on a time interval $I$ with $t_0 \in I$ and $u(t_0) = u_0$. Call a pair $(q, r)$ of exponents admissible if $2 \leq q, r \leq \infty, \frac{1}{q} + \frac{1}{r} = \frac{1}{2}$ and $(q, r) \neq (2, \infty)$. Let $(q, r)$ and $(\tilde{q}, \tilde{r})$ be admissible pairs of exponents. Then

$$
\|u\|_{L^{\infty}_{t \in I} L^{2}_{x \in \mathbb{R}^{2}}} + \|u\|_{L^{\tilde{q}}_{t \in I} L^{r}_{x \in \mathbb{R}^{2}}} \leq C\|u_0\|_{L^{2}_{x \in \mathbb{R}^{2}}} + \|f\|_{L^{q}_{t \in I} L^{r}_{x \in \mathbb{R}^{2}}},
$$

where the prime denotes the dual exponent, i.e., $\frac{1}{q'} = 1 - \frac{1}{q}$.

Theorem 2.4. (Local existence). Let $\phi_{0} \in H^{1}(\mathbb{R}^{2})$, then there exists a unique solution $\phi \in C([-T, T], H^{1}(\mathbb{R}^{2}))$ to (1.19)-(1.23) such that $\phi(0) = \phi_{0}$, where $|T|$ depends on $\|\phi_{0}\|_{H^{1}(\mathbb{R}^{2})}$.

Proof. We assume $|T| \leq 1$. We define the space $Y(I)$ and the map $T$ by

$$
Y(I) := \{ \phi \in C(I, H^{1}(\mathbb{R}^{2})) | \|\phi\|_{S_{I}(I)} \leq 2C\|\phi_{0}\|_{H^{1}} \},
$$

$$
T(\phi) := e^{i\Delta} \phi_{0} - i \int_{0}^{t} e^{i(t-s)\Delta} \left( \frac{2m}{r^2} A_{\theta} \phi + A_{0} \phi + \frac{1}{r^{2}} A_{\theta}^{2} \phi - \lambda |\phi|^{p-2} \phi \right)(s) ds.
$$

We can verify that $Y(I)$ becomes a metric space with the metric $\rho(\phi_{1}, \phi_{2}) := \|\phi_{1} - \phi_{2}\|_{S_{I}}$.

By using Lemma 2.2, we have

$$
\|T(\phi)\|_{S_{I}} \leq C\|\phi\|_{L^{2}} + C\frac{2m}{r^{2}} A_{\theta} \|\phi\|_{L^{\frac{4}{5}}_{t \in I} L^{\frac{5}{4}}_{x}} + C\|A_{0} \phi\|_{L^{\frac{4}{5}}_{t \in I} L^{\frac{5}{4}}_{x}} + \|\frac{1}{r^{2}} A_{\theta}^{2} \phi\|_{L^{\frac{4}{5}}_{t \in I} L^{\frac{5}{4}}_{x}}
$$

$$
+ C\|\phi\|_{L^{2}} + \|A_{0}\|_{L^{2}} \|\phi\|_{L^{\frac{4}{5}}_{t \in I} L^{\frac{5}{4}}_{x}} + \|A_{0}\|_{L^{2}} \|\phi\|_{L^{\frac{4}{5}}_{t \in I} L^{\frac{5}{4}}_{x}} + \|\frac{1}{r^{2}} A_{\theta}^{2} \phi\|_{L^{\frac{4}{5}}_{t \in I} L^{\frac{5}{4}}_{x}}
$$

$$
+ C\|\phi\|_{L^{2}} + C\|\phi\|_{L^{\frac{4}{5}}_{t \in I} L^{\frac{5}{4}}_{x}} + C\|\phi\|_{L^{\frac{4}{5}}_{t \in I} L^{\frac{5}{4}}_{x}} + \|\phi\|_{L^{\frac{4}{5}}_{t \in I} L^{\frac{5}{4}}_{x}}
$$

$$
+ C\|\phi\|_{L^{2}} + C\|\phi\|_{L^{\frac{4}{5}}_{t \in I} L^{\frac{5}{4}}_{x}} + C\|\phi\|_{L^{\frac{4}{5}}_{t \in I} L^{\frac{5}{4}}_{x}} + \|\phi\|_{L^{\frac{4}{5}}_{t \in I} L^{\frac{5}{4}}_{x}}
$$

$$
+ C\|\phi\|_{L^{2}} + C\|\phi\|_{L^{\frac{4}{5}}_{t \in I} L^{\frac{5}{4}}_{x}} + C\|\phi\|_{L^{\frac{4}{5}}_{t \in I} L^{\frac{5}{4}}_{x}} + \|\phi\|_{L^{\frac{4}{5}}_{t \in I} L^{\frac{5}{4}}_{x}}
$$

$$
+ C\|\phi\|_{L^{2}} + C|I|^{\frac{3}{2}} \|\phi\|_{S_{I}} + C|I|^{\frac{3}{2}} \|\phi\|_{S_{I}} + C|I|^{\frac{3}{2}} \|\phi\|_{S_{I}}.
$$

(2.25)
Similarly, Theorem 2.5. 

By combining (2.25) and (2.26), we have

\[
\|\nabla T(\phi)\|_{S_I} \leq C\|\nabla \phi_0\|_{L^2} + C\|A_0\|_{L^\infty} \|\phi\|^2_{L^\infty} \|\nabla \phi\|_{L^4_{t, I} L^4} + C\|A_0\|_{L^\infty} \|\nabla \phi\|_{L^4_{t, I} L^4} + C \|\phi^3\|_{L^4_{t, I} L^4} + C \|\phi^4\|_{L^4_{t, I} L^4} + C \|\phi^4\|_{L^4_{t, I} L^4} + C \|\phi^4\|_{L^4_{t, I} L^4} + C \|\phi^4\|_{L^4_{t, I} L^4} \leq C\|\nabla \phi_0\|_{L^2} + C\|\phi^3\|_{L^\infty} \|\nabla \phi\|_{L^4_{t, I} L^4} + C [I^{3/2}] \|\phi^3\|_{S_I} + C [I^{3/2}] \|\phi^3\|_{S_I} + C [I^{3/2}] \|\phi^3\|_{S_I}.
\]  

(2.26)

By combining (2.25) and (2.26), we have

\[
\|T(\phi)\|_{S_I} \leq C\|\phi_0\|_{H^1} + C [I^{3/2}] \|\phi^3\|_{S_I} + C [I^{3/2}] \|\phi^3\|_{S_I} + C [I^{3/2}] \|\phi^3\|_{S_I} + C [I^{3/2}] \|\phi^3\|_{S_I}.
\]  

(27.27)

Similarly,

\[
\|T(\phi_1) - T(\phi_2)\|_{S_I} \leq C [I^{3/2}] \|\phi_1\|_{S_I} + \|\phi_2\|_{S_I} \|\phi_1 - \phi_2\|_{S_I} + C [I^{3/2}] \|\phi_1\|_{S_I} + \|\phi_2\|_{S_I} \|\phi_1 - \phi_2\|_{S_I} + C [I^{3/2}] \|\phi_1\|_{S_I} + \|\phi_2\|_{S_I} \|\phi_1 - \phi_2\|_{S_I}.
\]  

(28)

So, when \(|I| \leq \delta \min\{\|\phi_0\|_{H^{4-8}}^4, \|\phi_0\|_{H^{4-2p}}^4, 1\}\), where \(\delta > 0\) is a sufficiently small constant, we have \(\|T(\phi)\|_{S_I} \leq 2C\|\phi_0\|_{H^1}\), and \(|T(\phi_1) - T(\phi_2)|_{S_I} \leq \frac{\delta}{2} \|\phi_1 - \phi_2\|_{S_I}\). We have shown that \(T\) is contraction in \((Y(I), \rho)\), so we get the solution \(\phi\) on \(I\) by the contraction mapping principle.

**Theorem 2.5.** (Global existence). The solution \(\phi\) in Theorem 2.4 can be extended globally, i.e., \(\phi \in C((-\infty, +\infty), H^1(\mathbb{R}^2))\). The solution \(\phi(t)\) conserves mass and energy, for \(t \in \mathbb{R}\), we have

\[
M(\phi(t)) = \int_{\mathbb{R}^2} |\phi(t)|^2 dx = M(\phi_0),
\]
E(ϕ(t)) = \frac{1}{2} \sum_{j=1}^{2} \int_{\mathbb{R}^2} |D_j \phi(t, \cdot)|^2 dx - \lambda \frac{1}{p} \int_{\mathbb{R}^2} |\phi(t, \cdot)|^p dx = E(\phi_0).

Proof. Mass and energy conservation is standard.

We show the global existence. By Theorem 2.3, it suffices to show the boundedness of \(\|\phi(t)\|_{H^1}\) on its existence time interval. By (1.12), we have From the energy conservation, we have

\[
\sum_{j=1}^{2} \|A_j\|_{L^4} \lesssim \|\nabla|^{-1}(|\phi|^2)\|_{L^4} \lesssim \|\phi^2\|_{L^4} \lesssim \|\phi\|_{L^2}\|\phi\|_{L^4}. \tag{2.29}
\]

\[
\|\nabla \phi(t)\|_{L^2} \leq \sum_{j=1}^{2} \|D_j \phi(t)\|_{L^2} + \sum_{j=1}^{2} \|A_j \phi\|_{L^2} \leq E(t)^{\frac{1}{2}} + \sum_{j=1}^{2} \|A_j\|_{L^4}\|\phi\|_{L^4} \lesssim E(0)^{\frac{1}{2}} + \|\phi(t)\|_{L^2}^{1+2}\|\phi(t)\|_{L^p}^{2-2} \lesssim E(0)^{\frac{1}{2}} + \|\phi_0\|_{L^2}^{1+2}\|\phi_0\|_{L^p}^{2-2}, \tag{2.30}
\]

where \(\frac{1}{2} = \frac{6}{2} + \frac{6}{p}\). So \(\|\nabla \phi(t)\|_{L^2}\) is bounded on its existence time interval, which combines the conservation of mass, shows \(\|\phi(t)\|_{H^1}\) is bounded on its existence time interval, and prove the theorem.\]

Now, we give a sufficient condition to guarantee the scattering of the global solution \(\phi(t)\).

**Theorem 2.6.** If the solution \(\phi(t)\) satisfies

\[
\|\phi\|_{L^4_{t,x} \cap L^{2(p-2)}_{t,x}((\infty, \infty) \times \mathbb{R}^2)} < \infty, \tag{2.31}
\]

then there exists \(\phi_{\pm} \in H^1(\mathbb{R}^2)\), such that

\[
\lim_{t \to \pm \infty} \|\phi(t) - e^{it\Delta} \phi_{\pm}\|_{H^1} = 0. \tag{2.32}
\]

Proof. Since \(\phi(t)\) satisfies (2.31), we can divide \([0, +\infty)\) into the union of disjoint intervals \(I_j = [t_j, t_j+1)\), for \(0 \leq j \leq k\), \([0, +\infty) = \bigcup_{j=0}^{k} [t_j, t_{j+1})\), such that, for \(0 \leq j \leq k\), \(\|\phi\|_{L^4_{t,x} \cap L^{2(p-2)}_{t,x}((t_j, t_{j+1}) \times \mathbb{R}^2)} \leq \delta\).

Now in \(I_j\), \(0 \leq j \leq k\),

\[
\|\phi\|_{S^1(I_j)} \leq \|\phi(t_j)\|_{H^1} + \int_{t_j}^{t_{j+1}} e^{i(t-s)\Delta} \frac{2m}{r^2} A_\theta \phi + A_0 \phi + A_2 \frac{1}{r^2} \phi - \lambda |\phi|^{p=2}\phi(s) ds \|S^1(I_j)\).
\]

\[
\leq \|\phi(t_j)\|_{H^1} + C \|\phi\|_{L^4_{t \in I_j} L^4_{x}} + C \|\phi\|_{L^4_{t \in I_j} L^4_{x}} + C \|\phi\|_{L^4_{t \in I_j} L^4_{x}}
\]

\[
+ C \|\phi\|_{L^4_{t \in I_j} L^4_{x}} \|\nabla \phi\|_{L^4_{t \in I_j} L^4_{x}} + C \|\phi\|_{L^4_{t \in I_j} L^4_{x}} \|\nabla \phi\|_{L^4_{t \in I_j} L^4_{x}}
\]

\[
\leq \|\phi(t_j)\|_{H^1} + C \|\phi\|_{S^1(I_j)}, \tag{2.33}
\]

we have

\[
\|\phi\|_{S^1(I_j)} \leq 2 \|\phi(t_j)\|_{H^1} \leq C. \tag{2.34}
\]
By combining (2.34) for $0 \leq j \leq k$, we have
\[
\|\phi\|_{S^1([0, +\infty))} < +\infty. \tag{2.35}
\]

Now, we define
\[
\phi_+ = \phi_0 + \int_0^{+\infty} e^{-i s \Delta} \left( \frac{2m}{r^2} A_0 \phi + A_0 \phi + \frac{A_0^2}{r^2} \phi - \lambda |\phi|^{p-2} \phi \right)(s) ds.
\tag{2.36}
\]
Since
\[
\|\phi(t) - e^{it \Delta} \phi_+\|_{H^1}
\leq \int_t^{+\infty} e^{-i s \Delta} \left( \frac{2m}{r^2} A_0 \phi + A_0 \phi + \frac{A_0^2}{r^2} \phi - \lambda |\phi|^{p-2} \phi \right)(s) ds.
\]

By using the conservation laws of the system, we have
\[
\|\phi\|_{S^1(\mathbb{R})} \lesssim \|\langle \nabla \rangle e^{it \Delta} \phi_0\|_{L^4_{t,x}}, \quad \text{and there exists } \phi_\pm \in H^1(\mathbb{R}^2), \text{ such that }
\lim_{t \to \pm \infty} \|\phi(t) - e^{it \Delta} \phi_\pm\|_{H^1} = 0. \tag{2.37}
\]

The construction of $\phi_-$ is similar. \hfill \square

**Theorem 2.7.** *(Small data Scattering).* If $\|\langle \nabla \rangle e^{it \Delta} \phi_0\|_{L^4_{t,x}}$ is sufficiently small, then $\|\phi\|_{S^1(\mathbb{R})} \lesssim \|\langle \nabla \rangle e^{it \Delta} \phi_0\|_{L^4_{t,x}}$, and there exists $\phi_\pm \in H^1(\mathbb{R}^2)$, such that
\[
\lim_{t \to \pm \infty} \|\phi(t) - e^{it \Delta} \phi_\pm\|_{H^1} = 0. \tag{2.39}
\]

**Proof.** By using the conservation laws of the system, $\|\nabla \phi(t)\|_{L^4_{t,x}} \leq C$.

By using Lemma 2.3, we have
\[
\|\phi\|_{S^1(\mathbb{R})} \leq C \|\langle \nabla \rangle e^{it \Delta} \phi_0\|_{L^4_{t,x}} + C \|\phi\|_{S^1(\mathbb{R})}^3 + C \|\phi\|_{L^4_{t,x}}^{p-2} \|\phi\|_{S^1(\mathbb{R})}. \tag{2.40}
\]

While
\[
\|\phi\|_{L^4_{t,x}}^{p-2} \leq \|\langle \nabla \rangle \phi\|_{L^4_{t,x}}^{2(p-2)} \|\langle \nabla \rangle \phi\|_{L^4_{t,x}}^{1-\frac{p-2}{2}}, \tag{2.41}
\]
so
\[
\|\phi\|_{S^1(\mathbb{R})} \leq C \|\langle \nabla \rangle e^{it \Delta} \phi_0\|_{L^4_{t,x}} + C \|\phi\|_{S^1(\mathbb{R})}^3. \tag{2.42}
\]

By continuity of $S^1$ norm,
\[
\|\phi\|_{S^1(\mathbb{R})} \lesssim \|\langle \nabla \rangle e^{it \Delta} \phi_0\|_{L^4_{t,x}}. \tag{2.43}
\]

Similarly as Theorem 2.5, we can define $\phi_\pm$, and prove (2.39). \hfill \square

**Remark 2.8.** If $\|\phi_0\|_{H^1}$ is sufficiently small, then $\|\langle \nabla \rangle e^{it \Delta} \phi_0\|_{L^4_{t,x}} \lesssim \|\phi_0\|_{H^1}$ is also sufficiently small, and we can use Theorem 2.7 to deduce that $\|\phi\|_{S^1(\mathbb{R})} \lesssim \|\langle \nabla \rangle e^{it \Delta} \phi_0\|_{L^4_{t,x}} \lesssim \|\phi_0\|_{H^1}$ and the solution scatters.
Theorem 2.9. (Existence of wave operator) Suppose $\psi_\pm \in H^1$, then there exists $\phi_{0, \pm} \in H^1$ such that the corresponding solution $\phi_\pm$ solving (1.19)-(1.23) with initial data $\phi_{0, \pm}$ is global, and satisfies
\[
\lim_{t \to \pm \infty} \|\phi_\pm(t) - e^{i\Delta t} \psi_\pm\|_{H^1} = 0. \tag{2.44}
\]

The proof of Theorem 2.9 is standard, see e.g., [13], we omit the details here.

3. Perturbation theory.

Theorem 3.1. (Short-time perturbation). Let $I$ be a time interval and let $w$ be an approximate solution to (1.19)-(1.23) on $I \times \mathbb{R}^2$,
\[
i \partial_t w + \Delta w - \frac{2m}{r^2} A_\theta(w)w - A_\theta(w)w - \frac{1}{r^2} A_\theta^2(w)w + \lambda |w|^{p-2} w = e, \tag{3.1}
\]
for some function $e$. Assume $e = g + h$, $\sum_{1=1}^2 \partial_i e = g_1 + h_1$, for some function $g$, $h$, $g_1$, and $h_1$.

Suppose we also have the energy bound
\[
\|w\|_{L^\infty_{t, x} L^1_x} \leq A_1 \tag{3.2}
\]
for some constant $A_1 > 0$.

Let $t_0 \in I$ and let $u(t_0) \in H^1$ be close to $w(t_0)$,
\[
\|u(t_0) - w(t_0)\|_{H^1} \leq A_2 \tag{3.3}
\]
for some $A_2 > 0$.

Moreover, assume the smallness conditions
\[
\|\langle \nabla \rangle w\|_{L^4_{t \in I, x \in \mathbb{R}^2}} + \|w\|_{L^{2(p-2)}_{t \in I, x \in \mathbb{R}^2}} \leq \delta_0, \tag{3.4}
\]
\[
\|\langle \nabla \rangle e^{i(t-t_0)} \Delta (u(t_0) - w(t_0))\|_{L^4_{t \in I, x \in \mathbb{R}^2}} \leq \delta, \tag{3.5}
\]
\[
\|g\|_{L^\frac{4}{3}_{t \in I, x \in \mathbb{R}^2}} + \|g_1\|_{L^\frac{4}{3}_{t \in I, x \in \mathbb{R}^2}} \leq \delta, \tag{3.6}
\]
\[
\|h\|_{L^1_{t \in I, x \in \mathbb{R}^2}} + \|h_1\|_{L^1_{t \in I, x \in \mathbb{R}^2}} \leq \delta.
\]

for some $0 < \delta \leq \delta_0$, where $\delta_0 = \delta_0(A_1, A_2) > 0$ is a small constant.

Then, there exists a solution $u$ to (1.19)-(1.23) on $I \times \mathbb{R}^2$ with the initial data $u(t_0)$ at time $t = t_0$ that satisfies
\[
\|u - w\|_{L^4_{t \in I, x \in \mathbb{R}^2} \cap L^{2(p-2)}_{t \in I, x \in \mathbb{R}^2}} \lesssim \delta^\alpha, \tag{3.7}
\]
where $0 < \alpha < \frac{2}{p-1}$.

Also, there exist $g_2$, $h_2$ such that $(i \partial_t + \Delta)(u - w) + e + \sum_{k=1}^2 \partial_x ((i \partial_t + \Delta)(u - w) + e) = g_2 + h_2$, and
\[
\|g_2\|_{L^\frac{4}{3}_{t \in I, x \in \mathbb{R}^2}} + \|h_2\|_{L^1_{t \in I, x \in \mathbb{R}^2}} \lesssim \delta^\alpha. \tag{3.8}
\]

Proof. By the wellposedness theory, the solution $u$ exists on $I$. By time symmetry, we may assume $t_0 = \inf I$.

Let $v = u - w$, then $v$ satisfies
\[
i \partial_t v + \Delta v = \frac{2m}{r^2} A_\theta(v + w)(v + w) - \frac{2m}{r^2} A_\theta(w)w + A_\theta(w)v + A_\theta(v)(v + w) - A_\theta(w)w.
\]
\[ + \frac{1}{r^2} A_\phi^2(v + w)(v + w) - \frac{1}{r^2} A_\phi^2(w)w - \lambda |v + w|^{p-2}(v + w) + \lambda |w|^{p-2}w \]
\[ - \epsilon, \quad (3.9) \]

with \( u(t_0) = u(t_0) - w(t_0) \).

For the term \( \frac{2m}{r^2} A_\phi(v + w)(v + w) - \frac{2m}{r^2} A_\phi(w)w \), we have
\[ = \frac{2m}{r^2} A_\phi(v + w)v + \frac{2m}{r^2} (A_\phi(v + w) - A_\phi(w))w, \quad (3.10) \]

\[ \lesssim \left| \frac{A_\phi(v + w)}{r^2} \right| |\nabla v| + \left| \frac{A_\phi(v + w) - A_\phi(w)}{r^2} \right| |\nabla w| + |v|^2|\nabla v| + |w|^2|\nabla w|. \quad (3.11) \]

For the term \( A_\phi(v + w)(v + w) - A_\phi(w)w \), we have
\[ = A_\phi(v + w)(v + w) - A_\phi(w)w \]
\[ = A_\phi(v + w)v + (A_\phi(v + w) - A_\phi(w))w, \quad (3.12) \]

\[ \lesssim \left| \frac{A_\phi(v + w)}{r} \right| |v + w|^2|v| + \left| \frac{A_\phi(v + w) - A_\phi(w)}{r} \right| |v + w|^2|w| \]
\[ + \left| \frac{A_\phi(w)}{r} \right| (|v|^2 + |v||w|)|w| + (|v|^2 + |w|^2)|\nabla v| \]
\[ + |A_\phi(v + w)||\nabla v| + |A_\phi(v + w) - A_\phi(w)||\nabla w|. \quad (3.13) \]

For the term \( \frac{1}{r^2} A_\phi^2(v + w)(v + w) - \frac{1}{r^2} A_\phi^2(w)w \), we have
\[ = \frac{1}{r^2} A_\phi^2(v + w)(v + w) - \frac{1}{r^2} A_\phi^2(w)w \]
\[ = \frac{1}{r^2} A_\phi^2(v + w)v + \frac{A_\phi^2(v + w) - A_\phi^2(w)}{r^2}w, \quad (3.14) \]

\[ \lesssim \left| \frac{A_\phi^2(v + w)}{r^2} \right| |\nabla v| + \left| \frac{A_\phi^2(v + w) - A_\phi^2(w)}{r^2} \right| |\nabla w| \]
\[ + \left| \frac{A_\phi(v + w)}{r} \right| (|w|^2|v| + |v|^2|w| + |v|^3) + \left| \frac{A_\phi(v + w) - A_\phi(w)}{r} \right| |w|^3. \quad (3.15) \]

For the term \( -\lambda |w + v|^p - 2(w + v) + \lambda |w|^{p-2}w \), we have
\[ \lesssim (|w|^{p-2} + |v|^{p-2})v, \quad (3.16) \]

\[ \lesssim |v + w|^p |\nabla v| + (|v|^{p-3} + |w|^{p-3})|v||\nabla w|. \quad (3.17) \]
By using the Strichartz estimate for (3.8), we have

\[
\|e^{i(t-t_0)\Delta}v(t_0)\|_{L^{2(p-2)}_{t,x}} \lesssim \left\| e^{i(t-t_0)\Delta}v(t_0) \right\|_{L^{2(p-2)}_{t,x}} + \left\| e^{i(t-t_0)\Delta}v(t_0) \right\|_{L^{2(p-2)}_{t,x}} ^{1/2} + \left\| e^{i(t-t_0)\Delta}v(t_0) \right\|_{L^{2(p-2)}_{t,x}} ^{1/2} + \left\| e^{i(t-t_0)\Delta}v(t_0) \right\|_{L^{2(p-2)}_{t,x}} ^{1/2} + \left\| e^{i(t-t_0)\Delta}v(t_0) \right\|_{L^{2(p-2)}_{t,x}} ^{1/2} + \left\| e^{i(t-t_0)\Delta}v(t_0) \right\|_{L^{2(p-2)}_{t,x}} ^{1/2}
\]

(3.18)
\[ \| A_0(v + w)(\nabla)v \|_{L^4_{t,x}} \]
\[ \lesssim \| A_0^{(1)}(v + w)\|_{L^4_{t,x}}^2 \| (\nabla)v \|_{L^4_{t,x}}^2 + \| A_0^{(2)}(v + w)\|_{L^4_{t,x}} \| (\nabla)v \|_{L^4_{t,x}} \]
\[ \lesssim \| v + w \|_{L^\infty_t L^2_x}^2 \| (\nabla)v \|_{L^4_{t,x}}^2 + \| v + w \|_{L^4_{t,x}}^2 \| (\nabla)v \|_{L^4_{t,x}}^2, \]
(3.23)
\[ \| (A_0^{(1)}(v + w) - A_0^{(1)}(w)) (\nabla)v \|_{L^4_{t,x}} \]
\[ \lesssim \| (A_0^{(1)}(v + w) - A_0^{(1)}(w))\|_{L^1_t L^\infty_x} \| (\nabla)v \|_{L^\infty_t L^2_x} \]
\[ \lesssim (\| v + w \|_{L^4_{t,x}}^2 + \| w \|_{L^4_{t,x}}^2) \| (\nabla)v \|_{L^4_{t,x}} \| (\nabla)w \|_{L^4_{t,x}}, \]
(3.23)
\[ \| A_0(v + w) \|_{L^\infty_t L^2_x}^2 + \| A_0(w) \|_{L^\infty_t L^2_x}^2 \| (\nabla)v \|_{L^4_{t,x}} \]
\[ \lesssim \| A_0(v + w) \|_{L^\infty_t L^2_x}^2 + \| A_0(w) \|_{L^\infty_t L^2_x}^2 \| (\nabla)v \|_{L^4_{t,x}}^2 \]
(3.23)
\[ \| \frac{A_0(v + w)}{r} \|_{L^4_{t,x}} \]
\[ \lesssim (\| A_0(v + w) \|_{L^\infty_t L^2_x}^2 + \| A_0(w) \|_{L^\infty_t L^2_x}^2) \| (\nabla)v \|_{L^4_{t,x}} \]
(3.23)
\[ \| \frac{A_0(v + w)}{r} \|_{L^4_{t,x}} \]
\[ \lesssim (\| A_0(v + w) \|_{L^\infty_t L^2_x}^2 + \| A_0(w) \|_{L^\infty_t L^2_x}^2) \| (\nabla)v \|_{L^4_{t,x}}^2 \]
(3.23)
\[ \| \frac{A_0(v + w)}{r^2} \|_{L^4_{t,x}} \]
\[ \lesssim (\| A_0(v + w) \|_{L^\infty_t L^2_x}^2 + \| A_0(w) \|_{L^\infty_t L^2_x}^2) \| (\nabla)v \|_{L^4_{t,x}}^2 \]
(3.23)
By combining (3.18)-(3.32), we have so, we have (3.8), and prove the Theorem.

(Long-time perturbation). Let Theorem 3.2.

approximate solution to (1.19)-(1.23) on

for some constant \( B > 0 \).

Let \( t_0 \in I \) and let \( u(t_0) \in H^1 \) be close to \( w(t_0) \),

for some constant \( A_2 > 0 \).
Assume also the smallness conditions
\begin{equation}
\|(\nabla) e^{i(t-t_0)\Delta} (u(t_0) - w(t_0))\|_{L^4_{t \in I, x \in \mathbb{R}^2}} \leq \delta, \tag{3.40}
\end{equation}
\begin{equation}
\|g\|_{L^4_{t \in I, x \in \mathbb{R}^2}} + \|g_1\|_{L^4_{t \in I, x \in \mathbb{R}^2}} \leq \delta, \|h\|_{L^4_{t \in I, x \in \mathbb{R}^2}} + \|h_1\|_{L^4_{t \in I, x \in \mathbb{R}^2}} \leq \delta, \tag{3.41}
\end{equation}
for some $0 < \delta \leq \delta_0$, where $\delta_0 = \delta_0(A_1, A_2) > 0$ is a small constant.

Then, there exists a solution $u$ to (1.19)-(1.23) on $I \times \mathbb{R}^2$ with the initial data
\begin{equation}
u(t_0)
\end{equation}
at time $t = t_0$ which satisfies
\begin{equation}
\|u - w\|_{L^4_{t \in I, x \in \mathbb{R}^2} \cap L^{2(p-2)}_{t \in I, x \in \mathbb{R}^2}} \lesssim \delta^\alpha, \tag{3.42}
\end{equation}
where $0 < \alpha < \frac{2}{p-1}$.

By (3.39), we can divide $I$ into into finite subintervals $I_k$, such that
\begin{equation}
\|w\|_{L^4_{t \in I_k, x \in \mathbb{R}^2}} + \|w\|_{L^{2(p-2)}_{t \in I_k, x \in \mathbb{R}^2}} \leq \delta_0, \tag{3.43}
\end{equation}
then apply Theorem 3.1 on $I_k$ sucessively to deduce the Theorem, see e.g., [13]. Since the procedure is standard, we omit the details.

4. Scattering. In this section, we will argue by contradiction to prove the scattering part of Theorem 1.1. If the scattering fails for some solution of (1.19)-(1.23), we will use the concentration compactness method, which is introduced in [17], to show there is a nontrivial critical element of the system, see Theorem 4.4. Then in section 4.3, following the computation in [20], we establish the localized virial identity of the system. Finally in section 4.4, we show the nontrivial critical element cannot exist. So the assumption that the scattering fails for some solution of (1.19)-(1.23) cannot be true, and prove the scattering part of Theorem 1.1.

Now we use the linear profile decomposition Theorem 4.2 in [22], and adapt it to the equivariant case. This type profile decomposition originates in [1] for the wave equation, for the Schrödinger equation it appeared e.g. in [22], [26]. It captures the failure of compactness of the sequence of bounded solutions to the free equation in terms of the non-compact symmetries of the equation and the superposition of profiles.

4.1. Linear profile decomposition.

**Theorem 4.1.** Let $\{\phi_n\}$ be a bounded radial sequence in $\mathcal{V}_m$, then up to passing to a subsequence of $\{\phi_n\}$, there exists a sequence $\varphi^j \in \mathcal{V}_m$ and $(\theta^j_n, t^j_n) \subset \mathbb{R}/2\pi \mathbb{Z} \times \mathbb{R}$, with
\begin{equation}
|t^j_n - t^m_n| \to \infty, \forall j \neq m, \text{ as } n \to \infty, \tag{4.1}
\end{equation}
such that for any $k \in \mathbb{N}$, there exists $w^k \in \mathcal{V}_m$,
\begin{equation}
\phi_n = \sum_{j=1}^{k} e^{i\theta^j_n} e^{-it^j_n\Delta} \varphi^j (x) + w^k_n. \tag{4.2}
\end{equation}
The remainder $w^k_n$ satisfies
\begin{equation}
\lim_{k \to \infty} \limsup_{n \to \infty} \|(\nabla) e^{it\Delta} w^k_n\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^2)} = 0, \tag{4.3}
\end{equation}
where $(q, r)$ is $L^2$-admissible and $2 < q < \infty$. 

Moreover, we have the following decoupling properties: \( \forall k \in \mathbb{N} \),

\[
\|\nabla^s \phi_n \|^2_{L^2} - \sum_{j=1}^{k} \|\nabla^s e^{-it_n^j \Delta} \varphi^j \|^2_{L^2} - \|\nabla^s w_n^k \|^2_{L^2} \to 0, \quad s = 0, 1, \tag{4.4}
\]

\[
E(\phi_n) - \sum_{j=1}^{k} E(e^{-it_n^j \Delta} \varphi^j) - E(w_n^k) \to 0, \tag{4.5}
\]

\[
M(\phi_n) - \sum_{j=1}^{k} M(e^{-it_n^j \Delta} \varphi^j) - M(w_n^k) \to 0, \quad \text{as} \quad n \to \infty. \tag{4.6}
\]

4.2. **Extraction of a critical element.** Now we aim to prove the scattering part of Theorem 1.1. By Theorem 2.6, it suffices to show that any solution \( \phi \) to (1.19)-(1.23) satisfies

\[
\|\phi\|_{L^4_{t,x} \cap L^{2(p-2)}_{t,x}((-\infty, +\infty) \times \mathbb{R}^2)} < \infty, \tag{4.7}
\]

To this end, for \( m > 0 \), let

\[
\Gamma(m) = \sup \{ \|\phi\|_K : \phi \text{ is a solution to (1.19)-(1.23), with } E(\phi) + M(\phi) \leq m \}, \tag{4.8}
\]

where \( \|\phi\|_K := \|\phi\|_{L^4_{t,x} \cap L^{2(p-2)}_{t,x}((-\infty, +\infty) \times \mathbb{R}^2)} \), and define

\[
(E + M)_c = \sup \{ m > 0 : \Gamma(m) < \infty \}. \tag{4.9}
\]

If \( E(\phi_0) + M(\phi_0) \leq m \) sufficiently small, then \( \|\phi_0\|_{H^1} << 1 \). Hence, Remark 2.8 implies \( (E + M)_c > 0 \).

Now our aim is to show \((E + M)_c = \infty\). Suppose by contradiction that \((E + M)_c < \infty\), we will show the existence of the critical element. In fact, by the definition of \((E + M)_c\), we can take a sequence \( \{\phi_n\} \) of solutions (up to time translations) to (1.19)-(1.23) such that \( E(\phi_n) + M(\phi_n) \to (E + M)_c \) and \( \|\phi_n\|_{K([0, +\infty))} \to \infty \) as \( n \to \infty \).

Applying Theorem 4.1 to \( \{\phi_n(0, x)\} \) and obtaining some subsequence of \( \{\phi_n(0, x)\} \) (still denoted by the same symbol), then there exists \( \varphi^j \in H^1(\mathbb{R}^2) \) and \( (\theta^j_n, t^j_n)_{n \geq 1} \) of sequences in \( \mathbb{R}/2\pi \mathbb{Z} \times \mathbb{R} \), with

\[
t^j_n \to t^j_\infty \in [-\infty, +\infty], |t^j_n - t^m_n| \to \infty, \forall j \neq m, \quad \text{as} \quad n \to \infty, \tag{4.10}
\]

such that for any \( k \in \mathbb{N} \), there exists \( w_n^k \in H^1(\mathbb{R}^2) \),

\[
\phi_n(0, x) = \sum_{j=1}^{k} e^{i\theta^j_n} e^{-it^j_n \Delta} \varphi^j(x) + w_n^k. \tag{4.11}
\]

The remainder \( w_n^k \) satisfies

\[
\lim_{k \to \infty} \limsup_{n \to \infty} \| \nabla e^{it \Delta} w_n^k \|_{L^4_{t} L^q_{x}(\mathbb{R}\times\mathbb{R}^2)} = 0, \tag{4.12}
\]

where \( (q, r) \) is \( L^2 \)-admissible, and \( 2 < q < \infty \).
Moreover, we have the following decoupling properties: \( \forall k \in \mathbb{N}, \)

\[
\left\| \nabla \phi_n(0) \right\|^2_{L^2} = \sum_{j=1}^{k} \left\| \nabla e^{-it\Delta} \varphi^j \right\|^2_{L^2} = \left\| \nabla w_n^k \right\|^2_{L^2} \to 0, \text{ as } n \to 0, \quad (4.13)
\]

\[
E(\phi_n(0)) - \sum_{j=1}^{k} E(e^{-it\Delta} \varphi^j) - E(w_n^k) \to 0, \quad (4.14)
\]

\[
M(\phi_n(0)) - \sum_{j=1}^{k} M(e^{-it\Delta} \varphi^j) - M(w_n^k) \to 0. \quad (4.15)
\]

Using Strichartz estimate, we have

\[
\sup_{k \in \mathbb{N}} \limsup_{n \to \infty} \left\| e^{it\Delta} w_n^k \right\|_{S^1(\mathbb{R})} \lesssim \sup_{k \in \mathbb{N}} \limsup_{n \to \infty} \left\| w_n^k \right\|_{H^1} < \infty. \quad (4.16)
\]

Now, we construct the nonlinear profile. We define the nonlinear profile \( u^j \in C \left( (-\infty, +\infty), H^1(\mathbb{R}^2) \right) \) to be the solution of (1.19)-(1.23) such that

\[
\left\| u^j(t) - e^{-it\Delta} \varphi^j \right\|_{H^1} \to 0, \quad \text{as } n \to \infty. \quad (4.17)
\]

Let \( u_n^j = e^{it\Delta} u^j(t - t_n^j) \). For the linear profile decomposition, we can give the corresponding nonlinear profile decomposition

\[
u_n^{\leq k}(t) = \sum_{j=1}^{k} u_n^j(t). \quad (4.18)
\]

We will show \( u_n^{\leq k} \) is a good approximation for \( u_n \), provided that each nonlinear profile has a finite global Strichartz norm, which is the key to show the existence of the critical element.

**Lemma 4.2.** There exists \( j_0 \in \mathbb{N} \), such that for \( j > j_0 \),

\[
\sum_{j > j_0} \left\| u^j \right\|^2_{S^1(\mathbb{R})} \lesssim \sum_{j > j_0} \left\| \varphi^j \right\|^2_{H^1} < \infty. \quad (4.19)
\]

**Proof.** By (4.13), we have

\[
\sum_{j=1}^{\infty} \left\| (\nabla) e^{-it\Delta} \varphi^j \right\|^2_{L^2} < \infty \quad (4.20)
\]

which shows \( \sum_{j=1}^{\infty} \left\| \varphi^j \right\|^2_{H^1} < \infty \), and therefore \( \left\| \varphi^j \right\|^2_{H^1} \to 0 \), as \( j \to \infty \). By Remark 2.7, we have when \( j \) is large enough, \( \left\| u^j \right\|_{S^1(\mathbb{R})} \lesssim \left\| \varphi^j \right\|_{H^1} \), and prove the Lemma. \( \square \)

**Lemma 4.3.** In the nonlinear profile decomposition (4.18), if

\[
\left\| u^j \right\|_{L^4_{t,x} \cap L^{2(p-2)}_{t,x}} \left( (-\infty, +\infty) \times \mathbb{R}^2 \right) < \infty \quad \text{for } 1 \leq j \leq k, \quad (4.21)
\]

then

\[
\left\| u_n^j \right\|_{S^1(\mathbb{R})} = \left\| u^j \right\|_{S^1(\mathbb{R})} \lesssim 1 \quad \text{for } 1 \leq j \leq k, \quad (4.22)
\]

and there exists \( B > 0 \) such that

\[
\limsup_{n \to \infty} \left( \left\| u_n^{\leq k} \right\|_{L^4_{t,x} \cap L^{2(p-2)}_{t,x}} + \left\| (\nabla) u_n^{\leq k} \right\|_{L^8_{t,x} \cap L^{8(2-p)}_{t,x}} \right) \leq B. \quad (4.23)
\]
Proof. By

$$\left| \sum_{1 \leq j \leq k} u_{n}^{j} - \sum_{1 \leq j \leq k} |u_{n}^{j}| \right| \leq C_{k,q} \sum_{1 \leq j \neq m \leq k} |u_{n}^{j}|^{-1} |u_{n}^{m}|, 1 < q < \infty,$$

we have

$$\left\| \sum_{1 \leq j \leq k} u_{n}^{j} \right\|_{L_{t,x}^{2(p-2)}(\mathbb{R} \times \mathbb{R}^{2})} \leq \sum_{1 \leq j \leq k} \left\| u_{n}^{j} \right\|_{L_{t,x}^{2(p-2)}(\mathbb{R} \times \mathbb{R}^{2})} + C_{k} \sum_{1 \leq j \neq m \leq k} \int_{\mathbb{R} \times \mathbb{R}^{2}} |u_{n}^{j}|^{2(p-2)-1} |u_{n}^{m}| \, dx \, dt$$

$$\lesssim \sum_{1 \leq j \leq k_{0}} \| u_{n}^{j} \|_{L_{t,x}^{2(p-2)}(\mathbb{R} \times \mathbb{R}^{2})} + \left( \sum_{j > j_{0}} \| \varphi^{j} \|_{L^{2}}^{2} \right)^{p-2} + \| u_{n}^{j} u_{n}^{m} \|_{L_{t,x}^{p-2}} \| u_{n}^{j} \|_{L_{t,x}^{2(p-3)}}.$$ 

In the above, \( \| u_{n}^{j} u_{n}^{m} \|_{L_{t,x}^{p-2}} = \| u^{j}(t - (t_{n}^{j} - t_{m}^{j}), \cdot) u^{m}(t, \cdot) \|_{L_{t,x}^{p-2}} \rightarrow 0 \), so by Lemma 4.2, we obtain that there is \( B_{1} > 0 \) such that

$$\limsup_{n \rightarrow \infty} \| u_{n}^{k} \|_{L_{t,x}^{2(p-2)}} \leq B_{1}. \quad (4.26)$$

Similarly,

$$\limsup_{n \rightarrow \infty} \| \langle \nabla \rangle u_{n}^{k} \|_{L_{t,x}^{4}} \leq B_{1}. \quad (4.27)$$

By using (4.13), and boundedness of \( \| \phi_{0}(0) \|_{L_{t,x}^{2}} \), we have

$$\limsup_{n \rightarrow \infty} \| \langle \nabla \rangle u_{n}^{k} \|_{L_{t,x}^{4}} \leq B_{1}. \quad (4.28)$$

By combining (4.26)-(4.28), we prove the Lemma. \( \square \)

**Theorem 4.4.** (Existence of the critical element) Assume \((E + M)_{c} < \infty\), then there exists a global solution \( \phi_{c} \in H^{1}(\mathbb{R}^{2}) \) to (1.19)-(1.23) such that

$$\| \phi_{c} \|_{L_{t,x}^{4} \cap L_{t,x}^{2(p-2)}((0, +\infty) \times \mathbb{R}^{2})} = \infty,$$

and \( E(\phi_{c}) + M(\phi_{c}) = (E + M)_{c} \).

**Proof.** Now, \( u_{n}^{k} \) satisfies

$$i \partial_{t} u_{n}^{k} + \Delta u_{n}^{k} - \frac{2m}{r^{2}} A_{\theta}(t_{n}^{j}) u_{n}^{k} - A_{0}(t_{n}^{j}) u_{n}^{k} - \frac{1}{r^{2}} A_{0}^{2}(t_{n}^{j}) u_{n}^{k} = -\lambda |u_{n}^{k}|^{p-2} u_{n}^{k} + \varepsilon_{n}^{k}, \quad (4.30)$$

where

$$\varepsilon_{n}^{k} := \sum_{j=1}^{k} \frac{2m}{r^{2}} A_{\theta}(u_{n}^{j}) u_{n}^{j} - \frac{2m}{r^{2}} A_{0}(u_{n}^{k}) u_{n}^{k} + \sum_{j=1}^{k} A_{0}(u_{n}^{j}) u_{n}^{j} - A_{0}(u_{n}^{k}) u_{n}^{k} + \sum_{j=1}^{k} \frac{1}{r^{2}} A_{0}^{2}(u_{n}^{j}) u_{n}^{j} - \frac{1}{r^{2}} A_{0}^{2}(u_{n}^{k}) u_{n}^{k} - \lambda \sum_{j=1}^{k} |u_{n}^{j}|^{p-2} u_{n}^{j} + \lambda |u_{n}^{k}|^{p-2} u_{n}^{k}. \quad (4.31)$$

**Case 1.** More than one \( \varphi_{j} \neq 0 \). By (4.14)-(4.15), we have \( M(\varphi_{j}) + E(\varphi_{j}) < (M + E)_{c} \), so

$$\| u_{j} \|_{L_{t,x}^{4} \cap L_{t,x}^{2(p-2)}((-\infty, +\infty) \times \mathbb{R}^{2})} < \infty, \forall j \geq 1.$$ 

(4.32)
By Lemma 4.3, there is a large constant $A_1, B$ (independent of $k$) with the property that for any $k$, there exists $n_0 = n_0(k)$ such that for $n > n_0$,

$$\|\nabla u_n^k(t)\|_{L^\infty_t L^2_x} \leq A_1,$$

$$\|u_n^k\|_{L^4_t L^{2(p-2)}_x ((-\infty, +\infty) \times \mathbb{R}^2)} \leq B.$$  \hfill (4.33)

Also, by (4.13), there is a large constant $A_2$ (independent of $k$) with the property that for any $k$, there exists $n_1 = n_1(k)$ such that for $n > n_1$,

$$\|\phi_n(0) - u_n^k(0)\|_{H^1} \leq \|\phi_n(0)\|_{H^1} + \sum_{j=1}^k \|\varphi_j\|_{H^1} \leq A_2.$$  \hfill (4.34)

Moreover, we verify that for each $k$ and $\delta > 0$ there exists $n_2 = n_2(k, \delta)$ such that for $n > n_2$, $e_n^k = f_n^k + g_n^k,$ \sum_{i=1}^2 \partial_i e_n^k = f_{n1}^k + g_{n1}^k, for some functions $f_n^k, g_n^k, f_{n1}^k, \text{ and } g_{n1}^k$.

$$\|f_n^k\|_{L^4_t \mathbb{R}, x \in \mathbb{R}^2}^4 + \|f_{n1}^k\|_{L^4_t \mathbb{R}, x \in \mathbb{R}^2}^4 \leq \delta, \|g_n^k\|_{L^1_t \mathbb{R}, L^2_x} + \|g_{n1}^k\|_{L^1_t \mathbb{R}, L^2_x} \leq \delta.$$  \hfill (4.35)

Note that since $u_n^{\leq k}(0) - u_n(0) = w_n^k,$ there exists $k_1 = k_1(\delta)$ sufficiently large, such that for each $k > k_1$ there exists $n_2 = n_2(k)$ such that $n > n_2$

$$\|\nabla e^{\mu \Delta} (u_n^{\leq k}(0) - u_n(0))\|_{L^4_t \mathbb{R}, x \in \mathbb{R}^2} \leq \delta.$$  \hfill (4.36)

Thus, we may apply long-time perturbation theory to obtain that for $n > \max\{n_1, n_2, n_3\}$ and $k > k_1$ sufficiently large,

$$\|u_n\|_{L^4_t \mathbb{R}, x \in \mathbb{R}^2 \cap \mathbb{R}^{2(p-2)}} < \infty,$$ a contradiction.

Now we verify (4.35).

$$\sum_{j=1}^k \frac{2m}{r^2} A_\theta (u_n^j) u_n^j - \frac{2m}{r^2} A_\theta (u_n^{\leq k}) u_n^{\leq k}$$

$$= \sum_{j=1}^k \frac{2m}{r^2} (A_\theta (u_n^j)) - A_\theta (u_n^{\leq k})) u_n^j$$

$$= \sum_{j=1}^k \sum_{1 \leq j_1, j_2 \leq k} \frac{m}{r^2} \int_0^r u_n^j u_n^{j_1} \overline{u_n^{j_2}} s ds u_n^{j_1} + \sum_{j=1}^k \sum_{1 \leq j_1, j_2 \leq k} \frac{m}{r^2} \int_0^r u_n^j u_n^{j_1} \overline{u_n^{j_2}} s ds u_n^{j_1}.$$  \hfill (4.37)

If $j_1 \neq j_2$,

$$\frac{m}{r^2} \int_0^r u_n^j u_n^{j_1} \overline{u_n^{j_2}} s ds u_n^{j_1} \|_{L^4_r \mathbb{R}, x} \leq \|u_n^j(t - (t_n^{j_1} - t_n^{j_2}), \cdot) \overline{u_n^{j_1}(t, \cdot)}\|_{L^4_r \mathbb{R}, x} \|u_n^{j_1}\|_{L^4_r \mathbb{R}, x}.$$  \hfill (4.38)

Since $|t_n^{j_1} - t_n^{j_2}| \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\|u_n^j(t - (t_n^{j_1} - t_n^{j_2}), \cdot) \overline{u_n^{j_1}(t, \cdot)}\|_{L^4_r \mathbb{R}, x} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

so,

$$\frac{m}{r^2} \int_0^r u_n^j u_n^{j_1} \overline{u_n^{j_2}} s ds u_n^{j_1} \|_{L^4_r \mathbb{R}, x} \rightarrow 0 \text{ as } n \rightarrow \infty.$$  \hfill (4.39)
If $j_1 \neq j_2$, 

$$
\| \frac{m}{r^2} \int_0^r u_n^{j_1 \to j_2} sds u_n^j \|_{L^q t, s}^{\frac{4}{q}} = \| \frac{2m}{r^2} A_{\phi}(u_n^{j_1})(t - (t_n^{j_1} - t_n^j), \cdot) u_n^j(t, \cdot) \|_{L^q t, s}^{\frac{4}{q}}. 
$$

(4.40)

Since $|t_n^{j_1} - t_n^j| \to 0$ as $n \to \infty$, we have 

$$
\| \frac{2m}{r^2} A_{\phi}(u_n^{j_1})(t - (t_n^{j_1} - t_n^j), \cdot) u_n^j(t, \cdot) \|_{L^q t, s}^{\frac{4}{q}} \to 0 \text{ as } n \to \infty,
$$

so, 

$$
\| \frac{m}{r^2} \int_0^r u_n^{j_1 \to j_2} sds u_n^j \|_{L^q t, s}^{\frac{4}{q}} \to 0 \text{ as } n \to \infty. 
$$

(4.41)

By combining (4.37)-(4.41), we have 

$$
\| \sum_{j=1}^k \frac{2m}{r^2} A_{\phi}(u_n^{j}) u_n^j - \frac{2m}{r^2} A_{\phi}(u_n^{\leq k}) u_n^{\leq k} \|_{L^q t, s}^{\frac{4}{q}} \to 0, \text{ as } n \to \infty. 
$$

(4.42)

Similarly, we also have 

$$
\| \nabla(\sum_{j=1}^k \frac{2m}{r^2} A_{\phi}(u_n^{j}) u_n^j - \frac{2m}{r^2} A_{\phi}(u_n^{\leq k}) u_n^{\leq k}) \|_{L^q t, s}^{\frac{4}{q}} \to 0, \text{ as } n \to \infty. 
$$

(4.43)

$$
\sum_{j=1}^k A_0(u_n^{j}) u_n^j - A_0(u_n^{\leq k}) u_n^{\leq k} = \sum_{j=1}^k (A_0(u_n^{j}) - A_0(u_n^{\leq k})) u_n^j 
$$

$$
= \sum_{j=1}^k (A_0^{(1)}(u_n^{j}) - A_0^{(1)}(u_n^{\leq k})) u_n^j + \sum_{j=1}^k (A_0^{(2)}(u_n^{j}) - A_0^{(2)}(u_n^{\leq k})) u_n^j 
$$

$$
= -\sum_{j=1}^k (\int_r^\infty A_{\phi}(u_n^{j}) \left|\frac{|u_n^j|^2}{s} - |u_n^{\leq k}|^2\right| + \frac{A_{\phi}(u_n^{j})}{s} |u_n^{\leq k}|^2 ds) u_n^j 
$$

$$
- \sum_{j=1}^k (\int_r^\infty \frac{m}{s} |u_n^j|^2 - |u_n^{\leq k}|^2) ds u_n^j. 
$$

(4.44)

$$
- \sum_{j=1}^k \int_r^\infty A_{\phi}(u_n^{j}) \left|\frac{|u_n^j|^2}{s} - |u_n^{\leq k}|^2\right| ds u_n^j 
$$

$$
= \sum_{j=1}^k \sum_{j_1 \neq j_2, j_1, j_2 \leq k} \int_r^\infty A_{\phi}(u_n^{j_1}) \frac{u_n^{j_1 \to j_2}}{u_n^{j_1 \to j_2}} ds u_n^{j_2} 
$$

$$
+ \sum_{j=1}^k \sum_{j_1 = j_2 \neq j} \int_r^\infty A_{\phi}(u_n^{j_1}) \frac{u_n^{j_1 \to j_2}}{u_n^{j_1 \to j_2}} ds u_n^{j_2}. 
$$

(4.45)

If $j_1 \neq j_2$, 

$$
\| \int_r^\infty A_{\phi}(u_n^{j}) \frac{u_n^{j_1 \to j_2}}{u_n^{j_1 \to j_2}} ds u_n^j \|_{L^q t, s}^{\frac{4}{q}} 
$$

$$
\leq \| u_n^j \|_{L^q \to L^q} \| u_n^{j_1 \to j_2}(t - (t_n^{j_1} - t_n^{j_2})), \cdot u_n^{j_2}(t, \cdot) \|_{L^q t, s} \| u_n^{j_1 \to j_2} \|_{L^q \to L^q}. 
$$

(4.46)
If $j_1 = j_2 \neq j$,

\[
\left\| \int_r^\infty \frac{A_0(u_n^j)}{s} u_n^j u_n^j \, dsu_n^j \right\|_{L^4_{t,x}}^\frac{1}{2} \lesssim \|A_0(u_n^j)(t - (t_n^j - t_n^h), \cdot)\|_{L^2_{t,x}} \|u_n^j\|_{L^2_{t,x}} \cdot \tag{4.47}
\]

\[
- \sum_{j=1}^k \int_r^\infty \frac{A_0(u_n^j) - A_0(u_n^k)}{s} |u_n^k|^2 \, dsu_n^j \\
= - \frac{1}{2} \sum_{j=1}^k \sum_{j_1 \neq j_2} \int_r^\infty \frac{f_n^j u_n^j u_n^{j_2} t \, dt}{s} |u_n^k|^2 \, dsu_n^j \\
- \frac{1}{2} \sum_{j=1}^k \sum_{j_1 = j_2 \neq j} \int_r^\infty \frac{f_n^j u_n^j u_n^{j_2} t \, dt}{s} |u_n^k|^2 \, dsu_n^j. \tag{4.48}
\]

If $j_1 \neq j_2$,

\[
\left\| \int_r^\infty \frac{f_0^j u_n^j u_n^{j_2} t \, dt}{s} |u_n^k|^2 \, dsu_n^j \right\|_{L^4_{t,x}} \lesssim \|u_n^j(t - (t_n^j - t_n^h), \cdot)\|_{L^2_{t,x}} \|u_n^j\|_{L^2_{t,x}} \|u_n^j\|_{L^\infty_{t,x}} \cdot \tag{4.49}
\]

\[
- \frac{1}{2} \sum_{j=1}^k \sum_{j_1 = j_2 \neq j} \left( \sum_{j_3 \neq j_4} + \sum_{j_3 = j_4 \neq j_1} + \sum_{j_3 = j_4 = j_1} \right) \int_r^\infty \frac{f_n^j u_n^j u_n^{j_2} t \, dt}{s} |u_n^k|^2 \, dsu_n^j. \tag{4.50}
\]

If $j_1 = j_2 \neq j$, $j_3 \neq j_4$,

\[
\left\| \int_r^\infty \frac{f_0^j u_n^j u_n^{j_2} t \, dt}{s} |u_n^k|^2 \, dsu_n^j \right\|_{L^4_{t,x}} \lesssim \|u_n^j\|_{L^2_{t,x}}^2 \|u_n^j(t - (t_n^j - t_n^h), \cdot)u_n^j(t, \cdot)\|_{L^2_{t,x}} \|u_n^j\|_{L^\infty_{t,x}}. \tag{4.51}
\]

If $j_1 = j_2 \neq j$, $j_3 = j_4 \neq j$,

\[
\left\| \int_r^\infty \frac{f_0^j u_n^j u_n^{j_2} t \, dt}{s} |u_n^k|^2 \, dsu_n^j \right\|_{L^4_{t,x}} \lesssim \|A_0(u_n^j)(t - (t_n^j - t_n^h), \cdot)\|_{L^2_{t,x}} \|u_n^j\|_{L^2_{t,x}} \cdot \tag{4.52}
\]

If $j_1 = j_2 = j_3 = j_4 \neq j$,

\[
\left\| \int_r^\infty \frac{f_0^j u_n^j u_n^{j_2} t \, dt}{s} |u_n^k|^2 \, dsu_n^j \right\|_{L^4_{t,x}} \lesssim \|A_0(u_n^j)(t - (t_n^j - t_n^h), \cdot)\|_{L^2_{t,x}}. \tag{4.53}
\]
\begin{equation}
- \sum_{j=1}^{k} \left( \int_{r}^{\infty} \frac{m}{s} \left( |u_n^j|^2 - |u_n^{\leq k}|^2 \right) ds u_n^j \right) = \sum_{j=1}^{k} \sum_{j_1 \neq j_2} \int_{r}^{\infty} \frac{m}{s} u_n^{j_1} \overline{u_n^{j_2}} ds u_n^j + \sum_{j=1}^{k} \sum_{j_1 \neq j} \int_{r}^{\infty} \frac{m}{s} u_n^{j_1} \overline{u_n^{j_2}} ds u_n^j. \tag{4.54}
\end{equation}

If $j_1 \neq j_2$,
\begin{equation}
\| \int_{r}^{\infty} \frac{m}{s} u_n^{j_1} \overline{u_n^{j_2}} ds u_n^j \|_{L_t^4 L_x^4} \lesssim \| u_n^{j_1} (t - (t_n^j - t_n^{j_2}), \cdot) \overline{u_n^{j_2} (t, \cdot)} \|_{L_t^4 L_x^2} \| u_n^{j_1} \|_{L_t^4 L_x^4}. \tag{4.55}
\end{equation}

If $j_1 = j_2 \neq j$,
\begin{equation}
\| \int_{r}^{\infty} \frac{m}{s} u_n^{j_1} \overline{u_n^{j_2}} ds u_n^j \|_{L_t^4 L_x^4} \lesssim \| \int_{r}^{\infty} \frac{m}{s} |u_n^{j_1}|^2 ds (t - (t_n^j - t_n^{j_2}), \cdot) u_n^{j_2} (t, \cdot) \|_{L_t^4 L_x^4}. \tag{4.56}
\end{equation}

Let
\begin{equation}
g_n^k = -\frac{1}{2} \sum_{j=1}^{k} \sum_{j_1 \neq j_2} \int_{r}^{\infty} \frac{\int_{0}^{s} u_n^{j_1} \overline{u_n^{j_2}} dt}{s} |u_n^{\leq k}|^2 ds u_n^j - \frac{1}{2} \sum_{j=1}^{k} \sum_{j_1 \neq j_2} \sum_{j_3 \neq j_4} \int_{r}^{\infty} \frac{\int_{0}^{s} u_n^{j_1} \overline{u_n^{j_2}} dt}{s} u_n^{j_3} \overline{u_n^{j_4}} ds u_n^j. \tag{4.57}
\end{equation}

By combining (4.44)-(4.56), and using (4.10), we have
\begin{equation}
\| g_n^k \|_{L_t^4 L_x^4} \to 0, \text{ as } n \to \infty, \tag{4.58}
\end{equation}
and
\begin{equation}
\| \sum_{j=1}^{k} A_0(u_n^j) u_n^j - A_0(u_n^{\leq k}) u_n^{\leq k} - g_n^k \|_{L_t^4 L_x^4} \to 0, \text{ as } n \to \infty. \tag{4.59}
\end{equation}

For $i = 1, 2$,
\begin{equation}
\partial_{x_i} \left( \sum_{j=1}^{k} A_0(u_n^j) u_n^j - A_0(u_n^{\leq k}) u_n^{\leq k} \right) = \sum_{j=1}^{k} \left( \frac{A_0(u_n^j)}{r} \left( |u_n^j|^2 - |u_n^{\leq k}|^2 \right) \frac{x_i}{r} u_n^j + \frac{A_0(u_n^j) - A_0(u_n^{\leq k})}{r} |u_n^{\leq k}|^2 \frac{x_i}{r} u_n^j \right)
\end{equation}
\begin{align*}
&+ \left( \frac{m}{r} \left( |u_n^j|^2 - |u_n^{\leq k}|^2 \right) \frac{x_i}{r} u_n^j \right) - \sum_{j=1}^{k} \left( \int_{r}^{\infty} \frac{A_0(u_n^j)}{s} \left( |u_n^j|^2 - |u_n^{\leq k}|^2 \right) ds \right) (u_n^j)_{x_i}
&+ \frac{A_0(u_n^j) - A_0(u_n^{\leq k})}{s} |u_n^{\leq k}|^2 + \frac{m}{s} \left( |u_n^j|^2 - |u_n^{\leq k}|^2 \right) ds (u_n^j)_{x_i}. \tag{4.60}
\end{align*}
\begin{equation}
\| \sum_{j=1}^{k} \left( \frac{A_0(u_n^j)}{r} \left( |u_n^j|^2 - |u_n^{\leq k}|^2 \right) \frac{x_i}{r} u_n^j \right) \|_{L_t^4 L_x^4}
\end{equation}
and

\[ \sum_{j=1}^{k} \sum_{1 \leq j_1 \leq j_k} (t - (t^{j_1}_n - t^{j_2}_n), \cdot) |u^{j_1}_{n}|^2(t, \cdot) \|L^2_t L^2_x \| \nabla u^{j_1}_n \|L^4_t L^4_x. \]  

(4.61)

If \( j_1 \neq j_2 \),

\[ \sum_{j=1}^{k} \frac{A_0(u^{j}_n)}{r} u^{j \leq k}_n \|L^4_t L^4_x \| \|u^{j \leq k}_n\|L^4_t L^4_x. \]  

(4.62)

If \( j_1 = j_2 \neq j \),

\[ \sum_{j=1}^{k} \int_{t}^{s} \frac{u^{j}_n u^{j}_n}{r} ds \|u^{j \leq k}_n\|L^4_t L^4_x. \]  

(4.63)

These terms are treated like (4.44)-(4.56), so

\[ (4.65) \rightarrow 0, \text{ as } n \rightarrow \infty. \]  

(4.66)

Let

\[ g^{k}_{n_1} = -\frac{1}{2} \sum_{i=1}^{2} \sum_{j_1 \neq j_2} \int_{t}^{s} \frac{u^{i}_n u^{j}_n}{s} ds \|u^{i \leq k}_n\|L^4_t L^4_x. \]  

(4.67)

By combining (4.60)-(4.66), and using (4.10), we have

\[ \|g^{k}_{n_1}\|L^2_t L^2_x \rightarrow 0, \text{ as } n \rightarrow \infty, \]  

(4.68)

and

\[ \| \sum_{i=1}^{2} \sum_{j=1}^{k} A_0(u^{j}_n) u^{j}_n - A_0(u^{j \leq k}_n) u^{j \leq k}_n \|L^4_t L^4_x. \]  

(4.69)
If \( j_1 \neq j_2 \),
\[
\left\| \frac{1}{r^2} (A_\theta(u_n^j) + A_\theta(u_n^{k,j})) \int_0^r u_n^j u_n^{k,j} sds \right\|_{L_t^4}^{\frac{4}{3}} \lesssim \left\| A_\theta(u_n^j) + A_\theta(u_n^{k,j}) \right\|_{L_t^\infty L_x^\infty} \left( \left\| A_\theta(u_n^{k,j}) \right\|_{L_t^\infty L_x^\infty} \left\| \frac{A_\theta(u_n^j)}{r^2} (t - (t_n^j - t_n^{k,j}), \cdot) u_n^j(t, \cdot) \right\|_{L_t^4 L_x^4} \right). \tag{4.71}
\]

If \( j_1 = j_2 \neq j \),
\[
\left\| \frac{1}{r^2} (A_\theta(u_n^j) + A_\theta(u_n^{k,j})) \int_0^r u_n^j u_n^{k,j} sds \right\|_{L_t^4}^{\frac{4}{3}} \lesssim \left\| A_\theta(u_n^j) + A_\theta(u_n^{k,j}) \right\|_{L_t^\infty L_x^\infty} \left( \left\| A_\theta(u_n^{k,j}) \right\|_{L_t^\infty L_x^\infty} \left\| \frac{A_\theta(u_n^j)}{r^2} (t - (t_n^j - t_n^{k,j}), \cdot) u_n^j(t, \cdot) \right\|_{L_t^4 L_x^4} \right). \tag{4.72}
\]

By combining (4.70)-(4.72), and using (4.10), we have
\[
\left\| \sum_{j=1}^k \frac{1}{r^2} A_\theta^2(u_n^j) u_n^j - \frac{1}{r^2} A_\theta^2(u_n^{k,j}) u_n^{k,j} \right\|_{L_t^4 L_x^4} \to 0, \text{ as } n \to \infty. \tag{4.73}
\]

Similarly, we also have
\[
\left\| \nabla \left( \sum_{j=1}^k \frac{1}{r^2} A_\theta^2(u_n^j) u_n^j - \frac{1}{r^2} A_\theta^2(u_n^{k,j}) u_n^{k,j} \right) \right\|_{L_t^4 L_x^4} \to 0, \text{ as } n \to \infty. \tag{4.74}
\]

\[
\left| u_n^{k,j} \right|^{p-2} u_n^{k} - \left| u_n^{j} \right|^{p-2} u_n^{j} = \left| \sum_{j=1}^k (|u_n^{k,j}|^{p-2} - |u_n^{j}|^{p-2}) u_n^j \right|
\approx \sum_{j=1}^k \sum_{l=1, l \neq j}^k (|u_n^{k,j}|^{p-3} + |u_n^{j}|^{p-3}) |u_n^l||u_n^j|, \tag{4.75}
\]

\[
\left| \nabla \left( u_n^{k,j} \right)^{p-2} u_n^{k,j} - \left( \sum_{j=1}^k (|u_n^{k,j}|^{p-2} - |u_n^{j}|^{p-2}) \nabla u_n^j \right) \right| + \frac{p}{2} \left( |u_n^{k,j}|^{p-4} u_n^{k} - |u_n^{j}|^{p-4} u_n^{j} \right) \nabla u_n^k
\lesssim \sum_{j=1}^k \sum_{l=1, l \neq j}^k (|u_n^{k,j}|^{p-3} + |u_n^{j}|^{p-3}) |u_n^l||\nabla u_n^j|. \tag{4.76}
\]
When \( l \neq j \),
\[
\left\| \left( u_n^k \right|^{p-3} \left( u_n^j \right|^{p-3} \right) u_n^l \right\|_{L_t^4 L_x^2}^4
\leq \left\| \left( u_n^k \right|^{p-3} \left( u_n^j \right|^{p-3} \right) u_n^l \right\|_{L_t^4 L_x^2}^4
\leq \left\| \left( u_n^k \right|^{p-3} \right) \left( u_n^j \right|^{p-3} \right) u_n^l \right\|_{L_t^4 L_x^2}^4
\leq \left\| \left( u_n^k \right|^{p-3} \right) \left( u_n^j \right|^{p-3} \right) u_n^l \right\|_{L_t^4 L_x^2}^4.
\] (4.77)

Since \( \left\| u_n^j \right\|_{L_t^2 L_x^2} \to 0 \), as \( n \to \infty \), so
\[
\left\| (u_n^k \right|^{p-3} \left( u_n^j \right|^{p-3} \right) u_n^l \right\|_{L_t^4 L_x^2}^4 \to 0, \quad \text{as} \quad n \to \infty.
\] (4.78)

Similarly,
\[
\left\| \nabla \left( u_n^k \right|^{p-3} u_n^j - \sum_{j=1}^k \left| u_n^j \right|^{p-2} u_n^j \right\|_{L_t^4 L_x^2}^4 \to 0, \quad \text{as} \quad n \to \infty.
\] (4.79)

By combining (4.30), (4.31), (4.42), (4.43), (4.58), (4.59), (4.68), (4.69), (4.73), (4.74), (4.79), (4.80), and letting \( \sum_{j=1}^k \nabla \left( u_n^k \right|^{p-3} u_n^j = \sum_{j=1}^k \left| u_n^j \right|^{p-2} u_n^j \), we get (4.35).

Caes 2. \( \varphi_1 \neq 0 \), and \( \varphi_j = 0 \) for all \( j \geq 2 \).

Similarly as in case 1, we have \( \left\| u \right\|_{L_t^4 L_x^2 \cap (L_t^2 + \infty) \times \mathbb{R}^2} = \infty \). On one hand, by the definition of \( (E+M) \), we have \( E(U^1) + M(U^1) \geq (E+M) \), on the other hand, by (4.14)-(4.15), we have \( E(U^1) + M(U^1) \leq (E+M) \), so \( E(U^1) + M(U^1) = (E+M) \).

Let \( \phi_c = u^1 \), we prove the Theorem.

**Theorem 4.5. (Compactness of the critical element)** With \( \phi_c \) as in Theorem 4.4, \( \{ \phi_c(t) : t \in [0, +\infty) \} \) is precompact in \( H^1(\mathbb{R}^2) \).

**Proof.** For \( t_n \geq 0 \), if \( t_n \to t^* \), as \( n \to \infty \), then we see by the continuity of \( \phi_c(t) \) in \( H^1(\mathbb{R}^2) \) with \( t \) that
\[
\phi_c(t_n) \to \phi_c(t^*) \quad \text{in} \quad H^1(\mathbb{R}^2), \quad \text{as} \quad n \to \infty.
\]

If \( t_n \to +\infty \). Similarly as in deriving the existence of critical element, we have
\[
\phi_c(t_n) = e^{-t_n \Delta} \varphi_1 + w_n^1,
\] (4.81)
\[
M(\varphi^1) + E(e^{-t_n \Delta} \varphi^1) = (E + M) \chi, \quad \text{lim}_{n \to \infty} (M(w_n^1) + E(w_n^1)) = 0
\] (4.82)

So, we have \( \lim_{n \to \infty} \left\| w_n^1 \right\|_{H^1} = 0 \), i.e.,
\[
\phi_c(t_n) - e^{-t_n \Delta} \varphi_1 \to 0 \quad \text{in} \quad H^1(\mathbb{R}^2), \quad \text{as} \quad n \to \infty.
\] (4.83)

(i) If \( t_n \to -\infty \), then we have
\[
\left\| (\nabla) e^{it \Delta} \phi_c(t_n) \right\|_{L_t^4 L_x^2(0, +\infty) \times \mathbb{R}^2}
= \left\| (\nabla) e^{it \Delta} \varphi_1 \right\|_{L_t^4 L_x^2(-t_n + +\infty) \times \mathbb{R}^2} + o_n(1) \to 0, \quad \text{as} \quad n \to \infty.
\] (4.84)
Similarly as in proof of Theorem 2.7, for a fixed small \( \delta > 0 \), when \( n \) is sufficiently large, \( \| \phi_c \|_{L^4_t(x) \cap L^{2(p-2)}_x((t_n, +\infty) \times \mathbb{R}^2)} \leq \| (\nabla) e^{it\Delta} \phi_c(t_n) \|_{L^4_t(x) \cap L^{2(p-2)}_x((0, +\infty) \times \mathbb{R}^2)} \leq \delta \), which contradicts

\[
\| \phi_c \|_{L^4_t(x) \cap L^{2(p-2)}_x((0, +\infty) \times \mathbb{R}^2)} = \infty. \tag{4.85}
\]

(ii) If \( t_n^1 \to +\infty \), then we have

\[
\| (\nabla) e^{it\Delta} \phi_c(t_n) \|_{L^4_t(x) \cap L^{2(p-2)}_x((0, +\infty) \times \mathbb{R}^2)} \leq \| (\nabla) e^{it\Delta} \phi_c(t_n) \|_{L^4_t(x) \cap L^{2(p-2)}_x((0, +\infty) \times \mathbb{R}^2)} \leq \delta. \tag{4.87}
\]

Since \( t_n \to +\infty \), which contradicts

\[
\| \phi_c \|_{L^4_t(x) \cap L^{2(p-2)}_x((0, +\infty) \times \mathbb{R}^2)} = \infty. \tag{4.88}
\]

Thus, \( t_n^1 \) converges to some finite \( t^1 \). Since \( e^{-it^1 \Delta} \phi^1 \to e^{-it^1 \Delta} \phi^1 \), (4.83) shows that \( \phi_c(t_n^1) \) converges in \( H^1 \).

By the compactness of the critical element, we have

**Corollary 4.6.** Let \( \phi_c \) be the critical element in Theorem 4.4, we have for any \( \epsilon > 0 \), there exists \( R_0(\epsilon) > 0 \) such that when \( R > R_0(\epsilon) \),

\[
\int_{|x| \geq R} \sum_{i=1}^2 |D_i\phi_c(t)|^2 + |\phi_c|^2 + |\phi_c|^p dx \leq \epsilon. \tag{4.89}
\]

4.3. **Virial identities.** Since \( dF = d^2A = 0 \), we have

\[
\partial_t F_{\theta} - \partial_r F_{\theta0} + \partial_\theta F_{\theta0} = 0. \tag{4.90}
\]

To rewrite (4.90) in terms of a natural stress-energy tensor, let

\[
T_{00} = \frac{1}{2} r|\phi|^2, T_{0r} = rIm(\bar{\phi} D_r \phi), T_{0\theta} = \frac{1}{r} Im(\bar{\phi} D_\theta \phi) \tag{4.91}
\]

Then, (4.90) can be rewritten as \( \sum_{\alpha=0}^2 \partial_\alpha T_{00} = 0 \).

**Lemma 4.7.** we have

\[
\partial_r T_{0r} = -(2 + 2r \partial_r)|D_r \phi|^2 + \frac{p-2}{p} 2 \lambda r \partial_r |\phi|^p + \frac{1}{r} \partial_r |D_\theta \phi|^2 - \frac{2}{r} \partial_\theta Re(D_\theta D_r \phi) + r \partial_r [\frac{1}{r^2} (\frac{1}{2} \partial_\theta^2 |\phi|^2 - |D_\theta \phi|^2)] + \frac{1}{2} \partial_r^2 + \frac{1}{2} \partial_\theta^2 - \frac{1}{2r} \partial_r) |\phi|^2. \tag{4.92}
\]

**Proof.** By the proof of Lemma 5.1 in [20], we have

\[
\partial_r T_{0r} = rIm(D_r \phi D_r \phi) + rIm(\bar{\phi} D_r D_r \phi) + 2F_{\theta r} F_{\theta r}, \tag{4.93}
\]

and calculate each term separately.
For the first term, we have
\[ r \text{Im}(\bar{D_r}D_r \phi) = -r \text{Re}(\bar{D_r}^2 D_r \phi) - |D_r \phi|^2 - \frac{1}{r} \left( \bar{D_r} \phi D_r \phi \right) - r \lambda |\phi|^{p-2} \text{Re}(\bar{\phi} D_r \phi) \]
\[ = -(1 + \frac{1}{2 r} \partial_r) |D_r \phi|^2 - \frac{1}{r} \partial_\theta \text{Re}(\bar{D_\theta} D_r \phi) - F_{\theta r} F_{0 r} + \frac{1}{2 r} \partial_r |D_\theta \phi|^2 - \frac{\lambda}{p} r \partial_r |\phi|^p. \]  
(4.94)

For the second term, we have
\[ r \text{Re}(\bar{\phi} D^2_r \phi) = r \text{Re}(\bar{\phi} D_r (\frac{1}{r} D_r \phi)) + r \text{Re}(\bar{\phi} D_r (\frac{1}{r^2} D^2_\theta \phi)) \]
\[ + \lambda r \text{Re}(\bar{\phi} D_r (|\phi|^{p-2} \phi)) \]
\[ = -(1 + \frac{3}{2 r} \partial_r) |D_r \phi|^2 + \left( \frac{1}{2} \partial^3_r + \frac{1}{2} \partial^2_r - \frac{1}{2 r} \partial_r \right) |\phi|^2 + \frac{1}{r} \partial_r |D_\theta \phi|^2 \]
\[ + r \partial_r \left( \frac{1}{2} \partial^2_r |\phi|^2 - |D_\theta \phi|^2 \right) - \frac{1}{r} \partial_\theta \text{Re}(\bar{D_\theta} D_r \phi) \]
\[ - F_{\theta r} F_{0 r} + \frac{p-1}{p} \lambda r \partial_r |\phi|^p. \]  
(4.95)

For more on the derivation of (4.94) and (4.95), we refer the reader to the proof of Lemma 5.1 in [20].

Combining (4.93), (4.94) and (4.95), prove the Lemma. \( \square \)

**Remark 4.8.** Under the equivariant ansatz, the identity is
\[ \partial_t T_{0 r} \]
\[ = -(2 + 2 r \partial_r) |D_r \phi|^2 + \frac{p-2}{p} r \lambda \partial_r |\phi|^p + \frac{1}{r} \partial_r |D_\theta \phi|^2 - r \partial_r \left( \frac{1}{r^2} |D_\theta \phi|^2 \right) \]
\[ + \left( \frac{1}{2} \partial^3_r + \frac{1}{2} \partial^2_r - \frac{1}{2 r} \partial_r \right) |\phi|^2. \]  
(4.96)

Let \( \chi : \mathbb{R}_+ \to [0, 1] \) be a smooth cut-off function equal to one on \([0, 1]\) and zero on \([2, \infty)\).

For any given \( R > 0 \), define \( \chi_R(r) := \chi(r/R) \). Set
\[ I_R(\phi) := \int_0^\infty T_{0 r} \chi_R r dr. \]

By using the pointwise identity (4.96) and integrating by parts, we have the following identity:

**Lemma 4.9.** (*Localized virial identity*)
\[ \frac{d}{dt} I_R(\phi) \]
\[ = 4 E(\phi) + 2 \int_0^\infty \left( |D_r \phi|^2 + \frac{1}{r^2} |D_\theta \phi|^2 - \frac{p-2}{p} \lambda |\phi|^p \right) (\chi_R - 1) r dr \]
\[ + \frac{2(4-p)}{p} \lambda \int |\phi|^p r dr + 2 \int_0^\infty \left( |D_r \phi|^2 - \frac{3}{4 r^2} \right) r^2 dr - \frac{p-2}{2p} \lambda |\phi|^p r \chi_R' r dr \]
\[ - \frac{5}{2} \int_0^\infty \frac{|\phi|^2}{r^2} r^2 \chi_R'' r dr - \frac{1}{2} \int_0^\infty \frac{|\phi|^2}{r^2} r^3 \chi_R''' r dr. \]  
(4.97)
4.4. Extinction of the critical element.

**Theorem 4.10.** There does not exist the critical element $\phi_c$ of (1.19)-(1.23) in Theorem 4.4.

**Proof.** On one hand

$$|I_R(\phi_c)| \lesssim R\|\phi_c\|_{L^2} \|\nabla \phi_c\|_{L^2} \lesssim R. \quad (4.98)$$

On the other hand,

$$\frac{d}{dt} I_R(\phi_c) \gtrsim 4E(\phi_c) - C \int_{\mathbb{R}} (|D_r \phi_c|^2 + \frac{1}{r^2} |D_{\theta} \phi_c|^2 + |\phi_c|^p) r dr. \quad (4.99)$$

By (4.89) and $\sum_{i=1}^2 |D_i \phi_c(t)|^2 = |D_r \phi_c|^2 + \frac{1}{r^2} |D_{\theta} \phi_c|^2$, we can take $R = R(\eta)$ sufficiently large, such that for sufficiently small $\eta > 0$,

$$\int_{\mathbb{R}} (|D_r \phi_c|^2 + \frac{1}{r^2} |D_{\theta} \phi_c|^2 + |\phi_c|^p) r dr \leq \eta E(\phi_c), \quad (4.100)$$

so when $R > R(\eta)$, we have $\frac{d}{dt} I_R(\phi_c) \gtrsim E(\phi_c)$.

Therefore, for $T > 0$,

$$T E(\phi_c) \lesssim \left| \int_0^T \frac{d}{dt} I_R(\phi_c) dt \right| = |I_R(\phi_c)(T) - I_R(\phi_c)(0)|, \quad (4.101)$$

which contradicts (4.98) when $T$ sufficiently large. \qed

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