Vortex nucleation by collapsing bubbles in Bose-Einstein condensates

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The nucleation of vortex rings accompanies the collapse of ultrasound bubbles in superfluids. Using the Gross-Pitaevskii equation for a uniform condensate we elucidate the various stages of the collapse of a stationary spherically symmetric bubble and establish conditions necessary for vortex nucleation. The minimum radius of the stationary bubble, whose collapse leads to vortex nucleation, was found to be $28 \pm 1$ healing lengths. The time after which the nucleation becomes possible is determined as a function of bubble’s radius. We show that vortex nucleation takes place in moving bubbles of even smaller radius if the motion made them sufficiently oblate.

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In this Letter we establish a new mechanism of vortex nucleation in a uniform condensate. Previously, the nucleation of vortices in a uniform condensate has been connected to critical velocities \cite{1, 2, 3}, instabilities of the initial states \cite{5} or to a transfer of energy among the solitary waves \cite{6}. Moving positive \cite{2} and negative \cite{3} ions were shown to generate vortex rings on their surface where the speed of sound was exceeded. Experiments in superfluid helium have demonstrated long time ago increasing attention \cite{11}. In a Bose- Einstein condensate, a topic which is receiving interest \cite{9} and phase imprinting \cite{10}. Moreover, our studies in BEC systems, alongside rotation \cite{8}, the decay of solitons \cite{5} and phase imprinting \cite{10}. Moreover, our work illustrates a new aspect of vortex-sound interaction in a Bose- Einstein condensate, a topic which is receiving increasing attention \cite{11}.

We write the GP equation in dimensionless form as

$$-2i\frac{\partial \psi}{\partial t} = \nabla^2 \psi + (1 - |\psi|^2 - V(x, t))\psi,$$  \hspace{1cm} (1)

in dimensionless variables such that the unit of length corresponds to the healing length $\xi$, the speed of sound is $c = 1/\sqrt{\xi}$, and the density at infinity is $\rho_{\infty} = |\psi_{\infty}|^2 = 1$. To convert the dimensionless units into values applicable to superfluid helium-4, we take the number density as $\rho = 2.18 \times 10^{28} \text{m}^{-3}$, the quantum of circulation as $\kappa = h/m = 9.92 \times 10^{-8} \text{m}^2\text{s}^{-1}$, and the healing length as $\xi = 0.128 \text{nm}$. This gives a time unit $2\pi \xi^2/\kappa \approx 1\text{s}$. Whereas for a sodium condensate with $\xi \approx 0.14 \mu\text{m}$, the time unit is about $8\text{s}$. $V(x, t)$ is the potential of interaction between a boson and a bubble. We will assume that the bubble acts as an infinite potential barrier to the condensate, so that no bosons can be found inside the bubble ($\psi = 0$) before the collapse. This is achieved by setting $V$ to be large inside the bubble and zero outside.

First we consider the case of a stationary spherically symmetrical bubble. The spherical symmetry allows us to reduce the problem to dimension one, so that the equation (1) for $\psi = \psi(r, t)$ becomes

$$-2i\psi_t = \psi'' + 2\psi'/r + (1 - |\psi|^2)\psi,$$  \hspace{1cm} (2)

where $r^2 = x^2 + y^2 + z^2$. Equation (2) is numerically integrated using fourth order finite differences discretization in space and fourth order Runge-Kutta method in time. Before the collapse the field around the bubble is a uniform condensate, $\psi_t = 0$. The boundary conditions are $\psi(a, t) = 0$ stating that the bubble surface is an infinite potential barrier to the condensate and $\psi(\infty, t) = 1$. The stationary solutions for various $a$ were found by the Newton-Raphson iterations. The solutions are $\psi(r) = (0, 0)$ if $r \leq a$ and $\psi(r) = (R_a(r), 0)$ if $r > a$, with the graphs of $R_a(r + a)$ for $a = 1, 2, 10, 30$ given on Figure 1. If the radius of the bubble, $a$, is sufficiently large, then we can set $r = a + \xi$ and to the leading order get $R''(\xi) + [1 - R(\xi)^2]R(\xi) = 0$ which has the solution, satisfying the boundary conditions, $R(\xi) = \tan(\xi/\sqrt{2})$. The total energy of the system \cite{12},

$$E = \frac{\pi}{2} \int |\nabla \psi|^2 \text{d}V + \frac{\pi}{2} \int (1 - |\psi|^2)^2 \text{d}V$$

depends on the radius of the bubble, and therefore, on the form of $R_a$:

$$E = \frac{\pi a^3}{3} + 2\pi \int_a^\infty \left[ R_a'(r)^2 + \frac{1}{2}(1 - R_a(r)^2)^2 \right] r^2 \text{d}r.$$  \hspace{1cm} (3)

The insert on Figure 1 shows the loglog plot of the energy vs radius of the bubble together with the linear fit. For $a > 20$ the energy depends on the radius as $E \sim 1.65 a^{2.9}$. \protect
From the energy conservation it is clear that after the bubble collapses and the condensate fills the cavity the necessary (but not sufficient) condition for vortex nucleation is that the energy has to be greater than that of one vortex ring. The minimal energy of the vortex solution was found in [12] to be about $E \sim 55 \pm 1$ which corresponds to the minimum radius of $a = 2.2$ with $E = 55.7$. As the condensate fills the cavity, most of the energy will be emitted via the sound waves, so the energy of the bubble has to be sufficiently greater than the energy of a single vortex ring to allow for such an emission.

The time-dependent evolution of the condensate after the bubble collapses involves several stages. The overall picture is complicated by a complex interplay between dispersive and nonlinear effects. Dispersive effects become important on the wavelengths of order of the healing length with the group velocity approximately given by $\partial(\sqrt{k^2/2 + k^2/4})/\partial k$ for the perturbation propagating along the uniform state $\psi = 1$ towards infinity and with the group velocity approximated by $\partial(k^2 - 1)/2\partial k$ for the perturbation moving along the uniform state $\psi = 0$ towards the centre of the cavity. The wavetrain generated by the nonlinearity is moving slower with the larger wavelengths than the dispersive wavetrain. During the first stage dispersive and nonlinear wavetrains are generated near the areas of the largest curvature of the stationary profile. The Fourier components propagate at different velocities generating wave packets moving in opposite directions. This stage of the evolution is characterised by a flux of particles towards the centre of the cavity as the oscillations of the growing amplitude are being formed on the real and imaginary parts of the wave function and the slope of the steep density front is getting smaller; see Figure 2. To follow the evolution of the particle flux we calculated the density per unit volume averaged over spheres of different radii, centred at the origin, as functions of time. Figure 3 shows these functions for the radius of the cavity $a = 128$ and the radii of the spheres over which the averaging of the density is performed being $4, 8, 16, 32,$ and $64$. From Figure 3 it is clear that there is a definite moment of time $t^* \approx 97$ when the density per unit volume reaches its maximum at the same time (and taking the same value) for two smallest spheres. This moment indicates the start of a qualitatively new stage of the evolution in which there is an outward flux of particles as the condensate that overfilled the cavity began to expand. The density gradually approaches the uniform state $\rho = 1$.

It is after the time $t^*$ that we expect the instability to set in and to give rise to vortex nucleation as the outward flow can support outward moving vortex rings. We calculated $t^*$ for various radii of the bubble and determined that for $a > 10$ the time of the start of the outflow from the centre of the cavity is approximated quite well by the linear function $t^* \sim 1.96 + 0.72a$.

As it seen on the insets of Figure 2, some of the surfaces of zero real and zero imaginary parts may be just a healing length apart (for instance, at $t = 100$ for the radius of the cavity $a = 128$, one of the surfaces of zero of the real part has radius $16.02$ and the nearest surface of the zero of imaginary part has radius $17.11$). Small radial perturbations on these zero surfaces can lead to their overlap which creates topological zero curves of the wavefunction which quickly adjust their form by emitting sound waves to become axisymmetrical vortex rings.

The smaller $a$ is, the larger the distances between the zero surfaces are during the condensate expansion, so it would require a much larger perturbation to bring these
FIG. 3: (colour online) The plots of the density per unit volume as function of time for various radii of spheres over which the averaging is performed. The initial radius of the cavity in this case is $a = 128$. The radii of averaging spheres are $b = 4, 8, 16, 32, 64$. The average density is calculated as $ar{\rho}_b = \frac{3}{b^3} \int_0^b \rho(r)r^2dr/b^3$. Surfaces to intersect. In this case the instability mechanism would be somewhat different taking longer time to develop. The radial 'dips' of the density of the expanding condensate, noticeable in Figure 2, are unstable to non-spherically symmetrical perturbations, similar to the instability of the Kadomtsev-Petviashvili 2D solitons in 3D [4]. Depending on the energy carried by these 'dips,' they evolve into either vortex solutions or sound waves. From these considerations we expect three possible outcomes after bubble collapses: (1) if the radius of the bubble is smaller then some critical radius $a^*$, the density 'dips' generated by the expanding condensate have rather small amplitude that decreases even further as they travel away from the centre quickly becoming sound waves before the instability has time to develop; (2) after the collapse of a bubble of an intermediate size, say, of the radius $a^* < a < \hat{a}$, the waves of sufficiently large amplitudes are generated and the instability of these waves develops in time inversely proportional to the radius $a$; (3) if the radius is sufficiently large, $a > \hat{a}$, the time of the first vortex nucleation is approximately given by the moment of the start of the outward flux of the particles $t^*$ approximated above. As a condensate continues to expand the instability mechanism described in (2) is further facilitated by the broken symmetry resulting from the previous nucleation events which leads to even more vortex rings being nucleated.

To confirm the scenario outlined above we performed full three-dimensional calculations for cavities of various radii in a computational box of the side 200 healing lengths [12]. We determined that there is a critical radius of the bubble for which vortex ring nucleate $a^* \sim 28 \pm 1$. The borderline radius between regime (2) and (3) was found as $\hat{a} \sim 45$. Figure 4 shows the density isoplots at the various time snapshots after the bubble of radius $a = 50$ collapsed. Notice, that the non-symmetry of the field that was created at the time of nucleation ($t \approx 40$) continues to produce even more vortex rings as condensate expands. Each vortex ring at the moment of its birth has zero radius, as the surfaces of zero real and imaginary parts touch each other, and gradually evolve into a vortex ring of increasingly larger and radius, as clearly seen on Figure 4. The process in which solitary waves evolve into states of a higher energy was elucidated in [5]. A finite amplitude sound wave that moves behind a vortex ring transfers its energy to it, allowing the vortex ring to grow in size. The radius of the vortex ring stabilises only when it travelled sufficiently far from the center of the collapsing bubble, where the flow became almost uniform. The larger the radius of the bubble, the more finite amplitude sound waves will be generated at shorter distances, the larger the size of the final ring is going to be.

So far we considered the collapse of the stationary bubble, where the vortex nucleation is connected to the instabilities developed in the spherically symmetric flow. There are situations when the nucleation is facilitated by an initial lack of the symmetry in the flow as in the case of a moving bubble or a bubble in the nonuniform (trapped) condensate. The surrounding helium exerts a net inward pressure across the surface, which is balanced.
by the pressure inside the bubble. In was shown in [3] by asymptotic analysis of the GP equation coupled with the equation of the motion for the wavefunction of an electron that a moving bubble becomes oblate in the direction of its motion. This flattening is created by the difference in pressure between the poles and equator associated with the greater condensate velocity at the latter than at the former. How oblate the bubble becomes during its motion will depend on the velocity and pressure inside the bubble. The non-uniformity of the flow in the collapsing oblate bubble leads to vortex ring nucleation for bubble sizes much smaller than in the case of a stationary spherically symmetric bubble. Figure 5 shows snapshots of the density plots of the cross-section of the collapsing bubble that prior to $t = 0$ was moving with a constant velocity $U = 0.2$ and acquired an oblate form given by $x^2 + \frac{1}{2}(y^2 + z^2) = 100$. As the result of bubble's collapse four vortex rings of different radii were created. Three of them are moving in the same direction as the bubble before the collapse and one vortex ring is moving in the opposite direction. In the model used, the oblateness of a bubble and the velocity of its propagation, $U$, are independent parameters. A useful quantity that can be determined from further numerical simulations is the critical radius of the bubble as a function of the oblateness and $U$. This will be further analysed in our future research.

In summary, we have suggested and studied a new mechanism of vortex nucleation as a result of the collapse of stationary and moving bubbles in the context of the GP equation. This is related to the experiments in superfluid helium in which the cavitated bubbles are generated by ultrasound in the megahertz frequency range. Our findings suggest that a sufficiently large or deformed stationary bubble in a trapped condensate will produce vortex rings as a result of its collapse. This will also be a subject of our future studies.

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[13] In 3D we used the same numerical method as in work 2. The faces of the computational box were open to allow sound waves to escape. The flow around a moving bubble was obtained by letting the potential of the bubble - boson interactions, $V$, to move with a constant velocity $U$, so that $V(x - Ut)$ in 4.