STOCHASTIC CONSERVATION LAWS: WEAK-IN-TIME FORMULATION AND STRONG ENTROPY CONDITION

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Abstract. This article is an attempt to complement some recent developments on conservation laws with stochastic forcing. In a pioneering development, Feng & Nualart[7] have developed the entropy solution theory for such problems and the presence of stochastic forcing necessitates introduction of strong entropy condition. However, the authors’ formulation of entropy inequalities are weak-in-space but strong-in-time. In the absence of a-priori path continuity for the solutions, we take a critical outlook towards this formulation and offer an entropy formulation which is weak-in-time and weak-in-space.

1. Introduction

Let \((\Omega, P, F, \{F_t\}_{t \geq 0})\) be a filtered probability space which satisfies the usual hypothesis. We are interested in finding an \(L^2(\mathbb{R}^d)(\text{or an appropriate function space})\)-valued predictable process \(u(t)\) which satisfies the stochastic partial differential equation

\[
du(t, x) + \text{div}_x F(u(t, x)) \, dt = \sigma(x, u(t, x)) \, dW(t) \quad t > 0, \ x \in \mathbb{R}^d, \tag{1.1}
\]

with the initial condition

\[
u(0, x) = u_0(x), \quad x \in \mathbb{R}^d. \tag{1.2}
\]

In the above, \(W(t)\) is a one-dimensional standard Brownian motion, \(F: \mathbb{R} \to \mathbb{R}^d\) is the flux function, and \(\sigma(x, u)\) is real valued function defined on the domain \(\mathbb{R}^d \times \mathbb{R}\).

In the case where \(\sigma = 0\), the equation (1.1) becomes a standard scalar conservation law with spatial dimension \(d\). It is well-known for conservation laws that solutions (that are obtained by method of characteristics) may develop discontinuities in finite time even when the initial data is smooth. In other words, the problem (1.1)-(1.2) do not have smooth solutions in general, even when the right hand side is zero. In this situation one has to invoke the notion of weak solutions, but the issues would persist as there could be infinitely many weak solutions to a given problem. It was a huge step forward for analytical understanding for scalar conservation laws when Kruzkov came up with his idea of entropy solutions. Kruzkov’s notion of entropy condition correctly isolates the physically relevant solution in a unique way, and there is a plethora literature (see [8, 2] and references therein) that has emerged on this subject.

Stochastic conservation laws is a relatively new area of pursuit. It is only recently that conservation laws with stochastic forcing have attracted the attention of many authors ([7, 11, 9, 3, 21, 1, 4]), and resulted in significant momentum in the theoretical development of such problems. As its deterministic counterpart, it is required to have a weak formulation coupled with an entropy criterion to establish the wellposedness for such problems. The equations of type (1.1) could be interpreted as the equation that describes conservation of physical quantities that are subjected to random force fields modeled by diffusion noise. One of the early work in this direction was [9], where one dimensional stochastic balance laws were studied where \(\sigma\) is independent of \(x\). The authors employed the splitting method to construct approximate solutions, and the approximations were shown to converge to a weak (possibly non-unique) solution. At around the same time, Khanin et al. [19] published a very influential article describing some statistical properties of Burger’s equations with...
stochastic forcing. When the noise term on the right hand side is of additive nature i.e. \( \sigma \equiv \sigma(t, x) \), J. U. Kim\textsuperscript{[13]} extended Kruzkov’s entropy formulation and established the wellposedness for one dimensional problems under the assumption that \( \sigma \in C([0, \infty) : W^{1, \infty}) \) and has compact support. The straight forward adaptation of the deterministic entropy inequalities fails to capture the noise-noise interaction, and the standard mechanism to derive the \( L^1 \)-contraction principle does not apply for general \( \sigma \). This issue was finally resolved by Feng & Nualart\textsuperscript{[7]} with the introduction of the notion of \textit{strong entropy} solution. In\textsuperscript{[7]}, the authors established the uniqueness of strong entropy solution in \( L^p \)-framework for several space dimension. The existence, however, was restricted to one space dimension. We also refer to the recent articles by Vovelle & Debussche\textsuperscript{[3]} and by Chen \textit{et al.}\textsuperscript{[1]} for the existence in the multi dimensional case. In\textsuperscript{[3]}, the authors obtain the existence via kinetic formulation. In\textsuperscript{[1]}, the authors use the BV solution framework. In this paper, we offer a weaker entropy formulation for\textsuperscript{[3, 14]} and establish wellposedness in the \( L^p \)-framework. In addition, we refer to\textsuperscript{[22, 23, 18, 17, 16, 15, 6]} for additional details relevant to the topic.

In our view, the article\textsuperscript{[7]} by Feng & Nualart is no less than a milestone in the subject and presents a comprehensive theory of entropy solutions for stochastic conservation laws. We draw our primary motivation from\textsuperscript{[7]}, but take a critical outlook to the approach and raise a few objections to some of the methods and offer an alternative which we perceive as better suited to the problem. The ordinary entropy inequalities in the stochastic case do not fully capture the noise-noise interactions and it may not be possible to replicate Kruzkov’s approach to get the \( L^1 \)-contraction principle. This issue is resolved by Feng & Nualart by introducing an additional condition called \textit{strong entropy condition}. However, the entropy inequalities in\textsuperscript{[7]} could be described as weak in space but strong in time. Moreover, the strong entropy condition is related to this formulation and reflects the strong-in-time picture. This formulation easily leads to the \( L^1 \) contraction principle, and uniqueness for such formulation naturally follows. However, the question of existence becomes much more subtle under this formulation. As its deterministic counterpart, the existence is settled via vanishing viscosity method in\textsuperscript{[7]} and this is where our viewpoint deviates from that of\textsuperscript{[7]}. The vanishing viscosity limit finds a perfect match with the entropy formulation which is weak in time and space both, and which would coincide with entropy formulation of\textsuperscript{[7]} if the solution process have continuous sample paths. In\textsuperscript{[7]}, the authors make an attempt to establish path-continuity for the vanishing viscosity limit but there are flaws in the proof. We have added a separate section in this article where we explain these flaws and describe the implications in details. With this apparent inconsistency in mind, in the absence of path-continuity, it is necessary that entropy inequalities are formulated weak in time and space both, and the strong entropy condition has to be accordingly specified to capture noise-noise interaction. In this article we set out to exactly do that.

The rest of the paper is organized as follows. In the next section, we describe the technical framework, define the notion of strong entropy solutions for\textsuperscript{[13, 14]} and state the main theorems. In Section 3, we establish the uniqueness of strong entropy solutions of\textsuperscript{[13, 14, 12]} by deriving the \( L^1 \) contraction property. In Section 4, we briefly discuss the wellposedness of vanishing viscosity approximation of\textsuperscript{[13]} and establish the existence of entropy solution for\textsuperscript{[13, 14, 12]}. In the Section 5, we show that the vanishing viscosity solution is indeed a strong entropy solution. Finally, in the last section, we describe the issues related to the path continuity of vanishing viscosity limit and its implications. We close this section with a description of the notations and symbols and the list of assumptions.

By \( C, K \) etc., we mean various constants which may change from line to line. The Euclidean norm on any \( \mathbb{R}^d \)-type space is denoted by \(| \cdot |\). Furthermore, let \( \Omega_T = (0, T] \times \mathbb{R}^d \). In the rest of the paper, the following assumptions hold:

\( \textbf{A.1} \) For every \( k = 1, 2, \ldots, d \), the function \( F_k(s) \in C^2(\mathbb{R}) \), and \( F_k(s), F_k^u(s) \) and \( F_k^v(s) \) have at most polynomial growth in \( s \).

\( \textbf{A.2} \) There exists a positive constant \( C > 0 \) such that

\[ |\sigma(y, v) - \sigma(x, u)| \leq C(|u - v| + |x - y|). \]

\( \textbf{A.3} \) There exists a nonnegative function \( g \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) such that

\[ |\sigma(x, u)| \leq g(x)(1 + |u|), \]
(A.4) The set \( \{ r \in \mathbb{R} : F''(r) \neq 0 \} \) is dense in \( \mathbb{R} \).

2. TECHNICAL FRAMEWORK AND STATEMENTS OF THE MAIN RESULTS

The notion of entropy solution is built around the so-called entropy flux pairs. We begin this section with the definition of entropy flux pairs.

**Definition 2.1** (entropy flux pair). \((\beta, \zeta)\) is called an entropy flux pair if \( \beta \in C^2(\mathbb{R}) \) and \( \beta \geq 0 \), and \( \zeta = (\zeta_1, \zeta_2, ..., \zeta_d) : \mathbb{R} \to \mathbb{R}^d \) is a vector field satisfying

\[
\zeta'(r) = \beta''(r)F'(r).
\]

An entropy flux pair \((\beta, \zeta)\) is called convex if \( \beta''(s) \geq 0 \).

As in the deterministic case, the primary motivation behind the notion of entropy solution comes from parabolic regularization. However, it requires considerable amount of work (cf. [4]) to show that perturbation by small diffusion will indeed regularize the solutions. To proceed, we assume that \( u \) is a smooth predictable solution of the parabolic perturbation of (1.1) i.e. \( u \) satisfies

\[
du(t, x) + \text{div}_x F(u(t, x)) dt = \sigma(x, u(t, x)) dW(t) + \varepsilon \Delta u(t, x) dt,
\]

where \( \varepsilon > 0 \) is a small positive number. As compared to the deterministic case, we need to replace the deterministic chain rule for derivatives by Itô chain rule to derive the entropy inequalities. Let \((\beta, \zeta)\) be a convex entropy flux pair. Then, by Itô formula, we have

\[
d\beta(u(t, x)) + \text{div}_x \zeta(u(t, x)) dt = \sigma(x, u(t, x))\beta'(u(t, x)) dW(t) + \frac{1}{2} \sigma^2(x, u(t, x))\beta''(u(t, x)) dt
\]

\[
+ \left( \varepsilon \Delta_{xx} \beta(u(t, x)) - \varepsilon \beta''(u(t, x)) |\nabla_x u(t, x)|^2 \right) dt.
\]

For each \( 0 \leq \psi \in C^{1,2}_c([0, \infty) \times \mathbb{R}^d) \), we apply Itô product rule to obtain

\[
d(\beta(u(t, x))\psi(t, x)) = \partial_t \psi(t, x)\beta(u(t, x)) dt - \psi(t, x)\text{div}_x \zeta(u(t, x)) dt
\]

\[
+ \psi(t, x)\sigma(x, u(t, x))\beta'(u(t, x)) dW(t) + \frac{1}{2} \psi(t, x) \sigma^2(x, u(t, x))\beta''(u(t, x)) dt
\]

\[
+ \psi(t, x) \left( \varepsilon \Delta_{xx} \beta(u(t, x)) - \varepsilon \beta''(u(t, x)) |\nabla_x u(t, x)|^2 \right) dt.
\]

It is to be kept in mind that \( \beta \) is non-negative and convex and \( \psi \) is non-negative. Therefore, for every \( T > 0 \), we have

\[
0 \leq \langle \beta(u(T, .)), \psi(T, .) \rangle
\]

\[
\leq \langle \beta(u(0, .)), \psi(0, .) \rangle + \int_0^T \langle \zeta(u(r, .)), \nabla_x \psi(r, .) \rangle dr
\]

\[
+ \int_{[0, T]} \langle \beta(u(r, .)), \partial_t \psi(r, .) \rangle dr + \int_{[0, T]} \langle \sigma(\cdot, u(r, .))\beta'(u(r, .)), \psi(r, .) \rangle dW(r)
\]

\[
+ \frac{1}{2} \int_{[0, T]} \langle \sigma^2(\cdot, u(r, .))\beta''(u(r, .)), \psi(r, .) \rangle dr + O(\varepsilon).
\]

Both the left-hand and right-hand sides of the inequality are stable under \( \varepsilon \to 0 \), provided we have \( L^p_{loc} \) type stability of (2.1) as \( \varepsilon \to 0 \). The above inequality leads to the entropy inequalities which are weak in time and space both.

**Definition 2.2** (stochastic entropy solution). An \( L^2(\mathbb{R}^d) \)-valued \( \{ \mathcal{F}_t : t \geq 0 \} \)-predictable stochastic process \( u(t) = u(t, x) \) is called a stochastic entropy solution of (1.1) provided

(1) For each \( T > 0, p = 2, 3, 4, \ldots \)

\[
\sup_{0 \leq t \leq T} E[\|u(t)\|_p^p] < \infty.
\]
and for each \( N = 1, 2, 3, \ldots \) fixed
\[
\int_0^T E \left[ \int_{|x| \leq N} |\sigma(x, u(r, x))|^4 \, dx \right] \, dr < \infty.
\]

(2) For \( 0 \leq \psi \in C^{1,2}_c([0, \infty) \times \mathbb{R}^d) \) and each convex entropy pair \((\beta, \zeta)\),
\[
0 \leq (\psi(0, \cdot), \beta(u(0, \cdot))) + \int_0^T \langle \partial_t \psi(t, \cdot), \beta(u(t, \cdot)) \rangle \, dt + \int_0^T \langle \beta'(u(r, \cdot)), \nabla_x \psi(r, \cdot) \rangle \, dr
\]
\[
+ \int_0^T (\psi(\cdot, u(r, \cdot)) \beta'(u(r, \cdot)), \psi(r, \cdot)) \, dW(r) + \frac{1}{2} \int_0^T (\sigma^2(u, u(r, \cdot)), \beta''(u(r, \cdot)), \psi(r, \cdot)) \, dr.
\]
(2.3)

(3) The process \( u(t, \cdot) \) satisfies the initial condition in the following sense: for every nonnegative test function \( \psi \in C^{1,2}_c(\mathbb{R}^d) \) such that \( \text{supp}(\psi) = K \)
\[
\lim_{t \to 0} E \left[ \int_K |u(t, x) - u_0(x)| \psi(x) \, dx \right] = 0.
\]
(2.4)

In the deterministic case, the entropy inequalities lead to the \( L^1 \)-contraction principle which implies uniqueness. In the stochastic case, however, the entropy inequalities alone do not seem to give rise to desired \( L^1 \)-contraction principle. The definition (2.2) does not reveal much about the noise-noise interaction when one tries to compare two solutions of the same problem. We refer to [1] for detailed mathematical description of this issue. However, to ensure uniqueness, we need to arrive at a version of so-called strong entropy condition which is compatible with the weak-in-time formulation.

Let \( \rho \) and \( g \) be the standard mollifiers on \( \mathbb{R} \) and \( \mathbb{R}^d \) respectively such that \( \text{supp}(\rho) \subset [-1, 0] \) and \( \text{supp}(g) = B_1(0) \). For \( \delta > 0 \) and \( \delta_0 > 0 \), let \( \rho_{\delta_0}(r) = \frac{1}{\delta_0 \rho(\frac{r}{\delta_0})} \) and \( g_{\delta}(x) = \frac{1}{\delta \rho(\frac{x}{\delta})} \). For a nonnegative test function \( \psi \in C^{1,2}_c([0, \infty) \times \mathbb{R}^d) \) and two positive constants \( \delta, \delta_0 \), define
\[
\phi_{\delta, \delta_0}(t, x; s, y) = \rho_{\delta_0}(t - s) g_{\delta}(x - y) \psi(s, y).
\]
(2.5)

Note that \( \rho_{\delta_0}(t - s) \neq 0 \) only if \( s - \delta_0 \leq t \leq s \), and therefore \( \phi_{\delta, \delta_0}(t, x; s, y) = 0 \) outside \( s - \delta_0 \leq t \leq s \).

**Definition 2.3** (stochastic strong entropy solution). An \( L^2(\mathbb{R}^d) \)-valued \( \{ \mathcal{F}_t : t \geq 0 \} \)-predictable stochastic process \( v(t) = v(t, x) \) is called a stochastic strong entropy solution of (1.1) provided

(i) it is a stochastic entropy solution,

(ii) For each \( L^2(\mathbb{R}^d) \)-valued \( \{ \mathcal{F}_t : t \geq 0 \} \)-adapted stochastic process \( \tilde{u}(t, x) \) satisfying, for \( T > 0 \),
\[
\sup_{0 \leq t \leq T} E \left[ ||\tilde{u}(t)||_p^p \right] < \infty,
\]

and for each \( \beta \in C^\infty(\mathbb{R}) \) such that \( \beta'' \) and \( \beta''' \) are of compact support and \( 0 \leq \psi \in C^\infty_c([0, \infty) \times \mathbb{R}^d) \), and
\[
h(r, s; v, y) = \int_x \sigma(x, \tilde{u}(r, x)) \beta'(\tilde{u}(r, x) - v) \phi_{\delta, \delta_0}(r, x; s, y) \, dx,
\]
where \( \phi_{\delta, \delta_0} \) is defined by in (2.5),
\[
E \left[ \int_0^T \int_0^T h(r, s; v, y) \, dW(r) \right]_{v=v(s,y)} \, dy \, ds
\]
\[
\leq -E \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma(x, \tilde{u}(r, x)) \sigma(y, v(r, y)) \beta''(\tilde{u}(r, x) - v(r, y)) \times \phi_{\delta, \delta_0}(r, x; s, y) \, dx \, dy \, ds \right] + A(\delta, \delta_0),
\]

where \( A(\delta, \delta_0) \) is a function depending on \( \beta, \psi \) such that \( A(\delta, \delta_0) \to 0 \) as \( \delta_0 \to 0 \).

**Remark.** The weak-in-time formulation is also manifested in the strong entropy condition. In our formulation the function \( A(\delta, \delta_0) \) plays a similar role as that of \( A(s, t) \) in Feng & Nualart.
where $\beta$ and approximate $|||\beta|||$ for $\leq (1.1)$. Let $0$ evolution of $\text{of doubling the variables to the stochastic case.}$ The central idea of the proof is to analyze the

$$\text{Theorem 2.1 (uniqueness). Let the assumptions } [\text{A.1}]-[\text{A.3}] \text{ be true, and that } \cap_{p=1,2} \text{, } L^p(\mathbb{R}^d)\text{-valued and } F_0\text{-measurable random variable } u_0 \text{ satisfies}$$

$$E\left[||u_0||_p^p + ||u_0||_2^2\right] < \infty \text{ for } p = 1, 2, \ldots .$$

Suppose that $u, v$ be two stochastic entropy solutions of (1.1) with the same initial condition $u(0) = u_0 = v(0)$, and that at least one of $u, v$ is a strong stochastic entropy solution. Then almost surely $u(t) = v(t)$ for almost every $t \geq 0$.

We further assume that $d = 1$, and state the existence theorem of strong entropy solutions.

**Theorem 2.2 (existence). Let the assumptions } [\text{A.1}]-[\text{A.4}] \text{ be true and } d = 1. \text{ Furthermore, } \cap_{p=1,2} \text{, } L^p(\mathbb{R}^d)\text{-valued } F_0\text{-measurable random variable } u_0 \text{ satisfies}$$

$$E\left[||u_0||_p^p + ||u_0||_2^2\right] < \infty \text{ for } p = 1, 2, \ldots .$$

Then there exists a strong entropy solution for (1.1) - (1.2).

3. Proof of uniqueness

adapts The proof of uniqueness follows a line argument that suitably adapts Kruzkov’s method of doubling the variables to the stochastic case. The central idea of the proof is to analyze the evolution of $||u(t) - v(t)||_{L^1(\mathbb{R}^d)}$ as a random quantity, and then arrive at the conclusion that $E(||u(t) - v(t)||_{L^1(\mathbb{R}^d)})$ decreases as a function of time. In our context also we use doubling of variables, and approximate $||u(s) - v(s)||_{L^1(\mathbb{R}^d)}$ by $\int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} \beta(u(t), x) - v(t, y) \psi(t)(x, y) \ dx \ dy$, where $\beta(r)$ is a suitable smooth convex approximation of $|r|$ and $\psi(x, y)$ is a smooth approximation for $\delta_{y}(t)$. We will, however, have to handle additional difficulties due to the stochastic forcing.

Let $u$ be a stochastic entropy solution and $v$ be a stochastic strong entropy solution to equation (1.1). Let $0 \leq \psi \in C^{1,2}_{\delta}(0, \infty) \times \mathbb{R}^d$ be given and $\beta \equiv \beta_\epsilon$ (as described above). For a fixed real
number $k \in \mathbb{R}$, $\beta(\cdot - k)$ is a convex smooth function. Therefore $(\beta(\cdot - k), F^\beta(\cdot, k))$ could be chosen as the corresponding convex entropy flux pair where $F^\beta(a, b)$ is described above. Next, we lay down the entropy inequality for $u(t, x)$ relative to the convex entropy pair $(\beta(\cdot - k), F^\beta(\cdot, k))$ and substitute $k$ by $v(s, y)$ and integrate with respect to $s$, $y$ to get

\[
\int_{\Omega_T} \int_{\mathbb{R}^d} \beta(u_0(x) - v(s, y)) \phi_{\delta, \delta_0}(0, x, s, y) \, dx \, dy \, ds + \int_{\Omega_T} \int_{\mathbb{R}^d} \beta(u(t, x) - v(s, y)) \partial_t \phi_{\delta, \delta_0} \, dy \, dx \, dt \, ds \\
+ \int_0^T \int_{\Omega_T} \int_{\Omega_T} \left[ h(r, s; v, y) \, dW(r) \right]_{v=v(s, y)} \, dy \, ds \\
+ \frac{1}{2} \int_{\Omega_T} \int_{\mathbb{R}^d} \int_{\Omega_T} \sigma^2(x, u(t, x)) \beta''(u(t, x) - v(s, y)) \phi_{\delta, \delta_0}(t, x; s, y) \, dy \, dx \, dt \, ds \\
+ \int_{\Omega_T} \int_{\mathbb{T}} F^\beta(u(t, x), v(s, y)) \nabla_x \phi_{\delta, \delta_0} \, dx \, dy \, ds \geq 0, \tag{3.1}
\]

where $\Omega_T = [0, T] \times \mathbb{R}^d$ and \( h(r, s; v, y) = \int_x \sigma(x, u(r, x)) \beta'(u(r, x) - v) \phi_{\delta, \delta_0}(r, x; s, y) \, dx \).

Similarly, since $v(s, y)$ is also a stochastic entropy solution, by substituting $k = u(t, x)$ and integrating with respect to $(t, x)$ we have

\[
\int_{\Omega_T} \int_{\mathbb{R}^d} \beta(v_0(y) - u(t, x)) \phi_{\delta, \delta_0}(t, x, 0, y) \, dx \, dy \, dt + \int_{\Omega_T} \int_{\mathbb{R}^d} \beta(v(s, y) - u(t, x)) \partial_t \phi_{\delta, \delta_0} \, dy \, dx \, dt \, ds \\
+ \int_{\Omega_T} \int_0^T \int_{\mathbb{R}^d} \sigma(y, v(s, y)) \beta'(v(s, y) - u(t, x)) \phi_{\delta, \delta_0} \, dy \, dx \, dW(s) \, dt \\
+ \frac{1}{2} \int_{\Omega_T} \int_{\mathbb{R}^d} \int_{\Omega_T} \sigma^2(y, v(s, y)) \beta''(v(s, y) - u(t, x)) \phi_{\delta, \delta_0}(t, x; s, y) \, dy \, dx \, dt \, ds \\
+ \int_{\Omega_T} \int_{\mathbb{T}} F^\beta(v(s, y), u(t, x)) \nabla_y \phi_{\delta, \delta_0} \, dx \, dy \, ds \geq 0 \tag{3.2}
\]

Adding the two inequalities (3.1) and (3.2) and using the fact that $\mathrm{supp} \, \rho_{\delta_0} \subset [-\delta_0, 0]$, we get

\[
\int_{\Omega_T} \int_{\mathbb{R}^d} \beta(u_0(x) - v(s, y)) \psi(s, y) \rho_{\delta_0}(-s) \phi_{\delta}(x - y) \, dx \, dy \, ds \\
+ \int_{\Omega_T} \int_{\mathbb{R}^d} \beta(v(s, y) - u(t, x)) \partial_{s} \psi(s, y) \rho_{\delta_0}(t - s) \phi_{\delta}(x - y) \, dy \, dx \, dt \, ds \\
+ \int_{\Omega_T} \int_{\mathbb{R}^d} F^\beta(v(s, y), u(t, x)) \nabla_y \psi(s, y) \rho_{\delta_0}(t - s) \phi_{\delta}(x - y) \, dy \, dx \, dt \, ds \\
+ \int_{\Omega_T} \int_{\mathbb{R}^d} F^\beta(u(t, x), v(s, y)) \nabla_x \phi_{\delta}(x - y) \psi(s, y) \rho_{\delta_0}(t - s) \, dx \, dt \, ds \\
+ \int_{\Omega_T} \int_{\mathbb{T}} F^\beta(v(s, y), u(t, x)) \nabla_y \phi_{\delta}(x - y) \psi(s, y) \rho_{\delta_0}(t - s) \, dx \, dt \, ds \\
+ \int_0^T \int_{\mathbb{R}^d} \left[ h(r, s; v, y) \, dW(r) \right]_{v=v(s, y)} \, dy \, ds \\
+ \int_{\Omega_T} \int_t^{t+\delta_0} \int_{\mathbb{R}^d} \sigma(y, v(s, y)) \beta'(v(s, y) - u(t, x)) \phi_{\delta, \delta_0}(t, x; s, y) \, dy \, dx \, dW(s) \, dt \\
+ \frac{1}{2} \int_{\Omega_T} \int_0^T \int_{\mathbb{R}^d} \sigma^2(y, v(s, y)) \beta''(v(s, y) - u(t, x)) \phi_{\delta, \delta_0}(t, x; s, y) \, dy \, dx \, dt \, ds \\
+ \frac{1}{2} \int_{\Omega_T} \int_0^T \int_{\mathbb{R}^d} \sigma^2(x, u(t, x)) \beta''(u(t, x) - v(s, y)) \phi_{\delta, \delta_0}(t, x; s, y) \, dy \, dx \, dt \, ds \\
\geq 0. \tag{3.3}
\]
We now take expectation on both sides and use the property of $v(s, y)$ as a strong entropy solution to have

\[
E \left[ \int_{\mathbb{R}^d} \int \beta(u_0(x) - v(s, y)) \psi(s, y) \rho_\delta(-s) g_\delta(x - y) \, dx \, dy \, ds \right] \\
+ E \left[ \int_{\mathbb{R}^d} \int \beta(v(s, y) - u(t, x)) \partial_s \psi(s, y) \rho_\delta(t - s) g_\delta(x - y) \, dx \, dy \, dt \, ds \right] \\
+ E \left[ \int_{\mathbb{R}^d} \int \int F^\delta(v(s, y), u(t, x)) \nabla_y \psi(s, y) \rho_\delta(t - s) g_\delta(x - y) \, dx \, dt \, dy \, ds \right] \\
+ E \left[ \int_{\mathbb{R}^d} \int \int F^\delta(u(t, x), v(s, y)) \nabla_x g_\delta(x - y) \psi(s, y) \rho_\delta(t - s) \, dx \, dt \, dy \, ds \right] \\
+ E \left[ \int_{\mathbb{R}^d} \int \int F^\delta(v(s, y), u(t, x)) \nabla_y g_\delta(x - y) \psi(s, y) \rho_\delta(t - s) \, dx \, dt \, dy \, ds \right] \\
+ \frac{1}{2} E \left[ \int_{\mathbb{R}^d} \int_T \int_{\mathbb{R}^d} \sigma^2(u(t, x)) \beta''(u(t, x) - v(s, y)) \phi_\delta(t; s, y) \, dy \, dt \, ds \right] \\
+ \frac{1}{2} E \left[ \int_{\mathbb{R}^d} \int_T \int_{\mathbb{R}^d} \sigma^2(v(s, y)) \beta''(v(s, y) - u(t, x)) \phi_\delta(t; s, y) \, dy \, dt \, ds \right] \\
- E \left[ \int_{\mathbb{R}^d} \int \sigma(u(t, x)) \sigma(v(t, y)) \beta''(u(t, x) - v(t, y)) \psi(s, y) \right. \\
\left. \times \rho_\delta(t - s) g_\delta(x - y) \, dy \, dt \, ds \right] + A(\delta, \delta_0) \\
\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + A(\delta, \delta_0) \\
\geq 0 \tag{3.4}
\]

Now, we estimate each of the terms above as $\delta_0, \delta \to 0$ and $\beta \to | \cdot |$. We start with $I_1$.

**Lemma 3.1.**

\[
\lim_{\delta_0 \to 0} I_1 = E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \beta(u_0(x) - v_0(y)) \psi(0, y) g_\delta(x - y) \, dx \, dy
\]

and

\[
\lim_{(\epsilon, \delta) \to (0, 0)} E \int_{\mathbb{R}^d \times \mathbb{R}^d} \beta_\epsilon(u_0(x) - v_0(y)) g_\delta(x - y) \psi(0, y) \, dx \, dy = E \int_{\mathbb{R}^d} |u_0(x) - v_0(x)| \, \psi(0, x) \, dx.
\]

**Proof.** The proof is divided into two steps, and in each step, we will justify the passage to the corresponding limit.

**Step 1:** In this step we consider the passage to the limit as $\delta_0 \to 0$. Let

\[
A_1 := E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \beta(u_0(x) - v(s, y)) \psi(s, y) \rho_\delta(-s) g_\delta(x - y) \, dx \, dy \, ds \\
- E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \beta(u_0(x) - v_0(y)) \psi(0, y) g_\delta(x - y) \, dx \, dy \\
= E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \beta(u_0(x) - v(s, y)) \psi(s, y) - \psi(0, y) \rho_\delta(-s) g_\delta(x - y) \, dx \, dy \, ds \\
+ E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[ \beta(u_0(x) - v(s, y)) - \beta(u_0(x) - v_0(y)) \right] \\
\psi(0, y) g_\delta(x - y) \rho_\delta(-s) \, dx \, dy \, ds.
\]

Since support $\psi(s, \cdot) \subset K$, we have

\[
|A_1| \leq \|\psi_t\|_\infty E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_K(y) \beta(u_0(x) - v(s, y)) \rho_\delta(-s) g_\delta(x - y) \, dx \, dy \, ds \\
+ \|\beta'\|_\infty E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v(s, y) - v_0(y)| \psi(0, y) g_\delta(x - y) \rho_\delta(-s) \, dx \, dy \, ds.
\]
\[
\begin{align*}
\leq & \|\psi\|_{\infty} \|\beta\|_{\infty}\delta_0 E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_K(y)(|u_0(x) - v(s, y)|)\rho_{\delta_0}(-s)\varphi_s(x - y) \, dx \, dy \, ds \\
& + \|\beta\|_{\infty} E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v(s, y) - v_0(y)|\bar{\psi}(0, y)\rho_{\delta_0}(-s) \, dx \, dy \, ds \\
\leq & \|\psi\|_{\infty} \|\beta\|_{\infty}\delta_0 E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_K(y)(|u_0(x) - v(s, y)|)\rho_{\delta_0}(-s)\varphi_s(x - y) \, dx \, dy \, ds \\
& + \|\beta\|_{\infty} E \int_0^T \int_K \psi(0, y)|v(s, y) - v_0(y)|\rho_{\delta_0}(-s) \, dy \, ds \\
\leq & \|\psi\|_{\infty} \|\beta\|_{\infty}\delta_0 \left[ \|u_0(x)\|_{L^1(\mathbb{R}^d)} + E \int_0^T \int_K |v(s, y)|\rho_{\delta_0}(-s) \, dy \, ds \right] \\
& + \|\beta\|_{\infty} E \int_0^T \int_K \psi(0, y)|v(s, y) - v_0(y)|\rho_{\delta_0}(-s) \, dy \, ds \\
= & \|\psi\|_{\infty} \|\beta\|_{\infty}\delta_0 \left[ \|u_0(x)\|_{L^1(\mathbb{R}^d)} + \sup_{0 \leq s \leq T} E[|v(s, -)|_{L^1}] \right] \\
& + \|\beta\|_{\infty} \int_0^1 E \left( \int_K \psi(0, y)|v(\delta_0 r, y) - v_0(y)| \, dy \right) \rho(-r) \, dr.
\end{align*}
\]

Note that, for a fixed \( r \in [0, 1], E \int_K \psi(0, y)|v(\delta_0 r, y) - v_0(y)| \, dy \to 0 \) as \( \delta_0 \to 0 \). Therefore by bounded convergence theorem, \( \lim_{\delta_0 \to 0} A_1 = 0 \).

**Step 2:** In this step, we now establish the second half of the lemma. Note that the sequence \( (\beta_r)_r \) is a sequence of functions that satisfies \( |\beta_r(r) - |r| \leq C \delta \) for any \( r \in \mathbb{R} \). Therefore

\[
\begin{align*}
|E \int_{\mathbb{R}^d \times \mathbb{R}^d} & \beta_r(u_0(x) - v_0(y))\varphi_s(x - y)v(0, y) \, dx \, dy - E \int_{\mathbb{R}^d} |u_0(y) - v_0(y)|\psi(0, y) \, dy| \\
\leq & E \int_{\mathbb{R}^d \times \mathbb{R}^d} |\beta_r(u_0(x) - v_0(y)) - |u_0(x) - v_0(y)||\varphi_s(x - y)v(0, y) \, dx \, dy \\
& + E \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( |u_0(x) - v_0(y)| - |u_0(y) - v_0(y)||\varphi_s(x - y)v(0, y) \, dx \, dy \\
\leq & \text{Const}(\delta) E \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( \int_{|z| \leq \delta} |u_0(x) - u_0(x + \delta z)||\varphi(z) \, dz \right) \, dx \, dy \\
\leq & \text{Const}(\delta) E \int_{|z| \leq \delta} \left( \int_{\mathbb{R}^d} \left|u_0(x) - u_0(x + \delta z)\right| \, dx \right) \, dz.
\end{align*}
\]

Note that \( \lim_{\delta \to 0} \int_{\mathbb{R}^d} \left|u_0(x) - u_0(x + \delta z)\right| \, dx \to 0 \) for all \( |z| \leq 1 \), therefore by bounded convergence theorem we have \( \lim_{\delta \to 0} E \int_{\mathbb{R}^d} \left|u_0(x) - u_0(x + \delta z)\right| \, dz = 0 \). This allows us to pass to the limit \( (\delta, \epsilon) \to (0, 0) \) in the last line and establish the second part of the claim. \( \square \)

**Lemma 3.2.** It follows that

\[
\lim_{\delta_0 \to 0} I_2 = E \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \beta(v(s, y) - u(s, x))\partial_s\psi(s, y)\varphi_s(x - y) \, dy \, ds
\]

and

\[
\lim_{(\epsilon, \delta) \to (0, 0)} E \left[ \int_{\mathbb{R}^d} \int_x \beta_r(v(s, y) - u(s, x))\partial_s\psi(s, y)\varphi_s(x - y) \, dy \, ds \right] = E \left[ \int_{\mathbb{R}^d} |v(s, x) - u(s, x)|\partial_s\psi(s, x) \, dx \, ds \right].
\]

**Proof.** As before, the proof is divided into two steps and in each of these steps we will justify the corresponding passage to the limit.
Step 1: Firstly, we consider the passage to the limit as $\delta_0 \to 0$. For this, let

$$
G_1 := |E \int_{\Pi_T} \int_{\Pi_T} \beta(v(s, y) - u(t, x)) \partial_s \psi(s, y) \rho_{\delta_0}(t - s) g_3(x - y) dy ds dt \\
- E \int_{\Pi_T} \int_{\mathbb{R}^d} \beta(v(s, y) - u(s, x)) \partial_s \psi(s, y) g_3(x - y) dy ds dt |
$$

$$
= |E \int_{s=\delta_0}^{T} \int_{\mathbb{R}^d} \beta(v(s, y) - u(t, x)) \partial_s \psi(s, y) \rho_{\delta_0}(t - s) g_3(x - y) dy ds dt \\
- E \int_{s=\delta_0}^{T} \int_{0}^{t} \int_{\mathbb{R}^d} \beta(v(s, y) - u(s, x)) \partial_s \psi(s, y) g_3(x - y) \rho_{\delta_0}(t - s) dy ds dt | + O(\delta_0)
$$

$$
\leq E \int_{s=\delta_0}^{T} \int_{\mathbb{R}^d} |\beta(v(s, y) - u(t, x)) - \beta(v(s, y) - u(s, x))| |\partial_s \psi(s, y)|
$$

$$
\times g_3(x - y) \rho_{\delta_0}(t - s) dy ds dt + O(\delta_0)
$$

$$
G_1 \leq C(\beta) \|\partial_s \psi\|_{\infty} E \left[ \int_{s=\delta_0}^{T} \int_{\Pi_T} |u(s, x) - u(t, x)| \rho_{\delta_0}(t - s) dy ds dt \right] + O(\delta_0)
$$

$$
\leq C(\beta) \|\partial_s \psi\|_{\infty} E \left[ \int_{r=0}^{1} \int_{s=\delta_0}^{T} \int_{\mathbb{R}^d} |u(t + \delta_0 r, x) - u(t, x)| \rho_1(-r) dy ds dt \right] + O(\delta_0). \quad (3.5)
$$

Note that, $\lim_{\delta_0 \to 0} \int_{\mathbb{R}^d} |u(t + \delta_0 r, x) - u(t, x)| dt dx \to 0$ almost surely for all $r \in [0, 1]$. Therefore, by bounded convergence theorem, $\lim_{\delta_0 \to 0} E \left[ \int_{0}^{1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(t + \delta_0 r, x) - u(t, x)| \rho_1(-r) dt dx dr \right] = 0$, and therefore the first step follows.

Step 2: In this step, we establish the second part of the lemma. For this, let

$$
G_2(\varepsilon, \delta) := |E \int_{\Pi_T} \int_{\mathbb{R}^d} \beta(v(s, y) - u(s, x)) \partial_s \psi(s, y) g_3(x - y) dx dy ds |
$$

$$
- E \int_{\Pi_T} \int_{\mathbb{R}^d} |v(s, y) - u(s, x)| \partial_s \psi(s, y) g_3(x - y) dx dy ds |
$$

$$
\leq \|\partial_s \psi\|_{\infty} E \int_{\text{supp}(\psi(s, y))} \int_{\mathbb{R}^d} \int_{0}^{T} |\beta(v(s, y) - u(s, x)) - |v(s, y) - u(s, x)||
$$

$$
\times g_3(x - y) dy ds dt.
$$

As before, note that the sequence $(\beta_\varepsilon)_\varepsilon$ is a sequence of functions that satisfies

$$
|\beta_\varepsilon(r) - |r|| \leq C \varepsilon \quad \text{for any } r \in \mathbb{R},
$$

we have

$$
G_2(\varepsilon, \delta) \leq \|\partial_s \psi\|_{\infty} \varepsilon C(\psi, T). \quad (3.6)
$$

Once again, let

$$
G_3(\delta) := |E \int_{\Pi_T} \int_{\mathbb{R}^d} |v(s, y) - u(s, x)| \partial_s \psi(s, y) g_3(x - y) dx dy ds |
$$

$$
- E \int_{0}^{1} \int_{\mathbb{R}^d} |v(s, y) - u(s, x)| \partial_s \psi(s, y) dy ds |
$$

$$
\leq E \int_{\Pi_T} \int_{\mathbb{R}^d} |u(s, y) - u(s, x)| \partial_s \psi(s, y) g_3(x - y) dx dy ds |
$$

$$
\to 0 \quad \text{as } \delta \to 0, \text{ (as in Lemma 3.1)}
$$

Now

$$
|E \left[ \int_{\Pi_T} \int_{x} \beta_\varepsilon(v(s, y) - u(s, x)) \partial_s \psi(s, y) g_3(x - y) dx dy ds \right] |
$$

and therefore the first step follows. Next we estimate the limit of $I_3$ as $\delta_0 \to 0$ and $(\epsilon, \delta) \downarrow (0, 0)$.

**Lemma 3.3.**

$$\lim_{\delta_0 \to 0} I_3 = E \int_{\mathbb{R}^d} \int_{\Pi_T} F^3(v(s, y), u(s, x)) \nabla_y \psi(s, y) g_3(x - y) dy dx ds$$

and

$$\lim_{(\epsilon, \delta) \to (0, 0)} E \int_{\Pi_T} \int_{\mathbb{R}^d} F^3(v(s, y), u(s, x)) \nabla_y \psi(s, y) g_3(x - y) dx dy ds$$

$$= E \left[ \int_{\Pi_T} \sum_{k=1}^d \text{sign}(u(s, y) - v(s, y))(F_k(u(s, y)) - F_k(v(s, y))) \partial_{y_k} \psi(s, y) dy ds \right].$$

**Proof.** The proof is divided into two steps.

**Step 1:** We first verify the passage to the limit as $\delta_0 \to 0$. Note that there exists $p \in \mathbb{N}$ such that, for all $a, b, c \in \mathbb{R}$,

$$|F^3(a, b) - F^3(a, c)| \leq K|b - c|(1 + |b|^{p} + |c|^{p}).$$

Therefore, upon denoting

$$B_1 := E \int_{\Pi_T} \int_{\mathbb{R}^d} F^3(v(s, y), u(t, x)) \nabla_y \psi(s, y) \rho_0(t - s) g_3(x - y) dy ds dx dt - E \int_{\Pi_T} \int_{\mathbb{R}^d} F^3(v(s, y), u(s, x)) \nabla_y \psi(s, y) g_3(x - y) dy dx ds,$$

we have

$$B_1 \leq E \left| \int_{s=0}^{T} \int_{\mathbb{R}^d} F^3(v(s, y), u(t, x)) \nabla_y \psi(s, y) \rho_0(t - s) g_3(x - y) dy ds dx dt \right|$$

$$- E \left| \int_{s=0}^{T} \int_{\mathbb{R}^d} F^3(v(s, y), u(s, x)) \nabla_y \psi(s, y) g_3(x - y) \rho_0(t - s) dy ds dx dt \right| + O(\delta_0)$$

$$\leq K|\nabla_y \psi(s, y)||_\infty E \left[ \int_{s=0}^{T} \int_{\mathbb{R}^d} \int_{t=0}^{T} |u(s, x) - u(t, x)| (1 + |u(s, x)|^{p} + |u(t, x)|^{p}) \right.$$

$$\times \rho_0(t - s) dx dt ds + O(\delta_0)$$

(by Cauchy-Schwartz inequality)

$$\leq C|\nabla_y \psi(s, y)||_\infty \left[ E \left| \int_{s=0}^{T} \int_{\mathbb{R}^d} \int_{t=0}^{T} |u(s, x) - u(t, x)|^{2} \rho_0(t - s) dx dt ds \right|^{1/2} + O(\delta_0) \right.$$}

$$\leq C|\nabla_y \psi(s, y)||_\infty \left[ E \left| \int_{r=0}^{1} \int_{\mathbb{R}^d} \int_{t=0}^{T} |u(t + \delta_0 r, x) - u(t, x)|^{2} \rho(-r) dx dt dr \right|^{1/2} + O(\delta_0) \right].$$

Note that, $\lim_{\delta_0 \to 0} I_0^{T} \int_{\mathbb{R}^d} |u(t + \delta_0 r, x) - u(t, x)|^{2} dx dt \to 0$ almost surely for all $r \in [0, 1]$. Therefore, by bounded convergence theorem, $\lim_{\delta_0 \to 0} E \left[ \int_{r=0}^{1} \int_{\mathbb{R}^d} |u(t + \delta_0 r, x) - u(t, x)|^{2} \rho(-r) dx dt dr \right] = 0$, and therefore the first step follows.

**Step 2:** In this step we establish the second part of the lemma. Note that $F^3_k(s)$ has at most polynomial growth in $s \in \mathbb{R}$. It follows from direct computation that there exists $p \in \mathbb{N}$ such that for all $u, v \in \mathbb{R}$ and $\beta = \beta_\epsilon$,

$$|F^3_k(v, u) - \text{sign}(u - v)((F_k(u) - F_k(v))| \leq \epsilon C_p (1 + |u|^{p} + |v|^{p}). \quad (3.7)$$
Lemma 3.4. It holds that

Therefore

\[ \left| -E \left[ \int_{\mathbb{R}^d} \int_{\Pi_T} F^\beta_k(v(s, y), u(s, x)) \nabla_y \psi(s, y) g_\delta(x - y) \, dy \, dx \, ds \right] \right| \]
\[ + \int_{\mathbb{R}^d} \int_{\Pi_T} \frac{d}{k=1} \sum \text{sign}(u(s, x) - v(s, y))(F_k(u(s, x)) - F_k(v(s, y))) \partial_{y_k} \psi(s, y) g_\delta(x - y) \, dy \, dx \, ds \right] \right| \]
\[ \leq E \left[ \int_{\mathbb{R}^d} \int_{\Pi_T} \left( F^\beta_k(v(s, y), u(s, x)) - \text{sign}(u(s, x) - v(s, y))(F_k(u(s, x)) - F_k(v(s, y))) \right) \right. \]
\[ \times \left| \partial_{y_k} \psi(s, y) \right| g_\delta(x - y) \, dy \, dx \, ds \right] \]
\[ \leq \text{Const}(\psi) \varepsilon \left[ 1 + \sup_{0 \leq s \leq T} E \|v(s)\|_p^p + \sup_{0 \leq s \leq T} E \|u(s)\|_p^p \right] \]
\[ \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (3.8) \]

Since \( \psi \) is smooth test function and \( F_k \)'s are smooth and have polynomially growing derivatives, it is easy to verify that \( F(u, v) = \text{sign}(u - v)(F(u) - F(v)) \) is locally Lipschitz and

\[ |F(u, v) - F(\bar{u}, v)| \leq C|u - \bar{u}|(1 + |u|^p + |\bar{u}|^p). \]

Therefore, we can employ dominated convergence theorem and conclude

\[ \left| E \left[ \int_{\mathbb{R}^d} \int_{\Pi_T} \frac{d}{k=1} \sum \text{sign}(u(s, x) - v(s, y))(F_k(u(s, x)) - F_k(v(s, y))) \partial_{y_k} \psi(s, y) g_\delta(x - y) \, dx \, dy \, ds \right] \right| \]
\[ - E \left[ \int_{\Pi_T} \frac{d}{k=1} \sum \text{sign}(u(s, y) - v(s, y))(F_k(u(s, y)) - F_k(v(s, y))) \partial_{y_k} \psi(s, y) \, dy \, ds \right] \right| = O(\delta). \]

Therefore

\[ \left| E \left[ \int_{\mathbb{R}^d} \int_{\Pi_T} F^\beta_k(v(s, y), u(s, x)) \nabla_y \psi(s, y) g_\delta(x - y) \, dx \, dy \, ds \right] \right| \]
\[ - E \left[ \int_{\Pi_T} \frac{d}{k=1} \sum \text{sign}(u(s, y) - v(s, y))(F_k(u(s, y)) - F_k(v(s, y))) \partial_{y_k} \psi(s, y) \, dy \, ds \right] \right| \]
\[ \leq \text{Const}(\psi) \varepsilon + O(\delta) \rightarrow 0 \text{ as } (\varepsilon, \delta) \rightarrow (0, 0). \]

\[ \square \]

Lemma 3.4. It holds that

\[ \lim_{\varepsilon \downarrow 0, \delta \downarrow 0} \lim_{\delta_0 \downarrow 0} (I_4 + I_5) = 0. \quad (3.9) \]

Proof. We can use the same reasoning as before and pass to the limit \( \delta_0 \downarrow 0 \) and conclude

\[ \lim_{\delta_0 \downarrow 0} (I_4 + I_5) \]
\[ = E \left[ \int_{\mathbb{R}^d} \int_{\Pi_T} \left\{ F^\beta(\nabla_y \psi_\delta(x - y)) \nabla_y g_\delta(x - y) + \nabla_y g_\delta(x - y) \right\} \psi(s, y) \, dx \, dy \, ds \right] \]
\[ = E \left[ \int_{\mathbb{R}^d} \int_{\Pi_T} \left\{ - F^\beta_\delta(\nabla_y \psi_\delta(x - y)) + \nabla_y g_\delta(x - y) \psi(s, y) \right\} \, dx \, dy \, ds \right]. \]

Note that, there exists \( p \in \mathbb{N} \), such that for all \( a, b \in \mathbb{R} \)

\[ \left| F^\beta_\delta(a, b) - F^\beta_\delta(b, a) \right| \]
\[ \leq \left| F^\beta_\delta(a, b) - \text{sign}(a - b)(F_\delta(a) - F_\delta(b)) \right| + \left| F^\beta_\delta(b, a) - \text{sign}(b - a)(F_\delta(b) - F_\delta(a)) \right| \]
\[ \leq C\varepsilon(1 + |a|^p + |b|^p). \]
Therefore,
\[
\left| E \left[ \int_{\mathbb{R}^d} \int_{\Pi_T} \left\{ F^2 (u(s,x), v(s,y)) - F^2 (v(s,y), u(s,x)) \right\} \nabla_y g_\delta (x-y) \psi (s,y) \, dx \, dy \, ds \right] \right|
\leq \varepsilon C E \left[ \int_{\mathbb{R}^d} \int_{\Pi_T} (1 + |u(s,x)|^p + |v(s,y)|^p) |\nabla_y g_\delta (x-y)| \psi (s,y) \, dx \, dy \, ds \right] 
\leq \frac{\varepsilon}{\delta} C \quad \text{to} \quad (\varepsilon, \frac{\varepsilon}{\delta}, \delta) \to (0,0,0).
\]

Hence the lemma follows. \( \square \)

**Lemma 3.5.** The following holds:

\[
\lim_{\delta_0 \to 0} I_6 = \frac{1}{2} E \int_{\Pi_T} \int_{\mathbb{R}^d} \sigma^2 (x,u(t,x)) \beta'' (u(t,x) - v(t,y)) \psi (t,y) g_\delta (x-y) \, dx \, dy \, dt \quad (3.10)
\]
and

\[
\lim_{\delta_0 \to 0} I_7 = \frac{1}{2} E \int_{\Pi_T} \int_{\mathbb{R}^d} \sigma^2 (y,v(s,y)) \beta'' (v(s,y) - u(s,x)) \psi (s,y) g_\delta (x-y) \, dx \, dy \, ds. \quad (3.11)
\]

**Proof.** We will rigorously establish (3.10) and (3.11). Note that

\[
I_6 = \frac{1}{2} E \int_{\Pi_T} \int_{\mathbb{R}^d} \sigma^2 (x,u(t,x)) \beta'' (u(t,x) - v(s,y)) \psi (s,y) \rho_{\delta_0} (t-s) g_\delta (x-y) \, dx \, dy \, ds \, dt.
\]

Therefore,

\[
|I_6 - \frac{1}{2} E \int_{\Pi_T} \int_{\mathbb{R}^d} \sigma^2 (x,u(t,x)) \beta'' (u(t,x) - v(t,y)) \psi (t,y) g_\delta (x-y) \, dx \, dy \, dt| 
\leq E \int_{s=\delta_0}^T \int_{t=0}^T \int_{\mathbb{R}^d \times \mathbb{R}^d} \sigma^2 (x,u(t,x)) \beta'' (u(t,x) - v(t,y)) \psi (t,y) \psi (s,y) \rho_{\delta_0} (t-s) g_\delta (x-y) \, dx \, dy \, ds \, dt + O(\delta_0) 
\]

\[
\leq E \int_{s=\delta_0}^T \int_{t=0}^T \int_{\mathbb{R}^d \times \mathbb{R}^d} \sigma (x,u(t,x))^2 \beta'' (u(t,x) - v(t,y)) \psi (t,y) \psi (s,y) \rho_{\delta_0} (t-s) g_\delta (x-y) \, dx \, dy \, ds \, dt + O(\delta_0) 
\]

\[
\leq \left| \beta'' \right| \| \psi \| \| \psi \| E \int_{s=\delta_0}^T \int_{t=0}^T \int_{\mathbb{R}^d \times \mathbb{R}^d} \sigma^2 (x,u(t,x)) |v(s,y) - v(t,y)| \psi (s,y) \rho_{\delta_0} (t-s) g_\delta (x-y) \, dx \, dy \, ds \, dt + O(\delta_0) 
\]

\[
(\text{By Cauchy-Schwartz}) 
\leq \text{Const} (\beta, \eta) \sqrt{E \int_{s=\delta_0}^T \int_{t=0}^T \int_{\mathbb{R}^d \times \mathbb{R}^d} g^2 (x) (1 + |u(t,x)|^4) \psi (s,y) \rho_{\delta_0} (t-s) g_\delta (x-y) \, dx \, dy \, ds \, dt} 
\]

\[
\leq \text{Const} (\beta, \eta, \psi) \sqrt{E \int_{s=\delta_0}^T \int_{t=0}^T \int_{\mathbb{R}^d \times \mathbb{R}^d} |v(s,y) - v(t,y)|^2 \psi (s,y) \rho_{\delta_0} (t-s) g_\delta (x-y) \, dx \, dy \, ds \, dt + O(\delta_0)} 
\]

\[
\leq \text{Const} (\beta, \eta, \psi) \sqrt{E \int_{s=\delta_0}^T \int_{t=0}^T \int_{\mathbb{R}^d \times \mathbb{R}^d} |v(s,y) - v(t,y)|^2 \rho_{\delta_0} (t-s) \, dy \, ds \, dt + O(\delta_0)} 
\]
Proof.

Once again we use the fact that \( \lim_{\delta_0 \to 0} \int_0^T \int_{\mathbb{R}^d} |u(t + \delta_0 r, x) - u(t, x)|^2 \, dt \, dr = 0 \), therefore, by dominated convergence theorem, \( E \int_0^1 \int_{\mathbb{R}^d} |v(t, y) - v(t + r\delta_0, y)|^2 \rho(-r) \, dy \, dt \, dr \to 0 \) as \( \delta_0 \to 0 \).

Additionally

\[
I_7 = \frac{1}{2} E \int_{\Omega_T} \sigma^2(y, v(s, y)) \beta''(v(s, y) - u(t, x)) \psi(s, y) \rho_{\delta_0}(t-s) g_3(x-y) \, dx \, dy \, ds \, dt,
\]
and

\[
|I_7| = \frac{1}{2} E \left| \int_{\Omega_T} \sigma^2(y, v(s, y)) \beta''(v(s, y) - u(s, x)) \psi(s, y) g_3(x-y) \, dx \, dy \, ds \right|
\]

\[
= \frac{1}{2} E \left| \int_0^T \int_{\mathbb{R}^d} \sigma^2(y, v(s, y)) \beta''(v(s, y) - u(s, x)) \psi(s, y) g_3(x-y) \, dx \, dy \, ds \right|
\]

\[
\leq \left| \int_0^T \int_{\mathbb{R}^d} \sigma^2(y, v(s, y)) \beta''(v(s, y) - u(s, x)) \psi(s, y) g_3(x-y) \, dx \, dy \, ds \right|
\]

\[
\leq ||\beta''||_{\infty} E \left( \int_0^T \int_{\mathbb{R}^d} \sigma^2(y, v(s, y)) |u(t, x) - u(s, x)| \psi(s, y) g_3(x-y) \, dx \, dy \, ds \right)
\]

\[
\leq \text{Const}(\beta, \eta) E \int_0^T \int_{\mathbb{R}^d} g^2(y) (1 + |v(s, y)|^2) |u(t, x) - u(s, x)| \psi(s, y) g_3(x-y) \, dx \, dy \, ds \right)
\]

\[
\times \psi(s, y) \rho_{\delta_0}(t-s) g_3(x-y) \, dx \, dy \, ds \, dt + O(\delta_0)
\]

(By Cauchy-Schwartz)

\[
\leq \text{Const}(\beta, \eta) \left( \int_0^T \int_{\mathbb{R}^d} g^4(y) (1 + |v(s, y)|^4) \psi(s, y) g_3(x-y) \, dx \, dy \, ds \, dt \right)
\]

\[
\times \int_0^T \int_{\mathbb{R}^d} |u(t, x) - u(s, x)|^2 \psi(s, y) \rho_{\delta_0}(t-s) g_3(x-y) \, dx \, dy \, ds \, dt + O(\delta_0)
\]

\[
\leq \text{Const}(\beta, \eta, \psi) \left( \int_0^T \int_{\mathbb{R}^d} |u(t, x) - u(s, x)|^2 \rho_{\delta_0}(t-s) \, dx \, ds \, dt \right)
\]

\[
\leq \text{Const}(\beta, \eta, \psi) \left( \int_0^1 \int_0^T \int_{\mathbb{R}^d} |u(t, x) - u(t + r\delta_0, x)|^2 \rho(-r) \, dy \, dt \, dr \right)
\]

As before \( \lim_{\delta_0 \to 0} \int_0^T \int_{\mathbb{R}^d} |u(t + \delta_0 r, x) - u(t, x)|^2 \, dt \, dr = 0 \). Therefore, by dominated convergence theorem, \( E \int_0^1 \int_{\mathbb{R}^d} |u(t, x) - u(t + r\delta_0, x)|^2 \rho(-r) \, dy \, dt \, dr \to 0 \) as \( \delta_0 \to 0 \).

Lemma 3.6. It holds that

\[
\lim_{\delta_0 \to 0} I_8 = - E \int_{\Omega_T} \int_{\mathbb{R}^d} \sigma(x, u(t, x)) \sigma(y, v(t, y)) \beta''(u(t, x) - v(t, y)) \psi(t, y) g_3(x-y) \, dy \, dx \, dt.
\]

Proof. Recall that

\[
I_8 = - E \int_{\Omega_T} \int_{\mathbb{R}^d} \sigma(x, u(t, x)) \sigma(y, v(t, y)) \beta''(u(t, x) - v(t, y)) \psi(s, y) \rho_{\delta_0}(t-s) g_3(x-y) \, dy \, dx \, ds.
\]
Therefore, as before,
\[
\left| I_8 + E \int_{\Pi_T} \int_{\mathbb{R}^d} \sigma(x, u(t, x))\sigma(y, v(t, y))\beta''(u(t, x) - v(t, y))\psi(t, y)g_\delta(x - y) \, dy \, dx \, dt \right|
\leq E \int_{s=0}^{T} \int_{\Pi_T} \int_{\mathbb{R}^d} |\sigma(x, u(t, x))\sigma(y, v(t, y))|\beta''(u(t, x) - v(t, y))|\psi(s, y) - \psi(t, y)| \times \rho_\delta(t - s)g_\delta(x - y) \, dy \, dx \, dt \, ds + O(\delta_0)
\leq \delta_0||\partial_t\psi||_\infty||\beta''||_\infty E \int_{\Pi_T} \int_{\mathbb{R}^d} |\sigma(x, u(t, x))\sigma(y, v(t, y))|g_\delta(x - y) \, dy \, dx \, dt + O(\delta_0)
\leq \delta_0||\partial_t\psi||_\infty||\beta''||_\infty C \left( 1 + \sup_{0 \leq t \leq T} E||u(t, \cdot)||_2^2 + \sup_{0 \leq t \leq T} E||v(t, \cdot)||_2^2 \right) + O(\delta_0).
\]
Hence the lemma follows by simply letting \( \delta_0 \downarrow 0 \) in the last line. \( \square \)

**Lemma 3.7.** Assume that \( \varepsilon \to 0^+ \), \( \delta \to 0^+ \) and \( \varepsilon^{-1}\delta^2 \to 0^+ \), then
\[
\limsup_{\varepsilon \to 0^+, \delta \to 0^+, \varepsilon^{-1}\delta^2 \to 0^+} \lim_{\delta_0 \to 0} \left( I_6 + I_7 + I_8 \right) = 0
\]

**Proof.** Since \( \beta'' \) is even function, we have from Lemma 3.5 and Lemma 3.6 that
\[
\lim_{\delta_0 \to 0} \left( I_6 + I_7 + I_8 \right) = \frac{1}{2} E \left[ \int_{\Pi_T} \int_{\mathbb{R}^d} \left( \sigma(x, u(s, x)) - \sigma(y, v(s, y)) \right)^2 \beta''(u(s, x) - v(s, y)) \psi(s, y) g_\delta(x - y) \, dx \, dy \, ds \right]
\]
Now, by our assumption on \( \sigma \), we have
\[
\left( \sigma(x, u(s, x)) - \sigma(y, v(s, y)) \right)^2 \beta''(u(s, x) - v(s, y)) \leq C \left( |x - y|^2 + |u(s, x) - v(s, y)|^2 \right) \beta''(u(s, x) - v(s, y)) \]
\[
\leq C \left( \varepsilon + \frac{|x - y|^2}{\varepsilon} \right)
\]
Therefore,
\[
E \left[ \int_0^T \int_{\mathbb{R}^d} \left( \sigma(x, u(s, x)) - \sigma(y, v(s, y)) \right)^2 \beta''(u(s, x) - v(s, y)) \psi(s, y) g_\delta(x - y) \, dx \, dy \, ds \right]
\leq C_1 \left( \varepsilon + \varepsilon^{-1}\delta^2 \right) T,
\]
and letting \( \varepsilon \to 0^+ \), \( \delta \to 0^+ \) and \( \varepsilon^{-1}\delta^2 \to 0^+ \) gets us to the desired conclusion. \( \square \)

**Theorem 3.8.** Assume \([A.1] - [A.3]\) Suppose \( u \) is a stochastic entropy solution of \([1.1]\) and \( v \) is a stochastic strong entropy solution of the same equation. Then
\[
E[||(u(t) - v(t))||_1] \leq E[||(u(0) - v(0))||_1],
\]
for almost every \( t > 0 \).

**Proof.** First we pass to the limit \( \delta_0 \downarrow 0 \) in \([3.3] \) and then let \( \delta = \varepsilon^{\frac{1}{2}} \) and finally let \( \varepsilon \downarrow 0 \). We use the Lemmas 3.1, 3.7 along with the preceding inequality \([3.4] \) and obtain
\[
E \left[ \int_{\mathbb{R}^d} |u_0(x) - v_0(x)| \psi(0, x) \, dx \right] + E \left[ \int_{\Pi_T} |v(t, x) - u(t, x)| \partial_x \psi(t, x) \, dt \, dx \right]
+ E \left[ \int_{\Pi_T} F(u(t, x), v(t, x)) \cdot \nabla_x \psi(t, x) \, dt \, dx \right] \geq 0
\]
(3.13)
For each \( n \in \mathbb{N}, \) define
\[
\phi_n(x) = \begin{cases} 
1, & \text{if } |x| \leq n \\
2(1 - \frac{|x|}{2n}), & \text{if } n < |x| \leq 2n \\
0, & \text{if } |x| > 2n.
\end{cases}
\]
For each $h > 0$ and fixed $t \geq 0$, define

$$
\psi_h(s) = \begin{cases} 
1, & \text{if } s \leq t \\
1 - \frac{s-t}{h}, & \text{if } t \leq s \leq t+h \\
0, & \text{if } s > t+h.
\end{cases}
$$

By standard approximation, truncation and mollification argument, \ref{A.13} holds with $\psi(s, x) = \phi_n(x)\psi_n(s)$. Define

$$
A(s) = E\left[ \int_{\mathbb{R}^d} |u(s, x) - v(s, x)| \, dx \right],
$$

then $A \in L^1_{\text{loc}}([0, \infty))$. It is trivial to check that any right Lebesgue point of $A(t)$ is also a right Lebesgue point of

$$
A_n(s) = E\left[ \int_{\mathbb{R}^d} \phi_n(x)|u(s, x) - v(s, x)| \, dx \right]
$$

for all $n$. Let $t$ be a right Lebesgue point of $A$. We choose this $t$ in the definition of $\psi_h(s)$. Thus, from \ref{A.13} we have

$$
\frac{1}{h} \int_{t}^{t+h} E\left[ \int_{\mathbb{R}^d} |v(s, x) - u(s, x)|\phi_n(x) \, dx \right] \, ds 
\leq E\left[ \int_{\Pi_T} F(u(s, x), v(s, x)) \nabla_x \phi_n(x) \psi_h(s) \, ds \, dx \right] 
+ E\left[ \int_{\mathbb{R}^d} |u_0(x) - v_0(x)|\phi_n(x) \, dx \right].
$$

Taking limit as $h \to 0$, we obtain

$$
E\left[ \int_{\mathbb{R}^d} |v(t, x) - u(t, x)|\phi_n(x) \, dx \right] 
\leq E\left[ \int_{\mathbb{R}^d} \int_{0}^{t} F(u(s, x), v(s, x)) \nabla_x \phi_n(x) \, ds \, dx \right] 
+ E\left[ \int_{\mathbb{R}^d} |u_0(x) - v_0(x)|\phi_n(x) \, dx \right] 
\leq C(T) \frac{1}{n} \left[ 1 + \sup_{0 \leq s \leq T} E||u(s)||^p_p + \sup_{0 \leq s \leq T} E||v(s)||^p_p \right] 
+ E\left[ \int_{\mathbb{R}^d} |u_0(x) - v_0(x)|\phi_n(x) \, dx \right].
$$

Letting $n \to \infty$, we have from \ref{A.14}

$$
E||v(t) - u(t)||_1 \leq E||v(0) - u(0)||_1.
$$

\hfill \square

**Theorem 3.9** (comparison principle). Assume \ref{A.1} and \ref{A.3}. Suppose $u$ is a stochastic entropy solution of \ref{1.1} and $v$ is a stochastic strong entropy solution. Then

$$
E||v(t) - u(t)||_1 \leq E||v(0) - u(0)||_1.
$$

Consequently, if $v(0, x) \leq u(0, x)$ a.e in $x$ holds almost surely, and that $E\left[ ||(u(0, .) - v(0, .))_+||_1 \right] < \infty$, then almost surely $v(t, x) \leq u(t, x)$ a.e. in $x$.

**Proof.** The proof follows exactly same as that of Theorem 3.8 if we choose $(\beta_\epsilon(r))_\epsilon$ to be a smooth convex approximation of $r_+ = \max(0, r)$.

**Proof of Theorem 2.1.** It is given that $u$ is a stochastic entropy solution of \ref{1.1} and $v$ is a stochastic strong entropy solution and $r_{p=1, 2, \ldots, L^p(\mathbb{R}^d)}$-valued random variable $u_0$ satisfies

$$
E\left[ ||u_0||^p_p + ||u_0||^p_1 \right] < \infty, \quad p = 1, 2, \ldots.
$$
Therefore by Theorem 3.8 as \( u(0) = v(0) \) almost surely, we have \( u(t) = v(t) \) for almost every \( t \). Hence the uniqueness is proved.

4. Vanishing viscosity and existence of entropy solutions

Here, we briefly outline the mechanism to prove the existence of entropy solutions. Just as its deterministic counterpart, one can employ the vanishing viscosity method for the stochastic scenario to establish the existence. This is indeed the approach adopted by Feng and Nualart [7] and perfectly suits the weak-in-time formulation as well. In [7], the authors offer a rigorous study of the wellposedness question of vanishing viscosity approximation along with a few a priori estimates which allow stochastic compensated compactness to work, leading to the existence of entropy solutions. In this section, we state (without proof) the key statements related to vanishing viscosity approximation that are necessary for existence. However, it would take considerable effort to show that the vanishing viscosity limit satisfies the strong entropy condition.

To this end, let \( J \in C^\infty_0(\mathbb{R}) \) be the one dimensional mollifier and \( \varphi \in C^\infty_0(\mathbb{R}) \) be a cut-off function satisfying

\[
\varphi(r) = \begin{cases} 
0 & \text{for } |r| \geq 2 \\
1 & \text{for } |r| \leq 1.
\end{cases}
\]

For \( \varepsilon > 0 \), define the approximates \( F_\varepsilon \) and \( \sigma_\varepsilon(t, x) \) as

\[
F_\varepsilon(r) = \varphi(|r|^2) F(r) * J_\varepsilon(r)
\]

\[
\sigma_\varepsilon(x, u) = \int \int \left( \prod_{k=1}^d J_\varepsilon(x_k - y_k) J_\varepsilon(u - v) \right) \varphi(|y|^2 + |v|^2) \sigma(y, v) dy dv,
\]

and introduce the viscous perturbation of (1.1):

\[
d\zeta(t, x) + div_x F_\varepsilon(\zeta(t, x)) dt = \sigma_\varepsilon(x, \zeta(t, x)) dW(t) + \varepsilon \Delta_x \zeta(t, x) dt \quad t > 0, \ x \in \mathbb{R}^d,
\]

with the regularized initial condition

\[
\zeta(0, x) = \int \left( \prod_{k=1}^d J_\varepsilon(x_k - y_k) (u_0(y) \varphi(\varepsilon |y|^2)) \right) dy.
\]

As expected, the perturbation (1.1) are uniquely solvable and has smooth solution. We have the following proposition, a proof of which could be found in [7].

**Proposition 4.1.** Let (A.1)-(A.3) hold and \( \varepsilon > 0 \) be a positive number. Then there is a unique \( C^2(\mathbb{R}^d) \)-valued predictable process \( \zeta(t, \cdot) \) which solves the initial value problem (4.1)-(4.2). Moreover,

1. For even positive integer \( p = 2, 4, 6, \ldots \)

\[
\sup_{\varepsilon > 0} \sup_{0 \leq t \leq T} E \left[ ||\zeta(t, \cdot)||^p_p \right] < +\infty
\]

2. For \( \phi \in C^2(\mathbb{R}) \) with \( \phi, \phi', \phi'' \) having at most polynomial growth

\[
\sup_{\varepsilon > 0} E \left[ \varepsilon^{p-1} \int_0^T \int_{\mathbb{R}^d} \phi''(\zeta(t, x)) |\nabla_x \zeta(t, x)|^2 dx dt \right] < \infty, \ p = 1, 2, \ldots, T > 0.
\]

However, the convergence of \( \{\zeta(t, \cdot)\} \) could only be established in a fairly larger space of measure valued processes. We add the following relevant details from [7].

Let \( \mathcal{M}_0 = \mathcal{M}(\mathbb{R}^d \times \mathbb{R}) \) be the space of non-negative Radon measure \( \mu \) on \( \mathbb{R}^d \times \mathbb{R} \) with \( \mu(dx, \mathbb{R}) = dx \). Let \( \tau_0 \) be a topology on \( \mathcal{M}_0 \) such that \( \mu_n \rightarrow \mu \in \mathcal{M}_0 \) if and only if \( (f, \mu_n) \rightarrow (f, \mu) \) for all \( f \in C_b(\mathbb{R}^d \times \mathbb{R}) \) satisfying \( f(x, 0) = 0 \) when \( |x| > K \) for some \( K > 0 \). Denote

\[
\Pi^{\mu_1, \mu_2} = \left\{ \pi \in \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}) : \pi(dx, du; \mathbb{R}^d \times \mathbb{R}) = \mu_1(dx, du) \right\}
\]
\[ \pi(\mathbb{R}^d \times \mathbb{R}; dy, dv) = \mu_2(dy, dv), \quad \mu_1, \mu_2 \in \mathcal{M}_0 \]

and define
\[ r(\mu_1, \mu_2) = \sum_{k=1}^{\infty} \frac{1}{\pi^k} \frac{q_k(\mu_1, \mu_2)}{1 + q_k(\mu_1, \mu_2)}, \]

where
\[ q_k^2(\mu_1, \mu_2) = \inf \left\{ \int_{|x| \leq k, |y| \leq k} (|x - y|^2 + |u - v|^2) \land 1 \right\} \pi(dx, du; dy, dv) : \pi \in \Pi(\mu_1, \mu_2) \}, \]

It can be shown that \((\mathcal{M}_0, r)\) is a complete separable metric space. Denote by \(\mathcal{M}(0, \infty); \mathcal{M}_0)\), the space of Borel-measurable and \(\mathcal{M}_0\)-valued processes on \([0, \infty)\). Define a metric on \(\mathcal{M}(0, \infty); \mathcal{M}_0)\)
\[ d(\mu_1(\cdot), \mu_2(\cdot)) = \int_0^\infty e^{-t} (1 \land r(\mu_1(t), \mu_2(t))) dt. \]

Under this metric, \((\mathcal{M}(0, \infty); \mathcal{M}_0), d)\) is a complete separable metric space. We refer to [10, 11, 12, 13] for more on measure valued processes.

We identify \(u_\varepsilon(t, x)\) as a random measure-valued function
\[ \mu_\varepsilon(t, x, du) = \delta_{u_\varepsilon(t, x)}(du) \quad (4.5) \]
and
\[ \mu_\varepsilon(t) = \mu_\varepsilon(t, dx, du) = \mu_\varepsilon(t, x, du) dx \quad (4.6) \]
and view it as a random measure-valued process in \((\mathcal{M}(0, \infty); \mathcal{M}_0), d)\). To this end, we quote the following results from [7].

**Proposition 4.2.** Let the assumptions \([\text{A.1]} \& [\text{A.3}]\) be true. There exists an \(\{F_t\}\)-predictable measure valued process \(\mu_0(\cdot)\) with trajectories in \((\mathcal{M}(0, \infty); \mathcal{M}_0), d)\) such that
\[ \lim_{\varepsilon \to 0} \mathbb{E}[d(\mu_\varepsilon(\cdot), \mu_0(\cdot))] = 0. \]

Also, by the existence of slicing measures (cf. [7]), \(\mu_0(t) \in \mathcal{M}(0, \infty); \mathcal{M}_0)\) also admits a representation
\[ \mu_0(t, dx, du) = \mu_0(t, x; du) dx. \]

At this point, one needs to employ the stochastic version of compensated compactness method to show that \(\mu_0(t, x; du)\) is supported at a single point. To this end, the following proposition holds. Let
\[ \bar{u}(t, x) = \int_{u \in \mathbb{R}} u \mu_0(t, x; du). \]

**Proposition 4.3.** Let \([\text{A.1]} \& [\text{A.3}]\) hold with \(d = 1\). Then
\[ \int u F(u) \mu_0(t, x, du) = F(\bar{u}(t, x)) dt dx \quad a.s. \quad (4.7) \]
Moreover, if \([\text{A.4}]\) holds, then
\[ \mu_0(t, dx, du) = \delta_{\bar{u}(t, x)}(du) dx. \]

**Proof.** See Lemma 4.15 and Lemma 4.17 of [7].

It is now fairly simple to argue that \(\bar{u}(t, x)\) obeys the entropy inequalities and satisfies the initial condition.

**Lemma 4.4.** The vanishing viscosity limit \(\bar{u}(t, x)\) is an entropy solution of \((1.1)\).
Proof. **Step 1.** We first verify that $\bar{u}(t, x)$ satisfies the initial condition. Let $\psi \in C_c^\infty(\mathbb{R})$ be nonnegative test function, with supp $\psi = K$, and we want to show that

$$\lim_{t \to 0} E\left[ \int_K |\bar{u}(t, x) - u_0(x)| \psi(x) \, dx \right] = 0. \quad (4.8)$$

We may assume, without loss of generality, that $K = [-M, M]$ for some $M > 0$. Note that since $K$ is compact, it suffices to prove that

$$\lim_{t \to 0} E\left[ \int_K |\bar{u}(t, x) - u_0(x)|^2 \psi(x) \, dx \right] = 0. \quad (4.9)$$

We claim that

$$t \mapsto E\left[ \int_K |\bar{u}(t, x) - u_0(x)|^2 \psi(x) \, dx \right] \text{ is right continuous in } t. \quad (4.10)$$

Let

$$A(t) := \int_K |\bar{u}(t, x) - u_0(x)|^2 \psi(x) \, dx = \int_u \int_K |u - u_0(x)|^2 \mu_0(t, x, du) \psi(x) \, dx. \quad (4.11)$$

It holds from [7, Lemma 4.23] that

$$\lim_{t \to s^+} E\left[ r(\mu_0(t), \mu_0(s)) \right] = 0, \quad s \geq 0. \quad (4.12)$$

Therefore, by properties of the metric $r$, we have

$$\lim_{t \to s^+} E(A(t)) - E(A(s)) = \lim_{t \to s^+} E\left[ \int_u \int_K |u - u_0(x)|^2 \mu_0(t, x, du) \psi(x) \, dx - \int_u \int_K |u - u_0(x)|^2 \mu_0(s, x, du) \psi(x) \, dx \right] = 0 \quad (4.13)$$

For $1 > \delta > 0$, let $K_\delta = [-M - \delta, M + \delta]$. Note that, for any $\delta > 0$,

$$E \int_K |\bar{u}(t, x) - u_0(x)|^2 \psi(x) \, dx = E \int_{y \in K_\delta} \int_{x \in K} |\bar{u}(t, x) - u_0(x)|^2 \psi(x) g_\delta(x - y) \, dx \, dy$$

$$\leq 2E \int_{y \in K_\delta} \int_{x \in K} |\bar{u}(t, x) - u_0(y)|^2 \psi(x) g_\delta(x - y) \, dx \, dy$$

$$+ 2E \int_{y \in K_\delta} \int_{x \in K} |u_0(y) - u_0(x)|^2 \psi(x) g_\delta(x - y) \, dx \, dy. \quad (4.14)$$

Moreover, for all most all $t \in [0, \infty)$,

$$E \int_{y \in K_\delta} \int_{x \in K} |\bar{u}(t, x) - u_0(y)|^2 \psi(x) g_\delta(x - y) \, dx \, dy$$

$$= \lim_{\varepsilon \to 0} E \int_{y \in K_\delta} \int_{x \in K} |u_\varepsilon(t, x) - u_0(y)|^2 \psi(x) g_\delta(x - y) \, dx \, dy. \quad (4.15)$$

Now, let $\beta(u) = (u - u_0(y))^2$ and

$$\xi(u) = \int_0^u 2(r - u_0(y)) F'(r) \, dr = 2 \int_0^u r F'(r) \, dr - 2u_0(y)(F(u) - F(0)) \leq C(1 + |u_0(y)|^2 + |u|^p)$$

for a suitable positive integer $p$. We now apply Itô formula on $\beta(u_\varepsilon(t, x)) = (u_\varepsilon(t, x) - u_0(y))^2$ and conclude that

$$E \int_{y \in K_\delta} \int_{x \in K} |u_\varepsilon(t, x) - u_0(y)|^2 \psi(x) g_\delta(x - y) \, dx \, dy$$

$$\leq E \int_{y \in K_\delta} \int_{x \in K} |u_\varepsilon(0, x) - u_0(y)|^2 \psi(x) g_\delta(x - y) \, dx \, dy$$

$$+ C\delta^{-2} \int_0^t E \int_{y \in K_\delta} \int_{x \in K} \left(1 + |u_\varepsilon(r, x)|^p + |u_0(y)|^2 \right) \, dx \, dy$$
Hence, by (4.14), (4.15) and (4.16), along a suitable subsequence

\[ \lim_{\varepsilon \to 0} E \int_{y \in K_s} \int_{x \in K} \left( |u_x(t, x) - u_0(y)|^2 \psi(x) g_0(x - y) \right) dx dy \leq 0. \]  

We now simply invoke (4.10) and conclude this step.

**Step 2.** We now verify that \( \bar{u}(t, x) \) obeys the entropy inequalities.

**Claim:** It holds that

\[ \lim_{\varepsilon \to 0^+} E \left[ \int_0^T \int_u \int_x \sigma_{x}(x, u) \beta' \mu_{x}(r, x, du) \psi(r, x) dx dW(r) - \int_0^T \int_u \int_x \sigma(u) \beta' \mu_{u}(r, x, du) \psi(r, x) dx dW(r) \right]^2 = 0. \]  

**Justification:**

\[
E \left[ \int_0^T \int_u \int_x \sigma_{x}(x, u) \beta' \mu_{x}(r, x, du) \psi(r, x) dx dW(r) - \int_0^T \int_u \int_x \sigma(u) \beta' \mu_{u}(r, x, du) \psi(r, x) dx dW(r) \right]^2 \leq CE \left[ \int_0^T \int_u \int_x \sigma_{x}(x, u) \beta' \mu_{x}(r, x, du) dx \right]^2 + CE \left[ \int_0^T \int_u \int_x \sigma(u) \beta' \mu_{u}(r, x, du) dx \right]^2 \leq \hat{C} \left[ \int_0^T \int_u \int_x \sigma_{x}(x, u) \beta' \mu_{x}(t, x, dx) - \int_0^T \int_u \int_x \sigma(u) \beta' \mu_{u}(t, x, dx) \right] dr, \]

by (4.19).
The second inequality follows from Itô isometry and the last one follows by uniform moment estimates. Note that $E\left[d\left(\mu_{\varepsilon}(\cdot), \mu_{0}(\cdot)\right)\right]$

$= \int_{0}^{\infty} e^{-t} E\left(r\left(\mu_{\varepsilon}(t), \mu_{0}(t)\right)\right) \to 0$ as $\varepsilon \to 0$. Therefore, for almost every $t \in [0, \infty)$,

$E\left(r\left(\mu_{\varepsilon}(t), \mu_{0}(t)\right)\right) \to 0$ as $\varepsilon \to 0$.

As a result, for almost every $t \in [0, \infty)$,

$E\left[\int_{0}^{T} \int_{x} \sigma(x, u)\beta'(u)\mu_{\varepsilon}(t, x, du)dx - \int_{0}^{T} \int_{x} \sigma(x, u)\beta'(u)\mu_{0}(t, x, du)dx\right] \to 0$ as $\varepsilon \downarrow 0$. \hspace{1cm} (4.20)

Moreover, we have

$|\sigma(x, u) - \sigma(x, u)| \leq C\varepsilon(1 + |x| + |u|)$. \hspace{1cm} (4.21)

We now apply bounded convergence theorem along with the uniform moment estimates and use (4.20) and (4.21) in (4.19) to have the claim.

Note that each of the $u_{\varepsilon}$ satisfies the inequality (2.2). We now simply pass to the limit $\varepsilon \downarrow 0$ in probability and arrive at the entropy inequality for $\bar{u}$.

\[\Box\]

5. Existence of Strong Entropy Solution

In this section we establish that the vanishing viscosity limit $v(t, x) = \bar{u}(t, x)$ is indeed a strong entropy solution. To this end, let $\bar{u}(t) = \bar{u}(t, x)$ be an $F_{t}$-predictable and $L^{2}(\mathbb{R})$-valued process with

$$\sup_{0 \leq t \leq T} E\left[\|\bar{u}(t)\|_{p}^{p}\right] < \infty, \text{ for all } T > 0, p = 2, 4, ... \hspace{1cm} (5.1)$$

Furthermore, let $\beta$ be a smooth convex function approximating the absolute value in $\mathbb{R}$ and $\psi \in C_{c}^{\infty}(\{0, \infty\} \times \mathbb{R})$ be a nonnegative test function. For constants $\delta > 0$, $\delta_{0} > 0$, define

$$\phi_{\delta, \delta_{0}}(t, x, s, y) = \rho_{\delta_{0}}(t - s)\rho_{\delta}(x - y)\psi(s, y).$$

**Lemma 5.1.** For each $T > 0$, there exists a deterministic function $A(\delta, \delta_{0})$ such that

$$E\left[\int_{0}^{T} \int_{x} \sigma(x, \bar{u}(r, x))\beta'(\bar{u}(r, x) - v)\phi_{\delta, \delta_{0}}(r, x, s, y) dx dW(r)\right] \leq \int_{0}^{T} \int_{x} \sigma(x, \bar{u}(r, x))\sigma(y, v(r, y))\beta''(\bar{u}(r, x) - v(r, y)) dx dW(r) \times \phi_{\delta, \delta_{0}}(r, x, s, y) dr dy ds + A(\delta, \delta_{0}).$$

Furthermore, for fixed $\delta$, $\psi$ and $\beta$, the function $A(\delta, \delta_{0})$ has the property that

$$\lim_{\delta_{0} \to 0} A(\delta, \delta_{0}) = 0.$$

A significant part of the proof is built on ideas borrowed from [7], and the proof requires some preparation. Given a nonnegative test function $\phi \in C_{c}^{\infty}(\Pi_{\infty} \times \Pi_{\infty})$ and $\beta \in C^{\infty}(\mathbb{R})$ such that $\beta^{\prime}, \beta^{\prime\prime} \in C_{b}(\mathbb{R})$, define

$$J[\beta, \phi](s; y, v) := \int_{0}^{T} \int_{x} \sigma(x, \bar{u}(r, x))\beta'(\bar{u}(r, x) - v)\phi(r, x, s, y)dx dW(r)$$

where $0 \leq s \leq T$, $(y, v) \in \mathbb{R} \times \mathbb{R}$.

Since the test function $\psi$ has compact support, there exists $c_{\phi} > 0$ such that $J[\beta, \phi](s; y, v) = 0$ if $y > c_{\phi}$ and $0 \leq s \leq T$.

**Lemma 5.2.** The following identities hold:

$$\partial_{v}J[\beta, \phi](s; y, v) = J[-\beta^{\prime}, \phi](s; y, v)$$

$$\partial_{y}J[\beta, \phi](s; y, v) = J[\beta, \partial_{y}\phi](s; y, v).$$
Proof. The proof is similar to that of Leibniz integral rule.

**Lemma 5.3.** Let \( \beta \in C^\infty(\mathbb{R}) \) be a function such that \( \beta', \beta'' \in C^\infty_c(\mathbb{R}) \). Then there exists a constant \( C = C(\beta', \psi) \) such that

\[
\sup_{0 \leq t \leq T} \left( E\left[ |J[\beta, \phi_{\delta, \delta_0}](s; \cdot, \cdot)|^2 \right] \right)^{1/2} \leq \frac{C(\beta', \psi)}{\delta_0^3}. \tag{5.2}
\]

**Proof.** We intend to establish (5.2) with the help of appropriate Sobolev embedding theorem. To this end, we begin with

\[
E \left[ |J[\beta, \phi_{\delta, \delta_0}](s; \cdot, \cdot)|^4 \right] = E \left[ \int_y \int_x |J[\beta, \phi_{\delta, \delta_0}](s; y, v)|^4 \, dv \, dy \right]
\]

\[
= E \left[ \int_y \int_x \int_0^T \int_0^T \sigma(x, \tilde{u}(r, x)) \beta(\tilde{u}(r, x) - v) \rho_{\delta_0}(r - s) \rho_{\delta_0}(x - y) \psi(s, y) \, dx \, dW(r) \right]^4 \, dv \, dy
\]

(By BDG inequality)

\[
\leq C \int_y \int_x E \left[ \int_0^T \int_x \sigma(x, \tilde{u}(r, x)) \beta(\tilde{u}(r, x) - v) \rho_{\delta_0}(r - s) \rho_{\delta_0}(x - y) \psi(s, y) \, dx \, dW(r) \right]^4 \, dv \, dy
\]

\[
\leq C \int_y \int_x E \left[ \int_0^T \int_x \sigma^2(x, \tilde{u}(r, x)) \beta^2(\tilde{u}(r, x) - v) \rho_{\delta_0}(r - s) \rho_{\delta_0}(x - y) \psi^2(s, y) \, dx \, dW(r) \right] \, dv \, dy
\]

\[
\leq C \int_y \int_x E \left[ \int_0^T \int_x \sigma^4(x, \tilde{u}(r, x)) \beta^4(\tilde{u}(r, x) - v) \rho_{\delta_0}^4(r - s) \rho_{\delta_0}(x - y) \psi^4(s, y) \, dx \, dW(r) \right] \, dv \, dy
\]

\[
\leq C E \left[ \int_{|y| < C \delta_0} \int_0^T \int_x \int_{|x| \leq C \beta + |\tilde{u}(r, x)|} \sigma^4(x, \tilde{u}(r, x)) \beta^4(\tilde{u}(r, x) - v) \rho_{\delta_0}^4(r - s) \rho_{\delta_0}(x - y) \psi^4(s, y) \, dx \, dW(r) \right] \, dv \, dy
\]

\[
\leq C(\beta, \psi) E \left[ \int_0^T \int_x \left( g^4(x) (1 + |\tilde{u}(r, x)|^4) (C_{\beta} + (1 + |\tilde{u}(r, x)|)) \rho_{\delta_0}^4(r - s) \, dx \, dr \right]
\]

\[
\leq C(\beta, \psi) E \left[ \int_0^T \int_x \left( g^4(x) (1 + |\tilde{u}(r, x)|^4) \rho_{\delta_0}^4(r - s) \, dx \, dr \right]
\]

\[
= C(\beta, \psi) \int_0^T \left( 1 + \sup_{0 \leq r \leq T} E[|\tilde{u}(r, \cdot)|^2] \right) \rho_{\delta_0}^4(r - s) \, dr
\]

\[
\leq C(\beta, \psi)(1 + \sup_{0 \leq r \leq T} E[|\tilde{u}(r, \cdot)|^2]) \rho_{\delta_0}^4(r - s) \, dr
\]

\[
\leq C(\beta, \psi)(1 + \sup_{0 \leq r \leq T} E[|\tilde{u}(r, \cdot)|^2]) \rho_{\delta_0}^4(r - s) \, dr
\]

\[
\leq C(\beta, \psi)(1 + \sup_{0 \leq r \leq T} E[|\tilde{u}(r, \cdot)|^2]) \rho_{\delta_0}^4(r - s) \, dr
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\[
\leq C(\beta, \psi)(1 + \sup_{0 \leq r \leq T} E[|\tilde{u}(r, \cdot)|^2]) \rho_{\delta_0}^4(r - s) \, dr
\]

\[
\leq C(\beta, \psi)(1 + \sup_{0 \leq r \leq T} E[|\tilde{u}(r, \cdot)|^2]) \rho_{\delta_0}^4(r - s) \, dr
\]

\[
\leq C(\beta, \psi)(1 + \sup_{0 \leq r \leq T} E[|\tilde{u}(r, \cdot)|^2]) \rho_{\delta_0}^4(r - s) \, dr
\]

(5.3)

Similarly, we have

\[
E \left[ |\partial_x J[\beta, \phi_{\delta, \delta_0}](s; \cdot, \cdot)|^4 \right] \leq \frac{C(\beta'', \psi)}{\delta_0^3}. \tag{5.4}
\]

\[
E \left[ |\partial_y J[\beta, \phi_{\delta, \delta_0}](s; \cdot, \cdot)|^4 \right] \leq \frac{C(\beta', \partial_y \psi)}{\delta_0^3}. \tag{5.5}
\]

Therefore, by (5.3), (5.4), and (5.5),

\[
E \left[ |J[\beta, \phi_{\delta, \delta_0}](s; \cdot, \cdot)|^4 \right] \leq \frac{C(\beta', \psi)}{\delta_0^3}. \tag{5.5}
\]
We simply now use Sobolev embedding along with Cauchy-Schwartz inequality and conclude
\[
\sup_{0 \leq s \leq T} \left( E \left[ \left| J[\beta', \phi_{s,\delta_0}](s; \cdot, \cdot) \right|^2 \right] \right) \leq \frac{C(\beta', \psi)}{\delta_0^2} \tag{5.6}
\]

Our primary aim is to estimate the expected value of \( J[\beta', \phi_{s,\delta_0}](s; y, v(s, y)) \), which we do by estimating the same for \( J[\beta', \phi_{s,\delta_0}](s; y, u_c(s, y)) \) and then passing to the limit. Note that if we directly substitute \( v = v(s, y) \) in the formula for \( J[\beta', \phi_{s,\delta_0}] \), the integrand would no-longer be nonanticipative, and therefore standard methods Itô integrals would no-longer apply. To work around this problem, we proceed as follows.

Let \( \{ \rho_l \}_{l \geq 0} \) be the standard sequence of mollifiers in \( \mathbb{R} \) and define
\[
Z_{\varepsilon, \delta, \delta_0, l} := \int_{\mathbb{R}} \int_{\Pi_T} J[\beta', \phi_{s,\delta_0}](s; y, v) \rho_l(u_c(s, y) - v) \, dv \, dy \, ds. \tag{5.7}
\]
We would like to find an upper bound on \( E[ Z_{\varepsilon, \delta, \delta_0, l} ] \) as \( l, \varepsilon \to 0 \). To this end, we claim that for two constants \( T_1, T_2 \geq 0 \) with \( T_1 < T_2 \),
\[
E \left[ X_{T_1} \int_{T_1}^{T_2} J(t) \, dW(t) \right] = 0 \tag{5.8}
\]
where \( J \) is a predictable integrand and \( X(\cdot) \) is an adapted process. The conclusion follows trivially if \( J \) is a simple predictable integrand. The general case could be argued by standard approximation technique.

If necessary, we extend the process \( u_c(\cdot, y) \) for negative time simply by \( u_c(s, y) = u_c(0, y) \) if \( s < 0 \). With this convention, it follows from (5.8) that
\[
E \left[ \int_{\mathbb{R}} \int_{\Pi_T} J[\beta', \phi_{s,\delta_0}](s; y, v) \rho_l(u_c(s, y) - v) \, dv \, dy \, ds \right] = 0.
\]
Hence
\[
E[ Z_{\varepsilon, \delta, \delta_0, l} ] = E \left[ \int_{\mathbb{R}} \int_{\Pi_T} J[\beta', \phi_{s,\delta_0}](s; y, v) \left( \rho_l(u_c(s, y) - v) - \rho_l(u_c(s - \delta_0, y) - v) \right) \, dv \, dy \, ds \right]. \tag{5.9}
\]
Given \( y \in \mathbb{R} \), \( u_c(\cdot, y) \) satisfies
\[
du_c(s, y) = -\text{div} F_c(u_c(s, y)) \, ds + \varepsilon \Delta u_c(s, y) \, ds + \sigma_c(y, u_c(s, y)) \, dW(s).
\]
Next, apply Itô-formula and obtain
\[
\rho_l(u_c(s, y) - v) - \rho_l(u_c(s - \delta_0, y) - v) \\
= \int_{s-\delta_0}^s \rho'_{l}(u_c(\tau, y) - v) \left( -\text{div} F_c(u_c(\tau, y)) + \varepsilon \Delta u_c(\tau, y) \right) \, d\tau \\
+ \int_{s-\delta_0}^s \sigma_c(y, u_c(\tau, y)) \rho_{l}(u_c(\tau, y) - v) \, dW(\tau) + \frac{1}{2} \int_{s-\delta_0}^s \sigma_c^2(y, u_c(\tau, y)) \rho'_{l}(u_c(\tau, y) - v) \, d\tau \\
= -\frac{\partial}{\partial v} \left[ \int_{s-\delta_0}^s \rho_l(u_c(\tau, y) - v) \left( -\text{div} F_c(u_c(\tau, y)) + \varepsilon \Delta u_c(\tau, y) \right) \, d\tau \right. \\
+ \int_{s-\delta_0}^s \sigma_c(y, u_c(\tau, y)) \rho_{l}(u_c(\tau, y) - v) \, dW(\tau) \left. + \frac{1}{2} \int_{s-\delta_0}^s \sigma_c^2(y, u_c(\tau, y)) \rho'_{l}(u_c(\tau, y) - v) \, d\tau \right].
\]
From (5.9), we now have
\[
E[ Z_{\varepsilon, \delta, \delta_0, l} ] \\
= E \left[ \int_{\mathbb{R}} \int_{\Pi_T} J[\beta', \phi_{s,\delta_0}](s; y, v) \left\{ -\frac{\partial}{\partial v} \left( \int_{s-\delta_0}^s \rho_l(u_c(\tau, y) - v) \left( -\text{div} F_c(u_c(\tau, y)) \right) \\
+ \varepsilon \Delta u_c(\tau, y) \right) \, d\tau \right. \\
+ \int_{s-\delta_0}^s \sigma_c(y, u_c(\tau, y)) \rho_{l}(u_c(\tau, y) - v) \, dW(\tau) \left. + \frac{1}{2} \int_{s-\delta_0}^s \sigma_c^2(y, u_c(\tau, y)) \rho'_{l}(u_c(\tau, y) - v) \, d\tau \right].
\]
It is straightforward to check that there is a positive integer

where

Moreover, we can argue as in Lemma 5.3 and find a constant $C = C(\beta, \psi)$ such that

\[
\sup_{\varepsilon > 0} \sup_{0 \leq s \leq T} \left( E \left[ \| X_\varepsilon [\partial_y \phi_{\delta, \delta_0}] (s; \cdot, \cdot) \|^2_{L^2(\mathbb{R} \times \mathbb{R})} \right] \right) \leq \frac{C(\beta, \psi)}{\delta_0^2},
\]
Claim:
\[ A_1^*(\delta, \delta_0) = -E \left[ \int_{\Pi_T} \int_{s-\delta_0}^s X_e [\partial_y \phi_{\delta, \delta_0}](s; y, u_e(\tau, y)) \, d\tau \, ds \right] \quad (5.14) \]

Proof of the claim: We repeatedly use integration by parts and have
\[
\int v \int_{\Pi_T} \int_{s-\delta_0}^s J[\beta''(\phi_{\delta, \delta_0})](s; y, u_e(\tau, y)) \left( \int_{s-\delta_0}^s \rho(y, \phi_{\delta, \delta_0}) \, d\tau \right) \, ds \, dy \, dv \\
= \int v \int_{\Pi_T} \int_{s-\delta_0}^s X_e [\phi_{\delta, \delta_0}](s; y, u_e(\tau, y)) \, d\tau \, ds \, dy \, dv \\
= - \int v \int_{\Pi_T} \int_{s-\delta_0}^s \partial_y X_e [\phi_{\delta, \delta_0}](s; y, u_e(\tau, y)) \, d\tau \, ds \, dy \, dv \\
= - \int v \int_{\Pi_T} \int_{s-\delta_0}^s X_e [\partial_y \phi_{\delta, \delta_0}](s; y, u_e(\tau, y)) \, d\tau \, ds \, dy \\
\] (5.15)
We simply let \( t \to 0 \) in both sides of (5.15) and obtain
\[
\int_{\Pi_T} \int_{s-\delta_0}^s J[\beta''(\phi_{\delta, \delta_0})](s; y, u_e(\tau, y)) \partial_y F_e(u_e(\tau, y)) \, d\tau \, ds \, dy \\
= - \int_{\Pi_T} \int_{s-\delta_0}^s X_e [\partial_y \phi_{\delta, \delta_0}](s; y, u_e(\tau, y)) \, d\tau \, ds \, dy \\
\] (5.16)
We take expectation in both sides of (5.16) and the claim follows.

Now
\[
\lim_{\varepsilon \to 0} \left| A_1^*(\delta, \delta_0) \right| = \lim_{\varepsilon \to 0} \left| E \left[ \int_{\Pi_T} \int_{s-\delta_0}^s X_e [\partial_y \phi_{\delta, \delta_0}](s; y, u_e(\tau, y)) \, d\tau \, ds \right] \right| \\
\leq C\delta_0 \sup_{0 \leq s \leq T} \sup_{\epsilon > 0} E \left[ \|X_e [\partial_y \phi_{\delta, \delta_0}](s; \cdot, \cdot)\|_\infty \right] \\
\leq C\delta_0 \sup_{0 \leq s \leq T} \sup_{\epsilon > 0} \left( E \left[ \|X_e [\partial_y \phi_{\delta, \delta_0}](s; \cdot, \cdot)\|_\infty^2 \right] \right)^{\frac{1}{2}} \\
\leq C\delta_0 \frac{C(\beta, \phi)}{\delta_0^{\frac{1}{2}}} \\
\leq C_1(\beta, \phi)\delta_0^{\frac{1}{2}}. \\
\]
In other words, \( A_1(\delta, \delta_0) \leq C_1(\beta, \phi)\delta_0^{\frac{1}{2}} \), and therefore
\[ A_1(\delta, \delta_0) \to 0 \quad \text{as} \quad \delta_0 \to 0. \]

Next, we define
\[ A_2(\delta, \delta_0) := \lim_{\varepsilon \to 0} \left| A_2^*(\delta, \delta_0) \right| \quad \text{where} \quad A_2^*(\delta, \delta_0) := \lim_{i \to 0} A_2^i(\delta, \delta_0) \quad (5.17) \]

Lemma 5.5. It holds that
\[ A_2(\delta, \delta_0) \to 0 \quad \text{as} \quad \delta_0 \to 0. \] (5.18)
Proof. From the definition of $A_2^\varepsilon(\delta, \delta_0)$, it follows that

$$A_2^\varepsilon(\delta, \delta_0) := \lim_{\varepsilon \to 0} A_2^{\varepsilon}(\delta, \delta_0)$$

$$= - E \left[ \int_{\Pi_T} \int_{s-\delta_0}^s \int_x^T \varepsilon \sigma(x, \tilde{u}(r, x)) \Delta y \beta'(\tilde{u}(r, x) - u_x(r, y)) \rho_\delta(r - s) \psi(s, y) \, d\tau \, ds \, dy \right].$$

Hence

$$A_2^\varepsilon(\delta, \delta_0)$$

$$= E \left[ \int_{\Pi_T} \int_{s-\delta_0}^s \int_x^T \varepsilon \sigma(x, \tilde{u}(r, x)) \Delta y \beta'(\tilde{u}(r, x) - u_x(r, y)) \rho_\delta(r - s) \psi(s, y) \, d\tau \, ds \, dy \right]$$

$$- E \left[ \int_{\Pi_T} \int_{s-\delta_0}^s \int_x^T \varepsilon \sigma(x, \tilde{u}(r, x)) \Delta y \beta'(\tilde{u}(r, x) - u_x(r, y)) \rho_\delta(r - s) \psi(s, y) \, d\tau \, ds \, dy \right]$$

$$= E \left[ \int_{\Pi_T} \int_{s-\delta_0}^s \int_x^T \varepsilon \sigma(x, \tilde{u}(r, x)) \Delta y \beta'(\tilde{u}(r, x) - u_x(r, y)) \rho_\delta(r - s) \psi(s, y) \, d\tau \, ds \, dy \right]$$

$$- E \left[ \int_{\Pi_T} \int_{s-\delta_0}^s \int_x^T \varepsilon \sigma(x, \tilde{u}(r, x)) \Delta y \beta'(\tilde{u}(r, x) - u_x(r, y)) \rho_\delta(r - s) \psi(s, y) \, d\tau \, ds \, dy \right]$$

$$\equiv I_1^\varepsilon + I_2^\varepsilon.$$

Now, we use the uniform moment estimates and conclude that

$$\lim_{\varepsilon \to 0} \sup \{ I_1^\varepsilon \} = \lim_{\varepsilon \to 0} \sup \{ I_1^\varepsilon \} = 0. \quad (5.19)$$

Thus

$$\lim_{\varepsilon \to 0} \sup \{ A_2^\varepsilon(\delta, \delta_0) \} \leq \lim_{\varepsilon \to 0} \sup \{ I_2^\varepsilon \}, \quad (5.20)$$

and need of the hour is to estimate $I_2^\varepsilon$. Define

$$M_{s-\delta_0}[\beta''', \psi, \delta](y, v) = \int_s^{t-s} \int_x^T \sigma(x, \tilde{u}(r, x)) \beta'''(\tilde{u}(r, x) - v) \psi(s, y) \, d\tau \, ds \, dy,$$

where $t \geq s - \delta_0$. We now invoke Itô-product rule and obtain

$$J[\beta''', \phi_\delta, \beta](s; y, v) = - \int_{s-\delta_0}^{s} \rho_\delta(t - s) M_{s-\delta_0}[\beta''', \psi, \delta](y, v) \, dt.$$

Therefore

$$||J[\beta''', \phi_\delta, \beta](s; \cdot, \cdot)||_{L^\infty(\mathbb{R} \times \mathbb{R})} \leq \frac{1}{\delta_0} \sup_{s-\delta_0 \leq t \leq s} ||M_{s-\delta_0}[\beta''', \psi, \delta](\cdot, \cdot)||_{L^\infty(\mathbb{R} \times \mathbb{R})}.$$

In other words

$$E \left[ \sup_{0 \leq t \leq T} ||J[\beta''', \phi_\delta, \beta](s; \cdot, \cdot)||_{L^\infty(\mathbb{R} \times \mathbb{R})} \right]$$

$$\leq \frac{1}{\delta_0} E \left[ \sup_{0 \leq t \leq T, s-\delta_0 \leq t \leq s} ||N_t[\beta''', \psi, \delta](\cdot, \cdot) - N_{s-\delta_0}[\beta''', \psi, \delta](\cdot, \cdot)||_{L^\infty(\mathbb{R} \times \mathbb{R})} \right] \quad (5.21)$$

where

$$N_t[\beta''', \psi, \delta](y, v) = \int_0^t \int_x^T \sigma(x, \tilde{u}(r, x)) \beta'''(\tilde{u}(r, x) - v) \psi(s, y) \, d\tau \, ds \, dy.$$
By a certain modulus of continuity estimate [7] Lemma 4.28, P 359 for paths of $N_t$, we have
\[
E \left[ \sup_{s,t \in [0,T]} \| N_t[\beta''', \psi, \delta](\cdot) - N_s[\beta''', \psi, \delta](\cdot) \|_{\mathcal{P}}^2 \right] \leq C\delta_0^2 \tag{5.22}
\]
for some $a > 0$ and $p > 8$. We combine (5.22) and (5.21) to have
\[
E \left[ \sup_{0 \leq s \leq T} \| J[\beta''', \phi_{\delta, \delta_0}](s; \cdot, \cdot) \|_{\mathcal{P}_\infty(\mathbb{R} \times \mathbb{R})}^2 \right] \leq C \frac{1}{\delta_0} \delta_0^a \tag{5.23}
\]
for some $a > 0$ and $p > 8$.

Next, we define
\[
A_\varepsilon(t) = \int_0^t \varepsilon \| \nabla u_\varepsilon(r) \|_{L^2}^2.
\]
From the moment estimate in Proposition [4.1] we have
\[
\sup_{\varepsilon > 0} E \left[ |A_\varepsilon(T)|^p \right] < \infty, \quad \text{for } p = 1, 2, \cdots, T > 0. \tag{5.24}
\]
Finally, we now focus on $I_2^\varepsilon$ and have
\[
|I_2^\varepsilon| \leq E \left[ \int_0^T \int_{|y| < C_\varepsilon} \int_{s-\delta_0}^{s+\delta_0} \sup_{0 \leq \tau \leq T} \| J[\beta''', \phi_{\delta, \delta_0}](s; \cdot, \cdot) \|_{\mathcal{P}} \varepsilon \| \nabla u_\varepsilon(\tau, y) \|_{L^2}^2 \, ds \, dy \right] \tag{5.25}
\]
(By Fubini theorem)
\[
= \delta_0 E \left[ \sup_{0 \leq s \leq T} \| J[\beta''', \phi_{\delta, \delta_0}](s; \cdot, \cdot) \|_{\mathcal{P}} \int_{|y| < C_\varepsilon} \int_{T=0}^T \left( \int_{s=\tau}^{s+\delta_0} \varepsilon \| \nabla u_\varepsilon(\tau, y) \|_{L^2}^2 \, d\tau \right) \, ds \, dy \right]
\]
\[
\leq \delta_0 E \left[ \sup_{0 \leq s \leq T} \| J[\beta''', \phi_{\delta, \delta_0}](s; \cdot, \cdot) \|_{\mathcal{P}} \int_{|y| < C_\varepsilon} \int_{T=0}^T \varepsilon \| \nabla u_\varepsilon(\tau, y) \|_{L^2}^2 \, d\tau \, ds \right]
\]
(By Hölder with $p > 8$)
\[
\leq \delta_0^a \left( E \sup_{0 \leq s \leq T} \| J[\beta''', \phi_{\delta, \delta_0}](s; \cdot, \cdot) \|_{\mathcal{P}}^p \right)^{\frac{a}{p}} \left( E[|A_\varepsilon(T)|^q] \right)^{\frac{1}{q}} \tag{5.26}
\]
(By (5.22) and (5.24))
\[
\leq C\delta_0^{\tilde{a}}, \tag{5.25}
\]
for some $\tilde{a} > 0$. In other words, there exists $\tilde{a} > 0$ such that
\[
\lim_{\varepsilon \downarrow 0} \sup |I_2^\varepsilon| \leq C(\beta, \psi)\delta_0^{\tilde{a}}
\]
and hence
\[
A_2(\delta, \delta_0) \to 0 \quad \text{as } \delta_0 \to 0. \tag{5.26}
\]

Finally, we define
\[
A_3(\delta, \delta_0) = \lim_{\varepsilon \downarrow 0} \sup_{\tau \to 0} |A_3^{\varepsilon}(\delta, \delta_0)| \tag{5.27}
\]

**Lemma 5.6.** It holds that
\[
A_3(\delta, \delta_0) \to 0 \quad \text{as } \delta_0 \to 0. \tag{5.27}
\]

**Proof.** By integration by parts, we have
\[
A_3^{\varepsilon}(\delta, \delta_0)
= \frac{1}{2} E \left[ \int_\mathbb{R} \int_{B_T} J[\beta''', \phi_{\delta, \delta_0}](s; y, v) \left\{ \int_{s-\delta_0}^{s} \sigma^2(y, u_\varepsilon(\tau, y)) \rho(u_\varepsilon(\tau, y) - v) \, d\tau \right\} \, dv \, dy \, ds \right].
\]
Therefore
\[
|A_3^{\varepsilon}(\delta, \delta_0)| \leq E \left[ \int_\mathbb{R} \int_{B_T} \int_{|y| < C_\varepsilon} \int_{s-\delta_0}^{s} \| J[\beta''', \phi_{\delta, \delta_0}](s; \cdot, \cdot) \|_{\mathcal{P}} \sigma^2(y, u_\varepsilon(\tau, y)) \right].
\]
Proof.

Since \( \epsilon \) the limit \( \lim_{\epsilon \to 0} \), we can use dominated convergence theorem and conclude

\[
\begin{align*}
\lim_{\epsilon \to 0} \sup_{0 \leq t \leq T} E[|u_\epsilon(t, \cdot)|^4_{L^4}] &\leq C(\beta, \psi) \delta_0^4 T \left[ 1 + \sup_{\epsilon > 0} \sup_{0 \leq t \leq T} E[|u_\epsilon(t, \cdot)|^4_{L^4}] \right]^{\frac{1}{2}}.
\end{align*}
\]

(5.28)

Thus

\[
\lim_{\epsilon \to 0} \lim_{t \to 0} \left| A^{\epsilon}_{1} (\delta, \delta_0) \right| \leq C(\beta, \psi, T) \delta_0^4
\]

and hence \( A_3(\delta, \delta_0) \) has the desired property. \( \square \)

**Lemma 5.7.** It holds that

\[
\lim_{\epsilon \downarrow 0} \lim_{t \downarrow 0} B^{\epsilon, t}(\delta, \delta_0) = -E \left[ \int_{\mathbb{P}_T} \int_{\Pi_T} \sigma(x, \bar{u}(r, x)) \sigma(y, v(r, y)) \beta''(r, \bar{u}(r, x) - v(r, y)) \right. \\
\left. \times \phi_{\delta, \delta_0}(r, x, s, y) \ dr \ dx \ dy \ ds \right]
\]

(5.29)

**Proof.** Since \( ||\beta''(\cdot)||_\infty < \infty \), we can use dominated convergence theorem and conclude

\[
\begin{align*}
\lim_{t \to 0} B^{\epsilon, t}(\delta, \delta_0) &= -E \left[ \int_{\mathbb{P}_T} \int_{\mathbb{P}_T} \beta''(\bar{u}(r, x) - u_\epsilon(r, y)) \sigma(x, \bar{u}(r, x)) \phi_{\delta, \delta_0}(r, x, s, y) \ dr \ dy \ dr \ ds \right] \\
&= -E \left[ \int_{\mathbb{P}_T} \int_{\mathbb{P}_T} \beta''(\bar{u}(r, x) - u_\epsilon(r, y)) \sigma(x, \bar{u}(r, x)) \phi_{\delta, \delta_0}(r, x, s, y) \ dr \ dy \ dr \ ds \right]
\end{align*}
\]

(5.30)

We use the uniform integrability conditions along with approximation properties of \( \sigma_\epsilon \) and pass to the limit \( \epsilon \downarrow 0 \) to obtain

\[
\lim_{\epsilon \downarrow 0} \lim_{t \downarrow 0} B^{\epsilon, t}(\delta, \delta_0) = -E \left[ \int_{\mathbb{P}_T} \int_{\mathbb{P}_T} \sigma(x, \bar{u}(r, x)) \sigma(y, v(r, y)) \beta''(r, \bar{u}(r, x) - v(r, y)) \right. \\
\left. \times \phi_{\delta, \delta_0}(r, x, s, y) \ dr \ dx \ dy \ ds \right]
\]

\( \square \)

**Proof of Lemma 5.1.** We now simply choose \( A(\delta, \delta_0) = A_1(\delta, \delta_0) + A_2(\delta, \delta_0) + A_3(\delta, \delta_0) \). Note that, in view of (5.7),

\[
\begin{align*}
E \left[ \int_{\mathbb{P}_T} \int_{\mathbb{P}_T} \int_{\mathbb{P}_T} \int_{\mathbb{P}_T} \sigma(x, \bar{u}(r, x)) \beta''(\bar{u}(r, x) - v) \phi_{\delta, \delta_0}(r, x, s, y) \ dr \ dx \ dy \ dr \ ds \right]_{v=v(s, y)} &\leq \lim_{\epsilon \downarrow 0} \lim_{t \downarrow 0} |A^{\epsilon, t}_1(\delta, \delta_0)| + \lim_{\epsilon \downarrow 0} \lim_{t \downarrow 0} |A^{\epsilon, t}_2(\delta, \delta_0)| + \lim_{\epsilon \downarrow 0} \lim_{t \downarrow 0} |A^{\epsilon, t}_3(\delta, \delta_0)| + \lim_{\epsilon \downarrow 0} \lim_{t \downarrow 0} B^{\epsilon, t}(\delta, \delta_0)
\end{align*}
\]
\[ A(\delta, \delta_0) - E\left[ \int_{\Omega_T} \int_{\Omega_T} \sigma(x, \bar{u}(r, x))\sigma(y, v(r, y)) \beta''(\bar{u}(r, x) - v(r, y)) \right. \]
\[ \times \phi_{\delta, \delta_0}(r, x, s, y) \, dr \, dx \, dy \, ds \]

where we have used Lemma 5.7. Furthermore, by Lemmas 5.4-5.6 the function \( A(\delta, \delta_0) \) has the desired property as \( \delta_0 \to 0. \]

We have seen from Lemma (4.1) that \( v(t, x) = \bar{u}(t, x) \) is a stochastic entropy solution. Moreover, we conclude from Lemma 5.1 that \( \bar{u}(t, x) \) is indeed a stochastic strong entropy solution of (1.1)-(1.2), which completes the proof of Theorem 2.2.

6. A critique on the strong-in-time formulation

In this final section, we will contest the suitability of strong-in-time formulation of [7] and try to make a case for weak-in-time formulation. However, the issues that we are going to raise are purely technical in nature and do not at any way disturb the broader message of [7]. We could not have emphasized more on the fact the article [7] is no less than a milestone in the area.

For any \( L^p \)-valued solution process \( u(\cdot, x) \) with continuous sample paths, it is easy to see that the strong-in-time and weak-in-time formulations are equivalent to each other. Furthermore, if it is not established that the solution process has continuous paths then weak-in-time formulation is certainly a more appropriate way to move forward. Just as in the deterministic case, the authors use vanishing viscosity method for existence in [7] and attempts have been made in [7] to justify that the vanishing viscosity limit has continuous sample paths when treated as a \( \mathcal{M}_0 \)-valued process. To be more precise, it is shown in [7] Lemma 4.23, P 355 that

\[ \lim_{t \to s} E\left[ r(\mu_0(t), \mu_0(s)) \right] = 0, \quad (6.1) \]

and a claim has been made that (6.1) implies that \( \mu_0(\cdot) \) has continuous sample paths as \( \mathcal{M}_0 \)-valued process. While we perfectly agree with the derivation of (6.1) in the proof of [7] Lemma 4.23, P 355, but strongly disagree with the claim that \( \mu_0(\cdot) \) has continuous sample paths because of (6.1).

In fact, we make a counter claim that an estimate of type (6.1) may not imply path continuity. To see this, let \( N_t \) be the usual Poisson process with parameter \( \lambda > 0 \). Then

\[ \lim_{t \to s} E\left[ d\sigma(N_t, N_s) \right] = \lim_{t \to s} E\left[ [N_t - N_s] \right] = \lim_{t \to s} \lambda|t - s| = 0, \quad (6.2) \]

but \( N_t \) clearly does not have continuous sample paths. Therefore, \( \mu_0(\cdot) \) cannot be claimed to have continuous sample paths on the basis of (6.1) alone.

This invalidates the claim in [7] Lemma 4.22, P 355 that \( \mu_0(t) \) has trajectories in \( C([0, \infty), \mathcal{M}_0) \), and puts a question mark next the entropy inequality [7] (74), P 355. To elaborate on this point, let us look at the proof [7] Lemma 4.15, P 343 where it is shown that

\[ \lim_{t \to 0} E[d\mu_e(\cdot), \mu_0(\cdot)] = \lim_{t \to 0} \int_0^\infty e^{-t} E\left[ \min \left\{ 1, r(\mu_e(t), \mu_0(t)) \right\} \right] dt = 0. \quad (6.3) \]

Clearly, (6.3) only implies that \( \lim_{t \to 0} E[r(\mu_e(t), \mu_0(t))] = 0 \) for almost every \( t \geq 0 \), contrary to the claim in [7] Lemma 4.15, P 343 that \( \lim_{t \to 0} E[r(\mu_e(t), \mu_0(t))] = 0 \) for every \( t \geq 0 \). This jeopardizes the claim that

\[ \lim_{t \to 0} \left( \mu_e(t_1), \ldots, \mu_e(t_m) \right) = \left( \mu_0(t_1), \ldots, \mu_0(t_m) \right) \quad \text{in probability} \]

for each \( 0 \leq t_1 \leq \cdots \leq t_m \). We object to the wording 'for each 0 \leq t_1 \leq \cdots \leq t_m'. In our view, the correct wording should be '0 \leq t_1 \leq \cdots \leq t_m where t_i's are chosen from a set of full Lebesgue measure in [0, \infty)'. Hence, one would only be allowed to pass to the limit in \( \varepsilon \) in [7] (73), P 354 for almost every \((t, s) \in [0, \infty) \times [0, \infty),\) and [7] (74), P 355 would be valid only for almost every \((t, s) \in [0, \infty) \times [0, \infty).\) However, this would mean that [7] (74), P 355 holds for all \( s, t \in [0, \infty) \) if \( \mu_0(\cdot) \) has trajectories in \( C([0, \infty), \mathcal{M}_0) \), but that is not shown to be the case.
Therefore, it is fair to say that the vanishing viscosity limit does not have sufficiently clear pointwise picture in time for its paths, and it is worthwhile go for the weak-in-time entropy formulation for (1.1).

REFERENCES

[1] G. Q. Chen, Q. Ding, and Kenneth H. Karlsen. On nonlinear stochastic balance laws. Arch. Ration. Mech. Anal. 204 (2012), no. 3, 707-743.

[2] C. M. Dafermos. Hyperbolic conservation laws in continuum physics, volume 325 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2000.

[3] A. Debussche and J. Vovelle. Scalar conservation laws with stochastic forcing. J. Funct. Analysis, 259 (2010), 1014-1042.

[4] Z. Dong and T. G. Xu. One-dimensional stochastic Burgers equation driven by Lévy processes. J. Funct. Anal., 249(2):631–678, 2007.

[5] Lawrence C. Evans. Weak convergence methods for nonlinear partial differential equations, volume 74 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1990.

[6] Lawrence C. Evans. Partial differential equations, volume 19 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, second edition, 2010.

[7] Jin Feng and David Nualart. Stochastic scalar conservation laws. J. Funct. Anal., 255(2):313–373, 2008.

[8] Edwige Godlewski and Pierre-Arnaud Raviart. Hyperbolic systems of conservation laws, volume 3/4 of Mathématiques & Applications (Paris) [Mathematics and Applications]. Ellipses, Paris, 1991.

[9] H. Holden and N. H. Risebro. Conservation laws with random source. Appl. Math. Optim, 36(1997), 229-241.

[10] J. Horowitz. Measure-valued random processes. Probability Theory and Related Fields, 70(1985), pp 213-236.

[11] J. Horowitz. Gaussian random measures Stochastic Process. Appl., 22(1986), pp 129-133.

[12] O. Kallenberg. Lp-intensities of random measures. Stochastic Process. Appl., 9 (1979), no. 2, 155-161.

[13] Olav Kallenberg. Foundations of modern probability. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

[14] J. U. Kim. On a stochastic scalar conservation law, Indiana Univ. Math. J. 52 (1) (2003) 227-256.

[15] J. Málek, J. Nečas, M. Rokyta, and M. Růžička. Weak and measure-valued solutions to evolutionary PDEs, volume 13 of Applied Mathematics and Mathematical Computation. Chapman & Hall, London, 1996.

[16] Michel Mémin. Semimartingales, volume 2 of de Gruyter Studies in Mathematics. Walter de Gruyter & Co., Berlin, 1982. A course on stochastic processes.

[17] K. R. Parthasarathy. Probability measures on metric spaces. Probability and Mathematical Statistics, No. 3. Academic Press Inc., New York, 1967.

[18] Philip Protter. Stochastic Integration and Differential Equations. Springer-Verlag, Berlin, 1990.

[19] Khanin, Sinai. Invariant measures for Burgers equation with random forcing Annals of Math (2). 151 (2000), no. 3, 877-906.

[20] Daniel W. Stroock and S. R. Srinivasa Varadhan. Multidimensional diffusion processes, volume 233 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1979.

[21] G. Vallet and P. Wittbold. On a stochastic first-order hyperbolic equation in a bounded domain. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 12 (2009), no. 4, 613-651.

[22] John B. Walsh. An introduction to stochastic partial differential equations. In École d’été de probabilités de Saint-Flour, XIV—1984, volume 1180 of Lecture Notes in Math., pages 265–439. Springer, Berlin, 1986.