LARGE STANDARD EXAMPLES IN POSETS WITH HIGH
DIMENSION AND FRACTIONAL DIMENSION

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ABSTRACT. Biró, Füredi, and Jahanbekam conjectured that if the dimension of a poset is just slightly less than half the number of points, it must contain large standard examples. In this paper we disprove this conjecture by explicitly constructing counterexamples using finite geometry. On the positive side, we prove that a similar statement holds for fractional dimension in bipartite posets.

1. Introduction

1.1. Dimension and standard examples. Let $P$ be a poset. A set of its linear extensions $\{L_1, \ldots, L_d\}$ forms a realizer, if $L_1 \cap \cdots \cap L_d = P$. The minimum cardinality of a realizer is called the dimension of the poset $P$, denoted by $\dim(P)$. This concept is also sometimes called the order dimension or the Dushnik-Miller dimension of the partial order, see [7].

The dimension is “monotone” (removal of a point can never increase the dimension), and “continuous” (removal of a point can never decrease the dimension by more than 1). The simplest examples of large dimensional posets are the so-called standard examples $S_n$. These are $2n$ element posets formed by considering the 1-element subsets, and the $(n-1)$-element subsets of a set of $n$ elements, ordered by inclusion. It is known that the dimension of $S_n$ is half of the number of elements of $S_n$, that is, $\dim(S_n) = n$. It is also known that there are posets of arbitrarily large dimension without an $S_3$ subposet [4]. For further study of the dimension of posets we refer the readers to the monograph [13], and to the survey article [14].

Let $P$ and $Q$ be two posets. $Q$ is a subposet of $P$, if there is an induced copy of $Q$ in $P$. More precisely, $Q$ is a subposet of $P$ if there is a bijection $f$ from a subset of the ground set of $P$ to the ground set of $Q$ with the property that $f(x) < f(y)$ if and only if $x < y$. If $P$ does not contain $Q$ as a subposet, $P$ is called $Q$-free. Note that some authors use the term “subposet” in the “non-induced” sense. We conform to the more standard usage, and we say “an extension of $Q$ is a subposet of $P$” if we ever need the non-induced meaning.

Hiraguchi [9] proved that any poset with $|P| \geq 4$ satisfy $\dim(P) \leq |P|/2$. ($|P|$ denotes the number of points of the ground set.) Bogart and Trotter [5] showed that if $|P| \geq 8$, then the only extremal examples of Hiraguchi’s theorem are the standard examples $S_n$ with $\dim(S_n) = |S_n|/2$. A natural question arises: if the dimension is just a bit less than the maximum, can we expect largely the same structure as in the extremal case?

At least in one situation this seems to be the case. Trotter and Kimble both claimed that if $|P| = 2n+1$ for some $n \geq 4$, then $P$ must contain $S_n$. However,
Kimble’s proof [10] appears to be incomplete, and Trotter never published his proof (due to the extreme tediousness of the proof).

Biró, Füredi, and Jahanbekam [1] conjectured the following.

**Conjecture 1.1.** For every \( t < 1 \), but sufficiently close to 1 there is a \( c > 0 \) and \( N \) positive integer, so that if \(|P| = 2n \geq N\), and \( \dim(P) \geq tn \), then \( P \) contains \( S_{\lfloor cn \rfloor} \).

In a previous paper [2] we proved a weaker version of this conjecture. We showed the following.

**Theorem 1.2.** Let \( k \geq 2 \) a fixed integer. Define \( D(n, k) = \max\{\dim(P) : |P| = n \text{ and } P \text{ is } S_k \text{-free}\} \). Then \( D(n, k) = o(n) \).

**Corollary 1.3.** For all \( k \geq 3 \) integer, and \( t < 1 \) there is an \( N \) integer, so that if \(|P| = 2n \geq N\), and \( \dim(P) \geq tn \), then \( P \) contains \( S_k \).

In this paper we show that Conjecture 1.1 is not true. We will construct explicit counterexamples with dimension arbitrarily close to the maximum but not containing any linear sized standard examples.

In the second part of the paper, we will prove a positive result of this nature.

A few more definitions are needed. The **height** of a poset is the size of a maximum chain. Height two posets are called bipartite. A **bipartition** of a bipartite poset \( P \) is a partition \((A, B)\) of its ground set such that \( x < y \) in \( P \) implies \( x \in A \) and \( y \in B \).

The **upset** of the element \( x \) is the set \( U(x) = \{y \in P : y > x\} \), and the **closed upset** is \( U[x] = U(x) \cup \{x\} \). Similarly, the **downset** of the element \( x \) is \( D(x) = \{y \in P : y < x\} \), and the **closed downset** is \( D[x] = D(x) \cup \{x\} \). The symbol \( x \parallel y \) is used to denote that \( x \) is incomparable with \( y \) in the poset.

A **critical pair** \((x, y)\) is an ordered pair of elements of \( P \) with the properties \( x \parallel y \), \( D(x) \subseteq D(y) \), and \( U(y) \subseteq U(x) \).

A linear extension \( L \) **reverses** the critical pair \((x, y)\), if \( y < x \) in \( L \). A set of linear extensions \( L \) **reverses** \((x, y)\) if there is a linear extension in \( L \) that reverses \((x, y)\). It is known [11] that a set of linear extensions is a realizer if and only if it reverses every critical pair of \( P \). This makes the investigation simpler in bipartite posets, where all incomparable minimum–maximum pairs are critical, and they are the only (interesting) critical pairs; the remaining critical pairs have a very special structure and can be usually handled in a simple way.

**1.2. Fractional dimension.** Determining the dimension of a poset can be regarded as a linear integer programming problem. Let \( P \) be a poset and \( \{L_1, \ldots, L_\ell\} \) the set of its linear extensions and \( \{(a_1, b_1), \ldots, (a_c, b_c)\} \) the set of its critical pairs.

Let \( A = [a_{ij}] \) be a \( c \times \ell \) binary matrix, where \( a_{ij} = 1 \) iff \((a_i, b_i)\) is reversed in \( L_j \). The optimal solution of the following integer program gives the dimension of \( P \).

\[
\begin{align*}
Ax & \geq 1 \\
x & \geq 0 \\
x & \in \mathbb{Z}^\ell \\
\min 1^T x
\end{align*}
\]

In 1992 Brightwell and Scheinerman [6] introduced the notion of fractional dimension of posets as the optimal solution of the linear relaxation of this integer program. They used the notation \( \text{fdim}(P) \) for fractional dimension, but to keep the
notation consistent with fractional graph theory, we will use \( \dim^*(P) \). A feasible solution of the linear program will be called an \( f \)-realizer.

If we consider the \( \ell \)-dimensional vector space generated by the abstract basis \( L_1, \ldots, L_\ell \), then an \( f \)-realizer is a linear combination \( \sum \alpha_i L_i \) with \( 0 \leq \alpha_i \leq 1 \). In fact, for a linear combination \( \sum \alpha_i L_i \), we will say that it reverses the critical pair \( (x, y) \) \( \alpha \) times, if \( \sum_{i : y < x} \alpha_i = \alpha \). If a critical pair is reversed at least once (1 times), we will simply say it is reversed. This way, a linear combination is an \( f \)-realizer if and only if it reverses every critical pair. The weight of an \( f \)-realizer is \( \sum_{i=1}^\ell \alpha_i \), and the fractional dimension of \( P \) is the minimum weight of an \( f \)-realizer.

Clearly, for all posets \( P \), \( \dim^*(P) \leq \dim(P) \). As shown by Brightwell and Scheinerman [6], the ordinary dimension and the fractional dimension can be arbitrarily far apart. Nevertheless, there exist posets with arbitrarily large fractional dimension; in particular, \( \dim^*(S_n) = n \).

In Section 3 we prove the following result.

**Theorem 1.4.** Let \( P \) be a bipartite poset with \( |P| = 2n \). If \( \dim^*(P) \geq n - c \) then \( P \) contains \( S_{\lfloor n - 13(c + 2) \rfloor} \).

**Corollary 1.5.** For every \( t \leq 1 \), if \(|P| = 2n \geq 8\), and \( \dim^*(P) \geq tn \), then \( P \) contains \( S_{\lfloor 39t - 38 \rfloor n} \).

**Proof.** By Theorem 1.4 \( P \) contains \( S_k \) with \( k \geq n - 13(n - tn + 2) \). If \( n - tn < 1 \) then \( \dim(P) \geq \dim^*(P) \) implies \( \dim(P) = n \), and so \( P = S_n \). Otherwise \( k \geq n - 39(n - tn) = n(39t - 38) \). \( \square \)

2. **Counterexample to Conjecture 1.1**

Let \( K \) be a finite field of order \( q \). Recall the construction of the finite projective plane (called field plane) \( \text{PG}(2, K) = \text{PG}(2, q) \), whose point set consists of lines in \( K^3 \) through the origin, and whose line set consists of planes in \( K^3 \) through the origin, with a point incident to a line if the corresponding line lies in the corresponding plane. \( \text{PG}(2, q) \) consists of \( q^2 + q + 1 \) points (and lines).

The following theorem is implicit from a result that is claimed to be folklore in [3] (Theorem 25). For completeness we outline the proof.

**Theorem 2.1.** Let \( \Pi \) be a set of points and \( \Lambda \) be a set of lines in \( \text{PG}(2, q) \) so that there is no incidence between them. Then \( |\Pi| \cdot |\Lambda| \leq q^3 \).

**Proof.** Let \( b = |\Pi| \). Let \( l_i \) be the number of lines of \( \text{PG}(2, q) \) with \( i \) points of \( \Pi \) on them. Then

\[
\sum_{i=0}^{q+1} l_i = q^2 + q + 1 - l_0,
\]

and

\[
\sum_{i=1}^{q+1} il_i = b(q + 1) \tag{1}
\]

Count the ordered pairs of distinct points by the lines they determine to get

\[
\sum_{i=1}^{q+1} i(i - 1)l_i = b(b - 1).
\]
From these,
\[ \sum_{i=1}^{q+1} i^2 l_i = b(b + q). \]
Squaring (1) and using the inequality of arithmetic and geometric means,
\[ b^2(q + 1)^2 \leq \left( \sum_{i=1}^{q+1} i^2 l_i \right) \left( \sum_{i=1}^{q+1} l_i \right) = b(b + q)(q^2 + q + 1 - l_0). \]
Hence
\[ bq = b(q + 1)^2 - b(q^2 + q + 1) \leq q(q^2 + q + 1) - bl_0 - ql_0, \]
so
\[ bl_0 \leq q(q^2 + q + 1 - b - l_0), \]
therefore
\[ bl_0 \leq q^3, \]
as desired, since \(|A| \leq l_0\).

The following lemma describes the counterexample.

**Lemma 2.2.** Let \( q \) be a prime power. Let \( A \) be the set of points of \( PG(2, q) \) and \( B \) be the set of lines of \( PG(2, q) \). Let \( P \) be the bipartite poset with bipartition \( (A, B) \) such that for \( a \in A \), \( b \in B \), \( a < b \) if and only if \( a \) is not on the line \( b \). Then
i) If \( P \) has an \( S_{2k} \) subposet then \( k \leq q^{3/2} \).
ii) \( \dim(P) > q^2 - q^{3/2} \).

**Proof.** Suppose that \( P \) has an \( S_{2k} \) subposet. Let the vertices of this be \( a_1, \ldots, a_{2k}, b_1, \ldots, b_{2k} \) with \( a_i < b_j \) for \( i \neq j \). Let \( \Pi = \{a_1, a_3, \ldots, a_{2k-1}\} \) and \( \Lambda = \{b_2, b_4, \ldots, b_{2k}\} \). Since for all \( a_i \in \Pi, b_j \in \Lambda, a_i < b_j \), we have that there is no incidence between \( \Pi \) and \( \Lambda \). Therefore, by Theorem 2.1 \( k^2 \leq q^3 \), and so \( k \leq q^{3/2} \).

For the second part, suppose that \( \dim(P) = d \), and consider a realizer \( \{L_1, \ldots, L_d\} \). For a given linear extension \( L_i \), let \( a_i \) be the element of \( A \) for which \( x > a_i \) in \( L_i \) implies \( x \in B \). Less formally, \( a_i \) is the highest element of \( A \) in \( L_i \). Similarly, let \( b_i \in B \) such that \( y < b_i \) implies \( y \in A \). Let \( A' = A \setminus \{a_1, \ldots, a_d\} \), and \( B' = B \setminus \{b_1, \ldots, b_d\} \).

We claim that if \( (x, y) \) is a critical pair, then \( x \notin A' \) or \( y \notin B' \). Suppose for a contradiction that \( x \in A' \) and \( y \in B' \). There is a linear extension \( L_i \) in the realizer for which \( x > y \) in \( L_i \). On the other hand, \( a_i > x \) and \( b_i < y \) in \( L_i \), so \( b_i \neq x \) and \( a_i \neq y \) in \( P \). In other words, \( \{x, y, a_i, b_i\} \) is an antichain. But this yields two distinct lines \( b_i \) and \( y \) passing through the same pair of points \( x \) and \( a_i \).

We obtained that for all \( a \in A' \) and \( b \in B' \), \( a < b \). Hence by Theorem 2.1, \( (q^2 + q + 1 - d)^2 \leq q^3 \), from which simple calculation shows \( d > q^2 - q^{3/2} \).

**Theorem 2.3.** For all \( \varepsilon > 0 \) there exists a (bipartite) poset \( P \) with \( |P| = 2n \), \( \dim(P) > (1 - \varepsilon)n \), and no \( S_{\lceil \varepsilon n \rceil} \) subposet.

**Proof.** Let \( \varepsilon > 0 \). Choose a \( q \) large enough prime power such that \( 2q^{3/2} < \varepsilon(q^2 + q + 1) \) and \( q^2 - q^{3/2} > (1 - \varepsilon)(q^2 + q + 1) \). Construct the poset \( P \) as in Lemma 2.2; part i) yields that there is no large standard example, and part ii) yields that the dimension is large, as required in the statement of this theorem.
After the discovery of this counterexample Trotter [12] pointed out that it is possible to show the existence of a counterexample in a probabilistic (non-constructive) way. This example uses random bipartite posets defined on \( n \) minimal and \( n \) maximal points, adding comparability between them with probability \( p \), independently at random.

**Theorem 2.4 ([8]).** For all \( \varepsilon > 0 \) there exist \( c > 0 \) so that if \( 1/\log n \leq p < 1 - n^{-1+\varepsilon} \), then \( \dim(P) > n - cn/(p \log n) \) for almost all \( P \).

Fix \( \varepsilon > 0 \), and choose \( p = \varepsilon \). If \( n \) is large enough, the condition on \( p \) is satisfied.

Let \( c \) be the constant in the theorem, and let \( c' = c/p \), still independent from \( n \). Then for almost all \( P \), \( \dim(P) > n(1 - c'/\log n) > (1 - \varepsilon)n \) for large \( n \). On the other hand, a standard calculation shows (using e.g. Chernoff bounds) that the maximum degree of the comparability graph of \( P \) is at most \( \varepsilon n + o(n) \) for almost all \( P \).

Note that our counterexample is dense: only \( O(n) \) comparabilities are missing from the poset, but this probabilistic example produces a sparse poset with only \( O(n) \) comparabilities.

Also note that choosing points and lines in \( \text{PG}(2, q) \) at random does not yield a counterexample; it can be shown that the dimension of such an example is likely to be low.

3. LARGE STANDARD EXAMPLES IN POSETS WITH HIGH FRACTIONAL DIMENSION

In this section we prove Theorem 1.4. To avoid some technical difficulties, we define a bipartite version of fractional dimension.

**Definition 3.1.** Let \( P \) be a bipartite poset with bipartition \( (A, B) \). Consider a linear combination of its linear extensions \( \mathcal{L} \). We say \( \mathcal{L} \) is a bipartite f-realizer if it reverses (at least once) each critical pair of the form \( (a, b) \), where \( a \in A \) and \( b \in B \). The minimum weight of a bipartite f-realizer is the fractional bipartite dimension of \( P \), denoted by \( \text{bdim}^*(P) \).

**Proposition 3.2.** For a bipartite poset \( P \)

\[
\dim^*(P) - 2 \leq \text{bdim}^*(P) \leq \dim^*(P)
\]

**Proof.** The second inequality is a direct consequence of the definition. To see the first inequality, consider a bipartite f-realizer, and add two linear extensions, in which order the minimal and maximal elements of \( P \) the opposite of each other. \( \square \)

**Proposition 3.3.** Let \( P \) be a bipartite poset, and \( x \in P \). Then \( \text{bdim}^*(P - x) \geq \text{bdim}^*(P) - 1 \).

**Proof.** The proof is the same as of the corresponding proposition for regular dimension and fractional dimension. \( \square \)

Let \( P \) be a bipartite poset with bipartition \( (A, B) \). Define \( G(P) \) as the bipartite graph with partite sets \( A \) and \( B \), and \( u \sim v \) if and only if \( u \in A, v \in B, \text{and } u \parallel v \) in \( P \).

Let \( a \in A \) and \( b \in B \) such that \( a \parallel b \). We will use the notation \( L_{ab} \) to denote a linear extension of \( P \) in which

\[
D(b) < b < P - (\{D[b] \cup U[a]\}) < a < U(a).
\]
Let $M = \{(a_1,b_1),\ldots,(a_m,b_m)\}$ be a matching in $G(P)$. Let $L_M = \sum_{i=1}^{m} L_{a_i,b_i}$ denote a linear combination (with all 1 coefficients).

**Lemma 3.4.** Let $P$ be a bipartite poset and let $M$ be a maximal matching in $G(P)$. Then $L_M$ is a bipartite $f$-realizer of $P$, so $\text{bdim}^*(P) \leq |M|$. Furthermore, all critical pairs $(a,b)$ such that $a$ and $b$ are incident with distinct edges of $M$ are reversed twice.

**Proof.** The maximality of $M$ implies that for every critical pair $(a,b)$, at least one of $a$ or $b$ is incident with an edge $a,b_i$ in $M$, and thereby reversed in the linear extension $L_{a,b_i}$. If $a$ and $b$ are incident with two distinct edges of $M$, there are two such linear extensions.

**□**

**Lemma 3.5.** Let $P$ be a bipartite poset with bipartition $(A,B)$, where $|A| = |B| = n$, and $G(P)$ has a perfect matching. If $\text{bdim}^*(P) \geq n - c$, then $P$ contains $S_k$ with $k \geq n - 6c$.

**Proof.** Let $M = \{(a_1,b_1),\ldots,(a_n,b_n)\}$ be a perfect matching in $G(P)$. Let $M'$ be a maximal matching in $G(P) - M$. First we will show that $\text{bdim}^*(P) \leq n - |M'|$ by constructing a bipartite $f$-realizer of the correct weight. For $i = 1,2,\ldots,n$, let

$$\alpha_i = \begin{cases} 
1 & \text{if neither } a_i \text{ nor } b_i \text{ is incident with } M', \\
2/3 & \text{if exactly one of } a_i \text{ and } b_i \text{ is incident with } M', \\
1/3 & \text{if both } a_i \text{ and } b_i \text{ are incident with } M'. 
\end{cases}$$

Let $L = \frac{1}{3}L_{M'} + \sum_{i=1}^{n} \alpha_i L_{a_i,b_i}$. The weight of $L$ is $\frac{1}{3}|M'| + \sum_{i=1}^{n} \alpha_i = \frac{1}{3}|M'| + n - \frac{|M'|}{3}$. We will show that $L$ is a bipartite $f$-realizer.

Consider a critical pair of the form $(a_i, b_i)$. If neither $a_i$ nor $b_i$ is incident with $M'$, then the pair is reversed 1 times in the term $\alpha_i L_{a_i,b_i}$. If exactly one of $a_i$ and $b_i$ is incident with $M'$, then the pair is reversed 2/3 times in $\alpha_i L_{a_i,b_i}$ and 1/3 times in the term $\frac{1}{3}L_{M'}$. If both $a_i$ and $b_i$ are incident with $M'$, then the pair reversed 1/3 times in $\alpha_i L_{a_i,b_i}$, and 2/3 times in $\frac{1}{3}L_{M'}$, because the pair is reversed twice in $M'$.

Now consider a critical pair of the form $(a_i, b_j)$ with $i \neq j$. The pair is reversed $\alpha_i$ times in $\alpha_i L_{a_i,b_i}$, $\alpha_j$ times in $\alpha_j L_{a_j,b_j}$, and at least 1/3 times in $\frac{1}{3}L_{M'}$, and $\alpha_i + \alpha_j + 1/3 \geq 1$.

We have shown $\text{bdim}^*(P) \leq n - \frac{|M'|}{3}$, and since $\text{bdim}^*(P) \geq n - c$, we have $|M'| \leq 3c$. So there are at least $n - 6c$ indices $i$ such that $M'$ is not incident with either $a_i$ or $b_i$. Those indices will determine a standard example of size at least $n - 6c$.

**□**

**Lemma 3.6.** Let $P$ be a bipartite poset with $|P| = 2n$. If $\text{bdim}^*(P) \geq n - c$ then $P$ contains $S_{|n-13c|}$.

**Proof.** Let $M$ be a maximal matching in $G(P)$, and suppose that $|M| = m$. By Lemma 3.4, $m = |M| \geq \text{bdim}^*(P) \geq n - c$. Let $P'$ be the poset induced by the vertices incident with $M$. Since we got $P'$ by deleting $2n - 2m$ points form $P$, by Proposition 3.3 $\text{bdim}^*(P') \geq n - c - (2n - 2m) = 2m - n - c$, we may use Lemma 3.5 to conclude that $P'$ (and therefore $P$) has an $S_k$ with

$$k \geq m - 6(m - (2m - n - c)) = 7m - 6n - 6c \geq 7(n - c) - 6n - 6c = n - 13c.$$ 

**□**
Finally, notice that Theorem 1.4 is a straightforward consequence of Lemma 3.6. Indeed, if $\text{dim}^*(P) \geq n - c$, then $\text{bdim}^*(P) \geq n - (c + 2)$, and the theorem follows.

The question remains open whether the statement of Theorem 1.4 is correct for all posets as opposed to just bipartite posets.

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