Origin of chaos in soft interactions and signatures of non-ergodicity

M. W. Beims\textsuperscript{1,2}, C. Manchein\textsuperscript{1} and J. M. Rost\textsuperscript{1,2}

\textsuperscript{1}Departamento de Física, Universidade Federal do Paraná, 81531-990 Curitiba, PR, Brazil
\textsuperscript{2}Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Strasse 38, D-01187 Dresden, Germany

The emergence of chaotic motion is discussed for hard, point-like, and soft collisions between two particles in a one-dimensional box. It is known that ergodicity may be obtained in point-like collisions for specific mass ratios $\gamma = m_2/m_1$ of the two particles, and that Lyapunov exponents are zero. However, if a Yukawa interaction between the particles is introduced, we show analytically that positive Lyapunov exponents are generated due to double collisions close to the walls. While the largest finite-time Lyapunov exponent changes smoothly with $\gamma$, the number of occurrences of the most probable one, extracted from the distribution of finite-time Lyapunov exponents over initial conditions, reveals details about the phase space dynamics. In particular the influence of the integrable and pseudointegrable dynamics without Yukawa interaction for specific mass ratios can be clearly identified and demonstrates the sensitivity of the finite-time Lyapunov exponents as a phase space probe. Being not restricted to two-dimensional problems such as Poincaré sections, the number of occurrences of the most probable Lyapunov exponents suggest itself as a suitable tool to characterize phase space dynamics in higher dimensions. This is shown for the problem of two interacting particles in a circular billiard.

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I. INTRODUCTION

The investigation of the origin of chaotic motion in standard billiard models (like the Sinai billiard \textsuperscript{1}, the Bunimovich stadium \textsuperscript{2} or the Annular billiard \textsuperscript{3}), has played a pioneering role since the very beginning of chaos theory. Usually in such models, ballistic chaotic motion (single particle dynamics) is a consequence of the spatial billiard geometry. For interacting many-particle systems, which appear in many areas of physics, chaotic motion can be generated from the combined effect of external forces and mutual interactions. In order to understand how chaotic motion emerges as a consequence of the interaction between particles, a simple billiard, namely a one-dimensional box, will be used. Since in this case the boundary alone cannot induce irregular motion of the two particles inside, the role of the interaction for the generation of chaotic motion becomes clear.

Interacting particles inside billiards can be used to model electrons in quantum dots. Electrons are confined inside a disk and are affected by the surrounding material which composes the semiconductor \textsuperscript{4}. In fact, the composition of the surrounding material may destroy the long Coulomb repulsion between electrons and also change the effective mass between particles \textsuperscript{5}. The influence of both effects on quantum energy levels and/or electrons dynamics is not obvious. However, they can be studied in detail for a physical model where kinetic and potential energy of the particles can be varied independently. This is achieved by a parameter which controls the range of interaction between the particles (electrons) and by varying the mass ratio $\gamma = m_2/m_1$ of the particles.

In this paper we will study the classical dynamics of two interacting particles inside a one-dimensional billiard as a function of $\gamma$ and as a function of interaction range between particles. A Yukawa interaction between particles is assumed. Such a system has been considered classically \textsuperscript{6} and quantum mechanically \textsuperscript{7} for the case of equal masses. In order to calculate the spectrum of Lyapunov Exponents (LEs), the dynamics in tangent space is determined explicitly. In the limit of very short range of interaction, this system should approach the hard-point collision case analyzed originally by Casati et al \textsuperscript{8}. Despite ergodic dynamics for the case of point-like collision at specific $\gamma$, vanishing LEs are a consequence of the linear instability of this system \textsuperscript{3}. Such linear unstable systems have become a topical problem in statistical mechanics \textsuperscript{10} (see also about the origin of diffusion in non-chaotic systems \textsuperscript{11}).

In the case of a Yukawa interaction, the repulsion between particles at collisions with the walls are shown to generate positive LEs. This is shown explicitly by determining the dynamics in tangent space. While the mean value of the largest finite-time LE only quantifies the degree of chaoticity of the system, the number of occurrences of the most probable LE, extracted from the distribution over initial conditions, is shown to give significant information about regular structures and sticky (or trapped) \textsuperscript{12} trajectories in phase space. This distribution is determined numerically as a function of $\gamma$.

The plan of the paper is as follows. Section \textsuperscript{11} reviews the main results from the problem of two hard point particles in a one-dimensional box. In section \textsuperscript{111} chaotic motion emerges with the introduction of the soft Yukawa interaction between the particles. Positive LEs can be generated from analytical expressions obtained for the dynamics in tangent space. The distribution of the
largest finite-time LE calculated over the phase space of initial conditions is used to reveal the underlying dynamics. Section III discusses the distribution of the largest finite-time LE for the case of two interacting particles in a circular billiard. The paper ends with conclusions in section V.

II. TWO PARTICLES IN A 1D-BOX WITH HARD POINT-LIKE COLLISIONS

Two particles in a 1D-box with hard point-like collisions, also called two-particle hard point gas, can also be treated as a particular case of the motion of three particles on a finite ring [13, 14], which can be mapped onto the motion of a particle in a triangle billiard [15]. In such systems the Lyapunov exponent is zero [9] and the whole dynamics can be monitored by changing the angles of the triangle billiard. These angles are functions of the masses ratio between particles. Such triangle billiards have also been applied to study energy diffusion in one-dimensional systems [8]. Although it is very useful to gain insights about collision properties of the particles, it is not needed for the purpose of the present work.

As follows we summarize the main results obtained by Casati [8], which are similar to those observed in the triangle billiard. Using Poincaré sections they [8] showed that the dynamics is non-ergodic if \( \theta \) is a rational multiple of \( \pi \), where

\[
\cos (\theta) = \frac{1 - m_2/m_1}{1 + m_2/m_1} \frac{1 - \gamma}{1 + \gamma} = \Delta .
\]

More specifically, writing \( \theta = \frac{m}{n} \pi \), where \( m \) and \( n \) are integers, at most \( 4n \) distinct velocity values occur. On the other hand, when \( \theta \) is an irrational multiple of \( \pi \), the velocities become uniformly dense [16] in velocity space. As a consequence, it is at least possible for the two-particle hard point gas to be ergodic in velocity space if \( \theta/\pi \) is irrational. Although Casati and Ford [8] did not show explicit results for irrational multiples of \( \pi \), they argued that their numerical results provide evidence supporting ergodic behavior for irrational \( \theta/\pi \) by demonstrating that an increasing number of velocities is observed for a sequence of rational \( \theta/\pi \) approaching an irrational \( \theta/\pi \)-value. Note however, that for irrational \( \theta/\pi \) infinite time may be required to observe all velocities.

Although there are infinitely many mass ratios which give rational values of \( \theta/\pi \), some of them are special. First, the integrable cases [17] \( \gamma = 1.3 \) (or 1/3), which have \( \theta = \frac{4}{3} \pi \) and \( \theta = \frac{2}{3} \pi \) (or \( \pi/3 \)), respectively. Relating Eq. (1) with results for the triangle billiard [18] (or even to the polygonal billiard [13, 20, 21]) it is possible to show that for the integrable cases the genus is equal \( g = 1 \) (the invariant surface of the billiard flow is a torus). For all other rational \( \theta/\pi \) the dynamics is pseudointegrable [14] and the invariant flow is not a torus (\( 1 \leq g < \infty \)). For genus \( g = 2 \), the possible values of \( \theta \) are [18]: \( \frac{1}{n} \pi, \frac{2}{n} \pi \) and the mass ratios are \( \gamma \sim 0.1 \) and \( \gamma \sim 1.9 \), respectively. As the genus increases, the invariant surface gets more and more complicated. Therefore, besides the integrable cases, the third special case which has a more “simpler” invariant surface, is expected for the pseudointegrable case \( \gamma \sim 1.9 \) (we do not use \( \gamma \sim 0.1 \) because we are interested in values of \( \gamma \) in the interval [1, 4]). Later we will come back to the special values \( \gamma = 1.0, 1.9, 3.0 \).

The momentum distribution looks quite different for irrational mass ratios. Although the motion can be ergodic, the number of momenta increases very slowly with longer system evolution [22]. However, this aspect is not in our present focus.

III. TWO PARTICLES IN A 1D-BOX WITH YUKAWA INTERACTION

It is adequate to introduce the center-of-mass and relative coordinates

\[
R = \frac{m_1 q_1 + m_2 q_2}{M}, \quad \text{and} \quad r = q_1 - q_2, \quad (2)
\]

respectively, with the total mass \( M = m_1 + m_2 \) and the reduced mass \( \mu = m_1 m_2/(m_1 + m_2) \). In these new coordinates, the equations of motion describe a single composite particle in the hyperspace \((r, R)\), called a hyperbilliard [6]. The Hamiltonian in relative coordinates is given by

\[
H = \frac{P^2}{2M} + \frac{p^2}{2\mu} + V_0 e^{-\alpha r}/r = E, \quad (3)
\]

where the Yukawa potential \( V(r) \) has strength \( V_0 \) and the parameter \( \alpha \geq 0 \) gives the interaction range \( r_0 = 1/\alpha \). Using scaled coordinates defined by \( (\alpha \neq 0) \)

\[
r = r_0 \tilde{r}, \quad p = \tilde{p} \sqrt{E}, \quad R = r_0 \tilde{R},
\]

\[
P = \tilde{P} \sqrt{E}, \quad dt = \frac{r_0}{\sqrt{E}} d\tau,
\]

and dividing Eq. (3) by \( E \), the scaled new Hamiltonian is

\[
\tilde{H} = \frac{\tilde{P}^2}{2M} + \frac{\tilde{p}^2}{2\mu} + \tilde{V}(\tilde{r}) = \epsilon = 1, \quad (4)
\]

with

\[
\tilde{V}(\tilde{r}) = \tilde{V}_0 e^{-\tilde{r}/\tilde{r}}, \quad \text{and} \quad \tilde{V}_0 = \frac{V_0}{r_0 E}.
\]

and scaled energy \( \epsilon = 1 \). When \( \alpha = 0 \) \((r_0 \to \infty)\) the transformation is independent of \( r_0 \) and \( \tilde{V}_0 = V_0/E \). Under \( \tilde{V}(\tilde{r}) \) the composite’s particle relative motion is subject to the force \( \tilde{Q}(\tilde{r}) = -\partial \tilde{V}/\partial \tilde{r} \) while its center of mass motion in \( \tilde{R} \) is free. For the case of equal masses the chaotic motion of [18] was already analyzed [6].
A. Dynamics in tangent space, Lyapunov exponents (LE), and the origin of chaotic motion

This section is dedicated to the analytical calculation of the Lyapunov spectrum. LEs are very useful to describe the dynamics in complex systems [23]. When the motion is chaotic, at least one LE is positive. Its value is determined through the dynamics in tangent space, as will be shown below.

Between collisions with the walls, equations of motion have the form

$$\tilde{F}(\tilde{\gamma}) = (\dot{\tilde{\gamma}}, \tilde{\dot{R}}, \tilde{\dot{v}}, \tilde{\dot{V}})^t = (\tilde{v}, \tilde{V}, \tilde{Q}(r), 0)^t, \tag{6}$$

and it is easy to see that center of mass momentum $M \tilde{\dot{R}}$ is a constant of motion. In relative coordinates, the composite particle moves under the influence of the force $\tilde{Q}(\tilde{r})$. This is a one-dimensional motion which is regular and integrable. Collision with left and right walls cause a breaking of the translational symmetry of the system, and as a consequence, center of mass momentum is not a constant of motion anymore. The effect of left (right) wall collisions lead to the following change in the phase space point $\tilde{\gamma}_0 = (\tilde{r}, \tilde{\dot{R}}, \tilde{\dot{v}}, \tilde{\dot{V}})$ before the collision to $\tilde{\gamma}_f = (\tilde{r}', \tilde{\dot{R}}', \tilde{\dot{v}}', \tilde{\dot{V}}')$ after the collision, $\tilde{\gamma}_f = \tilde{D}_k \tilde{\gamma}_i$, with

$$\tilde{D}_k = (-1)^k \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\Delta & 2 \\ 0 & 0 & 2\frac{\tilde{v}}{M} & -\Delta \end{pmatrix}, \tag{7}$$

where $\Delta = (m_1 - m_2)/M$. The label $k = 1(2)$ is used for particle 1(2). Since the two particles can never collide and they are moving in the one-dimensional billiard, it is assumed without loss of generality that particle 1(2) never collides with the right(left) wall. The complete time evolution in phase space can be formulated by integrating the equation of motion between collisions with the walls and by taking into account $\tilde{\gamma}_f$ from Eq. (7) each time the composite particle collides with the walls.

In order to calculate Lyapunov exponents, the time evolution of an infinitesimal path difference (the nearby trajectory) in the scaled tangent space $(\delta \tilde{\gamma})$ must be determined, given by

$$\delta \tilde{\gamma}(\tau) = \tilde{M}(\tau)\delta \tilde{\gamma}(\tau_0), \tag{8}$$

with the scaled monodromy matrix

$$\tilde{M}(\tau) = \frac{d\tilde{\gamma}(\tau)}{d\tilde{\gamma}(\tau_0)}. \tag{9}$$

Lyapunov exponents are the average rates of growth or shrinkage of such infinitesimal changes that are the eigenvectors of $\tilde{M}$,

$$\lambda_i = \lim_{\tau \to \infty} \frac{\log \tilde{\mu}_i(\tau)}{\tau}, \tag{10}$$

where $\tilde{\mu}_i(\tau)$ is the $i$-th eigenvalue of $\tilde{M}$. The matrix $\tilde{M}$ can be written itself as a product of matrices for small time steps. Since the motion between collisions is regular, Lyapunov exponents are zero. The situation is different for collisions of the composite particle with the walls. We follow the algorithm developed by Dellago et al. [24] to formulate the equations in the scaled tangent space according to

$$\delta \tilde{\gamma}_f = \frac{\partial \tilde{C}}{\partial \tilde{\gamma}_i} \delta \tilde{\gamma}_i + \left[ \frac{\partial \tilde{C}}{\partial \tilde{\gamma}_i} \tilde{F}(\tilde{\gamma}_i) - \tilde{F}(\tilde{C}(\tilde{\gamma}_i)) \right] \delta \tau_f \tag{11}$$

where $\tilde{C} = \tilde{D}_k \tilde{\gamma}_i$ gives the transformation at collisions with the walls with $\tilde{D}_k$ from Eq. (7). $\delta \tau_f$ is the delay in the collision time of the (infinitesimal) nearby trajectory with respect to the collision time of the reference trajectory. Between the collision time of the main trajectory and the collision time of the nearby trajectory, the composite particle moves under the influence of the force $\tilde{Q}(r)$ and the delay time can be determined from

$$t_k = \frac{\partial S_k}{\partial E} = m_k \int_{q_k^0}^{q_k} \frac{\partial q_k}{\partial E} dq_k = \int_{q_k^0}^{q_k} \frac{dq_k}{q_k}, \tag{12}$$

where $S_k$ is the action of the $k$ particle, $q_k^0$ is the position of the particle 1 in the reference trajectory, and $q_k^0$ is the collision point of the nearby trajectory. The energy dependence for $q_k$, $q_1 = \sqrt{2m_1(E - \frac{1}{2}m_2q_2^2 - V_0 - a \tau)}$, is obtained from the energy conservation. Using Eqs. (2) we have

$$q_{1(2)} = R - \frac{(-)^{1(2)}m_{2(1)}r}{M}, \tag{13}$$

$$q_{1(2)} = V - \frac{(-)^{1(2)}m_{2(1)}v}{M}, \tag{14}$$

and Eq. (12) can be written as

$$t_k = \int_{R_0}^{R_n} \frac{MdR}{MV - (-)^{1(2)}m_{2(1)}v} - \frac{(-)^{1(2)}m_{2(1)}}{MV - (-)^{1(2)}m_{2(1)}v}. \tag{15}$$

Since $V$ and $v$ does not depend on $R$, the integral in $R$ can be determined analytically. After the integration of Eq. (15), terms proportional to $\delta R^4 = R_0 - R_n$ and $\delta r^4 = r_0 - r_n$ will appear (quadratic terms $(\delta R)^2$ and $(\delta r)^2$ can be neglected in the linear analysis) and it can be written in scaled relative coordinates as

$$-\delta \tau_f = \tilde{t}_k = \frac{t}{\sqrt{E}} = \tilde{A}_k \delta \tilde{R}^i + \tilde{B}_k \delta \tilde{r}^i, \tag{16}$$

where $\tilde{A}_k$, $\tilde{B}_k$ are 2x2 matrices.
Finally, using Eqs. (19), (7) and (10), in (11), the collision of the composite particle in tangent space with left \((k = 1)\) and right walls \((k = 2)\) under a force \(\ddot{Q}(\tilde{r}) = -\partial V/\partial \tilde{r}\) takes the form \(\delta \gamma_f = \tilde{M}_k \delta \gamma_i\) with

\[
\tilde{M}_k = (-1)^k \begin{pmatrix}
-\Delta & 2 & 0 & 0 \\
2 & \Delta & 0 & 0 \\
-\Delta_k & -\Delta_k & -\Delta & 2 \\
-2 & -2 & -2 & \Delta \\
\end{pmatrix}
\]

where \(\Delta_k = -[\Delta + (-1)^k]\). The determinant of \(\tilde{M}_k\) is equal to 1 and eigenvalues are also equal to |1|. However, the matrix elements proportional to \(\Delta_k, Q, \dot{A}_k\) and \(\dot{B}_k\) generate positive LE when the global monodromy matrix \(\tilde{M}\) is constructed. Therefore, the presence of the interaction force \(Q\) and terms \(A_k\) and \(B_k\), related to the time delay in the tangent-space collision dynamics, are essential for the chaotic properties of the system. Otherwise, if \(Q = 0\), no positive LE can be obtained.

Two limiting situations can be discussed. a) The long range interaction \((\alpha = 0.0, r_0 \to \infty)\): at each collision with the wall, the interaction force \(\ddot{Q}(\tilde{r})\) is finite and positive LEs are generated; b) The short range interaction \((r_0 \ll 1)\): in general \(\ddot{Q}(\tilde{r})\) is close to zero if particles are sufficiently separated and the LEs are zero. However, the dynamics becomes chaotic due to a double collision process. For example, assuming that particles 1 and 2 are moving close together towards the left wall with a mutual repulsion close to zero due to the short range nature of the interaction. As particle 1 collides with the left wall, it changes its direction and moves towards particle 2, interacting with it. If such a double collision occurs infinitely close to the wall, the terms \(\dot{Q}\dot{B}_k\) and \(\ddot{Q}\dot{A}_k\) are not necessarily zero, a positive LE is generated and chaotic motion appears. This crucial role of double collisions to generate chaotic motion was also observed in another model of two interacting particles \[22\]. The chaotic motion induced by the soft potential will be discussed in more detail in the next section.

### B. Signatures of regular motion in the distribution of largest finite-time Lyapunov exponents

From the description of the dynamics by the monodromy matrices as constructed in the last section, it is clear that chaotic dynamics is generated in the presence of a soft interaction potential. The interesting question, however, is, if any signatures of non-ergodicity from the hard-point collision, or stickiness, can be identified. To this end we have investigated the distribution \(P(\Lambda_t, \gamma)\) of the finite-time largest Lyapunov exponents \[22\] \(\Lambda_t\) as a function of the mass ratio \(\gamma\). In general, for infinite time, the LEs \(\Lambda_\infty\) are well defined and do not depend on initial conditions. This holds also true for reasonably large finite times, if the motion is ergodic and the Lyapunov spectrum has good convergence properties. In quasi-regular systems, however, where the chaotic trajectory may approach a regular island and can be trapped there for a while, the value of the local LE can decrease. This will affect the convergence of \(\Lambda_t\) which depends now on the initial conditions. On the other hand, it implies that the distribution \(P(\Lambda_t)\), calculated over many initial conditions, contains information about the amount of regular motion (and sticky trajectories) in phase space. Usually, for fully chaotic systems \(P(\Lambda_t, \gamma)\) has a Gaussian distribution (see for example \[26\] and references therein).

![FIG. 1: Mean value of the finite-time largest Lyapunov exponent calculated over 400 trajectories up to time \(t = 10^4\) and at scaled energy \(\epsilon = 1\), for (a) long interaction range \((r_0 \to \infty)\), i.e. \(V_0 = 0.1\), and (b) short range interaction \((r_0 = 0.1)\), i.e. \(V_0 = 1.0\), with the hard walls located at \(q = \pm 1\). For each trajectory the largest LE is evaluated over \(10^5\) initial conditions.](image-url)
For mass ratio between \( \gamma \) a maximum as a function of \( \gamma \) is divided into two maxima, one close to 1 and the other close to 2. Therefore, a minimum of \( \Lambda_t^r \) appears in between \( (\gamma 
abla 1) \) from Fig. 2 (between \( \gamma = 1 \) and \( \gamma = 2 \)).

A more systematic way to uncover this trend is to follow \( P(\Lambda_t^p, \gamma) \equiv P(\Lambda, \gamma) \) as a function of the mass ratio \( \gamma \) shown in Fig. 3 (top, \( r_0 \to \infty \), black curve) close to \( \gamma \sim 1.9 \), which is closer to the limit of hard-point collisions, presents an additional little valley at \( \gamma \sim 1.9 \). This is the pseudointegrable case with genus \( g = 2 \) which appears in the hard-point collision. In other words, if \( \gamma \) is close to values for which the dynamics in the point-like gas is integrable (\( g = 1 \)) or “simpler” (\( g = 2 \)), then the dispersion around \( \Lambda_t^p \) increases so that \( P(\Lambda, \gamma) \) decreases, and signatures of non-ergodicity are expected under additional Yukawa interaction. Another interesting observation is that the minimum at \( \gamma \sim 1.9 \) (see gray curve from Fig. 3) disappears in the long interaction limit \( r_0 \to \infty \) (see black curve from Fig. 3). It means that the regular motion from the integrable cases of the hard-point collision survives longer under the perturbation of the soft interaction than the regular motion from the pseudointegrable case.

For fully chaotic systems the quantity \( P(\Lambda, \gamma) \) is just the maximum of a Gaussian distribution and it should increases linearly with \( t \), since the variance \( \sigma = ((\Lambda_t^2 - (\Lambda_t)^2)^{1/2} \) for such systems goes with \( 1/t \). This behavior of \( \sigma \) has been observed by studying ergodicity in high-dimensional symplectic maps [27], and used for the detection of small islands in the standard map [28]. For our case we found that the time dependence of \( P(\Lambda, \gamma) \)
peaks are not located at the integrable ($\gamma$) and pseudo-integrable ($\gamma$) regular islands which appear in the phase space, as for example the points related to zero LEs in Fig. 3 for $\gamma = 1.9$.

It is worth to mention that the maxima and minima from $P_\Lambda(\gamma)$ does not change significantly with the number of initial conditions.

We can conclude that signatures from regular structures in phase-space exist and are uncovered by the dispersion of the largest Lyapunov exponent $\Lambda_t$ most clearly visible along the cut $P_\Lambda(\gamma)$ defined by the number of occurrences of the most probable Lyapunov exponent $\Lambda^p_t$ (Eq. 13). No such signatures, however, are visible in less sensitive quantities such as the mean Lyapunov exponent $\langle \Lambda_t \rangle$ or the fluctuation $\sigma$.

The interpretation of the Lyapunov properties are supported by the relevant PSS. Figure 4 (top) shows PSSs for interaction $r_0 \to \infty$ and cases $\gamma = 3.0$ and $\gamma = 4.0$, where $P_\Lambda(\gamma)$ has a minimum and maximum (see black curve in Fig. 4), respectively. Although the system is more chaotic for $\gamma = 3.0$ than for $\gamma = 4.0$, trapped trajectories appear near the island for $\gamma = 3.0$, see the corresponding magnification in Fig. 4 (top, right). Such trapped motion near regular islands, which does not appear for $\gamma = 4.0$ (see Fig. 4 bottom, right), affects $P(\Lambda^p_t, \gamma)$ and consequently, $P_\Lambda(\gamma)$ has a minimum near $\gamma = 3.0$. Another example is shown in Fig. 4 for $r_0 = 0.1$. For $\gamma = 1.5$ (top, left), which has a maximum in Fig. 4 (top, gray curve), no trapped trajectories are found up to the time propagated. For $\gamma = 1.8$ (top, right) which has a minimum in Fig. 4 (top, gray curve), trapped trajectories start to appear. Moreover, the abrupt appearance of gray points for $\gamma \sim 2.7$ below the main curve Fig. 4 can be nicely explained using the PSS. Figure 5 compares the PSSs for $\gamma = 2.6$ (bottom, left) with $\gamma = 2.8$ (bottom, right), where gray points appear in Fig. 5. Clearly, it can be seen that when a regular island is born trapped trajectories start to appear around the island and, as a
FIG. 6: Magnification of the Poincaré Surfaces of Section with \( r_0 = 0.1 \) for a) \( \gamma = 1.5 \), b) \( \gamma = 1.8 \), c) \( \gamma = 2.6 \) and d) \( \gamma = 2.8 \).

consequence, many initial conditions with lower LE are obtained (gray points in Fig. 4). The regular island is related to a period-4 orbit which has the following property: for each second hit of particle 1 with the left wall, particle 2 is at rest. There is a similar periodic orbit for the hard-point collision case [20].

Note that although \( P_\Lambda(\gamma) \) is determined only from trajectories with positive \( \Lambda \), it provides information about the amount of regular structure in phase space through chaotic trajectories which are trapped close to regular islands. Since trapped trajectories are characteristic for mixing in phase space, \( P_\Lambda(\gamma) \) provides a tool to analyze phase space mixing.

IV. TWO PARTICLES IN A CIRCULAR BILLIARD WITH YUKAWA INTERACTION

In order to show the utility of \( P_\Lambda(\gamma) \) for systems with higher dimensions, we discuss now the case of two interacting particles in a circular billiard. The interaction between particles is still of Yukawa type. The chaotic motion is now generated by the combined effect of the curvature of the walls from the circular billiard and the double collisions discussed in last section. The phase space is 8-dimensional and it is not possible to construct an adequate PSS which allows to look at the underline dynamics. Trajectories will fill out any chosen PSS and no information about details of the dynamics can be obtained. Sticky trajectories, for example, which may cause non-ergodicity due to a partial focusing of trajectories [30], are difficult to detect. In such partial focusing, an infinitesimal family of nearby trajectories that starts out parallel will lead to Lyapunov exponents which converge very slowly in time. This is a typical behavior in high dimensional quasi-regular systems. In this section we show the effectiveness of \( P_\Lambda(\gamma) \) to obtain relevant informations in high-dimensional quasi-regular systems.

Figure 7h) shows the finite-time distribution of the largest LE for the case of long range interaction \( (r_0 \to \infty) \) in the circular billiard. The value of the the mean LE calculated of the 400 initial conditions, decreases growing the mass ratio and the regular motion increases. This is the only information we can get from the LE about the complicated dynamics of the two interacting particles inside the billiard. However, the gray points below

FIG. 7: (color online). a) Finite-time distribution of the largest Lyapunov exponent \( P(\Lambda, \gamma) \) calculated over 400 trajectories up to time \( t = 10^4 \) and for \( r_0 \to \infty \) for the circular billiard. With increasing \( P(\Lambda, \gamma) \) the color changes from light to dark (white over yellow and blue to black); b) Normalized distribution \( P_\Lambda(\gamma) \) of the number of occurrences of the most probable Lyapunov exponent \( \Lambda_p \).
V. CONCLUSIONS

Usually, chaotic motion is generated through nonlinear equations of motion. As a consequence, exponential divergence of nearby trajectories occurs which can be quantified by positive Lyapunov exponents. Another source of chaotic motion emerges even for linear equations of motion through boundary conditions. The advantage of such systems is the possibility to obtain rigorous mathematical results. One example is a point particle moving among high-dimensional cylindrical scatterers [29], which is similar to the high-dimensional Lorentz gas. In these systems, strongly chaotic motion is generated due to the convex curvature of hard disks or spheres. In fact, the collision time delay between nearby trajectories due to the curvature of the surface, is responsible for the chaotic motion.

Curved boundaries are not present in the one-dimensional confinement considered in this paper. For the hard point-like collision case non-ergodic motion is generated when the mass ratio gives a value \( \theta/\pi \) which is a rational multiple of \( \pi \). Ergodic motion, on the other hand, may be obtained for irrational multiples of \( \theta/\pi \). We have shown that for additional soft Yukawa interaction between the two particles, chaotic motion is obtained for any mass ratio. Double collisions of particles, which occur very close to the walls, are essential to generate positive LEs in the short interaction range limit. The collisional time delay in tangent space, together with the soft Yukawa interaction, are responsible for the chaotic motion. The mean of the largest finite-time LE \( \langle \Lambda \rangle \) decreases smoothly as the mass ratio increases and does not provide detailed information about the phase space structure. This type of information is provided by the probability distribution of the largest finite-time Lyapunov exponent \( P(\Lambda, \gamma) \). It reveals that the dispersion around \( \langle \Lambda \rangle \) increases when trapped trajectories are present in the phase space. We have shown that a cut through \( P(\Lambda, \gamma) \) along the number of occurrences of the most probable Lyapunov exponent, \( P_\Lambda(\gamma) \), gives a quantitative measure of the influence of regular motion in mixed phase space. Specifically for the system studied here, we have shown that \( P_\Lambda(\gamma) \) decreases when the structure in phase-space is more regular and the mass ratio is close to the integrable cases \( \gamma = 1, 3 \) (genus \( g = 1 \)) or to the “simpler” dynamics (pseudo-integrable at \( \gamma \sim 1.9 \), with \( g = 2 \)) from the hard point-like collision. We also observed that the regular motion of the integrable cases from the hard-point collision survives longer under the perturbation of the soft interaction than the regular motion from the pseudo-integrable case. Hence, the dynamics under the additional Yukawa interaction, although in principle chaotic, “remembers” the integrable and pseudo-integrable dynamics in the system without the soft Yukawa interaction. This is certainly a subtle effect and therefore, we expect that in general the number of occurrences of the most probable Lyapunov exponent provides a sensitive tool to probe details in phase space dynamics. We have also shown that this quantity is much more sensitive if compared with the mean square fluctuations of the LE. In contrast to Poincaré sections this tool is easily applicable in higher dimensional systems, where trapped trajectories may cause non-ergodicity due to a partial focusing of trajectories [30]. In order to show this we calculated \( P(\Lambda, \gamma) \) for two interacting particles in a circular billiard, where the phase space is 8-dimensional and it is not possible the construct an adequate Poincaré section to analyze the dynamics. We show that by increasing the mass ratio, the mean LE decreases and the system gets more regular. Furthermore, for a minimum of \( P_\Lambda(\gamma) \) at \( \gamma \sim 1.0 \) trapped trajectories are expected, and a maximum at \( \gamma \sim 3.0 \) ergodic-like motion is expected. Therefore, \( P_\Lambda(\gamma) \) can be used in higher-dimensional systems as a tool to characterize the dynamics.

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