Anomalous dimensions in gauge theories from rotating strings in $AdS_5 \times S^5$

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Abstract

Semi-classical soliton solutions for superstrings in $AdS_5 \times S^5$ are used to predict the dimension of gauge theory operators in $\mathcal{N} = 4$ SU($N$) SYM theory. We discuss the possible origin of scaling violations on the gauge theory side.

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1. Introduction

Understanding the AdS/CFT correspondence [1,2,3] to full string theory level can lead to important insights about the $1/N$ expansion and strong coupling physics in Yang-Mills theories. It is therefore of interest to develop methods to study string propagation in backgrounds with R-R gauge fields. Recently, some direct checks of the AdS/CFT correspondence beyond the supergravity level were performed by considering a special sector of states where some global quantum numbers (such as R-charge or spin) are large [4,5,6,7]. The idea of [4] is based on the observation [8,4,9,10] that string theory in certain R-R plane-wave backgrounds is solvable. Extensions in different directions were carried out in [13-29].

The strategy of [6], which was further elaborated in [30], is to identify certain semi-classical soliton solutions representing highly excited string states with gauge theory operators of some finite anomalous dimension. The classical energy in global $AdS$ coordinates is identified with the conformal dimension of the corresponding state in the dual gauge theory.

The soliton solutions investigated in [6] can be separated into three classes:

i) Strings spinning on $AdS_5$, stretched along the radial direction $\rho$, at fixed angles on $S^5$ (an earlier study of this classical string solution is in [31]). They represent string states with spin $S$, and they have an energy

$$E \cong S + \frac{\sqrt{\lambda}}{\pi} \log \frac{S}{\sqrt{\lambda}} , \quad S \gg \sqrt{\lambda} . \quad (1.1)$$

ii) Strings spinning on $S^5$, stretched along the radial direction $\rho$, representing string states carrying R-charge $J$. For them

$$E = J + \sum_{n=-\infty}^{\infty} N_n \sqrt{1 + \frac{\lambda n^2}{J^2}} , \quad J \gg \sqrt{\lambda} , \quad (1.2)$$

where the second term arises by considering small oscillations around the soliton state. This is the same formula obtained by [4] in the exactly solvable [8,9] pp-wave background [11,12].

iii) Strings spinning on $S^5$ (i.e. with R-charge $J$), sitting at $\rho = 0$ and stretched along an angular direction. Here one obtains [6]

$$E \cong J + \frac{2\sqrt{\lambda}}{\pi} . \quad (1.3)$$
In [30], a more general solution interpolating between the cases i) and ii) was investigated, leading to a general formula for the energy as a function of $S$ and $J$. Here we shall present a solution interpolating between the three cases i), ii), iii) described above. This will lead to a more general formula $E = E(S, J)$, which will reduce to the previously known cases by taking different limits. The string state (and thus the corresponding gauge operator) is not the same as the one studied in [30]. In a large $J$ limit, the energy approaches the energy of the state of [30]. The main virtue of the present approach is to incorporate the features of the formulas (1.1)-(1.3) into a single general expression for the conformal dimension.

2. More general soliton rotating in $AdS_5$ and $S^5$

We consider type IIB superstrings moving in the $AdS_5 \times S^5$ background, with metric:

$$ds^2 = \alpha' R^2 \left[ -dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho d\tilde{\Omega}^2_3 + d\psi^2 \sin^2 \theta + d\theta^2 + \cos^2 \theta d\Omega^2_3 \right],$$  

$$d\tilde{\Omega}^2_3 = \cos^2 \beta d\phi^2 + d\beta^2 + \sin^2 \beta d\phi^2,$$

$$d\Omega^2_3 = \cos^2 \psi_1 d\psi^2_2 + d\psi_1^2 + \sin^2 \psi_1 d\psi^2_3,$$

where $R^2 = \sqrt{\lambda}$ and $\lambda = g_{YM}^2 N$ is the 't Hooft coupling. We look for a soliton solution of the following form:

$$t = \kappa \tau , \quad \phi = \omega \tau , \quad \psi = \nu \tau ,$$  

$$\rho = \rho(\sigma) = \rho(\sigma + 2\pi) , \quad \theta = \theta(\sigma) = \theta(\sigma + 2\pi) ,$$  

$$\beta = \frac{\pi}{2} , \quad \psi_1 = \psi_2 = \psi_3 = \phi = 0 .$$

It describes a string rotating in $AdS_5$ and in $S^5$ with independent angular velocity parameters $\omega$ and $\nu$, which is stretched along the radial coordinate and along the angular coordinate $\theta$ of $S^5$. We will find below that this rotating soliton generalizes the solutions discussed in [3,30], and smoothly interpolates between all previously known cases.

The equations of motion and constraints become

$$\rho'' + (\omega^2 - \kappa^2) \cosh \rho \sinh \rho = 0 ,$$

$$\theta'' + \nu^2 \cos \theta \sin \theta = 0 ,$$

$$-\kappa^2 \cosh^2 \rho + \rho'^2 + \omega^2 \sin^2 \rho + \theta'^2 + \nu^2 \sin^2 \theta = 0 ,$$
where prime denotes derivative with respect to $\sigma$. The general solution is given by

$$\rho'^2 = \kappa^2 \cos^2 \alpha_0 - (\omega^2 - \kappa^2) \sinh^2 \rho, \quad (2.7)$$

$$\theta'^2 = -\nu^2 \sin^2 \theta + \kappa^2 \sin^2 \alpha_0. \quad (2.8)$$

where $\alpha_0$ is an integration constant.

We have chosen a convenient parametrization in terms of $\cos \alpha_0, \sin \alpha_0$ to take into account automatically that the solution describes a finite closed string stretching from $\rho = 0$ up to some $\rho = \rho_{\text{max}}$, which is finite provided $\omega > \kappa$. The string is folded onto itself, and the interval $0 \leq \sigma < 2\pi$ is split into four segments. The first segment starts with $\rho = 0$ at $\sigma = 0$ up to $\rho_{\text{max}}$ at $\sigma = \pi/2$.

Let us define parameters

$$a \equiv \frac{\nu^2}{\kappa^2 \sin^2 \alpha_0}, \quad b \equiv \frac{\omega^2 - \kappa^2}{\kappa^2 \cos^2 \alpha_0}. \quad (2.9)$$

We will assume $b > 0$. There are different situations according to the value of $a$.

The case $a = 1$ gives a solution with infinite energy, unless $\theta \equiv \pi/2$, which is the case discussed by [30]. This includes the cases discussed in sects. 2 and 3 of [6], corresponding to the pp wave limit and to the Regge string.

If $a < 1$, then $\theta \in [0, \pi]$. This describes a closed string stretched around the great circle of $S^5$. This solution is a generalization of a similar solution (but with $\rho \equiv 0$) discussed in [6] (it reduces to that particular constant $\rho$ case when $b \gg 1$). As pointed out in [6], the solution seems unstable, since a small perturbation of the string makes it to slide over the sides of the sphere due to the string tension.

Here we shall consider the interesting case of $a > 1$. Then $\theta \in [0, \theta_{\text{max}}]$, where $\theta_{\text{max}} = \arcsin(a^{-1/2})$, $\theta_{\text{max}} < \pi/2$. It generalizes the case discussed in sect. 4.1 of [6] (corresponding to the special case $\rho \equiv 0$, $a > 1$) to strings which also stretch along the radial direction. In the regime $a \sim 1$, the major contribution to the quantum numbers of the string will come from the region $\theta \sim \pi/2$. For this reason, in this regime we will recover the results of [30] for the anomalous dimension, with a small correction due to the fact that the string is also stretched along the $\theta$ direction.

Thus there is a general solution interpolating between the different known cases. This will give a more general formula relating the dimension of the operators with the R-charge and spin.
Note that there are no solutions in this class where $\theta \in \left[\frac{\pi}{2}, \frac{\pi}{2} + \epsilon\right]$. Either $\theta$ is stuck at $\frac{\pi}{2}$, or it goes all the way up to $\theta = 0$. However, one can consider time-dependent small fluctuations around $\theta = \frac{\pi}{2}$. This semiclassical quantization was carried out in [6], [30], and it leads to the full spectrum (1.2), including the contributions of oscillators. For the present soliton, the semiclassical quantization in terms of a small $\theta$ expansion, retaining only the harmonic oscillations, is meaningful only for small $J$, where $\theta_{\text{max}}$ is small, but not in the interesting region of large $J$ (corresponding to $a \sim 1$), where the angular variable $\theta$ takes all values from 0 to $\frac{\pi}{2}$.

From eqs. (2.7), (2.8), we obtain

$$\kappa \sin \alpha_0 \sigma = \int_0^\theta d\theta' \frac{1}{\sqrt{1 - a \sin^2 \theta'}} = \mathcal{F}(\theta, a), \quad (2.10)$$

$$\kappa \cos \alpha_0 \sigma = \int_0^\rho d\rho' \frac{1}{\sqrt{1 - b \sinh^2 \rho'}} = -i\mathcal{F}(i\rho, -b), \quad (2.11)$$

where $\mathcal{F}(x, m)$ represents the elliptic integral of the first kind. These equations define $\rho = \rho(\theta)$. In the present case of $a > 1$ and $b > 0$, there are maximum values of $\theta$ and $\rho$ taken by the string which are given by $\theta_{\text{max}} = \arcsin(a^{-1/2})$, $\rho_{\text{max}} = \arcsinh(b^{-1/2})$. From now on it is convenient to trade the parameters $\omega, \nu$ by $a, b$ (see (2.9)).

The periodicity condition (2.3) determines $\alpha_0, \kappa$ in terms of $a, b$. As a function of $\sigma$, $\rho(\sigma)$, $\theta(\sigma)$ start at $\rho = 0$, $\theta = 0$ at $\sigma = 0$ and reach a maximum value of $\rho$ and $\theta$ at the point where $\rho', \theta'$ vanish, which for the one-fold string is at $\sigma = \frac{\pi}{2}$. Therefore one has $\rho(\pi/2) = \rho_{\text{max}}$, $\theta(\pi/2) = \theta_{\text{max}}$. Demanding this, we get the conditions:

$$\frac{\pi}{2} \kappa \sin \alpha_0 = \int_0^{\theta_{\text{max}}} d\theta \frac{1}{\sqrt{1 - a \sin^2 \theta}} = \frac{\pi}{2\sqrt{a}} 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{a}\right), \quad (2.12)$$

$$\frac{\pi}{2} \kappa \cos \alpha_0 = \int_0^{\rho_{\text{max}}} d\rho \frac{1}{\sqrt{1 - b \sinh^2 \rho}} = \frac{\pi}{2\sqrt{b}} 2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; -\frac{1}{b}\right). \quad (2.13)$$

The hypergeometric function is related to $K(m)$, the complete elliptic integral of the first kind ($K(m) = \mathcal{F}(\pi/2, m)$).

The energy, spin, and R-charge of the soliton are given by the following formulas:

$$E = \frac{R^2}{2\pi} \kappa \int_0^{2\pi} d\sigma \cosh^2 \rho = \frac{2R^2}{\pi \cos \alpha_0} \int_0^{\rho_{\text{max}}} d\rho \frac{\cosh^2 \rho}{\sqrt{1 - b \sinh^2 \rho}}, \quad (2.14)$$

$$S = \frac{R^2}{2\pi} \int_0^{2\pi} d\sigma \omega \sinh^2 \rho = \frac{2R^2 \omega}{\pi \kappa \cos \alpha_0} \int_0^{\rho_{\text{max}}} d\rho \frac{\sinh^2 \rho}{\sqrt{1 - b \sinh^2 \rho}}, \quad (2.15)$$
\[ J = \frac{R^2}{2\pi} \int_0^{2\pi} d\sigma \nu \sin^2 \theta = \frac{2R^2\nu}{\pi \kappa \sin \alpha_0} \int_0^{\theta_{\text{max}}} d\theta \frac{\sin^2 \theta}{\sqrt{1 - a \sin^2 \theta}}. \]  

(2.16)

Computing the integrals, we find

\[ E = E(a, b) = \frac{R^2}{\sqrt{b \cos \alpha_0}} 2F_1\left(-\frac{1}{2}; \frac{1}{2}; 1; -\frac{1}{b}\right), \]  

(2.17)

\[ S = S(a, b) = \frac{R^2}{2b^{3/2} \cos \alpha_0} \sqrt{1 + b \cos \alpha_0^2} 2F_1\left(\frac{1}{2}, \frac{3}{2}; 2; -\frac{1}{b}\right), \]  

(2.18)

\[ J = J(a) = \frac{R^2}{2a} 2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; \frac{1}{a}\right), \]  

(2.19)

\[ \tan \alpha_0 = \sqrt{\frac{b}{a}} \frac{2F_1\left(\frac{1}{2}, \frac{3}{2}; 1; \frac{1}{a}\right)}{2F_1\left(\frac{1}{2}, \frac{3}{2}; 1; -\frac{1}{b}\right)}. \]

These formulas define parametrically \( E = E(J, S) \).

Let us now derive explicit analytic formulas for \( E = E(J, S) \) in four different regimes, according to the cases \( J \text{ or } S \ll \sqrt{\lambda} \), and \( J \text{ or } S \gg \sqrt{\lambda} \). In all cases, we will consider \( J, S \gg 1 \). The regimes of large spin \( S \) and large R-charge \( J \) (as compared to \( \sqrt{\lambda} \)) correspond to \( b \ll 1 \) and to \( a \sim 1 \) respectively, whereas small \( S \) and \( J \) correspond to \( b \gg 1 \) and to \( a \gg 1 \).

Define

\[ I_1(b) = \int_0^{\rho_{\text{max}}} d\rho \frac{1}{\sqrt{1 - b \sinh^2 \rho}}, \quad I_2(b) = \int_0^{\rho_{\text{max}}} d\rho \frac{\sinh^2 \rho}{\sqrt{1 - b \sinh^2 \rho}}, \]

\[ I_3(a) = \int_0^{\theta_{\text{max}}} d\theta \frac{1}{\sqrt{1 - a \sin^2 \theta}}, \quad I_4(a) = \int_0^{\theta_{\text{max}}} d\theta \frac{\sin^2 \theta}{\sqrt{1 - a \sin^2 \theta}}. \]

The basic expansion formulas we need are

\[ I_1(b) \approx -\frac{1}{2} \log b, \quad I_2(b) \approx \frac{1}{b} + \frac{1}{4} \log b, \quad \text{for } b \ll 1, \]  

(2.20)

\[ I_1(b) \approx \frac{\pi}{2\sqrt{b}} \left(1 - \frac{1}{4b}\right), \quad I_2(b) \approx \frac{\pi}{4b^{3/2}}, \quad \text{for } b \gg 1, \]  

(2.21)

and

\[ I_3(a) \approx -\frac{1}{2} \log \left(\frac{a - 1}{16}\right), \quad I_4(a) \approx -\frac{1}{2} \log \left(\frac{a - 1}{16}\right) - 1, \quad \text{for } a \approx 1, \]  

(2.22)

\[ I_3(a) \approx \frac{\pi}{2\sqrt{a}} \left(1 + \frac{1}{4a}\right), \quad I_4(a) \approx \frac{\pi}{4a^{3/2}}, \quad \text{for } a \gg 1. \]  

(2.23)
Let us now consider the different cases:

I) \( a \gg 1, b \gg 1, b/a \) fixed (short strings): This is the regime of small \( S \) and \( J \) with fixed \( J/S \). From eqs. (2.14), (2.15), (2.16), we get, to leading order,

\[
E \approx R^2 \sqrt{a^{-1} + b^{-1}}, \quad S \approx \frac{R^2}{2b}, \quad J \approx \frac{R^2}{2a},
\]

and \( \tan \alpha_0 \approx \sqrt{J/S} \). Thus

\[
E^2 \approx 2\sqrt{\lambda}(J + S), \quad \frac{J}{\sqrt{\lambda}} \ll 1, \quad \frac{S}{\sqrt{\lambda}} \ll 1,
\]

which is the usual Regge-type spectrum of string theory in flat spacetime. This is expected, since short strings do not feel the curvature of spacetime. In the case \( J/S \to 0 \), eq. (2.25) reduces to the case discussed in ref. [6]. Equation (2.25) differs from the formula \( E^2 \approx J^2 + 2\sqrt{\lambda} \) of [30]. The reason is the following: in the case of [30], the string is located at \( \theta = \pi/2 \); in the present case, the string is located near \( \theta = 0, \rho = 0 \), where there is a symmetry \( J \leftrightarrow S \) due to the symmetry of the solution at small \( \rho, \theta \) under \( \phi \leftrightarrow \psi \). From the gauge theory point of view, the soliton of [30] and the present solution correspond to different operators when \( J \neq 0 \).

II) \( a \sim 1, b \ll 1 \): This is the case of large R-charge \( J \) and large spin \( S \). Then the string is long with \( \theta_{\text{max}} \sim \frac{\pi}{2} \). Now eqs. (2.14) - (2.16) give

\[
E \approx \frac{2R^2}{\pi \cos \alpha_0} \left( \frac{1}{b} - \frac{1}{4} \log b \right),
\]

\[
S \approx \frac{2R^2}{\pi \cos \alpha_0} \left( \frac{1}{b} + \frac{1}{4} \log b \right),
\]

\[
J \approx -\frac{R^2}{\pi} \log \left( \frac{a - 1}{16} - \frac{2R^2}{\pi} \right),
\]

and \( \tan \alpha_0 = \frac{\log (a^{-1})}{\log b} \). Thus we obtain

\[
E - S = \frac{\sqrt{\lambda}}{\pi} \sqrt{\log^2 b + \left( \frac{\pi J}{\sqrt{\lambda}} + 2 \right)^2}.
\]

The parameter \( b \) is a function of \( S \) and \( J \), determined by the transcendental equation

\[
S = \frac{2\sqrt{\lambda}}{\pi b |\log b|} \sqrt{\log^2 b + \left( \frac{\pi J}{\sqrt{\lambda}} + 2 \right)^2}.
\]
For \( \frac{J}{\sqrt{\lambda}} \ll \log \frac{S}{\sqrt{\lambda}} \), we have \( b \approx \frac{2\sqrt{\lambda}}{\pi S} \). Thus we obtain

\[
E - S \approx \sqrt{(J + \frac{2\sqrt{\lambda}}{\pi})^2 + \frac{\lambda}{\pi^2} \log^2 \frac{\pi S}{2\sqrt{\lambda}}}
\]

\[
= \frac{\sqrt{\lambda}}{\pi} \log \frac{\pi S}{2\sqrt{\lambda}} + \frac{\pi(j + \frac{2\sqrt{\lambda}}{\pi})^2}{2\sqrt{\lambda} \log \frac{\pi S}{2\sqrt{\lambda}}},
\]  \hspace{1cm} (2.31)

\[
\frac{J}{\sqrt{\lambda}} \gg 1, \quad \frac{S}{\sqrt{\lambda}} \gg 1, \quad \frac{J}{\sqrt{\lambda}} \ll \log \frac{S}{\sqrt{\lambda}}.
\]

Corrections to this formula are of order \( J^2 / \log^2 \frac{S}{\sqrt{\lambda}} \).

For \( \frac{J}{\sqrt{\lambda}} \gg \log \frac{S}{\sqrt{\lambda}} \), we have \(-b \log b = \frac{2J}{S}\), so that

\[
E - S \approx J + \frac{2\sqrt{\lambda}}{\pi} + \frac{\lambda}{2\pi^2} \log^2 \frac{2J}{S},
\]  \hspace{1cm} (2.32)

\[
\frac{J}{\sqrt{\lambda}} \gg 1, \quad \frac{S}{\sqrt{\lambda}} \gg 1, \quad \frac{J}{\sqrt{\lambda}} \gg \log \frac{S}{\sqrt{\lambda}}.
\]

The formulas (2.31), (2.32) agree with the corresponding formulas in [30] for a rotating string fixed at \( \theta = \frac{\pi}{2} \). The reason is that in this regime \( a \approx 1 \), most of the contribution to the R-charge comes from the region \( \theta = \frac{\pi}{2} \). The only difference is the shift in \( J \) by \( \frac{2\sqrt{\lambda}}{\pi} \), which represents the subleading correction to the large \( J/\sqrt{\lambda} \) expansion. As mentioned above, the presence of this term reflects the fact that the string state we are considering is not the same as the state considered in [30].

III) \( b \ll 1, \ a \gg 1 \): In this case \( \frac{J}{\sqrt{\lambda}} \ll 1, \ \frac{S}{\sqrt{\lambda}} \gg 1 \); the string is long with \( \theta \) being nearly fixed at \( \theta \approx 0 \). Now \( \tan \alpha_0 = \frac{\pi}{\sqrt{\lambda} \log b} \). Using eqs. (2.26), (2.27), and \( J \) as in (2.24), we obtain

\[
E - S = \sqrt{\frac{\lambda}{\pi}} |\log b| (1 + \frac{\pi^2 J}{\sqrt{\lambda} \log^2 b}),
\]  \hspace{1cm} (2.33)

with \( b \approx \frac{2\sqrt{\lambda}}{\pi S} \). Thus

\[
E - S = \frac{\sqrt{\lambda}}{\pi} \log \frac{S}{\sqrt{\lambda}} + \frac{\pi J}{\log \frac{\pi S}{2\sqrt{\lambda}}},
\]  \hspace{1cm} (2.34)

\[
\frac{J}{\sqrt{\lambda}} \ll 1, \quad \frac{S}{\sqrt{\lambda}} \gg 1.
\]

This formula is new, and differs from the analog limit (small \( J \), large \( S \)) in [30]. Again, the reason is that the small \( J \) string spins in the region of small \( \theta \), whereas in [30] is always fixed at \( \theta = \frac{\pi}{2} \).
IV) $b \gg 1$, $a \sim 1$. Then the string extends from $\theta = 0$ to $\theta_{\text{max}} \sim \frac{\pi}{2}$ with $\rho$ being nearly fixed at $\rho \cong 0$. Now $\frac{J}{\sqrt{\lambda}} \gg 1$ and $\frac{S}{\sqrt{\lambda}} \ll 1$. Using eqs. (2.14), (2.13), (2.16), we obtain

$$E = \frac{2\sqrt{\lambda}}{\pi} I_3 \sqrt{1 + \frac{I_2^2}{I_3^2} (1 + \frac{I_2}{I_1})} ,$$

$$S = \frac{2\sqrt{\lambda}}{\pi} I_3 \frac{I_2}{I_1} \sqrt{1 + (1 + b) \frac{I_2^2}{I_3^2}} ,$$

$$J \cong \frac{2\sqrt{\lambda}}{\pi} (I_3 - 1) .$$

Combining these equations, we find

$$E \cong J + \frac{2\sqrt{\lambda}}{\pi} + S + \frac{\lambda S}{2J^2} ,$$

$$\frac{J}{\sqrt{\lambda}} \gg 1 , \quad \frac{S}{\sqrt{\lambda}} \ll 1 .$$

For $S = 0$, eq. (2.38) reduces to the result (1.3) of [6]. Equation (2.38) can be compared with the similar formula found in [30],

$$E \cong J + S + \frac{\lambda S}{2J^2} .$$

Note that in the present case the correction $\frac{2\sqrt{\lambda}}{\pi}$ is important, being larger than the next terms.

V) General case: Using the general formulas eqs. (2.17) - (2.19), one can do a numerical plot of $E = E(S, J)$ in the general case. We find that $E = E(S, J)$ smoothly interpolates between the different regimes described above, with no surprises at intermediates regimes.

3. Discussion

The regime IV) is closely related to strings in the pp-wave background. Indeed, this background is obtained as the limiting geometry seen by a particle moving along the $\psi$ direction with large momentum $J$, sitting near $\rho = 0$ and near $\theta = \frac{\pi}{2}$. To see that (2.39) can be understood from the string spectrum in the pp wave background, we start with eq. (1.2), and consider string states with spin $S$ associated with rotations in the plane.

\footnote{Note that our definition of $\theta$ in (2.1) differs by $\theta \to \frac{\pi}{2} - \theta$ from the notation of [4].}
1-2. Let us assume that it is a state of the Regge trajectory, so that only \( n = 1 \) oscillators in the directions 1-2 are excited (e.g. by acting \( S/2 \) times with \( a_{1+}^\dagger, \tilde{a}_{1+}^\dagger \), on the light-cone vacuum, where \( a_{n+}^\dagger = \frac{1}{\sqrt{2}}(a_{n+}^\dagger - ia_{n+}^\dagger) \)). Then we have \( N_n = S + N_n^T \), where \( N_n^T \) contains the oscillators corresponding to the other directions. Thus eq. (1.2) becomes

\[
E = J + S \sqrt{1 + \frac{\lambda}{J^2}} + \sum_{n=-\infty}^{\infty} N_n^T \sqrt{1 + \frac{\lambda n^2}{J^2}},
\]

For \( J \gg \sqrt{\lambda} \), this gives

\[
E \approx J + S + \frac{\lambda S}{2J^2} + \sum_{n=-\infty}^{\infty} N_n^T \sqrt{1 + \frac{\lambda n^2}{J^2}}.
\]

This agrees with eq. (2.39), up to the contribution of extra oscillators. This quantum part of the spectrum can be captured by a semiclassical quantization around the soliton \[6,30\].

Given the correspondence between physical string states and gauge theory operators, there must be operators in the dual gauge theory with dimension given by

\[
\Delta \approx J + S + \frac{\lambda S}{2J^2} + \ldots.
\]

Denoting by \( \phi^1, ..., \phi^6 \) the six scalars of \( \mathcal{N} = 4 \) Yang-Mills theory and \( Z = \phi^5 + i\phi^6 \), the unique single trace operator of bare dimension \( \Delta = J \) is given by \[4\] \( \text{tr}[Z^J] \). The spin \( S \) can be introduced by adding covariant derivatives to this operator, i.e. replacing \( S \) factors \( Z \) by \( D_i Z = \partial_i Z + [A_i, Z] \), \( i = 1, 2 \), adding phase factors of the form \( \exp(2\pi i l/J) \), and summing over all possible insertion points (\( l \) is the position of \( D_i \) along the string of \( Z \)'s).

We propose that the resulting operator:

\[
O_{i_1...i_S} = \sum_{l_1,...,l_s=1}^{J} \frac{1}{\sqrt{J^{N(J/2)}}} \text{tr}[...ZD_{i_1}Z...ZD_{i_S}Z...]e^{2\pi i(l_1+...+l_s)} , \quad S \ll J , \quad (3.4)
\]

should be identified with the soliton of \[30\] (which is fixed at \( \theta = \frac{\pi}{2} \)). One of the sums can be performed by using the cyclic property of the trace. For \( S \ll J \), this operator is almost BPS. The bare dimension is \( \Delta = J + S \). The correction \( \frac{\lambda S}{2J^2} \) arises by a one-loop calculation, by considering a subset of Feynman diagrams, as follows. Each insertion of \( D_i \) (corresponding to \( a_{1+}^\dagger \) acting on the light-cone vacuum) is treated similarly as the
insertions of $\phi^r$ operators computed in $[4]$. They give the contribution to the anomalous dimension

$$(\Delta - J)_1 = 1 + \frac{\lambda}{2J^2},$$

so that, for $S$ insertions, we get $\Delta - J = S + \frac{\lambda S}{2J^2}$, in agreement with eq. (3.3). This gives evidence that these operators are not decoupled in the large $\lambda$ limit, with $\lambda/J^2$ fixed.

Let us now consider the string of section 2, which is stretched from $\theta = 0$ up to $\theta = \theta_{\text{max}}$. Consider the case $S = 0$ and $J \gg \sqrt{\lambda}$. The correction to the energy by a shift $\frac{2\sqrt{\lambda}}{\pi}$ in (2.38) has a simple interpretation. For a string of large $J$ which is not stretched in the $\theta$ direction, its energy is $E = J$. When this string is stretched from $\theta = 0$ up to $\theta = \frac{\pi}{2}$, its energy must increase in a quantity approximately given by $E \approx J + \text{tension} \times \text{length}$. Indeed, using (2.10), the energy (2.14) can also be written as

$$E = \frac{2\sqrt{\lambda}}{\pi \sin \alpha_0} \int_{0}^{\theta_{\text{max}}} d\theta \frac{1}{\sqrt{1 - a \sin^2 \theta}} \approx J + \frac{\sqrt{\lambda}}{2\pi} \int_{0}^{\frac{\pi}{2}} d\theta \cos \theta,$$

where we have used (2.16) and $a \sim 1$. Thus we recover eq. (2.38) for $S = 0$, $E \approx J + \frac{2\sqrt{\lambda}}{\pi}$.

The possible decay of the string of section 2 into BPS states should be suppressed by powers of the string coupling $g_s$. From the gauge theory point of view, such string should correspond to a highly excited local operator. This operator should be a linear combination of operators containing $S$ insertions of $D_i$ into $\text{tr}[Z^J]$ – to account for the spin as above – and, in addition, insertions of various $\phi^r$, $r = 1, 2, 3, 4$. It remains an interesting question to identify the corresponding operator.$^3$

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$^2$ In the pp wave background there is a symmetry under the exchange of the directions 1234 and 5678, which in the gauge theory corresponds to the exchanges of $D_i$ with $\phi^r$.

$^3$ String states with $E - J \sim \sqrt{\lambda}$ can be explicitly constructed in the pp-wave background in terms of coherent states of length proportional to $R = \lambda^{1/4}$. 

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